"Doubled" generalized Landau-Lifshitz hierarchies and special quasigraded Lie algebras.

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Abstract

Using special quasigraded Lie algebras we obtain new hierarchies of integrable nonlinear vector equations admitting zero-curvature representations. Among them the most interesting is extension of the generalized Landau-Lifshitz hierarchy which we call "doubled" generalized Landau-Lifshitz hierarchy. This hierarchy can be also interpreted as an anisotropic vector generalization of "modified" Sine-Gordon hierarchy or as a very special vector generalization of so(3) anisotropic chiral field hierarchy.

Short title: "Doubled" generalized Landau-Lifshitz hierarchies.

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1 Introduction

Integrability of equations of 1 + 1 field theory and condensed matter physics is based on the possibility to represent them in the form of the so-called zero-curvature equations \[3\], \[1\]:

\[
\frac{\partial U(x, t, l)}{\partial t} - \frac{\partial V(x, t, l)}{\partial x} + [U(x, t, l), V(x, t, l)] = 0.
\]

(1)

The most productive interpretation of zero-curvature equations (see \[2\] and \[7\]) is to consider them as a consistency condition for a set of commuting Hamilton flows on a dual space to some infinite-dimensional Lie algebra \(\tilde{g}\) of matrix-valued function of \(l\) written in the Euler-Arnold (generalized Lax) form:

\[
\frac{\partial L(l)}{\partial t_k} = ad_{\nabla I_k(L(l))}^* L(l), \quad \frac{\partial L(l)}{\partial t_l} = ad_{\nabla I_l(L(l))}^* L(l),
\]

(2)

where \(L(l) \in \tilde{g}^*\) is the generic element of the dual space, \(\nabla I_k(L(l)) \in \tilde{g}\) is the algebra-valued gradient of \(I_k(L(l))\), and the ”Hamiltonians” \(I_k(L(l))\), \(I_l(L(l))\) belong to the set of mutually commuting with respect to the natural Lie-Poisson bracket functions on \(\tilde{g}^*\). The consistency condition of two commuting flows given by equations (2) yields equation (1) with \(U \equiv \nabla I_k, V \equiv \nabla I_l, t_k \equiv x, t_l \equiv t\). In such a way we obtain a lot of equations in partial derivatives that are indexed by two commuting Hamiltonians \(I_k\) and \(I_l\). The set of equations (1) with fixed index \(k\) and all indices \(l\) constitute so-called ”integrable hierarchy”. Hence, in order to construct new integrable hierarchies in the framework of the described approach it is necessary to have some infinite-dimensional Lie algebra \(\tilde{g}\) possessing infinite set of mutually commuting Hamiltonians \(\{I_k\}\) on its dual space. The main method, that provides such the set is a famous Kostant-Adler scheme and its extensions \([6],[2]\). Main ingredient of this scheme is an existence of the decomposition of the algebra \(\tilde{g}\) into sum of two subalgebras:

\[
\tilde{g} = \tilde{g}_+ + \tilde{g}_-.
\]

Although the described above approach was originally based on the graded loop algebras \(L(\mathfrak{g}) = \mathfrak{g} \otimes P(l, l^{-1})\) \([2],[7]\) that possess decompositions into sums of two subalgebras, in the papers \([8]-[9]\) it was shown that a special Lie algebra \(\mathfrak{g}_E\) on the elliptic curve \(E\) also possess the decomposition \(\mathfrak{g}_E = \mathfrak{g}^+_E + \mathfrak{g}^-_E\) and could be used in order to produce integrable systems. In our papers \([13],[15]\) we have generalized this construction onto the case of special quasigraded Lie algebras \(\mathfrak{g}_H\) on the algebraic curve \(H\). With their help we have obtained new integrable hamiltonian systems (both finite and infinite dimensional) \([14],[15]\). In papers \([16],[18]\) we gave Lie algebraic explanation of our previous semi-geometric construction of the Lie algebras \(\mathfrak{g}_H\). More explicitly, we have constructed a family of quasigraded Lie algebras \(\mathfrak{g}_A\) parametrized by some numerical matrices \(A\), such that loop algebras \(L(\mathfrak{g})\) correspond to the case \(A \equiv 0\) and quasigraded Lie algebras \(\mathfrak{g}_H\) correspond to the case \(A \in \text{Diag}(n)\).

In the present paper we generalize construction of \([16],[18]\) introducing even larger family of quasigraded Lie algebras \(\tilde{g}_{A_1,A_2}\) numbered by two numerical matrices \(A_1\) and \(A_2\) to which Kostant-Adler scheme may be applied. A family of Lie algebras \(\tilde{g}_A\) (see \([16],[18]\)) is embedded into the family of Lie algebras \(\tilde{g}_{A_1,A_2}\) as the algebras \(\tilde{g}_{1,A}\). We show that three types
of integrable hierarchies is associated with Lie algebras $\tilde{g}_{A_1,A_2}$: two small hierarchies are associated with algebras $\tilde{g}_{A_1,A_2}^\pm$ and large hierarchy is associated with the Lie algebra $\tilde{g}_{A_1,A_2}$. We show, that in the case when both matrices $A_i$ are degenerated, the algebras $\tilde{g}_{A_1,A_2}$ and $\tilde{g}_A$ are not isomorphic as quasigraded Lie algebras. This means that integrable hierarchies associated with $\tilde{g}_{A_1,A_2}$ such that det$A_i = 0$ are not equivalent to the integrable hierarchies associated with $\tilde{g}_A$ (see [15], [17], [18]). Moreover, we show, that when the matrices $A_i$ have the same matrix rank subalgebras $\tilde{g}_{A_1,A_2}$ and $\tilde{g}_{A_1,A_2}$ are isomorphic, and corresponding integrable hierarchies are also equivalent. That is why "large" integrable hierarchy associated with the whole Lie algebra $\tilde{g}_{A_1,A_2}$ could be viewed as the "double" of integrable hierarchy associated with $\tilde{g}_{A_1,A_2}^\pm$. The "doubling" consists in adding of "negative" flows and new dynamical variables to the integrable hierarchy associated with $\tilde{g}_{A_1,A_2}^\pm$.

We consider these hierarchies in the case $g = so(n)$ and rank$A_i = n - 1$ in detail. We show, that integrable hierarchy associated with $so(n)_{A_1,A_2}$ coincides with $(n - 1)$-component vector generalization of the ordinary 3-component Landau-Lifshiz hierarchy. For $n > 4$ this hierarchy was first obtained in [12] using technique of "dressing" and Lie algebra isomorphic to $\tilde{g}_{A_1,A_2}$ embedded into Lie algebra of formal power series. Simplest equation of this hierarchy has the form:

$$\frac{\partial \vec{s}}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\partial^2 \vec{s}}{\partial x^2} + 3/2 \left( \frac{\partial \vec{s}}{\partial x} \cdot \frac{\partial \vec{s}}{\partial x} \right) \vec{s} \right) + 3/2 (\vec{s}, J \vec{s}) \frac{\partial \vec{s}}{\partial x}, \quad (3)$$

where $\vec{s}$ is $n - 1$-component vector and tensor of anisotropy $J$ is expressed via $A_1$ and $A_2$.

"Double" of the generalized Landau-Lifshiz hierarchy is the "large" integrable hierarchy associated with $so(n)_{A_1,A_2}$. It is $2(n - 1)$-component hierarchy of vector equations satisfying two additional scalar constraints. The simplest equation of this hierarchy coincide with the two $(n - 1)$-component vector differential equations of the first order. We show that for these two equations two scalar constraints are easily solved and we obtain in the result two non-linear $(n - 2)$-component vector equations of the following form:

$$\partial_{x_+} \vec{s}_- = \left( c_- - (\vec{s}_-, \vec{s}_+) \right)^{1/2} J^{1/2} \vec{s}_+, \quad (4)$$

$$\partial_{x_-} \vec{s}_+ = \left( c_+ - (\vec{s}_+, \vec{s}_-) \right)^{1/2} J^{-1/2} \vec{s}_-. \quad (5)$$

where $\vec{s}_{\pm}$ are $(n - 2)$-component vectors, $c_{\pm}$ are arbitrary constants and anisotropy matrix $J$ is connected with matrices $A_i$ in a simple way (see section (3.4)).

Equations (4,5) is in a sense a "first negative equation" or a "first negative flow" of the generalized L-L hierarchy, $\vec{s}_+$ is an $n - 2$ independent components of its $n - 1$-component vector of dynamical variables: $\vec{s} = (s_1, \vec{s}_+)$, $x_+ \equiv x$ is a space coordinate and $x_-$ is a first "negative" time.

It is necessary to notice that in the $n = 3$ case equations (4,5) are equivalent to the "modified Sine-Gordon" equation [19], [20] and in the $n = 4$ case to the $so(3)$ anisotropic chiral field equations [21].
The structure of the present article is the following: in the second section we introduce algebras $\tilde{g}_{A_1,A_2}$ and describe their properties. In the third section we obtain integrable hierarchies associated with the Lie algebras $\tilde{g}_{A_1,A_2}$ and its subalgebras $\tilde{g}_{A_1,A_2}^\pm$. In the last its subsection we consider the examples of this construction: generalized Landau-Lifshiz hierarchy and its ”double”.

2 K-A admissible quasigraded Lie algebras.

2.1 Lie algebras $\tilde{g}_{A_1,A_2}$.

Let $g$ be a classical matrix Lie algebra of the type $gl(n)$, $so(n)$ and $sp(n)$ over the field of the complex or real numbers. We will realize algebra $so(n)$ as algebra of skew-symmetric matrices: $so(n) = \{X \in gl(n)|X = -X^\top\}$ and algebra $sp(n)$ as the following matrix algebra: $sp(n) = \{X \in gl(n)|X = sX^\top s\}$, where $n$ is an even number, $s \in so(n)$ and $s^2 = -1$.

Let us introduce the new Lie bracket into a loop space $L(g) = g \otimes Pol(l,l^{-1})$:

$$[X(l),Y(l)] = [X(l),Y(l)]_{A_1} - l[X(l),Y(l)]_{A_2},$$

where $X(l), Y(l) \in g \otimes Pol(l,l^{-1})$, $A_i$ are the numerical $n \times n$ matrices, $[X,Y]_{A_i} = XA_iY - YA_iX$.

Brackets $[X,Y]_{A_i} = XA_iY - YA_iX$ have arisen in the theory of consistent Poisson brackets on the finite-dimensional Lie algebras $g$ [10], [11]. In the present paper we use them in order to construct new Lie bracket on the infinite-dimensional space $g \otimes Pol(l,l^{-1})$ (see also [16]).

The following proposition holds true:

**Proposition 2.1** Let the numerical $n \times n$ matrices $A_i$, $i = 1, 2$ have the following form:

1) $A_i$ is arbitrary for $g = gl(n)$,

2) $A_i = A_i^\top$ for $g = so(n)$,

3) $A_i = -sA_i^\top s$ for $g = sp(n)$.

Then bracket (6) is a correctly defined Lie bracket on $g \otimes Pol(l,l^{-1})$.

**Definition.** We will denote infinite-dimensional space $g \otimes Pol(l,l^{-1})$ with the Lie bracket given by (6) by $\tilde{g}_{A_1,A_2}$.

**Remark 1.** Algebra $\tilde{g}_{A_1,A_2}$ could be realized also in the space of special matrix valued functions of $l$ with an ordinary Lie bracket $[\ , \ ]$. Nevertheless we consider realization in the space $g \otimes Pol(l,l^{-1})$ with the bracket (6) to be the most convenient.

Now we can introduce the convenient bases in the algebras $\tilde{g}_{A_1,A_2}$. Due to the fact, that we are dealing with matrix Lie algebras $g$, we will denote their basic elements as $X_{ij}$. Let $X_{ij}^m \equiv X_{ij} \otimes l^m$ be the natural basis in $\tilde{g}_{A_1,A_2}$. Commutation relations (6) in this basis have the following form:

$$[X_{ij}^r, X_{kl}^m] = \sum_{p,q} C_{ij,kl}^{pq}(A_1)X_{pq}^{r+m} - \sum_{p,q} C_{ij,kl}^{pq}(A_2)X_{pq}^{r+m+1}, \quad (7)$$

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where $C^{pq}_{ijkl}(A_i)$ are the structure constants of the Lie algebras $\mathfrak{g}_{A_i}$.

Remark 2. Note that contrary to the case of loop algebras our algebras $\tilde{\mathfrak{g}}_{A_1,A_2}$ admit only one type of decomposition $\tilde{\mathfrak{g}}_{A_1,A_2} = \tilde{\mathfrak{g}}^+_{A_1,A_2} + \tilde{\mathfrak{g}}^-_{A_1,A_2}$ compatible with quasigrading, where subalgebras $\tilde{\mathfrak{g}}^\pm_{A_1,A_2}$ are defined in the natural way:

$$\tilde{\mathfrak{g}}^+_{A_1,A_2} = \text{Span}_K \{ X^m_{ij} | m \geq 0 \}, \quad \tilde{\mathfrak{g}}^-_{A_1,A_2} = \text{Span}_K \{ X^m_{ij} | m < 0 \}. \quad (8)$$

Let us now find equivalences among the constructed Lie algebras. In particular, let us find conditions when $\tilde{\mathfrak{g}}_{A_1,A_2}$ is equivalent to the algebra $\tilde{\mathfrak{g}}_A \equiv \tilde{\mathfrak{g}}_{1,A}$ introduced in our previous papers [16]-[18]. All equivalences are understood in the sense of the isomorphisms of quasigraded Lie algebras. The following Proposition is true:

**Proposition 2.2** (i) The following isomorphisms hold: $\tilde{\mathfrak{g}}^+_{A_1,A_2} \simeq \tilde{\mathfrak{g}}^+_{A_2,A_1}$, $\tilde{\mathfrak{g}}^-_{A_1,A_2} \simeq \tilde{\mathfrak{g}}^-_{A_2,A_1}$.

(ii) If there exist matrix $C$ such that $CA_1C = A_2$ and $C^2 = 1$ then $\tilde{\mathfrak{g}}^+_{A_1,A_2} \simeq \tilde{\mathfrak{g}}^+_{A_2,A_1}$.

(iii) If $\det A_1 \neq 0$ or $\det A_2 \neq 0$ then $\tilde{\mathfrak{g}}_{A_1,A_2} \simeq \tilde{\mathfrak{g}}_A$.

**Remark 3.** Item (iii) of the Proposition means that for the algebra $\tilde{\mathfrak{g}}_{A_1,A_2}$ in order not to be equivalent to the algebra $\tilde{\mathfrak{g}}_A$ of [16], [18] matrices $A_1$ and $A_2$ should be degenerated. That is why we will consider the case $\det A_i = 0$ as the main case in the present paper.

### 2.2 Coadjoint representation and its invariants.

In this subsection we define dual spaces, coadjoint representations and their invariants for the Lie algebras $\tilde{\mathfrak{g}}_{A_1,A_2}$. At first we explicitly describe the dual space $\tilde{\mathfrak{g}}^*_{A_1,A_2}$ of $\tilde{\mathfrak{g}}_{A_1,A_2}$. For this purpose we define the pairing between $\tilde{\mathfrak{g}}_{A_1,A_2}$ and $\tilde{\mathfrak{g}}^*_{A_1,A_2}$ in the following way:

$$(X, L) = \text{rest}_{t=0} Tr(X(l)L(l)). \quad (9)$$

The generic element of the dual space $\tilde{\mathfrak{g}}^*_{A_1,A_2}$ with respect to this pairing is written as follows:

$$L(l) = \sum_{k \in \mathbb{Z}} \sum_{i,j=1,n} l^{(k)}_{ij} l^{-(k+1)} X^*_ij. \quad (10)$$

From the explicit form of the adjoint representation (6) and the pairing (9) it is easy to show that the coadjoint action of $\tilde{\mathfrak{g}}_{A_1,A_2}$ on $\tilde{\mathfrak{g}}^*_{A_1,A_2}$ has the form:

$$\text{ad}^*_{X(l)} \circ L(l) = \mathcal{A}(l) X(l)L(l) - L(l)X(l)\mathcal{A}(l), \quad (11)$$

where $X(l), Y(l) \in \tilde{\mathfrak{g}}_{A_1,A_2}$, $L(l) \in \tilde{\mathfrak{g}}^*_{A_1,A_2}$, $\mathcal{A}(l) = A_1 - lA_2$.

Having the explicit form of the coadjoint action it is easy to deduce the next Proposition:

**Proposition 2.3** Let $L(l)$ be the generic element of $\tilde{\mathfrak{g}}^*_{A_1,A_2}$. Then functions

$$I_k^m(L(l)) = 1/m \text{rest}_{t=0} l^{-(k+1)} Tr(L(l)\mathcal{A}(l)^{-1})^m. \quad (12)$$

are invariants of the coadjoint representation of the algebra $\tilde{\mathfrak{g}}_{A_1,A_2}$.
Remark 4. From the definition of the invariant functions it follows that in order to make algebra $\tilde{\mathfrak{g}}_{A_1,A_2}$ satisfy requirement (IR3) matrix $A(l)$ should be nondegenerated. This condition impose additional requirements on the matrices $A_i$.

2.3 Lie-Poisson structure.

Let us introduce Poisson structure in the space $\tilde{\mathfrak{g}}_{A_1,A_2}$ using the defined above pairing $\langle \ , \ \rangle$. It defines Lie-Poisson (Kirillov-Kostant) bracket on $P(\tilde{\mathfrak{g}}_{A_1,A_2}^*)$ in the following standard way:

$$\{F(L(l)), G(L(l))\} = \langle L(l), [\nabla F(L(l)), \nabla G(L(l))]_{A(l)} \rangle,$$

where $\nabla F(L(l)) = \sum_{k \in Z} \sum_{i,j=1}^n \frac{\partial F}{\partial l^{(k)}}(k)X^k_{ij}$, $\nabla G(L(l)) = \sum_{m \in Z} \sum_{k,l=1}^n \frac{\partial G}{\partial l^{(m)}}(l)X^m_{kl}$.

From the Proposition 2.4 and standard considerations the next statement follows:

**Proposition 2.4** Functions $I^n_k(L(l))$ are central for the Lie-Poisson bracket (13).

Let us explicitly calculate Poisson bracket (13). It is easy to show, that for the coordinate functions $I_{ij}^{(m)}$ these brackets will have the following form:

$$\{I_{ij}^{(m)}, I_{kl}^{(m)}\} = \sum_{p,q} C_{ijk,l}^{pq}(A_1)I^{(n+m)}_{pq} - \sum_{p,q} C_{ijkl}^{pq}(A_2)I^{(n+m+1)}_{pq}. \tag{14}$$

Lie bracket (14) determine the structure of the Lie algebra isomorphic to $\tilde{\mathfrak{g}}_{A_1,A_2}$ in the space of linear functions $\{l_{ij}^n\}$. That is why the corresponding Poisson algebra possess decomposition into direct sum of two Poisson subalgebras or, by other words subspaces $(\tilde{\mathfrak{g}}_{A_1,A_2}^*)^\pm$ are Poisson.

3 Integrable hierarchies associated with algebras $\tilde{\mathfrak{g}}_{A_1,A_2}$

In this section we construct two infinite sets of mutually Poisson-commuting functions on the Lie algebra $\tilde{\mathfrak{g}}_{A_1,A_2}$ and Lax type representation for the corresponding hamiltonian equations. We also derive zero-curvature equations as a compatibility condition of the above commuting hamiltonian flows and consider examples of the equations in partial derivatives from the corresponding integrable hierarchies.

3.1 Infinite-component hamiltonian systems on $\tilde{\mathfrak{g}}_{A_1,A_2}^*$

In this subsection we construct hamiltonian systems on the infinite-dimensional space $\tilde{\mathfrak{g}}_{A_1,A_2}^*$ possessing infinite number of independent, mutually commuting integrals of motion.

Let $L^\pm(l)$ be the generic element of the space $\tilde{\mathfrak{g}}_{A_1,A_2}^*$:

$$L^\pm(l) \equiv \sum_{i,j=1,n} L_{ij}^\pm(l)X_{ji} = \sum_{k \in Z} \sum_{i,j=1,n} l_{ij}^{(k)}(l)X_{ji}.$$
Let us consider the restriction of the invariant functions \( \{ I^m_k(L(l)) \} \) onto these subspaces. Note, that although Poisson subspaces \( \mathfrak{g}^*_{A_1,A_2} \) are infinite-dimensional, functions \( \{ I^m_k(L^\pm(l)) \} \) are polynomials, i.e. after the restriction onto \( \mathfrak{g}^*_{A_1,A_2} \) no infinite sums appear in their explicit expressions. Let us now consider functions \( I^m_k(L^\pm(l)) \) as functions on the whole space \( \mathfrak{g}^*_{A_1,A_2} \). We have two sets of hamiltonians \( \{ I^m_k(L(l)) \} \) and \( \{ I^m_k(L(l)) \} \) on \( \mathfrak{g}^*_{A_1,A_2} \) defined as follows:

\[
I^m_{k \pm}(L(l)) \equiv I^m_k(L^\pm(l)).
\]

Hamiltonian flows corresponding to hamiltonians \( I^m_{k \pm}(L(l)) \) are written in a standard way:

\[
\frac{\partial L_{ij}(l)}{\partial t_{m \pm}^k} = \{ L_{ij}(l), I^m_{k \pm}(L(l)) \}. \quad (15)
\]

The following theorem is true:

**Theorem 3.1** (i) Hamiltonian equations \((15)\) are written in the generalized Lax form:

\[
\frac{\partial L(l)}{\partial t_{m \pm}^k} = ad^*_{V^m_k(l)} L(l) = A(l)V^m_k(L(l))L(l) - L(l)V^m_k(L(l))A(l). \quad (16)
\]

where \( V^m_k(l) = \nabla I^m_k(L(l)) \equiv \sum_{s \in \mathbb{Z}_k} \sum_{i,j=1}^n \frac{\partial I^m_k}{\partial l_{ij}^s} X_{ij}^s \).

(ii) The functions \( \{ I^m_k(L(l)) \} \) form the commutative subalgebra in the algebra of polynomial functions on \( \mathfrak{g}^*_{A_1,A_2} \): \( \{ I^m_{k \pm}(L(l)), I^m_{l \pm}(L(l)) \} = \{ I^m_{k \pm}(L(l)), I^m_{l \pm}(L(l)) \} = 0 \), i.e. time flows defined by equations \((15)\) (or \((16)\)) mutually commute.

(iii) The functions \( I^m_{l \pm}(L(l)) \) are constant along all time flows: \( \frac{\partial I^m_{l \pm}}{\partial t_{m \pm}^k} = \frac{\partial I^m_{l \pm}}{\partial t_{m \pm}^l} = 0 \).

The proof of this theorem repeats the proof of the analogous theorem for the case of ordinary loop algebras (see [2] and references therein).

**Remark 5.** Due to the fact that the subspaces \( (\mathfrak{g}^*_{A_1,A_2})^* \) are Poisson equations \((15)\), generated by hamiltonians \( I^m_{k \pm}(L(l)) \), could be restricted onto them, i.e. it is correctly to consider the following hamiltonian equations:

\[
\frac{\partial L_{ij}^+(l)}{\partial t_{m \pm}^k} = \{ L_{ij}^+(l), I^m_k(L^+(l)) \}, \quad \frac{\partial L_{ij}^-(l)}{\partial t_{m \pm}^k} = \{ L_{ij}^-(l), I^m_k(L^-(l)) \}. \quad (17)
\]

In particular the following Corollary of the Theorem 3.1 holds true:

**Corollary 3.1** (i) Hamiltonian equations \((17)\) are written in the generalized Lax form:

\[
\frac{\partial L^\pm(l)}{\partial t_{m \pm}^k} = A(l)V^m_k(L^\pm(l))L^\pm(l) - L^\pm(l)V^m_k(L^\pm(l))A(l), \quad (18)
\]
where \( V^m_\pm (l) \) is defined as in theorem 3.1.

(ii) The functions \( \{ I^m_\pm (L^\pm (l)) \} \) form commutative subalgebra in the algebra of polynomial functions on \( \tilde{g}^*_A \cdot \cdot \cdot ^* \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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are described only by equations (19), while integrable hierarchies associated with \( \widetilde{g}_{A_1,A_2} \) are described by both equations (19) and (20), reflecting the fact, that we have in this case both positive and negative flows. In other words, integrable hierarchies associated with the algebras \( \widetilde{g}_{A_1,A_2} \) could be viewed as subhierarchies of the integrable hierarchy associated with algebra \( g_{A_1,A_2} \). Nevertheless, they are completely self-contained and could be considered separately. In particular, they do not depend on the ”large” algebra in which we embed corresponding subalgebra \( \widetilde{g}_{A_1,A_2} \) or \( g_{A_1,A_2} \).

For the case of integrable hierarchies, associated with algebras \( \widetilde{g}_{A_1,A_2} \), a choice of the one of the matrix gradients \( \nabla I^m_k \) to be \( U \)-operator yields fixation of dynamical variables that coincide with its matrix elements. For the case of integrable systems, connected with the algebras \( \widetilde{g}_{A_1,A_2} \) there are two types of hamiltonians and two types of flows. That is why in this case number of independent dynamical variables may be doubled: their role is played by the matrix elements of two \( U \) operators: \( U_+ = \nabla I^m_k (L^+(l)) \) and \( U_- = \nabla I^m_k (L^-(l)) \), where hamiltonians \( I^m_k (L^+(l)) \) and \( I^m_k (L^-(l)) \) generate evolution with respect to “times” \( x_+ \) and \( x_- \) — ”space” flows of the hierarchies associated with subalgebras \( \widetilde{g}_{A_1,A_2} \).

The number of the dynamical variables for the chosen integrable hierarchy coincide with the number of independent matrix elements of the \( U \)-operators, where \( U_+ = \nabla I^m_k (L^+(l)) \) and \( U_- = \nabla I^m_k (L^-(l)) \). In our case, when \( I^m_k (L^+(l)) \) depends on the additional parameters (matrix elements of the matrices \( A_i \)) we may decrease the number of the dynamical variables manipulating by these parameters (in particular tending some of them to zero). Hence, this provides us with simple procedure of reduction of the number of functional degrees of freedom. We will illustrate this in the next subsection on the \( g = so(n) \) example.

### 3.3 Integrable subhierarchy associated with subalgebra \( \widetilde{so(n)}_{A_1,A_2} \)

The aim of this subsection is a derivation of the equations of integrable hierarchy connected with the algebra \( \widetilde{g}_{A_1,A_2} \), where \( g = so(n) \), matrices \( A_i \) are degenerated: \( \det A_i = 0 \) and rank \( A_i = n - 1 \). We will start our consideration with nondegenerated case: rank \( A_i = n \) and obtain the case rank \( A_i = n - 1 \) as its continuous limit.

Let us now illustrate the procedure of obtaining of integrable equations in the partial derivatives starting from the Lie algebras \( \widetilde{g}_{A_1,A_2} \) where \( g \) and \( A_i \) are as the described above. For this purpose we have to describe the set of commuting integrals on \( (so(n)_{A_1,A_2})^* \). Let us at first note, that generic element of the dual space \( (so(n)_{A_1,A_2})^* \) has the following form:

\[
L^-(l) = l^{-1} L^{(0)} + l^{-2} L^{(1)} + l^{-3} L^{(2)} + l^{-4} L^{(3)} + \cdots,
\]

where \( L^{(k)} = \sum_{i<j=1,n} i^{(k)}_{ij}X_{ji} \). We will be interested in the second order integrals (hamiltonians). By the very definition they are written as follows:

\[
I^2_k (L(l)) = 1/2 \text{rest}_{l=0} l^{-(k+1)} Tr(L^-(l)A(l)^{-1})^2.
\]
In order for hamiltonians $I_k^\pm$ to be polynomials we will use the decomposition of the matrix $A(l)^{-1}$ in the formal power series in a neighbourhood of infinity:

$$I^2-(L(l)) = Tr((l^{-1}L(0) + l^{-2}L(1) + \cdots)(1 + A_1A_2^{-1}l^{-1} + \cdots)A_2^{-1}l^{-1})^2. \quad (23)$$

Commuting integrals of the series $I^2-(L(l))$ contain expression $A_2^{-1}$ and in the limit $\det A_2 = 0$ should be regularized in the appropriate way. We will calculate these hamiltonians for the case $\det A_2 \neq 0$ and then consider the continuous limit $\det A_2 \to 0$.

Simplest hamiltonians of the set (23) are the functions $I^2_{-4}(L(l))$ and $I^2_{-5}(L(l))$:

$$I^2_{-4}(L(l)) = 1/2 Tr(A_2^{-1}L(0))^2, \quad I^2_{-5}(L(l)) = Tr(A_2^{-1}L(0)A_1A_2^{-2}L(0)) + (A_2^{-1}L(1)A_2^{-1}L(0)). \quad (24)$$

We will hereafter consider the case of the diagonal matrices $A_i$: $A_1 = diag(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)})$, $A_2 = diag(a_1^{(2)}, a_2^{(2)}, \ldots, a_{n-1}^{(2)}, a_n^{(2)})$ and take in the previous formulas continuous limit $a_n^{(2)} \to 0$. Due to the fact that hamiltonians $I^2_{-4}(L(l))$ and $I^2_{-5}(L(l))$ are singular in this limit we have to rescale them, considering the limit $a_n^{(2)} \to 0$ of commuting integrals

$$I^2_{-4}'(L(l)) \equiv a_n^{(2)}I^2_{-4}(L(l)), \quad I^2_{-5}'(L(l)) \equiv ((a_n^{(2)}/a_n^{(1)})I^2_{-5}(L^-(l)) - I^2_{-4}(L^-(l))).$$

Taking this limit we obtain:

$$I^2_{-4}'(L(l)) = \sum_{i<n} \frac{(l^{(0)}_{in})^2}{a_i^{(2)}}, \quad I^2_{-5}'(L(l)) = \frac{1}{a_n^{(1)}} \sum_{i<n} \left(2\frac{l^{(1)}_{in}l^{(0)}_{in}}{a_i^{(2)}} + \frac{(l^{(0)}_{in})^2a_i^{(1)}}{(a_i^{(2)})^2} \right) - \sum_{0<i<j<n} \frac{(l^{(0)}_{ij})^2}{a_i^{(2)}a_j^{(2)}}. \quad (25)$$

Corresponding matrix gradients are written as follows:

$$1/2 \nabla I^2_{-4}' = \sum_{i<n} \frac{l^{(0)}_{in}}{a_i^{(2)}} X_{in}, \quad 1/2 \nabla I^2_{-5}' = \frac{1}{a_n^{(1)}} \sum_{i<n} \left(\frac{l^{(0)}_{in}}{a_i^{(2)}} X_{in} + \frac{l^{(1)}_{in}}{a_i^{(2)}} + \frac{a_i^{(1)}l^{(0)}_{in}}{(a_i^{(2)})^2} X_{in} \right) - \sum_{0<i<j<n} \frac{l^{(0)}_{ij}}{a_i^{(2)}a_j^{(2)}} X_{ij}. \quad (26)$$

Let us take for the hamiltonian that generate space flow function $I^2_{-4}'$. This fix our integrable hierarchy with $U \equiv \nabla I^2_{-4}'$. Taking into account the explicit form of $\nabla I^2_{-4}'$ we obtain that dynamical variables in the corresponding hierarchy are functions $l^{(0)}_{in}$, $i \in 1, n-1$. Hence, by taking the limit $a_n^{(2)} \to 0$, we have decreased the number of functional degrees of freedom from $n(n-1)/2$ (number of the independent components of $\nabla I^2_{-4}'$) to $n-1$ (number of the independent components of $\nabla I^2_{-4}'$).

In order to obtain all equations of this hierarchy it is necessary to obtain the regularized expression $\nabla I^2_{k-}$ for the all other hamiltonians $\nabla I^2_{k-}$ to express all coordinate functions
We will consider the simplest equation of the hierarchy \(27\) that correspond to the time flow of the hamiltonian \(I^2_{-5}'\), i.e. we will put \(V \equiv \nabla I^2_{-5}'\). Coordinate functions \(l_{ij}^{(0)}\) and \(l_{in}^{(1)}\) where \(i, j \in 1, n - 1\) enter in the explicit expression of the \(V\)-operator \(26\). They should be expressed via \(l_{in}^{(0)}\) and their derivatives in order to obtain wanted equation on the dynamical variables \(l_{in}^{(0)}\). This can be achieved by decomposing both sides of equation \(27\) in the powers of spectral parameter \(l\). Rescaling time variables \(x \rightarrow 2x, t \rightarrow 2t\) and introducing the following notations: \(m_i^{(1)} = l_{in}^{(1)} + \frac{a_{i}^{(1)} l_{in}^{(0)}}{a_i^{(2)}}\), we obtain that for the chosen \(U - V\) pair zero-curvature equation \(27\) is equivalent to the following system of differential equations:

\[
\frac{\partial l_{in}^{(0)}}{\partial t} - \frac{\partial m_i^{(1)}}{\partial x} = \sum_{k=1}^{n-1} \frac{l_{ik}^{(0)} a_k^{(1)} l_{kn}^{(0)}}{(a_k^{(2)})^2},
\]

\[
\frac{\partial l_{in}^{(0)}}{\partial x} = \sum_{k=1}^{n-1} \frac{l_{ik}^{(0)} l_{kn}^{(0)}}{a_k^{(2)}},
\]

\[
\frac{\partial l_{ij}^{(0)}}{\partial x} = a_n^{(1)} (m_i^{(1)} l_{jn}^{(0)} - m_j^{(1)} l_{in}^{(0)}).
\]

We will use equations \(29\) and \(30\) in order to to express \(m_i^{(1)}\) and \(l_{ij}^{(0)}\) via dynamical variables \(l_{jn}^{(0)}\) and their \(x\)-derivatives. From these equations it is easy to deduce, that the following equalities hold true:

\[
l_{ij}^{(0)} = \frac{\partial l_{in}^{(0)}}{\partial x} l_{jn}^{(0)} - \frac{\partial l_{jn}^{(0)}}{\partial x} l_{in}^{(0)},
\]

\[
m_i^{(1)} = \frac{1}{a_n^{(1)}} \frac{\partial^2 l_{in}^{(0)}}{\partial x^2} + c_2(L^{(0)}) l_{in}^{(0)},
\]

where \(c_2(L^{(0)})\) is some scalar function of the dynamical variables \(l_{in}^{(0)}\). We determine explicit form of the function \(c_2(L^{(0)})\) using the fact that hamiltonians \(I^2_{-4}'\) and \(I^2_{-5}'\) are constant along all flows and we may put \(I^2_{-4}' = 1, I^2_{-5}' = 0\). Using this and introducing vector \((\vec{t})_i = l_{in}^{(0)}/a_i^{(2)}\) and matrices \(A'_i \in \text{Mat}(n - 1), A'_i = \text{diag}(a_1^{(i)}, a_2^{(i)}, a_3^{(i)}, ..., a_{n-1}^{(i)})\) we obtain:

\[
c_2(\vec{t}) = 1/2(\vec{t}, A'_1 \vec{t}) + 1/a_n^{(1)}/2(\frac{\partial \vec{t}}{\partial x}, A'_2 \frac{\partial \vec{t}}{\partial x}).
\]
Using equality (31) we also deduce that:

\[ \sum_{k=1}^{n-1} \frac{i_k^{(0)} a_k^{(1)} l_n^{(0)} (a_k^{(2)})^2}{\alpha_i^{(2)} (a_k^{(2)})^2} = -1/2 \left( \frac{\partial^2 (\vec{\gamma}, A'_1 \vec{\tau})}{\partial x} (\vec{\tau})_i + (\vec{\tau}, A'_1 \vec{\tau}) \frac{\partial (\vec{\tau})_i}{\partial x} \right). \]

In the result we obtain the following differential equation in the partial derivatives:

\[ \frac{\partial \vec{\tau}}{\partial t} = \frac{1}{\alpha_n^{(1)}} \frac{\partial}{\partial x} \left( \frac{\partial^2 \vec{\tau}}{\partial x^2} + 3/2 (\frac{\partial \vec{\tau}}{\partial x}, A'_2 \frac{\partial \vec{\tau}}{\partial x}) \vec{\tau} \right) + 3/2 (\vec{\tau}, A'_1 \vec{\tau}) \frac{\partial \vec{\tau}}{\partial x}. \] (33)

In order to transform this equation to a more standard form it is necessary to introduce new notations: \( \vec{s} = (A'_2)^{1/2} \vec{\tau}, J \equiv A'_1 (A'_2)^{-1} \). Under such a replacement of variables constraint \( (\vec{\tau}', A'_2 \vec{\tau}) = 1 \) pass to the standard constraint \( (\vec{s}', \vec{s}) = 1 \) and equation (33) to the higher Landau-Lifshiz equation:

\[ \frac{\partial \vec{s}}{\partial t} = \frac{1}{\alpha_n^{(1)}} \frac{\partial}{\partial x} \left( \frac{\partial^2 \vec{s}}{\partial x^2} + 3/2 (\frac{\partial \vec{s}}{\partial x}, \frac{\partial \vec{s}}{\partial x}) \vec{s} \right) + 3/2 (\vec{s}, J \vec{s}) \frac{\partial \vec{s}}{\partial x}. \] (34)

**Remark 7.** In the case \( n = 4 \) this equation is the higher equation of the Landau-Lifshiz hierarchy. For \( n > 4 \) this equation was first obtained in [12] using the technique of ”dressing” and the embedding of the (specially realized) algebra \( \widetilde{so(n)}_A^+ \) into algebra \( so(n)((l)) \) of formal power series. Equation (34) was also obtained in our previous paper [18] using the algebra \( \widetilde{so(n)}_A^+ \) naturally embedded into algebra \( \widetilde{so(n)}_A \).

In the next subsection we will obtain the simplest equation of ”doubled” Landau-Lifshiz hierarchy. In order to do so it is necessary to use not the algebra of formal power series, nor the algebra \( \widetilde{so(n)}_A \) but its generalization — Lie algebra \( \widetilde{so(n)}_{A_1,A_2} \).

### 3.4 Integrable hierarchy associated with algebra \( \widetilde{so(n)}_{A_1,A_2} \).

In this subsection we will consider integrable hierarchies, admitting zero curvature type representation with \( U - V \) pairs taking values in the algebra \( \widetilde{so(n)}_{A_1,A_2} \). In order to obtain ”double” of Landau-Lifshiz hierarchies it is necessary to consider the case of the matrix \( \mathcal{A}(l) \) formed by the degenerated matrices \( A_i \), such that rank \( A_i = n - 1 \) but rank \( \mathcal{A}(l) = n \). As in the previous example of the hierarchies connected with \( \widetilde{so(n)}_{A_1,A_2}^\pm \) we will at first consider the case of the nondegenerated matrices \( A_i \): rank \( A_i = n \) and obtain the case rank \( A_i = n - 1 \) as its continuous limit.

Let us now illustrate the procedure of obtaining integrable equations in the partial derivatives associated with algebras \( \widetilde{so(n)}_{A_1,A_2} \). For this purpose we have to describe the set of commuting integrals on \( \widetilde{so(n)}_{A_1,A_2} \). Generic elements of the dual spaces to subalgebras \( \widetilde{so(n)}_{A_1,A_2}^- \) and \( \widetilde{so(n)}_{A_1,A_2}^+ \) have the following form:

\[ L^+(l) = L^{(1)} + llL^{(2)} + l^2L^{(3)} + l^3L^{(4)} + \cdots, \] (35)
\[ L^-(l) = l^{-1}L(0) + l^{-2}L(1) + l^{-3}L(2) + l^{-4}L(3) + \cdots, \]  
(36)

where \( L^{(\pm k)} \equiv \sum_{i<j=1,n} l^{(\pm k)}_{ij}X_{ji} \). Second order integrals (hamiltonians) by the very definition are written as follows:

\[ I^\pm_k(l) = 1/2 \, \text{res}_{l=0} l^{-(k+1)} Tr(L^\pm(l)A(l)^{-1})^2. \]  
(37)

Let us at first consider hamiltonians \( I^\pm_k \) in the case of the nondegenerated matrices \( A \). In order for hamiltonians \( I^\pm_k \) to be polynomials we will use two different decompositions of the matrix \( A(l)^{-1} \) in the formal power series — in the neighborhood of zero and infinity. Corresponding hamiltonians are calculated using their own decompositions:

\begin{align*}
I^+_k(l) &= 1/2 \, \text{res}_{l=0} l^{-(k+1)} Tr(A^{-1}_1(1 + A^{-1}_1A_2l + \cdots)(L^{-1} + lL^{(-2)} + \cdots))^2, \\
I^-_k(l) &= 1/2 \, \text{res}_{l=0} l^{-(k+1)} Tr((1 + A_1A^{-1}_2l^{-1} + \cdots)A^{-1}_2l^{-1}(l^{-1}L(0) + l^{-2}L(1) + \cdots))^2. \\
\end{align*}

(38)

(39)

Simplest hamiltonians of these sets are functions \( I^-_4(L(l)) \) and \( I^+_0(L(l)) \):

\[ I^-_4(L(l)) = 1/2Tr(A^{-1}_2lL(0))^2, \quad I^+_0(L(l)) = 1/2Tr(A^{-1}_1l^{-1}L^{-1})^2. \]  
(40)

Without loss of generality we will put that matrices \( A_i \) are diagonal: \( A_1 = \text{diag}(a_1^{(1)}, \ldots, a_n^{(1)}) \), \( A_2 = \text{diag}(a_1^{(2)}, \ldots, a_n^{(2)}) \) and consider the limits \( a_1^{(1)} \to 0 \), \( a_2^{(1)} \to 0 \) that correspond to the simplest degeneration of the matrices \( A_i \). Due to the fact that hamiltonians \( I^-_4 \) and \( I^+_0 \) are singular in this limit we will rescale them and consider integrals \( a_n^{(2)}I^-_4 \) and \( a_1^{(1)}I^+_0 \) instead. In the result we obtain the following hamiltonians:

\[ I^-_4(L(l)) = \lim_{a_n^{(2)} \to 0} a_n^{(2)}I^-_4 = 1/2 \sum_{i<n} \frac{l_{ii}^{(0)}a_i^{(2)}}{a_i^{(2)}}, \quad I^+_0(L(l)) = \lim_{a_1^{(1)} \to 0} a_1^{(1)}I^+_0 = 1/2 \sum_{i>1} \frac{l_{ii}^{(1)}a_i^{(1)}}{a_i^{(1)}}. \]  
(41)

Their matrix gradients are written as follows:

\[ \nabla I^-_4 = \sum_{i<n} \frac{l_{ii}^{(0)}a_i^{(2)}}{a_i^{(2)}} X_{in}, \quad \nabla I^+_0 = l^{-1} \sum_{i>1} \frac{l_{ii}^{(-1)}a_i^{(1)}}{a_i^{(1)}} X_{1i}. \]  
(42)

These are exactly \( U \)-operators of two independent generalized Landau-Lifshiz hierarchies. That is why we call this hierarchy to be “doubled” generalized Landau-Lifshiz hierarchy.

Corresponding zero-curvature condition:

\[ \frac{\partial \nabla I^-_4}{\partial x_-} - \frac{\partial \nabla I^+_0}{\partial x_+} + [\nabla I^-_4, \nabla I^+_0]_{A(l)} = 0. \]  
(43)

\(^2\)In the case of the nondegenerated matrices \( A \), corresponding matrix gradients produce anisotropic chiral field-type equations \([17]\).
yields the following equations:

\[ \partial_{x_+} l_{1i}^{(-1)} = -(a_i^{(1)}/a_i^{(2)}) l_{1in}^{(-1)}(0), \quad \partial_{x_-} l_{1n}^{(0)} = -a_i^{(2)} l_{1i}^{(-1)}, \quad (44) \]

\[ \partial_{x_+} l_{1n}^{(-1)} = a_i^{(1)} \sum_{k=2}^{n-1} l_{1kn}^{(-1)}/a_k^{(2)}, \quad \partial_{x_-} l_{1n}^{(0)} = a_i^{(2)} \sum_{k=2}^{n-1} l_{1kn}^{(-1)}/a_k^{(1)}. \quad (45) \]

Taking into account that functions

\[ J_0^{\pm} (L^+(l)) = 1/2 \sum_{i=1}^{n} \frac{(l_{1i}^{(-1)})^2}{a_i^{(1)}} = c_-, \quad J_0^{\pm} (L(l)) = 1/2 \sum_{i=1}^{n} \frac{(l_{1i}^{(0)})^2}{a_i^{(2)}} = c_+ \]

are constant along all time flows, we obtain that \( l_{1i}^{(-1)} \), \( l_{1n}^{(0)} \) are expressed via \( l_{1i}^{(-1)} \) and \( l_{1n}^{(0)} \):

\[ l_{1n}^{(-1)} = (a_i^{(1)})^{1/2} (c_- - \sum_{i=2}^{n-1} \frac{(l_{1i}^{(-1)})^2}{a_i^{(1)}})^{1/2}, \quad l_{1n}^{(0)} = (a_i^{(2)})^{1/2} (c_+ - \sum_{i=2}^{n-1} \frac{(l_{1i}^{(0)})^2}{a_i^{(2)}})^{1/2} \quad (46) \]

and equations (45) follows from the equations (44).

Introducing for convenience the following \((n - 2)\) component vectors:

\[ s_i^+ = \frac{l_{1i}^{(-1)}}{(a_i^{(1)})^{1/2}}, \quad s_i^+ = \frac{l_{1i}^{(0)}}{(a_i^{(2)})^{1/2}}, \quad i \in 2, n - 1. \]

and rescaling variables \( x_\pm \) we that our equations acquire the following form:

\[ \partial_{x_+} \tilde{s}_- = \left( c_- - (\tilde{s}_-, \tilde{s}_-)^T \right)^{1/2} \tilde{J}^{1/2} \tilde{s}_+, \quad (47) \]

\[ \partial_{x_-} \tilde{s}_+ = \left( c_+ - (\tilde{s}_+, \tilde{s}_+)^T \right)^{1/2} \tilde{J}^{-1/2} \tilde{s}_-, \quad (48) \]

where \((n - 2) \times (n - 2)\) matrix \( \tilde{J} \) is defined as follows: \( \tilde{J} = diag((a_2^{(2)})^{-1}a_2^{(1)}, \ldots, (a_{n-1}^{(2)})^{-1}a_{n-1}^{(1)}) \).

Remark 8. Note, that variables \( \tilde{s}_- \) could be expressed via \( \tilde{s}_+ \) and its derivatives with respect to "negative time" \( x_- \) using equation (48). In the result one obtains system of nonlinear differential equations of the second order on the vector \( \tilde{s}_+ \). Such procedure breaks simple form of the obtained equations and we prefer to leave them in the form of the system (47-48).

Let us now consider small \( n \) example of equations (47-48):

**Example 1.** Let \( n = 3 \). In this case we obtain the following two equations:

\[ \partial_{x_+} s_- = \left( c_- - s_-^2 \right)^{1/2} \tilde{J}^{1/2} s_+, \quad \partial_{x_-} s_+ = \left( c_+ - s_+^2 \right)^{1/2} \tilde{J}^{-1/2} s_- \]

Making substitution of variables: \( s_\pm = c_\pm sin\phi_\pm \), and rescaling variables \( x_\pm \) we obtain:

\[ \partial_{x_+} \phi_- = sin\phi_+, \quad \partial_{x_-} \phi_+ = sin\phi_- \]
Expressing $\phi_-$ via $\phi_+$ and putting it into the first equation we finally obtain:

$$\partial_{x+} \partial_{x-} \phi_+ = (1 - (\partial_{x-} \phi_+)^2)^{1/2} \sin \phi_+. \quad (49)$$

This is exactly so-called "modified Sine-Gordon equation" discovered by M. Kruskal and re-discovered later by H.Chen [19] (see also [20] and references therein).

**Example 2.** Let $n = 4$. In this case equations [17][18] define "the first negative flow" to standard Landau-Lifshitz equations. They have the following form:

$$\partial_{x_i} s^1_- = \left(c_+ - ((s^1_-)^2 + (s^2_-)^2)\right)^{1/2} J_1^{1/2} s^1_+, \quad \partial_{x_i} s^1_+ = \left(c_+ - ((s^1_+)^2 + (s^2_+)^2)\right)^{1/2} J_1^{-1/2} s^1_-,$$

$$\partial_{x_i} s^2_- = \left(c_+ - ((s^1_-)^2 + (s^2_-)^2)\right)^{1/2} J_2^{1/2} s^2_+, \quad \partial_{x_i} s^2_+ = \left(c_+ - ((s^1_+)^2 + (s^2_+)^2)\right)^{1/2} J_2^{-1/2} s^2_-.$$

Adding in third component to the vectors $\vec{s}_\pm$: $s^3_\pm = \left(c_+ - ((s^1_\pm)^2 + (s^2_\pm)^2)\right)^{1/2}$, rescaling one of the "times" $x'_i = (J_1 J_2)^{-1/2} x_-$ and making the following change of indices in vector $\vec{s}_-$: $s^2_\pm \leftrightarrow s^3_\pm$ we obtain that the above equations are written as follows:

$$\partial_{x_i} s^1_- = s^3_- J_2^{1/2} s^2_+, \quad \partial_{x_i'} s^1_+ = s^3_+ J_2^{1/2} s^2_-,$$

$$\partial_{x_i} s^2_- = s^3_- J_1^{1/2} s^1_+, \quad \partial_{x_i'} s^2_+ = s^3_+ J_1^{1/2} s^1_-,$$

$$\partial_{x_i} s^3_- = -(J_2^{1/2} s^1_- s^2_+ + J_1^{1/2} s^2_- s^1_+), \quad \partial_{x_i'} s^3_+ = -(J_2^{1/2} s^1_+ s^2_- + J_1^{1/2} s^2_+ s^1_-). \quad (52)$$

This system of equations coincide with anisotropic chiral field equations of Cherednik [21]:

$$\frac{\partial \vec{s}^-_+}{\partial x_+} = [\vec{s}^-_+ \times \vec{J}(s^2_+)], \quad \frac{\partial \vec{s}^-_-}{\partial x_-} = [\vec{s}^-_- \times \vec{J}(s^2_-)]$$

where a diagonal matrix $\vec{J}$ is defined as follows: $\vec{J} = \text{diag}(J_1^{1/2}, -J_2^{1/2}, 0)$.

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**References**

[1] Tahtadjan L. and Faddejev L. 1987 *Hamiltonian approach in the theory of solitons* (Berlin: Springer) 586 p

[2] Newell A. 1985 *Solitons in mathematics and physics* (University of Arizona: Society for industrial and Applied mathematics) 326 p

[3] Zaharov V, Shabat A 1979 *Funct An and appl* 13 No 3 13-21
[4] Kostant B 1979 *Adv Math* **34** 195-338

[5] Reyman A, Semenov-Tian-Shansky M 1979 *Invent. math.* **54** 81-100

[6] Reyman A, Semenov-Tian-Shansky M 1989 *VINITI: Fundamental trends* **6**, 145-147

[7] Holod P 1984 *Proceedings of the international conference" Nonlinear and turbulent Process in Physics"*, Kiev 1983 (Harwood Head. Publisher) **3** 1361-1367

[8] Holod P 1987 *Theoret and Math. Phys* **70** 11-19.

[9] Holod P 1987 *Soviet Phys Doklady* **32** 107-109

[10] Cantor I, Persits D 1988 *Proceedings of the IX USSR conference in Geometry, Kishinev, Shtinitsa* p.141

[11] Bolsinov A. 1988 *Trudy seminara po tenz. i vect. analizu* **23**, 18-28.

[12] Golubchik I, Sokolov V 2000 *Theoret and Math. Phys*, **124** No 1 62-71

[13] Holod P, Skrypnyk T 2000 *Naukovi Zapysky NAUKMA* ser phys-math sciences **18** 20-25

[14] Skrypnyk T 2001 *J Math Phys* **48** No 9 4570-4582

[15] Skrypnyk T, Holod P 2001 *J Phys A: Mathematics and General* **34** No 9 1123-1137

[16] Skrypnyk T 2002 *Czech J Phys* **52** No 11 1283-1288

[17] Skrypnyk T 2003 *Czech J Phys* **53** No 11 1119-1124

[18] Skrypnyk T.V. *J Math Phys* - to appear.

[19] Chen H 1974 *Phys Rev Lett* **33** No 15 925-930.

[20] Borisov A, Zykov S 1998 *Theor and Math. Phys* **115** No 2 199-214.

[21] I.V. Cherednik *Yadernaja Physica* **33** (1981) No 1, 278