Quantile-based Random Sparse Kaczmarz for Corrupted, Noisy Linear Inverse Systems

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Abstract

It is common for us to meet with large-scale corrupted and noisy linear systems in practical applications, where corruptions are that a fraction of measurements are largely corrupted while noise is that all measurements are damaged. Motivated by stefan steinerberger, we propose a quantile-based randomized sparse Kaczmarz method for obtaining sparse solutions of corrupted and noisy linear systems. We use Bregman projection for sparsity and quantile of residuals for detecting corrupted equations in each iterate. Theoretically, we show it converges to true solution of uncorrupted and noiseless linear system in expectation related to two key constants. Numerically, we demonstrate the superiority of our algorithm in terms of recovering sparse solutions through several experiments.

Keywords. Randomized Kaczmarz method, Linear convergence, Sparse solution, Bregman projection, Bregman distance

1 Introduction

Recently, there has been tremendous interest in developing Kaczmarz-type methods for large-scale linear inverse systems in many applied fields, such as compressed sensing [18], phase retrieval [28], tensor recovery [4, 5], medical imaging [7] and so on. When we apply Kaczmarz-type methods, a big challenge in many practical applications is to deal with corrupted data. In general, there are two types of corruptions: the first is random noise that every data is slightly damaged due to imprecision or processing, the second is corruption that a small fraction of data is largely damaged due to data collection, transmission, adversarial components, or modern storage systems [14].

Mathematically, a linear inverse problem in a finite-dimensional Euclidean space can be modeled by

\[ Ax = b, \]

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where the matrix $A \in \mathbb{R}^{m \times n}$ models the linear measurement procedure and the vector $b \in \mathbb{R}^m$ records the measurement data. Due to corruptions and noise in practical measure, the observed data $b$ is usually not exactly equal to $A \hat{x}$, where $\hat{x}$ stands for the unknown vector we want to recover. If we let $\tilde{b} = A \hat{x}$, the observed data $b$ can be modeled as

$$b = \tilde{b} + b^c + r,$$

where $b^c$ is a sparse vector caused by a fraction of corruptions and $r$ is a vector representing the noise. The work is motivated by the setting where the linear inverse system is corrupted and noisy.

As an iterative method, Kaczmarz method was proposed by Kaczmarz [16] in 1937. One views the overdetermined linear system as the intersection of hyperplanes and projects orthogonally iterate onto hyperplane in a cyclical manner

$$x_{k+1} = x_k - \frac{\langle a_i, x_k \rangle - b_i}{\|a_i\|^2} a_i,$$  \hfill (1.1)

where $a_i$ stands for the $i$-th row of coefficient matrix $A$, and $b_i$ denotes the $i$-th element of $b$. Later on, the Kaczmarz as algebraic reconstruction technique (ART) was rediscovered [7, 13], so that lots of works were devoted to quantify and improve the convergence rate of the Kaczmarz method [6, 11]. However, the convergence rate of standard Kaczmarz method is hard to estimate since the permutation of the hyperplane affects convergence rate [6]. Recently, until a randomized version of Kaczmarz method was analyzed in terms of overdetermined full rank matrix in [27], the linear convergence rate of the Kaczmarz variant in expectation was given by

$$\mathbb{E}\|x_{k+1} - \hat{x}\|^2 \leq (1 - \frac{1}{\kappa^2}) \mathbb{E}\|x_k - \hat{x}\|^2,$$  \hfill (1.2)

where $\kappa = \frac{\|A\|_F}{\sigma_{\min}(A)}$, $\|A\|_F$ is the Fröbenious norm of $A$, and $\sigma_{\min}(A)$ is the non-zero smallest singular value of $A$. The paper used a random probability $p_i = \frac{\|a_i\|^2}{\|A\|_F^2}$ to select hyperplane in each iterate and strictly proved the convergence rate of RK depends on the condition number of $A$ for the first time.

In view of the popularity of sparse solutions of linear systems in many fields, specially for underdetermined linear system, sparse Kaczmarz method (SK) was proposed to generate sparse solutions by introducing Bregman projection [18]. To obtain the convergence rate of SK, [23] presented randomized sparse Kaczmarz method (RaSK) by combining with randomization and sparsity of Kaczmarz method. In this context the iteration reads as

$$x^*_{k+1} = x^*_{k} - \frac{\langle a_i, x^*_{k} \rangle - b_i}{\|a_i\|^2} a_i,$$  \hfill (1.3)

$$x_{k+1} = S_\lambda(x^*_{k+1}),$$  \hfill (1.4)

where $S_\lambda$ is the soft shrinkage operator given by $S_\lambda(x) := \max\{|x| - \lambda, 0\} \text{sign}(x)$ aiming to generate sparse solutions. It has been shown in [17] that RaSK converges to the unique solution of regularized basis pursuit problem

$$\min_{x \in \mathbb{R}^n} \lambda \|x\|_1 + \frac{1}{2} \|x\|^2, \text{ subject to } Ax = b.$$  \hfill (1.5)

In terms of the rules of sampling hyperplanes, a lot of work have been done [1, 15, 21, 25, 29, 30] for accelerating convergence rate of Kaczmarz-type methods.
In addition to noisy data that has little impact on each measurement, we may encounter cor-
rupted data that a portion of measurements are significantly affected. In this case, if current iterate 
is projected onto corrupted hyperplane unfortunately, then that makes iterate far away from true 
solution so that the convergence is severely destroyed. To deal with corruptions, windowed Kacz-
marz method was proposed, which was consisted by $W$ windows of $k$ RK iterate beginning with 
$x_0 = 0$. Then it preserved $d$ indices of the sequence with the largest sum of residuals for each win-
dow. Finally, the linear subsystem removing these preserved indices was solved to obtain solution 
of the corrupted linear system \cite{8,9}.

Next, quantile-based Kaczmarz (QuantileRK) was proposed to analyze the convergence rate of 
Kaczmarz-type methods designed for corrupted linear system \cite{10,14}. For current iterate $x_k$, given 
acceptable probability $q$, QuantileRK used the $q$-quantile of sequence $|\langle a_i, x_k \rangle - b_i|$, $1 \leq i \leq m$ to 
divide all hyperplanes into corrupted and uncorrupted. Considering large residual means that cor-
responding equation is likely to be corrupted, QuantileRK views the indices with residuals greater 
than the $q$-quantile as corrupted, and indices with errors less than as uncorrupted. When solv-
ing corrupted and noisy linear systems, QuantileRK converges linearly for certain type of random 
matrix $A \in \mathbb{R}^{m \times n}$ \cite{14}. However, the convergence results strictly depend on the liminations of 
measurement matrix. In order to relax the restrictions on matrix $A$, a more general analysis about 
QuantileRK was introduced by \cite{26}, which only required that matrix $A$ be a largely overdetermined 
matrix. Another main difference from QuantileRK in \cite{10,14} is that relaxed version of QuantileRK 
samples uniformly at random from acceptable subset instead of overall set.

Given the popularity of sparse solutions for linear systems, we seek to design an algorithm for 
solving sparse solution of a damaged linear system, in the sense that it may be damaged by noise and 
corruptions. Naturally, we want to study whether randomized sparse Kaczmarz still behaves well 
in both corrupted and noisy cases. Thus we propose a quantile-based randomized sparse Kaczmarz 
(Quantile-RaSK) mainly motivated by paper \cite{26}. And the objective of this paper is twofold. The 
first depends on the advantage of QuantileRK which detects corruptions by using the quantile of 
residuals. The other is to use Bregman projection with respect to regularized basis pursuit function 
for obtaining sparse solutions. In other words, QuantileRaSK combines quantile-based Kaczmarz 
and randomized sparse Kaczmarz for identifying corrupted equations. We prove that Quantile-
RaSK converges linearly in corrupted and noisy case, whose convergence rate is affected by the 
condition numbers of submatrices of measurement matrix. The main difficulty in the analysis of 
convergence rate is how to quantify the relationship of consecutive iterate in the sense of Bregman 
distance in both corrupted and noisy cases.

An outline of this paper is as follows. In the next section, we recall some basic notions about 
convex analysis and Bregman projections. The detail Quantile-RaSK method and the analysis of 
its convergence rate are shown in Section 3. Numerical experiments are then presented in Section 
4, showing that Quantile-RaSK has better performance than QuantileRK in terms of recovering 
sparse solutions. Section 5 is our remarks of possible future work and conclusion.

2 Preliminaries

First of all, we recall some basic tools about convex analysis \cite{20,22}.
2.1 Convex analysis tool

**Definition 2.1.** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a convex function. The subdifferential of \( f \) at \( x \in \mathbb{R}^n \) is
\[
\partial f(x) := \{ y \in \mathbb{R}^n | f(y) \geq f(x) + \langle y - x, x - y \rangle, \forall y \in \mathbb{R}^n \}.
\]

If the convex function \( f \) is assumed to be differentiable, then \( \partial f(x) = \{ \nabla f(x) \} \).

**Definition 2.2.** We say that a convex function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is \( \alpha \)-strongly convex if there exists \( \alpha > 0 \) such that \( \forall x, y \in \mathbb{R}^n \) and \( x^* \in \partial f(x) \), we have
\[
f(y) \geq f(x) + \langle x^*, y - x \rangle + \frac{\alpha}{2} \| y - x \|^2.
\]

Given a convex function \( f \). Then, \( f(\cdot) + \frac{1}{\alpha} \| \cdot \|^2 \) must be \( \lambda \)-strongly convex if \( \lambda > 0 \).

**Theorem 2.1.** If \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is \( \alpha \)-strongly convex, then the conjugate function \( f^*(x^*) = \sup_{x \in \mathbb{R}^n} \{ \langle x, x \rangle - f(x) \} \) is \( \frac{1}{\alpha} \)-smooth in the sense that
\[
\| \nabla f^*(x^*) - \nabla f^*(y^*) \| \leq \frac{1}{\alpha} \| x^* - y^* \|, \forall x^*, y^* \in \mathbb{R}^n.
\]

2.2 Bregman distance and projection

Here we will introduce Bregman distance in [2] and Bregman projection for preparing our method.

**Definition 2.3 (17).** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a convex function. The Bregman distance between \( x, y \in \mathbb{R}^n \) with respect to \( f \) and \( x^* \in \partial f(x) \) is defined as
\[
D_f^x(y) := f(y) - f(x) - \langle x^*, y - x \rangle.
\]

**Example 2.1.** When \( f(x) = \frac{1}{2} \| x \|^2 \), we have \( D_f^x(y) = \frac{1}{2} \| y - x \|^2 \).

When \( f(x) = \lambda \| x \|_1 + \frac{1}{2} \| x \|^2 \), its subdifferential is given by \( \partial f(x) = \{ x + \lambda \cdot s | s_i = \text{sign}(x_i), x_i \neq 0 \text{ and } s_i \in [-1, 1], x_i = 0 \} \). We have \( x^* = x + \lambda \cdot s \in \partial f(x) \) and \( D_f^{x^*}(x, y) = \frac{1}{2} \| y - x \|^2 + \lambda (\| y \|_1 - \langle s, y \rangle) \).

The following lemma describes the important properties of Bregman distance, which will be used in the convergence analysis of Quantile-RaSK.

**Lemma 2.1 (23).** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be \( \alpha \)-strongly convex. Then for all \( x, y \in \mathbb{R}^n \) and \( x^* \in \partial f(x), y^* \in \partial f(y) \), we have
\[
\frac{\alpha}{2} \| x - y \|^2 \leq D_f^{x^*}(x, y) \leq \langle x^* - y^*, x - y \rangle \leq \| x^* - y^* \| \cdot \| x - y \|,
\]
and hence \( D_f^{x^*}(x, y) = 0 \iff x = y \).

**Definition 2.4 (17).** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be \( \alpha \)-strongly convex and \( C \subset \mathbb{R}^n \) be a nonempty closed convex set. The Bregman projection of \( x \) onto \( C \) with respect to \( f \) and \( x^* \in \partial f(x) \) is the unique point \( \Pi_C^{x^*}(x) \in C \) such that
\[
D_f^{x^*}(x, \Pi_C^{x^*}(x)) := \min_{y \in C} D_f^{x^*}(x, y). \quad (2.1)
\]
The Bregman projection generalizes the orthogonal projection by introducing different convex functions to obtain solutions of linear systems with different properties. The next lemma characterizes the Bregman projection by a variational inequality.

**Lemma 2.2** ([17]). Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\alpha$-strongly convex and $C$ be a nonempty convex set. The point $z \in C$ is the Bregman projection of $x$ onto $C$ with respect to $f$ and $x^* \in \partial f(x)$ iff there exists some $z^* \in \partial f(z)$ such that one of the following equivalent conditions is satisfied

$$
\langle z^* - x^*, y - z \rangle \geq 0, \forall \ y \in C,
$$

(2.2)

$$
D_f^z(z, y) \leq D_f^x(x, y) - D_f^{x^*}(x, z), \forall \ y \in C.
$$

(2.3)

where we call such $z^*$ an admissible subgradient for $z = \Pi_C^x(x)$.

In general, it is hard to compute the Bregman projection since we need to solve a constrained nonsmooth convex optimization. The next lemma presents a method to obtain the Bregman projection onto affine subspaces.

**Lemma 2.3** ([17]). Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\alpha$-strongly convex, $a \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$. The Bregman projection of $x \in \mathbb{R}^n$ onto the hyperplane $H(a, \gamma) = \{x \in \mathbb{R}^n | \langle a, x \rangle = \gamma \}$ with $a \neq 0$ is

$$
z := \Pi_{H(a, \gamma)}^x(x) = \nabla f^*(x^* - at),
$$

where $\hat{t} \in \mathbb{R}$ is a solution of $\min_{t \in \mathbb{R}} f^*(x^* - at) + t\gamma$. Moreover, $z^* = x^* - \hat{t}a$ is an admissible subgradient for $z$ and for all $y \in H(a, \gamma)$ we have

$$
D_f^{z^*}(z, y) \leq D_f^x(x, y) - \frac{\alpha}{2} \frac{(\langle a, x \rangle - \gamma)^2}{\|a\|^2}.
$$

(2.4)

If $x \notin H_{\leq (a, \gamma)} := \{x \in \mathbb{R}^n | \langle a, x \rangle \leq \gamma \}$, then $\Pi_{H_{\leq (a, \gamma)}}^x(x) = \Pi_{H(a, \gamma)}^x(x)$.

### 2.3 Two key constants

Define $\beta$ to be the fraction of corruptions, that is, $\beta = \|b^c\|_0/m$. During our analysis about convergence rate of Quantile-RaSK, we need two key constants about matrix $A$, which are analogously introduced by [10][26].

**Lemma 2.4.** Given matrix $A \in \mathbb{R}^{m \times n}$, the acceptable parameter $q$ and the corrupted parameter $\beta$. Denote $A_{I,J}$ as the submatrix generated by the rows of $A$ indexed by $I$ and the columns of $A$ indexed by $J$. Define the smallest singular value of $A_{I,J}$ by $\sigma_{\text{min}}(A_{I,J})$. Then

$$
\tilde{\sigma}_{q-\beta, \text{min}}(A) := \min\{\sigma_{\text{min}}(A_{I,J}) \mid |I| = (q - \beta)m, I \subset \{1, \ldots, m\}, J \subset \{1, \ldots, n\}\}.
$$

The largest singular value of $A$ is denoted by

$$
\sigma_{\text{max}}(A) := \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}.
$$
It is difficult to compute $\sigma_{\max}(A)$ and $\tilde{\sigma}_{q-\beta,\min}(A)$ actually, however, we can bound these constants by restricting matrix $A$ as certain types of random matrices. The following results can be deduced by [10].

**Lemma 2.5.** Assume that $A \in \mathbb{R}^{m \times n}$ is a normalized Gaussian matrix, that is, a matrix whose rows are sampled uniformly over the sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ in $\mathbb{R}^n$. Let $0 < q - \beta \leq 1$, then there exists absolute constant $C_K, C_1, C_2, c_1, c_2$ so that if $A$ is tall enough, namely,

$$\frac{m}{n} > C_1 \frac{\log C_2}{q - \beta},$$

then the first key constant can be bounded with probability at least $1 - 3\exp(-c_1(q - \beta)m)$

$$\tilde{\sigma}_{q-\beta,\min}(A) \gtrsim (q - \beta)^{\frac{3}{2}} \sqrt{\frac{m}{n}}. \quad (2.5)$$

Additionally, the second key constant has lower bound with probability at least $1 - 2\exp(-c_2m)$

$$\sigma_{\max}(A) \leq C_K \sqrt{\frac{m}{n}}. \quad (2.6)$$

**Remark 2.1.** The restrictions on measurement matrix $A$ in Lemma 2.5 can be relaxed, we refer to [10] for a certain class of measurement matrix. Proceeding as in the proof of Proposition 1 in [10], thus we proved (2.5).

### 2.4 Notations

We use $[\cdot]$ for the integral function. Define the quantile of sequence $S$ as

$$Q_q(x, S) := q - \text{quantile}(\langle x_k, a_i \rangle - b_i : i \in S).$$

For a sequence $y_i, 1 \leq i \leq n$, which is sorted from small to large as $y(i), 1 \leq i \leq n$, the $q$-quantile of $y_i, 1 \leq i \leq n$ is defined by

$$y_q := \begin{cases} y(\lfloor nq \rfloor + 1), & \text{if } \lfloor nq \rfloor \text{ is not an integer} \\ (y(nq) + y(nq + 1))/2, & \text{if } nq \text{ is an integer.} \end{cases} \quad (2.7)$$

Moreover, we denote $E_k$ as the expectation under the condition of indices $i_0, \cdots, i_{k-1}$ in $k$-th iterate. For arbitrary set $C$, $|C|$ represents number of elements in set $C$. If without a declaration, then norm $\|\cdot\|$ refers to the $l_2$ norm. Throughout the paper, $c_{A,\beta,q}, c_{\beta,q}, C_1, C_2, C_K, c_1, c_2$ represent absolute values and may vary line by line.

### 3 Quantile-RaSK method

In this section, we formally propose the quantile-based randomized sparse Kaczmarz, abbreviated by Quantile-RaSK. Moreover, we give convergence results in both noisy and noiseless case under the existence of $\beta m$ corruptions.
3.1 Proposed method

We are motivated by papers [10,14,26], which make use of an important statistical variable, quantile, to choose the corrupted rows. Concretely, we compute the quantile of residuals in each iterate, and then treat a row with residuals less than the quantile as acceptable rows. Finally, project onto an acceptable hyperplane sampled uniformly.

In this work, we use the Bregman projection with regularized augmented $l_1$ norm function to generate sparse solution of linear system. For generality, all rows $a_i, 1 \leq i \leq m$ are normalized. Note that there are two options for step size, that is, inexact step and exact step. The inexact step, $t_k = \langle a_k, x_k \rangle - b_k$, can be seen as a relaxation of exact step. According to Lemma 2.3, the exact step should solve the minimization problem

$$t_k := \arg \min_{t \in \mathbb{R}} f^*(x_k^* - ta_k) + tb_k.$$

The pseudocode is as follows.

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**Algorithm: Quantile-RaSK(q)**

**Input:** Given $A \in \mathbb{R}^{m \times n}, x_0 = x_0^* = 0 \in \mathbb{R}^n, b \in \mathbb{R}^m$, and parameter $q, N$

**Output:** solution of $\min_{x \in \mathbb{R}^n} \lambda \|x\|_1 + \frac{1}{2}\|x\|_2^2$ subject to $Ax = b$

1: normalize $A$ by row
2: for $k = 0, 1, 2, \ldots, N$ do
3: compute $N_1 = \{1 \leq i \leq m : |\langle x_k, a_i \rangle - b_i|\}$
4: compute the $q$-quantile of $N_1$: $Q_k = Q_q(x_k, 1 \leq i \leq m)$
5: consider $N_2 = \{1 \leq i \leq m : |\langle x_k, a_i \rangle - b_i| \leq Q_k\}$
6: sample $i \sim p_i = \frac{1}{[qm]}, i \in N_2$
7: **Switch** Type of step:
   **Case1:** Inexact step:
   $$t_k = \langle a_i, x_k \rangle - b_i$$
   **Case2:** Exact step:
   $$t_k = \arg \min_{t \in \mathbb{R}} f^*(x_k^* - ta_i) + t \cdot b_i$$
   end switch
8: $x_{k+1}^* = x_k^* - t_k \cdot a_i$
9: $x_{k+1} = S_\lambda(x_{k+1})$
10: increment $k = k + 1$
11: return $x_N$

The exact-step quantile-based randomized sparse Kaczmarz (Quantile-ERaSK) method needs an exact linesearch, which costs an $O(n \log n)$-sorting procedure while the inexact step costs $O(n)$. For the detail implementation of exact step, readers can refer to [17]. It is worth noting that the stopping criterion is up to the maximum number of iterations $N$, but we could also use the norm of residual less than a given precision.

3.2 Quantile-RaSK method for corrupted linear system

In this subsection, we consider a corrupted linear system

$$Ax = \tilde{b},$$

(3.1)
where \( b = \tilde{b} + b' \), where \( b' \) represents the corruptions satisfying \( \|b'\|_0 = \beta m \). The corrupted linear system can be solved by using the Quantile-RaSK. Before deducing the convergence theory in this case, we first introduce a bound on the quantile \( Q_k \) in [26], which mainly decided by the maximum singular value \( \sigma_{\max}(A), m, q, \beta \).

**Lemma 3.1** ([26]). Let \( \beta < q < 1 - \beta \), \( x_k \) be the \( k \)-th iterate, \( Q_k \) be the \( q \)-quantile of all residuals in \( k \)-th iterate, \( Ax = \tilde{b} \) and \( \|b'\|_0 = \beta m \). Let \( \hat{x} \) be the true solution of the linear system. Then

\[
Q_k \leq \frac{\sigma_{\max}(A)}{\sqrt{m\sqrt{1 - q - \beta}}} \|x_k - \hat{x}\|. \tag{3.2}
\]

With the bound of the quantile of residuals, now we can give the first main result.

**Theorem 3.1** (Corrupted case). Let the linear system be defined by normalized matrix \( A \in \mathbb{R}^{m \times n} \). Assume that \( \beta < q < 1 - \beta \), if

\[
\frac{2q}{q - \beta} \cdot \left( \frac{2\sqrt{\beta}}{\sqrt{1 - q - \beta}} + \frac{\beta}{1 - q - \beta} \right) \cdot \frac{|\hat{x}|_{\text{min}} + 2\lambda}{|\hat{x}|_{\text{min}}} \leq \frac{\sigma_{q - \beta, \min}^2}{\sigma_{\max}^2(A)},
\]

then the Quantile-RaSK and Quantile-ERaSK converge linearly for all \( \beta \)-corruptions of the linear system (3.1), in the sense that

(a) The iterates \( x_k \) generated by Quantile-RaSK satisfy

\[
\mathbb{E}(D_{f}^{x_{k+1}}(x_{k+1}, \hat{x})) \leq (1 - c_{A,\beta,q})\mathbb{E}(D_{f}^{x_k}(x_k, \hat{x}));
\]

(b) The iterates \( x_k \) generated by Quantile-ERaSK satisfy

\[
\mathbb{E}(D_{f}^{x_{k+1}}(x_{k+1}, \hat{x})) \leq (1 - c_{A,\beta,q})\mathbb{E}(D_{f}^{x_k}(x_k, \hat{x})) + \frac{2\lambda}{qm} \|b - \tilde{b}\| \cdot \|A\|_{1,2},
\]

where

\[
c_{A,\beta,q} = \frac{1}{2}(q - \beta) \frac{\sigma_{q - \beta, \min}^2}{q^2 m} \cdot \frac{|\hat{x}|_{\text{min}} + 2\lambda}{|\hat{x}|_{\text{min}}} - \frac{\sigma_{\max}^2(A)}{qm} \left( \frac{2\sqrt{\beta}}{\sqrt{1 - q - \beta}} + \frac{\beta}{1 - q - \beta} \right).
\]

In view of the proof of Theorem 3.1 is quite involved, it will be given in Appendix.

**Remark 3.1.** There are some estimates that can be improved for more exact bound [26]. The estimate

\[
\|A_S(x_k - \hat{x})\|^2 \leq \|A(x_k - \hat{x})\|^2 \leq \sigma_{\max}(A)^2 \|x_k - \hat{x}\|^2
\]

may be tightened by using \( |S| \leq \beta m \) to define

\[
\sigma_{\beta, \max} := \sup_{|S| = \beta m, S \subseteq 1, \ldots, m} \max_{x \neq 0} \frac{\|A_Sx\|}{\|x\|}.
\]

Similarly,

\[
\|A_S\|_{1,2} \leq \|A\|_{1,2}
\]

can be improved by defining

\[
\|A\|_{\beta, 1,2} := \sup_{|S| = \beta m, S \subseteq 1, \ldots, m} \|A_S\|_{1,2}.
\]
Remark 3.2. For a normalized Gaussian matrix $A$, we have the Lemma 2.5 holds. If we plug (2.6) and (2.5) into the result in Theorem 3.1, we obtain that $c_{A, \beta, q}$ is bounded by the reciprocal of $n$, that is,

$$c_{A, \beta, q} := \frac{C_{\beta, q}}{n},$$

it follows that the convergence conclusion of Quantile-RaSK can be simplified into

$$\mathbb{E}(D_{x_{k+1}}(x_{k+1}, \hat{x})) \leq (1 - \frac{c_{\beta, q}}{n})\mathbb{E}(D_{x_k}^*(x_k, \hat{x})).$$

Remark 3.3. In the case $\lambda = 0$, we can recover the convergence rate of Quantile-RK in [26], and obtain least-norm solutions of corrupted linear systems, which aims to solving the optimization problem

$$\min_{x \in \mathbb{R}^n} \|x\|_2, \text{ subject to } Ax = b.$$ 

3.3 Quantile-RaSK method for corrupted, noisy linear system

In Section 3.2, we consider a corrupted linear system. It follows from noisy data is common that now we aim to solve a corrupted, noisy linear system

$$Ax = \tilde{b},$$

where $b = \tilde{b} + b^c + r$, $b^c \in \mathbb{R}^m$ is the corruptions satisfying $\|b^c\|_0 = \beta m$, $r \in \mathbb{R}^m$ represents noise, and $\tilde{b} \in \mathbb{R}^m$ is an ideal vector construting consistent linear system. The corrupted, noisy linear system can be solved by using Quantile-RaSK.

Before giving the convergence result, we first bound the quantile of residuals in corrupted and noisy case, which is motivated by Lemma 1 in [14].

Lemma 3.2. Let $\beta < q < 1 - \beta$, $x_k$ be the $k$-th iterate, and $Q_k$ be the $q$-quantile of all the residuals in $k$-th iterate. Let $Ax = \tilde{b}$, $b = \tilde{b} + b^c + r$ and $\|b^c\|_0 = \beta m$. Then

$$Q_k \leq \frac{\sqrt{1 - \beta}}{(1 - \beta - q)\sqrt{m}}\sigma_{\max}(A)\|x_k - \hat{x}\| + \frac{1 - \beta}{1 - \beta - q}\|r\|_\infty.$$ 

Proof. Denote the set of all the indices of corrupted equations as $C$, and the set of indices other than $C$ is denoted as $\notin C$. For all $i \notin C$, i.e. $b^c_i = 0$, we have $b_i = \tilde{b}_i + r_i$ and $\langle a_i, \hat{x} \rangle = \tilde{b}_i = b_i - r_i$. Hence,

$$|\langle a_i, x_k - \hat{x} \rangle| = |\langle a_i, x_k \rangle - b_i + r_i| \geq |\langle a_i, x_k \rangle - b_i| - \|r\|_\infty. \quad (3.3)$$

Thus,

$$|\langle a_i, x_k \rangle - b_i| \leq |\langle a_i, x_k - \hat{x} \rangle| + \|r\|_\infty. \quad (3.4)$$
Recall that $Q_k = Q_q(x_k, 1 \leq i \leq m)$ and $|\not \in C| = (1 - \beta)m$, meaning that at least $(1 - \beta - q)m$ equations are at least $Q_k$. Then,

$$(1 - \beta - q)mQ_k \leq \sum_{i \not \in C} |(a_i, x_k) - b_i| \leq \sum_{i \not \in C} (|(a_i, x_k - \hat{x})| + \|r\|_{\infty}) \leq (\sum_{i \not \in C} |(a_i, x_k - \hat{x})|^2)^{\frac{1}{2}} \sqrt{|\not \in C|} + |\not \in C| \cdot \|r\|_{\infty} = \|A_{\not \in C}(x_k - \hat{x})\|\sqrt{(1 - \beta)m + (1 - \beta)m}\|r\|_{\infty} \leq \sigma_{\text{max}}(A)\|x_k - \hat{x}\|\sqrt{(1 - \beta)m + (1 - \beta)m}\|r\|_{\infty}. \quad (3.5)$$

Rewrite (3.5) to deduce that

$$Q_k \leq \frac{\sqrt{1 - \beta}}{(1 - \beta - q)\sqrt{m}} \sigma_{\text{max}}(A)\|x_k - \hat{x}\| + \frac{1 - \beta}{1 - \beta - q}\|r\|_{\infty}. \quad (3.6)$$

The proof is completed.

The Lemma 3.2 can be seen as a modified version of Lemma 3.1 by taking noise into consideration. According to Lemma 3.2, we can see that the $q$-quantile of residuals is mainly bounded by the largest singular value of measurement matrix $\sigma_{\text{max}}(A)$ and component with the largest absolute value of noise $\|r\|_{\infty}$.

Now we give the second main result.

**Theorem 3.2** (corrupted, noisy case). Let the linear system (3.3) be defined by standardized matrix $A \in \mathbb{R}^{m \times n}$. Assume that $\beta < q < 1 - \beta$, if

$$\sigma_{\text{max}}^2 \left( \frac{2\sqrt{1 - \beta}}{(1 - \beta - q)\sqrt{\beta m}} + \frac{1 - \beta}{(1 - \beta - q)^2 m} \right) + \sigma_{\text{max}} \left( \frac{2(1 - \beta)}{(1 - \beta - q)\sqrt{\beta m}} + \frac{2(1 - \beta)^{\frac{3}{2}}}{(1 - \beta - q)^2 \sqrt{m n}} \right) \leq \frac{1}{2} \cdot \frac{q - \beta}{\beta} \frac{\hat{x}_{\text{min}}}{|\hat{x}|_{\text{min}}} + 2\lambda \cdot \sigma_{q - \beta, \min}^2,$$

then the iterates $x_k$ generated by both Quantile-RaSK and Quantile-ERaSK converge for all $\beta$-corruptions of the linear system.

(a) The iterates $x_k$ generated by Quantile-RaSK satisfy

$$\mathbb{E} \left( D_f^{\tau_k + 1}(x_{k+1}, \hat{x}) \right) \leq (1 - C) \mathbb{E} \left( D_f^{\tau_k}(x_k, \hat{x}) \right) + E_{A, \beta, q}\|r\|_{\infty}^2;$$

(b) The iterates $x_k$ generated by Quantile-ERaSK satisfy

$$\mathbb{E} \left( D_f^{\tau_k + 1}(x_{k+1}, \hat{x}) \right) \leq (1 - C) \mathbb{E} \left( D_f^{\tau_k}(x_k, \hat{x}) \right) + E_{A, \beta, q}\|r\|_{\infty}^2 + \frac{2}{\sqrt{m}}\|r\|_{\infty} \cdot \|A\|_{1, 2} + \frac{2A}{|B|} \|b - \hat{b}\| \cdot \|A\|_{1, 2},$$

where

$$c_{A, \beta, q} = \frac{2\sigma_{\text{max}}^2(A)\sqrt{1 - \beta}}{\sqrt{m}(1 - \beta - q)} + \frac{2\sigma_{\text{max}}(A)(1 - \beta)}{\sqrt{n}(1 - \beta - q)}, C_{A, \beta, q} = \frac{\sigma_{\text{max}}^2(A)(1 - \beta)}{m(1 - \beta - q)^2} + \frac{2(1 - \beta)^{\frac{3}{2}}\sigma_{\text{max}}(A)}{\sqrt{mn}(1 - \beta - q)^2},$$
\[ d_{A,\beta,q} = \frac{(1 - \beta)\sqrt{n}}{1 - \beta - q}\sigma_{\max}(A), \quad D_{A,\beta,q} = \frac{(1 - \beta)^2\sqrt{n}}{(1 - \beta - q)^2\sqrt{m}}\sigma_{\max}(A) + \frac{1}{2}\left(\frac{1 - \beta}{1 - \beta - q}\right)^2, \]

\[ C = \frac{1}{2} \cdot \frac{\tilde{\sigma}_{q-\beta,\min}}{qm} \cdot \frac{\hat{x}_{\min}}{\|\hat{x}\|_{\min}} + \frac{\beta}{q} \left(1 - \frac{\tilde{\sigma}_{q-\beta,\min}}{qm} \cdot \frac{\hat{x}_{\min}}{\|\hat{x}\|_{\min}} + \frac{\sigma}{\sqrt{m}} + C_{A,\beta,q}\right), \]

\[ E_{A,\beta,q} = \frac{\beta}{q} \left(d_{A,\beta,q} + D_{A,\beta,q}\right) + \frac{1}{2}. \]

A proof of the theorem will be given in Appendix.

**Remark 3.4.** Clearly, parameters \( q, \beta \) should satisfy \( \beta < q < 1 - \beta \). It is natural for us to explore the best relationships between \( \beta \) and \( q \) for more precise convergence rate. However, it is difficult for us to provide a theoretical analysis, hence we will give an empirical result in numerical part.

**Remark 3.5.** Recall that we take normalized rows \( \|a_i\| = 1, 1 \leq i \leq m \) for simplicity. In fact, we can directly consider \( Ax = b \), which the values of \( \|a_i\| \) could affect the sampling probability \( p_i = \frac{\|a_i\|^2}{\|A\|_F} \) even if it have little influence on iterate.

**Remark 3.6.** For a normalized Gaussian matrix \( A \), we have the Lemma 2.5. Then the convergence result of Quantile-RaSK in corrupted and noisy case can be simplified into

\[ \mathbb{E}(D_f^{x_k+1}(x_{k+1}, \hat{x})) \leq (1 - \frac{C_{\beta,q}}{n})\mathbb{E}(D_f^{x_k}(x_k, \hat{x})) + C_{\beta,q}\|r\|_\infty^2, \]

which is similar to the main theorem in [14].

### 3.4 The convergence proof of Quantile-RaSK method

The trick of convergence proof of Quantile-RaSK method mainly follows from paper [26]. We denote the set consisting of all acceptable rows as

\[ B = \{1 \leq i \leq m : |\langle x_k, a_i \rangle| \leq Q_k\}. \]

Denote the corrupted equations in \( B \) as subset \( S \), and then the uncorrupted equations in \( B \) is subset \( B \setminus S \). Since all the sampling indices come from the acceptable set \( B \), we directly use Theorem 2 in [27] for uncorrupted indices in \( B \setminus S \), while we will employ Lemma 3.1 to obtain the conditional expectation of corrupted equations in \( S \). Naturally, the convergence theory will be given mainly by the following equality

\[ \mathbb{E}_k(D_f^{x_k+1}(x_{k+1}, \hat{x})) = P(i \in S)\mathbb{E}_k \left(D_f^{x_k+1}(x_{k+1}, \hat{x})|i \in S\right) + P(i \in B \setminus S)\mathbb{E}_k \left(D_f^{x_k+1}(x_{k+1}, \hat{x})|i \in B \setminus S\right). \tag{3.7} \]

Please refer to Appendix for detail proof.
4 Numerical experiments

In this section, we will carry out four experiments to test the performance of Quantile-RaSK. In experiments, we construct two types of matrices. In the first type, we generate the measurement matrix $A$ by MATLAB function ‘randn’. In the second type, the matrix $A$ is constructed by ATRtools toolbox [12]. Now we construct a consistent linear system, in the sense that the true vector $\hat{x}$ is created by the MATLAB function ‘sparserandn’, where has a parameter about sparsity to be set, and then let $A\hat{x} = \tilde{b}$. Furthermore, we turn to construct corrupted and noisy linear systems by damaging data, that is, $b = \tilde{b} + b^c + r$, where noise $r \in \mathbb{R}^m$ comes from uniform distribution $U(-0.02, 0.02)$ and corruptions $b^c \in \mathbb{R}^m$ are also taken from uniform distribution. We take the median error of 60 trials in every experiments, where we use

$$error(k) = \frac{\|x_k - \hat{x}\|}{\|\hat{x}\|}$$

to quantify the relative error at $k$-th iterate.

All experiments are performed with MATLAB (version R2021b) on a personal computer with 2.80-GHZ CPU(Intel(R) Core(TM) i7-1165G7), 16-GB memory, and Windows operating system (Windows 10).

4.1 The effect of corruptions

Let $m = 500, n = 200, q = 0.7, \beta = 0.2, \text{sparsity} = 35, \lambda = 1$, and the corruptions come from $U(-k, k)$ for a range of $k$. We use the first method to generate linear systems. We respectively compare the performance of QuantileRaSK with both inexact step and exact step for solving corrupted and noisy linear systems.

As shown in Figure 1, we have that both Quantile-RaSK and Quantile-ERaSK have faster convergence speeds as parameter $k$ increases. Empirically, the value of parameter $k$ has little effect on the convergence rate of Quantile-RaSK and Quantile-ERaSK as $k$ increases.

![Figure 1: The performance of Quantile-RaSK with corruptions from $U(-k, k)$](image)
4.2 The effect of parameter $\beta$, $q$

As demonstrated above, no theoretical guarantees about the choice of parameters $q, \beta$ in our analysis, thus we suggest empirical values of parameters through experiments. We take $m = 600, n = 200, \text{sparsity} = 30, \lambda = 1$, and let the corruptions come from $U(-5,5)$. We use the first method to construct consistent linear systems. Moreover, we use Quantile-RaSK and Quantile-ERaSK to solve the corrupted linear systems with 10% relative error respectively. For each $\beta$, we plot the log relative error at 2000-th iterate as a function of $q$, and each plotted point is the median over 10 trials.

Recall that we require $\beta < q < 1 - \beta$. As indicated in Figure 2, fixed $\beta$ then the best parameter $q$ takes $1 - \beta$, which holds for both Quantile-RaSK and Quantile-ERaSK. Note that $q$ should smaller than $1 - \beta$ theoretically, and we observe that the relative errors rapidly increase when $q > 1 - \beta$, which also validates our theory well. In views of this phenomenon, we prefer to take $q$ slightly smaller than the suggested value $1 - \beta$ in practical applications.

4.3 Comparing different Kaczmarz variants

In this part, we analyze performances of different Kaczmarz variants. Let $m = 600, n = 200, q = 0.7, \beta = 0.2, \text{sparsity} = 30, \lambda = 1$, and corruptions come from $U(-5,5)$. In this case, we use the first method to construct linear systems. We carry out experiments in noiseless and noisy case respectively, where the noisy linear system with 10% relative noise.

According to Figure 3, we can see that Quantile-ERaSK achieves convergence faster than Quantile-RaSK which is faster than Quantile-RK. Moreover, RaSK fails to obtain convergence due to the existence of big corruptions in some hyperplanes. It is clear that the superiority of Quantile-based Kaczmarz variants when solving corrupted linear systems.
4.4 Real world data

We construct a tomography problem by using the Matlab Regularization Toolbox by [12]. In test, we create a 2D X-ray tomography test matrix \( A \in \mathbb{R}^{m \times n} \) with \( m = fN^2 \) and \( n = N^2 \) by function ‘paralleltomo’ in AIRtools toolbox. Let \( N = 20 \) and \( f = 3 \), after removing the zero-norm rows of the matrix, we yield a matrix \( A \in \mathbb{R}^{1200 \times 400} \). Set sparsity = 50, we construct linear system \( q = 0.7, \beta = 0.2, \lambda = 1 \), we construct corrupted and noisy signal \( b \) by adding noise \( r \sim U(-0.02, 0.02) \) and corruptions coming from uniform distribution \( U(-50, 50) \).

As shown in Figure 4, Quantile-ERaSK obtains better reconstructions than Quantile-RaSK and Quantile-RK. Apparently, the relative error gradually increases with the number of iterations. Note that the relative errors of Quantile-ERaSK increases at the beginning of iterates, which maybe caused by Quantile-ERaSK pursues accuracy in each iterate so that breaks the superiority of algorithm.
5 Conclusion

In this paper, we propose a sparse variant of Quantile-RK to detect the corruptions in linear system. The Quantile-RaSK inherits the advantages of quantile-based randomized Kaczmarz method and sparse Kaczmarz method. Most important of all, we prove the linear convergence of Quantile-RaSK in corrupted and noisy case. Besides of this, we numerically verify the superiority of Quantile-RaSK generating sparse solutions comparing to other Kaczmarz variants when facing with corrupted, noisy linear systems especially.

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Appendix

We present the detail proof of Theorem 3.1 and Theorem 3.2.

Proof. (of Theorem 3.1) The proof is divided into three steps. First, we consider the uncorrupted equations indexed by \( i \in B \setminus S \). Since \( b^c_{B \setminus S} = 0 \), the equations indexed by \( i \in B \setminus S \) satisfy
\[
A_{B \setminus S} x = b_{B \setminus S}.
\] (5.1)

According to Theorem 3.2 in [23], we have
\[
\mathbb{E}_k(D^x_{f,k+1}(x_{k+1}, \hat{x}) | i \in B \setminus S) \leq \left( 1 - \frac{1}{2} \cdot \frac{1}{\tilde{\sigma}_2(A_{B \setminus S})} \frac{\| \hat{x} \|_{min}}{|x|_{min} + 2\lambda} \right) D^x_{f,k}(x_k, \hat{x}).
\] (5.2)

Recall that the definition of \( \tilde{\sigma}_{min}(A_{B \setminus S}) \) in [23]
\[
\tilde{\sigma}_{min}(A_{B \setminus S}) = \min \{ \sigma_{min}(A_{B \setminus S,J}) | J \subset \{1, \ldots, n\}, A_{B \setminus S,J} \neq 0 \},
\]
combining with \( |B \setminus S| \geq (q - \beta)m \), we have
\[
\tilde{\sigma}_{min}(A_{B \setminus S}) \geq \tilde{\sigma}_{q-\beta, min}.
\]

Together with \( \|A_{B \setminus S}\|_F^2 \leq |B| = qm \), the condition number of \( A_{B \setminus S} \) can be bounded by
\[
\kappa^2(A_{B \setminus S}) = \frac{\|A_{B \setminus S}\|_F^2}{\tilde{\sigma}_{min}^2(A_{B \setminus S})} \leq \frac{qm}{\tilde{\sigma}_{q-\beta, min}^2},
\] (5.3)

Thus we reformulate (5.2)
\[
\mathbb{E}_k(D^x_{f,k+1}(x_{k+1}, \hat{x}) | i \in B \setminus S) \leq \left( 1 - \frac{1}{2} \cdot \frac{\tilde{\sigma}_{q-\beta, min}^2}{qm} \frac{|\hat{x}|_{min}}{|x|_{min} + 2\lambda} \right) D^x_{f,k}(x_k, \hat{x}).
\] (5.4)
Second, we consider the corrupted equations indexed in $S$. Denote the orthogonal projection of true solution $\hat{x}$ in uncorrupted hyperplane $H(a_{i_k}, b_{i_k})$ to corrupted hyperplane $H(a_{i_k}, b_{i_k})$ as $x_k^c$ in $k$-th iterate $19$, then we observe that

$$x_k^c := \hat{x} + (b_{i_k} - \bar{b}_{i_k})a_{i_k} \in H(a_{i_k}, b_{i_k}). \quad (5.5)$$

Note that $x_{k+1}$ is obtained by projecting onto hyperplane $H(a_{i_k}, b_{i_k})$. And the optimal solution $\hat{x}$ lies in $H(a_{i_k}, \bar{b}_{i_k})$. According to Lemma 2.3, it follows that

$$D_f^{x_{k+1}}(x_{k+1}, \hat{x}) = \frac{1}{2} \langle (a_{i_k}, x_k) - b_{i_k} \rangle^2 - \langle x_k^s - x_k^c - \hat{x} \rangle,$$

Now we fix $i_0, \cdots, i_{k-1}$ and view $i_k$ as random variable. According to $|\langle a_i, x_k \rangle - b_i| \leq Q_k$, we have

$$D_f^{x_{k+1}}(x_{k+1}, \hat{x}) \leq D_f^{x_k}(x_k, \hat{x}) + \frac{1}{2} |\langle a_{i_k}, x_k \rangle - b_{i_k}|^2 - \langle x_{k+1} - x_k^s - x_k^c - \hat{x} \rangle,$$

Now we bound the inner product term $\langle x_{k+1}^s - x_k^s, x_k^c - \hat{x} \rangle$ in two cases to finish the proof.

(a) For Quantile-RaSK with inexact step, we have $x_{k+1}^s - x_k^s = -(a_{i_k}, x_k) - b_{i_k})a_{i_k}$ obtained from the 8-th step of the Algorithm. Hence the (5.7) can be rewritten as

$$D_f^{x_{k+1}}(x_{k+1}, \hat{x}) \leq D_f^{x_k}(x_k, \hat{x}) - \frac{1}{2} \langle (a_{i_k}, x_k) - b_{i_k} \rangle^2 + \langle x_k^s - x_k^c - \hat{x} \rangle,$$

(b) Now we fix $i_0, \cdots, i_{k-1}$ and view $i_k$ as random variable. According to $|\langle a_i, x_k \rangle - b_i| \leq Q_k$, we have

$$E_k \left( D_f^{x_{k+1}}(x_{k+1}, \hat{x}) \right) = \frac{1}{|S|} \sum_{i \in S} \langle a_i, x_k - \hat{x} \rangle \leq \frac{1}{|S|} \| A_S(x_k - \hat{x}) \| \leq \frac{1}{|S|} \sigma_{\text{max}}(A) \cdot \| x_k - \hat{x} \|,$$

thus,

$$E_k \left( D_f^{x_{k+1}}(x_{k+1}, \hat{x}) \right) \leq D_f^{x_k}(x_k, \hat{x}) + \frac{1}{2} \cdot \frac{\sigma_{\text{max}}^2(A)}{m(1-q-\beta)} \| x_k - \hat{x} \|^2 + \frac{\sigma_{\text{max}}^2(A)}{\sqrt{m \sqrt{1-q-\beta} \sqrt{|S|}}} \| x_k - \hat{x} \|^2 \leq \left( 1 + \frac{\sigma_{\text{max}}^2(A)}{m(1-q-\beta)} + \frac{2\sigma_{\text{max}}^2(A)}{\sqrt{m \sqrt{1-q-\beta} \sqrt{|S|}}} \right) D_f^{x_k}(x_k, \hat{x}) \leq \left( 1 + \frac{\sigma_{\text{max}}^2(A)}{\sqrt{|S|m}} \left( \frac{2}{\sqrt{1-q-\beta}} + \frac{\sqrt{\beta}}{1-q-\beta} \right) \right) D_f^{x_k}(x_k, \hat{x}).$$

(5.11)
the second inequality holds with the strong convexity of $f$. Moreover, $|S| \leq \beta m$ leads to the last inequality. In the third step, Combining (5.7), (5.4) with (5.11) we have

$$
\mathbb{E}_k \left(D_f^{\tau_{k+1}}(x_{k+1}, \hat{x})\right) \\
\leq |S| \left(1 + \frac{\sigma^2_{\max}(A)}{\sqrt{|S|m}} \left(\frac{2}{1-q-\beta} + \frac{\sqrt{\beta}}{1-q-\beta}\right)\right) D_f^{\tau_k}(x_k, \hat{x}) \\
+ \left(1 - \frac{|S|}{|B|}\right) \left(1 - \frac{1}{2} \cdot \frac{\sigma^2_{q-\beta, \min}}{q^2 m} \cdot \frac{\hat{x}_i^{|\min|+2\lambda}}{|\hat{x}_i^{|\min|}}\right) D_f^{\tau_k}(x_k, \hat{x}) \\
\leq \left[1 + \frac{\sigma^2_{\max}(A)}{\sqrt{|S|m}} \left(\frac{2}{1-q-\beta} + \frac{\sqrt{\beta}}{1-q-\beta}\right)\right] D_f^{\tau_k}(x_k, \hat{x}) \\
+ \frac{1}{2} \cdot \frac{\sigma^2_{q-\beta, \min}}{q^2 m} \cdot \frac{\hat{x}_i^{|\min|+2\lambda}}{|\hat{x}_i^{|\min|}} \left(\frac{|S|}{|B|} - 1\right) D_f^{\tau_k}(x_k, \hat{x}).
$$

(5.12)

The value of (5.12) is monotonically increasing with respect to $|S|$. Thus when $|S| = \beta m$, (5.12) obtains the maximum, yielding

$$
\mathbb{E}_k \left(D_f^{\tau_{k+1}}(x_{k+1}, \hat{x})\right) \leq (1 - c_{A, \beta, q}) D_f^{\tau_k}(x_k, \hat{x}),
$$

(5.13)

where

$$
c_{A, \beta, q} = \frac{1}{2} (q - \beta) \frac{\sigma^2_{q-\beta, \min}}{q^2 m} \cdot \frac{\hat{x}_i^{|\min|+2\lambda}}{|\hat{x}_i^{|\min|}} \cdot \frac{\sigma^2_{\max}(A)}{q^2 m} \left(\frac{2\sqrt{\beta}}{1-q-\beta} + \frac{\beta}{1-q-\beta}\right).
$$

Now viewing all indices $i_0, \ldots, i_k$ as random variables and taking full expectation on both sides, we get

$$
\mathbb{E}(D_f^{\tau_{k+1}}(x_{k+1}, \hat{x})) \leq (1 - c_{A, \beta, q}) \mathbb{E}(D_f^{\tau_k}(x_k, \hat{x})).
$$

(5.14)

Furthermore, to ensure decay in expectation, we require

$$
\frac{2q}{q - \beta} \cdot \left(\frac{2\sqrt{\beta}}{1-q-\beta} + \frac{\beta}{1-q-\beta}\right) \cdot \frac{\hat{x}_i^{|\min|+2\lambda}}{|\hat{x}_i^{|\min|}} \leq \frac{\sigma^2_{q-\beta, \min}}{\sigma^2_{\max}(A)}.
$$

(5.15)

(b) For Quantile-RaSK with exact step, we have $x^*_k = x_k + \lambda s_k$ with $\|s_k\|_\infty \leq 1$ and $\|s_k\|_\infty \leq 1$, then $x^*_k - x^*_k = (x_{k+1} - x_k) + \lambda (s_{k+1} - s_k)$. Note that the exact linesearch guarantees $(a_{i_k}, x_{k+1}) = b_{i_k}$, thus (5.7) can rewrite as

$$
D_f^{\tau_{k+1}}(x_{k+1}, \hat{x}) \leq D_f^{\tau_k}(x_k, \hat{x}) + \lambda (s_{k+1} - s_k, a_{i_k}) (b_{i_k} - \hat{b}_{i_k}) \\
+ \frac{1}{2} ((a_{i_k}, x_k) - b_{i_k})^2 - ((a_{i_k}, x_k) - b_{i_k})((a_{i_k}, x_k) - \hat{b}_{i_k}).
$$

(5.16)

Using the conclusion in (a) and view $i_k$ as a random variable under $i_0, \ldots, i_{k-1}$ is fixed,

$$
\mathbb{E}_k(D_f^{\tau_{k+1}}(x_{k+1}, \hat{x})|i \in S) \\
\leq \left(1 + \frac{\sigma^2_{\max}(A)}{\sqrt{|S|m}} \left(\frac{2}{1-q-\beta} + \frac{\sqrt{\beta}}{1-q-\beta}\right)\right) D_f^{\tau_k}(x_k, \hat{x}) \\
+ \frac{2\lambda}{|S|} \|b - \hat{b}\| \cdot |A|_{1, 2},
$$

(5.17)
where
\[
\mathbb{E}_k(\lambda (s_{k+1} - s_k, a_i)(b_i - \tilde{b}_i)|i \in S) \leq 2\lambda \mathbb{E}_k(\|a_i\|_1 \cdot |b_i - \tilde{b}_i||i \in S) \\
\leq \frac{2\lambda}{|S|} \sum_{i \in S}(\|a_i\|_1 \cdot |b_i - \tilde{b}_i|) \\
\leq \frac{2\lambda}{|S|}\|b - \tilde{b}\| \cdot \|A\|_{1,2}. \quad (5.18)
\]
The last inequality uses the estimate \(\sum_{i \in S}(\|a_i\|_1 \cdot |b_i - \tilde{b}_i|) \leq \|b_S - \tilde{b}_S\| \cdot \|A_S\|_{1,2}.\) Similar to the inexact case, combining with (3.7), (5.4) and (5.17) we obtain
\[
\mathbb{E}_k(D_{f,k}^{q,k+1}(x_{k+1}, \hat{x})) \leq (1 - c_{A,\beta,q})D_{f,k}^{q,k}(x_k, \hat{x}) + 2\frac{\lambda}{\eta_{\lambda}} b - \tilde{b} \cdot \|A\|_{1,2}. \quad (5.19)
\]
To ensure the convergence of generated sequences in expectation, we also require (5.15) to hold. The proof is completed.

**Proof.** (of Theorem 3.2) The proof of corrupted, noisy case is almost identical to that corrupted case, the major difference is the substitution of Theorem 3.2 in [23] for deducing \(\mathbb{E}_k(D_{f,k}^{q,k+1}(x_{k+1}, \hat{x}))|i \in S\) and the bound of quantiles in Lemma 3.2 for obtaining \(\mathbb{E}_k(D_{f,k}^{q,k+1}(x_{k+1}, \hat{x}))|i \in B\setminus S\).

For clarity, we take respectively both inexact step and exact step into account, which are all divided into three steps.

(a) In the inexact case, first of all, we consider the uncorrupted equations indexed \(i \in B\setminus S\). We have \(b_{C,B\setminus S} = 0\), then the equations indexed \(i \in B\setminus S\) satisfy
\[
A_{B\setminus S}x = b_{B\setminus S} = b_{B\setminus S} - r_{B\setminus S}.
\]
Recall Theorem 3.4 in [23] that
\[
\mathbb{E}_k(D_{f,k}^{q,k+1}(x_{k+1}, \hat{x}))|i \in B\setminus S) \\
\leq \left(1 - \frac{1}{2} \cdot \frac{1}{\lambda r_{B\setminus S}} \cdot \frac{\|x\|_{\min}}{|x|_{\min} + 2\lambda} \right) D_{f,k}^{q,k}(x_k, \hat{x}) + \frac{\|r_{B\setminus S}\|^2}{\|A_{B\setminus S}\|_F} + \frac{1}{2}\|A_{B\setminus S}\|_F \|r\|^2_\infty. \quad (5.20)
\]
The last inequality holds with \(\|r_{B\setminus S}\|^2 \leq q m \cdot \|r\|_\infty \), \(\|A_{B\setminus S}\|_F = |B\setminus S|\).

Second, we consider the conditional expectation in \(S\). In this case, we have
\[
A_Sx = b_S, b_S = b_{\tilde{S}} + b_S + r_S.
\]
Recall (5.8) we see
\[
\mathbb{E}_k(\mathbb{E}_k(D_{f,k}^{q,k+1}(x_{k+1}, \hat{x}))|i \in S) \leq D_{f,k}^{q,k}(x_k, \hat{x}) + \frac{1}{2}Q_{k}^2 + Q_k \mathbb{E}_k \left(\langle a_i, x_k \rangle - \tilde{b}_i|i \in S\right),
\]
combining (5.10) with (3.6), we have that
\[
\mathbb{E}_k \left(\mathbb{E}_k(D_{f,k}^{q,k+1}(x_{k+1}, \hat{x}))|i \in S\right) \\
\leq D_{f,k}^{q,k}(x_k, \hat{x}) + \frac{1}{2} \left(\frac{1 - \beta}{1 - \beta - q}\right)^2 \|r\|_\infty^2 \\
+ \frac{1 - \beta}{1 - \beta - q} \left(\frac{1}{\sqrt{|S|} \sqrt{1 - \beta - q}} + \frac{1}{\eta}\right) \sigma_{\max}^2(A) \|x_k - \hat{x}\|^2 \\
+ \frac{1 - \beta}{1 - \beta - q} \left(\frac{\sqrt{1 - \beta}}{(1 - \beta - q) \sqrt{m}} + \frac{1}{\sqrt{|S|}}\right) \sigma_{\max}^2(A) \|x_k - \hat{x}\| \cdot \|r\|_\infty^2. \quad (5.21)
\]
Now we discuss the size relationship between noise and error in a similar manner with that \cite{14}. It is easy to verify that whatever the relationship of noise and corruptions, we have that

\[
\mathbb{E}_k \left( D_f^{x_{k+1}}(x_{k+1}, \hat{x}) \middle| i \in S \right) \leq \left( 1 + \frac{c_{A,\beta,q}}{\sqrt{|S|}} + C_{A,\beta,q} \right) D_f^{x_k}(x_k, \hat{x}) + \left( \frac{d_{A,\beta,q}}{\sqrt{|S|}} + D_{A,\beta,q} \right) \|r\|_\infty^2.
\]

Finally, combining with \eqref{3.7}, \eqref{5.20}, we have

\[
\mathbb{E}_k(D_f^{x_{k+1}}(x_{k+1}, \hat{x})) \leq (1 - C) D_f^{x_k}(x_k, \hat{x}) + E_{A,\beta,q} \|r\|_\infty^2,
\]

where

\[
E_{A,\beta,q} = \frac{\beta}{q} \left( \frac{d_{A,\beta,q}}{\sqrt{\beta m}} + D_{A,\beta,q} \right) + \frac{1}{2},
\]

\[
C = \frac{1}{2} \frac{\tilde{\sigma}^2_{q-\beta,\min}}{qm} \frac{|\hat{x}|_{\min}}{|\hat{x}|_{\min} + 2\lambda} - \frac{\beta}{q} \left( \frac{1}{2} \frac{\tilde{\sigma}^2_{q-\beta,\min}}{qm} \frac{|\hat{x}|_{\min}}{|\hat{x}|_{\min} + 2\lambda} + \frac{c_{A,\beta,q}}{\sqrt{\beta m}} + C_{A,\beta,q} \right).
\]

(b) In the exact-step case. First, we consider the uncorrupted equations \( i \in B \setminus S \). We have \( b_{B \setminus S} = 0 \), then the equations \( i \in B \setminus S \) satisfy

\[
A_{B \setminus S}x = \tilde{b}_{B \setminus S} = b_{B \setminus S} - r_{B \setminus S}.
\]

Recall the Theorem 3.4 in \cite{23} that

\[
\mathbb{E}_k(D_f^{x_{k+1}}(x_{k+1}, \hat{x}) \middle| i \in B \setminus S) \leq \left( 1 - \frac{1}{2} \frac{\tilde{\sigma}^2_{q-\beta,\min}}{qm} \frac{|\hat{x}|_{\min}}{|\hat{x}|_{\min} + 2\lambda} \right) D_f^{x_k}(x_k, \hat{x}) + \frac{1}{2} \frac{qm}{|B \setminus S|} \cdot \|r\|_\infty^2
+ \frac{2\sqrt{qm}}{|B \setminus S|} \|r\|_\infty \cdot \|A\|_{1,2}.
\]

Second, we consider the conditional expectation in \( S \). In this case, we have

\[
A_{S}x = \tilde{b}_{S}, b_S = \tilde{b}_{S} + b_{S}^C + r_S.
\]

Recall \eqref{5.16} that

\[
D_f^{x_{k+1}}(x_{k+1}, \hat{x}) \leq D_f^{x_k}(x_k, \hat{x}) - \langle a_{i_k}, x_k \rangle - b_{i_k} + \lambda (s_{k+1} - s_k, a_{i_k})(b_{i_k} - \tilde{b}_{i_k}) + \frac{1}{2} (\langle a_{i_k}, x_k \rangle - b_{i_k})^2.
\]

Hence,

\[
\mathbb{E}_k(D_f^{x_{k+1}}(x_{k+1}, \hat{x}) \middle| i \in S) \leq \left( 1 + \frac{c_{A,\beta,q}}{\sqrt{|S|}} + C_{A,\beta,q} \right) D_f^{x_k}(x_k, \hat{x}) + \left( \frac{d_{A,\beta,q}}{\sqrt{|S|}} + D_{A,\beta,q} \right) \|r\|_\infty^2
+ \frac{2\lambda}{|S|} \|b - \tilde{b}\| \cdot \|A\|_{1,2}.
\]
Finally, we turn to the convergence result by the equality (3.7). Combining with (5.23) and (5.25), similar to the process of (a), we obtain that

\[ \mathbb{E}_k(D^{x_{k+1}}(x_{k+1}, \hat{x})) \]
\[ \leq (1 - C) D^{x_k}(x_k, \hat{x}) + E_{A,\beta,q} \|r\|_\infty^2 + \frac{2}{\sqrt{q}} \|r\|_\infty \cdot \|A\|_{1,2} + \frac{2\lambda}{|B|} \|b - \tilde{b}\| \cdot \|A\|_{1,2}. \]

(c) To ensure the decay in expectation whatever the relationship between noise and corruptions, we should promise

\[ -\frac{1}{2} \cdot \frac{\tilde{\sigma}_{q-\beta,\min}^2}{qm} \cdot \frac{|\hat{x}|_{\min}}{|\hat{x}|_{\min} + 2\lambda} + \frac{\beta}{q} \left( \frac{1}{2} \cdot \frac{\tilde{\sigma}_{q-\beta,\min}^2}{qm} \cdot \frac{|\hat{x}|_{\min}}{|\hat{x}|_{\min} + 2\lambda} + \frac{c_{A,\beta,q}}{\sqrt{\beta m}} + C_{A,\beta,q} \right) \leq 0. \]

Thus we arrive at the conclusion.