Scale-invariance in gravity and implications for the cosmological constant

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Recently a scale invariant theory was constructed by imposing a conformal symmetry on general relativity. The imposition of this symmetry changed the configuration space from superspace - the space of all Riemannian 3-metrics modulo diffeomorphisms - to conformal superspace - the space of all Riemannian 3-metrics modulo diffeomorphisms and conformal transformations. However, despite numerous attractive features, the theory suffers from at least one major problem: the volume of the universe is no longer a dynamical variable. In attempting to resolve this problem a new theory is found which has several surprising and attractive features from both quantisation and cosmological perspectives. Furthermore, it is an extremely restrictive theory and thus may provide testable predictions quickly and easily. One particularly interesting feature of the theory is the resolution of the cosmological constant problem.

1 Introduction

Despite many promising features the scale invariant gravity theory - conformal gravity - recently proposed in [1] there is at least one major drawback. We can find the time derivative of the volume quite easily and get that it is proportional to $tr \pi$ and thus is zero. That is, the volume does not change and so the theory predicts a static universe and we cannot have expansion. This is quite a serious problem as the prediction of expansion in GR is considered to be one of the theory’s greatest achievements. We are left with the following options:

(a) Abandon the theory;
(b) Find a new explanation of the red-shift (among other things);
(c) Amend the theory to recover expansion.

The first option seems quite drastic and the second, while certainly the most dramatic, also seems to be the most difficult. Thus, let’s consider option (c).

1.1 Resolving The problem(s)

The notation used here will be the same as that used in [2]. In this notation the Lagrangian of the original theory is

$$\mathcal{L} = N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + B_{ab} B^{ab} - (tr B)^2 \right)$$

where $B_{ab}$ is the analogue of the extrinsic curvature. It is given by

$$B_{ab} = - \frac{1}{2N} \left( \partial g_{ab} \partial t - (KN)_{ab} - \theta g_{ab} \right)$$

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The constraint

\[ tr\pi = 0 \]  

arises from variation with respect to \( \theta \) and so it is here that we shall make a change. Let’s naively change the form of \( B_{ab} \) to

\[ B_{ab} = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} + (KN)_{ab} - \nabla_c \xi^c g_{ab} \right) \]  

but we keep the original form for the Lagrangian equation (1).

Let’s vary the action with respect to \( \xi^c \). We get

\[ \delta L = 2N \sqrt{\psi} g^{4} \left( 2B^{ab} - 2trBg^{ab} \right) \delta B_{ab} \]

\[ = 2N \sqrt{\psi} g^{4} \left( B^{ab} - 2trBg^{ab} \right) \left( -\frac{1}{2N} \right) \left( -\nabla_c \delta \xi^c g_{ab} \right) \]

\[ = -2N \sqrt{\psi} g^{4} trB \nabla_c \delta \xi^c \]

Integrating by parts gives

\[ \delta L = 2N \sqrt{\psi} \nabla_c (trB^{4}) \delta \xi^c \]

and so

\[ \nabla_c (trB^{4}) = 0 \]  

(This will become the constant mean curvature (CMC) condition \( \nabla_c tr\pi = 0 \) later.)

There is still one more equation which is found by varying with respect to \( \psi \) - the so-called lapse fixing equation. However, since we have the same form for \( L \) as in the original theory our lapse-fixing equation is unchanged and as a result, the constraint is not propagated unless \( tr\pi = 0 \). Thus we haven’t gained anything. We need a further change.

It will prove instructive to split \( B_{ab} \) into its trace and tracefree parts. (The reason for this will become clear quite soon.) We label the tracefree part as \( S_{ab} \). Thus we have

\[ B_{ab} = S_{ab} + \frac{1}{3} g_{ab} trB \]  

We shall retain the new form of \( B_{ab} \) as defined above in (11) all the same. The Lagrangian now reads

\[ L = N \sqrt{\psi} g^{4} \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S_{ab} S^{ab} - \frac{2}{3} (trB)^2 \right) \]  

We still need to make one further change. We’ll simply stick in an additional \( \psi \) term to the \( trB \) part. (This is equivalent to redefining our conformal transformation so that \( S_{ab} \) and \( trB \) transform in different ways.) The Lagrangian takes the form

\[ L = N \sqrt{\psi} g^{4} \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S_{ab} S^{ab} - \frac{2}{3} \psi^{n} (trB)^2 \right) \]  

Before we continue, one interesting point about \( S_{ab} \) is the following. We have

\[ S_{ab} = B_{ab} - \frac{1}{3} g_{ab} trB \]  

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Let’s write this out explicitly. We have

\[ S_{ab} = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} - \nabla_c \xi^c g_{ab} \right) + \frac{1}{3} \frac{1}{2N} g_{ab} \left( g^{cd} \frac{\partial g_{cd}}{\partial t} - g^{cd} (KN)_{cd} - 3 \nabla_c \xi^c \right) \]  

(12)

Splitting this up further gives

\[ S_{ab} = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} \right) + \frac{1}{2N} g_{ab} \nabla_c \xi^c - \frac{1}{3} g_{ab} \left( g^{cd} \frac{\partial g_{cd}}{\partial t} - g^{cd} (KN)_{cd} \right) - \frac{1}{2N} g_{ab} \nabla_c \xi^c \]  

(13)

and with a simple cancellation

\[ S_{ab} = -\frac{1}{2N} \left( \frac{\partial g_{ab}}{\partial t} - (KN)_{ab} \right) - \frac{1}{3} g_{ab} \left( g^{cd} \frac{\partial g_{cd}}{\partial t} - g^{cd} (KN)_{cd} \right) \]  

(14)

Of course, this is

\[ S_{ab} = K_{ab} - \frac{1}{3} g_{ab} \text{tr} K \]  

(15)

That is, \( S_{ab} \) is the tracefree part of the extrinsic curvature and is independent of any conformal fields.

Let us find \( \pi^{ab} \). This is done as usual by varying with respect to \( \frac{\partial g_{ab}}{\partial t} \). We get

\[ \delta \mathcal{L} = 2N \sqrt{\psi} \psi^4 \left( 2 S^{ab} \delta S_{ab} - \frac{4}{3} \psi^n \text{tr} B g^{ab} \delta B_{ab} \right) \]

\[ = 2N \sqrt{\psi} \psi^4 \left( S^{ab} \left( \delta B_{ab} - \frac{1}{3} g_{ab} g^{cd} \delta B_{cd} \right) - \frac{2}{3} \psi^n \text{tr} B g^{ab} \delta B_{ab} \right) \]

\[ = 2N \sqrt{\psi} \psi^4 \left( S^{ab} - \frac{2}{3} \psi^n \text{tr} B g^{ab} \right) \delta B_{ab} \]

\[ = -\sqrt{\psi} \psi^4 \left( S^{ab} - \frac{2}{3} \psi^n \text{tr} B g^{ab} \right) \delta \frac{\partial g_{ab}}{\partial t} \]

Thus,

\[ \pi^{ab} = -\sqrt{\psi} \psi^4 S^{ab} + \frac{2}{3} \sqrt{\psi} \psi^{n+4} g^{ab} \text{tr} B \]

(17)

Splitting \( \pi^{ab} \) into its trace and tracefree parts will further clear things up. We’ll label the split as

\[ \pi^{ab} = \sigma^{ab} + \frac{1}{3} g^{ab} \text{tr} \pi \]  

(18)

Thus the tracefree part of \( \pi^{ab} \) is

\[ \sigma^{ab} = -\sqrt{\psi} \psi^4 S^{ab} \]  

(19)

and the trace is given by

\[ \text{tr} \pi = 2\psi^{n+4} \text{tr} B \]  

(20)

Note that our value of \( n \) is undefined as yet.

The constraints are found by varying with respect to \( \xi^c, \psi, N \) and \( N^a \). The conformal constraint and the lapse-fixing equation are given by varying with respect to \( \xi^c \) and \( \psi \) respectively. These give

\[ \nabla_c \text{tr} \pi = 0 \]  

(21)

and

\[ N \psi^3 \left( R - 7 \nabla^2 \psi \frac{\psi^3}{\psi^4} \right) - \nabla^2 (N \psi^3) + \frac{(\text{tr} p)^2}{4} = 0 \]  

(22)
respectively. From the variation with respect to \( N \) we get

\[
S_{ab}S^{ab} - \frac{2}{3}\psi^n (trB)^2 - g\psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0
\]  

(23)

which in terms of the momentum is

\[
\sigma_{ab}\sigma^{ab} - \frac{1}{6}\psi^{-n} (tr\pi)^2 - g\psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0
\]  

(24)

and finally, from the variation with respect to \( N^a \) we get

\[
\nabla_b \pi^{ab} = 0
\]  

(25)

We require conformal invariance in our constraints. Under what conditions is the momentum constraint (25) invariant? The tracefree part of the momentum, \( \sigma^{ab} \), has a natural weight of \(-4\) (from the original theory). That is

\[
\sigma^{ab} \rightarrow \omega^{-4} \sigma^{ab}
\]  

(26)

If \( tr\pi = 0 \) then we have conformal invariance. If not however, we require various further conditions. We need

\[
\nabla_b \sigma^{ab} = 0
\]  

(27)

\[
\nabla_c tr\pi = 0
\]  

(28)

and that

\[
trp = \frac{tr\pi}{\sqrt{g}} \rightarrow trp
\]  

(29)

under a conformal transformation. In our theory we have the first two conditions emerging directly and naturally from the variation. Thus we simply define \( trp \) to transform as a conformal scalar as required. With this done our momentum constraint is conformally invariant.

Transforming the constraint (24) gives

\[
\sigma_{ab}\sigma^{ab} - \frac{1}{6}\psi^{-n} g(trp)^2\omega^{12+n} - g\psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0
\]  

(30)

and so we must have \( n = -12 \) for conformal invariance. The constraint then becomes

\[
\sigma_{ab}\sigma^{ab} - \frac{1}{6}\psi^{12} g(trp)^2 - g\psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0
\]  

(31)

This is exactly the Lichnerowicz-York equation from GR. However, we have found it directly from a variational procedure.

Thus we have determined the unique value of \( n \) and our constraints are

\[
\sigma_{ab}\sigma^{ab} - \frac{1}{6}\psi^{12} (tr\pi)^2 - g\psi^8 \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0
\]  

(32)

\[
\nabla_b \pi^{ab} = 0
\]  

(33)
\[ \nabla_c tr\pi = 0 \]  

(34)

Thus we have found the exact constraints of the York method [3] directly from a variational procedure. This is quite novel. We also have a lapse fixing equation

\[ N\psi^3 \left( R - \frac{7}{8} \nabla^2 \psi \right) - \nabla^2 (N\psi^3) + \frac{(tr\pi)^2}{4} = 0 \]  

(35)

It turns out (as we shall show later) that this condition enforces propagation of the constraint (34).

Let’s proceed to the Hamiltonian formulation.

2 The Hamiltonian Formulation

The earlier expression for \( \pi_{ab} \) can be inverted to get \( \frac{\partial g_{ab}}{\partial t} \). We get

\[ \frac{\partial g_{ab}}{\partial t} = \frac{2N}{\sqrt{g} \psi^4} \left( \sigma_{ab} - \frac{1}{6} g_{ab} tr\pi \psi^{12} \right) + (KN)_{ab} + g_{ab} \nabla_c \xi_c \]  

(36)

The Hamiltonian may then be found in the usual way. We get

\[ H = \frac{N}{\sqrt{g} \psi^4} \left[ \sigma^{ab} \sigma_{ab} - \frac{1}{6} (tr\pi)^2 \psi^{12} - g_{ab} \left( R - \frac{8}{8} \nabla^2 \psi \right) \right] - 2N_a \nabla_b \pi^{ab} - \xi^c \nabla_c tr\pi \]  

(37)

As a consistency check let’s find \( \frac{\partial g_{ab}}{\partial t} \) from this by varying with respect to \( \pi_{ab} \). We get exactly equation (36) again. Thus, all is well. We may do all the usual variations here to get the constraints. Varying the Hamiltonian with respect to \( g_{ab} \) gives us our evolution equation for \( \pi_{ab} \). We get

\[ \frac{\partial \pi^{ab}}{\partial t} = -N \sqrt{g} \psi^4 \left( R^{ab} - g^{ab} \left( R - \frac{8}{8} \nabla^2 \psi \right) \right) \]

\[ - \frac{2N}{\sqrt{g} \psi^4} \left( \pi^{ac} \pi^b_c - \frac{1}{3} \pi^{ab} tr\pi - \frac{1}{6} \pi^{ab} tr\pi \psi^{12} \right) \]

\[ + \sqrt{g} \psi \left( \nabla^a \nabla^b \left( N\psi^3 \right) - g^{ab} \nabla^2 \left( N\psi^3 \right) \right) \]

\[ + N \sqrt{g} \psi^3 \left( \nabla^a \nabla^b \psi + 3g^{ab} \nabla^2 \psi \right) \]

\[ + 4g^{ab} \sqrt{g} \nabla_c \left( N\psi^3 \right) \nabla^c \psi - 6 \sqrt{g} \nabla^a \left( N\psi^3 \right) \nabla^b \psi \]

\[ + \nabla_c \left( \pi^{ab} N^c \right) - \pi^{ab} \nabla_c N^a - \pi^{ac} \nabla_c N^b \]

\[ - \frac{\left( \pi^{ab} - \frac{1}{2} g^{ab} tr\pi \right)}{g^{ab}} \nabla_c \xi^c \]  

(38)

We may use the evolution equations to find \( \frac{\partial \pi^{ab}}{\partial t} \) quite easily. (Of course, we need the evolution equations to propagate all of the constraints. We will deal with the others later.) We find that

\[ \frac{\partial tr\pi}{\partial t} = 0 \]  

(39)

using the lapse-fixing equation. Thus we have that \( tr\pi = \text{constant both spatially and temporally}!! \) (It is constant spatially since the densitised momentum \( \pi_{ab} \) is covariantly constant.

\[ 0 = \nabla_c tr\pi = \frac{1}{\sqrt{g}} \nabla_c tr\pi \]  

(40)
Thus

\[ trp_{c} = 0. \]  \hspace{1cm} (41)

We could proceed to check propagation of the constraints here but it will be easier and more instructive to do a little more work first.

Since \( trp \) is identically a constant our dynamical data consists of \( g_{ab} \) and \( \sigma^{ab} \). Thus, it may prove useful to have an evolution equation for \( \sigma^{ab} \) rather than the full \( \pi^{ab} \). It is reasonably straightforward to do this. Firstly we note that

\[
\frac{\partial \sigma^{ab}}{\partial t} = \frac{\partial \pi^{ab}}{\partial t} - \frac{1}{3} \frac{\partial g^{ab}tr\pi}{\partial t} \]  \hspace{1cm} (42)

Working through the details gives us

\[
\frac{\partial \sigma^{ab}}{\partial t} = -N\sqrt{g}\psi^{4}\left(R^{ab} - \frac{1}{3}g^{ab}\left(R - 8\frac{\nabla^{2}\psi}{\psi}\right)\right) - \frac{2N}{\sqrt{g}\psi^{4}}\sigma^{ac}\sigma^{bc} \\
+ \sqrt{g}\psi\left(\nabla^{a}\nabla^{b}(N\psi^{3}) - \frac{1}{3}g^{ab}\nabla^{2}(N\psi^{3})\right) \\
+ N\sqrt{g}\psi^{3}\left(\nabla^{a}\nabla^{b}\psi + \frac{7}{3}g^{ab}\nabla^{2}\psi\right) \\
+ 4g^{ab}\sqrt{g}\nabla^{c}(N\psi^{3})\nabla^{b}\psi - 6\sqrt{g}\nabla^{(a}(N\psi^{3})\nabla^{b)}\psi \\
+ \nabla_{c}(\sigma^{ab}N^{c}) - \sigma^{bc}\nabla_{c}N^{a} - \sigma^{ac}\nabla_{c}N^{b} \\
- \sigma^{ab}\nabla_{c}\xi^{c} + \frac{N\psi^{8}}{3\sqrt{g}}\sigma^{ab}tr\pi \]  \hspace{1cm} (43)

3 Jacobi Action

In 1962 Baierlein, Sharp and Wheeler (BSW) \[4\] constructed a Jacobi action for G.R. It was of the form

\[
I = \int d\lambda \int \sqrt{g} \sqrt{R} \sqrt{T} d^{3}x, \]  \hspace{1cm} (44)

where the ‘kinetic energy’ \( T \) is

\[
T = (g^{ac}g^{bd} - g^{ab}g^{cd}) \left( \frac{\partial g_{ab}}{\partial \lambda} - (KW)_{ab} \right) \left( \frac{\partial g_{cd}}{\partial \lambda} - (KW)_{cd} \right). \]  \hspace{1cm} (45)

This action reproduces the standard Einstein equations in the Arnowitt-Deser-Misner form \[5\] with lapse \( N = \sqrt{T/4R} \). We can also find the Jacobi action for the new theory. Recall the (3+1) Lagrangian,

\[
\mathcal{L} = N\sqrt{g}\psi^{4}\left(R - 8\frac{\nabla^{2}\psi}{\psi} + S^{ab}S_{ab} - \frac{2}{3}\psi^{-12}(trB)^{2}\right) \]  \hspace{1cm} (46)

We can write this as

\[
\mathcal{L} = \sqrt{g}\psi^{4}\left[N\left(R - 8\frac{\nabla^{2}\psi}{\psi}\right) + \frac{1}{4N}\left(\Sigma^{ab}\Sigma_{ab} - \frac{2}{3}\psi^{-12}(tr\beta)^{2}\right)\right] \]  \hspace{1cm} (47)

where \( \Sigma_{ab} = -2NS_{ab} \) and \( \beta_{ab} = -2NB_{ab} \). We now extremise with respect to \( N \). This gives us,

\[
N = \frac{1}{\psi^{4}}\left(\Sigma^{ab}\Sigma_{ab} - \frac{2}{3}\psi^{-12}(tr\beta)^{2}\right)^{1/2} \left(R - 8\frac{\nabla^{2}\psi}{\psi}\right)^{-1/2} \]  \hspace{1cm} (48)
Substituting this back into the action gives us

\[ S = \pm \int d\lambda \int \sqrt{g} \psi^4 \sqrt{R - 8 \frac{\nabla^2 \psi}{\psi}} \sqrt{T} d^3 x \]  

(49)

where \( T = \left( \Sigma^{ab} \Sigma_{ab} - \frac{2}{3} \psi^{-12} (tr \beta)^2 \right) \).

We can do all the usual variations here: \( N^a, \xi^c \) and \( \psi \). These give the momentum constraint, the conformal constraint and the lapse-fixing condition respectively. When we find the canonical momentum \( \pi^{ab} \) we can “square” it to give the “Hamiltonian constraint.”

So far, so good. We shall rarely use the Jacobi form of the action here but from a thin-sandwich point of view it is important and may well be of use in future work. Let’s move on.

4 Conformally Related Solutions

In conformal superspace conformally related metrics are equivalent. Thus, conformally related solutions of this theory must be physically equivalent and so it is crucial that we have a natural way to relate such solutions. Suppose we start with initial data \( \{ g_{ab}, \sigma^{ab}, trp \} \) obeying the initial data conditions (33) and (34). We then solve (32) for \( \psi \).

Suppose instead that we start with the conformally related initial data \( \{ h_{ab}, \rho^{ab}, trp \} = \{ \alpha^4 g_{ab}, \alpha^{-4} \sigma^{ab}, trp \} \). These automatically satisfy the initial data conditions by the conformal invariance. We now solve the Hamiltonian constraint for the conformal “field” \( \chi \), say. Exactly as in [2] it can be shown that \( \chi = \frac{\psi}{\alpha} \).

Thus,

\[ \psi^4 g_{ab} = \chi^4 h_{ab} \]  

(50)

and

\[ \psi^{-4} \sigma^{ab} = \chi^{-4} \rho^{ab} \]  

(51)

Let us label these as \( \tilde{g}_{ab} \) and \( \tilde{\rho}^{ab} \) and put a hat over \( trp \) also (for clarity). Thus a tilde over a quantity denotes the physical value of this quantity. It is very remarkable that we find not only a physical momentum, which is precisely analogous to the physical gauge-invariant electric field in Maxwellian theory, but also to a physical \( g_{ab} \). This has no analogue in Maxwell and Yang–Mills, in which the vector-potential velocity \( \dot{A}_k \) is gauge-corrected by the scalar potential, yielding the gauge-invariant electric field \( E \), but \( A_k \) itself retains irremovable gauge degeneracy.

The constraints become

\[ \tilde{\sigma}_{ab} \tilde{\rho}^{ab} - \frac{1}{6} (tr \pi)^2 - \tilde{g} \tilde{R} = 0 \]  

(52)

\[ \tilde{\nabla}^b \tilde{\pi}^{ab} = 0 \]  

(53)
\[
\n\tilde{\nabla} r \pi = 0 \tag{54}
\]
\[
\tilde{N} \tilde{R} - \tilde{\nabla}^2 \tilde{N} + \frac{(trp)^2}{4} = 0 \tag{55}
\]

Consider GR in the CMC gauge. The constraints are
\[
\sigma_{ab} \sigma^{ab} - \frac{1}{6} (tr\pi)^2 - gR = 0 \tag{56}
\]
\[
\nabla_b \pi^{ab} = 0 \tag{57}
\]
\[
\nabla_c tr\pi = 0 \tag{58}
\]

Evolution of the CMC condition gives
\[
NR - \nabla^2 N + \frac{(trp)^2}{4} = C \tag{59}
\]

The similarities are quite striking.

4.1 What of \(\xi^c\)?

Precious little has been revealed about what \(\xi^c\) may be or even how it transforms. This needs to be addressed. First let’s recall that we demanded that
\[
trB \rightarrow \omega^{-8} trB \tag{60}
\]
under a conformal transformation. This will be enough to reveal the transformation properties of \(\xi^c\).

Taking the trace gives us
\[
trB = -\frac{1}{2N} \left( g^{ab} \frac{\partial g_{ab}}{\partial t} - g^{ab} (KN)_{ab} - 3 \nabla_c \xi^c \right) \tag{61}
\]

Under a conformal transformation we get
\[
\omega^{-8} trB = -\frac{1}{2\omega^2 N} \left( g^{ab} \frac{\partial g_{ab}}{\partial t} + 12 \frac{\dot{\omega}}{\omega} - \omega^{-4} g^{ab} \left( \omega^4 (KN)_{ab} + 4 \omega^3 \omega_{,c} N^c g_{ab} \right) - 3 \tilde{\nabla}_c \xi^c \right) \tag{62}
\]
\[
= -\frac{1}{2\omega^2 N} \left( g^{ab} \frac{\partial g_{ab}}{\partial t} - g^{ab} (KN)_{ab} - 3 \nabla_c \xi^c \right)
\[= \frac{3}{2\omega^2 N} \nabla_c \xi^c + \frac{3}{2\omega^2 N} \tilde{\nabla}_c \tilde{\xi}^c - \frac{6}{\omega^3 N} \left( \dot{\omega} - \omega_{,c} N^c \right) \tag{62}
\]
\[= \omega^{-2} trB + \frac{3}{2\omega^2 N} \left( \tilde{\nabla}_c \tilde{\xi}^c - 3 \nabla_c \xi^c - \frac{4}{\omega} \left( \dot{\omega} - \omega_{,c} N^c \right) \right) \]

Thus,
\[
\frac{3}{2\omega^2 N} \left( \tilde{\nabla}_c \tilde{\xi}^c - 3 \nabla_c \xi^c - \frac{4}{\omega} \left( \dot{\omega} - \omega_{,c} N^c \right) \right) = -\frac{1}{\omega^2 N} trB \left( 1 - \omega^{-6} \right) \tag{63}
\]
and so
\[ \bar{\nabla}_c \bar{\xi}_c = \nabla_c \xi_c + \frac{4}{\omega} (\dot{\omega} - \omega_c N^c) - \frac{2N}{3} trB \left( 1 - \omega^{-6} \right) \] (64)

This tells us how things transform but not what \( \xi_c \) itself actually is. We can find this though.

Let’s write the evolution equations in the physical representation. It can be verified that they are

\[ \frac{\partial \bar{g}_{ab}}{\partial t} = \frac{2\bar{N}}{\sqrt{\bar{g}}} \left( \sigma_{ab} - \frac{1}{6} \bar{g}_{ab} \bar{tr} \bar{\pi} \right) + \left( \bar{K} \bar{N} \right)_{ab} + \bar{g}_{ab} \bar{\nabla}_c \bar{\xi}_c \] (65)

and

\[ \frac{\partial \bar{\sigma}_{ab}}{\partial t} = -\bar{N} \sqrt{\bar{g}} \left( \bar{R}_{ab} - \frac{1}{3} \bar{g}_{ab} \bar{R} \right) - \frac{2\bar{N}}{\sqrt{\bar{g}}} \bar{\sigma}_{ac} \bar{\sigma}^c_b \\
+ \sqrt{\bar{g}} \left( \bar{\nabla}_c \bar{\nabla}^b \bar{N} - \frac{1}{3} \bar{g}_{ab} \bar{\nabla}^2 \bar{N} \right) \]
\[ + \bar{\nabla}_c (\bar{\sigma}_{ab} \bar{\nabla}^c \bar{N}) - \bar{\sigma}_{bc} \bar{\nabla}_c \bar{N}^a - \bar{\sigma}_{ac} \bar{\nabla}_c \bar{N}^b \]
\[ + \frac{\bar{N}}{3\sqrt{\bar{g}}} \bar{\sigma}_{ab} \bar{tr} \bar{\pi} - \bar{\sigma}_{ab} \bar{\nabla}_c \bar{\xi}_c \] (66)

We require the evolution equations to propagate the constraints. However, when we check this it turns out that we are forced to set \( \bar{\nabla}_c \bar{\xi}_c \) to zero. However, this means that we have

\[ \nabla_c \xi_c + \frac{4}{\omega} (\dot{\psi} - \psi_c N^c) - \frac{2N}{3} trB (1 - \psi^{-6}) = 0 \] (67)

by (64). Thus we have

\[ \nabla_c \xi_c = -\frac{4}{\psi} (\dot{\psi} - \psi_c N^c) + \frac{2N}{3} trB (1 - \psi^{-6}) \] (68)

That is,

\[ \nabla_c \xi_c = \theta + \frac{2N}{3} trB (1 - \psi^{-6}) \] (69)

where \( \theta \) is as in the original theory. Thus, the exact form of \( \xi_c \) is determined. We needed \( \nabla_c \xi_c \) to be zero in the physical representation for constraint propagation and so we should check that this is the case with our newly found expression for \( \nabla_c \xi_c \). We can check this easily. In the physical representation \( \theta = 0 \) and \( \psi = 1 \). Thus, we do have that \( \nabla_c \xi_c \) is zero.

It is vital to note that this is strictly a POST-VARIATION identification. If we use this form for \( \xi_c \) in the action we will run into problems, not least an infinite sequence in the variation of \( trB \) with respect to \( \xi_c \). (This is because we would have \( trB \) defined in terms of \( trB \) itself.) We see that \( \xi_c \) is intimately related with how \( \psi \) changes from slice to slice.

Our constraints in the physical representation are

\[ \sigma_{ab} \sigma^{ab} - \frac{1}{6} (tr \pi)^2 - gR = 0 \] (70)

\[ \nabla_b \pi^{ab} = 0 \] (71)
\[ \nabla c \pi = 0 \quad (72) \]

\[ NR - \nabla^2 N + \frac{N(trp)^2}{4} = 0 \quad (73) \]

and our evolution equations are

\[ \frac{\partial g_{ab}}{\partial t} = \frac{2N}{\sqrt{g}} \left( \sigma_{ab} - \frac{1}{6} g_{ab} tr \pi \right) + (KN)_{ab} \quad (74) \]

and

\[ \frac{\partial \sigma^{ab}}{\partial t} = -N \sqrt{g} \left( R^{ab} - \frac{1}{3} g^{ab} R \right) - \frac{2N}{\sqrt{g}} \sigma^{ac} \sigma^{b}_c 
+ \sqrt{g} \left( \nabla^a \nabla^b N - \frac{1}{3} g^{ab} \nabla^2 N \right) 
+ \nabla_c (\sigma^{ab} N^c) - \sigma^{bc} \nabla_c N^a - \sigma^{ac} \nabla_c N^b 
+ \frac{N}{3 \sqrt{g}} \sigma^{ab} tr \pi \quad (75) \]

(The hats are removed for simplicity.) These are identical to those in GR in the CMC gauge (with \( trp \) a temporal constant).

## 5 Topological Considerations

In the original theory it was found that if the manifold is compact without boundary we get frozen dynamics. In this problematic case we can resolve the issue in much the same manner as with the original theory although, it is a little more complicated this time.

### 5.1 Integral Inconsistencies

The root of the integral inconsistency is in the lapse-fixing equation. If we integrate this equation we find that the only solution is \( N \equiv 0 \). That is, we have frozen dynamics. The resolution to this in the original conformal theory was to introduce a volume term in the denominator of the Lagrangian. Actually, the key is to keep the Lagrangian homogeneous in \( \psi \) using different powers of the volume. The volume of a hypersurface here is given by

\[ V = \int \sqrt{g} \psi^6 \, d^3 x \quad (76) \]

In the original theory the Lagrangian has an overall factor of \( \psi^4 \) and so we need to divide by \( V^{2/3} \) to keep homogeneity in \( \psi \). There is no such overall factor in the new theory and so it is not as straightforward. The key is to treat the two parts of the Lagrangian separately. We try

\[ \mathcal{L}_1 = \frac{N \sqrt{g} \psi^4}{V^n} \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S^{ab} S_{ab} \right) \quad (77) \]

and

\[ \mathcal{L}_2 = -\frac{2N \sqrt{g}}{3 V^m} (tr B)^2 \quad (78) \]
and we determine \( n \) and \( m \) from the homogeneity requirement. Thus we have that \( n = \frac{2}{3} \) and \( m = -\frac{4}{3} \).

Using this result our Lagrangian is now

\[
L = \frac{N\sqrt{g} \psi^4}{V^{2/3}} \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S^a b S_{ab} - \frac{2}{3} \psi^{-12} (tr B)^2 V^2 \right)
\]

(79)

### 5.2 New Constraints

We go about things in exactly the same manner as before. The momentum is found to be

\[
\pi^{ab} = -\frac{\sqrt{g} \psi^4}{V^{2/3}} S^{ab} + \frac{2}{3} \sqrt{g} \psi^{-8} V^{4/3} g^{ab} tr B
\]

(80)

The constraints are (almost) unchanged. They are

\[
\sigma_{ab} \sigma^{ab} - \frac{\psi^{12} (tr \pi)^2}{6 V^2} - \frac{g \psi^8}{V^{4/3}} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) = 0
\]

(81)

\[
\nabla_b \pi^{ab} = 0
\]

(82)

\[
\nabla_c tr \pi = 0
\]

(83)

The lapse-fixing equation is

\[
\frac{N\sqrt{g} \psi^3}{V^{2/3}} \left( R - 7 \frac{\nabla^2 \psi}{\psi} \right) - \frac{\sqrt{g}}{V^{2/3}} \nabla^2 (N \psi^3) - \frac{\sqrt{g}}{2 V^{2/3}} \nabla G N \psi^{-9} (tr B)^2 V^{4/3} - \frac{2}{3} \sqrt{g} D \psi V^{4/3} = 0
\]

(84)

where

\[
C = \int \frac{N\sqrt{g} \psi^3}{V} \left( R - 8 \frac{\nabla^2 \psi}{\psi} + S^{ab} S_{ab} \right) d^3 x
\]

(85)

and

\[
D = \int \frac{N\sqrt{g} \psi^{-8}}{V} (tr B)^2 d^3 x
\]

(86)

The \( C \) and \( D \) terms result from the variations of the volume. Rearranging the lapse-fixing equation we get

\[
\frac{N\sqrt{g} \psi^3}{V^{2/3}} \left( R - 7 \frac{\nabla^2 \psi}{\psi} \right) - \frac{\sqrt{g}}{V^{2/3}} \nabla^2 (N \psi^3) - \frac{\sqrt{g}}{2 V^{2/3}} \nabla G N \psi^{-9} (tr B)^2 V^{4/3} = \frac{\sqrt{g} \psi^5}{2 V^{2/3}} \left( C + \frac{4}{3} D V^2 \right)
\]

(87)

Integrating across this expression gives no problem. The inconsistency has been removed.

### 6 The Hamiltonian Formulation

We should consider the evolution equations again now that we have changed the action. First of all the momentum is now given by (80). The new Hamiltonian is

\[
H = \frac{N V^{2/3}}{\sqrt{g} \psi^4} \left[ \sigma^{ab} \sigma_{ab} - \frac{\psi^{12} (tr \pi)^2}{6 V^2} - \frac{g \psi^8}{V^{4/3}} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \right] - 2 N a \ nabla_b \pi^{ab} - \xi^c \ nabla_c tr \pi
\]

(88)

The evolution equations are then

\[
\frac{\partial g_{ab}}{\partial t} = \frac{2 N V^{2/3}}{\sqrt{g} \psi^4} \left( \sigma_{ab} - \frac{g_{ab} (tr \pi)^2}{6 V^2} \right) + \left( KN \right)_{ab} + g_{ab} \nabla_c \xi^c
\]

(89)
\begin{align}
\frac{\partial \sigma_{ab}}{\partial t} &= -\frac{N}{\sqrt{g}V^{2/3}} \left( R_{ab} - g_{ab} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \right) - 2NV^{2/3} \sqrt{g} \frac{2}{3g_{ab} tr \pi} - \frac{\nabla a \nabla b \left( N \psi^3 \right) - g_{ab} \nabla^2 \left( N \psi^3 \right) 12}{6V^2} \\
&+ \frac{\sqrt{g} \psi}{V^{2/3}} \left( \nabla a \nabla b \psi + 3g_{ab} \nabla^2 \psi \right) + 4\frac{\sqrt{g}}{V^{2/3}} g^{ab} \nabla c \left( N \psi^3 \right) \nabla c \psi - 6\frac{\sqrt{g}}{V^{2/3}} \nabla (a \left( N \psi^3 \right) \nabla b) \psi \\
&+ \nabla c \left( \sigma^{ab} N^c \right) - \sigma^{bc} \nabla c N^a - \sigma^{ac} \nabla c N^b \\
&- \frac{1}{2}g^{ab} tr \pi \nabla c \xi^c - \frac{2}{3} \frac{\sqrt{g} \psi^6 g_{ab}}{V^{2/3}} C \\
\end{align}

where

\begin{align}
C &= \left\langle N \sqrt{g} \psi^4 \left( R - 8 \frac{\nabla^2 \psi}{\psi} + \frac{\psi^4 (tr p)^2}{4 V^{2/3}} \right) \right\rangle \\
\end{align}

and \( \left\langle A \right\rangle \) is the usual notion of global average given by

\begin{align}
\left\langle A \right\rangle &= \frac{\int \sqrt{g} A \, d^3x}{\int \sqrt{g} \, d^3x} \\
\end{align}

We can again take the time derivative of \( tr p \) and find yet again that

\begin{align}
\frac{\partial tr p}{\partial t} &= 0
\end{align}

Thus, our dynamic data will once again be \( \{ g_{ab}, \sigma^{ab} \} \) and so we want to find the evolution equation for \( \sigma^{ab} \) again. Slogging through we get

\begin{align}
\frac{\partial \sigma_{ab}}{\partial t} &= -\frac{N}{\sqrt{g}V^{2/3}} \left( R_{ab} - g_{ab} \left( R - 8 \frac{\nabla^2 \psi}{\psi} \right) \right) - 2NV^{2/3} \sqrt{g} \frac{2}{3g_{ab} tr \pi} - \frac{\nabla a \nabla b \left( N \psi^3 \right) - g_{ab} \nabla^2 \left( N \psi^3 \right) 12}{6V^2} \\
&+ \frac{\sqrt{g} \psi}{V^{2/3}} \left( \nabla a \nabla b \psi + 3g_{ab} \nabla^2 \psi \right) + 4\frac{\sqrt{g}}{V^{2/3}} g^{ab} \nabla c \left( N \psi^3 \right) \nabla c \psi - 6\frac{\sqrt{g}}{V^{2/3}} \nabla (a \left( N \psi^3 \right) \nabla b) \psi \\
&+ \nabla c \left( \sigma^{ab} N^c \right) - \sigma^{bc} \nabla c N^a - \sigma^{ac} \nabla c N^b \\
&- \frac{1}{2}g^{ab} tr \pi \nabla c \xi^c + \frac{N \psi^8}{3\sqrt{g}V^{4/3}} \sigma^{ab} tr \pi \\
\end{align}

Note that the term with \( C \) has dropped out.

The physical representation is achieved either by the naive substitution of \( \psi = 1 \) and \( \nabla c \xi^c = 0 \) or by doing it the longer more correct way. The result is the same in either case. The momentum is

\begin{align}
\pi^{ab} &= -\frac{\sqrt{g}}{V^{2/3}} S^{ab} + \frac{2}{3} \frac{\sqrt{g} g^{ab} tr \pi K V^{4/3}}{V^{2/3}} \\
\end{align}
Thus
\[ \sigma^{ab} = -\sqrt{g} \frac{V^2}{2^{1/3}} S^{ab} \text{ and } tr\pi = 2\sqrt{g}(trK)V^{4/3} \] (96)

The constraints are
\[ \sigma^{ab}\sigma_{ab} - \frac{1}{6} (tr\pi)^2 = gR \frac{V^{4/3}}{V^4} \] (97)
\[ \nabla_b \pi^{ab} = 0 \] (98)
\[ \nabla_c trp = 0 \] (99)
\[ NR - \nabla^2 N + \frac{N(trp)^2}{4V^{2/3}} = C \] (100)

where we now have \( C = \left\langle N \left( R + \frac{(trp)^2}{4V^{2/3}} \right) \right\rangle \). The evolution equations are
\[ \frac{\partial g_{ab}}{\partial t} = \frac{2NV^{2/3}}{\sqrt{g}} \left( \sigma_{ab} - \frac{g_{ab} tr\pi}{6V^2} \right) + (KN)_{ab} \] (101)
and
\[ \frac{\partial \sigma^{ab}}{\partial t} = -\frac{N}{V^{2/3}} \left( R^{ab} - \frac{1}{3} \delta^{ab} R \right) - \frac{2NV^{2/3}}{\sqrt{g}} \sigma^{ac} \sigma_{bc} 
+ \frac{\sqrt{g}}{V^{2/3}} \left( \nabla^a \nabla^b N - \frac{1}{3} \delta^{ab} \nabla^2 N \right) 
+ \nabla_c \left( \sigma^{ab} N^c \right) - \sigma^{bc} \nabla_c N^a - \sigma^{ac} \nabla_c N^b 
+ \frac{N}{3\sqrt{g}V^{4/3}} \sigma^{ab} tr\pi \] (102)

7 The Volume

This theory was inspired by the need to recover expansion. After all this work, have we succeeded? The time derivative of the volume is
\[ \frac{\partial V}{\partial t} = \int \frac{1}{2} \sqrt{g} g^{ab} \frac{\partial g_{ab}}{\partial t} \, d^3x 
= -\int \frac{1}{2} N \frac{\sqrt{g} trp}{V^{4/3}} \, d^3x 
= -\frac{trp \langle N \rangle}{2V^{1/3}} \] (103)

Thus, we have recovered expansion. The big test of the compact without boundary theory presented here will be to study the cosmological solutions and this will be the focus of later work. Later in this paper we shall examine the consequences for the cosmological constant.
8 Jacobi Action

For completeness let’s find the Jacobi action for the compact theory. Without going through each step let’s simply require homogeneity in $\psi$. Recall that the Jacobi action for the non-compact theory was given by

$$S = \pm \int d\lambda \int \sqrt{g} \psi^4 \sqrt{R - 8 \frac{\nabla^2 \psi}{\psi}} \sqrt{T} d^3x$$

(49) where $T = \left( \Sigma^{ab} \Sigma_{ab} - \frac{2}{3} \psi^{-12} (tr \beta)^2 \right)$ where $\Sigma^{ab} = -2N S^{ab}$ and $\beta^{ab} = -2N B^{ab}$. Applying the homogeneity requirement gives

$$S = \pm \int d\lambda \int \sqrt{g} \psi^4 \sqrt{R - 8 \frac{\nabla^2 \psi}{\psi}} \sqrt{T} d^3x$$

(104)

where $T = \left( \Sigma^{ab} \Sigma_{ab} - \frac{2}{3} \psi^{-12} (tr \beta)^2 V^2 \right)$. Everything else emerges as before.

9 Comparison with GR

In the earlier “static” conformal theory we saw that the labelling

$$\hat{\pi}^{ab} = V^{2/3} \pi^{ab}$$

(105)

made the theory appear incredibly similar to GR. A similar labelling is possible here. Define

$$\hat{\sigma}^{ab} = V^{2/3} \sigma^{ab}$$

(106)

and

$$\hat{tr} \pi = \frac{tr \pi}{V^{1/3}}$$

(107)

With this rebelling the constraints are

$$\hat{\sigma}^{ab} \hat{\sigma}_{ab} - \frac{1}{6} (tr \hat{\pi})^2 = gR$$

(108)

$$\nabla_b \hat{\pi}^{ab} = 0$$

(109)

$$\nabla_c \hat{\pi}^{ab} = 0$$

(110)

and the lapse-fixing equation is

$$NR - \nabla^2 N + \frac{N (tr \hat{\pi})^2}{4} = C$$

(111)

where $C = \left\langle N \left( R + \frac{(tr \hat{\pi})^2}{4} \right) \right\rangle$. These are identical to GR in the CMC gauge. The evolution equations are

$$\frac{\partial g_{ab}}{\partial t} = \frac{2N}{\sqrt{g}} \left( \hat{\sigma}_{ab} - \frac{g_{ab} tr \hat{\pi}}{6V} \right) + (KN)_{ab}$$

(112)
and
\[
\frac{\partial \hat{\sigma}_{ab}}{\partial t} = V^{2/3} \frac{\partial \sigma_{ab}}{\partial t} + \frac{2}{3V^{1/3}} \frac{\partial V}{\partial t} = -N \sqrt{g} \left( R_{ab} - \frac{1}{3} g_{ab} R \right) - \frac{2N}{\sqrt{g}} \hat{\sigma}_{ac} \hat{\sigma}_{bc} \\
+ \sqrt{g} \psi \left( \nabla^a \nabla^b N - \frac{1}{3} g_{ab} \nabla^2 N \right) \\
+ \nabla_c (\sigma_{ab} N^c) - \sigma_{bc} \nabla_c N^a - \sigma_{ac} \nabla_c N^b \\
+ \left( N - \langle N \rangle \right) \langle \hat{\sigma}_{ab} \rangle \pi^{tr}_{\pi} 
\]
(113)

There are very few differences between these and those of GR (74) and (75).

10 Constraint Propagation

Of course, for consistency, we need the constraints to be preserved in time. It turns out that we have this here with one final restriction. The scalar curvature must be spatially constant. This condition is enough then for full constraint preservation.

11 Time

In his work on the initial value formulation of general relativity York [3] introduced the following time parameter (the York time)
\[
\tau = \frac{2}{3} tr p 
\]
(114)

In this theory we have that tr p is identically constant. Thus it cannot be used as a notion of time. We note now though that unlike in GR, for us the volume is monotonically increasing. Of course, the volume is constant on any hypersurface by definition and so the volume provides a good notion of time in this theory. This may be extremely beneficial in a quantisation program.

12 Light Cones

So far the theory is quite promising. There are a number of things that must carry over from GR though if it is to be taken seriously. One of these is that the speed of propagation of the wave front must be unity (the speed of light). The easiest way to check this is to consider the evolution equations. Let’s consider the case in GR briefly. The corresponding case in the conformal theory will work in almost exactly the same way.

The evolution equation for \( g_{ab} \) in GR is
\[
\frac{\partial g_{ab}}{\partial t} = \frac{2N}{\sqrt{g}} \left( \pi_{ab} - \frac{1}{2} g_{ab} tr \pi \right) + (KN)_{ab} 
\]
(115)
Inverting this we find that
\[ \pi_{ab} = \frac{\sqrt{g}}{2N} \frac{\partial g_{ab}}{\partial t} \] (116)
We will be working here to leading order in the derivatives which is the reason for only omitting the other terms. Differentiating both sides gives
\[ \frac{\partial \pi_{ab}}{\partial t} = \frac{\sqrt{g}}{2N} \frac{\partial^2 g_{ab}}{\partial t^2} \] (117)
Now substituting this into the evolution equation for \( \pi_{ab} \) gives us
\[ \frac{\sqrt{g}}{2N} \frac{\partial^2 g_{ab}}{\partial t^2} = -N \sqrt{g} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) \] (118)
(Note: The alternate form of the evolution equation is used here with the factor of \( \frac{1}{2} \) on \( R \).)

Now,
\[ \left( R_{ab} - \frac{1}{2} g_{ab} R \right) = \frac{1}{2} g^{cd} \left[ g_{bd,ac} + g_{ac,bd} - g_{ab,cd} - g_{cd,ab} - g_{ab} g^{ef} \left( g_{ec,fd} - g_{ef,cd} \right) \right] \] (119)
again only using leading order in the derivatives. We are concerned with the transverse traceless part of \( g_{ab} \) which we’ll label as \( g_{ab}^{TT} \). The only relevant part is then
\[ -\frac{1}{2} g^{cd} g_{ab,cd} \] (120)
which we’ll write as
\[ -\frac{1}{2} \frac{\partial^2 g_{ab}^{TT}}{\partial x^2} \] (121)
All the other terms are canceled either through the transverse or traceless properties. Using only the \( TT \) part in the time derivatives also gives us
\[ \frac{1}{2N^2} \frac{\partial^2 g_{ab}^{TT}}{\partial t^2} = \frac{1}{2} \frac{\partial^2 g_{ab}^{TT}}{\partial x^2} \] (122)
This is a wave equation with wave speed 1. Thus we get gravitational radiation! Various details are omitted here but the essence of the idea is quite clear. Let’s consider the conformal theory. We’ll use the compact without boundary theory (that is, the one with the volume terms).

The evolution equation for \( g_{ab} \) can be inverted to get
\[ \sigma_{ab} = \frac{\sqrt{g}}{2N V^{2/3}} \frac{\partial g_{ab}}{\partial t} + \ldots \] (123)
Differentiating both sides gives
\[ \frac{\partial \sigma_{ab}}{\partial t} = \frac{\sqrt{g}}{2N V^{2/3}} \frac{\partial^2 g_{ab}}{\partial t^2} \] (124)
again, working only to leading order in the derivatives. Substituting this into the evolution equation for \( \sigma_{ab} \) gives us
\[ \frac{\sqrt{g}}{2N V^{2/3}} \frac{\partial^2 g_{ab}}{\partial t^2} = \frac{N \sqrt{g}}{V^{2/3}} \left( R_{ab} - \frac{1}{2} g_{ab} R \right) \] (125)
The volume terms cancel and we are left with the same equation as (118) above. In exactly the same way this becomes
\[ \frac{1}{N^2} \frac{\partial^2 g_{ab}^{TT}}{\partial t^2} = \frac{\partial^2 g_{ab}^{TT}}{\partial x^2} \] (126)
Yet again, we have found a wave equation with speed 1. Thus we have recovered gravitational radiation with wavefronts propagating at the speed of light. All is still well.

13 Matter and Cosmology

The issues of coupling of matter and of cosmology will be treated in detail in forthcoming articles. Clearly, the theory here is incredibly restrictive cosmologically. We need a spatially constant scalar curvature an identically constant trp and a monotonically changing volume. However, there is one interesting result which is easily found here regarding the cosmological constant.

13.1 The Cosmological Constant Problem

This is probably the best known problem of the so called standard cosmology. In GR we have the following. Taking the interpretation of the cosmological constant $\Lambda$ as a vacuum energy there is a discrepancy of at least $10^{120}$ orders of magnitude between the theoretically predicted value and the measured value today. That is

$$\frac{\Lambda_{\text{Pl}}}{\Lambda_0} \geq 10^{120} \tag{127}$$

where the subscripts Pl and 0 refer to Planck scales and today respectively. In GR the cosmological constant appears with the scalar curvature in the form $R + \Lambda$. However, in the new theory here it appears with a volume coefficient in the form $R + \frac{\Lambda}{V^{2/3}}$. Thus we are concerned with the ratio of $\frac{\Lambda}{V^{2/3}}$ at the Planck scale and today. We take the value of $\Lambda$ to be identically constant and so we wish to consider $\left(\frac{V_0}{V_{\text{Pl}}}\right)^{2/3}$. We take the radius of the universe at the Planck scale to be the Planck length which is approximately $10^{-35}\text{m}$. Today, the radius of the universe today is at least $10^{26}\text{m}$ (the radius of the observed universe). Thus the ratio we are considering is

$$\left(\frac{V_0}{V_{\text{Pl}}}\right)^{2/3} \geq \left(\frac{10^{26}}{10^{-35}}\right)^2 = 10^{122} \tag{128}$$

There is no cosmological constant problem!

It should be pointed out that this is fundamentally different from postulating a “time-varying cosmological constant”. The constant enters at the same level in his theory as in GR and it is the behaviour of the scalar curvature which changes things.

As stated earlier, the full cosmological implications will be treated elsewhere. Such a restrictive theory is in general quite attractive providing definite testable predictions with relative ease. Indeed, looking at the above treatment of the cosmological constant in reverse as a prediction on the magnitude of $\Lambda$ is already one new prediction which seems to be satisfied experimentally!
14 Derivations

Naturally, as one would expect there exist alternative derivations of the theory. There are two in particular which are quite interesting. One of these will form part of a future paper. The other is quite straightforward and can be described quite easily.

Consider the Hamiltonian of GR. It is

\[ H_{GR} = \frac{N}{\sqrt{g}} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} (tr\pi)^2 - gR \right) - 2 N_a \nabla^b \pi^{ab} \]  \hspace{1cm} (129)

We wish to construct a theory which is invariant under both diffeomorphisms and conformal transformations. Consider now the momentum constraint

\[ \nabla^b \pi^{ab} = 0 \]  \hspace{1cm} (130)

This implements diffeomorphism invariance. We need it to be conformally invariant also under the transformation

\[ g_{ab} \rightarrow \omega^4 g_{ab} \]  \hspace{1cm} (131)

There are only two possibilities which satisfy this.

(i) \( \sigma^{ab} \rightarrow \omega^{-4} \sigma^{ab} \) and \( tr\pi = 0 \);

(ii) \( \sigma^{ab} \rightarrow \omega^{-4} g^{ab} \), \( \nabla_c tr\pi = 0 \) and \( trp = \frac{tr\pi}{\sqrt{g'}} \rightarrow trp \).

The first case here leads to the original conformal theory. The second is exactly the conformal transformation required to reproduce the new theory. Performing this transformation on the Hamiltonian constraint of GR (with conformal factor \( \psi \)) leads to the desired Hamiltonian constraint. Then finally, in a Dirac type procedure [6] we add the new constraint \( \nabla_c tr\pi = 0 \) to the Hamiltonian with a Lagrange multiplier to obtain the full Hamiltonian of the theory. Thus, the theory is found in a simple and natural way.

14.2 Quantisation

The theory has several attractive features from a quantisation point of view. The configuration space is no longer simply superspace but has been reduced to conformal superspace plus a constant \( trp \). There is a physically preferred slicing and the volume of the universe emerges as a good notion of time. These point to possible benefits of a quantisation program for the theory. In particular, the theory may shed light on the problem of time in quantum gravity. Thus, regardless of its fate as a competitor to GR this theory may teach some valuable lessons.

14.3 Recent Developments

Since this was written further work by the author and collaborators has led to a first principles derivation of the full York method of general relativity [7]. This is accomplished by only allowing invariance with respect to volume preserving conformal tranformations rather than general conformal transformations as in this work.
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