Fast Track Communication

Vectorial AdS$_5$/CFT$_4$ duality for spin-one boundary theory

Matteo Beccaria$^1$ and Arkady A Tseytlin$^{2,3}$

$^1$Dipartimento di Matematica e Fisica Ennio De Giorgi, Università del Salento & INFN, Via Arnesano, I-73100 Lecce, Italy
$^2$The Blackett Laboratory, Imperial College, London SW7 2AZ, UK

E-mail: matteo.beccaria@le.infn.it and tseytlin@imperial.ac.uk

Received 16 October 2014
Accepted for publication 21 October 2014
Published 12 November 2014

Abstract

We consider an example of vectorial AdS$_5$/CFT$_4$ duality when the boundary theory is described by free $N$ complex or real Maxwell fields. It is dual to a particular (‘type C’) higher spin theory in AdS$_5$ containing fields in special mixed-symmetry representations. We extend the study of this theory by Beccaria and Tseytlin (2014 Higher spins in AdS$_5$ at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT arXiv:1410.3273) by deriving the expression for the large $N$ limit of the corresponding singlet-sector partition function on $S^3 \times S^1$. We find that in both complex $U(N)$ and real $O(N)$ invariant cases the form of the one-particle partition function is as required by the AdS/CFT duality. We also discuss the matching of the Casimir energy on $S^3$ by assuming an integer shift in the bulk theory coupling.

Keywords: AdS/CFT duality, higher spins, conformal symmetry

PACS numbers: 04.50.-h, 11.25.Tq, 04.62.+v

1. Introduction

In addition to ‘adjoint’ AdS/CFT duality between gauge theory at the boundary and string theory in the bulk there is a simpler example of ‘vectorial’ AdS/CFT relating the singlet sector of conserved currents of a CFT described by $N$ free massless fields in a vector representation of $U(N)$ or $O(N)$ to a Vasiliev-type theory in AdS. This duality is not restricted to original examples in $d = 3$ [1, 2]: generalizations to $d > 3$ were studied, e.g., in [3–8]. While in $d = 3$ the only option to get a unitary theory with a higher spin symmetry is to use free scalars or

3 Also at Lebedev Institute, Moscow.

4 An early discussion of $d = 4$ case based on gauged $O(N)$ model appeared in [9].
spin $\frac{1}{2}$ fermions [10] in the $d = 4$ case that we will be interested in here one is also allowed to consider free spin $1\frac{1}{2}$ fields [11–13].

The corresponding conserved currents that appear in the product of two spin 1 doubletons (as in [16–21]) are in specific mixed-symmetry representations of $SO(2, 4)$.

The singlet sector of a theory of $N$ complex or real Maxwell vectors invariant under $U(N)$ or $O(N)$ should then be dual to a particular version of higher spin theory in $AdS_5$, involving mixed-symmetry fields. This theory was called ‘type C’ in [25] by analogy with type A theory dual to $N$ boundary scalars and type B theory dual to $N$ spin $1\frac{1}{2}$ boundary fermions. Here we will elaborate on its discussion in [25] by directly computing the singlet-sector partition in this theory and explaining the matching of the $AdS_5$ vacuum energy in the bulk and the $S^3$ Casimir energy at the boundary.

2. Representation content and relations between characters

Let us first review the case when the boundary theory is described by $N$ complex or real scalars. We shall use the notation $(\Delta; j_1, j_2)$ for generic ‘massive’ representation of $SO(2, 4)$, with the ‘massless’ case corresponding to $\Delta = 2 + j_1 + j_2$ with $j_1 j_2 > 0$.

Also, $(j_0, 0)$ and $(0, j)$ with $\Delta = 1 + j$ will denote spin $j$ doubleton representation. The spectrum of states in ‘non-minimal’ type A theory dual to complex $U(N)$ scalar theory may be found from the decomposition of the product of two $j = 0$ doubletons [19] (generalizing the $d = 3$ relation of [26])

$$\text{non-minimal A: } (0, 0) \otimes (0, 0) = (2; 0, 0) + \bigoplus_{s=1}^{\infty} \left( 2 + s; \frac{s}{2}, \frac{s}{2} \right).$$

Here the representations $(2 + s; \frac{s}{2}, \frac{s}{2})$ correspond to conserved currents dual to massless totally symmetric spin $s$ fields in $AdS_5$. In the ‘minimal’ type A theory case dual to the real scalar $O(N)$ theory one is to project out all odd spin fields (the corresponding currents become trivial), i.e. to ‘symmetrize’ the product

$$\text{minimal A: } [0, 0) \otimes [0, 0)]_{\text{sym}} = (2; 0, 0) + \bigoplus_{s=2, 4, \ldots}^{\infty} \left( 2 + s; \frac{s}{2}, \frac{s}{2} \right).$$

Let us define the (‘blind’) characters for the basic representations $(\Delta_0 = 2 + j_1 + j_2)$ [20]

$$\text{‘massive’: } Z(\Delta; j_1, j_2) = \frac{q^4}{(1 - q)^2} \left( 2j_1 + 1 \right) \left( 2j_2 + 1 \right),$$

$$\text{‘massless’: } Z(\Delta_0; j_1, j_2) = \frac{q^{\Delta_0}}{(1 - q)^2} \left[ \left( 2j_1 + 1 \right) \left( 2j_2 + 1 \right) - 4 q j_1 j_2 \right].$$

Higher-spin symmetries of such boundary models and their higher-spin and super-extensions were found in [14].

More generally, one may also attempt to use a higher-spin doubleton at the boundary; this possibility was noticed in [15] where the corresponding higher spin algebras were studied.

Mixed-symmetry fields in $AdS_5$ and the associated currents were discussed, e.g., in [22–24].

Here $(j_1, j_2)$ are $SU(2) \times SU(2)$ weights (with total spin being $s = j_1 + j_2$) and we shall use the notation $(\Delta; j_1, j_2) = (\Delta; j_1, j_2) = (2 + j_1 + j_2)$.

We follow the notation of [25]. Equation (2.3) applies also to the case of massive self-dual representations that appear when $j_1 j_2 = 0$ and $\Delta > 1 + j_1 + j_2$. 

5 Higher-spin symmetries of such boundary models and their higher-spin and super-extensions were found in [14].

More generally, one may also attempt to use a higher-spin doubleton at the boundary; this possibility was noticed in [15] where the corresponding higher spin algebras were studied.

6 Mixed-symmetry fields in $AdS_5$ and the associated currents were discussed, e.g., in [22–24].

7 Here $(j_1, j_2)$ are $SU(2) \times SU(2)$ weights (with total spin being $s = j_1 + j_2$) and we shall use the notation $(\Delta; j_1, j_2) = (\Delta; j_1, j_2) = (2 + j_1 + j_2)$.

8 We follow the notation of [25]. Equation (2.3) applies also to the case of massive self-dual representations that appear when $j_1 j_2 = 0$ and $\Delta > 1 + j_1 + j_2$. 

2
Here (2.3)/(2.4) has the interpretation of one-particle partition function \( Z(\Delta_0; j_1, j_2) \) for the corresponding massive/massless 5d field in AdS4 with standard boundary conditions. The character identities which are the counterparts of (2.1) and (2.2) are [20]

\[
\text{non-minimal A: } [Z(\{0, 0\})]^2 = Z(2; 0, 0) + \sum_{s=1}^{\infty} Z\left(2 + s; \frac{s}{2}, \frac{s}{2}\right)
\] (2.6)

and [8]

\[
\text{minimal A: } \frac{1}{2}[Z(\{0, 0\})]^2 + \frac{1}{2} [Z(\{0, 0\})]_{q\to q^2} = Z(2; 0, 0) + \sum_{s=2,4,\ldots}^{\infty} Z\left(2 + s; \frac{s}{2}, \frac{s}{2}\right)
\] (2.7)

The validity of (2.7) can be verified directly, but the group-theoretic origin of the combination in the lhs is not obvious. The lhs parts of (2.6) and (2.7) have the interpretation of the large \( N \) limit of singlet-sector one-particle partition functions of the \( U(N) \) [27] and the \( O(N) \) [8] scalar theories. The rhs parts are interpreted as the total contribution to the one-particle partition function of corresponding higher spin theory defined on thermal quotient of AdS4 [28, 29].

The same construction can be repeated for spin \( \frac{1}{2} \) boundary theory when the role of \( j = 0 \) doubleton is played by the \( j = \frac{1}{2} \) one, leading to the spectrum of type B higher spin theory in AdS4 [8, 25]. Similarly, taking the product of two \( j = 1 \) doubletons \( \{1, 0\} \times \{1, 0\} \) (that may be associated with the self-dual and anti self-dual parts of Maxwell field strength \( F_{mn} \)) we get the spectrum of conserved currents and other primary fields and thus the content of the corresponding type C theory in AdS4 [25]. In general, the products of two doubleton representations decompose as follows [19, 20]

\[
\{j, 0\} \otimes \{j', 0\} = \bigoplus_{k=0}^{j+j'} (2 + j + j'; k, 0) \\
\quad \quad + \bigoplus_{k=1}^{\infty} \left(2 + j + j' + k; j + j' + \frac{k}{2}, \frac{k}{2}\right)
\] (2.8)

\[
\{0, j\} \otimes \{0, j'\} = \bigoplus_{k=0}^{j+j'} (2 + j + j'; 0, k) \\
\quad \quad + \bigoplus_{k=1}^{\infty} \left(2 + j + j' + k; \frac{k}{2}, j + j' + \frac{k}{2}\right)
\] (2.9)

\[
\{j, 0\} \otimes \{0, j'\} = \bigoplus_{k=0}^{\infty} \left(2 + j + j' + k; j + \frac{k}{2}, j' + \frac{k}{2}\right)
\] (2.10)

For an ‘unsymmetrized’ product of two spin 1 doubletons one then finds the spectrum of the non-minimal type C theory dual to the singlet sector of \( N \) complex Maxwell fields at the boundary
non-minimal C: \( \{1, 0\}_c \otimes \{1, 0\}_c \equiv (\{1, 0\} + \{0, 1\}) \otimes (\{1, 0\} + \{0, 1\}) \)
\[= 2(4; 0, 0) + (4; 1, 0)_c + (4; 2, 0)_c + 2 \sum_{k=0}^\infty \left(4 + k; \frac{k + 2}{2}, \frac{k + 2}{2}\right) + \sum_{k=1}^\infty \left(4 + k; \frac{2 + k}{2}, \frac{k}{2}\right) + 2(4; 0, 0) + (4; 1, 0)_c \]
\[= 2(4; 0, 0) + (4; 1, 0)_c + 2 \sum_{s=2}^\infty \left(2 + s; \frac{s}{2}, \frac{s}{2}\right) + \sum_{s=2}^\infty \left(2 + s; \frac{s + 2}{2}, \frac{s - 2}{2}\right)_c. \] (2.11)

In addition to two infinite series of massless spin \(s \geq 2\) fields it contains also a massive scalar
and pseudoscalar in representation \((4; 0, 0)\) (dual to \(F_m^m F^{num}\) and \(\tilde{F}_m^m \tilde{F}^{num}\))
and the rank 2
antisymmetric tensor in self-dual and anti self-dual massive representation \((4; 1, 0)_c\) (dual to \(F_m^{[nu]} F_k^{[nm]}\)).
The spectrum of minimal type C theory dual to real \(O(N)\) Maxwell theory is found
by projecting out one parity-odd symmetric tensor states and odd-spin mixed-symmetry
states:
minimal C:
\[\left[ (1, 0)_c \otimes (1, 0)_c \right]_{\text{sym}} \]
\[= 2(4; 0, 0) + \sum_{s=2}^\infty \left(2 + s; \frac{s}{2}, \frac{s}{2}\right) + \sum_{s=2,4, \ldots} \left(2 + s; \frac{s + 2}{2}, \frac{s - 2}{2}\right)_c. \] (2.12)

The corresponding character relations are counterparts of (2.6) and (2.7) in type A case:
non-minimal C:
\[\left[ Z(1, 0)_c \right]^2 \]
\[= 2 Z(4; 0, 0) + \sum_{s=2}^\infty Z\left(2 + s; \frac{s}{2}, \frac{s}{2}\right) + \sum_{s=2}^\infty Z\left(2 + s; \frac{s + 2}{2}, \frac{s - 2}{2}\right)_c. \] (2.13)

minimal C:
\[\frac{1}{2}\left[ Z((1, 0)_c) \right]^2 + \frac{1}{2}\left[ Z((1, 0)_c) \right]_{q \rightarrow q^2} \]
\[= 2 Z(4; 0, 0) + \sum_{s=2}^\infty Z\left(2 + s; \frac{s}{2}, \frac{s}{2}\right) + \sum_{s=2,4, \ldots} Z\left(2 + s; \frac{s + 2}{2}, \frac{s - 2}{2}\right)_c. \] (2.14)

Like in the type A case and the type B cases [8] these relations between characters have again
a field theory or AdS/CFT interpretation. As we shall show below, the lhs of (2.13) is the one-particle partition function representing the leading term in the large \(N\) limit of the singlet-sector partition function of the theory of \(N\) complex Maxwell fields, while the lhs of (2.14)
corresponds to the real \(O(N)\) Maxwell theory case.
3. Singlet-sector partition function in the theory of $N$ Maxwell fields

Let us first recall the expression for the partition function of one scalar field on $S^1 \times S^3$, then consider $N$ scalars and impose the singlet-sector constraint and, finally, generalize to the case of Maxwell vectors instead of scalars.

For a conformal scalar we get the partition function (we assume $S^3$ to have unit radius and length of $S^1$ to be $\beta$)

$$Z_0 = \left( \det \mathcal{O}_0 \right)^{-1/2}, \quad \mathcal{O}_0 = -D^2 + \frac{1}{6} R = -\partial_0^2 - \mathbf{D}^2 + 1. \quad (3.1)$$

Using the eigenvalues of the Laplacian $-D^2$ on $S^3$ we get the eigenvalues and multiplicities of $\mathcal{O}_0$ ($k = 0, \pm 1, \ldots, n = 0, 1, 2, \ldots$)

$$\lambda_{k,n} = w_k^2 + \omega_n^2, \quad w_k = 2\pi k \beta^{-1}, \quad \omega_n = n + 1, \quad d_n = (n + 1)(n + 2). \quad (3.2)$$

Thus log $Z_0 = -\frac{1}{2} \log \det \mathcal{O}_0 = -\frac{1}{2} \sum d_n \log \lambda_{k,n}$.

In general, for the bosonic partition function we have ($q \equiv e^{-\beta}$)

$$\Gamma = -\log Z = \beta E_c + F(\beta), \quad E_c = \frac{1}{2} \sum d_n \omega_n, \quad (3.3)$$

$$F = \sum_n d_n \log \left( 1 - e^{-\beta \omega_n} \right) = -\sum_{m=1}^\infty \frac{1}{m} \mathcal{Z}(q^m), \quad \mathcal{Z}(q) = \sum_n d_n e^{-\beta \omega_n}, \quad (3.4)$$

where $E_c$ is Casimir energy on $S^3$ and $\mathcal{Z}$ is one-particle partition function. For the massless 4d scalar $\mathcal{Z}$ thus has the same expression as the character for spin 0 doubleton in (2.5)

$$\mathcal{Z}_0 = \mathcal{Z}(0, 0) = \sum_{n=0}^\infty (n + 1)(n + 2) q^{n+1} = \frac{q(1 + q)}{(1 - q)^2}. \quad (3.5)$$

Next, let us consider the singlet partition function for $N$ complex scalars in $d = 4$ (see [8, 27] and also [30–32] for relevant earlier work). One way to define it is to gauge the $U(N)$ symmetry and consider the coupling of $N$ scalars $\Phi$ to $U(N)$ gauge field $A_m$ with strength $F_{mun}$

$$L = D_m \Phi^* D^m \Phi + \frac{1}{4g^2} \tr F_{mun} F^{mun}. \quad (3.6)$$

We shall understand the limit $g \to 0$ as restricting the path integral to pure-gauge fields $A_m$. In the case of $S^1 \times S^3$ there is a non-trivial holonomy $U$ of $A_0$ along $S^1$ that cannot be gauged away. The remaining path integral will then be over $\Phi$ and $U$ or phases $\alpha_r$ in

$$A_0 = U^{-1} \partial_0 U, \quad U = \text{diag} \left( e^{i \tau \alpha_1}, \ldots, e^{i \tau \alpha_N} \right), \quad \tau = \beta^{-1} \chi_0 \in (0, 1). \quad (3.7)$$

We thus get the following expression for the singlet partition function of $U(N)$ scalars

$$\hat{Z} = \int \prod_{r=1}^N d\alpha_r \ e^{-\Gamma(\alpha; \beta)}, \quad (3.8)$$

$$\Gamma(\alpha; \beta) = \mathcal{F}(\alpha) + \hat{F}(\alpha; \beta), \quad \mathcal{F}(\alpha) = -\frac{1}{2} \sum_{r \neq s=1}^N \ln \sin^2 \frac{\alpha_r - \alpha_s}{2}, \quad (3.9)$$
\[ F = \ln \det \left[ -\left( D_0 + A_0 \right)^2 - D^2 + 1 \right] = \sum_{r=1}^{N} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} d_n \ln \left[ (2\pi k + \alpha_r)^2 \beta^{-2} + \omega_n^2 \right]. \] (3.10)

Here \( \mu(\alpha) \) is the contribution of the \( U(N) \) invariant measure \( [U^{-1}dU] \) and \( d_n \) and \( \omega_n \) are as in (3.2) and (3.5). To take the large \( N \) limit let us introduce the normalized \( \int d\alpha \rho(\alpha) = 1 \) eigenvalue density \( \rho(\alpha) \) and replace the integral over \( \alpha_r \) by the path integral over the periodic field \( \rho(\alpha) \) defined on a circle

\[ \hat{Z} \big|_{N \rightarrow \infty} = \int [d\rho] e^{-F(\rho, \hat{\beta})}, \] (3.11)

\[ F = N^2 \int d\alpha d\alpha' K(\alpha - \alpha') \rho(\alpha) \rho(\alpha') + 2N \int d\alpha \rho(\alpha) Q(\alpha; \beta), \] (3.12)

\[ K(\alpha) = -\frac{1}{2} \ln (2 - 2 \cos \alpha), \quad Q(\alpha; \beta) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} d_n \ln \left[ (2\pi k + \alpha)^2 \beta^{-2} + \omega_n^2 \right]. \] (3.13)

Here the \( N^2 \) term came from the measure term \( \mu \) in (3.9). Isolating the Casimir energy part and rearranging the sum we get

\[ Q(\alpha; \beta) = \beta E_c + \hat{Q}, \quad E_c = \frac{1}{2} \sum_{n=0}^{\infty} \omega_n, \quad \hat{Q}(\alpha; \beta) = \sum_{m=1}^{\infty} c_m(\beta) \cos (m\alpha), \] (3.14)

\[ c_m(\beta) = -\frac{1}{m} \hat{Z}_0(m\beta), \quad \hat{Z}_0(\beta) = \sum_{n=0}^{\infty} e^{-\beta \omega_n}. \] (3.15)

Splitting the normalized periodic function \( \rho(\alpha) \) into the constant and non-constant parts, \( \rho(\alpha) = \frac{1}{2\xi} + \tilde{\rho}(\alpha) \), we can write (3.11) as (using that \( \hat{Q} \) in (3.14) does not couple to the constant part of \( \rho \))

\[ F = 2N\beta E_c + N^2 \int d\alpha d\alpha' K(\alpha - \alpha') \tilde{\rho}(\alpha) \tilde{\rho}(\alpha') + 2N \int d\alpha \tilde{\rho}(\alpha) \hat{Q}(\alpha; \beta). \] (3.16)

Integrating over \( \tilde{\rho} \) (or, equivalently, evaluating the path integral at the large \( N \) saddle point) gives finally

\[ \hat{F} = -\log \hat{Z} \big|_{N \rightarrow \infty} = 2N\beta E_c + \hat{F}(\beta), \] (3.17)

\[ \hat{F} = -\sum_{m=1}^{\infty} m \left[ c_m(\beta) \right]^2 = -\sum_{m=1}^{\infty} \frac{1}{m} \hat{Z}(m\beta), \quad \hat{Z}(\beta) = \left[ \hat{Z}_0(\beta) \right]^2. \] (3.18)

Thus the one-particle partition function corresponding to the large \( N \) limit of the singlet-sector partition function is given by the square of the free scalar one in (3.5) [27]. Repeating this argument in the real \( O(N) \) scalar case one finds a similar result [8], i.e.

\[ \hat{Z}_{U(N)}(\beta) = \left[ \hat{Z}_0(\beta) \right]^2, \] (3.19)

\[ \hat{Z}_{O(N)}(\beta) = \frac{1}{2} \left[ \hat{Z}_0(\beta) \right]^2 + \frac{1}{2} \hat{Z}_0(2\beta). \] (3.20)

These are exactly the expressions that appear in the lhs parts of the character relations (2.6) and (2.7).
The leading Casimir energy term in (3.17) is the same as in (3.3) for \( 2N \) real scalars, while the non-trivial \( \beta \)-dependent part \( \hat{F}(\beta) \) is of order \( N^0 \) rather than of order \( N \) when the singlet constraint is not imposed.

While the scalar one-particle partition function \( \hat{Z} \) (3.15) counts all operators built out of one scalar and its derivatives modulo equations of motion, the singlet partition function \( \hat{Z} \) counts all bilinear spin 1 currents modulo the conservation condition. Equivalently, it is the total one-particle partition function for the theory of massless higher spin fields in thermal cover of AdS5 [8, 27]. Similar results are found [8] in the spin \( \frac{1}{2} \) case with \( Z_0 \to Z_\perp = \frac{-i g^2}{(1-q)} \) and the sign plus in (3.7) replaced by the sign minus.

Let us now repeat the same computation using the Maxwell action

\[
S_1 = \frac{1}{2} \int d^4x \sqrt{g} \ F_{\mu\nu} F^{\mu\nu}
\]

instead of the scalar one. For one real Maxwell field we get the well-known expression for the curved-space partition function in the usual covariant Feynman gauge

\[
L_1 = \frac{1}{2} \partial_\mu A_i \partial^\mu A^i + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} D^i A_i D_0 A_0.
\]

The contribution of the integral over \( A_0 \) cancels against the ghost or measure determinant giving back (3.22).

From the spectrum of transverse three-vector Laplacian \( (-D^2)_{\perp} \) on \( S^3 \) the spectrum of \( \mathcal{O}_{1\perp} \) in (3.22) is found to be

\[
\lambda_{i,n} = \omega_i^2 + \omega_n^2, \quad \omega_n = n + 2, \quad d_\nu = 2(n + 1)(n + 3).
\]

The resulting \( Z_1 \) has the form (3.3) where the one-particle partition function is thus equal to the character of spin 1 doubleton representation in (2.5) (see (3.22))

\[
Z_1 = \sum_{n=0}^{\infty} d_n e^{-\lambda_{i,n}} = \frac{2 q^2 (3 - q)}{(1 - q)^3}.
\]

Let us now consider \( N \) complex Maxwell vectors and impose the singlet constraint. One way to do this is to start with \( U(N + 1) \) YM theory and split the \( U(N + 1) \) field into the \( U(N) \) one \( A_m \), \( N \) complex vectors \( A_m \) in the fundamental of \( U(N) \) and a singlet. Then the YM action can be written as in (3.6)

\[
L = \frac{1}{2} F_{r m} F_{r}^{m n} + \frac{1}{4 g^2} \text{tr} F_{m n} F^{m n}, \quad F_{r m}^{m n} = D_m A_r^r - D_r A_m^r,
\]

\[
L = \frac{1}{2} F_{r m} F_{r}^{m n} + \frac{1}{4 g^2} \text{tr} F_{m n} F^{m n}, \quad F_{r m}^{m n} = D_m A_r^r - D_r A_m^r,
\]

\[
L = \frac{1}{2} F_{r m} F_{r}^{m n} + \frac{1}{4 g^2} \text{tr} F_{m n} F^{m n}, \quad F_{r m}^{m n} = D_m A_r^r - D_r A_m^r,
\]

\[
L = \frac{1}{2} F_{r m} F_{r}^{m n} + \frac{1}{4 g^2} \text{tr} F_{m n} F^{m n}, \quad F_{r m}^{m n} = D_m A_r^r - D_r A_m^r,
\]
where \( D_n = D_n(A) \), \( r = 1, \ldots, N \). We rescaled \( A_m \) by \( g \) and ignored the decoupled singlet. Taking the limit \( g \to 0 \) (for fixed \( N \)) understood as localizing the path integral over \( A_m \) on \( F_{mn} = 0 \) or pure gauge configurations we get then the direct analog of (3.6) and (3.7), i.e. the following generalization of (3.23)

\[
L_\lambda = D_0 A_\lambda^r D_0 A_\lambda^r + \frac{1}{2} F_{r,i}^\lambda F_{r,i}^\lambda,
\]

(3.27)

where \( D_0 \) contains \( A_0 \) in (3.7) (we ignored the trivial decoupled contribution of \( A_0 \)). Integrating over \( A_r \) and \( U \) we then get again the relations (3.8)–(3.10) where now \( d_n \) and \( o_n \) are given by (3.24). The remaining derivation of the large \( N \) limit of the singlet partition function is then literally the same as above in the scalar case.

As a result, we get exactly the same expressions for the singlet-sector partition function as in (3.19) and (3.7) with \( \hat{Z}_0 \) replaced by \( \hat{Z}_1 \), i.e.\(^9\)

\[
\hat{Z}_{U(N)}(\beta) = [Z(\beta)]^2 = \left[ \frac{2q^2(3 - q)}{(1 - q)^3} \right]^2,
\]

(3.28)

\[
\hat{Z}_{O(N)}(\beta) = \frac{1}{2} [Z(2\beta)]^2 + \frac{1}{2} \hat{Z}_1(2\beta)
\]

\[
= \frac{1}{2} \left[ \frac{2q^2(3 - q)}{(1 - q)^3} \right]^2 + \frac{1}{2} \left( \frac{2q^2(3 - q^2)}{(1 - q^2)^3} \right).
\]

(3.29)

These expressions are indeed consistent with the AdS/CFT, i.e. with counting of conserved currents or counting of states in the dual type C theory in AdS_5 as follows from the comparison with the identities for the corresponding characters in (2.11) and (2.12).

### 4. Casimir energy

The Casimir energy is determined by the same spectrum as the one-particle partition function \( Z \) in (3.3) and (3.4) and they may be directly related as

\[
E_c = \frac{1}{2} \sum_n d_n o_n = \frac{1}{2} \zeta_F(-1), \quad \zeta_F(z) = \sum_n d_n o_n = \frac{1}{\Gamma(z)} \int_0^\infty d\beta \beta^{z-1} Z(e^{-\beta}).
\]

(4.1)

One can show that the Casimir energy vanishes if the partition function \( Z \) obeys the property \( Z(q) = Z(q^{-1}) \) \(^8\). This property is true for the square of the scalar partition function in (3.19) (see (3.5)) but is not valid for the second term in (3.7) in the \( O(N) \) case. Since the Casimir energy is given by the linear in \( \beta \) part of the log of the total partition function in (3.3) it then follows that \( E_c \) for the spectrum corresponding to the singlet partition function (3.7) is the same as for one real 4d scalar, i.e. in the scalar or type A case of AdS/CFT we have

\[
\text{scalar:} \quad \left( \hat{E}_c \right)_{U(N)} = 0, \quad \left( \hat{E}_c \right)_{O(N)} = E_c \left( \{0, 0\} \right) = \frac{1}{240}.
\]

(4.2)

In the vector case (3.28) there is no \( q \to q^{-1} \) invariance already in the \( U(N) \) case. Since the combination \( [Z(\beta)]^2 - 2Z(\beta) = \frac{4(3 - q)(1 - 3q)q^2}{(1 - q)^3} \) is symmetric under \( q \to q^{-1} \) in this case

\(^9\) Note that the resulting canonical partition function in (3.17) is different from the canonical single-trace partition function of one-loop \( SU(N) \) YM theory on \( S^4 \times S^3 \) \([30, 31, 34]\) counting single-trace operators built out of any number of fields, not just two. It is given at large \( N \) by \( Z_{YM} = -\sum_{k=1}^{\infty} \frac{1}{k} \ln [1 - Z(ik\beta)] \), where \( \varphi \) is Euler’s totient function.
the singlet-sector Casimir energy is same as of 2 real Maxwell 4d vectors. In the \(O(N)\) case the contribution of the first term in (3.29) is half of the \(U(N)\) case one and the second term gives the same \(E_c\) contribution. Thus we find

\[
\left( \hat{E}_c \right)_{O(N)} = \left( \hat{E}_c \right)_{U(N)} = 2E_c = 2E_c([1, 0], e) = 2 \times \frac{11}{120}. \tag{4.3}
\]

The relation between the characters or one-particle partition functions in (2.13), (2.14) then implies the corresponding relations for the total Casimir energy of the AdS\(_5\) theory: in non-minimal and minimal type C theory we then get, respectively,

\[
2E_c(4; 0, 0) + E_c(4; 1, 0) + 2 \sum_{s=2}^{\infty} E_c\left(2 + s; \frac{s}{2}, \frac{s}{2} \right) + \sum_{s=2}^{\infty} E_c\left(2 + s; \frac{s}{2}, \frac{s}{2} \right) = 2E_c, \tag{4.4}
\]

\[
2E_c(4; 0, 0) + \sum_{s=2}^{\infty} E_c\left(2 + s; \frac{s}{2}, \frac{s}{2} \right) + \sum_{s=2,4 \ldots}^{\infty} E_c\left(2 + s; \frac{s}{2}, \frac{s}{2} \right) = 2E_c. \tag{4.5}
\]

Note that here one does not need to worry about regularization of the sums over spins provided one uses the \(\zeta\)-function prescription to define \(E_c\) as in (4.1). Namely, one is first to compute the sum over spins of all partition functions for finite \(q\) or the sum of their Mellin transforms \(\zeta_z(E)\) and then to continue the result to the required \(z = -1\) value. In view of (2.11), (2.13), and (4.3) the relation (4.4) expresses the equality of \(E_c\) computed for the product of two spin 1 doubleton representations to twice \(E_c\) for the single doubleton representation, i.e.

\[
E_c([1, 0], e \otimes [1, 0], e) = 2E_c([1, 0], e). \tag{4.6}
\]

Remarkably, the relations (4.4) or (4.6) and (4.5) are true also for the boundary theory conformal anomaly coefficients \(a\) and \(c\) [25] (these correspond to partition functions for \(S^4\) or Ricci flat space instead of \(S^4 \times S^1\)).

To interpret (4.6) from the point of view of AdS/CFT we observe that the full boundary CFT contribution to the Casimir energy is simply proportional to \(N\) (see (3.17)). This should correspond to the classical type C theory contribution in AdS\(_5\). Then the duality requires the one-loop bulk contribution to \(E_c\) to vanish, but it does not according to (4.4) and (4.5). To reconcile this with AdS/CFT duality we follow the suggestion made in the real scalar and fermion cases [6, 8] and conjecture that the coefficient in front of the classical bulk theory action should be shifted by an integer from its naive value \(N\). Namely, let us assume that the classical actions of the non-minimal type C theory dual to complex \(U(N)\) Maxwell theory and the minimal type C theory dual to real \(O(N)\) Maxwell theory have the form

\[
S_{\text{non-min } U(N)} = (2N - 2)(S_0 + \ldots), \quad S_{\text{min } O(N)} = (N - 2)S_0. \tag{4.7}
\]

Here \(S_0\) stands for the common sector of the two type C theories (see (2.11) and (2.12)) Then the factor of two difference between the leading large \(N\) terms in the two actions will be consistent with the fact that the boundary theory Casimir energy of \(N\) complex fields is the same as that of the \(2N\) real fields. The equal negative subleading terms will be required to

10 This procedure is equivalent to using an exponential cutoff \(\exp \left[-e(s + \frac{1}{2})\right]\) in the sum over \(s\) and dropping all terms that are singular in the \(\epsilon \to 0\) limit, see [8].
cancel the equal one-loop corrections (4.4) and (4.5) of the quantum fluctuations of the bulk fields, in agreement with the absence of the subleading in $N$ term in $E$, in the boundary theory.

It is interesting to note that the structure of the two actions in (4.7) is consistent also with the fact that $U(N)$ theory should be equivalent to $O(2N)$ one as far as ‘non-singlet’ properties like Casimir energy or partition function on a space with trivial holonomy (e.g., $S^4$) are concerned: one complex field should be the same as 2 real ones. The reason for the triviality of the bulk action in the $U(1)$ ($N = 1$) case implied by (4.7) remains to be clarified further.

The above discussion has a straightforward generalization to the case when the boundary theory is represented by a combination of vectors, fermions and scalars and, in particular, to the supersymmetric case. For example, the relation (4.6) holds also if one replaces the spin 1 doubleton by a combination $[1, 0]_c + n_1^c [1/2, 0]_c + n_0 [0, 0]$ which represents a super-doubleton for special choices of $n_1^c$ and $n_0$ [25]. In the case of $\mathcal{N} = 4$ superdoubleton ($n_1^c = 4$, $n_0 = 6$) one gets the duality between the singlet sector of the theory of $N$ copies of $\mathcal{N} = 4$ supersymmetric Maxwell multiplet and a special $\mathcal{N} = 4$ supersymmetric higher spin theory in AdS$_5$ generalizing maximally supersymmetric 5d gauged supergravity [16, 18, 35].

Acknowledgments

We thank K Alkalaev, R Metsaev and E Skvortsov for important discussions. This work was supported by the Russian Science Foundation grant 14-42-00047 associated with Lebedev Institute. The work of AAT was also partially supported by the ERC Advanced grant no.290456.

References

[1] Klebanov I and Polyakov A 2002 AdS dual of the critical O(N) vector model Phys. Lett. B 550 213–9
[2] Sezgin E and Sundell P 2005 Holography in 4D (super) higher spin theories and a test via cubic scalar couplings J. High Energy Phys. JHEP07(2005)044
[3] Didenko V and Skvortsov E 2013 Towards higher-spin holography in ambient space of any dimension J. Phys. A: Math. Theor. 46 214010
[4] Bekaert X and Grigoriev M 2013 Higher order singletons, partially massless fields and their boundary values in the ambient approach Nucl. Phys. B 876 667
Bekaert X and Grigoriev M 2013 Notes on the ambient approach to boundary values of AdS gauge fields J. Phys. A: Math. Theor. 46 214008
[5] Giombi S, Klebanov I R, Pufu S S, Safdi B R and Tarnopolsky G 2013 AdS description of induced higher-spin gauge theory J. High Energy Phys. JHEP10(2013)016
[6] Giombi S and Klebanov I R 2013 One loop tests of higher spin AdS/CFT J. High Energy Phys. JHEP12(2013)068
[7] Giombi S, Klebanov I R and Safdi B R 2014 Higher spin AdS$_{d+1}$/CFT$_d$ at one loop Phys. Rev. D 89 084004
[8] Giombi S, Klebanov I R and Tseytlin A A 2014 Partition functions and Casimir energies in higher spin AdS$_{d+1}$/CFT$_d$ Phys. Rev. D 90 024048
[9] Schnitzer H J 2004 Gauged vector models and higher spin representations in AdS(5) Nucl. Phys. B 695 283–300
[10] Maldacena J and Zhiboedov A 2013 Constraining conformal field theories with a higher spin symmetry J. Phys. A: Math. Theor. 46 214011
[11] Boulanger N, Ponomarev D, Skvortsov E and Taronna M 2013 On the uniqueness of higher-spin symmetries in AdS and CFT Int. J. Mod. Phys. A 28 1350162
[12] Stanev Y S 2013 Constraining conformal field theory with higher spin symmetry in four dimensions Nucl. Phys. B 876 651–66
[13] Alba V and Diab K 2013 Constraining conformal field theories with a higher spin symmetry in $d = 4$ arXiv:1307.8092
[14] Vasiliev M A 2002 Conformal higher spin symmetries of 4-d massless supermultiplets and osp $(L_2 M)$ invariant equations in generalized (super)space Phys. Rev. D 66 066006
[15] Bekaert X and Grigoriev M 2010 Manifestly conformal descriptions and higher symmetries of bosonic singletons SIGMA 6 038
[16] Gunaydin M, Minic D and Zagermann M 1998 4-D doubleton conformal theories, CPT and IIB string on $\text{AdS}_5 \times S^5$ Nucl. Phys. B 534 96–120
[17] Ferrara S and Fronsdal C 1998 Gauge fields as composite boundary excitations Phys. Lett. B 433 19–28
[18] Sezgin E and Sundell P 2001 Doubletons and 5-D higher spin gauge theory J. High Energy Phys. JHEP09(2001)036
[19] Vasiliev M 2004 Higher spin superalgebras in any dimension and their representations J. High Energy Phys. JHEP12(2004)046
[20] Dolan F 2006 Character formulae and partition functions in higher dimensional conformal field theory J. Math. Phys. 47 062303
[21] Boulanger N and Skvortsov E 2011 Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime J. High Energy Phys. JHEP09(2011)063
[22] Metsaev R R 1995 Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time Phys. Lett. B 354 78–84
[23] Alkalaev K, Shaynkman O and Vasiliev M 2004 On the frame-like formulation of mixed symmetry massless fields in $(\text{AdS}_d)$ Nucl. Phys. B 692 363–93
[24] Alkalaev K 2013 Mixed-symmetry tensor conserved currents and AdS/CFT correspondence J. Phys. A: Math. Theor. 46 214007
[25] Beccaria M and Tseytlin A 2014 Higher spins in AdS$_5$ at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT arXiv:1410.3273
[26] Flato M and Fronsdal C 1978 One massless particle equals two Dirac singletons Lett. Math. Phys. 2 421–6
[27] Shenker S H and Yin X 2011 Vector models in the singlet sector at finite temperature arXiv:1109.3519
[28] Gopakumar R, Gupta R K and Lal S 2011 The heat kernel on AdS J. High Energy Phys. JHEP11(2011)010
[29] Gupta R K and Lal S 2012 Partition functions for higher-spin theories in AdS J. High Energy Phys. JHEP07(2012)071
[30] Sundborg B 2000 The Hagedorn transition, deconfinement and $N = 4$ SYM theory Nucl. Phys. B 573 349–63
[31] Aharony O, Marsano J, Minwalla S, Papadodimas K and van Raamsdonk M 2004 The Hagedorn—deconfinement phase transition in weakly coupled large $N$ gauge theories Adv. Theor. Math. Phys. 8 603–96
[32] Schnitzer H J Confinement/deconfinement transition of large $N$ gauge theories in perturbation theory with $N(\phi)$ fundamentals: $N(\phi)/N$ finite arXiv:hep-th/0612099
[33] Beccaria M, Bekaert X and Tseytlin A A 2014 Partition function of free conformal higher spin theory J. High Energy Phys. JHEP08(2014)113
[34] Polyakov A M 2002 Gauge fields and space-time Int. J. Mod. Phys. A 17S1 119–36
[35] Sezgin E and Sundell P 2002 Massless higher spins and holography Nucl. Phys. B 644 303 Sezgin E and Sundell P 2003 Nucl. Phys. B 660 403 (erratum)