Neumann and Bargmann Systems Associated with an Extension of the Coupled KdV Hierarchy

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Abstract

An eigenvalue problem with a reference function and the corresponding hierarchy of nonlinear evolution equations are proposed. The bi-Hamiltonian structure of the hierarchy is established by using the trace identity. The isospectral problem is nonlinearized as to be finite-dimensional completely integrable systems in Liouville sense under Neumann and Bargmann constraints.

1 Introduction

A major difficulty in theory of integrable systems is that there is to date no completely systematic method for choosing properly an isospectral problem \( \psi_x = M \psi \) so that the zero-curvature representation \( M_t - N_x + [M, N] = 0 \) is nontrivial. By inserting a reference function into AKNS and WKI isospectral problems, we have obtained successfully two new hierarchies [1, 2].

The coupled KdV hierarchy associated with the isospectral problem

\[
\psi_x = M \psi, \quad M = \begin{pmatrix}
-\frac{1}{2} \lambda + \frac{1}{2} u & -v \\
1 & \frac{1}{2} \lambda - \frac{1}{2} u
\end{pmatrix}
\] (1.1)

is discussed by D. Levi, A. Sym and S. Wojciechowsk [3]. The isospectral problem (1.1) has been nonlinearized as finite-dimensional completely integrable systems in Liouville sense [4].

In this paper, we introduce the eigenvalue problem

\[
\psi_x = M \psi, \quad M = \begin{pmatrix}
-\frac{1}{2} \lambda + \frac{1}{2} u & -v \\
f(v) & \frac{1}{2} \lambda - \frac{1}{2} u
\end{pmatrix},
\] (1.2)

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where $u$ and $v$ are two scalar potentials, $\lambda$ is a constant spectral parameter and $f(v)$ called reference function is an arbitrary smooth function. The bi-Hamiltonian structure of the corresponding hierarchy is established by using the trace identity [5, 6]. Since the reference function $f(v)$ in (1.2) can be chosen arbitrarily, many new hierarchies and their Hamiltonian forms are obtained. When $f = (-v)^\beta \ (\beta \geq 0)$, the isospectral problem (1.2) is nonlinearized as finite-dimensional completely integrable systems in Liouville sense under Neumann and Bargmann constraints between the potentials and eigenfunctions.

2 Preliminaries

Consider the adjoint representation of (1.2)

$$N_x = MN - NM, \quad N = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{j=0}^\infty \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-j} \quad (2.1)$$

which leads to

$$c_0 = b_0 = 0, \quad a_0 = -\frac{1}{2} \alpha \quad \text{(constant)}, \quad (2.2)$$

$$c_1 = \alpha f(v), \quad b_1 = -\alpha v, \quad a_1 = 0, \quad (2.3)$$

$$c_2 = \alpha(f'(v)v_x + uf(v)), \quad b_2 = \alpha(v_x - uv), \quad a_2 = -\alpha v f(v), \quad (2.4)$$

$$a_j = -\partial^{-1}(vc_j + f(v)b_j), \quad (2.5)$$

$$\begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = \alpha \begin{pmatrix} f(v) \\ -v \end{pmatrix}, \quad \begin{pmatrix} c_{j+1} \\ b_{j+1} \end{pmatrix} = L \begin{pmatrix} c_j \\ b_j \end{pmatrix}, \quad j = 1, 2, \ldots, \quad (2.6)$$

where $\partial = \frac{d}{dx}, \partial\partial^{-1} = \partial^{-1}\partial = 1,$

$$L = \begin{pmatrix} \partial + u + 2f\partial^{-1}v & 2f\partial^{-1}f \\ -2v\partial^{-1}v & -\partial + u - 2v\partial^{-1}f \end{pmatrix}. \quad (2.8)$$

It is easy from (1.2) and (2.1) to calculate that

$$\text{tr} \left( N \frac{\partial M}{\partial \lambda} \right) = -a, \quad \text{tr} \left( N \frac{\partial M}{\partial u} \right) = a, \quad \text{tr} \left( N \frac{\partial M}{\partial v} \right) = -c + f'(v)b.$$

Noticing the trace identity [5, 6]

$$\left( \frac{\delta}{\delta u}, \frac{\delta}{\delta v} \right) (-a) = \frac{\partial}{\partial \lambda}(a, -c + f'(v)b),$$

hence we deduce that

$$\left( \frac{\delta}{\delta u}, \frac{\delta}{\delta v} \right) H_j = \left( G_{j-2}^{(1)}, G_{j-2}^{(2)} \right), \quad H = \frac{a_{j+1}}{j}, \quad (2.7)$$

where

$$G_{j-2}^{(1)} = a_j, \quad G_{j-2}^{(2)} = -c_j + f'(v)b_j. \quad (2.8)$$
Let \( \psi \) satisfy the isospectral problem (1.2) and the auxiliary problem
\[
\psi_t = N \psi, \quad N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},
\quad (3.1)
\]
where
\[
A = A_m + \sum_{j=0}^{m-1} a_j \lambda^{m-j}, \quad B = \sum_{j=1}^{m} b_j \lambda^{m-j}, \quad C = \sum_{j=1}^{m} c_j \lambda^{m-j}.
\]
The compatible condition \( \psi_{xt} = \psi_{tx} \) between (1.1) and (3.1) gives the zero-curvature representation
\[
M_t - N_x + [M, N] = 0,
\]
from which we have
\[
A_m = w(\partial + u)c_m + wf'(v)(\partial - u)b_m,
\quad (3.2)
\]
where
\[
w = \frac{1}{2}(vf'(v) + f)^{-1},
\quad \theta_0 = \begin{pmatrix} 2\partial w & -2\partial w f'(v) \\ 2wv & 2wf \end{pmatrix}.
\quad (3.3)
\]
By (2.6) we know that Eqs. (3.2) are equivalent to the hierarchy of nonlinear evolution equations
\[
\left( \begin{array}{c} u_t \\ v_t \end{array} \right) = \theta_0 L^m \left( \begin{array}{c} c_m \\ b_m \end{array} \right) = \theta_0 \left( \begin{array}{c} c_{m+1} \\ b_{m+1} \end{array} \right),
\quad m = 1, 2, \ldots. \quad (3.4)
\]
Let the potentials \( u \) and \( v \) in (1.2) belong to the Schwartz space \( S(-\infty, +\infty) \) over \( (-\infty, +\infty) \). Noticing (2.5) and (2.8) we get
\[
\left( \begin{array}{c} c_j \\ b_j \end{array} \right) = \theta_1 \begin{pmatrix} G_{j-2}^{(1)} \\ G_{j-2}^{(2)} \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} -2wf'(v)\partial & -2wf \\ -2w\partial & 2wv \end{pmatrix}.
\quad (3.5)
\]
Then the recursion relations (2.5), (2.6) and the hierarchy (3.2) can be written as
\[
G_{-2} = -\frac{1}{2}\alpha(1, 0)^T, \quad G_{-1} = -\alpha(0, vf'(v) + f)^T, \quad G_0 = -\alpha(u, uf + uvf'(v))^T,
\]
\[
KG_{j-1} = JG_j,
\quad (3.6)
\]
\[
(u_t, v_t)^T = JG_{m-1} = KG_{m-2},
\quad (3.7)
\]
where \( J = \theta_0 \theta_1 \) and \( K = \theta_0 L \theta_1 \) are two skew-symmetric operators,
in which
\[
\begin{align*}
K_{11} &= -2\partial - 4\partial w(\partial f'(v) + f'(v)\partial)w\partial, \\
K_{12} &= -2\partial w + 4\partial w(f'(v)\partial v - \partial f)w, \\
K_{21} &= -2wu\partial + 4w(\partial f - v\partial f'(v))w\partial, \\
K_{22} &= -4w(\partial f + f\partial v)w.
\end{align*}
\]

From (2.7) we obtain the desired bi-Hamiltonian form of (3.7)
\[
\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J \begin{pmatrix} \delta \\ \delta \end{pmatrix} \frac{\delta}{\delta u} H_{m+1} = K \begin{pmatrix} \delta \\ \delta \end{pmatrix} \frac{\delta}{\delta v} H_m. \tag{3.8}
\]

4 Nonlinearization of the isospectral problem

Let $\lambda_j$ and $\psi(x) = (q_j(x), p_j(x))^T$ be eigenvalue and the associated eigenfunction of (1.2). Through direct verification we know that the functional gradient $\nabla_{(u,v)}\lambda_j = \begin{pmatrix} \frac{\delta}{\delta u} \lambda_j \\ \frac{\delta}{\delta v} \lambda_j \end{pmatrix}$ satisfies
\[
\nabla_{(u,v)}\lambda_j = (q_j p_j, -p_j^2 - f'(v)q_j^2), \tag{4.1}
\]
\[
\theta_1 \nabla \lambda_j = \begin{pmatrix} p_j^2 \\ -q_j^2 \end{pmatrix}, \quad L \begin{pmatrix} p_j^2 \\ -q_j^2 \end{pmatrix} = \lambda_j \begin{pmatrix} p_j^2 \\ -q_j^2 \end{pmatrix}, \tag{4.2}
\]
in view of (1.2). Substituting the first expression of (4.2) into the second expression and acting with $\theta_0$ upon once, we have
\[
K \nabla \lambda_j = \lambda_j J \nabla \lambda_j. \tag{4.3}
\]

So, the Lenard operator pair $K, J$ and their gradient series $G_j$ satisfy the basic conditions (3.6) and (4.3) given in Refs. [7, 8] for the nonlinearization of the eigenvalue problem (1.2).

**Proposition 4.1.** When $f(v) = (-v)^\beta$ ($\beta \geq 0$), the isospectral problem (1.2) can be nonlinearized as to be a Neumann system.

In fact, the Neumann constraint $G_{-1}|_{\alpha=1} = \sum_{j=1}^{N} \nabla \lambda_j$ gives
\[
(q, p) = 0, \quad (p, p) = \beta + 1)(-v)^\beta + \beta(-1)^{\beta-1}\langle q, q \rangle. \tag{4.4}
\]

By differentiating (4.4) with respect to $x$ and using (1.2), we have
\[
\begin{align*}
u &= \frac{1}{\beta + 1} \left( \frac{\langle \Lambda p, p \rangle}{\langle p, p \rangle} + \beta \frac{\langle \Lambda q, q \rangle}{\langle q, q \rangle} \right), \\
v &= \langle q, q \rangle. \tag{4.5}
\end{align*}
\]
Substituting (4.5) into the equations for the eigenfunctions

\[
\begin{pmatrix}
    q_jx \\
p_jx
\end{pmatrix} = \begin{pmatrix}
    -\frac{1}{2}\lambda_j + \frac{1}{2}u & -v \\
    -v & 1/2\lambda_j - 1/2u
\end{pmatrix} \begin{pmatrix}
    q_j \\
p_j
\end{pmatrix}, \quad j = 1, \ldots, N,
\]

we obtain the Neumann system

\[
\begin{aligned}
    q_x &= -\frac{1}{2}\Lambda q - \langle q, q \rangle p + \frac{1}{2(\beta + 1)} \left( \frac{\langle \Lambda p, p \rangle}{\langle p, p \rangle} + \beta \frac{\langle \Lambda q, q \rangle}{\langle q, q \rangle} \right) q, \\
p_x &= \frac{1}{2}\Lambda p + \langle p, p \rangle q - \frac{1}{2(\beta + 1)} \left( \frac{\langle \Lambda p, p \rangle}{\langle p, p \rangle} + \beta \frac{\langle \Lambda q, q \rangle}{\langle q, q \rangle} \right) p,
\end{aligned}
\]

(4.7)

where \( p = (p_1, \ldots, p_N)^T, q = (q_1, \ldots, q_N)^T, \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N), \) and \( \langle , \rangle \) stands for the canonical inner product in \( \mathbb{R}^N \).

**Proposition 4.2.** When \( f(v) = (-v)^\beta (\beta \geq 0) \), the isospectral problem (1.2) can be nonlinearized as to be a Bargmann system.

In fact, the Bargmann constraint \( G_0|_{\alpha=1} = \sum_{j=1}^N \nabla \lambda_j \) gives

\[
\begin{aligned}
u &= \frac{1}{\beta + 1} \langle p, p \rangle \langle q, p \rangle^{-\frac{\beta}{\beta + 1}} - \frac{\beta}{\beta + 1} \langle q, q \rangle \langle q, p \rangle^{-\frac{1}{\beta + 1}}, \\
v &= -\langle q, p \rangle^\frac{1}{\beta + 1}.
\end{aligned}
\]

(4.8)

Substituting (4.8) into (4.6), we obtain the finite-dimensional Hamiltonian system

\[
\begin{aligned}
    q_x &= -\frac{1}{2}\Lambda q + \langle q, p \rangle \langle q, p \rangle^{-\frac{1}{\beta + 1}} p + \frac{1}{2(\beta + 1)} \langle p, p \rangle \langle q, q \rangle^{-\frac{\beta}{\beta + 1}} q \\
        &\quad - \frac{\beta}{2(\beta + 1)} \langle q, q \rangle \langle q, p \rangle^{-\frac{1}{\beta + 1}} q = \frac{\partial H}{\partial p}, \\
p_x &= \frac{1}{2}\Lambda p + \langle q, p \rangle \langle q, p \rangle^{-\frac{\beta}{\beta + 1}} p + \langle q, q \rangle \langle q, q \rangle^{-\frac{1}{\beta + 1}} q \\
        &\quad + \frac{\beta}{2(\beta + 1)} \langle q, q \rangle \langle q, p \rangle^{-\frac{1}{\beta + 1}} p = -\frac{\partial H}{\partial q}.
\end{aligned}
\]

(4.9)

The Hamiltonian is

\[
H = -\frac{1}{2} \langle \Lambda q, p \rangle + \frac{1}{2} \langle p, p \rangle \langle q, q \rangle^{-\frac{1}{\beta + 1}} - \frac{1}{2} \langle q, q \rangle \langle q, p \rangle^{-\frac{\beta}{\beta + 1}}.
\]

5 Integrability of the Neumann system

The Poisson brackets of two functions in symplectic space \( (\mathbb{R}^{2N}, dp \wedge dq) \) are defined as

\[
(F, G) = \sum_{j=1}^N \left( \frac{\partial F}{\partial q_j} \frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j} \frac{\partial G}{\partial q_j} \right) = \langle F_q, G_p \rangle - \langle F_p, G_q \rangle.
\]
The functions defined by \((m = 0, 1, 2, \ldots)\)

\[
F_m = -\frac{1}{2} \langle \Lambda^{m+1} q, p \rangle - \frac{1}{2} \sum_{i+j=m} \left| \begin{array}{cc} \langle \Lambda^i q, q \rangle & \langle \Lambda^i q, p \rangle \\ \langle \Lambda^j p, q \rangle & \langle \Lambda^j p, p \rangle \end{array} \right|
\]

are in involution in pairs (see, [9]).

Consider the Moser constraint on the tangent bundle

\[
T S^{N-1} = \left\{ (p, q) \in \mathbb{R}^{2N} \mid F = \langle q, p \rangle = 0, \quad G = \frac{1}{2(\beta + 1)}(\langle p, p \rangle - (-1)^{\beta} \langle q, q \rangle^\beta) = 0 \right\}.
\]

Through direct calculations we have

\[
(F, F_m) = 0, \quad (F, G) = \langle p, p \rangle, \quad (F_m, G) = -\frac{1}{2(\beta + 1)} \left( \langle \Lambda^{m+1} p, p \rangle + (-1)^{\beta} \beta \langle q, q \rangle^{\beta-1} \langle \Lambda^{m+1} q, q \rangle \right).
\]

Thus the Lagrangian multipliers are

\[
\mu_m = \frac{(F_m, G)}{(F, G)} = -\frac{1}{(\beta + 1)} \left( \frac{\langle \Lambda^{m+1} p, p \rangle}{\langle p, p \rangle} + (-1)^{\beta} \beta \frac{\langle q, q \rangle^{\beta-1}}{\langle p, p \rangle} \langle \Lambda^{m+1} q, q \rangle \right).
\]

Since \(F = 0\) on the tangent bundle \(T S^{N-1}\), the restriction of the canonical equation of \(H^* = F_0 - \mu_0 F\) on \(T S^{N-1}\) is

\[
\begin{cases}
q_x = F_{0,p} - \mu_0 F_p|_{T S^{N-1}}, \\
p_x = -F_{0,q} + \mu_0 F_q|_{T S^{N-1}}
\end{cases}
\]

which is exactly the Neumann system (4.7).

**Theorem 5.1.** The Neumann system (4.7) \((T S^{N-1}, dp \land dq|_{T S^{N-1}}, H^* = F_0 - \mu_0 F)\) is completely integrable in Liouville sense.

**Proof.** Let \(F^*_m = F_m - \mu_m F, m = 1, \ldots, N - 1\), then it is easy to verify \((F^*_k, F^*_l) = 0\) on \(T S^{N-1}\). Hence \(\{F^*_m\}\) is an involutive system.

6 Integrability of the Bargmann system

Let

\[
\Gamma_k = \sum_{\substack{j = 1 \\&\& j \neq k}}^{N} \frac{B^2_{kj}}{\lambda_k - \lambda_j},
\]

where \(B_{kj} = p_k q_j - p_j q_k\), we have (see Refs. [9, 10])

**Lemma 6.1.**

\[
(\langle q, p \rangle, p_i^2) = 2p_i^2, \quad (\langle q, p \rangle, q_i^2) = -2q_i^2,
\]

(6.2)
Proposition 6.1.

\[ (p_k^2, \Gamma_l) = -\frac{4B_{lk}}{\lambda_l - \lambda_k} pk p_l, \quad (q_k^2, \Gamma_l) = -\frac{4B_{lk}}{\lambda_l - \lambda_k} q_k q_l, \]
\[ (q_k p_k, \Gamma_l) = -\frac{2B_{lk}}{\lambda_l - \lambda_k} (pk q_l + q_k p_l). \] (6.3)

Lemma 6.2.

\[ (\Gamma_k, \Gamma_l) = (\langle q, p \rangle, \Gamma_l) = (\langle q, p \rangle, q_l p_l) = 0, \] (6.4)
\[ (p_k^2, p_l^2) = (q_k^2, q_l^2) = (q_k p_k, q_l p_l) = 0, \] (6.5)
\[ (q_k p_k, p_l^2) = 2p_k p_l \delta_{kl}, \quad (q_k^2, p_l^2) = 4q_k p_l \delta_{kl}, \quad (q_k^2, p_l q_l) = 2q_k q_l \delta_{kl}. \] (6.6)

Proposition 6.1. Let

\[ E_k = \frac{1}{2} \langle q, p \rangle^{\beta+1} p_k^2 - \frac{1}{2} \langle q, p \rangle^{\beta+1} q_k^2 - \frac{1}{2} \lambda_k q_k p_k - \frac{1}{2} \Gamma_k, \]

the \( E_1, \ldots, E_N \) constitute an \( N \)-involutive system.

Proof. Obviously \( (E_k, E_l) = 0 \) for \( k = l \). Suppose \( k \neq l \), in virtue of (6.4)–(6.6) and the property of Poisson bracket in \( (\mathbb{R}^{2N}, dp \wedge dq) \), we have

\[
4(E_k, E_l) = \frac{1}{\beta + 1} p_k^2 (\langle q, p \rangle, p_l^2) + \frac{1}{\beta + 1} p_l^2 (\langle q, p \rangle, p_k^2) \\
- \frac{1}{\beta + 1} p_k^2 (\langle q, p \rangle, q_l^2) - \frac{\beta}{\beta + 1} q_k^2 (p_k^2, \langle q, p \rangle) - \langle q, p \rangle^{\beta+1} (p_k^2, \Gamma_l) \\
- (\langle q, p \rangle)^{\beta+1} (\Gamma_k, p_l^2) - \frac{\beta}{\beta + 1} q_k^2 (\langle q, p \rangle, p_l^2) - \frac{1}{\beta + 1} p_l^2 (q_k^2, \langle q, p \rangle) \\
+ \frac{\beta}{\beta + 1} q_k^2 (q_l^2, \langle q, p \rangle) + \frac{\beta}{\beta + 1} q_l^2 (q_k^2, \langle q, p \rangle) \\
+ (\langle q, p \rangle)^{\beta+1} (q_k^2, \Gamma_l) + (\langle q, p \rangle)^{\beta+1} (q_l^2, \Gamma_k) + \lambda_k (q_k p_k, \Gamma_l) + \lambda_l (\Gamma_k, q_l p_l).
\]

Substituting (6.2) and (6.3) into the above equation yields \( (E_k, E_l) = 0 \).

Consider a bilinear function \( Q_\zeta(\xi, \eta) \) on \( \mathbb{R}^N \):

\[ Q_\zeta(\xi, \eta) = (z - \Lambda)^{-1} \xi \eta = \sum_{k=1}^{N} \frac{\xi_k \eta_k}{z - \lambda_k} = \sum_{m=0}^{\infty} z^{-m} \langle \Lambda^m \xi, \eta \rangle. \]

The generating function of \( \Gamma_k \) is (see, [9, 10])

\[
\begin{vmatrix}
Q_\zeta(q, q) & Q_\zeta(q, p) \\
Q_\zeta(p, q) & Q_\zeta(p, p)
\end{vmatrix} = \sum_{k=1}^{N} \frac{\Gamma_k}{z - \lambda_k}.
\]

Hence the generating function of \( E_k \) is

\[
\frac{1}{2} \langle q, p \rangle^{\beta+1} Q_\zeta(p, p) - \frac{1}{2} \langle q, p \rangle^{\beta+1} Q_\zeta(q, q) - \frac{1}{2} Q_\zeta(\Lambda q, p) \\
- \frac{1}{2} \begin{vmatrix}
Q_\zeta(q, q) & Q_\zeta(q, p) \\
Q_\zeta(p, q) & Q_\zeta(p, p)
\end{vmatrix} = \sum_{k=1}^{N} \frac{E_k}{z - \lambda_k}. \] (6.7)
Substituting the Laurent expansion of $Q_z$ and
\[(z - \lambda_k)^{-1} = \sum_{m=0}^{\infty} z^{-m-1} \lambda_k^m\]
in to both sides of (6.7) respectively, we have

**Proposition 6.2.** Let
\[F_m = \sum_{k=1}^{N} \lambda_k^m E_k, \quad m = 0, 1, 2, \ldots\]
then
\[F_0 = \frac{1}{2} \langle q, p \rangle^{\beta+1} \langle p, p \rangle - \frac{1}{2} \langle q, p \rangle^{\beta+1} \langle q, q \rangle - \frac{1}{2} \langle q, p \rangle^{\beta+1} \langle q, p \rangle,
F_m = \frac{1}{2} \langle q, p \rangle^{\beta+1} \langle \Lambda^m p, p \rangle - \frac{1}{2} \langle q, p \rangle^{\beta+1} \langle \Lambda^m q, q \rangle - \frac{1}{2} \langle \Lambda^m q, p \rangle - \frac{1}{2} \sum_{j=1}^{m} \langle \Lambda^{m-j} q, q \rangle \langle \Lambda^{m-j} p, p \rangle.
\]
Moreover, $(F_k, F_l) = 0$.

Hence we arrive at the following theorem.

**Theorem 6.1.** The Bargmann system defined by (4.9) is completely integrable in Liouville sense in the symplectic manifold $(\mathbb{R}^{2N}, dp \wedge dq)$.

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