THE YAMABE PROBLEM ON STRATIFIED SPACES

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Abstract. We introduce new invariants of a Riemannian singular space, the local Yamabe and Sobolev constants, and then go on to prove a general version of the Yamabe theorem under that the global Yamabe invariant of the space is strictly less than one or the other of these local invariants. This rests on a small number of structural assumptions about the space and of the behavior of the scalar curvature function on its smooth locus. The second half of this paper shows how this result applies in the category of smoothly stratified pseudomanifolds, and we also prove sharp regularity for the solutions on these spaces. This sharpens and generalizes the results of Akutagawa and Botvinnik (GAFA 13:259–333, 2003) on the Yamabe problem on spaces with isolated conic singularities.

Introduction

Our aim in this paper is to study a version of the Yamabe problem on a class of compact Riemannian singular spaces satisfying a small list of general structural axioms which we call ‘almost smooth metric-measure spaces’. This approach emphasizes the centrality of Sobolev inequality, and indeed relies on little else. Our main existence result is the analogue of that part of the resolution of this problem on compact smooth manifolds \((M, g)\) obtained through the work of Yamabe, Trudinger and Aubin, [Aub76], [Tru68], [Yam60]. In that original setting, the work of these authors established the existence of a smooth positive function minimizing the Yamabe functional

\[
Q_g(u) = \frac{\int_M (|\nabla u|^2 + \frac{n-2}{4(n-1)} \text{Scal}_g u^2) \, dV_g}{\left(\int_M u^{\frac{2n}{n-2}} \, dV_g\right)^{\frac{n-2}{n}}},
\]

where \(\text{Scal}_g\) is the scalar curvature of the metric \(g\), provided the infimum of this functional, the so-called Yamabe invariant \(Y(M, [g])\) (sometimes also called the Yamabe constant or conformal Yamabe invariant) of that conformal class \((M, [g])\), is strictly less than the corresponding invariant of the round sphere. In some papers on this subject, the energy \(Q_g\) is replaced by \((4(n-1)/(n-2))Q_g\). The geometric meaning of this functional is that if \(u\) is a minimizer, or indeed any critical point, then the conformally related metric \(\tilde{g} = u^{\frac{4}{n-2}} g\) has constant scalar curvature on any open set.
where \( u > 0 \). We refer to the well-known survey paper by Lee and Parker [LP87], as well as [Sch84], [SY88], for all details on the complete existence theory in the setting of smooth compact manifolds.

The singular spaces \((M, g, \mu)\) we are interested in here are typically Riemannian pseudomanifolds, and in particular Riemannian smoothly stratified spaces with iterated edge metrics, endowed with a measure which is a smooth positive multiple of the Riemannian volume form \( dV_g \). However, as indicated already, we require only a few structural assumptions and so our main existence theorem holds in much more general settings. In a companion to this paper [ACM13] we explore this direction further, extending this method to the setting of Dirichlet spaces. The ability to allow for a more general measure \( \mu \) is perhaps useful, but plays essentially no role in any of the arguments below, and for reasons of notational simplicity, we often omit \( \mu \) altogether from the discussion. The spaces here are ‘mostly smooth’ in that they possess an open dense set \( \Omega \) which is a smooth \( n \)-dimensional manifold carrying a Riemannian metric. Infinitesimally, every point in \( \Omega \) looks the same as every other. However, that is not true if one includes the singular points. To accomodate this, we replace the global Yamabe invariant by a new invariant which we call the local Yamabe invariant \( Y(\ell)(M, [g]) \). Briefly, this is just the infimum over all points \( p \in M \) of the Yamabe invariants of arbitrarily small balls around \( p \), where we minimize the standard energy functional amongst functions on these balls which vanish on the outer boundaries, but not necessarily near the singular set of \( M \). We also introduce the corresponding local Sobolev invariant \( S(\ell)(M, g) \). Our main existence theorem states that under various sets of conditions on the scalar curvature \( \text{Scal}_g \) (which we regard as a function computed in the usual way on the smooth domain \( \Omega \)), if the global Yamabe invariant \( Y(M, [g]) \) is strictly less than \( Y(\ell)(M, [g]) \) (or, in some versions of the result, than \( S(\ell)(M, g) \)), then \( Q_g \) admits a strictly positive minimizer \( u \). In certain cases we prove that this minimizer \( u \) is strictly positive, but show by example that this need not be the case if the hypotheses are relaxed.

Lest this criterion seem too abstract, observe that by conformal invariance, the local Yamabe invariant at a smooth point is equal to the Yamabe invariant of the round sphere; this is essentially what is known as Aubin’s inequality. It is important that \( Y(\ell) \) involves the limits as \( r \to 0 \) of the Yamabe invariants \( Y(B_r(p), g) \), rather than their values at any fixed \( r > 0 \); this means that local curvature invariants play a smaller role in \( Y(\ell) \). An invariant of this nature has been used previously for spaces \((M, g)\) with isolated conic singularities. In that setting, if \( p \) is a conic point, so that some neighbourhood \( U \) of \( p \) in \( M \) is modelled by a cone over a compact smooth Riemannian manifold \((Z, h)\), then the local Yamabe invariant at \( p \) is the same as the so-called cylindrical Yamabe invariant \( Y(\mathbb{R} \times Z, [dt^2 + h]) \) which plays an important role in the work of the first author and Botvinnik [AB03], see also [Aku12] for a discussion of this problem on orbifolds. It is proved there that \( Q_g \) has a minimizer provided

\[
- \infty < Y(M, [g]) < \min_j \{ Y(S^n, [g_0]), Y(\mathbb{R} \times Z_j, [dt^2 + h_j]) \}, \quad (0.2)
\]