Ryu-Takayanagi Area as an Entanglement Edge Term

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Abstract

I argue that an analogy between emergent gauge theories and the bulk in AdS/CFT suggests that the Ryu-Takayanagi area term is a boundary entropy, counting the number of ways UV degrees of freedom in the bulk can satisfy an emergent gauge constraint for gravity at the entangling surface. Throughout this note, I review various definitions of EE in lattice gauge theories.
1 Introduction

In recent years, it has been suggested that “spacetime emerges from quantum entanglement” of some pre-geometric degrees of freedom [1, 2]. The most precise realization of this statement to date is the Ryu-Takayanagi formula in AdS/CFT [3], that relates the entanglement entropy of a spatial subregion in a holographic CFT to the area of the minimal homologous surface in the bulk:

\[ S_{EE} = \frac{A_{\text{min}}}{4G_N} + O(G_N^0). \]  

(1.1) reduces to the Bekenstein-Hawking formula in some [4] (though not all [5]) cases. With it, one can derive some nice results in AdS/CFT such as the equality of the linearized Einstein equations and the entanglement first law around vacuum AdS [6], and entanglement wedge reconstruction [7, 8], which quantifies how nonlocal the support of a local bulk operator is on the boundary.

The progress so far raises the natural question: what is the microscopic meaning of the Ryu-Takayanagi formula (and its cousin, the Bekenstein-Hawking entropy)? Can we understand what they are counting from the bulk point of view? For example, can we find an \(\alpha'\)-exact formula for the RT entropy on the worldsheet, that reduces to “\(A/4G_N\)” in the Einstein gravity limit?

In this note, I point out that when one compares EE in an emergent gauge theory to the RT formula with the first subleading correction, the area term “\(A/4G_N\)” looks like the gravity analog of a boundary term in the EE of an emergent nonabelian gauge theory (sometimes called an “edge mode”), which counts the number of ways that UV degrees of freedom at the entangling surface can satisfy the emergent gauge constraint. Namely, in a factorizable, lattice-regulated UV theory that is isomorphic to a lattice gauge theory in a low-energy subspace, one can write the non-universal, UV-exact EE as the algebraic EE of gauge-invariant operators from the IR point of view plus a boundary term (up to a state-independent constant), \(^1\) and the RT area term looks like the analog of this boundary term in AdS/CFT.

Related remarks have appeared elsewhere. Ref. [8] explained that the RT area operator is in the center of the algebra associated with the entanglement wedge in bulk effective field theory, suggesting that it is an edge mode; the goal of this note is to clarify which one it is. Ref. [9] first pointed out that the Bekenstein-Hawking area resembles the “\(\log \dim R\)”-type boundary term that one finds when computing EE in a lattice gauge theory using the “extended Hilbert space” definition for the EE [10, 11], which I will review below. The extra points made here are that the definition in [10, 11] is (up to the aforementioned constant and fine-print) a physical definition of EE in an emergent gauge theory, with the UV Hilbert space replacing the extended Hilbert space, and that the bulk EFT limit of AdS/CFT resembles an emergent gauge theory with the CFT as the UV theory, so replacing the B-H with the RT area term lets us sharpen their conjecture.

This note is organized as follows. In section 2.1, I review the extended Hilbert space definition of EE in lattice gauge theories [10, 11] and introduce the types of edge modes that one finds there.

\(^1\)As should become clear below, the lattice regularization is not actually so important, but I will use this language throughout the note.
In section 2.2, I argue that this definition applies to the UV-exact EE of a subregion in an emergent
gauge theory, up to a constant. In section 3, I compare to the situation in AdS/CFT. Finally in
section 4, I speculate about how to interpret the RT area term as a boundary entropy in string
theory.

There are two appendices. Appendix A compares the extended Hilbert space definition to
the algebraic definition of EE in lattice gauge theories [12]. Appendix B gives some examples of
emergent gauge theories.

2  Entanglement entropy in emergent gauge theories

2.1  Extended Hilbert space definition of EE in lattice gauge theory

Traditionally when we study entanglement entropy, we assume that the Hilbert space factorizes,
then we trace out part of it and take the EE to be the von Neumann entropy of the reduced density
matrix. In a gauge theory, the Hilbert space doesn’t factorize, so something new is needed. This
has motivated several authors to suggest definitions of EE in lattice gauge theory in recent years.
In this section, I will review a definition by Donnelly [10, 11]. The presentation in this section
closely follows [11]. Other definitions are reviewed later on in this file, particularly the definition
of [13, 14, 15] in section 2.2.1, and the algebraic definition of Casini et al. [12] in Appendix A.

The proposal of [10, 11] is the following. In a lattice gauge theory, suppose that we pick out
a subregion $A$ by drawing a boundary $\partial A$ that cuts some lattice links. Let the “extended Hilbert
space” be the minimal Hilbert space that factorizes across $\partial A$,

$$
\mathcal{H} \subset \mathcal{H}_{\text{ext.}} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}},
$$

constructed formally by lifting the gauge constraint at sites added to the lattice at the intersection
points (see Figure 2 below). In practice, we add surface charges to every intersection of $\partial A$ with a
lattice link, so that extended operators in the gauge theory (i.e. Wilson loops) can factorize into a
sum of tensor products of Wilson lines. The EE of region $A$ is then defined as the von Neumann
entropy of the reduced density matrix on region $A$ in the extended Hilbert space.

When one computes EE’s with this definition, one finds various boundary terms, that I’ll now
introduce by example.

2.1.1  Example 1: 2d Electrodynamics on $S^1$

First, consider a 2d $U(1)$ gauge theory on spatial $S^1$. The gauge-invariant operator algebra has only
one canonically conjugate pair, the holonomy $\oint A$ and the electric field $E(x)$, which is constant
everywhere by Gauss’s law. A convenient basis for the Hilbert space are the eigenstates of $E$, which
are quantized because the gauge group is compact: $\mathcal{H} : \{|n\rangle\}$ with $E|n\rangle = n|n\rangle$ for $n \in \mathbb{Z}$.

To compute the EE of an interval in the extended Hilbert space prescription, we’re instructed to
embed $\mathcal{H}$ into the minimal larger Hilbert space $\mathcal{H}_{\text{ext.}}$ that factorizes across the interval. Formally,
this should be done by lifting the gauge symmetry at the endpoints of the interval and gauge-fixing again. An intuitive way to understand the answer is that for the Hilbert space to factorize, we must be able to cut all extended operators that cross the entangling surface – in this case, the unique Wilson loop operator – by adding charges that allow it to break into a pair of Wilson lines. In this case, the extended Hilbert space doubles the physical one,

$$H_{\text{ext}} = H \otimes H,$$

with the unique embedding of states $$|n\rangle \rightarrow |n\rangle \otimes |n\rangle.$$ We now take the most general state and proceed to compute the EE. For

$$|\psi\rangle = \sum_n \psi_n |n\rangle \in H = \sum_n \psi_n |n\rangle \otimes |n\rangle \in H_{\text{ext}},$$

one finds that the reduced density matrix is the diagonal probability distribution over the electric field eigenstates,

$$\rho_A = \sum_n p_n |n\rangle \langle n|, \quad p_n = |\psi_n|^2,$$

and hence

$$S_{\text{EE}} = -\sum_n p_n \log p_n.$$

What this “EE” is computing is the perfect correlation of the electric field operator in regions $$A$$ and $$\bar{A}$$ due to Gauss’s law. This type of entropy, that measures kinematic correlations of gauge-invariant operators, is called a “Shannon edge mode”.

2.1.2 Example 2: 2d Yang-Mills on $$S^1$$

Next, consider 2d Yang-Mills with gauge group $$G$$ on $$S^1$$. The gauge-invariant algebra now contains Wilson loops in all representations $$\text{Tr}_R \exp(i \oint A)$$, and all Casimirs $$E^a E^a \ldots$$ built out of the electric field. One can show from a careful gauge fixing (see e.g. [16]) that a convenient basis for the Hilbert space is labeled by representations $$R$$ of $$G$$: $$H : \{ |R\rangle \}$$.

To compute the EE of an interval, we again extend the Hilbert space by formally lifting the
gauge symmetry at the endpoints of the interval and gauge-fixing again. Intuitively, we need to add charges in every representation to cut the loop operators in every representation. This leads to a much larger extended Hilbert space than in the abelian case, with a subspace of size \((\dim R)^2\) (the size of the matrix in each representation) assigned to each state \(|R\rangle\):

\[
\mathcal{H}_{\text{ext.}} = \bigoplus_R \{ |R, i, j\rangle \otimes \{ |R, i, j\rangle \}, \quad i, j \in 1, \ldots, \dim R. \tag{2.6}
\]

The unique embedding of the physical state \(|R\rangle\) into \(\mathcal{H}_{\text{ext.}}\) is

\[
|R\rangle \rightarrow |R, i, j\rangle \otimes |R, j, i\rangle. \tag{2.7}
\]

Now for the most general state in the physical Hilbert space,

\[
|\psi\rangle = \sum_R \psi_R |R\rangle \in \mathcal{H} = \sum_R \psi_R |R, i, j\rangle \otimes |R, j, i\rangle \in \mathcal{H}_{\text{ext.}}, \tag{2.8}
\]

the normalized reduced density matrix is

\[
\rho_A = \sum_R p_R (\dim R)^{-2} \sum_{i,j} |R, i, j\rangle \langle R, j, i|, \quad p_R = |\psi_R|^2, \tag{2.9}
\]

and one finds

\[
S_{EE} = - \sum_R p_R \log p_R + 2 \sum_R p_R \log \dim R. \tag{2.10}
\]

The Shannon edge term appears with the same interpretation as before, but there is a new term that I’ll call the “\(\log \dim R\)” edge term. It counts the perfect correlation of the surface charges along the boundaries of \(A\) and \(\bar{A}\) in the extended Hilbert space, in order to make a state in the physical Hilbert space, when the dimensions of representations are greater than 1.

The “\(\log \dim R\)” term can be written as the expectation value of a gauge-invariant operator in the center of the operator algebra on regions \(A\) and \(\bar{A}\): i.e. there exists some \(\mathcal{L}_A\), built from the Casimirs, s.t. \(\langle R|\mathcal{L}_A|R\rangle = \log \dim R\). But from this point of view, \(\mathcal{L}_A\) is a seemingly arbitrary, group-dependent combination of the Casimirs. \(^2\) A canonical counting interpretation of the “\(\log \dim R\)” term does not exist until we introduce the fictitious extended Hilbert space, at which point the canonical interpretation becomes obvious, as the correlation of boundary charges in \(\mathcal{H}_{\text{ext.}}\) to recover states in \(\mathcal{H}\), (2.7). See [17] for a related discussion.

2.1.3 Higher-dimensional lattice gauge theory and comments

In a lattice gauge theory in \(d > 2\) dimensions, one assigns the Hilbert space for an interval on \(S^1\) to each lattice link, and the gauge-invariant Hilbert space is the tensor product of the Hilbert spaces

\(^2\)E.g. \(\sqrt{4E^6E^8+1}\) for \(G = SU(2)\).
on the links, modded out by a Gauss constraint at sites; schematically,

\[ H_{\text{link}} = \bigoplus_R \{ |i, j\rangle \}, i, j \in 1, \ldots, \dim R; \quad \mathcal{H} = \bigotimes \frac{H_{\text{link}}}{\text{Gauss}}. \]  

(2.11)

At each lattice site, the Gauss constraint can be implemented by demanding that a Gauss operator \( \mathcal{G} \), acting on the Hilbert spaces of the adjacent links, acts as the identity. The details are discussed in e.g. [18] but not overly important here.

To compute the EE for a subregion \( A \) of the lattice, whose boundary \( \partial A \) cuts a collection of links \( \{ e \} \), we extend the Hilbert space at each intersection of \( \partial A \) with a link. The EE according to the extended Hilbert space definition now will be the sum of the two edge terms that we found previously (generalized to receive contributions from all boundary links), along with quantum entanglement between interior degrees of freedom. Letting \( R_\partial \) be the vector of representations labeling the state for all boundary links, one finds

\[ S_{\text{EE}} = -\sum_{R_\partial} p_{R_\partial} \log p_{R_\partial} + \sum_{R_\partial} p_{R_\partial} \sum_{e \in \{ e \}} \log \dim R_e + \text{interior EE}. \]  

(2.12)

Comments:

(*) The edge terms in (2.12) are non-universal. Their universal part may be relevant for entanglement c-theorems [19] and as order parameters for phases of matter (e.g. one needs the Shannon term to recover the topological EE [20, 21], see [14, 15]), but I won’t explore this here.

(*) Although the Shannon term gets a contribution from each boundary link, it is not quite extensive with the area of the entangling surface, due to the fact that net flux through closed regions is zero. The analog of this in the \((1 + 1)d\) example is that it does not depend on the number of intervals.

(*) The “\( \log \dim R \)”-type edge term is local to the entangling surface and extensive with area. However, it is state-dependent and can appear in physical quantities that depend on the difference of the EE between states, such as the relative entropy.

(*) Only the “interior EE” in (2.12) is distillable [14].

(*) An alternative definition for EE in a lattice gauge theory is the “electric center” definition of Casini et al. [12], which takes the EE of a set of lattice links to be the algebraic EE of the maximal gauge-invariant subalgebra supported on the links (see Appendix A for a review). A fact which will be useful later is that the algebraic definition differs from the extended Hilbert space one \(^3\) by the “\( \log \dim R \)” edge term [14]:

\[ S_{\text{alg,ginv}}(A) = \text{Shannon edge term} + \text{interior EE}. \]  

(2.13)

\(^3\)Where we take the extended Hilbert space for a set of lattice links, instead of an entangling region that cuts through links, to be the lattice Hilbert space with the Gauss law lifted at the boundary sites. See section 2.2.1.
This is not surprising. While the Shannon EE and interior EE both describe correlations between gauge-invariant operators in regions $A$ and $\bar{A}$, the “log dim $R$” edge term describes correlations of fictitious Wilson line operators in regions $A$ and $\bar{A}$, that are not part of the gauge-invariant operator algebra.

(*) So far, the extended Hilbert space definition of the EE in a lattice gauge theory is completely formal. At this level, the main reason to prefer it to other definitions (e.g. the algebraic one mentioned above) is that both edge terms in (2.12) are needed to agree with the replica trick answer in topological gauge theories, where one can compute the partition function on replicated manifolds without worrying about the coupling to the conical singularity. For example, consider the EE of an interval in the de Sitter Hartle-Hawking state of 2d YM on $S^1$ [11]. This state corresponds to a particular set of coefficients $\psi(R)$ in (2.8), that one can plug into (2.9), (2.10). On the other hand, one can compute the EE by the replica trick, where $\text{Tr} \rho_A$ is $Z_{S^2}$, and the EE is a derivative wrt the area of the $S^2$, since 2d YM is a TQFT. The answers agree only when both edge terms are included.

I’ll now argue that for a lattice gauge theory that emerges from a factorizable UV theory (as defined by the conditions listed below), one can write the UV-exact EE in the form (2.12), up to a state-independent constant. (2.12) is a “more IR” way of writing the UV-exact answer.

2.2 EE in emergent gauge theories

In this section, I’ll take an emergent gauge theory to have the following properties.

1. The UV Hilbert space factorizes.

2. The choice of Hamiltonian is such that a low-energy subspace of the UV Hilbert space is isomorphic to the Hilbert space of a lattice gauge theory (possibly with matter at the lattice sites). In particular,

(a) All operators of the lattice gauge theory can be identified with UV operators s.t. their actions on states respects the duality map.

(b) Given region $\mathcal{E}_A$ of the lattice gauge theory, there is a region $A$ in the UV theory s.t. all gauge-invariant operators with support only on $\mathcal{E}_A$ can be reconstructed as UV operators with support only on $A$, and all gauge-invariant operators with support only on $\mathcal{E}_A$ can be reconstructed as UV operators with support only on $\bar{A}$.

(c) In addition, IR operators not fully contained in either $\mathcal{E}_A$ or $\mathcal{E}_A$ cannot be reconstructed in just $A$ or $\bar{A}$.  

Note that this is not one of the assumptions in [8] which we’ll later compare to. (The analogous assumption in a related argument of [22] is that the reconstruction of the Wilson line anchored to both boundaries in the TFD state of AdS/CFT has support on both CFT’s by the extrapolate dictionary.)
In practice, we are assuming that the scale of corrections to the IR gauge theory can be made parametrically small by tuning some external parameter. Examples of such theories are given in Appendix B.

For now, I assume that the IR gauge theory is lattice-regulated to compare to (2.12), but this isn’t really needed for the argument; I’ll discuss the regulator-independent interpretation below.

Suppose that we’re handed a state in the low-energy subspace of such a theory, and a region $A$. The goal of this section is to argue that, up to a constant that does not depend on the choice of the state, the microscopic EE of region $A$ equals (2.12) for region $\mathcal{E}_A$.

2.2.1 Example: Lattice gauge theory without gauge constraints

As a warm-up, consider the following simple class of models. Suppose that the UV Hilbert space is the Hilbert space of a lattice gauge theory but without imposing gauge constraints at the sites, $\mathcal{H} = \otimes \mathcal{H}_{\text{link}}$ (compare with (2.11)), and the Hamiltonian comes with a potential that makes violations of the would-be Gauss law energetically costly: i.e., it contains something like

$$H \supset U \sum_i |\mathcal{G}_i - 1|$$

(2.15)

where the index $i$ runs over all sites, $\mathcal{G}_i$ is the Gauss operator at site $i$, and $U$ is large. In this example, the IR gauge theory lives on the same lattice as the UV theory, and for a region $A = \mathcal{E}_A$ specified by a collection of lattice links, the UV Hilbert space is almost the extended Hilbert space of region $\mathcal{E}_A$ in the IR, except that it is much larger away from the entangling surface, since the Gauss law is not imposed in the interiors of regions $A$ and $\bar{A}$. This situation was studied in refs [13, 14, 15].

We would prove the claim for this example if we can show that the EE does not care that the UV Hilbert space is larger than the extended Hilbert space in the interiors of regions $A$ and $\bar{A}$. But this is true because we can choose a basis for the interiors of $A$ and $\bar{A}$ that divide their Hilbert spaces into a direct sum of subspaces satisfying the emergent gauge constraint at interior sites and not, $^5$ and the latter sectors of the Hilbert space cannot contribute to the trace in either $\text{Tr}_{A\rho}$ or $\text{Tr}_{A\rho} \log \rho_A$, by the assumption that the initial state was in the low-energy subspace.

2.2.2 General case

In general, the UV and IR theories will be less obviously related, not living on the same lattice. Nonetheless, we argue that the UV EE takes the form (2.12) up to a state-independent constant. The observations needed are the following.

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$^5$ To do this in practice, we’d reconstruct all the Gauss operators of the lattice gauge theory in the interiors of regions $\mathcal{E}_A, \mathcal{E}_{\bar{A}}$ and use their UV images to pick bases for $A$ and $\bar{A}$. 

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Figure 2: The left-hand side illustrates the extended Hilbert space definition of EE in a lattice gauge theory. We take the entangling region $A$ (in blue) to go through a set of links, and extend the Hilbert space at each intersection of $\partial A$ with a link (in red). The middle picture depicts the situation in section 2.2.1, where we specify an entangling region by a collection of links. The right-hand side illustrates the generic situation in an emergent gauge theory, where the UV Hilbert space might be the tensor product of microscopic Hilbert spaces at the sites of a UV lattice.

1. **Relating the extended Hilbert space to an extended operator algebra.** We first observe that, as mentioned above, the extended Hilbert space of section 2.1 can be thought of as a representation of a formal extended operator algebra, that contains in addition to the gauge-invariant operator algebra, Wilson lines in every representation ending on the entangling surface. In particular, the “$\log \dim R$” edge term comes from the correlation of the Wilson lines in a gauge-invariant state. From this point of view, the extended Hilbert space definition of the EE is the algebraic EE of the maximal subalgebra supported on the entangling region, including the fictitious Wilson lines (see appendix A for the definition of algebraic EE).

2. **Wilson loops factorize in emergent gauge theories.** The second point is that in an emergent gauge theory, such Wilson lines are not so fictitious; this is the argument of [22]. Since the UV Hilbert space factorizes, we must be able to write any operator, including the UV reconstruction of a Wilson loop in any representation with support on both $E_A$ and $\bar{E}_A$, as a sum of tensor products of operators in $A$ and $\bar{A}$. This means that in the UV or at some intermediate scale, there are charged fields to cut the Wilson loop, and the Wilson loop looks at that scale like an entangled product of Wilson lines. I.e. factorizability in the UV implies that the extended Hilbert space of the lattice gauge theory, not just the physical Hilbert space, is equivalent to a subspace of the UV Hilbert space, with an isomorphism between all states and operators. Of course, the UV Hilbert space is generally much larger than the extended Hilbert space.

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6The situation in the previous example 2.2.1 most clearly illustrates this. Imagine building up a gauge-invariant state in the lattice gauge theory by acting with Wilson loops on a ground state with zero entanglement. Each loop operator is defined by tracing over the holonomies $U_{ij}$ on the associated links, which are physical operators in the UV. The indices of the $U_{ij}$’s will be maximally entangled across boundary sites by matrix multiplication. This gives the “$\dim R$” degeneracy.
3. *EE’s of isomorphic subalgebras are equal up to a constant.* We wanted to show that the EE of region $A$ in the UV, algebraically the EE of the maximal subalgebra on region $A$, is the same as the EE of region $\mathcal{E}_A$ in the extended Hilbert space of the IR gauge theory, algebraically the maximal “extended operator subalgebra” on region $\mathcal{E}_A$, up to a state-independent constant. By the assumption that we are in the low-energy subspace, the reduced density matrix whose von Neumann entropy gives the algebraic EE of region $A$ in the UV is the image of the reduced density matrix of region $\mathcal{E}_A$ in the IR extended Hilbert space, with zero support on other UV operators (generalizing the argument in 2.2.1). But now we are comparing the EE’s of isomorphic subalgebras, which can differ only by a constant from the sizes of the representations of the algebras.

To summarize, in an emergent lattice gauge theory, (2.12) holds up to a state-independent constant $c$. Schematically,

$$S_{EE,UV}(A) = S_{EE,IR}(\mathcal{E}_A) + c = \text{Shannon edge} + \log \text{dim } R \text{ edge} + \text{interior EE} + c. \quad (2.16)$$

Comments:

(*) Regarding the state-independent constant, one way to see that we need it is that to any UV theory, we can add a lattice-regulated free massive scalar field with mass $m \gg$ the crossover scale from the UV to IR theory. It will not affect the map between states on the low-energy manifold of the UV Hilbert space and the Hilbert space of the lattice gauge theory. Each state in the low-energy manifold of the UV Hilbert space now simply comes tensored with the ground state of the scalar field. For a given subregion, the EE of this decoupled sector adds a nonzero constant to the EE relative to what it was before. However, such a state-independent constant should be contrasted with the edge terms in (2.12), which are non-universal but state-dependent. E.g. both edge terms will in general affect the relative entropy of states on the low-energy manifold, while state-independent constants will not.

(*) The edge terms in (2.16) were defined for a lattice regularization, but the argument did not really depend on this: the main thrust of the argument is that in an emergent gauge theory, the UV EE has a contribution from the correlation of extended IR operators cut by the entangling surface. In regulator-independent language, the “$\log \text{dim } R$” edge term quantifies these correlations of UV degrees of freedom at the entangling surface due to the emergent gauge constraint, that are not visible in the IR.
3 Analogy to AdS/CFT

AdS/CFT is an emergent gauge theory with the CFT as the factorizable UV theory, and bulk EFT (including perturbative gravity) on an AdS background as the IR-emergent description in a low energy subspace. Let us compare the Ryu-Takayanagi formula with its $1/N$ correction [23] to (2.16) and see where it leads.

It is not obvious that the argument of the previous section, stated in terms of factorizability of Wilson loop operators, carries over to emergent gravity (though see [24] for a discussion of gauge-invariant gravitational observables), so this comparison is a suggestive analogy, not a proof.

For this purpose, a recent repackaging of the RT formula + $1/N$ correction by Harlow [8] is convenient as it separates out the algebraic EE of gauge-invariant bulk operators in the entanglement wedge from the rest of the RT formula. The punchline of [8] is that RT + the $1/N$ correction and entanglement wedge reconstruction are equivalent; this was previously shown in [7], so our comparison does not rely on [8], but it makes our comparison more straightforward.

Ref. [8] proves the following theorem for quantum systems. Suppose that we have a (finite-dimensional) Hilbert space that factorizes, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$; a subspace $\mathcal{H}_{IR} \subseteq \mathcal{H}$, and a subalgebra $\mathcal{A}_{ginv}$ of the operator algebra, whose action on states in $\mathcal{H}_{IR}$ keeps us inside $\mathcal{H}_{IR}$. Then, the following statements are equivalent:

1. There is a subalgebra $\mathcal{A}_{ginv,A} \subset \mathcal{A}_{ginv}$ s.t. $\forall |\tilde{\psi}\rangle \in \mathcal{H}_{IR}$ and $\forall \tilde{O} \in \mathcal{A}_{ginv,A}$, there exists an operator $O_A$ acting on $\mathcal{H}_A$ s.t. $O_A|\tilde{\psi}\rangle = \tilde{O}|\tilde{\psi}\rangle$. Likewise, for all operators in the commutant of $\mathcal{A}_{ginv,A}$ on $\mathcal{H}_{IR}$, which I will denote $\mathcal{A}_{ginv,\bar{A}}$, there exists an operator supported on $\mathcal{H}_{\bar{A}}$ that reproduces its action on $\mathcal{H}_{IR}$.

2. There exists an operator $L_A$ in $\mathcal{A}_{ginv,A} \cap \mathcal{A}_{ginv,\bar{A}}$ s.t. $\forall \rho \in \mathcal{H}_{IR}$,

$$S_{EE}(\rho_A) = \text{Tr}(\rho L_A) + S_{alg}(\rho, \mathcal{A}_{ginv,A}),$$ (3.1)

where $S_{alg}(\rho, \mathcal{A}_{ginv,A})$ is the algebraic EE of the subalgebra $\mathcal{A}_{ginv,A}$ in state $\rho$, as defined in Appendix A.

To interpret this in AdS/CFT, we take $\mathcal{H}$ to be the CFT (UV) Hilbert space, $\mathcal{H}_{IR}$ to be the low-energy subspace of effective field theory on AdS (or whatever one chooses as the “code subspace”), $\mathcal{A}_{ginv}$ to be the gauge-invariant operators of bulk effective field theory, and $\mathcal{A}_{ginv,A} (\mathcal{A}_{ginv,\bar{A}})$ to be the operators of bulk effective field theory with support entirely on the bulk entanglement wedge $\mathcal{E}_A (\mathcal{E}_{\bar{A}})$ of boundary regions $A (\bar{A})$. Then statement 1 is entanglement wedge reconstruction with complementary recovery [7], which identifies region $A (\bar{A})$ of the CFT with the entanglement wedge

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[7] As generally assumed in applications of the RT formula. I stress that we are making an analogy between the IR-emergent gauge theory and the bulk, and we are not interested in the edge terms arising from the gauge symmetry of the CFT.

[8] A third statement, an algebraic version of “bulk relative entropy = boundary relative entropy” [25], was also shown to be equivalent, but we won’t need to use it here.
of the bulk EFT on AdS (and its complement), and statement 2 looks like the Ryu-Takayanagi formula with the $1/N$ correction [23], where

$$\mathcal{L}_A = \frac{A}{4G_N} + \ldots , \quad \text{(3.2)}$$

the ellipses contain some of the “Wald-like terms” of the $1/N$ correction [23], and $S_{alg}(\rho, A_{ginv, A})$ contains the other “Wald-like terms” as well as \( S_{bulk-ent} \) of the $1/N$ correction [23].

Assuming that the formula (2.16) for EE in an emergent gauge theory holds, if we now compare eqs. (2.13), (2.16),

$$S_{EE,UV}(A) = S_{alg,ginv}(E_A) + \log \text{dim } R \text{ edge term}, \quad \text{(3.3)}$$

to (3.1), we conclude that the RT area term is a “$\log \text{dim } R$”-type edge term for the bulk. This observation is the main point of this note.

Comments:

(*) Eq. (2.16) was ambiguous up to a state-independent constant. However, the RT area is state-dependent with a large enough code subspace, so this does not pollute the identification of the RT area term with the “$\log \text{dim } R$”-type edge term.\(^9\)

(*) In the lattice gauge theory, the “$\log \text{dim } R$” edge term literally counted the dimensions of the gauge group representations labeling the boundary links of the lattice in a given state. Accordingly, [9] suggested that perhaps the universal origin of the area term can be understood from the representation theory of the diffeomorphism group. One can also try to interpret the edge term in string theory; I speculate about this in the next section.

(*) There are related results in the literature. One can study edge terms in holographic tensor networks (e.g. [26]). Previously, [27] interpreted the Bekenstein-Hawking entropy of a BTZ black hole as a boundary entropy.

(*) As an aside, the bulk Shannon entropy is the entropy of mixing when one considers a state dual to a superposition of classical geometries [8, 28]. This fact was not needed for the discussion here, since the Shannon term was absorbed into $S_{alg,ginv}(E_A)$.

(*) The renormalization of $G_N$ [29] and the dependence on the choice of code subspace for how bulk EE is distributed between the RT area term and the $1/N$ correction [8], seems related to the simpler fact that the distribution of the entanglement between the interior EE and edge terms in lattice gauge theory depends on the lattice spacing. It would be interesting to make this analogy precise.

\(^9\)Alternatively, perhaps one can evade the appearance of state-independent constants in section 2 by assuming that the crossover scale for the emergence of the gauge theory is higher than all other scales in the problem. In AdS/CFT, gravity presumably emerges at the cutoff scale for the effective field theory, although it may emerge simultaneously with all gauge fields.
4 Discussion

To summarize, so far we’ve argued by analogy that the RT area term “$A/4G_N$” looks like a bulk edge mode, counting the number of ways that UV degrees of freedom in the bulk can satisfy an emergent gauge constraint for gravity at the entangling surface.

If the RT area is indeed a boundary entropy counting UV correlations that are invisible in bulk effective field theory, what is it counting? I will conclude with some speculative comments.

Does “$A/4G_N$” count Chan-Paton factors in string theory? To continue pushing the analogy between lattice gauge theory and the bulk side of AdS/CFT, it is tempting to make an analogy between a Wilson loop on the lattice and a closed string in the bulk. Could it be that at high energies in string theory, closed strings can “factorize” on the pre-geometric degrees of freedom that make up spacetime, and “$A/4G_N$” counts their Chan-Paton factors? A bit more precisely, “$A/4G_N$” was identified with the log of the number of boundary states, so should be the number of ways to glue two open strings into a closed one, since the background is a string gas. Could it be that the scaling with area comes from the fact that we can do the gluing all along the entangling surface, and the “$O(1/G_N)$” from there being $O(N)$ Chan-Paton factors at each of the two ends?

There is actually an independent argument that a fundamental string cut by an entangling surface comes with an $O(N)$ degeneracy. We can add a single string to the bulk of empty AdS with a space-filling flavor brane by putting a color singlet quark-antiquark pair at antipodal points of the boundary $S^1$ [30]. Lewkowycz and Maldacena computed the extra boundary entanglement from the $q\bar{q}$ pair and found that it always comes with a “$\log N$”, basically from cutting the color flux tube on the boundary [31]. But in the bulk, all we did was cut one string with an entangling surface.

What is the string dual of a density matrix? The naive cartoon in Figure 3 suggests that the Lorentzian string dual of a density matrix $\rho_A$, for a subregion $A$ in a state of a holographic CFT with a geometric dual description, is the entanglement wedge of $A$ with a stack of $O(N)$ end-of-
the-world branes at the RT surface, and with open strings on those branes being in a superposition of the Chan-Paton factors (see Figure 4). A related picture was advocated in [32] at the level of bulk effective field theory, who argued that the gravity dual of Rindler microstates in holographic CFT’s should be bulk spacetimes that are indistinguishable from the AdS-Rindler wedge except in a region of size ε around the horizon where geometry breaks down, ε being the cutoff for the bulk effective field theory. However, we cannot test this speculation because we have not independently defined what these objects are at the entangling surface.

A Euclidean-signature conjecture. To try to reach a more testable conjecture, suppose that ρ_A is the Rindler density matrix of the vacuum state of the boundary CFT, whose modular Hamiltonian generates a geometric flow. In this special case, analytically continuing the Rindler time coordinate on both the boundary and bulk leads to the following Euclidean version of the conjecture. There is some string background which is the answer to the question: the perturbative string partition sum on which Euclidean background equals log Tr ρ_A of the boundary CFT? The analytic continuation of our Lorentzian cartoon suggests that the answer is not the smooth cigar, but rather, is the cigar with a codimension-2 object of tension $T \sim 1/g_s^2$ at the tip of the cigar.  

There is a recent piece of evidence for this Euclidean proposal [33]. 2d Yang-Mills on $S^1$, at large $N$, looks like a string field theory: one can choose a basis for the Yang-Mills Hilbert space labeled

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10 Note that the answer to a superficially related question in field theory, “the QFT partition sum on which Euclidean manifold equals Trρ_{Rindler}?” for the Rindler wedge of a QFT on $\mathbb{R}^{d,1}$ (and the obvious generalization to other situations where the modular Hamiltonian generates geometric flow), is the smooth Euclidean manifold without any defect at the origin by the usual path integral argument. Namely, we set up the density operator on region $A$ by a Euclidean path integral with two open cuts along $A$ at $\tau = 0$, and Tr$\rho_A$ sews together the open cuts. But it does not follow that the Euclidean string background whose partition function gives log Tr$\rho_{\text{Rindler}}$ on the boundary must be the smooth cigar. We cannot trace out half a string background as a worldsheet operation, and quantum gravity versions of this argument are only defined holographically, with the smooth cigar being the dominant saddle in Einstein gravity.
by elements of the symmetric group, and interpret the cycles as closed strings wound around the $S^1$ [34]. The analog of the Rindler wedge for a QFT on $S^1$, whose modular Hamiltonian generates a geometric flow, is an interval $A$ in the $dS_2$ Hartle-Hawking state. For 2d Yang-Mills, the Euclidean partition sum that reproduces $\text{Tr} \rho_A$ is the partition sum on the smooth sphere, $\text{Tr}_A \rho_A = Z_{S^2}^M$, as it had to be. On the other hand, in the dual interpretation, the string-field partition sum that reproduces $\text{Tr} \rho_A$ is not the partition sum on the smooth sphere, but rather, the partition sum on the sphere with two $T \sim 1/g_s^2$ defects at the ends of the interval $A$. Namely, one can re-write $Z_{S^2}^M$ as a sum over all branched coverings of the $S^2$ that wrap two pointlike defects at the ends of the interval, with a factor of $N$ associated to each closed cycle around the defects [35]. The two points at the ends of the interval are the analog of the tip of the cigar in higher-dimensional examples.

Can we test this? We’d like to find a way to test this in higher-dimensional examples. 11 As usual, we are limited by the lack of solvable worldsheet CFT’s. One CFT that may perhaps be relevant is the $SL(2)_k/U(1)$ CFT [36], which in the limit $k \to \infty$, is believed to be the worldsheet CFT of string theory on 2d Euclidean Rindler space. In a QFT, the Euclidean Rindler wedge for flat space is a smooth plane. On the other hand, in our speculative discussion above, string diagrams in whichever background reproduces $\text{Tr} \rho_{\text{Rindler}}$ for the vacuum state of a holographic CFT as $R_{\text{AdS}} \to \infty$ (modulo all the subtleties in taking such a limit) would differ from string diagrams in flat space by couplings to a defect at the origin, with a boundary state that scales as $N$.

Therefore, a sign that the $SL(2)_k/U(1)$ CFT at large $k$ is relevant to our discussion would be if it comes with extra states relative to the flat space sigma model. In fact, Giveon and Itzhaki have recently argued that this is true, with various pieces of circumstantial evidence [37, 38]. For example, they found that contributions to the elliptic genus of the $(N = 2$ version of the) $SL(2)/U(1)$ CFT from normalizable discrete states at finite $k$, survive the large $k$ limit [38]. 12 Does the theory contain extra states relative to the flat space sigma model, and if so, could it be that they reflect a $T \sim 1/g_s^2$ defect at the origin? It would be interesting to understand this better.

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11Perhaps it’s worth emphasizing that at this point in the discussion, we’re looking for a way to count these conjectured Chan-Paton factors: although EE led us here, we are not here interested in defining spacetime EE on the worldsheet, hence we do not want to differentiate wrt the conical angle in Euclidean space, as in [29].

12In [37], they also argued that the contribution from discrete states in the $T^2$ partition function of the $N = 2$ $SL(2)_k/U(1)$ CFT, as isolated following [39], survives the large $k$ limit. However, this computation is more subtle than the elliptic genus one in that the contribution from the continuum sector diverges and can contribute at order $k^0$ as well. Hence it need not follow that the $T^2$ partition function of the full theory at large $k$ is different than in flat space. Indeed, with a different regulator, they later found a different answer [38].
A Algebraic definition of EE in lattice gauge theories

Given a state $|\psi\rangle$ in the Hilbert space of a quantum system and a subalgebra $A_0$ of the operator algebra, one can define an EE for the subalgebra (see e.g. [40] and a review in [8]). The starting point is that there is in general a unique element $\rho \in A_0$ s.t.

$$\text{Tr}_H(\rho O) = \langle \psi | O | \psi \rangle, \quad \forall O \in A_0 ,$$  \hspace{1cm} (A.1)

since one can expand $\rho = \sum_{O_i \in A_0} p_i O_i$ in a basis for $A_0$, and the condition (A.1) gives one equation for each unknown $p_i$. The von Neumann entropy of $\rho$ is well-defined, and it is tempting to take it to be the EE of the subalgebra,

$$S_{EE}(A_0) = \text{Tr}_H \rho \log \rho .$$  \hspace{1cm} (A.2)

However, this concise definition has the shortcoming that if we also take the Hilbert space $H$ to be the global one, the EE of the maximal subalgebra on a region $A$ in a factorizable QFT will not equal the EE one obtains from the partial trace, differing by a constant related to the ratio of the dimension of the global Hilbert space to the dimension of the Hilbert subspace on region $A$ (see e.g. [41]). One can always pick out the appropriate representation by hand, or define (A.2) wrt the global Hilbert space and keep this constant in mind for applications, but to automatically land on the standard result when the Hilbert space factorizes, one can proceed by the following algorithm [12]:

Given a subalgebra $A_0$ with a center $Z$, choose a basis that diagonalizes $Z$ in the global Hilbert space. In this basis, the elements of $A_0$ will take a block-diagonal form, and the algebra generated by $A_0$ and its commutant $A'_0$ will have the form

$$
\begin{pmatrix}
    A_1 \otimes A'_1 & 0 & \ldots & 0 \\
    0 & A_2 \otimes A'_2 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & A_m \otimes A'_m
\end{pmatrix},
$$  \hspace{1cm} (A.3)

with each $A_k, A'_k$ included in $A_0, A'_0$ respectively.

When $Z$ is nontrivial, $A_0$ and its commutant do not generate the entire algebra $A$, and the global density matrix $\rho$ may have off-diagonal elements. The algorithm instructs us to erase the off-diagonal elements of $\rho$. Then in each block, we partial trace over $A'_k$. The von Neumann entropy of the resulting density matrix agrees by construction with the standard result in a factorizable theory, with $A_0$ the maximal subalgebra on a factor, and with a trivial center. This algorithm is what I will refer to as the “algebraic EE” in this paper.

Casini et al. [12] suggested to define the EE of a collection of links in lattice gauge theory as the algebraic EE of a gauge-invariant subalgebra supported on the region. Actually, ref. [12] offered multiple definitions, corresponding to different choices of the subalgebra for a given region. In their “electric center” choice, one takes the EE of a collection of links in lattice gauge theory...
Figure 5: The electric center choice of [12] defines the EE for the collection of links marked in black above as the algebraic EE of the maximal subalgebra supported on those links. The magnetic center choice defines the EE for the collection of links marked in black to be the algebraic EE of the maximal subalgebra supported on the red links.

...to be the algebraic EE of the maximal gauge-invariant subalgebra supported on the region. In their “magnetic center” choice, one takes the EE on a collection of links to be the algebraic EE of the maximal gauge-invariant subalgebra supported in the interior of the region, excluding the boundary links. The claim to fame of the magnetic center choice is that it is related to the electric center under duality (so maximal algebras dualize to non-maximal ones; see e.g. [42, 43]). In this file, I refer specifically to the electric center choice, or choice of the maximal subalgebra on a region as the “algebraic definition”.

As mentioned above, (2.13), the algebraic definition of EE in a lattice gauge theory and the extended Hilbert space definition of section 2.1 turn out to disagree in nonabelian gauge theories by the “log dim $R$” edge term [14]. From the point of view of emergent gauge theory, the reason is that the extended Hilbert space definition gives the fine-grained EE WRT a UV observer, while the algebraic EE for gauge-invariant operators is a coarse-grained EE for an observer below the UV/IR crossover scale, that does not see the correlations of the UV degrees of freedom along the entangling edge. From this point of view, the different center choices in [12] are different coarse-grainings.

B Examples of emergent gauge theories

In this section, I review some examples of emergent gauge theories. This section is basically a brief summary of the references given below.

B.1 Toric code

The Kitaev toric code [44] is the simplest example of an emergent gauge theory. It belongs to the class of model discussed in Section 2.2.1, where the UV Hilbert space is the Hilbert space of a lattice gauge theory without the Gauss constraints at the vertices. Namely, one assigns a qubit to each lattice link, and the UV Hilbert space is the tensor product of the Hilbert spaces on the links,
The Hamiltonian is

\[ H = -U \sum_i \prod \sigma^x - g \sum_p \prod \sigma^z, \]  

(B.1)

where \( i \) runs over the lattice sites; \( p \) runs over the plaquettes; one takes the product over all links that end on site \( i \) in the first term, and over all links in plaquette \( p \) in the second term. Because the two terms in (B.1) commute, we can diagonalize the Hamiltonian by independently minimizing each. The Gauss operators in \( \mathbb{Z}_2 \) lattice gauge theory are \( \mathcal{G} = \prod \sigma^x \) over all links adjacent to a site, with eigenvalue +1 on gauge-invariant states and −1 otherwise. Hence as we make \( U \) parametrically large (\( \gg g \) and the lattice spacing), (B.1) is minimized on gauge-invariant states, and the low-energy Hilbert space of the toric code coincides with that of \( \mathbb{Z}_2 \) lattice gauge theory.

This model is solvable and has interesting properties outside the scope of the discussion here. On topologically nontrivial manifolds, the ground state is degenerate, exhibiting topological order. The phase diagram of the theory as one tunes the relative strengths of perturbations to the Hamiltonian is understood. See [45, 46] for further discussion.

### B.2 Example with continuum limit

The previous example is somewhat unsatisfactory from the high energy theorist’s point of view because there is no continuum version of the UV theory. In continuum examples, the duality map is often less obvious, and one sees the emergent gauge theory in two steps: first a kinematic change of variable makes an auxiliary vector field appear, then one must show that a kinetic term for it is generated by dynamics. The latter is easier to see with continuum methods (see e.g. the recent discussion of the \( \mathbb{C} \mathbb{P}^{N-1} \) model in [22]). Given a priori knowledge that a kinetic term is generated, the change of variable explains how to explicitly “reconstruct operators” of the IR theory.

Here is an example taken from ref. [47]. Consider a 4d Euclidean cubic lattice with \( N(N-1)/2 \) bosonic quantum variables at each site, each valued on an \( S^1 \) target. Let us label them by \( e^{i\theta_{ab}^i} \) with \( \theta_{ab}^i = -\theta_{ba}^i \) and \( a, b, \in 1, \ldots, N \). We take the Hamiltonian to be

\[ H = -t \sum_{\langle i, j \rangle} \sum_{a,b} \cos(\theta_{i}^{ab} - \theta_{j}^{ab}) + K \sum_i \sum_{a,b,c} \cos(\theta_{i}^{ab} + \theta_{i}^{bc} + \theta_{i}^{ca}), \]  

(B.2)

where the index \( i \) labels lattice sites and \( \langle i, j \rangle \) denotes the sum over nearest-neighbor sites. The first term is the kinetic term for the \( \theta_{ab}^i \)’s in the continuum, and the second is a potential for them. Now we take the large \( K \) limit. The potential imposes the dynamical constraint

\[ \theta_{i}^{ab} + \theta_{i}^{bc} + \theta_{i}^{ca} = 0. \]  

(B.3)

Solutions of (B.3) can be parametrized by new variables

\[ \theta_{i}^{ab} = \phi_{i}^{a} - \phi_{i}^{b}, \]  

(B.4)

that are unrestricted on the low-energy manifold. The effective Hamiltonian on the low energy
manifold is
\[
H = -t \sum_{\langle i,j \rangle} \left( \sum_a e^{i(\phi^a_i - \phi^a_j)} \right) \left( \sum_b e^{-i(\phi^b_i - \phi^b_j)} \right).
\] (B.5)

With another change of variable
\[
\eta_{ij} = \sum_b e^{-i(\phi^b_i - \phi^b_j)},
\] (B.6)

(B.5) becomes
\[
H = t \sum_{\langle i,j \rangle} \left[ |\eta_{ij}|^2 - |\eta_{ij}| \sum_a e^{i(\phi^a_i - \phi^a_j - a_{ij})} - \text{cc.} \right]
\] (B.7)

with \(a_{ij}\) the phase of \(\eta_{ij}\), which is invariant under
\[
\begin{align*}
\phi^a_i & \rightarrow \phi^a_i + \varphi_i, \\
a_{ij} & \rightarrow a_{ij} + \varphi_i - \varphi_j
\end{align*}
\]
for any \(\varphi\). Hence the low-energy effective Hamiltonian has a local \(U(1)\) symmetry.

Gauge-invariant IR operators, e.g. products of \(\eta_{ij}\)’s around closed loops on the lattice, can be mapped explicitly to the UV \(\theta\)’s (B.4): this is built into the redefinition. Since the gauge theory in this case is \(U(1)\), this example is too simple to support a “log \(\text{dim } R\)”-type edge term, but illustrates the general idea.
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