A pseudo - perturbative expansions method; nonpolynomial, cutoff - Coulomb, and Coulomb plus logarithmic potentials

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Abstract

We propose a new analytical method to solve for nonexactly soluble Schrödinger equation via expansions through some existing quantum numbers. Successfully, it is applied to the rational nonpolynomial oscillator potential. Moreover, a conclusion reached by Scherrer et al. [2], via matrix continued fractions method, that the shifted large N expansion method leads to dubious accuracies is investigated. The cutoff - Coulomb and Coulomb plus logarithmic potentials are also investigated.
1 Introduction

The importance of rational nonpolynomial oscillator (NPO) potential,

\[ V(q) = a_0q^2 + \frac{aq^2}{1+bq^2}, \quad b > 0, \]  

arises in nonlinear Lagrangian field theory, laser theory, and elementary particle physics [1-4]. Only a class of exact analytical solutions for a certain parameter dependence \( a = a(b) \) obtains [3,5-7]. Hence, it has been a subject of several investigations [1-15] (exhaustive lists of these could be found in Ref.s[2,15]).

On the other hand, the cutoff - Coulomb potential

\[ V(q) = -\frac{1}{q+c}, \quad c > 0, \]  

with the truncation parameter \( c \), avoids the singularity at \( q = 0 \) in the Coulomb potential (the crux of divergence difficulties in quantum field theory [16-18]). It has been suggested [18] that if gravitational interactions of elementary particles are taken into account, there would be a gravitational cutoff of Coulomb interaction. Equation (2) represents a nonrelativistic expression of this idea. It also serves as an approximation to the potential due smeared charge rather than a point charge.

The Schrödinger equation with such interactions, (1) and (2), belongs to the class of quantum mechanical systems which are nonexactly soluble in general. However, the solutions of exactly solvable potentials (an interesting field of mathematical physics in itself [19,20]) can be used in perturbation and pseudoperturbation theories, or they can be combined with numerical calculations. Nevertheless, in the simplest case, analytical calculations can
aid numerical studies in areas where numerical techniques might not be safely controlled. For example, when bound - state wave functions with arbitrary nodal zeros are required for certain singular potentials (a next level of complexity), analytical solutions can supply a basis for numerical calculations. Moreover, in many problems the Hamiltonian does not contain any physical parameter suitable for a perturbation expansion treatment. More often, the Hamiltonian contains physical parameters, but, typically, zeroth - order solutions for special values of these are not tractable or good starting approximations. Alternatively, one would resort to apparently artificial conversions to perturbation problems which have been shown to make progress [21-35].

The three - dimensional (3D) spherically symmetric NPO potential has been investigated [8] by means of the shifted large - N expansion technique (SLNT). However, Scherrer and co-workers [2] have employed a matrix continued fractions (MCF) method and concluded that SLNT [8] leads to dubiously accurate results in the critical range $a^{1/2}/b \approx 0.1 - 30$. Handy et al. [14] have used the eigenvalue moment method (EMM) to obtain upper and lower energy bounds, for $l \neq 0$, and compared their results with those of Roy et al. [12] by the supersymmetric quantum mechanics (SSQM). Singh et al [17] have used a numerical integration method (NIM) to find the eigenvalues for the 3D cutoff - Coulomb potential. Hall and Saad [35] have used a smooth transformation (STM) and a numerical integration methods to obtain bound - states for the Coulomb plus a logarithmic perturbation term. Hence, apart from those of SLNT, results from convincingly powerful methods exist for comparison purposes.

Recently, we have introduced a pseudoperturbative shifted - $l$ ( $l$ is the angular momentum quantum number) expansion technique ( PSLET) to solve for nodeless states of Schrödinger equation. It simply consists of using $1/\bar{l}$ as a pseudoperturbation parameter, where $\bar{l} = l - \beta$ and $\beta$ is a suitable shift. The
shift $\beta$ is vital for it removes the poles that would emerge, at lowest orbital states with $l=0$, in our proposed expansions below. Our analytical, or often semianalytical, methodical proposal PSLET has been successfully applied to quasi-relativistic harmonic oscillator [34], spiked harmonic oscillator [32], and to anharmonic oscillators [35] potentials.

Encouraged by its satisfactory performance in handling nodeless states, we generalize PSLET recipe (in section 2) for states with arbitrary number of nodal zeros, $k \geq 0$. Moreover, in the underlying ”radical” time-independent radial Schrödinger equation, in $\hbar = m = 1$ units,

$$\left[-\frac{1}{2} \frac{d^2}{dq^2} + \frac{l(l+1)}{2q^2} + V(q)\right] \Psi_{k,l}(q) = E_{k,l} \Psi_{k,l}(q), \quad (3)$$

the isomorphism between orbital angular momentum $l$ and dimensionality $D$ invites interdimensional degeneracies to obtain [36-39] (for more details the reader may refer to refs [36,38]). Hence, the symmetry of an attendant problem obviously manifests the admissibility of the quantum number $l$: In one-dimension (1D), $l$ specifies parity, $(-1)^{l+1}$, with the permissible values -1 and/or 0 (even and/or odd parity, respectively) where $q = x \in (-\infty, \infty)$.

For two-dimensional (2D) cylindrically symmetric Schrödinger equation one sets $l = |m| - 1/2$, where $m$ is the magnetic quantum number and $q = (x^2 + y^2)^{1/2} \in (0, \infty)$. Finally, for three-dimensional (3D) spherically symmetric Schrödinger equation, $l$ denotes the angular momentum quantum number with $q = (x^2 + y^2 + z^2)^{1/2} \in (0, \infty)$.

In section 3, we investigate PSLET recipe and consider, for the sake of diversity, the potentials; (i) $V(q) = A^2q^2/2$, the harmonic oscillator (the limit of (1) when $b \rightarrow \infty$ and/or $a \rightarrow 0$, with $a_o = A^2/2$), (ii) $V(q) = -1/q$, the Coulomb (the limit of (2) when $c \rightarrow 0$, (iii) the NPO (2), (iv) the cutoff-Coulomb (2), and (v) the Coulomb perturbed by a logarithmic term,
which has no experimental evidence, to the best of our knowledge, thus our calculations are only of academic interest. The last section is reserved for summary and remarks.

2 The generalization of PSLET

The construction of our PSLET starts with shifting the angular momentum in (3) to obtain

$$\left\{ -\frac{1}{2} \frac{d^2}{dq^2} + \frac{\bar{l}^2 + (2\beta + 1)\bar{l} + \beta(\beta + 1)}{2q^2} + \frac{\bar{l}^2}{Q} V(q) \right\} \Psi_{k,l}(q) = E_{k,l} \Psi_{k,l}(q), \quad (4)$$

where $Q$ is a constant that scales the potential $V(q)$ at large $l$ limit (the pseudoclassical limit [36]) and is set, for any specific choice of $l$ and $k$, equal to $\bar{l}^2$ at the end of the calculations. Next, we shift the origin of the coordinate system through $x = \bar{l}/2(q - q_o)/q_o$, where $q_o$ is currently an arbitrary point to be determined below. Expansions about this point, $x = 0$ (i.e. $q = q_o$), yield

$$\frac{1}{q^2} = \sum_{n=0}^{\infty} (-1)^n \frac{(n + 1)}{q_o^2} x^n \bar{l}^{-n/2}, \quad (5)$$

$$V(x(q)) = \sum_{n=0}^{\infty} \left( \frac{d^n V(q_o)}{dq_o^n} \right) \frac{(q_o x)^n}{n!} \bar{l}^{-n/2}. \quad (6)$$

Obviously, the expansions in (5) and (6) localize the problem at an arbitrary point $q_o$ and the derivatives, in effect, contain information not only at $q_o$ but also at any point on $q$-axis, in accordance with Taylor’s theorem. It is then convenient to expand $E_{k,l}$ as
\[ E_{k,l} = \sum_{n=-2}^{\infty} E_{k,l}^{(n)} \bar{l}^{-n}. \] (7)

Equation (4) thus becomes
\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{q_o^2}{l} \bar{V}(x(q)) \right] \Psi_{k,l}(x) = \frac{q_o^2}{l} E_{k,l} \Psi_{k,l}(x),
\] (8)

with
\[
\frac{q_o^2}{l} \bar{V}(x(q)) = q_o^2 \left[ \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} \right] + \bar{l}/2B_1 x + B_2 x^2 + \frac{(2\beta + 1)}{2} + (2\beta + 1) \sum_{n=1}^{\infty} (-1)^n \frac{(n + 1)}{2} x^n \bar{l}^{-n/2} + \sum_{n=3}^{\infty} B_n x^n \bar{l}^{-(n-2)/2} + \beta(\beta + 1) \sum_{n=0}^{\infty} (-1)^n \frac{(n + 1)}{2} x^n \bar{l}^{-(n+2)/2},
\] (9)

\[ B_n = (-1)^n \frac{(n + 1)}{2} + \left( \frac{d^n V(q_o)}{dq_o^n} \right) \frac{q_o^{n+2}}{n!Q}. \] (10)

Equation (8), along with (9) and (10), is evidently the one-dimensional Schrödinger equation for a perturbed harmonic oscillator
\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} w^2 x^2 + \varepsilon_o + P(x) \right] X_k(x) = \lambda_k X_k(x),
\] (11)

where \( w^2 = 2B_2 \),
\[
\varepsilon_o = \bar{l} \left[ \frac{1}{2} + \frac{q_o^2 V(q_o)}{Q} \right] + \frac{2\beta + 1}{2} + \frac{\beta(\beta + 1)}{2\bar{l}}.
\] (12)
and $P(x)$ represents the remaining terms in eq.(9) as infinite power series perturbations to the harmonic oscillator. One would then imply that

$$
\lambda_k = \bar{l} \left[ \frac{1}{2} + q_o^2 V(q_o) \right] + \left[ \frac{2\beta + 1}{2} + (k + \frac{1}{2})w \right] \\
+ \frac{1}{\bar{l}} \left[ \frac{\beta(\beta + 1)}{2} + \lambda_k^{(0)} \right] + \sum_{n=2}^{\infty} \lambda_k^{(n-1)} \bar{l}^{-n},
$$

(13)

and

$$
\lambda_k = q_o^2 \sum_{n=-2}^{\infty} E_{k,l}^{(n)} \bar{l}^{-(n+1)}. 
$$

(14)

Hence, equations (13) and (14) yield

$$
E_{k,l}^{(-2)} = \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} 
$$

(15)

$$
E_{k,l}^{(-1)} = \frac{1}{q_o^2} \left[ \frac{2\beta + 1}{2} + (k + \frac{1}{2})w \right] 
$$

(16)

$$
E_{k,l}^{(0)} = \frac{1}{q_o^2} \left[ \frac{\beta(\beta + 1)}{2} + \lambda_k^{(0)} \right] 
$$

(17)

$$
E_{k,l}^{(n)} = \lambda_k^{(n)}/q_o^2 ; \quad n \geq 1. 
$$

(18)

Where $q_o$ is chosen to minimize $E_{k,l}^{(-2)}$, i.e.
\[
\frac{dE_{k,l}^{(-2)}}{dq_o} = 0 \quad \text{and} \quad \frac{d^2E_{k,l}^{(-2)}}{dq_o^2} > 0.
\] (19)

Hereby, \( V(q) \) is assumed to be well behaved so that \( E_{k,l}^{(-2)} \) has a minimum \( q_o \) and there are well-defined bound states. Equation (19) in turn gives, with \( \bar{l} = \sqrt{Q} \),

\[
l - \beta = \sqrt{q_o^2 V'(q_o)}.
\] (20)

Consequently, the second term in Eq. (9) vanishes and the first term adds a constant to the energy eigenvalues. It should be noted that the energy term \( \bar{l}^2E_{k,l}^{(-2)} \) corresponds roughly to the energy of a classical particle with angular momentum \( L_z=\bar{l} \) executing circular motion of radius \( q_o \) in the potential \( V(q_o) \). It thus identifies the zeroth-order approximation, to all eigenvalues, as a classical approximation and the higher-order corrections as quantum fluctuations around the minimum \( q_o \), organized in inverse powers of \( \bar{l} \). The next correction to the energy series, \( \bar{l}E_{k,l}^{(-1)} \), consists of a constant term and the exact eigenvalues of the harmonic oscillator \( w^2x^2/2 \). The shifting parameter \( \beta \) is determined by choosing \( \bar{l}E_{k,l}^{(-1)}=0 \). This choice is physically motivated. In addition to its vital role in removing the singularity at \( l = 0 \), it also requires the agreements between PSLET eigenvalues and eigenfunctions with the exact well known ones for the harmonic oscillator and Coulomb potentials. Hence

\[
\beta = - \left[ \frac{1}{2} + \left( k + \frac{1}{2} \right) w \right] , \quad w = \sqrt{3 + \frac{q_o V''(q_o)}{V'(q_o)}}
\] (21)

where primes of \( V(q_o) \) denote derivatives with respect to \( q_o \). Then equation (9) reduces to
\[
\frac{q_0^2}{l} \tilde{V}(x(q)) = q_o^2 \left[ \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} \right] + \sum_{n=0}^{\infty} v^{(n)}(x) \bar{l}^{-n/2}, \tag{22}
\]

where

\[
v^{(0)}(x) = B_2 x^2 + \frac{2\beta + 1}{2}, \tag{23}
\]

\[
v^{(1)}(x) = -(2\beta + 1)x + B_3 x^3, \tag{24}
\]

\[
v^{(n)}(x) = B_{n+2} x^{n+2} + (-1)^n (2\beta + 1) \frac{(n+1)}{2} x^n \\
+ (-1)^n \frac{\beta(\beta + 1)}{2} (n-1) x^{(n-2)}, \quad n \geq 2. \tag{25}
\]

Equation (8) thus becomes

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \sum_{n=0}^{\infty} v^{(n)} \bar{l}^{-n/2} \right] \psi_{k,l}(x) = \left[ \sum_{n=1}^{\infty} q_0^2 E_{k,l}^{(n-1)} \bar{l}^{-n} \right] \psi_{k,l}(x). \tag{26}
\]

Up to this point, one would conclude that the above procedure is nothing but an imitation of the eminent shifted large-N expansion (SLNT) [24-29,35]. However, because of the limited capability of SLNT in handling large-order corrections via the standard Rayleigh-Schrödinger perturbation theory, only low-order corrections have been reported, sacrificing in effect its preciseness. Therefore, one should seek for an alternative and proceed by setting the wave
functions with any number of nodes as

\[ \Psi_{k,l}(x(q)) = F_{k,l}(x) \exp(U_{k,l}(x)). \]  \hspace{1cm} (27)

In turn, equation (26) readily transforms into the following Riccati equation:

\[
F_{k,l}(x) \left[ -\frac{1}{2} \left( U_{k,l}''(x) + U_{k,l}'(x)U_{k,l}'(x) \right) + \sum_{n=0}^{\infty} v^{(n)}(x) \bar{l}^{-n/2} \\
- \sum_{n=1}^{\infty} q_o^2 v^{(n-1)}(x) \bar{l}^{-n} \right] - F_{k,l}'(x)U_{k,l}'(x) - \frac{1}{2} F_{k,l}''(x) = 0,
\]  \hspace{1cm} (28)

where the primes denote derivatives with respect to \( x \). It is evident that this equation admits solution of the form

\[
U_{k,l}'(x) = \sum_{n=0}^{\infty} U_k^{(n)}(x) \bar{l}^{-n/2} + \sum_{n=0}^{\infty} G_k^{(n)}(x) \bar{l}^{-(n+1)/2},
\]  \hspace{1cm} (29)

\[
F_{k,l}(x) = x^k + \sum_{n=0}^{\infty} \sum_{p=0}^{k-1} q_p^{(n)} x^p \bar{l}^{-n/2},
\]  \hspace{1cm} (30)

where

\[
U_k^{(n)}(x) = \sum_{m=0}^{n+1} D_{m,n,k} x^{2m-1}; \quad D_{0,n,k} = 0
\]  \hspace{1cm} (31)

\[
G_k^{(n)}(x) = \sum_{m=0}^{n+1} C_{m,n,k} x^{2m}.
\]  \hspace{1cm} (32)

Substituting equations (29) - (32) into equation (28) implies
\[ F_{k,l}(x) \left[ -\frac{1}{2} \sum_{n=0}^{\infty} \left( U_k^{(n)} \bar{I}^{-n/2} + G_k^{(n)} \bar{I}^{-(n+1)/2} \right) \right. \]

\[ - \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( U_k^{(m)} U_k^{(n-m)} \bar{I}^{-n/2} + G_k^{(m)} G_k^{(n-m)} \bar{I}^{-(n+2)/2} \right) \]

\[ + 2U_k^{(m)} G_k^{(n-m)} \bar{I}^{-(n+1)/2} + \sum_{n=0}^{\infty} v^{(n)} \bar{I}^{-n/2} - \sum_{n=1}^{\infty} q_k^{(n-1)} E_k^{(n-1)} \bar{I}^{-n} \]

\[ - F'_{k,l}(x) \left[ \sum_{n=0}^{\infty} \left( U_k^{(n)} \bar{I}^{-n/2} + G_k^{(n)} \bar{I}^{-(n+1)/2} \right) \right] - \frac{1}{2} F''_{k,l}(x) = 0 \] (33)

The above procedure obviously reduces to the one described by Mustafa and Odeh [33,34,36,37], for \( k = 0 \). Moreover, the solution of equation (33) follows from the uniqueness of power series representation. Therefore, for a given \( k \) we equate the coefficients of the same powers of \( \bar{I} \) and \( x \), respectively. For example, when \( k = 1 \) one obtains

\[ D_{1,0,1} = -w, \quad U_1^{(0)}(x) = -w x, \] (34)

\[ C_{1,0,1} = -\frac{B_3}{w}, \quad a_{0,1}^{(1)} = -\frac{C_{0,0,1}}{w}, \] (35)

\[ C_{0,0,1} = \frac{1}{w} (2C_{1,0,1} + 2\beta + 1), \] (36)

\[ D_{2,2,1} = \frac{1}{w} \left( \frac{C_{1,0,1}^2}{2} - B_4 \right), \] (37)
\[ D_{1,2,1} = \frac{1}{w} \left( \frac{5}{2} D_{2,2,1} + C_{0,0,1} C_{1,0,1} - \frac{3}{2} (2\beta + 1) \right), \quad (38) \]

\[ E_{1,l}^{(0)} = \frac{1}{q_a^2} \left( \beta (\beta + 1) - a_{0,1}^{(1)} C_{1,0,1} - \frac{3}{2} D_{1,2,1} - \frac{C_{0,0,1}^2}{2} \right), \quad (39) \]

e tc. Here, we reported the nonzero coefficients only. One can then calculate the energy eigenvalues and eigenfunctions from the knowledge of \( C_{m,n,k} \), \( D_{m,n,k} \), and \( a_{p,k}^{(n)} \) in a hierarchical manner. Nevertheless, the procedure just described is suitable for a software package such as MAPLE to determine the energy eigenvalue and eigenfunction corrections up to any order of the pseudoperturbation series, (7) and (29)-(30).

Although the energy series, equation (7), could appear divergent, or, at best, asymptotic for small \( \bar{l} \), one can still calculate the eigenenergies to a very good accuracy by forming the sophisticated \([N,M]\) Padé approximation [21]

\[ P_N^M(1/\bar{l}) = (P_0 + P_1/\bar{l} + \cdots + P_M/\bar{l}^M)/(1 + q_0/\bar{l} + \cdots + q_N/\bar{l}^N) \]

to the energy series (7). The energy series is calculated up to \( E_{k,l}^{(8)}/\bar{l}^8 \) by

\[ E_{k,l} = \bar{l}^2 E_{k,l}^{(-2)} + E_{k,l}^{(0)} + \cdots + E_{k,l}^{(8)}/\bar{l}^8 + O(1/\bar{l}^9), \quad (40) \]

and with the \( P_4^4(1/\bar{l}) \) Padé approximant it becomes

\[ E_{k,l}[4,4] = \bar{l}^2 E_{k,l}^{(-2)} + P_4^4(1/\bar{l}). \quad (41) \]

Our recipe is therefore well prescribed.
3 Some applications

We begin with the harmonic oscillator potential \( V(q) = A^2 q^2 / 2 \) and find \( q_o \) from (20), along with (21). Once \( q_o \) is determined, Eq.(15) reads

\[
E_{k,l}^{(-2)} \tilde{l}^2 = A \tilde{l} \; ; \; \tilde{l} = 2k + l + 3/2 , \tag{42}
\]

\[
E_{k,l}^{(-1)} \tilde{l} = E_{k,l}^{(0)} = E_{k,l}^{(1)} \tilde{l}^{-1} = E_{k,l}^{(2)} \tilde{l}^{-2} = \cdots = 0 , \tag{43}
\]

Hence, the corresponding energies are

\[
E_{0,l} = A \left( l + \frac{3}{2} \right) , \tag{44}
\]

the well known exact results. For \( k = 0 \)

\[
U_{0,l}(x) = -\frac{1}{2} \left( y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6} + \cdots \right) \\
+ \tilde{l} \left( y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \frac{y^6}{6} + \cdots \right) \\
- \tilde{l} \frac{y^2}{2} - \tilde{l} y \; ; \; y = x \tilde{l}^{-1/2} . \tag{45}
\]

Obviously the terms between brackets in equation (44) are the infinite geometric series expansions for ln(1 + y). Equation (44) can be recast as

\[
U_{0,l}(x) = \ln(1 + y)^{-1/2} + \ln(1 + y)^\tilde{l} - \tilde{l} y - \tilde{l} \frac{y^2}{2} , \tag{46}
\]
which in turn implies the exact wave functions

$$\Psi_{0,l}(q) = N_{0,l}q^{\bar{l} - 1/2}e^{-Aq^2/2} ; \quad \bar{l} = Aq_0^2,$$

(47)

where $N_{0,l}$ are the normalization constants.

Next we consider the Coulomb potential $V(q) = -1/q$. In this case, Eq.(15) reads

$$E_{k,l} = -\frac{1}{2\bar{l}^2} ; \quad \bar{l} = k + l + 1 = q_0^2.$$  

(48)

The reminder corrections are identically zeros, Eq.(43). Hence, one obtains the eigenvalues

$$E_{k,l} = \frac{-1}{2(k + l + 1)^2},$$  

(49)

the well known exact energies for the Coulomb potential. For $k = 0$

$$U_{0,l}(x) = -\bar{l}y + ln(1 + y)^\bar{l},$$  

(50)

which in turn implies the exact wave functions

$$\Psi_{0,l}(q) = N_{0,l}q^{\bar{l}}e^{-\bar{l}q/q_0}.$$  

(51)
Now let us consider the NPO potential (1), for which Eq.(21) implies

\[
\beta = -\frac{1}{2}[1 + (2k + 1)w] \quad w = 2 \sqrt{\frac{(1 + 3bq_o^2 + b^2q^4_o + b^3q^6_o + a)}{(1 + 2bq^2_o + b^2q^4_o + a)(1 + bq^2_o)}}.
\]

(52)

In turn Eq.(20) reads

\[
l + \frac{1}{2}[1 + (2k + 1)w] = \frac{q_o^2}{1 + bq^2_o} \sqrt{1 + a + bq^2_o + b^2q^4_o}.
\]

(53)

Equation (43) is explicit in \(q_o\) and evidently a closed form solution for \(q_o\) is hard to find, though almost impossible. However, numerical solutions are feasible. Once \(q_o\) is determined the coefficients \(C_{m,n}\) and \(D_{m,n}\) are obtained in a sequel manner. Consequently, the eigenvalues, Eq.(40), and eigenfunctions, Eqs.(29)-(32), are calculated in the same batch for each value of \(a, b,\) and \(l\).

In order to make remediable analysis of our results we have calculated the first ten terms of the energy series. The effect of each term has been taken into account. We have also computed the Padé approximants \(E[N,M]\) for \(N = 2, 3, 4\) and \(M = 2, 3, 4\). Therefore, the stability of the sequence of the Padé approximants was in point.

In tables 1 and 2 we list PSLET results \(E_p\), Eq.(40), along with the [4,4] Padé approximants, Eqs.(41). The results of Roy et al.[12], via SSQM, and Handy et al.[14], via EMM, are also displayed for comparison. In tables 3-5 we compare our results with those of Scherrer et al.[2], via MCF, and Varshni [8], via SLNT.

In tables 1 and 2 we have observed that, if the last two digits are neglected, the energy series Eq.(40) stabilizes at \(E_5\) up to \(E_{10}\). Where \(E_5\) and \(E_{10}\) denote that the energy is computed by the first five and first ten terms of the energy series, respectively. Yet for \(l = 5−20\) it stabilizes at \(E_3\) up to \(E_{10}\). While the Padé approximants for \(l = 5−20\) had no effect on the energy series their effect
is not dramatic for $l = 1 - 3$. Also it should be reported that the sequence of Padé approximants stabilizes at $E[3, 3]$ for $l = 1 - 3$ and at $E[2, 2]$ for $l = 5 - 20$. Therefore, one could confidently conclude that the results from the Padé approximants are exact provided that some uncertainty lies in the last two digits for $l = 1 - 3$. Nevertheless, our results do not contradict with the upper and lower bounds computed from EMM [14]. To a satisfactory extent they also agree with those from SSQM [12].

In tables 3 - 5 we list our results along with those of Scherrer [2], from MCF, and Varshni [8], from SLNT. Our results compare better with those of Scherrer than the results of SLNT. However, collecting only the first four terms of the energy series Eq.(40) we found that PSLET results are in exact accord with SLNT. Moreover, the trends of convergence of the energy series and the sequence of Padé approximants are similar to those for $l = 1 - 3$ in tables 1 and 2.

In table 6 we display our results along with those from STM and direct numerical integration (DNI) reported in [35] for the Coulomb plus logarithmic potential. In table 7 we list our predictions for the cutoff - Coulomb and compare them with those of Singh et al. [17]. Finally, we report the k=0 eigenvalues for the NPO and cutoff - Coulomb potentials in table 8 and 9, respectively.

4 Summary and Remarks

In this work we have introduced a pseudoperturbative shifted - $l$ expansion technique (PSLET) to deal with the calculation of the energy eigenvalues and eigenfunctions of Schrödinger equation in one batch. We have shown that it is an easy task to implement PSLET without having to worry about ranges of couplings and forms of perturbations in the potential involved. In contrast to
the textbook Rayleigh - Schrödinger perturbation theory, an easy feasibility of computation of the eigenvalues and eigenfunctions has been demonstrated, and satisfactory accuracies have been obtained. Perhaps it should be noted that for each entry in tables 1-5 one can construct the wavefunction from the knowledge of $C_{m,n}$ and $D_{m,n}$. However, such a study lies beyond the scope of our methodical proposal.

The conclusion reached by Scherrer [2], via MCF method for the NPO potential Eq.(1), that SLNT [8] leads to dubiously accurate results in the critical range $a^{1/2}/b \approx 0.1 - 30$ has been confirmed in the present work. However, the natural extensions of SLNT or its variants [8,16-21], represented by PSLET or the modified SLNT [22,23], show that one could get convincingly reliable results. Moreover, the dubious accuracies of SLNT [8] should be attributed mainly to the limited capability of SLNT to calculate the energy series beyond the fourth - order term. Our results for the cutoff - Coulomb and Coulomb plus logarithmic potentials are also very satisfactory.

A final remark is in point. The attendant method could be applied to systems at lower dimensions. Here is the recipe. Rewrite the centrifugal term in Eq.(2) as $\Lambda(\Lambda + 1)/2q^2$. Shift $\Lambda$ through $\bar{\Lambda} = \Lambda - \beta$ and expand in inverse powers of $\bar{\Lambda}$ following exactly the same procedure described in section 2. In this case, $\Lambda = l$ in three dimensions ( with $q \geq 0$), $\Lambda = m - 1/2$ in two dimensions where $m$ is the magnetic quantum number ( with $q \geq 0$), and $\Lambda = -1$ and/or 0 in one dimension ( with $-\infty < q < \infty$) [23-28].
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Table 1: Bound-state energies of the NPO potential (1). Where $E_P$ represents PSLET results, Eq.(51), $E_{SS}$ from SSQM [12], $E_E$ from EMM [14]. The Padé approximant $E[3,4]$ and the upper bound of EMM are obtained, respectively, by replacing the last $j$ digits of $E[3,3]$ and of the lower bound of EMM with the $j$ digits in parentheses.

| $l$ | $a$ | $b$ | $E_P$ | $E[3,3] \& (E[3,4])$ | $E_{SS}$ | $E_E$ |
|-----|-----|-----|-------|---------------------|-------|-------|
| 1   | 0.1 | 0.1 | 5.1863731 | 5.1863730 (29) | 5.186338 | 5.1863730 (30) |
|     | 0.5 |     | 5.100883  | 5.100858 (61) | 5.100976 | 5.100842 (65)  |
|     | 1   |     | 5.065556  | 5.065563 (71) | 5.065610 | 5.06428 (609)  |
| 0.5 | 0.1 |     | 5.8935959 | 5.8935951 (27) | 5.893494 | 5.8935952 (52) |
| 1   | 0.1 |     | 6.7042393 | 6.704239 (46) | 6.704090 | 6.7042389 (89) |
|     | 1   |     | 5.65138  | 5.65113 (40)  | 5.652112 | 5.6503 (21)    |
| 10  | 0.1 |     | 15.8137089 | 15.81370943 (48) | 15.813628 | 15.81370943 (43) |
| 100 | 0.1 |     | 49.38979430 | 49.389794297 (97) | 49.389615 | 49.38979427 (34) |
| 10  | 14.371 |     | 14.3622 | 14.36322 (64) | 14.363739 | 1.7 (14.609) |
| 100 | 5.9934347 |     | 5.993450 | 5.993450 (59) | 5.993565 | — (6.389) |
| 2   | 0.1 | 0.1 | 7.24396166 | 7.243961847 (50) | 7.243927 | 7.243961840 (40) |
|     | 0.5 |     | 7.118983  | 7.1189816 (10) | 7.119005 | 7.11890 (901)  |
|     | 1   |     | 7.0737228 | 7.0737258 (70) | 7.073713 | 7.0730 (44)    |
| 0.5 | 0.1 |     | 8.17787177 | 8.17787168 (66) | 8.177754 | 8.17787169 (69) |
| 1   | 0.1 |     | 9.2619150 | 9.26191476 (42) | 9.261812 | 9.26191478 (78) |
|     | 1   |     | 7.73479  | 7.734821 (33)  | 7.734778 | 7.734 (36)     |
| 10  | 0.1 |     | 21.83609251 | 21.83609251 (58) | 21.836043 | 21.83609247 (54) |
| 100 | 0.1 |     | 68.802061155 | 68.802061155 (55) | 68.801562 | 68.8020606 (15) |
| 10  | 16.61083 |     | 16.61081 | 16.611028 | 16.61097 (6540) |
| 100 | 7.9960247 |     | 7.99602475 (91) | 7.996048 | 7.9947 (8.0378) |
Table 2: Same as table 1.

| $l$ | $a$ | $b$ | $E_P$   | $E[3, 3] \& (E[3, 4])$             | $E_E$            |
|-----|-----|-----|---------|------------------------------------|------------------|
| 3   | 0.1 | 0.1 | 9.2943590 | 9.294359116 (16) | 9.2943591109 (09) |
|     | 0.5 |     | 9.1318120 | 9.131807 (12)     | 9.131799 (838)   |
|     | 1   |     | 9.0789113 | 9.0789116 (13)    | 9.0787 (92)      |
|     | 0.5 | 0.1 | 10.4292039 | 10.429204129 (33) | 10.42920412 (12) |
|     | 1   | 0.1 | 11.7606210 | 11.760620955 (50) | 11.76062096 (96) |
|     | 1   |     | 9.787665  | 9.787668 (73)     | 9.7875 (81)      |
| 10  | 0.1 |     | 27.68830286 | 27.6883029 (31)  | 27.6883028 (28) |
| 100 | 0.1 |     | 88.01806590 | 88.018065906 (07) | 88.0180658 (60) |
|     | 10  |     | 18.719989  | 18.720001 (21)    | 18.7186 (307)    |
|     | 100 |     | 9.99715366 | 9.997153642 (55)  | 9.9969 (1.0113)  |
| 5   | 200 | 0.1 | 179.48311321 | 179.48311321 (21) | 179.483107 (16) |
| 500 | 0.1 |     | 286.13076490 | 286.130764905 (05) | 286.13073 (81)  |
| 1000| 0.2 |     | 401.608033488 | 401.608033488 (89) | 401.6078 (81)   |
| 10000| 0.4 |     | 1280.6255249 | 1280.62552494 (95) | 1280.6254 (56)  |
| 10000| 0.5 |     | 1275.7839677 | 1275.7839677 (77) | 1275.7838 (42)  |
| 10  | 200 | 0.1 | 311.86088089 | 311.86088089 (89) | 311.8601371 (16266) |
| 1000| 0.1 |     | 713.36153440 | 713.36153440 (40) | 713.36081 (321) |
| 1000| 0.2 |     | 699.10562257 | 699.10562257 (57) | 699.10424 (909) |
| 10000| 0.4 |     | 2242.79417589 | 2242.79417589 (89) | 2242.7891995 (3867) |
| 10000| 0.5 |     | 2228.5184345 | 2228.5184345 (45) | 2228.513255 (30746) |
| 20  | 500 | 0.1 | 914.36631099 | 914.36631099 (09) | 914.36540 (851) |
| 1000| 0.1 |     | 1312.251674809 | 1312.251674809 (09) | 1312.25006 (333) |
Table 3: Bound - state energies of the NPO potential (1) for $a = 10$. $E_P$, $E[3, 3]$, and $E[3, 4]$ are the same as in table 1. $E_M$ from MCF [2] and $E_{SL}$ from SLNT [8].

| $l$ | $b$ | $E_P$   | $E[3, 3]$ & $(E[3, 4])$ | $E_M$   | $E_{SL}$ |
|-----|-----|---------|--------------------------|---------|----------|
|     |     |         |                          |         |          |
| 0   | 1   | 7.41837 | 7.417532 (12)            | 7.417506| 7.4056   |
|     | 10  | 3.885   | 3.864 (75)               | 3.879037| 3.8732   |
|     | 100 | 3.09826 | 3.097906 (26)            | 3.089317| 3.0984   |
|     | 1000| 3.00998 | 3.0099799 (801)          | 3.009981| 3.0100   |
| 1   | 1   | 11.0714 | 11.07326 (27)            | 11.07330| 11.0714  |
|     | 10  | 5.94089 | 5.94057 (23)             | 5.940860| 5.9408   |
|     | 100 | 5.09934 | 5.099355 (64)            | 5.099344| 5.0994   |
|     | 1000| 5.00999 | 5.009993346 (41)         | 5.009993| 5.0100   |
| 2   | 1   | 14.0861 | 14.08540 (31)            | 14.08538| 14.0900  |
|     | 10  | 7.9622256| 7.962228 (57)           | 7.96230 | 7.9622   |
|     | 100 | 7.09960269| 7.099602614 (30)         | 7.099603| 7.0996   |
|     | 1000| 7.009996003| 7.00999600266 (38)     | 7.009996| 7.0100   |
| 3   | 1   | 16.719284| 16.71910 (33)            | 16.71932| 16.7200  |
|     | 10  | 9.9724344| 9.9724556 (62)           | 9.972455| 9.9724   |
|     | 100 | 9.09971542| 9.09971542 (42)         | 9.099715| 9.0998   |
|     | 1000| 9.00999714406| 9.00999714399 (92) | 9.009997| 9.0100   |
| 4   | 1   | 19.137789| 19.137816 (24)           | 19.137821| 19.1376  |
|     | 10  | 11.97836435| 11.97836459 (62)        | 11.978365| 11.9784  |
|     | 100 | 11.099778408| 11.099778407 (07)      | 11.099778| 11.0998  |
|     | 1000| 11.00999777842| 11.009997778412 (64) | 11.009998| 11.0100  |
Table 4: Same as table 3 for $a = 100.$

| $l$ | $b$ | $E_P$     | $E[3, 3] \& (E[3, 4])$ | $E_M$     | $E_{SL}$  |
|-----|-----|-----------|-------------------------|-----------|-----------|
| 0   | 1   | 26.70597  | 26.705966 (75)          | 26.705966 | 26.706    |
|     | 10  | 10.6      | 11.55 (76)              | 11.572197 | 11.6112   |
|     | 100 | 3.9825    | 3.9790 (17)             | 3.983098  | 3.9844    |
|     | 1000| 3.099813  | 3.0997996 (814)         | 3.099811  | 3.0998    |
| 1   | 1   | 42.23757  | 42.2375612 (88)         | 42.237560 | 42.236    |
|     | 10  | 14.370    | 14.3622 (64)            | 14.368811 | 14.3638   |
|     | 100 | 5.9934346 | 5.9934500 (91)          | 5.993439  | 5.9936    |
|     | 1000| 5.09993354| 5.09993346 (41)         | 5.099933  | 5.1000    |
| 2   | 1   | 55.97780  | 55.9778047 (70)         | 55.977804 | 55.976    |
|     | 10  | 16.61084  | 16.61081 (54)           | 16.610869 | 16.6110   |
|     | 100 | 7.99602470| 7.99602475 (91)         | 7.996025  | 7.9960    |
|     | 1000| 7.099960046| 7.0999600264 (37)     | 7.099960  | 7.1000    |
| 3   | 1   | 67.9608094| 67.96080 (81)           | 64.960806 | 67.9600   |
|     | 10  | 18.719989 | 18.720001 (21)          | 18.719999 | 18.7202   |
|     | 100 | 9.997153662| 9.997153641 (55)       | 9.997154  | 9.9972    |
|     | 1000| 9.0999714406| 9.099971440 (39)    | 9.099971  | 9.1000    |
| 4   | 1   | 78.2383822| 78.23838049 (45)        | 78.238380 | 78.2380   |
|     | 10  | 20.7814134| 20.7814161 (67)         | 20.781416 | 20.7820   |
|     | 100 | 11.9977838| 11.9977838311 (34)      | 11.997784 | 11.9978   |
|     | 1000| 11.099977784| 11.099977784 (84)   | 11.099978 | 11.10000  |
Table 5: Same as table 3 for $a = 1000.$

| $l$ | $b$ | $E_P$ | $E[3,3] \& (E[3,4])$ | $E_M$ | $E_{SL}$ |
|-----|-----|-------|----------------------|-------|----------|
| 0   | 1   | 91.2566 | 91.25661112 (32) | 91.256611 | 91.2560 |
|     | 10  | 64.833  | 64.8244 (51)      | 64.825083 | 64.7400 |
|     | 100 | 12.8162 | 12.784 (80)       | 12.823345 | 12.8366 |
|     | 1000| 3.99813 | 3.99800 (01)      | 3.998107  | 3.9984  |
| 1   | 1   | 149.6563194 | 149.65631949 (81) | 149.656319 | 149.6560 |
|     | 10  | 89.126  | 89.12349 (37)   | 89.123452 | 89.0400 |
|     | 100 | 14.933728 | 14.93398 (406)  | 14.933774 | 14.9350 |
|     | 1000| 5.9993353 | 5.99933458 (11) | 5.999335  | 5.9994  |
| 2   | 1   | 206.1068055 | 206.1068055 (59) | 206.106805 | 206.1000 |
|     | 10  | 100.60  | 100.7043 (64)    | 100.703996 | 100.7260 |
|     | 100 | 16.9601067 | 16.9601077 (91) | 16.960106 | 16.9604 |
|     | 1000| 7.99960031 | 7.999600250 (23) | 7.999600 | 7.99960 |
| 3   | 1   | 260.6091863 | 260.6091863 (71) | 260.609186 | 260.6200 |
|     | 10  | 105.5097 | 105.5054 (70)   | 105.50759  | 105.4960 |
|     | 100 | 18.9714849 | 18.97148472 (84) | 18.971485 | 18.9716 |
|     | 1000| 9.99971440 | 9.999714394 (86) | 9.999714 | 9.9998 |
| 4   | 1   | 313.16466655 | 313.164666 (72) | 313.164667 | 313.1600 |
|     | 10  | 108.52780 | 108.527802 (18) | 108.527834 | 108.5280 |
|     | 100 | 20.9778139 | 20.977813876 (97) | 20.977814 | 20.9780 |
|     | 1000| 11.999777840 | 11.999777839 (44) | 11.999778 | 11.9998 |
Table 6: $k = 0$ eigenvalues for $V(q) = -1/2q + (\mu/2)\ln(q^2 + q)$.

| $\mu$ | $E_P$     | E[4,4]     | STM        | DNI        |
|-------|-----------|------------|------------|------------|
| 0.0001| 0.2497779 | 0.2497779  | 0.24975    | 0.24978    |
| 0.0005| 0.248890  | 0.248890   | 0.24875    | 0.24889    |
| 0.001 | 0.247782  | 0.247782   | 0.24752    | 0.24778    |
| 0.005 | 0.238976  | 0.238973   | 0.23765    | 0.23897    |
| 0.01  | 0.228105  | 0.228098   | 0.22545    | 0.22810    |
| 0.05  | 0.145676  | 0.145681   | 0.13227    | 0.14568    |
| 0.1   | 0.051363  | 0.051499   | 0.02456    | 0.05153    |
| 0.5   | 0.521641  | 0.520529   | 0.65413    | 0.52033    |
Table 7: CUTT OFF POTENTIAL.

| \( b \) | \( E_P \)       | \( 1s \)         | \( 2p \)         | \( 3d \)         | \( 4f \)         |
|---------|----------------|------------------|------------------|------------------|------------------|
| 0.3     | 0.2938592647   | 0.1061734109     | 0.0516188327     | 0.029960986      |
|         | 0.2935835059   | 0.1061736042     | 0.0516188327     | 0.0299960986     |
|         | 0.29354528     | 0.10617351       | 0.05161883       | 0.02999610       |
| 0.5     | 0.2446947653   | 0.0976555780     | 0.0515016542     | 0.0292431052     |
|         | 0.2445430901   | 0.0976556895     | 0.0515016549     | 0.0292431052     |
|         | 0.24453144     | 0.09765562       | 0.04943696       | 0.02924311       |
| 5       | 0.0706714094   | 0.0434584151     | 0.0287056893     | 0.0200000000     |
|         | 0.0706701591   | 0.0434584053     | 0.0287056891     | 0.0200000000     |
|         | 0.07067028     | 0.04345840       | 0.02870569       | 0.02000000       |
| 100     | 0.0067420934   | 0.0056337380     | 0.0048124934     | 0.0041687844     |
|         | 0.0067420738   | 0.0056337379     | 0.0048124934     | 0.0041687844     |
|         | 0.00674208     | 0.00563374       | 0.00481249       | 0.00416878       |
Table 8: NPO for $k = 1$ and $a = 10$.

| $b$  | $l$  | $E_{PSLET}$  | $E_{44}$  | $E_{1/N}$  | $E_{CFM}$  |
|------|------|--------------|----------|------------|------------|
| 1    | 0    | 13.3557146   | 13.388417| 13.496     | 13.388323  |
| 10   |      | 7.9029106    | 7.9654956| 7.9180     | 7.903154   |
| 100  |      | 7.0984744    | 7.0979219| 7.0990     | 7.098449   |
| 1000 |      | 7.0099838    | 7.0099799| 7.0100     | 7.009982   |
| 1    | 1    | 16.0066200   | 16.0156839| 15.963     | 16.016128  |
| 10   |      | 9.9447832    | 9.9452031| 9.9504     | 9.944898   |
| 100  |      | 9.0993550    | 9.0993626| 9.0994     | 9.099352   |
| 1000 |      | 9.0099934    | 9.0099934| 9.0100     | 9.009993   |
| 1    | 2    | 18.5442860   | 18.5432358| 18.498     | 18.543473  |
| 10   |      | 11.9633289   | 11.9633409| 11.965     | 11.963343  |
| 100  |      | 11.0996043   | 11.0996041| 11.010     | 11.099604  |
| 1000 |      | 11.0099960   | 11.0099960| 11.010     | 11.009996  |