N=1 supersymmetric sigma model with boundaries, II

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ABSTRACT

We consider the $N = 1$ supersymmetric two-dimensional non-linear sigma model with boundaries and nonzero $B$-field. By analysing the appropriate currents we describe the full set of boundary conditions compatible with $N = 1$ superconformal symmetry. Using this result the problem of finding a correct action is discussed. We interpret the supersymmetric boundary conditions as a maximal integral submanifold of the target space manifold, and speculate about a new geometrical structure, the deformation of an almost product structure.
1 Introduction

The conditions that arise from the two-dimensional non-linear sigma model when imposing $N = 1$ worldsheet supersymmetry on the boundary have some interesting implications. These boundary conditions may be interpreted in terms of the target space manifold, where they put restrictions on the way in which D-branes may be embedded.

The present paper is a continuation and generalisation of the results presented in our previous paper [2]. There we considered a background of general metric and zero $B$-field, and showed how worldsheet supersymmetry on the boundary leads to the appearance of Riemannian submanifolds of the target space $\mathcal{M}$. Here we extend the analysis to a background $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$, where both the metric $g_{\mu\nu}$ and the antisymmetric field $B_{\mu\nu}$ are general. It turns out that Riemannian submanifolds arise here again, but in a slightly different way. In a sense, the $B \neq 0$ case is a deformation of the $B = 0$ case.

The bulk dynamics of the non-linear sigma model is given by the superfield action (for definitions, see Appendix A)

$$S = \int d^2\sigma d^2\theta \ D_+ \Phi^\mu D_- \Phi^\nu E_{\mu\nu}(\Phi).$$

(1.1)

The conserved currents can be consistently derived from this action. However, we do not know a priori whether or not (1.1) is the correct action for deriving the boundary dynamics. Therefore our starting point is to derive the boundary conditions by looking at the appropriate boundary conditions for the bulk conserved currents,

$$[T_{++} - T_{--}]_{\sigma=0,\pi} = 0, \quad [G_+ - \eta G_-]_{\sigma=0,\pi} = 0,$$

(1.2)

where $T_{\pm\pm}$ and $G_{\pm}$ correspond to, respectively, the stress tensor and the supersymmetric current. We then define the correct action to be such that it reproduces the full set of conditions resulting from (1.2). We show that this action can be obtained by adding a unique boundary term to (1.1). The present discussion serves as a clarification of the work [3] where the problem with the standard action was pointed out.

This paper is organised as follows. In Section 2 we introduce and motivate the set of axioms that serve as our starting point for the analysis. In Section 3 we derive the boundary conditions that follow from $N = 1$ superconformal invariance at the level of the conserved currents. Section 4 discusses the correct action in elementary terms, while Section 5 provides the general derivation of this action. In Section 6 we interpret the supersymmetric boundary conditions as a maximal integral submanifold of the target space, and we also study the effect the $B$-field has on the boundary conditions. In Section 7 we speculate about the possible

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Footnote: For a related discussion treating BRST invariance see [1].
global interpretation of the boundary conditions in the presence of a $B$-field. In our view this leads to an interesting geometrical structure which we call a deformation of an almost product structure. Finally, in Section 8 we discuss some possible directions for future work and the relation of our results to those usually adopted in the literature.

2 Algebraic considerations

We begin by setting the stage for our investigation, by adopting some definitions. To motivate the first assumption that we will make, let us first recall some definitions adopted in our analysis of the $B = 0$ case [2]. There we introduced two orthogonal projectors, $P^\mu_\nu$ and $Q^\mu_\nu$, which were written in terms of a $(1,1)$ “tensor” $R^\mu_\nu$ that represented the worldsheet boundary conditions$^5$ ($\eta \equiv \pm 1$),

$$\left[ \psi^-_\mu - \eta R^\mu_\nu \psi^\nu_+ \right]_{\sigma = 0, \pi} = 0,$$

and which squared to one, $R^\mu_\nu R^\nu_\rho = \delta^\mu_\rho$. (From now on we shall drop the notation $[\cdots]_{\sigma = 0, \pi}$, since all conditions in this paper are understood to hold on the boundary.) $P^\mu_\nu$ and $Q^\mu_\nu$ could be thought of as projectors onto the Neumann and Dirichlet directions, respectively, so that, e.g., the covariant Dirichlet condition is written $Q^\mu_\nu \partial_\nu X^\nu = 0$. Moreover, we introduced so-called adapted coordinates, a basis in which $P$, $Q$ and $R$ take the form

$$P^\mu_\nu = \begin{pmatrix} \delta^n_m & \emptyset \\ 0 & 0 \end{pmatrix}, \quad Q^\mu_\nu = \begin{pmatrix} 0 & \delta^i_j \\ 0 & \delta^i_j \end{pmatrix}, \quad R^\mu_\nu = \begin{pmatrix} \delta^n_m & \emptyset \\ 0 & -\delta^i_j \end{pmatrix}, \quad \text{(2.2)}$$

where $n$ and $m$ are spacetime indices in the Neumann directions, $i$ and $j$ are Dirichlet indices, and $\delta^\mu_\nu$ is the Kronecker delta. In particular, the Dirichlet boundary conditions in this basis assume their familiar form,

$$\psi^-_i + \eta \psi^+_i = 0.$$  \quad \text{(2.3)}

Our starting point in the present study is the Dirichlet projector $Q^\mu_\nu$. The Dirichlet conditions are by definition the same as for $B = 0$, so we may again assume the existence of a projector $Q$ such that

$$Q^\mu_\nu \partial_\nu X^\nu = 0, \quad Q^\mu_\rho Q^\rho_\nu = Q^\mu_\nu. \quad \text{(2.4)}$$

In addition, we make the same ansatz (2.1) for the fermionic boundary conditions as we did for $B = 0$, and we require that $R^\mu_\nu$ satisfy

$$Q^\mu_\rho R^\rho_\nu = R^\mu_\rho Q^\rho_\nu = -Q^\mu_\nu. \quad \text{(2.5)}$$

$^5 R^\mu_\nu$ is not a proper tensor field since it is not required to be defined on the entire spacetime manifold.
This property is reasonable from a physical point of view; in adapted coordinates we write

\[ Q_\nu^\mu = \begin{pmatrix} 0 & 0 \\ 0 & \delta^i_j \end{pmatrix}, \quad R_\nu^\mu = \begin{pmatrix} R_n^m & 0 \\ 0 & -\delta^i_j \end{pmatrix}, \tag{2.6} \]

which clearly satisfy (2.5). We had the same property for \( B = 0 \), but note that here \( R_\nu^\mu \) does not square to one.

We now proceed to introduce the object \( P_\nu^\mu \),

\[ P_\nu^\mu = \frac{1}{2}(\delta_\nu^\mu + R_\nu^\mu), \tag{2.7} \]

which, unlike in the \( B = 0 \) case, is not a projector. However, we may still derive some important results in the same way as in [2], as follows. Applying the off-shell supersymmetry transformations (A.4) to the ansatz (2.1), we obtain

\[ \partial_\mu X^{\mu} - R_\nu^\mu \partial_\nu X^{\nu} - 2i\eta P_\nu^\mu F_+^{\nu} + 2iR_\nu^\mu P_\rho^\gamma R^{\rho} R^{\nu} \psi^\rho \psi^\nu = 0. \tag{2.8} \]

Inserting the \( F \)-field equations

\[ F_+^{\rho} + \Gamma_{\lambda\sigma}^\rho \psi^\lambda \psi^\sigma = 0, \tag{2.9} \]

where \( \Gamma_{\lambda\sigma}^\rho \) is the connection with torsion defined in Appendix B, (2.8) reduces to its on-shell version,

\[ \partial_\mu X^{\mu} - R_\nu^\mu \partial_\nu X^{\nu} + 2i(P_\rho^\sigma \nabla_\sigma R^{\rho} + P_\gamma^\mu g^{\gamma\delta} H_{\delta\sigma\rho} R^{\rho}) \psi^\rho \psi^\nu = 0. \tag{2.10} \]

Here \( H_{\mu\nu\rho} \) is the field strength of the background \( B \)-field. We next contract (2.10) with \( Q_\mu^\lambda \), and use that \( Q_\mu^\lambda \partial_\nu X^{\nu} = 0 \), to obtain\(^6\)

\[ P_\nu^\mu P_\sigma^\rho \nabla_\nu Q_\rho^\delta = 0. \tag{2.11} \]

This equation looks exactly like the condition we found in the \( B = 0 \) case, and which led to the integrability condition for \( P \). However, as \( P \) is not a projector, this interpretation does not apply here.

To find out what (2.11) means, we define a (Neumann) projector \( \pi_\nu^\mu \equiv \delta_\nu^\mu - Q_\nu^\mu \) with the properties

\[ \pi_\nu^\mu \pi_\rho^\nu = \pi_\rho^\nu, \quad \pi_\nu^\mu Q_\rho^\nu = 0. \tag{2.12} \]

We now have two orthogonal projectors \( \pi \) and \( Q \), corresponding to the \( P \) and \( Q \) of the \( B = 0 \) case. They may be written in terms of a (1,1) “tensor” \( r_\nu^\mu \) that squares to one, in analogy with the “tensor” \( R_\nu^\mu \) for \( B = 0 \):

\[ \pi_\nu^\mu = \frac{1}{2}(\delta_\nu^\mu + r_\nu^\mu), \quad Q_\nu^\mu = \frac{1}{2}(\delta_\nu^\mu - r_\nu^\mu), \quad r_\nu^\mu r_\rho^\nu = \delta_\nu^\mu. \tag{2.13} \]

\(^6\)Note that the property (2.5) is crucial to finding (2.11). Conversely, one might take the view that contracting (2.10) with \( Q \) should lead to \( Q_\nu^\mu \partial_\nu X^{\nu} = 0 \) in the same way as it did for \( B = 0 \). To achieve this, (2.5) is required, and thus it is implied by our assumption of Dirichlet conditions.
Among the many useful relations that follow from all the definitions above, there is the property that \( \pi \) leaves \( P \) invariant,

\[
\pi^\mu P^\nu_{\rho} = P^\mu_{\rho},
\]

which is due to \( Q^\mu P^\nu_{\rho} = 0 \). Also note that the two boundary conditions (2.1) and (2.8) may be written in terms of 1D superfields analogously to what we did in Appendix B in reference [2]: \( DK^\mu = \pi^\mu (K) DK^\nu \) and \( S^\mu = Q^\mu (K) S^\nu \).

Returning now to condition (2.11), we may use (2.14) to rewrite it as

\[
P^\gamma_{\nu} P^\phi_{\sigma} \pi^\rho_{\gamma} \nabla_{\nu} Q^\delta = 0,
\]

which implies that \( \pi \) satisfies the integrability condition,

\[
\pi^\mu_{\gamma} \pi^\rho_{\phi} \nabla_{\nu} Q^\delta = 0,
\]

since \( P \) is invertible on the \( \pi \)-subspace. We thus have a situation completely analogous to that of the \( B = 0 \) case, in that the Neumann projector must be integrable. This integrability condition turns out to be essential in the geometrical interpretation of the supersymmetric boundary conditions, as we will see in Section 6.

Note that the ansatz (2.1) for the fermionic boundary conditions is a rather simple one, \( R^\mu_{\nu} \) depending only on \( X^\mu \) and not on \( \psi^\mu_{\nu} \). The relation of this linear ansatz to more general boundary conditions is discussed in Section 8.

\section{N=1 superconformal symmetry}

In this section we derive the full set of boundary conditions using conserved currents. The bulk action (1.1) yields the following supercurrents (the main steps in deriving them were sketched in Appendix C in reference [2]),

\[
T^\pm_{\mp} = D_+ \Phi^\mu \partial_+ \Phi^\nu g_{\mu\nu} - \frac{i}{3} D_+ \Phi^\mu D_+ \Phi^\nu D_+ \Phi^\rho H_{\mu\nu\rho},
\]

\[
T^\pm_\mp = D_- \Phi^\mu \partial_- \Phi^\nu g_{\mu\nu} + \frac{i}{3} D_- \Phi^\mu D_- \Phi^\nu D_- \Phi^\rho H_{\mu\nu\rho},
\]

which obey the corresponding conservation laws,

\[
D_+ T^\pm_\mp = 0, \quad D_- T^\pm_{\mp} = 0.
\]

The components of the supercurrents (3.1) and (3.2) correspond to the supersymmetry current and stress tensor as follows,

\[
G^\pm = T^\pm_{\mp} = \psi^\mu_+ \partial_+ X^\nu g_{\mu\nu} - \frac{i}{3} \psi^\mu_+ \psi^\nu_+ \psi^\rho_+ H_{\mu\nu\rho},
\]

5
\[ G_- = T_\pm^+ = \psi^\mu \partial_\pm X^\nu g_{\mu\nu} + \frac{i}{3} \psi^\mu \psi^\nu \psi^\rho H_{\mu\nu\rho}, \] (3.5)

\[ T_{++} = -iD_+ T_{-}^- = \partial_+ X^\mu \partial_+ X^\nu g_{\mu\nu} + i\psi^\mu \nabla^{(+)}_+ \psi^\nu g_{\mu\nu}, \] (3.6)

\[ T_{--} = -iD_- T_{-}^+ = \partial_- X^\mu \partial_- X^\nu g_{\mu\nu} + i\psi^\mu \nabla^{(-)}_- \psi^\nu g_{\mu\nu}, \] (3.7)

where the covariant derivatives acting on the worldsheet fermions are defined by

\[ \nabla^{(+)}_+ \psi^\nu = \partial_+ \psi^\nu + \Gamma^\nu_{\rho\sigma} \partial_+ X^\rho \psi^\sigma, \quad \nabla^{(-)}_- \psi^\nu = \partial_- \psi^\nu + \Gamma^\nu_{\rho\sigma} \partial_- X^\rho \psi^\sigma. \] (3.8)

To ensure superconformal symmetry on the boundary we need to impose the following boundary conditions on the currents (3.4)–(3.7),

\[ G_+ - \eta G_- = 0, \quad T_{++} - T_{--} = 0. \] (3.9)

Classically these conditions make sense only on-shell, which means that we may (and should) make use of the field equations in our analysis. Thus we use the fermionic equations of motion,

\[ g_{\mu\nu}(\psi^\mu \nabla^{(+)}_+ \psi^\nu - \psi^\mu \nabla^{(-)}_- \psi^\nu) = 0, \] (3.10)

to rewrite the stress tensor condition as

\[ 0 = T_{++} - T_{--} = \partial_+ X^\mu \partial_+ X^\nu g_{\mu\nu} - \partial_- X^\mu \partial_- X^\nu g_{\mu\nu} + i(\psi_+^\mu - \eta \psi^\mu_+) \nabla_0 (\psi^\nu_+ + \eta \psi^\nu_+) g_{\mu\nu} + i(\psi_+^\mu + \eta \psi^\mu_+) \nabla_0 (\psi^\nu_+ - \eta \psi^\nu_+) g_{\mu\nu} + 2i(\psi_+^\mu \psi^\nu_+ + \psi_+^\mu \psi^\nu_-) \partial_0 X^\rho H_{\mu\nu\rho}, \] (3.11)

where \( \nabla_0 \) is the covariant \( \tau \)-derivative without torsion,

\[ \nabla_0 \psi^\nu_\pm = \partial_0 \psi^\nu_\pm + \Gamma^\nu_{\rho\sigma} \partial_0 X^\rho \psi^\sigma_\pm. \] (3.12)

Substituting the fermionic ansatz (2.1) in (3.11) we get

\[ 0 = T_{++} - T_{--} = 2i\psi_\pm^\sigma \partial_0 \psi^\lambda_\pm \left[ g_{\sigma\lambda} - R^\mu_\sigma g_{\mu\nu} R^\nu_\lambda \right] + 2\partial_0 X^\delta \pi^\rho_\delta \left[ g_{\rho\nu}(\partial_+ X^\nu - \partial_- X^\nu) - 2B_{\rho\nu} \pi^\nu_\chi \partial_0 X^\lambda \right] + i \left( R^\mu_\gamma \Gamma_{\mu\rho\nu} R^\nu_\rho - \Gamma_{\gamma\rho\sigma} - R^\sigma_\rho g_{\mu\nu} R^\nu_\gamma \right) + H_{\sigma\rho\gamma} R^\mu_\sigma H_{\mu\nu\rho} R^\nu_\gamma \psi^\sigma_+ \psi^\gamma_+. \] (3.13)

The extra term involving an antisymmetric field \( B_{\mu\nu} \) and \( \partial_0 X^\lambda \) represents the arbitrariness due to contraction with \( \partial_0 X^\delta \) (the factors in front of \( B_{\mu\nu} \) are fixed for convenience) and we add it to find the most general boundary conditions. Note that \( B_{\mu\nu} \) in this term is a priori not necessarily the same as the background \( B \)-field (i.e., the \( B \)-field whose field strength is
$H_{\mu\nu\rho}$); we just need a general antisymmetric field. However, our choice is justified by the fact that our physical setup provides the $B$-field as the only available antisymmetric field. Moreover, supersymmetry will impose a relation between this field and $H_{\mu\nu\rho}$ (see Section 3.1) which strongly suggests that the extra term in (3.13) indeed contains $B_{\mu\nu}$.

Requiring that the first term in (3.13) vanish independently we obtain the following condition,

$$R^\mu_\sigma g_{\mu\nu} R^\nu_\rho = g_{\sigma\rho}.$$  \hspace{1cm} (3.14)

This condition, that $R^\mu_\nu$ preserves the metric, is our first condition on $R^\mu_\nu$. It arose also for $B = 0$, so we see that turning on the background $B$-field does not change this requirement.

We now substitute (3.14) into (3.13), and find that the stress tensor condition reduces to

$$\begin{cases} 
\pi^\rho_\delta E_{\nu\rho} \eta_\lambda \partial_{\perp} X^\lambda - \pi^\rho_\delta E_{\mu\rho} \eta_\lambda \partial_{\perp} X^\lambda - i \pi^\rho_\delta (R^\mu_\sigma g_{\mu\nu} \nabla_\rho R^\nu_\gamma - H_{\sigma\mu\gamma} - R^\mu_\sigma H_{\mu\rho\nu} R^\nu_\gamma) \psi_+^\rho \psi_+^\gamma = 0, \\
Q^\mu_\lambda (\partial_{\perp} X^\lambda + \partial_{\perp} X^\lambda) = 0,
\end{cases}$$

(3.15)

where $E_{\mu\nu} \equiv g_{\mu\nu} + B_{\mu\nu}$, and the second equation is just the assumption (2.4). Moreover, the condition on the supersymmetry current in (3.9) takes the form (inserting the ansatz (2.1) for $\psi^\mu$)

$$0 = G_+ - \eta G_- = \psi^\rho_+ (g_{\sigma\nu} \partial_{\perp} X^\nu - R^\mu_\sigma g_{\mu\nu} \partial_{\perp} X^\nu)$$

$$- i \frac{3}{4} (H_{\mu\nu\rho} + R^\rho_\mu R^\lambda_\nu R^\gamma_\rho H_{\sigma\mu\gamma}) \psi_+^\mu \psi_+^\nu \psi_+^\rho.$$  \hspace{1cm} (3.16)

Using (3.15) in (3.16)$^7$ and requiring the first term to vanish we find

$$\pi^\rho_\delta E_{\nu\rho} \eta_\lambda R^\gamma_\lambda = \pi^\rho_\delta E_{\nu\rho} \eta_\lambda R^\gamma_\lambda.$$  \hspace{1cm} (3.17)

This is our second condition on $R^\mu_\nu$. \hspace{1cm} (3.17)

Using (3.15) in (3.16)$^7$ and requiring the first term to vanish we find

$$\pi^\rho_\delta E_{\nu\rho} \eta_\lambda R^\gamma_\lambda = \pi^\rho_\delta E_{\nu\rho} \eta_\lambda R^\gamma_\lambda.$$  \hspace{1cm} (3.18)

Our third boundary condition is obtained by using (3.17) to rewrite the remaining three-fermion term in (3.16) as

$$P^\rho_\nu R^\mu_\sigma g_{\mu\nu} \nabla_\rho R^\nu_\gamma + \frac{4}{3} P^\rho_\nu P^\nu_\mu P^\mu_\gamma H_{\mu\nu\rho} \psi_+^\rho \psi_+^\sigma \psi_+^\gamma = 0.$$  \hspace{1cm} (3.19)

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$^7$Eq. (3.16) is first rewritten using $\delta^\rho_\nu = \pi^\rho_\nu + Q^\rho_\nu$ and $Q^\rho_\nu \partial_\lambda X^\lambda = 0$.

$^8$In the case of a space-filling D-brane, i.e., when there are no Dirichlet directions, this condition amounts to $R = E^{-1} E$, where $t$ denotes transposition (see Appendix C for a discussion of purely Neumann boundary conditions).
which implies that

\[ P^\rho_\tau R^\mu_\sigma g_{\mu\nu} \nabla_\rho R^\nu_\gamma + P^\rho_\sigma R^\mu_\gamma g_{\mu\nu} \nabla_\rho R^\nu_\tau + P^\rho_\gamma R^\mu_\nu g_{\mu\nu} \nabla_\rho R^\nu_\sigma + 4 P^\mu_\tau P^\nu_\rho P^\rho_\sigma H_{\mu\nu\rho} = 0. \tag{3.20} \]

Contracting (3.20) with \( Q^\delta_\tau \) one arrives at the condition

\[ P^\gamma_\nu P^\phi_\sigma \pi^\mu_\phi \nabla_{[\rho} Q^\delta_{\mu]} = 0, \tag{3.21} \]

which is equivalent to integrability of \( \pi^\mu_\nu \), as we saw in Section 2.

We summarise the conditions on \( R^\mu_\nu \) we have obtained:

\[
\begin{align*}
\pi^\rho_\delta E_{\nu\rho} \pi^\nu_\gamma &= \pi^\rho_\delta E_{\rho\mu} \pi^\mu_\lambda R^\lambda_\gamma, \\
\pi^\mu_\gamma \pi^\rho_\nu \nabla_{[\rho} Q^\delta_{\mu]} &= 0, \\
R^\mu_\rho g_{\mu\nu} R^\nu_\sigma &= g_{\rho\sigma}.
\end{align*}
\tag{3.22}
\]

### 3.1 The B-field

We next investigate whether there are further conditions to be found, by returning to the condition (3.19) and applying our newly found conditions (3.22). The goal is to rewrite (3.19) entirely in terms of covariant derivatives of \( B^\mu_\nu \), to see what it says about the torsion. To do this, we need some relations between derivatives of \( R^\mu_\nu \) and derivatives of \( B^\mu_\nu \), and such relations are provided by (3.22) as follows.

A general \( B \)-field may be expanded in its \( \pi \)- and \( Q \)-constituents as

\[ B^\mu_\nu = B^D^\mu_\nu + B^O^\mu_\nu, \]

where the “diagonal” (\( D \)) and “off-diagonal” (\( O \)) parts are defined as

\[
\begin{align*}
B^D^\mu_\nu &\equiv \pi^\sigma_\mu B^\sigma_\rho \pi^\rho_\nu + Q^\sigma_\mu B^\sigma_\rho Q^\rho_\nu, \\
B^O^\mu_\nu &\equiv \pi^\sigma_\mu B^\sigma_\rho Q^\rho_\nu + Q^\sigma_\mu B^\sigma_\rho \pi^\rho_\nu.
\end{align*}
\tag{3.23}
\tag{3.24}
\]

Inserting (3.23) and (3.24) into (3.17) we obtain the relation

\[ g_{\mu\nu} \left( R^\nu_\gamma - r^\nu_\gamma \right) = -2 B^D^\mu_\nu P^\nu_\gamma, \tag{3.25} \]

the covariant derivative of which is

\[ g_{\mu\nu} \left( \nabla_\rho R^\nu_\gamma - \nabla_\rho r^\nu_\gamma \right) = -2 P^\nu_\gamma \nabla_\rho B^D^\mu_\nu - B^D^\mu_\nu \nabla_\rho R^\nu_\gamma. \tag{3.26} \]

Moreover, using (3.17) one may write

\[ R^\mu_\rho B^\mu_\nu R^\nu_\sigma = B^D^\sigma_\rho + R^\mu_\sigma B^O^\mu_\rho R^\nu_\rho. \tag{3.27} \]
which is equivalent to

$$R^\mu_\tau B^\rho_\mu R^\nu_\gamma = B^\rho_\tau.$$  \hfill (3.28)

Acting on this equation with $P^\rho_\sigma \nabla_\rho$ yields

$$P^\rho_\sigma B^D_\mu R^\mu_{[\nu \nabla_\rho \gamma]} = P^\rho_\sigma \nabla_\rho B^D_\tau - P^\rho_\tau R^\mu_\gamma \nabla_\rho B^D_\mu.$$  \hfill (3.29)

Thus we have the relations we were looking for, namely (3.26) and (3.29), and we use them, together with the definition $P^\mu_\nu = \frac{1}{2} (\delta^\mu_\nu + R^\mu_\nu)$, to rewrite (3.19) as

$$\psi^\mu_\tau \psi^\nu_\rho \psi^\rho_\gamma \left[ 2 P^\rho_\sigma P^\mu_\nu \nabla_\rho B^D_\mu + \frac{4}{3} P^\mu_\sigma P^\nu_\nu B^D_\mu \right] = 0.$$  \hfill (3.30)

This implies that

$$\pi^\mu_\tau \pi^\nu_\sigma \pi^\rho_\gamma H_{\mu\nu\rho} = \frac{1}{2} \pi^\mu_\tau \pi^\nu_\sigma \pi^\rho_\gamma (\nabla_\mu B^D_\nu + \nabla_\nu B^D_\mu + \nabla_\rho B^D_\mu).$$  \hfill (3.31)

This condition gives us information about the way in which our ad hoc introduced antisymmetric tensor $B_{\mu\nu}$ (see Eq. (3.13)) is related to the torsion $H_{\mu\nu\rho}$. We see that the fully $\pi$-projected torsion equals the corresponding part of the torsion of the “diagonal” part of the introduced tensor $B_{\mu\nu}$. This lends strong support to our earlier assumption that the introduced tensor in (3.13) is in fact the background $B$-field. Assuming that this is indeed the case, (3.31) reduces to

$$\pi^\mu_\tau \pi^\nu_\sigma \pi^\rho_\gamma (\nabla_\mu B^O_\nu + \nabla_\nu B^O_\mu + \nabla_\rho B^O_\mu) = 0,$$  \hfill (3.32)

which is automatically satisfied since $\pi$ is integrable. Thus we find no further conditions in addition to (3.22).

As an aside one may consider some special cases of D-branes, to see what happens to the relations derived above. For example, take $B_{\mu\nu} = B^O_{\mu\nu}$, i.e., the diagonal part is zero. Then it follows immediately from (3.25) that $R^\mu_\nu = \tau^\mu_\nu$. As a consequence (because $\pi^\mu_\rho r^\rho_\nu = \pi^\mu_\nu$ and $Q^\mu_\rho r^\rho_\nu = -Q^\mu_\nu$) the definition of $B^O_{\mu\nu}$ implies that

$$R^\mu_\sigma B_{\mu\nu} R^\nu_\rho = -B_{\sigma\rho}.$$  \hfill (3.33)

This D-brane is oriented in such a way that effectively it does not feel the $B$-field. In addition, it follows trivially from (3.32) that

$$\pi^\mu_\tau \pi^\nu_\sigma \pi^\rho_\gamma H_{\mu\nu\rho} = 0,$$  \hfill (3.34)

i.e., the fully $\pi$-projected torsion vanishes.

On the other hand, if $B_{\mu\nu} = B^D_{\mu\nu}$, then (3.28) yields

$$R^\mu_\sigma B_{\mu\nu} R^\nu_\rho = B_{\sigma\rho}.$$  \hfill (3.35)
In this case, however, there is no additional information about the torsion.

If $B$ is a symplectic form on $\mathcal{M}$ (i.e., $\det B \neq 0$ and $dB = 0$), then the solution (3.33) corresponds to a Lagrangian submanifold of $\mathcal{M}$, and (3.35) corresponds to a symplectic submanifold.

### 3.2 Compatibility with the algebra

In the interest of consistency, the previously derived results should be compatible with the supersymmetry algebra (A.4). This is verified by showing that the supersymmetry partner (2.10) of the fermionic ansatz (2.1) is equivalent to the bosonic boundary condition (3.15) when the conditions on $R$ are imposed. We do this by using the properties (2.5) of $Q$, together with the ansatz (2.1) and the conditions (3.22), to rewrite (3.15) as

$$\partial_\pm X^\mu - R^\mu_\nu \partial_\pm X^\nu + 2i(P^\sigma_\rho \nabla_\sigma R^\rho_\nu + P^\mu_\gamma g^{\gamma \delta} H_{\delta \sigma \rho} R^\sigma_\nu)\psi^\rho_+ \psi^\nu_+ = 0,$$

which is precisely (2.10). Thus we may safely conclude that the derived boundary conditions are indeed compatible with the supersymmetry algebra (A.4).

### 4 The N=1 $\sigma$-model action revised

It would be nice to rederive the results of Section 3 from a different approach, e.g., from an action. However, in the presence of a $B$-field there is a problem with the standard superfield action (1.1). In this section we explain this problem and show how to cure it. This is in effect a clarification and extension of the discussion in [3].

We start from the action (1.1) and for the sake of simplicity we take $E_{\mu \nu}$ to be constant. The field variation of the action produces a boundary term

$$\delta S = -i \int d\tau \left[ D_- (\delta \Phi^\mu D_- \Phi^\nu E_{\mu \nu}) - D_+ (D_+ \Phi^\mu \delta \Phi^\nu E_{\mu \nu}) \right],$$

or in components,

$$\delta S = -i \int d\tau \left[ i \delta X^\mu (E_{\mu \nu} \partial_- X^\nu - E_{\nu \mu} \partial_+ X^\nu) + (\delta \psi_-^\mu \psi_-^\nu E_{\mu \nu} - \delta \psi_+^\mu \psi_+^\nu E_{\mu \nu}) \right].$$

Assuming that we are interested in the free open string (i.e., $\delta X^\mu$ is arbitrary), we see from (4.2) that the bosonic boundary condition should be

$$E_{\mu \nu} \partial_- X^\nu - E_{\nu \mu} \partial_+ X^\nu = 0.$$
Now applying the supersymmetry transformation (A.4) to (4.3), we obtain the fermionic condition
\[ E_{\mu\nu}\psi_\nu - \eta E_{\nu\mu}\psi_\nu' = 0, \] (4.4)
where we have assumed that the left- and rightmoving supersymmetry parameters are related as \( \epsilon^+ = \eta \epsilon^- \). However, inserting the two conditions (4.3) and (4.4) into (4.2) gives us a nonzero variation,
\[ \delta S = -i \int d\tau 2B_{\mu\nu}\delta \psi_\nu' \psi'_\nu. \] (4.5)
(Alternatively one could write the integrand as \( 2B_{\mu\nu}\delta \psi_\nu' \psi'_\nu \) or \( B_{\mu\nu}\delta \psi_\nu' \psi'_\nu + B_{\mu\nu}\delta \psi'_\nu \psi_\nu' \).) We conclude that the supersymmetry algebra is incompatible with the requirement that (4.2) vanish.

The problem with the action (1.1) is thus that it does not allow supersymmetric freely moving open strings. The most obvious way to cure the problem is to add to the action an extra boundary term, to compensate for (4.5), e.g., \( i \int d\tau B_{\mu\nu}\psi_\nu' \psi'_\nu \) or \( i \int d\tau B_{\mu\nu}\psi_\nu' \psi'_\nu \). In [3] the latter term was added, and it was shown that the new action does admit supersymmetric freely moving open strings.

However, if one tries to analyse the problem in all generality, i.e., to find an action which produces all solutions that we obtained from the analysis of currents, then the boundary term \( i \int d\tau B_{\mu\nu}\psi_\nu' \psi'_\nu \) is not the right one. After some trial and error one realises that the correct boundary term is \( \frac{i}{2} \int d\tau B_{\mu\nu}(\psi_\nu' \psi'\nu + \psi_\nu' \psi'\nu) \), so that the modified action is
\[ S = \int d^2\xi \ d^2\theta \ D_+ \Phi^\mu \ D_- \Phi^\nu E_{\mu\nu}(\Phi) - \frac{i}{2} \int d^2\xi \ \partial_+(B_{\mu\nu}\psi_\nu' \psi'_\nu + B_{\mu\nu}\psi'_\nu \psi_\nu'). \] (4.6)
In the following section we show that this action indeed reproduces all the solutions previously obtained from the currents. The extra boundary term in (4.6) has also appeared in [4].

We may have fixed the supersymmetry problem, but now there is another puzzle associated with the new action, concerning the coupling to a \( U(1) \) field. If we shift the \( B \)-form by an exact two-form \( d\Lambda \), then (4.6) transforms as
\[ S(B + d\Lambda) = S(B) + 2 \int d\tau \ A_\mu \partial_0 X^\mu, \] (4.7)
where we expect the \( A \)-term to be absorbed by a \( U(1) \) coupling, because there should be a shift symmetry
\[ B \rightarrow B + d\Lambda, \quad A \rightarrow A - \Lambda. \] (4.8)
However, the last term in (4.7) is purely bosonic and differs from the standard supersymmetric \( U(1) \) coupling
\[ \int d\tau \left[ 2A_\mu \partial_0 X^\mu - i\partial_\mu A_\nu (\psi^\mu_+ + \eta \psi^\mu_-)(\psi^\nu_+ + \eta \psi^\nu_-) \right]. \] (4.9)
As a final general remark, using the formal rules for 2D superfields one can write the action (4.6) entirely in terms of superfields, as follows,

\[
S = \int d^2 \xi d^2 \theta \left[ D_+ \Phi^\mu D_- \Phi^\nu E_{\mu\nu} + \frac{1}{2D_+} D_- (B_{\mu\nu} D_+ \Phi^\mu D_+ \Phi^\nu + B_{\mu\nu} D_- \Phi^\mu D_- \Phi^\nu) \right].
\]

(4.10)

The last “non-local” term appears only when \(B \neq 0\). This form of the action is perhaps formally acceptable, but we lack a physical interpretation for this kind of non-locality in superspace.

5 Boundary conditions from the action

Our aim in this section is to rederive the full set of boundary conditions from the action

\[
S = \int d^2 \xi d^2 \theta \ D_+ \Phi^\mu D_- \Phi^\nu E_{\mu\nu}(\Phi) - \frac{i}{2} \int d^2 \xi \ \partial_\pm (B_{\mu\nu} \psi^\mu_+ \psi^\nu_+ + B_{\mu\nu} \psi^\mu_- \psi^\nu_-),
\]

(5.1)

where now \(E_{\mu\nu}\) is general. We will show that the result coincides with the conditions obtained from the currents in Section 3, and hence that the above action is the correct one for describing the boundary dynamics.

The boundary term in the field variation of \(S\) is given by

\[
\delta S = i \int d\tau \left[ (\delta \psi^\mu_+ \psi^\nu_+ - \delta \psi^\mu_- \psi^\nu_-) g_{\mu\nu} + \delta X^\mu (i \partial_+ X^\nu E_{\nu\mu} - i \partial_- X^\nu E_{\nu\mu} + \Gamma^-_{\nu\mu\rho} \psi^\nu_- \psi^\rho_- - \Gamma^+_{\nu\mu\rho} \psi^\nu_+ \psi^\rho_+) \right].
\]

(5.2)

When we insert the fermionic ansatz (2.1) and use the properties (2.5), cancellation of the fermionic variation requires that

\[
R^\mu_{\sigma\rho} g_{\mu\nu} R^\nu_{\rho} = g_{\sigma\rho}.
\]

(5.3)

This is the first of our conditions, preservation of the metric.

Using (5.3), the variation (5.2) collapses to

\[
\delta S = \int d\tau \ \partial_\pm X^\nu \pi^\mu_\delta E_{\mu\nu} - \partial_+ X^\nu E_{\nu\mu} - \partial_- X^\nu E_{\nu\mu} - i (R^\nu_{\rho\sigma} g_{\gamma\sigma} \nabla_\mu R^\rho_{\nu} + H_{\nu\mu\rho} + H_{\gamma\mu\sigma} R^\rho_{\nu} R^\sigma_{\rho}) \psi^\nu_- \psi^\rho_+ \psi^\mu_+ \psi^\nu_+ \psi^\rho_+.
\]

(5.4)

implying the following bosonic boundary condition,

\[
\partial_\pm X^\nu \pi^\mu_\delta E_{\mu\nu} = \partial_+ X^\nu \pi^\mu_\delta E_{\nu\mu} - \partial_- X^\nu \pi^\mu_\delta E_{\nu\mu} - i \pi^\mu_\delta \left( R^\nu_{\rho\sigma} g_{\gamma\sigma} \nabla_\mu R^\rho_{\nu} + H_{\nu\mu\rho} + H_{\gamma\mu\sigma} R^\rho_{\nu} R^\sigma_{\rho} \right) \psi^\nu_- \psi^\rho_+ \psi^\mu_+ \psi^\nu_+ \psi^\rho_+ = 0.
\]

(5.5)
where we have assumed that \( Q^\mu_\nu \delta X^\nu = 0 \), or equivalently,

\[
Q^\mu_\nu (\partial_+ X^\nu + \partial_- X^\nu) = 0. \tag{5.6}
\]

(Note that the condition that (5.4) be zero gives rise to the same kind of subtlety as did the vanishing of (3.13); the terms involving \( B^\mu_\nu \) vanish due to contraction with \( \delta X^\mu \). However, here this arbitrariness is automatically represented by \( B^\mu_\nu \), as a physical and logical consequence of the action, and we need not introduce an ad hoc field.) The properties (5.3) and (5.6) may now be used to rewrite (5.5) as

\[
\partial_\pm X^\nu \pi^\mu_\delta E_{\mu\lambda} \pi^\lambda_\nu - \partial_+ X^\nu \pi^\mu_\delta E_{\lambda\mu} \pi^\lambda_\nu - i \pi^\mu_\delta (R^\gamma_\mu g_\gamma \sigma \nabla_\mu R^\sigma_\nu + H_\nu \mu \rho + H_\gamma \mu \rho R^\gamma_\nu R^\sigma_\rho) \psi^\nu_+ \psi^\rho_+ = 0, \tag{5.7}
\]

which is identical to (3.15).

Requiring supersymmetry on the boundary means, in terms of the action, that the boundary conditions on the worldsheet fields must be such that the field and supersymmetry variations vanish simultaneously. Our next step is therefore to examine the supersymmetry variation of (5.1), which is given by

\[
\delta s S = \epsilon^- \int d\tau \left[ \partial_+ X^\mu \psi^\nu_+ E_{\mu\nu} - \eta \psi^\nu_+ \partial_+ X^\nu E_{\mu\nu} + \eta \partial_+ X^\mu \psi^\nu_+ B_{\mu\nu} + \partial_- X^\mu \psi^\nu_+ B_{\mu\nu} - \frac{i}{3} \eta H_{\mu\nu\rho} \psi^\mu_+ \psi^\nu_+ - \frac{i}{3} H_{\mu\nu\rho} \psi^\mu_+ \psi^\nu_+ 
+ i F^\mu_-(\eta \psi^\nu_+ + \psi^\nu_+) g_{\mu\nu} + i \Gamma^-_{\nu\mu\rho} \psi^\mu_+ \psi^\rho_+ + i \Gamma^-_{\nu\mu\rho} \psi^\mu_+ \psi^\rho_+ \right]. \tag{5.8}
\]

Inserting the \( F \)-equation (2.9), (5.8) becomes

\[
\delta s S = \epsilon^- \int d\tau \left[ \partial_+ X^\mu \psi^\nu_+ E_{\mu\nu} - \eta \psi^\nu_+ \partial_+ X^\nu E_{\mu\nu} + \eta \partial_+ X^\mu \psi^\nu_+ B_{\mu\nu} + \partial_- X^\mu \psi^\nu_+ B_{\mu\nu} - \frac{i}{3} \eta H_{\mu\nu\rho} \psi^\mu_+ \psi^\nu_+ - \frac{i}{3} H_{\mu\nu\rho} \psi^\mu_+ \psi^\nu_+ 
+ i F^\mu_-(\eta \psi^\nu_+ + \psi^\nu_+) g_{\mu\nu} + i \Gamma^-_{\nu\mu\rho} \psi^\mu_+ \psi^\rho_+ + i \Gamma^-_{\nu\mu\rho} \psi^\mu_+ \psi^\rho_+ \right]. \tag{5.9}
\]

Now we plug in the fermionic ansatz (2.1), the bosonic conditions (5.5), and the property (5.6), into (5.9). The \( \partial X \psi \)-terms that remain after this operation cancel only if

\[
\pi^\mu_\delta E_{\nu\rho} \pi^\rho_\gamma = \pi^\mu_\delta E_{\rho\nu} \pi^\rho_\gamma R^\lambda_\gamma. \tag{5.10}
\]

Imposing (5.10) thus reduces the condition \( \delta s S = 0 \) to the following requirement for the three-fermion term,

\[
\left[ P^\gamma_\rho R^\mu_\sigma g_{\mu\nu} \nabla_\rho R^\sigma_\gamma + \frac{4}{3} P^\mu_\tau P^\nu_\sigma P^\rho_\gamma H_{\mu\nu\rho} \right] \psi^\tau_+ \psi^\rho_+ \psi^\gamma_+ = 0, \tag{5.11}
\]

which is precisely the condition (3.19). We know from Section 3 that this equation produces the integrability condition for \( \pi \),

\[
\pi^\mu_\sigma \pi^\nu_\gamma Q^\rho_{[\mu,\nu]} = 0, \tag{5.12}
\]

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as well as the requirement that
\[ \pi^\mu_\tau \pi^\nu_\sigma \pi^\rho_\gamma H_{\mu\nu\rho} = \frac{1}{2} \pi^\mu_\tau \pi^\nu_\sigma \pi^\rho_\gamma (\nabla_\mu B^D_{\nu\rho} + \nabla_\nu B^D_{\rho\mu} + \nabla_\rho B^D_{\mu\nu}). \] (5.13)

In Section 3.1 this condition gave us information about the way in which the ad hoc introduced antisymmetric tensor \( B_{\mu\nu} \) is related to \( H_{\mu\nu\rho} \). Within the present analysis, however, we already know that \( H = dB \) (by definition), and therefore (5.13) leads directly to
\[ \pi^\mu_\tau \pi^\nu_\sigma \pi^\rho_\gamma (\nabla_\mu B^O_{\nu\rho} + \nabla_\nu B^O_{\rho\mu} + \nabla_\rho B^O_{\mu\nu}) = 0, \] (5.14)
i.e., condition (3.32). Due to (5.12), this is identically satisfied.

Thus we have obtained the same set of boundary conditions as from the currents, namely (5.3), (5.10) and (5.12) (cf. Eq. (3.22)). The check that the above boundary conditions are compatible with the supersymmetry algebra (A.4) is now identical to that of Section 3.2, showing equivalence between (5.7) and (2.10).

6 Geometric interpretation

In this section we discuss the geometrical interpretation of our results. Some background information on submanifolds of Riemannian manifolds is given in Appendix D.

6.1 D-branes as submanifolds

Let us first summarise the formal results we have derived in previous sections. We found that supersymmetry requires the worldsheet fields to obey the boundary conditions

\[ \begin{align*}
\psi^-_\mu - \eta R^\mu_\nu \psi^\nu_+ &= 0, \\
\partial^- X^\mu - R^\mu_\nu \partial^+ X^\nu + 2i(P^\sigma \nabla_\sigma R^\mu_\nu + P^\gamma g^{\gamma\delta} H_{\delta\sigma\rho} R^\rho_\nu) \psi^\rho_+ \psi^\nu_+ &= 0,
\end{align*} \] (6.1)

where \( 2P^\mu_\nu = \delta^\mu_\nu + R^\mu_\nu \), and \( R^\mu_\nu \) satisfies

\[ \begin{align*}
\pi^\rho_\delta E_{\nu\rho} \pi^\nu_\gamma &= \pi^\rho_\delta E_{\rho\mu} \pi^\mu_\lambda R^\lambda_\gamma, \\
Q^\mu_\rho R^\rho_\nu &= R^\mu_\rho Q^\rho_\nu = -Q^\nu_\nu, \\
R^\mu_\rho g_{\mu\nu} R^\rho_\sigma &= g_{\rho\sigma}, \\
\pi^\mu_\rho \pi^\nu_\sigma Q^{\lambda}_{[\mu,\nu]} &= 0.
\end{align*} \] (6.2)

At first sight these results look rather formal. However, it turns out they have a simple geometrical interpretation.
In mathematical terms, the Dirichlet projector $Q^\mu_\nu(X)$ is a differentiable distribution\(^9\) which assigns to a point $X$ in the $d$-dimensional spacetime manifold $\mathcal{M}$ a $(d - p - 1)$-dimensional subspace\(^10\) of the tangent space $T_X(\mathcal{M})$. This subspace consists of all vectors $v^\mu(X) \in T_X(\mathcal{M})$ such that

$$Q^\mu_\nu(X)v^\nu(X) = v^\mu(X). \tag{6.3}$$

The complementary distribution is defined as $\pi^\mu_\nu = \delta^\mu_\nu - Q^\mu_\nu$, and assigns to $X$ a $(p + 1)$-dimensional space that consists of vectors $v^\mu(X) \in T_X(\mathcal{M})$ such that

$$\pi^\mu_\nu(X)v^\nu(X) = v^\mu(X). \tag{6.4}$$

Now we ask when the vector fields defined by (6.4) span a submanifold. To see the answer, note that the Lie bracket of two vector fields $v$ and $w$ in $\pi$-space is

$$\{v, w\}^\nu = \pi^\nu_\sigma \{v, w\}^\sigma + v^\rho w^\sigma \pi^\mu_\rho \pi^\lambda_\nu \tilde{Q}^{\lambda,\mu}. \tag{6.5}$$

If the last term vanishes (i.e., if $\pi^\mu_\nu$ is integrable), then the distribution $\pi^\mu_\nu$ is involutive, and due to the classical theorem of Frobenius there is a unique maximal integral submanifold corresponding to $\pi^\mu_\nu$. This submanifold of $\mathcal{M}$ is the worldvolume of a $Dp$-brane. We emphasise that worldsheet supersymmetry plays a crucial role in this interpretation. Also note that the discussion is completely local and therefore directly applicable to our boundary conditions; there is no need to extend our objects $R^\mu_\nu, \pi^\mu_\nu$, etc., to be globally defined.

Having established that supersymmetric boundary conditions define $Dp$-branes as submanifolds of the spacetime Riemann manifold $\mathcal{M}$, we may proceed to identify some of the objects associated with such a submanifold. In particular, there is an induced metric, an induced connection, a second fundamental form, and an associated second fundamental form. The induced metric may be taken to be the $\pi$-projected part of $g_{\mu\nu}$. To identify the other structures, take two vector fields $v$ and $w$ in the $\pi$-space. Denoting by $\nabla$ the connection on $\mathcal{M}$, we may write

$$v^\mu \nabla_\mu w^\nu = \pi^\nu_\rho v^\mu \nabla_\mu w^\rho + Q^\nu_\rho v^\mu \nabla_\mu w^\rho, \tag{6.6}$$

where we have decomposed the derivative into its tangential (to the $\pi$-space) and normal parts by using $\delta^\mu_\nu = \pi^\mu_\nu + Q^\mu_\nu$. The tangential component is the induced connection, and the normal component is the second fundamental form (the definitions are given in Appendix D). The latter may be rewritten, using that $\pi^\mu_\nu v^\nu = v^\mu$ and $\pi^\mu_\nu w^\nu = w^\mu$, as

$$v^\delta w^\sigma B^{\lambda}_{\delta\sigma} \equiv -v^\delta w^\sigma \pi^\mu_\rho \pi^\nu_\sigma \nabla_\mu \tilde{Q}^{\lambda,\nu}. \tag{6.7}$$

\(^9\)We need to assume differentiability to be able to do the calculations. However one should keep in mind that one can construct such brane configurations where this property is lost (e.g., a brane ending on a brane).

\(^10\)We take $\text{rank}(Q) = d - p - 1$. 

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Note that $B^\lambda_{\delta\sigma}$ is symmetric in indices $\delta$ and $\sigma$, as a second fundamental form must be, because $\pi^\mu_\nu$ is integrable.

Performing the same decomposition for the derivative of a vector field $u$ in the $Q$-space, we have

$$v^\mu \nabla_\mu u^\nu = \pi^\nu_\rho v^\mu \nabla_\mu u^\rho + Q^\nu_\rho v^\mu \nabla_\mu u^\rho,$$

(6.8)

where $Q^\nu_\rho u^\rho = u^\nu$ and $v$ is still in the $\pi$-space. The associated second fundamental form is then defined as the tangential part, which we can rewrite as

$$v^\delta u^\sigma A^\lambda_{\delta\sigma} \equiv -v^\delta u^\sigma \pi^\mu_\delta \pi^\lambda_\nu \nabla_\mu Q^\nu_\sigma.$$

(6.9)

For the case $B_{\mu\nu} = 0$ the bosonic boundary conditions in (6.1) can be expressed using the associated second fundamental form, as

$$\partial_\pm X^\mu - R^\mu_\nu \partial_\pm X^\nu + 4\,i\,A^\mu_\nu \psi^\dagger_\gamma \psi^\nu_+ = 0. $$

(6.10)

Thus the properties of the two-fermion term are closely related to the properties of the second fundamental form. For instance, when the submanifold is totally geodesic (i.e., $A = B = 0$), the two-fermion term vanishes and the bosonic boundary conditions reduce to

$$\partial_\pm X^\mu - R^\mu_\nu \partial_\pm X^\nu = 0,$$

(6.11)

a commonly assumed condition in the literature (see, e.g., [5]).

### 6.2 Effects of the B-field

When $B_{\mu\nu} \neq 0$, the two-fermion term will, in addition to the associated second fundamental form, involve structures depending on the $B$-field. To study the effects of this on the boundary conditions we introduce a parameter $z$ which measures the size of the $B$-field, and then vary this parameter within its allowed range. We define $B_{\mu\nu} \equiv z b_{\mu\nu}$, where $b_{\mu\nu}$ is a general antisymmetric field, and the physically relevant situation is when $z$ can take any real value, $z \in \mathbb{R}$. One may now view $R(z)$ as a matrix-valued function of $z$ for a given point $X$ in $\mathcal{M}$. The behaviour of this function is determined by the first two of Eqs. (6.2),

$$\begin{cases} 
\pi^\rho_\delta (g_{\rho\nu} - z b_{\rho\nu}) \pi^\nu_\gamma = \pi^\rho_\delta (g_{\rho\nu} + z b_{\rho\nu}) \pi^\nu_\gamma R^\lambda_\gamma(z), \\
Q^\mu_\rho R^\gamma_\nu(z) = R^\mu_\rho(z) Q^\rho_\nu = -Q^\mu_\nu.
\end{cases}$$

(6.12)

An immediate consequence of (6.12) is the property

$$R^\mu_\nu(z) R^\nu_\gamma(-z) = \delta^\mu_\gamma.$$

(6.13)
showing that $R(-z)$ is the inverse of $R(z)$.

To simplify the analysis of $R(z)$, we extend $z$ to the complex plane, $z \in \mathbb{C}$. Then (6.12) implies that $R(z)$ is a meromorphic function with poles given by the condition $\det(\pi^\rho_\phi (g_{\rho\nu} + z b_{\rho\nu}) \pi^\phi_\lambda) = 0$ (the determinant is understood to be taken in the $\pi$-space). It is clear from this condition that the poles are finite in number and located away from the real axis if the metric is positive definite.\footnote{If the metric has Minkowski signature, there may be a pole on the real axis away from the origin, associated with the critical “electric” field [6]. In this case one cannot go continuously from $z = 0$ to $z = \infty$ while keeping $z$ real.}

For the purpose of studying the limits $z \to 0$ and $z \to \infty$, we write $R(z)$ as a Taylor expansion around each limit. The first series is one in positive powers of $z$ around zero,

$$R^\mu_\nu(z) = r^\mu_\nu + \sum_{k=1}^{\infty} R^{(k)\mu}_\nu z^k, \quad |z| < |z_{\text{min}}|$$ (6.14)

where $r^\mu_\nu = R^\mu_\nu(0)$ is independent of $z$, and $z_{\text{min}}$ is the pole closest to the origin. Due to (6.13), $r$ squares to one, $r^\mu_\nu r^\nu_\rho = \delta^\mu_\rho$, a property that we discussed in Section 2.

The expansion around infinity can similarly be written as a power series in $1/z$,

$$R^\mu_\nu(z) = \tilde{r}^\mu_\nu + \sum_{k=1}^{\infty} \tilde{R}^{(k)\mu}_\nu z^{-k}, \quad |z| > |z_{\text{max}}|$$ (6.15)

where $z_{\text{max}}$ is the pole most distant from the origin. Again we see that the constant $\tilde{r}$ squares to one.

The discussion so far has been restricted to a given point $X$ in $\mathcal{M}$, but as one moves on $\mathcal{M}$, the poles of $R(z)$ move in $\mathbb{C}$, since their location is determined by the metric, by the projector $\pi^\mu_\nu$ and by $b_{\mu\nu}$, all of which depend on $X$. In principle it is possible for a pole to move out to infinity, rendering (6.15) useless. Thus, note that $\tilde{r}$ has a quite different status from that of $r$; it is not necessarily well-defined in the same way that $r$ is. In particular, it may not be a continuous function of $X$. This is easily seen for the special case when $\pi^\mu_\rho b_{\mu\nu} \pi^\nu_\sigma$ changes rank; a small perturbation of $b$ causes a jump in $\tilde{r}$.

The expansion (6.14) is well-defined and safe to use as it stands for analysing the $z \to 0$ limit; the only possible subtlety would be when there is a pole at the origin, but that never happens. Taking $z \to 0$ just means that $B_{\mu\nu} = 0$, reducing our boundary conditions to the ones derived in [2].

Using the expansion (6.15), however, requires some care. First, one needs to make sure that $\tilde{r}$ is well-defined and differentiable on the neighbourhood in $\mathcal{M}$ where we want to perform the analysis; second, there must exist a region of convergence. The first issue depends on
the rank of $\pi_\rho b_{\mu\nu} \pi^\nu_\sigma$, as indicated above. Thus as long as we keep to a fixed rank, $\tilde{r}$ is differentiable. The second issue is related to the location of the poles, which depends on the spacetime coordinates.

Hence we restrict our attention to a situation where $\pi_\rho b_{\mu\nu} \pi^\nu_\sigma$ has a fixed rank and where $g_{\mu\nu}, b_{\mu\nu}$ and $\pi_\mu^\nu$ do not vary significantly in some neighbourhood in $\mathcal{M}$. Then the expansion (6.15) makes sense in this neighbourhood and can be used to study the $z \to \infty$ limit of our boundary conditions. We proceed by looking at the boundary conditions on the form (3.15) and (3.18), in which we substitute the definition $B_{\mu\nu} = zb_{\mu\nu}$ and the expansion (6.15). Then as $z \to \infty$, (3.18) collapses to fermionic Dirichlet conditions along $\pi_\rho b_{\mu\nu} \pi^\nu_\sigma$, whereas the bosonic conditions (3.15) deviate from Dirichlet ones by a two-fermion term involving the field strength $H_{\mu\nu\rho}$. This deviation poses a problem for the physical interpretation; if we want to picture the boundary conditions in terms of D-branes, the bosonic condition must also be pure Dirichlet. Hence we demand that the two-fermion vanish, obtaining

$$\pi^\mu_\delta \left( H_{\mu\nu\rho} + \tilde{r}_\rho^\sigma \tilde{r}_\rho^\lambda H_{\mu\sigma\lambda} \right) = 0$$  \hspace{1cm} (6.16)

in the limit $z \to \infty$.

In conclusion we see that, provided that $H_{\mu\nu\rho}$ satisfies (6.16) there is a flow between $r$ and $\tilde{r}$ in the sense that those boundary conditions along $B_{\mu\nu}$ which are Neumann for zero $B$-field flow to Dirichlet conditions for very large $B$-field.

## 7 Deformation of almost product structures

In our previous paper [2] we pointed out that, in the absence of a $B$-field, globally defined supersymmetric boundary conditions lead to the appearance of a partially integrable almost product manifold, and vice versa. In the present general case, with arbitrary $B_{\mu\nu}$, we find an interesting generalisation of the almost product structure which is worth discussing.

Let $\mathcal{M}$ be a $d$-dimensional manifold with a $(1, 1)$ tensor $r_\mu^\nu$ such that, globally,

$$r_{\rho}^\mu r_{\nu}^\rho = \delta_{\rho}^\mu.$$  \hspace{1cm} (7.1)

Then $\mathcal{M}$ is an almost product manifold [10] with almost product structure $r_\mu^\nu$. Assuming that there is a $(1, 1)$ tensor $b_\mu^\nu$ such that $b_\nu^\mu = -b_\mu^\nu$, one can define a whole set of new $(1, 1)$ tensors $R_\nu^\mu(z)$ as follows,

$$\pi_\delta^\mu \left( \delta_\nu^\rho - zb_\nu^\rho \right) \pi^\nu_\gamma = \pi_\delta^\mu \left( \delta_\nu^\rho + zb_\nu^\rho \right) \pi^\nu_\lambda R^\lambda_\gamma(z),$$

$$Q_\mu^\nu R^\mu_\nu(z) = R_\mu^\nu(z)Q_\mu^\nu = -Q_\mu^\nu,$$  \hspace{1cm} (7.2)
where \( z \in \mathbb{R} \), and \( Q_\nu^\mu = \frac{1}{2}(\delta^\mu_\nu - r^\mu_\nu) \) is a globally defined distribution. It follows from (7.2) that \( R^\mu_\nu(z) \) satisfies
\[
R^\mu_\nu(z)R^\nu_\gamma(-z) = \delta^\mu_\gamma.
\]
(7.3)

At any given point in \( \mathcal{M} \) one can go to adapted coordinates, where \( R^\mu_\nu(z) \) can be symbolically written as
\[
R(z) = \begin{pmatrix}
I - zb & 0 \\
I + zb & -I
\end{pmatrix},
\]
(7.4)
where \( b = (b^m_n) \) (i.e., \( b \) is along the \( \pi \)-directions) and the inverse of \( I + zb \) is understood to be taken in the \( \pi \)-space. Since \( R^\mu_\nu(0) = r^\mu_\nu \), one may think of \( R^\mu_\nu(z) \) as a deformation of the almost product structure \( r^\mu_\nu \).

Following the logic of Section 6 one can extend \( z \) to the complex plane and study its analytic behaviour. For a given point \( X \) in \( \mathcal{M} \), \( R^\mu_\nu(z) \) is then a meromorphic function with a finite number of poles given by \( \det(I + zb) = 0 \) (as before the determinant is understood to be taken in the \( \pi \)-space). Again we assume that the number of poles does not change (i.e., the rank of \( b \) does not change) and that they do not move much in \( \mathbb{C} \) as \( X \) moves in \( \mathcal{M} \), so that we can use the expansions (6.14) and (6.15). Then there will be a flow between \( r \) and \( \tilde{r} \), just like in the previous section, which in the present context is a flow between two almost product structures.

8 Discussion

Starting from the fermionic ansatz
\[
\psi^\mu_- = \eta R^\mu_\nu(X)\psi^\nu_+ \tag{8.1}
\]
and assuming that the bosonic coordinate is confined to some region of the tangent manifold of spacetime (i.e., we take \( Q^\mu_\nu \partial_0 X^\nu = 0 \)), we have derived the supersymmetric boundary conditions for the general \( N = 1 \) non-linear sigma model. The problem is analysed in two different ways: by studying the conserved currents and by studying requirements for invariance of the action. In essence we find that the D-brane has to be a submanifold of the target space, and that the \( B \)-field and torsion have significant effects on the boundary conditions.

It is natural to ask to what extent the present results are general. In principle one could start from the most generic form of fermionic boundary conditions [7, 8],
\[
\psi^\mu_- = \mathcal{R}^\mu(X, \psi_+) = \eta R^\mu_\nu(X)\psi^\nu_+ + \eta R^\mu_\nu(X)\psi^\nu_+ \psi^\rho_+ \psi^\sigma_+ + ..., \tag{8.2}
\]
19
where the dots stand for terms with five or more fermions. After plugging (8.2) into the stress tensor condition $T_{++} - T_{--} = 0$ on the form (3.11), one realises without much effort that the three-fermion term in (8.2) gives rise to a four-fermion term in the bosonic boundary conditions. However, the two-fermion term in the bosonic condition would stay exactly the same as in (3.15). In fact, since we keep the property that $Q_\mu^\nu \partial_0 X^\nu = 0$, the general problem can be solved order by order in the fermions. Thus in the present paper we have derived the general supersymmetric boundary conditions up to two-fermion terms, and the proposed geometrical interpretation in terms of maximal integral manifolds will still be valid for the generic case. However, it is unclear to us what could be the geometrical interpretation of $R_\mu^{\nu\rho\sigma}$ and other higher rank objects. It is to be expected that the four- and higher-order fermion terms must be included in the bosonic boundary condition in the quantum theory. This problem would be interesting to study further.

In the light of the above discussion the two-fermion terms in the bosonic boundary conditions are important for consistency with the supersymmetry algebra (see also Section 2). However, the boundary conditions usually adopted in the literature (see, e.g., [5, 9]) do not have two-fermion terms. As we have explained, the two-fermion terms can be absent but only in very specific situations, for instance when the D-brane is a totally geodesic submanifold. We find this point confusing and think that this issue deserves further investigation. To resolve the problem one has to generalise the present results to the $N = 2$ supersymmetric sigma model, as this is the model studied in the above references.

Another aspect of our work which calls for further study is the issue of the correct action for the $N = 1$ supersymmetric sigma model with boundaries. In particular, the problem of the supersymmetric $U(1)$ coupling and the shift symmetry needs to be solved.

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**Note added in Proof:** Due to editorial problems at a different journal, the publication of this article has been delayed considerably. There has thus been some subsequent development along the lines described here. More precisely, the techniques have been applied to $N=1,2$ WZW-models in [11] and [12]. In these articles we analyze the restrictions on the gluing conditions of the affine currents and relate them to our boundary conditions. We reproduce known result from this geometric point of view and generalize them. In particular
we find that the gluing map between the left and right affine currents may be generalized in a very specific way allowing for non-constant Lie algebra automorphisms.

A (1, 1) supersymmetry

Throughout the paper we use \( \mu, \nu, ... \) as spacetime indices, \((+,=)\) as worldsheet indices (in lightcone coordinates \( \xi^\pm \equiv \tau \pm \sigma \), where \( \tau, \sigma \) are the usual worldsheet coordinates), and \((+,-)\) as two-dimensional spinor indices. We also use superspace conventions where the pair of spinor coordinates are labelled \( \theta^\pm \), and the covariant derivatives \( D^\pm \) and supersymmetry generators \( Q^\pm \) satisfy

\[
D^2_+ = i \partial_+, \quad D^2_- = i \partial_-, \quad \{D_+, D_-\} = 0,
\]

where \( \partial_{\pm} = \partial_0 \pm \partial_1 \) (\( \partial_{0,1} \equiv \partial_{\tau,\sigma} \)). In terms of the covariant derivatives, a supersymmetry transformation of a superfield \( \Phi \) is then given by

\[
\delta \Phi \equiv (\epsilon^+ Q_+ + \epsilon^- Q_-) \Phi = -(\epsilon^+ D_+ + \epsilon^- D_-) \Phi + 2i(\epsilon^+ \theta^+ \partial_+ + \epsilon^- \theta^- \partial_-) \Phi
\]

The components of a superfield \( \Phi \) are defined via projections as follows,

\[
\Phi| \equiv X, \quad D_\pm \Phi| \equiv \psi_\pm, \quad D_+ D_- \Phi| \equiv F_{+-},
\]

where a vertical bar denotes “the \( \theta = 0 \) part of ”. Thus, in components, the (1, 1) supersymmetry transformations are given by

\[
\begin{align*}
\delta X^\mu &= -\epsilon^+ \psi^\mu_+ - \epsilon^- \psi^\mu_-
\delta \psi^\mu_+ &= -i \epsilon^+ \partial_+ X^\mu - \epsilon^- F^\mu_+
\delta \psi^\mu_- &= -i \epsilon^- \partial_- X^\mu - \epsilon^+ F^\mu_-
\delta F^\mu_{+-} &= -i \epsilon^+ \partial_+ \psi^\mu_- + i \epsilon^- \partial_- \psi^\mu_+ \end{align*}
\]

B Affine connection with torsion

Here we collect our conventions for the affine connection with and without torsion. The covariant derivatives are defined as follows,

\[
\nabla^{(\pm)}_{\rho} R^{\mu}_{\nu} = R^{\mu}_{\nu,\rho} + \Gamma^{\pm\mu}_{\rho\nu} R^{\sigma}_{\nu} - \Gamma^{\pm\sigma}_{\rho\nu} R^{\mu}_{\sigma},
\]

\[
\nabla_{\rho} R^{\mu}_{\nu} = R^{\mu}_{\nu,\rho} + \Gamma^{\nu}_{\rho\sigma} R^{\sigma}_{\nu} - \Gamma^{\sigma}_{\rho\nu} R^{\mu}_{\sigma},
\]
where the comma stands for the partial derivative, so that $R^{\mu}_{\nu,\rho} \equiv \partial_{\rho}R^{\mu}_{\nu}$. The functions $\Gamma$ are defined as

\[
\Gamma^{\pm \nu}_{\rho \sigma} = \Gamma^{\nu}_{\rho \sigma} \pm g^{\nu \mu} H_{\mu \rho \sigma}, \\
\Gamma^{\nu}_{\rho \mu} = g_{\nu \sigma} \Gamma^{\sigma}_{\mu \rho} = \frac{1}{2}(g_{\nu \mu, \rho} + g_{\rho \nu, \mu} - g_{\rho \mu, \nu}),
\]

with $H$ being the torsion three form,

\[
H_{\mu \rho \sigma} = \frac{1}{2}(B_{\mu \rho, \sigma} + B_{\rho \sigma, \mu} + B_{\sigma \mu, \rho}).
\]

It follows that $\Gamma^{+ \mu \nu \rho} = \Gamma^{- \mu \rho \nu}$.

**C Boundary conditions in different coordinate systems**

By going to a specific coordinate system, the boundary conditions can sometimes be simplified considerably. As an illustration of this idea we discuss below the purely Neumann boundary conditions in different coordinate systems. The generalisation to other cases is straightforward.

The Neumann boundary conditions (freely moving string), in the presence of arbitrary metric and $B$-field, are

\[
\begin{cases}
E_{\mu \nu} \psi_{-}^{\nu} - \eta E_{\nu \mu} \psi_{+}^{\nu} = 0, \\
E_{\mu \nu} \partial_\pm X^{\nu} = - E_{\nu \mu} \partial_\pm X^{\nu} + i\eta \psi_{-}^{\nu} \psi_{+}^{\nu} E_{\rho \nu \mu} + i \psi_{+}^{\nu} \psi_{+}^{\nu} E_{\mu \nu, \rho} - i \psi_{-}^{\nu} \psi_{+}^{\nu} E_{\nu \mu, \rho} = 0,
\end{cases}
\]

where $E_{\mu \nu} \equiv g_{\mu \nu} + B_{\mu \nu}$. The bosonic conditions can be rewritten as

\[
\begin{align*}
E_{\mu \nu} \partial_\pm X^{\nu} - E_{\nu \mu} \partial_\pm X^{\nu} + i \Gamma_{\nu \rho \mu} (\psi_{+}^{\rho} - \eta \psi_{-}^{\rho}) (\psi_{+}^{\nu} + \eta \psi_{-}^{\nu}) + \\
+ i \eta \psi_{-}^{\rho} \psi_{+}^{\nu} B_{\rho \nu \mu} + i (\psi_{+}^{\nu} \psi_{+}^{\nu} + \psi_{-}^{\nu} \psi_{+}^{\nu}) B_{\mu \nu, \rho} = 0.
\end{align*}
\]

If we consider the case when the $B$-field is covariantly constant (i.e., $\nabla_\rho B_{\mu \nu} = 0$ with $\nabla_\rho$ being the Levi-Civita connection), then in normal coordinates the $\Gamma$-part and derivatives of $B_{\mu \nu}$ vanish. Thus in this case there is always a coordinate system where the boundary conditions take the simple form

\[
\begin{cases}
E_{\mu \nu} \psi_{-}^{\nu} - \eta E_{\nu \mu} \psi_{+}^{\nu} = 0, \\
E_{\mu \nu} \partial_{\pm} X^{\nu} - E_{\nu \mu} \partial_{\pm} X^{\nu} = 0.
\end{cases}
\]

If the $B$-form is closed ($dB = 0$), then locally it can be written as an exact form $B_{\mu \nu} = \partial_{[\mu} A_{\nu]}$, and (C.2) becomes

\[
\begin{align*}
E_{\mu \nu} \partial_{\pm} X^{\nu} - E_{\nu \mu} \partial_{\pm} X^{\nu} + i \Gamma_{\nu \rho \mu} (\psi_{+}^{\rho} - \eta \psi_{-}^{\rho}) (\psi_{+}^{\nu} + \eta \psi_{-}^{\nu}) + \\
+ i (\psi_{+}^{\nu} + \eta \psi_{-}^{\nu}) (\psi_{+}^{\rho} + \eta \psi_{-}^{\rho}) \partial_{\rho} \partial_{\mu} A_{\nu} = 0.
\end{align*}
\]

In this case there is always a coordinate system (Darboux-like coordinates) where we can get rid of the derivatives of $B$ (i.e., where the last term in (C.4) vanishes).
D Submanifolds of Riemannian manifolds

In this appendix we summarise the relevant mathematical details on submanifolds of Riemannian manifolds. In our use of terminology we closely follow [10].

We first give the definition of a distribution on a manifold (or neighbourhood) \( M \). A distribution \( \pi \) of dimension \((p + 1)\) on \( M \) is an assignment to each point \( X \in M \) of a \((p + 1)\)-dimensional subspace \( \pi_X \) of the tangent space \( T_X(M) \). The assignment can be done in different ways, for instance by means of an appropriate projection operator. A distribution \( \pi \) is called differentiable if every point \( X \) has a neighbourhood \( U \) and \((p + 1)\) differentiable vector fields, which form a basis of \( \pi_X \) at every \( Y \in U \). Furthermore, \( \pi \) is called involutive if for any two vector fields \( v_i, v_j \in \pi_X \) their Lie bracket \( \{v_i, v_j\} \in \pi_X \) for all \( X \in M \).

A connected submanifold \( D \) of \( M \) is called an integral manifold of \( \pi \) if \( f^*(T_X(D)) = \pi_X \) for all \( X \in D \), where \( f \) is the embedding of \( D \) into \( M \). If there is no other integral manifold of \( \pi \) which contains \( D \), then \( D \) is called a maximal integral manifold of \( \pi \).

**Frobenius theorem:** Let \( \pi \) be an involutive distribution on a manifold \( M \). Then through every point \( X \in M \), there passes a unique maximal integral manifold \( D(X) \) of \( \pi \). Any other integral manifold through \( X \) is an open submanifold of \( D(X) \).

If the manifold \( M \) is Riemannian, then various structures may be induced on the submanifold \( D \). For instance, \( D \) is automatically Riemannian. If one defines the Levi-Civita connection \( \nabla_v \equiv v^\mu \nabla_\mu \) on \( M \), and takes two vector fields \( v \) and \( w \) in the tangent space \( T(D) \) of \( D \), then the covariant derivative \( \nabla_v w \) can be decomposed as

\[
\nabla_v w = \hat{\nabla}_v w + B(v, w),
\]

where \( \hat{\nabla}_v w \) is the tangential component (i.e., it is in \( T(D) \)) and \( B(v, w) \) is the normal component. One can show that \( \hat{\nabla}_v \) can serve as the induced connection on the submanifold \( D \). \( B \) is called the second fundamental form of \( D \). Sometimes it is also useful to introduce the associated second fundamental form, \( A \), which is defined as follows. Taking \( u \) to be a normal vector field on \( D \) and \( v \) a tangent vector field on \( D \) we write

\[
\nabla_v u = -A_u v + D_v u \tag{D.2}
\]

where \( -A_u v \) and \( D_v u \) are, respectively, the tangential and the normal components of \( \nabla_v u \). Using the metric \( g \) on \( M \) one can prove the following simple identity,

\[
g(B(v, w), u) = g(A_u v, w). \tag{D.3}
\]

Eqs. (D.1) and (D.2) are called the Gauss formula and the Weingarten formula, respectively.
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