Widths of weighted Sobolev classes on a domain with a peak: some limiting cases*

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1 Introduction

In this paper, order estimates for Kolmogorov, Gelfand and linear widths of weighted Sobolev classes $W^r_{p,g}$ on a domain with a peak in a weighted Lebesgue space $L^q,v$ are obtained. In particular, it is proved that if the peak is defined by the function $\varphi(t) = t^\sigma \log t^\theta$, $r + (\sigma(d-1)+1)\left(\frac{1}{q} - \frac{1}{p}\right) = 0$, $g \equiv 1$, $v \equiv 1$, then this singularity may have effect on the orders of widths. This supplements the following result of Besov [11]: if $\varphi(t) = t^\sigma$ and $r + (\sigma(d-1)+1)\left(\frac{1}{q} - \frac{1}{p}\right) > 0$, then the orders of Kolmogorov widths are the same as for domains with the Lipschitz boundary.

Let $d \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^d$ be a bounded domain (an open connected set), and let $g, v : \Omega \to (0, \infty)$ be measurable functions. For each measurable vector-valued function $\psi : \Omega \to \mathbb{R}^m$, $\psi = (\psi_k)_{1 \leq k \leq m}$, $p \in [1, \infty]$, we put

$$
\|\psi\|_{L^p(\Omega)} = \left(\frac{1}{p} \int_\Omega \max_{1 \leq k \leq m} |\psi_k(x)|^p \, dx\right)^{1/p}.
$$

Let $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}^d_+ := (\mathbb{N} \cup \{0\})^d$, $|\beta| = \beta_1 + \ldots + \beta_d$. For any distribution $f$ defined on $\Omega$ we write $\nabla^rf = \left(\partial^rf/\partial x^\beta\right)_{|\beta|=r}$ (here partial derivatives are taken in the sense of distributions), and denote by $m_r$ the number of components of the vector-valued distribution $\nabla^rf$. We set

$$
W^r_{p,g}(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \exists \psi : \Omega \to \mathbb{R}^{m_r} : \|\psi\|_{L^p(\Omega)} \leq 1, \nabla^rf = g \cdot \psi \},
$$

(we denote the corresponding function $\psi$ by $\nabla^rf/g$),

$$
\|f\|_{L^q,v(\Omega)} = \|f\|_{q,v} = \|f\|_{L^q(\Omega)}, \quad L^q,v(\Omega) = \{ f : \Omega \to \mathbb{R} \mid \|f\|_{q,v} < \infty \}.
$$

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We call the set \( W^r_{p,q}(\Omega) \) a weighted Sobolev class. Notice that \( W^r_p(\Omega) = W^r_{p,1}(\Omega) \) is the non-weighted Sobolev class.

For properties of weighted Sobolev spaces and their generalizations, see the books [19][22][40][70][71][73] and the survey paper [39]. Mazya [52] obtained the necessary and sufficient condition for the embedding of \( W^r_p(\Omega) \) in \( L^q(\Omega) \) in terms of isoperimetric and capacity inequalities. Reshetnyak [57, 58], Besov [4] and Bojarski [14] showed that for a John domain \( \Omega \) or a domain with a flexible cone property, the condition of continuous embedding of \( W^r_p(\Omega) \) in \( L^q(\Omega) \) is the same as for a domain with the Lipschitz boundary; in [3] this result was generalized for domains with decaying flexible cone condition. The problem on embedding of weighted Sobolev classes on domains with the irregular boundary that has zero angles was also intensively studied (see, e.g., [4, 6–10, 17, 24, 33, 38, 42, 43, 53]).

Denote by \( AC[t_0, t_1] \) the space of absolutely continuous functions on an interval \([t_0, t_1]\).

Let \( d \geq 2 \). Denote by \( B_a(x) \) the closed Euclidean ball of radius \( a \) in \( \mathbb{R}^d \) centered at the point \( x \). Given \( x = (y, z) \), \( y \in \mathbb{R}^{d-1} \), \( z \in \mathbb{R} \), we denote by \( B_a^{d-1}(y) \) the closed Euclidean ball of radius \( a \) in \( \mathbb{R}^{d-1} \) centered at the point \( y \). Set \( B_0^{d-1} = B_1^{d-1}(0) \).

**Definition 1.** Let \( G \subseteq \mathbb{R}^d \) be a bounded domain, \( a > 0 \), \( x_* \in G \). We say that \( G \in FC(a, x_*) \), if for any \( x \in G \) there exists a curve \( \gamma_x : [0, T(x)] \to G \) with the following properties:

1. \( \gamma_x \in AC[0, T(x)], \ |\gamma_x| = 1 \) a.e.,
2. \( \gamma_x(0) = x, \ \gamma_x(T(x)) = x_* \),
3. \( B_{a\epsilon}(\gamma_x(t)) \subseteq G \) for any \( t \in [0, T(x)] \).

We write \( G \in FC(a) \) if \( G \in FC(a, x_*) \) for some \( x_* \in G \). If \( G \in FC(a) \) for some \( a > 0 \), then we say that \( G \) satisfies the John condition (and call \( G \) a John domain).

For a bounded domain, the John condition is equivalent to the flexible cone condition (see the definition in [12]).

Let \( X, Y \) be sets, \( f_1, f_2 : X \times Y \to \mathbb{R}_+ \). We write \( f_1(x, y) \lesssim f_2(x, y) \) (or \( f_2(x, y) \gtrsim f_1(x, y) \)) if, for any \( y \in Y \), there exists \( c(y) > 0 \) such that \( f_1(x, y) \leq c(y)f_2(x, y) \) for each \( x \in X ; f_1(x, y) \gtrsim f_2(x, y) \) if \( f_1(x, y) \gtrsim f_2(x, y) \).

For \( G \subseteq \mathbb{R}^d \), \( x_* \in G \) we denote

\[
\overline{R}_{x_*}(G) = \sup_{x \in G} \|x - x_*\|_{l^2}, \quad R_{x_*}(G) = \inf_{x \in \partial G} \|x - x_*\|_{l^2}
\]

(here \( \partial G \) is the boundary of the set \( G \)). Notice that if \( G \in FC(a, x_*) \), then

\[
\overline{R}_{x_*}(G) \approx_R R_{x_*}(G).
\]
For \( z \in \mathbb{R} \) we set \( \eta_z = (0, \ldots, 0, z) \in \mathbb{R}^d \).

Let \( \varphi : (0, 1) \to (0, \infty) \) be an increasing Lipschitz function such that \( \lim_{z \to +0} \varphi(z) = \lim_{z \to +0} \varphi'(z) = 0 \).

**Definition 2.** Let \( a > 0 \), \( \tau_* > 0 \). We write \( \Omega \in FC_{\varphi, \tau_*}(a) \) if \( \Omega = \bigcup_{z \in (0, \tau_*)} \Omega_z \), where \( \Omega_z \in FC(a, \eta_z) \).

\[
\kappa_\Omega := \sup_{z \in (0, 1)} \frac{R_{\eta_z}(\Omega_z)}{z} < 1
\]  

(2)

and

\[
\zeta\varphi(z) \leqslant R_{\eta_z}(\Omega_z) \leqslant \tau\varphi(z), \quad z \in (0, \tau_*]
\]  

(3)

for some \( 0 < \zeta < \tau < \infty \).

In [53] a criterion for the continuous embedding of \( W_p^r(D_{\varphi, G}) \) in \( L_q(D_{\varphi, G}) \) was obtained, where

\[
D_{\varphi, G} = \{ x = (y, z) \in \mathbb{R}^d : z \in (0, 1), y/\varphi(z) \in G \}
\]

and \( G \subset \mathbb{R}^{d-1} \) is a bounded domain satisfying the cone condition. Notice that \( D_{\varphi, G} \in FC_{\varphi, \tau_*}(a) \) for some \( a > 0 \), \( \tau_* > 0 \). This result will be generalized for weighted Sobolev spaces (with weights depending only on \( z \)) and domains from the class \( FC_{\varphi, \tau_*}(a) \).

Without loss of generality we may assume that \( \tau_* = \frac{1}{2} \). Throughout this paper we denote \( FC_{\varphi}(a) = FC_{\varphi, 1/2}(a) \).

Let \( g_0 : (0, \infty) \to (0, \infty) \), \( v_0 : (0, \infty) \to (0, \infty) \) be measurable functions, \( g, v : \Omega \to (0, \infty) \), \( g(y, z) = g_0(z), v(y, z) = v_0(z) \). In addition, we suppose that there is \( C_* > 0 \) such that

\[
\frac{g_0(t)}{g_0(s)} \leqslant C_*, \quad \frac{v_0(t)}{v_0(s)} \leqslant C_*, \quad z \in (0, 1),
\]

\[t, s \in [\max\{z/2, z - \varphi(z)\}, z + \varphi(z)]\].  

(4)

Notice that \( \max\{z/2, z - \varphi(z)\} = z - \varphi(z) \) for sufficiently small \( z \).

Let \( 1 < p \leq q < \infty \), \( r \in \mathbb{N} \), \( \delta := r + \frac{d}{q} - \frac{d}{p} > 0 \). We write

\[ \mathcal{Z}_1 = (p, q, r, d, a, \varphi, \kappa_\Omega, \zeta, \tau, C_*) \].

For \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) we set \( x' = (x_1, \ldots, x_{d-1}) \).

Denote \( R(z) = R_{\eta_z}(\Omega_z) \), \( \overline{R}(z) = \overline{R}_{\eta_z}(\Omega_z) \).

For \( 0 \leq \tau_- < \tau_+ \leq \frac{1}{2} \) we set \( \Omega_{[\tau_-, \tau_+]} = \bigcup_{z \in [\tau_-, \tau_+]} \Omega_z \) (with \( \Omega_0 = \emptyset \)). Observe that \( \Omega_{[\tau_-, \tau_+]} \) is a domain.
Theorem 1. Let $\Omega \in \text{FC}_p(a)$, $0 \leq \tau_- < \tau_+ \leq \frac{1}{2}$,

$$\tau_- < \tau_+ - \frac{R}{\tau_+}, \quad 0 < \lambda < 1, \quad R = \lambda \frac{R}{\tau_+},$$

(5)

$$W^r_{p,g}(\Omega_{[\tau_-, \tau_+]}, \Gamma^R_0) = \{ f \in W^r_{p,g}(\Omega_{[\tau_-, \tau_+]}) : f|_{B^R_0(\tau_+)} = 0 \}.$$  (6)

Then the set $W^r_{p,g}(\Omega_{[\tau_-, \tau_+]}, \Gamma^R_0)$ is bounded in $L_{q,v}(\Omega_{[\tau_-, \tau_+]})$ if and only if

$$A_{[\tau_-, \tau_+]} := \max\{ A_{0,[\tau_-, \tau_+]}, A_{1,[\tau_-, \tau_+]} \} < \infty,$$

with

$$A_{0,[\tau_-, \tau_+]} = \sup_{t \in (\tau_-, \tau_+)} \left( \int_{\tau_-}^t \varphi^{d-1}(z)v_0^g(z) \, dz \right)^{1/q} \left( \int_{t}^{\tau_+} (z-t)^{p(r-1)} g_0^p(z) \varphi^{\frac{1}{q-1}}(z) \, dz \right)^{1/p'},$$

$$A_{1,[\tau_-, \tau_+]} = \sup_{t \in (\tau_-, \tau_+)} \left( \int_{\tau_-}^t (t-z)^{q(r-1)} \varphi^{d-1}(z)v_0^g(z) \, dz \right)^{1/q} \left( \int_{t}^{\tau_+} g_0^p(z) \varphi^{\frac{1}{q-1}}(z) \, dz \right)^{1/p'}.$$  More over, if $I : \text{span} W^r_{p,g}(\Omega_{[\tau_-, \tau_+]}, \Gamma^R_0) \to L_{q,v}(\Omega_{[\tau_-, \tau_+]})$ is the embedding operator, then $\|I\| \lesssim A_{[\tau_-, \tau_+]}. $

Let $(X, \| \cdot \|_X)$ be a normed space, let $X^*$ be its dual, and let $\mathcal{L}_n(X)$, $n \in \mathbb{Z}_+$, be the family of subspaces of $X$ of dimension at most $n$. Denote by $L(X, Y)$ the space of continuous linear operators from $X$ into a normed space $Y$. Also, by $\text{rk} A$ denote the dimension of the image of an operator $A \in L(X, Y)$, and by $\|A\|_{X \to Y}$, its norm.

By the Kolmogorov $n$-width of a set $M \subset X$ in the space $X$, we mean the quantity

$$d_n(M, X) = \inf_{L \in \mathcal{L}_n(X)} \sup_{x \in M} \inf_{y \in L} \|x - y\|_X,$$

by the linear $n$-width, the quantity

$$\lambda_n(M, X) = \inf_{A \in L(X, X)} \sup_{x \in M} \|x - Ax\|_X,$$

and by the Gelfand $n$-width, the quantity

$$d^n(M, X) = \inf_{x_1^*, \ldots, x_n^* \in X^*} \sup\{\|x\| : x \in M, x_j^*(x) = 0, 1 \leq j \leq n\} =$$

$$= \inf_{A \in L(X, \mathbb{R}^n)} \sup\{\|x\| : x \in M \cap \ker A\}.$$  In [55] the definition of strict $s$-numbers of a linear continuous operator was given. In particular, Kolmogorov numbers of an operator $A : X \to Y$ coincide with

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Kolmogorov widths \( d_n(A(B_X), Y) \) (here \( B_X \) is the unit ball in the space \( X \)); if the operator is compact, then its approximation numbers coincide with linear widths \( \lambda_n(A(B_X), Y) \) (see the paper of Heinrich [34]). If \( X \) and \( Y \) are both uniformly convex and uniformly smooth and \( A : X \to Y \) is a bounded linear map with trivial kernel and range dense in \( Y \), then Gelfand numbers of \( A \) are equal to \( d^n(A(B_X), Y) \) (see the paper of Edmunds and Lang [21]).

In the 1960–1970s problems concerning the values of the widths of function classes in \( L_q \) and of finite-dimensional balls \( B^n_\nu \) in \( l^n_q \) were intensively studied (see [25, 26, 35–37, 50, 51, 63–66, 69] and also [67], [68] and [56]). Here \( l^n_q (1 \leq q \leq \infty) \) is the space \( \mathbb{R}^n \) with the norm

\[
\|(x_1, \ldots, x_n)\|_q = \begin{cases} 
|x_1|^q + \cdots + |x_n|^q/\gamma, & \text{if } q < \infty, \\
\max\{|x_1|, \ldots, |x_n|\}, & \text{if } q = \infty,
\end{cases}
\]

\( B^n_\nu \) is the unit ball in \( l^n_\nu \). For \( p \geq q \), Pietsch [55] and Stesin [62] found the precise values of \( d_n(B^n_\nu, l^n_\nu') \) and \( \lambda_n(B^n_\nu, l^n_\nu') \). For \( p < q \), Kashin [36], Gluskin [30] and Garnaev, Gluskin [29] determined the order values of widths of finite-dimensional balls up to quantities depending only on \( p \) and \( q \).

Order estimates for widths of non-weighted Sobolev classes on an interval were obtained by Tikhomirov, Ismagilov, Makovoz and Kashin [35, 36, 51, 66, 69]. For multidimensional cube, the upper estimate of widths was first obtained by Birman and Solomyak [13]. After publication of Kashin’s result in [36], estimates for widths of Sobolev classes on a \( d \)-dimensional torus and their generalizations were obtained by Temlyakov and Galeev [25, 26, 63, 65]. Kashin [37] (for \( d = 1 \)), Kulanin [41] and Galeev [27, 28] have obtained estimates for widths of Sobolev classes with dominating mixed smoothness in the case of “small-order smoothness”. Here the upper estimate was not precise for \( d > 1 \) (involving a logarithmic factor). Order estimates for widths of \( W^r_p([0, 1]^d) \) in the case of “small-order smoothness” were obtained by DeVore, Sharpley and Riemenschneider [18]. The result of Vybiral [78] on order estimates for widths of Besov classes on a cube is also worth mentioning.

Besov in [11] proved the result on the coincidence of orders of widths

\[
d_n(W^r_p(K_\sigma), L_q(K_\sigma)) \asymp_{p,q,r,d,\sigma} d_n(W^r_p([0, 1]^d), L_q([0, 1]^d));
\]

here

\[
K_\sigma = \{(x_1, \ldots, x_{d-1}, x_d) : |(x_1, \ldots, x_{d-1})|^{1/\sigma} < x_d < 1\},
\]

\( \sigma > 1, r - [\sigma(d-1) + 1] \left(1 - \frac{1}{p} - \frac{1}{q}\right) > 0 \) and some conditions on the parameters \( p, q, r, d \) hold (see Theorem [10] below). For \( r = 1, p = q \) and more general ridged domains, estimates of approximation numbers were obtained by W.D. Evans and D.J. Harris [24].

The problem of estimating Kolmogorov widths of weighted Sobolev classes and other weighted functional classes and the problem of estimating approximation numbers of the corresponding embedding operators was also extensively examined.
The case $d = 1$ was considered by Lifshits and Linde, Edmunds, Lang, Lomakina and Stepanov and other authors \[20,41,45,48,49\]; the authors in question have found different sufficient conditions under which orders of approximation numbers of the embedding operator are the same as in the non-weighted case for a finite interval. In \[24\] the weights of a special form were considered; these weights had a singularity at a point, which affected the asymptotics of Kolmogorov and linear widths.

An upper estimate of Kolmogorov widths of Sobolev classes on a cube in a weighted $L_p$-space was first obtained by Birman and Solomyak \[13\] (for $q > \max\{p, 2\}$, the orders of this bound are not sharp). In \[23\], El Kolli had found the orders of the quantities $d_n(W^r_{p,g}(\Omega), L_{q,v}(\Omega))$ for $p \leq q$. For intersections of some weighted Sobolev classes on a cube with weights that are powers of the distance from the boundary, order estimates of widths were obtained by Boykov \[15,16\]. In \[72\] Triebel obtained estimates of approximation numbers for weighted Sobolev classes with weights that have a singularity at a point; this result was generalized in \[76\]. For general weights, the Kolmogorov’s and approximation numbers of an embedding operator of Sobolev classes in $L_p$ were estimated by Lizorkin, Otelbaev, Aitenova and Kusainova \[3,47,54\].

It is worth noting recent results on estimates of approximation and entropy numbers of embedding operators of Besov and Triebel–Lizorkin classes (see, e.g., \[31,32,77\]).

Suppose that for any $0 < z \leq \frac{1}{2}$

$$g_0(z) = z^\beta \left| \log z \right|^{-\alpha} \rho_g(\left| \log z \right|), \quad v_0(z) = z^\beta \left| \log z \right|^{-\alpha} \rho_v(\left| \log z \right|),$$

$$\varphi(z) = z^\sigma \left| \log z \right|^\theta \omega(\left| \log z \right|),$$

where $\rho_g$, $\rho_v$, $\omega$ are absolutely continuous functions such that

$$\lim_{t \to +\infty} \frac{t \rho'_g(t)}{\rho_g(t)} = \lim_{t \to +\infty} \frac{t \rho'_v(t)}{\rho_v(t)} = \lim_{t \to +\infty} \frac{t \omega'(t)}{\omega(t)} = 0,$$

$$\sigma > 1, \quad r + (\sigma(d - 1) + 1) \left( \frac{1}{q} - \frac{1}{p} \right) = \beta_g + \beta_v, \quad \sigma(d - 1) + 1 - \beta_v q > 0,$$

$$\alpha := \alpha_g + \alpha_v + \theta(d - 1) \left( \frac{1}{p} - \frac{1}{q} \right) > 0.$$
We set
\[ \rho(s) := \rho_g(s)\rho_v(s)\lvert \omega(s) \rvert^\left((d-1)(\frac{1}{q}-\frac{1}{p})\right), \] (12)

3 = (31, g, v). Observe that \( \lim_{t \to +\infty} \frac{t^\alpha \rho(t)}{t^{\alpha\rho(t)}} = 0 \) and the function \( t^{-\alpha} \rho(t) \) is decreasing for large \( t > 0 \).

We set, respectively, \( \vartheta_t(M, X) = d_t(M, X) \) and \( \hat{q} = q, \vartheta_t(M, X) = \lambda_t(M, X) \) and \( \hat{q} = \min\{q, p'\}, \vartheta_t(M, X) = d^p_t(M, X) \) and \( \hat{q} = p' \) in estimating Kolmogorov, linear and Gelfand widths, respectively.

**Theorem 2.**
1. Let \( p = q \) or \( p < q, \hat{q} \leq 2, \alpha \neq \frac{\hat{q}}{2} \). Then
\[ \vartheta_n(W_{p,q}(\Omega), L_{q,v}(\Omega)) \lesssim \frac{1}{n^{\frac{\alpha\rho}{\alpha\rho + \hat{q}}} \rho(n^{\frac{\alpha\rho}{\alpha\rho + \hat{q}}})}. \]

2. Let \( p < q \) and \( \hat{q} > 2 \). We set \( \theta_1 = \frac{\hat{q}}{2} + \min\{\frac{1}{2} - \frac{1}{q}, \frac{1}{p} - \frac{1}{q}\}, \theta_2 = \frac{\hat{q}}{2p}, \theta_3 = \alpha + \min\{\frac{1}{2} - \frac{1}{q}, \frac{1}{p} - \frac{1}{q}\}, \theta_4 = \frac{\hat{q} \sigma}{2}, \sigma_1 = \sigma_2 = 0, \sigma_3 = 1, \sigma_4 = \frac{\hat{q}}{2} \). Suppose that there exists \( j_* \in \{1, 2, 3, 4\} \) such that \( \theta_j < \min_{j \neq j_*} \theta_j \). Then
\[ \vartheta_n(W_{p,q}(\Omega), L_{q,v}(\Omega)) \lesssim \frac{1}{n^{\theta_j} \rho(n^{\theta_j})}. \]

2 Preliminary results

Let \( t_0 < t_1, r > 0, \) and let \( u, w : [t_0, t_1] \to \mathbb{R}_+ \) be measurable functions. Set

\[ \tilde{I}_{r,u,w,t_1} f(t) = w(t) \int_t^{t_1} (t-s)^{r-1} u(s) f(s) \, ds. \]

The criterion of continuity for the operator \( \tilde{I}_{r,u,w,t_1} : L_p[t_0, t_1] \to L_q[t_0, t_1] \) is proved by V.D. Stepanov [61]. Let us formulate this result for the case \( p \leq q \).

**Theorem A.** Let \( r \geq 1, 1 < p \leq q < \infty \). Then \( \| \tilde{I}_{r,u,w,t_1} \|_{L_p \to L_q} \preceq B_0 + B_1, \) where

\[
B_0 = \sup_{t \in (t_0, t_1)} \left( \int_{t_0}^{t} (t-x)^{(r-1)\theta} u^\theta(x) \, dx \right)^{1/q} \left( \int_{t}^{t_1} u^{p'}(x) \, dx \right)^{1/p'},
\]

\[
B_1 = \sup_{t \in (t_0, t_1)} \left( \int_{t_0}^{t} u^\theta(x) \, dx \right)^{1/q} \left( \int_{t}^{t_1} (x-t)^{(r-1)\theta} u^{p'}(x) \, dx \right)^{1/p'}.
\]
Denote by \( \text{mes} \Omega \) the Lebesgue measure of a set \( \Omega \subset \mathbb{R}^d \).

Reshetnyak [57,58] constructed the integral representation for smooth functions defined on a John domain \( \Omega \) in terms of their derivatives of order \( r \). This together with the result of Sobolev and Adams [1,2] implies the following theorem.

**Theorem B.** Let \( \Omega \in FC(a) \), \( r \in \mathbb{N} \), \( 1 < p < q < \infty \), \( \frac{1}{q} + \frac{1}{r} - \frac{1}{p} > 0 \). Then for any function \( f \in W^r_p(\Omega) \) there exists a polynomial \( P_f \) of degree not exceeding \( r-1 \) such that

\[
\|f - P_f\|_{L_q(\Omega)} \lesssim_{p,q,r,d,a} (\text{mes} \Omega)^{\frac{1}{r}} \|\nabla^r f\|_{L_p(\Omega)};
\]

in addition,

\[
\|f\|_{L_q(\Omega)} \lesssim_{p,q,r,d,a} (\text{mes} \Omega)^{\frac{1}{r}} \|\nabla^r f\|_{L_p(\Omega)} + (\text{mes} \Omega)^{\frac{1}{r} - \frac{1}{p}} \|f\|_{L_p(\Omega)}.
\]

Kashin and Gluskin [30,36] obtained order estimates for \( d_n(B^\nu_p, l^\nu_q) \), \( d^n(B^\nu_p, l^\nu_q) \) and \( \lambda_n(B^\nu_p, l^\nu_q) \).

**Theorem C.** Let \( 1 < p < q < \infty \). Then

\[
d_n(B^\nu_p, l^\nu_q) \asymp_{q,p} \Phi(n, \nu, p, q), \tag{13}
\]

\[
\lambda_n(B^\nu_p, l^\nu_q) \asymp_{q,p} \Psi(n, \nu, p, q), \tag{14}
\]

\[
d^n(B^\nu_p, l^\nu_q) \asymp_{q,p} \Phi(n, \nu, q', p'), \tag{15}
\]

with

\[
\Phi(n, \nu, p, q) = \begin{cases} 
\min \left\{ 1, \left( \nu^{1/q} n^{-1/2} \right)^{\left( \frac{1}{q} - \frac{1}{p} \right) (\frac{1}{2} - \frac{1}{r})} \right\}, & 2 \leq p < q < \infty, \\
\max \left\{ \nu^{1 - \frac{1}{q}}, \min \left( \nu^{\frac{1}{2} - \frac{1}{q}}, \min \left( 1, \nu^{\frac{1}{q}} n^{-\frac{1}{2}} \right) \left( \frac{p}{q} - 1 \right)^{1/2} \right) \right\}, & 1 < p < 2 < q < \infty, \\
\max \left\{ \nu^{1 - \frac{1}{q}}, (1 - \frac{p}{q})^{\left( \frac{1}{2} - \frac{1}{r} \right) (1 - \frac{1}{q})} \right\}, & 1 < p < q \leq 2,
\end{cases}
\]

\[
\Psi(n, \nu, p, q) = \begin{cases} 
\Phi(n, \nu, p, q), & \text{if } q \leq q', \\
\Phi(n, \nu, q', p'), & \text{if } p' < q.
\end{cases}
\]

Let us formulate the result on estimates of widths \( d_n(W^r_p([0, 1]^d), L_q([0, 1]^d)) \), \( \lambda_n(W^r_p([0, 1]^d), L_q([0, 1]^d)) \) and \( d^n(W^r_p([0, 1]^d), L_q([0, 1]^d)) \) (see [18,26,35,36,50,65,66,69,78]).

**Theorem D.** Let \( r \in \mathbb{N} \), \( 1 \leq p \leq q \leq \infty \), \( \frac{r}{d} + \frac{1}{q} - \frac{1}{p} > 0 \). Set

\[
\theta_{p,q,r,d} = \begin{cases} 
\frac{\delta}{2}, & \text{if } p \geq q \text{ or } p < q, \hat{q} \leq 2, \\
\min \left\{ \frac{\delta}{2} + \min \left\{ \frac{1}{2} - \frac{1}{q}, \frac{1}{p} - \frac{1}{q}, \frac{q}{2d} \right\} \right\}, & \text{if } p < q, \hat{q} > 2.
\end{cases}
\]

In addition, suppose that \( \frac{\delta}{2} + \min \left\{ \frac{1}{2} - \frac{1}{q}, \frac{1}{p} - \frac{1}{q} \right\} \neq \frac{q}{2d} \) in the case \( p < q, \hat{q} > 2 \). Then

\[
\vartheta_n(W^r_p([0, 1]^d), L_q([0, 1]^d)) \asymp_{r,d,p,q} n^{-\theta_{p,q,r,d}};
\]
The following lemma was proved in [77].

**Lemma 1.** Let \( \Lambda : (0, +\infty) \to (0, +\infty) \) be an absolutely continuous function such that \( \lim_{y \to +\infty} \frac{\Lambda(y)}{\Lambda'(y)} = 0 \). Then for any \( \varepsilon > 0 \)

\[
t^{-\varepsilon} \lesssim \frac{\Lambda(ty)}{\Lambda(y)} \lesssim t^{\varepsilon}, \quad 1 \leq y < \infty, \quad 1 \leq t < \infty. \tag{16}
\]

3 The embedding theorem for weighted Sobolev classes on a domain with a peak

**Lemma 2.** Let \( 0 < \tau_0 < \tau_1 \leq \frac{1}{2} \) and \( c > 0 \) be such that

\[
\tau_1 - \tau_0 \leq c \varphi(\tau_0), \tag{17}
\]

and let \( L \) be a Lipschitz constant of the function \( \varphi|_{[\tau_0, \tau_1]} \). Then \( \bigcup_{z \in [\tau_0, \tau_1]} \Omega_z \in FC(b) \), with \( b = b(a, d, c, L, c) > 0 \).

**Proof.** For any \( t \in [\tau_0, \tau_1] \)

\[
\varphi(\tau_0) \leq \varphi(t) \leq \varphi(\tau_0) + L(t - \tau_0) \leq (L \cdot c + 1)\varphi(\tau_0).
\]

Therefore,

\[
R_t \gtrsim a, d, R_t \gtrsim c, L, c \varphi(\tau_0). \tag{18}
\]

Let \( z \in [\tau_0, \tau_1] \), \( x \in \Omega_z \), and let \( \gamma_x : [0, T(x)] \to \Omega_z \) be the curve from Definition 1 \( \gamma_x(T(x)) = \eta_z \). Then \( T(x) \leq a^{-1}R_z \). Extend the curve \( \gamma_x \) by connecting \( \eta_z \) and \( \eta_{\tau_0} \) by a segment. It remains to apply (18) and (17).

Set \( G_z = \{ y \in \mathbb{R}^{d-1} : (y, z) \in \Omega \} \). Then

\[
G_z \supset \{ y \in \mathbb{R}^{d-1} : (y, z) \in \Omega_z \} \supset B^{d-1}_{\frac{1}{2}R(z)}(0),
\]

which implies

\[
\text{dist}(0, \partial G_z) \geq R(z), \quad \text{diam } G_z \geq 2R(z). \tag{19}
\]

From (11) and Definition 2 it follows that

\[
R(z) \gtrsim a, d, R(z) \gtrsim c \varphi(z). \tag{20}
\]

**Lemma 3.** The following order equalities hold:

\[
\text{dist}(0, \partial G_z) \asymp \varphi(z), \quad \text{diam } G_z \asymp \varphi(z). \tag{21}
\]
Recall that $\zeta$ (indeed, \(\{z \mid z \in \mathbb{R} \\})\) is interrupted. If $\zeta > 0$, let, now, $\frac{z}{2\lambda} \geq z - \lambda \varphi(z)$ and for any $z \in (0, \frac{1}{2}]$ and for any

\[
\frac{z}{2\lambda} \geq z - \lambda \varphi(z), \quad \frac{z}{2\lambda} \leq t \leq \min \left\{ \frac{z + \lambda \varphi(z)}{2} \right\}.
\]

Proof. First prove that for $\lambda > 0$

\[
g_0(z) \geq g_0(t), \quad v_0(z) \geq v_0(t), \quad 0 < z \leq \frac{1}{2}, \quad z \leq t \leq \min \left\{ \frac{z + \lambda \varphi(z)}{2} \right\}.
\]

To this end we construct a sequence $\{z(k)\}$ by induction. Set $z(0) = z$. Suppose that $z(k) \leq \frac{1}{2}$ is constructed for some $k \in \mathbb{N}$. If $z(k) > \frac{1}{2}$, then the construction is interrupted. If $z(k) < \frac{1}{2}$, then we set $z(k+1) = \min \left\{ z(k) + \varphi(z(k)), \frac{1}{2} \right\}$. Let $k \in \mathbb{N}$, $z(k) < \frac{1}{2}$. Since the function $\varphi$ is non-decreasing, we get $z(k) \geq z(k+1) + \varphi(z)$. Hence, $z(k) \geq \frac{z + k\varphi(z)}{2}$, and for $k = \left\lfloor \lambda \right\rfloor$ we obtain $z(k) \geq \frac{z + \lambda \varphi(z)}{2}$. It remains to take into account that $g_0(t) \geq g_0(z(k+1))$, $v_0(t) \geq v_0(z(k+1))$ for any $t \in [z(k+1), z(k)]$.

Let, now, $\max \left\{ \frac{z}{2\lambda}, z - \lambda \varphi(z) \right\} \leq t \leq z$. Since $\lim_{s \to 0^+} \varphi'(s) = 0$, there exists $z_\ast \geq \frac{1}{2}$ such that $\frac{z}{2} \geq z - \lambda \varphi(z)$ and $|\varphi'(z)| \leq \frac{1}{2\lambda}$ for any $z \leq z_\ast$. Then for
each \( z \leq z_* \) we get \( \varphi(z-\lambda_* \varphi(z)) \geq \frac{\varphi(z)}{2} \). Therefore, \( z \leq z-\lambda_* \varphi(z)+2\lambda_* \varphi(z-\lambda_* \varphi(z)) \).

It remains to apply (24). Let \( z > z_* \). Prove that \( g_0(z) \sim g_0(t), \ v_0(z) \sim v_0(t) \) for any \( t \in \left[ \frac{z}{2\lambda_*}, z \right] \). We have \( \varphi \left( \frac{z}{2\lambda_*} \right) \geq \varphi \left( \frac{z}{2\lambda_*} \right) \sim 1 \). Consequently, there is \( c_{3,1,\lambda_*} > 0 \) such that \( z \leq \frac{z}{2\lambda_*} + c_{3,1,\lambda_*} \varphi \left( \frac{z}{2\lambda_*} \right) \). Apply (24) once again and obtain the desired estimate. \( \square \)

We say that sets \( A, B \subset \mathbb{R}^d \) do not overlap if \( A \cap B \) is a Lebesgue nullset.

Let \( E, E_1, \ldots, E_m \subset \mathbb{R}^d \) be measurable sets. We say that \( \{E_i\}_{i=1}^m \) is a covering of \( E \) if the set \( E \setminus (\bigcup_{i=1}^m E_i) \) is a Lebesgue nullset. We say that \( \{E_i\}_{i=1}^m \) is a partition of \( E \) if the sets \( E_i \) do not overlap pairwise and the set \( (\bigcup_{i=1}^m E_i) \triangle E \) is a Lebesgue nullset.

Let \( T \) be a covering of a set \( G \). Put

\[
N_{T,E} = \text{card}\{E' \in T : \text{mes}(E \cap E') > 0\}, \quad E \in T,
\]

\[
N_T = \sup_{E \in T} N_{T,E}.
\]

Lemma 5. Let \( 0 \leq \tau_k < \tau_{k-1} < \cdots < \tau_0 \leq \frac{1}{2}, \ 0 < \hat{c} < 1, \)

\[
\tau_{j-1} - \tau_j \geq \hat{c}\varphi(\tau_{j-1}), \quad (25)
\]

\( G_{(j)} = \Omega_{[\tau_j, \tau_{j-1}]}, \ 1 \leq j \leq k \), \( T = \{G_{(j)}\}_{j=1}^k \). Then \( \text{card}N_T \leq 1. \)

**Proof.** Since the function \( \varphi \) is Lipschitz and \( \lim_{z \to 0} \varphi'(z) = 0 \), there exists \( z_0 = z_0(3_1) \in (0, \frac{1}{2}] \) such that for any \( z \in (0, z_0] \)

\[
\varphi(z-2\varphi(\tau)) \geq \frac{\varphi(z)}{2}, \quad \varphi(z+2\varphi(\tau)) \leq 2\varphi(z), \quad |\varphi'(z)| \leq \frac{1}{4\hat{c}} \quad (26)
\]

Let us estimate

\[
\text{card}\{i \in [1, k], j : G_{(i)} \cap G_{(j)} \neq \emptyset\}, \quad j \in [1, k].
\]

Since the sets \( G_{(i)} \) are open, the condition \( G_{(i)} \cap G_{(j)} \neq \emptyset \) is equivalent to the inequality \( \text{mes} \left( G_{(i)} \cap G_{(j)} \right) > 0 \). Notice that if \( \tau_{i-1} \geq z_0 \), then \( \tau_{i-1} - \tau_i \geq \hat{c}\varphi(\tau_{i-1}) \sim 1 \). Hence,

\[
\text{card}\{i \in [1, k], j : \tau_{i-1} \geq z_0\} \leq 1.
\]

Therefore, it is sufficient to estimate

\[
\text{card}\{i \in [1, k] \setminus \{j\} : G_{(i)} \cap G_{(j)} \neq \emptyset, \tau_{i-1} < z_0\}.
\]

If \( G_{(i)} \cap G_{(j)} \neq \emptyset \), then there exist \( z \in [\tau_j, \tau_{j-1}], t \in [\tau_i, \tau_{i-1}] \) such that

\[
B_{\Omega_2}(\eta_2) \cap B_{\Omega_1}(\eta_1) \neq \emptyset. \quad (27)
\]
Let $i < j$. Then from (27) it follows that $z + R(t) \geq t$. Therefore,

$$t - \tau_{j-1} \leq t - z \leq R(z) + R(t) \leq \tau(z) + \varphi(t) \leq 2\varphi(t),$$

i.e., $t - 2\varphi(t) \leq \tau_{j-1}$. By the monotonicity of the function $\varphi$ and the inequality $t \leq \tau_{i-1} \leq z_0$, we have

$$\frac{z(t)}{2} \leq \varphi(t - 2\varphi(t)) \leq \varphi(\tau_{j-1}).$$

Applying (28) once again, we get

$$\tau_{i-1} - \tau_{j-1} \leq 2\varphi(t) \leq 4\varphi(\tau_{j-1}).$$

On the other hand, $\tau_{i-1} - \tau_i \geq \hat{c}\varphi(\tau_{i-1}) \geq \hat{c}\varphi(\tau_{j-1})$. This yields the desired estimate.

Let, now, $i > j$. Then from (27) it follows that

$$t + R(t) \geq z - R(z).$$

Let $z \geq z_0$. Then

$$(1 + \kappa\Omega)\tau_{i-1} \geq (1 + \kappa\Omega)t \geq t + R(t) \geq z - R(z) \geq (1 - \kappa\Omega)z \geq (1 - \kappa\Omega)z_0.$$}

Hence, $\tau_{i-1} \geq 1$, $\varphi(\tau_{i-1}) \geq 1$, $\tau_{i-1} - \tau_i \geq 1$ and

$$\text{card}\{i > j : G(i) \cap G(j) \neq \emptyset\} \lesssim 1.$$

Let $z < z_0$. From (29) it follows that

$$z - R(z) \leq t + R(t) \leq t + \varphi(t) \leq \tau_{i-1} + \varphi(z).$$

Set

$$\hat{z} = \begin{cases} 
  z, & \text{if } z - R(z) \leq \tau_j - R(\tau_j), \\
  \tau_j, & \text{otherwise}.
\end{cases}$$

Then

$$\hat{z} - R(\hat{z}) \leq \tau_{i-1} + \varphi(\hat{z}).$$

Indeed, if $\hat{z} = z$, then it follows from (30). If $\hat{z} = \tau_j$, then by (30), (31) and the inequalities $t \leq \tau_{i-1} \leq \tau_j$ we get

$$\hat{z} - R(\hat{z}) \leq z - R(z) \leq t + \varphi(t) \leq \tau_{i-1} + \varphi(\tau_j) = \tau_{i-1} + \varphi(\hat{z}).$$

Estimate $\hat{z} - \tau_j$ from above. Taking into account the condition $\hat{z} \leq z < z_0$, we get

$$\hat{z} - \tau_j \leq R(\hat{z}) - R(\tau_j) \leq \varphi(\hat{z}) - \varphi(\tau_j) =$$

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\[ \begin{align*}
&= (\overline{c} - c) \phi(\tau_j) + \overline{c} \int_{\tau_j}^{\tilde{z}} \phi'(s) \, ds \quad \text{(26)} \\
&\quad \leq (\overline{c} - c) \phi(\tau_j) + \frac{\tilde{z} - \tau_j}{4},
\end{align*} \]
which implies
\[ \tilde{z} - \tau_j \leq 2(\overline{c} - c) \phi(\tau_j). \quad \text{(33)} \]

From (32) we obtain that
\[ \tilde{z} - \tau_{i-1} \leq \overline{R}(\tilde{z}) + \overline{c} \phi(\tilde{z}) \quad \text{(33)} \]
Therefore, \( \phi(\tau_{i-1}) \geq \phi(\tilde{z} - 2c \phi(\tilde{z})) \geq \frac{\phi(\tilde{z})}{2}. \) Consequently,
\[ \tau_{i-1} - \tau_i \geq \hat{c} \phi(\tau_{i-1}) \geq \frac{\hat{c}}{2} \phi(\tilde{z}) \geq \frac{\hat{c}}{2} \phi(\tau_j). \]

On the other hand,
\[ \tau_j - \tau_{i-1} \leq \tilde{z} - \tau_{i-1} \leq 2 \phi(\tilde{z}) \quad \text{(26)} \]
This yields the desired estimate.

Proof of Theorem 1. The arguments are almost the same as in [53]. Here we give the sketch of the proof.

In order to obtain the lower estimate, we take functions
\[ \psi_f(y, z) = \int_{\tau}^{\tau + R} (t - z)^{r-1} g_0(t) f(t) \, dt, \]
where \( f \) is such that \( \|f\|_{L_p(\Omega)} = 1 \) for \( f_s(y, z) = f(z) \). By Theorem A,
\[ \sup_{\|f\|_{L_p(\Omega)} = 1} \|\psi_f\|_{L_{q,v}(\Omega)} \geq \max\{A_0[\tau_-, \tau_+ - R], A_1[\tau_-, \tau_+ - R]\}. \]

Applying (1), the fact that \( \phi \) is Lipschitz, the inequalities \( R \leq \overline{R}_{\tau_+} \leq \overline{c} \phi(\tau_+) \) and
\[ \tau_+ - R - \tau_- = \tau_+ - \overline{R}_{\tau_+} - \tau_- + (\overline{R}_{\tau_+} - R) \geq \overline{R}_{\tau_+} - R \geq \left( \frac{1}{\lambda} - 1 \right) R, \]
we get that
\[ \max\{A_0[\tau_-, \tau_+ - R], A_1[\tau_-, \tau_+ - R]\} \geq \max\{A_0[\tau_-, \tau_+], A_1[\tau_-, \tau_+]\}. \]

Prove the upper estimate. By Lemma 2.2 from [53], we may assume that
\[ \phi \in C^\nu(0, 1), \quad |\phi^{(k)}(z)| \leq c \phi(z)^{1-k}, \quad k \in \mathbb{N}, \quad z \in (0, 1) \quad \text{(35)} \]
for some \( c > 0 \) not depending on \( z \). From \([19]\) and \([20]\) it follows that there is \( c_* = c_*(3_1) > 0 \) such that \( \text{dist}(0, \partial G_z) \geq R_z \geq c_* \phi(z) \). Applying some homothetic transformation of \( y \), we may assume that \( \frac{\|y\|}{\phi(z)} = 1 \). Let \( K \in C_0^\infty(B^{d-1}) \), \( \int_{B^{d-1}} K(y) \, dy = 1 \), \( \int_{B^{d-1}} K(y) y^\mu \, dy = 0 \), \( \mu \in \mathbb{Z}^{d-1}_+ \), \( 1 \leq |\mu| \leq r - 1 \),

\[
u_\mu(z) = \phi(z)^{1-d} \int_{|y| < \phi(z)} K \left( \frac{y}{\phi(z)} \right) \frac{\partial^{|\mu|} u(z)}{\partial y^\mu}(y, z) \, dy, \quad z \in (0, 1), \quad u \in W_p^r(\Omega),
\]

\[
Q(x) = \sum_{|\mu| \leq r-1} \nu_\mu(z) \frac{y^\mu}{\mu!}, \quad x = (y, z) \in \Omega
\]

(see \([53\text{, page 112}]\)).

In \([53\text{, Lemma 3.1}]\) it is proved that if \([35]\) holds, then the function \( u_\mu(z) \) is absolutely continuous, as well as its derivatives of order not exceeding \( r - 1 \), \( u_\mu(r) \in L_p^\text{loc} \), and for \( r - |\mu| \leq s \leq r \), \( z \in (0, 1) \)

\[
|u_\mu^{(s)}(z)| \leq \frac{\phi(z)^{r-s-\frac{d-1}{p}}}{3_1} \left( \int_{|y| < \phi(z)} \left| \nabla^r u(y, z) \right|^p \, dy \right)^{\frac{1}{p}}.
\]

Let \( \mu \in \mathbb{Z}^{d-1}_+ \), \( |\mu| \leq r - 1 \), \( Q_\mu(x) = u_\mu(z) y^\mu \). Estimate \( \|Q_\mu\|_{L_q^\text{loc}(\Omega_{[\tau_-, \tau_+])}} \) from above. Since \( u_\mu(z) = 0 \) in some neighborhood of \( \tau_+ \) (see \([36]\)), we have

\[
u_\mu(z) = \frac{(-1)^r}{(r-1)!} \int_0^{\tau_+} (t-z)^{r-1} u_\mu^{(r)}(t) \, dt, \quad z \in [\tau_-, \tau_+].
\]

Applying \([36]\) with \( s = r \), we obtain

\[
\|Q_\mu\|_{L_q^\text{loc}(\Omega_{[\tau_-, \tau_+])}} \leq \left( \int_{\tau_-}^{\tau_+} u_\mu(z)\left( |\phi(z)|^{q|\mu|+d-1} \right) \, dz \right)^{1/q} \leq C_{\tau_-, \tau_+} \left( \int_{\tau_-}^{\tau_+} \frac{u_\mu^{(r)}(z)}{g_0(z)} \phi(z)^{p|\mu|+d-1} \, dz \right)^{\frac{1}{p}} \leq C_{\tau_-, \tau_+} \left( \int_{\tau_-}^{\tau_+} \int_{|y| < \phi(z)} \left| \nabla^r u(y, z) \right|^p \, dy \, dz \right)^{\frac{1}{p}}.
\]

The value \( C_{\tau_-, \tau_+} \) is estimated by applying Theorem \([A]\). Since \( \phi \) is non-decreasing, it gives the desired estimate (see the proof in \([53\text{, page 113}]\)).

Let us estimate \( \|u - Q\|_{L_q^\text{loc}(\Omega_{[\tau_-, \tau_+])}} \). Set \( z_0 = \tau_+, \ z_{k+1} + \phi(z_{k+1}) = z_k \), \( k \in \mathbb{Z}_+ \), \( k_* = \min\{k \in \mathbb{Z}_+ : z_{k+1} < \tau_-, \} \), \( z_{k+1} = \max\{z_{k+1}, \tau_-, \} \), \( \Omega_k = \bigcup_{z \in [z_{k+1}, z_k]} \). By
Lemma \[2\] there is \( b = b(\mathfrak{3}_1) > 0 \) such that \( \Omega(\mathfrak{k}) \in \mathcal{F}(b) \), \( k \in \mathbb{Z}_+ \). Set \( g(\mathfrak{k}) = g_{0}(z_{\mathfrak{k}}) \), \( v(\mathfrak{k}) = v_{0}(z_{\mathfrak{k}}) \). By \[4\], \( g(x) \asymp g_{\mathfrak{k}} \) on \( 3_{\mathfrak{i}} \) for any \( \mathfrak{x} \in \Omega(\mathfrak{k}) \). By Theorem \[13\] and \[3\],
\[
\| u \|_{L_q(\Omega(\mathfrak{k})))} \leq \varphi(z_k)^{\frac{d}{q} - \frac{d}{p}} \left( \| u \|_{p(\Omega(\mathfrak{k})))} + \varphi(z_k)^{r} \| \nabla^{r} u \|_{p(\Omega(\mathfrak{k})))} \right),
\]
Repeating arguments in \[53\] page 114, we can prove that
\[
\| u - Q \|_{L_q(\Omega(\mathfrak{k})))} \leq \varphi(z_k)^{r + \frac{d}{q} - \frac{d}{p}} \| \nabla^{r} u \|_{p(\Omega(\mathfrak{k})))}.
\]
By construction, \( z_k - z_{k+1} = \varphi(z_{k+1}) \asymp \varphi(z_k) \). Therefore, from Lemma \[5\] it follows that
\[
\| u - Q \|_{L_q,v(\Omega_{[\tau_-, \tau_+]})} \leq \max_{0 \leq k \leq k_{\mathfrak{s}}} g(\mathfrak{k})v(\mathfrak{k}) \varphi(z_k)^{r + \frac{d}{q} - \frac{d}{p}} \left( \| \nabla^{r} u \|_{g} \right)_{\Omega(\mathfrak{k})_{[\tau_-, \tau_+]}} \leq \mathfrak{s}.
\]
\[
\leq \sup_{t \in [\tau_-, \tau_+]} A_{1, [\tau_-, \tau_+]}(t) \left( \| \nabla^{r} u \|_{g} \right)_{\Omega(\mathfrak{k})_{[\tau_-, \tau_+]}}
\]
(the last relation is similar to the inequality (4.7) in \[53\]; it follows from the monotonicity of \( \varphi \)). \( \square \)

Denote by \( \mathcal{P}_{r-1}(\mathbb{R}^d) \) the space of polynomials on \( \mathbb{R}^d \) of degree not exceeding \( r - 1 \). For each measurable subset \( E \subset \mathbb{R}^d \) we put
\[
\mathcal{P}_{r-1}(E) = \{ f|_{E} : f \in \mathcal{P}_{r-1}(\mathbb{R}^d) \}.
\]

**Corollary 1.** Let \( \Omega \in \mathcal{F}_{\varphi}(a) \), \( 0 < \lambda_{\mathfrak{s}} < 1 \), \( 0 \leq \tau_- < \tau_+ \leq \frac{1}{2} \), and let
\[
\tau_- < \tau_+ - \lambda_{\mathfrak{s}} \varphi(\tau_+). \quad (37)
\]
Suppose that \( A_{[\tau_-, \tau_+]} < \infty \). Then \( W^{r}_{p,q}(\Omega_{[\tau_-, \tau_+]}) \subset L_{q,v}(\Omega_{[\tau_-, \tau_+]}) \) and there exists a linear continuous operator
\[
P : L_{q,v}(\Omega_{[\tau_-, \tau_+]}) \to \mathcal{P}_{r-1}(\Omega_{[\tau_-, \tau_+]})
\]
such that for any function \( f \in W^{r}_{p,q}(\Omega_{[\tau_-, \tau_+]}) \)
\[
\| f - Pf \|_{L_{q,v}(\Omega_{[\tau_-, \tau_+]})} \leq A_{[\tau_-, \tau_+]} \left( \| \nabla^{r} f \|_{g} \right)_{\Omega(\mathfrak{k})_{[\tau_-, \tau_+]}}. \quad (38)
\]
**Proof.** Let \( \tau_- \geq \tau_+ - \mathcal{P}_{r-1} \). Then \( \tau_+ - \tau_- \leq c_{0} \varphi(\tau_+) \leq \varphi(\tau_-) \) (here \( c_{0} = c_{0}(\mathfrak{3}_1) \)).

Prove the last inequality. Since \( \varphi'(t) \to 0 \), there exists \( \hat{\tau} = \hat{\tau}(\mathfrak{3}_1) > 0 \) such that for
such that \( r_+ = \frac{r_+}{1 - \kappa_0} = \frac{\tilde{r}_+}{1 - \kappa_0} \). Therefore, \( \varphi(t_-) < 1 < \varphi(t_+) \).

Thus, \( t_+ - t_- \leq \frac{\varphi(t_-)}{3_1} \). By Lemma 2, \( \Omega_{[t_-, t_+]} \in \mathcal{F}(\tilde{b}) \), \( \tilde{b} \approx 1 \). It remains to apply Lemma 1, Theorem 1 and the definition of \( A_{[t_-, t_+]} \).

Suppose that \( t_- < t_+ - R_{r_+} \). Let \( R_{t_0} : \text{Span}(\mathcal{B}_{t_0}(\eta_{r_+})) \to \mathcal{P}_{t_0-1}(\mathcal{B}_{t_0}(\eta_{r_+})) \) be a linear continuous projection. Then for any \( 0 \leq k \leq r \) and for each \( q_k \in [1, +\infty) \) such that \( r - k + \frac{d}{q_k} - \frac{d}{p} > 0 \) we have

\[
\| f - P_{t_0} f \|_{L_{q_k}(B_{r_+}(\eta_{r_+}))} \lesssim \| R_{t_0} f \|_{L_{q_k}(B_{r_+}(\eta_{r_+}))} \lesssim \frac{1}{3_1} \| \nabla f \|_{L_p(\mathcal{P}_{t_0-1}(\mathcal{B}_{t_0}(\eta_{r_+}))}
\]

(see (1)).

In order to define \( P_{t_0} \), we take the orthogonal projection in \( L_2(B_{r_+}(\eta_{r_+})) \) onto the space \( \mathcal{P}_{t_0-1}(\mathcal{B}_{t_0}(\eta_{r_+})) \), extend it on \( L_1(B_{r_+}(\eta_{r_+})) \) by continuity, and then restrict it to \( L_{q_k}(B_{r_+}(\eta_{r_+})) \). The function \( P f \) is defined as the extension of the polynomial \( P_{t_0}(f) \) on the domain \( \Omega_{[t_-, t_+]} \). Then the image of \( P \) is contained in \( \mathcal{P}_{t_0-1}(\Omega_{[t_-, t_+]})) \). From the condition \( A_{[t_-, t_+]} < \infty \) it follows that \( v \in L_{q}(\Omega) \) and the operator \( P : L_{q_0}(\Omega_{[t_-, t_+]})) \to L_{q_0}(\Omega_{[t_-, t_+]})) \) is continuous.

Let \( \psi_0 \in C_0^\infty(\mathbb{R}^d) \), \( \supp \psi_0 \subset B_1(0), \psi_0|_{B_{1/2}(0)} = 1, \psi_0(x) \in [0, 1] \) for any \( x \in \mathbb{R}^d \). Set \( R = \frac{R_{r_+}}{2} \), \( \psi(x) = \psi_0 \left( \frac{x - \eta_{r_+}}{R_{r_+}} \right) \). Then \( \psi \in B_{r_+}(\eta_{r_+}) \), \( \supp (1 - \psi) \subset \Omega \setminus B_R(\eta_{r_+}) \). By (2), (3), (4) and Lemma 1

\[
\frac{g(x)}{g(y)} \approx 1, \quad \frac{v(x)}{v(y)} \approx 1, \quad x, y \in B_{r_+}(\eta_{r_+}).
\]

Hence,

\[
\| (\psi(f - P f)) \|_{L_{q_0}(\Omega_{[t_-, t_+]})} \leq \frac{\| f - P f \|_{L_{q_0}(B_{r_+}(\eta_{r_+}))}}{3_1} \lesssim \sup_{x \in B_{r_+}(\eta_{r_+})} \| \nabla f \|_{L_{q_0}(B_{r_+}(\eta_{r_+}))} \lesssim A_{[t_-, t_+]} \| \nabla f \|_{L_{q_0}(\Omega_{[t_-, t_+]})}
\]

(the last inequality follows from the monotonicity of the function \( \varphi \); see the end of proof of Theorem 1). The value \( \| (1 - \psi)(f - P f) \|_{L_{q_0}(\Omega_{[t_-, t_+]})} \) is estimated by Theorem 1 (the arguments are the same as in (75)).

Let (7), (8), (9), (10), (11), (12) hold. We claim that

\[
A_{[t_-, t_+]} \lesssim A_{[0, t_+]} \lesssim \frac{\log t_+}{\gamma_0} \rho(\log t_+).
\]
Indeed, the integrals in the definition of $A_{i,[0,r_+)}$ ($i = 0, 1$) can be easily estimated (to this end we employ Lemma 1 and replace $z - t$ by $z$ and $t - z$ by $t$). Then we apply the fact that the function $t^{-a} \rho(t)$ is decreasing for large $t$.

This estimate will be employed for $\tau_+ < \frac{t}{2}$. If $\tau_+ \geq \frac{t}{2}$, then $\varphi(t) \approx \varphi(\tau_+)$, $g_0(t) \approx g_0(\tau_+)$, $v_0(t) \approx v_0(\tau_+)$ for any $t \in [\tau_-, \tau_+]$. Therefore,

$$
A_{[\tau_-, \tau_+]} \lesssim g_0(\tau_+)v_0(\tau_+)[\varphi(\tau_+)](d-1)(\frac{1}{2} - \frac{1}{p})(\tau_+ - \tau_+)^{\frac{1}{q} - \frac{1}{p}} = \left| \log \tau_+ \right|^a \rho(\left| \log \tau_+ \right|) \tau_+^{-\frac{1}{q} + \frac{1}{p} - 1} (\tau_+ - \tau_+)^{\frac{1}{q} - \frac{1}{p}}.
$$

(42)

## 4 Estimates of widths

In this section, we suppose that $p \leq q$.

Let $G \subset \Omega$ be a measurable set, and let $T = \{G_i\}_{i=1}^{i_0}$ be a finite partition of $G$. Denote

$$S_{r,T}(G) = \{S : \Omega \to \mathbb{R} : S|G_i \in \mathcal{P}_{r-1}(G_i), 1 \leq i \leq i_0, S|_{\Omega \setminus G} = 0\};$$

(43)

for each $f \in L_{q,v}(G)$ we set

$$\|f\|_{p,q,T,v} = \left( \sum_{i=1}^{i_0} \|f\|_{L_{q,v}(G_i)}^p \right)^{\frac{1}{p}}.$$

By $L_{p,q,T,v}(G)$ we denote the space $L_{q,v}(G)$ equipped with the norm $\|\cdot\|_{p,q,T,v}$. Notice that $\|f\|_{p,q,T,v} \geq \|f\|_{L_{q,v}(G)}$.

The following assertion (in fact, a more general result for weighted spaces) was proved in [75]. For the non-weighted case, Besov [11] later put forward a more simple proof.

**Lemma 6.** Let $a > 0$, $G \subset \mathbb{R}^d$, $G \in \mathbf{FC}(a)$, $n \in \mathbb{N}$. Then there exists a family of partitions $\{T_{m,n}(G)\}_{m \in \mathbb{Z}_+}$ with the following properties:

1. $\text{card } T_{m,n}(G) \lesssim 2^m n$;

2. for any $E \in T_{m,n}(G)$ there exists a linear continuous operator $P_E : L_q(E) \to \mathcal{P}_{r-1}(E)$ such that for any function $f \in W^r_p(E)$

$$
\| f - P_E f \|_{L_q(E)} \lesssim \sum_{r,q,d,a} (2^m n)^{-\frac{1}{2} - \frac{1}{q} + \frac{1}{p} (\text{mes } G)} \| \nabla^r f \|_{L_p(E)};
$$

(45)

3. for any $m \in \mathbb{Z}_+$, $E \in T_{m,n}(G)$

$$
\text{card } \{E' \in T_{m+1,n}(G) : \text{mes } (E \cap E') > 0\} \lesssim 1,
$$

$$
\text{card } \{E' \in T_{m-1,n}(G) : \text{mes } (E \cap E') > 0\} \lesssim 1, \text{ if } m \geq 1.
$$

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Let $X$, $Y$ be normed spaces, $B \subset X$, $A \in L(X, Y)$. Then
\[
d_n(A(B), Y) \leq \|A\|d_n(B, X).
\]
If $A$ is an isomorphism of $X$ and $Y$, then
\[
\lambda_n(A(B), Y) \leq \|A\|\lambda_n(B, X), \quad d^n(A(B), Y) \leq \|A\|d^n(B, X).
\]
The same inequalities hold if $Y$ is a subspace in $X$, $B \subset Y$ and $A$ is a linear projection onto $Y$. These assertions follow from definitions of Kolmogorov, Gelfand and linear widths.

Denote by $\chi_E$ the indicator function of a set $E$.

**Lemma 7.** Let $\Omega \subset \mathbb{R}^d$ be a domain, let $G_1, \ldots, G_m \subset \Omega$ be pairwise non-overlapping domains, and let $\psi_1, \ldots, \psi_m \in W_{p,g}(\Omega)$, $\left\| \sum_{j=1}^m \psi_j \right\|_{L_p(\Omega)} = 1$, supp $\psi_j \subset G_j$, $\|\psi_j\|_{L_{q,v}(G_j)} \geq M$, $1 \leq j \leq m$. Then
\[
\vartheta_n(W_{p,g}(\Omega), L_{q,v}(\Omega)) \geq M \cdot \vartheta_n(B_p^m, l_q^m)
\]
(see notations on page 7).

**Proof.** Let $X = \text{span} \{\psi_j\}_{j=1}^m \subset L_{q,v}(\Omega)$. From the definition of Gelfand widths it follows that
\[
d^n(W_{p,g}(\Omega), L_{q,v}(\Omega)) \geq d^n(W_{p,g}(\Omega) \cap X, X).
\]
Prove that
\[
d_n(W_{p,g}(\Omega), L_{q,v}(\Omega)) \geq d_n(W_{p,g}(\Omega) \cap X, X),
\]
\[
\lambda_n(W_{p,g}(\Omega), L_{q,v}(\Omega)) \geq \lambda_n(W_{p,g}(\Omega) \cap X, X).
\]
To this end we construct the projection $Q : L_{q,v}(\Omega) \rightarrow X$ such that $\|Q\| \leq 1$ and apply (46), (47). Let $X_j = \text{span} \{\psi_j\}$. Denote by $L_{q,v}^{(j)}(G)$ the set of functions in $L_{q,v}(G)$ whose support is contained in $G_j$. Since $\dim X_j = 1$, there is a projection $Q_j : L_{q,v}^{(j)}(G) \rightarrow X_j$, $\|Q_j\| \leq 1$. Let $Q(f) = \sum_{j=1}^m Q_j(f \chi_{G_j})$. Since the set $G_j$ do not overlap pairwise, we get $\|Q\| \leq 1$.

Define the isomorphism $T : X \rightarrow \mathbb{R}^m$ by
\[
T \left( \sum_{j=1}^m c_j \psi_j \right) = (c_1, \ldots, c_m).
\]
Since $\left\| \sum_{j=1}^m c_j \psi_j \right\|_{L_p(\Omega)} = 1$, $\sum_{j=1}^m c_j \psi_j \in W_{p,g}(\Omega)$ holds if and only if $(c_1, \ldots, c_n) \in B_p^m$. Therefore, $T(W_{p,g}(\Omega) \cap X) = B_p^m$. Prove that $\|T\|_{L_{q,v}(G) \rightarrow l_q^m} \leq M^{-1}$. Indeed, if
\[
f = \sum_{j=1}^m c_j \psi_j,
\]
then $\|Tf\|_{l_q^m}^{1/q} = \left( \sum_{j=1}^m |c_j|^q \right)^{1/q}$,
\[
\|f\|_{L_{q,v}(G)} = \left( \sum_{j=1}^m |c_j|^q \|\psi_j\|^q_{L_{q,v}(G_j)} \right)^{1/q} \geq M \|Tf\|_{l_q^m}.$
Proof. Let assertion 2.

By (46) and (47),

\[
\varphi_n(B_p^m, l_q^m) = \varphi_n(T(W_p^r(\Omega) \cap X), l_q^m) \leq \|T\|_{L_q^q(G)} \|\varphi_n(W_p^r(\Omega) \cap X, X) \leq M^{-1} \varphi_n(W_p^r(\Omega) \cap X, X).
\]

This implies the desired estimates. \(\square\)

**Proposition 1.** Let \(M > 0, m_0 \in \mathbb{Z}_+ \cup \{+\infty\}\) and let \(\{T_m\}_{m=0}^{m_0}\) be a family of finite coverings of a domain \(G\) with the following properties:

1. \(N_{\hat{T}_m} \leq M\) for any \(m \in \overline{0, m_0};\)

2. for any \(m \in \overline{0, m_0}, E \in T_m\) there exists a linear continuous operator \(P_{E,m} : L_{q,v}(E) \rightarrow \mathcal{P}_{r-1}(E)\) such that for any function \(f \in W_{p,q}^r(G)\) the inequality

\[
\|f - P_{E,m}f\|_{L_{q,v}(E)} \leq C_m \left\|\frac{\nabla f}{g}\right\|_{L_p(E)} \quad \text{holds};
\]

3. \(\text{card } \{E' \in T_{m+1} : \text{mes}(E \cap E') > 0\} \leq M\) for any \(E \in T_m.\)

Let \(U \subset G\) be a measurable subset. Then there exists a sequence of partitions \(\{\hat{T}_m\}_{m=0}^{m_0}\) of the set \(U\) such that

1. \(\text{card } \hat{T}_m \leq \text{card } T_m;\)

2. there exists a linear continuous operator \(P_m : L_{q,v}(G) \rightarrow \mathcal{S}_{r,\hat{T}_m}(U)\) such that for any function \(f \in W_{p,q}^r(G)\) the inequality

\[
\|f - P_m f\|_{L_{q,v}(G)} \leq C_m \left\|\frac{\nabla f}{g}\right\|_{L_p(G)} \quad \text{holds};
\]

3. \(\text{card } \{E' \in \hat{T}_{m+1} : \text{mes}(E \cap E') > 0\} \leq M\) for any \(E \in \hat{T}_m;\)

4. there are injective mappings \(F_m : \hat{T}_m \rightarrow T_m\) such that for any \(E \in \hat{T}_m\) the inclusion \(E \subset F_m(E)\) holds.

**Proof.** Let \(T_m = \{E_{i,m}\}_{i=1}^{k_m}.\) Denote

\[
\hat{E}_{i,m} = U \cap E_{i,m}, \quad \hat{E}_{i,m} = U \cap E_{i,m} \setminus \bigcup_{j=1}^{i-1} E_{j,m}, \quad i \geq 2,
\]

\[
\hat{T}_m = \{\hat{E}_{i,m}\}_{i \in I_m}, \quad I_m = \{i \in \overline{1, k_m} : \text{mes } \hat{E}_{i,m} > 0\},
\]

\(F_m(\hat{E}_{i,m}) = E_{i,m}, i \in I_m.\) Then \(\hat{T}_m\) satisfies assertions 1, 3 and 4. Define the operator \(P_m\) by \(P_m f|_{\hat{E}_{i,m}} = P_{E_{i,m},m} f|_{\hat{E}_{i,m}}, i \in I_m.\) Apply the condition \(N_{\hat{T}_m} \leq M\) and obtain assertion 2. \(\square\)

**Proof of Theorem 2.** Proof of the upper estimate. It suffices to consider the case \(n = 2^{N_d}, N \in \mathbb{N}.\) Denote

\[
D_1 = \Omega_{[2^{-2}, 2^{-1}]}, \quad D_j = \Omega_{[2^{j-1}, 2^{-j}]} \setminus \bigcup_{i=1}^{j-1} \Omega_{[2^{j-1}, 2^{-i}]} \setminus \Omega_{[2^{j-1}, 2^{-i}]}, \quad j \geq 2.
\]
Step 1. Let \( j \in \mathbb{N}, j \geq 2, l \in \mathbb{Z}_+ \). Construct the covering \( \mathcal{R}_{j,l} \) of \( \Omega_{[2^{-j},2^{-j+1}]} \). Without loss of generality we may assume that
\[
\varphi(z) \leq z \quad \text{for any } z \in (0, 1).
\] (48)
Choose \( l_j \in \mathbb{Z}_+ \) so that
\[
2^{-j-l_j-1} < \varphi(2^{-j}) \leq 2^{-j-l_j}.
\] (49)
For \( 0 \leq l \leq l_j, 1 \leq i \leq 2^l \) we set \( \tau_{j,l}(i) = 2^{-j} + i \cdot 2^{-j-l} \), \( U_{i,j,l} = \Omega_{[\tau_{j,l}(i-1), \tau_{j,l}(i)]} \). Then card \( \mathcal{R}_{j,l} = 2^l \); by Lemma 5
\[
\mathcal{N}_{\mathcal{R}_{j,l}} \leq 1;
\] (50)
this together with the definition of \( \mathcal{R}_{j,l} \) implies that
\[
\text{card } \{ U' \in \mathcal{R}_{j,l+1} : \text{mes } (U' \cap U) > 0 \} \leq \frac{1}{3}.
\]
By Corollary 1 and (42), for any \( i \in \mathbb{N}, i \leq 2^l \) there exists a linear continuous operator \( P^i_{l,j} : L_{q,v}(U_{i,j,l}) \to P_{r-1}(U_{i,j,l}) \) such that for any function \( f \in W^r_{p,q}(\Omega) \)
\[
\| f - P^i_{l,j} f \|_{L_{q,v}(U_{i,j,l})} \leq j^{-a} \rho(j) \cdot 2^{-l(r+\frac{1}{q}-\frac{1}{p})} \left\| \frac{\nabla f}{y} \right\|_{L_p(U_{i,j,l})} \leq j^{-a} \rho(j) \cdot 2^{-l(r+\frac{1}{q}-\frac{1}{p})} \left\| \frac{\nabla f}{y} \right\|_{L_p(U_{i,j,l})}.\] (51)
By Proposition 1, for any \( i \in \mathbb{N}, i \leq 2^l \) there exists a partition \( \mathcal{R}_{j,l} \) of \( D_{j-1} \), an injective mapping \( \mathcal{F}_{j,l} : \mathcal{R}_{j,l} \to \mathcal{R}_{j,l} \) and a linear continuous operator \( \mathbb{P}_{j,l} : L_{q,v}(\Omega_{[2^{-j},2^{-j+1}])} \to S_{r,\mathcal{R}_{j,l}}(D_{j-1}) \) such that
\[
\text{card } \mathcal{R}_{j,l} = 2^l,
\] (52)
\[
\text{card } \{ U' \in \mathcal{R}_{j,l+1} : \text{mes } (U' \cap U) > 0 \} \leq \frac{1}{3}
\] (53)
and for any \( f \in W^r_{p,q}(\Omega_{[2^{-j},2^{-j+1}])} \)
\[
\| f - \mathbb{P}_{j,l} f \|_{p,q,\mathcal{R}_{j,l+1}} \leq j^{-a} \rho(j) \cdot 2^{-l(r+\frac{1}{q}-\frac{1}{p})} \left\| \frac{\nabla f}{y} \right\|_{L_p(\Omega_{[2^{-j},2^{-j+1}])}}.\] (54)
By Lemma 2 and (49), \( U_{i,j,l} \in \mathcal{FC}(b_\rho) \) with \( b_\rho = b_\rho(3) > 0 \) and
\[
\text{mes } U_{i,j,l} \leq \frac{\varphi^d(2^{-j})}{3}.
\] (55)
Hence, by Lemma 3, for any \( l \geq l_j \) there exists a partition \( T_{i,j,l} = T_{l_j-1,l}(U_{i,j,l}) \) such that
\[
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1. card $T_{i,j,i} \lesssim 2^{l_{i,j,i}}$

2. for any $E \in T_{i,j,i}$ there exists a linear continuous operator $P_E : L_q(E) \to \mathcal{P}_r(E)$ such that for any function $f \in W^r_p(\Omega)$

$$
\| f - P_E f \|_{L_q(E)} \lesssim 2^{-(l_{i,j,i} - 1)} \left(\frac{2l_{i,j,i} + 1}{2} \right)^2 \left(\frac{1}{2} \right)^{l_{i,j,i}} \| \nabla^r f \|_{L_p(E)};
$$

(56)

3. for any $l \geq l_j$, $G \in T_{i,j,i}$

$$
\text{card} \{ G' \in T_{i+1,j,i} : \text{mes} (G \cap G') > 0 \} \lesssim 1,
$$

$$
\text{card} \{ G' \in T_{i-1,j,i} : \text{mes} (G \cap G') > 0 \} \lesssim 1, \quad \text{if } l \geq l_j + 1.
$$

From (7), (8), (9), (10), (11), (12), (49), (55), (56) and the inequality $2^{l_{i,j,i}} \leq 2^{(r+\frac{1}{q} - \frac{1}{p})l_j}$ it follows that for any $E \in T_{i,j,i}$

$$
\| f - P_E f \|_{L_{q,v}(E)} \lesssim j^{-\alpha} \rho(j) \cdot 2^{-l_{i,j,i}} \left(\frac{2l_{i,j,i} + 1}{2} \right)^2 \left(\frac{1}{2} \right)^{l_{i,j,i}} \| \nabla^r f \|_{L_p(E)}.
$$

(57)

Let $D \in R_{j,l_j}$. Denote by $i(D)$ the number $i$ such that $U_{i,j,l_j} = \mathcal{F}_{j,l}(D)$. For $l > l_j$ we set

$$
R_{j,l} = \{ D \cap E : D \in R_{j,l_j}, \ E \in T_{i,j,i}(D) \}.
$$

(58)

Then $R_{j,l}$ is a partition of $D_{j-1}$. From (52) and property 1 of the partition $T_{i,j,i}$ it follows that

$$
\text{card} R_{j,l} \lesssim 2^l.
$$

(59)

Prove that

$$
\text{card} \{ U' \in R_{j,l+1} : \text{mes} (U' \cap U) > 0 \} \lesssim 3.
$$

(60)

Indeed, let $D, D' \in R_{j,l_j}, E \in T_{i,j,i}(D), \ E' \in T_{i+1,j,i}(D')$, $\text{mes} (D \cap E \cap D' \cap E') > 0$. Since $R_{j,l_j}$ is a partition, we get $D' = D$. It remains to apply property 3 of the partition $T_{i,j,i}$.

Let $f \in L_{q,v}(\Omega_{[2^{-j},2^{j-1}])}$. Set $\mathbb{P}_{j,l}f|_{D \cap E} = P_E f|_{D \cap E}$ with $D, E$ from (58) and $P_E$ from property 2. Denote $C_{j,l} = j^{-\alpha} \rho(j) \cdot 2^{-l_{i,j,l_j} + \frac{1}{2}}$. Then $\mathbb{P}_{j,l} : L_{q,v}(\Omega_{[2^{-j},2^{j-1}])} \to S_{p,q,r}(D_{j-1})$ is a linear continuous operator and for any function $f \in W_{p,q,r}^r(\Omega_{[2^{-j},2^{j-1}])}$

$$
\| f - \mathbb{P}_{j,l} f \|_{L_{q,v}(D \cap E)} \lesssim 3.
$$

(57)
From (52), (59) and (64) it follows that 

$$\hat{\varepsilon}$$

Then we used that $F$ covering $Q$.

Then $t \approx C f$ (here we used that $P$ is an injection). Thus,

$$\|f - P_j f\|_{p,q,R_j,t,v} \lesssim j^{-\alpha} p(j) \cdot 2^{-l((\frac{1}{2} + \frac{1}{q} - \frac{1}{p}) \frac{1}{2} + \frac{1}{q} - \frac{1}{p})} \left\|\nabla^r f \right\|_{L^p(U, \Omega)}.$$  \hspace{1cm} (61)

**Step 2.** For $0 \leq t \leq Nd$ we set $G_t := \Omega_{[2^{l+1}, 2^l]}$. 

$$\hat{G}_0 = G_0, \quad \hat{G}_t = G_t \setminus \bigcup_{s=0}^{j-1} G_s.$$ Then $\cup_{j=0}^\infty G_t = \cup_{t=0}^\infty \hat{G}_t = \Omega$. The family of domains $\{\Omega_{[2^{l+1}, 2^l]}\}_{l=0}^{t+1} \in 2^{t+1}$ forms a covering $Q_i$ of the set $G_t$. By Lemma $5$ 

$$N_{Q_t} \lesssim 3.$$  \hspace{1cm} (62)

The family $\{D_{l-1}\}_{l=0}^{t+1} \in 2^{t+1}$ forms a partition of $\hat{G}_t$.

Let $t_+ = t_+(N) = 0$ or $t_+ = t_+(N) = Nd$ for each $N$, and let $\varepsilon > 0$. The choice of $\varepsilon$ and $t_+$ (both dependent on $3$) will be made later. Denote 

$$m_+ = \max\{[t - Nd + \varepsilon|t - t_+]|, 0\},$$

$$l_{m,t} = [Nd - t - \varepsilon|t - t_+]| + m, \quad m \in \mathbb{Z}_+, \quad m \geq m_+.$$  \hspace{1cm} (63)

$$\hat{T}_{m,t,n} = \{U \in R_{j,l,m,t}, \quad 2^j + 1 \leq j \leq 2^l\}.$$  \hspace{1cm} (64)

Then $\hat{T}_{m,t,n}$ is a partition of the set $\hat{G}_t$. By (54) and (60), for any $U \in \hat{T}_{m,t,n}$ 

$$\text{card} \{U' \in \hat{T}_{m+1,t,n} : \text{mes} (U \cap U') \neq 0\} \lesssim 3.$$  \hspace{1cm} (65)

From (52), (59) and (64) it follows that 

$$\text{card} \hat{T}_{m,t,n} \lesssim 2^t \cdot 2^l \cdot n \cdot 2^m - |t - t_+|.$$  \hspace{1cm} (66)

For $f \in L_{q,x}(\Omega)$ we set 

$$P_{m,t,n} f \mid_{D_{j-1}} = P_{j,l,m,t} f \mid_{D_{j-1}}, \quad 2^j + 1 \leq j \leq 2^{l+1}, \quad P_{m,t,n} f \mid_{\Omega \setminus \hat{G}_t} = 0.$$
Then \( P^1_{m,t,n} : L_{q,v}(\Omega) \to S, T^1_{m,t,n}(G_t) \) is a linear continuous operator. For any function \( f \in W^r_{p,g}(\Omega) \) we have
\[
\| f - P^1_{m,t,n} f \|_{p,q,T^1_{m,t,n,v}} \leq \left( \sum_{j=2^{t+1}}^{2^{t+1}} \left\| f - \mathbb{P}_{j,t,n} f \|_{p,q,R_{j,t,n,v}}^p \right\| \right)^{\frac{1}{p}} \leq \delta \frac{1}{3}
\]
\[
\leq 2^{-2\alpha \rho(2^t)} 2^{-\frac{4}{p} m,t,} \left( \sum_{j=2^{t+1}}^{2^{t+1}} \left\| \nabla^r f \|_{p,q,} \right\| \right)^{\frac{1}{p}} \leq \delta \frac{1}{3}
\]
Thus, for any function \( f \in W^r_{p,g}(\Omega) \)
\[
\| f - P^1_{m,t,n} f \|_{p,q,T^1_{m,t,n,v}} \leq 2^{-2\alpha \rho(2^t)} 2^{-\frac{4}{p} m,t}.
\]

**Step 3.** Given \( t \in \mathbb{Z}_+ \), we set \( U_t = \Omega_{[0,2^{-t}]} \). By Corollary 1 and 41, there exists a linear continuous operator \( P^t : L_{q,v}(U_t) \to \mathcal{P}_{r-1}(U_t) \) such that for any function \( f \in W^r_{p,g}(\Omega) \)
\[
\| f - P^t f \|_{L_{q,v}(U_t)} \leq 2^{-2\alpha \rho(2^t)}.
\]

**Step 4.** Consider the cases \( p = q \) and \( p < q, \hat{q} \leq 2 \). Denote \( \hat{U}_t = U_t \cup_{s=0}^{t-1} G_s \), \( \mathcal{T} \in N \). We set \( P^1_{m,t,n} f|_{\hat{G}_t} = P^1_{m,t,n} f|_{\hat{G}_t} \) for \( 0 \leq t \leq Nd, P^1_{m,t,n} f|_{\hat{U}_{Nd+1}} = P^{Nd+1} f|_{\hat{U}_{Nd+1}} \). Denote \( T^1_n = \left( \bigcup_{t=0}^{Nd} \mathcal{T}^1_{m,t,n} \right) \cup \{ \hat{U}_{Nd+1} \} \). Then \( T^1_n \) is a partition of \( \Omega \) and
\[
\text{card} T^1_n \leq 3 \sum_{0 \leq t \leq Nd, m_t = 0} n \cdot 2^{-\varepsilon|t-t'|} + \sum_{0 \leq t \leq Nd, m_t > 0} 2^t \leq n.
\]
In order to estimate Kolmogorov and linear widths, it is sufficient to obtain an upper bound for \( \| f - P^1 f \|_{L_{q,v}(\Omega)} \). In estimating Gelfand widths, we obtain an upper bound of \( \| f \|_{L_{q,v}(\Omega)} \) for \( f \in W^r_{p,g}(\Omega) \) such that \( P^1 f = 0 \). We have
\[
\| f - P^1 f \|^q_{L_{q,v}(\Omega)} = \sum_{t=0}^{Nd} \| f - P_{m,t,n} f \|^q_{L_{q,v}(G_t)} + \| f - P^{Nd+1} f \|^q_{L_{q,v}(U_{Nd+1})} \leq \delta \frac{1}{3}
\]
\[
\leq \sum_{t=0}^{Nd} 2^{-2\alpha \rho(2^t)} 2^{-\frac{4}{p} (Nd-t)\varepsilon|t-t'|} + n^{-\alpha q \rho(\hat{q})} =: S^q
\]
(we used the fact that if \( m_t^* > 0 \), then \( Nd - t - \varepsilon|t-t_*| < 0 \). Set \( \varepsilon = \frac{d}{\delta} |\alpha - \frac{\delta}{d}| \). If \( \alpha > \frac{\delta}{d} \), then we put \( t_* = 0 \) and get \( S \leq n^{-\frac{d}{3}} \); if \( \alpha < \frac{\delta}{d} \), then we take \( t_* = Nd \) and obtain \( S \leq n^{-\alpha}\rho(n) \).

**Step 5.** Consider the case \( p < q, \hat{q} > 2 \). Denote \( \hat{t}_N = \left[ \frac{\delta}{2} Nd \right] \). Then for sufficiently large \( N \) we have \( Nd + 1 < \hat{t}_N \).

For each \( Nd + 1 \leq t < \hat{t}_N \), \( m \in \mathbb{Z}^+ \) we construct the partition \( \hat{T}_{m,t,n}^2 \) of \( \hat{G}_t \).

**Substep 5.1.** Let \( \gamma = 2^{1+\varepsilon}, m_t = \left[ (1 + \varepsilon)(t - Nd) \right], 0 \leq m \leq m_t \). Set
\[
j_{m,t}(s) = \left\lfloor \gamma^{t-Nd}2^{-m}s \right\rfloor, \quad \tau_{m,t}(s) = 2^{-j_{m,t}(s)}, \quad s \in \mathbb{Z}^+,
\]
\[
J_{m,t} = \{ s \in \mathbb{Z}^+ : j_{m,t}(s) \leq 2^{t+1}, j_{m,t}(s + 1) \geq 2^t \}.
\]
Then \( \text{card} \ J_{m,t} \lesssim 2^{m_n} \cdot 2^{-(t-Nd)} \) for sufficiently small \( \varepsilon \).

Denote by \( T_{m,t,n}^2 \) the covering of \( G_t \) by the sets \( \Omega_{\tau_{m,t}(s+1),\tau_{m,t}(s)} \), \( s \in J_{m,t}, \tau_{m,t}(s) \neq \tau_{m,t}(s + 1) \). Then \( \text{card} \ T_{m,t,n}^2 \lesssim 2^{m_n} \cdot 2^{-(t-Nd)} \). If \( \tau_{m,t}(s) \neq \tau_{m,t}(s + 1) \), then \( \tau_{m,t}(s) - \tau_{m,t}(s + 1) \approx \varphi(\tau_{m,t}(s)) \). Therefore, from the definition of \( \tau_{m,t}(s) \) and from Lemma 3 it follows that for any set \( U \in T_{m,t,n}^2 \)
\[
\text{card} \{ U' \in T_{m,t,n}^2 : \text{mes} \ (U \cap U') > 0 \} \lesssim 1, \quad l = 0, 1, -1.
\] (69)

By Corollary 1 and (41), for any set \( E \in T_{m,t,n}^2 \) there exists a linear continuous operator \( P_E : L_{q,v}(E) \to \mathcal{P}_{r-1}(E) \) such that for any function \( f \in W_{r,q}^{r}(\Omega) \)
\[
\| f - P_E f \|_{L_{q,v}(E)} \lesssim 2^{-t_0}\rho(2^t) \left\| \frac{\nabla^r f}{g} \right\|_{L_p(E)}.
\] (70)

By Proposition 1 there exist a partition \( \hat{T}_{m,t,n}^2 \) of \( \hat{G}_t \) and an injection \( \mathcal{F}_{m,t,n} : \hat{T}_{m,t,n}^2 \to T_{m,t,n}^2 \) such that
\[
\text{card} \ T_{m,t,n}^2 \lesssim 2^{m_n} \cdot 2^{-(t-Nd)}; \quad \text{in particular,} \quad \text{card} \ T_{m,t,n}^2 \lesssim 2^t; \quad (71)
\]
for any \( U \in \hat{T}_{m,t,n}^2 \)
\[
\text{card} \{ U' \in \hat{T}_{m,t,n}^2 : \text{mes} \ (U \cap U') > 0 \} \lesssim 1
\] (72)
and \( U \subset \mathcal{F}_{m,t,n}(U) \). In addition, there exists a linear continuous operator \( P_{m,t,n}^2 : L_{q,v}(\Omega) \to \mathcal{S}_{r,\hat{T}_{m,t,n}^2}(\hat{G}_t) \) such that for any function \( f \in W_{r,q}^{r}(\Omega) \)
\[
\left\| f - P_{m,t,n}^2 f \right\|_{p,g,\hat{T}_{m,t,n}^2} \lesssim 2^{-t_0}\rho(2^t). \quad (73)
\]
Notice that $\hat{T}_{m,t,n}^2$ can be defined as follows:

$$E \in \hat{T}_{m,t,n}^2 \iff \text{mes } E > 0 \quad \text{and} \quad \exists s \in J_{m,t} : \quad E = \hat{G}_t \cap \Omega_{(n,m,(s+1),\tau_{m,t}(s))} \setminus \bigcup_{s' \in J_{m,t}, s' < s} \Omega_{[\tau_{m,t}(s'),\tau_{m,t}(s)]};$$

(74)

in this case, $\mathcal{F}_{m,t,n}(E) = \Omega_{[\tau_{m,t}(s+1),\tau_{m,t}(s)]}$

(see the proof of Proposition 1).

**Substep 5.2.** Construct the partition $\hat{T}_{m,t,n}^2$ for $m > m_t$. Observe that for any $s \in J_{m,t}$ the inequality $|j_{m,t}(s+1) - j_{m,t}(s)| \leq 1$ holds. Therefore, each element of the covering $T_{m,t,n}^2$ coincides with $\Omega_{[2^{-j}, 2^{-j}]}$ for some $j \in \{2^t, \ldots, 2^{t+1} + 1\}$. Recall that $\hat{G}_t \subset G_t = \Omega_{[2^{-j+1}, 2^{-j}]}$. This together with (74) implies that

$$\hat{T}_{m,t,n}^2 = \{D_j : 2^t + 1 \leq j \leq 2^{t+1}\}$$

and $\mathcal{F}_{m,t,n}(D_j) = \Omega_{[2^{-j}, 2^{-j+1}]}$.

Put

$$\hat{T}_{m,t,n}^2 = \{E \in \mathbb{R}_{j,m-m_t} : 2^t + 1 \leq j \leq 2^{t+1}\}.$$

Then $\hat{T}_{m,t,n}^2$ is a partition of $\hat{G}_t$. From (52), (59) and (71) it follows that

$$\text{card } \hat{T}_{m,t,n}^2 \leq 2^{m-m_t} \cdot 2^t \times 2^{m_t} \cdot 2^{(t-Nd)}.$$

(75)

By (53) and (61), for any $E \in \hat{T}_{m,t,n}^2$

$$\text{card } \{E' \in \hat{T}_{m,t,n}^{2} \pm 1 : \text{mes } (E \cap E') > 0 \} \leq 1.$$

(76)

For any $f \in L_{q,u}(\Omega)$ set $P_{m,t,n}^2 f \big|_{D_j} = \mathbb{P}_{j,m-m_t} f \big|_{D_j}$, $2^t + 1 \leq j \leq 2^{t+1}$, $P_{m,t,n}^2 f \big|_{\Omega \setminus \hat{G}_t} = 0$. Then $P_{m,t,n}^2 : L_{q,u}(\Omega) \to \mathcal{S}_{r,\hat{T}_{m,t,n}^2}(\hat{G}_t)$ is a linear continuous operator and for any $f \in W_{p,g}^r(\Omega)$

$$\|f - P_{m,t,n}^2 f\|_{p,q,\hat{T}_{m,t,n}^2} = \left( \sum_{j=2^t+1}^{2^{t+1}} \|f - \mathbb{P}_{j,m-m_t} f\|_{p,q,\mathbb{R}_{j,m-m_t}, v}^p \right) \frac{1}{2^t} \lesssim \left( \sum_{j=2^t+1}^{2^{t+1}} j^{-p\alpha} p(j) : 2^{-2\alpha(m-m_t)} \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega_{[2^{-j}, 2^{-j+1}]}), v}^p \right) \frac{1}{2^t} \lesssim \left( \sum_{j=2^t+1}^{2^{t+1}} \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega_{[2^{-j}, 2^{-j+1}]}), v}^p \right) \frac{1}{2^t} \lesssim 2^{-\alpha(t-m-m_t)} \frac{1}{2^t} \rho(2^t).$$

Thus, for any function $f \in W_{p,g}^r(\Omega)$

$$\|f - P_{m,t,n}^2 f\|_{p,q,\hat{T}_{m,t,n}^2} \lesssim 2^{-\alpha(t-m-m_t)} \frac{1}{2^t} \rho(2^t).$$

(77)
Substep 5.3. Let $f \in W^r_{p,g}(\Omega)$. Then
\[ \|f - P_N f\|_{L^q(U_n)} \lesssim \frac{n^{-\alpha/2} \rho(n^{\hat{q}/2})}{3}. \] (78)

The further arguments are the same as in [76]; here we apply (65), (66), (67), (61), (62), (63), (64), (65) and (78).

Proof of the lower estimate. Observe that $\Omega$ contains a cube $Q$ with the side of length $\lambda \geq 1$, such that $g(x) \geq 1, v(x) \geq 1$ for any $x \in Q$. Therefore,
\[ \partial_n(W^r_{p,g}(\Omega), L^q(\Omega)) \gtrsim \partial_n(W^r_p([0, 1]^d), L^q([0, 1]^d)). \]
Applying Theorem 12 we get
\[ \partial_n(W^r_{p,g}(\Omega), L^q(\Omega)) \gtrsim \frac{n^{-\hat{q}}}{3} \text{ for } p = q \text{ or } p < q, \hat{q} \leq 2, \]
\[ \partial_n(W^r_{p,g}(\Omega), L^q(\Omega)) \gtrsim n^{-\min(\theta_1, \theta_2)} \text{ for } p < q, \hat{q} > 2. \]

Let $\psi \in C^\infty_0((0, 1), \psi \geq 0, \int_0^1 |\psi(r)|^p dx = 1$. Given $j \in \mathbb{N}$, we set $\psi_j(x', x_d) = c_j \psi(2^j x_d - 1)$; here $c_j > 0$ is such that $\left\| \frac{\nabla \psi_j}{\psi} \right\|_{L^p(\Omega)} = 1$. Then
\[ \supp \psi_j \subset \{(x', x_d) \in \Omega : 2^{-j} < x_d < 2^{-j+1}\}. \]
From (21) and order equalities $g(x) \gtrsim g(2^{-j}), v(x) \gtrsim v(2^{-j}), 2^{-j} < x_d < 2^{-j+1}$, it follows that $c_j \gtrsim g(2^{-j}) \varphi(2^{-j})^{-\frac{d-1}{q}} 2^\frac{j}{q} 2^{-jr}$ and
\[ \|\psi_j\|_{L^q(\Omega)} \gtrsim c_j \cdot v(2^{-j}) \varphi(2^{-j})^{-\frac{d-1}{q}} 2^\frac{j}{q} \gtrsim \frac{j^{-\alpha} \rho(j)}{3} \]
(the last relation follows from (7), (8), (9), (10), (11), (12)). If $2^j \leq j < 2^{j+1}$, then
\[ \|\psi_j\|_{L^q(\Omega)} \gtrsim 2^{-\alpha} \rho(2^j). \]
Applying Lemma 7 we get
\[ \partial_n(W^r_{p,g}(\Omega), L^q(\Omega)) \gtrsim 2^{-\alpha} \rho(2^j) \partial_n(B^2_{2^j}, \psi). \]
Let $t = Nd + 1$. Then from (13), (14) and (15) it follows that
\[
\partial_n(W^r_{p,g}(\Omega), L^q(\Omega)) \gtrsim n^{-\alpha} \rho(n) d_n(B^2_{2^j}, \psi)_{p,q} \lesssim \begin{cases} n^{-\alpha} \rho(n), & p = q \text{ or } p < q, \hat{q} \leq 2, \\ n^{-\alpha-m\left(\frac{1}{p} - \frac{1}{q} + \frac{1}{\hat{q}}\right)}, & \hat{q} > 2. \end{cases}
\]
Let $t = \left[\frac{\hat{q}}{2} Nd\right], \hat{q} > 2$. Then
\[ \partial_n(W^r_{p,g}(\Omega), L^q(\Omega)) \gtrsim n^{-\alpha/2} \rho(n^{\hat{q}/2}) \partial_n(B^{[\hat{q}/2]}_{\hat{q}}, \psi)_{p,q} \gtrsim n^{-\alpha/2} \rho(n^{\hat{q}/2}). \]
This completes the proof. \(\square\)
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