On the weak Deligne-Simpson problem for index of rigidity 2 *

Vladimir Petrov Kostov

To the memory of my mother

Appendix by Ofer Gabber

1 Introduction

1.1 Regular and Fuchsian linear systems

Consider the linear system of ordinary differential equations defined on Riemann’s sphere:

\[ \frac{dX}{dt} = A(t)X \]  

(1)

where the \( n \times n \)-matrix \( A \) is meromorphic on \( \mathbb{C}P^1 \), with poles at \( a_1, \ldots, a_{p+1} \); the dependent variables \( X \) form an \( n \times n \)-matrix. Without loss of generality we assume that \( \infty \) is not among the poles \( a_j \) and not a pole of the 1-form \( A(t)dt \).

**Definition 1** System (1) is called regular at the pole \( a_j \) if its solutions have a moderate growth rate there, i.e. for every sector \( S \) centered at \( a_j \) and of sufficiently small radius and for every solution \( X \) restricted to the sector there exists \( N_j \in \mathbb{R} \) such that \( ||X(t - a_j)|| = O(|t - a_j|^{N_j}) \) for all \( t \in S \). System (1) is regular if it is regular at all poles \( a_j \).

System (1) is Fuchsian if its poles are logarithmic. Every Fuchsian system is regular.

A Fuchsian system admits the presentation

\[ \frac{dX}{dt} = (\sum_{j=1}^{p+1} \frac{A_j}{t - a_j})X , \quad A_j \in gl(n, \mathbb{C}) \]  

(2)

The sum of its matrices-residua \( A_j \) equals 0, i.e.

\[ A_1 + \ldots + A_{p+1} = 0 \]  

(3)

When one performs a linear change of the dependent variables

\[ X \mapsto W(t)X \]  

(4)

\( W \) being meromorphic on \( \mathbb{C}P^1 \), then system (1) changes as follows:

\[ A \rightarrow -W^{-1}(dW/dt) + W^{-1}AW \]  

(5)

*Research partially supported by INTAS grant 97-1644
(i.e. the system undergoes the gauge transformation). This transformation preserves regularity but, in general, it does not preserve being Fuchsian. The only invariant under the group of linear transformations is the monodromy group of the system.

Set \( \Sigma = \mathbb{CP}^1 \backslash \{a_1, \ldots, a_{p+1}\} \). Fix a base point \( a_0 \in \Sigma \) and a matrix \( B \in GL(n, \mathbb{C}) \).

**Definition 2** A monodromy operator of system (4) defined by the class of homotopy equivalence in \( \Sigma \) of a closed contour \( \gamma \) with base point \( a_0 \) and bypassing the poles of the system is a linear operator \( M \) acting on the solution space of the system which maps the solution \( X \) with \( X|_{t=a_0} = B \) into the value of its analytic continuation along \( \gamma \). Notation: \( X \stackrel{\gamma}{\mapsto} XM \).

The monodromy group is the subgroup of \( GL(n, \mathbb{C}) \) generated by all monodromy operators.

**Remark 3** The monodromy group is an antirepresentation \( \pi_1(\Sigma) \rightarrow GL(n, \mathbb{C}) \) because one has
\[
X \stackrel{\gamma_1}{\mapsto} XM_1 \stackrel{\gamma_2}{\mapsto} XM_2M_1 \quad (*),
\]
i.e. the concatenation \( \gamma_1 \gamma_2 \) of the two contours defines the monodromy operator \( M_2M_1 \).

One usually chooses a standard set of contours \( \gamma_j \), \( j = 1, \ldots, p+1 \) defining the generators \( M_j \) of the monodromy group as follows. One connects \( a_0 \) with the points \( a'_j \) (where \( a'_j \) is close to \( a_j \)) by simple Jordan curves \( \delta_j \) which intersect two by two only at \( a_0 \). The contour \( \gamma_j \) consists of \( \delta_j \), of a small circumference centered at \( a_j \) and passing through \( a'_j \) (run counterclockwise) and of \( \delta_j \) run from \( a'_j \) to \( a_0 \). Thus \( \gamma_j \) is freely homotopic to a small loop circumventing counterclockwise \( a_j \) (and no other pole \( a_i \)). The indices of the poles are chosen such that the indices of the contours increase from 1 to \( p+1 \) when one turns around \( a_0 \) clockwise.

For the standard choice of the contours the generators \( M_j \) satisfy the relation
\[
M_1 \ldots M_{p+1} = I \quad (6)
\]
which can be thought of as a multiplicative analog of (3) if the system is Fuchsian. Equality (3) results from (*) (see Remark 3) – the concatenation of contours \( \gamma_{p+1} \ldots \gamma_1 \) is homotopy equivalent to 0.

**Remarks 4**

1) The monodromy group is correctly defined only up to conjugacy due to the freedom to choose \( a_0 \) and \( B \).

2) For a Fuchsian system the generator \( M_j \) defined as above is conjugate to \( \exp(2\pi i A_j) \) if \( A_j \) has no eigenvalues differing by a non-zero integer.

3) The generators \( M_j \) of the monodromy group when defined after a standard set of contours \( \gamma_j \), are conjugate to the corresponding operators \( L_j \) of local monodromy, i.e. when the poles \( a_j \) are circumvented counterclockwise along small loops. The operators \( L_j \) of a regular system can be computed (up to conjugacy) algorithmically – one first makes the system Fuchsian at \( a_j \) by means of a change (4) as explained in [Mo] and then carries out the computation as explained in [Wa].

1.2 The Deligne-Simpson problem and its weak version

A natural question to ask is whether for given local monodromies (around the poles \( a_j \)) defined up to conjugacy there exists a Fuchsian system with such local monodromies; this is a realization problem. The difficulty is that one must have (3). A similar question can be asked for matrices \( A_j \) whose sum is 0 (see (4)). The question can be made more precise:
Give necessary and sufficient conditions on the choice of the conjugacy classes \( C_j \subset GL(n, \mathbb{C}) \) or \( c_j \subset gl(n, \mathbb{C}) \) so that there exist irreducible \((p + 1)\)-tuples of matrices \( M_j \in C_j \) or \( A_j \in c_j \) satisfying respectively (6) or (3).

This is the Deligne-Simpson problem (DSP). “Irreducible” means “with no common proper invariant subspace”. In technical terms, impossible to bring the \((p + 1)\)-tuple to a block upper-triangular form with the same sizes of the diagonal blocks for all matrices \( M_j \) or \( A_j \) by simultaneous conjugation.

**Remarks 5**

1) A priori one does not expect the problem to include the requirement of irreducibility. However, for almost all possible eigenvalues of the conjugacy classes the monodromy group is indeed irreducible and for such eigenvalues (called generic, see the next definition) the answer to the problem depends actually not on the conjugacy classes but only on the Jordan normal forms which they define, see Theorem 7.

2) In the multiplicative version (i.e. for matrices \( M_j \)) the DSP was stated by P. Deligne (in the additive, i.e. for matrices \( A_j \), presumably by the author) and C. Simpson was the first to obtain important results towards its resolution, see [5]. The multiplicative version is more important because the monodromy group is invariant under the action of the group of linear changes [4] while the matrices-residua of a Fuchsian system are not, see rule (5) and the lines following it. The additive version is technically easier to deal with and one can deduce corollaries about the multiplicative one due to 2) of Remarks 4.

We consider only such conjugacy classes \( C_j \) (resp. \( c_j \)) for which the necessary condition \( \prod \det(C_j) = 1 \) (resp. \( \sum \text{Tr}(c_j) = 0 \)) holds. (These conditions result from (6) and (3) respectively.) In terms of the eigenvalues \( \sigma_{k,j} \) (resp. \( \lambda_{k,j} \)) of the matrices from \( C_j \) (resp. \( c_j \)) repeated with their multiplicities, these conditions read

\[
\prod_{k=1}^{n} \prod_{j=1}^{p+1} \sigma_{k,j} = 1, \quad \text{resp.} \quad \sum_{k=1}^{n} \sum_{j=1}^{p+1} \lambda_{k,j} = 0 \tag{7}
\]

**Definition 6** An equality of the form

\[
\prod_{j=1}^{p+1} \prod_{k \in \Phi_j} \sigma_{k,j} = 1, \quad \text{resp.} \quad \sum_{j=1}^{p+1} \sum_{k \in \Phi_j} \lambda_{k,j} = 0 \tag{8}
\]

is called a non-genericity relation; the non-empty sets \( \Phi_j \) contain one and the same number \(< n\) of indices for all \( j \). Eigenvalues satisfying none of these relations are called generic. Reducible \((p + 1)\)-tuples exist only for non-generic eigenvalues (the eigenvalues of each diagonal block of a block upper-triangular \((p + 1)\)-tuple satisfy some non-genericity relation).

**Remark 7** In the case of matrices \( A_j \), if the greatest common divisor \( d \) of the multiplicities of all eigenvalues of all \( p + 1 \) matrices is \( > 1 \), then a non-genericity relation results automatically from \( \sum \text{Tr}(c_j) = 0 \). In the case of matrices \( M_j \) the equality \( \prod \sigma_{k,j} = 1 \) implies that if one divides by \( d \) the multiplicities of all eigenvalues, then their product would equal \( \exp(2\pi ik/d) \), \( 0 \leq k \leq d - 1 \), not necessarily 1, and a non-genericity relation might or might not hold (see Example 8).
Definition 8 Call Jordan normal form (JNF) of size $n$ a family $J^n = \{b_{i,l}\}$ ($i \in I_l$, $I_l = \{1, \ldots, s_l\}$, $l \in L$) of positive integers $b_{i,l}$ whose sum is $n$. Here $L$ is the set of indices of eigenvalues (all distinct) and $I_l$ is the set of Jordan blocks with the $l$-th eigenvalue, $b_{i,l}$ is the size of the $i$-th block with this eigenvalue. E.g. the JNF $\{\{2,1\}\{4,3,1\}\}$ is of size 11 and with two eigenvalues to the first (resp. second) of which there correspond two (resp. three) Jordan blocks, of sizes 2 and 1 (resp. 4, 3 and 1). An $n \times n$-matrix $Y$ has the JNF $J^n$ (notation: $J(Y) = J^n$) if to its distinct eigenvalues $\lambda_l, l \in L$, there belong Jordan blocks of sizes $b_{i,l}$. We denote by $J(C)$ the JNF defined by the conjugacy class $C$.

Example 9 Let $p + 1 = n = 4$, let for each $j$ the JNF $J^n_j$ consist of two Jordan blocks $2 \times 2$, with equal eigenvalues. If the eigenvalues of $M_1, \ldots, M_4$ equal $i, 1, 1$ and 1 (resp. $-1, 1, 1, 1$), then they are generic (resp. they are non-generic) – when their multiplicities are reduced twice, then their product equals $-1$, a primitive root of 1 of order 2 (resp. their product equals 1).

When the eigenvalues are not generic, then the $(p+1)$-tuples of matrices $M_j \in C_j$ or $A_j \in c_j$ (if it exists) is not automatically irreducible. Therefore we give the following

Definition 10 The formulation of the weak Deligne-Simpson problem is obtained when in the one of the DSP one replaces the requirement of irreducibility by the weaker requirement the centralizer of the $(p+1)$-tuple of matrices $A_j$ or $M_j$ to be trivial, i.e. reduced to scalars.

Remark 11 If one states the problem asking only the matrices $M_j \in C_j$ or $A_j \in c_j$ to satisfy respectively condition (3) or (4) and with no requirement of irreducibility or triviality of the centralizer, then solving the problem becomes much harder and the answer to it depends essentially on the eigenvalues (not only on the JNFs). E.g., suppose that $p = n = 2$ and that two of the matrices $M_j$ (resp. $A_j$) have distinct eigenvalues $\sigma_{1,j} \neq \sigma_{2,j}$, $j = 1, 2$ (resp. $\lambda_{1,j} \neq \lambda_{2,j}$) while the third must be scalar (i.e. $\sigma_{1,3} = \sigma_{2,3}$, resp. $\lambda_{1,3} = \lambda_{2,3}$). Then such triples exist exactly if $\sigma_{1,1}\sigma_{1,2}\sigma_{1,3} = 1$ or $\sigma_{1,1}\sigma_{2,2}\sigma_{3,3} = 1$ (resp. $\lambda_{1,1} + \lambda_{1,2} + \lambda_{1,3} = 0$ or $\lambda_{1,1} + \lambda_{2,2} + \lambda_{1,3} = 0$). Hence, such triples exist exactly if the eigenvalues are not generic.

Definition 12 Denote by $\{J^n_j\}$ a $(p+1)$-tuple of JNFs, $j = 1, \ldots, p + 1$. We say that the DSP (resp. the weak DSP) is solvable for a given $\{J^n_j\}$ and given eigenvalues if there exists an irreducible $(p+1)$-tuple (resp. a $(p+1)$-tuple with trivial centralizer) of matrices $M_j$ satisfying (4) or of matrices $A_j$ satisfying (3), with $J(M_j) = J^n_j$ or $J(A_j) = J^n_j$ and with the given eigenvalues. By definition, the DSP is solvable for $n = 1$. Solvability of the DSP implies the one of the weak DSP.

For generic eigenvalues the DSP is solved – the result is formulated in [Ko3] and proved in [Ko4] and [Ko2]. In the next subsection we recall this result (Theorem 15). The result is a necessary and sufficient condition on the JNFs $J(C_j)$ or $J(c_j)$.

The aim of the present paper is to show an example of a large class of $(p+1)$-tuples of conjugacy classes $C_j$ or $c_j$ (such that the conditions of Theorem 15 hold for the JNFs $J(C_j)$ or $J(c_j)$), with non-generic eigenvalues, for which the weak DSP is not solvable.
1.3 The results known up to now

**Notation 1** For a conjugacy class $C$ in $GL(n, \mathbb{C})$ or $gl(n, \mathbb{C})$ denote by $d(C)$ its dimension and for a matrix $Y$ from $C$ set $r(C) := \min_{\lambda \in \mathbb{C}} \text{rank}(Y - \lambda I)$. The integer $n - r(C)$ is the maximal number of Jordan blocks of $J(Y)$ with one and the same eigenvalue. Set $d_j := d(C_j)$ (resp. $d(c_j)$), $r_j := r(C_j)$ (resp. $r(c_j)$). The quantities $r(C)$ and $d(C)$ depend only on the JNF $J(Y) = J^n$, not on the eigenvalues, so we write sometimes $r(J^n)$ and $d(J^n)$.

**Proposition 13** (C. Simpson, see [Si].) The following two inequalities are necessary conditions for the solvability of the DSP:

\[
d_1 + \ldots + d_{p+1} \geq 2n^2 - 2 \quad (\alpha_n)
\]
\[
\text{for all } j, \ r_1 + \ldots + r_j + \ldots + r_{p+1} \geq n \quad (\beta_n)
\]

The following condition is not necessary and in most cases it is sufficient for the solvability of the DSP, see [Ko3] and [Ko1]:

\[
(r_1 + \ldots + r_{p+1}) \geq 2n \quad (\omega_n)
\]

For a given $\{J^n_j\}$ with $n > 1$, which satisfies conditions $(\alpha_n)$ and $(\beta_n)$ and doesn’t satisfy condition $(\omega_n)$ set $n_1 = r_1 + \ldots + r_{p+1} - n$. Hence, $n_1 < n$ and $n - n_1 \leq n - r_j$. Define the $(p+1)$-tuple $\{J'^{n_1}_j\}$ as follows: to obtain the JNF $J'^{n_1}_j$ from $J^n_j$ one chooses one of the eigenvalues of $J^n_j$ with greatest number $n - r_j$ of Jordan blocks, then decreases by 1 the sizes of the $n - n_1$ smallest Jordan blocks with this eigenvalue and deletes the Jordan blocks of size 0. For the above construction we use the notation $\Psi : \{J^n_j\} \mapsto \{J'^{n_1}_j\}$.

**Definition 14** A $(p+1)$-tuple $\{J^n_j\}$ with $n > 1$ is good if

1) it satisfies conditions $(\alpha_n)$ and $(\beta_n)$ and

2) either $\{J^n_j\}$ satisfies condition $(\omega_n)$ or the construction $\Psi$ iterated as long as it is defined stops at a $(p+1)$-tuple $\{J'_{n'}^j\}$ either with $n' = 1$ or satisfying condition $(\omega_{n'})$.

By definition, a $(p+1)$-tuple of JNFs with $n = 1$ is good.

**Theorem 15** (see [Ko3], Theorem 8.) Let $n > 1$. The DSP is solvable for the conjugacy classes $C_j$ or $c_j$ (with generic eigenvalues, defining the JNFs $J^n_j$) if and only if the $(p+1)$-tuple $\{J^n_j\}$ is good.

**Remark 16** The quantity $\kappa = 2n^2 - \sum_{j=1}^{p+1} d_j$ (called index of rigidity, see [Ko4]) is invariant for the construction $\Psi$, see [Ko3]. Therefore one can drop condition $(\alpha_n)$ in the definition of a good $(p+1)$-tuple -- condition $(\alpha_{n'})$ always holds for the $(p+1)$-tuple of JNFs $\{J'_{n'}^j\}$, see [Ko3] and [Ko4]; if $n' = 1$, then $(\alpha_{n'})$ is an equality, if there holds $(\omega_{n'})$, then $(\alpha_{n'})$ holds and is a strict inequality.

1.4 The basic result

In the present paper we consider the case when the index of rigidity equals 2.
Remark 17 In this case if there exist irreducible \((p + 1)\)-tuples, then they are unique up to conjugacy and the coexistence of irreducible and reducible \((p + 1)\)-tuples is impossible, see [Bo] and [Si] for the multiplicative version.

In the additive version this is also true – if there exist an irreducible and a reducible \((p + 1)\)-tuples of matrices \(A_j \in c_j\), then one can multiply them by a non-zero scalar \(h\) so that there should be no non-zero integer differences between any two eigenvalues of a given matrix \(hA_j\) and any integer sum of eigenvalues of the matrices should be 0. The monodromy operators of a Fuchsian system with matrices-residua \(hA_j\) are conjugate to \(\exp(2\pi ihA_j)\). The monodromy group of the system with (ir)reducible \((p + 1)\)-tuple of matrices-residua \(hA_j\) is (ir)reducible as well; in the reducible case this is evident, in the irreducible one this follows from Theorem 5.1.2 from [Bo] (the latter states that if the monodromy group is reducible and if the sum of the exponents relative to an invariant subspace is 0, then the matrices-residua can be simultaneously conjugated to a block upper-triangular form; the sum of the exponents for the system with residua \(hA_j\) is a sum of eigenvalues of these matrices; thus the irreducibility of the \((p + 1)\)-tuple of matrices-residua implies the one of the monodromy group). This is a contradiction with the non-coexistence of irreducible and reducible \((p + 1)\)-tuples in the multiplicative case.

Definition 18 We say that the conjugacy class \(c'\) (in \(gl(n, C)\) or in \(GL(n, C)\)) is subordinate to the conjugacy class \(c\) if \(c'\) belongs to the closure of \(c\). This means that the eigenvalues of \(c\) and \(c'\) are the same and of the same multiplicities and for each eigenvalue \(\lambda_i\) and for each \(j \in \mathbb{N}\) one has \(\text{rk}(A - \lambda_i I)^j \geq \text{rk}(A' - \lambda_i I)^j\) for \(A \in c, A' \in c'\). If \(c' \neq c\), then at least one inequality is strict.

Definition 19 Let \(n = ln_1, l, n_1 \in \mathbb{N}^*, n_1 > 1\). The \((p + 1)\)-tuple of conjugacy classes \(C_j\) or \(c_j\) with \(\kappa = 2\) is called \(l\)-special if for each class \(C_j\) (or \(c_j\)) there exists a class \(C'_j\) (or \(c'_j\)) subordinate to it which is a direct sum of \(n_1\) copies of a conjugacy class \(C''_j \subset GL(l, C)\) (or \(c''_j \subset gl(l, C)\)) where the \((p + 1)\)-tuple of JNFs \(J(C''_j)\) (or \(J(c''_j)\)) is good and the product of the eigenvalues of the classes \(C''_j\) equals 1 (see Remark \[3\] and Example \[3\], for the classes \(c''_j\) the sum of their eigenvalues is automatically 0).

Remarks 20 1) The index of rigidity of the \((p + 1)\)-tuple of conjugacy classes \(c''_j\) or \(C''_j\) equals 2. Indeed, one has \(d(c_j) \geq d(c'_j) = (n_1)^2 d(c''_j)\) and if \(\sum_{j=1}^{p+1} d(c''_j) \geq 2l^2\), then \(\sum_{j=1}^{p+1} d(c_j) \geq 2n^2\), i.e. the index of rigidity of the \((p + 1)\)-tuple of conjugacy classes \(c_j\) must be non-positive – a contradiction. The reasoning holds in the case of classes \(C_j\) as well.

2) It follows from the above definition that in the case of matrices \(A_j\) the eigenvalues of an \(l\)-special \((p + 1)\)-tuple of JNFs cannot be generic – their multiplicities are divisible by \(n_1\) and, hence, they satisfy a non-genericity relation, see Remark \[4\]. Notice that in the case of matrices \(M_j\) the divisibility by \(n_1\) alone of the multiplicities does not imply that the eigenvalues are not generic, see Remark \[5\] and Example \[5\]. Therefore the requirement the product of the eigenvalues of the classes \(C''_j\) to equal 1 (see the definition) is essential.

Definition 21 A \((p + 1)\)-tuple of conjugacy classes in \(gl(n, C)\) or \(GL(n, C)\) is called special if it is \(l\)-special for some \(l\). If in addition for this \(l\) the classes \(c''_j\) or \(C''_j\) are diagonalizable, then the \((p + 1)\)-tuple is called special-diagonal.
Example 22 The triple of conjugacy classes $C_1, C_2, C_3 \subset GL(4, \mathbb{C})$ (or $c_1, c_2, c_3 \subset gl(4, \mathbb{C})$) defining the JNFs $J^4_1 = \{\{4\}\}, J^4_2 = \{\{1,1\}, \{2\}\}, J^4_3 = \{\{1,1\}, \{1,1\}\}$, is good (to be checked directly).

This triple is also 2-special – one can choose as subordinate classes ones in which $C_1'$ (resp. $c_1'$) has two Jordan blocks of size 2 while $C_2'$ and $C_3'$ (resp. $c_2'$ and $c_3'$) define the JNF $J^4_3$. For such a choice the triple $(C_1', C_2', C_3')$ (resp. $(c_1', c_2', c_3')$) will be 2-special and the conjugacy classes $C_j'' \subset GL(2, \mathbb{C})$ (resp. $c_j'' \subset gl(2, \mathbb{C})$) have two distinct eigenvalues for $j = 2, 3$ and one Jordan block of size 2 for $j = 1$.

Example 23 For $n > 1$ a good $(p + 1)$-tuple of unipotent conjugacy classes in $GL(n, \mathbb{C})$ or of nilpotent conjugacy classes in $gl(n, \mathbb{C})$ is 1-special, hence, it is special.

Example 24 For $n = 9$ the triple of conjugacy classes $c_j$ defining the JNFs $\{\{2,2,1,1\}, \{1,1,1\}\}$ for $j = 1, 2$ and $\{\{2,2,1,1\}, \{2,1\}\}$ for $j = 3$ is good. Although the multiplicities of all eigenvalues are divisible by 3, the triple is not special (a priori if it is special, then it is 3-special). Indeed, the JNFs are such that the conjugacy classes $c_j''$ from the definition of a special $(p + 1)$-tuple must be diagonalizable (for each eigenvalue of $c_j$ there are at most two Jordan blocks of size $> 1$ and this size is actually 2). But then $c_j''$ must have each two eigenvalues, of multiplicities 1 and 2, which means that the triple $J(c_1''), J(c_2''), J(c_3'')$ is not good.

The basic result of the paper is the following

Theorem 25 The weak DSP is not solvable for special-diagonal $(p + 1)$-tuples of conjugacy classes.

Remark 26 The $(p + 1)$-tuple of conjugacy classes to be good is a necessary condition for the solvability of the weak DSP for index of rigidity 2, see the proof of this in Remark 37. (In fact, it is necessary for any index of rigidity $\leq 2$ but we do not need the proof of this statement in the present paper.)

Remark 27 The theorem raises the following two natural questions:

1) whether it remains true for special (not necessarily special-diagonal) $(p + 1)$-tuples of conjugacy classes;

2) whether for index of rigidity 2 and for $(p + 1)$-tuples of conjugacy classes defining good $(p + 1)$-tuples of JNFs the weak DSP is unsolvable only when the $(p + 1)$-tuples of conjugacy classes are special.

It seems that the answer to the first of them is positive although the author was unable to find a complete proof of it. The answer to the second question is not known to the author.

The next subsection contains the plan of the proof of the theorem. The rest of the proof is given in Section 3.
1.5 Plan of the proof of Theorem 25

We consider first the particular case when the conjugacy classes $C_j''$, resp. $c_j''$, are diagonalizable and with generic eigenvalues; we assume also that all non-genericity relations for the classes $c_j$ or $C_j$ are obvious ones, i.e. multiples of the fact that the sum of the eigenvalues of the classes $c_j''$ is 0 or that the product of the eigenvalues of the classes $C_j''$ is 1. Call this case and such special-diagonal $(p + 1)$-tuples of conjugacy classes quasi-generic. The proof in the general case is deduced from the quasi-generic one using a method for deforming analytically $(p + 1)$-tuples of matrices $A_j$ or $M_j$ (the method is called the basic technical tool, it is developed in Section 2). Namely, if there exists a $(p + 1)$-tuple with trivial centralizer of matrices $A_j$ or $M_j$ from a special-diagonal $(p + 1)$-tuple of conjugacy classes, then it can be analytically deformed into one defining the same JNFs, for which the classes $c_j''$ or $C_j''$ have generic eigenvalues and which is with trivial centralizer. The deformation changes the eigenvalues but not their multiplicities. The possibility to deform the eigenvalues into ones from the quasi-generic case follows from the fact that the classes $c_j''$ or $C_j''$ are semisimple and the index of rigidity of their $(p + 1)$-tuple is 2; hence, the multiplicities of their eigenvalues are not divisible by an integer $> 1$. Non-solvability of the weak DSP for quasi-generic special-diagonal $(p + 1)$-tuples implies its non-solvability for any special-diagonal $(p + 1)$-tuples.

The following proposition was suggested by the author and proved by Ofer Gabber (see the proof in the Appendix).

**Proposition 28** Suppose that the index of rigidity of the $(p + 1)$-tuple of conjugacy classes $C_j \subset GL(n, \mathbb{C})$ or $c_j \subset gl(n, \mathbb{C})$ equals 2. Suppose that the conjugacy classes $C_j^*$ (or $c_j^*$) are subordinate to the respective classes $C_j$ (or $c_j$), with $C_j^* \neq C_j$ (or $c_j^* \neq c_j$) for at least one value of $j$. Then the existence of matrices $M_j \in C_j^*$ satisfying condition (4) (resp. of matrices $A_j \in c_j^*$ satisfying condition (4)) implies that the DSP is not solvable for the $(p + 1)$-tuple of conjugacy classes $C_j$ (resp. $c_j$).

As it was mentioned in Remark 17, coexistence of irreducible and reducible $(p + 1)$-tuples of matrices $A_j$ or $M_j$ with index of rigidity 2 is impossible. Therefore the above proposition implies

**Corollary 29** The DSP is not solvable for special $(p + 1)$-tuples of conjugacy classes $C_j$ or $c_j$.

Indeed, for such $(p + 1)$-tuples one can construct block-diagonal $(p + 1)$-tuples of matrices $M_j \in C_j'$ or $A_j \in c_j'$ satisfying respectively (4) or (4); this follows from the $(p + 1)$-tuple of JNFs $J(C_j'')$ or $J(c_j'')$ being good.

Further we need the following

**Proposition 30** For index of rigidity $\kappa$ the dimension of the variety $V$ (when it is not empty) of $(p + 1)$-tuples with trivial centralizers of matrices $A_j \in c_j$ or $M_j \in C_j$ equals $n^2 + 1 - \kappa$.

The proposition is proved in Subsection 3.1.

**Proposition 31** A reducible $(p + 1)$-tuple with trivial centralizer of matrices $A_j \in c_j$ or $M_j \in C_j$, where the index of rigidity of the $(p + 1)$-tuple of conjugacy classes $c_j$ or $C_j$ equals 2, is up to conjugacy block upper-triangular with diagonal blocks defining irreducible representations each with index of rigidity 2.
The proposition was suggested by the author and proved by O. Gabber, see the proof in the Appendix. Note that the \((p + 1)\)-tuple of conjugacy classes is not supposed to be special. The proposition implies

**Corollary 32** All (if any) quasi-generic special-diagonal \((p + 1)\)-tuples of matrices \(A_j \in c_j\) or \(M_j \in C_j\) which solve the weak DSP are (up to conjugacy) block upper-triangular and their diagonal blocks define irreducible representations \(P_i\) each with index of rigidity 2.

**Lemma 33** If a quasi-generic special-diagonal \((p + 1)\)-tuple of matrices \(A_j \in c_j\) or \(M_j \in C_j\) is (up to conjugacy) block upper-triangular with diagonal blocks defining irreducible representations each with index of rigidity 2, then these representations are equivalent and of rank \(l\).

The lemma is proved in Subsection 3.2. It implies

**Corollary 34** A \((p + 1)\)-tuple of matrices \(A_j \in c_j\) or \(M_j \in C_j\) satisfying the conditions of Lemma 33 is with non-trivial centralizer.

Indeed, block-decompose the matrices from \(gl(n, \mathbb{C})\) or \(GL(n, \mathbb{C})\) into blocks \(l \times l\). Hence, the centralizer contains the matrix having a block equal to \(I\) in position \((1, n/l)\) and zeros elsewhere. \(\Box\)

Hence, the weak DSP is not solvable for quasi-generic special-diagonal \((p + 1)\)-tuples of conjugacy classes. Indeed, if it were solvable, then it would be solved only by reducible \((p + 1)\)-tuples of matrices which up to conjugacy are block upper-triangular, with diagonal blocks defining equivalent representations of rank \(l\) and of index of rigidity 2, see Corollary 32 and Lemma 33. By Corollary 34, these \((p + 1)\)-tuples have non-trivial centralizers.

This proves Theorem 25 in the quasi-generic case.

2 The basic technical tool

**Definition 35** Call basic technical tool the procedure described below whose aim is to deform analytically a given \((p + 1)\)-tuple of matrices \(A_j\) or \(M_j\) with trivial centralizer by changing their conjugacy classes in a desired way.

Set \(A_j = Q_j^{-1}G_jQ_j\), \(G_j\) being Jordan matrices. Look for a \((p + 1)\)-tuple of matrices \(\tilde{A}_j\) (whose sum is 0) of the form

\[
\tilde{A}_j = (I + \varepsilon X_j(\varepsilon))^{-1}Q_j^{-1}(G_j + \varepsilon V_j(\varepsilon))Q_j(I + \varepsilon X_j(\varepsilon))
\]

where \(\varepsilon \in (\mathbb{C}, 0)\) and \(V_j(\varepsilon)\) are given matrices analytic in \(\varepsilon\); they must satisfy the condition \(\text{tr}(\sum_{j=1}^{p+1} \varepsilon V_j(\varepsilon)) \equiv 0\); set \(N_j = Q_j^{-1}V_jQ_j\). The existence of matrices \(X_j(\varepsilon)\) is deduced from the triviality of the centralizer, using the following proposition (see its proof and the details in [Ko4]):

**Proposition 36** The centralizer of the \(p\)-tuple of matrices \(A_j\) \((j = 1, \ldots, p)\) is trivial if and only if the mapping \((sl(n, \mathbb{C}))^p \rightarrow sl(n, \mathbb{C}), (X_1, \ldots, X_p) \mapsto \sum_{j=1}^{p} [A_j, X_j]\) is surjective.
Notice that one has $\tilde{A}_j = A_j + \varepsilon [A_j, X_j(0)] + \varepsilon N_j + o(\varepsilon)$. The proposition assures the existence in first approximation w.r.t. $\varepsilon$ of the matrices $X_j$, the existence of true matrices $X_j$ analytic in $\varepsilon$ follows from the implicit function theorem.

If for $\varepsilon \neq 0$ small enough the eigenvalues of the matrices $\tilde{A}_j$ are generic, then their $(p+1)$-tuple is irreducible. In a similar way one can deform analytically $(p+1)$-tuples depending on a multi-dimensional parameter.

Given a $(p+1)$-tuple of matrices $M_j$ with trivial centralizer and satisfying condition (3), look for matrices $M_j$ (whose product is $I$) of the form

$$M_j = (I + \varepsilon X_j(\varepsilon))^{-1} (M_j^1 + \varepsilon N_j(\varepsilon)) (I + \varepsilon X_j(\varepsilon))$$

where the given matrices $N_j$ depend analytically on $\varepsilon \in (C, 0)$ and the product of the determinants of the matrices $M_j$ is 1; one looks for $X_j$ analytic in $\varepsilon$. (Like in the additive version one can set $M_j^1 = Q_j^{-1} G_j Q_j$, $N_j = Q_j^{-1} V_j Q_j$.) The existence of such matrices $X_j$ follows again from the triviality of the centralizer, see [Ko4].

**Remark 37** As it was mentioned in Remark [24], for index of rigidity 2 the $(p+1)$-tuple of conjugacy classes to be good is a necessary condition for the solvability of the weak DSP. Indeed, using the basic technical tool one can deform a $(p+1)$-tuple of matrices $A_j$ or $M_j$ with trivial centralizer into a nearby one (say, of matrices $A'_j$ or $M'_j$) defining the so-called corresponding diagonal JNFs (see the definition in [Ko3] or in [Ko4]); both $(p+1)$-tuples are simultaneously good or not and the latter $(p+1)$-tuple is also of index of rigidity 2. Hence, the multiplicities of the eigenvalues of the matrices $A'_j$ or $M'_j$ are not divisible by an integer $> 1$ and one can choose the eigenvalues to be generic. But then one applies Theorem [13] and finds out that the $(p+1)$-tuple of initial conjugacy classes must be good.

### 3 Proof of Theorem 25

#### 3.1 Proof of Proposition 30

10. Consider first the case of matrices $A_j$. Without restriction one can assume that $c_j \subset \text{sl}(n, C)$. To find the dimension of the variety $V$ one has first to consider the cartesian product $(c_1 \times \ldots \times c_p) \subset (\text{sl}(n, C))^p$. Define the mapping $\tau : (c_1 \times \ldots \times c_p) \rightarrow \text{sl}(n, C)$ by the rule $\tau : (A_1, \ldots, A_p) \mapsto A_{p+1} = -A_1 - \ldots - A_p$ (recall that there holds (3)).

20. The algebraic variety $V$ is the intersection of the two varieties in $c_1 \times \ldots \times c_p \times \text{sl}(n, C)$: the graph of the mapping $\tau$ and $c_1 \times \ldots \times c_p \times \text{sl}(n, C)$. This intersection is transversal which implies the smoothness of the variety $V$. Transversality follows from Proposition 30 - the tangent space to the conjugacy class $c_j$ at $A_j$ equals $\{[A_j, X] \mid X \in gl(n, C)\}$.

30. Recall that $d_j$ denotes $\text{dim} c_j$. Hence,

$$\dim V = (\sum_{j=1}^{p} d_j) - \text{codim}_{\text{sl}(n, C)} c_{p+1} = (\sum_{j=1}^{p} d_j) - [(n^2 - 1) - d_{p+1}].$$

Hence, $\dim V = \sum_{j=1}^{p+1} d_j - n^2 + 1 = 2n^2 - \kappa - n^2 + 1 = n^2 + 1 - \kappa$.

40. The only difference in the proof in the case of matrices $M_j$ is that the mapping $(A_1, \ldots, A_p) \mapsto A_{p+1} = -A_1 - \ldots - A_p$ from 20 has to be replaced by the mapping $(M_1, \ldots, M_p) \mapsto M_{p+1} = (M_1 \ldots M_p)^{-1}$.

The reader will be able to restitute the missing technical details after examining the more detailed description of the basic technical tool given in [Ko4]. The proposition is proved. $\Box$
3.2 Proof of Lemma 33

1°. Recall that in the quasi-generic case the conjugacy classes \( c''_j \) or \( C''_j \) are diagonalizable, the DSP is solvable for them, and the irreducible representation \( Q \) they define is with index of rigidity 2; recall that \( Q \) is unique up to conjugacy, see Remark 17. We denote by \( P_i \) also the diagonal block defining the representation \( P_i \).

A priori in the quasi-generic case every representation \( P_i \) is of rank \( \ell q_i \), \( q_i \in \mathbb{N}^+ \), and the multiplicity of every eigenvalue of the matrix \( A_j \) (or \( M_j \)) restricted to the block \( P_i \) equals \( q_i \) times its multiplicity as eigenvalue of \( c''_j \) (or of \( C''_j \)). This follows from the fact that the eigenvalues of the conjugacy classes \( c''_j \) or \( C''_j \) are generic.

2°. Denote by \( c''_j \) (resp. \( C''_j \)) the conjugacy class of the restriction of the matrix \( A_j \) (resp. \( M_j \)) to the block \( P_i \). For every \( i \) the index of rigidity of the \((p + 1)\)-tuple of conjugacy classes \( c''_j \) or \( C''_j \) equals 2 (Corollary 29). Suppose that a given block \( P_i \) is of size \( q_i \ell \) with \( q_i > 1 \). Hence, the conjugacy class which is \( q_i \) times the conjugacy class \( c''_j \) (resp. \( q_i \) times \( C''_j \)) is subordinate to \( c''_j \) (resp. to \( C''_j \)) for \( j = 1, \ldots, p + 1 \). Indeed, the classes \( q_i c''_j \) (resp. \( q_i C''_j \)) and \( c''_j \) (resp. \( C''_j \)) have the same eigenvalues, of the same multiplicities and \( c''_j \) (resp. \( C''_j \)) is diagonalizable.

This means that the \((p + 1)\)-tuple of conjugacy classes \( c''_j \) or \( C''_j \) is quasi-generic \( l \)-special and of size \( q_i \ell \).

3°. The DSP is not solvable for the \((p + 1)\)-tuple of conjugacy classes \( c''_j \) or \( C''_j \) if \( q_i > 1 \), see Corollary 29. Hence, one must have \( q_i = 1 \), i.e. all diagonal blocks are of equal size. Hence, the conjugacy classes of the restrictions of the matrices \( M_j \) (resp. \( A_j \)) to them equal \( C''_j \) (resp. \( c''_j \)). Indeed, the eigenvalues and the multiplicities are the ones of the classes \( C''_j \) (resp. \( c''_j \)) – recall that we are in the quasi-generic case. The presence of Jordan blocks of size > 1 would mean that the sum of the dimensions \( d(C(M_j|P_i)) \) (resp. \( d(C(A_j|P_i)) \)) is > \( 2n^2 - 2 \), i.e. the index of rigidity of the \((p + 1)\)-tuple of matrices \( M_j|P_i \) (resp. \( A_j|P_i \)) is non-positive – a contradiction with Proposition 31.

4°. The index of rigidity of the \((p + 1)\)-tuple of conjugacy classes \( C''_j \) (resp. \( c''_j \)) being 2, the diagonal blocks define equivalent representations (see Remark 17). \( \square \)

Appendix (Ofer Gabber)

Proof of Proposition 28.

We prove the proposition in the multiplicative version. The proof for the additive one can be deduced by means of a reasoning completely analogous to the one from Remark 17 and we leave it for the reader.

Consider distinct points \( a_0, a_1, \ldots, a_{p+1} \) in \( \mathbf{P}^1_\mathbb{C} \). Set \( U = \mathbf{P}^1_\mathbb{C} \setminus \{a_1, \ldots, a_{p+1}\} \) and fix usual generators \( \gamma_0 \) of \( \pi_1(U, a_0) \). Set \( j : U \hookrightarrow \mathbf{P}^1_\mathbb{C} \). If \( L \) is a local system (of finite dimensional \( \mathbb{C} \) vector spaces) on \( U \) we have

\[
\chi(\mathbf{P}^1_\mathbb{C}, j_*L) = 2rkL - \sum_{i=1}^{p+1} \text{drop}_{a_i}(j_*L)
\]

with \( \text{drop}_{a_i}(j_*L) = rkL - \dim(j_*L)_{a_i} \). The drop depends only on the conjugacy class of the local monodromy at \( a_i \) and decreases under specialization.

We have a map

\[
\text{hom} : \text{conjugacy classes in } GL(n) \times \text{conjugacy classes in } GL(m) \rightarrow \text{conjugacy classes in } GL(nm)
\]
\(([A], [B]) \mapsto \text{conjugacy class of } (T \mapsto BTA^{-1})\)

where \(T \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^m)\). If \([A'] \subset \text{closure of } [A], [B'] \subset \text{closure of } [B]\), then \(\text{hom}([A'], [B']) \subset \text{closure of } \text{hom}([A], [B])\).

Let \(L\) be a local system with local monodromies in \(C_j\), \(L^*\) a local system with local monodromies in \(C_j^*\), \(L\) being irreducible. By assumption \(\chi_{P_1, j_*\text{Hom}(L, L^*)} = 2\). Now the local monodromies of \(\text{Hom}(L, L^*)\) have conjugacy classes in the closures of the conjugacy classes of the corresponding local monodromies of \(\text{Hom}(L, L)\), so

\[
\forall i \quad \text{drop}_a \text{Hom}(L, L^*) \leq \text{drop}_a \text{Hom}(L, L),
\]

so \(\chi_{P_1, j_*\text{Hom}(L, L^*)} \geq 2\) and when the centralizer is reduced to scalars “rigid” \(\Leftrightarrow\) “index of rigidity = 2”. We deduce the proposition from the more general one:

**Proposition 38** For a rigid \((p + 1)\)-tuple, all irreducible subquotients of the representation are rigid.

**Proof of Proposition 31:**

Consider the multiplicative version (the proof in the additive one is performed by analogy). Suppose we are given the conjugacy classes \(C_i \subset GL(n, \mathbb{C}), 1 \leq i \leq p + 1\), and we are interested in solutions of

\[
M_1 \ldots M_{p+1} = \text{id}, \ M_i \in C_i
\]

We say that a solution \(M = (M_1, \ldots, M_{p+1})\) is rigid if every solution \(M'\) in some neighbourhood of \(M\) is \(GL(n, \mathbb{C})\)-conjugate to \(M\). Here “neighbourhood” can be taken in the classical or in the Zariski topology. Excluding the case \(n = 0\), for a rigid \((p + 1)\)-tuple the index of rigidity is \(\geq 1 + \text{dim(centralizer)} \geq 2\) and when the centralizer is reduced to scalars “rigid” \(\Leftrightarrow\) “index of rigidity = 2”. We deduce the proposition from the more general one:

**Proposition 38** For a rigid \((p + 1)\)-tuple, all irreducible subquotients of the representation are rigid.

We say that a finite dimensional representation of \(\pi_1\) is isotypical if all its irreducible subquotients are isomorphic. To prove Proposition 38 we consider first the case of an isotypical representation \(\rho : \pi_1 \to GL(V)\) of dimension \(n = n_1l\) where \(n_1, l > 0\), \(\tau : \pi_1 \to GL(W)\) is an \(l\)-dimensional irreducible representation and the semi-simplification of \(\rho\) is \(\tau \oplus \ldots \oplus \tau\) \((n_1\) times). Since the index of rigidity increases under specialization we have

\[
(n_1)^2 \text{index rig}(\tau) = \text{index rig}(\tau \oplus \ldots \oplus \tau) \geq \text{index rig}(\rho) \geq 2.
\]

Since \(\tau\) is irreducible, it must have index rig = 2.

To do the general case of Proposition 38 we suppose that \(\rho\) is not isotypical. Then one can find an exact sequence of non-zero representations

\[
0 \to V_1 \to V \to V_2 \to 0
\]

where \(V_1\) is isotypical and \(\text{Hom}_{\pi_1}(V_1, V_2) = 0\). \(V_1\) is the sum of all \(\tau\)-isotypical subrepresentations of \(V\) where \(\tau\) is an irreducible representation of \(V\).
Claim. For such an exact sequence if $V$ is rigid, then $V_1$ and $V_2$ are rigid. (This will complete the proof by induction.)

Lemma 39 If $\rho' : \pi_1 \to P$ is a representation sufficiently close to $\rho$, $GL(V)$-conjugate to $\rho$, then $\rho'$ is $P$-conjugate to $\rho$. Here $P = \{g \in GL(V) | gV_1 \subset V_1\}$.

Proof:

By general facts on algebraic group actions the conjugacy class of $\rho$ is a locally closed subvariety (of the variety of solutions of (9)) isomorphic to $GL(V)/\text{stabilizer}(\rho)$. Hence, $\rho'$ is conjugate to $\rho$ by $g \in GL(V)$ sufficiently close to $1 - \rho' = g\rho g^{-1}$, so $g^{-1}(V_1)$ is $\rho$-invariant.

If $g^{-1}(V_1) \neq V_1$ we get $\text{Hom}_{\pi_1}(g^{-1}(V_1), V_2) \neq 0$ since the projection $V \to V_2$ defines a non-zero $\pi_1$ map. In other words $\text{Hom}_{\pi_1}(\rho'|_{V_1}, V_2) \neq 0$. If this holds for a sequence of $\rho'$'s converging to $\rho$, then since the above is a question of solving a homogeneous system of linear equations, i.e. of vanishing of certain minors, we get that there is also a non-zero solution in the limit case, i.e. $\text{Hom}_{\pi_1}(V_1, V_2) \neq 0$ which contradicts the assumption. □

Proof of the claim: Suppose that $\rho_1 : \pi_1 \to GL(V)$ is not rigid. Then one has a one-parameter analytic deformation $\rho_{1,\phi}$, $\phi \in (C,0)$, s.t. $\rho_{1,0} = \rho_1$, $\rho_{1,\phi}$ is not conjugate to $\rho_{1,0}$ for $\phi \neq 0$. (In general one cannot say that the $\rho_{1,\phi}$ for $\phi \neq 0$ are non-equivalent.) Thus given analytic deformations of $\rho_1$, $\rho_2$ (within given conjugacy classes of local monodromies) it suffices in view of the lemma to extend them to a deformation of $\rho$ (within the conjugacy classes).

One can find suitable deformations of $\rho$ of the generators that do not necessarily satisfy (9) and then one tries to correct them by conjugation by maps from the parameter space to the unipotent subgroup

$$U = \{\gamma \in GL(V) | (\gamma - 1)V \subset V_1, \gamma|_{V_1} = 1\} = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

in terms of block matrices. The correction is possible by the following analogue of Proposition 36.

The map

$$U^{p+1} \to U, \ (u_1, \ldots, u_{p+1}) \mapsto (u_1 M_1 u_1^{-1}) \ldots (u_{p+1} M_{p+1} u_{p+1}^{-1})$$

has surjective differential.

We note that $U$ can be identified with $\text{Lie}(U) = \text{Hom}_C(V_2, V_1)$ and the map above is linear, namely (in additive notation for $U$, viewed as representation of $P$)

$$(u_1, \ldots, u_{p+1}) \mapsto u_1 - M_1(u_1) + M_1(u_2 - M_2(u_2)) + \ldots + M_1 \ldots M_p(u_{p+1} - M_{p+1}(u_{p+1})) .$$

Thus surjectivity of this map means that the coinvariant space $U_{\pi_1}$ is 0, but its dual is $\text{Hom}_{\pi_1}(V_1, V_2)$ whose vanishing is exactly the assumption.

References

[Bo] A.A. Bolibrukh, Hilbert’s twenty-first problem for linear Fuchsian systems, Proceedings of the Steklov Inst. Math. No. 5, 206 (1995).

[Ka] N.M. Katz, Rigid local systems, Annals of Mathematics, Studies Series, Study 139, Princeton University Press, 1995.
[Ko1] V.P. Kostov, On the existence of monodromy groups of fuchsian systems on Riemann’s sphere with unipotent generators. Journal of Dynamical and Control Systems, vol. 2, N° 1, p. 125 – 155.

[Ko2] V.P. Kostov, On some aspects of the Deligne-Simpson problem, manuscrit, 48 p., à paraître dans un volume de “Trudy Seminara Arnol’da”. Electronic preprint math.AG/0005016.

[Ko3] V.P. Kostov, The Deligne-Simpson problem, C.R. Acad. Sci. Paris, t. 329, Série I, p. 657-662, 1999.

[Ko4] V.P. Kostov, On the Deligne-Simpson problem, manuscrit, 47 p. Electronic preprint math.AG/0011013, to appear in Proceedings of the Steklov Institute, v. 238 (2002), Monodromy in problems of algebraic geometry and differential equations.

[L] A.H.M. Levelt, Hypergeometric functions, Indagationes Mathematicae, vol. 23 (1961), pp. 361 – 401.

[Mo] J. Moser, The order of a singularity in Fuchs’ theory, Math. Zeitschrift 72, 379-398 (1960).

[Si] C.T. Simpson, Products of matrices, Department of Mathematics, Princeton University, New Jersey 08544, published in “Differential Geometry, Global Analysis and Topology”, Canadian Math. Soc. Conference Proceedings 12, AMS, Providence (1992), p. 157 – 185.

[Wa] W.R. Wasow, Asymptotic expansions for ordinary differential equations. Huntington, New York, Krieger, 1976.