Jets in Effective Theory:
Summing Phase Space Logs.

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Abstract

We demonstrate how to resum phase space logarithms in the Sterman-Weinberg (SW) dijet decay rate within the context of Soft Collinear Effective theory (SCET). An operator basis corresponding to two and three jet events is defined in SCET and renormalized. We obtain the RGE of the two and three jet operators and run the operators from the scale $\mu^2 = Q^2$ to the phase space scale $\mu_\delta^2 = \delta^2 Q^2$. This phase space scale, where $\delta$ is the cone half angle of the jet, defines the angular region of the jet. At $\mu_\delta^2$ we determine the mixing of the three and two jet operators. We combine these results with the running of the two jet shape function, which we run down to an energy cut scale $\mu_\beta^2$. This defines the resumed SW dijet decay rate in the context of SCET. The approach outlined here demonstrates how to establish a jet definition in the context of SCET. This allows a program of systematically improving the theoretical precision of jet phenomenology to be carried out.

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I. INTRODUCTION

The phenomenology of QCD jets has been examined over the years with increasingly sophisticated techniques \[1, 2, 3, 4, 5, 6\]. Recently, an effective field theory (EFT) of QCD containing collinear degrees of freedom interacting with soft and ultrasoft gluons, SCETi \[7\], was formulated. Using SCET, nonperturbative corrections to dijet event shapes have been examined in a series of papers \[8, 9, 10, 11\].

In this paper, we use SCET to sum large phase space logarithms\(^1\) present in the perturbative expansion of the dijet decay of the Z boson. Our approach is general and addresses the problem of establishing a jet definition within SCET. We have chosen our initial state to be a Z boson to avoid complications from QCD interactions with the initial state. The logarithms resummed have the form \(\alpha_s^n \log^{n+1}\) in fixed order perturbation theory. At \(\mathcal{O}(\alpha_s)\) we are resumming the \(\alpha_s \log (2 \beta) \log (\delta)\) double log in terms of the jet parameters \(\delta\) and \(\beta\). We also resum a class of \(\alpha_s^n \log^n\) logarithms given by \(\alpha_s^n \log^n (\delta)\) terms in the fixed order perturbative expansion. Here \(\delta\) and \(\beta\) are parameters that impose phase space cuts according to the SW jet definition.\(^2\)

The SW jet definition defines the dijet decay rate as an integration of the triple differential decay rate over a phase space restricted by cuts. The cuts on the energy and angular integrations define the dijet decay rate to be the sum of three partonic final states. The first state (SW1), is comprised of a quark and anti-quark contained within different jets, where the jets are angularly defined as cones of half angle \(\delta\). The second state (SW2), is comprised of a quark and anti-quark each contained within a different jet, plus a gluon (unrestricted in direction) with energy \(E_g < \beta M_z\). The final state (SW3), is defined as a quark and an anti-quark within different jets and a gluon of energy \(E_g > \beta M_z\) within the jet cones. \(^3\)

In this paper, we match the amplitude for the decay of the Z onto an operator basis defined in SCET. This defines contributions of two and three jet final states at the scale \(M_z^2\). We run down from the scale \(M_z^2\) to the scale defining the dijet through the angular

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\(^1\) It has been shown that phase space logs can to be resumed for the decay \(\bar{B} \to X_c \ell \bar{\nu}\) in \[12\].

\(^2\) The SW jet definition is not the only possible definition of a jet. In fact, SW jets are somewhat experimentally disfavored compared to the \(k_\perp\) algorithm \[13, 14\]. The relative theoretical simplicity of the SW jet definition motivates us to demonstrate the origin of the relevant logarithms using this definition.

\(^3\) Note that both \(\delta\) and \(\beta\) are small \((\ll 1)\) in this definition; however one must also have \(\beta\) less than \(\delta\) such that \(\sin(\delta) > \beta/(1 - \beta)\) \[15\].
cut on phase space. This cut scale is given by $\mu_2^2 = \delta^2 M_Z^2$. The usoft degrees of freedom are matched onto a dijet shape function as in [8, 9, 10, 11]. This dijet shape function is restricted to emit gluons that take away energy $E_g < \beta M_z$. This requires the introduction of a further phase space scale corresponding to this energy cut. This cut scale is given by $\mu_3^2 = \delta^4 M_Z^2 / B^{1-\delta}$, where $B$ is an $\mathcal{O}(1)$ parameter. We fix $B$ by matching onto the SW dijet decay rate after we run down to the jet scales. We discuss the physical significance of these jet scales in Section II in more detail.

The resummation of Sudakov [16] logs for jets has been investigated in [5, 17, 18, 19, 20, 21] for various jet definitions. The leading logarithm results reported here have been produced in these other contexts. We approach the re-summation with the EFT formalism as this approach offers several advantages. For example, the EFT construction includes nonperturbative effects through the matrix elements of the usoft degrees of freedom. Within the effective theory, one can calculate to high orders in RGE improved perturbation theory in a clearly defined and easy manner. As an example of the utility of the effective theory approach, we have determined the mixing of the three and two jet operators. This mixing determines a class of NLL contributions in the RGE improved perturbative expansion.

It is important to note that although the SCET final states discussed here are an inclusive sum over the set of hadronic final states, the sum is not a colour singlet. Gluons are exchanged between the low energy partons comprising the jets in the hadronization process. This is a difference in calculating dijet production compared to the use of the operator product expansion in other inclusive decays, such as semileptonic $B \rightarrow X_c \ell \nu$ decay [22]. Experimental results [23] indicate that this issue does not invalidate the SW jet definition.

The outline of this paper is as follows. In the remaining parts of the introduction, we calculate the $\mathcal{O}(\alpha_s)$ phase space logarithms of the SW jet definition. We then briefly review SCET. In Section II we formulate the phase space cut scales corresponding to the SW jet definition. In Section III we discuss the matching of $\Gamma(Z \rightarrow q \bar{q})$ onto the dijet operator. We renormalize the operator basis and run down to the the scale $\mu_3^2$. We match the contribution of collinear gluon emission in $Z \rightarrow q \bar{q} g$ onto three jet operators in Section IV and renormalize the three jet operator basis. Section IV B is devoted to the running of these operators. In Section V the mixing of the three and two jet operators is discussed. In Section VII we introduce the ‘Phase Space Renormalization Group’ and run to the phase space cut scales. Finally, we obtain the leading log resumed dijet decay rate in SCET.
A. Phase space Logarithms at $O(\alpha_s)$

First, we review how the phase space logarithms present in the QCD calculation of $\Gamma(Z \to J\bar{J})$ are generated by the SW jet definition. Recall that to $O(\alpha_s)$ the amplitude for $\Gamma(Z \to J\bar{J})$ is given by the following expression, grouped in terms is $q\bar{q}$ and $q\bar{q}g$ final states

$$\mathcal{A}_{Z,JJ}^2 = |A_0 + A_V + A_{W1} + A_{W2}|^2 + |A_{B1} + A_{B2}|^2_R.$$  \hfill (1)

The subscript R refers to the restricted set of $q\bar{q}g$ final states in the jet definition. The amplitudes above are the tree level amplitude $A_0$, the vertex correction amplitude $A_V$ and the wavefunction amplitudes $A_{Wi}$. The $A_{Bi}$ are the bremsstrahlung amplitudes.

Consider first $A_0$ and the corrections $A_V, A_{W1}, A_{W2}$. These contributions to $\Gamma(Z \to q\bar{q})$ can be determined by calculating the imaginary part of the forward scattering diagrams in Fig. (1). We calculate in the massless quark limit throughout this paper in the rest frame

FIG. 1: The imaginary part of the forward scattering amplitude in QCD gives the $q\bar{q}$ final state.

The operator insertion is the weak neutral current $j_{nc} = \bar{q}\Gamma^\sigma q\epsilon^\sigma$ with $\Gamma^\sigma = \gamma^\sigma (g_V + g_A \gamma_5)$. The wavefunction renormalization graphs are not drawn as Landau gauge can be used so that the wavefunction contributions vanish. The sum of imaginary part of the diagrams is gauge independent at $O(\alpha_s)$.

of the $Z$, and always in $d = 4 - 2\epsilon$ dimensions. We average over the initial polarization of the $Z$ in $d - 1$ dimensions, and obtain the decay rate

$$\Gamma(Z \to q\bar{q})(\mu) = \frac{N_C}{32\pi^2} (g_V^2 + g_A^2) [M_Z^{1-2\epsilon}(4\pi)^2 \epsilon^2 \frac{2-2\epsilon}{3-2\epsilon} \Omega_{3-2\epsilon}] C_{qq}^{\overline{\text{MS}}} (\mu).$$  \hfill (2)

We have presented our results in the form suggested in [2] and agree with their result. The coefficient in $\overline{\text{MS}}$ is

$$C_{qq}^{\overline{\text{MS}}} (\mu) = 1 + \frac{\alpha_s(\mu)}{\pi} C_F \left( \frac{1}{\epsilon} + \frac{3}{2} \left( \ln \left[ \frac{-2p_q \cdot p_{\bar{q}}}{\mu^2} \right] - \frac{1}{\epsilon} \right) - 4 + \frac{\pi^2}{12} - \frac{1}{2} \ln^2 \left[ \frac{-2p_q \cdot p_{\bar{q}}}{\mu^2} \right] \right) + O(\alpha_s^2).$$  \hfill (3)
The $q\bar{q}g$ final state amplitude is $A_{Zq\bar{q}} = |A_{B1} + A_{B2}|^2$ and is obtained by calculating the bremstrahlung graphs. The double differential rate is

$$
\frac{d\Gamma_{q\bar{q}g}}{dx_{q}dx_{\bar{q}}} = \frac{N_{C} C_{F} (g_{V}^{2} + g_{A}^{2})}{256 \pi^{4}} (M_{Z}^{2-2\epsilon}) (4\pi)^{4\epsilon} \frac{\Omega_{3-2\epsilon} \Omega_{2-2\epsilon}}{(3-2\epsilon)} x_{q}^{(6-4)} x_{\bar{q}}^{(6-4)} (1 - z^{2})^{(\epsilon)} F[x_{q}, x_{\bar{q}}, \mu].
$$

where the energies of the decay product quarks are $x_{q} = 2 E_{q}/M_{Z}$ and $x_{\bar{q}} = 2 E_{\bar{q}}/M_{Z}$. The angle between the quarks has been fixed by a delta function to be $z = 1 + 2(1 - x_{q} - x_{\bar{q}})/(x_{q} x_{\bar{q}})$. The function $F$ is kept to order $\epsilon^{2}$ and is given by

$$
F[x_{q}, x_{\bar{q}}, \mu] = \alpha_{s}(\mu) \left( \frac{\mu}{M_{Z}} \right)^{(1-2\epsilon)} [(1 + \epsilon) x_{q} + x_{\bar{q}} (1 - x_{q}) + 2 \epsilon (1 + \epsilon) (1 - x_{q} - x_{\bar{q}})/(1 - x_{q})(1 - x_{\bar{q}})].
$$

Integrating this expression over the remaining three body phase space in $d$ dimensions gives

$$
\Gamma(Z \to q\bar{q}g)(\mu) = \frac{N_{C} C_{F} (g_{V}^{2} + g_{A}^{2})}{32 \pi^{2}} (M_{Z}^{2-2\epsilon}) (4\pi)^{2\epsilon} \frac{2 - 2\epsilon}{3 - 2\epsilon} \Omega_{3-2\epsilon} |C_{q\bar{q}g}^{MS}(\mu)|^{2},
$$

with

$$
C_{q\bar{q}g}^{MS}(\mu) = -\frac{\alpha_{s}(\mu) C_{F}}{\pi} \left( \frac{1}{\epsilon} + \frac{3}{2} (\ln \frac{M_{Z}^{2}}{\mu^{2}}) - \frac{19}{4} - \frac{7\pi^{2}}{12} - \frac{1}{2} \ln^{2} \frac{M_{Z}^{2}}{\mu^{2}} \right) + O(\alpha_{s}^{2}).
$$

The full decay rate to $O(\alpha_{s})$ is the result

$$
\Gamma_{Z}(M_{Z}) = \frac{N_{C} C_{F} (g_{V}^{2} + g_{A}^{2})}{12 \pi} \left( 1 + \frac{3 \alpha_{s}(M_{Z}) C_{F}}{4 \pi} \right) + O(\alpha_{s}^{2}).
$$

The SW jet definition introduces cuts on the energy and angular integrations of Eq (4). The resulting phase space integrations are difficult, some details can be found in [9]. Retaining the terms that diverge as the energy fraction $\beta$ and the cone angle $\delta$ go to zero, one obtains

$$
C_{q\bar{q}g}^{SW \, cuts}(\mu) = -\frac{\alpha_{s}(\mu) C_{F}}{\pi} \left( \frac{1}{\epsilon} + \frac{3}{2} (\ln \frac{M_{Z}^{2}}{\mu^{2}}) - \frac{1}{2} \ln^{2} \frac{M_{Z}^{2}}{\mu^{2}} \right)
$$

$$
+ \frac{\alpha_{s}(\mu) C_{F}}{\pi} \left( -4 \ln(2\beta) \ln(\delta) - 3 \ln(\delta) + \frac{13}{2} - \frac{11\pi^{2}}{12} \right).
$$
Combining this result with Eq. (3), one obtains the SW result; we find
\[
\Gamma(Z \rightarrow J\bar{J})(M_z) = \frac{N_C M_z (g_V^2 + g_A^2)}{12 \pi} \left( 1 + \frac{\alpha_s C_F}{\pi} \left( \frac{5}{2} - \frac{\pi^2}{3} - 3 \ln \delta - 4 \ln(2\beta) \ln(\delta) \right) \right). \tag{11}
\]
Note that the result is free of explicit IR singularities. This is expected as the SW jet definition is constructed to satisfy the KLN theorem [32, 33]. However, the result does depend on logs of the cuts used to partition phase space.

We will now examine this decay in SCET. The different final state phase space configurations will be separated out into different operators using the invariant \( p_g \cdot p_i \). By matching and running, the renormalization group allows us to sum the Sudakov logarithms and define a dijet amplitude.

### B. Jets in SCET

SCET contains collinear and ultrasoft degrees of freedom that are relevant for QCD jet studies.\(^4\) Consider a final state gluon produced by the decay products of the Z. The gluon is collinear in direction \( n_i \) when it has scaling \( p_c^{(n_i)} = M_Z (\lambda^2, 1, \lambda) \), where \( \lambda \) is a small parameter in the effective theory. We have used the lightcone coordinates \( L^\mu = (L^+, L^-, L_\perp) \) such that \( L^+ = n_i \cdot L, L^- = \bar{n}_i \cdot L \) and \( L_{\perp}^\mu = L^\mu - L^+ \bar{n}_i^\mu/2 - L^- n_i^\mu/2 \). The lightcone vectors used here are defined as \( n_i = (1, \bar{n}_i) \) and \( \bar{n}_i = (1, -\bar{n}_i) \) with \( \bar{n}_i \) a unit vector.

Alternatively, the gluon could have ultrasoft scaling \( p_{us} = M_Z (\lambda^2, \lambda^2, \lambda^2) \). When \( \lambda \sim \sqrt{\Lambda_{QCD}/M_Z} \), one cannot perturbatively expand the decay rate in terms of usoft gluons. The nonperturbative effects of these gluons are given by the matrix elements of ultrasoft degrees of freedom, which give nonperturbative corrections to the decay rate [8, 9, 10, 11]. For this reason, we take our final state gluon radiation to be collinear in our perturbative expansion. Expanding the decay current \( j^\mu \) in terms of collinear and ultrasoft degrees of freedom, the SCET jet fields are given by
\[
\bar{\chi}_{nq} = [\bar{\xi}_{nq} W_{nq}], \quad \chi_{n\bar{q}} = [W_{n\bar{q}}^\dagger \xi_{n\bar{q}}]. \tag{12}
\]
\(^4\) See [7] for more detailed reviews.
The $\xi$ field is defined with the large momenta $\tilde{p}$ in the direction $n_i$ removed

$$\xi_{n_i}(x) = \sum_{\tilde{p}} e^{-i\tilde{p}\cdot x} P_{n_i}\psi(x),$$

using the projector

$$P_{n_i} = \frac{\gamma_i\gamma_i}{4}.$$  \hspace{1cm} (14)

Also present to preserve collinear gauge invariance are Wilson lines,

$$W_{n_i}(q) = \left[ \sum_{\text{perms}} \exp \left( -\frac{1}{\not{p}} \tilde{n}_i \cdot A_{n_i,q} \right) \right],$$

$$W_{n_i}^\dagger(q) = \left[ \sum_{\text{perms}} \exp \left( -\frac{\tilde{n}_i \cdot A_{n_i,q}^\dagger}{\not{p}} \right) \right].$$

\hspace{1cm} (15) \hspace{1cm} (16)

II. THE SCALES OF THE SW CUTS

Three small parameters exist in the construction of SW jets $\Lambda_{QCD}/M_Z$, $\delta$ and $\beta$. We take $\delta \gg \sqrt{\Lambda_{QCD}/M_Z}$ for the cuts to be above the hadronization scale. To sum large phase space logs with the renormalization group, we must associate the phase space cuts with scales $\mu^2_{\text{cut}}$.

For the angular region defining the jet ($\delta$), we utilize the approach of \[2, 3, 24\] and take a collinear momenta comprising the jet to have collinear scaling $P_{j}^{(n_i)} \sim M_Z(\delta^2, 1, \delta)$. This gives the cut scale of

$$\mu^2_\delta = \delta^2 M_Z^2.$$  \hspace{1cm} (17)

We set the invariant mass of the initial off shell quark and anti-quark to this scale, $P^2_{1,2} = \delta^2 M_Z^2$. Physically this scaling has a simple interpretation. The jet cone is defined by the half angle $\delta$. For all momenta making up the jet, when $z$ is taken as the axis of the jet cone,

$$\tan (\delta) \sim \frac{|p_z|}{|p_z|}.$$  \hspace{1cm} (18)

The usoft gluons radiate into the final state and deposit a small fraction of center of mass energy, $\beta M_Z$, isotropically in the detector. For these gluons, we utilize the fact that this radiation cannot disturb the back to back configuration of the jet cones. The sum of usoft gluon emissions is restricted to carry a total energy ($P^0_u$) out of the jet cones that satisfies $\sin(\delta) > P^0_u/(1 - P^0_u)$ \[15\]. This requires the momenta emitted outside the jet cones to have
scaling $P_u \sim M_Z(\delta^2, \delta^2, \delta^2)$. We take the energy cut scale of the usoft degrees of freedom to be

$$\mu^2_\beta = B^{\delta-1} \frac{\mu^4_\delta}{M^2_Z}, \quad (19)$$

where $B$ is an $O(1)$ parameter. The correct resummation of leading logarithms in the SW jet definition requires that $\mu^2_\beta/\mu^2_\delta \propto \delta^2$. Physically, this corresponds to the usoft radiation not being able to cause the quark and anti-quark to lie outside the jet cones. Note that the results of the SW phase space integrals in Eq. (11) are expanded in $\delta$ and $\beta$. All dependence on the jet parameters $\delta$ and $\beta$ that does not diverge as $\beta, \delta \to 0$ is neglected in Eq. (11). In a similar manner, the dependence on $B, \delta$ in our RGE is expanded. Dependence on these parameters that does not lead to large logs as $\beta, \delta \to 0$ is neglected. In this manner, any $\mu^2_\beta$ dependence on $B$ that gives the same retained contribution to the RGE will lead to the same double Sudakov logarithm. The quark and anti-quark $B$ dependence is sub leading in the parameter $\delta$ and neglected except for the component that scales as $\delta^2$. How the particular components of the usoft gluons depend on $B$ is fixed when we match onto the SW jet definition. We will demonstrate how these phase space scales will establish the SW jet definition in SCET in Section [VI]

III. $Z \to q\bar{q}$ IN SCET

At the scale $\mu^2 = M_Z^2$ we match $\Gamma(Z \to q\bar{q})$ in QCD onto the dijet operator [25]

$$O_2' = \bar{\chi}_{n_q} \Gamma^\sigma \chi_{n_q}. \quad (20)$$

The EFT loops are scaleless and vanish in dimensional regularization at $O(\alpha_s)$. The Wilson coefficient $C_2$ and the remormalization factor $Z_2$ depend on the large label momenta components $2 p_q \cdot p_{\bar{q}} \sim M_Z^2$ of the quark and anti-quark [25]. The Wilson coefficient is given by the one loop matching condition

$$\langle q\bar{q}|A|Z_\sigma \rangle(\mu) = \frac{\langle O_2' \rangle(\mu) C_2(\mu) Z_2(\mu)}{Z_2(\mu)}, \quad (21)$$

determining

$$C_2(\mu) = 1 + \frac{\alpha_s(\mu^2)}{2\pi} C_F \left( \frac{3}{2} \ln \left[ - \frac{2 p_q \cdot p_{\bar{q}}}{\mu^2} \right] - 4 + \frac{\pi^2}{12} - \frac{1}{2} \ln \left[ - \frac{2 p_q \cdot p_{\bar{q}}}{\mu^2} \right] \right) + O(\alpha_s^2),$$

$$Z_2(\mu) = 1 + \frac{\alpha_s(\mu^2)}{2\pi} C_F \left( \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} - \frac{1}{\epsilon} \ln \left[ - \frac{2 p_q \cdot p_{\bar{q}}}{\mu^2} \right] \right) + O(\alpha_s^2). \quad (22)$$
The contribution from this operator at the scale $\mu^2 = M_Z^2$ is given by Eq. (21) with

$$C_2(M_Z) = 1 + \frac{\alpha_s(M_Z)}{2\pi} C_F \left( 3i - 4 + \frac{7\pi^2}{12} \right) + \mathcal{O}(\alpha_s^2).$$

(23)

Note that contribution to the decay rate from the imaginary part of the Wilson coefficient vanishes at $\mathcal{O}(\alpha_s)$. The imaginary term can contribute at $\mathcal{O}(\alpha_s^2)$.

To determine the counter terms in the effective theory, we use offshellness to regulate the IR. Both soft and collinear loops contribute to the UV divergences that determine the counter term. If one adopts two stage running$^5$, the subtraction points of the usoft and collinear loops are taken to be the same, i.e. $\mu^2_s = \mu^2_c$. With this choice, the anomalous dimension of $O_2$ is given by

$$\gamma_2 = -2\alpha_s \frac{dZ_2^s}{d\alpha_s},$$

$$= -\frac{\alpha_s(\mu)}{\pi} C_F \left( \log \left( \frac{\mu^2}{M_Z^2} \right) + \frac{3}{2} \right) + \mathcal{O}(\alpha_s^2).$$

(24)

Note that we have used the $\overline{\text{MS}}$ result$^2$ for the epsilon poles of the counter term $Z_2^s$.

The Wilson coefficient of the operator at this lower scale is obtained by solving the RGE, with $\beta_0 = 11 - 2n_f/3$. For two stage running we find

$$C_2^{\text{two}}(\mu) = \left( \frac{\alpha_s(\mu)}{\alpha_s(M_Z^2)} \right) \left( \frac{C_F}{3} \right) \left( \frac{3 - \frac{8\pi}{\alpha_s(M_Z^2)}}{\beta_0} \right) \left( \frac{\alpha_s(\mu)}{\beta_0} \right) \left( \frac{2 C_F}{\beta_0} \right) C_2(M_Z).$$

(25)

**IV. $Z \rightarrow q\bar{q}g$ IN SCET**

Beyond tree level, we match the contribution of $\Gamma(Z \rightarrow q\bar{q}g)$ onto the two and three jet operator basis. The QCD amplitude is matched onto a two jet operator when the collinear gluon emission can be described by $L_{\text{SCET}}$ and $P_{qq}^2 < \delta^2 M_Z^2$ or $P_{qg}^2 < \delta^2 M_Z^2$, where $P_{ig} = P_i^\alpha + P_g^\alpha$. Conversely, when $P_{iq}^2 \geq \delta^2 M_Z^2$ for both $i = q, \bar{q}$, the collinear gluon emission is matched onto a three jet operator. At the scale $\mu^2 = M_Z^2$, we have the following amplitude depicted in Fig. (2)

$$A_{B1} + A_{B2} = \bar{u}(p_q) \gamma^\nu \left[ (-ig_s\gamma^\mu T^a) \left( i\frac{P_{qq}}{P_{qq}^2} \right) \Gamma^\mu - \Gamma^\nu \left( i\frac{P_{qg}}{P_{qg}^2} \right) (-ig_s\gamma^\nu T^a) \right] v(p_q) \epsilon^\lambda \epsilon_{\mu \nu}. \quad (26)$$

$^5$ We find it convenient to adopt one stage running in Section VI. However we utilize two stage running initially to demonstrate the benefit of one stage running in Section VI.
For the three jet operators, we take $2p_\gamma \cdot p_i \sim M_Z^2$ and the intermediate propagator is not resolved between the scales $M_Z^2$ and $\delta^2 M_Z^2$. Our matching to order $\lambda^0 M_Z^2$ is given by

$$A_{B1} + A_{B2} = \sum_{i=3,4,5} \frac{C_i(\mu)}{Z_i(\mu)} (q \bar{q} A^\gamma_i | O^i_\gamma | \epsilon_\alpha)(\mu),$$

(27)

where we have suppressed the summations over the light cone directions $n_i$. For different phase space configurations, $2p_\gamma \cdot p_i$ takes on different values between $M_Z^2$ and $\delta^2 M_Z^2$. In running between $M_Z^2$ and $\delta^2 M_Z^2$ the logs generated by the separation of scales depend on the phase space configuration of the event. Our formalism takes this into account by $n_i$ dependence in the anomalous dimensions of the three jet operators.

To decompose the amplitude into operators with field content defined in SCET, we first rearrange the three parton amplitude into the form

$$A_{B1} + A_{B2} = \tilde{u}^2(p_q) \left[ \frac{P^\alpha q^\beta (P g_\alpha \epsilon^a_\beta - \epsilon^a_\alpha P g_\beta)}{(p \cdot P_q)(P_q \cdot P)} \right] \Gamma^\sigma(g_s T_\alpha) v^{s1}(p_q) \epsilon_\sigma$$

$$+ \tilde{u}^2(p_q) \left[ \frac{P \cdot (P_q - P_\gamma) (P g_\sigma \epsilon^a_\beta - \epsilon^a_\alpha P g_\beta)}{2 (p \cdot P_q)(P_q \cdot P)} \right] \Gamma^\beta(g_s T_\alpha) v^{s1}(p_q) \epsilon_\sigma$$

$$+ \tilde{u}^2(p_q) \left[ \frac{P \cdot (P_q - P_\gamma) i \epsilon^{\alpha \beta \sigma \eta} \epsilon^a_\beta \epsilon'_a P g_\eta \gamma_5}{2 (p \cdot P_q)(P_q \cdot P)} \right] \Gamma^\eta(g_s T_\alpha) v^{s1}(p_q).$$

(28)

We now expand such that the partons are collinear particles in the directions $n_1, n_2, n_3$ for $p_q, p_q$ and $p_\gamma$ respectively. The operator basis in SCET is dictated by the constraint of collinear gauge invariance in the three directions $n_1, n_2, n_3$. We find the following operators,

$$O_3^\sigma = \bar{\chi}_{n_q} C_3^{\sigma \beta}(\bar{n}_3 \cdot p_3, n_i) \left[ W_{n_3}^l g_s C_{n_3}^{\alpha \beta} W_{n_3} \right] \Gamma^\sigma \chi_{n_q},$$

$$O_4^\sigma = \bar{\chi}_{n_q} C_4(\bar{n}_i \cdot p_i, n_i) \left[ W_{n_i}^l g_s C_{n_i}^{\sigma} W_{n_3} \right] \Gamma^\alpha \chi_{n_q},$$

$$O_5^\sigma = \bar{\chi}_{n_q} C_5(\bar{n}_i \cdot p_i, n_i) \left[ W_{n_3}^l g_s C_{n_3}^{\sigma \eta} \gamma_5 W_{n_3} \right] \Gamma^\eta \chi_{n_q}.$$  

(29)

The Wilson coefficients are of the form

$$C_3^{\sigma \beta}(\mu, \bar{n}_3 \cdot p_3, n_i) = \frac{4 n_i^{\beta} n_2^\alpha}{(n_1 \cdot n_3)(n_2 \cdot n_3)(\bar{n}_3 \cdot p_3)^2} C_3(\mu),$$

$$C_4,5(\mu, \bar{n}_i \cdot p_i, n_i) = \left( \frac{2}{(n_1 \cdot n_3)(\bar{n}_i \cdot p_1)(\bar{n}_3 \cdot p_3)} - \frac{2}{(n_2 \cdot n_3)(\bar{n}_2 \cdot p_2)(\bar{n}_3 \cdot p_3)} \right) C_{4,5}(\mu)$$

(30)

where the non label part of the Wilson coefficient is

$$C_i(\mu) = \left[ C_i^0 + \alpha_s(\mu) C_i(\mu) + \mathcal{O}(\alpha_s^2) \right].$$

(31)
with $C_1^0 = 1$. We have used the $\mathcal{O}(\lambda^0)$ collinear gluon field strength $g_s G_{n_3}^{\alpha,\beta} \equiv [iD_{n_3}^\alpha + g_s A_{n_3,p_g}^\alpha, iD_{n_3}^\beta + g_s A_{n_3,p_g}^\beta]$ and $\tilde{G}_{n_3}^{\sigma,\eta} = i \epsilon^{\alpha\beta\sigma\eta} G_{n_3}^{\alpha,\beta}$ is the corresponding dual field. The amplitude expressed in terms of these operators requires a summation over lightcone directions. This summation is such that when $n_i$ and $n_j$ label distinct collinear directions, they lie in different jet cones. This constraint along with momentum conservation is

$$\hat{C}_3(\mu^2) = \sum_{n_i} C_3(\mu^2) \theta [n_i \cdot \bar{n}_j - (1 + \cos 2\delta)] \delta(n_1 \cdot n_2 + n_1 \cdot n_3 + n_2 \cdot n_3 - 2).$$

We suppress these summations and constraints until Section V.

The three jet operators contain the same field content. In SCET these operators are expected to have the same renormalization \[30, 34\]. We present the operators in the form of Eq. (29) as they have a clear field content and Dirac structure. We also find this form convenient for the calculations in Section V.

A collinear gluon with large angular separation from both the quark and anti-quark can also be produced when the quark and anti-quark are collinear. In the original calculation of SW, these states did not contribute at $\mathcal{O}(\alpha_s)$. In the effective field theory, the corresponding operators vanish at leading order in $\lambda$. This is another example of the utility of SCET.

### A. Renormalization of the 3-Jet Operators

The three jet operators are renormalized by determining the UV divergences in SCET using the Feynman rules stated in the Appendix. The UV divergences are regulated using dimensional regularization and \text{MS}. We utilize offshellness to separate out the IR divergences.$^6$ For some related details on one gluon external state renormalization calculations, see \[27\]. We also note that we use the background field method \[7, 28\].

As we examine perpendicular polarized collinear gluons in the $n_3$ direction, this causes the contributions from a number of diagrams to vanish in the effective theory. The remaining diagrams are shown in Figure 3.

Calculating in the rest frame of the $Z$, the perturbative expansion of the operator can be determined. We obtain the following for diagrams (a) and (b)

$$\langle \xi_{n_1} \bar{\xi}_{n_2} A_{n_3}^{\perp,\mu}|\mathcal{O}_3^\sigma|0 \rangle_a = -\frac{g_s^2 C_F C_v^\sigma}{8 \pi^2} \left[ -\frac{1}{\epsilon_{uv}^2} - \frac{1}{\epsilon_{uv}} + \frac{1}{\epsilon_{uv}} \log \left[ -\frac{P_{q u}^2}{\mu^2} \right] \right]$$

$^6$ We have used Feynman gauge.
FIG. 3: One gluon external state diagrams for the operators $O_i$ in the effective theory. Collinear gluons are drawn as springs with lines, usoft gluons as springs.

\[-g_s^2 C_F C_{\nu \sigma} \left[ -\frac{1}{2} \log \left( \frac{-P_q^2}{\mu^2} \right) + \log \left( \frac{-P_q^2}{\mu^2} \right) - 2 + \frac{\pi^2}{12} \right], \quad (33)\]

\[\langle \xi_{n_1} \bar{\xi}_{n_2} A_{\nu}^{\perp} | O_3^\sigma | 0 \rangle_b = -g_s^2 C_F C_{\nu \sigma} \left[ -\frac{1}{2} \log \left( \frac{-P_q^2}{\mu^2} \right) - \log \left( \frac{-P_q^2}{\mu^2} \right) - \log \left( \frac{-P_q^2}{\mu^2} \right) - 2 + \frac{\pi^2}{12} \right]. \quad (34)\]

We have defined $C_{\nu \sigma} = \langle \xi_{n_1} \bar{\xi}_{n_2} A_{\nu}^{\perp} | O_3^\sigma | 0 \rangle$ in the above. Defining the following function

\[K \left[ (P_i, n_i), (P_j, n_j) \right] \equiv \log \left( \frac{-2 P_i \cdot P_j}{\mu^2} \right) - \log \left( \frac{-P_j^2}{\mu^2} \right) - \log \left( \frac{-P_i^2}{\mu^2} \right) - \log \left( \frac{1}{\mu^2} \left( \frac{P_i^2}{P_j^2} \right) \right), \quad (35)\]

the loop results for $(c, d, e)$ can be expressed as functions of a master loop integral that gives the following form

\[I_1 \left[ (P_i, n_i), (P_j, n_j) \right] = \int \frac{d k}{2 \pi} \frac{1}{k^2 + i \epsilon} \frac{1}{P_i^2/(n_i \cdot P_i) - n_i \cdot k + i \epsilon} \frac{1}{P_j^2/(n_j \cdot P_j) - n_j \cdot k + i \epsilon}. \]
have the wavefunction renormalization factor graphs \([7]\) must also be added to this result. For each effective field \(\xi\) to the gluon vertex \([28]\) and the two gluon feynman rule for the operator. For this diagram one obtains the following results:

\[
\langle \xi_{n_1} \bar{q} A_{\alpha}^{+ \nu} | 0 \rangle = \frac{-g_s^2 C_A C_{\nu}^\sigma n_2 \cdot n_3 I_1 [(P_g, n_3), (P_q, n_2)],}{2}
\]

\[
\langle \xi_{n_1} \bar{q} A_{\alpha}^{+ \nu} | 0 \rangle = \frac{-g_s^2 C_A C_{\nu}^\sigma n_1 \cdot n_3 I_1 [(P_g, n_3), (P_q, n_1)].}{2}
\] (36)

In obtaining these results, one must properly account for the zero bin to avoid introducing IR poles into the anomalous dimension. For this purpose, one can utilize the method outlined in \([29]\). Diagram (e) can be calculated in a manner similar to \([25]\) where the anomalous dimension of \(O^\tau\) was determined. We obtain

\[
\langle \xi_{n_1} \bar{q} A_{\alpha}^{+ \nu} | 0 \rangle = \frac{-g_s^2 (C_F - C_A/2) C_{\nu}^\sigma n_2 \cdot n_2 I_1 [(P_q, n_2), (P_q, n_1)].}{2}
\] (37)

The remaining diagram (f) can be determined by utilizing the full background field three gluon vertex \([28]\) and the two gluon feynman rule for the operator. For this diagram one finds

\[
\langle \xi_{n_1} \bar{q} A_{\alpha}^{+ \nu} | 0 \rangle = \frac{-g_s^2 C_A C_{\nu}^\sigma n_2 \cdot n_3 I_1 [(P_g, n_3), (P_q, n_2)],}{16 \pi^2}
\]

\[
\langle \xi_{n_1} \bar{q} A_{\alpha}^{+ \nu} | 0 \rangle = \frac{-g_s^2 C_A C_{\nu}^\sigma n_1 \cdot n_3 I_1 [(P_g, n_3), (P_q, n_1)]}{16 \pi^2}
\] (38)

The remaining graphs are wave function graphs. For a soft gluon on the external quark field, the wavefunction graph vanishes as \(n_i^2 = 0\). The collinear wavefunction renormalization graphs \([7]\) must also be added to this result. For each effective field \(\xi\), with momenta \(P_i\), we have the wavefunction renormalization factor

\[
Z_{\xi}(P_i, \mu) = 1 - \frac{\alpha_s(\mu) C_F}{4 \pi} \left[ \frac{1}{\epsilon_{uv}} - \log \left( \frac{-P_g^2}{\mu^2} \right) + 1 \right].
\] (39)

Combining these results and adding the corresponding counter term leads to the renormalized perturbative expansion to \(O(\alpha_s)\). The renormalized operator is given by \(O^{(r)}_3 = \sqrt{Z_3 \sqrt{Z_q O^{(0)}_3}} / Z_3\). As we utilize the background field method, the combination \(g A_\mu\) is not renormalized \([28]\). The renormalization factor of the operator is given by

\[
Z_{\xi}(\mu) = 1 + \frac{\alpha_s(\mu) C_F}{4 \pi} \left[ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2}{\epsilon} \left( \log \left( \frac{-P_g^2}{\mu^2} \right) + \log \left( \frac{-P_g^2}{\mu^2} \right) + K [(P_g, n_1), (P_q, n_2)] \right) \right],
\]

\[
+ \frac{\alpha_s(\mu) C_A}{4 \pi} \left[ \frac{1}{\epsilon} K [(P_g, n_1), (P_q, n_2)] - \frac{1}{\epsilon} K [(P_q, n_2), (P_g, n_3)] - \frac{1}{\epsilon} K [(P_q, n_1), (P_g, n_3)] \right],
\]

\[
+ \frac{\alpha_s(\mu) C_A}{4 \pi} \left[ \frac{1}{\epsilon^2} + \frac{1}{\epsilon} - \frac{2}{\epsilon} \log \left( \frac{-P_g^2}{\mu^2} \right) \right].
\] (40)
We scale the results by the one gluon matrix element \( \langle \tilde{O}_3^{(r)} \rangle = \langle \xi_{n_1} \xi_{n_2} A_{n_3}^{\pm \nu} | (O_3^2)^{(r)} | 0 \rangle / C^{\nu \sigma} \).

The perturbative expansion of the operator in the effective theory is

\[
\langle \tilde{O}_3^{(r)} \rangle = 1 + \frac{\alpha_s(\mu) C_F}{4 \pi} \left[ 7 - \frac{5 \pi^2}{6} - \frac{3}{2} \log \left( \frac{P_\perp^2}{\mu^2} \right) - \frac{3}{2} \log \left( \frac{P_\perp^2}{\mu^2} \right) + \log^2 \left( \frac{P_\perp^2}{\mu^2} \right) \right],
\]

\[
- \frac{\alpha_s(\mu) C_F}{4 \pi} K \left[ (P_q, n_1), (P_q, n_2) \right]^2 + \frac{\alpha_s(\mu) C_A}{4 \pi} \left( 2 - \frac{5 \pi^2}{12} - \log \left( \frac{P_\perp^2}{\mu^2} \right) + \log^2 \left( \frac{P_\perp^2}{\mu^2} \right) \right),
\]

\[
+ \frac{\alpha_s(\mu) C_A}{8 \pi} \left( -K \left[ (P_q, n_1), (P_q, n_2) \right]^2 - K \left[ (P_q, n_2), (P_q, n_3) \right]^2 + K \left[ (P_q, n_2), (P_q, n_1) \right]^2 \right). \tag{41}
\]

The anomalous dimension can be determined from the renormalization factor. The scaling for the inner product \( P_i \cdot P_j \) in this operator basis was defined to be \( 2 P_i \cdot P_j \sim M_Z^2 \). Using this scaling, and taking the quark and anti-quark lightcone vectors along the jet direction, we have \( \langle \bar{n}_j \cdot P_j \rangle / \langle \bar{n}_i \cdot P_i \rangle \left( P_i^2 / P_j^2 \right) = \langle n_i \cdot P_i \rangle / \langle n_j \cdot P_j \rangle \). Using these results, we determine

\[
K \left[ (P_i, n_i), (P_j, n_j) \right] = - \log \left[ \frac{\mu^2}{M_Z^2} \right] - \log \left( \frac{P_\perp^2}{\mu^2} \right) - \log \left( \frac{P_\perp^2}{\mu^2} \right) - \log \left[ \frac{P_\perp^2}{\mu^2} \right] - L[(P_i, n_i), (P_j, n_j)] \tag{42}
\]

where

\[
L[(P_i, n_i), (P_j, n_j)] = \log \left[ \frac{2 n_i \cdot P_j + n_i \cdot \bar{n}_j (n_i \cdot P_i)}{2 \langle n_i \cdot n_j \rangle n_j \cdot P_j} \right]. \tag{43}
\]

The anomalous dimension is determined for two stage running to be

\[
\gamma_3(\mu, n_i) = -\frac{\alpha_s(\mu) C_F}{4 \pi} \left[ 4 \log \left( \frac{\mu^2}{M_Z^2} \right) + 6 + 4L[(P_1, n_1), (P_2, n_2)] \right] - \frac{\alpha_s(\mu) C_A}{4 \pi} \left[ 2 \log \left( \frac{\mu^2}{M_Z^2} \right) + 2 \right] - \frac{\alpha_s(\mu) C_A}{4 \pi} \left( L[(P_1, n_1), (P_3, n_3)] + L[(P_2, n_2), (P_3, n_3)] - L[(P_1, n_1), (P_2, n_2)] \right). \tag{44}
\]

Calculating the renormalization of the operators \( O_4, O_5 \) in a similar manner, one obtains the same result without any mixing complications, ie \( \gamma_3 = \gamma_4 = \gamma_5 \). The contribution of each field to the anomalous dimension is the Casimir invariant of the field’s representation times the factor \(-\alpha_s(\mu) \log \left[ \mu^2/M_Z^2 \right] / (2\pi)\). This result for the renormalization scale dependence agrees with \([21, 30]\). The remaining logs depend on labels which give the phase space configuration of the operator.

**B. Running the Three Jet Operators**

We have matched onto the operators \( O_3, O_4, O_5 \) at the scale \( \mu^2 = M_Z^2 \). One can run the Wilson coefficients of the three jet operators using the standard result

\[
\frac{C_i(\mu_2)}{C_i(\mu_1)} = \exp \left[ \int_{\mu_1}^{\mu_2} \frac{d \mu}{\mu} \gamma_3(\mu) \right]. \tag{45}
\]
We retain the logs dependent on \( n_i \) when solving the RGE as they are \( \mathcal{O}(1) \). Consider the logarithms dependent on \( n_i \). Expressing the inner products in terms of the angle between the vectors one finds a divergent log as \( \theta_{12} \to 0 \),

\[
L[(P_1, n_1), (P_2, n_2)] = \log \left[ \frac{2 n_j \cdot P_j + (1 + \cos \theta_{ij}) (n_i \cdot P_i)}{2 (1 - \cos \theta_{ij}) n_j \cdot P_j} \right].
\] (46)

This limit is not of concern for the \( C_F \) phase space logs, as when the gluon becomes collinear with the quark or anti-quark the fermion jets are back to back. The \( C_A \) phase space logs do diverge in this limit.

We find that the Wilson coefficients \( C_i \), for two stage running, are

\[
C_i(\mu, n_i) = \left( \frac{\alpha_s(\mu^2)}{\alpha_s(M_Z^2)} \right)^{\frac{1}{\beta_0} V_1(n_i)} \left( \frac{\mu^2}{M_Z^2} \right)^{\frac{-2 C_F}{\beta_0}} \left( \frac{\mu^2}{M_Z^2} \right)^{\frac{-C_A}{\beta_0}} C_i(M_Z),
\] (47)

where

\[
V_1(M_Z, n_i) = C_F \left( 3 + 2 L[(P_1, n_1), (P_2, n_2)] \right) - \frac{8 \pi (C_F + C_A/2)}{\beta_0 \alpha_s(M_Z^2)} \left( \frac{1 + \cos (2 \delta + \phi)}{1 - \cos (2 \delta + \phi)} \right) + C_A \left( 1 + L[(P_1, n_1), (P_3, n_3)] + L[(P_2, n_2), (P_3, n_3)] - L[(P_1, n_1), (P_2, n_2)] \right).
\] (48)

V. THE MIXING OF \( O_2 \) AND \( O_3 \)

As one runs the jet operators down to a lower scale, eventually the phase space scale defining the jet cone is reached. When one runs below this scale, a third collinear direction can be resolved in the dijet. Consequently the two and three jet operators mix beginning at the scale \( \mu_3^2 \). The following diagrams are defined in the effective theory by taking the limit before the loop integrals are performed. For the interacting collinear directions \( (n_i, n_j) \) this limit is given by

\[
\lim_{\phi \to 0} A \left( n_i \cdot \bar{n}_j, n_i \cdot n_j, n_k \cdot n_{i/j} \right) = \lim_{\phi \to 0} A \left( 1 + \cos (2 \delta + \phi), 1 - \cos (2 \delta + \phi), 1 - \cos (\pi - \delta - \phi/2) \right)
\]

where \( 0 \ll \phi \ll \delta \). This determines the mixing of the two and three jet operators at leading order in \( \lambda \) and fixes a SW jet cone in the effective theory. We find the following mixing for \( O_3 \) and \( O_2 \)

\[
\langle \xi_{n_1} \bar{\xi}_{n_2} | O_3^\sigma | 0 \rangle_k \sim \frac{g_s^2}{4 \pi} C_F 2 \langle \xi_{n_1} \bar{\xi}_{n_2} | O_2^\sigma | 0 \rangle (\mu_3^2) \int \left( \frac{dk}{2 \pi} \right)^d \lim_{\phi \to 0} \frac{1}{\bar{n}_3 \cdot k + i \epsilon} \frac{\bar{n}_1 \cdot P_2 + \bar{n}_1 \cdot k}{(P_{q+k})^2 + i \epsilon} \frac{1}{k^2 + i \epsilon}
\]

\footnote{We thank A. Manohar for conversations on this point.}
where the matrix element requires the following counter term

\[
\langle \xi_{n_1} \bar{\xi}_{n_2} | O_2^g | 0 \rangle (\mu_3^2) C_F g_s^2 \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} - \left( \frac{1}{\epsilon} + 1 \right) \log \left[ \frac{-P_q^2}{\mu_3^2} \right] + \frac{1}{2} \log^2 \left[ \frac{-P_q^2}{\mu_3^2} \right] \right)
\]

\[
+ \langle \xi_{n_1} \bar{\xi}_{n_2} | O_2^g | 0 \rangle (\mu_3^2) C_F g_s^2 \left( 2 - \frac{\pi^2}{12} \right).
\]

(49)

Diagram (g) is given by this result with \( P_q \to P_\bar{q} \). Note that these results agree with the form obtained in Eq. (53). Implementing the zero bin subtractions in these integrals ensures that the divergences present are all UV [29]. The mixing is given by the sum of (g, h)

\[
\langle \xi_{n_1} \bar{\xi}_{n_2} | O_3^g | 0 \rangle^{(0)} (\mu_3^2) = \langle \xi_{n_1} \bar{\xi}_{n_2} | O_3^g | 0 \rangle^{r_1} (\mu_3^2)
\]

\[
= \langle \xi_{n_1} \bar{\xi}_{n_2} | O_2^g | 0 \rangle (\mu_3^2) C_F g_s^2 \left( \frac{2}{\epsilon^2} + \frac{2}{\epsilon} - \left( \frac{1}{\epsilon} + 1 \right) \left( \log \left[ \frac{-P_q^2}{\mu_3^2} \right] + \log \left[ \frac{-P_\bar{q}^2}{\mu_3^2} \right] \right) \right)
\]

\[
+ \langle \xi_{n_1} \bar{\xi}_{n_2} | O_2^g | 0 \rangle (\mu_3^2) C_F g_s^2 \left( 4 - \frac{\pi^2}{6} + \frac{1}{2} \log^2 \left[ \frac{-P_q^2}{\mu_3^2} \right] + \frac{1}{2} \log^2 \left[ \frac{-P_\bar{q}^2}{\mu_3^2} \right] \right).
\]

(50)

This renormalized matrix element is defined by \( \langle \xi_{n_1} \bar{\xi}_{n_2} | O_3^g | 0 \rangle^{(r)} \) being free of UV divergences where

\[
\langle \xi_{n_1} \bar{\xi}_{n_2} | O_3^g | 0 \rangle^{(r)} (\mu_3^2) = \sqrt{Z_{\bar{q}}} \sqrt{Z_\bar{q}} \langle \xi_{n_1} \bar{\xi}_{n_2} | O_3^g | 0 \rangle (\mu_3^2)
\]

\[
= \langle \xi_{n_1} \bar{\xi}_{n_2} | O_2^g | 0 \rangle (\mu_3^2) C_F g_s^2 \left( \frac{2}{\epsilon^2} + \frac{3}{2\epsilon} - \left( \frac{1}{\epsilon} + \frac{3}{4} \right) \left( \log \left[ \frac{-P_q^2}{\mu_3^2} \right] + \log \left[ \frac{-P_\bar{q}^2}{\mu_3^2} \right] \right) \right)
\]

\[
+ \langle \xi_{n_1} \bar{\xi}_{n_2} | O_2^g | 0 \rangle (\mu_3^2) C_F g_s^2 \left( \frac{9}{2} - \frac{\pi^2}{6} + \frac{1}{2} \log^2 \left[ \frac{-P_q^2}{\mu_3^2} \right] + \frac{1}{2} \log^2 \left[ \frac{-P_\bar{q}^2}{\mu_3^2} \right] \right).
\]

(51)

The matrix element requires the following counter term

\[
Z_{3,2}(\mu_3^2) = 1 + \frac{\alpha_s(\mu_3^2) C_F}{2 \pi} \left( \frac{2}{\epsilon^2} + \frac{3}{2\epsilon} - \frac{1}{\epsilon} \left( \log \left[ \frac{-P_q^2}{\mu_3^2} \right] + \log \left[ \frac{-P_\bar{q}^2}{\mu_3^2} \right] \right) \right).
\]

(52)

Taking \( P_q^2 = P_\bar{q}^2 = \delta^2 M_\rho^2 \) we find the off diagonal term of the anomalous dimension matrix,

\[
\gamma_{3,2}(\mu_3) = -\frac{3 \alpha_s(\mu_3) C_F}{2 \pi}.
\]

(53)
For $O_4$ and $O_5$, the mixing of these operators into the two jet operator vanishes at $O(\alpha_s)$. This occurs due to the operator Dirac structure, demonstrating the utility of simplifying the Dirac structure of the three jet operators.

We can now determine the renormalized contribution of the three parton amplitude to the dijet decay rate. The contribution is the finite real part of Eq.(51), given by

$$\langle \xi_{n_1} \overline{\xi}_{n_2} | (O_3^r) | 0 \rangle (\mu_3^2) = \langle \xi_{n_1} \overline{\xi}_{n_2} | O_2^r | 0 \rangle (\mu_3^2) \frac{\alpha_s(\mu_3^2) C_F}{2 \pi} \left( \frac{9}{2} - \frac{7 \pi^2}{6} \right).$$  \tag{54}

VI. PHASE SPACE RENORMALIZATION GROUP

To determine $\Gamma(Z \to J \overline{J})(\mu_Z^2)$ we integrate over the two body phase space in $d$ dimensions. The tree level result is

$$\sigma_0(\epsilon) = \int \frac{d p_1^3}{(2 \pi)^3 (2 E_1)} \frac{d p_2^3}{(2 \pi)^3 (2 E_2)} \sum_{\text{states,pol}} (\langle O_2^r \rangle)^2$$

$$= \frac{N_C}{32 \pi^2} (g_V^2 + g_A^2) [M_Z^{1-2\epsilon} (4\pi)^{2\epsilon} \left( 2 \frac{2-2\epsilon}{3-2\epsilon} \Omega_{3-2\epsilon} \right) + \mathcal{O}(\epsilon)],$$

so that

$$\Gamma(Z \to J \overline{J})(\mu_Z^2) = \frac{N_C M_Z}{12 \pi} (g_V^2 + g_A^2) \left( C_2^{\text{tree}}(\mu_3^2) + \hat{C}_3(\mu_3^2) \frac{\alpha_s(\mu_3^2) C_F}{2 \pi} \left( \frac{9}{2} - \frac{7 \pi^2}{6} \right) \right)^2.$$   \tag{55}

Reexpanding this expression into fixed order perturbation theory, neglecting non logarithmic $O(\alpha_s(M_Z))$ terms, gives

$$\Gamma(Z \to J \overline{J})(M_Z^2) = \frac{N_C M_Z}{12 \pi} (g_V^2 + g_A^2) \left( 1 + \frac{\alpha_s(M_Z^2) C_F}{\pi} \left( -3 \log(\delta) - 2 \log^2(\delta) \right) \right).$$ \tag{56}

Comparing this result to Eq.(11), we see that we produce the purely collinear contribution $-3 \log[\delta]$ correctly, however the double Sudakov log in not correctly reproduced. The reason for this is that only part of the double log is present. We cannot run the collinear degrees of freedom of the dijet operator below the $\mu_3^2$ scale. However, the usoft degrees of freedom run below this scale and are restricted by the SW cuts to lie below the scale $\mu_3^3$. These scales are related by $\mu_3^3 \propto \delta^2 \mu_3^3$. To obtain the correct fixed order perturbative expansion for $\Gamma(Z \to J \overline{J})$, this restriction of the usoft degrees of freedom must be included. Including this restriction, we find the correct expression to be given by Eq.(72).
To determine the correct result, we implement these constraints with a one stage running formalism using a Phase Space Renormalization Group (PSRG). Running in this one stage formalism, the collinear degrees of freedom of the dijet run to the scale $\mu_2^2$; simultaneously, the usoft degrees of freedom run to the scale $\mu_2^2$.

The usoft matrix element for the dijet is defined in [8, 24]. The matrix element depends on the momenta $k$ of the usoft degrees of freedom where

$$\sum_{X_u} |\langle X_u(k)|O_2^\prime|0\rangle|^2 = \frac{S(k)}{(Z_2^2(\mu))^2}.$$  

(58)

We have indicated the required renormalization of the dijet shape function. The usoft radiation has momenta $k = M_Z (\lambda^2, \lambda^2, \lambda^2)$ where $\lambda \sim \sqrt{\Lambda_{QCD}/M_Z}$ and the cuts are taken so that $\delta \gg \Lambda_{QCD}/M_Z$. The cuts being larger than a typical component of an usoft momentum, in the effective theory, no cut restriction holds on the sum over $X_u$. We have

$$S(k) = \frac{1}{N_C} \int \frac{d\xi u}{2\pi} e^{ik \cdot u} \langle 0|T \left[ Y_n d^\dagger Y_n^a \right](\frac{\xi n u}{2}) T \left[ Y_n a^\dagger Y_n c \right](0)|0\rangle.$$  

(59)

using the notation of [8].

The renormalization of the dijet shape function can be determined from $Z_2$. The BPS field redefinition [7] is given for all collinear directions $n$ by

$$\xi_n \rightarrow Y^\dagger_n \xi'_n;$$  

(60)

$$\bar{\xi}_n \rightarrow \xi'_n Y_n;$$  

$$A_n \rightarrow Y^\dagger_n A'_n Y_n;$$  

(61)

where the path-ordered Wilson line of usoft gluons in the $n$ direction is

$$Y_n(z) = P \exp \left[ i g \int_0^\infty ds \, n \cdot A_u(ns + z) \right].$$  

(62)

This field redefinition removes all couplings in the Lagrangian of the redefined usoft and collinear fields ($\xi'_n, \xi'_n, A'_n$). This allows the matrix element to be factorized

$$\sum_{\text{final states}} |\langle J_n J_n X_u|\bar{X}_n \Gamma^\sigma \chi_{nq}(0) \epsilon_\sigma|0\rangle|^2 = \sum_{J_n J_n} |\langle J_n J_n|\bar{X}_n \Gamma^\sigma \chi_{nq}(0) \epsilon_\sigma|0\rangle|^2 \times \sum_{X_u} |\langle X_u|T[Y_n Y'_n](0)|0\rangle|^2.$$  

(63)

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8 The PSRG is inspired by the velocity renormalization group, although some important differences in implementation exist. See [35] for the VRG in NRQCD. The arguments of [36, 37, 38, 39] on the equivalence of one stage and two stage indicate that two stage running could be used to obtain the same results.

9 However, $|X_u|$ is still not a color singlet.
The renormalization of the initial matrix element must be reproduced by the product of the renormalized matrix elements after the field redefinition. Thus $Z_2 = Z_2^c Z_2^u$, where $Z_2^c$ is the renormalization for the matrix element of only collinear degrees of freedom, while $Z_2^u$ gives the renormalization for the usoft matrix element. These renormalization factors are

$$Z_2^c(\mu) = 1 + \frac{\alpha_s(\mu^2)}{2\pi} C_F \left( \frac{2}{\epsilon^2} + \frac{3}{2\epsilon} - \frac{1}{\epsilon} \ln \left[ \frac{p_q^2 p_\bar{q}^2}{\mu^4_c} \right] \right) + O(\alpha_s^2),$$  

(64)

and

$$Z_2^u(\mu) = 1 + \frac{\alpha_s(\mu^2)}{2\pi} C_F \left( - \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \left[ \frac{n \cdot p_q \tilde{n} \cdot p_\bar{q}}{\mu^4_s} \right] \right) + O(\alpha_s^2).$$  

(65)

Note that the convolution of the variable $k$ between the jet and usoft matrix element is neglected as we are considering the total decay rate. The anomalous dimensions of the factorized matrix elements are given by

$$\gamma^c_2(\mu) = -\frac{\alpha_s(\mu^2)}{\pi} C_F \left( - \ln \left[ \frac{p_q^2 p_\bar{q}^2}{\mu^4_c} \right] + \frac{3}{2} \right) + O(\alpha_s^2),$$  

(66)

and

$$\gamma^s_2(\mu) = -\frac{\alpha_s(\mu^2)}{\pi} C_F \left( \ln \left[ \frac{n \cdot p_q \tilde{n} \cdot p_\bar{q}}{\mu^4_s} \right] \right) + O(\alpha_s^2).$$  

(67)

The phase space scales that we will run the usoft and collinear factorized matrix elements to are related by $\mu^2_\beta = B^{\delta-1} \mu^4_\beta/(M^2_Z)$. We relate both $\mu_s$ and $\mu_c$ to a phase space parameter $\phi$ that runs from 1 to $\delta$ using

$$\mu^2_c = \phi^2 M^2_Z,$$

$$\mu^2_s = \phi^4 M^2_Z B^{(\phi^{-1})}.$$  

(68)

We can express the running of $C_2$ in the PSRG as

$$\phi \frac{d}{d\phi} C_2(\phi) = (\gamma^c_2(\phi) + \left( 2 + \phi \log(B) \right) \gamma^s_2(\phi)) C_2(\phi).$$  

(69)

We take the invariants to be $P^2_q, P^2_\bar{q} = \delta^2 M^2_Z$ and $n \cdot p_q \tilde{n} \cdot p_\bar{q} = f(B) \delta^4 M^2_Z$ giving

$$\phi \frac{d}{d\phi} C_2(\phi) = -\frac{\alpha_s(M_Z \phi)}{\pi} C_F \left( \frac{3}{2} - 2 \log \left[ \frac{\delta^2}{\phi^2} \right] \right) C_2(\phi)$$

$$- \alpha_s(M_Z \phi) C_F \left( 2 + \phi \log(B) \right) \left( \log \left( \frac{f(B)\delta^4}{\phi^4 B^{\phi^{-1}}} \right) \right) C_2(\phi).$$  

(70)

(71)

The particular functional dependence on $B$ in the logarithm is fixed by matching onto the full theory. Note that we must refer to QCD with the SW jet definition at the collinear scale.
as we matched the amplitude in QCD onto our SCET operator basis. As this matching was performed before the phase space integrals are performed we must refer to the full theory to establish the jet definition at the collinear scale. As in SW, we expand in the dependence on $B$ and $\delta$ and neglect the dependence on these parameters that does not contribute to large logarithms.

This RGE is solved using standard methods to give

$$
C_{SW}^2(\mu^2, \mu_3^2) = C_2(M_Z) \left( \frac{\alpha_s(\delta^2 M_Z^2)}{\alpha_s(M_Z^2)} \right)^{\frac{C_F}{\beta_0}} \left( \frac{3+16\pi}{\beta_0} \alpha_s(\delta^2 M_Z^2) \right)^{\frac{1}{2}} \left( \delta^2 \right)^{\left( \frac{\delta}{\delta_0} \right)}.
$$

(72)

The contributions from the running of the three jet operators contribute at $O(\alpha_s^2)$. We choose to retain two stage running for the three jet contribution to the Sudakov resumed dijet decay rate.

VII. TWO JET DECAY RATE

Using $C_{SW}^2(\mu_3^2, \mu_3^2)$ we obtain the total decay rate

$$
\Gamma(Z \rightarrow J \bar{J})(\mu_3^2, \mu_3^2) = \int \sum_{\text{states}} |\langle O_2^2 \rangle|^2(\mu_3^2) \left( C_{SW}^2(\mu_3^2, \mu_3^2) + \frac{\alpha_s(\mu_3^2) C_3 C_F}{2 \pi} \frac{9}{2} - \frac{7 \pi^2}{6} \right)^2.
$$

(73)

where we have indicated the dependence on the perturbative expansion of $|\langle O_2^2 \rangle|^2$. The contributions to this expression at $O(\alpha_s)$ come from the running of $O_2$ in the PSRG and the mixing of $O_3$ and $O_2$ beginning at the collinear scale. This is sufficient to resum the double Sudakov logarithm and the class of sub leading logs we are interested in. The running of $O_3$ contributes logarithms that first contribute at $O(\alpha_s^2)$. To extend the resummation consistently to sub-leading logarithms, the running of $O_3$ must be reexpressed in the PSRG. This is beyond the scope of this paper.

The renormalized perturbative expansion of $|\langle O_2^2 \rangle|^2$ is IR finite, as the SW jet definition was constructed to satisfy the KLN theorem [32, 33]. In our effective theory construction, the KLN theorem for the SW states ensures that the dijet partial cross section is free of explicit IR singularities, as the IR of the full theory is reproduced in the effective theory.

The expansion of the operator gives the collinear gluon emission contained within the dijet cones. However, the fixing of the SW definition within the effective theory removes gluons that are excluded by the SW jet definition from the jet cones. This results in constant
terms due to the degrees of freedom removed by the jet definition that have to be accounted for by matching in the effective theory. We introduce the phase space Wilson coefficient $S_2(\delta^2 M_Z)$ to account for this further matching onto the jet definition in the full theory. As a result, we have

$$C_2^{SW}(\mu^2, \mu^2) = C_2(M_Z) S_2(\delta^2 M_Z) \left( \frac{\alpha_s(\delta^2 M_Z^2)}{\alpha_s(M_Z^2)} \right) \frac{C_F}{\pi} \left( 3 + \frac{16 \pi}{3} \right) \left( \frac{4 \pi^2}{\alpha_s} \right).$$

(74)

We determine $S_2(\delta^2 M_Z)$ by calculating the SW dijet decay rate in full QCD, and in SCET with the PSRG. We match onto the SW jet definition in the full theory to determine the perturbative expansion of $S_2(\delta^2 M_Z)$.

Computing the SCET graphs in dimensional regularization we find the following $O(\alpha_s)$ corrections to the matrix element $^{10}$

$$\int \sum_{states} |O^2_2| \sigma_0(\epsilon) \left( 1 + \frac{1}{2} \frac{Z_s^2(\mu^2)}{Z_s^2(\mu^2)} \right) \int_0^1 dx_1 \int_0^1 dx_2 \int_{(1-x_1)}^{1} S(x_1, x_2, \mu^2) A(x_1, x_2) \right) \right)$$

(75)

with

$$S(x_1, x_2, \mu^2) = \frac{\alpha_s(\mu^2)}{2 \pi} \frac{1}{\Gamma(1-\epsilon)(1-x_1)^{\epsilon} (1-x_2)^{\epsilon} (x_1 + x_2 - 1)^{\epsilon}},$$

$$A(x_1, x_2) = \frac{1 + x_2^2}{(1-x_1)(1-x_2)} - \frac{1 + x_2}{(1-x_1)} - \frac{1 - \epsilon}{2(1-x_1)}. \right)$$

(76)

Integrating these expressions we find

$$\int \sum_{states} |O^2_2| \sigma_0(\epsilon) \left( 1 + \frac{\alpha_s(\mu^2)}{2 \pi} \frac{C_F}{\pi} \left( \frac{23}{2} - \frac{7 \pi^2}{6} \right) \right).$$

(77)

To check this result we re-expand the Sudakov factors out into fixed order perturbation theory in terms of $\alpha_s(M_Z)$, we find

$$\Gamma(Z \to J\bar{J})(M_Z) = \frac{N_C M_Z (g_\mu^2 + g_\lambda^2)}{12 \pi} \left( 1 + \frac{\alpha_s(M_Z)}{\pi} \left[ -3 \ln \delta - 4 \ln(f(\mathcal{B} \delta)) \ln(\delta) \right] \right)$$

(78)

$$+ \frac{N_C M_Z (g_\mu^2 + g_\lambda^2)}{12 \pi} \left( 1 + \frac{\alpha_s(M_Z)}{\pi} \left[ -\frac{57}{4} - \frac{7 \pi^2}{3} \right] + 2 \alpha_s(M_Z) s_1 \right) + O(\alpha_s^2).$$

We match onto the SW dijet decay rate to obtain $f(\mathcal{B}) \delta \equiv 2 \beta$. The phase space Wilson coefficient $S_2(\delta^2 M_Z)$ has the perturbative expansion

$$S_2(M_Z) = 1 + \alpha_s(M_Z) s_1 + O(\alpha_s^2),$$

(79)

$^{10}$ See [29] for details on these effective theory calculations and the requirement of zero bin subtractions. Note that as we are integrating over the full phase space. To avoid double counting we find the zero bin subtracted contribution to be $R_F + R_B - R_A - 1/2 R_C - 1/2 R_E$ in the notation of [29].
where
\[ s_1 = C_F \left( \pi - \frac{47}{8} \pi \right). \] (80)

This is the main result of our paper. The Sudakov resumed dijet decay rate in SCET is given by
\[ \Gamma(Z \rightarrow J \bar{J}) (\mu^2_\beta, \mu^2_\delta) = \sigma_0 \left( 1 + \frac{\alpha_s(\mu^2_\delta)}{2 \pi} \left( \frac{23}{2} - \frac{7 \pi^2}{6} \right) C_F \right) \]
\[ \times \left( C^\text{SW}_2 (\mu^2_\beta, \mu^2_\delta) + \frac{\alpha_s(\mu^2_\delta)}{2 \pi} \left( \frac{9}{2} - \frac{7 \pi^2}{6} \right) C^3_C C_F \right)^2. \] (81)

Comparing to the literature, we note that a leading log resummation was determined in [5] in full QCD for the SW jet definition. The exponentiation of the double Sudakov logarithm agrees in both formalisms. Comparing the results, we consider the resummation in SCET with the PSRG to be conceptually clearer and easier to extend to non leading logarithms.

VIII. CONCLUSIONS

We have determined the RGE improved decay rate \( \Gamma(Z \rightarrow J \bar{J}) \) using SCET. Recalling the SW jet definition, we can now state the corresponding definition of the dijet final states in our effective theory.

The dijet decay rate comes from the direct matching of \( O^2_2 \) onto QCD and from the mixing of the three jet operators at the scale defining the angular cut \( \mu^2_\delta \). These contributions correspond to the energetic quarks of SW1 and the gluons of collinear energy emitted inside the jets, SW3.

The nonperturbative corrections given by matrix elements of the ultrasoft final state gluons are restricted by the scale \( \mu^2_\beta \). These final state gluons correspond to the emission of gluons unrestricted in direction with low energy. The ultrasoft matrix element with this restriction corresponds to the state SW2.

Treating the decay rate in SCET allowed the dijet decay amplitude to be defined in terms of SCET jet fields. The EFT formalism allowed this definition to take place without large logarithms. The fraction of three parton events contributing to the dijet rate in the SW jet definition, corresponds to the three jet operators mixing into the dijet operators at the scale.
\( \mu_3^2 \), and the perturbative expansion of \( O_2^\sigma \) at the scale \( \mu_3^2 \). The general version of this result is that \( n + i \) jet operators mix into \( n \) jet cross sections and decay rates at order \( \alpha_i^s(\mu_3^2) \).

To obtain the resummed dijet decay rate, we introduced a one stage running formalism for the dijet operator. This allowed us to run to the phase space cut scales defining the dijet corresponding to the SW jet definition. We also introduced phase space Wilson coefficients. These are required to fix the SW jet definition onto the jet operators in SCET.

This approach establishes a jet definition in SCET and resums the large phase space logarithms that come about as the result of the jet definition. This allows a program of systematically improving the perturbative behavior of jet observables to be carried out in SCET.

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As this paper was approaching completion the work [40], which is a more detailed accounting of the work reported on in [30] appeared. In [40] the calculation of the renormalization of three jet operators equivalent to the three jet operators examined in this paper, Eq. (10), was reported. We find that [40], which was not calculated in the background field method, is inconsistent in its inclusion of the wavefunction renormalization of the gluon while not including the QCD coupling constant renormalization. Once [30, 40] is corrected for this inconsistency, their results agree with ours.

**APPENDIX A: FEYNMAN RULES OF THE 3-JET OPERATORS**

For completeness, we state the one and two gluon Feynman rules of the operators \( O_3, O_4, O_5 \) to \( \mathcal{O}(\lambda^0 M_Z^2) \) in this Appendix, we utilize these rules to renormalize the operators in the Section IV A. The following Feynman rules are for emitted collinear gluons.

The one gluon Feynman rules where the gluon is emitted in the direction \( n_3 \) requires the collinear gluon field strength to emit a gluon. Here \( P_g, P_q, P_{\bar{q}} \) are outgoing collinear
momentum:

\[
\langle \xi_{n_1} \xi_{n_2} A_{n_3}^\mu | O_3^\sigma | 0 \rangle = \xi_{n_1} 2 g_s T^a \left( \frac{n_1^\mu}{(n_1 \cdot n_3)(\bar{n}_3 \cdot P_g)} - \frac{n_2^\mu}{(n_2 \cdot n_3)(\bar{n}_3 \cdot P_g)} \right) \Gamma^\sigma \xi_{n_2},
\]

\[
\langle \xi_{n_1} \xi_{n_2} A_{n_3}^\mu | O_4^\sigma | 0 \rangle = \xi_{n_1} g_s T^a \left( \frac{1}{(n_1 \cdot n_3)(\bar{n}_1 \cdot P_q)} - \frac{1}{(n_2 \cdot n_3)(\bar{n}_2 \cdot P_q)} \right) (n_3^\sigma \Gamma^\mu - n_3 \cdot \Gamma g^{\sigma \mu}) \xi_{n_2},
\]

\[
\langle \xi_{n_1} \xi_{n_2} A_{n_3}^\mu | O_5^\sigma | 0 \rangle = \xi_{n_1} g_s T^a \left( \frac{1}{(n_1 \cdot n_3)(\bar{n}_1 \cdot P_q)} - \frac{1}{(n_2 \cdot n_3)(\bar{n}_2 \cdot P_q)} \right) i e^{\alpha \beta \sigma \eta} (n_3^\beta \gamma_5 \Gamma_\eta) \xi_{n_2}.
\]

The two gluon Feynman rules with \( P_{g1} \) and \( P_{g2} \) the two outgoing collinear gluon momenta, associated with \( A_{g1}^{\mu} \) and \( A_{g2}^{\mu} \) are

\[
\langle \xi_{n_1} \xi_{n_2} A_{n_3}^{\alpha, \mu} A_{n_3}^{\nu, \beta} | O_3^\sigma | 0 \rangle = \xi_{n_1} \frac{2 g_s^2 (T^a T^b)}{(n_3 \cdot P_{g1})(n_3 \cdot P_{g2})} \left[ \frac{n_3^\nu n_1^\mu - n_1^\mu n_3^\nu}{n_1 \cdot n_3} - \frac{n_3^\mu n_2^\nu - n_2^\nu n_3^\mu}{n_2 \cdot n_3} \right] \Gamma^\sigma \xi_{n_2}
+ \xi_{n_1} \frac{2 g_s^2 (T^b T^a)}{(n_3 \cdot P_{g1})(n_3 \cdot P_{g2})} \left[ \frac{n_3^\nu n_1^\mu - n_1^\mu n_3^\nu}{n_1 \cdot n_3} - \frac{n_3^\mu n_2^\nu - n_2^\nu n_3^\mu}{n_2 \cdot n_3} \right] \Gamma^\sigma \xi_{n_2}
+ \xi_{n_1} \frac{4 g_s^2 f^{abc} T_c (n_2^\mu n_1^\nu - n_1^\mu n_2^\nu)}{(n_3 \cdot (P_{g1} + P_{g2}))^2 (n_1 \cdot n_3)(n_2 \cdot n_3)} \Gamma^\sigma \xi_{n_2} \tag{A1}
\]

\[
\langle \xi_{n_1} \xi_{n_2} A_{n_3}^{\alpha, \mu} A_{n_3}^{\nu, \beta} | O_4^\sigma | 0 \rangle = -\xi_{n_1} \frac{g_s^2 i f^{abc} T_c n_3^\nu}{n_3 \cdot P_{g2}} \left( \frac{n_3^\sigma \Gamma^\nu - n_3 \cdot \Gamma g^{\sigma \nu}}{(n_1 \cdot n_3)(\bar{n}_1 \cdot P_q)} - \frac{n_3^\sigma \Gamma^\mu - n_3 \cdot \Gamma g^{\sigma \mu}}{(n_2 \cdot n_3)(\bar{n}_2 \cdot P_q)} \right) \xi_{n_2}
+ \xi_{n_1} \frac{g_s^2 i f^{abc} T_c n_3^\nu}{n_3 \cdot P_{g2}} \left( \frac{n_3^\sigma \Gamma^\nu - n_3 \cdot \Gamma g^{\sigma \nu}}{(n_1 \cdot n_3)(\bar{n}_1 \cdot P_q)} - \frac{n_3^\sigma \Gamma^\mu - n_3 \cdot \Gamma g^{\sigma \mu}}{(n_2 \cdot n_3)(\bar{n}_2 \cdot P_q)} \right) \xi_{n_2}
+ \xi_{n_1} \frac{2 g_s^2 f^{abc} T_c n_3^\nu}{(n_3 \cdot (P_{g1} + P_{g2})) (n_1 \cdot n_3)(n_2 \cdot n_3)} \Gamma^\mu g^{\sigma \nu} - \Gamma^\nu g^{\sigma \mu} \right) \xi_{n_2} \tag{A2}
\]

\[
\langle \xi_{n_1} \xi_{n_2} A_{n_3}^{\alpha, \mu} A_{n_3}^{\nu, \beta} | O_5^\sigma | 0 \rangle = \xi_{n_1} g_s^2 i f^{abc} T_c \left( i e^{\alpha \beta \sigma \eta} (n_3^\beta \gamma_5 \Gamma_\eta) \right) \left( \frac{g_{\alpha \mu} n_3^\nu}{\bar{n}_3 \cdot P_{g1}} - \frac{g_{\alpha \mu} n_3^\nu}{\bar{n}_3 \cdot P_{g2}} \right) \xi_{n_2}
\times \left( \frac{1}{(n_1 \cdot n_3)(\bar{n}_1 \cdot P_q)} - \frac{1}{(n_2 \cdot n_3)(\bar{n}_2 \cdot P_q)} \right) \xi_{n_2}
+ \xi_{n_1} \frac{2 g_s^2 f^{abc} T_c}{(n_3 \cdot (P_{g1} + P_{g2}))} \left( i e^{\alpha \beta \sigma \eta} (n_3^\beta \gamma_5 \Gamma_\eta) \right) \left( \frac{1}{(n_1 \cdot n_3)(n_1 \cdot P_q)} - \frac{1}{(n_2 \cdot n_3)(n_2 \cdot P_q)} \right) \xi_{n_2} \tag{A3}
\]

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