RANK OF THE FUNDAMENTAL GROUP OF A COMPONENT OF A FUNCTION SPACE

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Abstract. We compute the rank of the fundamental group of an arbitrary connected component of the space map(X, Y) for X and Y nilpotent CW complexes with X finite. For the general component corresponding to a homotopy class f : X → Y, we give a formula directly computable from the Sullivan model for f. For the component of the constant map, our formula expresses the rank in terms of classical invariants of X and Y. Among other applications and calculations, we obtain the following: Let G be a compact simple Lie group with maximal torus T^n. Then π_1(map(S^2, G/T^n; f)) is a finite group if and only if f : S^2 → G/T^n is essential.

1. Introduction

Given a finitely generated abelian group G, the rank of G is defined to be the cardinality of a basis of the free abelian group G/T where T is the torsion subgroup of G. This notion can be extended to finitely generated nilpotent groups Γ by considering the central series Γ = Γ_0 ⊃ Γ_1 ⊃ · · · ⊃ Γ_n = {1} where Γ_n = [Γ_1, Γ]. Specifically, the rank of Γ is defined to be the sum of the ranks of the finitely generated abelian quotients Γ_k/Γ_k−1 for k = 1, ..., n. Alternately, rank(Γ) = dim_{Q}(Γ_0) where Γ_0 is the rationalization of the nilpotent group Γ as described in [11, Ch. 1].

Given connected spaces X and Y let map(X, Y) denote the function space of all (not-necessarily based) maps from X to Y with the compact-open topology. Given a particular map f : X → Y, write map(X, Y; f) for the connected component of map(X, Y) containing f, that is, the space of maps freely homotopic to f. Under reasonable hypotheses on X and Y, the fundamental groups of the connected components of map(X, Y) are finitely generated nilpotent groups. To be precise, say a space X is nilpotent if π_1(X) is a nilpotent group and the action of π_1(X) on the higher homotopy groups of X is a nilpotent action. If X is a finite CW complex and Y is a nilpotent CW complex of finite type then the components of map(X, Y) are themselves nilpotent CW complexes of finite type ([13] and [11 Th.II.2.5]). Thus with these hypotheses π_1(map(X, Y; f)) is a finitely generated nilpotent group. Our purpose in this paper is to give a formula for the rank of this group expressed in terms of accessible invariants of the map f : X → Y.

In [12], we describe the higher rational homotopy groups of map(X, Y; f). Our description uses Sullivan minimal models, and is in terms of the homology of chain
complexes of derivations. Specifically, for each $n \geq 2$ we construct an isomorphism

$$\pi_n(\text{map}(X,Y;f)) \otimes \mathbb{Q} \cong H_n(\text{Der}(M_Y,M_X;M_f))$$

where $M_f : M_Y \to M_X$ is the Sullivan model of the map $f : X \to Y$ and $\text{Der}_*(M_Y,M_X;M_f)$ is a differential graded vector space of (generalized) algebra derivations (see Section 4 below for precise definitions). In [12]—and its companion [13], in which we give corresponding results using differential graded Lie algebra minimal models—our main goal was to establish a framework within which we could study evaluation subgroups of $\pi_n(Y) \otimes \mathbb{Q}$ for $n \geq 2$. Now within that framework, we may also compute $H_1(\text{Der}(M_Y,M_X;M_f))$, which is a rational homotopy invariant of the map $f$. Heuristically, one would expect this vector space to be related to rationalization of the fundamental group $\pi_1(\text{map}(X,Y;f))$. However, the latter group is generally non-abelian, and so the above isomorphism obviously cannot be extended in the naive way. As the main result of this paper, we establish

**Theorem 1.** Let $X$ and $Y$ be connected nilpotent CW complexes of finite type with $X$ finite. Let $f : X \to Y$ be a given map. Then

$$\text{rank}(\pi_1(\text{map}(X,Y;f))) = \dim_\mathbb{Q}(H_1(\text{Der}(M_Y,M_X;M_f))).$$

In the special case of a null component, that is, $\text{map}(X,Y;0)$, we obtain an identification of the rank of the fundamental group directly in terms of classical invariants of $X$ and $Y$. Let $b_n(X)$ denotes the $n$th Betti number of $X$. If $n \geq 2$ let $\rho_n(Y)$ denote the integer $\dim_\mathbb{Q}(\pi_n(Y) \otimes \mathbb{Q})$—sometimes called the $n$th *Hurewicz number* of $Y$. We have

**Theorem 2.** Let $X$ and $Y$ be nilpotent spaces, with $X$ a finite complex of dimension $N$. Then

$$\text{rank}(\pi_1(\text{map}(X,Y;0))) = \text{rank}(\pi_1(Y)) + \sum_{n=2}^{N+1} \rho_n(Y) \cdot b_{n-1}(X).$$

**Remark 1.1.** So far as the rational homotopy groups of function spaces are concerned, these results form an essentially complete complement to the results of [12]. Of course, they leave open the more general problem of describing the full structure of the rationalized fundamental group $\pi_1(\text{map}(X,Y;f))_\mathbb{Q}$. In principle, such a description is already available via the Sullivan models for $\text{map}(X,Y;f)$ described by Haefliger [5] and Brown-Szczarba [2]. In practice, however, there are so many technical issues involved in first assembling these models, and then extracting from them the necessary information, that these identifications are not helpful for our purposes. By focussing purely on the rank of $\pi_1(\text{map}(X,Y;f))$, we are able to give a “closed form” description that proceeds directly from the ordinary Sullivan model of the map $f : X \to Y$ and is relatively easy to handle in practice. We emphasize that our results and methods here are entirely independent of those of [5] and [2]. At the end of the paper, we consider some particular cases where further structure of the rationalized fundamental group can be described.

**Remark 1.2.** Theorem 2 extends to the fundamental group an isomorphism for higher homotopy groups of the form

$$\pi_n(\text{map}(X,Y;0)) \otimes \mathbb{Q} \cong \bigoplus_{k \geq n} H^{k-n}(X,\pi_n(Y) \otimes \mathbb{Q})$$
n ≥ 2. The isomorphism for n ≥ 2 has appeared in a number of places (see [15, 3, 12]). It is worth noting that the assertions of both [15, Lem.3.2] and [3] need to be adjusted so as to exclude the π₁ case, even under the hypothesis that Y is 1-connected.

We give several interesting computations using our results. In Example 6.2, we compute the rank of the fundamental group of any arbitrary component of the free loop space for Y. In Theorem 6.3, we specialize our results to a class of formal spaces which includes all homogeneous spaces G/H of equal rank Lie pairs. We express the rank of an arbitrary component corresponding to f in this case directly in terms of the map induced by f on rational cohomology. As a particular consequence, in Example 6.4 we show the following: Let G be a compact simple Lie group of rank n ≥ 2, T ⊆ G a maximal torus and f: S² → G/Tⁿ any map. Then π₁(map(S², G/Tⁿ; f)) is finite – that is, has rank 0 – if and only if f is essential.

We now give an outline of the paper. Section 2 consists of a concrete example that illustrates a component of a function space may have fundamental group that is non-abelian after rationalization. The example suggests that rationalized fundamental groups of function space components form a rich subject for investigation. In Section 3 we review some properties of minimal models of nilpotent spaces and in Section 4 we introduce the framework of derivation chain complexes in preparation for the proof of the main results in Section 5. In Section 6 we deduce several consequences of our results and give some further examples.

We end this section by fixing some notation and terminology for the sequel. We denote the trivial group by {1}, the trivial vector space by {0}, and the space consisting of a single point by {∗}. We also use 0 to denote a trivial homomorphism of groups or vector spaces, or a trivial map of spaces. We use H(g), respectively g#, to denote a homomorphism induced by g on either cohomology or homology, respectively homotopy. Whether we intend cohomology or homology, or the use of particular coefficients, will be clear from context.

We assume basic familiarity with the localization of nilpotent groups and spaces, as discussed in [11]. In particular, we recall that a nilpotent space Y (respectively, group G) admits a rationalization which we denote e_Y: Y → Y_Q (respectively, e_G: G → G_Q). Recall that if G is abelian then G_Q ≅ G ⊗ Q. If f: X → Y is a map into a nilpotent space, we write f_Q: X → Y_Q for the composotion e_Y ∘ f and likewise for a group homomorphism. A map of spaces g: Y → Z induces a map g_: map(X, Y; f) → map(X, Z; g ∘ f) induced by post-composition with g. By [11, Th.II.3.1], if f: X → Y is a map between spaces X and Y satisfying the hypotheses in the theorems above, then the induced map (e_Y)_*: map(X, Y; f) → map(X, Y_Q; f_Q) is a rationalization of the nilpotent space map(X, Y; f). In particular, under these hypotheses

\[ π_n(map(X, Y; f))_Q ≅ π_n(map(X, Y_Q; f_Q)) \]

for all n ≥ 1.

2. An Example

We begin with a concrete example. It illustrates that map(X, Y; 0) may have fundamental group that is non-abelian after rationalization, even though both X and Y are simply connected. In the course of our discussion, we establish a basic
ingredient of our further developments. This section does not require any familiarity with minimal models.

The following is a particular phrasing of a well-known result.

**Lemma 2.1.** Suppose given a pullback square of topological spaces and maps

\[
\begin{array}{ccc}
A & \xrightarrow{\tilde{k}} & B \\
\downarrow{\tilde{h}} & & \downarrow{h} \\
C & \xrightarrow{k} & D.
\end{array}
\]

If \( f: X \to A \) is any map, then

\[
\begin{array}{ccc}
\text{map}(X, A; f) & \xrightarrow{(\tilde{k})_*} & \text{map}(X, B; \tilde{k} \circ f) \\
(\tilde{h})_* & & (h)_* \downarrow \\
\text{map}(X, C; \tilde{h} \circ f) & \xrightarrow{k_*} & \text{map}(X, D; k \circ \tilde{h} \circ f = h \circ \tilde{k} \circ f).
\end{array}
\]

is a pullback square.

**Proof.** Suppose given based maps \( \beta: Z \to \text{map}(X, B; \tilde{k} \circ f) \) and \( \gamma: Z \to \text{map}(X, C; \tilde{h} \circ f) \) such that \( (h)_* \circ \beta = (k)_* \circ \gamma \). We must show that there exists a unique based map \( \alpha: Z \to \text{map}(X, A; f) \) that satisfies \( (\tilde{k})_* \circ \alpha = \beta \) and \( (\tilde{h})_* \circ \alpha = \gamma \). To this end, let \( b: Z \times X \to B \) and \( c: Z \times X \to C \) be the adjoints of \( \beta \) and \( \gamma \), respectively. Because of the choice of components for each function space, the existence and uniqueness of \( \alpha \) is equivalent to the existence and uniqueness, respectively, of its adjoint \( a \) in the following commutative diagram:

![Diagram](image)

The result follows. \( \square \)

We say a fibration \( p: E \to B \) is principal if \( p \) is obtained as a pullback

\[
\begin{array}{ccc}
E & \xrightarrow{k} & PK \\
p & & q \downarrow \\
B & \xrightarrow{k} & K,
\end{array}
\]

where \( q: PK \to K \) is the usual path fibration over some space \( K \) and \( k: B \to K \) is some map.

**Corollary 2.2.** Suppose given a map \( f: X \to E \) and a principal fibration \( p: E \to B \). Then the induced map \( p_*: \text{map}(X, E; f) \to \text{map}(X, B; p \circ f) \) is a principal fibration.
Proof. Applying Lemma 2.1 to the diagram above, we obtain a pullback

\[
\begin{array}{ccc}
\text{map}(X, E; f) & \xrightarrow{(k)_*} & \text{map}(X, PK; \bar{k} \circ f) \\
p_* & & q_* \\
\text{map}(X, B; p \circ f) & \xrightarrow{k_*} & \text{map}(X, K; k \circ p \circ f).
\end{array}
\]

Since \( PK \) is contractible, \( \text{map}(X, PK) \) is connected and we have a natural identification \( P\text{map}(X, K; k \circ p \circ f) = \text{map}(X, PK) \).

Now take \( X = CP^2 \) and \( Y \) the total space of the principal fibration \( K(Z, 5) \to Y \to K(Z \times Z, 3) \) whose \( k \)-invariant is \( k: K(Z \times Z, 3) \to K(Z, 6) \), corresponding to the cup-product \( x \cup y \in H^6(K(Z \times Z, 3); \mathbb{Z}) \). Here, \( x \) and \( y \) denote the generators in \( H^3(K(Z \times Z, 3); \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z} \). Apply Corollary 2.2 to the principal fibration \( K(Z, 5) \to Y \to K(Z \times Z, 3) \), and the trivial map \( 0: X \to Y \). We obtain a principal fibration \( p_*: \text{map}(X, Y; 0) \to \text{map}(X, K(Z \times Z, 3); 0) \). Since \( \text{map}(X, PK(Z \times Z, 3); 0) \) is contractible, we have a homotopy pullback

\[
\begin{array}{ccc}
\text{map}(X, Y; 0) & \xrightarrow{p_*} & \text{map}(X, K(Z \times Z, 3); 0) \\
\downarrow & & \downarrow \\
\text{map}(X, K(Z \times Z, 3); 0) & \xrightarrow{k_*} & \text{map}(X, K(Z, 6); 0)
\end{array}
\]

and consequently a fibre sequence

(2) \[
\begin{array}{ccc}
\text{map}(X, Y; 0) & \xrightarrow{p_*} & \text{map}(X, K(Z \times Z, 3); 0) \\
& & \xrightarrow{k_*}
\end{array}
\]

From [16, Th.2] (see also [8]), we have homotopy equivalences

\[
\text{map}(X, K(Z \times Z, 3); 0) \cong K(Z, 1) \times K(Z, 1) \times K(Z, 3) \times K(Z, 3),
\]

and

\[
\text{map}(X, K(Z, 6); 0) \cong K(Z, 2) \times K(Z, 4) \times K(Z, 6).
\]

In particular, we have \( H^2(\text{map}(X, K(Z \times Z, 3); \mathbb{Q})) \cong H^2(\text{map}(X, K(Z, 6); 0); \mathbb{Q}) \cong \mathbb{Q} \). Now a careful reading of [3, Sec.1.2] shows that

\[
H(k_*): H^2(\text{map}(X, K(Z, 6); 0); \mathbb{Q}) \to H^2(\text{map}(X, K(Z \times Z, 3); \mathbb{Q})
\]

is an isomorphism. We now show from this that \( \pi_1(\text{map}(X, Y; 0))_{\mathbb{Q}} \) is non-abelian. From the long exact sequence induced by \( \partial \) in rational homotopy, the Serre exact sequence induced by \( \text{map}(X, Y; 0) \) in rational homology, and the rational Hurewicz homomorphism between them, we obtain a commutative ladder as follows:

\[
\begin{array}{ccc}
\{1\} & \xrightarrow{(k)_*} & \mathbb{Q} \\
& \xrightarrow{\partial_w} & \pi_1(\text{map}(X, Y; 0))_{\mathbb{Q}} \\
& \xrightarrow{h} & \pi_1(\text{map}(X, Y; 0))_{\mathbb{Q}} \oplus \mathbb{Q} \\
\mathbb{Q} & \xrightarrow{H(k_*)} & \mathbb{Q} \\
& \xrightarrow{0} & H_1(\text{map}(X, Y; 0); \mathbb{Q}) \\
& \xrightarrow{H_1(p_*)} & \mathbb{Q} \oplus \mathbb{Q} \\
& & \{0\}
\end{array}
\]

From the top row, \( \pi_1(\text{map}(X, Y; 0))_{\mathbb{Q}} \) has rank 3 (as a nilpotent group), but from the bottom row \( H_1(\text{map}(X, Y; 0); \mathbb{Q}) \) has rank (or dimension) 2. Therefore, the
Hurewicz homomorphism \( h: \pi_1(\text{map}(X,Y;0))_\mathbb{Q} \to H_1(\text{map}(X,Y;0); \mathbb{Q}) \) is not an isomorphism, and hence \( \pi_1(\text{map}(X,Y;0))_\mathbb{Q} \) is not abelian.

3. MINIMAL MODELS IN THE NON-SIMPLY CONNECTED SETTING

We assume familiarity with rational homotopy theory using the DG (differential graded) algebra minimal models introduced by Sullivan. Our main reference for this material is [4], although that book restricts to the simply connected case. References that treat the non-simply connected case in some detail include [1, 6]. Here we briefly review some of this material in the nilpotent setting, and take this opportunity to establish some notation.

In general, we use the standard notation and terminology for minimal models as in [4]. The basic facts that we rely on are as follows: Each nilpotent space \( X \) has a Sullivan minimal model \( (\mathcal{M}_X, d_Y) \) in the category of nilpotent DG algebras over \( \mathbb{Q} \). This DG algebra is unique up to isomorphism and is of the form \( \mathcal{M}_X = \Lambda W \), a free graded commutative algebra generated by a positively graded vector space \( W \) of finite type. The differential \( d_Y \) is decomposable, in that \( d_Y(W) \subseteq \Lambda^{j+1}W \), and satisfies a certain “nilpotency” condition. A map \( f: X \to Y \) of nilpotent spaces has a Sullivan minimal model which is a DG algebra map \( \mathcal{M}_f: \mathcal{M}_Y \to \mathcal{M}_X \). The Sullivan minimal model is a complete rational homotopy invariant for a space or a map. Since the minimal model is determined by the rational homotopy type, the minimal models of \( Y \) and \( Y \), and more generally those of \( f_Q \) and \( f \), agree. The homomorphism of rational homotopy groups induced by a map \( f: X \to Y \) of nilpotent spaces may be identified with the homomorphism induced by \( \mathcal{M}_f \) of the (quotient) modules of indecomposables \( Q(\mathcal{M}_f): Q(\mathcal{M}_Y) \to Q(\mathcal{M}_X) \).

We now recall the structure of the minimal model of a nilpotent space \( Y \). By [11 Th.2.9], the Postnikov decomposition of \( Y \) admits a principal refinement at each stage. Precisely, \( p_r: Y^{(r)} \to Y^{(r-1)} \), the \( r \)th stage of the Postnikov decomposition of \( Y \), factors into a finite sequence of principal fibrations

\[
Y^{(r)} = Y^{(r)}_{c_r} \to Y^{(r)}_{c_{r-1}} \to \cdots \to Y^{(r)}_1 \to Y^{(r)}_0 = Y^{(r-1)},
\]

each induced from a path-loop fibration by a \( k \)-invariant of the form \( k^r_j: Y^{(r)}_{j-1} \to K(G^1_j, r+1) \).

The minimal model \( \mathcal{M}_Y \) of \( Y \) may be constructed by a sequence of so-called elementary extensions in a way that mirrors this principal refinement. Thus, for example, consider the 1-minimal model of \( Y \), that is, the sub-DG algebra \( \mathcal{M}_{Y(1)} \) of \( \mathcal{M}_Y \) generated in degree 1. Let \( V^1_j \) and \( \overline{\nabla}^1_j \) denote vector spaces isomorphic to \( G^1_j \otimes \mathbb{Q} \) and concentrated in degree 2 and 1 respectively. Then the 1-minimal model \( \mathcal{M}_{Y(1)} = \Lambda(W^{(1)}, d) \) is \( c_1 \)-stage in the sense that \( W^{(1)} = \oplus_{j=1}^{c_1} V^1_j \), with \( d(\overline{\nabla}^1_j) = 0 \) and \( d(\overline{\nabla}^1_j) \subseteq \Lambda(\overline{\nabla}^1_{j+1} \oplus \cdots \oplus \overline{\nabla}^1_{j-1}) \) for \( j = 2, \ldots, c_1 \). For each stage of the 1-minimal model we have an elementary K-S extension

\[
\mathcal{M}_{Y(j-1)} \to \mathcal{M}_{Y(j)} \to (\Lambda \overline{\nabla}^1_j, d = 0)
\]

that is a K-S model of the principal fibration \( K(G^1_j, 1) \to Y^{(1)}_1 \to Y^{(1)}_{j-1} \). This extension is elementary in the sense that \( \mathcal{M}_{Y(j)} = (\mathcal{M}_{Y(j-1)} \otimes \Lambda \overline{\nabla}^1_j, D) \) and \( D(\overline{\nabla}^1_j) \subseteq \mathcal{M}_{Y(j-1)} \). We write \( V^1_j = \langle v^1_{j,1}, \ldots, v^1_{j,n_j} \rangle \) and \( \overline{\nabla}^1_j = \langle \overline{v}^1_{j,1}, \ldots, \overline{v}^1_{j,n_j} \rangle \). Then the extension (4) is minimal in the sense that, for each \( i \), \( D(\overline{v}^1_{j,i}) = \xi^1_{j,k} \) with each
\[ \xi_{j,k}^1 \text{ decomposable in } \mathcal{M}_{Y_{j-1}} \]. In fact, since we only have generators of degree 1 so far, each \( \xi_{j,k}^1 \) is of homogeneous length 2. The relation between \( k \)-invariants of the principal fibrations of (3) and the elementary extensions (4) is as follows: Each \( \xi_{j,k}^1 \) is a cycle that represents a class in \( H^2(\mathcal{M}_{Y_{j-1}}) \). On the other hand, the \( k \)-invariant \( k_j^r \) has minimal model \( \mathcal{M}_{k_j^r} : (AV_j^r, d = 0) \rightarrow \mathcal{M}_{Y_{j-1}} \), with \( \mathcal{M}_{k_j^r}(v_{j,k}^r) = \xi_{j,k}^1 \) for each \( j \) and \( k \). Finally, we remark that the 1-minimal model \( \mathcal{M}_{Y_{(1)}} \) is a complete rational invariant for the fundamental group of \( Y \), in the case in which \( Y \) is a nilpotent group. In particular, the rank of \( \pi_1(Y) \) equals the number of generators of \( \mathcal{M}_{Y_{(1)}} \). Other aspects of \( \pi_1(Y) \) are determined in different, often less direct, ways by \( \mathcal{M}_{Y_{(1)}} \). The nilpotency class of \( \pi_1(Y) \), for example, is determined as the smallest \( c_1 \) for which \( \mathcal{M}_{Y_{(1)}} \) is \( c_1 \)-stage in the above sense.

A similar situation pertains for the higher dimensional parts of the minimal model \( \mathcal{M}_Y \). We can extend the preceding notation as follows: The \( r \)-minimal model of \( Y \), written \( \mathcal{M}_{Y_{(r)}} \), is the sub-DG algebra of \( \mathcal{M}_Y \) generated in degrees \( \leq r \). In fact it is a minimal model for the \( r \)th stage of the Postnikov decomposition \( Y^{(r)} \) of \( Y \). Let \( V_j^r \) and \( \nabla_j^r \) denote vector spaces isomorphic to \( H_j^r \otimes \mathbb{Q} \) and concentrated in degree \( r + 1 \) and \( r \) respectively. Corresponding to (3) we have \( c_r \)-minimal, elementary extensions

\[ \mathcal{M}_{Y_{j-1}} \rightarrow \mathcal{M}_{Y_{(r)}} \rightarrow (\Lambda \nabla_j^r, d = 0) \]

Write \( V_j^r = (v_{j,1}^r, \ldots, v_{j,n_j^r}) \) and \( \nabla_j^r = (\nabla_{j,1}^r, \ldots, \nabla_{j,n_j^r}) \). For each \( k \), \( D(\nabla_{j,k}^r) = \xi_{j,k}^r \in \mathcal{M}_{Y_{j-1}} \). In the general case, each \( \xi_{j,k}^r \) need not be of homogeneous length but is decomposable. Each \( \xi_{j,k}^r \) is a cycle that represents a class in \( H^{r+1}(\mathcal{M}_{Y_{j-1}}) \). The \( k \)-invariant \( k_j^r \) has minimal model \( \mathcal{M}_{k_j^r} : (AV_j^r, d = 0) \rightarrow \mathcal{M}_{Y_{j-1}} \), and we have \( \mathcal{M}_{k_j^r}(v_{j,k}^r) = \xi_{j,k}^r \) for each \( j \) and \( k \). Roughly speaking, the way in which generators of degree 1 are involved with the higher degree generators of \( \mathcal{M}_Y \) corresponds to the way in which the fundamental group of \( Y \) is involved with the topology of \( Y \). For instance, a differential from (3) of the form

\[ D(\nabla_{j,k}^r) = \sum c_{j,k}^r v_{j,s}^r \nabla_{j,p}^r + \text{quadratic terms not involving } \mathcal{M}_{Y_{(1)}} \]

+ length \( \geq 3 \) terms,

with at least one coefficient \( c_{j,k}^r \) non-zero, occurs when \( \pi_1(Y) \) acts non-trivially on \( \pi_r(Y) \).

4. Derivations and Function Space Components

In this section, we describe the framework of chain complexes of derivations mentioned in the introduction. We then construct a commutative ladder linking the long exact homotopy sequence of a principal fibre sequence of function spaces to a long exact homology sequence within this framework.

Fix two DG algebras \( (A,d_A) \) and \( (B,d_B) \) and a DG algebra map \( \phi : A \rightarrow B \) between them. Define a \( \phi \)-derivation to be a linear map \( \theta : A \rightarrow B \) that reduces degree by \( n \) and satisfies the derivation law

\[ \theta(xy) = \theta(x)\phi(y) + (-1)^n|\phi(x)|\phi(x)\theta(y). \]
Let $\text{Der}_n(A, B; \phi)$ denote the vector space of $\phi$-derivations of degree $n \geq 0$. Define a linear map $\delta: \text{Der}_n(A, B; \phi) \to \text{Der}_{n-1}(A, B; \phi)$ by $\delta(\theta) = d_B \circ \theta - (-1)^{\phi}(\theta) \circ d_A$.

A standard check now shows that $\delta^2 = 0$ and thus $(\text{Der}_*(A, B; \phi), \delta)$ is a chain complex.

The construction is the obvious generalization of the standard DG Lie algebra of derivations of a DG Lie algebra complex, which we would denote here by $\text{Der}_* (B, B; 1)$. Of course, in the general case of interest here there is no bracket. In another special case, when $(A, d_A) = (AV, 0)$, we have $\text{Der}_n(AV, B; 0, \delta) \cong \text{Hom}_n(V, B)$ where $\text{Hom}_n$ denotes the space of homomorphisms that reduce degree $n$. While it is sometimes convenient to truncate these complexes taking only cycles in degree 1, we do not take this convention here. To reduce notation, we will suppress the differential when writing the homology of a DG algebra. Also, we will only write the outermost degree in the homology of a DG space. In particular, $H_n(\text{Der}(A, B; \phi))$ will denote the homology in degree $n$ of the chain complex $(\text{Der}_*(A, B; \phi), \delta)$.

The derivation complexes enjoy the same functoriality as function spaces. We will be interested in the chain map $\psi^*: \text{Der}_*(A, B; \phi) \to \text{Der}_*(A', B; \phi')$ induced by pre-composition with a DG algebra map $\psi: A' \to A$ where $\phi' = \phi \circ \psi$. By a standard construction, the map $\psi^*$ gives rise to a long exact homology sequence on homology of the form

$$
\cdots \to H_{n+1}(\text{Der}(A', B; \phi')) \xrightarrow{J_{n+1}} H_{n+1}(\text{Rel}(\psi^*)) \xrightarrow{P_n} H_n(\text{Der}(A, B; \phi)) \xrightarrow{H(\psi^*)} H_n(\text{Der}(A', B; \phi')) \to \cdots
$$

Here the “relative” term $\text{Rel}_*(\psi^*)$ is the DG space $\text{Der}_*(A, B; \phi) \oplus \text{Der}_{*-1}(A', B; \phi')$ with differential $D: \text{Rel}_*(\psi^*) \to \text{Rel}_{*-1}(\psi^*)$ defined by the rule

$$D(\theta, \varphi) = (\delta(\varphi) - \psi^*(\theta), \delta'(\theta)).$$

We have written $\delta$ and $\delta'$ for the differentials in $\text{Der}_*(A, B; \phi)$ and $\text{Der}_*(A', B; \phi')$, respectively. The maps $J_n$ and $P_n$ are induced by the inclusion $j_n: \text{Der}_n(A, B; \phi) \to \text{Rel}_n(\psi^*)$ and the projection $p_n: \text{Rel}_n(\psi^*) \to \text{Der}_{n-1}(A', B; \phi')$.

In [12], we showed that the homology theory of derivation complexes may be used to model the rational homotopy theory of function spaces at the level of the higher homotopy groups. The link is provided by a map

$$\Phi_f: \pi_n(\text{map}(X, Y; f)) \to H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$$

whose construction originates with Thom [16]. Suppose $\beta \in \pi_n(\text{map}(X, Y; f))$ has adjoint $B: S^n \times X \to Y$. Then $B$ induces a DG algebra map $\mathcal{A}_B: \mathcal{M}_Y \to H^*(S^n, \mathcal{Q}) \otimes \mathcal{M}_X$, that is of the form

$$\mathcal{A}_B(\chi) = 1 \otimes \mathcal{M}_f(\chi) + u \otimes \theta_B(\chi),$$

where $u \in H^n(S^n, \mathcal{Q})$ denotes a generator and $\chi \in \mathcal{M}_X$ is of positive degree. Here $H^*(S^n, \mathcal{Q})$ is viewed as a DG algebra with zero differential. This expression defines a linear map $\theta_B: \mathcal{M}_Y \to \mathcal{M}_X$ that reduces degree by $n$. A standard check shows $\theta_B$ is an $\mathcal{M}_f$-derivation cycle. Set $\Phi_f(\beta) = [\theta_B] \in H_n(\text{Der}(\mathcal{M}_Y, \mathcal{M}_X; \mathcal{M}_f))$. 

Proposition 4.1. Let \( f: X \to Y \) be a map between nilpotent CW complexes with \( X \) finite. Then for \( n \geq 2 \) the map

\[
\Phi_f: \pi_n(\text{map}(X,Y;f)) \to H_n(\text{Der}(\mathcal{M}_Y,\mathcal{M}_X;\mathcal{M}_f))
\]

is a well-defined natural homomorphism and a rational equivalence. If \( Y \) is a rational \( H \)-space, this holds for \( n \geq 1 \) also. Finally, in the general case for \( n = 1 \), \( \Phi_f \) is a well-defined natural map of sets.

Proof. The result for \( Y \) a rational \( H \)-space is a direct generalization of the identification \( \pi_n(\text{map}(X, K(G,n); 0) \otimes \mathbb{Q} \cong \text{Hom}_n(G \otimes \mathbb{Q}, H^*(X, \mathbb{Q})) \) for \( G \) abelian due to Thom [16 Th. 2]. The result for \( n \geq 2 \) is [12 Th. 2.1]. The proof given there assumes \( X \) and \( Y \) are simply connected. However, all that is needed is that the map \( f \) have a minimal model and that a rationalization of \( Y \) induces a rationalization of \( \text{map}(X,Y;f) \) as in [1]. For the case \( n = 1 \), we observe that the proof that \( \Phi_f \) is a well-defined function [12 Th. A.2] for \( n \geq 2 \) goes through unchanged for \( n = 1 \). \( \square \)

Remark 4.2. By naturality, we mean the following: Suppose given maps of spaces \( g: Y \to Z \). Then we have induced maps

\[
(g_*): \pi_n(\text{map}(X,Y;f)) \to \pi_n(\text{map}(X,Z;g \circ f))
\]

and

\[
H(M^*_g): H_n(\text{Der}(\mathcal{M}_Y,\mathcal{M}_X;\mathcal{M}_f)) \to H_n(\text{Der}(\mathcal{M}_Z,\mathcal{M}_X;\mathcal{M}_{g \circ f})).
\]

Then \( H(M^*_g) \circ \Phi_f = \Phi_{g \circ f} \circ (g_*) \). Note that this identity still makes sense when \( n = 1 \), but we must think of both sides as maps of sets as opposed to groups.

We now return to the geometric situation of Section 3. Let \( Y \) be the total space of a principal fibration of nilpotent spaces \( K(G,n) \xrightarrow{j} Y \xrightarrow{\pi} B \) with \( k \)-invariant \( k: B \to K(G,n+1) \). We assume \( n \geq 1 \) and so \( G \) is abelian. Let \( f: X \to Y \) be given and write \( g = p \circ f: X \to B \). We have in mind a Postnikov section of \( Y \) but we only need here that \( k \) vanishes on homotopy groups. By Corollary 3.2 we have a fibre sequence

\[
\text{map}(X,Y;f) \xrightarrow{p_*} \text{map}(X,B;g) \xrightarrow{k_*} \text{map}(X,K(G,n+1);k \circ g)
\]

and so a long exact homotopy sequence

\[
\cdots \xrightarrow{} \pi_{n+1}(\text{map}(X,K(G,n+1),k \circ g)) \xrightarrow{\partial_{n+1}} \pi_n(\text{map}(X,Y;f)) \xrightarrow{(p_*)_*} \pi_n(\text{map}(X,B;g)) \xrightarrow{} \cdots
\]

\[
\cdots \xrightarrow{} \pi_1(\text{map}(X,B;g)) \xrightarrow{(k_*)_1} \pi_1(\text{map}(X,K(G,n+1);k \circ g)) \xrightarrow{} \cdots
\]

On the level of Sullivan minimal models, we may write \( \mathcal{M}_Y = \mathcal{M}_B \otimes \Lambda V \) where \( V \) is concentrated in degree \( n \) and isomorphic to \( G \otimes \mathbb{Q} \). The differential \( d_Y \) restricts to \( d_B \) on the factor \( \mathcal{M}_B \) while \( d_Y(\Lambda V) \subseteq \mathcal{M}_B \) is contained in the decomposables. The minimal model \( \mathcal{M}_f: \mathcal{M}_B \to \mathcal{M}_Y \) may be taken to be the inclusion. The minimal model for \( k \) is a map \( \mathcal{M}_k: \Lambda V \to \mathcal{M}_B \) where \( V \) is isomorphic to \( G \otimes \mathbb{Q} \) concentrated in degree \( n + 1 \) and \( \Lambda V \) has trivial differential. We may assume \( \mathcal{M}_k(v) = d_Y(\overline{v}) \)
where \( v \in V \) and \( v \mapsto \pi \) is the obvious degree +1 identification of \( V \) with \( \pi \). We consider the long exact homology sequence of the map

\[
\mathcal{M}'_k : \text{Der}_s(\mathcal{M}_B, \mathcal{M}_X; \mathcal{M}_g) \rightarrow \text{Der}_s(\Lambda V, \mathcal{M}_X; \mathcal{M}_{k \circ g}).
\]

Given a pair \((\theta, \varphi) \in \text{Rel}_n(\mathcal{M}'_k)\) we obtain \( \overline{\theta} \in \text{Der}_{n-1}(\mathcal{M}_B, \mathcal{M}_X; \mathcal{M}_f) \) by setting \( \overline{\theta}(\chi) = (-1)^{n-1} \varphi(\chi) \) for \( \chi \in \mathcal{M}_B \) and \( \overline{\theta}(\overline{\mathcal{M}}) = \theta(v) \). The assignment \((\theta, \varphi) \mapsto \overline{\theta} \) gives a chain equivalence and thus an isomorphism

\[
\Psi : H_n(\text{Rel}(\mathcal{M}'_k)) \cong H_{n-1}(\text{Der}(\mathcal{M}_B, \mathcal{M}_X; \mathcal{M}_f)).
\]

Observe \( \Psi \circ \phi_n = \mathcal{M}'_k \) and write \( \Delta_n = J_n \circ \Psi \). We then have a long exact sequence

\[
\cdots \longrightarrow H_{n+1}(\text{Der}(\Lambda V, \mathcal{M}_X; \mathcal{M}_{k \circ g})) \longrightarrow H_n(\text{Der}(\mathcal{M}_B, \mathcal{M}_X; \mathcal{M}_g)) \longrightarrow \cdots
\]

\[
\vdots
\]

\[
\cdots \longrightarrow H_1(\text{Der}(\mathcal{M}_B, \mathcal{M}_X; \mathcal{M}_g)) \longrightarrow H_1(\text{Der}(\Lambda V, \mathcal{M}_X; \mathcal{M}_{k \circ g})) \cdots
\]

Applying \( \Phi \) from (7) to (8) yields a ladder of long exact sequences of groups with each vertical map a homomorphism except in the third and second to last terms.

\[
\cdots \longrightarrow \pi_n(\text{map}(X, Y; f)) \longrightarrow \pi_n(\text{map}(X, B; g)) \longrightarrow \cdots
\]

\[
\Phi_f
\]

\[
\cdots \longrightarrow H_n(\text{Der}(\mathcal{M}_B, \mathcal{M}_X; \mathcal{M}_f)) \longrightarrow H_n(\text{Der}(\mathcal{M}_B, \mathcal{M}_X; \mathcal{M}_g)) \longrightarrow \cdots
\]

\[
\Phi_g
\]

\[
\cdots \longrightarrow \pi_1(\text{map}(X, B; g)) \longrightarrow \pi_1(\text{map}(X, K(G, n+1); k \circ g)) \longrightarrow \cdots
\]

\[
\Phi_g
\]

\[
\cdots \longrightarrow H_1(\text{Der}(\mathcal{M}_B, \mathcal{M}_X; \mathcal{M}_g)) \longrightarrow H_1(\text{Der}(\Lambda V, \mathcal{M}_X; \mathcal{M}_{k \circ g})) \cdots
\]

**Theorem 4.3.** The ladder (8) is commutative.

**Proof.** We need to check commutativity of three types of squares. However, for two of the three, those involving \( p \) and \( k \), commutativity is a direct consequence of the naturality of \( \Phi \). Commutativity of the third type of square, namely

\[
\pi_{n+1}(\text{map}(X, K(G, n+1); k \circ g)) \longrightarrow \pi_n(\text{map}(X, Y; f))
\]

\[
\Phi_{k \circ g}
\]

\[
H_{n+1}(\text{Der}(\Lambda V, \mathcal{M}_X; \mathcal{M}_{k \circ g})) \longrightarrow H_n(\text{Der}(\mathcal{M}_B, \mathcal{M}_X; \mathcal{M}_f))
\]

can deduced from naturality also, this time using the fibre inclusion \( j : K(G, n) \rightarrow Y \). For, by Thom [16, p. 32], in our situation the fibre of the induced fibration
Proof of Theorem 1. We first establish the desired formula for an arbitrary Postnikov section \( Y^{(r)} \) of \( Y \). That is, we prove
\[
\text{rank} \left( \pi_1 \left( \text{map}(X, Y^{(r)}; f^r) \right) \right) = \dim_{\mathbb{Q}} \left( H_1 \left( \text{Der} \left( M_{Y^{(r)}}, M_X; M_f \right) \right) \right)
\]
for any given \( r \geq 0 \) where \( f^r : X \to Y^{(r)} \) denotes the map induced by \( f : X \to Y \). We prove this formula by induction on \( r \geq 0 \). Since \( Y^{(0)} = * \), the base case is trivial.

For the induction step, we rely on our notation from Section 3. The \( r \)th Postnikov section \( p_r : Y^{(r)} \to Y^{(r-1)} \) of \( Y \) factors into a sequence of principal fibrations
\[
Y^{(r)} = Y_{c_1}^{(r)} \to Y_{c_1-1}^{(r)} \to \cdots \to Y_1^{(r)} \to Y_0^{(r)},
\]
with each fibration induced from a path-loop fibration by a map of the form \( k_j^r : Y_j^{(r)} \to K(G_j^{(r)}; r+1) \). Let \( f_j^r : X \to Y_j^{(r)} \) denote the map induced by \( f : X \to Y \). Write \( Z_j^{(r)} = \text{map}(X, Y_j^{(r)}; f_j^r) \). Then Corollary 2.2 gives a corresponding sequence of principal fibrations
\[
\text{map}(X, Y^{(r)}; f^r) = Z_{c_1}^{(r)} \to Z_{c_1-1}^{(r)} \to \cdots \to Z_1^{(r)} \to Z_0^{(r)} = \text{map}(X, Y^{(r-1)}; f^{r-1}),
\]
with each stage induced by a $k$-invariant of the form
\[(k^r_{j})_{*}: Z^r_{j-1} \to \text{map}(X, K(G^r_j, r+1); k^r_{j} \circ f^r_{j-1}).\]
Write $k^r_{j} = k^r_{j} \circ f^r_{j-1}: X \to K(G^r_j, r+1)$. The long exact homotopy sequence of the fibration sequence
\[Z^r_{j} \to Z^r_{j-1} \to \text{map}(X, K(G_j, r+1); k^r_{j-1}),\]
gives an exact sequence
\[\cdots \rightarrow \pi_2(Z^r_{j-1}) \xrightarrow{k^r_{j-1}} \pi_2(\text{map}(X, K(G_j, r+1); k^r_{j-1})) \rightarrow \pi_1(Z^r_{j}) \xrightarrow{k^r_{j}} \pi_1(\text{map}(X, K(G_j, r+1); k^r_{j-1})) \rightarrow \cdots\]
This displays $\pi_1(Z^r_{j})$ as a central extension of the kernel of $k^r_{j-1}$ in degree one by the cokernel of $k^r_{j-1}$ in degree two. Write
\[C^r_{j} = \pi_2(\text{map}(X, K(G_j, r+1); k^r_{j-1}))/k^r_{j}(\pi_2(Z^r_{j-1}))\]
for the cokernel of $k^r_{j-1}$ in degree two and
\[I^r_{j} = \text{im}(k^r_{j-1}: \pi_1(Z^r_{j-1}) \rightarrow \pi_1(\text{map}(X, K(G_j, r+1); k^r_{j-1})))\]
for its image on fundamental groups. Then
\[\text{(11)} \quad \text{rank}(\pi_1(Z^r_{j})) = \text{rank}(\pi_1(Z^r_{j-1})) + \text{rank}(C^r_{j}) - \text{rank}(I^r_{j}).\]

We can follow the preceding line of reasoning, analogously, within the framework of derivation complexes of minimal models. Write $D^r_{j} = \text{Der}(M^{(r)}_Y, M_X; M_{f^r j})$.

We have a sequence
\[\text{Der}_*(M^{(r)}_Y, M_X; M_{f^r j}) = D^r_{j} \rightarrow D^r_{j-1} \rightarrow \cdots \rightarrow D^r_{1} \rightarrow D^r_{0} = \text{Der}_*(M^{(r-1)}_Y, M_X; M_{f^r j-1}).\]

The long exact homology sequence corresponding to the map
\[M_{k^r_j}: D^r_{j} \rightarrow \text{Der}_*(AV^{(r)}_j, M_X; M_{k^r_j})\]
takes the form
\[\cdots \rightarrow H_2(D^r_{j-1}) \xrightarrow{H(M^r_{j-1})} H_2(\text{Der}(AV^{(r)}_j, M_X; M_{k^r_j})) \rightarrow \cdots \rightarrow H_1(D^r_{j}) \xrightarrow{H(M^r_{j})} H_1(\text{Der}(AV^{(r)}_j, M_X; M_{k^r_j})) \rightarrow \cdots\]
Writing
\[C^r_{j} = H_2(\text{Der}(AV^{(r)}_j, M_X; M_{k^r_j}))/H(M^r_{j})(H_2(D^r_{j-1}))\]
for the cokernel and
\[I^r_{j} = \text{im}(H(M^r_{j}): H_1(D^r_{j-1}) \rightarrow H_1(\text{Der}(AV^{(r)}_j, M_X; M_{k^r_j})))\]
for the image of $H(M_{ij}^r)$ as above, we obtain

\[(12) \quad \dim_Q(H_j(Z_j^r)) = \dim_Q(H_j(Z_j^{r-1})) + \dim_Q(C_j^r) - \dim_Q(I_j^r).\]

Now, combining Proposition 11 and Theorem 13 we see

$$\text{rank}(C_j^r) = \dim_Q(C_j^r) \quad \text{and} \quad \text{rank}(I_j^r) = \dim_Q(I_j^r).$$

An easy induction on $j = 1, \ldots, c_r$ using (11) and (12) completes the main induction step and establishes (10) for all $r \geq 0$.

To complete the proof of Theorem 11 we observe that, by obstruction theory, the $r$-equivalence $p_r: Y \to Y^{(r)}$ induces an $(r - N)$-equivalence $(p_r)_*: \map(X, Y; f) \to \map(X, Y^{(r)}; f^r)$ where $N$ is the dimension of the finite complex $X$. On the derivation complex side, it is straightforward to prove

$$M_{p_r}^*: \Der_*(M_{Y^{(r)}}, M_X; M_Y) \to \Der_*(M_Y, M_X; M_f)$$

induces an $(r - N)$-homology equivalence. Thus, for $r \geq N + 1$,

$$\text{rank}(\pi_1(\map(X, Y; f))) = \text{rank}(\pi_1(\map(X, Y^{(r)}; f^r))) = \dim_Q(H_1(\Der(M_{Y^{(r)}}, M_X; M_Y))) = \dim_Q(H_1(\Der(M_Y, M_X; M_f)))$$

\[\Box\]

**Proof of Theorem 2** Observe that

$$H_1(\Der(M_Y, M_X; 0)) \cong \text{Hom}_1(Q(M_Y), H(M_X))$$

where we recall $Q(M_Y)$ denotes the quotient space of indecomposables of the minimal model of $Y$. The result now follows directly from the fact that $Q_n(M_Y) \cong \pi_n(Y)_Q$ for $n \geq 2$ while $\dim_Q(Q_1(M_Y)) = \text{rank}(\pi_1(Y))$ by the results mentioned in Section 3. \[\Box\]

### 6. Consequences and Examples

First we illustrate the role of the map $f: X \to Y$ for the rank of the fundamental group of $\map(X, Y; f)$. As a direct consequence of Theorem 11 we obtain the following basic fact.

**Theorem 6.1.** Let $f: X \to Y$ be a map between nilpotent CW complexes of finite type with $X$ finite. Then

$$\text{rank}(\pi_1(\map(X, Y; f))) \leq \text{rank}(\pi_1(\map(X, Y; 0))).$$

**Proof.** It suffices to note $\dim_Q(H_n(\Der(A, B; \phi))) \leq \dim_Q(H_n(\Der(A, B; 0)))$ for any DG algebra map $\phi: A \to B$. \[\Box\]

The fundamental group can distinguish components of $\map(X, Y)$ even in a simple case such as $Y = K(G, 1)$ for $G$ non-abelian. For in this case, by a result of Gottlieb, $\map(X, K(G, 1)) \simeq C(f_2)$ where $C(f_2)$ is the centralizer of the map $f_2: \pi_1(X) \to G$. In particular, Gottlieb’s result describes the structure of the fundamental group of $\map(X, Y; f)$ directly in terms of $f_2$. We complement this result with the following example.
Example 6.2. Let $Y$ be a nilpotent CW complex and $f : S^1 \to Y$. We compute the rank of the fundamental group of the component map $(S^1, Y; f)$ of the free loop space in terms of the rational homotopy class $\alpha \in \pi_1(Y)_\mathbb{Q}$ represented by $f$. If $\pi_2(Y) \otimes \mathbb{Q} = 0$ then, by [1] and the argument of Hansen [10] Prop.1, $\pi_1(\text{map}(S^1, Y; f))_\mathbb{Q} = C(\alpha)$ where $C(\alpha)$ denotes the centralizer of $\alpha$ in $\pi_1(Y)_\mathbb{Q}$. In general, we prove

\begin{equation}
\text{rank}(\pi_1(\text{map}(S^1, Y; f))) = \rho_2(Y) + \text{rank}(C(\alpha)).
\end{equation}

To begin, as in the last step of the proof of Theorem [11] we have

$$H_1(\text{Der}(\mathcal{M}_Y, \mathcal{M}_{S^1}; \mathcal{M}_f)) \cong H_1(\text{Der}(\mathcal{M}_{Y(2)}, \mathcal{M}_{S^1}; \mathcal{M}_{f^2}))$$

where $\mathcal{M}_{Y(2)}$ is the 2-minimal model for $Y$ and $f^2 : S^1 \to Y^{(2)}$ the induced map. Fixing notation as in Section 3, write $$(\mathcal{M}_{Y(2)}, d_{Y(2)}) = (\Lambda(W^{(1)} \oplus W^{(2)}), D)$$

where $W^{(m)} = \bigoplus_{c_m} \nu^{(m)}_c$ for $m = 1, 2$. Recall $D(\nu^{(m)}_1) = 0$ while $D(\nu^{(m)}_j) \subseteq \Lambda(W^{(m-1)} \oplus \bigoplus_{c_{m-1}} \nu^{(m-1)}_j)$ for $j = 2, \ldots, c_m$, $m = 1, 2$ and where $W^{(0)} = \{0\}$. Fix bases $\nu^{(m)}_j = Q(\nu^{(m)}_{j,1}, \ldots, \nu^{(m)}_{j,n_j})$.

Write $\mathcal{M}_{S^1} = \Lambda(t)$ for $t$ of degree one. Without loss of generality, we assume the homotopy class $\alpha$ corresponds to a basis element $\nu^{(1)}_{j_0,k_0}$. That is, we assume $\mathcal{M}_{f^2}(\nu^{(1)}_{j_0,k_0}) = t$ while $\mathcal{M}_{f^2}(\nu^{(1)}_{j,k}) = 0$ for $(j, k) \neq (j_0, k_0)$. The centralizer $C(\alpha)$, in this set-up, corresponds to the space $C(\nu^{(1)}_{j_0,k_0})$ spanned by those basis vectors $\nu^{(1)}_{j,k}$ such that no non-zero multiple of the product $\nu^{(1)}_{j_0,k_0} \cdot \nu^{(1)}_{j,k}$ appears as a summand in $D(\nu^{(1)}_{j',k'})$ for any $j', k'$.

We observe that there are no boundaries in degree one in the chain complex $\text{Der}_*(\mathcal{M}_{Y(2)}, \mathcal{M}_{S^1}; \mathcal{M}_{f^2})$. For given an $\mathcal{M}_{f^2}$-derivation $\theta$ of degree 2 and $\chi \in \mathcal{M}_{Y(2)}$, we have $\delta(\theta)(\chi) = (-1)^n \theta(D(\chi)) = 0$ since $d_{S^1}$ is trivial while $D(\chi)$ is decomposable and $\mathcal{M}_{f^2}$ vanishes above degree 1.

As for degree one cycles, write $\theta_{j,k}$ for the $\mathcal{M}_{f^2}$-derivation which carries $\nu^{(1)}_{j,k}$ to $1 \in \mathcal{M}_{S^1}$ and vanishes on the other basis elements of $\mathcal{M}_{Y(2)}$. Similarly, let $t_{j,k}$ carry $\nu^{(1)}_{j,k}$ to $t$ and all other basis elements to 0. It is easy to check that $\delta(t_{j,k}) = 0$ for all $j, k$ which accounts for the $\rho_2(Y)$ term above. We also see that if $v^{(1)}_{j,k} \in Z(\nu^{(1)}_{j_0,k_0})$ then $\delta(\theta_{j,k}) = 0$. Finally, suppose $\nu^{(1)}_{j,k} \notin Z(\nu^{(1)}_{j_0,k_0})$ for some $j, k$. Then we may choose $\nu^{(1)}_{j',k'}$ such that

$$D(\nu^{(1)}_{j',k'}) = c \cdot \nu^{(1)}_{j_0,k_0} \cdot \nu^{(1)}_{j,k} + \text{other quadratic terms}$$

for $c \neq 0$. We then see

$$\delta(\theta_{j,k})(\nu^{(1)}_{j',k'}) = \theta_{j,k}(D(\nu^{(1)}_{j',k'})) = c \cdot \mathcal{M}_{f^2}(v^{(1)}_{j_0,k_0}) \cdot \theta_{j,k}(\nu^{(1)}_{j,k}) = c \cdot t \neq 0,$$

and so $\theta_{j,k}$ is not a cycle. This establishes (13).

We next consider a class of examples for which the rank of the fundamental group of map $(X, Y; f)$ depends only on the map $H(f) : H^*(Y, \mathbb{Q}) \to H^*(X, \mathbb{Q})$. We say a simply connected CW complex $Y$ is an $F_0$-space if $Y$ is rationally elliptic (rational homology and rational homotopy both finite-dimensional) with positive Euler characteristic. Equivalently, an $F_0$-space is any elliptic complex with vanishing rational cohomology in odd degrees. Examples of $F_0$-spaces include (products of) even dimensional spheres, complex projective spaces and, more generally, any
homogeneous spaces $G/H$ with $H$ a closed subgroup of maximal rank. Following Grivel [7], we can compute the rank of $\pi_1(\map(X,Y;f))$ for $f: X \to Y$ a map between $F_0$-spaces directly in terms of the degree 2 cohomology derivation space $\Der_2(H^*(Y,\mathbb{Q}),H^*(X,\mathbb{Q});H(f))$.

**Theorem 6.3.** Let $f: X \to Y$ be a map between $F_0$-spaces where $H^*(Y,\mathbb{Q})$ has top degree $2N$. Let $D_2(f) = \dim_\mathbb{Q}(\Der_2(H^*(Y,\mathbb{Q}),H^*(X,\mathbb{Q});H(f)))$. Then

$$\text{rank}(\pi_1(\map(X,Y;f))) = D_2(f) + \sum_{i=1}^{N} \rho_{2i+1}(Y) \cdot b_{2i}(X) - \sum_{i=0}^{N} \rho_{2i+2}(Y)b_{2i}(X).$$

**Proof.** By results of Halperin [10], the minimal model for $Y$ takes the form $(\mathcal{M}_Y, d_Y) = (\Lambda V_0 \otimes \Lambda V_1, d_Y)$ where $V_0$ and $V_1$ are graded spaces of equal (finite) dimension with $V_0$ evenly graded, $V_1$ oddly graded. The differential satisfies $d_Y(V_0) = \{0\}$ while $d_Y$ maps $V_1$ into the decomposables of $\Lambda V_0$. Write $\rho_Y: \Lambda V_0 \otimes \Lambda V_1 \to H^*(Y,\mathbb{Q})$ for the map which sends elements of $V_0$ to their corresponding cohomology class and elements of $V_1$ to zero. The map $\rho_Y$ then represents a formalization of $(\mathcal{M}_Y, d_Y)$.

The needed result follows directly from Theorem 1 and the existence of an exact sequence of the form

$$0 \to \Der_2(H^*(Y,\mathbb{Q}),H^*(X,\mathbb{Q});H(f)) \xrightarrow{\rho_Y^*} \Der_2(\Lambda V_0, H^*(X,\mathbb{Q});H(f) \circ \rho_Y) \xrightarrow{d_Y} \Der_1(\Lambda V_1,H^*(X,\mathbb{Q});0) \xrightarrow{\mathcal{H}} H_1(\Der(\mathcal{M}_Y,\mathcal{M}_X;\mathcal{M}_f)) \to 0.$$

The latter represents an extension of Grivel’s [7, Th.4.4] to our framework of generalized derivations. Here $\rho_Y^*$ and $d_Y^*$ are the maps induced by pre-composition with the formalization and the differential of the minimal model of $Y$. Given $\theta \in \Der_1(\Lambda V_1,H^*(X,\mathbb{Q});0)$ define $H(\theta) \in \Der_1(\mathcal{M}_Y,\mathcal{M}_X;\mathcal{M}_f)$, as follows. Set $H(\theta)(x) = 0$ for $x \in V_0$ and $H(\theta)(y) = P$ for $y \in V_1$ where $P \in \mathcal{M}_X$ is any cycle representative of the class $\theta(y) \in \mathcal{M}_X$ where $y \in V_1$. Extend by the $\mathcal{M}_f$-derivation law. The result is a cycle derivation $H(\theta)$. The map $\mathcal{H}: \Der_1(\Lambda V_1,H^*(X,\mathbb{Q});0) \to H_1(\Der(\mathcal{M}_Y,\mathcal{M}_X;\mathcal{M}_f))$ is then defined to be the linear map carrying $\theta$ to the homology class of $H(\theta)$. Grivel’s proof of the above cited result is directly adapted to show $\mathcal{H}$ is well-defined and the sequence is exact. \hfill $\square$

As a corollary, we deduce, for instance, the following example.

**Example 6.4.** Let $G$ be a compact simple Lie group of rank $n > 1$ and $T^n \subseteq G$ a maximal torus. Let $f: S^2 \to G/T^n$ be any given map. The space $G/T^n$ is an $F_0$-space with rational cohomology generated in degree 2. It is classical that $\rho_2(G/T^n) = 1$. Thus by Theorem 6.3, we have

$$\text{rank}(\pi_1(\map(S^2,G/T^n;f))) = 1 - n + D_2(f).$$

If $f$ is rationally trivial, then $D_2(f) = n$ and so $\text{rank}(\pi_1(\map(S^2,G/T^n;f))) = 1$. Now suppose $f$ is rationally essential. Fix an additive basis $\{t_1, \ldots, t_n\}$ for $H^2(Y,\mathbb{Q})$ and suppose, say, $H(f)(t_i) \neq 0$. Let $\theta_i$ be dual to $t_i$ in the space $\Hom_2(H^*(Y,\mathbb{Q}),H^*(X,\mathbb{Q}))$: that is, $\theta_i(t_i) = 1$ while $\theta_i(t_j) = 0$ for $j \neq i$. Suppose $\theta_i$ extends to an $H(f)$-derivation. Since the Weyl group of $G$ is a finite reflection
group, the cohomology class $t_1^2 + \cdots + t_n^2$ vanishes in $H^2(G/T^n, \mathbb{Q})$. On the other hand,

$$\theta_i(t_1^2 + \cdots + t_n^2) = 2H(f)(t_i)\theta_i(t_i) = 2H(f)(t_i) \neq 0.$$ 

This contradiction implies $D_2(f) = n - 1$ and so $\pi_1(\text{map}(S^2, G/T^n; f))$ is a finite group in this case.

Finally, by the result of E. Cartan, $\pi_2(G) = 0$ and so $\pi_2(G/T^n)$ is free abelian. Thus $f: S^2 \to G/T^n$ is essential if and only if it is rationally essential. Summarizing, we have shown

$$(14) \quad \pi_1(\text{map}(S^2, G/T^n; f))$$

is a finite group if and only if $f$ is essential.

We conclude with some results concerning further structure of the rationalized fundamental group of function space components. First we observe that, in the case of the null component, our inductive procedure for computing rank can also be applied to analyze the nilpotency class, and the ranks of successive quotients in the lower central series. For instance, we can phrase the following result.

**Theorem 6.5.** Let $X$ and $Y$ be nilpotent CW complexes with $X$ finite of dimension $N$. Suppose the Postnikov decomposition of $Y$ admits a principal refinement of length $c_r$ at the $r$th stage. Then, for each $r \geq 1$, the $r$th stage of the Postnikov decomposition of $\text{map}(X, Y; 0)$ admits a principal refinement of length $\leq \sum_{j=r}^{N+r} c_j$.

**Proof.** Each principal fibration $K(G_j^r, r) \to Y_j^{(r)} \to Y^{(r)}$ in a Postnikov decomposition of $Y$ leads to a principal fibration $\text{map}(X, Y_j^{(r)}, 0) \to \text{map}(X, Y^{(r)}_{j-1}; 0)$ by Corollary 6.6. Since we are working with null-components, the fibre can be identified with $\text{map}(X, K_1; 0)$. The result now follows by induction, as in the proof of Theorem 6.

Recall a space $Y$ is simple if $\pi_1(Y)$ is abelian and acts trivially on the higher homotopy groups of $Y$. In this case, $c_r = 1$ for all $r$. Thus we have

**Corollary 6.6.** Suppose $X$ is a nilpotent finite complex and $Y$ is a simple space. Then the Postnikov decomposition of $\text{map}(X, Y; 0)_\mathbb{Q}$ admits a principal refinement that is of length $\leq N$ at each stage. In particular, $\pi_1(\text{map}(X, Y; 0))_\mathbb{Q}$ is of nilpotency class $\leq N$.

Finally, we have seen above that the space of maps into a (simply connected) two-stage Postnikov piece $Y$ can have non-abelian rationalized fundamental group. We conclude with a simple result going the other way. Say a nilpotent space $Y$ is a rational two-stage space if its rationalization $Y_\mathbb{Q}$ appears as the total space of a principal fibration $K_1 \to Y_\mathbb{Q} \xrightarrow{p} K_0$ with $K_0$ and $K_1$ rationalized $H$-spaces. Many spaces of interest are rational two-stage including, for instance, homogeneous spaces $G/H$ for $G$ and $H$ compact Lie groups with $H$ now of arbitrary rank. Write $H^*(K_1, \mathbb{Q}) = \Lambda W_i$ for graded rational vector spaces $W_i$, $i = 0, 1$. We have

**Theorem 6.7.** Let $X$ be a finite, nilpotent CW complex and $Y$ a rational two-stage space as above. Suppose $\text{Hom}_1(W_0, H^*(X, \mathbb{Q}))$ and $\text{Hom}(W_1, H^*(X, \mathbb{Q}))$ are both trivial. Then the rationalization of $\text{map}(X, Y; f)$ is a simple space and, in particular, the group $\pi_1(\text{map}(X, Y; f))_\mathbb{Q}$ is abelian for all $f: X \to Y$.

**Proof.** In this case, we obtain $p_\times: \text{map}(X, Y_\mathbb{Q}; f_\mathbb{Q}) \to \text{map}(X, K_0; p \circ f_\mathbb{Q})$, a principal fibration with fibre homeomorphic to the full function space $\text{map}(X, K_1)$ as in
the proof of Theorem 4.3 above. However, $[X, K_1] \cong \text{Hom}(W_1, H^*(X, Q)) = 0$ by hypothesis and so $\text{map}(X, K_1)$ is actually connected. Thus $p_*$ is a principal fibration expressing $\text{map}(X, Y; f_0)$ as a generalized two-stage rational space and determining a (possibly non-minimal) model for $Y$. Again, by hypothesis, $\pi_1(\text{map}(X, K_0; p \circ f_0)) \cong \text{Hom}_r(W_0, H^*(X, Q)) = 0$. This means the differential in the induced model for $\text{map}(X, Y; f)$ has linear or trivial differential in degree one. It follows that the minimal model for $\text{map}(X, Y; f)$ has trivial differential in degree one.

We deduce directly the following result which, combined with Theorem 6.3 above, gives the full structure the rationalized fundamental group of $\text{map}(X, Y; f)$ for $X$ and $Y F_0$-spaces.

**Corollary 6.8.** Let $X$ and $Y$ be $F_0$-spaces. Then all components of $\text{map}(X, Y)$ have abelian rationalized fundamental group.

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