Alternative evaluation of a $\ln \tan$ integral arising in quantum field theory

Mark W. Coffey  
Department of Physics  
Colorado School of Mines  
Golden, CO 80401  
(Received 2008)

November 9, 2008

Abstract

A certain dilogarithmic integral $I_7$ turns up in a number of contexts including Feynman diagram calculations, volumes of tetrahedra in hyperbolic geometry, knot theory, and conjectured relations in analytic number theory. We provide an alternative explicit evaluation of a parameterized family of integrals containing this particular case. By invoking the Bloch-Wigner form of the dilogarithm function, we produce an equivalent result, giving a third evaluation of $I_7$. We also alternatively formulate some conjectures which we pose in terms of values of the specific Clausen function $\text{Cl}_2$.

Key words and phrases

Clausen function, dilogarithm function, Hurwitz zeta function, functional equation, duplication formula, triplication formula

AMS classification numbers

33B30, 11M35, 11M06
The particular integral

\[ I_7 \equiv \frac{24}{7 \sqrt{7}} \int_{\pi/3}^{\pi/2} \ln \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| \, dt, \tag{1} \]

occurs in a number of contexts and has received significant attention in the last several years \[3, 4, 5, 6\]. This and related integrals arise in hyperbolic geometry, knot theory, and quantum field theory \[6, 7, 8\]. Very recently \[9\] we obtained an explicit evaluation of (1) in terms of the specific Clausen function \( \text{Cl}_2 \). However, much work remains.

This is due to the conjectured relation between a Dirichlet \( L \) series and \( I_7 \) \[6\],

\[ I_7 \equiv L_{-7}(2) = \sum_{n=0}^{\infty} \left[ \frac{1}{(7n+1)^2} + \frac{1}{(7n+2)^2} - \frac{1}{(7n+3)^2} + \frac{1}{(7n+4)^2} - \frac{1}{(7n+5)^2} - \frac{1}{(7n+6)^2} \right]. \tag{2} \]

The \( ? \) here indicates that numerical verification to high precision has been performed but that no proof exists, the approximate numerical value of \( I_7 \) being \( I_7 \simeq 1.1519254705449104710169 \). The statement (2) is equivalent to the conjecture, with \( \theta_7 \equiv 2 \tan^{-1} \sqrt{7} \),

\[ \frac{1}{2} \left[ 3\text{Cl}_2(\theta_7) - 3\text{Cl}_2(2\theta_7) + \text{Cl}_2(3\theta_7) \right] \equiv \frac{1}{4} Z_{Q(\sqrt{-7})} = \frac{7}{4} \left[ \text{Cl}_2 \left( \frac{2\pi}{7} \right) + \text{Cl}_2 \left( \frac{4\pi}{7} \right) - \text{Cl}_2 \left( \frac{6\pi}{7} \right) \right], \tag{3} \]

relating triples of Clausen function values. Here, we alternatively evaluate \( I_7 \) directly in terms of the left side of (3). In addition, we present another evaluation of \( I_7 \), based upon a property of the Bloch-Wigner form of the dilogarithm function.

We recall that the \( L \) series \( L_{-7}(s) \) has occurred in several places before, including hyperbolic geometry \[19\] and Dedekind sums of analytic number theory \[2\]. Let \( \zeta_{Q(\sqrt{-p})} \) denote the Dedekind zeta function of an imaginary quadratic field \( Q(\sqrt{-p}) \).
Then indeed we have [2, 19, 20]

\[ \zeta_Q(\sqrt{-7})(s) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(m^2 + mn + 2n^2)^s} \]  

where \((\frac{\nu}{7})\) is a Legendre symbol, \(\zeta(s, a)\) is the Hurwitz zeta function, and \(\zeta(s) = \zeta(s, 1)\) is the Riemann zeta function.

The series \(L_{-7}(s)\) is an example of a Dirichlet L function corresponding to a real character \(\chi_k\) [here, modulo 7] with \(\chi_k(k - 1) = -1\). Such \(L\) functions, extendable to the whole complex plane, satisfy the functional equation [20]

\[ L_{-k}(s) = \frac{1}{\pi} (2\pi)^s k^{-s+1/2} \cos \left( \frac{s\pi}{2} \right) \Gamma(1 - s) L_{-k}(1 - s). \]  

Owing to the relation \(\Gamma(1 - s)\Gamma(s) = \pi/\sin(\pi s)\), this functional equation may also be written in the form

\[ L_{-k}(1 - s) = 2(2\pi)^{-s} k^{-s-1/2} \sin \left( \frac{\pi s}{2} \right) \Gamma(s) L_{-k}(s). \]  

Integral representations are known for these \(L\)-functions [20, 10]. From the functional equation (6) we find

\[ \frac{\partial}{\partial s} L_{-k}(s) \bigg|_{s=-1} = \frac{k^{3/2}}{4\pi} L_{-k}(2). \]  

In turn, we have

\[ \zeta_Q(\sqrt{-7})(-1) = -\frac{k^{3/2}}{48\pi} L_{-k}(2), \]  

where we used \(\zeta(-1) = -1/12\) and \(L_{-k}(-1) = 0\).
We have

**Proposition 1.** We have

\[
I_7 = \frac{4}{7\sqrt{7}} [3\text{Cl}_2(\theta_7) - 3\text{Cl}_2(2\theta_7) + \text{Cl}_2(3\theta_7)].
\] (10)

In fact, we treat integrals

\[
I(a) \equiv \int_{\pi/3}^{\pi/2} \ln \left| \tan t + a \right| \frac{dt}{\tan t - a},
\] (11)

and more general ones with varying limits. For (11), we assume that \(\pi/3 < \varphi = \tan^{-1} a < \pi/2\). These other integrals permit us to explicitly write other conjectures directly in terms of linear combinations of specific Clausen function values.

The Clausen function \(\text{Cl}_2\) can be defined by (e.g., [14, 16])

\[
\text{Cl}_2(\theta) \equiv -\int_0^\theta \ln \left| 2 \sin \frac{t}{2} \right| dt = \int_0^1 \tan^{-1} \left( \frac{x \sin \theta}{1 - x \cos \theta} \right) \frac{dx}{x}
\] (12)

\[
= -\sin \theta \int_0^1 \frac{\ln x}{x^2 - 2x \cos \theta + 1} dx = \sum_{n=1}^\infty \frac{\sin(n\theta)}{n^2}.
\] (13)

When \(\theta\) is a rational multiple of \(\pi\) it is known that \(\text{Cl}_2(\theta)\) may be written in terms of the trigamma and sine functions [11, 13]. This Clausen function is odd and periodic, \(\text{Cl}_2(\theta) = -\text{Cl}_2(-\theta)\), and \(\text{Cl}_2(\theta) = \text{Cl}_2(\theta + 2\pi)\). It also satisfies the duplication

\[
\frac{1}{2} \text{Cl}_2(2\theta) = \text{Cl}_2(\theta) - \text{Cl}_2(\pi - \theta),
\] (14)

triplication

\[
\frac{1}{3} \text{Cl}_2(3\theta) = \text{Cl}_2(\theta) + \text{Cl}_2 \left( \theta + \frac{2\pi}{3} \right) + \text{Cl}_2 \left( \theta + \frac{4\pi}{3} \right),
\] (15)

and quadriplication

\[
\frac{1}{4} \text{Cl}_2(4\theta) = \text{Cl}_2(\theta) + \text{Cl}_2 \left( \theta + \frac{\pi}{2} \right) + \text{Cl}_2 (\theta + \pi) + \text{Cl}_2 \left( \theta + \frac{3\pi}{2} \right),
\] (16)
formulas, as well as a more general multiplication formula \[14\]. We recall the specific relation
\[
\sum_{j=1}^{6} \text{Cl}_2 \left( \frac{2\pi}{7} j \right) = 0, \tag{17}
\]
that arises as a special case of \[14\] (pp. 95, 253)
\[
\sum_{j=1}^{n-1} \text{Cl}_2 \left( \frac{2\pi}{n} j \right) = 0. \tag{18}
\]
In (17), pairwise cancellation occurs, as \( \text{Cl}_2(\theta) = -\text{Cl}_2(2\pi - \theta) \).

Further information on the special functions that we employ may readily be found elsewhere \[15, 16, 18, 10\]. In particular, with
\[
\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}, \quad |z| \leq 1, \tag{19}
\]
or
\[
\text{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t} dt, \tag{20}
\]
the dilogarithm function, we have the relation
\[
\text{Li}_2(e^{i\theta}) = \frac{\pi^2}{6} - \frac{1}{4} \theta(2\pi - \theta) + i\text{Cl}_2(\theta), \quad 0 \leq \theta \leq 2\pi. \tag{21}
\]
We omit discussion of further relations between the Clausen function \( \text{Cl}_2 \) and the dilogarithm function.

For the proof of Proposition 1 we repeatedly rely on \[14\] (pp. 227, 272)
\[
\int_0^\theta \ln(\tan \theta + \tan \varphi) d\theta = -\theta \ln(\cos \varphi) - \frac{1}{2} \text{Cl}_2(2\theta + 2\varphi) + \frac{1}{2} \text{Cl}_2(2\varphi) - \frac{1}{2} \text{Cl}_2(\pi - 2\theta). \tag{22}
\]
We may split the integral in (11), writing
\[
I(a) = \int_{\pi/3}^\varphi \ln \left( \frac{a + \tan t}{a - \tan t} \right) dt + \int_{\varphi}^{\pi/2} \ln \left( \frac{\tan t + a}{\tan t - a} \right) dt
\]
\[ I(a) = \frac{1}{2} \left[ \text{Cl}_2 \left( \frac{2\varphi + 2\pi}{3} \right) + \text{Cl}_2 \left( 2\varphi - \frac{2\pi}{3} \right) \right] \text{Cl}_2 \left( \frac{2\pi}{3} + 2\varphi \right) - \text{Cl}_2 \left( \pi + 2\varphi \right). \]  

(25)

Then we apply both the duplication formula (14) and the triplication formula (15) wherein \( \text{Cl}_2(\theta + 4\pi/3) = \text{Cl}_2(\theta - 2\pi/3) \) as \( \text{Cl}_2(\theta) = \text{Cl}_2(\theta - 2\pi) \) by the 2\( \pi \)-periodicity of \( \text{Cl}_2 \). We find

\[ I(a) = \frac{1}{6} \left[ \text{Cl}_2(6\varphi) - 3\text{Cl}_2(4\varphi) + 3\text{Cl}_2(2\varphi) \right]. \]  

(26)

When \( \varphi = \tan^{-1} \sqrt{7} \), the case (10) follows.

We next present some reference integrals. We then apply them to write expressions for combinations of the integrals

\[ I_n \equiv \int_{n\pi/24}^{(n+1)\pi/24} \ln \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \, dt, \]  

(27)
where \( n \geq 0 \) is an integer. We supplement (22) with

\[
\int_y^y \ln(a - \tan t)dt = -(y - x) \ln \cos \varphi + \frac{1}{2} [\text{Cl}_2(2\varphi - y) - \text{Cl}_2(2\varphi - x)] - \frac{1}{2} [\text{Cl}_2(\pi - 2y) - \text{Cl}_2(\pi - 2x)],
\]

(28)

where \( a = \tan \varphi \). We also have

\[
\int_y^y \frac{\ln \left| \tan t + a \right|}{\tan t - a} dt = \frac{1}{2} \left[ \text{Cl}_2(2x + 2\varphi) - \text{Cl}_2(2x - 2\varphi) + \text{Cl}_2(2y - 2\varphi) - \text{Cl}_2(2y + 2\varphi) \right],
\]

(29)

with \( x < \varphi = \tan^{-1} a < y \). We now write expressions for the linear combinations

\[
C_1 \equiv -2(I_2 + I_3 + I_4 + I_5) + I_8 + I_9 - (I_{10} + I_{11}) \equiv 0,
\]

(30)

and

\[
C_2 \equiv I_2 + 3(I_3 + I_4 + I_5) + 2(I_6 + I_7) - 3I_8 - I_9 \equiv 0.
\]

(31)

These relations have been detected with further PSLQ computations [3]. A similar conjecture for integrals \( I_n \) with increments \( n\pi/60 \) has also been written [4] (p. 508).

The latter linear combination may also be expressed in terms of \( \text{Cl}_2 \) values, but here we concentrate on (30) and (31). We decompose the left side of (30) as indicated, and find for these contributions

\[
2(I_2 + I_3 + I_4 + I_5) = \text{Cl}_2 \left( \frac{\pi}{6} + 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{6} - 2\varphi \right) + \text{Cl}_2 \left( \frac{\pi}{2} - 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{2} + 2\varphi \right),
\]

(32)

where \( \varphi = \tan^{-1} \sqrt{7} \),

\[
2I_8 = \text{Cl}_2 \left( \frac{2\pi}{3} + 2\varphi \right) - \text{Cl}_2 \left( \frac{3\pi}{4} + 2\varphi \right) + \text{Cl}_2 \left( 2\varphi - \frac{2\pi}{3} \right) - \text{Cl}_2 \left( 2\varphi - \frac{3\pi}{4} \right),
\]

(33)
\[ 2I_9 = \text{Cl}_2 \left( \frac{3\pi}{4} + 2\varphi \right) - \text{Cl}_2 \left( \frac{3\pi}{4} - 2\varphi \right) + \text{Cl}_2 \left( \frac{5\pi}{6} - 2\varphi \right) - \text{Cl}_2 \left( \frac{5\pi}{6} + 2\varphi \right), \]  
\[ \text{(34)} \]

and

\[ I_{10} + I_{11} = -\text{Cl}_2(\pi + 2\varphi) + \frac{1}{2} \left[ \text{Cl}_2 \left( \frac{5\pi}{6} + 2\varphi \right) - \text{Cl}_2 \left( \frac{5\pi}{6} - 2\varphi \right) \right]. \]

\[ \text{(35)} \]

Therefore, we obtain

\[ C_1 = -\text{Cl}_2 \left( \frac{\pi}{6} + 2\varphi \right) + \text{Cl}_2 \left( \frac{\pi}{6} - 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{4} - 2\varphi \right) + \text{Cl}_2 \left( \frac{\pi}{4} + 2\varphi \right) \]

\[ \frac{1}{2} \left[ \text{Cl}_2 \left( \frac{2\pi}{3} + 2\varphi \right) + \text{Cl}_2 \left( 2\varphi - \frac{2\pi}{3} \right) \right] + \text{Cl}_2 \left( \frac{5\pi}{6} - 2\varphi \right) - \text{Cl}_2 \left( \frac{5\pi}{6} + 2\varphi \right) \]

\[ + \text{Cl}_2(\pi + 2\varphi). \]

\[ \text{(36)} \]

For the combination \( C_2 \) we have

\[ 2I_2 = \text{Cl}_2 \left( \frac{\pi}{6} + 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{6} - 2\varphi \right) + \text{Cl}_2 \left( \frac{\pi}{4} - 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{4} + 2\varphi \right), \]

\[ \text{(37)} \]

\[ -2(I_6 + I_7) = \text{Cl}_2 \left( \frac{2\pi}{3} + 2\varphi \right) - \text{Cl}_2 \left( \frac{2\pi}{3} - 2\varphi \right) + \text{Cl}_2 \left( \frac{\pi}{2} - 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{2} + 2\varphi \right), \]

\[ \text{(38)} \]

and

\[ 2(I_3 + I_4 + I_5) = \text{Cl}_2 \left( \frac{\pi}{4} + 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{4} - 2\varphi \right) + \text{Cl}_2 \left( \frac{\pi}{2} - 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{2} + 2\varphi \right). \]

\[ \text{(39)} \]

Therefore, we find

\[ 2C_2 = \text{Cl}_2 \left( \frac{\pi}{6} + 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{6} - 2\varphi \right) + \text{Cl}_2 \left( \frac{\pi}{2} - 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{2} + 2\varphi \right) \]

\[ -5\text{Cl}_2 \left( \frac{2\pi}{3} + 2\varphi \right) - 5\text{Cl}_2 \left( 2\varphi - \frac{2\pi}{3} \right) - \text{Cl}_2 \left( \frac{5\pi}{6} - 2\varphi \right) + \text{Cl}_2 \left( \frac{5\pi}{6} + 2\varphi \right) \]

\[ + 2 \left[ \text{Cl}_2 \left( \frac{\pi}{4} + 2\varphi \right) - \text{Cl}_2 \left( \frac{\pi}{4} - 2\varphi \right) + \text{Cl}_2 \left( \frac{3\pi}{4} + 2\varphi \right) + \text{Cl}_2 \left( 2\varphi - \frac{3\pi}{4} \right) \right]. \]

\[ \text{(40)} \]
By a combination of the quadriplication formula (16) and the duplication formula (14) we may write

\[
\frac{1}{4} \text{Cl}_2(4\theta) = \text{Cl}_2 \left( \theta + \frac{\pi}{2} \right) + \text{Cl}_2 \left( \theta - \frac{\pi}{2} \right) + \frac{1}{2} \text{Cl}_2(2\theta).
\]  

(41)

This enables other expressions for \(C_1\) and \(C_2\). Similarly, one may use the 6- and 12-fold multiplication formulas.

In regard to the combination on the right side of (3), we comment on an observation given previously [9]. We have that \(\pm \sin(2\pi/7)\), \(\pm \sin(4\pi/7)\), and \(\pm \sin(6\pi/7)\) are the nonzero roots of the Chebyshev polynomial \(T_7(x)\). Indeed, if we write the cubic polynomials

\[
p_1(x) = \left( x - \sin \frac{2\pi}{7} \right) \left( x - \sin \frac{4\pi}{7} \right) \left( x + \sin \frac{6\pi}{7} \right) = x^3 - \frac{\sqrt{7}}{2} x^2 + \frac{\sqrt{7}}{8}, \tag{42a}
\]

and

\[
p_2(x) = \left( x - \sin \frac{6\pi}{7} \right) \left( x + \sin \frac{2\pi}{7} \right) \left( x + \sin \frac{4\pi}{7} \right) = x^3 + \frac{\sqrt{7}}{2} x^2 - \frac{\sqrt{7}}{8}, \tag{42b}
\]

we then have the factorization \(p_1(x)p_2(x) = T_7(x)/64x\). This invites questions as to whether scaled versions of these or other Chebyshev polynomials could be useful in developing identities underlying (3), (30), (31), or the like.

Given the close relation of the Clausen function \(\text{Cl}_2\) and the dilogarithm function, one wonders if a set of ladder relations for the latter may be carried over to explain (3) and relations amongst the integrals \(I_n\). In developing ladder relations, cyclotomic equations for the base have proven very useful. It would be of interest to see if \(\text{Cl}_2\) relations with \(\theta_7\) could be discovered in this way.
We remark on using Kummer’s relation [14] (pp. 107, 254) to rewrite the right side of (3) in terms of the dilogarithm of complex argument. We have

\[ \frac{1}{4} Z_{Q(\sqrt{-7})} = \frac{7}{2} \left[ \text{Im \, Li}_2(Re^{i\phi}) - b \ln R \right], \tag{43} \]

where

\[ R = \frac{\tan b}{\sin \phi + \tan b \cos \phi}. \tag{44} \]

Here, we may take \( \phi = \pi/7 \) and \( b = 2\pi/7 \), or vice versa. Then by Proposition 2 of [9] we have the integral representation

\[ \frac{\sqrt{7}}{2} I_7 = \text{Cl}_2 \left( \frac{2\pi}{7} \right) + \text{Cl}_2 \left( \frac{4\pi}{7} \right) - \text{Cl}_2 \left( \frac{6\pi}{7} \right) = 2 \sin \left( \frac{\pi}{7} \right) \int_{d}^{\infty} \frac{\ln y \, dy}{y^2 - 2y \cos(\pi/7) + 1}, \tag{45} \]

where \( d = [2 \cos(\pi/7) - 1]^{-1} \).

Finally, we use relations from [16] (Appendix A) and [19] to write a third evaluation of the integral \( I_7 \). For this we introduce the angle \( \theta_{75} \equiv 2 \tan^{-1}(\sqrt{7}/5) \) and the Bloch-Wigner dilogarithm [17]

\[ D(z) = \text{Im} \left[ \text{Li}_2(z) \right] + \text{arg}(1 - z) \ln |z|, \tag{46} \]

for which we have [16] (p. 246)

\[ D(z) = \frac{1}{2} \left[ \text{Cl}_2(2\theta) + \text{Cl}_2(2\omega) - \text{Cl}_2(2\theta + 2\omega) \right], \tag{47} \]

where \( \theta = \text{arg} \, z \) and \( \omega = \text{arg} \, (1 - \bar{z}) \). We note the interpretation that for \( z \in C \), the volume of the asymptotic simplex with vertices 0, 1, \( z \), and \( \infty \) in 3-dimensional
hyperbolic space is given by \(|D(z)|\) \cite{16} (p. 271). We then rewrite the expression \cite{16, p. 384 or 19, p. 246}.

\[
\zeta_Q(\sqrt{-7})(2) = \zeta(2)L_{-7}(2) = \frac{4\pi^2}{21\sqrt{7}} \left[ 2D \left( \frac{1+i\sqrt{7}}{2} \right) + D \left( \frac{-1+i\sqrt{7}}{4} \right) \right].
\]

(48)

We apply (47), giving

\[
I_7 \equiv L_{-7}(2) = \frac{8}{7\sqrt{7}} \left[ 2D \left( \frac{1+i\sqrt{7}}{2} \right) + D \left( \frac{-1+i\sqrt{7}}{4} \right) \right]
\]

\[
= \frac{4}{7\sqrt{7}} \left[ 4\text{Cl}_2(\pi - \theta_7) - \text{Cl}_2(\theta_7) + \text{Cl}_2(\theta_{75}) + \text{Cl}_2(\theta_7 - \theta_{75}) \right].
\]

(49)

In the case of \(D[(1 + i\sqrt{7})/2]\) we used the duplication formula (14). In contrast to (49), the expression in \cite{9} for \(I_7\) involves \(\theta_+ \equiv \tan^{-1}(\sqrt{7}/3)\). With the various analytic evaluations now known for \(I_7\) or \(L_{-7}(2)\), we have enlarged the set of possible relations amongst \(\text{Cl}_2\) values. From (10), (14), and (49) we obtain the conjecture

\[
\text{Cl}_2(3\theta_7) - \text{Cl}_2(2\theta_7) \equiv \text{Cl}_2(\theta_{75}) + \text{Cl}_2(\theta_7 - \theta_{75}).
\]

(50)

In fact, we have \(\theta_7 - \theta_{75} = 2\theta_+\), and we conclude by proving (50), and thereby (49). We quickly show that both

\[
\text{Cl}_2(3\theta_7) = \text{Cl}_2(\theta_{75})
\]

(51)

and

\[
\text{Cl}_2(2\theta_7) = -\text{Cl}_2(\theta_7 - \theta_{75}),
\]

(52)

for we have \(3\theta_7 - 2\pi = \theta_{75}\) and \(\theta_7 - \pi = -\theta_+\). The latter relations require nothing more than the identity \(\tan(x/2) = \sin x/(1 + \cos x)\).
We have similarly found many other angular pairs \((\theta_1, \theta_2)\) satisfying \(3\theta_1 - 2\pi = \pm\theta_2\), immediately giving \(\text{Cl}_2(3\theta_1) = \pm\text{Cl}_2(\theta_2)\). As these may be useful elsewhere \([6, 16]\), we record several of them in the first Appendix. We also relegate to this Appendix a possibly new \(\log\) trigonometric integral in terms of \(\text{Cl}_2\). In the second Appendix, we develop new series and integral representations of the Clausen function.

Appendix A

We let \(\theta_k \equiv 2 \tan^{-1} \sqrt{k}\), and \(\theta_{k,j} \equiv 2 \tan^{-1} \sqrt{k/j}\). We find the relations

\[
3\theta_2 - 2\pi = -\theta_{2,5}, \quad 3\theta_5 - 2\pi = \theta_{5,7}, \tag{A.1}
\]

\[
3\theta_{11} - 2\pi = -\theta_{11,4}, \quad 3\theta_{13} - 2\pi = -2 \tan^{-1} \left(\frac{5\sqrt{13}}{19}\right), \tag{A.2}
\]

and

\[
3\theta_{91,3} - 2\pi = -2 \tan^{-1} \left(\frac{8\sqrt{91}}{99}\right), \quad 3\theta_{91,5} - 2\pi = 2 \tan^{-1} \left(\frac{2\sqrt{91}}{155}\right), \quad 3\theta_{91,7} - 2\pi = -\theta_{91,28}. \tag{A.3}
\]

With \(\theta_{32} \equiv 2 \tan^{-1} \sqrt{3/2}, \theta_{53} \equiv 2 \tan^{-1} \sqrt{5/3}, \theta_{133} \equiv 2 \tan^{-1} \sqrt{13/3}, \theta_{73} \equiv 2 \tan^{-1} \sqrt{7/3}\), we have

\[
3\theta_{32} - 2\pi = -2 \tan^{-1} \left(\frac{3 \sqrt{3}}{7}\right), \quad 3\theta_{53} - 2\pi = -2 \tan^{-1} \left(\frac{1 \sqrt{5}}{3}\right), \tag{A.4}
\]

and

\[
3\theta_{133} - 2\pi = 2 \tan^{-1} \left(\frac{1}{9} \sqrt{13/3}\right), \quad 3\theta_{73} - 2\pi = -2 \tan^{-1} \left(\frac{1}{9} \sqrt{7/3}\right). \tag{A.5}
\]

Based upon the trigonometric identity \(3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2\), we have found the integral

\[
\int_0^x \ln(3 + 4 \cos \theta + \cos 2\theta) d\theta = -x \ln 2 + 4\text{Cl}_2(\pi - x), \quad 0 \leq x \leq \pi. \tag{A.6}
\]
This obviously provides an integral expression for the Catalan constant \( G = \sum_{k \geq 0} (-1)^k/(2k + 1)^2 = \text{Cl}_2(\pi/2) \) when \( x = \pi/2 \).

The function \( \text{Cl}_2(t) \) for \( t \in (0, \pi) \) has its only maximum at \( t = \pi/3 \), when \( \text{Cl}_2(\pi/3) \simeq 1.014941606409653625021 \). We mention that near this value \( \text{Cl}_2 \) has a fixed point, \( \text{Cl}_2(y) = y \) for \( y \simeq 1.01447193895251725798414 \).

Related to the equality of expressions (10) and (49) for \( I_7 \) we have the relation

\[
6 \left[ \text{Li}_2 \left( \frac{1 - 3i\sqrt{7}}{8} \right) + \text{Li}_2 \left( \frac{1 + 3i\sqrt{7}}{8} \right) \right] = 3(\pi - 2\theta_+)^2 - \pi^2
\]

\[
= 3(\theta_7 - \theta_+)^2 - \pi^2 = 3[\pi - \tan^{-1}(3\sqrt{7})]^2 - \pi^2. \tag{A.7}
\]

Such relations follow readily from (21) as we have

\[
6[\text{Li}_2(e^{i\theta}) + \text{Li}_2(e^{-i\theta})] = 2\pi^2 + 3\theta^2, \quad 0 \leq \theta \leq 2\pi. \tag{A.8}
\]
Appendix B

We have

**Proposition B1.** We have for $\theta < \pi/n$ and $n \geq 1$ an integer

$$
\frac{1}{2} \left[ \frac{1}{n} \text{Cl}_2(2n\theta) - \text{Cl}_2(2\theta) \right] = \sum_{j=1}^{\infty} \frac{\zeta(2j) (n^{2j} - 1)}{j^{2j} (2j + 1)} \theta^{2j+1} - \theta \ln n. \quad (B.1)
$$

This result gives several Corollaries, including

**Corollary (i)**

$$
\frac{1}{2} \sum_{k=1}^{n-1} \text{Cl}_2 \left(2\theta + \frac{2\pi}{n}k\right) = \sum_{j=1}^{\infty} \frac{\zeta(2j) (n^{2j} - 1)}{j^{2j} (2j + 1)} \theta^{2j+1} - \theta \ln n, \quad (B.2)
$$

**Corollary (ii)**

$$
\frac{1}{2} \left[ \frac{1}{n} \text{Cl}_2(2n\theta) - \text{Cl}_2(2\theta) \right] = -\theta \ln n \\
+ \frac{2\theta}{n} \int_0^\infty \frac{1}{x^2} [\sinh(nx) - n \sinh x] \frac{dx}{(e^{\pi x/\theta} - 1)}, \quad (B.3)
$$

**Corollary (iii)**

$$
\frac{1}{2} \left[ \frac{1}{n} \text{Cl}_2(2n\theta) - \text{Cl}_2(2\theta) \right] = -\theta \ln n \\
+ \frac{\theta}{2\pi} \left[ 2n\theta \tanh^{-1} \left( \frac{n\theta}{\pi} \right) - 2\theta \tanh^{-1} \left( \frac{\theta}{\pi} \right) + \pi \ln \left( \frac{\pi^2 - n^2\theta^2}{\pi^2 - \theta^2} \right) \right] \\
+ 2\pi \int_1^{\infty} \left[ \tanh^{-1} \left( \frac{\theta}{\pi x} \right) - \frac{1}{n} \tanh^{-1} \left( \frac{n\theta}{\pi x} \right) \right] P_1(x) dx. \quad (B.4)
$$

In the last equation, $P_1(x) = x - [x] - 1/2$ is the first periodized Bernoulli polynomial.

**Proof.** The Proposition is based upon the relation [1] (p. 75)

$$
\ln \left( \frac{n \sin x}{\sin nx} \right) = \sum_{j=1}^{\infty} \frac{\zeta(2j) (n^{2j} - 1) x^{2j}}{j^{2j}}; \quad |x| < \pi/n, \quad (B.5)
$$
wherein we have used the relation between $\zeta(2j)$ and the Bernoulli numbers $B_{2j}$. (For more details, see the end of this Appendix.) With the series of (B.5) being boundedly convergent, we may integrate term-by-term over any finite interval avoiding the singularity at $x = \pi/n$. Doing so, integrating over $[0, \theta]$, and using the first integral representation for $\text{Cl}_2$ on the right side of (12) gives (B.1).

Corollary (i) follows from the multiplication formula for $\text{Cl}_2$ [14] (pp. 94, 253). Corollary (ii) uses a standard integral representation of the Riemann zeta function. With the interchange of summation and integration, with the integral being absolutely convergent, the Corollary follows.

Corollary (iii) uses the representation for $\Re s > -1$,

$$
\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s \int_1^\infty \frac{P_1(x)}{x^{s+1}} \, dx.
$$

Again, the interchange of summation and integration is employed.

Remarks. In connection with (B.5) we may note the relation with the Chebyshev polynomials $U_n$ of the second kind [12] (p. 1032),

$$
U_{n-1}(\cos \phi) = \frac{\sin n\phi}{\sin \phi}.
$$

For $\theta = \pi/4$ and other values, Proposition B1 gives many relations involving the Catalan constant $G$. More generally, for $\theta$ a rational multiple of $\pi$, the results are expressible in terms of $\psi'$, the trigamma function [11, 13]. If we let

$$
r(\theta, n) \equiv \sum_{j=1}^{\infty} \frac{\zeta(2j)}{j \pi^{2j} (2j + 1)} \theta^{2j+1}, \quad \theta \leq \pi/n,
$$

15
we may write several simple examples:

\[
\begin{align*}
  r \left( \frac{\pi}{3}, 2 \right) &= \pi \left[ \frac{1}{18} (\sqrt{3}\pi + 6 \ln 2) - \frac{\psi'(1/3)}{4\sqrt{3}\pi} \right], & (B.9a) \\
  r \left( \frac{\pi}{3}, 3 \right) &= \pi \left[ \frac{1}{27} (\sqrt{3}\pi + 9 \ln 3) - \frac{\psi'(1/3)}{6\sqrt{3}\pi} \right], & (B.9b) \\
  r \left( \frac{\pi}{4}, 2 \right) &= \frac{1}{2} G + \frac{\pi}{4} \ln 2, & (B.10a) \\
  r \left( \frac{\pi}{4}, 3 \right) &= \frac{2}{3} G + \frac{\pi}{4} \ln 3, & (B.10b) \\
  r \left( \frac{\pi}{4}, 4 \right) &= \frac{1}{2} G + \frac{\pi}{2} \ln 2, & (B.10c)
\end{align*}
\]

and

\[
\begin{align*}
  r \left( \frac{\pi}{6}, 2 \right) &= \pi \left[ \frac{1}{54} (2\sqrt{3}\pi + 9 \ln 2) - \frac{\psi'(1/3)}{6\sqrt{3}\pi} \right], & (B.11a) \\
  r \left( \frac{\pi}{6}, 3 \right) &= \pi \left[ \frac{1}{18} (\sqrt{3}\pi + 3 \ln 3) - \frac{\psi'(1/3)}{4\sqrt{3}\pi} \right], & (B.11b) \\
  r \left( \frac{\pi}{6}, 4 \right) &= \pi \left[ \frac{1}{108} (7\sqrt{3}\pi + 36 \ln 2) - \frac{7\psi'(1/3)}{24\sqrt{3}\pi} \right], & (B.11c) \\
  r \left( \frac{\pi}{6}, 5 \right) &= \pi \left[ \frac{1}{30} (2\sqrt{3}\pi + 5 \ln 5) - \frac{\sqrt{3}\psi'(1/3)}{10\pi} \right], & (B.11d) \\
  r \left( \frac{\pi}{6}, 6 \right) &= \pi \left[ \frac{1}{18} (\sqrt{3}\pi + 3 \ln 2 + 3 \ln 3) - \frac{\psi'(1/3)}{4\sqrt{3}\pi} \right]. & (B.11e)
\end{align*}
\]

Finally, we supply a derivation of (B.5). We have

\[
\begin{align*}
  \frac{d}{dx} \ln \left( \frac{n \sin x}{\sin nx} \right) &= \cot x - n \cot nx \\
  &= \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} (n^{2k} - 1)x^{2k-1}, \quad \frac{n|x|}{\pi} < 1, & (B.12)
\end{align*}
\]
where we used a series representation for \( \cot \) (p. 35). Since

\[
\frac{2^{2k} |B_{2k}|}{(2k)!} = \frac{2 \zeta(2k)}{\pi^{2k}},
\]

we have

\[
\cot x - n \cot nx = 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} (n^{2k} - 1)x^{2k-1}.
\]

Integrating both sides of this relation gives (B.5).
References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards (1972).

[2] G. Almkvist, Asymptotic formulas and generalized Dedekind sums, Exptl. Math. 7, 343-359 (1994).

[3] D. H. Bailey et al., Experimental Mathematics in Action, A. K. Peters, Wellesley, MA (2007).

[4] D. H. Bailey and J. M. Borwein, Experimental mathematics: Examples, methods and implications, Notices Amer. Math. Soc. 52, 502-514 (2005).

[5] D. H. Bailey and J. M. Borwein, Computer-assisted discovery and proof, in: Tapas in Experimental Mathematics, Contemp. Math., T. Amdeberhan and V. Moll, eds., Amer. Math. Soc. (2008), pp. 21-52; preprint http://crd.lbl.gov/~dhbailey/dhbpapers/comp-disc-proof.pdf (2007).

[6] J. M. Borwein and D. J. Broadhurst, Determination of rational Dedekind-zeta invariants of hyperbolic manifolds and Feynman knots and links, arxiv:hep-th/9811173 (1998).

[7] D. J. Broadhurst, Massive 3-loop Feynman diagrams reducible to SC* primitives of algebras of the sixth root of unity, Eur. Phys. J. C 8, 311-333 (1999).
[8] D. J. Broadhurst, Solving differential equations for 3-loop diagrams: relation to hyperbolic geometry and knot theory, arxiv/hep-th/9806174v2 (1998).

[9] M. W. Coffey, Evaluation of a ln tan integral arising in quantum field theory, J. Math. Phys. 49, 093508-1-15 (2008).

[10] M. W. Coffey, On a three-dimensional symmetric Ising tetrahedron, and contributions to the theory of the dilogarithm and Clausen functions, J. Math. Phys. 49, 043510-1-32 (2008).

[11] P. J. de Doelder, On the Clausen integral $\text{Cl}_2(\theta)$ and a related integral, J. Comput. Appl. Math. 11, 325-330 (1984).

[12] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York (1980); 7th ed. (2007), eds. A. Jeffrey and D. Zwillinger.

[13] C. C. Grosjean, Formulae concerning the computation of the Clausen integral $\text{Cl}_2(\theta)$, J. Comput. Appl. Math. 11, 331-342 (1984).

[14] L. Lewin, Dilogarithms and associated functions, Macdonald (1958).

[15] L. Lewin, Polylogarithms and associated functions, North Holland (1981).

[16] L. Lewin, ed., Structural properties of polylogarithms, American Mathematical Society (1991).

[17] Apparently in (11.22) in [16], the lower limit of the integral for $D(z)$ is intended to be 0, consistent with our (20).
[18] H. M. Srivastava and J. Choi, Series associated with the zeta and related functions, Kluwer (2001).

[19] D. Zagier, The dilogarithm function in geometry and number theory, in: Number theory and related topics, Bombay Tata Institute of Fundamental Research, 231-249 (1988).

[20] I. J. Zucker and M. M. Robertson, Some properties of Dirichlet $L$-series, J. Phys. A 9, 1207-1214 (1976).