Rational approximations in Analytic QCD

Gorazd Cvetić\textsuperscript{1,2,*} and Héctor E. Martínez\textsuperscript{1,†}

\textsuperscript{1}Department of Physics, Universidad Técnica Federico Santa María, Valparaíso, Chile
\textsuperscript{2}Center of Subatomic Studies and Scientific-Technological Center of Valparaíso, Chile

(Dated: October 15, 2009)

[In comparison with v1, in v2 Figs.6-8 are corrected due to a programming error; analysis extended to two different IR cutoffs; Introduction rewritten; to appear in J.Phys.G.]

We consider the “modified Minimal Analytic” (mMA) coupling that involves an infrared cut to the standard MA coupling. The mMA coupling is a Stieltjes function and, as a consequence, the paradiagonal Padé approximants converge to the coupling in the entire $Q^2$-plane except on the time-like semiaxis below the cut. The equivalence between the narrow width approximation of the discontinuity function of the coupling, on the one hand, and this Padé (rational) approximation of the coupling, on the other hand, is shown. We approximate the analytic analogs of the higher powers of mMA coupling by rational functions in such a way that the singularity region is respected by the approximants. Several comparisons, for real and complex arguments $Q^2$, between the exact and approximate expressions are made and the speed of convergence is discussed. Motivated by the success of these approximants, an improvement of the mMA coupling is suggested, and possible uses in the reproduction of experimental data are discussed.

PACS numbers:

I. INTRODUCTION

In perturbative QCD (pQCD) calculations, the coupling $a(Q^2) \equiv \alpha_s(Q^2)/\pi$ shows non-physical singularities at low energy ($-q^2 \equiv Q^2 \lesssim 0.1$ GeV$^2$). The aim of analytic QCD (aQCD) is to give a coupling $A_1(Q^2)$ which is analytic at low $Q^2 (> 0)$ and reproduces the high energy behavior of $a(Q^2)$. Using the Cauchy theorem we can write QCD running coupling in the integral form

$$a(Q^2) = \frac{1}{\pi} \int_{\sigma=\Lambda^2-\eta}^{\infty} d\sigma \frac{\rho_1(\sigma)}{\sigma + Q^2},$$  

(1)

where $\eta \to +0$ and $\rho_1(\sigma)$ is the discontinuity function of $a(Q^2)$ along the cut axis in the complex $Q^2$-plane at $n$-loop approximation given by $\rho_1^{(n-\ell)}(\sigma) = \text{Im}[a^{(n-\ell)}(-\sigma - i\epsilon)]$. The Minimal Analytic (MA) procedure of Shirkov and Solovtsov \cite{1} removes the pQCD contribution of the unphysical cut, $0 < -\sigma \leq \Lambda^2$, keeping the discontinuity elsewhere unchanged

$$A_1^{(MA)}(Q^2) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} d\sigma \frac{\rho_1(\sigma)}{\sigma + Q^2}. $$  

(2)

This expression doesn’t have singularities for $Q^2 > 0$ and, as required, reproduces the high energy behavior of $a(Q^2)$. In fact, $A_1^{(MA)}(Q^2)$ is analytic in the entire complex plane of $Q^2$ with the exception of negative semiaxis, reflecting the analyticity properties of the space-like observables.

We will consider a modification to the MA coupling \cite{2} of Eq. (2). Following Ref. \cite{3}, the lower limit of integration is increased to a certain value $M^2_\rho$ which we set to be $\sim M^2_\pi (\sim 10^{-2}-10^{-1}$ GeV$^2$). Thus, the modified MA coupling is given by the following dispersion relation:

$$A_1^{(mMA)}(Q^2) = \frac{1}{\pi} \int_{M^2_\rho}^{\infty} d\sigma \frac{\rho_1(\sigma)}{\sigma + Q^2}. $$  

(3)

One of the motivations for modifying the MA in this way is to include the point $Q^2 = 0$ and its vicinity in the analyticity region of $A_1(Q^2)$, something reflected by analyticity properties of the space-like observables, among them

*Electronic address: gorazd.cvetic@usm.cl
†Electronic address: hector.martinez@usm.cl
the V-type Adler function, where the infrared (IR) cutoff \( M_0^2 \) is: \( M_0^2 \approx M_z^2 \) (\( \sim \Lambda_{\text{QCD}}^2 \)).\(^1\) When \( M_0^2 > 0 \), the analytic coupling \( A_1 \) is a Stieltjes function, with the radius \( R \) of convergence for its Taylor expansion around \( Q^2 = 0 \) given by \( R = M_0^2 \). Stieltjes functions have the attractive property that their paradiagonal Padé approximants must converge to such functions at any complex value of the argument (except on the cut) when the order index of the approximants increases. Such a behavior is not guaranteed in MA model because there \( M_0^2 = 0 \) and, consequently, \( Q^2 = 0 \) is not a point of analyticity and \( A_1^{(\text{mA})} \) is not a Stieltjes function.

In this work we investigate the behavior of the coupling \( A_1^{(\text{mA})}(Q^2) \) (with \( M_0^2 \sim M_z^2 \)) of the mMA model \( \text{(3)} \) and its paradiagonal Padé approximants. In Sec. II the basic definitions of rational (Padé) approximants and of Stieltjes functions are presented. In Sec. III we demonstrate that the narrow width approximations for the discontinuity function \( \rho_1(\sigma) \) of \( A_1^{(\text{mA})} \) are equivalent to approximating \( A_1^{(\text{mA})}(Q^2) \) by paradiagonal Padé approximants; we investigate the convergence of the latter approximants to the “exact” function values \( A_1^{(\text{mA})}(Q^2) \) for \( Q^2 > 0 \). In Sec. IV we extend our analysis to the higher power analogs \( A_1^{(\text{mA})} \) of \( a^n \) \( (n \geq 2) \), construct the approximants for \( A_1^{(\text{mA})} \) based on the aforementioned paradiagonal Padé approximants for \( A_1^{(\text{mA})} \) and investigate their behavior for \( Q^2 > 0 \). In Sec. V we investigate the behavior of the aforementioned approximants for complex values of \( Q^2 \). In Sec. VI we propose an extension (improvement) of the mMA model, motivated by the success of the narrow width approximations of the discontinuity function \( \rho_1(\sigma) \) in mMA; further, we point out the prospects for applications of such simple analytic QCD models to fitting experimental data. Section VII represents the summary of our results.

II. PADÉ APPROXIMANTS AND STIELTJES FUNCTIONS

Consider a function \( f(z) \), of a complex variable \( z \), with power expansion about the origin given by

\[
f(z) \approx \sum_{n=0}^{N+M} f_n z^n.
\]

The Padé (or: rational) approximant, \( R_M^N(z) \), is defined as the rational function

\[
R_M^N(z) = \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{M} b_n z^n},
\]

satisfying the condition that its expansion about \( z = 0 \) matches \( N + M + 1 \) terms in the power series expansion in \( \text{(4)} \). The paradiagonal Padé \( R_{M-1}^M \) can be written (using partial fractions) in the following way:

\[
R_{M-1}^M(z) = \sum_{n=1}^{M} \frac{d_n}{z + z_n},
\]

where \( z_n \) are zeros of the denominator in Eq. \( \text{(5)} \).

If the function \( f(z) \) is a Stieltjes function with a finite radius of convergence \( R > 0 \), i.e.

\[
f(z) = \frac{1}{\pi} \int_{0}^{1/R} d\tau \frac{g(\tau)}{1 + \tau z},
\]

with \( g(\tau) \) a nonnegative function, thus \( f(z) \) having the real coefficients of its series expansion around \( z = 0 \) given by

\[
f_n = \frac{(-1)^n}{\pi} \int_{0}^{1/R} \tau^n g(\tau) d\tau,
\]

then a strong convergence theorem in the theory of Padé approximants applies which says: in the limit \( M \rightarrow \infty \), \( R_{M+J}^M(z) \) \( (J \geq -1) \) is equal to \( f(z) \), and the poles of \( R_{M+J}^M(z) \) are simple poles which lie on the interval of the negative real axis given by \( -\infty < z < -R \) (Ref. \( \text{[3]} \), Sections 5.2, 5.4).

In particular, when \( f(z) \) is a Stieltjes function, the Padé \( R_{M-1}^M(z) \) of \( f(z) \) has the form \( \text{(8)} \) with \( d_n > 0 \) and \( z_n > 0 \), the poles \( -z_n \) having the following ordering:

\(-\infty < -z_M < -z_{M-1} < \cdots < -z_1 < -R\).

\(^1\) We note that \( M_0 \) is a renormalization scheme (RSch) dependent quantity, just like \( \Lambda \) is (e.g., \( \Lambda \) in V-RSch and \( \text{MS} \) RSch differ); \( s_0 \equiv \frac{M_0^2}{\Lambda^2} \) is RSch-independent. Therefore, while we expect \( M_0 \sim M_n \), \( M_0 \) \( \Leftrightarrow s_0 \) is a free parameter of the model.
III. APPROXIMATION OF THE COUPLING BY RATIONAL FUNCTIONS

In order to establish a relationship between $A_{1}^{(mMA)}(Q^2)$ and a Padé approximant, we first note that the modified MA coupling in Eq. (3) is a Stieltjes function, as defined in Eq. (7) by identifying: $Q^2 = z$, $\sigma = 1/\tau$ and $\rho_1(\sigma) = \tau \tilde{g}(\tau)$. It has the convergence radius $M_0^2$, and the series expansion about $Q^2 = 0$ given by

$$A_{1}^{(mMA)}(Q^2) = \frac{1}{\pi} \int_0^{1/M_0^2} d\tau \frac{\tilde{g}(\tau)}{(1 + \tau Q^2)} = \sum_{n=0}^{\infty} \tilde{L}_n(Q^2)^n, \tag{9}$$

where

$$\tilde{L}_n = \frac{(-1)^n}{\pi} \int_0^{1/M_0^2} \tau^n \tilde{g}(\tau) d\tau, \tag{10}$$

and $\tilde{g}(\tau) = \sigma \rho_1(\sigma)$. For simplicity, we want the expansion coefficients to be dimensionless. To fulfill this, we introduce dimensionless variables $s = \sigma/\Lambda^2 = 1/\tau$ and $u = Q^2/\Lambda^2$ in Eq. (3), where $\Lambda^2 = \Lambda_{QCD}^2$ (in MS convention; $\Lambda_{QCD}^2 \approx 10^{-1}$ GeV$^2$). Consequently, Eqs. (9) and (10) obtain the form

$$A_{1}^{(mMA)}(u\Lambda^2) = \frac{1}{\pi} \int_0^{1/s_0} dt \frac{g(t)}{1+tu} = \sum_{n=0}^{\infty} L_n u^n, \quad \tag{11}$$

with

$$L_n = \frac{(-1)^n}{\pi} \int_0^{1/s_0} t^n g(t) d\tau, \tag{12}$$

where $u = Q^2/\Lambda^2 = z/\Lambda^2$, $s_0 = M_0^2/\Lambda^2$, $L_n = \tilde{L}_n(\Lambda^2)^n$ are dimensionless, and $g(t) \equiv \rho_1(\Lambda^2/\tau)/\tau$.

An important consequence of the fact that $A_{1}^{(mMA)}$ is a Stieltjes function is that the series in Eq. (9) only converges within the finite disc in the $Q^2$-plane ($Q^2 < M_0^2$), elsewere the series is divergent. However, we will see that, because of the aforementioned theorem [3], only few coefficients of this divergent series are needed in order to evaluate $A_{1}^{(mMA)}(Q^2)$ in regions well beyond the convergence disc, via approximate analytic continuation in the form of Padé approximants $R_{M-1}^{M-1}(z)$.

Following Ref. [4], we can approximate $\theta(s) \equiv g(t=1/s) \equiv s \rho_1(\sigma=\Lambda^2 s)$ by a singular series expansion

$$\frac{s \rho_1(s\Lambda^2)}{\pi} \equiv \frac{g(1/s)}{\pi} \equiv \frac{\theta(s)}{\pi} \approx \sum_{n=1}^{M} f_n^2 s_n \delta(s-s_n), \tag{13}$$

where $s = \sigma/\Lambda^2$, $s_n = M_n^2/\Lambda^2$ and $f_n^2$ are all positive dimensionless quantities. In Ref. [4], this type of approximation was applied to the spectral function of the vector channel vacuum polarization function $\Pi_V(Q^2)$. It was motivated there by the highly singular behavior of the vector spectral function in the limit $N_c \to \infty$. In our case we aren’t working in that limit, but we note that the spectral function $\rho_1(\sigma)$ of $A_{1}^{(mMA)}(Q^2)$ is positive; although $\theta(s)$ has finite values, such a function can be well approximated as a sum of positively weighted Dirac deltas for the purpose of integration. We will see that this approximation is equivalent to approximating the mMA coupling $A_{1}^{(mMA)}(Q^2)$ as the Padé approximant $R_{M-1}^{M-1}(u)$.

Using the singular approximation (13) for $\theta(s)$ in Eq. (11), we obtain

$$A_{1}^{(mMA)}(u\Lambda^2) \approx \sum_{n=1}^{M} \frac{f_n^2}{u + s_n} = \sum_{n=1}^{M} \frac{F_n^2}{Q^2 + M_n^2}, \tag{14}$$

where $F_n^2 = f_n^2 \Lambda^2$. This relation is the same as $R_{M-1}^{M-1}(z)$ Padé of $A_{1}^{(mMA)}$ given in Eq. (6), with $z = u$, $z_n = s_n$ and $d_n = f_n^2$.
Therefore, by approximating $\theta(s)$ by positive Dirac deltas we obtain $A_1^{(mA)}$ written in a (paradiagonal) Padé form. Further, the following relation holds \[3\]:

$$|A_1^{(mA)}(u\Lambda^2) - R_{M}^{M-1}(u)| \leq K \left| \frac{\sqrt{s_0 + u} - \sqrt{s_0}}{\sqrt{s_0 + u} + \sqrt{s_0}} \right|^{2M},$$ \hspace{1cm} (16)

where $K$ is a constant. Therefore $R_{M}^{M-1}(u) \rightarrow A_1^{(mA)}(u\Lambda^2)$ when $M \rightarrow \infty$.

In our evaluations we will use the perturbative coupling given by the solution of the two-loop renormalization group (RG) equation

$$a^{(2-\epsilon)}(u\Lambda^2) = \frac{-1}{c_1(1 + W_{\mp 1}(z_{\pm}))},$$ \hspace{1cm} (17)

where $W_{\mp 1}(z)$ is the Lambert function (branches $n = \mp 1$), and the argument $z_{\pm}$ is given by

$$z_{\pm} = \left(\frac{|u|^{-\beta_0/c_1}}{c_1 e}\right) e^{i(\pm \pi - \phi \beta_0/c_1)},$$ \hspace{1cm} (18)

where $Q^2 = |Q|^2 e^{i\phi}$, $|u| = |Q|^2/\Lambda^2$, $\beta_0 = (11 - 2n_f/3)/4$, $c_1 = \beta_1/\beta_0$, $\beta_1 = (102 - 38n_f/3)/16$. For $0 < \phi < \pi$, the branch $W_{-1}(z_+)$ is chosen, for $-\pi < \phi < 0$, the branch $W_{+1}(z_-)$ is chosen. At low energies ($|Q|^2 \lesssim 10$ GeV$^2$), the number of active quark flavors is $n_f = 3$. For details see Ref. \[3\]. With the coupling given in Eq. (17), the discontinuity function becomes

$$\rho_{1}^{(2-\epsilon)}(s\Lambda^2) = \text{Im} \left[ \frac{-1}{c_1(1 + W_{+1}(z_-(s)))} \right],$$ \hspace{1cm} (19)

with $\phi = -\pi$ and $|u| = \sigma/\Lambda^2 = s$ in Eq. (18). These expressions are in fact valid at any $n$-loop level ($n \geq 2$) in the ’t Hooft renormalization scheme.

![FIG. 1: $A_1^{(mA)}(Q^2)$ (continuous), and $A_1^{(mA)}(Q^2)$ at two-loop level for various values of $M_0$ cut. Throughout this paper we use $n_f = 3$ and $\Lambda_{\overline{MS}} = 0.35$ GeV.](image)

We set $\Lambda = 0.35$ GeV (at $n_f = 3$).\footnote{In Ref. \[3\], the value $\Lambda_{(n_f = 3)} \approx 0.4$ GeV was obtained in MA by requiring that the MA model reproduce measured values of QCD observables at higher energies ($|Q| \gtrsim 10$ GeV) in $\overline{\text{MS}}$ scheme.} This value can be changed later when we fit experimental data. Here we will compare numerically the accuracy of the rational approximants with the “exact” numbers, i.e., those obtained by evaluating numerically integrals Eq. (3).
from \( \text{A} \) and (11): respectively. We recall that these values are much higher than the convergence radius of the Taylor series, Eqs. (9).

In Fig. 2: The full line is the “exact” mMA coupling, Eq. (3). In both Figures, the dashed lines (starting from the bottom) are all Padé approximants \( R_{M}^{M-1}: R_{1}^{0}, R_{2}^{0}, ..., R_{10}^{0} \). The cut value \( M_0 \) is: (a) \( M_0 = 2M_\pi \); (b) \( M_0 = 3M_\pi \).

In order to give an idea of how the approximants work, we list in Table I the values of \( R \) at various values of the cutoff \( M_0 \) with the standard MA coupling \(^3\) at low positive \( Q^2 \)’s. These values are calculated by performing integrals in the dispersive relations (3) and (2) for each \( Q^2 \).

The agreement is even better.

All the rational approximants \( R_{M}^{M-1}(u) \) up to \( R_{10}^{0}(u) \) for \( A_1^{(mMA)}(uA^2) \) are shown in the Figs. (2) for two choices of the IR cutoff in Eq. (3): \( M_0 = 2M_\pi, 3M_\pi \). We recall that \( M_0^2 \) is the convergence radius of the Stieltjes function \( A_1^{(mMA)}(Q^2) \). Since this radius is larger in Fig. (2b), the convergence of the Padé approximants is also faster there. The rational function \( R_{10}^{0}(Q^2) \) matches \( A_1^{(mMA)}(Q^2) \) in Fig. (2a) quite well for all \( 0 < Q^2 < 10 \text{ GeV}^2 \), and in Fig. (2b) the agreement is even better.

In order to give an idea of how the approximants work, we list in Table I the values of \( Q^2 \) at which the deviation from \( A_1^{(mMA)}(Q^2) \) reaches 0.1% (the deviation increases with increasing \( Q^2 \)), for the case \( M_0 = 2M_\pi \). For the Padé approximant \( R_{M}^{M-1}(u) \) with \( M = 10 \) the deviation is less than 0.1% for all \( Q^2 < 2.5 \text{ GeV}^2 \), and less than 1% for all \( Q^2 < 5.4 \text{ GeV}^2 \) (when the Padé index is \( M = 20 \), these values increase to 10.33 GeV\(^2\) and 22.63 GeV\(^2\), respectively). We recall that these values are much higher than the convergence radius of the Taylor series, Eqs. (9) and (11): \( R = Q^2 = M_0^2 = 0.0784 \text{ GeV}^2 \). We see that the first pole \( -s_1 \) converges from below to the IR cut...

---

\(^3\) \( A_1^{(MA)}(0) = 1/\beta_0 \approx 0.44 \); however, the slope \( dA_1^{(MA)}(Q^2)/dQ^2 \) at \( Q^2 = 0 \) is infinite.
integrals of the (perturbative) discontinuity function $\rho$ is:

$$\int \rho \, ds.$$ 

The correspondence between the powers $Q^2 = \Lambda^2$ and $s_\pi$ is:

$$s_\pi \equiv M^2 / \Lambda^2$$

where $s_\pi = M^2 / \Lambda^2$ ($-s_\pi = -0.64$ in our case) when index $M$ increases, as expected. For example, when $M = 10$, we obtain: $s_1 = 0.6492$ ($\approx s_0 = 0.6400$), $s_2 = 0.6907$, etc. When the cut $M_0$ is increased from $2M_\pi$ to $3M_\pi$ ($\Rightarrow -s_0 = -1.44$), we obtain for $M = 10$: $s_1 = 1.4605$ ($\approx s_0 = 1.44$), $s_2 = 1.5525$, etc.

### IV. APPROXIMANTS FOR THE HIGHER POWER ANALOGS OF THE ANALYTIC COUPLING

To obtain the analogs of the higher powers of the analytic coupling in this formalism, we use relations given in Ref. [7]. At the 3-loop level truncated series for the analytic coupling, we have:

$$A_2(Q^2) = A_2(Q^2) - c_1A_3(Q^2), \quad A_3(Q^2) = A_3(Q^2),$$

(20)

where

$$A_n(Q^2) = (-1)^{n-1} \frac{\partial^{n-1} A_1(Q^2)}{\beta_0^{n-1}(n-1)!} \frac{1}{\partial (\ln Q^2)^{n-1}}.$$

(21)

The correspondence between the powers $a^k$ of the perturbative coupling $a(Q^2) = \alpha_s(Q^2)/\pi$ and the above quantities is: $a^k \rightarrow A_k$. The couplings $A_k$ are the analytic versions ("analogs") of higher powers needed for evaluation of observables. We note that in general $A_k \neq A_k^k$ (for further discussion, c.f. Sec. III of Ref. [7]).

Using the dispersive integral expression Eq. (3) in Eqs. (21), we obtain explicit expressions for $A_k$’s in terms of integrals of the (perturbative) discontinuity function $\rho_1$:

$$A_2^{(mMA)}(u \Lambda^2) = \frac{u}{\beta_0 \pi} \int_{s_0}^{\infty} \frac{\rho_1(s \Lambda^2)}{(s + u)^2} \, ds - c_1A_3^{(mMA)}(u \Lambda^2),$$

(22)

$$A_3^{(mMA)}(u \Lambda^2) = \frac{u}{2 \beta_0 \pi} \int_{s_0}^{\infty} \frac{\rho_1(s \Lambda^2)}{(s + u)^3} \, ds.$$

(23)

We see from here that the quantities $A_k^{(mMA)}(Q^2)$ have the same location of the singularities as $A_1^{(mMA)}$: $Q^2 < -M_0^2$ ($= -\Lambda^2 s_0$). On the other hand, the poles $Q^2 = -M_0^2 = -s_0 \Lambda^2$ ($n = 1, \ldots, M$) of the Padé approximants $R_{M-1}^M$ for $A_1$, Eq. (15), do reflect, in a discretized manner, the singularity cut of $A_1^{(mMA)}$; namely, these poles appear on the negative real axis in the range: $-\infty < -M_0^2 < -M_{M-1}^2 < \cdots < -M_1^2 < M_0^2$. Such a pole structure of $R_{M-1}^M(u)$ is guaranteed because $A_1^{(mMA)}$ is a Stieltjes function. On the other hand, the analytic higher power analogs $A_k^{(mMA)}(Q^2)$ ($k \geq 2$) are no longer Stieltjes functions. Therefore, if we construct the Padé approximants $R_{M-1}^M(u)$ for $A_k^{(mMA)}(Q^2)$ ($k \geq 2$) in the usual way, i.e., based on the first $2M$ coefficients of their Taylor expansion around $Q^2 = 0$, there is no longer the guarantee that all the poles of the obtained rational function lie on the negative real axis (and below $-M_0^2$). For instance, the Padé approximant $R_{10}^1$ computed from the series of $A_3$ has a non-physical pole on the positive real axis. One way to avoid this problem is to compute the rational approximants of $A_k$'s directly from the derivatives of the Padé approximant of $A_1^{(mMA)}$. If $R_{M-1}^M(u)$ are the paradiagonal Padés that approximate $A_1^{(mMA)}(u \Lambda^2)$, we shall define the rational approximants $R_2(u)$ and $R_3(u)$ for $A_2^{(mMA)}(u \Lambda^2)$ and $A_3^{(mMA)}(u \Lambda^2)$, respectively.
respectively, in the following way:

\[ R_2(u) = \tilde{R}_2(u) - c_1 \tilde{R}_3(u), \quad R_3(u) = \tilde{R}_3(u), \]  

(24)

where

\[ \tilde{R}_n(u) = \frac{(-1)^{n-1} \beta_0^{n-1} \partial^{n-1} R^{M-1}_M(u)}{(n-1)!}. \]  

(25)

A comparison between \( R_k \) and their respective \( A_k \), computed from \( R^{10}_M(u) \) (i.e., Padé index \( M = 10 \)), is shown in Figs. 3.

The accuracy of these approximants decreases notably compared to the \( R^{M-1}_M \) approximants of \( A^{(mMA)}_1 \). The accuracy of the approximants increases when the cut \( M_0 \) increases, e.g., from \( 2M_\pi \) to \( 3M_\pi \).

**V. COMPLEX ARGUMENTS**

In evaluation of observables, sometimes evaluation of the analytic coupling and their power analogs at complex values of arguments is needed (e.g., see App. C of Ref. [7]). For some complex arguments \( Q^2 = |Q^2| \exp(i\phi) \mid u \equiv Q^2/A^2 = |u| \exp(i\phi) \)], the results based on the approximant \( R^{10}_M(u) \) (Padé index \( M = 10 \)), are shown in Figs. 4-7. When \( \phi = \pi/4 \), Figs. 4 (a), (b) show that the approximation for \( A^{(mMA)}_1(Q^2) \) works with less than 1% error up to \( |Q^2| \approx 10 \text{ GeV}^2 \) for the real part, and \( |Q^2| \approx 3 \text{ GeV}^2 \) for the imaginary part. The approximants \( R_2(u) \) and \( R_3(u) \) for \( A_2 \) and \( A_3 \), (see Figs. 6 (a), (b) fail already at lower \( |Q^2| \). Figs. 5 (a), (b) show the approximant for \( A^{(mMA)}_1(Q^2) \) when \( \phi = 3\pi/4 \). Figs. 6 (a) indicate that the relative accuracy of our approximants decreases: (I) when the index \( n \) of the power analog \( A_n \) increases; (II) when the considered ray \( Q^2 = |Q^2| \exp(i\phi) \) comes closer to the time-like semiaxis \( |\phi| \) closer to \( \pi \). Aspect (I) can be understood from the fact that, by our construction, \( R_n \) involves \( (n-1) \) derivatives of the Padé approximant \( R_1 = R^{M-1}_M \). Aspect (II) is also to be expected, because rays with \( \phi \approx \pm\pi \) are close to the singularity cut of \( A_n \)’s.

The accuracy can be increased if we compute \( R_1 = R^{M-1}_M \) as a higher order Padé (higher index \( M \)). For instance, Figs. 5 (a), (b) show the approximants for \( A^{(mMA)}_2 \), with \( \phi = \pi/4 \), which are calculated from \( R^{10}_M(u) \) (i.e., Padé index \( M = 20 \)). The latter is calculated from the first 40 coefficients of the Taylor series of \( A^{(mMA)}_1(u\Lambda^2) \). We see that in the case of \( M_0 = 2M_\pi \), the deviation of \( \text{Re}[R_2] \) from \( \text{Re}[A_2] \) becomes discernible to the eye (1% deviation) only for \( Q^2 > 5 \text{ GeV}^2 \); this is to be compared with Fig. 6 (a) (where Padé index \( M = 10 \)). If we used \( \phi = 0 \) instead (see Figs. 3 where \( M = 10 \)), the deviations of \( \text{Re}[R_2] \) from \( \text{Re}[A_2] \), for \( M = 20 \) and \( M_0 = 2M_\pi \), on the scale of Figs. 5 would turn out to be discernible to the eye only for \( Q^2 > 6 \text{ GeV}^2 \), i.e., the convergence turns out to be even better than in the case \( \phi = \pi/4 \). The deviations of \( \text{Im}[R_2] \) from \( \text{Im}[A_2] \) in Fig. 5 (b) are not discernible to the eye. We also see that the deviations of \( R_2 \) from \( A_2 \) cannot be seen by the eye in Figs. 5 when the cut value \( M_0 \) increases to \( 3M_\pi \).

---

4 At the level \( M = 20 \), it is important that the 40 Taylor coefficients \( L_n \) and the coefficients of the Padé approximant \( R^{M-1}_M \) be calculated.
Both aforementioned aspects (I, II) that decrease accuracy are, however, not very important in practice when evaluating observables. Namely, the higher order contributions ($\sim A_n$) are very suppressed in anQCD (even when $|Q^2|$ is low), and the contributions of $Q^2$ near the time-like axis in the contour-type of integrations (e.g., for the semihadronic $\tau$ decay ratio $r_\tau$) are usually suppressed by the integrand.

![Figure 4](image1.png)

**FIG. 4**: (a) Real parts of $A_1^{(mMA)}$ and $R_0^{(0)}$ with complex arguments $Q^2 = |Q^2| \exp(i\pi/4)$; (b) same as in (a), but for imaginary parts.

![Figure 5](image2.png)

**FIG. 5**: (a) Real parts of $A_1^{(mMA)}$ and $R_0^{(0)}$ with complex arguments $Q^2 = |Q^2| \exp(i3\pi/4)$; (b) same as in (a), but for imaginary parts.

**VI. PROSPECTS OF APPLICATIONS IN FITTING EXPERIMENTAL DATA**

In Eq. (3) we considered the dispersive relation for $A_1$ with an IR $\sigma$-cutoff $\sigma_{\text{cut}} = M_0^2$ ($\sim M_2^2$) imposed on the perturbative QCD discontinuity function $\rho_1(\sigma) = \text{Im}[a(Q^2 = -\sigma - i\epsilon)]$. Such a cutoff results in the analyticity of $A_1$ around $Q^2 = 0$, thus reflecting the analyticity of space-like observables $D(Q^2)$ in the complex plane excluding the time-like semiaxis but including a regime around $Q^2 = 0$.

To high accuracy (to at least about 20 and 30 digits, respectively), in order to avoid numerical instabilities connected with cancellation of large numbers.
In MA model, the scale $\Lambda$ can be fixed so that it reproduces measured values of QCD observables at higher energies $Q \gtrsim 10$ GeV ($\Rightarrow \Lambda_{n_f=3} \approx 0.4$ GeV) \cite{6}. However, then MA gives too low value of the massless strangeless $(\Delta S = 0)$ semihadronic $\tau$-decay ratio: $r_\tau \approx 0.14$ \cite{7,8}. The experimentally measured value of this quantity is: $r_\tau = 0.204 \pm 0.005$ \cite{9}. The latter value can be reproduced in MA with $\Lambda_{n_f=3} \approx 0.4$ GeV only if the current masses of light quarks ($m_u, m_d \sim 1$ MeV) are replaced by much larger (constituent) masses ($m_u, m_d \approx 0.25$ GeV) \cite{10} and the threshold effects become very important.

By introducing IR cutoff $\sigma_{\text{cut}} = M_0^2$, the coupling $A_1$ gets further diminished at low $Q^2$, and thus further diminishes the value of $r_\tau$. To remedy this situation, we can, in the simplest way, simulate the unknown behavior of $\rho_1(\sigma)$ (Fig. 9) at $\sigma \lesssim M_0^2$ by adding a simple positive Dirac delta peak: $\delta \rho_1(s\Lambda^2) = \pi f_1^2 \delta(s - s_{-1})$, where $0 < s_{-1} \lesssim s_0 (\equiv M_0^2/\Lambda^2)$. This would then allow us to achieve, in the model, the correct value of $r_\tau$ while still maintaining the analyticity of $A_1$ around $Q^2 = 0$. Thus, the full discontinuity function in such a “delta-modified” MA model (dmMA) is

$$\rho_1^{(\text{dmMA})}(s\Lambda^2) = \Theta(s - s_0)\rho_1(s\Lambda^2) + \pi f_1^2 \delta(s - s_{-1}) ,$$

where $\Theta(x)$ is the Heaviside step function ($+1$ for $x > 0$, zero otherwise), and $\rho_1(s\Lambda^2)$ is the pertubative QCD discontinuity function: $\rho_1(s\Lambda^2) = a(-s\Lambda^2 - i\epsilon)$. This leads to the following $A_1$:

$$A_1^{(\text{dmMA})}(u\Lambda^2) = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{\rho_1(s\Lambda^2)}{s + u} + \frac{f_1^2}{u + s_{-1}} .$$

In Fig. 6, (a) Real parts of $A_2^{(mMA)}$ and $R_2$ with complex arguments $Q^2 = |Q^2| \exp(i\pi/4)$; (b) same as in (a), but for imaginary parts.

In Fig. 7, (a) Real parts of $A_3^{(mMA)}$ and $R_3$ with complex arguments $Q^2 = |Q^2| \exp(i\pi/4)$; (b) same as in (a), but for imaginary parts.
Applying the Padé $R_{M}^{M-1}(u)$ approximation to this analytic coupling we obtain

$$A_{1}^{(d_mMA)}(u\Lambda^2) \approx R_{M}^{M-1}(u) = \sum_{n=-1(n\neq 0)}^{M} \frac{f_n}{u + s_n} = \sum_{n=-1(n\neq 0)}^{M} \frac{F_n^2}{Q_n^2 + M_n^2}. \tag{28}$$

This has the same form as the Padé $R_{M}^{M-1}(u)$ applied to $A_{1}^{(mMA)}$ (MA with IR cut), Eq. (15), but just with one more term $(n = -1)$. This model has three dimensionless model parameters: $s_0$, $s_{-1}$, $f_{-1}$. All are positive and $\sim 1$. As presented, the model is considered in the ’t Hooft scheme ($\beta_j = 0$ for all $j \geq 2$). The scale parameter $\Lambda_n_{\beta=2}$ is fixed by fitting the model to experimental values of observables at high energies ($|Q| \geq 10$ GeV), such as $\Upsilon$ decay, $e^+e^- \rightarrow$ hadrons, $Z \rightarrow$ hadrons. The values of low energy QCD observables ($|Q| \sim 1$ GeV), such as $r_t$ and Bjorken polarized sum rule, are sensitive to the values of parameters $s_0$, $s_{-1}$ and $f_{-1}$; therefore, the latter are to be fixed by fitting to the experimental values of such observables.

![Diagram](image-url)
We see that even with this modification (dmMA) of the MA, the evaluation of the couplings $A_n (u A^2)$ ($n = 1, 2, \ldots$) is made simple and efficient by using the Padé approximant \[25\]; for the evaluation, it suffices to know the (three) parameters $s_0, s_{-1}$ and $f_{-1}$ and the first few coefficients $L_k$ [Eqs. 111-112] and Table II.

VII. SUMMARY

We worked with the minimal analytic (MA) model modified (mMA) by an IR cutoff $\sigma_{\text{cut}} = M_0^2 \sim M_0^2$ for the perturbative discontinuity function. In such a model, the analytic coupling $A_1 (Q^2)$ is analytic in the entire $Q^2$-complex plane excluding the semi-axis $Q^2 < -M_0^2$. The analytic properties of such (mMA) coupling reflect the analytic properties of space-like QCD observables $D(Q^2)$, among them analyticity in the point $Q^2 = 0$ and its vicinity. Further, such a (mMA) coupling is a Stieltjes function of $Q^2$. This implies that it will be efficiently approximated by (paradiagonal) Padé approximants $R_{M}^{M-1}(Q^2)$, i.e., $R_{M}^{M-1}(Q^2)$ converges to $A_1(Q^2)$ at any point of analyticity when index $M$ increases ($M \to \infty$). The coupling $A_1(Q^2)$ in the form of $R_{M}^{M-1}(Q^2)$ can be easily and efficiently evaluated [i.e., without performing time-consuming dispersion-type integrations \[3\] for each $Q^2$] just by knowing the first few coefficients of Taylor expansion of $A_1(Q^2)$ in powers of $Q^2$.

We showed that for real and complex arguments the paradiagonal Padé approximants of the analytic coupling $A_1^{(mMA)}(Q^2)$ are precise at low positive $Q^2$'s. This high precision range of positive $Q^2$'s increases fast when the order index $M$ of the Padé approximant increases. When $Q^2$'s are complex, the precision range of $|Q^2|$'s decreases when $Q^2$ approaches the singularity cut. The analytic analogs $A_1^{(mMA)}$ of higher powers $\sigma^n \equiv (\alpha_s / \pi)^n (n \geq 2)$ are then evaluated as combinations of logarithmic derivatives of the approximant $R_{M}^{M-1}(Q^2) \approx A_1(Q^2)$). The approximants obtained in this way for $A_n^{(mMA)}$ show less precision when $n$ increases and/or when $Q^2$ approaches the singularity cut. These approximants still work fine if we increase the order index $M$ of the Padé. However, high precision is needed only for the $n = 1$ case, because the higher couplings $A_n^{(mMA)}$ get strongly suppressed (even at low $|Q^2|$) in analytic QCD when $n$ increases. Further, when evaluation of observable involves contour integration (such as, for example, in the case of $r_\tau$), the contributions of $A_n^{(Q^2)}$ get suppressed by the rest of the integrand when $Q^2$ comes close to the singularity cut. While we generally used for the IR cutoff $\sigma_{\text{cut}} \sim M_0^2$ the specific value $\sigma_{\text{cut}} \equiv M_0^2 = 4M_0^2$, we also showed that the conclusions in this work are independent of the specific value chosen, by comparing various results for $M_0 = 2M_0$ and $M_0 = 3M_0$.

We further suggested an inclusion of one additional Dirac delta function to the mMA discontinuity function $\theta (\sigma - M_0^2) \rho_1 (\sigma)$ at low energies where the precise behavior of $\rho_1$ is unknown - Dirac modified MA model (dmMA). Such a modification maintains the analyticity at $Q^2 = 0$ and its vicinity, and allows us to reproduce the experimental value of the semihadronic $\tau$ decay ratio $r_\tau$. Such a modification keeps the same form of the Padé approximants $R_{M}^{M-1}(Q^2)$ of $A_1^{(dmMA)}$ as in the mMA case and allows us to evaluate them (and the higher power analogs) in an easy and efficient manner. The (three) parameters of such a model can be determined by requiring that the model reproduces the measured values of low energy QCD observables [11].

Acknowledgments

This work was supported by FONDECYT Grant No. 1095196 (G.C.) and a PIIC-USM grant (H.M.)

[1] D. V. Shirkov and I. L. Solovtsov, hep-ph/9604363; Phys. Rev. Lett. 79, 1209 (1997) hep-ph/9704333.
[2] A. V. Nesterenko and J. Papavassiliou, Phys. Rev. D 71, 016009 (2005) hep-ph/0410406; Nucl. Phys. Proc. Suppl. 164, 304 (2007) [arXiv:hep-ph/0507320].
[3] G. A. Baker and P. Graves-Morris, Padé Approximants, Encyclopedia of Mathematics and its Applications (Cambridge University, Cambridge, England, 1996).
[4] S. Peris, Phys. Rev. D 74, 054013 (2006) arXiv:hep-ph/0603190.
[5] E. Gardi, G. Grunberg and M. Karliner, JHEP 9807, 007 (1998) arXiv:hep-ph/9806462.
[6] D. V. Shirkov, Theor. Math. Phys. 127, 409 (2001) hep-ph/0012283; Eur. Phys. J. C 22, 331 (2001) hep-ph/0107282.
[7] G. Cvetić and C. Valenzuela, Phys. Rev. D 74, 144030 (2006) [arXiv:hep-ph/0608250].
[8] K. A. Milton, I. L. Solovtsov, O. P. Solovtsova and V. I. Yasnov, Eur. Phys. J. C 14, 495 (2000) hep-ph/0003030.
[9] R. Barate et al. [ALEPH Collaboration], Eur. Phys. J. C 4, 409 (1998); K. Ackerstaff et al. [OPAL Collaboration], Eur. Phys. J. C 7, 571 (1999) hep-ex/0008019; S. Schael et al. [ALEPH Collaboration], Phys. Rept. 421, 191 (2005) hep-ex/0506072; M. Davier, A. Höcker and Z. Zhang, hep-ph/0507078.
[10] K. A. Milton, I. L. Solovtsov, O. P. Solovtsova, Phys. Rev. D 64, 016005 (2001) [hep-ph/0102254]; Mod. Phys. Lett. A 21, 1355 (2006) [hep-ph/0512200].

[11] G. Cvetić and H. E. Martínez, work in progress.