Dynamical scaling analysis of the optical Hall conductivity in the graphene quantum Hall system with various types of disorder

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Abstract. Dynamical scaling of the optical Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega)$ at the $n = 0$ Landau level in graphene is analyzed for the 2D effective Dirac fermion and honeycomb lattice models with various types of disorder. In the Dirac fermion model with potential disorder, $\sigma_{xy}(\varepsilon_F, \omega)$ obeys a well-defined dynamical scaling, characterized by the localization exponent $\nu$ and the dynamical critical exponent $z$. In sharp distinction, scaling behavior of $\sigma_{xy}(\varepsilon_F, \omega)$ in the honeycomb lattice model with bond disorder (preserving chiral symmetry), becomes anomalous.

1. Motivation

Study of the quantum Hall effect (QHE) has been primarily focused on static properties. Yet, dynamical aspects are expected to display a wealth of interesting phenomena. For the QHE in graphene, investigation of dynamical response is rather scarce. Two of the present authors (in collaboration with Hatsugai), studied the ac Hall effect in the THz frequency regime. The Hall conductance (although not quantized in ac), is shown to display plateau structure resulting from the localization of states. [1] While the localization dominates the QHE already in the static case, here we are talking about the ac Hall conductivity, so that the most reliable way to examine the dynamics of localization is to apply the dynamical scaling argument. [2]

The QHE in graphene is peculiar due to the occurrence of $n = 0$ Landau level (LL) around the Dirac point where electron and hole branches intersect. It has no counterpart in the ordinary QHE\cite{3, 4}, and hence, static and dynamic scaling are of special interest in the $n = 0$ LL. Analysis of localization delocalization transition in QHE has been mostly focused on the scaling behavior of the dc longitudinal conductivity $\sigma_{xx}(\varepsilon_F)$, where $\varepsilon_F$ is the Fermi energy. [5] On the other hand, dynamical scaling properties of the longitudinal ac conductivity $\sigma_{xx}(\varepsilon_F, \omega)$ and ac Hall conductivity $\sigma_{xy}(\varepsilon_F, \omega)$ have not yet been thoroughly addressed, less so in the graphene QHE.

Experimentally, scaling properties of $\sigma_{xx}(\varepsilon_F, \omega)$ was investigated in frequency range $\omega = 0$ [6] up to the GHz regime [7], while the relevant (i.e., $\sim$ cyclotron energy) frequency range is THz. Recent advances in THz measurements (e.g., Faraday rotation in magnetic fields) enable us to study the dynamical response functions \cite{8, 9}. Study of optical properties of graphene develops rather intensely, including experimental transmission spectra\cite{10}, or theoretical examination of the cyclotron emission\cite{11}. Thus, the physics of dynamical scaling in graphene QHE is experimentally accessible in the THz regime.
Theoretically, an intriguing question is how the static Hall conductivity, which may be regarded as a topological quantity\cite{12, 13}, evolves into the optical Hall conductivity, especially in the THz regime where the relevant energy scale is the cyclotron energy.\cite{14, 1, 15} It has recently been shown that the plateau structure of $\sigma_{xy}(\omega)$ is retained in the ac ($\sim$ THz) regime in both the ordinary two-dimensional electron gas and the massless Dirac model, although the plateau height in the ac conductivity deviates from the quantized values.\cite{1} The plateaus remain remarkably robust against disorder, which can be attributed to localization. To gain a deeper understanding of the robust step structures in the ac regime, we need to elucidate the static and dynamical scaling behavior of the response functions near the plateau to plateau transition. For finite frequencies we have a new length scale, $L_{\omega} \sim \omega^{-\frac{1}{2}}$, where $z$ is the dynamical critical exponent. A proper quantity for carrying scaling analysis is the width $W$ of the transition region, which generically depends on $\omega$ through $L_{\omega}$ and on the sample size $L$.

An interest in graphene QHE also comes from the relationship between the chiral symmetry or sublattice symmetry in graphene and the anomalous $n = 0$ LL. If the disorder respects the chiral symmetry which is the case with bond disorder or random magnetic fields, $n = 0$ LL remains anomalously sharp \cite{16}. Hence, for the first time in this paper, we also discuss what happens to the scaling behavior in the honeycomb lattice (tight-binding) model, with bond disorder that preserves chiral symmetry.

2. Formalism
To study graphene QHE we first employ the two-dimensional effective 2D Dirac fermion model in a magnetic field with random potential,

$$H = v_F \sigma \cdot \pi + V(r),$$

(1)

where $\sigma = (\sigma_x, \sigma_y)$ is a two-component vector of Pauli matrices, and $\pi = p + eA$ is the covariant momentum. Disorder is introduced by a random potential $V(r)$ composed of Gaussian scattering centers of range $d$ and amplitude $\pm u$ with a density $n_{\text{imp}}$ placed at randomly chosen positions $R_j$. A measure of disorder (the Landau level broadening) is $\Gamma = 2u[n_{\text{imp}}/2\pi(\ell^2 + d^2)]^{1/2}$, where $\ell$ is the magnetic length and we chose $d = 0.7\ell$.

Diagonalization of the Hamiltonian is carried out within subspace spanned by the five LLs around $n = 0$ LL. The system’s geometry is a square of size $L \times L$ with $L/\ell = 25, 30, 35, 40$. In terms of the eigenfunctions and the energy eigenvalues $\epsilon_a$, the optical Hall conductivity is evaluated in terms of the Kubo formula,

$$\sigma_{xy}(\epsilon_F, \omega) = \frac{i\hbar e^2}{L^2} \sum_{\epsilon_a < \epsilon_F} \sum_{\epsilon_b \geq \epsilon_F} \frac{1}{\epsilon_b - \epsilon_a} \left( \frac{j_{ab}^{\tau \tau} j_{ba}^{\tau \tau}}{\epsilon_b - \epsilon_a - \hbar \omega} - \frac{j_{ab}^{\tau \tau} j_{ba}^{\tau \tau}}{\epsilon_b - \epsilon_a + \hbar \omega} \right),$$

(2)

where $j_{ab}^{\tau \tau}$ is the current matrix element\cite{1}. The conductance is then averaged over a few thousands samples with different disorder potential realizations.

The optical Hall conductivity $\sigma_{xy}(\epsilon_F, \omega)$ as a function of the Fermi energy $\epsilon_F$ and frequency $\omega$ is displayed for the graphene (Fig.1(a)) QHE systems, which clearly exhibits plateau structure discussed in \cite{1}.

3. Dynamical Scaling Analysis
For a finite system size $L$ and a finite frequency $\omega$, dynamical scaling analysis of the optical Hall conductivity $\sigma_{xy}(\epsilon_F, \omega, L)$ and the width $W(\omega, L)$ of the plateau to plateau transition are carried out by inspecting the ac step structure as follows:\cite{1, 2} The steepness of the transition
can be quantified by fitting $\sigma_{xy}(\varepsilon_F,\omega,L)$ around the transition region for a given LL to a smooth function of $\varepsilon_F$. For the transition $\sigma_{xy}/(e^2/h) = -1 \rightarrow 1$ in graphene QHE we chose

$$\sigma_{xy}(\varepsilon_F,\omega,L)_{\text{graphene}} = \tanh \left( \frac{\varepsilon_F - \varepsilon_0}{W(\omega,L)} \right),$$

where $\varepsilon_0$ is a very small shift of the center of the tanh function to achieve a better fit. The quality of fitting of the plateau to plateau transition by the tanh function is quite satisfactory as can be deduced from Fig.1(b).

Dynamical scaling analysis for $\sigma_{xy}(\varepsilon_F,\omega,L)$ is carried out in a similar manner as that for the longitudinal conductivity [17]. In this ansatz the optical Hall conductivity depends on the Fermi energy and frequency only through the ratios $L/\xi$ and $L_\omega/\xi$. The relevant exponents are the localization exponent $\nu$ that describes the localization length $\xi \sim 1/|\varepsilon_F - \varepsilon_c|^{\nu}$ where $\varepsilon_c$ is the critical energy which coincides with the center of the LL, while a frequency dependent length scale $L_\omega$ behaves like $L_\omega \sim 1/\omega^{1/z}$ with the dynamical critical exponent $z$. Then the dynamical scaling ansatz for the optical Hall conductivity reads

$$\sigma_{xy}(\varepsilon_F,\omega,L) = \frac{e^2}{h} F((\varepsilon_F - \varepsilon_c) L^{1/\nu}, \omega L^z),$$

where $F$ is a universal scaling function. This implies that the width of the plateau to plateau transition scales as,

$$W(\omega,L) = L^{-1/\nu} f(\omega L^z),$$

where $f$ is another universal function deduced from $F$. The first factor, which is responsible for the plateau structure in the static QHE, makes the step sharper for larger systems, while the second factor $f$ describes the dynamical behaviour.

Fitting the conductivity with eqn.(3) yields $W$ for each system size $L$ and frequency $\omega$. In Fig.2 we show the scaling of the inverse transition width for the graphene QHE system. If we first inspect the inverse width $1/W(\omega = 0)$ for the static case as a function of $L$, the localization critical exponent is obtained from the expression $\log 1/W(\omega = 0) = 1/\nu \log L + f(0)$, with the result $\nu = 2.1 \pm 0.1$. This agrees with (albeit slightly smaller than) the accepted value of the static critical exponent in the integer 2DEG QHE.
The frequency dependence of the inverse width for a fixed system size $L$ is displayed in (Fig.2(a)). We can immediately notice that there are two regions: In the first one, $1/W$ stays nearly constant up to some critical frequency that depends on the system size $L$, while in the higher-frequency region it begins to decrease monotonically with $\omega$. In the latter range, $1/W$ assumes similar values for all the sample sizes considered here as shown in (Fig.2(a)). Roughly speaking then, in the first region $L < L_\omega$ ($L_\omega$ is a typical distance over which an electron travels during one cycle of the ac field), $W$ is almost independent on $\omega$, while in the second region $L > L_\omega$ $W$ is virtually independent of $L$. Inspecting eqn.5, and assuming a power-law for the scaling function $f$, the $L$-independent inverse width in this region takes a form $1/W(\omega) \propto \omega^{-1/z\nu}$.

Evaluation of the dynamical exponent should be carried out near the critical point at the crossing region where $1/W$ begins to decrease. This happens, typically, for $\omega < 0.002\omega_c$. With a least-square fitting, $\log 1/W(\omega) = \text{const} - \frac{1}{z\nu} \log \omega$, the dynamical critical exponent for the graphene QHE system is obtained as $z = 1.8 \pm 0.2$.

Having derived the static and dynamic exponents, we are in position to look at the scaling plot against the rescaled inverse width against rescaled frequency in Fig.2(b) for graphene QHE. It can be judged that the scaling is obeyed within the numerical accuracy, which indicates that the form of the universal function assumed in eqn(5) is adequate. In the first region (with small $\omega L^z$) we basically have a constant value of $1/W(\omega L^z)$, while in the second region we have monotonic decrease of $1/W$ for increasing $\omega L^z$.

4. Bond disorder, chiral symmetry and anomalous $n=0$ LL
In order to consider the effect of another type of disorder, i.e. bond disorder, we now study the honeycomb tight-binding model with a Hamiltonian,

$$H = \sum_{\langle ij \rangle} (-t + \delta t_{ij}) e^{-2\pi i \theta_{ij}} c_i^\dagger c_j + \text{h.c.}.$$ 

Here $\delta t_{ij}$ is the random component in the transfer energy, for which we assume a spatially correlated disorder to represent the ripples in graphene. The random-bond disorder respects the chiral symmetry[16].

Figure 3 displays $\sigma_{xy}(\varepsilon_F, \omega)$ for a bond disorder $\sqrt{\langle \delta t^2 \rangle} = 0.1t$. While the overall structure of the $\sigma_{xy}(\varepsilon_F, \omega)$ is similar to the case of potential disorder, a closer look at the $n=0$ step (Fig.3(b)) immediately reveals that $\sigma_{xy}(\varepsilon_F, \omega)$ has an anomalously sharp step structure at the
$\sigma_{xy}(\varepsilon_F, \omega)$ plotted against Fermi energy $\varepsilon_F$ and frequency $\omega$ for the honeycomb tight-binding model with random hopping $\sqrt{\langle \delta t^2 \rangle} = 0.1t$ which respects the chiral symmetry. (b) $\sigma_{xy}(\varepsilon_F, \omega)$ with $\omega = 0, 0.5t/\hbar$ is displayed to show the anomalous step structure at $n = 0$ LL in the ac Hall conductivity. The parameters are $\eta/a = 2.0$, $\phi/\phi_0 = 1/18$, $(L_x/2, L_y/2) = (27, 24)$.

$n = 0$ LL. So the step-function like anomaly persists for the optical Hall conductivity as in the dc Hall conductivity for the chiral-symmetric disordered honeycomb system. Indeed, Fig.3(a) shows that $\sigma_{xy}(\varepsilon_F = 0, \omega)$ continues to be a sharp step for all the value of $\omega$, which may be thought of as coming from the sharp (delta-function like) $n = 0$ LL due to the chiral symmetry. This means that $W(\omega, L) = 0$ for all the value of $\omega$ for $n = 0$ LL, which implies that the dynamical scaling becomes ill-defined in this case. So the applicability of the dynamical scaling argument depends on the type of disorder, and the presence or otherwise of the chiral symmetry in the disorder in particular.

To summarize, a dynamical scaling analysis of the optical Hall conductivity in graphene quantum Hall systems has been carried out in the presence of potential disorder, and the effect of bond disorder which is chiral symmetric is also argued in terms of anomalous $n = 0$ LL.

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