Quantum tomography under perturbed Hamiltonian evolution and scrambling of errors - a quantum signature of chaos

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How do quantum simulations of many-body chaotic dynamics behave under noise that will lead to a rapid scrambling of quantum information as well as errors across the system? We employ continuous measurement quantum tomography as a paradigm to study this question. The measurement record is generated as a sequence of expectation values of a Hermitian observable evolving under the repeated application of the Floquet map of the quantum kicked top. Interestingly, we find that the reconstruction fidelity initially increases regardless of the degree of chaos or the strength of perturbations in the dynamics. For random states, when the measurement record is obtained from a random initial observable, the subsequent drop in the fidelity obtained is inversely correlated to the degree of chaos in the dynamics. More importantly, this also gives us an operational interpretation of Loschmidt echo for operators by connecting it to the performance of quantum tomography. We define a quantity to capture the scrambling of errors, an out-of-time-ordered correlator (OTOC) between two operators under perturbed and unperturbed system dynamics that serves as a signature of chaos and quantifies the spread of errors. Our results demonstrate not only a fundamental link between Loschmidt echo and scrambling of errors as captured by OTOCs but that such a link can have operational consequences in quantum information processing.

Chaos, classically as well as in quantum mechanics, has an intimate connection with complexity. Classically, chaos implies unpredictability. Time-evolved trajectories twist and wind away from each other at an exponential rate and then fold back to remain confined in a bounded phase space respecting ergodicity. The flip side of these complex trajectories is the potential information that can be obtained if one tracks these trajectories. The perspective, quantified by the KS (Kolmogorov-Sinai) entropy [1], gives the rate of information gain, at increasingly fine scales, about the missing information in classical chaos - the initial conditions [2].

The central goal of quantum chaos is to inform us about the properties of quantum systems whose classical counterpart is chaotic. How does chaos manifest in the quantum world, and what notions of complexity might be suitable to quantify it? Quantum theory comes with another layer of complexity hitherto unknown in our classical description of reality, the Hilbert space, which is a big space [3]. Quantum theory permits the state of the system to be any vector in this space, even permitting a coherent superposition of possibilities considered mutually exclusive in the classical world. Therefore, while classically chaotic dynamics generate classical information in the form of classical trajectories, quantum chaotic dynamics generate quantum information in the form of pseudo-random vectors in the Hilbert space, which typically have a high entropy. Vigorous thrust in the understanding of quantum many-body dynamical systems through dynamically generated entanglement [4–10] and quantum correlations [11, 12], deeper studies in the ergodic hierarchy of quantum dynamical systems [13–16] have been some recent milestones. Also, the out-of-time-ordered correlators (OTOCs) that attempt to capture operator growth and scrambling of quantum information have been very useful as a probe for chaos in quantum systems [17–24]. These, coupled with the traditional approach to studies of level statistics [25] and Loschmidt echo [26–29] and complemented by the ability to coherently control and manipulate many-body quantum systems in the laboratory [30–34], have brought us to a fork in our path. On the one hand, this is a harbinger of the possibility of building quantum simulators, an important milestone in our quest for the holy grail - a many-body quantum computer. On the other hand, the same properties that make quantum systems generate complexity will make them sensitive to errors that naturally occur in implementing many-body Hamiltonians.

Quantum simulators are broadly regarded as one of the most promising near-term applications of quantum technologies [35–40]. However, in the era of noisy, intermediate-scale quantum (NISQ) devices [41], the accuracy of an analog quantum simulator will decay after just a few time steps. The reliability of such analog quantum simulators is highly questionable even for state-of-the-art architecture when it is likely to exhibit quantum chaos [42, 43]. On the contrary, the digital quantum simulation is often associated with the inherent Trotter errors [44] because of the discretization of the time evolution of a quantum many-body system as a sequence of quantum gates. It is observed that in specific models like long-range Ising spin chain model [45–47] above a sharp threshold value of the Trotter step size, chaos sets in, and that ultimately leads to the proliferation of Trotter errors [48]. Moreover, specific regimes are known as dynamical structural instabilities where the Floquet map underlying the Trotter decomposition faces abrupt
changes even for small variations in the simulation step size [49]. Thus, a better understanding of errors in simulating many-body quantum systems and information processing protocols that exploit such rich dynamics is paramount.

While chaotic dynamics is a source of information quantified by the positive KS entropy, it is sensitive to errors, as captured by Loschmidt echo. In many body systems, quantum or classical, we must expect the presence of both chaos and errors. In this letter, we address this scenario: we go on to discover quantum signatures of chaos while shedding light on the larger question of many-body quantum simulations under unavoidable perturbations. While the KS entropy enables a rapid information gain, Loschmidt echo will cause a rapid accumulation of errors, or error scrambling as we quantify. This interplay between KS entropy and Loschmidt echo is a generic feature of any many-body system, and we identify and quantify the crossover between these two competing effects.

Quantum tomography gives us a window to study sensitivity to errors in quantum simulations of chaotic Hamiltonians [35, 37]. Quantum tomography uses the statistics of measurement records on an ensemble of identical systems in order to make the best estimate of the actual state \( \rho_0 \). Here we consider continuous weak measurement tomography protocol [33, 50–52], and the time series of operators can be generated by the Floquet map of a quantum dynamical system to investigate the role of chaos on the information gain in tomography [53–55].

An ensemble of \( N \) identical systems \( \rho_0 \) undergo separable time evolution by a unitary \( U(t) \). A weakly coupled probe will generate the measurement record by performing weak continuous measurement of an observable \( \mathcal{O} \). For sufficiently weak coupling, the randomness of the measurement outcomes is dominated by the quantum noise in the probe rather than the measurement uncertainty, i.e., the projection noise. In this case, the quantum backaction is negligible, and the state remains approximately separable. Thus, we get the stochastic measurement record

\[
M(t) = \text{Tr}(\mathcal{O}(t)\rho_0) + W(t),
\]

where \( \mathcal{O}(t) = U(t)\mathcal{O}U(t)^\dagger \) is the time evolved operator in the Heisenberg picture, and \( W(t) \) is a Gaussian white noise with spread \( \sigma/N \).

Any density matrix of Hilbert-space dimension \( d \) can be realized as a generalized Bloch vector \( \mathbf{r} \) by expanding \( \rho_0 = I/d + \sum_{\alpha=1}^{d^2-1} r_\alpha E_\alpha \) in an orthonormal basis of traceless Hermitian operators \( \{E_\alpha\} \). We consider the measurement record at discrete times as \( M_n = M(t_n) = \text{Tr}(\mathcal{O}_n\rho_0) + W_n \), that allows one to express the measurement history

\[
M = \tilde{\mathbf{O}} \mathbf{r} + \mathbf{W},
\]

where \( \tilde{\mathbf{O}}_{\alpha\alpha} = \text{Tr}(\mathcal{O}_n E_\alpha) \). Thus, the problem of quantum tomography is reduced to linear stochastic state estimation of \( \rho_0 \) given \( \{M_n\} \). In the limit of negligible backaction, the probability distribution associated with measurement history \( M \) for a given state vector \( \mathbf{r} \) is [50, 51]

\[
p(M|\mathbf{r}) \propto \exp\left\{-\frac{N^2}{2\sigma^2} \sum_i \left[ M_i - \sum_\alpha \tilde{\mathbf{O}}_{\alpha i} r_\alpha \right]^2 \right\}
\]

\[
\propto \exp\left\{-\frac{N^2}{2\sigma^2} \sum_{\alpha,\beta} (\mathbf{r} - \mathbf{r}_{\text{ML}})_\alpha C_{\alpha\beta}^{-1} (\mathbf{r} - \mathbf{r}_{\text{ML}})_\beta \right\}.
\]

(3)

In the weak backaction limit, the fluctuations around the mean are Gaussian distributed, and hence the maximum likelihood estimate of the Bloch vector components is the least-squared fit as

\[
\mathbf{r}_{\text{ML}} = C\tilde{\mathbf{O}}^T \mathbf{M},
\]

(4)

where \( C = (\tilde{\mathbf{O}}^T \tilde{\mathbf{O}})^{-1} \) is the covariance matrix and the inverse is Moore-Penrose pseudo inverse [56] in general. The estimated Bloch vector \( \mathbf{r}_{\text{ML}} \) may not always represent a physical density matrix with non-negative eigenvalues because of the finite signal-to-noise ratio. Therefore we impose the constraint of positive semidefiniteness on the reconstructed density matrix and obtain the physical state closest to the maximum-likelihood estimate. To do this, we employ a convex optimization procedure [57] where the final estimate of the Bloch vector \( \tilde{\mathbf{r}} \) is obtained by minimizing the argument

\[
||\mathbf{r}_{\text{ML}} - \tilde{\mathbf{r}}||^2 = (\mathbf{r}_{\text{ML}} - \tilde{\mathbf{r}})^T C^{-1}(\mathbf{r}_{\text{ML}} - \tilde{\mathbf{r}})
\]

subject to the constraint

\[
I/d + \sum_{\alpha=1}^{d^2-1} \tilde{r}_\alpha E_\alpha \geq 0.
\]

The above description represents an ideal scenario where the experimentalist has complete knowledge of the true dynamics (which is symbolized as unprimed variables describing the observables, \( \mathcal{O}_n \), and covariance matrix, \( C \), thus generated over time) and they can properly reconstruct the state using Eq. (4). However, in reality, one never knows the true underlying dynamics, and there is always a departure from the ideal case due to inevitable errors and perturbations to the true dynamics. Thus, the experimentalist, oblivious to these, models their estimation using a covariance matrix, \( C' = (\tilde{\mathbf{O}}'^T \tilde{\mathbf{O}}')^{-1} \).

Here the primed variables represent the experimentalist’s knowledge of the dynamics in the laboratory, and as a result, they end up reconstructing an incorrect state as \( \tilde{\rho} \) from

\[
\mathbf{r}'_{\text{ML}} = C'\tilde{\mathbf{O}}'^T \mathbf{M}.
\]

(6)

In the above equation, the measurement record is obtained from the measurement device (probe), and the experimentalist is ignorant about the true dynamics (which
is accompanied by perturbations relative to the idealised dynamics as assumed by the experimentalist), given by the unitary $U(t)$, that has generated this record. However, the covariance matrix is uniquely determined from the experimentalist’s version of the dynamics given by the unitary $U'(t)$ and the initial observable $\mathcal{O}$. Thus, the ignorance about the error in the dynamics directs the operator trajectory away from the actual one, leading to an improper reconstruction of the state $\rho_0$.

Our goal is to study the effect of the perturbation on the information gain in tomography in the presence of chaos. To accomplish this, we implement the quantum kicked top [25, 58, 59] described by the Floquet map $F_{\text{QKT}} = e^{-i\lambda J_z^2/2} e^{-i\alpha J_z}$ as the unitary for a period $\tau$ for simplicity, and the unitary at $n^{th}$ time step is $U(n\tau) = U^\tau$. The measurement record generated by such periodic application of the Floquet map is not informationally complete, and it leaves out a subspace of dimension $\geq d - 2$, out of $d^2 - 1$ dimensional operator space. For our current work, we fix the linear precision angle $\alpha = 1.4$ and choose the kicking strength $\lambda$ as the chaoticity parameter. The classical dynamics change from highly regular to fully chaotic as we vary $\lambda$ from 0 to 7. The dynamics that represents the true evolution is perturbed relative to the idealised dynamics given by $F_{\text{QKT}}$, and we choose a small variation in the kicking strength, $\lambda + \delta \lambda$, and the perturbed unitary becomes $U^\tau_r = e^{-i(\lambda+\delta\lambda)J_z^2/2} e^{-i\alpha J_z}$. For our analysis, we consider the dynamics of quantum kicked top for a spin $j = 10$, and perturbation strength $\delta \lambda = 0.01$.

The connection between chaos and information gain depends on the localisation properties of the state, i.e. their participation ratio, the degree of chaos, as well as how well the state is aligned with time-evolved measurement observables [55]. Therefore, to study the effect of the degree of chaos on the performance of noisy tomography purely, we consider random initial states measured via random initial observables (generated by rotating $J_z$ through random unitary) picked from the appropriate Haar measure. We apply our reconstruction protocol on an ensemble of 100 random pure states sampled from the Haar measure on SU(d), where $d = 2j + 1 = 21$. We choose one random initial observable and generate the measurement record from the repeated application of the Floquet map of the quantum kicked top. The fidelity of the reconstructed state $\tilde{\rho}$ obtained from Eq. (6) is determined relative to the actual state $|\psi_0\rangle$, $F = \langle \psi_0 | \tilde{\rho} | \psi_0 \rangle$ as a function of time. We notice that the reconstruction fidelity increases in the beginning despite the errors, and after a certain period of time, it starts decaying. The rise in fidelity during the initial time period is because any information, even if partially inaccurate, about a completely unknown random state offsets the presence of errors in its estimation. However, as time progresses, the effect of errors becomes significant. Beyond a certain time, we observe a decline in fidelity as the dynamics continues to accumulate errors that dominate the archive of information present in the measurement record. Most interestingly, the rate of this fidelity decay is inversely correlated with the degree of chaos in the dynamics. This is the analog of an interplay between the rapid information gain due to Lyapunov divergence, a “quantum” analog of the classical KS entropy, and Loschmidt echo leading to errors that cause fidelity decay.

We now quantify the role of chaos in tomography when the error in the dynamics influences our ability to reconstruct the random quantum states. It is evident from Fig. 1a that the rate of drop in fidelity decreases with an increase in the strength of chaos for small perturbations in the dynamics. To understand the foregoing discussion, we define the operator Loschmidt echo $\mathcal{F}_{\mathcal{O}}$ as the Hilbert-Schmidt inner product of the operators $\mathcal{O}$, and $\mathcal{O}'$ generated from repeated application of the Floquet map for true (perturbed) dynamics $U^\tau$ and ideal (unperturbed) dynamics $U'_r$ of the kicked top on the initial observable $\mathcal{O}$

$$\mathcal{F}_{\mathcal{O}}(t) = \frac{\text{Tr}(\mathcal{O}^\dagger_r \mathcal{O}'_r)}{\text{Tr}(\mathcal{O}^2)}.$$  

The operator Loschmidt echo that captures the overlap of the operators $\mathcal{O}_n$ and $\mathcal{O}'_n$, decays with time. We can see from Fig. 1b that the operator Loschmidt echo decays much slower when the dynamics is chaotic than when it is regular. This behavior of the operator Loschmidt echo correlates positively with the rate of drop in reconstruction fidelity as demonstrated in Fig. 1a and Fig. 1b. The greater the distance between the operators at a given time, the greater the difference between the expectation values with respect to the state and the archive of the measurement record obtained through the time series. Our results give an operational interpretation of the operator Loschmidt echo by connecting it to a concrete physical task of continuous measurement quantum tomography. This also points to a beneficial way to probe these quantities in experiments using current techniques. Quantum relative entropy is a measure of distance between two quantum states. Here we use this metric to measure the distance between two operators $\mathcal{O}_n$ and $\mathcal{O}'_n$. To treat both observables as density operators, we regularize them as follows. We construct a positive operator from an observable by retaining its eigenvectors and taking the absolute value of its eigenvalues. To normalize this operator, we divide it by its trace. Now we can determine the quantum relative entropy

$$D_{\text{KL}}(\rho_{\mathcal{O}_n} \|ho_{\mathcal{O}'_n}) = \text{Tr}(\rho_{\mathcal{O}_n} (\log \rho_{\mathcal{O}_n} - \log \rho_{\mathcal{O}'_n})).$$  

where $\rho_{\mathcal{O}_n}$ and $\rho_{\mathcal{O}'_n}$ are positive operators of unit trace obtained from the regularization of operators $\mathcal{O}_n$ and $\mathcal{O}'_n$ respectively. We can see clearly from Fig. 1c that the distance between the two operators increases rapidly when the level of chaos is less in the dynamics. This indicates
the operator becomes less prone to error in the Hamiltonian with the rise in the level of chaos. Ultimately, this makes quantum state tomography more immune to error in the presence of chaos, as we see in Fig. 1a.

To further elucidate the decline rate of reconstruction fidelity, we connect the operator incompatibility to the information gain. We quantify the incompatibility of two operators $O_n$ and $O'_n$ with time as $I_O(t_n) = \frac{1}{2J^2} \text{Tr} \left( [O_n, O'_n] \right)$.

The growth of OTOC has been studied extensively as a quantifier for information scrambling under chaotic dynamics [17–24]. Similarly, growth of $I_O$ implies scrambling of errors with time. It is apparent from Fig. 1d that the rate of error scrambling decreases with an increase in the value of the chaoticity parameter $\lambda$. This signifies that the measurement record is less affected by the error in the dynamics when one approaches a greater extent of chaos. In Eq. (6), the measurement record $M$ is obtained from the true (perturbed) dynamics, but the covariance matrix $C'$ and $O'$ are determined from the experimentalist’s version of the dynamics (ideal or unperturbed). Thus, a higher rate of error scrambling for regular dynamics leads to a faster decay of reconstruction fidelity as the measurement record is more vulnerable. How errors scramble across a chaotic system, as given by Eq. (9), is itself an interesting quantifier of quantum chaos. Here we notice the correlation between scrambling of errors as captured by the incompatibility between the operator and its time evolution through the error unitary in Eq. (9) and operator Loschmidt echo, as viewed from the lens of quantum tomography under chaotic dynamics. This links two fundamental quantifiers of quantum chaos, complements findings in [60] and provides a different but more intuitive connection.

In conclusion, we find dynamical signatures of chaos that quantify the scrambling of errors across a many-body quantum system that has consequences on the performance of quantum information and simulation protocols. We also give an operational interpretation of the operator Loschmidt echo by connecting it to the growth of distance between operators evolved in continuous measurement quantum tomography. Our results linking Loschmidt echo, error scrambling, and OTOCs will be helpful to the condensed matter community as well and in addressing broader issues involving non-integrable quantum systems [61]. These signatures of chaos can be further explored using state-of-the-art experimental techniques involving cold atoms interacting with lasers and magnetic fields [59]. In future work, we hope to further build upon our results to develop quantum analogs of the “classical shadowing lemma” that guarantee a true classical trajectory in the neighbourhood of any arbitrary simulated trajectory of a chaotic system in the presence of truncation errors due to finite precision [62–66].

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Supplementary Material

Understanding information gain in tomography with errors

Here, we describe information gain and the general behavior of reconstruction fidelity as shown in the main text. Later in this section, we will demonstrate the effect of the magnitude of perturbation on the fidelity obtained in tomography. We observe that independent of the degree of chaos in the dynamics, the fidelity initially rises despite errors and then starts to decline after attaining a peak. As described in the main text, we consider the density matrix of Hilbert-space dimension $d$ can be realized as a generalized Bloch vector $r$ by expanding $\rho_0 = I/d + \sum_{\alpha=1}^{d^2-1} r_\alpha E_\alpha$ in an orthonormal basis of traceless Hermitian operators $\{E_\alpha\}$.

The probability of reconstructing a state $\rho_0$ is

$$p(\rho_0|\mathcal{M},\mathcal{L},\mathcal{M}) = A \ p(\mathcal{M}|\rho_0,\mathcal{L},\mathcal{M}) \ p(\rho_0|\mathcal{L},\mathcal{M}) \ p(\mathcal{L},\mathcal{M}),$$

where $A$ is a normalization constant. The first term $p(\mathcal{M}|\rho_0,\mathcal{L},\mathcal{M})$, is the probability of acquiring a measurement record $\mathcal{M}$, given an initial state $\rho_0$, the dynamics $\mathcal{L}$ (choice of unitaries), and the measurement process $\mathcal{M}$ (choice of operators $\mathcal{O}$ for generating measurement record). This term contains the errors due to shot noise and helps one to quantify the signal-to-noise ratio in various directions in the operator space independent of the state to be estimated. Thus, in the limit of negligible backaction $p(\mathcal{M}|\rho_0,\mathcal{L},\mathcal{M})$ is identical to the probability distribution corresponding to the measurement history $\mathcal{M}$ for a given Bloch vector $r$,

$$p(\mathcal{M}|r) \propto \exp \left\{ -\frac{N^2}{2\sigma^2} \sum_i [M_i - \sum_{\alpha} \mathcal{O}_{i\alpha} r_\alpha]^2 \right\}$$

$$\propto \exp \left\{ -\frac{N^2}{2\sigma^2} \sum_{\alpha,\beta} (r - r_{\text{ML}})_\alpha C^{-1}_{\alpha\beta} (r - r_{\text{ML}})_\beta \right\}.$$

Therefore, this term estimates the information gained, given a density matrix, in different directions in the operator space. Now we have the second term $p(\rho_0|\mathcal{L},\mathcal{M})$, which is the posterior probability distribution relating the knowledge of the dynamics and the measurement operators. On that account, in the limit of vanishing shot noise and with complete knowledge of the system dynamics for given measurement observables $\{E_\alpha\}$, this conditional probability is continuously updated and ultimately becomes a product of Dirac-delta functions. Once we obtain an informationally complete measurement record, each Dirac-delta function identifies a particular Bloch vector component. The term $p(\mathcal{L},\mathcal{M})$ in Eq. (10) can be absorbed in the constant as it gives the prior information about the choice of dynamics and measurement operators. Thus, Eq. (10) separates the probability of quantum state estimation into a product of two terms (up to a constant).

$$p(\rho_0|\mathcal{M},\mathcal{L},\mathcal{M}) \propto \exp \left\{ -\frac{N^2}{2\sigma^2} \sum_i [M_i - \sum_{\alpha} \mathcal{O}_{i\alpha} r_\alpha]^2 \right\} \ p(\rho_0|\mathcal{L},\mathcal{M})$$

$$\propto \exp \left\{ -\frac{N^2}{2\sigma^2} \sum_{\alpha,\beta} (r - r_{\text{ML}})_\alpha C^{-1}_{\alpha\beta} (r - r_{\text{ML}})_\beta \right\} \ p(\rho_0|\mathcal{L},\mathcal{M})$$

In the limit of zero shot-noise, the errors due to the first term are zero, and we may purely focus on the conditional probability distribution, $p(\rho_0|\mathcal{L},\mathcal{M})$. In terms of the observables in continuous measurement tomography, one can express $p(\rho_0|\mathcal{L},\mathcal{M}) = p(\mathcal{L}|\mathcal{O}_1,\mathcal{O}_2,\ldots,\mathcal{O}_n)$, giving the conditional probability of the density matrix parameters $r$ till the time step $n$. For example, consider the measurement operator at the first $k$ time steps are the ordered set $\{E_1, E_2, \ldots, E_k\}$, giving precise information about Bloch vector components $\{r_1, r_2, \ldots, r_k\}$. The conditional probability distribution at time $k$ is,

$$p(r|E_1, E_2, \ldots, E_k) = \delta(r_1 - \text{Tr}[E_1 \rho_0]) \ \delta(r_2 - \text{Tr}[E_2 \rho_0]) \ldots \ \delta(r_k - \text{Tr}[E_k \rho_0]) \ \delta\left( \sum_{\alpha \neq 1,2,\ldots,k} r_\alpha^2 = 1 - 1/d - r_1^2 - r_2^2 - \ldots - r_k^2 \right).$$

(13)

Each noiseless measurement above gives us complete information in one of the orthogonal directions. For ex-
Here, after the first measurement,

\[ p(r|E_1) = \delta(r_1 - \text{Tr}[E_1 \rho_0]) \delta(\sum_{\alpha \neq 1}^{d^2-1} r_\alpha^2 - 1 - 1/d - r_1^2). \] (14)

Therefore, once \( r_1 \) is determined, the rest of the \( d^2 - 2 \) Bloch vector components are constrained to reside on a surface given by the equation \( \sum_{\alpha \neq 1}^{d^2-1} r_\alpha^2 = 1 - 1/d - r_1^2 \). The state estimation procedure shall select a state based on incomplete information consistent with \( r_1 \) as determined precisely by the first measurement and the remaining Bloch vector components from a point on this surface. Therefore, qualitatively, the average fidelity of the estimated state is proportional to the area of this surface. After \( k \) time steps, the error is proportional to the area of the surface consistent with the equation \( \sum_{\alpha \neq 1}^{d^2-1} r_\alpha^2 = 1/d - r_1^2 - r_2^2 - \ldots - r_k^2 \). This area, quantifying the average error, decreases with each subsequent measurement.

To see it in another way, consider the fidelity between the actual and reconstructed state. The fidelity \( F = \langle \psi_0 | \tilde{\rho} | \psi_0 \rangle \),

\[ F = 1/d + \sum_{\alpha=1}^{d^2-1} \bar{r}_\alpha r_\alpha \] (15)

Here \( r_\alpha \) and \( \bar{r}_\alpha \) are the Bloch vectors for \( \rho_0 \) and \( \tilde{\rho} \) respectively. As one makes measurements, \( E_1, E_2, \ldots, E_k \) and gets information about the corresponding Bloch vector components (with absolute certainty in the case of zero noise for example), one can express the fidelity as

\[ F = 1/d + \sum_{i=1}^{k} r_i^2 + \sum_{\alpha \neq 1,2,\ldots,k}^{d^2-1} \bar{r}_\alpha r_\alpha \] (16)

The term \( \sum_{i=1}^{k} r_i^2 \) puts a lower bound on the fidelity obtained after \( k \) measurements. Here \( r_\alpha \) for \( \alpha \neq 1,2,\ldots,k \) represent the state estimator’s guess for the unmeasured Bloch vector components consistent with the constraint \( \sum_{\alpha \neq 1,2,\ldots,k}^{d^2-1} r_\alpha^2 = 1/d - r_1^2 - r_2^2 - \ldots - r_k^2 \). It is this guess that picks a point from the surface with area consistent with the above constraint.

Consider the same scenario but now with perturbations to the system dynamics. The estimate of the density matrix gets modified as

\[ p(r|E'_1,E'_2,\ldots,E'_k) = \delta(r'_1 - \text{Tr}[E'_1 \rho_0]) \delta(r'_2 - \text{Tr}[E'_2 \rho_0]) \ldots \delta(r'_k - \text{Tr}[E'_k \rho_0]) \delta(\sum_{\alpha \neq 1,2,\ldots,k}^{d^2-1} r'^2_\alpha - 1 - 1/d - r'_1^2 - r'_2^2 - \ldots - r'_k^2). \] (17)

Here, \( E'_1, E'_2, \ldots, E'_k \) are the perturbed operators leading

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FIG. 2. Reconstruction fidelity as a function of time in the limit of vanishing shot noise for an increase in the perturbation of \( \{E_\alpha\} \). The measurement operators are the perturbed ordered set \( \{E'_1, E'_2, \ldots, E'_k\} \) with ordered Bloch vector components of the initial state \( \rho_0 \) (i.e. that corresponds to the Bloch vector components, \( r_\alpha = \text{Tr}[\rho_0 E_\alpha] \) in a particular order of their magnitudes). The perturbed operators \( E'_\alpha \) are generated by applying a fractional power \( \eta \) of a random unitary \( U_\eta \). The inset figure shows the Euclidean norm of the difference between \( U_\eta \) and Identity \( I \), which increases with an increase in the value \( \eta \).

FIG. 3. Reconstruction fidelity as a function of time for an increase in perturbation strength. The measurement record is generated for spin \( j = 10 \). Here we consider rotation angle \( \alpha = 1.4 \) and kicking strength \( \lambda = 7.0 \) for the quantum kicked top.
ponents \(r_1', r_2', ..., r_k'\) respectively. The operators \(\{E'_\alpha\}\) are obtained by rotating \(\{E_\alpha\}\) by a unitary \(U'_\eta\), where \(U_r\) is a random unitary and \(\eta\) is a fractional power which makes \(U'_\eta - I\) close to identity. The Euclidean norm of the operator \(U'_\eta - I\) is less when \(\eta\) is small. Thus, \(\eta\) serves as the strength of perturbation in this analysis of Bloch vector components.

In spite of the perturbation, the uncertainty of the Bloch vector components \(r_\alpha\) for \(\alpha \neq 1, 2, ..., k\) reduces to the area of the surface consistent with the equation

\[
\sum_{\alpha \neq 1, 2, ..., k} r_\alpha^2 = 1 - 1/d - r_1'^2 - r_2'^2 - ... - r_k'^2.
\]

The fidelity between the original and the estimated state now reads as

\[
\mathcal{F} = 1/d + \sum_{i=1}^{k} r_i' r_i + \sum_{\alpha \neq 1, 2, ..., k} \bar{r}_\alpha r_\alpha, \quad (18)
\]

that we can see in Fig. 2. We know that the overlap between two Bloch vectors is maximum only when they are exactly aligned in the same direction, and the overlap decreases when they move far from each other. Comparing the second terms of Eq. (16) and Eq. (18) it is now clear why with an increase in perturbation, the initial rise in fidelity is less. Therefore, the drop in fidelity is more, and the fidelity saturates at a lower value if the perturbation is more, as illustrated in Fig. 3. For relatively weaker perturbations, the fidelity will continue to increase when there is an information gain despite such errors to the measurement operators. The partially inaccurate information about the \(j\)th Bloch vector owing to perturbations to the dynamics still offsets the estimator’s guess of the Bloch components of the unmeasured \(j\)th direction in the operator space determined by \(E_j\).