On the topological and crosscap entropies in non-oriented RCFTs

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Abstract We establish a relation between the boundary and the topological entropies for the conformal minimal models in some of the simplest models of the unitary A–A series. We show that in these models the boundary entropy is a difference of topological entropies. Furthermore, we define the crosscap entropy as the analog to the boundary entropy in non-oriented theories. The crosscap entropy is defined as the logarithm of the degeneracy of the ground state due to the presence of crosscaps, and it can be expressed in terms of the crosscap coefficients. This crosscap entropy has not an explicit relation to the topological entropy as the boundary entropy. However, we propose a new quantity, $\hat{S} = \ln P_{0i}$ defined in terms of the modular transformation $P$ between the open and closed channel of the Möbius partition function. With this quantity, the crosscap entropy can be regarded as a difference of entropies, very similar to the boundary entropy. We also compute the left-right entanglement entropy (LREE) for crosscap states, and we express it in terms of $\hat{S}$. An explicit example of the LREE of the crosscap state in Wess–Zumino–Witten is carried out.

1 Introduction

Entanglement is a quantum feature that has become central in studies of several phenomena in string theory, quantum chaos and condensed matter physics. It has recently been observed experimentally that there are new phase transitions of matter that cannot be explained with the theoretical framework of the symmetries of the Hamiltonian. Several investigations point out that these new phase transitions have their explanation in terms of the topological order. Topological order can be manifested in several ways, in particular as the entanglement of the ground state of the Hamiltonian in the limit of large system size. In a many-body system, this entanglement is quantified by the topological (entanglement) entropy $S_{\text{top}} = \ln (d_i / D)$, where $d_i$ is the quantum dimension of a particle of charge $i$ and $D$ is the total quantum dimension of the medium [1, 2]. In unitary 1+1 Conformal Field Theory (CFT), these quantum numbers are defined in terms of the modular-$S$ matrix, and $i$ labels the kind of primary operator.
When boundary conditions are implemented into the CFT, there is an entropy associated with the ground state degeneracy $g$ due to the presence of the boundaries. Specifically, for a 1+1 Boundary CFT (BCFT) defined on a cylinder of length $L$ and temperature $T = 1/\beta$, with a boundary $a$ at one of the ends of the cylinder, the free energy has the contribution $\ln Z = \ln g_a$ at the large system limit $L \gg \beta$. Therefore, the boundary entropy associated with the universal non-integer ground states degeneracy at the boundary $a$ is $S_a^B = \ln g_a$ [3]. Because of the open-closed duality, $g$ is expressed in two equivalent forms, in the closed or the open channel. In the closed channel, $g$ is determined by the coefficients of the boundary states. Along this work, we consider only Cardy’s boundary states, where the boundary index $a$ agrees with the index of the representations $i$. In this case, the boundary coefficients are expressed in terms of the $S$-matrix and hence, in terms of the quantum dimensions.

Recently, another entropy has been defined in 1+1 BCFT, this is the left-right entanglement entropy (LREE). It measures the amount of quantum correlation between the left (holomorphic) and right (anti-holomorphic) modes of the CFT. We recall that in this entropy the division of the system is not geometric, but it is done as a factorization of the Hilbert space like a tensor product of subspaces. Another entropy based on the decomposition of the Hilbert space is the momentum-space entanglement given in [4]. The same idea of non-geometrical division was also applied to study entanglement entropy for two conformal field theories separated by conformal interfaces [5,6]. The LREE was first studied for bosonic boundary states in [7] and later for D-branes in [8]. Furthermore, this entropy was explored in [9] for boundary states in $1+1$-Rational Conformal Field Theories. Also, LREE for $1+1$ Wess–Zumino–Witten (WZW) models and the consequences of the level-rank duality on this entropy was determined in [10]. One of the interesting issues of this entropy is that the LREE for an Ishibashi state of a diagonal CFT over a circle of circumference $\ell$ was found to be $S = \pi c \ell/24\epsilon + \log(d_j/D)$, where $\epsilon$ is a UV cut-off. Therefore, the LREE of a 2-$d$ CFT can be useful to study the entanglement entropy of 2+1 Topological Field Theories through a bulk-boundary correspondence [9,11]. Further generalizations involving Hilbert spaces of the 2+1 theory constructed from states in an arbitrary superposition of Ishibashi states was proposed in [12].

Entanglement is a quantum feature that has become central. Until now, the topological and the boundary entropy, as well as the LREE, have been studied for CFT’s in the cylinder which is an oriented surface. By gauging the CFT by parity symmetries (preserving conformal invariance) of the two-dimensional space-time, one obtains a CFT on non-oriented surfaces, such as the Klein bottle. In this case, there are physical states called crosscap states associated with the parity symmetries that contain the information on the non-orientability of the space-time. There is one crosscap state for each parity symmetry. Such theories are usually referred to as non-oriented Conformal Field Theories and have a lot of interest in string phenomenology [13] and recently in modern advances in condensed matter physics [14–16].

In this work, we explore the relation between the entropies described above in some oriented unitary and non-unitary CFT’s and in unitary non-oriented. Our motivation comes from the fact that the LREE of boundary states as well as the boundary entropy are determined by the boundary coefficients. Then, the immediate question here is if the LREE can detect the ground state degeneracy at a boundary. Also, we want to understand how susceptible is this degeneracy when one changes the topology of the 1+1 space-time when going from the cylinder to the Klein bottle. To address this work, we first analyze the relation between the boundary and topological entropies in unitary oriented CFT. An analog to the boundary entropy in non-oriented theories has been given in [17,18]. There, the “boundary entropy” in non-oriented surfaces is defined in terms of the quantum dimensions $d$ and $D$ of the theory, and their results were obtained from the analysis of the Klein bottle partition function in the
open-channel. However, from their results, a relation between their entropy and a topological entropy is not evident. This leads us to study the “boundary entropy” in non-oriented theories in the closed-channel. By the open-closed duality, we give an equivalent definition of this entropy in terms of the crosscap coefficients; hence, we call it the crosscap entropy. However, compared to the boundary states, the crosscap coefficients are expressed in terms of both the modular transformation \( P \) and \( S \) matrix; consequently, the crosscap entropy is not explicitly related to the topological entropy. Therefore, based on the behavior of the character \( \hat{\chi} \) of the Möbius strip at the long-distance limit, we propose two new quantities \( \hat{d} \) and \( \hat{D} \) given in terms of the modular transformation \( P \) between the open and closed channel of the Möbius partition function. In a similar way to the oriented case, we define \( \hat{S} = \ln P_{0i} = \ln(\hat{d}/\hat{D}) \).

With this new entropy, the crosscap entropy can be regarded as a difference of entropies, very similar to the boundary entropy. Finally, we compute the LREE for 1+1 non-oriented CFT and we see that this entanglement entropy codifies the crosscap as well as subleading terms.

The organization of the article is as follows. In Sect. 2, we give a short review of boundary and crosscap states in oriented and non-oriented surfaces. The definitions of boundary and topological entropies are also outlined in this section. In Sect. 3, a relation between the boundary and the topological entropies is described for the oriented case. It is found that the boundary entropy is related to the difference between the topological entropy for the non-trivial representation and one-half of the topological entropy for the identity representation. In the non-oriented case, a similar formula for the crosscap entropy is given. We compute the boundary and crosscap entropies using our new definitions for three minimal models of A–A series: the critical Ising model, the Tricritical Ising model, and Tetracritical Ising model. Our results agree with those using explicitly the form of the boundary and crosscap states. Section 4 is devoted to computing the LREE for crosscap states. We also consider crosscap entropy and the LREE for \( SU(2)_k \) WZW models. In the end, the final remarks are given in Sect. 5.

2 Overview of entropies in rational conformal field theory

In this section, we start with a brief overview of the two-dimensional Rational Conformal Field Theory (RCFT). We focus basically on two-dimensional conformal field theories on surfaces with boundaries and crosscaps. (For an extensive study of the subject see [19–22]). We then revisit the definitions of boundary and topological entropies and apply these definitions to some minimal models.

2.1 Boundaries and crosscaps in RCFT

A RCFT is a realization of a symmetry algebra \( \mathcal{A} \otimes \bar{\mathcal{A}} \) where the number of primary fields is finite. The holomorphic (chiral) algebra \( \mathcal{A} \) is an extension of the Virasoro symmetry algebra. The generators are denoted as \( W^{(r)} \) with \( r = 0, 1, 2, \ldots \) labeling the different chiral fields with the case \( r = 0 \) representing the Virasoro generators \( L_n = W^{(0)} \). The antiholomorphic (anti-chiral) algebra \( \bar{\mathcal{A}} \) is generated by \( \bar{W}^{(r)} \). In the following, the isomorphism between \( \mathcal{A} \) and \( \bar{\mathcal{A}} \) is assumed. The Hilbert space is organized into irreducible representations of this symmetry algebra, where each irreducible representation \( \mathcal{H}_i \) of \( \mathcal{A} \) consists of a tower of states constructed from a primary field \( \phi_i \) of conformal weight \( h_i \) and their descendants. Here, we will denote the identity representation by 0.

The presence of boundaries in the theory breaks the two-dimensional conformal algebra \( \mathcal{A} \otimes \bar{\mathcal{A}} \). However, one can introduce them in such a way that a diagonal subalgebra \( \mathcal{A}_{\text{diag}} \) is
preserved. Thus, the boundary conditions can be described by certain coherent states called boundary states, where all information of the boundaries is codified. For each boundary $a$, there is a boundary state defined as

$$|B_a\rangle = \sum_i B_{ai} |i\rangle\rangle,$$

(2.1)

where $|i\rangle\rangle$ is the Ishibashi state of the representation $i$.

On the other hand, non-oriented theories are constructed by gauging the oriented theory with a parity symmetry. The simplest case is the standard operator $\Omega_1$ which acts on the spatial coordinates as an inversion and it acts on the representations as $\Omega : i \rightarrow \bar{i}$. Moreover, if the theory contains an internal symmetry $G$, the parity symmetry operator is defined as $\mathcal{P} = \Omega h$ with $h$ an internal transformation. These theories are defined on non-oriented surfaces which naturally include crosscaps. The crosscap boundary conditions are implemented in the theory as coherent states called crosscap states, and for each parity symmetry operator $\mathcal{P}_\mu$ with $\mathcal{P}_\mu^2 = 1$ ($\mu$ labels the different parity symmetries) there is a crosscap state defined as

$$|C_\mu\rangle = \sum_i \Gamma_{\mu i} |C_i\rangle\rangle,$$

(2.2)

with $|C_i\rangle$ the Ishibashi crosscap state [23]. The coefficients $B_{ai}$ in (2.1) and $\Gamma_{\mu i}$ in (2.2) are determined by the sewing constrains [24,25]. Nevertheless, solving the constrains requires a detailed analysis of the $n$-point functions in the presence of boundaries and crosscaps. Only in few cases, the constrains can be solved.

There is another constraint for the boundary and the crosscap coefficients that can be used to determine these coefficients. This is the duality open-closed (direct-transverse) channel for the partition functions on the annulus $C$, the Möbius strip $M$ and the Klein bottle $K$. This duality establishes that, for each surface, the partition function $Z$ defined in the open channel is the same as the partition function described in the closed channel. The fundamental region for these surfaces is a strip. In the open (direct) channel, the width $L$ of the strip lies on the spatial direction and the height lies along the temporal direction with period $\beta$. The open-closed duality interchanges spatial and temporal directions; therefore, in the closed (transverse) channel the width $L$ runs along a temporal direction and the height $\beta$ is periodic along the spatial direction. This duality is very well known, and we only mention the relations between the partition functions following the notation given in [22]:

$$Z^C := \text{Tr}_{\mathcal{H}_{ab}} e^{-\beta H_a(L)} = \langle B_a | e^{-L H_b(\beta)} | B_b \rangle,$$

$$Z^K := \text{Tr}_{\mathcal{H}_{g\mu v}} \mathcal{P}_v e^{-\beta H_v(L)} = \langle C_\mu | e^{\frac{L}{2} H_t(2\beta)} | C_v \rangle,$$

(2.3)

$$Z^M := \text{Tr}_{\mathcal{H}_{a\mu(a)}} \mathcal{P}_\mu e^{-\beta H_\mu(L)} = \langle B_a | e^{\frac{L}{2} H_t(2\beta)} | C_\mu \rangle.$$

The left-hand side of these equations represent the partition functions $Z$ on $C$, $K$ and $M$ in the open channel, respectively. The traces are on the Hilbert space with respective boundary and/or crosscap conditions: $\mathcal{H}_{ab}$ is the Hilbert space with boundary conditions $a$ and $b$. $\mathcal{H}_{g\mu v}$ is the space of states with boundary conditions twisted by $g \in G$ and by the parity symmetries $\mathcal{P}_\mu$. When transforming to the closed channel, $g = \mathcal{P}_\mu(\mathcal{P}_v)^{-1}$ with $\mathcal{P}_\mu$ and $\mathcal{P}_v$ determining the crosscap boundary conditions. Finally, $\mathcal{H}_{a\mu(a)}$ denotes the space of states with a boundary $a$ and the image $\mu(a)$ of it under $\mathcal{P}_\mu$. On the right-hand side, $Z^C$, $Z^K$ and $Z^M$ denote the propagator between boundary states, crosscap states and a boundary and a crosscap state, respectively.
The equalities in (2.3) are established by modular transformations. In terms of the characters, \( Z^C \) is expressed as \( \sum_i n_{ab} \chi_i(\tau) = \sum_i B^*_{ai} B_{bi} \chi_i(-1/\tau) \) where \( \chi \) satisfies the modular transformation \( S \): \( \chi_j(-1/\tau) = \sum_i \chi_i(\tau) S_{ij} \) and the coefficients in the open channel \( n_{ab} \) are nonnegative integers. For the Klein bottle \( Z^K \), one has \( \sum_i k_{\mu\nu} \chi(2\tau) = \sum_i \Gamma^*_{\mu i} \Gamma_{\nu i} \chi_i(-1/2\tau) \). The relation between both sides is determined by the \( T \) modular transformation \( \chi_i(\tau + 1) = T_{ij} \chi_j(\tau) \) followed by an \( S \) transformation. The open channel coefficient \( k^i_{\mu\nu} \) has to be an integer satisfying

\[
|k^i_{\mu\nu}| \leq h^{ii}_{\mu\nu} \quad \text{and} \quad k^i_{\mu\nu} \equiv h^{ii}_{\mu\nu} \pmod{2}, \tag{2.4}
\]

where \( h_{ij} \) are the multiplicities of \( \mathcal{H}_i \otimes \mathcal{H}_j \) in the space \( \mathcal{H}_g \) of \( g \)-twisted closed string states. This condition projects out states from the oriented one. The consistency of the theory with boundaries and crosscaps is determined by \( Z^M \). The relation (2.3) in this case is \( \sum_i m^i_{\mu\nu} \hat{\chi}(\tau) = \sum_i B^*_{ai} \Gamma_{\mu i} \hat{\chi}_i(-1/4\tau) \) with \( \hat{\chi}(\tau) = e^{-\pi i(h-c/24)} \chi(\tau + 1/2) \). It is transformed as \( \hat{\chi}_j(-1/4\tau) = \sum_i \chi_i(\tau) P_{ij} \) by the modular transformation \( P = \sqrt{T} S T^2 S \sqrt{T} \).

The open-channel Möbius partition function is interpreted as the projection of states in the annulus \( Z^C \). This is achieved requiring the coefficient \( m^i \) to be an integer such that

\[
m^i_{\mu\nu} \leq |n^i_{\mu\nu(a)}| \quad \text{and} \quad m^i_{\mu\nu} \equiv n^i_{\mu\nu(a)} \pmod{2}. \tag{2.5}
\]

After the modular transformations described above, the open-closed duality sets the following constraints

\[
Z^C : \quad n^i_{ab} = \sum_j B^*_{ai} B_{bi} S_{ij},
\]
\[
Z^K : \quad k^i_{\mu\nu} = \sum_j \Gamma^*_{\mu i} \Gamma_{\nu i} S_{ij},
\]
\[
Z^M : \quad m^i_{\mu\nu} = \sum_j B^*_{ai} \Gamma_{\mu i} P_{ij}.
\]

For conformal field theories with isomorphic left- and right-algebras, some solutions to these constraints have been found. We will consider the solution in which the internal symmetry is the identity, and the multiplicity \( h^{ii} \) is a charge conjugation invariant. In such a case, the boundary labels \( a \) take the same values of the representations \( i \); therefore, one can set \( n^i_{ab} = N^i_{ab} \) with \( N^i_{ab} \) the fusion coefficients in the Verlinde formula. In this way, the solution to the first equation in (2.6) is

\[
B_{ai} = \frac{S_{ai}}{S_{bi}}, \tag{2.7}
\]

This is known as the Cardy solution [26]. With the same considerations, the relevant parity operator is the standard operator \( \Omega \) for which the index \( \mu \) agrees with the index of the identity representation \((\mu = 0)\). The solutions to the last two equations in (2.6) are found using the integrality conditions (2.4) and (2.5). In [27], Pradisi–Sagnotti–Stanev (PSS) realized that the matrices \( S \) and \( P \) satisfy a similar relation to the Verlinde formula: \( Y^k_{ij} = \sum_l S_{ij} P_{kl} B^*_{li} \) and that the solutions to (2.4) can be written in terms of this matrix as

\[
m^i_{00} = Y^i_{00},
\]
\[
k^i_{00} = Y^i_{00}. \tag{2.8}
\]
Using these expressions in Eq. (2.6), the PPS solution for the crosscap coefficients is given by

$$\Gamma_{0i} = \frac{P_{0i}}{\sqrt{S_{0i}}}. \quad (2.9)$$

If the theory has an internal symmetry associated with simple currents $J$ of order $N$, the crosscap coefficients are given by [28]

$$\Gamma_{[J^n]}_{0i} = \frac{P_{J^n i}}{\sqrt{S_{0i}}}, \quad n = 0, \ldots , N - 1, \quad (2.10)$$

with the condition (2.8) modified by replacing 0 by the index running on the algebra representations.

2.2 Boundary and topological entropy

The topological entropy is a measure of topological order manifested as entanglement in the ground state of a system [1,2]. It is defined as

$$S_{i}^{top} = \ln \left( \frac{d_i}{D} \right) = \ln (S_{0i}), \quad (2.11)$$

where $d_i$ is the relative quantum dimension of the representation $\mathcal{H}_i$ with respect to the identity representation $\mathcal{H}_0$ and $D = \sqrt{\sum_i d_i^2}$ is the total quantum dimension [29,30]. These quantum numbers are related to the matrix $S$ as $d_i = S_{0i}/S_{00}$ with $D = 1/S_{00}$.

On the other hand, the boundary entropy is associated with the ground state degeneracy due to the boundaries. It can be computed from the partition function on the cylinder with boundary conditions $a$ and $b$. In the closed channel and in the thermodynamic limit $L \gg \beta$, the ground state degeneracy is $g = g_a g_b$. The contribution of the boundary $a$ to the ground state degeneracy is $g_a = \langle 0 | B_a \rangle = B_{a0}$ with $|0\rangle$ the ground state of $\mathcal{H}_c$. Due to the open-closed duality, the ground state degeneracy can also be computed in the loop-channel, it is given as $g_a g_b = \sum_i n_{a}^i S_{0i}$, where the coefficient $n_{a}^i$ are the nonnegative integers described above and $S_{0i}$ a nonnegative number [3].

The boundary entropy associated with the ground state degeneracy $g_a$ is defined as

$$S_a^{B} = \ln g_a \quad \text{with} \quad g_a = \begin{cases} \sqrt{\sum_i n_{a}^i S_{0i}} & \text{open channel} \\ B_{a0} & \text{closed channel} \end{cases}. \quad (2.12)$$

The ground state degeneracy is not only related to the boundary entropy; it was observed in [31] that the regularized dimension of $\mathcal{H}_{aa} = \oplus_i n_{a}^i \mathcal{H}_i$ is precisely the ground state degeneracy

$$\dim \mathcal{H}_{aa} = \lim_{q_o \to 1} q_c^{c/24} Z_{aa}^c(q_o) = g_a^2 = \sum_i n_{a}^i d_i / D, \quad (2.13)$$

here $Z_{aa}^c(q_o)$ is the partition function in the open channel described in (2.3).
3 Relation between boundary entropy and topological entropy

So far, we have discussed the boundary entropy and topological entropy separately. In this section, we study the relation between them for some unitary minimal models, specifically in some of the A–A series of the minimal models.

3.1 Oriented case

The relation between the boundary entropy and the topological entropy comes from the following observation. A boundary state is an entangled system in its left- and right modes [7, 9]; therefore, one could expect that the boundary entropy defined by the coefficients of the boundary states must be associated with some measure of entanglement of the ground state. In the closed channel, the boundary entropy is given as $S^B_a = \ln (B_{a0})$. Using the definition (2.7) for $B_{a0}$, we have $S^B_a = \ln (S_{a0}/\sqrt{S_{00}}) = \ln (d_a/D) - \ln \sqrt{1/D}$, where in the last equation we have taken into account that $S$ is symmetric and that $S_{0i} = d_i/D$. Therefore, by equation (2.11) we have

$$S^B_a = S^\text{top}_a - \frac{1}{2} S^\text{top}_0. \quad (3.1)$$

Note that for the identity representation $d_0 = 1$ and the topological entropy in this case is $S^\text{top}_0 = -\ln D = \ln (S_{00})$. The boundary entropy associated with the boundary $a$ can be computed easily from the relation (3.1), since it is the difference between the topological entropy for the irreducible representation $a$ and the topological entropy of the identity representation. In particular, $S^B_0 = \frac{1}{2} S^\text{top}_0$. While the topological entropy is always negative (since $d_i < D$), the sign of the boundary entropy depends on the ground state degeneracy which is $g_a \geq 1$ for $d_a \geq \sqrt{D}$ and $g_a < 1$ for $d_a < \sqrt{D}$, as can be inferred from (3.1).

In the open channel, we substitute $S_{0i} = d_i/D$ in the first relation of (2.12) to have $g_a = \sqrt{\sum_i n^k_{ai} d_i/D}$. Using the fact that the Cardy solution sets $n^k_{ai} = N^k_{aa}$ and that the quantum dimension satisfies the fusion algebra $d_i d_j = \sum_k N^k_{ij} d_k$ [32], we get Eq. (3.1).

We apply Eq. (3.1) to some minimal models of the A–A series where the $S$-matrix is given in [32,33]. The Ising model with $c = 1/2$ and $D = 2$ contains three primary operators: the identity operator 1 (labeled by 0), the energy field $\epsilon$ and the spin field $\sigma$. For the operators 1 and $\epsilon$ which have the same relative dimension $d = 1$, the topological entropy is $S^\text{top}_0 = -\ln 2$. The associated boundary entropy is $S^B_{0,\epsilon} = -\frac{1}{2} \ln 2$. For the spin operator, the relative dimension is $d_{\sigma} = \sqrt{2}$ and $S^\text{top}_{\sigma} = -\frac{1}{2} \ln 2$; hence, $S^B_{\sigma} = 0$. Since $d_{\sigma}$ is the square of the total quantum dimension, the boundary entropy is zero. These results agree with those given in [3].

In the Tricritical model with $c = 7/10$, there are six primary fields ($1, \epsilon, \epsilon', \epsilon'', \sigma, \sigma'$); the relative dimensions are $d_0 = d_{\epsilon'} = 1$, $d_{\epsilon} = d_{\epsilon'} = s_1/s_2$, $d_{\sigma} = \sqrt{2}s_1/s_2$ and $d_{\sigma'} = \sqrt{2}$ and $D = \sqrt{40/(5 - \sqrt{5})}$.

The entropies are

$$S^\text{top}_0 = -\ln D = -1.3361 \quad S^B_0 = \ln \sqrt{s_2} = -0.6681
$$

$$S^\text{top}_\epsilon = \ln (s_1) = -0.8549 \quad S^B_\epsilon = \ln \left(\frac{s_1}{\sqrt{s_2}}\right) = -0.1868
$$

$$S^\text{top}_\sigma = \ln (\sqrt{2}s_1) = -0.5083 \quad S^B_\sigma = \ln \left(\frac{2}{\sqrt{s_2}}\right) = 0.1597$$
with \(s_1 = \sqrt{(5 + \sqrt{5})/4}\) and \(s_2 = \sqrt{(5 - \sqrt{5})/4}\). Also, \(S_{\sigma''} = S_0\) and \(S_{\sigma'} = S_e\) since such fields have the same relative quantum dimension, respectively. There is not a \(d_i\) equal to \(\sqrt{D}\), so all boundary entropies are different from zero. There is a positive boundary entropy corresponding to the primary field \(\sigma\), since its relative dimension satisfies \(d_\sigma > \sqrt{D}\). These expressions for the boundary entropies are in complete agreement with those computed directly from the boundary states in this model.

In the Tetracritical Ising Model with \(c = 4/5\), the primary fields have conformal weight \([0, 2/5, 1/40, 7/5, 21/40, 1/15, 3, 13/8, 2/3, 1/8]\) and \(D = \sqrt{6(5 + \sqrt{5})}\). As in the Tricritical case, all boundary entropies are different from zero and the positive boundary entropies correspond to the operators with conformal weight 7/5, 1/40 and 21/40. All other boundary entropies are negative.

### 3.2 Non-unitary minimal models

In this subsection, we study the relation between the boundary entropy and the topological entropy in non-unitary Virasoro minimal models.

These theories contain fields with negative conformal weights. The ground state in these theories is not the conformal vacuum but the state with lowest conformal weight. The representation of the field with smallest conformal weight is denoted as min. When boundary conditions are included, the boundary coefficients for the A-series are \([34]\):

\[
g_a = B_a \text{min} = \frac{S_{a \text{min}}}{\sqrt{S_{0 \text{min}}}}. \tag{3.3}
\]

On the other hand, a generalized quantum dimension for these models was defined in \([35–37]\) as:

\[
\tilde{d}_i = \lim_{q_i \to 1} \frac{\chi_i(\tau)}{\chi_0(\tau)} = \lim_{q_i \to 0} \frac{\sum_j S_{ji} \chi_j(-1/\tau)}{\sum_j S_{j0} \chi_j(-1/\tau)} = \frac{S_{i \text{min}}}{S_{0 \text{min}}}. \tag{3.4}
\]

The elements \(S_{i \text{min}}\) of the matrix \(S\) are positive for any \(i\)-representation. We define the total generalized quantum dimension as \(\tilde{D} = \sqrt{\sum_i \tilde{d}_i^2}\) and since \(S\) is unitary we can express this quantity as \(\tilde{D} = 1/S_{0 \text{min}}\). The topological entropy is then defined as

\[
S^{\text{top}}_i = \ln\left(\frac{\tilde{d}_i}{\tilde{D}}\right) = \ln(S_{i \text{min}}). \tag{3.5}
\]

Substituting the expression (3.3) into (2.12) and using Eq. (3.5), we get the relation (3.1) between the boundary and the topological entropies.

As an example, we apply Eq. (3.1) to the Yang–Lee model. This is a non-unitary minimal model of the A-series with central charge \(c = -22/5\). It has only two primary fields denoted as \(1\) and \(\phi\) with conformal weight \(h = 0\) and \(h_{\text{min}} = -1/5\), respectively. The matrix \(S\) for this model can be found in \([37]\), the relevant matrix elements for our study are \(S_{0 \text{min}} = (2/\sqrt{5})\sin(\pi/5)\) and \(S_{\text{min min}} = (2/\sqrt{5})\sin(2\pi/5)\).
The quantum dimensions are \( \tilde{d}_0 = 1, \tilde{d}_{\text{min}} = \sqrt{(3 + \sqrt{5})/2} = 1.61803 \) and \( \tilde{D} = \sqrt{(3 + \sqrt{5})/2} = 1.90211 \). The topological and boundary entropies are:

\[
\begin{align*}
S_0^{\text{top}} &= -0.64297, \\
S_0^{\text{min}} &= -0.16175, \\
S_0^B &= -0.32148, \\
S_{\text{min}}^B &= 0.15973
\end{align*}
\] (3.6)

The results for the boundary entropy coincide with those in [38], which were obtained computing the correlation functions on a disk. As before, we observe that \( S_0^B < 0 \) as long as \( \tilde{d} < \sqrt{\tilde{D}} \) and \( S_{\text{min}}^B > 0 \) since \( \tilde{d} \geq \sqrt{\tilde{D}} \).

It is worth mentioning that in the Yang–Lee model, their characters can be represented both in the product and in the sum representations. In the last one, they are written as [39]

\[
\chi_{A, B, C}(\tau) = \sum_{M} a_M q^M = \sum_{n \in \mathbb{Z}^r} q^{n \cdot A n + B \cdot n + C} (q)_n,
\] (3.8)

where \((q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)\). A more general structure of these characters is given by the so-called Nahm’s sum

\[
\chi_{A, B, C}(\tau) = \sum_{M} a_M q^M = \sum_{n \in \mathbb{Z}^r} q^{n' \cdot A n + B \cdot n + C} (q)_n,
\] (3.9)

where \((q)_n = (q)_{n_1} \cdots (q)_{n_r}\) and \((q)_{n_1} = \prod_{i=1}^{n_r} (1 - q^i)\) with \( q = e^{2\pi i \tau} \) and \( n = (n_1, \ldots, n_r) \). Moreover, \( A \) is a \( r \times r \) symmetric and positive matrix, \( B \) is a vector and \( B \cdot n = \sum_{i=1}^{r} b_i n_i \) and \( C \) is a scalar. The specific form of the matrix \( A \), the vector \( B \) and the scalar \( C \) is related to a certain CFT. For instance, for the Yang–Lee model, their characters can be obtained from this general form (3.8) for specific values of \( r, A, B \) and \( C \).

According to Nahm’s conjecture [39], the characters (3.8) are modular forms for certain values of \( A, B \) and \( C \), the entries of the matrix \( A \) satisfy certain equations associated with a certain integrable model. Two-dimensional integrable models described, for instance, by a massive scalar quantum field theory can be regarded as a CFT at high energies, i.e., in the limit when the mass of the field can be considered very small compared with the characteristic energy of the relevant process. Thus, integrable models can be interpreted as some deformations of CFTs preserving certain aspects of the theory, for instance, the infinite hierarchy of conserved quantities. In these integrable models, the Thermodynamic Bethe Ansatz (TBA) constitutes a bridge between integrable models and CFTs [35,36,39]. Some properties of the CFT as the conformal weights and the central charge can be written in terms of some functions as the dilogarithm formula [35,36].

In Ref. [36], it was shown that after an asymptotic treatment of a continuous version of Eq. (3.8) and the use of the saddle point method, the TBA-type equation realization of the partial generalized quantum dimension (mentioned above) is given by [36]

\[
\tilde{\lambda}_i \equiv \frac{S_{i, \text{min}}}{S_{0, \text{min}}} = \prod_{l=1}^{r} \left( 1 - w_l \right)^{b_{I, l} - b_{I, 0}},
\] (3.10)

where

\[
S_{i, \text{min}} = \exp \left\{ \sum_{l} \left( b_{I, l} - \frac{1}{2} \right) \log(1 - w_l) \right\}.
\]
with $w_i$ are real parameters with $0 < w_i < 1$ for all $I = 1, \ldots, r$. This is an important point for our description since at least for these classes of systems is possible to define a corresponding analogue of the topological entropy

$$S_i^{\text{top}} = \ln(S_{i, \text{min}}) = \sum_{I=1}^{r} \left( b_{I,i} - \frac{1}{2} \right) \ln(1 - w_I),$$

(3.11)

which evidently constitutes a finite value.

3.3 Non-oriented case

In non-oriented theories, the presence of parity symmetries modifies the space of states since the states can have positive or negative parity eigenvalues. For the Klein bottle surface, the physical states are those which are symmetric under parity symmetries. In particular, if the ground state is degenerated, the antisymmetric part is projected out [40]. Therefore, one could expect an entropy associated with such degeneracy. In [17], the boundary entropy associated with the standard parity projection was defined using $Z^K_{\text{in}}$ in the open channel. There, the ground state degeneracy is given as $g = \sum_i k_i d_i / D$ where $k_i$ are the nonnegative coefficients of the Klein bottle partition function in the open channel subjects to the duality constraints. We want to compute this entropy in the closed channel where we have the crosscap state formalism at hand. In the next, we prefer to call the entropy due to the presence of crosscaps, the crosscap entropy.

To obtain the crosscap entropy, we follow the prescription given for the boundary states in [3]. The Klein bottle amplitude in the transverse channel is given as [22]

$$Z^K = \langle C_\mu | e^{-\frac{\pi L}{2\beta} (L_0 + L_0 - \frac{c}{24})} | C_\nu \rangle = \sum_i \Gamma_{\mu i} \Gamma_{\nu i} \chi_i(-1/2\tau).$$

(3.12)

In the limit $L \gg \beta$, only the ground state dominates and $Z^K \sim \langle C_\mu |0\rangle \langle 0| C_\nu \rangle e^{\frac{\pi L}{24\beta}} = \Gamma_{\mu0} \Gamma_{\nu0} e^{\frac{\pi L}{24\beta}}$, where $\langle 0| C_\mu \rangle = \Gamma_{\mu0}$ with $|0\rangle$ the ground state of $H_c$. The thermodynamical entropy is $S_{\text{ther}}^{\mu} = \frac{cL}{12\beta} + \ln g_{\mu} + \ln g_{\nu}$, where $g_{\mu} = \Gamma_{\mu0}$ is the degeneracy of the ground states due to the crosscap associated with the parity symmetry $P_\mu$. Summarizing, we have that the crosscap entropy is

$$S_{\mu}^C = \ln g_{\mu} \quad \text{with} \quad g_{\mu} = \begin{cases} \sqrt{\sum_i k_{i\mu} d_i / D} & \text{open channel} \\ \Gamma_{\mu0} & \text{closed channel} \end{cases}$$

(3.13)

Now, we want to address if this crosscap entropy can be expressed as a difference of topological entropies as in Eq. (3.1). In the open channel, the ground state degeneracy defined in (3.13) is given in terms of the quantum dimensions, but it is not evident how it can be expressed in terms of topological entropies. On the other hand, we are not aware if from the topological quantum field theory point of view, the topological entanglement entropy (2.11) can be consistently defined for no-oriented surfaces, therefore, we cannot define the crosscap entropy as a difference of topological entropies as in (3.1).

We know from Eqs. (2.9) and (2.10) that $\Gamma_{\mu i} = P_{\mu i} / \sqrt{S_{\mu i}}$. We recall that the label $\mu$ denotes the kind of parity operators and for each operator there is a crosscap entropy. For the standard parity operator, the index $\mu$ agrees with the index of the identity representation ($\mu = 0$), and for non-standard parities associated with simple currents $\mu$ takes the label of the simple currents. Then, the crosscap entropy in the closed channel is $S_{\mu}^C = \ln \Gamma_{\mu0} =$
\[ \ln(P_{\mu 0}) - \ln(\sqrt{S_{00}}). \]

Now, we define the quantity
\[ \hat{S}_i = \ln P_{i0} = \ln \left( \frac{\hat{d}_i}{\hat{D}} \right) \quad \text{for} \quad P_{i0} \neq 0. \] (3.14)

This is defined for each irreducible representation in which \( P_{i0} \) is not zero since primary fields \( \phi_i \) which are not invariant under the parity symmetry have \( P_{i0} = 0 \).

The term \( P_{i0} \) appears naturally in the Möbius partition function \( Z^M \). In terms of this, the regularized dimension of \( H_{a\mu} \) is
\[
\dim \mathcal{H}_{a\mu} = \lim_{q_o \to 1} q_c^{c/96} Z^M(q_o) = \lim_{q_o \to 1} q_c^{c/96} \text{Tr} \mathcal{H}_{a\mu} \mathcal{P}_\mu e^{2\pi i \tau H_o} = \sum_i m^i_{a\mu} \hat{\chi}_i(\tau) = \sum_i m^i_{a\mu} P_{i0},
\] (3.15)

where in the first line, \( H_o \) is the Hamiltonian in the open sector. In the second row, \( m^i_{a\mu} \) are the integers whose values are restricted by the open-closed duality. Here, \( \hat{\chi}_i(\tau) \) is defined up to a factor of 1/2 which is induced by the relative orientation of the horizontal sides of the cylinder with a boundary and a crosscap and the ends. Due to this 1/2, the signs of the states alternate in the character and the signs depend on the levels. In the last equality, we have used the modular transformation \( \hat{\chi}_i(-1/4\tau) = \sum_j \hat{\chi}_j(\tau) P_{ij} \). From this modular transformation, the regularized dimension is expressed in terms of the elements of the matrix \( P \) and therefore, it cannot be expressed in terms of the quantum dimensions \( d \) and \( D \).

Looking at the character of the Möbius partition functions \( \hat{\chi}_i \), we note that in the limit \( q \to 1 \), the contributions to \( \hat{\chi}_i(\tau) \) at each level are the differences between the number of even and odd states under the parity operator. Hence, we define a relative parity index of the representation \( i \) compared to the identity representation 0 as
\[
\hat{d}_i = \lim_{q_o \to 1} \frac{\hat{\chi}_i(\tau)}{\hat{\chi}_0(\tau)} = \lim_{q_o \to 0} \frac{\sum_j P_{ji} \hat{\chi}_j(-1/4\tau)}{\sum_j P_{j0} \hat{\chi}_j(-1/4\tau)} = \frac{P_{i0}}{P_{00}}.
\] (3.16)

For the minimal models here analyzed, \( P_{i0} \geq 0 \) and \( P_{00} > 0 \). Since \( P^\dagger P = 1 \), then \( \sum_j |P_{ji}|^2 = 1 \) and one can define a total parity index as \( \hat{D} = \sqrt{\sum_i \hat{d}_i^2} = \frac{1}{P_{00}}. \) In terms of these numbers, the regularized dimension becomes
\[
\dim \mathcal{H}_{a\mu} = \sum_i m^i_{a\mu} \frac{\hat{d}_i}{\hat{D}}.
\] (3.17)

This expression is very similar to the regularized dimension in the cylinder (2.13).

With the definition (2.11) and (3.14), we can write the crosscap entropy as
\[
S^\mu_C = \hat{S}_\mu - \frac{1}{2} S^\text{top}_0.
\] (3.18)

In this way, the crosscap entropy has the same form as the boundary entropy given in (3.1). For the standard parity, \( \hat{S}_0 = -\ln \hat{D} = \ln P_{00} \).

From the relation (3.18), we can see that \( S^\mu_C > 0 \) if \( \hat{d}_\mu > \hat{D}/\sqrt{\hat{D}}. \) In some cases, the equality to zero is possible [17]; however, for the minimal models here studied with standard parity, this inequality is true. This is because \( \hat{D} \) and \( D \) are related to \( P_{00} \) and \( S_{00} \), respectively.
and $\sqrt{s_{00}} < P_{00}$. We have tested this last inequality for several values of the parameter $m$ of the minimal models in the A–A series and it is satisfied; therefore, we can assert that for such models the crosscap ground state degeneracy is $g > 1$. We will see in the next chapter that in $SU(2)$ WZW models it is possible to have models where the crosscap ground state degeneracy $g < 1$.

We have computed the crosscap entropies for the same models studied in 3.1 using the definition (3.18). For the Ising and the Tricritical model (with real representations), there is only one Klein bottle partition function associated with the standard parity symmetry (where $\mu = 0$), then the crosscap coefficients are given by the PSS solution and the crosscap entropy is determined by the coefficient $\Gamma_{00} = P_{00}/\sqrt{s_{00}}$. We summarize the results in Table 1.

We could also compute the crosscap entropy using the crosscap states. In such case, the crosscap coefficients for the Ising model were computed in [25] and the crosscap entropy for the Ising model here computed agrees with the one given in [17, 18], where it was obtained working in the open channel. For the Tricritical model, only the rate between the coefficients was given in [41]. We have computed these coefficients in the Tricritical and Tetracritical model and we give the explicit expressions in Appendix A. In both cases, the crosscap entropy coincides with those given in Table 1, respectively.

Finally, we go back to the Möbius partition function in the closed sector which in the limit $L \gg \beta$ becomes $Z_{\mu(a)}^{M} \sim B_{a0} \Gamma_{\mu0} \exp(\pi L c/24\beta)$. As in the cylinder and Klein bottle case, we can define the thermodynamic entropy in the Möbius sector, and the contribution to this entropy due to the boundary state $|B_{a}\rangle$ and to the crosscap state $|C_{\mu(a)}\rangle$ is given by $S^{M} = \ln(B_{a0}) + \ln(\Gamma_{\mu0}) = S_{a}^{B} + S_{\mu}^{C}$. This equation agrees with the one given in [42] up to a factor of 1/2, this is because the authors considered only the standard parity operator and define the ground states degeneracy as $g_{\mu}^{2}$. In our case, we allow the possibility of other parity operators like in theories with simple currents and then the ground state degeneracy will have the form $g_{\mu} g_{\nu}$.

### Table 1

For the Ising model $\hat{D} = 2/(\sqrt{2} + \sqrt{2})$ and $\mathcal{D} = 2$. In the Tricritical model, $\hat{D} = (\sqrt{2} + \sqrt{2}/s_{1})^{-1}$ and $\mathcal{D} = 1/s_{2}$. In the Tetracritical model, $\hat{D} = \sqrt{3}/(1 + \sqrt{3} s_{1})$ and $\mathcal{D} = \sqrt{6(5 + \sqrt{3})}$ where $s_{1} = \sqrt{5 \pm \sqrt{5}}/40$ and $s_{2} = \sqrt{5 - \sqrt{5}/40}$.

| Model       | $S_{0}$         | $S_{0}^{\text{top}}$ | $S_{0}^{C}$      |
|-------------|-----------------|----------------------|-----------------|
| Ising       | $-\ln(2/(\sqrt{2} + \sqrt{2}))$ | $-\ln(2)$           | $\ln(2 + \sqrt{2})/2$ |
|             | $= -0.07917$    | $= -0.69315$         | $= 0.2673$      |
| Tricritical | $\ln(\sqrt{2 + \sqrt{2}} s_{1})$ | $-\ln(s_{2})$       | $\ln(\sqrt{2 + \sqrt{2}} s_{1}/\sqrt{32})$ |
|             | $= -0.24093$    | $= -1.33611$         | $= 0.4271$      |
| Tetracritical | $\ln((1 + \sqrt{3}) s_{1}/\sqrt{3})$ | $-\ln(6(5 + \sqrt{3}))$ | $\ln((1 + \sqrt{3}) s_{1}/\sqrt{32})$ |
|             | $= -0.3992$     | $= -1.8854$          | $= 0.54355$     |

### 4 Left-right entanglement entropy for crosscaps in RCFT

The methodology to compute the LREE for crosscaps is very similar to the LREE for boundary states given in [9]; here, we rewrite those computations adapted to crosscaps. In the
following discussion, the label $\mu$ denoting different crosscap states related to the different parity operators is intentionally omitted. In order to define the density matrix, one has to consider instead the regularized crosscap state $e^{-\frac{i}{2} H(2\ell)} |C\rangle$. This is the same prescription employed in the case of entanglement entropy for boundary states due to the fact that the boundary Ishibashi states are non-normalizable. The parameter $\varepsilon$ is an UV cutoff, and $H(2\ell) = \frac{\pi}{2}(L_0 + \bar{L}_0 - c/12)$ is the Hamiltonian defined on a circle of circumference $2\ell$ that generates time translations in the tree-channel. In this configuration, $\tau = i\ell/2\varepsilon$. The density matrix is defined as

$$\rho = \frac{e^{-\frac{i}{2} H} |C\rangle \langle C| e^{-\frac{i}{2} H}}{\mathcal{N}},$$

where the normalization constant $\mathcal{N} = \sum_j |\Gamma_j|^2 e^{-2\pi i(h_j - c/24)\chi_j(-1/2\tau + 1)}$ is fixed by the condition $\text{Tr}\rho = 1$. After performing the $T$-transformation of the character, this normalization constant becomes $\mathcal{N} = \sum_j |\Gamma_j|^2 \chi_j(-1/2\tau)$. This constant is in fact the Klein bottle partition function between the same crosscap states [21].

Let us now denote the holomorphic sector by $A$ and the anti-holomorphic sector by $B$. The reduced density matrix $\rho_A = \text{Tr}_B \rho$, which is obtained by tracing on the anti-holomorphic modes is given by

$$\rho_A = \frac{1}{\mathcal{N}} \text{Tr}_B (e^{-\frac{i}{2} H} |C\rangle \langle C| e^{-\frac{i}{2} H}).$$ (4.2)

It is easy to see that this matrix is diagonal. Using the replica trick, and after performing the $T$-transformation on the characters, we find

$$\text{Tr} \rho^n_A = \frac{1}{\mathcal{N}^n} \sum_j |\Gamma_j|^{2n} \chi_j(-n/2\tau).$$ (4.3)

After this transformation, the computation of the entanglement entropy is very similar to the boundary states given in [7,9]. We will not repeat those computations here, and we give just the relevant results. One has to perform an $S$-transformation of the characters and take the limit $\varepsilon/\ell \rightarrow 0$ to obtain the leading contribution in the characters which comes from the lowest conformal dimension. In unitary theories, the identity representation has the lowest value and it is zero. Then,

$$\text{Tr}_A \rho^n_A = e^{\frac{2\pi c}{24}\left(\frac{1}{n}-1\right)} \left[\sum_j |\Gamma_j|^{2n} S_{j0} \right]^n.$$ (4.4)

Using the definition for the entanglement entropy $S_A = -\lim_{n \rightarrow 1} \partial_n \text{Tr} \rho^n_A$, the LREE for each crosscap state associated with $\mathcal{P}_\mu$ is given by

$$S_{C,\mathcal{P}_\mu} = \frac{\pi \ell c}{6\varepsilon} - \frac{\sum_j |\Gamma_{\mu j}|^2 S_{j0} \ln |\Gamma_{\mu j}|^2}{\sum_j |\Gamma_{\mu j}|^2 S_{j0}} + \ln \sum_j |\Gamma_{\mu j}|^2 S_{j0}.$$ (4.5)

Using the symmetric and unitary properties of the matrix $P$, the LREE becomes

$$S_{C,\mathcal{P}_\mu} = \frac{\pi \ell c}{6\varepsilon} - 2\left(\frac{\hat{d}_\mu}{\hat{D}}\right)^2 S^C_\mu - \sum_{j \neq 0} |P_{\mu j}|^2 \ln \left(\frac{|P_{\mu j}|^2}{S_{0j}}\right).$$ (4.6)

with $S^C_\mu$ given in (3.18).
For the particular case of the standard parity, we have
\[
S_{\langle C_{\mathcal{P}0} \rangle} = \frac{\pi \ell c}{6\varepsilon} - \frac{2}{\mathcal{D}^2} S_0^C - 2 \sum_{j \neq 0} \left( \frac{\hat{d}_j}{\mathcal{D}} \right)^2 \left[ \hat{S}_j - \frac{1}{2} S_{\text{top}}^j \right]. \tag{4.7}
\]

4.1 Wess–Zumino–Witten models

The crosscap states in WZW models are also determined by the crosscap coefficients (2.9) and (2.10) \cite{27,43–45}. In this case, the parity operator is given as \( \mathcal{P} = \mathcal{I} \Omega \) which is a combination of \( \Omega \) with a target involution \( \mathcal{I} \) on the group manifold. For any WZW model, one can consider the involution \( \mathcal{I} : h \rightarrow \gamma^n h^{-1} \) where \( h \) is the map from the world-sheet to the group manifold and \( \gamma \) generates the \( \mathbb{Z}_n \) center of group manifold and \( n = 0, \ldots, N - 1 \) \cite{46–48}. For \( n = 0 \), the associated crosscap state is the PSS solution (2.9). On the other hand, for \( n \neq 0 \), the center of the group is isomorphic to an abelian group formed by simple currents; therefore, the related crosscap state is given by (2.10). The matrix \( S \) for WZW model is given in \cite{32}, while the matrix \( P \) can be found in \cite{47}. In the following, we compute for simplicity the crosscap entropy and the LREE for diagonal \( SU(2)_k \) with \( k \) even. The explicit \( P \) matrix for this case is given in the appendix, although the explicit crosscap states are given in \cite{46,48}.

We recall that the crosscap entropy is \( S^C_\mu = \ln \Gamma_{\mu0} = \ln (P_{\mu0}/\sqrt{\mathcal{S}_{00}}) \). For the involution \( \mathcal{I} : h \rightarrow h^{-1} \), the crosscap state is a PSS state (where \( \mu = 0 \)) and it corresponds to the fixed points of the center of \( SU(2) \):
\[
S_0^C = \ln \left( \frac{2}{k + 2} \right)^{\frac{1}{4}} \tan \frac{\pi}{2(k + 2)}. \tag{4.8}
\]

For the other involution \( \mathcal{I} : h \rightarrow -h^{-1} \), the simple current is in the representation \( k/2 \) with the center isomorphic to \( \mathbb{Z}_2 \). The corresponding crosscap coefficient is given by (2.10), then
\[
S_{k/2}^C = \ln \left( \frac{2}{k + 2} \right)^{\frac{1}{4}} \cot \frac{\pi}{2(k + 2)}. \tag{4.9}
\]

It is interesting to note that for \( S_0^C \), the crosscap ground state degeneracy \( g_0 < 1 \) due to the fact that \( \hat{D}/\sqrt{\mathcal{D}} > \hat{d}_0 = 1 \), while for \( S_{k/2}^C \) the ground state degeneracy \( g_{k/2} > 1 \) since \( \hat{d}_{k/2} > \hat{D}/\sqrt{\mathcal{D}} \) where the expressions for these quantities are given in (A.9).

The central charge in these models is \( c = k/(k + 2) \), the LREE are
\[
S_{\langle C_0 \rangle} = \frac{\pi \ell c}{6\varepsilon} - \sum_j \frac{4}{k + 2} \sin^2 \frac{\pi(2j + 1)}{2(k + 2)} \ln \left[ \sqrt{\frac{2}{k + 2}} \tan \frac{\pi(2j + 1)}{2(k + 2)} \right]. \tag{4.10}
\]

For the standard parity and for the non-standard, one has
\[
S_{\langle C_{k/2} \rangle} = \frac{\pi \ell c}{6\varepsilon} - \sum_j \frac{4}{k + 2} (-1)^{2j} \cos^2 \frac{\pi(2j + 1)}{2(k + 2)} \ln \left[ \sqrt{\frac{2}{k + 2}} (-1)^{2j} \cot \frac{\pi(2j + 1)}{2(k + 2)} \right]. \tag{4.11}
\]

5 Final remarks

LREE has been exhaustively studied for 2d theories defined on oriented surfaces. In this context, it is known that LREE of the edge theory is deeply connected to entanglement entropy

\[ \square \]
of the 2+1 bulk theory. However, very few is known on LREE on non-oriented surfaces. The extension of the edge/bulk correspondence mentioned in the oriented case to the non-oriented case is also unknown. Therefore, we find interesting to study the crosscap entropy (which is explained in the article) and LREE for crosscap states and their interrelations with other entropies, as the topological entropy. We hope that our results bring some light into these directions. In what follows, we describe some of our results.

A relation between the topological entropy and the boundary entropy does not necessarily exist. However, in this work, we have found that, for some unitary and non-unitary minimal models in the A–A series, the change of topological order from the identity representation to the one associated with the boundary is manifested as a boundary entropy.

For the crosscap entropy, we cannot have such interpretation, as the crosscap entropy includes also the matrix \( P \). However, we defined \( \hat{S} \) as an auxiliary tool to express the crosscap entropy in a similar fashion as the boundary entropy. Thus, the crosscap entropy can be written as the difference between \( \hat{S} \) and \( \frac{1}{2} S_{\text{top}} \). We could interpret \( \hat{S} \) as a topological entropy due to the topological order when the system is defined on a non-oriented surface.

We have also computed the left-right entanglement entropy for crosscap states that preserve a diagonal subalgebra. Similarly, as in the boundary case, the leading term in the LREE (4.7) depends on the underlying CFT and it diverges as the UV cut-off \( \epsilon \) goes to zero. The subleading terms in the LREE are the crosscap entropy and the sum over the non-trivial representations. This was shown explicitly for the case of the \( SU(2)_k \) WZW model.

Finally, we want to address some concerns about the explicit expressions of the crosscap coefficients in some of the minimal models. In [41,49], the authors considered the correspondence between quantum gravity in AdS\(_3\) and 1+1 non-oriented CFT. In particular, they study pure quantum gravity (which includes only smooth saddles). Then, they restrict their analysis to minimal models in the CFT side. They conclude that in the dual minimal models, the crosscap coefficients \( \Gamma_{i0} = 0 \) must vanish for operators with conformal dimension \( h_i < c/12 \) to have dual smooth saddles. They were concerned about the existence of models with such condition. Here, we have computed the crosscap coefficients for the Tetracritical model (see (A.7)) and we found that there is a primary field with \( h = 1/40 = 0.025 \) and the corresponding crosscap coefficient is \( \Gamma_{02} = 0 \). We expect that this result could contribute to the understanding of holography on non-orientable surfaces.

Finally, it would be very interesting to develop the formalism of the topological entropy and its relation to the boundary entropy for the oriented and unoriented cases for characters given in terms of the Nahm sum (3.8) with the precise structure of a mock modular forms (see for instance, [50]). We leave this subject for a future work.

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**A Appendix**

The matrix \( S \) for minimal models in the A–A series is given in [32,51]. The matrix \( P \) can be constructed by means of its definition in terms of the matrix \( S \) and \( T \), although an simple expression for it has been given in [33]. For the Ising model, the crosscap coefficients are [24]
\[ \Gamma_{00} = \sqrt{\frac{2 + \sqrt{2}}{2}}, \quad \Gamma_{0e} = \sqrt{\frac{2 - \sqrt{2}}{2}}, \quad \Gamma_{0\sigma} = 0. \] (A.1)

In this case, there is only a crosscap corresponding to the parity symmetry \( \mathcal{P}_0 \)

\[ |C_{\mathcal{P}_0} \rangle = \Gamma_{00} |C_0 \rangle + \Gamma_{0e} |C_e \rangle. \] (A.2)

From (4.7), the LREE is

\[ S_{|C_{\mathcal{P}_0} \rangle} = \frac{\pi \ell}{12 \varepsilon} - \frac{2 + \sqrt{2}}{4} \ln \left( \frac{2 + \sqrt{2}}{2} \right) - \frac{2 - \sqrt{2}}{4} \ln \left( \frac{2 - \sqrt{2}}{2} \right) \] (A.3)

For the Tricritical model, the primary fields the crosscap coefficients are

\[ \Gamma_{00} = \sqrt{\frac{2 + \sqrt{2}}{2}}, \quad \Gamma_{0e} = \sqrt{\frac{2 - \sqrt{2}}{2}}, \quad \Gamma_{0\sigma} = 0, \quad \Gamma_{0\sigma'} = 0. \] (A.4)

These results agree with the quotients between them found using the sewing constraints in [41].

The crosscap state is

\[ |C_{\mathcal{P}_0} \rangle = \Gamma_{00} |C_0 \rangle + \Gamma_{0e} |C_e \rangle + \Gamma_{0e'} |C_{e'} \rangle + \Gamma_{0e''} |C_{e''} \rangle. \] (A.5)

The corresponding LREE is given by

\[ S_{|C_{\mathcal{P}_0} \rangle} = \frac{7\pi \ell}{60 \varepsilon} - 4 \left[ s_2^2 \ln \left( \frac{s_2^2}{s_1^2} \right) - s_1^2 \ln \left( \frac{s_1^2}{s_2^2} \right) \right] + (s_1^2 + s_2^2) \left[ (2 - \sqrt{2}) \ln (2 - \sqrt{2}) + (2 + \sqrt{2}) \ln (2 + \sqrt{2}) \right]. \] (A.6)

For the Tetracritical model, we label the different conformal fields with \( i = 0, \ldots, 9 \) in the order given for the conformal weights in 3.1. The crosscap coefficients are:

\[ \Gamma_{00} = \sqrt{3} + \frac{1}{3^{1/4}} \frac{s_1}{\sqrt{s_2}}, \quad \Gamma_{01} = \sqrt{3} - \frac{1}{3^{1/4}} \frac{s_2}{\sqrt{s_1}}, \quad \Gamma_{03} = \sqrt{3} + \frac{1}{3^{1/4}} \frac{s_2}{\sqrt{s_1}}, \]
\[ \Gamma_{05} = \sqrt{3} + \frac{1}{3^{1/4}} \frac{s_1}{\sqrt{s_2}}, \quad \Gamma_{06} = \sqrt{3} - \frac{1}{3^{1/4}} \frac{s_2}{\sqrt{s_1}}, \quad \Gamma_{08} = \sqrt{3} + \frac{1}{3^{1/4}} \frac{s_1}{\sqrt{s_2}}, \]

and \( \Gamma_{02} = \Gamma_{04} = \Gamma_{07} = \Gamma_{09} = 0. \)

The crosscap state and its corresponding LREE can be obtained also for the Tetracritical model. This can be performed in a straightforward way and will not be carried out here.

The matrix \( P \) for the \( SU(2)_k \) WZW model for \( k \) even is [48], then

\[ P_{jj} = \frac{2}{\sqrt{k + 2}} \sin \left( \frac{\pi (2j + 1)(2l + 1)}{2(k + 2)} \right), \quad j + l \in \mathbb{Z} \] (A.8)

\[ \hat{D} = \frac{1}{P_{00}} = \frac{\sqrt{k + 2}}{2} \csc \left( \frac{\pi}{2(k + 2)} \right), \quad D = \frac{1}{S_{00}} = \sqrt{\frac{k}{k + 2}} \csc \left( \frac{\pi}{k + 2} \right) \]

\[ \hat{d}_j = \frac{P_{0j}}{P_{00}} = \csc \left( \frac{\pi}{2(k + 2)} \right) \sin \left( \frac{(2j + 1)\pi}{2(k + 2)} \right) \] (A.9)
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