Bisimilarity of diagrams

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Abstract
In this paper, we investigate diagrams, namely functors from any small category to a fixed category, and more particularly, their bisimilarity. Initially defined using the theory of open maps of Joyal et al., we prove several equivalent characterizations: it is equivalent to the existence of a relation, similar to history-preserving bisimulations for event structures and it has a logical characterization similar to the Hennessy-Milner theorem. We then prove that we capture many different known bisimilarities, by considering the category of executions and extensions of executions, and by forming the functor that maps every execution to the information of interest for the problem at hand. We then look at the particular case of finitary diagrams with values in real or rational vector spaces. We prove that checking bisimilarity and satisfiability of a positive formula by a diagram are both decidable by reducing to a problem of existence of invertible matrices with linear conditions, which in turn reduces to the existential theory of the reals.

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1 Introduction

Our main motivation for the use of diagrams comes from our previous work on directed algebraic topology, namely the use of tools from algebraic topology in the study of geometric models of true concurrency. We invite interested readers to take a look at [4, 6, 8] for detailed exposition of this problem. We provide here a (rough) explanation of [4] to illustrate what motivates us in the present paper.

Given a geometric model of a truly concurrent system (e.g., PV-programs [3], Higher Dimensional Automata [13]), it is possible to produce a directed space, namely a topological space equipped with specified paths that we call directed. Intuitively, the points of this space are the states of the systems, and the directed paths correspond to executions (‘directed’ stands for the fact that executions must follow the direction of time, and that time is not reversible). The goal of directed algebraic topology is to extend tools from algebraic topology by adding considerations with respect to direction of time. Those tools are, in general, algebraic structures (groups, vector spaces, categories, ...) obtained from the geometry of a topological space. The main contribution of [4] was to nicely extend one of those tools, namely the homology, to the directed case as follows:

1) We first constructed the category of traces whose objects are directed paths modulo reparametrisations, and morphisms are ‘extensions’ of such traces. Intuitively, this category stands for the category of executions of the system.

2) We then form the functor that maps every trace to the homology of the trace space between its end points. Intuitively, this functor maps every execution to a vector space which describes the default of true concurrency encountered along this execution (by counting the number of holes of the trace space).
This new algebraic structure is then a functor from a category of executions (the category of traces) to a fixed category of values (the category of vector spaces). This is what we called a diagram.

Ultimately, the goal would be to decide whether two truly concurrent systems have the same behaviours. The main idea of directed algebraic topology is to declare that this corresponds to the fact that the induced directed spaces have the same geometry. The natural question was then what does that mean from a diagrammatic point of view: what kind of equivalences of diagrams captures the fact that systems have the same behaviours, the directed spaces have the same geometry. In [4], it was observed that isomorphism of diagrams is too strict of an equivalence: what matters is not the fact that two diagrams are precisely the same, but that the information contained in those have the same evolution with time. This idea was formalized using the general framework of bisimilarity from [10], using open maps, namely, morphisms that have lifting properties with respect to executions.

We then proved that given a simple truly concurrent system, typically a PV-program, it was possible to compute a finitary diagrams with values in a module of finite type (e.g., an Abelian group of finite type, or a finite dimensional vector space) bisimilar to the one defined previously. One open question remained: given two such finitary diagrams, how can we prove/decide that they are bisimilar or not?

**Contributions of the paper**

In Section 2 we start by recalling the definitions from [4] of a diagram and of bisimilarity of diagrams using open maps. We generalise them to diagrams with values in any category, not only Abelian groups. We also generalise the criterion for a map to be open from [4]. This criterion provides a convenient theoretical way to prove that diagrams are bisimilar, as, in many cases (much as the main theorem from [4]), constructions of diagrams come together with nice morphisms (typically, projections) that can be proved open. We finally prove a generalisation of the characterisation from [6] using bisimulation relations. This will turn out to be one the main ingredients to decide bisimilarity: to decide it, we will guess most of the information of a bisimulation and the remaining part will just be a problem of isomorphisms in the category of values, which becomes a problem of matrices for vector spaces. In Section 3 we introduce the diagrammatic path logic, similar to path logics in [10] and we prove that it completely characterises bisimilarity of diagrams. This will provide a simple criterion to prove that two diagrams are not bisimilar. Indeed, in many cases, two diagrams are not bisimilar because one contains a kind of evolution that the other does not. This evolution can be turned into a path formula that discriminates those two diagrams. In Section 4 we describe precise relationships between known bisimilarities and bisimilarity of diagrams. We first show how to encode strong path bisimilarity and path bisimilarity from [14] of any category with a small subcategory of paths as a special case of bisimilarity of diagrams following the same pattern: construct the category of executions, and form the functor that maps an execution to its interesting information (e.g., labels). We also show that this general pattern is not limited to the framework of [10], and that we can also capture several bisimilarities of Higher Dimensional Automata from [17], by choosing the right notion of ‘category of executions’ and the ‘interesting information’. In Section 5 we investigate the second main ingredient for deciding bisimilarity for finitary diagrams: the existential theory of invertible matrices. As the existence of a bisimulation between such diagrams can be reduced to the existence of some invertible matrices in reals or in rationals, that satisfy some linear conditions, we prove that this existence can be decided. This is done by reducing this problem to the existential theory of the reals which
is known to be decidable. Finally, in Section 7 we wrap up by providing an algorithm to decide whether two diagrams are bisimilar or not, by non-deterministically constructing a problem of the existential theory of invertible matrices. This provides an algorithm in NEXPSPACE, proving that this problem is in EXPSPACE. Following the same ideas, we provide an algorithm in NPSPACE deciding whether a diagram satisfies a positive path formula, proving that this problem is in PSPACE.

Related work
When dealing with abstract categorical formalisations of bisimilarity, it is necessary to compare with existing frameworks. Such comparison is possible (as we show in Section 4), but requires that the category of diagrams we are considering is with values in a small category (which implies that the path category is also small). We are going beyond this smallness with categories of modules, so it is impossible to fully compare this framework with other existing frameworks, like presheaves and coalgebras.

When the category of values is small, there is an adjunction between the category of diagrams and the category of presheaves over the category of values, which is a particular case of the study in Section 4. Another possible relation could be to remark that presheaves are particular diagrams with values in Set. Nevertheless, both notions of bisimilarities (the diagram’s and the presheave’s ones) accounts for very different kind of behaviours: bisimilarity of diagrams works on the domain category and use the values of functors to discriminate states; bisimilarity of presheaves works on the values of the functors (which are sets) to discriminate elements of those sets.

Another important categorical framework is coalgebras. A possible comparison can be provided by [11], where the author proved that, when the path category is small, open maps can be seen as particular morphisms of coalgebras for some endofunctor on graded sets. But, in this context, bisimilarity using open maps is stronger than bisimilarity using morphisms of coalgebras.

2 Bisimilarity and bisimulations of diagrams
Diagrams with values in a fixed category $\mathcal{A}$ are functors $F : \mathcal{C} \rightarrow \mathcal{A}$ from any small category to $\mathcal{A}$. If you think $\mathcal{A}$ as a category of “information” and $\mathcal{C}$ as the category of executions of a system, a diagram encodes the information along every execution (typically, a label), and its actions on morphisms of $\mathcal{C}$ encodes how this information evolves when the system evolves. It was used in [4] in the context of directed algebraic topology, in the case where $\mathcal{A}$ is a category of modules on a ring. Those modules intuitively count the number of local holes of a space (which model the default of true concurrency of the system), and evolution represents how holes appear and disappear with time. Spaces were then compared by comparing those diagrams up-to a notion of bisimilarity, which is a particular case of bisimilarity via open maps from [10]. In this section, we describe the original form of the bisimilarity from [4], defined as the existence of a span of particular morphisms of diagrams having some lifting properties. We then develop an equivalent characterization using relations, similar to bisimulations of event structures as introduced in [13].

2.1 Category of diagrams
As announced, bisimilarity will be defined using particular morphisms of diagrams. Such a morphism, say from the diagram $F : \mathcal{C} \rightarrow \mathcal{A}$ to the diagram $G : \mathcal{D} \rightarrow \mathcal{A}$ is a pair
\( (\Phi, \sigma) \) with \( \Phi : C \rightarrow D \) a functor and \( \sigma \) is a natural isomorphism from \( F \) to \( G \circ \Phi \). The composition \( (\Psi, \tau)\circ(\Phi, \sigma) \) is defined as \( (\Psi\circ\Phi, (\tau\circ\Phi)\circ\sigma) \). We denote this category by \( \text{Diag}(A) \).

**Example 1.** Throughout the next two sections, we will develop a particular example of diagrams in which transition systems can be encoded. This example will allow us to relate constructions in diagrams to classical constructions in concurrency theory. From now, we fix a set \( L \) called **alphabet**. Such a set induces a poset (which can be seen as a category) \( A_L \) whose elements are the words on \( L \) and whose order is the prefix order. A transition system \( T \) on \( L \) produces a diagram \( F_T : C_T \rightarrow A_L \) as follow. The category \( C_T \) is formed by considering as objects the runs of \( T \), that is, sequences \( i \xrightarrow{a_1} q_1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} q_n \) of transitions of \( T \) where \( i \) is the initial state, and by ordering them by prefix. \( F_T \) then maps a run to its sequence of labels. This construction extends to a functor \( \Pi \) from the category \( \text{TS}(L) \) of transition systems on \( L \) to the category \( \text{Diag}(A_L) \). Conversely, a diagram \( F : C \rightarrow A_L \) produces a transition system \( T \) as follow. First, such a diagram can be identified with a diagram with values in \( \text{TS}(L) \) by identifying a word \( a_1, a_2, \ldots, a_n \) with the finite linear transition system \( 0 \xrightarrow{a_1} 1 \xrightarrow{a_2} \ldots \xrightarrow{a_n} n \). \( T \) is then obtained by forming the colimit of this diagram in \( \text{TS}(L) \). This extends to a functor \( \Gamma \) from \( \text{Diag}(A_L) \) to \( \text{TS}(L) \). Note that \( \Gamma \circ \Pi \) is the unfolding of transition systems and that \( \Gamma \) is the left adjoint of \( \Pi \).

The reason why we need natural isomorphisms in the definition of a morphism of diagram is not clear yet, as the only isomorphisms in the category \( A \) are those conditions mean that the restriction of \( B' \) to \( n \) is \( B \) and that the morphism \( (\Pi, \theta) \) is the inclusion of \( B \) in \( B' \).

### 2.2 Open morphisms of diagrams

The original idea from [10] was to compare diagrams similarly to transition systems using the theory of [10]. Let us call **branch** a diagram from \( n \) to \( A \) for \( n \in \mathbb{N} \), where \( n \) is the poset (seen as a category) \( \{1, \ldots, n\} \) with the usual ordering. An **evolution** of a diagram \( F : C \rightarrow A \) is then a morphism from any branch to \( F \). Much as transition systems and executions, a morphism of diagrams \( (\Phi, \sigma) \) from \( F : C \rightarrow A \) to \( G : D \rightarrow A \) maps evolutions of \( F \) to evolutions of \( G \): if \((\Psi, \tau)\) is an evolution of \( F \), i.e., a morphism from a branch to \( F \), then \( (\Phi, \sigma)\circ(\Psi, \tau) \) is an evolution of \( G \). Then morphisms act as particular simulations of diagrams. The idea from [10] was to provide conditions on morphisms for them to act as particular bisimulations. The general idea is that a morphism induces a bisimulation if it lifts evolutions of \( G \) to evolutions of \( F \). In the context of diagrams, this will be defined using **extensions of branches**. An extension of a branch \( B : n \rightarrow A \) is a morphism of diagrams \((\Pi, \theta)\) from \( B : n \rightarrow A \) to a branch \( B' : n' \rightarrow A \), with \( n' \geq n \) such that:

- for every \( i \leq n \), \( B(i) = B'(i) \),
- for every \( i \leq j \leq n \), the morphism \( B'(i \leq j) \) of \( A \) is equal to \( B(i \leq j) \),
- for every \( i \leq n \), \( \Pi(i) = i \),
- for every \( i \leq n \), \( \theta_i = \text{id}_{B(i)} \).

\[
\begin{array}{c}
B(1) \\
\downarrow^\text{id} \\
B'(1)
\end{array}
\xrightarrow{B(1 \leq 2)} \cdots \xrightarrow{B(n-1 \leq n)} B(n)
\xrightarrow{\Pi(i)}
\begin{array}{c}
B'(1) \\
\downarrow^\text{id}
\end{array}
\xrightarrow{B'(1 \leq 2)} \cdots \xrightarrow{B'(n-1 \leq n)} B'(n)
\xrightarrow{B'(n \leq n+1)} B'(n')
\]

Those conditions mean that the restriction of \( B' \) to \( n \) is \( B \) and that the morphism \((\Pi, \theta)\) is the inclusion of \( B \) in \( B' \).
Branches and extensions form a path category in the sense of [10], and we then say that a morphism \((\Phi, \sigma)\) from \(F : C \to A\) to \(G : D \to A\) is open if for every diagram of the form (in plain):

\[
\begin{array}{c}
\text{\(B\)} \\
\text{\(\Phi, \tau\)}
\end{array}
\begin{array}{c}
\text{\(F\)} \\
\text{\(\sigma, \rho\)}
\end{array}
\begin{array}{c}
\text{\(B'\)} \\
\text{\(\Psi, \tau'\)}
\end{array}
\]

where \((\Pi, \theta)\) is an extension of branches, there is an evolution of \(F\) (in dots) which makes the two triangles commute. This means that if we can extend an evolution of \(F\), mapped on an evolution of \(G\) by \((\Phi, \sigma)\), as a longer evolution of \(G\), then we can extend it as a longer evolution of \(F\) that is mapped to this longer evolution of \(G\). This means in particular that \(F\) and \(G\) have exactly the same evolutions. As observed in [4], the definition of an open map can be simplified as follows:

\begin{itemize}
  \item \textbf{Theorem 1.} A morphism \((\Phi, \sigma)\) is open if and only if:
    \begin{itemize}
      \item \(\Phi\) is surjective on objects, i.e., for every object \(d\) of \(D\), there is an object \(c\) of \(C\) such that \(\Phi(c) = d\),
      \item \(\Phi\) is a fibration, i.e., for every morphism of \(D\) of the form \(j : \Phi(c) \to d'\), there is a morphism \(i : c \to c'\) of \(C\) such that \(\Phi(i) = j\).
    \end{itemize}

Following [10], we say that two diagrams \(F : C \to A\) and \(G : D \to A\) are \textbf{bisimilar} if there is a span of open morphisms between them, that is, a diagram \(H : E \to A\) and two open morphisms, one from \(H\) to \(F\), one from \(H\) to \(G\).

\begin{itemize}
  \item \textbf{Example 2.} In the case of diagrams in \(\mathcal{A}_L\), the notion of open morphisms is related to the notion of open morphisms of transition systems as defined in [10]. First, an open morphism \(f : T \to S\) between transition systems always induces an open morphism \(\Pi(f) : \Pi(T) \to \Pi(S)\) between the associated diagrams. In particular, if two transition systems are bisimilar then their diagrams are bisimilar. The converse also holds but proving it using open morphisms is hard (the reason will be explained later). For example, we may expect that an open morphism between diagrams of the form \(\Phi : F \to \Pi(T)\) induces an open morphism between transition systems \(\Gamma(\Phi) : \Gamma(F) \to \Gamma \circ \Pi(T)\), but that is not true in general.
\end{itemize}

\section*{2.3 Bisimulations of diagrams}

In the general context of [10], two notions of bisimulations, the \textbf{path bisimulations} and the \textbf{strong path bisimulations}, were defined as relations between evolutions. However, they assume that the path category (in the case of diagrams, the category of branches and extensions) is small to avoid problems of foundation, which is not the case of our path category for general categories \(\mathcal{A}\) of values (much as categories of modules as we will use next). Also, even if (strong) path bisimulations and bisimilarity using open morphisms are related, they are not equivalent in general [5]. In this section, we propose a notion of bisimulations as relations in the case of diagrams, which is equivalent to the existence of a span of open morphisms.

A \textbf{bisimulation} \(R\) between two diagrams \(F : C \to A\) and \(G : D \to A\) is a set of triples \((c, f, d)\) where \(c\) is an object of \(C\), \(d\) is an object of \(D\) and \(f : F(c) \to G(d)\) is an isomorphism of \(\mathcal{A}\) such that:

- for every \((c, f, d)\) in \(R\) and \(i : c \to c' \in C\), there exist \(j : d \to d' \in D\) and \(g : F(c') \to G(d') \in A\) such that \(g \circ F(i) = G(j) \circ f\) and \((c', g, d') \in R\),

\begin{itemize}
  \item for every \((c, f, d)\) in \(R\) and \(i : c \to c' \in C\), there exist \(j : d \to d' \in D\) and \(g : F(c') \to G(d') \in A\) such that \(g \circ F(i) = G(j) \circ f\) and \((c', g, d') \in R\),
\end{itemize}
Bisimilarity of diagrams

\[ c \xrightarrow{F(c)} f \xrightarrow{G(d)} d \]
\[ c' \xrightarrow{F(i)} \xrightarrow{G(j)} j \]
\[ c' \xrightarrow{G(d')} \xrightarrow{d'} \]

- symmetrically, for every \((c, f, d)\) in \(R\) and \(j : d \rightarrow d' \in \mathcal{D}\), there exist \(i : c \rightarrow c' \in \mathcal{C}\), \(g : F(c') \rightarrow G(d') \in \mathcal{A}\) such that \(g \circ F(i) = G(j) \circ f\) and \((c', g, d') \in R\).
- for all \(c \in \mathcal{C}\), there exists \(d\) and \(f\) such that \((c, f, d) \in R\).
- for all \(d \in \mathcal{D}\), there exists \(c\) and \(f\) such that \((c, f, d) \in R\).

**Theorem 2.** Two diagrams are bisimilar if and only if there is a bisimulation between them.

**Example 3.** In the case of diagrams in \(A_L\), a bisimulation between diagrams \(\Pi(T)\) and \(\Pi(S)\) is just a rephrasing for a path bisimulation in the sense of [10] between the transition systems \(T\) and \(S\). In the particular case of transition systems, the existence of a path bisimulation is equivalent to the existence of a strong path bisimulation and is equivalent to the existence of a bisimulation. Consequently:

**Proposition 1.** Two transition systems \(T\) and \(S\) are bisimilar if and only if the diagrams \(\Pi(T)\) and \(\Pi(S)\) are bisimilar.

### 3 diagrammatic path logic

In this section, we focus on a logical characterization of bisimilarity of diagrams. The logic used, that we call **diagrammatic path logic**, is similar to the logic introduced in [9] for transition systems, or to path logics developed in [10]. A formula in this logic allows to express that a diagram has some kind of evolutions or not.

The formulae used are generated by the following grammar:

**Object formulae:** 
\[ S ::= [x]P \quad x \in \text{Ob}(A) \]

**Morphism formulae:** 
\[ P ::= \langle f \rangle P | ?S | \neg P | \bigwedge_{i \in I} P_i \quad f \in \text{Mor}(A) \text{ and } I \text{ a set} \]

where \(\text{Ob}(A)\) is the class of objects of \(A\) and \(\text{Mor}(A)\) is its class of morphisms.

Intuitively, the object formula \([x]P\) means that the current object is isomorphic to \(x\), and the morphism formula \(\langle f \rangle P\) means that from the current object, one can fire a transition labelled by a morphism equivalent (in the sense of matrices, or conjugate in the language of group theory) to \(f\). Observe that we have arbitrary conjunctions, in particular infinite and empty (we will denote the empty conjunction by \(\top\)).

**Example 4.** In the case of diagrams in \(A_L\), \([w]P\) means that the current run is labeled by the word \(w\) and \(\langle w \leq w' \rangle P\) means that the current run is labeled by \(w\) and that it can be extended to a run labeled by \(w'\). The idea is very similar to Hennessy-Milner logic [9] and forward path logic [10]. The next theorem proves that, for two transition systems, satisfying the same Hennessy-Milner formulae, forward path formulae or path formulae is the same as their diagrams satisfying the same diagrammatic formulae.

For a diagram \(F : \mathcal{C} \rightarrow \mathcal{A}\), an object \(c\) of \(\mathcal{C}\), and an isomorphism \(f\) of \(\mathcal{A}\) of the form \(f : F(d) \rightarrow x\) for some \(d\) and \(x\), we define \(F, c \models S\) for an object formula \(S\) and \(F, f, d \models P\).
for a morphism formula $P$ by induction on $S$ (resp. $P$) as follows:
- $F, f, c \models \top$ always,
- more generally, $F, f, c \models \bigwedge_{i \in I} P_i$ iff for all $i \in I$, $F, f, c \models P_i$,
- $F, c \models [x]P$ iff there exists an isomorphism $f : F(c) \longrightarrow x$ of $A$ such that $F, f, c \models P$,
- for every $g : x \longrightarrow x'$, $F, f, c \models (g)P$ iff there exists $i : c \longrightarrow c'$ in $C$ and an isomorphism $h : F(c') \longrightarrow x'$ such that $h \circ F(i) = g \circ f$ and $F, h, c' \models P$,
- $F, f, c \models S$ iff $F, c \models S$,
- $F, f, c \models \neg P$ iff $F, f, c \not\models P$.

We say that a diagram $F : C \longrightarrow A$ is **logically simulated** by another diagram $G : D \longrightarrow A$ if for every object $c$ of $C$, there exists an object $d$ of $D$ such that for all object formula $S$, $F, c \models S$ iff $G, d \models S$. Two diagrams $F$ and $G$ are **logically equivalent** if $F$ is logically simulated by $G$ and vice-versa.

**Theorem 3.** Two diagrams are bisimilar iff they are logically equivalent.

### 4 Relation to other bisimilarities

#### 4.1 Relation to path and strong path bisimulations

We have seen in the previous two sections that path bisimilarities of transition systems can be seen as bisimulations of diagrams. Actually, this is general to any category of models $M$ with a specified small path category $P$ as in [10]. Given an object $X$ of $M$, it is possible to construct a diagram $F_X : C_X \longrightarrow P$ with values in $P$ as follow:
- $C_X$ is the slice category $P \downarrow X$,
- $F_X$ maps a morphism $f : P \longrightarrow X$ to $P$.

Much as in the case of transition systems, a bisimulation between the diagrams $F_X$ and $F_Y$ is just a rephrasing for a path bisimulation between $X$ and $Y$. However, in general, path bisimilarity and strong path bisimilarity do not coincide, but it is also possible to see the latter using bisimilarity of diagrams as follow. First, for a category $C$, define the category $\overline{C}$ as the category whose objects are those of $C$ and whose morphisms are generated by:
- $f : X \longrightarrow Y$ morphisms of $C$,
- $\overline{f} : Y \longrightarrow X$ for every $f : X \longrightarrow Y$ morphism of $C$,

with the following relations:
- $f, g = g \circ f$ for $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ morphisms of $C$,
- $\overline{f}, \overline{g} = \overline{g} \circ \overline{f}$ for $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ morphisms of $C$.

This means that $\overline{C}$ is obtained by adding formal “rebuses” of morphisms of $C$ (but $\overline{f}$ is not the inverse of $f$). The functor $F_X$ can be extended to a functor $\overline{F_X} : C_X \longrightarrow \overline{P}$, and a strong path bisimulation between $X$ and $Y$ is the same as a bisimulation between the diagrams $\overline{F_X}$ and $\overline{F_Y}$. Finally, bisimilarity using open morphisms, called $P$-bisimilarity in [10], is not equivalent to either of the other two bisimilarities in general. However, in many cases (see [5] for example), including presheaves models, $P$-bisimilarity coincides with strong path bisimilarity.
4.2 Relation to several bisimilarities of Higher Dimensional Automata

We have seen in the previous section that (strong) path bisimulations can be seen as bisimulations of diagrams. This idea can be extended to other kinds of bisimulations defined as relations on evolutions, instead of states. A typical example is bisimulations of Higher Dimensional Automata (HDA for short) [17]. In this case, the evolutions cannot be well expressed using a subcategory of paths and it is a challenge to describe classical bisimulations of HDA from [17] using open morphisms (see for example, [7]). One reason is that the natural notion of path is outside of the category of HDA, and that the corresponding natural notion of bisimilarity using open maps does not seem to coincide with any known bisimilarity. In the present section, we will see that those classical bisimulations are, in fact, bisimulations of diagrams, constructed in a similar way as in the previous section.

A **precubical set** is a sequence \((Q_n)_{n \in \mathbb{N}}\) of sets together with functions:
\[
\partial^\alpha_i : Q_n \to Q_{n-1}
\]
for \(n \in \mathbb{N}, 1 \leq i \leq n\) and \(\alpha \in \{0, 1\}\) satisfying for every \(1 \leq i < j \leq n\) and \(\alpha, \beta \in \{0, 1\}\):
\[
\partial^\alpha_i \circ \partial^\beta_j = \partial^{\beta}_{j-1} \circ \partial^\alpha_i.
\]

A **Higher Dimensional Automata** is a tuple \((Q, \partial, i, \lambda)\) with:
- \((Q, \partial)\) a precubical set,
- \(i \in Q_0\) (initial state),
- \(\lambda : Q_1 \to L\) (labelling), such that for every \(c \in Q_2\) and \(i \in \{1, 2\}\):
\[
\lambda(\partial^0_i(c)) = \lambda(\partial^1_i(c)).
\]

The objects of higher dimensions are just a way to express that some actions are independent and so that they can be made at the same time. The objects playing the role of the evolutions of an HDA are called **paths**. They are sequences \((t_1, c_1), \ldots, (t_n, c_n)\) where:
- \(c_k \in Q\),
- \(t_k\) is of the form \(\partial^\alpha_k\),
- if \(\alpha_k = 0\), then \(t_k(c_{k-1}) = c_k\), with \(t_k(c_k) = c_{k-1}\) otherwise.

The main difference with the evolutions of a transition system is that in paths, it is possible to start a transition (modeled as an \(\alpha_k\) being 0), and not to finish it right away (modeled as an \(\alpha_k\) being 1). Their are two different objects that play the role of the label of a path [17]:
- the **split trace**, which is a word in \(L \times \{+, -\}\), which represents the succession of starting and ending of actions along the path,
- the **ST-trace**, which is a word in \(L \times \{+, n \in \mathbb{N}\}\), which also represents the succession of starting and ending, except that we keep the information of which action is ended (which is important for auto-concurrency).

In this context, there are two important transformations of paths to consider:
- **extensions**, that is, extending a path to a longer path. The action on the traces is just the concatenation of letters.
- **homotopies**, that is, permuting independent actions. The action on the traces is precisely to permute two consecutive letters.

From this, it is possible to construct some diagrams from an HDA \(X\). For example, defining \(C^{hhp}_X\) as the category of paths of \(X\) and whose morphisms are generated by the homotopies, the extensions and the reverses of extensions, \(A^{hhp}_L\) as the category whose objects are the words on \(L \times \{+, n \in \mathbb{N}\}\) and whose morphisms are generated by permutations, concatenations by a letter, and deconcatenation of a letter, and \(F^{hhp}_X : C^{hhp}_X \to A^{hhp}_L\) as
the functor that maps a path to its ST-trace, a hereditary history-preserving bisimulation as defined in [17] between two HDA $X$ and $Y$ is precisely a bisimulation between the diagrams $F_X^{hbp}$ and $F_Y^{hbp}$. Similarly, it is possible to describe the three other bisimulations defined in [17], namely the split bisimulations, the ST-bisimulations and the history-preserving bisimulations, as bisimulations of diagrams constructed in a similar way (by removing the reverses, the homotopies or by considering the split trace instead).

5 Interlude

The second part of the paper is dedicated to decidability. We will focus on the following two problems:

- **bisimilarity**: given two diagrams, are they bisimilar?
- **diagrammatic model-checking**: given a diagram $F$, an object $c$ of its domain and a state formula $S$, does $F, c \models S$ hold?

The general idea of those problems is that the difficulty lies in the possibility to decide that systems are isomorphic in the category $\mathcal{A}$. For example, it would be easy to decide those problems in the category of finite sets, while it would be undecidable in the category of finite presentations of groups and group morphisms because it is undecidable whether two presentations present isomorphic groups. In this paper, we will focus on the category of finite dimensional real or rational vector spaces and matrices. This is the case needed in [4] to decide whether two simple truly concurrent systems have equivalent directed homology. This is also a nice example of what we meant by “bisimilarity becomes a problem of isomorphisms in $\mathcal{A}$”, since this problem reduces to a problem of invertible matrices.

More precisely, we will stick to finitary diagrams and finitary positive formulae defined as follow. By a **finitary diagram** $F$, we mean the following data:

- a finite poset $(\mathcal{C}, \leq)$, the **domain**,
- for every element $c$ of $\mathcal{C}$, a natural number $F(c)$ (which stands for the real vector space $\mathbb{R}^{F(c)}$),
- for every pair $c \leq c'$ of $\mathcal{C}$, a matrix $F(c \leq c')$ of size $F(c) \times F(c')$, with coefficients in rationals, presented as the list of all its elements, such that:
  - $F(c \leq c)$ is the identity matrix,
  - for every triple $c \leq c' \leq c''$, $F(c \leq c'') = F(c' \leq c'').F(c \leq c')$, where \( \cdot \) denotes the matrix multiplication.

In short, a finitary diagram is a functor from a finite poset to the category of matrices in rationals. One may argue that those assumptions are not reasonable, because they are not satisfied by the diagrams from Section 4 as soon as there is a loop. The reason is that when deciding this bisimilarity, there are two problems: finding out how to relate the executions and constructing the bisimulation, in particular, the isomorphism part. Loops make the first part difficult, because this relation is necessary infinite in this case. In this paper, we want to focus on the second problem because: 1) describing the fact that the existence of a bisimulation is deeply related to problems of isomorphisms in the category of values is interesting, 2) this case is what we need to address the open question from [4].

We call **finitary formulae**, the formulae generated by the following grammar:

**Object formulae:** $S ::= [n]P \quad n \in \mathbb{N}$

**Morphism formulae:** $P ::= \langle M \rangle P \mid ?S \mid \neg P \mid \top \mid P_1 \land P_2 \quad M$ matrix in rationals
Here, $\left[ n \right] P$ stands for $\left[ \mathbb{R}^n \right] P$ which makes finitary formulae diagrammatic formulae in real vector spaces. This time, since we only have finitely branching diagrams, we only consider finite conjunctions. We will more particularly consider positive formulae, i.e., formulae without any occurrences of the negation. For example, a formula of the form $\langle M_1 \rangle \ldots \langle M_k \rangle \top$ means that there is a sequence of matrices $N_1, \ldots, N_k$ in the diagrams where $N_i$ is equivalent to $M_i$, and those equivalences are natural (in the categorical meaning).

In this case, bisimilarity and model checking problems become a problem of existence of invertible matrices satisfying some linear conditions, as we will see in Section 7. In Section 6, we will start by proving that this problem of matrices can be encoded in the existential theory of the reals, which is known to be decidable.

### 6 Existential theory of invertible matrices

In the present section, we focus on an existential theory of matrices. We first recall the case of the existential theory of the reals, which is known to be decidable. We then introduce the existential theory of invertible matrices in $\mathbb{R}$ and $\mathbb{Q}$ and we finally prove the decidability of their satisfiability problems.

#### 6.1 The existential theory of some rings

Designing algorithms for finding solutions of equations is an old problem in mathematics. The famous Hilbert’s tenth problem posed the problem for polynomial equations in integers, but the question can be asked for other rings. Tarski in [16] solved this question for the reals: the first-order logic of real closed fields is decidable, although the solution being of non-elementary complexity. Several improvements have been made: it was proved to be in EXPSPACE in [11] and that the existential theory of the reals is in PSPACE in [2]. On the contrary, Matiyasevich’s negative answer of the tenth problem [12], means that the existential theory of the integers is undecidable. In particular, since it is possible to express that a rational is an integer (using possibly universal quantifiers), the full first-order logic of the rationals is undecidable. However, it is still an open question whether its existential fragment is decidable or not.

#### 6.2 Theory of matrices

In this section, we will consider a logic of matrices that will be expressible in the existential theory of the reals. It will be the main ingredient to decide some problems in diagrams with values in vector spaces. Namely, we consider formulae of the form:

$$\exists_{n_1}X_1 \ldots \exists_{n_k}X_k, \bigwedge_{j=1}^m P_j(X_1, \ldots, X_k)$$

where:
- $n_i \geq 0$, is a natural number,
- $X_i$ is a variable ranging over invertible matrices of dimension $n_i$,
- $P_j$ is a predicate of the form $A.X_i = X_k.B$ for some $i, k$ and matrices $A, B$ with coefficients in rationals, $A$ and $B$ are of size $n_k \times n_i$, and $\cdot$ denotes the matrix multiplication.

We call it the existential theory of invertible matrices.

We will consider the following decision problem: given such a formula, is it satisfiable, that is, are there matrices $M_1, \ldots, M_k$, with $M_i$ of size $n_i \times n_i$, invertible such that for every $j$, $P_j(M_1, \ldots, M_k)$ is true?
We may ask this question for matrices $M_i$ in reals or rationals. We will prove that both problems actually coincide and are decidable in PSPACE.

6.3 Decidability in $\mathbb{R}$

We stick here to the case of reals. We prove that we have a reduction to the existential theory of the reals. Given a formula

$$\Phi = \exists n_1 X_1. \ldots . \exists n_k X_k . \bigwedge_{j=1}^{m} P_j(X_1, \ldots , X_k)$$

we will construct a formula $\Psi$ in the existential theory of the reals which is satisfiable if and only if $\Phi$ is.

First, for every variable $X_i$, check if it appears in some $P_j$. If not, forget it. Indeed, if it does not appear in any predicate, then we can just choose an identity. Then, for every other quantifier $\exists n_i X_i$, we fix $2n_i^2$ fresh first-order variables $x_{r,s}^{i,r}$ and $y_{r,s}^{i,r}$ for $r, s \in \{1, \ldots , n_i\}$. Let $X_i$ be the matrix of size $n_i \times n_i$ whose coefficients are $x_{r,s}^{i,r}$, and $Y_i$ whose coefficients are $y_{r,s}^{i,r}$. Developing $A.X_i = X_j.B$ leads to $n_j n_i$ linear equations on the variables $x_{r,s}^{i,r}$ and $x_{r,s}^{j,r}$. So every predicate $P_j$ induces a set $L_j$ of linear equations. It remains to express that $X_i$ is invertible in the first-order logic. The idea is to express that $Y_i$ is its inverse. Developing $X_i Y_i = \text{Id}$ and $Y_i X_i = \text{Id}$, leads to $2n_i^2$ polynomial equations on the variables $x_{r,s}^{i,r}$ and $y_{r,s}^{i,r}$. Let $S_i$ be this set. We denote by $\Psi$ the formula:

$$\exists x_1^{1,1}. \ldots . \exists x_k^{n_k,n_k}. \exists y_1^{1,1}. \ldots . \exists y_k^{n_k,n_k}. \bigwedge_{i=1}^{k} S_i \land \bigwedge_{j=1}^{m} L_j$$

$\Psi$ is of polynomial size on the size of $\Phi$: indeed, the only problem might be that we fix $2n_i^2$ variables while $n_i$ is of size $\log(n_i)$, which may say that we fix an exponential number of variables. The point is that if we fixed those $2n_i^2$ variables, then it means that $X_i$ appears in some $P_j$, and that the matrices appearing in $P_j$ has a polynomial size in $n_i$. Consequently, we fix only a polynomial number of variables.

**Theorem 4.** $\Psi$ is satisfiable in the existential theory of the reals iff $\Phi$ is satisfiable in the existential theory of invertible matrices in reals. Consequently, the existential theory of invertible matrices in reals is decidable in PSPACE.

6.4 The rational case

As we have seen previously, first-order theories of rationals are harder in general. But there are some algebraic problems that are known to coincide when considering reals and rationals. Given a linear system with coefficients in rationals, gaussian elimination works independently of the coefficient field. Consequently, the real subspace $F_R$ of solutions of this system has the same dimension as the rational subspace $F_Q$ of solutions of the system. Actually, $F_R \cap \mathbb{Q}^n = F_Q$ and they have a common basis whose vectors are in rationals. Similarly, the problem of equivalence of matrices coincide in reals and rationals. Given two matrices $A$ and $B$ with coefficients in rationals, $A$ and $B$ are equivalent if there are two invertible matrices $X$ and $Y$ such that $A.X = Y.B$. This problem is also solvable using gaussian elimination by computing the rank of $A$ and $B$, which is independent of the coefficient field. Our problem is a generalization of the equivalence problem and it is not surprising that the same kind of results hold:
Theorem 5. A formula $\Phi$ is satisfiable in the existential theory of invertible matrices in reals if and only if it is satisfiable in rationals.

7 Decidability in diagrams

Finally, we prove a few decidability results for bisimilarity of diagrams and diagrammatic logic using the existential theory of invertible matrices. In this section, we consider diagrams with values in real vector spaces (or rational, but as we have seen in the previous section, both theories will coincide). We prove the decidability of the following two problems:

- **bisimilarity**: given two finitary diagrams, are they bisimilar?
- **diagram model-checking**: given a finitary diagram $F$, an object $c$ of its domain and a positive finitary state formula $S$, does $F, c \models S$ hold?

7.1 Decidability of bisimilarity

We start with the bisimilarity problem. Assume given two finitary diagrams $F$ and $G$, with domain $(C, \leq)$ and $(D, \preceq)$ respectively. The idea is to non-deterministically construct a bisimulation $R$, that is, a set of triples $(c, M, d)$ where $M$ is a matrix in reals (or rationals) satisfying the properties of a bisimulation from Section 2. The only exception is that we will not guess explicitly the matrices $M$, but a formula in the existential theory of invertible matrices that encodes the fact that there exist some matrices $M$ such that the bisimulation constructed satisfies those properties.

Consider the algorithm written in pseudo-code. It maintains the bisimulation $R$ and two sets $\text{var}$, encoding the variables of the formula we are constructing and $\text{lin}$, encoding its predicates.

The algorithm always terminates. First, the innermost while loop terminates since after every loop an element $(c, X, d)$ is marked and only elements of the form $(c', X', d')$ with either $c < c'$ and $d \preceq d'$ or $c \leq c'$ and $d \prec d'$ are added. The outer loop terminates since after every loop at least one element of $S$ is removed.

Assume that there is an execution of the algorithm that answers Yes. Let $R$ and $\Phi$ constructed during this execution. Since the algorithm answers Yes, the formula $\Phi$ is satisfiable, that is, for every $(X, n) \in \text{var}$, there is an invertible matrix $M_X$ of size $n \times n$ such that for every equation $A.X = X'.B$ in $\text{lin}$, $A.M_X = M_{X'}B$ holds. Let $R'$ be the set $\{(c, M_X, d) | (c, X, d) \in R\}$. Then by construction of $R$ and $\Phi$, $R'$ is a bisimulation between $F$ and $G$.

Assume that there is a bisimulation $R'$ between $F$ and $G$. We show that there are non-deterministic choices that lead to the answer Yes. The idea is to ensure that every $(c, X, d)$ that belongs to $R$ at some point corresponds to an element $(c, f, d)$ of $R'$. To ensure this, we must:

1. when choosing $d$ in line 7, choose it such that there is $(c, f, d) \in R'$. It exists by definition of a bisimulation.
2. when choosing $Q$ in line 17, choose it in such a way that for every $(c', d') \in Q$, there is $(c', f', d')$ in $R'$ and that the element $(c, f, d) \in R'$ corresponding to $(c, X, d)$ satisfies that $G(d \preceq d') \circ f = f' \circ F(c \leq c')$. Such a $Q$ always exists since $R'$ is a bisimulation.

With this, the algorithm does not FAIL and the formula $\Phi$ is valid: the assignment that map $X$ to the corresponding $f$ satisfies $\Phi$. Consequently, the algorithm answers Yes. Finally, this algorithm non-deterministically construct in exponential space a formula of exponential size in the size of the data. By Theorem 5, this algorithm is in NEXPSPACE. Consequently, by Savitch’s theorem [15], since NEXPSPACE = EXPSPACE:
Theorem 6. Knowing if two finitary diagrams are bisimilar in reals or in rationals is decidable in \( \text{EXPSPACE} \).

Example 5. Consider the two finitary diagrams at the end of this Section, \( F \) on the left, \( G \) on the right. Let us apply a few steps of the algorithm on those two diagrams:

1. Pick \( a \) and choose 0. At this point \( S = \{1, 2, b, c, d\} \), \( \text{var} = \{[X_1, 1]\} \) and \( R = \{[0, X_1, a]\} \) (we will only write the unmarked elements).

2. Pick \( (0, X_1, a) \) and choose \( Q = \{(1, c), (2, d), (0, b)\} \). At this point, \( S = \emptyset \), \( \text{var} = \{[X_1, 1]; (X_2, 2); (X_3, 1); (X_4, 1)\} \), \( R = \{(1, X_2, c); (2, X_3, d); (0, X_4, b)\} \) and \( \text{lin} = \{[\frac{3}{2}].X_1 = X_2, \frac{1}{2} \}; 6.X_1 = X_3; 2X_1 = X_4 \).
3. Pick $(2, X_3, d)$ and choose $Q = \varnothing$. At this point, $R = [(1, X_2, c); (0, X_4, b)]$.

4. Pick $(1, X_2, c)$ and choose $Q = \{(2, d)\}$. At this point, $var = [(X_1, 1); (X_2, 2); (X_3, 1); (X_4, 1); (X_5, 1)]$, $\bar{R} = [(0, X_4, b), (2, X_5, d)]$ and $lin = [(\frac{3}{2}), X_1 = X_2, (\frac{1}{2})]; 6.X_1 = X_3; 2X_1 = X_4; (\frac{4}{3}).X_2 = X_5; (\frac{1}{3})]$. The induced matrix problem or reals problem is satisfiable, which means that both diagrams are bisimilar.

\[
\begin{aligned}
\begin{array}{c|c}
\text{d} & \frac{3}{2} \\
\hline
1 & \frac{1}{2} \\
2 & 1 \\
0 & 0
\end{array}
\end{aligned}
\]

\[
\begin{aligned}
\begin{array}{c|c}
\text{c} & \frac{4}{3} \\
\hline
\text{a} & 2 \\
\text{b} & 1 \\
\text{1} & \frac{1}{3}
\end{array}
\end{aligned}
\]

5. ... At the end, the algorithm produces $var = [(X_1, 1); (X_2, 2); (X_3, 1); (X_4, 1); (X_5, 1); (X_6, 2); (X_7, 1); (X_8, 1)]$ and their $lin = [(\frac{3}{2}), X_1 = X_2, (\frac{1}{2})]; 6.X_1 = X_3; 2X_1 = X_4; (\frac{4}{3}).X_2 = X_5; (\frac{1}{3}); (\frac{7}{3}).X_4 = X_6; (\frac{1}{3}); 3.X_4 = X_7; (\frac{4}{3}).X_6 = X_8; (\frac{1}{3})]$. The induced matrix problem or reals problem is satisfiable, which means that both diagrams are bisimilar.

### 7.2 Decidability of the model checking

#### 7.2.1 Positive case

We start with the positive fragment. So starting with a finitary diagram $F$, an element $c$ of its domain, and a positive finitary object formula $S$, we inductively construct two lists, initially empty, as previously:

- $var$ of pairs $(X, n)$ where $X$ is a variable and $n$ an integer. This will stand for $\exists_n X$.
- $lin$ of equations $A.X = Y.B$ where $X$ and $Y$ are variables and $A$ and $B$ are matrices.

The formula $S$ is of the form $[n]P$. We first check if $n = F(c)$. If it is not the case then we fail. Otherwise, let $X$ be a fresh variable. Add the pair $(X, n)$ to $var$. Continue with $F$, $c$, $X$, and $P$.

Now, assume that we consider the following data: a finitary diagram $F$, an element of its domain $c$, an $X$ with $(X, n)$ in $var$ for some integer $n$ and a positive finitary morphism formula $P$. Several cases:

- if $P = \exists^n S'$, continue with $F$, $c$ and $S'$,
- if $P = \neg$, stop,
- if $P = P_1 \land P_2$, first continue with $F$, $c$, $X$ and $P_1$. When this part terminates, continue with $F$, $c$, $X$ and $P_2$,
- if $P = \langle M \rangle P'$, with $M$ of size $n_1 \times n_2$. If $n_1 \neq F(c)$, then we fail. Otherwise, non-deterministically choose an element $c' \geq c$, with $F(c') = n_2$. If such a $c'$ does not exist, then we fail. Finally, create a fresh variable $X'$, add $(X', n_2)$ to $var$ and $M.X = X'.F(c \leq c')$ to $lin$.

If the algorithm does not fail, construct a formula $\Phi$ from $var$ and $lin$ as previously and check if it is satisfiable using the existential theory of invertible matrices. The formula $\Phi$ is non-deterministically constructed in polynomial time and so is of polynomial size. So, this algorithm is in NPSPACE and again, by Savitch’s theorem [15], since NPSPACE = PSPACE:

> **Theorem 7.** Knowing if a finitary diagram satisfies a positive finitary formula (either in reals or in rationals) is decidable in PSPACE.
Example 8. Let us consider the following positive finitary formula \( \phi = [1][ \langle 1 0 \rangle][ \langle 1 1 \rangle] \top \). It is not hard to check that \( F, 0 \models \phi \), and so that \( G, a \models \phi \) (you can unroll the algorithm, the identities will give a solution of the problem of matrices). Let \( H \) be the following diagram:

\[
\begin{array}{c|c}
\downarrow & \downarrow \\
2 \ldots, 1 & (0 1) \\
\uparrow & \uparrow \\
1 \ldots, 2 & (\frac{1}{2}) \\
\downarrow & \downarrow \\
0 \ldots, 1 & \\
\end{array}
\]

We will show that \( H, 0 \not\models \phi \), and that \( H \) is not bisimilar to \( F \) and \( G \). Let us unroll the algorithm on \( H, 0 \), and \( \phi \). We are in the first case, and we create a fresh variable \( X_1 \) and \( \text{var} := [(X_1, 1)] \). We then continue the algorithm with \( H, 0, X_1 \) and \( (\frac{1}{2}) \langle (1 1) \rangle \top \). We are then in the last case, and we can only choose 1 without failing. So, \( \text{var} = [(X_1, 1); (X_2, 2)] \) and \( \text{lin} = [(\frac{1}{2}).X_1 = X_2.(\frac{1}{2})] \). We continue with \( H, 1, X_2 \) and \( (\langle 1 1 \rangle) \top \). We still are in the last case and we can only choose 2 without failing. So, \( \text{var} = [(X_1, 1); (X_2, 2); (X_3, 1)] \) and \( \text{lin} = [(\frac{1}{2}).X_1 = X_2.(\frac{1}{2}); (1 1).X_2 = X_3.(0 1)] \). Let us prove that we cannot solve this problem of invertible matrices. If we could, we would have that:

\[
X_1 = (1 1).(\frac{1}{2}).X_1 = (1 1).X_2.(\frac{1}{2}) = X_3.(0 1).(\frac{1}{2}) = 0
\]

which is impossible since \( X_1 \) must be invertible.

7.2.2 Full case

The full case is also decidable for the reals. The idea is similar, except that, because of the negation, it is not possible to encode our problem in the existential fragment. However, using the same ideas, it is still possible to encode it in the full first-order theory of real closed fields. There are two counter-parts:

- First, since the full first-order theory is decidable in \( \text{EXPSPACE} \), the full model-checking in reals is in \( \text{EXPSPACE} \),

- Secondly, theorem 6 does not hold anymore and nothing can be said about the rational case.

8 Conclusion and future work

We investigated bisimilarity of diagrams, more particularly of diagrams with values in real and rational vector spaces. While the same intuition as for bisimilarity of transition systems holds – bisimilarity has equivalent characterizations involving relations and logic – deciding bisimilarity involves arguments from the category in which diagrams take values, e.g., algebraic arguments when values are in vector spaces. We introduced a class of problems of matrices for which the answer in rationals coincides with the answer in reals and reduced it to the existential theory of the reals. This allows us to prove that bisimilarity is decidable in \( \text{EXPSPACE} \) and the satisfaction of a positive formula by a diagram is decidable in \( \text{PSPACE} \).

As a future work, we would like to investigate the case of diagrams with values in Abelian groups, i.e., matrices with values in integers, for which the existential theory is undecidable, but for which we can still decide some problems of matrices.
23:16 Bisimilarity of diagrams

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A Full proof of Theorem 2

⇒ Assume that there is a span:
Assume now that there is a bisimulation $\rho$. We define $R = \{(\Phi(e), \tau_e \circ \sigma_e^{-1}, \Psi(e)) \mid e \in \mathcal{E}\}$ and show that this is a bisimulation. First, it is well defined because $\tau$ and $\sigma$ are isomorphisms. The third condition of a bisimulation comes from the surjectivity of $\Phi$. Idem for the forth and the surjectivity of $\Psi$. The first condition comes from the fibrationality of $\Phi$: let $(\Phi(e), \tau_e \circ \sigma_e^{-1}, \Psi(e))$ in $R$ and $i : \Phi(e) \longrightarrow e'$ in $\text{Mor}(\mathcal{C})$. Since $\Phi$ is a fibration, there exists $k : e \longrightarrow e'$ in $\text{Ob}(\mathcal{E})$ such that $\Psi(k) = i$. Then define $j = \Psi(k)$, $d' = \Psi(e')$ and $g = \tau_e \circ \sigma_e^{-1}$. $(\Phi(e'), g, d')$ belongs to $R$ by construction and $g \circ F(i) = G(j) \circ \tau_e \circ \sigma_e^{-1}$ by naturality of $\sigma$ and $\tau$. Idem for the second condition of a bisimulation.

Assume now that there is a bisimulation $R$ between $F$ and $G$. We will construct a span of open maps. Let $\mathcal{E}$ be the small category whose objects are elements of $R$, and whose morphisms from $(c, f, d)$ to $(c', f', d')$ are pairs $(i, j)$ of a morphism $i : c \longrightarrow c'$ in $\mathcal{C}$ and of a morphism $j : d \longrightarrow d'$ in $\mathcal{D}$, such that the following diagram commutes:

$$
\begin{array}{ccc}
F(c) & \xrightarrow{f} & G(d) \\
F(i) \downarrow & & \downarrow G(j) \\
F(c') & \xleftarrow{f'} & G(d')
\end{array}
$$

Define the tip $H$ of the span between $F$ and $G$ as the functor $H : \mathcal{E} \longrightarrow \mathcal{A}$ that maps every object $(c, f, d) \in R$ to $F(c)$, and every morphism $(i, j) : (c, f, d) \longrightarrow (c', f', d')$ to $F(i) : F(c) \longrightarrow F(c')$.

We now build a morphism $(\Phi, \sigma)$ from $H$ to $F$. We start by building $\Phi : \mathcal{E} \longrightarrow \mathcal{C}$. We define $\Phi$ as the functor that maps every object $(c, f, d)$ to $c$ and every morphism $(i, j) : (c, f, d) \longrightarrow (c', f', d')$ to $i : c \longrightarrow c'$. We verify that $\Phi$ satisfies the condition of the previous theorem:

1. $\Phi$ is surjective on objects: this is the third condition of a bisimulation.

2. Let $i : \Phi(e) \longrightarrow c'$ be a morphism of $\mathcal{C}$. The object $e$ must be a triple $(c, f, d) \in R$, and $i$ is a morphism from $c$ to $c'$ in $\mathcal{C}$. By the first condition of a bisimulation, there is a triple $(c', f', d') \in R$ and a morphism $j : d \longrightarrow d'$ of $\mathcal{D}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
F(c) & \xrightarrow{f} & G(d) \\
F(i) \downarrow & & \downarrow G(j) \\
F(c') & \xleftarrow{f'} & G(d')
\end{array}
$$

In particular, $(i, j)$ is a morphism of $\mathcal{E}$, from $(c, f, d)$ to $(c', f', d')$. Moreover, $\Phi(i, j) = i$.

For every $(c, f, d) \in R$, let $\sigma_{(c, f, d)} = \text{id}_{F(c)} : H(c, f, d) = F(c) \longrightarrow F \circ \Phi(c, f, d) = F(c)$. Those are isomorphisms, and define a natural transformation $\sigma : H \longrightarrow F \circ \Phi$. It follows that $(\Phi, \sigma)$ is an open map from $H$ to $F$.

We define the open map $(\Psi, \tau)$ from $H$ to $G$ similarly.
B Full proof of Theorem 3

B.1 Open maps imply logical equivalence

Let us suppose that $F$ and $G$ are bisimilar. We can restrict to the case where there exists an open map $(\Phi, \sigma) : F \to G$, the general case ensuing, since logical equivalence is an equivalence relation.

We prove that:
1. $F, c \models S$ iff $G, \Phi(c) \models S$ for every object formula $S$ and for every object $c$ of $\mathcal{C}$,
2. $F, f, d \models P$ iff $G, f \circ \sigma_c^{-1}, \Phi(d) \models P$ for every morphism formula $P$ and for every isomorphism $f : F(d) \to x$ of $\mathcal{A}$,

by induction on $S$ (resp. $P$).

⋆ If $F, c \models [x]P$ then there exists an isomorphism $f : F(c) \to x$ of $\mathcal{A}$ such that $F, f, c \models P$.

By induction hypothesis, $G, f \circ \sigma_c^{-1}, \Phi(c) \models P$ and so $G, \Phi(c) \models [x]P$.

Conversely, if $G, \Phi(c) \models [x]P$ then there exists an isomorphism $f : G(\Phi(c)) \to x$ of $\mathcal{A}$ such that $G, f, \Phi(c) \models P$. By induction hypothesis, $F, f \circ \sigma_c, c \models P$ and so $F, c \models [x]P$.

⋆ If $F, f, c \models \langle g \rangle P$ then there exists $i : c \to c'$ in $\mathcal{C}$ and an isomorphism $h : F(c') \to x'$ such that $h \circ F(i) = g \circ f$ and $F, h, c' \models P$.

By induction hypothesis, $G, h \circ \sigma_{c'}^{-1}, \Phi(c') \models P$. By naturality of $\sigma$:

So $G, f \circ \sigma_c^{-1}, \Phi(c) \models \langle g \rangle P$.

Conversely, if $G, f \circ \sigma_c^{-1}, \Phi(c) \models \langle g \rangle P$ then there exists $j : \Phi(c) \to d'$ in $\mathcal{D}$ and an isomorphism $h : G(d') \to x'$ such that $h \circ G(j) = g \circ f \circ \sigma_c^{-1}$ and $G, h, d' \models P$.

Since $(\Phi, \sigma)$ is open, there exists $i : c \to c'$ in $\mathcal{C}$ such that $\Phi(i) = j$ and $\Phi(c') = d'$. So $G, h, \Phi(c') \models P$ and by induction hypothesis, $F, h \circ \sigma_{c'}, c' \models P$. Moreover, by naturality of $\sigma$:
We prove that $R\vdash (g)P$.

* $F, f, c \models \neg S$ iff $F, c \models S$ iff $G, \Phi(c) \models S$ iff $G, f \circ \sigma_c^{-1}, \Phi(c) \models \neg S$
* $F, f, c \models \neg P$ iff $F, f, c \not\models P$ iff $G, f \circ \sigma_c^{-1}, \Phi(c) \not\models P$ iff $G, f \circ \sigma_c^{-1}, \Phi(c) \models \neg P$
* $F, f, c \models \bigwedge_{i \in I} P_i$ iff for all $i \in I$, $F, f, c \models P_i$ iff for all $i \in I$, $G, f \circ \sigma_c^{-1}, \Phi(c) \models P_i$ iff $G, f \circ \sigma_c^{-1}, \Phi(c) \models \bigwedge_{i \in I} P_i$.

From this and the surjectivity of $\Phi$, we deduce the first part of the theorem.

### B.2 Bisimulation induced by logical equivalence

Suppose that $F$ and $G$ are logically equivalent. Define the relation:

$$R = \{(c, f, d) \mid \forall S. (F, c \models S \iff G, d \models S) \land f : F(c) \to G(d) \text{ iso } \land \exists h_1, h_2 \text{ isos. } f = h_2^{-1} \circ h_1 \land \forall P. (F, h_1, c \models P \iff G, h_2, d \models P)\}$$

We prove that $R$ is a bisimulation:

* Let $c$ be an object of $\mathcal{C}$. We exhibit an object $d$ of $\mathcal{D}$ and an iso $f : F(c) \to G(d)$ such that $(c, f, d) \in R$. Let $d$ such that for every object formula $S$, $F, c \models S \iff G, d \models S$ (there exists at least one such a $d$ by hypothesis). Let :

$$Z = \{h \mid h : G(d) \to F(c) \text{ iso}\}$$

$Z$ is non empty : $F, c \models [F(c)]\top$ because $id_{F(c)} : F(c) \to F(c)$ is an iso. So, $G, d \models [F(c)]\top$ and there exists an isomorphism $h : G(d) \to F(c)$.

Now, assume that there is no $h \in Z$ such that for all morphism formula $F, id_{F(c)}, c \models P$ iff $G, h, d \models P$. Then for all $h \in Z$, let $P_h$ be a formula such that $F, id_{F(c)}, c \models P_h$ and $G, h, d \not\models P_h$ (we can always assume that we are in this case because we have negation). Then $F, c \models [F(c)] \land_{h \in Z} P_h$ and $G, d \not\models [F(c)] \land_{h \in Z} P_h$ which is absurd. So there is an isomorphism $h : G(d) \to F(c)$ such that for every morphism formula $P$, $F, id_{F(c)}, c \models P$ iff $G, h, d \models P$. Then $(c, h^{-1}, d) \in R$.

* Assume that we have :

$$\begin{array}{ccc}
  c & F(c) & h_1 \downarrow \\
  \downarrow & \downarrow & \downarrow \\
  F(i) & x & G(d) \downarrow h_2 \\
  \end{array}$$

with $h_1, h_2$ isos and for every morphism formula $P$, $F, h_1, c \models P$ iff $G, h_2, d \models P$ (that is $(c, h_2^{-1} \circ h_1, d) \in R$). First, this diagram is commutative :
Bisimilarity of diagrams

There is then an isomorphism

\[ \text{Inv}^{n} \subset S \]

image of \( R \)

the constraints that some matrices are invertible. Since \( \Psi \) is satisfiable, the homogeneous satisfiability problem reduces to solving a homogeneous system

**Proof.**

It remains to prove that \( F, d \) is an open set of \( S \) if and only if \( F, d \) is a non-empty open set of \( R \). In particular, for every object formula \( S, F, id_{F(c')}, c' \models P \) if \( G, d' \models P \). In particular, for every object formula \( S, F, id_{F(c')}, c' \models P \) if \( G, d' \models P \), i.e., \( F, c' \models S \) if \( G, d' \models S \) and so \((c', h^{-1}, d') \in R \).

\( * \) the other two conditions are symmetric.

### C Full proof of Theorem 5

**Proof.** Let \( \Psi \) be a formula which is satisfiable in reals. It is enough to prove that the formula \( \Psi \) constructed in the previous subsection has a model in rationals. We have seen that the satisfiability problem reduces to solving a homogeneous system \( \bigwedge_{i} L_{j} \in \mathbb{R}^{n_{1} + \cdots + n_{k}} \), with the constraints that some matrices are invertible. Since \( \Psi \) is satisfiable, the homogeneous system \( \bigwedge_{i} L_{j} \) has a non-trivial subspace \( F \) of solution in reals. Let \( p \) be its dimension and \( t_{1}, \ldots, t_{p} \) its basis (which is in rationals, since the system is with coefficients in rationals).

There is then an isomorphism \( \kappa : F \to \mathbb{R}^{p} \). The set of solution of \( \bigwedge_{i} S_{i} \cap \bigwedge_{j} L_{j} \) in reals is a subset \( S \) of \( F \). It is enough to prove that \( S \) is a non-empty open set of \( F \) (with any topology coming from a norm). Indeed, in this case, the image \( \kappa(S) \) is then a non-empty open set of \( \mathbb{R}^{p} \). Since, \( Q^{p} \) is dense in \( \mathbb{R}^{p} \), \( \kappa(S) \) intersects \( Q^{p} \) and there is \((s_{1}, \ldots, s_{p}) \in \kappa(S) \cap Q^{p} \). Then, \( s_{1}, t_{1} + \ldots + s_{p}, t_{p} \) is a vector of rationals which is solution of \( \bigwedge_{i} S_{i} \cap \bigwedge_{j} L_{j} \).

It remains to prove that \( S \) is open in \( F \). So it is enough to prove that the set of solutions \( T_{i} \) of \( S_{i} \) is an open set of \( \mathbb{R}^{n_{1} + \cdots + n_{k}} \). \( T_{i} \) is of the form \( \mathbb{R}^{n_{1} + \cdots + n_{k}} \times \text{Inv}_{n_{i}} \times \mathbb{R}^{n_{i+1} + \cdots + n_{k}} \), where \( \text{Inv}_{n_{i}} \) is the set of invertible matrices in reals of size \( n_{i} \times n_{i} \). \( \text{Inv}_{n_{i}} \) is the inverse image of \( \mathbb{R} \setminus \{0\} \) by the determinant function, which is continuous. Consequently, \( \text{Inv}_{n_{i}} \) is open, and \( T_{i} \) is open.
An example of an encoding of a matrix formula to existential theory of reals

Example 6. The matrix formula \( \exists X_1. \exists X_2 . (a b). X_1 = X_2 . (c d) \) is encoded as:

\[
\begin{align*}
\exists x_1^{1,1}, x_1^{1,2}, x_1^{2,1}, x_1^{2,2}.
\exists x_2^{1,1}.
\exists y_1^{1,1}, y_1^{1,2}, y_1^{2,1}, y_1^{2,2}.
\exists y_2^{1,1}.
\end{align*}
\]

\[
\begin{align*}
& a.x_1^{1,1} + b.\overline{x}_1^{1,1} = c.x_2^{1,1} \\
& a.x_1^{1,2} + b.\overline{x}_1^{1,2} = d.x_2^{1,1} \\
& x_1^{1,1}.y_1^{1,1} + x_1^{1,2}.\overline{y}_1^{1,1} = 1 \\
& x_1^{1,1}.y_1^{1,2} + x_1^{1,2}.\overline{y}_1^{2,1} = 0 \\
& x_1^{2,1}.y_1^{1,1} + x_1^{2,2}.\overline{y}_1^{1,1} = 0 \\
& x_1^{2,1}.y_1^{1,2} + x_1^{2,2}.\overline{y}_1^{2,1} = 1 \\
& \overline{y}_1^{1,1}.x_1^{1,1} + \overline{y}_1^{1,2}.\overline{x}_1^{2,1} = 1 \\
& \overline{y}_1^{1,1}.x_1^{1,2} + \overline{y}_1^{1,2}.\overline{x}_1^{2,2} = 0 \\
& \overline{y}_1^{2,1}.x_1^{1,1} + \overline{y}_1^{2,2}.\overline{x}_1^{2,1} = 0 \\
& \overline{y}_1^{2,1}.x_1^{1,2} + \overline{y}_1^{2,2}.\overline{x}_1^{2,2} = 1 \\
& x_2^{1,1}.\overline{y}_2^{1,1} = 1 \\
& x_2^{1,1}.x_2^{1,1} = 0 \\
& x_2^{2,1}.\overline{y}_2^{1,1} = 0 \\
& x_2^{2,1}.x_2^{1,1} = 1 \\
\end{align*}
\]

(coefficients of \( X_1 \))

(coefficients of \( X_2 \))

(coefficients of \( X_1^{-1} \))

(coefficients of \( X_2^{-1} \))