Reliability-based Robust Optimization Design for Rubbing Rotor System

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Abstract. Based on the reliability-based optimization design theory, the reliability sensitivity technique and the robust design method, the reliability-based robust design of rubbing rotor system is extensively discussed and a numerical method for reliability-based robust design is proposed. The reliability sensitivity is added to the reliability-based optimization design model and the reliability-based robust design is described as a multi-objective optimization. On the condition of known first four moments of basic random variables, the respective program can be used to obtain the reliability-based robust design information of rubbing rotor system accurately and quickly using the fourth moment technique. According to the numerical results, the approach proposed is a convenient and practical reliability-based robust design method.

Introduction

Rotating machinery is taking more and more important role in many fields such as power system, transportation, national defense and chemical industry, etc., consequently. The rubbing phenomenon occurs when a rotating element eventually hits a stationary part of the rotating machinery. Increasing the rotor speed and decreasing the radial clearance between the rotating and the non-rotating parts can enhance the performance of the rotating machinery. This leads to an increased risk of rubbing contact [1].

Reliability optimization design is that in the situation of product solutions determined, reliability expected and distribution are taken into reliability design. The results may have larger quality fluctuation. Robust design is to design produce meeting users’ requirements and minimize the quality fluctuation. Reliability design can improve the reliability of the structure by increasing costs, but robust design is to obtain optimal design and reduce the cost by parameter design. For a thorough summary of the research in the reliability-based robust optimization design, see the papers [2–9].

In the paper, the reliability-based robust optimization design method of rubbing rotor system is presented by using the reliability theory, fourth-order moment method and robust design theory. The reliability sensitivity is added to the reliability-based optimization model and the reliability-based robust optimization is described as a multi-objective optimization. The reliability-based robust optimization design makes the reliability of rotor system not sensitive to the change of random parameters. The safety and robust stability of rotor system are improved by the robust design.

Random Response Analyses

The generalized formulate of rotor dynamic model are developed below.

\[ M\ddot{q} + C\dot{q} + (K + Kr)q = F(t) \]  \hspace{1cm} (1)

where the matrices \( M, C, K, Kr \) and \( F \) are respectively the mass, damping, stiffness, stator radial stiffness matrices of the system, and the external force vector.
To derive the matrix equation for nonlinear structural dynamics, the matrices of the both sides of Eq. 1 are expanded about $\overline{B}$ via Taylor series, then equating the terms with the same order, the zeroth, first and second order equation corresponding to Eq. 1 are obtained. Thus, by solving the equation, the mean value and variance of the responses are represented as

$$E(\mathbf{q}) = \overline{q} + \overline{q}^2$$  \hspace{1cm} (2)

$$\text{Var}(\mathbf{q}) = E(\mathbf{q}_1 \otimes \mathbf{q}_1) = \left[ \frac{\partial \overline{q}}{\partial (\text{cs} \mathbf{B})} \right]^{[2]} \text{Var} (\text{cs} \mathbf{B})$$  \hspace{1cm} (3)

$$\text{Tm}(\mathbf{q}) = E(\mathbf{q}_1 \otimes \mathbf{q}_1 \otimes \mathbf{q}_1) = \left[ \frac{\partial \overline{q}}{\partial (\text{cs} \mathbf{B})} \right]^{[3]} \text{Tm} (\text{cs} \mathbf{B})$$  \hspace{1cm} (4)

$$\text{Fm}(\mathbf{q}) = E(\mathbf{q}_1 \otimes \mathbf{q}_1 \otimes \mathbf{q}_1 \otimes \mathbf{q}_1) = \left[ \frac{\partial \overline{q}}{\partial (\text{cs} \mathbf{B})} \right]^{[4]} \text{Fm} (\text{cs} \mathbf{B})$$  \hspace{1cm} (5)

where $q_i = \frac{\partial \overline{q}}{\partial (\text{cs} \mathbf{B})^T}[\text{cs} (\mathbf{B} - \overline{B})]$ \hspace{1cm} $\overline{q}_2 = \frac{1}{2} \frac{\partial^2 \overline{q}}{\partial (\text{cs} \mathbf{B})^{T^2}} [\text{Var} (\text{cs} \mathbf{B})]$

### Reliability Analysis

The reliability of rotor system can be computed by the stress-strength interference theory.

$$R = \int_{g(\delta, r) > 0} f(Z) dz$$  \hspace{1cm} (6)

where $f(Z)$ denotes the joint probability density function of the random response, $\delta$ is the radial clearance between the rotor and the ring. $g(\delta, r)$ defines the state function, representing the safe state and failure state.

$$g(\delta, r) = \delta - r$$  \hspace{1cm} (7)

The first fourth order moments of the state function $g(\delta, r)$ are determined as

$$\begin{align*}
\sigma_{g}^2 &= \text{Var}[g(\delta, r)] = \sigma_{\delta}^2 + \sigma_{r}^2 \\
\mu_{g} &= E[g(\delta, r)] = E(\delta) - E(r) = \mu_{\delta} - \mu_{r} \\
\theta_{g} &= E[g(\delta, r) - \overline{g}(\delta, r)]^3 = \theta_{\delta} - \theta_{r} \\
\eta_{g} &= E[g(\delta, r) - \overline{g}(\delta, r)]^4 = \eta_{\delta} + \eta_{r} + 6 \sigma_{\delta}^2 \sigma_{r}^2
\end{align*}$$  \hspace{1cm} (8)

The reliability index is defined as

$$\beta = \frac{3(\alpha_{4G} - 1) u_{g}/\sigma_{g} + \alpha_{3G} \mu_{g}^2/\sigma_{g}^2 - 1}{\sqrt{9 \alpha_{4G} - 5 \alpha_{3G}^2 - 9)(\alpha_{4G} - 1)}}$$  \hspace{1cm} (9)

where $\alpha_{3G} = \theta_{g}/\sigma_{g}^3, \alpha_{4G} = \eta_{g}/\sigma_{g}^4$ respectively denote the coefficients of skewness and of kurtosis.

The transient reliability of the system is represented as

$$R(\beta) = 1 - \Phi(-\beta)$$  \hspace{1cm} (10)

where $\Phi(\cdot)$ is the standard normal distribution function.

### Reliability Sensitivity Analysis

The reliability sensitivity with respect to the mean value and the variance of the rotor system random parameter are approximately derived as follows:

$$\frac{DR(\beta)}{\partial \overline{B}} = \frac{\partial R(\beta)}{\partial \beta} \frac{\partial \beta}{\partial \overline{B}} \frac{\partial \overline{u}_g}{\partial \overline{u}}$$  \hspace{1cm} (11)

$$\frac{DR(\beta)}{\partial \text{Var}(\mathbf{B})} = \frac{\partial R(\beta)}{\partial \beta} \frac{\partial \sigma_{g}}{\partial \overline{u}} \frac{\partial \sigma_{r}}{\partial \overline{u}}$$  \hspace{1cm} (12)
where \( \frac{\partial R(\beta)}{\partial \beta} = \varphi(\beta) \frac{\partial \beta}{\partial u_g} \frac{\partial u_g}{\partial u_y} \frac{\partial u_y}{\partial \beta} \) and \( \frac{\partial u_y}{\partial \beta} = \frac{u_x}{\partial B} + u_y \frac{\partial u_y}{\partial B} \).

\[
\frac{\partial \sigma_r}{\partial \text{Var}(B)} = \frac{u_x}{\sigma_r} \frac{\partial u_x}{\partial \text{Var}(B)} + \frac{u_y}{\sigma_r} \frac{\partial u_y}{\partial \text{Var}(B)} \frac{\partial \sigma_r}{\partial \sigma_r} = \frac{\sigma_r}{\sigma_r} \\
\frac{\partial \beta}{\partial \sigma_r} = \frac{3(\alpha_{4g} - 1)}{\sigma_r^2} + 2\alpha_{3g} \frac{\sigma_r^2}{\sigma_r^2} + \frac{12\alpha_{3g}}{\sigma_r^2} \frac{\sigma_r^2}{\sigma_r^2} - \frac{3\alpha_{3g}}{\sigma_r^2} \frac{\sigma_r^2}{\sigma_r^2} - 1 \\
\frac{\partial \sigma_g}{\partial \text{Var}(B)} = \frac{3(\alpha_{4g} - 1)}{\sigma_g^2} + \frac{\sigma_r}{\sigma_g} \frac{\partial \sigma_r}{\partial \text{Var}(B)} \\
\frac{\partial \sigma_r}{\partial \sigma_g} = \frac{\sigma_r}{\sigma_g} \\
\frac{\partial \beta}{\partial \sigma_g} = \left( \frac{36\alpha_{4g}}{\sigma_g^2} + \frac{30\alpha_{3g}}{\sigma_g^2} \right) (\alpha_{4g} - 1) - \frac{9\alpha_{4g} - 5\alpha_{3g}}{\sigma_g^2} - 9 \frac{4\alpha_{4g}}{\sigma_g^2} \right) \left( \frac{3(\alpha_{4g} - 1)}{\sigma_g^2} + \alpha_{3g} \frac{\sigma_r^2}{\sigma_g^2} - 1 \right) \\
\frac{\partial \sigma_g}{\partial \text{Var}(B)} = \frac{1}{2\sigma_g} \left[ \frac{\partial u_x}{\partial \text{Var}(B)} \frac{\partial u_x}{\partial \text{Var}(B)} \right] \\
\frac{\partial \sigma_g}{\partial \text{Var}(B)} = \frac{1}{2\sigma_g} \left[ \frac{\partial u_y}{\partial \text{Var}(B)} \frac{\partial u_y}{\partial \text{Var}(B)} \right]
\]

Reliability-based Robust Optimization Design

A typical reliability-based robust design problem can be formulated as the following form

\[
\begin{align*}
\min f(\vec{X}) &= \sum_{k=1}^{n} w_k f_k(\vec{X}) \\
\text{S.t.} & \quad R - R_0 \geq 0 \\
& \quad q_j(\vec{X}) \geq 0 \quad (i=1,\ldots,l) \\
& \quad h_j(\vec{X}) = 0 \quad (j=1,\ldots,m)
\end{align*}
\]

where \( w_k \) is the weighting factor, the value is determined by the importance degree of the sub-objective function[11]. \( f_k(\cdot) \) is the objective function. \( R_0 \) is the given reliability. \( q_j(\cdot), h_j(\cdot) \) is respectively the inequality constrains and equality constrains.

Numerical Example

Consider a one-disk symmetric rotor-bearing system supported by two oil film bearing. \( m \) is the mass of disk. \( k \) is the stiffness of shaft. \( e \) is the damping of shaft. \( e \) is the radial clearance. \( k_r \) is the radial stiffness of stator. Random parameter vector \( B = [k \ c \ e \ k_r]^T \). The mean values of the random parameters are \( k=4.6\times10^3\text{[N/mm]}, c=120\text{[Ns/mm]}, k_r=1.2\times10^5\text{[N/mm]}, e=0.2\text{[mm]}, \) the coefficient of variables are equal to 0.05. \( \omega=350\text{rad/s} \). The first four order moments of the gap \( \delta \) are \( 4\text{[mm]}, 0.2\text{[mm]}, 0.017\text{[mm}^3\text{]}, 0.0025\text{[mm}^4\text{]} \).

![Fig. 1 Reliability curve](image1.png)  ![Fig. 2 DR/Dµe](image2.png)
According to the presented method, the reliability and reliability sensitivity results of the rotor system are showed in Fig. 1-5. It is observed from Fig. 1-5 that the influence of the radial clearance is the most, that of the shaft stiffness and damping are medial, and that of the stator radial stiffness is the least.
The objective function is made up of two functions, \( f_1(x) \) is to make the mass of rotor disk lightest, i.e. the value of \( m \) minimum; \( f_2(x) \) is to make the reliability sensitivity to the random parameter vector \( B \) minimum at the mean value of \( B \). They are respectively represented as
\[
\begin{align*}
  f_1(x) &= m \\
  f_2(x) &= \sqrt{\sum_{i=1}^{n} \left( \frac{\partial R}{\partial b_i} \right)^2}
\end{align*}
\]

The needed reliability \( R_0 = 0.999 \). The constraint condition of robust design is established as follows
\[
R - R_0 \geq 0 \quad 4 \leq m \leq 10
\]

The initial values, \( m = 7 \text{kg} \) is given, and the solution for \( m \) of optimization is \( m = 5.05 \text{kg} \).

According to the results computed by the reliability-based robust design method, the reliability and reliability sensitivity of the rubbing rotor system are showed in Fig.6-10. It can be found that reliability of rubbing rotor system is greater than 0.999, and the reliability sensitivity to the mean value of \( B \) are significantly smaller. That is to say, the reliability and robustness of rotor system are improved.

**Summary**

This paper presents a practical and effective method for reliability-based robust optimization of rubbing rotor system. Techniques from reliability optimization theory, reliability sensitivity technique and the robust design method are employed to systematically resolve the reliability-based robust design problem of rotor system. The presented numerical method provides the theoretic basis for improving the reliability and robustness of rotor system.

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