The minimal polynomials of powers of cycles in the ordinary representations of symmetric and alternating groups

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Abstract

Denote the alternating and symmetric groups of degree $n$ by $A_n$ and $S_n$ respectively. Consider a permutation $\sigma \in S_n$ all of whose non-trivial cycles are of the same length. We find the minimal polynomials of $\sigma$ in the ordinary irreducible representations of $A_n$ and $S_n$.

1 Introduction

The question of finding the minimal polynomials of elements in representations for a given group goes back to the famous work of P. Hall and G. Higman [1].

Assume that $\mathbb{F}$ is an algebraically closed field and $G$ is a finite group. Consider some irreducible representation $\rho$ of $G$ over $\mathbb{F}$. For $g \in G$ denote by $\deg(\rho(g))$ the degree of the minimal polynomial of the matrix $\rho(g)$ and by $o(g)$ the order of $g$ modulo $Z(G)$. The general problem was formulated in [2] as follows.

**Problem 1.** Determine all possible values for $\deg(\rho(g))$, and if possible, all triples $(G, \rho, g)$ with $\deg(\rho(g)) < o(g)$, in the first instance under the condition that $o(g)$ is a $p$-power.

There are plenty of publications in this area and we mention only the results on symmetric and alternating groups of degree $n$, which will be denoted by $A_n$ and $S_n$ respectively. The minimal polynomials of prime order elements of $A_n$ and $S_n$ in the ordinary or projective representations were found in [3]. For the algebraically closed fields of positive characteristic $p$, Kleshchev and Zalesski [4] described the minimal polynomials of order $p$ elements in the irreducible representations of covering groups of $A_n$.

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In this paper we describe the minimal polynomials of some elements in the ordinary representations of $A_n$ and $S_n$. Namely, we consider permutations with the cycle decomposition consisting of cycles of a fixed length and cycles of length one. This set includes the set of permutations of prime order and consists of powers of single cycles.

Recall that a partition of $n$ is a nonincreasing finite sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ whose sum equals $n$. It is well-known that there exists a one-to-one correspondence between the set of equivalence classes of ordinary irreducible representations of $S_n$ and the set of partitions of $n$. The symmetric group $S_n$ has several named representations. The alternating representation is the representation of degree one mapping each permutation to its sign. Furthermore, if $V$ is a vector space of dimension $n$ over a field of characteristic zero with base $e_1, e_2, \ldots, e_n$ then $S_n$ acts naturally on $V$: for every $\sigma \in S_n$ and $i \in \{1, \ldots, n\}$ we have $\sigma(e_i) = e_{\sigma(i)}$. There are two irreducible constituents in this representation: the line $l = \langle e_1 + e_2 + \ldots + e_n \rangle$ and its orthogonal complement $l^\perp = \langle e_1 - e_i \mid 2 \leq i \leq n \rangle$. This action of $S_n$ on $l^\perp$ is called the standard irreducible representation of $S_n$. It corresponds to the partition $(n-1,1)$. Every irreducible representation corresponding to the partition $(2,1^{n-1}) = (2,1,\ldots,1)$ is called the associated representation with the standard representation.

We say that a permutation $\sigma \in S_n$ is of shape $[a_1^{b_1}a_2^{b_2} \ldots a_k^{b_k}]$, where $a_i$ are distinct integers, if the cycle decomposition of $\sigma$ is comprised of $b_i$ cycles of length $a_i$ for $i = 1 \ldots k$. Denote the field of complex numbers by $\mathbb{C}$. The main result of this paper is the following.

**Theorem 1.** Given positive integers $n$, $r$ and $m$ with $n \geq 3$, $r \geq 2$ and $rm \leq n$, assume that $\sigma \in S_n$ is of shape $[r^m1^{n-rm}]$, i.e. a product of $m$ cycles of length $r$, and $\rho : S_n \to GL(V)$ is a nontrivial irreducible representation of $S_n$ over $\mathbb{C}$. Denote the minimal polynomial of $\rho(\sigma)$ by $\mu_{\rho(\sigma)}(x)$. Then $\mu_{\rho(\sigma)}(x) \neq x^r - 1$ if and only if one of the following statements holds.

(i) We have $r = n$, $m = 1$, and $\rho$ is the standard representation; in this case $\mu_{\rho(\sigma)}(x) = \frac{x^n-1}{x-1}$.

(ii) We have $r = n$, $m = 1$, and $\rho$ is associated with the standard representation; in this case $\mu_{\rho(\sigma)}(x) = \frac{x^n-1}{x+1}$.

(iii) We have $r = n = 6$, $m = 1$, and $\rho$ corresponds to $\lambda = (3,3)$; in this case $\mu_{\rho(\sigma)}(x) = \frac{x^6-1}{x^3+x+1}$.

(iv) We have $r = n = 6$, $m = 1$, and $\rho$ corresponds to $\lambda = (2,2,2)$; in this case $\mu_{\rho(\sigma)}(x) = \frac{x^6-1}{x^2-x+1}$. 

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(v) We have $n = 4$, $\rho$ corresponds to $\lambda = (2, 2)$; and $(r, m, \mu_{\rho(\sigma)}(x)) \in \{(4, 1, x^2 - 1), (3, 1, x^2 + x + 1), (2, 2, x - 1)\}$.

(vi) $\rho$ is the alternating representation and $\mu_{\rho(\sigma)}(x) = x - (-1)^{\text{sgn}(\sigma)}$.

Recall that the standard representation of $A_n$ is the restriction of the standard representation of $S_n$ to $A_n$. As a corollary of Theorem 1 we prove the following statement.

**Corollary 1.** Given positive integers $n \geq 5$, $r \geq 2$ and $rm \leq n$, assume that $\sigma \in A_n$ is of shape $[r^m 1^{n-rm}]$, and $\rho : A_n \to GL(V)$ is a nontrivial irreducible representation of $A_n$ over $\mathbb{C}$. Denote the minimal polynomial of $\rho(\sigma)$ by $\mu_{\rho(\sigma)}(x)$.

Then $\mu_{\rho(\sigma)}(x) \neq x^r - 1$ if and only if one of the following statements holds.

(i) We have $r = n$ is odd, $m = 1$, and $\rho$ is the standard representation; in this case $\mu_{\rho(\sigma)}(x) = \frac{x^n - 1}{x - 1}$.

(ii) We have $r = n = 5$, $m = 1$, and $\rho$ corresponds to $\lambda = (3, 1, 1)$; in this case $\mu_{\rho(\sigma)}(x) = \frac{x^5 - 1}{(x - \eta)(x - \eta^2)}$, where $\eta$ is a primitive 5th root of unity.

There are a few other examples of permutations $\sigma \in S_n$ such that 1 is not an eigenvalue of the image of $\sigma$ in an irreducible representation of $S_n$. We list them in the following proposition.

**Proposition 1.** For an integer $n \geq 3$, consider the irreducible representation $\rho : S_n \to GL(V)$ corresponding to a partition $\lambda$ of $n$. Assume that a pair $(\lambda, \sigma)$ is one of the following:

(i) $((2, 1^{n-4}), [(n - 2)^1 1^1])$, where $n$ is odd,

(ii) or $\lambda = (1^n)$ and $\sigma$ is odd,

(iii) or $\sigma$ is $n$-cycle and either $\lambda = (n - 1, 1)$ or $n$ is odd and $\lambda = (2, 1^{n-2})$,

(iv) or $((2^3), [3^1 2^1 1^1]), ((4^2), [5^1 3^1]), ((2^4), [5^1 3^1]), ((2^5), [5^1 3^1 2])$.

Then 1 is not an eigenvalue of $\rho(\sigma)$.

**Remark 1.** Using GAP [3], we verify that for $3 \leq n \leq 20$ Theorem 1 and Proposition 1 include all examples of permutations $\sigma \in S_n$ and corresponding irreducible representations such that 1 is not an eigenvalue of $\rho(\sigma)$.

This paper is organized as follows. In Section 2 we recall notation and basic facts about ordinary representations of symmetric and alternating groups. In Section 3 we formulate and prove some auxiliary results used for proving Theorem 1 and Corollary 1. Sections 4–6 are devoted to the proofs of the main results.
2 Ordinary representations: notation and basic facts

Our notation for the ordinary representations of $S_n$ and $A_n$ follows [6]. Let $n$ be a positive integer. We write $\lambda \vdash n$ for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, that is, a sequence of positive integers $\lambda_i$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$. If $\lambda \vdash n$ then by $T^\lambda$ we denote the Young diagram of shape $\lambda$, consisting of $k$ rows with $\lambda_1$, $\lambda_2$, $\ldots$, $\lambda_k$ square boxes, respectively (see Figure 1).

**Definition 1.** The column lengths of $T^\lambda$ form a partition of $n$ which is called the partition associated with $\lambda$ and denoted by $\lambda'$. In other words, $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$, where $\lambda'_i = \sum_{j, \lambda_j \geq i} 1$. If $\lambda = \lambda'$ then $\lambda$ is called self-associated.

We naturally numerate the boxes of $T^\lambda$ by the pairs $(i, j)$ meaning $j$ in row $i$.

**Definition 2.** The set of boxes to the right or below box $(i, j)$ in $T^\lambda$ together with the latter forms the hook $H^\lambda_{ij}$ (see Figure 2). The number of boxes in $H^\lambda_{ij}$ is denoted by $h^\lambda_{ij}$ and called the length of $H^\lambda_{ij}$. Using the elements of $\lambda'$, we can express $h^\lambda_{ij}$ as $\lambda_i - j + \lambda'_j - i + 1$.

**Definition 3.** Let $H^\lambda_{ij}$ be a hook in $T^\lambda$. The number $l_{ij} = \lambda'_j - i$ is called the leg length of $H^\lambda_{ij}$. The part of $T^\lambda$ consisting of the boxes on the rim between the lower left and the upper right boxes of $H^\lambda_{ij}$ is denoted by $R^\lambda_{ij}$ and called a rim hook (see Figure 3).

It is well-known that there exists a bijection between the set of equivalence classes of ordinary irreducible representations of $S_n$ and the set of partitions of $n$. The equivalence class of irreducible representations corresponding to a partition $\lambda \vdash n$ is denoted by $[\lambda]$ and its character, by $\chi^\lambda$. By $[\lambda] \downarrow A_n$ we denote the restriction of $[\lambda]$ to $A_n$. 4
We collect some basic facts about the ordinary irreducible representations and characters of symmetric and alternating groups in the following three propositions.

**Proposition 2.** For every integer $n > 1$, the following statements hold.

(i) The splitting field for $S_n$ is the field of rationals $\mathbb{Q}$ and the values of all characters of $S_n$ are integers.

(ii) We have the hook-formula $\chi^\lambda(1) = \frac{n!}{\prod_{(i,j) \in T^\lambda} h^\lambda_{ij}}$.

(iii) If $\sigma \in S_n$ then $\chi^\lambda(\sigma) = \text{sgn}(\sigma) \cdot \chi^\lambda(\sigma)$.

**Proposition 3.** Suppose that $\lambda$ is a partition of $n > 1$. Then the following statements hold.
(i) If $\lambda \neq \lambda'$ then $[\lambda] \downarrow A_n = [\lambda'] \downarrow A_n$ is irreducible.

(ii) If $\lambda = \lambda'$ then $[\lambda] \downarrow A_n = [\lambda'] \downarrow A_n$ splits into two irreducible and conjugate representations $[\lambda]^{\pm}$, i.e., $[\lambda]^{+(12)}$, defined by $[\lambda]^{+(12)}((12)\sigma(12)) := [\lambda]^{+}(\sigma)$, where $\sigma \in A_n$, is equivalent to $[\lambda]^{-}$.

(iii) The set $\{[\lambda] \downarrow A_n \mid \lambda \neq \lambda'\} \cup \{[\lambda]^{\pm} \mid \lambda = \lambda' \vdash n\}$ is a complete system of equivalence classes of ordinary irreducible representations of $A_n$.

Proposition 4. \cite[Theorem 2.5.13]{6} Suppose that $\lambda = \lambda'$ is a self-associated partition of $n > 1$ and denote by $\chi^{\lambda \pm}$ the characters of $[\lambda]^{\pm}$. Assume that the main diagonal of $T^\lambda$ has $l$ boxes and denote by $a_1, a_2, \ldots, a_l$ the lengths of hooks along the main diagonal. Then the following statements hold.

(i) If $\sigma \in A_n$ and $\sigma$ is not of shape $[a_1^1 a_2^1 \ldots a_l^1]$ then $\chi^{\lambda \pm}(\sigma) = \frac{1}{2}\chi^{\lambda}(\sigma)$ and this number is an integer.

(ii) If $\sigma \in A_n$ and $\sigma$ is of shape $[a_1^1 a_2^1 \ldots a_l^1]$ then

$$
\chi^{\lambda \pm}(\sigma) \in \left\{ \frac{1}{2}(\chi^{\lambda}(\sigma) + \sqrt{\chi^{\lambda}(\sigma) \cdot \Pi_i a_i}), \frac{1}{2}(\chi^{\lambda}(\sigma) - \sqrt{\chi^{\lambda}(\sigma) \cdot \Pi_i a_i}) \right\}.
$$

Finally we formulate three famous rules that help to evaluate characters on the elements of $S_n$.

Proposition 5. \cite[Theorem 2.4.3]{6} (The Branching Law) If $\lambda = (\lambda_1, \lambda_2, \ldots,)$ is a partition of $n$ then for the restriction of $[\lambda]$ to the stabilizer $S_{n-1}$ of the point $n$ we have

$$
[\lambda] \downarrow S_{n-1} = \sum_{i: \lambda_i > \lambda_i+1} [\lambda_i^-],
$$

where $\lambda_i^-$ is a partition of $n - 1$ equal to $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots)$.

Proposition 6. \cite[2.4.7]{6} (The Murnaghan–Nakayama formula) Let $n$ and $k$ be integers with $k \leq n$. Consider $\sigma \in S_n$ such that $\sigma = \pi \cdot \rho$, where $\pi$ fixes the symbols $1, 2, \ldots, n - k$ and $\rho$ is the cycle $(n - k + 1, n - k + 2, \ldots, n)$ of length $k$. Then

$$
\chi^{\lambda}(\sigma) = \sum_{i: j, h^\lambda = k} (-1)^{i^\lambda \sigma^0} \chi^{\lambda \setminus R_{ij}}(\pi),
$$

where $\lambda \setminus R_{ij}$ denotes the partition obtained from $\lambda$ by removing the rim hook $R_{ij}$.
Proposition 7. (4.10) (Frobenius’s Formula) Let \( n \) be a positive integer and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be a partition of \( n \). Put \( l_1 = \lambda_1 + k - 1, l_2 = \lambda_1 + k - 2, \ldots, l_k = \lambda_k \). Introduce the power sum polynomials \( P_j(x) = x_1^j + x_2^j + \ldots + x_k^j \) and the discriminant polynomial \( \Delta(x) = \prod_{i<j} (x_i - x_j) \). If \( \sigma \) is of shape \([1^n 2^m \ldots n^m] \) then

\[
\chi^\lambda(\sigma) = [\Delta(x) \cdot \prod_j P_j(x)^{l_j}]_{(l_1, l_2, \ldots, l_k)},
\]

where \([f(x)]_{(l_1, l_2, \ldots, l_k)} \) stands for the coefficient of \( x_1^{l_1} x_2^{l_2} \ldots x_k^{l_k} \) in \( f(x) \).

3 Preliminaries

Here is our main tool to estimate character values on permutations.

Lemma 1. Theorem 1.1 Suppose that \( n = rm \), where \( r \) and \( m \) are integers, and \( \sigma \) is of shape \([r^m] \). Assume that \( \rho : S_n \to V \) is an ordinary irreducible representation of \( S_n \) and \( \chi_V \) is the character of \( \rho \). Then

\[
|\chi_V(\sigma)| \leq \frac{m^! \cdot r^m}{(n!)^{1/r}} (\chi_V(1))^{1/r}.
\]

To estimate factorials, we use the following well-known result due to H. Robbins.

Lemma 2. Let \( n \geq 1 \) be an integer. Then

\[
\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n} \cdot e^{1/(12n+1)} < n! < \sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n} \cdot e^{1/(12n)}.
\]

We apply these inequalities to the conclusion of Lemma 1.

Lemma 3. Suppose that \( r \) and \( m \) are integers, \( n = rm \), and \( \sigma \) is a permutation of shape \([r^m] \). For every irreducible character \( \chi^\lambda \) of \( S_n \) we have

\[
\frac{|\chi^\lambda(\sigma)|}{(\chi^\lambda(1))^{1/r}} < \frac{(2\pi)^{-1/2} \cdot n^{-1/2} \cdot e^{-1/12n}}{r^{-1/2} \cdot n^{-1/2} \cdot e^{-1/12m}}.
\]

Proof. Lemma 1 implies that

\[
\frac{|\chi^\lambda(\sigma)|}{(\chi^\lambda(1))^{1/r}} \leq \frac{m^! \cdot r^m}{(n!)^{1/r}}.
\]

Using Lemma 2 we obtain

\[
\frac{1}{n!} < \frac{1}{\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n} \cdot e^{1/(12n+1)}} < \frac{1}{\sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n}}
\]

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and
\[ m! < \sqrt{2\pi} \cdot m^{m+1/2} \cdot e^{-m} \cdot e^{1/(12m)} = (2\pi)^{1/2} \cdot \left( \frac{n}{r} \right)^{m+1/2} \cdot e^{-n/r} \cdot e^{1/(12m)}. \]

Then
\[ \frac{|\chi^\lambda(\sigma)|}{(\chi^\lambda(1))^{1/r}} < \frac{(2\pi)^{1/2} \cdot \left( \frac{n}{r} \right)^{m+1/2} \cdot r^m \cdot e^{-n/r} \cdot e^{1/(12m)}}{(2\pi)^{1/2r} \cdot n^{n/r+1/2r} \cdot e^{-n/r}} = (2\pi)^{-\frac{1}{2r}} \cdot r^{-1/2} \cdot n^{-1/2} \cdot e^{1/12m}, \]
as required.

**Lemma 4.** [2.3.17] Suppose that \( \lambda \) is a partition of \( n \), \( \chi^\lambda \) is the character of \( [\lambda] \), and \( \sigma \) is a permutation of shape \([n^1]\), i.e. a length \( n \) cycle. Then
\[ \chi^\lambda(\sigma) = \begin{cases} (-1)^r & \text{if } \lambda = (n-r, 1^r) \text{ with } 0 \leq r \leq n-1; \\ 0 & \text{otherwise.} \end{cases} \]

We need the following lemma to evaluate the characters of hook shape partitions on the permutations of shape \([r^m]\).

**Lemma 5.** Suppose that \( n \) is an odd integer and \( n = r(k+l+1) \), where \( k \geq 0, l \geq 0, \) and \( r \) are integers. Consider an integer \( a \) satisfying \( 1 \leq a \leq r \). Denote by \( \lambda \) the partition \((a+rk, 1^{r-a+rl})\) of \( n \), so that \( T^\lambda \) coincides with the hook \( H^\lambda_{11} \). If \( \sigma \in S_n \) is of shape \( [r^{k+l+1}] \) then \( \chi^\lambda(\sigma) = (-1)^{(r-a)} \binom{k+l+1}{k} \), where \( \binom{k+l+1}{k} \) is a binomial coefficient.

**Proof.** Induct on \( k+l \). Denote by \( \chi(k,l) \) the partition \((a+rk, 1^{r-a+rl})\) of \( n \). If \( k+l = 0 \) then \( \chi^\lambda(\sigma) = (-1)^{(r-a)} \) by Lemma 4.

Consider a permutation \( \sigma' \in S_{n-r} \) of shape \( [r^{k+l}] \). Assume now that \( k = 1 \) and \( l = 0 \). Applying the Murnaghan-Nakayama formula, we find that \( \chi^{(1,0)}(\sigma) = \chi^{(0,0)}(\sigma') = (-1)^{(r-a)} = (-1)^{(r-a)} \binom{1}{0} \). The case \( k = 0, l = 1 \) is similar. Assume now that \( k, l \geq 1 \). The Murnaghan–Nakayama formula implies that
\[ \chi^{(k,l)}(\sigma) = \chi^{(k-1,l)}(\sigma') + \chi^{(k,l-1)}(\sigma') = (-1)^{(r-a)} \left( \binom{k+l-1}{k-1} + \binom{k+l-1}{l-1} \right) = (-1)^{(r-a)} \left( \binom{k+l-1}{k-1} + \binom{k+l-1}{l-1} \right) = (-1)^{(r-a)} \binom{k+l}{k}, \]
as required.

**Lemma 6.** [11 Result 3, 14 Lemma 2.1] Suppose that \( \lambda \vdash n \).

(a) If \( n \geq 15 \) then the first six nontrivial minimal character degrees of \( S_n \) are:
1. $d_1(S_n) = n - 1$ and $\lambda \in \{(n - 1, 1), (2, 1^{n-2})\}$;

2. $d_2(S_n) = \frac{1}{2}n(n - 3)$ and $\lambda \in \{(n - 2, 2), (2^2, 1^{n-4})\}$;

3. $d_3(S_n) = d_2(S_n) + 1 = \frac{1}{2}(n - 1)(n - 2)$ and $\lambda \in \{(n - 2, 1^2), (3, 1^{n-3})\}$;

4. $d_4(S_n) = \frac{1}{2}n(n - 1)(n - 5)$ and $\lambda \in \{(n - 3, 3), (2^3, 1^{n-6})\}$;

5. $d_5(S_n) = \frac{1}{2}(n - 1)(n - 2)(n - 3)$ and $\lambda \in \{(n - 3, 1^3), (4, 1^{n-4})\}$;

6. $d_6(S_n) = \frac{1}{2}n(n - 2)(n - 4)$ and $\lambda \in \{(n - 3, 2, 1), (3, 2, 1^{n-5})\}$;

(b) If $n \geq 22$ then the next five smallest character degrees are:

1. $d_7(S_n) = (n - 1)(n - 2)(n - 7)/24$ and $\lambda \in \{(n - 4, 4), (2^4, 1^{n-8})\}$;

2. $d_8(S_n) = (n - 1)(n - 2)(n - 3)(n - 4)/24$ and $\lambda \in \{(n - 4, 1^4), (5, 1^{n-5})\}$;

3. $d_9(S_n) = n(n - 1)(n - 4)(n - 5)/12$ and $\lambda \in \{(n - 4, 2^2), (3^2, 1^{n-6})\}$;

4. $d_{10}(S_n) = (n - 1)(n - 3)(n - 6)/8$ and $\lambda \in \{(n - 4, 3, 1), (3, 2^2, 1^{n-7})\}$;

5. $d_{11}(S_n) = n(n - 2)(n - 3)(n - 5)/8$ and $\lambda \in \{(n - 4, 2, 1^2), (4, 2, 1^{n-6})\}$.

For the representations in Lemma 6(a) we need explicit values of their characters.

Lemma 7. Suppose that $n$ is an integer and take $\sigma \in S_n$. Assume that $\sigma$ has $i_1$ cycles of length 1, $i_2$ cycles of length 2 and $i_3$ cycles of length 3. Then the following holds.

(i) $\chi^{(n-1,1)}(\sigma) = i_1 - 1$ if $n \geq 2$;

(ii) $\chi^{(n-2,2)}(\sigma) = \frac{1}{2}(i_1 - 1)(i_1 - 2) + i_2 - 1$ if $n \geq 4$;

(iii) $\chi^{(n-2,1,1)}(\sigma) = \frac{1}{2}(i_1 - 1)(i_1 - 2) - i_2$ if $n \geq 3$;

(iv) $\chi^{(n-3,3)}(\sigma) = \frac{1}{6}i_1(i_1 - 1)(i_1 - 5) + i_2(i_1 - 1) + i_3$ if $n \geq 6$;

(v) $\chi^{(n-3,1^3)}(\sigma) = \frac{1}{6}(i_1 - 1)(i_1 - 2)(i_1 - 3) - i_2(i_1 - 1) + i_3$ if $n \geq 4$;

(vi) $\chi^{(n-3,2,1)}(\sigma) = \frac{1}{3}i_1(i_1 - 2)(i_1 - 4) - i_3$ if $n \geq 5$. 

Proof. For claims (i) – (iii), see [7 Exercise 4.15].

Now we consider the remaining cases:

\[ \lambda \in \{(n - 3, 3), (n - 3, 1^3), (n - 3, 2, 1)\} . \]

We often use further that the coefficient of \( x_1^{i_1}x_2^{i_2} \ldots x_k^{i_k} \) in the expansion of \( (x_1 + x_2 + \ldots + x_k)^n \) equals the multinomial coefficient \( \binom{n}{i_1, i_2, \ldots, i_k} = \frac{n!}{i_1!i_2!\ldots i_k!} \).

Case \( \lambda = (n - 3, 3) \). Proposition 7 yields

\[
\chi^\lambda(\sigma) = [(x_1 - x_2)(x_1 + x_2)i_1(x_1^2 + x_2^2)i_2(x_1^3 + x_2^3)i_3x_1^{n-2i_2-3i_3}]_{(n-2,3)} ,
\]
i.e. the coefficient of \( x_1^{n-2}x_3^3 \) in this product. Four factors involve \( x_2 \), namely \((x_1 - x_2), (x_1 + x_2)i_1, (x_1^2 + x_2^2)i_2, (x_1^3 + x_2^3)i_3 \). The sum of powers of \( x_2 \) must be equal to 3. It is easy to see that there are five possibilities: \( 3 = 0 + 0 + 0 + 3, 3 = 0 + 1 + 2 + 0, 3 = 1 + 0 + 2 + 0, 3 = 0 + 3 + 0 + 0, 3 = 1 + 2 + 0 + 0 \).

The corresponding factors containing \( x_2 \) are \( x_3i_3 \cdot x_1^{3i_3-3}x_2^3, i_1i_2i_3 \cdot x_2^{i_1}x_1^{i_2-2}x_2^2, -i_2 \cdot x_2x_1^{-2}x_2^2, (\frac{1}{2}) \cdot x_1^{3i_1-3}x_2^3, -i_2 \cdot x_2x_1^{-2}x_2^2 \). Therefore, the coefficient of \( x_1^{n-2}x_3^3 \) equals \( i_3 + i_1i_2 - i_2 + \frac{i_1(i_1-1)(i_1-2)}{6} - \frac{i_2(i_2-1)}{2} = i_3 + i_1i_2 - i_2 + \frac{i_1(i_1-1)(i_1-2)-3i_1(i_1-1)}{6} = i_3 + i_2(i_1)\cdot i_2(i_2(i_1-1)) + i_3 + i_2(i_1-1), \) as claimed.

Case \( \lambda = (n - 3, 1^3) \). Proposition 7 yields

\[
\chi^\lambda(\sigma) = [\Delta(x) \cdot (x_1 + x_2 + x_3 + x_4)i_1(x_1^2 + x_2^2 + x_3^2 + x_4^2)i_2(x_1^3 + x_2^3 + x_3^3 + x_4^3)i_3]_{(n,3,2,1)},
\]
and we need to find the coefficient of \( x_1^n x_2 x_3^2 x_4 \) in this product. In particular, if \( i_1 = i_2 = i_3 = 0 \) then \( \chi^\lambda(\sigma) = 0 \) and this is consistent with the claimed formula. So we can assume that at least one of the numbers \( i_1, i_2, \) and \( i_3 \) is nonzero.

Furthermore, we continue our calculations using the Murnaghan–Nakayama formula rather than the Frobenius formula because the latter in this case requires the consideration of a large number of cases.

Rim hooks are removed in the same way for \( n \geq 7 \), so first we suppose that \( 4 \leq n \leq 6 \). If \( n = 4 \) then \( \lambda = (1^4) \) corresponds to the alternating representation. If \( n = 5 \) then \( \lambda' = (4, 1) \) corresponds to the standard representation. If \( n = 6 \) then \( \lambda' = (4, 1^2) \) is the representation of claim (iii) of the lemma. In particular, we know the formulas for characters in these cases. This is a routine check that in all cases their values coincide with \( \frac{1}{n!}(i_1 - 1)(i_1 - 2)(i_1 - 3) - i_2(i_1 - 1) + i_3 \).

Assume now that \( n \geq 7 \). We consider three cases: for \( i_1 \neq 0, i_2 \neq 0, \) and \( i_3 \neq 0 \). If \( i_1 \neq 0 \) then we can assume that \( \sigma \) fixes \( n \) and denote by \( \sigma_1 \) the permutation in \( S_{n-1} \) acting on \( \{1, 2, \ldots, n-1\} \) as \( \sigma \). Observe that only the hooks \( H^\lambda_{1,n-3} \) and \( H^\lambda_{2,1} \) of \( T^\lambda \) are of length one, so Proposition 6 implies that \( \chi^\lambda(\sigma) = \chi^{(n-4,1^3)}(\sigma_1) + \chi^{(n-3,1^2)}(\sigma_1) \). By induction, we can assume that
\(\chi^{(n-2,1^3)}(\sigma_1) = \frac{1}{6}(i_1 - 2)(i_1 - 3)(i_1 - 4) - i_2(i_1 - 2) + i_3\), and by (ii) we know that\(\chi^{(n-3,1^3)}(\sigma_1) = \frac{1}{2}(i_1 - 2)(i_1 - 3) - i_2\). Therefore,

\[
\chi^\lambda(\sigma) = \frac{1}{6}(i_1 - 2)(i_1 - 3)(i_1 - 4) - i_2(i_1 - 2) + i_3 + \frac{1}{2}(i_1 - 2)(i_1 - 3) - i_2 = \\
= \frac{1}{6}(i_1 - 2)(i_1 - 3)(i_1 - 4 + 3) - i_2(i_1 - 1) + i_3 = \frac{1}{6}(i_1 - 1)(i_1 - 2)(i_1 - 3) - i_2(i_1 - 1) + i_3,
\]
as required.

Suppose that \(i_2 \neq 0\) and \(\sigma\) switches the symbols \(n - 1\) and \(n\). Denote by \(\sigma_1\) the permutation in \(S_{n-2}\) acting on \(\{1, 2, \ldots, n - 2\}\) as \(\sigma\). Observe that only the hooks \(H^\lambda_{1,n-4}\) and \(H^\lambda_{3,1}\) of \(T^\lambda\) are of length two, so Proposition 6 implies that \(\chi^\lambda(\sigma) = \chi^{(n-5,1^3)}(\sigma_1) - \chi^{(n-3,1)}(\sigma_1)\). By induction and (i), we find that

\[
\chi^\lambda(\sigma) = \frac{1}{6}(i_1 - 1)(i_1 - 2)(i_1 - 3) - (i_2 - 1)(i_1 - 1) + i_3 - i_1 + 1 = \\
= \frac{1}{6}(i_1 - 1)(i_1 - 2)(i_1 - 3) - i_2(i_1 - 1) + i_3,
\]
as required.

Finally, suppose that \(i_3 \neq 0\) and \(\sigma\) includes the cycle \((n - 2, n - 1, n)\) in its cycle decomposition. Denote by \(\sigma_1\) the permutation in \(S_{n-3}\) acting on \(\{1, 2, \ldots, n - 3\}\) as \(\sigma\). Observe that only the hooks \(H^\lambda_{1,n-5}\) and \(H^\lambda_{2,1}\) of \(T^\lambda\) are of length three, so Proposition 6 implies that \(\chi^\lambda(\sigma) = \chi^{(n-6,1^3)}(\sigma_1) + \chi^{(n-3)}(\sigma_1)\). Clearly, \(\chi^{(n-3)}(\sigma_1) = 1\). By induction, we find that

\[
\chi^\lambda(\sigma) = \frac{1}{6}(i_1 - 1)(i_1 - 2)(i_1 - 3) - i_2(i_1 - 1) + i_3 - 1 + 1 = \\
= \frac{1}{6}(i_1 - 1)(i_1 - 2)(i_1 - 3) - i_2(i_1 - 1) + i_3,
\]
as required.

**Case** \(\lambda = (n - 3, 2, 1)\). Proposition 7 yields

\[
\chi^\lambda(\sigma) = [\Delta(x)(x_1 + x_2 + x_3)^{i_1}(x_1^2 + x_2^2 + x_3^2)^{i_2}(x_1^3 + x_2^3 + x_3^3)^{i_3}]_{(n-1,3,1)},
\]
and we need to find the coefficient of \(x_1^{n-1}x_2^3x_3\) in this product. First we note that

\[
\Delta(x) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = x_1^2x_2 - x_1x_2^2 + x_1x_3^2 - x_2x_3^2 + x_1x_3^2 - x_1^2x_3.
\]
Since \(x_2^3\) does not divide \(x_1^{n-1}x_2^3x_3\), we can use only \(x_1^2x_2 - x_1x_2^2 + x_1x_3^2 - x_1x_3^2\) to obtain the term \(x_1^{n-1}x_2^3x_3\) in the product.
Assume that we avoid \( x_i^{3i_3} \) in the factor \((x_1^3 + x_2^3 + x_3^3)^{i_3}\). Then we can take only the term \( i_3 x_i^{3(i_3 - 1)} x_2^2 \). Note that \( \Delta(x) \) contains the factor \((x_2 - x_3)\) and since we chose \( x_i^3 \), we use \(-x_3\) in this factor. Now from all the other factors in the product we can use only the powers of \( x_i \); hence, in this case we get \(-i_3 x_i x_2^2 x_3\). So we can assume now that we use \( x_i^{3i_3} \) in \((x_1^3 + x_2^3 + x_3^3)^{i_3}\).

Assume that we avoid \( x_i^{2i_2} \) in the factor \((x_1^2 + x_2^2 + x_3^2)^{i_2}\). Then we can use only the term \( i_2 x_1^{2(i_2 - 1)} x_2^2 \). Now we need to find the factors where we take \( x_2 \) and \( x_3 \) to obtain the product \( x_2 x_3 \). Observe that in \( \Delta(x) \) we can use either \( x_i^2 x_2 \) or \(-x_i^2 x_3\). In the first case we must take \( i_1 x_i^{i_1-1} x_3 \) in \((x_1 + x_2 + x_3)^{i_1}\) and in the second case \( i_1 x_i^{i_1-1} x_2 \). Therefore, in the product for \( x_i^{2i_2} \) we get \( i_2 (i_1 - i_1) \cdot (-i_3 x_i x_2^2 x_3) = 0 \). So we can assume now that we use \( x_i^{2i_2} \) in \((x_1^2 + x_2^2 + x_3^2)^{i_2}\).

It remains to find the coefficient of \( x_i^{n-2i_2-3i_3} x_2^2 x_3 \) in the product \((x_1^2 x_2 - x_1 x_2^2 + x_2^2 x_3 - x_1^2 x_3)(x_1 + x_2 + x_3)^{i_1}\). Once we fix a term in the first factor, the term in \((x_1 + x_2 + x_3)^{i_1}\) is determined uniquely. Namely, for \( x_1^2 x_2 \) we take \( i_1(i_1-1)(i_1-2) x_1^{i_1-3} x_2^2 x_3 \), for \( -x_1 x_2^2 \) we take \( i_1(i_1-1) x_1^{i_1-2} x_2 x_3 \), for \( x_2^2 x_3 \) we take \( i_1 x_1^{i_1-1} x_2 \), and for \(-x_1^2 x_3 \) we take \( \frac{i_1(i_1-4)}{6} x_1^{i_1-3} x_2^3 \). Thus, in this case we obtain

\[
\left( \frac{i_1(i_1 - 1)(i_1 - 2)}{3} - i_1(i_1 - 1) + i_1 \right) x_1^{n-1} x_2^2 x_3 =
\]

\[
= \frac{i_1}{3} (i_1^2 - 3i_1 + 2 - 3i_1 + 3 + 3)x_1^{n-1} x_2^2 x_3 = \frac{i_1(i_1 - 2)(i_1 - 4)}{3} x_1^{n-1} x_2^3 x_3.
\]

So the final coefficient of \( x_1^{n-1} x_2^3 x_3 \) is \( \frac{i_1(i_1-2)(i_1-4)}{3} - i_3 \), as claimed.

\[\square\]

The following result helps us to find the dimensions of eigenspaces of the elements using the character values.

**Lemma 8.** Consider a finite group \( G \) and an ordinary representation of \( \rho : G \to GL(V) \). Take \( g \in G \) with \( |g| = n \) and \( \eta \in \mathbb{C} \) with \( \eta^n = 1 \). If \( \chi \) is the character of \( \rho \) then

\[
n \cdot \dim(V^n(\rho(g))) = \sum_{i=0}^{n} \chi(g^i) \eta^i,
\]

where \( V^n(\rho(g)) = \ker(\rho(g) - \eta \cdot \text{Id}_V) \), i.e. the eigenspace of \( \rho(g) \) associated to \( \eta \).
Proof. Denote by $\rho'$ the restriction of $\rho$ to $H = \langle g \rangle$ and by $\chi'$ the character of $\rho'$. Consider an irreducible representation of $H$ which maps $g$ to $\eta g$ and denote by $\psi$ its character. Then the inner product

$$\langle \chi', \psi \rangle = \frac{1}{|H|} \sum_{i=0}^{i<n} \chi'(g^i) \psi(g^i) = \frac{1}{n} \sum_{i=0}^{i<n} \chi(g^i) \eta^i$$

equals the number of irreducible constituents of $\chi'$ with character $\psi$. Clearly, the direct sum of these constituents is the eigenspace of $\rho'(g)$ associated to $\eta$. Therefore, $\sum_{i=0}^{i<n} \chi(g^i) \eta^i = n \cdot \dim(V^\eta(\rho(g)))$, as claimed. \qed

Lemma 9. Suppose that $n \geq 23$. Then the following inequalities hold.

(i) $\sqrt{n(n-2)(n-7)} > 54 \cdot (n-1)$;
(ii) $\sqrt{n(n-2)(n-7)^2} > 8 \cdot (24)^2 \cdot (n-1)$.

Proof. To prove (i), we show that $\sqrt{n(n-2)(n-7)} > 54 n$. This inequality is equivalent to $\frac{n-2}{\sqrt{n}} (n-7) > 54$. Since $\sqrt{n} > 4$, we have $\frac{n-2}{\sqrt{n}} = \sqrt{n} - \frac{2}{\sqrt{n}} > 4 - 1/2 = 7/2$. Then $\frac{n-2}{\sqrt{n}} (n-7) > \frac{7}{2} \cdot 16 = 56$. So $\sqrt{n(n-2)(n-7)} > 54 n > 54(n-1)$.

Now we prove (ii). By (i), we have $\sqrt{n(n-2)(n-7)^2} > 54(n-2)(n-7)(n-1)$. So it suffices to prove that $(n-2)(n-7) \geq \frac{8(24)^2}{54}$. This inequality is true for $n = 23$ and hence it is true for all $n \geq 23$. \qed

4 Proof of Theorem 1

In this section we prove Theorem 1. Throughout, we suppose that $n \geq 3$ and $\sigma$ is a permutation of shape $[r \cdot 1^n 1^{n-r}]$ in $S_n$, where $r \geq 2$. We fix a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$ and an irreducible $S_n$-module $V$ corresponding to $\lambda$. The proof is split into several lemmas.

Lemma 10. If $\lambda \in \{(1^n), (n-1, 1), (2, 1^{n-2})\}$ then Theorem 1 holds.

Proof. Assume first that $\lambda$ gives the alternating representation, i.e. $\lambda = (1^n)$ and $V$ is one-dimensional. Then we have $\chi^\lambda(\sigma) = \text{sgn}(\sigma)$. If $\sigma$ is even then $|\sigma| \cdot \dim(C_V(\rho(\sigma))) = \sum_{i=0}^{i<|\sigma|} \chi^\lambda(\sigma^i) = \sum_{i=0}^{i<|\sigma|} 1 = |\sigma|$. Hence, $\sigma$ acts trivially on $V$ and the minimal polynomial equals $x - 1$. If $\sigma$ is odd then $|\sigma| = r$ is
even. Therefore, $|\sigma| \cdot \dim V^{-1}(\rho(\sigma)) = \sum_{i=0}^{[r]} |\chi^\lambda(\sigma^i)|(-1)^i = |\sigma|$. In this case $V = V^{-1}(\rho(\sigma))$ and the minimal polynomial equals $x + 1$.

Now take $\lambda = (n-1,1)$, i.e. $\lambda$ corresponds to the standard representation. Denote by $i_1 = n - rm$ the number of length one cycles in the cycle decomposition of $\sigma$. Lemma 7 implies that $\chi^\lambda(\sigma^0) = n - 1$ and $\chi^\lambda(\sigma^j) = i_1 - 1$ for $1 \leq j \leq r - 1$. If $\eta$ is a root of the polynomial $x^r - 1$ then $r \cdot \dim V^\eta(\rho(\sigma)) = n - 1 + (i_1 - 1)(\eta + \eta^2 + \ldots + \eta^{r-1}) = n - i_1 + (i_1 - 1)(1 + \eta + \ldots + \eta^{r-1})$.

Thus, $r \cdot \dim V^\eta(\rho(\sigma)) = 0$ if and only if $i_1 = 0$, $\eta = 1$, and $r = n$. Consequently, if $\sigma$ is not of shape $[n^1]$ then the minimal polynomial equals $x^r - 1$, and for $\sigma = [n^1]$ the minimal polynomial equals $\frac{x^n - 1}{x - 1}$.

Finally, take $\lambda = (2, 1^{n-2})$. If $\sigma$ is even then $\chi^\lambda(\sigma) = \chi^{(n-1,1)}(\sigma)$ and this case is similar to the previous one. If $\sigma$ is odd then $r$ is even. If $\eta$ is a root of the polynomial $x^n - 1$ then
\[
r \cdot \dim V^\eta(\rho(\sigma)) = n - 1 + (i_1 - 1)(-\eta + \eta^2 - \ldots - \eta^{r-1}) = n - i_1 + (i_1 - 1)(1 - \eta + \ldots - \eta^{r-1}).
\]
Therefore,
\[
r \cdot \dim V^\eta(\rho(\sigma)) = \begin{cases} n - i_1 + (i_1 - 1)r & \text{for } \eta = 1, \\ n - i_1 & \text{for } \eta \neq 1. \end{cases}
\]
Thus, $r \cdot \dim V^\eta(\rho(\sigma)) = 0$ if and only if $i_1 = 0$, $\eta = -1$, and $r = n$.

\[\Box\]

**Lemma 11.** Theorem 7 holds if $n \leq 22$ and $\sigma$ is of shape $[r^m]$, where $rm = n$.

**Proof.** By Lemma 10 we can assume that $\lambda \not\in \{(1^n), (n-1,1), (2, 1^{n-2})\}$. We need to prove that the minimal polynomial of $\sigma$ is $x^n - 1$. Since $\sigma$ is a power of a $n$-cycle, we can assume that $\sigma = [n^1]$. Now we apply GAP to establish the assertion. If $\eta$ is a root of $x^n - 1$ and $V$ is an irreducible representation of $S_n$ then we find $\dim V^\eta(\rho(\sigma))$ with the aid of Lemma 8. \[\Box\]

**Lemma 12.** Let $n \geq 7$. Assume that either $\lambda$ or $\lambda'$ belongs to $M = \{(n-2,2), (n-2,1,1), (n-3,3), (n-3,1^3), (n-3,2,1)\}$.

If $\sigma$ is of shape $[r^m]$, where $n = rm$ and $r \geq 2$, then $\chi^\lambda(1) > \sum_{i=1}^{r} |\chi^\lambda(\sigma^i)|$.  

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Proof. Since $\chi^\lambda(\rho) = \sgn(\rho) \cdot \chi^\lambda(\rho)$, we can assume that $\lambda \in M$. Now we consider each possibility for $\lambda$.

Suppose that $\lambda = (n-2,2)$. Note that $\sigma^i$ is of shape $[k^2]$, where $2 \leq k \leq r$ and $k \geq n$. Lemma 7 implies that $\chi^\lambda(\sigma^i) = n/2$ if $k = 2$ and $\chi^\lambda(\sigma^i) = 0$ if $k > 2$. Observe that $k = 2$ for at most one value of $i$. Therefore, $\sum_{i \leq r} |\chi^\lambda(\sigma^i)| \leq n/2$. On the other hand, Lemma 6 implies that $\chi^\lambda(1) = n/(2(n-3))$ and hence $\chi^\lambda(1) > n/2$ for $n \geq 5$. So $\chi^\lambda(1) > \sum_{i \leq r} |\chi^\lambda(\sigma^i)|$.

Suppose that $\lambda = (n-2,1,1)$. Lemma 7 implies that $\chi^\lambda(\sigma^i) = 1 - n/2 = (2-n)/2$ if $\sigma^i$ is of shape $[2n/2]$ and $\chi^\lambda(\sigma^i) = 1$ otherwise. Again, the permutation $\sigma^i$ is of shape $[2n/2]$ only for at most one value of $i$. On the other hand, we know that $\chi^\lambda(1) = (n-1)(n-2)/2$. Therefore, if $n \geq 6$ then $\chi^\lambda(1) > n-1 + (n-2)/2 \geq \sum_{i \leq r} |\chi^\lambda(\sigma^i)|$.

Suppose that $\lambda = (n-3,3)$. By Lemma 7 if $1 \leq i \leq r-1$ then

$$\chi^\lambda(\sigma^i) = \begin{cases} \chi^\lambda(\sigma^i) = -n/2 & \text{if $\sigma^i$ is of shape $[2n/2]$}, \\ \chi^\lambda(\sigma^i) = n/3 & \text{if $\sigma^i$ is of shape $[3n/3]$}, \\ 0 & \text{otherwise}. \end{cases}$$

Observe that $\sigma^i$ is of shape $[2n/2]$ for at most one value of $i$ and $\sigma^i$ is of shape $[3n/3]$ for at most two values of $i$. Therefore, we have $\sum_{i \leq r} |\chi^\lambda(\sigma^i)| \leq n/2 + 2n/3 = 7n/6$. On the other hand, $\chi^\lambda(1) = \frac{1}{6}n(n-1)(n-5)$. Consequently, if $n \geq 7$ then $(n-1)(n-5) > 7$ and hence $\chi^\lambda(1) > \sum_{i \leq r} |\chi^\lambda(\sigma^i)|$.

Suppose that $\lambda = (n-3,1^3)$. By Lemma 7 if $1 \leq i \leq r-1$ then

$$\chi^\lambda(\sigma^i) = \begin{cases} \chi^\lambda(\sigma^i) = n-2 - 1 & \text{if $\sigma^i$ is of shape $[2n/2]$}, \\ \chi^\lambda(\sigma^i) = n/3 - 1 & \text{if $\sigma^i$ is of shape $[3n/3]$}, \\ -1 & \text{otherwise}. \end{cases}$$

Therefore, we have $\sum_{i \leq r} |\chi^\lambda(\sigma^i)| \leq n/2 + 2n/3 + n - 3 = 13n/6 - 3$. On the other hand, $\chi^\lambda(1) = \frac{1}{6}(n-1)(n-2)(n-3)$. Since $n \geq 7$, we have $\frac{1}{6}(n-1)(n-2)(n-3) \geq 20(n-1)/6 > 13n/6$ and hence $\chi^\lambda(1) > \sum_{i \leq r} |\chi^\lambda(\sigma^i)|$.

Finally, suppose that $\lambda = (n-3,2,1)$. By Lemma 7 if $\chi^\lambda(\sigma^i) \neq 0$ then $\chi^\lambda(\sigma^i) = -n/3$ and $\sigma^i$ is of shape $[3n/3]$. Therefore, $\sum_{i \leq r} |\chi^\lambda(\sigma^i)| \leq 2n/3$. On the other hand, $\chi^\lambda(1) = \frac{1}{3}n(n-2)(n-4)$ and hence $\chi^\lambda(1) > \sum_{i \leq r} |\chi^\lambda(\sigma^i)|$ for $n \geq 5$.

Lemma 13. Suppose that $n \geq 23$ and $\sigma$ is of shape $[r^m]$, where $rm = n$ and $r \geq 2$. If $\chi$ is an irreducible character of $S_n$ and $\chi(1) > n - 1$ then $\chi(1) > \sum_{i \leq r} |\chi(\sigma^i)|$. \qed

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Lemma 9. Therefore, we infer that \( \chi \).

Theorem 1 holds if \( \lambda \) and \( \sigma' \) is of shape \([k']\) for some integer \( k \) and \( t \) with \( 2 \leq k \leq r \) and \( kt = n \). Lemma 3 implies that

\[
|\chi(\sigma')| < (2\pi)^{1/2} k^{-1/2} n^{k-1} e^{1/12t}.
\]

Assume that \( k = 2 \). Then \( (2\pi)^{1/2} k^{-1/2} n^{k-1} e^{1/12t} = (\frac{\pi}{2})^{1/2} n^{1/2} e^{1/12t} < \frac{3}{2} n^{1/2} \).

Therefore, in this case

\[
|\chi(\sigma')| < \frac{3}{2} n^{1/2} \sqrt{\chi(1)}
\]

Assume now that \( k \geq 3 \). Then we have \( (2\pi)^{1/2} k^{-1/2} n^{k-1} e^{1/12t} < \frac{3}{2} n^{1/2} \), \( n^{k-1/2} \leq \sqrt{n} \), and \( e^{1/12t} \leq 1.1 \). Moreover, \( \chi(1)^{1/k} \leq \chi(1)^{1/2} \) and hence

\[
|\chi(\sigma')| \leq 1.1 \sqrt{n} \sqrt{\chi(1)} \leq 2 \sqrt{n} \chi(1)^{1/2}.
\]

Now we prove that \( \chi(\sigma') < \chi(1)/(n-1) \). For \( k = 2 \), it suffices to show that \( \frac{3}{2} n^{1/2} \sqrt{\chi(1)} < \chi(1)/(n-1) \). This is equivalent to \( \frac{3}{2} \sqrt{n} (n-1)^2 < \chi(1) \).

If \( \chi(1) \geq n(n-1)(n-2)(n-7)/24 \) then Lemma 9 implies that \( \chi(1) \geq \sqrt{n} (n-1)^2 54/24(n-1) = 9/4 \sqrt{n} (n-1)^2 \), as required.

For \( k \geq 3 \), it suffices to show that \( 2n^{1/2} \chi(1)^{1/2} < \chi(1)/(n-1) \). This is equivalent to \( 8n \sqrt{n} (n-1)^3 < \chi(1)^2 \). We verify that

\[
(n(n-1)(n-2)(n-7)/24)^2 \geq 8n \sqrt{n} (n-1)^3.
\]

This is equivalent to \( \sqrt{n} (n-2)(n-7)^2 \geq 8(24)^2(n-1) \), which is true by Lemma 9. Therefore, we infer that \( (\chi(1))^2 > (n(n-1)(n-2)(n-7)/24)^2 \geq 8n \sqrt{n} (n-1)^3 \), as required.

Lemma 14. Theorem 4 holds if \( \lambda \notin \{(1^n, (n-1, 1), (2, 1^{n-2})\} \).

Proof. Proceed by induction on \( n \). Assume that \( \sigma \) is of shape \([r^{m}1^{n-rm}]\).

Suppose that \( rm < n \) and \( m \neq 5, 7 \). Then we may assume that \( \sigma \) belongs to the stabilizer \( S_{n-1} \) of the point \( n \). The Branching Law yields \( [\lambda] \downarrow S_{n-1} = \sum \lambda_i > \lambda_{i+1} |\lambda^-| \). Take some partition \( \mu \) of \( n-1 \) occurring in the latter sum. If \( \mu \notin \{(1^{n-1}), (n-2, 1), (2, 1^{n-2})\} \) then by the induction hypothesis \( \sigma \) has an eigenvector for every root \( \eta \) of the polynomial \( x^{r} - 1 \) and hence the minimal polynomial equals \( x^{r} - 1 \). If \( \mu = (1^{n-1}) \) then \( \lambda \) is either \( (1^n) \) or \( (2, 1^{n-2}) \); a
contradiction. If \( \mu = (n - 2, 1) \) then, by assumption, \( \lambda \) is either \((n - 2, 2)\) or \((n - 2, 1, 1)\). In the first case \([\lambda] \downarrow S_{n-1}\) has a summand for the partition \((n - 3, 2)\) and in the second case for \((n - 3, 1, 1)\). Using the induction hypothesis for these summands, we infer that the minimal polynomial is \(x^r - 1\). If \(n = 5\) or \(7\) then we apply the same argument avoiding the exceptional cases of the theorem.

Therefore, we can assume that \(n = rm\). By Lemma 11, it remains to consider the cases \(n \geq 23\). If \(\eta\) is a root of the polynomial \(x^r - 1\) then \(\eta^r = 1\) and hence the minimal polynomial of \(\sigma\) equals \(x^r - 1\), as required.

5 Proof of Corollary

In this section we prove Corollary 1. Throughout, we assume that \((\rho, V)\) is an irreducible representation of \(A_n\), where \(n \geq 5\), and \(\chi\) is the character of \(\rho\). By \(\sigma\) we denote a permutation of \(A_n\) of shape \([r m 1 n - rm]\), where \(r \geq 2\) and \(rm \leq n\). The minimal polynomial of \(\rho(\sigma)\) on \(V\) is denoted by \(\mu(\rho(\sigma))(x)\).

**Lemma 15.** If \(\rho\) is the standard representation then \(\mu(\rho(\sigma))(x) \neq x^r - 1\) if and only if \(r = n\), where \(n\) is odd.

**Proof.** Take the partition \(\lambda = (n - 1, 1)\) of \(n\) corresponding to the standard representation of \(S_n\). Since \(n \geq 5\), we have \(\lambda \neq \lambda'\) and hence \([\lambda] \downarrow A_n\) is the standard representation of \(A_n\). For every \(\tau \in A_n\) we have \(\chi(\tau) = \chi^\lambda(\tau)\). Then for every \(\eta\) with \(\eta^r = 1\) the dimensions of eigenspaces of \(\sigma\) for \(A_n\) and \(S_n\) coincide. Now Theorem implies that \(\mu(\rho(\sigma))(x) \neq x^r - 1\) only if \(\sigma\) is a length \(n\) cycle. If \(\sigma\) is such a cycle then \(n\) is odd and hence \(\mu(\rho(\sigma))(x) = \frac{x^n - 1}{x - 1}\) by Theorem 1.

**Lemma 16.** Corollary 1 holds if \(\rho\) is not the standard representation.
Proof. Induct on $n$. If $n = 5$ or $6$ then the assertion can be verified using the character tables of $A_5$ and $A_6$. Observe that there are two irreducible characters of $A_5$ that correspond to the self-associated partition $(3, 1, 1)$. In these cases if $η$ is a primitive 5th root of unity then either $η$ and $η^2$ or $η^2$ and $η^3$ are not roots of $μ_{ρ((1,2,3,4,5))}(x)$.

Assume now that $ρ$ is obtained from the irreducible representation of $S_n$ corresponding to a partition $λ$ of $n$. If $λ \neq λ'$ then we can argue as in Lemma 15 and the assertion follows from Theorem 1. Suppose that $λ = λ'$. If $χ(σ) = χ^λ(σ)/2$ then the assertion follows from Theorem 1. Denote the lengths of hooks along the main diagonal of $T^λ$ by $l_1, l_2, \ldots, l_k$. It follows from Proposition 4 that $χ(σ) \neq χ^λ(σ)/2$ only if $σ$ is of shape $[l_1^2 \cdots l_k^2]$. Since $l_i$ are always distinct, we infer that either $σ = [(n - 1)^2]$ and $n$ is even or $σ = [n^2]$ and $n$ is odd.

Assume that $σ = [(n - 1)^2]$, that is, a length $n - 1$ cycle. In this case $ρ$ results from the representation of $S_n$ corresponding to the self-associated partition $λ = (\frac{n-1}{2}, 2, 1^{n-1})$. By Proposition 3, we have $[λ] \downarrow A_n = V_1 \oplus V_2$. Consider the subgroups $A_{n-1}$ and $S_{n-1}$ containing $σ$ and stabilizing the point $n$. By the Branching Law, $[λ] \downarrow S_{n-1} = [(\frac{n-2}{2}, 2, 1^{n-1})] + [(\frac{n}{2}, 2, 1^{n-2})] + [(\frac{n}{2}, 1^{n-2})]$. All the summands are not equivalent to the standard representation of $S_{n-1}$ and only the partition $(\frac{n}{2}, 1^{n-2})$ is self-associated. By Proposition 3, restricting these representations to $A_{n-1}$, yields four summands. By induction, in each of them the minimal polynomial of $σ$ equals $x^{n-1} - 1$. Therefore, each irreducible constituent of the restrictions of the representations of $A_n$ to $A_{n-1}$ equals one of the four mentioned above and hence the minimal polynomials of $σ$ on $V_1$ and $V_2$ equal $x^{n-1} - 1$.

Assume that $σ = [n^2]$, that is, a length $n$ cycle with odd $n$. In this case $ρ$ results from a representation of $S_n$ corresponding to the self-associated partition $λ = (\frac{n+1}{2}, 1^{n-1})$. By Proposition 4, if $τ$ is not a length $n$ cycle then $χ(τ) = χ^λ(τ)/2$. Moreover, Lemma 4 and Proposition 4 imply that $χ(σ) = \frac{1}{2}((-1)^{\frac{n-1}{2}} \pm \sqrt{(-1)^{\frac{n-1}{2}}})$.

We prove that $|χ(1)| \geq \sum_{i=1}^{n-1} |χ(σ^i)|$. The hook-length formula yields $χ(1) = χ^λ(1)/2 = \frac{n}{2} \cdot (\frac{n-1}{2})$. We show that $|χ(σ^i)| \leq \frac{1}{2} \left(\frac{n-1}{2}\right)$ for each $i \in \{1, \ldots, n-1\}$. If $σ^i$ is not a cycle then $σ^i$ is of shape $[r^m]$, where $r, m \geq 2$. Now Lemma 5 yields $χ^λ(σ^i) = (-1)^{(r-1)/2} \cdot (\frac{m}{2})$ and hence $|χ(σ^i)| = \frac{1}{2}|χ^λ(σ^i)| \leq \frac{1}{2} \left(\frac{n-1}{2}\right)$, as claimed. If $σ^i$ is a cycle then $χ(σ^i) = \frac{1}{2}((-1)^{\frac{n-1}{2}} \pm \sqrt{(-1)^{\frac{n-1}{2}}})$. It follows that $|χ(σ^i)| \leq \frac{1}{2}(1 + \sqrt{n})$. Since $\left(\frac{n-1}{2}\right) > \left(\frac{n-1}{2}\right) = n - 1$, we have $\left(\frac{n-1}{2}\right) \geq n \geq 2\sqrt{n} - 1 > 1 + \sqrt{n}$, and hence $|χ(σ^i)| < \frac{1}{2} \left(\frac{n-1}{2}\right)$, as claimed.
Now consider a complex number $\eta$ such that $\eta^n = 1$. Since
\[ \text{Re}(\chi(\sigma^i)\eta^i) \geq -|\chi(\sigma^i)\eta^i| = -|\chi(\sigma^i)|, \]
we have
\[ n \cdot \dim(V^\eta(\rho(\sigma))) = \sum_{i=0}^{n-1} \chi(\sigma^i)\eta^i = \sum_{i=0}^{n-1} \text{Re}(\chi(\sigma^i)\eta^i) \geq \chi(1) - \sum_{i=1}^{n-1} |\chi(\sigma^i)| > 0. \]
This implies that the minimal polynomial of $\rho(\sigma)$ equals $x^n - 1$.

6 Proof of Proposition 1

In this section we prove Proposition 1. The proof is divided into two
lemmas.

Lemma 17. For an odd integer $n \geq 5$, consider $\sigma \in S_n$ of shape $[(n-2)^12^1]$ and an irreducible representation $\rho : S_n \to GL(V)$ of $S_n$ corresponding to the
partition $(2^2, 1^{n-4})$. Then $\sigma$ lacks nontrivial fixed vectors in $V$.

Proof. Put $\lambda = (2^2, 1^{n-4})$ and denote by $\chi^\lambda$ the character of $\rho$. Proposition 2(iii) implies that $\chi^\lambda(\tau) = \text{sgn}(\tau)\chi^{(n-2,2)}(\tau)$ for every $\tau \in S_n$. Therefore, applying Lemma 7, we can find $\chi^\lambda(\sigma^j)$ for every $0 \leq j < 2(n-2)$:
\[ \chi^\lambda(\sigma^j) = \begin{cases} \frac{1}{2}(n-1)(n-2) - 1 = \frac{n(n-3)}{2} & \text{if } j = 0, \\ -1 & \text{if } j \neq 0 \text{ is even,} \\ -\frac{1}{2}(n-3)(n-4) & \text{if } j = n-2, \\ -1 & \text{if } j \neq n-2 \text{ and } j \text{ is odd.} \end{cases} \]

Since among the numbers $0, \ldots, 2(n-2) - 1$ exactly $n-2$ are even and
$n-2$ are odd, we infer that $|\sigma| \cdot \dim C_V(\rho(\sigma)) = \sum_{j=0}^{n-3} \chi^\lambda(\sigma^j) = \frac{n(n-3)}{2} - (n-3) - \frac{(n-3)(n-4)}{2} - (n-3) = 0$. □

Lemma 18. Suppose that the pair $(\lambda, \sigma)$ is one of the following.
(i) $\lambda = (1^n)$ and $\sigma$ is odd,
(ii) or $\sigma$ is $n$-cycle and either $\lambda = (n-1, 1)$ or $n$ is odd and $\lambda = (2, 1^{n-2}),$
(iii) or $((2^3), [3^12^11^1]), ((4^2), [5^13^1]), ((2^4), [5^13^1]), ((2^5), [5^13^12^1]).$

Then $\sigma$ has no nontrivial fixed vectors in $V$.

Proof. The claims corresponding to (i) and (ii) are considered in Theorem 1.
The last claim can be verified using characters tables of $S_6$, $S_8$, and $S_{10}$ or GAP. □
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