Particle motion in circularly polarized vacuum pp waves

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Abstract

Bialynicki-Birula and Charzyński argued that a gravitational wave emitted during the merger of a black hole binary may be approximated by a circularly polarized wave which may in turn trap particles [1]. In this paper we consider particle motion in a class of gravitational waves which includes, besides circularly polarized periodic waves (CPP) [2], also the one proposed by Lukash [3] to study anisotropic cosmological models. Both waves have a 7-parameter conformal symmetry which contains, in addition to the generic 5-parameter (broken) Carroll group, also a 6th isometry. The Lukash spacetime can be transformed by a conformal rescaling of time to a perturbed CPP problem. Bounded geodesics, found both analytically and numerically, arise when the Lukash wave is of Bianchi type VI. Their symmetries can also be derived from the Lukash-CPP relation. Particle trapping is discussed.

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I. INTRODUCTION

Bialynicki-Birula and Charzynski (BBC) [1] argued that the gravitational waves emitted during the merger of compact binaries may trap massive particles. Their clue is that in the vicinity of the wave axis a gravitational wave carrying angular momentum can be approximated by a Bessel beam. Then for small deviations the geodesic deviation equations yield a coupled system, their eqn. # (14), which admits bounded solutions shown in their fig. 1 – just like their electromagnetic counterparts do [4–6].

Their result is consistent with what was found before for Circularly Polarized Periodic (CPP) gravitational waves [2], whose geodesic equation can be reduced to similar equations [7, 8]. Our paper extends and amplifies these findings by studying, both analytically and numerically, geodesic motion in Lukash plane gravitational waves considered before in the study of the isotropy/anisotropy of cosmological models [2, 3, 9].

Gravitational plane waves have a generic 5-parameter isometry group [2, 10, 11], identified as a “broken Carroll” group [12, 13]. CPP and Lukash waves are special in that they have an additional 6th “screw” (or “helical” [14]) isometry [2, 5, 7]. The isometry group extends to a 7-parameter conformal symmetry [9, 15, 16].

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The 6th isometry, which is a sort of “spiralling time translation”, arises for our circularly polarized waves. Moreover, according to Table 24.2 of [2] p.385, CPP and Lukash are the only vacuum pp waves with this property.

The properties of CPP waves are widely known [2, 9]; our principal interest here is to study the analogous but less-known Lukash waves. Our clue is to reduce the geodesic motion in a Lukash metric to one in a perturbed CPP wave by a clever rescaling ot time, (II.13) below. Exact analytic solutions can be found by following a “road map” outlined in sec. II B. The solvability comes from transforming the system to one with constant coefficients, (II.16) – which implies also the extra 6th symmetry. Bounded motions arise when the Lukash metric has Bianchi type VI.

Our strategy fits into the framework proposed by Gibbons [18]. A bonus obtained from our CPP ↔ Lukash correspondence is to relate their 6th isometries, see sec.IV.

The subtle notion of particle trapping is discussed in sec.V.

Before starting our study we fix our conventions. Lower-case latin letters as \( \mathbf{x} = (x^1, x^2) \), \( u, v \) refer to generic pp waves. Latin capitals \( \mathbf{X} = (X^1, X^2) \), \( U, V, Z = X^1 + iX^2 \sqrt{2} \) refer to Lukash waves with profile functions \( A_+ (U) \) and \( A_\times (U) \), respectively; greek letters \( \xi = (\xi, \eta), \zeta = \frac{\xi + i\eta}{\sqrt{2}} \), completed with \( T \) and \( \nu \) and profile \( B_+ (T) \) and \( B_\times (T) \) refer to CPP(-type) waves.

### II. CIRCULARLY POLARIZED GRAVITATIONAL WAVES

Both CPP and Lukash waves are plane gravitational waves described, in Brinkmann coordinates, in terms of a symmetric and traceless \( 2 \times 2 \) matrix \( K_{ij}(u) \) [2, 3, 9, 17],

\[
\mathrm{d}s^2 = \mathrm{d}x^2 + 2\mathrm{d}udv + K_{ij}(u)x^i x^j \mathrm{d}u^2,
\]

(II.1)

where \( \mathbf{x} = (x^1, x^2) \). The vacuum Einstein equations reduce to \( \Delta (K_{ij}(u)x^i x^j) = 0 \), where \( \Delta \equiv \Delta_2 \) is the transverse-space Laplacian. (II.1)-(II.2) is therefore an exact plane wave for any profile \( K_{ij}(u) \).

The profile is decomposed into + and \( \times \) polarization states,

\[
K_{ij}(u)x^i x^j = \frac{1}{2} A_+ (u) ((x^1)^2 - (x^2)^2) + A_\times (u)x^1 x^2.
\]

(II.2)

A spin-zero particle moves along a geodesic, described by,

\[
\mathbf{x}'' - K(u) \mathbf{x} = 0 \quad \text{where} \quad K(u) = \frac{1}{2} \begin{pmatrix} A_+ & A_\times \\ A_\times & -A_+ \end{pmatrix},
\]

(II.3a)

\[
v'' + A_+ ( (x^1)^2 - (x^2)^2 ) + A_+ (x^1 (x^1)' - x^2 (x^2)') + A_\times x^1 x^2 + A_\times (x^2 (x^1)' + x^1 (x^2)') = 0,
\]

(II.3b)

1. Our earlier investigations [17] concern Lukash waves with a different range of the parameters and of different Bianchi type.
where the prime means derivative w.r.t. the affine parameter $u$, \( \{ \cdot \}' = d/du \).

For any affine parameter $\sigma$ the quantity $-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = m^2$ where the “dot” means $d/d\sigma$ is a constant of the motion, identified with the relativistic mass-square. The transverse motion described by (II.3a) does not depend on $m$, and the solution of the $V$-equation differs from that for $m = 0$ by the simple shift $m^2 u$ [19], allowing us to restrict our attention at \textit{massless geodesics}. After solving the transverse equations (II.3a) the $v$-equation is solved by,

$$v(u) = v_0 - S(u),$$

(II.4)

where $S(u)$ is the Hamiltonian action calculated along the transverse trajectory [19, 22, 23]. Therefore it is enough to solve the transverse equations (II.3a).

In what follows, we restrict our attention at circularly polarized waves. A CPP wave has, for example, the profile (with a slight change of notations $A \rightarrow B$ and $u \rightarrow T$).

$$B_+(T) = -C \cos(\omega T), \quad B_\times(T) = C \sin(\omega T), \quad \text{where } C, \omega = \text{const.}$$

(II.5)

The geodesic motion in a CPP can be determined both numerically and analytically [7, 8].

A. Lukash Geodesics

A Lukash plane gravitational wave is described, in complex form [20], by

$$ds_L^2 = 2dUdV + 2dZd\bar{Z} - 2C\text{Re}[U^2(i\kappa - 1)Z^2]dU^2,$$

where \( Z = \frac{X_1 + iX_2}{\sqrt{2}} \).

(II.6)

The coordinates are well-defined for either $U > 0$ or $U < 0$ but break down at $U = 0$. The nature of the singularity at $U = 0$ was the subject of intensive investigations [3, 20, 21]. The constant $C$ determines the strength of the wave; $C \geq 0$ can be chosen with no loss of generality. $\kappa$ is the frequency (inverse wavelength) ; its sign is the (right or left) polarisation. In what follows we shall choose $\kappa > 0$. Then for an arbitrary affine parameter $\sigma$ we have the null-geodesic equations,

$$\frac{d^2 Z}{d\sigma^2} + CU^{-2(i\kappa + 1)}\dot{Z} = 0,$$

(II.7a)

$$\frac{d^2 V}{d\sigma^2} - 2C \left\{ \frac{d}{d\sigma} \text{Re}[U^2(i\kappa - 1)Z^2] \frac{dU}{d\sigma} - \text{Re}[(i\kappa - 1)U^2(i\kappa - 1)Z^2U^{-2}Z^2] \left( \frac{dU}{d\sigma} \right)^2 U^{-1} \right\} = 0.$$

(II.7b)

$$\frac{dU}{d\sigma^2} = 0,$$

implying that $U$ is an affine parameter itself. Reversing the sign of the light-cone coordinates,

$$U \rightarrow -U, \quad V \rightarrow -V,$$

(II.8)
leaves the Lukash metric (II.6), and consequently also the equations (II.7) invariant. Choosing $U > 0$ henceforth, we can switch to more familiar real coordinates, in terms of which the Lukash (II.6) is,

$$ds^2_L = dX^2 + 2dUdV - \left\{ \frac{C}{U^2} \cos \left(2\kappa \ln(U)\right) \left[(X^1)^2 - (X^2)^2\right] - \frac{2C}{U^2} \sin \left(2\kappa \ln(U)\right) X^1 X^2 \right\} dU^2. \quad (\text{II.9})$$

The transverse equations form a complicated coupled Sturm-Liouville system,

$$\begin{pmatrix} (X^1)'' \\ (X^2)'' \end{pmatrix} = -\frac{C}{U^2} \begin{pmatrix} \cos(2\kappa \ln U) - \sin(2\kappa \ln U) \\ -\sin(2\kappa \ln U) - \cos(2\kappa \ln U) \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \quad (\text{II.10})$$

where now $\{ \cdot \}' = d/dU$.

Let us recall, for the sake of comparison, “ordinary” CPP : the transverse equations of motion of the metric (II.1) with profile (II.5) are

$$\begin{pmatrix} \ddot{\xi} \\ \ddot{\eta} \end{pmatrix} = -C \begin{pmatrix} \cos \omega T + \sin \omega T \\ \sin \omega T - \cos \omega T \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad (\text{II.11})$$

where $\omega = \text{const.}$ (We changed notation, $x^1 \to \xi$, $x^2 \to \eta$, $u \to T$, $\{ \cdot \}' = d/dT$ on purpose). Having chosen $C > 0$, bounded solutions arise when

$$\left(\frac{\omega}{2}\right)^2 - C > 0. \quad (\text{II.12})$$

Lukash and CPP are thus similar but still different.

### B. Solution of the Sturm-Liouville equations

Sturm-Liouville equations are notoriously difficult to solve. Analytic solutions can be found in our case, though, by the following steps.

- Our clue is to switch to “logarithmic time” and introduce new transverse coordinates,

$$U = e^T, \quad X = e^{T/2} \xi, \quad \text{where} \quad \xi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \quad (\text{II.13})$$

in terms of which (II.10) becomes,

$$\frac{d^2}{dT^2} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \text{linear} + \text{CPP} = \frac{1}{4} \begin{pmatrix} \xi \\ \eta \end{pmatrix} - C \begin{pmatrix} \cos(2\kappa T) - \sin(2\kappa T) \\ -\sin(2\kappa T) - \cos(2\kappa T) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \quad (\text{II.14})$$

Comparison with (II.11) shows that the projected non-relativistic dynamics is that of a repulsive linear force with spring constant $\frac{1}{4}$, combined with a periodic “CPP” force. The latter may be attractive or repulsive depending on the amplitude $C > 0$ and the frequency, $-2\kappa$ [2, 7, 8].
Our second step is to change again to new position coordinates, \((\xi, \eta)\). The rotation with half-of-the-angle [1, 7] (suggested to us by Piotr Kosinski),

\[
\begin{pmatrix} \xi \\ \eta \end{pmatrix} = R_{\kappa T} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad R_{\kappa T} = \begin{pmatrix} \cos \kappa T + \sin \kappa T \\ -\sin \kappa T \cos \kappa T \end{pmatrix}
\]

converts (II.14) into a coupled Coriolis-type system with \textit{constant coefficients},

\[
\ddot{\alpha} + 2\kappa \dot{\beta} - \Omega^2 \alpha = 0, \tag{II.16a}
\]
\[
\ddot{\beta} - 2\kappa \dot{\alpha} - \Omega^2 \beta = 0, \tag{II.16b}
\]

where the dot \(\dot{\cdot}\) means, henceforth, \(d/dT\),

\[
\Omega_-^2 = \left(\kappa^2 + \frac{1}{4} - C\right) \quad \text{and} \quad \Omega_+^2 = \left(\kappa^2 + \frac{1}{4} + C\right).
\]

Our oscillator is anisotropic, \(\Omega_-^2 - \Omega_+^2 = 2C\); the lower frequency may become negative, depending on the parameters.

Remarkably, the system (II.16) coincides with the equations \#(14) of Bialynicki-Birula and Charzynski [1], who obtained it after a series of approximations. In the Eisenhart-Duval framework [22, 23] it would describe a \textit{charged anisotropic linear oscillator} in the plane with frequencies \(\Omega_{\pm}\), put into a uniform magnetic field \(B = -2\kappa\). Its behavior is determined by the subtle competition between the magnetic and oscillating terms, as it will be illustrated by our figures below.

Our last step comes from that the equations (II.16) are up to the frequency-shift 1/4 those for a CPP [1, 2, 8] and can thus be solved analytically [1, 8, 24]. We start with the Hamiltonian and symplectic form,

\[
\mathcal{H} = \frac{1}{2}p^2 - \frac{1}{2}(\Omega_-^2 \alpha^2 + \Omega_+^2 \beta^2), \tag{II.18a}
\]
\[
\sigma = dp^i \wedge d\alpha^i - \kappa \varepsilon^{ij} d\alpha^i \wedge d\alpha^j \tag{II.18b}
\]

and introduce four phase-space coordinates \(w_\pm^1, w_\pm^2\) by setting

\[
p^1 = \mu_+ w_+^2 + \mu_- w_-^2, \quad p^2 = -\nu_+ w_+^1 - \nu_- w_-^1, \tag{II.19a}
\]
\[
\alpha = w_+^1 + w_-^1, \quad \beta = w_+^2 + w_-^2. \tag{II.19b}
\]

Then choosing the coefficients as,

\[
\mu_+ = \frac{1}{2\kappa} \left(\sqrt{C^2 - \kappa^2} - 2\kappa^2 - C\right), \quad \nu_+ = \frac{1}{2\kappa} \left(\sqrt{C^2 - \kappa^2} - 2\kappa^2 + C\right), \tag{II.20a}
\]
\[
\mu_- = -\frac{1}{2\kappa} \left(\sqrt{C^2 - \kappa^2} + 2\kappa^2 + C\right), \quad \nu_- = -\frac{1}{2\kappa} \left(\sqrt{C^2 - \kappa^2} + 2\kappa^2 - C\right), \tag{II.20b}
\]
decomposes the system into two uncoupled 1D Hamiltonian systems with opposite relative signs,

\[ \sigma = \sigma_+ - \sigma_-, \quad H = H_+ - H_- , \quad \text{(II.21)} \]

where

\[ \sigma_+ = -\sqrt{C_2 - \kappa^2} \kappa dw_1^+ \wedge dw_2^+ , \quad H_+ = \sqrt{C_2 - \kappa^2} \left[ \nu_+ w_1^+ w_1^+ + \mu_+ w_2^+ w_2^+ \right] , \quad \text{(II.22a)} \]

\[ \sigma_- = -\sqrt{C_2 - \kappa^2} \kappa dw_1^- \wedge dw_2^-, \quad H_- = \sqrt{C_2 - \kappa^2} \left[ \nu_- w_1^- w_1^- + \mu_- w_2^- w_2^- \right] , \quad \text{(II.22b)} \]

respectively.

**Strong but slow perturbation: \( C > \kappa \).**

For \( C > \kappa \) \( \sigma_\pm \) and \( H_\pm \) in (II.22) are real and the Hamiltonian system is regular. The associated equations of motion are

\[ \ddot{w}_1^+ + \lambda_+^2 w_1^+ = 0, \quad \ddot{w}_2^+ = 0, \quad \text{(II.23a)} \]

\[ \ddot{w}_1^- + \lambda_-^2 w_1^- = 0, \quad \ddot{w}_2^- = 0, \quad \text{(II.23b)} \]

where the two effective frequency-squares are,

\[ \lambda_+^2 = \mu_+ \nu_+ = \kappa^2 - \frac{1}{4} - \sqrt{C_2 - \kappa^2} \quad \text{and} \quad \lambda_-^2 = \mu_- \nu_- = \kappa^2 - \frac{1}{4} + \sqrt{C_2 - \kappa^2} . \quad \text{(II.24)} \]

The equations (II.23) are not independent, though: \( w_+ \) and \( w_- \) have to satisfy \( \dot{w}_1^+ = \mu_+ w_2^+ \) and \( \dot{w}_1^- = \mu_- w_2^- \), which eliminates half of the integration constants and we end up with two independent oscillations with frequencies \( \lambda_\pm \),

\[ w_1^+ = a \cos \lambda_+ T + b \sin \lambda_+ T, \quad w_2^+ = \sqrt{\frac{\nu_+}{\mu_+}} (-a \sin \lambda_+ T + b \cos \lambda_+ T) , \quad \text{(II.25a)} \]

\[ w_1^- = c \cos \lambda_- T + d \sin \lambda_- T, \quad w_2^- = -\sqrt{\frac{\nu_-}{\mu_-}} (c \sin \lambda_- T - d \cos \lambda_- T) , \quad \text{(II.25b)} \]

where \( a, b, c, d \) are free real constants. These equations show that bounded solutions arise when both lambdas are real, as illustrated below in figs.2,3 and 5. In view of (II.19b) the chiral decomposition can be interpreted as follows: in the rotating frame the transverse-space trajectory is decomposed into a sum,

\[ \begin{pmatrix} \alpha(T) \\ \beta(T) \end{pmatrix} = \begin{pmatrix} w_1^+(T) \\ w_2^+(T) \end{pmatrix} + \begin{pmatrix} w_1^-(T) \\ w_2^-(T) \end{pmatrix} , \quad \text{(II.26)} \]

where the first term represents a sort of “guiding center” and the second describes a sort of “epicycle” around it. Proceeding backwards, from (II.15) we infer \( \left( \frac{\xi}{\eta} \right) = R_{(\kappa T)} (\alpha_{\beta}) \) and then \( X(U) \) can be deduced
from (II.13). Bounded \((\alpha_\beta)\) [or \((\xi_\eta)\)] motions arise when both frequency-squares are positive, which happens when \(\lambda_\pm^2 > 0\) that requires

\[
\kappa < C < C_{\text{crit}} = \kappa^2 + \frac{1}{4} \quad \text{with} \quad \kappa > 1/2, \tag{II.27}
\]

studied before by Siklos [20] and illustrated in fig.1 below, strongly reminiscent of fig.1 in [1]. See also [25, 26]. Expressed in Brinkmann terms

\[
X(U) = \sqrt{U} R_{(\kappa,lnU)} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \tag{II.28}
\]

and therefore the trajectories escape. Boundedness/unboudness will be further discussed in sec.V.

The metric admits a 3-parameter group of motions acting on spacelike hypersurfaces if and only if either \(\kappa \leq \frac{1}{2}\) and \(0 \leq C \leq \kappa^2 + \frac{1}{4}\), or \(\kappa \geq \frac{1}{2}\) and \(0 \leq C \leq \kappa\) [20]. The group type (with some overlaps) is :

- **Bianchi VII:** if

  \[
  \text{either} \quad 0 \leq C < \kappa \quad \text{or} \quad C = \kappa > \frac{1}{2} \tag{II.29}
  \]

- **Bianchi VI:** if

  \[
  \text{either} \quad \kappa < C \leq \kappa^2 + \frac{1}{4} \quad \text{or} \quad 0 < C = \kappa < \frac{1}{2} \tag{II.30}
  \]

- **Bianchi IV:** if

  \[
  C = \kappa \tag{II.31}
  \]

It follows that bound motion arise only when the wave is of Bianchi-VI type.

Below we study the behavior for various values of the parameters.

**Behaviour at the critical values.**

At the upper critical value \(C = C_{\text{crit}}\) the effective frequencies (II.24) become \(\lambda_+ = 0\) and \(\lambda_-^2 = 2(\kappa^2 - 1/4)\). The motion is unbounded, \(w_+^1 = a + bT, \quad w_+^2 = c + dT\), while \(w_-\) either oscillates or escapes, depending on \(\kappa\) being above or below \(1/2\). The behavior for \(C\) close to \(C_{\text{crit}}\) is illustrated in figs. 2, 3 5. For \(C > C_{\text{crit}}\) the motion escapes rapidly \(^2\).

\(^2\) The motion can remain bounded even for \(C > C_{\text{crit}}\), though. If \(\lambda_+^2 < 0\) but \(\lambda_-^2 > 0\) then the trigonometric functions in (II.25a) become hyperbolic, while those in (II.25b) remain oscillatory. However when \(b = -a\) the terms \(w_i^1 \propto \exp(-\lambda_i T)\) \((i = 1, 2)\) fall off exponentially and the bounded (oscillatory) terms dominate.
FIG. 1: In terms of logarithmic position coordinates \((\xi, \eta)\) (II.13) (or their rotated versions \((\alpha, \beta)\) (II.15)) one gets bounded Lukash trajectories in the parameter range (II.27). The magenta curve near the origin is the contribution of the “CPP” term in (II.14), which is pushed outwards by the repulsive force. However the CPP term keeps the motion bounded.

At the lower limiting value \(C = \kappa\) in (II.31) both the symplectic forms and Hamiltonians (II.22) vanish and we have to return to the equations (II.16) with \(\Omega_{\pm}^2 = (\kappa \mp \frac{1}{2})^2\). Then we consider three subcases, illustrated in figs.4 and 5.

FIG. 2: Trajectories shown in \((\alpha, \beta)\) coordinates unfolded in logarithmic time \(T\) when close to the upper critical value \(C_{\text{crit}}\). The motion of the guiding center \(w_+\) is determined by \(\lambda_+^2\) and the “epicycle” around it is described by \(w_-\), determined by \(\lambda_-^2 = \lambda_+^2 + 2\sqrt{C^2 - \kappa^2}\). (a) when \(\kappa < C < C_{\text{crit}} = \kappa^2 + \frac{1}{4}\) (which has Bianchi type VI), the trajectory remains bounded; (b) for \(C = C_{\text{crit}}\) with \(\kappa > \frac{1}{2}\) (which is Bianchi VI), the radius is constant in logarithmic time \(T\) (grows linearly in \(U\)); (c) for \(C > C_{\text{crit}}\) the trajectory escapes exponentially.

At the lower limiting value \(C = \kappa\) in (II.31) both the symplectic forms and Hamiltonians (II.22) vanish and we have to return to the equations (II.16) with \(\Omega_{\pm}^2 = (\kappa \mp \frac{1}{2})^2\). Then we consider three subcases, illustrated in figs.4 and 5.
For $0 < C = \kappa < \frac{1}{2}$ (which is Bianchi VI) we get in the rotating coordinates $(^\alpha _\beta )$ in (II.15),

$$
\alpha = \left( a - \kappa \left( \frac{2b + (1 + 2\kappa)c}{1 - 2\kappa} \right) T \right) \cosh \bar{\omega} T + \left( \frac{b + (1 + 2\kappa)\kappa c}{1 - 2\kappa} - \frac{\kappa}{2} \left( (1 - 2\kappa)a + 2d \right) T \right) \frac{\sinh \bar{\omega} T}{\bar{\omega}},
$$

(II.32)

$$
\beta = \left( c + \kappa \left( \frac{(1 - 2\kappa)a + 2d}{1 + 2\kappa} \right) T \right) \cosh \bar{\omega} T + \left( \frac{-(1 - 2\kappa)\kappa a + d}{1 + 2\kappa} + \kappa \left( b + (1 + 2\kappa)2c \right) T \right) \frac{\sinh \bar{\omega} T}{\bar{\omega}},
$$

(II.33)

(II.34)

where

$$
\bar{\omega} = \sqrt{\frac{1}{4} - \kappa^2}.
$$

(II.35)

The trajectory is exponentially escaping: the rotation is too weak to keep the motion bounded.

For $C = \kappa > \frac{1}{2}$ (which is Bianchi VII) we get the same formulae up to replacing hyperbolic functions by trigonometric ones,

$$
cosh \bar{\omega} \to \cos \tilde{\omega}, \quad \sinh \bar{\omega} \to \sin \tilde{\omega} \quad \text{where} \quad \tilde{\omega} = \sqrt{\kappa^2 - \frac{1}{4}},
$$

(II.36)

see fig.4. Exponential escape is eliminated, however terms which are linear in $T = \ln U$ may remain.

In the Bianchi IV case $C = \kappa = \frac{1}{2}$, the trajectories are polynomial functions of $T$,

$$
\alpha = a + bT - \frac{d}{2} T^2 - \frac{(b+c)}{6} T^3, \quad \beta = c + dT + \frac{(b+c)}{2} T^2,
$$

(II.37)

which remain bounded only in the trivial case $b = c = d = 0$. 
The value (II.31) separates our present domain of investigations (II.27) from the Bianchi VII\(_h\) range

\[ 0 < C < \kappa \tag{II.38} \]

we studied in [17] using a rather different (“Siklos” [20]) technique. The results are consistent, though.

In the range (II.38) the chiral decomposition (II.22) would yield imaginary symplectic forms and Hamiltonians.

**FIG. 4:** The components of (a) CPP and (b) Lukash geodesics at the lower critical value \( C = \kappa \) (which is Bianchi VII and Bianchi IV). The colors refer to the respective cases. Caveat: the CPP and Lukash scales in (a) and (b) are different. Fig.(c) shows both trajectories in the transverse plane.

**FIG. 5:** (a) CPP and (b) Lukash trajectories just above the lower critical value \( C = \kappa \). The colors and labels refer to the respective cases. Caveat: the CPP and Lukash scales are different. (c) combines the two curves.
III. LIFT TO 4D

Further insight is provided by lifting the coordinate transformation to 4D Lukash spacetime. Completing (II.13) with $V \to V$ allows us to write the Lukash metric as

$$ds^2_L = e^T d\Sigma^2,$$

which is conformal to

$$d\Sigma^2 = d\xi^2 + d\eta^2 + 2dTdV + (\xi d\xi + \eta d\eta)dT + \left(\frac{1}{4}(\xi^2 + \eta^2) - C(\xi^2 - \eta^2) \cos 2\kappa T + 2C\xi\eta \sin 2\kappa T\right)dT^2.$$

From the Barmann point of view, the geodesics of this metric project to motion in a uniform magnetic field combined with an anisotropic oscillator. The term $(\xi d\xi + \eta d\eta)dT$ can actually be absorbed by a redefinition of the vertical coordinate,

$$V = \nu - \frac{1}{4}(\xi^2 + \eta^2),$$

allowing us to present (III.2) as,

$$d\Sigma^2 = d\xi^2 + 2dTd\nu + \left(\frac{1}{4}\xi^2 - C(\xi^2 - \eta^2) \cos 2\kappa T + 2C\xi\eta \sin 2\kappa T\right)dT^2,$$

which differs from the CPP metric only in an additional $\frac{1}{4}\xi^2$ perturbation of the potential (where we wrote obviously $\xi = (\xi, \eta)$). In conclusion,

$$(X, U, V) \to (\xi, T, \nu),$$

is a conformal mapping from Brinkmann to coordinates with logarithmic time, $T = \ln U$. We shall call it “perturbed CPP metric” in what follows. The latter is not a vacuum solution due to the additional $\xi^2$ in the potential.

Conformally related metrics have identical null geodesics (and intertwined respective affine parameters) [27]; the identity of the null geodesics of $ds^2_L$ and $d\Sigma^2$, respectively, can be checked directly. Henceforward we work with $d\Sigma^2$ and we choose $T$ as affine parameter.

We record for later use that applying again the rotational trick (II.15) and putting $A_\alpha = \kappa \beta$, $A_\beta = -\kappa \alpha$ allows us to rewrite $d\Sigma^2$ as

$$d\Sigma^2 = d\alpha^2 + d\beta^2 + 2\left(d\nu + A_\alpha d\alpha + A_\beta d\beta\right)dT + \left(\Omega_\alpha^2 + \Omega_\beta^2\right)dT^2.$$

$\frac{1}{2}(\alpha d\alpha + \beta d\beta) = d\left(\frac{1}{4}(\xi^2 + \eta^2)\right)$ is a vector potential for a constant magnetic field $B = -2\kappa$ which can be removed by redefining the vertical coordinate, cf. (III.3).
IV. ISOMETRIES AND CONFORMAL TRANSFORMATIONS

Now we show that the Lukash metric carries a 6-parameter group of isometries which extends to a 7 parameter conformal group. We follow first our road map set out in sec.II.B.

1. The Lukash metric admits the generic 5-parameter broken Carroll symmetry of gravitational plane waves [2, 12, 13, 16, 28, 29] spanned by the covariantly constant vector $\partial_V$ plus of 4 “U-dependent translations” [30],

$$X^1 \rightarrow X^1 + \beta^1(U), \quad X^2 \rightarrow X^2 + \beta^2(U) \quad \text{(IV.1)}$$

that carry solutions into solutions. Then by the linearity of (II.3a) the $\beta^i$ have to satisfy the same equations as the transverse coordinates do. Thus the time-dependent symmetries of (II.3a) are the projections of geodesics [13]. They lift to 4D to isometries by (II.4) and are analogous to what one obtains for an isotropic oscillator by pulling back the translations and boosts of a free particle by Niederer’s map [31]. They span a Newton-Hooke group structure [30, 32].

2. The remarkable property of the Lukash system is its additional 6th isometry [2, 17], recovered as follows : in terms of the new coordinates $T = \ln U$, $\left(\begin{array}{c} \alpha \\ \beta \end{array}\right)$ and $\nu$ in (II.15)–(III.3) none of the coefficients in the rotated metric (III.6) [or of the equations of motion (II.16)] depends on $T$; therefore $T$-translations,

$$T \rightarrow T + \tau, \quad \left(\begin{array}{c} \alpha \\ \beta \end{array}\right) \rightarrow \left(\begin{array}{c} \alpha \\ \beta \end{array}\right), \quad \nu \rightarrow \nu \quad \text{(IV.2)}$$

are manifest isometries for any real constant $\tau$. Working backwards, the perturbed CPP equations (II.14) are invariant w.r.t. the “screw” obtained by combining a $T$-translation with a transverse rotation with angle $(-\kappa \tau)$ [7, 8, 13],

$$T \rightarrow T + \tau, \quad \left(\begin{array}{c} \xi \\ \eta \end{array}\right) \rightarrow R(-\kappa \tau) \left(\begin{array}{c} \xi \\ \eta \end{array}\right), \quad \nu \rightarrow \nu. \quad \text{(IV.3)}$$

In Brinkmann terms this isometry is implemented as

$$U \rightarrow e^{\tau} U, \quad X \rightarrow R(-\kappa \tau) X, \quad V \rightarrow e^{-\tau} V \quad \text{(IV.4)}$$

and is generated by combining an U-V boost with a transverse rotation $^3$,

$$Y_\kappa = (U \partial_U - V \partial_V) - \kappa (X^1 \partial_2 - X^2 \partial_1) \quad \text{(IV.5)}$$

we propose to call “expanding screw”. For $\kappa = 0$ the Lukash profile reduces to $U^{-2}$ and the results in [15, 16] are recovered.

$^3$ $Y_\kappa$ is actually a symmetry for any $\kappa$ and $C$, and is consistent with eqn. (3.19) in [17] valid in the Bianchi VII case.
3. In addition to isometries, we also have the homotheties,

\[ U \rightarrow U, \quad X \rightarrow \lambda X, \quad V \rightarrow \lambda^2 V \quad (\lambda = \text{const.} > 0), \quad (\text{IV.6}) \]

which are conformal transformations with conformal factor \( \lambda^2 \) [16, 33]. (IV.6) is generated by,

\[ Y_h = X^i \frac{\partial}{\partial X^i} + 2V \frac{\partial}{\partial V}. \quad (\text{IV.7}) \]

Now we relate the 6th isometry (IV.5) for Lukash to the “screw”, known before for CPP. The latter combines a \( T \)-translation with a transverse rotation,

\[ Y_s = \partial_T + \frac{\omega}{2} \epsilon_{ij} \xi^i \partial_j. \quad (\text{IV.8}) \]

We first recall how it goes for CPP. Using again the notations \( x \rightarrow \xi = (\xi, \eta), u \rightarrow T, v \rightarrow \nu \), the CPP metric (II.1)-(II.2)-(II.5) is written, in complex notation \( \zeta = (\xi + i \eta)/\sqrt{2} \),

\[ ds_{\text{CPP}}^2 = 2d\zeta d\bar{\zeta} + 2d\nu dT + A_0 \text{Re}[e^{-i\omega T} \zeta^2]dT^2. \quad (\text{IV.9}) \]

Then for any \( \tau = \text{const} \).

\[ T \rightarrow T + \tau, \quad \zeta \rightarrow e^{i\tau \omega/2} \zeta, \quad \nu \rightarrow \nu \quad (\text{IV.10}) \]

leaves the metric (IV.9) invariant [2, 7]. Infinitesimally, this is generated by

\[ Y_s = \partial_T + i\frac{\omega}{2} \left( \zeta \partial_\zeta - \bar{\zeta} \partial_{\bar{\zeta}} \right), \quad (\text{IV.11}) \]

cf. (IV.8). Thus CPP has a 6-parameter isometry group [2, 10, 13] \(^4\), extended to 7-parameter conformal symmetry by the homothety

\[ Y_h = \xi^i \frac{\partial}{\partial \xi^i} + 2\nu \partial_\nu. \quad (\text{IV.12}) \]

Turning to Lukash waves, we assume \( U > 0 \) and start with the real form (II.9) which is conformal to the “perturbed CPP metric” \( d\Sigma^2 \) in (III.2) [or in (III.4)], \( ds_L^2 = e^T d\Sigma^2 \) cf. (III.1). \( d\Sigma^2 \) is readily seen to admit also the “screw” isometry

\[ Y_s = \partial_T - \kappa \epsilon_{ij} \xi^i \partial_j, \quad (\text{IV.13}) \]

cf. (IV.8) with \( \omega \rightarrow -2\kappa \), as well as the homothety (IV.12). These symmetries have to be pulled back to Lukash, which involves the conformal factor \( e^T \) see (III.1). However we find

\[ L_{Y_h}(e^T g_{\mu\nu}(\xi, T, V)) = 2e^T g_{\mu\nu}(\xi, T, V), \quad (\text{IV.14}) \]
\[ L_{Y_s}(e^T g_{\mu\nu}(\xi, T, V)) = e^T g_{\mu\nu}(\xi, T, V) \quad (\text{IV.15}) \]

\(^4\) The half-angle rotation (II.15) reduces (IV.10) to a mere \( T \)-translation — it “unscrews the screw”. 
which are now both conformal. But combining them as,

\[ Y_\kappa = Y_s - \frac{1}{2} Y_h = \partial_T - V \partial_V - \left( \frac{1}{2} \xi^j + \kappa \epsilon_{ij} \xi^i \right) \frac{\partial}{\partial \xi^j}, \tag{IV.16} \]

the conformal factors compensate, providing us finally with an \textit{isometry} \( Y_\kappa \),

\[ L_{Y_\kappa} \left( e^T g_{\mu \nu}(\xi, T, V) \right) = 0. \tag{IV.17} \]

Then returning to our original Brinkmann coordinates by (II.13), the Lukash isometry (IV.5) and the homothety (IV.7) are recovered,

\[ Y_\kappa = U \partial_U - V \partial_V - \kappa \epsilon_{ij} X^i \partial_j, \tag{IV.18} \]
\[ Y_h = X^i \frac{\partial}{\partial X^i} + 2V \frac{\partial}{\partial V}. \tag{IV.19} \]

We note for completeness that the “expanding screw” (IV.4)–(IV.5) can also be obtained along the same lines as for the CPP screw (IV.10) : the complex forms (II.6)–(II.7) are manifestly invariant under the “expanding screw” transformation

\[ U \rightarrow e^\tau U, \quad V \rightarrow e^{-\tau} V, \quad Z \rightarrow e^{-i\kappa \tau} Z. \tag{IV.20} \]

V. COORDINATE DEPENDENCE OF BOUNDED/UNBOUNDED PROPERTY

Now, as we realised while answering a question of our referee, we argue that the \textit{very notion of “particle trapping”} (which appears in the title of ref. [1], and also in the previous version of our paper) may be \textit{coordinate dependent} : the motion can appear bounded in one coordinate system and unbounded in another one, as we illustrate it on various examples.

1. Let us first consider the Niederer correspondence between a harmonic oscillator of frequency \( \omega = \text{const.} \) and a free particle [23, 31, 35–37]. The mapping

\[ T = \frac{\tan \omega t}{\omega}, \quad X = \frac{x}{\cos \omega t} \tag{V.1} \]

carries the half oscillator-period \(-\frac{\pi}{2\omega} < t < \frac{\pi}{2\omega}\) into a full free motion with \(-\infty < T < \infty\), whereas the bounded oscillations become unbounded.

2. A similar behavior is observed also for planetary motion with a time-dependent gravitational constant as suggested by Dirac [38], e.g.,

\[ G(t) = G_0 \frac{t_0}{t}. \tag{V.2} \]
For $t$ close to $t_0$, $t \approx t_0(1 + \epsilon)$. The Bargmann manifolds with $G = G_0 = \text{const.}$ and $G = G(t)$, respectively, are conformally related with conformal factor $\Omega^2(t) = \left(\frac{t_0}{t}\right)^2$ [23]. A circular transverse trajectory $\zeta(T) = e^{iT}$ for $G = G_0$ becomes

$$t \to T = -t_0^2/t, \quad Z(t) = \frac{t}{t_0} e^{-it_0^2/t} \approx (1 + \epsilon) e^{-it_0(1-\epsilon)}.$$  \hspace{1cm} (V.3)

Thus for $G(t)$ in (V.2) the orbit of planetary spirals outwards, as shown in FIG. 6: $G(t)$ decreases with increasing $t$. The gravitational pull weakens, the particle escapes and its rotation slows down.

Let us note that the relativistic coordinate transformation (V.3) lifted to Bargmann space transforms the two different underlying non-relativistic systems with $G(t)$ and $G_0$ into each other.

3. Our last example is the motion in the expanding universe as seen by different observers [39]. Since Lukash metric is related with the open Friedmann universe, we consider the flat ($k = 0$) FRW metric

$$ds^2 = -dt^2 + a(t)^2 \left( d\rho^2 + \rho^2 d\Omega^2 \right).$$  \hspace{1cm} (V.4)

With the conformal time

$$dT = \frac{dt}{a(t)},$$  \hspace{1cm} (V.5)

the above metric can be expressed using co-moving coordinates $(T, \rho, \theta, \phi)$,

$$ds^2 = a(T)^2 \left( -dT^2 + d\rho^2 + \rho^2 d\Omega^2 \right).$$  \hspace{1cm} (V.6)

For the scale factor $a(t) = \sqrt{t}$, for example, one gets the radial trajectory $\theta, \phi = \text{const.}$,

$$\rho(T) = \rho_1 + \frac{\alpha}{2} \ln \left( 2T + \sqrt{4T^2 + \alpha^2} \right), \quad \rho_1 = \text{const}.$$  \hspace{1cm} (V.7)
Thus for a co-moving observer staying in the co-moving coordinates \((T, \rho, \theta, \phi)\), the free particle can stay in rest: \(\rho(T) = \rho_1 = \text{const.}\) when \(\alpha = 0\).

Next, we consider global coordinates \((t, R, \theta, \phi)\), with global time \(t\) and position

\[
R(t) = a(t) \rho = \sqrt{t} \rho .
\]  
(V.8)

Then the FRW metric (V.6) can be expressed as

\[
ds^2 = -\left(1 - \frac{R^2}{4t^2}\right) dt^2 - \frac{R}{t} dR dt + dR^2 + R^2 d\Omega^2 \]  
(V.9)

with the radial trajectory becoming

\[
R(t) = \sqrt{t} \left\{ \rho_1 + \frac{\alpha}{2} \ln \left(4\sqrt{t} + 4\sqrt{t + \frac{\alpha^2}{16}}\right) \right\} .
\]  
(V.10)

Thus in the global observer’s coordinates \((t, R, \theta, \phi)\) the motion of the free particle is unbounded: for a global observer in an expanding universe there is no bounded (localized) motion which, however, may exist for a co-moving observer.

We conclude that the motion being bounded or unbounded may depend on the coordinates we choose.

VI. CONCLUSION

Bialynicki-Birula and his collaborators argue that gravitational waves emitted during the merger of a compact binary system may trap particles [1]. Their statement is consistent with our previous study for a CPP wave [7, 8] for which we had found bounded geodesics. An analogy is provided by a rotating saddle [8, 25, 26], illustrated in FIG.7.

Lukash gravitational waves were proposed to study anisotropic models [3, 20, 21]. Their profile in (II.9) is reminiscent of but still different from (in fact more complicated) than that of CPP waves, (II.5). They were studied in [17] along the lines set out by Siklos [20] in the parameter domain (II.38), where they are of Bianchi type VIIh. In this paper we consider instead what happens in the adjacent but different range, (II.27) where problems are solved using different techniques, but, reassuringly, the results agree on the boundary, (II.31), which separates the parameter ranges.

One of our main results here is that the time redefinition (II.13) relates CPP and Lukash waves schematically as,

\[
\text{Lukash} \equiv \text{CPP + linear force term} .
\]  
(VI.1)

Then a sequence of clever transformations carries the time-dependent Sturm-Liouville problem to a system with constant coefficients, (II.16) - (II.17), we solve by chiral decomposition [8, 24]. Our results
FIG. 7: A ball put to the point \( C \) will fall under the slightest perturbation when the saddle is fixed, but its position will be stabilized when the saddle is rotated.

confirm that particles can, in a suitable range of parameters, be trapped by Lukash waves, as shown in figs.2, 3, 4, 5. Bounded geodesics arise when the wave is of Bianchi type VI. The approximations used by Bialynicki-Birula and Charzynski lead to equations similar to our (II.16).

Our findings are exact in two respects:

- We deal with exact plane gravitational waves, and do not use any weak field approximation.
- We solve the equations of motion exactly.

Our time redefinition (II.13) fits into the framework advocated by Gibbons [18]. Introducing new coordinates \((T, \xi)\) by,

\[
U = f(T), \quad X = \left( \frac{df}{dT} \right)^{1/2} \xi
\]

extends the trick used in [34, 35, 37] in \( d = 1 \) space dimension. Completing (II.13) by (III.3) yields a conformal map between the perturbed CPP (II.1)-(II.2)-(II.5) to Lukash, \( ds^2_L \leftrightarrow d\Sigma^2 \), with conformal factor \( \Omega^2(U) = e^{T} \). This explains why (II.13) works: conformally related space-times have, up to reparametrization, identical null geodesics.

As an additional bonus, the CPP ↔ Lukash relation allows us to derive the symmetries of Lukash from those of CPP [2, 7, 9]. The additional 6th isometry (IV.5) we call “screw” is reproduced by following backwards our “road map” outlined in sec. II.B and summarized, again schematically, by

\[
(X, U, V) \leftrightarrow (\xi, T, \nu) \leftrightarrow \left( \frac{\alpha}{\beta}, T, \nu \right).
\]

In our examples in sec.V the respective manifolds are conformally related, — however being
bounded or not is not invariant under a conformal redefinition of time, (VI.2)\(^5\) : it is a coordinate dependent statement.

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