Proper Weyl Collineations in Kantowski-Sachs and Bianchi Type III Space-Times

Ghulam Shabbir
Faculty of Engineering Sciences
GIK Institute of Engineering Sciences and Technology
Topi Swabi, NWFP, Pakistan
Email: shabbir@giki.edu.pk

Abstract

A study of proper Weyl collineations in Kantowski-Sachs and Bianchi type III space-times is given by using the rank of the $6 \times 6$ Weyl matrix and direct integration techniques. Studying proper Weyl collineations in each of the above space-times, it is shown that there exists no such possibility when the above space-times admit proper Weyl collineations.

1 Introduction

The aim of this paper is to study proper Weyl collineations (WCS) in Kantowski-Sachs and Bianchi type-III space-times by using the rank of the $6 \times 6$ Weyl metric and direct integration techniques. Throughout $M$ denotes a (4-dimensional Connected, Hausdorff) smooth space-time manifold with Lorentz metric $g$ of signature $(-,+,+,+)$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol $L$, respectively. The curvature tensor associated with $g_{ab}$, through the Levi-Civita connection, is denoted in component form where $R_{abcd}$, the Ricci tensor components are $R_{ab} = R^{c}_{abc}$, the Weyl tensor components are $C^{a}_{bcd}$, and the Ricci scalar is $R = g^{ab} R_{ab}$. Round and square brackets denote the usual symmetrization and skew-symmetrization.

Let $X$ be a smooth vector field on $M$ then in any coordinate system on $M$, one may decompose $X$ in the form
\[ X_{a;b} = \frac{1}{2} h_{ab} + F_{ab} \]  

(1)

where \( h_{ab} = L_x g_{ab} \) and \( F_{ab} (=-F_{ba}) \) are symmetric and skew symmetric tensor on \( M \), respectively. If \( h_{ab} = f g_{ab} \) and \( f(f : M \to R) \) is a real valued function on \( M \) then \( X \) is called a conformal vector field where \( F_{ab} \) is called the conformal bivector. The vector field \( X \) is called a proper conformal vector field if \( f \) is not constant on \( M \). For a conformal bivector \( F_{ab} \) one can show that [1]

\[ F_{ab;c} = R_{abcd} X^d - 2 f_{[a} g_{b]c} \]  

(2)

and

\[ f_{a,b} = -\frac{1}{2} L_{ab;c} X^c - f L_{ab} + R_{c(a} F_{b)}^c \]  

(3)

where \( L_{ab} = R_{ab} - (1/6) R g_{ab} \). If \( X \) is a conformal vector field on \( M \) then by using (3) one can show that

\[ L_X R_{ab} = -2 f_{a;b} - (f^{c} X_{c} ) g_{ab} \].

Further, the conformal vector field \( X \) also satisfies [3]

\[ L_X C^{a}_{bcd} = 0 \]  

(4)

which can be written equivalently as

\[ C^{a}_{bcd;f} X^{f} + C^{a}_{b;cf} X^{f}_{;d} + C^{a}_{b;df} X^{f}_{;c} + C^{a}_{c;fd} X^{f}_{;b} - C^{f}_{bcd} X^{a}_{;f} = 0. \]

The vector field \( X \) satisfying the above equation is called a Weyl collineation (WC). The vector field \( X \) is called a proper WC if it is not conformal [2]. The vector field \( X \) is called a homothetic vector field if \( f \) is constant and a proper homothetic vector field if \( f = cons \tan t \neq 0 \). If \( f = 0 \) on \( M \) then vector field \( X \) is called a Killing vector field.

2 Main Results

It has been shown [2,4,5] that much information on the solutions of (4) can be obtained without integrating it directly. To see this let \( p \in M \) and consider the following algebraic classification of the Weyl tensor as a linear map \( \beta \) from the vector space of bivectors to itself; \( \beta : F_{ab} \to F_{cd} C^{cd}_{ab} \), for any bivector \( F_{ab} \) at \( p \). The range of
the Weyl tensor at \( p \) is then the range of \( \beta \) at \( p \) and its dimension is the Weyl rank at \( p \). It follows from [4] that the rank of the \( 6 \times 6 \) Weyl matrix is always even i.e. 6, 4, 2 or 0. If the rank of the \( 6 \times 6 \) Weyl matrix is 6 or 4 then every Weyl symmetry is a conformal symmetry [4,5]. For finding proper WCS, we restrict attention to those cases of rank 2 or less.

2.1 Proper WCS in Bianchi type III space-times

Consider a Bianchi type III space-time in the usual coordinate system \((t, r, \theta, \phi)\) (labeled by \((x^0, x^1, x^2, x^3)\), respectively) with line element [6,8]

\[
ds^2 = -dt^2 + A(t)dx^2 + B(t)(d\theta^2 + \sinh^2 \theta d\phi^2),
\]

where \( A(t) \) and \( B(t) \) are nowhere zero function of \( t \). The non-zero independent components of Weyl tensor are

\[
\begin{align*}
C_{0101} & = \frac{1}{12AB^2} K(t) \equiv F1, \\
C_{0202} & = -\frac{1}{24A^2B} K(t) \equiv F2, \\
C_{0303} & = \sinh^2 \theta F2 \equiv F3, \\
C_{1212} & = \frac{1}{24AB} K(t) \equiv F4, \\
C_{1313} & = \sinh^2 \theta F4 \equiv F5, \\
C_{2323} & = -\frac{\sinh^2 \theta}{12A^2} K(t) \equiv F6,
\end{align*}
\]

where \( K(t) = (B^2 (-2\ddot{A}A + \dot{A}^2) + A\dot{B} - \dot{B}^2) + 4A^2 B \) and dot denotes the derivative with respect to \( t \). The Weyl tensor of \( M \) can be described by components \( C_{abcd} \) written in a well known way [7]

\[
C_{abcd} = \text{diag}(F1, F2, F3, F4, F5, F6).
\]

We restrict attention to those cases of rank 2 or less, since by theorem [4] no proper WCS can exist when the rank of the \( 6 \times 6 \) Weyl matrix is \( > 2 \). For the rank less or equal to two one may set four components of Weyl tensor in (6) to be zero. One gets \( A \) and \( B \) to be zero which gives contradiction to our assumption that \( A \) and \( B \) are nowhere zero functions on \( M \) this implies that there exists no such possibility when the rank of the \( 6 \times 6 \) Weyl matrix is less or equal to zero. Hence no proper Weyl collineations exist in the above space-time (5).
2.2 Proper WCS in Kantowski-Sachs space-times

Consider a Kantowski-Sachs space-time in the usual coordinate system \((t,r,\theta,\phi)\) (labeled by \((x^0,x^1,x^2,x^3)\), respectively) with line element [6,8]

\[ ds^2 = -dt^2 + A(t)dx^2 + B(t)(d\theta^2 + \sin^2 \theta d\phi^2), \] (7)

where \(A(t)\) and \(B(t)\) are nowhere zero function of \(t\). The non-zero independent components of Weyl tensor are

\[ C_{0101} = \frac{1}{12AB^2} K(t) \equiv E1, \quad C_{0202} = -\frac{1}{24A^2B} K(t) \equiv E2, \]
\[ C_{0303} = \sin^2 \theta E2 \equiv E3, \quad C_{1212} = \frac{1}{24AB} K(t) \equiv E4, \] (8)
\[ C_{1313} = \sin^2 \theta E4 \equiv E5, \quad C_{2323} = -\frac{\sin^2 \theta}{12A^2} K(t) \equiv E6, \]

where \(K(t) = (B^2(-2\ddot{A}A + \dot{A}^2) + ABA\bar{B} + 2A^2(\ddot{B}B - \dot{B}^2) + 4A^2B)\) and dot denotes the derivative with respect to \(t\). The Weyl tensor of \(M\) can be described by components \(C_{abcd}\) written in a well known way [7]

\[ C_{abcd} = \text{diag}(E1,E2,E3,E4,E5,E6). \]

Here we will restrict our attention to those cases when the rank of the \(6\times6\) Weyl matrix is 2 or less, since by theorem [4] no proper WCS can exist when the rank of the \(6\times6\) Weyl matrix is \(> 2\). For the rank less or equal to two one may set four components of Weyl tensor in (8) to be zero. One gets \(A\) and \(B\) to be zero which gives contradiction to our assumption that \(A\) and \(B\) are nowhere zero functions on \(M\) this implies that there exists no possibility when the rank of the \(6\times6\) Weyl matrix is less or equal to zero. Hence again no proper Weyl collineations exist in the above space-time (7).

Summary

In this paper a study of proper Weyl collineations in Kantowski-Sachs and Bianchi type III space-times is given by using the rank of the \(6\times6\) Weyl matrix and direct integration techniques and the theorem given in [4]. Studying proper Weyl collineations in the Kantowski-Sachs and Bianchi type III space-times, it is shown that the above space-times do not admit proper Weyl collineations.
References

[1] G. S. Hall, Gen. Rel. Grav., 22 (1990) 203.
[2] G. Shabbir, Ph. D. Thesis University of Aberdeen, 2001.
[3] G. S. Hall, “Proceedings of The Hungarian Relativity Workshops”, Tihany, Hungary, 1989.
[4] G. S. Hall, Gravitation and Cosmology, 2 (1996) 270.
[5] G. S. Hall, Curvature and Physics, Kazan, 1998.
[6] H. Baofa, Int. J. Theor. Phys., 30 (1991) 1121.
[7] G. Shabbir, Class. Quantum Grav., 21 (2004) 339.
[8] J. D. Lorenz, Phys. A, 15 (1982) 2809.