THE LAS VERGNAS POLYNOMIAL FOR EMBEDDED GRAPHS

JOANNA A. ELLIS-MONAGHAN AND IAIN MOFFATT

This paper is dedicated to the memory of Michel Las Vergnas in gratitude for not only so much beautiful mathematics, but also many instances of kind and very insightful correspondence.

Abstract. The Las Vergnas polynomial is an extension of the Tutte polynomial to cellularly embedded graphs. It was introduced by Michel Las Vergnas in 1978 as special case of his Tutte polynomial of a morphism of matroids. While the general Tutte polynomial of a morphism of matroids has a complete set of deletion-contraction relations, its specialisation to cellularly embedded graphs does not. Here we associate a new matroid with an embedded graph. The rank function of this matroid measures how a graph separates the surface it is embedded in. This matroid enables us to extend the Las Vergnas polynomial to graphs arbitrarily embedded in surfaces. Furthermore, in this setting we can define deletion and contraction for embedded graphs consistently with the deletion and contraction of the underlying matroid perspective, thus yielding a version of the Las Vergnas polynomial with complete recursive definition. This also enables us to obtain a more complete understanding of the relationships among the Las Vergnas polynomial, the Bollobás-Riordan polynomial, and the Krushkal polynomial (which assimilates both of the preceding polynomials). We also take this opportunity to extend some of Las Vergnas’ results on Eulerian circuits from graphs in surfaces of low genus to surfaces of arbitrary genus.

1. Introduction

In [13, 15] (see also [12, 14]), Michel Las Vergnas introduced a polynomial $L_G(x, y, z)$ that extends the classical Tutte polynomial to cellularly embedded graphs. This topological Tutte polynomial, now called the Las Vergnas polynomial, is the first extension of the Tutte polynomial to embedded graphs that the authors are aware of. Michel Las Vergnas was ahead of his time in his investigation as not until many years later did other mathematicians and physicists initiate the serious attention now paid to embedded graph polynomials. More recent embedded graph polynomials, such as the ribbon graph polynomial of Bollobás and Riordan, $R_G$ (see [3, 4]), and Krushkal’s polynomial, $K_G$ (see [11]), have led in turn to renewed interest in $L_G$, for example in [1, 2].

The Las Vergnas polynomial was first defined in terms of the combinatorial geometry of an embedded graph (i.e., via circuit matroids). It arises as a special case in his much larger body of work on the Tutte polynomial of a morphism of matroids (see [2, 15, 16, 17, 18, 19]). Although it has its origins in matroid theory, $L_G$ is of independent interest as a tool for extracting both combinatorial and topological information from graphs embedded in surfaces. Accordingly, one of the aims of this work is to provide a formulation of $L_G$ that is readily accessible to topological graph theorists without reference to matroid theory, and so encouraging further investigation into it. (Also see [1] for such a discussion.)

Here we are especially interested in deletion-contraction definitions of graph polynomials. A very desirable property of such recursive definitions is that it reduces any graph to a linear combination...
of edgeless graphs. Las Vergnas gave this type of deletion-contraction definition for the Tutte polynomial of a morphism of matroids. This definition, however, does not hold for his cellularly embedded graph polynomial \( L_G \).

Here we show that by extending the Las Vergnas polynomial to all embedded graphs (not just cellularly embedded ones) it is possible to obtain a deletion-contraction definition of \( L_G \) in the language of topological graph theory. Furthermore, this recursive definition for the embedded graph polynomial is consistent with that for the Tutte polynomial of a morphism of matroids. Our approach is associate a new matroid with embedded graphs. To do this we introduce a rank function for embedded graphs that measures how a graph separates the surface it is embedded in. This matroid allows us to extend the Las Vergnas polynomial to the broader class of arbitrarily embedded graphs. Moreover, this extended polynomial arises as a special case Tutte polynomial of a morphism of matroids, just as the original cellularly embedded graph polynomial did. By carefully defining deletion and contraction for embedded graphs in a way that is compatible with deletion and contraction for their associated matroid perspectives, we are able to give complete deletion-contraction relations for the Las Vergnas polynomial.

Given the two extensions \( L_G \) and \( R_G \) of the Tutte polynomial to embedded graphs, it is natural to ask how they are related. The Krushkal polynomial, \( K_G \), of \([11]\), another generalisation of the Tutte polynomial, contains both \( L_G \) and \( R_G \) as specialisations (see \([1, 2]\) ), but does not yet provide a full understanding of the connection between the two polynomials. We show that, similarly, the Las Vergnas polynomial of an arbitrarily embedded graph arises as a specialisation of \( K_G \). We also discuss some the connections among these three topological Tutte polynomials, and show how differences among various definitions of deletion and contraction explain disagreements among them.

We also take the opportunity here to revisit some of Las Vergnas’ work on Eulerian circuits. In \([14]\), Las Vergnas gave a number of formulae for enumerating Eulerian circuits of 4-regular graphs in surfaces. However, most of the formulas only apply for graphs in the sphere, torus, or real projective plane. Now, with recently developed language and tools for ribbon graphs we are able to extend these results to all surfaces.

## 2. Background

One of our primary goals is describing the Las Vergnas polynomial in terms of embedded graphs instead of matroid perspectives, as this reveals its connections with other topological graph polynomials, exposes nuances of deletion and contraction, and facilitates future research. (See also \([1]\) for work in this direction.) We begin by reviewing some necessary concepts from topological graph theory and matroid perspectives.

### 2.1. Embedded graphs.

#### 2.1.1. Notation and terminology. We first establish some notation, assuming familiarity with basic graph theory and topological graph theory. Further details may be found in \([7, 10]\).

As usual, if \( G \) is a graph, then \( V(G) \) is its vertex set, and \( E(G) \) its edge set, with \( v(G) := |V(G)| \) and \( e(G) := |E(G)| \). We denote the number of components of \( G \) by \( c(G) \), and if \( A \subseteq E(G) \), then \( v(A), e(A), \) and \( c(A) \) are the number of vertices, edges, and components, respectively, of the spanning subgraph \( (V(G), A) \) of \( G \). In addition to the spanning subgraph determined by an edge set \( A \), we will also consider the induced subgraph \( G|_A \), which has edge set \( A \), and vertex set consisting of all of the vertices of \( G \) that are incident to an edge in \( A \) (so \( G|_A \) is not in general spanning). The rank of \( G \) is \( r(G) := v(G) - c(G) \), and the nullity of \( G \) is \( n(G) := e(G) - r(G) \). These agree with the rank and nullity of the cycle matroid of the graph discussed in Section \([2, 3]\).
Let $\Sigma$ be a surface, possibly with boundary. The *Euler genus*, $\gamma(\Sigma)$, of $\Sigma$ is its genus if it is non-orientable, and twice its genus if it is orientable. Recall that the *Euler characteristic*, $\chi(\Sigma)$, of $\Sigma$ can be obtained as $\chi(\Sigma) = v_t - e_t + f_t$, where $v_t$, $e_t$, and $f_t$ are the numbers of vertices, edges, and faces, respectively, in any triangulation (or more generally, cellulation) of $\Sigma$. *Euler’s formula* gives that $\gamma(\Sigma) = 2k(\Sigma) - b(\Sigma) - \chi(\Sigma)$, where $k(\Sigma)$ is the number of components of $\Sigma$ and $b(\Sigma)$ the number of its boundary components. For a subset $X$ of $\Sigma$, we let $N(X)$ denote a regular neighbourhood of $X$.

An *embedded graph*, $G \subset \Sigma$, consists of a graph $G$ and a drawing of $G$ on a closed surface $\Sigma$ such that the edges only intersect at their ends. The components remaining after removing the points of $G$ from $\Sigma$ are called *regions*. An embedded graph $G \subset \Sigma$ is *cellularly embedded* if each of its regions is homeomorphic to an open disc.

To reduce notational clutter, we establish the following convention.

**Notation 2.1.** Let $G \subset \Sigma$ be an embedded graph and $A \subseteq E(G)$. Then by $\Sigma \setminus A$ we mean the surface with boundary obtained by deleting the interior of a regular neighbourhood of the embedded spanning subgraph $(V(G), A) \subset \Sigma$ of $G \subset \Sigma$.

Observe that with this notational convention $\Sigma \setminus \emptyset$ consists of $\Sigma$ with $v(G)$ discs removed.

We note that the above notation is in keeping with the standard use of $v(A)$, $e(A)$, $c(A)$, etcetera to refer to the spanning subgraph $(V(G), A)$ of $G$.

**2.1.2. Surface deletion and contraction.** Deletion of an edge of an embedded graph is straightforward. If $G \subset \Sigma$ is an embedded graph and $e \in E(G)$ then $G \setminus e \subset \Sigma$ is the embedded graph obtained by removing the edge $e$ from the drawing of $G \subset \Sigma$ (without removing the points of $e$ from $\Sigma$). However, in contrast with the deletion defined below via ribbon graphs, here if $G$ is cellularly embedded then $G \setminus e$ need not be.

Contraction of a non-loop edge of an embedded graph is simply contraction in the drawing of the graph on the surface, moving the two end point vertices within the surface until they merge into a single vertex, smoothly deforming any other incident edges along with them, thus forming $G/e \subset \Sigma/e$. In the case of a non-loop edge, $\Sigma/e = \Sigma$. This may be viewed as deleting the interior of a regular neighbourhood of $e$ from $\Sigma$, which creates a new boundary component, then contracting this boundary component to a point, carrying the drawing of $G \setminus e$ along with the surface, and then placing a new vertex on the resulting point.

This second construction of the contraction of an edge also applies to loops. If $e$ is a loop incident with a vertex $v$, delete the interior of a regular neighbourhood of $e$ from $\Sigma$, creating either one or two new boundary components. Contract each boundary component to a point, carrying the drawing of $G \setminus e$ along with the surface. This creates a new surface which we denote $\Sigma/e$. Place a new vertex on each of these one or two newly created points, thus forming $G/e \subset \Sigma/e$, as in Figure 1.

As an example, if $G \subset S^2$ consists of one vertex and one edge $e$ in the 2-sphere, then $G/e \subset S^2/e$ consists of two 2-spheres each containing an isolated vertex. Similarly, if $G \subset \mathbb{RP}^2$ is one vertex and one edge cellularly embedded in the real projective plane, then $G/e \subset \mathbb{RP}^2$ is an isolated vertex in

\[ G \subset \Sigma \quad \text{delete} \quad G/e \subset \Sigma/e \]
a 2-sphere. Note that as with the case of deletion, the ribbon graphs $G$ and $G/e$ may correspond to cellularly embedded graphs in different surfaces.

Furthermore, while the notation $G/e \subset \Sigma/e$ is appropriate for our purposes here, it is important to remember that if $\tilde{G}$ is the underlying abstract graph of $G \subset \Sigma$, and if $e$ is a loop, then the underlying abstract graph of $G/e \subset \Sigma/e$ may not in general be $\tilde{G}\backslash e$. To minimise any confusion, we adopt the common convention that loops in abstract graphs are never contracted, only deleted, while we do have a mechanism for contracting loops in embedded graphs.

2.2. Ribbon graphs.

2.2.1. Notation and terminology. At times it will be convenient to describe cellularly embedded graphs as ribbon graphs, particularly when discussing deletion and contract for cellularly embedded graphs. We refer the reader to [7] for a more detailed discussion of ribbon graphs. A ribbon graph $G = (V(G), E(G))$ is a surface with boundary, represented as the union of two sets of discs: a set $V(G)$ of vertices and a set of edges $E(G)$ such that: (1) the vertices and edges intersect in disjoint line segments; (2) each such line segment lies on the boundary of precisely one vertex and precisely one edge; (3) every edge contains exactly two such line segments.

Ribbon graphs arise as regular neighbourhoods of cellularly embedded graphs, with the vertex set of the ribbon graph arising from regular neighbourhoods of the vertices of the embedded graph, and the edge set of the ribbon graph arising from regular neighbourhoods of the edges of the embedded graph. On the other hand, if $G$ is a ribbon graph, then topologically it is a surface with boundary. Filling in each hole by identifying its boundary component with the boundary of a disc results in a ribbon graph embedded in a closed surface. (This in a band decomposition with the vertex discs called 0-bands, the edge discs called 1-bands and the face discs are called 2-bands.) A deformation retract of the ribbon graph in the surface yields a graph cellularly embedded in the surface. Thus ribbon graphs, band decompositions, and cellularly embedded graphs are equivalent.

If $G$ is a ribbon graph, then $G^*$ is the ribbon graph corresponding to the geometric dual when $G$ is viewed as a cellularly embedded graph. Because of the duality between $G$ and $G^*$ in the case of cellularly embedded graphs, it is often useful to think of them in the setting of band decompositions. In this setting $G$ and $G^*$ are represented by the same topological object, the only difference being that the sets designated as vertices (0-bands) and face discs (2-bands) are reversed. See [7] for details.

If $G$ is a ribbon graph, then $v(G)$, $e(G)$, $c(G)$, $r(G)$, and $n(G)$ are all as defined for the underlying abstract graph of $G$ (we use $c$ for the components of a graph and $k$ for the components of a surface, as these need not coincided if $G$ is not cellularly embedded). Furthermore, $f(G)$ is the number of boundary components of the surface defining the ribbon graph, and the Euler genus, $\gamma(G)$, of $G$ equals the Euler genus of the surface defining the ribbon graph. Euler’s formula gives that $v(G) − e(G) + f(G) = 2c(G) − \gamma(G)$. A ribbon graph $G$ is plane if it is connected and $\gamma(G) = 0$. A ribbon graph $H$ is a ribbon subgraph of $G$ if $H$ can be obtained by deleting vertices and edges of $G$. If $H$ is a ribbon subgraph of $G$ with $V(H) = V(G)$, then $H$ is a spanning ribbon subgraph of $G$. We let $r(A)$, $c(A)$, $n(A)$, $f(A)$, $g(A)$, $\gamma(A)$ each refer to the spanning subgraph $(V(G), A)$ of $G$ (where $G$ is given by context).

2.2.2. Ribbon graph deletion and contraction. Deletion for ribbon graphs just removes an edge: if $G$ is a ribbon graph, and $e \in E(G)$, then $G − e$ is the spanning ribbon subgraph on edge set $E(G)\backslash\{e\}$. (Note that we use “−” for ribbon graph edge deletion, and “\” for embedded graph edge deletion.) An important aspect of deletion as defined via ribbon graphs is that the result is again a ribbon graph. Remembering ribbon graphs correspond to cellularly embedded graphs, ribbon graph edge deletion is appropriate for cellularly embedded graphs as $G − e$ remains in the class of cellularly embedded graphs. However, the surface associated with $G$ by filling in the holes with discs may
not be the same surface associated with \( G - e \). For example, if \( e \) is a bridge or a non-orientable loop, then deleting the former will increase the number of components of the surface, and deleting the latter may change its orientability.

Contraction requires a bit more care than deletion. Let \( u \) and \( v \) be the (not necessarily distinct) incident vertices of \( e \). Then \( G/e \) denotes the ribbon graph obtained as follows: consider the boundary component(s) of \( e \cup u \cup v \) as curves on \( G \). For each resulting curve, attach a disc (which will form a vertex of \( G/e \)) by identifying its boundary component with the curve. Delete the interior of \( e \cup u \cup v \) from the resulting complex. We say that \( G/e \) is obtained from \( G \) by contracting \( e \).

A very important point here is that ribbon graph deletion and embedded graph deletion are incompatible operations. For example, if \( G \) is the genus 1 orientable ribbon graph consisting of two loops and one vertex, and \( e \in E(G) \), then \( G - e \) is of genus zero. However if \( G' \subset T^2 \) is the cellularly embedded graph corresponding to the ribbon graph \( G \), then \( G' \setminus e \) is not equivalent to \( G/e \), since \( G' \setminus e \subset T^2 \), while \( G - e \) has genus zero.

In contrast, the two definitions of contraction are consistent. If \( G \) is cellularly embedded in \( \Sigma \), and corresponds to the ribbon graph \( G' \), then \( G/e \subset \Sigma/e \) is a cellularly embedded graph and corresponds to \( G'/e \). This follows since \( N(e) \) in the cellular embedding corresponds to the union of the discs for \( u \), \( v \), and \( e \) when \( G' \) is viewed as embedded in the surface resulting from filling in its boundaries. Contracting the boundaries of \( N(e) \) to points and calling the points vertices corresponds to putting vertex discs in the boundaries of \( e \cup u \cup v \).

2.3. Matroids and matroid perspectives. Since the Las Vergnas polynomial was originally defined in terms of the Tutte polynomial of a matroid perspective, we review the essential concepts of matroids and matroid perspectives, and recall the definition of the Tutte polynomial of a matroid perspective. We work with matroids in terms of rank functions.

A matroid \( M = (E, r) \) consists of a set \( E \) and a rank function, \( r : \mathcal{P}(E) \to \mathbb{Z}_{\geq 0} \) from the power set of \( E \) to the non-negative integers such that for each \( A \subseteq E \) and \( e, f \in E \) we have

\[
\begin{align*}
(2.1) & \quad r(\emptyset) = 0 \\
(2.2) & \quad r(A \cup \{e\}) \in \{r(A), r(A) + 1\} \\
(2.3) & \quad r(A) = r(A \cup \{e\}) = r(A \cup \{f\}) \implies r(A \cup \{e, f\}) = r(A)
\end{align*}
\]

A set \( A \subseteq E \) is independent if \( r(A) = |A| \), and is a circuit if \( r(A) = |A| - 1 \). An element \( e \in E \) is an isthmus (or loop) if for each independent set \( A \), we have that \( A \cup \{e\} \) is also independent; and \( e \) is a loop if \( \{e\} \) is a circuit.

If \( M = (E, r) \) is a matroid and \( e \in E \), then \( M \setminus e = (E \setminus \{e\}, r|_{E \setminus \{e\}}) \) is the matroid obtained by deleting \( e \); and \( M/e = (E \setminus \{e\}, r') \), where \( r'(A) := r(A \cup \{e\}) - r(\{e\}) \), is the matroid obtained by contracting \( e \). The dual of \( M \) is the matroid given by \( M^* = (E, r^*) \), where \( r^*(A) := |A| + r(E \setminus A) - r(E) \).

If \( G \) is a graph, its cycle matroid is \( C(G) := (E(G), r_{C(G)}) \), where \( r_{C(G)}(A) := v(A) - c(A) \); and its bond matroid is \( B(G) := (C(G))^* \). When \( G \) is a plane graph (i.e. a graph cellularly embedded a sphere) \( B(G^*) = (C(G^*))^* = C((G^*))^* = C(G) \). However, this identity does not hold in general when \( G \) cellularly embedded in a higher genus surface.

A matroid perspective is a triple \((M, M', \varphi)\) where \( M = (E, r) \) and \( M' = (E', r') \) are matroids, and \( \varphi : E \to E' \) is a bijection such that for all \( A \subseteq B \subseteq E \), we have

\[
(2.4) \quad r(B) - r(A) \geq r'(\varphi(B)) - r'(\varphi(A)).
\]

Following the usual convention in the area, at times we suppress the bijection \( \varphi \) and use \( M \to M' \) to denote a matroid perspective \((M, M', \varphi)\), especially since we will primarily be interested in matroid perspectives of the form \((M, M', \text{id})\).
We note that the condition given by (2.4) can be equivalently formulated as the requirement that each circuit of \( M \) is mapped to a union of circuits of \( M' \), or as the requirement that every flat of \( M' \) corresponds to a flat of \( M \).

Deletion and contraction for a matroid perspective \( (M, M', \varphi) \) are defined by, for \( e \in E \), setting \( (M, M', \varphi)\backslash e := (M\backslash e, M'\backslash e, \varphi|\backslash e) \) and \( (M, M', \varphi)/e := (M/e, M'/e, \varphi|\backslash e) \). We will denote these matroid perspectives by \( M\backslash e \rightarrow M'\backslash e \) and \( M/e \rightarrow M'/e \), respectively.

2.4. The Tutte polynomial of a matroid perspective. Let \( M = (E, r) \) and \( M' = (E', r') \) be matroids. As defined in [13, 15], the Tutte polynomial of the matroid perspective \( M \rightarrow M' = (M, M', \varphi) \) is defined by

\[
T_{M \rightarrow M'}(x, y, z) = \sum_{X \subseteq E} (x - 1)^{r'(E') - r'(\varphi(X))}(y - 1)^{|X| - r(X)}z^{r(E) - r(X) - (r'(E') - r'(\varphi(X)))}.
\]

(2.5) As noted in [15], the classical Tutte polynomial \( T_M(x, y) = \sum_{X \subseteq E} (x - 1)^{r(E) - r(X)}(y - 1)^{|X| - r(X)} \) of a matroid \( M \) can be recovered from the more general polynomial:

\[
\begin{align*}
T_M(x, y) &= T_{M \rightarrow M}(x, y, z) \\
T_M(x, y) &= T_{M \rightarrow M'}(x, y, x - 1) \\
T_M(x, y) &= (y - 1)^{r'(M') - r(M')}T_{M \rightarrow M'}(x, y, 1/(y - 1))
\end{align*}
\]

Las Vergnas (Theorem 5.3 of [16]) showed that \( T_{M \rightarrow M'} \) satisfies deletion-contraction relations that provide a complete recursive definition of the polynomial.

**Theorem 2.2.** Let \( M \rightarrow M' \) be a matroid perspective on a set \( E \). The following relations hold:

1. if \( e \in E \) is neither an isthmus nor a loop of \( M \), then

\[
T_{M \rightarrow M'}(x, y, z) = T_{M \backslash e \rightarrow M' \backslash e}(x, y, z) + T_{M/e \rightarrow M'/e}(x, y, z);
\]

2. if \( e \in E \) is an isthmus of \( M' \), and hence also an isthmus of \( M \), then

\[
T_{M \rightarrow M'}(x, y, z) = xT_{M \backslash e \rightarrow M' \backslash e}(x, y, z);
\]

3. if \( e \in E \) is a loop of \( M \), and hence also a loop of \( M' \), then

\[
T_{M \rightarrow M'}(x, y, z) = yT_{M \backslash e \rightarrow M' \backslash e}(x, y, z);
\]

4. if \( e \in E \) is an isthmus of \( M \), and is not an isthmus of \( M' \), then

\[
T_{M \rightarrow M'}(x, y, z) = xT_{M \backslash e \rightarrow M' \backslash e}(x, y, z) + T_{M/e \rightarrow M'/e}(x, y, z);
\]

5. if \( E = \emptyset \), then \( T_{M \rightarrow M'}(x, y, z) = 1 \).

3. Las Vergnas' topological Tutte polynomial

The Las Vergnas polynomial, \( L_G \), was first defined in terms of the combinatorial geometry of an embedded graph, that is, \( B(G^*) \) and \( C(G) \), the bond and circuit geometries, or equivalently bond and cycle matroids, of \( G^* \) and \( G \) from Subsection 2.3. In Proposition 3.2 we describe \( L_G \) in graph theoretical terms. (This approach was also taken in [1].) In subsection 3.1 like Las Vergnas, we assume that \( G \) is cellularly embedded, so that \( G^* \) is also cellularly embedded in the same surface as \( G \). In this setting, we use the rank function of \( C(G) \) which is given by \( r(G) = v(G) - c(G) \). However, to realize the full power of the Tutte polynomial of a matroid perspective, in particular its deletion-contraction reductions, we need to extend the domain of the Las Vergnas polynomial to graphs that are not necessarily cellularly embedded. We do this in Subsection 3.2 using a new rank function that measures how the graph separates the surface.
3.1. The Las Vergnas polynomial for cellurally embedded graphs.

**Definition 3.1.** Let $G$ be a graph cellurally embedded in a surface $\Sigma$. Let $B(G^*) \rightarrow C(G)$ denote the matroid perspective $(B(G^*), C(G), \text{id})$, where id is the natural identification of the edges of $G$ and $G^*$ and so suppressed in the following. Then the Las Vergnas polynomial, $L_G$, is defined by

$$L_G(x, y, z) := T_{B(G^*) \rightarrow C(G)}(x, y, z).$$

By translating the notation and using Euler’s formula, we can rewrite Las Vergnas’ topological Tutte polynomial in a form that more clearly reveals how it encodes topological information (see also [1]).

**Proposition 3.2.** Let $G$ be a ribbon graph. Then

$$L_G(x, y, z) = \sum_{A \subseteq E(G)} (x - 1)^{r(G) - r(G)(A)} (y - 1)^{\mu(G)(A)} - \frac{\gamma(G) + \gamma(G)(A) - \gamma(G^*)(A^*)}{2} z^{\gamma(G) - \gamma(G)(A) - \gamma(G^*)(A^*)}/2,$$

where $A^c := E(G) - A$.

**Proof.** By definition,

$$L_G(x, y, z) = \sum_{A \subseteq E(G)} (x - 1)^{r(G) - r(G)(A)} (y - 1)^{|A| - r(B(G^*))(A) z^{r(B(G^*)) - r(C(G)) - (r(B(G^*))(A) - r(C(G))(A))}}.$$

Note that $r(C(G)) = r(G)$ and $r_G(A) = r_G(A)$, and recall that $r_M^G(A) = |A| + r_M(\Sigma \setminus A) - r(M)$. Then, since $B(G) = (C(G))^*$, we have $r(B(G^*)) = r(C(G^*)) = e(G^*) + r_G(\emptyset) - r(G^*) = n(G^*)$, and $r_B(G^+)(A) = r(C(G^+))^*(A) = |A| + r_G(A) - r(G^*)$. (Recall that if $G$ is not plane, then $C(G^+)^*$ and $C(G)$ are not generally equal.)

By Euler’s formula and the facts that $f_G^*(A^*) = f_G(A)$ and $f_G(A) = v(G)$, we have

$$2r_B(G^+)(A) = 2|A| - 2k_G(A^*) + 2e(G^*)$$

$$= 2|A| - v_G(A^*) + |A^c| - f_G(A^c) - \gamma_G(A^c) + v(G^*) - e(G^*) + f(G^*) + \gamma(G^*)$$

$$= |A| - v(G^*) + e(G^*) - f_G(A^*) - \gamma_G(A^c) + v(G^*) - e(G^*) + f(G^*) + \gamma(G^*)$$

$$= |A| - f_G(A) - \gamma_G(A^c) + v(G) + \gamma(G^*)$$

$$= v_G(A) - 2k_G(A) + \gamma_G(A) - \gamma_G(A^c) + v(G) + \gamma(G^*)$$

$$= 2r_G(A) + \gamma(G) + \gamma_G(A) - \gamma_G(A^c).$$

Also, using this computation for the exponent of $z$, we have:

$$r(B(G^*)) - r(C(G)) - (r_B(G^+)(A) - r_C(G)(A))$$

$$= n(G^*) - r(G) - r_G(A) + r_G(A) - (\gamma(G) + \gamma_G(A) - \gamma_G(A^c))/2$$

$$= e(G^*) - v(G^*) + c(G^*) - v(G) + c(G) - (\gamma(G) + \gamma_G(A) - \gamma_G(A^c))/2$$

$$= e(G) - f(G) + 2c(G^*) - v(G) - (\gamma(G) + \gamma_G(A) - \gamma_G(A^c))/2$$

$$= (\gamma(G) - (\gamma(G) + \gamma_G(A) - \gamma_G(A^c))/2)$$

$$= (\gamma(G) - \gamma_G(A) + \gamma_G(A^c))/2.$$

If $G$ is plane, so that $\gamma(G) = \gamma(A) = 0$ for all $A \subseteq E(G)$, then is is easily seen from Equation (3.1) that $L_G(x, y, z) = T_G(x, y)$. Furthermore, Las Vergnas showed in [15] that, for any cellurally embedded graph $G$, the Tutte polynomial of the underlying abstract graph of $G$ can be recovered from $L_G$ as

$$L_G(x, y, z) = T_G(x, y).$$
Collecting the topological contributions in the expression for \(L_G\) given in Equation 3.1 gives the following particularly simple form of \(L_G\), which facilitates comparison with other topological graph polynomials.

\[
(3.3) \quad (z(y - 1))^{n(G)} L_G \left( x, y, \frac{1}{z^2(y - 1)} \right) = \sum_{A \subseteq E(G)} (x - 1)^{r_G(A) - r_G(A')} (y - 1)^{n_G(A)} z^{\gamma_G(A) - \gamma_G(A')}. 
\]

It is also informative to compare the following form of \(L_G\), which is obtained by expanding the Euler genus terms using Euler’s formula, to the dichromatic polynomial, \(Z_G(x, y) := \sum_{A \subseteq E(G)} x^{e(A)} y^{|A|} = (x/y)^{e(G)} y^n G \gamma G((x + y)/x, y + 1):

\[
L_G(x, y, z) = (1/(x - 1)(y - 1))^{c(G)} z^{n(G^*)} \sum_{A \subseteq E(G)} ((x - 1)/z)^{e(A)} ((y - 1)z)^{\gamma_G(A')}(1/z)^{|A|}.
\]

We note that in [5] it is shown that \(L_G\) is also determined by the delta-matroid of \(G\), but we do not pursue this perspective here.

3.2. The Las Vergnas polynomial for arbitrarily embedded graphs. Although the Tutte polynomial of a matroid perspective, \(T_{M \rightarrow M'}\), has a deletion-contraction relation that applies to all types of edges, the Las Vergnas polynomial for cellularly embedded graphs does not (although it does have a deletion-contraction relation for some special types of edges). The difficulty is that deleting certain types of edge of a cellularly embedded graph can result in a graph that is not cellularly embedded. Taking an example from [1] a little further, if \(G\) is the theta graph cellularly embedded on the torus, then \(L_G(x, y, z) = 3z + 2z^2 + xz^2 + 1\). Of the 17 cellularly embedded graphs on two edges, none have an \(x^2z\) term, and so \(L_G = 3z + 2z^2 + xz^2 + 1\) can not satisfy the deletion-contraction identities of Theorem 2.2 for any notion of edge deletion or contraction. Thus although \(L_G\) is defined in terms of \(T_{M \rightarrow M'}\), it does not have a complete recursive definition even though \(T_{M \rightarrow M'}\) does. However, here we will see that by extending the Las Vergnas polynomial to the class of (not necessarily cellularly) embedded graphs we can get a recursive definition for \(L_G\).

3.3. A matroid associated with embedded graphs. The construction of the matroid \(B(G^*)\) used in the original definition of \(L_G\) requires that \(G\) be cellularly embedded (as otherwise the construction of the geometric dual is not a well-defined process). To extend \(L_G\) to all embedded graphs we begin with an observation about \(B(G^*)\), or more specifically \(r^*\), in the case that \(G\) is cellularly embedded.

Proposition 3.3. If \(G\) is cellularly embedded and \(G^*\) is its cellularly embedded geometric dual, then \(r_{B(G^*)}(A) = |A| - k(\Sigma \setminus A) + k(\Sigma)\).

Proof.

\[
(3.4) \quad r_{B(G^*)}(A) = r_{(C(G^*))^*}(A) = |A| + r_{C(G^*)}(E \setminus A) - r_{C(G^*)}(E) = |A| + v_{G^*}(E \setminus A) - c_{G^*}(E \setminus A) - v_{G^*}(E) + c_{G^*}(E) = |A| - c_{G^*}(E \setminus A) + c_{G^*}(E) = |A| - k(\Sigma \setminus A) + k(\Sigma),
\]

All the equalities except the last follow from the definitions. The last equality may be seen in the setting of band decompositions. Consider a band decomposition for \(G\), and denote the sets of 0-, 1-, and 2-bands by \(V\), \(E\), and \(F\), respectively. Then \(c_{G^*}(E \setminus A)\) is the number of components of the ribbon graph \(G^* - A = F \cup (E \setminus A)\). On the other hand, \(k(\Sigma \setminus A)\) is the number of components of \(\Sigma \setminus A = (V \cup E \cup F) \setminus (V \cup A) = F \cup (E \setminus A)\). It follows that \(c_{G^*}(E \setminus A) = k(\Sigma \setminus A)\). The equality \(c_{G^*}(E) = k(\Sigma)\) follows since \(G\) is cellularly embedded. \(\square\)
The quantity \(|A| - k(\Sigma \setminus A) + k(\Sigma)| may be computed even for non-cellularly embedded graphs. This leads to the following definition of a rank function that we will see associates a matroid to all embedded graphs, not just cellularly embedded ones.

**Definition 3.4.** Let \( G \subseteq \Sigma \) be a graph embedded in a surface (this embedding need not be cellular), and let \( A \subseteq E(G) \). We define

\[
    r_{(G, \Sigma)}(A) := |A| + k(\Sigma) - k(\Sigma \setminus A).
\]

**Proposition 3.5.** Let \( G \subseteq \Sigma \) be a graph embedded in a surface. Then

1. the pair \( M(G, \Sigma) := (E(G), r_{(G, \Sigma)}) \) forms a matroid, and
2. \((M(G, \Sigma), C(G), \text{id})\) is a matroid perspective.

**Proof.** To reduce notational clutter, for the first part proof (while there is only one rank function involved) we will use \( r \) to denote \( r_{(G, \Sigma)} \). We first show that \( M(G, \Sigma) \) is a matroid by showing it satisfies Equations \((2.1)-(2.3)\). Let \( A \subseteq E(G) \) and \( e, f \in E(G) \). Trivially, \( r(A) \in \mathbb{Z}_{\geq 0} \).

Equation \((2.1)\), which specifies the value of the rank of the empty set, is satisfied since \( r(\emptyset) = |\emptyset| + k(\Sigma) - k(\Sigma \setminus \emptyset) = 0 + k(\Sigma) - k(\Sigma) = 0 \). This uses the fact that \( \Sigma \setminus \emptyset \) consists of \( \Sigma \) with \( v(G) \) discs removed.

Next, for Equation \((2.2)\), we have

\[
    r(A \cup \{e\}) - r(A) = 1 - k(\Sigma \setminus (A \cup \{e\})) + k(\Sigma \setminus A),
\]

but removing \( N(e) \) from \( \Sigma \setminus A \) can create at most one additional component. (To see this note that \( e \subset \Sigma \setminus A \) is a simple curve between one or two holes of \( \Sigma \setminus A \); deleting a neighbourhood of this arc, giving \( (\Sigma \setminus A) \setminus N(e) \), can create at most one additional component; and that the closure of the resulting surface is \( \Sigma \setminus (A \cup \{e\}) \).) Thus, \( k(\Sigma \setminus (A \cup \{e\})) - k(\Sigma \setminus A) \in \{0, 1\} \) and it follows that \( r(A \cup \{e\}) \in \{r(A), r(A) + 1\} \).

Finally, for Equation \((2.3)\), by Equation \((3.5)\), we have

\[
    r(A) = r(A \cup \{e\}) \iff k(\Sigma \setminus (A \cup \{e\})) = k(\Sigma \setminus A) + 1,
\]

and

\[
    r(A) = r(A \cup \{f\}) \iff k(\Sigma \setminus (A \cup \{f\})) = k(\Sigma \setminus A) + 1.
\]

Now suppose that \( r(A) = r(A \cup \{e\}) = r(A \cup \{f\}) \) and consider \( \Sigma' := \Sigma \setminus A \). Then by \((3.6)\) and \((3.7)\), deleting \( e \) from \( \Sigma' \) creates an extra component, and deleting \( f \) from \( \Sigma' \) also creates an extra component. It follows that deleting \( \{e, f\} \) from \( \Sigma' \) must create two extra components. (This is since if it created no extra components then deleting \( e \) from \( \Sigma' \) would not; and if it created one extra component, and deleting \( e \) from \( \Sigma' \) also created one component, then deleting \( f \) from \( \Sigma' \) could not create an extra component.) Thus

\[
    r(A \cup \{e, f\}) - r(A) = 2 - k(\Sigma \setminus (A \cup \{e, f\})) + k(\Sigma \setminus A) = 0,
\]

and so \( r(A \cup \{e, f\}) = r(A) \), as required.

To prove that \((M(G, \Sigma), C(G), \text{id})\) is a matroid perspective, we must show that the rank functions satisfy the condition from Equation \((2.4)\). By telescopic summations, it is suffices to do this one edge at a time, that is, to show that

\[
    r_{(G, \Sigma)}(A \cup \{e\}) - r_{(G, \Sigma)}(A) \geq r_{C(G)}(A \cup \{e\}) - r_{C(G)}(A),
\]

for each \( A \subseteq E \) and \( e \in E \setminus A \). By Equation \((3.5)\), this reduces to,

\[
    1 - k(\Sigma \setminus (A \cup \{e\})) + k(\Sigma \setminus A) \geq c(A) - c(A \cup \{e\}),
\]

so we need to show that

\[
    c(A) - c(A \cup \{e\}) = 1 \implies k(\Sigma \setminus (A \cup \{e\})) - k(\Sigma \setminus A) = 0.
\]
If \( c(A) - c(A \cup \{e\}) = 1 \), then \( e \) is a bridge of \((V(G), A)\), so it lies on exactly one region in the drawing of \((V(G), A)\) in \(\Sigma\). Furthermore, the disc formed by the intersection of a regular neighborhood of \( e \) with \(\Sigma \setminus A\) must meet two distinct boundary components of the same component of \(\Sigma \setminus A\) (since \( e \) is not a loop). Removing this disc to form \(\Sigma \setminus (A \cup e)\) merges these two boundary components into a single boundary component, which does not separate the component of \(\Sigma\) into two components (since \( e \) is adjacent to exactly one region of \((V(G), A) \subset \Sigma\)). Thus, \( k(\Sigma \setminus (A \cup \{e\})) - k(\Sigma \setminus A) = 0 \), as needed. \(\square\)

We now extend the Las Vergnas polynomial to all embedded graphs, giving an expression for it in purely topological graph theory terms.

**Definition 3.6.** Let \( G \) be a graph embedded in a surface \(\Sigma\), and write \( E \) for \(E(G)\). Then the Las Vergnas polynomial, \( L_{G \subset \Sigma} \), is defined by

\[
(3.8) \quad L_{G \subset \Sigma}(x, y, z) := T_{(M(G,\Sigma),C(G),\text{id})}(x, y, z) = \sum_{A \subseteq E} (x - 1)^{c(A) - c(E)}(y - 1)^{k(\Sigma \setminus A) - k(\Sigma)}z^{|E| - |A| - k(\Sigma \setminus E) + k(\Sigma \setminus A) + c(E) - c(A)}.
\]

If \( G \subset \Sigma \) is cellulary embedded then Definition 3.6 reduces to the classical case as follows.

**Proposition 3.7.** If \( G \subset \Sigma \) is a cellulary embedded graph, then

1. \( M(G,\Sigma) = B(G^*) \).
2. \( (M(G,\Sigma),C(G),\text{id}) = (B(G^*),C(G),\text{id}) \), and
3. \( L_{G \subset \Sigma}(x, y, z) = L_G \).

**Proof.** The first item follows from Equation (3.4). The second item follows from the first item, and the third from the second. \(\square\)

### 3.4. Deletion and contraction.

**Lemma 3.8.** Let \( G \subset \Sigma \) be an embedded graph, and \( e \in E(G) \). Then

1. \( M(G,\Sigma)\setminus e = M(G\setminus e,\Sigma) \), and
2. \( M(G,\Sigma)/e = M(G/e,\Sigma/e) \).

**Proof.** That \( M(G,\Sigma)\setminus e = M(G\setminus e,\Sigma) \) follows since both matroids are on the same set, and since \( r(G,\Sigma)(A) = r(G\setminus e,\Sigma)(A) \), when \( e \notin A \).

For the second item, if \( M(G,\Sigma) = (E, r(G,\Sigma)) \) then \( M(G,\Sigma)/e = (E, r') \) where

\[
(3.9) \quad r'(A) = r(G,\Sigma)(A \cup \{e\}) - r(G,\Sigma)(\{e\}) = |A| + k(\Sigma \setminus e) - k(\Sigma \setminus (A \cup \{e\}))
\]

On the other hand,

\[
r_{G/e,\Sigma/e}(A) := |A| + k(\Sigma/e) - k((\Sigma/e)\setminus A).
\]

Recall that \( G/e \subset \Sigma/e \) is formed by deleting a regular neighbourhood of \( e \) from \(\Sigma\), and then contracting the boundary components, which created one or two points. From this construction, it is clear that \( k(\Sigma/e) = k(\Sigma \setminus N(e)) = k(\Sigma \setminus e) \) (noting for \(\Sigma \setminus e\) that boundary components created by isolated vertices do not change the number of connected components), and that \( k((\Sigma/e)\setminus A) = k((\Sigma \setminus N(e))\setminus A) = k((\Sigma \setminus (A \cup \{e\})) \). Substituting these into (3.9) gives that \( r_{G/e,\Sigma/e}(A) = r'(A) \). Since \( M(G/e,\Sigma/e) \) and \( M(G,\Sigma)/e \) are over the same set it follows that the matroids are equal. \(\square\)

For the cycle matroid \( C(G) \) of a graph \( G \), we have that \( e \) is a loop in \( G \) if and only if it is a loop in \( C(G) \), and that \( e \) is a bridge (cut-set of size one) in \( G \) if and only if it is an isthmus in \( C(G) \). This however does not hold for the matroid \( M(G,\Sigma) \). To find the the types of edges of an embedded graph that correspond to loops and isthmuses in \( M(G,\Sigma) \) we generalise loops and bridges by extracting one key feature of each:
**Definition 3.9.** Let $G \subset \Sigma$ be an embedded graph, and $e \in E(G)$. Then we say that $e$ is a quasi-loop if $k(\Sigma \setminus e) > k(\Sigma)$, and we say that $e$ is a quasi-bridge if it is adjacent to exactly one region of $G \subset \Sigma$.

A quasi-loop in $G$ is a loop, and a bridge in $G$ is a quasi-bridge. However, a loop is not necessarily a quasi-loop and quasi-bridge need not be a bridge. Consider for example a longitudinal loop on a torus. This is not a quasi-loop, and is in fact a quasi-bridge.

**Lemma 3.10.** Let $G \subset \Sigma$ be an embedded graph, and $e \in E(G)$. Then the following hold.

1. $e$ is a quasi-loop in $G$ if and only if $e$ is a loop in $M(G, \Sigma)$.
2. $e$ is a quasi-bridge in $G$ if and only if $e$ is an isthmus of $M(G, \Sigma)$.

**Proof.** For the first item, $e$ is a loop in $M(G, \Sigma)$ if and only if $r_{(G, \Sigma)}(e) = 0$ if and only if $1 + k(\Sigma) - k(\Sigma \setminus e) = 0$ if and only if $k(\Sigma \setminus e) > k(\Sigma)$, since deleting $e$ cannot create more than one additional component.

For the second item, given $G \subset \Sigma$, we consider a slight generalisation of a polygonal decomposition of a surface. Recall that if $G$ is cellurally embedded in a surface $\Sigma$, we may create a polygonal decomposition of $\Sigma$ by arbitrarily orienting the edges of $G$ and giving them distinct labels (we identify the labels with the edge names). The faces of $G$ are then polygons with directed labeled sides, and thus give a polygonal decomposition of $\Sigma$, and $\Sigma$ may be recovered from this set of labeled polygons by identifying edges of polygons with the same labels consistently with the directions of the arrows. This can be thought of as “cutting” the surface along the directed edges of $G$ to form the polygons with directed sides labeled by the edge names. We can generalise this construction to arbitrarily embedded graphs. Again we direct and distinctly label the edges of $G$, and cut the surface along edges. The resulting regions are no longer necessarily polygons, but surfaces with labeled directed curves forming on their boundary components. Again, the original surface, with the graph embedded in it, may be recovered by identifying curves with like labels so that the directions align.

If $G$ is embedded in a surface $\Sigma$ and we arbitrarily direct and distinctly label its edges, then we denote the set of surfaces that result from cutting along $G$ by $\text{Cut}(E)$, and call these surfaces the bricks of $\text{Cut}(E)$. We note that the bricks of $\text{Cut}(E)$ correspond to the regions of $G$, and that $|\text{Cut}(E)| = k(\Sigma \setminus E)$. If $A \subseteq E$, then $\text{Cut}(A)$ refers to the spanning subgraph $(V(G), A) \subset \Sigma$. Again we have $|\text{Cut}(A)| = k(\Sigma \setminus A)$ since removing regular neighbourhoods of isolated vertices does not change the number of components.

Observe that a set of bricks for $\text{Cut}(A)$ may be found from $\text{Cut}(E)$ by identifying brick boundary curves labeled by all $e \in E \setminus A$, and in particular the bricks of $\text{Cut}(A)$ may be found from the bricks of $\text{Cut}(A \cup e)$ by identifying the two curves labeled by $e$.

We are now ready to prove the second item. We need to show if $e$ is an isthmus of $M(G, \Sigma)$ if and only if $e$ is a quasi-bridge of $G \subset \Sigma$. That is, we need to show

$$[k(\Sigma \setminus A) = k(\Sigma) \implies k(\Sigma \setminus (A \cup \{e\})) = k(\Sigma)] \iff [\text{e a quasi-bridge}].$$

By restricting to the component of $\Sigma$ that contains $e$, we can assume without loss of generality that $\Sigma$ is connected, and so it is enough to show that

$$[(\Sigma \setminus A \text{ connected}) \implies (\Sigma \setminus (A \cup \{e\}) \text{ connected})] \iff [\text{e a quasi-bridge}].$$

An edge $e \in E$ is a quasi-bridge if and only if $e$ is on one region of $G$, if and only if $e$ is on one brick of $\text{Cut}(E)$, if and only if $e$ is on one brick of $\text{Cut}(A \cup e)$ for all $A \subseteq E$. Also $|\text{Cut}(A)| = |\text{Cut}(A \cup e)|$ if and only if $k(\Sigma \setminus A) = k(\Sigma \setminus (A \cup e))$.

Thus, if $e$ is a quasi-bridge and $\Sigma \setminus A$ is connected, then $e$ is on one brick of $\text{Cut}(A \cup e)$. But $\Sigma \setminus A$ is connected, so there is only one brick in $\text{Cut}(A)$. This one brick can be formed by identifying the
two curves labeled $e$ on a single brick of $\text{Cut}(A \cup e)$, so $|\text{Cut}(A \cup e)| = 1$ and $\Sigma \setminus (A \cup e)$ is also connected.

On the other hand, if whenever $\Sigma \setminus A$ is connected then $\Sigma \setminus (A \cup e)$ is connected, then, by considering just the component containing $e$, we see that $e$ is on one brick of $\text{Cut}(B \cup e)$ for all $B \subseteq E$. Thus $e$ is a quasi-bridge.

Theorem 2.2 together with Lemmas 3.8 and 3.10 now give the desired complete deletion and contraction relations for the Las Vergnas polynomial:

**Theorem 3.11.** Let $G \subseteq \Sigma$ be an embedded graph. Then the following relations hold:

1. if $e \in E$ is neither a quasi-loop or quasi-bridge of $G$, then
   $$L_{G \subseteq \Sigma}(x, y, z) = L_{G \setminus \{e\} \subseteq \Sigma}(x, y, z) + L_{G / \{e\} \subseteq \Sigma}(x, y, z);$$
2. if $e \in E$ is a bridge of $G$, then
   $$L_{G \subseteq \Sigma}(x, y, z) = xL_{G \setminus \{e\} \subseteq \Sigma}(x, y, z);$$
3. if $e \in E$ is a quasi-loop of $G$, then
   $$L_{G \subseteq \Sigma}(x, y, z) = yL_{G \setminus \{e\} \subseteq \Sigma}(x, y, z);$$
4. if $e \in E$ is a quasi-bridge but not a bridge of $G$, then
   $$L_{G \subseteq \Sigma}(x, y, z) = zL_{G \setminus \{e\} \subseteq \Sigma}(x, y, z) + L_{G / \{e\} \subseteq \Sigma}(x, y, z);$$
5. if $E(G) = \emptyset$, then $L_{G \subseteq \Sigma}(x, y, z) = 1$.

4. **Relations with other topological Tutte polynomials**

We consider two other notable topological Tutte polynomials, that is, polynomials of embedded graphs that generalize the classical Tutte polynomial. These are the 2002 ribbon graph polynomial of Bollobás and Riordan, $R_G$, from [3] (which subsumes the 2001 version for orientable ribbon graphs from [3]), and the 2011 Krushkal polynomial, $K_G$, from [11] for graphs arbitrarily embedded in orientable surfaces (which was extended to non-orientable surfaces by Butler in [2]). We now determine the relations between these polynomials and the Las Vergnas polynomial, both the original version for cellularly embedded graphs (which was first done in [3]), and the new version for arbitrarily embedded graphs. We begin by recalling the definitions of $R_G$ and $K_G$.

**Definition 4.1.** Let $G$ be an cellularly embedded graph, or, equivalently, a ribbon graph. Then the ribbon graph polynomial or Bollobás-Riordan polynomial, $R_G(x, y, z) \in \mathbb{Z}[x, y, z]$, is defined by

$$R_G(x, y, z) = \sum_{A \subseteq E(G)} (x - 1)^{r_G(A) - r_G(A) - \gamma_G(A)} y^{n_G(A)} z^{\gamma_G(A) - \gamma_G(A) + n_G(A)}.$$

Noting that exponent of $z$ is equal to the Euler genus $\gamma(A)$, the ribbon graph polynomial may be rewritten as

$$R_G(x, y, z) = \sum_{A \subseteq E(G)} (x - 1)^{r_G(E) - r_G(A)} y^{n_G(A)} z^{\gamma_G(A)}.$$

Although that $R_G$ often appears with a fourth variable that records the orientability of each spanning ribbon subgraph, here we omit the fourth variable as it plays no role in our results. Note that the classical Tutte polynomial, $T_G$, is a specialisation of $R_G$ as $T_G(x, y) = R_G(x, y - 1, 1)$, and that $T_G(x, y) = R_G(x, y - 1, z)$ when $G$ is a plane graph (since when $G$ is plane the Euler genus of all of its spanning ribbon subgraphs is zero).

Comparing the state sums for $L_G$ and $R_G$ from Equations (3.3) and (4.1) illuminates the key differences and similarities between these two topological Tutte polynomials in the case of cellularly
Definition 4.2. Let $A$ be a polynomial, here as Las Vergnas polynomial for cellularly embedded graphs can also be recovered from the Kruskal polynomial.

Furthermore, it was shown in [1] for the orientable case, and [2] for the non-orientable case, that the homological definition given in [11] suggesting that it is unlikely that either of $R_G$ or $L_G$ may be recovered from the other.

We now turn our attention to the Kruskal polynomial which was defined in [11] for graphs (not necessarily cellularly) embedded in orientable surfaces, and in [2] for graphs in non-orientable surfaces. For this, recall from Section 2.1 that $N(X)$ denotes a regular neighbourhood of a subset $X$ of a surface $\Sigma$. Observe that $N(X)$ is itself a surface and so we can consider topological properties of this surface. Note that if $G \subset \Sigma$ is an embedded graph and $A \subset E(G)$, then $\gamma(N(A)) = \gamma(N(V(G) \cup A))$, and so $\gamma(N(A))$ can be thought of as referring to either a regular neighbourhood of the subgraph by $A$ or a regular neighbourhood of the spanning subgraph on $A$. This observation does not hold for all topological properties of the neighbourhoods, for example $k(N(A))$ and $k(N(V(G) \cup A))$ need not be equal.

Definition 4.2. Let $G \subset \Sigma$ be an embedded graph. Then the Kruskal polynomial, $K_{G \subset \Sigma}(x, y, a, b) \in \mathbb{Z}[a, b, x^{1/2}, y^{1/2}]$, is defined by

$$K_{G \subset \Sigma}(x, y, a, b) := \sum_{A \subset E(G)} x^{c(G) - c(A)} y^{k(\Sigma \setminus A) - k(\Sigma)} a^{\frac{1}{2} \gamma(N(A))} b^{\frac{1}{2} \gamma(\Sigma \setminus A)}.$$

We follow [2] and use the form of the exponent of $y$ from the proof of Lemma 4.1 of [1] rather than the homological definition given in [11].

Kruskal showed for orientable surfaces [11], and Butler [2] for non-orientable surfaces, that when $G \subset \Sigma$ is cellularly embedded, then the ribbon graph polynomial $R_G$ can be recovered from $K_{G \subset \Sigma}$ as

$$R_G(x, y, z) = y^{\frac{1}{2} \gamma(G)} K_{G \subset \Sigma}(x - 1, y, yz^2, y^{-1}).$$

Furthermore, it was shown in [11] for the orientable case, and [2] for the non-orientable case, that the Las Vergnas polynomial for cellularly embedded graphs can also be recovered from the Kruskal polynomial, here as

$$(4.2) \quad L_G(x, y, z) = z^{\frac{1}{2} \gamma(G)} K_{G \subset \Sigma}(x - 1, y - 1, z^{-1}, z).$$

Now, with the Las Vergnas polynomial for arbitrarily embedded graphs, we can fully expose the relation between the Las Vergnas polynomial and the Kruskal polynomial.

Theorem 4.3. If $G \subset \Sigma$ is an embedded graph, and write $E$ for $E(G)$. Then

$$(4.3) \quad L_{(G, \Sigma)}(x, y, z) = z^{\frac{1}{2} \gamma(N(E)) - \gamma(\Sigma \setminus E)} K_{G \subset \Sigma}(x - 1, y - 1, z^{-1}, z).$$

Proof. Let $V := V(G)$. We proceed by comparing the exponents in the expression for $L_G$ from Definition 3.6 with those on the right-hand side of Equation 4.3 which is

$$(4.4) \quad \sum_{A \subset E} (x - 1)^{c(A) - c(E)} (y - 1)^{k(\Sigma \setminus A) - k(\Sigma)} z^{\frac{1}{2} \gamma(\Sigma \setminus A) - \gamma(N(A)) + \gamma(N(G)) - \gamma(\Sigma \setminus E))}.$$

The exponents of $x - 1$ and $y - 1$ in Equations 3.8 and 4.4 are the same, so we examine the exponents of $z$.

Noting that $c(E) = k(N(V \cup E))$ and $c(A) = k(N(V \cup A))$, by Euler’s formula, the exponent of $z$ in $L_{(G, \Sigma)}(x, y, z)$ is

$$(4.5) \quad |E| - |A| - k(\Sigma \setminus E) + k(\Sigma \setminus A) + k(N(V \cup E)) - k(N(V \cup A)).$$
On the other hand, expanding the \( z \) exponent in Equation 4.4 in terms of the Euler characteristic gives

\[
\begin{align*}
&k(N(V \cup E)) - \frac{1}{2}\chi(N(V \cup E)) - \frac{1}{2}b(N(V \cup E)) \\
&- k(N(V \cup E)) + \frac{1}{2}\chi(E) + \frac{1}{2}b(E) \\
&- k(N(V \cup A)) + \frac{1}{2}\chi(N(V \cup A)) + \frac{1}{2}b(N(V \cup A))' \\
&+ k(G) - \frac{1}{2}\chi(G) - \frac{1}{2}b(G).
\end{align*}
\]

The \( b \) terms in this expression cancel since \( \Sigma \backslash A \) and \( N(V \cup A) \) have identical boundary components for each \( A \subseteq E(G) \). To show that the above sum is equal to Equation 4.5 we show that \( \chi(N(V \cup A)) - \chi(N(V \cup E)) = |E| - |A| \) and that \( \chi(\Sigma \backslash E) - \chi(\Sigma \backslash A) = |E| - |A| \). It suffices to show this one edge at a time, i.e. to show that \( \chi(N(V \cup (X \cup \{e\}))) = \chi(N(V \cup X)) - 1 \), and \( \chi(\Sigma \backslash (X \cup \{e\})) = \chi(\Sigma \backslash X)) + 1 \), for any \( X \subseteq E \). This follows by recalling that \( \chi(\Sigma) = v - e + f \), where \( v, e, \) and \( f \) are the numbers of vertices, edges, and faces, respectively, in any cellulation of \( \Sigma \), and then noting that extending a cellulation of \( N(V \cup X) \) to \( N(V \cup X \cup \{e\}) \) changes the Euler characteristic by \( 1 \), as can easily be seen from Figure 2. Similarly, \( \chi(\Sigma \backslash E) - \chi(\Sigma \backslash A) = |E| - |A| \), and thus the exponents of \( z \) in Equations 3.8 and 4.4 agree, completing the proof.

Observe that when \( G \subseteq \Sigma \) is cellularly embedded then \( \gamma(\Sigma \backslash E) = 0 \), and Equations 4.3 and 4.2 agree.

Theorem 4.3 together with Theorem 3.11 imply that the Krushkal polynomial has a full deletion-contraction reduction when the variables are restricted as in Theorem 4.3. This, together with the partial deletion-contraction reductions given in [2] which distinguish loops that separate the surface, suggest that quasi-loops and quasi-bridges might be appropriate objects to explore in the context of the Krushkal polynomial as well.

5. New perspectives on Las Vergnas’ low genus work with Eulerian circuits

Michel Las Vergnas also worked with cellularly embedded graphs via their Tait graph. We now use some tools recently developed to study twisted duality (see [7, 6]) to build on Las Vergnas’ foundations in this area.

In this section we will work entirely with cellularly embedded graphs and ribbon graphs (which are equivalent). We recall that if \( G = (V, E) \) is a ribbon graph and \( A \subseteq E \) then \( f(A) \) is the number of boundary components of the spanning ribbon subgraph \( (V, A) \), and \( \gamma(A) \) is its Euler genus. The parameters \( f(A) \) and \( \gamma(A) \) are most easily described in terms of ribbon graphs, but they can be computed in terms of cellularly embedded graphs: given \( G \subseteq \Sigma \), describe \( G \) as a ribbon graph, construct its spanning ribbon subgraph \( G' = (V, A) \), then translate back to the language of cellularly embedded graphs to get \( G' \subseteq \Sigma' \). Then \( f(A) \) is the number of faces of \( G' \), and \( \gamma(A) = \gamma(\Sigma') \). In particular, it is important to remember that \( f(A) \) may not be the number of regions of \( G' \backslash A' \), and similarly \( \gamma(A) \) need not equal \( \gamma(\Sigma) \).
5.1. **Graph states and Tait graphs.** We first briefly recall some terminology. Further details, including definitions of vertex and graph states, as well as medial and Tait graphs, relevant to this context, may be found in [6, 7].

A **vertex state** at a vertex $v$ of an abstract 4-regular graph $F$ is a partition, into pairs, of the edges incident with $v$. If $F$ is an cellularly embedded 4-regular graph, a vertex state is simply a choice of one of the configurations in Figure 3 in a neighbourhood of the vertex $v$ to replace a small neighbourhood of $v$.

![Figure 3. The vertex states of a vertex $v$ of a graph.](image)

If $G$ is a cellularly embedded graph and $G_m$ its medial graph, checkerboard coloured so that faces containing a vertex of $G$ are colored black, then we may use the checkerboard colouring to distinguish among the vertex states, naming them a **white split**, a **black split** or a **crossing**, as in Figure 4.

![Figure 4. The three vertex states of a vertex $v$ of a checkerboard coloured medial graph.](image)

A **graph state** $s$ of any 4-regular graph $F$ is a choice of vertex state at each of its vertices. Each graph state corresponds to a specific family of edge-disjoint cycles in $F$. We call these cycles the components of the state, denoting the number of them by $c(s)$.

A cellularly embedded graph $G$ is a **Tait graph** of a cellularly embedded 4-regular graph $F$ if $F$ is the medial graph of $G$. A cellularly embedded checkerboard colourable 4-regular graph is always a medial graph and will have exactly two (possibly isomorphic) Tait graphs, one corresponding to each colour in the checkerboard colouring as in the following definition. We will generally view Tait graphs as ribbon graphs.

**Definition 5.1.** Let $F$ be a checkerboard coloured 4-regular cellularly embedded graph. Then

1. the **blackface graph**, $F_{bl}$, of $F$ is the embedded graph constructed by placing one vertex in each black face and adding an edge between two of these vertices whenever the corresponding regions meet at a vertex of $F$;
2. the **whiteface graph**, $F_{wh}$, is constructed analogously by placing vertices in the white faces.

Note that $F_{bl}$, $F_{wh}$ are the two Tait graphs of $F$, and that choosing the other checkerboard colouring just switches the names of $F_{bl}$ and $F_{wh}$. An example is given in Figure 5.

5.2. **Circuits in medial graphs.** We begin with the main theorem of [14] which is a formula for the number of components in a graph state without crossings of a checkerboard coloured 4-regular graph (or equivalently, a checkerboard coloured medial graph) cellularly embedded in the sphere, torus, or real projective plane. We note that in the language of [14], a graph state with $k$-components is called an Eulerian $k$-partition. Also, the labelling of vertex states as black or white in [14] is the reverse from that used in this paper.
\textbf{Theorem 5.2} (Las Vergnas [14]). Let $F$ be a checkerboard coloured 4-regular graph cellularly embedded in the sphere, torus, or real projective plane; and let $s$ be a graph state without crossings. Then the number of circuits of $s$ is equal to

$$\min\{|B| + r(F_{wh}) - 2r_{F_{wh}}(B) + 1, \quad v(F) - |B| + r(F_{bl}) - 2r_{F_{bl}}(W) + 1\},$$

where $B$ is the set of edges of $F_{wh}$ corresponding to vertices of $F$ with a black split in the graph state, and where $W$ is the set of edges of $F_{bl}$ corresponding to vertices of $F$ with a white split in the graph state when we view $F$ as the medial graph of both $F_{wh}$ and $F_{bl}$.

The strength of this formula is that it computes a topological property from readily attainable combinatorial quantities.

We now give a related formula for the number of components of a graph state, with a much shorter proof than the original, that does hold for every surface. We then use it to explain why the formula of Theorem 5.2 fails on surfaces other than the sphere, torus, or real projective plane.

\textbf{Proposition 5.3}. Let $F$ be a 4-regular connected checkerboard coloured cellularly embedded graph, and let $s$ be a graph state without crossings. Then the number of circuits in $s$ is

$$f_{F_{bl}}(W) = 2c_{F_{bl}}(W) - \gamma_{F_{bl}}(W) + |W| - v(F_{bl}),$$

where $F_{bl}$ is viewed as a ribbon graph, and $W$ is the set of edges of $F_{bl}$ corresponding to vertices of $F$ with a white split in the graph state $s$.

\textbf{Proof}. This is nearly a tautology. We see in Figure 5.2 an edge of $F_{bl}$ (realised as a ribbon graph) together with the corresponding vertex of $F$, which shows that black splits essentially ‘snip through’ the corresponding edges, effectively deleting them as indicated in Figures 6(a)–6(c). Thus, the circuits of the graph state $s$ of $F_{bl}$ just follow the face boundaries when the edges corresponding to
black splits are deleted. The number of circuits in a state with no crossings is then just \( f(F_{bl} - B) = f_{F_{bl}}(W) \). The right-hand side of Equation (5.2) follows from Euler’s formula.

Although, since practically tautological, Proposition 5.3 may be less useful than Theorem 5.2, it does lead us to rewrite Theorem 5.2 in a form that reveals why it does not generalise to other surfaces.

**Theorem 5.4.** Let \( F \) be a connected checkerboard coloured 4-regular graph cellularly embedded in the sphere, torus, or real projective plane. Then the number of components of a graph state without crossings is equal to

\[
\min\{f_{F_{bl}}(W) + \gamma_{F_{wh}}(B), \quad f_{F_{bl}}(W) + \gamma_{F_{bl}}(W)\},
\]

where \( F_{bl} \) and \( F_{wh} \) are viewed as ribbon graphs, \( B \) is the set of edges of either \( F_{bl} \) or \( F_{wh} \) corresponding to vertices of \( F \) with a black split in the graph state, and where \( W \) is the set of edges of either \( F_{bl} \) or \( F_{wh} \) corresponding to vertices of \( F \) with a white split in the graph state, when we view \( F \) as the medial graph of both \( F_{wh} \) and \( F_{bl} \).

**Proof.** Viewing \( F_{bl} \) and \( F_{wh} \) as ribbon graphs, Euler’s formula states that \( v(G) - e(G) + f(G) = 2c(G) - \gamma(G) \). With this,

\[
|B| + r(F_{wh}) - 2r_{F_{wh}}(B) + 1 = |B| + v(F_{wh}) - 2v(F_{wh}) + 2c_{F_{wh}}(B) = f_{F_{wh}}(B) + \gamma_{F_{wh}}(B) = f_{F_{bl}}(W) + \gamma_{F_{bl}}(B),
\]

where the last equality follows by noting that since \( B \) and \( W \) are complementary sets in dual graphs, \( f_{F_{bl}}(W) = f_{F_{wh}}(B) \). A similar calculation shows that \( v(F) - |B| + r(F_{bl}) - 2r_{F_{bl}}(W) + 1 = f_{F_{bl}}(W) + \gamma_{F_{bl}}(W) \), and the result then follows by Theorem 5.2.

In the proof of Corollary 5.5 we can now see the importance of low genus in Theorem 5.2.

**Corollary 5.5.** If \( F \) is a connected checkerboard coloured 4-regular graph cellularly embedded in the sphere, torus, or projective plane, then

\[
\min\{f_{F_{bl}}(W) + \gamma_{F_{wh}}(B), \quad f_{F_{bl}}(W) + \gamma_{F_{bl}}(W)\} = f_{F_{bl}}(W),
\]

where \( F_{bl} \) and \( F_{wh} \) are viewed as ribbon graphs, and \( B \) and \( W \) are as in the statement of Theorem 5.4.

**Proof.** For the plane, torus, or projective plane, we note that \( \gamma_{F_{wh}}(B) \) and \( \gamma_{F_{bl}}(W) \) are in \( \{0, 1, 2\} \). For the plane, both are 0, so the result follows immediately. On the torus and the projective plane, since \( (F_{wh} - W) \) and \( (F_{bl} - B) \) are edge disjoint (if we identify the edges of \( F_{wh} \) and \( F_{bl} \)), both cannot contain fundamental cycles. Thus, one or the other of \( \gamma_{F_{wh}}(B) \) and \( \gamma_{F_{bl}}(W) \) must be 0, from which the result follows. This is not the case on surfaces of higher genus.

The tools of twisted duality from [7, 6] allow us to extend the enumeration formula in Proposition 5.3 to all graph states, not just those without crossings. We will not review those tools in detail here, but only note that an edge in a ribbon graph may be given a “half-twist”, i.e. detach one end of an ribbon from an incident vertex, give the ribbon a half twist, and then reattach it. If \( G \) is a ribbon graph, and \( A \subseteq E(G) \), then \( G^{\tau(A)} \) is the ribbon graph resulting from giving a half-twist to all the edges in \( A \).

**Proposition 5.6.** Let \( F \) be a connected checkerboard coloured 4-regular cellularly embedded graph. Then the number of circuits in any graph state is

\[
f((F_{bl})^{\tau(C)} - B),
\]

where
where $F_m$ is viewed as a ribbon graph, $B$ is the set of edges of $F_m$ corresponding to vertices of $F$ with a black split in the graph state, and $C$ is the set corresponding to crossings.

The proof, based on Figure 6, is nearly a tautology, so we omit the details.

Las Vergnas provided, in Theorem 5.7 below, an application of Theorem 5.2 which relates Eulerian circuits and spanning trees. By using the language of ribbon graphs and the quasi-bridges introduced in the previous section, we can now extend this result and give new perspectives on circuits in medial graphs.

**Theorem 5.7 (Las Vergnas [14]).** Let $F$ be a checkerboard coloured 4-regular graph embedded in the sphere, torus or real projective plane. Let $s$ be a graph state without crossings of $F$, let $B$ be the set of edges of $F_{wh}$ corresponding to vertices of $F$ with a black split in the graph state $s$, and $W$ be the set of edges of $F_m$ corresponding to vertices of $F$ with a white split in the graph state $s$. Then $s$ defines an Euler circuit of $F$ if and only if $F_{wh} - W$ is a spanning tree of $F_{wh}$, or $F_m - B$ is a spanning tree of $F_m$.

The language of ribbon graphs allows us to extend Theorem 5.7 to all cellularly embedded graphs. If we let $G$ denote the whiteface graph $F_{wh}$ and view it as a ribbon graph, then an Eulerian circuit without crossings in $F$ corresponds to a quasi-tree of $G$, which is a ribbon subgraph of $G$ that has exactly one face (so all the edges of a quasi-tree are quasi-bridges). In addition, the ribbon graph $F_{wh} - W$ corresponds to a ribbon subgraph $G - A$ of $G$, and $F_m - B$ corresponds to a ribbon subgraph $G^* - A^c$ of $G^*$, where we identify the edges of $G$ and $G^*$, and $A^c = E(G) - A = E(G^*) - A$. Thus, Theorem 5.7 is equivalent to the statement that if $G$ is a ribbon graph homeomorphic to a punctured sphere, torus or real projective plane, then $G - A$ is a quasi-tree if and only if $G^* - A^c$ is a spanning tree of $G$. It is clear that this statement, and hence Las Vergnas’ Theorem 5.7, is completed by Theorem 5.8 below.

**Theorem 5.8.** Let $G$ be a ribbon graph and $A \subseteq E(G)$. Then $G - A$ is a quasi-tree if and only if $G^* - A^c$ is a quasi-tree. Moreover, if $G - A$ is a quasi-tree, then

$$\gamma_G(A^c) + \gamma_{G^*}(A) = \gamma(G).$$

**Proof.** Since the ribbon subgraphs $G - A$ and $G^* - A^c$ of $G$ have the same boundary components, $G - A$ has exactly one face if and only if $G^* - A^c$ has exactly one face. This proves the first part of the theorem.

For the second statement, suppose that $G - A$ is a quasi-tree. It then follows that $G$, $G - A$ and $G^*|_A$ are all connected. By Euler’s formula we then have

\[
\begin{align*}
\gamma_G(A^c) + \gamma_{G^*}(A) &= \gamma(G - A) + \gamma(G^* - A^c) \\
&= e(G - A) - v(G - A) - f(G - A) + 2c(G - A) \\
&\quad + e(G^* - A^c) - v(G^* - A^c) - f(G^* - A^c) + 2c(G^* - A^c) \\
&= e(G) - v(G) - f(G) + 2 \\
&= e(G) - v(G) - f(G) + 2c(G) = \gamma(G),
\end{align*}
\]

where the second equality follows since $e(G) = e(G - A) + e(G^* - A^c)$, $v(G - A) = v(G)$, $v(G^* - A^c) = v(G^*) = f(G)$, and $f(G - A) = f(G^* - A^c) = e(G - A) = e(G^* - A^c) = e(G) = 1$ (as $G - A$ is a quasi-tree).

**5.3. A curious relation.** In [14], Las Vergnas also gave interpretations for evaluations of $L_G$ for graphs cellularly embedded in the plane, torus, or real projective plane in terms of the medial graph of $G$. We conclude by showing that this now yields a very different kind of relationship between the Las Vergnas polynomial and the Bollobás-Riordan polynomial than that given previously in
This identity uses circuits in medial graphs to give a relation between one variable specialisations of $L_G$ and $R_G$ on low genus graphs.

To do so, we first note the following evaluation of $R_G$.

**Proposition 5.9.** Let $G$ be a connected cellulary embedded graph and let $f_k(G_m)$ be the number of $k$-component graph states of its medial graph $G_m$ without crossings. Then 

$$tR_G(t + 1, t, 1/t) = \sum_{k \geq 1} f_k(G_m) t^k.$$ 

**Proof.** This result is immediate from the relation between the topological transition polynomial and $R_G$ in [8], but can also be seen as follows. If $G$ is connected, then $R_G(t + 1, t, 1/t) = t^{-1} \sum_{A \subseteq E(G)} t^{|E(A)|}$. Note that there is a one-to-one correspondence between the boundary components of the spanning ribbon subgraphs of $G$ and the components of states of $G_m$ with no crossings. This correspondence is given by a white split at the vertex corresponding to an edge $e$ if $e \in A$, and a black split otherwise. Thus, $R_G(t + 1, t, 1/t) = t^{-1} \sum t^{l(s)}$, where the sum is over all non-crossing states $s$ of $G_m$, and collecting like terms gives the result. □

**Theorem 5.10.** If $G$ is a graph embedded on the plane or real projective plane. Then

$$L_G(t + 1, t + 1, 1) = R_G(t + 1, t, 1/t);$$

and if $G$ is embedded in the torus, then

$$L_{2,G}(t + 1, t + 1, 1) + tL_{1,G}(t + 1, t + 1, 1) + L_{0,G}(t + 1, t + 1, 1) = R_G(t + 1, t, 1/t),$$

where, if we view $L_G(x, y, z)$ as a polynomial in $(\mathbb{Z}[x, y])[z]$, then $L_{i,G}$ is the coefficient of $z^i$ in $L_G(x, y, z)$.

**Proof.** Let $F$ be the checkerboard coloured medial graph of $G$ so that $F_{bl} = G$. Las Vergnas proved in Proposition 4.1 of [14] that $tL_{F_{bl}}(t + 1, t + 1, 1) = \sum_{k \geq 1} f_k(F) t^k$ when $F$ is on the sphere or real projective plane; and that $L_{2,F_{bl}}(t + 1, t + 1, 1) + tL_{1,F_{bl}}(t + 1, t + 1, 1) + L_{0,F_{bl}}(t + 1, t + 1, 1) = \sum_{k \geq 1} f_k(F) t^{k-1}$, when $F$ is on the torus. The results then follow by Proposition 5.9. □

**References**

1. R. Askanazi, S. Chmutov, C. Estill, J. Michel, and P. Stollenwerk, Polynomial invariants of graphs on surfaces, Quantum Topol. 4 (2013) 77–90.
2. C. Butler, A quasi-tree expansion of the Krushkal polynomial, preprint arxiv.org/abs/1205.0298.
3. B. Bollobás, O. Riordan, A polynomial for graphs on orientable surfaces, Proc. London Math. Soc. 83 (2001) 513–531.
4. B. Bollobás, O. Riordan, A polynomial of graphs on surfaces, Math. Ann. 323 (2002) 81–96.
5. C. Chun, I. Moffatt, S. Noble, R. Rueckriemen, Matroids, delta-matroids and embedded graphs, preprint.
6. J. Ellis-Monaghan, I. Moffatt, Twisted duality and polynomials of embedded graphs, Trans. Amer. Math. Soc. 364 (2012) 1529–1569.
7. J. Ellis-Monaghan, I. Moffatt, Graphs on Surfaces: Twisted Duality, Polynomials, and Knots, Springer, New York, 2013.
8. J. Ellis-Monaghan, I. Sarmiento, A recipe theorem for the topological Tutte polynomial of Bollobás and Riordan, European J. Combin. 32 (2011) 782–794.
9. G. Etienne, M. Las Vergnas, The Tutte polynomial of a morphism of matroids, III: Vectorial matroids. Special issue on the Tutte polynomial, Adv. in Appl. Math. 32 (2004) 198–211.
10. J. Gross, T. Tucker, Topological graph theory, Wiley-interscience publication, 1987.
11. V. Krushkal, Graphs, links, and duality on surfaces, Combin. Probab. Comput. 20 (2011) 267–287.
12. M. Las Vergnas, Extensions normales d’un matroide, polynôme de Tutte d’un morphisme, C. R. Acad S. Paris, 280 (1975) Série A, 1479–1482.
13. M. Las Vergnas, Sur les activités des orientations d’une géométrie combinatoire, Colloque Mathématiques Discrètes: Codes et Hypergraphes (Brussels, 1978), Cahiers Centre Études Rech. Opér. 20 (1978) 293–300.
14. M. Las Vergnas, Eulerian circuits of 4-valent graphs imbedded in surfaces, in: Algebraic methods in graph theory, Vol. I, II (Szeged, 1978), Colloq. Math. Soc. János Bolyai, 25, North-Holland, Amsterdam-New York, 1981, pp. 451–477.
15. M. Las Vergnas, On the Tutte polynomial of a morphism of matroids, Ann. Discrete Math. 8 (1980) 7–20.
16. M. Las Vergnas, The Tutte polynomial of a morphism of matroids, I: Set-pointed matroids and matroid perspectives. Symposium à la mémoire de Franois Jaeger (Grenoble, 1998), Ann. Inst. Fourier (Grenoble) 49 (1999), no. 3, 973–1015.
17. M. Las Vergnas, The Tutte polynomial of a morphism of matroids, II: Activities and orientations, in: J. A. Bondy and U.S.R. Murty (Eds.), Progress in Graph Theory, Academic Press, 1984, pp. 367–380.
18. M. Las Vergnas, The Tutte polynomial of a morphism of matroids, IV: Computational complexity, Port. Math. (N.S.) 64 (2007), 303–309.
19. M. Las Vergnas, The Tutte polynomial of a morphism of matroids, V: Derivatives as generating functions of Tutte activities, European J. Combin. 34 (2013) 1390–1405.
20. M. Las Vergnas, Le polynôme de Martin d’un Graphe Eulerien, Ann. Discrete Math. 17 (1983) 397–411.

Department of Mathematics, Saint Michael’s College, 1 Winooski Park, Colchester, VT 05439, USA.

E-mail address: jellis-monaghan@smcv.t.edu

Department of Mathematics, Royal Holloway, University of London, Egham, Surrey, TW20 0EX, United Kingdom.

E-mail address: iain.moffatt@rhul.ac.uk