5D Lovelock gravity: new exact solutions with torsion

B. Cvetković and D. Simić
Institute of Physics, University of Belgrade
Pregrevica 118, 11080 Belgrade, Serbia
October 27, 2016

Abstract

Five-dimensional Lovelock gravity is investigated in the first order formalism. A new class of exact solutions is constructed: the BTZ black rings with and without torsion. We show that our solution with torsion exists in a different sector of the Lovelock gravity, as compared to the Lovelock Chern-Simons sector or the one investigated by Canfora et al. The conserved charges of the solutions are found using Nester’s formula, and the results are confirmed by the canonical method. We show that the theory linearized around the background with torsion possesses two additional degrees of freedom with respect to general relativity.

1 Introduction

General theory of relativity introduced a revolution in our understanding of space-time and gravity, the influence of which on modern physics can hardly be emphasized enough—almost all present investigations in high-energy physics are, in certain way, related to it. On one hand, General theory of relativity has been very successful in explaining experimental results, but on the other, it produced a lot of problems for physicists to solve. The first of them is the problem of singularities, appearing quite often in gravitational solutions; there are theorems which show that singularities must appear under certain physically reasonable assumptions [1]. This situation inspired research in the direction of alternative theories of gravity, with an idea of finding a singularity free theory that reproduces experimental results equally well as general relativity.

The second problem is quantization of general theory of relativity. Inability to quantize general relativity in a standard way, like Yang-Mills theories, motivated physicists to search for alternatives, on one side for different quantization procedure (LQG) and on the other for modifications of the original theory (extra dimensions, SUSY, string theory, alternative theories of gravity) [2][3][4][5]. In this paper we shall focus on an alternative theory of gravity with one extra dimension – Lovelock gravity in five dimensions (5D).

Lovelock gravity is one of many generalizations of general relativity, physically appealing because of its similarity to the former. It possesses equations of motion which are the second
order differential equations, it is ghost free etc. But beyond this, most of its basic properties
are not well known, and as the old saying says ”Devil is in the details”. Firstly, not many
solutions are known and those constructed usually are torsionless or belong to some special
point in the parameter space [6, 7, 8, 9, 10]. Secondly, symmetries and local degrees of
freedom of the theory are not known for the generic choice of parameters, but only for the
special case of Lovelock-Chern-Simons gravity [11].

In this paper we shall introduce new solutions with(out) torsion within Lovelock gravity
in 5D by using the first order formulation. The most interesting of them are the BTZ black
rings with(out) torsion, the properties of which can be analyzed by using the canonical
formalism. The canonical analysis is a powerful tool for studying gauge theories, but it is
not limited solely to them. It gives a well-defined procedure for determining symmetries of
a theory, construction of the symmetry generators, and for counting the number of local
degrees of freedom. Applying the canonical analysis to a theory is extremely rewarding
because of the already mentioned results it gives. Note, in particular, that the most reliable
approach to conserved charges in gravity is based on the canonical analysis [12, 13]. The
main aspect of this approach consists in demanding the canonical generators to have well-
defined functional derivatives. For a given asymptotic behavior of the fields, this condition
usually requires the form of the generators to be improved by adding suitable surface terms.

The paper is organized as follows. Section 2 contains a short review of Poincaré gauge
theory of gravity and Lovelock gravity. Third section is devoted to the new solutions of 5D
Lovelock gravity – the BTZ black rings with(out) torsion. The conserved charges for these
solutions are computed by using Nester formula [14]. In section 4, we construct the canonical
generator of gauge transformations, local translations and Lorentz rotations, and compute
the canonical conserved charges for the solutions constructed in section 3, confirming the
results obtained in section 3. In section 5, we investigate canonical structure of the theory
linearized around the solution with torsion, and conclude that in this sector, the theory
exhibits additional degrees of freedom.

Our conventions are given by the following rules: the Latin indices refer to the lo-
cal Lorentz frame, the Greek indices refer to the coordinate frame; the first letters of
both alphabets (a, b, c, ..., α, β, γ, ...) run over 1, 2, ... D − 1 the middle alphabet letters
(i, j, k, ..., μ, ν, λ, ...) run over 0, 1, 2, ... D − 1; the signature of spacetime is η = (+, −, ..., −);
totally antisymmetric tensor ε_{01...D} and the related tensor density ε^μ μν...D are both nor-
malized so that ε_{01...D−1} = 1. The symbol ”∧” of exterior (wedge) product between forms
is omitted for simplicity.

2 Lovelock gravity

PGT in brief. The basic gravitational variables in PGT are the vielbein e^i and the
Lorentz connection ω^{ij} = −ω^{ji} (1-forms). The field strengths corresponding to the gauge
potentials e^i and ω^{ij} are the torsion T^i and the curvature R^{ij} (2-forms): T^i = de^i + ω^i_m ∧ e^m
R^{ij} = dω^{ij} + ω^i_m ∧ ω^mj. Gauge symmetries of the theory are local translations and local
Lorentz rotations, parametrized by ξ^μ and ε^{ij}.

In local coordinates x^μ, we can expand the vielbein and the connection 1-forms as e^i =
\( e^i_\mu dx^\mu, \omega^i = \omega^i_\mu dx^\mu. \) Gauge transformation laws have the form
\[
\delta_0 e^i_\mu = \xi^i_j e^j_\mu - (\partial_\mu \xi^\rho) e^i_\rho - \xi^\rho \partial_\rho e^i_\mu =: \delta_{\text{PGT}} e^i_\mu, \\
\delta_0 \omega^i_\mu = \nabla_\mu \xi^j - (\partial_\mu \xi^\rho) \omega^j_\rho - \xi^\rho \partial_\rho \omega^j_\mu =: \delta_{\text{PGT}} \omega^j_\mu, 
\]
and the field strengths are given as
\[
T^i = \nabla e^i = d e^i + \omega^{ij} \wedge e^j = \frac{1}{2} T^i_{\mu \nu} dx^\mu \wedge dx^\nu, \\
R^{ij} = d \omega^{ij} + \omega^{ik} \wedge \omega^k_j = \frac{1}{2} R^{ij}_{\mu \nu} dx^\mu \wedge dx^\nu,
\]
where \( \nabla = dx^\mu \nabla_\mu \) is the covariant derivative.

To clarify the geometric meaning of the above structure, we introduce the metric tensor as a specific, bilinear combination of the vielbeins
\[
g = \eta_{ij} e^i \otimes e^j = g_{\mu \nu} dx^\mu \otimes dx^\nu, \\
g_{\mu \nu} = \eta_{ij} e^i_\mu e^j_\nu, \quad \eta_{ij} = (+, -, -, -, -),
\]
Although metric and connection are in general independent dynamical/geometric variables, the antisymmetry of \( \omega^{ij} \) in PGT is equivalent to the so-called metricity condition, \( \nabla g = 0 \). The geometry whose connection is restricted by the metricity condition (metric-compatible connection) is called Riemann-Cartan geometry. Thus, PGT has the geometric structure of Riemann-Cartan space.

The connection \( \omega^{ij} \) determines the parallel transport in the local Lorentz basis. Being a true geometric operation, parallel transport is independent of the basis. This property is incorporated into PGT via the so-called vielbein postulate, the vanishing of the total covariant derivative of \( e^i_\mu \)
\[
D_\mu (\omega + \Gamma) e^i_\nu := \partial_\mu e^i_\nu + \omega^{ij}_\mu e^j_\nu - \Gamma^{\rho}_{\nu \mu} e^i_\rho = 0,
\]
where \( \Gamma^{\rho}_{\nu \mu} \) is affine connection and torsion is defined by \( T^{\rho}_{\mu \nu} = \Gamma^{\rho}_{\nu \mu} - \Gamma^{\rho}_{\mu \nu}. \) Previous relation implies the identity
\[
\omega_{ijk} = \Delta_{ijk} + K_{ijk},
\]
where \( \Delta \) is Riemannian (Levi-Civita) connection, and \( K_{ijk} = -\frac{1}{2} (T_{ijk} - T_{kij} + T_{jki}) \) is the contortion. Latin indices are changed into Greek and vice versa by means of vielbeins (and its inverse). Namely, \( X^i = e^i_\mu X_\mu \) and \( X^\mu = e^\mu_i X^i. \) For the details see [13].

**Lovelock action and equations of motion.** Lovelock gravity can also be considered in the framework of PGT. Dimensionally continued Euler density \( L_p \) in \( D \) dimensions is defined as
\[
L_p = \varepsilon^{i_1 i_2 \ldots i_D} R^{i_1 i_2} \ldots R^{i_2 \ldots i_{2p-1} i_{2p}} e^{i_2 \ldots i_{2p+1}} \ldots e^{i_1 D},
\]
where \( p \) is the number of curvature tensors in Euler density. In the previous relation we omitted the wedge product for simplicity. General form of the Lovelock gravity Lagrangian [13] in 5D is a linear combination of all dimensionally continued Euler densities in 5 dimensions:
\[
I = \frac{\alpha_0}{5} I_0 + \frac{\alpha_1}{3} I_1 + \alpha_2 I_2,
\]
(2.5a)
where:

\[ I_0 = \int \varepsilon_{ijkl} e^i e^j e^k e^l e^n, \]
\[ I_1 = \int \varepsilon_{ijkl} R^{ij} e^k e^l e^n, \]
\[ I_2 = \int \varepsilon_{ijkl} R^{ij} R^{kl} e^n. \]  \hfill (2.5b)

**Field equations.** Variation of the action with respect to vielbein \( e^i \) and connection \( \omega^{ij} \) yields the gravitational field equations:

\[ \varepsilon_{ijkl} \left( \alpha_0 e^j e^k e^l e^n + \alpha_1 R^{jk} e^l e^n + \alpha_2 R^{jk} R^{ln} e^n \right) = 0, \]  \hfill (2.6a)
\[ \varepsilon_{ijkl} \left( \alpha_1 e^k e^l e^n + 2 \alpha_2 R^{kl} e^n \right) T^n = 0. \]  \hfill (2.6b)

Let us note that in the generic case, the field equations \((2.6)\) imply that torsion can be non-vanishing.

For later convenience, let us present the tensor form of the field equations:

\[ \varepsilon_{ijkln}^{\mu\nu\rho\sigma\tau} \left( \alpha_0 e^j e^k e^l e^n \varepsilon_{\rho\sigma\tau} + \frac{1}{2} \alpha_1 R^{jk} e^l e^n \varepsilon_{\rho\sigma\tau} + \frac{1}{4} \alpha_2 R^{jk} R^{ln} e^n \varepsilon_{\rho\sigma\tau} \right) = 0, \]  \hfill (2.7a)
\[ \varepsilon_{ijkln}^{\mu\nu\rho\sigma\tau} \left( \alpha_1 e^k e^l e^n \varepsilon_{\rho\sigma\tau} + \alpha_2 R^{kl} e^n \varepsilon_{\rho\sigma\tau} \right) T^n \varepsilon_{\rho\sigma\tau} = 0, \]  \hfill (2.7b)

where \( \varepsilon_{ijkln}^{\mu\nu\rho\sigma\tau} := \varepsilon_{ijkln}^{\mu\nu\rho\sigma\tau} \varepsilon_{ijkl} \).

**Consequences of field equations.** If we take covariant derivative of the \((2.6a)\) make use of the Bianchi identities and multiply \((2.6b)\) with \( e^j \) we get the following system:

\[ \varepsilon_{ijkl} \left( 2 \alpha_0 e^j e^k e^l e^n + \alpha_1 R^{jk} e^l e^n \right) T^n = 0, \]
\[ \varepsilon_{ijkl} \left( \alpha_1 e^k e^l e^n + 2 \alpha_2 R^{kl} e^n \right) T^n = 0. \]

In the case \( 4 \alpha_0 \alpha_2 - \alpha_1^2 \neq 0 \) the previous set of equations reduces to the the following conditions:

\[ v_i := T^j_{ji} = 0, \]  \hfill (2.8a)
\[ R^{jk} \varepsilon_{ir} T^r_{jk} - 2 \varepsilon_{ij} \varepsilon^{kl} R^{jk} R^{kl} = 0, \]  \hfill (2.8b)

where \( \text{Ric}^j_k := R^{jk}_{kl} \) is Ricci tensor.

Therefore, in the generic case torsion is *traceless* and second irreducible component of torsion \( (2) T_i \) vanishes. For details on irreducible decomposition of torsion and curvature in PGT see \cite{16}. Let us note that the condition \( 4 \alpha_0 \alpha_2 - \alpha_1^2 \neq 0 \) is violated in the case of Lovelock Chern-Simons gravity.

**Maximally symmetric solution.** The field equation admit the existence of maximally symmetric Riemannian solution (maximally symmetric Riemannian background) defined by:

\[ \bar{R}^{ij} = -\Lambda e^i e^j, \quad \bar{T}^i = 0, \]  \hfill (2.9)
where $\Lambda$ is the effective cosmological constant iff:

$$\alpha_0 - \alpha_1 A + \alpha_2 A^2 = 0.$$  \hfill (2.10)

This equation can be solved for $\Lambda$:

$$\Lambda_{\pm} = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_0 \alpha_2}}{2\alpha_2}. \hfill (2.11)$$

The solution is unique for $\alpha_1^2 - 4\alpha_0 \alpha_2 = 0$, which is the case in Lovelock Chern-Simons gravity.

Let us note that in terms of $\Lambda_{\pm}$ equations of motion (2.6) take an elegant form:

$$\varepsilon_{ijkln} \left( R^{jk} + \Lambda_+ e^j e^k \right) \left( R^{ln} + \Lambda_- e^l e^n \right) = 0, \hfill (2.12a)$$

$$\varepsilon_{ijkln} \left( R^{kl} + \frac{A_+ + A_-}{2} e^k e^l \right) T^n = 0. \hfill (2.12b)$$

In obtaining these equations we assumed that $\alpha_2 \neq 0$, and this condition will be used in the rest of the paper, because for $\alpha_2 = 0$ the theory reduces to general relativity.

3 New class of solutions

The search for a new class of solutions is inspired by Canfora et al. [17], who found a solution of the type $\text{AdS}_2 \times S^3$ when the coupling constants satisfy the relation

$$\alpha_1^2 = 12\alpha_0 \alpha_2,$$  \hfill (3.1)

which is different from the one satisfied in Lovelock Chern-Simons gravity. We shall now construct another class of solutions of the ”complementary” type $\Sigma_3 \times \Gamma_2$, where $\Sigma_3$ and $\Gamma_2$ are three- and two-dimensional manifolds, determined by solving the equations of motion.

We start from the following anzatz for curvature:

$$R^{ab} = Ae^a e^b, \hfill (3.2)$$

and torsion:

$$T^a = p \varepsilon^{abc} e_b e_c, \hfill (3.3)$$

In the anzatz we used the following notation: $a, b, c, \cdots \in \{0, 1, 2\}$ and $\varepsilon^{abc} := \varepsilon^{ab34}$, and $A$, $B$ and $p$ are some functions restricted by the equations of motion. Note that torsion is totally antisymmetric and thus only third irreducible component $(^3T^i)$ is non-vanishing, see [16]. Let us now check whether anzatz solves the equations of motion [2.12]. From (2.12b) we obtain:

$$\left( B + \frac{A_+ + A_-}{2} \right) p = 0.$$
Thus, one can have a vanishing torsion for \( p = 0 \), or a non-vanishing torsion for
\[
B = -\frac{\Lambda_- + \Lambda_+}{2}.
\]

From (2.12a) we obtain:
\[
A(\Lambda_- + \Lambda_+) + 2\Lambda_- \Lambda_+ = 0,
\]
\[
4\Lambda_- \Lambda_+ + (A + B)(\Lambda_- + \Lambda_+) + 2AB = 0.
\]
If \( \Lambda_- + \Lambda_+ = 0 \), which is equivalent to \( \alpha_1 = 0 \), Eq. (3.5) implies \( \alpha_0 = 0 \) whereas \( A \) remains undetermined; otherwise, for \( \alpha_1 \neq 0 \), we have
\[
A = -\frac{2\Lambda_- \Lambda_+}{\Lambda_- + \Lambda_+}.
\]

Let us first analyze the case with non-vanishing torsion and \( \alpha_1 \neq 0 \), when \( A \) and \( B \) are both determined. By combining equations (3.4), (3.5), (3.6) and using Vieta’s formulas: \( \Lambda_- + \Lambda_+ = \frac{2\alpha_1}{\alpha_2} \) and \( \Lambda_- \Lambda_+ = \frac{\alpha_0}{\alpha_2} \) we obtain that solution exists in the sector:
\[
\alpha_1^2 = 8\alpha_0 \alpha_2.
\]
This sector is different from the one in (17) and the above solution is the first one in this sector. Using Eqs. (3.4), (3.10) and (3.8) we obtain
\[
A = \frac{B}{2}.
\]

Now we turn to the solution with vanishing torsion and \( \alpha_1 \neq 0 \). In this case \( A \) is determined and \( B \) is arbitrary, which can be used to insure the validity of (3.6), which takes the form
\[
2\frac{\alpha_0}{\alpha_2} + B \left( \frac{\alpha_1}{\alpha_2} - 4\frac{\alpha_0}{\alpha_1} \right) = 0.
\]
We see that if \( \alpha_1^2 - 4\alpha_0 \alpha_2 = 0 \), which is the Lovelock Chern-Simons gravity, for the validity of (3.10), one must have \( \alpha_0 = 0 \). These two conditions imply \( \alpha_1 = 0 \), which is in contradiction with our assumption; hence, the solution does not exist in the Lovelock-Chern-Simons case. If \( \alpha_1^2 - 4\alpha_0 \alpha_2 \neq 0 \) and \( \alpha_1 \neq 0 \) (recall that we are not interested in General relativity so \( \alpha_2 \neq 0 \) also) we can choose any value of parameters obeying this conditions and get a solution. So this class of solutions exist generically i.e. for almost any choice of parameters.

For clarity of the exposure, we devote next few sections to the most interesting solutions which belong to the class derived in this section.

### 3.1 BTZ black ring with torsion

For this case, the curvature takes the following form:
\[
R_{ab} = q e^a e^b,
\]
\[
R^3_{\alpha} = R^{4\alpha} = 0,
\]
\[
R^{34} = -\frac{1}{r_0^2} e^3 e^4,
\]
(3.11)
while the torsion is given by
\[ T^a = p \varepsilon^{abc} e_b e_c, \]
\[ T^3 = T^4 = 0. \]  
(3.12)

The Bianchi identity implies that \( p \) is constant, and the Riemannian curvature reads:
\[ \tilde{R}^{ab} = \left( q + \frac{p^2}{4} \right) e^a e^b, \]
\[ \tilde{R}^{3a} = \tilde{R}^{4a} = 0, \]
\[ \tilde{R}^{34} = -\frac{1}{r_0} e^3 e^4, \]  
(3.13)

Therefore, we can introduce the AdS\(_3\) radius \( \ell \) as
\[ \frac{1}{\ell^2} := q + \frac{p^2}{4}. \]

Identity (3.9) implies the following relation
\[ \frac{1}{\ell^2} = -\frac{1}{2r_0^2} + \frac{p^2}{4}. \]  
(3.14)

In the AdS\(_3\) sector, the anzatz for curvature and torsion is solved by the AdS\(_3\) solution with torsion, as well as by the Bañados, Teitelboim, Zanelli (BTZ) black hole \([18]\) with torsion. In the latter, physically more appealing case, the 5D vielbein reads:
\[ e^0 = N dt, \quad e^1 = N^{-1} dr, \quad e^2 = r d\varphi + N\varphi dt, \]
\[ e^3 = r_0 d\theta, \quad e^4 = r_0 \sin \theta d\chi, \]  
(3.15a)

where
\[ N^2 = -2m + \frac{r^2}{\ell^2} + \frac{j^2}{r^2}, \quad N\varphi = \frac{j}{r^2}, \]
where \( m \) and \( j \) are (dimensionless) parameters. The Cartan connection is given by:
\[ \omega^{ab} = \tilde{\omega}^{ab} - \varepsilon^{abc} p e_c, \]
\[ \tilde{\omega}^{01} = -\frac{r}{\ell^2} dt - \frac{j}{r} d\varphi, \quad \tilde{\omega}^{12} = N d\varphi, \]
\[ \tilde{\omega}^{20} = N^{-1} \frac{j}{r^2} dr, \]
\[ \tilde{\omega}^{34} = \tilde{\omega}^{34} = -\cos \theta d\chi, \]  
(3.15b)

where \( \tilde{\omega}^{ij} \) is Riemannian connection. Let us note that the coordinate ranges are:
\[ -\infty < t < +\infty, \quad 0 \leq r < +\infty, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \chi \leq 2\pi. \]

**Killing vectors.** The maximal number of Killing vectors of the solution with field strengths (3.11), (3.12) and (3.13), is 9 = 6 + 3, since the AdS\(_3\) solution with(out) torsion has 6 Killing
vectors, see [19]. The solution (3.15) has five Killing vectors, since the BTZ solution possesses 2 Killing vectors. They are given by:

\[ \xi^{(1)} = \ell \frac{\partial}{\partial t}, \quad \xi^{(2)} = \frac{\partial}{\partial \varphi}, \quad \xi^{(3)} = \frac{\partial}{\partial \chi}, \]
\[ \xi^{(4)} = \sin \chi \frac{\partial}{\partial \theta} + \cot \theta \cos \chi \frac{\partial}{\partial \chi}, \quad \xi^{(5)} = \cos \chi \frac{\partial}{\partial \theta} - \cot \theta \sin \chi \frac{\partial}{\partial \chi}. \] (3.16)

### 3.2 Riemannian BTZ ring

For this case, the curvature (Riemannian) takes the following form:

\[ R^{ab} = \frac{1}{\ell^2} \epsilon^a b^b, \]
\[ R^{3a} = R^{4a} = 0, \]
\[ R^{34} = -\frac{1}{r_0^2} \epsilon^3 \epsilon^4, \] (3.17)

while the torsion equals zero, \( T^i = 0 \).

Let us note that since torsion is zero, there are no further constraints on \( B \), so that we can chose \( B = -\frac{1}{r_0^2} \). In terms of the action constants we get:

\[ \frac{1}{\ell^2} = \frac{2\alpha_0}{\alpha_1}, \quad \frac{1}{r_0^2} = \frac{2\alpha_0 \alpha_1}{\alpha_1^2 - 4\alpha_0 \alpha_2} \] (3.18)

The solution exists provided that \( \alpha_0 \alpha_1 < 0 \) and \( \alpha_1^2 - 4\alpha_0 \alpha_2 < 0 \). Let us note this solution does not solve equations of motion in Lovelock Chern-Simons gravity.

The vielbein fields and connection take the same form as in (3.15) with \( p = 0 \), while Killing vectors are identical and given by (3.16).

### 3.3 Conserved charges

In order to compute conserved charges, we shall make use of Nester formula. Let us denote the difference between any variable \( X \) and its reference value \( \bar{X} \) by \( \Delta X = X - \bar{X} \). In 5D, the boundary term \( B \) is a 3-form. With a suitable set of boundary conditions for the fields, the proper boundary term reads [14]:

\[ B = (\xi \wedge b^i) \Delta \tau_i + \Delta b^i (\xi \wedge \bar{\tau}_i) + \frac{1}{2} (\xi \wedge \omega^i_j) \Delta \rho_{ij} + \frac{1}{2} \Delta \omega^i_j (\xi \wedge \bar{\rho}_i^j), \] (3.19)

where \( \xi \) is an asymptotically Killing vector, while \( \tau_i \) and \( \rho_{ij} \) are covariant momenta corresponding to torsion and curvature, respectively. The covariant momenta for the Lovelock action (2.5) are given by:

\[ \tau_i := \frac{\partial L}{\partial T^i} = 0, \] (3.20)
\[ \rho_{ij} = \frac{\partial L}{\partial R^{ij}} = 2\varepsilon_{ijklm} \left( \frac{\alpha_1}{3} \epsilon^k \epsilon^l + 2\alpha_2 R^{kl} \right) \epsilon^m. \] (3.21)
Consequently, we obtain:

\[
\rho_{ab} = 4\varepsilon_{abc} \left( \alpha_1 - \frac{2\alpha_2}{r_0^2} \right) e^c e^d e^e,
\]
\[
\rho_{a3} = 2\varepsilon_{abc} (\alpha_1 + 2\alpha_2 r_0^2) e^b e^c e^e = \alpha_1 \varepsilon_{abc} e^b e^c e^e,
\]
\[
\rho_{a4} = 2\varepsilon_{abc} (\alpha_1 + 2\alpha_2 q) e^b e^c e^e = \alpha_1 \varepsilon_{abc} e^b e^c e^e,
\]
\[
\rho_{34} = 2\varepsilon_{abc} \left( \frac{\alpha_1}{3} + 2\alpha_2 q \right) e^a e^b e^c e^e = -\frac{\alpha_1}{3} e^a e^b e^c. \tag{3.22}
\]

In our calculations of the boundary integrals, we use the coordinates \( x^\mu = (t, r, \varphi, \theta, \chi) \).

The background configuration is the one defined by zero values of the solution parameters \( m = 0 \) and \( j = 0 \). For the solutions with Killing vectors \( \partial_t \) and \( \partial_\varphi \), the conserved charges are energy and angular momentum, respectively:

\[
E = \int_{\partial\Sigma} B(\partial_t) = \int_{\partial\Sigma} e^t \Delta r_1 t_1 + \Delta e^t r_1 t_1 + \frac{1}{2} \omega^{ij} \Delta \rho_{ij} \Delta \rho_{ij}, \tag{3.23a}
\]
\[
J = \int_{\partial\Sigma} B(\partial_\varphi) = \int_{\partial\Sigma} e^\varphi \Delta r_1 \varphi_1 + \Delta e^\varphi r_1 \varphi_1 + \frac{1}{2} \omega^{ij} \Delta \rho_{ij} \Delta \rho_{ij}, \tag{3.23b}
\]

where \( \partial\Sigma \) is a boundary \( S^1 \times S^2 \), located at infinity, described by coordinates \( \varphi, \theta, \chi \).

Thus, conserved charges for for the black ring with torsion and the Riemannian black ring are given by:

\[
E = 8\pi^2 r_0^2 \left( \alpha_1 - \frac{2\alpha_2}{r_0^2} \right) m, \quad J = 8\pi^2 r_0^2 \left( \alpha_1 - \frac{2\alpha_2}{r_0^2} \right) j. \tag{3.24}
\]

Let us note that the solution with torsion exists in the sector \( \alpha_1^2 = 8\alpha_0 \alpha_2 \), where both conserved charges vanish.

### 4 Canonical gauge generator

As an important step in our examination of the asymptotic structure of spacetime, we are going to construct the canonical gauge generator, which is our basic tool for studying asymptotic symmetries and conserved charges of 5D Lovelock gravity.

#### 4.1 Hamiltonian and constraints

The best way to understand the dynamical content of gauge symmetries is to explore the canonical generator, which acts on the basic dynamical variables via the Poisson bracket (PB) operation. To begin the canonical analysis, we rewrite the action (2.5) as

\[
I = \int d^5 x \mathcal{L}
\]
\[
\mathcal{L} = \varepsilon_{ijklm} \int_M d^4 x \left( \frac{\alpha_0}{5} e^i \epsilon^j e^k \epsilon^l e^m + \frac{\alpha_1}{6} R_{ijkl} e^k e^l e^m e^n + \frac{\alpha_2}{4} R_{ijkl} R_{kln} + \frac{\alpha_3}{4} R_{ijkl} R_{kln} \right) e^n. \tag{4.1}
\]
1. The basic Lagrangian variables \((e^i_\mu, \omega^{ij}_\mu)\) and the corresponding canonical momenta \((\pi^i_\mu, \pi^{ij}_\mu)\) are related to each other through the set of primary constraints:

\[
\begin{align*}
\phi^i_0 := \pi^i_0 &\approx 0, \\
\phi^{ij}_0 := \pi^{ij}_0 &\approx 0, \\
\phi^i_\alpha := \pi^i_\alpha &\approx 0, \\
\phi^{ij}_\alpha := \pi^{ij}_\alpha &\approx 0, \\
\end{align*}
\]

Going over to the total Hamiltonian,

\[
\begin{align*}
\mathcal{H}_c &= \mathcal{H}_i + \frac{1}{2} \omega^{ij}_0 \mathcal{H}_{ij} + \partial_\alpha D^\alpha, \\
\mathcal{H}_i &= -\varepsilon_{ijklm} \left( \alpha_0 e^i_\alpha e^j_\beta e^k_\gamma e^l_\delta + \frac{1}{2} \alpha_1 R_{ijk}^{\alpha \beta \gamma \delta} e^l_\gamma e^m_\delta + \frac{1}{4} \alpha_2 R_{ijklm}^{\alpha \beta \gamma \delta} \right), \\
\mathcal{H}_{ij} &= -\varepsilon_{ijklm} \left( \alpha_1 e^i_\beta e^j_\gamma + \alpha_2 R_{ij}^{\beta \gamma} \right) T^m_\delta, \\
D^\alpha &= \varepsilon_{ijklm} \omega^{ij}_0 \left( \alpha_1 e^k_\beta e^l_\gamma + \alpha_2 R_{kl}^{\beta \gamma} \right) e^n_\delta.
\end{align*}
\]

2. Going over to the total Hamiltonian,

\[
\mathcal{H}_T = \mathcal{H}_c + u^i_\mu \phi^i_\mu + \frac{1}{2} u^{ij}_\mu \phi^{ij}_\mu,
\]

we find that the consistency conditions of the primary constraints \(\pi^i_0\) and \(\pi^{ij}_0\) yield the secondary constraints:

\[
\mathcal{H}_i \approx 0, \quad \mathcal{H}_{ij} \approx 0.
\]

Let us note that these constraints reduce to the \(\mu = 0\) components of the Lagrangian field equations (2.7).

The consistency of the remaining primary constraints \(\phi^i_\alpha\) and \(\phi^{ij}_\alpha\) leads to the relations for multipliers \(u^i_\beta\) and \(u^{ij}_\beta\):

\[
\begin{align*}
\varepsilon_{ijklm} \left[ R_{ijk}^{\alpha \beta \gamma \delta} \left( \alpha_1 e^l_\gamma e^n_\delta + \alpha_2 R^{m \gamma}_n \right) + \left( \alpha_1 R_{ijk}^{\alpha \beta \gamma} + 4 \alpha_0 e^i_\beta e^j_\gamma \right) e^k_\delta e^n_\delta \right] &= 0, \\
\varepsilon_{ijklm} \left[ \tilde{T}^k_0 \left( \alpha_1 e^l_\gamma e^n_\delta + \alpha_2 R^{m \gamma}_n \right) + \alpha_2 \tilde{T}^{kl}_0 \right] T^n_\delta + \alpha_1 e^k_\beta e^l_\gamma T^n_\delta &= 0,
\end{align*}
\]

where \(\tilde{T}^i_0 = T^i_0 (\dot{e}^i_\alpha \rightarrow u^i_\alpha)\) and \(\tilde{R}^{ij}_0 = R^{ij}_0 (\dot{\omega}^{ij}_\alpha \rightarrow u^{ij}_\alpha)\). Using the Hamiltonian equations of motion \(\dot{e}^i_\alpha = u^i_\beta\) and \(\dot{\omega}^{ij}_\alpha = u^{ij}_\alpha\) these relations reduce to the \(\mu = \alpha\) components of the Lagrangian field equations (2.7).

\textbf{Further consistency procedure.} Some of the relations (4.6) can be solved in terms of the multipliers \(u^i_\alpha\) and \(u^{ij}_\alpha\), while the others may lead to ternary constraints, whose consistency has to be examined as well. However, this procedure is extremely sensitive to the particular sector of the theory as we shall illustrate in the next section (for the pure Lovelock theory.
see \cite{20}). The final form of the total Hamiltonian is given by:

\[ H_T = \bar{H}_T + u^0 \pi_i^0 + \frac{1}{2} u^{ij} \pi_{ij}^0 + (u \cdot \phi), \]

\[ \bar{H}_T = \epsilon_i^0 \bar{H}_i + \frac{1}{2} \omega^{ij}_0 \bar{H}_{ij} + \partial_\alpha \bar{D}^\alpha, \]

\[ \bar{H}_i = \bar{H}_i + (\bar{u} \cdot \phi), \]

\[ \bar{H}_{ij} = \bar{H}_{ij} + (\bar{u} \cdot \phi), \]

\[ \bar{D}^\alpha = D^\alpha + (\bar{u} \cdot \phi), \tag{4.7} \]

where by \((u \cdot \phi)\) we denoted terms stemming from the undetermined multipliers and belonging to the set \((u^i, u^{ij})\) and by \((\bar{u} \cdot \phi)\) we denoted terms stemming from the determined multipliers belonging to the same set.

### 4.2 Canonical generator and charges

The sure symmetries of the theory are local translations and local Lorentz rotations. The general form of the canonical generator of the local Poincaré transformations constructed by the Castellani procedure \cite{21} is given by:

\[ G = -G_1 - G_2, \]

\[ G_1 = \xi^\rho \left( \epsilon^i_\rho \pi_i^0 + \frac{1}{2} \omega^{ij}_\rho \pi_{ij}^0 \right) + \xi^\rho \left( \epsilon_i^\rho \bar{H}_i + \frac{1}{2} \omega^{ij}_\rho \bar{H}_{ij} + C_{PFC} \right), \]

\[ G_2 = \frac{1}{2} \varepsilon^{ij} \pi_{ij}^0 + \frac{1}{2} \varepsilon^{ij} (\bar{H}_{ij} + C_{PFC}), \]

where \(C_{PFC}\) are terms proportional to sure primary first class constraints \((\pi_i^0, \pi_{ij}^0)\).

The canonical generator acts on dynamical variables via the PB operation, hence, it should have well-defined functional derivatives. In order to ensure this property, we have to improve the form of \(G\) by adding a suitable surface term \(\Gamma\), such that \(\tilde{G} = G + \Gamma\) is a well-defined canonical generator. In this process, the asymptotic conditions play a crucial role, see for instance \cite{22, 23}. Though we didn’t construct the exact form of the canonical generator it still allows us to compute canonical charges for the solutions found in section 3. Namely, if we adopt the general principle that the quantities that vanish on shell have an arbitrary fast asymptotic decrease, we obtain that the on shell variation of the generator takes the following form:

\[ \delta \bar{G}(\xi^t = \ell, \xi^\phi = 1) \approx \delta \Gamma = -\ell \delta E_c - \delta J_c, \tag{4.8} \]

where

\[ E_c = 8 \pi^n r_0^2 \left( \alpha_1 - \frac{2 \alpha_2}{r_0^2} \right) m, \quad J_c = 8 \pi^n r_0^2 \left( \alpha_1 - \frac{2 \alpha_2}{r_0^2} \right) j, \tag{4.9} \]

are the canonical conserved charges, which are identical to the expressions \cite{3:24}, obtained from the Nester formula.
5 Linearized theory

The canonical structure of the full non-linear theory crucially depends on the relations (1.6), as we already mentioned in the previous section. In order to get a deeper insight into the structure of the Lovelock gravity in the sector \( \alpha_1^2 = 8 \alpha_0 \alpha_2 \) we shall consider the theory linearized around around the BTZ black ring with torsion (3.15). The linearization is based on the expansion of the basic dynamical variables \((e^i_\mu, \omega^{ij}_\mu)\) and the related conjugate momenta \((\pi_i^\mu, \pi_{ij}^\mu)\) denoted shortly by \( Q_A \):

\[
Q_A = \tilde{Q}_A + \hat{Q}_A,
\]

where \(\tilde{Q}_A\) refers to the background (solution (3.15) with \( m = j = 0 \) and \( p \neq 0 \)), while \(\hat{Q}_A\) denotes small excitations.

From the linearized form of the 60 relations (4.6) we conclude that out of \( 60 = 5 \times 4 + 10 \times 4 \) multipliers \((\tilde{u}^i_\alpha, \tilde{u}^{ij}_\alpha)\), 46 are determined, while among 14 remaining relations there are 12 new constraints (since two pairs of them are identical), whose explicit form is given by:

\[
\begin{align*}
\alpha_1 \tilde{R}^{24}_r + \alpha_1 \sin \theta \tilde{R}^{23}_r + 4 \alpha_0 r_0 \sin \theta \tilde{e}^2_r & \approx 0, \quad (5.2a) \\
\alpha_1 \tilde{R}^{14}_r + \alpha_1 \sin \theta \tilde{R}^{13}_r + 4 \alpha_0 r_0 \sin \theta \tilde{e}^2_r & \approx 0, \quad (5.2b) \\
r^2 \left( \alpha_1 \tilde{R}^{14}_r + \alpha_1 \sin \theta \tilde{R}^{13}_r + 2 \alpha_0 r_0 \sin \theta \tilde{e}^1_r \right) & - \alpha_1 \tilde{R}^{24}_r - \alpha_1 \sin \theta \tilde{R}^{23}_r - 2 \alpha_0 r_0 \sin \theta \tilde{e}^2_\varphi & \approx 0, \quad (5.2c)
\end{align*}
\]

and

\[
\begin{align*}
\tilde{T}^{4}_r + \sin \theta \tilde{T}^{3}_r & \approx 0, \quad (5.3a) \\
p \left( \alpha_1 r_0 \left( e^4_\chi + \sin \theta \tilde{e}^3_\theta \right) + 2 \alpha_2 \tilde{R}^{34}_\theta \right) & \approx 0, \quad (5.3b) \\
\tilde{T}^{4}_r + \sin \theta \tilde{T}^{3}_r & \approx 0, \quad (5.3c) \\
\alpha_1 \frac{r}{\ell} r_0 \sin \theta \tilde{T}^{2}_r - 2 p \left( \alpha_1 r_0 \sin \theta \tilde{e}^0_\theta + 2 \alpha_2 \tilde{R}^{04}_\theta \right) & \approx 0, \quad (5.3d) \\
\alpha_1 \frac{r}{\ell} \tilde{T}^{2}_r - 2 p \left( \alpha_1 r_0 \sin \theta \tilde{e}^0_\chi - 2 \alpha_2 \tilde{R}^{03}_\theta \right) & \approx 0, \quad (5.3e) \\
\alpha_1 r_0 \tilde{T}^{1}_r + 2 p r \left( \alpha_1 r_0 \tilde{e}^0_\chi - 2 \alpha_2 \tilde{R}^{03}_\theta \right) & \approx 0, \quad (5.3f) \\
\tilde{R}^{03}_{rr} & \approx 0, \quad (5.3g) \\
\tilde{R}^{02}_\theta \chi & \approx 0, \quad (5.3h) \\
\tilde{R}^{01}_\theta \chi & \approx 0. \quad (5.3i)
\end{align*}
\]

Let us denote 12 constraints (5.2a) and (5.3a) by \( \tilde{\psi}_A \). The consistency conditions of \( \tilde{\psi}_A \) leads to determination of 12 additional multipliers thus finishing the consistency procedure. Thus, out of 60 multipliers \((\tilde{u}^i_\alpha, \tilde{u}^{ij}_\alpha)\), 58 are determined, while 2 remain undetermined. By using the PB algebra from appendix A we find:

\[
\begin{align*}
\{ \tilde{\phi}_{12}^r, \tilde{\phi}_A^\alpha \} & \approx 0, \quad \{ \tilde{\phi}_{12}^r, \tilde{\phi}_{ij}^\alpha \} \approx 0, \\
\{ \tilde{\phi}_{12}^r, \tilde{\phi}_{ij}^\alpha \} & \approx 0, \quad \{ \tilde{\phi}_{12}^r, \tilde{\phi}_{ij}^\alpha \} \approx 0, \\
\{ \tilde{\phi}_{12}^r, \tilde{\phi}_{ij}^\alpha \} & \approx 0, \quad \{ \tilde{\phi}_{12}^r, \tilde{\phi}_{ij}^\alpha \} \approx 0, \\
\{ \tilde{\phi}_{12}^r, \tilde{\psi}_A^\alpha \} & \approx 0.
\end{align*}
\]
The undetermined multipliers correspond to the constraints $\tilde{\phi}_{12}^r$ and $\tilde{\phi}_{12}^\phi$ which are FC. The final classification of constraints is given by:

| Primary        | First class                  | Second class                  |
|----------------|------------------------------|-------------------------------|
|                | $\tilde{\phi}_i^0, \tilde{\phi}_{ij}^0, \tilde{\phi}_{12}^r, \tilde{\phi}_{12}^\phi$ | $\tilde{\phi}_i^\alpha, \tilde{\phi}_{ij}^\alpha$ $ij \neq 12 \land \alpha \neq r, \phi$ |
| Secondary      | $\tilde{\mathcal{H}}_i, \tilde{\mathcal{H}}_{ij}$                  | $\tilde{\psi}_A$             |

Table 1. Classification of constraints

In total there are $N_1 = 32$ FC constraints and $N_2 = 70$ SC constraints. The number of propagating degrees of freedom in phase space is:

$$N^* = 2N - 2N_1 - N_2 = 150 - 64 - 70 = 14.$$  

In the configuration space there are 7 degrees of freedom, 5 of them correspond to GR in $D = 5$ and two are additional degrees of freedom. The presence of two primary FC constraints $\tilde{\phi}_{12}^r, \tilde{\phi}_{12}^\phi$ implies that there is an additional gauge symmetry in the theory, as a consequence of the fact that variables $\tilde{\omega}_{12}^r$ and $\tilde{\omega}_{12}^\phi$ do not appear in the linearized equations of motion.

6 Conclusion

In this paper we found a new class of solutions of Lovelock gravity in 5D, in the first order formalism. The most interesting solutions are the BTZ black rings with(out) torsion. It is shown that the solution with torsion exists provided that the parameters of the theory satisfy the relation $\alpha_1^2 = 8\alpha_0\alpha_2$. This sector of the parameter space is different from the one of Lovelock Chern-Simons gravity, as well as from the sector investigated by Canfora et al [17]. Restricting our attention to the basic properties of the solutions, we calculated the values of conserved charges by using Nester's formula and the canonical method. The canonical structure of the theory linearized around the background with torsion shows that there are two additional degrees of freedom, compared to general relativity.

Acknowledgements

We thank Milutin Blagojević for useful remarks and suggestions. This work was supported by the Serbian Science foundation, Serbia, grant 171031.

A The algebra of constraints

The structure of the PB algebra of constraints is an important ingredient in the analysis of the Hamiltonian consistency conditions. Starting from the fundamental PB $\{e^i_\mu, \pi^j_\nu\} = \delta^i_j \delta^\mu_\nu (x - x')$ and $\{\omega^i_\mu, \pi_{kl}^\nu\} = 2\delta^i_l \delta^\mu_\nu \delta(x - x')$ we find PB between primary constraints:

$$\{\phi_i^\alpha, \phi_j^\beta\} = -2\varepsilon^{i00\alpha\gamma\delta}_{ijklm} (\alpha_1 e^l_1 e^n_\delta + \alpha_2 R^l_\gamma\delta n) \delta,$$

$$\{\phi_{ij}^\alpha, \phi_{kl}^\beta\} = -8\alpha_2 \varepsilon^{i0\alpha\beta\gamma\delta}_{ijklm} T^n_\gamma\delta \delta. \quad \text{(A.4)}$$
References

[1] S.W. Hawking and G.F.R. Ellis, The large scale structure of space-time Cambridge monographs on mathematical physics, Cambridge University Press, Cambridge, 1973.

[2] J. Zanelli, Lecture notes on Chern-Simons (Super-)Gravities, [hep-th/0502193].

[3] C. Csaki, TASI Lectures on Extra Dimensions and Branes [hep-ph/0404096].

[4] C. Rovelli and F. Vidotto, Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2014.

[5] K. Becker, M. Becker and J.H. Schwarz, String theory and M-theory: A modern introduction, Cambridge University Press, 2006.

[6] S. Ohashi and M. Nozawa, Lovelock black holes with non-constant curvature horizon, Phys. Rev. D 92 (2015) 064020, arXiv:1507.04496 [hep-th].

[7] G. Dotti, J. Oliva and R. Troncoso, Static solutions with nontrivial boundaries for the Einstein-Gauss-Bonnet theory in vacuum, Phys. Rev. D 82(2010) 024002, arXiv: 1004.5287 [hep-th].

[8] N. Dadhich and J. M Pons, Static pure Lovelock black hole solutions with horizon topology $S^n \times S^n$, JHEP 1505 (2015) 067, arXiv:1503.00974 [gr-qc].

[9] Rong-Gen Caia, Li-Ming Caob and N. Ohta, Black Holes without Mass and Entropy in Lovelock Gravity, Phys. Rev. D 81(2010)024018, arXiv: 0911.0245 [hep-th].

[10] S. Ray, Birkhoffs theorem in Lovelock gravity for general base manifolds, Class. Quant. Grav. 32(2015) 195022, arXiv:1505.03830 [gr-qc].

[11] M. Banados, L.J. Garay and M. Henneaux, The Dynamical structure of higher dimensional Chern-Simons theory, Nucl. Phys. B 476 (1996) 611-635. [hep-th/9605159].

[12] T. Regge and C. Teitelboim, Role of Surface Integrals in the Hamiltonian Formulation of General Relativity, Annals of Physics 88 (1974) 286.

[13] M. Blagojević, Gravitation and gauge symmetries (Institute of Physics, Bristol, 2002).

[14] J. M. Nester, A covariant Hamiltonian for gravity theories, Mod. Phys. Lett. A 6 (1991) 2655.

C.-M. Chen, J. M. Nester and R.-S. Tung, Gravitational energy for GR and Poincare gauge theories: A covariant Hamiltonian approach, Int. J. Mod. Phys. D 24 (2015) 1530026

[15] D. Lovelock, The Einstein tensor and its generalizations, J. Math. Phys.12, 498 (1971).

[16] Yu. N. Obukhov, Poincare gauge gravity: Selected topics, Int. J. Geom. Meth. Mod. Phys. 3 (2006) 95.
[17] F. Canfora, A. Giacomini and S. Wilinson, Some exact solutions with torsion in 5-D Einstein-Gauss-Bonnet gravity, \texttt{arXiv:0706.2891[gr-qc]}

[18] M. Banados, C. Teitelboim and J. Zanelli, The Black hole in three-dimensional space-time, Phys. Rev. Lett. \textbf{69} (1992) 1849-1851, \texttt{[hep-th/9204099]}

[19] M. Blagojevic and B. Cvetkovic, Canonical structure of 3D gravity with torsion \texttt{[gr-qc/0412134]}

[20] N. Dadhich, R. Durka, N. Merino and O. Miskovic, Dynamical structure of Pure Lovelock gravity, Phys. Rev. D \textbf{93} (2016) 064009, \texttt{arXiv: 1511.02541 [hep-th]}

[21] L. Castellani, Symmetries of constrained Hamiltonian systems, Ann. Phys. (N.Y.) \textbf{143} (1982) 357.

[22] M. Blagojević and B. Cvetković, Conserved charges in 3D gravity, Phys. Rev. D \textbf{81} (2010) 124024.

[23] M. Blagojevic and B. Cvetkovic, Black hole entropy in 3D gravity with torsion, Class. Quant. Grav. \textbf{23} (2006) 4781, \texttt{[gr-qc/0601006]}