Spinor representation of Maxwell’s equations

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Abstract. Spinors are more special objects than tensor. Therefore possess more properties than the more generic objects such as tensors. Thus, the group of Lorentz two-spinors is the covering group of the Lorentz group. Since the Lorentz group is a symmetry group of Maxwell’s equations, it is assumed to reasonable to use when writing the Maxwell equations Lorentz two-spinors and not tensors. We describe in detail the representation of the Maxwell’s equations in the form of Lorentz two-spinors. This representation of Maxwell’s equations can be of considerable theoretical interest.

1. Introduction
Maxwell’s equations have a large number of representations [1–4]. The principle of the introduction of the following: every representation must simplify the concrete theoretical and practical study. In this paper, we consistently describe the Lorentz two-spinor [5] representation of Maxwell’s equations. It is supposed that this form will be interested in theoretical studies [6–8].

The structure of the article is as follows. In the section 2 basic notations and conventions are introduced. Section 3 gives a brief description of the Maxwell equations. Section 4 gives the spinors of the electromagnetic field. Further, section 5 gives the Lorenz two-spinor representation of Maxwell’s equations.

2. Notations and conventions
(i) The abstract indices notation [9] is used in this work. Under this notation a tensor as a whole object is denoted just as an index (e.g., $x^i$), components are denoted by underlined index (e.g., $x_i$).
(ii) We will adhere to the following agreements. Greek indices ($\alpha, \beta$) will refer to the four-dimensional space, in component form it looks like: $\alpha = 0, 3$. Latin indices from the middle of the alphabet ($i, j, k$) will refer to the three-dimensional space, in the component form it looks like: $i = 1, 3$. 
(iii) The comma in the index denotes partial derivative with respect to corresponding coordinate \((f_i := \partial f_i)\); semicolon denotes covariant derivative \((\dot{f}_i := \dot{\nabla} f_i)\).

(iv) To write the equations of electrodynamics in the article is used CGS symmetrical system.

3. Maxwell’s Equations

Maxwell’s equations in 3-dimensional form are as follows:

\[
\begin{align*}
\nabla_0 B^i &= -\epsilon^{ijk} \nabla_j E_k; \\
\nabla_i D^i &= 4\pi \rho; \\
\n\nabla_0 D^i &= \epsilon^{ijk} \nabla_j H_k - \frac{4\pi}{c} j^i; \\
\n\nabla_i B^i &= 0.
\end{align*}
\]

where \(\epsilon^{ijk}\) is the alternating tensor expressed by Levi-Civita symbol \(\epsilon_{ijk}\):

\[
\epsilon_{ijk} = \sqrt{3g} \epsilon_{ijk}, \quad \epsilon^{ijk} = \frac{1}{\sqrt{3g}} \epsilon_{ijk}.
\]

Let’s rewrite (1) with the help of electromagnetic field tensors \(F_{\alpha\beta}\) and \(G_{\alpha\beta}\) [10]:

\[
\begin{align*}
\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} &= F_{[\alpha\beta\gamma]} = 0, \\
\n\nabla_\alpha G^{\alpha\beta} &= \frac{4\pi}{c} j^\beta, \\
\end{align*}
\]

where

\[
F_{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \quad F^{\alpha\beta} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & B_3 & 0 & B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix},
\]

\[
G^{\alpha\beta} = \begin{pmatrix} 0 & -D_1 & -D_2 & -D_3 \\ D_1 & 0 & H_3 & H_2 \\ D_2 & H_3 & 0 & H_1 \\ D_3 & -H_2 & H_1 & 0 \end{pmatrix}, \quad G_{\alpha\beta} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & -H_3 & H_2 \\ -D_2 & H_3 & 0 & -H_1 \\ -D_3 & -H_2 & H_1 & 0 \end{pmatrix},
\]

\(E_i, H^i\) are components of electric and magnetic fields intensity vectors; \(D_i, B^i\) are components of vectors of electric and magnetic induction.

4. Spinors of electromagnetic field

Spinors are used in physics quite extensively. The following spinors are mainly used: Dirac four-spinors; Pauli three-spinors; quaternions. If Dirac four-spinors are used, the main difficulty is \(\gamma\)-matrices. The essence of these objects is that they serve to connect the spinor and tensor spaces and therefore have two types of indices: spinor and tensor ones. It would be logical to perform calculations in one of these spaces only. In this paper we use semispinors of Dirac spinors, Lorentz two-spinors.

The tensor of electromagnetic field \(F_{\alpha\beta}\) and its components \(F_{\alpha\beta}\), \(\alpha\) may be considered in spinor form (and similarly for \(G_{\alpha\beta}\)):

\[
\begin{align*}
F_{\alpha\beta} &= F_{A_1 A_2 B_1 B_2}, \\
F_{\alpha\beta} &= F_{A_1 A_2 B_1 B_2} g_{\alpha\beta} A^{A_1} B^{B_1} A^{A_2} B^{B_2},
\end{align*}
\]
where $g_A^{\hat{A}\hat{A}}$ are Infeld–van der Waerden symbols defined in real spinor basis $\varepsilon_{AB}$ in the following way [9]:

$$g_A^{\hat{A}\hat{A}} := g_A^{\alpha\hat{A}}\varepsilon_{\hat{A}A}, \quad g_A^{\hat{A}\hat{A}} := g_A^{\alpha\hat{A}}\varepsilon_{\hat{A}A},$$

$$\varepsilon_{AB} = \varepsilon_{\hat{A}\hat{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_{AB} = \varepsilon_{\hat{A}\hat{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_A^\alpha \varepsilon_{\hat{A}B} = \varepsilon_{\hat{A}B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4)$$

Let's $g_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$ is the Minkowski space metric. We use (4) as spinor space metric. Then the Infeld–van der Waerden symbols will have the following coordinate representation:

$$g_{\hat{A}\hat{A}} = g_{\alpha\hat{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{\hat{A}A} = g_{\hat{A}A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$g_{\hat{B}B} = g_{\alpha\hat{B}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_{\hat{B}B} = g_{\hat{B}B} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

The tensor $F_{\alpha\beta}$ is real and antisymmetric, it can be represented in the form

$$F_{\alpha\beta} = \varphi_{AB} \varepsilon_{AB} + \varepsilon_{AB} \varphi_{AB},$$

$$*F_{\alpha\beta} = -i \varphi_{AB} \varepsilon_{AB} + i \varepsilon_{AB} \varphi_{AB}. \quad (5)$$

where $\varphi_{AB}$ is a spinor of electromagnetic field:

$$\varphi_{AB} := \frac{1}{2} F_{ABC'} C' = \frac{1}{2} F_{AB\hat{B}} \varepsilon_{\hat{B}B} = \frac{1}{2} F_{\alpha\beta} \varepsilon_{\hat{A}B}. \quad \varepsilon_{\hat{A}B} = \varepsilon_{\hat{A}B}.$$  

The components of electromagnetic field spinor:

$$\varphi_{AB} = \frac{1}{2} F_{A\alpha B} \varepsilon_{\hat{A}B} g_{\alpha\hat{A}} g_{\hat{B}B}. \quad \varepsilon_{\hat{A}B} = \varepsilon_{\hat{A}B}.$$  

Using the equations (3), (4) and notation $F_i = E_i - iB^i$, we will get:

$$\varphi_{00} = \frac{1}{2} (F_{31} + F_{01} - iF_{32} - iF_{02}) = \frac{1}{2} (F_1 - iF_2),$$

$$\varphi_{01} = \varphi_{10} = \frac{1}{2} (-F_{03} - iF_{12}) = -\frac{1}{2} F_3,$$

$$\varphi_{11} = \frac{1}{2} (F_{31} - F_{01} + iF_{32} - iF_{02}) = -\frac{1}{2} (F_1 + iF_2).$$

Similarly

$$G_{\alpha\beta} = \eta_{AB} \varepsilon_{AB} + \varepsilon_{AB} \eta_{\hat{A}B},$$

$$*G_{\alpha\beta} = -i \eta_{AB} \varepsilon_{\hat{A}B} + i \varepsilon_{AB} \eta_{\hat{A}B}.$$  

where $\eta_{AB}$ is a Minkowski spinor:

$$\eta_{AB} := \frac{1}{2} G^{ABC'} C' = \frac{1}{2} G_{AB\hat{B}} \varepsilon_{\hat{B}B} = \frac{1}{2} G_{\alpha\beta} \varepsilon_{\hat{A}B}. \quad \varepsilon_{\hat{A}B} = \varepsilon_{\hat{A}B}.$$  

The components of the spinor $\eta_{AB}$:

$$\eta_{AB} = \frac{1}{2} G_{\alpha\beta} \varepsilon_{\hat{A}B} g_{\alpha\hat{A}} g_{\hat{B}B}. \quad \varepsilon_{\hat{A}B} = \varepsilon_{\hat{A}B}.$$
Using the equations (3), (4) and notation $G^i = D^i - iH^i$, we will get:

$$
\eta^{00} = \frac{1}{2} (G^{31} + G^{01} + iG^{32} + iG^{02}) = \frac{1}{2} (G^1 - iG^2),
$$

$$
\eta^{01} = \eta^{10} = \frac{1}{2} (-G^{03} + iG^{12}) = -\frac{1}{2} G^3,
$$

$$
\eta^{11} = \frac{1}{2} (G^{31} - G^{01} - iG^{32} + iG^{02}) = -\frac{1}{2} (G^1 + iG^2).
$$

5. Spinor Form of Maxwell’s Equations

Let’s write Maxwell’s equations using the spinors.

Replacing in (2) abstract indices $\alpha$ by $A\dot{A}$ and $\beta$ by $B\dot{B}$, we can write:

$$
\nabla_{A\dot{A}} G^{AABB} = \frac{4\pi}{c} J^{BB}.
$$

Using (6) we will get

$$
\nabla^{A\dot{B}} \eta_{A\dot{A}}^{B} + \nabla^{B\dot{A}} \eta_{B\dot{B}}^{A} = \frac{4\pi}{c} J^{BB}.
$$

Similarly, from (5) it follows

$$
\nabla \dot{A} \varphi_{B}^{A} - \nabla \dot{B} \varphi_{A}^{B} = 0.
$$

In so doing the system of Maxwells equations can be written as

$$
\nabla^{A\dot{B}} \varphi_{B}^{A} - \nabla^{A\dot{B}} \varphi_{A}^{B} = 0,
$$

$$
\nabla^{A\dot{B}} \eta_{A\dot{A}}^{B} + \nabla^{B\dot{A}} \eta_{B\dot{B}}^{A} = \frac{4\pi}{c} J^{BB}. \tag{7}
$$

In the vacuum case (no medium), we can put $\eta_{A\dot{B}} = \varphi_{A\dot{B}}$. Then we can write the equations (7) as follows:

$$
\nabla^{A\dot{B}} \varphi_{B}^{A} = \nabla^{A\dot{B}} \varphi_{A}^{B},
$$

$$
\nabla^{A\dot{B}} \varphi_{A}^{B} + \nabla^{B\dot{A}} \varphi_{A}^{B} = \frac{4\pi}{c} J^{BB}.
$$

Thus, the spinor form of Maxwell’s equations system in vacuum can be written in the form of one equation:

$$
\nabla^{A\dot{B}} \varphi_{A}^{B} = \frac{2\pi}{c} J^{BB}.
$$

6. Conclusions

Thus, in the article, we have proposed a representation of Maxwell’s equations in the form of Lorentz 2-spinors. We consider that the given representation might be interested in in theoretical studies.

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