Geometry of antimatroidal point sets

Yulia Kempner 1
1 Holon Institute of Technology, ISRAEL
yuliak@hit.ac.il

Vadim E. Levit 1,2
2 Ariel University Center of Samaria, ISRAEL
levitv@ariel.ac.il

Abstract

The notion of "antimatroid with repetition" was conceived by Bjorner, Lovasz and Shor in 1991 as a multiset extension of the notion of antimatroid [2]. When the underlying set consists of only two elements, such two-dimensional antimatroids correspond to point sets in the plane. In this research we concentrate on geometrical properties of antimatroidal point sets in the plane and prove that these sets are exactly parallelogram polyominoes. Our results imply that two-dimensional antimatroids have convex dimension 2. The second part of the research is devoted to geometrical properties of three-dimensional antimatroids closed under intersection.

Keywords: antimatroid, convex dimension, lattice animal, polyomino.

1 Preliminaries

An antimatroid is an accessible set system closed under union [3]. An algorithmic characterization of antimatroids based on the language definition was introduced in [6]. Another algorithmic characterization of antimatroids that depicted them as set systems was developed in [12]. While classical examples of antimatroids connect them with posets, chordal graphs, convex
geometries, etc., game theory gives a framework in which antimatroids are interpreted as permission structures for coalitions [11]. There are also rich connections between antimatroids and cluster analysis [13]. In mathematical psychology, antimatroids are used to describe feasible states of knowledge of a human learner [10].

In this paper we investigate the correspondence between antimatroids and polyominoes. In the digital plane $\mathbb{Z}^2$, a polyomino [11] is a finite connected union of unit squares without cut points. If we replace each unit square of a polyomino by a vertex at its center, we obtain an equivalent object named a lattice animal [5]. Further, we use the name polyomino for the two equivalent objects.

A polyomino is called column-convex (resp. row-convex) if all its columns (resp. rows) are connected (in other words, each column/row has no holes). A convex polyomino is both row- and column-convex. The parallelogram polyominoes [7, 8, 9], sometimes known as staircase polygons [4, 17], are a particular case of this family. They are defined by a pair of monotone paths made only with north and east steps, such that the paths are disjoint, except at their common ending points.

In this paper we prove that antimatroidal point sets in the plane and parallelogram polyominoes are equivalent.

Let $E$ be a finite set. A set system over $E$ is a pair $(E, F)$, where $F$ is a family of sets over $E$, called feasible sets. We will use $X \cup x$ for $X \cup \{x\}$, and $X - x$ for $X - \{x\}$.

**Definition 1.1** [15] A finite non-empty set system $(E, F)$ is an antimatroid if

(A1) for each non-empty $X \in F$, there exists $x \in X$ such that $X - x \in F$

(A2) for all $X, Y \in F$, and $X \not\subseteq Y$, there exists $x \in X - Y$ such that $Y \cup x \in F$.

Any set system satisfying (A1) is called accessible.

In addition, we use the following characterization of antimatroids.

**Proposition 1.2** [15] For an accessible set system $(E, F)$ the following statements are equivalent:

(i) $(E, F)$ is an antimatroid

(ii) $F$ is closed under union $(X, Y \in F \Rightarrow X \cup Y \in F)$

Consider another property of antimatroids.
Definition 1.3 A set system \((E, \mathcal{F})\) satisfies the chain property if for all \(X, Y \in \mathcal{F}\), and \(X \subset Y\), there exists a chain \(X = X_0 \subset X_1 \subset \ldots \subset X_k = Y\) such that \(X_i = X_{i-1} \cup x_i\) and \(X_i \in \mathcal{F}\) for \(0 \leq i \leq k\).

It is easy to see that the chain property follows from \((A2)\), but these properties are not equivalent.

A poly-antimatroid [16] is a generalization of the notion of the antimatroid to multisets. A poly-antimatroid is a finite non-empty multiset system \((E, S)\) that satisfies the antimatroid properties \((A1)\) and \((A2)\).

Let \(E = \{x, y\}\). In this case each point \(A = (x_A, y_A)\) in the digital plane \(\mathbb{Z}^2\) may be considered as a multiset \(A\) over \(E\), where \(x_A\) is a number of repetitions of an element \(x\), and \(y_A\) is a number of repetitions of an element \(y\) in multiset \(A\). Consider a set of points in the digital plane \(\mathbb{Z}^2\) that satisfies the properties of an antimatroid. That is a two-dimensional poly-antimatroid.

Definition 1.4 A set of points \(S\) in the digital plane \(\mathbb{Z}^2\) is an antimatroidal point set if

(A1) for every point \((x_A, y_A) \in S\), such that \((x_A, y_A) \neq (0, 0)\),

either \((x_A - 1, y_A) \in S\) or \((x_A, y_A - 1) \in S\)

(A2) for all \(A \not\subset B \in S\),

if \(x_A \geq x_B\) and \(y_A \geq y_B\) then either \((x_B + 1, y_B) \in S\) or \((x_B, y_B + 1) \in S\)

if \(x_A \leq x_B\) and \(y_A \geq y_B\) then \((x_B, y_B + 1) \in S\)

if \(x_A \geq x_B\) and \(y_A \leq y_B\) then \((x_B + 1, y_B) \in S\)

Accessibility implies that \(\emptyset \in S\).

For example, see an antimatroidal point set in Figure 1.

Figure 1: An antimatroidal point set.
A three-dimensional point set is defined similarly to a two-dimensional set.

2 Two-dimensional antimatroidal point sets and polyominoes

In this section we consider a geometric characterization of two-dimensional antimatroidal point sets. The following notation [14] is used. If \( A = (x, y) \) is a point in a digital plane, the 4-neighborhood \( N_4(x, y) \) is the set of points

\[
N_4(x, y) = \{(x - 1, y), (x, y - 1), (x + 1, y), (x, y + 1)\}
\]

and 8-neighborhood \( N_8(x, y) \) is the set of points

\[
N_8(x, y) = \{(x - 1, y), (x, y - 1), (x + 1, y), (x, y + 1), (x - 1, y - 1), (x - 1, y + 1), (x + 1, y - 1), (x + 1, y + 1)\}.
\]

Let \( m \) be any of the numbers 4 or 8. A sequence \( A_0, A_1, ..., A_n \) is called an \( N_m \)-path if \( A_i \in N_m(A_{i-1}) \) for each \( i = 1, 2, ...n \). Any two points \( A, B \in S \) are said to be \( N_m \)-connected in \( S \) if there exists an \( N_m \)-path \( A = A_0, A_1, ..., A_n = B \) from \( A \) to \( B \) such that \( A_i \in S \) for each \( i = 1, 2, ...n \). A digital set \( S \) is an \( N_m \)-connected set if any two points \( P, Q \) from \( S \) are \( N_m \)-connected in \( S \). An \( N_m \)-connected component of a set \( S \) is a maximal subset of \( S \), which is \( N_m \)-connected.

An \( N_m \)-path \( A = A_0, A_1, ..., A_n = B \) from \( A \) to \( B \) is called a monotone increasing \( N_m \)-path if \( A_i \subset A_{i+1} \) for all \( 0 \leq i < n \), i.e.,

\[
(x_{A_i} < x_{A_{i+1}}) \land (y_{A_i} \leq y_{A_{i+1}}) \text{ or } (x_{A_i} \leq x_{A_{i+1}}) \land (y_{A_i} < y_{A_{i+1}}).
\]

The chain property and the fact that the family of feasible sets of an antimatroid is closed under union mean that for each two points \( A, B \): if \( B \subset A \), then there is a monotone decreasing \( N_4 \)-path from \( B \) to \( A \), and if \( A \) is non-comparable with \( B \), then there is a monotone increasing \( N_4 \)-path from both \( A \) and \( B \) to \( A \cup B = (\max(x_A, x_B), \max(y_A, y_B)) \). In particular, for each \( A \in S \) there is a monotone decreasing \( N_4 \)-path from \( A \) to \( 0 \). So, we can conclude that an antimatroidal point set is an \( N_4 \)-connected component in the digital plane \( \mathbb{Z}^2 \).
Definition 2.1 A point set $S \subseteq \mathbb{Z}^2$ is defined to be orthogonally convex if, for every line $L$ that is parallel to the $x$-axis ($y = y^*$) or to the $y$-axis ($x = x^*$), the intersection of $S$ with $L$ is empty, a point, or a single interval $\left( ([x_1, y^*], (x_2, y^*)) = \{(x_1, y^*), (x_1 + 1, y^*), \ldots, (x_2, y^*)\} \right)$.

It follows immediately from the chain property that any antimatroidal point set $S$ is an orthogonally convex connected component.

Thus, antimatroidal point sets are convex polyominoes.

In the following we prove that antimatroidal point sets in the plane closed not only under union, but under intersection as well.

Lemma 2.2 An antimatroidal point set in the plane is closed under intersection, i.e., if two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ belong to an antimatroidal point set $S$, then the point $A \cap B = (\min(x_A, x_B), \min(y_A, y_B)) \in S$.

Proof. The proposition is evident for two comparable points. Consider two non-comparable points $A$ and $B$, and assume without loss of generality that $x_A < x_B$ and $y_A > y_B$. Then there is a monotone decreasing $N_4$-path from $A$ to $0$, and so there is a point $C = (x_C, y_B) \in S$ on this path with $x_C \leq x_A$. Hence, the point $A \cap B$ belongs to $S$, since it is located on the monotone increasing $N_4$-path from $C$ to $B$. ■

Lemma 2.2 implies that every antimatroidal point set is a union of rectangles built on each pair of non-comparable points. The following theorem shows that antimatroidal point sets in the plane and parallelogram polyominoes are equivalent.

Theorem 2.3 A set of points $S$ in the digital plane $\mathbb{Z}^2$ is an antimatroidal point set if and only if it is an orthogonally convex $N_4$-connected set that is bounded by two monotone increasing $N_4$-paths between $(0, 0)$ and $(x_{\text{max}}, y_{\text{max}})$.

To prove the "if" part of Theorem 2.3 it remains to demonstrate that every antimatroidal point set $S$ is bounded by two monotone increasing $N_4$-paths between $(0, 0)$ and $(x_{\text{max}}, y_{\text{max}})$. To give a definition of the boundary we begin with the following notions.

A point $A$ in set $S$ is called an interior point in $S$ if $N_8(A) \subseteq S$. A point in $S$ which is not an interior point is called a boundary point. All boundary points of $S$ constitute the boundary of $S$. We can see an antimatroidal point set with its boundary in Figure 2.
Since antimatroidal sets in the plane are closed under union and under intersection, there are six types of boundary points that we divide into two sets – lower and upper boundary:

\[ B_{\text{lower}} = \{ (x, y) \in S : (x+1, y) \notin S \lor (x, y-1) \notin S \lor (x+1, y-1) \notin S \} \]

\[ B_{\text{upper}} = \{ (x, y) \in S : (x-1, y) \notin S \lor (x, y+1) \notin S \lor (x-1, y+1) \notin S \} \]

It is possible that \( B_{\text{lower}} \cap B_{\text{upper}} \neq \emptyset \). For example, the point \( B_{10} \) in Figure 2 belongs to both the lower and upper boundaries.

**Lemma 2.4** The lower boundary is a monotone increasing path between \((0,0)\) and \((x_{\text{max}}, y_{\text{max}})\).

**Proof.** Let \( B_0 = (0,0), B_1, ..., B_k = (x_{\text{max}}, y_{\text{max}}) \) be a lexicographical order of \( B_{\text{lower}} \). We will prove that it is a monotone increasing path. Suppose the opposite. Then there is a pair of non-comparable points \( B_i \) and \( B_j \), such that \( i < j \), \( x_{B_i} < x_{B_j} \) and \( y_{B_i} > y_{B_j} \). Then Lemma 2.2 implies that there is a rectangle built on the pair of non-comparable points. Hence, the point \( B_i \) does not belong to the lower boundary, which is a contradiction.

**Lemma 2.5** For each \( B_k = (x, y) \in B_{\text{lower}} \) holds:

(i) if \( A = (x, z) \in S \) and \( A \notin B_{\text{lower}} \) then \( z > y \);

(ii) if \( A = (z, y) \in S \) and \( A \notin B_{\text{lower}} \) then \( z < x \).

**Proof.** Suppose the opposite, i.e., there is \( A = (x, z) \in S \) such that \( A \notin B_{\text{lower}} \) and \( z < y \). Then the point \( (x+1, z) \in S \), so there is the last right point \((x^*, z) \in S\), such that \( x^* > x \). Hence, the point \((x^*, z)\) belongs to \( B_{\text{lower}} \), contradicting Lemma 2.4. The Property (ii) is proved similarly.
Lemma 2.6  The lower boundary is an $N_4$-path between $(0,0)$ and $(x_{\text{max}}, y_{\text{max}})$.

Proof. Prove that $|B_{i+1} - B_i| = 1$ for every $1 \leq i < k$. If $x_{B_i} = x_{B_{i+1}}$, then from the chain property and Lemma 2.5 it immediately follows that $y_{B_{i+1}} = y_{B_i} + 1$.

If $x_{B_i} < x_{B_{i+1}}$, then the chain property implies that either $(x_{B_i} + 1, y_{B_i}) \in S$ or $(x_{B_i}, y_{B_i} + 1) \in S$. Since the point $(x_{B_i}, y_{B_i} + 1) \notin B_{\text{lower}}$, we can conclude that $(x_{B_i} + 1, y_{B_i}) \in S$ in any case. Now Lemma 2.4 implies that this point belongs to $B_{\text{lower}}$, i.e., $B_{i+1} = (x_{B_i} + 1, y_{B_i})$. ■

The upper boundary case is validated in the same way.

Thus an antimatroidal point set is an orthogonally convex $N_4$-connected set bounded by two monotone increasing $N_4$-paths.

The following lemma is the “only-if” part of Theorem 2.3.

Lemma 2.7  An orthogonally convex $N_4$-connected set $S$ that is bounded by two monotone increasing $N_4$-paths between $(0,0)$ and $(x_{\text{max}}, y_{\text{max}})$ is an antimatroidal point set.

Proof. By Definition 1.4 we have to check the two properties (A1) and (A2):

(A1) Let $A = (x, y) \in S$. If $A$ is an interior point in $S$ then $(x - 1, y) \in S$ and $(x, y - 1) \in S$. If $A$ is a boundary point, then the previous point on the boundary $((x, y - 1) \text{ or } (x - 1, y))$ belongs to $S$.

(A2) Let $A \notin B \in S$. Consider two cases:

(i) $x_a \geq x_b$ and $y_a \geq y_b$. If $B$ is an interior point in $S$ then $(x_b + 1, y_b) \in S$ and $(x_b, y_b + 1) \in S$. If $B$ is a boundary point, then the next point on the boundary $((x_b, y_b + 1) \text{ or } (x_b + 1, y_b))$ belongs to $S$.

(ii) $x_a \leq x_b$ and $y_a \geq y_b$. We have to prove that $(x_b, y_b + 1) \in S$. Suppose the opposite. Then the point $B$ is an upper boundary point. Since $y_a \geq y_b$, there exists an upper boundary point $(x_a, y)$ with $y \geq y_b$ that contradicts the monotonicity of the boundary. ■

Corollary 2.8  Any antimatroidal point set $S$ may be represented by its boundary in the following form:

$$S = B_{\text{lower}} \lor B_{\text{upper}} = \{X \cup Y : X \in B_{\text{lower}}, Y \in B_{\text{upper}}\}$$

This result shows that the convex dimension of a two-dimensional polyantimatroid is two.
Definition 2.9 Convex dimension $c\dim(S)$ of any antimatroid $S$ is the minimum number of maximal chains 

$$\emptyset = X_0 \subset X_1 \subset ... \subset X_k = X_{\max} \text{ with } X_i = X_{i-1} \cup x_i$$

whose union gives the antimatroid $S$.

The boundary of any poly-antimatroid may be found by using the following algorithm. Starting with the maximum point $(x_{\max}, y_{\max})$ it follows the upper boundary down to point $(0, 0)$ in the first pass, and it follows the lower boundary in the second pass.

**Algorithm 2.10 Upper boundary tracing algorithm**

1. $i := 0; x := x_{\max}; y := y_{\max}$;
2. $B_i := (x, y)$
3. do
   3.1 if $(x - 1, y) \in S$ then $x := x - 1$, else $y := y - 1$;
   3.2 $i := i + 1$;
   3.3 $B_i := (x, y)$;
   until $B_i = (0, 0)$
4. Return the sequence $B = B_i, B_{i-1}, ..., B_0$

It is easy to check that Algorithm 2.10 returns a monotone increasing $N_4$-path that only passes over the upper boundary points from $B_{\text{upper}}$ (for each $i$ point $B_i = (x^i, y^i)$ has the maximum $y$-coordinate, i.e., $y^i = \arg \max\{(x^i, y) \in S\}$). Hence, the Upper boundary tracing algorithm returns the upper boundary of poly-antimatroids.

The Lower boundary tracing algorithm differs from Algorithm 2.10 on search order only: $(y, x)$ instead of $(x, y)$. Step 3.1 of Algorithm 2.10 will be as follows:

3.1 if $(x, y - 1) \in S$ then $y := y - 1$, else $x := x - 1$;

Correspondingly, the Lower boundary tracing algorithm returns the lower boundary of poly-antimatroids.

It turned out that a two-dimensional poly-antimatroid known in this section as an antimatroidal point set is equivalent to special cases of polyominoes or staircase polygons and is defined by a pair of monotone paths made only with north and east steps. In the next section we research the path structure of three-dimensional antimatroidal point sets.
3 Three-dimensional antimatroidal point set

In this section we consider a particular case of three-dimensional antimatroidal point sets.

**Definition 3.1** A finite non-empty set $C$ of points in digital 3D space is a digital cuboid if

$$C = \{(x, y, z) \in \mathbb{Z}^3 : x_{\text{min}} \leq x \leq x_{\text{max}} \land y_{\text{min}} \leq y \leq y_{\text{max}} \land z_{\text{min}} \leq z \leq z_{\text{max}}\} \text{ and } |C| > 1$$

A digital cuboid is specified by the coordinates of opposite corners.

Consider the following construction of $n$ cuboids:

**Definition 3.2** The sequence of $n$ cuboids $C_1, C_2, ..., C_n$ is called regular if

(a) $x_0 = y_0 = z_0 = 0$

(b) $x_i \leq x_{i+1} \land y_i \leq y_{i+1} \land z_i \leq z_{i+1}$

and at least one of the inequality is strong for each $1 \leq i \leq n - 1$

(c) $x_{i+1} \leq x_{\text{max}} \land y_{i+1} \leq y_{\text{max}} \land z_{i+1} \leq z_{\text{max}}$ for each $1 \leq i \leq n - 1$

(d) $x_{i} \leq x_{\text{max}} \land y_{i} \leq y_{\text{max}} \land z_{i} \leq z_{\text{max}}$ for each $1 \leq i \leq n - 1$

**Definition 3.3** The union of elements of regular sequence $C = C_1 \cup C_2 \cup ... \cup C_n$ is called an $n$-step staircase.

An example of a 2-step staircase is depicted in Figure 3.

![Figure 3: 2-step staircase $C$.](image)

Each point $A = (x_A, y_A, z_A)$ in the digital space $\mathbb{Z}^3$ may be considered as a multiset $A$ over $\{x, y, z\}$, where $x_A$ is the number of repetitions of an
element \( x \), and \( y_A \) is the number of repetitions of an element \( y \), and \( z_A \) is the number of repetitions of an element \( z \) in multiset \( A \). Then a 1-step staircase (cuboid) is a poly-antimatroid, since the family of points is accessible and closed under union ((\( (x^1, y^1, z^1) \), \( (x^2, y^2, z^2) \) \( \in \) \( C \) \( \Rightarrow \) \( (x^1, y^1, z^1) \cup (x^2, y^2, z^2) = (\max(x^1, x^2), \max(y^1, y^2), \max(z^1, z^2)) \in \) \( C \)).

**Lemma 3.4** An \( n \)-step staircase \( C \) is a poly-antimatroid.

**Proof.** Consider some point \( A \neq (0, 0, 0) \in \) \( C \). Then there exists \( i \) such that \( A \in C_i \). If \( A \neq (x^i_{\min}, y^i_{\min}, z^i_{\min}) \), then, without lost of generality, \( x^i_{\min} < x_A \), and so \( (x_A - 1, y_A, z_A) \in C_i \subseteq \) \( C \). If \( A = (x^i_{\min}, y^i_{\min}, z^i_{\min}) \), then, from Definition 3.2 (b,c) it follows that \( A \in C_{i-1}\) and \( A \neq (x^{i-1}_{\min}, y^{i-1}_{\min}, z^{i-1}_{\min}) \). Hence, \( (x_A - 1, y_A, z_A) \in C_{i-1} \subseteq \) \( C \) or \( (x_A, y_A - 1, z_A) \in C_{i-1} \subseteq \) \( C \) or \( (x_A, y_A, z_A - 1) \in C_{i-1} \subseteq \) \( C \). Thus, an \( n \)-step staircase \( C \) is accessible.

To prove that \( C \) is closed under union consider two points \( A, B \in \) \( C \). Let \( A \in C_i \), \( B \in C_j \), and \( i \leq j \). From Definition 3.2 (b) it follows that \( x^i_{\min} \leq \max(x_A, x_B) \). From Definition 3.2 (d), \( \max(x_A, x_B) \leq x^i_{\max} \). So, \( A \cup B = (\max(x_A, x_B), \max(y_A, y_B)) \in C_j \), i.e., \( A \cup B \in \) \( C \). \( \blacksquare \)

Our goal is to find the minimum number of maximal chains which describe an \( n \)-step staircase.

First, consider a cuboid \( C \) given by two points \( (x_{\min}, y_{\min}, z_{\min}) \) and \( (x_{\max}, y_{\max}, z_{\max}) \). We will denote by \( P_X \) the maximal chain connecting the points \( (x_{\min}, y_{\min}, z_{\min}) \) and \( (x_{\max}, y_{\min}, z_{\min}) \), i.e.,

\[
P_X = (x_{\min}, y_{\min}, z_{\min}), (x_{\min} + 1, y_{\min}, z_{\min}), \ldots, (x_{\max}, y_{\min}, z_{\min}).
\]

\( P_Y \) and \( P_Z \) are defined correspondingly. It is easy to see that

\[
C = P_X \cup P_Y \cup P_Z = \{A_1 \cup A_2 \cup A_3 : A_1 \in P_X, A_2 \in P_Y, A_3 \in P_Z\}
\]

So, if we have three maximal chains \( B_1, B_2, B_3 \) from the cuboid \( C \), which pass over all the points of \( P_X \), \( P_Y \) and \( P_Z \), then \( C \subseteq B_1 \cup B_2 \cup B_3 \). On the other hand, since \( C \) is closed under union, we have \( B_1 \cup B_2 \cup B_3 \subseteq C \). Thus, \( B_1 \cup B_2 \cup B_3 = C \).

Now, consider the three-dimensional version of Algorithm 2.10 for search order \( (x, y, z) \). Let \( (x_{\max}, y_{\max}, z_{\max}) \) be the maximum point of \( n \)-step staircase \( C \). The algorithm builds the chain connecting the maximum point and the point \( (0, 0, 0) \).
Algorithm 3.5  \textit{XYZ-Boundary tracing algorithm}
1. \(i := 0; x := x_{\text{max}}; y := y_{\text{max}}; z := z_{\text{max}};\)
2. \(B_i := (x, y, z)\)
3. do
   \hspace{1cm} 3.1 if \((x-1, y, z) \in S\) then \(x := x - 1,\) else if \((x, y-1, z) \in S\) then \(y := y - 1,\) else \(z := z - 1;\)
   \hspace{1cm} 3.2 \(i := i + 1;\)
   \hspace{1cm} 3.3 \(B_i := (x, y, z);\)
   until \(B_i := (0, 0, 0)\)
4. Return the sequence \(B = B_i, B_{i-1}, ..., B_0\)

Repeat Algorithm 3.5 for search order \((y, z, x)\) and for search order \((z, x, y)\).
As a result, we obtain three monotone increasing \(N_6\)-paths \([14]\) from \((0, 0, 0)\) to \((x_{\text{max}}, y_{\text{max}}, z_{\text{max}})\), denoted by \(B_Z, B_X\) and \(B_Y\), respectively. The length of any of these chains is equal to \((x_{\text{max}} + y_{\text{max}} + z_{\text{max}})\).

\textbf{Theorem 3.6} Any \(n\)-step staircase \(C = B_X \lor B_Y \lor B_Z\).

To show this we prove a stronger statement.
Change Algorithm 3.5 in the following way. Let the XYZ-Boundary tracing algorithm begin from some point \((x, y, z_{\text{max}}) \in C\) and return chain \(H_Z\); the YZX-Boundary tracing algorithm begins from some point \((x_{\text{max}}, y, z) \in C\) and returns chain \(H_X\); and the ZXY-Boundary tracing algorithm begins from some point \((x, y_{\text{max}}, z) \in C\) and returns chain \(H_Y\).

\textbf{Lemma 3.7} Any \(n\)-step staircase \(C = H_X \lor H_Y \lor H_Z\).

\textbf{Proof.} Let us proceed by induction on \(n\).
For \(n = 1\) the XYZ-Boundary tracing algorithm beginning from the point \((x, y, z_{\text{max}}) \in C\) returns chain \(H_Z\) that begins from the point \((0, 0, 0)\), moves to \((x_{\text{min}}, y_{\text{min}}, z_{\text{max}})\), then to \((x_{\text{min}}, y_{\text{max}}, z_{\text{max}})\), and finishes at the point \((x, y, z_{\text{max}})\).
So, this chain passes over all the points of \(P_Z\). In the same way, the YZX-Boundary tracing algorithm returns chain \(H_X\) that passes over all the points of \(P_X\), and the ZXY-Boundary tracing algorithm returns chain \(H_Y\) that passes over all the points of \(P_Y\). Thus, \(C = H_X \lor H_Y \lor H_Z\).
Assume that the proposition is correct for all \(k < n\) and prove it for \(n\). It is easy to see that the XYZ-Boundary tracing algorithm begins from some point \((x, y, z_{\text{max}}) \in C_n\), reaches the point \((x_{\text{min}}, y_{\text{min}}, z_{\text{max}})\), moves down
to the point \((x_{\min}^n, y_{\min}^n, z_{\min}^{n-1}) \in C_{n-1}\), and then continues to build chain \(H_Z\). In the same way, the YZX-Boundary tracing algorithm moves through the point \((x_{\max}^n, y_{\min}^n, z_{\min}^1) \in C_{n-1}\), and the ZXY-Boundary tracing algorithm moves through the point \((x_{\min}^n, y_{\min}^{n-1}, z_{\min}^n) \in C_{n-1}\). The induction assumption implies \(C_1 \cup C_2 \cup \ldots \cup C_{n-1} \subseteq H_X \vee H_Y \vee H_Z\).

It remains to show that \(C_n \subseteq H_X \vee H_Y \vee H_Z\). To this end we prove that \(P_X^n \subseteq H_X \vee H_Y \vee H_Z\), \(P_Y^n \subseteq H_X \vee H_Y \vee H_Z\), and \(P_Z^n \subseteq H_X \vee H_Y \vee H_Z\). We have already seen that the XYZ-Boundary tracing algorithm returns chain \(H_Z\) that covers the part \([((x_{\min}^n, y_{\min}^n, z_{\max}^n), (x_{\min}^n, y_{\min}^n, z_{\min}^{n-1})] \subseteq C_1 \cup C_2 \cup \ldots \cup C_{n-1}\). So \(P_Z^n \subseteq H_X \vee H_Y \vee H_Z\). It is easy to verify two other cases.

Finally, \(C = C_1 \cup C_2 \cup \ldots \cup C_n \subseteq (H_X \vee H_Y \vee H_Z) \vee (H_X \vee H_Y \vee H_Z) \vee (H_X \vee H_Y \vee H_Z) = H_X \vee H_Y \vee H_Z\), since each \(H\) is a chain.

On the other hand, \(H_X \vee H_Y \vee H_Z \subseteq C\), and so \(C = H_X \vee H_Y \vee H_Z\). 

Theorem 3.6 shows that it is enough to know only three maximal chains \(B_X, B_Y, \) and \(B_Z\) to describe an \(n\)-step staircase.

**Corollary 3.8** An \(n\)-step staircase has convex dimension at most 3.

Note that the three points \((0, 0, 1)\), \((0, 1, 0)\) and \((1, 0, 0)\) cannot be covered by two chains only, since each such point (multiset) cannot be formed as a union of smaller multisets. So, there exist \(n\)-step staircases of convex dimension 3.

However, the convex dimension of an arbitrary three-dimensional poly-antimatroid may be arbitrarily large [10]. Let \(S\) be a set of points:

\[
S = \{(x, y, z) : (0 \leq x, y \leq N) \land (0 \leq z \leq 1) \land (z = 1 \Rightarrow x + y \geq N)\}
\]

It is easy to check that \(S\) is a three-dimensional poly-antimatroid. Consider \(N + 1\) points \((x, y, 1)\) with \(x + y = N\). Since each of these points cannot be represented as a union of any points from \(S\) with smaller coordinates, the convex dimension of \(S\) is at least \(N + 1\) [10].

In order to characterize the family of three-dimensional poly-antimatroids of convex dimension 3, consider a particular case of antimatroids called **poset antimatroids** [15]. A poset antimatroid has as its feasible sets the lower sets of a poset (partially ordered set). The poset antimatroids can be characterized as the unique antimatroids which are closed under intersection [15]. We extend this definition to poly-antimatroids.
Definition 3.9 A poly-antimatroid is called a poset poly-antimatroid if it is closed under intersection.

Now, note that $n$-step staircases are closed under intersection too. Indeed, consider two points $A, B \in \mathcal{C}$. Let $A \in C_i, B \in C_j$, and $i \leq j$. From Definition 3.2 (b) it follows that $\min(x_A, x_B) \geq x_{i \min}$. On the other hand, (Definition 3.2 (d)), $\min(x_A, x_B) \leq x_{i \max}$. So, $A \cap B = (\min(x_A, x_B), \min(y_A, y_B)) \in C_i$, i.e., $A \cap B \in \mathcal{C}$.

So, our conjectures are as follows.

Conjecture 3.10 A three-dimensional poset poly-antimatroid is a step staircase.

Corollary 3.11 The convex dimension of a three-dimensional poset poly-antimatroid is at most 3.

Moreover,

Conjecture 3.12 The convex dimension of an $n$-dimensional poset poly-antimatroid is at most $n$.

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