Critical Anisotropies of a Geometrically-Frustrated Triangular-Lattice Antiferromagnet

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This work examines the critical anisotropy required for the local stability of the collinear ground states of a geometrically-frustrated triangular-lattice antiferromagnet (TLA). Using a Holstein-Primakoff expansion, we calculate the spin-wave frequencies for the 1, 2, 3, 4, and 8-sublattice (SL) ground states of a TLA with up to third neighbor interactions. Local stability requires that all spin-wave frequencies are real and positive. The 2, 4, and 8-SL phases break up into several regions where the critical anisotropy is a different function of the exchange parameters. We find that the critical anisotropy is a continuous function everywhere except across the 2-SL/3-SL and 3-SL/4-SL phase boundaries, where the 3-SL phase has the higher critical anisotropy.

Introduction. Geometrically-frustrated systems exhibit many novel characteristics including non-collinear ground states and multiferroic properties\textsuperscript{1}–\textsuperscript{5}. One of the best realizations of a geometrically-frustrated triangular-lattice antiferromagnet (TLA) is CuFeO\textsubscript{2}, which contains stacked hexagonal planes of spin-5/2 Fe\textsuperscript{3+} ions. Accompanied by a phase transition from a collinear 4-sublattice (SL) ground state to a non-collinear phase\textsuperscript{2,3,4,5}, CuFeO\textsubscript{2} exhibits multiferroic properties above a critical magnetic field or above a critical concentration of non-magnetic Al\textsuperscript{3+} impurities, which substitute for the Fe\textsuperscript{3+} ions\textsuperscript{6,7}. Inelastic neutron-scattering experiments\textsuperscript{8,9,10} on CuFeO\textsubscript{2} have reported a spin-wave (SW) gap of about 0.9 meV, which decreases with Al doping and may vanish\textsuperscript{11} upon the appearance of multiferroic behavior. Similar behavior is produced in a model TLA as the anisotropy is reduced\textsuperscript{12} and spin fluctuations about the 4-SL collinear phase become stronger. In this paper, we evaluate the critical anisotropies required for the local stability of the collinear magnetic phases in a model TLA with up to third nearest neighbors. As shown elsewhere\textsuperscript{12}, the wave-vector of the dominant SW instabilities of a collinear phase coincide with the dominant wave-vector of the non-collinear phase that appears with decreasing anisotropy. Therefore, an analysis of the critical anisotropies and wave-vectors of a frustrated TLA can provide useful information about the non-collinear phases that appear at small anisotropy.

The collinear ground states of a TLA with strong anisotropy were first obtained by Takagi and Makata\textsuperscript{14}, who examined an Ising model with interactions up to third nearest neighbors. The ground-state phase diagram consists of the five phases sketched in Fig. 1, where the energies of these five states are given in Table I. Using a Holstein-Primakoff (HP) expansion, we have calculated the SW frequencies and critical anisotropies for each of these phases.

The Hamiltonian for a TLA is given by

\[ H = -\frac{1}{2} \sum_{i \neq j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - D \sum_i S_{iz}^2, \]

where \( \mathbf{S}_i \) is the local moment on site \( i \), \( J_{ij} \) is the interaction between sites \( i \) and \( j \), and \( D \) is the single-ion anisotropy. Employing a HP transformation, the spin operators are given by \( S_{iz} = S - a_i^\dagger a_i, \ S_{i+} = \sqrt{2S} a_i \), and \( S_{i-} = \sqrt{2S} a_i^\dagger \). Expanded about the classical limit in powers of \( 1/\sqrt{S} \), the Hamiltonian can be written as \( H = E + H_1 + H_2 + \ldots \). The first-order term \( H_1 \) vanishes when the spin configuration minimizes the energy \( E \). The second-order term \( H_2 \) provides the dynamics of non-interacting SWs. Higher-order terms \( H_{n>2} \) reflect the interactions between SWs. They are unimportant

![FIG. 1: (Color online) The 1, 2, 3, 4, and 8-SL phases for the ground states of the geometrically-frustrated TLA. The solid black lines denote the magnetic unit cell of each phase. Up and down spins are designated by red and blue circles, respectively.](image-url)
at low temperature and for large 1/S. Similar to Takagi and Makata, we consider nearest neighbor \( J_1 \), next-nearest neighbor \( J_2 \), and next-next-nearest-neighbor \( J_3 \) exchange interactions, as sketched in Fig. 1.

To determine the SW frequencies \( \omega_k \), we solve the equation-of-motion for the vectors \( v_k = [a_k^{(1)}, a_k^{(2)}, a_k^{(3)}, \ldots] \), which may be written in terms of the \( 2N \times 2N \) matrix \( M(k) \) as \( i d v_k / dt = -[H_{2S}, v_k] = \hat{M}(k)v_k \), where \( N \) is the number of spin sites in the unit cell. The SW frequencies are then determined from the condition \( \text{Det}[\hat{M}(k) - \omega_k I] = 0 \).

Two conditions are required for the local stability of any magnetic phase: all SW frequencies must be real and positive and all SW weights must be positive. The SW weights \( W_k^{(s)} \) are coefficients of the spin-spin correlation function:

\[
S(k, \omega) = \frac{1}{N} \int dt \, e^{-i\omega t} \sum_{i,j} e^{ik_{ij}} \left\{ \langle S_i^+ S_j^- \rangle (t) \right\} + \langle S_i^- S_j^+ \rangle (t) \right\} = \sum_s \omega_k^{(s)} \delta(\omega - \omega_k^{(s)}),
\]

where \( s \) denotes a branch of the SW spectrum and \( d_{ij} \) is defined as the vector pointing from site \( i \) to site \( j \). The weights \( W_k^{(s)} \) were evaluated within the HP formalism by solving the equations-of-motion for coupled spin Green’s functions.\(^{13,16} \) In zero field, the condition that the SW weights are positive for all \( k \) is equivalent to the condition that all SW frequencies are positive.

We obtained analytic expressions for the SW frequencies for all phases shown in Fig. 1 with the exception of the 8-SL phase, which was solved numerically. Analysis of the SW frequencies yields the critical anisotropy \( D_c \) and the critical wave-vectors \( k \) where the SW frequencies vanish. To simplify the following discussion, the SW and anisotropy coefficients are provided in the appendix.

1 - Sublattice. The 1-SL phase (Fig. 1(a)) is a ferromagnet with SW frequencies

\[
\omega_k^{(1)} = 2S \left( D + A_{1k} \right).
\]

Since the 1-SL phase is locally stable for any positive value of the anisotropy, \( D_c = 0 \). The SW intensity \( W_k^{(1)} \) is constant throughout \( k \) for all interactions.

2 - Sublattice. For the 2-SL phase (shown in Fig. 1(b)), the SW frequencies are given by

\[
\omega_k^{(2)} = 2S \sqrt{A_{2k}^2 - A_{3k}^2}.
\]

The SW weights for the 2-SL phase are

\[
W_k^{(2)} = \frac{A_{2k}^2 + A_{3k}^2}{A_{2k} - A_{3k}}.
\]

From Eq. (1), the condition for the local stability of a 2-SL phase is \( A_{2k}^2 - A_{3k}^2 > 0 \). At \( D_c \), \( A_{2k}^2 = A_{3k}^2 \).

| Table I: Classical Energies and Critical Anisotropies for TLA Sublattices |
|-----------------|-----------------|-----------------|
| SL              | Energy          | \( D_c \)       |
| 1-SL            | \( E^{(1)}_{NS} \) = -3J_1 - 3J_2 - 3J_3 - D | \( D_c^{(1)} \) = 0 |
| 2-SL            | \( E^{(2)}_{NS} \) = J_1 + J_2 - 3J_3 - D | \( D_c^{(2)} \) (Eq. (5)) |
|                  |                 | \( D_c^{(2II)} \) (Eq. (9)) |
| 3-SL            | \( E^{(3)}_{NS} \) = J_1 - 3J_2 + J_3 - D | \( D_c^{(3)} \) (Eq. (13)) |
| 4-SL            | \( E^{(4)}_{NS} \) = J_1 - J_2 + J_3 - D | \( D_c^{(4)} \) (Eq. (18)) |
| 8-SL            | \( E^{(8)}_{NS} \) = J_2 + J_3 - D | \( D_c^{(8)} \) (Eq. (20)) |

This condition is satisfied when \( D_c = 0 \) in most of the 2-SL phase. But approaching the 3, 4, and 8-SL phase boundaries, nonzero anisotropy is required for local stability. As shown in Fig. 3(a), the critical anisotropy is continuous across the 4-SL and 8-SL boundaries, but is discontinuous across the 3-SL boundary.

Upon closer examination (Fig. 3(c)), we find that \( D_c \) depends differently on the exchange parameters in the three regions designated by Roman numerals. In region 2I (bounded by \( J_3 = J_2/2, J_3 = (9J_2 - J_1)/12 \), and \( J_3 = J_2^2/(J_1 - 2J_2) \)),

\[
D_c^{(2II)} = \frac{1}{(4J_3)^3} \left\{ -272J_3^4 + 64J_3^4 J_2 + 48J_3^3 J_1 \\
+72J_3^2 J_2^2 - 48J_3 J_2 J_1 - 8J_2^3 J_1 \\
+36J_3 J_2 J_1 - 27J_2^2 - (2J_3 - J_2)C^3 \right\},
\]

where

\[
C = \sqrt{(2J_3 + 3J_2)^2 - 8J_3 J_1}
\]

In region 2II (bounded by \( J_3 = J_2/2, J_3 = J_2, J_3 = (8J_2 - J_1)/9 \), and \( J_3 = J_2^2/(J_1 - 2J_2) \)),

\[
D_c^{(2II)} = 4J_2 - \frac{9}{2} J_3 - \frac{1}{2} J_1.
\]

Finally, in region 2III (bounded by \( J_3 = J_2/2, J_3 = J_1/4 \),
and \( J_3 = (J_1 - J_2)/4 \),
\[
D_c^{(2III)} = -\frac{(4J_3 + J_2 - J_1)^2}{2(J_2 + 2J_3)}.
\]

The region with no critical anisotropy \( D_c^{(2IV)} \) is bounded by \( J_2 = 0 \), \( J_3 = (8J_2 - J_1)/9 \), and \( J_3 = (J_1 - J_2)/4 \) as shown in Figs. 3(b) and (c).

The critical wave-vectors, \( k \) for the SW instabilities in region 2I are:
\[
k_x^{(2I,a)} = 2 \arccos \left\{ \frac{3J_2 - 2J_3 - C}{8J_3} \right\},
\]
\[
k_y^{(2I,a)} = 0.
\]
Two other instabilities \( k^{(2I,b)} \) and \( k^{(2I,c)} \) are related to \( k^{(2I,a)} \) by \( \pm \pi/3 \) rotations and can be considered “twins” of the \( k^{(2I,a)} \) instabilities. All three instabilities occur at the same critical anisotropy \( D_c^{(2I)} \). For regions 2II and 2III, the SW instabilities occur at
\[
k_x^{(2II)} = \pi \pm \pi/3,
\]
\[
k_y^{(2II)} = 0,
\]
and
\[
k_x^{(2III)} = 0,
\]
\[
k_y^{(2III)} = \frac{2}{\sqrt{3}} \arccos \left\{ \frac{J_2 + J_1}{2(J_2 + 2J_3)} \right\},
\]
along the \( k_x \) and \( k_y \) axis, respectively.
Notice that $D_3^{(3)} = 0$ along the 3-SL/1-SL boundary. Again, $D_c$ is discontinuous along the 2-SL/3-SL and 3-SL/4-SL boundaries: the anisotropy required for the local stability of the 3-SL phase is three times the critical anisotropy of the 2 or 4-SL phases. As discussed further below, the discontinuities at the 2-SL/3-SL and 3-SL/4-SL phase boundaries are related to the distinction between the conditions for global and local stability.

In Fig. 3(b), we plot a SW dispersion in the 3-SL phase with interaction parameters $J_2/|J_1| = 0.5$, $J_3/|J_1| = -0.5$, and $D_c/|J_1| = 2.25$. Since the 3-SL phase has a net moment, the SW frequencies are quadratic functions of $k$ near the instability wave-vectors.

4. - Sublattice. The SW frequencies for the 4-SL phase (shown in Fig. 3(c)) were evaluated in Ref. [12] and are given by

$$\omega^4_k = 2S \left( \frac{A_{6k}^2 - A_{7k}^2}{2} \right) \left( \frac{F_{2k}^2 - F_{2k}^*2}{2} \right)^{1/2} + 4|A_{6k}F_{2k} - A_{7k}F_{2k}^*|^2 \right)^{1/2}. \quad (16)$$

The SW weights of the 4-SL phase are

$$W^4_k = \left[ R_{5k}(A_{7k} - A_{6k}) + (F_{2k} + F_{2k}^*)(A_{6k} - A_{7k}) \right] + (F_{2k}^* - F_{2k})^2(F_{2k} + F_{2k}^* - A_{6k} - A_{7k}) \times \left[ R_{5k}\sqrt{A_{6k}^2 - A_{7k}^2 - R_{5k}} \right]^{-1}. \quad (17)$$

As for the 2-SL phase, the critical anisotropy $D_c$ for the 4-SL phase depends differently on the interaction parameters in two regions, again denoted by Roman numerals I and II (Fig. 3(b)). In region 4I (bounded by $J_3 = J_2/2$, $J_2 = J_1/2$, and $J_3 = J_2^2/(J_1 - J_2)$),

$$D_c^{(4)}(a) = \frac{1}{2} \left\{ -16J_2^2 - 64J_3^2J_2 + 48J_3^3 J_1 \right\} + 72J_2^2J_3^2 - 8J_3^2J_2^2 - 48J_3^2J_2J_1 \quad (18)$$

and in region 4II (bounded by $J_3 = J_2/2$, $J_2 = 0$, and $J_3 = J_2^2/(J_1 - J_2)$),

$$D_c^{(4)(II)} = 2J_2 - \frac{1}{2}J_3 - \frac{1}{2}J_1. \quad (19)$$

The critical wave-vectors for the 4-SL phase are the same as those in the respective region of the 2-SL phase, including the multiple instabilities in region 2I: $k^{(4I,a)} = k^{(2I,a)}$, $k^{(4I,b)} = k^{(2I,b)}$, and $k^{(4I)} = k^{(2I)}$. Figure 3(c) shows two representative SWs for regions 4I and 4II with $k_0 a = 0$. The interactions parameters for region 4I are $J_2/|J_1| = -0.439$, $J_3/|J_1| = -0.570$, and $D_c/|J_1| = 0.105$. For region 4II, they are $J_2/|J_1| = -0.25$, $J_3/|J_1| = -0.5$, and $D_c/|J_1| = 0.25$. 

Figure 4(a) shows three representative SWs for all 2-SL regions. The interaction parameters for region 2I are $J_2/|J_1| = -0.25$, $J_3/|J_1| = -0.12$, and $D_c/|J_1| = 0.04$. For region 2II, $J_2/|J_1| = -0.10$, $J_3/|J_1| = -0.05$, and $D_c/|J_1| = 0.325$. For region 3II, $J_2/|J_1| = -0.75$, $J_3/|J_1| = -0.125$, and $D_c/|J_1| = 0.031$. Finally, for region IV, the interaction parameters are $J_2/|J_1| = -1.0$, $J_3/|J_1| = -0.125$, and $D_c/|J_1| = 0.0$. Regions I, II and IV were evaluated with $k_0 a = 0$, while region III was evaluated at $k_0 a = 0.186\pi$ as explained above.

In Figs. 3(b) and (c), we examine the critical anisotropy of the 2-SL along the $J_3/|J_1| = 0$ axis. The critical anisotropy vanishes for $-1 < J_2/|J_1| < -1/8$ but is nonzero outside this region. Therefore, noncollinear phases should appear for $J_2/|J_1| < -1$ and $J_2/|J_1| > -1/8$ when $D < D_c$. This agrees with Jolli-coeur et al., who studied a TLA with nearest and next-nearest neighbor exchange interactions and $D = 0$. They obtain a Néel state up to $J_2/|J_1| = -1/8$ and an incommensurate spiral for $J_2/|J_1| < -1$. Similar results have been obtained on square lattices.

3. - Sublattice. For the 3-SL phase (shown in Fig. 3(c)), the SW frequencies are

$$\omega^{(3)}_k = 6S \sqrt{R_{1k} \cos(\theta/3 + 2m/3\pi) + R_{2k}/3}, \quad (13)$$

where $m$ is an integer (0,1,2) distinguishing the three separate SW dispersion relations and

$$\theta = \arccos \left\{ \frac{2R_{2k}^3 - 9R_{2k}R_{3k} - 27R_{4k}}{1458R_{1k}^{3/2}} \right\}. \quad (14)$$

The critical anisotropy of the 3-SL phase is independent of $J_2$ and given by

$$D^{(3)}_c = \frac{3}{2}(J_1 + J_3). \quad (15)$$
the midpoint of the Brillouin zone.

Since the SW frequencies are symmetric about
function of \( k \) around the wave-vectors of the instabilities.

The 4-SL phase is of particular interest since it is the
known ground state of \( \text{CuFeO}_2 \). Fits of the experimental SW frequencies of \( \text{CuFeO}_2 \) have determined the ratios of exchange parameters \( J_2/|J_1| \approx -0.44 \) and \( J_3/|J_1| \approx -0.57 \), which lies within region 4I. Consequently, we have studied the SW frequencies of the 4-SL phase more closely. Figure 5 shows the behavior of \( k_{21}^{(a)} \) along various cuts through region 4I of phase space. Since the SW frequencies are symmetric about the midpoint of the Brillouin zone \( a \pi \), we consider the quantity \( \Delta = a |k_x - \pi| \). As \( J_3/|J_1| \) increases in region 4I, \( \Delta \) asymptotically approaches \( \pi/3 \), which is the constant value of \( \Delta \) in region 4II. For small values of \( J_3/|J_1| \), the wave-vector instabilities approach \( \pi \) as \( J_2/|J_1| \) increases, equal \( \pi \) for \( J_2/|J_1| = -1/3 \), and then move away from \( \pi \) as \( J_2/|J_1| \) approaches zero; this behavior is shown along the 2-SL/4-SL boundary in Fig. 5(b).

8 - Sublattice. For the 8-SL phase (shown in Fig. 1(c)), we have determined SW dispersion relations numerically. The critical anisotropy values for this phase are shown in Fig. 1(a). Notice that \( D_c \) has a cusp dividing the phase into regions 8I and 8II (Fig. 1(b)), separated by \( J_3 = J_2/2 \). Looking more closely at the numerical results, the critical anisotropies in the 8-SL regions are closely related to those of their respective neighbors and are given by

\[
D_c^{(8I)} = D_c^{(21I)} + 4 J_3 - J_1, \tag{20}
\]

\[
D_c^{(8II)} = D_c^{(4I)} + 2 J_2 - J_1, \tag{21}
\]

which clearly show that the critical anisotropies are continuous across the phase boundaries. In region 8II, the wave-vector instabilities occur for \( k_y = 0 \) (as in region 4I); in region 8I, the wave-vector instabilities occur for non-zero \( k_y \) (as in region 2III). Figure 2(d) shows two representative SWs for regions 8I and 8II. The interactions parameters for region 8I are \( J_2/|J_1| = -1.5, J_3/|J_1| = -0.50, \) and \( D_c/|J_1| = 0.25. \) For region 8II, they are \( J_2/|J_1| = -0.75, J_3/|J_1| = -0.50, \) and \( D_c/|J_1| = 0.62. \) Whereas \( k_y a = 0 \) for region 8II, \( k_y a = 0.382 \pi \) for region 8I as explained above.

To better understand the discontinuities along the 2-SL/3-SL and 3-SL/4-SL phase boundaries, we consider the relationship between local and global stability. Our SW calculations only guarantee the local stability of each collinear phase. But even when a phase is locally stable, it can still be globally unstable to a lower-energy spin configuration. Hence, the critical anisotropy \( \tilde{D}_c \) for global stability must be greater than or equal to the critical anisotropy \( D_c \) for local stability. Unlike \( D_c \), \( \tilde{D}_c \) must also be a continuous function of \( J_1, J_2, \) and \( J_3 \). So when \( D_c \) is discontinuous, the phase with the lower critical anisotropy cannot be globally stable. Since the 3-SL has a higher critical anisotropy along the 2 and 4-SL boundaries, the 2-SL and 4-SL phases cannot be globally stable along those boundaries when \( D_c^{(21II)} < D < D_c^{(3)} \) or \( D_c^{(4)} < D < D_c^{(3)} \). Therefore, our results for the local stability of the collinear phases also has implications for the global stability of those phases.

Conclusion. We have examined the critical anisotropy for a geometrically-frustrated TLA. Based on the Takagi-Makata phase diagram, we calculated the SW frequencies

\[
\Delta / |k| = \Delta / \pi
\]

FIG. 5: (Color online) (a) Location of SW instability \( \Delta = a |k_x - \pi| \) along \( k_y = 0 \) in region 4I for fixed values of \( J_2/|J_1| \). As \( J_3/|J_1| \) increases along \( J_2/|J_1| = -0.5, \Delta \) asymptotically approaches \( \pi/3 \). (b) Plot of \( \Delta \) in region 4II along the 2-SL/4-SL boundary \( J_2/|J_1| = 1/2 \). The cusp in \( \Delta \) occurs at \( J_2/|J_1| = -1/3 \) where the SW instability occurs at \( \pi \).
for all five phases. Imposing the two conditions for local stability, we obtained the critical anisotropies and wave-vector instabilities for all phases as functions of the exchange interactions. Surprisingly, these results are highly dependent on the longer-range exchange interactions and most phases break into several regions where the anisotropy has a distinct dependence on the exchange parameters. As discussed for the 2-SL and 4-SL phases, the critical anisotropies and wave-vectors for the local stability of the collinear phases provides useful information about the non-collinear phases that appear at small anisotropy. We have also shown that the discontinuity of the critical anisotropy at the 2-SL/3-SL and 3-SL/4-SL phase boundaries has implications for the global stability of the 2-SL and 4-SL phases with the smaller critical anisotropies.

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**APPENDIX A: SPIN-WAVE AND ANISOTROPY COEFFICIENTS**

This Appendix provides the coefficients that enter the SW frequencies and weights for each phase. The coefficients for the 1-SL or ferromagnetic phase are

\[ A_{1k} = 3(J_1 + J_2 + J_3) \]

\[ -J_1 \left( \cos(k \cdot d_1) + \cos(k \cdot d_2) + \cos(k \cdot d_3) \right) \]

\[ -J_2 \left( \cos(k \cdot d_4) + \cos(k \cdot d_5) + \cos(k \cdot d_6) \right) \]

\[ -J_3 \left( \cos(2k \cdot d_1) + \cos(2k \cdot d_2) + \cos(2k \cdot d_3) \right) \]

where \( d_1 = ax, d_2 = 1/2 ax + \sqrt{3}/2 ay, d_3 = -1/2 ax + \sqrt{3}/2 ay, d_4 = 3/2 ax + \sqrt{3}/2 ay, d_5 = \sqrt{3} ay, \) and \( d_6 = -3/2 ax + \sqrt{3}/2 ay. \)

The 2-SL phase coefficients are

\[ A_{2k} = D + 3J_3 \]

\[ -J_1 \left( \cos(k \cdot d_1) + 1 \right) - J_2 \left( \cos(k \cdot d_5) + 1 \right) \]

\[ -J_3 \left( \cos(2k \cdot d_1) + \cos(2k \cdot d_2) + \cos(2k \cdot d_3) \right) \]

\[ A_{3k} = J_1 \left( \cos(k \cdot d_2) + \cos(k \cdot d_3) \right) \]

\[ + J_2 \left( \cos(k \cdot d_4) + \cos(k \cdot d_6) \right). \]

The 3-SL phase coefficients are

\[ R_{1k} = R_{2k}^2 - 3R_{3k}, \]

\[ R_{2k} = 2A_{4k} + A_{5k}, \]

\[ R_{3k} = A_{4k}^2 + 2A_{4k}A_{5k} + |F_{1k}|^2, \]

\[ R_{4k} = (A_{5k} - 2A_{4k})|F_{1k}|^2 - A_{4k}^2A_{5k} - F_{1k}^2 - F_{1k}^2, \]

\[ A_{4k} = 2D + 2J_2(3 - \cos(k \cdot d_4) - \cos(k \cdot d_5) - \cos(k \cdot d_6)), \]

\[ A_{5k} = 6J_1 + 6J_3 - A_{3k}. \]

\[ F_{1k} = J_1(e^{-ik \cdot d_2} + e^{ik \cdot d_2}e^{i2k \cdot d_3}) \]

\[ + J_2(1 + e^{-2ik \cdot d_1} + e^{-2ik \cdot d_3}). \]

As in Ref. [12], the 4-SL phase coefficients are

\[ R_{5k} = (F_{2k}^2 + F_{4k}^2 - 2(F_{2k}^2 + 2A_{6k}A_{7k})F_{2k}^2) \]

\[ + 4(A_{6k}^2 + A_{7k}^2)|F_{2k}|^2 - 4A_{6k}A_{7k}F_{2k}^2)^{1/2}, \]

\[ A_{6k} = D - J_1 + J_2(1 - \cos(k \cdot d_5)) \]

\[ - J_3(1 + \cos(2k \cdot d_1)), \]

\[ A_{7k} = -\cos(k \cdot d_1) \left( J_1 + 2J_3 \cos(\sqrt{3}k \cdot d_3) \right), \]

\[ F_{2k} = -\cos(k \cdot d_5/2) \left( J_1 e^{i(k \cdot d_1/2} + J_2 e^{-3i(k \cdot d_1)/2} \right). \]
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