Sufficient Dimension Reduction for Classification

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Supplementary Material

The supplementary material includes all the theoretical proof of the main paper.

S1 Proof of Theorem 1

Proof. The following lemma is required.

Lemma 1. Under the linearity and coverage condition, it holds that

if \( Y \perp \beta_{0i}^T X \), then \( \beta_{0i} \perp S_{X|Y} \). \hfill (S1.1)

Proof. We use the method of contradiction to prove the argument in (S1.1).

Without loss of generality, we assume \( E(X) = 0 \) and \( \text{Cov}(X) = I_p \) in the following proof. If \( \beta_{0i} \not\perp S_{X|Y} \), then \( \beta_{0i} = \gamma_{0i} + \lambda_{0i} \), where \( \gamma_{0i} \in S_{X|Y} \), \( \gamma_{0i} \neq 0 \), and \( \lambda_{0i} \in S_{X|Y}^\perp \). We consider

\[
E(\beta_{0i}^T X|Y) = E(\gamma_{0i}^T X|Y) + E(\lambda_{0i}^T X|Y) =: T_1 + T_2.
\]

Let \( \mu_Y = E(X|Y) \) and \( \Sigma_{E(X|Y)} = \text{Cov}\{E(X|Y)\} \). For \( T_1 \), noting that
\( E(\mu_Y) = E(X) = 0 \) and \( \gamma_{0i} \in S_{X|Y} \), we have

\[
\text{Var}\{E(\gamma_{0i}^T X|Y)\} = \text{Var}(\gamma_{0i}^T E(\mu_Y) \gamma_{0i}) = \alpha_i^T \Gamma^T \Sigma_{E(X|Y)} \Gamma \alpha_1,
\]

where \( \Gamma \in \mathbb{R}^{p \times d^*} \) is an orthonormal basis of the central subspace \( S_{X|Y} \) and \( \alpha_1 \in \mathbb{R}^{d^*} \) does not equal 0. If the linearity condition and the coverage condition hold, we know that (Li, 1991; Tan et al., 2020)

\[
\Sigma_{E(X|Y)} = \sum_{i=1}^{d^*} \lambda_i \gamma_i \gamma_i^T = \Gamma \Lambda \Gamma^T, \quad \text{where } \Gamma = (\gamma_1, \ldots, \gamma_{d^*}) \text{ and } \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{d^*}) \text{ with } \lambda_1 > \ldots > \lambda_{d^*} > 0.
\]

Therefore,

\[
\Gamma^T \Sigma_{E(X|Y)} \Gamma = \Gamma^T \Gamma \Lambda \Gamma^T \Gamma = \Lambda.
\]

Combining \((S1.2)\), we obtain \( \text{Var}\{E(\gamma_{0i}^T X|Y)\} = \alpha_i^T \Lambda \alpha_1 \geq \lambda_{d^*} \|\alpha_1\|^2 > 0 \), which implies that

\[
P\{E(\gamma_{0i}^T X|Y) = E(\gamma_{0i}^T X)\} < 1. \quad (S1.3)
\]

For \( T_2 \), we have

\[
\text{Var}\{E(\lambda_{0i}^T X|Y)\} = \text{Var}(\lambda_{0i}^T E(\mu_Y) \mu_{Y}^T) \lambda_{0i} = \alpha_2^T \Gamma_{\perp}^\top \Sigma_{E(X|Y)} \Gamma_{\perp} \alpha_2,
\]

where \( \Gamma_{\perp} \in \mathbb{R}^{p \times (p-d^*)} \) is an orthonormal basis of \( S_{X|Y}^\perp \) such that \( \Gamma_{\perp}^\top \Gamma = 0 \).

Recall the linearity condition and coverage condition imply that \( \Sigma_{E(X|Y)} = \Gamma \Lambda \Gamma^T \), and then it holds that

\[
\text{Var}\{E(\lambda_{0i}^T X|Y)\} = \alpha_2^T \Gamma_{\perp}^\top \Sigma_{E(X|Y)} \Gamma_{\perp} \alpha_2 = \alpha_2^T \Gamma_{\perp}^\top \Gamma \Lambda \Gamma^T \Gamma_{\perp} \alpha_2 = 0,
\]
which leads to

\[ P\{E(\lambda^T X|Y) = E(\lambda^T X)\} = 1. \quad \text{(S1.4)} \]

Combining (S1.3) and (S1.4), it holds that

\[
P\{E(\beta^T X|Y) = E(\beta^T X)\} \\
= P\{E(\gamma^T X|Y) + E(\lambda^T X|Y) = E(\gamma^T X) + E(\lambda^T X)\} \\
= P\{E(\gamma^T X|Y) + E(\lambda^T X|Y) = E(\gamma^T X) + E(\lambda^T X), E(\lambda^T X|Y) = E(\lambda^T X)\} \\
+ P\{E(\gamma^T X|Y) + E(\lambda^T X|Y) = E(\gamma^T X) + E(\lambda^T X), E(\lambda^T X|Y) \neq E(\lambda^T X)\} \\
\leq P\{E(\gamma^T X|Y) = E(\gamma^T X)\} + P\{E(\lambda^T X|Y) \neq E(\lambda^T X)\} \\
< 1.
\]

From the relationship between mean independence and independence, we obtain

\[ P\{E(\beta^T X|Y) = E(\beta^T X)\} < 1 \Rightarrow Y \not\perp \perp \beta^T X. \]

We complete the proof for the argument in (S1.1). \[\square\]

Now we begin the proof. By the property of the MV index, we have \(\beta^T X \perp Y\) for \(i = d + 1, \ldots, p\). From Lemma 1, it holds that \(\beta_{0i} \in S_{Y|X}^\perp\) for \(i = d + 1, \ldots, p\), where \(S_{Y|X}^\perp\) denotes the orthogonal complement space of \(S_{Y|X}\). Noting that \(\beta^T \beta = 1\), \(\beta^T \beta = 0\) for \(i, j = 1, \ldots, p\) and \(i \neq j\) and \(\text{span}\{\beta_{0(d+1)}, \ldots, \beta_{0p}\} \subseteq S_{Y|X}^\perp\), we easily obtain \(S_{Y|X}^\perp \subseteq \).
span\{\beta_{01}, \ldots, \beta_{0d}\}.

For any $k < d$, if span\{\beta_{01}, \ldots, \beta_{0k}\} \supseteq S_{Y|X}$, then $Y \perp \perp X|\beta_{01}^T X, \ldots, \beta_{0k}^T X$.

Since $X \sim N(0, I)$, it holds that $\beta_{0d}^T X \perp \perp (\beta_{01}^T X, \ldots, \beta_{0k}^T X)$. The above two arguments imply that $Y \perp \beta_{0d}^T X$ (Cook (1994), Corollary 5.1), which contradicts the assumption that $\text{MV}(\beta_{0d}^T X|Y) \neq 0$. We complete the proof.

\[\square\]

**S2 Proof of Corollary 2**

*Proof.* Let $Z$ denote the variable $\beta^T X$ and $Z_i$ denote the variable $\beta^T X|Y = i$ ($i = 1, -1$) for $\beta \in \mathbb{R}^p$ satisfying $\|\beta\| = 1$. Then, $Z_1 \sim N(\beta^T \mu, \beta^T \Sigma \beta)$, $Z_{-1} \sim N(-\beta^T \mu, \beta^T \Sigma \beta)$ and

\[
F(Z) = p_1 \cdot F_1(Z) + p_{-1} \cdot F_{-1}(Z)
\]

\[= p_1 \cdot \Phi((Z - \beta^T \mu)/\sqrt{\beta^T \Sigma \beta}) + p_{-1} \cdot \Phi((Z + \beta^T \mu)/\sqrt{\beta^T \Sigma \beta}),\]
where $F_i(Z)$ denotes the conditional distribution function of $Z|Y = i$ for $i = 1, -1$. Then,

$$
\text{MV}(\beta) = p_1 \int [F_1(z) - F(z)]^2 dF(z) + p_{-1} \int [F_{-1}(z) - F(z)]^2 dF(z)
$$

$$
= (p_1p_{-1}^2 + p_{-1}p_1^2) \left\{ p_1 \int [F_1(z) - F_{-1}(z)]^2 dF_1(z) + p_{-1} \int [F_1(z) - F_{-1}(z)]^2 dF_{-1}(z) \right\}
$$

$$
= p_1p_{-1} \left\{ p_1 \int \left[ \Phi(t) - \Phi(t + 2\beta^T \mu/\sqrt{\beta^T \Sigma \beta}) \right]^2 d\Phi(t)
+ p_{-1} \int \left[ \Phi(t) - \Phi(t - 2\beta^T \mu/\sqrt{\beta^T \Sigma \beta}) \right]^2 d\Phi(t) \right\}.
$$

Hence, we only need to maximize $\beta^T \mu/\sqrt{\beta^T \Sigma \beta}$. The solution is exactly the optimal weight of LDA, i.e. $\beta_{01} \propto \Sigma^{-1} \mu$, where $\beta_{01}$ is defined in Theorem 1. Thus, if we choose $d = 1$ in the MMV procedure, then MMV+LDA equals LDA at the population level.

Then we try to find $\beta_{02}$. We always assume $\Sigma$ is positive definite, and thus for any non-zero $\beta$, $\beta^T \Sigma \beta > 0$. Since now we maximize $(\beta^T \Sigma \beta)^{-1/2} \beta^T \mu$ subject to $\beta^T \beta = 1$ and $\beta^T \Sigma^{-1} \mu = 0$, let

$$
L = (\beta^T \Sigma \beta)^{-1/2} \beta^T \mu + \lambda(\beta^T \beta - 1) + \pi \beta^T \Sigma^{-1} \mu.
$$

Take first partial derivative w.r.t. $\beta$ and set it to zero. We then have

$$
\frac{\partial L}{\partial \beta} = (\beta^T \Sigma \beta)^{-1/2} \mu^T - \beta^T \mu (\beta^T \Sigma \beta)^{-3/2} \beta^T \Sigma + 2\lambda \beta^T + \pi \mu^T \Sigma^{-1} = 0.
$$
Then
\[
0 = \frac{\partial L}{\partial \beta}
\]
\[
= (\beta^T \Sigma \beta)^{-1/2} \mu^T \beta - (\beta^T \Sigma \beta)^{-1/2} \beta^T \mu + 2 \lambda \beta^T \beta + \pi \mu^T \Sigma^{-1} \beta
\]
\[
= 2 \lambda,
\]
i.e. \( \lambda = 0 \). Now \((\beta^T \Sigma \beta)^{3/2} \frac{\partial L}{\partial \beta} = 0\) is reduced to
\[
0 = (\beta^T \Sigma \beta) \mu^T - \beta^T \mu \beta^T \Sigma + (\beta^T \Sigma \beta)^{3/2} \pi \mu^T \Sigma^{-1} \beta. \tag{S2.1}
\]

Multiply both sides of above equation by \( \Sigma^{-1} \beta \) from right, together with \( \beta^T \beta = 1 \) and \( \beta^T \Sigma^{-1} \mu = 0 \), we have
\[
0 = (\beta^T \Sigma \beta) \mu^T \Sigma^{-1} \beta - \beta^T \mu \beta^T \Sigma \Sigma^{-1} \beta + (\beta^T \Sigma \beta)^{3/2} \pi \mu^T \Sigma^{-1} \Sigma^{-1} \beta
\]
\[
= 0 - \beta^T \mu + \pi (\beta^T \Sigma \beta)^{3/2} \mu^T \Sigma^{-2} \beta
\]
\[
= \pi (\beta^T \Sigma \beta)^{3/2} \mu^T \Sigma^{-2} \beta - \mu^T \beta, \tag{S2.2}
\]
and thus
\[
\pi = (\beta^T \Sigma \beta)^{-3/2} (\mu^T \Sigma^{-2} \beta)^{-1} (\mu^T \beta). \tag{S2.3}
\]

Note that none of \( \pi, \mu^T \Sigma^{-2} \beta \) and \( \mu^T \beta \) can be 0. If \( \pi = 0 \), then (S2.2) gives \( \mu^T \beta = 0 \). Plugging this and \( \pi = 0 \) back into (S2.1) gives \( \beta^T \Sigma \beta \mu^T = 0 \), i.e. \( \mu = 0 \), which contradicts the assumption \( \mu \neq 0 \) and thus \( \pi \neq 0 \). If \( \mu^T \Sigma^{-2} \beta = 0 \), then (S2.2) gives \( \mu^T \beta = 0 \). Multiplying \( \Sigma^{-2} \beta \) from the right to (S2.1) and plugging in \( \mu^T \Sigma^{-2} \beta = 0 \) and \( \mu^T \beta = 0 \), we have \( \mu^T \Sigma^{-3} \beta = 0 \).
as $\pi \neq 0$. If we repeat this deduction then we have $\mu^T \Sigma^{-i} \beta = 0$, $i = 0, 1, \cdots$. This can not hold for general $\mu$ and $\Sigma$ unless $\beta = 0$, which contradicts $\beta = 1$, and thus $\mu^T \Sigma^{-2} \beta \neq 0$. If $\mu^T \beta = 0$, then $\pi = 0$ which contradicts $\pi \neq 0$. Therefore (S2.3) holds, is well-defined and $\pi \neq 0$. Now plug the $\pi$ back into (S2.1) and we have

$$0 = \beta^T (\Sigma \mu^T) - \beta^T (\mu^T \Sigma) + (\mu^T \Sigma^{-2} \beta)^{-1} (\mu^T \beta) \mu^T \Sigma^{-1}$$

(S2.4)

and thus $\beta^T U = \beta^T U^T - W$. By multiplying this equation from the right by $\beta$ and applying it again, we have

$$\beta^T U \beta = \beta^T U^T \beta - W \beta$$

$$= \beta^T (\beta^T U)^T - W \beta$$

$$= \beta^T (\beta^T U^T - W)^T - W \beta$$

$$= \beta^T U \beta - \beta^T W^T - W \beta,$$

i.e. $0 = W \beta + \beta^T W^T = (\mu^T \Sigma^{-2} \beta)^{-1} (\mu^T \beta) [\mu^T \Sigma^{-1} \beta + \beta^T \Sigma^{-1} \mu]$. Since $\mu^T \beta \neq 0$, we have $\mu^T \Sigma^{-1} \beta + \beta^T \Sigma^{-1} \mu = 0$. Its solution is of the form $\beta = \Sigma (V - V^T) \mu$ with $V$ a $p \times p$ matrix. Plugging this $\beta$ back into (S2.4), we have

$$0 = \mu^T (V^T - V) \Sigma^3 (V - V^T) \mu \mu^T - \mu^T (V^T - V) \Sigma \mu \mu^T (V^T - V) \Sigma^2$$

$$+ [\mu^T \Sigma^{-1} (V - V^T) \mu]^{-1} \mu^T \Sigma (V - V^T) \mu \mu^T \Sigma^{-1}.$$
By the arbitrariness of $\mu$, the above equation gives

$$0 = (V^T - V) \Sigma^3 (V - V^T) \mu \mu^T - (V^T - V) \Sigma \mu \mu^T (V^T - V) \Sigma^2$$

$$+ \left[ \mu^T \Sigma^{-1} (V - V^T) \mu \right]^{-1} \Sigma (V - V^T) \mu \mu^T \Sigma^{-1}.$$

Multiplying it by $\mu^T \Sigma^{-2}$ from the left, we have

$$0 = \mu^T \Sigma^{-2} (V^T - V) \Sigma^3 (V - V^T) \mu \mu^T - \mu^T \Sigma^{-2} (V^T - V) \Sigma \mu \mu^T (V^T - V) \Sigma^2 + \mu^T \Sigma^{-1},$$

from which, by the arbitrariness of $\mu$ and $\Sigma$ again, we have

$$0 = \Sigma^{-1} (V^T - V) \Sigma^3 (V - V^T) \mu \mu^T - \Sigma^{-1} (V^T - V) \Sigma \mu \mu^T (V^T - V) \Sigma^2 + I.$$

Thus

$$I = \Sigma^{-1} (V^T - V) \Sigma \left\{ \mu \mu^T (V^T - V) \Sigma^2 - \left[ \mu \mu^T (V^T - V) \Sigma^2 \right]^T \right\} = \Sigma^{-1} (V^T - V) \Sigma S,$$ say.

If we take transpose it follows that

$$I = S^T \Sigma (V - V^T) \Sigma^{-1}$$

$$= -S \Sigma \left[ -(V^T - V) \right] \Sigma^{-1}$$

$$= S \Sigma (V^T - V) \Sigma^{-1}.$$

Combining the above two equations gives $\Sigma^{-1} (V^T - V) \Sigma S = S \Sigma (V^T - V) \Sigma^{-1}$ which has the solutions $S = \Sigma^{-2}$ and $S = 0$. If $S = \Sigma^{-2}$, then by the fact $S^T = -S$ we have $\Sigma^{-2} = -\Sigma^{-2}$ which is not valid. Thus we have $S = 0$. From this we have $\mu \mu^T (V^T - V) \Sigma^2 = \left[ \mu \mu^T (V^T - V) \Sigma^2 \right]^T$ which has solutions $V^T - V = \Sigma^{-2}$ and $V^T - V = 0$. If $V^T - V = \Sigma^{-2}$,
then \((V^T - V)^T = (\Sigma^{-2})^T\), i.e. \(V - V^T = \Sigma^{-2}\), i.e. \(-\Sigma^{-2} = \Sigma^{-2}\) is a contradiction. Thus \(V^T - V = 0\) which gives \(\beta = \Sigma(V - V^T)\mu = 0\).

Therefore \(\beta_{02} = 0\) for general \(\mu\) and \(\Sigma\). For some specific \(\mu\) and \(\Sigma\), there might exist nonzero \(\beta_{02}\), but always \(\beta_{02}^T\mu = 0\). This gives the conclusion that for multivariate normal, the true \(d = 1\) and only the first \(\beta_{01}\) contributes to the variation among different classes.

\[\square\]

**S3  Proof of Corollary 3**

*Proof.*** From \(Y \perp \perp X|BX\) and \(B\gamma = 0\), we get \(Y \perp \perp \gamma^T X|BX\) (Proposition 4.3 in [Cook (1998)]). If we also have \(\gamma^T X \perp \perp BX\), then by Lemma 4.3 of [Dawid (1979)] and Proposition 4.6 of [Cook (1998)], we obtain \(\gamma^T X \perp \perp Y\) which implies \(\text{MV}(\gamma^T X|Y) = 0\) according to the property of the MV index.

When \(X \sim N(\mu, \Sigma)\), \(\gamma^T X \perp \perp BX\) if and only if

\[
\text{Cov}(\gamma^T X, BX) = \gamma^T \Sigma B^T = 0. \tag{S3.1}
\]

If \(\Sigma = I\), then (S3.1) holds by \(B\gamma = 0\). Let \(\Gamma\) denote the vector space spanned by all \(\gamma\) satisfying (S3.1) and \(B\gamma = 0\), then we can easily get \(\dim(\Gamma) = p - k\) which implies \(d = k\) for \(d\) defined in Theorem 1. If \(\Sigma \neq I\) and \(2k \leq p\), then \(\dim(\Gamma) \geq p - 2k\) by noting that \(\Sigma\) is positive definite.
Hence $d \leq 2k$. We complete the proof.

S4 Proof of Proposition 4

The following lemma is needed.

**Lemma 2.** Under Conditions 1, 3 and 4, for any $\beta \in B(k_0)$, it holds that

1. if $R$ is fixed, $\text{MV}_n(\beta^T X | Y) \rightarrow \text{MV}(\beta^T X | Y)$ in probability as $n \rightarrow \infty$.

2. if $R$ is diverging with $n$ and satisfies $R = O(n^{\delta})$ with $0 < \delta \leq 1/2$, $\text{MV}_n(\beta^T X | Y) \rightarrow \text{MV}(\beta^T X | Y)$ in probability as $n \rightarrow \infty$.

**Proof.** Denote $\beta^T X$ by $X$ with support $\mathbb{R}_X$ and the transformed samples $\{\beta^T X_j\}_{j=1}^n$ by $\{X_j\}_{j=1}^n$. By the definitions of $\text{MV}(\beta^T X | Y)$ and $\text{MV}_n(\beta^T X | Y)$,
we have

\[
\text{MV}_n(\beta^T X | Y) - \text{MV}(\beta^T X | Y) = \text{MV}_n(X | Y) - \text{MV}(X | Y)
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \sum_{r=1}^{R} \hat{p}_r [\hat{F}_{hr}(X_j) - \hat{F}_h(X_j)]^2 - \sum_{r=1}^{R} p_r \int [F_r(x) - F(x)]^2 dF(x)
\]

\[
= \sum_{r=1}^{R} \hat{p}_r \left( \int [\hat{F}_{hr}(x) - \hat{F}_h(x)]^2 d\hat{F}(x) - \int [F_r(x) - F(x)]^2 dF(x) \right)
\]

\[
+ \sum_{r=1}^{R} (\hat{p}_r - p_r) \int [F_r(x) - F(x)]^2 dF(x)
\]

\[
= \sum_{r=1}^{R} \hat{p}_r \int (|\hat{F}_{hr}(x) - \hat{F}_h(x)|^2 - [F_r(x) - F(x)]^2) d\hat{F}(x)
\]

\[
+ \sum_{r=1}^{R} \hat{p}_r \int [F_r(x) - F(x)]^2 d[\hat{F}(x) - F(x)]
\]

\[
+ \sum_{r=1}^{R} (\hat{p}_r - p_r) \int [F_r(x) - F(x)]^2 dF(x)
\]

\[=: A_1 + A_2 + A_3.\]

For the first term \(A_1\),

\[
|A_1| \leq 2 \max_{1 \leq r \leq R} \int |[\hat{F}_{hr}(x) - F_r(x)] - [\hat{F}_h(x) - F(x)]| d\hat{F}(x)
\]

\[
\leq 2 \max_{1 \leq r \leq R} \sup_{x \in \mathbb{R}_x} (|\hat{F}_{hr}(x) - F_r(x)| + |\hat{F}_h(x) - F(x)|)
\]

\[=: 2(B_1 + B_2),\]

where the second inequality is obtained by \(\int d\hat{F}(x) = 1\). We then consider
the term $B_1$,

$$B_1 = \max_{1 \leq r \leq R} \sup_{x \in \mathbb{R}_X} |\hat{F}_{hr}(x) - F_r(x)| = \max_{1 \leq r \leq R} O_p(n_r^{-\alpha}) = O_p(n^{-\alpha(1-\delta)}),$$

where the second equality is implied by Theorem 2.2 of Cheng (2017) with any $0 < \alpha < 1/2$ under Conditions 3 and 4, and the last equality is given by Condition 1 and the fact that $|\hat{p}_r - p_r| = O_p(n^{-1/2})$ holds uniformly for $r = 1, \ldots, R$. For the second term $B_2$, also by Theorem 2.2 of Cheng (2017), we obtain $B_2 = \sup_{x \in \mathbb{R}_X} |\hat{F}_h(x) - F(x)| = O_p(n^{-\alpha}).$

We turn to the term $A_2$,

$$|A_2| = \sum_{r=1}^{R} \hat{p}_r \int [F_r(x) - F(x)]^2 d\hat{F}(x) - F(x)|$$

$$\leq \max_r \int [F_r(x) - F(x)]^2 d\hat{F}(x) - F(x)|$$

$$\leq \int d\hat{F}(x) - F(x)|$$

$$\leq 2 \sup_{x \in \mathbb{R}_X} |\hat{F}(x) - F(x)|$$

$$= O_p(n^{-\alpha}),$$

where the last equality is based on the extended Glivenko-Cantelli lemma (Fabian, 1985) with $\alpha$ defined above.

For the last term $A_3$, we have $|A_3| = O_p(n^{-\alpha})$ with any $0 < \alpha < 1/2$ by Lemma A.4 of Cui, Li and Zhong (2015). To sum up, $|\text{MV}_n(\beta^T X|Y) - \text{MV}(\beta^T X|Y)| = |A_1 + A_2 + A_3| \leq O_p(n^{-\alpha(1-\delta)}) + O_p(n^{-\alpha}) = O_p(n^{-\alpha(1-\delta)}).$
S4. PROOF OF PROPOSITION 4

Thus, we complete the proof.

We first prove $\hat{\beta}_1 \to_p \beta_{01}$. From Lemma 2, we obtain that for any $\beta_1 \in B(\kappa_1)$ with $\kappa_1 \in (0, \kappa_{01}]$, $\text{MV}_n(\beta^T_1X|Y) \to_p \text{MV}(\beta^T_1X|Y)$. For any $\epsilon > 0$, let $\{\beta^1, \ldots, \beta^M\}$ be an $\epsilon/\sqrt{p}$-net of $B(\kappa_{01})$ with $M = (2\kappa_{01}\sqrt{p}/\epsilon + 1)^p$. Since $M$ is fixed, by Lemma 2 we obtain

$$\max_{1 \leq j \leq M} |\text{MV}_n(\beta^j) - \text{MV}(\beta^j)| \to_p 0,$$

as $n \to \infty$. For any $\beta \in B(\kappa_{01})$, there exists a $m \in \{1, 2, \ldots, M\}$ such that $||\beta - \beta^m|| \leq \epsilon/\sqrt{p}$. Then by Condition 5, $|\text{MV}_n(\beta) - \text{MV}_n(\beta^m)| = o_p(1)$ and $|\text{MV}(\beta) - \text{MV}(\beta^m)| \to 0$. Therefore, combining the above three equations, we obtain

$$\sup_{\beta \in B(\kappa_{01})} |\text{MV}_n(\beta) - \text{MV}(\beta)| \to_p 0.$$

Then by Condition 2, it is easy to see that for any sufficiently small $\kappa_1 > 0$,

$$\mathbb{P}\left(\sup_{\beta_1 \in \partial B(\kappa_1) \cap \Gamma_1} \text{MV}_n(\beta_1) \leq \text{MV}_n(\beta_{01})\right) \to 1$$

as $n \to \infty$. Thus, there exists a local maximum local point $\hat{\beta}_1 \in \partial B(\kappa_1) \cap \Gamma_1$ with probability approaching to 1, which means that $\mathbb{P}(||\hat{\beta}_1 - \beta_{01}|| < \kappa_1) \to 1$.

Next, Let $p$ be diverging with $n$ and $p^{p/2}n^{-\alpha(1-\delta)} = o(1)$. Notice that Lemma 2 still holds when $p$ is diverging, because the dimension of $\beta^T X$ stays to be 1 whether the dimension of $X$ diverges or not. Recall that
\[ |\text{MV}_n(\beta) - \text{MV}(\beta)| = O_p(n^{-\alpha(1-\delta)}) \] (See Lemma 2). Then

\[
\max_{1 \leq i \leq M} |\text{MV}_n(\beta_j) - \text{MV}(\beta_j)| \\
\leq \sum_{i=1}^{M} |\text{MV}_n(\beta_j) - \text{MV}(\beta_j)| \\
= O_p(Mn^{-\alpha(1-\delta)}) \\
= o_p(1).
\]

By the same arguments for the fixed \( p \) case, we complete the proof.

\[ \square \]

S5 Proof of Theorem 5

The following lemma is needed.

Lemma 3. Under Conditions 1, 3 and 4, for any \( \beta \in \mathbb{C}^p \), it holds that

1. if \( R \) is fixed, \( \text{MV}_n(\beta^T X|Y) \to \text{MV}(\beta^T X|Y) \) in probability as \( n \to \infty \).
2. if \( R \) is diverging with \( n \) and satisfies \( R = O(n^{\delta}) \) with \( 0 < \delta \leq 1/2 \), \( \text{MV}_n(\beta^T X|Y) \to \text{MV}(\beta^T X|Y) \) in probability as \( n \to \infty \).

Proof. The proof is similar to that of Lemma 2. The only difference comes from the complex \( \beta \) and the corresponding complex random variable \( X =: \beta^T X \) with support \( C_X \). Denote \( X =: a + ib \) where \( Z := (a, b)^T \) is a real random vector. By the definition of the cumulative distribution function of a complex variable, that is, the joint distribution function of the real part
and the imaginary part of the variable, we obtain

\[
\sup_{z \in \mathbb{R}^2} |\hat{F}_h(z) - F(z)| = O_{a.s.}(n^{-1/2}(\log n)^{1/2}).
\]

This convergence rate comes from Theorem 3 of Liu and Yang (2008). Then, by the same arguments of Lemma 2, we complete the proof. \(\square\)

The following gives the proof of Theorem 5.

For simplicity, we omit \(i = 1\) in the subscripts of \(\beta\) and \(\theta\). Then \(\theta = (\beta^T, \lambda)^T \in \mathbb{R}^{p+1}\), and \(\hat{\theta} = (\hat{\beta}^T, \hat{\lambda})^T\) is the maximizer of \(L_{nh}(\theta)\). Hence, \(\hat{\theta} = (\hat{\beta}^T, \hat{\lambda})^T\) is a stationary point of \(L_{nh}(\theta)\), that is, \(L'_{nh}(\hat{\theta}) = 0\). Similarly, since \(\theta_0 = (\beta_0^T, \lambda_0)^T\) is the maximizer of \(L(\theta)\), then \(\theta_0 = (\beta_0^T, \lambda_0)^T\) is a stationary point of \(L(\theta)\).

We then prove \(n^{1/2}(\hat{\theta} - \theta_0) \rightarrow_d N(0, V)\) where the covariance matrix \(V\) will be given in the proof below. By the Taylor expansion, we have

\[
0 = L'_{nh}(\hat{\theta}) = L'_{nh}(\theta_0) + L''_{nh}(\theta_0)(\hat{\theta} - \theta_0) + R(\theta^*), \quad (S5.1)
\]

where \(\theta^*\) satisfies \(||\theta^* - \theta_0|| \leq ||\hat{\theta} - \theta_0||\) and \(\theta^* = (\beta^* T, \lambda^*)^T\). With regular calculation, we obtain

\[
L'_{nh}(\theta_0) = \begin{pmatrix}
MV'_{nh}(\beta_0) + 2\lambda_0 \beta_0 \\
\beta_0^T \beta_0 - 1
\end{pmatrix},
\]
\[ L''_{nh}(\theta_0) = \begin{pmatrix} MV''_n(\beta_0) + 2\lambda_0I_p & 2\beta_0 \\ 2\beta_0^T & 0 \end{pmatrix}, \]

where \( I_p \) denotes the identity matrix of dimension \( p \times p \). The remainder term \( R(\theta^*) \) contains the third derivative of \( L_{nh}(\theta) \) at \( \theta = \theta^* \). Let \( T_n = L''_{nh}(\theta^*) \), an array of dimension \((p+1) \times (p+1) \times (p+1)\), and for each \( j = 1, \ldots, (p+1) \), \( T_n(j,.;.;) \) is a matrix of dimension \((p + 1) \times (p + 1)\). Hence, we can write

\[
R(\theta^*) = \frac{1}{2} \begin{pmatrix} (\tilde{\theta} - \theta_0)^T T_n(1,.;.;)(\tilde{\theta} - \theta_0) \\ (\tilde{\theta} - \theta_0)^T T_n(2,.;.;)(\tilde{\theta} - \theta_0) \\ \vdots \\ (\tilde{\theta} - \theta_0)^T T_n(p + 1,.;.;)(\tilde{\theta} - \theta_0) \end{pmatrix}.
\]

Then, based on the explicit expressions of the derivatives given above and
(S5.1), we obtain

\[
- \left( \begin{array}{cc}
MV''(\beta_0) + 2\lambda_0 I_p & 2\beta_0 \\
2\beta_0^T & 0
\end{array} \right)^{-1} \times \sqrt{n} \left( \begin{array}{c}
MV'(\beta_0) + 2\lambda_0 \beta_0 \\
\beta_0^T \beta_0 - 1
\end{array} \right) = \\
\left[ \begin{array}{c}
I_p + 1 + \frac{1}{2} \left( \begin{array}{cc}
MV''(\beta_0) + 2\lambda_0 I_p & 2\beta_0 \\
2\beta_0^T & 0
\end{array} \right)^{-1} \times \left( \begin{array}{c}
(\hat{\theta} - \theta_0)^T T_n(1, \ldots ,) \\
(\hat{\theta} - \theta_0)^T T_n(2, \ldots ,)
\end{array} \right) \\
\sqrt{n}(\hat{\theta} - \theta_0)
\end{array} \right] \\
\left( \begin{array}{c}
(\hat{\theta} - \theta_0)^T T_n(p + 1, \ldots ,)
\end{array} \right)
\]

Next, our proof is divided into two parts:

Part 1:

\[
\left( \begin{array}{cc}
MV''(\beta_0) + 2\lambda_0 I_p & 2\beta_0 \\
2\beta_0^T & 0
\end{array} \right)^{-1} \times \sqrt{n} \left( \begin{array}{c}
MV'(\beta_0) + 2\lambda_0 \beta_0 \\
\beta_0^T \beta_0 - 1
\end{array} \right) \rightarrow N(0, V),
\]

where \( V \) will be defined later.
Part 2:

\[
L_{nh}''(\theta_0)^{-1} = \begin{pmatrix} \text{MV}_n''(\beta_0) + 2 \lambda_0 I_p & 2 \beta_0 \\ 2 \beta_0^T & 0 \end{pmatrix}^{-1} \rightarrow_p L''(\theta_0)^{-1} =: A. \quad (S5.3)
\]

Denote \( \beta = (b_1, \ldots, b_p)^T \) and \( T_{jn} = \sqrt{n} \partial \text{MV}_n(\beta)/\partial b_j |_{\beta = \beta_0} \). Notice that \( \beta_0^T \beta_0 - 1 = 0 \). Then it suffices to prove the asymptotic normality of \( T_{jn} \) for each \( j = 1, \ldots, p \). It is worth noting that the elements of \(-A \sqrt{n} L_{nh}'(\theta_0)\) are linear combinations of \( T_{jn}s \). We consider the \( j = 1 \) case.
S5. PROOF OF THEOREM 5

By the definition of \( \text{MV}_n(\beta) \), we have

\[
T_{1n} = -\sqrt{n} \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial b_1} \left\{ \sum_{r=1}^{R} \hat{p}_r \left( \hat{F}_{hr}(\beta^T X_i) - \hat{F}_h(\beta^T X_i) \right)^2 \right\} \right]_{\beta = \beta_0}.
\]

By Lemma 3,

\[
\sum_{i=1}^{n} \left\{ \sum_{r=1}^{R} \hat{p}_r \left[ \left( \hat{F}_{hr}(\beta^T X_i) - \hat{F}_h(\beta^T X_i) \right)^2 \right] \right\} = \sum_{i=1}^{n} \left\{ \sum_{r=1}^{R} p_r \left[ F_r(\beta^T X_i) - F(\beta^T X_i) \right]^2 \right\} (1 + u_n(\beta) + iv_n(\beta)),
\]

where \( \beta \in C(\kappa_0) \), \( i^2 = -1 \), and \( u_n(\beta) + iv_n(\beta) = o_p(1) \) is uniform in \( \beta \in C(\kappa_0) \) when \( n \to \infty \) with \( u_n(\beta) \) and \( v_n(\beta) \) being real functions of \( \beta \).

By Cauchy’s residue theorem, we have

\[
T_{1n} = \frac{1}{\sqrt{n}} \frac{1}{2\pi i} \oint_{C_1} \sum_{i=1}^{n} \sum_{r=1}^{R} \frac{\hat{p}_r \left[ \hat{F}_{hr}(\beta^T X_i) - \hat{F}_h(\beta^T X_i) \right]^2}{(b_1 - b_{01})^2} db_1,
\]

where \( \tilde{\beta} = (b_1, b_{02}, \ldots, b_{0p})^T \) with \( \beta_0 = (b_{01}, \ldots, b_{0p})^T \), and \( C_1 \) satisfies \( \{b_1 \in \mathbb{C} : \|b_1 - b_{01}\| = r\} \) with \( r < \kappa_0 \). Define

\[
S_{1n} = \frac{1}{\sqrt{n}} \frac{1}{2\pi i} \oint_{C_1} \sum_{i=1}^{n} \sum_{r=1}^{R} \frac{p_r \left[ F_r(\beta^T X_i) - F(\beta^T X_i) \right]^2}{(b_1 - b_{01})^2} db_1.
\]

Then,

\[
T_{1n} - S_{1n} = \frac{1}{\sqrt{n}} \frac{1}{2\pi i} \oint_{C_1} \sum_{i=1}^{n} \sum_{r=1}^{R} \frac{p_r \left[ F_r(\beta^T X_i) - F(\beta^T X_i) \right]^2 \left( u_n(\tilde{\beta}) + iv_n(\tilde{\beta}) \right)}{(b_1 - b_{01})^2} db_1.
\]

Let

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{r=1}^{R} p_r \left[ F_r(\beta^T X_i) - F(\beta^T X_i) \right]^2 =: R_n(b_1) + iI_n(b_1).
\]
Noting that the left-hand side of (S5.4) is real, we consider the real part of the other hand, that is

$$\frac{1}{2\pi} \int_0^{2\pi} \left( R_n \cos \mu + I_n \sin \mu \right) u_n + \left( R_n \sin \mu - I_n \cos \mu \right) v_n \, d\mu,$$

(S5.5)

where the arguments of $R_n, I_n, u_n, v_n$ are $b_{01} + re^{i\mu}$. By the mean value theorem, we obtain

$$\text{(S5.5)} = \frac{1}{r} \left( R_{0n} \cos \mu_0 + I_{0n} \sin \mu_0 \right) u_{0n} + \left( R_{0n} \sin \mu_0 - I_{0n} \cos \mu_0 \right) v_{0n},$$

(S5.6)

where the arguments of $R_{0n}, I_{0n}, u_{0n}, v_{0n}$ are all $b_{01} + re^{i\mu_0}$ with $\mu_0 \in [0, 2\pi]$.

By the definition of $S_{1n}$, we have

$$S_{1n} = -\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{\partial}{\partial b_1} \sum_{r=1}^R p_r [F(r\beta_0^i X_i) - F(\beta_0^i X_i)]^2 = -\frac{\partial}{\partial b_1} (R_{0n}(b_{01}) + iI_{0n}(b_{01})).$$

(S5.7)

By central limit theorem, $S_{1n}$ is asymptotically normally distributed. Then, noticing $u_n(\beta) + iv_n(\beta) = o_p(1)$ for $\beta \in C(\kappa_0)$ and letting $r \to 0$, $|T_{1n} - S_{1n}| \leq 2/r(\|R_{0n}\| + \|I_{0n}\|)(|u_{0n}| + |v_{0n}|) = o_p(1)$ as $n \to \infty$. Using Slutsky's theorem, we complete the proof for the asymptotic normality of $T_{1n}$. From (S5.7), we can further derive the form of $V$:

$$V = \begin{pmatrix} V_1 & * \\ * & * \end{pmatrix},$$
where \( V_1 = A_1 \Sigma A_1 \in \mathbb{R}^{p \times p}, \)

\[
A = \begin{pmatrix}
A_1 & * \\
* & *
\end{pmatrix}
\]

(S5.8)

for \( A \) defined in (S5.3), \( \Sigma = \mathbb{E}\{\alpha(X_i)X_iX_i^T\} + 4\lambda_0 \beta_0 \mathbb{E}\{\alpha(X_i)X_i^T\} + 4\lambda_0^2 \beta_0^T \beta_0 \)

with \( \alpha(X_i) = 2 \sum_{r=1}^R p_r \{F(\beta_0^T X_i) - F_r(\beta_0^T X_i)\}\{f(\beta_0^T X_i) - f_r(\beta_0^T X_i)\} \). We complete the proof of Part 1.

**Proof of Part 2.** By Conditions 6-7, we can easily obtain

\[
I_{p+1} + \frac{1}{2} \left( \begin{array}{cc}
 MV_n''(\beta_0) + 2\lambda_0 I_p & 2\beta_0 \\
 2\beta_0^T & 0
\end{array} \right)^{-1} \left( \begin{array}{c}
(\hat{\theta} - \theta_0)^T T_n(1,\ldots,\cdot) \\
(\hat{\theta} - \theta_0)^T T_n(2,\ldots,\cdot) \\
\vdots \\
(\hat{\theta} - \theta_0)^T T_n(p + 1,\ldots,\cdot)
\end{array} \right) \rightarrow_p I_{p+1}.
\]
Hence, by Slutsky’s theorem,

$$
\sqrt{n}(\hat{\theta} - \theta_0) = d \sqrt{n}(\hat{\theta} - \theta_0).
$$

Therefore, combining Part 1 and Part 2, we have $\sqrt{n}(\hat{\theta} - \theta_0) \to_d N(0, V)$. Because $\hat{\theta} = (\hat{\beta}^T, \hat{\lambda})^T$, by the property of multivariate normal distribution, we complete the proof for the case $i = 1$ with the covariance matrix $V_1$ being the $p \times p$ sub-matrix at the top right-hand corner of $V$. □

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