Random numbers are an important resource for applications such as numerical simulation and secure communication. However, it is difficult to certify whether a physical random number generator is truly unpredictable. Here, we exploit the phenomenon of quantum nonlocality in a loophole-free photonic Bell test experiment for the generation of randomness that cannot be predicted within any physical theory that allows one to make independent measurement choices and prohibits superluminal signaling. To certify and quantify the randomness, we describe a new protocol that performs well in an experimental regime characterized by low violation of Bell inequalities. Applying an extractor function to our data, we obtained 256 new random bits, uniform to within 0.001.

Random numbers have many uses. A motivating application for our experiment is a public randomness beacon that broadcasts certified random bits at predetermined times [1]. For certain applications, such as sampling and numerical simulation, algorithmically generated pseudorandom strings are often sufficient. However, the predictability of pseudorandom strings makes them unsuitable for other applications such as secure communication. For such purposes, the
theoretical unpredictability of quantum mechanical experiments make them good candidates for random number generation [2].

A simple quantum random number generator may consist of a device that measures a pure state of a two-level system in a basis that is not aligned with the input state. To assert the presence of randomness in the output of such a device, one must assume that the state and measurement are properly characterized. However, this assumption can be compromised in a potentially undetectable manner. For example, if predictable sources of noise infect the device, the output may become increasingly predictable without inducing the failure of statistical tests of randomness [3, 4]. This issue has inspired the development of the field of device-independent quantum randomness generation in recent years [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In the device-independent paradigm, randomness is generated through an experiment called a Bell test [13]. In its simplest form, a Bell test performs measurements on an entangled system located in two physically separated measurement stations, where at each station there are two types of measurements that can be made. After multiple experimental trials with varying measurement choices, if the measurement data violates conditions known as “Bell inequalities,” then the data can be certified to contain randomness under very weak assumptions.

Here, we report the generation of 256 new random bits uniform to within 0.001 with a “loophole-free” Bell test, which notably is characterized by high detection efficiency and space-like separation of the measurement stations during each experimental trial. The bits are unpredictable assuming that (1) the choices of measurement settings are independent of the experimental devices and pre-existing classical information about them and (2) in each experimental trial, the measurement outcomes at each station are independent of the settings choices at the other station. The first assumption is ultimately untestable, but the premise that it is possible to choose measurement settings independently of a system being measured is often tacitly invoked in the interpretation of many scientific experiments and laws of physics [14]. The second assumption can only be violated if one admits a theory that permits sending signals faster than the speed of light, given space-like separation of the stations. We trust the recording and timing electronics to accurately verify the space-like separation of the relevant events in the experiment, and that the classical computing equipment used to process the data operates according to specification. Under these assumptions, the output randomness is certified to be unpredictable with respect to a real or hypothetical actor “Eve” in possession of the pre-existing classical information and physically isolated from the devices while they are under our control. The bits remain unpredictable to Eve if she learns the settings at any time after her last interaction with the devices. If the devices are trusted, which is reasonable if we built them, it may be the case that Eve has no pre-existing information. The settings can then come from public randomness generated externally at any time [3]. Our framework encompasses any quantum description of the system being measured, as well as more general theoretical possibilities [15].

The only previous experimental production of certified randomness from Bell test data was reported in the ground-breaking paper by Pironio et al. [6]. Their Bell test was implemented with ions in two separate ion-traps, closing the detection loophole [16] but without space-like separation. Indeed, Bell tests achieving space-like separation without leaving other experimen-
tal loopholes open have been performed only recently [17, 18, 19, 20]. Under more restrictive assumptions than ours, the maximum amount of randomness in principle available in the data of Pironio et al. was quantified as 42 bits with an error parameter of 0.01. However, they did not extract a uniformly distributed bit string from their data set.

We generated randomness using a photonic loophole-free Bell test, illustrated in Fig. 1. The experiment consisted of a source of entangled photons and two measurement stations named “Alice” and “Bob”. During an experimental trial, at each station a random choice was made between two measurement settings labeled 0 and 1, after which a measurement outcome of detection (+) or nondetection (0) was recorded. Each station’s choice was space-like separated from the other station’s measurement event. For trial $i$, we model Alice’s settings choices with the random variable $X_i$ and Bob’s with $Y_i$, both of which take values in the set $\{0, 1\}$. Alice’s and Bob’s measurement outcome random variables are respectively $A_i$ and $B_i$, both of which take values in the set $\{+, 0\}$. When referring to a generic single trial, we omit indices. With this notation, a general Bell inequality for our scenario can be expressed in the form [21]

$$\sum_{abxy} s_{ab}^{xy} P(A = a, B = b | X = x, Y = y) \leq \beta,$$

(1)

where the $s_{ab}^{xy}$ are fixed real coefficients indexed by $a, b, x, y$ that range over all possible values of $A, B, X, Y$. The upper bound $\beta$ is required to be satisfied whenever the settings-conditional outcome probabilities are induced by a model satisfying “local realism” (LR). LR distributions, which cannot be certified to contain randomness, are those for which $P(A = a, B = b | X = x, Y = y)$ is of the form $\sum_\lambda P(A = a | X = x, \Lambda = \lambda)P(B = b | Y = y, \Lambda = \lambda)P(\Lambda = \lambda)$ for a random variable $\Lambda$ representing local hidden variables. The Bell inequality is non-trivial if there exists a quantum-realizable distribution that can violate the bound $\beta$.

Experimental violations of Eq. [2] indicate the presence of randomness in the data [2]. To quantify randomness with respect to Eve, we represent Eve’s initial classical information by a random variable $E$. We formalize the assumption that measurement settings can be generated independently of the system being measured and Eve’s information with the following condition:

$$P(X_i = x, Y_i = y | E = e, \text{past}_i) = P(X_i = x, Y_i = y) = \frac{1}{4}, \quad \forall x, y, e,$$

(2)

where past, represents events in the past of the $i$'th trial, specifically including the trial settings and outcomes for trial 1 through $i - 1$. Our other assumption, that measurement outcomes are independent of remote measurement choices, is formalized as follows:

$$P(A_i = a | X_i = x, Y_i = y, E = e, \text{past}_i) = P(A_i = a | X_i = x, E = e, \text{past}_i)$$

$$P(B_i = b | X_i = x, Y_i = y, E = e, \text{past}_i) = P(B_i = b | Y_i = y, E = e, \text{past}_i) \quad \forall x, y, e.$$  (3)

These equations are commonly referred to as the “non-signaling” assumptions, although they are often stated without the conditionals $E$ and past$_i$. Our space-like separation of choices and remote measurements provide assurance that the experiment obeys Eqs. [3] We make no
Figure 1: Schematic layout of the loophole-free Bell test \cite{18}. The setup consists of a source located at the corner of an “L” and two measurement stations, Alice and Bob, at the ends of the “L”. The source generates entangled pairs of photons that travel through fiber optic cables to the respective measurement stations. At each station, schematically shown in the box labeled Alice/Bob, a random number generator (RNG) governs polarization rotators (PR) that implement one of two possible measurement settings prior to detection or nondetection of an arriving photon by a photon detector (PD), achieving a system efficiency of about 75\%. Each station’s measurement setting, outcome, and synchronization signal are then recorded by its timetagger. See Ref. \cite{18} for details.
other assumptions on the physics of the devices used in the experiment, but remark that if one
constrains the devices to quantum physics, constraints stronger than non-signaling are possible [22].

Given Eqs. 2 and 3 our protocol produces random bits in two sequential parts. For the first
part, “entropy production”, we implement $n$ trials of the Bell test, from which we compute a
statistic $V$ related to a Bell inequality (Eq. 1). $V$ quantifies the Bell violation and determines
whether or not the protocol passes or aborts. If the protocol passes, we can certify an amount of
randomness in the outcome string even conditioned on the setting string and $E$. In the second
part, “extraction,” we process the outcome string into a shorter string of bits whose distribution
is close to uniform. We used our customized implementation of the Trevisan extractor [23]
derived from the framework of Mauerer, Portmann and Scholz [24] and the associated open
source code. We call this the TMPS algorithm, see Supplementary Text (ST) S.4 for details.

We developed a new method for the entropy production part of our protocol, as previous
methods [3, 4, 5, 6, 7, 8, 9, 25, 26] are ineffective in our experimental regime (ST S.7), which
is characterized by a small per-trial violation of Bell inequalities. Recent papers that explore
how to effectively certify randomness from a wider range of experimental regimes assume that
measured states are independent and identically distributed (i.i.d.) or that the regime is asymp-
totic [10, 11, 12, 27]. Our method, which does not require these assumptions, builds on the
Prediction-Based Ratio (PBR) method for rejecting LR [28]. Applying this method to training
data (see below), we obtain a real-valued “Bell function” $T$ with arguments $A, B, X, Y$ that
satisfies $T(A, B, X, Y) > 0$ with expectation $\mathbb{E}(T) \leq 1$ for any LR distribution satisfying
Eq. 2. From $T$ we determine the maximum value $1 + m$ of $\mathbb{E}(T)$ over all distributions satisfy-
ing Eqs. 2 and 3 where we require that $m > 0$. Such a function $T$ induces a Bell inequality
(Eq. 1) with $\beta = 4$ and $s_{xy}^{ab} = T(a, b, x, y)$. Define $T_i = T(A_i, B_i, X_i, Y_i)$ and $V = \prod_{i=1}^{n} T_i$. If the experimenter observes a value of $V$ larger than 1, this indicates a violation of the Bell
inequality and the presence of randomness in the data. The randomness is quantified by the
following theorem, proven in the ST S.2.

Entropy Production Theorem. Suppose $T$ is a Bell function satisfying the above conditions.
Then in an experiment of $n$ trials obeying Eqs. 2 and 3, the following inequality holds for all
$\epsilon_p \in (0, 1)$ and $v_{\text{thresh}}$ satisfying $1 \leq v_{\text{thresh}} \leq (1 + (3/2)m)^n\epsilon_p^{-1}$:

$$\Pr_e(\Pr_e(AB|XY) > \delta \text{ AND } V \geq v_{\text{thresh}}) \leq \epsilon_p$$

where $\delta = \left[1 + \left(1 - \sqrt{\epsilon_p v_{\text{thresh}}}/(2m)^n\right)\right]^n$ and $\Pr_e$ denotes the probability distribution conditioned
on the event $\{E = e\}$, where $e$ is arbitrary. The expression $\Pr_e(AB|XY)$ denotes the random
variable that takes the value $\Pr_e(AB = ab|XY = xy)$ when $ABXY$ takes the value $abxy$.

In words, the theorem says that with high probability, if $V$ is at least as large as $v_{\text{thresh}}$, then
the output $AB$ is unpredictable, in the sense that no individual outcome $\{AB = ab\}$ occurs
with probability higher than \( \delta \), even given the information \( \{XYE = xye\} \). The theorem supports a protocol that aborts if \( V \) takes a value less than \( v_{\text{thresh}} \), and passes otherwise. In order for the guarantee of the theorem to be meaningful, it is necessary to have a lower bound \( \kappa \) on \( \mathbb{P}(\text{pass}) = \mathbb{P}(V \geq v_{\text{thresh}}) \). While we cannot be sure of such a lower bound, an assumption that the probability of passing exceeds some positive value is necessary, because for any implementation of the protocol there is always a completely predictable LR theory with positive passing probability, however small. If \( \kappa \) were 1, then \( -\log_2(\delta) \) would be a so-called “smoothed min-entropy”, a quantity that characterizes the number of uniform bits of randomness that are in principle available in \( AB \). We show in the ST S.3 that for constant \( \epsilon_p \) and \( \kappa \), \( -\log_2(\delta) \) is proportional to the number of trials. How many bits we can actually extract depends on \( \kappa \) and \( \epsilon_{\text{fin}} \), the output’s maximum allowed distance from uniform.

To extract the available randomness in \( AB \), we use the TMPS algorithm to obtain an extractor, specifically a function \( \text{Ext} \) that takes as input the string \( AB \) and a length \( d \) “seed” bit string \( S \), where \( S \) is uniform and independent of \( ABXY \). Its output is a length \( t \) bit string. Note that \( S \) can be obtained from \( d \) additional instances of the random variables \( X_i \), so Eq. 2 ensures the needed independence and uniformity conditions on \( S \). In order for the output to be within a distance \( \epsilon_{\text{fin}} \) of uniform independent of \( XY \) and \( E \), the entropy production and extractor parameters must satisfy the constraints given in the next theorem, proven in the ST S.5. The measure of distance used is the “total variation distance”, expressed by the left-hand side of Eq. 6 in the theorem.

**Protocol Soundness Theorem.** Let \( 0 < \epsilon_{\text{ext}} < 1 \). Suppose that the protocol parameters satisfy

\[
t + 4 \log_2 t \leq -\log_2 \delta + \log_2 \kappa + 5 \log_2 \epsilon_{\text{ext}} - 11
\]

with \( \kappa \leq \mathbb{P}(\text{pass}) \). Then the function \( \text{Ext} \) obtained by the TMPS algorithm satisfies

\[
\frac{1}{2} \sum_{z,x\text{yes}} \left| \mathbb{P}(\text{Ext}(AB, S) = z, XYE = xye|\text{pass}) - \mathbb{P}_{\text{unif}}(Z = z)\mathbb{P}(XYE = xye|\text{pass}) \mathbb{P}_{\text{unif}}(S = s) \right| \\
\leq \epsilon_{\text{fin}} = \frac{\epsilon_p}{\kappa} + \epsilon_{\text{ext}},
\]

where \( \mathbb{P}_{\text{unif}} \) denotes the uniform probability distribution.

The number of seed bits \( d \) required satisfies \( d = O(\log(t) \log(nt/\epsilon)^2) \). An explicit upper bound on \( d \) for our extractor implementation is given in the ST S.4.

As the primary demonstration of our protocol, we applied it to the main data set analyzed in [18], which is titled “XOR 3” and consists of a total of 182,161,215 trials, acquired in 30 min of running the experiment, improving on the approximately one month duration of data acquisition reported in Ref. [6]. Before starting the protocol, we set aside the first \( 5 \times 10^7 \) trials as training data, which we used to choose parameters needed by the protocol. With the training data removed, the number \( n \) of trials used by the protocol was 132,161,215. We used the training data to determine a Bell function \( T \) with statistically strong violation of LR on
Table 1: The Bell function $T$ obtained from the training data. We used a maximum likelihood method to infer a non-signaling distribution based on the raw counts of the training trials, namely the first $5 \times 10^7$ trials of data set XOR 3. We then determined the function $T$ that maximizes $\mathbb{E}(\ln T)$ according to this distribution, subject to the constraints that $\mathbb{E}(T)_{LR} \leq 1$ for all LR distributions and $T(0, 0, x, y) = 1$ for all $x, y$. The latter constraint improves the signal-to-noise for our data. The function $T$ yields $m = 0.0120275$, and $\mathbb{E}(T) = 1.0000003928$ for the non-signaling distribution inferred from the training data. One can also interpret the numbers below as the coefficients $s_{xy}^{ab}$ in Eq. 1, which defines a Bell inequality with $\beta = 4$. The values of $T$ are rounded down at the tenth digit.

| $xy$ | $ab = ++$ | $ab = +0$ | $ab = 0+$ | $ab = 00$ |
|------|-----------|-----------|-----------|-----------|
| 00   | 1.0244479364 | 0.9643897947 | 0.9638375026 | 1         |
| 01   | 1.0315040078 | 0.9393895435 | 0.9958939908 | 1         |
| 10   | 1.0317342738 | 0.9955719750 | 0.9399418138 | 1         |
| 11   | 0.9123069953 | 1.0044279882 | 1.0041059756 | 1         |

the training data according to the PBR method [28]; see ST S.3. The function $T$ obtained is given in Table 1. A sample of $n$ i.i.d. trials from the distribution inferred from the training data would have an approximate 0.95 probability for $V$ to exceed $1.66 \times 10^6$. Based on this estimate, we chose $v_{\text{thresh}} = 1.66 \times 10^6$ and then numerically studied the constraints on the number $t$ of bits extracted and the final error $\epsilon_{\text{fin}}$ as a function of $\epsilon_p$, $\epsilon_{\text{ext}}$, and $\kappa$. Based on this study we chose a benchmark goal of $t = 256$ and $\epsilon_{\text{fin}} = 0.001$, with the constraints satisfied for $\epsilon_p = 3.1797 \times 10^{-4}$, $\epsilon_{\text{ext}} = 3.533 \times 10^{-5}$, and $\kappa = 0.33$. We chose $\epsilon_p$ and $\epsilon_{\text{ext}}$ in the ratio 9:1, which was generally found to be near-optimal when numerically maximizing $t$ in Eq. 5 for fixed values of $\epsilon_{\text{fin}}$. We chose $\kappa$ to be safely below 0.95 (the estimate of the probability to exceed $v_{\text{thresh}}$ based on the training data) so that the protocol would be more robust against drifts in trial statistics or the possibility of mid-experiment equipment malfunction. For commissioning purposes, we examined how the protocol behaves in six earlier runs of the experiment [18]. Including XOR 3 and the two blind data sets described below, the value of $v_{\text{thresh}}$ chosen in the same manner is exceeded seven times, suggesting that our choice of $\kappa$ is reasonable.

Running the protocol on XOR 3 with these parameters, the running product $\prod_{i=1}^{c} T_i$ exceeded $v_{\text{thresh}}$ at trial number $c = 67, 173, 533$. At this point it is consistent with Eqs. 2 and 3 to change all outcomes to 0 for the remainder of the run, ensuring that the remaining values of $T_i$ are $T_i = 1$. This is justified because our assumptions allow for Alice and Bob to cooperatively make arbitrary changes to the experiment in advance of a trial based on the past, which includes the current running product. Turning off the detectors to guarantee outcomes of 0 is one such change. In principle there was sufficient time (at least 5 $\mu$s) for the necessary communication to take place after the previous trial. We thereby preserve randomness accumulated at trial $c$ even if statistical fluctuations or experimental drift would otherwise cause $V$ to be less than $\prod_{i=1}^{c} T_i$. In the ST S.2, soundness of this procedure is established by proving a past-parametrized version of the Entropy Production Theorem. Applying the extractor to the resulting output string $AB$.
with a seed of length $d = 73,947$, we extracted 256 bits, certified to be uniform to within 0.001. They are, in hexadecimal form:

$$D731F577BC44F4993E28A84E44EEBD7824C09D203772F876F67D13D3C974FBC2$$

The largest running product of the $T_i$’s observed during the protocol without changing outcomes to 0 was $2.76 \times 10^9$. If we had chosen this value for $v_{\text{thresh}}$, then 256 bits could have been extracted to within 0.001 of uniform for any value of $\kappa$ exceeding $4.85 \times 10^{-4}$. Figure 2 displays the nominal smoothed min-entropy $-\log_2(\delta)$ and extractable bits for alternative choices of $\epsilon_{\text{fin}}$.

The data set XOR 3 was previously subject to statistical analysis [18], and hence the above protocol was run with advance knowledge of its features, in particular, that it performed well as a test against LR. However, two “blind” data sets, “Blind 1” and “Blind 2” were recorded as part of the original experiment [18] and saved for future analysis. After the development of the protocol of this paper, these data sets were analyzed for the first time according to the new methods, and the results are summarized in Figure 3. Further details on all data sets considered, analyses applied and results are in the ST S.6.

In conclusion, we have demonstrated a protocol that extracts randomness from experimentally feasible low-violation photonic experiments. We obtained 256 new near-uniform random bits from $1.32 \times 10^8$ trials. In the process we used $2.64 \times 10^8$ uniform bits to choose the settings and $7.34 \times 10^4$ for the extractor seed. Because the extractor used is a “strong” extractor, the seed bits are still uniform conditional on passing, so they can be recovered at the end of the protocol for uses elsewhere. However, this is not the case for the settings-choice bits because the probability of passing is less than 1. To reduce the entropy used for the settings, our protocol can be modified to use highly biased settings choices [6]. It is also helpful to observe that the output bits are unknown to external observers such as Eve even if the settings and seed bits are public [3, 11]; the only requirement is that the experiment’s physical isolation ensures that these public bits are random relative to the experimental devices before they are used and, if Eve ever had access to the devices, to Eve at the time of access. In particular, if there never was a physical connection to Eve, there is no restriction on settings and seed bit knowledge. When applied in this way, the protocol is a method for private randomness generation.

For future work, we hope to take advantage of the adaptive capabilities of the Entropy Production Theorem (ST S.2) to dynamically compensate for experimental drift during run time, and in view of advances toward practical quantum computing it is desirable to study the protocol in the presence of quantum side information. We also look forward to technical improvements in experimental equipment for larger violation and higher trial rates. These will enable faster generation of random bits with lower error and support the use of biased settings choices.

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Figure 2: Entropy and extractable bits as a function of error for data set XOR 3. The figure shows the tradeoff between final error and number of extractable bits. The horizontal axis shows the error $\epsilon$, which is $\epsilon = \epsilon_p$ for the dashed curves and $\epsilon = \epsilon_{\text{fin}}$ for the solid curves. The vertical axis shows $-\log_2(\delta)$ (referred to as “entropy”, a bound on the available number of random bits) for the dashed curves and $t$ (the number of bits extractable by the TMPS algorithm) for the solid curves. The dashed curves are based on the Entropy Production Theorem and the solid ones on the Protocol Soundness Theorem. For the solid curves, we set $\kappa = 1$, $\epsilon_p = 0.9 \epsilon_{\text{fin}}$, and $\epsilon_{\text{ext}} = 0.1 \epsilon_{\text{fin}}$. The upper curves labeled by “maximum” show the maximum number of extractable bits, obtained if we had set $v_{\text{thresh}} = 2.76 \times 10^9$, which is the maximum running product of the $T_i$. The lower curves labeled by “actual” use $v_{\text{thresh}} = 1.66 \times 10^6$, which we chose before running the protocol. The “+” symbol corresponds to $\epsilon_{\text{fin}} = 0.001$ and 256 extracted bits, and lies off the lower solid curve because this value of $\epsilon_{\text{fin}}$ was calculated with $\kappa = 0.33 < 1$. 
Figure 3: Entropy and extractable bits as a function of error for Blind 1 and 2. The figure shows the tradeoff between final error and number of extractable bits for the blind data sets. The axes and distinction between dashed and solid curves are as in Fig. 2, but only the curves for $v_{\text{thresh}}$ given by the maximum observed running product of the $T_i$’s is shown. Data sets Blind 1 and Blind 2 have parameters $m = 0.00540, n = 306, 464, 574, v_{\text{thresh}} = 6.88 \times 10^7$ and $m = 0.01231, n = 162, 837, 253, v_{\text{thresh}} = 17, 528$, respectively. We did not explicitly extract bits for these data sets.
Experimentally Generated Random Numbers Certified by the Impossibility of Superluminal Signaling
(Supporting Online Material)
Peter Bierhorst, Emanuel Knill, Scott Glancy,
Alan Mink, Stephen Jordan, Andrea Rommal, Yi-Kai Liu,
Bradley Christensen, Sae Woo Nam, Lynden K. Shalm

After preliminaries to establish notation and summarize needed properties of total variation distance and non-signaling distributions in S.1, we give the proof of the Entropy Production Theorem in S.2. We explain how we chose the Bell function $T$, whose product determines whether we obtained the desired amount of randomness, in S.3. We then discuss the parameters of the extractors obtained by the TMPS algorithm (S.4) and prove the Protocol Soundness Theorem (S.5). Details on how we analyzed the experimental data sets are in S.6. Justification for our claim that previous methods do not obtain any randomness from our weakly violating data is given in S.7.

S.1 Preliminaries

We use the standard convention that capital letters refer to random variables (RVs) and corresponding lowercase letters refer to values that the RVs can take. All our RVs take values in finite sets such as the set of bit strings of a given length or a finite subset of the reals, so that our RVs can be viewed as functions on a finite probability space. We usually just work with the induced joint distributions on the sets of values assumed by the RVs. When working with conditional probabilities, we implicitly exclude points where the conditioner has zero probability whenever appropriate. We use $\mathbb{P}(\ldots)$ to denote probabilities and $\mathbb{E}(\ldots)$ for expectations. Inside $\mathbb{P}(\ldots)$ and when used as conditioners, logical statements involving RVs are event specifications to be interpreted as the event for which the statement is true. For example, $\mathbb{P}(R > \delta)$ is equivalent to $\mathbb{P}(\{\omega : R(\omega) > \delta\})$, which is the probability of the event that the RV $R$ takes a value greater than $\delta$. The same convention applies when denoting events with $\{\ldots\}$. For example, the event in the previous example is written as $\{R > \delta\}$. While formally events are sets, we commonly use logical language to describe relationships between events. For example, the statement that $\{R > \delta\}$ implies $\{S > \epsilon\}$ means that as a set, $\{R > \delta\}$ is contained in $\{S > \epsilon\}$. When they appear outside the the mentioned contexts, logical statements are constraints on RVs. For example, the statement $R > \delta$ means that all values $r$ of $R$ satisfy $r > \delta$, or equivalently, for all $\omega$, $R(\omega) > \delta$. As usual, comma separated statements are combined conjunctively (with “and”). (In the main text, for clarity, we have used an explicit “AND” for this purpose.)

If there are free RVs inside $\mathbb{P}(\ldots)$ or in the conditioner of $\mathbb{E}(\ldots | \ldots)$ outside event specifications, the final expression defines a new RV as a function of the free RVs. An example from the Entropy Production Theorem is the expression $\mathbb{P}(AB|XY)$, which defines the RV that takes the value $\mathbb{P}(AB = ab|XY = xy)$ when the event $\{ABXY = abxy\}$ occurs. Values of RVs such as $x$ appearing by themselves in $\mathbb{P}(\ldots)$ denote the event $\{X = x\}$. Thus we abbreviate
expressions such as $P(AB = ab|XY = xy)$ by $P(ab|xy)$. Sometimes it is necessary to dis-ambiguate the probability distribution with respect to which $E(\ldots)$ is to be computed. In such cases we use a subscript at the end of the expression consisting of a symbol for the probability distribution, so $E(T)_Q$ is the expectation of $T$ with respect to the distribution $Q$. In a few instances, we use $[\phi]$ for logical expressions $\phi$ to denote the $\{0, 1\}$-valued function evaluating to 1 iff $\phi$ is true.

The amount of randomness that can be extracted from an RV $R$ is quantified by the min-entropy, defined as $-\log_2 \max_x P(R = r)$. The error of the output of an extractor is given as the total variation (TV) distance from uniform. Given two probability distributions $P_1$ and $P_2$ for $R$, the TV distance between them is given by

$$TV(P_1, P_2) = \frac{1}{2} \sum_r |P_1(R = r) - P_2(R = r)|$$

$$= \sum_{r: P_1(r) > P_2(r)} (P_1(R = r) - P_2(R = r))$$

$$= \sum_r [P_1(r) > P_2(r)] (P_1(R = r) - P_2(R = r)).$$  \hspace{1cm} (7)

We sometimes compute TV distances for distributions of specific RVs, conditional or unconditional ones. For this we introduce the notation $P_X$ for the distribution of values of $X$ according to $P$, and $P_{X|Y=y}$ for the distribution of $X$ conditioned on the event $\{Y = y\}$. With this notation, $P_X P_Y$ refers to the product distribution that assigns probability $P_X(X = x)P_Y(Y = y)$ to the event $\{X = x, Y = y\}$.

For the proof of the Protocol Soundness Theorem, we need three results involving the TV distance. The first is the triangle inequality for TV distances, see Ref. [31] for this and other basic properties of TV distances.

$$TV(P_1, P_3) \leq TV(P_1, P_2) + TV(P_2, P_3).$$  \hspace{1cm} (8)

According to the second result, if $P$ and $Q$ are joint distributions of RVs $V$ and $W$, where the marginals of $W$ satisfy $P(w) = Q(w)$, then the distance between them is given by the average conditional distance. This is explicitly calculated as follows:

$$TV(P_{VW}, Q_{VW}) = \sum_{w} \sum_{v} [P(v, w) > Q(v, w)] (P(v, w) - Q(v, w))$$

$$= \sum_{w} \sum_{v} [P(v|w)P(w) > Q(v|w)Q(w)] (P(v|w)P(w) - Q(v|w)Q(w))$$

$$= \sum_{w} \sum_{v} [P(v|w) > Q(v|w)] (P(v|w) - Q(v|w)) P(w)$$

$$= \sum_{w} TV(P_{V|W=w}, Q_{V|W=w}) P(w).$$  \hspace{1cm} (9)
The third result is a special case of the data-processing inequality for TV distance. See Ref. [32] for this and many other data-processing inequalities. Let \( V \) be a random variable taking values in a finite set \( \mathcal{V} \), and let \( F : \mathcal{V} \to \mathcal{W} \) be a function so that \( F(V) \) is a random variable taking values in the set \( \mathcal{W} \). Then if \( P \) and \( Q \) are two distributions of \( V \),

\[
\text{TV}(P \| Q) \geq \text{TV}(P_{F(V)} \| Q_{F(V)}).
\]

Here is a proof of this inequality. Write \( \mathcal{W} = \{s_1, \ldots, s_c\} \), and for each \( i \in \{1, \ldots, c\} \), define \( \mathcal{V}_i = \{v : f(v) = s_i\} \). The \( \mathcal{V}_i \) form a partition of \( \mathcal{V} \). Then we have

\[
\text{TV}(P_{F(V)} \| Q_{F(V)}) = \frac{1}{2} \sum_{i=1}^c \left| P(V \in \mathcal{V}_i) - Q(V \in \mathcal{V}_i) \right|
\]

\[
= \frac{1}{2} \sum_{i=1}^c \left| \sum_{v \in \mathcal{V}_i} \left[ P(V = v) - Q(V = v) \right] \right|
\]

\[
\leq \frac{1}{2} \sum_{i=1}^c \sum_{v \in \mathcal{V}_i} \left| P(V = v) - Q(V = v) \right|
\]

\[
= \text{TV}(P \| Q).
\]

We sometimes need to refer to the sequences of RVs associated with the first \( i - 1 \) trials. To do this we use notation such as \( (AB)_{<i} \) for the outcome sequence \( A_1 B_1 A_2 B_2 \ldots A_{i-1} B_{i-1} \), \( (XY)_{<i} \) for the settings sequence \( X_1 Y_1 \ldots X_{i-1} Y_{i-1} \), and \( (ABXY)_{<i} \) for the joint outcomes and settings sequence \( A_1 B_1 X_1 Y_1 \ldots A_{i-1} B_{i-1} X_{i-1} Y_{i-1} \). In general we often juxtapose RVs to indicate the “joint” RV. From our assumptions Eqs. 2 and 3 and the fact that past, subsumes the trial settings and outcomes from trials 1 through \( i - 1 \), we obtain

\[
\forall i \in \{1, \ldots, n\}, \quad P_e(X_i Y_i | (ABXY)_{<i}) = P_e(X_i Y_i) = 1/4,
\]

and

\[
P_e(A_i | X_i Y_i, (ABXY)_{<i}) = P_e(A_i | X_i, (ABXY)_{<i})
\]

\[
P_e(B_i | X_i Y_i, (ABXY)_{<i}) = P_e(B_i | Y_i, (ABXY)_{<i}).
\]

These are the forms of our assumptions used in the proof of the Entropy Production Theorem.

For a generic trial of a two station Bell test, a distribution is defined to be non-signaling if

\[
P(A | XY) = P(A | X) \quad \text{and} \quad P(B | XY) = P(B | Y).
\]

Such distributions form a convex polytope and include the local realist (LR) distributions. Using the conventions of [21], these are defined as follows: Let \( \lambda \) range over the set of sixteen four-element vectors of the form \( (a_0, a_1, b_0, b_1) \) with elements in \( \{+, 0\} \). Each \( \lambda \) induces settings-conditional deterministic distributions according to

\[
P_{\lambda}(ab | xy) = \begin{cases} 1, & \text{if } a = a_x \text{ and } b = b_y, \\ 0, & \text{otherwise}. \end{cases}
\]
Then a probability distribution $\mathbb{P}$ is LR iff its conditional probabilities $\mathbb{P}(ab|xy)$ can be written as a convex combination of the $\mathbb{P}_\lambda(ab|xy)$. That is

$$\mathbb{P}(ab|xy) = \sum_\lambda q_\lambda \mathbb{P}_\lambda(ab|xy), \quad (16)$$

with $q_\lambda$ a $\lambda$-indexed set of nonnegative numbers summing to 1. This definition agrees with the one given in the main text.

The eight “Popescu-Rohrlich (PR) boxes” [15] are examples of non-signaling distributions that are not LR. One of the PR boxes is defined by

$$\mathbb{P}_{PR}(ab|xy) = \begin{cases} 1/2 & \text{if } xy \neq 11 \text{ and } a = b, \text{ or if } xy = 11 \text{ and } a \neq b, \\ 0 & \text{otherwise,} \end{cases} \quad (17)$$

and the other seven are obtained by relabeling settings or outcomes. We take advantage of the facts that a PR box contains one bit of randomness conditional on the settings and that the PR boxes together with the 16 deterministic LR distributions of Eq. [15] form the set of extreme points of the non-signaling polytope [33].

S.2 Proof of the Entropy Production Theorem

The conditions on $T$ given in the main text are that $T > 0$, $\mathbb{E}(T)_P \leq 1$ for every LR distribution $P$ and $\mathbb{E}(T)_Q \leq 1 + m$ for every non-signaling distribution $Q$, where the bound $1 + m$ is achievable. In the following, we call these the Bell-function conditions with bound $m$. Here we prove the Entropy Production Theorem in the more general form where the $T_i$ can be chosen based on $(abxy)_{<i}$. We call $T_i$ a past-parametrized family of Bell functions if for all $(abxy)_{<i}$, $T_i(a_i,b_i,x_i,y_i,(abxy)_{<i})$ satisfies the Bell-function conditions with bound $m' \leq m$ when considered as a function of the results $a_i,b_i,x_i,y_i$ from the $i$'th trial. The theorem and its proof can also be directly applied to the special case where $T_i$ is the same function for all trials $i$.

**Theorem.** Let $T_i$ be a past-parametrized family of Bell functions as defined in the previous paragraph. Then in an experiment of $n$ trials obeying Eqs. 2 and 3 the following inequality holds for all $\epsilon_p \in (0, 1)$ and $v_{thresh}$ satisfying $1 \leq v_{thresh} \leq (1 + (3/2)m)^n \epsilon_p^{-1}$:

$$\mathbb{P}_e(\mathbb{P}_e(AB|XY) > \delta, V \geq v_{thresh}) \leq \epsilon_p \quad (18)$$

where $\delta = [1 + (1 - \sqrt{\epsilon_p v_{thresh}})/2m]^n$ and $\mathbb{P}_e$ represents the probability distribution conditioned on the event $\{E = e\}$.

We include the constraint $v_{thresh} \leq (1 + (3/2)m)^n \epsilon_p^{-1}$ for technical reasons. Higher values of $v_{thresh}$ are unreasonably large and result in pass probabilities that are too low to be relevant. Note that this bound ensures $\delta \geq 2^{-2n}$, a fact that will be useful in proving the Protocol Soundness Theorem in (S.5).
Proof. Since the condition on \( \{ E = e \} \) appears uniformly throughout, in this proof we omit the subscript on \( \mathbb{P}_e \) specifying conditioning on \( \{ E = e \} \).

The strategy of the proof is to first obtain an upper bound on the one-trial outcome probabilities from the expectations of Bell functions \( T \). This bound can be chained to give a bound on the probabilities of the outcome sequence as a monotonically decreasing function of the product of the conditional expectations of the \( T_i \). That is, a larger product of expectations yields a smaller maximum probability and therefore more extractable randomness. This product cannot be directly observed, so we relate it to the observed product \( V \) of the \( T_i \) via the Markov inequality applied to an associated positive, mean-1 martingale. In the following, we suppress the arguments \( a, b, x, y \) and \( \langle ABXY \rangle_{<i} \) of \( T_i \).

The one-trial outcome probabilities are bounded by means of the following lemma:

**Lemma.** Let \( T \) satisfy the Bell-function conditions with bound \( m > 0 \). For any non-signaling distribution \( \mathbb{P} \) with equiprobable settings (Eqs. [12] and [14]),

\[
\max_{abxy} \mathbb{P}(ab|xy) \leq 1 + \frac{1 - \mathbb{E}[T(A, B, X, Y)]_{\mathbb{P}}}{2m}.
\]  

(19)

**Proof.** As \( \mathbb{P} \) is a non-signaling distribution with equiprobable settings, it can be obtained as a convex combination of extremal such distributions. In particular, \( \mathbb{P} \) can be expressed as such a convex combination containing at most one PR box ([34], Corollary 2.1), so we can write \( \mathbb{P} = p\mathbb{Q} + (1 - p)\mathbb{Q}' \), where \( \mathbb{Q} \) is the PR box and \( \mathbb{Q}' \) is LR. By assumption, \( \mathbb{E}(T) \leq 1 \) for all LR distributions with uniform settings choices, so \( \mathbb{E}(T)_{\mathbb{Q}'} \leq 1 \) and \( \mathbb{E}(T)_{\mathbb{Q}} \leq 1 + m \). Consequently, \( \mathbb{E}(T)_{\mathbb{P}} = p\mathbb{E}(T)_{\mathbb{Q}} + (1 - p)\mathbb{E}(T)_{\mathbb{Q}'} \leq p(1 + m) + (1 - p) = 1 + pm \), or equivalently, \( p \geq (\mathbb{E}(T)_{\mathbb{P}} - 1)/m \). The PR box assigns \( xy \)-conditional probability 1/2 to at least one outcome different from \( ab \). It follows that the \( xy \)-conditional probability relative to \( \mathbb{P} \) of an outcome different from \( ab \) is at least \( p/2 \). Therefore, \( \mathbb{P}(ab|xy) \leq 1 - p/2 \leq 1 - (\mathbb{E}(T)_{\mathbb{P}} - 1)/(2m) \). Since \( ab \) and \( xy \) are arbitrary, this gives the inequality in the lemma.

The inequality in the lemma holds if \( T \) has bound \( m' \leq m \). If \( \mathbb{E}(T)_{\mathbb{P}} \leq 1 \) this is trivial. If \( 1 < \mathbb{E}(T)_{\mathbb{P}} \leq m' \), the lemma holds with \( m \) substituted by \( m' \), giving a lower upper bound on the maximum probability. With this observation, and the fact that by assumption, \( \mathbb{P}(a,b_i|(abxy)_{<i}, x_iy_i) \) is non-signaling with respect to \( a_i, b_i, x_i, \) and \( y_i \), we can establish a bound on \( \mathbb{P}(ab|xy) \) as follows:

\[
\mathbb{P}(ab|xy) = \prod_{i=1}^{n} \mathbb{P}(a_i,b_i|(abxy)_{<i}, xy) = \prod_{i=1}^{n} \mathbb{P}(a_i,b_i|(abxy)_{<i}, x_iy_i) \leq \prod_{i=1}^{n} \left[ 1 + \frac{1 - \mathbb{E}(T_i|(abxy)_{<i})}{2m} \right].
\]  

(20)
Here, the first identity is the chain rule for conditional probabilities, and the second follows from Eq. [12]. By twice using the fact that the geometric mean of a set of positive numbers is always less than or equal to their arithmetic mean, we continue from the last line of Eq. 20:

\[
\prod_{i=1}^{n} \left[ 1 + \frac{1 - \mathbb{E}(T_i|(abxy)_{<i})}{2m} \right] = \left( \prod_{i=1}^{n} \left[ 1 + \frac{1 - \mathbb{E}(T_i|(abxy)_{<i})}{2m} \right] \right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} \left( 1 + \frac{1 - \mathbb{E}(T_i|(abxy)_{<i})}{2m} \right)}{n} \leq \left( \frac{1 + \frac{1}{2m} - \frac{\sum_{i=1}^{n} \mathbb{E}(T_i|(abxy)_{<i})}{2m}}{n} \right)^{n} \leq \left( 1 + \frac{1}{2m} - \left[ \prod_{i=1}^{n} \frac{\mathbb{E}(T_i|(abxy)_{<i})}{2m} \right]^{\frac{1}{n}} \right)^{n} = \left( 1 + \frac{1 - \left[ \prod_{i=1}^{n} \mathbb{E}(T_i|(abxy)_{<i}) \right]^{\frac{1}{2m}}}{2m} \right)^{n}.
\] (21)

Referring back to the statement of the theorem, we see that \( \delta \) can be expressed as \( f(\epsilon_p v_{\text{thresh}}) \) where \( f(x) = [1 + (1 - \sqrt{x}/2m)]^{n} \). Expressing Eq. 21 in terms of this same function \( f \), we see that the event \( \{ \mathbb{P}(AB|XY) > \delta \} \) implies the event \( \{ f\left( \prod_{i=1}^{n} \mathbb{E}(T_i|(ABXY)_{<i}) \right) > \delta \} \). The latter event is the same as \( \{ \prod_{i=1}^{n} \mathbb{E}(T_i|(ABXY)_{<i}) < f^{-1}(\delta) = \epsilon_p v_{\text{thresh}} \} \), since \( f^{-1} \) is strictly decreasing. Conjoining the event \( \{ V \geq v_{\text{thresh}} \} \) to both sides of the implication, we have \( \{ \mathbb{P}(AB|XY) > \delta, V \geq v_{\text{thresh}} \} \) implies \( \{ \prod_{i=1}^{n} \mathbb{E}(T_i|(ABXY)_{<i}) < \epsilon_p v_{\text{thresh}}, V \geq v_{\text{thresh}} \} \), and so by the monotonicity of probabilities,

\[
\mathbb{P}(\mathbb{P}(AB|XY) > \delta, V \geq v_{\text{thresh}}) \leq \mathbb{P}\left( \prod_{i=1}^{n} \mathbb{E}(T_i|(ABXY)_{<i}) < \epsilon_p v_{\text{thresh}}, V \geq v_{\text{thresh}} \right).
\] (22)

The event \( \{ \Phi \} \) whose probability appears on the left-hand side of this equation is the event in the theorem statement whose probability we are required to bound. For any values of the RVs, the two inequalities in the event on the right-hand side imply the inequality in the event \( \{ \Psi \} = \{ V/\prod_{i=1}^{n} \mathbb{E}(T_i|(ABXY)_{<i}) \geq 1/\epsilon_p \} \). Hence \( \mathbb{P}(\Phi) \leq \mathbb{P}(\Psi) \). It remains to show that \( \mathbb{P}(\Psi) \leq \epsilon_p \). For this purpose we define the sequence \( \{ W_c \}_{c=1}^{n} \) of RVs by

\[
W_c = \prod_{i=1}^{c} \frac{T_i \mathbb{E}(T_i|(ABXY)_{<i})}{1},
\] (23)

so that \( \{ \Psi \} = \{ W_n \geq 1/\epsilon_p \}. \)
By definition, \( W_c > 0 \) and the factors \( T_i / \mathbb{E}(T_i | (ABXY)_{<i}) \) have expectation 1 conditional on the past. Sequences of RVs with these properties are referred to as test martingales \cite{35} and satisfy that \( \mathbb{E}(W_n) = 1 \), which can be verified directly by induction:

\[
\mathbb{E}(W_c | (ABXY)_{<c}) = \mathbb{E} \left( \prod_{i=1}^{c} \frac{T_i}{\mathbb{E}(T_i | (ABXY)_{<i})} \left| (ABXY)_{<c} \right. \right)
\]

\[
= \mathbb{E} \left( \prod_{i=1}^{c-1} \frac{T_i}{\mathbb{E}(T_i | (ABXY)_{<i})} \right) \frac{1}{\mathbb{E}(T_c | (ABXY)_{<c})} \mathbb{E}(T_c | (ABXY)_{<c})
\]

\[
= \left( \prod_{i=1}^{c-1} \frac{T_i}{\mathbb{E}(T_i | (ABXY)_{<i})} \right) \frac{1}{\mathbb{E}(T_c | (ABXY)_{<c})} \mathbb{E}(T_c | (ABXY)_{<c})
\]

\[
= W_{c-1}, \quad (24)
\]

where in the second last line, we pulled out factors that are functions of the conditioner \( (ABXY)_{<c} \) by applying the rule that if \( F \) is a function of \( H \), then \( \mathbb{E}(FG|H) = F\mathbb{E}(G|H) \). Taking the unconditional expectation of both sides of Eq. 24 and invoking the law of total expectation, we have \( \mathbb{E}(W_c) = \mathbb{E}(W_{c-1}) \), and so inductively, \( \mathbb{E}(W_n) = \mathbb{E}(W_1) \). Since \( \mathbb{E}(W_1) = 1 \), the claim follows. To finish the proof of the Entropy Production Theorem, we apply Markov’s inequality to obtain

\[
P(W_n \geq 1/\epsilon_p) \leq \epsilon_p
\]

and consequently \( P(\Phi) \leq \epsilon_p \).

Now that we have proved the Entropy Production Theorem for any past-parametrized family of Bell functions, we can justify the strategy described in the main text and used in our protocol, where we set outcomes to 0 after \( v_{\text{thresh}} \) is exceeded by the running product. Given the constraints on \( T \) used in the protocol, this strategy is equivalent to setting the remaining Bell functions to \( T_i = 1 \) as allowed by past-parametrization. Formally, since the running product \( V_{i-1} = \prod_{j=i}^{i-1} T_j \) is a function of \( (ABXY)_{<i} \), we can define \( T_i = T \) conditional on \( \{V_{i-1} < v_{\text{thresh}}\} \) and \( T_i = 1 \) conditional on the complement.

### S.3 Choosing the Bell Function \( T \)

The Entropy Production Theorem does not indicate how to find functions \( T \) satisfying the specified conditions. We seek a high typical value of \( V = \prod_{i=1}^{n} T_i \), as this permits larger values of \( v_{\text{thresh}} \) and consequently more extractable randomness at the same values of \( \epsilon_p \) and \( m \). Here, we describe a procedure for constructing a function \( T \) that can be expected to perform well if the trial results are i.i.d. with known distribution. We estimate the distribution from an initial portion of the run that we set aside as training data, and in a stable experiment we expect that the trial results’ statistics are i.i.d. to a good approximation. Note however that the optimistic i.i.d. assumption is only used as a heuristic to construct \( T \); once \( T \) is chosen the guarantees of the Entropy Production Theorem hold regardless of whether the trial results are actually i.i.d.
The observed measurement outcome frequencies for training data generally yield a weakly signaling distribution that does not exactly satisfy the non-signaling constraints in Eq. [14] due to statistical fluctuation. Hence one can obtain an estimated distribution by determining the maximum likelihood non-signaling distribution for the observed measurement outcomes frequencies as described in Ref. [28]. Let \( N(xy) \) be the number of training trials at setting \( xy \) and \( f(ab|xy) = N(ab|xy)/N(xy) \) be the empirical frequencies of outcome \( ab \) given setting \( xy \). Let \( Q(a,b,x,y) \) be a candidate for the probability distribution from which these frequencies were sampled. Then up to an additive term independent of \( Q \) accounting for the settings probabilities, the log-likelihood of \( f \) given \( Q \) is
\[
L(Q) = \sum_{a,b,x,y} N(xy) f(ab|xy) \ln(Q(a,b|x,y)).
\]
We maximized a variant of this function to find our estimated distribution \( Q(a,b,x,y) \):

\[
\text{Maximize } \sum_{a,b,x,y} f(ab|xy) \ln Q(a,b,x,y) \quad (25)
\]

Subject to
\[
Q(x,y) = 1/4 \quad \text{for } x,y \in \{0,1\} \\nonumber
\]
\[
Q(a|x,y) = Q(a|x) \quad \text{for } x,y \in \{0,1\}, \ a \in \{+,0\} \\nonumber
\]
\[
Q(b|x,y) = Q(b|y) \quad \text{for } x,y \in \{0,1\}, \ b \in \{+,0\}. \nonumber
\]

The first group of constraints encode our knowledge that all settings combinations are equiprobable, and the remaining constraints are the non-signaling constraints. Note that the conditional expressions in these constraints are equivalently expressed as linear functions of \( Q(a,b,x,y) \) after using the identities \( Q(x,y) = 1/4 \).

Once the estimated distribution \( Q \) is obtained, we maximize the typical values of \( V \) by taking advantage of the observation that the conditions on \( T \) imply that \( V^{-1} \) is a conservative \( p \)-value against local realism [28]. Such \( p \)-values were studied in Ref. [28], which gives a general strategy, the PBR method, for maximizing \( E(\ln(V))_Q \). This is useful because typical values of \( V \) are close to \( \exp(nE(\ln(V))_Q) \): Since \( \ln(V) = \sum_{i=1}^n \ln(T_i) \) is a sum of i.i.d. bounded terms (given our optimistic assumption), the central limit theorem ensures that \( \ln V \) is approximately normally distributed with mean \( nE(\ln(V))_Q \). We therefore perform the following optimization problem to find \( T \):

\[
\text{Maximize } E(\ln(T))_Q \quad (26)
\]

Subject to
\[
E(T)_{P,\lambda} \leq 1 \quad \forall \lambda \nonumber
\]
\[
T(0,0,x,y) = 1 \quad \forall x,y, \nonumber
\]

where \( P,\lambda \) refers to the 16 conditionally deterministic LR distributions in Eq. [15]. This ensures that \( E(T)_{P_{LR}} \leq 1 \) for all LR distributions \( P_{LR} \). The second constraint is motivated by the fact that in our experiments, an overwhelming fraction of the trials have no detections for both stations. While it is possible that a better \( E(\ln(T))_Q \) can be obtained without this constraint, we have found that the improvement is small and likely not statistically significant given the amount of training data used to determine the results distribution. Since the objective functions are concave and the constraints are linear, the optimization problems given in Eq. [25] and Eq. [26] are readily solved numerically with standard tools.
Given the assumption that the trial results are i.i.d., the previous paragraph shows that the typical values for \( V \) are exponential in the number of trials, \( V = e^{-n\mathbb{E}(\ln(T))-o(n)} \). If the experiment is successful in showing violation of local realism, \( \mathbb{E}(\ln(T)) \) is positive. Neglecting the contribution from \( o(n) \), with \( \nu_{\text{thresh}} = e^{n\mathbb{E}(\ln(T))} \), we can bound \(-\ln(\delta)\) as

\[
-\ln(\delta) = -n \ln(1 + (1 - (\epsilon_p e^{n\mathbb{E}(\ln(T))} + \ln(n)^1)/2m)) \\
\geq n(1 - e^{\mathbb{E}(\ln(T)) + \ln(n)/n} - 1)/(2m) \\
\geq \left(n\mathbb{E}(\ln(T)) + \ln(\epsilon_p)/2m\right).
\]

where we used \(-\ln(1 + x) \geq -x \) and \( e^x - 1 \geq x \). This shows asymptotically (with \( \epsilon_p \) and \( \kappa \) constant) we get at least \( \mathbb{E}(\ln(T)) \log_2(e)/(2m) = \mathbb{E}(\log_2(T))/(2m) \) bits of randomness per trial. For the empirical distribution obtained from the XOR 3 trials used for the protocol according to Eq. 25, we obtain \( \mathbb{E}(\log_2(T))/(2m) = 1.19 \times 10^{-5} \). The bound in Eq. 27 shows that we can get an asymptotically positive number of bits of randomness per trial even with \( \epsilon_p \) exponentially small in \( n \).

S.4 The TMPS Algorithm

A strong randomness extractor with parameters \((\sigma, \epsilon, q, d, t)\) is a function \( \text{Ext} : \{0, 1\}^q \times \{0, 1\}^d \rightarrow \{0, 1\}^t \) with the property that for any random string \( R \) of length \( q \) and min-entropy at least \( \sigma \), and an independent, uniformly distributed seed string \( S \) of length \( d \), the distribution of the concatenation \( \text{Ext}(RS) \) with \( S \) of length \( t + d \) is within TV distance \( \epsilon \) of uniform. There are constructions of extractors that extract most of the input min-entropy \( \sigma \) with few seed bits. For a review of the achievable asymptotic tradeoffs, see Ref. 36, chapter 6. For explicit extractors that perform well if not optimally, we used a version of Trevisan’s construction 23 implemented by Mauerer, Portmann and Scholz 24, which we adapted1 to make it functional in our environment and to incorporate recent constructions achieving improved parameters 37. We call this construction the TMPS algorithm. For a fixed choice of \( \sigma, \epsilon, \) and \( q \), the TMPS algorithm can construct a strong randomness extractor for any value \( t \) obeying the following bound:

\[
t + 4 \log_2 t \leq \sigma - 6 + 4 \log_2(\epsilon). \tag{28}
\]

Given \( t \), the length of the seed satisfies

\[
d \leq w^2 \cdot \max \{2, 1 + [(\log_2(t - e) - \log_2(w - e))/\log_2 e - \log_2(1/e)]\} \tag{29}
\]

where \( w \) is the smallest prime larger than \( 2 \times \lceil \log_2(4q^2/\epsilon^2) \rceil \). We note that the TMPS extractors are secure against classical and quantum side information 24, and this security is reflected in

\footnote{1Our adapted source code is available at https://github.com/usnistgov/libtrevisan}
the parameter constraints. Since we do not take direct advantage of this security, it is in principle possible to improve the parameters in the Protocol Soundness Theorem.

For the bound on the number of seed bits given after the Protocol Soundness Theorem in the main text, we have \( q = 2n \) and \( \epsilon = \epsilon_{\text{ext}} \). Since for any \( r \), there is a prime \( w \leq 2r \), \( w = O(\log(n) + \log(t/\epsilon)) = O(\log(nt/\epsilon)) \), were we pulled out exponents from the \( \log \), and dropped and arbitrarily increased the implicit constants in front of each term to match summands. The coefficient of \( w^2 \) in the bound on \( d \) is \( O(\log(t)) \), because of the “minus” sign in front of the term containing \( w \). Multiplying gives \( d = O(\log(t) \log(nt/\epsilon)^2) \).

S.5 Proof of the Protocol Soundness Theorem

The distinction between the stations was needed to establish the inequality in the Entropy Production Theorem and plays no further role in this section. We therefore simplify the notation by abbreviating \( C = AB \) and either \( Z = XY \) or \( Z = X YE \). In the former case \( P(\ldots) \) refers to probabilities conditional on \( \{E = e\} \). Otherwise, \( P(\ldots) \) involves no implicit conditions. The Protocol Soundness Theorem holds regardless of which definition of \( Z \) is in force. We write \( R_{\text{pass}} \) to refer to the RV that takes value 1 conditional on the passing event \( \{V \geq v_{\text{thresh}}\} \) and 0 otherwise. Here is a restatement of the Protocol Soundness Theorem. The constants \( \epsilon_p \) and \( \delta \) appearing below are the same as in the Entropy Production Theorem.

**Theorem.** Let \( 0 < \epsilon_{\text{ext}} < 1 \) and \( p(R_{\text{pass}} = 1) \geq \kappa > 0 \). Suppose \( t \) is a positive integer satisfying

\[
t + 4 \log_2 t \leq -\log_2 \delta + \log_2 \kappa + 5 \log_2 \epsilon_{\text{ext}} - 11. \tag{30}
\]

Then if Ext : \( \{0, 1\}^{2n} \times \{0, 1\}^d \to \{0, 1\}^t \) is obtained by the TMPS algorithm with parameters \( \sigma = -\log_2(2\delta/(\kappa \epsilon_{\text{ext}})) \) and \( \epsilon = \epsilon_{\text{ext}}/2 \), and \( S \) is a random bit string of length \( d \) independent of the joint distribution of \( C, Z, R_{\text{pass}} \), we have

\[
\text{TV}(P_{\text{unif}}^{R_{\text{pass}} = 1}, P_{\text{unif}}^S P_{\text{unif}}^{Z|R_{\text{pass}} = 1}) \leq \epsilon_{\text{fin}} = \epsilon_p/\kappa + \epsilon_{\text{ext}}, \tag{31}
\]

where \( U = \text{Ext}(CS) \) and \( P_{\text{unif}} \) denotes the uniform probability distribution.

At this point it is tempting to just apply an extractor to \( AB \) with parameter \( \sigma \) given by the nominal \( \epsilon_p \)-smoothed min-entropy \( \sigma = -\log_2(\delta) \). However, this does not guarantee the strong condition Eq. \([31]\). Specifically, there are three reasons that Eq. \([18]\) of the Entropy Production Theorem does not immediately support the application of an extractor to \( AB \). The first is that as specified, the extractor input should have min-entropy \( -\log_2 \max_{ab} P(AB = ab) = \sigma \) with no smoothness error. The second is that the settings-conditional smoothed min-entropies can be substantially smaller than the nominal one. The third is that the min-entropy is also affected by the probability of passing \( \kappa \) being less than 1. Accounting for these effects requires an analysis of the settings- and pass-conditional distributions and the extractor parameters specified in the theorem.
Proof. The proof proceeds in two main steps inspired by the corresponding arguments in Ref. [3]. In the first we determine a probability distribution $P^*$ that is within $\epsilon_p$ of $P$ but satisfies an appropriate bound on the conditional probabilities of $C$ with probability 1 rather than $1 - \epsilon_p$. The distribution $P^*$’s marginals agree with those of $P$ on $ZS$. The probabilities conditional on aborting also agree, and uniformity and independence of $S$ is preserved. In the second, we apply a proposition from Ref. [38] on applying extractors to distributions such as $P^*$ whose average maximum conditional probabilities satisfy a specified bound. The proposition enables us to determine the extractor parameters that achieve the required final distance $\epsilon_{\text{fin}}$ in the theorem.

The Entropy Production Theorem guarantees that $P(P(C|Z) > \delta, R_{\text{pass}} = 1) \leq \epsilon_p$. In the case where $E$ is included in $Z$, this follows by the uniformity in $\{E = e\}$ of the theorem’s conclusion:

$$P(P(C|Z, E) > \delta, R_{\text{pass}} = 1) = \sum_e P(P(C|Z, E) > \delta, R_{\text{pass}} = 1|E = e)P(E = e)$$

$$= \sum_e P(P(C|Z = e) > \delta, R_{\text{pass}} = 1|E = e)P(E = e)$$

$$\leq \sum_e \epsilon_p^eP(e)$$

$$= \epsilon_p.$$ (32)

Using the following construction, one may observe that for any random variable $U$ with values in a set of cardinality $K$ and $\gamma$ satisfying $1/K \leq \gamma$, and any distribution $P'$ of $U$, there exists $P''$ such that $P''(U) = \gamma$ and $P''$ is within TV distance $P'(P'(U) > \gamma)$ of $P'$. To construct $P''$, for $u$ such that $P'(u) > \gamma$, set $P''(u) = \gamma$. To compensate for the reduced probabilities, increase the values of $P'$ to obtain those of $P''$ without exceeding $\gamma$ on the set $\{u : P'(u) \leq \gamma\}$ so that $P''$ is a normalized probability distribution. This is possible because in constructing $P''$ from $P'$, the total reduction in probability on $\{u : P'(u) > \gamma\}$ given by $r_- = \sum_{u : P'(u) > \gamma}(P'(u) - \gamma)$ is less than the maximum total increase possible given by $r_+ = \sum_{u : P'(u) \leq \gamma}(\gamma - P'(u))$, as a consequence of $\gamma \geq 1/K$. To see this, compute $r_- - r_+ = \sum_{u}(\gamma - P'(u)) \geq \sum_{u}(1/K - P'(u)) = 0$. The distance $TV(P', P'')$ is given by $\sum_{u : P'(u) > \gamma}(P'(u) - \gamma) \leq P'(P'(U) > \gamma)$.

We can now construct $P^*$ by defining its conditional distributions on $C$. For this, substitute $U \leftarrow C(z)$, $P'(U) \leftarrow P(C|z, R_{\text{pass}} = 1)$, $\gamma \leftarrow \delta/P(R_{\text{pass}} = 1|z)$ and $P''(U) \leftarrow P^*(C|z, R_{\text{pass}} = 1)$. The constraint on $\gamma$ is satisfied because the upper bound on $\nu_{\text{thresh}}$ in the statement of the Entropy Production Theorem ensures that $\delta \geq 2^{-2n}$. Each conditional distribution satisfies $P^*(C|z, R_{\text{pass}} = 1) \leq \delta/P(R_{\text{pass}} = 1|z)$, which is equivalent to $P^*(C, R_{\text{pass}} = 1|z) \leq \delta$, and is within TV distance $P(P(C|z, R_{\text{pass}} = 1) > \delta/P(R_{\text{pass}} = 1|z) \leq \delta/P(R_{\text{pass}} = 1|z, R_{\text{pass}} = 1)$ of $P(C|z, R_{\text{pass}} = 1)$. The joint probability distribution $P^*$ is determined pointwise from the already assigned values of $P^*(c|z|r_{\text{pass}})$ for $r_{\text{pass}} = 1$ as

$$P^*(czs|r_{\text{pass}}) = \begin{cases} P^*(c|z|r_{\text{pass}})P(zs|r_{\text{pass}}) & \text{if } r_{\text{pass}} = 1 \\ P(czs|r_{\text{pass}}) & \text{otherwise.} \end{cases}$$ (33)
Since the marginal distribution of $ZSR_{\text{pass}}$ is unchanged, the full TV distance between $\mathbb{P}$ and $\mathbb{P}^*$ is given by the average conditional TV distance with respect to $ZSR_{\text{pass}}$, see Eq. (9). Since the conditional TV distance is zero when $R_{\text{pass}} = 0$ and from independence of $S$, we obtain

$$\text{TV}(\mathbb{P}^*_{CZSR_{\text{pass}}}, \mathbb{P}_{CZSR_{\text{pass}}}) = \sum_{zSR_{\text{pass}}} \text{TV}(\mathbb{P}^*_{C|zSR_{\text{pass}}}, \mathbb{P}_{C|zSR_{\text{pass}}}) \mathbb{P}(zSR_{\text{pass}})$$

$$= \sum_{zSR_{\text{pass}}} \text{TV}(\mathbb{P}^*_{C|zSR_{\text{pass}}}, \mathbb{P}_{C|zSR_{\text{pass}}}) [r_{\text{pass}} = 1] \mathbb{P}(zSR_{\text{pass}})$$

$$\leq \sum_{zSR_{\text{pass}}} \mathbb{P}(\mathbb{P}(C, R_{\text{pass}} = 1|z) > \delta|z, R_{\text{pass}} = 1) [r_{\text{pass}} = 1] \mathbb{P}(zSR_{\text{pass}})$$

$$= \sum_{zSR_{\text{pass}}} [\mathbb{P}(c_{r_{\text{pass}}}|z) > \delta] \mathbb{P}(c|zSR_{\text{pass}}) [r_{\text{pass}} = 1] \mathbb{P}(zSR_{\text{pass}})$$

$$= \mathbb{P}(\mathbb{P}(CR_{\text{pass}}|Z) > \delta, R_{\text{pass}} = 1)$$

$$\leq \mathbb{P}(\mathbb{P}(C|Z) > \delta, R_{\text{pass}} = 1)$$

$$\leq \epsilon_p. \quad (34)$$

At this point we can also bound the TV distance conditional on passing. Since $\mathbb{P}^*(R_{\text{pass}}) = \mathbb{P}(R_{\text{pass}})$, we can apply Eq. (9) and the above bound on the distance to get

$$\epsilon_p \geq \text{TV}(\mathbb{P}^*_{CZSR_{\text{pass}}}, \mathbb{P}_{CZSR_{\text{pass}}}) = \sum_r \text{TV}(\mathbb{P}^*_{CZSR_{\text{pass}}}, \mathbb{P}_{CZSR_{\text{pass}}}) \mathbb{P}(R_{\text{pass}} = r)$$

$$= \text{TV}(\mathbb{P}^*_{CZSR_{\text{pass}} = 1}, \mathbb{P}_{CZSR_{\text{pass}} = 1}) \mathbb{P}(R_{\text{pass}} = 1). \quad (35)$$

We conclude that

$$\text{TV}(\mathbb{P}^*_{CZR_{\text{pass}} = 1}, \mathbb{P}_{CZR_{\text{pass}} = 1}) \leq \epsilon_p / \mathbb{P}(R_{\text{pass}} = 1) \leq \epsilon_p / \kappa. \quad (36)$$

For the second main step, we need the average “guessing probability” of $C$ given $Z$ conditional on $\{R_{\text{pass}} = 1\}$. This is given by

$$\sum_z \max_c (\mathbb{P}^*(c|z, R_{\text{pass}} = 1)) \mathbb{P}(z|R_{\text{pass}} = 1) \leq \sum_z \frac{\delta}{\mathbb{P}(R_{\text{pass}} = 1|z)} \mathbb{P}(z|R_{\text{pass}} = 1)$$

$$= \delta \sum_z \frac{\mathbb{P}(z)}{\mathbb{P}(R_{\text{pass}} = 1)} \leq \delta / \kappa \quad (37)$$
Now we can apply Proposition 1 of Ref. [38]. The next lemma extracts the conclusion of this proposition in the form we need. It is obtained by substituting the variables and expressions in the reference as follows: \( X \leftarrow C, Y \leftarrow S, E \leftarrow Z, E(X, Y) \leftarrow \text{Ext}(CS), k \leftarrow -\log_2(\delta/\kappa) - \log_2(2/\epsilon_{\text{ext}}), \epsilon \leftarrow \epsilon_{\text{ext}}/2 \) and the distributions are replaced with the corresponding ones that are conditional on \( \{R_{\text{pass}} = 1\} \). The guessing entropy in the reference is the negative logarithm of the guessing probability computed above.

**Lemma.** Suppose that \( \text{Ext} \) is a strong extractor with parameters \((-\log_2(2\delta/(\kappa\epsilon_{\text{ext}})), \epsilon_{\text{ext}}/2, 2n, d, t)\). Write \( U = \text{Ext}(CS) \). Then we have the following bound:

\[
TV\left(\mathbb{P}_{UZS|R_{\text{pass}}=1}, \mathbb{P}_{U}^{\text{unif}}\mathbb{P}_{S}\mathbb{P}_{Z|R_{\text{pass}}=1}\right) \leq \epsilon_{\text{ext}}. \tag{38}
\]

To apply the lemma, we obtain \( \text{Ext} \) by the TMPS algorithm with the parameters in the lemma. Expanding the logarithms as \( \sigma = -\log_2(\delta) + \log_2(\kappa) + \log_2(\epsilon_{\text{ext}}) - 1 \) and substituting in Eq. 28 gives the requirement

\[
t + 4\log_2 t \leq -\log_2(\delta) + \log_2(\kappa) + 5\log_2(\epsilon_{\text{ext}}) - 11, \tag{39}
\]

as asserted in the Protocol Soundness Theorem. The number of seed bits \( d \) is obtained from Eq. 29.

It remains to determine the overall TV distance conditional on passing. Applying Eq. 10 with \( V = C, Z, S \) and \( F \) defined as \( F(C, Z, S) = (\text{Ext}(C, S), Z, S) \), and applying Eq. 36 we have

\[
TV\left(\mathbb{P}_{UZS|R_{\text{pass}}=1}, \mathbb{P}_{UZS|R_{\text{pass}}=1}\right) \leq TV\left(\mathbb{P}_{CZS|R_{\text{pass}}=1}, \mathbb{P}_{CZS|R_{\text{pass}}=1}\right) \leq \epsilon_{p}/\kappa. \tag{40}
\]

Then by Eq. 8, Eq. 38 and Eq. 40 we have

\[
TV\left(\mathbb{P}_{UZS|R_{\text{pass}}=1}, \mathbb{P}_{U}^{\text{unif}}\mathbb{P}_{S}\mathbb{P}_{Z|R_{\text{pass}}=1}\right) \leq \epsilon_{\text{ext}} + \epsilon_{p}/\kappa. \tag{41}
\]

As \( \mathbb{P}_{Z|R_{\text{pass}}=1} = \mathbb{P}_{Z|R_{\text{pass}}=1} \), the statement of the theorem follows.

\[\square\]

**S.6 Protocol Application Details**

Ref. [18] reported data sets from six experiments obtained under space-like separation. Four additional data sets were obtained: two in which an extra delay was purposefully implemented so that space-like separation was not enforced, and two space-like separated “blind” data sets initially not subject to any analysis. In these data sets, each trial involved multiple pulses of the source laser, each detected separately. For the analyses in [18], the trial outcomes were determined by aggregating a consecutive sequence of \( k \) pulses, with outcome “+” if there was a detection in any one of these pulses and “0” otherwise. For our work, all analyses used \( k = 7 \). This was the largest group of pulses certified to be space-like separated in the data runs where space-like separation was enforced.

We first investigated training data from seven of the non-blind sets, labeled 03_43, 19_45, 21_15, 22_20, 23_55 (XOR 1), 00_25 (XOR 2), and 02_31 (XOR 3) in the online repository of
[18], running the full protocol only on XOR 3. We then applied the protocol to the blind data sets. One of the six experiments reported in Ref. [18] was temporarily unavailable in the online repository and was omitted from our investigation.

For each of the non-blind data sets, we determined the Bell function $T$ from training data consisting of the first $5 \times 10^7$ trials as explained in S.3. We chose $5 \times 10^7$ trials so that we could obtain a $T$ using an accurate estimate of the experimental distribution of measurement outcomes without sacrificing too much data that could be used for randomness extraction. Assuming i.i.d. trials and Gaussian statistics according to the central limit theorem, we then inferred the expected value $n\mu$ and variance $n\sigma^2$ of $\sum_{i=1}^{n} \ln(T_i)$ on the remaining trials, where $n$ and $\mu$ were calculated according to the distribution obtained from the optimization problem of Eq. 25. Note that under these assumptions, we treat $\sum_{i=1}^{n} \ln(T_i)$ as if it were a sum of independent and bounded RVs. Since $V = \exp(\sum_{i=1}^{n} \ln(T_i))$ we can then calculate $v_{\text{thresh}}$ according to the 0.95 rule described in the main text. That is, we set $v_{\text{thresh}} = e^{n\mu-1.645\sqrt{n}\sigma}$. Based on the results, we found that only the last data set, XOR 3, was anticipated to yield sufficient randomness at low error, consistent with expectations based on the $p$-values against LR given in Ref. [18]. We therefore applied the full protocol only to XOR 3. We note that while we had prior knowledge of general statistical features of XOR 3 from the analysis in Ref. [18], once the relevant parameters were determined from the training data, the protocol was run only once on the remaining data of XOR 3.

We next give details for our analysis of XOR 3, then describe how the protocol performed on the two blind data sets, and finish with a discussion of results from tests for non-uniformity in the settings and for signaling in the settings-conditional outcomes.

Data set XOR 3 consists of 182,161,215 trials. The counts for each trial result from the first $5 \times 10^7$ trials are shown in Table 2. The maximum likelihood non-signaling distribution corresponding to these counts is shown in Table 3. We determined $T$ from this distribution, the values of $T$ are shown in Table 1 of the main text. The parameter $m$ for $T$ is 0.0120275.

Table 2: Result counts for the first $5 \times 10^7$ trials of XOR 3.

| $xy$   | $ab = ++$ | $ab = +0$ | $ab = 0+$ | $ab = 00$ |
|--------|-----------|-----------|-----------|-----------|
| $xy = 00$ | 2483      | 1341      | 1266      | 12496049  |
| $xy = 01$ | 2645      | 1113      | 9095      | 12489487  |
| $xy = 10$ | 2602      | 8295      | 1076      | 12483646  |
| $xy = 11$ | 44        | 10869     | 11768     | 12478221  |

The 0.95 rule for determining $v_{\text{thresh}}$ given that there are 132,161,215 trials for the protocol yields $v_{\text{thresh}} = 1.66 \times 10^6$. While this suggests that we can set $\kappa = 0.95$, the i.i.d. assumption cannot be met exactly due to experimental drift and intermittent faults. This suggests a more conservative choice for $\kappa$. For $v_{\text{thresh}} = 1.66 \times 10^6$ and $\epsilon_p$ and $\epsilon_{\text{ext}}$ in the fixed ratio 9:1, one can choose $\kappa$ to be as low as 0.33 while still meeting the benchmark that Eq. 30 allows for an output string of at least $t = 256$ bits within $\epsilon_{\text{fin}} = \epsilon_p/\kappa + \epsilon_{\text{ext}} = 0.001$ of uniform. We observed that for the six earlier data sets for which we computed $T$ from training, the corresponding value of
Table 3: Maximum likelihood non-signaling distribution according to the counts in Table 2

| xy  | ab = ++ | ab = +0 | ab = 0+ | ab = 00 |
|-----|---------|---------|---------|---------|
| 00  | 0.000049006 | 0.00026663 | 0.00025112 | 0.249899219 |
| 01  | 0.000053304 | 0.00022364 | 0.00182341 | 0.249741991 |
| 10  | 0.000052435 | 0.00165906 | 0.00021683 | 0.249759976 |
| 11  | 0.000000876 | 0.00217465 | 0.00234769 | 0.249546890 |

$v_{\text{thresh}}$ was exceeded five times by the running product of the $T_i$’s on the remaining trials. Hence a choice of $\kappa = 0.33$ does not seem unreasonably high. Our choice to fix the ratio 9:1 for $\epsilon_p$ and $\epsilon_{\text{ext}}$ was based on numerical studies optimizing $t$ in Eq. 30 with various fixed values of $\kappa$ and $\epsilon_{\text{fin}}$. This ratio generally performed well, so we used it for all instances of the protocol.

Throughout, we did not consider the length $d$ of the seed in making our choices and determined $d$ from the other parameters according to Eq. 29. For applying the extractor to XOR 3, we generated 73,947 seed bits. The seed bits were obtained after the experiment from a random number generator similar to one used to select the settings [18]. The independence assumption $P(S, C, Z, R_{\text{pass}}) = P(S)P(C, Z, R_{\text{pass}})$ required by the Protocol Soundness Theorem is therefore justified, and this is consistent with our suggestion in the main text that the seed bits can be obtained from additional instances of the RVs $X_i$ (in which case the needed independence follows from Eq. 2). It took 128 seconds for our computer to construct the extractor according to the TMPS algorithm and generate the explicit final output string.

As reported at the online data repository of [18], two “blind” data runs were taken at the time of the original experiment and set aside for analysis at a later date. No analysis was performed on these data sets until the work reported here. We refer to these data sets as “Blind 1” and “Blind 2,” according to the order in which they were taken.

Blind 1 consists of 356,464,574 trials. We followed the training procedure described above on the initial $5 \times 10^7$ trials. For the computed value of $v_{\text{thresh}}$, our benchmark values of $t$ and $\epsilon_{\text{fin}}$ could not be met, so we did not formally run the protocol on the remaining trials. Upon examining these trials, we found that the final product $V$ of the $T_i$ was $2.023 \times 10^7$. The maximum running product was $6.884 \times 10^7$, and the curves for randomness that could have been extracted from Blind 1 in Fig. 3 are based on this value.

Blind 2 consists of 182,837,253 trials. The first $2 \times 10^7$ trials were set aside for training. This smaller number was chosen because Blind 2 was taken directly after XOR 3 and expected to be consistent with this prior data set. The function $T$ obtained from training performed well when retroactively applied to XOR 3, prompting us to apply the protocol to the remainder of the data set without further training. From the training procedure, we obtained $v_{\text{thresh}} = 8.515 \times 10^8$, which was predicted to be more than sufficient for extracting 256 random bits within 0.001 of uniform. We then ran the protocol, but found that the running product of the $T_i$ never exceeded $v_{\text{thresh}}$. This is therefore an instance where the protocol aborted. The failure to exceed $v_{\text{thresh}}$ is explained by a dramatic change of the results statistics after approximately $1.08 \times 10^8$ trials, entering a non-violating regime that drives $V$ below $10^{-300}$. However, prior to this change,
the statistics demonstrate violation, reaching a maximum running product of 17528.87 at trial number 73,057,106. The curves for randomness that could have been extracted from Blind 2 in Fig. [3] are based on this value.

Tests for non-uniformity of the settings distribution were performed and reported in Ref. [18]. The tests showed that a combination of uncontrolled environmental variables and the synchronization electronics introduced small biases, particularly in Alice’s settings. The sizes of the biases were found to be inconsistent over time, and so their existence or magnitude in any particular data run cannot be precisely inferred based on results from other data runs or post-experiment testing. Nevertheless, it is informative to perform statistical consistency checks for uniformity of the settings distribution (Eq. 2) within each run, especially in consideration of the known tendency for small biases. Using the tests described in Ref. [18], we examined the individual unbiasedness of $X$ and $Y$, and then we performed a separate test of the independence of $X$ and $Y$. For these tests we used statistics whose asymptotic distributions would approach the standard normal with mean 0 and variance 1, if the trials were i.i.d. For Blind 1, these statistics yielded two-tailed p-values for unbiasedness of $X$ and $Y$ of $9.9 \times 10^{-6}$ and 0.65 respectively, and 0.33 for independence. For Blind 2, these p-values were 0.04, 0.42, and 0.41, and for XOR 3, they were 0.05, 0.27, and 0.90.

The observed bias for Alice’s setting $X$ yielding the notably significant result in Blind 1 was small: the inferred probability of setting 1 is $0.50012 \pm 0.00003$. A possible concern is that the settings bias may have been in a direction to systematically bias $V$ upward. To check for this possibility, we computed $E(\ln(T))_{F} = 5.49 \times 10^{-8}$ according to the distribution $F$ obtained directly from the observed frequencies. We then obtained a normalized distribution $F'$ by the transformation $F'(a, b, x, y) = (1/4)F(a, b, x, y)/F(x, y)$, so $F'$ has the same settings-conditional probabilities as $F$ but $F'(x, y) = 1/4$ for all settings $x, y$. We found $E(\ln(T))_{F'} = 5.52 \times 10^{-8} > E(\ln(T))_{F}$, supporting the conclusion that the empirical overall direction of the bias in Blind 1 did not favor larger $V$. Nonetheless, this motivates potential future work to strengthen the protocol to allow for a relaxed version of Eq. [2] where the settings distribution is only assumed to be within some $\epsilon$ of uniform, such as is done in the statistical arguments of [17, 18, 19, 20] for falsifying LR. We do not pursue here a precise quantification of how such a relaxation would decrease the certifiable entropy in the entropy production theorem.

Tests for signaling were also reported in Ref. [18]. There are four signaling equalities that can be independently tested: $P(A|X = 0, Y) = P(A|X = 0)$, $P(A|X = 1, Y) = P(A|X = 1)$, $P(B|X, Y = 0) = P(B|Y = 0)$, and $P(B|X, Y = 1) = P(B|Y = 1)$. These tests performed on Blind 1 and Blind 2 showed only expected statistical variation. Specifically, the p-values for these statistics were 0.72, 0.03, 0.50, and 0.84 for XOR 3, 0.98, 0.09, 0.87 and 0.83 for Blind 1, and 0.01, 0.50, 0.04, and 0.13 for Blind 2.

S.7 Performance of Previous Protocols.

Other protocols in the literature could not be used for our data sets for the following reasons. Protocols in Refs. [4, 5, 9, 25, 27] either apply to different experimental setups or provide only
asymptotic security results as the number of trials \( n \) approaches infinity. The analysis of \([12]\) applies to i.i.d. scenarios, and the protocol of \([7]\) requires systems that achieve Bell violations much higher than ours. In contrast, the protocols of Refs. \([6, 26]\) would yield randomness for our results’ distribution given a sufficiently large number of trials. However, they are ineffective for the numbers of trials in our data sets, which we illustrate with a heuristic argument. Both protocols are based on the Clauser-Horne-Shimony-Holt (CHSH) Bell function \([39]\)

\[
T^c(a, b, x, y) = \begin{cases} 
1 & \text{if } (x, y) \neq (1, 1) \text{ and } a = b \\
1 & \text{if } (x, y) = (1, 1) \text{ and } a \neq b \\
0 & \text{otherwise.}
\end{cases}
\]  

(42)

The statistic \( T^c = n^{-1} \sum_{i=1}^{n} T^c_i \) used by these protocols for witnessing accumulated violation satisfies \( E(T^c) \leq 0.75 \) under LR, while \( E(T^c) = 0.750008165 \) for the distribution in Table 3. The completely predictable LR theory that only produces “00” outcomes regardless of the settings satisfies \( E(T^c) = 0.75 \), but in an experiment of \( n = 132, 161, 215 \) trials, this theory can produce a value of \( T^c \) exceeding 0.750008165 with probability above 0.4. Thus, based on this statistic alone, we cannot infer the presence of any low-error randomness.

The protocol of Ref. \([6]\) (the PM protocol for short, see \([3, 8]\) for amendments), can be modified to work with any Bell function, and there are methods for obtaining better Bell functions \([10, 11]\) or simultaneously using a suite of Bell functions \([40]\). Here, we demonstrate that for any choice of Bell function, the method of \([6]\) as refined in \([3]\) cannot be expected to effectively certify randomness from an experiment distributed according to Table 3 unless the number of trials exceeds \( 2.4 \times 10^{10} \), which is much larger than the number of trials in our experiments.

For the most informative comparison to our protocol, we consider the PM protocol without their additional constraint that the distribution be induced by a quantum state. To derive a bound on the performance of the PM protocol, we refer to Theorem 1 of \([3]\). This theorem involves a choice of Bell function denoted by \( I \) (analogous to our \( T \)), a threshold \( J_m \) (analogous to our \( v_{thres} \)) to be exceeded by the Bell estimator \( \bar{I} = n^{-1} \sum_{i=1}^{n} I_i \), and a function \( f \) that we discuss below. To be able to extract some randomness, the theorem requires that

\[
nf(J_m - \mu) > 0. 
\]  

(43)

The parameter \( \mu \) is given by \( (I_{\text{max}} + I_{\text{NS}})\sqrt{(2/n) \ln(1/\epsilon)} \) where \( I_{\text{max}} \) is the largest value in the range of the Bell function \( I \), \( I_{\text{NS}} \leq I_{\text{max}} \) is the largest possible expected value of \( I \) for non-signaling distributions, and \( 0 < \epsilon \leq 1 \) is a free parameter that is added to the TV distance from uniform for the final output string. Smaller choices of \( \epsilon \), which is analogous to our \( \epsilon_p \), are desirable but require larger \( n \) for the constraint Eq. 43 to be positive as we will see below. We also note that Eq. 43 is a necessary but not sufficient condition for extracting randomness; in particular, we ignore the negative contribution from the parameter \( \epsilon' \) of \([3]\) (somewhat analogous to our \( \kappa \)) as well as any error introduced in the extraction step.

For Eq. 43 we can without loss of generality consider only Bell functions for which \( 0 \leq I_L < I_{\text{NS}} \leq I_{\text{max}} \), where \( I_L \) is the maximum expectation of \( I \) for LR distributions. Further,
because the relevant quantities below are invariant when the Bell function is rescaled, we can assume $I_L = 1$. According to Ref. [3]’s Eq. 8 and the following paragraph, we can write $f(x) = -\log_2(g(x))$, where $g$ is monotonically decreasing and concave, and satisfies
\[
\max_{ab} \mathbb{P}(ab|xy) \leq g(\mathbb{E}(I)_\mathbb{P})
\]  
for all $xy$ and non-signaling distributions $\mathbb{P}$. (Recall that we are not using the stronger constraint that $\mathbb{P}$ be induced by a quantum state.) According to Eq. [19] we can define $g(x) = 1 + (1 - x)/(2(I_{NS} - 1))$. Later we argue that this definition of $g$ cannot be improved. Substituting into Eq. [43] we get the inequality
\[
-n \log_2 \left[ 1 + \frac{1 - J_m + (I_{max} + I_{NS}) \sqrt{\frac{2}{n} \ln \frac{1}{\epsilon}}}{2(I_{NS} - 1)} \right] > 0.
\]  
Since $2(I_{NS} - 1)$ is positive, this is equivalent to
\[
\sqrt{\frac{2}{n} \ln \frac{1}{\epsilon}} < \frac{J_m - 1}{2I_{NS}}.
\]  
Noting that $I_{max} + I_{NS} \geq 2I_{NS}$, this implies
\[
\sqrt{\frac{2}{n} \ln \frac{1}{\epsilon}} < \frac{J_m - 1}{2I_{NS}}.
\]  
Thus, the number of trials needed to extract randomness by the PM protocol is bounded below according to
\[
n > 8 \frac{\ln(1/\epsilon) I_{NS}^2}{(J_m - 1)^2}.
\]  
For a given anticipated experimental distribution $\mathbb{P}_\text{ant}$, $J_m$ is best chosen to be at most $\mathbb{E}(I)_{\mathbb{P}_\text{ant}}$. Otherwise, the probability that $\bar{I}$ exceeds $J_m$ is small. However, for the maximum amount of extractable randomness, $J_m$ should be close to $\mathbb{E}(I)_{\mathbb{P}_\text{ant}}$. Consider the inferred distribution of XOR 3 shown in Table 3. By following the procedure given in Section 2 of [34], we can write this distribution as a convex combination of a PR box with weight $p = 3.266 \times 10^{-5}$ and an LR distribution with weight $1 - p$. From this we see that one should choose $J_m \leq \mathbb{E}(I)_{\mathbb{P}_\text{ant}} = pI_{NS} + (1-p) \leq pI_{NS} + 1$. Substituting into Eq. [48] and using $\epsilon \leq 0.05$ (a rather high bound on the allowable TV distance from uniform) gives
\[
n > 8 \frac{\ln(1/\epsilon) I_{NS}^2}{p^2} \geq 2.4 \times 10^{10},
\]  
which is substantially larger than the number of trials in our data sets.

To finish our argument that the PM protocol cannot improve on this bound under our assumptions, consider the definition of $g$. If we could find a function $g' \leq g$ with $g'(x) < g(x)$
for some $x \in (1, I_{NS}]$, then $f = -\log_2(g')$ might yield a smaller lower bound on $n$. Note that for $x \leq 1$, $g'(x) \geq g'(1)$ and $g'(1)$ must be at least 1 because, referring to Eq. 44, there is a conditionally deterministic LR distribution $\mathbb{P}$ satisfying $\mathbb{E}(I)_{\mathbb{P}} = 1$ and $\max_{ab} \mathbb{P}(ab|xy) = 1$. Hence Eq. 43 cannot be satisfied for arguments $x$ of $f(x) = -\log_2(g'(x))$ with $x \leq 1$. Given $x \in (1, I_{NS}]$, write $x = (1 - p) + pI_{NS}$. Let $\mathbb{Q}$ be the PR box achieving $\mathbb{E}(I)_{\mathbb{Q}} = I_{NS}$ and $\mathbb{Q}'$ a conditionally deterministic LR theory achieving $\mathbb{E}(I)_{\mathbb{Q}'} = 1$. Then $\mathbb{E}(I)_{(1-p)\mathbb{Q}'+p\mathbb{Q}'} = x$. Furthermore, there is a setting $xy$ at which the LR theory’s outcome is inside the support of the PR box’s outcomes. To see this, by symmetry it suffices to consider the PR box of Eq. 17. Its outcomes are opposite at setting 11 and identical at the other three. A deterministic LR theory’s outcomes are opposite at an even number of settings, so either it is opposite at setting 11, or it is identical at one of the others. For setting $xy$, the bound in Eq. 44 is achieved for our definition of $g$. Hence any other valid replacement $g'$ for $g$ must satisfy $g'(x) \geq g(x)$ for $x \in (1, I_{NS}]$, and so Eq. 43 with $f(x) = -\log_2(g'(x))$ implies Eq. 43 with $f(x) = -\log_2(g(x))$. Thus the lower bound on $n$ derived above will apply to $g'$ as well.

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