LENGTH-MINIMIZING LEVEL CURVES VIA CALIBRATIONS

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ABSTRACT. We present an elementary criterion to show the length-minimizing property of geodesics for a large class of conformal metrics. In particular, we prove the length-minimizing property of level curves of harmonic functions and the length-minimizing property of a family of the conic sections with the eccentricity $\varepsilon$ in the upper half plane endowed with the conformal metric $\left( e^z + \frac{1}{e^z} \right) (dz^2 + dy^2)$.

Inspired by two neat proofs due to G. Lawlor, briefly sketched in [6, p. 247-248] (see also his paper [7]), of the length-minimizing property of the cycloid in the lower half plane endowed with the conformal metric $\frac{1}{\varepsilon^2} (dz^2 + dy^2)$, we provide an elementary first order criterion to determine the length-minimizing property of geodesics for a large class of conformal metrics. The key idea is to construct two families of orthogonal level curves by integrating exact differential equations.

Theorem 1. Let $\Omega \subset \mathbb{R}^2$ denote an open domain. Assume that two $C^1$ functions $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ satisfy the following three conditions:

\[
\nabla f(x, y) \cdot \nabla g(x, y) = 0, \quad \| \nabla f(x, y) \| > 0, \quad \| \nabla g(x, y) \| > 0.
\]

The given $C^1$ parameterized curve $X_*$ connecting from $(x_1, y_1) \in \Omega$ to $(x_2, y_2) \in \Omega$ lies in the level curve $C_* = \{ (x, y) \in \Omega \subset \mathbb{R}^2 \mid g(x, y) = g(x_1, y_1) = g(x_2, y_2) \}$.

Then, it becomes the geodesic with respect to the conformal metric $\| \nabla f(x, y) \|^2 (dx^2 + dy^2)$. Moreover, for any $C^1$ parameterized curve $X \subset \Omega$ connecting from $(x_1, y_1)$ to $(x_2, y_2)$, we have the inequality

\[
\int_X \| \nabla f(x, y) \| \, ds \geq \int_X \| \nabla f(x, y) \| \, ds.
\]

In other words, the curve $X_*$ becomes a weighted length-minimizer with respect to the density $\| \nabla f(x, y) \|$.

Proof. On the domain $\Omega$, we introduce the unit vector field $T_*(x, y)$ and angle $\Theta_*(x, y)$ by

\[
T_*(x, y) = \pm \frac{1}{\| \nabla f(x, y) \|} \nabla f(x, y) = \left[ \frac{\cos \Theta_*(x, y)}{\sin \Theta_*(x, y)} \right],
\]

where the sign will be chosen later, and introduce the unit vector field

\[
N_*(x, y) = \frac{1}{\| \nabla g(x, y) \|} \nabla g(x, y).
\]

According to the assumption $\nabla f \cdot \nabla g = 0$ on the domain $\Omega$, we find that, on the level curve $C_* \subset \Omega$,

\[
T_*(x, y) \cdot N_*(x, y) = 0.
\]

We choose the sign in (1) so that $T_*(x, y)$ is the unit tangent vector field on the oriented curve $X_*, C_*$. The vector field $T_*(x, y)$ is a natural extension of the unit tangent vector field on the curve $X_*$ to the whole domain $\Omega$. Along the oriented competing curve $X(s) = (x(s), y(s))$, where $s$ is the arc-length parameter starting from the initial point $(x_1, y_1)$, we define the unit tangent vector field $T(x(s), y(s))$ and the angle $\Theta(x(s), y(s))$ with

\[
T(x(s), y(s)) = \left[ \frac{\frac{dx}{ds}(s)}{\frac{dy}{ds}(s)} \right] = \left[ \frac{\cos \Theta(x(s), y(s))}{\sin \Theta(x(s), y(s))} \right].
\]

On the competitor $X(s) = (x(s), y(s))$ lying in the domain $\Omega$, we consider the geometric quantity

\[
T_*(x(s), y(s)) \cdot T(x(s), y(s)) = \cos \left( \Theta_*(x(s), y(s)) - \Theta(x(s), y(s)) \right).
\]

The line integral of the quantity $\| \nabla f \| \cos \left( \Theta_*(x) - \Theta \right)$ along the competitor $X(s)$ is constant:

\[
\int_X \| \nabla f \| \cos \left( \Theta_*(x) - \Theta \right) \, ds = \int_X \nabla f \cdot T \, ds_x = \int_X df = \int_{X_*.} df = \int_{X_*.} \nabla f \cdot T_x \, ds_{x_*.} = \int_{X_*.} \| \nabla f \| \, ds.
\]

It follows from this and the estimation $1 \geq \cos \left( \Theta_*(x) - \Theta \right)$ that

\[
\int_X \| \nabla f(x, y) \| \, ds \geq \int_X \| \nabla f \| \cos \left( \Theta_*(x) - \Theta \right) \, ds = \int_X \| \nabla f(x, y) \| \, ds.
\]

\[ \square \]
The key idea in the above proof of Theorem 1 is to employ the method of calibrations illustrated in a beautiful paper [2] by R. Harvey and H. B. Lawson. For more recent resources, we refer the interested readers to D. Joyce’s book [4] and J. Lotay’s article [8]. Neat expositions on the calibrations, manifolds with density, isoperimetric problems with density are given by F. Morgan [9, 10, 11].

Example 1 (Astroid length density $\sqrt{x^\frac{2}{3} + y^\frac{2}{3}}$). We consider the one parameter family of (a part of) the astroids in the quadrant $\Omega = (0, \infty) \times (0, \infty)$:

$$C_{\lambda>0} = \left\{(x, y) \in \Omega \mid \lambda = g(x, y) = x^\frac{2}{3} + y^\frac{2}{3}\right\}.$$  

To construct a calibration function $f(x, y)$ in Theorem 1, we need to solve the orthogonality condition

$$0 = \nabla g(x, y) \cdot \nabla f(x, y) = \frac{2}{3} \left[\frac{x^\frac{1}{3}}{y^\frac{1}{3}} \right] \cdot \left[f_x f_y \right] = \frac{2}{3} \left(x^\frac{1}{3} f_x + y^\frac{1}{3} f_y \right).$$

Integrating the induced exact differential equation $df = f_x dx + f_y dy = x^\frac{1}{3} dx - y^\frac{1}{3} dy$ gives the function $f(x, y) = C \left(x^\frac{2}{3} - y^\frac{2}{3}\right)$ for a constant $C \in \mathbb{R}$. We take the function $f : \Omega \to \mathbb{R}$ defined by

$$f(x, y) = \int \frac{3}{4} \left(x^\frac{2}{3} - y^\frac{2}{3}\right)$$

and compute the length density $\|
abla f(x, y)\| = \sqrt{x^\frac{2}{3} + y^\frac{2}{3}} = g(x, y)$. We conclude that the astroid $x^\frac{2}{3} + y^\frac{2}{3} = \lambda$ becomes an weighted length-minimizer with respect to the length density $\sqrt{x^\frac{2}{3} + y^\frac{2}{3}}$.

Example 1 can be generalized in various ways. We consider geodesics in the quadrant endowed with the metric $(x^{2p} + y^{2q}) (dx^2 + dy^2)$. For each $\lambda \in \mathbb{R}$, we introduce the function $\Psi_\lambda : (0, \infty) \to \mathbb{R}$ by

$$\Psi_\lambda(t) = \left\{\begin{array}{ll}
\frac{1}{1-\lambda} t^{1-\lambda}, & \lambda \neq 1, \\
\ln t, & \lambda = 1.
\end{array}\right.$$  

Given a pair $(p, q)$ of real constants, take $g(x, y) = \Psi_p(x) + \Psi_q(y)$ and $f(x, y) = \Psi_{-p}(x) - \Psi_{-q}(y)$. The level curves of the function $g$ are length-minimizers with respect to the density $\int g^{\frac{1}{2}} = \sqrt{x^{2p} + y^{2q}}$.

Example 2 (Brachistochrone length density $\frac{1}{\sqrt{y}}$). For various solutions and generalizations to Johann Bernoulli’s time-minimizing curve problem, we refer to, for instance, [1, 3, 5, 6]. We present details how to construct the calibration function $f(x, y)$ in the lower half plane with respect to the density $\frac{1}{\sqrt{y}} = \frac{1}{\sqrt{x^2 + y^2}}$.

Our construction here will be generalized in Corollary 2. Integrating the first variation formula for the weighted length functional with respect to $\frac{1}{\sqrt{y}} ds$ gives

$$\frac{1}{\sqrt{y}} \frac{dx}{ds} = \text{constant}.$$  

indeed, letting $L(y, x, \dot{x}) = \frac{1}{\sqrt{y}} \sqrt{1 + \dot{x}^2}$ with $\dot{x} = \frac{dx}{ds}$, the Euler-Lagrange equation for the weighted length functional with respect to the weighted length element

$$\frac{1}{\sqrt{y}} ds = \frac{1}{\sqrt{y}} \sqrt{dx^2 + dy^2} = L(y, x, \dot{x}) dy$$

reads

$$0 = \frac{\partial L}{\partial x} - \frac{d}{dy} \left( \frac{\partial L}{\partial \dot{x}} \right) = -\frac{d}{dy} \left( \frac{1}{\sqrt{y}} \sqrt{1 + \dot{x}^2} \right),$$

which guarantees that, for some constant $c \in \mathbb{R}$,

$$c = \frac{1}{\sqrt{y}} \sqrt{1 + \dot{x}^2} = \frac{1}{\sqrt{y}} \frac{dx}{ds},$$

or equivalently, $0 = \pm dx + \frac{c \sqrt{y}}{\sqrt{1 - c^2 y}} dy$.

Integrating the first integral of the geodesic equation indicates why the level set formulation of geodesics is natural. In the case when $c = 0$, this reduces to $0 = dx$, which gives the vertical ray $x = \text{constant}$ as a geodesic. From now on, we consider the case $c^2 > 0$ and take the plus sign in the above equation. Recall the factor $v(y) = \sqrt{-y}$. Letting $\rho = \frac{1}{\sqrt{y}} > 0$, it becomes the exact differential equation

$$dg = g_v dx + g_y dy = dx + \sqrt{\frac{-c^2 y}{1 + c^2 y}} dy = dx + \sqrt{\frac{-2 \rho y}{1 + 2 \rho y}} dy,$$

which gives the level set formulation of the geodesic

$$C = \left\{(x, y) \in \mathbb{R}^2 \mid 0 = g(x, y) = x - \rho \arccos \left(1 + \frac{y}{\rho} \right) + \rho \sqrt{1 - \left(1 + \frac{y}{\rho} \right)^2}\right\}.$$
(The observation that it admits the cycloid path \((x, y) = (\rho(t - \sin t), -\rho(1 - \cos t))\) will not be used in the minimization part.) We consider the strip \(\Omega = \mathbb{R} \times (-2\rho, 0)\). To obtain the weighted length-minimizing property of the half cycloid \(C \cap \Omega\), we find the function \(f(x, y)\) satisfying the orthogonality condition

\[
0 = \nabla g(x, y) \cdot \nabla f(x, y) = \left[ \frac{1}{\sqrt{\frac{x}{2p} + y}} \right] \cdot \left[ \frac{f_x}{f_y} \right] = f_x + \sqrt{\frac{-y}{2p + y}} f_y.
\]

It induces the exact differential equation

\[
df = f_x dx + f_y dy = -dx + \sqrt{\frac{2p + y}{y}} dy.
\]

Taking the calibration function \(f : \Omega \to \mathbb{R}\) defined by

\[
f(x, y) = \frac{1}{\sqrt{2p}} \left( -x + \rho \arcsin \left( 1 + \frac{y}{p} \right) - \rho \sqrt{1 - \left( 1 + \frac{y}{p} \right)^2} \right)
\]

in Theorem 1 gives the weighted length-minimizing property of the half cycloid \(C \cap \Omega\) with respect to the length density \(\|\nabla f(x, y)\| = \frac{1}{\sqrt{2p}}\). One may ask the reason why the calibration function obtained by integrating the orthogonality condition \(f_x g_x + f_y g_y = 0\) gives the desired density \(\sqrt{f_x^2 + f_y^2} = \frac{1}{\sqrt{2p}}\).

It is not a coincidence. It is due to the symmetry of the density, as indicated in the proof of Corollary 2.

Example 2 can be extended to the case when the density function has a symmetry.

**Corollary 2** (Generalized brachistochrone length density \(\frac{1}{\sqrt{v(y)}} > 0\)). Let \(v(y) > 0\) be a \(C^1\) function. Given a constant \(c \in \mathbb{R}\), we consider a \(C^1\) solution \(g : \Omega \to \mathbb{R}\) of the exact differential equation

\[
dg = g_x dx + g_y dy = dx + \frac{cv(y)}{\sqrt{1 - c^2 v(y)^2}} dy.
\]

Here, we choose a simply connected domain \(\Omega \subset \mathbb{R}^2\) such that \(1 - c^2 v(y)^2 > 0\) on \(\Omega\). Then, the above differential equation is well-defined and solvable on \(\Omega\). The level curve (assuming that it is non-empty)

\[
C_* = \{ (x, y) \in \Omega \mid g(x, y) = 0 \}
\]

becomes a weighted length-minimizing geodesic in \(\Omega\) with respect to the density function \(\frac{1}{\sqrt{v(y)}}\).

**Proof.** In Example 2, we noted that the differential equation

\[
dg = dx + \frac{cv(y)}{\sqrt{1 - c^2 v(y)^2}} dy
\]

is the first integral of the weighted geodesic equation for the length density \(\frac{1}{\sqrt{v(y)}} > 0\). We observe that the curve \(C_*\) is regular due to the inequality

\[
\|\nabla g(x, y)\|^2 = \frac{1}{1 - c^2 v(y)^2} > 0.
\]

We use Poincaré Lemma to find a \(C^1\) solution \(f : \Omega \to \mathbb{R}\) of the exact differential equation

\[
df = f_x dx + f_y dy = -dx + \frac{1 - c^2 v(y)^2}{v(y)} dy.
\]

We check the orthogonality condition

\[
\nabla g(x, y) \cdot \nabla f(x, y) = \left[ \frac{cv(y)}{\sqrt{1 - c^2 v(y)^2}} \right] \cdot \left[ \frac{-c}{\sqrt{1 - c^2 v(y)^2}} \right] = -c + c = 0
\]

and compute the length density

\[
\|\nabla f(x, y)\| = \sqrt{f_x^2 + f_y^2} = \sqrt{c^2 + \frac{1 - c^2 v(y)^2}{v(y)^2}} = \frac{1}{\sqrt{v(y)}}.
\]

Theorem 1 shows the length-minimizing property of the level curve \(C_*\) for the density \(\frac{1}{\sqrt{v(y)}}\). \(\square\)

We deform the hyperbolic metric in the Poincaré half-plane to construct a one parameter family of conformal metrics in the upper half plane so that a family of the conic sections are geodesics. In Examples 4, 5, 6, 7, we use Theorem 1 or Corollary 2 to show the length-minimizing property of such curves.
Example 3 (Conic sections with the eccentricity $\varepsilon \geq 0$). We preview curves in Examples 5, 6, 7. Given a constant $\varepsilon \in \mathbb{R}$, we define the curve

$$D_\varepsilon = \left\{(x, y) \in \mathbb{R}^2 \mid 1 - \varepsilon x = \sqrt{x^2 + y^2}\right\}.$$  

The curve $D_\varepsilon$ becomes a conic section with the eccentricity $\varepsilon$ and the focus $(0, 0)$. The curve $D_1$ is a parabola $2x + y^2 = 1$. For $\varepsilon \neq 1$, translating the curve $D_\varepsilon$ horizontally gives the curve

$$C_\varepsilon = \left\{(x, y) \in \mathbb{R}^2 \mid 1 - \varepsilon \left(x - \frac{\varepsilon}{1 - \varepsilon^2}\right) = \sqrt{\left(x - \frac{\varepsilon}{1 - \varepsilon^2}\right)^2 + y^2}\right\} = \left\{(x, y) \in \mathbb{R}^2 \mid (1 - \varepsilon^2)^2 x^2 + (1 - \varepsilon^2) y^2 = 1\right\}.$$

Example 4 (Conic section length density $\sqrt{\varepsilon^2 + \frac{1}{\varepsilon^2}}$ with $\varepsilon = 0$). We consider the hyperbolic length density $\frac{1}{\varepsilon^2}$. Given two constants $x_0 \in \mathbb{R}$ and $\rho > 0$, we take the quarter circle $C_\varepsilon$ in the strip $\Omega = \mathbb{R} \times (0, \rho)$:

$$C_\varepsilon = \left\{(x, y) \in \Omega \mid 0 = g(x, y) = x - x_0 - \sqrt{\rho^2 - y^2}\right\}.$$  

To obtain the weighted length-minimizing property of the curve $C_\varepsilon$, in the domain $\Omega$ with respect to the length density $\|\nabla f(x, y)\| = \frac{1}{\varepsilon^2}$, we introduce the function $f : \Omega \to \mathbb{R}$ defined by

$$f(x, y) = -\frac{x}{\rho} + \sqrt{1 - \left(\frac{y}{\rho}\right)^2} - \tanh^{-1}\left(\sqrt{1 - \left(\frac{y}{\rho}\right)^2}\right).$$  

The level curves of two functions of $f(x, y)$ and $g(x, y)$ are orthogonal:

$$\nabla g(x, y) \cdot \nabla f(x, y) = \left[\frac{1}{\sqrt{\rho^2 - y^2}}\right] \cdot \left[-\frac{1}{\sqrt{\rho^2 - y^2}}\right] = 0.$$  

Example 5 (Conic section length density $\sqrt{\varepsilon^2 + \frac{1}{\varepsilon^2}}$ with $\varepsilon \in (0, 1)$). Let $\varepsilon \in (0, 1)$ and $x_0 \in \mathbb{R}$ be constants. The level curve $C_{\varepsilon, x_0}$ in the strip $\Omega = \mathbb{R} \times \left(0, \frac{1}{1 - \varepsilon} \right)$ defined by

$$C_{\varepsilon, x_0} = \left\{(x, y) \in \Omega \mid 0 = g(x, y) = (1 - \varepsilon^2)(x - x_0) - \sqrt{1 - (1 - \varepsilon^2) y^2}\right\}$$

is a part of the ellipse with the eccentricity $\varepsilon \in (0, 1)$. To obtain the weighted length-minimizing property of the curve $C_{\varepsilon, x_0}$ in the domain $\Omega$ with respect to the length density $\|\nabla f(x, y)\| = \sqrt{\varepsilon^2 + \frac{1}{\varepsilon^2}}$, we introduce the function $f : \Omega \to \mathbb{R}$ defined by

$$f(x, y) = -x + \sqrt{1 - (1 - \varepsilon^2) y^2} - \tanh^{-1}\left(\sqrt{1 - (1 - \varepsilon^2) y^2}\right).$$  

The level curves of two functions of $f(x, y)$ and $g(x, y)$ are orthogonal:

$$\nabla g(x, y) \cdot \nabla f(x, y) = (1 - \varepsilon^2) \left[\frac{1}{\sqrt{1 - (1 - \varepsilon^2) y^2}}\right] \cdot \left[-\frac{1}{\sqrt{1 - (1 - \varepsilon^2) y^2}}\right] = 0.$$  

Example 6 (Conic section length density $\sqrt{\varepsilon^2 + \frac{1}{\varepsilon^2}}$ with $\varepsilon = 1$). Let $x_0 \in \mathbb{R}$ be a constant. The level curve $C_{x_0}$ in the upper-half plane $\Omega = \mathbb{R} \times (0, \infty)$ defined by

$$C_{x_0} = \left\{(x, y) \in \Omega \mid 0 = g(x, y) = x - x_0 + \frac{1}{2} y^2\right\}$$

is a part of the parabola with the eccentricity $\varepsilon = 1$. To obtain the weighted length-minimizing property of the curve $C_{x_0}$ in the upper-half plane $\Omega$ with respect to the length density $\|\nabla f(x, y)\| = \sqrt{1 + \frac{1}{\varepsilon^2}},$ we introduce the function $f : \Omega \to \mathbb{R}$ defined by

$$f(x, y) = -x + \ln y.$$  

The level curves of two functions of $f(x, y)$ and $g(x, y)$ are orthogonal:

$$\nabla g(x, y) \cdot \nabla f(x, y) = \left[\frac{1}{y}\right] \cdot \left[-\frac{1}{y}\right] = 0.$$
Example 7 (Conic section length density \(\sqrt{e^2 + \frac{1}{y^2}}\) with \(e \in (1, \infty)\)). Let \(e \in (1, \infty)\) and \(x_0 \in \mathbb{R}\) be constants. The level curve \(C_{e,x_0}\) in the upper-half plane \(\Omega = \mathbb{R} \times (0, \infty)\) defined by

\[
C_{e,x_0} = \left\{ (x,y) \in \Omega \mid 0 = g(x,y) = (1-e^2) (x-x_0) - \sqrt{1-(1-e^2)y^2} \right\}
\]

is a part of one branch of the hyperbola with the eccentricity \(e \in (1, \infty)\). To obtain the weighted length-minimizing property of the curve \(C_{e,x_0}\) in the domain \(\Omega\) with respect to the length density \(\| \nabla f(x,y) \| = \sqrt{e^2 + \frac{1}{y^2}}\), we introduce the function \(f: \Omega \to \mathbb{R}\) defined by

\[
f(x,y) = -x + \sqrt{1-(1-e^2)y^2} - \tanh^{-1}\left(\sqrt{1-(1-e^2)y^2}\right).
\]

Example 8 (Grim reaper length density \(e^\alpha\)). It is well-known that the grim reaper \(y = -\ln(\cos x)\) in \((-\frac{\pi}{2}, \frac{\pi}{2}) \times [0, \infty)\) is a translating soliton for the curve shortening flow in the Euclidean plane \(\mathbb{R}^2\). We consider the right half of the grim reaper lying in the domain \(\Omega = (0, \frac{\pi}{2}) \times (0, \rho)\):

\[
C_\alpha = \left\{ (x,y) \in \mathbb{R}^2 \mid y = -\ln(\cos x), x \in \left(0, \frac{\pi}{2}\right) \right\} = \left\{ (x,y) \in \Omega \mid g(x,y) = 0 \right\},
\]

where the function \(g: \Omega \to \mathbb{R}\) is defined by

\[
g(x,y) = x - \arccos(e^{-y}).
\]

To obtain the weighted length-minimizing property of the half grim reaper \(C_\alpha\) in the domain \(\Omega\) with respect to the length density \(\| \nabla f(x,y) \| = e^\alpha\), we prepare the function \(f: \Omega \to \mathbb{R}\) defined by

\[
f(x,y) = x + \sqrt{e^{2\alpha} - 1} - \arccos(e^{-y}).
\]

The level curves of two functions of \(f(x,y)\) and \(g(x,y)\) are orthogonal:

\[
\nabla g(x,y) \cdot \nabla f(x,y) = \left[ \frac{1}{\sqrt{e^{2\alpha} - 1}} \right] \cdot \left[ \frac{1}{\sqrt{e^{2\alpha} - 1}} \right] = 0.
\]

Corollary 3 (Level curves of harmonic functions). Let \(p(x,y) + iq(x,y)\) be a holomorphic function such that \(\| \nabla p(x,y) \| > 0\) on an open domain \(\Omega \subset \mathbb{R}^2\). We consider the one parameter family \(\{C_\alpha\}_{\alpha \in \mathbb{R}}\) of the level curves of harmonic functions:

\[
C_\alpha = \left\{ (x,y) \in \Omega \subset \mathbb{R}^2 \mid (\sin \alpha) p(x,y) + (\cos \alpha) q(x,y) = 0 \right\}.
\]

Assume that the \(C^1\) parameterized curve \(X_\alpha\) connecting from \((x_1,y_1) \in C_\alpha\) to \((x_2,y_2) \in C_\alpha\) lies in the level curve \(C_\alpha\). For any \(C^1\) parameterized curve \(X \subset \Omega\) connecting from \((x_1,y_1)\) to \((x_2,y_2)\), we have

\[
\int_X \| \nabla p(x,y) \| ds \geq \int_X \| \nabla p(x,y) \| ds.
\]

In other words, the curve \(X_\alpha\) becomes a weighted length-minimizer with respect to the density \(\| \nabla p(x,y) \|\).

Proof. We take the holomorphic function \(f(x,y) + iq(x,y) = e^{\alpha\theta}(p(x,y) + iq(x,y))\) in Theorem 1 to deduce the inequality. It follows from the Cauchy-Riemann equations \(\partial_v = q_y - p_x = 0\) that

\[
\| \nabla f(x,y) \| = \| \nabla g(x,y) \| = \| \nabla p(x,y) \| = \| \nabla q(x,y) \| > 0,
\]

which implies that the level curve \(C_\alpha = \left\{ (x,y) \in \Omega \subset \mathbb{R}^2 \mid g(x,y) = 0 \right\}\) is regular. \(\square\)

Example 9 (Logarithmic spiral length density \(\frac{1}{\sqrt{e^{2\alpha} + y^2}}\)). We work with the polar coordinates \((r, \theta)\). Let \(r_0 > 0, \theta_0 \in \mathbb{R}, \alpha \in (0, \pi), \lambda = -\cot \alpha\) be constants. Taking the holomorphic function \(p(x,y) + iq(x,y) = \ln \left(\frac{1}{r_0} + i (\theta - \theta_0)\right)\) in Corollary 3 gives the weighted length-minimizing property of the logarithmic spiral \(r = r_0 e^{\lambda(\theta-\theta_0)}\) with respect to the length density \(\| \nabla p(x,y) \| = \frac{1}{\sqrt{e^{2\alpha} + y^2}}\). See also Shehupsky’s argument [12, p. 785] based on a variation of Snell’s Law for the radial density.

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