THE VECTOR $k$–CONSTRAINED KP HIERARCHY AND SATO’S GRASSMANNIAN

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ABSTRACT. We use the representation theory of the infinite matrix group to show that (in the polynomial case) the $n$–vector $k$–constrained KP hierarchy has a natural geometrical interpretation on Sato’s infinite Grassmannian. This description generalizes the the $k$–reduced KP or Gelfand–Dickey hierarchies.

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§1. Introduction.

It is well-known that the $k$–th Gelfand–Dickey hierarchy, which generalizes the Korteweg–de Vries (KdV) hierarchy, can be obtained as a reduction of the Kadomtsev–Petviashvili (KP) hierarchy. The latter is defined as the set of deformation equations

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L],$$

for the first order pseudo-differential operator

$$L \equiv L(t, \partial) = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2} + \cdots,$$

here $\partial = \frac{\partial}{\partial t_1}$, $t = (t_1, t_2, \ldots)$ and $(L^k)_+$ stands for the differential part of $L^k$. Now $L$ dresses as $L = P\partial P^{-1}$ with

$$P \equiv P(t, \partial) = 1 + a_1(t)\partial^{-1} + a_2(t)\partial^{-2} + \cdots.$$

One can choose $P$ in such a way that

$$P(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)},$$

where $\tau(t) = \tau(t_1, t_2, t_3, \ldots)$ is the famous $\tau$–function, introduced by the Kyoto group [DJKM1-3] and $[z] = (z, z^2, z^3, \ldots)$. Sato [S] showed that such a $\tau$–function corresponds to a point of some infinite Grassmannian $Gr$ (see e.g. [S,SW]). Let $H$ be the space of formal Laurent series $\sum a_n t^n$, such that $a_n = 0$ for $n >> 0$. The points of $Gr$ are those linear subspaces $W \subset H$ for which the projection $\pi_+ \,$ of $W$ into $H_+ = \{ \sum a_n t^n \in H | a_n = 0 \text{ for all } n < 0 \}$ is a Fredholm operator. The $k$–th reduction or $k$–th Gelfand–Dickey hierarchy is obtained by assuming that

$$L^k = (L^k)_+, \,$$

which corresponds to a $\tau$–function for which

$$\frac{\partial \tau}{\partial t_k} = \lambda \tau \quad \text{for some } \lambda \in \mathbb{C}.$$

In the polynomial case, i.e. $\tau$ is a polynomial, clearly $\lambda = 0$. The point in the Grassmannian that corresponds to such a reduced $\tau$–function satisfies

$$t^k W \subset W.$$
In recent years a lot of attention has been drawn to a new kind of reduction of the KP hierarchy, viz. the so-called $k$--constrained KP hierarchies [AFGZ,C,CWZ,CZ,D,DS,OS] (and references therein). Here one assumes that

\begin{equation}
L^k = (L^k)_+ + q \partial^{-1} r,
\end{equation}

$q = q(t), r = r(t)$ being functions. Under this condition the KP hierarchy is constrained to

\begin{equation}
\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \quad \frac{\partial q_j}{\partial t_k} = (L^k)_+ q_j, \quad \frac{\partial r_j}{\partial t_k} = -(L^k)_+^* r_j.
\end{equation}

Here $A^*$ stands for the adjoined operator of $A$ (see e.g. [KV] for more details about pseudo--differential operators). The AKNS, Yajima–Oikawa and Melnikov hierarchies are some of the examples that appear amongst these constrained KP families.

In this paper we consider the generalization of this $k$--constrained KP hierarchy, which was introduced by Sidorenko and Strampp in [SS], the $n$--vector $k$--constrained hierarchy. We assume that

\begin{equation}
L^k = (L^k)_+ + \sum_{j=1}^{n} q_j \partial^{-1} r_j,
\end{equation}

then one obtains the following integrable system:

\begin{equation}
\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \quad \frac{\partial q_j}{\partial t_k} = (L^k)_+ q_j, \quad \frac{\partial r_j}{\partial t_k} = -(L^k)_+^* r_j \quad \text{for } 1 \leq j \leq n.
\end{equation}

For $k = 1$ this hierarchy contains the coupled vector non--linear Schrödinger. Zhang and Cheng showed in [ZC] that if one assumes that

\begin{equation}
q_j(t) = \frac{\rho_j(t)}{\tau(t)}, \quad \text{and} \quad r_j(t) = \frac{\sigma_j(t)}{\tau(t)},
\end{equation}

then $L, q_j$ and $r_j, 1 \leq j \leq n$ satisfy the $n$--vector $k$--constrained hierarchy if and only if $\tau(t), \rho_j(t)$ and $\sigma_j(t)$ satisfy the following set of equations:

\begin{align}
\text{Res}_{z=0} e^{-\eta(t,z)} \tau(t)e^{\xi(t,z)} e^{\eta(t',z)} \tau(t') e^{-\xi(t',z)} = 0, \\
\text{Res}_{z=0} z^k e^{-\eta(t,z)} \tau(t)e^{\xi(t,z)} e^{\eta(t',z)} \tau(t') e^{-\xi(t',z)} = \sum_{j=1}^{n} \rho_j(t) \sigma_j(t'), \\
\text{Res}_{z=0} z^{-1} e^{-\eta(t,z)} \tau(t)e^{\xi(t,z)} e^{\eta(t',z)} \rho_j(t') e^{-\xi(t',z)} = \rho_j(t) \tau(t'), \\
\text{Res}_{z=0} z^{-1} e^{-\eta(t,z)} \sigma_j(t)e^{\xi(t,z)} e^{\eta(t',z)} \tau(t') e^{-\xi(t',z)} = \tau(t) \sigma_j(t').
\end{align}
where

\begin{equation}
\eta(t, z) = \sum_{i=1}^{\infty} \frac{\partial}{\partial t_i} z^{-i}, \quad \xi(t, z) = \sum_{i=1}^{\infty} t_i z^i
\end{equation}

and \( \text{Res}_{z=0} \sum_{i} a_i z^i = a_{-1} \).

In the case that \( n = 1 \), Loris and Willox [LW] show that one can deduce some additional bilinear identities, but now involving \( \frac{\partial \tau}{\partial \mu_k} \). It is unclear if this is possible for \( n > 1 \), but we will not need these extra bilinear identities.

We will show in this paper that in fact \( L \) satisfies the \( n \)-vector \( k \)-constrained KP hierarchy (1.3-4) if and only if the corresponding point \( W \) in \( Gr \) has a linear subspace \( W' \subset W \) of codimension \( n \) such that

\begin{equation}
t^k W' \subset W.
\end{equation}

We will proof this only in the polynomial case, i.e. polynomial \( \tau, \rho_j \) and \( \sigma_j \), but we expect that this is still true in the non-polynomial case. We use the representation theory of the infinite-dimensional matrix group \( GL_\infty \), developed by Kac and Peterson [KP1-2] (see also [KR]), to achieve this result.

Notice that in this way we get a filtration of hierarchies, i.e., the \( n \)-vector \( k \)-constrained hierarchy is a subsystem of the \( (n + 1) \)-vector \( k \)-constrained hierarchy, \( n = 0 \) being the \( k \)-reduced KP or Gelfand–Dickey hierarchies.

§2. The semi-infinite wedge representation of the group \( GL_\infty \) and Sato’s Grassmannian.

Consider the infinite complex matrix group

\[ GL_\infty = \{ A = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} | A \text{ is invertible and all but a finite number of } a_{ij} - \delta_{ij} \text{ are } 0 \} \]

and its Lie algebra

\[ gl_\infty = \{ a = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} | \text{all but a finite number of } a_{ij} \text{ are } 0 \} \]

with bracket \([a, b] = ab - ba\). The Lie algebra \( gl_\infty \) has a basis consisting of matrices \( E_{ij}, i, j \in \mathbb{Z} + \frac{1}{2} \), where \( E_{ij} \) is the matrix with a 1 on the \((i, j)\)-th entry and zeros elsewhere. Let \( \mathbb{C}^\infty = \bigoplus_{j \in \mathbb{Z} + \frac{1}{2}} \mathbb{C} v_j \) be an infinite dimensional complex vector space with fixed basis \( \{ v_j \}_{j \in \mathbb{Z} + \frac{1}{2}} \). Both the group \( GL_\infty \) and its Lie algebra \( gl_\infty \) act linearly on \( \mathbb{C}^\infty \) via the usual formula:

\[ E_{ij}(v_k) = \delta_{jk}v_i. \]

The well-known semi–infinite wedge representation is constructed as follows [KP2] (see also [KR] and [KV]). The semi-infinite wedge space \( F = \Lambda^{\frac{1}{2}} \mathbb{C}^\infty \) is the vector space
with a basis consisting of all semi-infinite monomials of the form \(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \cdots\), where \(i_1 > i_2 > i_3 > \ldots\) and \(i_{\ell+1} = i_\ell - 1\) for \(\ell \gg 0\). We can now define representations \(R\) of \(GL_\infty\) and \(r\) of \(gl_\infty\) on \(F\) by

\[
R(A)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) = Av_{i_1} \wedge Av_{i_2} \wedge Av_{i_3} \wedge \cdots, \\
r(a)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) = \sum_k v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{k-1}} \wedge av_{i_k} \wedge v_{i_{k+1}} \wedge \cdots.
\]

These equations are related by the usual formula:

\[
\exp(r(a)) = R(\exp a) \text{ for } a \in gl_\infty.
\]

In order to perform calculations later on, it is convenient to introduce a larger group

\[
\overline{GL}_\infty = \{A = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} | A \text{ is invertible and all but a finite number of } a_{ij} \text{ with } i \geq j \text{ are 0}\}
\]

and its Lie algebra

\[
\overline{gl}_\infty = \{a = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} | \text{ all but a finite number of } a_{ij} \text{ with } i \geq j \text{ are 0}\}.
\]

Both \(\overline{GL}_\infty\) and \(\overline{gl}_\infty\) act on a completion \(\overline{\mathbb{C}}^\infty\) of the space \(\mathbb{C}^\infty\), where

\[
\overline{\mathbb{C}}^\infty = \{\sum_j c_j v_j | c_j = 0 \text{ for } j \gg 0\}.
\]

It is easy to see that the representations \(R\) and \(r\) extend to representations of \(\overline{GL}_\infty\) and \(\overline{gl}_\infty\) on the space \(F\).

The representation \(r\) of \(gl_\infty\) and \(\overline{gl}_\infty\) can be described in terms of wedging and contracting operators in \(F\) (see e.g. [KP2,KR]). Let \(v_j^*\) be the linear functional on \(\mathbb{C}^\infty\) defined by \(\langle v_j^*, v_j \rangle := v_j^*(v_j) = \delta_{ij}\) and let \(\mathbb{C}^{\infty*} = \bigoplus_{j \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}v_j^*\) be the restricted dual of \(\mathbb{C}^\infty\), then for any \(w \in \mathbb{C}^\infty\), we define a wedging operator \(\psi^+(w)\) on \(F\) by

\[
\psi^+(w)(v_{i_1} \wedge v_{i_2} \wedge \cdots) = w \wedge v_{i_1} \wedge v_{i_2} \cdots.
\]

Let \(w^* \in \mathbb{C}^{\infty*}\), we define a contracting operator

\[
\psi^-(w^*)(v_{i_1} \wedge v_{i_2} \wedge \cdots) = \sum_{s=1}^\infty (-1)^{s+1} \langle w^*, v_{i_s} \rangle v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{s-1}} \wedge v_{i_{s+1}} \wedge \cdots.
\]

For simplicity we write

\[
\psi^+_j = \psi^+(v_{-j}), \quad \psi^-_j = \psi^-(v_j^*) \quad \text{for } j \in \mathbb{Z} + \frac{1}{2}.
\]
These operators satisfy the following relations \((i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -)\):

\[
\psi^\lambda_i \psi^\mu_j + \psi^\mu_j \psi^\lambda_i = \delta_{\lambda, -\mu} \delta_{i, -j},
\]
hence they generate a Clifford algebra, which we denote by \(\mathcal{C}\).

Introduce the following elements of \(F\) \((m \in \mathbb{Z})\):

\[
|m\rangle = v_{m - \frac{1}{2}} \wedge v_{m - \frac{3}{2}} \wedge v_{m - \frac{5}{2}} \wedge \cdots.
\]

It is clear that \(F\) is an irreducible \(\mathcal{C}\)-module generated by the vacuum \(|0\rangle\) such that

\[
\psi^\pm_j |0\rangle = 0 \text{ for } j > 0.
\]

It is straightforward that the representation \(r\) is given by the following formula:

\[
(2.5) \quad r(E_{ij}) = \psi^+_{-i} \psi^-_{j}.
\]

Define the charge decomposition

\[
(2.6) \quad F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}
\]
by letting

\[
(2.7) \quad \text{charge}(|0\rangle) = 0 \text{ and } \text{charge}(\psi^\pm_j) = \pm 1.
\]

It is clear that the charge decomposition is invariant with respect to \(r(g\ell_\infty)\) (and hence with respect to \(R(GL_\infty))\). Moreover, it is easy to see that each \(F^{(m)}\) is irreducible with respect to \(g\ell_\infty\) (and \(GL_\infty\)). Note that \(|m\rangle\) is its highest weight vector, i.e.

\[
\begin{align*}
& r(E_{ij})|m\rangle = 0 \text{ for } i < j, \\
& r(E_{ii})|m\rangle = 0 \text{ (resp. } = |m\rangle) \text{ if } i > m \text{ (resp. if } i < m).
\end{align*}
\]

Let \(w \in F\), we define the Annihilator space \(Ann(w)\) of \(w\) as follows:

\[
(2.8) \quad Ann(w) = \{v \in \mathbb{C}^\infty | v \wedge w = 0\}.
\]

Notice that \(Ann(w) \neq 0\), since \(v_j \in Ann(w)\) for \(j << 0\). This Annihilator space for perfect (semi–infinite) wedges \(w \in F^{(m)}\) is related to the \(GL_\infty\)-orbit

\[
\mathcal{O}_m = R(GL_\infty)|m\rangle \subset F^{(m)}
\]
of the highest weight vector \(|m\rangle\) as follows. Let \(A = (A_{ij})_{i, j \in \mathbb{Z}} \in GL_\infty\), denote by \(A_j = \sum_{i \in \mathbb{Z}} A_{ij} v_i\) then by (2.8)

\[
(2.9) \quad \tau_m = R(A)|m\rangle = A_{m - \frac{1}{2}} \wedge A_{m - \frac{3}{2}} \wedge A_{m - \frac{5}{2}} \wedge \cdots,
\]
with $A_{-j} = v_{-j}$ for $j >> 0$. Notice that since $\tau_m$ is a perfect (semi-infinite) wedge

$$Ann(\tau_m) = \sum_{j<m} \mathbb{C}A_j \subset \mathbb{C}^\infty.$$  

By identifying $v_i = t^{-i-\frac{1}{2}}$ for $i \in \mathbb{Z} + \frac{1}{2}$, we can write $A_j = A_j(t) = \sum_{i \in \mathbb{Z} + \frac{1}{2}} A_{ij} t^{-i-\frac{1}{2}}$ as a Laurent polynomial in $t$. In this way we can identify $Ann(\tau_m)$ with a subspace $W_{\tau_m} = \sum_{j<m} \mathbb{C}A_j(t)$ of the space $H$ of all Laurent polynomials. Notice that this space $H$ differs from the one described in section 1. So from now on let $Gr$ consist of all linear subspaces of $H$ which contain

$$H_j := \sum_{i=-j}^\infty \mathbb{C}t^i$$

for $j >> 0$ and let $Gr = \bigcup_{m \in \mathbb{Z}} Gr_m$ (disjoint union) with

$$Gr_m = \{ W \in Gr | H_j \subset W \text{ and } \dim W/H_j = m - j \text{ for } j << 0 \},$$

then we can construct a canonical map

$$\phi : O_m \to Gr_m, \quad \phi(\tau_m) = W_{\tau_m} := \sum_{i<m} \mathbb{C}A_i(t).$$

It is clear that $\phi(|m\rangle) = H_m$ and that $\phi$ is surjective with fibers $\mathbb{C}^\times$. This construction is due to Sato [S]; we call $Gr$ the polynomial Grassmannian. From now on we will call a perfect wedge also a $\tau$–function (N.B. $\tau = 0$ is also a $\tau$–function).

§3. The boson-fermion correspondence.

Introduce the fermionic fields ($z \in \mathbb{C}^\times$):

(3.1) \[ \psi^\pm[z] \defeq \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^\pm_k z^{-k-\frac{1}{2}}. \]

Next we introduce bosonic fields:

(3.2) \[ \alpha[z] \equiv \sum_{k \in \mathbb{Z}} \alpha_k z^{-k-1} \defeq :\psi^+[z] \psi^-[z]:, \]

where $:\colon:\colon$ stands for the normal ordered product defined in the usual way ($\lambda, \mu = +$ or $-$):

$$:\psi^\lambda_k \psi^\mu_\ell:\colon \begin{cases} 
\psi^\lambda_k \psi^\mu_\ell & \text{if } \ell \geq k \\
-\psi^\mu_\ell \psi^\lambda_k & \text{if } \ell < k.
\end{cases}$$
One checks (using e.g. Wick’s formula) that the operators $\alpha_k$ satisfy the commutation relations of the associative oscillator algebra, one has:

$$[\alpha_k, \alpha_\ell] = k\delta_{k,-\ell} \quad \text{and} \quad \alpha_k|m\rangle = 0 \text{ if } k > 0.$$  

In order to express the fermionic fields $\psi^\pm(z)$ in terms of the bosonic operators $\alpha_\ell$, we need some additional operator $Q$. This operators is uniquely defined as follows:

$$Q(v_{i_1} \wedge v_{i_2} \wedge \cdots) = (v_{i_1+1} \wedge v_{i_2+1} \wedge \cdots).$$

So

$$Q|0\rangle = |1\rangle, \quad Q\psi^\pm_k = \psi^\pm_{k+1}Q$$

and $Q$ satisfies the following commutation relations with the $\alpha$’s:

$$[\alpha_k, Q] = \delta_{k0}Q.$$  

In this paper the operator $Q^{-k}$ will play an important role. If $w_{m-\frac{1}{2}} \wedge w_{m-\frac{3}{2}} \wedge \cdots$ is a perfect wedge then

$$Q^{-k}(w_{m-\frac{1}{2}} \wedge w_{m-\frac{3}{2}} \wedge \cdots) = \Lambda^k w_{m-\frac{1}{2}} \wedge \Lambda^k w_{m-\frac{3}{2}} \wedge \cdots,$$

where $\Lambda = \sum_{j \in \mathbb{Z} \cap \frac{1}{2}} E_{j,j+1}$.

**Theorem 3.1.** ([DJKM1], [JM])

$$\psi^\pm[z] = Q^{\pm 1}z^{\pm \alpha_0} \exp(\mp \sum_{k<0}^{1 \alpha_k z^{-k}}) \exp(\mp \sum_{k>0}^{1 \alpha_k z^{-k}}).$$

**Proof.** See [TV].

The operators on the right-hand side of (3.6) are called vertex operators. They made their first appearance in string theory (cf. [FK]).

We now describe the boson-fermion correspondence. Let $\mathbb{C}[t]$ be the space of polynomials in indeterminates $t = (t_1, t_2, t_3, \ldots)$. Let $B = \mathbb{C}[q, q^{-1}, t] = \mathbb{C}[t] \otimes_{\mathbb{C}} \mathbb{C}[q, q^{-1}]$ be the tensor product of algebras. Then the boson-fermion correspondence is the vector space isomorphism

$$\sigma : F \cong B,$$

given by

$$\sigma(\alpha_{-m_1} \cdots \alpha_{-m_s}|k\rangle) = m_1 \cdots m_s t_{m_1} \cdots t_{m_s} q^k.$$
Notice that the power of $q$ is the value of the charge. The transported action of the operators $\alpha_m$ and $Q$ looks as follows:

\begin{equation}
\sigma Q \sigma^{-1} = q, \quad \sigma \alpha_m \sigma^{-1} = \begin{cases} 
-m t_m & \text{if } m < 0, \\
\frac{\partial}{\partial t_m} & \text{if } m > 0, \\
\frac{\partial}{\partial q} & \text{if } m = 0.
\end{cases}
\end{equation}

Hence

\begin{equation}
\sigma \psi^{\pm}[z] \sigma^{-1} = q^{\pm 1} z^{\pm} e^{\pm \xi(t,z)} e^{\mp \eta(t,z)},
\end{equation}

with $\eta(t,z)$ and $\xi(t,z)$ given by (1.10)

§4. Identification of the bilinear identities.

From now on we assume that $\tau \in F^{(m)}$, hence that $\tau$ is the inverse image under $\sigma$. Using the boson–fermion correspondence of the previous section, we rewrite the bilinear identities (1.6-9) of Zhang and Cheng now as equations in $F \otimes F$. Notice first the following equality of operators on $F \otimes F$:

\[ Res_{z=0} \psi^+ [z] \otimes \psi^- [z] = \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \otimes \psi^-_i \]

Now (1.6-9) turn into the following equations:

\begin{align}
(4.1) \quad & \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \tau \otimes \psi^-_i \tau = 0, \\
(4.2) \quad & \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \tau \otimes \psi^-_i Q^{-k} \tau = \sum_{j=1}^{n} \rho_j \otimes \sigma_j, \\
(4.3) \quad & \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \tau \otimes \psi^-_i \rho_j = \rho_j \otimes \tau, \\
(4.4) \quad & \sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \sigma_j \otimes \psi^-_i Q^{-k} \tau = Q^{-k} \tau \otimes \sigma_j.
\end{align}

Here $Q^{-k} \tau \in F^{(m-k)}$, $\rho_j \in F^{(m+1)}$ and $\sigma_j \in F^{(m-k-1)}$ for all $1 \leq j \leq n$. Equation (4.1) is called the KP hierarchy in the fermionic picture, it characterizes the $GL_{\infty}$-orbit $O_m$, i.e.

**Proposition 4.1.** ([KP2]) A non-zero element $\tau$ of $F^{(m)}$ lies in $O_m$ if and only if $\tau$ satisfies the equation (4.1).

If $\tau \in O_m$, then we can write $\tau$ as a perfect wedge

\begin{equation}
\tau = w_{m-\frac{1}{2}} \wedge w_{m-\frac{3}{2}} \wedge w_{m-\frac{5}{2}} \wedge w_{m-\frac{7}{2}} \wedge \cdots,
\end{equation}

such that $w_{-\ell} = v_{-\ell}$ for $\ell \gg 0$. The corresponding point $W_\tau \in Gr_m$ is then given by

\begin{equation}
W_\tau = \langle w_{m-\frac{1}{2}}, w_{m-\frac{3}{2}}, w_{m-\frac{5}{2}}, w_{m-\frac{7}{2}}, \cdots \rangle.
\end{equation}

The geometrical interpretation of (4.3-4) is given by the following proposition.
Proposition 4.2. Let \( \tau \in O_m \), \( \rho \in F^{(m+1)} \) and \( \sigma \in F^{(m-1)} \), then

(1) \( \tau \) and \( \rho \) satisfy

\[
\sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \tau \otimes \psi_i^- \rho = \rho \otimes \tau,
\]

if and only if \( \rho \in O_{m+1} \) and \( W_\tau \subset W_\rho \).

(2) \( \tau \) and \( \sigma \) satisfy

\[
\sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \sigma \otimes \psi_i^- \tau = \tau \otimes \sigma,
\]

if and only if \( \sigma \in O_{m-1} \) and \( W_\sigma \subset W_\tau \).

Proof. Without loss of generality we may assume (since the operator \( \sum_i \psi_i^+ \otimes \psi_i^- \) commutes with the action of \( R(GL_\infty) \otimes R(GL_\infty) \)) that \( \tau = |m \rangle \). Then (4.7) is equivalent to

\[
\sum_{i > m} v_i |m \rangle \otimes \psi_i^- \rho = \rho \otimes |m \rangle.
\]

Since all elements \( v_i |m \rangle \), for \( i > m \), are linearly independent, we deduce that \( \psi_i^- \rho = \lambda_i |m \rangle \) and that \( \rho \in \langle v_i \wedge |m \rangle | i > m \rangle \). Hence \( \rho = w \wedge |m \rangle \) for some \( w \in \mathbb{C}^\infty \) and thus \( \rho \in O_{m+1} \) and \( W_\tau \subset W_\rho \).

The converse, since \( W_\tau \subset W_\rho \), \( \rho = w \wedge |m \rangle \) for some \( w \in \mathbb{C}^\infty \). Then

\[
\sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \tau \otimes \psi_i^- (w \wedge \tau) = (w \wedge \tau) \otimes \tau - (1 \otimes \psi^+(w)) (\sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \tau \otimes \psi_i^- \tau)
\]

\[
= (w \wedge \tau) \otimes \tau
\]

For \( \tau = |m \rangle \), (4.8) is equivalent to

\[
\sum_{i < m} (v_i \wedge \sigma) \otimes \psi_i^- |m \rangle = |m \rangle \otimes \sigma.
\]

Since the elements \( \psi_i^- |m \rangle \) for \( i < m \) are all linearly independent, we conclude that \( v_i \wedge \sigma = \lambda_i |m \rangle \) and that \( \sigma \in \langle \psi_i^- |m \rangle | i < m \rangle \). Hence \( \sigma = \sum_{i = -\infty}^{m-\frac{1}{2}} a_i \psi_i^- |m \rangle \). Since \( \sigma \in F^{(m-1)} \), \( a_i = 0 \) for all \( i < -N << 0 \). We now calculate \( \text{Ann}(\sigma) \). Clearly \( \text{Ann}(\sigma) \subset \langle v_i | i < m \rangle = \text{Ann}(|m \rangle) \), so let \( v = \sum_{i < m} (-)^i b_i v_i \), then \( \sum_{i = -N + \frac{1}{2}}^{m-\frac{1}{2}} a_i b_i = 0 \). Hence, if \( \sigma \neq 0 \), we only find one restriction for the collection of \( b_i \)'s, from which we conclude that \( \sigma \) is a perfect wedge. The converse of this statement follows immediately by writing \( \tau = w \wedge \sigma \).

We next proof the following
Proposition 4.3. Let $\tau \in \mathcal{O}_m$, $\rho_j \in \mathcal{O}_{(m+1)}$ and $\sigma_j \in \mathcal{O}_{(m-k-1)}$, $1 \leq j \leq n$, be related by
\begin{equation}
W_\tau \subset W_{\rho_j}, \quad W_{\sigma_j} \subset \Lambda^kW_\tau,
\end{equation}
then $\tau$ satisfies equation (4.2) if and only if there exists a subspace $W' \subset W_\tau$ of codimension $n$ such that $\Lambda^kW' \subset W_\tau$.

Proof. Notice first that $\Lambda^kW_\tau = W_{Q-k\tau}$. We assume that $n$ is minimal, so that all $\sigma_j$ and $\rho_j$ are nonzero perfect wedges, and that $\tau$ is of the form (4.5). Then
\[
\sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \tau \otimes \psi_i^- Q^{-k} \tau
\]
\[
= \sum_{\ell = 0}^{\infty} (-)^\ell \Lambda^k w_{m-\ell-\frac{1}{2}} \wedge \tau \otimes \Lambda^k w_{m+\frac{1}{2}} \wedge \cdots \wedge \Lambda^k w_{m-\ell+\frac{1}{2}} \wedge \cdots
\]
\[
= \sum_{j=1}^{n} u_j \wedge \tau \otimes \sigma_j,
\]
where $\rho_j = u_j \wedge \tau$. Since all vectors $\Lambda^k w_{m-\ell-\frac{1}{2}} \wedge \cdots \wedge \Lambda^k w_{m-\ell+\frac{1}{2}} \wedge \cdots$ are linearly independent, we deduce that
\[
\Lambda^k w_{m-\ell-\frac{1}{2}} \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_n \wedge \tau = 0,
\]
for all $\ell = 0, 1, 2, \ldots$. Since we have assumed that $n$ is minimal, also all $u_j$'s are linearly independent and moreover $u_1 \wedge u_2 \wedge \cdots \wedge u_n \wedge \tau \neq 0$, hence
\[
\Lambda^k w_{m-\ell-\frac{1}{2}} \in \langle u_1, u_2, \ldots, u_n, w_{m-\frac{1}{2}}, w_{m-\frac{3}{2}}, \ldots \rangle,
\]
so there exists a subspace $W' \subset W_\tau$ of codimension $n$ such that $\Lambda^kW' \subset W_\tau$.

For the converse, choose a basis $w_{m-n-\frac{1}{2}}, w_{m-n-\frac{3}{2}}, \ldots$ of $W'$ and extend it to a basis $w_{m-\frac{1}{2}}, w_{m-\frac{3}{2}}, \ldots$ of $W_\tau$, then
\[
\sum_{i \in \mathbb{Z} + \frac{1}{2}} \psi_i^+ \tau \otimes \psi_i^- Q^{-k} \tau
\]
\[
= \sum_{\ell = 0}^{\infty} (-)^\ell \Lambda^k w_{m-\ell-\frac{1}{2}} \wedge \tau \otimes \Lambda^k w_{m-\ell+\frac{1}{2}} \wedge \cdots \wedge \Lambda^k w_{m-\ell+\frac{1}{2}} \wedge \cdots
\]
\[
= \sum_{\ell = 0}^{n-1} (-)^\ell \Lambda^k w_{m-\ell-\frac{1}{2}} \wedge \tau \otimes \Lambda^k w_{m-\ell+\frac{1}{2}} \wedge \cdots \wedge \Lambda^k w_{m-\ell+\frac{1}{2}} \wedge \cdots.
\]
So choose
\[
\rho_j = \Lambda^k w_{m-j+\frac{1}{2}} \wedge \tau,
\]
\[
\sigma_j = \Lambda^k w_{m-j+\frac{1}{2}} \wedge \cdots \wedge \Lambda^k w_{m-j+\frac{3}{2}} \wedge \Lambda^k w_{m-j-\frac{1}{2}} \wedge \cdots,
\]
then $W_\tau$, $\Lambda^kW_\tau$, $W_{\sigma_j}$ and $W_{\rho_j}$ clearly satisfy the equations (4.9). \qed

From this proposition we deduce the main Theorem of this paper.
Theorem 4.4. The pseudo-differential operator

\[ L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots, \]

satisfies the \( n \)-vector \( k \)-constrained KP hierarchy if and only if the corresponding point \( W \in Gr_m \) has a subspace \( W' \) of codimension \( n \) such that \( t^k W' \subset W \).

As an easy consequence we obtain

Corollary 4.5. Let \( \tau \) be a polynomial \( \tau \)-function of the \( n \)-vector \( k \)-constrained KP hierarchy, then \( \frac{\partial \tau}{\partial t_k} = \sum_{\ell=1}^{n} \tau_\ell \) where every \( \tau_\ell \) satisfies the KP hierarchy, i.e. equation (4.1).

Proof. The proof follows immediately by taking the same basis for \( W_\tau \) as in the converse part of the proof of Proposition 4.3. \( \square \)

If \( n = 1 \), one can proof [V] that every polynomial \( \tau \)-function \( \tau \), for which \( \frac{\partial \tau}{\partial t_k} \) is again \( \tau \)-function, is a solution of the \( k \)-constrained KP hierarchy.

Notice that we have constructed a natural filtration on the space \( Gr_m \), which is determined by the \( n \)-vector \( k \)-constrained KP hierarchy for \( n = 0, 1, 2, \ldots \). Let

\[ Gr_m^{(n,k)} = \{ W \in Gr_m | \text{there exists a subspace } W' \subset W \text{ of codimension } n \text{ such that } t^k W' \subset W \}, \]

then

\[ Gr_m^{(0,k)} \subset Gr_m^{(1,k)} \subset \cdots \subset Gr_m^{(n,k)} \subset Gr_m^{(n+1,k)} \subset \cdots. \]

It is obvious that every point \( W \in Gr_m \) (in this polynomial case) is contained in \( Gr_m^{(n,k)} \) for \( n >> 0 \), in other words

\[ Gr_m = \bigcup_{n \in \mathbb{Z}_+} Gr_m^{(n,k)}. \]

So for every \( \tau \)-function of the KP hierarchy there exists a non-negative integer \( n \) such that for all \( m \geq n \), \( \tau \) is also a \( \tau \)-function of the \( m \)-vector \( k \)-constrained KP hierarchy. In other words, for every \( L \), corresponding to a polynomial \( \tau \)-function, one can find a non-negative integer \( n \) such that \( L \) satisfies (1.3).

§5. Polynomial solutions of the \( n \)-vector \( k \)-constrained KP hierarchy..

We will now state an immediate consequence of the boson–fermion correspondence, viz., we calculate the image under \( \sigma \) of a perfect wedge of the form (2.9). One finds the following result
Proposition 5.1. Let $S_i$ be the elementary Schur functions, defined by \( \exp \sum_{i=1}^{\infty} t_i z^i \) ($S_i = 0$ for $i < 0$) and let $\tau_m \in O_m$ be of the form (2.9), i.e.,

$$\tau_m = A_{m-\frac{1}{2}} \wedge A_{m-\frac{3}{2}} \wedge A_{m-\frac{5}{2}} \wedge \cdots$$

with $A_j = \sum_{i \in \mathbb{Z} + \frac{1}{2}} A_{ij} v_i$ and $A_{-k} = v_{-k}$ for all $k > N > 0$. Set $A = (A_{ij})_{i \in \mathbb{Z} + \frac{1}{2}, m > j \in \mathbb{Z} + \frac{1}{2}}$ and let $\Lambda = \sum_{i \in \mathbb{Z} = \frac{1}{2}} E_{i,i+1} \in gl_{\infty}$. Then

$$\sigma(\tau_m) = \det \left( \sum_{i,j=-n+\frac{1}{2}}^{m-\frac{1}{2}} \left( \sum_{\ell=-n+\frac{1}{2}}^{\infty} S_{\ell-i} A_{\ell,j} \right) E_{ij} \right) q^m.$$  

Proof. The proof of this proposition is the same as the proof of Theorem 6.1 of [KR]. One computes

$$\sigma(\exp \left( \sum_{i=1}^{\infty} t_i \Lambda^i \right) \tau_m)$$

and takes the coefficient of $q^m$. One thus obtains (see also [DJKM1,M]):

$$\sigma(\tau_m) = \det \left( \left( \exp \left( \sum_{i=1}^{\infty} t_i \Lambda^i \right) A \right)_{<m} \right) q^m,$$

where $B_{<m}$ denotes the submatrix of $B$ where one only takes the rows $j \in \mathbb{Z} + \frac{1}{2}$ with $j < m$. Notice that $\sum_i t_i \Lambda^i \in gl_{\infty}$ and $\exp(\sum_i t_i \Lambda^i) \in GL_{\infty}$. Here we calculate the determinant of an infinite matrix, however there is no problem, since the matrix is of the form $(B_{ij})_{m > j \in \mathbb{Z} + \frac{1}{2}}$ with all but a finite number of $B_{ij} - \delta_{ij}$ with $i \geq j$ are zero.

It is clear that one can subtract $\sum_{i<-N} A_{ij} v_i$ from every $A_j$, with $j > -N$, in $\tau_m$, this will not change $\tau_m$. Then the new $A$ is of the form

$$A = \sum_{-N < i, -N < j < m} A_{ij} E_{ij} + \sum_{i <- N} E_{ii},$$

it is then straightforward, using the elementary Schur functions, to calculate the right-hand-side of (5.2). One finds formula (5.1). \(\square\)

We will use this proposition to obtain all polynomial solutions of the $n$–vector $k$–constrained KP hierarchy. Notice that our approach is different from the one in [ZC]. Instead of taking $\tau_m$ of the form (2.9), we may choose another basis of $W_{\tau_m}$ and construct the corresponding perfect wedge, it is clear that this will be a multiple of $\tau_m$. We can choose this basis in such a way

$$W_{\tau_m} = \langle A_{-\frac{1}{2}}, A_{-\frac{3}{2}}, A_{-\frac{5}{2}}, \ldots, A_{-N+\frac{1}{2}}, v_{-N+\frac{1}{2}}, v_{-N-\frac{1}{2}}, \ldots \rangle,$$
such that $A_j = \sum_{i=-N}^{\infty} A_{ij} v_i$ and that, except for at most $n$ vectors $A_j$, all $A_j$ satisfy the following condition

$$\Lambda^k A_j \left\{ \begin{array}{ll}
= A_\ell & \text{for some } -N + \frac{1}{2} \leq \ell \leq m - \frac{1}{2}, \\
\in \langle v_{-N-\frac{1}{2}}, v_{-N-\frac{3}{2}}, \ldots \rangle.
\end{array} \right.$$ 

Of course every $A_j$ is bounded, i.e., there exists an integer $M$ such that $A_j = \sum_{i=-N}^{M} A_{ij} v_i$.

Now making a shift in the index and permuting the columns we obtain the following result:

**Proposition 5.2.** Let $M,N \in \mathbb{Z}$ such that $M > N > 0$ and let $e_j, 1 \leq j \leq M$ be an orthonormal basis of $\mathbb{C}^M$. Let $R$ be the $M \times M$-matrix $R = \sum_{i=1}^{M-k} E_{i,i+k}$ and let $A = (A_{ij})_{1 \leq i \leq M, 1 \leq j \leq N}$ be an $M \times N$-matrix of rank $N$. Denote by $A_j = \sum_{i=1}^{M} A_{ij} e_i$. If all $A_j$ satisfy the condition that $RA_j \neq A_i$ for all $1 \leq i < j$ and if all $A_j$, except for at most $n$, satisfy the condition that

$$RA_j = \left\{ \begin{array}{ll}
A_{j+1} & \text{or} \\
0,
\end{array} \right.$$ 

then

$$(5.3) \quad \tau = \det \left( \sum_{i,j=1}^{N} \left( \sum_{\ell=1}^{M} S_{\ell-i} A_{\ell j} \right) E_{ij} \right)$$

is a $\tau$-function of the $n$-vector $k$-constrained KP hierarchy. All polynomial solutions can be obtained in this way.

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