ALMOST COMMUTING VARIETIES FOR THE SYMPLECTIC LIE ALGEBRAS

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Abstract. In this note we introduce and study the almost commuting varieties for the symplectic Lie algebras.

1. Introduction

The commuting schemes of semisimple Lie algebras is a classical object of study in Lie theory and Invariant theory. Let \( g \) be a semisimple Lie algebra. Consider the subscheme \( C_2(g) \subset g^{\otimes 2} \) defined by the equation \([x, y] = 0\). This is the commuting scheme for \( g \). Note that we can define \( C_k(g) \subset g^{\otimes k} \) analogously but in this note we restrict ourselves to the case of \( k = 2 \).

The algebro-geometric properties of \( C_2(g) \) are largely unknown. It is known that this scheme is irreducible, \([R]\). But it is not known whether \( C_2(g) \) is reduced (or whether \( C_2(g) \) with reduced scheme structure is normal) even in the case of \( g = sl_n \). However, for \( g = sl_n \) there is a related variety, the almost commuting variety. The version we need was introduced and studied in \([GG]\).

We now recall the definition. Consider the vector space \( R := sl_n^{\otimes 2} \oplus \mathbb{C}^n \oplus (\mathbb{C}^n)^* \). We write \((x, y, i, j)\) for a typical element of \( R \). We have the subscheme \( M_n \subset R \) defined by \([x, y] + ij = 0\). The following is the main result of \([GG]\).

**Theorem 1.1.** (Theorem 1.1.2 in \([GG]\)). The scheme \( M_n \) is a reduced complete intersection. It has \( n + 1 \) irreducible components, all of which have dimension \( n^2 + 2n - 2 \).

The goal of this note is to define the analog of \( M_n \) for the symplectic Lie algebra \( g = sp_{2n} \), study its properties and apply to the study of the categorical quotient \( C_2(g)/\!/G \) (where \( G = Sp_{2n} \)).

We will need a notation. Let \( C^{2n} \) be the tautological representation of \( sp_{2n} \). We can identify \( S^2(C^{2n}) \) with \( sp_{2n} \) in the standard way (the Lie bracket on \( S^2(C^{2n}) \) is the restriction of the Poisson bracket). So for \( i \in C^{2n} \) we can view \( i^2 \in S^2(C^{2n}) \) as an element of \( sp_{2n} \). The almost commuting scheme for \( sp_{2n} \) is defined by

\[
X_n = \{(x, y, i) \in sp_{2n}^{\otimes 2} \oplus C^{2n} | [x, y] + i^2 = 0\}.
\]

Here is our main result concerning \( X_n \). Somewhat surprisingly, its algebro-geometric properties are even better than those of \( M_n \):

**Theorem 1.2.** The scheme \( X_n \) is a reduced complete intersection of dimension \( 2n^2 + 3n \). It is irreducible.

Here is an application of \( X_n \) to the study of the commuting scheme \( C_2(g) \) (with \( g = sp_{2n} \)). Note that \( G \) naturally acts on \( X_n \). Let \( \mathfrak{h}, W \) be the Cartan subalgebra and the Weyl group of \( g \).
**Theorem 1.3.** We have scheme isomorphisms $X_n//G \isom C_2(\mathfrak{g})//G \isom \mathfrak{h}^\oplus//W$, where $W$ acts on $\mathfrak{h}^\oplus$ diagonally.

Note that the isomorphism $C_2(\mathfrak{g})//G \isom \mathfrak{h}^\oplus//W$ is a special case (for $k = 2$) of the main result of [CN].

**Acknowledgements.** I would like to thank Tsao-Hsien Chen for his talk on [CN] at Yale which motivated this work. My work was partially supported by the NSF under grant DMS-2001139.

2. Properties of almost commuting variety

2.1. Upper triangularity property. Here we prove the following easy result.

**Lemma 2.1.** Let $(x, y, i) \in X_n$. Then there is a Borel subalgebra of $\mathfrak{sp}_{2n}$ containing both $x$ and $y$.

**Proof.** We note that $\text{rk}[x, y] = \text{rk}(i^2) \leq 1$. So $x, y$ have a common eigenvector, see, e.g., [EG] Lemma 12.7. Denote this vector by $v$ and let $v^\perp$ denote its skew-orthogonal complement. The subspace $v^\perp$ is $x$- and $y$-stable. Let $x_1, y_1$ be the operators on $v^\perp/\mathbb{C}v$ induced by $x, y$. Then we have $\text{rk}([x_1, y_1]) \leq \text{rk}([x, y]) \leq 1$. So we can argue by induction to show that $x, y$ preserve a lagrangian flag. Equivalently, they are contained in some Borel subalgebra of $\mathfrak{g}$.

Here is a standard corollary.

**Corollary 2.2.** Let $(x, y, i) \in X_n$ be such that $G(x, y, i)$ is closed. Then $[x, y] = 0, i = 0$ and $x, y$ are semisimple. Conversely, the orbit of such a triple is closed.

**Remark 2.3.** One can ask for which simple Lie algebras $\mathfrak{g}$ the claim Lemma 2.1 holds, where we consider the condition $[x, y] \in \mathfrak{O}_{\text{min}}$ for the minimal nilpotent orbit $\mathfrak{O}_{\text{min}}$. By [EG] Lemma 12.7, it holds for $\mathfrak{g} = \mathfrak{sl}_n$. And it also holds for $\mathfrak{g} = \mathfrak{sp}_{2n}$. In fact, it does not hold for any other simple Lie algebra. Indeed, $\dim \mathfrak{O}_{\text{min}} > 2 \dim \mathfrak{h}$ for all $\mathfrak{g}$ different from $\mathfrak{sl}_n, \mathfrak{sp}_{2n}$. Assuming the inequality holds, consider a regular element $x \in \mathfrak{h}$. For any element $z \in \mathfrak{O}_{\text{min}} \cap \mathfrak{h}^\perp$ we can find $y \in \mathfrak{g}$ with $[x, y] = z$. On the other hand, if $x, y$ lie in a Borel subalgebra $\mathfrak{b}$, then $\mathfrak{b}$ is one of $\mathfrak{h}$ Borel subalgebras of $\mathfrak{g}$ containing $x$. For any such $\mathfrak{b}$, we have $\dim(\mathfrak{O}_{\text{min}} \cap \mathfrak{b}) = \frac{1}{2} \dim \mathfrak{O}_{\text{min}}$, see, e.g., [CG] Theorem 3.3.7. If $\dim(\mathfrak{O}_{\text{min}} \cap \mathfrak{h}^\perp) > \dim(\mathfrak{O}_{\text{min}} \cap \mathfrak{b})$, then we can find $y$ such that $x$ and $y$ do not lie in the same Borel subalgebra but $[x, y] \in \mathfrak{O}_{\text{min}}$.

2.2. Local structure. The goal of this section is to describe the structure of $X_n$ near a closed $G$-orbit. Recall that the closed orbits are described by Corollary 2.2.

Let $p := (x, y, 0)$ be a point with closed $G$-orbit. Then $x, y$ are commuting semisimple elements. The common centralizer, $L$, of $x$ and $y$ is a Levi subgroup in $G$, hence has the form $\prod_{i=1}^k GL_{n_i} \times \mathfrak{Sp}_{2n_0}$ for a partition $n = n_0 + n_1 + \ldots + n_k$ into the sum of positive integers. Consider the subscheme $\mathbb{C}^{2k} \times \prod_{i=1}^k M_{n_i} \times X_{n_0} \subset \mathbb{C}^{2k} \oplus \mathbb{C}^{2n}$, where $M_{n_i}$ was defined in Section A and $\mathbb{C}^{2k}$ is identified with $\mathfrak{u}(l)^{\oplus 2}$. It comes with an action of $L$. We can form the homogeneous bundle $G \times_L (\mathbb{C}^{2k} \oplus \prod_{i=1}^k M_{n_i} \times X_{n_0})$. Let $X^n_{n_0\mathfrak{c}G\mathfrak{p}}$ denote the spectrum of the completion of $\mathbb{C}[X_n]$ at the ideal of all functions vanishing at $Gp$. Similarly, we can consider the scheme $\left(G \times_L (\mathbb{C}^{2k} \times \prod_{i=1}^k M_{n_i} \times X_{n_0})\right)^{\wedge_G//L}$, here we complete with respect to the ideal of all functions on $G \times_L (\mathbb{C}^{2k} \times \prod_{i=1}^k M_{n_i} \times X_{n_0})$ vanishing at the orbit $G \times L \{0\}$. 
Lemma 2.4. We have a $G$-equivariant scheme isomorphism

$$X_n^G \cong \left( G \times L \left( \mathbb{C}^{2k} \times \prod_{i=1}^{k} M_{n_i} \times X_{n_0} \right) \right)^{\wedge_{G/L}}.$$  

Proof. Note that the action of $G$ on $\mathfrak{g}^{\oplus 2} \oplus \mathbb{C}^{2n}$ is Hamiltonian with moment map $\mu : \mathfrak{g}^{\oplus 2} \oplus \mathbb{C}^{2n} \to \mathfrak{g}$ given by $\mu(x, y, i) = [x, y] + i^2$. So we can apply the main result of [L] to this action and the point $p$. Note that this result is stated in [L] for neighborhoods in the usual topology, but it works for formal neighborhoods as well. We get the isomorphism we have $\mu$ locus of the moment map in $(\mathfrak{g}^{\oplus 2} \oplus \mathbb{C}^{2n})^G$. On the other side, we have $\left( T^*(G \times L) \times \mathbb{C}^{2n} \right)^{\wedge_{G/L}}$ with natural symplectic form and moment map. The zero locus of the moment map in $(\mathfrak{g}^{\oplus 2} \oplus \mathbb{C}^{2n})^G$ (as a scheme) is $X_n^G$. The analogous locus in $\left( T^*(G \times L) \times \mathbb{C}^{2n} \right)^{\wedge_{G/L}}$ is $\left( G \times L \left( \mathbb{C}^{2k} \times \prod_{i=1}^{k} M_{n_i} \times X_{n_0} \right) \right)^{\wedge_{G/L}}$. This yields the required isomorphism. □

Remark 2.5. Using Lemma [2.4] and the fact that $M_n$ is not irreducible, one sees that $X_n$ is not normal.

2.3. Dimension bound. Here we are going to prove a technical lemma. Consider the map $\pi : X_n \to \mathfrak{g}$ given by projection to the first component. Let $\mathcal{O} \subset \mathfrak{g}$ be an adjoint orbit.

Lemma 2.6. We have $\dim \pi^{-1}(\mathcal{O}) < 2n^2 + 3n$.

Proof. Fix $x \in \mathcal{O}$. We need to show that $X_{n,x} := \{(y, i)|[x, y] = i^2\}$ has dimension less then $\dim \mathfrak{g}(x) + 2n$. Also consider the varieties

$$X_{n,x} := \{(y, z)|[x, y] = z, z \in \mathcal{O}_{\min}\}, Y_{n,x} := \mathcal{O}_{\min} \cap [\mathfrak{g}, x].$$

We have the natural maps $\rho_1 : X_{n,x} \hookrightarrow Y_{n,x}$, $\rho_2 : X_{n,x} \to Y_{n,x}$, $(y, z) \mapsto z$. We note that $\rho_1$ is finite, while $\rho_2$ is an affine bundle with fiber of dimension $\dim \mathfrak{g}(x)$. So we reduce to proving that $\dim Y_{n,x} < 2n = \dim \mathcal{O}_{\min}$, equivalently, that $Y_{n,x} \neq \mathcal{O}_{\min}$, equivalently, $\mathcal{O}_{\min} \not\subset [\mathfrak{g}, x]$. Note that $\mathcal{O}_{\min}$ is $G$-stable. So if $\mathcal{O}_{\min} \subset [\mathfrak{g}, x]$, then $[\mathfrak{g}, x]$ contains a nonzero $G$-stable subspace. Since $\mathfrak{g}$ is a simple Lie algebra, this is impossible. This contradiction finishes the proof. □

Remark 2.7. Note that a direct analog of this lemma holds for $M_n$: if $\pi$ denotes the projection $(x, y, i, j) \mapsto x$, then $\dim \pi^{-1}(\mathcal{O}) < n^2 + 2n - 2$. The proof essentially repeats that of Lemma [2.6].

2.4. Proof of Theorem [1.2]. The proof requires two technical statements. Consider the regular semisimple locus $\mathfrak{g}_{reg} \subset \mathfrak{g}$ and set $X_{n,reg} := \pi^{-1}(\mathfrak{g}_{reg})$. This is an open subscheme in $X_n$.

1. We will show that $X_{n,reg}$ is dense in $X_n$.
2. We will show that $X_{n,reg}$ is irreducible.

These two statements are proved in the lemmas below. After that we will easily finish the proof of Theorem [1.2].

Lemma 2.8. $X_{n,reg}$ is dense in $X_n$. 
Proof. The subscheme $X_n$ is defined by $\dim \mathfrak{g} + 2n$. So the dimension of every irreducible component of $X_n$ is at least $\dim \mathfrak{g} + 2n$. For the sake of contradiction, let $Z$ be a component that does not intersect $X_n^{reg}$. Let $p := (x, y, 0)$ be a point in a closed orbit in $Z$ that maps to a Zariski generic point in the image of $Z$ in $\mathfrak{g}//G$ (via $Z \xrightarrow{\sim} \mathfrak{g} \rightarrow \mathfrak{g}//G$). Let $L$ be the centralizer of $x$ and $y$. Recall, Lemma 2.4 that locally near $Gp$ the scheme $X_n$ looks like $(G \times L (\mathbb{C}^{2k} \times \prod_{i=1}^{k} M_{n_i} \times X_{n_0}))^{^\wedge G/L}$. The stabilizer of every closed orbit in $Z$ contains a conjugate of $L$ by the construction of the latter. It follows that the image of $(G \times L (\mathbb{C}^{2k} \times \prod_{i=1}^{k} M_{n_i} \times X_{n_0}))^{^\wedge G/L} \cap Z$ under taking the categorical quotient lies in $(\mathbb{C}^{2k})^{^\wedge 0}$. So the intersection of $Z$ with $(\prod_{i=1}^{k} M_{n_i} \times X_{n_0})^{^\wedge 0}$ lies in the nilpotent locus. Using Lemma 2.6 (for the $X_{n_0}$-factor) and Remark 2.7 (for the $M_{n_i}$-factor) we conclude that the dimension of the intersection does not exceed $\sum_{i=1}^{k} (n_i^2 + 2n_i - 2 - 1) + 2n_0^2 + 3n_0 - 1$. Therefore the dimension of $Z$ does not exceed

$$\dim G/L + 2k + \sum_{i=1}^{k} (n_i^2 + 2n_i - 2 - 1) + 2n_0^2 + 3n_0 - 1 = \dim \mathfrak{g} + 2n - 1.$$ 

This contradicts the observation that $\dim Z \geq \dim \mathfrak{g} + 2n$. \hfill \Box

Lemma 2.9. The scheme $X_n^{reg}$ is irreducible.

Proof. Recall that $\mathfrak{g}^{reg} = G \times N_G(T) \mathfrak{h}^{reg}$. Set $X_n^0 = \{(x, y, i) | x \in \mathfrak{t}^{reg}, [x, y] = i^2\}$ so that $X_n^{reg} = G \times N_G(T) X_n^0$. We need to show that the Weyl group $W = N_G(T)/T$ acts transitively on the irreducible components of $X_n^0$. Set $Y_n = \{i \in \mathbb{C}^{2n} | i^2 \in \mathfrak{h}^\bot\}$. We have a forgetful map $X_n^0 \rightarrow Y_n \times \mathfrak{h}^{reg}$ forgetting $y$. This map is an affine bundle. So we need to show that $W$ transitively acts on the set of irreducible components of $Y_n$. Let $p_1, \ldots, p_n, q_1, \ldots, q_n$ be Darboux coordinates on $\mathbb{C}^{2n}$. Then $Y_n$ is given by

$$\{(p_1, \ldots, p_n, q_1, \ldots, q_n) | p_i q_i = 0, \forall i = 1, \ldots, n\}.$$ 

So there are $2^n$ irreducible components of $Y_n$: we need to choose if $p_i = 0$ or $q_i = 0$ for all $i = 1, \ldots, n$. It is clear that $W$ permutes these components transitively – in fact, $\{\pm 1\}^n \subset W$ acts simply transitively on them. \hfill \Box

Proof of Theorem 1.3. Lemmas 2.8 and 2.9 imply that $X_n$ is irreducible. Note that $\mu$ is a submersion at points with free $G$-orbit. There is a point in $X_n$ with free orbit: for example we can take $(x, 0, i)$, where $x \in \mathfrak{h}^{reg}$ and $i$ is given by (in the notation of the proof of Lemma 2.9) by $p_1 = \ldots = p_n = 1, q_1 = \ldots = q_n = 0$. It follows that $\dim X_n = \dim \mathfrak{g} + 2n$ and that $X_n$ is generically reduced. Since $X_n$ is a complete intersection, we see that $X_n$ is reduced. \hfill \Box

3. Application to commuting scheme

The goal of this section is to prove Theorem 1.3. Namely, we have inclusions $\mathfrak{h}^{\otimes 2} \hookrightarrow C_2(\mathfrak{g}) \hookrightarrow X_n$ that give rise to morphisms of categorical quotients

$$\mathfrak{h}^{\otimes 2}/W \rightarrow C_2(\mathfrak{g})//G \rightarrow X_n//G.$$ 

Proof of Theorem 1.3. We need to prove that the morphisms in (3.1) are isomorphisms. We note that the second morphism is a closed embedding. Also thanks to Corollary 2.2 it is bijective. Thanks to Theorem 1.2 $X_n$ is reduced, hence so is $X_n//G$. It follows that the second morphism in (3.1) is an isomorphism, and, in particular, $C_2(\mathfrak{g})//G$ is reduced.
The first morphism in (3.1) is bijective. It remains to show that it is a full embedding, equivalently, that the pullback homomorphism $\mathbb{C}[C_2(\mathfrak{g})]^G \rightarrow \mathbb{C}[\mathfrak{h}^{\oplus 2}]^W$ is an isomorphism. Note that both algebras are Poisson. For the source this holds because $\mathbb{C}[C_2(\mathfrak{g})]^G$ is obtained from $\mathbb{C}[\mathfrak{g}^{\oplus 2}]$ by Hamiltonian reduction. The homomorphism $\mathbb{C}[C_2(\mathfrak{g})]^G \rightarrow \mathbb{C}[\mathfrak{h}^{\oplus 2}]^W$ intertwines the Poisson brackets. To see this note that this homomorphism becomes a Poisson isomorphism after tensoring with $\mathbb{C}[\mathfrak{g}^{\text{reg}}]^G \cong \mathbb{C}[\mathfrak{h}^{\text{reg}}]^W$ (in the first coordinate) and $\mathbb{C}[\mathfrak{h}^{\oplus 2}]^W \hookrightarrow \mathbb{C}[\mathfrak{h}^{\text{reg}}]^W \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}^{\oplus 2}]^W$. By results of [W], the Poisson algebra $\mathbb{C}[\mathfrak{h}^{\oplus 2}]^W$ is generated by the two subalgebras $\mathbb{C}[\mathfrak{h}]^W$ (in the first and the second coordinates). To finish the proof it remains to notice that the homomorphism $\mathbb{C}[C_2(\mathfrak{g})]^G \rightarrow \mathbb{C}[\mathfrak{h}^{\oplus 2}]^W$ restricts to $\mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{h}]^W$ (for both the first and the second copy). □

References

[CN] T.-H. Chen, B.C. Ngo, Invariant theory for the commuting scheme of symplectic Lie algebras. arXiv:2102.01849.
[CG] N. Chriss, V. Ginzburg, Representation theory and complex geometry. Birkhäuser Boston, Inc., Boston, MA, 1997.
[GG] W.L. Gan, V. Ginzburg, Almost-commuting variety, D-modules, and Cherednik algebras. With an appendix by Ginzburg. IMRP Int. Math. Res. Pap. 2006, 26439, 1–54.
[EG] P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. Invent. Math. 147 (2002), no. 2, 243–348.
[L] I.V. Losev. Symplectic slices for reductive groups. Mat. Sbornik 197(2006), N2, p. 75-86 (in Russian). English translation in: Sbornik Math. 197(2006), N2, 213-224.
[R] R.W. Richardson, Commuting varieties of semisimple Lie algebras and algebraic groups. Compositio Math. 38 (1979), no. 3, 311–327.
[W] N. Wallach, Invariant differential operators on a reductive Lie algebra and Weyl group representations. J. Amer. Math. Soc. 6 (1993), no. 4, 779–816.