TIGHTNESS FOR THICK POINTS IN TWO DIMENSIONS

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Abstract. Let $W_t$ be Brownian motion in the plane started at the origin and let $\theta$ be the first exit time of the unit disk $D_1$. Let

$$\mu_\theta(x, \epsilon) = \frac{1}{\pi \epsilon^2} \int_0^\theta \mathbb{1}_{B(x, \epsilon)}(W_t) \, dt,$$

and set $\mu_\theta^*(\epsilon) = \sup_{x \in D_1} \mu_\theta(x, \epsilon)$. We show that

$$\sqrt{\mu_\theta^*(\epsilon)} - \sqrt{2/\pi} \left( \log \epsilon^{-1} - \frac{1}{2} \log \log \epsilon^{-1} \right)$$

is tight.

1. Introduction

Let $W_t$ be Brownian motion in the plane started at the origin and let $\theta$ be the first exit time of the unit disk $D_1$. In [6] we showed that

$$\lim_{\epsilon \to 0} \sup_{x \in D_1} \frac{1}{\epsilon^2 \log^2(\epsilon)} \int_0^\theta \mathbb{1}_{B(x, \epsilon)}(W_t) \, dt = 2, \text{ a.s.},$$

where $B(x, \epsilon)$ is the ball of radius $\epsilon$ centered at $x$. The integral above is the occupation measure of $B(x, \epsilon)$, and points $x$ with large occupation measure are referred to as thick points. Taking square roots we can write this as

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon \log(\epsilon^{-1})} \sqrt{\sup_{x \in D_1} \frac{1}{\pi \epsilon^2} \int_0^\theta \mathbb{1}_{B(x, \epsilon)}(W_t) \, dt} = \sqrt{2/\pi}, \text{ a.s.}$$

Let

$$\mu_\theta^*(\epsilon) = \frac{1}{\pi \epsilon^2} \int_0^\theta \mathbb{1}_{B(x, \epsilon)}(W_t) \, dt,$$

and set $\mu_\theta^*(\epsilon) = \sup_{x \in D_1} \mu_\theta(x, \epsilon)$. Then (1.2) says that $\sqrt{\mu_\theta^*(\epsilon)} \sim \sqrt{2/\pi} \log \epsilon^{-1}$, as $\epsilon \to 0$. In this paper we obtain more detailed asymptotics. Let

$$m_\epsilon = \sqrt{2/\pi} \left( \log \epsilon^{-1} - \frac{1}{2} \log \log \epsilon^{-1} \right).$$
We will say that the thick points in \( D_1 \) are tight if \( \sqrt{\mu^*_\theta(\epsilon)} - m_\epsilon \) is a tight family of random variables. That is,
\[
(1.5) \quad \lim_{K \to \infty} \lim_{\epsilon \to 0} \mathbb{P} \left( \left| \sqrt{\mu^*_\theta(\epsilon)} - m_\epsilon \right| > K \right) = 0.
\]

**Theorem 1.1.** The thick points in \( D_1 \) are tight.

In fact we obtain the following improvement on the right tail of (1.5), where \( c^* := 2\sqrt{2} \).

**Theorem 1.2.** On \( D_1 \), for some \( 0 < C, C', z_0 < \infty \) and all \( z \geq z_0 \),
\[
(1.6) \quad \lim_{\epsilon \to 0} \mathbb{P} \left( \sqrt{\mu^*_\theta(\epsilon)} - m_\epsilon \geq z \right) \leq C z e^{-c^{*} \sqrt{\pi} z},
\]
and
\[
(1.7) \quad \lim_{\epsilon \to 0} \mathbb{P} \left( \sqrt{\mu^*_\theta(\epsilon)} - m_\epsilon \geq z \right) \geq C' z e^{-c^{*} \sqrt{\pi} z}.
\]

It follows from Brownian scaling that Theorems 1.1 and 1.2 hold if \( D_1 \) is replaced by any disc centered at the origin.

For reasons of symmetry it is easier to work on the sphere \( S^2 \), and derive our results for thick points in \( D_1 \) from results for thick points on \( S^2 \). We use \( B_d(x, r) \) for the ball centered at \( x \) of radius \( r \), in the spherical metric \( d \).

Let \( X_t \) be Brownian motion on \( S^2 \) started at some point \( v \) (the ‘South Pole’). For some (small) \( r^* \) let \( \tau \) be the first hitting time of \( \partial B_d(v, r^*) \) (the ‘Antarctic Circle’). Let \( \omega_\epsilon = 2\pi (1 - \cos \epsilon) \), the area of \( B_d(x, \epsilon) \), and set
\[
(1.8) \quad \mu_\tau(x, \epsilon) = \frac{1}{\omega_\epsilon} \int_0^\tau 1_{\{B_d(x, \epsilon)\}}(X_t) \, dt,
\]

With \( \bar{\mu}^*_\tau(x, \epsilon) = \sup_{x \in S^2} \mu_\tau(x, \epsilon) \) we will say that the thick points on \( S^2 \) are tight if \( \sqrt{\bar{\mu}^*_\tau_{\tau,x} - m_\epsilon} \) is a tight family of random variables.

**Theorem 1.3.** The thick points on \( S^2 \) are tight.

As in Theorem 1.2 we obtain the following improvement for the right tail.

**Theorem 1.4.** On \( S^2 \), for some \( 0 < C, C', z_0 < \infty \) and all \( z \geq z_0 \),
\[
(1.9) \quad \lim_{\epsilon \to 0} \mathbb{P} \left( \sqrt{\bar{\mu}^*_\tau_{\tau,x} - m_\epsilon} \geq z \right) \leq C z e^{-c^{*} \sqrt{\pi} z},
\]
and
\[
(1.10) \quad \lim_{\epsilon \to 0} \mathbb{P} \left( \sqrt{\bar{\mu}^*_\tau_{\tau,x} - m_\epsilon} \geq z \right) \geq C' z e^{-c^{*} \sqrt{\pi} z}.
\]

Theorems 1.3 and 1.4 are stated and first proven for \( r^* \) sufficiently small. In Section 8 we show that they hold for any \( 0 < r^* < \pi \).

In analogy with [6], rather than work directly with occupation measures, we work with excursion counts. To define this let \( h_l = 2 \arctan \left( r_0 e^{-l/2} \right) \) with \( r_0 \) small. For some \( d_0 \leq 1/1000 \) let \( F_l \) be the centers of a \( d_0 h_l \) covering of \( S^2 \).

Let \( T_{x,l}^\tau \) be the number of excursions from \( \partial B_d(x, h_{l-1}) \) to \( \partial B_d(x, h_l) \) prior to \( \tau \). We will obtain the following result for \( \sup_{x \in F_l} T_{x,l}^\tau \).
Theorem 1.5. On $S^2$, for some $0 < z_0, C, C' < \infty$, all $L$ large and all $z_0 \leq z \leq \log L$,

\begin{equation}
C'ze^{-2z} \leq \mathbb{P}\left( \sup_{x \in F_L} \sqrt{2T_{x,L}} - (2L - \log L) \geq z \right) \leq Cze^{-2z}.
\end{equation}

Equivalently

\begin{equation}
C'ze^{-2z} \leq \mathbb{P}\left( \sup_{x \in F_L} T_{x,L} \geq 2L \left( \log L + \frac{1}{2} \right) \right) \leq Cze^{-2z}.
\end{equation}

Since $L \sim \log h^{-1}$, Theorem 1.5 is then suggestive of Theorem 1.4 if we knew that on average the occupation measure of $B_d(x, h_L)$ during an excursion from $\partial B_d(x, h_L)$ to $\partial B_d(x, h_L-1)$ was ‘about’ $h_L^2$. While this is basically known for our choices of $h_L, h_L-1$, see [6, Lemma 6.2], it is more delicate to get the precision necessary to show the equivalence of Theorem 1.5 with (1.9).

We now write (1.11) in a more convenient form. Set

\begin{equation}
\rho_L = 2 - \frac{\log L}{L}.
\end{equation}

We will prove the following version of Theorem 1.5.

Theorem 1.6. On $S^2$, for some $0 < z_0, C, C' < \infty$, all $L$ large and all $z_0 \leq z \leq \log L$,

\begin{equation}
C'ze^{-2z} \leq \mathbb{P}\left( \sup_{x \in F_L} \sqrt{2T_{x,L}} \geq \rho_L + z \right) \leq Cze^{-2z}.
\end{equation}

1.1. Background. This paper is based in many ways on my work [4] with Belius and Zeituni on tightness for the cover time of $S^2$. The general approach is similar, and whenever results of that paper could be used directly I did so. However, the mathematics often necessitated different arguments.

The family

\begin{equation}
\{ \bar{\mu}_x(x, \epsilon); x \in B_d(v, r^*), \epsilon > 0 \}
\end{equation}

is associated with a second order Gaussian chaos $H(x, \epsilon), x \in B_d(v, r^*), \epsilon > 0$ by an isomorphism theorem of Dynkin [12]. Intuitively,

\begin{equation}
H(x, \epsilon) = \int_{B_d(x, \epsilon)} G^2_y dm(y) - E \left( \int_{B_d(x, \epsilon)} G^2_y dm(y) \right)
\end{equation}

where $G_x$ is the mean zero Gaussian process with covariance $u(x, y)$, the Green’s function for $B_d(v, r^*)$ and $m$ denotes the standard surface measure on $S^2$. Since $u(x, x) = \infty$ for all $x$, (1.15) is not a priori well defined. Nevertheless, this would suggest that there is a close relationship between $\bar{\mu}^*_{x, \epsilon} = \sup_{x \in R^2} \bar{\mu}_x(x, \epsilon)$ and the supremum of Gaussian fields. For details on $H$ and the isomorphism theorem see [19, Section 2].

For related work see [1, 15, 16, 17, 18].
1.2. **Open problems.**

1. Based on the analogy with the extrema of Branching random walks and log-correlated Gaussian fields, one expects that Theorem 1.1 should be replaced by the statement

\[
\text{The sequence of random variables } \sqrt{\mu^*_\theta(\epsilon)} - m_\epsilon \text{ converges in distribution to a randomly shifted Gumbel random variable.}
\]

A key step in proving such convergence would be the improvement of the tail estimates in Theorems 1.2 and 1.4 for \(z\) large, which in turn would require a corresponding improvement of Theorem 1.6.

2. In [6] we also proved a conjecture of Erdős and Taylor concerning the number \(L^*_n\) of visits to the most visited site for simple random walk in \(Z^2\) up to step \(n\). It was shown there that

\[
\lim_{n \to \infty} \frac{L^*_n}{(\log n)^2} = \frac{1}{\pi} \quad \text{a.s.}
\]

(1.16)

The approach in that paper was to first prove (1.1) for planar Brownian motion and then to use strong approximation. Subsequently, in [22], we presented a purely random walk method to prove (1.16) for simple random walk. A natural problem is to prove tightness for \(\sqrt{L^*_n}\).

3. Following [6] we analyzed thick points for several other process. See [7], [8] and [9]. One can ask about tightness or some analog for these processes.

1.3. **Structure of the paper.** In Section 2 we obtain the upper bounds for excursion counts in Theorem 1.6 and in Section 3 we derive the lower bounds. These sections employ many of the tools developed in [4]. In Section 4 we show how to go from results on excursion counts to Theorems 1.3 and 1.4 which involve \(\tilde{\mu}_\tau(x, \epsilon)\) in \(S^2\). Here we have to deal with a new problem for the upper bounds: \(\tilde{\mu}_\tau(x, \epsilon)\) in \(S^2\) is not in general monotone in \(\epsilon\). This requires interpolation and a continuity estimate which are developed in Sections 5 and 7. In the short Section 8 we derive our results on thick points for the unit disc in the plane from our results on thick points for \(S^2\), and use this to show that Theorems 1.3 and 1.4 hold for any \(0 < r^* < \pi\). The last section is an Appendix containing the barrier estimates we need for Sections 2 and 8.
1.4. **Index of Notation.** The following are frequently used notation, and a pointer to the location where the definition appears.

| Notation | Page |
|----------|------|
| $\mu_0(x, \epsilon)$ | (1.3) |
| $m_\epsilon$ | (1.4) |
| $c^*$ | (1.6) |
| $\mu_\tau(x, \epsilon), B_d(x, r)$ | (1.8) |
| $\rho_L$ | (1.13) |
| $T_{x,l}$ | (1.11) |
| $r_l, h_l$ | (2.2) |
| $F_l$ | (2.4) |
| $T_{y,l}^{k\rightarrow 0}$ | (2.6) |
| $l_L$ | (2.14) |
| $\alpha_{z,+}(l)$ | (2.15) |
| $k_y$ | (2.16) |
| $F_L^+$ | (2.17) |
| $F_L^m$, $\mathcal{H}_{m,l}$ | (2.47) |
| $B_{m,l}$ | (2.49) |
| $C_{m,l}$ | (2.52) |
| $D_{m,l}(j)$ | (2.55) |
| $B_{m,l}^{\gamma,k}$ | (2.63) |
| $T_{y,l}^{m,r_l-2,n}$ | (2.66) |
| $\beta_{z}^l(l)$ | (3.2) |
| $\alpha_{z,-}(l)$ | (3.3) |
| $T_{y,l}^{1}, T_{y,l}^{1,x^2}$ | (3.3) |
| $F_L^0$ | (3.4) |
| $\mathcal{W}_{y,k}(n)$ | (3.10) |
| $N_{k,a}$ | (3.11) |
| $N_{k,l}, I_u$ | (3.12) |
| $\mathcal{H}_{k,a}$ | (3.13) |
| $k^+, k^{++}$ | (3.49) |
| $T_{y,z}$ | (3.14) |
| $N_{k,a}$ | (3.15) |
| $N_{k}$ | (3.16) |
| $I_{y,z}$ | (3.13) |
| $\mathcal{H}_{k,a}$ | (3.18) |
| $J_{y,k}$ | (3.27) |
| $B_{y,k,a}$ | (3.28) |
| $\mathcal{M}_{x,e,a,b}(n)$ | (4.1) |
| $t_r(z)$ | (4.5) |
| $\mathcal{M}_{y,t_{xy},a,b}(n)$ | (4.27) |
| $D_*$ | (8.2) |
2. Upper bounds for excursions

Let
\[ h(r) = 2 \arctan(r/2) \]
and let
\[ r_l = r_0 e^{-l}, l = 0, 1, \ldots \quad \text{and} \quad h_l = h(r_l) \]
for some \( r_0 < 1 \). We can take \( r_0 < 1 \) sufficiently small that for all \( 0 \leq x \leq r_0 \)
\[ x - x^3 \leq h(x) \leq x \quad \text{and} \quad |h'(x) - 1| \leq x^2. \]

For some \( d_0 \leq 1/1000 \) let \( F_l \) be the centers of an \( d_0 h_l \) covering of \( \mathbb{S}^2 \). We record for future use that
\[ |F_l| \asymp c r_l^{-2} = c r_0^{-2} e^{2l}, l \geq 0. \]

Recall that \( T_{x,l}^T \) is the number of excursions from \( \partial B_d(x, h_{l-1}) \) to \( \partial B_d(x, h_l) \) prior to \( \tau \). In this section we will assume that \( 2r^* \leq h_0 \) so that for all \( y \in B_d(v, r^*) \) we have \( B_d(v, r^*) \subseteq B_d(y, h_0) \).

The reason for using \( h(r) \) is due to the following result for \( \mathbb{S}^2 \), see [4, (2.6)]. For any \( u_1 < u_2 < u_3 \)
\[ P_{x \in \partial B_d(0, h(u_2))} \left[ H_{\partial B_d(0, h(u_1))} < H_{\partial B_d(0, h(u_3))} \right] = \frac{\log(u_2)}{\log(u_3)}. \]

The next Lemma provides simple bounds which will be adequate to handle points which are close to the ‘South Pole’ \( v \).

**Lemma 2.1.** For \( L \) large, any \( y \in B_d(v, h_k) \) and all \( |z| \leq \log L \),
\[ P \left[ \sqrt{2T_{y,L}^T} \geq \rho L + z \right] \leq c k e^{-2L^2} e^{-2z}. \]

**Proof:** For \( k \leq l - 1 \), let \( T_{y,l}^{k \rightarrow 0} \) be the number of excursions from \( \partial B_d(y, h_{l-1}) \) to \( \partial B_d(y, h_l) \) between \( H_{\partial B_d(y,h_k)} \) and \( H_{\partial B_d(y,h_0)} \). We first estimate probabilities involving \( T_{y,l}^{k \rightarrow 0} \). Using (2.5), an excursion from \( \partial B_d(y, h_k) \) hits \( B_d(y, h_{l-1}) \) before exiting \( B_d(y, h_0) \) has probability \( k/(l - 1) \), and then the probability to hit \( \partial B_d(y, h_l) \) before exiting \( B_d(y, h_0) \) is \( 1 - \frac{1}{L} \). Thus, using the strong Markov property,
\[ P \left[ T_{y,l}^{k \rightarrow 0} \geq n \right] = \frac{k}{l - 1} \left( 1 - \frac{1}{L} \right)^n \leq \frac{k}{l - 1} e^{-\frac{n}{L}}. \]
for \( n \) large. Since (recall (1.13))
\[
(\rho_L L + z)^2 = (\rho L L)^2 + 2z\rho L L + z^2
\]
(2.9)
\[
= 4L^2 - 4L \log L + 4zL + z^2 - 2z \log L + \log^2 L,
\]
it follows that for \( L \) large
\[
\mathbb{P} \left[ \sqrt{2T^{k\to 0}_{y,L}} \geq \rho_L L + z \right] \leq c e^{-2L} L^{-2z}.
\]
(2.10)
follows since for \( y \in B^c_d(v, h_k) \) we have \( T^{y}_{y,L} \leq T^{k\to 0}_{y,L} \).

For (2.7) we note that for \( y \in B_d(v, h_{L-1}) \) we have \( T^{y}_{y,L} \leq T^{L-1\to 0}_{y,L} \).

\[\Box\]

**Lemma 2.2.** For \( L \) large and all \( 0 \leq z \leq \log L \),
\[
\mathbb{P} \left[ \sup_{y \in F \cap B_d(v, h_{\log L})} \sqrt{2T^{y}_{y,L}} \geq \rho_L L + z \right] \leq c e^{-2z},
\]
(2.11)
for some \( c < \infty \).

**Proof:** By Lemma 2.1 the probability in (2.11) is bounded by
\[
\sum_{k=\log L}^{\infty} \mathbb{P} \left[ \sup_{y \in F \cap B_d(v, h_k) \cap B^c_d(v, h_{k+1})} \sqrt{2T^{y}_{y,L}} \geq \rho_L L + z \right] \\
\leq \sum_{k=\log L}^{\infty} \left| F \cap B_d(v, h_k) \cap B^c_d(v, h_{k+1}) \right| c e^{-2L} L^{-2z} \\
\leq c L e^{-2z} \sum_{k=\log L}^{\infty} k e^{-2k} \leq c e^{-2z}.
\]
(2.12)

Thus we only need deal with \( y \in B^c_d(v, h_{\log L}) \). However, Lemma 2.1 would give, for example, that
\[
\mathbb{P} \left[ \sup_{y \in F \cap B^c_d(v, h_1)} \sqrt{2T^{y}_{y,L}} \geq \rho_L L + z \right] \leq c L e^{-2z},
\]
(2.13)
which would be disastrous if we let \( L \to \infty \). To deal with this we introduce a barrier. Let
\[
l_L = l \wedge (L - l).
\]
(2.14)
Fix \( z \) and set
\[
\alpha_{z, +}(l) = \alpha(l, L, z) = \rho L l + z + l^{1/4}_L.
\]
(2.15)
Let
\[
k_y = \inf \{ k \mid y \in B^c_d(v, h_k) \},
\]
(2.16)

and
\[ F_L^* = F_L \cap B_d^c(v, h_{\log L}). \]  

Since \( \alpha_{z,+}(L) = \rho_L L + z \) and \( k_L \leq \log L \) for \( y \in F_L^* \), our desired upper bound will follow from the next Lemma.

**Lemma 2.3.** There exists \( z_0 > 0 \) such that for all \( z_0 \leq z \leq \log L \) and all \( L \) large
\[ P \left[ \exists y \in F_L^*, l \in \{k_y + 1, \ldots, L\} \text{ s.t. } T_{y,l}^\tau \geq \alpha_{z,+}^2(l)/2 \right] \leq c e^{-2z}. \]  

Although this formulation looks more complicated and demanding than our desired upper bound, it will allow us to proceed level by level and to eventually use a barrier estimate. The next Lemma will be used in our proof.

**Lemma 2.4.** For \( L \) large, any \( y \in B_d^c(v, h_{k}) \) and all \( 0 \leq z \leq \log L \),
\[ P \left[ T_{y,l}^\tau \geq \alpha_{z,+}^2(l)/2 \right] \leq c k e^{-2l} e^{-2(z + l^{1/4}) - (z + l^{1/4})^2/2l}, \]  
for some \( c < \infty \) independent of \( k \geq 1 \) and \( l \in \{k + 1, \ldots, L\} \).

**Proof:** As in (2.8) \[ P \left[ T_{y,l}^\tau \geq \alpha_{z,+}^2(l)/2 \right] \leq c k e^{-2l} e^{-2(z + l^{1/4}) - (z + l^{1/4})^2/2l}, \]  
and
\[ \alpha_{z,+}^2(l) = l^2 \rho_L^2 + 2(z + l^{1/4}) l \rho_L + (z + l^{1/4})^2, \]
\[ = l^2 \left( 4 - \frac{4 \log L}{L} + \frac{\log^2 L}{L^2} \right) + 2(z + l^{1/4}) l \left( 2 - \frac{\log L}{L} \right) \]
\[ = (z + l^{1/4})^2 \]
using the concavity of the logarithm. Our result follows.\\

2.1. **Proof of (2.18) for \( l \) not too large.**

**Proposition 2.5.** There exists \( z_0 > 0 \) such that for all \( z_0 \leq z \leq \log L \) and all \( L \) large
\[ P \left[ \exists y \in F_L^*, l \in \{k_y + 1, \ldots, L - (4 \log L)^4 \} \text{ s.t. } T_{y,l}^\tau \geq \alpha_{z,+}^2(l)/2 \right] \leq c e^{-2z}. \]
Proof: We use a packing argument. Let \( \phi(l) = e^{-25 l^{1/4} L} \). Considering separately the case of \( l \leq L/2 \) and \( L/2 < l \leq L - (4 \log L)^4 \), we see that for some \( m_0 \)
\[
(2.24) \quad l^{1/4} \geq 4 \log l, \quad m_0 \leq l \leq L - (4 \log L)^4,
\]
so that
\[
(2.25) \quad \frac{l}{\phi(l)} = \frac{l}{e^{25 l^{1/4} L^{1/4}}} \leq 1, \quad m_0 \leq l \leq L - (4 \log L)^4.
\]
We define modified radii by
\[
(2.26) \quad r_{l-1}^- = \left(1 - \frac{1}{\phi(l-1)}\right) r_{l-1} \quad \text{and} \quad r_{l+} = \left(1 + \frac{1}{\phi(l)}\right) r_l \quad \text{for } l \geq 1.
\]
Note that
\[
(2.27) \quad h(r_{l+\log \phi(l)}) \leq r_{l+\log \phi(l)} \quad \text{and} \quad \frac{r_{l-1}}{\phi(l)} = \frac{r_{l-1}}{e \phi(l)} \leq \frac{r_{l-1}}{\phi(l-1)}.
\]
Using this and (2.3) we have for \( \phi(l) \) large enough
\[
(2.28) \quad h(r_{l-1}^-) + \frac{1}{10^3} h(r_{l+\log \phi(l)}) \leq h(r_{l-1}) \quad \text{and} \quad h(r_l) + \frac{1}{10^3} h(r_{l+\log \phi(l)}) \leq h(r_{l+}).
\]
For each \( y \in S^2 \) let \( y_l \) denote the point in \( F_l \) closest to \( y \) (breaking ties in some arbitrary way). By the definition of \( F_{l+\log \phi(l)} \) we have
\[
(2.29) \quad d(y, y_{l+\log \phi(l)}) \leq \frac{1}{10^3} h(r_{l+\log \phi(l)}),
\]
so that using (2.28) we see that for all \( y \in S^2 \)
\[
(2.30) \quad B_d(y, h_1) \subset B_d(y_{l+\log \phi(l)}, h(r_{l+})) \subset B_d(y_{l+\log \phi(l)}, h(r_{l-1})) \subset B_d(y, h_{l-1}).
\]
Now for \( k \leq l - 1 \) set
\[
(2.31) \quad r_{k,l}^- = \left(1 - \frac{1}{\phi(l)}\right) r_k \quad \text{and} \quad r_{0,l}^+ = \left(1 + \frac{1}{\phi(l)}\right) r_0 \quad \text{for } l \geq 0.
\]
As in the proof of (2.28) we have
\[
(2.32) \quad h(r_{k,l}^-) + \frac{1}{10^3} h(r_{l+\log \phi(l)}) \leq h(r_k) \quad \text{and} \quad h(r_0) + \frac{1}{10^3} h(r_{l+\log \phi(l)}) \leq h(r_{0,l}^+),
\]
so that (2.29) also implies that
\[
(2.33) \quad B_d(y_{l+\log \phi(l)}, h(r_{k,l}^-)) \subset B_d(y, h(r_k)) \subset B_d(y, h(r_0)) \subset B_d(y_{l+\log \phi(l)}, h(r_{0,l}^+)).
\]
For each \( y \in F_{l+\log \phi(l)} \) let \( \overline{T}_{y,l}^{k \to 0} \) be the number of excursions from \( \partial B_d(y, h(r_{l-1})) \) to \( \partial B_d(y, h(r_{l+})) \) prior to the first excursion from \( \partial B_d(y, h(r_{k,l}^-)) \) to \( \partial B_d(y, h(r_{0,l}^+)) \). Then define
\[
\overline{T}_{y,l}^{k \to 0} = \overline{T}_{y_i,l}^{k \to 0} \quad \text{for } y_i \in S^2 \setminus F_{l+\log \phi(l)} \quad \text{for all } l \geq k + 1.
\]
It follows from (2.30) and (2.32) that \( \widetilde{T}_{y,l} \to 0 \geq \mathcal{T}_{y,l} \to 0 \geq \mathcal{T}_{r,l} \) for all \( l \geq k_y + 1 \). Thus

**Lemma 2.6.** For all \( y \in \mathbb{S}^2, l \geq k_y + 1 \) we have that

\[
(2.33) \quad \mathbb{P} \left[ \widetilde{T}_{y,l} \to 0 \geq \mathcal{T}_{y,l} \right].
\]

Because of this the probability in (2.23) is bounded above by

\[
(2.34) \quad \sum_{k=1}^{\log L} L^{-4(4 \log L)^4} \sum_{l=k+1}^{\log L} \sum_{y \in \mathcal{B}_d(v, h(r_{l-1})) \cap \mathcal{F}_l+ \log \phi(l)} \mathbb{P} \left[ \widetilde{T}_{y,l} \to 0 \geq \alpha_{z,l}^2 + (l)/2 \right]
\]

\[
= \sum_{k=1}^{\log L} L^{-4(4 \log L)^4} \sum_{l=k+1}^{\log L} \mathbb{P} \left[ \mathcal{B}_d(v, h(r_{l-1})) \cap \mathcal{F}_l+ \log \phi(l) \right] \mathbb{P} \left[ \widetilde{T}_{y,l} \to 0 \geq \alpha_{z,l}^2 + (l)/2 \right]
\]

\[
\leq c \sum_{k=1}^{\log L} L^{-4(4 \log L)^4} \sum_{l=k+1}^{\log L} e^{5l/4} e^{2(l-k)} \mathbb{P} \left[ \widetilde{T}_{y,l} \to 0 \geq \alpha_{z,l}^2 + (l)/2 \right],
\]

for some arbitrary \( y \in \mathcal{F}_l+ \log \phi(l) \). We show below that for all \( k \leq l \)

\[
(2.35) \quad \mathbb{P} \left[ \widetilde{T}_{y,l} \to 0 \geq \alpha_{z,l}^2 + (l)/2 \right] \leq c e^{-2l-l^4/4} e^{-2z},
\]

and since

\[
\sum_{k=1}^{\log L} \sum_{l=1}^{\log L} e^{5l/4} e^{2(l-k)} e^{-2l-l^4/4} e^{-2z} \leq c e^{-2z},
\]

this will complete the proof of (2.23).

We now turn to the proof of (2.35). Let

\[
(2.36) \quad p_l = \frac{\log \left( \frac{r_{l-1}^+}{r_{0,l}} \right)}{\log \left( \frac{r_{l-1}^+}{r_{0,l}} \right)} = \frac{\log \left( \frac{1 - \frac{1}{\phi(l-1)}}{1 + \frac{1}{\phi(l)}} \right)}{\log (e-l)} = \frac{l - 1 + 2/\phi(l) + O(\phi(l)^{-2})}{l} = 1 - \frac{1 - 2/\phi(l) + O(\phi(l)^{-2})}{l},
\]

and

\[
q_l = \frac{\log \left( \frac{r_{l-1}^+/r_{0,l}}{r_{l-1}^-/r_{0,l}} \right)}{\log \left( \frac{r_{l-1}^+}{r_{0,l}} \right)} = \frac{\log \left( \frac{1 - \frac{1}{\phi(l)}}{1 + \frac{1}{\phi(l)}} \right)}{\log (1 - \frac{1}{\phi(l-1)}) (1 + \frac{1}{\phi(l)})^{-1} e^{-k}} = \frac{k + 2/\phi(l) + O(\phi(l)^{-2})}{l - 1 + 2/\phi(l) + O(\phi(l)^{-2})} = \frac{k + O(\phi(l)^{-1})}{l - 1}.
\]

Using the fact that \( p_l < 1 \) together with (2.36) we can write

\[
(2.37) \quad p_l = 1 - \frac{1 - b_1/\phi(l)}{l}.
\]
with \(1 - b_l / \phi(l) > 0\). In addition, using (2.23) and possibly increasing \(m_0\),

\[(2.38) \quad \frac{lb_l}{\phi(l)} \leq 3, \quad l \geq m_0.\]

Since \(q_l\) is the probability for an excursion from \(\partial B_d (y, h(r_{-1}))\) before \(\partial B_d (y, h(r_{0,l}))\), and \(p_l\) is the probability for an excursion from \(B_d (y, h(r_{-1}))\) to hit \(B_d (y, h(r_{0,l}))\), we see that as in (2.38)

\[(2.39) \quad \mathbb{P} \left[ \bar{T}^k \geq \alpha_{z,+}(l)/2 \right] \leq \frac{ck}{l} e^{-\frac{\alpha_{z,+}(l)}{2l} \left(1 - \frac{b_l}{\phi(l)} \right)}.\]

By (2.22) we have that

\[(2.40) \quad \frac{\alpha_{z,+}(l)}{2l} \geq (2l - 2 \log l) + 2(z + l^{1/4}) + z^2/2l + o_L(1),\]

so that, for \(k \leq l\)

\[(2.41) \quad \mathbb{P} \left[ \bar{T}^k \geq \alpha_{z,+}(l)/2 \right] \leq c l^2 e^{-\left(2l + 2(z + l^{1/4}) + z^2/2l \right) \left(1 - \frac{b_l}{\phi(l)} \right)}.\]

We claim that

\[\frac{z^2}{2l} \left(1 - \frac{b_l}{\phi(l)} \right) - 2z b_l / \phi(l) > 0,\]

that is

\[z \left(1 - \frac{b_l}{\phi(l)} \right) > 4lb_l / \phi(l)\]

for \(z \geq z_0\) sufficiently large. For \(l > m_0\), this follows from (2.38), and for \(l \leq m_0\) we can just increase \(z\) further. Thus for such \(z\),

\[(2.42) \quad \mathbb{P} \left[ \bar{T}^k \geq \alpha_{z,+}(l)/2 \right] \leq c l^2 e^{-\left(2l + 2l^{1/4} \right) \left(1 - \frac{b_l}{\phi(l)} \right) e^{-2z}}.\]

For \(k \leq l \leq m_0\) this already proves (2.35) with \(c\) sufficiently large. For \(l > m_0\), using (2.38) again we now have

\[(2.43) \quad \mathbb{P} \left[ \bar{T}^k \geq \alpha_{z,+}(l)/2 \right] \leq c l^2 e^{-\left(2l + 2l^{1/4} \right) e^{-2z}}.\]

and (2.42) completes the proof of (2.35). \(\square\)

2.2. Proof of (2.18) for \(l\) very large. It suffices to show that for some small but fixed constant \(\tilde{c}\) to be chosen later we have that for all \(L\) sufficiently large and all \(z_0, 0 \leq z \leq \log L\)

\[(2.44) \quad \mathbb{P} \left[ \exists y \in F_L^* \cap B_d (0, \tilde{c}h_0), \text{ and } k_y + 1 \leq l \leq L \text{ such that } \sqrt{2T_{y,l}^*} \geq \alpha_{z,+}(l) \right] \leq cze^{-2z}.\]

Here 0, the center of \(B_d (0, \tilde{c}h_0)\), is used to denote an arbitrary point in \(S^2\).
Now consider
\[ G_l = \left\{ \sqrt{2T_{y,l'}} \leq \alpha_{z,+}(l') \text{ for all } l' = k_y + 1, \ldots, l \text{ and } \forall y \in F^*_L \cap B_d(0, \tilde{c}h_0) \right\}. \]

Let \( L' = L - (4 \log L)^4 \). With
\[ (2.45) \quad \mathcal{H}_l = \left\{ \exists y \in F^*_L \cap B_d(0, \tilde{c}h_0) \text{ s.t. } \sqrt{2T_{y,l'}} \geq \alpha_{z,+}(l), k_y < l \right\}, \]
we will prove that for all \( l > L' \)
\[ (2.46) \quad \mathbb{P}[\mathcal{H}_l \cap G_{l-2}] \leq cze^{-l_{L-2}^1/4 - 2z}, \]
so that we have
\[ \mathbb{P}[G_c^c_L] \leq \sum_{m=L+1}^L \mathbb{P}[G_c^c_l \cap G_{l-1}] + \mathbb{P}[G_c^c_L] \]
\[ \leq \sum_{m=L+1}^L \mathbb{P}[\mathcal{H}_m \cap G_{l-2}] + \mathbb{P}[G_c^c_L] \]
\[ \leq \sum_{m=L+1}^L cze^{-l_{L-2}^1/4 - 2z} + \mathbb{P}[G_c^c_L] \leq cze^{-2z}, \]
by Proposition 2.5, which will prove (2.44).

Setting \( F^m_L = F^*_L \cap B^*_d(v, h_m) \cap B_d(v, h_{m-1}) \), so that \( k_y = m \) for \( y \in F^m_L \), and for any \( l > m \)
\[ (2.47) \quad \mathcal{H}_{m,l} = \left\{ \exists y \in F^m_L \cap B_d(0, \tilde{c}h_0) \text{ s.t. } \sqrt{2T_{y,l'}} \geq \alpha_{z,+}(l) \right\}, \]
we will prove that for all \( l > L' \)
\[ (2.48) \quad \sum_{m=1}^{\log L} \mathbb{P}[\mathcal{H}_{m,l} \cap G_{l-2}] \leq cze^{-l_{L-2}^1/4 - 2z}, \]
which gives (2.46).

For any \( l > m \) let
\[ (2.49) \quad \mathcal{B}_{m,l} = \left\{ \exists x \in F^m_L \cap B_d(u_m, \tilde{c}h_l) \text{ s.t. } \sqrt{2T_{x,l'}} \geq \alpha_{z,+}(l) \right\}, \]
where we can assume that \( u_m \in F^m_L \). By a union bound, \( \mathbb{P}[\mathcal{H}_{m,l} \cap G_{l-2}] \) is bounded above by
\[ (2.50) \quad cze^{2(l-m)} \times \mathbb{P}[\mathcal{B}_{m,l} \cap G_{l-2}]. \]
Hence it suffices to show that
\[ (2.51) \quad \sum_{m=1}^{\log L} cze^{-2m} \mathbb{P}[\mathcal{B}_{m,l} \cap G_{l-2}] \leq cze^{-2l_{L-2}^1/4 - 2z}. \]

Since \( G_{l-2} \subset C_{m,l} \), where
\[ (2.52) \quad C_{m,l} = \left\{ \sqrt{2T_{u_m,l'}} \leq \alpha_{z,+}(l') \text{ for all } l' = m + 1, \ldots, l - 2 \right\}, \]
it suffices to show that

\[ \sum_{m=1}^{\log L} e^{-2m} \mathbb{P}[B_{m,l} \cap C_{m,l}] \leq cze^{-2l-1/4-2z}. \]  

We show in the next Section that for all \( l \geq L - (4 \log L)^4 \)

\[ \mathbb{P}[B_{m,l} \cap \left\{ \sqrt{2T_{T_i}^\tau} \leq \frac{1}{2} \alpha_z, (l-2) \right\}] \leq ce^{-c' L^2}. \]

It follows from this that with

\[ \sum_{m=1}^{\log L} e^{-2m} \sum_{j=0}^{1 \frac{1}{2} \alpha_z, (l-2)} \mathbb{P}[B_{m,l} \cap C_{m,l} \cap D_{m,l}(-j)] \leq cze^{-2l-1/4-2z}. \]

We also show in the next Section that we can find a fixed \( j_0 \) such that for all \( j_0 \leq j \leq \frac{1}{2} \alpha_z, (l-2) \), uniformly in \( 1 \leq m \leq \log L \) and \( z_0 \leq z \leq \log L \), for any \( \tilde{C}_{m,l} \in \mathcal{F}(T_{T_i}^\tau, k = 1, \ldots, l-2) \)

\[ \mathbb{P}[B_{m,l} \mid \tilde{C}_{m,l} \cap D_{m,l}(-j)] \leq Ce^{-4j}, \]

by taking \( \tilde{c} > 0 \) sufficiently small.

It follows from the barrier estimate (9.5) that for \( 0 \leq j \leq \frac{1}{2} \alpha_z, (l-2) \),

\[ \mathbb{P}[C_{m,l} \cap D_{m,l}(-j)] \leq ce^{-2l-2z-2l_1^{1/4}+2j} \times m^2 \left( 1 + \alpha_z, (l-2) \right) (1 + j). \]

Combining the last 2 displays we can bound the left hand side of (2.57) by

\[ Cze^{-2l-2z-l_1^{1/4}} \sum_{j=0}^{\infty} e^{-4j} (1 \geq \alpha_z, (l-2)) (1 + j), \]

which proves (2.56).

2.3. Proof of the conditional barrier estimate (2.57) and (2.54). We first prove that for some \( j_0 \) fixed and all \( j_0 \leq j \leq \frac{1}{2} \alpha_z, (l) \),

\[ \mathbb{P}[B_{m,l} \mid \tilde{C}_{m,l} \cap D_{m,l}(-j)] \leq Ce^{-4j}. \]

Proof: For each \( \gamma \in (0, 1] \) and \( y \), let \( T_{y, \tilde{r}_l}^\tau \) be the number of excursions from \( \partial B(y, h(\tilde{r}_l)) \) to \( \partial B(y, h(\tilde{r}_l)) \) prior to \( \tau \), where

\[ \tilde{r}_{l-1} = r_{l-1} (1 - \gamma), \quad \tilde{r}_l = r_l (1 + \gamma). \]

Note that

\[ T_{y, \tilde{r}_l}^\tau \geq T_{y, \tilde{r}_l}^\tau \quad \text{for all } y' \text{ such that } d(y, y') \leq \frac{\gamma T_i^\tau}{2}. \]
We will show that with these values
\[ B_d(y, h_{t-1}) \supset B_d(y', h(r_{t-1} (1 - \gamma))) \supset B_d(y', h(r_t (1 + \gamma))) \supset B_d(y, h_t). \]

Let
\[ (2.63) \quad B_{m,l}^\gamma \{ \exists y \in F_{k}^m \cap B_d(u_m, c h_t) \text{ such that } \sqrt{2T_{y,r_t}^\gamma} \geq \alpha_{+z}(l) \}. \]

Note \( F_{k}^m \) not \( F_{L}^m \). From now on we fix
\[ (2.64) \quad \gamma = \frac{1}{\alpha_{+z}(l) - j}, \quad \text{and} \quad k = \log (2(\alpha_{+z}(l) - j)) + l. \]

We will show that with these values
\[ (2.65) \quad \mathbb{P} \left[ B_{m,l}^\gamma \cap \mathcal{D}_{m,l}(-j) \right] \leq C e^{-4j}. \]

Using (2.62) this will imply (2.60), since for each \( y \in F_{L}^m \cap B_d(u_m, c h_t) \) there exists a representative \( y' \in F_{k}^m \cap B_d(u_m, c h_t) \) such that
\[ d(y, y') \leq r_k = \frac{1}{2(\alpha_{+z}(l) - j)} r_t = \frac{\gamma r_t}{2}. \]

To show (2.65), we first show that for some \( c_3 > 0 \)
\[ (2.66) \quad \mathbb{P} \left[ \sqrt{2T_{y,r_t}^\gamma} \geq \alpha_{+z}(l) - \frac{j}{2} \right] \leq c' e^{-c_3 j^2}. \]

Let \( T_{y,r_t}^{u_m,r_{t-2}} \) be the number of excursions from \( \partial B(y, h(r_{t-1})) \) to \( \partial B(y, h(r_t)) \) during the first \( n \) excursions from \( \partial B(u_m, h_{t-2}) \) to \( \partial B(u_m, h_{t-3}) \). Using the Markov property we have that
\[ (2.67) \quad \mathbb{P} \left[ \sqrt{2T_{y,r_t}^\gamma} \geq \alpha_{+z}(l) - \frac{j}{2} \right] \leq \mathbb{P} \left[ \sqrt{2T_{y,r_t}^\gamma} \geq \alpha_{+z}(l) - \frac{j}{2} \right] \mathbb{P} \left[ \sqrt{2T_{y,r_t}^\gamma} \geq \alpha_{+z}(l) - \frac{j}{2} \right] \leq \mathbb{P} \left[ \sqrt{2T_{y,r_t}^\gamma} \geq \alpha_{+z}(l) - \frac{j}{2} \right]. \]

To prove (2.66) it suffices to show that
\[ (2.68) \quad \mathbb{P} \left[ \sqrt{2T_{y,r_t}^{u_m,r_{t-2}}} \geq \alpha_{+z}(l) - \frac{j}{2} \right] \leq c' e^{-c_3 j^2}. \]

To see this, let \( s = \alpha_{+z}(l) - j + \zeta \), where \( 0 \leq \zeta \leq 1 \). It follows from Lemma 4.6, that
\[ (2.69) \quad \mathbb{P} \left[ \sqrt{2T_{y,r_t}^{u_m,r_{t-2}}} \geq \alpha_{+z}(l) - \frac{j}{2} \right] \leq e^{-\left( \sqrt{\mathbb{P}(\alpha_{+z}(l) - j + \zeta)} - \sqrt{\mathbb{P}(\alpha_{+z}(l) - \frac{j}{2})} \right)^2 / 2}. \]
where
\[
q = \mathbb{P}_{u \in \partial B_d(u_m, h_{l-2})} \left[ \log r_{l-3} - \log r_{l-2} \leq \frac{1}{3 + O(\gamma)} \right],
\]
and
\[
p = \mathbb{P}_{u \in \partial B_d(u_m, h_{l-1}(\gamma))} \left[ \log r_{l-2} - \log r_{l-3} \leq \frac{1}{3 + O(\gamma)} \right].
\]

To apply [3, Lemma 4.6] it suffices to show that
\[
\alpha_{z,+}(l) - \frac{j}{2} \geq (\alpha_{z,+}(l) - j + 1)\sqrt{q/p},
\]
and the right hand side
\[
= (\alpha_{z,+}(l) - j + 1)(1 + O(\gamma)) = (\alpha_{z,+}(l) - j) + O(1),
\]
since \(\gamma (\alpha_{z,+}(l) - j) = 1\). Thus we can use (2.69) for all \(j \geq c_3\) for some \(c_3 < \infty\). For such \(j\) we therefore have
\[
\mathbb{P} \left[ \sqrt{2T_{u_m, \tilde{r}_l}} \geq \alpha_{z,+}(l) - \frac{j}{2} \right] \leq ce^{-\frac{1}{6}(\frac{q}{4} + \zeta + O(\gamma (\alpha_{z,+}(l) - \frac{q}{4})))^2},
\]
and since \(\gamma (\alpha_{z,+}(l) - j) = 1\) and \(j \leq \frac{1}{2}\alpha_{z,+}(l)\) so that \(j \leq (\alpha_{z,+}(l) - j)\), it follows that
\[
\gamma \left( \alpha_{z,+}(l) - \frac{j}{2} \right) = \gamma (\alpha_{z,+}(l) - j) + \gamma \frac{j}{2} \leq 2,
\]
so that we obtain (2.68) for all \(j \geq c_3\). By enlarging \(c'\) we then have (2.68) for all \(j\).

We now bound
\[
\mathbb{P} \left[ B_{m,l}^{\gamma,k} \cap D_{m,l}(-j) \right] \leq \mathbb{P} \left[ \sqrt{2T_{u_m, \tilde{r}_l}} \geq \alpha_{z,+}(l) - \frac{j}{2} \right] \mathbb{P} \left[ \tilde{C}_{m,l} \cap D_{m,l}(-j) \right]
\]
\[
+ \mathbb{P} \left[ B_{m,l}^{\gamma,k} \cap \left\{ \sqrt{2T_{u_m, \tilde{r}_l}} < \alpha_{z,+}(l) - \frac{j}{2} \right\} \right] \mathbb{P} \left[ \tilde{C}_{m,l} \cap D_{m,l}(-j) \right].
\]

Because of the bound (2.66), to prove (2.65) it suffices to show that
\[
\mathbb{P} \left[ B_{m,l}^{\gamma,k} \cap \left\{ \sqrt{2T_{u_m, \tilde{r}_l}} < \alpha_{z,+}(l) - \frac{j}{2} \right\} \right] \mathbb{P} \left[ \tilde{C}_{m,l} \cap D_{m,l}(-j) \right] \leq Ce^{-Aj}.
\]

Assign to each \(y \in F_{m+1}^{m} \cap B_d(u_m, \tilde{c}h_t)\) a unique “parent” \(\tilde{y} \in F_{m+1}^{m} \cap B_d(u_m, \tilde{c}h_t)\) such that \(d(\tilde{y}, \tilde{y}) \leq r_{l+i}\). In particular, for \(i = 1\) we set \(\tilde{y} = u_m\). Let \(q = q(\tilde{y}, y) = d(\tilde{y}, y)/r_l\) and set
\[
A_i = \left\{ \sup_{y \in F_{m+1}^{m} \cap B_d(u_m, \tilde{c}h_t)} \left| T_{\tilde{y}, \tilde{r}_l} - T_{\tilde{y}, \tilde{r}_l} \right| \leq d_0 j \left( \alpha_{z,+}(l) - j \right) \sqrt{q} \right\},
\]
where \( d_0 \) will be chosen later, but small enough that \( d_0 \sum_{i \geq 1} i e^{-i/2} \leq \frac{1}{8} \).

We now show that

\[
\bigcap_{i=1}^{k-l} A_i \cap \left\{ \sqrt{2 \gamma_{ym,\tilde{r_f}}} < \alpha_{+,z}(l) - \frac{j}{2} \right\} \subseteq \left\{ \sqrt{2 \gamma_{y,\tilde{r_f}}} < \alpha_{+,z}(l), \forall y \in F_k^m \cap B_d(u_m, \tilde{c}h_l) \right\}.
\]

For this, we use \( q = d(\tilde{y}, y)/r_f \leq r_{l+i}/r_f = e^{-i} \) for \( y \in F_{l+i}^m \) to see that for any trajectory in the left hand side of (2.75) and all \( y \in F_k^m \cap B_d(u_m, \tilde{c}h_l) \)

\[
T_{\gamma,\tilde{r_f}} \leq \left( \alpha_{+,z}(l) - \frac{j}{2} \right)/2 + j (\alpha_{+,z}(l) - j) \frac{d_0}{i \geq 1} i e^{-i/2},
\]

which, since \( d_0 \sum_{i \geq 1} i e^{-i/2} \leq \frac{1}{8} \), implies that

\[
T_{\gamma,\tilde{r_f}} \leq \left( \alpha_{+,z}(l) - \frac{j}{2} \right)/2 + \frac{1}{8} j (\alpha_{+,z}(l) - j) < \frac{1}{8} j (\alpha_{+,z}(l))/2.
\]

This establishes (2.75) and taking complements we see that

\[
B_{\gamma,k}^{x,\tilde{r_f}} = \left\{ \exists y \in F_k^m \cap B_d(u_m, \tilde{c}h_l) \text{ such that } \sqrt{2 \gamma_{y,\tilde{r_f}}} \geq \alpha_{+,z}(l) \right\} \subseteq \bigcup_{i=1}^{k-l} A_i \cup \left\{ \sqrt{2 \gamma_{ym,\tilde{r_f}}} \geq \alpha_{+,z}(l) - \frac{j}{2} \right\}.
\]

It follows that

\[
B_{\gamma,k}^{x,\tilde{r_f}} \cap \left\{ \sqrt{2 \gamma_{ym,\tilde{r_f}}} \geq \alpha_{+,z}(l) - \frac{j}{2} \right\} \subseteq \bigcup_{i=1}^{k-l} A_i.
\]

We can thus bound

\[
\mathbb{P} \left[ B_{\gamma,k}^{x,\tilde{r_f}} \cap \left\{ \sqrt{2 \gamma_{ym,\tilde{r_f}}} < \alpha_{+,z}(l) - \frac{j}{2} \right\} \bigg| \tilde{c}_{m,l} \cap D_{m,l}(-j) \right]
\]

by

\[
\sum_{i=1}^{k-l} \mathbb{P} \left[ \sup_{y \in F_{l+i}^m \cap B_d(u_m, \tilde{c}h_l)} \left| T_{y,\tilde{r_f}} - T_{\tilde{y},\tilde{r_f}} \right| \geq d_0 ji (\alpha_{+,z}(l) - j) \sqrt{q} \bigg| \tilde{c}_{m,l} \cap D_{m,l}(-j) \right].
\]

Since \( |F_{l+i}^m \cap B_d(u_m, \tilde{c}h_l)| \leq ce^{2i} \), a union bound gives that (2.78) is at most

\[
\sum_{i=1}^{k-l} ce^{2i} \sup_{y \in F_{l+i}^m \cap B_d(u_m, \tilde{c}h_l)} \mathbb{P} \left[ \left| T_{y,\tilde{r_f}} - T_{\tilde{y},\tilde{r_f}} \right| \geq d_0 ji (\alpha_{+,z}(l) - j) \sqrt{q} \bigg| \tilde{c}_{m,l} \cap D_{m,l}(-j) \right].
\]

(2.79)
We can write the last probability as

$$P\left[ T_{\tilde{y},\tilde{r}}^{m,\tau -2} - T_{\tilde{y},\tilde{r}}^{m,\tau -2} \geq d_0ji (\alpha_{+,-}(l) - j) \sqrt{q} \mid \tilde{C}_{m,l} \cap D_{m,l}(-j) \right].$$

Using [4, Lemma 5.6] with \( \theta = d_0ji \) and \( n = (\alpha_{+,-}(l) - j)^2/2 \), we find that for an appropriate choice of \( d_0, \tilde{c} \), the last probability is bounded by \( ce^{-d(j+l)} \).

To apply [4, Lemma 5.6] we must verify several points.

First, we need to verify that for some small \( \tilde{c}_0 \) we have \( \theta \leq \tilde{c}_0(n-1) \), that is \( d_0ji \leq \tilde{c}_0((\alpha_{+,-}(l) - j)^2 \). For this it suffices to note that for \( j, l \) in our range \( i/(\alpha_{+,-}(l) - j) \leq (k - l)/(\alpha_{+,-}(l) - j) = (\log 2(\alpha_{+,-}(l) - j))/(\alpha_{+,-}(l) - j) \) goes to 0 as \( L \to \infty \).

Secondly, we need to show that \( \theta \leq ((n-1)q)^2 \). Since we have already seen that \( \theta \leq \tilde{c}_0(n-1) \), it suffices to show that \( (n-1)q^2 \geq c_2^2 \) for some \( c_2 > 0 \), or equivalently that \( \sqrt{2nq} \geq \tilde{c}_2 \). That is, \( (\alpha_{+,-}(l) - j) / r_l \geq \tilde{c}_2 \). Assume that \( d(\tilde{y}, y) / r_l \geq c_3r_k \) for a small \( c_3 > 0 \), then that, see (2.64),

\[ (\alpha_{+,-}(l) - j) / r_l \geq c_3(\alpha_{+,-}(l) - j)e^{-(k-l)} = c_3/2. \]

With the \( F_l \) constructed appropriately we can indeed assume that either \( d(\tilde{y}, y) / r_l \geq c_3r_k \) for a small \( c_3 > 0 \), or that \( y = \tilde{y} \), in which case the corresponding term in the sum in (2.79) is zero. Also, by taking \( \tilde{c} = q_0/2 \) we will have \( d(\tilde{y}, y) / r_l \leq q_0 \).

Thus we see that (2.79) is at most

$$c \sum_{i=1}^{k-l} e^{2i} e^{-d(j+l)} \leq Ce^{-d(j+l)}.$$

This completes the proof of (2.73).

**Proof of (2.54):** As in (2.62)

$$T_{u_m,\tilde{r}}^T \geq T_{\tilde{y},\tilde{r}}^T$$

for all \( y \) such that \( d(y, u_m) \leq \frac{\gamma r_j}{2} \), where we take \( \gamma \) to be some fixed small number. Hence under \( B_{m,l} \) we have \( \sqrt{2T_{u_m,\tilde{r}}^T} \geq \alpha_{z,+}(l) \). The fact that for all \( l \geq L - (4\log L)^4 \)

\[ \Pr \left[ \sqrt{2T_{u_m,\tilde{r}}^T} \geq \frac{1}{2} \alpha_{z,+}(l - 2), \sqrt{2T_{u_m,\tilde{r}}^T} \geq \alpha_{z,+}(l) \right] \leq ce^{-c' L^2} \]

then follows easily as in the proof of (2.68). (In fact, the proof uses the same ideas but is much easier). 

**3. LOWER BOUNDS FOR EXCURSIONS**

In this section we will prove the following.
Lemma 3.1. There exist $0 < c_1, c_2 < \infty$ such that for all $L$ large and all $0 \leq z \leq \log L$,

$$
P \left[ \sup_{y \in F_L} \sqrt{2T_{y,L}} \geq \rho_L L + z \right] \geq c_1 \frac{(1 + z)e^{-2z}}{(1 + z)e^{-2z} + c_2}. $$

This will immediately give the lower bounds in Theorems 1.6 and 1.5 and hence complete the proofs of those Theorems.

Note that for any $z_0$ it suffices to show that (3.1) holds for all $0 \leq z \leq \log L$, since by adjusting $c_1$ we then get (3.1) for all $0 \leq z \leq \log L$.

Let

$$
\beta_z (l) = \rho_L l + z,
$$

and

$$
\alpha_z (l) = \alpha_z (l, L, z) = \rho_L l + z - l^{4/4} L.
$$

For each $k \geq 1$ we define $T_{y,l}^{k,m}$ to be the number of excursions from $\partial B_d (y, h_{l-1})$ to $\partial B_d (y, h_l)$ during the first $m$ excursions from $\partial B_d (y, h_k)$ to $\partial B_d (y, h_{k-1})$. We abbreviate $T_{y,l}^{1,x^2} = T_{y,l}^{1,x^2}$ with $x$ fixed.

Choose $r_0$ in (2.2) sufficiently small that $4h(r_{-1}) \leq r^*$. Let $\tilde{r} = h_{1/20}$, and with $F^0 := B_d (v, \tilde{r})$ we set

$$
F_0^0 = F^0 \cap F_L, \quad \text{so that} \quad |F_0^0| \asymp e^{2L}.
$$

That is, $|F_0^0|$ does not depend on $r_0$.

In this section we show that

Lemma 3.2. There exists a $0 < c < \infty$ such that for all $0 < r_0$ sufficiently small, $L$ large and all $0 \leq z \leq \log L$,

$$
P \left[ \sup_{y \in F_L} \sqrt{2T_{y,L}} \geq \rho_L L + z \right] \geq (1 + z)e^{-2z} \frac{(1 + z)e^{-2z}}{(1 + z)e^{-2z} + c}. $$

Since the probability of $x^2$ excursions from $\partial B_d (v, h_{l-1} - \tilde{r})$ to $\partial B_d (v, h_{l} - \tilde{r})$ before $\tau$ is greater than 0 and does not depend on $L$, (3.5) will imply Lemma 3.1. We note that the $r_0$ used in this Lemma, and hence all $h_l$, are smaller than the corresponding quantities used until now. This is for notational convenience and, as can easily be seen, does not affect Lemma 3.1 which concerns large $L$. We could have kept the original $r_0$ and in place of $h_l$ used $h_{l+k}$ for some fixed $k$, but this would have made the notation cumbersome.

The proof of Proposition 3.2 uses a modified second moment method and occupies the rest of this section.

We introduce the events $I_{y,z}$, beginning with a barrier event. Let

$$
I_{y,z} = \left\{ \sqrt{2T_{y,L}^1} \leq \alpha_{z,-} (l) \right\} \text{ for } l = 1, \ldots , L - 1 \text{ and } \sqrt{2T_{y,L}^1} \geq \rho_L L + z \right\},
$$

for $y \in F_L$. As discussed in [4], we need to augment $I_{y,z}$ by information on the angular increments of the excursions. Instead of keeping track of
individual excursions, we track the empirical measure of the increments, by comparing it in Wasserstein distance to a reference measure. Recall that the Wasserstein $L^1$-distance between probability measures on $\mathbb{R}$ is given by

$$d_{1,W}^1(\mu, \nu) = \inf_{\xi \in \mathcal{P}^2(\mu, \nu)} \left\{ \int |x - y| \, d\xi(x, y) \right\},$$

where $\mathcal{P}^2(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R} \times \mathbb{R}$ with marginals $\mu, \nu$. If $\mu$ is a probability measure on $\mathbb{R}$ with finite support and if $\theta_i, 1 \leq i \leq n$ denotes a sequence of i.i.d $\mu$-distributed random variables then it follows from [13, Theorem 2] that for some $c_0 = c_0(\mu)$

$$\text{Prob} \left\{ d_{1,W}^1 \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{\theta_i, \mu} \right) > c_0 x \frac{\log(L - k)}{\sqrt{n}} \right\} \leq 2e^{-x^2}.$$

Let $W_t$ be Brownian motion in the plane. For each $k$ let $\nu_k$ be the probability measure on $[0, 2\pi]$ defined by

$$\nu_k(dx) = P^{(r_k, 0)}(\text{arg} W_{H \partial B(0, r_k - 1)} \in dx),$$

where $\text{arg} x$ for $x \in \mathbb{R}^2$ is the argument of $x$ measured from the positive $x$-axis and $P^w$ is the law of $W$ started from $w$.

Returning to $X_t$, our Brownian motion on the sphere, and using isothermal coordinates, see [4, Section 2], let $0 \leq \theta_{k,i} \leq 2\pi, i = 1, 2, \ldots$ be the angular increments centered at $y$, mod $2\pi$, from $X_{H \partial B(y, h_k)}$ to $X_{H \partial B(y, h_k - 1)}$, the endpoints of the $i$'th excursion between $\partial B(y, h_k)$ and $\partial B(y, h_k-1)$. By the Markov property the $\theta_{k,i}, i = 1, 2, \ldots$ are independent, and using [4, Section 2] we see that each $\theta_{k,i}$ has distribution $\nu_k$. We set, for $n$ a positive integer,

$$W_{y,k}(n) = \left\{ d_{1,W}^1 \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{\theta_{k,i}, \nu_k} \right) \leq c_0 \frac{\log(L - k)}{2\sqrt{n}} \right\}.$$

We are ready to define the good events $I_{y,z}$. For $a \in \mathbb{Z}_+$ let

$$N_{k,a} = [(\rho_L k + z - a + 1)^2 / 2].$$

We set

$$N_k = N_{k,a} \text{ if } \sqrt{2T_{y,k}^{1}} \in I_{\rho_k k + z - a},$$

where $I_s = [s, s + 1]$. With $L_k = L - (500 \log L)^4$ and $d^*$ a constant to be determined below, let

$$I_{y,z} = \tilde{I}_{y,z} \cap \bigcap_{k=L_k}^{L- d^*} W_{y,k}(N_k),$$

and define the count

$$J_z = \sum_{y \in I_L^0} 1_{I_{y,z}}.$$
To obtain (3.5), we need a control on the first and second moments of $J_z$, which is provided by the next two lemmas. In fact, (3.5) will follow directly from these two Lemmas as in the proof of [4, Proposition 4.2], taking into account that $|F_0^L|$ does not depend on $r_0$. Most of this section is devoted to their proof. We emphasize that in the statements of the lemma, the implied constants are uniform in $r_0$ smaller than a fixed small threshold.

**Lemma 3.3** (First moment estimate). There is a large enough $d^*$, such that for all $L$ sufficiently large, all $0 \leq z \leq \log L$, and all $y \in F_0^L$,

(3.15) \[ P(I_{y,z}) \propto (1 + z)e^{-2L}e^{-2z}. \]

Let

(3.16) \[ G_0 = \{(y,y') : y, y' \in F_L \text{ s.t. } d(y,y') > 2h_0\}, \]

\[ G_k = \{(y,y') : y, y' \in F_L \text{ s.t. } 2h_k < d(y,y') \leq 2h_{k-1}\} \text{ for } 1 \leq k < L, \]

\[ G_L = \{(y,y') : y, y' \in F_L \text{ s.t. } 0 < d(y,y') \leq 2h_{L-1}\}. \]

**Lemma 3.4** (Second moment estimate). There are large enough $d^*, c'$, such that for all $L$ sufficiently large, all $0 \leq z \leq \log L$ and all $(y,y') \in G_k$, $1 \leq k \leq L$,

(3.17) \[ P(I_{y,z} \cap I_{y',z}) \leq c'(1 + z)e^{-4L}e^{-2z}e^{-ck/L^2}. \]

Before turning to the proofs, we introduce some notation and record some simple estimates that will be useful in calculations. Recall (3.2), (3.11)-(3.12) and for $a \in \mathbb{Z}_+$ let

(3.18) \[ \mathcal{H}_{k,a} = \{\sqrt{2T_{y,k}^L} \in I_{\rho L, k+z-a}\} = \{\sqrt{2T_{y,k}^L} \in I_{\beta z(k)-a}\}. \]

Note that on $\mathcal{H}_{k,a}$ we have $N_k = N_{k,a}$.

Before proceeding we need to state a deviation inequality of Gaussian type for the Galton-Watson process $T_l, l \geq 0$ under $P_{GW}^n$, the law of a critical Galton-Watson process with geometric offspring distribution with initial offspring $n$. The proof is very similar to [3, Lemma 4.6], and is therefore omitted.

**Lemma 3.5.** For all $n = 1, 2, 3, \ldots$,

(3.19) \[ P_{GW}^n \left( \left| \sqrt{2T_l} - \sqrt{2T_0} \right| \geq \theta \right) \leq ce^{-\theta^2/2}, \quad \theta \geq 0. \]

In the proof of our moment estimates we will need the following.

**Lemma 3.6.** For any $a, b \leq L/\log L$

(3.20) \[ P \left[ \sqrt{2T_{y,L}^k(\beta z(k)-a)^2} \geq \rho L + z - b \right] \leq ce^{-2(L-k-2(a-b)-\frac{(a-b)^2}{L-\rho L})/L^2}. \]
**Proof:** By (3.19) we have that for all \( \theta > n \geq 1 \)

\[
\mathbb{P} \left[ T_{y,k}^{n/2} \geq \theta^2/2 \right] \leq c \exp \left( -\frac{(\theta - n)^2}{2(L-k)} \right).
\]

We apply this with \( \theta = \rho L z - a = \rho L k + z - a \)

so that

\[
\theta - n = \rho L (L - k) + a - b,
\]

and hence

\[
\frac{(\theta - n)^2}{2(L-k)} \geq 2(L-k) - 2\frac{(L-k)}{L}(\log L) + 2(a-b) + \frac{(a-b)^2}{2(L-k)} + O_L(1).
\]

This gives (3.20). \( \square \)

3.1. **First moment estimate.** In this subsection we prove Lemma 3.3.

For the lower bound we have that

\[
\mathbb{P} \left[ \mathcal{I}_{y,z} \right] \geq \mathbb{P} \left[ \hat{\mathcal{I}}_{y,z} \right] - \sum_{k=L+}^{L-d^*} \mathbb{P} \left[ \hat{\mathcal{I}}_{y,z} \cap W_{y,k}^c (N_k) \right]
\]

\[
\geq c(1 + z)e^{-2L}e^{-2z} - \sum_{k=L+}^{L-d^*} \mathbb{P} \left[ \hat{\mathcal{I}}_{y,z} \cap W_{y,k}^c (N_k) \right],
\]

where for \( \mathbb{P} \left[ \hat{\mathcal{I}}_{y,z} \right] \) we have used the barrier estimate (9.12) of Appendix I.

We note that

\[
\mathbb{P} \left[ \hat{\mathcal{I}}_{y,z} \cap W_{y,k}^c (N_k) \right] \leq \sum_{a \geq k^{1/4}} \mathbb{P} \left( \hat{\mathcal{I}}_{y,z}^{k,a} \right),
\]

where

\[
\hat{\mathcal{I}}_{y,z}^{k,a} = \hat{\mathcal{I}}_{y,z} \cap \mathcal{H}_{k,a} \cap W_{y,k}^c (N_{k,a}).
\]

We show below that for all \( L_+ \leq k \leq L - d^* \) and \( 0 \leq z \leq \log L \),

\[
\sum_{a \geq k^{1/4}} \mathbb{P} \left( \hat{\mathcal{I}}_{y,z}^{k,a} \right) \leq c' (1 + z) \left( e^{-2L} e^{-2z} \right) e^{-c \log^2(L-k)},
\]

which will finish the proof of the lower bound for (3.15) for \( d^* \) sufficiently large.

Furthermore, it is easily seen using (3.20) and the fact that \( L - k \leq (500 \log L)^4 \) that the sum in (3.25) over \( a \geq L^{3/4} \) is much smaller than the right hand side of (3.25), hence it suffices to show that

\[
\sum_{a \geq k^{1/4}} \mathbb{P} \left( \hat{\mathcal{I}}_{y,z}^{k,a} \right) \leq c' (1 + z) \left( e^{-2L} e^{-2z} \right) e^{-c \log^2(L-k)},
\]

(3.26)
We now turn to the proof of (3.26). Let
\[
J_{y,k}^\uparrow = \left\{ \sqrt{2T_{y,l}^1} \leq \rho_L l + z \text{ for } l = 1, \ldots, k \right\},
\]
and
\[
\mathcal{B}_{y,k,a} = \left\{ \sqrt{2T_{y,L}^{k,(\beta_z(k)-a)^2/2}} \geq \rho_L L + z \right\}.
\]
Then with
\[
K_{k,p,a} = J_{y,k-3}^\uparrow \cap \mathcal{H}_{k-3,p} \cap \mathcal{H}_{k,a} \cap W_{y,k}^{c} (N_{k,a}) \cap \mathcal{B}_{y,k,a}
\]
we have
\[
\mathbb{P} \left( \hat{I}_{y,z}^{k,a} \right) \leq \sum_{p \geq (k-3)^{1/4}} \mathbb{P} (K_{k,p,a}),
\]
plus a term which is much smaller than the right hand side of (3.25).

Let
\[
W_{y,k}(n) = \left\{ \epsilon_{\text{ws}}^1 \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{y,k,i}, \nu_k \right) \in \frac{c_0}{\sqrt{n}} I_{x} \right\},
\]
so that
\[
W_{y,k}^c (N_{k,a}) \subseteq \bigcup_{m=\log(L-k)}^{\infty} W_{y,k}^{e_m} (N_{k,a}),
\]
and consequently, setting
\[
\mathcal{L}_{k,m,p,a} = K_{k,p,a} \cap W_{y,k}^{e_m} (N_{k,a}),
\]
we have
\[
\mathbb{P} (K_{k,p,a}) \leq \sum_{m=\log(L-k)}^{\infty} \mathbb{P} (\mathcal{L}_{k,m,p,a}).
\]

Let
\[
\mathcal{L}_{k,m,p,a}^\prime = J_{y,k-3}^\uparrow \cap \mathcal{H}_{k-3,p} \cap \mathcal{H}_{k,a} \cap W_{y,k}^{e_m} (N_{k,a}).
\]
To prove (3.25) it suffices to prove that for all \( m \geq \log(L-k) \),
\[
\sum_{a \geq k_3^{1/4}} \sum_{p \geq (k-3)^{1/4}} \mathbb{P} (\mathcal{B}_{y,k,a} \cap \mathcal{L}_{k,m,p,a}^\prime) \leq c' (1 + z) \left( e^{-2L} e^{-2z} \right) e^{-cm^2}.
\]

Lemma 3.7.
\[
\mathbb{P} (\mathcal{L}_{k,m,p,a}) = \mathbb{P} \left( J_{y,k-3}^\uparrow \cap \mathcal{H}_{k-3,p} \cap \mathcal{H}_{k,a} \cap W_{y,k}^{e_m} (N_{k,a}) \right) \leq C (1 + z) (1 + p) e^{-2k-2(z-p)} e^{-c(p-a)^2} e^{-m^2}.
\]
Proof: By (3.8)

(3.36) \[ P \left( W_{y,k}^m \left( N_{k,a} \right) \mid \mathcal{H}_{k,a} \right) \leq e^{-m^2}. \]

By (3.20)

(3.37) \[ P \left( \mathcal{H}_{k,a} \mid \mathcal{H}_{k-3,p} \right) \leq ce^{-(p-a)^2}, \]

and by (9.14) we see that

(3.38) \[ P \left( J_{y,k-3} \cup \mathcal{H}_{k-3,p} \right) \leq C(1 + z)(1 + p)e^{-2k-2(z-p)}. \]

It follows as in the proof of [4, Lemma 4.7] that for some \( M_0 < \infty \)

(3.39) \[ P \left( B_{y,k,a} \cap \mathcal{H}_{k,m,p,a} \right) \leq \mathbb{P} \left\{ \sqrt{2T_{y,L}^{k,(\beta_4(k)-a)^2/2}} \geq \rho_L L + z - M_0m \right\} \times P \left( \mathcal{L}_{k,m,p,a}^l \right) + e^{-4L}. \]

We note that by (3.20)

(3.40) \[ \mathbb{P} \left\{ \sqrt{2T_{y,L}^{k,(\beta_4(k)-a)^2/2}} \geq \rho_L L + z - M_0m \right\} \leq ce^{-2(L-k)-2(a-M_0m)-\frac{(a-M_0m)^2}{\pi(L-k)}}, \]

Putting this all together with (3.35), and using \( |a - p| \leq 1 + (p-a)^2 \) we find that

(3.41) \[ P \left( B_{y,k,a} \cap \mathcal{H}_{k,m,p,a} \right) \leq C(1 + z)e^{-2L-2z}e^{-m^2/2}(1 + p)e^{-c(p-a)^2}e^{-\frac{a^2}{\pi(L-k)}}. \]

Summing first over \( p \) and then over \( a \) it is easy to see, using a fraction of the exponent \( m^2/2 \), that (3.34) holds for all \( m \geq \log(L-k) \). This completes the proof of the lower bound in (3.15).

Since \( I_{y,z} \subseteq \hat{I}_{y,z} \) the upper bound in (3.15) follows from the barrier estimate (9.11) of Appendix I.

3.2. Second moment estimate: branching in the bulk. We prove the second moment estimate for \( y, y' \in F_L \) with

\[ 2h_{k-1} < d(y, y') \leq 2h_{k-2}. \]

In this subsection we prove Lemma 3.4 for

(3.42) \[ (500 \log L)^4 < k \leq L - (500 \log L)^4. \]

We need to “give ourselves a bit of space”, and we therefore define

(3.43) \[ k^+ = k + \lceil 100 \log L \rceil. \]

Let

\[ \hat{I}_{y,z; k\pm 3} = \left\{ \sqrt{2T_{y,l}^{1}} \leq \rho_L l + z; l = 1, \ldots, k-4, k+4, \ldots, L-1 \right\} \]
(3.44) \[ \cap \left\{ \sqrt{T_{y,L}^1} \geq \rho L + z \right\}, \]

where we have skipped the barrier condition for \( k - 3, \ldots, k + 3 \). To obtain the two point bound for the range (3.42) we will bound the probability of

(3.45) \[ \tilde{I}_{y,z; k \pm 3} \cap \left\{ \sqrt{2T_{y',L}^{k+, \alpha_2^2, -(k+)/2}} \geq \rho L + z \right\}, \]

which contains the event \( I_{y,z} \cap I_{y',z}. \)

Let \( G_{y'} \) denote the \( \sigma \)-algebra generated by the excursions from \( \partial B_d(y', h_{k-1}) \) to \( \partial B_d(y', h(r_k)) \). Note that \( \tilde{I}_{y,z; k \pm 3} \in G_{y'} \). Since

\[ \left\{ \sqrt{2T_{y',L}^{k+, \alpha_2^2, -(k+)/2}} \geq \rho L + z \right\} \]

is measurable with respect to the first \( \alpha_2^2 \) excursions from \( \partial B_d(y', h_{k-1}) \) to \( \partial B_d(y', h(r_k+1)) \), it follows from the basic ideas in [3, sub-section 6.2] that

(3.46) \[ P \left[ \tilde{I}_{y,z; k \pm 3}, \sqrt{2T_{y',L}^{k+, \alpha_2^2, -(k+)/2}} \geq \rho L + z \right] \]

\[ \leq cP \left[ \tilde{I}_{y,z; k \pm 3} \right] P \left[ \sqrt{2T_{y',L}^{k+, \alpha_2^2, -(k+)/2}} \geq \rho L + z \right]. \]

By Lemma 9.3

(3.47) \[ P \left[ \tilde{I}_{y,z; k \pm 3} \right] \leq c(1 + z)e^{-2L}e^{-2z}. \]

Using (3.20) for the last term in (3.46) together with the fact that in the range (3.42) we have \( (k^+)_{L^{1/4}} \geq 500 \log L \), we find that (3.46) is bounded by

(3.48) \[ c(1 + z)e^{-2L}e^{-2z}e^{-2(L - k)^{1/4} - k^{1/4}} L^{2}, \]

\[ \leq c(1 + z)e^{-2L}L^{202}e^{-2(L - k)^{1/4} - k^{1/4}} e^{-2z}. \]

\[ \leq c(1 + z)e^{-2L}e^{-2(L - k - k^{1/4})} e^{-2z}. \]

3.3. **Second moment estimate: early branching.** In this subsection we prove Lemma [3.4] for

(3.49) \[ 1 \leq k < (500 \log L)^4. \]

Since we no longer have \( k^{1/4}_{L} \geq \log L \) we will have to use barrier estimates to control the factors of \( L \) such as arise in the first line of (3.48). On the other hand, since the number of excursions at lower levels is not so great we don't need such a large separation. Let

(3.50) \[ \tilde{k} = k + \lfloor 100 \log k \rfloor, \quad k_z = k + \lfloor 100 \log z \rfloor. \]
For \( v \in \{y, y'\}\)

\[
J_{v,s,k}^+ = \begin{cases} \sqrt{2 T_{v,l}^2} \leq \rho_L l + z & \text{for } l = \tilde{k} + 1, \ldots, L - 1; \\ \sqrt{2 T_{v,L}^2} \geq \rho_L L + z & \end{cases}
\]

with the barrier condition applied only for \( l \geq \tilde{k}\).

We first consider the case where \( z \leq 100k \). Then

(3.50) \[ P(I_{y,z} \cap I_{y',z}) \leq \alpha z, \]

Let \( G^y \) denote the \( \sigma \)-algebra generated by the excursions from \( \partial B_d(y, h_{k-1}) \) to \( \partial B_d(y, h(r_{\tilde{k}})) \). Note that \( \tilde{I}_{y,z;k;\pm 3} \in G^y \). Since, under our assumption that \( z \leq 100k \), the number of excursions from \( \partial B_d(y, h_{k-1}) \) to \( \partial B_d(y, h(r_{\tilde{k}})) \) is dominated by \( n = O(k^2) \), it follows as before that

(3.51) \[ P(J_{y,n,k}^+ \cap \tilde{I}_{y';z;k;\pm 3}) \leq cP(J_{y,n,k}^+) \]

By the barrier estimate (9.16) of Appendix I, with \( n = \beta_z(\tilde{k}) - t \)

(3.52) \[ P(J_{y,n,k}^+) \leq c t n^{1/2} e^{-2(L-\tilde{k})-2t} \leq c k^{202} e^{-2(L-k)-2k^{1/4}}. \]

where the last step followed from the fact that \( k^{1/4} \leq \tilde{k}_L^{1/4} \leq t \leq \beta_z(\tilde{k}) \leq c k \). Since, under our assumption that \( z \leq 100k \), the number of terms in (3.50) is \( \leq c k^2 \), and using (3.47), we find that (3.50) is bounded by

(3.53) \[ c k^{204} e^{-2(L-k)-2k^{1/4}} (1 + z) e^{-2L} e^{-2z} \]

\[ \leq c(1 + z) e^{-4L + 2k^{1/4}} e^{-2z}. \]

Thus we can assume that

(3.54) \[ z \geq 100k. \]

We have

(3.55) \[ P(I_{y,z} \cap I_{y',z}) \]

\[ = \sum_{n=1}^{\alpha_z - (k_z)} 1_{n=\beta_z(k_z)-t; t \geq z/2} P \left( \left\{ \sqrt{2 T_{y,k_z}^1} = n \right\} \cap I_{y,z} \cap I_{y',z} \right) \]

\[ + \sum_{n=1}^{\alpha_z - (k_z)} 1_{n=\beta_z(k_z)-t; t < z/2} P \left( \left\{ \sqrt{2 T_{y,k_z}^1} = n \right\} \cap I_{y,z} \cap I_{y',z} \right) \]
Since in the above sums \( n \leq cz \) in view of (3.54), we can bound the first sum in (3.55) by

\[
\sum_{n=1}^{\alpha_{z,-}(k_z)} 1_{\{n=\beta_z(k_z) - t; t \geq z/2\}} \mathbb{P} \left( J_{y,n,k_z}^{1} \cap \tilde{I}_{y',z;k_z+3} \right)
\]

\[
\leq c \sum_{n=1}^{\alpha_{z,-}(k_z)} 1_{\{n=\beta_z(k_z) - t; t \geq z/2\}} \mathbb{P} \left( J_{y,n,k_z}^{1} \right) \mathbb{P} \left( \tilde{I}_{y',z;k_z+3} \right)
\]

\[
\leq c \sum_{n=1}^{\alpha_{z,-}(k_z)} 1_{\{n=\beta_z(k_z) - t; t \geq z/2\}} \mathbb{P} \left( J_{y,n,k_z}^{1} \right) (1 + z)e^{-2L}e^{-2z},
\]
as before. Instead of (3.52) we now have

\[
\mathbb{P} \left( J_{y,n,k_z}^{1} \right) \leq ct \frac{1}{2} e^{-2(L-k_z)-2t} \leq cz \frac{202}{2} e^{-2(L-k)-z/2},
\]

where the last inequality used \( t \geq z/2 \). In view of (3.54) and the fact that the number of terms in the sum is \( \leq cz \), this gives the desired bound for the first sum in (3.55).

Note next that if \( t < z/2 \) then we must have \( n = \beta_z(k_z) - t \geq z/2 \), (but we still have \( n \leq cz \) by (3.54)). Thus we can bound the second sum in (3.55) by

\[
\sum_{n,n'=1}^{\alpha_{z,-}(k_z)} 1_{\{n \geq z/2\}} \mathbb{P} \left( \left\{ \sqrt{2T_{y,k_z}^{-1}} = n \right\} \cap J_{y,n,k_z}^{1} \cap J_{y',n',k_z}^{1} \right)
\]

\[
\leq c \sum_{n,n'=1}^{\alpha_{z,-}(k_z)} 1_{\{n \geq z/2\}} \mathbb{P} \left( \left\{ \sqrt{2T_{y,k_z}^{-1}} = n \right\} \cap J_{y,n,k_z}^{1} \right) \mathbb{P} \left( J_{y',n',k_z}^{1} \right).
\]
as before. Then by the Markov property, this is bounded by

\[
\sum_{n,n'=1}^{\alpha_{z,-}(k_z)} 1_{\{n \geq z/2\}} \mathbb{P} \left( \left\{ \sqrt{2T_{y,k_z}^{-1}} = n \right\} \right) \mathbb{P} \left( J_{y,n,k_z}^{1} \right) \mathbb{P} \left( J_{y',n',k_z}^{1} \right).
\]

By (3.19) with \( n \geq z/2 \) and then (3.54)

\[
\mathbb{P} \left( \left\{ \sqrt{2T_{y,k_z}^{-1}} = n \right\} \right) \leq e^{-z^2/4k_z} \leq e^{-10z},
\]

while now, instead of (3.57), we use

\[
\mathbb{P} \left( J_{y,n,k_z}^{1} \right) \leq ct z \frac{1}{2} e^{-2(L-k_z)-2t} \leq cz \frac{202}{2} e^{-2(L-k)},
\]

and a similar bound for \( \mathbb{P} \left( J_{y',n',k_z}^{1} \right) \). Thus (3.59) is bounded by

\[
c \sum_{n,n'=1}^{\alpha_{z,-}(k_z)} e^{-10z} e^{-4(L-k)} \leq ce^{-10z} e^{40k} e^{-4L+2k}.
\]

In view of (3.54), this gives the desired bound for the second sum in (3.55).
3.4. Second moment estimate: late branching. In this subsection we prove Lemma 3.4 for \( L - (500 \log L)^4 \leq k < L - 1 \).

Consider first the case
\[
L - (500 \log L)^4 \leq k < L - d^*.
\]
We will bound the probability of
\[
A = \left\{ \sqrt{2T_{y,L}^{k,\alpha_{z,-(k)/2}}} \geq \rho_L L + z \right\} \cap W_{y,k} (N_k) \cap \tilde{I}_{y',z; k \pm 3},
\]
(which contains the event \( I_{y,z} \cap I_{y',z} \)).

It follows as in the proof of [4, Lemma 4.7] that for some \( M_0 < \infty \)
\[
\mathbb{P}(A) \leq \mathbb{P} \left\{ \sqrt{2T_{y,L}^{k,\alpha_{z,-(k)/2}}} \geq \rho_L L + z - M_0 \log(L - k) \right\} \times \mathbb{P} \left( \tilde{I}_{y',z; k \pm 3} \right) + e^{-4L}.
\]
Using (3.20) and (3.47) this shows that
\[
\mathbb{P}(A) \leq c e^{-2(L-k) + 2M_0 \log(L-k) - 2k^{1/4} \frac{1}{100} \log \left( 1 + \frac{1}{100} \right)} e^{-2L} e^{-2z} + e^{-4L}.
\]
By taking \( d^* \) sufficiently large we will have \( M_0 \log(L - k) \leq k^{1/4}/2 \), which then gives (3.15).

For \( L - d^* \leq k < L - 1 \) we simply bound the term \( \mathbb{P}(I_{y,z} \cap I_{y',z}) \) by \( \mathbb{P}(I_{y,z}) \) and obtain from (3.15) the following upper bound
\[
\mathbb{P}(I_{y,z} \cap I_{y',z}) \leq c(1+z)e^{-2L} e^{-2z} \leq c_d^* (1+z) e^{-(4L-2k) - ck^{1/4} \frac{1}{100} \log \left( 1 + \frac{1}{100} \right)} e^{-2z}.
\]

4. Excursion counts and occupation measure on \( S^2 \).

In this section we prove Theorems 1.3 and 1.4.

For \( 0 \leq \epsilon < a < b < \pi \), let \( \mathcal{M}_{x,\epsilon,a,b}(n) \) be the total occupation measure of \( B_d(x, \epsilon) \) until the end of the first \( n \) excursions from \( \partial B_d(x, a) \) to \( \partial B_d(x, b) \).

With \( \omega_\epsilon = 2\pi(1 - \cos(\epsilon)) \), the area of \( B_d(x, \epsilon) \), let
\[
\overline{M}_{x,\epsilon,a,b}(n) = \frac{1}{\omega_\epsilon} \mathcal{M}_{x,\epsilon,a,b}(n).
\]
In particular, when starting from \( \partial B_d(x, a) \),
\[
\overline{M}_{x,\epsilon,a,b}(1) = \frac{1}{\omega_\epsilon} \int_0^{H_{B_d(x, \epsilon)}} 1_{\{B_d(x, \epsilon)\}}(X_t) \, dt.
\]
The following Lemma is proven in Section 6

**Lemma 4.1.** For some \( c > 0 \), uniformly in \( x \in S^2 \), and \( h_k/100 \leq \epsilon \leq h_k \),
\[
\mathbb{P} \left( \overline{M}_{x,\epsilon,h_k,h_{k-1}}(n) \leq \frac{1}{n}(1 - \delta) n \right) \leq e^{-cn^2}.
\]
and

\[ P \left( \bar{M}_{x, \epsilon, h_k, h_{k-1}}(n) \geq \frac{1}{\pi} (1 + \delta) n \right) \leq e^{-c \delta^2 n} \]  

Recall \( \bar{\mu}_\tau(y, \epsilon) \) from (1.8) and set

\[ t_L(z) = 2L (L - \log L + z) . \]

Lemma 4.2. We can find \( 0 < c, c', z_0 < \infty \) such that for \( L \) large, all \( z_0 \leq z \leq \log L \), and all \( h_L/100 \leq \epsilon_y \leq h_L \)

\[ c' e^{-2z} \leq P \left( \exists y \in F_L \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \leq c e^{-2z} . \]

For the sphere, it suffices to take \( \epsilon_y = \epsilon \) independent of \( y \). The present formulation is needed for the plane, as we will see in Section 8. To clarify the connection with (1.9) we note that for some \( 0 < c_\ast = c_\ast(r_0) < \infty \),

\[ (m_{h_L} + z)^2 = \frac{1}{\pi} t_L \left( \sqrt{2\pi z} + c_\ast + o_L(1) \right) . \]

4.1. The upper bound. We first show that, with \( F_L = F_L \cap B(d(v, h \log L)) \),

\[ P_1 =: P \left( \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \leq c e^{-2z} . \]

If

\[ \hat{A}_{L,z} = \left\{ \sup_{y \in F_L^+} \sqrt{2T_{y,L}^\tau} \geq \rho_L L + z \right\} , \]

then by (2.11)

\[ P_1 \leq P \left( \hat{A}_{L,z} \right) + P \left( \hat{A}_{L,z}^c, \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \leq c e^{-2z} + P \left( \hat{A}_{L,z}^c, \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) . \]

Recalling the notation \( F_L^m = F_L \cap B_d(v, h_m) \cap B_d(v, h_{m-1}) \), we then bound

\[ P \left( \hat{A}_{L,z}^c, \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \]

\[ \leq \sum_{m=\log L}^{L} c e^{2(L-m)} \sup_{y \in F_L^m} P \left( \sqrt{2T_{y,L}^\tau} \leq \rho_L L + z, \bar{\mu}_\tau(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) . \]
We can write

\begin{align}
(4.10) \quad P \left( \sqrt{2T_{y,L}^r} \leq \rho_L z + \mu_r (y, \epsilon_y) \geq \frac{1}{\pi} t_L (z) \right) \\
= \sum_{j=1}^{z+ML^{1/2}} P \left( \sqrt{2T_{y,L}^r} \in I_{\rho_L, L} + z - j \text{ and } \mu_r (y, \epsilon_y) \geq \frac{1}{\pi} t_L (z) \right) \\
+ P \left( \sqrt{2T_{y,L}^r} \leq \rho_L L - ML^{1/2} \text{ and } \mu_r (y, \epsilon_y) \geq \frac{1}{\pi} t_L (z) \right).
\end{align}

**Lemma 4.3.** For all \( y \in F^m_L \), \( \log L \leq m \leq L \) and \( j \leq z + ML^{1/2} \)

\begin{align}
(4.11) \quad P \left( \sqrt{2T_{y,L}^r} \in I_{\rho_L, L} + z - j \text{ and } \mu_r (y, \epsilon_y) \geq \frac{1}{\pi} t_L (z) \right) \\
\leq c m e^{-2L} L e^{-2(z-j)} e^{-c'j^2},
\end{align}

and

\begin{align}
(4.12) \quad P \left( \sqrt{2T_{y,L}^r} \leq \rho_L L - ML^{1/2} \text{ and } \mu_r (y, \epsilon_y) \geq \frac{1}{\pi} t_L (z) \right) \\
\leq c m e^{-2L} L e^{-2(z-j)} e^{-4L}.
\end{align}

**Proof of Lemma 4.3.** By (2.9)

\begin{align}
(4.13) \quad \left( \rho_L L + z - j \right)^2 / 2 \leq t_L \left( z - j + 2M^2 \right)
\end{align}

for all \( j \leq z + ML^{1/2} \). Hence for such \( j \)

\begin{align}
P \left( \sqrt{2T_{y,L}^r} \in I_{\rho_L, L} + z - j \text{ and } \mu_r (y, \epsilon_y) \geq \frac{1}{\pi} t_L (z) \right) \\
\leq P \left( \sqrt{2T_{y,L}^r} \in I_{\rho_L, L} + z - j, \overline{\mathcal{M}}_{y, \epsilon_y, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \geq \frac{1}{\pi} t_L (z) \right),
\end{align}

Using the Markov property and then (2.6), we have for \( y \in F^m_L \) this is

\begin{align}
= P \left( \sqrt{2T_{y,L}^r} \in I_{\rho_L, L} + z - j \right) P \left( \overline{\mathcal{M}}_{y, \epsilon_y, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \geq \frac{1}{\pi} t_L (z) \right) \\
\leq c m e^{-2L} L e^{-2(z-j)} P \left( \overline{\mathcal{M}}_{y, \epsilon_y, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \geq \frac{1}{\pi} t_L (z) \right).
\end{align}

Consider first the case of \( 4M^2 \leq j \). We now apply (4.1) with

\begin{align}
n = t_L \left( z - j + 2M^2 \right) = t_L \left( z - 2(j - 2M^2) L \right) \sim L^2
\end{align}

and

\begin{align}
\delta = 2(j - 2M^2) L / t_L \left( z - j + 2M^2 \right) << 1
\end{align}
for $4M^2 \leq j \leq z + ML^{1/2}$ to see that

\begin{equation}
\mathbb{P} \left( \overline{M}_{y, \epsilon_y, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \geq \frac{1}{\pi} t_L \left( z \right) \right)
\end{equation}

\begin{align*}
= & \mathbb{P} \left( \overline{M}_{y, \epsilon_y, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \right.
\geq & \frac{1}{\pi} \left( t_L \left( z - j + 2M^2 \right) + 2(j - 2M^2)L \right)
= & \mathbb{P} \left( \overline{M}_{y, \epsilon_y, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \right.
\geq & \frac{1}{\pi} \left( 1 + \frac{2(j - 2M^2) t_L}{t_L \left( z - j + 2M^2 \right)} \right) t_L \left( z - j + 2M^2 \right)
\leq & e^{-\frac{(j - 2M^2)L}{L}} \leq e^{-c'j^2}.
\end{align*}

Similarly, for (4.12) we use

\begin{equation}
\mathbb{P} \left( \sqrt{2T_{y,L}} \leq \rho_L L - ML^{1/2} \text{ and } \bar{\mu}_\tau \left( y, \epsilon_y \right) \geq \frac{1}{\pi} t_L \left( z \right) \right)
\end{equation}

\begin{align*}
\leq & \mathbb{P} \left( \overline{M}_{y, \epsilon_y, h_L, h_{L-1}} \left( t_L \left( -ML^{1/2} + 2M^2 \right) \right) \geq \frac{1}{\pi} t_L \left( z \right) \right) \leq e^{-4L}
\end{align*}

by (4.14) for $M$ sufficiently large.

For $j < 4M^2$ we simply bound the probability in (4.14) by 1 which we can bound by $Ce^{-c'j^2}$ for $C$ sufficiently large.

Then using (4.9) and Lemma 4.3 we see that

\begin{equation}
\mathbb{P} \left( \hat{A}_{L,z}, \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau \left( y, \epsilon_y \right) \geq \frac{1}{\pi} t_L \left( z \right) \right)
\end{equation}

\begin{align*}
\leq & C \sum_{m = \log L}^{L} \text{cmLe}^{2(L-m)} \sum_{j=1}^{z+ML^{1/2}} e^{-2Le^{-2(z-j)}e^{-c'j^2}}
+ & \sum_{m = \log L}^{L} \text{cmLe}^{2(L-m)}e^{-4L}.
\end{align*}

This is easily seen to be bounded by the right hand side of (4.8).

Recalling the notation $F_L^+ = F_L \cap B_0^2 (v, h_{\log L})$ from (2.17), to complete the proof of the upper bound it remains to show that

\begin{equation}
\mathcal{P}_2 =: \mathbb{P} \left( \exists y \in F_L^+ \text{ s.t. } \bar{\mu}_\tau \left( y, \epsilon_y \right) \geq \frac{1}{\pi} t_L \left( z \right) \right) \leq cze^{-2z}.
\end{equation}

Note that with $k_y$ as in (2.16), if

\begin{equation}
\mathcal{A}_{L,z} = \left\{ \exists y \in F_L^+, l \in \{k_y + 1, \ldots, L\} \text{ s.t. } T_{y,l} \geq \alpha_{z,+}^2 \left( l / 2 \right) \right\},
\end{equation}
then

\[ P_2 \leq P(A_{L,z}) + P \left( A_{L,z}^c, \exists y \in F_L^c \text{ s.t. } \mu_L(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \]

\[ \leq cze^{-c^*z} + P \left( A_{L,z}^c, \exists y \in F_L^c \text{ s.t. } \mu_L(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right), \]

by (2.18). Recalling again the notation \( F_L^m = F_L \cap B^c_d(v, h_m) \cap B_d(v, h_{m-1}) \), we have that

\[ (4.19) \]

\[ \sum_{m=1}^{\log L} P \left( A_{L,z}^c, \exists y \in F_{mL} \text{ s.t. } \mu_L(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \]

\[ = \sum_{m=1}^{\log L} P \left( A_{L,z}^c, \exists y \in F_{mL}^c \text{ s.t. } \mu_L(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \]

Since

\[ (4.20) \]

\[ A_{L,z}^c = \left\{ \sup_{y \in F_{mL}^c} T_{y,L}^+ \leq \alpha_{z+}^2(l)/2, k_y + 1 \leq l \leq L \right\} \]

and \( k_y = m \) for \( y \in F_{mL}^c \), we see that

\[ (4.21) \]

\[ P \left( A_{L,z}^c, \exists y \in F_{mL}^c \text{ s.t. } \mu_L(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \]

\[ \leq c e^{2(L-m)} \]

\[ \sup_{y \in F_{mL}^c} P \left( T_{y,L}^+ \leq \alpha_{z+}^2(l)/2, m + 1 \leq l \leq L \text{ and } \mu_L(y, h_L) \geq \frac{1}{\pi} t_L(z) \right). \]

With

\[ (4.22) \]

\[ B_{L,m,z}^y = \left\{ T_{y,L}^+ \leq \alpha_{z+}^2(l)/2, m + 1 \leq l \leq L - 1 \right\} \]

we have for \( y \in F_{mL}^c \),

\[ (4.23) \]

\[ P \left( T_{y,L}^+ \leq \alpha_{z+}^2(l)/2, m + 1 \leq l \leq L \text{ and } \mu_L(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \]

\[ = \sum_{j=1}^{z+ML^{1/2}} P \left( B_{L,m,z}^y, \sqrt{2T_{y,L}^+} \in I_{\alpha_{z+}^2(l)-j} \text{ and } \mu_L(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \]

\[ + P \left( B_{L,m,z}^y, \sqrt{2T_{y,L}^+} \leq \alpha_{z+}^2(L) - z - ML^{1/2} \text{ and } \mu_L(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right). \]

Here, \( M \geq 1 \) is a fixed constant to be chosen shortly.

Recalling, see (2.15), that \( \alpha_{z+}^2(L) = \rho_LL + z \), and using (4.13) we see that

\[ \left( \alpha_{z+}^2(L) - j \right)^2 / 2 \leq t_L(z - j + 2M^2) \]
for all \( j \leq z + ML^{1/2} \). It follows that for such \( j \)

\[
P\left( B^y_{L,m,z}, \sqrt{2T_{y,L}^*} \in I_{\alpha_{z+}(L) - j} \text{ and } \bar{\mu}_F(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right)
\]

\[
\leq P\left( B^y_{L,m,z}, \sqrt{2T_{y,L}^*} \in I_{\alpha_{z+}(L) - j}, \bar{\mathcal{M}}_{y,\epsilon_y,s_L,s_{L-1}} \left( t_L(z) - j + 2M^2 \right) \geq \frac{1}{\pi} t_L(z) \right)
\]

\[
= P\left( \sqrt{2T_{y,l}} \leq \alpha_{z+}(l), m + 1 \leq l \leq L - 1, \sqrt{2T_{y,L}^*} \in I_{\alpha_{z+}(L) - j} \right)
\]

\[
\times P\left( \bar{\mathcal{M}}_{y,\epsilon_y,s_L,s_{L-1}} \left( t_L(z) - j + 2M^2 \right) \geq \frac{1}{\pi} t_L(z) \right),
\]

by the Markov property. Using the barrier estimate (9.5) of Appendix I, and recalling that \( m = k_y < \log L \), this is bounded by

\[
(4.24) \quad ce^{-2L}e^{-2(z-j)}m^2j(z+m)P\left( \bar{\mathcal{M}}_{y,\epsilon_y,s_L,s_{L-1}} \left( t_L(z) - j + 2M^2 \right) \geq \frac{1}{\pi} t_L(z) \right).
\]

The rest of the proof of (4.17) follows as in the proof of (4.8). This completes the proof of the upper bound in Lemma 4.2.

We now remove the restriction that \( y \in F_L \) in the upper bound, subject to a continuity restriction on \( \epsilon_y \).

**Lemma 4.4.** We can find \( 0 < c, C, z_0 < \infty \) such that for \( L \) large, all \( z_0 \leq z \leq \log L \), and all \( h_L/20 \leq \epsilon_y \leq h_{L+1} \) such that \( |\epsilon_y - \epsilon_y'| \leq C d(y,y')/L \) for all \( y,y' \in S^2 \),

\[
(4.25) \quad P\left( \exists y \text{ s.t. } \bar{\mu}_F(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z) \right) \leq cze^{-2z}.
\]

**Proof of Lemma 4.4:** Let \( F'_L \) be the centers of a \( \frac{d_0}{L} h_L \) covering of \( S^2 \) which contains \( F_L \). For any \( y \in S^2 \) we can find \( y' \in F'_L \) such that \( d(y,y') \leq \frac{d_0}{L} h_L \), so that by our assumptions \( |\epsilon_y - \epsilon_y'| \leq C \frac{d_0}{L} h_L \). If we set \( \bar{\epsilon}_y = (1 + \frac{1}{L}) \epsilon_y \) for all \( y \in S^2 \) we see that for \( L \) large \( h_L/30 \leq \bar{\epsilon}_y \leq 2h_{L+1} \) and \( |\bar{\epsilon}_y - \bar{\epsilon}_y'| \leq \frac{d_0}{L} h_L \). It follows from Lemma 5.1 below that it suffices to prove that

\[
(4.26) \quad P\left( \exists y \in F'_L \text{ s.t. } \bar{\mu}_F(y, \bar{\epsilon}_y) \geq \frac{1}{\pi} t_L(z) \right) \leq cze^{-2z}.
\]

For \( 0 < \epsilon < a < b < \pi \), let \( \mathcal{M}_{y,\epsilon_y,y_0,a,b}(n) \) be the total occupation measure of \( B_d(y,\epsilon_y) \) during the first \( n \) excursions from \( \partial B_d(y_0,a) \) to \( \partial B_d(y_0,b) \). With \( \omega_\epsilon = 2\pi(1 - \cos(\epsilon)) \), the area of \( B_d(y,\epsilon) \), let

\[
(4.27) \quad \bar{\mathcal{M}}_{y,\epsilon_y,y_0,a,b}(n) = \frac{1}{\omega_\epsilon \bar{\epsilon}_y} \mathcal{M}_{y,\epsilon_y,y_0,a,b}(n).
\]

For \( y_0 \in F_L \) let

\[
(4.28) \quad D_{y_0} = \{ y \in F'_L \mid d(y,y_0) \leq d_0 h_L/2 \}.
\]
Following the proof of the upper bound for Lemma 4.2, to prove (4.26) it suffices to show that

\[(4.29) \quad \mathbb{P} \left( \sup_{y \in D_{y_0}} \overline{M}_{y,\epsilon, y_0, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \geq \frac{1}{\pi} t_L \left( z \right) \right) \leq c e^{-c'j^2} \]

for \( j \leq z + ML^{1/2} \) sufficiently large. Setting \( \epsilon = \sup_{y \in D_{y_0}} \bar{\epsilon}_y \) and using our condition on \( |\epsilon_y - \bar{\epsilon}_{y'}| \) to control the denominator in (4.27), we see that it suffices to show that

\[(4.30) \quad \mathbb{P} \left( \sup_{y \in D_{y_0}} \overline{M}_{y,\epsilon, y_0, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \geq \frac{1}{\pi} t_L \left( z - M^2/2 \right) \right) \leq c e^{-c'j^2} \]

Abbreviating \( Y_y^{(n)} = \overline{M}_{y,\epsilon, y_0, h_L, h_{L-1}} (n) \) where \( n = t_L \left( z - j + 2M^2 \right) \) we have that

\[(4.31) \quad \mathbb{P} \left( \sup_{y \in D_{y_0}} \overline{M}_{y,\epsilon, y_0, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \geq \frac{1}{\pi} t_L \left( z - M^2/2 \right) \right) \leq \mathbb{P} \left( \overline{M}_{y_0,\epsilon, h_L, h_{L-1}} \left( t_L \left( z - j + 2M^2 \right) \right) \geq \frac{1}{\pi} t_L \left( z - j/2 - M^2/2 \right) \right) + \mathbb{P} \left( \sup_{y \in D_{y_0}} |Y_y^{(n)} - Y_{y_0}^{(n)}| \geq jL/2 \right) . \]

As in the proof of Lemma 4.3, the first term on the right hand side is bounded by \( c e^{-c'j^2} \) for \( j \leq z + ML^{1/2} \) sufficiently large. We then bound

\[(4.32) \quad \mathbb{P} \left( \sup_{y \in D_{y_0}} |Y_y^{(n)} - Y_{y_0}^{(n)}| \geq jL/2 \right) \leq \sum_{l=1}^{\log_2 L} \mathbb{P} \left( \sup_{y,y' \in D_{y_0}, \bar{d}(y,y') \approx 2^{-1}d_0,h_L} |Y_y^{(n)} - Y_{y'}^{(n)}| \geq jL/2^l \right) \leq \sum_{l=1}^{\log_2 L} 2^{2l} \mathbb{P} \left( |Y_y^{(n)} - Y_{y'}^{(n)}| \geq jL/2^l \right) . \]

It follows from Lemma 7.2 with \( n = t_L \left( z - j + 2M^2 \right) \sim 2L^2 \) as above and \( \theta = j/2^{3/2}L^2, \bar{d}(y,y') = 2^{-l}d_0 \) that for some \( C_0 > 0 \)

\[(4.33) \quad 2^{2l} \sup_{y,y' \in D_{y_0}, \bar{d}(y,y') \approx 2^{-1}d_0,h_L} \mathbb{P} \left( |Y_y^{(n)} - Y_{y'}^{(n)}| \geq jL/2^l \right) \leq 2^{2l} \exp \left( -C_0^2 j^{3/2}d_0^{1/2}L^2 \right) \]

whose sum over \( l \) is bounded by \( c e^{-c'j^2} \). In order to apply Lemma 7.2 we have to verify that \( \theta \leq \sqrt{\bar{d}(y,y')n}/2 \). In our situation this means that \( j/2^{3/2}L^2 \leq 2^{-l/2}d_0^{1/2}L/2 \), for all \( j \leq 2ML^{1/2} \). Thus we have to verify that
$2^{1/2}M2^{1/2}/l^2 \leq d_0^{1/2}L^{1/2}$, which follows from the fact that $l \leq \log L$, $L$ is large and $d_0, M$ are fixed.

4.2. The lower bound. Recall the notation $T_{y,l}^1 = T_{y,l}^{x^2,1}$ from the beginning of Section 3. Let $\tau_y$ be the time needed to complete $x^2$ excursions from $\partial B_d(y, h_1)$ to $\partial B_d(y, h_0)$, and set

$$\mu_\tau(y, \epsilon) = \frac{1}{\omega \epsilon} \int_0^{\tau_y} 1_{\{B_d(y, \epsilon)\}}(X_t) \, dt.$$ \hspace{1cm} (4.34)

Recall $F_0^L$ from (3.4). We will prove the following analogue of Lemma 3.2

Lemma 4.5. There exists a $0 < c < \infty$ such that for all $0 < r_0$ sufficiently small, $L$ large, all $0 \leq z \leq \log L$, and all $h_L/100 \leq \epsilon \leq h_L$

$$P\left[ \sup_{y \in F_0^L} \mu_\tau(y, \epsilon) \geq \frac{1}{\pi} l_L(z) \right] \geq \frac{(1 + z)e^{-2z}}{(1 + z)e^{-2z} + c}. \hspace{1cm} (4.35)$$

As before, the lower bound in (4.6) will follow from this, and hence combined with (4.25) we see that for some $0 < z_0$, and all $z_0 \leq z \leq \log L$

$$c'z e^{-2z} \leq P \left( \exists y \text{ s.t. } \mu_\tau(y, \epsilon) \geq \frac{1}{\pi} l_L(z) \right) \leq cze^{-2z}. \hspace{1cm} (4.36)$$

Combined with (4.7) it is easy to check that this implies Theorem 1.4.

To prove (4.35) set

$$\tilde{I}_{y,z+d} = I_{y,z+d} \cap \{ \mu_\tau(y, \epsilon) \geq \frac{1}{\pi} l_L(z) \} \hspace{1cm} (4.37)$$

for some $d < \infty$ to be chosen shortly. We use the second moment method used in the proof of Lemma 3.2. Indeed, since $\tilde{I}_{y,z+d} \subseteq I_{y,z+d}$ all upper bounds needed follow from those used in the proof of Lemma 3.2 and it only remains to prove the appropriate lower bound for $\tilde{I}_{y,z+d}$.

As in (3.22)-(3.23) we have

$$P\left( \tilde{I}_{y,z+d} \right) \geq P\left( \tilde{I}_{y,z+d}, \mu_\tau(y, h_L) \geq \frac{1}{\pi} l_L(z) \right) \hspace{1cm} (4.38)$$

$$- \sum_{k=L^+}^{L-d^*} \sum_{a \geq k^{1/4}} P \left[ \tilde{I}_{y,z+d} \cap H_{k,a} \cap W_{y,k}^c(N_{k,a}) \right].$$
Using the Markov property and then the barrier estimate (9.12) of Appendix I,
\begin{equation}
\tag{4.39}
\mathbb{P}\left( \mathcal{T}_{y,z+d}, \bar{\mu}_{\tau_y} (y, \epsilon_y) \geq \frac{1}{t_L} (z) \right)
\end{equation}
\begin{align*}
&\geq \mathbb{P}\left( \mathcal{T}_{y,z+d} \text{ and } \mathcal{M}_{y,\epsilon_y,h_L,h_{L-1}} (t_L(z+d)) \geq \frac{1}{t_L} (z) \right) \\
&= \mathbb{P}\left( \mathcal{T}_{y,z+d} \right) \mathbb{P}\left( \mathcal{M}_{y,\epsilon_y,h_L,h_{L-1}} (t_L(z+d)) \geq \frac{1}{t_L} (z) \right) \\
&\geq c(1+z)e^{-2L}e^{-2(z+d)} \\
&\geq c(1+z)e^{-2L}e^{-2(z+d)} \left( 1 - e^{-c'' \frac{(dL)^2}{t_L(z+d)}} \right),
\end{align*}
where the last line used (4.33). It should be clear from the structure of $t_L(z+d)$ that we can choose some $d < \infty$ so that $e^{-c'' \frac{(dL)^2}{t_L(z+d)}} \leq 1/2$ uniformly in $0 \leq z \leq \log L$. Finally, after fixing such a $d$, we can show as in the proof of the first moment estimate in Section 3.1, that for $d''$ large enough, the last line in (4.38) is much smaller than the last line of (4.39). \hfill \square

4.3. The left tail.

**Lemma 4.6.** There exists a $0 < c < \infty$ such that for all $0 < r_0$ sufficiently small, $L$ large, all $0 \leq z \leq \log L$, and all $h_L/100 \leq \epsilon_y \leq h_L$
\begin{equation}
\tag{4.40}
\mathbb{P}\left[ \sup_{y \in F^0_L} \bar{\mu}_{\tau_y} (y, \epsilon_y) \geq \frac{1}{t_L} (z) \right] \geq \frac{e^{2z}}{e^{2z} + c}.
\end{equation}

This will complete the proof of Theorem 1.3 since, as discussed right after the statement of Lemma 3.2, the probability of completing $x^2$ excursions from $\partial B_d (y,h_1)$ to $\partial B_d (y,h_0)$ before time $\tau$ for all $y \in F^0_L$ is a strictly positive function of $r_0$ which goes to 1 as $r_0 \to 0$.

The proof of Lemma 4.6 is very similar to our proof of the lower bound on the right tail, except we now have to change the upper barrier to allow for negative $z$. Fix $|z| \leq \log L$. We fix $\bar{x} > 0$ once and for all. We abbreviate,
\begin{equation}
\tag{4.41}
\hat{\beta}_z (l) = f_{\bar{x},\rho_L L^+ z} (l; L) = \bar{x} \left( 1 - \frac{l}{L} \right) + \left( \rho_L l + z \frac{l}{L} \right),
\end{equation}
and
\begin{equation}
\tag{4.42}
\hat{\gamma}_{z,-} (l) = \hat{\gamma}_{z,-} (l, L, z) = \hat{\beta}_z (l) - l^{1/4}.
\end{equation}
The barrier estimates needed are given in Lemma 9.6. We point out that the factors \((1 + z)\) which appear on the right hand side of (4.35) but not (4.40) come from the difference in the initial points of the barriers.

5. Interpolation

Recall, (1.8), that

\[
\bar{\mu}_\tau(y, \epsilon_y) = \frac{1}{\omega_{\epsilon_y}} \int_0^\tau 1_{\{B_d(y, \epsilon_y)\}}(X_t) \, dt,
\]

where \(\omega_{\epsilon_y} = 2\pi(1 - \cos \epsilon_y)\), the area of \(B_d(y, \epsilon_y)\), and, (4.5),

\[
t_L(z) = 2L(L - \log L + z).
\]

Lemma 5.1. Assume that \(d(y, y') \leq a h_L L, |\epsilon_y - \epsilon_{y'}| \leq b h_L, \) and \( h_L / 30 \leq \epsilon_{y}, \epsilon_{y'} \leq 2h_L + 1 \). We can find a \(d_1 < \infty\) such that for all \(L\) large and \(z \leq \log L\), if

\[
\bar{\mu}_\tau(y', \epsilon_{y'}) \geq \frac{1}{\pi} t_L(z),
\]

then for any \(c_1 \geq 30(a + b)\),

\[
\bar{\mu}_\tau(y, (1 + c_1 / L) \epsilon_y) \geq \frac{1}{\pi} t_L(z - d_1).
\]

**Proof:** Under our assumptions, for any \(z \in B_d(y', \epsilon_{y'})\) we have \(d(z, y) \leq d(z, y') + d(y, y') \leq \epsilon_{y'} + a h_L L \leq (1 + \frac{1}{2} \epsilon_{y}) \epsilon_y\) so that

\[
B_d(y', \epsilon_{y'}) \subseteq B_d(y, (1 + c_1 / L) \epsilon_y).
\]

It follows that

\[
\bar{\mu}_\tau(y', \epsilon_{y'}) = \frac{1}{\omega_{\epsilon_{y'}}} \int_0^\tau 1_{\{B_d(y', \epsilon_{y'})\}}(X_t) \, dt \leq \frac{1}{\omega_{\epsilon_{y'}}} \int_0^\tau 1_{\{B_d(y, (1 + c_1 / L) \epsilon_y)\}}(X_t) \, dt = \frac{\omega_{(1 + c_1 / L) \epsilon_y}}{\omega_{\epsilon_{y'}}} \bar{\mu}_\tau(y, (1 + c_1 / L) \epsilon_y).
\]

Hence

\[
\bar{\mu}_\tau(y', \epsilon_{y'}) \geq \frac{1}{\pi} t_L(z)
\]

implies that

\[
\bar{\mu}_\tau(y, (1 + c_1 / L) \epsilon_y) \geq \frac{\omega_{\epsilon_{y'}}}{\omega_{(1 + c_1 / L) \epsilon_y}} \frac{1}{\pi} t_L(z).
\]

But under our assumptions

\[
\frac{\omega_{\epsilon_{y'}}}{\omega_{(1 + c_1 / L) \epsilon_y}} = 1 + O(1/L).
\]

This gives (5.4).
6. Green’s functions and proof of Lemma 4.1

Let $G_a(x, y)$ denote the potential density for Brownian motion killed the first time it leaves $B_e(0, a)$, that is, the Green’s function for $B_e(0, a)$. We have, see [9, Section 2] or [11, Chapter 2, (1.1)],

$$G_a(x, y) = -\frac{1}{\pi} \log |x - y| + \frac{1}{\pi} \log \left(\frac{|y|}{a} |x - y_a^*| \right), \quad y \neq 0,$$

where

$$y_a^* = \frac{a^2 y}{|y|^2},$$

and

$$G_a(x, 0) = -\frac{1}{\pi} \log |x| + \frac{1}{\pi} \log a.$$

Let $v$ denote the south pole of $S^2$. We claim that in the isothermal coordinates induced by stereographic projection $\sigma$, the Green’s function for $\sigma(B_d(v, h(a))) = B_e(0, a)$ is just $G_a(x, y)$. To see this we must show that if $\Delta_{S^2}$ is the Laplacian for $S^2$ in isothermal coordinates and $dV(y)$ is the volume measure, then

$$\frac{1}{2} \Delta_{S^2} \int G_a(x, y) f(y) dV(y) = -f(x)$$

for all continuous $f$ compactly supported in $B_e(0, a)$.

For $x = (x_1, x_2)$, let

$$g(x) = \frac{1}{(1 + \frac{1}{4}(x_1^2 + x_2^2))^2}.$$

As shown in [23] Chapter 7, p. 6-9, the stereographic projection $\sigma$ is an isometry if we give $R^2$ the metric

$$g(x) (dx_1 \otimes dx_1 + dx_2 \otimes dx_2).$$

Because of (6.6) the Laplace-Beltrami operator takes the form

$$\frac{1}{g(x)} \left(\partial_{x_1}^2 + \partial_{x_2}^2\right).$$

Thus, $\Delta_{S^2} = \frac{1}{g} \Delta$ and $dV(y) = g(y) dy$, so that (6.4) holds.

**Proof of Lemma 4.1** Let $\epsilon = h(\alpha)$ so that $h(\alpha) \leq h_k$. If $\tau_{h_k-1}$ is the first exit time of $B_d(v, h_{k-1})$ and $\rho_{h_k}$ is uniform measure on $\partial B_d(v, h_k)$, then by symmetry, for any $z \in \partial B_d(v, h_k)$

$$J_1 =: \mathbb{E}^z \left( \int_0^{\tau_{h_k-1}} 1_{\{B_d(v, \epsilon)\}}(X_t) \, dt \right)$$

$$= \mathbb{E}^{\rho_{h_k}} \left( \int_0^{\tau_{h_k-1}} 1_{\{B_d(v, \epsilon)\}}(X_t) \, dt \right).$$
Since uniform measure \( \rho_{h_k} \) on \( \partial B_d(v, h_k) \) goes over to uniform measure \( \gamma_{r_k} \) on \( \partial B_e(0, r_k) \), using the discussion at the beginning of this section we have

\[
J_1 = \int_{B_e(0, \alpha)} \int_{\partial B_e(0, r_k)} G_{r_{k-1}}(x, y) \, d\gamma_{r_k}(x) \, g(y) \, dy.
\]

We recall, [20, Chapter 2, Prop. 4.9] or [11, Chapter 1, (5.4), (5.5)], that

\[
\int_{\partial B_e(0, b)} \log(|x - y|) \, d\gamma_b(x) = \log(b \vee |y|).
\]

This shows that for \( y \in B_e(0, r_k) \)

\[
\int_{\partial B_e(0, r_k)} G_{r_{k-1}}(x, y) \, d\gamma_{r_k}(x)
= \frac{1}{\alpha} \int_{\partial B_e(0, r_k)} \left( -\log |x - y| + \log \left( \frac{|y|}{r_{k-1}} |x - y_{r_{k-1}}| \right) \right) \, d\gamma_{r_k}(x)
= \frac{1}{\alpha} \left( -\log r_k + \log \left( \frac{|y|}{r_{k-1}} |y_{r_{k-1}}| \right) \right)
= \frac{1}{\alpha} (-\log r_k + \log r_{k-1}) = \frac{1}{\alpha} \log(r_{k-1}/r_k) = \frac{1}{\alpha}.
\]

Thus

\[
J_1 = \frac{1}{\alpha} \int_{B_e(0, \alpha)} g(y) \, dy = \frac{1}{\alpha} \text{Area} \left( B_d(v, h(\alpha)) \right) = \frac{1}{\alpha} \omega_{h(\alpha)} = \frac{1}{\alpha} \omega_e.
\]

It follows that for any \( z \in \partial B_d(v, h_k) \)

\[
\mathbb{E}^z \left( \mathcal{M}_{v, \epsilon, h_k, h_{k-1}}(1) \right) = \frac{1}{\alpha}.
\]

By the Kac moment formula, for any \( z \in \partial B_d(v, h_k) \), with \( x = \sigma(z) \)

\[
\mathbb{E}^z \left( \left( \int_0^{r_{k-1}} 1_{\{B_d(v, \epsilon)\}}(X_t) \, dt \right)^n \right)
= n! \int_{B_0^2(0, \alpha)} G_{r_{k-1}}(x, y_1) \prod_{j=2}^n G_{r_{k-1}}(y_{j-1}, y_j) \prod_{i=1}^n g(y_i) \, dy_i
\leq c^n n! \alpha^{2n} (\log(r_{k-1}/\alpha) + c_0)^n
\]

where the last inequality follows as in the proof of [9, Lemma 2.1]. It follows that for any \( z \in \partial B_d(v, h_k) \)

\[
\mathbb{E}^z \left( \left( \mathcal{M}_{v, \epsilon, h_k, h_{k-1}}(1) \right)^n \right) \leq c^n n! (\log(r_{k-1}/\alpha) + c_0)^n.
\]

By (2.3), our assumption that \( h_k/100 \leq \epsilon \leq h_k \) implies that \( \epsilon \leq r_{k-1}/\alpha \leq 200\epsilon \). Using (6.13) and (6.15), our Lemma then follows as in the proof of [10, Lemma 2.2]. \( \square \)
7. Continuity Estimates

For fixed $u \in S^2$, let $\tau_u$ be the first exit time of $B_d(u,a)$ and let $\rho_m$ be uniform measure on $\partial B_d(u,m)$.

Lemma 7.1. If $d(u,v), d(u,\bar{v}) \leq d_0 h_L/2$, $d_0/L \leq \bar{d} =: d(v,\bar{v})/h_L \leq d_0$, and $h_L/20 \leq \epsilon \leq h_{L+1}$, then

$$\mathbb{E}^{\rho_m} \left( \int_0^{\tau_{h_L-1}} 1_{\{B_d(v,\epsilon)\}}(X_t) \, dt - \int_0^{\tau_{h_L-1}} 1_{\{B_d(\bar{v},\epsilon)\}}(X_t) \, dt \right) = 0,$$

(7.1)

$$\mathbb{E}^{\rho_m} \left( \left( \int_0^{\tau_{h_L-1}} 1_{\{B_d(v,\epsilon)\}}(X_t) \, dt - \int_0^{\tau_{h_L-1}} 1_{\{B_d(\bar{v},\epsilon)\}}(X_t) \, dt \right)^2 \right) \leq c \epsilon^4 \bar{d}^2,$$

(7.2)

and

$$\sup_{x \in \partial B_d(u,h_L)} \mathbb{E}^x \left( \left( \int_0^{\tau_{h_L-1}} 1_{\{B_d(v,\epsilon)\}}(X_t) \, dt - \int_0^{\tau_{h_L-1}} 1_{\{B_d(\bar{v},\epsilon)\}}(X_t) \, dt \right)^2 \right) \leq c \epsilon^4 \bar{d}^2.$$

(7.3)

Proof of Lemma 7.1 As in (6.8)-(6.9) we have

$$J_2 = \frac{1}{\pi} \int d\mu_{v,\bar{v}}(y)$$

(7.4)

where

$$d\mu_{v,\bar{v}}(y) = (1_{\{\sigma(B_d(v,\epsilon))\}} - 1_{\{\sigma(B_d(\bar{v},\epsilon))\}})(y)g(y) \, dy.$$

Then by (6.11)-(6.12) we have that

$$J_2 = \frac{1}{\pi} \int d\mu_{v,\bar{v}}(y)$$

(7.5)

since all balls of radius $\epsilon$ on the sphere have area $\omega_\epsilon = 2\pi(1 - \cos \epsilon)$. This completes the proof of (7.1).

We next observe that

$$\mathbb{E}^{\rho_m} \left( \left( \int_0^{\tau_{h_L-1}} 1_{\{B_d(v,\epsilon)\}}(X_t) \, dt - \int_0^{\tau_{h_L-1}} 1_{\{B_d(\bar{v},\epsilon)\}}(X_t) \, dt \right)^2 \right)$$

(7.6)

$$= 2 \int \int G_{r_{L-1}}(x,y)G_{r_{L-1}}(y,z) \, d\gamma_{r_{L}}(x) \, d\mu_{v,\bar{v}}(y) \, d\mu_{v,\bar{v}}(z)$$

$$= \frac{2}{\pi} \int \int G_{r_{L-1}}(y,z) \, d\mu_{v,\bar{v}}(y) \, d\mu_{v,\bar{v}}(z)$$

as above.
We note that for $b < a$

\[(7.8)\]

\[G_a(bx, by) = -\frac{1}{\pi} \log (b|x - y|) + \frac{1}{\pi} \log \left( b \frac{|y|}{a/b} |x - y|_a/b \right) = G_{a/b}(x, y), \]

since

\[(7.9)\]

\[(by)_a^* = \frac{a^2by}{b^2|y|^2} = by_{a/b}^*.\]

Using this to scale by $r_L$ we see that

\[(7.10)\]

\[\int \int G_{r_L-1}(y, z) d\mu_{v, \bar{v}}(y) d\mu_{v, \bar{v}}(z) = r_L^4 \int \int G_e(y, z) d\mu_{L, v, \bar{v}}(y) d\mu_{L, v, \bar{v}}(z),\]

where

\[(7.11)\]

\[d\mu_{L, v, \bar{v}}(y) = \left( 1\{\sigma(B_d(v, \epsilon))\} - 1\{\sigma(B_d(\bar{v}, \epsilon))\} \right) (r_L y)g(r_L y) dy\]

\[= \left( 1\{\frac{r_L}{L} \sigma(B_d(v, \epsilon))\} - 1\{\frac{r_L}{L} \sigma(B_d(\bar{v}, \epsilon))\} \right) (y)g(r_L y) dy.\]

For $y$ in our range we have $g(r_L y) = 1 + O(\epsilon)$, and it is easy to check that up to errors of order $\epsilon$, $\frac{r_L}{L} \sigma(B_d(v, \epsilon))$ and $\frac{r_L}{L} \sigma(B_d(\bar{v}, \epsilon))$ can be replaced by $B_e(v', \epsilon/r_L)$ and $B_e(v' - (0, \delta \epsilon/r_L), \epsilon/r_L)$ for some $v'$ with $|v'| \leq d_0$ and $0 < \delta \leq c_2 d$.

Hence with

\[(7.12)\]

\[d\mu_{v'}(y) = \left( 1\{B_e(v', \epsilon/r_L)\} - 1\{B_e(v' - (0, \delta \epsilon/r_L), \epsilon/r_L)\} \right) (y) dy,\]

it remains to show that

\[(7.13)\]

\[\int \int G_e(y, z) d\mu_{v'}(y) d\mu_{v'}(z) \leq C\delta^2.\]

The symmetric difference of $B_e(v', s)$ and $B_e(v' - (0, \delta s), s)$ consist of two disjoint pieces we denote by $A, B$. They have the same area

\[(7.14)\]

\[\text{Area}(A) = 2s^2 \arcsin \left( \frac{\delta}{2} + \frac{\delta s}{2} \sqrt{4 - \delta^2} \right) \approx \delta s^2.\]

We observe that

\[(7.15)\]

\[\lim_{y \to 0} \frac{|y|}{a} |z - y|_a^* = \lim_{y \to 0} \frac{|y|}{a} (|y| + O(1)) = \lim_{y \to 0} \frac{|y|}{a} (\frac{a^2|y|}{|y|^2} + O(1)) = a.\]

It follows that for $y, z$ in our range, $\log \left( \frac{|y|}{\epsilon} |z - y|_a^* \right)$ is bounded, hence to prove (7.13) it suffices to show that

\[(7.16)\]

\[\int \int \log |y - z| d\mu_{v'}(y) d\mu_{v'}(z) \leq C\delta^2.\]

It is then easy to see that we need only show that

\[(7.17)\]

\[\int_A \int_A \log |y - z| dy dz \leq C\delta^2.\]
It is also clear that we only need to consider $|y - z| \leq 1/2$. Writing $y = (y_1, y_2), z = (z_1, z_2)$ we see that

$$\int_A \int_A |\log |y - z||1_{\{|y-z| \leq 1/2\}}\, dy\, dz$$

$$\leq \int_{[0,1] \times [0,\delta]} \int_{[0,1] \times [0,\delta]} |\log |y - z||1_{\{|y-z| \leq 1/2\}}\, dy_1\, dy_2\, dz_1\, dz_2$$

$$\leq \int_{[0,1] \times [0,\delta]} \int_{[0,1] \times [0,\delta]} |y_1 - z_1|\, dy_1\, dy_2\, dz_1\, dz_2 \leq C\delta^2,$$

which completes the proof of (7.17).

To obtain (7.3), arguing as before we need to show that

$$K_1 = \sup_{x \in \partial B_r(0, r_L)} \int \int G_{r_{L-1}}(x, y)G_{r_{L-1}}(y, z)\, d\mu_{y, \bar{v}}(y)\, d\mu_{v, \bar{v}}(z) \leq c\epsilon^2\delta^2.$$

Scaling in $r_L$ as before shows that

$$K_1 = r_L^4 \sup_{x \in \partial B_r(0, 1)} \int \int G_{c}(x, y)G_{c}(y, z)\, d\mu_{v, \bar{v}}(y)\, d\mu_{v, \bar{v}}(z)$$

But for $y$ in our range, $G_c(x, y)$ is bounded uniformly in $x \in \partial B_c(0, 1)$, so that (7.3) follows as before. \[\Box\]

The same proof shows that

$$\sup_{x \in \partial B_d(u, h_L)} \mathbb{E}^x \left( \left( \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v, \epsilon)\}}(X_t)\, dt - \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(\bar{v}, \epsilon)\}}(X_t)\, dt \right)^2 \right)^n$$

$$\leq (2n)!c_1^2 n^4 \epsilon n^2.$$

and hence by the Cauchy-Schwarz inequality

$$\sup_{x \in \partial B_d(u, h_L)} \mathbb{E}^x \left( \left| \frac{1}{\omega_c} \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(v, \epsilon)\}}(X_t)\, dt - \frac{1}{\omega_c} \int_0^{\tau_{h_{L-1}}} 1_{\{B_d(\bar{v}, \epsilon)\}}(X_t)\, dt \right|^n \right)^n$$

$$\leq n!c_2^n n^2.$$

Recall (4.27) and set

$$Y^{(n)}_{y} = \overline{M}_{y, \epsilon, u, h_L, h_{L-1}}(n)$$

**Lemma 7.2.** For some $d_0 > 0$ we can find $C_0 > 0$ such that, if $d(u, v), d(u, \bar{v}) \leq d_0 h_L/2, d_0/L \leq d(v, \bar{v}) =: d(v, \bar{v})/h_L \leq d_0, h_L/20 \leq \epsilon \leq h_{L+1}$, and

$$\theta \leq \sqrt{d(v, \bar{v})/n},$$

then

$$\mathbb{P} \left( |Y^{(n)}_{y} - Y^{(n)}_{\bar{v}}| \geq \theta \sqrt{n} \right) \leq \frac{c_0 \theta^2}{\sqrt{d(v, \bar{v})} n^2}.$$
so that
\[(7.25) \quad Y_v^{(n)} = \sum_{i=1}^{n} Y_{v,i}.\]

Let $J$ be a geometric random variable with success parameter $p_3 > 0$, independent of $\{Y_{v,i}, Y_{\tilde{v},i}\}$. It follows from (7.22) and the proof of [4, Corollary 5.3] that, abbreviating $\bar{d} = \bar{d}(v, \tilde{v})$, if $c_2 \bar{d} \lambda \leq p_3/2$ then for some $c_4$

\[(7.26) \quad \sup_{x \in \partial B_d(u, h_l)} \mathbb{E}^x \left( \exp \left( \lambda \sum_{i=1}^{J-1} \left| (Y_{v,i} - Y_{\tilde{v},i}) \right| \right) \right) \leq e^{c_4 \bar{d} \lambda / p_3},\]

and from (7.22) together with (7.1) and the proof and notation of [4, Lemma 5.5] it follows that, after perhaps enlarging $c_4$

\[(7.27) \quad \mathbb{E} \left( \exp \left( \lambda \sum_{i=J_1}^{J_2-1} (Y_{v,i} - Y_{\tilde{v},i}) (X^i) \right) \right) \leq e^{c_4 (\bar{d} \lambda / p_3)^2}.\]

Instead of [4, (5.33)] we set
\[
\delta = \theta / \sqrt{dn} \leq 1/2.
\]

It then follows from the proof of [4, Lemma 5.1] that for $c_2 \bar{d} \lambda \leq p_3/2$

\[(7.28) \quad \mathbb{P} \left( |Y_v^{(n)} - Y_{\tilde{v}}^{(n)}| \geq \theta \sqrt{n} \right) \leq e^{-c_4 \theta^2 / \bar{d}} + \exp \left( c_4 \lambda^2 \bar{d}^2 n / p_3 + 2c_4 \lambda \theta \sqrt{dn} - \lambda \theta \sqrt{n} \right)\]

By taking $d_0$ sufficiently small we can be sure that $c_2 \bar{d} \leq 1$, so the above holds for any $\lambda \leq p_3/2$. If we set
\[
\lambda = p_3 \theta / \sqrt{dn} \leq p_3/2
\]

we see that
\[
\exp \left( c_4 \lambda^2 \bar{d}^2 n / p_3 + 2c_4 \lambda \theta \sqrt{dn} - \lambda \theta \sqrt{n} \right) = \exp \left( c_4 p_3 \theta^2 \bar{d} + 2c_4 p_3 \theta^2 - p_3 \theta^2 / \sqrt{\bar{d}} \right),
\]

which completes the proof of (7.23) for $d_0$ sufficiently small. \(\square\)

8. From the sphere to the plane, and back

Using (6.7) it follows from [21, Chapter 5, Theorem 1.9], that we can find a planar Brownian motion $W_t$ such that in the isothermal coordinates induced by stereographic projection,

\[(8.1) \quad X_t = W_{U_t}, \quad \text{where} \quad U_t = \int_0^t \frac{1}{g(X_s)} \, ds.\]

We take the $v$ of this paper to be $v = (0, 0, 0)$. Let
\[(8.2) \quad D_s = \sigma (B_d (v, r^*)) = B_e ((0, 0), 2 \tan(r^*/2)).\]
For the last equality see [4, (2.4)]. If θ is the first hitting time of ∂Dx by \( W_t \), then under the coupling (8.1) we see that θ = \( U_\tau \). Set \( \mu_\theta (x, \epsilon) = \frac{1}{\pi \epsilon^2} \int_0^\theta 1_{\{B_\epsilon (x, \epsilon)\}} (W_t) dt \).

**Lemma 8.1.** For some \(-\infty < d_1, d_2, d_3, d_4 < \infty \), all \( x \in D_\ast \) and all \( \epsilon \) sufficiently small

\[
\mu_\theta (x, \epsilon) \leq (1 + d_1 \epsilon) \bar{\mu}_\tau (x, g^{1/2}(x) \epsilon (1 + d_2 \epsilon)),
\]

and

\[
\mu_\theta (x, \epsilon) \geq (1 + d_3 \epsilon) \bar{\mu}_\tau (x, g^{1/2}(x) \epsilon (1 + d_4 \epsilon)).
\]

**Proof of Lemma 8.1** We first note that for \( \epsilon \) sufficiently small, we can find \( c_1 < c_2 \) such that uniformly in \( x' \in B_\epsilon (x, 2\epsilon) \) and \( x \in D_\ast \)

\[
g(x) (1 + c_1 \epsilon) \leq g(x') \leq g(x) (1 + c_2 \epsilon).
\]

For \( x' \in B_\epsilon (x, \epsilon) \), with \( x_t = x + t(x' - x) \)

\[
d(x, x') \leq \int_0^1 g^{1/2}(x_t) |x' - x| \, dt \leq g^{1/2}(x) |x' - x| (1 + c_3 \epsilon).
\]

Hence

\[
B_\epsilon (x, \epsilon) \subseteq B_d (x; g^{1/2}(x) \epsilon (1 + c_3 \epsilon)).
\]

Similarly, for some \( c_4 < c_3 \)

\[
B_\epsilon (x, \epsilon) \supseteq B_d (x; g^{1/2}(x) \epsilon (1 + c_4 \epsilon)).
\]

Consider

\[
\int_0^\tau 1_{\{B_d (x, g^{1/2}(x) \epsilon (1 + c_3 \epsilon))\}} (W_t) \, dt.
\]

By the nature of \( U_\tau \) in [4, (8.1)] it follows that whenever the path \( W_{U_\tau} \) enters \( B_d (x, g^{1/2}(x) \epsilon (1 + c_3 \epsilon)) \) it is slowed by a variable factor between \( \frac{1}{g(x)(1 + c_5 \epsilon)} \) and \( \frac{1}{g(x)(1 + c_6 \epsilon)} \). Hence the amount of time spent in \( B_d (x, g^{1/2}(x) \epsilon (1 + c_3 \epsilon)) \) during each incursion is multiplied by a variable factor between \( g(x)(1 + c_7 \epsilon) \) and \( g(x)(1 + c_8 \epsilon) \). Thus

\[
\int_0^\tau 1_{\{B_d (x, g^{1/2}(x) \epsilon (1 + c_6 \epsilon))\}} (W_t) \, dt \geq g(x)(1 + c_7 \epsilon) \int_0^g 1_{\{B_d (x, g^{1/2}(x) \epsilon (1 + c_6 \epsilon))\}} (W_t) \, dt,
\]

and

\[
\int_0^\tau 1_{\{B_d (x, g^{1/2}(x) \epsilon (1 + c_6 \epsilon))\}} (W_t) \, dt \leq g(x)(1 + c_8 \epsilon) \int_0^g 1_{\{B_d (x, g^{1/2}(x) \epsilon (1 + c_6 \epsilon))\}} (W_t) \, dt,
\]
It follows from (8.8) and (8.11) that

\[
\int_0^\theta 1_{\{B_e(x,\epsilon)\}}(W_t) \, dt \\
\leq \int_0^\theta 1_{\{B_d(x,g^{1/2}(x)(1+c_3\epsilon))\}}(W_t) \, dt \\
\leq \frac{1}{g(x)(1+c_7\epsilon)} \int_0^\tau 1_{\{B_d(x,g^{1/2}(z)(1+c_3\epsilon))\}}(W_U_t) \, dt.
\]

Since \(\omega_\delta = 2\pi(1 - \cos(\delta))\), we see that if we set \(\delta_x = g^{1/2}(x)\epsilon(1+c_3\epsilon)\), then uniformly in \(x \in D^2\) and sufficiently small \(\epsilon\)

\[
(1 + f_0'\epsilon) \leq \frac{\omega_\delta_x}{\pi g(x) \epsilon^2} \leq (1 + f_0\epsilon),
\]

so that by (8.8) and (8.13)

\[
\mu_\theta(x,\epsilon) = \frac{1}{\pi \epsilon^2} \int_0^\theta 1_{\{B_e(x,\epsilon)\}}(W_t) \, dt \\
\leq \frac{1}{\pi \epsilon^2 g(x)(1+c_7\epsilon)} \int_0^\tau 1_{\{B_d(x,g^{1/2}(z)(1+c_3\epsilon))\}}(W_U_t) \, dt \\
= \frac{1}{(1+c_7\epsilon)} \frac{\omega_\delta_x}{\pi g(x) \epsilon^2} \bar{\mu}_\tau(x, g^{1/2}(x)\epsilon(1+c_3\epsilon)) \\
\leq (1 + \hat{d} \epsilon) \bar{\mu}_\tau(x, g^{1/2}(x)\epsilon(1+c_3\epsilon)),
\]

where

\[
(1 + \hat{d} \epsilon) = \frac{1 + f_0\epsilon}{1 + c_7\epsilon}.
\]

This completes the proof of (8.4).

The lower bound (8.5) is proven similarly using (8.9) and (8.12).

**Lemma 8.2.** We can find \(0 < c, c', z_0 < \infty\) such that for \(L\) large and all \(\frac{1}{12}h_L \leq \epsilon \leq \frac{1}{2}h_L\) and \(z_0 \leq z \leq \log L\),

\[
c' z e^{-c' \sqrt{\pi z}} \leq \mathbb{P} \left( \sup_y \sqrt{\mu_\theta(y,\epsilon)} \geq m_\epsilon + z \right) \leq c z e^{-c' \sqrt{\pi z}}.
\]

**Proof of Lemma 8.2.** We consider the upper bound. By (8.14) it suffices to show that

\[
\mathbb{P} \left( \sup_y \sqrt{\bar{\mu}_\tau(y, g^{1/2}(y)\epsilon(1+d_2\epsilon))} \geq m_\epsilon + z \right) \leq c z e^{-c' \sqrt{\pi z}}.
\]

Since for \(\epsilon\) in our range

\[
(m_\epsilon + z)^2 = \frac{1}{\pi} t L \left(\sqrt{2\pi} \ z + O(1)\right),
\]

(compare (4.7)), (8.16) follows from Lemma 4.4 once we verify the condition that \(|\epsilon_y - \epsilon_y'| \leq C d(y, y')/L\), where now \(\epsilon_y = g^{1/2}(y)\epsilon(1+d_2\epsilon))\). This follows
easily since $g$ is smooth and we can assume that $\frac{4}{5} \leq g^{1/2}(y) \leq 1$. We also point out that for $\frac{1}{12}h_L \leq \epsilon \leq \frac{1}{3}h_L$ and $L$ large we have $\frac{1}{20}h_L \leq \epsilon_y \leq h_{L+1}$. The lower bound is similar.

We note that $\frac{1}{3}h_{L+1} = \frac{1}{3}h_L \geq \frac{1}{12}h_L$, so all $\epsilon$ are covered by Lemma 8.2.

Lemma 8.2 is the analog of Theorem 1.2, but where now $\theta$ is the first hitting time of $\partial D^*$, see (8.2). Theorem 1.2 then follows by Brownian scaling.

To spell this out for later use, let $\theta_a$ be the first hitting time of $\partial B_{e(a, a)}$ and set

$$
\mu_a(x, \epsilon) = \frac{1}{\pi \epsilon^2} \int_0^{\theta_a} 1_{B(x, \epsilon)}(W_t) dt.
$$

Then it follows from Brownian scaling that for any $a, b > 0$,

$$
\mu_a(x, \epsilon; x, \epsilon_x) \overset{\text{law}}{=} \mu_{ba}(bx, b\epsilon_x; x, \epsilon_x).
$$

The left tail and then Theorem 1.1 can be proven similarly.

### 8.1. $0 < r^* < \pi$

We first note the following extension of Lemma 8.2.

**Lemma 8.3.** We can find $0 < c, c', z_0 < \infty$ such that for $L$ large and all $\frac{1}{12}h_L \leq \epsilon_y \leq \frac{1}{3}h_L$ with $|\epsilon_y - \epsilon_y'| \leq C|y - y'|/L$ and $z_0 \leq z \leq \log L$,

$$
c' \epsilon \epsilon^{-2z} \leq P\left(\sup_y \mu_\theta(y, \epsilon_y) \geq \frac{1}{\pi} t_L(z)\right) \leq c \epsilon \epsilon^{-2z}.
$$

This follows as in the proof of Lemma 8.2 once we observe that in Lemma 8.1 we can allow the $\epsilon$ to depend on $x$.

It follows from (8.18) that for any fixed $a > 0$, Lemma 8.3 holds with $\theta$ replaced by $\theta_a$.

We now show that Theorem 1.4 holds for any $0 < r^* < \pi$. This is done by using Lemma 8.4. That is, with $a = 2 \tan(r^*/2)$ we have that for some $-\infty < d_1, d_2, d_3, d_4 < \infty$, all $x \in D_a$ and all $\epsilon$ sufficiently small

$$
\bar{\mu}_r(x, \epsilon) \leq (1 + d_1 \epsilon) \mu_{\theta_a}(x, g^{-1/2}(x)\epsilon(1 + d_2 \epsilon)),
$$

and

$$
\bar{\mu}_r(x, \epsilon) \geq (1 + d_3 \epsilon) \mu_{\theta_a}(x, g^{-1/2}(x)\epsilon(1 + d_4 \epsilon)),
$$

Theorem 1.4 then follows from Lemma 8.3 just as Lemma 8.2 followed from Lemma 4.4. Theorem 1.3 can be proven similarly.

### 9. Appendix I: Barrier estimates

In what follows, we use the notation $H_{y, \delta} = [y, y + \delta]$ from [5]. The following is a variant of [5, Theorem 1.1], which can be proven similarly. Re-set

$$
f_{a,b}(l; L) = a + (b - a)\frac{l}{L}.
$$
Theorem 9.1. a) For all fixed \( \delta > 0, C \geq 0, \eta > 1 \) and \( \varepsilon \in (0, 1/2) \) we have, uniformly in \( \sqrt{2} \leq x, y \leq \eta L \) such that \( x^2/2 \in \mathbb{N} \), any \( 0 \leq x \leq a \), \( 0 \leq y \leq b \), that

\[
P_{x^2/2} \left( \sqrt{2T_l} \leq f_{a,b} (l; L) + \mathcal{C} \left[ \frac{1}{L} \right] - \varepsilon, l = 1, \ldots, L - 1, \sqrt{2T_L} \in H_{y,\delta} \right)
\]

\[
\leq c \frac{(1 + a - x)(1 + b - y)}{L} \sqrt{\frac{x}{yL}} e^{-\frac{y(x+y)^2}{2}}.
\]

(9.2)

b) Let \( \text{Tube}_{C,\delta}(l; L) = \left[ f_{x,y} (l; L) - \mathcal{C} \left[ \frac{1}{L} \right] + \varepsilon, f_{a,b} (l; L) - \mathcal{C} \left[ \frac{1}{L} \right] - \varepsilon \right] \). If, in addition to the conditions in part a), we also have \( (1 + a - x)(1 + b - y) \leq \eta L \), \( \max(xy, |y - x|) \geq L/\eta \) and \( [y, y + \delta] \cap \sqrt{2N} \neq \emptyset \), and \( \text{Tube}_{C,\delta}(l; L) \cap \sqrt{2N} \neq \emptyset \) for all \( l = 1, \ldots, L - 1 \), then

\[
P_{x^2/2} \left( \sqrt{2T_l} \in \text{Tube}_{C,\delta}(l; L), l = 1, \ldots, L - 1, \sqrt{2T_L} \in H_{y,\delta} \right)
\]

\[
\geq c \frac{(1 + a - x)(1 + b - y)}{L} \left( \sqrt{\frac{x}{yL}} \wedge 1 \right) e^{-\frac{y(x+y)^2}{2}},
\]

(9.3)

and the estimate is uniform in such \( x, y, a, b \) and all \( L \).

Similar results hold if we delete the barrier condition on some fixed finite interval.

For the last statement, we simply note that following the proof of [5, Lemma 2.3] we can show that the analogue of [5, Theorem 1.1] holds where we skip some fixed finite interval.

Recall that

\[
p_L = 2 - \frac{\log L}{L}, \quad \alpha_{z,\pm}(l) = \alpha(l, L, z) = \rho_L l + z \pm l_{1/4}.
\]

(9.4)

Lemma 9.2. Let \( m = k_y + 1 \leq \log L \). For any \( k \geq L - (\log L)^4 \), \( 0 \leq j \leq \alpha_{z,\pm}(k)/2 \) and \( z \leq \log L \)

\[
P \left[ \sqrt{2T_{y,l}} \leq \alpha_{z,\pm}(l), l = m, \ldots, k - 1; \sqrt{2T_{y,k}} \in I_{\alpha_{z,\pm}(k) - j} \right] \leq c e^{-2k - 2z - 2k_{1/4} + 2j} \times m^2 \left( 1 + z + m + k_{1/4} \right) (1 + j).
\]

(9.5)

Proof of Lemma (9.2): Using the Markov property, the probability in (9.3) is bounded by

\[
\sum_{s=0}^{\alpha_{z,\pm}(m)} P \left[ \sqrt{2T_{y,m}} \in I_s \right] \times \sup_{x \in I_s} P \left[ \sqrt{2T_{y,l}^{m,x^2/2}} \leq \alpha_{z,\pm}(l), l = m + 1, \ldots, k - 1; \sqrt{2T_{y,k}} \in I_{\alpha_{z,\pm}(k) - j} \right],
\]

(9.6)
and using the fact that $T_{y,m}^r \leq T_{y,m}^{m,0}$ and (2.3), we see that (9.6) is bounded by

$$
\sum_{s=0}^{\alpha_{z,+}(m)} e^{-s^2/2m} \sup_{x \in I_s} \mathbb{P} \left[ \sqrt{2T_{y,l}^{m,x^2/2}} \leq \alpha_{z,+}(l), l = m + 1, \ldots, k - 1; \sqrt{2T_{y,k}^{m,x^2/2}} \in I_{\alpha_{z,+}(k) - j} \right].
$$

(9.7)

Recall that $\alpha_{z,+}(l) = \rho_L l + z + l_L^{1/4}$. Using this we can write the last probability as

$$
K_{1,s} := \mathbb{P} \left[ \sqrt{2T_{y,l}^{m,x^2/2}} \leq \rho_L l + z + l_L^{1/4} \text{ for } l = m + 1, \ldots, k - 1; \sqrt{2T_{k-m}^{m,x^2/2}} \in I_{\rho_L k + z + k_L^{1/4} - j} \right].
$$

(9.8)

Using the fact that for all $1 \leq l \leq k - 1$

$$
l_L^{1/4} \leq l_k^{1/4} + k_L^{1/4},
$$

see [4], it follows that

$$
K_{1,s} \leq P_{x^2/2}^{GW} \left[ \sqrt{2T_l} \leq \rho_L (m + l) + z + (m + l)_k^{1/4} + k_L^{1/4}, \right.
$$

\text{for } l = 1, \ldots, k - m - 1; \sqrt{2T_{k-m}^{m,x^2/2}} \in I_{\rho_L k + z + k_L^{1/4} - j} \right].
$$

(9.9)

Thus using (9.2), with $a = \rho_L m + z + (m_k^{1/4} + k_L^{1/4})$ and $b = \rho_L k + z + k_L^{1/4}$,

$$
y = \rho_L k + z + k_L^{1/4} - j,
$$

$$
K_{1,s} \leq c \frac{(1 + a - x)(1 + j)}{k - m} \frac{\sqrt{x}}{y(k - m)} e^{-\frac{(\rho_L k + z + k_L^{1/4} - j - x)^2}{2(k-m)}}.
$$

We have

$$
e^{-\frac{(\rho_L k + z + k_L^{1/4} - j - x)^2}{2(k-m)}} \leq c e^{-\frac{(\rho_L k)^2}{2(k-m)}} e^{-2(z + k_L^{1/4} - j - x)} \leq c e^{2k \log L} e^{-2(k+m)-2z-2k_L^{1/4}+2j+2x},
$$

$$
\frac{1}{k-m} \sqrt{\frac{x}{y(k-m)}} e^{2k \log L} \times \sqrt{x}, \text{ and by assumption,}
$$

$$
a - x \leq c \left( k_L^{1/4} + m + z \right).
$$

Hence we can bound (9.7) by

$$
c \left( 1 + k_L^{1/4} + m + z \right)(1+j) e^{-2k-2z-2k_L^{1/4}+2j} \sum_{s=0}^{\alpha_{z,+}(m)} \sqrt{s} e^{-(s-2m)^2/2m}.
$$

Our Lemma follows. □
Lemma 9.3. For all $L$ sufficiently large, and all $0 \leq z \leq \log L$,

$$P \left[ \sqrt{2T^{1}_{y,l}} \leq \alpha_{z,-}(l) \text{ for } l = 1, \ldots, L - 1; \sqrt{2T^{1}_{y,L}} \geq \rho_{L} + z \right]$$

$$\leq P \left[ \sqrt{2T^{1}_{y,l}} \leq \rho_{L}l + z \text{ for } l = 1, \ldots, L - 1; \sqrt{2T^{1}_{y,L}} \geq \rho_{L}L + z \right]$$

(9.11)

and

$$P \left[ \sqrt{2T^{1}_{y,l}} \leq \alpha_{z,-}(l) \text{ for } l = 1, \ldots, L - 1; \sqrt{2T^{1}_{y,L}} \in I_{\rho_{L}L + z} \right]$$

(9.12)

Similar results hold if we delete the barrier condition on some fixed finite interval.

Proof of Lemma 9.3. The first inequality in (9.11) is obvious. Theorem 9.1 requires that $y \leq b$ which we will not have if we go all the way to $L$. Instead, using the Markov property at $l = L - 1$ and (3.19) we bound

$$P \left[ \sqrt{2T^{1}_{y,l}} \leq \rho_{L}l + z \text{ for } l = 1, \ldots, L - 2; \sqrt{2T^{1}_{y,L}} \in I_{\rho_{L}(L-1) + z} \right]$$

$$\leq c \sum_{j=1}^{\rho_{L}(L-1) + z} P \left[ \sqrt{2T^{1}_{y,l}} \leq \rho_{L}l + z \text{ for } l = 1, \ldots, L - 2; \sqrt{2T^{1}_{y,L}} \in I_{\rho_{L}(L-1) + z} \right] e^{-j^{2}/2}.$$  

(9.13)

If $j \geq L/2$, then $e^{-j^{2}/2} \leq e^{-L^{2}/8}$ so we get a bound much smaller than (9.11). Thus we need only bound the sum over $1 \leq j \leq L/2$.

It follows from (9.2), with $a = z, b = \rho_{L}(L-1) + z, y = \rho_{L}(L-1) + z - j$ that the last probability is bounded by

$$c \frac{(1 + z)(1 + j)}{L} \sqrt{\frac{1}{L^{2}}e^{-(\rho_{L}(L-1) + z - j)^{2}/2(L-1)}}$$

$$\leq c(1 + z)(1 + j) e^{-2L-2(z-j)-(z-j)^{2}/4L},$$

and our upper bound follows after summing over $j$.

The lower bound follows similarly using (9.3). The last statement in our Lemma comes from the last statement in Theorem 9.1.

Lemma 9.4. If $k \geq L/2$, $0 \leq z \leq \log L$ and $L$ is sufficiently large, then uniformly in $0 \leq p \leq k$,

$$P \left[ \sqrt{2T^{1}_{y,l}} \leq \rho_{L}l + z \text{ for } l = 1, \ldots, k - 1; \sqrt{2T^{1}_{y,k}} \in I_{\rho_{L}k + z-p} \right]$$

$$\leq C(1 + z)(1 + p) e^{-2k-2(z-p)-(z-p)^{2}/4k}.$$  

(9.14)
Proof of Lemma 9.4: Using Theorem 9.1 with \(a = z, y = \rho_L k + z - p, b = \rho_L k + z\) this is bounded by
\[
e^{(1+z)(1+p)/k} \sqrt{\frac{1}{k^2}} e^{-(\rho_L k + z - p)^2/2k} \leq C \frac{(1+z)(1+p)}{k^2} e^{2\log(L)k/L} e^{-2k-2(z-p)-(z-p)^2/4k}.
\]

(9.15) follows since by the convexity of log we have \(e^{2\log(L)k/L} \leq e^{2\log(k)} = k^2\).

Lemma 9.5. If \(k \leq \log^5 L, 0 \leq z \leq \log L\) and \(L\) is sufficiently large, then uniformly in \(m = \rho_L k + z - t\),
\[
\mathbb{P}\left[ \sqrt{2T_{y,l}^{k,m^2/2} \leq \rho_L l + z \text{ for } l = k + 1, \ldots, L - 1; \sqrt{2T_{y,L}^{k,m^2}} \geq \rho_L L + z} \right] \\
\leq c(1 + t)m^{1/2} e^{-2(L-k)-2t-t^2/4(L-k)}.
\]

(9.16)

Proof of Lemma 9.5 As before, we can bound the probability by
\[
\sum_{j=0}^{\rho_L(L-1)+z} \mathbb{P}\left[ \sqrt{2T_{y,l}^{k,m^2/2} \leq \rho_L l + z \text{ for } l = k + 1, \ldots, L - 2; \sqrt{2T_{y,L-1}^{k,m^2}} \in I_{\rho_L(L-1)+z-j} \right] e^{-j^2/2}.
\]

(9.17)

Also, as before, we need only bound the sum over \(1 \leq j \leq L/2\).

It follows from (9.2), with the \(x\) of that estimate given by \(m = a - t\) and \(a = \rho_L k + z, b = \rho_L(L-1) + z, y = \rho_L(L-1) + z - j\) that the last probability is bounded by
\[
e^{(1+t)(1+j)/L-k} \sqrt{\frac{m}{L(L-k)}} e^{-(\rho_L(L-k-1)+t-j)^2/2(L-k)}.
\]

(9.18)

and
\[
e^{-\frac{(\rho_L(L-k-1)+t-j)^2}{2(L-k)}} \\
\leq c e^{-\frac{(\rho_L(L-k-1))^2}{2(L-k)}-2(t-j)-(t-j)^2/2L} \\
\leq C e^{2\log(L)(L-k)/L} e^{-2(L-k)-2(t-j)-(t-j)^2/2(L-k)}.
\]

(9.19)

This gives
\[
\mathbb{P}\left[ \sqrt{2T_{y,l}^{k,m^2/2} \leq \rho_L l + z \text{ for } l = k + 1, \ldots, L - 1; \sqrt{2T_{y,L}^{k,m^2}} \geq \rho_L L + z} \right] \\
\leq C \sum_{j=0}^{\rho_L(L-1)+z} \frac{(1+t)(1+j)}{L-k} \sqrt{\frac{m}{L(L-k)}} e^{2(L-k)/L} \\
e^{-2(L-k)-2(t-j)-(t-j)^2/2(L-k)} e^{-j^2/2}.
\]
(9.16) follows since by the convexity of log, \( e^{2 \log(L-k)/L} \leq e^{2 \log((L-k))} = (L-k)^2 \).

The following Lemma states the barrier estimates needed for the proof of the left tail estimates in Theorem 1.3. For notation see Section 4.3. The proof of this Lemma is similar to the proofs of Lemmas 9.3 - 9.5.

**Lemma 9.6.** For all \( L \) sufficiently large, and all \( |z| \leq \log L \),

\[
\mathbb{P} \left[ \sqrt{2T_{y,l}^1} \leq \beta_{z,-}(l) \quad \text{for } l = 1, \ldots, L - 1; \quad \sqrt{2T_{y,L}^1} \geq \rho_L L + z \right] \\
\leq c e^{-2L - 2z - z^2/(4L)}.
\]  

(9.20)

and

\[
\mathbb{P} \left[ \sqrt{2T_{y,l}^1} \leq \beta_{z,-}(l) \quad \text{for } l = 1, \ldots, L - 1; \quad \sqrt{2T_{y,L}^1} \in I_{\rho_L L + z} \right] \\
\geq c e^{-2L - 2z - z^2/(4L)}.
\]  

(9.21)

Similar results hold if we delete the barrier condition on some fixed finite interval.

If \( k \geq L/2, \ |z| \leq \log L, \) and \( L \) is sufficiently large, then uniformly in \( p \leq k \),

\[
\mathbb{P} \left[ \sqrt{2T_{y,l}^1} \leq \beta_z(l) \quad \text{for } l = 1, \ldots, k - 1; \quad \sqrt{2T_{y,k}^1} \in I_{\beta_z(k) - p} \right] \\
\leq C (1 + p) e^{-2k - 2(z-p)-(z-p)^2/4k}.
\]  

(9.22)

If \( k \leq \log^5 L, \ |z| \leq \log L, \) and \( L \) is sufficiently large, then uniformly in \( m = \beta_z(k) - t \),

\[
\mathbb{P} \left[ \sqrt{2T_{y,l}^{k,m}} \leq \beta_z(l) \quad \text{for } l = k + 1, \ldots, L - 1; \quad \sqrt{2T_{y,L}^{k,m}} \geq \rho_L + z \right] \\
\leq c(1 + t)m^{1/4} e^{-2(L-k) - 2(t^2)/(4(L-k))}.
\]  

(9.23)

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