Convergence to diffusion waves for solutions of Euler equations with time-depending damping on quadrant

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Abstract: This paper is concerned with the asymptotic behavior of the solution to the Euler equations with time-depending damping on quadrant \((x, t) \in \mathbb{R}^+ \times \mathbb{R}^+\),

\[
\partial_t v - \partial_x u = 0, \quad \partial_t u + \partial_x p(v) = -\frac{\alpha}{(1 + t)^\lambda} u,
\]

with null-Dirichlet boundary condition or null-Neumann boundary condition on \(u\). We show that the corresponding initial-boundary value problem admits a unique global smooth solution which tends time-asymptotically to the nonlinear diffusion wave. Compared with the previous work about Euler equations with constant coefficient damping, studied by Nishihara and Yang (1999, J. Differential Equations, 156, 439-458), and Jiang and Zhu (2009, Discrete Contin. Dyn. Syst., 23, 887-918), we obtain a general result when the initial perturbation belongs to the same space. In addition, our main novelty lies in the facts that the cut-off points of the convergence rates are different from our previous result about the Cauchy problem. Our proof is based on the classical energy method and the analyses of the nonlinear diffusion wave.

Key Words: Euler equations with time-depending damping, nonlinear diffusion waves, initial-boundary value problem, decay estimates.

AMS Subject Classification : 35L65, 76N15, 35B45, 35B40.

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1 Introduction

In this paper, we consider the asymptotic behavior and the convergence rates of solutions to the one-dimensional compressible Euler equations with time-depending damping:

\[ \begin{aligned}
\partial_t v - \partial_x u &= 0, \\
\partial_t u + \partial_x p(v) &= -\frac{\alpha}{(1+t)^\lambda} u, \\
\end{aligned} \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.1} \]

with initial data

\[ (v, u) \mid_{t=0} = (v_0, u_0)(x) \to (v_+, u_+), \quad \text{as} \quad x \to +\infty \quad \text{and} \quad v_+ > 0, \tag{1.2} \]

and the null-Dirichlet boundary condition

\[ u(0, t) = 0, \tag{1.3} \]

or the null-Neumann boundary condition

\[ \partial_x u(0, t) = 0, \tag{1.4} \]

where \( v = v(x, t) > 0 \) is the specific volume, \( u = u(x, t) \) is the velocity and the pressure \( p(v) > 0 \) is a smooth function with \( p'(v) < 0 \) for any \( v \in \mathbb{R}^+ \). The external term \(-\frac{\alpha}{(1+t)^\lambda} u\) with physical coefficients \( \alpha > 0 \) and \( \lambda \geq 0 \), is called a time-depending damping. \( v_+ > 0 \) and \( u_+ \) are constant states.

The system (1.1) is not only a mathematical model of the wave equation with time-depending dissipation [23, 24]

\[ \omega_{tt} - \omega_{xx} + b(t)\omega_t = 0, \]

but it models the compressible flow through porous media with unsteady drag force. For more information about this model, see for instance [2] and references cited therein for related models.

When \( \alpha = 0 \), the system (1.1) reduces to the standard compressible Euler equations which is an extremely important equation to describe the motion of compressible ideal fluids. There have been many important developments and extensive studies on the Euler equations in the past few decades.

When \( \alpha > 0, \lambda = 0 \), the system (1.1) becomes the compressible Euler equations with constant coefficient damping. The global existence and large time behaviors of smooth solutions

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to the Cauchy problem or initial-boundary problem (1.1) have been investigated by many authors.

(i) For the Cauchy problem, the global existence of solution has been investigated by many authors (see [15] and references therein). Hsiao and Liu [6] firstly considered the large time behavior of solution. Precisely, they showed that the solution of (1.1) tended time-asymptotically to the nonlinear diffusion waves. And a better convergence rate was obtained by Nishihara [16]. In the case of large initial data, Zhao in [25] showed that for a certain class of given large initial data, the system (1.1) admits a unique global smooth solution and such a solution tends time-asymptotically to the strong diffusion wave. For other results, see [8, 9, 17, 22, 26] and some references therein.

(ii) For the initial-boundary value problem on a half line \( \mathbb{R}^+ \), we refer to [10, 13, 14, 18]. Precisely, Marcati and Mei in [13] studied the system (1.1) with boundary condition on \( v \) as follows:

\[
 v(0, t) = g(t), \quad t > 0.
\]

Nishihara and Yang in [18] considered the asymptotic behavior of solution to the system (1.1) with the Dirichlet boundary condition \( (1.3) \) or the Neumann boundary condition \( (1.4) \). More precisely, for the Dirichlet boundary condition, they got the global existence and convergence rates of \( (v - \bar{v}, u - \bar{u}) \) as \( x \to +\infty \), which satisfied

\[
 \left\{ \begin{array}{l}
 \partial_t \bar{v} - \partial_x \bar{u} = 0, \\
 p'(v_+) \partial_x \bar{v} = -\alpha \bar{u}, \\
 (\bar{v}, \bar{u})(0) = (\bar{v}_0, \bar{u}_0)(x) \to (v_+, 0), \quad \text{as} \quad x \to +\infty, \\
 \bar{u}(0, t) = 0 \quad (\text{or} \quad \partial_x \bar{v}(0, t) = 0), \\
 \bar{u}(\infty, t) = 0,
\end{array} \right.
\]

provided the initial perturbation belonged to \( H^3(\mathbb{R}^+) \times H^2(\mathbb{R}^+) \). In [14], for the Dirichlet boundary condition, Marcati, Mei and Rubino improved the convergence rates to \( \| (v - \bar{v}, u - \bar{u}) \|_{L^\infty} \leq C(t^{-1}, t^{-\frac{\lambda}{2}}) \) by perturbing the initial value around the nonlinear diffusion waves \( (\bar{v}, \bar{u})(x, t) \) which satisfied

\[
 \left\{ \begin{array}{l}
 \partial_t \bar{v} - \partial_x \bar{u} = 0, \\
 \partial_x p(\bar{v}) = -\alpha \bar{u}, \\
 (\bar{v}, \bar{u})(0) = (\bar{v}_0, \bar{u}_0)(x) \to (v_+, 0), \quad \text{as} \quad x \to +\infty, \\
 \bar{u}(0, t) = 0 \quad (\text{or} \quad \partial_x \bar{v}(0, t) = 0), \\
 \bar{u}(\infty, t) = 0,
\end{array} \right.
\]

when the initial perturbation additionally belonged to \( L^1(\mathbb{R}^+) \). Later, Jiang and Zhu in [10] obtained the same convergence rates as in [14] under a rather weaker small assumption on the initial disturbance. For the initial-boundary value problem on the bounded domain \([0, 1]\), we refer to [7]. For initial-boundary value problem to the compressible Euler equations with nonlinear damping, we refer to [11, 12] and some references therein.

When \( \alpha > 0, \lambda > 0 \), the system (1.1) is the compressible Euler equations with time-depending damping. The authors [3, 4] considered the global existence of smooth solutions
when \( u_0 \in C_0^\infty(\mathbb{R}^3) \) for \( 0 \leq \lambda \leq 1 \) in multi-dimensions. And they proved the solutions will blow up in finite time for \( \lambda > 1 \). For more results about this direction, we can refer to \([19, 20]\) and references cited therein. Recently, the authors \([1]\) considered the Cauchy problem for the system \((1.1)\) and proved that the solution time-asymptotically converged to the nonlinear diffusion waves with the initial perturbation around the diffusion wave in \( H^3(\mathbb{R}) \times H^2(\mathbb{R}) \). For more literature on this model, see \([5, 21]\) and the references therein.

However, to our knowledge, there are very few results on the large-time behavior of solutions near nonlinear diffusion waves to the initial-boundary value problem \((1.1)-(1.4)\) for \( 0 < \lambda < 1 \). It is very interesting and challenging to study this problem because it has more physical meanings and of course some new mathematical difficulties will arise due to the boundary effect and time-depending damping. In this paper, we will consider the initial-boundary value problem of \((1.1)\) on a half line \( \mathbb{R}^+ \) and obtain the convergence to diffusion waves for classical solution compared with previous results about Euler equations with constant coefficient damping.

Now, we give the main ideas used in deducing our results. In the case of Dirichlet boundary condition, the main difficulty of this paper lies in obtaining the decay rates of the diffusion waves. The general strategy is to construct the self-similar solution. However, in our diffusion waves system, the equations don’t possess self-similar solution. One possible way to get around these issues is to explicitly write out the solution by Green function as in Nishihara and Yang \([18]\). But, in our case, we cannot achieve the expected results because the diffusion waves is nonlinear. Another option would be to construct nonlinear diffusion wave by iteration. However, achieving the Green function of the Dirichlet type IBVP to the nonlinear diffusion waves share the same difficulty as directly solving the solution. Our strategy, inspired by the work of Jiang and Zhu \([11]\), is to employ extension the initial data to the real line and consider the corresponding linearized problem \((2.2.2)\). Based on some delicate energy estimates, we can get the decay rates of nonlinear diffusion waves \((\bar{v}, \bar{u})(x, t)\) indirectly. Because of the different decay rates of the diffusion waves, we obtain the cut-off point of the convergence rate is \( \lambda = \frac{3}{5} \) while the Cauchy problem \((1)\) is \( \lambda = \frac{1}{7} \). On the other hand, for the case of Neumann boundary condition, we construct the self-similar diffuse waves. Therefore, the cut-off point of the convergence rate is \( \lambda = \frac{1}{7} \) if \( v_0(0) \neq v_+ \). However, there is no cut-off point of the convergence rate if \( v_0(0) = v_+ \) because the diffusion waves are constant states. Finally, we also take full use of the weight function \((1 + t)^\beta\) to overcome the time-depending damping.

The rest of the paper is organized as follows. In Section 2, we derive the convergence in the case of Dirichlet boundary condition. In Section 2.1, the problem with null-Dirichlet boundary condition is reformulated and the main results will be stated. In Section 2.2, we will obtain the dissipative properties of the nonlinear diffusion waves \((\bar{v}, \bar{u})(x, t)\) in Section 2.3, the proofs of Theorem will be given, much of that is base on the papers \([1]\). In Section 3, we will study the null-Neumann boundary problem.

**Notations:** In the following, \( C \) and \( c(C_i, c_i) \) denote the generic positive constants depending only on the initial data and the physical coefficients \( \alpha, \lambda \), but independent of the time. For two quantities \( a \) and \( b \), \( a \sim b \) means \( \frac{1}{C} |b| \leq |a| \leq C |b| \) for a generic constant \( C \). \( \| \cdot \|_{L^p} \) and \( \| \cdot \|_{H^l} \) are denote by \( \| \cdot \|_{L^p(\mathbb{R}^+)} \), \( \| \cdot \|_{H^l(\mathbb{R}^+)} \), for \( 1 \leq p \leq \infty \), \( l \geq 0 \), respectively.
2 The case of Dirichlet boundary condition

2.1 Reformulation of the problem and main results

We firstly consider the problem \(1.1\)-\(1.2\) with the Dirichlet boundary condition \(1.3\).

Hints by \(1.1\) and the initial data \(1.2\), we suppose for any \(t \geq 0\)
\[u(x, t) \rightarrow u_+ \beta(t), \quad \text{as} \quad x \rightarrow +\infty,\]
where
\[
\beta(t) = \begin{cases} 
  e^{-\frac{\alpha}{1+\lambda}[(1+t)^{1-\lambda}-1]}, & \text{if } \lambda \in [0, 1), \\
  (1+t)^{-\alpha}, & \text{if } \lambda = 1.
\end{cases}
\]

Denote
\[B(t) = -\int_{t}^{\infty} \beta(\tau)d\tau.\]

From Darcy’s law and asymptotic analysis, it is well-known that the first term \(u_+\) of \(1.1\) decay to zero, as \(t \to \infty\), faster than the term \(-\frac{\alpha}{(1+t)^{\lambda}}u\) for some \(0 \leq \lambda < 1\). Therefore, we expect the solution \((v, u)(x, t)\) of \(1.1\)-\(1.3\) time-asymptotically behaves as the solutions \((\bar{v}, \bar{u})(x, t)\) of
\[
\begin{align*}
\partial_t \bar{v} - \partial_x \bar{u} &= 0, \\
\partial_x p(\bar{v}) &= -\frac{\alpha}{(1+t)^{\lambda}} \bar{u}, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
(\bar{v}, \bar{u}) |_{t=0} &= (\bar{v}_0, \bar{u}_0)(x) \rightarrow (v_+, 0), \quad \text{as} \quad x \rightarrow +\infty, \\
\bar{u}(0, t) &= 0 \quad (\text{or } \bar{v}_x(0, t) = 0), \quad \bar{u}(\infty, t) = 0,
\end{align*}
\]

where \(\bar{v}_0(x)\) satisfies
\[\bar{v}_0(x) > 0,\]
and
\[
\int_0^{\infty} (\bar{v}_0(y) - v_+) dy = \int_0^{\infty} (v_0(y) - v_+) dy - u_+ B(0). \tag{2.1.2}
\]

Therefore
\[
\int_0^{\infty} (v_0(y) - \bar{v}_0(y)) dy = u_+ B(0). \tag{2.1.3}
\]

Next, as in \[18\], we define a pair of correction functions
\[
\hat{v}(x, t) = u_+ m_0(x) B(t) \tag{2.1.4}
\]
and
\[
\hat{u}(x, t) = u_+ \beta(t) \int_0^x m_0(y) dy, \tag{2.1.5}
\]
where \(m_0(x)\) is a smooth function with compact support such that
\[
\int_0^{\infty} m_0(x) dx = 1, \quad \text{supp } m_0(x) \subset \mathbb{R}^+.\]
Therefore, \((\hat{v}, \hat{u})(x, t)\) satisfies
\[
\begin{cases}
\partial_t \hat{v} - \partial_x \hat{u} = 0, \\
\partial_t \hat{u} = -\frac{\alpha}{(1 + t)^\lambda} \hat{u}, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
\hat{u}(0, t) = 0, \quad (\hat{v}, \hat{u})(\infty, t) = (0, u_+(t)).
\end{cases}
\] (2.1.6)

Combining (1.1), (2.1.1) and (2.1.6), we have
\[
\begin{cases}
\partial_t (v - \bar{v} - \hat{v}) - \partial_x (u - \bar{u} - \hat{u}) = 0, \\
\partial_t (u - \bar{u} - \hat{u}) + \partial_x (p(v) - p(\bar{v})) + \partial_t \hat{u} + \frac{\alpha}{(1 + t)^\lambda} (u - \bar{u} - \hat{u}) = 0 \\
(u - \bar{u} - \hat{u})(0, t) = 0, \quad (u - \bar{u} - \hat{u})(\infty, t) = 0.
\end{cases}
\] (2.1.7)

By (2.1.3) and (2.1.7), the integration of (2.1.7) over \(\mathbb{R}^+ \times [0, t]\) yields
\[
\int_0^\infty (v - \bar{v} - \hat{v})dx = \int_0^\infty (v_0(x) - \bar{v}_0(x))dx - u_+B(0) = 0,
\]
and hence we reach the setting of perturbation
\[
\omega(x, t) = -\int_x^\infty (v(y, t) - \bar{v}(y, t) - \hat{v}(y, t))dy,
\] (2.1.8)
and
\[
z(x, t) = u(x, t) - \bar{u}(x, t) - \hat{u}(x, t). \quad (2.1.9)
\]

By (2.1.7), we have the reformulated problem
\[
\begin{cases}
\omega_t - z = 0, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
z_t + (p(\omega_x + \bar{v} + \hat{v}) - p(\bar{v}))_x + \frac{\alpha}{(1 + t)^\lambda} z = -\bar{u}_t, \\
\omega(0, t) = 0, \quad z(0, t) = 0.
\end{cases}
\] (2.1.10)

with initial data
\[
(\omega, z) \mid_{t=0} = (\omega_0, z_0)(x),
\]
where
\[
\begin{cases}
\omega_0(x) = -\int_x^\infty (v_0(y) - \bar{v}_0(y) - \hat{v}(y, 0))dy, \\
z_0(x) = u_0(x) - \bar{u}_0(x) - \hat{u}(x, 0).
\end{cases}
\]

Rewrite (2.1.10) and (2.1.11) as
\[
\begin{cases}
\omega_{tt} + (p'(\bar{v})\omega_x)_x + \frac{\alpha}{(1 + t)^\lambda} \omega_t = F, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
\omega(0, t) = 0, \quad \omega_t(0, t) = 0.
\end{cases}
\] (2.1.12)

with initial data
\[
(\omega, \omega_t) \mid_{t=0} = (\omega_0, z_0)(x),
\] (2.1.13)
where
\[
F = \frac{1}{\alpha} (1 + t)^\lambda p(\bar{v})_{xt} + \frac{\lambda}{\alpha} (1 + t)^{\lambda-1} p(\bar{v})_x - (p(\omega_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})\omega_x)_x. \quad (2.1.14)
\]
Theorem 2.1. (Dirichlet boundary for $0 \leq \lambda < \frac{3}{8}$). For $\alpha > 0$, suppose that $v_0(x) - v_+ \in L^1(\mathbb{R}^+)$, if we assume further that both $\delta = \|v_0 - v_+\|_{L^1(\mathbb{R}^+)} + |u_+| + \|V_0\|_{H^5(\mathbb{R})} + \|Z_0\|_{H^4(\mathbb{R})}$ and $\|\omega_0\|_{H^3(\mathbb{R}^+)} + \|z_0\|_{H^2(\mathbb{R}^+)}$ are sufficiently small. Then, there exists a unique time-global solution of the initial-boundary value problem (2.1.12), (2.1.13) satisfying

$$\omega \in C^k((0, \infty), H^{3-k}(\mathbb{R}^+)), \quad k = 0, 1, 2, 3,$$
$$\omega_t \in C^k((0, \infty), H^{2-k}(\mathbb{R}^+)), \quad k = 0, 1, 2,$$

furthermore, we have

$$\sum_{k=0}^{3}(1 + t)^{(\lambda+1)k} \|\partial_x^k \omega(\cdot, t)\|_{L^2}^2 + \sum_{k=0}^{2}(1 + t)^{(\lambda+1)k+2} \|\partial_x^k \omega_t(\cdot, t)\|_{L^2}^2$$

$$+ \int_0^t \left[ \sum_{j=1}^{3}(1 + s)^{(\lambda+1)j-1} \|\partial_x^j \omega(\cdot, s)\|_{L^2}^2 + \sum_{j=0}^{2}(1 + s)^{(\lambda+1)j+1} \|\partial_x^j \omega_t(\cdot, s)\|_{L^2}^2 \right] ds$$

$$\leq C(\|\omega_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta),$$

where the initial error function $(V_0, Z_0)(x)$ will be defined in (2.2.8).

Theorem 2.2. (Dirichlet boundary for $\frac{3}{8} < \lambda < 1$). For $\alpha > 0$, suppose $v_0(x) - v_+ \in L^1(\mathbb{R}^+)$, if we assume further that both $\delta = \|v_0 - v_+\|_{L^1(\mathbb{R}^+)} + |u_+| + \|V_0\|_{H^5(\mathbb{R})} + \|Z_0\|_{H^4(\mathbb{R})}$ and $\|\omega_0\|_{H^3(\mathbb{R}^+)} + \|z_0\|_{H^2(\mathbb{R}^+)}$ are sufficiently small. Then, there exists a unique time-global solution of the initial-boundary value problem (2.1.12), (2.1.13) satisfying

$$\omega \in C^k((0, \infty), H^{3-k}(\mathbb{R}^+)), \quad k = 0, 1, 2, 3,$$
$$\omega_t \in C^k((0, \infty), H^{2-k}(\mathbb{R}^+)), \quad k = 0, 1, 2,$$

furthermore, we have

$$\sum_{k=0}^{3}(1 + t)^{(\lambda+1)k+\frac{\beta}{2} - \frac{\alpha}{2}} \|\partial_x^k \omega(\cdot, t)\|_{L^2}^2 + \sum_{k=0}^{2}(1 + t)^{(\lambda+1)k+\frac{\beta}{2} - \frac{\alpha}{2}} \|\partial_x^k \omega_t(\cdot, t)\|_{L^2}^2$$

$$\leq C(\|\omega_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta),$$

and for any $\beta \in \left(\frac{3}{8} - \frac{\alpha}{2}, \lambda\right)$, we have

$$\int_0^t \left[ \sum_{j=0}^{3}(1 + s)^{(\lambda+1)(j-1)+\beta} \|\partial_x^j \omega(\cdot, s)\|_{L^2}^2 + \sum_{j=0}^{2}(1 + s)^{(\lambda+1)j+\beta-\lambda+1} \|\partial_x^j \omega_t(\cdot, s)\|_{L^2}^2 \right] ds$$

$$\leq C(1 + t)^{\beta + \frac{\alpha}{2} - \frac{3}{8}} (\|\omega_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta).$$

Theorem 2.3. (Dirichlet boundary for $\lambda = \frac{3}{7}$). For $\alpha > 0$, suppose $v_0(x) - v_+ \in L^1(\mathbb{R}^+)$, if we assume further that both $\delta = \|v_0 - v_+\|_{L^1(\mathbb{R}^+)} + |u_+| + \|V_0\|_{H^5(\mathbb{R})} + \|Z_0\|_{H^4(\mathbb{R})}$ and $\|\omega_0\|_{H^3(\mathbb{R}^+)} + \|z_0\|_{H^2(\mathbb{R}^+)}$ are sufficiently small. Then, there exists a unique time-global solution of the initial-boundary value problem (2.1.12), (2.1.13) satisfying

$$\omega \in C^k((0, \infty), H^{3-k}(\mathbb{R}^+)), \quad k = 0, 1, 2, 3,$$
\[ \omega_t \in C^k((0, \infty), H^{2-k}(\mathbb{R}^+)), \quad k = 0, 1, 2, \]

furthermore, for any sufficiently small \( \varepsilon > 0 \) we have

\[
\sum_{k=0}^{3}(1 + t)^{\frac{8k}{5}} \| \partial_x^k \omega(\cdot, t) \|^2_{L^2} + \sum_{k=0}^{2}(1 + t)^{\frac{8k}{5}+2} \| \partial_x^k \omega_t(\cdot, t) \|^2_{L^2} \\
+ \int_0^{t} \left[ \sum_{j=1}^{3}(1 + s)^{\frac{8k}{7}-1} \| \partial_x^j \omega(\cdot, s) \|^2_{L^2} + \sum_{j=0}^{2}(1 + s)^{\frac{8k}{5}+1} \| \partial_x^j \omega_t(\cdot, s) \|^2_{L^2} \right] ds \\
\leq C(1 + t)^{\varepsilon} (\| \omega_0 \|_{H^3} + \| z_0 \|_{H^2} + \delta).
\]

Notice that \( \omega_x = v - \bar{v} - \hat{\omega}, z = u - \bar{u} - \hat{u} \), and use Proposition 2.2 in the next subsection and Sobolev inequality, we immediately obtain the following convergence rates.

**Corollary 2.1.** Under the assumptions of Theorem 2.1-2.3, the system (1.1)-(1.3) possesses a uniquely global solution \((v, u)(x, t)\) satisfying

\[
\|(v - \bar{v})(\cdot, t)\|_{L^\infty(\mathbb{R}^+)} \leq \begin{cases} 
C(1 + t)^{-\frac{3(\lambda+1)}{4}}, & 0 \leq \lambda < \frac{3}{5}, \\
C(1 + t)^{-\frac{3}{4}+\varepsilon}, & \lambda = \frac{3}{5}, \\
C(1 + t)^{\frac{3}{2}}, & \frac{3}{5} < \lambda < 1,
\end{cases}
\]

and

\[
\|(u - \bar{u})(\cdot, t)\|_{L^\infty(\mathbb{R}^+)} \leq \begin{cases} 
C(1 + t)^{-\frac{\lambda+5}{4}}, & 0 \leq \lambda < \frac{3}{5}, \\
C(1 + t)^{-\frac{5}{8}+\varepsilon}, & \lambda = \frac{3}{5}, \\
C(1 + t)^{\lambda-2}, & \frac{3}{5} < \lambda < 1.
\end{cases}
\]

**Remark 2.1.** It should be noted that the cut-off point of the convergence rate in this paper is \( \lambda = \frac{3}{5} \), while the Cauchy problem in [11] is \( \lambda = \frac{1}{3} \). This is caused by the diffusion wave constructed in this paper is not be self-similar solution. It is worth pointing out that the time-depending damping could reveal more phenomena about the wave equation.

**Remark 2.2.** For the case of \( \lambda = 0 \), the convergence rates shown in Theorem 2.1-2.3 and Corollary 2.1 are the same as all existing convergence rates obtained in the previous works [10, 13, 14, 18].

### 2.2 Preliminaries

In this subsection, we will establish some fundamental dissipative properties of the solution \((\bar{v}, \bar{u})(x, t)\) to the system (2.1.1). The equations (2.1.1) can also be written as

\[
\begin{cases} 
\partial_t \bar{v} + \frac{(1 + t)^\lambda}{\alpha} \partial_{xx} p(\bar{v}) = 0, & (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
\bar{v} \big|_{t=0} = \bar{v}_0(x) \to v_+, & x \to +\infty, \\
\bar{v}_x(0, t) = 0, & \bar{v}(\infty, t) = v_+.
\end{cases}
\]
Let \((\bar{v}_0, \bar{u}_0)(x)\) denote the even and odd extensions of \((v_0, u_0)(x)\) in the whole space \(\mathbb{R}\), respectively, i.e.,

\[
\bar{v}_0(x) = \begin{cases} 
\bar{v}_0(x), & \text{if } x \geq 0, \\
\bar{v}_0(-x), & \text{if } x < 0,
\end{cases} \quad \bar{u}_0(x) = \begin{cases} 
\bar{u}_0(x), & \text{if } x \geq 0, \\
-\bar{u}_0(-x), & \text{if } x < 0.
\end{cases}
\]

Then, we study the properties of \((\bar{v}, \bar{u})(x, t)\) by investigating the following Cauchy problem:

\[
\begin{align*}
\partial_t \bar{v} - \partial_x \bar{u} &= 0, \\
\partial_t p(\bar{v}) - \frac{\alpha}{(1+t)^{\delta}} \bar{u}, & \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
(\bar{v}, \bar{u}) |_{t=0} &= (\bar{v}_0, \bar{u}_0)(x) \to (v_+, 0), \quad \text{as } x \to \pm \infty.
\end{align*}
\]  \(\tag{2.2.1}\)

Different from the Cauchy problem in \([1]\), the initial value problem \(\text{(2.2.1)}\) does not possess self-similar solution. Therefore, we can’t directly get the decay rates of \((\bar{v}, \bar{u})(x, t)\). Therefore, as in \([1]\), we study the corresponding linearized problem of \(\text{(2.2.1)}\) around \(v_+\)

\[
\begin{align*}
\partial_t \tilde{v} - \partial_x \tilde{u} &= 0, \\
p'(v_+) \partial_x \tilde{v} &= -\frac{\alpha}{(1+t)^{\delta}} \tilde{u}, & \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
\tilde{v}(x, 0) &= v_+ + \frac{\delta_0(\lambda+1)^2}{(4\kappa \pi)^2} e^{-\frac{(\lambda+1)x^2}{4\kappa}}, \\
\tilde{u}(x, 0) &= -\frac{2}{\kappa} \frac{\delta_0(\lambda+1)^2}{(4\kappa \pi)^2} e^{-\frac{(\lambda+1)x^2}{4\kappa}},
\end{align*}
\]  \(\tag{2.2.2}\)

where \(\kappa = -\frac{p'(v_+)}{\alpha} > 0\) and \(\delta_0 = 2 \int_0^\infty (\bar{v}_0(y) - v_+)dy\).

The solution \((\tilde{v}, \tilde{u})(x, t)\) of the Cauchy problem \(\text{(2.2.2)}\) can be written explicitly as

\[
\begin{align*}
\tilde{v}(x, t) &= v_+ + \frac{\delta_0(\lambda+1)^2}{(4\kappa \pi)^2 (1+t)^{\delta}} e^{-\frac{(\lambda+1)x^2}{4\kappa(1+t)^{\delta}}}, \\
\tilde{u}(x, t) &= \kappa(1+t)^{\lambda} \tilde{v}_2(x, t), & \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+.
\end{align*}
\]

By direct calculations, we have the following estimates.

**Lemma 2.1.** For each \(p \in [1, \infty]\), we know

\[
||\partial_t^k \partial_x^k (\tilde{v}(t) - \tilde{v}_+)||_{L^p(\mathbb{R})} \leq C|\delta_0|(1+t)^{-\frac{(\lambda+1)k}{2} + \frac{(\lambda+1)\delta}{2} - 1}, \quad k, l = 0, 1, 2, \ldots.
\]

Furthermore, for each \(p \in [1, \infty]\), let \(h(x, t) = -(p'(v_+) - p'(\bar{v}))\tilde{v}_x\), we have

\[
\int_{-\infty}^{\infty} |\partial_t^k \partial_x^k h(x, t)|^2 dx \leq C\delta_0^4 (1+t)^{-\frac{3(\lambda+1)}{4} - (\lambda+1)k-2l}, \quad k, l = 0, 1, 2, \ldots.
\]

Combining \(\text{(2.2.1)}\) and \(\text{(2.2.2)}\) leads to

\[
\begin{align*}
\partial_t (\bar{v} - \tilde{v}) - \partial_x (\bar{u} - \tilde{u}) &= 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
\partial_x (p(\bar{v}) - p(\tilde{v})) + \frac{\alpha}{(1+t)^{\lambda}}(\bar{u} - \tilde{u}) &= (p'(v_+) - p'(\bar{v}))\tilde{v}_x.
\end{align*}
\]  \(\tag{2.2.3}\)
Integrating (2.2.3) with respect to \(x\) and \(t\) over \(\mathbb{R} \times (0, t)\) and using (2.1.2), we can get
\[
\int_{-\infty}^{\infty} (\bar{v}(x, t) - \tilde{v}(x, t)) dx = \int_{-\infty}^{\infty} (\bar{v}_0(x) - \tilde{v}(x, 0)) dx
\]
\[
= 2 \int_{0}^{\infty} (\bar{v}_0(x) - \tilde{v}(x, 0)) dx
\]
\[
= 2 \int_{0}^{\infty} (\bar{v}_0 - v_+) dx - \delta_0
\]
\[
= 0.
\]
Hence we define the new variables
\[
V(x, t) = \int_{-\infty}^{x} (\bar{v}(y, t) - \tilde{v}(y, t)) dy,
\]
(2.2.4)
and
\[
Z(x, t) = \bar{u}(x, t) - \tilde{u}(x, t).
\]
(2.2.5)
By (2.2.3), we have the reformulated problem
\[
\begin{cases}
V_t - Z = 0, & (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
(p(V_x + \tilde{v}) - p(\tilde{v}))_x + \frac{\alpha}{(1 + t)^{\lambda}} Z = (p'(v_+) - p'(\tilde{v}))\tilde{v}_x.
\end{cases}
\]
(2.2.6)
The corresponding initial data are given by
\[
(V, Z) \big|_{t=0} = (V_0, Z_0)(x),
\]
(2.2.7)
where
\[
\begin{cases}
V_0(x) = \int_{-\infty}^{x} (\bar{v}_0(y) - \tilde{v}(y, 0)) dy, \\
Z_0(x) = \bar{u}_0(x) - \tilde{u}(x, 0).
\end{cases}
\]
(2.2.8)
Rewrite (2.2.6) and (2.2.7) as
\[
(p'(\tilde{v})V_x)_x + \frac{\alpha}{(1 + t)^{\lambda}} V_t = -F_1, & (x, t) \in \mathbb{R} \times \mathbb{R}^+,
\]
(2.2.9)
with initial data
\[
(V, V_t) \big|_{t=0} = (V_0, Z_0)(x),
\]
(2.2.10)
where
\[
F_1 = h(x, t) + (p(V_x + \tilde{v}) - p(\tilde{v}) - p'(\tilde{v})V_x)_x.
\]
(2.2.11)
Noticing that, by (2.1.2) and the definition of \(\delta_0\), we have
\[
|\delta_0| \leq C(\|v_0 - v_+\|_{L^1(\mathbb{R}^+)} + |u_+|).
\]
(2.2.12)
In order to get the dissipative properties of \((\bar{v}, \bar{u})(x, t)\), now we investigate the properties of \((V, Z)(x, t)\). In fact, we could obtain the following theorem.
Proof. First, multiplying (2.2.9) by \((1 + t)\) gives
\[
\sum_{k=0}^{3} (1 + t)^{\lambda+k} \|\partial_x^{k} V(\cdot, t)\|_{L^2}^2 + \sum_{k=0}^{2} (1 + t)^{\lambda+k+2} \|\partial_x^{k} V_{t}(\cdot, t)\|_{L^2}^2 \leq C(\|V_{0}\|_{H^3}^2 + \|Z_{0}\|_{H^2}^2 + |\delta_0|^2),
\]
and
\[
(1 + t)^{\lambda} \|Z_{t}(\cdot, t)\|_{L^2}^2 + (1 + t)^{\lambda+5} \|Z_{xt}(\cdot, t)\|_{L^2}^2 \leq C(\|V_{0}\|_{H^3}^2 + \|Z_{0}\|_{H^2}^2 + |\delta_0|^2).
\]

Notations: For the sake of simplicity, throughout this subsection, we denote \(\| \cdot \|_{L^p} := \| \cdot \|_{L^p(\mathbb{R})}, 1 \leq p \leq \infty, \| \cdot \| := \| \cdot \|_{L^2(\mathbb{R})}, \| \cdot \|_{H^l} := \| \cdot \|_{H^l(\mathbb{R})}, l \geq 0,\) and we also use
\[
\int_{\mathbb{R}} f(\xi) \, d\xi \equiv \int_{\mathbb{R}} f(x) \, dx.
\]

Now, we begin to estimate the solution \((V, Z)(x, t), 0 < t < T < \infty,\) to the Cauchy problem (2.2.9)-(2.2.11) under the \textit{a priori} assumption
\[
N_1(T) := \sup_{0 < t < T} \left\{ \|V_{x}(\cdot, t)\|_{L^\infty} + (1 + t)\|V_{xt}(\cdot, t)\|_{L^\infty} + (1 + t)^{\lambda+1 \over 2} \|V_{xx}(\cdot, t)\|_{L^\infty} \right\} \leq \epsilon \quad (2.2.13)
\]
for some \(0 < \epsilon \ll 1.\)

Then we will establish some necessary \textit{a priori} bounds for \((V, Z).\) The first result is the lower order energy estimates.

**Lemma 2.2.** Under the assumptions of Theorem 2.4, if \(\epsilon, |\delta_0|\) are small, it holds that
\[
\|V\|^2 + (1 + t)^{\lambda+1} \|V_{x}\|^2 + \int_{0}^{t} \left\{ (1 + s)^{\lambda} \|V_{t}\|^2 + (1 + s)^{\lambda} \|V_{x}\|^2 \right\} ds \leq C(\|V_{0}\|_{H^1}^2 + |\delta_0|^2).
\]

**Proof.** First, multiplying (2.2.9) by \((1 + t)^{\lambda}V,\) integrating the resulting equality with respect to \(x\) over \(\mathbb{R}\) give
\[
{d \over dt} \int_{\mathbb{R}} {\alpha \over 2} V^2 \, dx - \int (1 + t)^{\lambda} p'(\bar{v}) V_x^2 \, dx = - \int (1 + t)^{\lambda} F_1 V \, dx. \tag{2.2.14}
\]
Then, to estimate the last term in the right hand of (2.2.14), one obtains that
\[
- \int (1 + t)^{\lambda} F_1 V \, dx
\]
\[
= - \int (1 + t)^{\lambda} h(x, t) V \, dx + \int (1 + t)^{\lambda} \left( p(V_{x} + \bar{v}) - p(\bar{v}) - p'(\bar{v}) V_{x} \right) V \, dx
\]
\[
\leq C \int (1 + t)^{-\kappa} V^2 \, dx + \int (1 + t)^{2\lambda+\kappa} h^2 \, dx + \int (1 + t)^{\lambda} p''(\theta_{1} V_{x} + \bar{v}) V_{x}^3 \, dx
\]
\[
\leq C \int (1 + t)^{-\kappa} V^2 \, dx + C|\delta_0|^2 (1 + t)^{\kappa-\frac{1}{2}-\frac{3}{2}} + C\epsilon \int (1 + t)^{\lambda} V_{x}^2 \, dx, \tag{2.2.15}
\]
where $0 < \theta_1 < 1$.

Substituting (2.2.15) into (2.2.14), using the smallness of $\epsilon$ and for some positive constant $C_0$ satisfying $-p'(\tilde{v}) \geq C_0 > 0$, we have

$$\frac{d}{dt} \int \frac{\alpha}{2} V^2 dx + \frac{C_0}{2} \int (1 + t)^{\lambda} V_x^2 dx \leq C \int (1 + t)^{-\kappa} V^2 dx + C|\delta_0|^2 (1 + t)^{\kappa - \frac{1}{2} - \frac{5}{2}}.$$  

Then taking $1 < \kappa < \frac{1}{2} + \frac{3}{2}$ and using Gronwall's inequality lead to

$$\|V\|^2 + \int_0^t (1 + s)^{\lambda} \|V_x\|^2 ds \leq C (\|V_0\|^2 + |\delta_0|^2). \quad (2.2.16)$$

Next, multiplying (2.2.9) by $(1 + t)^{2\lambda} V_t$, and integrating the resulting equality over $\mathbb{R}$, one yields

$$-\frac{1}{2} \frac{d}{dt} \int (1 + t)^{2\lambda} p'(\tilde{v}) V_x^2 dx + \int \alpha (1 + t)^{\lambda} V_t^2 dx$$

$$= -\frac{1}{2} \int (1 + t)^{2\lambda} p''(\tilde{v}) \tilde{v}_x V_x^2 dx - \lambda \int (1 + t)^{2\lambda - 1} p'(\tilde{v}) V_x^2 dx - \int (1 + t)^{2\lambda} F_1 V_t dx$$

$$\leq C \int (1 + t)^{2\lambda - 1} V_x^2 dx - \int (1 + t)^{2\lambda} F_1 V_t dx. \quad (2.2.17)$$

Now we estimate the last term in the right hand of (2.2.17) as follows:

$$-\int (1 + t)^{2\lambda} F_1 V_t dx = -\int (1 + t)^{2\lambda} h(x, t) V_t dx$$

$$+ \int (1 + t)^{2\lambda} (p(V_x + \tilde{v}) - p(\tilde{v}) - p'(\tilde{v}) V_x) V_x dx. \quad (2.2.18)$$

Firstly, applying Lemma 2.1, one gets

$$\int (1 + t)^{2\lambda} h(x, t) V_t dx$$

$$\leq \frac{\alpha}{2} \int (1 + t)^{\lambda} V_t^2 dx + C \int (1 + t)^{3\lambda} h^2 dx$$

$$\leq \frac{\alpha}{2} \int (1 + t)^{\lambda} V_t^2 dx + C|\delta_0|^2 (1 + t)^{\frac{3}{2} - \frac{5}{2}}. \quad (2.2.19)$$
Hence we complete the proof of Lemma 2.2. 

Proof. Differentiating (2.2.9) with respect to 

\[ \frac{d}{dt} (\int_{\tilde{\theta}}^{1} p(s)ds - p(\tilde{\varphi})V_{x} - \frac{p'(\tilde{\varphi})}{2}V_{x}^2) \]

Since \( \int_{\tilde{\theta}}^{1} p(s)ds = p(\tilde{\theta})V_{x} + \frac{1}{2}p'(\theta_2 V_{x} + \tilde{\varphi})V_{x}^2 \),

where \( 0 < \theta_2 < 1 \). Then, putting (2.2.18)-(2.2.20) into (2.2.17), we have

\[
\leq \frac{d}{dt} \int (1 + t)^{2\lambda} \left( \int_{\tilde{\theta}}^{1} p(s)ds - p(\tilde{\varphi})V_{x} - \frac{p'(\tilde{\varphi})}{2}V_{x}^2 \right) dx
\]

\[
- 2\lambda \int (1 + t)^{2\lambda-1} \left( \int_{\tilde{\theta}}^{1} p(s)ds - p(\tilde{\varphi})V_{x} - \frac{p'(\tilde{\varphi})}{2}V_{x}^2 \right) dx
\]

\[
+ C|\delta_0| \int (1 + t)^{\frac{3\lambda}{2}-\frac{\lambda}{2}}|V_x|^3 dx
\]

\[
\leq \frac{d}{dt} \int (1 + t)^{2\lambda} \left( \int_{\tilde{\theta}}^{1} p(s)ds - p(\tilde{\varphi})V_{x} - \frac{p'(\tilde{\varphi})}{2}V_{x}^2 \right) dx
\]

\[
+ C(|\delta_0| + \epsilon) \int (1 + t)^{2\lambda-1}V_{x}^2 dx. \tag{2.2.20}
\]

Since

\[
\int_{\tilde{\theta}}^{1} p(s)ds = p(\tilde{\theta})V_{x} + \frac{1}{2}p'(\theta_2 V_{x} + \tilde{\varphi})V_{x}^2 ,
\]

where \( 0 < \theta_2 < 1 \). Then, putting (2.2.18)-(2.2.20) into (2.2.17), we have

\[
- \frac{1}{2} \frac{d}{dt} \int (1 + t)^{2\lambda} p'(\theta_2 V_{x} + \tilde{\varphi})V_{x}^2 dx + \frac{\alpha}{2} \int (1 + t)^{\lambda} V_{x}^2 dx
\]

\[
\leq C \int (1 + t)^{2\lambda-1}V_{x}^2 dx + C|\delta_0|^2(1 + t)^{\frac{3\lambda}{2}-\frac{\lambda}{2}}. \tag{2.2.21}
\]

It follows from (2.2.16) and \( 0 \leq \lambda < 1 \) that

\[
(1 + t)^{2\lambda}||V_x||^2 + \int_0^t (1 + s)^{\lambda}||V_{s}||^2 ds \leq C(||V_0||^2_{H^1} + ||\delta_0||^2).
\]

Finally, multiplying (2.2.21) by \( (1 + t)^{1-\lambda} \), and using (2.2.16) again, we know

\[
(1 + t)^{\lambda+1}||V_x||^2 + \int_0^t (1 + s)||V_{s}||^2 ds \leq C(||V_0||^2_{H^1} + ||\delta_0||^2).
\]

Hence we complete the proof of Lemma 2.2. \( \square \)

Lemma 2.3. Under the assumptions of Theorem 2.4, if \( \epsilon, |\delta_0| \) are small, it holds that

\[
(1 + t)^{2\lambda+2}||V_{xx}||^2 + \int_0^t \left( (1 + s)^{\lambda+2}||V_{xt}||^2 + (1 + s)^{2\lambda+1}||V_{xxt}||^2 \right) ds \leq C(||V_0||^2_{H^2} + ||\delta_0||^2).
\]

Proof. Differentiating (2.2.39) with respect to \( x \), one yields

\[
(p'(\tilde{\varphi})V_{xx} + \frac{\alpha}{1 + t}V_{x}^2 = -F_{1x}. \tag{2.2.22}
\]
We utilize Cauchy-Schwarz's inequality and Lemma 2.1 to addres the following estimates:

Now we turn to estimate \( I_1 \)

\[
-I_1 \leq C \int (1 + t)^{2\lambda - 1} V_{xx}^2 dx,
\]

\( I_2 \leq C|\delta_0| \int (1 + t)^{3\lambda + \frac{3}{2}} V_{xx}^2 dx,
\]

\[
I_3 \leq \frac{\alpha}{16} \int (1 + t)^{4\lambda} V_{xt}^2 dx + C \int (1 + t)^{3\lambda} |x|^{4} V_x^2 dx
\]

\[
\leq \frac{\alpha}{16} \int (1 + t)^{4\lambda} V_{xt}^2 dx + C|\delta_0|^2 \int (1 + t)^{-4} V_x^2 dx,
\]

\[
I_4 \leq \frac{\alpha}{16} \int (1 + t)^{4\lambda} V_{xt}^2 dx + C \int (1 + t)^{3\lambda} |x|^{2} V_x^2 dx
\]

\[
\leq \frac{\alpha}{16} \int (1 + t)^{4\lambda} V_{xt}^2 dx + C|\delta_0|^2 \int (1 + t)^{-3} V_x^2 dx,
\]

and

\[
I_5 \leq \frac{\alpha}{16} \int (1 + t)^{4\lambda} V_{xt}^2 dx + C \int (1 + t)^{3\lambda} |x|^{2} V_x^2 dx
\]

\[
\leq \frac{\alpha}{16} \int (1 + t)^{4\lambda} V_{xt}^2 dx + C|\delta_0|^2 \int (1 + t)^{-2} V_x^2 dx.
\]

Now we turn to estimate \( I_6 \) as follows:

\[
I_6 = \int - (1 + t)^{2\lambda} h_x - (1 + t)^{2\lambda} (pV_x + \bar{v}) - p(\bar{v}) - p'(\bar{v}) V_x xx \] \( V_{xt} dx.
\]

By employing Lemmas 2.1 and the a priori assumption (2.2.13), we can get

\[
\int (1 + t)^{2\lambda} h_x V_{xt} dx
\]

\[
\leq \frac{\alpha}{16} \int (1 + t)^{4\lambda} V_{xt}^2 dx + C \int (1 + t)^{3\lambda} h_x^2 dx
\]

\[
\leq \frac{\alpha}{16} \int (1 + t)^{4\lambda} V_{xt}^2 dx + C|\delta_0|^2 (1 + t)^{-\frac{1}{2} - \frac{7}{2}},
\]
and

\[- \int (1 + t)^{2\lambda} \left( p(V_x + \tilde{v}) - p(\tilde{v}) - p'(\tilde{v})V_x \right) V_{xx}dx \]

\[= \int (1 + t)^{2\lambda} \left( p(V_x + \tilde{v}) - p(\tilde{v}) - p'(\tilde{v})V_x \right) V_{xx}dx \]

\[= (1 + t)^{2\lambda} \frac{1}{2} \frac{d}{dt} \int \left( p'(V_x + \tilde{v}) - p'(\tilde{v}) \right) V_{xx}^2 dx \]

\[\leq \frac{1}{2} \int (1 + t)^{2\lambda} \left( p''(V_x + \tilde{v})(V_{xx} + \tilde{v}_t) - p''(\tilde{v})\tilde{v}_t \right) V_{xx}^2 dx \]

\[+ \int (1 + t)^{2\lambda} \tilde{v} V_{xx} \left( p'(V_x + \tilde{v}) - p'(\tilde{v}) - p''(\tilde{v})V_x \right) dx \]

\[\leq \frac{1}{2} \frac{d}{dt} \int (1 + t)^{2\lambda} \left( p'(V_x + \tilde{v}) - p'(\tilde{v}) \right) V_{xx}^2 dx \]

\[\leq C \int (1 + t)^{2\lambda} -1 V_{xx}^2 dx + C|\delta_0|^2 \int (1 + t)^{-3} V_{xx}^2 dx + C|\delta_0|^2 (1 + t)^{-\frac{3}{2} - \frac{7}{2}} \]

\[\leq C \int (1 + t)^{2\lambda} -1 V_{xx}^2 dx + C(1 + t)^{-\frac{3}{4} - \frac{7}{2} \frac{3}{2} + |\delta_0|^2}. \]

Substituting \[2.2.24]-\[2.2.31] into \[2.2.23]\], using Lemma 2.2, \(0 \leq \lambda < 1\) and taking \(\epsilon, |\delta_0|\) sufficiently small, we derive

\[- \frac{1}{2} \frac{d}{dt} \int (1 + t)^{2\lambda} p'(V_x + \tilde{v})V_{xx}^2 dx + \frac{\alpha}{2} \int (1 + t)^{-1} V_{xx}^2 dx \]

\[\leq C \int (1 + t)^{2\lambda} -1 V_{xx}^2 dx + C|\delta_0|^2 \int (1 + t)^{-3} V_{xx}^2 dx + C|\delta_0|^2 (1 + t)^{-\frac{3}{2} - \frac{7}{2}} \]

\[\leq C \int (1 + t)^{2\lambda} -1 V_{xx}^2 dx + C(1 + t)^{-\frac{3}{4} - \frac{7}{2} \frac{3}{2} + |\delta_0|^2}. \]

Next, multiplying \(2.2.9\) by \(-(1 + t)^{\lambda} V_{xx}\), integrating the resulting equality with respect to \(x\) over \(\mathbb{R}\), and using \(0 \leq \lambda < 1\), we obtain

\[\frac{d}{dt} \int \frac{\alpha}{2} V_{xx}^2 dx - \int (1 + t)^{\lambda} p'(\tilde{v})V_{xx}^2 dx = \int (1 + t)^{\lambda} p''(\tilde{v})\tilde{v}_x V_{xx} dx + \int (1 + t)^{\lambda} F_1 V_{xx} dx. \]

(2.2.33)

By Lemmas 2.1, 2.2, it is easy to see that

\[\int (1 + t)^{\lambda} p''(\tilde{v})\tilde{v}_x V_{xx} dx \leq \frac{C_0}{10} \int (1 + t)^{\lambda} V_{xx}^2 dx + C \int (1 + t)^{\lambda} |\tilde{v}_x|^2 V_{xx}^2 dx \]

\[\leq \frac{C_0}{10} \int (1 + t)^{\lambda} V_{xx}^2 dx + C|\delta_0|^2 \int (1 + t)^{-2} V_{xx}^2 dx \]

\[\leq \frac{C_0}{10} \int (1 + t)^{\lambda} V_{xx}^2 dx + C(1 + t)^{-2\lambda - 3} (\|V_0\|^2_{H^1} + |\delta_0|^2). \]

(2.2.34)

From \(2.2.11\), one has

\[\int (1 + t)^{\lambda} F_1 V_{xx} dx = \int \left[ (1 + t)^{\lambda} h + (1 + t)^{\lambda} \left( p(V_x + \tilde{v}) - p(\tilde{v}) - p'(\tilde{v})V_x \right) \right] V_{xx} dx. \]

(2.2.35)
By using Lemmas 2.4 and the a priori assumption (2.2.15), we have
\[ \int (1 + t)^\lambda h(x, t)V_{xx}^2 \, dx \leq \frac{C_0}{10} \int (1 + t)^\lambda V_{xx}^2 \, dx + C \int (1 + t)^\lambda h^2 \, dx. \]
\[ \leq \frac{C_0}{10} \int (1 + t)^\lambda V_{xx}^2 \, dx + C|\delta_0|^2(1 + t)^{-\frac{3\lambda}{2} - \frac{5}{4}}, \]  
(2.2.36)
and
\[ \int (1 + t)^\lambda \left( p(V_x + \tilde{v}) - p(\tilde{v}) \right) V_{xx}^2 \, dx \]
\[ = \int (1 + t)^\lambda \left( p'(V_x + \tilde{v}) - p'(\tilde{v}) \right) V_{xx}^2 \, dx \]
\[ + \int (1 + t)^\lambda \left( p'(V_x + \tilde{v}) - p'(\tilde{v}) \right) V_{xx}^2 \, dx \]
\[ \leq C \int (1 + t)^\lambda |V_x|^2 V_{xx}^2 \, dx + \frac{C_0}{10} \int (1 + t)^\lambda V_{xx}^2 \, dx + C \int (1 + t)^\lambda |\tilde{v}_x|^2 |V_x|^4 \, dx \]
\[ \leq C\varepsilon \int (1 + t)^\lambda V_{xx}^2 \, dx + \frac{C_0}{10} \int (1 + t)^\lambda V_{xx}^2 \, dx + C\varepsilon^2 |\delta_0| \int (1 + t)^{-\lambda - 2} V_{xx}^2 \, dx \]
\[ \leq C\varepsilon \int (1 + t)^\lambda V_{xx}^2 \, dx + \frac{C_0}{10} \int (1 + t)^\lambda V_{xx}^2 \, dx + C(1 + t)^{-2\lambda - 3}(|V_0|^2_{H1} + |\delta_0|^2). \]  
(2.2.37)
Substituting (2.2.34)-(2.2.37) into (2.2.33), and 0 ≤ λ < 1 and the smallness of ε, |δ_0|, we have
\[ \frac{d}{dt} \int \frac{\alpha}{2} V_x^2 \, dx + \frac{C_0}{2} \int (1 + t)^\lambda V_{xx}^2 \, dx \leq C(1 + t)^{-\frac{3\lambda}{2} - \frac{5}{4}}(|V_0|^2_{H1} + |\delta_0|^2), \]  
(2.2.38)
which together with (2.2.32) yields
\[ (1 + t)^{2\lambda} \|V_{xx}\|^2 + \int_0^t (1 + s)^\lambda (\|V_{xx}\|^2 + \|V_{xx}\|^2) \, ds \leq C(|V_0|^2_{H2} + |\delta_0|^2). \]
Finally, performing (2.2.38)×(1 + t)^{1+\lambda}, (2.2.32)×(1 + t)^2, respectively, and using the above inequality, we complete the proof of Lemma 2.3.

Similar to the proof of Lemma 2.3, performing (2.2.9)_{xx} × (1 + t)^{3\lambda}V_{xxt}, (2.2.9)_{xx} × (1 + t)^{\lambda}(-V_{xxx}), integrating the resulting equations with respect to x over \mathbb{R}, we can get the following lemma:

**Lemma 2.4.** Under the assumptions of Theorem 2.4, if ε, |δ_0| are small, it holds that
\[ (1 + t)^{3\lambda + 3} \|V_{xxx}\|^2 + \int_0^t \left( (1 + s)^{2\lambda + 3} \|V_{xxt}\|^2 + (1 + s)^{3\lambda + 2} \|V_{xxx}\|^2 \right) \, ds \leq C(|V_0|^2_{H3} + |\delta_0|^2). \]

**Lemma 2.5.** Under the assumptions of Theorem 2.4, if ε, |δ_0| are small, it holds that
\[ (1 + t)^2 \|Z\|^2 + (1 + t)^{\lambda + 3} \|Z_x\|^2 + \int_0^t \left( (1 + s)^3 \|Z_t\|^2 + (1 + s)^{\lambda + 2} \|Z_x\|^2 \right) \, ds \]
\[ \leq C(|V_0|^2_{H2} + \|Z_0\|^2_{H1} + |\delta_0|^2). \]  
(2.2.39)
Proof. Differentiating (2.2.30) with respect to \(t\) leads to
\[
(p'(\bar{v})V_x)_{xt} + \frac{\alpha}{(1 + t)\lambda} Z_t = \frac{\alpha \lambda}{(1 + t)^{\lambda+1}} Z - F_{1t}.
\]
(2.2.40)

Multiplying (2.2.40) by \((1 + t)^\lambda Z\), from Lemma 2.2 and after complicated calculations, we know
\[
\frac{d}{dt} \int \frac{\alpha}{2} Z^2 dx + \frac{C_0}{2} \int (1 + t)^\lambda Z_t^2 dx
\]
\[
\leq C \int (1 + t)^{-1} Z^2 dx + C|\delta_0|^2 \int (1 + t)^{-3} V_x^2 dx + C|\delta_0|^2 (1 + t)^{-\frac{1}{2} - \frac{\lambda}{2}}
\]
\[
\leq C \int (1 + t)^{-1} Z^2 dx + C (1 + t)^{-\frac{1}{2} - \frac{\lambda}{2}} (\|V_0\|_{H^1} + |\delta_0|^2).
\]
(2.2.41)

Integrating \((2.2.41) \times (1 + t)^2\), and using Lemma 2.2, we get
\[
(1 + t)^2 \|Z\|^2 + \int_0^t (1 + s)^{\lambda+2} \|Z_x\|^2 ds \leq C (\|V_0\|^2_{H^1} + \|Z_0\|^2 + |\delta_0|^2).
\]

Similarly, multiplying (2.2.40) by \((1 + t)^{2\lambda} Z_t\), integrating the resulting equality in \(x\) over \(\mathbb{R}\), using Lemmas 2.2-2.4 the a priori assumption (2.2.3), and after tedious calculations, we have
\[
- \frac{1}{2} \frac{d}{dt} \int (1 + t)^{2\lambda} p'(V_x + \bar{v}) Z_x^2 dx + \frac{\alpha}{2} \int (1 + t)^{\lambda} Z_t^2 dx
\]
\[
\leq C \int (1 + t)^{\lambda-2} Z^2 dx + C \int (1 + t)^{2\lambda-1} Z_x^2 dx + C|\delta_0|^2 \int (1 + t)^{\lambda-4} V_x^2 dx
\]
\[
+ C|\delta_0|^2 \int (1 + t)^{2\lambda-3} V_{xx}^2 dx + C|\delta_0|^2 (1 + t)^{\lambda - \frac{9}{2}}
\]
\[
\leq C \int (1 + t)^{\lambda-2} Z^2 dx + C \int (1 + t)^{2\lambda-1} Z_x^2 dx + C (1 + t)^{\frac{\lambda}{2} - \frac{9}{2}} (\|V_0\|^2_{H^1} + \|Z_0\|^2_{H^1} + |\delta_0|^2).
\]
(2.2.42)

Integrating (2.2.42) \times (1 + t)^{3-\lambda}, and using Lemma 2.2-2.4, we complete the proof of Lemma 2.5.

Similar to the proof of Lemma 2.5, performing \(\text{(2.2.40)}_{tx} \times (1 + t)^{\lambda} Z_x, \text{(2.2.40)}_{tx} \times (1 + t)^{2\lambda} Z_{xt}\) integrating the resulting equations with respect to \(x\) over \(\mathbb{R}\), we can get the following estimate.

Lemma 2.6. Under the assumptions of Theorem 2.4, if \(\epsilon, |\delta_0|\) are small, it holds that
\[
(1 + t)^{2\lambda+4} \|Z_{xx}\|^2 + \int_0^t \left( (1 + s)^{\lambda+4} \|Z_{xt}\|^2 + (1 + s)^{2\lambda+3} \|Z_{xx}\|^2 \right) ds
\]
\[
\leq C (\|V_0\|^2_{H^3} + \|Z_0\|^2_{H^2} + |\delta_0|^2).
\]

Lemma 2.7. Under the assumptions of Theorem 2.4, if \(\epsilon, |\delta_0|\) are small, it holds that
\[
(1 + t)^4 \|Z_t\|^2 + (1 + t)^{\lambda+5} \|Z_{xt}\|^2 + \int_0^t \left( (1 + s)^{\lambda+4} \|Z_{xt}\|^2 + (1 + s)^5 \|Z_{tt}\|^2 \right) ds
\]
\[
\leq C (\|V_0\|^2_{H^3} + \|Z_0\|^2_{H^2} + |\delta_0|^2).
\]
Proof. Differentiating (2.2.40) with respect to $t$, we get

$$(p' (\tilde{v}) V_x)_{xtt} + \frac{\alpha}{(1 + t)^{\lambda}} Z_{ttt} + \frac{\alpha \lambda (\lambda + 1)}{(1 + t)^{\lambda + 2}} Z = \frac{2 \alpha \lambda}{(1 + t)^{\lambda + 1}} Z_t - F_{ttt}. \quad (2.2.43)$$

Multiplying $\text{(2.2.43)}$ by $(1 + t)^{\lambda} Z_t$, from Lemma 2.5 and after complicated calculations, we have

$$\frac{d}{dt} \int \left( \frac{\alpha}{2} Z_t^2 + \frac{\alpha \lambda (\lambda + 1)}{2} (1 + t)^{-2} Z^2 \right) dx + \frac{C_0}{2} \int (1 + t)^{\lambda} Z_{xt}^2 dx \leq C \int (1 + t)^{-1} Z_t^2 dx + C \int (1 + t)^{-3} Z^2 dx + C |\delta_0|^2 \int (1 + t)^{-5} Z_x^2 dx + C |\delta_0|^2 \int (1 + t)^{-2} Z_x^2 dx$$

$$+ C(1 + t)^{-\frac{\lambda}{2} - \frac{11}{2}} \left( \|V_0\|_{H^2}^2 + \|Z_0\|_{H^1}^2 + |\delta_0|^2 \right). \quad (2.2.44)$$

Integrating $\text{(2.2.44)} \times (1 + t)^4$, and using Lemmas 2.5, 2.6, we get

$$(1 + t)^4 \|Z_t\|^2 + \int_0^t (1 + s)^{\lambda + 4} \|Z_{xt} \|^2 ds \leq C \left( \|V_0\|_{H^3}^2 + \|Z_0\|_{H^2}^2 + |\delta_0|^2 \right).$$

Similarly, multiplying $\text{(2.2.43)}$ by $(1 + t)^{2\lambda} Z_{tt}$, integrating the resulting equality in $x$ over $\mathbb{R}$, using Lemmas 2.2, 2.6, and after tedious calculations, we have

$$\frac{1}{2} \frac{d}{dt} \int (1 + t)^{2\lambda} p'(V_x + \tilde{v}) Z_{xt}^2 dx + \frac{\alpha}{2} \int (1 + t)^{\lambda} Z_{ttt}^2 dx \leq C \int (1 + t)^{\lambda - 2} Z_t^2 dx + C \int (1 + t)^{2\lambda - 1} Z_{xt}^2 dx$$

$$+ C |\delta_0|^2 \int (1 + t)^{-2} V_x^2 dx + C |\delta_0|^2 \int (1 + t)^{-4} V_{xt}^2 dx + C |\delta_0|^2 \int (1 + t)^{-2} Z_{xt}^2 dx$$

$$+ C |\delta_0|^2 \int (1 + t)^{-2} V_{xx}^2 dx + C |\delta_0|^2 (1 + t)^{\frac{\lambda}{2} - \frac{11}{2}}$$

$$\leq C \int (1 + t)^{\lambda - 1} Z_t^2 dx + C \int (1 + t)^{\lambda - 2} Z_t^2 dx + C \int (1 + t)^{2\lambda - 1} Z_{xt}^2 dx$$

$$+ C(1 + t)^{\frac{\lambda}{2} - \frac{13}{2}} \left( \|V_0\|_{H^2}^2 + \|Z_0\|_{H^1}^2 + |\delta_0|^2 \right). \quad (2.2.45)$$

Integrating $\text{(2.2.45)} \times (1 + t)^{5 - \lambda}$, and using Lemmas 2.3, 2.6, we complete the proof of Lemma 2.7. \hfill \Box

With Lemmas 2.2, 2.7 in hand, by the Sobolev inequality and $0 \leq \lambda < 1$, it’s easy to know that

$$\|V_x (\cdot, t)\|_{L^\infty} \leq C (1 + t)^{-\frac{3(\lambda + 1)}{4}} \left( \|V_0\|_{H^3} + \|Z_0\|_{H^2} + |\delta_0| \right) \leq \frac{\epsilon}{2},$$

$$\|V_{xt} (\cdot, t)\|_{L^\infty} \leq C (1 + t)^{-\frac{3(\lambda + 7)}{4}} \left( \|V_0\|_{H^3} + \|Z_0\|_{H^2} + |\delta_0| \right) \leq \frac{\epsilon}{2} (1 + t)^{-1},$$
and

$$\|V_{xx}(\cdot,t)\|_{L^\infty} \leq C(1 + t)^{-\frac{5\lambda + 5}{4}} (\|V_0\|_{H^3} + \|Z_0\|_{H^2} + |\delta_0|) \leq \frac{\epsilon}{2} (1 + t)^{-\frac{\lambda + 1}{2}},$$

provided $\|V_0\|_{H^3} + \|Z_0\|_{H^2} + |\delta_0| \ll 1$. Up to now, we thus close the a priori assumption (2.2.1) about $(V_x, V_{xt}, V_{xx})$ from Lemmas 2.1, 2.2, 2.6. The proof of Theorem 2.4 is completed.

Furthermore, if $V_0 \in H^5(\mathbb{R})$, $Z_0 \in H^4(\mathbb{R})$, we can get the following lemmas.

**Lemma 2.8.** Under the assumptions of Theorem 2.4, if $\epsilon, |\delta_0|$ are small, it holds that

$$(1 + t)^{4\lambda + 4} \|\partial_{xx}^4 V\|^2 + \int_0^t \left((1 + s)^{3\lambda + 4} \|\partial_{xx}^3 Z\|^2 + (1 + s)^{4\lambda + 3} \|\partial_{xx}^4 V\|^2\right)ds \leq C(\|V_0\|_{H^4}^2 + |\delta_0|^2).$$

**Lemma 2.9.** Under the assumptions of Theorem 2.4, if $\epsilon, |\delta_0|$ are small, it holds that

$$(1 + t)^{5\lambda + 5} \|\partial_{xx}^5 V\|^2 + \int_0^t \left((1 + s)^{4\lambda + 5} \|\partial_{xx}^4 Z\|^2 + (1 + s)^{5\lambda + 4} \|\partial_{xx}^5 V\|^2\right)ds \leq C(\|V_0\|_{H^5}^2 + |\delta_0|^2).$$

**Lemma 2.10.** Under the assumptions of Theorem 2.4, if $\epsilon, |\delta_0|$ are small, it holds that

$$(1 + t)^{3\lambda + 5} \|\partial_{xx}^3 Z\|^2 + \int_0^t \left((1 + s)^{2\lambda + 5} \|Z_{xxt}\|^2 + (1 + s)^{3\lambda + 4} \|\partial_{xx}^3 Z\|^2\right)ds \leq C(\|V_0\|_{H^4}^2 + \|Z_0\|_{H^3}^2 + |\delta_0|^2).$$

**Lemma 2.11.** Under the assumptions of Theorem 2.4, if $\epsilon, |\delta_0|$ are small, it holds that

$$(1 + t)^{4\lambda + 6} \|\partial_{xx}^4 Z\|^2 + \int_0^t \left((1 + s)^{3\lambda + 6} \|\partial_{xx}^3 Z_t\|^2 + (1 + s)^{4\lambda + 5} \|\partial_{xx}^4 Z\|^2\right)ds \leq C(\|V_0\|_{H^5}^2 + \|Z_0\|_{H^4}^2 + |\delta_0|^2).$$

**Lemma 2.12.** Under the assumptions of Theorem 2.4, if $\epsilon, |\delta_0|$ are small, it holds that

$$(1 + t)^{2\lambda + 6} \|Z_{xxt}\|^2 + (1 + t)^{3\lambda + 7} \|\partial_{xx}^3 Z_t\|^2 \leq C(\|V_0\|_{H^5}^2 + \|Z_0\|_{H^4}^2 + |\delta_0|^2).$$

Restricting the solution $(\bar{v}, \bar{u})(x,t)$ of the Cauchy problem (2.2.1) on $\mathbb{R}^+$, we get the solution of the Dirichlet initial-boundary value problem (2.1.1). From (2.2.4), Lemmas 2.1, 2.3, 2.8, 2.12 and Minkowski’s inequality $\|u + v\|_{L^p} \leq \|u\|_{L^p} + \|v\|_{L^p}$, we have

**Proposition 2.1.** (Decay rates of the nonlinear diffusion waves). For any $\alpha > 0$, $(V_0, Z_0)(x) \in H^5(\mathbb{R}) \times H^4(\mathbb{R})$, assume that $\|v_0 - v_+\|_{L^1(\mathbb{R}^+)} + |u_+| + \|V_0\|_{H^3(\mathbb{R})} + \|Z_0\|_{H^2(\mathbb{R})}$ is sufficiently small. Then, for any $0 \leq \lambda < 1$, let $\delta = \|v_0 - v_+\|_{L^1(\mathbb{R}^+)} + |u_+| + \|V_0\|_{H^3(\mathbb{R})} + \|Z_0\|_{H^4(\mathbb{R})}$, there exists a unique time-global solution $(\bar{v}, \bar{u})(x,t)$ of the initial-boundary value problem (2.1.1) satisfying

$$\|\partial_{xx}^k (\bar{v}(t) - v_+)\|_{L^p(\mathbb{R}^+)} \leq C\delta(1 + t)^{-\frac{\lambda + 1}{2} \frac{(1 - \frac{1}{p}) - (\lambda + 1)}{2}},$$

where $k \leq 3$, if $p \in (2, \infty]$; $k \leq 4$, if $p = 2$,

$$\|\partial_{xx}^k \bar{v}(t)\|_{L^p(\mathbb{R}^+)} \leq C\delta(1 + t)^{-\frac{\lambda + 1}{2} \frac{(1 - \frac{1}{p}) - (\lambda + 1)}{2} - 1},$$

where $k \leq 2$, if $p \in (2, \infty)$; $k \leq 3$, if $p = 2$,

$$\|\partial_{xx}^k \partial_{xx}^l v(t)\|_{L^p(\mathbb{R}^+)} \leq C\delta(1 + t)^{-\frac{\lambda + 1}{2} \frac{(1 - \frac{1}{p}) - (\lambda + 1)}{2} - 2},$$

where $k \leq 1$, if $p \in (2, \infty]$; $k \leq 2$, if $p = 2$.  

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Finally, we give the following dissipative property of the correction function \( \hat{v}(x,t) \) as in \([1]\).

**Proposition 2.2.** For any \( 0 \leq \lambda < 1 \), then there exist constants \( 0 < \vartheta < 1 - \lambda \), \( c \) and \( C > 0 \), the correction function \( \hat{v}(x,t) \) defined in (2.1.4) satisfies the following dissipative estimates:

\[
\|\hat{v}\|_{L^1(\mathbb{R}^+)} + \|\hat{v}\|_{H^\infty(\mathbb{R}^+)} + \|\hat{v}_t\|_{H^\infty(\mathbb{R}^+)} \leq C|u_+|e^{-\alpha t^\theta}.
\]

### 2.3 Proofs of Theorems 2.1-2.3

In this subsection, we prove Theorems 2.1-2.3. To begin with, we give the local (in time) estimates of the initial boundary problem (2.1.12)-(2.1.14) for any \( 0 \leq \lambda < 1 \). The proofs are quite similar to those in Section 3 of \([1]\). Thus we omit the details here.

**Proposition 2.3. (Locally estimates).** Under the conditions of Theorems 2.1-2.3, for any given \( T > 0 \), \( 0 \leq \lambda < 1 \), \( \alpha > 0 \), the solution \((\omega, \omega_t)(x,t)\) to the initial boundary problem (2.1.12)-(2.1.14) on \([0,T] \) satisfying

\[
\|\omega\|^2 + \|\omega_t\|^2 + \|\omega_x\|^2 + \int_0^t (\|\omega_t\|^2 + \|\omega_x\|^2) \, ds \leq C(T)(\|\omega_0\|_{H^1}^2 + \|z_0\|^2 + \delta),
\]

\[
\|\omega_{xx}\|^2 + \|\omega_{xt}\|^2 + \int_0^t (\|\omega_{xx}\|^2 + \|\omega_{xt}\|^2) \, ds \leq C(T)(\|\omega_0\|_{H^2}^2 + \|z_0\|_{H^1}^2 + \delta),
\]

and

\[
\|\omega_{xxx}\|^2 + \|\omega_{xxt}\|^2 + \int_0^t (\|\omega_{xxx}\|^2 + \|\omega_{xxt}\|^2) \, ds \leq C(T)(\|\omega_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta).
\]

Now we devote ourselves to the estimates of the global solution \((\omega, z)(x,t)\) under the a priori assumption

\[
N(T) := \sup_{0 < t < T} \left\{ \|\omega_x(\cdot,t)\|_{L^\infty} + (1 + t)\|\omega_{xt}(\cdot,t)\|_{L^\infty} + (1 + t)\frac{\|\omega_{xxx}\|_{L^\infty}}{\|\omega_{xx}\|_{L^\infty}} \right\} \leq \epsilon, \quad (2.3.1)
\]

for some \( 0 < \epsilon \ll 1 \) and \( 0 < T < \infty \). Throughout this subsection all estimates are independent of \( T \).

It can be checked that

\[
\begin{cases}
\omega(0,t) = \omega_{xx}(0,t) = \omega_t(0,t) = \omega_{txx}(0,t) = 0, & \text{etc}, \\
\omega(\infty,t) = \omega_x(\infty,t) = \omega_t(\infty,t) = \omega_{xx}(\infty,t) = 0, & \text{etc}. \quad (2.3.2)
\end{cases}
\]

We need the following lemma concerning the lower order estimates on \((\omega, \omega_t)\).

**Lemma 2.13.** Under the assumptions of Theorems 2.1-2.3, if \( \epsilon, \delta \) are small, it holds that

\[
\|\omega\|^2 + (1 + t)^{2\lambda}(\|\omega_t\|^2 + \|\omega_x\|^2) + \int_0^t (1 + s)^{\lambda}(\|\omega_t\|^2 + \|\omega_x\|^2) \, ds \leq C(\|\omega_0\|_{H^1}^2 + \|z_0\|^2 + \delta), \quad \text{for any} \quad 0 \leq \lambda < \frac{3}{5}, \quad (2.3.3)
\]
and for $\frac{3}{2} < \lambda < 1$

\[
\begin{cases}
(1 + t)^{\frac{3}{2} - \frac{3\lambda}{2}} \|\omega\|^2 + (1 + t)^{\frac{3}{2} - \frac{3\lambda}{2}} (\|\omega_x\|^2 + \|\omega_t\|^2) \leq C(\|V_0\|^2_{H^1} + \|z_0\|^2 + \delta), \\
\int_0^t [(1 + s)^{-\lambda - 1} \|\omega\|^2 + (1 + s)^{\beta} (\|\omega_x\|^2 + \|\omega_t\|^2)] ds \\
\leq C(1 + t)^{\beta + \frac{3\lambda}{2} - \frac{3}{2}} (\|\omega_0\|^2_{H^1} + \|z_0\|^2 + \delta), \text{ for any } \frac{3}{2} - \frac{3\lambda}{2} < \beta < \lambda.
\end{cases}
\]  
\tag{2.3.4}

Proof. Multiplying (2.1.12) by $(1 + t)^\beta \omega$, integrating the resulting equality with respect to $x$ over $\mathbb{R}^+$, using the boundary condition (2.3.2), one obtains

\[
\frac{d}{dt} \int_0^\infty [(1 + t)^\beta \omega_t + \frac{\alpha}{2} (1 + t)^{-\lambda - 1} \omega^2] dx - \int_0^\infty (1 + t)^\beta p'(\bar{v})\omega^2 dx \\
+ \frac{\alpha (\lambda - \beta)}{2} \int_0^\infty (1 + t)^{-\lambda - 1} \omega^2 dx \\
= \int_0^\infty (1 + t)^\beta \omega_t^2 dx + \frac{d}{dt} \int_0^\infty \frac{\beta}{2} (1 + t)^{-\lambda - 1} \omega^2 dx + \frac{\beta (1 - \beta)}{2} \int_0^\infty (1 + t)^{-2} \omega^2 dx \\
+ \int_0^\infty (1 + t)^\beta F \omega dx.
\]  
\tag{2.3.5}

Now we estimate the last term in the right hand of (2.3.5) as follows:

\[
\int_0^\infty (1 + t)^\beta F \omega dx = \int_0^\infty \left[ \frac{1}{\alpha} (1 + t)^{\beta + \lambda} p(\bar{v})_{xt} + \frac{\lambda}{\alpha} (1 + t)^{\beta + \lambda - 1} p(\bar{v})_x \\
- (1 + t)^\beta \left(p(\omega_x + \bar{v} + \bar{v}) - p(\bar{v}) - p'(\bar{v})\omega_x\right)_x \right] \omega dx.
\]  
\tag{2.3.6}

Firstly, from Proposition 2.1 and (2.3.2), we have

\[
\int_0^\infty \frac{\lambda}{\alpha} (1 + t)^{\beta + \lambda - 1} p(\bar{v})_x \omega dx \leq \nu \int_0^\infty (1 + t)^{-\kappa} \omega^2 dx + C \int_0^\infty (1 + t)^{2\beta + 2\lambda + \kappa - 2} |\bar{v}_x|^2 dx \\
\leq \nu \int_0^\infty (1 + t)^{-\kappa} \omega^2 dx + C \delta^2 (1 + t)^{2\beta + \frac{3\lambda}{2} - \frac{3}{2}},
\]  
\tag{2.3.7}

and for some constants $\kappa > 1, \nu > 0$ which will be determined below

\[
\int_0^\infty \frac{\lambda}{\alpha} (1 + t)^{\beta + \lambda - 1} p(\bar{v})_x \omega dx \leq \nu \int_0^\infty (1 + t)^{-\kappa} \omega^2 dx + C \int_0^\infty (1 + t)^{2\beta + 2\lambda + \kappa - 2} |\bar{v}_x|^2 dx \\
\leq \nu \int_0^\infty (1 + t)^{-\kappa} \omega^2 dx + C \delta^2 (1 + t)^{2\beta + \frac{3\lambda}{2} - \frac{3}{2}}.
\]  
\tag{2.3.8}

Secondly, by using Propositions 2.1, 2.2, 2.3, 2.4 and the a priori assumption (2.3.1), we have

\[
- \int_0^\infty (1 + t)^\beta \left(p(\omega_x + \bar{v} + \bar{v}) - p(\bar{v}) - p'(\bar{v})\omega_x\right)_x \omega dx \\
= \int_0^\infty (1 + t)^\beta \left(p(\omega_x + \bar{v} + \bar{v}) - p(\bar{v}) - p'(\bar{v})\omega_x\right)_x \omega dx \\
\leq \frac{C_0}{8} \int_0^\infty (1 + t)^{\beta} \omega_x^2 dx + C \int_0^\infty (1 + t)^{\beta} (|\bar{v}|^2 + |\omega_x|^4) dx \\
\leq \frac{C_0}{8} \int_0^\infty (1 + t)^{\beta} \omega_x^2 dx + C \epsilon^2 \int_0^\infty (1 + t)^{\beta} \omega_x^2 dx + C \delta^2 (1 + t)^{\beta + \frac{3\lambda}{2} - \frac{3}{2}}.
\]  
\tag{2.3.9}
Substituting (2.3.6)-(2.3.9) into (2.3.5), and using the smallness of \( \epsilon \), we have

\[
\frac{d}{dt} \int_0^\infty [(1 + t)^\beta \omega_t + \frac{\alpha}{2} (1 + t)^{\beta - \lambda} \omega^2] \, dx + \frac{C_0}{2} \int_0^\infty (1 + t)^\beta \omega^2_x \, dx \\
+ \frac{\alpha(\lambda - \beta)}{2} \int_0^\infty (1 + t)^{\beta - \lambda - 1} \omega^2 \, dx \\
\leq \int_0^\infty (1 + t)^\beta \omega^2_t \, dx + \frac{d}{dt} \int_0^\infty \frac{\beta}{2} (1 + t)^{\beta - 1} \omega^2 \, dx + \frac{\beta(1 - \beta)}{2} \int_0^\infty (1 + t)^{\beta - 2} \omega^2 \, dx \\
+ \nu \int_0^\infty (1 + t)^{-\kappa} \omega^2 \, dx + C \delta^2 (1 + t)^{\beta + \frac{3\lambda - 2}{2}} + \frac{C^2}{\nu} (1 + t)^{2\beta + \frac{3\lambda - 2}{2}}. \tag*{(2.3.10)}
\]

Multiplying (2.1.12) by \((1 + t)^{\beta + \lambda} \omega_t\), and integrating the resulting equality over \(\mathbb{R}^+\), we see that

\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty \left[ (1 + t)^{\beta + \lambda} \omega^2_t - (1 + t)^{\beta + \lambda} p' (\bar{v}) \omega^2_x \right] \, dx + \int_0^\infty \alpha (1 + t)^\beta \omega^2_t \, dx \\
= -\frac{1}{2} \int_0^\infty (1 + t)^{\beta + \lambda} p' (\bar{v}) \bar{v}_t \omega^2 dx - \frac{\beta + \lambda}{2} \int_0^\infty (1 + t)^{\beta + \lambda - 1} p' (\bar{v}) \omega^2 dx \\
+ \frac{\beta + \lambda}{2} \int_0^\infty (1 + t)^{\beta + \lambda - 1} \omega^2_t \, dx + \int_0^\infty (1 + t)^{\beta + \lambda} F \omega_t \, dx \\
\leq C \int_0^\infty (1 + t)^{\beta + \lambda} (\omega^2_x + \omega^2_t) \, dx + \int_0^\infty (1 + t)^{\beta + \lambda} F \omega_t \, dx. \tag*{(2.3.11)}
\]

Now we estimate the last term in the right hand of (2.3.11) as follows:

\[
\int_0^\infty (1 + t)^{\beta + \lambda} F \omega_t \, dx = \int_0^\infty \left[ \frac{1}{\alpha} (1 + t)^{\beta + 2\lambda} p(\bar{v})_x t + \frac{\lambda}{\alpha} (1 + t)^{\beta + 2\lambda - 1} p(\bar{v}) \right] \omega_t \, dx \\
- (1 + t)^{\beta + \lambda} \left( p(\omega_x + \bar{v} + \bar{v}) - p(\bar{v}) - p(\bar{v}) \omega_x \right) \omega_t \, dx. \tag*{(2.3.12)}
\]

Firstly, applying Proposition 2.1, one gets

\[
\int_0^\infty \frac{1}{\alpha} (1 + t)^{\beta + 2\lambda} p(\bar{v})_x t \omega_t \, dx \\
\leq \frac{\alpha}{4} \int_0^\infty (1 + t)^{\beta} \omega^2_t \, dx + C \int_0^\infty (1 + t)^{\beta + 4\lambda} (|\bar{v}_x|^2 + |\bar{v}_x|^2 |\bar{v}_t|^2) \, dx \\
\leq \frac{\alpha}{4} \int_0^\infty (1 + t)^{\beta} \omega^2_t \, dx + C \delta^2 (1 + t)^{\beta + \frac{5\lambda - 2}{2}}, \tag*{(2.3.13)}
\]

and

\[
\int_0^\infty \frac{\lambda}{\alpha} (1 + t)^{\beta + 2\lambda - 1} p(\bar{v})_x \omega_t \, dx \\
\leq \frac{\alpha}{4} \int_0^\infty (1 + t)^{\beta} \omega^2_t \, dx + C \int_0^\infty (1 + t)^{\beta + 4\lambda - 2} |\bar{v}_x|^2 \, dx \\
\leq \frac{\alpha}{4} \int_0^\infty (1 + t)^{\beta} \omega^2_t \, dx + C \delta^2 (1 + t)^{\beta + \frac{5\lambda - 2}{2}}. \tag*{(2.3.14)}
\]
Next by using Propositions 2.1, 2.2, and the \textit{a priori} assumption (2.3.11), we have

\[
\begin{align*}
&= \int_0^\infty (1 + t)^{\beta + \lambda} \left( p(\omega_x + \bar{v} + \bar{\omega}) - p(\bar{v}) - p'(\bar{v})\omega_x \right) \omega_t \, dx \\
&= \frac{d}{dt} \int_0^\infty \left( \int_{\omega_x + \bar{v} + \bar{\omega}} p(s) \, ds - p(\bar{v})\omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) \, dx \\
&\leq \frac{d}{dt} \int_0^\infty (1 + t)^{\beta + \lambda} \left( \int_{\omega_x + \bar{v} + \bar{\omega}} p(s) \, ds - p(\bar{v})\omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) \, dx \\
&\leq \frac{d}{dt} \int_0^\infty (1 + t)^{\beta + \lambda} \left( \int_{\omega_x + \bar{v} + \bar{\omega}} p(s) \, ds - p(\bar{v})\omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) \, dx \\
&\quad + C(\delta + \epsilon) \int_0^\infty (1 + t)^{\beta + \lambda - 1} \omega_x^2 \, dx + C\delta(1 + t)^{\beta + \frac{3}{2} - \frac{7}{2}}. \quad (2.3.15)
\end{align*}
\]

Putting (2.3.12) - (2.3.15) into (2.3.11) implies

\[
\begin{align*}
&\leq \frac{d}{dt} \int_0^\infty (1 + t)^{\beta + \lambda} \left( \int_{\omega_x + \bar{v} + \bar{\omega}} p(s) \, ds - p(\bar{v})\omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) \, dx \\
&\quad + C \delta \int_0^\infty (1 + t)^{\beta + \lambda - 1} \omega_x^2 \, dx + C\delta(1 + t)^{\beta + \frac{3}{2} - \frac{7}{2}}. \quad (2.3.16)
\end{align*}
\]

Multiplying (2.3.16) by \( h \), and adding up the resulting inequality and (2.3.10), we get

\[
\begin{align*}
&\frac{d}{dt} \int_0^\infty \left[ (1 + t)^{\beta} \omega_t + \frac{\alpha}{2} (1 + t)^{\beta - \lambda} \omega^2 + \frac{h}{2} (1 + t)^{\beta + \lambda} \omega_t^2 - \frac{h}{2} (1 + t)^{\beta + \lambda} p'(\bar{v})\omega_x \right] \, dx \\
&\quad + \frac{C_0}{2} \int_0^\infty (1 + t)^{\beta + \lambda} \omega_x^2 \, dx + \frac{\alpha h}{2} \int_0^\infty (1 + t)^{\beta + \lambda} \omega_t^2 \, dx + \frac{\alpha(h - 1)}{2} \int_0^\infty (1 + t)^{\beta - \lambda - 1} \omega^2 \, dx \\
&\leq \frac{d}{dt} \int_0^\infty (1 + t)^{\beta + \lambda} \left( \int_{\omega_x + \bar{v} + \bar{\omega}} p(s) \, ds - p(\bar{v})\omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) \, dx \\
&\quad + \beta \frac{d}{dt} \int_0^\infty (1 + t)^{\beta - 1} \omega^2 \, dx \\
&\quad + \nu \int_0^\infty (1 + t)^{-\kappa} \omega^2 \, dx + Ch \int_0^\infty (1 + t)^{\beta + \lambda - 1} (\omega_x^2 + \omega_t^2) \, dx + \frac{\beta(1 - \beta)}{2} \int_0^\infty (1 + t)^{\beta - 2} \omega^2 \, dx \\
&\quad + C\delta^2 (1 + t)^{\beta + \frac{3}{2} - \frac{7}{2}} + C\delta^2 \nu (1 + t)^{2\beta + \frac{3}{2} + \kappa - \frac{7}{2}}. \quad (2.3.17)
\end{align*}
\]

\textbf{Case 1.} \( 0 \leq \lambda < \frac{3}{5} \)
It is easy to know that \(1 < \frac{\beta}{2} - \frac{5\lambda}{2}\). Therefore we can take \(\beta = \lambda, \nu = 1\), and there exists constant \(\kappa\) satisfying \(1 < \kappa < \frac{\beta}{2} - \frac{5\lambda}{2}\). Then we have

\[
\frac{d}{dt} \int_0^\infty \left[ (1 + t)^{\lambda} \omega_t + \frac{\alpha}{2} \omega^2 + \frac{h}{2}(1 + t)^{2\lambda} \omega_t^2 - \frac{\alpha h}{2} (1 + t)^{2\lambda} p'(\bar{v}) \omega_x^2 \right] dx \\
+ \frac{C_0}{2} \left(1 + t\right)^{\lambda} \omega_t^2 dx + \left( \frac{\alpha h}{2} - 1 \right) \int_0^\infty (1 + t)^{\lambda} \omega_t^2 dx \\
\leq \frac{d}{dt} \left( \int_0^\infty (1 + t)^{2\lambda} \left( \int_0^{\omega + \bar{v}} p(s) ds - p(\bar{v}) \omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) dx \right) + \frac{\lambda}{2} \frac{d}{dt} \int_0^\infty (1 + t)^{\lambda-1} \omega^2 dx \\
+ \int_0^\infty (1 + t)^{-\kappa} \omega^2 dx + \frac{\lambda(1 - \lambda)}{2} \int_0^\infty (1 + t)^{\lambda-2} \omega^2 dx + C \int_0^\infty (1 + t)^{2\lambda-1} (\omega_x^2 + \omega_t^2) dx \\
+ C\delta(1 + t)^{\frac{5\lambda}{2} - \frac{7}{2} + \kappa}.
\]

(2.3.18)

Let \(T_0\) sufficiently large such that

\[
\begin{cases}
Ch(1 + t)^{\lambda-1} \leq \frac{C_0}{4}, \\
Ch(1 + t)^{\lambda-1} \leq \frac{1}{2} \left( \frac{\alpha h}{2} - 1 \right), \\
\frac{1}{2}(1 + t)^{\lambda-1} \leq \frac{1}{4},
\end{cases}
\]

if \(t \geq T_0\), and fixing \(h = \frac{6}{\alpha}\), we have

\[
\frac{d}{dt} \mathcal{H}(t) + \frac{C_0}{4} \int_0^\infty (1 + t)^{\lambda} \omega_t^2 dx + \frac{1}{2} \left( \frac{\alpha h}{2} - 1 \right) \int_0^\infty (1 + t)^{\lambda} \omega_t^2 dx
\]

\[
\leq C \frac{d}{dt} \left( \int_0^\infty (1 + t)^{2\lambda} \left( \int_0^{\omega + \bar{v}} p(s) ds - p(\bar{v}) \omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) dx \right) + \frac{\lambda}{2} \frac{d}{dt} \int_0^\infty (1 + t)^{\lambda-1} \omega^2 dx \\
+ C \int_0^\infty (1 + t)^{-\kappa} \omega^2 dx + \frac{\lambda(1 - \lambda)}{2} \int_0^\infty (1 + t)^{\lambda-2} \omega^2 dx + C\delta(1 + t)^{\frac{5\lambda}{2} - \frac{7}{2} + \kappa},
\]

where

\[\mathcal{H}(t) \sim ||\omega||^2 + (1 + t)^{2\lambda} ||\omega_x||^2 + (1 + t)^{2\lambda} ||\omega_t||^2.\]

Then, we have

\[
\frac{d}{dt} \mathcal{H}(t) + \frac{C_0}{4} \int_0^\infty (1 + t)^{\lambda} \omega_t^2 dx + \int_0^\infty (1 + t)^{\lambda} \omega_t^2 dx
\]

\[
\leq C \frac{d}{dt} \left( \int_0^\infty (1 + t)^{2\lambda} \left( \int_0^{\omega + \bar{v}} p(s) ds - p(\bar{v}) \omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) dx \right) + \frac{\lambda}{2} \frac{d}{dt} \int_0^\infty (1 + t)^{\lambda-1} \omega^2 dx \\
+ C(1 + t)^{-\kappa} \mathcal{H}(t) + C(1 + t)^{\lambda-2} \mathcal{H}(t) + C\delta(1 + t)^{\frac{5\lambda}{2} - \frac{7}{2} + \kappa}, \quad \text{for any} \quad t \in [T_0, \infty).
\]

Using Gronwalls inequality on \([T_0, t]\), one has by \(1 < \kappa < \frac{5}{2} - \frac{5\lambda}{2}\) and \(0 \leq \lambda < \frac{3}{5}\),

\[
\mathcal{H}(t) + c \int_{T_0}^t (1 + s)^{\lambda} \left( ||\omega_x||^2 + ||\omega_t||^2 \right) ds
\]

\[
\leq C \int_0^\infty (1 + t)^{2\lambda} \left( \int_0^{\omega + \bar{v}} p(s) ds - p(\bar{v}) \omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) dx + \frac{\lambda}{2} \int_0^\infty (1 + t)^{\lambda-1} \omega^2 dx \\
+ C(\mathcal{H}(T_0) + \delta)
\]

\[
\leq C \int_0^\infty (1 + t)^{2\lambda} ||\omega_x||^2 dx + \frac{1}{4} \int_0^\infty \omega^2 dx + C(\mathcal{H}(T_0) + \delta),
\]

(2.3.19)
which together with Proposition 2.3 deduce (2.3.3) in view of the smallness of $\epsilon$.

**Case 2.** $\frac{3}{5} < \lambda < 1$

In this case, from $\lambda > \frac{3}{2} - \frac{3\lambda}{2}$, we can take $\frac{3}{2} - \frac{3\lambda}{2} < \beta < \lambda$, $\kappa = \lambda - \beta + 1$, and $\nu = \frac{\alpha(\lambda - \beta)}{4} > 0$. Therefore from (2.3.17), we know

$$
\frac{d}{dt}\int_0^\infty \left[ (1 + t)^\beta \omega_t + \frac{\alpha}{2} (1 + t)^{\beta - \lambda} \omega^2 + \frac{h}{2} (1 + t)^{\beta + \lambda} \omega_t^2 - \frac{h}{2} (1 + t)^{\beta + \lambda} p'(\bar{v}) \omega_x^2 \right] dx
+ \frac{\alpha(\lambda - \beta)}{4} \int_0^\infty (1 + t)^{\beta - \lambda} \omega^2 dx + \frac{C_0}{2} \int_0^\infty (1 + t)^{\beta} \omega_t^2 dx + \left( \frac{\alpha h}{2} - 1 \right) \int_0^\infty (1 + t)^{\beta} \omega_t^2 dx
\leq \frac{h}{2} \frac{d}{dt}\int_0^\infty (1 + t)^{\beta + \lambda} \left( \int_0^\infty p(s) ds - p(\bar{v}) \omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) dx + \frac{\beta(1 - \beta)}{2} \int_0^\infty (1 + t)^{\beta - 2} \omega^2 dx + C \int_0^\infty (1 + t)^{\beta + \lambda - 1} (\omega_x^2 + \omega_t^2) dx + C \delta (1 + t)^{\beta + \lambda - \frac{5}{2}}.
$$

(2.3.20)

Let $h = \frac{\alpha}{\alpha + T_1}$, and $T_1$ sufficiently large such that

$$
\begin{align*}
&\{ Ch(1 + t)^{\lambda - 1} \leq \frac{C_0}{4}, \\
&Ch(1 + t)^{\lambda - 1} \leq \frac{1}{2} \left( \frac{\alpha h}{2} - 1 \right), \\
&\frac{\beta(1 - \beta)}{2} (1 + t)^{\lambda - 1} \leq \frac{1}{4},
\end{align*}
$$

for $t \geq T_1$, then we have

$$
\frac{d}{dt} \mathcal{H}_1(t) + \frac{\alpha(\lambda - \beta)}{4} \int_0^\infty (1 + t)^{\beta - \lambda - 1} \omega^2 dx + \frac{C_0}{4} \int_0^\infty (1 + t)^{\beta} \omega_t^2 dx + \left( \frac{\alpha h}{2} - 1 \right) \int_0^\infty (1 + t)^{\beta} \omega_t^2 dx
\leq C \frac{d}{dt} \int_0^\infty (1 + t)^{\beta + \lambda} \left( \int_0^\infty p(s) ds - p(\bar{v}) \omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) dx + \frac{\beta(1 - \beta)}{2} \int_0^\infty (1 + t)^{\beta - 2} \omega^2 dx + C \delta (1 + t)^{\beta + \lambda - \frac{5}{2}},
$$

where

$$
\mathcal{H}_1(t) \sim (1 + t)^{\beta - \lambda} \| \omega \|^2 + (1 + t)^{\beta + \lambda} \| \omega_t \|^2 + (1 + t)^{\beta + \lambda} \| \omega_x \|^2.
$$

Therefore, we know

$$
\frac{d}{dt} \mathcal{H}_1(t) + \frac{C_0}{4} \int_0^\infty (1 + t)^{\beta} \omega_x^2 dx + C \int_0^\infty (1 + t)^{\beta} \omega_t^2 dx + C \int_0^\infty (1 + t)^{\beta - \lambda - 1} \omega^2 dx
\leq C \frac{d}{dt} \int_0^\infty (1 + t)^{\beta + \lambda} \left( \int_0^\infty p(s) ds - p(\bar{v}) \omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) dx + \frac{\beta d}{dt} \int_0^\infty (1 + t)^{\beta - 1} \omega^2 dx
+ C (1 + t)^{\lambda - 2} \mathcal{H}_1(t) + C \delta (1 + t)^{\beta + \lambda - \frac{5}{2}}, \quad \text{for any} \quad t \in [T_1, \infty).
$$

Using Gronwall’s inequality on $[T_1, t]$, Proposition 2.3 $\beta + \frac{3\lambda}{2} - \frac{5}{2} > -1$ and $\frac{3}{5} < \lambda < 1$, one
Proof. We only prove (2.3.23), and the proof of (2.3.22) is similar. In the case of multiplying \((2.3.16)\) by \((1 + t)^{\beta - \lambda - 1}\), which together with Proposition 2.3 deduce \((2.3.4)\) in view of the smallness of \(\epsilon\) and \(\beta + \frac{3\lambda}{2} - \frac{3}{2} > 0\).

Hence we complete the proof of Lemma 2.13.

Furthermore, we can get the better decay rate of the functions \(\omega_x\) and \(\omega_t\) as follows:

**Lemma 2.14.** Under the assumptions of Theorems 2.1-2.3, if \(\epsilon, \delta\) are small, it holds that
\[
(1 + t)^{\lambda + 1} (\|\omega_x\|^2 + \|\omega_t\|^2) + \int_0^t (1 + s) \|\omega_t\|^2 ds 
\leq C(\|\omega_0\|^2_{H^1} + \|z_0\|^2 + \delta), \quad \text{for} \quad 0 \leq \lambda < \frac{3}{5},
\]
and for \(\frac{3}{2} < \lambda < 1\)
\[
\left\{ \begin{array}{l}
(1 + t)^{\frac{3}{2} - \frac{3\lambda}{2}} (\|\omega_x\|^2 + \|\omega_t\|^2) \leq C(\|\omega_0\|^2_{H^1} + \|z_0\|^2 + \delta), \\
\int_0^t (1 + s)^{\beta - \lambda + 1} \|\omega_t\|^2 ds 
\leq C(1 + t)^{\beta + \frac{3\lambda}{2} - \frac{3}{2}} (\|\omega_0\|^2_{H^1} + \|z_0\|^2 + \delta), \quad \text{for any} \quad \frac{3}{2} - \frac{3\lambda}{2} < \beta < \lambda.
\end{array} \right.
\]

**Proof.** We only prove (2.3.23), and the proof of (2.3.22) is similar. In the case of \(\frac{3}{2} < \lambda < 1\), multiplying \((2.3.14)\) by \((1 + t)^{1-\lambda}\) leads to
\[
\frac{1}{2} \frac{d}{dt} \int_0^\infty \left[ (1 + t)^{\beta + 1} \omega_t^2 - (1 + t)^{\beta + 1} p'(\bar{v}) \omega_x^2 \right] dx + \frac{\alpha}{2} \int_0^\infty (1 + t)^{\beta - \lambda + 1} \omega_t^2 dx 
\leq \frac{d}{dt} \int_0^\infty (1 + t)^{\beta + 1} \left( \int_0^{\omega_x + \bar{v} + \bar{v}} p(s) ds - p(\bar{v}) \omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) dx 
+ C \left| \int_0^\infty (1 + t)^{\beta} \left( \int_0^{\omega_x + \bar{v} + \bar{v}} p(s) ds - p(\bar{v}) \omega_x - \frac{p'(\bar{v})}{2} \omega_x^2 \right) dx \right| 
+ C \int_0^\infty ((1 + t)^{\beta} \omega_t^2 - (1 + t)^{\beta} p'(\bar{v}) \omega_x^2) dx 
+ C \int_0^\infty (1 + t)^{\beta} (\omega_x^2 + \omega_t^2) dx + C \delta (1 + t)^{\beta + \frac{3\lambda}{2} - \frac{3}{2}}.
\]

Integrating the above inequality in \(t\) over \((0, t)\), using Lemma 2.13 and \(\beta + \frac{3\lambda}{2} > \frac{3}{2}\), we get
\[
(1 + t)^{\beta + 1} (\|\omega_x\|^2 + \|\omega_t\|^2) + \int_0^t (1 + s)^{\beta - \lambda + 1} \|\omega_t\|^2 ds 
\leq C(1 + t)^{\beta + \frac{3\lambda}{2} - \frac{3}{2}} (\|\omega_0\|^2_{H^1} + \|z_0\|^2 + \delta).
\]
This completes the proof of Lemma 2.14

Similarly, we can derive decay rates on the higher derivatives of the global solution \( \omega(x, t) \).

**Lemma 2.15.** Under the assumptions of Theorems 2.1-2.3, if \( \epsilon, \delta \) are small, it holds that

\[
(1 + t)^{2\lambda + 2}(\|w_{xx}\|^2 + \|w_{xt}\|^2) + \int_0^t ((1 + s)^{2\lambda + 1}\|w_{xx}\|^2 + (1 + s)^{2\lambda + 2}\|w_{xt}\|^2)ds
\]

\[
\leq C(\|\omega_0\|^2_{H^2} + \|z_0\|^2_{H^1 + \delta}), \quad \text{for any} \quad 0 \leq \lambda < \frac{3}{5},
\]

and for \( \frac{3}{5} < \lambda < 1 \)

\[
\begin{align*}
(1 + t)^{\frac{2}{\lambda} - \frac{4}{7}}(\|w_{xx}\|^2 + \|w_{xt}\|^2) & \leq C(\|\omega_0\|^2_{H^2} + \|z_0\|^2_{H^1 + \delta}), \\
\int_0^t ((1 + s)^{\beta + \lambda + 1}\|w_{xx}\|^2 + (1 + s)^{\beta + 2}\|w_{xt}\|^2)ds & \leq C(1 + t)^{\beta + \frac{2\lambda}{5} - \frac{4}{7}}(\|\omega_0\|^2_{H^2} + \|z_0\|^2_{H^1 + \delta}), \quad \text{for any} \quad \frac{3}{2} - \frac{3\lambda}{2} < \beta < \lambda.
\end{align*}
\]

**Lemma 2.16.** Under the assumptions of Theorems 2.1-2.3, if \( \epsilon, \delta \) are small, it holds that

\[
(1 + t)^{3\lambda + 3}(\|w_{xx}\|^2 + \|w_{xt}\|^2) + \int_0^t ((1 + s)^{3\lambda + 2}\|w_{xx}\|^2 + (1 + s)^{2\lambda + 3}\|w_{xt}\|^2)ds
\]

\[
\leq C(\|\omega_0\|^2_{H^3} + \|z_0\|^2_{H^2 + \delta}), \quad \text{for any} \quad 0 \leq \lambda < \frac{3}{5},
\]

and for \( \frac{3}{5} < \lambda < 1 \)

\[
\begin{align*}
(1 + t)^{\frac{2}{\lambda} + \frac{1}{2}}(\|w_{xx}\|^2 + \|w_{xt}\|^2) & \leq C(\|\omega_0\|^2_{H^3} + \|z_0\|^2_{H^2 + \delta}), \\
\int_0^t ((1 + s)^{\beta + 2\lambda + 2}\|w_{xx}\|^2 + (1 + s)^{\beta + \lambda + 3}\|w_{xt}\|^2)ds & \leq C(1 + t)^{\beta + \frac{2\lambda}{5} + \frac{1}{2}}(\|\omega_0\|^2_{H^3} + \|z_0\|^2_{H^2 + \delta}), \quad \text{for any} \quad \frac{3}{2} - \frac{3\lambda}{2} < \beta < \lambda.
\end{align*}
\]

With Lemmas 2.13-2.16 in hand, by the Sobolev inequality and \( 0 \leq \lambda < 1 \), it’s easy to know that

\[
\|w_x(\cdot, t)\|_{L^\infty} \leq C(1 + t)^{-\frac{4}{7}}(\|\omega_0\|_{H^3} + \|z_0\|_{H^2 + \delta}) \leq \frac{\epsilon}{2},
\]

\[
\|w_{xt}(\cdot, t)\|_{L^\infty} \leq C(1 + t)^{-\frac{4}{7}}(\|\omega_0\|_{H^3} + \|z_0\|_{H^2 + \delta}) \leq \frac{\epsilon}{2}(1 + t)^{-1},
\]

and

\[
\|w_{xx}(\cdot, t)\|_{L^\infty} \leq C(1 + t)^{-\frac{4}{7}}(\|\omega_0\|_{H^3} + \|z_0\|_{H^2 + \delta}) \leq \frac{\epsilon}{2}(1 + t)^{-\frac{\lambda + 1}{2}}
\]

provided \( \|\omega_0\|_{H^3} + \|z_0\|_{H^2 + \delta} \ll 1 \). Up to now, we thus close the a priori assumption (2.3.1) about \( (\omega_x, \omega_xt, \omega_{xx}) \) from Lemmas 2.13-2.16.

In next two lemmas we want to pay an attention to \( z = \omega_t \) which has the improved decay rates.
Lemma 2.17. Under the assumptions of Theorems 2.1-2.3 if $\epsilon, \delta$ are small, it holds that

\[
(1 + t)^2 \|z\|^2 + (1 + t)^{\lambda^+3}(\|z_x\|^2 + \|z_t\|^2) + \int_0^t ((1 + s)^{\lambda+2}\|z_x\|^2 + (1 + s)^3\|z_t\|^2)ds
\]

\[
\leq C(\|\omega_0\|_{\dot{H}^2}^2 + \|z_0\|_{\dot{H}^1}^2 + \delta), \quad \text{for any} \quad 0 \leq \lambda < \frac{3}{5},
\]

and for $\frac{3}{5} < \lambda < 1$

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(1 + t)^{\frac{7}{2} - \frac{\lambda}{2}} \|z\|^2 + (1 + t)^{\frac{7}{2} - \frac{\lambda}{2}}(\|z_x\|^2 + \|z_t\|^2) \leq C(\|V_0\|_{\dot{H}^2}^2 + \|z_0\|_{\dot{H}^1}^2 + \delta), \\
\int_0^t ((1 + s)^{\beta+2}\|z_x\|^2 + (1 + s)^{\beta-\lambda+3}\|z_t\|^2)ds \\
\leq C(1 + t)^{\beta + \frac{3\lambda}{2} - \frac{\lambda}{2}}(\|\omega_0\|_{\dot{H}^2}^2 + \|z_0\|_{\dot{H}^1}^2 + \delta), \quad \text{for any} \quad \frac{3}{2} - \frac{3\lambda}{2} < \beta < \lambda.
\end{array} \right.
\end{aligned}
\]

Similar to the proof of Lemma 2.17, we can get the following estimates.

Lemma 2.18. Under the assumptions of Theorems 2.1-2.3 if $\epsilon, \delta$ are small, it holds that

\[
(1 + t)^{2\lambda^+4}(\|z_{xx}\|^2 + \|z_{xt}\|^2) + \int_0^t ((1 + s)^{2\lambda^+3}\|z_{xx}\|^2 + (1 + s)^{\lambda^+4}\|z_{xt}\|^2)ds
\]

\[
\leq C(\|\omega_0\|_{\dot{H}^3}^2 + \|z_0\|_{\dot{H}^2}^2 + \delta), \quad \text{for any} \quad 0 \leq \lambda < \frac{3}{5},
\]

and for $\frac{3}{5} < \lambda < 1$

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(1 + t)^{\frac{11}{2} - \frac{\lambda}{2}}(\|z_{xx}\|^2 + \|z_{xt}\|^2) \leq C(\|\omega_0\|_{\dot{H}^1}^2 + \|z_0\|_{\dot{H}^2}^2 + \delta), \\
\int_0^t ((1 + s)^{\beta+\lambda+3}\|z_{xx}\|^2 + (1 + s)^{\beta+4}\|z_{xt}\|^2)ds \\
\leq C(1 + t)^{\beta + \frac{3\lambda}{2} - \frac{\lambda}{2}}(\|\omega_0\|_{\dot{H}^3}^2 + \|z_0\|_{\dot{H}^2}^2 + \delta), \quad \text{for any} \quad \frac{3}{2} - \frac{3\lambda}{2} < \beta < \lambda.
\end{array} \right.
\end{aligned}
\]

Recalling Lemmas 2.13-2.18 we complete the proofs of Theorems 2.1-2.2. The proof of Theorem 2.3 is similar.

3 The case of Neumann boundary condition

In this section, we consider the problem (1.1)-(1.2) with the null-Neumann boundary condition (1.4), i.e.

\[
\begin{aligned}
\frac{\partial}{\partial \tau}v - \frac{\partial}{\partial x} u &= 0, \\
\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} p(v) &= -\frac{\alpha}{(1 + t)^{\lambda}} u, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
u_x(0, t) &= 0,
\end{aligned}
\]

and the initial data is

\[
(v, u) \mid_{t=0} = (v_0, u_0)(x) \to (v_+, u_+), \quad \text{as} \quad x \to +\infty \quad \text{and} \quad v_+ > 0.
\]

From (3.1) and the boundary condition (3.13), we have $v(0, t) = v_0(0)$ for any $t > 0$. 

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3.1 The case of \(v_0(0) \neq v_+\)

To construct the diffusion waves \((\bar{v}, \bar{u})(x, t)\) to the corresponding boundary condition, from \[1, 6, 18\], it is known that for any two constants \(v_+ > 0\) there exists a unique self-similar solution \(\tau(x, t) = \phi\left(\frac{x}{(1 + t)^{\frac{1}{\lambda}}}\right)\) satisfying

\[
\begin{cases}
\tau_t - \frac{1}{\alpha} p(\tau)_{xx} = 0, & (x, t) \in \mathbb{R} \times \mathbb{R}^+, \\
\tau(\pm \infty, t) = v_{\pm}.
\end{cases}
\]

Therefore, for any constant \(v_0(0) > 0\) between \(v_-\) and \(v_+\), there exists a unique \(\bar{v}(x, t)\) in the form of \(\phi\left(\frac{x}{(1 + t)^{\frac{1}{\lambda}}}\right)|_{t \geq 0}\) satisfying

\[
\begin{cases}
\bar{v}_t - \bar{v}_{xx} = 0, \\
p(\bar{v})_x = -\frac{\alpha}{(1 + t)^{\lambda}} \bar{v}, \\
(\bar{v}, \bar{u}_x)(0, t) = (v_0(0), 0), & (\bar{v}, \bar{u})(\infty, t) = (v_+, 0), \\
(\bar{v}, \bar{u})|_{t=0} = (\bar{v}_0, \bar{u}_0)(x) \to (v_+, 0), \quad \text{as } x \to +\infty.
\end{cases}
\]

According to the idea in \[18\], similar to that in the Dirichlet boundary problem, the correction functions are defined by

\[
\hat{v}(x, t) = -(u_0(0) - u_+ m_0(x)) B(t),
\]

and

\[
\hat{u}(x, t) = \left[u_+ + (u_0(0) - u_+) \int_x^\infty m_0(y) dy\right] \beta(t),
\]

where \(\beta(t), B(t)\) have been defined in Section 2.1, and \(m_0(x)\) is a smooth function with compact support such that

\[
\int_0^\infty m_0(x) dx = 1, \quad \text{supp } m_0 \subset \mathbb{R}^+.
\]

Therefore, \((\hat{v}, \hat{u})(x, t)\) satisfies

\[
\begin{cases}
\partial_t \hat{v} - \partial_x \hat{u} = 0, \\
\partial_t \hat{u} = -\frac{\alpha}{(1 + t)^{\lambda}} \hat{u}, & (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
(\hat{v}, \hat{u}_x)(0, t) = (0, 0), & \hat{u}(0, t) = u_0(0) \beta(t), \\
(\hat{v}, \hat{u})(\infty, t) = (0, u_+ \beta(t)).
\end{cases}
\]
Combining (3.1), (3.1.1) and (3.1.4), it is easy to see that
\[
\begin{align*}
\partial_t(v - \bar{v} - \hat{v}) - \partial_x(u - \bar{u} - \hat{u}) &= 0, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
\partial_t(u - \bar{u} - \hat{u}) + \partial_x(p(v) - p(\bar{v})) + \partial_t \bar{u} + \frac{\alpha}{(1 + t)^\lambda} (u - \bar{u} - \hat{u}) &= 0, \\
(u - \bar{u} - \hat{u})(x, 0) &= 0.
\end{align*}
\] (3.1.5)

Hence defining the perturbation by
\[
\begin{align*}
\omega(x, t) &= -\int_x^\infty (v(y, t) - \bar{v}(y, t) - \hat{v}(y, t))dy, \\
z(x, t) &= u(x, t) - \bar{u}(x, t) - \hat{u}(x, t).
\end{align*}
\] (3.1.6) (3.1.7)

It follows from (3.1.5) that the reformulated problem is
\[
\begin{align*}
\omega_t - z &= 0, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
z_t + (p(\omega_x + \bar{v} + \hat{v}) - p(\bar{v}))_x + \frac{\alpha}{(1 + t)^\lambda} z &= -\bar{u}_t, \\
\omega_x(0, t) &= 0, \quad z_x(0, t) = 0,
\end{align*}
\] (3.1.8)

with initial data
\[
(\omega, z) \mid_{t=0} = (\omega_0, z_0)(x),
\] (3.1.9)

where
\[
\begin{align*}
\omega_0(x) &= -\int_x^\infty (v_0(y) - \bar{v}(y, 0) - \hat{v}(y, 0))dy, \\
z_0(x) &= u_0(x) - \bar{u}(x, 0) - \hat{u}(x, 0).
\end{align*}
\]

Rewrite (3.1.8) and (3.1.9) as
\[
\begin{align*}
\omega_{tt} + (p'(\bar{v})\omega_x)_x + \frac{\alpha}{(1 + t)^\lambda} \omega_t &= F_2, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\
\omega_x(0, t) &= 0,
\end{align*}
\] (3.1.10)

with initial data
\[
(\omega, \omega_t) \mid_{t=0} = (\omega_0, z_0)(x),
\] (3.1.11)

where
\[
F_2 = \frac{1}{\alpha}(1 + t)^\lambda p(\bar{v})_{xx} + \frac{\lambda}{\alpha}(1 + t)^{\lambda-1} p(\bar{v})_x - (p(\omega_x + \bar{v} + \hat{v}) - p(\bar{v}) - p'(\bar{v})\omega_x)_x.
\] (3.1.12)

Note that, from the boundary condition, we have
\[
\omega_x(0, t) = \omega_{tx}(0, t) = \omega_{ttx}(0, t) = (p(\omega_x + \bar{v} + \hat{v}) - p(\bar{v})) \mid_{x=0} = 0, \quad \text{etc.}
\]

The nonlinear diffusion wave $\bar{v}(x, t)$ defined in (3.1.1) has the same behavior as in [1]. It’s worth noting that their convergence rates are different from Proposition 2.1. Therefore, our main result about Neumann boundary condition in this section is different from the case of Dirichlet boundary condition (see Theorems 2.1, 2.3).
Theorem 3.1. (The case of $v_0(0) \neq v_+$ and $0 \leq \lambda < \frac{1}{\alpha}$) For any $\alpha > 0$ and $v_0 - v_+ \in L^1(\mathbb{R}^+)$, assume that both $\delta_1 = |v_+ - v_0(0)| + |u_+ - u_0(0)|$ and $\|\omega_0\|_{H^3(\mathbb{R}^+)} + \|z_0\|_{H^2(\mathbb{R}^+)}$ are sufficiently small. Then, there exists a unique time-global solution of the problem \((3.1.10)-(3.1.11)\) satisfying

$$\omega \in C^k((0, \infty), H^{3-k}(\mathbb{R}^+)), \quad k = 0, 1, 2, 3,$$

$$\omega_t \in C^k((0, \infty), H^{2-k}(\mathbb{R}^+)), \quad k = 0, 1, 2,$$

furthermore, we have

$$\sum_{k=0}^3 (1 + t)^{(\lambda + 1)k} \|\partial_x^k \omega(\cdot, t)\|_{L^2}^2 + \sum_{k=0}^2 (1 + t)^{(\lambda + 1)k + 2} \|\partial_x^k \omega_t(\cdot, t)\|_{L^2}^2$$

$$+ \int_0^t \left[ \sum_{j=1}^3 (1 + s)^{(\lambda + 1)j - 1} \|\partial_x^j \omega(\cdot, s)\|_{L^2}^2 + \sum_{j=0}^2 (1 + s)^{(\lambda + 1)j + 1} \|\partial_x^j \omega_t(\cdot, s)\|_{L^2}^2 \right] ds$$

$$\leq C(\|\omega_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta_1).$$

Theorem 3.2. (The case of $v_0(0) \neq v_+$ and $\frac{1}{\alpha} < \lambda < 1$) For any $\alpha > 0$ and $v_0 - v_+ \in L^1(\mathbb{R}^+)$, assume that both $\delta_1 = |v_+ - v_0(0)| + |u_+ - u_0(0)|$ and $\|\omega_0\|_{H^3(\mathbb{R}^+)} + \|z_0\|_{H^2(\mathbb{R}^+)}$ are sufficiently small. Then, there exists a unique time-global solution of the problem \((3.1.10)-(3.1.11)\) satisfying

$$\omega \in C^k((0, \infty), H^{3-k}(\mathbb{R}^+)), \quad k = 0, 1, 2, 3,$$

$$\omega_t \in C^k((0, \infty), H^{2-k}(\mathbb{R}^+)), \quad k = 0, 1, 2,$$

furthermore, we have

$$\sum_{k=0}^3 (1 + t)^{(\lambda + 1)k + \frac{1}{2} - \frac{2\lambda}{\alpha}} \|\partial_x^k \omega(\cdot, t)\|_{L^2}^2 + \sum_{k=0}^2 (1 + t)^{(\lambda + 1)k + \frac{1}{2} - \frac{2\lambda}{\alpha}} \|\partial_x^k \omega_t(\cdot, t)\|_{L^2}^2$$

$$\leq C(\|\omega_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta),$$

and for any $\beta \in (\frac{3}{2} - \frac{2\lambda}{\alpha}, \lambda)$, we have

$$\int_0^t \left[ \sum_{j=0}^3 (1 + s)^{(\lambda + 1)(j - 1) + \beta} \|\partial_x^j \omega(\cdot, s)\|_{L^2}^2 + \sum_{j=0}^2 (1 + s)^{(\lambda + 1)j + 1 - \lambda - 1} \|\partial_x^j \omega_t(\cdot, s)\|_{L^2}^2 \right] ds$$

$$\leq C(1 + t)^{\lambda + \frac{1}{2} - \frac{2\lambda}{\alpha}} (\|\omega_0\|_{H^3}^2 + \|z_0\|_{H^2}^2 + \delta_1).$$

Theorem 3.3. (The case of $v_0(0) \neq v_+$ and $\lambda = \frac{1}{\alpha}$) For any $\alpha > 0$ and $v_0 - v_+ \in L^1(\mathbb{R}^+)$, assume that both $\delta_1 = |v_+ - v_0(0)| + |u_+ - u_0(0)|$ and $\|\omega_0\|_{H^3(\mathbb{R}^+)} + \|z_0\|_{H^2(\mathbb{R}^+)}$ are sufficiently small. Then, there exists a unique time-global solution of the problem \((3.1.10)-(3.1.11)\) satisfying

$$\omega \in C^k((0, \infty), H^{3-k}(\mathbb{R}^+)), \quad k = 0, 1, 2, 3,$$

$$\omega_t \in C^k((0, \infty), H^{2-k}(\mathbb{R}^+)), \quad k = 0, 1, 2,$$
Furthermore, we have for any sufficiently small $\varepsilon > 0$

\[
\sum_{k=0}^{3} (1 + t)^{k} \| \partial_{x}^{k} \omega(\cdot, t) \|_{L^{2}}^{2} + \sum_{k=0}^{2} (1 + t)^{k+2} \| \partial_{x}^{k} \omega_{t}(\cdot, t) \|_{L^{2}}^{2} \\
+ \int_{0}^{t} \left[ \sum_{j=1}^{3} (1 + s)^{j-1} \| \partial_{x}^{j} \omega(\cdot, s) \|_{L^{2}}^{2} + \sum_{j=0}^{2} (1 + s)^{j+1} \| \partial_{x}^{j} \omega_{t}(\cdot, s) \|_{L^{2}}^{2} \right] ds \\
\leq C(1 + t) \varepsilon (\| \omega_{0} \|_{H^{3}}^{2} + \| z_{0} \|_{H^{2}}^{2} + \delta_{1}).
\]

Theorems 3.1-3.3 can be proved by using the similar method as Theorems 2.1-2.3 and the details of the proofs are omitted here.

### 3.2 The case of $v_{0}(0) = v_{+}$

Let

\[
(v, u)(x, t) \equiv (v_{+}, 0).
\]

(3.2.1)

Similar to the case as above, the correction functions are defined by

\[
\hat{v}(x, t) = -(u_{0}(0) - u_{+}) m_{0}(x) B(t),
\]

and

\[
\hat{u}(x, t) = \left[ u_{+} + (u_{0}(0) - u_{+}) \int_{x}^{\infty} m_{0}(y) dy \right] \beta(t),
\]

where $m_{0}(x)$ is a smooth function with compact support such that

\[
\int_{0}^{\infty} m_{0}(x) dx = 1, \quad \text{supp} \, m_{0} \subset \mathbb{R}^{+}.
\]

Therefore, $(\hat{v}, \hat{u})(x, t)$ satisfies

\[
\begin{aligned}
\partial_{t} \hat{v} - \partial_{x} \hat{u} &= 0, \\
\partial_{t} \hat{u} &= -\frac{\alpha}{(1 + t)^{\lambda}} \hat{u}, \\
(\hat{v}, \hat{u}_{x})(0, t) &= (0, 0), \quad \hat{u}(0, t) = u_{+} \beta(t), \\
(\hat{v}, \hat{u})(\infty, t) &= (0, u_{+} \beta(t)).
\end{aligned}
\]

(3.2.2)

It follows from (3.1) and (3.2.2) that

\[
\begin{aligned}
\partial_{t}(v - v_{+} - \hat{v}) - \partial_{x}(u - \hat{u}) &= 0, \\
\partial_{t}(u - \hat{u}) + \partial_{x}(p(v) - p(v_{+})) + \frac{\alpha}{(1 + t)^{\lambda}} (u - \hat{u}) &= 0.
\end{aligned}
\]

(3.2.3)

The definition of

\[
\omega(x, t) = -\int_{x}^{\infty} (v(y, t) - v_{+} - \hat{v}(y, t)) dy,
\]

and

\[
z(x, t) = u(x, t) - \hat{u}(x, t),
\]

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remark 3.1.

It should be noted that there is no cut-off point of the convergence rate in this case.

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