The sextic oscillator and $E(5)$ type nuclei

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Abstract. The sextic oscillator is applied as a $\gamma$-independent potential to describe the energy spectrum of nuclei in regions where the occurrence of $E(5)$ symmetry has been assumed. Experimental levels of Ru, Pd, Cd, Te, Xe and Ba isotopes have been assigned to model states characterized by the $\xi = 1, 2$ quantum numbers, and the two model parameters $a$ and $b$ have been determined from a least square fit procedure. The resulting potentials have a spherical or deformed minimum. Most of the typical $E(5)$ candidates, i.e. $^{104}$Ru, $^{102}$Pd, $^{106,108}$Cd and $^{124}$Te correspond to parameters lying close to the parabola separating the two domains in the $a - b$ plane.

1. Introduction
The collective motion of nuclei results in characteristic shapes near the equilibrium states of nuclei. The most well-known shapes, such as the spherical vibrator, the axially deformed rotor and the $\gamma$-unstable rotor are linked with the U(5), SU(3) and O(6) symmetries, respectively [1].

A rather effective method of the consistent quantum mechanical description of these collective features on the phenomenological level is the application of the Bohr Hamiltonian, which uses a potential picture to describe quadrupole collective excitations in terms of the intrinsic shape variables $\beta$, $\gamma$. The minima of the $V(\beta, \gamma)$ potential define the equilibrium shapes, while the potential itself depends on various parameters characterizing the given nucleus, e.g. the proton and neutron number, etc. Changing these parameters gradually (e.g. by proceeding along an isotope chain) the potential shape and its minima change and may give rise to transition from one equilibrium shape (and symmetry) to another one. This procedure can be interpreted as a shape phase transition of some order that goes through a critical point.

In the simplest case the potential is independent from the $\gamma$ variable, in which case the Bohr Hamiltonian reduces to a Shrödinger-like radial equation. Such a situation can be used to describe the transition from the spherical (U(5)) to the $\gamma$-unstable (O(6)) shape phases. Iachello approximated this second-order phase transition with the infinite square well [2] and introduced the concept of the $E(5)$ critical point symmetry. This problem can be solved exactly in terms of Bessel functions, so parameter-free analytical results are available for the key spectroscopic properties (energy spectrum, electric quadrupole transition rates). Comparison of these quantities with the spectroscopic data of actual nuclei revealed that several nuclei (e.g. $^{134}$Ba) are close to the $E(5)$ critical point symmetry. Later the $X(5)$ symmetry was also introduced [3] to give account of the first-order phase transition between the the spherical and axially deformed shape phases. In this case the $\gamma$ shape variable also has to be incorporated into the formalism through certain approximation schemes.
In certain cases the Bohr Hamiltonian (or the one-dimensional equation derived from it) can be solved exactly. These cases are typically adapted to the five-dimensional problem from solvable radial problems of three-dimensional potentials. In order to obtain a complete picture, potentials with centrifugal-like term have to be considered. The list of exactly solvable problems of this type is finite (see e.g. [4]). Besides the infinite square well only the Kratzer and Davidson potentials can be solved for arbitrary quantum numbers. Recently we introduced a rather flexible potential [5], which is quasi-exactly solvable meaning that the closed analytical solutions are available for a finite subset of states for certain potential parameters. However, this set of states contains the most important low-lying levels necessary to characterize the collective excitations of the nucleus within the scheme of the Bohr Hamiltonian. This potential, the sextic oscillator can describe situations in which a nucleus has spherical minimum (at \( \beta = 0 \)), deformed minimum (at \( \beta > 0 \)) or both. This feature is certainly advantageous when one wishes to describe transitions between different shape phases. In its first application the sextic oscillator was considered as a \( \gamma \)-independent potential describing the transition between the spherical and the \( \gamma \)-unstable shape phases. Here we employ this potential to describe the energy spectrum of a number of nuclei that have been proposed as candidates for \( E(5) \) symmetry.

2. The sextic oscillator

The general Bohr Hamiltonian is written as [6]

\[
H = -\frac{\hbar^2}{2B} \left( \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} - \frac{1}{4\beta^2} \sum_k \frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)} \right) + V(\beta, \gamma). \tag{1}
\]

If the potential is assumed to be independent of the \( \gamma \) variable, i.e. \( V(\beta, \gamma) = U(\beta) \), then the \( \beta \)-dependent part can be separated by the substitution \( \Psi(\beta, \gamma, \theta) = \beta^{-2} \phi(\beta) \Phi(\gamma, \theta) \), leading to a form similar to the usual radial Schrödinger equation

\[
-\frac{d^2\phi}{d\beta^2} + \left( \frac{(\tau + 1)(\tau + 2)}{\beta^2} + u(\beta) \right) \phi = \epsilon \phi, \tag{2}
\]

where \( \epsilon = \frac{2B}{\hbar^2} \) and \( u(\beta) = \frac{2B}{\hbar^2} U(\beta) \). In (2) \( \tau \) originates from the angular equation and essentially plays the role of the angular momentum in five spatial dimensions.

In Ref. [5] the \( u(\beta) \) potential was chosen as the sextic oscillator

\[
u(\beta) = (b^2 - 4ac^2)\beta^2 - 2ab\beta^4 + a^2\beta^6 + u_0^\pm \tag{3},
\]

which is a quasi-exactly solvable potential [7] meaning that only a subset of its lowest-energy solutions can be obtained exactly for certain combinations of the potential parameters. \( u(\beta) \) in (3) depends on three parameters, \( a, b \) and \( c^\pm \), but \( c^\pm \) is related to the \( \tau \) parameter via \( 2c^\pm = (\tau + 2M + \frac{2}{3}) \) in order to account for the “centrifugal” term. Here \( M \) is a non-negative integer number that sets the number \( M + 1 \) of solutions that can be obtained exactly. In typical applications of the sextic oscillator in the Bohr Hamiltonian it is sufficient to consider \( M = 0 \) and \( M = 1 \): with this the most characteristic levels are included in the description. The potential is slightly different for even and odd values of \( \tau \) (or the \( \tau + 2M \) combination): in the former case \( c^+ = 11/4 \), while in the latter one \( c^- = 13/4 \) corresponding to \( \tau + 2M = 2 \) and \( 3 \), respectively. This slight difference manifests itself near \( \beta = 0 \) in the coefficient of the \( \beta^2 \) term. In Ref. [5] this slight ambiguity was handled by selecting the constant terms \( u_0^+ \) and \( u_0^- \) such that the minima of the two potentials are at the same energy.

The solutions are written as

\[
\phi_{n,\tau}(\beta) = N_{n,\tau} P_n(\beta^2)(\beta^2)^{\frac{\tau}{2}} \exp \left( -\frac{a}{4} \beta^4 - \frac{b}{2} \beta^2 \right), \tag{4}
\]
Explicit form of the lowest few energy eigenvalues and wavefunctions. Here \( \lambda_{\pm} = 2b \pm 2(b^2 + 10a)^{1/2} \), \( \lambda_{+} = 2b + 2(b^2 + 14a)^{1/2} \), and \( u_0 = u_0(c = 11/4) - u_0(c = 13/4) \) hold.

| \( \xi \) | \( \tau \) | \( M \) | \( E_{\xi,\tau} \) | \( E_{\xi,\tau} \) | \( \phi_{\xi,\tau} \) | \( \phi_{\xi,\tau} \) |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 5b + \lambda_\_ | 0 | \( N_{10}(b^2 - \lambda_- / 10) \) |
| 1 | 1 | 1 | 7b + \lambda_+ + u_0 | 2b + \lambda_+ - \lambda_- + u_0 | \( N_{11}(b^2 - \lambda_- / 14) \) |
| 1 | 2 | 0 | 9b | 4b - \lambda_- | \( N_{12}(b^2 - \lambda_- / 14) \) |
| 1 | 3 | 0 | 11b + u_0 | 6b - \lambda_- + u_0 | \( N_{13}(b^5 \) |
| 2 | 0 | 1 | 5b + \lambda_+ | \lambda_+ - \lambda_- | \( N_{20}(b^2 - \lambda_+ / 10) \) |
| 2 | 1 | 1 | 7b + \lambda_+ + u_0 | 2b + \lambda_+ - \lambda_- + u_0 | \( N_{21}(b^2 - \lambda_+ / 14) \) |

**Table 1.**

Figure 1. Excitation energies \( E_{\xi,\tau} = E_{\xi,\tau} - E_{1,0} \) as with \( a = 40000 \) fixed as a function of \( b \) (left panel) and with \( b = 200 \) fixed as a function of \( a \) (right panel).

where \( P_{n,\tau}(\beta^2) \) is a polynomial of the order \( n = 0, 1 \). The \( N_{n,\tau} \) normalization constants and the matrix elements necessary to calculate \( B(E2) \) values can be expressed exactly in terms of parabolic cylinder functions [5]. Note that normalizability requires \( a \geq 0 \) and that \( a = 0 \) corresponds to the harmonic oscillator limit. Table 1 displays the energy eigenvalues and the corresponding (unnormalized) eigenfunctions, while Figure 1 shows the trend of the energy eigenvalues for some fixed values of \( a \) and \( b \).

Note that the \( a \) and \( b \) parameters can be fitted exactly to two energy eigenvalues as \( a = E_{1,2}^2[E_{2,0}^2 - E_{1,2}^2] / 40, b = [2E_{1,2}^2 - E_{2,0}^2] / 4 \). In practice this means that knowing the energies of the 0\(^+\) state with \( \xi = 2 \) and that of the 2\(^+_1\) or 4\(^+_1\) state (with \( \xi = 1 \) and \( \tau = 2 \)) immediately provides us with a first approximation of the model parameters.

Obviously, the extrema of the potential depend on the sign of \( b \) and \( b^2 - 4ac^2 \), i.e. on the coefficients of the \( \beta^2 \) and \( \beta^4 \) terms in (3). If \( b^2 > 4ac^2 \) and \( b > 0 \) hold then the potential has a minimum at \( \beta = 0 \) and it increases with \( \beta \). When \( b^2 < 4ac^2 \) (irrespective of the sign of \( \beta \) the minimum shifts to a finite value of \( \beta \), while for \( b^2 > 4ac^2 \) and \( b < 0 \) (i.e. \( b < -2(ac^2)^{1/2} \) there are two minima (one at \( b = 0 \) and one at \( \beta > 0 \)) separated by a local maximum. The \( b^2 = 4ac^2 \) parabola thus separates the \( ab \) plane into three domains: to the right of it the minimum is spherical (\( \beta_{\text{min}} = 0 \)), above the parabola the minimum is deformed (\( \beta_{\text{min}} > 0 \)), while to the left there are two minima such that (\( \beta_{\text{min2}} > \beta_{\text{min1}} = 0 \). Crossing the right leg of the parabola thus corresponds to the transition from the spherical to the deformed phase, i.e. it is expected to be close to the \( E(5) \) critical point. The domain to the left of the parabola corresponds to
potential shapes typical for the $X(5)$ symmetry, so the sextic oscillator may also be appropriate to describe the $\beta$-dependent part of the $V(\beta, \gamma)$ potential of such systems.

3. Application to some $E(5)$ candidates

The signatures of the $E(5)$ symmetry originate from the parameter-free calculations with the five-dimensional infinite well potential in the $\beta$ shape variable [2] mimicking the situation in which a second-order phase transition occurs between the spherical and $\gamma$-unstable phases. The ratios of certain energy eigenvalues and electric quadrupole transition rates have been compared with experimental data and several nuclei have been proposed as candidates for the $E(5)$ symmetry. The first example was the $^{134}$Ba nucleus [8]. Later the $^{104}$Ru [9] and $^{102}$Pd [10] nuclei have also been discussed in terms of this scheme. Later in a systematic search for $E(5)$ candidates the $^{106}$Cd, $^{108}$Cd, $^{124}$Te and $^{128}$Xe nuclei have also been proposed [11]. More recently the $^{124}$Xe isotope has also been mentioned as a promising candidate [12].

From these nuclei the sextic oscillator has already been applied to $^{134}$Ba, although not in a full scale fit, rather as an illustration of the model [5]. It turned out that the sextic oscillator was able to reproduce the main signatures of the $E(5)$ symmetry by applying model parameters corresponding to a potential with a deformed minimum. As another application the energy spectrum of the chain of Ru isotopes has been fitted and the resulting potentials have been analyzed [13, 14]. It was shown that starting from $^{108}$Ru the spectrum is close to the $U(5)$ harmonic limit, then at $^{104}$Ru a flat potential develops, which evolves further to a potential with a deformed minimum for $^{106}$Ru and $^{108}$Ru.

Here we extend these studies to a number of isotopes near those proposed recently as $E(5)$ candidates. For this we first associate experimental levels, notably $2^+_1; 4^+_1, 2^+_2; 0^+_2, 3^+_1, 4^+_2, 6^+_1$ and $0^+_5$ with model states corresponding to quantum numbers $(\xi, \tau) = (1, 1), (1, 2), (1, 3)$ and $(2, 0)$. The identification of these levels is not too complicated, except for the second and third $0^+$ levels. In identifying these levels the predictions of the $E(5)$ scheme on the $B(E2)$ rates can be used: the $0^+(2, 0)$ state is known to decay strongly to the $2^+(1, 1)$ level, while the $0^+(1, 3)$ state prefers to decay to the $2^+(1, 2)$ level. In $^{134}$Ba, for example, the $0^+_5$ experimental level is thought to be the one with $(\xi, \tau) = (2, 0)$. According to the analysis of Ref. [11] the situation is similar in the case of $^{102}$Pd and $^{128}$Xe too, while in $^{106,108}$Cd and $^{124}$Te the $0^+_2$ experimental state is associated with the $(\xi, \tau) = (2, 0)$ level. The situation with the $B(E2)$ rates justifies the same treatment of $^{100,102,104}$Ru too [17].

In our analysis we considered altogether 6 Ru, 3 Pd, 2 Cd, 3 Te, 6 Xe and 2 Ba isotopes for which the data set was reasonably complete. Their energy levels are associated with the model states in the upper half Tables 2, 3 and 4. The experimental data set [17] was sometimes incomplete: in some cases only the $0^+_2$ level was known, and often there is no information on the $B(E2)$ rates. In identifying the $0^+$ levels thus we followed the arguments discussed above for each member of the given isotope chain.

In the next step we performed a least square fit of the $a$ and $b$ parameters such that states with ambiguous $J^\pi$ assignment were considered with weight 0.5. One further special feature of the $E(5)$ symmetry is that the $2^+_3$ and $4^+_1$ levels are degenerate, as are the $3^+_1$, $4^+_2$, $6^+_1$ and $0^+_5$ (or $0^+_7$) levels. Obviously, this degeneracy is not fulfilled in the experimental spectrum, so the fit concerned an average value of these energy levels.

In the case of the light Ru isotopes $a$ turned out to be a small negative value, which is clearly not allowed in the wave functions, so in these cases we chose $a = 0$ corresponding to the harmonic oscillator limit. Indeed, the energy spectrum of these nuclei is close to a harmonic vibrator. The resulting parameters are displayed in the middle section of the tables, while the calculated energy eigenvalues are displayed in the bottom section. The last line in the Tables indicates whether the corresponding potentials have spherical or deformed minimum. As discussed before, this is clearly determined by the sign of $b^2 - 4ac^+ = b^2 - 11c$. 

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Table 2. Experimental levels (in keV) of Ru and Pd nuclei corresponding to the $\phi_{\xi, \tau}$ model states, the $a$ and $b$ model parameters and the $E_{\xi, \tau}$ energy eigenvalues calculated from them. Experimental energies in parentheses indicate ambiguous $J^\pi$ assignment. The last row displays the minimum of the corresponding $V(\beta)$ potential.

| $A$ | $^{98}\text{Ru}$ | $^{100}\text{Ru}$ | $^{102}\text{Ru}$ | $^{104}\text{Ru}$ | $^{106}\text{Ru}$ | $^{102}\text{Pd}$ | $^{104}\text{Pd}$ | $^{106}\text{Pd}$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $N$ | 54              | 56              | 58              | 60              | 62              | 64              | 56              | 58              | 60              |
| $E_{\text{Exp}}^1(2^+)$ | 652             | 540             | 475             | 358             | 270             | 242             | 556             | 556             | 512             |
| $E_{\text{Exp}}^1(2^+)$ | 1414            | 1362            | 1103            | 893             | 792             | (708)           | 1534            | 1341            | 1128            |
| $E_{\text{Exp}}^1(4^+)$ | 1398            | 1226            | 1106            | 888             | (715)           | 665             | 1276            | 1323            | 1129            |
| $E_{\text{Exp}}^2(0^+)$ | 1741            | 1837            |                 |                 |                 |                 | 1593            | 1334            | 1134            |
| $E_{\text{Exp}}^2(3^+)$ | 1797            | 1522            | 1242            | (1092)          | (945)           | 2112            | 1821            | 1558            |                 |
| $E_{\text{Exp}}^2(4^+)$ | 2267            | (2075)          | (1799)          | (1183)          | 2138            | 2082            | 1932            |                 |                 |
| $E_{\text{Exp}}^2(6^+)$ | 2223            | 2076            | 1873            | 1556            | (1296)          | 1240            | 2111            | 2249            | 2076            |
| $E_{\text{Exp}}^3(0^+)$ | 1322            | 1130            | 944             | 988             | 991             | 976             | 1658            | 1793            | 1706            |
| $a$ | 0               | 0               | 0               | 1496            | 4190            | 5154            | 8918            | 15692           | 15812           |
| $b$ | 347             | 318             | 283             | 216             | 143             | 114             | 283             | 191             | 134             |
| $E_{\text{Th}}^1$ | 695             | 635             | 566             | 409             | 308             | 297             | 528             | 511             | 461             |
| $E_{\text{Th}}^2$ | 1389            | 1270            | 1131            | 930             | 786             | 735             | 1388            | 1261            | 1108            |
| $E_{\text{Th}}^3$ | 2084            | 1905            | 1697            | 1362            | 1157            | 1108            | 1999            | 1906            | 1708            |
| $\beta_{\text{min.}}$ | H.O.            | H.O.            | H.O.            | 0               | > 0             | > 0             | > 0             | > 0             | > 0             |

4. Discussion

Although the degree of completeness of the data sets is different for the members of the isotope chains, the parameters $a$ and $b$ change rather smoothly. The only exception is $^{122}$Xe, which strays away from the rest of the chain occupying a rather straight line with small $b$ and $a$ increasing with increasing mass number $A$. This behaviour may be explained by the fact that the $0^+_2$ level is not known in $^{122}$Xe, so the fit might be distorted. The other heavy isotopes, $^{134}$Ba and $^{136}$Ba are in the same region of the $a - b$ parameter space, although there $a$ decreases with increasing $A$. It is also remarkable that the parameters chosen in Ref. [5] to generate the spectroscopic properties of $^{134}$Ba ($a = 40000$, $b = 200$) are not too far from the fitted values.

The evolution of the Ru chain is also rather smooth: $a$ and $b$ move monotonously with increasing $A$. Furthermore, in the process the chain crosses the parabola separating the shape phases with spherical and deformed potential minimum. In fact, it is the $^{104}$Ru nucleus that is closest to the parabola, which is in agreement with the findings of [9]. The Te chain also crosses this boundary, to which $^{124}$Te (and also $^{122}$Te) lies rather close. The former isotope was proposed as another example for $E(5)$ symmetry in [11]. The Pd chain also behaves in a similar way, and although it does not cross the parabola, the $^{102}$Pd isotope, another candidate in [11] gets to it rather close from the deformed side. Both the $^{106}$Cr and $^{108}$Cr, the two remaining $E(5)$ candidates of [11] are also close to the parabola from the side of the spherical minimum.
Table 3. The same as Table 2 for Cd, Te and Ba nuclei.

| A      | $^{106}$Cd | $^{108}$Cd | $^{122}$Te | $^{124}$Te | $^{126}$Te | $^{134}$Ba | $^{136}$Ba |
|--------|------------|------------|------------|------------|------------|------------|------------|
| N      | 58         | 60         | 70         | 72         | 74         | 78         | 80         |
| $E_{1,1}^\text{Exp.}$ $(2^+)$ | 633        | 633        | 564        | 603        | 666        | 605        | 819        |
| $E_{1,2}^\text{Exp.}$ $(2^+)$ | 1717       | 1602       | 1257       | 1326       | 1420       | 1170       | 1551       |
| $E_{1,2}^\text{Exp.}$ $(4^+)$ | 1494       | 1508       | 1181       | 1249       | 1361       | 1401       | 1867       |
| $E_{1,3}^\text{Exp.}$ $(0^+)$ | 2144       | 1913       | (1750)     | 1883       | 1761       | 1579       |            |
| $E_{1,3}^\text{Exp.}$ $(3^+)$ | (2254)     | 2146       | 2483       | 2128       | 1643       | 2431       |            |
| $E_{1,3}^\text{Exp.}$ $(4^+)$ | 2104       | 2239       | 1909       | 1958       | 2013       | 1970       | 2054       |
| $E_{1,3}^\text{Exp.}$ $(6^+)$ | 2492       | 2541       | 1751       | 1747       | 1776       | 2118       | 2344       |
| $E_{2,0}^\text{Exp.}$ $(0^+)$ | 1795       | 1720       | 1357       | 1657       | 1873       | (2160)     | 2141       |
| a      | 7920       | 6989       | 2921       | 11917      | 17131      | 27308      | 20510      |
| b      | 350        | 339        | 286        | 251        | 206        | 65         | 223        |
| $E_{1,1}^\text{Th.}$       | 633        | 616        | 539        | 529        | 540        | 516        | 589        |
| $E_{1,2}^\text{Th.}$       | 1600       | 1538       | 1240       | 1355       | 1336       | 1184       | 1455       |
| $E_{1,3}^\text{Th.}$       | 2301       | 2217       | 1812       | 1989       | 2015       | 1890       | 2195       |
| $E_{2,0}^\text{Th.}$       | 1798       | 1720       | 1334       | 1707       | 1849       | 2107       | 2019       |
| $\beta_{\text{min.}}$      | 0          | 0          | 0          | >0         | >0         | >0         | >0         |

It is remarkable that the best $E(5)$ candidates are rather close to the boundary separating the shape phases corresponding to spherical and deformed minimum. The two exceptions are $^{128}$Xe and $^{134}$Ba, which differ from the remaining $E(5)$ candidates in that it is the $0^+_2$ state that is assigned to $\xi = 2$ level, and the $0^+_3$ level is substantially lower (400 keV). (Although $0^+_2$ plays the same role in $^{102}$Pd, the energy difference between the $0^+_2$ and $0^+_3$ states is much smaller.)

It has to be noted that we did not adjust the $B$ inertia parameter in (1), although it is known to vary with the mass $A$ approximately as $B \sim A^{5/3} [6]$. Neglecting this effect does not influence the results significantly within an isotope chain. The effect is more significant within the whole mass range considered here, but it can be handled by a rescaling of $a$, $b$ and $c$.

Due to the lack of space we did not consider the electric quadrupole transition rates, although they can be calculated in closed form [5]. This task will be completed elsewhere. Another possible task for the future is developing the formalism to treat nuclei with $X(5)$ symmetry. For this the $\gamma$ degrees of freedom also have to be incorporated into the potential. There are several ways to approximate this effect [3, 15, 16], so the application of the sextic oscillator to $X(5)$ nuclei seems not only desirable but also possible. For this potential shapes with two minima (corresponding to $b < 0$ and $b^2 > 4ac^2$) seem suitable.

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Table 4. The same as Table 2 for Xe nuclei.

| $A$ | $^{120}\text{Xe}$ | $^{122}\text{Xe}$ | $^{124}\text{Xe}$ | $^{126}\text{Xe}$ | $^{128}\text{Xe}$ | $^{130}\text{Xe}$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $N$ | 66 | 68 | 70 | 72 | 74 | 76 |
| $E_{1,1}^{\text{Exp.}}(2^+)$ | 322 | 331 | 354 | 389 | 443 | 536 |
| $E_{1,2}^{\text{Exp.}}(2^+)$ | 876 | (843) | 846 | 880 | 969 | 1122 |
| $E_{1,2}^{\text{Exp.}}(4^+)$ | 796 | 828 | 879 | 942 | 1033 | 1205 |
| $E_{1,3}^{\text{Exp.}}(0^+)$ | 909 | 1149 | 1269 | 1314 | 1583 | 1793 |
| $E_{1,3}^{\text{Exp.}}(3^+)$ | 1271 | 1214 | 1248 | 1318 | 1430 | 1633 |
| $E_{1,3}^{\text{Exp.}}(4^+)$ | 1401 (1402) | 1438 | 1488 | 1604 (1808) | |
| $E_{1,3}^{\text{Exp.}}(6^+)$ | 1397 | 1467 | 1549 | 1634 | 1737 | 1944 |
| $E_{2,0}^{\text{Exp.}}(0^+)$ | 1623 | 1690 | 1760 | 1877 | 2017 |
| $a$ | 16148 | 9521 | 17760 | 19239 | 21875 | 24822 |
| $b$ | -17 | 95 | 2 | 5 | 25 | 65 |
| $E_{1,1}^{\text{Th.}}$ | 350 | 349 | 380 | 398 | 437 | 493 |
| $E_{1,2}^{\text{Th.}}$ | 770 | 835 | 846 | 888 | 986 | 1134 |
| $E_{1,3}^{\text{Th.}}$ | 1267 | 1293 | 1380 | 1447 | 1594 | 1809 |
| $E_{2,0}^{\text{Th.}}$ | 1609 | 1291 | 1686 | 1755 | 1873 | 2010 |
| $\beta_{\text{min.}}$ | $>0$ | $>0$ | $>0$ | $>0$ | $>0$ | $>0$ |

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