Logarithmic stability for a coefficient inverse problem of coupled Schrödinger equations

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Abstract
In this paper, we study an inverse coefficient problem for two coupled Schrödinger equations with an observation of one component of the solution. The observation is carried out in a nonempty open subset of the domain where the equations hold. A logarithmic-type stability result is obtained. The main method is based on the Carleman estimate for coupled Schrödinger equations and coupled heat equations, and the Fourier–Bros–Iagolnitzer transform.

Keywords: logarithmic stability, coefficient inverse problem, coupled Schrödinger equations, Carleman estimate

1. Introduction
Let $\Omega \subset \mathbb{R}^3$ be a nonempty bounded domain with smooth boundary $\Gamma := \partial \Omega$ and let $i = \sqrt{-1}$. Consider the following coupled Schrödinger equations:

$$\begin{cases}
i \partial_t y_1 + \Delta y_1 + a_{11}(x)y_1 + a_{12}(x)y_2 = 0 & \text{in } \Omega \times (0, \infty), \\
i \partial_t y_2 + \Delta y_2 + a_{21}(x)y_1 + a_{22}(x)y_2 = 0 & \text{in } \Omega \times (0, \infty), \\
y_1 = 0, y_2 = 0 & \text{on } \Gamma \times (0, \infty), \\
y_1(x,0) = y_{10}, y_2(x,0) = y_{20} & \text{in } \Omega.
\end{cases}$$

System (1) is a useful model for describing molecular multiphoton transitions induced by a laser (for example, [1, 13]), where $a_{11}(x)$ and $a_{22}(x)$ are field-free molecular electronic potentials, and $a_{12}(x)$ and $a_{21}(x)$ are radiation–molecule interactions. In physical models, the
radiation–molecule interactions can usually be deduced \textit{a priori} while the field-free molecular electronic potentials should be determined \textit{a posteriori}. This is a background for our inverse problem. More precisely, in this paper, we study the following inverse problems:

\textbf{Problem (IP).} Let \( \omega \) be a nonempty open subset of \( \Omega \). Can one recover the field-free molecular electronic potentials \((a_{11}, a_{22})\) from a suitable observation of \( y_1 \) on \([0, T] \times \omega\)?

Here, the word ‘recover’ refers to two issues: one is that the observation determines the potentials uniquely. The other is to find an algorithm to compute the potentials efficiently.

For a non-negative continuous function \( \eta(\xi) \) defined for \( \xi \geq 0 \) satisfying \( \eta(0) = 0 \), a stability estimate

\[
|||(a_{11}, a_{22})||| \leq \eta(|||y_1|||) \tag{2}
\]

with conditions on boundedness of suitable norms is not only important theoretically but also essential for the second issue: it can guarantee the convergence of the numerical algorithm for computing \((a_{11}, a_{22})\). In particular, Cheng and Yamamoto [9] show that the stability rate described by the function \( \eta \) is a quasi-optimal convergence rate of Tikhonov regularization with a suitable \textit{a priori} choice of the regularizing parameters according to noise levels in data norms \( |||y_1||| \). We note that if \( \eta(\xi) = C\xi \) and \( \eta(\xi) = C|\ln \xi| \) with constants \( C > 0 \), then the estimate (2) indicates Lipschitz-type stability and logarithmic-type stability, respectively.

Inequalities in the type of (2) for Schrödinger equations were studied extensively (for example, [2–5, 7, 8, 11, 15–19]). Roughly speaking, the existing works fall into two categories: one is Lipschitz-type stability when the observation domain fulfills some geometric condition (for example, [2, 3, 7, 8, 11, 15, 20, 23]), while the other is logarithmic-type stability when the observation domain is a general nonempty open subset of the domain or its boundary (for example, [4, 5]). For the latter case, some \textit{a priori} knowledge about the potential on a suitable subdomain should be known (see [5]).

The main method for establishing Lipschitz-type stability is based on the Carleman estimate. On the other hand, the key method for proving logarithmic-type stability is a combination of the Carleman estimate and the Fourier–Bros–Iagolnitzer (FBI) transformation. For readers who are not familiar with the FBI transform, we refer them to [10] for an introduction and to [22] for the application of FBI transform to establish the observability estimate for Schrödinger equations.

To the best of our knowledge, although there are several interesting works concerning inverse problems for a parabolic system with two components by measurements of one component (see [6] as an example), there is no work on the inverse coefficient problem for coupled Schrödinger equations with an observation on one component of the solution. Due to the essential difference between these two equations, we have to argue independently of [6] in the case of parabolic systems. In this paper, we will study this problem by the Carleman estimate for the Schrödinger equation, coupled heat equations and the FBI transform. Although we borrow some ideas in [5] to prove our main result, since we study the inverse problem for coupled Schrödinger equations with a single observation on one component of the solution, we cannot simply mimic the method in [5] to obtain the desired logarithmic-type stability. Some technical obstacles should be overcome, as is seen in the proof.

The rest of this paper is organized as follows. Section 2 is devoted to presenting the main result while section 3 is devoted to the proof of the main result.
2. Statement of the main result

Let \( \omega_0 \) be an open subset of \( \Omega \) such that there exists a function \( \phi \in C^1(\overline{\Omega}) \) satisfying

\[
\begin{align*}
\nabla \phi & \neq 0 \text{ in } \Omega \setminus \omega_0, \\
\frac{\partial \phi}{\partial \nu} & \leq 0 \text{ on } \partial \Omega, \\
|\nabla \phi(x) \cdot \xi|^2 + \sum_{j=1}^{3}(\partial_j \phi(x))\xi_j^2 & > 0, \quad \forall x \in \Omega \setminus \omega_0, \forall \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, (3) \\
\phi(x) & > \frac{2}{3}\|\phi\|_{L^\infty(\Omega)}, \quad \forall x \in \Omega.
\end{align*}
\]

Here, \( \nu = \nu(x) \) denotes the outward normal vector of \( \Omega \).

There are plenty of choices of \( \omega_0 \) satisfying the above condition. A typical example can be constructed as follows.

Let \( x_0 \in \mathbb{R}^3 \setminus \overline{\Omega} \) and

\[
\Gamma_0 = \{ x \in \Gamma \mid (x - x_0) \cdot \nu(x) \geq 0 \}.
\]

Let \( \delta > 0 \). Put

\[
\omega_0 = \Omega_\delta \triangleq \{ x \in \Omega \mid \text{dist}(x, \Gamma_0) < \delta \}.
\]

Let \( \tilde{\phi}(\cdot) \in C^1(\overline{\Omega}) \) be a nonnegative function such that \( \tilde{\phi}(x) = \|x - x_0\|^2 \) for \( x \in \Omega \setminus \omega_0 \) and \( \tilde{\phi}(x) > 0 \) for \( x \in \Omega_\delta/2 \) and \( \tilde{\phi} = 0 \) on \( \Gamma_0 \). Then \( \phi(x) = \tilde{\phi}(x) + 2\|\phi\|_{L^\infty(\Omega)} \) is an example satisfying the above condition (3).

More examples of \( \omega_0 \) and \( \phi \) can be found in example 3.4 in [20].

Clearly, if (3) holds, then there exists \( \omega_1 \subset \subset \omega_0 \) such that

\[
\begin{align*}
\nabla \phi & \neq 0 \text{ in } \Omega \setminus \omega_1, \\
|\nabla \phi(x) \cdot \xi|^2 + \sum_{j=1}^{3}(\partial_j \phi(x))\xi_j^2 & > 0 \quad \forall x \in \Omega \setminus \omega_1, \forall \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
\end{align*}
\]

Let \( l > 0 \). Let \( \widetilde{\omega} \subset \Omega \) be a neighborhood of \( \omega_1 \) such that \( \omega_1 \subset \subset \widetilde{\omega} \) and \( \partial \widetilde{\omega} \) is \( C^2 \). Set

\[
\mathcal{H}_l = C^1([0, l]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^2([0, l]; H_0^1(\Omega)),
\]

where \( H^l(\Omega) \) is the usual Sobolev space. The Banach space \( \mathcal{H}_l \) is equipped with its natural norm

\[
|z|^2_{\mathcal{H}_l} = |z|^2_{C^1([0, l]; H^2(\Omega) \cap H_0^1(\Omega))} + |z|^2_{C^2([0, l]; H_0^1(\Omega))}, \quad \forall z \in \mathcal{H}.
\]

Let \( \omega \subset \omega_1 \subset \Omega \) be an arbitrary nonempty open subset. Suppose that \( \{a_k\}_{k=1}^2 \subset W^{2, \infty}(\Omega) \) and we can choose a constant \( a_0 > 0 \) such that

\[
a_{21} \geq a_0 \text{ or } -a_{21} \geq a_0 \text{ in } \omega.
\]

Remark 2.1. Condition (7) means that the coupling between \( y_1 \) and \( y_2 \) does not degenerate. More precisely, \( y_1 \) can effect \( y_2 \) adequately. Without (7), one cannot obtain information regarding \( y_2 \) from \( y_1 \).

Let us now define the admissible set of unknown coefficients. Fix a constant \( M > 0 \) and two functions \( \omega_1, \omega_2 \in W^{2, \infty}(\overline{\omega}; \mathbb{R}) \). Let \( T > 0 \) and \( \mathcal{A}(\omega; M) \) be an admissible set of pairs of real-valued functions \( (a_{11}, a_{22}) \) such that
\[ A(\tilde{\omega}, M) \triangleq \{(a_{11}, a_{22}) \in W^{2,\infty}(\Omega; \mathbb{R})^2 \mid |a_{ij}|_{L^\infty(\Omega)} \leq M, a_j(x) = y_j(x) \text{ on } \tilde{\omega}, \]

equation (1) has a unique solution \((y_1, y_2) \in \mathcal{H}_\infty\) satisfying

\[ ||y_j||_{\mathcal{H}_T} \leq \tilde{C}(M, T) \text{ for some constant } \tilde{C}(M, T) \text{ depending on } M \]

and \(T, j = 1, 2\).

(8)

Remark 2.2. The set \(A(\tilde{\omega}, M)\) is not empty when \(C(M) > 0\). Indeed, by a standard operator semigroup theory (for example, [21, section 4.2]), one can show that equation (1) has a unique solution \((y_1, y_2) \in \mathcal{H}_\infty\) when \((y_{10}, y_{20}) \in [H_0^2(\Omega)]^2\). Further, one has that

\[ |y_1|_{\mathcal{H}_T} + |y_2|_{\mathcal{H}_T} \leq C(|y_{10}|_{H^1(\Omega)} + |y_{20}|_{H^1(\Omega)}), \]

where \(C\) depends on \(T\) and \(a_{jk}, j, k = 1, 2\). Thus, if one chooses \((y_{10}, y_{20})\) satisfying \(|y_{10}|_{H^1(\Omega)} + |y_{20}|_{H^1(\Omega)} \leq \frac{\tilde{C}(M, T)}{C}\), then \(|y_j||_{\mathcal{H}_T} \leq \tilde{C}(M, T), j = 1, 2\).

Remark 2.3. There are mainly two restrictions on an element in \(A(\tilde{\omega}, M)\). The first one is that there is an \textit{a priori} bound \(M\). This is reasonable since in a physical model, one can assume to know some preliminary upper bound on unknown potentials. The second one is that we know the value of \((a_{11}(x), a_{22}(x))\) for \(x \in \tilde{\omega}\). This is technically restrictive but is acceptable because we may be able to directly measure potentials near the boundary. Furthermore, we note that compared with [20], we need less information on unknown potentials.

In what follows, in order to emphasize the dependence of the solution to (1) on the unknown potentials, we write \((y_1(a_{11}, a_{22}), y_2(a_{11}, a_{22}))\) for the solution to (1).

We choose the initial data \((y_{10}, y_{20})\) which satisfy all conditions ensuring that \(A(\tilde{\omega}, M)\) is nonempty. Also, for \(j = 1, 2\), they fulfill

\[ \begin{align*}
&y_{10}(x) \in \mathbb{R} \text{ or } iy_{10}(x) \in \mathbb{R} \text{ for a.a. } x \in \Omega, \\
&|y_{10}(x)| \geq r > 0 \text{ for a.a. } x \in \Omega, \\
&y_j(a_{11}, a_{22}) \in H^1(0, T; L^\infty(\Omega)).
\end{align*} \]

(9)

Remark 2.4. Condition (9) means that we have to choose initial data suitably, and is a technical restriction. Similarly to appendix B in [5], we can verify that such \((y_{10}, y_{20})\) exists.

The main result of this paper is stated as follows.

Theorem 2.1. There exists a constant \(C > 0\) depending on \(A(\tilde{\omega}, M)\) such that

\[ \begin{align*}
&||a_{11} - \tilde{a}_{11}, a_{22} - \tilde{a}_{22}||_{L^1(\Omega)} \\
&\leq C \left( |\ln ||y_1(a_{11}, a_{22}) - \tilde{y}_1(\tilde{a}_{11}, \tilde{a}_{22})||_{W^{1,\infty}(\Omega)}|^{-1} \\
&+ ||y_1(a_{11}, a_{22}) - \tilde{y}_1(\tilde{a}_{11}, \tilde{a}_{22})||_{W^{1,\infty}(\Omega)} \right),
\end{align*} \]

(10)

for all \((a_{11}, a_{22}), (\tilde{a}_{11}, \tilde{a}_{22}) \in A(\tilde{\omega}, M)\).

Remark 2.5. One can consider the problem that all the coefficients \(\{a_{jk}\}_{j,k=1}^2\) are unknown. In this case, the following three conditions are needed: (a) the unknown coefficient \(a_{21}\) must be nonzero in a nonempty open subset \(\omega\); (b) the functions \(a_{11}, a_{22}\) and \(a_{21}\) must be linearly independent, respectively; (c) two observations with different suitable chosen initial data of \(y_1\) are required. As pointed out in remark 2.1, condition (a) cannot be removed since we only observe a single component of the solutions. Condition (b) is reasonable since what we can observe
is only the linear combination of the coefficients. Condition (c) cannot be deleted because, for each observation data, we can determine only the linear combinations and have to determine the coefficients from these combinations so that we need to observe the system twice.

**Remark 2.6.** From the proof of theorem 2.1, one can see that it can be generalized to a system coupled by more than two Schrödinger equations with an observation on some components of the solution. In this paper, in order to present the key idea in a simple way, we do not pursue the full technical generality.

**Remark 2.7.** It is usual that we discuss the numerical reconstruction of coefficients by assuming the existence of solutions to the corresponding forward problem, such as an initial-boundary value problem. Accordingly, not based on the well-posedness for the initial-boundary value problem, by assuming the existence of solutions to (1) satisfying suitable a priori boundedness and regularity, the study for stability is very common for the inverse problem.

**Remark 2.8.** Here we adopt an internal observation to recover the unknown potentials. It is quite interesting to obtain the same result by boundary observation. For a single equation, this is done in [5]. As yet, we do not know how to do that for our systems.

3. Proof of theorem 2.1

Before giving the proof, we prove a preliminary result.

**Lemma 3.1.** For all \((a_{11}, a_{22}), (\tilde{a}_{11}, \tilde{a}_{22}) \in \mathcal{A}(\tilde{\omega}, M),\)

\[
\sum_{j=1}^{2} ||a_{jj} - \tilde{a}_{jj}||_{L^2(\Omega)}^2 \leq C \sum_{j=1}^{2} ||y_j(a_{11}, a_{22}) - \tilde{y}_j(\tilde{a}_{11}, \tilde{a}_{22})||_{H^1(\Omega)}^2.
\]

(11)

In order to obtain the Lipschitz stability in (11), the subdomain \(\omega_1\) cannot be arbitrarily small and must satisfy (4). Lemma 3.1 should be a known result. However, since we failed to find an exact reference, we provide it here for the sake of completeness and the reader’s convenience.

**Proof of lemma 3.1.** Let \(\phi \in C^4(\Omega)\) be the function satisfying (3) and (4). Set

\[
\tilde{\phi}(x, t) \equiv \frac{e^{\eta \phi(x)}}{(T + t)(T - t)}, \quad \tilde{\alpha}(x, t) \equiv \frac{\varepsilon^{2\beta}||\phi||_{L^\infty(\Omega)} - e^{\eta \phi(x)}}{(T + t)(T - t)}, \quad \forall (x, t) \in \Omega \times (0, T),
\]

(12)

where \(\eta\) denotes some positive number which can be specified later.

For \(j = 1, 2,\) let

\[
z_j = y_j(a_{11}, a_{22}) - \tilde{y}_j(\tilde{a}_{11}, \tilde{a}_{22}), \quad f_j(x) = a_{jj}(x) - \tilde{a}_{jj}(x), \quad R_j(x, t) \equiv \tilde{y}_j(x, t).
\]

Then \((z_1, z_2) \in [C([0, \infty); H_0^1(\Omega))]^2\) is the solution of the following system:

\[
\begin{aligned}
i \tilde{\partial}_t z_1 + \Delta z_1 + a_{11}z_1 + a_{12}z_2 &= f_1(x)R_1(x, t) \quad \text{in } \Omega \times (0, \infty), \\
i \tilde{\partial}_t z_2 + \Delta z_2 + a_{21}z_1 + a_{22}z_2 &= f_2(x)R_2(x, t) \quad \text{in } \Omega \times (0, \infty), \\
z_1(x, 0) = z_2(x, 0) = 0 \quad \text{in } \Omega, \\
z_1 &= z_2 = 0 \quad \text{on } \Gamma \times (0, \infty).
\end{aligned}
\]

(13)
Take the even-conjugate extensions of \((z_1, z_2)\) to the interval \((-\infty, \infty)\), i.e. set
\[
(z_1(x, t), z_2(x, t)) = \overline{(z_1(x, -t), z_2(x, -t))} \quad \text{for } t \in (-\infty, 0).
\]

If \((R_1(x, 0), R_2(x, 0)) \in \mathbb{R}^2\) for a.e. \(x \in \Omega\), then we set
\[
(R_1(x, t), R_2(x, t)) = (\overline{R_1(x, -t)}, \overline{R_2(x, -t)}) \quad \text{for } t \in (-\infty, 0).
\]

If \((iR_1(x, 0), iR_2(x, 0)) \in \mathbb{R}^2\) for a.e. \(x \in \Omega\), then we set
\[
(R_1(x, t), R_2(x, t)) = (-\overline{R_1(x, -t)}, -\overline{R_2(x, -t)}) \quad \text{for } t \in (-\infty, 0).
\]

In such a context, we have that \((R_1, R_2) \in H^1(-\infty, \infty; L^\infty(\Omega))^2\), and \((z_1, z_2)\) solves the system (13) in \(\Omega \times (-\infty, \infty)\).

Assume \((u_1, u_2) = (\delta z_1, \delta z_2)\). We have
\[
\begin{aligned}
i\partial_t u_1 + \Delta u_1 + a_{11}u_1 + a_{12}u_2 = f_1(x)\partial_t R_1(x, t) & \quad \text{in } \Omega \times (-\infty, \infty), \\
i\partial_t u_2 + \Delta u_2 + a_{21}u_1 + a_{22}u_2 = f_2(x)\partial_t R_2(x, t) & \quad \text{in } \Omega \times (-\infty, \infty), \\
u_1(x, 0) = -i f_1(x)R_1(x, 0), & \quad u_2(x, 0) = -i f_2(x)R_2(x, 0) \quad \text{in } \Omega, \\
u_1 = u_2 = 0 & \quad \text{on } \Gamma \times (-\infty, \infty).
\end{aligned}
\]

From (6), (13) and (14), by using the standard technique of operator semigroup theory (for example, [21, section 4.2]), we have that \((u_1, u_2) \in [C^1((-\infty, \infty); H_0^1(\Omega)) \cap C((-\infty, \infty); H^2(\Omega))]^2\).

Further, there exists a constant \(\tilde{C} = \tilde{C}(M, T) > 0\) such that
\[
\left\|\left(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\right)\right\|_{L^2(-T, T; H^2(\Omega))^2}^2 + \left\|\left(\frac{\partial \tilde{u}_1}{\partial t}, \frac{\partial \tilde{u}_2}{\partial t}\right)\right\|_{L^2(-T, T; H^2(\Omega))^2}^2 \leq \tilde{C}.
\]

For \(j = 1, 2\) and \(\tau > 0\), let \(\tilde{u}_j = e^{-\tau \tilde{\alpha}} u_j\) and
\[
\begin{aligned}
M_1^j & \triangleq i(2 \tau \nabla \tilde{\alpha} \cdot \nabla u_j + \tau \Delta \tilde{\alpha} \tilde{u}_j) + \tau \tilde{\alpha} \partial_t \tilde{u}_j, \\
M_2^j & \triangleq \partial_t \tilde{u}_j + i(\Delta \tilde{u}_j + \tau \nabla \tilde{\alpha} \cdot \nabla \tilde{u}_j).
\end{aligned}
\]

Regarding \(f_1 \partial_t R_1 - a_{12}u_2\) and \(f_2 \partial_t R_2 - a_{21}u_1\) respectively in the first and second equations in (14) as nonhomogeneous terms, applying proposition 3.1 in [20] and adding it, we know that \(\tau_0 > 0\) and \(\eta_0(\tau_0) > 0\) exist such that for all \(\tau > \tau_0\) and \(\eta > \eta_0(\tau_0)\), it holds that
\[
\begin{aligned}
f_T^\tau \Omega \sum_{j=1}^2 |\partial_t \tilde{u}_j|^2 + f_T^\tau \Omega \sum_{j=1}^2 |\partial \tilde{u}_j|^2 & \leq C \left\{ f_T^\tau \Omega e^{-2\tau \tilde{\alpha}} \tilde{\alpha} \tilde{\phi} (|u_1|^2 + |u_2|^2) dx dt + f_T^\tau \Omega \sum_{j=1}^2 |M_2^j|^2 dx dt \right. \\
& \left. + f_T^\tau \Omega e^{-2\tau \tilde{\alpha}} \left( |f_1(x)\partial_t R_1(x, t)|^2 + |f_2(x)\partial_t R_2(x, t)|^2 \right) dx dt \right\}.
\end{aligned}
\]

Put
\[
J = -\int_0^T \int_{\Omega} e^{-\tau \tilde{\alpha}} M_{12} \tilde{u}_1 dx dt - \int_0^T \int_{\Omega} e^{-\tau \tilde{\alpha}} M_{22} \tilde{u}_2 dx dt.
\]
Then

\[
\text{Re} (J) = -\text{Re} \left[ \int_0^T \int_\Omega \partial_t \bar{u}_t \bar{u}_t \, dx \, d\tau + i \int_0^T \int_\Omega \left( -|\nabla \bar{u}_t|^2 + \tau^2 |\nabla \bar{\alpha}|^2 |\bar{u}_t|^2 \right) \, dx \, d\tau \right] - \text{Re} \left[ \int_0^T \int_\Omega \partial_t \bar{u}_t \bar{u}_t \, dx \, d\tau + i \int_0^T \int_\Omega \left( -|\nabla \bar{u}_t|^2 + \tau^2 |\nabla \bar{\alpha}|^2 |\bar{u}_t|^2 \right) \, dx \, d\tau \right] = \frac{1}{2} \int_\Omega \left( |\bar{u}_1(x, 0)|^2 + |\bar{u}_2(x, 0)|^2 \right) \, dx = \frac{1}{2} \int_\Omega e^{-2\tau \bar{\alpha}(x, 0)} \left( |f_1(x)|^2 |R_1(x, 0)|^2 + |f_2(x)|^2 |R_2(x, 0)|^2 \right) \, dx .
\]

This, together with (9) and the definitions of \(R_1(x, 0)\) and \(R_2(x, 0)\), imply that

\[
\text{Re} (J) \geq \frac{r^2}{2} \int_\Omega e^{-2\tau \bar{\alpha}(x, 0)} \left( |f_1(x)|^2 + |f_2(x)|^2 \right) \, dx .
\]

On the other hand, it follows from (18) that

\[
|J| \leq \left( \int_0^T \int_\Omega e^{-2\tau \bar{\alpha}} |u_1|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left( \int_0^T \int_\Omega |M_1|^2 \, dx \, d\tau \right)^{\frac{1}{2}} + \left( \int_0^T \int_\Omega e^{-2\tau \bar{\alpha}} |u_2|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left( \int_0^T \int_\Omega |M_2|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \leq \tau \frac{1}{2} \int_0^T \int_\Omega e^{-2\tau \bar{\alpha}} \left( |u_1|^2 + |u_2|^2 \right) \, dx \, d\tau + \tau \frac{1}{2} \int_0^T \int_\Omega \left( |M_1|^2 + |M_2|^2 \right) \, dx \, d\tau .
\]

From the choice of \(\bar{\alpha}\), we find that

\[
\int_0^T \int_\Omega e^{-2\tau \bar{\alpha}} \left( |f_1(x)|^2 \partial_t R_1(x, t) \right)^2 + |f_2(x)| \partial_t R_2(x, t) \right)^2 \right) \, dx \, d\tau \leq C \int_\Omega e^{-2\tau \bar{\alpha}(x, 0)} \left( |f_1(x)|^2 + |f_2(x)|^2 \right) \, dx .
\]

This, together with (17), (19) and (20), implies that

\[
\tau \frac{1}{2} \int_0^T \int_\Omega e^{-2\tau \bar{\alpha}(x, 0)} \left( |f_1(x)|^2 + |f_2(x)|^2 \right) \, dx \, d\tau \leq C \tau \frac{1}{2} \eta \left\{ \int_0^T \int_\Omega e^{-2\tau \bar{\alpha}} \left[ \tau^2 \bar{\alpha}^3 (|u_1|^2 + |u_2|^2) + \nabla \bar{\alpha} \right] \, dx \, d\tau + \int_\Omega e^{-2\tau \bar{\alpha}(x, 0)} \left( |f_1(x)|^2 + |f_2(x)|^2 \right) \, dx \right\} .
\]

Thus, there is a \(\tau_1 > 0\) such that for all \(\tau \geq \max\{\tau_0, \tau_1\}\) and \(\eta \geq \eta_0(\tau_0)\),

\[
\int_\Omega e^{-2\tau \bar{\alpha}(x, 0)} \left( |f_1(x)|^2 + |f_2(x)|^2 \right) \, dx \, d\tau \leq C \tau \frac{1}{2} \eta \left( \int_0^T \int_\Omega e^{-2\tau \bar{\alpha}} \left[ \tau^2 \bar{\alpha}^3 (|u_1|^2 + |u_2|^2) + \nabla \bar{\alpha} \right] \, dx \, d\tau \right) .
\]

This concludes (11) and completes the proof of lemma 3.1. 

Next, in order to retain self-containment, we give a brief introduction to FBI transformation here [10, 22]. Let

\[
F(z) = \frac{1}{2\pi} \int_\mathbb{R} e^{iz\phi} e^{-z^2} \, d\phi .
\]
Then
\[ F(z) = \frac{\sqrt{\pi}}{2\pi} e^{\frac{1}{2}(|\text{Im}z|^2 - |\text{Re}z|^2)} e^{-\frac{1}{4}(|\text{Im}z| + |\text{Re}z|)}. \]

For every \( \lambda \geq 1 \), define
\[ F_\lambda(z) \triangleq \lambda F(\lambda z) = \frac{1}{2\pi} \int_R e^{i\tau} e^{-\frac{1}{2}(|\tau|^2)} d\tau. \]

Then,
\[ |F_\lambda(z)| = \frac{\sqrt{\pi}}{2\pi} \lambda e^{\frac{1}{8}(|\text{Im}z|^2 - |\text{Re}z|^2)}. \]

Let \( \Phi \in C^\infty_0(\mathbb{R}) \) satisfy the following conditions:
\[
\begin{cases}
\Phi \in C^\infty_0((0, 1) \times \left(-\frac{L}{4}, \frac{L}{4}\right)), \\
\Phi = 1 \text{ on } \left[-\frac{L}{4}, \frac{L}{4}\right], \\
|\nabla \Phi(x)| > 0, \forall x \in \tilde{\omega} \setminus \omega.
\end{cases}
\]

where \( L > 4T \) is a positive number and will be chosen later.

Take
\[ K = \left[-\frac{L}{2}, -\frac{L}{4}\right] \cup \left[\frac{L}{4}, \frac{L}{2}\right], \quad K_0 = \left[-\frac{L}{8}, \frac{L}{8}\right]. \]

Then \( l_0 \in K_0 \) in (23).

Let \( s, l_0 \in \mathbb{R} \), the FBI transformation \( F_\lambda \) for \( u \in S(\mathbb{R}^{n+1}) \) is defined as follows:
\[
(F_\lambda u)(x, s) = \int_R F_\lambda(l_0 + is - l) \Phi(l) u(x, l) dl.
\]

Now we are in a position to prove theorem 2.1.

**Proof of theorem 2.1.** Since the proof is long, we divide it into four steps.

**Step 1.** In this step, we introduce an equation on \((-T, T) \times \tilde{\omega}\).

Recall that \( \omega \) is an arbitrary fixed nonempty subset of \( \tilde{\omega} \) such that \( \overline{\omega} \subset \tilde{\omega} \). By [12, lemma 1.1], there exists a function \( \psi \in C^2(\overline{\omega}) \) such that
\[
\begin{cases}
\psi(x) > 0, \quad \forall x \in \tilde{\omega}, \\
\psi(x) = 0, \quad \forall x \in \partial \tilde{\omega}, \\
|\nabla \psi(x)| > 0, \quad \forall x \in \tilde{\omega} \setminus \omega.
\end{cases}
\]

We can conclude from (24) that there exists a constant \( \beta > 0 \) and \( \omega_1 \subset \subset \omega_2 \subset \subset \tilde{\omega} \) such that
\[ \psi(x) \leq \beta, \quad \forall x \in \tilde{\omega} \setminus \omega_2 \]
and that
\[ \psi(x) \geq 2\beta, \quad \forall x \in \omega_1. \]

It follows from the last condition in (24) that the maximum value of \( \psi \) can only be attained in \( \omega \), i.e. there exists a point \( x_0 \in \omega \) such that
\[ \psi(x_0) = \max_{x \in \tilde{\omega}} \psi(x). \]  

(27)

Let \( \chi \in C^\infty_0(\tilde{\omega}) \) be a cut-off function, which satisfies \( 0 \leq \chi \leq 1 \) and

\[
\chi(x) = \begin{cases} 
1, & \text{if } x \in \omega_2, \\
0, & \text{if } x \in \tilde{\omega} \setminus \omega_3,
\end{cases}
\]

(28)

where \( \omega_3 \) is a subset of \( \tilde{\omega} \) such that \( \omega_2 \subset \subset \omega_3 \).

Let \((w_1, w_2) = (\chi a_1, \chi a_2)\). Then by (8) and (14), we have that

\[
\begin{align*}
&\begin{cases} 
i \partial w_1 + \Delta w_1 + a_{11} w_1 + a_{12} w_2 = [\Delta, \chi] a_1 & \text{in } \tilde{\omega} \times (-\infty, \infty), \\
i \partial w_2 + \Delta w_2 + a_{21} w_1 + a_{22} w_2 = [\Delta, \chi] a_2 & \text{in } \tilde{\omega} \times (-\infty, \infty), \\
w_1(0) = w_2(0) = 0 & \text{in } \tilde{\omega}, \\
w_1 = w_2 = 0 & \text{on } \partial \tilde{\omega} \times (-\infty, \infty).
\end{cases}
\end{align*}
\]

(29)

By (15), there exists \( \tilde{C} = \tilde{C}(M, T) > 0 \) such that

\[
||w_1, w_2||^2_{L^2(-T,T; H^2(\tilde{\omega}))} + ||(\partial w_1, \partial w_2)||^2_{L^2(-T,T; H^2(\tilde{\omega}))} \leq \tilde{C}.
\]

(30)

**Step 2.** In this step, we introduce a system of parabolic equations related to (29) and a Carleman estimate to the parabolic system.

For \( j = 1, 2 \), let \( W_j(x, s) = \int_R F_{\lambda}(l_0 + is - l) \Phi(l) w_j(x, l) dl \). Since

\[
\begin{align*}
&\partial_s W_j(x, s) = \int_R -i \partial_l F_{\lambda}(l_0 + is - l) \Phi(l) w_j(x, l) dl \\
&= i \int_R F_{\lambda}(l_0 + is - l) \left( \Phi'(l) w(x, l) + \Phi(l) w_j(x, l) \right) dl,
\end{align*}
\]

we follow that

\[
\begin{align*}
&\begin{cases} 
\partial_s W_1 + \Delta W_1 + a_{11} W_1 + a_{12} W_2 = E_1 + G_1 & \text{in } \tilde{\omega} \times (-\infty, \infty), \\
\partial_s W_2 + \Delta W_2 + a_{21} W_1 + a_{22} W_2 = E_2 + G_2 & \text{in } \tilde{\omega} \times (-\infty, \infty), \\
W_1 = W_2 = 0 & \text{on } \partial \tilde{\omega} \times (-\infty, \infty),
\end{cases}
\end{align*}
\]

(31)

where for \( j = 1, 2 \),

\[
\begin{align*}
E_j(x, s) &= i \int_R F_{\lambda}(l_0 + is - l) \Phi'(l) w_j(x, l) dl, \\
G_j(x, s) &= \int_R F_{\lambda}(l_0 + is - l) \Phi(l) |\Delta, \chi| a_j dl.
\end{align*}
\]

Let

\[
\varphi(x, t) = \frac{e^{\eta \psi(x)}}{(T + t)(T - t)}, \quad \alpha(x, t) = \frac{e^{\eta \psi(x)} - e^{\eta \psi(x) - \eta|\psi|_{L^\infty(\tilde{\omega})}}}{(T + t)(T - t)}, \quad \forall (x, t) \in \tilde{\omega} \times (-T, T),
\]

(32)

where \( \eta > 0 \).

Following the proof for theorem 1.1 in [14] step-by-step, we can show that there exists a positive function \( a_0 \in C^2(\tilde{\omega}) \) (only depending on \( \tilde{\omega} \) and \( \omega \)), two positive constants \( C_0 \) (only depending on \( \tilde{\omega}, \omega, a_0 \) and \( M_0 \)) and \( \sigma_0 = \sigma_0(\tilde{\omega}, \omega, M_0) \) such that the solution \((W_1, W_2) \in C([-T, T]; L^2(\tilde{\omega})) \cap L^2([-T, T]; H^1(\tilde{\omega}))^2 \) of (31) satisfies that

\[
\begin{align*}
&\int_T T \int_{\tilde{\omega}} (\sigma^3 \gamma(s)^4 |\nabla W_1|^2 + \sigma \gamma(s) |\nabla W_2|^2 + \sigma^6 \gamma(s)^6 |W_1|^2 + \sigma^3 \gamma(s)^3 |W_2|^2) e^{2\sigma \alpha} dx ds \\
&\leq C_0 \left[ \int_T T \int_{\tilde{\omega}} (\sigma^3 \gamma(s)^3 E_1(x, s) + G_1(x, s))^2 + |E_2(x, s) + G_2(x, s)|^2 \right] e^{2\sigma \alpha} dx ds + \sigma^3 \int_T T \int_{\tilde{\omega}} e^{2\alpha \gamma(s)^3 |W_1|^2 dx ds},
\end{align*}
\]

(33)

where \( \gamma(s) = \frac{1}{(T + s)(T - s)} \) and \( \sigma \geq \sigma_0 \).
Step 3. In this step, we estimate all the terms in the right-hand side of (33).

Let

\[ \mu_2 = \frac{e^{2\gamma_\alpha(s_0)} - e^{\gamma_\alpha(s_0)}}{T^2}. \]  

(34)

There exists \( \delta_2 > 0 \) such that

\[ \max_{x \in \omega \cap [-T, T]} \gamma(s)^2 e^{2\sigma_\alpha} \leq e^{-(2-\delta_2)\sigma_\mu_2}. \]

By the property of FBI transformation, we have that

\[
\int_{-T}^{T} \int_{\omega} \gamma(s)^2 |W_1|^2 e^{2\sigma_\alpha} \, dx \, ds \\
\leq \max_{x \in \omega \cap [-T, T]} (\gamma(s)^2 e^{2\sigma_\alpha}) \int_{-T}^{T} \int_{\omega} |W_1(x, s)|^2 \, dx \, ds \\
\leq e^{-(2-\delta_2)\sigma_\mu_2} \int_{-T}^{T} \int_{\omega} \left|\int_{\mathbb{R}} F_x(l_0 + is - l) \Phi(l) w_1(x, l) \, dl\right|^2 \, dx \, ds \\
\leq e^{-(2-\delta_2)\sigma_\mu_2} \int_{-T}^{T} \int_{\omega} \left|\int_{\mathbb{R}} \frac{\sqrt{\pi}}{2\pi} \lambda e^{\frac{\pi}{4} (r^2 - |l_0 - l|^2)} \Phi(l) w_1(x, l) \, dl\right|^2 \, dx \\
\leq \frac{\lambda^2}{\pi} e^{-(2-\delta_2)\sigma_\mu_2} \int_{-T}^{T} e^{\frac{\pi^2}{2T} x^2} dx \sup \Phi l^2 \int_{\omega} \left|\int_{\mathbb{R}} \frac{\sqrt{\pi}}{2\pi} \lambda e^{\frac{\pi}{4} (r^2 - |l_0 - l|^2)} \Phi(l) w_1(x, l) \, dl\right|^2 \, dx \\
\leq \int_{-T}^{T} \int_{\omega} \left|E_j(x, l)\right|^2 \, dx \, dl. 
\]

(35)

From the definition of \( E_j \), we see that

\[
\int_{-T}^{T} \int_{\omega} |E_j(x, s)\|^2 \, dx \, ds \\
= \int_{-T}^{T} \int_{\omega} \left|\int_{\mathbb{R}} F_x(l_0 + is - l) \Phi'(l) w_j(x, l) \, dl\right|^2 \, dx \, ds \\
\leq \int_{-T}^{T} \int_{\omega} \left|\int_{\mathbb{R}} \frac{\sqrt{\pi}}{2\pi} \lambda e^{\frac{\pi}{4} (r^2 - |l_0 - l|^2)} \Phi'(l) w_j(x, l) \, dl\right|^2 \, dx \\
\leq \frac{1}{2\pi} \lambda^2 e^{\frac{\pi^2}{4T} x^2} T \max_k |\Phi'(l)|^2 \int_{\omega} \left|\int_{\mathbb{R}} e^{\frac{\pi}{4} (r^2 - |l_0 - l|^2)} w_j(x, l) \, dl\right|^2 \, dx \\
\leq \frac{1}{2\pi} \lambda^2 e^{\frac{\pi^2}{4T} x^2} T e^{-\frac{\pi}{4} (x^2)} \max_k |\Phi'(l)|^2 \frac{L}{2} \int_{\omega} \int_{K} |w_j(x, l)|^2 \, dl \, dx \\
\leq \frac{1}{2\pi} \lambda^2 e^{\frac{\pi^2}{4T} x^2} T e^{-\frac{\pi}{4} (x^2)} \left(\frac{2}{L}\right)^2 \frac{L}{2} \int_{\omega} \int_{K} |w_j(x, l)|^2 \, dl \, dx \\
\leq \frac{\lambda^2}{\pi L} e^{\frac{\pi^2}{4T} x^2} \frac{L}{2} \int_{\omega} \int_{K} |w_j(x, l)|^2 \, dl \, dx. 
\]

(36)

Since \( \text{supp} \chi' \subset \omega \setminus \omega_1 \) and \( G_j(x, s) = 0 \) in \( \omega_1 \), it holds that
Set
\[ \mu_1 = \frac{e^{2\eta|\varphi|_\infty} - e^{\eta\beta}}{T^2}. \]  

By (25), we know that there exists \( \delta_1 > 0 \) such that
\[ \max_{x \in \mathcal{D}, \sigma \in [-T,T]} \gamma(s)^{3} \sigma^{3\alpha} \leq e^{-(2-\delta_1)\sigma\mu_2}, \quad \max_{x \in \mathcal{D}, \sigma \in [-T,T]} \gamma(s)^{3} \sigma^{3\alpha} \leq e^{-(2-\delta_1)\sigma\mu_1}. \]

Consequently,
\[ \int_{-T}^{T} \int_{\Omega} e^{2\sigma\alpha} \gamma(t)^{3} |E_{1}(x,s)|^{2} dx ds \leq e^{-(2-\delta_1)\sigma\mu_2} \frac{\lambda^2 e^{\frac{\sigma}{T} \left( (\frac{T}{2})^2 - \frac{1}{4} \right)}}{2\pi} \int_{\Omega} \int_{\mathcal{K}} |w_1(x,l)|^2 dldx \]  

and
\[ \int_{-T}^{T} \int_{\Omega} e^{2\sigma\alpha} \gamma(t)^{3} (G_{1}(x,s))^2 dx ds \leq e^{-(2-\delta_1)\sigma\mu_1} \frac{\lambda^2 L e^{\frac{\sigma}{T} \left( (\frac{T}{2})^2 - \frac{1}{4} \right)}}{2\pi} \int_{\Omega} \int_{\mathcal{K}} \max(|\nabla \chi_1|^2, |\Delta \chi_1|^2) \int_{\frac{1}{2}}^{\frac{2}{3}} \int_{\Omega} \left\{ |u_1(x,l)|^2 + |\nabla u_1(x,l)|^2 \right\} dldx. \]

Substituting (35), (39) and (40) into (33), we obtain that
\[ \int_{-T}^{T} \int_{\Omega} \left[ \sigma^4 \gamma(s)^{4} |\nabla W_1|^2 + \sigma^4 \gamma(s)^{4} |\nabla W_2|^2 + \sigma^6 \gamma(s)^{6} |W_1|^2 + \sigma^3 \gamma(s)^{3} |W_2|^2 \right] e^{2\sigma\alpha} dx ds \leq C_0 \left[ e^{-(2-\delta_1)\sigma\mu_2} \frac{2\lambda^2 T e^{\frac{\sigma}{T} \left( (\frac{T}{2})^2 - \frac{1}{4} \right)}}{\pi L} \int_{\Omega} \int_{\mathcal{K}} \left[ \sigma^3 (|w_1(x,l)|^2 + |\nabla w_1(x,l)|^2) \right] dldx + e^{-(2-\delta_1)\sigma\mu_1} \frac{\lambda^2 L e^{\frac{\sigma}{T} \left( (\frac{T}{2})^2 - \frac{1}{4} \right)}}{\pi} \int_{\Omega} \int_{\mathcal{K}} \max(|\nabla \chi_1|^2, |\Delta \chi_1|^2) \int_{\frac{1}{2}}^{\frac{2}{3}} \int_{\Omega} \left\{ |u_1(x,l)|^2 + |\nabla u_1(x,l)|^2 \right\} dldx \right] \]

In order to reduce the computation complexity of the proof, and without loss of generality, in the following steps, we assume \( T = 1 \). Let \( A > 1 \). By choosing \( L = 8AT = 8A \), we have
\( e^{-2\eta_1} \sigma^4 \int_{1-\epsilon}^{1} \int_{\Omega_1} (|\nabla W_1|^2 + |W_1|^2) \, dx \, dt \leq \int_{1-\epsilon}^{1} \int_{\Omega} [\sigma^2 \gamma(s) |\nabla W_2|^2 + \sigma^3 \gamma(s) |W_2|^2] e^{-2\eta_2} \, dx \, dt \)

\( \leq C_0 \left[ e^{-(2-\delta_1)\sigma \mu_2} \frac{2\lambda^2}{8\pi \sigma} e^{\frac{\lambda^2}{\pi} (1-\epsilon^2)} \int_{\Omega} |\nabla W_2|^2 + |W_2|^2 \right] \, dx \, dt \)

\( + e^{-(2-\delta_3)\sigma \mu_1} \frac{8\lambda^2 \mu_2}{\pi} \max\{(|\nabla \chi|^2, |\Delta \chi|^2\} \int_{-4\lambda}^{4\lambda} \int_{\omega} [\sigma^3 (|u_1| + |\nabla u_1|) + |\nabla W_2|^2] \, dx \, dt \)

\( + (|u_2(x, l)|^2 + |\nabla u_2(x, l)|^2 \right] \, dx \, dt \right\}.

(42)

where

\( \mu_3 = \frac{e^{2\eta_1} ||\psi||_{\infty} - e^{2\eta_3}}{\epsilon(2-\epsilon)}. \)  

(43)

Similarly, we can get that

\( e^{-2\eta_1} \sigma^4 \int_{1-\epsilon}^{1} \int_{\Omega_1} (|\nabla W_1|^2 + |W_1|^2) \, dx \, dt \leq \int_{1-\epsilon}^{1} \int_{\Omega} [\sigma^2 \gamma(s) |\nabla W_2|^2 + \sigma^3 \gamma(s) |W_2|^2] e^{-2\eta_2} \, dx \, dt \)

\( \leq C_0 \left[ e^{-(2-\delta_1)\sigma \mu_2} \frac{2\lambda^2}{8\pi \sigma} e^{\frac{\lambda^2}{\pi} (1-\epsilon^2)} \int_{\Omega} |\nabla W_2|^2 + |W_2|^2 \right] \, dx \, dt \)

\( + e^{-(2-\delta_3)\sigma \mu_1} \frac{8\lambda^2 \mu_2}{\pi} \max\{(|\nabla \chi|^2, |\Delta \chi|^2\} \int_{-4\lambda}^{4\lambda} \int_{\omega} [\sigma^3 (|u_1| + |\nabla u_1|) + |\nabla W_2|^2] \, dx \, dt \)

\( + (|u_2(x, l)|^2 + |\nabla u_2(x, l)|^2 \right] \, dx \, dt \right\}.

(44)

Fix \( \epsilon \in (0, 1) \) such that

\( \tau = \epsilon(2-\epsilon) \frac{2 - \delta_1}{2} \frac{e^{2\eta_1} ||\psi||_{\infty} - e^{2\eta_3}}{\epsilon(2-\epsilon)} - 1 > 0. \)

This is equivalent to saying that

\( (2 - \delta_1)\mu_1 - 2\mu_3 = 2\tau\mu_3 > 0. \)

Hence,
\[
\int_{-1+\epsilon}^{1-\epsilon} \int_{\Omega} (|\nabla W_1|^2 + |W_1|^2) \, dx \, dt + \frac{1}{\sigma^3} \int_{-1+\epsilon}^{1-\epsilon} \int_{\Omega} (|\nabla W_2|^2 + |W_2|^2) \, dx \, dt \leq C_0 \frac{1}{\sigma^2} \left[ \frac{\lambda^3 T}{4\pi A} \frac{e^{\frac{2}{3}(1-\lambda^2)}}{f(\lambda)} \left(1 - \sigma \right) \int_K \int_{\L} \left( |u_1|_{L^2(\Omega \setminus \{0\})} \right)^2 \, dx \right]
\]

\[
\begin{align*}
&+ \sigma \frac{8\lambda^2 A}{\pi} e^{-2\sigma \mu \epsilon} \frac{e^{\frac{2}{3}(1-\lambda^2)}}{f(\lambda)} \max \{ |\nabla \chi|^2, |\Delta \chi|^2 \} \left( |u_1|_{L^2(\Omega \setminus \{0\})} \right)^2 \\
&+ |u_2|_{L^2(\Omega \setminus \{0\})}^2 \max \{ |\nabla \chi|^2, |\Delta \chi|^2 \} \left( |u_1|_{L^2(\Omega \setminus \{0\})} \right)^2 \\
&+ \sigma \frac{4\lambda^2 A}{\pi} e^{-2\sigma \mu \epsilon} e^{\frac{2}{3}(1-\lambda^2)} |\Delta \chi|^2 \left( |u_1|_{L^2(\Omega \setminus \{0\})} \right)^2
\end{align*}
\]

Step 4. By lemma 3.1,

\[
\sum_{j=1}^{3} |a_j - \tilde{a}_j|_{L^2(\Omega)} \leq C \left( |(y_1(a_{11}, a_{22}) - \tilde{y}_1(a_{11}, a_{22}), y_2(a_{11}, a_{22}) - \tilde{y}_2(a_{11}, a_{22}))|_{\mu_t(0,T,H^p(\omega))} \right)^2
\]

\[
\leq C \left( \left( |\nabla w_1, \nabla w_2|_{L^2(\Omega \setminus \{0\})} \right)^2 \right) \leq \left( |(\nabla (\Phi w_1), \nabla (\Phi w_2))|_{L^2(\Omega \setminus \{0\})} \right)^2 \\
\leq C \left( |(\Phi w_1, \Phi w_2)|_{L^2(\Omega \setminus \{0\})} \right)^2
\]

At the last inequality, we used \( \frac{T}{\lambda} > T \) and \( \Phi = 1 \) on \( \left[ -\frac{T}{\lambda}, \frac{T}{\lambda} \right] \). It follows from Young’s inequality and Parseval’s identity that

\[
\left( |\Phi w_1|^2_{L^2(\Omega \setminus \{0\})} \right)^2 = \frac{1}{T} \int_{-\frac{T}{\lambda}}^{\frac{T}{\lambda}} \int_{\Omega \setminus \{0\}} |\Phi(t)w_i(x,t)|^2 \, dx \, dt
\]

\[
= \int_{\mathbb{R}} \int_{\Omega \setminus \{0\}} |\Phi(t)w_i(x,t)|^2 \, dx \, dt = \left( \frac{1}{T} \right) \int_{\mathbb{R}} \int_{\Omega \setminus \{0\}} |\Phi(t)w_i(x,t)|^2 \, dx \, dt + \frac{1}{T} \int_{\mathbb{R}} \int_{\Omega \setminus \{0\}} |F_x (\Phi(t)w_i(x,t))|^2 \, dx \, dt
\]

\[
\leq \frac{1}{T} \int_{\mathbb{R}} \int_{\Omega \setminus \{0\}} |(1-F_\lambda) (\Phi(t)w_i(x,t))|^2 \, dx \, dt + \frac{1}{T} \int_{\mathbb{R}} \int_{\Omega \setminus \{0\}} |F_x (\Phi(t)w_i(x,t))|^2 \, dx \, dt
\]

\[
\leq \frac{1}{T} \int_{\mathbb{R}} \int_{\Omega \setminus \{0\}} |(1-F_\lambda) (\Phi(t)w_i(x,t))|^2 \, dx \, dt + 2 \int_{\mathbb{R}} \int_{\Omega \setminus \{0\}} |F_x (\Phi(t)w_i(x,t))|^2 \, dx \, dt
\]

Here, \( \Phi(t)w_i(x,t) \) is the Fourier transform of the product of \( \Phi \) and \( w_i \). In the last line of the above formula, we have used the fact that \( F_x (\Phi(t)w_i(x,t)) = \int_{\mathbb{R}} e^{-ixz} F_x (\phi(z)) \) for
Fourier transform of $F_{\lambda}$. The coefficient before the second term of the last line in the above formula is $2\pi \times \frac{1}{2} = 2$ based on the Parseval identity.

The first term on the right-hand side of (47) reads

$$\frac{1}{\pi} \int \int_{\omega_1} |(1 - F_{\lambda}) \Phi(l_0) w_j(x, l_0)(t)|^2 \, dx \, dt$$

$$= \frac{1}{\pi} \int \int_{\omega_1} (1 - e^{-\left(\frac{t}{4}\right)^2}) |\Phi(l_0) w_j(x, l_0)(t)|^2 \, dx \, dt$$

$$\leq \frac{2}{\pi \lambda^2} \int \int_{\omega_1} |\Phi(l_0) w_j(x, l_0)(t)|^2 \, dx \, dt$$

$$\leq \frac{4}{\lambda^2} \int \int_{\omega_1} |\Phi'(l_0) w_j(x, l_0) + \Phi(l_0) \partial_\rho w_j(x, l_0)|^2 \, dx \, dl_0$$

$$\leq \frac{8}{\lambda^2} \int \int_{\omega_1} \left( |\Phi'(l_0) w_j(x, l_0)|^2 + |\Phi(l_0) \partial_\rho w_j(x, l_0)|^2 \right) \, dx \, dl_0$$

$$\leq \frac{8}{\lambda^2} \left( \frac{1}{4AT} \right)^2 \int \int_{\omega_1} |w_j(x, l_0)|^2 \, dx \, dl_0 + \int_0^L \int_{\omega_1} |\partial_\rho w_j(x, l_0)|^2 \, dx \, dl_0 \right]$$

$$\leq \frac{8}{\lambda^2} \left( \frac{1}{4AT} \right)^2 \int \int_{\omega_1} |w_j(x, l_0)|^2 \, dx \, dl_0 + \int_0^{8AT} \int_{\omega_1} |\partial_\rho w_j(x, l_0)|^2 \, dx \, dl_0 \right].$$

Let

$$W_{j, \lambda}(x, l_0) \triangleq W_j(x, 0) = \int_\mathbb{R} F_{\lambda}(l_0 - l) \Phi(l) w_j(x, l) \, dl = F_{\lambda} * \Phi(\cdot) w_j(x, \cdot)(l_0).$$

(49)

For $z = l_0 + is$, we set $z = \kappa + \rho e^{i\eta}$ with $\rho \in (0, T - \epsilon)$. By applying the Cauchy integral formula and changing the polar coordinate, we have

$$W_{j, \lambda}(x, \kappa) = \frac{1}{2\pi i} \int_{|z| = \rho} \frac{W_{j, \lambda}(z_0)}{z_0 - \kappa} \, dz = \frac{1}{2\pi i} \int_0^{2\pi} W_{j, \lambda}(x, \kappa + \rho e^{i\eta}) \, d\eta \rho$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{(T - \epsilon)^2} \int_0^{T - \epsilon} \sqrt{(T - \epsilon - \frac{s}{2})^2 + \frac{\rho^2}{4}} \, W_{j, \lambda}(x, l_0 + is) \, d\det J(l_0, s) \, ds \, dl_0$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{(T - \epsilon)^2} \int_T^{T + \epsilon} \sqrt{(T - \epsilon - \frac{s}{2})^2 + \frac{\rho^2}{4}} \, W_j(x, s) \, d\det J(l_0, s) \, ds \, dl_0.$$

(50)

where $\det J(l_0, s)$ is the Jacobian determinant.

Thus,

$$|W_{j, \lambda}(x, \kappa)|^2 = \frac{1}{\pi^2} \int_{T - \epsilon}^{T + \epsilon} \int_{T - \epsilon}^{T + \epsilon} |W_j(x, s)|^2 \, dx \, ds.$$  

(51)

Integrating (51) with respect to $x$ over $\omega_1$ and with respect to $\kappa$ over $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$, we get that

$$\int_{-4AT}^{4AT} \int_{\omega_1} |W_{j, \lambda}(x, \kappa)|^2 \, dx \, d\kappa$$

$$\leq \frac{1}{\pi (1 - \epsilon)^2} \int_{-4AT}^{4AT} \int_{-T + \epsilon}^{T - \epsilon} \left( \int_{T - \epsilon}^{T + \epsilon} \int_{\omega_1} |W_j(x, s)|^2 \, dx \, ds \right) \, dl_0 \, d\kappa$$

$$\leq \frac{16AT(T - \epsilon)}{\pi} \int_{-T + \epsilon}^{T - \epsilon} \int_{\omega_1} |W_j(x, s)|^2 \, dx \, ds.$$  

(52)

Substituting (48) and (52) into (47) and noting that $T = 1$, we find that
\[
\|\Phi w_j\|_{L^2(\omega_t \times (-4A, 4A))}^2 \\
\leq \frac{8}{\lambda^2} \left[ \frac{1}{16A^2} \int_{K_0} \int_{\omega_t} \left( |w_j(x, t_0)|^2 + |\nabla w_j(x, t_0)|^2 \right) dx dt_0 \right] \\
+ \frac{128}{\lambda^2} \int_{-1+\epsilon}^1 \int_{\omega_t} \left| W_j(x, s) \right|^2 dxds.
\]  

Similarly, we can obtain that
\[
\|\nabla (\Phi w_j)\|_{L^2(\omega_t \times (-4A, 4A))}^2 \\
\leq \frac{8}{\lambda^2} \left[ \frac{1}{16A^2} \int_{K_0} \int_{\omega_t} \left( |\nabla w_j(x, t_0)|^2 + |\nabla \nabla w_j(x, t_0)|^2 \right) dx dt_0 \right] \\
+ \frac{128}{\lambda^2} \int_{-1+\epsilon}^1 \int_{\omega_t} \left| \nabla W_j(x, s) \right|^2 dxds.
\]

Let \( \sigma = \frac{\lambda^2}{2\pi^2 \mu}, \) and \( C_2 = \max\{||\nabla \chi|^2, |\Delta \chi|^2\} \) such that
\[
e^{\sigma(2\mu_s - (2-\delta_2)\mu_2)} e^{-\frac{A\lambda^2}{2}} \leq 1.
\]

From (45), (47), (53) and (54), we have
\[
\sum_{j=1}^{2} \left| a_{ij} - \tilde{a}_{ij} \right|^2 \leq C \left\{ \frac{16}{\lambda^2} \left[ \frac{1}{16A^2} \int_{K_0} \int_{\omega_t} \left( |w_j(x, t_0)|^2 + |\nabla w_j(x, t_0)|^2 \right) dx dt_0 \right] \\
+ \frac{2}{\pi^2} \left( 1 + \frac{1}{\epsilon} \right)^2 C_0 \frac{256A^2 \lambda^2}{\pi \sigma} \left( \frac{1}{32A^2} e^{\sigma(2\mu_s - (2-\delta_2)\mu_2)} e^{\frac{1}{16A^2}} \lambda^2 \\
+ e^{-2\sigma \mu_s} e^{\frac{2A}{\pi^2}} \max\{||\nabla \chi|^2, |\Delta \chi|^2\} \hat{C}(M, 4A) \\
+ \frac{\sigma^2}{2} e^{\sigma(2\mu_s - (2-\delta_2)\mu_2)} e^{\frac{2A}{\pi^2}} \left| w_1 \right|^2_{L^2(-4A, 4A; H^1(\omega))} \right) \right\}
\]
\[
\leq C \left\{ \frac{16}{\lambda^2} \left[ \frac{512A^2 \lambda^2}{\pi^4 \sigma} e^{-2\sigma \mu_s} \left( \frac{1}{32A^2} e^{-\frac{2A}{\pi^2}} \lambda^2 + C_2 e^{\frac{2A}{\pi^2}} \right) \right] \hat{C}(M, 4A) \\
+ C_0 \frac{256A^2 \lambda^2 \sigma^3}{\pi^3} e^{\sigma(2\mu_s - (2-\delta_2)\mu_2)} e^{\frac{2A}{\pi^2}} \left| w_1 \right|^2_{L^2(-4A, 4A; H^1(\omega))} \right\}
\]
\[
\leq C \left\{ \frac{16}{\lambda^2} \left[ \frac{1024A^2 \mu_s \lambda^2}{\pi^4 \mu_2} e^{\frac{2A}{\pi^2}} \lambda^2 + C_2 e^{-\frac{A}{\pi^2}} \right] \hat{C}(M, 4A) \\
+ C_0 \frac{32A^2 \mu_s \lambda^2}{\pi^3 \lambda^2} e^{\frac{2A}{\pi^2}} \lambda^2 \left| w_1 \right|^2_{L^2(-4A, 4A; H^1(\omega))} \right\}.
\]

Let \( \lambda \geq \lambda_0 \) satisfy
\[
\sum_{j=1}^{2} \left| a_{ij} - \tilde{a}_{ij} \right|^2 \leq C_3 \hat{C}(M, 4A) + e^{C_4 \lambda^2} \left| w_1 \right|^2_{L^2(-T, T; H^1(\omega))}.
\]

where the two constants \( C_3 \) and \( C_4 \) are independent of \( \lambda \). We take
\[ \lambda = \max \left\{ \lambda_0, \left( \frac{\ln \|w_1\|_{L^2(-T,T,H'((\omega)))}}{C_4} \right)^{\frac{1}{2}} \right\} . \]

If \( \|w_1\|_{L^2(0,T,H'((\omega)))} \) is small enough, then

\[ \sum_{j=1}^2 \|a_j - \hat{a}_j\|_{L^2(\Omega)}^2 \leq \frac{C_4}{\min \{\ln \|w_1\|_{L^2(-T,T,H'((\omega)))} \}} \tilde{C}(M,4A) + \|w_1\|_{L^2(0,T,H'((\omega)))} \leq C \left( \ln \|w_1\|_{L^2(-T,T,H'((\omega)))} \right)^{-1} \tilde{C}(M,4A) + \|w_1\|_{L^2(0,T,H'((\omega)))} \]  \hspace{1cm} (57)

Otherwise, there exists a constant \( m > 0 \) such that \( \|w_1\|_{L^2(-T,T,H'((\omega)))} \geq m \). Thus, by (30) we have

\[ \sum_{j=1}^2 \|a_j - \hat{a}_j\|_{L^2(\Omega)}^2 \leq \tilde{C}(M,4A) = \frac{\tilde{C}(M,4A)}{m} m \] \hspace{1cm} (58)

\[ \square \]

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