A SHARP AFFINE POISSON-SOBOLEV TRACE INEQUALITY FOR THE FRACTIONAL-ORDER ANTIDERIVATIVES

NICO LOMBARDI AND JIE XIAO

ABSTRACT. This paper presents a novel affine Sobolev trace inequality with the sharp constant generated by the Poisson extension of a Sobolev function with the fractional antiderivative.

1. INTRODUCTION

The Sobolev trace inequalities play a fundamental role in Analysis and Geometry as well as PDE. Especially, the sharp $L^2$ inequality for the half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ says that for a function $f: \mathbb{R}^n_+ \to \mathbb{R}$, being smooth and decaying fast at infinity, one has

$$
\left(\int_{\mathbb{R}^{n-1}_+} |f(0, x)|^{2(n-1)} \, dx \right)^{\frac{n-2}{n-1}} \leq A(n) \int_{\mathbb{R}^n_+} |\nabla f(t, x)|^2 \, dx \, dt,
$$

where

$$A(n) = \frac{1}{\sqrt{\pi(n-2)}} \left( \frac{\Gamma(n)}{(n-1)\Gamma(n-1/2)} \right)^{\frac{1}{n-1}}$$

is the best constant.

This result was proved by Beckner [11] and Escobar [9] independently. Although both of them employed the conformal invariance of the inequality to link this problem with that one over the unit sphere $S^{n-2}$ of $\mathbb{R}^{n-1}$, their proofs were quite different. Escobar used the Obata’s rigidity theorem, while Beckner deduced his result from the spectral form of Lieb’s sharp Hardy-Littlewood-Sobolev inequality. They also proved that equality in (1) holds if and only if

$$f(t, x) = \gamma((t + \delta)^2 + |x - x_0|^2)^{-\frac{n+2}{n-1}}$$

for some

$$(\gamma, \delta, x_0) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{n-1}.$$
We also refer to Lions’ work [14] on finding the extremal functions of (1) via a concentration-compactness principle.

Moreover, [18] and [8] contain some extended counterparts of (1) for the fractional homogeneous Sobolev space. For

\[ 1 - n \leq -1 < \alpha < 1 \leq n - 1 \]

let \( \dot{H}^\alpha(\mathbb{R}^{n-1}) \) be the completion of \( C_c^\infty(\mathbb{R}^{n-1}) \) - the space of all smooth functions with compact support in \( \mathbb{R}^{n-1} \) under the fractional homogeneous Sobolev norm

\[
\|f\|_{\dot{H}^\alpha(\mathbb{R}^{n-1})} = \left( \int_{\mathbb{R}^{n-1}} (2\pi|x|)^{2\alpha} |\hat{f}(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]

In [18] Xiao proved a sharp fractional Sobolev trace inequality in the sense of Beckner and Escobar for a particular class of functions on \( \mathbb{R}_+^n \). To be more precise, let \( f : \mathbb{R}^{n-1} \to \mathbb{R} \) be a function in \( \dot{H}^\alpha(\mathbb{R}^{n-1}) \) and consider its Poisson extension onto \( \mathbb{R}_+^n \), i.e.,

\[
\begin{align*}
  f(t, x) &= P_t * f(x); \\
  P_t(x) &= \pi^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) t(t^2 + |x|^2)^{-\frac{n}{2}}; \\
  \Gamma(\cdot) &= \text{the standard gamma function}.
\end{align*}
\]

Regarding this case Xiao proved in [18, Theorem 1] that if

\[ 0 < \alpha < \min\left\{1, \frac{n-1}{2}\right\} \quad \& \quad f \in \dot{H}^\alpha(\mathbb{R}^{n-1}), \]

then

\[
(2) \quad \left( \int_{\mathbb{R}^{n-1}} |f(0, x)|^{2(n-1)} dx \right)^{\frac{n-1-2\alpha}{n}} \leq B(n - 1, \alpha) \int_{\mathbb{R}_+^n} |\nabla f(t, x)|^2 \frac{dxdt}{t^{2\alpha - 1}},
\]

where

\[
B(n, \alpha) = \frac{2^{1-4\alpha}}{\pi^n \Gamma(2(1-\alpha))} \left( \frac{\Gamma(n)}{\Gamma(n/2 + \alpha)} \right) \left( \frac{\Gamma(n/2)}{\Gamma(n/2)} \right)^{2\alpha/n}
\]

is optimal. We refer also to the work made by Einav and Loss in [8], where they proved a sharp fractional Sobolev trace inequality in a more general setting. In short - given a function \( f \in \dot{H}^\alpha(\mathbb{R}^n) \), instead of taking its Poisson’s extension onto \( \mathbb{R}_+^{n+1} \) they considered its restriction to \( \mathbb{R}^{n-m} \), in terms of the trace of \( f \), where \( m \) is a positive integer obeying \( 0 \leq \frac{m}{2} < \alpha < \frac{n}{2} \).

The aim of this paper is to obtain a stronger dual version of (2). Still working on the Poisson extension of an arbitrary function \( f \in \dot{H}^\alpha(\mathbb{R}^{n-1}) \)
we fortunately find a possibility to shrink the right-hand-integral of (2):
\[
\int_{\mathbb{R}^n_+} |\nabla f(t, x)|^2 \frac{dx dt}{t^{2\alpha - 1}} = \int_{\mathbb{R}^n_+} \left( \frac{\partial f}{\partial t}(t, x) \right)^2 + \sum_{i=1}^{n-1} \left( \frac{\partial f}{\partial x_i}(t, x) \right)^2 \frac{dx dt}{t^{2\alpha - 1}}
\]

in accordance with the following elementary rule: if \( \nabla_x f(t, x) \) stands for the gradient of \( f \) at the point \((t, x)\) with respect to the variable \( x \in \mathbb{R}^{n-1} \) only, then
\[
\int_{\mathbb{R}^n_+} |\nabla f(t, x)|^2 t^{1-2\alpha} dx dt
\]
\[= \int_{\mathbb{R}^n_+} \left| \frac{\partial f}{\partial t}(t, x) \right|^2 t^{1-2\alpha} dx dt + \int_{\mathbb{R}^n_+} |\nabla_x f(t, x)|^2 t^{1-2\alpha} dx dt
\]
\[\geq 2 \left( \int_{\mathbb{R}^n_+} \left| \frac{\partial f}{\partial t}(t, x) \right|^2 t^{1-2\alpha} dx dt \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n_+} |\nabla_x f(t, x)|^2 t^{1-2\alpha} dx dt \right)^{\frac{1}{2}},
\]

and hence the energy of \( f \) is split in two terms - one depending on the derivative in \( t \) and the other in \( x \). Actually, the above splitting idea suggests us to discover an affine and fractional counterpart of the Escobar and Beckner result. To justify this affine counterpart, we introduce briefly the work made by De Nápoli, Haddad, Jiménez and Montenegro in [7], where they proved a stronger version of (1). They established a sharp affine \( L^2 \) Sobolev trace inequality involving the \( L^2 \)-affine energy
\[
\mathcal{E}_2(f) = c_{n-1} \left( \int_{\mathbb{S}^{n-2}} \|\nabla_{\xi} f(t, x)\|_{L^2(\mathbb{R}^n_+)}^{1-n} d\xi \right)^{-\frac{1}{n-1}},
\]
where
\[
c_n = n \frac{n^2}{n-2} \sqrt{\omega_n},
\]
and \( \omega_n \) denotes the volume of the Euclidean ball \( B^n \) in \( \mathbb{R}^n \). It is known that
\[
\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}, \quad \forall \ n \geq 2,
\]
and we can extend it to every positive real number \( s > 0 \),
\[
\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s+2}{2})}.
\]
Moreover we have that \( \xi \in \mathbb{S}^{n-2} \) - the unit sphere of \( \mathbb{R}^{n-1} \), and \( \nabla_{\xi} f(t, x) \) is the directional derivative with respect to the direction \( \xi \) at the point \((t, x)\), and
\[
\nabla_{\xi} f(t, x) = < \nabla_x f(t, x), \xi >.
\]
Remarkably, the \( L^2 \)-affine energy and its extension to \( p \in (1, n) \) have appeared in many results concerning the affine Sobolev inequalities (cf.
(15, 19)) and the affine version of the Polya-Szego principle (cf. [5, 10]). We also remark that the $L^2$ (and $L^p$)-affine energy is invariant under volume preserving affine transformations of $\mathbb{R}^n_+$. In [7, Theorem 1], De Nápoli, Haddad, Jiménez and Montenegro recently showed that for any $f \in C^\infty_c(\mathbb{R}^n)$ one has

\[
\left( \int_{\mathbb{R}^n} |f(0, x)|^{\frac{2(n-1)}{n-2}} \frac{dx}{n-1} \right)^{\frac{n-2}{n-1}} \leq 2A(n) \mathcal{E}_2(f) \left\| \frac{\partial f}{\partial t} \right\|_{L^2(\mathbb{R}^n_+)} ,
\]

and this is invariant under affine transformations of $\mathbb{R}^n_+$. Moreover equality holds if and only if

\[
f(t, x) = \pm \left( (\lambda t + \delta)^2 + |B(x - x_0)|^2 \right)^{-\frac{n-2}{2}}
\]

for some triple

\[
(\lambda, \delta, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n-1}
\]

and a matrix $B \in GL_{n-1}$ - the set of all invertible real $(n - 1) \times (n - 1)$-matrices.

We note also that they proved, always in [7], an $L^p$ version of sharp affine Sobolev trace inequality for $p \in (1, n)$. Indeed Escobar, in his paper [9], conjectured a possible extension of his result (1) from $p = 2$ to any $p \in (0, n)$, proved by Nazareth in [16]. De Nápoli, Haddad, Jiménez and Montenegro proved an affine and stronger version of results in [1], [9] and [16]. With stronger we mean, for instance in the case $p = 2$, that it holds

\[
\mathcal{E}_2(f) \leq \left( \int_{\mathbb{R}^n_+} |\nabla_x f(t, x)|^2 dx dt \right)^{\frac{1}{2}},
\]

as we can find in [7] and [15]. Hence we have

\[
\left( \int_{\mathbb{R}^n} |f(x)|^{\frac{2(n-1)}{n-2}} dx \right)^{\frac{n-2}{n-1}} \leq 2A(n) \mathcal{E}_2(f) \left( \int_{\mathbb{R}^n_+} \left| \frac{\partial f}{\partial t} (t, x) \right|^2 dx dt \right)^{\frac{1}{2}} \\
\leq 2A(n) \left( \int_{\mathbb{R}^n_+} |\nabla_x f(t, x)|^2 dx dt \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n_+} \left| \frac{\partial f}{\partial t} (t, x) \right|^2 dx dt \right)^{\frac{1}{2}} \\
\leq A(n) \left( \int_{\mathbb{R}^n_+} |\nabla f(t, x)|^2 dx dt \right) .
\]

So we can say that inequality (3) implies (1).
For our goal we utilize the weighted $L^p$-affine energy

$$E_p(f, \sigma) = c_{n-1,p} \left( \int_{S^{n-2}} \| \nabla \xi f \|_{L^p(\mathbb{R}^n_+, \sigma)}^{\frac{1}{p-1}} d\xi \right)^{-\frac{1}{n-1}},$$

where

$$c_{n,p} = \left( n\omega_n \right)^{\frac{1}{n}} \left( \frac{n\omega_n}{2\omega_{n+p-2}} \right)^{\frac{1}{p}}.$$

$\sigma: \mathbb{R}^n_+ \to \mathbb{R}_+$ is a measurable function, and $L^p(\mathbb{R}^n_+, \sigma)$ is the weighted Lebesgue space of all measurable functions $f: \mathbb{R}^n_+ \to \mathbb{R}$ satisfying

$$\| f \|_{L^p(\mathbb{R}^n_+, \sigma)} = \left( \int_{\mathbb{R}^n_+} |f(t, x)|^p \sigma(t, x) dx dt \right)^{\frac{1}{p}} < \infty.$$

We discover a sharp affine Poisson-Sobolev inequality for the fractional-order antiderivatives as seen below.

**Theorem 1.1.** Let

$$\begin{cases}
   n \geq 3; \\
   \frac{1}{2} \leq \alpha < 1; \\
   p = \frac{2(n-1+2\alpha)}{n+1+2\alpha}; \\
   p' = \frac{p}{p-1} = \frac{2(n+2\alpha-1)}{n+2\alpha-3}; \\
   \sigma(t, x) = t^{2\alpha-1} \quad \forall \quad (t, x) \in \mathbb{R}^n_+.
\end{cases}$$

Then there is a sharp constant $D(n, p, \alpha)$ depending only on the triple $(n, p, \alpha)$ such that for any $g \in C^\infty_c(\mathbb{R}^{n-1})$ and its Poisson extension

$$g(t, x) = P_t \ast g(x) \quad \forall \quad (t, x) \in \mathbb{R}^n_+$$

one has

$$\| g \|_{H^{-\alpha}(\mathbb{R}^{n-1})} \leq D(n, p, \alpha) \left( E_p(g, \sigma) \right)^{\frac{n-1}{n+2\alpha}} \left\| \frac{\partial g}{\partial t} \right\|_{L^p(\mathbb{R}^n_+, \sigma)}^{\frac{2\alpha}{n-1+2\alpha}}$$

Indeed,

$$D(n, p, \alpha) = \left( \frac{2^{2\alpha}}{\Gamma(2\alpha)} \right)^{\frac{1}{2}} \left( \pi^{-\frac{n-1}{2(n+2\alpha-1)}} (2\alpha)^{-\frac{2\alpha}{p(n+2\alpha-1)}} \right) \times (n-1)^{-\frac{n-1}{p(n+2\alpha-1)}} \left( \frac{n+2\alpha-1-p}{p-1} \right)^{-\frac{1}{p}} \times \left( \frac{p'\Gamma(n+2\alpha)}{\Gamma(n+2\alpha-1)} \right)^{\frac{1}{n+2\alpha-1}}.$$
Moreover, the equality in (4) holds if and only if

\[ g(t, x) = c \left( 1 + |\lambda t|^{\frac{p+1}{p}} + |B(x - x_0)|^{\frac{p+1}{p}} \right)^{\frac{p+1-n-2\alpha}{p}} \]

for some quadruple

\[ (c, |\lambda|, x_0, B) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{n-1} \times GL_{n-1}. \]

Here \( \|g\|_{H^{-\alpha}(\mathbb{R}^{n-1})} \) is the norm in the fractional homogenous Sobolev space with negative exponent \(-\alpha\). As we will see in Section 2, we can define it with the Riesz potential.

The main ingredient for the proof of our statement is a sharp affine weighted \( L^p \) Sobolev inequality proved in [11] by Haddad, Jiménez and Montenegro.

**Theorem 1.2.** Let

\[
\begin{cases}
  a \geq 0; \\
  1 \leq p < n + a; \\
  p^*_a = \frac{p(n+a)}{n-p+a}; \\
  \sigma(t, x) = t^\alpha \forall \ (t, x) \in \mathbb{R}^n_+.
\end{cases}
\]

Then there is a sharp constant \( J(n, p, a) \) depending only on the triple \((n, p, a)\) such that if \( f \in C_c^\infty(\mathbb{R}^n) \) then

\[ \|f\|_{L^{p^*_a}(\mathbb{R}^n_+, \sigma)} \leq J(n, p, a) \left( E^a_p(f, \sigma) \right)^{\frac{n-a}{n+a}} \left\| \frac{\partial f}{\partial t} \right\|_{L^p(\mathbb{R}^n_+, \sigma)}. \]

The equality in (5) holds if and only if

\[ f(t, x) = \begin{cases}
  c(1 + |\gamma t|^{\frac{p+1}{p}} + |B(x - x_0)|^{\frac{p+1}{p}})^{-\frac{n+a-p}{p}} & \text{as } p > 1; \\
  c \chi_{B^n}(\gamma t, B(x - x_0)) & \text{as } p = 1,
\end{cases} \]

for some quadruple

\[ (c, |\gamma|, x_0, B) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{n-1} \times GL_{n-1}, \]

where \( \chi_{B^n} \) is the characteristic function of the unit ball \( B^n \) in \( \mathbb{R}^n \).

We will manipulate the left-hand side of inequality (5) to establish (4). We observe that also inequality (5) is invariant under affine transformations of \( \mathbb{R}^n_+ \), and this allows us to say that inequality (4) is affine invariant.

The rest of this paper is organized as follows. In Section 2 we present some essentials about \( \dot{H}^\alpha \) and the fractional Laplacian \((-\Delta)^{\frac{\alpha}{2}}\), while in Section 3 we will give the detail of the proof of Theorem 1.1.
2. Basics on $\dot{H}^\alpha$ and $(-\Delta)^{\frac{\alpha}{2}}$

For an integer $n \geq 1$ we define the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ as
\[
\hat{f}(x) = \mathcal{F}(f)(x) = \int_{\mathbb{R}^n} f(\xi) \exp(-2\pi i < x, \xi >) d\xi.
\]

It is well known that we can extend, by Plancherel’s theorem, the Fourier transform to every $L^2(\mathbb{R}^n)$-function.

**Definition 2.1.** Given $0 < \alpha < 1$, the fractional homogeneous Sobolev space $\dot{H}^\alpha(\mathbb{R}^n)$ is defined as the completion of all functions $f \in C_\infty^c(\mathbb{R}^n)$ under the norm
\[
\|f\|_{\dot{H}^\alpha(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (2\pi |x|)^{2\alpha} |\hat{f}(x)|^2 dx \right)^{\frac{1}{2}}.
\]

This definition allows us to link it with the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$. More precisely, if $\mathcal{F}^{-1}$ is the inverse Fourier transform and $f \in C_\infty^c(\mathbb{R}^n)$, then the fractional Laplacian of $f$ is
\[
(-\Delta)^{\frac{\alpha}{2}} f(x) = \mathcal{F}^{-1}\left( (2\pi |x|)^{\alpha} \hat{f}(x) \right),
\]
and hence we can say that the fractional Laplacian is an operator enjoying
\[
(-\Delta)^{\frac{\alpha}{2}} f(x) = (2\pi |x|)^{\alpha} \hat{f}(x)
\]
and the fractional homogeneous Sobolev norm of $f$
\[
\|f\|_{\dot{H}^\alpha(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{2}} f(x)|^2 dx \right)^{\frac{1}{2}},
\]
exists as a consequence of Plancherel’s theorem.

**Proposition 2.2.** $\dot{H}^\alpha(\mathbb{R}^n)$ endowed with the inner product
\[
(f, g)_{\dot{H}^\alpha(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (2\pi |x|)^{\alpha} \hat{f}(x) \hat{g}(x) dx
\]
is an Hilbert space.

We have the following fractional Sobolev inequality; see e.g. [4, 6, 8, 12, 18] for more details about the fractional Laplacian and its connections with PDE.

**Theorem 2.3.** If $f \in \dot{H}^{0 < \alpha < \frac{n}{2}}(\mathbb{R}^n)$, then
\[
\left( \int_{\mathbb{R}^n} |f(x)|^{\frac{2n}{n-2\alpha}} dx \right)^{\frac{n-2\alpha}{n}} \leq B(n, \alpha)\Gamma(2(1-\alpha))2^{2\alpha-1}\|f\|_{\dot{H}^\alpha(\mathbb{R}^n)}^2.
\]
where $B(n, \alpha)$ is the best constant that appears in (2). Moreover, equality in the inequality of (7) holds if and only if
\[ f(x) = A(\gamma^2 + |x - a|^2)^{-\frac{2}{2} - \alpha}, \]
for some triple
\[ (A, |\gamma|, a) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n. \]

The previous theorem indicates that the fractional homogeneous Sobolev space may be treated as a function space - more precisely - we have the following corollary.

**Corollary 2.4.** Under $0 < \alpha < \frac{n}{2}$ one has
\[ \dot{H}^\alpha(\mathbb{R}^n) = \{ f \in L^{\frac{2n}{n-2\alpha}}(\mathbb{R}^n) : (-\Delta)^{\frac{\alpha}{2}} f \in L^2(\mathbb{R}^n) \}. \]

Now, we want to introduce fractional homogeneous Sobolev space and fractional Laplacian with negative exponent: we use the Riesz potential. Let $f$ be a smooth function on $\mathbb{R}^n$ with a fast decay at infinity. According to [17, Section 5.1], under $0 < \alpha < \frac{n}{2}$ we define
\[
\left\{ \begin{array}{l}
I_{2\alpha} f(x) = \frac{1}{\gamma(2\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2\alpha}} dy; \\
\gamma(2\alpha) = \frac{\pi^{\frac{n}{2}} 2^{2\alpha} \Gamma(\alpha)}{\Gamma(\frac{n}{2} - \alpha)}.
\end{array} \right.
\]

$I_{2\alpha} f$ is the Riesz potential of $f$, hence we get
\[ (-\Delta)^{-\frac{\alpha}{2}} f = I_{\alpha} f. \]

To justify this last identification, recall the following result (cf. [17 Section 5.1, Lemma 2] and [13 Corollary 5.10]).

**Lemma 2.5.** Let $0 < \alpha < \frac{n}{2}$.

- For every $\phi \in C_c^\infty(\mathbb{R}^n)$ one has
  \[ \int_{\mathbb{R}^n} |x|^{2\alpha - n} \phi(x) dx = \gamma(2\alpha) \int_{\mathbb{R}^n} \frac{\phi(x)}{(2\pi |x|)^{2\alpha}} dx. \]

- The Fourier transform of $I_{\alpha} f$
  \[ \widehat{I_{2\alpha} f}(x) = (2\pi |x|)^{-2\alpha} \widehat{f}(x) \]
  holds in the sense of
  \[ \int_{\mathbb{R}^n} I_{2\alpha} f(x) \overline{g(x)} dx = \int_{\mathbb{R}^n} \frac{\widehat{f(x)} \overline{\widehat{g(x)}}}{(2\pi |x|)^{2\alpha}} dx \quad \forall \ f, g \in S(\mathbb{R}^n). \]

If we take $f = g$ in the last statement of Lemma 2.5 then we have
\[ \int_{\mathbb{R}^n} (2\pi |x|)^{-2\alpha} |\widehat{f(x)}|^2 dx = \frac{1}{\gamma(2\alpha)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x) \overline{f(y)}}{|x - y|^{n-2\alpha}} dxdy. \]
We present now the following lemma, a powerful tool for the proof of fractional Sobolev inequality and inequality (2) and also useful to treat fractional Laplacian with negative exponent.

**Lemma 2.6.** Let \( \alpha \in (0, \frac{n}{2}) \) and \( f, g \in L^{\frac{2n}{n+2\alpha}}(\mathbb{R}^n) \). Then we have

\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{n-2\alpha}} \, dx \, dy \right| \leq \pi^{\frac{n}{2}-\alpha} \frac{\Gamma(\alpha)}{\Gamma(\frac{n}{2}+\alpha)} \left( \frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right)^{\frac{2\alpha}{n}} \times \frac{\|f\|_{L^{\frac{2n}{n+2\alpha}}(\mathbb{R}^n)}}{\|f\|_{\dot{H}^{-\alpha}(\mathbb{R}^n)}},
\]

where the constant is sharp and equality holds if and only if

\[
f(x) = c(\gamma^2 + |x-x_0|^2)^{-\frac{n+2\alpha}{2}}
\]

for some triple

\[(c, \gamma, x_0) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n\]

and \( g \) is equal to \( f \) times some constant.

Lemma 2.6 is a particular case of Hardy-Littlewood-Sobolev inequality and we refer to [13] for its general setting. For our purpose we apply Lemma 2.6 to (8) and we get exactly

\[
\int_{\mathbb{R}^n} (2\pi |x|)^{-2\alpha} |\hat{f}(x)|^2 \, dx \leq B(n, \alpha) \Gamma(2(1-\alpha)) 2^{2\alpha-1} \|f\|_{L^{\frac{2n}{n+2\alpha}}(\mathbb{R}^n)}^2
\]

and that is the dual version of fractional Sobolev inequality (2.3) and inequality (2). We recall that \( B(n, \alpha) = \left( \frac{2^{1-4\alpha}}{\pi \Gamma(2(1-\alpha))} \right) \left( \frac{\Gamma(\frac{n}{2}-\alpha)}{\Gamma(\frac{n}{2}+\alpha)} \right) \left( \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right)^{\frac{2\alpha}{n}} \).

In terms of the fractional Laplacian we have

\[
||(-\Delta)^{-\frac{\alpha}{2}} f||_{L^2(\mathbb{R}^n)} \leq B(n, \alpha) \Gamma(2(1-\alpha)) 2^{2\alpha-1} ||f||_{L^{\frac{2n}{n+2\alpha}}(\mathbb{R}^n)}^2
\]

and that explains the reason we can consider

\[
\dot{H}^{-\alpha}(\mathbb{R}^n) = \{ f \in L^{\frac{2n}{n+2\alpha}}(\mathbb{R}^n) \mid (-\Delta)^{-\frac{\alpha}{2}} f \in L^2(\mathbb{R}^n) \}
\]

as we have done for \( \dot{H}^\alpha \).

3. **Proof of Theorem 1.1**

First of all, we recall Poisson kernel, Poisson extension and their properties that we will use along the proof.

**Definition 3.1.** For

\[
\begin{cases}
  n \geq 2; \\
  t \in \mathbb{R}_+,
\end{cases}
\]

...
let
\[ P_t : \mathbb{R}^{n-1} \rightarrow \mathbb{R}, \]
be the Poisson kernel, i.e.,
\[ P_t(x) = \Gamma\left(\frac{n}{2}\right) \pi^{-\frac{n}{2}} t (t^2 + |x|^2)^{-\frac{n}{2}} \forall \ x \in \mathbb{R}^{n-1}. \]
Then the Poisson extension or harmonic extension of \( f \in \dot{H}^\alpha(\mathbb{R}^{n-1}) \) is the convolution of \( P_t \) with \( f \):
\[
f(t, x) = P_t * f(x) = \Gamma\left(\frac{n}{2}\right) \pi^{-\frac{n}{2}} \int_{\mathbb{R}^{n-1}} t (t^2 + |x - z|^2)^{-\frac{n}{2}} f(z) \, dz.
\]

**Remark 3.2.** Below is a short list of the Poisson kernel properties (cf. [17, Chapter 3, Section 2.1]).

- The reason why \( f(t, x) \) is called the harmonic extension of \( f \) is that \( P_t \) is harmonic in \( \mathbb{R}_+^n \).
- \( f \in C^\infty(\mathbb{R}_+^n) \Rightarrow f(t, \cdot) \in C^\infty(\mathbb{R}^{n-1}) \forall t \in \mathbb{R}_+ \).
- \[
\begin{align*}
P_t &\in L^p(\mathbb{R}^{n-1}) \forall (t, p) \in \mathbb{R}_+ \times [1, \infty); \\
\|P_t\|_{L^1(\mathbb{R}^{n-1})} &= 1 \forall t \in \mathbb{R}_+.
\end{align*}
\]
- \( f \in L^p(\mathbb{R}^{n-1}) \Rightarrow f(t, \cdot) \in L^p(\mathbb{R}^{n-1}) \forall t > 0. \)
- \[
\lim_{t \to 0^+} f(t, x) = f(x) \text{ for a.e. } x \in \mathbb{R}^{n-1}.
\]
- \[
\lim_{t \to 0^+} \|f(t, \cdot) - f(\cdot)\|_{L^p(\mathbb{R}^{n-1})} = 0.
\]

Now we employ Theorem 1.2 to verify (4). As a matter of fact, if
\[
\begin{align*}
a &\geq 0; \\
p &\in (1, n + a); \\
p_a^* &= 2; \\
\sigma(t, x) &= t^a \forall \ (t, x) \in \mathbb{R}_+^n,
\end{align*}
\]
then
\[
p_a^* = \frac{p(n + a)}{n - p + a} = 2 \iff p = \frac{2(n + a)}{n + a + 2} < n + a.
\]
Meanwhile, if
\[
g \in L^p(\mathbb{R}^{n-1}) \ \& \ g(t, x) = P_t * g(x).
\]
then an application of
\[
P_t \in L^q(\mathbb{R}^{n-1}) \ \forall \ 1 \leq q < \infty
\]
and Young’s inequality with

\[ q = \frac{n + a}{n + a - 1} \]

derives

\[ g(t, \cdot) \in L^2(\mathbb{R}^{n-1}) \quad \forall \quad t \in \mathbb{R}_+. \]

Accordingly, we can evaluate the left-hand side of (5) with \( p^*_a = 2 \), thereby finding (via Fubini’s and Planehrel’s theorems)

\[
\| g \|_{L^2_2(\mathbb{R}^+_n, \sigma)}^2 = \int_0^\infty \int_{\mathbb{R}^{n-1}} |g(t, x)|^2 t^a \, dx \, dt \\
= \int_0^\infty \left( \int_{\mathbb{R}^{n-1}} |g(t, x)|^2 \, dx \right) t^a \, dt \\
= \int_0^\infty \left( \int_{\mathbb{R}^{n-1}} |\hat{g}(t, x)|^2 \, dx \right) t^a \, dt.
\]

Upon utilizing \( g(t, x) = P_t \ast g(x) \) and the convolution property of Fourier’s transform, we obtain

\[
\| g \|_{L^2_2(\mathbb{R}^+_n, \sigma)}^2 = \int_0^\infty \left( \int_{\mathbb{R}^{n-1}} |\hat{P}_t(x)|^2 |\hat{g}(x)|^2 \, dx \right) t^a \, dt \\
= \int_{\mathbb{R}^{n-1}} \left( \int_0^\infty |\hat{P}_t(x)|^2 t^a \, dt \right) |\hat{g}(x)|^2 \, dx.
\]

At the same time, we can take advantage of the explicit expression of the Fourier transform of \( P_t \) to evaluate

\[
\int_0^\infty |\hat{P}_t(x)|^2 t^a \, dt
\]

via the gamma function - more precisely - a simple calculation gives

\[
\int_0^\infty |\hat{P}_t(x)|^2 t^a \, dt = \frac{\Gamma(a + 1)}{(4\pi |x|)^{a+1}} \quad \forall \quad |x| > 0.
\]

This in turn implies

\[
\| g \|_{L^2_2(\mathbb{R}^+_n, \sigma)}^2 = \int_{\mathbb{R}^{n-1}} |\hat{g}(x)|^2 \frac{\Gamma(a + 1)}{(4\pi |x|)^{a+1}} \, dx \\
= \frac{\Gamma(a + 1)}{2^{a+1}} \int_{\mathbb{R}^{n-1}} \frac{|\hat{g}(x)|^2}{(2\pi |x|)^{a+1}} \, dx \\
= \frac{\Gamma(a + 1)}{2^{a+1}} \| \mathcal{L}^{\frac{a+1}{4}} g \|_{L^2(\mathbb{R}^{n-1})}^2.
\]
Consequently, we can apply Theorem 1.2 to the fractional Laplacian with exponent $-2^{2}(a + 1)$, thereby obtaining that if
\[
\begin{cases}
  a = 2\alpha - 1 \geq 0; \\
  \sigma(t, x) = t^{2\alpha - 1}; \\
  g \in C_{c}^{\infty}(\mathbb{R}^{n-1}); \\
  g(t, x) = P_{t} * g(x) \quad \forall \quad (t, x) \in \mathbb{R}^{n}_{+},
\end{cases}
\]
then
\[
\|g\|_{H^{-\alpha}(\mathbb{R}^{n-1})} = \left\|(-\Delta)^{-\frac{\alpha + 1}{4}} g\right\|_{L^{2}(\mathbb{R}^{n-1})}
= \left(\frac{2^{a+1}}{\Gamma(a + 1)}\right)^{\frac{1}{2}} \left\|g\right\|_{L^{2}(\mathbb{R}^{n-1}^{+})}
\leq \left(\frac{2^{a+1}}{\Gamma(a + 1)}\right)^{\frac{1}{2}} J(n, p, a) \left(\mathcal{E}_{p}(g, \sigma)\right)^{\frac{\alpha + 1}{n + a}} \left\|\frac{\partial g}{\partial t}\right\|_{L^{p}(\mathbb{R}^{n}_{+}, \sigma)}.
\]
Note that
\[
D(n, p, \alpha) = \left(\frac{2^{2\alpha}}{\Gamma(2\alpha)}\right)^{\frac{1}{2}} J(n, p, 2\alpha - 1).
\]
So, (4) follows up.

Next, in order to compute $D(n, p, \alpha)$, recall that $J(n, p, 2\alpha - 1)$, as the best constant in Theorem 1.2 under $\sigma(t, x) = t^{2\alpha - 1}$, has been evaluated in [11]. To see this evaluation, for a generic $a \geq 0$ we use [11, Appendix A] to write
\[
J(n, p, a) = L(n, p, a) M(n, p, a),
\]
where $L(n, p, a)$ is the best value found in [2, 3] concerning a weighted Sobolev inequality for the $L^{p}$-norm of the gradient - that is -
\[
L(n, p, a) = \left(\frac{(p - 1)^{p-1}}{(n + a)(n - p + a)^{p-1}}\right)^{\frac{1}{p}}
\times \left(\frac{2\pi^{\frac{a + 2 - n}{2}}}{\Gamma(n + a)\Gamma(\frac{n + a}{2})}\right)^{\frac{1}{n + a}}
\times \left(\frac{\Gamma\left(\frac{a + 2 - n}{2}\right)}{\Gamma\left(\frac{(n + a)(p - 1)}{p} + 1\right)\Gamma\left(\frac{a + 2}{p}\right)\Gamma\left(\frac{1 + a}{2}\right)}\right)^{\frac{1}{n + a}},
\]
while $M(n, p, a)$ appears in [11] Lemma 4.1 - namely - if $p' = \frac{p}{p - 1}$ then
\[
M(n, p, a) = (p')^{\frac{1}{p'}} \pi^{\frac{n - 1}{2(n + a)}} (1 + a)^{-\frac{1 + a}{p(n + a)}} (n - 1)^{-\frac{n - 1}{p(n + a)}}
\times \left(\frac{n + a}{p}\right)^{\frac{1}{p'}} \left(\frac{p'\Gamma\left(\frac{n + 1}{2}\right)\Gamma\left(\frac{n + a + 2}{p'}\right)}{\Gamma\left(\frac{1 + a}{p}\right)\Gamma\left(\frac{1 + a}{p'}\right)}\right)^{\frac{1}{n + a}}.
\]
Putting together $L(n, p, a)$, $M(n, p, a)$ and $a = 2\alpha - 1$, one has
\[
J(n, p, 2\alpha - 1) = \pi^{-\frac{n-1}{2(n+2\alpha-1)}} (2\alpha)^{-\frac{2\alpha}{p(n+2\alpha-1)}} \\
\times (n-1)^{-\frac{n-1}{p(n+2\alpha-1)}} \left( \frac{n + 2\alpha - 1 - p}{p - 1} \right)^{-\frac{1}{p}} \\
\times \left( \frac{p' \Gamma(n+1)}{\Gamma(\frac{n}{p'}) \Gamma(n+2\alpha-1-p') \Gamma(\frac{n+2\alpha-1}{p'})} \right)^{\frac{n+1}{n+2\alpha-1}},
\]
where
\[
p = \frac{2(n+2\alpha-1)}{n+2\alpha+1} \quad \text{and} \quad p' = \frac{2(n+2\alpha-1)}{n+2\alpha-3}.
\]

Finally, an application of the equality part of Theorem 1.2 derives that the equality in (4) holds if and only if the equality in (6) is valid under
\[
p = \frac{2(n+a)}{n+a+2} = \frac{2(n+2\alpha-1)}{n+2\alpha+1}.
\]
In other words, (5) takes its equality if and only if
\[
g(t, x) = c \left( 1 + |\lambda t|^{\frac{p+1}{p}} + |B(x-x_0)|^{\frac{p+1}{p}} \right)^{\frac{p+1-n-2\alpha}{p}}
\]
for some quadruple
\[
(c, |\lambda|, x_0, B) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{n-1} \times GL_{n-1}.
\]
Of course, the desired function $g$ on $\mathbb{R}^{n-1}$ for the equality of (4) can be obtained via taking the Fourier transform in $x$ on the last equation for $g(t, x)$ - in fact -
\[
\left\{
\begin{aligned}
g(x) &= \mathcal{F}^{-1}(\hat{g})(x); \\
\hat{g}(x) &= \frac{\mathcal{F} \left( c \left( 1 + |\lambda t|^{\frac{p+1}{p}} + |B(x-x_0)|^{\frac{p+1}{p}} \right)^{\frac{p+1-n-2\alpha}{p}} \right)(x)}{\mathcal{F}(f_t)(x)}.
\end{aligned}
\right.
\]

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DIPARTIMENTO DI MATEMATICA E INFORMATICA “U.DINI”, VIALE MORGAGNI 67/A, 50134, FIRENZE, ITALY

*E-mail address:* nico.lombardi@unifi.it

DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY, ST. JOHN’S, NL A1C 5S7, CANADA

*E-mail address:* jxiao@mun.ca