An analytic KAM-Theorem

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Abstract

We prove an analytic KAM-theorem, which is used in [1], where the differential part of KAM-theory is discussed. Related theorems on analytic KAM-theory exist in the literature (e. g., among many others, [7], [8], [13]). The aim of the theorem presented here is to provide exactly the estimates needed in [1].

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1 Formulation of the main theorem

We consider Hamiltonian systems of the form

\[ \dot{x} = H_y, \quad \dot{y} = -H_x. \tag{1.1} \]

Here \( x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n) \), \( \dot{x} \) and \( \dot{y} \) are vectors in \( \mathbb{R}^n \) \((n \geq 2)\) and \( H = H(x, y) \) is a function from \( \mathbb{R}^{2n} \) to \( \mathbb{R} \). We try to prove the existence of solutions of a system (1.1) under the assumption, that it can be written as a sum \( H = N + \tilde{R} \) with a function

\[ N(x, y) = a + \langle \omega, y \rangle + \frac{1}{2} \langle y Q(x), y \rangle + \mathcal{O}(|y|^3), \]

\((a \in \mathbb{R}, \omega \in \mathbb{R}^n, Q(x) \in \mathbb{R}^{n \times n}, \langle x, y \rangle := x_1 y_1 + \ldots + x_n y_n)\)

which we call normal form, and a remainder \( \tilde{R} \). The dynamics of \( N \) read

\[ \dot{x} = N_y = \omega + \mathcal{O}(|y|), \quad \dot{y} = -N_x = \mathcal{O}(|y|), \]

and are solved by

\[ t \mapsto (\omega t + \text{const.}, 0). \]

In case the frequencies \( \omega_1, \ldots, \omega_n \) are rationally independent, such a solution is called quasi-periodic and it covers the Torus \( \mathbb{R}^n/(2\pi \mathbb{Z}^n) \times \{0\} \) densely. KAM-Theory provides the means to prove, that many quasiperiodic solutions survive the perturbation of the Hamiltonian. In our notation, the perturbed Hamiltonian is given by

\[ H(x, y) = a + \langle \omega, y \rangle + \frac{1}{2} \langle y Q(x), y \rangle + R(x, y), \]

where \( R \) denotes the sum of the terms of higher order of \( N \) and the remainder \( \tilde{R} \). We prove the existence of quasiperiodic solutions of (1.1) for Hamiltonians of this kind.

Notations and Definitions

For vectors \( z = (z_1, \ldots, z_\ell) \in \mathbb{C}^\ell \) we use the \( \ell_\infty \)-norm \( |z| := \max_{1 \leq i \leq \ell} |z_i| \). For matrices \( Q = (q_{ij}) \in \mathbb{C}^{k \times \ell} \) we use the row-sum norm

\[ |Q| := \max_{1 \leq i \leq k} \sum_{j=1}^\ell |q_{ij}|. \]

For arbitrary matrices \( Q \in \mathbb{C}^{k \times \ell} \) and \( P \in \mathbb{C}^{\ell \times m} \) the inequality \( |Q P| \leq |Q| |P| \) holds. Transposed vectors and matrices are denoted with a superscript “\( T \)”. For transposed matrices we have the estimate \( |Q^T| \leq k |Q| \), in which \( Q \) has \( k \) rows. The product of two vectors \( x, y \in \mathbb{C}^\ell \) is defined by

\[ \langle x, y \rangle := \sum_{j=1}^\ell x_j y_j. \]
Then we have $\| (x, y) \| \leq \ell |x||y|$. For the product of a vector $x \in \mathbb{C}^\ell$ and a matrix $Q \in \mathbb{C}^{k \times \ell}$ the estimate $\| xQ^T \| \leq |x||Q|$ holds. Finally we have

$$|Q| = \max_{|z| \leq 1} |zQ^T|.$$  

(1.2)

**Domains and functions.**

**Definition 1.1.** Let $r$ and $s$ be positive numbers. We define

$$\mathcal{D}(r, s) := \{ z = (x, y) \in \mathbb{C}^{2n} | |\text{Im} \ x| < r, |y| < s \},$$

$$\mathcal{S}(r) := \{ x \in \mathbb{C}^n | |\text{Im} \ x| < r \},$$

$$\mathcal{S}'(r) := \{ z \in \mathbb{C}^{2n} | |\text{Im} z| < r \}.$$

Let $\mathcal{P}_m(r, s)$ be the set of all functions $f : \mathcal{D}(r, s) \longrightarrow \mathbb{C}^m$, $z = (x, y) \mapsto f(z)$, which are analytic, map real vectors to real values, and have period $2\pi$ in the variables $x_1, \ldots, x_n$.

The set of all functions $f : \mathcal{S}(r) \rightarrow \mathbb{C}^m$, which are analytic, map real vectors to real values and have period $2\pi$ in every variable, is denoted by $\mathcal{P}_m(r)$.

The set of all functions $f : \mathcal{S}'(r) \rightarrow \mathbb{C}^m$, which are analytic, map real vectors to real values and have period $2\pi$ in every variable, is denoted by $\mathcal{P}_m'(r)$.

The definition shall hold for $m = n \times n$ as well. In case $m = 1$ we write $\mathcal{P}(r, s) := \mathcal{P}_1(r, s)$, $\mathcal{P}(r) := \mathcal{P}_1(r)$, and $\mathcal{P}'(r) := \mathcal{P}_1'(r)$.

We denote the restriction of a function $f$ to a subset $\mathcal{M}$ of its domain with $f|_{\mathcal{M}}$.

**Notation of derivatives.** Derivatives are denoted with a subscript, for example

$$f_{x_1} = \frac{\partial f}{\partial x_1}, \quad f_x = (f_{x_1}, f_{x_2}, \ldots, f_{x_n}).$$

Hence, for a function $f = (f_1, \ldots, f_m) \in \mathcal{P}_m(r)$, $f_x$ is the Jacobian. Finally we write for functions $t \mapsto (x_1(t), \ldots, x_n(t))$ depending on a single variable only

$$\frac{dx}{dt} = \dot{x} = (\dot{x}_1, \ldots, \dot{x}_n) = (x_{1t}, \ldots, x_{nt}).$$

By our definition of the Jacobian we have $\dot{x} = x_t^T$.

**Frequency vectors.** The vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$, which comes into play as the first derivative of the Hamiltonian, is called frequency vector. To prove theorem 1.6 one has to assume that it satisfies a sequence of Diophantine inequalities.

That means, it has to be an element of a set of the following type:

**Definition 1.2.** For $n \geq 2$, $\tau > 0$, and $\gamma > 0$ let

$$\Omega(\gamma, \tau) := \left\{ \omega \in \mathbb{R}^n \bigg| |\langle \omega, k \rangle| \geq \frac{\gamma}{|k|^\tau} \quad \forall \quad k \in \mathbb{Z}^n \setminus \{0\} \right\}.$$
Remark 1.3. The following assertions hold (see [10] and the literature given there):

1. In case \(0 < \tau < n - 1\), all sets \(\Omega(\gamma, \tau), \gamma > 0\), are empty.

2. In case \(\tau = n - 1\), the \(n\)-dimensional Lebesgue measure of the set \(\Omega(n - 1) := \bigcup_{\gamma > 0} \Omega(\gamma, n - 1)\) is 0. However, the intersection of every open subset of \(\mathbb{R}^n\) with \(\Omega(n - 1)\) has the cardinality of \(\mathbb{R}\).

3. In case \(\tau > n - 1\), there exists a \(\gamma = \gamma(\omega) > 0\) with \(\omega \in \Omega(\gamma, \tau)\) for almost every \(\omega \in \mathbb{R}^n\).

Simple canonical transformations

Definition 1.4. Let \(U\) and \(V \subseteq \mathbb{C}^n\) be open connected sets. Let \(J\) be the matrix

\[
J = \begin{pmatrix}
0 & E_n \\
-E_n & 0
\end{pmatrix} \in \mathbb{C}^{2n \times 2n}, \quad E_n \text{ the } (n \times n) \text{ identity matrix.}
\]

We call a differentiable map

\[Z : U \times V \rightarrow \mathbb{C}^n, \quad \zeta = (\xi, \eta) \mapsto z = Z(\zeta)\]

symplectic transformation, if for all \(\zeta\) in \(U \times V\) the equation

\[Z_\zeta(\zeta)^T \cdot J \cdot Z_\zeta(\zeta) = J \quad (1.3)\]

holds.

Definition 1.5. Let \(U, V \subseteq \mathbb{C}^n\) be open connected sets. We call an analytic symplectic transformation

\[Z : U \times V \rightarrow \mathbb{C}^n, \quad \zeta = (\xi, \eta) \mapsto (x, y) = Z(\zeta) = (X(\zeta), Y(\zeta))\]

simple canonical transformation, if the map \(\zeta = (\xi, \eta) \mapsto X(\zeta)\) does not depend on \(\eta\), which means \(X = X(\xi)\).

Whenever the composition of two simple canonical transformations \(Z_1\) and \(Z_2\) is possible, \(Z_1 \circ Z_2\) is a simple canonical transformation as well. If \(Z_1\) and \(Z_2\) have the property, that \((\xi, \eta) \mapsto Z_i(\xi, \eta) - (\xi, 0)\) has the period \(2\pi\) in \(\xi_1, \ldots, \xi_n\) \((i = 1, 2)\), so has \((\xi, \eta) \mapsto Z_2 \circ Z_1(\xi, \eta) - (\xi, 0)\).

Theorem 1.6. Analytic KAM-theorem. Let \(\tau \geq n - 1 \geq 1, \gamma > 0, \text{ and } 0 < s \leq r^{\tau+1} \leq 1\). We consider the Hamiltonian \(H \in \mathcal{P}(r, s)\),

\[H(x, y) = a + \langle \omega, y \rangle + \frac{1}{2} \langle y \cdot Q(x), y \rangle + R(x, y), \quad (1.4)\]

where \(a \in \mathbb{R}, \omega \in \Omega(\gamma, \tau), Q \in \mathcal{P}_{n \times n}(r), \text{ and } R \in \mathcal{P}(r, s)\). Let \(C \in \mathbb{R}^{n \times n}\) be a non-singular matrix with

\[|Q - C|_{S(r)} \leq \frac{1}{4|C^{-1}|}. \quad (1.5)\]
Then there exist positive constants $c_1, c_2, \ldots, c_5$ depending on $n$, $\tau$, $\gamma$, and $C$ only, such that for all $\vartheta$, $0 < \vartheta \leq c_1$, and

$$M := |R|_{\mathcal{D}(r,s)} \leq c_2 s^2 \vartheta$$

(1.6)

the following holds: There exists a simple canonical transformation

$$W = (U, V) : \mathcal{D}(r/2, s/2) \rightarrow \mathcal{D}(r, s), \quad W - \text{id} \in \mathcal{P}_{2n}(r/2, s/2)$$

with the estimate

$$|W_\zeta - E_{2n}|_{\mathcal{D}(r/2, s/2)} \leq c_3 \vartheta.$$  

(1.7)

The transformed Hamiltonian $H_+ := H \circ W$ is an element of $\mathcal{P}(r/2, s/2)$ and has the form

$$H_+(\xi, \eta) = a_+ + \langle \omega, \eta \rangle + \frac{1}{2} \langle \eta, Q_+(\xi), \eta \rangle + R^*(\xi, \eta),$$

(1.8)

where $a_+ \in \mathbb{R}$, $Q_+ \in \mathcal{P}_{n \times n}(r/2)$, and $R^* \in \mathcal{P}(r/2, s/2)$. The functions $Q_+$ and $R^*$ fulfill the estimates

$$|Q_+ - Q|_{\mathcal{S}(r/2)} \leq c_4 \vartheta,$$

(1.9)

$$|R^*(\xi, \eta)| \leq c_5 M \frac{|\eta|^3}{s^3} \text{ for all } (\xi, \eta) \in \mathcal{D}(r/2, s/2).$$

(1.10)

Assertion (1.10) means, that we can find solutions to the canonical equations given by the Hamiltonian $H_+ = H \circ W$,

$$\dot{\xi} = H_+ \eta, \quad \dot{\eta} = -H_+ \xi.$$  

(1.11)

Indeed, using the Landau symbol $\mathcal{O}$ we have $R^* = \mathcal{O}(|\eta|^3)$, therefore (1.8) is the Taylor expansion of $H_+$. So the equations (1.11) can be written like this:

$$\dot{\xi} = \omega + \mathcal{O}(|\eta|), \quad \dot{\eta} = \mathcal{O}(|\eta|^2).$$

We find the solution $\eta = 0, \xi = \omega t + \text{const}$. It can be used to find a solution for the canonical equations corresponding to the original Hamiltonian $H$,

$$\dot{x} = H_y, \quad \dot{y} = -H_x.$$  

Namely, the solution is $W(\xi, \eta) = W(\omega t + \text{const}, 0)$.

The trick of theorem 1.6 is to get $\vartheta$ independent of $s$ in the estimates (1.7) and (1.9). This is essential to apply the theorem in differential KAM-theory.

The fact, that $\omega$ can be kept fixed, is due to assumption (1.5), for it causes $Q$ to be non-singular.
2 Motivation of the linearized equation

We prove theorem 1.6 with Newton’s method, for its rapid convergence overcomes the influence of the so-called small divisors, see remarks 3.3 (page 9), 4.2 (page 17), and 4.4 (page 22). To this end we have to establish a suitable linearised equation, which we now motivate. We write the Hamiltonian (1.4) as a sum

\[ H = N + R. \]

The summands are the normal form

\[ N(x, y) = a + \langle \omega, y \rangle + O(|y|^2), \]

and the small remainder \( R(x, y) \). We have to find a sequence \((Z_k)_{k \in \mathbb{N}}\) of symplectic transformations, such that the remainder gets smaller after every transformation. Write for \( k \in \mathbb{N}_0 \)

\[ H = H_0, \quad H_k = N_k + R_k, \quad H_{k+1} := H_k \circ Z_{k+1}, \]

where \( N_k \) again is a normal form (with \( a_k \) instead of \( a \) and with the same \( \omega \)), and \( R_k \) is the remainder after the \( k \)-th step. When we set

\[ W_k := Z_1 \circ \ldots \circ Z_k, \quad W_0 := \text{id} \quad (k \in \mathbb{N}), \]

we get \( H_k = H \circ W_k = N_k + R_k \). In case the limits

\[ R_k \to 0, \quad W_k \to W_\infty, \quad N_k \to N_\infty \quad (k \to \infty) \]

exist with some symplectic transformation \( W_\infty \) and normal form \( N_\infty \),

\[ H \circ W_\infty = N_\infty \]

follows and we are successful. In other words, we look for a root of the function

\[ \mathcal{R}(W, N) := H \circ W - N, \]

which is given by a pair of functions \((W, N)\). According to the above considerations, we try to find this root as a limit

\[ (W_\infty, N_\infty) = \lim_{k \to \infty} (W_k, N_k). \]

This leads to the problem to improve an approximate solution \((W_k, N_k)\) to a better approximate solution \((W_{k+1}, N_{k+1})\). For \( k \in \mathbb{N}_0 \) we set

\[ W := W_k, \quad N := N_k, \quad W_+ = W + \Delta W := W_{k+1}, \quad N_+ = N + \Delta N := N_{k+1}, \quad (2.1) \]

and obtain the new remainder as

\[ \mathcal{R}(W_+, N_) = H \circ (W + \Delta W) - N - \Delta N \]

\[ = \mathcal{R}(W, N) + H_z(W)\Delta W - \Delta N + \text{terms of higher order}. \]

6
Linearisation means to solve the equation
\[ R(W, N) + H_z(W)\Delta W - \Delta N = 0. \tag{2.2} \]
However, due to the term \( H_z(W)\Delta W \) this is not possible in general. We have to separate further terms of higher order to get (2.2) simple enough. – The following considerations are a simplified version of the approach presented in [12]. (The situation in [12] is more complicated than the situation here because in [12] the assumption (1.5) is avoided.) We construct the symplectic transformations as flows of certain Hamiltonian systems. So we work with a function \( \Delta S = \Delta S(x, y) \) and consider the Hamiltonian system
\[ \dot{x} = \Delta S_y, \quad \dot{y} = -\Delta S_x. \tag{2.3} \]
The solution of the respective initial value problem is denoted with
\[ z = (x, y) = (X(t, \xi, \eta), Y(t, \xi, \eta)) = Z(t, \xi, \eta), \quad Z(0, \xi, \eta) = (\xi, \eta) = \zeta. \]
Then, \( t \) fixed, provided existence, the map \( \zeta \mapsto Z(t, \zeta) \) is a symplectic transformation (see appendix A.3).

**Definition 2.1.** Let \( f, g \in \mathcal{P}(r, s) \) or \( f, g : \mathbb{R}^2n \to \mathbb{R} \) be differentiable functions. Then we define the Poisson bracket of \( f \) and \( g \) by
\[ \{f, g\} := \langle f_x, g_y \rangle - \langle f_y, g_x \rangle. \]
For the moment let \( F \) be a real valued, differentiable function. Then using (2.3) we can replace a derivative with respect to time by a Poisson bracket as follows:
\[ \frac{d}{dt} F(Z(t, \zeta)) = \langle F_x(Z(t, \zeta)), Z_t(t, \zeta) \rangle = \langle F_x(Z(t, \zeta)), X_t(t, \zeta) \rangle + \langle F_y(Z(t, \zeta)), Y_t(t, \zeta) \rangle = \langle F_x(Z(t, \zeta)), \Delta S_y(Z(t, \zeta)) \rangle - \langle F_y(Z(t, \zeta)), \Delta S_x(Z(t, \zeta)) \rangle = \{F, \Delta S\}(Z(t, \zeta)). \tag{2.4} \]
Now assume the existence of a map \( \zeta \mapsto Z(t, \zeta) \) for all \( 0 \leq t \leq 1 \) and a set of allowed \( \zeta \). The new transformation \( W_+ = W + \Delta W \) (see (2.1)) shall be given by \( W_+(\zeta) := W(Z(1, \zeta)) \). \( W \) being a symplectic transformation, \( W_+ \) will be a symplectic transformation as well. With (2.4) we get for \( \Delta W \) the equation
\[ \Delta W(\zeta) = W_+(\zeta) - W(\zeta) = W(Z(1, \zeta)) - W(\zeta) = \int_0^1 \frac{d}{dt} W(Z(t, \zeta)) dt = \int_0^1 (\{W_1, \Delta S\}, \ldots, \{W_{2n}, \Delta S\})(Z(t, \zeta)) dt. \tag{2.5} \]
Let us calculate \( R(W_+, N_+) \) once more using (2.4).
\[ \begin{align*}
R(W_+, N_+)(\zeta) &= H \circ W_+(\zeta) - N_+(\zeta) = H \circ W(Z(1, \zeta)) - N(\zeta) - \Delta N(\zeta) \\
&= R(W, N)(Z(1, \zeta)) + N(Z(1, \zeta)) - N(\zeta) - \Delta N(\zeta) \\
&= (R(W, N) + \{N, \Delta S\} - \Delta N)(\zeta) \\
&\quad + R(W, N)(Z(1, \zeta)) - \Delta N(\zeta) \\
&\quad + N(Z(1, \zeta)) - N(\zeta) - \frac{d}{dt} N(Z(t, \zeta)) \bigg|_{t=0}.
\end{align*} \]
\( t = 0 \) means that the function has to be evaluated in the point \( t = 0 \). Like in (2.5) we get
\[
\mathcal{R}(W, N)(Z(1, \zeta)) - \mathcal{R}(W, N)(\zeta) = \int_0^1 \{\mathcal{R}(W, N), \Delta S\} (Z(t, \zeta)) dt. \tag{2.6}
\]
Taylor’s formula yields
\[
N(Z(1, \zeta)) - N(\zeta) - \frac{d}{dt}N(Z(t, \zeta)) \bigg|_{t=0} = \int_0^1 (1 - t) \frac{d^2}{dt^2}N(Z(t, \zeta)) dt. \tag{2.7}
\]
When we use this, we obtain
\[
\mathcal{R}(W_+, N_+)(\zeta) = (\mathcal{R}(W, N) + \{N, \Delta S\} - \Delta N)(\zeta) + \\
+ \int_0^1 \left(\{\mathcal{R}(W, N), \Delta S\} (Z(t, \zeta)) + (1 - t) \frac{d^2}{dt^2}N(Z(t, \zeta))\right) dt.
\]
The time derivatives can be handled with (2.4),
\[
\frac{d^2}{dt^2}N(Z(t, \zeta)) = \frac{d}{dt} \{N, \Delta S\} (Z(t, \zeta)) = \{\{N, \Delta S\}, \Delta S\} (Z(t, \zeta)),
\]
\[
\Rightarrow \mathcal{R}(W_+, N_+)(\zeta) = (\mathcal{R}(W, N) + \{N, \Delta S\} - \Delta N)(\zeta) + \\
+ \int_0^1 \{\mathcal{R}(W, N) + (1 - t) \{N, \Delta S\}, \Delta S\} (Z(t, \zeta)) dt.
\]
Hence we obtain the simplified linearised equation:
\[
\boxed{\mathcal{R}(W, N) + \{N, \Delta S\} - \Delta N = 0} \tag{2.8}
\]
This equation determines \( \Delta N \) and \( \Delta S \). Then \( Z \) has to be calculated as the flow of (2.3). This in turn determines \( W_+ = W \circ Z(1, \cdot) \). (2.8) being solved, the new remainder reads
\[
\mathcal{R}(W_+, N_+)(\zeta) = \int_0^1 \{\mathcal{R}(W, N) + (1 - t) \{N, \Delta S\}, \Delta S\} (Z(t, \zeta)) dt.
\]
The inner Poisson bracket can be transformed with (2.8), for now
\[
(1 - t) \{N, \Delta S\} = (1 - t)\Delta N - (1 - t)\mathcal{R}(W, N)
\]
holds. So we can write
\[
\boxed{\mathcal{R}(W_+, N_+)(\zeta) = \int_0^1 \{t \mathcal{R}(W, N) + (1 - t)\Delta N, \Delta S\} (Z(t, \zeta)) dt} \tag{2.9}
\]
3 Solution of the linearized equation

The solution of (2.8) is based on the following theorem 3.2 from [11] (in [11] it is theorem 9.7).

**Definition 3.1.** Let \( r > 0 \) and \( f : \mathcal{S}(r) \subseteq \mathbb{C}^n \to \mathbb{C}^m, x \mapsto f(x) \), be a continuous function with period \( 2\pi \) in \( x_1, \ldots, x_n \). We define the mean \( [f] \) of \( f \) to be

\[
[f] := \left( \frac{1}{2\pi} \right)^{n} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} f(x) \, dx_1 \ldots dx_n.
\]

**Theorem 3.2.** Let \( \tau \geq n - 1 \geq 1, \gamma > 0, r > 0, M > 0 \) and \( g : \mathcal{S}(r) \subseteq \mathbb{C}^n \to \mathbb{C} \) a 2\( \pi \)-periodic, analytic function with \( |g|_{\mathcal{S}(r)} \leq M \) and \( [g] = 0 \). Let \( \omega \in \Omega(\gamma, \tau) \) (compare definition 7.2). Then there exists one and only one 2\( \pi \)-periodic analytic function \( u : \mathcal{S}(r) \to \mathbb{C} \) with \( [u] = 0 \) and

\[
\langle u_\xi, \omega \rangle = g.
\]

In addition there is a constant \( c_0 = c_0(n, \tau) > 0 \) with

\[
|u|_{\mathcal{S}(r-\delta)} \leq \frac{c_0 M}{\gamma \delta^{\tau}} \quad \forall \ \delta \in (0, r).
\]

In case \( g \) maps real vectors to real values, so does \( u \).

**Remark 3.3.** Small divisors. Let us expand the given function \( g \) and the solution \( u \) into their Fourier series. These read, with coefficients \( g_k \) and \( u_k \in \mathbb{C} \) \((k \in \mathbb{Z}^n \setminus \{0\})\), respectively,

\[
g(\xi) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} g_k e^{i \langle k, \xi \rangle} \quad \text{and} \quad u(\xi) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} u_k e^{i \langle k, \xi \rangle} \quad \forall \ \xi \in \mathcal{S}(r).
\]

The vanishing means of \( g \) and \( u \) amount to \( g_0 = 0 \) and \( u_0 = 0 \), respectively. The function \( u \) can be differentiated term by term, so in \( \mathcal{S}(r) \) we get

\[
\langle u_\xi(\xi), \omega \rangle = \left\langle \sum_{k \in \mathbb{Z}^n \setminus \{0\}} i k u_k e^{i \langle k, \xi \rangle}, \omega \right\rangle = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} i \langle k, \omega \rangle u_k e^{i \langle k, \xi \rangle}.
\]

Comparing coefficients with \( g \) shows \( i \langle k, \omega \rangle u_k = g_k \) for all \( k \in \mathbb{Z}^n \setminus \{0\} \). Hence

\[
u(\xi) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{g_k}{i \langle k, \omega \rangle} e^{i \langle k, \xi \rangle} \quad \forall \ \xi \in \mathcal{S}(r).
\]

So, if we took (3.3) as an ansatz for the solution of the equation \( \langle u_\xi, \omega \rangle = g \), we had to proof convergence of this series. However, there is a serious obstacle: The divisors \( i \langle k, \omega \rangle \) become very small – in case the entries of \( \omega \) are not linear independent over \( \mathbb{Q} \), there even exists some \( k \in \mathbb{Z}^n \setminus \{0\} \), such that \( \langle k, \omega \rangle \) vanishes: Therefore in this case there doesn’t exist a 2\( \pi \)-periodic analytic solution of (3.1).

The meaning of theorem 3.2 now is, that the series (3.3) indeed converges. The influence of the small divisors is represented by the factor \( c_0/(\gamma \delta^{\tau}) \) in estimate (3.2).
Theorem 3.4. Let $\tau \geq n - 1 \geq 1, \gamma > 0, r > 0, 0 < \delta < r/4$ and $0 < s \leq \delta^{\tau+1} \leq 1$ be given. Suppose there is a constant $M > 0$ such that the function $f \in \mathcal{P}(r,s)$ fulfills

$$|f|_{\mathcal{D}(r,s)} \leq M. \tag{3.4}$$

Let $N \in \mathcal{P}(r,s)$ be a function with

$$N(x,0) = N(0) \text{ and } N_y(x,0) = \omega \in \Omega(\gamma, \tau) \quad \forall \ x \in S(r). \tag{3.5}$$

Finally, let $C \in \mathbb{R}^{n \times n}$ be a non-singular matrix with

$$|N_{yy} - C|_{\mathcal{D}(r,s)} \leq \frac{1}{2|C^{-1}|}. \tag{3.6}$$

Then the equation

$$f + \{N, \Delta S\} - \Delta N = 0 \tag{3.7}$$

possesses a solution, that is a pair of functions $(\Delta S, \Delta N)$, with the properties:

It is $\Delta S(x,y) = \langle \lambda, x \rangle + U(x) + \langle V(x), y \rangle$ with $\lambda \in \mathbb{R}^n$ and $U \in \mathcal{P}(r), V \in \mathcal{P}_n(r)$. Especially the function $(x,y) \mapsto \Delta S(x,y) - \langle \lambda, x \rangle$ lies in $\mathcal{P}(r,s)$. We have $\Delta N \in \mathcal{P}(r,s)$,

$$\Delta N(x,0) = \Delta N(0) \text{ and } \Delta N_y(x,0) = 0 \quad \forall \ x \in S(r). \tag{3.8}$$

There are constants $c_7, c_8, \tilde{c}_9, c_{10}$ and $c_{11} > 0$, such that the following estimates hold:

$$|\Delta S_x|_{\mathcal{D}(r-4\delta,s)} \leq c_7 \frac{M}{s}, \tag{3.9}$$

$$|\Delta S_y|_{\mathcal{S}(r-3\delta)} \leq c_8 \frac{M}{s\delta^{\tau}}, \tag{3.10}$$

$$|\Delta N(0)| \leq \tilde{c}_9 \frac{M}{s}, \tag{3.11}$$

$$|\Delta N - \Delta N(0)|_{\mathcal{D}(r-4\delta,s/2)} \leq c_{10} M, \tag{3.12}$$

$$|\Delta N_{yy}|_{\mathcal{D}(r-4\delta,s/4)} \leq c_{11} \frac{M}{s^2}. \tag{3.13}$$

The constants $c_j$ ($j \neq 9$) only depend on $n$, $\tau$, $\gamma$, and $C$. The constant $\tilde{c}_9$ depends in addition on $|\omega|$.

Proof. For $\Delta S$ we make the ansatz

$$\Delta S(x,y) = \langle \lambda, x \rangle + U(x) + \langle V(x), y \rangle. \tag{3.14}$$

Here we try to obtain $U \in \mathcal{P}(r)$ and $V \in \mathcal{P}_n(r)$ with $[U] = 0$ and $[V] = 0$. The vector $\lambda \in \mathbb{R}^n$ has to be chosen suitable. We proceed in five steps.
1. Establish an equation to determine $U$.
2. Solve this equation.
3. Establish an equation to determine $V$.
4. Define $\lambda$ and solve the equation for $V$.
5. Define $\Delta N$ and prove the properties of $\Delta S$ and $\Delta N$.

(1) We deduce an equation for $U$. To this end we put $y = 0$ in (3.7). Assuming $\Delta N(x, 0) = \Delta N(0)$ for $x \in S(r)$ (see (3.8)) we obtain with (3.5)

$$f(x, 0) + \{N, \Delta S\}(x, 0) - \Delta N(x, 0) =$$

$$= f(x, 0) + \langle N_x, \Delta S_y \rangle(x, 0) - \langle N_y, \Delta S_x \rangle(x, 0) - \Delta N(0)$$

$$= f(x, 0) - \langle \Delta S_x(x, 0), \omega \rangle - \Delta N(0).$$

This has to be zero. By (3.14) that means for $\Delta S$

$$f(x, 0) - \langle \lambda, \omega \rangle - \langle U_x(x), \omega \rangle - \Delta N(0) = 0. \tag{3.15}$$

Well, with the help of theorem 3.2 we can solve the equation

$$\langle U_x(x), \omega \rangle = f(x, 0) - [f(\cdot, 0)]. \tag{3.16}$$

We take this equation to determine $U$.

Remark on the connection between equations (3.15) and (3.16): Clearly (3.15) and (3.16) are equivalent, if

$$\Delta N(0) = [f(\cdot, 0)] - \langle \lambda, \omega \rangle. \tag{3.17}$$

In step (4) we will have to fix $\lambda$ in such a way that the equation for $V$ is solvable, and then in step (5) define $\Delta N$ such that (3.17) holds.

(2) Solution of equation (3.16). The right hand side of (3.16) is bounded by $2M$ because of (3.4). Hence Theorem 3.2 yields a solution $U \in P(r)$ with $[U] = 0$ and

$$|U|_{S(r-\delta)} \leq \frac{c_6 2M}{\gamma \delta^2} \quad \forall \quad \delta \in (0, r).$$

With Cauchy's estimate (see lemma A.3 in the appendix) we obtain

$$|U|_{S(r-2\delta)} \leq \frac{2c_6 M}{\gamma \delta^{3+1}} \quad \forall \quad \delta \in (0, r/2). \tag{3.18}$$

(3) Now we have to find an equation for $V$. To this end we differentiate (3.7) with respect to $y$ and put $y = 0$ to get

$$0 = f_y(x, 0) + \{N, \Delta S\}_y(x, 0) - \Delta N_y(x, 0)$$

$$= f_y(x, 0) + \langle N_x, \Delta S_y \rangle_y(x, 0) - \langle N_y, \Delta S_x \rangle_y(x, 0) - \Delta N_y(x, 0)$$

$$= f_y(x, 0) + \Delta S_y(x, 0) \cdot N_{xy}(x, 0) + N_x(x, 0) - \Delta S_y(x, 0)$$

$$- \Delta S_x(x, 0) \cdot N_{xy}(x, 0) - N_y(x, 0) \cdot \Delta S_{xy}(x, 0) - \Delta N_y(x, 0).$$
The second summand vanishes because of (3.5). The third summand is zero as well by construction (3.14). Therefore (3.7) implies

\[ f_y(x, 0) - \Delta S_x(x, 0) \cdot N_{yy}(x, 0) - N_y(x, 0) \cdot \Delta S_{xy}(x, 0) - \Delta N_y(x, 0) = 0. \]  

(3.19)

Supposing \( \Delta N_y(x, 0) = 0 \) for \( x \in S(r) \) (compare (3.8)) we get with (3.5) and (3.14)

\[ f_y(x, 0) - (\lambda + U_x(x)) \cdot N_{yy}(x, 0) - \omega \cdot V_x^T(x) = 0 \]

\[ \Leftrightarrow \omega \cdot V_x^T(x) = f_y(x, 0) - (\lambda + U_x(x)) \cdot N_{yy}(x, 0). \]  

(3.20)

This is a system of \( n \) equations which can be solved separately by theorem 3.2, provided

\[ \lambda = \frac{[f_y(\cdot, 0) - (\lambda + U_x(\cdot)) \cdot N_{yy}(\cdot, 0)]}{[N_{yy}(\cdot, 0)]^{-1}}. \]  

(3.21)

This equation has to be solved for \( \lambda \).

(4) Definition of \( \lambda \) and solution of (3.20). When \( [N_{yy}(\cdot, 0)] \) is non-singular, equation (3.21) can be solved for \( \lambda \). We apply Lemma A.1 to \( [N_{yy}(\cdot, 0)] \). By (3.6)

\[ |[N_{yy}(\cdot, 0)] - C| \leq \frac{1}{2|C^{-1}|} \]

holds. So we can set \( S = C, P = [N_{yy}(\cdot, 0)] \), and \( h = 1/2 \) in the assumptions of lemma A.1. It follows, that \( [N_{yy}(\cdot, 0)]^{-1} \) exists and that we have the estimate

\[ |[N_{yy}(\cdot, 0)]^{-1}| \leq 2|C^{-1}|. \]  

(3.22)

Therefore \( \lambda \) can be defined as

\[ \lambda := ([f_y(\cdot, 0) - (U_x \cdot N_{yy}(\cdot, 0))] \cdot [N_{yy}(\cdot, 0)]^{-1}. \]

This choice guarantees, that the mean of the right hand side of (3.20) vanishes. In order to apply theorem 3.2 to (3.20), we have to find an estimate for the right hand side of (3.20). To begin with, (3.4) and Cauchy’s estimate yield

\[ |f_y(\cdot, 0)|_{S(r)} \leq \frac{M}{s}. \]

With respect to \( N_{yy} \) we observe

\[ 1 = |CC^{-1}| \leq |C||C^{-1}| \Rightarrow \frac{1}{|C^{-1}|} \leq |C|, \]

hence with (3.6) we see

\[ |N_{yy}|_{D(r,s)} \leq |N_{yy} - C|_{D(r,s)} + |C| \leq \frac{1}{2|C^{-1}|} + |C| \leq 2|C|. \]  

(3.23)
Together with (3.18) and \( s \leq \delta^{r+1} \)
\[
|f_y(\cdot, 0) - U_x \cdot N_{yy}(\cdot, 0)|_{S(r-2\delta)} \leq \frac{M}{s} + \frac{2c_6 M}{\gamma \delta^{r+1}} 2|C|
\]
\[
\leq \left(1 + \frac{4c_6 |C|}{\gamma}\right) \frac{M}{s} = c_{12} \frac{M}{s}
\]
(3.24)
follows, where
\[
c_{12} := 1 + \frac{4c_6 |C|}{\gamma}
\]
(3.25)
is a positive constant. This and (3.22) give an estimate for \( \lambda \), namely
\[
|\lambda| \leq 2|C|^{-1} c_{12} \frac{M}{s}.
\]
(3.26)
The desired estimate for the right hand side of (3.20) can be found using (3.23), (3.24), and (3.26):
\[
|f_y(\cdot, 0) - (\lambda + U_x) \cdot N_{yy}(\cdot, 0)|_{S(r-2\delta)} \leq |f_y(\cdot, 0) - U_x \cdot N_{yy}(\cdot, 0)|_{S(r-2\delta)} + |\lambda| |N_{yy}(\cdot, 0)|_{S(r)} \leq c_{12} \frac{M}{s} + 4|C| |C|^{-1} c_{12} \frac{M}{s}.\]
(3.27)
Now we can solve (3.20). Observe
\[
V(x) = (V_1(x), \ldots, V_n(x)), \quad \omega \cdot V^T(x) = (\langle \omega, V_1x(x) \rangle, \ldots, \langle \omega, V_{nx}(x) \rangle).
\]
Estimates for every \( V_i \) (1 \( \leq i \leq n \)) become estimates for \( V \) for we use the maximum norm. The right hand side of (3.20) is bounded on every substrip \( S(r - \varepsilon) \) of \( S(r) \) (\( \varepsilon \in (0, r) \)), because \( f, U, \) and \( N \) are periodic in \( x \). Therefore the solution \( V \) exists on \( S(r) \) and we have \( V \in \mathcal{P}_n(r) \) with the estimate
\[
|V|_{S(r-3\delta)} \leq \frac{c_6}{\gamma \delta^r} \left(c_{12} \frac{M}{s} + 4|C| |C|^{-1} c_{12} \frac{M}{s}\right) = c_8 \frac{M}{s \delta^r}.
\]
(3.28)
Herein \( c_8 = c_8(n, \tau, \gamma, C) \) is a positive constant. Further Cauchy’s estimate yields
\[
|V_{xx}|_{S(r-4\delta)} \leq c_8 \frac{M}{s \delta^{r+1}}.
\]
(3.29)
(5) Now let us define \( \Delta S \) by (3.14). Then the assertions on the form of \( \Delta S \) are fulfilled automatically. The definition
\[
\Delta N := f + \{ N, \Delta S \}
\]
solves (3.7) and \( \Delta N \in \mathcal{P}(r, s) \) holds as well. Assertion (3.8) is on the form of \( \Delta N \). Using (3.5), (3.14), and (3.16) we get
\[
\Delta N(x, 0) = f(x, 0) + \{ N, \Delta S \}(x, 0)
\]
\[
= f(x, 0) + \langle N_x, \Delta S_y \rangle(x, 0) - \langle N_y, \Delta S_x \rangle(x, 0)
\]
\[
= f(x, 0) - \langle \omega, \Delta S_x(x, 0) \rangle
\]
\[
= f(x, 0) - \langle \lambda, \omega \rangle - \langle U_x(x), \omega \rangle
\]
\[
= [f(\cdot, 0)] - \langle \lambda, \omega \rangle.
\]
(3.30)
This is obviously independent of \( x \). So we may write \( \Delta N(x,0) = \Delta N(0) \) for all \( x \in S(r) \). Incidentally the calculation shows, that (3.17) is fulfilled and that solving \( (3.14) \) solves \( (3.30) \) as well. – In (3.19) we have seen, that equation (3.7), which we have proven in the meantime, implies

\[
\Delta N_y(x,0) = f_y(x,0) - \Delta S_x(x,0) \cdot N_{yy}(x,0) - N_y(x,0) \cdot \Delta S_{xy}(x,0).
\]

Therefore (3.14) and (3.20) yield

\[
\Delta N_y(x,0) = f_y(x,0) - (\lambda + U_x(x)) \cdot N_{yy}(x,0) - \omega \cdot V^T_x(x) = 0,
\]

and (3.8) is shown. We turn to the estimates for the derivatives of \( \Delta S \). By definition (3.14) \( \Delta S_y = V \), so (3.28) means

\[
|\Delta S_y|_{S(r-\delta)} \leq c_8 \frac{M}{s^{\delta+1}}.
\]

This is (3.10). We have \( \Delta S_x(x,y) = \lambda + U_x(x) + y \cdot V_x(x) \). With (3.26), (3.18), (3.29) and the assumption \( s \leq \delta^{r+1} \) we calculate

\[
|\Delta S_x|_{D(r-\delta,s)} \leq |\lambda| + |U_x|_{S(r-\delta)} + ns |V_x|_{S(r-\delta)} \\
\leq 2|C^{-1}|c_{12} \frac{M}{s} + 2c_9 M + \gamma s c_8 \frac{M}{s^2} = c_7 \frac{M}{s},
\]

where \( c_7 = c_7(\tau, \gamma, C) \) is a positive constant. This proves (3.9). The estimates for \( \Delta N \) and \( \Delta N_{yy} \) remain. In (3.30) we have seen \( \Delta N(0) = [f(\cdot,0)] - \langle \lambda, \omega \rangle \). According to (3.14) and (3.26) this yields

\[
|\Delta N(0)| \leq M + 2n|\omega||C^{-1}|c_{12} \frac{M}{s} \leq \tilde{c}_9 \frac{M}{s},
\]

where \( \tilde{c}_9 = \tilde{c}_9(\tau, \gamma, C, |\omega|) \) again is a positive constant. Hence (3.11) holds. In order to show (3.12) we use (3.14), (3.16) and (3.30) to get

\[
\langle \Delta S_x(x,y), \omega \rangle = \langle \lambda, \omega \rangle + \langle U_x(x), \omega \rangle + \langle y \cdot V_x(x), \omega \rangle \\
= \langle \lambda, \omega \rangle + f(x,0) - [f(\cdot,0)] + \langle y, \omega \cdot V^T_x(x) \rangle \\
= f(x,0) + \langle y, \omega \cdot V^T_x(x) \rangle - \Delta N(0).
\]

With (3.20) and (3.27) we obtain

\[
|\langle \Delta S_x, \omega \rangle + \Delta N(0)|_{D(r-2\delta,s)} \leq M + ns(c_{12} + 4|C| |C^{-1}|c_{12}) \frac{M}{s} = c_{13} M,
\]

where

\[
c_{13} := 1 + nc_{12} \left( 1 + 4|C| |C^{-1}| \right).
\]

Let us for the moment denote the function \( y \mapsto \langle \omega, y \rangle \) by \( g_\omega \). Then we can write

\[
\Delta N = f + \{ N, \Delta S \} = f + \langle N_x, \Delta S_y \rangle - \langle N_y, \Delta S_x \rangle \\
= f + \langle (N - g_\omega - N(0))_x, \Delta S_y \rangle \\
- \langle (N - g_\omega - N(0))_y, \Delta S_x \rangle - \langle \omega, \Delta S_x \rangle \\
= f + \{ N - g_\omega - N(0), \Delta S \} - \langle \omega, \Delta S_x \rangle.
\]
Let us have a closer look at the first entry of the Poisson bracket. We can write
\[ N(x, y) - \langle \omega, y \rangle = N(x, y) - \langle N_y(x, 0), y \rangle - N(x, 0) =: h(x, y) \tag{3.34} \]
for all \((x, y) \in \mathcal{D}(r, s)\) because of (3.5). This defines a function \(h \in \mathcal{P}(r, s)\) with \(h(x, 0) = 0\) and \(h_y(x, 0) = 0\) for all \(x \in S(r)\). Taylor’s formula yields
\[ |h(x, y)| \leq \int_0^1 \frac{(1 - \sigma)^2}{2} \langle y \cdot h_{yy}(x, \sigma y), y \rangle \, d\sigma \leq \frac{1}{2} n s^2 |h_{yy}(x, \cdot)|_{\{y \in \mathbb{C}^n | |y| < s\}} \]
for all \((x, y) \in \mathcal{D}(r, s)\), from which we conclude with (3.23)
\[ |h|_{\mathcal{D}(r,s)} \leq |C| ns^2. \tag{3.35} \]
Cauchy’s estimate results in
\[ |h_x|_{\mathcal{D}(r-\delta, s)} \leq \frac{|C| ns^2}{s}, \quad \text{and} \quad |h_y|_{\mathcal{D}(r,s/2)} \leq 2|C| ns. \tag{3.35} \]
Now, (3.33) and (3.34) show
\[ \Delta N - \Delta N(0) = f + \{N - g_\omega - N(0), \Delta S\} - \langle \omega, \Delta S_x \rangle - \Delta N(0) \]
\[ = f + \{h, \Delta S\} - \langle \omega, \Delta S_x \rangle + \Delta N(0) \]
\[ = f + \langle h_x, \Delta S_y \rangle - \langle h_y, \Delta S_x \rangle - \langle \omega, \Delta S_x \rangle + \Delta N(0) \].
When we put the estimates for \(f, \Delta S_y\) and \(\Delta S_x\), (3.35), (3.31), and (3.32) together, we get
\[ |\Delta N - \Delta N(0)|_{\mathcal{D}(r-4\delta, s/2)} \leq M + n \cdot \frac{|C| ns^2}{s} \cdot c_8 \frac{M}{s\delta^\tau} + n \cdot 2|C| ns \cdot c_7 \frac{M}{s} + c_{13} M \]
\[ \leq c_{10} M, \]
where
\[ c_{10} := 1 + n^2 |C| \left(2c_7 + c_8\right) + c_{13} \tag{3.36} \]
is a positive constant. This proves (3.12). Now (3.13) is a consequence of Lemma A.3
\[ |\Delta N_{yy}|_{\mathcal{D}(r-4\delta, s/4)} \leq \frac{8}{s} |\Delta N_y|_{\mathcal{D}(r-4\delta, (3/8)s)} \leq 64 \frac{Mc_{10}}{s^2}. \]
It remains only to set \(c_{11} = c_{11}(n, \tau, \gamma, C) := 64c_{10} > 0\) to finish the proof. \(\Box\)

### 4 The inductive lemma

In this section we construct a sequence of symplectic transformations and proceed in three steps. At first we prove theorem 4.1. It deals with a transformation \(Z\), which transforms a given Hamiltonian \(H\) into \(H_+ = H \circ Z\). Next we find sequences of numbers \((r_k), (\delta_k), (s_k),\) and \((M_k)\), such that theorem 4.1 can be applied repeatedly. That means that the obtained function \(H_+\) can be again inserted in the assumptions of theorem 4.1 as a new function \(H\). The third step is to summarize the results and describe the inductive process for all \(k \in \mathbb{N}_0\) in form of the inductive lemma 4.9
Theorem 4.1. Let \( \tau \geq n-1 \geq 1, \gamma > 0, r > 0, 0 < \delta < r/6, 0 < s \leq \delta^{\tau+1} \leq 1, \) and \( 0 < r \leq r - 6\delta \) and \( 0 < s \leq s/8 \). We consider a function \( H \in \mathcal{P}(r, s), H = N + R \) with \( N, R \in \mathcal{P}(r, s) \) and

\[
N(x, y) = a + \langle \omega, y \rangle + O(|y|^2),
\]

where \( a \in \mathbb{R} \) and \( \omega \in \Omega(\gamma, \tau) \) is assumed. Further we assume the existence of a non-singular matrix \( C \in \mathbb{R}^{n \times n} \) with

\[
|N_{yy} - C|_{D(r, s)} \leq \frac{1}{2|C^{-1}|}.
\]

The remainder \( R \) has to be bounded by a constant \( M > 0 \) with

\[
|R|_{D(r, s)} \leq M \leq \frac{1}{16} c_7 c_8 s^2.
\]

Herein the constants \( c_7 \) and \( c_8 \) are given by Theorem 3.4 (see (3.9) and (3.10)). Then there exists a simple canonical transformation (see definition 1.5)

\[
Z : D(r_+, s_+) \rightarrow D(r_+ - 5\delta, s/4), \quad Z - \text{id} \in \mathcal{P}_{2n}(r_+, s_+),
\]

such that the transformed Hamiltonian \( H_+ = H \circ Z \) is an element of \( \mathcal{P}(r_+, s_+) \) and \( H_+ = N_+ + R_+ \) holds, where \( N_+, R_+ \in \mathcal{P}(r_+, s_+) \), and

\[
N_+(\xi, \eta) = a_+ + \langle \omega, \eta \rangle + O(|\eta|^2)
\]

with some \( a_+ \in \mathbb{R} \). The following estimates hold:

\[
|Z|_{D(r_+, s_+)} \leq \exp \left( c_{14} \frac{M}{s^2} \right),
\]

\[
|Z - E_{2n}|_{D(r_+, s_+)} \leq c_{14} \frac{M}{s^2} \exp \left( c_{14} \frac{M}{s^2} \right),
\]

\[
|a_+ - a| \leq \tilde{c}_9 \frac{M}{s},
\]

\[
|N_{\eta\eta} - N_{\eta\eta}|_{D(r_+, s_+)} \leq c_{11} \frac{M}{s^2},
\]

\[
|R_+|_{D(r_+, s_+)} \leq c_{15} \frac{M^2}{s^2}.
\]

The constants \( \tilde{c}_9 \) and \( c_{11} \) are given by Theorem 3.4 (see (3.11) and (3.13)), and \( c_{14}, c_{15} \) are positive constants depending on \( n, \tau, \gamma, \) and \( C \) only. Finally, if the partial derivatives \( W_\xi \) and \( W_\eta \) of the function \( W = W(\xi, \eta) : D(r, s) \rightarrow \mathbb{C}^{2n} \) are continuous and bounded by \( K_1 > 0 \), then \( \Delta W := W \circ Z - W \) satisfies

\[
|\Delta W|_{D(r_+, s_+)} \leq nK_1(c_7 + c_8) \frac{M}{s\delta^\tau}.
\]
Remark 4.2. We see the success of our approach in estimate (4.10), for the magnitude \( M \) of the old remainder enters quadratically. This is due to Newton’s method. The disturbing influence of the small divisors (compare remark 3.3) is seen in the factor \( 1/s^2 \).

Proof of theorem 4.1. We solve the linearized equation

\[
R + \{N, \Delta S\} - \Delta N = 0 \tag{4.12}
\]

by means of theorem 3.4. Let us check the assumptions of that theorem. We apply the constants \( \tau, \gamma, \delta, r, s, \) and \( M \) as they are in theorem 3.4, such that the assumptions on those constants are fulfilled. Further we insert \( f = R \) and \( N = H - R \). Now, \( R, N \in \mathcal{P}(r, s) \) and from (4.1) \( N(x, 0) = N(0) = a \) and \( N_y(x, 0) = \omega \in \Omega(\gamma, \tau) \) hold for all \( x \in S(r) \). With (4.2) and (4.3) all assumptions of theorem 3.4 are met. Hence we obtain a solution \( (\Delta S, \Delta N) \) of (4.12) with all the properties asserted in theorem 3.4, especially the estimates (3.9) to (3.13).

The construction of \( Z \) proceeds like it is described in the appendix, see theorem A.17 in section A.3. Theorem A.17 can be applied with

\[
K = (c_7 + c_8) \frac{M\delta}{s} > 0, \tag{4.13}
\]

\( \varrho = r - 4\delta, \sigma = s/4, \) and \( F = \Delta S|_{D(\varrho, \sigma)} \in \mathcal{P}(\varrho, \sigma) \).

We have \( 2\delta < \varrho \) because of \( \delta < r/6 \) and \( 0 < \sigma \leq \delta \) from \( 0 < s \leq \delta^{\tau+1} \leq 1 \). (4.13) and (4.3) show

\[
\frac{\sigma\delta}{2K} = \frac{\sigma\delta}{2} \cdot \frac{s}{(c_7 + c_8)M\delta} = \frac{s^2}{8(c_7 + c_8)M} \geq 2 > 1.
\]

The function \( F \) is affine linear in \( y \), as is \( \Delta S \). We use (3.9) to get

\[
|F_x|_{D(\varrho, \sigma)} = |\Delta S_x|_{D(\varrho, \sigma)} \leq c_7 \frac{M}{s} \leq (c_7 + c_8) \frac{M\delta}{s} \cdot \frac{1}{\delta} = \frac{K}{\delta},
\]

and (3.10) yields

\[
|F_y|_{D(\varrho, \sigma)} \leq |\Delta S_y|_{S(r-3\delta)} \leq c_8 \frac{M}{s}\delta^{\tau} \leq (c_7 + c_8) \frac{M\delta}{\delta^{\tau+1}} \cdot \frac{1}{s} \leq \frac{K}{s} < \frac{4K}{s} = \frac{K}{\sigma}.
\]

So \( F \) fulfills the assumptions (A.19) of theorem A.17 which can be applied now. According to (A.22) we obtain simple canonical transformations

\[
Z(t, \cdot): \mathcal{D}(r - 6\delta, s/8) \rightarrow D(r - 5\delta, s/4),
\]

\[
Z(t, \cdot) - \text{id} \in \mathcal{P}_{2n}(r - 6\delta, s/8) \quad (0 \leq t < 2).
\]

With (4.13) we calculate

\[
\frac{2nK}{\delta\sigma} = \frac{2 \cdot 4n(c_7 + c_8)M}{s^2} = \frac{c_{14}}{s^2}.
\]
wherein $c_{14} = 8n(c_7 + c_8)$ is a positive constant. This can be put into the estimates (A.23) and (A.24) of theorem A.17 to infer

$$|Z_{ζ}(t, \cdot)|_{D(r−6δ,s/8)} ≤ \exp \left(c_{14} \frac{M}{s^2} t \right) \quad \forall \quad t \in [0, 2), \quad (4.15)$$

$$|Z_{ζ}(t, \cdot) - E_{2n}|_{D(r−6δ,s/8)} ≤ c_{14} \frac{M}{s^2} \exp \left(c_{14} \frac{M}{s^2} t \right) \quad \forall \quad t \in [0, 1] \quad (4.16)$$

for the maps given in (4.14). Now we define $Z$ to be the function $Z(1, \cdot)$ restricted to $D(r + 6, s + 6)$. Than $Z$ has the properties (4.4) because of (4.14), (4.15) and (4.16) cause $Z$ to meet the estimates (4.6) and (4.7).

We set for all $ζ ∈ D(r + 6, s + 6)$

$$H_{+}(ζ) := (H ∘ Z)(ζ), \quad N_{+}(ζ) := N(ζ) + ΔN(ζ), \quad R_{+}(ζ) := H_{+}(ζ) - N_{+}(ζ),$$

(observe $N = H - R$). We deduce the properties of $N_{+}$ from the properties of $ΔN$ formulated in theorem 3.4. $ΔN ∈ \mathcal{P}(r, s)$ implies $N_{+} ∈ \mathcal{P}(r + 6, s + 6)$. Furthermore,

$$N_{+}(ζ, 0) = N(ζ, 0) + ΔN(ζ, 0) = a + ΔN(0) =: a_{+} \quad \forall \quad ζ ∈ S(r + 6).$$

(4.8) is a consequence of (3.11):

$$|a_{+} - a| = |ΔN(0)| ≤ \widetilde{C}_0 \frac{M}{s}.$$  

Next we see

$$N_{+y}(ζ, 0) = N_{y}(ζ, 0) + ΔN_{y}(ζ, 0) = ω \quad \forall \quad ζ ∈ S(r + 6).$$

So the Taylor expansion of $N_{+}$ is given by

$$N_{+}(ζ, η) = a_{+} + η \cdot ω + O(η^2),$$

which is (4.5). Estimate (4.9) follows from (3.13):

$$|N_{+η} - N_{η}|_{D(r + 6, s + 6)} = |ΔN_{η}|_{D(r + 6, s + 6)} ≤ c_{11} \frac{M}{s^2}.$$  

Now we check $R_{+} ∈ \mathcal{P}(r + 6, s + 6):$ $R_{+}$ is an analytic function, which maps real vectors to real values, and we have for all $1 ≤ j ≤ n$

$$R_{+}(ζ + 2πe_j, η) = H(Z(ζ + 2πe_j, η)) - N_{+}(ζ + 2πe_j, η)$$

$$= H(Z(ζ, η) + (2πe_j, 0)) - N_{+}(ζ, η)$$

$$= H(Z(ζ, η)) - N_{+}(ζ, η) = R_{+}(ζ, η),$$

which is the desired periodicity. In order to prove (4.10) we recalculate (2.9) – we redo the calculations of section 2 with our functions, which are well-defined in the meantime,
and use (2.4), (2.6), (2.7), and (4.12):

\[ R_+ (\zeta) = H_+ (\zeta) - N_+ (\zeta) = H \circ Z (\zeta) - N_+ (\zeta) = H \circ Z (1, \zeta) - N (\zeta) - \Delta N (\zeta) \]

\[ = R (Z (1, \zeta)) + N (Z (1, \zeta)) - N (\zeta) - \Delta N (\zeta) \]

\[ = (R + \{N, \Delta S\} - \Delta N) (\zeta) + R (Z (1, \zeta)) - R (\zeta) \]

\[ + N (Z (1, \zeta)) - N (\zeta) - \frac{d}{dt} N (Z (t, \zeta)) \bigg|_{t=0} \]

\[ = \int_0^1 \{R, \Delta S\} (Z (t, \zeta)) \, dt + \int_0^1 (1 - t) \frac{d^2}{dt^2} N (Z (t, \zeta)) \, dt \]

\[ = \int_0^1 \{R + (1 - t) \{N, \Delta S\}, \Delta S\} (Z (t, \zeta)) \, dt \]

\[ = \int_0^1 \{tR + (1 - t) \Delta N, \Delta S\} (Z (t, \zeta)) \, dt \quad \forall \quad \zeta \in \mathcal{D}(r_+, s_+). \quad (4.17) \]

To estimate the integrand we set for \( t \in [0, 1] \)

\[ F (t) := tR + (1 - t)(\Delta N - \Delta N (0)) \in \mathcal{P}(r, s). \]

Then our assumption (4.13) and (3.12) lead to

\[ |F (t)|_{\mathcal{D}(r - 4\delta, s/2)} \leq tM + (1 - t)c_{10}M \leq (1 + c_{10})M \quad \forall \quad t \in [0, 1]. \]

We use Cauchy’s estimate to get for all \( t \in [0, 1] \)

\[ |F (t)_x|_{\mathcal{D}(r - 4\delta, s/2)} \leq (1 + c_{10}) \frac{M}{\delta}, \quad |F (t)_y|_{\mathcal{D}(r - 4\delta, s/4)} \leq 4(1 + c_{10}) \frac{M}{s}. \]

Together with (3.9) and (3.10) we obtain for all \( t \in [0, 1] \)

\[ |\{F (t), \Delta S\}|_{\mathcal{D}(r - 5\delta, s/4)} \leq n \left( |F (t)_x|_{\mathcal{D}(r - 5\delta, s/2)} |\Delta S_y|_{\mathcal{S}(r - 3\delta)} + \right. \]

\[ + \left. |F (t)_y|_{\mathcal{D}(r - 4\delta, s/4)} |\Delta S_x|_{\mathcal{D}(r - 4\delta, s)} \right) \]

\[ \leq n(1 + c_{10}) \left( \frac{M}{\delta} c_8 \frac{M}{s\delta^r} + \frac{4M}{s} c_7 \frac{M}{s} \right) \leq c_{15} \frac{M^2}{s^2}, \]

where

\[ c_{15} := n(1 + c_{10})(4c_7 + c_8) \quad (4.18) \]

is a positive constant. Now,

\[ \{tR + (1 - t)\Delta N, \Delta S\} = \{F (t), \Delta S\} \quad \forall \quad t \in [0, 1], \]

and we have \( Z (t, \zeta) \in \mathcal{D}(r - 5\delta, s/4) \) for all \( t \in [0, 1] \) and \( \zeta \in \mathcal{D}(r_+, s_+) \) by (4.14). So we can deduce the estimate (4.10) for \( R_+ \) from (4.17).

Finally we have to show (4.11). The estimates for \( W_\xi \) and \( W_\eta \) become estimates for


\[ W_{j\xi} \text{ and } W_{j\eta} \ (1 \leq j \leq 2n), \] because we use the row-sum norm. Hence our assumptions read

\[ |W_{j\xi}|_{D(r,s)} \leq K_1 \text{ and } |W_{j\eta}|_{D(r,s)} \leq K_1 \quad \forall \ 1 \leq j \leq 2n. \]

(2.5) implies for all \( 1 \leq j \leq 2n \) and \( \zeta \in D(r, s) \)

\[
\Delta W_j(\zeta) = \int_0^1 \{W_j, \Delta S\}(Z(t, \zeta)) \, dt.
\]

So, writing \( \Delta S_\xi := \Delta S_x \) and \( \Delta S_\eta := \Delta S_y \), we obtain with (3.9) and (3.10)

\[
|\Delta W_j|_{D(r, s)} = \max_{1 \leq j \leq 2n} |\Delta W_j|_{D(r, s)} \leq nK_1 (c_7 + c_8) \frac{M}{s\delta\tau}.
\]

The estimate

\[
|\Delta W|_{D(r, s)} = \frac{\max}{1 \leq j \leq 2n} |\Delta W_j|_{D(r, s)} \leq nK_1 (c_7 + c_8) \frac{M}{s\delta\tau}
\]

follows and the proof is finished.

Existence of the sequences

Our intention is to formulate theorem 4.1 universally for the \( k \)-th step and to connect it with the Hamiltonian (1.4). To do that we have to find suitable sequences \((r_k), (\delta_k), (s_k), \) and \((M_k)\). They shall allow it to use theorem 4.1 repeatedly with

\[ r = r_k, \ r = r_k+1, \ \delta = \delta_k, \ s = s_k, \ s = s_{k+1}, \ \text{and} \ M = M_k. \]

At first we make sure that \( r_k, \delta_k, \) and \( s_k \) mesh correctly. We set

\[
\delta_k := q^k \delta_0, \ s_k := \delta_k \tau^{k+1}, \ r_k := \frac{3}{4} r + 8\delta_k \quad \forall \ k \in \mathbb{N}_0,
\]

where \( r \) is given in the assumptions of Theorem 1.6 \( \delta_0 \in (0, 1) \) is to be determined later, and

\[
q := \frac{1}{4}.
\]

(4.19) yields immediately

\[
\delta_{k+1} = q^{k+1} \delta_0 = q\delta_k \text{ and } s_{k+1} = \delta_{k+1} \tau^{k+1} = q^{k+1} s_k \quad \forall \ k \in \mathbb{N}_0.
\]
Lemma 4.3. The sequences \((r_k)_{k=0}^\infty\), \((\delta_k)_{k=0}^\infty\) and \((s_k)_{k=0}^\infty\) of (4.19) and (4.20) are decreasing and fulfill

\[
\begin{align*}
\frac{3}{4} r_k & > r_k, \quad 0 < \delta_k < \frac{r_k}{6}, \quad 0 < s_k \leq \delta^{\tau+1}_k \leq 1, \\
0 < r_{k+1} & \leq r_k - 6\delta_k, \quad 0 < s_{k+1} \leq \frac{s_k}{8} \quad \forall \ k \in \mathbb{N}_0.
\end{align*}
\]

Proof. That the sequences decrease and that \(r_k > 3r/4\) for all \(k \in \mathbb{N}_0\) is clear. We have

\[
\delta_k < \frac{8}{6} \delta_k < \frac{1}{6} \left( \frac{3}{4} r + 8\delta_k \right) = \frac{r_k}{6} \quad \forall \ k \in \mathbb{N}_0.
\]

The definition of \(s_k\) and \(\delta_k\) \((k \in \mathbb{N}_0)\) imply \(0 < s_k \leq \delta^{\tau+1}_k \leq 1\). It is \(r_{k+1} = 3r/4 + 8\delta_{k+1}\) and \(r_k - 6\delta_k = 3r/4 + 2\delta_k\). Therefore \(r_{k+1} \leq r_k - 6\delta_k\) holds if and only if

\[
8\delta_{k+1} \leq 2\delta_k \iff 4q^{k+1}\delta_0 \leq q^k\delta_0 \iff 4q \leq 1,
\]

which is indeed true according to (4.20). From \(\tau + 1 \geq 2\) we infer

\[
s_{k+1} = (q^{k+1}\delta_0)^{\tau+1} = q^{\tau+1}s_k \leq q^2s_k = \frac{s_k}{16} < \frac{s_k}{8}.
\]

The Lemma is proved. \(\square\)

For the inductive lemma it is required to have sequences of functions \((H_k)\), \((N_k)\), and \((R_k)\) which can be inserted for \(H\), \(N\), and \(R\), respectively, in the assumptions of theorem 4.1. Let us suppose there are normal forms \(N_\ell\) defined on \(D(r_\ell, s_\ell)\) \((0 \leq \ell \leq k+1, k \in \mathbb{N}_0)\), which meet (4.9) and let us suppose \(N_0\) fulfills something like (1.5), namely

\[
|N_{0yy} - C|_{D(r_0, s_0)} \leq \frac{1}{4|C^{-1}|}.
\]

Then

\[
|N_{k+1yy} - C|_{D(r_{k+1}, s_{k+1})} \leq \sum_{\ell=0}^k |N_{\ell+1yy} - N_{\ellyy}|_{D(r_{\ell+1}, s_{\ell+1})} + |N_{0yy} - C|_{D(r_0, s_0)}
\]

is a consequence. Having (4.2) in mind we therefore require

\[
\sum_{k=0}^\infty \frac{M_k}{s_k^2} \leq c_{17}, \quad c_{17} = \frac{1}{4c_{11}|C^{-1}|}.
\]

From (4.3) and (4.10) the requirements

\[
c_{15} \frac{M_k^2}{s_k^2} \leq M_{k+1} \quad \text{and} \quad M_k \leq c_{18} s_k^2 \quad \forall \ k \in \mathbb{N}_0, \ c_{18} = \frac{1}{16(c_7 + c_8)}
\]

(4.22)
follow. Observe, that $c_{17}$ and $c_{18}$ depend on $n$, $\tau$, $\gamma$, and $C$ only. In order to fulfill \((4.22)\) we choose

$$M_k := \frac{s_k^2}{c_{15}}, \quad t_k := t_0^{\mu^k} \equiv t_0(\mu^k) \quad \forall \quad k \in \mathbb{N}_0,$$

with some $t_0 \in (0, 1)$, and

$$\mu := \frac{3}{2}. \quad (4.24)$$

\((4.23)\) gives promptly

$$t_{k+1} = t_0^{\mu^{k+1}} = t_0^{\mu \cdot \mu^k} = t_k^\mu \quad \forall \quad k \in \mathbb{N}_0.$$

**Remark 4.4.** In formulas \((4.20)\) and \((4.24)\) any other value of $q \in (0, 1/4]$ and $\mu \in (1, 2)$ would have done it equally well.

The parameter $\mu$ may be interpreted as the speed of convergence. However, $\mu = 2$ is not possible. This is due to the small divisors (compare remarks 3.3 (page 9) and 4.2 (page 17)).

**Lemma 4.5.** The inequality $c_{15} \cdot c_{18} \geq 1$ holds.

**Proof.** We do the proof by tracing back the definition of $c_{15}$. At first, \((3.25)\) determines

$$c_{12} = 1 + \frac{4c_6|C|}{\gamma} \geq 1.$$

Using $n \geq 2$, $|C| |C^{-1}| \geq |C C^{-1}| = 1$, and \((3.32)\) we obtain

$$c_{13} = 1 + n c_{12} \left(1 + 4|C| |C^{-1}|\right) \geq 1 + 5n \geq 11.$$

Hence we have for $c_{10}$ (see definition \((3.36)\))

$$c_{10} = 1 + n^2 |C| (2c_7 + c_8) + c_{13} \geq 12.$$

The constant $c_{15}$ was defined in \((4.18)\), this yields

$$c_{15} = n(1 + c_{10})(4c_7 + c_8) \geq 26(c_7 + c_8).$$

Now we calculate

$$c_{15} \cdot c_{18} = \frac{c_{15}}{16(c_7 + c_8)} \geq \frac{26}{16} \geq 1,$$

and the lemma is proven. \qed

**Lemma 4.6.** Let $m > 1$ and $0 < t < 1$. Then the estimate

$$\sum_{k=0}^{\infty} t^{m^k} \leq \frac{t}{1 - t^{m-1}}$$

holds.
We define the constants in the assumptions of Theorem 1.6 as follows: $c$ and theorem 1.6 will be determined later.

Proof. Because of the equality
$$\frac{t}{1 - t^{m-1}} = t \sum_{k=0}^{\infty} (t^{m-1})^k$$
it is sufficient to prove
$$t (t^{m-1})^k \geq t^m \iff k(m - 1) + 1 \leq m^k = (1 + (m - 1))^k \quad \forall \ k \in \mathbb{N}_0.$$  
This amounts to Bernoulli’s inequality, which implies the assertion. \hfill \square

Lemma 4.7. There exists a constant $c_{19} = c_{19}(n, \tau, \gamma, C) > 0$, such that the sequence $(M_k)_{k=0}^{\infty}$ defined in $[4.23]$ satisfies the conditions $[4.21]$ and $[4.22]$ for all $t_0 \in (0, c_{19}]$. Moreover
$$\sum_{k=0}^{\infty} \frac{M_k}{s_k^2} \leq \frac{2}{c_{15}} t_0$$
holds.

Proof. By definition of the $t_k$ we see $t_{k+1} = t_k^\mu \ (k \in \mathbb{N}_0)$. We require $c_{19} \leq q^{(2r+2)/(2-\mu)}$, than $t_0 \leq q^{(2r+2)/(2-\mu)}$ follows. The sequence of the $t_k$ decreases, so $t_k \leq q^{(2r+2)/(2-\mu)}$ for all $k \in \mathbb{N}_0$. This means $t_k^{2-\mu} \leq q^{2r+2} \ (k \in \mathbb{N}_0)$. Furthermore we have
$$s_{k+1} = \delta_{k+1}^{\tau+1} = (q \cdot \delta_k)^{\tau+1} = q^{\tau+1} s_k \quad \forall \ k \in \mathbb{N}_0.$$
Hence we obtain
$$c_{15} \frac{M_k^2}{s_k^2} = \frac{1}{c_{15}} s_k^2 t_k^2 = \frac{1}{c_{15}} s_{k+1}^2 t_{k+1}^{2-\mu} t_k^\mu \leq \frac{1}{c_{15}} s_{k+1}^2 t_{k+1} = M_{k+1} \quad \forall \ k \in \mathbb{N}_0.$$
This is the first inequality $[4.22]$. The second one $[4.22]$ is equivalent to
$$t_k \leq c_{15} \cdot c_{18} \quad \forall \ k \in \mathbb{N}_0.$$
This in turn is a consequence of lemma 4.5. $[4.20]$ and $[4.24]$ imply $c_{19} < q = 1/4 = (1/2)^{1/(\mu-1)}$. Hence $t_0^{\mu-1} \leq c_{19}^{\mu-1} \leq 1/2$, and with $[4.23]$ and lemma 4.6 we get
$$\sum_{k=0}^{\infty} \frac{M_k}{s_k^2} = \frac{1}{c_{15}} \sum_{k=0}^{\infty} t_0^{\mu k} \leq \frac{1}{c_{15}} \frac{t_0}{1 - t_0^{\mu-1}} \leq \frac{2}{c_{15}} t_0,$$
which is formula $[4.25]$. Let us diminish $c_{19}$ by setting
$$c_{19} := \min \left\{ q^{2r+2}, \frac{c_{15} c_{17}}{2} \right\},$$
then $t_0 \leq c_{19} \leq c_{15} c_{17}/2$ and $[4.25]$ imply $[4.21]$. All assertions are shown. \hfill \square

We define the constants in the assumptions of Theorem 1.6 as follows:
$$c_1 := \min \left\{ c_{19}, \frac{c_{15}}{32 n^2 (c_7 + c_8) \exp(c_{14} c_{17})} \right\}, \quad c_2 := \frac{1}{32 (\tau+1) c_{15}}.$$  
To remind: So far we encountered the positive constants $c_6$ to $c_{19}$. The constants $c_1$ and $c_2$ were defined right now, and the constants $c_3$, $c_4$, and $c_5$ from the assertions of theorem 1.6 will be determined later.

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Lemma 4.8. Let \( r, s, M, \) and \( \vartheta \) be the constants from theorem 1.6 and set
\[
\delta_0 := \frac{1}{32} s^{\tau+1}, \quad t_0 := \vartheta.
\]
Then \( r_0, s_0 \) given by (4.19), and \( M_0 \) from (4.23) with \( k = 0 \), satisfy
\[
 r_0 \leq r, \quad s_0 \leq s, \quad M_0 \geq M.
\]

Proof. The fact \( s \leq r^{\tau+1} \) and the definition of \( \delta_0 \) show
\[
r_0 = \frac{3}{4} r + 8 \delta_0 \leq \frac{3}{4} r + \frac{1}{4} s^{\tau+1} \leq r.
\]
Furthermore
\[
s_0 = \delta_0^{\tau+1} = \frac{s}{32^{\tau+1}} < s
\]
follows. For the claim \( M_0 \geq M \) it is sufficient to prove \( M_0 \geq c_2 s^2 \vartheta \) because of \( M \leq c_2 s^2 \vartheta \). We have
\[
c_2 s^2 \vartheta \leq \frac{1}{32^{2(\tau+1)} c_{15}} s^2 \vartheta = \frac{1}{c_{15}} \left( \frac{s^{\tau+1}}{32} \right)^{2(\tau+1)} \cdot \vartheta = \frac{1}{c_{15}} \delta_0^{2(\tau+1)} \vartheta = \frac{1}{c_{15}} s_0^2 t_0 = M_0,
\]
which proves the lemma. \( \square \)

Theorem 4.9. (inductive lemma) Under the assumptions of theorem 1.6 and with the sequences \( (r_k)_k \), \( (\delta_k)_k \), \( (s_k)_k \), and \( (M_k)_k \) fixed in (4.19), (4.20), (4.23), (4.24), and (4.27) the following holds for all \( k \in \mathbb{N}_0 \):
There exist simple canonical transformations
\[
Z_{k+1} : \mathcal{D}(r_{k+1}, s_{k+1}) \longrightarrow \mathcal{D}(r_k - 5 \delta_k, s_k/4), \quad Z_{k+1} - \text{id} \in \mathcal{P}_{2n}(r_{k+1}, s_{k+1}), \quad (4.28)
\]
such that the functions
\[
H_{k+1} := H_k \circ Z_{k+1} = H_0 \circ Z_1 \circ Z_2 \circ \ldots \circ Z_{k+1} \quad \text{with} \quad H_0 := H|_{\mathcal{D}(r_0, s_0)} \quad (4.29)
\]
are elements of the respective space \( \mathcal{P}(r_{k+1}, s_{k+1}) \) and can be written as \( H_{k+1} = N_{k+1} + R_{k+1} \) with \( N_{k+1}, R_{k+1} \in \mathcal{P}(r_{k+1}, s_{k+1}) \), and
\[
N_{k+1}(\xi, \eta) = a_{k+1} + \langle \omega, \eta \rangle + \mathcal{O}(\eta^2), \quad a_{k+1} \in \mathbb{R}. \quad (4.30)
\]
The following estimates hold for all \( k \in \mathbb{N}_0 \):
\[
|Z_{k+1, \xi}|_{\mathcal{D}(r_{k+1}, s_{k+1})} \leq \exp \left( \frac{c_{14} M_k}{s_k^2} \right), \quad (4.31)
\]
\[
|Z_{k+1, \xi} - E_{2n}|_{\mathcal{D}(r_{k+1}, s_{k+1})} \leq c_{14} \frac{M_k}{s_k^2} \exp \left( \frac{c_{14} M_k}{s_k^2} \right), \quad (4.32)
\]
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In theorem 4.1 we use
\[ |a_{k+1} - a_k| \leq \bar{c}_9 \frac{M_k}{s_k}, \]  
(4.33)
\[ |N_{k+1,\eta} - N_{k,\eta}|_{D(r_{k+1},s_{k+1})} \leq c_{11} \frac{M_k}{s_k^2} \left( N_0 := (H - R)|_{D(r_0,s_0)} \right), \]  
(4.34)
\[ |R_{k+1}|_{D(r_{k+1},s_{k+1})} \leq c_{15} \frac{M_k^2}{s_k^2}. \]  
(4.35)
Herein the constants \( \bar{c}_9 \) and \( c_{11} \) are given by Theorem 3.4, \( c_{14} \) and \( c_{15} \) by Theorem 4.1. Moreover \( W_{k+1} := Z_1 \circ \ldots \circ Z_{k+1} \) fulfills
\[ |W_{k+1}|_{D(r_{k+1},s_{k+1})} \leq \exp \left( c_{14} \sum_{\ell=0}^{k} \frac{M_{\ell}}{s_{\ell}^2} \right) \quad \forall \ k \in \mathbb{N}_0, \]  
(4.36)
and \( \Delta W_{k+1} := W_{k+1} - W_k \) (\( k \in \mathbb{N} \)), \( \Delta W_1 := W_1 - \text{id} \) satisfies
\[ |\Delta W_{k+1}|_{D(r_{k+1},s_{k+1})} \leq c_{20} \frac{M_k}{s_k \delta_k^2} \quad \forall \ k \in \mathbb{N}_0, \]  
(4.37)
where \( c_{20} = c_{20}(n, \tau, \gamma, C) \) is a positive constant.

**Proof.** Clearly the proof is to be done by repeated use of theorem 4.1. Lemma 4.8 shows \( D(r_0,s_0) \subseteq D(r,s) \). So \( H_0 \) can be defined as the restriction of the function
\[ H(x,y) = a + \langle \omega, y \rangle + \frac{1}{2} \langle y \cdot Q(x), y \rangle + R(x, y) \]
of (1.4) to \( D(r_0,s_0) \). We set \( a_0 := a \) and \( R_0 := R|_{D(r_0,s_0)} \) with \( a \) and \( R \) from (1.4). To summarize, we start the induction in accordance with (4.29) and (4.34) with
\[ H_0 = H|_{D(r_0,s_0)}, \ R_0 = R|_{D(r_0,s_0)}, \ a_0 = a \text{ and } N_0 = (H - R)|_{D(r_0,s_0)}, \]  
(4.38)
where \( H, R \) and \( a \) are given by (1.4).
We check the assumptions of theorem 4.1. The assumptions on the constants \( r, \delta, s, r_+ \) and \( s_+ \) are fulfilled by lemma 4.3. Apply the lemma for \( k = 0 \) and
\[ r = r_0, \ \delta = \delta_0, \ s = s_0, \ r_+ = r_1, \ s_+ = s_1. \]
In theorem 4.1 we use
\[ H = H_0, \ N = N_0 = H_0 - R_0, \ R = R_0 \text{ and } M = M_0 \]
with \( H_0, N_0, R_0 \) from (4.38) and \( M_0 \) from (4.28) for \( k = 0 \).
Then the function \( N \) of (4.1) has the form
\[ N(x,y) = a_0 + \langle \omega, y \rangle + \frac{1}{2} \langle y \cdot Q(x), y \rangle \quad \forall \ (x,y) \in D(r_0,s_0) \]
because of (1.4). So (1.5) implies (4.2). Lemma 4.8 and (1.6) show
\[ |R_0|_{D(r_0,s_0)} = |R|_{D(r_0,s_0)} \leq |R|_{D(r,s)} = M \leq M_0. \]
Moreover by the inequality (4.22), which holds according to Lemma 4.7, we have

\[ M_0 \leq c_{18} s_0^2 = \frac{1}{16(c_7 + c_8)} s_0^2. \]

Hence assumption (4.3) is met and we may apply theorem 4.1. It yields a transformation \( Z \) and a function \( H_+ \), as well as \( a_+ \), \( N_+ \), and \( R_+ \). Now we set

\[ Z_1 := Z, \quad H_1 := H_+ \in \mathcal{P}(r_1, s_1), \quad a_1 := a_+ \in \mathbb{R}, \]

\[ N_1 := N_+ \in \mathcal{P}(r_1, s_1) \text{ and } R_1 := R_+ \in \mathcal{P}(r_1, s_1). \]

Then assertions (4.28) to (4.35) follow for \( k = 0 \). In case \( k = 0 \) (4.36) is equivalent to (4.31) because of \( W_1 = Z_1 \). Hence (4.36) holds. To prove (4.37) for \( k = 0 \) we consider

\[ \Delta W_1 = Z_1 - \text{id} = \text{id} \circ Z_1 - \text{id}. \]

So let us put \( W = \text{id} \) and \( K_1 = 1 \) in theorem 4.1, then we obtain with (4.11)

\[ |\Delta W_1|_{\mathcal{D}(r_1, s_1)} \leq n(c_7 + c_8) \frac{M_0}{s_0 \delta_0}. \]

We define

\[ c_{20} := n(c_7 + c_8) \exp(c_{14} c_{17}), \quad (4.39) \]

then (4.37) holds for \( k = 0 \). (The reason for the factor \( \exp(c_{14} c_{17}) \) will become clear at the end of the proof.)

Now suppose the inductive Lemma is true for all \( \ell \), \( 0 \leq \ell \leq k - 1 \in \mathbb{N}_0 \). We want to apply theorem 4.1 with

\[ r = r_k, \quad \delta = \delta_k, \quad s = s_k, \quad r_+ = r_{k+1}, \quad s_+ = s_{k+1}. \]

Lemma 4.3 says that the assumptions on these constants are fulfilled. Next we have to put

\[ H = H_k, \quad a = a_k, \quad N = H_k - R_k, \quad \text{and } R = R_k. \]

By lemma 4.7 formula (4.21) holds, namely

\[ \sum_{k=0}^{\infty} \frac{M_k}{s_k^2} \leq \frac{1}{4c_{11} |C^{-1}|}. \]

Using (4.34) up to \( k - 1 \) we get

\[ |N_{k\eta \eta} - C|_{\mathcal{D}(r_k, s_k)} \leq \sum_{\ell=0}^{k-1} |N_{\ell+1\eta \eta} - N_{\ell\eta \eta}|_{\mathcal{D}(r_{\ell+1}, s_{\ell+1})} + |N_{0\eta \eta} - C|_{\mathcal{D}(r_0, s_0)} \]

\[ \leq c_{11} \frac{1}{4c_{11} |C^{-1}|} + \frac{1}{4|C^{-1}|} = \frac{1}{2|C^{-1}|}. \]
So assumption (4.2) is satisfied. (4.22) holds because of lemma 4.7, in particular we have
\[
\frac{M_{k-1}^2}{s_{k-1}^2} \leq M_k \text{ and } M_k \leq \frac{1}{16(c_7 + c_8)} s_k^2.
\]

Hence (4.35) for \( k - 1 \) shows
\[
|R_k|_{\mathcal{D}(r_k, s_k)} \leq c_{15} \frac{M_{k-1}^2}{s_{k-1}^2} \leq \frac{1}{16(c_7 + c_8)} s_k^2,
\]
this is assumption (4.3). Theorem 4.1 can be applied and yields a transformation \( Z \) and a function \( H+ \), as well as \( a+ \), \( N+ \), and \( R+ \). Now we set
\[
Z_{k+1} := Z, \quad H_{k+1} := H+ \in \mathcal{P}(r_{k+1}, s_{k+1}), \quad a_{k+1} := a_+ \in \mathbb{R},
\]
\[
N_{k+1} := N_+ \in \mathcal{P}(r_{k+1}, s_{k+1}), \quad R_{k+1} := R_+ \in \mathcal{P}(r_{k+1}, s_{k+1}).
\]

Assertions (4.28) to (4.35) follow for the index \( k \). To prove (4.36) we calculate
\[
W_{k+1, \zeta} = Z_{1, \zeta} (Z_2 \circ \cdots \circ Z_{k+1}) \cdot Z_{2, \zeta} (Z_3 \circ \cdots \circ Z_{k+1}) \cdots Z_{k+1, \zeta}.
\]

Formula (4.31) up to \( k \) implies
\[
|W_{k+1, \zeta}|_{\mathcal{D}(r_{k+1}, s_{k+1})} \leq |Z_{1, \zeta}|_{\mathcal{D}(r_1, s_1)} \cdot |Z_{2, \zeta}|_{\mathcal{D}(r_2, s_2)} \cdots |Z_{k+1, \zeta}|_{\mathcal{D}(r_{k+1}, s_{k+1})} \leq \prod_{\ell=0}^k \exp \left( c_{14} \frac{M_\ell}{s_\ell^2} \right) = \exp \left( c_{14} \sum_{\ell=0}^k \frac{M_\ell}{s_\ell^2} \right),
\]
so (4.36) is shown for the index \( k \). Furthermore (4.36) for \( k - 1 \) and (4.21), which holds by Lemma 4.7, give the estimate
\[
|W_k|_{\mathcal{D}(r_k, s_k)} \leq \exp \left( c_{14} \sum_{\ell=0}^{k-1} \frac{M_\ell}{s_\ell^2} \right) \leq \exp(c_{14} c_{17}).
\]

Therefore we can insert \( K = \exp(c_{14} c_{17}) \) in formula (4.11) and (4.37) follows for the index \( k \). Altogether the inductive lemma is proved.

5 Convergence of the iterative process

In this section we complete the proof of theorem 1.6. Henceforth we work with the general assumption:

\[
\text{Let the assumptions of theorem 1.6 be fulfilled. Let the sequences } (r_k)_{k=0}^\infty, \quad (\delta_k)_{k=0}^\infty, \quad (s_k)_{k=0}^\infty, \quad \text{and } (M_k)_{k=0}^\infty \text{ be defined according to (4.19), (4.20), (4.23), (4.24), and (4.27).}
\]

Especially lemmas 4.3, 4.7 and 4.8 and the inductive lemma 4.9 hold under this general assumption.

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Convergence of the symplectic transformations

**Theorem 5.1.** The maps

\[ W_k = Z_k \circ \ldots \circ Z_1 \quad (k \in \mathbb{N}) \]

provided by theorem 4.3 are simple canonical transformations. \( W_k - \text{id} \in \mathcal{P}_{2n}(r_k, s_k) \) holds.

**Proof.** The maps \( W_k \) are well-defined, for \( Z_{k+1} \) lies in the domain of \( Z_k \) for all \( k \in \mathbb{N} \) by (4.28). The \( W_k \) are simple canonical transformations. Moreover \( W_k - \text{id} \in \mathcal{P}_{2n}(r_k, s_k) \) holds for all \( k \in \mathbb{N} \). \( \square \)

Simple canonical transformations are affine-linear in \( \eta \), so they can always be defined for all \( \eta \in \mathbb{C}^n \). More precisely, if \( W_k = (U_k, V_k) \) is defined on \( \mathcal{D}(r_k, s_k) \) by

\[ W_k(\xi, \eta) = (U_k(\xi), V_k(\xi, 0) + \eta \cdot U_k(\xi)^{-1}) \quad \forall \ (\xi, \eta) \in \mathcal{D}(r_k, s_k), \]

as it is seen in theorem A.9, then there exists a simple canonical transformation \( \tilde{W}_k \) defined on \( \mathcal{S}(r_k) \times \mathbb{C}^n \) with \( \tilde{W}_k \big|_{\mathcal{D}(r_k, s_k)} = W_k \). The equation

\[ \tilde{W}_k(\xi, \eta) = (U_k(\xi), V_k(\xi, 0) + \eta \cdot U_k(\xi)^{-1}) \quad \forall \ (\xi, \eta) \in \mathcal{S}(r_k) \times \mathbb{C}^n \]

holds. Comparing (5.1) and (5.2) we notice that \( \tilde{W}_k(\cdot, 0) = W_k(\cdot, 0) \). When we write \( \tilde{W}_k = (\tilde{U}_k, \tilde{V}_k) \), we have \( \tilde{U}_k = U_k \) and \( \tilde{V}_{k\eta} = V_{k\eta} \). We will use this in the sequel.

**Theorem 5.2.** There exists a subsequence \( \left( \tilde{W}_{k_\ell} \right)_{\ell=1}^\infty \) which converges uniformly on compact subsets of \( \mathcal{S}(3r/4) \times \mathbb{C}^n \) to a simple canonical transformation \( W_\infty \) with \( W_\infty - \text{id} \in \mathcal{P}_{2n}(3r/4, s) \).

**Proof.** It is \( r_k > 3r/4 \) for all \( k \in \mathbb{N} \) by (4.19). Therefore all maps \( \tilde{W}_k \) are defined for \( \zeta \in \mathcal{S}(3r/4) \times \mathbb{C}^n \). Looking at the assumptions of theorem A.11 we calculate with (4.37), \( s_k \leq \delta_k \) (by lemma 4.3), and (4.21)

\[
\sum_{k=0}^{\infty} |W_{k+1} - W_k|_{\mathcal{S}(3r/4) \times \{0\}} \leq \sum_{k=0}^{\infty} |\Delta W_{k+1}|_{\mathcal{D}(r_{k+1}, s_{k+1})} \\
\leq \sum_{k=0}^{\infty} c_{20} \frac{M_k}{s_k \delta_k^2} \leq c_{20} \sum_{k=0}^{\infty} \frac{M_k}{s_k^2} \leq c_{17} c_{20}.
\]

This means, that the functions \( \tilde{W}_k(\cdot, 0) = W_k(\cdot, 0) \) converge uniformly on \( \mathcal{S}(3r/4) \), in particular they converge uniformly on compact subsets. We use the row-sum norm, so (4.36) and (4.21) show

\[ |V_{k\eta}|_{\mathcal{S}(3r/4)} \leq |W_{k\eta}|_{\mathcal{S}(3r/4) \times \{0\}} \leq \exp(c_{14} c_{17}) \quad \forall \ k \in \mathbb{N}. \]

Hence the theorem of Montel (see [9], theorem 1.6) tells us that there exists a subsequence \( (V_{k_\ell, \eta})_{\ell=1}^\infty \) which converges uniformly on compact subsets of \( \mathcal{S}(3r/4) \). Let us set

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\( \left( \tilde{W}_{k_\ell} \right)_ {\ell=1}^{\infty} \) and \( U = \mathcal{S}(3r/4) \) in the assumptions of theorem A.11. Then the theorem may be applied and predicates, that the sequence \( \left( \tilde{W}_{k_\ell} \right)_ {\ell=1}^{\infty} \) converges uniformly on compact subsets of \( \mathcal{S}(3r/4) \times \mathbb{C}^n \) against a simple canonical transformation

\[
W_\infty = (U_\infty, V_\infty) : \mathcal{S}(3r/4) \times \mathbb{C}^n \rightarrow \mathbb{C}^{2n}.
\]

The functions \( \tilde{W}_{k_\ell} \) map real vectors to real values and the \( \tilde{W}_{k_\ell} - \text{id} \) are \( 2\pi \)-periodic by (5.1), (5.2), and theorem 5.1. Therefore we obtain \( W_\infty - \text{id} \in \mathcal{P}_{2n}(3r/4, s) \) and the proof is finished.

\[ \text{Theorem 5.3.} \]

The function \( W_\infty \) of theorem 5.2 fulfills

\[
W_\infty(\zeta) \in \mathcal{D}(r, s) \quad \forall \quad \zeta \in \mathcal{D}(r/2, 5s/8). \tag{5.3}
\]

The restriction

\[
W = (U, V) := W_\infty|_{\mathcal{D}(r/2, s/2)}
\]

is a simple canonical transformation with

\[
W : \mathcal{D}(r/2, s/2) \rightarrow \mathcal{D}(r, s), \quad W - \text{id} \in \mathcal{P}_{2n}(r/2, s/2).
\]

There exists a positive constant \( c_3 \), which depends on \( n, \tau, \gamma, \) and \( C \) only, such that

\[
|W_\zeta - E_{2n}|_{\mathcal{D}(r/2, s/2)} \leq c_3 \vartheta.
\]

\[ \text{Proof.} \]

The definition of \( W \) and theorem 5.2 show that \( W - \text{id} \in \mathcal{P}_{2n}(r/2, s/2) \) and that \( W \) is a simple canonical transformation. By the definition in theorem 4.9 we have

\[
W_k = Z_1 \circ \ldots \circ Z_k \quad (k \in \mathbb{N}).
\]

Let us write \( W_k = (U_k, V_k) \). The functions \( Z_k = (X_k, Y_k) \) are simple canonical transformations, so

\[
U_k = U_k(\xi) = X_1 \circ \ldots \circ X_k(\xi).
\]

In particular the functions \( U_k \) map to \( \mathcal{S}(r_0 - 5\delta_0) \) by (4.28). The function \( U \) is the limit of a subsequence of the \( U_k \). Hence \( U \) is defined on \( \mathcal{S}(r/2) \) and maps to \( \mathcal{S}(r_0 - 4\delta_0) \). Because of lemma 4.8, \( r_0 \leq r \), so \( \mathcal{S}(r_0 - 4\delta_0) \subseteq \mathcal{S}(r) \), and consequently

\[
U : \mathcal{S}(r/2) \rightarrow \mathcal{S}(r).
\]

By definition of \( W \) we have \( U = U_\infty|_{\mathcal{S}(r/2)} \). This implies

\[
U_\infty(\xi) \in \mathcal{S}(r) \quad \forall \quad \xi \in \mathcal{S}(r/2).
\]

Next (notice (5.3)) we have to prove

\[
|V_\infty(\xi, \eta)| < s \quad \forall \quad (\xi, \eta) \in \mathcal{D}(r/2, 5s/8).
\]

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To that end we observe for \((\xi, \eta) \in \mathcal{D}(3r/4, 5s/8)\)
\[
V_\infty(\xi, \eta) = V_\infty(\xi, 0) + \eta U_\infty(\xi)^{-1} = V_\infty(\xi, 0) + \eta \left( U_\infty(\xi)^{-1} - E_n \right). \tag{5.4}
\]

We consider \(V(\cdot, 0)\). Each \(W_k (k \in \mathbb{N})\) maps \((\xi, 0) \in \mathcal{S}(r_k) \times \{0\}\) to \(\mathcal{D}(r_0 - 5\delta_0, s_0/4)\), for this is true for \(Z_1\). Therefore \(|V_k(\cdot, 0)|_{\mathcal{S}(r_k)} < s_0/4\) holds for all \(k \in \mathbb{N}\). This implies \(|V(\cdot, 0)|_{\mathcal{S}(r/2)} \leq s_0/4\), and with \(s_0 \leq s\) (by lemma 4.8) we obtain
\[
|V(\xi, 0)| \leq \frac{s}{4} \quad \forall \quad \xi \in \mathcal{S}(r/2). \tag{5.5}
\]

We need an estimate for \(U_\xi^{-1} - E_n\). It can be found with lemma A.1. Thereto we search for an inequality for \(U_\xi - E_n\). We have for all \(k \in \mathbb{N}\) and all \(\zeta \in \mathcal{D}(r_k, s_k)\)
\[
W_k(\zeta) - \zeta = \Delta W_1(\zeta) + \ldots + \Delta W_k(\zeta). \tag{5.6}
\]

\((4.37)\) and Cauchy’s estimate show for \(k \in \mathbb{N}_0\)
\[
|\Delta W_{k+1, \zeta}|_{\mathcal{D}(r_{k+1} - \delta_k, s_{k+1})} \leq c_{20} \frac{M_k}{s_k \delta_k} \leq c_{20} \frac{M_k}{s_k^2}.
\]

By \((4.19)\) and \((4.20)\) we see
\[
r_{k+1} - \delta_k = \frac{3r}{4} + 8\delta_{k+1} - \delta_k = \frac{3r}{4} + 8q\delta_k - \delta_k = \frac{3r}{4} + \delta_k > \frac{3r}{4} \quad \forall \quad k \in \mathbb{N}_0.
\]

So \((4.25)\) and \((5.6)\) yield the estimate
\[
\left| W_k - \begin{pmatrix} E_n \\ 0 \end{pmatrix} \right|_{\mathcal{S}(3r/4) \times \{0\}} \leq \sum_{\ell=0}^{\infty} |\Delta W_{\ell+1, \zeta}|_{\mathcal{D}(r_{\ell+1} - \delta_{\ell+1})} \leq \frac{2c_{20}}{c_{15}} t_0. \tag{5.7}
\]

Let us write \(\Delta W_k = (\Delta U_k, \Delta V_k)\). Then in particular
\[
|U_k - E_n|_{\mathcal{S}(3r/4)} \leq \sum_{\ell=0}^{\infty} |\Delta U_{\ell+1, \xi}|_{\mathcal{S}(r_{\ell+1} - \delta_{\ell+1})} \leq \frac{2c_{20}}{c_{15}} t_0
\]
follows (note that we use the row-sum norm). When we have a look at \((4.20)\) and \((4.39)\), we see
\[
c_1 \leq \frac{c_{15}}{32nc_{20}}.
\]

It is \(t_0 = \vartheta\) by \((4.27)\) and \(\vartheta \leq c_1\) by assumption of theorem 1.6 so
\[
|U_k - E_n|_{\mathcal{S}(3r/4)} \leq \frac{2c_{20}}{c_{15}} \vartheta \leq \frac{1}{16n} \leq \frac{1}{16n} \quad \forall \quad k \in \mathbb{N}. \tag{5.8}
\]

Now we can apply lemma A.1. Thereto we have to put \(S = E_n, P = U_k(\xi) (\xi \in \mathcal{S}(3r/4))\) and \(h = 2c_{20}\vartheta/c_{15}\). The lemma says that \(U_k(\xi)^{-1}\) satisfies the estimate
\[
|U_k(\xi)^{-1} - E_n| \leq \frac{2c_{20}}{c_{15}} \vartheta \frac{1}{1 - \frac{1}{16}} = \frac{16}{15} \frac{2c_{20}}{c_{15}} \vartheta \leq \frac{1}{15n} \quad \forall \quad \xi \in \mathcal{S}(3r/4). \tag{5.9}
\]
This implies
\[ |U^{-1}_{\infty \xi} - E_n|_{S(3r/4)} \leq \frac{1}{15n} \quad \text{and} \quad |U^{-1}_\xi - E_n|_{S(r/2)} \leq \frac{1}{15n}, \]
which in turn together with (5.4) and (5.5) leads to
\[ |V_\infty(\xi, \eta)| < \frac{s}{4} + \frac{5s}{8} + \frac{5s}{8} n \frac{1}{15n} = \frac{30 + 75 + 5}{120} s < s \quad \forall \ (\xi, \eta) \in \mathcal{D}(r/2, 5s/8). \]
We obtain
\[ W_\infty(\xi, \eta) \in \mathcal{D}(r, s) \quad \forall \ (\xi, \eta) \in \mathcal{D}(r/2, 5s/8), \]
as well as
\[ W : \mathcal{D}(r/2, s/2) \longrightarrow \mathcal{D}(r, s). \]
In order to find an inequality for \( |W_\xi - E_{2n}| \) we observe
\[ W_\xi - E_{2n} = \begin{pmatrix} U_\xi - E_n & 0 \\ V_\xi & (U^{-1}_\xi)^T - E_n \end{pmatrix}. \]
(5.8) gives
\[ |U_\xi - E_n|_{S(r/2)} \leq \frac{2c_{20}}{c_{15}} \vartheta, \quad (5.10) \]
and (5.9) shows
\[ \left| (U^{-1}_\xi)^T - E_n \right|_{S(r/2)} \leq n \left| U^{-1}_\xi - E_n \right|_{S(r/2)} \leq n \frac{16c_{20}}{15c_{15}} \vartheta < 3n \frac{c_{20}}{c_{15}} \vartheta. \quad (5.11) \]
Let’s turn to \( V_\xi \). By definition \( V = V_\infty|_{\mathcal{D}(r/2, s/2)} \) holds, and
\[ V_\infty(\xi, \eta) = V_\infty(\xi, 0) + (V_\infty(\xi, \eta) - V_\infty(\xi, 0)) \quad \forall \ (\xi, \eta) \in \mathcal{D}(3r/4, s). \]
Hence with (5.4) we obtain
\[ V_\infty(\xi, \eta) = V_\infty(\xi, 0) + \frac{\partial}{\partial \xi} (V_\infty(\xi, \eta) - V_\infty(\xi, 0) - \eta) \]
\[ = V_\infty(\xi, 0) + \frac{\partial}{\partial \xi} \left( \eta (U^{-1}_\xi(\eta) - E_n) \right) \quad \forall \ (\xi, \eta) \in \mathcal{D}(3r/4, s). \quad (5.12) \]
From (5.7) it follows with \( t_0 = \vartheta \)
\[ |V_{k\xi}(\cdot, 0)|_{S(3r/4)} \leq 2 \frac{c_{20}}{c_{15}} \vartheta \quad \forall \ k \in \mathbb{N}. \]
This inequality holds for the limit $V_\infty$ as well and hence for $V$ giving

$$|V_\xi(\cdot, 0)|_{S(r/2)} \leq \frac{2c_{20}}{c_{15}} \vartheta.$$ 

To get the second summand of (5.12) under control we define

$$u : \mathcal{D}(3r/4, s) \rightarrow \mathbb{C}^n, \quad (\xi, \eta) \mapsto u(\xi, \eta) = \eta (U_\infty(\xi)^{-1} - E_n).$$

From (5.9) we see

$$|u_{\xi}|_{\mathcal{D}(3r/4, s)} \leq sn |U_\infty^{-1} - E_n|_{S(3r/4)} \leq 3n \frac{c_{20}}{c_{15}} s \vartheta.$$ 

Hence Cauchy’s estimate and $s \leq r^{\tau+1} \leq r$ show

$$|u_{\xi}|_{\mathcal{D}(r/2, s)} \leq 3n \frac{c_{20}}{c_{15}} \frac{4s}{r} \vartheta \leq 12n \frac{c_{20}}{c_{15}} \vartheta.$$ 

Therefore we can conclude with (5.12) that

$$|V_\xi|_{\mathcal{D}(r/2, s/2)} \leq |V_\xi(\cdot, 0)|_{S(r/2)} + |u_{\xi}|_{\mathcal{D}(r/2, s)} \leq 2 \frac{c_{20}}{c_{15}} \vartheta + 12n \frac{c_{20}}{c_{15}} \vartheta \leq 13n \frac{c_{20}}{c_{15}} \vartheta.$$ 

For matrices we use the row-sum norm, so this estimate, (5.10), and (5.11) yield

$$|W_\xi - E_{2n}|_{\mathcal{D}(r/2, s/2)} \leq \left| \begin{pmatrix} U_\xi - E_n & 0 \\ V_\xi & (U_\xi^{-1})^T - E_n \end{pmatrix} \right|_{\mathcal{D}(r/2, s/2)} \leq \max \left\{ |U_\xi - E_n|_{\mathcal{D}(r/2, s/2)} , |V_\xi|_{\mathcal{D}(r/2, s/2)} + \left| (U_\xi^{-1})^T - E_n \right|_{\mathcal{D}(r/2, s/2)} \right\} \leq (3n + 13n) \frac{c_{20}}{c_{15}} \vartheta = c_3 \vartheta,$$

where

$$c_3 = 16n \frac{c_{20}}{c_{15}}$$

is a positive constant. The theorem is proved.

Proof of the properties of the transformed Hamiltonian

**Theorem 5.4.** The functions $R_k (k \in \mathbb{N})$ provided by theorem 4.9 fulfill

$$|R_k|_{S(r/2) \times \{0\}} \rightarrow 0, \quad |R_k\eta|_{S(r/2) \times \{0\}} \rightarrow 0 \quad \text{and} \quad |R_k\eta\eta|_{S(r/2) \times \{0\}} \rightarrow 0 \quad (k \rightarrow \infty).$$
Proof. The estimates (4.22) and (4.35) imply
\[
|R_k|_{D(r_k, s_k)} \leq c_{15} \frac{M_k - 1}{s_k - 1}^2 \leq M_k \quad \forall \ k \in \mathbb{N}.
\]
From this we conclude with Cauchy's estimates
\[
|R_{k\eta}|_{D(r_k, s_k/2)} \leq \frac{2M_k}{s_k}, \quad |R_{k\eta\eta}|_{D(r_k, s_k/4)} \leq \frac{8M_k}{s_k^2} \quad \forall \ k \in \mathbb{N}.
\]
The series \( \sum_{k=0}^{\infty} \frac{M_k}{s_k^2} \) is convergent, hence the sequences \( (M_k)_{k=0}^{\infty}, (2M_k/s_k)_{k=0}^{\infty} \), and \( (8M_k/s_k^2)_{k=0}^{\infty} \) tend to zero. This proves the theorem. \( \square \)

Theorem 5.5. Let \( H \) be the function of theorem 1.6. Then there exists a number \( a_+ \in \mathbb{R} \) and a function \( Q_+ \in \mathcal{P}_{n \times n}(r/2) \), such that the Taylor expansion of \( H \circ W : D(r/2, s/2) \rightarrow \mathbb{C} \) is given by
\[
H \circ W(\xi, \eta) = a_+ + \langle \omega , \eta \rangle + \frac{1}{2} \langle \eta \cdot Q_+(\xi) , \eta \rangle + O(|\eta|^3). \quad (5.13)
\]

Proof. By theorem 4.9 we have
\[
H_k = H \circ W_k = N_k + R_k \quad \forall \ k \in \mathbb{N}. \quad (5.14)
\]
So
\[
H \circ W(\xi, 0) = \lim_{\ell \to \infty} H \circ W_{k\ell}(\xi, 0) = \lim_{\ell \to \infty} (N_{k\ell}(\xi, 0) + R_{k\ell}(\xi, 0))
\]
holds for all \( \xi \in S(r/2) \). The sequence \( R_{k\ell}(\xi, 0) \) has the limit zero as we have seen in the theorem above. The sequence \( N_{k\ell}(\xi, 0) = a_{k\ell} \) is convergent because of (4.33), we call its limit
\[
a_+ := \lim_{\ell \to \infty} a_{k\ell}
\]
The number \( a_+ \) is a limit of real numbers, so it is a real number as well. We have
\[
H \circ W(\xi, 0) = a_+ \quad \forall \ \xi \in S(r/2).
\]
Moreover we obtain for all \( \xi \in S(r/2) \) by (5.14)
\[
(H \circ W)_\eta(\xi, 0) = H_z(W(\xi, 0)) \cdot W_\eta(\xi, 0) = \lim_{\ell \to \infty} H_z(W_{k\ell}(\xi, 0))W_{k\ell,\eta}(\xi, 0) = \lim_{\ell \to \infty} (N_{k\ell,\eta}(\xi, 0) + R_{k\ell,\eta}(\xi, 0)) = \omega.
\]
Now, the derivatives \( N_{k\ell,\eta} \) converge on \( S(r/2) \times \{0\} \) by (4.34) and we obtain a limit
\[
Q_+(\xi) := \lim_{\ell \to \infty} N_{k\ell,\eta}(\xi, 0) \quad \forall \ \xi \in S(r/2)
\]
This convergence is uniformly on \( S(r/2) \) and all functions \( N_{k\ell,\eta}(\cdot, 0) \) are elements of \( \mathcal{P}_{n \times n}(r/2) \), so \( Q_+ \in \mathcal{P}_{n \times n}(r/2) \). Theorem 5.2 implies
\[
W_{k\ell}(\cdot, 0) \longrightarrow W(\cdot, 0) \quad \text{uniformly on compact subsets of } S(r/2).
\]
Hence we conclude using the continuity of $W(\cdot,0)$ and $H$

$$H \circ W_{k}(\cdot,0) \longrightarrow H \circ W(\cdot,0)$$
uniformly on compact subsets of $S(r/2)$.

Hence (5.14) and the theorem of Weierstrass (see [4], (9.12.1)) show for all $\xi \in S(r/2)$

$$(H \circ W)_{\eta}(\xi,0) = \lim_{\ell \to \infty} (H \circ W_{k_{\ell}})_{\eta}(\xi,0) = \lim_{\ell \to \infty} (N_{k_{\ell},\eta}(\xi,0) + R_{k_{\ell},\eta}(\xi,0)) = Q_{+}(\xi),$$

which proves (5.13).

\[ \square \]

**Theorem 5.6.** There exists a constant $c_{4} = c_{4}(n, \tau, \gamma, C) > 0$, such that the function $Q_{+}$ meets inequality (1.9), namely

$$|Q_{+} - Q|_{S(r/2)} \leq c_{4} \vartheta.$$

**Proof.** With (4.34), (4.25), $t_{0} = \vartheta$, and the fact that $N_{0,\eta}(\xi,0) = Q(\xi)$ holds for all $\xi \in S(r/2)$ by definition of $N_{0}$ in theorem 4.9, we conclude that

$$|Q_{+} - Q|_{S(r/2)} \leq \sum_{k=0}^{\infty} c_{11} \frac{M_{k}}{s_{k}^{2}} \leq \frac{2c_{11}}{c_{15}} \vartheta.$$

So, with the definition

$$c_{4} = \frac{2c_{11}}{c_{15}},$$

(1.9) is shown.

\[ \square \]

**Theorem 5.7.** There exists a number $c_{5} = 512/25 > 0$, such that the function

$$R^{*}(\xi, \eta) := (H \circ W)(\xi, \eta) - \left( a_{+} + \langle \omega, \eta \rangle + \frac{1}{2} \langle \eta \cdot Q_{+}(\xi), \eta \rangle \right), \quad (5.15)$$

defined for all $(\xi, \eta) \in D(r/2, s/2)$, fulfills estimate (1.10).

**Proof.** At first we observe that $H \circ W_{\infty}(\xi, \eta)$ can be defined for all $(\xi, \eta) \in D(r/2, 5s/8)$
by theorem 5.3. This gives an analytic continuation of $H \circ W$ to the domain
$D(r/2, 5s/8)$. We call it $H^{**}$. Therefore we can enlarge definition (5.13) to $D(r/2, 5s/8)$
and obtain an analytic continuation $R^{**}$ of $R^{*}$. Clearly (1.10) is equivalent to

$$|R^{**}(\xi, \eta)| \leq c_{6} M \frac{|\eta|^{3}}{s^{3}} \quad \text{for all } (\xi, \eta) \in D(r/2, s/2),$$

which will be shown in the following. The derivatives with respect to $\eta$ of $H \circ W$ and
$H^{**}$ coincide for all $(\xi, 0) \in S(r/2) \times \{0\}$. So $R^{**}(\xi, \eta) = O(|\eta|^{3})$ holds by theorem 5.5.

Moreover $R^{**}$ is an analytic function. We fix an arbitrary $\xi \in S(r/2)$, set $N := H - R$, and consider

$$H^{**}(\xi, \eta) = H \circ W_{\infty}(\xi, \eta) = N \circ W_{\infty}(\xi, \eta) + R \circ W_{\infty}(\xi, \eta) \quad (|\eta| < 5s/8).$$
Well, $W(\xi, \eta)$ is a polynomial of degree one in $\eta$ and $N$ is, by (1.4), a polynomial of degree two in $\eta$. Therefore $N \circ W(\xi, \eta)$ has degree two in $\eta$ and the terms of order three and higher in $\eta$ of $H^*(\xi, \cdot)$ and $R \circ W(\xi, \cdot)$ coincide. Hence the same holds for $R^*(\xi, \cdot)$ and $R \circ W(\xi, \cdot)$. So we can apply lemma $A.5$ in which the function $\eta \mapsto R \circ W(\xi, \eta)$ is bounded by $M$ for $|\eta| < 5s/8$ because of (1.6) and theorem 5.3.

Putting $\sigma = 5s/8$, $f = R \circ W(\xi, \cdot)$ and $\varepsilon = 4/5$ in lemma $A.5$ we obtain

$$|R^*(\xi, \eta)| \leq 5M \frac{|\eta|^3}{(5s/8)^3} = \frac{512}{25} M \frac{|\eta|^3}{s^3} \quad \forall \quad |\eta| < \frac{45s}{5} = \frac{s}{2}.$$

Now, $\xi \in S(r/2)$ was arbitrary, so (1.10) holds with $c_5 = \frac{512}{25}$ and the theorem is proved.

Altogether theorems 5.3, 5.5, 5.6, and 5.7 prove theorem 1.6.

A Appendix

A.1 A lemma on non-singular matrices

Lemma A.1. Let $S \in \mathbb{C}^{n \times n}$ be an invertible matrix. Then each matrix $P \in \mathbb{C}^{n \times n}$ with

$$|P - S| \leq h \cdot \frac{1}{|S^{-1}|}, \quad 0 < h < 1,$$

is invertible as well. The inverse of $P$ fulfills

$$|P^{-1}| \leq \frac{|S^{-1}|}{1 - h} \quad \text{and} \quad |P^{-1} - S^{-1}| \leq \frac{h|S^{-1}|}{1 - h}.$$

Proof. We set $H := E_n - S^{-1}P$. The assumption leads to the estimate

$$|H| = |E_n - S^{-1}P| \leq |S^{-1}| |S - P| \leq h < 1.$$

Therefore the Neumann series

$$\sum_{k=0}^{\infty} H^k = (E_n - H)^{-1} = (S^{-1}P)^{-1}$$

converges, in particular $S^{-1}P$ is non-singular. Hence this is also true for $P = S \cdot S^{-1}P$. For $P^{-1} = (S^{-1}P)^{-1}S^{-1}$ we find the estimate

$$|P^{-1}| \leq |S^{-1}| \sum_{k=0}^{\infty} |H|^k \leq \frac{|S^{-1}|}{1 - h}.$$
For $P^{-1} - S^{-1} = (P^{-1}S - E_n)S^{-1}$ we calculate

$$P^{-1}S - E_n = \left( \sum_{k=0}^{\infty} H^k \right) - E_n = \sum_{k=1}^{\infty} H^k$$

to see

$$|P^{-1} - S^{-1}| \leq |S^{-1}| \sum_{k=1}^{\infty} |H|^k \leq \frac{h|S^{-1}|}{1 - h},$$

as was to be shown.

\[\square\]

### A.2 Estimates for analytic maps

**Definition A.2.** Let $z \in \mathbb{C}^n$ and $s > 0$. We set

$$\mathcal{B}(s; z) := \{ y \in \mathbb{C}^n \mid |y - z| < s \}.$$

The following lemma is Cauchy’s estimate for analytic functions of several variables.

**Lemma A.3.** Let $M > 0$ and $f : \mathcal{B}(s; 0) \subseteq \mathbb{C}^n \to \mathbb{C}^m$ be an analytic function with

$$|f|_{\mathcal{B}(s; 0)} \leq M.$$

The Jacobian of $f$ satisfies the estimate

$$|f_x|_{\mathcal{B}(s-\varepsilon; 0)} \leq \frac{M}{\varepsilon} \text{ for all } 0 < \varepsilon < s.$$

**Proof.** We fix an arbitrary $x_0 \in \mathcal{B}(s - \varepsilon; 0)$. Then \([1, 2]\) shows

$$|f_x(x_0)| = \max_{|y|=1} |y f_x^T(x_0)| = \max_{1 \leq k \leq m} \max_{|y|=1} |\langle f_kx(x_0), y \rangle|,$$

where $f_k$ denotes the $k$-th coordinate function of $f$. We give us arbitrary $k \in \{1, \ldots, m\}$ and $y \in \mathbb{C}^n$ with $|y| = 1$ and consider the auxiliary function

$$g : \mathcal{B}(\varepsilon; 0) \subseteq \mathbb{C} \to \mathbb{C}, \quad t \mapsto f_k(x_0 + ty).$$

We obtain

$$g_t(t) = \langle f_kx(x_0 + ty), y \rangle \quad \Rightarrow \quad g_t(0) = \langle f_kx(x_0), y \rangle,$$

and Cauchy’s estimate in one dimension says

$$|\langle f_kx(x_0), y \rangle| = |g_t(0)| \leq \frac{M}{\varepsilon},$$

which finishes the proof.

\[\square\]

We need an estimate for the remainder of order three relating to the Taylor expansion of an analytic function. At first we prove it in dimension one.
Lemma A.4. Let $\sigma > 0$ and $g : B(\sigma; 0) \subseteq \mathbb{C} \rightarrow \mathbb{C}$, $z \mapsto g(z)$ be an analytic function bounded by a constant $M > 0$. Then the remainder

$$h^{(g)}(z) := \sum_{k=3}^{\infty} \frac{1}{k!} \frac{\partial^k g}{\partial z^k}(0) z^k \quad (|z| < \sigma)$$

satisfies for all $\varepsilon \in (0, 1)$ the estimate

$$|h^{(g)}(z)| \leq \frac{M}{1 - \varepsilon} \frac{|z|^3}{\sigma^3} \quad \forall ~ |z| \leq \varepsilon \sigma.$$

Proof. By Cauchy’s formula we have for $0 < \tilde{\sigma} < \sigma$

$$\left| \frac{\partial^k g}{\partial z^k}(0) \right| = \left| \frac{k!}{2\pi i} \oint_{|z|=\tilde{\sigma}} \frac{g(z)}{z^{k+1}} \, dz \right| \leq \frac{Mk!}{\tilde{\sigma}^k}.$$

The limit $\tilde{\sigma} \rightarrow \sigma$ yields

$$\left| \frac{\partial^k g}{\partial z^k}(0) \right| \leq \frac{Mk!}{\sigma^k}.$$

Hence we get for the remainder, in case $|z| \leq \varepsilon \sigma$,

$$|h^{(g)}(z)| \leq \sum_{k=3}^{\infty} \frac{1}{k!} \left| \frac{\partial^k g}{\partial z^k}(0) \right| |z|^k \leq \sum_{k=3}^{\infty} \frac{1}{k!} \frac{Mk!}{\sigma^k} |z|^k = M \sum_{k=3}^{\infty} \left( \frac{|z|}{\sigma} \right)^k = M \left( \frac{|z|}{\sigma} \right)^3 \sum_{k=0}^{\infty} \left( \frac{|z|}{\sigma} \right)^k \leq M \left( \frac{|z|}{\sigma} \right)^3 \sum_{k=0}^{\infty} \varepsilon^k = M \frac{|z|^3}{1 - \varepsilon}.$$

as was to be shown.

Lemma A.5. Let $\sigma > 0$ and $f : B(\sigma; 0) \subseteq \mathbb{C}^n \rightarrow \mathbb{C}$, $y \mapsto f(y)$ analytic and bounded by $M > 0$. Then the remainder

$$h^{(f)}(y) = f(y) - \left( f(0) + \langle f_y(0), y \rangle + \frac{1}{2} \langle yf_{yy}(0), y \rangle \right),$$

fulfills for all $\varepsilon \in (0, 1)$ the estimate

$$|h^{(f)}(y)| \leq \frac{M}{1 - \varepsilon} \frac{|y|^3}{\sigma^3} \quad \forall ~ |y| \leq \varepsilon \sigma.$$ (A.2)

Proof. Let us fix an $\varepsilon$, $0 < \varepsilon < 1$ and $y \in \mathbb{C}^n$ with $|y| \leq \varepsilon \sigma$. In case $y = 0$ (A.2) is an immediate consequence of (A.1). In case $y$ does not vanish we set

$$y_0 := \varepsilon \sigma \frac{y}{|y|},$$

such that $|y_0| = \varepsilon \sigma$, and consider the function

$$g : B(\varepsilon^{-1}; 0) \subseteq \mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto g(z) := f(zy_0).$$
By construction \( g(0) = f(0) \) and with the chain rule we get
\[
g_z(z) = \langle f_y(zy_0), y_0 \rangle, \quad g_{zz}(z) = \langle y_0 f_{yy}(zy_0), y_0 \rangle \quad \forall \ |z| < \varepsilon^{-1}.
\]

Lemma A.4 yields
\[
|h^{(f)}(zy_0)| = \left| f(zy_0) - f(0) - \langle f_y(0), zy_0 \rangle - \frac{1}{2} \langle zy_0 f_{yy}(0), zy_0 \rangle \right|
\leq \frac{M}{1 - \varepsilon (\varepsilon^{-1})^3} = \frac{M}{1 - \varepsilon} |z|^3 \varepsilon^3 \quad \forall \ |z| \leq \varepsilon(\varepsilon^{-1}) = 1.
\]

It is allowed to put \( z = |y|/(\varepsilon \sigma) \) in this inequality, so
\[
|h^{(f)}(zy_0)| = \left| h^{(f)} \left( \frac{|y|}{\varepsilon \sigma} \varepsilon \sigma \frac{y}{|y|} \right) \right| = \left| h^{(f)}(y) \right| \leq \frac{M}{1 - \varepsilon} \frac{|y|^3}{\varepsilon^3 \sigma^3} = \frac{M}{1 - \varepsilon} \frac{|y|^3}{\sigma^3},
\]
and the proof is finished. \( \square \)

## A.3 Generating symplectic transformations

Auxiliary results on autonomous differential equations

**Theorem A.6.** Let \( \rho > 0, S(\rho) \subseteq \mathbb{C}^n, V \subseteq \mathbb{C}^m \) open and
\[
f : S(\rho) \times V \longrightarrow \mathbb{C}^{n+m}, \quad z = (x, y) \mapsto f(z)
\]
be continuous and such that
\[
\dot{z} = f(z) \quad (A.3)
\]
has unique solutions. The function \( f \) shall have the period \( T > 0 \) in \( z_1 = x_1, \ldots, z_n = x_n \). We assume that there are numbers \( a, b, \tilde{\delta} \), \( a \leq 0 < b, 0 < \tilde{\delta} < \rho \) and an open set \( U \subseteq V \), such that the flow \( \varphi \) of [A.3] exists on \( [a, b] \times S(\rho - \tilde{\delta}) \times U \). Then the function
\[
\varphi(t, \cdot) - \text{id} : S(\rho - \tilde{\delta}) \times U \longrightarrow S(\rho) \times V, \quad \zeta = (\xi, \eta) \mapsto \varphi(t, \zeta) - \zeta
\]
has the period \( T \) in \( \zeta_1 = \xi_1, \ldots, \zeta_n = \xi_n \) for all \( t \in [a, b] \).

The assumption on the existence of the flow \( \varphi \) means, that there is a map
\[
\varphi : [a, b] \times S(\rho - \tilde{\delta}) \times U \longrightarrow S(\rho) \times V
\]
with \( \varphi(0, \zeta) = \zeta \) and \( \varphi(\cdot, \zeta) \) solves the differential equation [A.3].

**Proof of theorem A.6.** We show for all \( (t, \zeta) \in [a, b] \times S(\rho - \tilde{\delta}) \times U \) that
\[
\varphi(t, \zeta) + T \cdot e_j = \varphi(t, \zeta + T \cdot e_j) \quad (1 \leq j \leq n). \quad (A.4)
\]
Let \( j \in \{1, \ldots, n\} \) be arbitrary and set \( h(t) := \varphi(t, \zeta) + T \cdot e_j \) and \( g(t) := \varphi(t, \zeta + T \cdot e_j) \). Then \( h(0) = g(0) = \zeta + T \cdot e_j \) and
\[
\begin{align*}
\dot{h}(t) &= \varphi(t, \zeta) = f(\varphi(t, \zeta)) = f(\varphi(t, \zeta) + T \cdot e_j) = f(h(t)), \\
\dot{g}(t) &= \varphi(t, \zeta + T \cdot e_j) = f(\varphi(t, \zeta + T \cdot e_j)) = f(g(t)).
\end{align*}
\]

Therefore both functions fulfill the differential equation. Hence they coincide. This proves \( (A.4) \). Now \( (A.4) \) shows for all \( 1 \leq k \leq n \)
\[
\varphi(t, \zeta + T \cdot e_j) - (\zeta + T \cdot e_j) = \varphi(t, \zeta) - \zeta,
\]
which proves the lemma. \( \square \)

**Lemma A.7.** Let \( a < b \) and \( f : (a, b) \to \mathbb{C}^m, m \in \mathbb{N} \) be an analytic function. Let \( a \leq a_0 < b_0 \leq b \) and suppose that the restriction of \( f \) to \((a_0, b_0)\) maps to \( \mathbb{R}^m \). Then \( f \) maps to \( \mathbb{R}^m \).

**Proof.** Without loss of generality we may assume \( m = 1 \), for in case \( f = (f_1, \ldots, f_m) : (a, b) \to \mathbb{C}^m \) is analytic, so is every coordinate function \( f_i, 1 \leq i \leq m \). Hence we can apply the lemma for \( m = 1 \) to each coordinate function and get the result for \( f \). So let us assume \( m = 1 \).

Let \( A \subseteq (a, b) \) be the biggest interval, which contains \((a_0, b_0)\), and on which \( f \) maps to \( \mathbb{R} \). \( A \) exists, because it can be constructed as the union of all intervals, which contain \((a_0, b_0)\) and on which \( f \) maps to \( \mathbb{R} \). \( A \) is not empty, for it contains \((a_0, b_0)\).

\( A \) is closed in \((a, b)\). To see that we consider a cluster point \( \alpha \) of \( A \) and choose a sequence \((x_i)_{i=1}^\infty \subseteq A \setminus \{\alpha\}\), which tends to \( \alpha \). \( f \) is in particular continuous on \((a, b)\), so the limit
\[
f(\alpha) = \lim_{i \to \infty} f(x_i)
\]
exists. It is a limit of real numbers, so it is real as well. Hence \( \alpha \in A \). So \( A \) contains its cluster points which means it is closed.

However, \( A \) is open in \((a, b)\). In order to see that consider an arbitrary \( \alpha \in A \). By assumption \( f \) may be expanded in a power series around the point \( \alpha \). The series is given by
\[
f(x) = \sum_{k=0}^\infty \frac{f^{(k)}(\alpha)}{k!} (x - \alpha)^k. \tag{A.5}
\]
Herein \( f^{(k)}(\alpha) \) denotes the \( k \)-th derivative of \( f \) in \( \alpha \). We show that \( f^{(k)}(\alpha) \) is a real number for all \( k \in \mathbb{N}_0 \). This is obvious for \( f^{(0)}(\alpha) = f(\alpha) \) because \( \alpha \in A \). If it is true for some \( k \in \mathbb{N}_0 \) then for \( k + 1 \) as well. Indeed, take a sequence \((x_i)_{i=1}^\infty \subseteq A \setminus \{\alpha\}\), which tends to \( \alpha \) and consider the limit
\[
f^{(k+1)}(\alpha) = \lim_{i \to \infty} \frac{f^{(k)}(x_i) - f^{(k)}(\alpha)}{x_i - \alpha}.
\]
Again, this is a limit of real numbers, hence a real number. So all coefficients of the series \( (A.5) \) are real and \( f \) maps to \( \mathbb{R}^m \) in a neighborhood of \( \alpha \). So \( \alpha \) is an inner point of \( A \) and \( A \) is open in \((a, b)\).

Altogether, \( A \) is not empty, open and closed in \((a, b)\), meaning \( A = (a, b) \). The lemma is proved. \( \square \)
**Theorem A.8.** Let \( \rho > 0, \sigma > 0 \) and \( f \in \mathcal{P}_{2n}(\rho, \sigma) \). Suppose there are \( 0 < \delta < \rho, 0 < \varepsilon < \sigma \) and \( a \leq 0 < b \) such that the flow \( \varphi \) of the differential equation
\[
\dot{z} = f(z) \quad \text{(A.6)}
\]
exists on \([a,b) \times D(\rho - \delta, \sigma - \varepsilon)\). If then \( f \) maps real vectors to real values, so does \( \varphi \).

**Proof.** We consider the restriction of \( f \) to real vectors, namely
\[
g : \mathbb{R}^n \times \{ y \in \mathbb{R}^n \mid |y| < \sigma \} \rightarrow \mathbb{R}^{2n}, \quad z \mapsto g(z) := f(z),
\]
and the differential equation
\[
\dot{z} = g(z). \quad \text{(A.7)}
\]
Observe that the domain of \( g \) coincides with \( D(\rho, \sigma) \cap \mathbb{R}^{2n} \). Now let \( \zeta \in \mathbb{R}^n \times \{ y \in \mathbb{R}^n \mid |y| < \sigma - \varepsilon \} \) be arbitrary. Then there are numbers \( a_1 < 0 < b_1 \) and a solution
\[
h : (a_1, b_1) \rightarrow \mathbb{R}^n \times \{ y \in \mathbb{R}^n \mid |y| < \sigma \}
\]
of (A.7). Clearly \( h \) is a solution of (A.6) as well. Therefore
\[
\varphi(t, \zeta) = h(t) \quad \forall \ t \in (a_1, b_1) \cap [a, b).
\]
The set of the \( t \) which can applied herein contains an open interval. So the preceding lemma shows that \( \varphi(\cdot, \zeta) \) maps to \( \mathbb{R}^{2n} \), which proves the assertion. \( \Box \)

**Simple canonical transformations**

**Theorem A.9.** Let \( \mathcal{U}, \mathcal{V} \subseteq \mathbb{C}^n \) be open and connected sets and \( Z = (X, Y) : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{C}^{2n} \) a simple canonical transformation (see definition 1.5). Then we have for all \( (\xi, \eta) \in \mathcal{U} \times \mathcal{V} \)
\[
det X_\xi(\xi) \neq 0, \quad \text{(A.8)}
\]
\[
Y(\xi, \eta) = Y(\xi, 0) + \eta X_\xi(\xi)^{-1}. \quad \text{(A.9)}
\]

**Proof.** \( X \) is independent of \( \eta \), so
\[
Z_\zeta = \begin{pmatrix} X_\xi & 0 \\ Y_\xi & Y_\eta \end{pmatrix}.
\]
Hence (1.3) implies
\[
\begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} = \begin{pmatrix} X_\xi^T & Y_\xi^T \\ 0 & Y_\eta^T \end{pmatrix} \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} \begin{pmatrix} X_\xi & 0 \\ Y_\xi & Y_\eta \end{pmatrix} = \begin{pmatrix} -Y_\eta^T X_\xi & X_\xi Y_\eta - Y_\eta^T X_\xi \\ -Y_\eta^T Y_\xi & X_\xi Y_\eta \end{pmatrix}.
\]

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We consider the right upper block on the left hand and right hand side of the equation and see
\[ X_\xi^T Y_\eta = E_n. \] (A.10)

Building determinants we obtain
\[ \det X_\xi(\xi) \det Y_\eta(\xi, \eta) = 1 \quad \forall \ (\xi, \eta) \in U \times V. \]

This yields (A.8). Moreover by (A.10) we get
\[ Y_\eta = (X_\xi^T)^{-1} = (X_\xi^{-1})^T. \] (A.11)

Therefore \( Y_\eta \) does not depend on \( \eta \) and consequently \( Y_{\eta\eta} = 0 \), such that \( Y \) is affine-linear in \( \eta \). The Taylor expansion of \( Y \) with respect to \( \eta \) therefore reads
\[ Y(\xi, \eta) = Y(\xi, 0) + \eta \cdot Y_\eta(\xi, 0)^T \quad \forall \ (\xi, \eta) \in U \times V. \]

Together with (A.11) we obtain (A.9) and the proof is finished. \( \square \)

Remark A.10. Theorem A.9 in particular implies, that simple canonical transformations are affine-linear in \( \eta \). So they may be defined for all \( \eta \in \mathbb{C}^n \). Moreover the functions \( Y_{k\eta} \) do not depend on \( \eta \).

Let us denote the uniform convergence of a sequence of functions \((f_k)\) on compact subsets of an open set \( U \) towards some limit function \( f \) by
\[ f_k \xrightarrow{U, \text{compact}} f \quad (k \to \infty). \]

Clearly, when \( U \subseteq \mathbb{C}^n \) or \( U \subseteq \mathbb{R}^n \), the uniform convergence on compact subsets of \( U \) is equivalent to the fact, that the sequence converges uniformly on bounded open subsets of \( U \).

Theorem A.11. Let \( U \subseteq \mathbb{C}^n \) be an open and connected set and
\[ Z_k = (X_k, Y_k) : U \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \times \mathbb{C}^n \quad (k \in \mathbb{N}) \] (A.12)
a sequence of simple canonical transformations with the property, that the sequences \((Z_k(\cdot, 0))_{k=1}^\infty\) and \((Y_{k\eta})_{k=1}^\infty\) converge uniformly on compact subsets of \( U \). Then \((Z_k)_{k=1}^\infty\) converges uniformly on compact subsets of \( U \times \mathbb{C}^n \) to a simple canonical transformation.

Proof. For all \( k \in \mathbb{N} \)
\[ Z_k(\xi, 0) = (X_k(\xi), Y_k(\xi, 0)) \]
holds. The functions \( Z_k \) are analytic. By assumption and the theorem of Weierstrass (see [4], (9.12.1)) there exist analytic functions \( X, V, \) and \( W, \) defined on \( U \), with
\[ Z_k(\cdot, 0) \xrightarrow{U, \text{compact}} (X, V) \text{ and } Y_{k\eta} \xrightarrow{U, \text{compact}} W, \quad (k \to \infty). \] (A.13)
The first limit means in particular
\[ X_k \xrightarrow{\mathcal{U}, \text{compact}} X \quad \text{and} \quad Y_k(\cdot, 0) \xrightarrow{\mathcal{U}, \text{compact}} V, \quad (k \to \infty). \quad \text{(A.14)} \]

By the theorem of Weierstrass we conclude
\[ X_{k\xi} \xrightarrow{\mathcal{U}, \text{compact}} X_{\xi}, \quad (k \to \infty). \]

Now by (A.9) we have \( Y_{k\eta} = ((X_{k\xi})^{-1})^T \), so the second limit in (A.13) yields for all \( \xi \in \mathcal{U} \)
\[ E_n = X_{k\xi}(\xi)Y_{k\eta}(\xi)^T \longrightarrow X_{\xi}(\xi)W(\xi)^T, \quad (k \to \infty). \]

Hence \( E_n = X_{\xi}W^T \) holds and \( (X_{\xi})^{-1} = W^T \) exists, where
\[ (X_{k\xi})^{-1} \xrightarrow{\mathcal{U}, \text{compact}} (X_{\xi})^{-1}, \quad (k \to \infty), \quad \text{(A.15)} \]
again because of (A.13). We set
\[ Y(\xi, \eta) := V(\xi) + \eta X_{\xi}(\xi)^{-1} \quad \forall \quad (\xi, \eta) \in \mathcal{U} \times \mathbb{C}^n, \]
and show for the functions \( Y_k(\xi, \eta) = Y_k(\xi, 0) + \eta X_{k\xi}(\xi)^{-1} \) that
\[ Y_k \xrightarrow{\mathcal{U} \times \mathbb{C}^n, \text{compact}} Y, \quad (k \to \infty). \quad \text{(A.16)} \]

For this purpose let \( \mathcal{K}_1 \subseteq \mathcal{U} \) and \( \mathcal{K}_2 \subseteq \mathbb{C}^n \) be compact and \( \varepsilon > 0 \). By (A.14) there exists a \( N_1 \in \mathbb{N} \) with
\[ |Y_k(\cdot, 0) - V|_{\mathcal{K}_1} < \frac{\varepsilon}{2} \quad \forall \quad k \geq N_1. \]

Because \( \mathcal{K}_2 \) is compact there exists a number \( K > 0 \), such that \( \mathcal{K}_2 \) is contained in the ball \( B(K; 0) \). From (A.16) we infer that there is a \( N_2 \in \mathbb{N} \) with
\[ |(X_{k\xi})^{-1} - (X_{\xi})^{-1}|_{\mathcal{K}_1} < \frac{\varepsilon}{2nK} \quad \forall \quad k \geq N_2. \]

So for all \( k \geq N_1 + N_2 \)
\[ |Y_k - Y|_{\mathcal{K}_1 \times \mathcal{K}_2} \leq |Y_k(\cdot, 0) - V|_{\mathcal{K}_1} + nK |(X_{k\xi})^{-1} - (X_{\xi})^{-1}|_{\mathcal{K}_1} < \varepsilon \]
holds and therefore (A.16) is true. We know from (A.14) and (A.16), that the sequence \( (Z_k) \) converges uniformly on compact subsets of \( \mathcal{U} \times \mathbb{C}^n \) to an analytic function \( Z := (X, Y) \). It remains to show that \( Z \) is a simple canonical transformation. We do already know that \( Z \) is analytic and that its component \( X \) does not depend on \( \eta \). Hence the only missing information is that \( Z \) is a symplectic transformation. Well, by the theorem of Weierstrass we see for all \( (\xi, \eta) \in \mathcal{U} \times \mathbb{C}^n \)
\[ J = Z_{k\xi}(\xi, \eta)^T \cdot J \cdot Z_{k\xi}(\xi, \eta) \longrightarrow Z_{\xi}(\xi, \eta)^T \cdot J \cdot Z_{\xi}(\xi, \eta), \quad (k \to \infty), \]
hence \( Z_{\xi}^T \cdot J \cdot Z_{\xi} = J \). The proof is finished. \( \square \)
Generating symplectic transformations

The discussion in this section is like the one given in [12]. However, we consider an
other class of Hamiltonians.

**Theorem A.12.** Let \( K > 0, \tilde{\rho} > 0, 0 < \delta < \tilde{\rho} \) and \( 0 < \sigma \leq \delta \). Let \( F : \mathcal{D}(\tilde{\rho}, \sigma) \to \mathbb{C}, F = F(x, y) \) be an analytic function fulfilling

\[
|F_x|_{\mathcal{D}(\tilde{\rho},\sigma)} \leq \frac{K}{\delta}, \quad |F_y|_{\mathcal{D}(\tilde{\rho},\sigma)} \leq \frac{K}{\sigma}.
\]

(A.17)

Then the Hamiltonian system

\[
\dot{x} = F_y, \quad \dot{y} = -F_x
\]

(A.18)

possesses an analytic flow

\[
Z : \left[0, \frac{\sigma \delta}{2K}\right] \times \mathcal{D}(\tilde{\rho} - \delta, \sigma/2) \to \mathcal{D}(\tilde{\rho}, \sigma), \quad (t, \zeta) \mapsto Z(t, \zeta),
\]

which is uniquely determined.

In particular \( Z(\cdot, \zeta) \) is the unique solution to (A.18) with respect to the initial
value \( Z(0, \zeta) = \zeta \in \mathcal{D}(\tilde{\rho} - \delta, \sigma/2) \). Using the matrix \( J \) from definition 1.4 we can write (A.18) in the form

\[
\dot{z} = F_z J^T.
\]

Proof of theorem A.12. The existence theorem of Cauchy (see [4], (10.4.5)) says,
that solutions \( t \mapsto Z(t, \zeta) \) to the initial value \( Z(0, \zeta) = \zeta \in \mathcal{D}(\tilde{\rho}, \sigma) \) exist locally and are uniquely determined. The flow \( Z \) is analytic in \( t \) and \( \zeta = (\zeta_1, \ldots, \zeta_{2n}) \) (see [4], (10.8.2)).

Each solution of (A.18) maps to \( \mathcal{D}(\tilde{\rho}, \sigma) \) by definition and it remains to show, that the
solutions to the initial values \( \zeta \in \mathcal{D}(\tilde{\rho} - \delta, \sigma/2) \) exist for all \( t \in [0, \sigma\delta/(2K)] \).

To this end let \( \zeta \in \mathcal{D}(\tilde{\rho} - \delta, \sigma/2) \) be arbitrary. We assume, that the solution \( Z(\cdot, \zeta) = (X(\cdot, \zeta), Y(\cdot, \zeta)) \) does only exist up to a \( b \in (0, \sigma\delta/(2K)) \). By (A.18) we have for all \( t \in [0, b) \)

\[
X(t, \zeta) - \xi = \int_0^t F_y(Z(\tau, \zeta)) \, d\tau,
\]

\[
Y(t, \zeta) - \eta = \int_0^t -F_x(Z(\tau, \zeta)) \, d\tau.
\]

Assumption (A.17) and \( 0 < b < \sigma\delta/(2K) \) imply

\[
|X(\cdot, \zeta) - \xi|_{[0,b]} \leq \sup_{t \in [0,b]} \int_0^t |F_y|_{\mathcal{D}(\tilde{\rho},\sigma)} \, d\tau \leq \frac{bK}{\sigma} < \frac{\delta}{2},
\]

\[
|Y(\cdot, \zeta) - \eta|_{[0,b]} \leq \sup_{t \in [0,b]} \int_0^t |F_x|_{\mathcal{D}(\tilde{\rho},\sigma)} \, d\tau \leq \frac{bK}{\delta} < \frac{\sigma}{2}.
\]

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Now let $(t_k)_{k=1}^\infty$ be an increasing sequence in $[0, b)$ with $\lim_{k \to \infty} t_k = b$. According to our assumption on $b$ the sequence $(Z(t_k, \zeta))_{k=1}^\infty$ cannot have a cluster point in $D(\tilde{\varrho}, \sigma)$ (see [5], Chapter 8, §5). On the other hand, the sequence is contained in the compact set

$\left\{(x, y) \in \mathbb{C}^n \mid |\text{Im} x| \leq \tilde{\varrho} - \delta + b \frac{K}{\sigma}, |y| \leq \frac{\sigma}{2} + b \frac{K}{\delta} \right\} \subseteq D(\tilde{\varrho}, \sigma),$

which implies the existence of a cluster point in $D(\tilde{\varrho}, \sigma)$. This contradiction shows $b \geq \sigma \delta / (2K)$ and therefore, that the solutions exist for all $t \in [0, \sigma \delta / (2K))$.

**Corollary A.13.** Let $K > 0$, $\varrho > 0$, $0 < 2\delta < \varrho$, and $0 < \sigma \leq \delta$. Let $F : D(\varrho, \sigma) \to \mathbb{C}$, $F = F(x, y)$ be analytic and such that

$$|F_x|_{D(\varrho, \sigma)} \leq \frac{K}{\delta}, \quad |F_y|_{D(\varrho, \sigma)} \leq \frac{K}{\sigma}$$

holds. Then the Hamiltonian system

$$\dot{x} = F_y, \quad \dot{y} = -F_x$$

possesses an analytic flow

$$Z : \left[0, \frac{\sigma \delta}{2K} \right] \times D(\varrho - 2\delta, \sigma/2) \longrightarrow D(\varrho - \delta, \sigma), \quad (t, \zeta) \mapsto Z(t, \zeta),$$

which is uniquely determined.

**Proof.** For the proof it suffices to put $\tilde{\varrho} = \varrho - \delta$ in the assumptions of the preceding theorem. \qed

When we fix the time $t$ and vary the initial value, (A.21) gives rise to the maps

$$Z(t, \cdot) : D(\varrho - 2\delta, \sigma/2) \longrightarrow D(\varrho - \delta, \sigma), \quad \left(0 \leq t < \frac{\sigma \delta}{2K}\right).$$

Let us analyze these maps in detail.

**Theorem A.14.** Let $K > 0$, $\varrho > 0$, $0 < 2\delta < \varrho$, and $0 < \sigma \leq \delta$ with

$$\frac{\sigma \delta}{2K} > 1.$$

Let $F : D(\varrho, \sigma) \to \mathbb{C}$, $F = F(x, y)$ be an analytic function, which is affine-linear in $y$ and fulfills (A.19). Then the functions (A.22) satisfy

$$|Z_\zeta(t, \cdot)|_{D(\varrho - 2\delta, \sigma/2)} \leq \exp\left(\frac{2nK}{\delta \sigma} t\right) \quad \forall \ t \in \left[0, \frac{\sigma \delta}{2K}\right),$$

$$|Z_\zeta(t, \cdot) - E_{2n}|_{D(\varrho - 2\delta, \sigma/2)} \leq \frac{2nK}{\delta \sigma} \exp\left(\frac{2nK}{\delta \sigma} t\right) \quad \forall \ t \in [0, 1].$$

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Proof. We make use of the lemma of Gronwall ([2], Corollary (6.2)). For this we have to find an estimate for \( F_{zz} \). Cauchy’s estimate and (A.19) give

\[
|F_{xx}|_{D(\rho-\delta, \sigma)} \leq \frac{K}{\delta^2} \leq \frac{K}{\delta \sigma}, \quad |F_{yx}|_{D(\rho-\delta, \sigma)} \leq \frac{K}{\delta \sigma}.
\]

The second inequality and the lemma of Schwarz yield

\[
|F_{xy}|_{D(\rho-\delta, \sigma)} = |F^T_{yx}|_{D(\rho-\delta, \sigma)} \leq n |F_{yx}|_{D(\rho-\delta, \sigma)} \leq \frac{nK}{\delta \sigma}.
\]

\( F \) is affine-linear in \( y \), so \( F_{yy} = 0 \). Altogether we obtain

\[
|F_{zz}|_{D(\rho-\delta, \sigma)} \leq (n + 1)\frac{K}{\delta \sigma}.
\] (A.25)

The equation

\[
Z^T(\tau, \zeta) = F^T_{z}(Z(\tau, \zeta)) J^T
\]

holds for all \( 0 \leq \tau < \sigma \delta/(2K) \), because \( Z(\cdot, \zeta) \) solves (A.20) for all \( \zeta \in D(\rho - 2\delta, \sigma/2) \). (On the left hand side we have to write \( Z^T \) because of our definition \( \dot{Z} = Z^T \).) Differentiating with respect to \( \zeta \) yields

\[
Z_{\zeta t}(t, \zeta) = (Z^T_t)_{\zeta}(t, \zeta) = JF_{zz}(Z(t, \zeta)) \cdot Z_{\zeta}(t, \zeta).
\] (A.26)

Now integration with respect to \( t \) gives

\[
Z_{\zeta}(t, \zeta) = E_{2n} + \int_0^t JF_{zz}(Z(\tau, \zeta)) \cdot Z_{\zeta}(\tau, \zeta) \, d\tau.
\] (A.27)

With (A.25) we obtain the estimate

\[
|Z_{\zeta}(t, \zeta)| \leq 1 + \int_0^t |F_{zz}|_{D(\rho-\delta, \sigma)} |Z_{\zeta}(\tau, \zeta)| \, d\tau \leq 1 + \frac{(n + 1)K}{\delta \sigma} \int_0^t |Z_{\zeta}(\tau, \zeta)| \, d\tau
\]

\[
\forall \quad \zeta \in D(\rho - 2\delta, \sigma/2), \quad t \in \left[0, \frac{\sigma \delta}{2K}\right).
\]

With the lemma of Gronwall

\[
|Z_{\zeta}(t, \zeta)| \leq \exp \left( \frac{(n + 1)K}{\delta \sigma} t \right) \leq \exp \left( \frac{2nK}{\delta \sigma} t \right)
\]

\[
\forall \quad \zeta \in D(\rho - 2\delta, \sigma/2), \quad t \in \left[0, \frac{\sigma \delta}{2K}\right),
\]

follows. To obtain the second estimate, we derive with (A.27) for \( t \in [0, \sigma \delta/(2K)) \) and \( \zeta \in D(\rho - 2\delta, \sigma/2) \)

\[
Z_{\zeta}(t, \zeta) - E_{2n} = \int_0^t JF_{zz}(Z(\tau, \zeta)) \, d\tau + \int_0^t JF_{zz}(Z(\tau, \zeta))(Z_{\zeta}(\tau, \zeta) - E_{2n}) \, d\tau.
\]
This together with (A.25) implies
\[ |Z_\zeta(t, \zeta) - E_{2n}| \leq \frac{(n+1)K}{\delta \sigma} t + \frac{(n+1)K}{\delta \sigma} \int_0^t |Z_\zeta(\tau, \zeta) - E_{2n}| \, d\tau \]
\[ \leq \frac{2nK}{\delta \sigma} + \frac{2nK}{\delta \sigma} \int_0^t |Z_\zeta(\tau, \zeta) - E_{2n}| \, d\tau \]
\[ \forall \ \zeta \in D(\varrho - 2\delta, \sigma/2), \ t \in [0, 1]. \]

Here the lemma of Gronwall says
\[ |Z_\zeta(t, \zeta) - E_{2n}| \leq \frac{2nK}{\delta \sigma} \exp \left( \frac{2nK}{\delta \sigma} t \right) \]
\[ \forall \ \zeta \in D(\varrho - 2\delta, \sigma/2), \ t \in [0, 1]. \]

The theorem is shown. \( \square \)

**Theorem A.15.** Let \( K > 0, \varrho > 0, 0 < 2\delta < \varrho, \) and \( 0 < \sigma \leq \delta. \) Let the function \( F : D(\varrho, \sigma) \to \mathbb{C}, \) \( F = F(x, y) \) be analytic and such that (A.19) holds. Then the maps (A.22) are symplectic transformations.

**Proof.** We meet the assumptions of corollary A.13. Therefore the flow (A.21) and the maps (A.22) exist. We have to prove:
\[ Z_\zeta(t, \zeta)^T J Z_\zeta(t, \zeta) = J \]
\[ \forall \ (t, \zeta) \in \left[ 0, \frac{\sigma \delta}{2K} \right] \times D(\varrho - 2\delta, \sigma/2). \] \( (A.28) \)

This equation is certainly true for \( t = 0, \) because \( Z(0, \cdot) \) is the identity and so \( Z(0, \zeta) = E_{2n} \) for all \( \zeta \in D(\varrho - 2\delta, \sigma/2). \)
To get the assertion for all \( t \in [0, \sigma \delta/(2K)) \) we show that the left hand side of (A.28) is constant with respect to \( t. \) To this end we calculate for \( (t, \zeta) \in [0, \sigma \delta/(2K)) \times D(\varrho - 2\delta, \sigma/2) \) with (A.26)
\[ \frac{\partial}{\partial t} (Z_\zeta(t, \zeta)^T J Z_\zeta(t, \zeta)) = (Z_\zeta)^T(t, \zeta) J Z_\zeta(t, \zeta) + Z_\zeta(t, \zeta)^T J Z_\zeta(t, \zeta) \]
\[ = Z_\zeta(t, \zeta)^T F_{zz}(Z(t, \zeta)) J Z_\zeta(t, \zeta) + Z_\zeta(t, \zeta)^T J F_{zz}(Z(t, \zeta)) Z_\zeta(t, \zeta) \]
\[ = Z_\zeta(t, \zeta)^T F_{zz}(Z(t, \zeta)) Z_\zeta(t, \zeta) - Z_\zeta(t, \zeta)^T F_{zz}(Z(t, \zeta)) Z_\zeta(t, \zeta) = 0. \]
This ends the proof. \( \square \)

**Theorem A.16.** Let \( K > 0, \varrho > 0, 0 < 2\delta < \varrho, \) and \( 0 < \sigma \leq \delta. \) Let \( F : D(\varrho, \sigma) \to \mathbb{C}, \) \( F = F(x, y) \) be analytic, so that (A.19) holds, and affine-linear in \( y. \) Then the maps (A.22) are simple canonical transformations.

**Proof.** The assumptions on the function \( F \) mean, that \( F \) can be written as
\[ F(x, y) = F_1(x) + \langle y, F_2(x) \rangle, \]
where \( F_1 : S(\varrho) \to \mathbb{C} \) and \( F_2 : S(\varrho) \to \mathbb{C}^n \) are analytic functions. System (A.20) reads in this case
\[ \dot{x} = F_2(x), \quad \dot{y} = -F_{1x}(x) - y \cdot F_{2x}(x). \]
The first equation possesses a unique solution \( \tilde{X}(\cdot, \xi), \tilde{X}(0, \xi) = \xi \) for all initial values \( \xi \in \mathcal{S}(\varrho - 2\delta) \). This solution exists for all \( 0 \leq t < \sigma\delta/(2K) \), as can be seen as above. Let us consider the system

\[
\dot{x} = F_2(x), \quad \dot{y} = 0.
\]

(A.29)

Obviously its solutions are given by \( \tilde{Z}(\cdot, \xi, \eta) = (\tilde{X}(\cdot, \xi), \eta) \). Now, let \( Z = (X, Y) \) be a solution of (A.20) with initial value \( Z(0, \zeta) = \zeta = (\xi, \eta) \). Then \( X(0, \zeta) = \xi \) holds and \( t \mapsto (X(t, \zeta), \eta) \) solves (A.29). Therefore \( X \) has the same values as \( \tilde{X} \), meaning

\[
X(t, \xi, \eta) = \tilde{X}(t, \xi) \quad \forall \quad (t, \xi, \eta) \in \left[0, \frac{\sigma\delta}{2K}\right] \times \mathcal{D}(\varrho - 2\delta, \sigma/2).
\]

Hence \( X \) is independent of \( \eta \) and the map (A.22) is a simple canonical transformation as was to be shown.

We resume the results of this appendix [A.3] in the following theorem.

**Theorem A.17.** Let \( K > 0, \varrho > 0, 0 < 2\delta < \varrho, \) and \( 0 < \sigma \leq \delta \) with

\[
\frac{\sigma\delta}{2K} > 1.
\]

Let the function \( F \in \mathcal{P}(\varrho, \sigma) \) fulfill estimates (A.19) and be affine-linear in \( y \). Then the maps (A.22) are simple canonical transformations, for all \( 0 \leq t < \sigma\delta/(2K) \) we have \( Z(t, \cdot) - \text{id} \in \mathcal{P}_{2n}(\varrho - 2\delta, \sigma/2) \), and the estimates (A.23) and (A.24) are fulfilled.

**Proof.** The maps (A.22) are well-defined and analytic by corollary [A.13]. They are simple canonical transformations by theorem [A.16]. The assumptions of theorem [A.6] are met, one has to put

\[
\mathcal{V} = \mathcal{B}(\sigma; 0), \quad f = F_2J^T, \quad T = 2\pi, \quad a = 0, \quad b = \sigma\delta/(2K),
\]

\[
\tilde{\delta} = 2\delta, \quad \tilde{\mathcal{U}} = \mathcal{B}(\sigma/2; 0) \text{ and } \varphi = Z.
\]

Therefore \( Z(t, \cdot) - \text{id} \) has period \( 2\pi \) in \( x \) for all \( 0 \leq t < \sigma\delta/(2K) \). The assumptions of theorem [A.8] are achieved with

\[
f = F_2J^T, \quad \tilde{\delta} = 2\delta, \quad \varepsilon = \sigma/2, \quad a = 0, \quad b = \sigma\delta/(2K) \text{ and } \varphi = Z.
\]

So \( Z \) maps real vectors to real values. This shows \( Z(t, \cdot) - \text{id} \in \mathcal{P}_{2n}(\varrho - 2\delta, \sigma/2) \). Finally (A.23) and (A.24) are a consequence of theorem [A.14]. This finishes the proof.

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