Darondeau–Pragacz formulas in complex cobordism

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Abstract
In this paper, we generalize the push-forward (Gysin) formulas for flag bundles in ordinary cohomology theory, which are due to Darondeau–Pragacz, to the complex cobordism theory. Then, we introduce the universal quadratic Schur functions, which are a generalization of the (ordinary) quadratic Schur functions introduced by Darondeau–Pragacz, and establish some Gysin formulas for the universal quadratic Schur functions as an application of our Gysin formulas.

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1 Introduction

In [8], Darondeau and Pragacz established push-forward (Gysin) formulas for flag bundles in ordinary cohomology theory, which are due to Darondeau–Pragacz, to the complex cobordism theory. Then, we introduce the universal quadratic Schur functions, which are a generalization of the (ordinary) quadratic Schur functions introduced by Darondeau–Pragacz, and establish some Gysin formulas for the universal quadratic Schur functions as an application of our Gysin formulas.

¹ D–P formulas for short.

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the ordinary Schur functions, and established some Gysin formulas involving these
functions. The purpose of the present paper is twofold: first, we generalize the D–
P formulas in the ordinary cohomology theory to the complex cobordism theory.
Since the complex cobordism theory is universal among complex-oriented (in the
sense of Adams [1]) generalized cohomology theories (see Quillen [23]), our formulas
readily yield the Gysin formulas for flag bundles in other complex-oriented generalized
cohomology theories under the specialization of the universal formal group law. For
instance, one can obtain the Gysin formulas for flag bundles in the complex K-theory.
Second, we generalize the quadratic Schur functions to the complex cobordism theory.
When performing this generalization, we utilize the Gysin formulas as a useful tool to
obtain the characterization of the quadratic Schur functions and their generalization.
To accomplish our first purpose, we need to formulate the following two items:
– Segre classes of a complex vector bundle in complex cobordism,
– Gysin formula for a projective bundle in complex cobordism.

Recently Hudson–Matsumura [14] introduced the Segre classes of a complex vector
bundle in the algebraic cobordism theory, and their definition applies to the complex
cobordism theory as well (see Sect. 3.1.1). As for the latter, Quillen [23] described
the Gysin homomorphism of a projective bundle using the residue symbol (see Sect.
3.1.2). Having prepared these two notions, we can establish the Gysin formulas for full
flag bundles along the same lines as in [8]. To derive the formulas for general partial
flag bundles from those of full flag bundles, we utilize the same idea as described in
Damon [7]. In this process, we noticed that the universal Schur functions (corresponding
to the empty partition) were needed. These functions are the complex cobordism
analogues of the ordinary Schur polynomials, and were first introduced by Fel’dman
[10] (see also Nakagawa–Naruse [18]). Using these functions, we can generalize the
D–P formulas in the ordinary cohomology theory to the complex cobordism theory.
Our main results are Theorems 3.3, 3.6, and 3.8. Furthermore, it turned out that most
of the results established in [8] can be generalized to the complex cobordism setting. In
fact, a characterization of the quadratic Schur functions via Gysin formulas immedi-
ately leads to the definition of the universal quadratic Schur functions (see Definition
4.2). Then, as applications of our Gysin formulas in complex cobordism, we give a
certain Gysin formula (Proposition 4.4), which is a complex cobordism analogue of
the Pragacz–Ratajski formula [22]. The generating function for the universal quadratic
Schur functions is also obtained (Theorem 4.6), which immediately yields a determin-
antal formula for the K-theoretic quadratic Schur functions under the specialization
(Theorem 4.7).

1.1 Organization of the paper

The remainder of this paper is organized as follows: Sect. 2 is a preliminary section,
that gives a brief account of the complex cobordism theory, universal formal group
law, universal Schur functions, and flag bundles associated with vector bundles, which

2 For further applications of Gysin formulas in complex cobordism and techniques of generating functions,
readers are referred to a companion paper [20].
will be used throughout this paper. Section 3 is the main body of the paper, and establishes the D–P formulas of types $A$, $B$, $C$, and $D$ in complex cobordism. As mentioned above, three items, namely, Segre classes in complex cobordism, Gysin formula for a projective bundle in complex cobordism, and universal Schur functions, play a significant role. The final section, Sect. 4, deals with applications of our Gysin formulas, and discusses various properties of the universal quadratic Schur functions in detail. In Appendix (Section 5), Quillen’s residue formula will be computed explicitly.

2 Notation and conventions

2.1 Complex cobordism theory

Complex cobordism theory $MU^*(-)$ is a generalized cohomology theory associated with the Milnor–Thom spectrum $MU$ (for a detailed account of the complex cobordism theory, readers are referred to e.g., Adams [1]). According to Quillen [24, Proposition 1.2], for a manifold $X$, $MU^q(X)$ can be identified with the set of cobordism classes of proper, complex-oriented maps of dimension $-q$. Thus, a map of manifolds $f: Z \rightarrow X$, which is complex-oriented in the sense of Quillen, determines a class denoted by $\left[ Z \stackrel{f}{\rightarrow} X \right]$, or simply $\left[ Z \right]$ in $MU^q(X)$. The coefficient ring of this theory is given by $MU^*: = MU^*(pt)$, where $pt$ is a space consisting of a single point. With this geometric interpretation of $MU^*(X)$, the Gysin map can be defined as follows: for a proper complex-oriented map $g: X \rightarrow Y$ of dimension $d$, the Gysin map

$$g^*: MU^q(X) \rightarrow MU^{q-d}(Y)$$

is defined by sending the cobordism class $\left[ Z \stackrel{f}{\rightarrow} X \right]$ into the class $\left[ Z \stackrel{g \circ f}{\rightarrow} Y \right]$.

Complex cobordism theory $MU^*(-)$ is equipped with the “generalized” Chern classes. More precisely, for a rank $n$ complex vector bundle $E$ over a space $X$, one can define the $MU^*$-theory Chern classes $c_i^{MU}(E) \in MU^{2i}(X)$ for $i = 0, 1, \ldots, n$, which have the usual properties of the ordinary Chern classes in cohomology (see Conner–Floyd [6, Theorem 7.6], Switzer [25, Theorem 16.2]).

Let $\mathbb{C}P^\infty$ be an infinite complex projective space, and $\eta_\infty: = x MU$ be the $MU^*$-theory first Chern class of the line bundle $\eta_\infty^\vee$, dual of $\eta_\infty$. Then, it is well-known that $MU^*(\mathbb{C}P^\infty) \cong MU^*[[x]]$. Denote the natural projection onto the $i$th factor ($i = 1, 2$) by $\pi_i: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$. Then, the product map $\mu: \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$ is defined as the classifying map of the line bundle $\pi_i^*\eta_\infty \otimes \pi_2^*\eta_\infty$ over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Applying the functor $MU^*(-)$ to the map $\mu$, we obtain

$$\mu^*: MU^*(\mathbb{C}P^\infty) \cong MU^*[[x]] \rightarrow MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong MU^*[[x_1, x_2]],$$

where $x_i = x_i^MU$ is the $MU^*$-theory first Chern class of the line bundle $\pi_i^*\eta_\infty^\vee (i = 1, 2)$. From this, one obtains the formal power series in two variables:
\[ \mu^*(x) = F^{MU}(x_1, x_2) = \sum_{i,j \geq 0} a_{i,j} x_1^i x_2^j \quad (a_{i,j} = a_{i,j}^{MU} \in MU^{2(1-i-j)}). \]

Therefore, for complex line bundles \( L \) and \( M \) over the same base space, the following formula holds:

\[ c_1^{MU}(L \otimes M) = F^{MU}(c_1^{MU}(L), c_1^{MU}(M)). \]

Quillen [23, §2] showed that the formal power series \( F^{MU}(x_1, x_2) \) is a formal group law over \( MU^* \). Moreover, he also showed that the formal group law \( F^{MU}(x_1, x_2) \) over \( MU^* \) is \textit{universal} in the sense that given any formal group law \( F(x_1, x_2) \) over a commutative ring \( R \) with unit, there exists a unique ring homomorphism \( \theta : MU^* \rightarrow R \) carrying \( F^{MU} \) to \( F \). Topologically, Quillen’s result claims that the complex cobordism theory \( MU^*(-) \) is \textit{universal} among complex-oriented generalized cohomology theories. It has been known since Quillen that a complex-oriented generalized cohomology theory gives rise to a formal group law in the same manner as that above. For instance, the ordinary cohomology theory (with integer coefficients) \( H^*(-) \) corresponds to the additive formal group law \( F^H(x_1, x_2) = x_1 + x_2 \), and the (topological) complex \( K \)-theory \( K^*(-) \) corresponds to the multiplicative formal group law \( F^K(x_1, x_2) = x_1 + x_2 - \beta x_1 x_2 \) for some unit \( \beta \in K^{-2} = K^{-2}(pt) = \tilde{K}^0(S^2) \).

Here, a comment concerning the \( K \)-theoretic Chern classes is in order: Following Levine–Morel [16, Example 1.1.5], for a complex line bundle \( L \rightarrow X \), we define the \( K \)-theoretic first Chern class \( c_1^K(L) \) to be \( \beta^{-1}(1 - L^\vee) \in K^2(X) \). Then, the corresponding formal group law is given as stated above.

\subsection*{2.2 Lazard ring \( \mathbb{L} \) and universal formal group law \( F_{\mathbb{L}} \)}

Quillen’s result in the previous subsection implies that the formal group law \( F^{MU} \) over \( MU^* \) is identified with the so-called \textit{Lazard’s universal formal group law} \( F_{\mathbb{L}}(u, v) \) over the \textit{Lazard ring} \( \mathbb{L} \), which we briefly recall from Levine–Morel’s book [16, Chapters 1 and 2]. Accordingly, the additive formal group law is denoted by \( F_{\mathbb{L}}(u, v) = u + v \) in place of \( F^H \), and the multiplicative formal group law by \( F_m(u, v) = u + v - \beta uv \) in place of \( F^K \) in the following. In [15], Lazard constructed the universal formal group law

\[ F_{\mathbb{L}}(u, v) = u + v + \sum_{i,j \geq 1} a_{i,j}^\mathbb{L} u^i v^j \in \mathbb{L}[[u, v]] \]

over the ring \( \mathbb{L} \), where \( \mathbb{L} \) is the \textit{Lazard ring}, and he showed that it is isomorphic to the polynomial ring in a countably infinite number of variables with integer coefficients. \( F_{\mathbb{L}}(u, v) \) is a formal power series in \( u, v \) with coefficients \( a_{i,j}^\mathbb{L} \in \mathbb{L} \) that satisfies the axioms of the formal group law:

- (i) \( F_{\mathbb{L}}(u, 0) = u, F_{\mathbb{L}}(0, v) = v \),
- (ii) \( F_{\mathbb{L}}(u, v) = F_{\mathbb{L}}(v, u) \),
- (iii) \( F_{\mathbb{L}}(u, F_{\mathbb{L}}(v, w)) = F_{\mathbb{L}}(F_{\mathbb{L}}(u, v), w) \).
For the universal formal group law, we shall use the notation

\[ u +_\mathbb{L} v = F_{\mathbb{L}}(u, v) \] (formal sum),
\[ \overline{u} = [-1]_{\mathbb{L}}(u) = \chi_{\mathbb{L}}(u) \] (formal inverse of \( u \)),
\[ u -_\mathbb{L} v = u + _\mathbb{L} [-1]_{\mathbb{L}}(v) = u + _\mathbb{L} \overline{v} \] (formal subtraction).

Furthermore, we define \([0]_{\mathbb{L}}(u) := 0\), and inductively, \([n]_{\mathbb{L}}(u) := [n-1]_{\mathbb{L}}(u) + _\mathbb{L} u\) for a positive integer \( n \geq 1 \). We also define \([-n]_{\mathbb{L}}(u) := [n]_{\mathbb{L}}([-1]_{\mathbb{L}}(u))\) for \( n \geq 1 \). We call \([n]_{\mathbb{L}}(u)\) the \( n \)-series in the sequel (we often drop \( \mathbb{L} \) from the notation, and simply write \([n](u)\) for simplicity). Denote the logarithm (see [16, Lemma 4.1.29]) of \( F_{\mathbb{L}} \) by \( \ell_{\mathbb{L}}(u) \in \mathbb{L} \otimes \mathbb{Q}[[u]] \), i.e., a unique formal power series with leading term \( u \) such that

\[ \ell_{\mathbb{L}}(u +_\mathbb{L} v) = \ell_{\mathbb{L}}(u) + \ell_{\mathbb{L}}(v). \]

The Lazard ring \( \mathbb{L} \) can be graded by assigning each coefficient \( a_{i,j}^\mathbb{L} \), the degree \( 1 - i - j \) \( (i, j \geq 1) \). This grading makes \( \mathbb{L} \) into the graded ring over the integers \( \mathbb{Z} \). Be aware that in topology, it is customary to give \( a_{i,j}^\mathbb{L} \) the cohomological degree \( 2(1-i-j) \).

For the complex \( K \)-theory, we shall use the following notation:

\[ u \oplus v = F_m(u, v) = u + v - \beta uv, \]
\[ \overline{u} = \frac{-u}{1 - \beta u}, \]
\[ u \circledast v = u \oplus \overline{v} = \frac{u - v}{1 - \beta v}. \]

### 2.3 Universal Schur functions

Throughout this paper, we use the notation concerning partitions as in Macdonald’s book [17, Chapter I]. A partition \( \lambda \) is a non-increasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) of non-negative integers such that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0 \). As is customary, we do not distinguish between two such sequences that differ only by a finite or infinite sequence of zeros at the end. The non-zero \( \lambda_i \)'s are called parts of \( \lambda \), and the number of parts is the length of \( \lambda \), denoted by \( \ell(\lambda) \). The sum of the parts is the weight of \( \lambda \), denoted by \( |\lambda| \). If \( |\lambda| = n \), then we say that \( \lambda \) is a partition of \( n \). If \( \lambda \) and \( \mu \) are partitions, then we write \( \lambda \subset \mu \) to mean that \( \lambda_i \leq \mu_i \) for all \( i \geq 1 \). In what follows, the set of all partitions of length \( \leq n \) is denoted by \( \mathcal{P}_n \). For a non-negative integer \( n \), we set \( \rho_0 := (n, n-1, \ldots, 2, 1) \) (\( \rho_0 \) as understood to be the unique partition of \( 0 \), which we denote by just \( 0 \) or \( \emptyset \)). The partition \((k, k, \ldots, k)\) is abbreviated as \((k^i)\), or just \( k^i \). Let \( \lambda, \mu \in \mathcal{P}_n \) be partitions. Then, \( \lambda + \mu \) is a partition defined by \( (\lambda + \mu)_i := \lambda_i + \mu_i \) \( (i = 1, 2, \ldots, n) \). Given a partition \( \lambda \in \mathcal{P}_n \) and a positive integer \( k \), we denote the partition \((k\lambda_1, \ldots, k\lambda_n)\) by \( k\lambda \).

Now, let us recall the definition of the universal Schur functions (see Nakagawa–Naruse [18, Definition 4.10]): Let \( x_n = (x_1, x_2, \ldots, x_n) \) be \( n \) independent variables. For a partition \( \lambda \in \mathcal{P}_n \), we use the notation \( x^\lambda := \prod_{i=1}^{n} x_i^{\lambda_i} \) \( (x_1^0 := 1) \). Then, the...
universal Schur function $s^L_\lambda(x_n)$ in the variables $x_n = (x_1, x_2, \ldots, x_n)$ corresponding to the partition $\lambda \in \mathcal{P}_n$ is defined as

$$s^L_\lambda(x_n) := \sum_{w \in S_n} w \cdot \frac{x^{\lambda+\rho_n-1}}{\prod_{1 \leq i < j \leq n} (x_i + \sum_j x_j)},$$  \hspace{1cm} (2.1)$$

where the symmetric group $S_n$ acts on the variables $x_n = (x_1, \ldots, x_n)$ by permutations. This is a generalization of the usual Schur polynomial $s_\lambda(x_n)$ (see e.g., Macdonald [17, Chapter I, (3.1)]), and the Grassmann Grothendieck polynomial $G_\lambda(x_n)$ (Buch [4, §2]). In fact, under the specialization from the universal formal group law $F_L(u, v) = u + L v$ to the additive one, $F_a(u, v) = u + v$ (resp. the multiplicative one, $F_m(u, v) = u \otimes v$), the function $s^L_\lambda(x_n)$ is reduced to $s_\lambda(x_n)$ (resp. $G_\lambda(x_n)$). Note that $s_\lambda(x_n)$ is a polynomial in $x_n$ with integer coefficients, and $G_\lambda(x_n)$ is also a polynomial in $x_n$ with coefficients in $\mathbb{Z}[\beta]$, whereas $s^L_\lambda(x_n)$ is a formal power series in $x_n$ with coefficients in $\mathbb{L}$. Moreover, unlike the Schur and Grothendieck polynomials, the function $s^L_\emptyset(x_n)$ corresponding to the empty partition $\emptyset = (0^n)$ is not equal to 1. For example, we have

$$s^L_\emptyset(x_2) = \frac{x_1}{x_1 + \sum x_2} + \frac{x_2}{x_2 + \sum x_1} = 1 + a_{1,2}x_1x_2 + \cdots \neq 1.$$

In the later section (Sect. 4), we need to extend the above definition (2.1) to arbitrary sequences of non-negative integers. For such a sequence $I = (I_1, \ldots, I_n) \in (\mathbb{Z}_{\geq 0})^n$, the corresponding universal Schur function $s^L_I(x_n)$ is defined by the same expression as (2.1).

2.4 Flag bundles associated with vector bundles

Darondeau–Pragacz formulas describe Gysin maps for flag bundles associated with complex vector bundles. In this subsection, we prepare the necessary notations and terminologies concerning flag bundles used in this paper (for more details, readers are referred to Darondeau–Pragacz [8, Sections 1, 2, and 3], Edidin–Graham [9, Section 6], Fulton–Pragacz [12, Section 6.1]). Let $E \longrightarrow X$ be a rank $N$ complex vector bundle of type $Y$, where $Y$ stands for $A$, $B$, $C$, or $D$. The situations considered here are as follows:

- Type $A$: $N = n$, and no conditions on $E$;
- Type $C$: $N = 2n$, and $E$ is equipped with a non-degenerate symplectic form $\langle -, - \rangle$ with values in a certain line bundle $L$;
- Type $BD$: $N = 2n + 1$ for type $B$, or $N = 2n$ for type $D$, and $E$ is equipped with a non-degenerate orthogonal form $\langle -, - \rangle$ with values in a certain line bundle $L$.

Let $0 < q_1 < q_2 < \cdots < q_m \leq n$ be a sequence of integers. We denote by $\varpi_{q_1,\ldots,q_m} = \varpi_{q_1,q_2,\ldots,q_m}^Y : \mathcal{F}_{q_1,q_2,\ldots,q_m}^Y(E) \longrightarrow X$ the corresponding partial flag bundle of type $Y$. On $\mathcal{F}_{q_1,\ldots,q_m}^Y(E)$, there exists a universal (isotropic for $Y = B, C$
or D) flag of subbundles of $\varpi_{q_1,\ldots,q_m}^*(E) = E$,

$$0 \subset U_{q_1} \subset U_{q_2} \subset \cdots \subset U_{q_m} \subset E,$$

where rank $U_{q_i} = q_i$ ($i = 1,\ldots,m$) (throughout the paper, the subscripts of the bundles will denote the rank unless otherwise specified). As a special case when $q_k = k$ ($k = 1,\ldots,m$), we obtain the full flag bundle $\varpi_{1,2,\ldots,m} = \varpi_{1,2,\ldots,m}^Y : \mathcal{F}\ell_{1,2,\ldots,m}^{Y} (E) \longrightarrow X$ of type $Y$. In particular, the full flag bundle $\varpi_{Y} = \varpi_{1,2,\ldots,n}^Y : \mathcal{F}\ell_{1,2,\ldots,n}^Y (E) \longrightarrow X$ is the complete flag bundle of type $Y$.\footnote{We adopted the terminology used in Darondeau–Pragacz \cite[§1.2]{DarondeauPragacz}.} On $\mathcal{F}\ell_{1}^Y (E)$, we have the universal (isotropic for $Y = B, C,$ or $D$) flag of subbundles

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n \subset E, \quad (2.2)$$

and we put

$$y_i := c_1^{MU} ((U_i/U_{i-1})^\vee) \in MU^2(\mathcal{F}\ell_{1}^Y (E)) \quad (i = 1,2,\ldots,n).$$

These are the $MU^*$-theory Chern roots of $U_n^\vee$. For $Y = B, C,$ or $D$, the universal isotropic flag of subbundles $(2.2)$ can be extended to the “ordinary” universal flag of subbundles of $E$,

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n \subset U_n^\perp \subset U_{n-1}^\perp \subset \cdots \subset U_2^\perp \subset U_1^\perp \subset E.$$

Here $U_i^\perp$ denotes its complement with respect to the given symplectic or orthogonal form $\langle -,- \rangle$. Then, using an isomorphism of vector bundles $U_i^\perp/U_{i-1}^\perp \sim (U_i/U_{i-1})^\vee \otimes L$ induced from the given symplectic or orthogonal form, the whole of Chern roots of $E^\vee$ are given as follows:

- Type C case: $y_1,\ldots,y_n, \overline{y}_1 + _L \overline{z},\ldots,\overline{y}_n + _L \overline{z},$
- Type B case: $y_1,\ldots,y_n, y_{n+1}, \overline{y}_1 + _L \overline{z},\ldots,\overline{y}_n + _L \overline{z},$
- Type D case: $y_1,\ldots,y_n, \overline{y}_1 + _L \overline{z},\ldots,\overline{y}_n + _L \overline{z},$

where we set $y_{n+1} := c_1^{MU} ((U_n^\perp/U_n)^\vee)$ and $z := c_1^{MU} (L)$.

For $Y = A$ or $C$, it is well-known that the full flag bundle $\mathcal{F}\ell_{1,2,\ldots,n}^Y (E)$ is constructed as a chain of projective bundles, and we apply the push-forward formula for a projective bundle (see (3.4)) repeatedly to this chain to obtain our Gysin formulas. For $Y = B$ or $D$, the full flag bundle $\mathcal{F}\ell_{1,\ldots,q}^Y (E)$ is constructed as a chain of quadric bundles of isotropic lines, and we apply the push-forward formula for a quadric bundle (see (3.12)) repeatedly to this chain to obtain our Gysin formulas. The quadric bundle of isotropic lines associated with an orthogonal bundle $E \longrightarrow X$ is denoted by $\rho_1 : Q(E) \longrightarrow X$ in the sequel. Let $\iota : Q(E) \hookrightarrow P(E)$ be the natural inclusion. Then, we have $\rho_1 = \varpi_1 \circ \iota$, where $\varpi_1 : P(E) \longrightarrow X$ is the associated projective bundle of lines.
3 Darondeau–Pragacz formulas in complex cobordism

In this section, we generalize the push-forward (Gysin) formulas for flag bundles due to Darondeau–Pragacz [8] to the complex cobordism theory. Their formulas will be referred to as the Darondeau–Pragacz formulas in the sequel. The original Darondeau–Pragacz formulas were formulated in the Chow theory, or the ordinary cohomology theory. Their formulas are obtained by iterating the classical push-forward formula for a projective bundle associated with a complex vector bundle, and are expressed by using the Segre classes, or the Segre polynomials of complex vector bundles. Therefore, to generalize their formulas to the complex cobordism theory, we need to generalize the notion of the Segre classes and the classical push-forward formula for a projective bundle in cohomology to complex cobordism. These generalizations were essentially established by Hudson–Matsumura [14] and Quillen [23] respectively, and are recalled in the next subsection.

3.1 From cohomology to complex cobordism

3.1.1 Segre classes in complex cobordism

To formulate the Darondeau–Pragacz formulas in complex cobordism, we need a notion of the Segre classes of complex vector bundles in complex cobordism. Fortunately such a notion was recently introduced by Hudson–Matsumura [14, Definition 4.3]. More precisely, they defined the Segre classes $\mathcal{S}_m(E) (m \in \mathbb{Z})$ in the algebraic cobordism theory $\Omega^*(-)$ of a complex vector bundle $E$ using the push-forward image of the projective bundle $G^1(E) \cong P(E^*)$. Notice that this definition is the exact analogue of the Segre classes in ordinary cohomology given by Fulton [11, §3.1].

$K$-theoretic generalization of Segre classes $G_m(E) (m \in \mathbb{Z})$ is also introduced in the same manner (Buch [5, Lemma 7.1]). The Segre classes of $E$ in the complex cobordism theory $MU^*(-)$, denoted by $\mathcal{S}_m(E) (m \in \mathbb{Z})$, can be defined in exactly the same manner as above. Denote the generating function of the Segre classes, which we call the Segre series in complex cobordism, by

$$\mathcal{S}_\mathcal{L}(E; u) := \sum_{m \in \mathbb{Z}} \mathcal{S}_m(E) u^m.$$\hspace{1cm}(3.1)

More explicitly, the following expression has been obtained (see [14, Theorem 4.6]): Let $x_1, \ldots, x_n$ be the $MU^*$-theory Chern roots of $E$. Then, the Segre series $\mathcal{S}_\mathcal{L}(E; u)$ of $E$ is given by

$$\mathcal{S}_\mathcal{L}(E; u) = \frac{1}{\mathcal{P}_\mathcal{L}(z)} \prod_{j=1}^{n} \frac{z}{z + L x_j} \bigg|_{z = u^{-1}} = \frac{1}{\mathcal{P}_\mathcal{L}(z)} \prod_{j=1}^{n} \frac{z^n}{z + L x_j} \bigg|_{z = u^{-1}}, \hspace{1cm}(3.1)$$

where $\mathcal{P}_\mathcal{L}(z) := 1 + \sum_{i=1}^{\infty} \alpha_i^{\mathcal{L}} z^i$.\hspace{1cm}$\blacksquare$
These Segre classes $\mathcal{S}^L_m(E)$ ($m \in \mathbb{Z}$) are natural generalizations of the (ordinary) Segre classes in cohomology. If we specialize the universal formal group law $F_L(u, v) = u + L v$ to $F_u(u, v) = u + v$, then $\mathcal{S}^L(u; u)$ reduces to

$$
\prod_{j=1}^{n} \left. \frac{z}{z - x_j} \right|_{z=u^{-1}} = \prod_{j=1}^{n} \frac{1}{1 - x_j u} = \frac{1}{c(E; -u)} = \frac{1}{c(E^v; u)} = s(E; u).
$$

Here, $c(E; u) := \sum_{i=0}^{n} c_i(E) u^i$ (resp. $s(E; u) = \sum_{i=0}^{n} s_i(E) u^i$) denotes the (ordinary) Chern polynomial (resp. Segre series) of $E$. This formula also implies that the $i$th Segre class $s_i(E)$ is identified with the $i$th complete symmetric polynomial $h_i(x_n)$, or the Schur polynomial $s_i(x_n)$ corresponding to the one-row $(i)$ in the variables $x_n = (x_1, \ldots, x_n)$. The classes $\mathcal{S}^L_m(E)$ are also generalizations of the $K$-theoretic Segre classes $G_m(E)$ ($m \in \mathbb{Z}$). In fact, if we specialize $F_L(u, v) = u + L v$ to $F_m(u, v) = u \oplus v$, then $\mathcal{S}^L_m(E; u)$ reduces to

$$
\frac{1}{1 - \beta z} \prod_{j=1}^{n} \left. \frac{z}{z \Theta x_j} \right|_{z=u^{-1}} = \frac{1}{1 - \beta u} \prod_{j=1}^{n} \frac{1 - \beta x_j}{1 - x_j u} = \frac{1}{1 - \beta u} c^K(E; -\beta) = \frac{1}{1 - \beta u} c^K(E; -u),
$$

(3.2)

which is the $K$-theoretic Segre series $G(E; u) = \sum_{m \in \mathbb{Z}} G_m(E) u^m$ of $E$ given by Hudson–Ikeda–Matsumura–Naruse [13, Theorem 2.8]. Here, $c^K(E; u) = \sum_{i=0}^{n} c_i^K(E) u^i$ is the $K$-theoretic Chern polynomial of $E$. For $m \geq 1$, the $m$th $K$-theoretic Segre class $G_m(E)$ is identified with the $m$th Grothendieck polynomial $G_m(x_n)$ corresponding to the one-row $(m)$. We remark that our Segre class $\mathcal{S}^L_m(E)$ ($m \geq 1$) in complex cobordism can be identified with the new universal Schur function $S_m^\mathbb{L}(x_n)$ (see Nakagawa–Naruse [19, Remark 5.10]).

**Remark 3.1** The formal power series $\mathcal{P}^L(z) \in \mathbb{L}[[z]]$ has the following geometric meaning: by the argument in Quillen [23, §1], we have $\frac{\partial F_L}{\partial v}(z, 0) = \mathcal{P}^L(z)$, and hence $\ell_L^0(z) = \frac{1}{\mathcal{P}^L(z)}$. As a result of Miščenko (see Adams [1, Chapter II, Corollary 9.2]), we know that

$$
\ell_L(z) = \sum_{m=0}^{\infty} \frac{[\mathbb{C}P^m]}{m + 1} z^{m+1},
$$

where $[\mathbb{C}P^m] \in MU^{-2m} = \mathbb{L}^{-2m}$ is the cobordism class of $\mathbb{C}P^m$. Therefore, we have

$$
\frac{1}{\mathcal{P}^L(z)} = \sum_{m=0}^{\infty} [\mathbb{C}P^m] z^m.
$$

**3.1.2 Fundamental Gysin formula for projective bundle in complex cobordism**

Let $E \to X$ be a complex vector bundle of rank $n$, and $\sigma_1 : P(E) \to X$ the associated projective bundle of lines in $E$. Denote the tautological line bundle on
\( P(E) \) by \( U_1 \). Put \( y_1 := c_1^{MU}(U^\vee_1) \in MU^2(P(E)) \). In [23], Quillen described the Gysin map \( \varpi_{1*} : MU^*(P(E)) \to MU^*(X) \). In our notation, his formula is stated as follows:

**Theorem 3.2** (Quillen [23], Theorem 1) For a polynomial \( f(t) \in MU^*(X)[t] \), the Gysin map \( \varpi_{1*} : MU^*(P(E)) \to MU^*(X) \) is given by the residue formula

\[
\varpi_{1*}(f(y_1)) = \text{Res}'_{t=0} \frac{f(t)}{\mathcal{P}^L(t) \prod_{i=1}^n (t + _L y_i)}, \tag{3.3}
\]

where \( y_1, \ldots, y_n \) denote the \( MU^* \)-theory Chern roots of \( E^\vee \).

Here, the symbol \( \text{Res}' \) \( F(t) \) is understood to be the coefficient of \( t^{-1} \) in the formal Laurent series \( F(t) \). However, we must be careful when we apply the operation \( \text{Res}'_{t=0} \) to the formal Laurent series. For example, let us consider the rational function \( f(t) = 1/(1 - t) \). Then, on the one hand, we have \( f(t) = 1 + t + t^2 + \cdots \) when expanded as a formal power series in \( t \), and therefore \( \text{Res}'_{t=0} f(t) = 0 \). On the other hand, one can expand \( f(t) \) as a formal power series in \( t^{-1} \) so that \( f(t) = -t^{-1} - t^{-2} - \cdots \). Therefore, we have \( \text{Res}'_{t=0} f(t) = -1 \). Thus, we must specify how to expand \( f(t) \) as a formal power series in \( t \) or \( t^{-1} \) when we apply \( \text{Res}'_{t=0} \) to the rational function \( f(t) \). In the above formula \( (3.3) \), we expand the rational function of the right-hand side in accordance with the following convention (for this interpretation of Quillen’s result, see also Naruse [21, Lemma 4]): As mentioned in Remark 3.1, we always treat \( 1/\mathcal{P}^L(t) \) as a formal power series in \( t \), that is, \( 1/\mathcal{P}^L(t) = \sum_{m=0}^\infty [C P^m] t^m \). For the product \( 1/\prod_{i=1}^n (t + _L y_i) \), we expand this as a formal power series in \( t^{-1} \) by using the following expansion:

\[
\frac{1}{t + _L y_i} = \mathcal{P}^L(t, y_i) = t^{-1} \times \mathcal{P}^L(t, y_i) \times \frac{1}{1 - y_i t^{-1}} = t^{-1} \times \mathcal{P}^L(t, y_i) \times \sum_{k=0}^\infty y_i^k t^{-k},
\]

where \( \mathcal{P}^L(t, y_i) := \frac{t - y_i}{t + _L y_i} \). For further calculations, readers are referred to Appendix 5.1.

Following Darondeau–Pragacz [8, p.2, (2)], we reformulate the above formula in a more convenient form. To do so, we use the same notation as in [8]. For a monomial \( m \) of a Laurent polynomial \( F \), we denote the coefficient of \( m \) in \( F \) by \( [m](F) \). With these conventions and the Segre series (3.1), the residue formula (3.3) becomes

\[
\varpi_{1*}(f(y_1)) = [t^{-1}] \left( f(t) \cdot t^{-n} \mathcal{P}^L(E^\vee; 1/t) \right) = [t^{n-1}](f(t) \mathcal{P}^L(E^\vee; 1/t)). \tag{3.4}
\]

This is the fundamental formula for establishing more general Gysin formulas for general flag bundles.
3.2 Darondeau–Pragacz formula of type A in complex cobordism

With the above preliminaries, we can extend Darondeau–Pragacz formula [8, Theorem 1.1] in the following manner: Let $E \longrightarrow X$ be a complex vector bundle of rank $n$. Given a sequence of integers $q_0 = 0 < q_1 < \cdots < q_m \leq n = q_{m+1}$, we set $q := q_m$. Then, the following Gysin formula holds for the partial flag bundle $\varpi_{q_1,\ldots,q_m-1} : \mathcal{F}_q \longrightarrow X$.

**Theorem 3.3** (Darondeau–Pragacz formula of type A in complex cobordism) For a polynomial $f(t_1, \ldots, t_q) \in MU^*(X)[t_1, \ldots, t_q]^{S_{q_1} \times S_{q_2} \times \cdots \times S_{q_m}}$, one has

$$\varpi_{q_1,\ldots,q_m-1} \cdot f(y_1, \ldots, y_q) = \left[ \prod_{k=1}^{m} \prod_{q_{k-1} < i \leq q_k} t_i^{(n-1)-(q_k-i)} \right] \left( f(t_1, \ldots, t_q) \times \prod_{k=1}^{m} \sum_{\ell=0}^{q_{k-1}} (t_{q_{k-1}+1}, \ldots, t_{q_k})^{-1} \prod_{1 \leq i < j \leq q} \prod_{i=1}^{q} Sq^\ell_i(\mathcal{E}^\vee; 1/t_i) \right).$$

(3.5)

Before starting the proof, we recall the following fact concerning the universal Schur class of a vector bundle: For the Gysin map: $\varpi_* : MU^*(\mathcal{F}(\ell(E))) \longrightarrow MU^*(X)$, the following formula holds (see Nakagawa–Naruse [19, Corollary 4.8]):

$$\varpi_*(y^{\lambda+\rho_n-1}) = s_\lambda^L(\mathcal{E}^\vee),$$

where $s_\lambda^L(\mathcal{E}^\vee) \in MU^{2|\lambda|}(X)$ is a cohomology class defined by $\varpi^*(s_\lambda^L(\mathcal{E}^\vee)) = s_\lambda^L(y_n)$. Hereafter, we often identify $s_\lambda^L(\mathcal{E}^\vee)$ with $s_\lambda^L(y_n)$ via monomorphism $\varpi^* : MU^*(X) \longrightarrow MU^*(\mathcal{F}(\ell(E)))$, and write $s_\lambda^L(\mathcal{E}^\vee) = s_\lambda^L(y_n)$. As a particular case, for $\lambda = \emptyset$, we have

$$\varpi_*(y^{\rho_n-1}) = s_\emptyset^L(y_n).$$

(3.6)

Since we know that $s_\emptyset^L(y_n) = (1 + \text{higher terms in } y_1, \ldots, y_n) \in MU^0(X)$ is an invertible element, we deduce that

$$\varpi_*(s_\emptyset^L(y_n)^{-1} y^{\rho_n-1}) = 1.$$

**Proof of Theorem 3.3** One can prove the theorem along the same lines as in Darondeau–Pragacz [8, Theorem 1.1]. First, we prove the case of full flag bundles $\varpi_{1,\ldots,q} : \mathcal{F}(\ell_1,\ldots,q(E)) \longrightarrow X$ by induction on $q \geq 1$. For the case $q = 1$, the result is simply the formula (3.4). Hence, we may assume the result for the case of $q-1$ with $q \geq 2$. Thus, for any polynomial $g(t_1, \ldots, t_{q-1}) \in MU^*(X)[t_1, \ldots, t_{q-1}]$, one has

$$\varpi_{1,\ldots,q-1} \cdot g(t_1, \ldots, y_q-1)) = [t_1^{n-1} \cdots t_{q-1}^{n-1}] \left( g(t_1, \ldots, t_{q-1}) \prod_{1 \leq i < j \leq q-1} (t_j + t_i) \prod_{i=1}^{q-1} Sq^\ell_i(\mathcal{E}^\vee; 1/t_i) \right).$$

(3.7)
Now, we consider the image of the Gysin map $\varpi_{1,...,q*}(f(y_1, \ldots, y_q))$. Since $\varpi_{1,...,q} : \mathcal{F}_{1,...,q}(E) \to X$ is the composite of $\varpi_q : \mathcal{F}_{1,...,q}(E) \to \mathcal{F}_{1,...,q-1}(E)$ and $\varpi_{1,...,q-1} : \mathcal{F}_{1,...,q-1}(E) \to X$, namely, $\varpi_{1,...,q} = \varpi_{1,...,q-1} \circ \varpi_q$, we have

$$\varpi_{1,...,q*}(f(y_1, \ldots, y_q)) = \varpi_{1,...,q-1*} \circ \varpi_q* (f(y_1, \ldots, y_q)).$$

By the construction recalled in Sect. 2.4, $\varpi_q : \mathcal{F}_{1,...,q}(E) \to \mathcal{F}_{1,...,q-1}(E)$ is the same as the projective bundle of lines $\varpi_q : P(E/U_{q-1}) \to \mathcal{F}_{1,...,q-1}(E)$. The rank of the quotient bundle $E/U_{q-1}$ is $n - q + 1$, and therefore, by the fundamental formula (3.4), we have

$$\varpi_q* (f(y_1, \ldots, y_{q-1}, y_q)) = [t^{n-q}_q](f(y_1, \ldots, y_{q-1}, t_q) \mathcal{S}^L((E/U_{q-1})^\vee; 1/t_q)).$$

Here, the vector bundle $(E/U_{q-1})^\vee$ has the Chern roots $y_q, \ldots, y_n$ as is easily seen by the definition of the Chern roots of $E^\vee$, and hence we deduce from (3.1),

$$\mathcal{S}^L((E/U_{q-1})^\vee; 1/t_q) = \frac{t^{n-q+1}_q}{\mathcal{S}^L(t_q) \prod_{i=q}^{n} (t_q + L \overline{y}_i)} = t^{-q-1}_q \prod_{i=1}^{q-1} (t_q + L \overline{y}_i) \mathcal{S}^L(E^\vee; 1/t_q).$$

Thus, we have

$$\varpi_{q*}(f(y_1, \ldots, y_{q-1}, y_q)) = [t^{n-q}_q] \left( f(y_1, \ldots, y_{q-1}, t_q) t^{-q-1}_q \prod_{i=1}^{q-1} (t_q + L \overline{y}_i) \mathcal{S}^L(E^\vee; 1/t_q) \right)$$

$$= [t^{n-1}_q] \left( f(y_1, \ldots, y_{q-1}, t_q) \prod_{i=1}^{q-1} (t_q + L \overline{y}_i) \mathcal{S}^L(E^\vee; 1/t_q) \right),$$

and hence,

$$\varpi_{1,...,q*}(f(y_1, \ldots, y_q)) = [t^{n-1}_q] \left[ \varpi_{1,...,q-1*} \left( f(y_1, \ldots, y_{q-1}, t_q) \prod_{i=1}^{q-1} (t_q + L \overline{y}_i) \mathcal{S}^L(E^\vee; 1/t_q) \right) \right]$$

$$= [t^{n-1}_q] \left[ \varpi_{1,...,q-1*} \left( f(y_1, \ldots, y_{q-1}, t_q) \prod_{i=1}^{q-1} (t_q + L \overline{y}_i) \mathcal{S}^L(E^\vee; 1/t_q) \right) \right].$$

Then, by the induction assumption (3.7), we have

$$\varpi_{1,...,q-1*} \left( f(t_1, \ldots, t_{q-1}, t_q) \prod_{i=1}^{q-1} (t_q + L \overline{y}_i) \right)$$

$$= [t^{n-1-1}_1 \ldots t^{n-1}_{q-1}] \left( f(t_1, \ldots, t_{q-1}, t_q) \prod_{i=1}^{q-1} (t_q + L \overline{T}_i) \times \prod_{1 \leq i < j \leq q-1} (t_j + L \overline{T}_i) \prod_{i=1}^{q-1} \mathcal{S}^L(E^\vee; 1/t_i) \right),$$

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and therefore, we obtain the desired formula.

From the result of full flag bundles, we can prove the case of general partial flag bundles $\sigma_{q_1, \ldots, q_m} : F_{q_1, \ldots, q_m}(E) \to X$. For simplicity, we set $F := F_{q_1, \ldots, q_m}(E)$, and $q := q_m$. On $F$, we have the universal flag of subbundles of $E$:

$$U_{q_0} = 0 \subset U_{q_1} \subset \cdots \subset U_{q_{m-1}} \subset U_q \subset U_{q_{m+1}} = E.$$ 

Let us consider the fiber product

$$\mathcal{Y} := F(\ell(U_{q_1}) \times F(\ell(U_{q_2}/U_{q_1}) \times F \cdots \times F(\ell(U_q/U_{q_{m-1}}))$$

with the natural projection map $\sigma_F : \mathcal{Y} \to F$. Then, by the definition of the full flag bundles $\sigma_F^i : F(\ell(U_{q_k}/U_{q_{k-1}}) \to F(k = 1, 2, \ldots, m)$, the variety $\mathcal{Y}$ is naturally isomorphic to $F_{1,2,\ldots,q}(E)$. We denote this isomorphism by $\theta : \mathcal{Y} \xrightarrow{\sim} F_{1,2,\ldots,q}(E)$.

Identifying $\mathcal{Y}$ with $F_{1,2,\ldots,q}(E)$ through $\theta$, we have $\sigma_{1,2,\ldots,q} = \sigma_{q_1,\ldots,q_{m-1},q} \circ \sigma_F$. Therefore, by the naturality of the Gysin map, we have $\sigma_{1,2,\ldots,q} = \sigma_{q_1,\ldots,q_{m-1},q} \circ \sigma_F$. By applying the formula (3.6) to the full flag bundles $\sigma_F^i : F(\ell(U_{q_k}/U_{q_{k-1}}) \to F$, we have

$$\sigma_{F}^i \left( \prod_{q_{k-1} < i \leq q_k} y_i^{q_k-i} \right) = s_{\mathcal{Y}}^\vee ((U_{q_k}/U_{q_{k-1}})^\vee) = s_{\mathcal{Y}}^\vee (y_{q_k-1+1}, \ldots, y_{q_k}),$$

and hence we obtain

$$\sigma_F^i \left( \prod_{k=1}^m \prod_{q_{k-1} < i \leq q_k} y_i^{q_k-i} \right) = \prod_{k=1}^m s_{\mathcal{Y}}^\vee (y_{q_{k-1}+1}, \ldots, y_{q_k}).$$

Since each $s_{\mathcal{Y}}^\vee (y_{q_{k-1}+1}, \ldots, y_{q_k})$ is an invertible element, we have

$$\sigma_F^i \left( \prod_{k=1}^m s_{\mathcal{Y}}^\vee (y_{q_{k-1}+1}, \ldots, y_{q_k})^{-1} \prod_{k=1}^m \prod_{q_{k-1} < i \leq q_k} y_i^{q_k-i} \right) = 1. \quad (3.8)$$

With these preliminaries, we proceed with the proof as follows: For a polynomial $f(t_1, t_2, \ldots, t_q) \in MU^*(X)[t_1, t_2, \ldots, t_q]^{S_q \times S_{q_2-q_1} \times \cdots \times S_{q-m-1}}$, we compute

$$\sigma_{q_1,\ldots,q_{m-1},q}(f(y_1, y_2, \ldots, y_q))$$

$$= \sigma_{q_1,\ldots,q_{m-1},q}(f(y_1, \ldots, y_q) \times 1)$$

$$= \sigma_{q_1,\ldots,q_{m-1},q} \left( f(y_1, \ldots, y_q) \times \sigma_F^i \left( \prod_{k=1}^m s_{\mathcal{Y}}^\vee (y_{q_{k-1}+1}, \ldots, y_{q_k})^{-1} \prod_{k=1}^m \prod_{q_{k-1} < i \leq q_k} y_i^{q_k-i} \right) \right)$$

$$= \sigma_{q_1,\ldots,q_{m-1},q} \circ \sigma_F^i \left( f(y_1, \ldots, y_q) \times \prod_{k=1}^m s_{\mathcal{Y}}^\vee (y_{q_{k-1}+1}, \ldots, y_{q_k})^{-1} \prod_{k=1}^m \prod_{q_{k-1} < i \leq q_k} y_i^{q_k-i} \right)$$

$$= \sigma_{q_1,\ldots,q_{m-1},q} \circ \sigma_F^i \left( f(y_1, \ldots, y_q) \times \prod_{k=1}^m s_{\mathcal{Y}}^\vee (y_{q_{k-1}+1}, \ldots, y_{q_k})^{-1} \prod_{k=1}^m \prod_{q_{k-1} < i \leq q_k} y_i^{q_k-i} \right).$$
\[ \sigma_{q_1, \ldots, q_m}(f(y_1, \ldots, y_q)) = \prod_{k=1}^{m} \prod_{1 \leq i \leq q_k} t_i^{(n-1)-(q_k-i)} \left( f(t_1, \ldots, t_q) \times \prod_{1 \leq i \leq j \leq q} \prod_{i=1}^{q} G(E^\vee; 1/t_i) \right). \] (3.9)

**Remark 3.5** It is known that Gysin maps for various flag bundles can be described as certain symmetrizing operators (see Bressler–Evens [2, Theorem 1.8], Nakagawa–Naruse [19, Theorem 2.5, Corollary 2.6] for complex cobordism). For the partial flag bundle \( \sigma_{q_1, \ldots, q_m} : MU^*(F^j q_1, \ldots, q_m(E)) \to MU^*(X) \), the formula is given as follows: For a symmetric polynomial \( f(t_1, \ldots, t_n) \in MU^*(X) \), the formula

\[ \sigma_{q_1, \ldots, q_m}(f(y_1, \ldots, y_n)) = \sum_{w \in S_n/S_{q_1} \times S_{q_2} \times \cdots \times S_{q_m}} w \cdot \left[ \prod_{k=1}^{m} \prod_{1 \leq i \leq q_k} t_i^{q_k} \prod_{q_k < j \leq n} (y_j + \bar{y}_j) \right]. \] (3.10)

### 3.3 Darondeau–Pragacz formula of type C in complex cobordism

The same technique can be applied to the type C case because type C flag bundles are also constructed as a chain of projective bundles of lines. Indeed, we can extend the Darondeau–Pragacz formula [8, Theorem 2.1] in the following manner: Let \( E \to X \) be a symplectic vector bundle of rank \( 2n \). Given a sequence of integers \( q_0 = 0 < q_1 < \cdots < q_m \leq n \), we set \( q := q_m \). Then, the following Gysin formula holds for the isotropic partial flag bundle \( \sigma_{q_1, \ldots, q_m-1, q} : F^j q_1, \ldots, q_{m-1}, q(E) \to X \).
Theorem 3.6 (Darondeau–Pragacz formula of type C in complex cobordism) For a polynomial \( f(t_1, \ldots, t_q) \in MU^*(X) [t_1, \ldots, t_q] S_q 1 \times S_{q_2} q_2 \times \cdots \times S_{q_m} q_m - 1 \), one has

\[
\sigma_{q_1, \ldots, q_m-1, q*}(f(y_1, \ldots, y_q)) = \left[ \prod_{k=1}^{m} \prod_{q_k-1 < i \leq q_k} t_i^{(2n-1)-(q_k-i)} \right] \left( f(t_1, \ldots, t_q) \times \prod_{k=1}^{m} s_{\emptyset}^q (t_{q_k-1}+1, \ldots, t_{q_k})^{-1} \times \prod_{1 \leq i < j \leq q} (t_j + \_ + t_i)(t_j + \_ + t_i + \_ + z) \prod_{i=1}^{q} \mathcal{F}^L(E^\vee; 1/t_i) \right).
\]

Proof As in the proof due to Darondeau–Pragacz [8, Theorem 2.1], we first consider the case of isotropic full flag bundles. We shall prove the formula for \( \sigma_{1,2,\ldots,q} : \mathcal{F} \ell_{1,2,\ldots,q}(E) \rightarrow X \) by induction on \( q \geq 1 \). For the case \( q = 1 \), the result follows from the formula (3.4) since \( \mathcal{F} \ell_{1,2}(E) = P(E) \). Hence, we may assume the result for the case of \( q - 1 \) with \( q \geq 2 \). Let us consider the projection \( \sigma_q : \mathcal{F} \ell_{1,2,\ldots,q}(E) \rightarrow \mathcal{F} \ell_{1,2,\ldots,q-1}(E) \), which is the same as the projective bundle of lines \( \sigma_q : P(U_{q-1}^\perp/U_{q-1}) \rightarrow \mathcal{F} \ell_{1,2,\ldots,q-1}(E) \). Here, the rank of the quotient bundle \( U_{q-1}^\perp/U_{q-1} \) is \( 2(n-q) + 2 \). Therefore, for a polynomial \( f(t_1, \ldots, t_q) \in MU^*(X) [t_1, \ldots, t_q] \), the fundamental formula (3.4) gives

\[
\sigma_{q*}(f(y_1, \ldots, y_q-1, y_q)) = [t_q^{2(n-q)+1}] \left( f(y_1, \ldots, y_q-1, t_q) \mathcal{F}^L((U_{q-1}^\perp/U_{q-1})^\vee; 1/t_q) \right).
\]

Since the vector bundle \( (U_{q-1}^\perp/U_{q-1})^\vee \) has the Chern roots \( y_q, \ldots, y_n, \overline{y}_q + \_ + \overline{z}, \ldots, \overline{y}_n + \_ + \overline{z} \) as seen in Sect. 2.4, we deduce from (3.1),

\[
\mathcal{F}^L((U_{q-1}^\perp/U_{q-1})^\vee; 1/t_q) = \frac{t_q^{2n+2}}{[t_q]^{2n} \prod_{i=1}^{q-1} (t_q + \_ + y_i)(t_q + \_ + y_i + \_ + z)}.
\]

Thus, we have

\[
\sigma_{q*}(f(y_1, \ldots, y_q-1, y_q)) = [t_q^{2(n-q)+1}] \left( f(y_1, \ldots, y_q-1, t_q) \mathcal{F}^L((U_{q-1}^\perp/U_{q-1})^\vee; 1/t_q) \right) = [t_q^{2n-1}] \left( f(y_1, \ldots, y_q-1, t_q) \prod_{i=1}^{q-1} (t_q + \_ + y_i)(t_q + \_ + y_i + \_ + z) \mathcal{F}^L(E^\vee; 1/t_q) \right).
\]

Therefore, we have

\[
\sigma_{1,2,\ldots,q*}(f(y_1, \ldots, y_q)) = \sigma_{1,2,\ldots,q-1*} \circ \sigma_{q*}(f(y_1, \ldots, y_q))
\]
be the natural inclusion. Then, we have

\[ E \text{ of isotropic lines in } \mathcal{V}. \]

Then, by the induction assumption, we obtain the required formula.

From the result of isotropic full flag bundles, we can prove the case of isotropic partial flag bundles \( \phi_{q,1,...,q_m} : \mathcal{F}^{C}_{1,...,q_m}(E) \rightarrow X \) by the same manner as the type A case. The space \( \mathcal{V} \), that is the type C analogue of the space used in the proof of Theorem 3.3, is naturally isomorphic to \( \mathcal{F}^{C}_{1,2,...,q}(E) \) because any flag inside an isotropic subbundle is also an isotropic flag. \( \square \)

In \( K \)-theory, the above D–P formula takes the following form:

**Corollary 3.7** (Darondeau–Pragacz formula of type C in \( K \)-theory)

\[
\phi_{q,1,...,q_m-1,q} \ast (f(y_1, \ldots, y_q)) = \prod_{k=1}^{m} \prod_{q_k \leq i \leq q_k \leq i} \left( f(t_{i,1}, \ldots, t_{q_n}) \times \prod_{1 \leq i < j \leq q} (t_j \oplus t_i)(t_j \oplus t_i \oplus z) \right) \times \prod_{i=1}^{q} G(E^\vee; 1/t_{i}).
\]

(3.11)

### 3.4 Darondeau–Pragacz formulas of types B and D in complex cobordism

#### 3.4.1 Fundamental Gysin formula for quadric bundle in complex cobordism

Let \( E \rightarrow X \) be an orthogonal vector bundle of rank \( N \) (\( 2n + 1 \) for \( Y = B \), or \( 2n \) for \( Y = D \)), and let \( \rho_1 : Q(E) \rightarrow X \) be the associated quadric bundle of isotropic lines in \( E \). Denote the tautological line bundle on \( Q(E) \) by \( U_1 \). Put \( y_1 = c_1^{MU}(U_1^\vee) \in MU^2(Q(E)) \) (We also denote the tautological line bundle on \( P(E) \) by \( U_1 \)) with the same symbol \( y_1 \) for \( c_1^{MU}(U_1^\vee) \in MU^2(P(E)) \). Thus, \( t^*(y_1) = y_1 \). Following Darondeau–Pragacz [8, §3.3], we shall describe the Gysin homomorphism \( \rho_1^* : MU^*(Q(E)) \rightarrow MU^*(X) \). First, we shall describe the class \([Q(E)] \in MU^2(P(E))\). By the definition of \( Q(E) \), it is given by the zero set of a section of the line bundle \( \text{Hom}(U_1 \otimes U_1, L) \cong U_1^\vee \otimes U_1^\vee \otimes L \). Therefore we have

\[
[Q(E)] = c_1^{MU}(U_1^\vee \otimes U_1^\vee \otimes L) = y_1 \oplus y_1 \oplus z = [2]_L(y_1) \oplus z.
\]

Let \( \phi_1 : P(E) \rightarrow X \) be the projective bundle of lines, and \( \iota : Q(E) \hookrightarrow P(E) \) be the natural inclusion. Then, we have \( \rho_1 = \phi_1 \circ \iota \), and therefore \( \rho_1^* = \phi_1^* \circ \iota^* : MU^*(Q(E)) \rightarrow MU^*(X) \). Then, using the fundamental formula (3.4), which still
holds for a formal power series \( f(t) \in MU^*(X)[[t]] \), one can compute for \( k \geq 0 \),

\[
\rho_1^*(y^k) = \varpi_1 \circ \iota_*(y^k) = \varpi_1 \circ \iota_*(t^n(y^k)) = \varpi_1^*(y^k \iota_*(1)) = \varpi_1^*(y_1^k Q(E)]](2L(t) + z) \varpi^L(E^\vee; 1/t)).
\]

Therefore, for a polynomial \( f(t) \in MU^*(X)[t] \), the Gysin homomorphism \( \rho_1^*: MU^*(Q(E)) \rightarrow MU^*(X) \) is given by

\[
\rho_1^*(f(y_1)) = [t^{N-1}] (f(t)([2L(t) + z) \varpi^L(E^\vee; 1/t))].
\]

This is the fundamental formula for establishing more general Gysin formulas for flag bundles of types \( B \) and \( D \), which will be given in the next subsection.

### 3.4.2 Darondeau–Pragacz formula of types \( B \) and \( D \) in complex cobordism

With the above preliminaries, we can extend the Darondeau–Pragacz formula [8, Theorem 3.1]) in the following manner: Let \( E \rightarrow X \) be an orthogonal vector bundle of rank \( N \). Given a sequence of integers \( q_0 = 0 < q_1 < \cdots < q_m \leq n \), we set \( q := q_m \). Then, the following Gysin formula holds for the isotropic partial flag bundle \( \rho_{q_1,\ldots,q_{m-1},q} : \mathcal{F}_q E_{q_1,\ldots,q_{m-1},q} \rightarrow X \).

**Theorem 3.8** (Darondeau–Pragacz formula of types \( B \) and \( D \) in complex cobordism)

*For a polynomial \( f(t_1, \ldots, t_q) \in MU^*(X)[t_1, \ldots, t_q] \) one has has

\[
\rho_{q_1,\ldots,q_{m-1},q}^*(f(y_1, \ldots, y_q)) = \prod_{k=1}^{m} \prod_{q_{h-1} < l \leq q_k} t_l^{(N-1)-(q_k-i)}
\]

\[
f(t_1, \ldots, t_q) \times \prod_{i=1}^{q} ([2L(t_i) + z) \times \prod_{k=1}^{m} s_q(t_{q_{h-1}+1}, \ldots, t_{q_k})^{-1}
\]

\[
\times \prod_{1 \leq i < j \leq q} (t_j + L t_i)(t_j + L t_i + z) \varpi^L(E^\vee; 1/t)) \).
\]

**Proof** One can prove the theorem in the same manner as the type \( C \) case, replacing the fundamental formula (3.4) for a projective bundle with the fundamental formula (3.12) for a quadric bundle. For example, the first step of the induction proceeds as follows: The projection \( \rho_q : \mathcal{F}_{q_1,\ldots,q} E \rightarrow \mathcal{F}_{q_1,\ldots,q} E \) is the same as the quadric bundle of isotropic lines \( \rho_q : Q(U_{q-1}/U_{q-1}) \rightarrow \mathcal{F}_{q_1,\ldots,q} E \). The rank of the quotient bundle \( U_{q-1}/U_{q-1} \) is \( N - 2(q - 1) \). Therefore, the fundamental formula (3.12) gives
\[ \rho_{q*}(f(y_1, \ldots, y_{q-1}, y_q)) = [t_q^{N-2q+1}](f(y_1, \ldots, y_{q-1}, t_q)([2]_L(t_q) + L z)\mathcal{S}^L((U_{q-1}^\perp/U_q)^\vee; 1/t_q)). \]

As with the type \( C \) case, we know
\[ \mathcal{S}^L((U_{q-1}^\perp/U_q-1)^\vee; 1/t_q) = t_q^{-2(q-1)} \prod_{i=1}^{q-1} (t_q + L \bar{y}_i)(t_q + L y_i + L z)\mathcal{S}^L(E^\vee; 1/t_q), \]
and hence we have
\[ \rho_{q*}(f(y_1, \ldots, y_{q-1}, y_q)) = [t_q^{N-1}](f(y_1, \ldots, y_{q-1}, t_q)([2]_L(t_q) + L z) \prod_{1 \leq i \leq q-1} (t_q + L \bar{y}_i)(t_q + L y_i + L z)\mathcal{S}^L(E^\vee; 1/t_q)). \]

The rest of the proof is entirely analogous to the type \( C \) case. \( \square \)

4 Applications: Universal quadratic Schur functions

4.1 Quadratic Schur functions

4.1.1 Definition of quadratic Schur functions

In [8, §4.2], Darondeau–Pragacz introduced the type \( BCD \) analogues of the usual Schur functions, called the quadratic Schur functions. First, we recall their definition:

Let \( E \to X \) be a symplectic vector bundle of rank \( 2n \), where we assume that the line bundle \( L \) is trivial. Then, using the given symplectic form, we have the isomorphism \( E \cong E^\vee \) as complex vector bundles, and the (cohomology) Chern roots of \( E \) are given by \( \pm y_1, \ldots, \pm y_n \). Thus, the Segre series of \( E^\vee \) is formally given by
\[ s(E^\vee; u) = \prod_{j=1}^{n} \frac{1}{1 - y_j u} \frac{1}{1 + y_j u} = \prod_{j=1}^{n} \frac{1}{1 - y_j^2 u^2} = \sum_{l \geq 0} h_l(y_n^2) u^{2l}, \]
where \( h_l(y_n^2) = h_l(y_1^2, \ldots, y_n^2) \) denotes the complete symmetric polynomial in \( y_1^2, \ldots, y_n^2 \). Hence, the Segre classes of \( E \) are given by
\[ s_{2k+1}(E^\vee) = 0 \quad \text{and} \quad s_{2k}(E^\vee) = h_k(y_n^2) \quad (k = 0, 1, 2, \ldots). \]

Then, for a sequence of integers \( I = (I_1, \ldots, I_n) \in \mathbb{Z}^n \), Darondeau–Pragacz defines the cohomology class \( s_I^{(2)}(E^\vee) \) to be
\[ s_I^{(2)}(E^\vee) := \det (s_{I,j+2(j-i)}(E^\vee))_{1 \leq i, j \leq n}. \]
By analogy with the usual Schur functions, they called $s_I^{(2)}(E^\vee) = s_I^{(2)}(y_n^2)$ the quadratic Schur function. Notice that their Segre classes $s_I(E)$’s are the same as those of our $s_I(E^\vee)$. However, this difference does not affect the above definition because of $E \cong E^\vee$ as complex vector bundles. From this definition, we observe that the variables $y_i$ are non-trivial only if all parts of $I$ are even numbers. If $I$ is such a partition, then $I$ is of the form $2J$ for some partition $J$, then we see that

$$s_I^{(2)}(E^\vee) = \det (s_{I_i+2(j-i)}(E^\vee)) = \det (h_{J_i+(j-i)}(y_n^2)) = s_J^{(2)}(E^\vee).$$

Here the class $s_J^{(2)}(E^\vee)$ is introduced in Pragacz–Ratajski [22, Theorem 5.13], and is defined as $s_J(y_n^2)$, the ordinary Schur polynomial corresponding to the partition $J$ in the variables $y_n^2 = (y_1^2, \ldots, y_n^2)$.

### 4.1.2 Gysin formulas for quadratic Schur functions

We shall describe the quadratic Schur functions in terms of Gysin maps of type $C$ full flag bundles. Let $\lambda = (\lambda_1, \ldots, \lambda_q)$ be a partition of length $\ell(\lambda) = q \leq n$. Consider the type $C$ full flag bundle $\sigma_{1,2,\ldots,q} : \mathcal{F}_{\ell_1,\ldots,\ell_q}(E) \rightarrow X$, and the induced Gysin map in cohomology,

$$\sigma_{1,\ldots,q}^* : H^*(\mathcal{F}_{\ell_1,\ldots,\ell_q}(E)) \rightarrow H^*(X).$$

Then, the following proposition is essentially given by Darondeau–Pragacz [8, Proposition 4.3]:

**Proposition 4.1** (Characterization of the quadratic Schur functions) For the Gysin map $\sigma_{1,\ldots,q}^* : H^*(\mathcal{F}_{\ell_1,\ldots,\ell_q}(E)) \rightarrow H^*(X)$, the following formula holds:

$$\sigma_{1,\ldots,q}^*(y^{\lambda + \rho_{2q-1}^{(2)}+(2(n-q))^q}) = \sigma_{1,\ldots,q}^* \left( \prod_{i=1}^{q} y_i^{\lambda_i + 2n-2i+1} \prod_{1 \leq i < j \leq q} (a_j^2 - t_i^2) \prod_{i=1}^{q} s(E^\vee; 1/t_i) \right) = s_\lambda^{(2)}(E^\vee),$$

where $\rho_{2q-1}^{(2)} := (2q - 1, 2q - 3, \ldots, 3, 1)$.

**Proof** For convenience of the readers, we shall provide the proof of the proposition along the same lines as in [8]: By the D–P formula of type $C$ [8, Theorem 2.1], one can compute

$$\sigma_{1,\ldots,q}^*(y^{\lambda + \rho_{2q-1}^{(2)}+(2(n-q))^q}) = \left[ \prod_{i=1}^{q} t_i^{2n-1} \right] \left( \prod_{i=1}^{q} t_i^{\lambda_i + 2n-2i+1} \prod_{1 \leq i < j \leq q} (a_j^2 - t_i^2) \prod_{i=1}^{q} s(E^\vee; 1/t_i) \right)$$

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Using this formula and Proposition 4.1, the quadratic Schur function Darondeau–Pragacz was defined as a determinantal form (4.1). It seems difficult to characterize the quadratic Schur functions via the Gysin map (Proposition 4.1 and (4.2)). Namely, we adopt the following definition.

### 4.2 Universal quadratic Schur functions

#### 4.2.1 Definition of universal quadratic Schur functions

As the (ordinary) Schur function has been generalized to the universal Schur function, we wish to consider the universal analogue of the (ordinary) quadratic Schur function. As we reviewed in the previous subsection, the quadratic Schur function due to Darondeau–Pragacz was defined as a determinantal form (4.1). It seems difficult to generalize this determinantal form directly to the complex cobordism theory. Instead, we shall utilize the characterization of the quadratic Schur functions via the Gysin map (Proposition 4.1 and (4.2)). Namely, we adopt the following definition.

**Definition 4.2** (Universal quadratic Schur functions) Let $E \rightarrow X$ be a symplectic vector bundle of rank $2n$. Given a partition $\lambda \in \mathcal{P}_n$ with length $\ell(\lambda) = q \leq n$, consider...
the type $C$ full flag bundle $\varpi_{1,\ldots,q} : \mathcal{F}_q^C \longrightarrow X$, and the induced Gysin map $\varpi_{1,\ldots,q} : \text{MU}^*(\mathcal{F}_q^C) \longrightarrow \text{MU}^*(X)$.

Then the universal quadratic Schur function $s^{\mathcal{L}_n(2)}_\lambda (E^\vee) = s^{\mathcal{L}_n(2)}_\lambda (y_n)$ corresponding to $\lambda$ is defined to be

$$s^{\mathcal{L}_n(2)}_\lambda (E^\vee) = \sum_{\varpi \in C_n/C_n-q} w \cdot \left[ \prod_{i=1}^{q} y_i^{\lambda_i + 2n - 2i + 1} \cdot \prod_{1 \leq i < j \leq n} (y_i + L y_j) (y_i + L y_j) \right].$$

(4.3)

It follows immediately from the definition that under specialization from $F_L(u, v) = u + L v$ to $F_a(u, v) = u + v$, the function $s^{\mathcal{L}_n(2)}_\lambda (E^\vee)$ reduces to the ordinary quadratic Schur function $s^{\mathcal{L}_n(2)}_\lambda (E^\vee)$. Under specialization from $F_L(u, v) = u + L v$ to $F_{\text{m}}(u, v) = u \oplus v$, we obtain the $K$-theoretic quadratic Schur function, denoted as $G^{\mathcal{L}_n(2)}_\lambda (E^\vee) = G^{\mathcal{L}_n(2)}_\lambda (y_n)$, which should be regarded as a type $BCD$ analogue of the Grothendieck polynomial $G_\lambda (E^\vee) = G_\lambda (y_n)$. The precise definition is given as follows:

**Definition 4.3** ($K$-theoretic quadratic Schur functions) In the same setting as in Definition 4.2, the $K$-theoretic quadratic Schur function corresponding to the partition $\lambda \in P_n$ of length $q \leq n$ is defined as

$$G^{\mathcal{L}_n(2)}_\lambda (E^\vee) = G^{\mathcal{L}_n(2)}_\lambda (y_n)$$

$$= \sum_{\varpi \in C_n/C_n-q} w \cdot \left[ \prod_{i=1}^{q} y_i^{\lambda_i + 2n - 2i + 1} \cdot \prod_{1 \leq i < j \leq n} (y_i + y_j) (y_i + y_j) \right].$$

4.2.2 Gysin formulas for universal quadratic Schur functions

By Definition 4.2, the analogue of [8, Proposition 4.7] in complex cobordism can be easily obtained: Let $E \longrightarrow X$ be a symplectic vector bundle of rank $2n$, where we assume that the line bundle $L$ is trivial. Consider the isotropic Grassmann bundle $\varpi_q : \mathcal{F}_q^C \longrightarrow X$ for $q = 1, \ldots, n$. The induced Gysin map $\varpi_q^* : \text{MU}^*(\mathcal{F}_q^C) \longrightarrow \text{MU}^*(X)$ has the following symmetrizing operator description:

$$\varpi_q^*(f) = \sum_{\varpi \in C_n/C_n-q} w \cdot \left[ \prod_{i=1}^{q} (y_i + \ast \ast y_j) \cdot \prod_{1 \leq i < j \leq n} (y_i + y_j) (y_i + y_j) \right].$$

for $f \in \text{MU}^*(\mathcal{F}_q^C)$. Next, we need to define the universal quadratic Schur functions for arbitrary sequences of positive integers. Namely, for a sequence of positive integers $I = (I_1, \ldots, I_q) \in (\mathbb{Z}_{>0})^q$ (in this case, we say that the length $\ell(I)$ of $I$ is...
The corresponding universal quadratic Schur function \( s_{I_1}^{(2)}(E^\vee) = s_{I_2}^{(2)}(y_n) \) is defined by the same expression as (4.3). With these preparations, one can obtain the following result:

**Proposition 4.4** For a sequence of non-negative integers \( I = (I_1, \ldots, I_q) \in (\mathbb{Z}_{\geq 0})^q \), which satisfies the condition \( I_i > 2n - q - i + 1 \) for \( i = 1, \ldots, q \), one has the following Gysin formula:

\[
\varpi_{q*}(s_{I_1}^{(2)}(U_q^\vee)) = s_{I_1 - \rho_q}^{(2)}(E^\vee),
\]

where \( \rho_q = (2n - q, 2n - q - 1, \ldots, 2n - q - i + 1, \ldots, 2n - 2q + 1) \).

**Proof** By (2.1) and the symmetrizing operator description of \( \varpi_{q*} \) mentioned above, one can compute

\[
\varpi_{q*}(s_{I_1}^{(2)}(U_q^\vee)) = \varpi_{q*}(s_{I_1}^{(2)}(y_q)) = \sum_{\varpi \in C_n/S_q \times C_{n-q}} w \cdot \left[ \sum_{v \in S_q} v \cdot \left[ \prod_{i=1}^{q} y_i^{I_i + q - i} \prod_{1 \leq i < j \leq q} (y_i + \underline{L}, y_j) \prod_{1 < j \leq q} (y_i + \underline{L}, y_j) \right] \right].
\]

Since \( \prod_{i=1}^{q} [2]_L(y_i) \), \( \prod_{q+1 < j \leq n} (y_i + \underline{L}, y_j) \), and \( \prod_{1 \leq i < j \leq n} (y_i + \underline{L}, y_j) \) are all \( S_q \)-invariant, the above expression is equal to

\[
\sum_{\varpi \in C_n/S_q \times C_{n-q}} \sum_{v \in S_q} w v \cdot \left[ \prod_{i=1}^{q} y_i^{I_i + q - i} \prod_{1 \leq i \leq q} (y_i + \underline{L}, y_j) \prod_{1 \leq i \leq q} (y_i + \underline{L}, y_j) \right]
\]

\[
= \sum_{\varpi \in C_n/S_q \times C_{n-q}} u \cdot \left[ \prod_{i=1}^{q} y_i^{I_i - 2n + q + i - 1} \prod_{1 \leq i \leq q} (y_i + \underline{L}, y_j) \prod_{1 \leq i \leq q} (y_i + \underline{L}, y_j) \right]
\]

\[= s_{I_1 - \rho_q}^{(2)}(E^\vee), \]

as desired. \( \square \)

As a special case of the above proposition \((q = n)\), one immediately obtains a complex cobordism analogue of the result of Pragacz–Ratajaki [22, Theorem 5.13]:

**Corollary 4.5** (Pragacz–Ratajaki formula in complex cobordism) Let \( I = (I_1, \ldots, I_n) \in (\mathbb{Z}_{\geq 0})^n \) be a sequence of non-negative integers satisfying \( I_i > n - i + 1 \) \((i = 1, \ldots, n)\). Then, the following Gysin formula holds for the Lagrangian Grassmann bundle \( \varpi_n : LG_n(E) = F_n^\ell(E) \longrightarrow X \):

\[
\varpi_{n*}(s_{I_1}^{(2)}(U_n^\vee)) = s_{I_1 - \rho_n}^{(2)}(E^\vee).
\]
If we reduce the universal formal group law \( F_L(u, v) = u + L v \) to \( F_a(u, v) = u + v \), then the formula (4.5) reduces to

\[
\varpi_n^*(s_1(U_n^{\vee})) = s^{(2)}_{I-\rho_n}(E^{\vee}).
\] (4.6)

If one of the parts of \( I - \rho_n \) is odd, then the right-hand side of (4.6) is zero. Therefore, the element \( s_1(U_n^{\vee}) \) has a nonzero image under \( \varpi_n^* \) only if \( I \) is of the form \( 2J + \rho_n \) for some \( J_n \in \mathbb{Z}_{\geq 0} \). If \( I = 2J + \rho_n \), then one obtains

\[
\varpi_n^*(s_1(U_n^{\vee})) = s^{(2)}_{2J}(E^{\vee}) = s^{[2]}_J(E^{\vee})
\]
as observed in Sect. 4.1.1. This is the original Pragacz–Ratajski formula.

### 4.2.3 Generating function for universal quadratic Schur functions

As an application of our Darondeau–Pragacz formulas in complex cobordism, one can derive the generating function for the universal quadratic Schur functions. According to Theorem 3.6, the D–P formula for the Gysin map \( \varpi_1,\ldots,q^* : MU^*(\bigwedge^C F_{1,\ldots,q}(E)) \to MU^*(X) \) takes the following form: For a polynomial \( f(t_1, \ldots, t_q) \in MU^*(X)[t_1, \ldots, t_q] \), one has

\[
\varpi_1,\ldots,q^*(f(y_1, \ldots, y_q)) = \left[ \prod_{i=1}^q t_i^{2n-1} \right] \left( f(t_1, \ldots, t_q) \prod_{1 \leq i < j \leq q} (t_j + L t_i)(t_j + L t_i) \times \prod_{i=1}^q \mathcal{L}(E^{\vee}; 1/t_i) \right).
\]

This formula, together with Definition 4.2, yields the following expression:

\[
s^{(2)}_\lambda(E^{\vee}) = \varpi_1,\ldots,q^*(y^{1+\rho_{2n-1}+(2(n-q))})
\]

\[
= \left[ \prod_{i=1}^q t_i^{2n-1} \right] \left( \prod_{i=1}^q \mathcal{L}_{i+2n-2i+1} \prod_{1 \leq i < j \leq q} \mathcal{L}(E^{\vee}; 1/t_i) \right)
\]

\[
= \left[ \prod_{i=1}^q t_i^{-\lambda_i} \right] \left( \prod_{1 \leq i < j \leq q} \left( t_j + L t_i \right) \mathcal{L}(E^{\vee}; 1/t_i) \right).
\]

Since the \( MU^* \)-theory Chern roots of \( E^{\vee} \) are given by \( y_i, \bar{y}_i \) (\( 1 \leq i \leq n \)), from (3.1), the Segre series of \( E^{\vee} \) is given as follows:

\[
\mathcal{L}_L(E^{\vee}; u) = \frac{1}{\prod_{j=1}^q (z + L y_j)(z + L \bar{y}_j)} \bigg|_{z=\mu^{-1}}.
\]
Let us reformulate the above result in terms of symmetric functions: For independent variables $y_n = (y_1, \ldots, y_n)$, we set

$$s_{L,(2)}^L(u) \coloneqq \frac{1}{P_{L,(2)}^L(u) \prod_{j=1}^n (u + \varrho_j)(u + \varpi_j)},$$

$$s_{L,(2)}^L(u_1, \ldots, u_q) \coloneqq \prod_{i=1}^q s_{L,(2)}^L(u_i) \prod_{1 \leq i < j \leq q} t_j + \varrho_i t_i \cdot \frac{t_j + \varpi_i t_i}{t_j}.$$

Then, we have the following result:

**Theorem 4.6** For a partition $\lambda = (\lambda_1, \ldots, \lambda_q)$ of length $\ell(\lambda) = q \leq n$, the universal quadratic Schur function $s_{L,(2)}^L(y_n)$ is the coefficient of $u_{1-\lambda_1} \cdots u_{q-\lambda_q}$ in $s_{L,(2)}^L(u_1, \ldots, u_q)$, that is,

$$s_{L,(2)}^L(y_n) = \left[u_{1-\lambda_1} \cdots u_{q-\lambda_q}\right] (s_{L,(2)}^L(u_1, \ldots, u_q)).$$

If we specialize $F_{L}(u, v) = u + \varrho v$ to $F_{u}(u, v) = u + v$, then the factor $\prod_{1 \leq i < j \leq r} t_j + \varrho_i t_i \cdot \frac{t_j + \varpi_i t_i}{t_j}$ reduces to

$$\prod_{1 \leq i < j \leq q} \frac{t_j - t_i}{t_j} \cdot \frac{t_j + t_i}{t_j} = \prod_{1 \leq i < j \leq q} \left(1 - \frac{\beta^2 t_i^2}{t_j^2}\right) = \det ((t_i^2 - t_j^2)^{1 \leq i, j \leq q}).$$

Therefore, by Theorem 4.6, the determinantal formula (4.1) can be recovered.

In $K$-theory, we can say more: Let $E$ be a symplectic vector bundle of rank $2n$ with $K$-theoretic Chern roots $y_i, \varpi_i = \varrho_j (i = 1, \ldots, n)$. Then, by (3.1), the $K$-theoretic Segre series $G(E^\vee; 1/u)$ is given by

$$G(E^\vee; 1/u) = \frac{1}{1 - \beta u \prod_{j=1}^n (u \oplus y_j)(u \oplus \varrho_j)}.$$

Then, by Theorem 4.6, we have

$$G_{\lambda}^{(2)}(E^\vee) = \left[\prod_{i=1}^q t_i^{-\lambda_i}\right] \left(\prod_{1 \leq i < j \leq q} \frac{t_j \oplus t_i}{t_j} \cdot \frac{t_j \oplus \varrho_i t_i}{t_j} \prod_{i=1}^q G(E^\vee; 1/t_i)\right).$$

(4.7)

Notice that the following identity holds:

$$\prod_{1 \leq i < j \leq q} \frac{t_j \oplus t_i}{t_j} \cdot \frac{t_j \oplus \varrho_i t_i}{t_j} = \prod_{1 \leq i < j \leq q} \left(1 - \frac{t_i^2}{1 - \beta t_i} \cdot \frac{t_j^2}{1 - \beta t_j}\right) = \det ((1 - \beta t_i)^{1-j} t_i^{2(j-i)})^{1 \leq i, j \leq q}.$$
Therefore, by (4.7), we have the following result:

**Theorem 4.7** (Determinantal formula for the $K$-theoretic quadratic Schur function)

$$G_{\lambda}^{(2)}(E^\vee) = \det \left( \sum_{k=0}^{\infty} \binom{i-j}{k} (-\beta)^k G_{\lambda_i+2(j-i)+k}(E^\vee) \right)_{1 \leq i, j \leq q}.$$  (4.8)

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5 Appendix

5.1 Quillen’s Residue Formula

As we mentioned in Sect. 3.1.2, we proceed with the calculation of Quillen’s residue formula, i.e., the right-hand side of (3.3) in the following manner: first, we consider the case where $f(t) = t^N$, a monomial in $t$ of degree $N \geq 0$. We expand $\mathcal{P}(t, y_j)$ as $\sum_{\ell_j=0}^{\infty} \mathcal{P}_{\ell_j}(y_j)t^{\ell_j}$ for $j = 1, \ldots, n$. Then, we compute

$$t^N \mathcal{P}(t) \prod_{j=1}^{n}(t + \cdots) = t^N \times \frac{1}{\mathcal{P}(t)} \times \frac{1}{\prod_{j=1}^{n}(t - y_j)}$$

$$= t^{N-n} \times \frac{1}{\mathcal{P}(t)} \times \prod_{j=1}^{n}(t, y_j) \times \prod_{j=1}^{n}\left(1 - y_j t^{-1}\right)$$

$$= t^{N-n} \times \left(\sum_{k=0}^{\infty} [\mathcal{P}k]t^k\right) \times \prod_{j=1}^{n} \left(\sum_{\ell_j=0}^{\infty} \mathcal{P}_{\ell_j}(y_j)t^{\ell_j}\right)$$

$$\times \left(\sum_{m=0}^{\infty} h_m(y_n)t^{-m}\right).$$

For brevity, we put

$$\prod_{j=1}^{n} \left\{ \sum_{\ell_j=0}^{\infty} \mathcal{P}_{\ell_j}(y_j)t^{\ell_j} \right\} = \sum_{\ell=0}^{\infty} \left( \sum_{\ell_1+\cdots+\ell_n=\ell} \prod_{j=1}^{n} \mathcal{P}_{\ell_j}(y_j) \right) t^\ell = \sum_{\ell=0}^{\infty} \mathcal{P}_\ell(y_n)t^\ell.$$
Then, the computation continues as

\[
i^{N-n} \times \left( \sum_{k=0}^{\infty} \left[ \mathbb{C} P^k \right]^k \right) \times \left\{ \sum_{r=-\infty}^{\infty} \left( \sum_{\ell=r}^{\infty} \mathcal{P}_\ell(y_n) h_{\ell-r}(y_n) \right) i^r \right\}
\]

\[
= i^{N-n} \times \left[ \sum_{k=0}^{\infty} \left( \sum_{r=-\infty}^{\infty} \left[ \mathbb{C} P^k \right]^k \right) \mathcal{P}_\ell(y_n) h_{\ell-r}(y_n) \right] i^r + \sum_{r=1}^{\infty} \left( \sum_{\ell=0}^{\infty} \mathcal{P}_\ell(y_n) h_{\ell+r}(y_n) \right) i^{-r}
\]

\[
= i^{N-n} \times \left[ \sum_{s=-\infty}^{\infty} \left( \sum_{k=s+1}^{\infty} \sum_{r=-\infty}^{\infty} \left[ \mathbb{C} P^k \right] \mathcal{P}_\ell(y_n) h_{k+\ell-r}(y_n) \right) i^s \right]
\]

Next, we extract the coefficient of \( t^{-1} \) in the above formal Laurent series. In the case of \( N \geq n \), one sees immediately that the coefficient of \( t^{-1} \) is

\[
\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \left[ \mathbb{C} P^k \right] \mathcal{P}_\ell(y_n) h_{k+\ell+N-n+1}(y_n).
\]

In the case of \( 0 \leq N < n \), the coefficient of \( t^{-1} \) is

\[
\sum_{k=0}^{n-N-1} \sum_{\ell=n-N-1-k}^{\infty} \left[ \mathbb{C} P^k \right] \mathcal{P}_\ell(y_n) h_{k+\ell+N-n+1}(y_n)
\]

\[
+ \sum_{k=n-N}^{\infty} \sum_{\ell=0}^{\infty} \left[ \mathbb{C} P^k \right] \mathcal{P}_\ell(y_n) h_{k+\ell+N-n+1}(y_n)
\]

\[
= \sum_{k=0}^{\infty} \sum_{\ell=n-N-1-k}^{\infty} \left[ \mathbb{C} P^k \right] \mathcal{P}_\ell(y_n) h_{k+\ell+N-n+1}(y_n).
\]

Summing up the above calculation, we get the following result:

\[
\text{Res}_{\ell=0}^{\mathscr{L}(t) \prod_{j=1}^{n} (t + L_j)} i^{N} = \begin{cases} 
\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \left[ \mathbb{C} P^k \right] \mathcal{P}_\ell(y_n) h_{k+\ell+N-n+1}(y_n) & (N \geq n), \\
\sum_{k=0}^{\infty} \sum_{\ell=n-N-1-k}^{\infty} \left[ \mathbb{C} P^k \right] \mathcal{P}_\ell(y_n) h_{k+\ell+N-n+1}(y_n) & (0 \leq N < n).
\end{cases}
\] (5.1)
For a general polynomial of the form $f(t) = \sum_{N=0}^{M} a_N t^N \in MU^*(X)[t]$, one has

$$\varpi_1^*(f(y_1)) = \varpi_1^* \left( \sum_{N=0}^{M} a_N y_1^N \right) = \sum_{N=0}^{M} a_N \varpi_1^*(y_1^N)$$

since the Gysin map $\varpi_1^*$ is an $MU^*(X)$-homomorphism. From this, we can calculate $\varpi_1^*(f(y_1))$.

References

1. Adams, J.F.: Stable Homotopy and Generalised Homology. Chicago Lectures in Mathematics. The University of Chicago Press, Chicago (1974)
2. Bressler, P., Evens, S.: The Schubert calculus, braid relations, and generalized cohomology. Trans. Am. Math. Soc. 317(2), 799–811 (1990)
3. Brion, M.: The push-forward and Todd class of flag bundles. Parameter Spaces 36, 45–50 (1996)
4. Buch, A.S.: A Littlewood–Richardson rule for the $K$-theory of Grassmannians. Acta Math. 189, 37–78 (2002)
5. Buch, A.S.: Grothendieck classes of quiver varieties. Duke Math. J. 115(1), 75–103 (2002)
6. Conner, P.E., Floyd, E.E.: The Relation of Cobordism to $K$-theories. Lecture Notes in Mathematics, vol. 28. Springer, Berlin (1966)
7. Damon, J.: The Gysin homomorphism for flag bundles. Am. J. Math. 95, 643–659 (1973)
8. Darondeau, L., Pragacz, P.: Universal Gysin formulas for flag bundles. Internat. J. Math. 28(11), 1750077 (2017)
9. Edidin, D., Graham, W.: Characteristic classes and quadric bundles. Duke Math. J. 78, 277–299 (1995)
10. Fel’dman, K.E.: An equivariant analog of the Poincaré–Hopf theorem. J. Math. Sci. 113, 906–914 (2003). (Translated from Zap. Nauchn. Sem. POMI 267(2001), 303–318)
11. Fulton, W.: Intersection Theory, 2nd edn. Springer, Berlin (1998)
12. Fulton, W., Pragacz, P.: Schubert Varieties and Degeneracy Loci. Lecture Notes in Mathematics, vol. 1689. Springer, Berlin (1998)
13. Hudson, T., Ikeda, T., Matsumura, T., Naruse, H.: Degeneracy loci classes in $K$-theory-determinantal and Pfaffian formula. Adv. Math. 320, 115–156 (2017)
14. Hudson, T., Matsumura, T.: Segre classes and Damon–Kempf–Laksov formula in algebraic cobordism. Math. Ann. 374, 1439–1457 (2019)
15. Lazard, M.: Sur les groupes de Lie formels à un paramètre. Bull. Soc. Math. Fr. 83, 251–274 (1955)
16. Levine, M., Morel, F.: Algebraic Cobordism. Springer Monographs in Mathematics. Springer, Berlin (2007)
17. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, Oxford (1995)
18. Nakagawa, M., Naruse, H.: Generalized (co)homology of the loop spaces of classical groups and the universal factorial Schur $P$- and $Q$-functions. Schubert Calculus-Osaka 2012, 337–417, Adv. Stud. Pure Math., 71, Math. Soc. Japan, Tokyo (2016)
19. Nakagawa, M., Naruse, H.: Universal Gysin formulas for the universal Hall–Littlewood functions. Contemp. Math. 708, 201–244 (2018)
20. Nakagawa, M., Naruse, H.: Generating functions for the universal factorial Hall–Littlewood $P$- and $Q$-functions. arXiv:1705.04791
21. Naruse, H.: Elementary proof and application of the generating functions for generalized Hall–Littlewood functions. J. Algebra 516, 197–209 (2018)
22. Pragacz, P., Ratajski, J.: Formulas for Lagrangian and orthogonal degeneracy loci; ˜$Q$-polynomial approach. Compos. Math. 107, 11–87 (1997)
23. Quillen, D.: On the formal group laws of unoriented and complex cobordism theory. Bull. Am. Math. Soc. 75(6), 1293–1298 (1969)
24. Quillen, D.: Elementary proofs of some results of cobordism theory using Steenrod operations. Adv. Math. 7, 29–56 (1971)
25. Switzer, R.: Algebraic Topology–Homology and Homotopy, Classics in Mathematics (Reprint of the 1975th edn). Springer, Berlin (2002)

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