An $O(k \log n)$ Time Fourier Set Query Algorithm

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Abstract—Fourier transformation is an extensively studied problem in many research fields. It has many applications in machine learning, signal processing, compressed sensing, and so on. In many real-world applications, approximated Fourier transformation is sufficient and we only need to do the Fourier transform on a subset of coordinates. Given a vector $x \in \mathbb{C}^n$, an approximation parameter $\epsilon$ and a query set $S \subseteq [n]$ of size $k$, we propose an algorithm to compute an approximate Fourier transform result $x'$ which uses $O(\epsilon^{-1}k \log(n/\delta))$ Fourier measurements, runs in $O(\epsilon^{-1}k \log(n/\delta))$ time and outputs a vector $x'$ such that $\| (x' - \hat{x})_S \|_2^2 \leq \epsilon \| \hat{x}_S \|_2^2 + \delta \| \hat{x} \|_2^2$ holds with probability of at least 9/10.

Index Terms—Sparse Recovery, Fourier Transform, Set Query.

I. INTRODUCTION

Fourier transform is ubiquitous in image and audio processing, telecommunications and so on. The time complexity of classical Fast Fourier Transform (FFT) algorithm proposed by Cooley and Turkey [1] is $O(n \log n)$. Optics imaging [2], magnetic resonance image (MRI) [4] and the physics [5] are benefits from this algorithm. The algorithm proposed by Cooley and Turkey [1] takes $O(n)$ samples to compute the Fourier transformation result.

The number of samples taken is an important factor. For example, it influences the amount of ionizing radiation that a patient is exposed to during CT scans. The amount of time a patient spends within the scanner can also be reduced by taking fewer samples. Thus, we consider the Fourier Transform problems in two computational aspects. Thus, two aspects of the Fourier Transform problems are taken into consideration by us. The first aspect is the reconstruction time which is the time of decoding the signal from the measurements. The second aspect is the sample complexity. Sample complexity is the number of noisy samples required by the algorithm. There is a long line of works optimizing the time and the sample complexity of Fourier Transform in the field of signal-processing and the field of TCS [1], [5], [4], [2], [6], [7].

As a result, we can anticipate that algorithms that leverage sparsity assumptions about the input and outperform FFT in applications will be of significant practical utility. In general, the two most significant factors to optimize are the sample complexity and the time complexity of obtaining the Fourier Transform result.

In many real world applications, computing the approximate Fourier transformation results for a set of selective coordinates is sufficient, and we can leverage the approximation guarantee to accelerate the computation. The set query is originally proposed by [8]. The original definition doesn’t have restriction on Fourier measurements. Then [9] generalizes the classical set query definition [8] into Fourier setting. In this paper we consider the set estimation based on Fourier measurement problem (defined by [9]) where given a vector $x \in \mathbb{C}^n$, approximation parameters $\epsilon, \delta \in (0, 1)$ and a query set $S \subseteq [n]$ and $|S| = k$, we want to compute an approximate Fourier transform result $x'$ in sublinear time and sample complexity and compared with the Fourier transform result $\hat{x}$, the following approximation guarantee holds:

$$\| (x' - \hat{x})_S \|_2^2 \leq \epsilon \| \hat{x}_S \|_2^2 + \delta \| \hat{x} \|_2^2$$

with probability at least 9/10. For a set $S \subseteq [n]$ and a vector $x \in \mathbb{R}^n$, we define $x_S$ by setting if $i \in S$, $(x_S)_i = x_i$ and otherwise $(x_S)_i = 0$.

| References | Samples | Time |
|------------|---------|------|
| [6] | $e^{-1}k \log^2(n)$ | $e^{-1}k \log^2(n)$ |
| [9] | $e^{-1}k$ | $e^{-1}k \log^2(n) \log(n)$ |
| Ours | $e^{-1}k \log(n)$ | $e^{-1}k \log(n)$ |

TABLE I  
SUMMARY OF THE HISTORY OF RESULTS

For this Fourier set query problem, there are two major prior works [9] and [6]. The [9] studies the problem explicitly and [6] implicitly provides a solution to Fourier set query, we will provide more details in the later paragraphs.

The work by [9] first explicitly define Fourier set query problem and studies it. [9] obtains an algorithm that has sample complexity $O(k/\epsilon)$ and running time $O(e^{-1}k \log^2(n) \log(R^*))$ for $\ell_2/\ell_2$ Fourier set query. Here, $R^*$ is an upper bound on the $\| \cdot \|_\infty$ norm of the vector. In most applications, $R^*$ are considered $\text{poly}(n)$. Our approach gives an algorithm with $O(e^{-1}k \log(n))$ running time. The running time of our result has no dependence on $\log R^*$, but our result do not achieve the optimal sample complexity.

The [6] didn’t study Fourier set query problem, instead they study Fourier sparse recovery problem. However, applying their algorithm [6] to Fourier set query, it provides an algorithm with time complexity of $O(e^{-1}k \log^2(n))$ and sample complexity of $O(e^{-1}k \log^2(n))$. 

Our main contributions are summarized as follows:

- We present a efficient algorithms for Fourier set query problem.
- We provide comprehensive theoretical guarantees to show the predominance of our algorithms over the existing algorithm.

Roadmap. We first present the related work about discrete Fourier transform, continuous Fourier transform and some applications of Fourier transform in Section II. We define our problem and present our main theorem in Section III. We present a high-level overview of our techniques in Section IV. We provide some definitions, notations and technique tools in Section V. And as our main result in this paper, our algorithm (See Algorithm 1.) and the analysis about the correctness and complexity of it is given in Section VI. Finally, we conclude our paper in Section VII.

II. RELATED WORK

a) Discrete Fourier Transform: For computational jobs, among the most crucial and often employed algorithms is the discrete Fourier transform (DFT). There is a long line of works focus on sparse discrete Fourier transforms. Results can be divided into two kinds: the first kind of results choose sub-linear measurements and achieve sublinear or linear recovery time. This kind of work includes [10], [6], [11], [12], [13], [14], [15], [9], [16]. The second kind of results randomly choose measurements and prove that a generic recovery algorithm succeed with high probability. A common generic recovery algorithm that this kind of works used is $\ell_1$ minimization. These results prove the Restricted Isometry Property [17], [18], [19]. Currently, the first kind of solutions have better theoretical guarantee in sample and time complexity. However, the second kind of algorithm has high success probabilities and higher capability in practice.

b) Continuous Fourier Transform: [20] studies sparse Fourier transforms on a continuous signals. They apply a discrete sparse Fourier transform algorithm, followed by a hill-climbing method to optimize their solution into a reasonable range. [21] presents an algorithm whose sample complexity is only linear to $k$ and logarithmic in the signal-to-noise ratio. Their frequency resolution is suitable for robustly computing sparse continuous Fourier transforms. [22] generalizes [21] into high-dimensional setting. [23] provide an algorithm that support the reconstruction of a signal without frequency gap. They present a solution to approximate the signal using a constant factor noise growth and takes samples polynomial in $k$ and logarithmic in the signal-to-noise ratio. Recently [24] improves the approximation ratio of [23].

c) Application of Fourier Transform: Fourier transformation has a wide application in many fields including physics, mathematics, signal processing, probability theory, statistics, acoustics, cryptography and so on.

Solving partial differential equations is one of the most important application of Fourier transformation. Some differential equations are simpler to understand in the frequency domain because the action of differentiation in the time domain corresponds to the multiplication by the frequency. Additionally, frequency-domain multiplication is equivalent to convolution in the time domain [25], [26], [27].

Various applications of the Fourier transform include nuclear magnetic resonance (NMR) [28], [29], [30] and other types of spectroscopy, such as infrared (FTIR) [31]. In NMR, a free induction decay (FID) signal with an exponential shape is recorded in the time domain and Fourier-transformed into a Lorentzian line-shape in the frequency domain. Mass spectrometry and magnetic resonance imaging (MRI) both employ the Fourier transform. The Fourier transform is also used in quantum mechanics [32].

For the spectrum analysis of time-series [33], [34], the Fourier transform is employed. The Fourier transformation is often not applied to the signal itself in the context of statistical signal processing. It has been discovered in practice that it is best to simulate a signal by a function (or, alternatively, a stochastic process) that is stationary in the sense that its distinctive qualities are constant across all time, even though a genuine signal is in fact transitory. It has been discovered that taking the Fourier transform of the function’s autocorrelation function is more advantageous for the analysis of signals since the Fourier transform of such a function does not exist in the conventional sense.

III. FOURIER SET QUERY

In Section III-A, We define the problem we focus on. In Section III-B, we provide our main result.

A. Fourier set query problem

In this section, we give a formal definition of the main problem focused on.

Definition III.1 (Sample Complexity). Given a vector $x \in \mathbb{C}^n$, we say the sample complexity of an algorithm is $c$ (an Algorithm takes $c$ samples), when $c$ is the number of the coordinates used and $c \leq n$.

Definition III.2 (Main problem). Given a vector $x \in \mathbb{C}^n$ and the $\hat{x}$ as the concrete Fourier transformation result, then for every $\epsilon, \delta \in (0, 1)$ and $k \geq 1$, any $S \subseteq [n]$, $|S| = k$, the goal is to design an algorithm that

- takes samples from $x \in \mathbb{C}^n$ (note that we treat one entry of $x$ as one sample)
- takes some time to output a vector $x' \in \mathbb{C}^n$ such that $\|x' - \hat{x}\|_2^2 \leq \epsilon \|\hat{x}\|_2^2 + \delta \|\hat{x}\|_1^2$

We want to optimize both sample complexity (which is the number of coordinates we need to access in $x$), and also the running time.

B. Our Result

We present our main theorem as follows:

Theorem III.3 (Main result). Given a vector $x \in \mathbb{C}^n$ and the $\hat{x}$ as the concrete Fourier transformation result, then for every
\(\epsilon, \delta \in (0, 1) \text{ and } k \geq 1, \text{ any } S \subseteq [n], \, |S| = k, \text{ there exists an algorithm (Algorithm I) that takes} O(\epsilon^{-1}k \log(n/\delta))\)
samples from \(x\), runs in \(O(\epsilon^{-1}k \log(n/\delta))\)
time, and outputs a vector \(x' \in \mathbb{C}^n\) such that
\[\|(x' - \tilde{x})_S\|_2^2 \leq \epsilon \|\tilde{x}_S\|_2^2 + \delta \|\tilde{z}\|_1^2,\]
holds with probability at least \(9/10\).

IV. Technique Overview

In this section, we will give an overview about the technique methods used on the proof of our main result and the complexity analysis about time and sample (See Definition III.1.). At first, we will give an introduction about main functions and their time complexity as well as other properties used in our algorithm. And based on the functions, then we will give the analysis about the correctness of our algorithm where with probability at least \(9/10\) it can finally produce a \(x'\) which satisfies
\[\|(x' - \tilde{x})_S\|_2^2 \leq \epsilon \|\tilde{x}_S\|_2^2 + \delta \|\tilde{z}\|_1^2.\]

The analysis of total complexity comes last, with \(O(\epsilon^{-1}k \log(n/\delta))\) as the sample complexity (See Definition III.1) and \(O(\epsilon^{-1}k \log(n/\delta))\) as the time complexity. And then we can make sure the algorithm solve the problem (See Definition III.2.) with better performance compared to the prior works [9] and [6] (See details in Table I).

a) Technique I: HashToBins: We use the same function HashToBins with the one in [6], which is one of the key part of the function EstimateValues. We can attain a \(\tilde{u}\), where the \(\tilde{u}_j\) for satisfies the following equation
\[\tilde{u}_j = \sum_{h_{\alpha,\beta}} (\tilde{x} - \tilde{z})_i (\tilde{G}_{B,\delta,\alpha}^{-1} - \alpha_{\alpha,\beta} \omega_{\alpha,\beta} \pm \delta \|\tilde{z}\|_1).\]
To help the analysis of the time complexity of our algorithm I, by Lemma V.15, the time complexity of the function above is \(O(\frac{|S|}{\|\tilde{z}\|_0} \log(n/\delta) + \|\tilde{z}\|_0 + \zeta \log(n/\delta))\) with
\[\zeta = |\{i \in \text{supp}(\tilde{z}) \mid E_{\text{off}}(i)\}|.\]

b) Technique II: EstimateValues: EstimateValues is an key function in loop (See Section VI-A). By using this function, we attain the new set \(T_i\) and the new value \(\tilde{w}^{(i)}\) to update \(S_i\) by
\[S_{i+1} \leftarrow S_i \backslash T_i,\]
and \(\tilde{z}^{(i+1)}\) by
\[\tilde{z}^{(i+1)} \leftarrow \tilde{z}^{(i)} + \tilde{w}^{(i)}.\]

c) Technique III: Query Set \(S\): We use \(S\) as the query set and \(S_i\) is the set attained by updating \(S\) with \(i - 1\) iterations. And we use \(k_i = k \gamma^{i-1}\) where \(\gamma \leq \frac{1}{10}\) and \(k \geq 1\).
We demonstrate that we can compress \(S_i\) to a small enough value where \(|S_i| \leq k_i\). Due to reason that \(S_i\) is a query set, the above sentence can be said as that we can finish the query of all the elements in \(S\) with a large enough number of the iterations.

In the proof of above statement, we bring some properties about \(t\) as follows (See Details in Definition V.9):
1) “Collision”
2) “Large offset”
3) “Large noise”

Given a vector \(x\) and \(t \in [n]\) as a coordinate of it, we also give the definition about “well-isolated” based on concepts above. And then we can prove that with probability at least \(1 - a_i\), we can have \(t\) is “well-isolated”.

Based on the statement above, we can have small enough \(|S_i|\) by \(|S_i| \leq k_i\) and a large enough \(R\) here.

d) Technique IV: Correctness and Complexity: By the upper bound of \(\|\tilde{z}^{(i+1)}_S\|_2^2\) which we attain in Section VI-A. We can demonstrate the error can satisfy the requirement in the problem. With probability 10\(a_i/\gamma\), we can have
\[\|\tilde{z}^{(i+1)}_S\|_2^2 \leq (1 + \epsilon_i)\|\tilde{x}_S\|_2^2 + \epsilon_i \delta^2 n\|\tilde{z}\|_1.\]
Then we can demonstrate
\[\|\tilde{x}_S - \tilde{z}^{(R+1)}_S\|_2^2 \leq \epsilon (\|\tilde{x}_S\|_2^2 + \delta^2 n\|\tilde{z}\|_1^2).\]

Notice that the \(\tilde{z}^{(R+1)}\) is the output of our Algorithm I which is also the \(x'\) in our problem (See Definition III.2.). The above inequalities demonstrate that the Algorithm I constructed by us can output a \(x'\) which satisfies
\[\|(x' - x)_S\|_2^2 \leq \epsilon \|\tilde{x}_S\|_2^2 + \delta \|\tilde{z}\|_1^2\]
and succeed probability 9/10. And we attain sample complexity and time complexity by
\[\sum_{i=1}^{R}(B_i/\alpha_i) \log(n/\delta) = \epsilon^{-1}k \log(n/\delta).\]

V. Preliminary

In this section, we first present some definitions and background for Fourier transform in Section V-A. We introduce some technical tools in Section V-B. Then we introduce spectrum permutations and filter functions in Section V-C. They are used as hashing schemes in the Fourier transform literature. In Section V-D, we introduce collision events, large offset events, and large noise events.

A. Notations

We use \(i\) to denote \(\sqrt{-1}.\) Note that \(e^{i\theta} = \cos(\theta) + i \sin(\theta).\) For any complex number \(z \in \mathbb{C},\) we have \(z = a + ib,\) where \(a, b \in \mathbb{R}.\) We define the complement of \(z\) as \(\overline{z} = a - ib.\) We define \(|z| = \sqrt{a^2 + b^2}\). For any complex vector \(x \in \mathbb{C}^n,\) we use \(\text{supp}(x)\) to denote the support of \(x,\) and then
We define $\omega = e^{2\pi i / n}$, which is the $n$-th unitary root i.e. $\omega^n = 1$.

The discrete convolution of functions $f$ and $g$ is given by,

$$(f * g)[n] = \sum_{m=0}^{\infty} f[m]g[n-m]$$

For a complex vector $x \in \mathbb{C}^n$, we use $\hat{x} \in \mathbb{C}^n$ to denote its Fourier spectrum,

$$\hat{x}_i = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{-2\pi i j/n} x_j, \forall i \in [n].$$

Then the inverse transform is

$$x_j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} e^{2\pi i j/n} \hat{x}_i, \forall j \in [n].$$

We define $x_S$ as a vector by setting if $i \in S$, $(x_S)_i = x_i$ and otherwise $(x_S)_i = 0$, for a vector $x \in \mathbb{R}^n$ and a set $S \subseteq [n]$.

B. Technical Tools

We show several technical tools and some lemmas in prior works we used in the following section.

**Lemma V.1** (Markov’s inequality). If $X$ is a nonnegative random variable and $a > 0$, then the probability that $X$ is at least $a$ is at most the expectation of $X$ divided by $a$:

$$\Pr[X \geq a] \leq \frac{\mathbb{E}(X)}{a}.$$  

Let $a = \bar{a} \cdot \mathbb{E}(X)$ (where $\bar{a} > 0$); then we can rewrite the previous inequality as

$$\Pr[X \geq \bar{a} \cdot \mathbb{E}(X)] \leq \frac{1}{\bar{a}}$$

The following two lemmas of complex number are standard. We prove the following two lemmas for the completeness of the paper.

**Lemma V.2.** Given a fixed vector $x \in \mathbb{R}^n$ and a pairwise independent random variable $\sigma_i$, where $\sigma_i = \pm 1$ with probability $1/2$ respectively. Then we have:

$$\mathbb{E}_\sigma \left( \sum_{i=1}^{n} \sigma_i x_i \right)^2 = \|x\|^2_2$$

**Proof.** We have:

$$\mathbb{E}_\sigma \left( \sum_{i=1}^{n} \sigma_i x_i \right)^2 = \mathbb{E} \left( \sum_{i=1}^{n} \sigma_i^2 x_i^2 \right) + \mathbb{E} \left( \sum_{i \neq j} \sigma_i x_i \sigma_j x_j \right) \begin{align*}
= \mathbb{E} \left( \sum_{i=1}^{n} \sigma_i^2 x_i^2 \right) + \sum_{i \neq j} \mathbb{E} [\sigma_i \sigma_j] x_i x_j \\
= \mathbb{E} \left( \sum_{i=1}^{n} x_i^2 \right) + \sum_{i \neq j} \mathbb{E} [\delta_{i,j}] x_i x_j \end{align*}$$

where the first step comes from the linearity of expectation, the second step follows the linearity of expectation, the third step follows the definition of $\| \cdot \|_2$ and $\sigma_i^2 = 1$.

**Lemma V.3.** Let $a \sim [n]$ uniformly at random. Given a fixed vector $x \in \mathbb{C}^n$ and $\omega^{a_i}$, then we have:

$$\mathbb{E}_a \left[ \sum_{i=1}^{n} x_i \sigma^{a_i} \right]^2 = \|x\|^2_2$$

**Proof.** For any fixed $i \in [n]$, we have the inequality as follows

$$\mathbb{E}_a [\omega^{a_i}] = \frac{1}{n} \sum_{a=1}^{n} \omega^{a_i} = \frac{1}{n} \cdot \frac{1 - \omega^{ni}}{1 - \omega} = 0 \quad (1)$$

where the first step comes from geometric sum, and the second step comes from We have:

$$\mathbb{E}_a \left[ \sum_{i=1}^{n} x_i \sigma^{a_i} \right]^2 \begin{align*}
= \mathbb{E}_a \left[ \sum_{i=1}^{n} x_i \sigma^{a_i} \right] \left[ \sum_{i=1}^{n} x_i \omega^{-a_i} \right] \\
= \mathbb{E}_a \left[ \sum_{i=1}^{n} x_i \bar{x}_i \right] + \mathbb{E}_a \left[ \sum_{i \neq j} x_i \sigma^{a_i} \bar{x}_j \omega^{-a_j} \right] \\
= \mathbb{E}_a \left[ \sum_{i=1}^{n} x_i \bar{x}_i \right] + \sum_{i \neq j} \mathbb{E}_a [\omega^{a(i-j)}] x_i \bar{x}_j \\
= 0 + \mathbb{E}_a \left[ \sum_{i=1}^{n} x_i \bar{x}_i \right] = \|x\|^2_2$$

where the first step follows that for a complex number $z$, $|z|^2 = z \bar{z}$, the second step follows the linearity of expectation, the third step follows the linearity of expectation, where the fourth step follows Eq.1, and the final step comes from the definition of $\| \cdot \|_2$.

C. Permutation and filter function

We use the same (pseudorandom) spectrum permutation as [6].

**Definition V.4.** Suppose $\sigma^{-1}$ exists mod $n$. We define the permutation $P_{\sigma,a,b}$ by

$$(P_{\sigma,a,b}x)_i = x_{\sigma(i-a)}e^{-2\pi i b i/n}.$$  

We also define $\pi_{\sigma,b} = \sigma(i-b) \pmod{n}$. Then we have
Claim V.5. We have that
\[ P^{\hat{\sigma}, \hat{b}}_\sigma x_{\pi, b}^{\hat{\sigma}}(i) = \hat{\sigma}_i e^{-2\pi i \sigma i / n}. \]
h_{\sigma, b}(i) is defined as the “bin” with the mapping of frequency \( i \) onto. We define \( o_{\sigma, b}(i) \) as the “offset”. We formally define them as follows:

Definition V.6. Let the hash function be defined as
\[ h_{\sigma, b}(i) := \text{round}(\pi_{\sigma, b}(i) B / n). \]

Definition V.7. Let the offset function be defined as
\[ o_{\sigma, b}(i) := \pi_{\sigma, b}(i) - h_{\sigma, b}(i) B / n. \]

We use the same filter function as [6], [21], [23].

Definition V.8. Given parameters \( B \geq 1, \delta > 0, \alpha > 0 \). We say that \((G, G') = (G_{B, \delta, \alpha}, G'_{B, \delta, \alpha}) \in \mathbb{R}^n \) is a filter function if it satisfies the following properties:
1. \( |\text{supp}(G)| = O(\alpha^{-1} B \log(n/\delta)). \)
2. if \( |i| \leq (1 - \alpha) n / (2 B) \), \( G_i = 1 \).
3. if \( |i| \geq n / (2 B) \), \( G_i = 0 \).
4. for all \( i \), \( G_{i} \in [0, 1] \).
5. \( \|G - \hat{G}\|_{\infty} < \infty \).

D. Collision event, large offset event, and large noise event

We use three types of events defined in [6] as basic building blocks for analyzing Fourier set query algorithms. For any \( i \in S \), we define three types of events associated with \( i \) and \( S \) and defined over the probability space induced by \( \sigma \) and \( b \):

Definition V.9 (Collision, large offset, large noise). The definition of three events are given as follow:
- We say “Large offset” event \( E_{\text{off}}(i) \) holds if \( n(1 - \alpha) / (2 B) \leq |o_{\sigma, b}(i)| \).
- We say “Large noise” event \( E_{\text{noise}}(i) \) holds if \( (\alpha B)^{-1} \cdot \text{Err}^2(\hat{\sigma}, k) \leq \mathbb{E}\left[\left\|\hat{\sigma}_{\sigma, b}(i) - \hat{\sigma}_{\sigma, b}(i)\right\|_2^2\right] \).
- We say “Collision” event \( E_{\text{coll}}(i) \) holds if \( h_{\sigma, b}(i) \in h_{\sigma, b}(S \setminus \{i\}) \).

Definition V.10 (Well-isolated). For a vector \( x \in \mathbb{R}^n \), we say a coordinate \( t \in [n] \) is “well isolated” when none of “Collision” event, “Large offset” and “Large noise” event holds.

Claim V.11 (Claim 3.1 in [6]). For all \( i \in S \), we have \( \mathbb{P}[E_{\text{coll}}(i)] \leq 4 |S| / B \).

Claim V.12 (Claim 3.2 in [6]). For all \( i \in S \), we have \( \mathbb{P}[E_{\text{off}}(i)] \leq \alpha \).

Claim V.13 (Claim 4.1 in [6]). For any \( i \in S \), the event \( E_{\text{noise}}(i) \) holds with probability at most \( 4 \alpha \)
\[ \mathbb{P}[E_{\text{noise}}(i)] \leq 4 \alpha. \]

Lemma V.14 (Lemma 4.2 in [6]). With \( B \) divide \( n \), a uniformly sampled from \([n]\) and the others without limitation in
\[ \tilde{u} = \text{HASHToBins}(P_{\sigma, a, b, \alpha, \hat{\sigma}, B, \delta, x}). \]
With all of \( E_{\text{off}}(i) \), \( E_{\text{coll}}(i) \) and \( E_{\text{noise}}(i) \) not holding and \( j = h_{\sigma, b}(i) \), we have for all \( i \in [n] \),
\[ \mathbb{E}\left[\|\hat{\sigma}_j e^{-2\pi i \hat{\sigma} i / n} - \tilde{u}_j\|^2\right] \leq 2 B^2 / \alpha B. \]

Lemma V.15 (Lemma 3.3 in [6]). Suppose \( B \) divides \( n \). The output \( \tilde{u} \) of HASHToBins satisfies
\[ \tilde{u}_j = \sum_{h_{\sigma, b}(i) = j} (x - z)_i (G'_{B, \delta, \alpha})_{\sigma, b}(i) \omega^{a \sigma i} \pm \delta \|\tilde{z}\|_1. \]
Let \( \zeta := \{i \in \text{supp}(\tilde{\sigma}) \mid E_{\text{off}}(i)\} \).

The running time of HASHToBins is
\[ O\left(B / \alpha \log(n/\delta) + \|\tilde{z}\|_0 + \zeta \log(n/\delta)\right). \]

VI. Analysis on Fourier Set Query Algorithm

In this section, we will give an total analysis about our Algorithm 1. First, we will provide the iterative loop analysis which is the main part of our main function FOURIERSET-QUERY in Section VI-A. By this analysis, we demonstrate an important property of the Algorithm 1 in Section VI-B. In Section VI-C, we prove the the correctness of the algorithm. We also provide the analysis of the complexity (sample and time) of Algorithm 1. Then we can give an satisfying answer to the problem (See Definition III.2) with Algorithm 1 attained by us whose performance (on sample and time complexity) is better than prior works (See Table I).

A. Iterative loop analysis

Iterative loop analysis for Fourier set query is more tricky than the classic set query, because in the Fourier case, hashing is not perfect, in the sense that by using spectrum permutation and filter function (as the counterpart of hashing techniques), one coordinate can non-trivially contribute to multiple bins. We give iterative loop induction in Lemma VI.4.

Lemma VI.1. Given a vector \( x \in \mathbb{R}^n \), \( \gamma \leq 1/1000 \), \( \alpha_i = 1/(200i^3) \), for a coordinate \( t \in [n] \) and \( i \in [R] \), with probability at least \( 1 - 6 \alpha_i \), we say that \( t \) is “well isolated” (See Definition V.10).

Proof. Collision. Using Claim V.11, for any \( t \in S_i \), the event \( E_{\text{coll}}(t) \) holds with probability at most
\[ 4 |S_i| / 2 \leq \frac{4 k_i}{C \epsilon_i^2 / (\alpha_i^2 \epsilon_i)} = 4 \epsilon_i / C \leq \alpha_i, \]
where the first step follows from the definition of \( B_i \) and the assumption on \( |S_i| \), the second step is straightforward, the third step follows from the definition of \( \epsilon_i, \alpha_i, \) and \( C \).
We consider a particular step \(E_{\text{off}}(t)\) holds with probability at most \(\alpha_t\), i.e.
\[
\Pr_{\sigma,b}[E_{\text{off}}(t)] \leq \alpha_t.
\]

**Large offset.** Using Claim V.12, for any \(t \in S_t\), the event \(E_{\text{off}}(t)\) holds with probability at least \(\alpha_t\), i.e.
\[
\Pr_{\sigma,b}[E_{\text{off}}(t)] \leq \alpha_t.
\]

**Large noise.** Using Claim V.13, for any \(t \in S_t\),
\[
\Pr_{\sigma,b}[E_{\text{noise}}(t)] \leq 4\alpha_t.
\]

By a union bound over the above three events, we have \(t\) is “well isolated” with probability at least \(1 - 6\alpha_t\).

**Lemma VI.2.** Given parameters \(C \geq 1000, \gamma \leq 1/1000\). For any \(k \geq 1, \epsilon \in (0,1), R \geq 1\). For each \(i \in [R]\), we define
\[
k_i := k^i, 
\epsilon_i := \epsilon(10\gamma)_i, 
\alpha_i := 1/(200i^3), 
B_i := C \cdot k_i/\alpha_i^2.
\]

For each \(i \in [R]\): If for all \(j \leq [i-1]\) we have
1) \(\supp(\hat{\omega}(i)) \subseteq S_j\),
2) \(|S_{j+1}| \leq k_{j+1}\),
3) \(\hat{\omega}(i) = \omega(i) + \hat{\omega}(i)_0\),
4) \(\hat{\omega}(i+1) = \hat{\omega}(i) - \hat{\omega}(i)_1\),
5) \(\|\hat{\omega}(i+1)\|_2^2 \leq (1 + \epsilon_j)\|\hat{\omega}(i)\|_2^2 + \epsilon_j \delta^2 n\|\hat{\omega}\|_2^2\).

Then, with probability \(1 - 10\alpha_i/\gamma\), we have
\[
|S_{i+1}| \leq k_{i+1}.
\]

**Proof.** We consider a particular step \(i\). We can condition on \(|S_i| \leq k_i\).

By Lemma VI.1, we have \(t\) is “well isolated” with probability at least \(1 - 6\alpha_t\).

Therefore, each \(t \in S_i\) lies in \(T_i\) with probability at least \(1 - 6\alpha_t\). We have Then by Markov’s inequality (See Lemma V.1) and assumption in the statement, we have
\[
|S_i \setminus T_i| \leq \gamma k_i
\]
with probability \(1 - 6\alpha_t/\gamma\). Then we know that
\[
|S_{i+1}| = |S_i \setminus T_i| 
\leq \gamma k_i 
\leq k_{i+1}.
\]

where the first step follows from the definition of \(S_{i+1} = S_i \setminus T_i\), the second step follows from Eq. (2), the third step follows from the definition of \(k_i\) and \(k_{i+1}\).

**Lemma VI.3.** Given parameters \(C \geq 1000, \gamma \leq 1/1000\). For any \(k \geq 1, \epsilon \in (0,1), R \geq 1\). For each \(i \in [R]\), we define
\[
k_i := k^i, 
\epsilon_i := \epsilon(10\gamma)_i, 
\alpha_i := 1/(200i^3), 
B_i := C \cdot k_i/\alpha_i^2.
\]

For each \(i \in [R]\): If for all \(j \leq [i-1]\) we have
1) \(\supp(\hat{\omega}(i)) \subseteq S_j\),
2) \(|S_{j+1}| \leq k_{j+1}\),
3) \(\hat{\omega}(j+1) = \hat{\omega}(j) + \hat{\omega}(j)_1\),
4) \(\hat{\omega}(j+1) = \hat{\omega} - \hat{\omega}(j)_1\),
5) \(\|\hat{\omega}(j+1)\|_2^2 \leq (1 + \epsilon_j)(\|\hat{\omega}(j)\|_2^2 + \epsilon_j \delta^2 n\|\hat{\omega}\|_2^2)\).

Then, with probability \(1 - 10\alpha_i/\gamma\), we have
\[
\Pr \left[ \|\hat{\omega}(i) - \hat{\omega}(i)_1\|_2^2 \leq \frac{\epsilon_i}{20}(\|\hat{\omega}(i)\|_2^2 + \delta^2 n\|\hat{\omega}\|_2^2) \right] \geq 1 - \alpha_i.
\]

**Proof.** We define \(\rho(i)\) and \(\mu(i)\) as follows
\[
\rho(i) = \|\hat{\omega}(i)\|_2^2 + \delta^2 n\|\hat{\omega}\|_2^2, 
\mu(i) = \frac{\epsilon_i}{k_i} \left(\|\hat{\omega}(i)\|_2^2 + \delta^2 n\|\hat{\omega}\|_2^2\right).
\]

For a fixed \(t \in S_i\), let \(J = h_{\sigma,b}(t)\). By Lemma V.15, we have
\[
\hat{u}_j - \hat{\omega}(t)\omega^{\sigma t} = \sum_{t' \in T_i} \hat{G}_{\sigma,b}(t') \hat{\omega}(t')\omega^{\sigma t'} \pm \delta\|\hat{\omega}\|_2
\]
for each \(t \in S_i\), we define set \(Q_{t,i} = h_{\sigma,b}(j) \setminus \{t\}\). Let \(T_i\) be the set of coordinates \(t \in S_i\) such that \(Q_{t,i} \cap S_i = \emptyset\). Then it is easy to observe that
\[
\sum_{t \in T_i} \left[ \sum_{t' \in Q_{t,i}} \hat{G}_{\sigma,b}(t') \hat{\omega}(t')\omega^{\sigma t'} \right]^2 
= \sum_{t \in T_i} \left[ \sum_{t' \in Q_{t,i}} \hat{G}_{\sigma,b}(t') \hat{\omega}(t')\omega^{\sigma t'} \right]^2 
\leq \sum_{t \in S_i} \left[ \sum_{t' \in Q_{t,i}} \hat{G}_{\sigma,b}(t') \hat{\omega}(t')\omega^{\sigma t'} \right]^2
\]
where the first step comes from \(Q_{t,i} \cap S_i = \emptyset\), and the second step follows that \(T_i \subseteq S_i\).

We can calculate the expectation of \(\|\hat{\omega}(i)\|_2^2\)

We first demonstrate that
\[
E_{\sigma,a,b} \left[ \|\hat{\omega}(i)\|_2^2 \right] = E_{\sigma,a,b} \left[ \sum_{t \in T_i} \|\hat{\omega}(t) - \hat{\omega}(i)_1\|_2^2 \right].
\]

then get the upper bound of
\[
E_{\sigma,a,b} \left[ \sum_{t \in T_i} \|\hat{\omega}(t) - \hat{\omega}(i)_1\|_2^2 \right]
\]
We have
\[
E_{\sigma,a,b} \left[ \|\hat{\omega}(i) - \hat{\omega}(i)_1\|_2^2 \right] = E_{\sigma,a,b} \left[ \sum_{t \in T_i} \|\hat{\omega}(t) - \hat{\omega}(i)_1\|_2^2 \right]
\]
where the first step follows that summation over $T_i$, the
second step comes from the definition of $\bar{\omega}_t^{(i)}$ (in Line 19 in
Algorithm 1), the third step follows that

$$|\bar{\omega}_t^{(i)} - \bar{\omega}_{h_\sigma,b}(t)\omega^{-\sigma t}| = |\omega^{-\sigma t} \cdot |\bar{\omega}_t^{(i)}\omega^{\sigma t} - \bar{\omega}_{h_\sigma,b}(t)|$$

and $|\omega^{-\sigma t}| = 1$, the fourth step comes from Eq. (4).

And then we have

$$E_{\sigma,a,b} \left[ \sum_{t \in T_i} |\bar{\omega}_t^{(i)}\omega^{\sigma t} - \bar{\omega}_{h_\sigma,b}(t)|^2 \right]$$

$$\leq \sum_{t \in S_i} E_{\sigma,a,b} \left[ \sum_{t' \in Q_{t',1}, S_i} \left| \bar{\omega}_t^{(i)}\omega^{\sigma t'} - \bar{\omega}_{h_\sigma,b}(t') \right|^2 + \delta^2 \|ar{x}^2\|_1^2 \right]$$

$$\leq \sum_{t \in S_i} E_{\sigma,a,b} \left[ \sum_{t' \in Q_{t',1}, S_i} \left| \bar{\omega}_t^{(i)}\omega^{\sigma t'} - \bar{\omega}_{h_\sigma,b}(t') \right|^2 + \delta^2 \|ar{x}^2\|_1^2 \right]$$

$$= \sum_{t \in S_i} E_{\sigma,a,b} \sum_{t' \in Q_{t',1}, S_i} 1(t' \in Q_{t',1}, S_i) \cdot \left| \bar{\omega}_t^{(i)}\omega^{\sigma t'} - \bar{\omega}_{h_\sigma,b}(t') \right|^2 + \delta^2 \|ar{x}^2\|_1^2$$

where the first step follows that the equation above, the second step follows Lemma V.3, the third step follows from expanding the squared sum, the fourth step follows that if $A_1 \subseteq A_2$, we have

$$\sum_{i \in A_1} f(i) = \sum_{i \in A_2} f(i)$$

the fifth step follows for two pairwise independent random variable $t$ and $t'$, we have $h_\sigma,b(t) = h_\sigma,b(t')$ holds with probability at most $1/B_i$, the sixth step comes from the summation over $S_i$, and the last step follows from $|S_i| \leq k_i$ and $B_i = C \cdot k_i/\alpha_i^2 \epsilon_i$.

Then, using Markov’s inequality, we have,

$$\Pr \left[ \left\| \bar{\omega}^{(i)} - \bar{\omega}^{(i)} \right\|_2^2 \geq \frac{\epsilon_i \alpha_i}{C} \|\bar{x}^{(i)}\|_2^2 + \delta^2 \frac{|S_i|}{\alpha_i} \|ar{x}\|_1^2 \right] \leq \alpha_i$$

Note that

$$\frac{\epsilon_i \alpha_i}{C} \|\bar{x}^{(i)}\|_2^2 + \delta^2 \frac{|S_i|}{\alpha_i} \|ar{x}\|_1^2 \leq \frac{\epsilon_i \alpha_i}{C} \|\bar{x}^{(i)}\|_2^2 + \delta^2 \frac{|S_i|}{\alpha_i} \|ar{x}\|_1^2 \leq \frac{\epsilon_i \alpha_i}{C} \|\bar{x}^{(i)}\|_2^2 + \frac{\epsilon_i \alpha_i^2}{C} \delta^2 B_i \|ar{x}\|_1^2$$

$$\leq \frac{\epsilon_i \alpha_i}{C} \|\bar{x}^{(i)}\|_2^2 + \frac{\epsilon_i \alpha_i^2}{C} \delta^2 n \|ar{x}\|_1^2$$

where the first step follows by $\alpha_i \leq 1$, the second step follows by $|S_i| \leq k_i = \epsilon_i B_i \alpha_i^2 / C$, the third step follows by $B_i \leq n$, the last step follows by $C \geq 1000$.

Thus, we have

$$\Pr \left[ \left\| \bar{\omega}^{(i)} - \bar{\omega}^{(i)} \right\|_2^2 \leq \frac{\epsilon_i \alpha_i}{C} \|\bar{x}^{(i)}\|_2^2 + \delta^2 n \|ar{x}\|_1^2 \right] \geq 1 - \alpha_i$$

\[\square\]

**Lemma VI.4.** Given parameters $C \geq 1000$, $\gamma \leq 1/1000$. For
any $k \geq 1$, $\epsilon \in (0, 1)$, $R \geq 1$. For each $i \in [R]$, we define

$$k_i := k \gamma^{-1}, \quad \epsilon_i := \epsilon \gamma^i, \quad \alpha_i := 1/(200 i^2), \quad B_i := C \cdot k_i/\alpha_i^2 \epsilon_i$$

For each $i \in [R]$: If for all $j \leq |i - 1| - 1$ we have

1) $\supp(\bar{\omega}^{(i)}) \subseteq S_j$, 2) $|S_{j+1}| \leq k_{j+1}$, 3) $\bar{\omega}^{(j+1)} = \bar{\omega}^{(j)} + \bar{\omega}^{(i)}$, 4) $\bar{\omega}^{(j+1)} = \bar{\omega}^{(j)} - \bar{\omega}^{(i)}$, 5) $\|\bar{\omega}^{(j+1)}\|_2^2 \leq (1 + \epsilon_i)\|\bar{\omega}^{(j)}\|_2^2 + \epsilon_i \delta^2 n \|ar{x}\|_1^2$.

Then, with probability $1 - 10 \alpha_i / \gamma$, we have

1) $\supp(\bar{\omega}^{(i)}) \subseteq S_i$, 2) $|S_{i+1}| \leq k_{i+1}$, 3) $\bar{\omega}^{(i+1)} = \bar{\omega}^{(i)} + \bar{\omega}^{(i)}$, 4) $\bar{\omega}^{(i+1)} = \bar{\omega}^{(i)} - \bar{\omega}^{(i)}$, 5) $\|\bar{\omega}^{(i+1)}\|_2^2 \leq (1 + \epsilon_i)\|\bar{\omega}^{(i)}\|_2^2 + \epsilon_i \delta^2 n \|ar{x}\|_1^2$.

**Proof.** We will prove the five results one by one.

**Part 1.**

Follows from Line 19 in the Algorithm 1, we have that

$$\supp(\bar{\omega}^{(i)}) \subseteq S_i$$

**Part 2.**

By Lemma VI.2, we have that

$$|S_{i+1}| \leq k_i$$

**Part 3.**

Follows from Line 7 in the Algorithm 1, we have that

$$\bar{\omega}^{(i+1)} = \bar{\omega}^{(i)} + \bar{\omega}^{(i)}$$

**Part 4.**

Follows from Line 28 in the Algorithm 1, we have that

$$\bar{\omega}^{(i+1)} = \bar{\omega}^{(i+1)}$$

**Part 5.**

By Lemma VI.3, we have that

$$\Pr \left[ \left\| \bar{\omega}^{(i)} - \bar{\omega}^{(i)} \right\|_2^2 \leq \frac{\epsilon_i \alpha_i}{C} \|\bar{x}^{(i)}\|_2^2 + \delta^2 n \|ar{x}\|_1^2 \right] \geq 1 - \alpha_i$$

(5)
Recall that
\[ \tilde{w}^{(i)} = z^{(i+1)} - z^{(i)} = \tilde{x}^{(i)} - \tilde{x}^{(i+1)}. \]

It is obvious that
\[ \text{supp}(\tilde{w}^{(i)}) \subseteq T_i. \]

Conditioning on all coordinates in \( T_i \) are well isolated and Eq. (5) holds, we have
\[ \|\tilde{x}^{(i+1)}_{S_{i+1}}\|_2^2 = \|\tilde{x}^{(i)}_{S_{i+1}} - \tilde{w}^{(i)}_{S_{i+1}}\|_2^2 \]
\[ = \|\tilde{x}^{(i)}_{S_{i+1}} - \tilde{w}^{(i)}_{S_{i+1}}\|_2^2 - \|\tilde{w}^{(i)}_{S_{i+1}}\|_2^2 \]
\[ = \|\tilde{x}^{(i)}_{S_{i+1}} - \tilde{w}^{(i)}_{S_{i+1}}\|_2^2 \]
\[ = \|\tilde{x}^{(i)}_{S_i \cup T_i} - \tilde{w}^{(i)}\|_2^2 \]
\[ = \|\tilde{x}^{(i)}_{S_i}\|_2^2 + \|\tilde{x}^{(i)}_{T_i} - \tilde{w}^{(i)}\|_2^2 \]
\[ \leq \|\tilde{x}^{(i)}_{S_i}\|_2^2 + \epsilon_i \|\tilde{x}^{(i)}_{T_i}\|_2^2 + \delta^2 n \|\tilde{x}\|_1^2 \]
\[ = (1 + \epsilon_i) \|\tilde{x}^{(i)}_{S_i}\|_2^2 + \epsilon_i \delta^2 n \|\tilde{x}\|_1^2. \]

where the first step comes from \( \tilde{x}^{(i+1)} = \tilde{x}^{(i)} - \tilde{w}^{(i)} \), the second step is due to rearranging the terms, the third step is due to \( \tilde{w}^{(i)} = \tilde{w}^{(i)}_{S_{i+1}} \), and the forth step comes from \( S_i = T_i \cup S_{i+1} \), the fifth step is due to rearranging the terms, the sixth step the comes from a Eq. (5), and the final step comes from merging the \( \|\tilde{x}^{(i)}_{S_i}\|_2^2 \) terms.

**Part 2.** Given that all coordinates \( t \in [n] \) in \( T_1 \) are well isolated, with probability at least \( 1 - 10\alpha_i / \gamma \), we have
\[ \|\tilde{x}^{(\ell)}_{S_\ell}\|_2^2 = \|\tilde{x}^{(\ell)} - \tilde{w}^{(\ell)}\|_{S_\ell}^2 \]
\[ = \|\tilde{x}^{(\ell)} - \tilde{w}^{(\ell)}\|_{S_\ell}^2 \]
\[ = \|\tilde{x}^{(\ell)} - \tilde{w}^{(\ell)}\|_{S_\ell}^2 \]
\[ = \|\tilde{x}^{(\ell)}_{S_\ell} + \|\tilde{x}^{(\ell)}_{T_\ell} - \tilde{w}^{(\ell)}\|_2^2 \]
\[ \leq \|\tilde{x}^{(\ell)}_{S_\ell}\|_2^2 + \epsilon_i \|\tilde{x}^{(\ell)}_{T_\ell}\|_2^2 + \delta^2 n \|\tilde{x}\|_1^2 \]
\[ = (1 + \epsilon_i) \|\tilde{x}^{(\ell)}_{S_\ell}\|_2^2 + \epsilon_i \delta^2 n \|\tilde{x}\|_1^2. \]

where the first step comes from \( \tilde{x}^{(\ell)} = \tilde{x}^{(\ell)} - \tilde{w}^{(\ell)} \), the second step is due to rearranging the terms, the third step is due to \( \tilde{w}^{(\ell)} = \tilde{w}^{(\ell)}_{S_\ell} \), and the forth step comes from \( S_\ell = T_\ell \cup S_\ell \), the fifth step is due to rearranging the terms, the sixth step the comes from expanding the terms, and the final step comes from merging the \( \|\tilde{x}^{(\ell)}_{S_\ell}\|_2^2 \) terms.

By Lemma VI.4, we have
\[ \|\tilde{x}^{(i+1)}_{S_{i+1}}\|_2^2 \leq (1 + \epsilon_i) \|\tilde{x}^{(i)}_{S_i}\|_2^2 + \epsilon_i \delta^2 n \|\tilde{x}\|_1^2. \]

**C. Main result**

In this subsection, we give the main result as the following theorem.

**Theorem VI.6** (Main result). Given a vector \( x \in \mathbb{C}^n \) and the \( \tilde{x} \) as the concrete Fourier transformation result, for every \( \epsilon, \delta \in (0, 1) \) and \( k \geq 1 \), any \( S \subseteq [n] \), \( |S| = k \), there exists an algorithm (Algorithm 1) that takes
\[ O(\epsilon^{-1} k \log(n/\delta)) \]
samples, runs in \[ O(\epsilon^{-1} k \log(n/\delta)) \]
time, and outputs a vector \( x' \in \mathbb{C}^n \) such that
\[ \|x' - \tilde{x}|S|\|_2^2 \leq \epsilon \|x'|_{S,2}^2 + \delta \|\tilde{x}|_1^2 \]
holds with probability at least \((9/10)\).

**Proof.** By the Setting in the Algorithm 1, we can make the assumption in Lemma VI.4 hold. And by induction on Lemma VI.4, the following conclusion can be attained by us.

By Lemma VI.4 and the parameters as follows
\[ k_i := k \gamma^{i-1}, \]
\[ \epsilon_i := \epsilon (10 \gamma)^i, \]
\[ \alpha_i = 1/(200 \delta^3), \]
\[ B_i := C \cdot k_i / (\alpha_i^2 \epsilon_i), \]
for \( i \in [R] \), we can have that with probability \( 1 - 10\alpha_i / \gamma \), we have
1) \( \text{supp}(\tilde{w}^{(i)}) \subseteq S_i. \)
Algorithm 1: Fourier set query algorithm

1: procedure FOURIERSETQUERY($x, S, \epsilon, k$) $\triangleright$ Theorem VI.6

2: $\gamma \leftarrow 1/1000$, $C \leftarrow 1000$, $\hat{z}^{(i)} \leftarrow 0$, $S_1 \leftarrow S$

3: for $i = 1 \rightarrow R$ do

4: $k_i \leftarrow k^{(i)}$, $\epsilon_i \leftarrow \epsilon(10\gamma)^i$, $\alpha_i \leftarrow 1/(100\gamma)^i$, $B_i \leftarrow C \cdot k_i/(\alpha_i^2 \epsilon_i)$

5: $\hat{w}^{(i)}, T_i \leftarrow \text{ESTIMATEVALUES}(x, \hat{z}^{(i)}, S_i, B_i, \delta, \alpha_i)$ $\triangleright \hat{w}^{(i)}$ is $|T_i|$-sparse

6: $S_{i+1} \leftarrow S_i \setminus T_i$

7: $\hat{z}^{(i+1)} \leftarrow \hat{z}^{(i)} + \hat{w}^{(i)}$

8: end for

9: return $\hat{z}^{(R+1)}$

10: end procedure

11: procedure ESTIMATEVALUES($x, \hat{z}, S, B, \delta, \alpha$) $\triangleright$ Lemma VI.4

12: Choose $a, b \in [n]$ uniformly at random

13: Choose $\sigma$ uniformly at random from the set of odd numbers in $[n]$

14: $\hat{w} \leftarrow \text{HASHTOBINS}(P_{\sigma, a, b}, \hat{z}, S, B, \delta, x)$

15: $\hat{w} \leftarrow 0$, $T \leftarrow \emptyset$

16: for $t \in S$ do

17: if $t$ is isolated from other coordinates of $S$ then $\triangleright h_{\sigma, b}(t) \notin h_{\sigma, b}(S \setminus \{t\})$

18: if no large offset then $\triangleright n(1 - \alpha)/2B > |o_{\sigma, b}(t)|$

19: $\hat{w}_t \leftarrow \hat{w}_{h_{\sigma, b}(t)} e^{-\frac{2\pi}{n}\sigma t}$

20: $T \leftarrow T \cup \{t\}$

21: end if

22: end if

23: end for

24: return $\hat{w}, T$

25: end procedure

26: procedure HASHTOBINS($P_{\sigma, a, b}, \hat{z}, S, B, \delta, x$)

27: Compute $\hat{y}_{jn/B}$ for $j \in [B]$, where $y = G_{B, a, \delta} \cdot (P_{\sigma, a, b}x)$

28: Compute $\tilde{y}_{jn/B} = \hat{y}_{jn/B} - (G_{B, a, \delta} \cdot P_{\sigma, a, b}\hat{z})_{jn/B}$

29: return $\tilde{y}_j = \tilde{y}_{jn/B}$

30: end procedure

2) $|S_{i+1}| \leq k_{i+1}$.
3) $\hat{z}^{(i+1)} = \hat{z}^{(i)} + \hat{w}^{(i)}$.
4) $\|\hat{z}^{(i+1)}\|_2^2 \leq (1 + \epsilon_i)\|\hat{z}^{(i)}\|_2^2 + \epsilon_i \delta^2 n \|\hat{z}\|_2^2$.

By Lemma VI.5, we can conclude that with $R = \log k$ iterations, we will attain the result we want. Then we will give the analysis about the time complexity and sample complexity.

Proof of Sample Complexity.

From analysis above, the sample needed in each iteration is $O((B_i/\alpha_i) \log(n/\delta))$ then we have the following complexity.

The sample complexity of ESTIMATION is

$$\sum_{i=1}^{R} (B_i/\alpha_i) \log(n/\delta) = O(\epsilon^{-1} k \log(n/\delta)).$$

The time in each iteration mainly from two parts. The EstimateValues and HashToBins functions. For the running time of EstimateValues, its running time is mainly from loop. The number of the iterations of the loop can be bounded by $O(B_i/\alpha_i \log(n/\delta))$.

By Lemma VI.15, we can attain the time complexity of HashToBins with the bound of $O(B_i/\alpha_i \log(n/\delta))$. This function is used only once at each iteration.

With $R = \log k$, we can have the following equation.

Proof of Time Complexity. The Time complexity of ESTIMATION is

$$\sum_{i=1}^{R} (B_i/\alpha_i) \log(n/\delta) = O(\epsilon^{-1} k \log(n/\delta)).$$

Proof of Success Probability.

The failure probability is $\sum_{i=1}^{R} 10 \alpha_i/\gamma < 1/10.$

Upper bound $\|\hat{z}^{(i)}\|_2^2$.

By Lemma VI.4, we have that

$$\|\hat{z}^{(i)}\|_2^2 \leq (1 + \epsilon)\|\hat{z}^{(i)}\|_2^2 + \epsilon \delta^2 n \|\hat{z}\|_2^2$$

$$\leq (1 + \epsilon)(1 + \epsilon_{i-1})\|\hat{z}^{(i-1)}\|_2^2 + ((1 + \epsilon_i)\epsilon_{i-1} + \epsilon_i)\delta^2 n \|\hat{z}\|_2^2$$

$$\leq \prod_{j=1}^{i} (1 + \epsilon_j)\|\hat{z}_{jn/B}\|_2^2 + \sum_{j=1}^{i} \epsilon_j \delta^2 n \|\hat{z}\|_2^2 \prod_{l=j+1}^{i} (1 + \epsilon_l)$$

$$\leq 8(\|\hat{z}_{jn/B}\|_2^2 + \delta^2 n \|\hat{z}\|_2^2).$$

where the first step comes from the assumption in Lemma VI.4, the second step comes from the assumption in
Lemma VI.4, the third step refers to recursively apply the second step, the last step follows by a geometric sum.

**Proof of Final Error.** We can bound the query error by:

\[
\|\tilde{x}_S - \hat{x}^{(R+1)}\|_2 \leq \sum_{i=1}^{R} k_i \cdot \frac{\mu(i)}{20} \leq \frac{1}{20} \sum_{i=1}^{R} \epsilon_1 + 10(\|\tilde{x}_S\|_2^2 + \delta^2 n \|\tilde{x}\|_1^2)/20 \leq \epsilon(\|\tilde{x}_S\|_2^2 + \delta^2 n \|\tilde{x}\|_1^2).
\]

where the first step follows that \( T_1 \) is well isolated (See Definition V.10.) and \( \hat{\mu}(i) = \tilde{z}^{(i+1)} - \tilde{z}^{(i)} \), the second step is by Eq. (5), the third step comes from the definition of \( \mu(i) \) in Eq. (3), the fourth step follows from the Eq.(6), and the final step follows from the geometric sum, \( \epsilon_1 = (e(10)^4) \) and \( \gamma \leq 1/1000. \)

VII. CONCLUSION

Fourier transformation is an intensively researched topic in a variety of scientific disciplines. Numerous applications exist within machine learning, signal processing, compressed sensing, etc. In this paper, we study the problem of Fourier set query. With an approximation parameter \( \epsilon \), a vector \( x \in \mathbb{C}^n \) and a query set \( S \subset [n] \) of size \( k \), our algorithm uses \( O(\epsilon^{-1} k \log(n/\delta)) \) Fourier measurements, runs in \( O(\epsilon^{-1} k \log(n/\delta)) \) time and outputs a vector \( x' \) such that \( \|x' - \tilde{x}\|_2 \leq \epsilon \|\tilde{x}_S\|_2 + \delta \|\tilde{x}\|_1^2 \) with probability of at least \( 9/10 \).

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