Yet More Ado About Nothing:
The Remarkable Relativistic Vacuum State*

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Abstract

An overview is given of what mathematical physics can currently say about the vacuum state for relativistic quantum field theories on Minkowski space. Along with a review of classical results such as the Reeh–Schlieder Theorem and its immediate and controversial consequences, more recent results are discussed. These include the nature of vacuum correlations and the degree of entanglement of the vacuum, as well as the striking fact that the modular objects determined by the vacuum state and algebras of observables localized in certain regions of Minkowski space encode a remarkable range of physical information, from the dynamics and scattering behavior of the theory to the external symmetries and even the space–time itself. In addition, an intrinsic characterization of the vacuum state provided by modular objects is discussed.

1 Introduction

For millenia, the concept of nothingness, in many forms and guises, has occupied reflective minds, who have adopted an extraordinary range of stances towards the notion — from holding that it is the Godhead itself, to rejecting it vehemently as a foul blasphemy. Even among more scientifically inclined thinkers there has been a similar range of views [49]. We have no intention here to sketch this vast richness of thought about nothingness. Instead, we shall more modestly attempt

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to explain what mathematical physics has to say about nothingness in its modern scientific guise: the relativistic vacuum state.

What is the vacuum in modern science? Roughly speaking, it is that which is left over after all which can possibly be removed has been removed, where “possibly” refers not to “technically possible” nor to “logically possible”, but to “physically possible” — that which is possible in light of (the current understanding of) the laws of physics. The vacuum is therefore an idealization which is only approximately realized in the laboratory and in nature. But it is a most useful idealization and a surprisingly rich concept.

We shall discuss the vacuum solely in the context of the relativistic quantum theory of systems in four spacetime dimensional Minkowski space, although we shall briefly indicate how similar states for quantum systems in other space–times can be defined and studied. In a relativistic theory of systems in Minkowski space, the vacuum should appear to be the same at every position, and in every direction, for all inertial observers. In other words, it should be invariant under the Poincaré group, the group of isometries of Minkowski space. And since one can remove no further mass/energy from the vacuum, it should be the lowest possible (global) energy state. In a relativistic theory, when one removes all mass/energy, the total energy of the resultant state is 0.

These desiderata of a vacuum are intuitively appealing, but it remains to give mathematical content to these intuitions. Once this is done, it will be seen that this state with “nothing in it” manifests remarkable properties, most of which have been discovered only in the past twenty years, and many of which are not intuitively appealing at first exposure. On the contrary, some properties of the vacuum state have proven to be decidedly controversial.

In order to formulate in a mathematically rigorous manner the notion of a vacuum state and to understand its properties, it is necessary to choose a mathematical framework which is sufficiently general to subsume large classes of models, is powerful enough to facilitate the proof of nontrivial assertions of physical interest, and yet is conceptually simple enough to have a direct, if idealized, interpretation in terms of operationally meaningful physical quantities. Such a framework is provided by algebraic quantum theory [3,12,13,40,52], also called local quantum physics, which is based on operator algebra theory, itself initially developed by J. von Neumann for the express purpose of providing quantum theory with a rigorous and flexible foundation [74,75]. This framework is briefly described in the next section, where a rigorous definition of a vacuum state in Minkowski space is given.

In Section 3 the earliest recognized consequences of such a definition are discussed, including such initially nonintuitive results as the Reeh–Schlieder Theorem. Rigorous results indicating that the vacuum is a highly entangled state are presented in Section 4. Indeed, by many measures it is a maximally entangled state. Though some of these results have been proven quite recently, readers who are familiar with the heuristic picture of the relativistic vacuum as a seething broth of virtual particle–antiparticle pairs causing wide-ranging vacuum correlations may
not be entirely surprised by their content. But there are concepts available in algebraic quantum field theory (AQFT) which have no known counterpart in heuristic quantum field theory, such as the mathematical objects which arise in the modular theory of M. Tomita and M. Takesaki [105], which is applicable in the setting of AQFT. As explained in Sections 5 and 6 the modular objects associated with the vacuum state encode a truly astonishing amount of physical information and also serve to provide an intrinsic characterization of the vacuum state which admits a generalization to quantum fields on arbitrary space–times. In addition, it is shown in Section 7 how these objects may be used to derive the space–time itself, thereby providing, at least in principle, a means to derive from the observables and their preparation (the state) a space–time in which the former can be interpreted as being localized and evolving without any a priori input on the nature or even existence of a space–time. We make some concluding remarks in Section 8.

2 The Mathematical Framework

The operationally fundamental objects in a laboratory are the preparation apparatus — devices which prepare in a repeatable manner the individual quantum systems which are to be examined — and the measuring apparatus — devices which are applied to the prepared systems and which measure the “value” of some observable property of the system. The physical notion of a “state” can be viewed as a certain equivalence class of such preparation devices, and the physical notion of an “observable” (or “effect”) can be viewed as a certain equivalence class of such measuring (or registration) devices [3, 70]. In principle, therefore, these quantities are operationally determined.

In algebraic quantum theory, such observables are represented by self–adjoint elements of certain algebras of operators, either $W^*$- or $C^*$-algebras\(^1\) In this paper we shall restrict our attention primarily to concretely represented $W^*$-algebras, which are commonly called von Neumann algebras in honor of the person who initiated their study [75]. The reader unfamiliar with these notions may simply think of algebras $\mathcal{M}$ of bounded operators \(^2\) on some (separable) Hilbert space $\mathcal{H}$ (or see [59, 60, 106–108] for a thorough background). We shall denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$. Physical states are represented by mathematical states $\phi$, i.e. linear, continuous maps $\phi : \mathcal{M} \to \mathbb{C}$ from the algebra of observables to the complex number system which take the value 1 on the identity map $I$ on $\mathcal{H}$ and are positive in the sense that $\phi(A^*A) \geq 0$ for all $A \in \mathcal{M}$. An important subclass of states consists of normal states; these are states such that $\phi(A) = \text{Tr}(\rho A)$, $A \in \mathcal{M}$, for some density matrix $\rho$ acting on $\mathcal{H}$, i.e. a bounded operator on $\mathcal{H}$ satisfying the conditions $0 \leq \rho = \rho^*$ and $\text{Tr}(\rho) = 1$. A special case

\(^1\)Other sorts of algebras have also been seriously considered for various reasons; see e.g. [40, 88, 89].

\(^2\)Technicalities concerning topology will be systematically suppressed in this paper. We therefore will not discuss the difference between $C^*$- and $W^*$-algebras.
of such normal states is constituted by the vector states: if \( \Phi \in \mathcal{H} \) is a unit vector and \( P_\Phi \in \mathcal{B}(\mathcal{H}) \) is the orthogonal projection onto the one dimensional subspace of \( \mathcal{H} \) spanned by \( \Phi \), the corresponding vector state is given by

\[
\phi(A) = \langle \Phi, A\Phi \rangle = \text{Tr}(P_\Phi A), \quad A \in \mathcal{M}.
\]

Generally speaking, theoretical physicists tacitly restrict their attention to normal states.

In AQFT the spacetime localization of the observables is taken into account. Let \( \mathbb{R}^4 \) represent four dimensional Minkowski space and \( \mathcal{O} \) denote an open subset of \( \mathbb{R}^4 \). Since any measurement is carried out in a finite spatial region and in a finite time, for every observable \( A \) there exist bounded regions \( \mathcal{O} \) containing this “localization” of \( A \). We say that the observable \( A \) is localized in any such region \( \mathcal{O} \) and denote by \( \mathcal{R}(\mathcal{O}) \) the von Neumann algebra generated by all observables localized in \( \mathcal{O} \). Clearly, it follows that if \( \mathcal{O}_1 \subset \mathcal{O}_2 \), then \( \mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2) \). This yields a net \( \mathcal{O} \mapsto \mathcal{R}(\mathcal{O}) \) of observable algebras associated with the experiment(s) in question. In turn, this net determines the smallest von Neumann algebra \( \mathcal{R} \) on \( \mathcal{H} \) containing all \( \mathcal{R}(\mathcal{O}) \). The preparation procedures in the experiment(s) then determine states \( \phi \) on \( \mathcal{R} \), the global observable algebra.

Given a state \( \phi \) on \( \mathcal{R} \), one can construct \([40, 59, 106]\) a Hilbert space \( \mathcal{H}_\phi \), a distinguished unit vector \( \Omega_\phi \in \mathcal{H}_\phi \) and a \((C^*-)\)homomorphism \( \pi_\phi: \mathcal{R} \rightarrow \mathcal{B}(\mathcal{H}_\phi) \), so that \( \pi_\phi(\mathcal{R}) \) is a \((C^*-)\)algebra acting on the Hilbert space \( \mathcal{H}_\phi \), the set of vectors \( \pi_\phi(\mathcal{R})\Omega_\phi = \{ \pi_\phi(A)\Omega_\phi \mid A \in \mathcal{R} \} \) is dense in \( \mathcal{H}_\phi \) and

\[
\phi(A) = \langle \Omega_\phi, \pi_\phi(A)\Omega_\phi \rangle, \quad A \in \mathcal{R}.
\]

The triple \( (\mathcal{H}_\phi, \Omega_\phi, \pi_\phi) \) is uniquely determined up to unitary equivalence by these properties, and \( \pi_\phi \) is called the GNS representation of \( \mathcal{R} \) determined by \( \phi \). Only if \( \phi \) is a normal state is \( \pi_\phi(\mathcal{R}) \) a von Neumann algebra and can \( (\mathcal{H}_\phi, \Omega_\phi, \pi_\phi) \) be identified with (a subrepresentation of) \( (\mathcal{H}, \Omega, \mathcal{R}) \) such that \( \Omega_\phi \in \mathcal{H} \). Hence, a state determines a concrete, though idealized, representation of the experimental setting in a Hilbert space.

In this setting, relativistic covariance is expressed through the presence of a representation \( \mathcal{P}_+ \ni \lambda \mapsto \alpha_\lambda \) of the identity component \( \mathcal{P}_+ \) of the Poincaré group.
by automorphisms $\alpha_\lambda : \mathcal{R} \to \mathcal{R}$ of $\mathcal{R}$ such that
\[ \alpha_\lambda(\mathcal{R}(\mathcal{O})) = \mathcal{R}(\lambda\mathcal{O}), \]
for all $\mathcal{O}$ and $\lambda$, where $\alpha_\lambda(\mathcal{R}(\mathcal{O})) = \{ \alpha_\lambda(A) \mid A \in \mathcal{R}(\mathcal{O}) \}$ and $\lambda\mathcal{O} = \{ \lambda(x) \mid x \in \mathcal{O} \}$. One says that a state $\phi$ is Poincaré invariant if $\phi(\alpha_\lambda(A)) = \phi(A)$ for all $A \in \mathcal{R}$ and $\lambda \in \mathcal{P}_+^\dagger$. In this case, there then exists a unitary representation $\mathcal{P}_+^\dagger \ni \lambda \mapsto U_\phi(\lambda)$ acting on $\mathcal{H}_\phi$, leaving $\Omega_\phi$ invariant, and implementing the action of the Poincaré group:
\[ U_\phi(\lambda)\pi_\phi(A)U_\phi(\lambda)^{-1} = \pi_\phi(\alpha_\lambda(A)), \]
for all $A$ and $\lambda$. If the joint spectrum of the self–adjoint generators of the translation subgroup $U_\phi(\mathbb{R}^4)$ is contained in the forward lightcone, then $U_\phi(\mathcal{P}_+^\dagger)$ is said to satisfy the (relativistic) spectrum condition. This condition is a relativistically invariant way of requiring that the total energy in the theory be nonnegative with respect to every inertial frame of reference and that the quantum system is stable in the sense that it cannot decay to energies below that of the vacuum state.

We can now present a standard rigorous definition of a vacuum state, which incorporates all of the intuitive desiderata discussed above.

**Definition 2.1** A vacuum state is a Poincaré invariant state $\phi$ on $\mathcal{R}$ such that $U_\phi(\mathcal{P}_+^\dagger)$ satisfies the spectrum condition. The corresponding GNS representation $(\mathcal{H}_\phi, \Omega_\phi, \pi_\phi)$ is called a vacuum representation of the net of observable algebras.

Note that after choosing an inertial frame of reference, the self–adjoint generator $H$ of the time translation subgroup $U_\phi(t)$, $t \in \mathbb{R}$, carries the interpretation of the total energy operator and that, by definition, $H\Omega_\phi = 0$, if $\phi$ is a vacuum state. Moreover, the total momentum operator $\vec{P}$ and the total mass operator $M \equiv \sqrt{H^2 - \vec{P}^2} \geq 0$ also annihilate the vacuum ($M\Omega_\phi = 0 = \vec{P}\Omega_\phi$).

Such vacuum states, and hence such vacuum representations, actually exist. In the case of four dimensional Minkowski space, vacuum representations for quantum field models with trivial $S$-matrix have been rigorously constructed by various means (cf. e.g. [2,8,17,48,114]) and, more recently, the same has been accomplished for quantum field models with nontrivial scattering matrices [32,33,50]. For two, resp. three, dimensional Minkowski space, fully interacting quantum field models in vacuum representations have been constructed, cf. e.g. [8,47,48,69]. Moreover, general conditions are known under which to a quantum field model without a vacuum state can be (under certain conditions uniquely) associated a vacuum representation which is physically equivalent and locally unitarily equivalent to it [18,20,38]. Hence, the mathematical existence of a vacuum state is often assured even in models which are not initially provided with one.

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6Some authors just require of a vacuum state that it be invariant under the translation group and satisfy the spectrum condition. For the purposes of this paper, it is convenient to adopt the more restrictive of the two standard definitions.
It will be useful in the following to describe two special classes of spacetime regions in Minkowski space. A double cone is a (nonempty) intersection of an open forward lightcone with an open backward lightcone. Such regions are bounded, and the set $\mathcal{D}$ of all double cones is left invariant by the natural action of $\mathcal{P}^+_{\uparrow}$ upon it. An important class of unbounded regions is specified as follows. After choosing an inertial frame of reference, one defines the right wedge to be the set

$$ W_R = \{ x = (t, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_1 > |t| \} $$

and the set of wedges to be

$$ \mathcal{W} = \{ \lambda W_R \mid \lambda \in \mathcal{P}^+_{\uparrow} \} $$

The set of wedges is independent of the choice of reference frame; only which wedge is designated the right wedge is frame-dependent.

### 3 Immediate Consequences

We now turn to some immediate consequences of the definition of a vacuum state. One of the most controversial was also one of the first to be noted. In order to avoid a too heavily laden notation, and since in this and the next section our starting point is a vacuum representation, we shall drop the subscript $\phi$ and the symbol $\pi_\phi$ (i.e. we identify $\mathcal{R}(\mathcal{O})$ and $\pi_\phi(\mathcal{R}(\mathcal{O}))$). A vacuum representation is said to satisfy weak additivity if for each nonempty $\mathcal{O}$ the smallest von Neumann algebra containing

$$ \{ U(x) \mathcal{R}(\mathcal{O}) U(x)^{-1} \mid x \in \mathbb{R}^4 \} $$

coincides with $\mathcal{R}$. This is a weak technical assumption satisfied in most models; for example, it holds in any theory in which there is a Wightman field locally associated with the observable algebras (see, e.g. [6, 7, 37]).

Let $\mathcal{O}$ be an open subset of $\mathbb{R}^4$ and let $\mathcal{O}'$ denote the interior of its causal complement, the set of all points in $\mathbb{R}^4$ which are spacelike separated from all points in $\mathcal{O}$. A net $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ is said to be local (or to satisfy locality) if whenever $\mathcal{O}_1 \subset \mathcal{O}_2'$ one has $\mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2)'$, where $\mathcal{R}(\mathcal{O})'$, the commutant of $\mathcal{R}(\mathcal{O})$, represents the set of all bounded operators on $\mathcal{H}$ which commute with all elements of $\mathcal{R}(\mathcal{O})$. Ordinarily, this property of locality is viewed as a manifestation of Einstein causality, which posits that signals and causal influences cannot propagate faster than the speed of light, and therefore spacelike separated quantum systems must be independent in some sense. As is the case with so many received notions, there is much more here than meets the eye initially; but this is not the place to address this matter (cf. [34, 96, 102] for certain aspects of this point). We wish to emphasize that locality will not be a standing assumption in this paper. If a net is not explicitly assumed to be local, then the property is not necessary for the respective result. And, in fact, locality will be derived in the settings discussed in Sections 6 and 7.

For vacuum representations of local nets in which weak additivity is satisfied, the Reeh–Schlieder Theorem holds (cf. [3, 8, 52, 58, 91]).

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7 For a very different derivation of locality, see [31].
Theorem 3.1 Consider a vacuum representation of a local net fulfilling the condition of weak additivity. For every nonempty region $\mathcal{O}$ such that $\mathcal{O}' \neq \emptyset$, the vector $\Omega$ is cyclic and separating for $\mathcal{R}(\mathcal{O})$, i.e. the set of vectors $\mathcal{R}(\mathcal{O})\Omega$ is dense in $\mathcal{H}$, resp. $A \in \mathcal{R}(\mathcal{O})$ and $A\Omega = 0$ entail $A = 0$.

There are two distinct aspects to this theorem. First of all, the fact that the vacuum is separating for local observables means exactly that no nonzero local observable can annihilate $\Omega$. Hence, any event represented by a nonzero projection $P \in \mathcal{R}(\mathcal{O})$ must have nonzero expectation in the vacuum state: $\langle \Omega, P\Omega \rangle > 0$. In the vacuum, any local event can occur! Moreover, $0 < C = C^* \in \mathcal{R}(\mathcal{O})$ entails the existence of an element $0 \neq A \in \mathcal{R}(\mathcal{O})$ such that $C = A^*A$; thus $\langle \Omega, C\Omega \rangle = \|A\Omega\|^2$, which also yields $\langle \Omega, C\Omega \rangle > 0$ in this more general case. Therefore the stress–energy density tensor $T(x)$ smeared with any test function with compact support cannot be a positive operator in a vacuum representation [41] (in fact, it is unbounded below), in contrast to the situation in classical physics, since its vacuum expectation is zero. Furthermore, in light of the fact that the vacuum state contains no real particles ($M\Omega = 0$), it follows that there can be no localized particle counters. Indeed, if $C \in \mathcal{R}$ is a particle counter for a particle described in the model, then $C = C^* > 0$ and $\langle \Omega, C\Omega \rangle = 0$. Therefore, $C$ cannot be an element of any algebra $\mathcal{R}(\mathcal{O})$ with $\mathcal{O}'$ nonempty. Hence the notion of particle in relativistic quantum field theory cannot be quite as simple as classical mechanics would have it. It has even been argued that the notion is nonsensical in relativistic quantum field theory, but this is not the place for further discussion of this point, either. (See, however, [19, 22, 45, 52, 55, 85].)

Second, there is the cyclicity of the vacuum for all local algebras: every vector state in the vacuum representation can be arbitrarily well approximated using vectors of the form $A\Omega$, $A \in \mathcal{R}(\mathcal{O})$, no matter how small in extent $\mathcal{O}$ may be. Hence, the class of all states resulting from the action of arbitrary operations upon the vacuum is effectively indistinguishable from the class of states resulting from operations performed in arbitrarily small spacetime regions upon the vacuum. Prima facie, such a state would seem to be different from the vacuum only in a region which one can make as small as one desires. In our view, a reasonable physical picture of this situation is indicated in this way: an experimenter in any given region $\mathcal{O}$ can, in principle, perform measurements designed to exploit nonlocal vacuum fluctuations (see the next section) in such a manner that any prescribed state can be reproduced with any given accuracy. These consequences of cyclicity also unleashed some controversy, some of which is well discussed in [54] (see also [82]). We shall not elaborate upon these matters here, except to point out the fact that the existing proposals to avoid Reeh–Schlieder by changing the notion of localization (1) are necessarily restricted to free quantum field models and (2) introduce at least as many problems as they “solve”, see e.g. [54].

We wish to emphasize that these (for some readers disturbing) properties are by no means unique to the vacuum — the Reeh–Schlieder Theorem is valid for

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8 Note that even if the net of observable algebras is not local, $\Omega$ is still cyclic for $\mathcal{R}(\mathcal{O})$. 7
any vector in the vacuum representation which is analytic for the energy [9]; in particular, it holds for any vector with finite energy content. So its conclusions and various consequences are true of all physically realizable vector states in the vacuum representation, since any preparation can only implement a finite exchange of energy!

4 Vacuum Correlations

We turn to what is rigorously known about the nature of vacuum correlations, preparing first some definitions to be used in this section. Given a pair \((\mathcal{M}, \mathcal{N})\) of algebras representing the observable algebras of two subsystems of a given quantum system, a state \(\phi\) is said to be a product state across \((\mathcal{M}, \mathcal{N})\) if \(\phi(MN) = \phi(M)\phi(N)\) for all \(M \in \mathcal{M}, N \in \mathcal{N}\). In such states, the observables of the two subsystems are not correlated and the subsystems manifest a certain kind of independence — see e.g. [96]. A normal state \(\phi\) on \(\mathcal{M} \cup \mathcal{N}\) is separable if it is in the norm closure of the convex hull of the normal product states across \((\mathcal{M}, \mathcal{N})\), i.e. it is a mixture of normal product states. Otherwise, \(\phi\) is said to be entangled (across \((\mathcal{M}, \mathcal{N})\)). From the point of view of what is now called quantum information theory, the primary difference between classical and quantum theory is the existence of entangled states in quantum theory. In fact, only if both subsystems are quantum, i.e. both algebras are noncommutative, do there exist entangled states on the composite system [79]. Although not understood at that time in this manner, some of the founders of quantum theory realized as early as 1935 [39, 84] that such entangled states were the source of the “paradoxical” behavior of quantum theory (as viewed from the vantage point of classical physics). Today, entangled states are regarded as a resource to be employed in order to carry out tasks which cannot be done classically, i.e. only with separable states — cf. [57, 62, 112].

Another direct consequence of the Reeh–Schlieder Theorem is that for all nonempty spacelike separated \(\mathcal{O}_1, \mathcal{O}_2\) with nonempty causal complements, no matter how far spacelike separated they may be, there exist many projections \(P_1 \in R(\mathcal{O}_1)\) which are positively correlated in the vacuum state, i.e. such that \(\phi(P_1P_2) > \phi(P_1)\phi(P_2)\).

**Theorem 4.1** Consider a vacuum representation of a local net fulfilling the condition of weak additivity, and let \(\mathcal{O}_1, \mathcal{O}_2\) be any nonempty spacelike separated regions with nonempty causal complements. Let \(\phi\) be any state induced by a vector analytic for the energy (e.g. the vacuum state). Then for any projection \(P_1 \in R(\mathcal{O}_1)\) with \(0 \neq P_1 \neq I\) there exists a projection \(P_2 \in R(\mathcal{O}_2)\) such that \(\phi(P_1P_2) > \phi(P_1)\phi(P_2)\).

This is an immediate consequence of Theorem 3.1 and the following lemma, the

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9 also termed decomposable, classically correlated, or unentangled by various authors

10 This terminology is becoming standard in quantum information theory, but there are still physicists who tacitly restrict their attention to vector states on mutually commuting algebras of observables which are isomorphic to full matrix algebras, i.e. they consider only pure states, which are entangled if and only if they are not product states.
proof of which is implicit in the proof of Theorem 5 in [82]. For the convenience
of the reader, we make this explicit here.

**Lemma 4.2** Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras on $\mathcal{H}$ with $\Omega \in \mathcal{H}$ a unit vector cyclic for $\mathcal{N}$ and separating for $\mathcal{M}$, and let $\omega$ be the corresponding state induced upon $\mathcal{B}(\mathcal{H})$. Then for any projection $P \in \mathcal{M}$ with $0 \neq P \neq I$, there exists a projection $Q \in \mathcal{N}$ such that $\omega(PQ) > \omega(P)\omega(Q)$.

**Proof.** Let $P \in \mathcal{M}$ be a projection with $0 \neq P \neq I$. It suffices to establish the existence of a projection $Q \in \mathcal{N}$ such that $\omega(PQ) \neq \omega(P)\omega(Q)$, since, if necessary, $Q$ can be replaced by $I - Q \in \mathcal{N}$ to yield the assertion. So assume for the sake of contradiction that $\omega(PQ) = \omega(P)\omega(Q)$, for all such $Q$. Then with $\hat{P} = P - \omega(P) \cdot I \in \mathcal{M}$, one has $\omega(\hat{P}Q) = 0$, for all projections $Q \in \mathcal{N}$. By the spectral theorem, this entails $\omega(\hat{P}N) = 0$, for all $N \in \mathcal{N}$, i.e. 

$$\langle \hat{P}\Omega, N\Omega \rangle = 0, \ N \in \mathcal{N}.$$ 

Since $\Omega$ is cyclic for $\mathcal{N}$, this yields $\hat{P}\Omega = 0$, so that $\hat{P} = 0$, i.e. $P = \omega(P) \cdot I$. Since $P = P^2$, this entails $\|P\Omega\|^2 = \langle P\Omega, P\Omega \rangle = \omega(P) \in \{0, 1\}$, i.e. either $P\Omega = 0$ or $P\Omega = \Omega$. Since $\Omega$ is separating for $\mathcal{M}$, this implies either $P = 0$ or $P = I$ holds, a contradiction in either case. \qed

The fact that vacuum fluctuations enable such generic “superluminal correlations” has also generated controversy, since they seem to challenge received notions of causality. This is another complex matter which we cannot go into here, but at least some forms of causality have been proven in AQFT (for recent discussions, see e.g. [34, 80]) and therefore are completely compatible with such correlations.

Of course, Theorem 4.1 entails that the vacuum is not a product state across $(\mathcal{R}(O_1), \mathcal{R}(O_2))$, but not yet that it is entangled across $(\mathcal{R}(O_1), \mathcal{R}(O_2))$. Much finer analyses of the nature and degree of the entanglement of the vacuum state have been carried out in the literature, and we shall explain some of these. A quantitative measure of entanglement is provided by using Bell correlations. The following definition was made in [93].

**Definition 4.3** Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be von Neumann algebras such that $\mathcal{M} \subset \mathcal{N}'$. The maximal Bell correlation of the pair $(\mathcal{M}, \mathcal{N})$ in the state $\phi$ is

$$\beta(\phi, \mathcal{M}, \mathcal{N}) \equiv \sup \frac{1}{2} \phi(M_1(N_1 + N_2) + M_2(N_1 - N_2)),$$

where the supremum is taken over all self-adjoint $M_i \in \mathcal{M}, N_j \in \mathcal{N}$ with norm less than or equal to 1.

As explained in e.g. [94], the CHSH version of Bell’s inequalities can be formulated in algebraic quantum theory as

$$\beta(\phi, \mathcal{M}, \mathcal{N}) \leq 1.$$

(4.1)

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If $\phi$ is separable across $(\mathcal{M}, \mathcal{N})$, then $\beta(\phi, \mathcal{M}, \mathcal{N}) = 1$ [94]. Hence states which violate Bell’s inequalities are necessarily entangled, though the converse is not true (cf. [112] for a discussion and references). Whenever at least one of the systems is classical, the bound (4.1) is satisfied in every state:

**Proposition 4.4 ([94])** Let $\mathcal{M}, \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be mutually commuting von Neumann algebras. If either $\mathcal{M}$ or $\mathcal{N}$ is abelian, then $\beta(\phi, \mathcal{M}, \mathcal{N}) = 1$ for all states $\phi$ on $\mathcal{B}(\mathcal{H})$.

If, on the other hand, both algebras are nonabelian, then there always exists a state in which the inequality (4.1) is (maximally) violated, as long as the Schlieder property holds, i.e. $MN = 0$ for $M \in \mathcal{M}$ and $N \in \mathcal{N}$ entail either $M = 0$ or $N = 0$ [68]. Because it is known [35, 94] that $1 \leq \beta(\phi, \mathcal{M}, \mathcal{N}) \leq \sqrt{2}$, for all states $\phi$ on $\mathcal{B}(\mathcal{H})$, one says that if $\beta(\phi, \mathcal{M}, \mathcal{N}) = \sqrt{2}$, then the pair $(\mathcal{M}, \mathcal{N})$ maximally violates Bell’s inequalities in the state $\phi$.

In [95] it is shown under quite general physical assumptions that in a vacuum representation of a local net one has $\beta(\phi, \mathcal{R}(W), \mathcal{R}(W^\prime)) = \sqrt{2}$, for every wedge $W$ and every normal state $\phi$. In particular, Bell’s inequalities are maximally violated in the vacuum state. In addition, under somewhat more restrictive but still general assumptions which include free quantum field theories and other physically relevant models, it is shown in [95] that $\beta(\phi, \mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2)) = \sqrt{2}$, for any two spacelike separated double cones whose closures intersect (i.e. tangent double cones) and all normal states $\phi$. Hence, such pairs of observable algebras also maximally violate Bell’s inequalities in the vacuum.

Commonly, physicists say that theories violating Bell’s inequalities are “nonlocal”; yet, here are fully local models maximally violating Bell’s inequalities. This linguistic confusion is probably so profoundly established by usage that it cannot be repaired, but the reader should be aware of the distinct meanings of these two uses of “local”. The former refers to nonlocalities in certain correlations (in certain states), while the latter refers to the commensurability of observables localized in spacelike separated spacetime regions. So the former is a property of states, while the latter is a property of observable algebras. The results discussed above establish the generic compatibility of the former sort of “nonlocality” with the latter kind of “locality”. The wary reader should always ascertain which sense of “local” is being employed by a given author.

In the now quite extensive quantum information theory literature, there are various attempts to quantify the degree of entanglement of a given state (cf. e.g. [57, 62]), but these agree that maximal violation of inequality (4.1) entails maximal entanglement. Thus, the vacuum state is maximally entangled and thereby describes a maximally non-classical situation.

The localization regions for the observable algebras which have been proven to manifest maximal violation of Bell’s inequality in the vacuum (indeed, in every state) are spacelike separated but tangent. If the double cones have nonzero
spacelike separation, any violation of Bell’s inequality in the vacuum cannot be maximal:

**Proposition 4.5 ([93, 94, 97])** Let \( \mathcal{O} \rightarrow \mathcal{R}(\mathcal{O}) \) be a local net in an irreducible vacuum representation with a lowest mass \( m > 0 \). Then for any pair \((\mathcal{O}_1, \mathcal{O}_2)\) of spacelike separated regions one has

\[
\beta(\phi, \mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2)) \leq \sqrt{2} - \frac{\sqrt{2}}{7 + 4\sqrt{2}}(1 - e^{-md(\mathcal{O}_1, \mathcal{O}_2)})
\]

(optimal for smaller \( d(\mathcal{O}_1, \mathcal{O}_2) \)) and

\[
\beta(\phi, \mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2)) \leq 1 + 2e^{-md(\mathcal{O}_1, \mathcal{O}_2)}
\]

(optimal for larger \( d(\mathcal{O}_1, \mathcal{O}_2) \)), where \( \phi \) is a vacuum state and \( d(\mathcal{O}_1, \mathcal{O}_2) \) is the maximal timelike distance \( \mathcal{O}_1 \) can be translated before it is no longer spacelike separated from \( \mathcal{O}_2 \).

Hence, if \( d(\mathcal{O}_1, \mathcal{O}_2) \) is much larger than a few Compton wavelengths of the lightest particle in the theory, then any violation of Bell’s inequality in the vacuum would be too small to be observed. As explained in [94], if there are massless particles in the theory, then the best decay in the vacuum Bell correlation one can expect is proportional to \( d(\mathcal{O}_1, \mathcal{O}_2)^{-2} \). Although the decay in the massless case is much weaker, experimental apparatus have nonzero lower bounds on the particle energies they can effectively measure. Such nonzero sensitivity limits would serve as an effective lowest mass, leading to an exponential decay once again [94]. Nonetheless, attempts have been made to obtain lower bounds on the Bell correlation \( \beta(\phi, \mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2)) \) as a function of \( d(\mathcal{O}_1, \mathcal{O}_2) \). As the published results have only treated some very special models and very special observables, we shall refrain from discussing these here (but cf. [83] and references given there).

Nonetheless, using properties of \( \beta(\phi, \mathcal{M}, \mathcal{N}) \) established by the author and R.F. Werner [97], H. Halvorson and R. Clifton have proven the following result, which entails that in a vacuum representation in which weak additivity and locality hold, the vacuum state (and any state induced by a vector analytic for the energy) is entangled across \((\mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2))\) for arbitrary nonempty spacelike separated regions \( \mathcal{O}_1, \mathcal{O}_2 \).

**Theorem 4.6 ([53])** Let \( \mathcal{M} \) and \( \mathcal{N} \) be nonabelian von Neumann algebras acting on \( \mathcal{H} \) such that \( \mathcal{M} \subset \mathcal{N}' \). If \( \Omega \in \mathcal{H} \) is cyclic for \( \mathcal{M} \) and \( \omega \) is the state on \( \mathcal{B}(\mathcal{H}) \) induced by \( \Omega \), then \( \omega \) is entangled across \((\mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2))\) for arbitrary nonempty spacelike separated regions \( \mathcal{O}_1, \mathcal{O}_2 \).

The proof does not provide a lower bound on \( \beta(\phi, \mathcal{M}, \mathcal{N}) \). For further discussion and references concerning the violation of Bell’s inequalities in algebraic quantum theory, see [53, 82, 97, 99].
Though model independent lower bounds on $\beta(\phi, \mathcal{R}(O_1), \mathcal{R}(O_2))$ are not yet available, R. Verch and Werner [111] have obtained model independent results on the nature of the entanglement of the vacuum state across nontangent pairs $(\mathcal{R}(O_1), \mathcal{R}(O_2))$ in terms of some further notions currently employed in quantum information theory, which go beyond Theorem 4.6. They proposed the following definition [111].

\textbf{Definition 4.7} Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras acting upon a Hilbert space $\mathcal{H}$. A state $\phi$ on $\mathcal{B}(\mathcal{H})$ has the ppt property if for any choice of finitely many $M_1, \ldots, M_k \in \mathcal{M}$ and $N_1, \ldots, N_k \in \mathcal{N}$, one has

$$\sum_{\alpha,\beta} \phi(M_\beta M_\alpha^* N_\alpha^* N_\beta) \geq 0.$$ 

They show that this generalizes the notion of states with positive partial transpose familiar from quantum information theory [76], a notion restricted to finite dimensional Hilbert spaces prior to [111]. They also show that if a state is ppt, then it satisfies Bell’s inequalities, and they prove that any separable state is ppt. Indeed, in general the class of ppt states properly contains the class of separable states.

Another notion from quantum information theory is that of distillability (of entanglement). Roughly speaking, this refers to being able to operate upon a given state in certain (local) ways to increase its entanglement across two subsystems. Separable states are not distillable; they are not entangled, and operating upon them in the allowable manner will not result in an entangled state. We refer the reader to [111] for a discussion of the general case and restrict ourselves here to a discussion of the following special case.

\textbf{Definition 4.8 ( [111])} Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras acting upon a Hilbert space $\mathcal{H}$. A state $\phi$ on $\mathcal{B}(\mathcal{H})$ is 1-distillable if there exist completely positive maps $T : \mathcal{B}(\mathbb{C}^2) \to \mathcal{M}$ and $S : \mathcal{B}(\mathbb{C}^2) \to \mathcal{N}$ such that the functional $\omega(X \otimes Y) \equiv \phi(T(X)S(Y)), X \otimes Y \in \mathcal{B}(\mathbb{C}^2) \otimes \mathcal{B}(\mathbb{C}^2)$ is not ppt.

Verch and Werner show that 1-distillable states are distillable and not ppt. They also prove the following theorem.

\textbf{Proposition 4.9 ( [111])} Let $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ be a local net in a vacuum representation satisfying weak additivity. Then if $\mathcal{O}_1$ and $\mathcal{O}_2$ are strictly spacelike separated double cones, the vacuum state is 1-distillable across the pair $(\mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2))$.

Hence, the vacuum is distillable and not ppt across $(\mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2))$ no matter how large $d(\mathcal{O}_1, \mathcal{O}_2)$ is. We remark that, once again, this theorem is valid also for states induced by vectors in the vacuum representation which are analytic for the energy [111]. For a discussion of some further aspects of the entanglement of the vacuum in AQFT, we refer the reader to [36].
5 Geometric Modular Action

We emphasize that nearly all of the remarkable properties of the vacuum state discussed to this point are shared by all vector states which are analytic for the energy. In the remainder of this paper we shall be dealing with properties unique to the vacuum.

A crucial breakthrough in the theory of operator algebras was the Tomita–Takesaki theory [105] (see also [60, 107]), which is proving itself to be equally powerful and productive for the purposes of mathematical quantum theory. One of the settings subsumed by this theory is a von Neumann algebra \( \mathcal{M} \) with a cyclic and separating vector \( \Omega \in \mathcal{H} \). The data \((\mathcal{M}, \Omega)\) then uniquely determine an antiunitary involution \( J \in \mathcal{B}(\mathcal{H}) \) and a strongly continuous group of unitaries \( \Delta^t, t \in \mathbb{R} \) such that \( J\Omega = \Omega = \Delta^t\Omega \), \( J\mathcal{M}J = \mathcal{M}' \) and \( \Delta^t\mathcal{M}\Delta^{-t} = \mathcal{M} \), for all \( t \in \mathbb{R} \), along with further significant properties. From the Reeh–Schlieder Theorem (Theorem 3.1), this theory is applicable to the pair \((\mathcal{R}(\mathcal{O}), \Omega)\), under the indicated conditions. Since, as explained above, the algebras and states are operationally determined (in principle), the corresponding modular objects \( J_\mathcal{O}, \Delta^t_\mathcal{O} \) are, as well.

In pathbreaking work [6, 7], J.J. Bisognano and E.H. Wichmann showed that for a net of von Neumann algebras \( \mathcal{O} \mapsto \mathcal{R}(\mathcal{O}) \) locally associated with a finite–component quantum field satisfying the Wightman axioms [8, 58, 91] (and therefore in a vacuum representation), the modular objects \( J_W, \Delta^t_W \) determined by the wedge algebras \( \mathcal{R}(W), W \in \mathcal{W}, \) and the vacuum vector \( \Omega \) have a geometric interpretation:

\[
\Delta^t_W = U(\lambda_W(2\pi t)),
\]

for all \( t \in \mathbb{R} \) and \( W \in \mathcal{W} \), where \( \{\lambda_W(2\pi t) \mid t \in \mathbb{R}\} \subset \mathcal{P}_+ \) is the one-parameter subgroup of boosts leaving \( W \) invariant. Explicitly for \( W = W_R \),

\[
\lambda_{W_R}(t) = \begin{pmatrix}
\cosh t & \sinh t & 0 & 0 \\
\sinh t & \cosh t & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The relation (5.2) has come to be referred to as modular covariance. Moreover, for scalar Boson fields [13], one has

\[
J_{W_R} = \Theta U_\pi,
\]

where \( \Theta \) is the PCT-operator associated to the Wightman field and \( U_\pi \) implements the rotation through the angle \( \pi \) about the 1-axis, with similar results for general wedge \( W \in \mathcal{W} \). Hence, one has

\[
J_{W_R} \mathcal{R}(\mathcal{O}) J_{W_R} = \mathcal{R}(\theta_R \mathcal{O}),
\]

11commonly called the modular conjugation or modular involution associated with \((\mathcal{M}, \Omega)\)
12\( \Delta \) is a certain, typically unbounded, positive operator called the modular operator associated with \((\mathcal{M}, \Omega)\)
13See also [37] for later advances in this particular setting.
14See [7] for arbitrary finite-component Wightman fields.
for all \( \mathcal{O} \), where \( \theta_R \in \mathcal{P}_+ \) is the reflection through the edge 
\[ \{ (0, 0, x_2, x_3) \mid x_2, x_3 \in \mathbb{R} \} \] of the wedge \( W_R \). This implies in turn that for all \( W \in \mathcal{W} \) one has
\[
J_W \{ \mathcal{R}(\tilde{W}) \mid \tilde{W} \in \mathcal{W} \} J_W = \{ \mathcal{R}(\tilde{W}) \mid \tilde{W} \in \mathcal{W} \}.
\]

Thus the adjoint action of the modular involutions \( J_W, W \in \mathcal{W} \), leaves the set 
\[ \{ \mathcal{R}(W) \mid W \in \mathcal{W} \} \] of observable algebras associated with wedges invariant, i.e. wedge algebras are transformed to wedge algebras by this adjoint action.

Although in the special case of the massless free scalar field [56] (and, more generally, for conformally invariant quantum field theories [14]) also the modular objects corresponding to \( (\mathcal{R}(\mathcal{O}), \Omega) \) for \( \mathcal{O} \in \mathcal{D} \) have geometric meaning, some explicit computations in the free massive field have indicated that this is not true in general. Moreover, as we shall see in the next section, only the vacuum vector \( \Omega \) yields modular objects having any geometric content. This fact yields an intrinsic characterization of the vacuum state.

But before we explore this noteworthy state of affairs, let us examine some of the more striking consequences of the above relations. For simplicity, we shall restrict these remarks to the case of nets of algebras locally associated with a scalar Bose field. Since every element \( \tilde{\lambda} \in \mathcal{L}_+ \setminus \mathcal{L}_+^1 \) of the complement of the identity component \( \mathcal{L}_+^1 \) of the Lorentz group in the proper Lorentz group \( \mathcal{L}_+ \) can be factored uniquely into a product \( \tilde{\lambda} = \theta_R \lambda \), with \( \lambda \in \mathcal{L}_+^1 \), it follows that by defining \( U(\tilde{\lambda}) = J_{W_R} U(\lambda) \) one obtains an (anti-)unitary representation of the proper Poincaré group \( \mathcal{P}_+ \) which acts covariantly upon the original net of observables. Moreover, denoting by \( \mathcal{J} \) the group generated by \( \{ J_W \mid W \in \mathcal{W} \} \) and \( \mathcal{J}_+ \) as the subgroup of \( \mathcal{J} \) consisting of products of even numbers of the generating involutions \( \{ J_W \mid W \in \mathcal{W} \} \), one has
\[
\mathcal{J} = U(\mathcal{P}_+) \quad \text{and} \quad \mathcal{J}_+ = U(\mathcal{P}_+^1).
\]

Hence, the modular involutions \( \{ J_W \mid W \in \mathcal{W} \} \) encode the isometries of the underlying space–time as well as a representation of the isometry group which acts covariantly upon the observables. So in particular, \( U(\mathbb{R}^4) \subset \mathcal{J}_+ \). Recalling that the subgroup of translations \( U(\mathbb{R}^4) \) determines the dynamics of the quantum field, one sees that the modular involutions also encode the dynamics of the model! The dynamics need not be posited, but instead can be derived from the observables and preparations of the quantum system, at least in principle, using the modular involutions.

If the quantum field model is such that a scattering theory can be defined for it and satisfies asymptotic completeness [3, 8, 58], then the original fields and the asymptotic fields act on the same Hilbert space and have the same vacuum. Letting \( \mathcal{R}^{(0)}(W), W \in \mathcal{W} \), denote the observable algebras associated with the free asymptotic field and \( J_W^{(0)} \) represent the modular involution corresponding to \( (\mathcal{R}^{(0)}(W), \Omega) \), one has, as was pointed out by B. Schroer [86],
\[
S = J_{W_R} J_W^{(0)}.
\]
where $S$ is the *scattering matrix* for the original field model. Hence, the modular involutions associated with the wedge algebras and the vacuum state also encode all information about the results of scattering processes in the given model.\footnote{Note that the same is *not* true about the modular unitaries, since both the original field and the asymptotic field are covariant under the same representation of $P^+_+$.

In addition, because of the connection between Tomita–Takesaki modular theory and KMS–states [13], modular covariance entails that when the vacuum state is restricted to $\mathcal{R}(W)$ for any wedge $W$, then with respect to the automorphism group on $\mathcal{R}(W)$ generated by the boosts $U(\lambda_W(t))$, it is an equilibrium state at temperature $1/2\pi$ (in suitable units). Hence, any uniformly accelerated observers find when testing the vacuum that it has a nonzero temperature [90]. This striking fact is called the Unruh effect [110]. Moreover, because KMS–states are passive [77], the vacuum satisfies the second law of thermodynamics with respect to boosts — an additional stability property.

Modular covariance and/or the geometric action of the modular conjugations \footnote{5.4} have also been derived under other sets of assumptions (in addition to those discussed in the next section) which do not refer to the Wightman axioms, *i.e.* purely algebraic settings in which no appeal to Wightman fields is made [15,65,72,109] (see [11] for a review). Thus, these properties and their many consequences hold quite generally. It is also of interest that some of these settings provide algebraic versions of the PCT Theorem and the Spin–Statistics connection [15,51,63,67,72], but we shall not enter upon this topic here. We now turn to those conditions which provide an intrinsic characterization of the vacuum state.

### 6 Intrinsic Characterization of the Vacuum State

Though the definition of a vacuum state given in Definition 2.1 is standard, it is not quite satisfactory, since it is not (operationally) *intrinsic*. It has been seen in Section 2 that the elements of quantum theory which are closest to its operational foundations are states and observables. However, in the definition of the vacuum state one finds such notions as the spectrum condition and automorphic (and unitary) representations of the Poincaré group, all of which are not expressed solely in terms of these states and observables. This may not disturb some readers, so let us step back and locate the notion of Minkowski space vacuum state in a larger context.

One of the primary roles of the vacuum state in quantum field theory has been to serve as a physically distinguished reference state with respect to which other physical states can be defined and referred. Let us recall as an example of this that perturbation theory is performed with respect to the vacuum state, *i.e.* computations performed for general states of interest in quantum field theory are carried out by suitably perturbing the vacuum. This role has proven to be so central that when theorists tried to formulate quantum field theory in space–
times other than Minkowski space\footnote{After all, the space–time in which we find ourselves is not Minkowski space.}, they tried to find analogous states in these new settings, thereby running into some serious conceptual and mathematical problems. This is not the place to explain the range and scope of these difficulties, but one noteworthy problem is indicated by the question: what could replace the large isometry group (the Poincaré group) of Minkowski space in the definition of “vacuum state”, in light of the fact that the isometry group of a generic space–time is trivial? A further point is that in the definition of “vacuum state” the spectrum condition serves as a stability condition; what could replace it even in such highly symmetric space–times as de Sitter space, where the isometry group, though large, does not contain any translations?

After much effort, a number of interesting selection criteria have been isolated and studied; see, \emph{e.g.} [10, 16, 24, 25, 29, 30, 43, 61, 66, 71, 78, 92]. Of these, all but one either select an entire folium of states — i.e. a representation, instead of a state — or are explicitly limited to a particular subclass of spacetimes (or both). Here we shall discuss the selection criterion provided by the Condition of Geometric Modular Action (CGMA), which in the special case of Minkowski space selects the vacuum state (as opposed to selecting the entire vacuum representation) but which can be formulated for general space–times.

As we now no longer have a vacuum state/representation given, we return to the notation of Section 2 and the initial data of a net $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ of observable algebras and a state $\phi$ on $\mathcal{R}$. The question we are now examining is: under which conditions, stated solely in terms of mathematical quantities completely determined by these initial data, is $\phi$ a vacuum state? Surprisingly, the core of the answer to this question is the relation (5.5). It will be convenient to introduce the notation $\mathcal{R}_\phi(\mathcal{O}) \equiv \pi_{\phi}(\mathcal{R}(\mathcal{O}))' = (\pi_{\phi}(\mathcal{R}(\mathcal{O})))'$. We consider a special case of the condition first discussed in [21] and subsequently further generalized in [25].

**Definition 6.1** A state $\phi$ on a net $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ satisfies the Condition of Geometric Modular Action if the vector $\Omega_\phi$ is cyclic and separating for $\mathcal{R}_\phi(W)$, $W \in \mathcal{W}$, and if the modular conjugation $J_W$ corresponding to $(\mathcal{R}_\phi(W), \Omega_\phi)$ satisfies

\[
J_W \{\mathcal{R}_\phi(\tilde{W}) \mid \tilde{W} \in \mathcal{W}\} J_W \subset \{\mathcal{R}_\phi(\tilde{W}) \mid \tilde{W} \in \mathcal{W}\},
\]

for all $W \in \mathcal{W}$.

Note that there is no \emph{prima facie} reason why (5.5) should imply (5.4). Indeed, why should the action (5.5) even be implemented by point transformations on $\mathbb{R}^4$, much less by Poincaré transformations? And since all Poincaré transformations map wedges to wedges, why should (5.4) be the only solution, even if one did find oneself in the latter, fortunate situation?

The following theorem was proven in [25,29]. The interested reader may consult [29] for the definition of the weak technical property referred to in hypothesis (c)
Theorem 6.2 ([25, 29]) Let $\phi$ be a state on a net $O \mapsto \mathcal{R}(O)$ which satisfies the following constraints:

(a) The map $W \ni W \mapsto \mathcal{R}_\phi(W) \in \{\mathcal{R}_\phi(W) \mid W \in \mathcal{W}\}$ is an order-preserving bijection.

(b) If $W_1 \cap W_2 \neq \emptyset$, then $\Omega_\phi$ is cyclic and separating for $\mathcal{R}_\phi(W_1) \cap \mathcal{R}_\phi(W_2)$. Conversely, if $\Omega_\phi$ is cyclic and separating for $\mathcal{R}_\phi(W_1) \cap \mathcal{R}_\phi(W_2)$, then $\overline{W_1 \cap W_2} \neq \emptyset$, where the bar denotes closure.

(c) The net $W \mapsto \mathcal{R}_\phi(W)$ is locally generated.

(d) The adjoint action of the modular conjugations $J_W$, $W \in \mathcal{W}$, acts transitively upon the set $\{\mathcal{R}_\phi(W) \mid W \in \mathcal{W}\}$, i.e. there exists a wedge $W_0 \in \mathcal{W}$ such that

$$\{J_W \mathcal{R}_\phi(W_0)J_W \mid W \in \mathcal{W}\} = \{\mathcal{R}_\phi(W) \mid W \in \mathcal{W}\}.$$

Then there exists a continuous (anti-)unitary representation $U$ of $\mathcal{P}_+$ which leaves $\Omega_\phi$ invariant and acts covariantly upon the net:

$$U(\lambda)\mathcal{R}_\phi(O)U(\lambda)^{-1} = \mathcal{R}_\phi(\lambda O),$$

for all $O$ and $\lambda \in \mathcal{P}_+$. Moreover, $J = U(\mathcal{P}_+)$, $J_+ = U(\mathcal{P}_+)$ and

$$J_{W_R}\mathcal{R}(O)J_{W_R} = \mathcal{R}(\theta_R O),$$

for all $O$. Furthermore, the wedge duality condition holds:

$$\mathcal{R}_\phi(W') = \mathcal{R}_\phi(W)'',$$

for all $W \in \mathcal{W}$, which entails that the net $W \mapsto \mathcal{R}_\phi(W)$ is local.

Hence, from the state and net are derived the isometry group of the space–time; a unitary representation of the isometry group formed from the modular involutions, leaving the state invariant and acting covariantly upon the net; the specific geometric action of the modular involutions found in a special case by Bisognano and Wichmann; the locality of the net; and even the dynamics etc. of the theory (see Section 5).

The conceptually crucial observation is that all conditions in the hypothesis of this theorem are expressed solely in terms of the initial net and state, or algebraic quantities completely determined by them. Condition (a) entails that the adjoint action of the modular involutions $J_W$ upon the net induces an inclusion preserving bijection on the set $\mathcal{W}$. Condition (b) assures that this bijection can be implemented by point transformations (indeed Poincaré transformations) [25], and (c)

\[17\] In fact, hypothesis (c) may be dispensed with if the Modular Stability Condition (see below) is satisfied [29].
implies that the representation \( U(\mathcal{P}_+) \) is continuous \(^{18}\). Condition (d) strengthens the Condition of Geometric Modular Action. Without this strengthening, the adjoint action of the \( J_W \) can still be shown to be implemented by Poincaré transformations \(^{25}\), but the group \( J \) can then be isomorphic to a proper subgroup of \( \mathcal{P}_+ \) \(^{44}\).

Although such a state \( \phi \) is clearly a physically distinguished state, the spectrum condition and modular covariance need not be fulfilled \(^{25}\). As an intrinsic stability condition, the Modular Stability Condition has been proposed.

**Definition 6.3** (\(^{25}\)) For any \( W \in \mathcal{W} \), the elements \( \Delta^n_{W}, t \in \mathbb{R} \), of the modular group corresponding to \( (\mathcal{R}_\phi(W), \Omega_\phi) \) are contained in \( J \).

Note that in this condition no reference is made to the space–time, its isometry group, or any representation of the isometry group. This condition can be posed for models on any space–time \(^{25}\). Together with the CGMA, this modular stability condition then yields both the spectrum condition and modular covariance \(^{5.2}\).

**Theorem 6.4** (\(^{25, 29}\)) If, in addition to the hypothesis of Theorem \(^{6.2}\), the Modular Stability Condition is satisfied, then after choosing suitable coordinates on \( \mathbb{R}^4 \), the spectrum condition is satisfied by \( U(\mathcal{P}_+) \) and modular covariance holds. The associated representation \( (\mathcal{H}_\phi, \pi_\phi, \Omega_\phi) \) is therefore a vacuum representation.

Of course, this is not, strictly speaking, a characterization of arbitrary vacuum states; this theorem provides an intrinsic characterization of those vacuum states which manifest further desirable properties, properties which are also manifested in the models in the special circumstances considered by Bisognano and Wichmann. But since these latter circumstances are precisely those expected to arise in standard quantum field theory, the vacuum states characterized in Theorems \(^{6.2}\) and \(^{6.4}\) are probably the vacuum states of most direct physical interest.

7 Deriving Space–Time From States and Observables

Although the hypothesis of Theorem \(^{6.4}\) makes no explicit or implicit reference to an underlying space–time, Theorem \(^{6.2}\) does so implicitly through use of the set of wedges \( W \).\(^{19}\) However, the results of the preceding section did suggest the possibility that, without any \textit{a priori} reference to a space–time, the space–time

\(^{18}\)Note that the continuity of the representation of the translation group follows without condition (c) \(^{23}\).

\(^{19}\)In fact, only a four dimensional real manifold with a coordinatization is required in order to formulate and prove the theorems in Section \(^{6}\) but it is nonetheless clear that the introduction of wedges as defined tacitly appeals to Minkowski space.
itself as well as an assignment of localization regions for the observable algebras, along with all of the above-mentioned results, could be derived from the modular conjugations associated with a collection of algebras and a suitable state, as long as the set of modular conjugations verifies certain purely algebraic relations. And if the Modular Stability Condition is also satisfied, the state would then be a vacuum state, and the CGMA and modular covariance would be satisfied. In fact, this program has been carried out for a few space–times in [101, 103, 104, 113]. In order to minimize technical complications which would distract attention away from the essential conceptual point to be made, we will only discuss this approach in the example of three dimensional Minkowski space.

To eliminate any reference to a space–time and to strengthen the purely operational nature of the initial data, we consider a collection \(\{A_i\}_{i \in I}\) of unital C\(^*\)–algebras indexed by “laboratories” \(i \in I\). \(A_i\) is interpreted as the algebra generated by all observables measurable in the laboratory \(i\). Since it makes sense to speak of one laboratory as being contained in another, the set \(I\) of laboratories is provided with a natural partial order \(\leq\). It is then immediate that if \(i \leq j\) then \(A_i \subset A_j\). Hence, the map \(I \ni i \mapsto A_i \in \{A_i\}_{i \in I}\) is order preserving. We shall assume that this map is a bijection, since otherwise there would be some redundancy in the description of the system. If \((I, \leq)\) is a directed set, then \(\{A_i\}_{i \in I}\) is a net and the inductive limit \(A\) of \(\{A_i\}_{i \in I}\) exists and may be used as a reference algebra. But even if \(\{A_i\}_{i \in I}\) is not a net, it is possible [46] to naturally embed \(A_i, i \in I\), into a C\(^*\)–algebra \(A\) so that the inclusion relations are preserved. It will not be necessary to distinguish between these cases in the results, and we shall refer to states \(\phi\) upon \(A\) as being states upon the net \(\{A_i\}_{i \in I}\).

Given such a state \(\phi\), we proceed to the corresponding GNS representation and define \(R_i = \pi_\phi(A_i)''\), \(i \in I\). We shall assume that the implementing vector \(\Omega_\phi\) is cyclic and separating for all \(R_i, i \in I\), and denote by \(J_i, \Delta_i\), the corresponding modular objects. Again, let \(J\) denote the group generated by the involutions \(J_i, i \in I\). Note that \(J\Omega_\phi = \Omega_\phi\), for all \(J \in J\). In this abstract context, the CGMA is the requirement that the adjoint action of each \(J_i\) upon the elements of \(\{R_i\}_{i \in I}\) leaves the set \(\{R_i\}_{i \in I}\) invariant [25]. Among other matters, the CGMA here entails that the set \(\{J_i\}_{i \in I}\) is an invariant generating set for the group \(J\) and such a structure is the starting point for the investigations of the branch of geometry known as absolute geometry, see e.g. [1, 4, 5]. From such a group and a suitable set of axioms to be satisfied by the generators of that group, absolute geometers derive various “metric” spaces such as Minkowski spaces and Euclidean spaces upon which the abstract group \(J\) now acts as the isometry group of the metric space. Different sets of axioms on the group yield different metric spaces. This affords us with the possibility of deriving a space–time from the group \(J\), so

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20 The index set can be naturally refined by further encoding the time (with respect to some reference clock in the laboratory) during which the measurement is carried out without changing the validity of the following assertions.

21 In other words, the smallest group containing \(\{J_i\}_{i \in I}\) is \(J\) and \(J\{J_i\}_{i \in I}J^{-1} \subset \{J_i\}_{i \in I}\) for all \(J \in J\).
that the operational data \((\phi, \{A_i\}_{i \in I})\) would determine the space–time in which the quantum systems could naturally be considered to be evolving. We emphasize that different groups \(\mathcal{J}\) would verify different sets of algebraic relations and would thus lead to different space–times.

For the convenience of the reader, we summarize our standing assumptions, which refer solely to objects which are completely determined by the data \((\phi, \{A_i\}_{i \in I})\).

**Standing Assumptions** For the net \(\{A_i\}_{i \in I}\) of nonabelian \(C^\ast\)-algebras and the state \(\phi\) on \(A\) we assume

(i) \(i \mapsto R_i\) is an order-preserving bijection;

(ii) \(\Omega_\phi\) is cyclic and separating for each algebra \(R_i, i \in I\);

(iii) the adjoint action of each \(J_i\) leaves the set \(\{R_i\}_{i \in I}\) invariant.

Already these assumptions restrict significantly the class of admissible groups \(\mathcal{J}\) [25]. In general, it may be necessary to pass to a suitable subcollection of \(\{R_i\}_{i \in I}\) in order for the Standing Assumptions to be satisfied [25] (if, indeed, they are satisfied at all) — see [101] for a brief discussion of this point.

We must introduce some notation in order to concisely formulate the algebraic requirements upon \(\mathcal{J}\) which lead to the construction of three dimensional Minkowski space. We use lower case Latin letters to denote arbitrary modular involutions \(J_i, i \in I\), upper case Latin letters to denote involutions in \(\mathcal{J}\) of the form \(ab\), and lower case Greek letters for arbitrary elements of \(\mathcal{J}\). By \(\xi | \eta\) we shall mean “\(\xi \eta\) is an involution”, and \(\alpha, \beta | \xi, \eta\) is shorthand for “\(\alpha | \xi, \beta | \xi, \alpha | \eta, \) and \(\beta | \eta\)”.

**Theorem 7.1 ([101])** Assume in the above setting that the following relations hold in \(\mathcal{J}\):

1. For every \(P, Q\) there exists a \(g\) with \(P, Q | g\).
2. If \(P, Q | g, h\), then \(P = Q\) or \(g = h\).
3. If \(a, b, c | P\), then \(abc \in \{J_i : i \in I\}\).
4. If \(a, b, c | g\), then \(abc \in \{J_i : i \in I\}\).
5. There exist \(g, h, j\) such that \(g | h\) but \(j | g, h, gh\) are all false.
6. For each \(P\) and \(g\) with \(P | g\) false, there exist exactly two distinct elements \(h_1, h_2\) such that \(h_1, h_2 | P\) is true and \(g, h_i | R, c\) are false for all \(R, c, i = 1, 2\).

Then there exists a model (based on \(\mathcal{J}\)) of three dimensional Minkowski space in which each \(J_i, i \in I\), is identified as a spacelike line (and every spacelike line is such an element) and on which each \(J_i, i \in I\), acts adjointly as the reflection about the spacelike line to which it corresponds. \(\mathcal{J}\) is isomorphic to \(P_\mathbb{R}^{22}\) and forms in a canonical manner a strongly continuous (anti)unitary representation \(U\)

\[\text{the proper Poincaré group for three dimensional Minkowski space}\]
of $\mathcal{P}_+$. Moreover, there exists a bijection $\chi : I \to \mathcal{W}^{23}$ such that after defining $R(\chi(i)) = R_i$, the resultant net $\{R(\chi(i))\}$ of wedge algebras on Minkowski space is covariant under the action of the representation $U(\mathcal{P}_+)$. Furthermore, one has $R(\chi(i))' = R(\chi(i)'')$ for all $i \in I$. Thus, if the map $\chi : I \to \mathcal{W}$ is order-preserving, then the net $\{R(\chi(i))\}$ is local.

If, further, $\Delta^it_j \in J$ for all $j \in I$, $t \in \mathbb{R}$, then modular covariance is satisfied and the state $\omega$ is a vacuum state on the net $\{R(\chi(i))\}$.

We emphasize that assumptions 1–6 are purely algebraic in nature and involve only the group $J$, which is completely determined by the initial data $(\phi, \{A_i\}_{i \in I})$. Although we do not propose the verification of such conditions as a practical procedure to determine space–time, it is, in our view, a noteworthy conceptual point that such a derivation is possible in principle. It is also noteworthy that the derived structure is so rigid and provides such a complete basis for physical interpretation. Indeed, from the observables and state can be derived a space–time, an identification of the localizations of the observables in that space–time and a continuous unitary representation of the isometry group of the space–time such that the resultant, re–interpreted net is covariant under the action of the isometry group and the re–interpreted state is a vacuum state. It is perhaps worth mentioning that the modular symmetry group $J$ of a theory on four dimensional Minkowski space as discussed in Section 6 does not verify assumptions 1–6 above. Moreover, models on three dimensional Minkowski space satisfying the CGMA do verify assumptions 1–6.

In [103, 104, 113] sets of algebraic conditions on $J$ have also been found so that the space derived is three dimensional de Sitter space, respectively four dimensional Minkowski space. We anticipate that similar results can be proven for other highly symmetric space–times such as anti–de Sitter space and the Einstein universe, but not for general space–times.

8 Concluding Remarks

It is a striking fact that, in the senses indicated above, the modular involutions associated with the vacuum state (and only the vacuum state) encode the following physically significant matters.

• the space–time in which the quantum systems may be viewed as evolving
• the isometry group of the space–time
• a strongly continuous unitary representation of this isometry group which acts covariantly upon the net of observable algebras and leaves the state invariant
• the locality, i.e. the Einstein causality, of the quantum systems

\[23\text{the set of wedges in three dimensional Minkowski space}\]
• the dynamics of the quantum systems
• the scattering behavior of the quantum systems
• the spin–statistics connection in the quantum systems
• the stability of the quantum systems
• the thermodynamic behavior of the quantum systems

It has also become clear through examples — quantum field theories on de Sitter space [10, 25, 44], anti-de Sitter space [24, 30], a class of positively curved Robertson–Walker space–times [26, 27], as well as others [92, 98] — that the encoding of crucial physical information by modular objects and the subsequent utility of this approach are not limited to Minkowski space theories.

It is necessary to distinguish between the, in some sense, maximal results of Section 7 and those of Section 6. The former cannot be expected to be reproducible in most space–times, since the isometry groups are not large enough to determine the space–time, and the arguments in Section 7 rely tacitly upon the possibility of interpreting the modular group \( J \) as (a suitably large subgroup of) the isometry group of some space–time. However, most of the results of Section 6, and hence most of the list above, can be expected to be attainable in more general space–times, without regard to the size of the isometry group of the space–time. As has been verified in a class of models in a family of Robertson–Walker space–times [26], the CGMA and the encoding of crucial physical information by modular involutions associated with certain observable algebras and select states can hold even when the modular symmetry group \( J \) is strictly larger than the isometry group of the space–time (in fact, in these examples a significant portion of \( J \) is not associated with any kind of pointlike transformations upon the space–time). In other words, it is quite possible that the fact that the modular symmetry group gives no more than (a subgroup of) the isometry group of the space–time in the presence of the CGMA for theories on Minkowski or de Sitter space is an accident due to the fact that these space–times are maximally symmetric. Moreover, it is possible that using the CGMA and Modular Stability Condition to select states of physical interest yields a modular symmetry group \( J \) containing, along with the standard symmetries expected from classical theory, new and purely quantum symmetries encoding unexpected physical information (further evidence for this speculation which goes beyond [26] can be adduced in [42]).

Finally, we mention that modular objects associated with privileged algebras of observables and states (usually the vacuum) are also proving to be useful in the construction of quantum field models in two, three and four dimensional Minkowski space, which cannot be constructed by previously known techniques of constructive quantum field theory [17, 28, 32, 33, 50, 69, 73, 87]. But such matters go well beyond the scope of this paper.
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