GLOBAL AND TOUCHDOWN BEHAVIOUR OF
THE GENERALIZED MEMS DEVICE EQUATION

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Abstract. We will prove the local and global existence of solutions of the generalized micro-electromechanical system (MEMS) equation $u_t = \Delta u + \lambda f(x)/g(u)$, $u < 1$, in $\Omega \times (0, \infty)$, $u(x,t) = 0$ on $\partial\Omega \times (0, \infty)$, $u(x,0) = u_0$ in $\Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\lambda > 0$ is a constant, $0 \leq f \in C^\alpha(\bar{\Omega})$, $f \neq 0$, for some constant $0 < \alpha < 1$, $0 < g \in C^2((-\infty,1))$ such that $g'(s) \leq 0$ for any $s < 1$ and $u_0 \in L^1(\Omega)$ with $u_0 \leq a < 1$ for some constant $a$. We prove that there exists a constant $\lambda^* = \lambda^*(\Omega, f, g) > 0$ such that the associated stationary problem has a solution for any $0 \leq \lambda < \lambda^*$ and has no solution for any $\lambda > \lambda^*$. We obtain comparison theorems for the generalized MEMS equation. Under a mild assumption on the initial value we prove the convergence of global solutions to the solution of the corresponding stationary elliptic equation as $t \to \infty$ for any $0 \leq \lambda < \lambda^*$. We also obtain various conditions for the existence of a touchdown time $T > 0$ for the solution $u$. That is a time $T > 0$ such that $\lim_{t \to T} \sup_{\Omega} u(\cdot, t) = 1$.

Micro-electromechanical systems (MEMS) are widely used nowadays in many electronic devices including accelerometers for airbag deployment in cars, inkjet printer heads, and the device for the protection of hard disk, etc. Interested readers can read the book, Modeling MEMS and NEMS [PB], by J.A.Pelesko and D.H. Berstein for the mathematical modeling and various applications of MEMS devices. Due to the importance of MEMS devices it is important to get a detail analysis of the mathematical models of MEMS devices. In recent years there is a lot of study on the evolution and stationary equations arising from MEMS devices by P. Esposito, N. Ghoussoub, Y. Guo, Z. Pan and M.J. Ward [EGhG],[GhG1],[GhG2],[GPW],[G], N.I. Kavallaris, T. Miyasita and T. Suzuki [KMS], F. Lin and Y. Yang [LY], L. Ma and J.C. Wei [MW] and J.A.Pelesko [P], etc.

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Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^2$ domain. Let
\[ 0 \leq f \in C^\alpha(\overline{\Omega}) \quad \text{for some constant } 0 < \alpha < 1 \quad \text{and } f \not\equiv 0 \quad \text{in } \Omega \] (0.1)
and let
\[ 0 < g \in C^2((-\infty, 1)) \quad \text{such that } g'(s) \leq 0 \quad \forall s < 1. \] (0.2)

In this paper we will study the generalized MEMS equation
\[
\begin{cases}
  u_t = \Delta u + \frac{\lambda f(x)}{g(u)} & \text{in } \Omega \times (0, T) \\
  u(x, t) = 0 & \text{on } \partial \Omega \times (0, T) \\
  u(x, 0) = u_0 & \text{in } \Omega
\end{cases}
\] (0.3)
and the associated stationary problem,
\[
\begin{cases}
  -\Delta v = \frac{\lambda f(x)}{g(v)} & \text{in } \Omega \\
  v(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\] (S\_\lambda)

When $g(u) = (1-u)^2$, (0.3) and (S\_\lambda) reduces to the evolution and stationary MEMS equations respectively which were studied extensively in [EGhG],[GhG1],[GhG2],[GPW],[G],[P]. An equation similar to (S\_\lambda) arising from the motion of thin films of viscous fluid is studied by H. Jiang and W.M. Ni in [JN]. The asymptotic and touchdown behaviour of solutions of (S\_\lambda) with $g(u) = (1-u)^2$ and $u_0 \equiv 0$ was studied in [GhG2] and [G]. When $g(u) = (1-u)^p$ with $p > 0$, (S\_\lambda) was studied by L. Ma, J.C. Wei, Z. Wang and L. Ruan [MW],[WR]. The equation (0.3) and (S\_\lambda) with $g(u) = (1-u)^p$ and $u_0 \in [0,1)$ were also studied by N.I. Kavallaris, T. Miyasita, T. Suzuki [KMS]. By the results of [GhG1],[GhG2], and [WR], when $g(u) = (1-u)^p$ with $p > 0$, there exists a constant $\lambda^* > 0$ such that (S\_\lambda) has a solution for any $0 \leq \lambda < \lambda^*$ and (S\_\lambda) has no solution for any $\lambda > \lambda^*$.

In this paper we will show that there exists a constant $\lambda^* > 0$ such that similar results hold for (S\_\lambda). The constant $\lambda^*$ is called the pull-in voltage of the equation (S\_\lambda) in the literature of MEMS. For any $u_0 \in L^1(\Omega)$ with $u_0 \leq a < 1$ for some constant $a$ we will prove the local existence and comparison theorems of solutions of (0.3). If $u$ is a global solution of (0.3) with $0 \leq \lambda < \lambda^*$, then under a mild assumption on the initial value we prove the convergence of the solution of (0.3) as $t \to \infty$. We also obtain various conditions for the solution $u$ of (0.3) to touchdown at a finite time. That is the existence of a time $T > 0$ such that
\[ \lim_{t \to T} \sup_{\Omega} u(\cdot, t) = 1. \]

The plan of the paper is as follows. In section 1 we will prove the existence of finite pull-in voltage $\lambda^* > 0$ of (S\_\lambda) and the existence and non-existence of solutions of (S\_\lambda). We will also prove the non-existence of bounded solution of the stationary problem in $\mathbb{R}^n$. In section 2 we will prove the existence of solutions and various comparison results for
solutions of (0.3). In section 3 we will prove the global convergence of solutions of (0.3) for \(0 \leq \lambda < \lambda^*\). We also obtain various conditions for the solutions of (0.3) to have finite touchdown time.

We start with a definition. We say that \(v\) is a solution (subsolution, supersolution respectively) of \((S_\lambda)\) if \(v \in C^2(\Omega) \cap C(\overline{\Omega})\), \(v < 1\) in \(\Omega\), satisfies

\[-\Delta v = \frac{\lambda f(x)}{g(v)} \quad \text{in } \Omega\]

(\(\leq\), \(\geq\) respectively) with \(v(x) = 0\) (\(\leq\), \(\geq\) respectively) on \(\partial \Omega\). Note that by the maximum principle for superharmonic function if \(v\) is a solution or supersolution of \((S_\lambda)\), then \(v \geq 0\) in \(\Omega\). We say that \(v\) is a minimal solution of \((S_\lambda)\) if \(v\) is a solution of \((S_\lambda)\) and \(v \leq \tilde{v}\) in \(\Omega\) for any solution \(\tilde{v}\) of \((S_\lambda)\).

For any \(u_0 \in L^1(\Omega)\) with \(u_0 \leq a\) on \(\Omega\)

\(\text{(0.4)}\)

we say that \(u\) is a solution (subsolution, supersolution respectively) of (0.3) in \(\Omega \times (0, T)\) if \(u \in C^{2,1}(\Omega \times (0, T)) \cap C(\overline{\Omega} \times (0, T))\) satisfies

\[u_t = \Delta u + \frac{\lambda f(x)}{g(u)} \quad \text{in } \Omega \times (0, T)\]

(\(\leq\), \(\geq\) respectively) with \(u(x,t) = 0\) (\(\leq\), \(\geq\) respectively) on \(\partial \Omega \times (0, T)\),

\[
\sup_{\overline{\Omega} \times (0, T')} u(x,t) < 1 \quad \forall 0 < T' < T
\]

and

\[
\|u(\cdot, t) - u_0\|_{L^1(\Omega)} \to 0 \quad \text{as } t \to 0.
\]

\(\text{(0.5)}\)

For any solution \(u\) of (0.3) we define the touchdown time \(T_\lambda = T_\lambda(\Omega, f, g) > 0\) as the time which satisfies

\[
\begin{aligned}
\sup_{\Omega} u(x,t) < 1 & \quad \forall 0 < t < T_\lambda \\
\lim_{t \to T_\lambda} \sup_{\Omega} u(x,t) = 1.
\end{aligned}
\]

We say that \(u\) has a finite touchdown time if \(T_\lambda < \infty\) and we say that \(u\) touchdowns at time infinity if \(T_\lambda = \infty\).

Let \(G(x, y, t), x, y \in \Omega, t > 0\), be the Dirichlet Green function of the heat equation in \(\Omega \times (0, \infty)\). That is for any \(y \in \Omega\),

\[
\begin{aligned}
\partial_t G = \Delta_x G & \quad \text{in } \Omega \times (0, \infty) \\
G(x, y, t) = 0 & \quad \forall x \in \partial \Omega, t > 0 \\
\lim_{t \to 0} G(x, y, t) = \delta_y
\end{aligned}
\]

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Moreover, the existence and non-existence of solutions of \((S)\) holds for some constant \(\lambda > \lambda^*\). Let \(\phi_\lambda\) be the first Dirichlet eigenfunction of \(-\Delta\) in \(\Omega\) and \(\psi_\lambda\) be the corresponding positive eigenfunction normalized such that
\[
\max_{\Omega} \psi_\lambda = 1 \quad \text{and} \quad s_\lambda = \min_{\overline{\Omega}} \psi_\lambda > 0.
\]

Let \(\mu_1 > 0\) be the first Dirichlet eigenvalue of \(-\Delta\) in \(\Omega\) and let \(\phi_1\) be the first positive Dirichlet eigenfunction of \(-\Delta\) in \(\Omega\) normalized such that \(\int_\Omega \phi_1 \, dx = 1\). Let
\[
\nu_\Omega = \sup_{\Omega_1 \in C} \mu_\lambda \, s_\lambda.
\]

Section 1

In this section we will prove the existence of finite pull-in voltage \(\lambda^* > 0\) of \((S_\lambda)\) and the existence and non-existence of solutions of \((S_\lambda)\). We also obtain various estimates for \(\lambda^*\).

Theorem 1.1. Suppose \(f\) satisfies (0.1) and \(g\) satisfies (0.2). Then there exists a constant \(\lambda^* = \lambda^*(\Omega, f, g) > 0\) such that

(i) \(\forall 0 \leq \lambda < \lambda^*\), there exists at least one solution of \((S_\lambda)\)

(ii) \(\forall \lambda > \lambda^*\), there exists no solution of \((S_\lambda)\).

Moreover
\[
\nu_\Omega \frac{\sup_{0 < s < 1} s g(s)}{\max_{\overline{\Omega}} f} \leq \lambda^* \leq \frac{\mu_1 g(0)}{\int_\Omega f \phi_1 \, dx}.
\]
Proof. Since the proof of the theorem is similar to the proof of Theorem 2.1 of [GhG1], we will sketch the argument here. Note that \( v \equiv 0 \) in \( \Omega \) is a solution of \((S_\lambda)\) when \( \lambda = 0. \)

Let \( D = \{ \lambda > 0 : (S_\lambda) \text{ has a solution} \} \) and

\[
\lambda^* = \lambda^*(\Omega, f, g) = \sup_{\lambda \in D} \lambda.
\]

We claim that \( D \neq \emptyset \). In order to prove the claim we first observe that \( v \equiv 0 \) on \( \Omega \) is a subsolution of \((S_\lambda)\) for any \( \lambda > 0. \) We will next construct a supersolution of \((S_\lambda)\). For any \( \Omega_1 \in \mathcal{C} \) and \( 0 < A < 1 \) let \( \psi = A\psi_{\Omega_1} \). Then by (0.2) for any

\[
0 \leq \lambda \leq \mu_{\Omega_1}s_{\Omega_1}|A|g(\Omega_1)\max_{\Omega_1}f,
\]

we have

\[
-\Delta \psi = A\mu_{\Omega_1}\psi_{\Omega_1} \geq \frac{\lambda f_{\Omega_1}}{g(A\psi_{\Omega_1})} \quad \text{in } \Omega.
\]

Hence \( \psi \) is a supersolution of \((S_\lambda)\). Let \( v_0 \equiv 0 \) in \( \Omega \) and for any \( k \geq 1 \), let \( v_k \) be the solution of

\[
\begin{cases}
-\Delta v_k = \frac{\lambda f(x)}{g(v_{k-1})} & \text{in } \Omega \\
v_k(x) = 0 & \text{on } \partial \Omega.
\end{cases}
\]

By (0.2) and an argument similar to that of [GhG1], \( 0 \leq v_k \leq v_{k+1} \leq \psi < 1 \) in \( \Omega \) for all \( k \geq 0 \) and \( v_k \) will converge to the minimal solution \( v \) of \((S_\lambda)\) as \( k \to \infty. \) Hence \( D \neq \emptyset \) and the left hand side inequality of (1.1) holds.

Suppose now \( v \) is a solution \((S_\lambda)\). Multiplying \((S_\lambda)\) by \( \phi_1 \) and integrating over \( \Omega \), by (0.2) we have

\[
\mu_1 \geq \mu_1 \int_\Omega v \phi_1 \, dx = -\int_\Omega v \Delta \phi_1 \, dx = -\int_\Omega \phi_1 \Delta v \, dx = \lambda \int_\Omega \frac{f \phi_1}{g(v)} \, dx \geq \frac{\lambda}{g(0)} \int_\Omega f \phi_1 \, dx.
\]

Hence

\[
\lambda \leq \frac{\mu_1 g(0)}{\int_\Omega f \phi_1 \, dx}.
\]

Thus the right hand side inequality of (1.1) and (ii) follows. For any \( 0 \leq \lambda < \lambda^* \), there exists \( \lambda < \lambda_1 < \lambda^* \) such that \((S_{\lambda_1})\) has a solution \( v_{\lambda_1} \). Then \( v_{\lambda_1} \) is a supersolution of \((S_\lambda)\).

By (0.2) and the monotone iteration scheme as before (cf. [GhG1]) \((S_\lambda)\) has a solution \( v \) satisfying \( 0 \leq v \leq v_{\lambda_1} \) in \( \Omega \) and (i) follows.

We will now let \( \lambda^* \) be given by Theorem 1.1 for the rest of the paper. The following result improves the upper bound of \( \lambda^* \) of Theorem 1.1.
**Proposition 1.2.** Suppose $f$ satisfies (0.1) and $g$ satisfies (0.2). Then

$$\lambda^* \leq \mu_1 \frac{\int_0^1 g(s) \, ds}{\int_\Omega f \phi_1 \, dx} \quad (1.2)$$

where

$$H(v) = \int_v^1 g(s) \, ds. \quad (1.3)$$

**Proof.** Suppose $v$ is a solution of $(S_\lambda)$. Multiplying $(S_\lambda)$ by $g(v)\phi_1$ and integrating over $\Omega$,

$$\lambda \int_\Omega f \phi_1 \, dx = - \int_\Omega g(v)\phi_1 \Delta v \, dx$$

$$= \int_\Omega \nabla (g(v)\phi_1) \cdot \nabla v \, dx - \int_{\partial \Omega} g(v)\phi_1 \frac{\partial v}{\partial \nu} \, d\sigma$$

$$= \int_\Omega g'(v)\phi_1 |\nabla v|^2 \, dx + \int_\Omega g(v)\nabla \phi_1 \cdot \nabla v \, dx$$

$$\leq - \int_\Omega \nabla \phi_1 \cdot \nabla H(v) \, dx$$

$$= \int_\Omega H(v)\Delta \phi_1 \, dx - \int_{\partial \Omega} H(v) \frac{\partial \phi_1}{\partial \nu} \, d\sigma$$

$$= - \mu_1 \int_\Omega H(v)\phi_1 \, dx - H(0) \int_{\partial \Omega} \frac{\partial \phi_1}{\partial \nu} \, d\sigma$$

$$= - \mu_1 \int_\Omega H(v)\phi_1 \, dx - H(0) \int_\Omega \Delta \phi_1 \, dx$$

$$= - \mu_1 \int_\Omega H(v)\phi_1 \, dx + \mu_1 H(0) \int_\Omega \phi_1 \, dx$$

$$= - \mu_1 \int_\Omega H(v)\phi_1 \, dx + \mu_1 H(0)$$

and (1.2) follows.

We will next prove a more computable bound for $\lambda^*$.

**Proposition 1.3.** Suppose $f \in C^1(\overline{\Omega})$ satisfies

$$\delta_1 = \inf_{\Omega} f > 0, \quad (1.4)$$

g satisfies (0.2), and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a strictly star-shape domain such that $x \cdot \nu \geq b > 0$ on $\partial \Omega$ where $\nu$ is the unit outward normal to $\partial \Omega$ at $x \in \partial \Omega$. Then

$$\lambda^* \leq \frac{(n+2)\|f\|_{L^\infty} + 2b_1)|\partial \Omega|}{\delta_1^2 b_1 |\Omega|} g(0)$$
where $b_1 = \sup_{\Omega} |x \cdot \nabla f|$. In particular if $\Omega = B_R$, then

$$
\lambda^* \leq \frac{n((n + 2)\|f\|_{L^\infty} + 2b_1)}{\delta^2 R} g(0).
$$

**Proof.** Suppose $\lambda > 0$ and $v$ is a solution of $(S_\lambda)$. By $(S_\lambda)$ and the Pohozaev identity [N],

$$
\frac{1}{2} \int_{\partial\Omega} (x \cdot \nu) \left( \frac{\partial v}{\partial \nu} \right)^2 d\sigma = \lambda n \int_\Omega f(x) \left( \int_0^{v(x)} \frac{ds}{g(s)} \right) dx - \lambda \frac{(n - 2)}{2} \int_\Omega \frac{vf(x)}{g(v)} dx
$$

$$
+ \lambda \int_\Omega (x \cdot \nabla f(x)) \left( \int_0^{v(x)} \frac{ds}{g(s)} \right) dx.
$$

(1.5)

By (0.2),

$$
\left( \int_0^v \frac{ds}{g(s)} \right) \leq \frac{v}{g(v)}.
$$

Hence the right hand side of (1.5) is less than

$$
\leq \lambda n \int_\Omega \frac{vf(x)}{g(v)} dx - \lambda \frac{(n - 2)}{2} \int_\Omega \frac{vf(x)}{g(v)} dx + \lambda b_1 \int_\Omega \frac{v}{g(v)} dx
$$

$$
\leq \lambda \left( \frac{(n + 2)\|f\|_{L^\infty} + b_1}{2} \right) \int_\Omega \frac{dx}{g(v)}.
$$

(1.6)

Now by the Holder inequality, the Green theorem and $(S_\lambda)$,

$$
\int_{\partial\Omega} (x \cdot \nu) \left( \frac{\partial v}{\partial \nu} \right)^2 d\sigma \geq \frac{b}{|\partial\Omega|} \left( \int_{\partial\Omega} \frac{\partial v}{\partial \nu} d\sigma \right)^2 \geq \frac{b}{|\partial\Omega|} \left( \int_\Omega \Delta v dx \right)^2 \geq \frac{b\lambda^2 \delta^2}{|\partial\Omega|} \left( \int_\Omega \frac{dx}{g(v)} \right)^2.
$$

(1.7)

By (1.5), (1.6) and (1.7),

$$
\frac{b\lambda^2 \delta^2}{2|\partial\Omega|} \left( \int_\Omega \frac{dx}{g(v)} \right)^2 \leq \lambda \left( \frac{(n + 2)\|f\|_{L^\infty} + b_1}{2} \right) \int_\Omega \frac{dx}{g(v)}
$$

$$
\Rightarrow \frac{(n + 2)\|f\|_{L^\infty} + 2b_1}{b} |\partial\Omega| \geq \lambda \delta^2 \int_\Omega \frac{dx}{g(v)} \geq \frac{\lambda \delta^2 |\Omega|}{g(0)}
$$

$$
\Rightarrow \lambda \leq \frac{(n + 2)\|f\|_{L^\infty} + 2b_1 |\partial\Omega|}{\delta^2 b |\Omega|} g(0)
$$

$$
\Rightarrow \lambda^* \leq \frac{(n + 2)\|f\|_{L^\infty} + 2b_1 |\partial\Omega|}{\delta^2 b |\Omega|} g(0)
$$

and the proposition follows.
Corollary 1.4. Let \( f \in C^1(\Omega) \) satisfy (1.4) such that \( \text{supp} \, \nabla f \subset B_{R_1} \) for some constant \( R_1 > 1 \) and let \( g \) satisfy (0.2). For any \( \lambda > 0 \) there does not exist any bounded solution for the problem,

\[
-\Delta w = \frac{\lambda f(x)}{g(w)}, \quad w < 1, \quad \text{in } \mathbb{R}^n
\]  

(1.8)

Proof. Suppose there exists \( \lambda > 0 \) such that (1.8) has a bounded solution \( w \). Without loss of generality we may assume that \( 0 \leq w < 1 \) in \( \mathbb{R}^n \). Let \( R_2 = \frac{2m((n+2)\|f\|_{L^\infty} + 2R_1\|\nabla f\|_{L^\infty})g(0)}{\delta_1^2 \lambda} \).

By Proposition 1.3 \( \lambda^*(B_{R_2}, f, g) \leq \lambda/2 \). On the other hand since \( w \) is a supersolution of \((S_\lambda)\) with \( \Omega = B_{R_2} \), by the construction of solutions of \((S_\lambda)\) in Theorem 1.1, there exists a solution \( v \) of \((S_\lambda)\) with \( \Omega = B_{R_2} \) satisfying \( 0 \leq v \leq w \). Hence \( \lambda^*(B_{R_2}, f, g) \geq \lambda \) and contradiction arises. Thus no such solution \( w \) exists.

Proposition 1.5. Let \( \Omega_1 \subset \Omega_2 \) and let \( f_1, f_2 \) satisfy (0.1) in \( \Omega_1, \Omega_2 \), respectively for some constant \( 0 < \alpha < 1 \) such that \( f_1 \leq f_2 \) in \( \Omega_1 \). Let \( g_1, g_2 \) satisfy (0.2) such that \( g_1(s) \geq g_2(s) > 0 \) for any \( s < 1 \). Then \( \lambda^*(\Omega_1, f_1, g_1) \geq \lambda^*(\Omega_2, f_2, g_2) \). If \( 0 \leq \lambda < \lambda^*(\Omega_2, f_2, g_2) \) and \( v_1, v_2 \), are the minimal solutions of \((S_\lambda)\) with \( \Omega = \Omega_1, \Omega_2 \), \( f = f_1, f_2, g = g_1, g_2 \), respectively, then \( v_1 \leq v_2 \) in \( \Omega_1 \). If moreover \( \Omega_1 = \Omega_2 = \Omega \) and \( f_1 \neq f_2 \), then \( v_1 < v_2 \) in \( \Omega \).

Proof. For any \( \lambda < \lambda^*(\Omega_2, f_2, g_2) \), let \( v_2 \) be the minimal solution of \((S_\lambda)\) with \( \Omega = \Omega_2, f = f_2, g = g_2 \). Then \( v_2 \) is a supersolution of \((S_\lambda)\) with \( \Omega = \Omega_1, f = f_1, g = g_1 \). Since \( 0 \) is a subsolution of \((S_\lambda)\) with \( \Omega = \Omega_1, f = f_1, g = g_1 \), by the monotone iteration scheme for the construction of solution of \((S_\lambda)\) as in the proof of Theorem 1.1 the minimal solution \( v_1 \) of \((S_\lambda)\) with \( \Omega = \Omega_1, f = f_1, g = g_1 \) satisfies \( 0 \leq v_1 \leq v_2 \) in \( \Omega_1 \). Hence \( \lambda^*(\Omega_1, f_1, g_1) \geq \lambda^*(\Omega_2, f_2, g_2) \).

We next suppose that \( \Omega_1 = \Omega_2 = \Omega \) and \( f_1 \neq f_2 \). Let \( G(x, y) \) be the Green function for \( \Delta \) in \( \Omega \). Then

\[
v_i(x) = \lambda \int_\Omega G(x, y) \frac{f_i(y)}{g_i(v_i(y))} \, dy \quad \forall i = 1, 2.
\]  

(1.9)

Since \( v_1 \leq v_2 \) in \( \Omega \), by (0.2) \( f_1(x)/g_1(v_1) \leq f_2(x)/g_2(v_2) \) in \( \Omega \). If \( f_1 \neq f_2 \), there exists a set \( A \subset \Omega \) of positive measure such that \( f_1(x)/g_1(v_1) < f_2(x)/g_2(v_2) \) in \( A \). Then by (1.9), \( v_1 < v_2 \) in \( \Omega \) and the proposition follows.

For any solution \( v \) of \((S_\lambda)\) we let

\[
L_{v, \lambda}w = -\Delta w + \lambda \frac{f(x)g'(v)}{g(v)^2} w
\]

be the linearized operator of \((S_\lambda)\) around the solution \( v \). Let

\[
\tilde{\mu}_1 = \tilde{\mu}_1(\lambda, v) = \inf_{w \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla w|^2 \, dx + \lambda \int_\Omega (fg'(v)/g(v)^2)w^2 \, dx}{\int_\Omega w^2 \, dx}
\]

and \( \tilde{\phi}_1 \) be the first eigenvalue and the corresponding first positive eigenfunction of \( L_{v, \lambda} \). We say that \( v \) is a stable solution of \((S_\lambda)\) if \( v \) is a solution of \((S_\lambda)\) with \( \tilde{\mu}_1(\lambda, v) > 0 \).
Theorem 1.6. Let $f$ satisfy (0.1) and $g$ satisfy (0.2) and

$$\left(\frac{1}{g}\right)''(s) \geq 0 \quad \forall s < 1.$$  \hspace{1cm} (1.10)

Suppose $v$ and $\tilde{v}$ are solution and supersolution of $(S_{\lambda})$ respectively. If $\tilde{\mu}_1 = \tilde{\mu}_1(\lambda, v) > 0$, then $\tilde{v} \geq v$ in $\Omega$. If $\tilde{\mu}_1 = 0$, then $\tilde{v} \equiv v$ in $\Omega$.

Proof. We will use a modification of the proof of Lemma 4.1 of [GhG1] to prove the theorem. Let

$$h(x, s) = -\Delta(s\tilde{v} + (1 - s)v) - \frac{\lambda f}{g(s\tilde{v} + (1 - s)v)} \quad \forall 0 \leq s \leq 1.$$

Then

$$h(x, 0) = 0. \hspace{1cm} (1.11)$$

By (1.10) and the Jensen inequality,

$$-\Delta(s\tilde{v} + (1 - s)v) = \lambda f \left( \frac{s}{g(\tilde{v})} + \frac{1 - s}{g(v)} \right) \geq \frac{\lambda f}{g(s\tilde{v} + (1 - s)v)} \quad \text{in } \Omega \quad \forall 0 \leq s \leq 1.$$

Hence

$$h(x, s) \geq 0 \quad \text{in } \Omega \quad \forall 0 \leq s \leq 1. \hspace{1cm} (1.12)$$

By (1.11) and (1.12),

$$\frac{\partial h}{\partial s}(x, 0) \geq 0 \quad \Rightarrow \quad -\Delta(\tilde{v} - v) + \lambda f \frac{g'(v)}{g(v)^2}(\tilde{v} - v) \geq 0 \quad \text{in } \Omega. \hspace{1cm} (1.13)$$

Suppose first $\tilde{\mu}_1 > 0$. Multiplying (1.13) by $(\tilde{v} - v)_-$ and integrating over $\Omega$,

$$0 \geq \int_{\Omega} |\nabla(\tilde{v} - v)_-|^2 \, dx + \lambda \int_{\Omega} \frac{f g'(v)}{g(v)^2} (\tilde{v} - v)_-^2 \, dx$$

$$\geq \tilde{\mu}_1 \int_{\Omega} (\tilde{v} - v)_-^2 \, dx$$

$$\Rightarrow \quad \tilde{v} \geq v \quad \text{in } \Omega. \hspace{1cm} (1.14)$$

Suppose now $\tilde{\mu}_1 = 0$. Multiplying (1.13) by $\tilde{\phi}_1$ and integrating over $\Omega$,

$$0 \leq -\int_{\Omega} \tilde{\phi}_1 \Delta(\tilde{v} - v) \, dx + \lambda \int_{\Omega} \tilde{\phi}_1 f \frac{g'(v)}{g(v)^2} (\tilde{v} - v) \, dx$$

$$= \int_{\Omega} (\tilde{v} - v) \left( -\Delta \tilde{\phi}_1 + \lambda f \frac{g'(v)}{g(v)^2} \tilde{\phi}_1 \right) \, dx$$

$$= 0. \hspace{1cm} (1.15)$$
Hence by \((1.13), (1.15)\) and the positivity of \(\tilde{\phi}_1\) in \(\Omega\),
\[
\frac{\partial h}{\partial s}(x, 0) = -\Delta(\tilde{v} - v) + \lambda f \frac{g'(v)}{g(v)^2}(\tilde{v} - v) = 0 \quad \text{in } \Omega. \quad (1.16)
\]

By \((1.10), (1.11), (1.12)\) and \((1.16)\),
\[
\frac{\partial^2 h}{\partial s^2}(x, 0) \geq 0 \quad \Rightarrow \quad -\lambda f \left(\frac{1}{g}\right)''(v)(\tilde{v} - v)^2 \geq 0 \quad \text{in } \Omega
\]
\[
\Rightarrow \quad \tilde{v} = v \quad \text{in } \Omega \setminus D_1
\]
where \(D_1 = \{x \in \Omega : f(x) = 0\}\). By \((1.16)\) \(\Delta(\tilde{v} - v) = 0\) in \(D_1\). Since \(\tilde{v} - v = 0\) on \(\partial D_1\), \(\tilde{v} \equiv v\) on \(D_1\). Hence \(\tilde{v} \equiv v\) in \(\Omega\) and the theorem follows.

By Theorem 1.6 and an argument similar to the proof of Theorem 4.2 of \([\text{GhG1}]\) we have the following theorem.

**Theorem 1.7.** Let \(f\) satisfy \((0.1)\) and \(g\) satisfy \((0.2)\) and \((1.10)\). For each \(0 < \lambda < \lambda^*\) let \(v_\lambda\) be the minimal solution of \((S_\lambda)\). Then \(v_\lambda(x)\) is a stable solution of \((S_\lambda)\) for any \(0 < \lambda < \lambda^*\). Moreover for each \(x \in \Omega\), \(v_\lambda(x)\) is differentiable and strictly increasing with respect to \(\lambda \in (0, \lambda^*)\) and \(\tilde{\mu}_1(\lambda, v_\lambda)\) is a decreasing function of \(\lambda \in (0, \lambda^*)\).

**Proposition 1.8.** Let \(f\) satisfy \((0.1)\) and \(g\) satisfy \((0.2)\) and \((1.10)\). For each \(0 < \lambda < \lambda^*\) let \(v_\lambda\) be the minimal solution of \((S_\lambda)\). Suppose \(v\) is a solution of \((S_\lambda)\) and \(v \not\equiv v_\lambda\). Then \(\tilde{\mu}_1(\lambda, v) < 0\) and the function \(w = v - v_\lambda\) is in the negative space of \(L_{v, \lambda}\).

**Proof.** Since \(v_\lambda\) is the minimal solution of \((S_\lambda)\), \(v \geq v_\lambda\) in \(\Omega\). Let \(D_1 = \{x \in \Omega : f(x) = 0\}\) and \(D_2 = \{x \in \Omega \setminus D_1 : v(x) \not\equiv v_\lambda(x)\}\). If \(v \equiv v_\lambda\) in \(\Omega \setminus D_1\), then \(\Delta(v - v_\lambda) = 0\) in \(D_1\) and \(v = v_\lambda\) on \(\partial D_1\). Thus \(v \equiv v_\lambda\) on \(\overline{\Omega}\). Contradiction arises. Hence \(v \not\equiv v_\lambda\) in \(\Omega \setminus D_1\) and \(D_2\) is a set of positive measure. By the mean value theorem,
\[
L_{v, \lambda}(v - v_\lambda) = -\Delta(v - v_\lambda) - \lambda f \left(\frac{1}{g}\right)'(v)(v - v_\lambda)
\]
\[
= \lambda f \left\{ \frac{1}{g(v)} - \frac{1}{g(v_\lambda)} - \left(\frac{1}{g}\right)'(v)(v - v_\lambda) \right\}
\]
\[
= \lambda f \left\{ \left(\frac{1}{g}\right)'(\xi_1) - \left(\frac{1}{g}\right)'(v) \right\}(v - v_\lambda)
\]
\[
= \lambda f \left(\frac{1}{g}\right)''(\xi_2)(v - v_\lambda)(\xi_1 - v) \quad \text{in } D_2 \quad (1.17)
\]
for some functions \(\xi_1(x) \in (v_\lambda(x), v(x))\), \(\xi_2(x) \in (v_\lambda(x), \xi_1(x))\). Hence by \((1.10)\) and \((1.17)\),
\[
< L_{v, \lambda}w, w > = \int_{D_2} \lambda f \left(\frac{1}{g}\right)''(\xi_2)(v - v_\lambda)^2(\xi_1 - v) \, dx < 0.
\]
Thus \( \tilde{\mu}_1(\lambda, v) < 0 \) and the proposition follows.

**Section 2**

In this section we will prove the local and global existence of solutions of (0.3). We also obtain various comparison results for the solutions of (0.3).

**Theorem 2.1.** Let \( u_{0,1}, u_{0,2} \in L^1(\Omega) \). Let \( f \in C(\bar{\Omega}) \) and \( 0 < g \in C^2((\infty, 1)) \). Suppose \( u_1, u_2 \), are subsolution and supersolution of (0.3) in \( \Omega \times (0, T) \) with initial value \( u_0 = u_{0,1}, u_{0,2} \), respectively such that

\[
a_1 = \max( \sup_{\bar{\Omega} \times (0,T)} u_1(x,t), \sup_{\bar{\Omega} \times (0,T)} u_2(x,t)) < 1. \tag{2.1}
\]

Suppose either (1.10) holds or there exists \( a_2 < 1 \) such that

\[
u_1(x,t), u_2(x,t) \geq a_2 \quad \text{on } \Omega \times (0, T). \tag{2.2}
\]

Then

\[
(i) \quad \int_{\Omega} (u_1 - u_2)_+ (x,t) \, dx \leq e^{bt} \int_{\Omega} (u_{0,1} - u_{0,2})_+ \, dx \quad \forall 0 \leq t < T
\]

hold for some constant \( b > 0 \) depending on \( \lambda, f, \) and \( a_1 \) if (1.10) holds and on \( \lambda, f, a_1 \) and \( a_2 \) if (2.2) holds. If both \( u_1 \) and \( u_2 \) are solutions of (0.3) in \( \Omega \times (0, T) \) with initial value \( u_0 = u_{0,1}, u_{0,2} \), respectively, then

\[
(ii) \quad \int_{\Omega} |u_1 - u_2| (x,t) \, dx \leq e^{bt} \int_{\Omega} |u_{0,1} - u_{0,2}| \, dx \quad \forall 0 \leq t < T.
\]

**Proof.** We will use a modification of the technique of Dahlberg and C. Kenig [DK] to prove the theorem. Let \( h \in C_0^\infty(\Omega) \) be such that \( 0 \leq h \leq 1 \). For any \( t_1 \in (0, T) \), let \( \eta \) be the solution of

\[
\begin{cases}
\eta_t + \Delta \eta + H\eta = 0 & \text{in } \Omega \times (0, t_1) \\
\eta = 0 & \text{on } \partial \Omega \times (0, t_1) \\
\eta(x, t_1) = h(x) & \text{in } \Omega
\end{cases} \tag{2.3}
\]

where

\[
H(x, t) = \begin{cases}
\lambda f(x) \left( \frac{(g(u_1)^{-1} - g(u_2)^{-1})}{u_1 - u_2} \right) & \text{if } u_1(x,t) \neq u_2(x,t) \\
\lambda f(x) \left( \frac{1}{g} \right)' (u_1) & \text{if } u_1(x,t) = u_2(x,t)
\end{cases} \tag{2.4}
\]
Then
\[ \int_{\Omega} (u_1 - u_2)(x, t_1) h(x) \, dx - \int_{\Omega} (u_{0,1} - u_{0,2}) \eta \, dx \]
\[ = \int_0^{t_1} \int_{\Omega} \frac{\partial}{\partial t} [(u_1 - u_2) \eta] \, dx \, dt \]
\[ = \int_0^{t_1} \int_{\Omega} [(u_1 - u_2)_t \eta + (u_1 - u_2) \eta_t] \, dx \, dt \]
\[ \leq \int_0^{t_1} \int_{\Omega} [\eta \Delta (u_1 - u_2) + \lambda \eta (g(u_1)^{-1} - g(u_2)^{-1}) f + (u_1 - u_2) \eta_t] \, dx \, dt \]
\[ = \int_0^{t_1} \int_{\Omega} (u_1 - u_2)[\eta_t + \Delta \eta + H \eta] \, dx \, dt \]
\[ = 0. \]

Hence
\[ \int_{\Omega} (u_1 - u_2)(x, t_1) h(x) \, dx \leq \int_{\Omega} (u_{0,1} - u_{0,2}) \eta \, dx. \] (2.5)

Let \( b = \sup_{\Omega \times (0, T)} |H(x, t)| \). By (2.1), (2.4) and either (1.10) or (2.2), \( b < \infty \). By the maximum principle \( \eta \geq 0 \). By (2.3),
\[ \eta_t + \Delta \eta + b \eta \geq 0 \quad \text{in } \Omega \times (0, t_1) \]
\[ (e^{bt} \eta)_t + \Delta (e^{bt} \eta) \geq 0 \quad \text{in } \Omega \times (0, t_1). \]

Hence by the maximum principle,
\[ \eta(x, 0) \leq \max_{\Omega} (e^{bt_1} \eta(x, t_1)) = e^{bt_1} \| h \|_{L^\infty} \leq e^{bt_1}. \] (2.6)

By (2.5) and (2.6),
\[ \int_{\Omega} (u_1 - u_2)(x, t_1) h(x) \, dx \leq e^{bt_1} \int_{\Omega} (u_{0,1} - u_{0,2})_+ \, dx. \] (2.7)

Let \( A = \{ x \in \Omega : u_1(x, t_1) > u_2(x, t_1) \} \). We now choose a sequence of function \( h_k \in C_0^\infty(\Omega) \), \( 0 \leq |h_k| \leq 1 \), such that \( h_k \to \chi_A \) a.e. as \( k \to \infty \). Putting \( h = h_k \) in (2.7) and letting \( k \to \infty \),
\[ \int_{\Omega} (u_1 - u_2)(x, t_1)_+ \, dx \leq e^{bt_1} \int_{\Omega} (u_{0,1} - u_{0,2})_+ \, dx. \]

Since \( t_1 \in (0, T) \) is arbitrary, (i) follows. Similarly if both \( u_1 \) and \( u_2 \) are solutions of (0.3) in \( \Omega \times (0, T) \) with initial value \( u_0 = u_{0,1}, u_{0,2} \), respectively, then
\[ \int_{\Omega} (u_1 - u_2)(x, t)_- \, dx \leq e^{bt_1} \int_{\Omega} (u_{0,1} - u_{0,2})_- \, dx \quad \forall 0 < t < T. \] (2.8)

By (i) and (2.8), (ii) follows.
Corollary 2.2. Let \( u_{0,1}, u_{0,2} \in L^1(\Omega) \) be such that \( u_{0,1} \leq u_{0,2} \) in \( \Omega \). Let \( f \in C(\overline{\Omega}) \) and \( 0 < g \in C^2((\infty,1)) \). Suppose \( u_1, u_2 \), are the subsolution and supersolution of (0.3) in \( \Omega \times (0,T) \) with initial value \( u_0 = u_{0,1}, u_{0,2} \), respectively. Suppose (2.1) holds and either (1.10) holds or (2.2) holds for some constant \( a_2 < 1 \). Then \( u_1 \leq u_2 \) in \( \overline{\Omega} \times (0,T) \).

Corollary 2.3. Let \( u_0 \in L^1(\Omega), f \in C(\overline{\Omega}) \) and \( 0 < g \in C^2((\infty,1)) \). Then the solution of (0.3) in \( \Omega \times (0,T) \) is unique.

Corollary 2.4. Let \( u_0 \in L^1(\Omega), f \in C(\overline{\Omega}) \) and \( 0 < g \in C^2((\infty,1)) \). Then the solution of (0.3) in \( \Omega \times (0,T) \) is unique in the class of functions on \( \overline{\Omega} \times (0,T) \) which are uniformly bounded below on \( \overline{\Omega} \times (0,T') \) for any \( 0 < T' < T \).

Theorem 2.5. Let \( u_0 \) satisfy (0.4) for some constant \( 0 < a < 1 \). Let \( f \) satisfy (0.1) and \( g \) satisfy (0.2). Then for any \( \lambda \geq 0 \) there exists \( T > 0 \) such that (0.3) has a solution which satisfies

\[
u(x,t) = \int_{\Omega} G(x,y,t)u_0(y)\,dy + \lambda \int_0^t \int_{\Omega} G(x,y,t-s)f(y)g(u(y,s))\,dy\,ds \quad \forall x \in \overline{\Omega}, 0 < t < T, \Omega \times (0,T).
\]

Proof. When \( \lambda = 0 \), (0.3) reduces to the heat equation and the theorem follows from standard theory for heat equation [F]. We next assume that \( \lambda > 0 \). We divide the proof into two cases.

Case 1: \( u_0 \in C_0^\infty(\Omega) \) and \( u_0 \) satisfies (0.4) for some constant \( 0 < a < 1 \).

Let

\[
T = \frac{(1-a)}{4\lambda\|f\|_{L^\infty}}g((1+a)/2),
\]

\[
w(x,t) = \int_{\Omega} G(x,y,t)u_0(y)\,dy,
\]

and

\[
u_1(x,t) = w(x,t) + \lambda \int_0^t \int_{\Omega} G(x,y,t-s)f(y)g(u(y))\,dy\,ds \quad \forall x \in \overline{\Omega}, 0 < t < T.
\]

Then \( w \) satisfies

\[
\begin{cases}
\partial_t w = \Delta w & \text{in } \Omega \times (0,\infty) \\
w(x,t) = 0 & \forall x \in \partial\Omega, t > 0 \\
w(x,0) = u_0(x) & \text{in } \Omega.
\end{cases}
\]

Let \( T_1 = \sup\{0 < t_1 < T : u_1(x,t) < (1+a)/2 \quad \forall x \in \overline{\Omega}, 0 < t \leq t_1\} \). Suppose \( T_1 < T \).

By (0.2), (0.4), (0.6), (2.10) and (2.12), \( \forall x \in \overline{\Omega}, 0 < t \leq T_1 \),

\[
u_1(x,t) \leq a + \lambda (\|f\|_{L^\infty}/g(a))t \leq a + \frac{(1-a)}{4}g((1+a)/2) < \frac{1+a}{2}.
\]
By continuity of $u_1$ there exists $0 < \delta < (T - T_1)/2$ such that

$$u_1(x, t) < \frac{1 + a}{2}$$

holds for all $x \in \overline{\Omega}$, $0 < t \leq T_1 + \delta$. This contradicts the maximality of $T_1$. Hence $T_1 = T$ and (2.12) holds for all $x \in \overline{\Omega}$, $0 < t \leq T$. Suppose $u_1, u_2, \ldots, u_k$, are defined. We define

$$u_{k+1}(x, t) = w(x, t) + \lambda \int_0^t \int_{\Omega} G(x, y, t - s) \frac{f(y)}{g(u_k(y, s))} \, dy \, ds \quad \forall x \in \overline{\Omega}, 0 < t < T. \quad (2.14)$$

Let $T_k = \sup\{0 < t_1 < T : u_k(x, t) < (1 + a)/2 \quad \forall x \in \overline{\Omega}, 0 < t \leq t_1\}$. We claim that $T_k = T$ for all $k \in \mathbb{Z}^+$. We will prove this claim by induction. Note that $T_1 = T$ is already proved before. Suppose $T_1 = T_2 = \cdots = T_k = T$ but $T_{k+1} < T$. Then

$$u_k(x, t) < \frac{1 + a}{2} \quad \forall x \in \overline{\Omega}, 0 < t < T. \quad (2.15)$$

By (0.2), (0.4), (2.10), (2.14) and (2.15),

$$u_{k+1}(x, t) \leq a + \lambda \left( \|f\|_{L^\infty} / g((1 + a)/2) \right) t \leq a + \frac{(1 - a)}{4} < \frac{1 + a}{2} \quad \forall x \in \overline{\Omega}, 0 < t \leq T_1.$$

By continuity of $u_{k+1}$ there exists $0 < \delta < (T - T_{k+1})/2$ such that

$$u_{k+1}(x, t) < \frac{1 + a}{2} \quad (2.16)$$

holds for all $x \in \overline{\Omega}$, $0 < t \leq T_{k+1} + \delta$. This contradicts the maximality of $T_{k+1}$. Hence $T_{k+1} = T$ and (2.16) holds for all $x \in \overline{\Omega}$, $0 < t \leq T$. Thus by induction $T_k = T$ for all $k \in \mathbb{Z}^+$. Hence (2.15) holds for all $k \in \mathbb{Z}^+$. Since

$$w(\cdot, t) \to u_0 \quad \text{in } L^1(\Omega) \quad \text{as } t \to 0, \quad (2.17)$$

by (0.6), (2.14) and (2.15),

$$u_k(\cdot, t) \to u_0 \quad \text{in } L^1(\Omega) \quad \text{as } t \to 0. \quad (2.18)$$

By (2.12) and (2.14),

$$w(x, t) \leq u_k(x, t) \quad \text{in } \Omega \times (0, T) \quad \forall k \in \mathbb{Z}^+. \quad (2.19)$$

By (2.12), $u_1$ is continuously differentiable in $x$ and $t$. Then by (2.14), (2.19) and standard parabolic theory [F], $u_k \in C^{2,1}(\overline{\Omega} \times (0, T])$ for all $k \geq 2$. Then by (2.14), (2.15) and (2.18), (2.19), $\forall k \geq 2$, $u_k$ satisfies

$$\begin{cases}
\frac{\partial u_k}{\partial t} - \Delta u_k = \frac{\lambda f}{g(u_{k-1})} & \text{in } \Omega \times (0, T) \\
u_k(x, t) = 0 & \text{on } \partial \Omega \times (0, T) \\
u_k(x, 0) = u_0(x) & \text{in } \Omega.
\end{cases} \quad (2.20)$$
By (2.15), (2.19), (2.20) and the parabolic Schauder estimates [LSU], the sequence \( \{u_k\}_{k=2}^{\infty} \) are uniformly Holder continuous on \( \overline{\Omega} \times [0, T] \). Then by (2.15), (2.19), (2.20) and the Schauder estimates for the heat equation ([F],[LSU]) \( \{u_k\}_{k=2}^{\infty} \) are uniformly bounded in \( C^{2+\beta,1+(eta/2)}(K) \) for any compact subset \( K \subset \overline{\Omega} \times (0, T] \) where \( 0 < \beta < 1 \) is some constant. By the Ascoli theorem and a diagonalization argument \( \{u_k\}_{k=2}^{\infty} \) has a subsequence which we may assume without loss of generality to be the sequence itself which converges uniformly in \( C^{2+\beta,1+(eta/2)}(K) \) to some function \( u \) for any compact subset \( K \subset \overline{\Omega} \times (0, T] \) as \( k \to \infty \). Then by (2.14), (2.15), (2.19) and (2.20) \( u \) satisfies (2.9),

\[
w(x,t) \leq u(x,t) \leq \frac{1+a}{2} \quad \forall x \in \overline{\Omega}, 0 < t \leq T,
\]

and

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = \frac{\lambda f}{g(u)} & \text{in } \Omega \times (0, T) \\
u(x,t) = 0 & \text{on } \partial \Omega \times (0, T)
\end{cases}
\]  

By (0.6), (2.9), (2.17) and (2.21), \( u \) satisfies (0.5). Hence \( u \) is a solution of (0.3) in \( \Omega \times (0, T) \).

**Case 2:** \( u_0 \) satisfies (0.4) for some constant \( 0 < a < 1 \).

We choose a sequence of function \( \{u_{0,k}\}_{k=1}^{\infty} \in C_0^\infty(\Omega) \) such that \( u_{0,k} \) converges to \( u_0 \) in \( L^1(\Omega) \) and a.e. as \( k \to \infty \). For any \( k \in \mathbb{Z}^+ \), by case 1 there exists a solution \( u_k \) of (0.3) in \( \Omega \times (0, T) \) with initial value \( u_{0,k} \) which satisfies

\[
u_k(x,t) = \int_{\Omega} G(x,y,t)u_{0,k}(y)\,dy + \lambda \int_0^t \int_{\Omega} G(x,y,t-s)\frac{f(y)}{g(u_{0,k}(y,s))}\,dy\,ds
\]  

for any \( x \in \overline{\Omega}, 0 < t < T \), and

\[
w(x,t) \leq u_k(x,t) \leq \frac{1+a}{2} \quad \forall x \in \overline{\Omega}, 0 < t \leq T.
\]

Since \( \{u_k\}_{k=1}^{\infty} \) satisfy (0.3) with initial value \( u_{0,k} \) in \( \Omega \times (0, T) \), by the parabolic Schauder estimates [LSU], the sequence \( \{u_k\}_{k=1}^{\infty} \) are uniformly Holder continuous on \( \overline{\Omega} \times (\delta_1, T) \) for any \( 0 < \delta_1 < T \). Then by the parabolic Schauder estimates ([F],[LSU]) \( \{u_k\}_{k=1}^{\infty} \) are uniformly bounded in \( C^{2+\beta,1+(eta/2)}(K) \) for any compact subset \( K \subset \overline{\Omega} \times (0, T] \) where \( 0 < \beta < 1 \) is some constant. By the Ascoli theorem and a diagonalization argument \( \{u_k\}_{k=1}^{\infty} \) has a subsequence which we may assume without loss of generality to be the sequence itself which converges uniformly in \( C^{2+\beta,1+(eta/2)}(K) \) to some function \( u \) for any compact subset \( K \subset \overline{\Omega} \times (0, T] \) as \( k \to \infty \). Then \( u \) satisfies (2.22). Letting \( k \to \infty \) in (2.23) and (2.24), we get (2.9) and (2.21). By (0.6), (2.9), (2.17) and (2.21), \( u \) satisfies (0.5). Hence \( u \) is a solution of (0.3) in \( \Omega \times (0, T) \) and the theorem follows.

By Corollary 2.2 and the Duhamel principle we have the following corollary.

**Corollary 2.6.** Let \( f \) satisfy (0.1), \( g \) satisfy (0.2) and \( u_0 \) satisfy (0.4) for some constant \( a < 1 \). Suppose \( u \) is a bounded solution of (0.3) in \( \Omega \times (0, T) \). Then \( u \) satisfies (2.9) in \( \overline{\Omega} \times (0, T) \).
Corollary 2.7. Let \( f \) satisfy (0.1) and \( g \) satisfy (0.2). Let \( u_{0,1}, u_{0,2} \in L^\infty(\Omega) \) be such that \( u_{0,1} \leq u_{0,2} \leq a < 1 \) for some constant \( 0 < a < 1 \) and \( u_{0,1} \neq u_{0,2} \). Suppose \( u_1, u_2 \), are bounded solutions of (0.3) in \( \Omega \times (0,T) \) with initial values \( u_{0,1}, u_{0,2} \) respectively. Then

\[
    u_1 < u_2 \quad \text{in} \quad \Omega \times (0,T).
\]

Proof. By Corollary 2.2 \( u_1 \leq u_2 \) in \( \Omega \times (0,T) \). By Corollary 2.6 both \( u_1 \) and \( u_2 \) satisfies (2.9) with \( u_0 = u_{0,1}, u_{0,2} \) respectively. By (2.9) for \( u_1, u_2 \), (0.2) and the positivity of the Green function for the heat equation the corollary follows.

By Corollary 2.2, Theorem 2.5 and a continuity argument we have the following theorem.

Theorem 2.8. Let \( f \) satisfy (0.1) and \( g \) satisfy (0.2). Let \( \lambda \geq 0 \). Suppose \((S_\lambda)\) has a supersolution \( v_\lambda \). Let \( u_0 \in L^\infty(\Omega) \) satisfy

\[
    u_0 \leq v_\lambda \quad \text{in} \quad \Omega.
\]

Then (0.3) has a unique bounded global solution which satisfies (2.9) and

\[
    \inf_\Omega u_0 \leq u(x,t) \leq v_\lambda(x) \quad \forall \Omega \times (0,\infty).
\]

Theorem 2.9. Let \( f \) satisfy (0.1) and \( g \) satisfy (0.2). Let \( 0 \leq \lambda \leq \lambda^* \) and let \( v_\lambda \) be a supersolution of \((S_\lambda)\). Let \( u_0 \in L^1(\Omega) \) satisfy

\[
    u_0 \leq v_\lambda \quad \text{in} \quad \Omega.
\]

Then (0.3) has a global solution \( u \) which satisfies (2.9) and

\[
    w(x,t) \leq u(x,t) \leq v_\lambda(x) \quad \forall \Omega \times (0,\infty)
\]

where \( w \) is given by (2.11). The solution is unique within the family of functions satisfying (2.25) if either (1.10) holds or

\[
    \sup_{s \leq a} \left( \frac{1}{g(s)} \right)'(s) < \infty \quad \forall a < 1.
\]

Proof. For any \( k \in \mathbb{Z}^+ \), let \( u_{0,k} = \max(u_0, -k) \). Then

\[
    u_{0,k+1} \leq u_{0,k} \quad \text{and} \quad -k \leq u_{0,k} \leq v_\lambda \quad \text{in} \quad \Omega \quad \forall k \in \mathbb{Z}^+.
\]

By Corollary 2.2 and Theorem 2.8 for any \( k \in \mathbb{Z}^+ \) there exists a global bounded solution \( u_k \) of (0.3) with initial value \( u_{0,k} \) which satisfies (2.23) in \( \Omega \times (0,\infty) \),

\[
    -k \leq u_k \leq v_\lambda \quad \text{in} \quad \Omega \times (0,\infty) \quad \forall k \in \mathbb{Z}^+.
\]
and

\[ u_{k+1} \leq u_k \quad \text{in } \Omega \times (0, \infty) \quad \forall k \in \mathbb{Z}^+. \tag{2.28} \]

By (2.23),

\[ w_k(x, t) \leq u_k \quad \text{in } \Omega \times (0, \infty) \quad \forall k \in \mathbb{Z}^+ \tag{2.29} \]

where

\[ w_k(x, t) = \int_\Omega G(x, y, t) u_{0,k}(y) \, dy \tag{2.30} \]

is the solution of (2.13) with initial value \( u_{0,k} \). Let \( w \) be given by (2.11). Since \( |u_{0,k}| \leq |u_0| \) in \( \Omega \), by (0.6), (2.30) and the Lebesgue dominated convergence theorem \( w_k \) converges uniformly to \( w \) on \( \overline{\Omega} \times [\delta_1, \infty) \) as \( k \to \infty \) for any \( \delta_1 > 0 \). Hence by (2.27) and (2.29), the sequence \( \{u_k\}_{k=1}^\infty \) are uniformly bounded on \( \overline{\Omega} \times [\delta_1, \infty) \) for any \( \delta_1 > 0 \). Since \( u_k \) satisfies (0.3) in \( \Omega \times (0, \infty) \) with initial value \( u_{0,k} \), by the Schauder estimates [LSU] \( \{u_k\}_{k=1}^\infty \) are uniformly bounded in \( C^{2+\beta,1+(\beta/2)}(\overline{\Omega} \times [\delta_1, \infty)) \) for any \( \delta_1 > 0 \) where \( 0 < \beta < 1 \) is some constant. By (2.28), the Ascoli theorem and a diagonalization argument \( \{u_k\}_{k=1}^\infty \) has a subsequence which we may assume without loss of generality to be the sequence itself which decreases and converges uniformly in \( C^{2+\beta,1+(\beta/2)}(\overline{\Omega} \times [\delta_1, \infty)) \) to some function \( u \) for any \( \delta_1 > 0 \) as \( k \to \infty \).

Then \( u \) satisfies (2.22) and (2.25). Letting \( k \to \infty \) in (2.23) we get (2.9). By (2.9) and (2.17) \( u \) satisfies (0.5). Hence \( u \) is a solution of (0.3) in \( \Omega \times (0, T) \). If (1.10) holds, by Corollary 2.3 the solution is unique.

Suppose (2.26) holds. Suppose \( u_1, u_2 \), are both solutions of (0.3) in \( \Omega \times (0, \infty) \). Then by (2.25) and the Duhamel principle, both \( u_1, u_2 \), satisfies (2.9). Putting \( u = u_1, u_2 \), in (2.9) and subtracting the resulting equations, we get

\[
\begin{align*}
\frac{\partial}{\partial t} u_1(x, t) - \frac{\partial}{\partial t} u_2(x, t) &= \lambda \int_0^t \int_\Omega G(x, y, t-s) f(y) \left( \frac{1}{g(u_1(y, s))} - \frac{1}{g(u_2(y, s))} \right) \, dy \, ds \\
&\leq \lambda \|f\|_{L^\infty} \int_0^t \int_\Omega G(x, y, t-s) \left( \frac{1}{g} \right)'(\xi(y, s))(u_1(y, s) - u_2(y, s))_+ \, dy \, ds \\
&\leq a_0 \lambda \|f\|_{L^\infty} \int_0^t \int_\Omega G(x, y, t-s)(u_1(y, s) - u_2(y, s))_+ \, dy \, ds \\
&\leq a_0 \lambda \|f\|_{L^\infty} T \sup_{\Omega \times (0, T)} (u_1 - u_2)_+ \quad \forall x \in \Omega, 0 < t < T
\end{align*}
\]

for any \( T > 0 \) where \( \xi(y, s) \) is some number between \( u_1(y, s) \) and \( u_2(y, s) \),

\[
a_0 = \sup_{s \leq \|v\|_{L^\infty}} \left( \frac{1}{g} \right)'(s).
\]

Hence

\[
\sup_{\Omega \times (0, T)} (u_1 - u_2)_+ \leq a_0 \lambda \|f\|_{L^\infty} T \sup_{\Omega \times (0, T)} (u_1 - u_2)_+. \tag{2.31}
\]
We now choose $T = 1/(1 + 2a_0\lambda\|f\|_{L^\infty})$. Then by (2.31),
\[
\sup_{\Omega \times (0, T)} (u_1 - u_2)_+ = 0 \implies u_1 \leq u_2 \text{ in } \overline{\Omega} \times (0, T).
\]
By interchanging the role of $u_1$ and $u_2$ we get
\[
u_2 \leq u_1 \text{ in } \Omega \times (0, T).
\]
Hence
\[
u_1 = u_2 \text{ in } \Omega \times (0, T).
\]
By dividing the time interval into disjoint intervals of length $T$ and repeating the above argument we get
\[
u_1 = u_2 \text{ in } \Omega \times (0, \infty)
\]
and the theorem follows.

**Theorem 2.10.** Let $g$ satisfy (0.2) and
\[
0 \leq f \in C^\alpha(\mathbb{R}^n) \quad \text{for some constant } 0 < \alpha < 1.
\]
Let $u_0 \in L^1(\mathbb{R}^n)$ be such that $u_0 \leq a$ in $\mathbb{R}^n$ for some constant $a < 1$. Then for any $\lambda \geq 0$ there exists a constant $T > 0$ such that the Cauchy problem
\[
\begin{cases}
  u_t = \Delta u + \frac{\lambda f(x)}{g(u)} & \text{in } \mathbb{R}^n \times (0, T) \\
  u(x, 0) = u_0 & \text{in } \mathbb{R}^n
\end{cases}
\]
has a solution $u$ which satisfies
\[
u(x, t) = \int_{\mathbb{R}^n} Z(x, y, t)u_0(y) \, dy + \lambda \int_0^t \int_{\mathbb{R}^n} Z(x, y, t-s) \frac{f(y)}{g(u(y, s))} \, dy \, ds
\]
in $\mathbb{R}^n \times (0, T)$ where $Z(x, y, t) = (4\pi)^{-\frac{n}{2}} e^{-|x-y|^2/4t}$.

**Proof.** If $\lambda = 0$ or $f \equiv 0$ in $\mathbb{R}^n$, (2.32) reduces to the heat equation and the result follows by standard results on heat equation [F]. Hence we may assume without loss of generality that $\lambda > 0$ and $f \not\equiv 0$ in $\mathbb{R}^n$. Let $T$ be given by (2.10). For any $R > 0$ let $G_R(x, y, t)$ be the Dirichlet Green function of the heat equation in $B_R \times (0, \infty)$. By the proof of Theorem 2.5 for any $k \geq 1$ there exists a solution $u_k$ of
\[
\begin{cases}
  u_t = \Delta u + \frac{\lambda f(x)}{g(u)} & \text{in } B_k \times (0, T) \\
  u(x, t) = 0 & \text{on } \partial B_k \times (0, T) \\
  u(x, 0) = u_0 & \text{in } B_k
\end{cases}
\]

which satisfies
\[ u_k(x, t) = \int_{B_k} G_k(x, y, t)u_0(y) dy + \lambda \int_0^t \int_{B_k} G_k(x, y, t-s) \frac{f(y)}{g(u_k(y, s))} dy \, ds \quad (2.35) \]
for any \((x, t) \in B_k \times (0, T)\) and
\[ w_k(x, t) \leq u_k(x, t) \leq \frac{1 + a}{2} \quad \text{in } B_k \times (0, T) \quad \forall k \geq 1 \quad (2.36) \]
where
\[ w_k(x, t) = \int_{B_k} G_k(x, y, t)u_0(y) dy \]
Since \(G_k(x, y, t) \leq G_{k+1}(x, y, t)\) in \(B_k \times (0, T)\) for any \(k \geq 1\), by the construction of solutions in Theorem 2.5, \(u_k \leq u_{k+1}\) in \(B_k \times (0, T)\) \(\forall k \geq 1\). (2.37)
Since \(w_k\) converges uniformly to \(w(x, t) = \int_{\mathbb{R}^n} Z(x, y, t)u_0(y) dy\) (2.38)
as \(k \to \infty\), by (2.36) the sequence \(\{u_k\}_{k=1}^\infty\) is uniformly bounded on every compact subset of \(\mathbb{R}^n \times (0, T)\). By (2.34) for \(u_k\), (2.36), and the parabolic Schauder estimates the sequence \(\{u_k\}_{k=1}^\infty\) is uniformly Holder continuous on every compact subset of \(\mathbb{R}^n \times (0, T)\). Then by (2.34) for \(u_k\), (2.36), and the parabolic Schauder estimates the sequence \(\{u_k\}_{k=1}^\infty\) is uniformly bounded in \(C^{2+\beta, 1+(\beta/2)}(K)\) for any compact subset \(K \subset \mathbb{R}^n \times (0, T)\) where \(0 < \beta < 1\) is some constant. Then by (2.35), (2.36), (2.37), the Ascoli Theorem and a diagonalization argument the sequence \(\{u_k\}_{k=1}^\infty\) has a subsequence which we may assume without loss of generality to be the sequence \(\{u_k\}_{k=1}^\infty\) itself which increases and converges uniformly in \(C^{2+\beta, 1+(\beta/2)}(K)\) for any compact subset \(K \subset \mathbb{R}^n \times (0, T)\) to a function \(u\) which satisfies (2.33),
\[ u_t = \Delta u + \frac{\lambda f(x)}{g(u)} \quad \text{in } \mathbb{R}^n \times (0, T), \]
and
\[ w(x, t) \leq u(x, t) \leq \frac{1 + a}{2} \quad \text{in } \mathbb{R}^n \times (0, T) \quad (2.39) \]
Since \(w(x, t) \to u_0\) as \(t \to 0\), by (2.33) and (2.39) \(u(x, t) \to u_0\) as \(t \to 0\). Hence \(u\) satisfies (2.32) in \(\mathbb{R}^n \times (0, T)\).

**Section 3**

In this section we will prove the convergence of solutions of (0.3) for any \(0 \leq \lambda < \lambda^*\) as \(t \to \infty\). We also obtain various conditions for the solutions of (0.3) to have finite touchdown time.
Theorem 3.1. Suppose $f$ satisfies (0.1) and $g$ satisfies (0.2). Let $0 \leq \lambda < \lambda^*$ and let $v_\lambda$ be the unique minimal solution of $(S_\lambda)$ given by Theorem 1.1. Let $u_0$ satisfies
\[ u_0 \leq v_\lambda \quad \text{in } \Omega \]
and let $u$ be the global solution of (0.3) constructed in Theorem 2.9. Then $u$ converges uniformly on $\Omega$ to $v_\lambda$ as $t \to \infty$.

Proof. Note that the theorem is proved by N. Ghoussoub and Y. Guo in [GhG2] for the case $g(s) = (1-s)^2$ and $u_0 = 0$ in $\Omega$ and by T. Suzuki, etc. in [KMS] for the case $g(s) = (1-s)^p$ and $0 \leq u_0 \leq v_\lambda$ in $\Omega$. Both are based on proving the positivity of $u_t$ in $\Omega \times (0, \infty)$ when $u_0 = 0$ using a modification of Fujita’s technique [Fu]. This approach is not applicable in our case and we will use a different proof for the convergence result.

By Theorem 2.9 $u$ satisfies (2.9) and (2.25) with $w$ being given by (2.11). Let $\{t_k\}_{k=1}^\infty$, $t_k \geq 1$ for all $k \geq 1$, be a sequence such that $t_k \to \infty$ as $k \to \infty$. By (2.25) and the parabolic Schauder estimates [LSU] $u(x, t)$ is uniformly bounded in $C^{2+\beta,1+(\beta/2)}(\bar{\Omega} \times [1, \infty))$ where $0 < \beta < 1$ is some constant. Then by the Ascoli theorem $\{t_k\}_{k=1}^\infty$ has a subsequence $\{t_{i_k}\}_{k=1}^\infty$ such that $u(x, t_{i_k} + t)$ converges uniformly in $C^{2,1}(\bar{\Omega} \times [0,1])$ to some function $v_1$ as $k \to \infty$. Let $v(x) = v_1(x, 0)$. Multiplying (0.3) by $u_t$ and integrating over $\Omega \times (1, t)$,
\[
\int_1^t \int_\Omega u_t^2 \, dx \, dt = \int_1^t \int_\Omega u_t \Delta u \, dx \, dt + \lambda \int_1^t \int_\Omega \frac{f(x)u_t}{g(s)} \, dx \, dt
\]
\[
= -\frac{1}{2} \int_1^t \frac{\partial}{\partial t} \left( \int_\Omega |\nabla u|^2 \, dx \right) \, dt + \lambda \int_\Omega f(x) \left( \int_{u(x,1)}^{u(x,t)} \frac{ds}{g(s)} \right) \, dx
\]
\[
\leq \frac{1}{2} \int_\Omega |\nabla u(x,1)|^2 \, dx + \lambda\|f\|_{L^\infty(\Omega)} \left( a_2 - a_1 \right) \max_{a_1 \leq s \leq a_2} \left( 1/g(s) \right)
\]
holds for all $t \geq 1$ where $a_1 = \min_\Omega u(x,1)$, $a_2 = \max_\Omega v_\lambda$. Letting $t \to \infty$,
\[
\int_1^\infty \int_\Omega u_t^2 \, dx \, dt \leq \frac{1}{2} \int_\Omega |\nabla u(x,1)|^2 \, dx + \lambda\|f\|_{L^\infty(\Omega)} \left( a_2 - a_1 \right) \max_{a_1 \leq s \leq a_2} \left( 1/g(s) \right).
\]
Hence
\[
\int_{t_{i_k}}^{t_{i_k}+1} \int_\Omega u_t^2 \, dx \, dt \to 0 \quad \text{as } k \to \infty.
\]
Thus
\[
\int_\Omega |u(x, t_{i_k} + t) - u(x, t_{i_k})| \, dx \leq \int_{t_{i_k}}^{t_{i_k}+1} \int_\Omega |u_t| \, dx \, dt
\]
\[
\leq |\Omega|^{1/2} \left( \int_{t_{i_k}}^{t_{i_k}+1} \int_\Omega u_t^2 \, dx \, dt \right)^{1/2}
\]
\[
\to 0 \quad \text{as } k \to \infty
\]
\[
\Rightarrow \quad \int_\Omega |v_1(x, t) - v(x)| \, dx = 0 \quad \forall 0 \leq t \leq 1
\]
\[
\Rightarrow \quad v_1(x, t) = v(x) \quad \forall x \in \bar{\Omega}, 0 \leq t \leq 1.
\]
Hence \( u(x, t_{ik} + t) \) converges uniformly to \( v(x) \) on \( \overline{\Omega} \times [0, 1] \) as \( k \to \infty \). Putting \( t = t_{ik} \) and letting \( k \to \infty \) in (2.25),

\[
0 \leq v(x) \leq v_\lambda(x) \quad \text{in} \ \overline{\Omega}.
\]

Integrating (0.3) over \( (t_{ik}, t_{ik} + 1) \),

\[
u(x, t_{ik} + 1) - u(x, t_{ik}) = \int_{t_{ik}}^{t_{ik} + 1} \Delta u(x, s) \, ds + \int_{t_{ik}}^{t_{ik} + 1} \frac{\lambda f(x)}{g(u(x, s))} \, ds \quad \text{on} \ \overline{\Omega}.
\]

Letting \( k \to \infty \) we get that \( v \) satisfies \((S_\lambda)\). Since \( v_\lambda \) is the minimal solution of \((S_\lambda)\), by (3.1),

\[
v(x) = v_\lambda(x) \quad \text{on} \ \overline{\Omega}.
\]

Since the sequence \( \{t_k\}_{k=1}^\infty \) is arbitrary, \( u(x, t) \) converges uniformly to \( v_\lambda \) on \( \overline{\Omega} \) as \( t \to \infty \) and the theorem follows.

By (ii) of Theorem 1.1 and an argument similar to the proof of Theorem 3.1 we have the following theorem.

**Theorem 3.2.** Suppose \( f \) satisfies (0.1) and \( g \) satisfies (0.2). Let \( \lambda > \lambda^* \) and let \( u \) be a solution of (0.3). Then either \( T_\lambda < \infty \) or \( u \) touchdowns at time infinity.

**Theorem 3.3.** Let \( f \) satisfy (0.1) and (1.4) and \( g \) satisfy (0.2) and (1.10). Let \( \lambda_1 = (\mu_1/\delta_1) \sup_{0 \leq s \leq 1} sg(s) \). Then for any solution \( u \) of (0.3) with initial value \( u_0 \) and \( \lambda > \lambda_1 \), we have

\[
T_\lambda \leq \frac{1}{(\lambda - \lambda_1)\delta_1} \int_0^1 g(s) \, ds \quad (3.2)
\]

where \( E(0) = \int_\Omega u_0 \phi_1 \, dx \). Moreover if \( g \) also satisfies

\[
g(s) \to 0 \quad \text{as} \ s \nearrow 1 \quad (3.3)
\]

then there exists a constant \( a_0 < 1 \) such that if

\[
\int_\Omega u_0 \phi_1 \, dx \geq a_0, \quad (3.4)
\]

then for any solution \( u \) of (0.3) with \( \lambda > 0 \) and initial value \( u_0 \) we have \( T_\lambda \leq (1 - a_0)/10 \).

**Proof.** We will use a modification of the argument of [GPW] and [KMS] to prove the theorem. Suppose \( u \) is a solution of (0.3) with \( \lambda > 0 \) and initial value \( u_0 \). Let

\[
E(t) = \int_\Omega u(x, t)\phi_1(x) \, dx.
\]
Multiplying (0.3) by $\phi_1$ and integrating over $\Omega$, by the Green theorem, (1.10) and the Jensen inequality,

$$
\frac{d}{dt}E(t) = \frac{d}{dt} \left( \int_{\Omega} u \phi_1 \, dx \right) = \int_{\Omega} \phi_1 \Delta u \, dx + \lambda \int_{\Omega} \frac{f \phi_1}{g(u)} \, dx \\
\geq -\mu \int_{\Omega} u \phi_1 \, dx + \lambda \delta_1 \int_{\Omega} \frac{\phi_1}{g(u)} \, dx \\
\geq -\mu E(t) + \lambda \frac{\delta_1}{g(E(t))}
$$

(3.5)

Note $E(t) \leq 1$ for any $t > 0$. We now divide the proof into two cases.

**Case 1:** $\lambda > \lambda_1$.

Then the right hand side is

$$
\geq (\lambda - \lambda_1) \frac{\delta_1}{g(E(t))}.
$$

(3.6)

Integrating (3.5) over $(0, t)$, by (3.6),

$$
t \leq \frac{1}{(\lambda - \lambda_1)\delta_1} \int_{E(0)}^{1} g(s) \, ds
$$

and (3.2) follows.

**Case 2:** $\lambda > 0$ and (3.3), (3.4), hold for some constant $a_0$ to be determined later.

By (3.3) there exists a constant $a_0 < 1$ such that

$$
-\mu y + \lambda \frac{\delta_1}{g(y)} \geq 10 \quad \forall a_0 \leq y < 1.
$$

(3.7)

Integrating (3.5) over $(0, t)$, by (3.4) and (3.7),

$$
10t \leq E(t) - E(0) \leq 1 - E(0) \quad \Rightarrow \quad T_\lambda \leq \frac{1 - E(0)}{10} \leq \frac{1 - a_0}{10}.
$$

By Corollary 2.2, Theorem 2.9, Theorem 3.3 and a comparison argument we have the following corollary.

**Corollary 3.4.** Let $f$ satisfy (0.1),

$$
\delta_\lambda = \inf_{B_\lambda(\rho_0)} f > 0
$$

for some $B_\lambda(\rho_0) \subset \Omega$, and let $g$ satisfy (0.2) and (1.10). Let $\mu_\lambda$ be the first eigenvalue of $-\Delta$ in $B_\lambda(\rho_0)$ and let $\phi_\lambda$ be the first positive eigenfunction of $-\Delta$ in $B_\lambda(\rho_0)$ normalized such that $\int_{B_\lambda(\rho_0)} \phi_\lambda \, dx = 1$. Let $\lambda_\lambda = (\mu_\lambda/\delta_\lambda) \sup_{0 \leq s \leq 1} g(s)$. Then for any solution $u$ of (0.3) with initial value $u_0 \geq 0$ and $\lambda > \lambda_\lambda$, we have

$$
T_\lambda \leq \frac{1}{(\lambda - \lambda_1)\delta_\lambda} \int_{E_1(0)}^{1} g(s) \, ds
$$
where $E_1(0) = \int_{B_R(x_0)} u_0 \phi_R \, dx$. Moreover if $g$ also satisfies (3.3), then there exists a constant $a_1 < 1$ such that if $u_0 \geq 0$ and

$$
\int_{B_R(x_0)} u_0 \phi_R \, dx \geq a_1,
$$

then for any solution $u$ of (0.3) with $\lambda > 0$ and initial value $u_0$ we have $T_\lambda \leq (1 - a_1)/10$.

**Theorem 3.5.** Let $f$ satisfy (0.1), $g$ satisfy (0.2) and $\lambda > \lambda^*$. Suppose $u_0$ satisfies (0.4) for some constant $a < 1$ and

$$
\overline{u}_0 \leq u_0 \quad \text{in } \Omega \tag{3.8}
$$

for some subsolution $\overline{u}_0 \in C^2(\Omega) \cap C(\Omega)$ of $(S_\lambda)$. If $u$ is the unique bounded solution of (0.3), then $T_\lambda < \infty$.

**Proof.** Suppose $u$ is a global bounded solution of (0.3). Let $\overline{u}$ be the unique bounded solution of (0.3) with initial value $\overline{u}_0$ given by Theorem 2.5 and Corollary 2.2. Then by Theorem 2.5, Corollary 2.2 and a continuity argument $\overline{u}$ can be extended to a global solution of (0.3) with initial value $\overline{u}_0$ which satisfies

$$
\overline{u} \leq u \quad \text{in } \Omega \times (0, \infty). \tag{3.9}
$$

By an argument similar to the proof on P.4–6 of [KMS] but with $(1 - u)^p$ there being replaced by $g(u)$ we get that there exists a time $T > 0$ such that

$$
\lim_{t \to T} \sup_{\Omega} \overline{u}(x, t) = 1. \tag{3.10}
$$

By (3.9) and (3.10), $\sup_\Omega u(x, t)$ will converge to 1 before the time $T$. Hence $T_\lambda < \infty$.

**Theorem 3.6.** Let $f$ satisfy (0.1) and $g$ satisfy (0.2). Let

$$
\lambda > \mu_1 \frac{\int_0^1 g(s) \, ds}{\int_\Omega f \phi_1 \, dx}
$$

and let $u$ be a solution of (0.3) with initial value $u_0$. Then

$$
T_\lambda \leq \frac{1}{(\lambda - \lambda')} \frac{\int_\Omega H(u_0) \phi_1 \, dx}{\int_\Omega f \phi_1 \, dx} \tag{3.11}
$$

where

$$
\lambda' = \mu_1 \frac{\int_0^1 g(s) \, ds}{\int_\Omega f \phi_1 \, dx}.
$$
Proof. Let $H(u)$ be given by (1.3). Then

$$\frac{d}{dt} \left( \int_{\Omega} H(u) \phi_1 \, dx \right) = - \int_{\Omega} \phi_1 g(u) u_t \, dx$$

$$= - \int_{\Omega} \phi_1 g(u) u \, dx - \lambda \int_{\Omega} f \phi_1 \, dx$$

$$= \int_{\Omega} \phi_1 g'(u) \left| \nabla u \right|^2 \, dx + \int_{\Omega} g(u) \nabla \phi_1 \cdot \nabla u \, dx - \lambda \int_{\Omega} f \phi_1 \, dx$$

$$\leq - \int_{\Omega} \nabla \phi_1 \cdot \nabla H(u) \, dx - \lambda \int_{\Omega} f \phi_1 \, dx$$

$$\leq \int_{\Omega} H(u) \Delta \phi_1 \, dx - \int_{\partial \Omega} H(u) \frac{\partial \phi_1}{\partial \nu} \, d\sigma - \lambda \int_{\Omega} f \phi_1 \, dx$$

$$\leq - \mu_1 \int_{\Omega} H(u) \phi_1 \, dx - H(0) \int_{\partial \Omega} \frac{\partial \phi_1}{\partial \nu} \, d\sigma - \lambda \int_{\Omega} f \phi_1 \, dx$$

$$= - \mu_1 \int_{\Omega} H(u) \phi_1 \, dx - H(0) \int_{\Omega} \Delta \phi_1 \, dx - \lambda \int_{\Omega} f \phi_1 \, dx$$

$$= - \mu_1 \int_{\Omega} H(u) \phi_1 \, dx - \mu H(0) \int_{\Omega} \phi_1 \, dx - \lambda \int_{\Omega} f \phi_1 \, dx$$

$$= - \mu_1 \int_{\Omega} H(u) \phi_1 \, dx - \mu H(0) \int_{\Omega} \phi_1 \, dx - \lambda \int_{\Omega} f \phi_1 \, dx$$

$$\leq - (\lambda - \lambda') \int_{\Omega} f \phi_1 \, dx$$

Integrating over $(0, t)$,

$$\int_{\Omega} H(u(x,t)) \phi_1(x) \, dx \leq \int_{\Omega} H(u_0) \phi_1 \, dx - (\lambda - \lambda') t \int_{\Omega} f \phi_1 \, dx.$$

Since the left hand side is positive while the right hand is negative for any

$$t > \frac{1}{(\lambda - \lambda')} \frac{\int_{\Omega} H(u_0) \phi_1 \, dx}{\int_{\Omega} f \phi_1 \, dx},$$

(3.11) follows.

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