TWO-DIMENSIONAL BADLY APPROXIMABLE VECTORS
AND SCHMIDT’S GAME

JINPENG AN

Abstract. We prove that for any pair \((s, t)\) of nonnegative numbers with \(s + t = 1\), the set of two-dimensional \((s, t)\)-badly approximable vectors is winning for Schmidt’s game. As a consequence, we give a direct proof of Schmidt’s conjecture using his game.

1. Introduction

1.1. Schmidt’s conjecture and Schmidt’s game. Given a pair \((s, t)\) of nonnegative numbers with \(s + t = 1\), a two-dimensional vector \((x, y)\) \(\in \mathbb{R}^2\) is said to be \((s, t)\)-badly approximable if

\[
\inf_{q \in \mathbb{N}} \max \{q^s \|qx\|, q^t \|qy\|\} > 0,
\]

where \(\| \cdot \|\) denotes the distance of a number to the nearest integer. As a natural generalization of badly approximable numbers, the set of \((s, t)\)-badly approximable vectors, denoted by \(\text{Bad}(s, t)\), is a fundamental object of study in simultaneous Diophantine approximation. It is well-known that \(\text{Bad}(s, t)\) has Lebesgue measure zero and full Hausdorff dimension in \(\mathbb{R}^2\) (see [17]). In the early 1980’s, W. M. Schmidt [21] conjectured that \(\text{Bad}(\frac{1}{3}, \frac{2}{3}) \cap \text{Bad}(\frac{2}{3}, \frac{1}{3}) \neq \emptyset\). Schmidt’s conjecture was recently proved by D. Badziahin, A. Pollington and S. Velani [2]. In fact, they proved a much stronger theorem, which states that certain countable intersection (in particular, any finite intersection) of \(\text{Bad}(s_n, t_n)\) has full Hausdorff dimension.

On the other hand, in the 1960’s, Schmidt [18] introduced a game played on a complete metric space by two players. Winning sets for Schmidt’s game has very nice properties. For example, a winning subset of an Euclidean space has full Hausdorff dimension. More importantly, a countable intersection of \(\alpha\)-winning sets is still \(\alpha\)-winning. Schmidt [18] [20] showed that \(\text{Bad}(\frac{1}{2}, \frac{1}{2})\) is \(1/2\)-winning. As such, it is natural to expect that \(\text{Bad}(s, t)\) is a winning set in general, and thus Schmidt’s conjecture can be proved directly using his game. This expectation was raised explicitly by Kleinbock [9] (see also [12, 15]). For similar questions and results for higher-dimensional vectors and matrices, see, for example, [9, 11, 13, 17, 19].

1.2. Proving Schmidt’s conjecture using his game. The goal of proving Schmidt’s conjecture using his game was partly achieved in [1]. It was proved there that if \(x \in \mathbb{R}\) is badly approximable, then the set of \(y \in \mathbb{R}\) such that \((x, y)\) is \((s, t)\)-badly approximable is a winning subset of \(\mathbb{R}\). As a consequence, any countable intersection of \(\text{Bad}(s_n, t_n)\) has full Hausdorff dimension. In this paper, we prove that \(\text{Bad}(s, t)\) itself is a winning subset of \(\mathbb{R}^2\), thus give a more direct proof of Schmidt’s conjecture. Our main theorem is as follows.

**Theorem 1.1.** For any \(s, t \geq 0\) with \(s + t = 1\), the set \(\text{Bad}(s, t)\) is \((24\sqrt{2})^{-1}\)-winning.

Theorem [13] implies stronger full dimension results. For example, since a countable intersection of images of \(\alpha\)-winning sets under uniformly bi-Lipschitz homeomorphisms is still winning (see [3, 18]), we obtain the following result.

Research supported by NSFC grant 10901005/11322101 and FANEDD grant 200915.
Corollary 1.2. Let \((s_n, t_n)_{n=1}^\infty\) be a sequence of pairs of nonnegative numbers with \(s_n + t_n = 1\), and let \((f_n)_{n=1}^\infty\) be a sequence of uniformly bi-Lipschitz homeomorphisms of \(\mathbb{R}^2\), that is, there exists \(M \geq 1\) such that
\[
M^{-1} |x_1 - x_2| \leq |f_n(x_1) - f_n(x_2)| \leq M |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^2, n \geq 1,
\]
where \(|\cdot|\) is the Euclidean norm. Then the set \(\bigcap_{n=1}^\infty f_n(\text{Bad}(s_n, t_n))\) has full Hausdorff dimension in \(\mathbb{R}^2\).

It should be noted that several stronger variants of Schmidt’s game have been defined and used to problems in Diophantine approximation (see, for example, [14, 3]). By using the main lemma in a previous version of this paper (a weaker form of Corollary 4.2 below), it has been proved in [16] that \(\text{Bad}(s, t)\) is hyperplane absolute winning in the sense of [3].

1.3. Relationship to homogeneous dynamics. As is well known, badly approximable vectors correspond to certain bounded trajectories on the homogeneous space \(\text{SL}_3(\mathbb{R})/\text{SL}_3(\mathbb{Z})\).

For \((x, y) \in \mathbb{R}^2\), we denote \(h_{(x,y)} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\). Let \(H \cong \mathbb{R}^2\) be the subgroup of \(G = \text{SL}_3(\mathbb{R})\) consisting of matrices of the form \(h_{(x,y)}\): By a ray \(G\), we mean a set of the form \(F^+ = \{g_u : u \geq 0\}\), where \(u \mapsto g_u\) is a one-parameter subgroup of \(G\). Consider rays of the form
\[
F^+_{(s,t)} = \{\text{diag}(e^{su}, e^{tu}, e^{-u}) : u \geq 0\}, \quad s, t \geq 0, s + t = 1.
\]

Then \((x, y)\) is \((s, t)\)-badly approximable if and only if the trajectory \(F^+_{(s,t)} h_{(x,y)} \Gamma\) is bounded in \(G/\Gamma\), where \(\Gamma = \text{SL}_3(\mathbb{Z})\) (see [14, 9]).

Let \(D\) be the group of diagonal matrices in \(G\), and consider its sub-semigroup
\[
D^+ = \{\text{diag}(e^{u_1}, e^{u_2}, e^{-u_1-u_2}) : u_1, u_2 \geq 0\}.
\]

Then any ray in \(D^+\) of the form (1.1). Thus Theorem 1.1 amounts to the statement that for any ray \(F^+\) in \(D^+\), the set of \(h \in H\) for which \(F^+ h \Gamma\) is bounded is \((24\sqrt{2})^{-1}\)-winning.

In a much more general context, the winning property for sets of this form with respect to a modified Schmidt game was established in [13].

It was proved in [7] that the set of \(h \in H\) for which \(D^+ h \Gamma\) is bounded has Hausdorff dimension zero (note that \(D^+ h_{(x,y)} \Gamma\) is bounded if and only if \((x, y)\) violates Littlewood’s conjecture \(\inf_{g \in G} q \|q x\| |q y| = 0\)). A conjecture from [8] states that for any two rays \(F^+_1, F^+_2\) in \(D^+\), there exists \(g \in G\) such that \(F^+_1 g \Gamma\) and \(F^+_2 g \Gamma\) are bounded but \(D g \Gamma\) is unbounded. D. Kleinbock observed that if \(F^+_1\) and \(F^+_2\) lie in opposite Weyl chambers, the arguments in [10] can be adapted to prove that the set of \(g \in G\) satisfying the conjecture full Hausdorff dimension. On the other hand, the main theorem in [2] implies that if \((F^+_n)_{n=1}^\infty\) is a sequence of rays in \(D^+\) satisfying a certain technical assumption, then the set
\[
\{h \in H : F^+_n h \Gamma\text{ is bounded}, \forall n \geq 1\}
\]
has full Hausdorff dimension in \(H\). It follows from Theorem 1.1 that without the technical assumption, the set (1.2) is winning.

1.4. On the proof of Theorem 1.1. Unlike previous proofs of the winning property (except for [1]), our proof of Theorem 1.1 is non-constructive. In other words, it does not give an explicit winning strategy, but only shows the existence of a winning strategy. This is reflected in the proof of Proposition 2.2 below, where we use König’s lemma in graph theory to show the existence of a certain subtree that corresponds to a winning strategy.

A crucial ingredient in establishing Theorem 1.1 is the height function on rational points given by (3.2). It relies not only on the rational point itself, but also on a rational line
We first prove that under the assumptions of the proposition, for every subtree $S$ of type (I) of $T$, there exists a subtree $S'$ of type (II) of $T$ such that $S' \cap S = \emptyset$. To prove this, we use the construction of the Cantor-like set, at the $n$-th step we need only to remove squares that intersect small neighborhoods of $n$ lines. This implies that the Cantor-like set is “fat” enough so that it is winning for Schmidt’s game.

In order to simplify the presentation and resort to König’s lemma directly, it is convenient to represent squares used in the construction of the Cantor-like set as vertices of a rooted tree, and color the vertices in a regular manner. In Section 2, we provide preliminaries on colored rooted trees. Theorem 1.1 is proved in Sections 3 and 4.

2. Regular colorings of rooted trees

We use the same notation and conventions for rooted trees as in [1]. For example, we identify a rooted tree $T$ with the set of its vertices, and denote the set of vertices of level $n$ by $T_n$. For $\tau \in T$, let $T(\tau)$ denote the rooted tree formed by the descendants of $\tau$, and $T_{\text{fix}}(\tau)$ denote the set of successors of $\tau$. For all $\tau \subset V$, denote $T_{\text{fix}}(\tau) = \bigcup_{\tau \in V} T_{\text{fix}}(\tau)$. By convention, a subtree has the same root as the ambient tree.

Let $D \in \mathbb{N}$. A $D$-coloring of a rooted tree $T$ is a map $\gamma : T \to \{1, \ldots, D\}$. For $\tau \subset T$ and $1 \leq i \leq D$, we denote $\gamma(i) = V \cap T_{\text{fix}}(\tau)$. Let $N \in \mathbb{N}$ be an integer multiple of $D$, and suppose that $T$ is $N$-regular, that is, $\#T_{\text{fix}}(\tau) = N$ for every $\tau \in T$. We say that a $D$-coloring of $T$ is regular if for any $\tau \in T$ and $1 \leq i \leq D$, we have $\#T_{\text{fix}}(\tau)(i) = N/D$. The following two types of subtrees are of interest to us.

**Definition 2.1.** Let $T$ be an $N$-regular rooted tree with a regular $D$-coloring, and let $S \subset T$ be a subtree.

- The subtree $S$ is of type (I) if for any $\tau \in S$ and $1 \leq i \leq D$, we have $\#T_{\text{fix}}(\tau)(i) = 1$.
- The subtree $S$ is of type (II) if for any $\tau \in S$, there exists $1 \leq i(\tau) \leq D$ such that $S_{\text{fix}}(\tau) = T_{\text{fix}}(\tau)(i(\tau))$.

Roughly speaking, in the proof of Theorem 1.1 the two types of subtrees correspond to strategies of the two players in Schmidt’s game. We need the following criterion for the existence of subtrees of type (I) in establishing Proposition 3.3 below, which is at the heart of the proof of Theorem 1.1.

**Proposition 2.2.** Let $T$ be an $N$-regular rooted tree with a regular $D$-coloring, and let $S \subset T$ be a subtree. Suppose that for every subtree $R \subset T$ of type (II), $S \cap R$ is infinite. Then $S$ contains a subtree of type (I).

**Proof.** We first prove that under the assumptions of the proposition, for every $h \geq 0$, there exists a subtree $F$ of $S$ such that for any $\tau \in F_n$ with $n < h$ and any $1 \leq i \leq D$, we have $\#T_{\text{fix}}(\tau)(i) = 1$. (2.1)

If $h = 0$, there is nothing to prove. Assume $h \geq 1$ and (2.1) holds if $h$ is replaced by $h - 1$. Let

$$S'_1 = \{ \tau \in S_1 : \text{the intersection of } S(\tau) \text{ with every subtree of } T(\tau) \text{ of type (II) is infinite} \}.$$  

By the induction hypothesis, if $\tau \in S'_1$, then $S(\tau)$ has a subtree $F_\tau$ such that for any $\tau' \in (F_\tau)_n$, with $n < h - 1$ and any $1 \leq i \leq D$, we have $\#(F_\tau)_n(\tau')(i) = 1$. Thus to prove (2.1), it suffices to prove that $(S'_1)(i) = \emptyset$ for every $1 \leq i \leq D$. Suppose on the contrary that $(S'_1)(i_0) = \emptyset$ for some $1 \leq i_0 \leq D$. Then for every $\tau \in T_{1}(i_0)$, $T(\tau)$ has a subtree $R_\tau$
of type (II) such that \( S(\tau) \cap R_\tau \) is finite whenever \( \tau \in S \). Let \( R \subset T \) be the subtree such that \( R_1 = T_1^{(0)} \) and \( R(\tau) = R_\tau \) for every \( \tau \in R_1 \). Then \( R \) is of type (II) and

\[
S \cap R = \{ \text{the root of } T \} \cup \bigcup_{\tau \in S_1^{(0)}} S(\tau) \cap R_\tau
\]

is finite. This contradicts the assumption of the proposition.

We now prove the proposition by considering the rooted tree \( \mathcal{F} \) constructed as follows. For \( h \geq 0 \), the set \( \mathcal{F}_h \) of vertices of level \( h \) consists of the subtrees \( \mathcal{F} \) of \( S \) such that \( \mathcal{F}_{h+1} = \emptyset \) and \( \# \mathcal{F}_{\text{Suc}}(\tau)(i) = 1 \) for any \( \tau \in \mathcal{F}_n \) with \( n < h \) and any \( 1 \leq i \leq D \). Define \( \mathcal{F} \in \mathcal{F}_{h+1} \) to be a successor of \( \mathcal{F}' \in \mathcal{F}_h \) whenever \( \mathcal{F}' = \bigcup_{n=0}^h \mathcal{F}_n \). In view of (2.1), we have \( \mathcal{F}_h \neq \emptyset \) for every \( h \geq 0 \). By König’s lemma (see [6, Lemma 8.1.2]), \( \mathcal{F} \) has an infinite path starting from the root. This means that there exists a family of subtrees \( \{ \mathcal{F}(h) \in \mathcal{F}_h : h \geq 0 \} \) such that \( \mathcal{F}(h) = \bigcup_{n=0}^h \mathcal{F}(h+1)_n \) for every \( h \). It follows that \( \bigcup_{n=0}^\infty \mathcal{F}(h) \) is a subtree of type (I) contained in \( S \).

3. The winning strategy

In this section, we review the notion of a winning set for Schmidt’s game, introduce a height function on rational points, and prove Theorem [1.1] from Proposition 3.3 below.

3.1. Winning sets for Schmidt’s game. Schmidt’s game was introduced in [18]. It involves two real numbers \( \alpha, \beta \in (0, 1) \) and is played by two players, say Alice and Bob. Restricting the attention to \( \mathbb{R}^2 \), Bob starts the game by choosing a closed disk \( B_0 \subset \mathbb{R}^2 \). After \( B_n \) is chosen, Alice chooses a closed disk \( A_n \subset B_n \) with \( \rho(A_n) = \alpha \rho(B_n) \), and Bob chooses a closed disk \( B_{n+1} \subset A_n \) with \( \rho(B_{n+1}) = \beta \rho(A_n) \), where \( \rho(\cdot) \) denotes the radius of a disk. A subset \( X \subset \mathbb{R}^2 \) is \( (\alpha, \beta) \)-winning if Alice can play so that the single point in \( \bigcap_{n=0}^\infty A_n = \bigcap_{n=0}^\infty B_n \) lies in \( X \), and is \( \alpha \)-winning if it is \( (\alpha, \beta) \)-winning for any \( \beta \in (0, 1) \).

3.2. A height function on rational points. We introduce a height function on \( \mathbb{Q}^2 \) that play a crucial role in proving Theorem 1.1. For this, we consider rational lines in \( \mathbb{R}^2 \) of the form

\[
L(A, B, C) = \{(x, y) \in \mathbb{R}^2 : Ax + By + C = 0\},
\]

where \( A, B, C \in \mathbb{Z} \) and \( (A, B) \neq (0, 0) \). It is natural to make the convention that when a rational line is expressed as above, then \( A, B, C \) are coprime. Thus the vector \( (A, B, C) \) is determined by \( L(A, B, C) \) up to a negative sign. We also assume that when a point in \( \mathbb{Q}^2 \) is expressed as \( \left( \frac{p}{q}, \frac{r}{s} \right) \), then \( q > 0 \) and the integers \( p, q, r \) are coprime. Let \( s, t \geq 0 \) be such that \( s + t = 1 \). The following simple lemma is a baby version of [2, Lemma 1].

**Lemma 3.1.** To each \( P = \left( \frac{p}{q}, \frac{r}{s} \right) \in \mathbb{Q}^2 \), one can attach a rational line \( L_P = L(A_P, B_P, C_P) \) passing through \( P \) such that

\[
|A_P| \leq q^s, \quad |B_P| \leq q^t.
\]

We now define the height function \( H : \mathbb{Q}^2 \to \mathbb{N} \) as follows.

**Definition 3.2.** The height of a rational point \( P = \left( \frac{p}{q}, \frac{r}{s} \right) \) is

\[
H(P) = q \max\{|A_P|, |B_P|\}.
\]

It follows from (3.1) that

\[
q \leq H(P) \leq q^{1+\max\{s, t\}}.
\]
3.3. **The winning strategy.** Let \( \alpha_0 = (24\sqrt{2})^{-1} \). To prove Theorem 1.1, we need to show that for any \( \beta \in (0, 1) \), Alice can win Schmidt’s \((\alpha_0, \beta)\)-game with target set \( \text{Bad}(s, t) \). In what follows, we describe a winning strategy for Alice.

In the first round of the game, for any choice of the closed disc \( B_0 \) made by Bob, Alice chooses the closed disc \( A_0 \subset B_0 \) with \( \rho(A_0) = \alpha_0 \rho(B_0) \) arbitrarily. Let
\[
l = 2 \rho(A_0), \quad R = (\alpha_0 \beta)^{-1},
\]
and let \( c > 0 \) be such that
\[
c < \min \left\{ \frac{1}{6} lR^{-1}, \frac{1}{16} R^{-12} \right\}.
\]
For \( P = \left( \frac{p}{q}, \frac{r}{q} \right) \in \mathbb{Q}^2 \), we denote
\[
\Delta(P) = \left\{ (x, y) \in \mathbb{R}^2 : \left| x - \frac{p}{q} \right| \leq \frac{c}{q^{1+s}}, \left| y - \frac{r}{q} \right| \leq \frac{c}{q^{1+t}} \right\}.
\]
Then it is easy to see that
\[
\mathbb{R}^2 \setminus \bigcup_{P \in \mathbb{Q}^2} \Delta(P) \subset \text{Bad}(s, t).
\]
We will show that for a suitable partition \( \mathbb{Q}^2 = \bigcup_{n=1}^{\infty} \mathcal{R}_n \), Alice has a strategy so that she can choose the closed disc \( A_n \) in \( \mathbb{R}^2 \setminus \bigcup_{P \in \mathcal{R}_n} \Delta(P) \). This will ensure that the single point in \( \bigcap_{n=0}^{\infty} A_n \) lies in the left hand side of \((3.12)\), hence in \( \text{Bad}(s, t) \).

To define the appropriate partition, we use the height function defined above. For \( n \geq 1 \), let
\[
H_n = 6cl^{-1} R^n,
\]
and let
\[
\mathcal{R}_n = \left\{ P = \left( \frac{p}{q}, \frac{r}{q} \right) \in \mathbb{Q}^2 : H_n \leq H(P) < H_{n+1} \right\}.
\]
It follows from \((3.4)\) that
\[
H_1 = 6cl^{-1} R \leq 1.
\]
So \( \mathbb{Q}^2 = \bigcup_{n=1}^{\infty} \mathcal{R}_n \). Starting from this, we construct a Cantor-like set using squares. By a *square* we mean a set of the form
\[
\Sigma = \{ (x, y) \in \mathbb{R}^2 : x_0 \leq x \leq x_0 + \ell(\Sigma), y_0 \leq y \leq y_0 + \ell(\Sigma) \},
\]
where \( \ell(\Sigma) > 0 \) is the side length of \( \Sigma \). Let \( \Sigma_0 \) be the circumscribed square of \( A_0 \). Then \( \ell(\Sigma_0) = l \). We represent certain subsquares of \( \Sigma_0 \) as vertices of a regular rooted tree with a regular coloring. Let
\[
m = 12,
\]
and let \( T \) be an \( m^2[R/m]^2 \)-regular rooted tree with a regular \([R/m]^2\)-coloring, where \([ \cdot ]\) denotes the integer part of a real number. We choose and fix an injective map \( \Phi \) from \( T \) to the set of subsquares of \( \Sigma_0 \) satisfying the following conditions:

- For any \( n \geq 0 \) and \( \tau \in T_n \), we have
  \[
  \ell(\Phi(\tau)) = lR^{-n}.
  \]
  In particular, the root of \( T \) is mapped to \( \Sigma_0 \).
- For \( \tau, \tau' \in T \), if \( \tau \) is a descendant of \( \tau' \), then \( \Phi(\tau) \subset \Phi(\tau') \).
- For any \( n \geq 1 \) and \( \tau \in T_{n-1} \), the interiors of the squares \( \{ \Phi(\tau') : \tau' \in T_{\text{Suc}}(\tau) \} \) are mutually disjoint, the union \( \bigcup_{\tau' \in T_{\text{Suc}}(\tau)} \Phi(\tau') \) is a square of side length \( m[R/m]lR^{-n} \), and for any \( 1 \leq i \leq [R/m]^2 \), the union \( \bigcup_{\tau' \in T_{\text{Suc}}(\tau)^c} \Phi(\tau') \) is a square of side length \( mlR^{-n} \).
It is easy to see that for any $\tau \in T_{n-1}$ with $n \geq 1$ and any subsquare $\Sigma$ of $\Phi(\tau)$ of side length $2mR^{-n}$, there exists $1 \leq i \leq [R/m]^2$ such that $\bigcup_{\tau' \in T_{suc}(\tau)^{(i)}} \Phi(\tau') \subset \Sigma$.

The Cantor-like set is constructed from the subtree $S$ of $T$ defined as follows. Let $S_0 = T_0$. If $n \geq 1$ and $S_{n-1}$ is defined, we let

$$S_n = \{ \tau \in T_{suc}(S_{n-1}) : \Phi(\tau) \cap \bigcup_{P \in \mathcal{P}_n} \Delta(P) = \emptyset \}.$$  \hspace{1cm} (3.12)

Then $S = \bigcup_{n=0}^{\infty} S_n$ is a subtree of $T$. This gives rise to a Cantor-like set

$$C = \bigcap_{n=1}^{\infty} \bigcup_{\tau \in S_n} \Phi(\tau).$$

Note that by (3.12), we have

$$\bigcup_{\tau \in S_n} \Phi(\tau) \subset \mathbb{R}^2 \setminus \bigcup_{P \in \mathcal{P}_n} \Delta(P), \quad \forall n \geq 1.$$  \hspace{1cm} (3.13)

Thus $C$ is contained in the left hand side of (3.10), and hence is contained in $\text{Bad}(s, t)$. The winning strategy for Alice will in fact enable her to choose $A_n$ to be the inscribed closed disc of $\Phi(\tau)$ for some $\tau \in S_n$. This will imply that the single point in $\bigcap_{n=0}^{\infty} A_n$ lies in $C$, hence in $\text{Bad}(s, t)$. Such a winning strategy corresponds to a subtree of $T$ of type (I) contained in $S$, whose existence is ensured by the following proposition.

**Proposition 3.3.** The tree $S$ contains a subtree of type (I).

Proposition 3.3 will be proved in the next section. In the rest of this section, we assume it and prove Theorem 1.1. The proof also reflects the idea that a strategy of Bob roughly corresponds to a subtree of $T$ of type (II).

**Proof of Theorem 1.1.** Let $S'$ be a subtree of $S$ of type (I). In view of the above analysis, it suffices to prove that for every $n \geq 0$,

Alice can choose $A_n$ to be the inscribed closed disc of $\Phi(\tau_n)$ for some $\tau_n \in S'_n$. \hspace{1cm} (3.14)

We prove this by induction. If $n = 0$, there is nothing to prove. Assume $n \geq 1$ and Alice has chosen $A_{n-1}$ as the inscribed closed disc of $\Phi(\tau_{n-1})$, where $\tau_{n-1} \in S'_{n-1}$. For any choice $B_n \subset A_{n-1}$ of Bob, the inscribed square of $B_n$ has side length

$$\sqrt{2} \rho(B_n) = \sqrt{2} \beta \rho(A_{n-1}) = \frac{\sqrt{2}}{2} \beta \ell(\Phi(\tau_{n-1})) = \frac{\sqrt{2}}{2} \beta l R^{-n+1} = 2m R^{-n}.$$  \hspace{1cm} (3.13)

So there exists $1 \leq i \leq [R/m]^2$ such that $\bigcup_{\tau \in T_{suc}(\tau_{n-1})^{(i)}} \Phi(\tau) \subset B_n$. Let $\tau_n$ be the unique vertex in $S'_{suc}(\tau_{n-1})^{(i)}$. Then $\Phi(\tau_n) \subset B_n$. Note that the radius of the inscribed closed disc of $\Phi(\tau_n)$ is equal to

$$\frac{1}{2} \ell(\Phi(\tau_n)) = \frac{1}{2} R^{-1} \ell(\Phi(\tau_{n-1})) = \alpha_0 \beta \rho(A_{n-1}) = \alpha_0 \rho(B_n).$$

Thus Alice can choose $A_n$ to be the inscribed closed disc of $\Phi(\tau_n)$. This proves (3.14). \hspace{1cm} \Box

4. **Proof of Proposition 3.3**

In this section we prove Proposition 3.3. Without loss of generality, we may assume that $s \leq t$. \hspace{1cm} (4.1)

In view of (3.3) and (3.3), for $P = (\frac{q}{q}, \frac{r}{q}) \in \mathcal{P}_n$ we have

$$H_n^{\frac{1}{q}} \leq q < H_{n+1}.$$  \hspace{1cm} (4.2)
We further divide each $\mathcal{P}_n$ into at most $n$ parts. Let
\[ \mathcal{P}_{n,1} = \{ P \in \mathcal{P}_n : H_n^{1/\tau} \leq q < H_n^{1/\tau} R^{10}\}, \tag{4.3} \]
and for $k \geq 2$, let
\[ \mathcal{P}_{n,k} = \{ P \in \mathcal{P}_n : H_n^{1/\tau} R^{2k+6} \leq q < H_n^{1/\tau} R^{2k+8}\}. \tag{4.4} \]
Note that if $k \geq n+1$, then by (3.3),
\[ H_n^{1/\tau} R^{2k+6} \geq H_n^{1/\tau} R^{2n+8} = H_1^{1/\tau} R^{2+6(n-1)+9} H_n \geq H_n, \]
and it follows from (4.2) that $\mathcal{P}_{n,k} = \emptyset$. Hence $\mathcal{P}_n = \bigcup_{k=1}^n \mathcal{P}_{n,k}$. The following lemma is a key step in the proof of Proposition 3.3. Roughly speaking, it states that those points in $\mathcal{P}_{n,k}$ which are “responsible” for the construction of the Cantor-like set $C$ lie on a single line.

**Lemma 4.1.** Let $n \geq 1$, $1 \leq k \leq n$, and $\tau \in S_{n-k}$. Then the map $P \mapsto L_P$ is constant on the set $\mathcal{P}_{n,k}(\tau) := \{ P \in \mathcal{P}_{n,k} : \Phi(\tau) \cap \Delta(P) \neq \emptyset\}$.

**Proof.** Let $P_1 = (\frac{p_1}{q_1}, \frac{r_1}{q_1})$ and $P_2 = (\frac{p_2}{q_2}, \frac{r_2}{q_2})$ be distinct points in $\mathcal{P}_{n,k}(\tau)$. We need to prove that $L_{P_1} = L_{P_2}$. Suppose $L_{P_i} = L(A_i, B_i, C_i)$, $i = 1, 2$. Consider the three-dimensional vectors $v_i = (\frac{p_i}{q_i}, \frac{r_i}{q_i}, 1)$ and $w_i = (A_i, B_i, C_i)$. Note that $\langle v_i, w_i \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^3$. We first verify that
\[ |\langle v_1, w_2 \rangle| \leq 4c_{q_1}^{-1} R^{\lambda_k} + 12c_{q_2}^{-1} R^{k+1}, \tag{4.5} \]
where
\[ \lambda_k = \begin{cases} 10, & k = 1, \\ 2, & k \geq 2. \end{cases} \]
In fact, since $\Phi(\tau) \cap \Delta(P_i) \neq \emptyset$, we have
\[ |\langle v_1, w_2 \rangle| = |\langle v_1 - v_2, w_2 \rangle| \]
\[ = |A_2 \left( \frac{p_1}{q_1} - \frac{p_2}{q_2} \right) + B_2 \left( \frac{r_1}{q_1} - \frac{r_2}{q_2} \right)| \]
\[ \leq |A_2| \left( \frac{c}{q_1^{1+\tau}} + \frac{c}{q_2^{1+\tau}} + l R^{-n+k} \right) \]
\[ + |B_2| \left( \frac{c}{q_1^{1+\tau}} + \frac{c}{q_2^{1+\tau}} + l R^{-n+k} \right) \] (by (3.5) and (3.11))
\[ \leq q_2 \left( \frac{c}{q_1^{1+\tau}} + \frac{c}{q_2^{1+\tau}} \right) + q_2 \left( \frac{c}{q_1^{1+\tau}} + \frac{c}{q_2^{1+\tau}} \right) \]
\[ + 2 \max\{ |A_2|, |B_2| \} l R^{-n+k} \] (by (3.1))
\[ = c_{q_1}^{-1} \left( \frac{q_2^2}{q_1^2} + \frac{q_1}{q_2} + \frac{q_2}{q_1} + \frac{q_1}{q_2} \right) + 2q_2^{-1} H(P_2) l R^{-n+k} \] (by (4.2))
\[ \leq 4c_{q_1}^{-1} R^{\lambda_k} + 12c_{q_2}^{-1} R^{k+1}. \] (by (4.3), (4.4) and (8.8))
This proves (4.5).

We now prove the lemma by considering two cases.

---

1In fact, it is easy to show that $\mathcal{P}_{n,k} = \emptyset$ for $k \geq \frac{4}{2(1+\tau)n}$. But for simplicity, we prefer to use the range of $k$ as $1 \leq k \leq n$. 

---
Case 1. Suppose \( k = 1 \). In this case, it follows from (4.5), (4.3) and (3.4) that
\[
q_1|\langle v_1, w_2 \rangle| \leq 4cR^{10} + 12c\frac{q_1}{q_2}R^2 \leq 16cR^{12} < 1.
\]

Note that \( q_1|\langle v_1, w_2 \rangle| \) is a nonnegative integer. Thus \( q_1|\langle v_1, w_2 \rangle| = 0 \). This implies that \( L_{P_1} \) passes through \( P_1 \), hence is the line passing through \( P_1 \) and \( P_2 \). Similarly, \( L_{P_1} \) is the line passing through \( P_1 \) and \( P_2 \). Hence \( L_{P_1} = L_{P_2} \). This proves the \( k = 1 \) case of the lemma.

Case 2. Suppose \( k \geq 2 \). It follows from (4.1) that \( s \leq \frac{1}{4} \). Thus, by (4.4), we have
\[
|A_i| \leq q_i^s \leq H_n^{-\frac{1}{1+\tau}}R^{k+4}. \tag{4.6}
\]

On the other hand, it follows from (4.4) and (3.8) that
\[
\max\{|A_i|, |B_i|\} = q_i^{-1}H(P_i) \leq H_n^{-\frac{1}{1+\tau}}R^{-2k-6} \cdot H_{n+1}^{-\frac{1}{1+\tau}}R^{-2k-5}. \tag{4.7}
\]

Consider the cross product
\[
(q_0, \tilde{r}_0, \tilde{q}_0) := w_1 \times w_2. \tag{4.8}
\]

By the triple cross product expansion, we have
\[
v_1 \times (w_1 \times w_2) = \langle v_1, w_2 \rangle w_1.
\]

Comparing the first two components of the vectors on both sides, we obtain
\[
\tilde{q}_0 \frac{p_1}{q_1} - \tilde{r}_0 = \langle v_1, w_2 \rangle A_1, \tag{4.9}
\]
\[
\tilde{q}_0 \frac{p_1}{q_1} - \tilde{p}_0 = -\langle v_1, w_2 \rangle B_1. \tag{4.10}
\]

Note that by (4.5) and (4.4), we have
\[
|\langle v_1, w_2 \rangle| \leq 4c q_1^{-1}R^2 + 12c q_2^{-1}R^{k+1} \leq 16c H_n^{-\frac{1}{1+\tau}}R^{-k-5}. \tag{4.11}
\]

We now prove that \( L_{P_1} = L_{P_2} \) by contradiction. Suppose the contrary. Then \( w_1 \times w_2 \) is a nonzero vector. We first consider the case where \( \tilde{q}_0 = 0 \), that is, \( L_{P_1} \) is parallel to \( L_{P_2} \). In this case, it follows from \( w_1 \times w_2 \neq 0 \) that \( \max\{|\tilde{p}_0|, |\tilde{r}_0|\} \geq 1 \). On the other hand, we have
\[
\max\{|\tilde{p}_0|, |\tilde{r}_0|\} = |\langle v_1, w_2 \rangle| \max\{|A_i|, |B_i|\} \text{ (by (4.9) and (4.10))}
\]
\[
\leq |\langle v_1, w_2 \rangle| \max\{|A_i|, |B_i|\} \frac{1}{s}
\]
\[
\leq 16c H_n^{-\frac{1}{1+\tau}}R^{-k-5} \cdot H_n^{-\frac{1}{1+\tau}}R^{-2k-5} \quad \text{ (by (4.11) and (4.7))}
\]
\[
\leq 16c < 1. \quad \text{ (by (3.4))}
\]

This is a contradiction.

Next, suppose that \( \tilde{q}_0 \neq 0 \), that is, \( L_{P_1} \) is not parallel to \( L_{P_2} \). Let \( P_0 = (p_0, q_0, r_0) \) be the intersection point of \( L_{P_1} \) and \( L_{P_2} \), where \( q_0 > 0 \) and the integers \( p_0, q_0, r_0 \) are coprime. Suppose \( \Delta(P_1) \subset \Delta(P_0) \). Firstly, note that the vector \( (\tilde{p}_0, \tilde{r}_0, \tilde{q}_0) \) is a nonzero integer multiple of \( (p_0, r_0, q_0) \). Thus
\[
q_0 \leq |\tilde{q}_0| = |A_1 B_2 - A_2 B_1| \quad \text{ (by (4.8))}
\]
\[
\leq |A_1 B_2| + |A_2 B_1| \quad \text{ (by (4.8))}
\]
\[
\leq 2H_n^{\frac{1}{1+\tau}}R^{k+4} \cdot H_n^{\frac{1}{1+\tau}}R^{-2k-5} \quad \text{ (by (4.6) and (4.7))}
\]
\[
= 2H_n^{\frac{1}{1+\tau}}R^{-k-1}. \tag{4.12}
\]
Suppose \((x, y) \in \Delta(P_1)\). In view of the fact that \(R = (\alpha_0\beta)^{-1} > 24\sqrt{2}\), it follows that
\[
q_0^{1+s}\left|x - \frac{p_0}{q_0}\right| \leq q_0^s \left|\frac{p_1}{q_1} - \frac{p_0}{q_0}\right| + q_0^{1+s}\left|x - \frac{p_1}{q_1}\right|
\]
\[
\leq q_0^s\langle v_1, w_2 \rangle |B_1| + q_0^{1+s} \frac{c}{q_1^{1+s}} \quad \text{(by (4.10) and (3.5))}
\]
\[
\leq 2H_n^{\frac{1}{s}} R^{-s(k+1)} \cdot 16cH_n^{\frac{1}{s}} R^{-k-5} \cdot H_n^{\frac{1}{s}} R^{-2k-5}
\]
\[
+ 4H_n^{\frac{1}{s}} R^{-(1+s)(k+1)} \cdot cH_n^{\frac{1}{s}} R^{-(1+s)(2k+6)} \quad \text{(by (4.12), (4.11), (4.7) and (4.4))}
\]
\[
\leq 36cR^{-2} \leq c
\]
and
\[
q_0^{1+t}\left|y - \frac{r_0}{q_0}\right| \leq q_0^t \left|\frac{r_1}{q_1} - \frac{r_0}{q_0}\right| + q_0^{1+t}\left|y - \frac{r_1}{q_1}\right|
\]
\[
\leq q_0^t\langle v_1, w_2 \rangle |A_1| + q_0^{1+t} \frac{c}{q_1^{1+t}} \quad \text{(by (4.9) and (3.5))}
\]
\[
\leq 2H_n^{\frac{1}{t}} R^{-t(k+1)} \cdot 16cH_n^{\frac{1}{t}} R^{-k-5} \cdot H_n^{\frac{1}{t}} R^{k+4}
\]
\[
+ 4H_n^{\frac{1}{t}} R^{-(1+t)(k+1)} \cdot cH_n^{\frac{1}{t}} R^{-(1+t)(2k+6)} \quad \text{(by (4.12), (4.11), (4.6) and (4.4))}
\]
\[
\leq 36cR^{-2} \leq c
\]

Thus \((x, y) \in \Delta(P_1)\). This proves \(\Delta(P_1) \subset \Delta(P_0)\).

Let \(n_0 \geq 1\) be the unique integer such that \(P_0 \in \mathcal{P}_{n_0}\). We claim that
\[
n_0 \geq n - k + 1. \quad (4.13)
\]
In fact, if \(n_0 \leq n - k\), then \(S_{n_0}\) contains an ancestor \(\tau'\) of \(\tau\). By (3.12), we have
\[
\Phi(\tau) \cap \Delta(P_1) \subset \Phi(\tau') \cap \Delta(P_0) = \emptyset.
\]
This contradicts \(P_1 \in \mathcal{P}_{n,k}(\tau)\). In view of (4.2) and (4.13), we have
\[
q_0 \geq H_n^{\frac{1}{s}} \geq H_{n-k+1}^{\frac{1}{s}} = H_n^{\frac{1}{s}} R^{-\frac{k+1}{s}}.
\]
This contradicts (4.12). Thus the proof of Lemma 4.1 is completed. \(\square\)

Let \(w > 0\). By a strip of width \(w\), we mean a subset of \(\mathbb{R}^2\) of the form
\[
\mathcal{L} = \{ x \in \mathbb{R}^2 : |x \cdot u - a| \leq w/2 \},
\]
where \(u \in \mathbb{R}^2\) is a unit vector, the dot denotes the standard inner product, and \(a \in \mathbb{R}\).

Lemma 4.1 implies the following statement.

**Corollary 4.2.** For any \(n \geq 1\), \(1 \leq k \leq n\) and \(\tau \in S_{n-k}\), there exists a strip of width \(\frac{2}{3} R^{-n}\) which contains all the rectangles \(\{ \Delta(P) : P \in \mathcal{P}_{n,k}(\tau) \}\).

**Proof.** By Lemma 4.1, there exists \((A, B, C) \in \mathbb{Z}^3\) with \((A, B) \neq (0,0)\) such that for any \(P = (\frac{p}{q}, \frac{r}{q}) \in \mathcal{P}_{n,k}(\tau)\), we have
\[
|A| \leq q^s, \quad |B| \leq q^t, \quad Ap + Br + Cq = 0, \quad q \max\{|A|, |B|\} \geq H_n.
\]
For such \(P\), if \((x, y) \in \Delta(P)\), then
\[
|Ax + By + C| = \left|A \left(x - \frac{p}{q}\right) + B \left(y - \frac{r}{q}\right)\right|
\]
\[
\leq |A| \left|x - \frac{p}{q}\right| + |B| \left|y - \frac{r}{q}\right| \leq q^s \frac{c}{q^{1+s}} + q^t \frac{c}{q^{1+t}} = \frac{2c}{q}.
\]
Thus it follows from (3.7) that
\[
|Ax + By + C| \leq \frac{2c}{q \max\{|A|, |B|\}} \leq \frac{2c}{\frac{H}{n\frac{1}{3}}} = \frac{1}{3}lR^{-n}.
\]
This implies that \(\Delta(P)\) is contained in the strip
\[
\left\{(x, y) \in \mathbb{R}^2 : \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}} \leq \frac{1}{3}lR^{-n}\right\},
\]
which has width \(\frac{2}{3}lR^{-n}\).

The following lemma gives an upper bound for the number of certain squares which intersect a thin strip.

**Lemma 4.3.** Let \(\mathcal{R} \subset \mathcal{T}\) be a subtree of type (II), let \(n \geq 1\), and let \(\mathcal{L}\) be a strip of width \(\frac{2}{3}lR^{-n}\). Then for any \(1 \leq k \leq n\) and \(\tau \in \mathcal{R}_{n-k}\), we have
\[
\#\{\tau' \in \mathcal{R}(\tau) : \mathcal{P}(\tau') \cap \mathcal{L} \neq \emptyset\} \leq (3m - 2)^k.
\]

**Proof.** For \(\mathcal{V} \subset \mathcal{R}\), we denote \(\mathcal{V}^\mathcal{L} = \{\tau' \in \mathcal{V} : \mathcal{P}(\tau') \cap \mathcal{L} \neq \emptyset\}\). We prove the lemma by showing that
\[
\#\mathcal{R}(\tau)^\mathcal{L}_{k'} \leq (3m - 2)^{k'} \text{, } \forall k' \in \{0, \ldots, k\}.
\]
(4.14)
Firstly, we note that if \(0 \leq n' \leq n - 1\) and \(\tau' \in \mathcal{R}_{n'}\), then
\[
\#\mathcal{R}_{\text{suc}}(\tau')^\mathcal{L} \leq 3m - 2.
\]
(4.15)
In fact, since the \(m^2\) squares \(\{\Phi(\tau'') : \tau'' \in \mathcal{R}_{\text{suc}}(\tau')\}\) have side lengths \(lR^{-n'-1}\), and their union is a square of side length \(mlR^{-n'-1}\), it is easy to see that a strip of width less than \(\frac{2}{3}lR^{-n'-1}\) intersects at most \(3m - 2\) squares \(\Phi(\tau'')\). We now prove (4.14) by induction on \(k'\). If \(k' = 0\), there is nothing to prove. Suppose that \(1 \leq k' \leq k\) and (4.14) holds if \(k'\) is replaced by \(k' - 1\). In view of
\[
\mathcal{R}(\tau)^\mathcal{L}_{k'} = \bigcup_{\tau' \in \mathcal{R}(\tau)^\mathcal{L}_{k'-1}} \mathcal{R}_{\text{suc}}(\tau')^\mathcal{L},
\]
it follows from (4.15) and the induction hypothesis that
\[
\#\mathcal{R}(\tau)^\mathcal{L}_{k'} = \sum_{\tau' \in \mathcal{R}(\tau)^\mathcal{L}_{k'-1}} \#\mathcal{R}_{\text{suc}}(\tau')^\mathcal{L} \leq (3m - 2)\#\mathcal{R}(\tau)^\mathcal{L}_{k'-1} \leq (3m - 2)^{k'}.
\]
This proves (4.14). 

Combining Corollary 4.2 and Lemma 4.3, we obtain

**Corollary 4.4.** Let \(\mathcal{R} \subset \mathcal{T}\) be a subtree of type (II). Then for any \(n \geq 1\), \(1 \leq k \leq n\) and \(\tau \in \mathcal{S}_{n-k} \cap \mathcal{R}_{n-k}\), we have
\[
\#\{\tau' \in \mathcal{R}(\tau)_k : \mathcal{P}(\tau') \cap \bigcup_{P \in \mathcal{P}_{n-k}(\tau)} \Delta(P) \neq \emptyset\} \leq (3m - 2)^k.
\]

We now prove Proposition 3.3 using Proposition 2.2 and Corollary 4.4.

**Proof of Proposition 3.3.** In view of Proposition 2.2, it suffices to prove that the intersection of \(\mathcal{S}\) with every subtree of type (II) is infinite. Let \(\mathcal{R} \subset \mathcal{T}\) be a subtree of type (II), and denote \(a_n = \#\mathcal{S}_n \cap \mathcal{R}_n\). Then \(a_0 = 1\). We prove the infinity of \(\mathcal{S} \cap \mathcal{R}\) by showing that for any \(n \geq 1\),
\[
a_n > 88a_{n-1}.
\]
(4.16)
It is easy to see from (3.12) that $R_{\text{suc}}(S_{n-1} \cap R_{n-1})$ is the disjoint union of $S_n \cap R_n$ and 

$$U_n := \{ \tau \in R_{\text{suc}}(S_{n-1} \cap R_{n-1}) : \Phi(\tau) \cap \bigcup_{P \in \mathcal{P}_n} \Delta(P) \neq \emptyset \}.$$ 

Thus

$$a_n = \# R_{\text{suc}}(S_{n-1} \cap R_{n-1}) - \# U_n = m^2 a_{n-1} - \# U_n. \quad (4.17)$$

But

$$U_n = \bigcup_{k=1}^n \{ \tau' \in R_{\text{suc}}(S_{n-1} \cap R_{n-1}) : \Phi(\tau') \cap \bigcup_{P \in \mathcal{P}_{n,k}} \Delta(P) \neq \emptyset \}$$

$$\subset \bigcup_{k=1}^n \bigcup_{\tau \in S_{n-k} \cap R_{n-k}} \{ \tau' \in R(\tau) : \Phi(\tau') \cap \bigcup_{P \in \mathcal{P}_{n,k}(\tau)} \Delta(P) \neq \emptyset \}.$$ 

Thus it follows from Corollary 4.4 that

$$\# U_n \leq \sum_{k=1}^n (3m - 2)^k a_{n-k}. \quad (4.18)$$

From (4.17), (4.18) and (3.10), we obtain

$$a_n \geq m^2 a_{n-1} - \sum_{k=1}^n (3m - 2)^k a_{n-k} = 144 a_{n-1} - \sum_{k=1}^n 34^k a_{n-k}. \quad (4.19)$$

By letting $n = 1$ in (4.19), we see that $a_1 \geq 110$. So (4.16) holds for $n = 1$. Assume $n \geq 2$ and (4.16) holds if $n$ is replaced by $1, \ldots, n-1$. Then for any $1 \leq k \leq n$, we have

$$a_{n-k} \leq 88^{-k+1} a_{n-1}.$$ 

Substituting this into (4.19), we obtain

$$a_n \geq \left( 144 - 88 \sum_{k=1}^n (34/88)^k \right) a_{n-1} > 88a_{n-1}.$$ 

This proves (4.16).  

\section*{Acknowledgments}

The author would like to thank Dmitry Kleinbock for helpful comments on an early version of this paper. He is also grateful to Dzmitry Badziahin, Nikolay Moshchevitin, Andrew Pollington, Sanju Velani and Barak Weiss for valuable conversations.

\section*{References}

[1] J. An, Badziahin-Pollington-Velani’s theorem and Schmidt’s game, Bull. Lond. Math. Soc. 45 (2013), no. 4, 721–733.

[2] D. Badziahin, A. Pollington, S. Velani, On a problem in simultaneous Diophantine approximation: Schmidt’s conjecture, Ann. of Math. (2) 174 (2011), no. 3, 1837–1883.

[3] R. Broderick, L. Fishman, D. Kleinbock, A. Reich, B. Weiss, The set of badly approximable vectors is strongly $C^1$ incompressible, Math. Proc. Cambridge Philos. Soc. 153 (2012), no. 2, 319–339.

[4] S. G. Dani, Divergent trajectories of flows on homogeneous spaces and Diophantine approximation, J. Reine Angew. Math. 359 (1985), 55–89.

[5] S. G. Dani, On badly approximable numbers, Schmidt games and bounded orbits of flows, in “Number theory and dynamical systems (York, 1987)”, London Math. Soc. Lecture Note Ser. 134, Cambridge Univ. Press, Cambridge, 1989, pp. 69–86.

[6] R. Diestel, Graph theory, 4th ed., Springer, Heidelberg, 2010.

[7] M. Einsiedler, A. Katok, E. Lindenstrauss, Invariant measures and the set of exceptions to Littlewood’s conjecture, Ann. of Math. (2) 164 (2006), no. 2, 513–560.
[8] A. Gorodnik, *Open problems in dynamics and related fields*, J. Mod. Dyn. 1 (2007), no. 1, 1–35.

[9] D. Kleinbock, *Flows on homogeneous spaces and Diophantine properties of matrices*, Duke Math. J. 95 (1998), no. 1, 107–124.

[10] D. Kleinbock, G. A. Margulis, *Bounded orbits of nonquasiunipotent flows on homogeneous spaces*, Amer. Math. Soc. Transl. 171 (1996), 141–172.

[11] D. Kleinbock, B. Weiss, *Dirichlet's theorem on Diophantine approximation and homogeneous flows* J. Mod. Dyn. 2 (2008), no. 1, 43–62.

[12] D. Kleinbock, B. Weiss, *Modified Schmidt games and Diophantine approximation with weights*, Adv. Math. 223 (2010), no. 4, 1276–1298.

[13] D. Kleinbock, B. Weiss, *Modified Schmidt games and a conjecture of Margulis*, J. Mod. Dyn. 7 (2013), no. 3, 429–460.

[14] C. T. McMullen, *Winning sets, quasiconformal maps and Diophantine approximation*, Geom. Funct. Anal. 20 (2010), no. 3, 726–740.

[15] N. Moshchevitin, *On some open problems in Diophantine approximation*, preprint, arXiv:1202.4539

[16] E. Nesharim, D. Simmons, *Bad\((s,t)\) is hyperplane absolute winning*, preprint, arXiv:1307.5037

[17] A. Pollington, S. Velani, *On simultaneously badly approximable numbers*, J. London Math. Soc. (2) 66 (2002), no. 1, 29–40.

[18] W. M. Schmidt, *On badly approximable numbers and certain games*, Trans. Amer. Math. Soc. 123 (1966), 178–199.

[19] W. M. Schmidt, *Badly approximable systems of linear forms*, J. Number Theory 1 (1969), 139–154.

[20] W. M. Schmidt, *Diophantine approximation*, Lecture Notes in Mathematics 785, Springer, Berlin, 1980.

[21] W. M. Schmidt, *Open problems in Diophantine approximation*, in “Diophantine approximations and transcendental numbers (Luminy, 1982)”, Progr. Math. 31, Birkhäuser, Boston, 1983, pp. 271–287.

LMAM, SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING, 100871, CHINA

E-mail address: anjinpeng@gmail.com