ON THE TWO- AND THREE DIMENSIONAL LENZ-ISING-ONSAGER PROBLEM IN PRESENCE OF MAGNETIC FIELD

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Abstract

In this paper a new approach to solving the Ising-Onsager problem in external magnetic field is investigated. The expression for free energy on one Ising spin in external field both for the twodimensional and threedimensional Ising model with interaction of the nearest neighbors are derived. The representations of free energy being expressed by multidimensional integrals of Gauss type with the appropriate dimensionality are shown. Possibility of calculating the integrals and the critical indices on the base of the derived representations for free energy is investigated.
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INTRODUCTION

It is well known that the Ising-Onsager problem [1], [2] in external magnetic field has not been solved till now, despite intensive efforts of a few generations of physicists and mathematicians. By the problem we mean the exact calculation of the statistical sum for the Ising model with interaction of the nearest neighbors both in the twodimensional case (in finite external field) and in the threedimensional case (in external field as well as without such a field). Therefore, we do not intend to present the variety of approximate methods and approaches to solving the Ising-Onsager problem. Detailed discussion of these matters could be found in numerous well known papers and monographs. We note here only the paper by C.N. Yang [3], where the problem was investigated for the case of infinitely small external field in two dimensions. In our opinion, the efforts to find an exact solution to the Ising-Onsager problem is of great interest till now. The reason is, the Ising models are connected with numerous other models of statistical mechanics and quantum field theory [4] – [7]. Therefore the present paper we treat as the next step in the direction described above.

I. THE PARTITION FUNCTION

Here we consider firstly the twodimensional case \( d = 2, H \neq 0 \), and then we will report on the results for the threedimensional case \( d = 3, H \neq 0 \) without going into details of derivations. Let us consider a rectangular lattice consisting of \( M \) columns and \( N \) lines, in nodes of which are given variables \( \sigma_{nm} \) which take values \( \pm 1 \).

These variables will be called below ”spins”. The collective index \( nm \) numbers nodes of the lattice; \( n \) numbers a line, \( m \) numbers a column. The Ising model with nearest neighbors interaction is given by the following form of the Hamiltonian:

\[
\mathcal{H} = -J_2 \sum_{nm} \sigma_{nm} \sigma_{n+1,m} - J_1 \sum_{nm} \sigma_{nm} \sigma_{n,m+1} - H \sum_{nm} \sigma_{nm}, \tag{1.1}
\]
which takes into account possible anisotropy of the interaction between nearest neighbors and also interaction of spins $\sigma_{nm}$ with external field $H$, which is directed ”upwards” ($\sigma_{nm} = +1$).

The investigated problem consists of calculation of the statistical sum for the system:

$$Z(h) = \sum_{\sigma_{11}=\pm 1} \cdots \sum_{\sigma_{NM}=\pm 1} \exp\left(-\beta H\right)$$

$$= \sum_{(\sigma_{nm}=\pm 1)} \exp \left[ \sum_{n,m=1}^{NM} \left( K_2 \sigma_{nm} \sigma_{n+1,m} + K_1 \sigma_{nm} \sigma_{n,m+1} + h \sigma_{nm} \right) \right], \quad (1.2)$$

where

$$K_{1,2} = \beta J_{1,2}, \quad h = \beta H, \quad \beta = 1/k_B T. \quad (1.3)$$

Typically periodic boundary conditions on variables $\sigma_{nm}$ are imposed and we will assume this everywhere below. Let us note that the statistical sum (1.2) is symmetric with respect to the change $h \to -h$.

As is known [8], the statistical sum (1.2) can be represented in the form of the trace of the $T$-operator ($T$-transfer matrix):

$$Z(h) = \text{Tr}(T)^M = \text{Tr} \left[ (2 \sinh 2K_1)^{(N/2)} T_1 T_2 T_h \right]^M, \quad (1.4)$$

where the matrices $T_{1,2,h}$ are of the form:

$$T_1 = \exp \left( K_1^* \sum_{n=1}^{N} \sigma^x_n \right), \quad (1.5)$$

$$T_2 = \exp \left( K_2 \sum_{n=1}^{N} \sigma^z_n \sigma^z_{n+1} \right), \quad \sigma^z_{N+1} = \sigma^z_1, \quad (1.6)$$

$$T_h = \exp \left( h \sum_{n=1}^{N} \sigma^z_n \right), \quad (1.7)$$

and $K_1^*$ and $K_1$ are connected by the relations:

$$\exp(-2K_1) = \tanh(K_1^*), \quad \sinh(2K_1) \sinh(2K_1^*) = 1. \quad (1.8)$$
In the formulae (1.5) – (1.7) the quantities \((\sigma_{n}^{x,y,z})\), \((n = 1, 2, ...N)\) - are well known from quantum mechanics \(2^N\)-dimensional matrices:

\[
\sigma_{n}^{x,y,z} = 1 \otimes 1 \otimes ... \otimes \sigma_{n}^{x,y,z} \otimes ... \otimes 1, \quad (N \text{ factors}),
\]

where \(\sigma^{x,y,z}\) — twodimensional spin Pauli matrices:

\[
\sigma_{x,y,z}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{x,y,z}^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{x,y,z}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

satisfying the standard transposition relations:

\[
(\sigma_k)^2 = 1, \quad \sigma_k \sigma_i + \sigma_i \sigma_k = 0, \quad (k, j) = x, y, z; \quad \sigma_x \sigma_y = i\sigma_z, \quad ...
\]

(1.10)

For example, spin matrices for the \(n\)-th electron in the system consisting of \(N\) nonrelativisitc electrons are exactly the matrices \(\sigma_{n}^{x,y,z}\). It is known that for \(n \neq n'\) spin matrices \(\sigma_{n}^{x,y,z}\) commute and for any given particular \(n\) they satisfy formally relations (1.10). It follows from this that matrices \(T_2\) and \(T_h\), (1.6 – 7) commute but they do not commute with the matrix \(T_1\), (1.5), i.e.

\[
[T_2, T_h] - = 0, \quad [T_h, T_1] - \neq 0, \quad [T_2, T_1] - \neq 0.
\]

(1.11)

It follows from (1.11) that under \(Tr(...)\) we can write for the statistical sum (1.4) the expression:

\[
Z(h) = Tr(\gamma T_1 T_h^{1/2} T_2 T_h^{1/2})^M = Tr(\gamma T_h^{1/2} T_1 T_h^{1/2} T_2)^M \equiv Tr(\gamma M),
\]

(1.12)

\[
P \equiv (2 \sinh 2K_1)^{N/2} T_h^{1/2} T_1 T_h^{1/2} T_2,
\]

(1.13)

where we used the identity \(Tr(AB) = Tr(BA)\).

Now we will consider in more details the matrix \(U \equiv T_h^{1/2} T_1 T_h^{1/2}\); since the matrices \(\sigma_{n}^{x}\) and \(\sigma_{n'}^{x}\), commute for \(n \neq n'\), we can write the matrix \(U\) in the form:

\[
U \equiv T_h^{1/2} T_1 T_h^{1/2} = \prod_{n=1}^{N} e^{(h/2)\sigma_n^x} e^{K_1 \sigma_n^z} e^{(h/2)\sigma_n^x} \equiv \prod_{n=1}^{N} U_n.
\]

(1.14)
Further the matrix $U_n$ can be represented in the following form:

$$U_n = e^{(h/2)\sigma_n^z} (\cosh K_1^* + \sigma_n^x \sinh K_1^*) e^{(h/2)\sigma_n^z} = \exp \left[ \omega \left( \frac{\cosh K_1^* \sinh h}{\sinh \omega} \sigma_n^z + \frac{\sinh K_1^*}{\sinh \omega} \sigma_n^x \right) \right],$$

(1.15)

where $\omega$ - is a positive root of the equation:

$$\cosh \omega = \cosh K_1^* \cosh h,$$

(1.16)

In determination of the expression (1.15) we used the identity:

$$\exp(\mu t) = \cosh \mu + t \sinh \mu, \quad t^2 = 1.$$  

(1.17)

It is easy to see that for $h = 0$ we obtain from (1.12) the standard expression for the statistical sum $Z$ for the twodimensional Ising model without external field [5, 6].

A. The 1D Ising model

Relatively easy one can show that (1.12) becomes:

$$Z(h) = Tr \left[ \left(2 \sinh 2K_1 \right)^{N/2} \prod_{n=1}^N U_n T_2 \right]^M,$$

(1.18)

where $U_n$ is given by the formula (1.15), and it describes correctly the transition to the onedimensional Ising model both in the constant $K_1$, and in the constant $K_2$. Indeed, if we take $K_2 = 0$ and $N = 1$, and this procedure corresponds to neglecting summation over $n$ we obtain:

$$Z_1(h) = Tr \left[ \left(2 \sinh 2K_1 \right)^{1/2} U_0 \right]^M,$$

(1.19)

where the matrix $U_0$ is given by (1.15), where the index $n$ was omitted. Eigenvalues of the matrix $U_0$ can be easily obtained:

$$\lambda^\pm = \exp(\pm \omega),$$
where \( \omega \) — is the positive root for the equation (1.16). As a result we obtain the following formula describing free energy on one spin in the thermodynamic limit:

\[
f(h) = -\frac{1}{\beta} \lim_{M \to \infty} \frac{1}{M} \ln Z_1(h) = -\frac{1}{\beta} \ln \left[ e^{K_1} \cosh h + (e^{2K_1} \sinh^2 h + e^{-2K_1})^{1/2} \right],
\]

(1.20)
i.e. the known classical expression \[1\].

Transition to the onedimensional Ising model limit in the constant \( K_1 \) could be done by taking \( K_1 \to 0 \), and \( M = 1 \), i.e. we neglect summation over the index \( m \) and go to the limit \( K_1 \to 0 \). As a result we get from (1.18):

\[
Z_1(h) = Tr \left[ \prod_{n=1}^{N} (1 + \sigma_n^z) T_2 T_h \right],
\]

(1.21)
where we used the transition to the limit:

\[
\lim_{K_1 \to 0} (2 \sinh 2K_1)^{1/2} \exp(K_1^* \sigma_n^z) = (1 + \sigma_n^z),
\]

where we took into account the relation (1.8) between \( K_1 \) and \( K_1^* \). It is reasonable to stress here that the factors \( (1 + \sigma_n^z) \), entering the expression (1.21), are simply necessary to get the correct result. For the aim of calculating the trace (1.21), it is convenient to go to the fermion representation \[6\]. Omitting some calculations we write for \( Z_1(h) \), (1.21) the expression:

\[
Z_1(h) = Tr(D T_2^\pm T_h),
\]

(1.22)
where the operators \( D, T_2^\pm \) and \( T_h \), expressed in terms of Fermi creation and annihilation operators \((c_n^+, c_n)\) are of the form:

\[
D = \prod_{n=1}^{N} \left[ 1 + (-1)^n c_n^+ c_n \right],
\]

(1.23)

\[
T_2^\pm = \exp \left[ K_2 \sum_{n=1}^{N} (c_n^+ - c_n)(c_{n+1}^+ + c_{n+1}) \right],
\]

(1.24)

\[
T_h = \exp \left\{ h \sum_{n=1}^{N} \exp \left[ i\pi \sum_{p=1}^{n-1} c_p^+ c_p \right] (c_n^+ + c_n) \right\},
\]

(1.25)
In the formula (1.24) the sign (+) is related to states that are even with respect to the operator of the complete number of particles \( \hat{N} = \sum_{n=1}^{N} c_n^+ c_n \), to which there correspond anticyclic boundary conditions, and the sign (−) to the odd states, to which there correspond cyclic boundary conditions. It is easy to see that because of the multiplicative character of the operator \( D \), (1.23), all diagonal matrix elements in (1.22) vanish with the exception of the vacuum-vacuum matrix element, i.e.:

\[
Z_1(h) = 2^N < 0 | (T_2^\pm T_h) | 0 >,
\]

where the operators \( T_2^\pm \) and \( T_h \) are defined by (1.24−25). Then, ”acting” with the operator \( T_h \) on the vacuum state \( | 0 > \), and using the Hausdorff-Baker formula \([10]\), \((\alpha, \beta = const)\):

\[
\exp(\alpha x) \exp(\beta y) = \exp(\alpha x + \beta y + (\alpha \beta / 2)[x, y]_-),
\]

the operator \( T_h \), (1.25) can be reduced to the ”effective” form (in the sense of action on \( | 0 > \)):

\[
T_h = \cosh^N(h) \exp \left[ \tanh^2 h \sum_{n=1}^{N} \sum_{p=1}^{N-n} c_n^+ c_{n+p}^+ \right].
\]

When developing the expression (1.27) we have taken into account the fact, that the diagonal matrix elements of the odd number of Fermi operators are equal to zero.

Finally, going to the momentum representation:

\[
c_n = \frac{\exp(-i\pi/4)}{\sqrt{N}} \sum_q e^{iqn} \eta_q,
\]

and computing the matrix element for a fixed \( q \) after some not complicated transformations we arrive in the case of even states at the expression for the statistical sum \([Z_1^+(h)]\) (1.26):

\[
Z_1^+(h) = [2 \cosh(h)]^N \prod_{0 \leq q \leq \pi} \left[ \cosh 2K_2 - \sinh 2K_2 \cos q + \alpha^2 \sinh 2K_2(1 + \cos q) \right]
\]

\[
= [2\cosh(h)\cosh K_2]^N \prod_{n=1}^{N} \left[ 1 + \frac{z^2}{2} + 2z_2z - 2z_2(1 - z) \cos \left( \frac{2\pi n}{N} \right) \right]^{1/2},
\]

\( \frac{1}{2} \)
where \( z_2 \equiv \tanh K_2 \) and \( z \equiv \alpha^2 = \tanh^2 h \). In the case of odd states it is easy to show that the sum \( Z_1^- (h) \), is equal to \( Z_1^+ (h) = 2Z_1^+ (h) \). Finally, we obtain in the thermodynamic limit again the formula (1.20), \( (M \to N, \ K_1 \to K_2) \) for free energy on one spin.

The 1D Ising model was discussed here with so many details because it was unexpectedly found, that \( Z_1^+ (h) \) such as represented in (1.28) can be applied in graph theory. Namely, using the representation (1.28) one can calculate the generating function for the Hamilton cycles on the simple square lattice \( (N \times M) \).

**B. The 2D Ising model**

For further purposes it is convenient to represent the matrix \( UT_2 \), entering the formula (1.18) and where \( U \) is defined by (1.14) in the form of a simple product of matrices \( P_n \) such that their diagonalization is relatively easy. The reason is that, as is known from [3] — [6], to calculate free energy on one spin in the thermodynamic limit it is sufficient to find the maximal eigenvalue of the matrix \( UT_2 \), which is \( 2^N \times 2^N \) dimensional. First of all we note that the matrix \( U \) (1.14) can be represented in the form of a simple product of matrices \( U_0 \):

\[
U = \prod_{n=1}^{N} U_n = U_0 \otimes U_0 \otimes \ldots \otimes U_0, \quad N - \text{factors}, \tag{1.29}
\]

where the matrix \( U_0 \) is defined by the formula (1.15), in which one should skip the index \( n \). In order to represent the matrix \( T_2 \) (1.6) in the form of a simple product we will use the well known identity [7], [8]:

\[
\exp(A^2) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \exp(-\xi^2 + 2A\xi) d\xi, \tag{1.30}
\]

where \( A \) — is a bounded operator (matrix). Writing \( \exp(K_2\sigma^z_n\sigma^z_{n+1}) \) in the form

\[
\exp(K_2\sigma^z_n\sigma^z_{n+1}) = \exp \left[ \frac{K_2}{2} (\sigma^z_n + \sigma^z_{n+1})^2 - K_2 \right],
\]

we can represent the matrix \( T_2 \) in the form:

\[
T_2 = \frac{e^{-NK_2}}{\pi^{N/2}} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp \left[ -\sum_{n=1}^{N} \xi_n^2 + (2K_2)^{1/2} \sum_{n=1}^{N} (\xi_n + \xi_{n+1})\sigma^z_{n+1} \right] \prod_{n=1}^{N} d\xi_n, \tag{1.31}
\]
where $\sigma_{N+1}^z = \sigma_1^z$ and $\xi_{N+1} = \xi_1$. After writing the matrix $T_2$, (1.31) this way we can represent it in the form of a simple product of matrices $\exp[(2K_2)^{1/2}(\xi_n + \xi_{n+1})\sigma^z]$ inside the integral:

$$\prod_{n=1}^{N} \exp[(2K_2)^{1/2}(\xi_n + \xi_{n+1})\sigma_{n+1}^z] = \prod_{n=1}^{N} \otimes \exp[(2K_2)^{1/2}(\xi_n + \xi_{n+1})\sigma^z],$$  \hspace{1cm} (1.32)

where on the right hand side of the formula there is a simple product of $2 \times 2$ matrices.

Next, we can write the matrix $U T_2$, using (1.29) and (1.31 – 32), in the form:

$$U T_2 = \frac{e^{-NK_2}}{\pi^{N/2}} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{n=1}^{N} d\xi_n \exp \left[ - \sum_{n=1}^{N} \xi_n^2 \right] \left[ \prod_{n=1}^{N} \otimes \exp[(2K_2)^{1/2}(\xi_n + \xi_{n+1})\sigma^z] \right], \hspace{1cm} (1.33)$$

where we included the constant matrix $U$ under the integral and we used the known theorem on simple product of matrices:

$$(A_1 \otimes A_2 \otimes \ldots)(B_1 \otimes B_2 \otimes \ldots) = (A_1B_1) \otimes (A_2B_2) \otimes \ldots.$$  \hspace{1cm}

Using the expression (1.33) enables calculation of all $2^N$ eigenvalues of the matrix $U T_2$. Eigenvalues of the matrix $U_0 \exp[\alpha(\xi_n + \xi_{n+1})\sigma^z]$ can be easily calculated and are equal to:

$$\lambda^\pm(n, n+1) = e^{\pm \omega(n,n+1)},$$  \hspace{1cm} (1.34)

where $\omega(n, n+1)$ is defined as a positive root of the equation:

$$\cosh[\omega(n, n+1)] = \cosh(K_1^z) \cosh[h + \alpha(\xi_n + \xi_{n+1})], \hspace{0.5cm} \alpha \equiv (2K_2)^{1/2}. \hspace{1cm} (1.35)$$

In the diagonal representation the matrix $V$ under the integral (1.33) can be represented in the form:

$$V = \left[ \prod_{n=1}^{N} \otimes S(n, n+1) \right] \prod_{n=1}^{N} \otimes \begin{pmatrix} \lambda^+(n, n+1) & 0 \\ 0 & \lambda^-(n, n+1) \end{pmatrix} \left[ \prod_{n=1}^{N} \otimes S'(n, n+1) \right], \hspace{1cm} (1.36)$$

where $S(n, n+1)S'(n, n+1) = 1$, and $\lambda^\pm(n, n+1)$ are defined above by (1.34). From this it follows that the eigenvalues $\Lambda_j$ of the matrix $V$ are equal to:

$$\Lambda_j = \lambda^\pm(1, 2)\lambda^\pm(3, 4)\ldots\lambda^\pm(N, 1), \hspace{0.5cm} (j = 1, 2, 3, \ldots, 2^N), \hspace{1cm} (1.37)$$
where to each $j$ there corresponds a combination of $(+)$ and $(-)$ eigenvalues $\lambda^{\pm}(n, n + 1)$.

Finally we can express the statistical sum (1.18) by the formula:

$$Z(h) = Tr \left[ (2 \sinh 2K_1)^{N/2} e^{-NK_2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{n=1}^{N} d\xi_n \exp \left( -\sum_{n=1}^{N} \xi_n^2 \right) V^M \right],$$

(1.38)

where the matrix $V$ is given by (1.36).

### II. THE FREE ENERGY

As we mentioned above, the free energy on one spin in the thermodynamic limit can be expressed by the maximal eigenvalue of the matrix $UT_2$, entering (1.18). We managed to express this matrix in the form of an $N$-type integral (1.38), where the matrix $V$ is defined by (1.36), and all the matrix elements of $V$ are positive. On the other hand, in accordance with the known Frobenius-Perron theorem, the matrix $B$, with all matrix elements positive has its maximal eigenvalue nondegenerate. Let us assign to the maximal eigenvalue of the matrix $V$ a letter $\Lambda_{\text{max}}$. In accordance with our definition of the eigenvalues $\lambda^{\pm}(n, n + 1)$, using (1.37) we obtain the following expression for $\Lambda_{\text{max}}$:

$$\Lambda_{\text{max}} = \prod_{n=1}^{N} \lambda^{+}(n, n + 1) = \prod_{n=1}^{N} \exp \left( \sum_{n=1}^{N} \omega(n, n + 1) \right),$$

(2.1)

where $\omega(n, n + 1)$ is defined as a positive root of the equation (1.35), and

$$\Lambda_{\text{max}} > \Lambda_j, \quad (j = 1, 2, ..., 2^N).$$

Further we denote the eigenvalues of the matrix $UT_2$ by $\tilde{\Lambda}_j$. Taking into account the dimension of the matrix $UT_2$, which is equal to $2^N$, we can write on the base of the relation (1.18) obvious inequalities:

$$\tilde{\Lambda}_{\text{max}}^M \leq Z(h) \leq 2^N \tilde{\Lambda}_{\text{max}}^M,$$

(2.2)

where $\tilde{\Lambda}_{\text{max}}$ — is the maximal eigenvalue in the set $\tilde{\Lambda}_j$, to which we included also the constant factor $(2 \sinh 2K_1)^{N/2}$. Taking the logarithm (2.2) of this expression and dividing by the nodes number $NM$, we arrive at the next system of inequalities:
\[
\frac{1}{N} \ln(\tilde{\Lambda}_{max}) \leq \frac{1}{NM} \ln Z(h) \leq \frac{1}{N} \ln(\tilde{\Lambda}_{max}) + \frac{1}{M} \ln 2,
\] (2.3)

in which the expression in the middle represents free energy on the node if we neglect the factor \(-\beta^{-1}\), where \(\beta = \frac{1}{k_B T}\), \(T\) is temperature. Going to the limit \((N, M) \to \infty\), we obtain the desired formula describing free energy on one spin in the thermodynamic limit:

\[
f_2(h) = -\frac{1}{\beta} \lim_{N,M \to \infty} \frac{1}{NM} \ln Z(h) = -\frac{1}{\beta} \lim_{N \to \infty} \frac{1}{N} \ln(\tilde{\Lambda}_{max}),
\] (2.4)

where \(\tilde{\Lambda}_{max}\) is, in accordance with (1.38) and (2.1) equal to:

\[
\tilde{\Lambda}_{max} = (2 \sinh 2K_1)^{N/2} e^{-NK_2 \pi N/2} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{n=1}^{N} d\xi_n \exp \left[ \sum_{n=1}^{N} (-\xi_n^2 + \omega(n, n+1)) \right].
\] (2.5)

Finally, using the Onsager identity:

\[
| x | = \frac{1}{\pi} \int_{0}^{\pi} dq \ln[2 \cosh(x) - 2 \cos(q)],
\] (2.6)

we obtain the following expression for the function \(\omega(n, n+1)\) (1.35)

\[
\omega(n, n+1) = \frac{1}{\pi} \int_{0}^{\pi} dq \ln[2 \cosh K_1^* \cosh(h + (2K_2)^{1/2}(\xi_n + \xi_{n+1})) - 2 \cos(q)],
\] (2.7)

Expressions (2.4) and (2.5) should describe properly at least the transition to the onedimensional Ising model. It is easy to show that the transition to the limit \(K_2 = 0\), gives the correct result (1.20) for the onedimensional Ising model. The analogous limit taken with respect to the constant \(K_1\) seems a little bit more complicated and has the form:

\[
\lim_{K_1 \to 0} \tilde{\Lambda}_{max}(K_1) = e^{-NK_2 \pi N/2} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{n=1}^{N} d\xi_n \exp \left[ -\sum_{n=1}^{N} \xi_n^2 \right] \prod_{n=1}^{N} 2 \cosh[h + \alpha(\xi_n + \xi_{n+1})],
\] (2.8)

where \(\alpha\) is defined above by (1.35). This is an integral of the Gauss type and it could be relatively easily calculated. For this purpose we apply the following formal procedure. Namely, let us write the expression \(2 \cosh(\ldots)\), entering the integral (2.8), in the form:

\[
2 \cosh[h + \alpha(\xi_n + \xi_{n+1})] = \sum_{\mu_n = \pm 1} \exp[\mu_n h + \alpha \mu_n (\xi_n + \xi_{n+1})], \quad (n = 1, 2, \ldots, N),
\] (2.9)
where we introduced a new variable $\mu_n$ of the Ising type. Therefore, we can represent the right hand side of the equality (2.8) in the form:

$$\sum_{(\mu_n=\pm 1)} \left\{ \frac{e^{-NK_2}}{\pi^{N/2}} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \prod_{n=1}^{N} d\xi_n \exp \left[ - \sum_{n=1}^{N} \xi_n^2 \right] \prod_{n=1}^{N} e^{h\mu_n} \exp \left[ \alpha\mu_n(\xi_n + \xi_{n+1}) \right] \right\} =$$

$$\sum_{(\mu_n=\pm 1)} \exp \left[ \sum_{n=1}^{N} (h\mu_n + K_2\mu_n\mu_{n+1}) \right], \quad (2.10)$$

where we took an integral over the variables $\xi_n$, and we imposed on variables $\mu_n$ cyclic boundary conditions ($\mu_{N+1} = \mu_1$). Calculation by standard methods \cite{6,7} of the sum (2.10), and following substitution of the expression (2.4), gives well known result (1.20).

Consideration of the expressions (2.4) and (2.5) for free energy of the twodimensional Ising model in external field we present in the end of this paper but now we go to the three-dimensional case.

III. THE THREE-DIMENSIONAL ISING MODEL

Hamiltonian for the three-dimensional Ising model in external field with nearest neighbors interaction we write in the form:

$$\mathcal{H} = - \sum_{(n,m,k)=1}^{NMK} (J_1\sigma_{n,m+1,k} + J_2\sigma_{nmk}\sigma_{n+1,mk} + J_3\sigma_{nmk}\sigma_{nm,k+1} + H\sigma_{nmk}), \quad (3.1)$$

where the collective index $(nmk)$ numbers nodes of the simple cubic lattice and $H$ is the external field. Constants $I_j$ take into account anisotropy of interaction of Ising spins. We impose on the variables $\sigma_{nmk}$, as it is commonly done, periodic boundary conditions. Quantities $N, M$ and $K$ are node numbers in corresponding directions of a cubic lattice. As is known \cite{8}, the statistical sum for the three-dimensional Ising model can be represented in the form of a trace of the $K$-th power of the fiber-fiber transfer matrix ($R$):

$$W(h) = Tr(R)^K \equiv Tr(T_3T_2T_1T_h)^K, \quad (3.2)$$

where the matrices $T_i$, ($i = 1, 2, 3, h$) of dimensions $2^{NM} \times 2^{NM}$ are of the form:
\[ T_1 = \exp \left( K_1 \sum_{nm} \sigma_{nm}^z \sigma_{n,m+1}^z \right), \quad T_2 = \exp \left( K_2 \sum_{nm} \sigma_{nm}^z \sigma_{n+1,m}^z \right), \]  
\[ T_3 = (2 \sinh 2K_3)^{NM/2} \exp \left( K_3^* \sum_{nm} \sigma_{nm}^x \right), \quad T_h = \exp \left( h \sum_{nm} \sigma_{nm}^z \right). \]  
(3.3)

Here \( K_i = \beta J_i, \ (i = 1, 2, 3); \) \( \beta = \left( \frac{1}{k_B T} \right), \ T - \text{temperature}, \ h = \beta H, \) and \( K_3 \) and \( K_3^* \) are connected by relations of type (1.8). In the formulae (3.3–4) the matrices \( \sigma_{nm}^{z,x} \) are Pauli matrices, which are defined analogously to (1.9), and have dimensions \( 2^{NM} \times 2^{NM} \).

Continuing considerations analogous to these in the twodimensional case we obtain the following formula describing free energy on one spin in the thermodynamic limit:

\[ f_3(h) = -\frac{1}{\beta} \lim_{N,M \to \infty} \frac{1}{NM} \ln \Lambda_{max}, \]  
(3.5)

where the maximal eigenvalue \( \Lambda_{max} \) of the matrix \( R, (3.2) \) is defined by:

\[ \Lambda_{max} = \]

\[ (2 \sinh 2K_3)^{NM/2} e^{-NM(K_1+K_2)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Pi_{nm} d\eta_{nm} d\xi_{nm} \exp \left[ \sum_{n,m=1}^{NM} (-\eta_{nm}^2 - \xi_{nm}^2 + \omega(n,m)) \right] , \]  
(3.6)

and \( \omega(n, m) \) is defined as the positive root of the equation:

\[ \cosh \omega(n, m) = \cosh K_3^* \cosh[h + \alpha_1 (\eta_{n+1,m} + \eta_{n+1,m+1}) + \alpha_2 (\xi_{n+1,m+1} + \xi_{n+1,m+1})], \]  
(3.7)

where \( \alpha_{1,2} = (2K_{1,2})^{1/2} \).

We impose on integration variables \( \eta_{nm} \) and \( \xi_{nm} \) cyclic boundary conditions, in accordance with periodic boundary conditions for the former variables:

\[ \sigma_{N+1,m}^z = +\sigma_{1,m}^z, \quad \sigma_{n,M+1}^z = +\sigma_{n,1}^z, \quad n(m) = 1, 2, 3, \ldots N(M). \]  
(3.8)

Similarly as in the twodimensional case, the function \( \omega(n, m) \) can be expressed explicitly in terms of variables \( \eta_{nm} \) and \( \xi_{nm} \), using for this aim the integral representation given by the Onsager identity (2.6).
It is a little bit more complicated matter to take a limit with respect to the constant of interaction $K_3$, although the derived formula is much simpler than the formulae (2.4 – 5). Imposition of the limit ($K_3 \to 0$) in the formulae (3.5 – 7), gives, after some simple transformations, the following representation of free energy on one spin of the twodimensional Ising model $f_2(h)$:

$$f_2(h) = -\frac{1}{\beta} \left( \lim_{K_3 \to 0, (N, M) \to \infty} \right) \frac{1}{NM} \ln \Lambda_{max} =$$

$$-\frac{1}{\beta} \lim_{(N, M) \to \infty} \frac{1}{NM} \ln \left\{ \frac{e^{-NM(K_1 + K_2)}}{\pi NM} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{nm} d\eta_{nm} d\xi_{nm} \exp \left[ -\sum_{nm} (\eta_{nm}^2 + \xi_{nm}^2) \right] \right\} \times \prod_{nm} 2 \cosh[h + \alpha_1(\eta_{n+1,m} + \eta_{n+1,m+1}) + \alpha_2(\xi_{n,m+1} + \xi_{n+1,m+1})] \right\} ; \quad (3.9)$$

where we used relations of the type (1.8), and $\alpha_{1,2}$ are defined above (3.7). The integrals in (3.9) are integrals of the Gauss type and, as it is easy to show applying the described above formal way of introducing a variable of Ising type $\mu_{nm} = \pm 1$, can be represented in the form (1.2).

Analogously, one can show rigorously that free energy on one spin for the three-dimensional Ising model can be represented in the form of a multiple integral of the Gauss type:

$$f_3(h) = -\frac{1}{\beta} \lim_{(N, M, K) \to \infty} \frac{1}{NMK} \ln \left\{ \frac{e^{-NMK(K_1 + K_2 + K_3)}}{\pi^{3NMK/2}} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{nmk} d\eta_{nmk} d\xi_{nmk} d\zeta_{nmk} \times \exp \left[ -\sum_{n,m,k} (\eta_{nmk}^2 + \xi_{nmk}^2 + \zeta_{nmk}^2) \right] \right\} \times \prod_{nmk} 2 \cosh[h + \alpha_1(\eta_{nmk} + \eta_{n+1,mk}) + \alpha_2(\xi_{nmk} + \xi_{n,m+1,k}) + \alpha_3(\zeta_{nmk} + \zeta_{nm,k+1})] \right\} ; \quad (3.10)$$

where $\alpha_i = (2K_i)^{1/2}$, ($i = 1, 2, 3$). The formulae (3.10) can be in obvious way generalized to describe d-dimensional Ising models but we will not stop on these matters here.

We will make here a few remarks. First of all, as far as it is known to the author, the representations for free energy on one spin in the forms (2.4 – 5) and (3.5 – 10) for
the twodimensional and threedimensional Ising models, respectively, did not appear in the literature. We believe that the known representations (see, e.g. [12–14]) are more complicated than the ones derived by us. For example, in the paper [14] was given the following integral representation for the Ising model:

\[ Z = \prod_m \int_{-\infty}^{\infty} \exp(-\frac{1}{2}b s_m^2 - us_m^4) \exp(K\sum_{n,i} s_n s_{n+i}) ds_m, \]  

(3.11)

where \((m, n, i)\) are vector indices (we use notation from the paper [14]). In the limit \(u \to \infty\) and \(b \to -\infty\), \((b = -4u)\) the Ising model is recovered (for this aim one should use additionally the fact that to every particular spin there is a factor \((u/\pi)^{1/2} \exp(-u))\). It is clear that the presence of the term \(us_m^4\) in \(\exp(...\) of the expression (3.11) considerably complicates analysis of this expression. Further, although the formulae (3.9) and (3.10) are in a sense obvious, the formulae (2.4 – 5) and (3.5 – 7) are not obvious. The integrals (2.5) and (3.6) can be represented as integrals of the "quasi Gauss" type, because the functions \(\omega(n, n + 1)\) and \(\omega(n, m)\), described by the relations (1.35) and (3.7), respectively, in accordance with the Onsager identity (2.6) are almost "linear" in their arguments \((\xi_n)\) and \((\eta_{nm}, \xi_{nm})\). This justifies our hopes we could learn now to calculate rigorously the integrals (2.5) and (3.6) in case \(h = 0\) using an Ising type variable \((\mu = \pm 1)\), described above. On the other hand, it seems to us that for the case \((h \neq 0)\) it is much simpler to deal with the expression (2.5), than with the expression (3.9), although it could sound paradoxically. For the threedimensional Ising model in external field \((h \neq 0)\) the situation is no longer so clear, for the infinitely small field \((h \sim 0)\), similarly as in the twodimensional case, it is easier to analyze the expression (3.6), than (3.10), that we shall be show in a next publication.

**IV. CONCLUSIONS**

The derived expressions (2.4 – 5) and (3.5) – (3.10) for free energy on one spin for the Ising model can be of some interest, we hope. The reason is we actually should learn how to calculate logarithmic asymptotics of multiple integrals of Gauss type, as it could be seen
from (2.4) and (3.5). It is known, that there exist a well developed formalism of calculation of logarithmic asymptotes for integrals of the Laplace type for the onedimensional as well as for the multidimensional cases [15]. In the case under consideration the situation is a little bit more complicated, because for the integrals of the kinds (2.4) and (3.5) it is not possible to transform them to a form of a multiple integral of the Laplace type at least in the framework of their classical definition. With the increase of the large parameter (λ → ∞) there changes also the number of variables, over which one integrates and this needs reformulation of the corresponding methods of asymptotic estimation of the considered integrals [15]. In future publications we intended to investigate in more details the expressions (2.4) and (3.5), obtained in this paper and to calculate critical indices for the Ising model.

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