1. Introduction

The standard model of choice in economics is the maximization of a complete and transitive preference relation over a fixed set of alternatives. While completeness of preferences is usually regarded as a strong assumption, weakening it requires care to ensure that the resulting model still has enough structure to yield interesting results. This paper takes a step in this direction by studying the class of “connected preferences”, that is, preferences that may fail to be complete but have connected maximal domains of comparability.

We offer four new results. Theorem 1 identifies a basic necessary condition for a continuous preference to be connected in the sense above, while Theorem 2 provides sufficient conditions. Building on the latter, Theorem 3 characterizes the maximal domains of comparability. Finally, Theorem 4 presents conditions that ensure that maximal domains are arc-connected.

Methodologically, our contribution provides an incomplete preference perspective on a theoretical literature relating basic assumptions on preferences and the space of alternatives over which these preferences are defined. For example, Schmeidler (1971) shows that every nontrivial preference on a connected topological space which satisfies seemingly innocuous continuity conditions must be complete. In a recent article, Khan and Uyank (2019) revisit Schmeidler’s theorem and link it to the results in Eilenberg (1941), Sonnenschein (1965), and Sen (1969), providing a thorough analysis of the logical relations between the form of continuity assumed by Schmeidler, completeness, transitivity, and the connectedness of the space.

In particular, Theorem 4 in Khan and Uyank (2019) implies a converse to Schmeidler’s theorem: if every strongly nontrivial Schmeidler preference is complete, the underlying space must be connected. This paper provides a different kind of converse: every compact space that admits at least one complete and gapless Schmeidler preference with connected indifference classes must be connected.

2. Preliminaries

Let $X$ be a (nonempty) set of alternatives equipped with some topology. A preference is a reflexive and transitive binary relation on $X$. For the rest of this paper, we consider a fixed preference $\succeq$.

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Gorno (2018) examines the maximal domains of comparability of a general preorder.
≿ is complete on a set $A \subseteq X$ if $A \times A \subseteq \succsim \cup \prec$. The set $A$ is a domain if $\succsim$ is complete on $A$. If $A$ is a domain such that there exists no larger domain containing it, then $A$ is a maximal domain.

≿ is continuous if $\{y \in X|y \succ x\}$ and $\{y \in X|x \succ y\}$ are closed sets for every $x \in X$. $\succsim$ has connected indifference classes if $\{y \in X|y \sim x\}$ is connected for every $x \in X$.

The set $A \subseteq X$ contains every indifferent alternative if $x \succsim A \succsim y$ implies $x, y \in A$. $A$ has no exterior bound if $x \succsim A \succsim y$ implies $x, y \in A$.

3. Connected preferences

The main concept of this paper is embedded in the following definition:

**Definition 1.** $\succsim$ is connected if every maximal domain is connected.

We will restrict attention to preferences that are not only connected, but also continuous. As a result, maximal domains will be necessarily closed (see Theorem 1 in Gorno (2018)).

### 3.1. A necessary condition

A natural first step towards a characterization of connected preferences is to obtain a simple necessary condition.

**Definition 2.** $\succsim$ is gapless if, for every $x, y \in X$, $x \succ y$ implies that there exists $z \in X$ such that $x \succ z \succ y$.

The notion of gapless preferences is not really new; its content coincides with a specific definition of order-denseness for sets. We now prove our first result:

**Theorem 1.** If $\succsim$ is continuous and connected, then $\succsim$ is gapless.

**Proof.** Suppose, seeking a contradiction, that $\succsim$ is not gapless. Then, there exist alternatives $x, y \in X$ such that $x \succ y$ and no $z \in X$ satisfies $x \succ z \succ y$. By Lemma 1 in Gorno (2018), there exists a maximal domain $D$ such that $\{x, y\} \subseteq D$. Define $A := \{z \in D|z \succsim x\}$ and $B := \{z \in D|y \succsim z\}$. Clearly, $A$ and $B$ are nonempty, $A \cap B = \emptyset$, and $A \cup B = D$. Moreover, since $\succsim$ is continuous, $A$ and $B$ are closed. It follows that $D$ is not connected, a contradiction.

It is easy to see that not every continuous and gapless preference is connected:

**Example 1.** Let $X = [-1, 1]$ and $\succsim = \{(x, y) \in X^2|x = y \lor x^2 = y^2 = 1\}$. Then, the preference $\succsim$ is continuous and gapless, but not connected (the maximal domain $\{-1, 1\}$ is not a connected set).

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2$X$ is said to be $\succsim$-dense if for every $x, y \in X$ satisfying $x \succ y$ there exists $z \in X$ such that $x \succ z \succ y$ (see Ok (2007), p. 92). Evidently, $X$ is $\succsim$-dense if and only if $\succsim$ is gapless. We should perhaps note that there are multiple distinct definitions of order-denseness in the literature and that the terminology has not necessarily been consistent.
3.2. A sufficiency theorem. We already know that every continuous and connected preference must be gapless. In this section, we provide a set of assumptions which constitute a sufficient condition for a preference to be connected.

**Theorem 2.** If \( X \) is compact and \( \succeq \) is a continuous and gapless preference with connected indifference classes, then \( \succeq \) is connected.

**Proof.** Assume first that \( \succeq \) is antisymmetric. Suppose, seeking a contradiction, that there is a maximal domain \( D \) that is not connected. Then, there exist disjoint nonempty closed sets \( A \) and \( B \) such that \( A \cup B = D \). Since \( \succeq \) is continuous and \( X \) is compact, \( D \) is compact by Proposition 1 in Gorno (2018). It follows that \( A \) and \( B \) are also compact. Let \( \underline{\pi}_A \) and \( \underline{\pi}_B \) be the best elements in \( A \) and \( B \), respectively. Since \( D \) is a domain and \( A \) and \( B \) are disjoint, either \( \underline{\pi}_A \succ \pi_B \) or \( \pi_B \succ \underline{\pi}_A \). Consider the first case (the other is symmetric). Define the set \( C := \{ x \in A \mid x \succeq \underline{\pi}_B \} \). \( C \) is nonempty (as \( \underline{\pi}_A \in C \)) and compact. Let \( \underline{\pi}_C \) be the worst element in \( C \). It is easy to check that \( \underline{\pi}_C \succ \underline{\pi}_B \). Since \( \succeq \) is gapless, there exists \( z \in X \) such that \( \underline{\pi}_C \succ z \succ \pi_B \). It is easy to verify that \( z \notin A \) and \( z \notin B \). Hence, \( z \notin D \). Moreover, \( D \cup \{ z \} \) is a domain, contradicting the assumption that \( D \) is a maximal domain.

The argument above assumes that \( \succeq \) is antisymmetric. If this is not the case, we can still apply it to the partial order \( \succeq^* \) induced by \( \succeq \) in the quotient space \( X/\sim \). To verify this claim, note that continuity and compactness are automatically inherited by \( \succeq^* \) and \( X/\sim \), respectively. Moreover, since all \( \succeq \)-indifference classes are connected and \( D \) must contain every indifferent alternative, \( D \) is connected if and only if \( D/\sim \) is connected. It follows that there is no loss of generality in assuming the antisymmetry of \( \succeq \) from the outset.

The following example identifies an important class of connected preferences:

**Example 2.** Let \( X \) be the set of Borel probability measures (lotteries) on a compact metric space of prizes \( Z \), equipped with the topology of weak convergence. Following Dubra, Maccheroni, and Oh (2004), we say that the preference \( \succeq \) is an expected multi-utility preference if there exists a set \( U \) of continuous functions \( Z \to \mathbb{R} \) such that \( x \succeq y \) if and only if

\[
\int_Z u dx \geq \int_Z u dy
\]

holds for all \( u \in U \). It is easy to verify that all the assumptions of Theorem 2 hold. Thus, \( \succeq \) is connected.

4. Characterization of maximal domains

Building on Theorem 2 we can offer a useful characterization of the maximal domains:

**Theorem 3.** Assume \( X \) is compact and \( \succeq \) is continuous, gapless, and has connected indifference classes. Then, a set \( A \subseteq X \) is a maximal domain if and only if it is a connected domain that contains every indifferent alternative and has no exterior bound.
Proof. We start establishing sufficiency through the following lemma:

**Lemma 1.** Every connected domain that contains every indifferent alternative and has no exterior bound is a maximal domain.

**Proof.** Suppose, seeking a contradiction, that $D$ is a domain that contains every indifferent alternative, has no exterior bound, but it is not a maximal domain, then by Lemma 1 in Gorno (2018) exists $D'$, a maximal domain, such that $D \subseteq D'$. Take $x \in D' \setminus D$. Since $D$ has no exterior bounds there are $y, z \in D$ such that $y \succ x \succ z$. Define $D_1 := \{ w \in D \mid w \succeq x \}$ and $D_2 := \{ w \in D \mid x \succeq w \}$. $D_1$ and $D_2$ are nonempty since $y \in D_1$ and $z \in D_2$. Also, $D_1 \cup D_2 = D$ because $x \in D'$ and $D'$ is a domain that contains $D$. Moreover, $D_1 \cap D_2 = \emptyset$. If this intersection was not empty, there would be $w \in D$ such that $x \sim w$, which would contradict that $D$ contains every indifferent alternative. Finally, $D_1$ and $D_2$ are closed. To see this, note that, since $\succsim$ is continuous, $D$ is closed, but also $\{ y \in X \mid y \succeq x \}$ and $\{ y \in X \mid x \succeq y \}$ are both closed. Thus, $D_1$ and $D_2$ are the intersection of closed sets. It follows that $\{ D_1, D_2 \}$ is a nontrivial partition of $D$ by closed sets. We conclude that $D$ is not connected, which is a contradiction. □

Now we turn to necessity. It is easy to show that every maximal domain contains every indifferent alternative and has no exterior bound. Moreover, since $\succsim$ satisfies the assumptions of Theorem 2, every maximal domain is connected. □

We finish this section, discussing the two additional assumptions employed in Theorem 3.

4.1. **$X$ is compact.** Compactness of $X$ cannot be dispensed with, as the following example shows.

**Example 3.** Let $X = \{-1\} \cup [0, 1]$ and $\succsim = \{(x, y) \in X^2 \mid x = -1 \lor x \geq y \geq 0\}$. Then, $X$ is bounded, locally compact and $\sigma$-compact, but fails to be compact. Moreover, $\succsim$ is complete, continuous, and gapless. However, the only maximal domain is $X$ itself and is not connected.

4.2. **Connected indifferent classes.** On the one hand, the assumption that indifferent classes are connected is not strictly necessary for the conclusion of Theorem 3. That is, there are examples failing this condition in which the equivalence in the theorem holds:

**Example 4.** Let $X = [-1, 1]$ and $\succsim = \{(x, y) \in X^2 \mid x^2 \geq y^2\}$.

On the other hand, it is a tight condition: there are examples that violate it, satisfy the remaining conditions, and for which the equivalence in the theorem fails to hold:

**Example 5.** Let $X = \{-1\} \cup [0, 1]$ and $\succsim = \{(x, y) \in X^2 \mid x^2 \geq y^2\}$.

There is a well-known axiom introduced by Dekel (1986) that ensures that indifference classes are connected. Assuming that $X$ is convex, we say that $\succsim$ satisfies *betweenness* if $x \succeq y$ implies $x \succeq \alpha x + (1 - \alpha)y \succeq y$ for all $x, y \in X$ and $\alpha \in [0, 1]$. 

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- **Gorno (2018)**
- **Dekel (1986)**
Prominent examples of preferences satisfying betweenness include preferences satisfying the independence axiom (such as expected utility or the expected multi-utility preferences studied in [Dubra, Maccheroni, and Ok (2004)]) and also preferences exhibiting disappointment aversion as in [Gul (1991)]. The following lemma shows that betweenness implies connected indifference classes.

**Lemma 2.** If $X$ is convex and $\succsim$ satisfies betweenness, then $\succsim$ has connected indifference classes.

**Proof.** Take any $x, y \in X$ such that $x \sim y$ and $\alpha \in [0,1]$. Define $z := \alpha x + (1 - \alpha)y$. Since $x \succsim y$ and $y \succsim x$, by betweenness, we have $x \succsim z \succsim y$ and $y \succsim z \succsim x$ and, so $z \sim y$. It follows that each indifference class is convex, thus connected. \hfill $\Box$

We should note that, if $X$ is convex and $\succsim$ is a continuous preference that satisfies betweenness, then $\succsim$ does not only possess connected indifferent classes, but is also necessarily gapless. This fact makes the application of Theorem 2 and Theorem 3 to preferences satisfying betweenness quite direct.

### 5. Arc-connected preferences

In some cases, it can be useful to strengthen the notion of connectedness to arc-connectedness:

**Definition 3.** $\succsim$ is arc-connected if every maximal domain is arc-connected.

Every arc-connected preference is connected, but the converse does not generally hold. To see this it suffices to take $X$ to be any space that is connected but not arc-connected and consider $\succsim = X \times X$, that is, universal indifference.

In the particular case of antisymmetric preferences (i.e., partial orders) on a metrizable space, we can strengthen the conclusion of Theorem 2:

**Theorem 4.** If $X$ is a compact metrizable space and $\succsim$ is a continuous, gapless, and antisymmetric preference, then $\succsim$ is arc-connected.

**Proof.** Let $D$ be a maximal domain. Since $\succsim$ is continuous and $X$ is compact and metrizable, Theorem 1 in [Gorno (2018)] implies that $D$ is compact and metrizable, hence second countable. Because $\succsim$ is complete and continuous on $D$, there exists a continuous utility representation $u : D \to \mathbb{R}$.

Since $\succsim$ is antisymmetric, its indifference classes are singletons, hence connected. By Theorem 2, $D$ is connected. It follows that $u(D)$ is connected and compact, thus a compact interval. Without loss of generality, we can assume that $u(D) = [0,1]$. Since $\succsim$ is antisymmetric, $u$ is a continuous bijection. Since $X$ is compact and $[0,1]$ is Hausdorff, $u$ is actually an homeomorphism between $D$ and $[0,1]$. It follows that $D$ is arc-connected, as desired. \hfill $\Box$

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3A well-known example is the closed topologist’s sine curve, which is also compact.
6. Applications

6.1. First-order stochastic dominance. Suppose $X$ is the set of cumulative distribution functions (CDFs) over a compact interval $[0, \bar{z}]$ (endowed with the topology of weak convergence of the associated probability measures). Let $\geq_1$ denote the first-order stochastic dominance relation on $X$, that is, $F \geq_1 G$ if and only if $F(z) \leq G(z)$ for all $z \in [0, \bar{z}]$.

**Proposition 1.** $\geq_1$ is arc-connected. Moreover, a subset of $X$ is a maximal domain of $\geq_1$ if and only if it is the image of an $\geq_1$-increasing arc joining the degenerate CDFs associated with 0 and $\bar{z}$.

**Proof.** $X$ is a compact metrizable space (it is metrized by the Lévy metric) and $\geq_1$ is continuous, gapless, and antisymmetric. Thus, by Theorem 4, $\geq_1$ is arc-connected. Since every arc-connected set is connected, Theorem 3 implies the desired equivalence. \hfill \Box

An analogous result holds for second-order stochastic dominance.

6.2. Schmeidler preferences. [Schmeidler (1971)] shows that, in a connected space, every nontrivial preference satisfying seemingly innocuous continuity conditions must be complete. In this section, we explore the implications of his assumptions in spaces that are not connected.

We start by formulating the class of preferences which are the subject of Schmeidler’s theorem:

**Definition 4.** A preference $\succeq$ is a Schmeidler preference if it is continuous and the sets $\{y \in X | x \succ y \}$ and $\{y \in X | y \succ x \}$ are open for all $x \in X$.

The following definition captures a property that generalizes the conclusion of Schmeidler’s theorem in terms of maximal domains:

**Definition 5.** A preference is decomposable if every maximal domain is either a connected component or an indifference class.

Note that, when $X$ is connected, every nontrivial decomposable preference is complete. More generally, any two distinct maximal domains of a decomposable preference must necessarily be disjoint. As a result, if a decomposable preference is locally nonsatiated, then no maximal domain can be trivial or, equivalently, every maximal domain must be a connected component.

We can now state the main result of this section:

**Proposition 2.** Let $X$ be compact and let $\succeq$ be a Schmeidler preference with connected indifference classes. Then, $\succeq$ is decomposable if and only if $\succeq$ is gapless.

**Proof.** To prove necessity, assume that $\succeq$ is gapless. Since $\succeq$ is a Schmeidler preference, Proposition 10 in [Gorno (2018)] implies that every nontrivial connected component is contained in a maximal domain. Moreover, because $\succeq$ is a gapless preference on a compact space, every maximal domain is connected by Theorem 3. It follows
that every nontrivial maximal domain is a connected component. Finally, since trivial maximal domains must be indifference classes, \( \succeq \) is decomposable.

For sufficiency, note that, since \( \succeq \) is decomposable and has connected indifference classes, every maximal domain is connected. Thus, Theorem 1 implies that \( \succeq \) is gapless.

Note that every continuous and complete preference is a Schmeidler preference. In that particular case, we have the following

**Corollary 1.** Let \( X \) be compact and let \( \succeq \) be a continuous and complete preference with connected indifference classes. Then, \( \succeq \) is gapless if and only if \( X \) is connected.

Schmeidler (1971) shows that if \( X \) is connected, then every nontrivial Schmeidler preference must be complete. Khan and Uyank (2019) prove the converse and obtain following characterization: \( X \) is connected if and only if every nontrivial Schmeidler preference is complete. The corollary above implies a different characterization for compact spaces: provided \( X \) is compact, \( X \) is connected if and only if there exists at least one complete, gapless, and continuous preference with connected indifference classes.  

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### References

E. Dekel. An axiomatic characterization of preferences under uncertainty: Weakening the independence axiom. *Journal of Economic Theory*, 40(2):304 – 318, 1986.

J. Dubra, F. Maccheroni, and E. A. Ok. Expected utility theory without the completeness axiom. *Journal of Economic Theory*, 115(1):118–133, 2004.

S. Eilenberg. Ordered topological spaces. *American Journal of Mathematics*, 63(1):39–45, 1941.

L. Gorno. The structure of incomplete preferences. *Economic Theory*, 66(1):159–185, 2018.

F. Gul. A theory of disappointment aversion. *Econometrica*, 59(3):667–686, 1991.

M. Khan and M. Uyank. Topological connectedness and behavioral assumptions on preferences: a two-way relationship. *Economic Theory*, pages 1–50, 2019.

E. A. Ok. *Real analysis with economic applications*. 2007.

D. Schmeidler. A condition for the completeness of partial preference relations. *Econometrica*, 39(2):403–404, 1971.

A. Sen. Quasi-transitivity, rational choice and collective decisions. *The Review of Economic Studies*, 36(3):381–393, 1969.

H. Sonnenschein. The relationship between transitive preference and the structure of the choice space. *Econometrica*, pages 624–634, 1965.

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Note that, if \( X \) is connected, the trivial preference that declares all alternatives indifferent satisfies all the desired properties.