HÖLDER STRONG METRIC SUBREGULARITY AND ITS APPLICATIONS TO CONVERGENCE ANALYSIS OF INEXACT NEWTON METHODS

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Abstract. In this paper we conduct local convergence analysis of the inexact Newton methods for solving the generalized equation $0 \in f(x) + F(x)$ under the assumption of Hölder strong metric subregularity, where $f : X \to Y$ is a single-valued mapping while $F : X \rightrightarrows Y$ is a set-valued mapping between arbitrary Banach spaces. Our work are proceeded as twofold: we first explore fully the property of Hölder strong metric subregularity by establishing a verifiable necessary and sufficient condition as well as discussing its stability under small perturbations, and secondly, with the help of aforementioned theoretical analysis, we conclude that every sequence generated by the inexact (quasi) Newton method and staying in a neighborhood of the solution $\bar{x}$ is convergent (superlinearly) of order $p(1 + q)$ where $p$ is the order of Hölder strong metric subregularity imposed on the mapping $f + F$ and $q$ is the order of Hölder calmness property for the derivative $Df$ while $p$ and $q$ complement each other as long as $p(1 + q) \geq 1$.

1. Introduction. The main objective of this paper is to study iterative methods of Newton type for solving the generalized equation

$$0 \in f(x) + F(x),$$

where $f : X \to Y$ is a single-valued mapping while $F : X \rightrightarrows Y$ is a set-valued mapping between arbitrary Banach spaces. It is well recognized that the general model (1) has been used to describe a vast variety of problems in a unified way, including equations and most notably variational inequalities, constraint systems, and optimality conditions in mathematical programming and optimal control.

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A Newton-type method for solving (1) utilizes the iteration
\[ f(x_k) + Df(x_k)(x_{k+1} - x_k) + F(x_{k+1}) \geq 0, \text{ for } k = 0, 1, \ldots, \] (2)
with a given starting point \( x_0 \), where \( Df \) is the derivative mapping of \( f \). When \( F \equiv 0 \), the iteration (2) becomes the standard Newton method for solving the equation \( f(x) = 0 \). Inexact Newton methods for solving smooth equation \( f(x) = 0 \) in finite dimensions (i.e., (2) with \( F \equiv 0 \) and \( X = Y = \mathbb{R}^n \)) were introduced by Dembo, Eisenstat, and Steihaug [5]. Specifically, for a given sequence \( \eta_k \in (0, +\infty) \) and a starting point \( x_0 \), they choose the \((k+1)\)st iterate to satisfy the condition
\[ (f(x_k) + Df(x_k)(x_{k+1} - x_k)) \cap \eta_k \|f(x_k)\| B_X \neq \emptyset, \] (3)
where \( B_X \) stands for the closed unit ball of the space \( X \). For solving generalized equation (1) in the Banach space setting, Dontchev and Rockafellar [10] proposed the following inexact Newton method:
\[ (f(x_k) + Df(x_k)(x_{k+1} - x_k) + F(x_{k+1})) \cap R_k(x_k, x_{k+1}) \neq \emptyset, \text{ for } k = 0, 1, \ldots, \] (4)
where \( Df \) is the Fréchet derivative mapping of \( f \) and \( R_k : X \times X \rightrightarrows Y \) is a sequence of set-valued mappings with closed graphs which represent the inexactness for the general model (2) and are not actually calculated in a specified manner. Under regularity properties of the mappings \( f + F \) and \( R_k \), it is shown in [10] that every sequence generated by method (4) is convergent either \( q \)-linearly, \( q \)-superlinearly or \( q \)-quadratically according to the particular assumptions, provided that the starting point is sufficiently close to the reference solution. In the case when \( F \equiv 0 \) and \( R_k(x_k, x_{k+1}) = \eta_k \|f(x_k)\| B_X \), the iteration (4) reduces to (3).
For the case when \( f \) is not necessarily differentiable, the authors in [3] introduced a mapping \( H : X \rightrightarrows L(X,Y) \) and applied a sequence of linear mappings \( A_k \in H(x_k) \) instead of \( Df(x_k) \) with \( R_k \) depending on \( x_k \) only in iteration (4) to study nonsmooth generalized equations (1).

The utilization of metric regularity, strong metric regularity and strong metric subregularity properties of set-valued mappings is the key to handle generalized equations in the aforementioned work. Strong metric subregularity plays a pivotal role in stability and sensitivity analysis of many optimization problems. For example, in the problem of minimizing a convex twice differentiable function over a convex polyhedral set, the strong metric subregularity of the optimality mapping is equivalent to the standard second-order sufficient condition. Since it is persistent with respect to small calm perturbations (see [7, Theorem 3I.6]), strong metric subregularity/regularity has been widely employed in convergence analysis of the method (2) or (4) as well as other forms of Newton method (cf. [1, 2, 3, 6, 8, 9, 10, 11]).

As a useful variant of (strong) metric subregularity, Hölder (strong) metric subregularity has aroused much attention in recent years and been extensively studied by many authors (cf. [12, 13, 14, 15, 16, 17, 18]). Li and Mordukhovich [13] considered the characterization of Hölder metric subregularity of order \( q \in (0, 1] \) and its application in convergence rate analysis of the proximal point method. In this paper, enlightened mainly by the work in [9, 10], we focus on studying Hölder strong metric subregularity of any positive order \( q \) and its applications in local convergence analysis of the inexact Newton method (4). We first established a verifiable necessary and sufficient condition for Hölder strong metric subregularity of any positive order, which is a generalization of [4, Theorem 5.3] for Hölder case. A class of compelling examples are provided to illustrate the importance and application of such theoretical extension. Then relying on the assumptions that \( f + F \) possessing the
Hölder strong metric subregularity, $Df$ being Hölder calm and the outer distance $d^+(0, R_k(u, \bar{x}))$ (the definition will be reviewed in Sect. 2) being bounded, we show that every sequence generated by the inexact (quasi) Newton method and staying close to the solution $\bar{x}$ is convergent of higher order. Exact relationships between the order of convergence and the order of Hölder strong metric subregularity/calmness property are established.

The rest of the paper is structured as follows. In Section 2, we start with the introduction of some basic notations and properties under consideration from variational analysis. Then we investigate the property of Hölder strong metric subregularity fully, with the help of which we conduct in section 3 the convergence rate analysis of inexact (quasi) Newton method.

2. Hölder strong metric subregularity. Firstly, we recall some basic definitions and preliminaries widely employed in what follows. Unless otherwise stated, all the spaces $X$, $Y$ and $Z$ under consideration are Banach spaces with the generic notation $\| \cdot \|$ for their norms. If no confusion arises, the symbol $B_X$ stands for the closed unit ball of the space $X$ while $B(x, \delta)$ indicates the open ball of radius $\delta \in (0, +\infty)$ centered at $x \in X$ in the space $X$. Given a point $x \in X$ and a subset $C \subset X$, define the inner distance from $x$ to $C$ by $d(x, C) := \inf \{ \| x - c \| : c \in C \}$ with the convention that $d(x, \emptyset) := +\infty$, while the outer distance by $d^\ast(x, C) = \sup \{ \| x - c \| : c \in C \}$. Let $F : X \rightrightarrows Y$ be a multifunction and its graph be defined as

$$\text{gph}(F) := \{ (x, y) \in X \times Y : y \in F(x) \}.$$ 

We say that $F$ is a closed multifunction if $\text{gph}(F)$ is closed in the product space $X \times Y$. The symbol $F^{-1} : Y \rightrightarrows X$ stands for the inverse mapping to $F$ such that: $(x, y) \in \text{gph}(F)$ if and only if $(y, x) \in \text{gph}(F^{-1})$.

Next we recall the notions of Hölder strong metric subregularity and isolated calmness properties of mappings with one component in our study.

**Definition 2.1.** Let $p \in (0, +\infty)$. Consider a multifunction $F : X \rightrightarrows Y$ and a pair $(\bar{x}, \bar{y}) \in \text{gph}(F)$.

(i) We say that $F$ is Hölder metrically subregular at $(\bar{x}, \bar{y})$ with constant $\kappa > 0$ of order $p$, if there exists $\delta \in (0, +\infty)$ such that

$$d(x, F^{-1}(\bar{y})) \leq \kappa d(\bar{y}, F(x))^p \quad \forall x \in B(\bar{x}, \delta). \tag{5}$$

The infimum of $\kappa > 0$ over all the combinations $(\kappa, \delta)$ for which (5) holds is called the exact subregularity bound of $F$ around $(\bar{x}, \bar{y})$ and is denoted by $\text{subreg}^p(F; (\bar{x}, \bar{y}))$.

(ii) We say that $F$ is Hölder strongly metrically subregular at $(\bar{x}, \bar{y})$ with constant $\kappa > 0$ of order $p$, if there exists $\delta \in (0, +\infty)$ such that

$$\| x - \bar{x} \| \leq \kappa d(\bar{y}, F(x))^p \quad \forall x \in B(\bar{x}, \delta). \tag{6}$$

It is clear that, $F$ is Hölder strongly metrically subregular at $(\bar{x}, \bar{y})$ of order $p$ if and only if $F$ is Hölder metrically subregular at $(\bar{x}, \bar{y})$ of order $p$ and $F^{-1}(\bar{y}) \cap B(\bar{x}, \delta) = \{ \bar{x} \}$ for some $\delta \in (0, +\infty)$, hence the infimum of $\kappa$ for which (6) holds is equal to $\text{subreg}^p(F; (\bar{x}, \bar{y}))$. When $p = 1$, the above definitions go back to the usual (strong) metric subregularity (cf. [7]). For a single-valued mapping $f$, the Hölder strong metric subregularity of order $p$ reduces to the $p$-th order growth condition (cf. [17, 18]). Let $X = Y = \mathbb{R}, p \in (0, +\infty)$ and $f(x) = |x|^p$ for all $x \in \mathbb{R}$, it is easy to see that $f$ is Hölder strongly metrically subregular at $(0, 0)$ of order $1/p$. Hence, the order of Hölder strong metric subregularity may be any positive number.
Definition 2.2. Let \( q \in [0, +\infty) \). Consider a multifunction \( F : X \rightrightarrows Y \) and a pair \((\bar{x}, \bar{y}) \in \text{gph}(F)\).

(i) We say that \( F \) has the H"older calmness property at \((\bar{x}, \bar{y})\) with constant \( l > 0 \) of order \( q \), if there exists \( \delta \in (0, +\infty) \) such that

\[
F(x) \cap B(\bar{y}, \delta) \subset F(\bar{x}) + l\|x - \bar{x}\|^q B_Y \quad \forall x \in B(\bar{x}, \delta). \tag{7}
\]

The infimum of \( l > 0 \) over all the combinations \((l, \delta)\) for which (7) holds is called the exact calmness bound of \( F \) at \((\bar{x}, \bar{y})\) and denoted by \( \text{clm}^q(F; (\bar{x}, \bar{y})) \):

(ii) We say that \( F \) has the H"older isolated calmness property at \((\bar{x}, \bar{y})\) with constant \( l > 0 \) of order \( q \), if there exists \( \delta \in (0, +\infty) \) such that

\[
F(x) \cap B(\bar{y}, \delta) \subset \{\bar{y}\} + l\|x - \bar{x}\|^q B_Y \quad \forall x \in B(\bar{x}, \delta). \tag{8}
\]

(iii) When \( F = f \) is a single-valued mapping, we say that \( f \) is H"older calm at \( \bar{x} \) of order \( q \) with constant \( l > 0 \) , if there exists \( \delta \in (0, +\infty) \) such that

\[
\|f(x) - f(\bar{x})\| \leq l\|x - \bar{x}\|^q \quad \forall x \in B(\bar{x}, \delta). \tag{9}
\]

The infimum of \( l > 0 \) over all the combinations \((l, \delta)\) for which (9) holds is called the exact calmness bound of \( f \) at \( \bar{x} \) and denoted by \( \text{clm}^q(f; \bar{x}) \).

It is easy to observe that, \( F \) has the H"older isolated calmness property at \((\bar{x}, \bar{y})\) with constant \( l > 0 \) of order \( q \) if and only if \( F \) has the H"older calmness property at \((\bar{x}, \bar{y})\) with constant \( l > 0 \) of order \( q \) and \( \bar{y} \) is an isolated point of \( F(\bar{x}) \). It is worth to mention that the order \( q \) could be zero under our consideration. For a single-valued mapping \( f : X \to Y \) and \( \bar{x} \in X \), it is clear that \( \text{clm}^0(f; \bar{x}) = 0 \) is equivalent as \( f \) being continuous at \( \bar{x} \). And also, the above property goes back to the usual notion of (isolated) calmness when \( q = 1 \).

In the case when \( F : X \times Y \rightrightarrows Z \) is a multifunction with two variables and \(((\bar{x}, \bar{y}), \bar{z}) \in \text{gph}(F)\), we say that \( F \) has the partial H"older calmness property at \((\bar{x}, \bar{y}), \bar{z}\) with respect to \( x \) uniformly in \( y \) of order \( q \), if there exist \( l, \delta \in (0, +\infty) \) such that

\[
F(x, y) \cap B(\bar{z}, \delta) \subset F(\bar{x}, \bar{y}) + l\|x - \bar{x}\|^q B_Z \quad \forall x \in B(\bar{x}, \delta), y \in B(\bar{y}, \delta). \tag{10}
\]

The infimum of \( l > 0 \) over all the combinations \((l, \delta)\) for which (10) holds is called the exact partial calmness bound of \( F \) at \((\bar{x}, \bar{y}), \bar{z}\) and is denoted by \( \text{clm}^q_x(F; ((\bar{x}, \bar{y}), \bar{z})) \).

When \( F = f \) is single-valued, \( \text{clm}^q_x(f; (\bar{x}, \bar{y})) \) is similarly defined.

The following proposition provides a characterization for H"older strong metric subregularity through H"older isolated calmness of its inverse, which follows from the lines used in [7, Theorem 3I. 2]. For completeness, we give its proof as follows.

Proposition 2.3. For \( p \in (0, +\infty) \), a mapping \( F : X \rightrightarrows Y \) is strongly metrically subregular at \((\bar{x}, \bar{y})\) with constant \( \kappa \) of order \( p \) if and only if its inverse \( F^{-1} \) has the isolated calmness property at \((\bar{y}, \bar{x})\) of order \( p \) with the same constant \( \kappa \). Furthermore, we have the equality

\[
\text{subreg}^p(F; (\bar{x}, \bar{y})) = \text{clm}^p(F^{-1}; (\bar{y}, \bar{x})). \tag{11}
\]

Proof. Firstly, we assume that \( F \) is strongly metrically subregular at \((\bar{x}, \bar{y})\) of order \( p \) and intend to show the isolated calmness property of its inverse. To this end, we take arbitrary \( \kappa \in (\text{subreg}^p(F; (\bar{x}, \bar{y})), +\infty) \) and let it be fixed. Then there exists \( \delta \in (0, +\infty) \) such that

\[
\|x - \bar{x}\| \leq \kappa d(\bar{y}, F(x))^p \quad \forall x \in B(\bar{x}, \delta). \tag{12}
\]
It then remains to show that for any $y \in B(\bar{y}, \delta)$, we have
\[ F^{-1}(y) \cap B(\bar{x}, \delta) \subseteq \{ \bar{x} \} + \kappa \| y - \bar{y} \|^p B_X. \] (13)

Pick any $x \in F^{-1}(y) \cap B(\bar{x}, \delta)$. If there is no such $x \in X$, then (13) holds trivially. If not, this entails $y \in F(x)$. It follows from (12) that
\[ \| x - \bar{x} \| \leq \kappa d(\bar{y}, F(x))^p \leq \kappa \| y - \bar{y} \|^p. \]

Thus, $x \in \{ \bar{x} \} + \kappa \| y - \bar{y} \|^p B_X$, which establishes (13) and hence we have
\[ \clm^p(F^{-1}; \bar{y}, \bar{x}) \subseteq \subreg^p(F; \bar{y}, \bar{x}). \]

For the converse, suppose there exists $\delta \in (0, +\infty)$ such that (13) holds with the indicated $\kappa$. Consider any $x \in B(\bar{x}, \min\{\delta, \kappa \delta^p\})$. If $F(x) \cap B(\bar{y}, \delta) = \emptyset$, the right side of (12) is no less than $\kappa \delta^p$ and then (12) holds automatically. If not, for any $y \in F(x) \cap B(\bar{y}, \delta)$, we must have $x \in F^{-1}(y) \cap B(\bar{x}, \delta)$ and it follows from (13) that $x \in \{ \bar{x} \} + \kappa \| y - \bar{y} \|^p B_X$, which indicates that $\| x - \bar{x} \| \leq \kappa \| y - \bar{y} \|^p$. By the arbitrary choice of $y \in F(x) \cap B(\bar{y}, \delta)$, we conclude that (12) holds with $\delta = \min\{\delta, \kappa \delta^p\}$ and in particular we have $\clm^p(F^{-1}; \bar{y}, \bar{x}) \supseteq \subreg^p(F; \bar{y}, \bar{x})$. Therefore the equality (11) is valid and our proof is completed.

Next we present a verifiable necessary and sufficient condition for H"older strong metric subregularity, followed by an example demonstrating the importance and application of such theoretical analysis.

**Theorem 2.4.** Let $F : X \rightrightarrows Y$ be a closed multifunction and $(\bar{x}, \bar{y}) \in \gph(F)$.

(i) Let $p, \tau, \delta, \varepsilon, \eta \in (0, +\infty)$. If for any $x \in B(\bar{x}, \min\{\delta, \kappa \delta^p\})$ and $y \in F(x) \cap B(\bar{y}, \delta)$ with $\| y - \bar{y} \| < d(\bar{y}, F(x)) + \varepsilon$, there exists $(u, v) \in \gph(F) \setminus \{(x, y)\}$ satisfying
\[ \| y - \bar{y} \| \geq \| v - \bar{y} \| + \tau \| x - \bar{x} \|^\frac{1}{p} \max\{\| u - x \|, \eta \| v - y \|\}. \] (14)

Then,
\[ \| x - \bar{x} \| \leq \kappa d(\bar{y}, F(x))^p \quad \forall x \in B(\bar{x}, r), \] (15)
where $\kappa := \frac{2}{r \tau^p}$ and $r := \min\{\frac{2 \delta}{\tau^p}, \kappa \delta^p\}$. Consequently $F$ is H"older strongly metrically subregular at $(\bar{x}, \bar{y})$ of order $p$.

(ii) Let $\kappa, \eta, r, \bar{r}, \bar{p} \in (0, +\infty)$ with $\eta^{\frac{1}{1-p}} \leq \kappa^{\frac{1}{p}}$ and $\bar{r} < r$. If (15) holds, then for any $x \in B(\bar{x}, \bar{r})$ and $y \in F(x)$, we have
\[ \| y - \bar{y} \| \geq \kappa^{-\frac{1}{p}} \| x - \bar{x} \|^\frac{1}{p} \max\{\| x - \bar{x} \|, \eta \| y - \bar{y} \|\}, \] (16)
i.e. (14) holds for $(u, v) = (\bar{x}, \bar{y})$ and $\tau = \kappa^{-\frac{1}{p}}$.

**Proof.** For (i), suppose to the contrary that there exists $x_0 \in B(\bar{x}, r)$ such that
\[ \| x_0 - \bar{x} \| > \kappa d(\bar{y}, F(x_0))^p. \]

Pick $y_0 \in F(x_0)$ to satisfy
\[ \frac{1}{\kappa^{\frac{1}{p}}} \| x_0 - \bar{x} \|^\frac{1}{p} > \| \bar{y} - y_0 \|. \] (17)

Choose $\eta' \in (0, \eta)$ sufficiently small such that
\[ 2\eta' r^{\frac{1}{1-p}} < \kappa^{\frac{1}{p}} \text{ and } \frac{4\eta' \delta r^{\frac{1}{1-p}}}{\kappa^{\frac{1}{p}} - 2\eta' r^{\frac{1}{1-p}}} < \varepsilon. \] (18)

Equip the product space $X \times Y$ with the following norm $\| \cdot \|_{\eta'}$
\[ \| (x, y) \|_{\eta'} := \max\{\| x \|, \eta' \| y \|\} \quad \forall (x, y) \in X \times Y \] (19)
and let $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be such that
\[
f(x, y) := \|y - \tilde{y}\| + \delta_{\text{gph}(F)}(x, y) \quad \forall (x, y) \in X \times Y.
\]
Then, $f$ is lower semicontinuous and
\[
f(x_0, y_0) = \|y_0 - \tilde{y}\| < \inf_{(x, y) \in X \times Y} f(x, y) + \frac{1}{\kappa^p} \|x_0 - \bar{x}\|^{\frac{1}{p}}
\]
(thanks to (17)). Applying to this function the Ekeland variational principle, we obtain $(\hat{x}, \hat{y}) \in \text{gph}(F)$ such that
\[
\|(\hat{x}, \hat{y}) - (x_0, y_0)\|_{q'} \leq \frac{1}{2} \|x_0 - \bar{x}\|, \quad f(\hat{x}, \hat{y}) \leq f(x_0, y_0)
\]
and
\[
f(\hat{x}, \hat{y}) < f(x, y) + \frac{\frac{1}{\kappa^p} \|x_0 - \bar{x}\|^{\frac{1}{p}}}{\frac{2}{\kappa^p} \|x_0 - \bar{x}\|} \|(x, y) - (\hat{x}, \hat{y})\|_{q'}
\]
\[
= f(x, y) + \frac{2}{\kappa^p} \|x_0 - \bar{x}\|^{\frac{1}{p}} \|(x, y) - (\hat{x}, \hat{y})\|_{q'} \quad \forall (x, y) \in X \times Y \setminus \{(\hat{x}, \hat{y})\}.
\]
Hence, \(\|\hat{x} - x_0\| \leq \|(\hat{x}, \hat{y}) - (x_0, y_0)\|_q \leq \frac{1}{2} \|x_0 - \bar{x}\| < \frac{r}{2}\),
\[
\|\hat{x} - x_0\| \leq \|\hat{x} - x_0\| + \|x_0 - \bar{x}\| < \frac{r}{2} + r \leq \delta
\]
and
\[
\|\hat{x} - \bar{x}\| \geq \|x_0 - \bar{x}\| - \|\hat{x} - x_0\| \geq \frac{1}{2} \|x_0 - \bar{x}\| > 0.
\]
This show that $\hat{x} \neq \bar{x}$. By (17) and the definition of $r$, we also have
\[
d(\tilde{y}, F(\hat{x})) \leq \|\tilde{y} - \tilde{y}\| = f(\hat{x}, \tilde{y}) \leq f(x_0, y_0) = \|y_0 - \tilde{y}\| < \frac{1}{\kappa^p} \|x_0 - \bar{x}\|^{\frac{1}{p}} < \left(\frac{r}{\kappa^p}\right)^{\frac{1}{p}} \leq \delta.
\]
Moreover, by (20) and (21), one has that for all $(x, y) \in \text{gph}(F) \setminus \{(\hat{x}, \hat{y})\},$
\[
\|\tilde{y} - \tilde{y}\| < \|y - \tilde{y}\| + \frac{2}{\kappa^p} \|x_0 - \bar{x}\|^{\frac{1}{p}} \max\{\|x - \hat{x}\|, \eta'\|y - \tilde{y}\|\}
\]
\[
\leq \|y - \tilde{y}\| + \frac{2}{\kappa^p} \|\hat{x} - \bar{x}\|^{\frac{1}{p}} \max\{\|x - \hat{x}\|, \eta'\|y - \tilde{y}\|\}.
\]
Therefore, for all $y \in F(\hat{x})$, we also have that
\[
\|\tilde{y} - \tilde{y}\| \leq \|y - \tilde{y}\| + \frac{2}{\kappa^p} \|x_0 - \bar{x}\|^{\frac{1}{p}} \eta'\|y - \tilde{y}\|
\]
\[
\leq \left(1 + \frac{2\eta' r^{\frac{1}{p}}}{\kappa^p}\right) \|y - \tilde{y}\| + \frac{2\eta' r^{\frac{1}{p}}}{\kappa^p} \|\tilde{y} - \tilde{y}\|.
\]
This and the first inequality of (18) imply that
\[
\|\tilde{y} - \tilde{y}\| \leq \frac{1 + 2\eta' r^{\frac{1}{p}}}{1 - 2\eta' r^{\frac{1}{p}}} \|y - \tilde{y}\| \quad \forall y \in F(\hat{x}).
\]
And then,
\[
\|\hat{y} - \hat{g}\| \leq \frac{1 + \frac{2\eta' r^{1-p}}{\kappa p}}{1 - \frac{2\eta' r^{1-p}}{\kappa p}} d(\hat{y}, F(\hat{x})) = d(\hat{y}, F(\hat{x})) + \frac{4\eta' r^{1-p}}{\kappa p - 2\eta' r^{1-p}} d(\hat{y}, F(\hat{x})).
\]

By the second inequality of (18) and (22), one has
\[
\|\hat{y} - \hat{g}\| < d(\hat{y}, F(\hat{x})) + \varepsilon.
\]

Note that \(\hat{x} \in B(\bar{x}, \delta) \setminus \{\bar{x}\}\) and \(\hat{y} \in B(\bar{y}, \delta)\), by (14), there exists \((u, v) \in \text{gph}(F)\) such that
\[
\|\hat{y} - \hat{g}\| \geq \|v - \hat{g}\| + \frac{2\eta \|\hat{x} - \hat{g}\|^{\frac{1}{p}}}{\kappa^{\frac{1}{p}}} \max\{\|u - \hat{x}\|, \eta\|v - \hat{g}\|\}.
\]

Since \(\eta' < \eta\), we obtain a contradiction to (23).

For (ii), fix any \(x \in B(\bar{x}, \bar{r})\) and \(y \in F(x)\), it follows from (15) that
\[
\frac{1}{\kappa^{\frac{1}{p}}} \|x - \bar{x}\|^{\frac{1-p}{p}} \|x - \bar{x}\| = \frac{1}{\kappa^{\frac{1}{p}}} \|x - \bar{x}\|^{\frac{1}{p}} \leq d(\bar{y}, F(x)) \leq \|y - \bar{y}\|. \tag{24}
\]

Note that \(\eta\|\bar{y} - \bar{y}\| \leq \kappa^{\frac{1}{p}}\), we also have that
\[
\frac{1}{\kappa^{\frac{1}{p}}} \|x - \bar{x}\|^{\frac{1-p}{p}} \eta\|y - \bar{y}\| \leq \frac{\eta\|y - \bar{y}\|^{\frac{1}{p}}}{\kappa^{\frac{1}{p}}} \|y - \bar{y}\| \leq \|y - \bar{y}\|.
\]

This and (24) show that (16) holds. The proof is complete. \(\square\)

Following the proof, it is easy to see that the above result also holds for the case when \(X\) and \(Y\) are complete metric spaces. In [4, Theorem 5.3], the authors established a necessary and sufficient condition for strong metric subregularity (i.e., \(p = 1\) in the Hölder case) similar to the statement in Theorem 2.4 by invoking the notion of steepest displacement rate.

We provide next an example illustrating the application of Theorem 2.4 while showing that it’s imperative to extend the analysis of strong metric subregularity to the Hölder case.

**Example 2.5.** Let \(f : \mathbb{R}^n \to \mathbb{R}\) denote a class of functions defined as \(f(x) = x^T Ax\) for all \(x \in \mathbb{R}^n\), where \(A\) is a \(n \times n\) symmetric, positive definite matrix. Let \(\lambda_1, \lambda_2, \ldots, \lambda_n\) be the eigenvalues of \(A\), then \(\lambda_i > 0\) for all \(i = 1, 2, \ldots, n\) and
\[
\frac{\Lambda}{2} \|x\|^2 \leq x^T Ax \leq \bar{\Lambda} \|x\|^2 \quad \forall x \in \mathbb{R}^n,
\]
where \(\|x\| = \sqrt{x^T x}, \Lambda = \min\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) and \(\bar{\Lambda} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\). Define \(F(x) = [f(x), +\infty)\) for all \(x \in \mathbb{R}^n\). It is easy to see that \(0 \in F(0)\) and \(F\) is Hölder strongly metrically subregular at \((0, 0)\) of order \(\frac{1}{2}\) but not strongly metrically subregular at \((0, 0)\) (of order one). Let \(p = 2, \delta \in (0, +\infty)\) and \(\varepsilon, \eta \in (0, 1)\) such that \(\eta\delta < 1\). For any \(x \in B(0, \delta) \setminus \{0\}\) and \(y \in F(x) \cap B(0, \delta)\), let \(h = \frac{1}{\|x\|}\) and \(t \in (0, \frac{\|x\|}{\delta})\), we have \(y \geq f(x)\) and
\[
f(x) = x^T Ax = (x - th)^T A(x - th) + 2tx^T Ah - t^2 h^T Ah \\
\quad \geq f(x - th) + 2A\|x\| - \bar{X}t^2 \geq f(x - th) + M\|x\|. \tag{25}
\]

Let \(u = x - th\) and \(v = f(x - th)\), it follows that \(v \in F(u), \|u - x\| = t\) and \(\|y - 0\| \geq f(x) \geq |v - 0| + \frac{M}{\Lambda}\|u - x\|\|x\|\). On the other hand, note that \(\eta\|x\| < \eta\delta < 1\)
and \(y \geq v \geq 0\), we also have \(|y - 0| \geq |v - 0| + \eta\|x\||y - v|\). This shows that (14) holds, and it follows from Theorem 2.4 that \(F\) is Hölder strongly metrically subregular at \((0, 0)\) of order \(\frac{1}{2}\).

The next result was established in [4, Theorem 4.1]. It implies that the property of Hölder strong metric subregularity is preserved under the perturbations of single-valued mappings with Hölder calmness property and extends [7, Theorem 3.1.6] to the Hölder case.

**Theorem 2.6.** Let \(p, q\) be any positive numbers such that \(pq \geq 1\). Consider a set-valued mapping \(F : X \rightrightarrows Y\) and a pair \((\bar{x}, \bar{y}) \in \text{gph}(F)\). Suppose that \(F\) is Hölder strongly metrically subregular at \((\bar{x}, \bar{y})\) of order \(p \geq 1\) and there exists \(\kappa, \mu \in (0, +\infty)\) such that

\[
\text{subreg}^p(F; (\bar{x}, \bar{y})) \leq \kappa \quad \text{and} \quad \kappa^\frac{1}{p}\mu < 1.
\]

Then for any Hölder calm function \(f : X \to Y\) of order \(q \geq 1\) with \(\text{clm}^q(f; \bar{x}) \leq \mu\), we have

\[
\text{subreg}^p(F + f; (\bar{x}, f(\bar{x}) + \bar{y})) \leq \frac{\kappa}{1 - \kappa^\frac{1}{p}\mu}.
\]  

(26)

The following simple example shows that the condition \(pq \geq 1\) in Theorem 2.6 cannot be dropped.

**Example 2.7.** Let \(X = Y = \mathbb{R}, F(x) = [x^2, +\infty)\) and \(f(x) = x/2\) for all \(x \in \mathbb{R}\). It is clear that \(F\) is Hölder strongly metrically subregular at \((0, 0)\) of order \(1/2\) and \(f\) is Hölder calm at \(0\) of order one. However, \((f + F)^{-1}(0) = [-1/2, 0]\) and hence \(f + F\) is not Hölder strongly metrically subregular at \((0, 0)\) of any order.

Enlightened by the first-order approximation of functions defined in [7], for \(q \in [0, +\infty)\) and function \(f : X \to Y\), we say that a function \(g : X \to Y\) is an approximation of \(f\) at \(\bar{x} \in X\) of order \(q\) with constant \(\lambda \in [0, +\infty)\), if

\[
g(\bar{x}) = f(\bar{x}) \quad \text{and} \quad \text{clm}^q(f - g; \bar{x}) \leq \lambda.
\]

It is worth to mention that in the case of \(q = 0\) and \(\lambda = 0\), \(q^{th}\)-order approximation simply indicates that the values of \(f\) and \(g\) are locally not far from each other around \(\bar{x}\) without the information of estimating the degree of closeness. By utilizing the the \(q^{th}\)-order approximation and applying Theorem 2.6, it is easy to get next result:

**Corollary 2.8.** Let \(p, q \in (0, +\infty)\) such that \(pq \geq 1\). Consider a set-valued mapping \(F : X \rightrightarrows Y\), a pair \((\bar{x}, \bar{y}) \in \text{gph}(F)\) and two functions \(f : X \to Y\) and \(g : X \to Y\). Suppose that \(g\) is a \(q^{th}\)-order approximation of \(f\) at \(\bar{x}\) with constant \(0\), then \(f + F\) is Hölder strongly metrically subregular at \((\bar{x}, f(\bar{x}) + \bar{y})\) of order \(p\) if and only if \(g + F\) is Hölder strongly metrically subregular at \((\bar{x}, g(\bar{x}) + \bar{y})\) of order \(p\). In specific, we have

\[
\text{subreg}^p(f + F; (\bar{x}, f(\bar{x}) + \bar{y})) = \text{subreg}^p(g + F; (\bar{x}, g(\bar{x}) + \bar{y})).
\]  

(27)

**Proof.** Note that \(f + F = g + F + (f - g)\) and, in fact, \(f\) and \(g\) are approximations to each other at \(\bar{x}\) of order \(q\) with constant \(0 = \text{clm}^q(f - g; \bar{x})\), it is easy to see that (27) directly follows from Theorem 2.6. \(\square\)

When the approximation is represented by a linearization, we have our next result:
Corollary 2.9. Let $q \in [0, +\infty)$ and $p \in \left[\frac{1}{1+q}, +\infty\right)$. Let $F : X \rightrightarrows Y$ be a set-valued function and $f : X \rightarrow Y$ be a single-valued function with $\bar{y} \in F(\bar{x})$. Suppose that $f$ is continuously differentiable near $\bar{x}$ and its derivative $Df$ is H"{o}lder calm at $\bar{x}$ of order $q$ with $\text{clm}^q(Df; \bar{x}) = 0$. Let $h(x) := f(\bar{x}) + Df(\bar{x})(x - \bar{x})$ for all $x \in X$. Then $f + F$ is H"{o}lder strongly metrically subregular at $(\bar{x}, f(\bar{x}) + \bar{y})$ of order $p$ if and only if $h + F$ is H"{o}lder strongly metrically subregular at $(\bar{x}, f(\bar{x}) + \bar{y})$ of order $p$. Moreover, $\text{subreg}^p(h + F; (\bar{x}, f(\bar{x}) + \bar{y})) = \text{subreg}^p(h + F; (\bar{x}, f(\bar{x}) + \bar{y}))$.

Proof. Since $\text{clm}^q(Df; \bar{x}) = 0$, then for any $\varepsilon \in (0, +\infty)$, there exists $\delta \in (0, +\infty)$ such that $\|Df(x) - Df(\bar{x})\| \leq \varepsilon \|x - \bar{x}\|^q$ for all $x \in B(\bar{x}, \delta)$. Therefore,

$$\|f(x) - h(x)\| = \left\| \int_0^1 Df(\bar{x} + t(x - \bar{x}))(x - \bar{x})dt - Df(\bar{x})(x - \bar{x}) \right\| \leq \frac{\varepsilon}{1 + q}\|x - \bar{x}\|^{1+q} \forall x \in B(\bar{x}, \delta).$$

(28)

Hence, we have that $\text{clm}^{1+q}(f - h; \bar{x}) = 0$. This shows that $h$ is an approximation of $f$ at $\bar{x}$ of order $1 + q$ with constant 0. Note that $p(1 + q) \geq 1$, then Corollary 2.8 directly implies the result. The proof is complete. \(\square\)

Next we provide the following parameterized result which we will use in section 3 for convergence analysis of inexact Newton methods.

Lemma 2.10. Let $q \in [0, +\infty)$ and $p \in \left[\frac{1}{1+q}, +\infty\right)$. Consider a set-valued mapping $F : X \rightrightarrows Y$ and a single-valued perturbation $f : X \rightarrow Y$. Suppose that $(\bar{x}, 0) \in \text{gph}(f + F)$, $f + F$ is H"{o}lder strongly metrically subregular at $(\bar{x}, 0)$ of order $p$ and $f$ is continuously differentiable near $\bar{x}$ with its derivative $Df$ being calm at $\bar{x}$ of order $q$. Let $u \in X$ and consider the mapping

$$G_u(x) := f(u) + Df(u)(x - u) + F(x) \quad \forall x \in X.$$  

(29)

Then, for every $\kappa > \text{subreg}^p(f + F; (\bar{x}, 0))$ and $\mu > \text{clm}^q(Df; \bar{x})$ with $\kappa \frac{\mu}{1 + q} < 1 + q$, there exists $\delta \in (0, +\infty)$ such that

$$\|x - \bar{x}\| \leq \frac{\kappa \frac{\mu}{1 + q}}{1 + q - \kappa \frac{\mu}{1 + q} - \mu \|u - \bar{x}\|^{q\|x - \bar{x}\|}} \quad \forall u, x \in B(\bar{x}, \delta).$$

(30)

Proof. Choose some positive $\kappa'$ and $\mu'$ such that $\kappa > \kappa' > \text{subreg}^p(f + F; (\bar{x}, \bar{y}))$ and $\mu > \mu' > \text{clm}^q(Df; \bar{x})$. Then, there exists $r \in (0, +\infty)$ such that

$$\|Df(x) - Df(\bar{x})\| \leq \mu'\|x - \bar{x}\|^q \quad \forall x \in B(\bar{x}, r).$$  

(31)

Let $g(x) := f(\bar{x}) + Df(\bar{x})(x - \bar{x}) - f(x)$ for all $x \in X$, it follows from (31) that

$$\|g(x) - g(\bar{x})\| = \left\| \int_0^1 Df(\bar{x} + t(x - \bar{x}))dt - Df(\bar{x})(x - \bar{x}) \right\| \leq \frac{\mu'}{1 + q}\|x - \bar{x}\|^{1+q} \quad \forall x \in B(\bar{x}, r).$$  

(32)

This shows that $g$ is H"{o}lder calm at $\bar{x}$ with $\text{clm}^{1+q}(g; \bar{x}) \leq \frac{\mu'}{1+q}$. It is clear that $p(1 + q) \geq 1$ and $\kappa \frac{\mu}{1 + q} < 1 + q$. Then, it follows from Theorem 2.6 that $G_{\bar{x}} = g + f + F$ is H"{o}lder strongly metrically subregular at $(\bar{x}, 0)$ of order $p$ and

$$\text{subreg}^p(G_{\bar{x}}; (\bar{x}, 0)) \leq \frac{\kappa'}{1 - \kappa' \mu'(1 + q)} < \frac{\kappa}{1 - \kappa \mu'(1 + q)}.$$
Hence, there exists $\delta \in (0, r)$ such that
\[
\|x - \bar{x}\| \leq \frac{\kappa}{(1 - \kappa \mu/(1 + q))^p} d(0, G_x(x))^p \quad \forall x \in B(\bar{x}, \delta).
\] (33)

Fix any $u \in B(\bar{x}, r)$ and let
\[
g_u(x) := f(u) + Df(u)(x - u) - f(\bar{x}) - Df(\bar{x})(x - \bar{x}) \quad \forall x \in X.
\]
By (31), we have that
\[
\|g_u(\bar{x}) - g_u(x)\| = \|Df(\bar{x})(x - \bar{x}) - Df(u)(x - \bar{x})\| \leq \mu\|u - \bar{x}\|^q\|x - \bar{x}\| \quad \forall x \in B(\bar{x}, r).
\]
It then follows from (33) that
\[
\|x - \bar{x}\|^{\frac{1}{\mu}} \leq \frac{\kappa^{\frac{1}{p}} (1 + q)}{1 + q - \kappa \mu} d(0, G_x(x))
\]
\[
= \frac{\kappa^{\frac{1}{p}} (1 + q)}{1 + q - \kappa \mu} d(g_u(x), g_u(x) + G_x(x))
\]
\[
= \frac{\kappa^{\frac{1}{p}} (1 + q)}{1 + q - \kappa \mu} d(g_u(x), G_x(x))
\]
\[
\leq \frac{\kappa^{\frac{1}{p}} (1 + q)}{1 + q - \kappa \mu} (d(g_u(\bar{x}), G_x(x)) + \|g_u(\bar{x}) - g_u(x)\|)
\]
\[
\leq \frac{\kappa^{\frac{1}{p}} (1 + q)}{1 + q - \kappa \mu} (d(g_u(\bar{x}), G_x(x)) + \mu\|u - \bar{x}\|^q\|x - \bar{x}\|) \quad \forall x \in B(\bar{x}, \delta).
\]
This shows (30) holds and our proof is completed. \qed

3. Convergence of the inexact Newton method under Hölder strong metric subregularity. In this section, we first consider the inexact Newton method (4) for solving the generalized equation (1) under Hölder strong metric subregularity assumption. For convergence analysis of such method, we need the following lemma.

Lemma 3.1. Let $q \in [0, +\infty), p \in [\frac{1}{1+q}, +\infty)$ and $\alpha, \beta \in (0, +\infty)$ with $\alpha + \beta < 1$. Assume that $\{s_k\} \subset [0, 1)$ satisfying
\[
s_{k+1}^{\frac{1}{p}} \leq \alpha s_k^{1+q} + \beta s_k^{q} s_{k+1} \quad \forall k \in \mathbb{N},
\] (34)
then $s_{k+1} \leq s_k$ and $s_{k+1} \leq (\alpha + \beta)^p s_k^{p(1+q)}$ for all $k \in \mathbb{N}$.

Proof. Note that if $s_{k+1} \leq s_k$ holds for all $k \in \mathbb{N}$, then (34) immediately implies that
\[
s_{k+1} \leq (\alpha + \beta)^p s_k^{p(1+q)} \quad \forall k \in \mathbb{N}.
\]
Hence it suffices to show that $s_{k+1} \leq s_k$ for all $k \in \mathbb{N}$. We use contradiction. Assume to the contrary that there exists $k_0 \in \mathbb{N}$ such that $s_{k_0+1} > s_{k_0}$, then it follows from (34) that
\[
s_{k_0+1}^{\frac{1}{p}} \leq \alpha s_{k_0}^{1+q} + \beta s_{k_0}^{q} s_{k_0+1}
\]
\[
< (\alpha + \beta) s_{k_0+1}^{1+q} \leq (\alpha + \beta) s_{k_0}^{\frac{1}{p}}
\]
whenever (due to (37)), it follows from Lemma 2.10 that there exists \( \delta \) is continuously differentiable near \( p \) is stronger than the properties of lower order. Indeed, when the convergence rate is clear that Hölder strong metric subregularity and isolated calmness of higher order Hölder strongly metrically subregular of order \( p \) iteration (4) converges of higher order (the last inequality holds due to 1 + \( q \)) is required in advance during application, the orders \( p \) and \( q \) in Theorem 3.2 complete them in a way that we may choose to impose stronger assumptions on one property and weaker ones on the other.

**Theorem 3.2.** Let \( q \in [0, +\infty) \) and \( p \in \left(\frac{1}{1+q}, +\infty\right) \). Suppose that the mapping \( f + F \) is Hölder strongly metrically subregular at \((\bar{x}, 0) \in \text{gph}(f + F)\) of order \( p \), \( f \) is continuously differentiable near \( \bar{x} \) and its derivative \( Df \) is calm at \( \bar{x} \) of order \( q \). In addition, assume that for each \( k \in \mathbb{N} \), \( d^+(0, R_k(u, x)) \to 0 \) as \((u, x) \to (\bar{x}, \bar{x})\) and the mapping \((u, x) \mapsto R_k(u, x)\) is partially calm at \((\bar{x}, \bar{x}), 0) \in \text{gph}(R_k)\) with respect to \( x \) uniformly in \( u \) of order \( 1/p \) on the same neighborhood while sharing the same positive constant \( \lambda \) for each \( k \in \mathbb{N} \), i.e. there exist \( a, b \in (0, +\infty) \) such that

\[
R_k(u, x) \cap B(0, b) \subseteq R_k(u, \bar{x}) + \lambda\|x - \bar{x}\|^\frac{1}{p} B_Y \forall u, x \in B(\bar{x}, a), k \in \mathbb{N}. \quad (35)
\]

Let \( \kappa > \text{subreg}^p(f + F; (\bar{x}, 0)) \) and \( \mu > \text{clm}^q(Df; \bar{x}) \) and let there exist positive \( \gamma \) and \( r \) such that

\[
d^+(0, R_k(u, \bar{x})) \leq \gamma\|u - \bar{x}\|^{1+q} \forall u \in B(\bar{x}, r), k \in \mathbb{N}. \quad (36)
\]

If

\[
\kappa^p((1 + q)(\gamma + \lambda) + (4 + 2q)\mu) < 1 + q, \quad (37)
\]

then there exists a neighborhood \( U \) of \( \bar{x} \) such that for any \( x_0 \in U \), every sequence \( \{x_k\} \) generated by the inexact Newton method (4) starting from \( x_0 \) and staying in \( U \) for all \( k \) satisfies:

\[
\|x_{k+1} - \bar{x}\| \leq \left( \frac{\gamma(1 + q) + \mu(3 + 2q)}{(\kappa^{-1} - \lambda)(1 + q) - \mu} \right)^p \|x_k - \bar{x}\|^{(1+q)}, k \in \mathbb{N}, \quad (38)
\]

that is, \( x_k \to \bar{x} \) of order \( p(1 + q) \).

**Proof.** Note that \( \kappa > \text{subreg}^p(f + F; (\bar{x}, 0)), \mu > \text{clm}^q(Df; \bar{x}) \) and \( \frac{\kappa^p}{p} \mu < 1 + q \) (due to (37)), it follows from Lemma 2.10 that there exists \( \delta \in (0, +\infty) \) such that (30) holds. We adjust \( \delta \) if necessary so that \( \delta < \min\{a, r, 1\}, R_k(u, x) \subseteq B(0, b) \) whenever \( u, x \in B(\bar{x}, \delta) \) (due to \( d^+(0, R_k(u, x)) \to 0 \) as \((u, x) \to (\bar{x}, \bar{x})\)) and

\[
\|Df(x) - Df(\bar{x})\| \leq \mu\|x - \bar{x}\|^q \forall x \in B(\bar{x}, \delta).
\]

And so, we have

\[
\|f(x) - Df(x)(x - \bar{x}) - f(\bar{x})\|
\leq \|f(x) - Df(\bar{x})(x - \bar{x}) - f(\bar{x})\| + \|Df(\bar{x})(x - \bar{x})\| + \|Df(x)(x - \bar{x})\|
\leq \|f(x) - Df(\bar{x})(x - \bar{x}) - f(\bar{x})\| + \|Df(\bar{x})(x - \bar{x})\| + \mu\|x - \bar{x}\|^{1+q} \quad (39)
\]

\[
\leq \frac{\mu(2 + q)}{1 + q} \|x - \bar{x}\|^{1+q} \forall x \in B(\bar{x}, \delta).
\]
Pick \( x_0 \in B(\bar{x}, \delta) \) and consider any sequence \( \{x_k\} \) generated by method (4) starting at \( x_0 \) and staying in \( B(\bar{x}, \delta) \). Then for each \( k \), there exists \( y_{k+1} \in G_{x_k}(x_{k+1}) \cap R_k(x_k, x_{k+1}) \), where \( G_{x_k}(x_{k+1}) \) is defined as in (29). It follows from (35) that there exists \( y'_{k+1} \in R_k(x_k, \bar{x}) \) such that

\[
\|y'_{k+1} - y_{k+1}\| \leq \lambda \|x_{k+1} - \bar{x}\|^\frac{1}{\mu}
\]

and moreover, (36) implies that \( \|y'_{k+1}\| \leq \gamma \|x_k - \bar{x}\|^1 + q \). By this, it follows from (30) and (39) that

\[
\|x_{k+1} - \bar{x}\|^\frac{1}{\mu} \leq \frac{\kappa^\frac{1}{\mu}(1 + q)}{1 + q - \kappa^\frac{1}{\mu} \mu}(d(f(x_k) - Df(x_k)(x_k - \bar{x}) - f(\bar{x}), G_{x_k}(x_{k+1}))
\]

\[
+ \mu \|x_k - \bar{x}\|^q \|x_{k+1} - \bar{x}\|
\]

\[
\leq \frac{\kappa^\frac{1}{\mu}(1 + q)}{1 + q - \kappa^\frac{1}{\mu} \mu}(\|y'_{k+1}\| + \|f(x_k) - Df(x_k)(x_k - \bar{x}) - f(\bar{x})\|
\]

\[
+ \mu \|x_k - \bar{x}\|^q \|x_{k+1} - \bar{x}\|
\]

\[
\leq \kappa^\frac{1}{\mu}(1 + q)
\]

\[
\frac{1}{1 + q - \kappa^\frac{1}{\mu} \mu} \|x_{k+1} - \bar{x}\|^\frac{1}{\mu} + \|y'_{k+1} - y_{k+1}\| + \frac{\mu(2 + q)}{1 + q} \|x_k - \bar{x}\|^1 + q
\]

\[
+ \mu \|x_k - \bar{x}\|^q \|x_{k+1} - \bar{x}\|
\]

This shows that

\[
\|x_{k+1} - \bar{x}\|^\frac{1}{\mu} \leq \frac{\kappa^\frac{1}{\mu}(1 + q)}{1 + q - \kappa^\frac{1}{\mu} \mu - \kappa^\frac{1}{\mu} \lambda(1 + q)}((\gamma + \frac{\mu(2 + q)}{1 + q}) \|x_k - \bar{x}\|^1 + q
\]

\[
+ \mu \|x_k - \bar{x}\|^q \|x_{k+1} - \bar{x}\|)
\]

(40)

Let \( \alpha := \frac{\kappa^\frac{1}{\mu}(1 + q)}{1 + q - \kappa^\frac{1}{\mu} \mu - \kappa^\frac{1}{\mu} \lambda(1 + q)} \) and \( \beta := \frac{\mu(2 + q)}{1 + q} \), then assumption (37) guarantees that \( \alpha + \beta < 1 \). Now we are ready to apply Lemma 3.1 with \( s_k = \|x_k - \bar{x}\| \) ensures that (38) holds, which completes our proof.

In [10, Theorem 5], the authors considered inexact Newton method (4) for solving generalized equations under strong metric subregularity of \( f + F \) with \( Df \) being continuous and Lipschitz continuous respectively. As a special case, when \( p = 1 \) and \( q = 0 \), Theorem 3.2 is a supplement of [10, Theorem 5(i)].

Next we provide an example in which \( f + F \) is Hölder strongly metrically subregular of order \( \frac{1}{2} \) but not strongly metrically subregular. It illustrates the application of Theorem 3.2 in a clear manner while in the same time exhibits the limitation of [10, Theorem 5(i)].

**Example 3.3.** Let \( X = Y = \mathbb{R}, \dot{x} = 0, f(x) = \frac{x^2}{2}, F(x) = [4x^2, +\infty) \) for all \( x \in \mathbb{R} \) and \( R_k(u, x) = [-\frac{u^2}{2}, \frac{u^2}{2}] \) for all \( u, x \in \mathbb{R} \) and \( k \in \mathbb{N} \). Let \( \kappa = \mu = p = \gamma = \lambda = \frac{1}{2}, q = 1 \). It is easy to see that \( f + F \) is Hölder strongly metrically subregular at \( (0, 0) \) with constant \( \frac{1}{2} \) of order \( \frac{1}{2} \), \( Df \) is calm at 0 with constant \( \frac{1}{2} \) of order 1, \( \cld_m^2(R_k; ((0, 0), 0)) = 0 \) and \( d^+(0, R_k(u, 0)) = \frac{u^2}{2} \leq \frac{1}{2}|u|^2 \) for all \( u \in \mathbb{R} \) and \( k \in \mathbb{N} \). It is also not difficult to verify that the other conditions in Theorem 3.2...
are all satisfied. By the inexact Newton method (4), we have $[-\frac{1}{4}x_k^4 + \frac{1}{2}x_kx_{k+1} + 4x_{k+1}^2, +\infty) \cap [-\frac{1}{4}x_k^4, x_k^2/4] \neq \emptyset$. And then $8x_{k+1}^2 + x_kx_{k+1} - x_k^2 \leq 0$ for all $k \in \mathbb{N}$. This shows that for any starting point $x_0$, the method (4) is surely executable and $x_{k+1} \in [-\frac{x_k - \sqrt{3x_k^2}}{16}, -\frac{x_k + \sqrt{3x_k^2}}{16}]$. Therefore $|x_{k+1}| \leq \frac{\sqrt{3} + 1}{16}|x_k|$ for all $k \in \mathbb{N}$ and hence $x_k \to 0$ at least linearly of order one.

In [4, Theorem 6.2], the authors also considered Newton method (2) under Hölder strong metric subregularity of $f + F$ and Hölder calmness of $Df$ with $p \in [1, +\infty)$ and $q \in (0, 1]$. In general, the Hölder strong metric subregularity assumption in Theorem 3.2 does not guarantee that the method (4) is surely executable. As a simple example, consider the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sqrt{x} + 1$ if $x \in [0, +\infty)$ and otherwise $f(x) = \emptyset$, the mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ with $F(x) = \emptyset$ while $x \neq 0$ and $F(0) = [-1, +\infty)$ and $R_k \equiv 0$ for any $k \in \mathbb{N}$. It is easy to see that $f+F$ is Hölder strongly metrically subregular at $(0, 0)$ of order 2 and subreg$(f+F; (0, 0)) = 0$. By $\lim_{x \to 0^+} Df(x) = Df(0) = 1/2$, we have that $f$ is Hölder calm at 0 of order 0, but from any point $x_0$ arbitrarily close to 0 there is no Newton step $x_1$.

For finding a zero of a smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$, The Dennis-Moré theorem [6] characterizes the superlinear convergence of quasi-Newton methods of the form

$$f(x_k) + B_k(x_{k+1} - x_k) = 0 \text{ for } k = 0, 1, \ldots,$$

where $B_k$ is a sequence of metrics approximating the Jacobian $Df(x)$ at a solution $x$. Under the assumption of strong metric subregularity, Dontchev [9] extended Dennis-Moré theorem to solving the generalized equation (1) via the approach of quasi-Newton methods

$$f(x_k) + B_k(x_{k+1} - x_k) + F(x_{k+1}) \ni 0, \text{ for } k = 0, 1, \ldots,$$

where $B_k$ is a sequence of linear and bounded mappings from $X$ to $Y$.

We consider next in this section the following inexact quasi-Newton methods for solving generalized equation (1):

$$(f(x_k) + B_k(x_{k+1} - x_k) + F(x_{k+1})) \cap R_k(x_k) \neq \emptyset, \text{ for } k = 0, 1, \ldots, \quad (41)$$

where $B_k$ is a sequence of linear and bounded mappings acting from $X$ to $Y$ and the mapping $R_k : X \rightrightarrows Y$ which represents inexactness now only depends on the current iteration $x_k$.

The following result shows that the iteration (41) converges superlinearly of order $p(1+q)$ where $p$ denotes the order of Hölder strong metric subregularity on $f + F$ and $q$ denotes the order of Hölder calmness property on $Df$.

**Theorem 3.4.** Let $q \in [0, +\infty), p \in \left[\frac{1}{1+q}, +\infty\right)$ and $\bar{x} \in X$. Suppose that $f$ is continuously differentiable in a neighborhood $U$ of $\bar{x}$ and its derivative $Df$ satisfies $\text{clm}^q(Df; \bar{x}) = 0$. And assume that the sequence $\{R_k\}$ satisfies

$$\lim_{x \to \bar{x}, x \neq \bar{x}} \sup_{u \in R_k(x)} \frac{1}{\|x - \bar{x}\|^{1+q}} \sup_{u \in R_k(x)} \|u\| = 0. \quad (42)$$

Let for some starting point $x_0$ in $U$ the sequence $\{x_k\}$ be generated by (41), remain in $U$ for all $k$, which converges to $\bar{x}$ and such that $x_k \neq \bar{x}$ for all $k \in \mathbb{N}$. And let $E_k = B_k - Df(x_k)$. If the sequence $\{x_k\}$ satisfies

$$\lim_{k \to \infty} \frac{\|E_k(x_{k+1} - x_k)\|}{\|x_{k+1} - x_k\|^{1+q}} = 0 \quad (43)$$

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$$\lim_{k \to \infty} \frac{\|E_k(x_{k+1} - x_k)\|}{\|x_{k+1} - x_k\|^{1+q}} = 0 \quad (43)$$
and $F$ is closed, then $\bar{x}$ is a solution of the generalized equation (1). If in addition, the mapping $f + F$ is Hölder strongly metrically subregular at $(\bar{x},0)$ of order $p$, then

$$
\lim_{k \to \infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|^{p(1+q)}} = 0,
$$

that is, $x_k \to \bar{x}$ superlinearly of order $p(1+q)$.

**Proof.** Pick any $\varepsilon \in (0,1)$ sufficient small and let it be fixed. According to the assumption $\lim\sup(Df;\bar{x}) = 0$, there exists $\delta \in (0,1)$ such that $\|Df(x) - Df(\bar{x})\| \leq \varepsilon \|x - \bar{x}\|^q$ for all $x \in B(\bar{x}, \delta)$. Writing $h(x) := f(\bar{x}) + Df(\bar{x})(x - \bar{x})$ as in Corollary 2.9, we have that inequality (28) holds. By the assumption (42), let $\delta$ be smaller if necessary, we also have that

$$
\|u\| \leq \varepsilon \|x - \bar{x}\|^{1+q} \forall x \in B(\bar{x}, \delta), u \in R_k(x), k \in \mathbb{N}.
$$

Since $x_k \to \bar{x}$, without loss of generality, we can assume that $x_k \in B(\bar{x}, \delta)$ for all $k \in \mathbb{N}$. From iterations (41) and (45), there exists $u_k \in R_k(x_k)$ such that

$$
u_k \in f(x_k) + B_k(x_k+1 - x_k) + F(x_k+1) \quad \text{and} \quad \|u_k\| \leq \varepsilon \|x_k - \bar{x}\|^{1+q} \forall k \in \mathbb{N}. \tag{46}
$$

Specifically, for each $k$ there exists $y_k \in F(x_k+1)$ such that $u_k = f(x_k) + B_k(x_k+1 - x_k) + y_k$. Due to the inequality in (46), we have that $\|u_k\| \to 0$ as $k \to \infty$ and by taking into account of assumption (43), we have that

$$
\|B_k(x_k+1 - x_k)\| \leq \|E_k(x_k+1 - x_k)\| + \|Df(\bar{x})(x_k+1 - x_k)\| \to 0 \quad \text{as} \quad k \to \infty.
$$

Therefore $y_k$ converges to $-f(\bar{x})$. Since the graph of $F$ is closed and $y_k \in F(x_k+1)$, we conclude that $0 \in f(\bar{x}) + F(\bar{x})$, i.e., $\bar{x}$ is a solution of generalized equation (1).

If, in addition, $f + F$ is Hölder strongly metrically subregular at $(\bar{x},0)$ of order $p$, then by Corollary 2.9, we have that $h + F$ is also Hölder strongly metrically subregular at $(\bar{x},0)$ of order $p$ and subreg$^p(f + F; (\bar{x},0)) = \text{subreg}^p(g + F; (\bar{x},0))$. Pick any $\kappa \geq \text{subreg}^p(g + F; (\bar{x},0))$, then there exists $\delta' \in (0, \delta)$, such that

$$
\|x - \bar{x}\| \leq \kappa \delta(0, (h + F)(x)) \kappa \|x - \bar{x}\| \forall x \in B(\bar{x}, \delta'). \tag{47}
$$

From the selection of $u_k$ in (46), we have that for all $k \in \mathbb{N},$

$$
u_k \in f(x_k) - h(x_k) + E_k(x_k+1 - x_k) + (h + F)(x_k+1).$$

And hence, by taking into account of (28), (43), (46) together with (47), we obtain that for $k$ sufficiently large,

$$
\|x_{k+1} - \bar{x}\|^p \leq \kappa \frac{\varepsilon}{1+q} \|x_k - \bar{x}\|^{1+q} + \kappa \varepsilon \|x_{k+1} - x_k\|^{1+q}
$$

Note that $\|x_{k+1} - \bar{x}\|^{1+q} \leq \|x_{k+1} - \bar{x}\|^p$ (due to $1 + q \geq 1/p$ and $\|x_{k+1} - \bar{x}\| < 1$), it then follows that

$$
\|x_{k+1} - \bar{x}\|^p \leq \frac{\kappa \frac{\varepsilon}{1+q} + \kappa \varepsilon 2^{1+q}}{1 - \kappa \frac{\varepsilon}{1+q}} \|x_k - \bar{x}\|^{1+q},
$$

for all $k$ large enough. Since $\varepsilon$ can be arbitrarily small, we conclude that (44) is true and hence completes our proof. \[\square\]
Note that Theorem 3.4 can be considered as a generalization of the second part of [9, Theorem 3] for both higher convergence rate and inexact quasi-Newton method. In [14, Theorem 5.1], the authors established higher convergence rate for quasi-Newton iterations by imposing Hölder strong metric subregularity of order $p \in [1, +\infty)$ on $F$ and $C^1$-smooth assumption on $f$, which is covered by Theorem 3.4 upon setting $R_k \equiv 0$ and $q = 0$. It is worth mentioning that the orders $p$ and $q$ in Theorem 3.4 also complement each other. Indeed, the order of Hölder strong metric subregularity imposed on the mapping $f + F$ can be less than 1 as long as the product $p(1 + q) \geq 1$. By introducing a mapping known as generalized set-valued derivative of the function $f$, iterations of such kind are investigated more elaborately for the case when the function $f$ is not necessarily differentiable; for more discussions see reference [3].

The following simple example illustrates that condition (43) is not necessary for superlinear convergence in general.

**Example 3.5.** Let $X = Y = \mathbb{R}$, $f(x) = x^5$, $F(x) = x^2$ and $R_k(x) \equiv 0$ for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$. It is clear that $F$ is Hölder strongly metrically subregular at $(0, 0)$ of order 1/2, and then $f + F$ is Hölder strongly metrically subregular at $(0, 0)$ of order 1/4. So, we take $p = 1/4$ and $q = 3$ and therefore all the assumptions of Theorem 3.4 are satisfied. Consider the inexact quasi-Newton method (41) with $B_k = \frac{(k!)^{-1} + ((k+1))^{-2}}{(k!)^{-1} - ((k+1))^{-1}}$ and $x_1 = 1$. Then (41) generates the sequence $x_k = (k!)^{-1}$, which is superlinearly convergent to 0 of order one; however, condition (43) does not hold since

$$
\frac{B_k(x_{k+1} - x_k)}{(x_{k+1} - x_k)^4} = \frac{(k!)^{-1} + (k + 1)^{-2}(k!)^2}{(1 - (k + 1)^{-1})^4} \to \infty, \text{ as } k \to \infty.
$$

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