All cyclic $p$-roots of index 3, found by symmetry-preserving calculations

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Introduction

When using a Groebner basis to solve the highly symmetric system of algebraic equations defining the cyclic $p$-roots, one has the feeling that much of the advantage of computerized symbolic algebra over hand calculation is lost through the fact that the symmetry is immediately “thrown out” by the calculations. In this paper, the problem of finding (for all relevant primes $p$) all cyclic $p$-roots of index 3 (as defined in Section 1) is treated with the symmetry preserved through the calculations. Once we had found the relevant formulas, using MAPLE and MATHEMATICA, the calculations could even be made by hand. On the other hand, with respect to a straightforward attack with Groebner basis, it is not even clear how this could be organized for a general $p$.

In other terminologies, our results involve listings of all bi-unimodular sequences constant on the cosets of the group $G_0$ of cubic residues, or equivalently all circulant complex Hadamard matrices related to $G_0$ (cf. [3]).

The corresponding problem for bi-unimodular sequences of index 2 was solved by the first named author in [2] and shortly after solved independently by de la Harpe and Jones [8] in the case $p \equiv 1 \pmod{4}$ and by Munemasa and Watatani [11] in the case $p \equiv 3 \pmod{4}$, see also [7], sect. 3.

The organization of the paper should be clear from the section headings with the understanding that “the main problem” refers to simple sequences of index 3 (cf. Definitions 1.2, 1.3, and 1.4).

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1 Notation, definitions, and problem formulation

We begin by quoting from [2] and [3] definitions of and relations between bi-unimodular \( p \)-sequences and cyclic \( p \)-roots for any positive integer \( p \). For any \( p \)-sequence \( x \), that is any sequence \( x = (x_0, \ldots, x_{p-1}) \) of \( p \) complex numbers, define its normalized Fourier transform by
\[
\hat{x}_\nu = \frac{1}{\sqrt{p}} \sum_{j=0}^{p-1} x_j \omega^{j\nu},
\]
where \( \omega = \exp\left(\frac{2\pi i}{p}\right) \). The sequence \( x \) is called unimodular if \( |x_j| = 1 \) for \( j = 0, 1, \ldots, p-1 \), and it is called bi-unimodular if both \( x \) and \( \hat{x} \) are unimodular.

Taking all indices modulo \( p \), we define the periodic autocorrelation coefficients \( \gamma_k \) by
\[
\gamma_k = \sum_{j \equiv k \pmod{p}} \bar{x}_j x_{j+k}.
\] (1.1)

Then, by the Parseval relation and an easy calculation,
\[
\hat{x} \text{ is unimodular } \iff (\gamma_0 = p \text{ and } \gamma_k = 0 \text{ when } k \not\equiv 0 \pmod{p}).
\] (1.2)

We will now express the property of bi-unimodularity with the help of a certain system of algebraic equations. Let \( z = (z_0, \ldots, z_{p-1}) \in \mathbb{C}^p \). We will call \( z \) a "cyclic \( p \)-root", if \( z \) satisfies the following system of \( p \) algebraic equations:
\[
\begin{align*}
    z_0 + z_1 + \cdots + z_{p-1} &= 0, \\
    z_0z_1 + z_1z_2 + \cdots + z_{p-1}z_0 &= 0, \\
    &\vdots \\
    z_0z_1 \cdots z_{p-2} + z_1z_2 \cdots z_{p-1} + \cdots + z_{p-1}z_0 \cdots z_{p-3} &= 0, \\
    z_0z_1 \cdots z_{p-1} &= 1.
\end{align*}
\] (1.3)

(Note that the sums are cyclic and contain just \( p \) terms and are in general not the elementary symmetric functions.) Let now \( x \in \mathbb{C}^p \) and \( z \in \mathbb{C}^p \) be related by
\[
z_j = x_{j+1}/x_j
\] (1.4)
(with \( x_p := x_0 \)). Clearly \( x \) is unimodular iff \( \bar{x}_j = 1/x_j (\forall j) \). In this case, (1.1) for \( k = 1, 2, \ldots, p-1 \) becomes the \( k \)'th equation of (1.3). Let us call \( x \) normalized if \( x_0 = 1 \). Then (1.2) can be expressed as follows:

**Proposition 1.1** A normalized \( x = (1, x_1, x_2, \ldots, x_{p-1}) \) is bi-unimodular if and only if the corresponding \( z \) is a unimodular cyclic \( p \)-root.

In the rest of the paper, \( p \) will be a prime \( \equiv 1 \pmod{6} \), and we will define \( s := (p-1)/3 \). The multiplicative group \( \mathbb{Z}_p^* \) on \( \mathbb{Z}_p \setminus \{0\} \) is cyclic (cf. [3]) and has a unique index-3 subgroup \( G_0 \) (the group of cubic residues modulo \( p \)). Let \( G_1 \) and \( G_2 \) be the other two cosets of \( G_0 \) in \( \mathbb{Z}_p^* \). (The choice of the subscripts 1 and 2 will be specified later.) We will now for \( p \)-sequences define a property of index 3 meaning "taking few values in a way governed by \( G_0 \):
Definition 1.2 We will say that $x \in \mathbb{C}^p$ is simple of index 3, if there are complex numbers, $c_0, c_1,$ and $c_2,$ such that

$$x_j = c_k \text{ when } 0 \neq j \in G_k \quad (k = 0, 1, 2).$$ \hfill (1.5)

Note that we have slightly changed the notation from [2] where index 3 was called “pre-index 3” and where “index 3” excluded the case of index 1. i.e. $c_0 = c_1 = c_2.$

Allowing shifts and multiplication by exponentials in a way familiar in Fourier transform theory, we make the following definition:

Definition 1.3 We will say that $x \in \mathbb{C}^p$ has index 3, if for some fixed elements $r \neq 0$ and $l$ of $\mathbb{Z}_p$ and some simple $y$ of index 3 we have

$$x_j = \omega^{rj} y_{j-l},$$ \hfill (1.6)

which amounts to

$$x_j = \omega^{rj} c_k \text{ when } 0 \neq j - l \in G_k \quad (k = 0, 1, 2).$$ \hfill (1.7)

We will now define simple and general cyclic $\mathbb{Z}_p$-roots of index 3:

Definition 1.4 By a cyclic $\mathbb{Z}_p$-root of index 3 we will mean a cyclic $\mathbb{Z}_p$-root $z$ such that the corresponding $x,$ as defined by (1.4) has index 3. We will also call a cyclic $\mathbb{Z}_p$-root $z$ simple of index 3, if the corresponding $x$ is simple of index 3.

Note that we do not require $x$ (and thus $z$) to be unimodular.

The purpose of the present paper is to find explicitly all cyclic $\mathbb{Z}_p$-roots of index 3 (for every relevant prime $p$) using a method which utilizes the symmetries of the system.

We will now show (following [2]), that if $z$ is a simple cyclic $\mathbb{Z}_p$-root of index 3 and its corresponding $x$ is normalized by $x_0 = 1,$ then the system (1.3) reduces to a system of three equations for $c_0, c_1$ and $c_2.$ (To help the reader, an example is given at the end of the section.) Let $g$ be a generator for $\mathbb{Z}_p^*$, and let $G_0, G_1, G_2$ be the cosets of $G_0,$ numbered in such a way that $G_k = \{g^{k+3m}; m = 0, 1, \ldots, s - 1\}.$ For every $i$ and $k = 0, 1, 2,$ and every $d = 1, \ldots, p - 1,$ we define the transition number $n_{ik}(d)$ as the number of elements $b$ in $\{1, 2, \ldots, (p - 1)\}$ for which $b \in G_i$ and $b + d \in G_k.$ (Subscripts are taken modulo 3. We do not count $b = p - d).$ Suppose now that $d \in G_a,$ i.e. that $d \equiv g^{a+3m}$ for some $m$ (congruences are modulo $p$). For each $b$ which contributes to $n_{ik}(1),$ we have $b \equiv g^{i+3u}$ and $b + 1 \equiv g^{k+3v}$ for some $u$ and $v.$ Thus, from $d(b + 1) = db + d$ we get

$$g^{k+a+3(m+v)} \equiv g^{i+a+3(m+u)} + d.$$ \hfill (1.8)

Writing $n_{ik}$ instead of $n_{ik}(1),$ we thus get

$$n_{i+a,k+a}(d) = n_{ik}.$$ \hfill (1.9)

Let us now consider a simple cyclic $\mathbb{Z}_p$-root of index 3, and let the corresponding $x$ be normalized by $x_0 = 1$ and have values given by (1.5). Fix $d$ such that $d \in G_a,$ and
consider the individual products in the degree \( d \) equation of (1.3). These products will take the values \((c_k + a)/(c_i + a)\) with the frequency \( n_{i+k+a}(d) \), the value \( c_a/1 \) once (since \( p - 1 \in G_0 \)), and the value \( 1/c_a \) once (since \( p - d \in G_a \)). Thus (1.9) implies that all equations whose degrees \( d \) belong to the same coset \( G_a \), are identical, and the system (1.3) consists of the following 3 equations (where \( n_{ik} = n_{ik}(1) \) are the transition numbers, and the \( c \) subscripts are counted modulo 3):

\[
\frac{c_a}{1} + \sum_{k=0}^{2} \sum_{i=0}^{2} n_{ik} \frac{c_k + a}{c_i + a} = 0, \quad (a = 0, 1, 2). \tag{1.10}
\]

We will now return to the choice of the subscripts in \( G_1 \) and \( G_2 \). Without loss of generality, we can (and do in fact from now on) suppose that \( n_{02} > n_{01} \). (1.11)

In fact, we must have \( n_{02} \neq n_{01} \) (see Corollary 2.3), and if \( n_{02} < n_{01} \), we replace the generator \( g \) by \( g' := g^{2+3j} \), for some \( j \) such that \( 2 + 3j \) is relatively prime to \( p - 1 \). Since \( g \in G_1 \) and \( g' \in G_2 \), this will interchange \( G_1 \) and \( G_2 \), and we have arrived at (1.11). Finally, we will give the promised example: Let \( p = 13 \), and take \( g = 2 \) or 11. Then \( G_0 = \{1, 5, 8, 12\} \), \( G_1 = \{2, 3, 10, 11\} \), \( G_2 = \{4, 6, 7, 9\} \), and we will have \( n_{00} = 0 \), \( n_{01} = n_{10} = n_{12} = n_{21} = n_{22} = 1 \), and \( n_{02} = n_{20} = n_{11} = 2 \).

2 Number theoretic results used

In this section we give some relations between the transition numbers \( n_{ik} \) defined in (1.9) and appearing in (1.10). These relations will lead to explicit formulas for the \( n_{ik} \).

The mapping \( b \rightarrow p - b \) from \( \mathbb{Z}_p \) to \( \mathbb{Z}_p \) will leave each one of the sets \( G_i \) invariant and thus we have

\[
n_{ij} = n_{ji}, \quad i, j = 0, 1, 2. \tag{2.1}
\]

Moreover
\[
\sum_{j=0}^{2} n_{ij} = \sharp((G_i \setminus \{p - 1\})), \quad \text{and thus (recall that we have defined } s = \frac{p-1}{3}\)
\]

\[
\sum_{j=0}^{2} n_{0j} = s - 1, \quad \sum_{j=0}^{2} n_{1j} = \sum_{j=0}^{2} n_{2j} = s. \tag{2.2}
\]

We will get one more linear relation between the \( n_{ik} \) in the following way: By (1.9), all \( n_{01}(d) \) with \( d \) belonging to the same \( G_a \) are equal. Thus, since \( \sharp(G_0) = \sharp(G_1) = \sharp(G_2) = s \), we get \( s \cdot s = \sum_{d=1}^{p-1} n_{01}(d) = \sum_{a=0}^{2} s \cdot n_{-a,1-a,} \), which becomes

\[
n_{01} + n_{12} + n_{20} = s. \tag{2.3}
\]
With the help of (2.1), (2.2) and (2.3) we can express all our nine transition numbers \( n_{ik} \) in terms of \( n_{01} \) and \( n_{02} \):

\[
\begin{align*}
  n_{00} &= s - 1 - n_{01} - n_{02}, \\
  n_{11} &= n_{20} = n_{02}, \\
  n_{22} &= n_{10} = n_{01}, \\
  n_{12} &= n_{21} = s - n_{01} - n_{02}.
\end{align*}
\]  

(2.4)

These relations are given in [2] and also in [5], Exercise 4.29 (d). There is, however, one further equation satisfied by the transition numbers. We first state this equation in terms of \( n_{12} \), \( n_{01} \) and \( n_{02} \):

**Proposition 2.1** Let \( p \) be a prime \( \equiv 1 \pmod{6} \), and let \( n_{12} \), \( n_{01} \) and \( n_{02} \) be the transition numbers defined in Section 1. Then

\[
n_{01}n_{02} + n_{01}n_{12} + n_{02}n_{12} = n_{01}^2 + n_{02}^2 + n_{12}^2 - n_{12}.
\]

We have proved this result by establishing the following explicit formulas for the convolutions \( F * G \) (defined by \( (F * G)(a) = \sum_{b \in \mathbb{Z}_p} F(a-b)G(b) \)) of certain complex-valued functions \( F \) and \( G \) on \( \mathbb{Z}_p \). Let \( \Gamma_j \) be the characteristic functions \( \chi_{G_j} \) of \( G_j \) \((j = 0, 1, 2)\), and let \( I = \chi_{\{0\}} \). Then, (with indices taken modulo 3):

\[
\begin{align*}
  \Gamma_i * \Gamma_i &= n_{i,i} \Gamma_0 + n_{i+2,i+2} \Gamma_1 + n_{i+1,i+1} \Gamma_2 + sI, \\
  \Gamma_i * \Gamma_{i+1} &= n_{i,i+1} \Gamma_0 + n_{i+2,i} \Gamma_1 + n_{i+1,i+2} \Gamma_2.
\end{align*}
\]

Our original proof of Proposition 2.1 used these formulas and the commutativity and associativity of the convolution. Also, the reader of [5] is encouraged in Exercise 4.29 (e) to prove this proposition. But it turns out that Proposition 2.1 is just a reformulation of a theorem of Gauss (in *Disquisitiones*, Article 358), which we give in a form a little more precise than in [10] or [13] or [5]:

**Proposition 2.2** Let \( p \) be a prime \( \equiv 1 \pmod{6} \), and let \( n_{12} \), \( n_{01} \) and \( n_{02} \) be the transition numbers defined in Section 1. Then there are integers \( A \) and \( B \) such that

\[
4p = A^2 + 27B^2.
\]

If we require that \( A \equiv 1 \pmod{3} \) and \( B > 0 \) (which is always possible and which we always do), then \( A \) and \( B \) are unique, and we have

\[
A = 9n_{12} - p - 1 \quad \text{and} \quad B = |n_{02} - n_{01}|.
\]

Since \( 4p \) is not a square, we must have \( B \neq 0 \), and hence we get the following corollary, which we needed at the end of Section 1:

**Corollary 2.3** Let \( p \) be a prime \( \equiv 1 \pmod{6} \), and let \( n_{01} \) and \( n_{02} \) be the transition numbers defined in Section 1. Then \( n_{01} \neq n_{02} \).
Recall that we have in fact chosen \( G_1 \) and \( G_2 \) in such a way that \( n_{02} > n_{01} \). Since \( B > 0 \), we thus have

\[
A = 9n_{12} - p - 1 \quad \text{and} \quad B = n_{02} - n_{01}.
\]

Solving the linear system given by (2.4) and (2.5) for \( n_{ik} \), we have proved the following corollary of Proposition 2.2:

**Corollary 2.4** Let \( p \) be a prime \( \equiv 1 \pmod{6} \), let \( n_{ik} \) be the transition numbers defined in Section 1, and let \( A \) and \( B \) be the numbers given in Proposition 2.2. Then

\[
\begin{align*}
n_{12} = n_{21} &= \frac{1}{9}(p + A + 1), \\
n_{02} = n_{20} = n_{11} &= \frac{1}{15}(2p - A + 9B - 4), \\
n_{01} = n_{10} = n_{22} &= \frac{1}{15}(2p - A - 9B - 4), \\
n_{00} + n_{11} + n_{22} &= \frac{1}{3}(p - 4).
\end{align*}
\]  

**Proof of Proposition 2.1:** Starting from Proposition 2.2 and replacing \( A \) and \( B \) by the expressions given there and then replacing \( p \) by the expression \( p = 3(n_{01} + n_{12} + n_{20}) + 1 \) from (2.3) we get

\[
0 = A^2 + 27B^2 - 4p = -36(n_{01}n_{02} + n_{01}n_{12} + n_{02}n_{12} - n_{01}^2 - n_{02}^2 - n_{12}^2 + n_{12})
\]

which completes the proof.

**Proof of Proposition 2.2:** The calculations needed are given very explicitly in [13]. In fact the theorem of Gauss stated there in Section IV.2 is our Proposition 2.2 except that the statement of the theorem does not contain the value of \( B \) and for \( A \) gives the value \( M_p - p - 1 \), where \( M_p \) is the number of solutions \( (x, y, z) \) in \( \mathbb{Z}_p^3 \) of \( x^3 + y^3 + z^3 = 0 \) in the projective sense. In the proof of the theorem, the formula \( mB = [STT] - [STS] \) is given where \( m \) is our \( s \), where \( R \) is our \( G_0 \), \( S \) and \( T \) are our \( G_1 \) and \( G_2 \) (in some order), and where finally the symbol \([XYZ]\) is defined for subsets \( X, Y, Z \) of \( \mathbb{Z}_p \) as the number of triples \( (x, y, z) \) such that \( x \in X \), \( y \in Y \), and \( z \in Z \) and \( x + y + z = 0 \). In the course of the proof it is also shown that \( mM_p = 9[RTS] \). Thus all that remains for us to have a proof of Proposition 2.2 is to check that \( [G_1G_2G_2] - [G_1G_2G_1] = s(n_2 - n_1) \) and \( [G_0G_2G_1] = sn \).

We write \( x + y + z = 0 \) as \( x + y = -z \), and since \( G_2 = -G_2 \), we have that

\[
[G_{i+2}G_{k+2}] = \sum_{y \in G_2} n_{i+2,k+2}(y) = sn_{ik},
\]

where we have used (1.9) with \( a = 2 \) and \( d = y \). Thus \( [G_1G_2G_2] - [G_1G_2G_1] = s(n_{20} - n_{22}) \) and \( [G_0G_2G_1] = sn_{12} \), and the result follows from (2.4), which completes the proof.
3 Reduction of the main problem

Let $p$ be a prime of the form $p = 3s + 1$, $s \in \mathbb{N}$ and let

$$4p = A^2 + 27B^2$$

be the Gauss decomposition of $4p$, i.e. $A, B \in \mathbb{Z}$, $A \equiv 1 \pmod{3}$ and $B > 0$ (cf. Proposition 2.2). Our main problem is to find all simple cyclic $p$-roots of index 3, i.e. to solve the set of equations (cf. 1.10 and Corollary 2.4)

\[
\begin{align*}
    c_0 + \frac{1}{c_0} &= -\frac{p-4}{3} - n_{12} \left( \frac{c_2}{c_1} + \frac{c_1}{c_2} \right) - n_{02} \left( \frac{c_0}{c_2} + \frac{c_2}{c_0} \right) - n_{01} \left( \frac{c_1}{c_0} + \frac{c_0}{c_1} \right) \\
    c_1 + \frac{1}{c_1} &= -\frac{p-4}{3} - n_{12} \left( \frac{c_0}{c_2} + \frac{c_2}{c_0} \right) - n_{02} \left( \frac{c_1}{c_0} + \frac{c_0}{c_1} \right) - n_{01} \left( \frac{c_2}{c_1} + \frac{c_1}{c_2} \right) \\
    c_2 + \frac{1}{c_2} &= -\frac{p-4}{3} - n_{12} \left( \frac{c_0}{c_0} + \frac{c_0}{c_1} \right) - n_{02} \left( \frac{c_1}{c_0} + \frac{c_0}{c_1} \right) - n_{01} \left( \frac{c_2}{c_1} + \frac{c_1}{c_2} \right)
\end{align*}
\]

with

\[
\begin{align*}
    n_{12} &= \frac{p + A + 1}{9}, & n_{02} &= \frac{2p - A + 9B - 4}{18}, & n_{01} &= \frac{2p - A - 9B - 4}{18}.
\end{align*}
\]

Proposition 3.1 Assume $(c_0, c_1, c_2)$ is a solution to (3.1). Then the numbers

\[
h_j = \frac{c_{j+2}}{c_{j+1}} + \frac{c_{j+1}}{c_{j+2}}, \quad j = 0, 1, 2,
\]

(index counted modulo 3) are up to a cyclic permutation given by

\[
h_j = \xi_1 + \eta_1 \cos \left( \frac{2\pi}{3} j \right), \quad j = 0, 1, 2,
\]

where $\theta = \frac{1}{3} \arccos \left( \frac{4}{2\sqrt{p}} \right)$ and the pair $(\xi_1, \eta_1)$ is one of the following 4 pairs:

\[
\begin{align*}
    \left\{ \begin{array}{l}
        \xi_1^{(0)} = 2 \\
        \eta_1^{(0)} = 0,
    \end{array} \right.
\end{align*}
\]

\[
\begin{align*}
    \left\{ \begin{array}{l}
        \xi_1^{(1)} = -\frac{p^2 - 6p + 2A}{p^2 - 3p - A} \\
        \eta_1^{(1)} = \frac{6\sqrt{p(p-4)}}{p^2 - 3p - A},
    \end{array} \right.
\end{align*}
\]

\[
\begin{align*}
    \left\{ \begin{array}{l}
        \xi_1^{(2)} = -\frac{-2pA - 9p - 4 + 3\sqrt{p(p+4A+16)}}{2(pA+3p-1)} \\
        \eta_1^{(2)} = \frac{3\sqrt{p(p+2)} - 3p\sqrt{p+4A+16}}{pA+3p-1},
    \end{array} \right.
\end{align*}
\]

\[
\begin{align*}
    \left\{ \begin{array}{l}
        \xi_1^{(3)} = -\frac{-2pA - 9p - 4 - 3\sqrt{p(p+4A+16)}}{2(pA+3p-1)} \\
        \eta_1^{(3)} = \frac{3\sqrt{p(p+2)} + 3p\sqrt{p+4A+16}}{pA+3p-1}.
    \end{array} \right.
\end{align*}
\]
Remark 3.2 a) Let us first check that all the above formulas give well-defined real numbers: Since $p > 4$ and $|A| < 2\sqrt{p}$ we have
\[
p^2 - 3p - A > p^2 - 3p - 2\sqrt{p} = \sqrt{p}(\sqrt{p} - 2)(\sqrt{p} + 1)^2 > 0.
\]
Moreover,
\[
p + 4A + 16 > p - 8\sqrt{p} + 16 = (\sqrt{p} - 4)^2 \geq 0
\]
and since $A \equiv 1 \pmod{3}$, we have $|A + 3| \geq 1$. Hence
\[
|pA + 3p - 1| \geq |(A + 3)p| - 1 \geq p - 1 > 0.
\]

b) We do not prove in this section that all four cases (3.5)–(3.8) actually occur. However this will follow from the proof of Theorem 4.1 in the next section.

Proof of Proposition 3.1: To make our method of proof more transparent, we first consider the case $p = 7$. In this case $A = B = 1$, $n_{12} = n_{02} = 1$, and $n_{01} = 0$. Put
\[
f_j = c_j + \frac{1}{c_j} \quad \text{and} \quad h_j = \frac{c_{j+2}}{c_{j+1}} + \frac{c_{j+1}}{c_{j+2}}.
\]
Then (3.1) becomes
\[
\begin{align*}
f_0 &= -1 - h_0 - h_1 \\
f_1 &= -1 - h_1 - h_2 \\
f_2 &= -1 - h_2 - h_0.
\end{align*}
\]
(3.9)

Consider now the matrix
\[
K = \begin{bmatrix}
2 & f_0 & f_1 & f_2 \\
f_0 & 2 & h_2 & h_1 \\
f_1 & h_2 & 2 & h_0 \\
f_2 & h_1 & h_0 & 2
\end{bmatrix}.
\]

Since
\[
K = \begin{bmatrix}
1 \\
c_0 \\
c_1 \\
c_2
\end{bmatrix} \begin{bmatrix}
1, \frac{1}{c_0}, \frac{1}{c_1}, \frac{1}{c_2}
\end{bmatrix} + \begin{bmatrix}
1 \\
\frac{1}{c_0} \\
\frac{1}{c_1} \\
\frac{1}{c_2}
\end{bmatrix} \begin{bmatrix}
1, c_0, c_1, c_2
\end{bmatrix},
\]
we get (considering $K$ as an operator on column vectors)
\[
\text{range}(K) = \text{span} \left\{ \begin{bmatrix}
1 \\
c_0 \\
c_1 \\
c_2
\end{bmatrix}, \begin{bmatrix}
1 \\
\frac{1}{c_0} \\
\frac{1}{c_1} \\
\frac{1}{c_2}
\end{bmatrix} \right\}.
\]
Hence rank($K$) ≤ 2, and thus all $3 \times 3$ submatrices of $K$ have determinant = 0.
Let $L = (\ell_{ij})_{i,j=1}^4$ be the co-factor matrix of $K$, i.e.

$$\ell_{ij} = (-1)^{i+j} \det(K_{ij}),$$

where $K_{ij}$ is the $3 \times 3$ minor of $K$ obtained by erasing the $i$'th row and the $j$'th column.

Put

$$\begin{aligned}
p_1 &= \ell_{11} \\
p_2 &= \ell_{12} + \ell_{13} + \ell_{14} \\
p_3 &= \ell_{22} + \ell_{33} + \ell_{44} \\
p_4 &= \ell_{23} + \ell_{34} + \ell_{42}.
\end{aligned}$$

(3.10)

Since $\ell_{ij} = 0$ for all $i$ and $j$, we have in particular

$$p_1 = p_2 = p_3 = p_4 = 0.$$ 

This gives four equations of degree three in $(f_0, f_1, f_2, h_0, h_1, h_2)$, but taking (3.9) into account, we can consider $p_1, p_2, p_3, p_4$ as polynomials in $(h_0, h_1, h_2)$ only, namely

$$\begin{aligned}
p_1 &= 8 - 2(h_0^2 + h_1^2 + h_2^2) + 2h_0h_1h_2 \\
p_2 &= 12 - 4(h_0 + h_1 + h_2) - 3(h_0^2 + h_1^2 + h_2^2) - 4(h_0h_1 + h_1h_2 + h_2h_0) \\
&\quad - (h_0^3 + h_1^3 + h_2^3) + 2(h_0h_1^2 + h_1h_2^2 + h_2h_0^2) + 3h_0h_1h_2, \\
p_3 &= 12 - 14(h_0 + h_1 + h_2) - 8(h_0^2h_1^2 + h_2^2) - 2(h_0h_1 + h_1h_2 + h_2h_0) \\
&\quad + 2(h_0h_1^2 + h_1h_2^2 + h_2h_0^2) + 4(h_0^2h_1 + h_1^2h_2 + h_2^2h_0)6h_0h_1h_2, \\
p_4 &= 6 + 3(h_0 + h_1 + h_2) + (h_0^2 + h_1^2 + h_2^2) + 5(h_0h_1 + h_1h_2 + h_2h_0) \\
&\quad - 2(h_0h_1^2 + h_1h_2^2 + h_2h_0^2) - 6h_0h_1h_2.
\end{aligned}$$

Let $s_1, s_2, s_3$ denote the three elementary symmetric polynomials in $h_0, h_1, h_2$:

$$\begin{aligned}
s_1 &= h_0 + h_1 + h_2 \\
s_2 &= h_0h_1 + h_1h_2 + h_2h_0 \\
s_3 &= h_0h_1h_2
\end{aligned}$$

(3.11)

and let $a$ denote the antisymmetric polynomial:

$$a = (h_0 - h_1)(h_1 - h_2)(h_2 - h_0).$$

(3.12)

Then,

$$\begin{aligned}
h_0^2 + h_1^2 + h_2^2 &= s_1^2 - 2s_2 \\
h_0^3 + h_1^3 + h_2^3 &= s_1^3 - 3s_1s_2 + 3s_3 \\
h_0h_1^2 + h_1h_2^2 + h_2h_0^2 &= \frac{1}{2}(s_1s_2 - 3s_3 + a) \\
h_0^2h_1 + h_1^2h_2 + h_2^2h_0 &= \frac{1}{2}(s_1s_2 - 3s_3 - a).
\end{aligned}$$
Hence $p_1, p_2, p_3, p_4$ can be expressed as polynomials in $s_1, s_2, s_3$ and $a$. One gets

\[
\begin{align*}
p_1 &= (8 - 2s_1^2) + 4s_2 + 2s_3 \\
p_2 &= (12 + 4s_1 - 3s_1^2 - s_1^3) + (2 + 4s_1)s_2 - 3s_3 + a \\
p_3 &= (12 - 14s_1 - 8s_1^2) + (14 + 3s_1)s_2 - 3s_3 - a \\
p_4 &= (6 + 3s_1 + s_1^2) + (3 - s_1)s_2 - 3s_3 - a.
\end{align*}
\]

Therefore the equations $p_1 = p_2 = p_3 = p_4 = 0$ can be rewritten in the form

\[
\begin{bmatrix}
8 - 2s_1^2 & 4 & 2 & 0 \\
12 + 4s_1 - 3s_1^2 - s_1^3 & 2 + 4s_1 & -3 & 1 \\
12 - 14s_1 - 8s_1^2 & 14 + 3s_1 & -3 & -1 \\
6 + 3s_1 + s_1^2 & 3 - s_1 & -3 & -1
\end{bmatrix}
\begin{bmatrix}
1 \\
s_2 \\
s_3 \\
a
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\tag{3.13}
\]

A necessary condition for the existence of solutions to this system of equations is that the determinant of the coefficient matrix $M$ is 0. One finds

\[
\det(M) = 8(s_1 - 6)(s_1 + 1)(s_1^2 + 9s_1 + 15).
\]

Thus $s_1$ must be one of the 4 numbers

\[
s_1^{(0)} = 6, \quad s_1^{(1)} = -1, \quad s_1^{(2)} = \frac{-9 + \sqrt{21}}{2} \text{ or } s_1^{(3)} = \frac{-9 - \sqrt{21}}{2}.
\]

Let $M^{(i)}$ be the matrix obtained by substituting $s_1 = s_1^{(i)}$ in $M$ ($i = 0, 1, 2, 3$). It is easy to compute the kernel for $M^{(i)}$, $i = 0, 1, 2, 3$. One finds $\dim(\ker(M^{(i)})) = 1$ in all cases, and (for convenience writing vectors in row form)

\[
\begin{align*}
\ker(M^{(0)}) &= \text{span}\{[1, 12, 8, 0]\} \\
\ker(M^{(1)}) &= \text{span}\{[1, -2, 1, 7]\} \\
\ker(M^{(2)}) &= \text{span}\{[1, -9 + \sqrt{21}, \frac{79 - 17\sqrt{21}}{2}, -189 + 42\sqrt{21}]\} \\
\ker(M^{(3)}) &= \text{span}\{[1, -9 - \sqrt{21}, \frac{79 + 17\sqrt{21}}{2}, -189 + 42\sqrt{21}]\}.
\end{align*}
\]

Hence there are exactly 4 solutions $(s_1, s_2, s_3, a)$ to (3.13):

\[
\begin{align*}
(s_1^{(0)} &= 6, \quad s_2^{(0)} = 12, \quad s_3^{(0)} = 8, \quad a^{(0)} = 0 ) \\
(s_1^{(1)} &= -1, \quad s_2^{(1)} = -2, \quad s_3^{(1)} = 1, \quad a^{(1)} = -7 ) \\
(s_1^{(2)} &= \frac{-9 + \sqrt{21}}{2}, \quad s_2^{(2)} = -9 + 2\sqrt{21}, \quad s_3^{(2)} = \frac{79 - 17\sqrt{21}}{2}, \quad a^{(2)} = -189 + 42\sqrt{21} ) \\
(s_1^{(3)} &= \frac{-9 - \sqrt{21}}{2}, \quad s_2^{(3)} = -9 - 2\sqrt{21}, \quad s_3^{(3)} = \frac{79 + 17\sqrt{21}}{2}, \quad a^{(3)} = -189 - 42\sqrt{21} )
\end{align*}
\tag{3.14}
\]
However, there is a hidden relation between $s_1, s_2, s_3$ and $a$, namely $a^2$ is a symmetric polynomial in $(h_0, h_1, h_2)$ and can therefore be expressed in terms of $s_1, s_2$ and $s_3$. One finds

$$a^2 = s_1^2 s_2^2 - 4 s_1^3 s_3 - 4 s_2^3 + 18 s_1 s_2 s_3 - 27 s_3^2. \tag{3.15}$$

It is elementary to check that this equality holds for each of the four sets $(s_1^{(i)}, s_2^{(i)}, s_3^{(i)}, a^{(i)})$ found above.

We must now in each case find $h_0, h_1, h_2$ by solving the 4 equations:

$$\begin{align*}
& h_0 + h_1 + h_2 = s_1^{(i)} \\
& h_0 h_1 + h_1 h_2 + h_2 h_0 = s_2^{(i)} \\
& h_0 h_1 h_2 = s_3^{(i)} \\
& (h_0 - h_1)(h_1 - h_2)(h_2 - h_0) = a^{(i)}.
\end{align*} \tag{3.16}$$

The solutions $(h_0, h_1, h_2)$ to the first 3 equations in (3.16) are exactly the three roots (in arbitrary order) to the polynomial

$$h^3 - s_1^{(i)} h^2 + s_2^{(i)} h - s_3^{(i)}.$$

Since (3.15) holds in each of the four cases $i = 0, 1, 2, 3$, we have

$$(h_0 - h_1)(h_1 - h_2)(h_2 - h_0) = \pm a^{(i)}.$$  

Hence the 4th coordinate in the solution to the equations (3.13) only determines the cyclic order of the three numbers $(h_0, h_1, h_2)$. For $i = 0$, (3.17) becomes

$$h^3 - 6h^2 + 12h - 8 = 0.$$  

Hence $h_0 = h_1 = h_2 = 2$ which corresponds to case (3.5) in Proposition 3.1.

In the cases $i = 1, 2, 3$ we solve (3.17) by the classical trigonometric formula in the form of Lemma 3.5 below, where we use (3.30) when $a < 0$ and (3.32) when $a > 0$. This will give the correct cyclic order of $(h_0, h_1, h_2)$. Note that Lemma 3.5 can be applied because in all 3 cases $(i = 1, 2, 3) s_1, s_2, s_3$ and $a$ are all real (being solutions to the real linear system (3.13)) and thus $a^2 > 0$, which by (3.15) means that $s_1^2 s_2^2 - 4 s_1^3 s_3 - 4 s_2^3 + 18 s_1 s_2 s_3 - 27 s_3^2 = a^2 > 0$. Hence, up to cyclic permutation of $(h_0, h_1, h_2)$ we have

$$h_j = \xi_1 + \eta_1 \cos \left( \theta - \frac{2\pi}{3} j \right)$$

where

$$\begin{align*}
\xi_1 &= \frac{1}{3} s_1 \\
\eta_1 &= -\text{sign}(a) \cdot \frac{2}{3}(s_1^2 - 3 s_2)^{\frac{1}{2}} \\
\theta &= \frac{1}{3} \text{Arccos}( -\text{sign}(a) \frac{2 s_1^3 - 9 s_1 s_2 + 27 s_3}{2(s_1^2 - 3 s_2)^{\frac{3}{2}}}).
\end{align*}$$

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It turns out that \( \theta^{(i)} = \frac{1}{3} \arccos \left( \frac{1}{2\sqrt{7}} \right) \) in all three cases \((i = 2, 3, 4)\), while

\[
\begin{align*}
(\xi^{(1)}_1, \eta^{(1)}_1) &= \left(-\frac{1}{3}, \frac{2\sqrt{7}}{3}\right) \\
(\xi^{(2)}_1, \eta^{(2)}_1) &= \left(-\frac{3}{2}, \frac{\sqrt{21}}{6}, \sqrt{7} - \frac{7}{3}\sqrt{3}\right) \\
(\xi^{(3)}_1, \eta^{(3)}_1) &= \left(-\frac{3}{2}, \frac{\sqrt{21}}{6}, \sqrt{7} + \frac{7}{3}\sqrt{3}\right).
\end{align*}
\]

This gives case (3.6), (3.7), and (3.8) respectively in Proposition 3.1 in the case \(p = 7\).

Consider now a general prime \(p, p \equiv 1 \pmod{3}\). This case is mathematically no more difficult than the case \(p = 7\) but a computer algebra language as MAPLE or MATHEMATICA is helpful for bookkeeping purpose. Using (3.2) and (3.1) instead of (3.9), the polynomials (3.11) again becomes polynomials in \(s_1, s_2, s_3, a\), namely

\[
\begin{bmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4
\end{bmatrix} =
\begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} \\
m_{21} & m_{22} & m_{23} & m_{24} \\
m_{31} & m_{32} & m_{33} & m_{34} \\
m_{41} & m_{42} & m_{43} & m_{44}
\end{bmatrix}
\begin{bmatrix}
1 \\
s_2 \\
s_3 \\
a
\end{bmatrix}
\]

(3.18)

where the \(m_{ij}\):s are the following 16 polynomials in \(s_1\):

\[
\begin{align*}
m_{11} &= -2s_1^2 + 8 \\
m_{12} &= 4 \\
m_{13} &= 2 \\
m_{14} &= 0 \\
m_{21} &= \frac{1}{6}(A + p + 1)s_1^3 + \frac{1}{6}(2A - 7p + 20)s_1^2 + 4s_1 + (4p - 16) \\
m_{22} &= \frac{4}{6}(A + p + 1)s_1 + \frac{1}{3}(4p - 2A - 20) \\
m_{23} &= -A - 2 \\
m_{24} &= B \\
m_{31} &= \frac{2}{81}(p^2 - pA - 7p + a^2 + 2A + A)s_1^3 - \frac{2}{27}(pA + 12p + 17)s_1^2 \\
&\quad - \frac{2}{3}p(p - 4)s_1 + \frac{4}{3}(2 - p^2 + 8p) \\
m_{32} &= \frac{1}{27}(-6A - 12p - 3A^3 - 8 + 2pA)s_1 + \frac{2}{9}(6p + pA + 14) \\
m_{33} &= \frac{1}{3}(2A + A^2 - 2p + 2) \\
m_{34} &= -\frac{1}{3}(A + 2)B \\
m_{41} &= \frac{1}{8}(7A - p^2 + 4p + 2A^2 + pA + 5)s_1^3 + \frac{1}{27}(-6A + 6p + pA - 16)s_1 \\
&\quad + \frac{8}{9}(p^2 - 4p - 12)s_1 + \frac{2}{3}(p^2 - 8p + 16) \\
m_{42} &= -\frac{1}{27}(9A + 3A^2 + pA + 8)s_1 + \frac{1}{6}(6A - pA + 28) \\
m_{43} &= \frac{1}{3}(2A + A^2 - 2p + 2) \\
m_{44} &= -\frac{1}{3}(A + 2)B.
\end{align*}
\]
Since \( p_1 = p_2 = p_3 = p_4 = 0 \), we must have \( \det M = 0 \) where \( M = (m_{ij})_{i,j=1}^{4} \). One finds
\[
\det M = \frac{8B}{729}(s_1 - 6)q(s_1)r(s_1),
\]
where
\[
\begin{cases}
q(s_1) &= \ (p^2 - 3p - A)s_1 + (6A + 3p^2 - 18p) \\
r(s_1) &= \ (pA + 3p - 1)s_1^2 + (6pA + 27p + 12)s_1 + (9pA + 54p - 36).
\end{cases}
\]

(3.19)

It is interesting that if \( \det M \) is considered as a polynomial in the independent variables \( s_1, p, A, B \), forgetting the relation \( 4p = A^2 + 27B^2 \), we will get an irreducible cubic polynomial instead of \( q(s_1)r(s_1) \). By Remark 3.2, \( p^2 - 3p - A \neq 0 \) and \( pA + 3p - 1 \neq 0 \), so the equation \( \det(M) = 0 \) has exactly 4 solutions (counted with multiplicity), namely
\[
\begin{align*}
    s^{(0)}_1 &= 6 \\
    s^{(1)}_1 &= \frac{18p - 3p^2 - 6A}{p^2 - 3p - A} \\
    s^{(2)}_1 &= \frac{-6pA - 27p - 12 + 9\sqrt{p(p + 4) + 16}}{2(p^2 + 3p - 1)} \\
    s^{(3)}_1 &= \frac{-6pA - 27p - 12 - 9\sqrt{p(p + 4) + 16}}{2(p^2 + 3p - 1)}.
\end{align*}
\]

(3.20)

Let \( M^{(i)} \) be the \( 4 \times 4 \)-matrix obtained by substituting \( s_1 = s^{(i)}_1 \) in \( M \). We next compute the kernel for \( M^{(i)} \) in each of the four cases. Let \( M^{(i)}_{j,k} \) be the \( 3 \times 3 \) minor of \( M^{(i)} \) obtained by erasing the \( j \)’th row and the \( k \)’th column of \( M^{(i)} \). Then
\[
\det(M^{(i)}_{11}) = -\frac{2B}{27}(A + p + 1)((pA + A + 4p)s^{(i)}_1 + 3pA - 6A + 12p).
\]

In particular
\[
\begin{align*}
    \det(M^{(0)}_{11}) &= -\frac{2B}{3}p(p + A + 1)(A + 4), \\
    \det(M^{(1)}_{11}) &= -\frac{2B}{3}p(p + A + 1)(4p - A^2), \\
    \det(M^{(2)}_{11}) \cdot \det(M^{(3)}_{11}) &= \frac{-4B^2(p + A + 1)^2(A + 4)(4p - A^2)}{9(pA + 3p - 1)}.
\end{align*}
\]

Since \( A \equiv 1 \pmod{3} \), we have \( A + 4 \neq 0 \). Moreover \( 4p - A^2 = 27B^2 > 0 \) and \( p + A + 1 > (\sqrt{p} - 1)^2 \geq 0 \). Hence \( \det(M^{(i)}_{11}) \neq 0 \) in all 4 cases. Together with \( \det(M^{(i)}) = 0 \), this shows that for all \( i, s, M^{(i)} \) has rank 3 and thus
\[
\dim(\ker(M^{(i)})) = 1, \quad i = 0, 1, 2, 3.
\]

Hence in each case \((i = 0, 1, 2, 3), s^{(i)}_2, s^{(i)}_3 \) and \( a^{(i)} \) are uniquely determined by (3.18). Applying Cramer’s rule to the last three equations in (3.18) we get
\[
\begin{align*}
    s^{(i)}_2 &= \frac{\det(M^{(i)}_{12})}{\det(M^{(i)}_{11})}, \quad s^{(i)}_3 = \frac{\det(M^{(i)}_{13})}{\det(M^{(i)}_{11})}, \quad a^{(i)} = -\frac{\det(M^{(i)}_{14})}{\det(M^{(i)}_{11})}.
\end{align*}
\]
where \( \frac{\pi}{2} \). 

For \( i = 0 \), \( (s_1^{(0)}, s_2^{(0)}, s_3^{(0)}, a^{(0)}) = (6, 12, 8, 0) \) as in the case \( p = 7 \) and for \( i = 1 \) we have

\[
\begin{align*}
\begin{cases}
    s_1^{(1)} &= \frac{18p^2 - 36A}{p^2 - 3p - A} \\
    s_2^{(1)} &= 34p^2 A - 24p A + 4A^2 - p^4 - 21p^3 + 108p^2 - 144p \\
    s_3^{(1)} &= 20p^2 A - 96p A + 8A^2 - p^4 + 4p^3 - 360p^2 + 864p \\
    a^{(1)} &= -\frac{729p(p - 1)^3 B}{p^2 - 3p - A}.
\end{cases}
\] (3.21)
\]

For \( i = 2, 3 \), it is more convenient to express the solutions in terms of \( u = \sqrt{p} \) and \( v = \sqrt{p + 4A + 16} \). We get

\[
\begin{align*}
\begin{cases}
    s_1^{(2)} &= -3u^2 + uv - 4 \\
    s_2^{(2)} &= 3(4uv + 6u - 4)(u^2 + uv - 4) \\
    s_3^{(2)} &= \frac{u^4 + 2uv - 176u^2 + uv^2 + 30uv - 32}{u^2 + uv + 2} \\
    a^{(2)} &= 5832 \frac{B u^2}{(u^2 + uv + 2)^3}.
\end{cases}
\] (3.22)
\]

and

\[
\begin{align*}
\begin{cases}
    s_1^{(3)} &= -3u^2 - uv - 4 \\
    s_2^{(3)} &= 3(4uv + 6u - 4)(u^2 - uv - 4) \\
    s_3^{(3)} &= \frac{u^4 - 2uv - 176u^2 + uv^2 - 30uv - 32}{u^2 - uv + 2} \\
    a^{(3)} &= 5832 \frac{B u^2}{(u^2 - uv + 2)^3}.
\end{cases}
\] (3.23)
\]

Note that all the numbers are well-defined because by Remark 3.2, \( p^2 - 3p - A > 0 \), \( p + 4A + 16 > 0 \) and

\[
(u^2 + uv + 2)(u^2 - uv + 2) = -4(pA + 3p - 1) \neq 0.
\]

For \( i = 0 \), we get as for \( p = 7 \) that \( h_0 = h_1 = h_2 = 2 \) which corresponds to (3.5) in Proposition 3.1. It is easy to check that the identity (3.15) is satisfied for the above sets \( (s_1^{(i)}, s_2^{(i)}, s_3^{(i)}, a^{(i)}) \), so as in the case \( p = 7 \) we can determine \( h_0, h_1, h_2 \) by Lemma 3.5 where we use (3.30) when \( a^{(i)} < 0 \) and (3.32), when \( a^{(i)} > 0 \) to obtain the correct cyclic ordering. Note that \( a^{(1)} < 0, a^{(2)} > 0 \) and \( \text{sign}(a^{(3)}) = \text{sign}(u^2 - uv + 2) = -\text{sign}(pA + 3p - 1) \). We obtain

\[
h_j = \xi_1^{(i)} + \eta_1^{(i)} \cos \left( 3\theta^{(i)} - \frac{2\pi}{3} j \right), \quad j = 0, 1, 2,
\]

where \( \theta^{(i)} = \frac{1}{3} \text{Arccos} \left( \frac{A}{\sqrt{p}} \right) \) in all three cases \( (i = 2, 3, 4) \), while

\[
\begin{align*}
\begin{cases}
    (\xi_1^{(1)}, \eta_1^{(1)}) &= \left( \frac{p^2 + 6A + 24}{p^2 - 3p - A}, \frac{6\sqrt{p}(p - 1)}{p^2 - 3p - A} \right) \\
    (\xi_1^{(2)}, \eta_1^{(2)}) &= \left( \frac{u^2 + uv - 4}{u^2 + uv + 2}, \frac{-12u}{u^2 + uv + 2} \right) \\
    (\xi_1^{(3)}, \eta_1^{(3)}) &= \left( \frac{u^2 - uv - 4}{u^2 - uv + 2}, \frac{-12u}{u^2 - uv + 2} \right).
\end{cases}
\] (3.24)
\]
Using \( u = \sqrt{p}, \ v = \sqrt{p + 4A + 16}, \) we get (3.6), (3.7) and (3.8) in Proposition 3.1. This completes the proof of Proposition 3.1.

**Remark 3.3** It easily follows from the proof that if \( c = (c_0, c_1, c_2) \) is a solution to the system (3.1) and two \( c_i \) are equal, then they are all equal. In fact, if e.g. \( c_1 = c_2, \) then with \( h \) as in (3.3) we get \( h_1 = h_2, \) which leads to \( a = 0. \) But since \( B \neq 0, \) it follows from (3.21), (3.22), and (3.23) that \( a \neq 0 \) in all cases except the case where all \( h_i = 2. \)

**Remark 3.4** (a) At a first glance it is surprising that the angle \( \theta \) in the solution formula above is the same for \( i = 1, 2, 3. \) However, this fact has a fairly simple explanation: Computing the linear combination

\[
(p - 1)p_1 - 2p_2 - p_3 - 2p_4
\]

of the polynomials \( p_i = p_i(s_1, s_2, s_3, a) \) given by (3.18) one gets

\[
\frac{4p - A^2}{27}(2s_1^3 - 9s_1s_2 + 27s_3) + ABa.
\]

Since \( p_1 = p_2 = p_3 = p_4 = 0 \) and \( B^2 = \frac{4p - A^2}{27}, \) we have the following identity

\[
B(2s_1^3 - 9s_1s_2 + 27s_3) + Aa = 0. \tag{3.25}
\]

But if \( h_j = \xi_1 + \eta_1 \cos(\theta - \frac{2\pi}{3}j), \) \( j = 0, 1, 2, \) and \( s_1, s_2, s_3, a \) are defined as in (3.11) and (3.12) one finds

\[
2s_1^3 - 9s_1s_2 + 27s_3 = \frac{27}{4} \eta_1^3 \cos 3\theta
\]

and

\[
a = -\frac{3\sqrt{3}}{4} \eta_1^3 \sin 3\theta.
\]

Hence, when \( \eta_1 \neq 0, \) (3.25) is equivalent to

\[
3\sqrt{3}B \cos 3\theta - A \sin 3\theta = 0.
\]

This has a unique solution \( \theta \in (0, \frac{\pi}{3}), \) namely

\[
\theta = \frac{1}{3} \text{Arccot} \left( \frac{A}{3\sqrt{3}B} \right) = \frac{1}{3} \text{Arccos} \left( \frac{A}{2\sqrt{p}} \right).
\]

(b) It is interesting to compare the solutions in Proposition 3.1 with the Gaussian cubic sum

\[
G = \sum_{j=0}^{p-1} e^{2\pi j^3/p}.
\]

It is known that (cf. [9] or Section IV.2 of [13]) that for \( p \) prime, \( p \equiv 1 \pmod{3}, \) \( G \) is a solution to the cubic equation

\[
x^3 - 3px - pA = 0.
\]
This equation has the 3 solutions

\[ x_j = 2\sqrt{p} \cos \left( \theta - \frac{2\pi j}{3} \right), \quad j = 0, 1, 2 \]

where \( \theta = \frac{1}{3} \arccos \left( \frac{A}{2\sqrt{p}} \right) \) as in Proposition 3.1.

It is a famous problem (the Problem of Kummer) to decide for each \( p \) which of the 3 solutions is equal to \( G \) (cf. [9] and Section 9.12 of [10] or Section IV.2 of [13]).

We conclude this section by stating as a lemma the classical trigonometric solution of a cubic equation with 3 real roots. For completeness, we recall an elementary proof (cf. e.g. §47 of [6])

**Lemma 3.5** Consider the cubic equation

\[ h^3 - s_1 h^2 + s_2 h - s_3 = 0 \]  \hspace{1cm} (3.26)

and assume that \( s_1, s_2, s_3 \in \mathbb{R} \) and

\[ s_1^2 s_2^2 - 4 s_1^3 s_3 - 4 s_2^3 + 18 s_1 s_2 s_3 - 27 s_3^2 > 0. \]  \hspace{1cm} (3.27)

Then

\[ s_1^2 - 3 s_2 > 0, \]  \hspace{1cm} (3.28)

\[ |2s_1^3 - 9s_1 s_2 + 27 s_3| < 2(s_1^2 - 3s_2)^{\frac{3}{2}}. \]  \hspace{1cm} (3.29)

Moreover (3.26) has 3 different real solutions. Listed in decreasing order \( h_0 > h_1 > h_2 \), the solutions are

\[ h_j = \frac{s_1}{3} + \frac{2}{3}(s_1^2 - 3s_2)^{\frac{1}{2}} \cos \left( \theta - \frac{2\pi j}{3} \right), \quad j = 0, 1, 2, \]  \hspace{1cm} (3.30)

where

\[ \theta = \frac{1}{3} \arccos \left( \frac{2s_1^3 - 9s_1 s_2 + 27 s_3}{2(s_1^2 - 3s_2)^{\frac{3}{2}}} \right), \]  \hspace{1cm} (3.31)

and listed in increasing order \( h_0' < h_1' < h_2' \), the solutions are

\[ h_j' = \frac{s_1}{3} + \frac{2}{3}(s_1^2 - 3s_2)^{\frac{1}{2}} \cos \left( \theta' - \frac{2\pi j}{3} \right), \quad j = 0, 1, 2 \]  \hspace{1cm} (3.32)

where

\[ \theta' = \frac{1}{3} \arccos \left( -\frac{2s_1^3 - 9s_1 s_2 + 27 s_3}{2(s_1^2 - 3s_2)^{\frac{3}{2}}} \right). \]  \hspace{1cm} (3.33)
Proof: Let \( h_0, h_1, h_2 \) be the solutions to (3.26) and define \( a \) by (3.12). Then from (3.15) follows that (the discriminant) \( a^2 > 0 \). Hence \( h_0, h_1, h_2 \) are real and different (since if e.g. \( h_1 = c + id, h_2 = c - id \) with \( d \neq 0 \) and \( h_0 \in \mathbb{R} \) we would have \( a^2 = -4d^2((h_0 - c)^2 + d^2)^2 < 0 \), whereas e.g. \( h_0 = h_1 \) would imply that \( a = 0 \) ). Substituting \( h = u + \frac{a}{3} \) in equation (3.26) we get

\[ u^3 + ru + q = 0 \]  

(3.34)

where \( r = -\frac{1}{3}(s_1^2 - 3s_2) \) and \( q = -\frac{1}{27}(2s_1^3 - 9s_1s_2 + 27s_3) \). Applying (3.15) with \( s_1 = 0, s_2 = r, s_3 = -q \) we get \( a^2 = -4r^3 - 27q^2 \). Since the transformation from \( h \) to \( u \) is a translation, the discriminant does not change and thus (3.27) becomes \( -4r^3 - 27q^2 > 0 \). Thus \( r < 0 \), which is (3.28). Next we consider (3.29). Squaring this relation and introducing \( r \) and \( q \) we give it the form \(| -27q|^2 < 4(-3r)^3 \), which we have just seen is true.

Taking \( u = mz \) in (3.34) we get the equation

\[ z^3 + \frac{r}{m^2}z + \frac{q}{m^3} = 0. \]  

(3.35)

We now start from the trigonometric identity \( \cos 3\theta = 4\cos^3\theta - 3\cos\theta \). Writing \( z = \cos\theta \) we give it the form

\[ z^3 - \frac{3}{4}z - \frac{1}{4}\cos 3\theta = 0, \]  

(3.36)

which clearly has the solutions

\[ z_j = \cos\left(\theta - \frac{2\pi j}{3}\right), \quad j = 0, 1, 2. \]  

(3.37)

We see that equation (3.35) will be identical with (3.36) if \( m = \sqrt{-\frac{4r}{3}} \) and \( \cos 3\theta = -\frac{27q}{\sqrt{-27r^3}} \). Returning to the variable \( h \) we see that (3.37) will lead to the solutions (3.30) to the original equation (3.26) and that we can choose \( \theta \) as in (3.31).

Since \( \theta \in (0, \frac{\pi}{3}) \), we have

\[-1 < \cos\left(\theta - \frac{4\pi}{3}\right) < -\frac{1}{2} < \cos\left(\theta - \frac{2\pi}{3}\right) < \frac{1}{2} < \cos\theta < 1.\]

Hence \( h_0 > h_1 > h_2 \). Finally, note that with the notation from (3.33) we have \( \theta' = \frac{\pi}{3} - \theta \) and therefore \( h_0' = h_2, \ h_1 = h_1', \ h_2' = h_0 \) and thus \( h_0' < h_1' < h_2' \).

4 Solution of the main problem

Theorem 4.1 The set of equations (3.1) has exactly 20 solutions in \( \mathbb{C}^3 \). The first two solutions are the “c-solutions”:

\[ c_0 = c_1 = c_2 = \frac{2 - p \pm \sqrt{p(p - 4)}}{2}. \]  

(4.1)
The remaining 18 solutions can be obtained from the three solutions listed below by the six transformations

\[
\begin{align*}
(c_0, c_1, c_2) & \rightarrow (c_k, c_{k+1}, c_{k+2}) \\
(c_0, c_1, c_2) & \rightarrow \left( \frac{1}{c_k}, \frac{1}{c_{k+1}}, \frac{1}{c_{k+2}} \right)
\end{align*}
\]

where \( k = 0, 1, 2 \) and indices are computed modulo 3. Put \( u = \sqrt{p}, \ v = \sqrt{p+4A+16} \) and \( \theta = \frac{1}{3} \arccos \left( \frac{A}{2\sqrt{p}} \right) \). The three solutions are \( c^{(i)} = (c_0^{(i)}, c_1^{(i)}, c_2^{(i)}), \ i = 1, 2, 3 \), where

\[
c_j^{(i)} = \alpha^{(i)} + \beta^{(i)} \cos(\theta - \frac{2\pi}{3} j) + \gamma^{(i)} \sin(\theta - \frac{2\pi}{3} j) \quad (4.2)
\]

and

\[
\begin{align*}
\alpha^{(1)} & = \frac{1}{2} pA^2 - 2p - 2A + i \frac{3\sqrt{3}}{2} \sqrt{p} \sqrt{p - 4B} \\
\beta^{(1)} & = -\frac{1}{2} \sqrt{p} (p - 4) (A + 2) + i \frac{3\sqrt{3}}{2} \sqrt{p} (p - 4) (p - 2) B \\
\gamma^{(1)} & = -\frac{3\sqrt{3}}{2} \sqrt{p} (p - 4) B + i \frac{\sqrt{p} (p - 4) (pA - 2p - 2A)}{2 p^2 - 3p - A} \\
\end{align*}
\]

\[
\begin{align*}
\alpha^{(2)} & = -\frac{1}{2} \frac{u^2 - uv - 4}{u^2 + uv + 2} - \frac{i}{2} \frac{u \sqrt{4 + u - v} \sqrt{4 - u + v}}{u^2 + uv + 2} \\
\beta^{(2)} & = -\frac{A + 2}{u^2 + uv + 2} + i \frac{(u^2 + uv + 4) \sqrt{u + v} \sqrt{4 - u + v}}{u^2 + uv + 2} \\
\gamma^{(2)} & = \frac{3\sqrt{3}Bu}{u^2 + uv + 2} + i \frac{(u^2 - uv - 4) \sqrt{u + v} \sqrt{4 - u + v}}{u^2 + uv + 2} \\
\end{align*}
\]

\[
\begin{align*}
\alpha^{(3)} & = -\frac{1}{2} \frac{u^2 + uv - 4}{u^2 - uv + 2} - \frac{i}{2} \frac{u \sqrt{u + v} \sqrt{4 - u + v}}{u^2 - uv + 2} \\
\beta^{(3)} & = -\frac{A + 2}{u^2 - uv + 2} - i \frac{(u^2 - uv + 4) \sqrt{u + v} \sqrt{4 + u - v}}{u^2 - uv + 2} \\
\gamma^{(3)} & = \frac{3\sqrt{3}Bu}{u^2 - uv + 2} + i \frac{(u^2 + uv - 4) \sqrt{u + v} \sqrt{4 - u + v}}{u^2 - uv + 2} \\
\end{align*}
\]

The solutions (4.2) given by (4.3) and (4.4) are unimodular while the \( \epsilon \)-solutions and the solution (4.2) given by (4.5) are real. Hence of the 20 solutions 12 are unimodular and 8 are real.

Remark 4.2
(a) Of course the choice of a “canonical” solution among six possible ones is arbitrary. Our choice is motivated by a wish to give the asymptotic results in Section 6 a simple form.
(b) It follows from the proof of Theorem 4.1 that the transformation

\[
(c_0, c_1, c_2) \rightarrow \left( \frac{1}{c_0}, \frac{1}{c_1}, \frac{1}{c_2} \right)
\]

can be obtained just by changing the sign of the second term in the above formulas for \( \alpha^{(i)} \), \( \beta^{(i)} \), and \( \gamma^{(i)} \).
(c) Since \( u = \sqrt{p} \) and \( v = \sqrt{p + 4A + 16} \) and \(|A| < 2\sqrt{p}\), we have

\[ |u - 4| < v < u + 4 \]

which means that the numbers \( u, v, 4 \) can be the lengths of the three sides in a non-degenerate triangle. Hence the 4 square roots

\[ \sqrt{u + v + 4}, \quad \sqrt{u + v - 4}, \quad \sqrt{4 + u - v}, \quad \sqrt{4 - u + v} \]

are well defined and strictly positive. Note also that

\[ A = \frac{v^2 - u^2 - 16}{4} \quad (4.6) \]

and

\[ B = \frac{1}{3\sqrt{3}} \sqrt{4p - A^2} = \frac{\sqrt{u + v + 4} \sqrt{u + v - 4} \sqrt{4 + u - v} \sqrt{4 - u + v}}{12\sqrt{3}}. \quad (4.7) \]

The proof of Theorem 4.1 relies on Proposition 3.1 and the following 3 lemmas:

**Lemma 4.3** Let \( a_0, a_1, a_2 \in \mathbb{C} \) and let \( \theta \in \mathbb{R} \). Then there are unique numbers \( \rho, \sigma, \tau \in \mathbb{C} \) such that

\[ a_j = \rho + \sigma \cos \left( \theta - \frac{2\pi}{3} j \right) + \tau \sin \left( \theta - \frac{2\pi}{3} j \right), \quad j = 0, 1, 2. \]

**Proof:** By an elementary computation one finds

\[
\det \begin{pmatrix}
1 & \cos \theta & \sin \theta \\
1 & \cos(\theta - \frac{2\pi}{3}) & \sin(\theta - \frac{2\pi}{3}) \\
1 & \cos(\theta - \frac{4\pi}{3}) & \sin(\theta - \frac{4\pi}{3})
\end{pmatrix} = -\frac{3\sqrt{3}}{2}.
\]

In particular the determinant is non-zero, which proves Lemma 4.3.

**Lemma 4.4** Let \( \theta \in \mathbb{R} \) and let \( \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in \mathbb{C} \), and put

\[
c_j = \frac{\alpha_1 + \alpha_2}{2} + \frac{\beta_1 + \beta_2}{2} \cos \left( \theta - \frac{2\pi}{3} j \right) + \frac{\gamma_1 + \gamma_2}{2} \sin \left( \theta - \frac{2\pi}{3} j \right)
\]

\[
\tilde{c}_j = \frac{\alpha_1 - \alpha_2}{2} + \frac{\beta_1 - \beta_2}{2} \cos \left( \theta - \frac{2\pi}{3} j \right) + \frac{\gamma_1 - \gamma_2}{2} \sin \left( \theta - \frac{2\pi}{3} j \right)
\]

for \( j = 0, 1, 2 \). Then the following two conditions are equivalent

(i) \( c_0 \tilde{c}_0 = c_1 \tilde{c}_1 = c_2 \tilde{c}_2 = 1 \),

(ii) \( t_1 = t_2 = t_3 = 0 \),

where

\[
t_1 = \left( \alpha_1^2 - \alpha_2^2 \right) + \frac{1}{2} (\beta_1^2 - \beta_2^2) + \frac{1}{2} (\gamma_1^2 - \gamma_2^2) - 4,
\]

\[
t_2 = 2 (\alpha_1 \beta_1 - \alpha_2 \beta_2) + \frac{1}{2} (\beta_1^2 - \beta_2^2 - \gamma_1^2 + \gamma_2^2) \cos 3\theta + (\beta_1 \gamma_1 - \beta_2 \gamma_2) \sin 3\theta,
\]

\[
t_3 = 2 (\alpha_1 \gamma_1 - \alpha_2 \gamma_2) + \frac{1}{2} (\beta_1^2 - \beta_2^2 - \gamma_1^2 + \gamma_2^2) \sin 3\theta - (\beta_1 \gamma_1 - \beta_2 \gamma_2) \cos 3\theta.
\]
Proof: Put

\[ f_j = c_j + \tilde{c}_j = \alpha_1 + \beta_1 \cos \left( \theta - \frac{2\pi}{3} j \right) + \gamma_1 \sin \left( \theta - \frac{2\pi}{3} j \right), \]
\[ g_j = c_j - \tilde{c}_j = \alpha_2 + \beta_2 \cos \left( \theta - \frac{2\pi}{3} j \right) + \gamma_2 \sin \left( \theta - \frac{2\pi}{3} j \right). \]

Then (i) is equivalent to

\[ f_j^2 - g_j^2 = 4, \quad j = 0, 1, 2. \]

By expressing \( \cos^2 \varphi, \sin^2 \varphi, \cos \varphi \sin \varphi \) in terms of \( \cos 2\varphi, \sin 2\varphi \) (\( \varphi = \theta - \frac{2\pi}{3} j \)) one finds

\[ f_j^2 = \left( \alpha_1^2 + \frac{\beta_1^2 + \gamma_1^2}{2} \right) + 2\alpha_1\beta_1 \cos \left( \theta - \frac{2\pi}{3} j \right) + 2\alpha_1\gamma_1 \sin \left( \theta - \frac{2\pi}{3} j \right) + \frac{\beta_1^2 - \gamma_1^2}{2} \cos \left( 2\theta - \frac{4\pi}{3} j \right) + \beta_1 \gamma_1 \sin \left( 2\theta - \frac{4\pi}{3} j \right). \]

Using \( \frac{4\pi}{3} j \equiv -\frac{2\pi}{3} j \pmod{2\pi} \) one gets

\[
\begin{align*}
\cos \left( 2\theta - \frac{4\pi}{3} j \right) &= \cos 3\theta \cos \left( \theta - \frac{2\pi}{3} j \right) + \sin 3\theta \sin \left( \theta - \frac{2\pi}{3} j \right), \\
\sin \left( 2\theta - \frac{4\pi}{3} j \right) &= \sin 3\theta \cos \left( \theta - \frac{2\pi}{3} j \right) - \cos 3\theta \sin \left( \theta - \frac{2\pi}{3} j \right).
\end{align*}
\]

Hence

\[ f_j^2 = \rho_1 + \sigma_1 \cos \left( \theta - \frac{2\pi}{3} j \right) + \tau_1 \sin \left( \theta - \frac{2\pi}{3} j \right), \quad (4.11) \]

where

\[
\begin{align*}
\rho_1 &= \alpha_1^2 + \frac{1}{2}(\beta_1^2 + \gamma_1^2) \\
\sigma_1 &= 2\alpha_1\beta_1 + \frac{\beta_1^2 - \gamma_1^2}{2} \cos 3\theta + \beta_1 \gamma_1 \sin 3\theta \\
\tau_1 &= 2\alpha_1\gamma_1 + \frac{\beta_1^2 - \gamma_1^2}{2} \sin 3\theta - \beta_1 \gamma_1 \cos 3\theta.
\end{align*}
\]

Similarly

\[ g_j^2 = \rho_2 + \sigma_2 \cos \left( \theta - \frac{2\pi}{3} j \right) + \tau_2 \sin \left( \theta - \frac{2\pi}{3} j \right), \]

where

\[
\begin{align*}
\rho_2 &= \alpha_2^2 + \frac{1}{2}(\beta_2^2 + \gamma_2^2) \\
\sigma_2 &= 2\alpha_2\beta_2 + 2\alpha_2\beta_2 \cos 3\theta + \beta_2 \gamma_2 \sin 3\theta \\
\tau_2 &= 2\alpha_2\gamma_2 + 2\alpha_2\beta_2 \sin 3\theta - \beta_2 \gamma_2 \cos 3\theta.
\end{align*}
\]

Since the coefficients in the decomposition

\[ f_j^2 - g_j^2 = (\rho_1 - \rho_2) + (\sigma_1 - \sigma_2) \cos \left( \theta - \frac{2\pi}{3} j \right) + (\tau_1 - \tau_2) \sin \left( \theta - \frac{2\pi}{3} j \right) \]

are unique by Lemma 4.3, we have \( f_j^2 - g_j^2 = 4, \ j = 0, 1, 2, \) if and only if

\[ \rho_1 - \rho_2 = 4, \quad \sigma_1 - \sigma_2 = 0, \quad \text{and} \quad \tau_1 - \tau_2 = 0. \]

This proves Lemma 4.4.
Lemma 4.5 Let \( \theta \in \mathbb{R} \) and let \( c_0, c_1, c_2 \in \mathbb{C} \setminus \{0\} \). Put
\[
  f_j = c_j + \frac{1}{c_j}, \quad g_j = c_j - \frac{1}{c_j}, \quad h_j = \frac{c_{j+2}}{c_{j+1}} + \frac{c_{j+1}}{c_{j+2}}, \quad k_j = \frac{c_{j+2}}{c_{j+1}} - \frac{c_{j+1}}{c_{j+2}},
\]
where \( j = 0, 1, 2 \) (counted modulo 3).

Let moreover \( \alpha_\nu, \beta_\nu, \gamma_\nu, \xi_\nu, \eta_\nu, \zeta_\nu \) (\( \nu = 1, 2 \)) be the coefficients in the decompositions
\[
\begin{cases}
  f_j &= \alpha_1 + \beta_1 \cos(\theta - \frac{2\pi}{3} j) + \gamma_1 \sin(\theta - \frac{2\pi}{3} j) \\
  g_j &= \alpha_2 + \beta_2 \cos(\theta - \frac{2\pi}{3} j) + \gamma_2 \sin(\theta - \frac{2\pi}{3} j)
\end{cases} \quad (4.12)
\]
\[
\begin{cases}
  h_j &= \xi_1 + \eta_1 \cos(\theta - \frac{2\pi}{3} j) + \zeta_1 \sin(\theta - \frac{2\pi}{3} j) \\
  k_j &= \xi_2 + \eta_2 \cos(\theta - \frac{2\pi}{3} j) + \zeta_2 \sin(\theta - \frac{2\pi}{3} j)
\end{cases} \quad (4.13)
\]
Then
\[
\begin{cases}
  \xi_1 &= \frac{3}{2} (\alpha_1^2 - \alpha_2^2) - 1 \\
  \eta_1 &= -\frac{3}{2} (\alpha_1 \beta_1 - \alpha_2 \beta_2) \\
  \zeta_1 &= -\frac{3}{2} (\alpha_1 \gamma_1 - \alpha_2 \gamma_2)
\end{cases} \quad (4.14)
\]
and
\[
\begin{cases}
  \xi_2 &= \frac{\sqrt{3}}{2} (\beta_2 \gamma_1 - \beta_1 \gamma_2) \\
  \eta_2 &= \frac{\sqrt{3}}{2} (\gamma_2 \alpha_1 - \gamma_1 \alpha_2) \\
  \zeta_2 &= \frac{\sqrt{3}}{2} (\alpha_2 \beta_1 - \alpha_1 \beta_2)
\end{cases} \quad (4.15)
\]

Proof: Clearly
\[
c_j = \frac{1}{2} (f_j + g_j), \quad \frac{1}{c_j} = \frac{1}{2} (f_j - g_j).
\]
Hence
\[
  h_j = \frac{1}{2} (f_j+1 f_{j+2} - g_{j+1} g_{j+2}) \\
  k_j = \frac{1}{2} (f_{j+1} g_{j+2} - g_{j+1} f_{j+2}).
\]
By expressing \( \cos(\theta - \frac{2\pi}{3} j) \), \( \sin(\theta - \frac{2\pi}{3} j) \), \( \cos(\theta - \frac{2\pi}{3} j) \), and \( \sin(\theta - \frac{4\pi}{3}) \) as linear combinations of \( \cos \theta \) and \( \sin \theta \) one gets
\[
f_1 f_2 = (\alpha_1^2 - \beta_1^2 + \gamma_1^2) - \alpha_1 \beta_1 \cos \theta - \alpha_1 \gamma_1 \sin \theta + \frac{\beta_1^2 - \gamma_1^2}{2} \cos 2 \theta + \beta_1 \gamma_1 \sin 2 \theta. \quad (4.16)
\]
Using now (4.11) from the proof of Lemma 4.4, we have
\[
f_1 f_2 - f_0^2 = -\frac{3}{4} (\beta_1^2 + \gamma_1^2) - 3 \alpha_1 \beta_1 \cos \theta - 3 \alpha_1 \gamma_1 \sin \theta.
\]
Repeating the same argument with \( \theta - \frac{2\pi}{3} j \) instead of \( \theta \), we have
\[
f_{j+1} f_{j+2} - f_j^2 = -\frac{3}{4} (\beta_1^2 + \gamma_1^2) - 3 \alpha_1 \beta_1 \cos \left( \theta - \frac{2\pi}{3} j \right) - 3 \alpha_1 \gamma_1 \sin \left( \theta - \frac{2\pi}{3} j \right). \quad (4.17)
\]
and in the same way we have
\[ g_{j+1}g_{j+2} - g_j^2 = -\frac{3}{4}(\beta_2^2 + \gamma_2^2) - 3\alpha_2\beta_2 \cos \left( \theta - \frac{2\pi}{3}j \right) - 3\alpha_2\gamma_2 \sin \left( \theta - \frac{2\pi}{3}j \right). \]  

(4.18)

By the definition of \( f_j \) and \( g_j \) we have
\[ f_j^2 - g_j^2 = \left( c_j + \frac{1}{c_j} \right)^2 - \left( c_j - \frac{1}{c_j} \right)^2 = 4. \]  

(4.19)

Hence, by (4.17), (4.18), and (4.19)
\[
2h_j = f_{j+1}f_{j+2} - g_{j+1}g_{j+2} = 4 - \frac{3}{4}(\beta_1^2 + \gamma_1^2 - \beta_2^2 - \gamma_2^2) - 3(\alpha_1\beta_1 - \alpha_2\beta_2) \cos(\theta - \frac{2\pi}{3}j)
- 3(\alpha_1\gamma_1 - \alpha_2\gamma_2) \sin(\theta - \frac{2\pi}{3}j).
\]

By uniqueness of this decomposition (Lemma 4.3) we can read off the coefficients \( \xi_1, \eta_1, \zeta_1 \) in (4.13) namely
\[
\xi_1 = 2 - \frac{3}{8}(\beta_1^2 + \gamma_1^2 - \beta_2^2 - \gamma_2^2),
\eta_1 = -\frac{3}{2}(\alpha_1\beta_1 - \alpha_2\beta_2),
\zeta_1 = -\frac{3}{2}(\alpha_1\gamma_1 - \alpha_2\gamma_2).
\]

However by (4.8) in Lemma 4.4, we have
\[
(\alpha_1^2 - \alpha_2^2) + \frac{1}{2}(\beta_1^2 - \beta_2^2) + \frac{1}{2}(\gamma_1^2 - \gamma_2^2) = 4.
\]

Hence the above formula for \( \xi_1 \) can be changed to
\[
\xi_1 = \frac{3}{4}(\alpha_1^2 - \alpha_2^2) - 1.
\]

This proves (4.14). A similar but much simpler computation gives
\[
k_j = \frac{1}{2}(f_{j+1}g_{j+2} - f_{j+2}g_{j+1})
= \frac{\sqrt{3}}{4}(\beta_2\gamma_1 - \beta_1\gamma_2) + \frac{\sqrt{3}}{2}(\gamma_2\alpha_1 - \gamma_1\alpha_2) \cos \left( \theta - \frac{2\pi}{3}j \right)
+ \frac{\sqrt{3}}{2}(\alpha_2\beta_1 - \alpha_1\beta_2) \sin \left( \theta - \frac{2\pi}{3}j \right),
\]

which proves (4.15).

**Proof of Theorem 4.1:** Assume that \((c_0, c_1, c_2)\) is a solution to the set of equations (3.1).

By Proposition 3.1, the numbers
\[
h_j = \frac{c_{j+2}}{c_{j+1}} + \frac{c_{j+1}}{c_{j+2}}, \quad j = 0, 1, 2,
\]

...
must be of the form
\[ h_j = \xi_1 + \eta_1 \cos \left( \theta - \frac{2\pi}{3} j \right), \quad j = 0, 1, 2, \quad (4.20) \]
where \( (\xi_1, \eta_1) \) is one of the four pairs \( (\xi_1^{(i)}, \eta_1^{(i)}) \), \( i = 0, 1, 2, 3 \), listed in (3.5)–(3.8). For \( i = 0 \), we have \( \xi_1 = 2 \) and \( \eta_1 = 0 \). Hence \( h_0 = h_1 = h_2 = 2 \) which implies that \( c_0 = c_1 = c_2 \), and in this case the only solutions to (3.1) are the 2 “\( \varepsilon \)-solutions” from [2], namely
\[ c_0 = c_1 = c_2 = \frac{2 - p \pm \sqrt{p(p - 4)}}{2}. \]
For \( i = 1, 2, 3 \) we can compute the numbers \( c_j \) from \( (\xi_1, \eta_1) \) by Lemma 4.5. Define
\[ f_j = c_j + \frac{1}{c_j}, \quad g_j = c_j - \frac{1}{c_j}, \quad h_j = \frac{c_j + 2}{c_{j+1}} + \frac{c_{j+1}}{c_j}, \quad k_j = \frac{c_j + 2}{c_{j+1}} - \frac{c_{j+1}}{c_j} \]
as in Lemma 4.5, and let \( \alpha_\nu, \beta_\nu, \gamma_\nu, \xi_\nu, \eta_\nu, \zeta_\nu, \nu = 1, 2 \) be the coefficients in the decompositions (4.12) and (4.13). Note that by Lemma 4.3 this new definition of \( \xi_1 \) and \( \eta_1 \) is consistent with (4.20). Moreover \( \zeta_1 = 0 \) by (4.20).

By (3.1)
\[ f_j = \frac{p - 4}{3} - \frac{p + A + 1}{9} h_j - \frac{2p - A - 9B - 4}{18} h_{j+1} - \frac{2p - A - 9B - 4}{18} h_{j+2}. \]
Since
\[ h_0 = \xi_1 + \eta_1 \cos \theta, \]
\[ h_1 = \xi_1 + \eta_1 \left( -\frac{1}{2} \cos \theta + i \frac{\sqrt{3}}{2} \sin \theta \right), \]
\[ h_2 = \xi_1 + \eta_1 \left( -\frac{1}{2} \cos \theta - i \frac{\sqrt{3}}{2} \sin \theta \right), \]
we have
\[ f_0 = \left( -\frac{p - 4}{3} - \frac{p - 1}{3} \xi_1 \right) - \frac{A + 2}{6} \eta_1 \cos \theta - \frac{\sqrt{3}}{2} B \eta_1. \]
Repeating the same computation with \( \theta \) replaced by \( \eta - \frac{2\pi}{3} j \), we get that the coefficients \( \alpha_1, \beta_1, \gamma_1 \) in the decomposition
\[ f_j = \alpha_1 + \beta_1 \cos \left( \theta - \frac{2\pi}{3} j \right) + \gamma_1 \sin \left( \theta - \frac{2\pi}{3} j \right) \]
are given by
\[ \left\{ \begin{array}{ll}
\alpha_1 & = -\frac{p - 4}{3} - \frac{p - 1}{3} \xi_1 \\
\beta_1 & = -\frac{A + 2}{6} \eta_1 \\
\gamma_1 & = -\frac{\sqrt{3}}{2} B.
\end{array} \right. \quad (4.21) \]
Provided $\alpha_1^2 - \frac{4}{3}(\xi_1 + 1) \neq 0$ we then get from (4.14)
\[
\begin{cases}
\alpha_2 = \pm \sqrt{\alpha_1^2 - \frac{4}{3}(\xi_1 + 1)} \\
\beta_2 = \frac{\alpha}{\alpha_2} (\alpha_1 \beta_1 + \frac{2}{3} \eta_1) \\
\gamma_2 = \frac{\alpha}{\alpha_2} (\alpha_1 \gamma_1 + \frac{2}{3} \xi_1).
\end{cases}
\tag{4.22}
\]

Inserting the values $(\xi_1^{(i)}, \eta_1^{(i)})$, $i = 1, 2, 3$ from (3.24) in (4.21) we find that $\alpha_1^2 - \frac{4}{3}(\xi_1 + 1) \neq 0$ in all the cases $i = 1, 2, 3$. Hence the numbers $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ given by (4.21) and (4.22) are unique up to simultaneous sign change of $(\alpha_2, \beta_2, \gamma_2)$. For $i = 1, 2$,
\[
\alpha_2 = \pm \sqrt{\alpha_1^2 - \frac{4}{3}(\xi_1 + 1)}
\]
is purely imaginary, and we choose the solution with $\Im(\alpha_2^{(i)}) > 0$ ($i = 1, 2$). For $i = 3$, $\alpha_2$ is real and we choose the solution with $\sign(\alpha_2^{(3)}) = -\sign(u^2 - uv + 2)$. It is now easy to compute $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$ explicitly from (3.24) in the 3 cases $i = 1, 2, 3$. One finds
\[
\begin{cases}
\alpha_1^{(1)} = \frac{pA - 2p - 2A}{p^2 - 3p - A}, & \alpha_2^{(1)} = i \frac{3\sqrt{3}\sqrt{p-4}B}{p^2 - 3p - A} \\
\beta_1^{(1)} = -\frac{\sqrt(4)(p-4)(A+2)}{p^2 - 3p - A}, & \beta_2^{(1)} = -i \frac{3\sqrt(4)(p-2)B}{p^2 - 3p - A} \\
\gamma_1^{(1)} = -\frac{3\sqrt(3)\sqrt(4)B}{p^2 - 3p - A}, & \beta_3^{(1)} = i \frac{\sqrt(4)(pA-2p-2A)}{p^2 - 3p - A}.
\end{cases}
\tag{4.23}
\]
\[
\begin{cases}
\alpha_1^{(2)} = \frac{-u^2 - uv - 4}{u^2 + uv + 2}, & \alpha_2^{(2)} = i \frac{u\sqrt(4+u-\sqrt(4-u+v)}}{u^2 + uv + 2} \\
\beta_1^{(2)} = \frac{2(A+2)u}{u^2 + uv + 2}, & \beta_2^{(2)} = i \frac{(u^2 + uv + 4)\sqrt(4+4u-\sqrt(4-u+v)}}{u^2 + uv + 2} \\
\gamma_1^{(2)} = \frac{6\sqrt(3)Bu}{u^2 + uv + 2}, & \gamma_2^{(2)} = i \frac{(u^2 - uv + 4)\sqrt(u+v+4u+v-4)}{u^2 + uv + 2}.
\end{cases}
\tag{4.24}
\]
\[
\begin{cases}
\alpha_1^{(3)} = \frac{-u^2 + uv - 4}{u^2 - uv + 2}, & \alpha_2^{(3)} = -\frac{u\sqrt(4+u+\sqrt(4-u-v)}}{u^2 - uv + 2} \\
\beta_1^{(3)} = \frac{2(A+2)u}{u^2 - uv + 2}, & \beta_2^{(3)} = -\frac{1}{2} \frac{(u^2 - uv + 4)\sqrt(u+v+4u+v-4)}{u^2 - uv + 2} \\
\gamma_1^{(3)} = \frac{3\sqrt(3)Bu}{u^2 - uv + 2}, & \gamma_2^{(3)} = \frac{1}{2} \frac{(u^2 + uv - 4)\sqrt(u-v+4u+v-4)}{u^2 - uv + 2}.
\end{cases}
\tag{4.25}
\]
Since
\[
c_j = \frac{1}{2} (f_j + g_j) = \frac{\alpha_1 + \alpha_2}{2} + \frac{\beta_1 + \beta_2}{2} \cos \left( \theta - \frac{2\pi}{3} j \right) + \frac{\gamma_1 + \gamma_2}{2} \sin \left( \theta - \frac{2\pi}{3} j \right),
\tag{4.26}
\]
we obtain (4.2) with $\alpha^{(i)}, \beta^{(i)}, \gamma^{(i)}$ given by (4.3), (4.4) and (4.5).
We still have to check that the \((c^{(i)}_0, c^{(i)}_1, c^{(i)}_2)\) given by (4.2)–(4.5) actually are solutions to (3.1). From Lemma 4.4 and Lemma 4.5 it follows that the only thing left to check is that \(c_j \neq 0, j = 0, 1, 2\) and that

\[
\frac{1}{c_j} = \frac{\alpha_1 \alpha_2}{2} + \frac{\beta_1 - \beta_2}{2} \cos \left( \theta - \frac{2\pi}{3} j \right) + \frac{\gamma_1 - \gamma_2}{2} \sin \left( \theta - \frac{2\pi}{3} j \right),
\]

which is equivalent to checking that the numbers \(t_1, t_2, t_3\) listed in (4.8)–(4.10) are zero.

Using

\[
\cos 3\theta = \frac{A}{2\sqrt{p}}, \quad \sin 3\theta = \frac{\sqrt{4p^2 - A^2}}{2\sqrt{p}} = \frac{3\sqrt{3}B}{2\sqrt{p}}
\]

it is elementary to check by MAPLE or MATHEMATICA that \(t_1 = t_2 = t_3 = 0\) in each of the 3 cases (4.23), (4.24) and (4.25) above. It is also possible to avoid a case by case check by relating \(t_1, t_2\) and \(t_3\) to the polynomials \(p_1, p_2, p_3, p_4\) used in the proof of Proposition 3.1 (see Remark 4.6 below).

Finally we have to show that we have found 20 distinct solutions: Since \(\eta^{(i)}_1 \neq 0, i = 1, 2, 3\), the 3 solutions given by (4.2)–(4.5) are distinct from the two \(\epsilon\)-solutions. This also implies that in each of the 3 cases, the 6 solutions given by

\[
\begin{align*}
\{ (c_j, c_{j+1}, c_{j+2}) & \quad j = 0, 1, 2 \\
\{ \frac{1}{c_j}, \frac{1}{c_{j+1}}, \frac{1}{c_{j+2}} & \quad j = 0, 1, 2
\end{align*}
\]

are all distinct. To check that there is no overlap between these 3 groups of 6 solutions it is sufficient to check that the 3 numbers \(s^{(i)}_1 = 3\xi^{(i)}_1\) are distinct because

\[
s_1 = h_0 + h_1 + h_2 = \frac{c_2}{c_1} + \frac{c_0}{c_2} + \frac{c_1}{c_0} + \frac{c_1}{c_2} + \frac{c_2}{c_0} + \frac{c_0}{c_1}
\]

is invariant under the 6 transformations listed in (4.28). From (3.20)

\[
\begin{align*}
\{ s^{(1)}_1 & = \frac{18p - 3p^2 - 6A}{p^2 - 3p - A} \\
\{ s^{(2)}_1 & = \frac{-6pA - 27 - 12 \pm 9\sqrt{p(p + 4A + 16)}}{2(pA + 2p - 1)}
\end{align*}
\]

Clearly \(s^{(2)}_1 \neq s^{(3)}_1\), since \(p + 4A + 16 > 0\) by Remark 3.2. Moreover \(s^{(2)}_1\) and \(s^{(3)}_1\) are the two zeros of the polynomial \(r\) from (3.19):

\[
r(s_1) = (pA + 3p - 1)s_1^2 + (6pA + 27p + 12)s_1 + (9pA + 54p - 36).
\]

We get

\[
r(s^{(1)}_1) = 81 \frac{(2p - A - 4)(4p - A^2)}{(p^2 - 3p - A)^2},
\]

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but \(2p - A - 4 > 2p - 2\sqrt{p} - 4 = 2(\sqrt{p} + 1)(\sqrt{p} - 2) > 0\), and \(4p - A^2 = 27B^2 > 0\). Hence \(s_1^{(1)} \neq s_1^{(2)}\) and \(s_1^{(1)} \neq s_1^{(3)}\). Therefore we have found altogether \(2 + 3 \cdot 6 = 20\) solutions. By (4.27), passing from \(c_j^{(i)}\) to \(\frac{1}{\sqrt{p}}\) in (4.2) corresponds to a change of sign of \(\alpha_2, \beta_2\) and \(\gamma_2\). Hence the 12 solutions generated by (4.3), (4.4) and the transformations (4.28) are all unimodular while the remaining 8 solutions clearly are real.

This completes the proof of Theorem 4.1.

**Remark 4.6** We sketch here a different proof of \(t_1 = t_2 = t_3 = 0\) for the values of \(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2\) listed in (4.23)–(4.25):

By (4.21) and (4.22), \(\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2\) can be expressed in terms of \((\xi_1, \eta_1)\) and hence \(t_1, t_2, t_3\) given by (4.8)–(4.9) can be expressed in terms of \(\xi_1, \eta_1, \) and \(\theta\). Next we observe that if

\[
h_j = \xi_1 + \eta_1 \cos \left(\theta - \frac{2\pi}{3}j\right), \quad j = 0, 1, 2,
\]

then

\[
s_1 = h_0 + h_1 + h_2 = 3\xi_1,
\]
\[
s_2 = h_0h_1 + h_1h_2 + h_2h_0 = 3\xi_1^2 - \frac{3}{4}\eta_1^2,
\]
\[
s_3 = h_0h_1h_2 = \xi_1^3 - \frac{3}{4}\xi_1\eta_1 + \frac{1}{4}\eta_1^3\cos 3\theta,
\]
\[
a = (h_0 - h_1)(h_1 - h_2)(h_2 - h_0) = -\frac{3\sqrt{3}}{4}\eta_1 \sin 3\theta.
\]

Inserting this into the 4 polynomials \(p_i = p_i(s_1, s_2, s_3, a)\) from the proof of Proposition 3.1 and comparing these new formulas for \(p_1, p_2, p_3\) and \(p_4\) with the formulas found above for \(t_1, t_2, t_3\) one discovers after some work that

\[
t_1 = \frac{4(p_3 - p_4)}{27\alpha_2^2},
\]
\[
t_2 = -\frac{4(3p_1 + (\xi_1 - p\xi_1 - p + 4)p_2 + (2\xi_1 - 1)p_3 + (\xi_1 + 4)p_4)}{27\xi_1\alpha_2^2},
\]
\[
t_3 = \frac{4\left(\xi_1 + p\xi_1 + A\xi_1 + p + A - 2\right)(p - 1)p_1 - 2p_2 - p_3 - 2p_4)}{3\sqrt{3}B\xi_1\alpha_2^2},
\]

and since \((\xi_1^{(i)}, \eta_1^{(i)}), i = 1, 2, 3\) were found by solving the equations \(p_1 = p_2 = p_3 = p_4 = 0\), it follows that \(t_1 = t_2 = t_3 = 0\) in all three cases.

## 5 Corollaries of the main result (Leaving the simple case)

In this section we will formulate and prove various consequences of the main result; in particular we will identify all bi-unimodular \(p\)-sequences and cyclic \(p\)-roots of index 3. We will give the \(c^{(i)}\) names:
Definition 5.1 We denote as the first, second and third canonical solution the solutions $c^{(1)}, c^{(2)},$ and $c^{(3)}$ defined in Theorem 4.1.

We will start by presenting all bi-unimodular $p$-sequence of index 3 (cf. Definition 1.3). Recall that $\omega = \exp(\frac{2\pi i}{p})$.

Proposition 5.2 Let $p$ be a prime $\equiv 1 \pmod{6}$, and let $x$ be a bi-unimodular $p$-sequence of index 3. Then there are a complex number $b$ of modulus one and integers $r$ and $l$ such that $x$ is given by $x_l = b$ and $x_j = b \cdot \omega^{rj} \cdot c_k$ when $0 \neq j - l \in G_k$ ($k = 0, 1, 2$), where $c = (c_0, c_1, c_2)$ is one of the 12 solutions to (3.1) coming from the first or second canonical solution $c^{(1)}, c^{(2)}$, as described in Theorem 4.1. If $p \neq 7$, there are $12p^2$ different normalized bi-unimodular $p$-sequences of index 3 (i.e. with $x_0 = 1$). There are 336 different normalized bi-unimodular 7-sequences. Of these, $6 \cdot 7^2$ come from the second canonical solution, whereas only $6 \cdot 7$ come from the first canonical solution. The last-mentioned sequences can be uniquely written in the form $x_j = \omega^{m_jj^2 + n}$, where $m$ and $n \in \mathbb{Z}_7$ and $m \neq 0$.

Next we formulate our result as a theorem bearing on cyclic $p$-roots rather than on bi-unimodular $p$-sequences:

Proposition 5.3 Let $p$ be a prime $\equiv 1 \pmod{6}$, and let $z = (z_0, \ldots, z_{p-1})$ be a cyclic $p$-root of index 3. Then there are integers $r$ and $l$ such that $z$ is given by $z_j = \omega^r \cdot c_k$ when $j + 1 - l \in G_k$ and $j - l \in G_k$, where $c = (c_0, c_1, c_2)$ is one of the 20 solutions to (3.1) as described in Theorem 4.1. If $p \neq 7$, there are $20p^2$ different cyclic $p$-roots of index 3, $(2p^2$ of which being in fact of index 1). There are only 434 different cyclic 7-roots of index 3. Of these, 42 come from the first canonical solution. These “Gaussian” cyclic 7-roots can be uniquely written in the form $z_j = \omega^{m_jj+n}$ where $m$ and $n \in \mathbb{Z}_7$ and $m \neq 0$.

Proof of Proposition 5.2 and Proposition 5.3 The first statements in these theorems are obvious reformulations of Theorem 4.1 in terms of the concepts introduced in Section 1, and we leave it to the reader to check this. We will only prove the statements about the number of different normalized bi-unimodular sequences of index 3 (NBUS3), the number of different cyclic $p$-roots of index 3, and the explicit forms given in the first canonical case for $p = 7$.

We start with the last topic. Since the 42 possible $\omega$-exponents in the $z_j$-formula in Proposition 5.3 form the set of all differences (as functions of $j$) of those in the $x_j$-formula in Proposition 5.2, it suffices to consider the latter (cf. (1.4) and Proposition 1.1). We start by taking $m = 1$ and $n = 0$, which gives $x = (1, \omega, \omega^4, \omega^2, \omega^2, \omega^4, \omega)$. Since for $p = 7$ we have $G_0 = \{1, 6\}$, $G_1 = \{3, 4\}$, and $G_2 = \{2, 5\}$, this means that this particular $x$ is in fact simple of index 3 with $c_0 = \omega$, $c_1 = \omega^2$, and $c_2 = \omega^4$ (cf. Definition 1.2). We claim that this $c = (c_0, c_1, c_2)$ is one of the six solutions coming from $c^{(1)}$ in Theorem 4.1. To prove this, we calculate $h_0 = c_0 + c_1 = \omega^2 + \omega^{-2}$, $h_1 = c_0 + c_2 = \omega^3 + \omega^{-3}$, and $h_2 = c_0 + c_1 + c_0 = \omega + \omega^{-1}$. Thus, using the relation $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$, we get $s_1 = h_0 + h_1 + h_2 = -1$, $s_2 = h_0h_1 + h_1h_2 + h_2h_0 = -2$, $s_3 = h_0h_1h_2 = 1$, and
\(a = (h_1 - h_0)(h_2 - h_1)(h_0 - h_{02}) = -7.\) Since these values agree with those of \(s_1^{(1)}, s_2^{(1)}, s_3^{(1)}\), and \(a^{(1)}\) in (3.14), our last claim is proved.

Next we keep \(n = 0\) but consider a general \(m\). But all we have used about \(\omega\) in our calculations is that \(\omega\) is a primitive seventh root of unity. So is \(\omega^m\). Thus, \(x_j = \omega^{m \cdot j^2} = (\omega^m)^{j^2}\) will also give a simple bi-unimodular 7-sequence of index 3. Of course the six possibilities for \(m\) correspond to the six transformations mentioned in Theorem 4.1. Finally, taking a general \(n\), we see by Definition 1.3 (with \(l = 0\) and \(h = n\)) that all our \(x\) are bi-unimodular 7-sequences of index 3. Clearly they are normalized.

It is clear that the 42 normalized bi-unimodular 7-sequences of index 3 we have found are different. Next we show that no other normalized bi-unimodular 7-sequence comes from the first canonical case. All we have to prove is that taking \(l \neq 0\) in Definition 1.3 does not give anything new when \(y\) is a simple bi-uninormal sequence of index 3 given by \(y_k = \omega^{mj^2}\). But this is trivial, since Definition 1.3 gives the unnormalized bi-uninormal sequence \(x\) of index 3 defined by \(x_j = \omega^{hj + m(j-l)^2} = \omega^{ml^2 + mj^2 - 2mlj}\) which is normalized through division by \(x_0 = \omega^{ml^2}\) and becomes \(\omega^{mj^2 - 2mlj} = \omega^{(-2ml)j}y_j\), which is of the desired form.

It remains to prove that the numbers of different NBUS3:s and different cyclic \(p\)-roots of index 3 given in our two propositions are correct, that is that no such “collapse” occurs except in the first canonical case for for \(p = 7\). Recall that in the end of the proof of Theorem 4.1 we showed that all the 20 solutions to the main problem are different. We now have to extend this from the simple to the general case and we start by considering the \(\epsilon\)-solutions. Every corresponding NBUS3 \(x\) has the form \(x_j = d_j \omega^{rj}\) with \(r \in \mathbb{Z}\) and \(d = (1, \epsilon, \epsilon, \ldots, \epsilon, \epsilon)\) or \(d = (\ldots, 1, 1, \epsilon, 1, 1, \ldots)\) with \(\epsilon = (2 - p \pm \sqrt{p(p - 4)})/2\). These \(p^2\) NBUS3:s are clearly distinct.

Let us when \(r \neq 0\) and \(l\) are in \(\mathbb{Z}/p\) and \(c = (c_0, c_1, c_2) \in \mathbb{C}^3\) is one of the 20 solutions mentioned in 4.1, define \(x(r, l, c)\) as the NBUS3 \(x = (x_0, x', \ldots, x_{p-1})\) given by the formulas:

\[
x_j = b \omega^{rj}(c)_k \quad \text{when} \ 0 \neq j - l \in G_k, \quad \text{(5.1)}
\]

\[
x_l = b \omega^{r'l}, \quad \text{(5.2)}
\]

where \(b\) is determined by the normalization

\[
x_0 = 1. \quad \text{(5.3)}
\]

Let us consider two coinciding NBUS3:s, \(x(r', l', c') = x(r'', l'', c'')\) which do not satisfy all the three equalities \(r' = r'', l' = l'', c' = c''\). We denote the two \(b\)’s defined by (5.3) by \(b'\) and \(b''\), respectively. We start by considering the possibility that \(l'' = l'\). Denote the common value by \(l\) and fix a \(k\). From (5.1) follows that \(b' \omega^{r'j}(c')_k = b'' \omega^{r''j}(c'')_k\) if \(j - l \in G_k\) and thus for at least two different non-zero \(j\), which leads to \(r' = r''\). Then (5.2) gives \(b' = b''\). Now (5.1) implies that we have also \(c' = c''\), which is against our hypothesis that at least one of \(r, l\) and \(c\) differs between the two NBUS3:s.

Thus we have \(l' \neq l''\). Let us now suppose that \(r' = r''\) (and \(l' \neq l''\)). Denote the common \(r\)-value by \(r\). Choose \(j_1\) such that \(j_1 \neq l'\) and \(j_1 \neq l''\) and define \(k_1\) and \(k_2\) by

\[
(j_1 - l') \in G_{k_1}, \quad (j_1 - l'') \in G_{k_2}. \quad \text{(5.4)}
\]
Consider the set $F := \{ j - l''; (j - l') \in G_{k_1} \} \cap G_{k_2}$. Taking $d = l'' - l'$ in (1.9), we see that if $d \in G_a$, then the cardinality of $F$ is a transition number: $\#(F) = n_{k_1 - a, k_2 - a}$. By (2.4), all transition numbers are $\leq s - 1$, and since $\#(G_{k_1}) = s$, there is at least one $j_2$ and one $k_3 \neq k_2$ such that

\[(j_2 - l') \in G_{k_1} \quad \text{and} \quad (j_2 - l'') \in G_{k_3}\]  

(5.5)

Now from (5.4) and (5.5) follows that (5.1) with $j = j_1$ and with $j = j_2$ gives

\[b' \omega^{r j_1} c'_{k_1} = b'' \omega^{r j_1} c''_{k_1},\]

\[b' \omega^{r j_2} c'_{k_1} = b'' \omega^{r j_2} c''_{k_3}.\]

This leads to $c''_{k_2} = c''_{k_3}$. Then it follows from from Remark 3.3 that $c''$ is an $\epsilon$-solution. Since $c'$ and $c''$ play the same part in our situation, the same must be true for $c'$. But we know already that there is no internal collapse among the NBUS3:s coming from $\epsilon$-solutions, so the case $r' = r''$ also leads to a contradiction. Now we know that $r' \neq r''$ and $l' \neq l''$. From (5.2) and (5.1) with $j = l'$ we get

\[x_{l'} = b' \omega^{r_{l'}} c'_{k_1} = b'' \omega^{r_{l''}} c''_{k},\]

(5.6)

where $k$ is determined by $(l'' - l') \in G_k$. Since $G_k$ has at least two elements we can choose $j \neq l''$ with $(j - l') \in G_k$. For this $j$ we get from (5.1)

\[x_{j} = b' \omega^{r_{j}} c'_{k} = b'' \omega^{r_{j}} c''_{k}.\]

(5.7)

From (5.6) and (5.7) we get by division

\[c'_{k} = \omega^{(r'' - r')(j - l')}.\]

Since the exponent of $\omega$ is not zero (modulo $p$), we have found a $c'_{k}$ which is a primitive $p'$th root of unity. But we have also proved that we must have $p = 7$. For if $p \geq 13$, there are more than two elements in $G_k$, and we can make two different choices of $j$, giving conflicting values to $c'_{k}$. To sum up, we know that to have collapse we must have $p = 7$, and some $c'_{k}$ must be a seventh root of unity. Again our symmetry argument says that also some $c''_{k}$ must be a seventh root of unity. The third canonical case is not of interest, since the absolute values are not one. We can also easily exclude the second canonic case e.g. with the following numerical argument: The imaginary part of the seventh power of the six values of the components of $c^{(2)}$ are approximately $\pm 0.92, \pm 0.94, \text{and} \pm 0.41$ rather than 0. So the collapse is an internal affair within the first canonical case, which we have already studied. This completes the proof of the two propositions.

6 Numerical and asymptotic results

In this section we will study the behavior for large $p$ of the solutions $c^{(i)}, i = 1, 2, 3$ defined in Theorem 4.1. We will give numerical data leading to educated guesses about this behavior (see Remark 6.3 and we will prove quantitative forms of these guesses.

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In Table 6.1 below we list the first few primes \( \equiv 1 \pmod{6} \) and corresponding numerical values of \( A, B, \theta, c_0^{(1)}, c_1^{(1)}, \text{ and } c_2^{(1)} \). In Table 6.2, we give the corresponding information for \( c^{(2)} \). We will also include an indication of the shape of the triangle formed by the three complex numbers \( c_0^{(i)}, c_1^{(i)}, c_2^{(i)}, \ (i = 1, 2) \), reasoning as follows:

In the corresponding situation for simple bi-unimodular sequences of index two (cf. [2]) we have two complex numbers \( c_0 \) and \( c_1 \) on the unit circle, and with increasing \( p \) their sum tends to zero. A natural guess in our situation might therefore be that the sum of the three numbers tends to zero or, equivalently, that the triangle becomes more and more equilateral when \( p \) grows. We prefer the latter description. To be able to give quantitative results we will revive the old noun scalarity, (cf. [1]) and give it a precise meaning:

**Definition 6.1** In the complex plane, let \( b = (b_0, b_1, b_2) \) be a triple of points on a circle \( C \) with center \( w \). Let \( \phi_i = \arg(b_i - w) \). Let the scalarity of \( b \) be

\[
\text{scal}(b) = \max \left| \frac{1}{2} + \cos(\phi_{j+2} - \phi_{j+1}) \right|,
\]

(indices counted modulo 3).

**Remark 6.2** Since \( \frac{1}{2} = -\cos 2\pi/3 \), the triangle with vertices \( b \) will be equilateral iff its scalarity is zero. Let us now consider the definition of \( h_j \) (in Proposition 3.1). If we take \( b = c^{(i)} \) with \( i = 1 \) or \( 2 \), we have all \( |b_j| = 1 \) and thus \( w = 0 \). Hence \( \text{scal}(c^{(1)}) = \frac{1}{2} \max_j |1 + h_j|, \) where \( h_j \) is given by (3.4) with \( c \) replaced by \( b \).

| \( p \) | \( A \) | \( B \) | \( \theta \) | \( c_0^{(1)} \) | \( c_1^{(1)} \) | \( c_2^{(1)} \) | \( \text{scal}(c^{(1)}) \) |
|---|---|---|---|---|---|---|---|
| 7 | 1 | 1 | 0.4602 | -0.9010 - 0.4339 i | 0.6235 + 0.7818 i | -0.2225 + 0.9749 i | 1.1235 |
| 13 | -5 | 1 | 0.7790 | -0.4822 - 0.8761 i | 0.3953 + 0.9185 i | -0.8132 + 0.5820 i | 0.7132 |
| 19 | 7 | 1 | 0.2129 | -0.9528 - 0.3037 i | 0.9838 - 0.1791 i | 0.3780 + 0.9258 i | 0.7061 |
| 31 | 4 | 2 | 0.4011 | -0.8023 - 0.5969 i | 0.9923 + 0.1235 i | -0.0963 + 0.9954 i | 0.5274 |
| 37 | -11 | 1 | 0.9001 | -0.0604 - 0.9982 i | 0.4630 + 0.8863 i | -0.9452 + 0.3265 i | 0.4127 |
| 43 | -8 | 2 | 0.7423 | -0.3124 - 0.9499 i | 0.7272 + 0.6864 i | -0.7742 + 0.6330 i | 0.3792 |
| 61 | 1 | 3 | 0.5022 | -0.6466 - 0.7628 i | 0.9759 + 0.2181 i | -0.3560 + 0.9345 i | 0.3564 |
| 67 | -5 | 3 | 0.6271 | -0.4569 - 0.8895 i | 0.8964 + 0.4433 i | -0.5999 + 0.8001 i | 0.3170 |
| 73 | 7 | 3 | 0.3829 | -0.7843 - 0.6204 i | 0.9988 - 0.0481 i | -0.1114 + 0.9938 i | 0.3409 |
| 79 | -17 | 1 | 0.9483 | 0.1286 - 0.9917 i | 0.4824 + 0.8759 i | -0.9765 + 0.2154 i | 0.3066 |
| 97 | 19 | 1 | 0.0890 | -0.9875 - 0.1576 i | 0.7708 - 0.6371 i | 0.4823 + 0.8760 i | 0.3137 |
| 103 | 13 | 3 | 0.2919 | -0.8668 - 0.4986 i | 0.9629 - 0.2697 i | 0.0653 + 0.9979 i | 0.2937 |
| 109 | -2 | 4 | 0.5566 | -0.5387 - 0.8425 i | 0.9666 + 0.2561 i | -0.4841 + 0.8750 i | 0.2562 |
| 127 | -20 | 2 | 0.8875 | 0.0580 - 0.9983 i | 0.6211 + 0.7837 i | -0.9411 + 0.4382 i | 0.2464 |
| 139 | -23 | 1 | 0.9731 | 0.2257 - 0.9742 i | 0.4899 + 0.8718 i | -0.9873 + 0.1588 i | 0.2837 |
| 151 | 19 | 3 | 0.2290 | -0.9138 - 0.4062 i | 0.9066 - 0.4219 i | 0.1770 + 0.9842 i | 0.2452 |
| 157 | -14 | 4 | 0.7212 | -0.2380 - 0.9713 i | 0.8495 + 0.5276 i | -0.7655 + 0.6434 i | 0.2146 |
| 163 | 25 | 1 | 0.0683 | -0.9922 - 0.1248 i | 0.7153 - 0.6988 i | 0.4898 + 0.8718 i | 0.2411 |
| 181 | 7 | 5 | 0.4359 | -0.6932 - 0.7207 i | 0.9996 + 0.0270 i | -0.2649 + 0.9643 i | 0.2092 |
We present the corresponding values for $c^{(3)}$ in Table 6.3. Since these values are real, we will save some space and we use this for giving the information also in another form, namely $\frac{c^{(3)}}{\sqrt{p}}$, which should shed some light on the surprising behaviour of the components.

Table 6.3 (Third canonical case)

| $p$ | $A$ | $B$ | $\theta$ | $c^{(3)}_0$ | $c^{(3)}_1$ | $c^{(3)}_2$ | $c^{(3)}_3$ | $c^{(3)}_4$ | $c^{(3)}_5$ |
|-----|-----|-----|---------|-----------|-----------|-----------|-----------|-----------|-----------|
| 7   | 1   | 1   | 0.4602  | -1.2221   | 9.4127    | 2.7389    | -0.4619   | 3.5577    | 1.0352    |
| 13  | -5  | 1   | 0.7790  | -1.4201   | -14.6415  | 2.1601    | -0.3939   | -4.0668   | 0.5991    |
| 19  | 1   | 1   | 0.2129  | -2.3201   | 8.4655    | 4.8488    | -0.5167   | 1.9421    | 1.1112    |
| 31  | 4   | 2   | 0.4011  | -2.8168   | 17.2938   | 4.6888    | -0.5695   | 3.1061    | 0.8421    |
| 37  | -11 | 1   | 0.9001  | -3.0328   | -7.1015   | 2.8445    | -0.4986   | -1.1675   | 0.4676    |
| 43  | -8  | 2   | 0.7423  | -3.2558   | -13.2707  | 2.6557    | -0.4965   | -1.0751   | 0.5578    |
| 61  | 1   | 3   | 0.5222  | -3.0014   | 50.9574   | 1.7586    | 0.5123    | 6.5244    | 0.6989    |
| 67  | -5  | 3   | 0.6271  | -4.2289   | 95.9688   | 5.0055    | 0.5166    | -11.7245  | 0.6109    |
| 73  | 3   | 2   | 0.3829  | 4.4100    | 25.6091   | 6.4107    | -0.5162   | 2.9937    | 0.7772    |
| 79  | -17 | 1   | 0.9483  | 5.6126    | -8.9422   | 4.1623    | -0.6315   | -1.0061   | 0.4683    |
| 97  | 1   | 1   | 0.0890  | 5.0982    | 13.5365   | 10.4124   | 0.5176    | 1.3144    | 1.0512    |
| 103 | 3   | 2   | 0.2919  | 5.2556    | 21.9500   | 8.4142    | -0.5179   | 2.1628    | 0.7891    |
| 109 | -2  | 4   | 0.5556  | 5.5506    | 337.8101  | 6.6180    | -0.5275   | 32.5363   | 0.6339    |
| 127 | -20 | 2   | 0.8875  | -7.1148   | 14.4873   | 5.5161    | -0.6313   | -1.2855   | 0.4895    |
| 139 | -23 | 1   | 0.9731  | 8.4417    | -11.5986  | 5.6060    | -0.7160   | -0.9838   | 0.4755    |
| 151 | 19  | 3   | 0.2209  | -3.5433   | 22.1314   | 10.5843   | -0.5171   | 1.8610    | 0.8613    |
| 157 | -14 | 4   | 0.7212  | -7.1235   | -36.5459  | 6.8056    | -0.5685   | -2.9167   | 0.5431    |
| 163 | 25  | 1   | 0.0683  | -6.5753   | 16.4057   | 13.3294   | -0.5150   | 1.2850    | 1.0440    |
| 181 | 7   | 5   | 0.4359  | -7.0841   | 58.2887   | 9.2448    | -0.5266   | 4.3326    | 0.6872    |

Our observations are summarized in the following remark:
Remark 6.3 Our numerical observations and our results are of five kinds:
(1) For each large \( p \), the first and second canonical solutions are approximately symmetric
to each other w.r.t. the origin.
(2) Even though two large primes may be close to each other without their canonical
solutions being close, large primes with approximately the same \( \theta \) will have approximately
the same first canonical solutions and approximately the same second canonical solutions
(even if the primes are not close to each other).
(3) For large \( p \), the first and second canonical solution each forms an approximately equi-
lateral triangle.
(4) For large \( p \), the approximate positions of the nearly equilateral triangles are simple
functions of \( \theta \).
(5) If \( p \) is large, then all components of \( |c(3)| \) are large. If in addition \( |A| \) is small, that is
if \( \theta \) is close to \( \pi/6 \), then \( |c(3)|_1 \) is very large.

To make it easier to guess quantitative results (making “approximately” more precise in
Remark 6.3) we present a few more numerical results in Table 6.4.

| \( p \)   | \( A \) | \( B \) | \( \theta \)         | \( \text{arg}(c^{(1)}_0) \) | \( 2\theta - \pi \) | \( \text{arg}(c^{(2)}_0) \) | \( 2\theta \) | \( \text{scal}(c^{(1)}) \) |
|---------|-------|-------|-------------------|-------------------|-------------------|-------------------|---------|-------------------|
| 1003273 | 973   | 337   | 0.354542          | -2.43320          | -2.43251          | 0.70803           | 0.709084 | 0.002810          |
| 1003279 | 1993  | 39    | 0.033775          | -3.07411          | -3.07404          | 0.06742           | 0.067555 | 0.002995          |
| 100205473 | 9733  | 3367  | 0.354372          | -2.43292          | -2.43285          | 0.70864           | 0.708744 | 0.000281          |

From Table 6.4 it seems that “approximately” means agreement in approximately \( \frac{n}{2} \)
decimals. Thus quantitative results in terms of \( O\left( \frac{1}{\sqrt{p}} \right) \) might seem plausible. In our quan-
titative results we will use the maximum norm to measure distances in \( \mathbb{C}^3 \). We will also
need a name for the equilateral “limit” triangle hinted at in Remark 6.3 (4), hopefully vis-
ible in Tables 6.2 and 6.3, and present in columns 5 and 8 of Table 6.4. Thus we make the
following two definitions:

Definition 6.4 Let \( a = (a_0, a_1, a_2) \in \mathbb{C}^3 \), then we define \( \|a\| = \max(|a_0|, |a_1|, |a_2|) \).

Definition 6.5 Let \( p \) be a prime \( \equiv 1 \pmod{6} \) and let \( \theta = \frac{1}{3} \text{Arccos} \left( \frac{A}{2\sqrt{p}} \right) \), where \( 4p = A^2 + 27B^2 \) and \( A \equiv 1 \pmod{3} \). We denote by \( d = d(p) = (d_0, d_1, d_2) \) the (equilateral)
triangle for which
\[
d_j = \exp \left( 2i(\theta - \frac{2j\pi}{3}) \right), \quad j = 0, 1, 2.
\]

We will now state four quantitative results for the first and second canonical cases,
where Proposition 6.j for \( j = 6, \ldots, 9 \) is of the kind \( (j - 5) \) listed in Remark 6.3. (The discus-
on of kind (5) starts after Corollary 6.11 below.)

Proposition 6.6 Let \( p \) be a prime \( \equiv 1 \pmod{6} \), and let \( c^{(1)} \) and \( c^{(2)} \) be the corresponding
first and second canonical solution. Then
\[
\|c^{(1)} + c^{(2)}\| \leq \frac{36}{3\sqrt{p}}.
\]
Proposition 6.7 Let $p'$ and $p''$ be primes $\equiv 1 \pmod{6}$, let $\theta'$ and $\theta''$ be their respective $\theta$-values and let $c'$ and $c''$ be their respective first canonical solutions. Then

$$\|c' - c''\| \leq 2|\theta' - \theta''| + \frac{3}{\sqrt{p'}} + \frac{3}{\sqrt{p''}}.$$ 

The same result, with the constants 3 replaced by $\frac{21}{5}$, holds if $c'$ and $c''$ are the respective second canonical solutions.

Proposition 6.8 Let $p$ be a prime $\equiv 1 \pmod{6}$, and let $c^{(1)}$ and $c^{(2)}$ be the corresponding first and second canonical solutions. Then

$$\text{scal}(c^{(1)}) \leq \frac{7}{2\sqrt{p}} \quad \text{and} \quad \text{scal}(c^{(2)}) \leq \frac{21}{5\sqrt{p}}.$$ 

Proposition 6.9 Let $p$ be a prime $\equiv 1 \pmod{6}$, let $c^{(1)}$ and $c^{(2)}$ be the corresponding first and second canonical solution, and let $d$ be as in Definition 6.5. Then

$$\|c^{(1)} + d\| \leq \frac{3}{\sqrt{p}} \quad \text{and} \quad \|c^{(2)} - d\| \leq \frac{21}{5\sqrt{p}}.$$ 

Remark 6.10 The constants in these propositions are not best possible but are chosen as compromises to make the proofs less cumbersome. Even if we restrict our claims to hold only for $p > M$ for some large $M$, the constants cannot always be significantly improved. For instance, for $p = 10^{10} + 279$ we have $\|c^{(2)} - d\| \approx \frac{4}{\sqrt{p}}$. For a kind of “best possible”, result, see Remark 6.13.

Since to each number $\theta$ (in the interval $[0, \pi/3]$) there corresponds at most one $p$-s with a common $\theta$. However, Proposition 6.9 obviously has the following corollary, where we have used the notation $\theta(p)$, $c^{(2)}(p)$ and $c^{(1)}(p)$ for the values of $\theta$ and the first and second canonical solutions corresponding to $p$:

Corollary 6.11 Let $\theta_0$ be a real number in the interval $[0, \pi/3]$. Denote by $d = (d_0, d_1, d_2)$ the (equilateral) triangle for which $d_j = \exp\left(2i(\theta - \frac{2j\pi}{3})\right)$, $j = 0, 1, 2$. Let $\{p_n\}_1^\infty$ be a sequence of primes $\equiv 1 \pmod{6}$ going to infinity in such a way that $\lim_{n \to \infty} \theta(p_n) = \theta_0$. Then $\lim_{n \to \infty} c^{(1)}(p_n) = -d$ and $\lim_{n \to \infty} c^{(2)}(p_n) = d$. 

Before proving our four Propositions we will comment item (5) of Remark 6.3. In Table 6.5 we present some more numerical values with focus on $\theta$-values close to 0, $\pi/6$ and $\pi/3$. 

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These and other numerical results make it plausible that "large" in item (5) of Remark 6.3, may be specified to mean "not much smaller than $\frac{1}{2}\sqrt{p}$", but it seems difficult to find $\theta$-independent estimates of "convergence rate" for the third canonical case. We are now ready to state a proposition:

**Proposition 6.12** If $\{p_n\}_1^\infty$ is any sequence of primes $\equiv 1 \pmod{6}$ going to infinity, then (with obvious notation) for $i = 0, 1, 2$,

$$\liminf_{n \to \infty} \frac{|c^{(3)}_i(p_n)|}{\sqrt{p_n}} \geq 0.5. \quad (6.1)$$

We remark that this proposition implies that for every normalized $x = (1, x_1, \ldots, x_{p-1}) \in \mathbb{R}^p$ of index 3 coming from the third canonical case for a large $p$, either all $|x_j|, j \neq 0$, are large or they are all small (leaving the canonical case via the transformations mentioned in Theorem 4.1 and leaving the simple case via Definition 1.3).

We will now prove our five propositions.

**Proof of Propositions 6.6 and 6.6** Proposition 6.6 follows from Proposition 6.9 via a straightforward application of the triangle inequality.

Similarly, Proposition 6.7 follows from Proposition 6.9 via the triangle inequality and the inequality $|\exp(2i\phi') - \exp(2i\phi'')| \leq 2|\phi' - \phi''|$. 

**Proof of Proposition 6.8.** From (3.4) and (3.6) and Remark 6.2 we get

$$\sqrt{p} \text{scal} (c^{(1)}) = \frac{1}{2} \sqrt{p} \max_j |h_j + 1| \leq \frac{1}{2} \sqrt{p} \left( |1 + \xi_1^{(1)}| + |\eta_1^{(1)}| \right) = \frac{3\sqrt{p}(p - A) + p(6p - 24)}{2(p^2 - 3p - A)}. \quad (6.2)$$

Since the right-hand side of (6.2) is a decreasing function of $A$ and $A > -2\sqrt{p}$, we get

$$\sqrt{p} \text{scal} (c^{(1)}) < \frac{3\sqrt{p}(p + 2\sqrt{p}) + p(6p - 24)}{2(p^2 - 3p + 2\sqrt{p})} = 3 + \frac{3(\sqrt{p} - 2)}{2(\sqrt{p} - 1)^2}. \quad (6.3)$$

The last member of (6.3) as a function of $p$ is decreasing for $p > 9$ and takes values $< 3.4$ for $p = 7$ and 13. This completes the proof of the first part of the proposition.

For the second part, we will use (3.7) and the identity

$$\left(\sqrt{p}\sqrt{p + 4A + 16 + p + 2}\right)\left(\sqrt{p}\sqrt{p + 4A + 16 - p - 2}\right) = 4(Ap + 3p - 1)$$

The following table summarizes the numerical results:

| $p$         | $A$     | $B$     | $\theta$ | $c_0^{(3)}/\sqrt{p}$ | $c_1^{(3)}/\sqrt{p}$ | $c_2^{(3)}/\sqrt{p}$ |
|-------------|---------|---------|----------|-----------------------|-----------------------|-----------------------|
| 67 521 601 729 | -2      | 100 016 | 0.523600 | -0.577349             | 779 550.5             | 0.577355              |
| 67 544 557 351 | 1       | 100 033 | 0.523598 | -0.577347             | 194 920.0             | 0.577353              |
| 250 004 500 027 | 1 000 009 | 1       | 0.000002 | -0.500000             | 1.000007              | 1.000001              |
| 250 018 500 349 | -1 000 037 | 1       | 1.047196 | -0.999995             | -0.999997            | 0.500000              |
to give the counterpart of (6.2) the form
\[
\sqrt{p} \text{scal} \left( c^{(2)} \right) \leq \frac{3}{2} \sqrt{p} \left( 1 + |\xi^{(1)}| + |\eta^{(1)}| \right) = \frac{3(2p + \sqrt{p})}{p + 2 + \sqrt{p}\sqrt{p} + 4A + 16}
\]

We can again take \( A = -2\sqrt{p} \). The resulting expression is easily seen to be \( < 4 \) for \( p > 100 \) and for the remaining \( p \) we enter the true value of \( A \) (given in Table 6.1) to get a maximum \( \approx 4.1966 \) for \( p = 37 \). This completes the proof.

**Remark 6.13** From (6.3) we easily get the following result: For each \( \epsilon \) with \( 0 < \epsilon < 1 \) we have
\[
\text{scal}(c^{(1)}) \leq 3 + \epsilon \sqrt{p} \text{ if } p > \left( 2 + \frac{3}{2\epsilon} \right)^2,
\]
which could be contrasted with the fact that for \( p = 10002900217 \) we have \( \sqrt{p} \text{scal}(c^{(1)}) \approx 3.000015 \).

In the proof of Proposition 6.9 we will work with \( \alpha, \beta \) and \( \gamma \) as given in Theorem 4.1 and \( \rho, \sigma, \) and \( \tau \) as given in Lemma 4.3. We will use the following lemma:

**Lemma 6.14** Let \( b' = (b'_0, b'_1, b'_2) \in \mathbb{C}^3 \) and \( b'' = (b''_0, b''_1, b''_2) \in \mathbb{C}^3 \) be given by
\[
\begin{align*}
b'_j &= \rho' + \sigma' \cos \left( \theta - \frac{2\pi}{3} j \right) + \tau' \sin \left( \theta - \frac{2\pi}{3} j \right), \quad j = 0, 1, 2, \\
b''_j &= \rho'' + \sigma'' \cos \left( \theta - \frac{2\pi}{3} j \right) + \tau'' \sin \left( \theta - \frac{2\pi}{3} j \right), \quad j = 0, 1, 2,
\end{align*}
\]
where \( \theta \in \mathbb{R} \) and \( \rho', \sigma', \tau', \rho'', \sigma'', \tau'' \in \mathbb{C} \), Then
\[
\|b' - b''\| \leq |\rho' - \rho''| + \sqrt{|\sigma' - \sigma''|^2 + |\tau' - \tau''|^2}.
\]

The proof of Lemma 6.14 is a straightforward application of the triangle inequality, the Cauchy inequality, and the identity \( \cos^2 + \sin^2 = 1 \).

**Proof of Proposition 6.9** In Lemma 6.14 we take \( b' = c^{(1)} \) and \( b'' = -\alpha \) (cf. Definitions 5.1 and 6.5). Then \( \alpha' = \beta^{(1)}, \beta' = \gamma^{(1)} \) as given in (4.3), whereas \( \rho'' = 0, \sigma'' = -\cos 3\theta - i \sin 3\theta, \) and \( \tau'' = -\sin 3\theta + i \cos 3\theta, \) as is easily checked by introducing these values in (6.4) and applying the addition theorems for sine and cosine. Since \( \cos 3\theta = \frac{A}{2\sqrt{p}} \) and \( \sin 3\theta = \frac{3B\sqrt{3}}{2\sqrt{p}} \), Lemma 6.14 shows that for the proof of the first half of Proposition 6.9 it only remains to check that with \( \alpha^{(1)}, \beta^{(1)}, \) and \( \gamma^{(1)} \) as in (4.3) we have
\[
\sqrt{p} |\alpha^{(1)}| + \left| p \beta^{(1)} + \frac{A + i3B\sqrt{3}}{2\sqrt{p}} \right|^2 + \left| p \gamma^{(1)} + \frac{3\sqrt{3}B - iA}{2\sqrt{p}} \right|^2 \leq 3. 
\]
Introducing the values of $\alpha^{(1)}, \beta^{(1)}$, and $\gamma^{(1)}$ and replacing $3B\sqrt{3}$ by $\sqrt{4p-A^2}$ we can after some calculation treat the first term of the left member of (6.5) as follows

$$\sqrt{\mathcal{P}} |\alpha^{(1)}| = \sqrt{\frac{p^2 - Ap}{p^2 - 3p - A}} < \sqrt{\frac{p^2 + 2p\sqrt{p}}{p^2 - 3p + 2\sqrt{p}} = \sqrt{\frac{p(2 + \sqrt{p})}{(p - 3)^2 + 2}},} \quad (6.6)$$

where the estimate comes from the facts that the second term of (6.6) is a decreasing function of $A$ and that $A > -2\sqrt{p}$. Let us denote by $Q$ the expression under the big root sign in (6.5). Since the last member of (6.6) is a decreasing function of $p$ with a value $< 1.5$ for $p = 31$, we can prove (6.5) for $p \geq 31$ by checking that

$$Q \leq (3 - 1.5)^2 = 2.25 \text{ for } p \geq 31 \quad (6.7)$$

Treating $Q$ in the same way as we did with first term of the left member of (6.5) we find

$$Q = \frac{2p^3 - (A + 6)p^2 + 2Ap - (2p - A - 4)\sqrt{p^3 - 4p^3}}{p^2 - 3p - A}. \quad (6.8)$$

Using a Taylor formula with rest term we have

$$\sqrt{p^3 - 4p^3} = p^2 \left(1 - \frac{4}{p}\right)^{\frac{1}{2}} = p^2 - 2p - 2 + R_3,$$

where $-\frac{6}{p} < R_3 < 0$ (since $p > 31$). Introducing this in (6.8) we get

$$Q = \frac{2p^2 - 4p - 2A - 8 - (2p - A - 4)R_3}{p^2 - 3p - A} < \frac{2p^3 - 2p^2 + Ap - 2p - 3A - 12}{p(p^2 - 3p - A)}. \quad (6.9)$$

Since the last member of (6.9) is an increasing function of $A$ we can estimate it with its value for $A = 2\sqrt{p}$, which is a decreasing function of $p$ and thus not larger than its value for $p = 31$, which turns out to be $\approx 2.07$ in agreement with (6.7). Finally, we check numerically the value of $\sqrt{\mathcal{P}} \|c(1) + d\|$ for $p = 7, 13, \text{ and } 19$. We find 2.59, 2.31, and 1.91, which are all $< 3$. This completes the proof of the first half of the proposition.

For the second part of the proof we proceed in the same way but let MATHEMATICA help us to get a good start, namely by telling us that defining $m(p) = \sqrt{\mathcal{P}} \|c(2) - d\|$ we have $m(p) \leq m(43) < 4.1$ if $p < 10000$. We get after some calculation

$$\sqrt{\mathcal{P}} |\alpha^{(2)}| = \sqrt{\frac{2p}{2 + p + \sqrt{p} \sqrt{p} + 4A + 16}} \leq \sqrt{\frac{2p}{2 + p + \sqrt{p} \sqrt{p} - 8\sqrt{p} + 16}} < \frac{100}{99}$$

if $p > 10000$. Thus to complete the proof is enough to prove that

$$p |\beta^{(2)} - \frac{A + i3\sqrt{3}B}{2\sqrt{p}}|^2 + p |\gamma^{(2)} - \frac{3\sqrt{3}B - iA}{2\sqrt{p}}|^2 \leq 10.17 \quad (6.10)$$
if $p > 10000$, e.g. by proving that the first term of (6.10) is $< 1.07$ and the second term is $< 9.1$. This can be done as in the proof of the first part, using (4.4). Just as we have studied functions of $A$ restricted to the interval $|A| < 2\sqrt{p}$, we will now with the help of (4.6) and (4.7) write the left member of (6.10) as a function of $u$ and $v$, where $|u - 4| < v < u + 4$. Again a certain square root can be estimated with a Taylor formula. We leave the details to the reader.

**Proof of Proposition 6.12** Inspired by the first two rows of Table 6.5 we expect infinities near $\theta = \pi/6$, and thus, to avoid zeros in the denominator, we “turn everything upside down”. Thus we want to prove that

$$\limsup_{n \to \infty} \frac{\sqrt{p_n}}{|c^{(3)}_i(p_n)|} \leq 2.$$  

Suppose this is not true. Then (by taking subsequences if needed) we can find a sequence $\{p_n\}_1^\infty$ of primes $\equiv 1 \pmod{6}$ going to infinity, such that

$$\lim_{n \to \infty} \frac{\sqrt{p_n}}{c^{(3)}_j(p_n)} \sqrt{p_n} = l_j,$$  \hspace{1cm} (6.11)

where these limits exist (finite or $+\infty$) and $|l_j| > 2$ for at least one $j$ (0, 1, or 2). Since the interval $[0, \pi/3]$ is compact, we can by again taking a subsequence (keeping the notation $\{p_n\}_1^\infty$) arrange that $\theta_0 = \lim_{n \to \infty} \theta(p_n)$ exists. Starting from (4.5) we replace $A$ by $2\sqrt{p} \cos 3\theta$ and $B$ by $2\sqrt{p} \sin 3\theta/\sqrt{27}$. Introducing the resulting expressions for $\alpha^{(3)}$, $\beta^{(3)}$, and $\gamma^{(3)}$ in (4.2), we get $\sqrt{p}/c^{(3)}$ as a function of $p$ and $\theta$, which we denote by $q(p, \theta)$. We now fix $\theta = \theta_0$ and study $q(p, \theta_0)$ as a function of $p$ when $p \to \infty$. Estimating various square roots with a Taylor formula, we get after a considerable amount of calculation:

$$\lim_{p \to \infty} q(p, \theta_0) = \left(-2 \cos \theta_0, -2 \sin(\theta_0 - \pi/6), 2 \sin(\theta_0 + \pi/6)\right).$$

A simple continuity argument (w.r.t. $\theta(p_n)$ and $\theta_0$) shows that with $l_j$ from (6.11) we have

$$l_0 = -\cos \theta_0, \ l_1 = -2 \sin(\theta_0 - \pi/6), \ l_2 = 2 \sin(\theta_0 + \pi/6).$$  \hspace{1cm} (6.12)

This is a contradiction, since we have supposed that $|l_j| > 2$ for at least one $j$. We have thus completed the proof and also substantiated the “very large” part of item (5) of Remark 6.3 (take $l_1$ from (6.12) and consider $|1/l_1|$ for $\theta_0$ close to $\pi/6$).

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