Constraints and Reality Conditions in the Ashtekar Formulation of General Relativity

Gen Yoneda† and Hisaaki Shinkai‡
† Department of Mathematics, Waseda University,
Okubo 3-4-1, Shinjuku, Tokyo 169, Japan
‡ Department of Physics, Waseda University,
Okubo 3-4-1, Shinjuku, Tokyo 169, Japan

Abstract

We show how to treat the constraints and reality conditions in the $SO(3)$-ADM (Ashtekar) formulation of general relativity, for the case of a vacuum spacetime with a cosmological constant. We clarify the difference between the reality conditions on the metric and on the triad. Assuming the triad reality condition, we find a new variable, allowing us to solve the gauge constraint equations and the reality conditions simultaneously.

Key Words: General Relativity, Connection Formulation

To appear in Classical and Quantum Gravity

†Electronic address: yoneda@cfi.waseda.ac.jp
‡Electronic address: shinkai@cfi.waseda.ac.jp
1 Introduction

The $SO(3)$-ADM (Ashtekar) formalism of general relativity\[1\] has many advantages in the treatment of gravity. In this formulation, the constraint equations are classified as the Hamiltonian constraint equation, $C_H$, and momentum constraint equations, $C_M$, with a new set of additional gauge constraint equations, $C_G$. These constraints become low-order polynomials and do not contain the inverses of the variables. This formulation enables us to treat a quantum description of gravity nonperturbatively.

In order to apply the Ashtekar formalism in classical general relativity, we need to solve the reality condition for the metric and the extrinsic curvature. In this paper, we show how to treat the constraints and reality conditions, for a vacuum spacetime with/without a cosmological constant. We stand on the point of pursuing the dynamics of spacetime, using evolutions of time-constant slices by fixing gauge (slicing) condition in each time step. After a brief review of the Ashtekar formulation (§2), we clarify the difference between the reality conditions on the metric and on the triad (§3). We show that the latter condition restricts a part of the gauge freedoms.

Assuming the triad reality condition, we find a new variable, which allows us to solve $C_G$ and the reality conditions simultaneously (§4). This technique is motivated by the works of Capovilla, Jacobson and Dell (CDJ) \[3\] and Barbero \[4\]. CDJ discovered that in the vacuum spacetime the introduction of an arbitrary traceless and symmetric $SO(3)$ tensor makes two constraints $C_H$ and $C_M$ trivial, leaving only the third constraint $C_G$ to be solved, while Barbero developed a technique for making $C_H$ and $C_G$ trivial. Our new variable is analogous to both CDJ’s and Barbero’s, but has advantage of clarifying the meaning of the additional constraint $C_G$ in terms of ADM variables.

We use greek letters ($\mu, \nu, \rho, \cdots$), which range over the four spacetime coordinates 0, ..., 3, while uppercase latin letters from the middle of the alphabet ($I, J, K, \cdots$) range over the four internal $SO(1,3)$ indices (0), ..., (3). Lower case latin indices from the middle of the alphabet ($i, j, k, \cdots$) range over the three spatial indices 1, ..., 3, while lower case latin indices from the beginning of the alphabet ($a, b, c, \cdots$) range over the three internal $SO(3)$ indices (1), ..., (3). We use volume forms $\epsilon^{abc}$; $\epsilon_{abc}\epsilon^{abc} = 3!$.

\[\text{§} \]We raise and lower $\mu, \nu, \rho$ by $g^{\mu\nu}$ and $g_{\mu\nu}$ (Lorenzian metric); $I, J, K$ by $\eta^{IJ} = \text{diag}(-1,1,1,1)$ and $\eta_{IJ}$; $i, j, k$ by $\gamma^{ij}$ and $\gamma_{ij}$ (3-metric).
2 Brief review of the Ashtekar formulation

The key feature of Ashtekar’s formulation of general relativity [1] is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write the metric $g_{\mu \nu}$ using the tetrad, $e_I^\mu$, and define its inverse, $g_I^\mu$, by $g_{\mu \nu} = e_I^\mu e_J^\nu \eta_{IJ}$ and $E_I^\mu := e_I^\nu g^{\mu \nu} \eta_{IJ}$. We define a SO(3,C) self-dual connection

$$A_a^i := \omega_{a0}^i - \frac{i}{2} e^a_b \omega_{b0}^{bc}, \quad (1)$$

where $\omega_{IJ}^\mu$ is a spin connection 1-form (Ricci connection), $\omega_{IJ}^\mu := E_I^\nu \nabla_\mu e_J^\nu$. Note that the extrinsic curvature, $K_{ij} := -(\delta_i^l + n_i n^l)\nabla_l n_j$ in the ADM formalism, where $\nabla$ is a covariant derivative on $\Sigma$, satisfies the relation $-K_{ij} E_j^a = \omega_{i0}^a$, when the gauge condition $E_0^a = 0$ is fixed. So $A_a^i$ is also expressed by

$$A_a^i = -K_{ij} E_j^a - \frac{i}{2} e^a_b \omega_{b0}^{bc}. \quad (2)$$

The lapse function, $N$, and shift vector, $N_i$, are expressed as $E_0^a = (1/N, -N_i/N)$. Ashtekar treated the set $(A_a^i, \tilde{E}_a^i)$ as basic dynamical variables, where $\tilde{E}_a^i$ is an inverse of the densitized triad defined by $\tilde{E}_a^i := e E_a^i$, and where $e := \det e^a_i$ is a density. This pair forms the canonical set

$$\{ \tilde{E}_a^i(x), \tilde{E}_b^i(y) \} = 0, \quad (3a)$$
$$\{ A_a^i(x), \tilde{E}_b^i(y) \} = i \delta_a^i \delta_b^i \delta(x-y), \quad (3b)$$
$$\{ A_a^i(x), A_b^j(y) \} = 0. \quad (3c)$$

The Hilbert action takes the form

$$S = \int d^4 x [\dot{A}_a^i \dot{\tilde{E}}_a^i + \frac{i}{2} N \dot{\tilde{E}}_a^i \dot{\tilde{E}}_b^j F_a^i e^{ab}_c - 2\Lambda N \det \tilde{E} - N^i F_{ij}^a \tilde{E}_a^j + A_0^a D_i \tilde{E}_a^i], \quad (4)$$

where $\tilde{N} := e^{-1} N$, $\Lambda$ is cosmological constant, $D_i \tilde{E}_a^i := \partial_i \tilde{E}_a^i - i e_{ab}^c A_b^i \tilde{E}_c^i$, and where $F_a^i$ is curvature 2-form, defined as $F_a^i := \partial_a A_a^i - \partial_0 A_a^i - \frac{i}{2} e^a_b \omega^i_{b0} (A^b \wedge A^c)_{i \mu \nu}$, and $\det \tilde{E}$ is defined to be $\det \tilde{E} = \frac{1}{6} e^{a b c} \xi_{i j k} \tilde{E}_a^i \tilde{E}_b^j \tilde{E}_c^k$, where $\epsilon_{i j k} := \epsilon_{a b c} e^a_i e^b_j e^c_k$ and $\xi_{i j k} := e^{-1} \epsilon_{i j k}$.

\[\epsilon_{x y z} = e, \quad \xi_{x y z} = 1, \quad \epsilon_{x y z} = e^{-1}, \quad \xi_{x y z} = 1.\]
Varying the action with respect to the non-dynamical variables \( N, N^i \) and \( A^a_0 \) yields the constraint equations,

\[
C_H = \frac{\partial L}{\partial N} = \frac{i}{2} \varepsilon^{abc} \tilde{E}^i_a \tilde{E}^j_b F^c_{ij} - 2\Lambda \det \tilde{E} \approx 0,
\]

\( (5a) \)

\[
C_{Mi} = \frac{\partial L}{\partial N^i} = -F^a_{ji} \tilde{E}^j_i \approx 0,
\]

\( (5b) \)

\[
C_{Ga} = \frac{\partial L}{\partial A^a_0} = \mathcal{D}_i \tilde{E}^i_a \approx 0.
\]

\( (5c) \)

The equations of motion for the dynamical variables (\( A^a_i \) and \( \tilde{E}^a_i \)) are

\[
\dot{A}^a_i = -i\varepsilon^{abc} N \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_j A^a_0 + 2\varepsilon \Lambda N c^a_i,
\]

\( (6a) \)

\[
\dot{\tilde{E}}^a_i = -i\mathcal{D}_j (\varepsilon^{cb} N \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j (N^j \tilde{E}^a_i) + i A^b_0 \varepsilon^{ab} \tilde{E}^c_i,
\]

\( (6b) \)

where \( \mathcal{D}_j T^a_{ji} := \partial_j T^a_{ji} - i\varepsilon^{a} b c \mathcal{A}^b_j T^c_{ji} \), for \( T^a_{ij} + T^a_{ji} = 0 \).

### 3 Reality conditions

To ensure the metric is real-valued, we need to impose two conditions; the first is that the doubly densitized contravariant metric \( \tilde{\gamma}^{ij} := \varepsilon^2 \gamma^{ij} \) is real,

\[
\Re (\tilde{E}_a^i \tilde{E}^a_j) = 0, \quad \text{metric reality condition} \quad (7a)
\]

and the second condition is that the time derivative of \( \tilde{\gamma}^{ij} \) is real,

\[
\Re \{ \partial_t (\tilde{E}_a^i \tilde{E}^a_j) \} = 0. \quad \text{second metric reality condition} \quad (7b)
\]

We denote these condition the “metric reality condition” and the “second metric reality condition” (extrinsic curvature reality condition), hereafter. Ashtekar et al. \[2\] discovered that, with the second metric reality condition \( (7b) \), the reality of the 3-metric and extrinsic curvature are automatically preserved under time evolution, as a consequence of the equations of motion. This means we need only solve both reality conditions \( (7a) \) and \( (7b) \) on the initial hypersurface. Immirzi\[3\] found that the reality conditions are consistent with the constraints, making the theory equivalent to Einstein’s.

Using the equations of motion for \( \tilde{E}^a_i \) \[3]\, the gauge constraint \( (5c) \) and the first reality condition \( (7a) \), we can replace the second reality condition \( (7b) \) with a different constraint (see Appendix)

\[
W^{ij} := \Re (\varepsilon^{abc} \tilde{E}^k_a \tilde{E}^i_b \mathcal{D}_k \tilde{E}^j_c) \approx 0,
\]

\( (8) \)
which fixes six components of $A^a_i$ and $\tilde{E}^i_a$. Moreover, in order to recover the original lapse function $N := \tilde{N}e$, we demand $\Im(N/e) = 0$, i.e. the density $e$ be real and positive. This requires that $e^2$ be positive, i.e.

$$\det \tilde{E} > 0.$$  \hfill (9a)

Note that the metric reality conditions only guarantee the reality of $e^4$. The secondary condition of (9a),

$$\Im[\partial_i(\det \tilde{E})] = 0,$$  \hfill (9b)

is automatically satisfied as a consequence of the equations of motion for $\tilde{E}^i_a$ (6b), the gauge constraint (5c), the metric reality conditions (7a), (7b) and the first condition (9a) (see Appendix). Therefore, in order to ensure that $e$ is real, we only require (9a).

Note that this condition does not remove any degrees of freedom for the variables and is analogous to making the implicit assumption of $\det \gamma_{ij} > 0$ in the ADM formulation.

We now show that rather stronger reality conditions are useful in Ashtekar’s formalism for recovering the real 3-metric and extrinsic curvature. These conditions are

$$\Im(\tilde{E}^i_a) = 0 \quad \text{first triad reality condition}$$  \hfill (10a)

and $\Im(\dot{\tilde{E}}^i_a) = 0$, \textit{second triad reality condition} (10b)

and we denote them the “first triad reality condition” and the “second triad reality condition”, hereafter. Using the equations of motion of $\tilde{E}^i_a$, the gauge constraint (5c), the metric reality conditions (7a), (7b) and the first condition (10a), we see (in Appendix) that (10b) is equivalent to

$$\Re(A^a_0) = \partial_i(N)\tilde{E}^a_i + \frac{1}{2}e^{-1}e^b N\tilde{E}^ja\partial_j\tilde{E}^i_b + N^{ij}\Re(A^a_i).$$  \hfill (11)

From this expression we see that the second triad reality condition restricts the three components of “triad lapse” vector $A^a_i$. Therefore (11) is not a restriction on the dynamical variables ($A^a_i$ and $\tilde{E}^i_a$) but on a part of slicing, which we should impose on each hypersurface. Thus the second triad reality condition does not restrict the dynamical variables any further than the second metric condition does.

\[\text{This “triad lapse” is named by A. Ashtekar in private communication.}\]
4 Solving the constraint equations

The equations we need to solve for $A^a_i$ and $\tilde{E}^i_a$ are the constraints (5a), (5b), (5c) and the reality conditions (8), (10a). CDJ solved $C_H$ and $C_M$ by introducing new variables. These reduced the 36 (real) independent components of $A^a_i$ and $\tilde{E}^i_a$ to 28, or in CDJ’s variables the 18 (real) independent components of $\psi_{ab}$ are reduced to 10 (a symmetric and traceless tensor), which corresponds to Weyl curvature $\Psi_i$. These are again restricted by $C_G$ and the reality condition.

In contrast to CDJ’s method, we make an alternative treatment of the gauge constraint (5c) and the second metric reality condition (8). For convenience, we assume that $\tilde{E}^i_a$ is real. This assumption restricts our choice of triad, but this constraint is not difficult to satisfy. We introduce the connection with double internal indices (note that here we do not use the densitized triad),

$$A_{ab} := A^a_i E^i_b,$$

and express all the constraints with $(A_{ab}, \tilde{E}^i_a)$ as the basic pair of variables. The real part of $C_G$ gives

$$\Re(C_G) = \partial_i \tilde{E}^{ia} + \epsilon^a_b c \Im(A^b_i) \tilde{E}^i_c = \partial_i \tilde{E}^{ia} + \epsilon^a_b \Re(A^{bc}) \Im(A^{bc}) = 0,$$

where $e = \sqrt{\det \tilde{E}}$. Thus the imaginary and anti-symmetric part of $A^{ab}$ is determined from

$$\Im(A^{[ab]}) = -\frac{1}{2e} \epsilon^{abc} \partial_i \tilde{E}^i_c.$$

(13)

The imaginary part of $C_G$ gives

$$\Im(C_G) = -\epsilon^a_b c \Re(A^b_i) \tilde{E}^i_c = -\epsilon^a_b \Re(A^{bc}) \Im(A^{bc}) = 0.$$

Thus the real and anti-symmetric part of $A^{ab}$ is

$$\Re(A^{[ab]}) = 0.$$

(14)

Thus we have confirmed that the 6 real constraints of $C_G$ are automatically satisfied if we impose (13) and (14).

Next the second metric reality condition (8) becomes

$$W^{ij} = \frac{1}{2} [\epsilon^{abe} \tilde{E}^k_a \tilde{E}^i_b \partial_k \tilde{E}^j_c + \epsilon^{abe} \tilde{E}^k_a \tilde{E}^j_b \partial_k \tilde{E}^i_c + 2 \tilde{E}^i_a \tilde{E}^j_b \Im(A^{[ab]}) - 2 \tilde{E}^i_a \tilde{E}^j_b \Im(A^{[ab]})] = 0.$$
Thus the imaginary and symmetric part of $A^{ab}$ is

$$\Im(A^{ab}) = \frac{1}{2}[E^j_d(e^{dca}e^b_i + e^{dbc}e^a_i)\partial_j\tilde{E}^i_c - \delta^{ab}E^j_d e^c_i\epsilon_i^j\partial_j\tilde{E}^i_c],$$

(15)

where $\epsilon_i^a$ is the inverse of $\tilde{E}^i_a$. From these expressions, we see that (5c) and (8) are satisfied if and only if $A^{ab}$ satisfies (13), (14) and (15), after assuming (10a). We note that the imaginary part of $A^{ab}$ consists of the triad and its spatial differential and that the real part of $A^{ab}$ is symmetric.

These results become clearer if we compare $\Re(A^{ab})$ and the extrinsic curvature $K_{ij}$ through the definition of $A^a_i$, (2). From (2) we derive

$$\Re(A^{ab}) = -K_{ij}E^i_aE^j_b,$$

(16a)

$$\Im(A^{ab}) = -\frac{1}{2}\epsilon^{a}{}_{cd}\omega_{i}{}^{cd}E^{i}_b.$$

(16b)

Since the extrinsic curvature is symmetric, we see $\Re(A^{ab})$ is also symmetric [(16a)]. After some calculation, we can see that (16b) is equivalent to (13) and (15). Moreover, we find from (2) that

$$K_{ij} = -A^a_i e^a_j - \frac{i}{2}\epsilon_{abc}\omega_{i}{}^{bc}e^a_j,$$

so the reality of the extrinsic curvature, $\Im(K_{ij}) = 0$, is equivalent to (16a). Consequently, $\Re(C_G) = 0$ [(13)] and $W^{ij} = 0$ [(13)] indicates that the extrinsic curvature is real and $\Im(C_G) = 0$ [(14)] indicates that the extrinsic curvature is symmetric.

When one has solved the 12 equations $C_G = 0$ and $W^{ij} = 0$ for the 27 variables $A^a_i$ (complex) and $\tilde{E}^i_a$ (real), 15 degrees of freedom remain. Introducing $A^{ab}$ clarifies this remaining freedom; these are 6 degrees of freedom for $\Re(A^{ab})$ and 9 for $\tilde{E}^i_a$. Our task is now reduced to solving the other constraints (5a) and (5b) for the variables $\Re(A^{ab})$ and $\tilde{E}^i_a$.

In terms of $\Re(A^{ab})$ and $\tilde{E}^i_a$, the constraints are given by substituting (13), (14) and (15) into (5a) and (5b). Then we see $\Im(C_H) = 0$ and $\Im(C_M) = 0$ are automatically satisfied [(3)], thus the equations which we need to solve are just four equations:

$$\Re(C_H) = e[\epsilon_{ab}E^i_c(\partial_i I^{ab}) + \frac{1}{2}\epsilon_{ab}E^i_c\partial_i E^j_d I^{ab} + \epsilon_{cd}E^i_c\partial_i E^j_d I^{ab} + \frac{1}{2}(R^2 - I^2 - R^{ab}R_{ba} + I^{ab}I_{ba}) - 2\Lambda] \approx 0,$$

(17a)

$$\Re(C_Mi) = e[-\partial_i R + \epsilon_{ia}E^j_d\partial_j R^{ka} - \frac{1}{2}\epsilon_{d}^{j}(\partial_i E^j_d)R + \epsilon_{jc}(\partial_i E^j_d)R^{bc}.$$
\[ + \frac{1}{2} \tilde{E}^k_b \varepsilon_{ia} \varepsilon^d_j (\partial_k \tilde{E}^j_a) R^{ba} - \tilde{E}^k_b \varepsilon^a_i \varepsilon_j c (\partial_k \tilde{E}^j a) R^{bc} - \epsilon_{bcd} e R^{ab} I^{cd} \xi_{ia} \approx 0 \quad (17b) \]

where \( R^{ab} = \Re(\mathcal{A}^{ab}) = \Re(\mathcal{A}(ab)), \ I^{ab} = \Im(\mathcal{A}^{ab}), \ R = R^a_a \) and \( I = I^a_a \). These are equivalent to the scalar and vector constraints in ADM formulation.

The Poisson bracket for this pair of the variables becomes

\[
\{ \tilde{E}^i_a(x), \tilde{E}^j_b(y) \} = 0, \quad (18a)
\]
\[
\{ \mathcal{A}^{ab}(x), \tilde{E}^i_a(y) \} = \frac{1}{e} \tilde{E}^{ab}(x) \delta^a_c \delta(x - y), \quad (18b)
\]
\[
\{ \mathcal{A}^{ab}(x), \mathcal{A}^{cd}(y) \} = 0. \quad (18c)
\]

We expect that our variables are convenient for expressing the data on each hypersurface if we impose a reality condition. However, we remark that, like CDJ’s variable, our variables \( (\Re(\mathcal{A}^{ab}), \tilde{E}^i_a) \) are not canonical. Therefore, when we describe the equations of motion of our variables, we transform those of canonical pair [e.g., (13a) and (13b)] into ours. Also note that the formulation is not polynomial. Consequently, our full set of equations consists of four constraint equations \((17a), (17b)\) [together with definitions \((13), (14)\) and \((15)\)], and the equations of motion \((13a), (13b)\).

### 5 Discussion

We have studied the \( SO(3)\)-Ashtekar formulation from the point of pursuing the dynamics of spacetime, using evolutions of time-constant slices. We examined the difference between the reality conditions on the metric and on the triad, and demonstrated that the latter condition restricts a part of the gauge freedoms \( \Re(\mathcal{A}^a_c) \). When we apply this condition in time evolution problems (based on 3+1 decompositions), this restriction of gauge variables must be imposed at every time step. Having assumed the triad reality condition, we find a new variable, allowing us to solve \( \mathcal{C}_G \) and the reality conditions simultaneously. Our variable clarifies the meanings of the additional constraint and the second reality condition, which express the reality and symmetry of the extrinsic curvature.

Let us now compare our variables and CDJ’s. CDJ’s \( \psi_{ab} \) is expressed by the Weyl scalar \( \Psi_i \) in the Newman-Penrose formulation \([3]\), so that \( \psi_{ab} \) has the same true degrees of freedom as the gravitational curvature. While our variable \( \mathcal{A}^{ab} \) is a “connection”, using it instead of \( A^a_i \) shows us the physical significance of the gauge constraints and the second
reality condition. In CDJ’s method, the remaining tasks are to solve the gauge constraints and the reality conditions for $A^a_i$ and $\Psi_{ab}$, note that this is not simple in the CDJ variables. In our method, we have only 4 equations to solve, as opposed to 21 equations in CDJ’s.

A practical application of this variable is expected to arise in numerical relativity. Recently, Salisbury et al [7] proposed the use of CDJ formulation to numerical relativity, in which they expect to improve the boundary conditions for gravitational waves. We are now preparing a new approach to numerical treatment of gravity, by combining our connection formulation together with the ADM, Ashtekar and CDJ formulations to express data on the 3-hypersurfaces. We expect that we can give new procedures in evolving data, fixing slicing conditions and/or including gauge field. Such a formulation and simulations will be presented elsewhere [8].

We thank R. Easther for a careful reading of our manuscript. This work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science, Sports and Culture No. 07854014 and by a Waseda University Grant for Special Research Projects.
Appendix  
Details of the reality conditions

In this appendix, we derive the second metric reality condition (8) and the second triad reality condition (11), and show that \( \Re[\partial_t(\det \tilde{E})] = 0 \) is automatically satisfied.

First we derive the second metric reality condition (8). We start from its original definition (7d):

\[ \Re\{\partial_t(\tilde{E}^a_i \tilde{E}^j_a)\} = 2\Re(\dot{\tilde{E}}^i_a \tilde{E}^j_a) \]

Using the equation of motion (6b) and gauge constraint (5c), we have

\[
\dot{\tilde{E}}^i_a \tilde{E}^j_a = [-i \mathcal{D}_k(\epsilon^{cba} N \tilde{E}_c^k \tilde{E}_b^i) + \mathcal{D}_k(N^k \tilde{E}^{ia}) - \mathcal{D}_k(N^i \tilde{E}^{ka}) - i \mathcal{A}^b_0 \epsilon^{ca} \tilde{E}^{i}_a] \tilde{E}^j_a 
\]

Thus we obtain

\[ \Re(\dot{\tilde{E}}^i_a \tilde{E}^j_a) = -i \epsilon^{cba} N \tilde{E}_c^k \mathcal{D}_k(\tilde{E}_b^i \tilde{E}_a^j) + \partial_k(N^k) \tilde{E}_a^i \tilde{E}^j_a + N^k \mathcal{D}_k(\tilde{E}_a^i \tilde{E}^j_a) - \tilde{E}_a^i \partial_k(N^i) \tilde{E}^j_a. \]

Thus we obtain

\[
\Re(\dot{\tilde{E}}^i_a \tilde{E}^j_a) = -N \epsilon^{cba} \Re[\mathcal{E}^k \mathcal{D}_k(\tilde{E}_b^i \tilde{E}_a^j)],
\]

where we use the metric first reality condition (7a). The vanishing of this gives (8). □

Second, we show that \( \Re[\partial_t(\det \tilde{E})] = 0 \) is automatically satisfied when we assume the first density reality condition \( \det \tilde{E} > 0 \). We have

\[
\partial_t(\det \tilde{E}) = \frac{1}{2} \xi_{ijk} \epsilon^{ade} \dot{\tilde{E}}^j_d \tilde{E}^k_e \dot{\tilde{E}}^i_a
\]

\[
= \frac{1}{2} \xi_{ijk} \epsilon^{ade} \dot{\tilde{E}}^j_d \tilde{E}^k_e [-i \epsilon^{cb}_a \mathcal{D}_l(N^l) \tilde{E}_c^i \tilde{E}_b^i - i \epsilon^{cb}_a N \tilde{E}_c^l \mathcal{D}_l(\tilde{E}_b^i) + \mathcal{D}_l(N^i) \tilde{E}_a^i - N^l \mathcal{D}_l(\tilde{E}_a^i)]
\]

\[ = \frac{1}{2} \xi_{ijk} \epsilon^{ade} \dot{\tilde{E}}^j_d \tilde{E}^k_e [-i \epsilon^{cb}_a N \tilde{E}_c^l \mathcal{D}_l(\tilde{E}_b^i) + \partial_l(N^i) \tilde{E}_a^i + N^l \mathcal{D}_l(\tilde{E}_a^i) - \partial_l(N^i) \tilde{E}_a^i] \]

where we use the gauge constraint again. The first term becomes

\[
-i \frac{1}{2} \xi_{ijk} \epsilon^{ade} \dot{\tilde{E}}^j_d \tilde{E}^k_e \epsilon^{cb}_a N \tilde{E}_c^l \mathcal{D}_l(\tilde{E}_b^i) = -\frac{i}{\epsilon} \xi_{ijk} \tilde{E}^j_c \tilde{E}^{kb} N \tilde{E}_c^l \mathcal{D}_l(\tilde{E}_b^i)
\]

\[ = i \epsilon N \xi_{ik} \tilde{E}^{kb} \mathcal{D}_l \tilde{E}_b^i = i \epsilon N \xi_{ik} \tilde{E}^{kb} \mathcal{D}_l \tilde{E}_b^i. \]
Now we have

\[ W^{ij} \gamma_{ij} = e^{abc} \Re(\tilde{E}_a^k \tilde{E}_b^l D_k \tilde{E}_c^j) e_i^d e_j^d = e \epsilon_a^c \epsilon_b^d \Re(\tilde{E}_a^k \tilde{E}_b^l D_k \tilde{E}_c^j) = e \epsilon_a^c \epsilon_b^d \Re(\tilde{E}_e^i \epsilon_i^c D_l \tilde{E}_b^l). \]

Since this vanishes by the second metric reality condition, we see that the imaginary part of the first term is zero. Thus we have

\[ \Im[\partial_t(\det \tilde{E})] = \Im \left\{ \frac{1}{2} \epsilon_{ijk} \epsilon^{ade} \tilde{E}_d^k \tilde{E}_e^i \left[ \partial_t (N^l) \tilde{E}_a^l + N^l \partial_t (\tilde{E}_a^l) - \partial_t (N^a) \tilde{E}_a^l \right] \right\} = \Im \left\{ 3 \epsilon^2 \partial_t (N^i) + N^i \partial_t (e^2) - e^2 \partial_t (N^l) \right\} = 0, \]

where we use the assumption \( \det \tilde{E} = e^2 > 0 \). Thus the second condition is automatically satisfied. □

Next we show the second triad reality condition is written in the form of (11) when we assume the second metric reality condition (7b) and the first triad reality condition \( \Im(\tilde{E}_a^i) = 0 \). Since \( \tilde{E}_a^i \) is non-degenerate, there exists \( P_{ab} \) such that

\[ \Im(\tilde{E}_a^i) = P_{ab} \tilde{E}^{ib}. \]

Using the metric second reality and the first triad reality, we have

\[ 0 = \Im(\tilde{E}_a^{(i)} \tilde{E}_b^{(j)}) = P_{ab} \tilde{E}^{(i)} \tilde{E}^{(j)} = P_{(ab)} \tilde{E}^{ib} \tilde{E}^{ja} \]

which implies \( P_{(ab)} = 0 \). Thus the second triad reality conditions is equivalent to \( P_{[ab]} = 0 \).

Let us derive \( P_{ab} \).

\[ P_{ab} = e^{-1} \Im(\tilde{E}_a^{(i)}) \epsilon_i^b \]

\[ = e^{-1} \epsilon_{ib} \Im \left\{ -i D_j (\epsilon^{cd} a N \tilde{E}_c^j \tilde{E}_d^i) + 2 D_j (N^i \tilde{E}_a^j) + i A_0^d \epsilon_{ad} \tilde{E}_c^i \right\} \]

\[ = e^{-1} \epsilon_{ib} \Im \left\{ -i \epsilon^{cd} a \partial_j (N) \tilde{E}_c^j \tilde{E}_d^i - i \epsilon^{cd} a N \tilde{E}_c^j (\partial_j \tilde{E}_d^i - i \epsilon_{de} f A_e^c \tilde{E}_j^i) + \partial_j (N^i) \tilde{E}_a^j + N^j (\partial_j \tilde{E}_a^i - i \epsilon_{de} f A_e^c \tilde{E}_j^i) \right\} \]

\[ = e^{-1} \epsilon_{ib} \left\{ -i \epsilon^{cd} a \partial_j (N) \tilde{E}_c^j \tilde{E}_d^i - i \epsilon^{cd} a N \tilde{E}_c^j \partial_j \tilde{E}_d^i - i \epsilon^{cd} a N \tilde{E}_c^j \epsilon_{de} f \Im(A_e^c) \tilde{E}_j^i \right. \]

\[ \left. - N^j \epsilon_{ae} f \Re(A_e^c) \tilde{E}_f^i + \Re(A_0^d) \epsilon_{ad} \tilde{E}_c^i \right\} \]

\[ = - \epsilon_{ba} \partial_j (N) \tilde{E}_c^j - e^{-1} \epsilon_{ib} \epsilon^{cd} a N \tilde{E}_c^j \partial_j \tilde{E}_d^i - N \tilde{E}_c^j \Im(A_{ja}) + N \tilde{E}_c^j \Im(A_e^c) \delta^{ab} \]

\[ - N^j \epsilon_{ae} f \Re(A_e^c) + \Re(A_0^d) \epsilon_{ad} \tilde{E}_c^i. \]
Thus $P_{[ab]} = 0$ becomes

$$
\epsilon^{abc} P_{ab} = \epsilon^{abc} [-\epsilon^d_{ba} \partial_j (N) \tilde{E}^j_d - e^{-1} \epsilon^d_{ab} N \tilde{E}^j_e \partial_j \tilde{E}^i_d - N \tilde{E}^j_e \mathbb{S}(A_{ja}) + N \tilde{E}^j_e \mathbb{S}(A^d_j) \delta^{ab} - N^j \epsilon_{ab} \mathbb{R}(A_j^e) + \mathbb{R}(A^d_j) \epsilon_{abcd}]
$$

$$
= 2 \partial_j (N) \tilde{E}^{jc} - e^{-1} \epsilon^b_i N \tilde{E}^{jc} \partial_i \tilde{E}^{ci} + e^{-1} \epsilon^b_i N \tilde{E}^{jc} \partial_i \tilde{E}^{ci} - \epsilon^{abc} N \tilde{E}^j_e \mathbb{S}(A_{ja})
$$

$$
+ 2 N^j \mathbb{R}(A_j^c) - 2 \mathbb{R}(A_0^c)
$$

This last equation vanishes, giving (11). \(\square\)

References

[1] A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986); Phys. Rev. D 36, 1587 (1987); Lectures on Non-Perturbative Canonical Gravity (Singapore, World Scientific, 1991).

[2] A. Ashtekar, J.D. Romano, and R.S. Tate, Phys. Rev. D40, 2572 (1989).

[3] R. Capovilla, T. Jacobson, and J. Dell, Phys. Rev. Lett. 63, 2325 (1989); Class. Quantum Grav. 8, 59 (1991).

[4] J.F. Barbero G, Class. Quantum Grav. 12, L5 (1995).

[5] G. Immirzi, Class. Quantum Grav. 10, 2347 (1993).

[6] R. Penrose and W. Rindler, Spinors and Space-time, vol.2, (Cambridge University Press, 1986).

[7] D.C Salisbury, L.C. Shepley, A. Adams, D. Mann, L. Turvan and B. Turner, Class. Quantum Grav. 11, 2789 (1994).

[8] H. Shinkai and G. Yoneda, in preparation.