NEW LOWER BOUND FOR THE RADIUS OF ANALYTICITY OF SOLUTIONS TO THE FIFTH ORDER KDV-BBM MODEL

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ABSTRACT. We show that the uniform radius of spatial analyticity \( \sigma(t) \) of solutions at time \( t \) to the fifth order KdV-BBM equation cannot decay faster than \( 1/\sqrt{t} \) for large \( t \), given initial data that is analytic with fixed radius \( \sigma_0 \). This improves a recent result by Belayneh, Teggen and the third author \([1]\), where they obtained a \( 1/t \) decay of \( \sigma(t) \) for large time \( t \).

1. Introduction

In this paper we consider the Cauchy problem for fifth order KdV-BBM equation

\[
\begin{aligned}
\partial_t \eta + \partial_x \eta - \gamma_1 \partial_t \partial_x^2 \eta + \gamma_2 \partial_x^3 \eta + \delta_1 \partial_t \partial_x^4 \eta + \delta_2 \partial_x^5 \eta \\
= -\frac{3}{4} \partial_x (\eta^2) - \gamma \partial_x^3 (\eta^2) + \frac{7}{48} \partial_x (\eta^3) + \frac{1}{8} \partial_x (\eta^3),
\end{aligned}
\tag{1}
\]

where \( \eta : \mathbb{R}^{1+1} \to \mathbb{R} \) is the unknown function, and \( \gamma, \gamma_1, \gamma_2, \delta_1, \delta_2 \) are constants satisfying certain constraints; see \([3,9]\) for more details. The fifth order KdV-BBM equation describes the unidirectional propagation of water waves, and was recently introduced by Bona et al. \([3]\) using the second order approximation in the two way model, the so-called \( abcd \)-system derived in \([3,4]\). In the case \( \gamma = 7/48 \), \( (1) \) satisfies the energy conservation

\[ E(t) := \frac{1}{2} \int_{\mathbb{R}} \eta^2 + \gamma_1 \eta_x^2 + \delta_1 \eta_{xx}^2 \, dx = E(0) \quad (t > 0). \]

The well-posedness theory for the Cauchy problem \((1)\) was studied by Bona et al. in \([2]\), where they established local well-posedness for the initial data \( \eta_0 \in H^s(\mathbb{R}) \) with \( s \geq 1 \). For \( \gamma_1, \sigma_1 > 0 \) and \( \gamma = 7/48 \), the authors \([2]\) used the conservation of energy to prove global well-posedness of \((1)\) for \( \eta_0 \in H^s(\mathbb{R}) \) with \( s \geq 2 \). Furthermore, they used the method of high-low frequency splitting to obtain global well-posedness for \( \eta_0 \in H^s(\mathbb{R}) \) with \( 3/2 \leq s < 2 \). The global well-posedness result was further improved in \([9]\) for \( \eta_0 \in H^s(\mathbb{R}) \) with \( s \geq 1 \).

The main concern of this paper is to study the property of spatial analyticity of the solution \( \eta(x, t) \) to \((1)\), given a real analytic initial data \( \eta_0(x) \) with uniform radius of analyticity \( \sigma_0 \), so that there is a holomorphic extension to a complex strip

\[ S_{\sigma_0} = \{ x + iy \in \mathbb{C} : |y| < \sigma_0 \}. \]

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Information about the domain of analyticity of a solution to a PDE can be used to gain a quantitative understanding of the structure of the equation, and to obtain insight into underlying physical processes. It is classical since the work of Kato and Masuda [17] that, for solutions of nonlinear dispersive PDEs with analytic initial data, the radius of analyticity, $\sigma(t)$, of the solution might decrease with $t$. Bourgain [7] used a simple argument in the context of Kadomtsev Petviashvili equation to show that $\sigma(t)$ decays exponentially in $t$.

Rapid progress has been made lately in obtaining an algebraic decay rate of the radius, i.e., $\sigma(t) \sim t^{-\alpha}$ for some $\alpha \geq 1$, to various nonlinear dispersive PDEs, see eg., [1, 15, 22–26]. The method used in these papers was first introduced by Selberg and Tesfahun [24] in the context of the Dirac-Klein-Gordon equations, which is based on an approximate conservation laws and Bourgain’s Fourier restriction method. For earlier studies concerning properties of spatial analyticity of solutions for a large class of nonlinear partial differential equations, see eg., [5–7, 11–14, 16, 17, 19–21].

By the Paley–Wiener Theorem, the radius of analyticity of a function can be related to decay properties of its Fourier transform. It is therefore natural to take initial data in Gevrey space $G^{\sigma,s}(\mathbb{R})$ defined by the norm

$$\|f\|_{G^{\sigma,s}(\mathbb{R})} = \left\| \exp(\sigma|\xi|) \xi^s f \right\|_{L^2_x(\mathbb{R})} \quad (\sigma \geq 0),$$

where $\langle \xi \rangle = \sqrt{1 + \xi^2}$. For $\sigma = 0$, this space coincides with the Sobolev space $H^s(\mathbb{R})$, with norm

$$\|f\|_{H^s(\mathbb{R})} = \left\| \langle \xi \rangle^s f \right\|_{L^2_x(\mathbb{R})},$$

while for $\sigma > 0$, any function in $G^{\sigma,s}(\mathbb{R})$ has a radius of analyticity of at least $\sigma$ at each point $x \in \mathbb{R}$. This fact is contained in the following theorem, whose proof can be found in [18] in the case $s = 0$; the general case follows from a simple modification.

**Paley-Wiener Theorem.** Let $\sigma > 0$ and $s \in \mathbb{R}$, then the following are equivalent

(a) $f \in G^{\sigma,s}(\mathbb{R})$,

(b) $f$ is the restriction to $\mathbb{R}$ of a function $F$ which is holomorphic in the strip

$$S_\sigma = \{ x + iy \in \mathbb{C} : |y| < \sigma \}.$$

Moreover, the function $F$ satisfies the estimates

$$\sup_{|y| < \sigma} \| F(\cdot + iy) \|_{H^s(\mathbb{R})} < \infty.$$

Recently, Carvajal and Panthee [8] used the Gevrey space to obtain an exponential decay on the radius of spatial analyticity $\sigma(t)$ for solution $\eta(x,t)$ to (1), i.e., $\sigma(t) \sim e^{-t}$ for large $t$. This was improved, more recently, to a linear decay rate, $\sigma(t) \sim 1/t$, by Belayneh, Tegzen and the third author [1], using the method of almost conservation law. In the present paper, we improve the decay rate further to $\sigma(t) \sim 1/\sqrt{t}$, by using a modified Gevrey space that was introduced recently in [10] and the method of almost conservation law.

The modified Gevrey space, denoted $H^{\sigma,s}(\mathbb{R})$, is obtained from the Gevrey space $G^{\sigma,s}(\mathbb{R})$ by replacing the exponential weight $\exp(\sigma|\xi|)$ with the hyperbolic...
weight \( \cosh(\sigma|\xi|) \), i.e.,

\[
\|f\|_{H^{\sigma,s}(\mathbb{R})} = \left\| \cosh(\sigma|\xi|)\xi^s f \right\|_{L^2_x(\mathbb{R})} \quad (\sigma \geq 0).
\]

Observe that

\[
\frac{1}{2} \exp(\sigma|\xi|) \leq \cosh(\sigma|\xi|) \leq \exp(\sigma|\xi|),
\]

and hence the \( G^{\sigma,s}(\mathbb{R}) \) and \( H^{\sigma,s}(\mathbb{R}) \)-norms are equivalent, i.e.,

\[
\|f\|_{H^{\sigma,s}(\mathbb{R})} \sim \|f\|_{G^{\sigma,s}(\mathbb{R})} = \left\| \exp(\sigma|\xi|)\xi^s f \right\|_{L^2_x(\mathbb{R})}.
\]

Therefore, the statement of Paley-Wiener Theorem still holds for functions in \( H^{\sigma,s}(\mathbb{R}) \).

Observe also that for \( \sigma \geq 0 \) the exponential weight \( \exp(\sigma|\xi|) \) satisfies the estimate

\[
1 - \exp(-\sigma|\xi|) \leq \sigma\frac{|\xi|}{|\xi|},
\]

whereas the hyperbolic weight \( \cosh(\sigma|\xi|) \) satisfies

\[
1 - \frac{|\cosh(\sigma|\xi|)|^{-1}}{|\xi|^2} \leq \sigma^2.
\]

Consequently, the decay rate \( \sigma(t) \sim 1/t \) that was obtained in [1] stems from the \( \sigma \)-factor on the r.h.s of (5) whereas the improved decay rate \( \sigma(t) \sim 1/\sqrt{t} \) obtained in this paper stems from the \( \sigma^2 \)-factor on the r.h.s of (6).

We state our main result as follows.

**Theorem 1** (Asymptotic lower bound for \( \sigma \)). Let \( \gamma_1, \delta_1 > 0, \gamma = 7/48 \), and \( \eta_0 \in H^{\sigma_0,2}(\mathbb{R}) \) for \( \sigma_0 > 0 \). Then the global \(^1\) solution \( \eta \) of (1) satisfies

\[ \eta(t) \in H^{\sigma,2}(\mathbb{R}) \quad \text{for all} \quad t > 0, \]

with the radius of analyticity \( \sigma \) satisfying the asymptotic lower bound

\[ \sigma := \sigma(t) \geq C/\sqrt{t} \quad \text{as} \quad t \to \infty, \]

where \( C > 0 \) is constant depending on the initial data norm \( \|\eta_0\|_{H^{\sigma_0,2}(\mathbb{R})} \).

So it follows from Theorem 1 that the solution \( \eta(t) \) at any time \( t \) is analytic in the strip \( S_{\sigma(t)} \) (due to (4) and the Paley-Wiener Theorem).

To prove Theorem 1 first we establish the following local well-posedness result, which states that for short time the radius of analyticity of solution remains constant.

**Theorem 2.** (Local well-posedness). Let \( \sigma_0 > 0 \) and \( \eta_0 \in H^{\sigma_0,2}(\mathbb{R}) \). Then there exist a unique solution

\[ \eta \in C([0,T];H^{\sigma_0,2}(\mathbb{R})) \]

of the Cauchy problem (1), where the existence time is

\[ T \sim \left( 1 + \|\eta_0\|_{H^{\sigma_0,2}(\mathbb{R})} \right)^{-2}. \]

---

\(^1\)As a consequence of the embedding \( H^{\sigma_0,2}(\mathbb{R}) \hookrightarrow H^2(\mathbb{R}) \) and the existing well-posedness theory in \( H^2(\mathbb{R}) \) (see [2]), the Cauchy problem (1) (with \( \gamma_1, \delta_1 > 0 \) and \( \gamma = 7/48 \)) has a unique, smooth solution for all time, given initial data \( \eta_0 \in H^{\sigma_0,2} \).
Moreover, the data to solution map \( \eta_0 \mapsto \eta \) is continuous from \( H^{\sigma_0,2}(\mathbb{R}) \) to \( C([0,T]; H^{\sigma_0,2}(\mathbb{R})) \).

Next, we derive an approximate energy conservation law for

\[ \nu_\sigma := \cosh(\sigma |D|)\eta, \]

where \( D = -i\partial_x \) and \( \eta \) is a solution to (1). To do this, we define a modified energy associated with \( \nu_\sigma \) by

\[ E_\sigma(t) = \frac{1}{2} \int_{\mathbb{R}} \nu_\sigma^2 + \gamma_1 (\partial_x \nu_\sigma)^2 + \delta_1 (\partial^2_x \nu_\sigma)^2 \, dx. \tag{8} \]

Note that since \( \nu_0 = \eta \), by (2) we have \( E_0(t) = E_0(0) \) for all \( t \).

**Theorem 3. (Almost conservation law).** Let \( \eta_0 \in H^{\sigma,2}(\mathbb{R}) \). Suppose that \( \eta \in C([0,T]; H^{\sigma,2}(\mathbb{R})) \) is the local-in-time solution to the Cauchy problem (1) from Theorem 2. Then

\[ \sup_{0 \leq t \leq T} E_\sigma(t) = E_\sigma(0) + \sigma^2 T O \left( \left[ 1 + (E_\sigma(0))^{1/2} \right] (E_\sigma(0))^{1/2} \right). \tag{9} \]

Observe that from (9), in the limit as \( \sigma \to 0 \), we recover the conservation \( E_0(t) = E_0(0) \) for \( 0 \leq t \leq T \). Applying the last two theorems repeatedly, and then by taking \( \sigma \) small enough we can cover any time interval \([0,T_x]\) and obtain the lower bound in Theorem 1.

**Notation:** For any positive numbers \( p \) and \( q \), the notation \( p \lesssim q \) stands for \( p \leq cq \), where \( c \) is a positive constant that may vary from line to line. Moreover, we denote \( p - q \) when \( p \lesssim q \) and \( q \lesssim p \).

In the next sections we prove Theorems 2, 3 and 1.

2. **Proof of Theorem 2**

Taking the spatial Fourier transform of the first equation in (1) we obtain

\[ \partial_t \hat{\eta} + i\xi \hat{\eta} + \gamma_1 \xi^2 \partial_x \hat{\eta} - i\gamma_2 \xi^3 \hat{\eta} + \delta_1 \xi^4 \partial_x \hat{\eta} + i\delta_2 \xi^5 \hat{\eta} \]

\[ = -\frac{3}{4} i\xi \hat{\eta}^2 + i\gamma_1 \xi^3 \hat{\eta}^2 + \frac{7}{48} i\xi \hat{\eta}^2 + \frac{1}{8} i\xi \hat{\eta}^3. \]

Arranging the terms we have

\[ \left( 1 + \gamma_1 \xi^2 + \delta_1 \xi^4 \right) \partial_x \hat{\eta} + i\xi \left( 1 - \gamma_2 \xi^2 + \delta_2 \xi^4 \right) \hat{\eta} \]

\[ = \frac{1}{4} i\xi \left( -3 + 4\gamma_1 \xi^2 \right) \hat{\eta}^2 + \frac{7}{48} i\xi \hat{\eta}^2 + \frac{1}{8} i\xi \hat{\eta}^3. \]

Dividing this equation by \( \varphi(\xi) := 1 + \gamma_1 \xi^2 + \delta_1 \xi^4 \) and multiplying by \( i \), we obtain

\[ i\partial_t \hat{\eta} - \phi(\xi) \hat{\eta} = \tau(\xi) \hat{\eta}^2 - \frac{7}{48} \psi(\xi) \hat{\eta}^2 - \frac{1}{8} \psi(\xi) \hat{\eta}^3, \tag{10} \]

where

\[ \phi(\xi) = \frac{\xi(1 - \gamma_2 \xi^2 + \delta_2 \xi^4)}{\varphi(\xi)}, \quad \tau(\xi) = \frac{\xi(3 - 4\gamma_1 \xi^2)}{4 \varphi(\xi)}, \quad \psi(\xi) = \frac{\xi}{\varphi(\xi)}. \]

In an operator form (10) can be rewritten as

\[ i\partial_t \eta - \phi(D) \eta = \tau(D) \eta^2 - \frac{7}{48} \psi(D) \eta^2 - \frac{1}{8} \psi(D) \eta^3, \tag{11} \]

\[ := F(\eta) \]
where \( \phi(D), \psi(D) \) and \( \tau(D) \) are Fourier multiplier operators defined as
\[
\mathcal{F}[\phi(D)f](\xi) = \phi(\xi)\hat{f}(\xi), \quad \mathcal{F}[\psi(D)f](\xi) = \psi(\xi)\hat{f}(\xi), \quad \mathcal{F}[\tau(D)f](\xi) = \tau(\xi)\hat{f}(\xi).
\]
Now given initial data \( \eta(0) = \eta_0 \), the integral representation of (11) is
\[
\eta(t) = e^{-it\phi(D)}\eta_0 - i \int_0^t e^{-i(t-s)\phi(D)}F(\eta)(s)\,ds. \tag{12}
\]

By combining the estimates in [8, Lemma 2.2–2.4] and (4), we obtain the following a priori estimate for the \( H^{\sigma,2}(\mathbb{R}) \)-norm of \( F(\eta) \).

**Lemma 1.** For \( \sigma \geq 0 \), we have nonlinear estimate
\[
\|F(\eta)\|_{H^{\sigma,2}(\mathbb{R})} \lesssim \|1 + \|\eta\|_{H^{\sigma,2}(\mathbb{R})}\| \|\eta\|^2_{H^{\sigma,2}(\mathbb{R})}
\]
for all \( \eta \in H^{\sigma,2}(\mathbb{R}) \).

Next, we use the contraction mapping techniques and Lemma 1 to prove Theorem 2. To this end, define the mapping \( \eta \mapsto \Gamma(\eta) \) by
\[
\Gamma(\eta)(t) := e^{-it\phi(D)}\eta_0 - i \int_0^t e^{-i(t-s)\phi(D)}F(\eta)(s)\,ds
\]
and the space \( X_T \) by
\[
X_T = C([0,T] : H^{\sigma,2}(\mathbb{R})) \quad \text{with norm} \quad \|u\|_{X_T} = \sup_{0 \leq t \leq T} \|u(t)\|_{H^{\sigma,2}(\mathbb{R})}.
\]
Then we look for a solution in the set
\[
S_\mathcal{R} = \{ \eta \in X_T : \|\eta\|_{X_T} \leq r \},
\]
where \( 2r = \|\eta_0\|_{H^{\sigma,2}(\mathbb{R})} \).

For \( \eta \in X_T \), we have by Lemma 1,
\[
\|\Gamma(\eta)\|_{X_T} \leq \|\eta_0\|_{H^{\sigma,2}(\mathbb{R})} + cT \left[ 1 + \|\eta\|_{X_T} \right]\|\eta\|_{X_T}^2 
\leq r/2 + cTr(1 + r)^2. \tag{13}
\]
Similarly, for \( \eta_1, \eta_2 \in X_T \), we obtain the difference estimate
\[
\|\Gamma(\eta_1) - \Gamma(\eta_2)\|_{X_T} \leq cT(1 + r)^2\|\eta_1 - \eta_2\|_{X_T}. \tag{14}
\]
By choosing
\[
T = \frac{1}{2c(1 + r)^2}
\]
in (13) and (14) we obtain
\[
\|\Gamma(\eta)\|_{X_T} \leq r \quad \text{and} \quad \|\Gamma(\eta_1) - \Gamma(\eta_2)\|_{X_T} \leq \frac{1}{2}\|\eta_1 - \eta_2\|_{X_T}.
\]
Therefore, \( \Gamma \) is a contraction on \( S_\mathcal{R} \) and therefore it has a unique fixed point \( \eta \in S_\mathcal{R} \) solving the integral equation (12) on \( \mathbb{R} \times [0,T] \). Continuous dependence on the initial data can be shown in a similar way, using the difference estimate. This concludes the proof of Theorem 2.
3. Proof of Theorem 3

Fix $\gamma_1, \delta_2 > 0$ and $\gamma = \frac{7}{3}$. Recall that $v_\sigma := \cosh(\sigma|\Delta|)\eta$, where $\eta$ is the solution to (1), and hence $\eta = \text{sech}(\sigma|\Delta|)v_\sigma$.

Applying the operator $\cosh(\sigma|\Delta|)$ to (1) we obtain
\[
\partial_t v_\sigma + \partial_x v_\sigma - \gamma_1 \partial_t \partial_x^2 v_\sigma + \gamma_2 \partial_3^2 v_\sigma + \delta_1 \partial_t \partial_3^4 v_\sigma + \delta_2 \partial_3^6 v_\sigma
\]
\[
= - \left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x (v_\sigma^2) - \gamma \partial_x (\partial_x v_\sigma)^2 + \frac{1}{8} \partial_x (v_\sigma^3) + N(v_\sigma),
\]
where
\[
N(v_\sigma) = \left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x N_1(v_\sigma) - \gamma \partial_x N_2(v_\sigma) - \frac{1}{8} \partial_x N_3(v_\sigma)
\]
with
\[
N_1(v_\sigma) = v_\sigma^2 - \cosh(\sigma|\Delta|) \text{sech}(\sigma|\Delta|)|v_\sigma|^2,
\]
\[
N_2(v_\sigma) = (\partial_x v_\sigma)^2 - \cosh(\sigma|\Delta|) \text{sech}(\sigma|\Delta|)|\partial_x v_\sigma|^2,
\]
\[
N_3(v_\sigma) = v_\sigma^3 - \cosh(\sigma|\Delta|) \text{sech}(\sigma|\Delta|)|v_\sigma|^3.
\]

Differentiating the modified energy, (8), and using (15)–(17) we obtain
\[
\frac{d}{dt} E_\sigma(t) = \int_R v_\sigma \partial_t v_\sigma + \gamma_1 \partial_x v_\sigma \partial_t \partial_x v_\sigma + \delta_1 \partial_3 v_\sigma \partial_t \partial_3 v_\sigma \partial_x v_\sigma \partial_t (\partial_3 v_\sigma) dx
\]
\[
= \int_R v_\sigma \partial_t (v_\sigma - \gamma_1 \partial_x^2 v_\sigma + \delta_1 \partial_3^4 v_\sigma) \partial_x v_\sigma + \left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x (v_\sigma^2) - \gamma \partial_x (\partial_x v_\sigma)^2 - \frac{1}{8} \partial_x (v_\sigma^3) \right] dx
\]
\[
+ \int_R v_\sigma N(v_\sigma) dx.
\]
However, the integral on the third line is zero due to integration by parts (assuming sufficiently regular solution) and the following identities:
\[
u \partial_x u = \frac{1}{2} \left( u^2 \right)_x,
\]
\[
u \partial_3^2 u = (uu_{xx})_x - \frac{1}{2} \left( u^2 \right)_x,
\]
\[
u \partial_3^5 u = \left( uu_4 \right)_x - \left( \partial_x uu_3 \right)_x + \frac{1}{2} \left( u^2 \right)_x
\]
and
\[
u \partial_x (u^2) = \frac{2}{3} \left( u^3 \right)_x,
\]
\[
u \partial_x (u^3) = \frac{3}{4} \left( u^4 \right)_x
\]
\[
u \partial_3^4 u^2 = 2 \left( u^2 u_{xx} \right)_x + u \left( u^2 \right)_x
\]
Therefore,
\[
\frac{d}{dt} E_\sigma(t) = \int_R v_\sigma \partial_t (v_\sigma(x, t)|N(v_\sigma(x, t)) dx.
\]
Consequently, integrating with respect to time we get
\[
E_\sigma(t) = E_\sigma(0) + \int_0^t \int_R v_\sigma(x, s)|N(v_\sigma(x, s)) dx ds.
\]
Combining (18) with the following key lemma, which will be proved in the last section, we obtain (9).
Lemma 2. For $N(v_\sigma)$ as in (16)–(17) we have
\[ \left| \int_R v_\sigma N(v_\sigma) \, dx \right| \leq c\sigma^2 \left[ 1 + \|v_\sigma\|_{H^2(R)}^2 \right] \|v_\sigma\|^2_{H^2(R)} \] (19)
for all $v_\sigma \in H^2(R)$.

Indeed, applying (19) to (18) we obtain
\[ \sup_{0 \leq t \leq T} E_\sigma(t) = E_\sigma(0) + \sigma^2 T \left[ 1 + \|v_\sigma\|_{L^2_\sigma H^2}^2 \right] \] (20)
where $L^2_\sigma H^2 := L^2([0, T] \times R)$ with $T$ is as in Theorem 2.

As a consequence of Theorem 2 we have the bound
\[ \|v_\sigma\|_{L^2_\sigma H^2(R)} = \|\eta\|_{L^2_\sigma H^2(R)} \leq c \|\eta_0\|_{H^2(R)} = c \|v_\sigma(\cdot, 0)\|_{H^2(R)}. \] (21)

On the other hand,
\[ E_\sigma(0) = \frac{1}{2} \int_R [v_\sigma(x, 0)]^2 + \gamma_1 [\partial_x v_\sigma(x, 0)]^2 + \delta_1 \left[ \partial_x^2 v_\sigma(x, 0) \right]^2 dx \]
\[ \sim \|v_\sigma(\cdot, 0)\|_{H^2(R)}^2. \] (22)

From (21) and (22) we get
\[ \|v_\sigma\|_{L^2_\sigma H^2(R)} \sim (E_\sigma(0))^{1/2}, \]
which can combined with (20) to obtain the desired estimate (9).

4. Proof of Theorem 1

Suppose that $\eta(\cdot, 0) = \eta_0 \in H^{\sigma_0-2}(R)$ for some $\sigma_0 > 0$. This implies $v_\sigma(\cdot, 0) = \cosh(\sigma_0 D) \eta_0 \in H^2$, and hence
\[ E_{\sigma_0}(0) \sim \|v_{\sigma_0}(\cdot, 0)\|_{H^2(R)}^2 < \infty. \]

Now following the argument in [24] (see also [22]) we can construct a solution on $[0, T_\sigma]$ for arbitrarily large time $T_\sigma$. This is achieved by applying the approximate conservation (9), so as to repeat the local result in Theorem 3 on successive short time intervals of size $T$ to reach $T_\sigma$, by adjusting the strip width parameter $\sigma \in (0, \sigma_0]$ of the solution according to the size of $T_\sigma$.

In what follows we prove that
\[ \sup_{0 \leq t \leq T_\sigma} E_\sigma(t) \leq 2E_{\sigma_0}(0) \quad \text{for} \quad \sigma \geq C/\sqrt{T_\sigma} \] (23)
for arbitrarily large $T_\sigma$ and $C > 0$ depending on $E_{\sigma_0}(0)$. This would in turn imply
\[ \sup_{0 \leq t \leq T_\sigma} \|\eta(t)\|_{H^{\sigma_0-2}(R)} < \infty \quad \text{for} \quad \sigma \geq C/\sqrt{T_\sigma} \]
which proves Theorem 1.

It remains to prove (23). To do this, first observe that for $\sigma \in (0, \sigma_0]$ and $\tau \in (0, T)$, we have by Theorems 2 and 3,
\[ \sup_{0 \leq t \leq \tau} E_\sigma(t) \leq E_\sigma(0) + c\sigma^2 \left[ 1 + (E_\sigma(0))^{1/2} \right] (E_\sigma(0))^{3/2} \]
\[ \leq E_{\sigma_0}(0) + c\sigma^2 \left[ 1 + (E_{\sigma_0}(0))^{1/2} \right] (E_{\sigma_0}(0))^{3/2}. \]
To get the second line we used the fact the \( \mathcal{E}_\sigma(0) \leq \mathcal{E}_{\sigma_0}(0) \) which holds for \( \sigma \leq \sigma_0 \) as \( \cosh \tau \) is increasing for \( \tau \geq 0 \). Thus,

\[
\sup_{0 \leq t \leq \tau} E_\sigma(t) \leq 2E_{\sigma_0}(0) \tag{24}
\]

provided that

\[
c_0^2T \left[ 1 + (E_{\sigma_0}(0))^{1/2} \right] (E_{\sigma_0}(0))^{3/2} \leq E_{\sigma_0}(0). \tag{25}
\]

Next, we apply Theorem 2 with initial time \( t = \tau \) and time-step size \( T \) as in (7) to extend the solution from \( [0, \tau] \) to \( [\tau, \tau + T] \). By Theorem 3 and (24) we obtain

\[
\sup_{\tau \leq t \leq \tau + T} E_\sigma(t) \leq E_\sigma(\tau) + c_0^2T \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2}. \tag{26}
\]

In this way we cover all time intervals \( [0, T], [T, 2T], \text{etc.} \), and obtain

\[
E_\sigma(T) \leq E_\sigma(0) + c_0^2T \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2}
\]

\[
E_\sigma(2T) \leq E_\sigma(T) + c_0^2T\left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2}
\]

\[
\leq E_\sigma(0) + 2c_0^2T\left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2}
\]

and so on.

\[
E_\sigma(nT) \leq E_\sigma(0) + nc_0^2T\left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2}.
\]

This argument can be continued as long as

\[
nc_0^2T \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2} \leq E_{\sigma_0}(0) \tag{27}
\]

as this would imply \( E_\sigma(nT) \leq 2E_{\sigma_0}(0) \).

Thus, the induction stops at the first integer \( n \) for which

\[
nc_0^2T \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2} > E_{\sigma_0}(0)
\]

and then we have reached the finite time \( T_\ast = nT \) when

\[
c_0^2T_\ast \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{1/2} > 1.
\]

This proves \( \sigma \geq C/\sqrt{T_\ast} \) for some \( C > 0 \) depending on \( E_{\sigma_0}(0) \).

5. Proof of Lemma 2

To prove (19) we need the following estimate from [10, Lemma 3] in the special cases of \( p = 2 \) and \( p = 3 \).

Lemma 3. Let \( \xi = \sum_{j=1}^{p} \xi_j \) for \( \xi_j \in \mathbb{R} \), where \( p \geq 1 \) is an integer. Then

\[
\left| 1 - \cosh |\xi| \prod_{j=1}^{p} \text{sech} |\xi_j| \right| \leq 2^p \sum_{j \neq k=1}^{p} |\xi_j||\xi_k|. \tag{28}
\]

Proof. For the readers convenience we include the proof in the case \( p = 2 \). Note that

\[
\cosh |\xi_1| \cosh |\xi_2| = \frac{1}{2} \left[ \cosh(|\xi_1| - |\xi_2|) + \cosh(|\xi_1| + |\xi_2|) \right]. \tag{29}
\]
On the other hand, we have (see [10, Lemma 2]),

\[ |\cosh b - \cosh a| \leq \frac{1}{2} |b^2 - a^2| (\cosh b + \cosh a). \tag{30} \]

for \( a, b \in \mathbb{R} \).

Then by (29) and (30),

\[
|\cosh |\xi_1| \cosh |\xi_2| - \cosh |\xi| | = \left| \frac{1}{2} \left( \sum_{\pm} \cosh (|\xi_1| \pm |\xi_2|) - \cosh |\xi| \right) \right|
\leq \frac{1}{2} \sum_{\pm} \frac{1}{2} |(|\xi_1| \pm |\xi_2|)^2 - |\xi|^2| (\cosh (|\xi_1| \pm |\xi_2|) + \cosh |\xi|)
\leq \frac{1}{2} \cdot 4|\xi_1||\xi_2| \cdot 4 \cosh(|\xi_1|) \cosh(|\xi_2|)
= 8|\xi_1||\xi_2| \cosh(|\xi_1|) \cosh(|\xi_2|).
\]

Dividing by \( \cosh(|\xi_1|) \cosh(|\xi_2|) \) yields the desired estimate (28) in the case \( p = 2 \). \qed

Next we prove (19). For \( N(v_{\sigma}) \) as in (16)–(17), we use Plancheral theorem to write

\[
\int_{\mathbb{R}} v_{\sigma} N(v_{\sigma}) \, dx = \int_{\mathbb{R}} v_{\sigma} \left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x N_1(v_{\sigma}) - \gamma v \partial_x N_2(v_{\sigma}) - \frac{1}{8} v \partial_x N_3(v_{\sigma}) \, dx
= \underbrace{\int_{\mathbb{R}} v_{\sigma} \left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x N_1(v_{\sigma}) \, dx + \gamma \int_{\mathbb{R}} \partial_x v_{\sigma} \cdot N_2(v_{\sigma}) \, dx}_{:= I_1} + \underbrace{\frac{1}{8} \int_{\mathbb{R}} \partial_x v_{\sigma} \cdot N_3(v_{\sigma}) \, dx}_{:= I_2}.
\]

So (19) follows from

\[
|I_j| \leq \sigma^2 \|v_{\sigma}\|_{H^2(\mathbb{R})}^3, \quad (j = 1, 2) \tag{31}
|I_3| \leq \sigma^2 \|v_{\sigma}\|_{H^2(\mathbb{R})}^4. \tag{32}
\]

5.1. **Proof of (31) when \( j = 1 \).** By Cauchy-Schwarz inequality,

\[
|I_1| \leq \left\| \left( \frac{3}{4} + \gamma \partial_x^2 \right) v_{\sigma} \right\|_{L^2(\mathbb{R})} \|\partial_x N_1(v_{\sigma})\|_{L^2(\mathbb{R})}
\leq \|v_{\sigma}\|_{H^2(\mathbb{R})} \|\partial_x N_1(v_{\sigma})\|_{L^2(\mathbb{R})}.
\]

So the proof reduces to

\[
\|\partial_x N_1(v_{\sigma})\|_{L^2(\mathbb{R})} \lesssim \sigma^2 \|v_{\sigma}\|_{H^2(\mathbb{R})}^2, \tag{33}
\]

where

\[ N_1(v_{\sigma}) = v_{\sigma}^2 - \cosh(\sigma|D|) |\text{sech}(\sigma|D|)|v_{\sigma}|^2. \]
Now taking the Fourier Transform of $\partial_x N_1(\nu_\sigma)$ and applying (28) with $p = 2$, we obtain
\[
|\mathcal{F}[\partial_x N_1(\nu_\sigma)](\xi)|
\leq 4\sigma^2 \int_{\xi = \xi_1 + \xi_2} |\xi| \left( 1 - \cosh(\sigma|\xi|) \right)^2 \sum_{j=1}^{2} \delta(\xi_j) \delta(\xi_{2j}) \, d\xi_1 \, d\xi_2.
\]

By symmetry, we may assume $|\xi_1| \leq |\xi_2|$. Then
\[
|\mathcal{F}[\partial_x N_1(\nu_\sigma)](\xi)| \leq 16\sigma^2 \int_{\xi = \xi_1 + \xi_2} |\xi_1| |\nu_\sigma(\xi_1)| \cdot |\xi_2| |\nu_\sigma(\xi_2)| \, d\xi_1 \, d\xi_2
= 16\sigma^2 \mathcal{F}[|D|w_\sigma|D|^2w_\sigma](\xi),
\]
where $w_\sigma = \mathcal{F}_x^{-1}(|\nu_\sigma|)$. Finally, by Plancheral, Hölder inequality and Sobolev embedding,
\[
\|\partial_x N_1(\nu_\sigma)\|_{L^2_\sigma(\mathbb{R})} \leq 16\sigma^2 \left\| |D|w_\sigma|D|^2w_\sigma \right\|_{L^2_\sigma(\mathbb{R})}
\leq \sigma^2 \left\| |D|w_\sigma \right\|_{L^4_\sigma(\mathbb{R})} \left\| |D|^2w_\sigma \right\|_{L^2_\sigma(\mathbb{R})}
\leq \sigma^2 \left\| w_\sigma \right\|_{H^2(\mathbb{R})} \leq \sigma^2 \left\| \nu_\sigma \right\|_{H^2(\mathbb{R})}^2,
\]
which proves (33).

5.2. Proof of (31) when $j = 2$. By Plancheral and Cauchy-Schwarz inequality,
\[
|I_2| = \left| \int_{\mathbb{R}} \partial_x v_\sigma \cdot N_2(\nu_\sigma) \, dx \right| = \left| \int_{\mathbb{R}} \langle D \rangle \partial_x v_\sigma \cdot (\langle D \rangle^{-1} N_2(\nu_\sigma)) \, dx \right|
\leq \| \langle D \rangle \partial_x v_\sigma \|_{L^2_\sigma(\mathbb{R})} \| (\langle D \rangle^{-1} N_2(\nu_\sigma)) \|_{L^2_\sigma(\mathbb{R})}
\leq \| \nu_\sigma \|_{H^2(\mathbb{R})} \| N_2(\nu_\sigma) \|_{H^{-1}(\mathbb{R})}.
\]

So the proof reduces to
\[
\| N_2(\nu_\sigma) \|_{H^{-1}(\mathbb{R})} \leq \sigma^2 \left| \nu_\sigma \right|_{H^2(\mathbb{R})}^2,
\]
where
\[
N_2(\nu_\sigma) = (\partial_x v_\sigma)^2 - \cosh(\sigma|D|) \left( \text{sech}(\sigma|D|) \partial_x v_\sigma \right)^2.
\]
Taking the spatial Fourier Transform of $N_2(\nu_\sigma)$ and using (28) with $p = 2$, we obtain

$$
\mathcal{F}[N_2(\nu_\sigma)](\xi) = \left| \int_{\xi = \xi_1 + \xi_2} \left( 1 - \cosh(\sigma|\xi|) \prod_{j=1}^{2} \text{sech}(\sigma|\xi|) \right) \hat{v}_\sigma(\xi_1) \hat{v}_\sigma(\xi_2) d\xi_1 d\xi_2 \right|
$$

$$
\leq 8\sigma^2 \int_{\xi = \xi_1 + \xi_2} |\xi_1|^2 |\xi_2|^2 |\hat{v}_\sigma(\xi_1)| |\hat{v}_\sigma(\xi_2)| d\xi_1 d\xi_2
$$

$$
\leq 8\sigma^2 \mathcal{F}_x \left[ |D|^2 \nu_\sigma |D|^2 \nu_\sigma \right](\xi),
$$

where $\nu_\sigma = \mathcal{F}_x^{-1}(|\hat{\nu}_\sigma|)$.

Then by Plancharal, the Sobolev embedding

$$
H^1_x(\mathbb{R}) \hookrightarrow L^\infty_x(\mathbb{R}) \iff L^1_x(\mathbb{R}) \hookrightarrow H^{-1}_x(\mathbb{R})
$$

and Cauchy-Schwarz, we obtain

$$
\|N_2(\nu_\sigma)\|_{H^1_x(\mathbb{R})} \leq 8\sigma^2 \left\| |D|^2 \nu_\sigma |D|^2 \nu_\sigma \right\|_{H^1_x(\mathbb{R})}
$$

$$
\leq \sigma^2 \left\| |D|^2 \nu_\sigma |D|^2 \nu_\sigma \right\|_{L^1_x(\mathbb{R})}
$$

$$
\leq \sigma^2 \left\| |D|^2 \nu_\sigma \right\|_{L^2_x(\mathbb{R})} \left\| |D|^2 \nu_\sigma \right\|_{L^2_x(\mathbb{R})}
$$

$$
\leq \sigma^2 \|\nu_\sigma\|^2_{L^2_x(\mathbb{R})},
$$

which proves (34).

5.3. **Proof of (32).** By Cauchy-Schwarz inequality,

$$
|I_3| = \frac{1}{8} \int_{\mathbb{R}} \partial_x \nu_\sigma N_3(\nu_\sigma) dx \leq \|\partial_x \nu_\sigma\|_{L^2_x(\mathbb{R})} \|N_3(\nu_\sigma)\|_{L^2_x(\mathbb{R})}
$$

$$
\leq \|\nu_\sigma\|_{H^1_x(\mathbb{R})} \|N_3(\nu_\sigma)\|_{L^2_x(\mathbb{R})}.
$$

So it remains to prove

$$
\|N_3(\nu_\sigma)\|_{L^2_x(\mathbb{R})} \lesssim \sigma^2 \|\nu_\sigma\|^3_{H^1_x(\mathbb{R})},
$$

where

$$
N_3(\nu_\sigma) = \nu_\sigma^3 - \cosh(\sigma|D|) |\text{sech}(\sigma|D|)| \nu_\sigma^3.
$$

Taking the Fourier Transform of $N_3(\nu_\sigma)$ and applying (28) with $p = 3$, we obtain

$$
\mathcal{F}_x [N_3(\nu_\sigma)](\xi) = \left| \int_{\xi = \xi_1 + \xi_2 + \xi_3} 1 - \cosh(\sigma|\xi|) \prod_{j=1}^{3} \text{sech}(\sigma|\xi|) \hat{\nu}_\sigma(\xi_1) \hat{\nu}_\sigma(\xi_2) \hat{\nu}_\sigma(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right|
$$

$$
\leq 8\sigma^2 \int_{\xi = \xi_1 + \xi_2 + \xi_3} \left( \sum_{j \neq k=1}^{3} |\xi_j| |\xi_k| \right) |\hat{\nu}_\sigma(\xi_1)| |\hat{\nu}_\sigma(\xi_2)| |\hat{\nu}_\sigma(\xi_3)| d\xi_1 d\xi_2 d\xi_3.
$$
By symmetry, we may assume $|\xi_1| \leq |\xi_2| \leq |\xi_3|$, which implies
\[
\left| \mathcal{F}_x \left[ N_3 (v_\sigma) \right] (\xi) \right| \leq 48 \sigma^2 \int_{\xi = \xi_1 + \xi_2 + \xi_3} \left| \hat{v}_\sigma (\xi_1) \right| \left| \hat{v}_\sigma (\xi_2) \right| \left| \hat{v}_\sigma (\xi_3) \right| d\xi_1 d\xi_2 d\xi_3
\]
\[
= 48 \sigma^2 \mathcal{F}_x (w_\sigma, w_\sigma, |D|^3 w_\sigma) (\xi),
\]
where $w_\sigma = \mathcal{F}_x^{-1} (|\hat{v}_\sigma|)$.

Then by Plancheral and Hölder inequality we get
\[
\left\| \mathcal{F}_x \left[ N_3 (v_\sigma) \right] (\xi) \right\|_{L_2^2 (\mathbb{R})} \lesssim \sigma^2 \left\| w_\sigma |D|^2 w_\sigma \right\|_{L_2^2 (\mathbb{R})} \lesssim \sigma^2 \left\| w_\sigma \right\|_{L_2^\infty (\mathbb{R})} \left\| |D|^2 w_\sigma \right\|_{L_2^2 (\mathbb{R})}
\]
\[
\lesssim \sigma^2 \left\| w_\sigma \right\|_{L_2^2 (\mathbb{R})} \left\| |D|^2 w_\sigma \right\|_{L_2^2 (\mathbb{R})}
\]
\[
\lesssim \sigma^2 \left\| w_\sigma \right\|_{L_2^2 (\mathbb{R})} \left\| |D|^3 w_\sigma \right\|_{L_2^2 (\mathbb{R})}
\]
which proves (35).

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