Comparing the sandpile model with targeted triggering and the Olami-Feder-Christensen model as models of seismicity using recurrence network analysis

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Abstract. Slowly driven sandpile models has found applications in modelling earthquakes due to the observed power law statistics in its magnitude distributions, like the behaviour of earthquakes. Adding a probability to target the most susceptible site in the grid, the sandpile model recovers even the spatio-temporal statistics of earthquake events. In this work, we compare the sandpile model with targeted triggering to the Olami-Feder-Christensen (OFC) model: a standard earthquake model that also exhibits self-organized criticality. The sandpile model captures the magnitude distributions of earthquake events at a value of targeted triggering probability \( p = [0.004,0.007] \). The triggering probability value \( p = 1.0 \), showing that the most susceptible site is always triggered, follows the magnitude distribution of the OFC model. A comparison was done by constructing a record-breaking recurrence network for the events. Spatial and magnitude criteria set the temporally directed links between events across the entire record. Both the models recover power-law exponents comparable to those previously obtained for earthquake data, which is 1.0 for recurrence distance and recurrence time distributions, and 2.1 for the in-degree distributions for the farthest recurrence criteria. The sandpile model with targeted triggering exhibits a behaviour in between a slowly driven sandpile and the OFC model.

1. Introduction
Earthquake modelling had attracted a lot of attention in the study of dynamical systems due to its self-organized critical behaviour [1-6]. A full understanding of earthquakes is essential since it may cause a lot of destruction in properties and lives [7-9].

Aside from earthquakes, many other systems in nature exhibit self-organized criticality (SOC), wherein the components move to a critical state independent of their initial conditions. Self-organized critical systems exhibit long-tailed distributions of some physical quantities [4]. One defining property of earthquake events are the observed power law distributions in the magnitude of the energy release in each event. This was defined as a universal law by Gutenberg and Richter, stating that the probability distribution of energy follows the scaling as shown in equation (1) [10]. This power law behavior in the distribution of the energy released during an earthquake is important to note since it indicates that earthquakes are SOC [11].

\[
p(E) \sim E^{-5/3}
\]  

(1)
An example of a self-organized critical system is the Bak Tang Wiesenfeld (BTW) sandpile model which is a 2-dimensional grid that receives a slow external perturbation and grows in height as if grains of sand in a pile. The small perturbation can cause avalanche events which may be bigger in size than the external trigger. The sandpile model has been used to model seismicity because the distribution of avalanche sizes captures the magnitude distributions of earthquakes defined in the GR Law [1].

Aside from the magnitude distributions, earthquakes are also observed to cluster in space and time [12]. Correlated earthquakes occur in proximity with one another, such that after a large earthquake, a series of aftershocks occur near the original mainshock. This spatial and temporal clustering of events is not easily reproduced in the original sandpile model. A modification on this was done by adding a parameter that targets the most susceptible site in the pile. The interevent statistics from the modified model capture the space and time distributions of earthquakes [2].

Another earthquake model is the Olami-Feder-Christensen which is associated to the two-dimensional Burridge-Knopoff spring block model. It simulates an earthquake fault system through the transfer of energy along a system of blocks. The blocks are connected by a spring, which moves back to a stable position when subjected to an external force [3]. This model not only follow the Gutenberg-Richter law [13], it also exhibits the clustering in space and time of events [11]. To investigate the clustering in space and time of correlated events, Davidsen et al. used a spatio-temporal structure of seismicity from an idea of the causal relationship of events beyond the simplistic idea that each event has at most one correlated event. They constructed a network where the nodes are the events and a temporally directed link connects the two events if the latter event is spatially closest to the previous event compared to all events that occurred in between the two [14]. This method scans through the whole record, wherein the latest spatial distance record will be broken if an event nearer than the current record occurred. The network grows every time a new event arrives since the links of previous records are retained in the network. Extending the idea that correlated events cluster in space, a previous work by Tarun et al. added a farthest distance criterion for connection, wherein connection is made between to events if they are farthest in space compared to all events in between [5].

Magnitudes of earthquakes are important aspects to fully understand earthquake characteristics. Previous studies showed that using the magnitude as a metric resulted to observations regarding the clustering of correlated earthquakes [15]. Aside from using the spatial criteria in constructing the network introduced by Davidsen et al., magnitude was incorporated by Janer et al. by segregating the UP connections, if the link is from an event of lower magnitude to an event of higher magnitude, from the DOWN connections, if the link is connecting to an event of lower magnitude [6].

In our work, we constructed a temporally directed network using the spatial near and far connections and the magnitude up and down connections for avalanche events from a sandpile model with targeted triggering and an Olami-Feder-Christensen (OFC) model. The statistics from the network was used to compare the two models with the basis on the observed behaviour in earthquakes. For our analyses, we note the power law behaviour observed in earthquake magnitudes and the spatiotemporal correlations in earthquake events as a basis for comparison with the models similar to what was done in previous works [2, 5, 11].

2. Methodology
A sandpile model with targeted triggering and an Olami-Feder-Christensen (OFC) model was constructed and compared using recurrence network analysis.

2.1. Sandpile Model with Targeted Triggering
Sandpile is a cellular automata model where an external triggering causes unstable sites in the grid form avalanches. The 2-dimensional grid follows the von-Neumann neighbourhood, \( n = \{(x,y), (x \pm \Delta x, y), (x, y \pm \Delta y)\} \), where \( \Delta x = \Delta y = 1 \), such that the cell at \((x, y)\) have four nearest neighbours in its right, left, top, and bottom. The state of each cell evolves in every discrete time step \( \Delta t = 1 \).

Compared to the Bak-Tang-Wiesenfeld (BTW) sandpile model wherein the possible states are discrete, \( s = \{0, 1, 2, 3, 4\} \) and the unstable state is at \( \sigma_{\text{max}} = 4 \), the sandpile model used has a
A continuous set of states $s = \{\sigma | \sigma \in [0, \sigma_{\text{max}}]\}$, where $\sigma_{\text{max}} = 1$. The update rule $f$ is due to the external triggering $v = 10^{-3}\sigma_{\text{max}}$. For each update, an external trigger $v$ was added to the state $\sigma_{xy}$ of a selected neighbourhood centered at $(x, y)$. This rule, shown as $f_1$ in equation (2), allows the system to receive a continuous but slow driving.

$$f_1 : \sigma_{xy} + v$$  \hspace{1cm} (2)

A conditional rule $f_2$ represents the redistribution done to the system when the triggering results to an unstable site with a state greater than or equal to the unstable state $\sigma_{xy} \geq \sigma_{\text{max}}$. The unstable state $\sigma^*$ is equally redistributed to the state $\sigma_{nn}$ of its four neighbours as shown in equation (3). The site with an unstable state will then have a state $\sigma_{xy} = 0$ after the redistribution. The redistribution rule causes the cascading nature of the model since local redistributions from the original neighbourhood may make other cells unstable, thus continuing the redistribution across other neighbourhoods outside of the initially collapsing site. Compared to other sandpile models, the driving rule uses a targeted triggering probability $p$ which is a probability that the driving term $v$ will be directed to the most susceptible site. This represents the high chance of rupturing of most susceptible locations.

$$f_2 : \sigma_{nn} = \sigma_{nn} + \sigma^* (4)^{-1}$$  \hspace{1cm} (3)

In our work, we used a sandpile grid with size $L = 1024$ with initial random states $\sigma \in (0, \sigma_{\text{max}})$. The driving was done for $10^7$ iteration times where the first $10^6$ iterations were discarded to remove the transient effects. The recorded locations were from the location of the initial toppling. The magnitude of the avalanche event from an initial external driving was measured from the number of sites that was affected by the toppling event. The minimum area of avalanche $A$ is 5 cells. A threshold magnitude was also applied wherein only the avalanche events greater than the threshold $A_{th} = 10$ is considered. This removes the very small single neighbourhood avalanches representing that in an actual catalogue, very weak earthquakes are not completely captured.

### 2.2. Olami-Feder-Christensen model

The Olami-Feder-Christensen (OFC) model is another model for seismicity which is a simplification of the Burridge-Knopoff spring block model, which considers a network or springs and blocks where blocks move to their balanced positions after being submitted to a force which made them unstable.

All states were initialized to have random values between 0 and the threshold force $F_{th}$. If any site $(x, y)$ has a force greater than the threshold force $F_{xy} \geq F_{th}$, the force $F_{xy}$ on the site is redistributed to the neighbours and the force on the site will be zero as shown in equations 4 and 5. The redistribution process is repeated until the earthquake is fully evolved. For the driving, compared to the sandpile model, a global perturbation was done such that the external driving was added to all the sites. Same threshold area $A_{th} = 10$ was used for the events recorded from the OFC model. The value for the dissipation parameter used is $\alpha = 1.5$.

$$F_{nn} \rightarrow F_{nn} + \alpha F_{xy}$$  \hspace{1cm} (4)

$$F_{xy} \rightarrow 0$$  \hspace{1cm} (5)

### 2.3. Recurrence Network Construction

A temporally directed network was constructed using the spatial coordinates and magnitudes of the avalanche events to characterize the spatio-temporal clustering of the system. A previous work by Davidsen et al. [14] used a generalized theory of records in finding the probable causal relationships among earthquake events.

Two spatial criteria were used for the network construction, the nearest and the farthest criteria. The nearest criteria connect the events if the later event is nearer in space to the earlier event compared to all intervening events. This is demonstrated in figure 1 (a) where a sample network for 5 events was constructed. In the figure, each circle or node represents an event. The numbers correspond to the interval at which the event occurred and the relative spaces among the nodes represents the relative distances between events. The magnitudes of events are shown through the radius of the circles. When
the second event arrived, a connection was made from the first event to the second event, since interevent connections are always done. In this specific network, the record connection of event 1 is the connection to event 2. When the third event arrived, the second event connected to the third event since the two are consecutive events. The spatial criterion is then applied when a link was made from event 1 and event 3. Since event 3 is nearer in space to event 1 compared to the connection of event 1 to event 2, the record connection was broken due to the nearest distance criterion. The new record of event 1 is a connection to event 3. Same rule was observed when creating connections to event 4 and event 5. For the farthest distance connection as shown in figure 1 (b), a similar rule with the nearest criterion of connection was applied. However, to create a connection, the farthest event compared to other intervening events, connects with a previous event. All interevent connections are made.

Figure 1. Network construction for a sample of 5 events using the (a) nearest criterion of connection and (b) the farthest criterion of connection. The numbers in the events correspond to each temporal coordinate, the relative distances correspond to the spatial aspect, and radii of the circles in each node represent the magnitudes of the events.

Figure 2. Metrics used in network construction. (a) Fully constructed network. (b) Separation distances \( t_{ij} \) and temporal separations \( t_{ij} \) between connected events. (c) The number of in going links \( k_{in,i} \) and out going links \( k_{out,i} \) an event. (d) Links were segregated into UP connections, when the connection was done from an event of lower magnitude to an event of higher magnitude, and DOWN connections, when the connection was done from a higher to a lower magnitude.

After the full construction of the network aver all events, as shown in figure 2 (a), the metrics obtained are the spatial separation \( t_{ij} \) and the temporal separation \( t_{ij} \) of two connected events \( i \) and \( j \) as shown in figure 2 (b). The number of in going links \( k_{in,i} \) and the number of out going links \( k_{out,i} \) from an event \( i \) was obtained to characterize the structure of the network as shown in figure 2 (c). Lastly, the magnitude was incorporated in the network through a segregation of the connections to
either labelled as UP connections, when the connection was going from an event of lower magnitude to an event of higher magnitude, or being labelled as DOWN connections, when the link is from an event of higher magnitude to an event of lower magnitude. Statistical analyses were done from the metrics.

3. Results and Discussion

The Gutenberg-Richter law states that the magnitude distribution of earthquakes follow a power law behaviour. As a first test, the magnitude distributions of the models should also follow the observed earthquake statistics. In figure 3, magnitude distributions of avalanche areas $p(A)$ obtained from the sandpile model and the Olami-Feder-Christensen (OFC) model was shown.

![Figure 3. Magnitude distribution of avalanche events from the sandpile model for different targeted triggering probability $p = \{0, 10^{-3}, 10^{-2}, 10^{-1}, 1\}$ and of the avalanche events from the OFC model.](image)

Due to the targeted triggering probability $p$, a bias in the spatial and temporal organization of events occurred compared to the original sandpile model. We investigate the cases for a broad range of values of the targeted triggering probability $p = \{0, 10^{-3}, 10^{-2}, 10^{-1}, 1\}$. The case where $p = 0$ is the original sandpile where the triggering is done on a random site and $p = 1$ is the completely targeted sandpile. From the magnitude distributions, the OFC model follows the distribution of the completely targeted sandpile model at $p = 1$ where the sites with highest states in the grid are driven. Compared to the power law behaviour observed in earthquakes, the completely targeted case shows a faster tail decay and an increase in the intermediate magnitude values $A$, this was observed since there is a slow accumulation of new susceptible sites, resulting to a small extent of avalanche magnitude produced.

Cases where targeted triggering was done only during a fraction of the intervals, such that $0 < p < 1$, the sandpile model captures the behaviour in between a simple sandpile and the OFC model. For the succeeding analyses, we focus on the case of $p = 0.004$, which shows characteristics similar to those obtained from the earthquake data.

3.1. Spatial and Temporal Distributions of Recurrences

The original sandpile recovers the power-law behaviour of the avalanche size distributions as observed in earthquakes since the stress in the system is released in a single cascade. However, the distribution of the spatial and temporal differences of correlated events results in random (Poisson) statistics. The effect of the targeted triggering is illustrated in the distributions of the separation distances and the waiting times between recurrences shown in figures 4(a) and 5(a).

The distribution of the separation distances from the OFC model and from the sandpile model with the nearest criterion of connection shows a similar behaviour. For the farthest criterion, the events with long separation in the OFC model closely follow the exponent of a power law observed in the behaviour of the original sandpile model. An abundance of events with small separation distances are observed which was not the case for earthquake catalogues. Temporal distributions from the two models behave in a similar trend as shown in figure 5.
3.2. *K*-degree Distributions of Recurrences

To complete the analysis and the comparison done, the structure of the constructed recurrence network was analysed from the distribution of the number of in and out going links from each event. Figure 6 presents the degree distributions \( p(k_{in}) \) and \( p(k_{out}) \) of the recurrence network for the nearest and farthest criteria.

![Degree Distributions](image)

**Figure 4.** The distribution of separation distances of connected events in the constructed recurrence network for (a) the sandpile model with targeted triggering probability \( p = 0.004 \) and (b) the OFC model for the nearest and farthest criteria of connection. The connections were also segregated into the up and down magnitude criteria.

![Temporal Separation Distributions](image)

**Figure 5.** The distribution of temporal separation of the links in the constructed recurrence network with the nearest and farthest spatial criteria and the up and down magnitude criteria for the events from (a) the sandpile model with targeted triggering probability \( p = 0.004 \) and (b) the Olami-Feder-Christensen (OFC) model with dissipation parameter \( \alpha = 0.15 \).

The distribution of the in going links \( (k_{in}) \) for the farthest criteria as shown in figure 6 (b) results to a scale-free distribution comparable to \( p(k_{in}) \sim k_{in}^{-1.8} \), where \( \beta = 2.1 \). The distribution of the outgoing links \( (k_{out}) \) from the sandpile model closely follow a Poisson distribution regardless of the criteria of connection used, as shown in figure 4 (c) and (d). The distributions from the OFC model, however, show nontrivial properties. This could be due to the faster generation of events since all the sites in the pile is driven for every time step.

4. Summary and Conclusions

A sandpile model with a targeted triggering probability to target the most susceptible site in the grid and the Olami-Feder-Christensen (OFC) model for earthquake system was compared using a
recurrence network analysis. The targeted triggering mechanism in the sandpile more closely captured the magnitude distribution of earthquake events, while the OFC model was observed to behave similar to a completely targeted sandpile model. The distribution of the temporal separation and the distribution of the spatial separation with the nearest criterion cannot be used as a basis for comparison of the two models due to the similar trend exhibited by both. The spatial separation of the OFC model using the farthest criterion of connection showed an abundance of short separation of connected events, which is not similar to the behaviour of earthquakes. The degree distributions of the out going links in the OFC model deviates a lot from a Poisson distribution showing non-trivial properties which is due to the faster generation of events.

Figure 6. The distribution of the degree connections going into \(k_{\text{in}}\): (a,b) and out \(k_{\text{out}}\): (c,d) of the events from the sandpile model with targeted triggering probability \(p = 0.004\) and the OFC model for the (a,c) nearest and (b,d) farthest criteria of connection.

5. References
[1] Bak P, Tang C and Wiesenfeld K 1988 Phys. Rev. A 38 364
[2] Batac R C, Paguirigan Jr. A A, Tarun A B and Longas A G 2017 Nonlin. Processes Geophys. 24 179
[3] Olami Z, Feder H J S and Christensen K 1992 Phys. Rev. Lett. 68 1244
[4] Bak P, Tang C and Wiesenfeld K 1987 Phys. Rev. Lett. 59 4
[5] Tarun A B, Paguirigan A A, and Batac R C 2015 Physica A 436 293
[6] Janer C D, Bitoon D C and Batac R C 2017 Acta Geophys. 65 1153
[7] Oteng-Ababio M 2012 J Hous Built Environ. 27 19
[8] Allen R B, Bellingham P J and Wiser S K 1999 Ecol. 80 708-14
[9] Nakagawa M, Saito M and Yamaga H 2009 Jpn. Econ. Rev. 60 208-22
[10] Gutenberg B and Richter C F 1949 Seismicity of the earth and associated phenomena (London: Princeton University Press) p 16–24
[11] Christensen K and Olami Z 1992 Phys. Rev. A 46 4
[12] Zaliapin I, Gabrilov A, Keilis-Borok V and Wong H 2008 Phys. Rev. Lett. 101 018501
[13] Lise S and Paczuski M 2001 Phys. Rev. E 63 036111
[14] Davidsen J, Grassberger P and Paczuski M 2006 Geophys. Res. Lett. 33 L11304; Davidsen J, Grassberger P and Paczuski M 2008 Phys. Rev. E 77 066104
[15] Baiiesi M and Paczuski M 2004 Phys. Rev. E 69 066106; Zaliapin I, Gabrilov A, Keilis-Borok V and Wong H 2008 Phys. Rev. Lett. 101 018501