Exceptional reductions

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Received 17 January 2011
Published 22 March 2011
Online at stacks.iop.org/JPhysA/44/155207

Abstract
Starting from basic identities of the group $E_8$, we perform progressive reductions, namely decompositions with respect to the maximal and symmetric embeddings of $E_7 \times SU(2)$ and then of $E_6 \times U(1)$. This procedure provides a systematic approach to the basic identities involving invariant primitive tensor structures of various irreps of finite-dimensional exceptional Lie groups. We derive novel identities for $E_7$ and $E_6$, highlighting the $E_8$ origin of some well-known ones. In order to elucidate the connections of this formalism to four-dimensional Maxwell–Einstein supergravity theories based on symmetric scalar manifolds (and related to irreducible Euclidean Jordan algebras, the unique exception being the triality-symmetric $N=2$ $stu$ model), we then derive a fundamental identity involving the unique rank-4 symmetric invariant tensor of the 0-brane charge symplectic irrep of $U$-duality groups, with potential applications in the quantization of the charge orbits of supergravity theories, as well as in the study of multi-center black hole solutions therein.

PACS numbers: 02.20.Tw, 04.65.+e, 04.50.Cd

1. Introduction

Supergravity theories have a rich algebraic structure, which also reflects into their scalar manifolds. A particularly remarkable class of scalar manifolds is given by the homogeneous spaces $G/H$, with $G$ a non-compact Lie group, and $H$ denoting its maximal compact subgroup. In particular, in maximal supergravities the $U$-duality\textsuperscript{3} groups $G$ belong to the so-called exceptional $E_{n(0)}$-sequence [3, 4] of symmetries of theories in $11-n$ dimensions. This sequence is encoded in the very-extended Kac–Moody algebra $E_{11}$ [5], and each theory corresponds to a decomposition with respect to each subalgebra $GL(11-n,\mathbb{R}) \times E_{n(0)}$. When $n = 9$, that is in two dimensions, the field equations of the theory possess an $E_{9(9)}$ symmetry, which is the infinite-dimensional affine extension of $E_{8(8)}$ [4, 6]. This is completely general: the (on-shell) symmetry of a two-dimensional theory obtained from the reduction

\textsuperscript{3} Here $U$-duality is referred to as the ‘continuous’ symmetries of [1]. Their discrete versions are the $U$-duality non-perturbative string theory symmetries introduced by Hull and Townsend [2].
of a three-dimensional theory whose scalars parametrize the manifold \( G/H \) is the infinite-dimensional affine extension of \( G \).

Since the process of dimensional reduction leads to infinite-dimensional symmetries in \( D \leq 2 \), when one confines his attention to finite-dimensional symmetry groups the endpoint of a chain of symmetries of theories related by dimensional reduction is \( D = 3 \). As shown in [7, 8], and further systematized in [9, 10] (elaborating on ideas and results on ‘group disintegrations’ of [3, 11]; see also [12]), one has a group-theoretic framework to determine which \( D = 3 \) theories can be conceived as dimensional reductions of higher-dimensional theories. In particular, in [10] the systematics of oxidations involving \textit{non-split} \( U \)-dualities, including the bosonic sectors of the theories with eight supersymmetries based on symmetric scalar manifolds, related by the \( r \)- and \( c \)- maps \([13–15]\), has been developed using the diagrammatic language of Tits–Satake diagrams (see e.g. [16] and references therein). This result was systematized in [17], where it was shown that starting from the Tits–Satake diagram of the three-dimensional theory one can construct a very-extended Kac–Moody algebra such that its Dynkin diagram encodes all the properties of the theory in various dimensions. The reconstruction procedure, which allows us to determine the higher-dimensional ancestor(s) of a lower-dimensional theory, is usually named ‘oxidation’. A remarkable aspect of oxidation is that, differently from the dimensional reduction, it is not unique, in the sense that it can admit different ‘branches’, namely distinct higher-dimensional theories (eventually related by string dualities) originating the same lower-dimensional theory upon dimensional reduction.

The investigation presented in this paper approaches the oxidations from the point of view of the fundamental identities involving invariant primitive tensor structures of the relevant (namely, fundamental and adjoint) irreps of the \( U \)-duality groups. Confining ourselves to finite-dimensional groups, we start from basic identities in the adjoint irrep \( 248 \) of \( E_8 \) and we perform progressive reductions, given by decompositions with respect to the maximal and symmetric \( E_8 \)-embedding of \( E_7 \times SU (2) \) and then to the maximal and symmetric \( E_7 \)-embedding of \( E_6 \times U (1) \). Within such a framework, this approach provides a systematic way to derive all basic identities describing the structure of \( E_8, E_7 \) and \( E_6 \) exceptional Lie groups. Indeed, we derive many novel identities involving the relevant invariant primitive tensors of such groups, and we also highlight the common origin (through iterated reduction) of some well-known identities. Furthermore, we also present some results on the further maximal and symmetric \( E_6 \)-embedding of \( SO (10) \times U (1) \), retrieving various well-known Fierz identities of \( SO (10) \).

Our procedure applies to the \( D = 3 \to 4 \to 5 \to 6 \) entries of the \( E_{n(\infty)} \) exceptional sequence of Cremmer–Julia, pertaining to split forms and thus to maximal supergravity in \( D = 3, 4, 5, 6 \), related to the irreducible Euclidean Jordan algebra over the split form of the octonions \( \mathbb{O}_s \):

\[
E_8(8) \to E_7(7) \to E_{6(6)} \to SO (5, 5) ; \tag{1.1}
\]

in particular, \( SO (5, 5) \equiv E_{5(5)} \) is the \( U \)-duality group of the \( (2, 2) \) non-chiral maximal \( D = 6 \) supergravity based on\(^4 \) \( J_2^{D_5} \sim \Gamma_{5,5} \). However, since we do not specify the non-compact real \( \mathbb{R} \) stands for the Jordan algebra of degree 2 with a quadratic form of Lorentzian signature \((5, 5)\), which is nothing but the Clifford algebra of \( O (5, 5) \) [21].

Also note the maximal (symmetric) algebraic embedding

\[
J_3^{D_5} \supset \max \mathbb{R} \oplus J_2^{D_5}.
\]

\(^4 \)
form of the groups under consideration, our procedure also applies to the following non-split version of the Cremmer–Julia sequence (1.1)

\[
E_{8(-24)} \xrightarrow{c} E_{7(-5)} \xrightarrow{r} E_{6(-26)} \longrightarrow SO(1, 9); \tag{1.2}
\]

in particular, \(SO(1, 9)\) is the \(U\)-duality group of the \((1, 0)\) chiral minimal \(D = 6\) magic supergravity based on\(^5\) \(J_3^A \sim \Gamma_{1,9}\). This sequence has an interpretation in terms of iterated oxidations pertaining to the ‘magic’ supergravity theory with eight supersymmetries in \(D = 3 \rightarrow 4 \rightarrow 5 \rightarrow 6\), related to the irreducible Euclidean Jordan algebra over the division algebra of the octonions \(\mathbb{O}\).

Remarkably, the results on the fundamental identities of \(E_8, E_7\) and \(E_6\) also hold for the first three elements of the sequence

\[
E_{8(-24)} \longrightarrow E_{7(-5)} \longrightarrow E_{6(2)} \longrightarrow F(4) \longrightarrow SO(4, 4) \longrightarrow G_{2(2)}, \tag{1.3}
\]

This sequence does not have an interpretation in terms of iterated oxidation, but, as reported in table 1, it rather describes a chain of embeddings of the \(D = 3\) \(U\)-duality groups of supergravity theories with symmetric scalar manifolds associated with irreducible Euclidean Jordan algebras, given in the second line of (1.3). The cases \(J_3^A\) (\(A = \mathbb{O}, \mathbb{H}, \mathbb{C}\) and \(\mathbb{R}\) being the four division algebras of octonions, quaternions, complex and real numbers) correspond to ‘magic’ supergravities \([18]\), whereas the cases \(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}\) and \(\mathbb{R}\) respectively pertain to the \(c\)-map of the so-called \(\mathcal{N} = 2, D = 4\) \(stu\) \([24]\) and \(r^3\) models. The corresponding cosets can all be obtained by a \(c\)-map \([13]\) of suitable symmetric special Kähler manifolds. Clearly, the rank-3 Jordan algebra \(\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}\) is not irreducible; however, it is \(su\) \(g\) \(ener\) \(s\)ic, because it enjoys the remarkable \textit{triality symmetry} \([24]\).

Also note that sequence (1.1) is related to sequence (1.3) by the maximal symmetric embedding \(E_{8(8)} \supset E_{7(-5)}\), which has the trivial supergravity interpretation that \(\mathcal{N} = 4, d = 3 J_3^A\)-magic theory is a consistent truncation of maximal \(\mathcal{N} = 16, d = 3\) supergravity.

It is worth remarking that in our treatment we will not restrict to dimensional reductions on purely spacelike internal manifolds (usually \textit{tori}). As resulting from the analyses of \([7, 25]\), the only group theoretical difference between timelike and spacelike reductions is the non-compact nature of the coset stabilizer \(H\). Recently, timelike reductions to \(D = 3\) have been used as an efficient tool to describe and classify spherically symmetric, asymptotically flat and stationary black hole solutions (and the corresponding scalar flows) of \(D = 4\) supergravity theories with symmetric scalar manifolds (see e.g. \([26]\) and references therein). Interestingly, this also turned out to be relevant within the so-called black hole/qubit correspondence \([27]\).

In general, our group-theoretical approach to oxidation can be considered as complementary to the one exploited in \([3, 7–11]\), because we deal with the reductions of the identities involving the invariant primitive tensors of the relevant irreps of the \(U\)-duality groups. This procedure provides a systematic derivation of a number of fundamental identities characterizing the \(U\)-duality groups of supergravity theories in various dimensions.

As an application of the results on the oxidations of group structure identities discussed above, we will then derive an identity involving the so-called \(K\)-tensor, namely the unique rank-4 symmetric invariant tensor of the irrep of \(G_4\) in which the black hole charges sit. When contracted with four charge vectors, the \(K\)-tensor gives rise to the \(G_2\)-invariant homogeneous

\(^5\) In the theories with eight supersymmetries, the so-called \(c\)-map \([13]\) and \(r\)-map (see e.g. \([15, 22]\) for more tables and a list of references) relate \(D = 4/D = 3\) and \(D = 5/D = 4\), respectively.

\(^6\) Note the maximal (symmetric) algebraic embedding (see e.g. \([23]\))

\[J_3^A \sim_{\text{max}} \mathbb{R} \oplus J_2^A,\]

where \(J_2^A \sim \Gamma_{1,q+1}\), with \(q = \dim_\mathbb{A} (\mathbb{A}) = 8, 4, 2, 1\) for the division algebras \(\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}\), respectively.
developments on the possible properties of the charge orbits of supergravity theories, which might be relevant in relation to recent investigations. Indeed, the possible structure of the charge orbits and 'moduli spaces' of extremal black hole attractors [28] in (Minkowskian) $\mathbb{R}^4$ is the Jordan triple system (not upliftible to $D = 5$) generated by $2 \times 2$ matrices over $\mathbb{O}$ [18]. Note that the $\text{stu}$ model, based on $\mathbb{O} \oplus \mathbb{O} \oplus \mathbb{O}$, is reducible, but triality symmetric. All cases pertain to models with eight supersymmetries, with the exception of $\text{stu}_5$, related to both 8 and 24 supersymmetries, because the corresponding supergravity theories share the very same bosonic sector [18–20]. Note that the dimensions $f$ and $d$ of all reported $G_4$'s satisfy the relation $d = \frac{1}{2} f(f+1)$. In particular, by considering $\mathcal{N} = 2$, $D = 4$ 'magic' supergravities (based on $J_4^{\mathbb{O}}$) as well as $\mathcal{N} = 8$, $D = 4$ maximal supergravity (based on $J_1^{\mathbb{O}}$), and defining $q = \dim_\mathbb{R} \mathcal{A} = 8, 4, 2, 1$, Hor$\mathcal{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, the relation $f = 6q + 8$ yields $d = \frac{1}{2} (6q+4)(q+2)$. Note that these relations also admit a limit $q = 0$, reproducing $d$ and $f$ of the reducible yet triality symmetric $\text{stu}$ model.

### Table 1. $U$-duality groups $G_5, G_4$ and $G_3$ related to irreducible rank-3 Euclidean Jordan algebras in (Minkowskian) $\mathbb{D} = 3, 4$ and 5, respectively. Also $d = \dim_\mathbb{R} \text{Ad} (G_4)$ and $f = \dim_\mathbb{O} \mathbb{R} (G_4)$ are given.

| $J_3$ | $G_5$ | $G_4$ | $G_3$ |
|-------|-------|-------|-------|
| $J_4^{\mathbb{D}}$ | $E_6(\mathbb{O})$ | $d = 133$, $f = 56$ | $E_8(8)$ |
| $J_4^{\mathbb{O}}$ | $E_6(\mathbb{O})$ | $d = 133$, $f = 56$ | $E_8(24)$ |
| $J_4^{\mathbb{S}}$ | $SU^+(6)$ | $SO^+(12)$ | $E_{7(-5)}$ |
| $J_4^{\mathbb{S}}$ | $SU(3, \mathbb{C})$ | $Sp(6, \mathbb{R})$ | $F_4(4)$ |
| $J_4^{\mathbb{D}}$ | $SL(3, \mathbb{C})$ | $SU(1, 5)$ | $G_2(2)$ |
| $J_4^{\mathbb{R}}$ | $SL(3, \mathbb{R})$ | $d = 3$, $f = 4$ | $G_2(2)$ |
| $\mathbb{R}$ | $[SO(1, 1)]^3$ | $d = 9$, $f = 8$ | $SO(4, 4)$ |

quartic polynomial $I_d$, which plays a prominent role in the algebraic classification of the charge orbits and 'moduli spaces' of extremal black hole attractors [28] in $d = 4$ Maxwell–Einstein supergravities (see e.g. [29–31] and references therein). Moreover, the identity we will derive has potential applications in at least two other frameworks, namely (i) the quantization of the charge orbits of supergravity theories, which might be relevant in relation to recent developments on the possible $UV$-finiteness of $\mathcal{N} = 8$, $D = 4$ supergravity (see e.g. [32] and references therein); (ii) the group-theoretical study of $U$-invariants relevant for multi-center black holes [33, 34].

The plan of the paper is as follows. In section 2, we introduce the notation and report some general results on the relation between various data characterizing some Lie groups, appearing, through suitable non-compact real forms, as $U$-duality groups of supergravity theories. We also consider identities involving up to four structure constants, holding true for all finite-dimensional exceptional Lie groups. Sections 3 and 4 exploit the approach based on the progressive oxidation of the starting $E_8$-identities involving up to four structure constants.
As discussed above, this amounts to decomposing such identities with respect to the following chain of maximal and symmetric group embeddings:

\[ E_8 \supset E_7 \times SU(2) \supset E_6 \times SU(2) \times U(1), \]  

and it provides a systematic way to derive all \( E_7 \)-identities and \( E_6 \)-identities originating from the basic starting relations for \( E_8 \). We name this method ‘exceptional reductions’. In section 5 we derive a fundamental identity involving the \( K \)-tensor. Besides the aforementioned importance of the \( K \)-tensor for the theory of extremal black hole attractors [28] in Maxwell–Einstein supergravities (see section 5), this (hitherto unknown) result has potential application in the issue of the classification of the orbits of the irrep \( R(G_4) \) in the presence of Dirac–Zwanziger–Schwinger charge quantization conditions, especially for \( N = 8 \), \( D = 4 \) supergravity (see e.g. [32, 35–37] and references therein), as well as in the study of multi-center black holes [33, 34].

Various details and further results are given in the three appendices which conclude the paper. In appendix A, we summarize our conventions for \( SU(2) \), crucial in order to perform the reduction of \( E_8 \)-identities in section 3. In appendix B, we further reduce some \( E_6 \)-identities obtained in section 4 with respect to the maximal and symmetric embedding

\[ E_6 \supset SO(10) \times U(1), \]  

retrieving some well-known \( SO(10) \) Fierz identities, whose common origin (through iterated reduction) is thus clarified. In appendix C, we derive a useful group theoretical decomposition used in section 5, holding at least for all \( G_4 \)'s reported in table 1.

2. Preliminaries

This section is aimed at introducing the notation used throughout the paper, and at discussing the general approach which we will follow. Furthermore, some basic identities for the exceptional Lie group \( E_8 \) will be derived, which will then be used in the analysis of sections 3 and 4, in turn leading to other basic identities for the exceptional groups \( E_7 \) and \( E_6 \), respectively.

For a generic simple Lie group \( G \) the Cartan–Killing metric \( \kappa_{\alpha\beta} \) is defined as

\[ C_{\text{Adj}}\kappa_{\alpha\beta} = f_{\alpha\gamma} f_{\beta\gamma}, \]  

where \( C_{\text{Adj}} \) is the quadratic Casimir in the adjoint irrep \( \text{Adj} \) (with lowercase Greek indices), and \( f_{\alpha\beta\gamma} \) are the structure constants of the corresponding Lie algebra \( g \).

We then consider the charge irrep \( R \) of the \( U \)-duality group \( G \). In \( D = 4 \), \( R = \text{Sympl} \), namely it is the smallest non-trivial symplectic irrep of \( G \) (e.g. \( R = \text{Fund} = 56 \) for \( E_7 \)), with a unique singlet \( C_{MN} \) (the symplectic metric) in its antisymmetric tensor product\(^7\)

\[ \exists! C_{MN} \equiv 1 \in \text{Sympl}^2. \]

In \( D = 5 \), \( R \) is not symplectic, but rather it splits into two (electric and magnetic) charge irreps (e.g. for \( E_6 \): \( R = \text{Fund} = 27 \) gradient, and \( \overline{R} = \overline{\text{Fund}} = 27 \) contragradient). The \( D = 3 \) case of \( E_8 \) stands on its own, because \( R = \text{Fund} = \text{Adj} = 248 \); see the treatment of sections 3, 4 and 5 for further elucidation.

The quadratic Casimir \( C_R \) of \( R \) (with uppercase Latin indices) is defined via

\[ C_R^{MN} = \kappa_{\alpha\beta} t_M^{\alpha} t_N^{\beta}, \]  

where \( t_M^{\alpha N} \) are the generators of \( g \) in \( R(G) \):

\[ [t_M^{\alpha}, t_N^{\beta}] = f_{\alpha\beta\gamma} t_M^{\gamma N}. \]  

\(^7\) The subscripts ‘s’ and ‘a’ respectively denote the symmetric and antisymmetric tensor products throughout.
In this paper we will adopt instead a different metric (see for instance the appendix of [38]), namely

\[ g^{\alpha\beta} = \text{Tr}(\alpha^\beta) = i_{M}^\alpha N_i^\beta M. \]  

(2.5)

This in turn implies

\[ g_{\alpha\beta} i^\alpha_M P^\beta_N = \frac{d}{f} \delta^N_M, \]  

(2.6)

where \( d \equiv \text{dim}_\mathbb{R} \text{Adj} (G) \) and \( f \equiv \text{dim}_\mathbb{R} \mathbb{R} (G) \) throughout.

Within this notation, equation (2.1) is replaced by (see e.g. [38, 39] and references therein)

\[ f_{\alpha\beta} f^{\alpha\gamma}_\beta = -\frac{d}{f} \frac{C_{\text{Adj}}}{C_{\mathbb{R}}} g_{\alpha\beta} = -\frac{g^\gamma}{T} g_{\alpha\beta}, \]  

(2.7)

where \( g^\gamma \) is the dual Coxeter number of \( G \), and \( T \) is the Dynkin index of \( \mathbb{R} (G) \).

The results obtained in this paper, and in particular in section 5, hold at least for all \( D = 4 \) \( U \)-duality groups \( G_4 \)'s related to irreducible rank-3 Euclidean Jordan algebras, with the only exception of \( \text{stu} \) model (related to the triality-symmetric, reducible rank-3 Jordan algebra \( \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \)). Such groups are reported in table 1, along with their corresponding \( R \)-duality group both in \( D = 4 \) and \( D = 5 \) counterparts\(^8\).

It is worth observing that the fourth column of table 1, pertaining to \( D = 3 \), is composed only by suitable non-compact, real forms of all exceptional (finite-dimensional, as understood throughout) Lie groups (once again, with the exception of \( \text{stu} \) model). Interestingly, all exceptional Lie groups share the property that there exists a unique singlet in the completely symmetric rank-4 tensor product of their adjoint irrep, namely

\[ \mathbb{I} \in (\text{Adj})^2_4, \]  

(2.8)

On the other hand, for all infinite sequences of classical Lie algebras (but the groups \( SO (8) \) and \( SU (3) \)), there instead exist two such singlets, i.e.

\[ \mathbb{I}_1, \mathbb{I}_2 \in (\text{Adj})^4_4. \]  

(2.9)

By looking at the \( G_4 \) and \( G_3 \) given in table 1, one can observe that the Lie groups for which result (2.8) is valid are nothing but, in suitable non-compact real forms, the \( G_4 \)'s of supergravity theories based on irreducible rank-3 Euclidean Jordan algebras (with the only exception of the reducible, but triality-symmetric, rank-3 Jordan algebra \( \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \)).

As mentioned, in the cases with eight supersymmetries, \( G_3 \)'s and \( G_4 \)'s are related through c-map [13]. It is also worth remarking that, as yielded by table 1, the unique, exceptional group which is a \( U \)-duality group both in \( D = 4 \) and in \( D = 3 \) is \( E_7 \), actually through all its possible non-compact, real forms, namely \( E_{7(7)} \) (split, i.e. maximally non-compact, form) for maximal (\( \mathcal{N} = 8 \)) theory in \( D = 4 \), \( E_{3(25)} \) for \( \mathcal{N} = 2 \) `magic` octonionic model in \( D = 4 \), and \( E_{7(-5)} \) for \( \mathcal{N} = 4 \) `magic`(dual to \( \mathcal{N} = 12 \)) quaternionic supergravity in \( D = 3 \).

As anticipated, exceptions to (2.9) are provided by the following classical groups:

\[ \text{SO} (8) : \mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_3 \in (28)^4_4, \]  

(2.10)

\[ \text{SU} (3) : \mathbb{I} \in (8)^4_4. \]  

(2.11)

The three singlets characterizing the case of \( \text{SO} (8) \), which appears as the \( D = 3 \) \( U \)-duality group of the \( \text{stu} \) model through its non-compact form \( \text{SO} (4, 4) \), can actually be traced back to the triality of \( \text{SO} (8) \) itself (related to the threefold symmetry of its Dynkin diagram).

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\(^8\) The trivial Jordan algebra related to the so-called \( i^3 \) model, namely \( \mathbb{R} \), has rank 1 (see the eighth row of table 1).
On the other hand, $SU(3)$, in its non-compact form $SU(2,1)$, is the $D = 3$ $U$-duality group of the so-called universal hypermultiplet scalar sector, parameterized by

$$SU(2,1) \times SU(2) \times U(1),$$

(2.12)

which is both a rank-1 special Kähler and quaternionic manifold of real dimension 4, obtained as the $c$-map of the 'pure' $N = 2$, $D = 4$ supergravity. It is an example of Einstein space with self-dual Weyl curvature [40].

Observation (2.8) allows us to prove a crucial identity involving four structure constants, holding for all exceptional groups:

$$f_{\alpha\epsilon\tau} f_{\beta\epsilon\rho} f_{\gamma\tau\sigma} f_{\delta\rho\sigma} = a[g_{\alpha\delta} g_{\beta\gamma} + 2g_{\alpha(\beta g_{\gamma})\delta}] + b[2f^\epsilon_{\alpha\gamma} f_{\epsilon\beta\delta} - f^\epsilon_{\alpha\delta} f_{\epsilon\beta\gamma}],$$

(2.13)

where $a$ and $b$ are real ($G$-dependent) constants to be determined.

In order to prove (2.13), we start by noting that the expression on its right-hand side is symmetric upon the exchanges $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$, as well as upon the simultaneous exchanges $\alpha \leftrightarrow \beta$, $\gamma \leftrightarrow \delta$. Therefore, the indices can either be completely symmetric or with mixed symmetry (such that the complete symmetrization of any three indices vanishes). The completely symmetric part is the term of the right-hand side of (2.13) proportional to $a$, and this is fixed by property (2.8). On the other hand, the mixed symmetry part is the term of the right-hand side of (2.13) proportional to $b$, which is then determined using the Jacobi identity

$$f_{\alpha\beta\gamma} f^\alpha_{\delta\beta} = 0.$$  

(2.14)

For later convenience, let us define the ($G$-dependent) constant

$$k = \frac{g^\gamma}{7},$$

(2.15)

such that e.g. identity (2.7) can be rewritten as

$$f_{\alpha\beta\gamma} f^\gamma_{\beta\delta} = -kg_{\alpha\delta}.$$  

(2.16)

By suitably contracting indices, it is then straightforward to obtain

$$a = \frac{5k^2}{6(d+2)}, \quad b = -\frac{1}{6}k,$$

(2.17)

which plugged into (2.13) leads to

$$f_{\alpha\epsilon\tau} f^\epsilon_{\beta\rho} f^\tau_{\gamma\rho} f_{\delta\sigma\sigma} = \frac{5k^2}{6(d+2)}[g_{\alpha\delta} g_{\beta\gamma} + 2g_{\alpha(\beta g_{\gamma})\delta}] - \frac{1}{6}k[2f^\epsilon_{\alpha\gamma} f_{\epsilon\beta\delta} - f^\epsilon_{\alpha\delta} f_{\epsilon\beta\gamma}].$$  

(2.18)

Identity (2.18) was originally determined for $E_8$ in [41] by using computer manipulations. The present analysis shows that the same identity applies to all exceptional Lie groups. The values of the constants $k$, $a$ and $b$ appearing in the above identities, as well as the values of $g^\gamma$, $I$, $d$ and $f$, are summarized for all exceptional groups in table 2. As is well known, $E_8$ is peculiar, because $\text{Adj} = \text{Fund} (= 248)$ for this group.

Another identity with three structure constants, exploited in sections 3 and 4, is

$$f_{\alpha\beta\epsilon} f^\epsilon_{\rho\delta} f^\rho_{\gamma\delta} = -\frac{k}{2} f_{\alpha\beta\gamma},$$  

(2.19)

which can be proved using the Jacobi identities (2.14).

Starting from $E_8$, in sections 3 and 4, identities (2.14), (2.16), (2.18) and (2.19) will be used to derive many relevant identities of $E_7$ and $E_6$ which involve the invariant tensors made out of the $\text{Fund}$ and $\text{Adj}$ irreps occurring in the reduction. Our procedure amounts to splitting the indices of $\text{Fund}$ and $\text{Adj}$ with respect to the relevant maximal symmetric group
Table 2. Table giving the dual Coxeter number $g^\vee$, the Dynkin index $\tilde{I}$ (and their ratio $k$), the dimensions $d$ and $f$, as well as the parameters $a$ and $b$ for all exceptional Lie groups.

| $G$  | $g^\vee$ | $\tilde{I}$ | $d$ | $f$ | $k$ | $a$ | $b$ |
|------|----------|-------------|-----|-----|-----|-----|-----|
| $G_2$ | 4        | 11          | 7   | 4   | $\frac{5}{6}$ | $-\frac{2}{3}$ |
| $F_4$ | 9        | 33          | 26  | 3   | $\frac{5}{12}$ | $-\frac{2}{3}$ |
| $E_6$ | 12       | 123         | 27  | 4   | $\frac{1}{6}$ | $-\frac{2}{3}$ |
| $E_7$ | 18       | 6           | 133 | 56  | $\frac{1}{14}$ | $-\frac{2}{3}$ |
| $E_8$ | 30       | 30          | 248 | 248 | 1   | $\frac{1}{300}$ | $-\frac{2}{3}$ |

embeddings, and then to analyzing the invariant tensor structures occurring in the branching of products of irreps.

As discussed above, such a progressive reduction of the $U$-duality groups corresponds to a progressive oxidation, namely to a progressive uplift of the spacetime dimension $D$ in which the corresponding supergravity theory is defined.

3. $E_8 \supset E_7 \times SU(2)$

The aim of this section and the next one is to determine all possible $E_7$ and $E_6$ identities that result from the $E_8$ identities listed in the previous section. Most of these identities are already known in the form we write them; in particular, in the supergravity literature they have been used to derive the constraints satisfied by the so-called embedding tensor, and thus determine all possible gaugings of maximal supergravity theories in any dimensions (see e.g. [38, 42, 43]). Most identities have also been derived in [39], where all the possible gauging have been determined from $E_11$, following the results of [44] (see also [45], where the so-called trombone gaugings of [43] were shown to result from $E_{11}$). Still, we are now aware of the appearance in the literature of some of the identities we list, like the $E_7$-identities (3.24) and (3.27), and the $E_6$-identity (4.23). Anyway, what we want to emphasize the most here is the straightforward $E_8$ origin of all identities derived in this section and in the next one.

In this section, we consider the maximal and symmetric group embedding

$$E_8 \supset E_7 \times SU(2),$$

and we derive all the $E_7$-identities arising from the corresponding branching of the $E_8$-identities (2.14), (2.16), (2.18) and (2.19).

As mentioned above, $E_8$ is a peculiar exceptional Lie group, because $\text{Adj} = \text{Fund} = 248$. From the theory of symmetric invariant tensors of the $\text{Adj}$ of Lie groups (see e.g. [46]), it is known that the 248 of $E_8$ admit eight invariant tensors of order 2, 8, 12, 14, 18, 20, 24 and 30. The order-2 and order-8 invariants correspond to primitive invariant tensors, in terms of which the higher ones should be expressible [47]. The quadratic one is nothing but the Cartan–Killing metric, whereas the octic one has been recently constructed (for $E_8$ and its split form $E_{8(8)}$, in a manifestly $Spin(16)/\mathbb{Z}_2$-covariant form) in [47]. To the best of our knowledge, explicit expressions of all other higher-order invariants (also in terms of the rank-2 and rank-8 invariants) are currently unavailable. However, this will not affect the subsequent analysis, in which only the $E_8$-invariant tensors given by the rank-2 (symmetric) Cartan–Killing metric and by the rank-3 (completely antisymmetric) structure constants are involved.

9 See in particular appendix A of [39].

10 Unless otherwise noted, all group embeddings considered in this paper are maximal and symmetric.
Under (3.1), the 248 of $E_8$ branches as
\[ 248 \to (133, 1) + (56, 2) + (1, 3), \]  
(3.2)
where 133 = $\text{Adj} \ (E_7)$ and 56 = $\text{Fund} \ (E_7)$.

We will denote the indices in 248 of $E_8$ with tilded Greek indices $\tilde{\alpha}, \tilde{\beta}, \ldots$, whereas the indices in 133 and 56 of $E_7$ will be denoted by Greek indices $\alpha, \beta, \ldots$ and capital Latin indices $M, N, \ldots$ respectively. The index $i = 1, 2, 3$ and the index $a = 1, 2$ respectively denote the 3 = $\text{Adj} \ (\text{spin } s = 1)$ and 2 = $\text{Fund} \ (\text{spin } s = 1/2)$ of $SU(2)$.

Within these notations, the index splitting induced by (3.2) reads
\[ \tilde{\alpha} \to (\alpha, Ma, i). \]
(3.3)

The Cartan–Killing metric $g_{\tilde{\alpha}\tilde{\beta}}$ of $E_8$ branches according to
\[ g_{\tilde{\alpha}\tilde{\beta}} \to (g_{\alpha\beta}, C_{MN}\epsilon_{ab}, g_{ij}). \]
(3.4)
where $C_{MN}$ is the symplectic invariant metric of $E_7$ (indeed, the 56 is symplectic; recall (2.2)), satisfying
\[ C_{MN}C_{NP} = -\delta^M_P. \]
(3.5)

Concerning the decomposition of the $E_8$ structure constants $f_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}$ according to (3.2), one should note that in this case the normalization in the reduction is not free, because equation (2.16) relates it with the normalization of the Cartan–Killing metric. Thus, the normalization used in (3.4) constrains the normalization in the reduction of $f_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}$:
\[ f_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} \to (a \ f_{\alpha\beta\gamma}, b \ D_{i,ab}C_{MN}, c \ t_{a[M}N\epsilon_{ab}, d \ e_{ijk}). \]
(3.6)
where $a, b, c$ and $d$ are real parameters to be determined, and where
\[ t_{a[M}N \equiv t_{a[M}P C_{PN}. \]
(3.7)
is symmetric in $M N$.

For clarity’s sake, let us recall here the various identities, discussed on general ground in section 2, and decomposed, in the case of $E_8$ under (3.1), in the treatment below.

1. **Jacobi identity:**
\[ f_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} f_{\tilde{\alpha}\tilde{\gamma}\tilde{\delta}} = 0; \]
(3.8)
2. definition of the Cartan–Killing metric:
\[ g_{\tilde{\alpha}\tilde{\beta}} = -f_{\tilde{\alpha}\tilde{\gamma}\tilde{\delta}} f_{\tilde{\beta}\tilde{\gamma}\tilde{\delta}}; \]
(3.9)
3. the identity with three structure constants:
\[ f_{\tilde{\alpha}\tilde{\beta}} f_{\tilde{\beta}\tilde{\gamma}} f_{\tilde{\gamma}\tilde{\delta}} = -\frac{1}{2} f_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}; \]
(3.10)
4. the identity with four structure constants:
\[ f_{\tilde{\alpha}\tilde{\beta}} f_{\tilde{\gamma}\tilde{\delta}} f_{\tilde{\gamma}\tilde{\delta}} f_{\tilde{\gamma}\tilde{\delta}} = \frac{1}{32}\{g_{\tilde{\alpha}\tilde{\beta}g_{\tilde{\gamma}}\tilde{\delta}} + 2g_{\tilde{\alpha}\tilde{\beta}g_{\tilde{\gamma}3}} - \frac{1}{2} [2f_{\tilde{\alpha}\tilde{\gamma}f_{\tilde{\beta}\tilde{\delta}} - f_{\tilde{\alpha}\tilde{\gamma}f_{\tilde{\beta}\tilde{\delta}}}]. \]
(3.11)

We now proceed with the reduction of such $E_8$-identities under the embedding (3.1)–(3.2).

---

11 A summary of our $SU(2)$ conventions is given in appendix **A**.
(1) Let us start with the reduction of the $E_8$-Jacobi identity (3.8). If all free indices are either in the $\mathbf{133}$ of $E_7$ or in the $\mathbf{3}$ of $(SU(2))$, one simply gets the corresponding Jacobi identity. If three indices are in the $\mathbf{133}$, this implies the previous case. The first non-trivial case corresponds to having two and only two indices in the $\mathbf{133}$, implying that the other two indices are in the $\mathbf{56}$ of $E_7$; thus, one obtains (2.4) for the generators of $E_7$, provided that

$$c = -a.$$  \hspace{1cm} (3.12)

Similarly, if two indices are in the $\mathbf{3}$ of $SU(2)$, one gets

$$d = -b.$$  \hspace{1cm} (3.13)

All the terms with only one index in the $\text{Adj}$ of either $E_7$ or $SU(2)$ identically vanish. Finally, the case in which all indices are in the $\mathbf{56}$ of $E_7$ should be considered. Using the Fierz identity

$$\epsilon_{[ab}\epsilon_{cd]} = 0,$$  \hspace{1cm} (3.14)

one gets

$$\epsilon_{ab}\epsilon_{cd} \left[ e^2 t_{[MNP]} t_{PQ} - e^2 t_{MNP} t_{[NP]} \right]$$

$$\left. - b^2 C_{MN} C_{PQ} - \frac{b^2}{2} C_{NP} C_{MQ} - \frac{b^2}{4} C_{PM} C_{NQ} \right]$$

$$+ \epsilon_{bc}\epsilon_{ad} \left[ e^2 t_{[MNP]} t_{PQ} - e^2 t_{MNP} t_{[NP]} \right]$$

$$\left. + \frac{b^2}{4} C_{NP} C_{MQ} - \frac{b^2}{2} C_{MN} C_{PQ} + \frac{b^2}{4} C_{PM} C_{NQ} \right] = 0.$$ \hspace{1cm} (3.15)

Given that the two terms in (3.15) are independent, this implies the $E_7$-identity

$$e^2 t_{[MNP]} t_{PQ} - \frac{b^2}{2} C_{MN} C_{[NP]} + \frac{b^2}{4} C_{NP} C_{MQ} = 0.$$ \hspace{1cm} (3.16)

(2) We now perform the exceptional reduction of (3.9), by recalling the branching (3.4). The $g_{ab}$ term yields identities (2.5) and (2.16) for $E_7$, provided that

$$a^2 = \frac{1}{3}.$$  \hspace{1cm} (3.17)

On the other hand, by using the $SU(2)$ conventions reported in appendix A, one can show that the $g_{ij}$ term yields

$$b^2 = \frac{1}{30}.$$  \hspace{1cm} (3.18)

Finally, the $C_{MN} g_{ab}$ term gives

$$\frac{1}{2} b^2 + \frac{19}{4} c^2 = 1,$$  \hspace{1cm} (3.19)

which is identically satisfied. By plugging (3.17) and (3.18) into (3.16), one obtains the following $E_7$-identity:

$$t_{[MNP]} t_{PQ} - \frac{1}{22} C_{MN} C_{PQ} + \frac{1}{22} C_{NP} C_{MQ} = 0.$$ \hspace{1cm} (3.20)

It should be pointed out that all the identities which can be obtained through the ‘exceptional reduction’ approach under consideration are invariant under simultaneous change of sign of the generators and of the structure constants, for both $E_7$ and $SU(2)$. Therefore, the coefficients in equation (3.6) can only be determined up to an independent sign in front of $a$ and $c$, and in front of $b$ and $d$. By assuming $a$ and $d$ to be positive,
the performed analysis fixes $a$, $b$, $c$, $d$ completely, and the consistent normalization of the branching (3.6) reads

$$f_{\bar{a} \bar{b} \gamma} \rightarrow \left( \frac{1}{\sqrt{2}} f_{ab} \delta_{\gamma}, -\frac{1}{\sqrt{30}} D_{(ab} C_{MN)}, -\frac{1}{\sqrt{3}} l_{a(MN} \epsilon_{ab), \frac{1}{\sqrt{30}} \epsilon_{ijk} \right).$$

(3.21)

(3) Next, let us consider the exceptional reduction of (3.10). If the free indices are $\alpha \beta \gamma$, by using the identity (2.19) for $E_7$, one gets

$$l_{\alpha|M} l_{\alpha|N} t_{\gamma|P} t_{\gamma|Q} = -\frac{1}{2} f_{\alpha \beta \gamma},$$

(3.22)

which can be proved by using the symmetry in $MN$ of the generators of $E_7$. The identity (3.10) is also trivially true if the free indices are $i j k$. The unique other non-trivial identity comes from setting one index in the 133 and the other two indices in the 56; in such a case, (3.10) implies

$$l_{\alpha|N} t_{\beta|M} t_{\beta|N} l_{\gamma|Q} = \frac{7}{8} l_{\alpha|P} l_{\gamma|Q}.$$ 

(3.23)

All other values of the free indices yield trivial relations.

(4) Finally, we consider the reduction of (3.11). If all four indices are in the 133, using the identity (2.19) for $E_7$, one obtains the $E_7$-identity

$$l_{\alpha|M} l_{\beta|N} l_{\gamma|P} l_{\gamma|Q} = \frac{1}{2} l_{\alpha | MK} g_{\beta | KM} + 2 g_{\alpha | (\beta | \gamma | k)} - 3 [2 f^* v_{\delta} f_{\delta \beta} f_{\delta \gamma}].$$

(3.24)

On the other hand, if all indices are in the 3 of $SU(2)$, (3.10) is identically satisfied using the properties of the Pauli matrices. Moreover, if one sets the indices $\bar{a}$ and $\bar{b}$ in the 133 and the indices $\tilde{\alpha}$ and $\tilde{\beta}$ in the 56, then the following $E_7$-identity is achieved:

$$f_{u^* v} f_{\beta | M} l_{\gamma | P} l_{\gamma | Q} + l_{\alpha | M} l_{\beta | P} l_{\gamma | Q} = -\frac{1}{3} g_{\alpha | M} C_{MN} + \frac{11}{8} l_{\alpha | M} t_{\gamma | P} t_{\gamma | Q} + \frac{5}{6} l_{\alpha | P} t_{\gamma | M} l_{\gamma | P},$$

(3.25)

which can equivalently be rewritten as

$$(t^* t_{\alpha}) Q_{\beta | R} = \frac{1}{2} \delta_{\beta | R} \delta_{\alpha} + \frac{11}{2} (t^* t_{\alpha}) P_{M} - \frac{5}{6} (t_{\alpha} t_{\gamma}) P_{M}.$$ 

(3.26)

Finally, if all indices are in the 56, by using the Fierz identity (3.14) the reduction of (3.11) leads to two expressions turning out to be identical after using (3.20), and resulting into the following $E_7$-identity:

$$l_{\alpha | M} l_{\beta | R} t_{\gamma | P} s_{\gamma | Q} = \frac{1}{2} l_{\alpha | M} l_{\beta | Q} + \frac{11}{8} l_{\alpha | M} t_{\beta | P} + \frac{5}{6} l_{\alpha | M} t_{\gamma | P}$$

$$- \frac{19}{576} C_{M P C_{N Q}} - \frac{47}{576} C_{M N C_{P Q}} + \frac{7}{576} C_{M Q C_{N P}} = 0.$$ 

(3.27)

These are all the non-trivial relations among $E_7$-invariant tensors that can be obtained by performing the reduction of the $E_8$-identities (3.8)–(3.11) under the embedding (3.1)–(3.2).

Apart from the identities (2.5), (2.6), (2.14), (2.16), (2.18) and (2.19), we also derived the $E_7$-identities (3.20), (3.22), (3.23), (3.24), (3.25) and (3.27).

4. $E_7 \supset E_6 \times U(1)$

In this section, we consider the group embedding

$$E_7 \supset E_6 \times U(1),$$

(4.1)

and we derive all the $E_6$-identities arising from the corresponding branching of the $E_7$-identities obtained in section 3, given by (3.20), (3.22), (3.23), (3.24), (3.25) and (3.27).

(3.26) corrects a typo in the identity (A.16) of [39].
In order to perform the reduction of the $E_7$-identities down to $E_6$, we start and decompose the $\text{Adj} = 133$ and $\text{Fund} = 56$ irreps of $E_7$ in terms of $E_6 \times U(1)$:

\[ 133 \rightarrow 78_6 + 27_{-2} + \overline{27}_{-2} + 1_0; \]  
\[ 56 \rightarrow 27_{+1} + \overline{27}_{-1} + 1_{+3} + \overline{1}_{-3}, \]  
(4.2)

where the subscripts denote the $U(1)$-charges, and the two singlets in (4.3) are written in such a way that their opposite $U(1)$-charges are manifest. Concerning the $E_6$-irreps, we will here denote the $\text{Adj}(E_6) = 78$ with hatted lowercase Greek indices $\hat{\alpha}$, and the $\text{Fund}(E_6) = 27$ and $\overline{\text{Fund}}(E_6) = \overline{27}$ with covariant (lower), respectively contravariant (upper), hatted uppercase Latin indices $\hat{M}$. Thus, decompositions (4.2) and (4.3) respectively yield the following index splittings:

\[ V_a \rightarrow (V_{\hat{G}}, V_{\hat{M}}, V_{\hat{\bar{G}}}, V_1); \]  
(4.4)

\[ W_M \rightarrow (V_{\hat{G}}, V_{\hat{\bar{M}}}, V_1, V_{\overline{\bar{G}}}). \]  
(4.5)

From (4.2), the Cartan–Killing metric of the $133$ of $E_7$ decomposes according to

\[ g_{\alpha\beta} \rightarrow (g_{\hat{G}\hat{G}}(78 \, 78), g_{\hat{M}\hat{\bar{M}}}(27 \, \overline{27}), 1 \, (1 \, 1)), \]  
(4.6)

whereas from (4.3) the symplectic invariant tensor $C_{MN}$ of the $56$ of $E_7$ branches as

\[ C_{MN} \rightarrow (\delta_{\hat{M}\hat{\bar{M}}}(27 \, \overline{27}), 1 \, (1 \, 1)). \]  
(4.7)

In (4.6) and (4.7), we notated in brackets the irreps to which the indices ($\alpha\beta$ respectively $MN$) belong in each term. Note that all terms occurring in the reduction must be invariant tensors of $E_6 \times U(1)$, and thus they trivially have vanishing $U(1)$-charge.

As also holding for the reductions considered in section 3, the normalization in the reduction of the structure constants and of the generators cannot be arbitrarily chosen, because it is related to the normalization of the Cartan–Killing metric and of the invariant tensor $\delta_M^N$.

We will fix such normalizations further below.

The generator $t_{a\bar{a}MN}$ of $E_7$, which is symmetric in $MN$, is thus decomposed according to

\[ t_{a\bar{a}MN} \rightarrow \begin{pmatrix} a \, t_{\hat{G}\hat{G}M} (78 \, 27 \, \overline{27}), \\ b \, d_{\hat{G}\hat{G}\bar{M}} (27 \, 27 \, 27), \quad -b \, d^{\hat{G}\hat{G}\bar{M}} (27 \, \overline{27} \, \overline{27}), \\ c \, \delta_{\hat{M}\hat{\bar{M}}}(1 \, 27 \, \overline{27}), \\ d \, \delta_{\hat{M}}^\bar{M}(27 \, \overline{27} \, 1), \quad d \, \delta_{\hat{M}}^\bar{M}(27 \, \overline{27} \, 1), \\ e \, (1 \, 1 \, 1) \end{pmatrix}, \]  
(4.8)

where $d_{\hat{M}\hat{\bar{M}}}^{\hat{G}}$ and $d_{\hat{M}\hat{\bar{M}}}^{\bar{M}}$ are the rank-3 completely symmetric invariant tensors of the 27 and $\overline{27}$ of $E_6$, namely the singlets:

\[ d_{\hat{M}\hat{\bar{M}}}^{\hat{G}} \equiv 1 \in (27)_3, \quad d_{\hat{M}\hat{\bar{M}}}^{\bar{M}} \equiv 1 \in (\overline{27})^3. \]  
(4.9)

In (4.8), we notated in brackets the representations to which the indices $aMN$ belong for each term. The real parameters $a, b, c, d, e$ will be determined in the following treatment. It should be here remarked that the terms proportional to $d_{\hat{M}\hat{\bar{M}}}^{\hat{G}}$ and $d_{\hat{M}\hat{\bar{M}}}^{\bar{M}}$ in (4.8) have opposite coefficients, for consistency with the condition

\[ d_{\hat{M}\hat{\bar{M}}}^{\hat{G}} d_{\hat{M}\hat{\bar{M}}}^{\bar{M}} = \delta_{\hat{G}}^{\bar{M}}. \]  
(4.10)
that we assume\footnote{This simplifying assumption changes the normalization of the $d$-tensors $d_{\tilde{M}\tilde{N}}$ and $d_{\hat{M}\hat{N}\hat{P}}$ with respect the one usually adopted in supergravity. For example, let us consider the embedding (particular non-compact real form of (4.1))

\[ E_{7(-25)} \supset E_{6(-26)} \times SO (1,1) \, . \]

pertaining to $N' = 2$, $D = 4$ octonionic ‘magic’ supergravity (based on $J^O_3$) branched with respect to its $D = 5$ $U$-duality group $E_{6(-26)}$. In this case, it holds that [48] (see also [49] for recent treatment)

\[ d_{\tilde{M}\tilde{N}\tilde{P}}d_{\tilde{N}\tilde{Q}\tilde{R}} = 10\hat{Q}_{\tilde{P}} \, . \]

Thus, assumption (4.10) amounts to changing such a normalization, and setting to 1 the coefficient of proportionality between $d_{\tilde{M}\tilde{N}\tilde{P}}d_{\tilde{M}\tilde{N}\tilde{P}}$ and $\delta_{\tilde{P}}^\tilde{Q}$.}. Again, as was the case for (4.6) and (4.7), the $U(1)$-charge of each term in the decomposition (4.8) vanishes.

Furthermore, the structure constants of $E_7$ decompose according to

\[ f_{\alpha\beta\gamma} \rightarrow \begin{pmatrix} f_{\alpha\beta\gamma} & (78 78 78) & , & g t_{\tilde{M}\tilde{N}} & (78 27 27) & , & h \delta_{\tilde{M}}^\tilde{N} & (27 27 1) \end{pmatrix}, \]

with the real parameters $f$, $g$, $h$ to be determined.

(1) We start by considering the reduction of the identities that do not involve the structure constants $f_{\alpha\beta\gamma}$ of $E_7$. The identities (2.5) and (2.6) specified for $E_7$ give the same relations for $E_6$, provided that $a^2 = \frac{1}{7}$, and that the other parameters in (4.8) satisfy the system

\[ \begin{align*}
1 &= b^2 + 2d^2; \\
1 &= 54c^2 + 2e^2; \\
\frac{19}{7} &= 27d^2 + e^2.
\end{align*} \]

which leaves one parameter undetermined.

(2) The reduction of identity (3.20) constrains the squares of all the parameters in the decomposition (4.8) to be

\[ b^2 = \frac{5}{6}, \quad c^2 = \frac{1}{12}, \quad d^2 = \frac{1}{12}, \quad e^2 = \frac{1}{8}, \]

and it also yields the further constraint

\[ ce = \frac{1}{12}. \]

On the other hand, the only non-trivial $E_6$-identity that it produces reads

\[ t_{\tilde{M}\tilde{N}}s_{\tilde{P}\tilde{Q}} = \frac{1}{6}s_{\tilde{M}\tilde{N}}s_{\tilde{P}\tilde{Q}} + \frac{1}{12} s_{\tilde{M}\tilde{N}}s_{\tilde{P}\tilde{Q}} - \frac{5}{4} d_{\tilde{M}\tilde{P}\tilde{R}}d_{\tilde{N}\tilde{Q}\tilde{R}}. \]

(3) We then consider the reduction of the identity (3.23), which gives rise to the following two $E_6$-identities:

\[ \begin{align*}
t_{\tilde{M}} & d_{\tilde{M}\tilde{N}\tilde{P}}d_{\tilde{N}\tilde{Q}\tilde{R}} = -\frac{1}{7} t_{\tilde{M}}; \\
t_{\tilde{M}} & t_{\tilde{M}}d_{\tilde{M}\tilde{N}\tilde{P}}d_{\tilde{N}\tilde{Q}\tilde{R}} = -\frac{13}{7} d_{\tilde{M}\tilde{P}\tilde{R}}.
\end{align*} \]

By using the values of the parameters obtained above, it can be checked that all other combinations of indices yield trivial relations.
(4) The remaining $E_7$-identities that do not involve the structure constants $f_{\alpha \beta \gamma}$ are given by (3.26) and (3.27). Identity (3.26) has two free indices in the $133$ and two other ones in the $56$ of $E_7$. Its reduction produces only two non-trivial $E_6$-identities, namely

$$\frac{1}{2}(t_{a \alpha}p_{\alpha})p_{\beta}p_{\gamma}p_{\Omega}t_{\tilde{p} \tilde{\Omega} \tilde{N}} + \frac{1}{12} (t_{\beta \gamma}t_{\Omega})p_{\rho}d_{\tilde{M} \tilde{O} \tilde{R}}d_{\tilde{N} \tilde{P} \tilde{R}} - \frac{1}{2} (t_{a \gamma}t_{\tilde{p}}p_{\tilde{\Omega} \tilde{N}}

+ \frac{1}{2} (t_{\beta \gamma}t_{\Omega})p_{\rho}d_{\tilde{M} \tilde{O} \tilde{R}} = 0,$$

(4.18)

where the free indices are $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{M}$ and $\tilde{N}$ (thus, two in the $78$, one in the $27$, and the other one in $\overline{27}$), and

$$\frac{5}{2} d_{\tilde{M} \tilde{O} \tilde{R}} d_{\tilde{P} \tilde{Q} \tilde{R}} d_{\tilde{O} \tilde{Q} \tilde{R}} + d_{\tilde{M} \tilde{O} \tilde{R}} d_{\tilde{P} \tilde{Q} \tilde{R}} - \frac{1}{2} (t_{\beta \gamma}t_{\Omega}p_{\rho}d_{\tilde{M} \tilde{O} \tilde{R}} = 0,$$

(4.19)

with two indices in the $27$ and two other ones in the $\overline{27}$. By using (4.16), one can show that all other identities resulting from the reduction of (3.26) and (3.27) are equivalent to the the ones given above.

(5) We then proceed to considering the reduction of the identities containing the structure constants $f_{\alpha \beta \gamma}$ of $E_7$. In order to derive the coefficients $f$, $g$ and $h$ in the decomposition (4.11), it is sufficient to consider the reduction of the $E_7$-identity (2.4). This yields (i) again (4.11) for $E_6$: (ii) the identity

$$t_{\beta \gamma}p_{\rho}d_{\tilde{P} \tilde{Q} \tilde{R}} = 0,$$

(4.20)

which is the condition of invariance of $d_{\tilde{P} \tilde{Q} \tilde{R}}$ itself; and (iii) the identity (4.16). Thus, the parameters $f$, $g$ and $h$ must satisfy

$$f = a, \quad g = -a, \quad hc = \frac{1}{3\sqrt{2}}.$$

(4.21)

(6) The reduction of the Jacobi identity of $E_7$ and the reduction of the $E_7$-identity (2.16) produce no further new $E_6$-identities, while the reduction of the $E_7$-identity (2.19) yields the $E_6$-identity

$$t_{\beta \gamma}p_{\rho}d_{\tilde{P} \tilde{Q} \tilde{R}} = \frac{1}{2} f_{\alpha \beta \gamma},$$

(4.22)

together with (2.19) for $E_6$. Also it can be checked that the reduction of (3.22) gives no additional $E_6$-identities.

(7) Finally, we consider the reduction of the $E_7$-identities (2.13), (3.24) and (3.25). These all give rise to the following two $E_6$-identities:

$$t_{\beta \gamma}p_{\rho}d_{\tilde{P} \tilde{Q} \tilde{R}} = \frac{1}{2} g_{\alpha \beta \gamma} + \frac{1}{2} (2 f_{\alpha \beta \gamma} f_{\tilde{Z} \tilde{\Omega} \tilde{\gamma}} - f_{\tilde{Z} \tilde{\Omega} \tilde{\gamma}} f_{\alpha \beta \gamma}) = 0;$$

(4.23)

$$f_{\alpha \beta \gamma} f_{\tilde{Z} \tilde{\Omega} \tilde{\gamma}} (t_{\beta \gamma}p_{\rho}d_{\tilde{P} \tilde{Q} \tilde{R}} - (t_{\alpha \gamma}t_{\rho})p_{\beta}d_{\tilde{P} \tilde{Q} \tilde{R}} + \frac{1}{12} (t_{\alpha \gamma}t_{\rho})d_{\tilde{M} \tilde{O} \tilde{R}}

- (t_{\beta \gamma}t_{\Omega})p_{\rho}d_{\tilde{M} \tilde{O} \tilde{R}} = 0.$$
where the adjoint representation of generators (made out of the structure constants $J. Phys. A: Math. Theor. 44 (2011) 155207 A Marrani et al$

\begin{align*}
  f_{a\beta\gamma} &\rightarrow \begin{pmatrix}
  \frac{1}{\sqrt{2}} f_{\alpha\beta\rho} (78\ 78\ 78), \\
  -\frac{1}{\sqrt{2}} f_{\alpha\beta\rho} (27\ 27\ 27), \\
  \frac{1}{\sqrt{27}} \delta_{\alpha\beta} (27\ 27\ 1)
  \end{pmatrix}.
\end{align*}

(4.27)

In appendix B we proceed further and consider some examples of reductions of $E_6$ identities with respect to its maximal subgroup $SO (10) \times U (1)$.

5. The $K$-tensor and its Master identity

As an application of the formalism developed in previous sections, in order to elucidate the connections to $D = 4$ Maxwell–Einstein supergravity theories based on symmetric scalar manifolds and related to irreducible Euclidean Jordan algebras (the unique exception being the triality-symmetric $\mathcal{N} = 2$ string model), we now derive a fundamental (dubbed ‘master’) identity, involving the unique rank-4 completely symmetric invariant tensor (named $K$-tensor) of the 0-brane (black hole) charge irrep. $\mathfrak{R}$ of $G_4$ (see also the treatment of [51]). Besides the importance of the $K$-tensor for the theory of extremal black hole attractors [28] in Maxwell–Einstein supergravities, this (hitherto unknown) identity has potential application in the issue of the classification of the orbits of $\mathfrak{R} (G_4)$ in the presence of Dirac–Zwanziger–Schwinger charge quantization conditions (especially for $\mathcal{N} = 8$, $D = 4$ supergravity, see e.g. [32, 35–37] and references therein), as well as in the study of multi-center black hole solutions [33, 34].

At least in $D = 4$ supergravities with symmetric scalar manifolds $\mathfrak{G}_4$, the $U$-duality groups $G_4$’s share the property that the generators $t_{a(MN)} (3.7)$ in $\mathfrak{R}$ are $G_4$-singlets:

\begin{align*}
  \exists ! t_{a(MN)} &\equiv 1 \in \text{Adj} \times (\mathfrak{R} \times s \mathfrak{R}).
\end{align*}

(5.1)

This can be proven explicitly by using Baker–Campbell–Hausdorff formula (see e.g. [50] and references therein) and acting on the generators (3.7) with a generic group element $S \equiv e^{t_{a(MN)}} \in G_4$. In fact, $G_4$ induces a transformation of the symplectic indices which is equivalent to an inverse transformation on the adjoint index:

\begin{align*}
  t_a \rightarrow S t_a S^{-1} &= t_a + \xi^{\beta_1} [t_{\beta_1}, t_a] + \frac{1}{2} \xi^{\beta_1} \xi^{\beta_2} [t_{\beta_1} [t_{\beta_2}, t_a]] + \cdots \\
  &= t_a - \xi^{\beta_1} \left( T_{\beta_1} \right)_a t_a + \frac{1}{2} \xi^{\beta_1} \xi^{\beta_2} \left( T_{\beta_1} T_{\beta_2} \right)_a t_a + \cdots \\
  &= (e^{-t_{a(MN)}})_a t_a \equiv (S^{-1})_a t_a,
\end{align*}

(5.2)

where the adjoint representation of generators (made out of the structure constants $f_{a\beta\gamma}$ of the Lie algebra $\mathfrak{g}_4$ of $G_4$)

\begin{align*}
  (T_a)_b \equiv - f_{ab}^c
\end{align*}

(5.3)

was used. Consequently, $S \equiv e^{t_{a(MN)}}$, defined by the last line of (5.2), is the adjoint representation of $S$ itself.

A key property of the symplectic representation $t^a_{MN}$ of the generators is the even symmetry of its symplectic indices: $t^a_{MN} = t^a_{(MN)}$. Besides (5.1) itself, this is also implied by the fact that $t^a_{[MN]}$ is the symplectic variation of $\mathcal{C}_{MN}$, which trivially vanishes, due to the fact that the symplectic metric $\mathcal{C}_{MN}$ itself is a $G_4$-singlet (recall (2.2)).

By exploiting such a symmetry, it is possible to construct a rank-4 completely symmetric $G_4$-invariant tensor of $\mathfrak{R}$, dubbed $K$-tensor:

\begin{align*}
  \exists ! \mathcal{K}_{MNPQ} &\equiv 1 \in (\mathfrak{R})^4,
\end{align*}

(5.4)

which is not a primitive invariant, since it is in general defined as follows:

\begin{align*}
  \mathcal{K}_{MNPQ} \propto t^a_{(MN)\{a\}} (PQ) = \frac{1}{3} (t^a_{[MN]a} Q_P + t^a_{MP\{a\}Q} + t^a_{MQ\{a\}P}).
\end{align*}

(5.5)
Such an invariant structure exists at least in $D = 4$ supergravities with symmetric scalar manifolds $G_4$. Furthermore, for all these theories but the $\mathcal{N} = 2$ model [52] and the $\mathcal{N} = 3$ theory [53], the $K$-tensor is irreducible in $\mathbb{R}$, namely it cannot be expressed in terms of lower-rank tensors with indices only in $\mathbb{R}$.

For instance, for $G_4 = E_7$ (corresponding to $\mathcal{N} = 8$ maximal and to $\mathcal{N} = 2$ ‘magic’ octonionic $D = 4$ supergravity, for $E_7(7)$ respectively $E_7(-25)$), it holds $\mathbb{R} = \text{Fund} = 56$, and

$$\bar{K}_{MNPQ}^\text{(56)} = 1 + 1463 + 1539 + 7371 + 150\,822 + 293\,930.$$  \hspace{1cm} (5.6)

In this case, the $K$-tensor defined in (5.5) can be characterized in a more useful way using the $E_7$-identity (3.20), which can be recast in the following form:

$$t^\nu_{MN[t^\alpha_{(\omega)]PQ]} = \frac{1}{6}[C_{M(P\bar{C}_{Q|N} - C_{M(N|P}\bar{C}_{Q}\pmb{]}}].$$ \hspace{1cm} (5.7)

After some algebra, the following identity is achieved:

$$t^\nu_{MN[t^\alpha_{(\omega)]PQ]} = t^\nu_{MN[t^\alpha_{(\omega)]PQ}} - \frac{1}{12}C_{M(P\bar{C}_{Q|N}].}$$ \hspace{1cm} (5.8)

By using (2.6) and (3.5), one can check (5.8) to be skew-traceless. Thus, for $G_4 = E_7$ the following fundamental relation is obtained:

$$\bar{K}_{MNPQ} = \xi t^\nu_{MN[t^\alpha_{(\omega)]PQ]} = \xi t^\nu_{MN[t^\alpha_{(\omega)]PQ}} - \frac{1}{2}C_{M(P\bar{C}_{Q|N}].}$$ \hspace{1cm} (5.9)

where the real proportionality constant $\xi$ has been introduced. At least in all $D = 4$ supergravities with symmetric scalar manifolds in which the $K$-tensor is irreducible in $\mathbb{R}$ (see the comment below (5.5)), the result (5.9) can be generalized as

$$\bar{K}_{MNPQ} = \xi(t^\nu_{MN[t^\alpha_{(\omega)]PQ}} - \tau C_{M(P\bar{C}_{Q|N}].}$$ \hspace{1cm} (5.10)

where $\tau$ is a real constant, in general depending on $d$ and $f$, determined by imposing the skew-tracelessness condition on $\bar{K}_{MNPQ}$ (recall identities (2.6) and (3.5)):

$$C_{NP}^{MN} \bar{K}_{MNPQ} = 0 \iff \tau = \frac{2d}{f(f+1)}.$$ \hspace{1cm} (5.11)

Thus, the following expression for the $K$-tensor is obtained:

$$\bar{K}_{MNPQ} = \xi(t^\nu_{MN[t^\alpha_{(\omega)]PQ}} - \frac{2d}{f(f+1)}C_{M(P\bar{C}_{Q|N}].}$$ \hspace{1cm} (5.12)

where the real proportionality constant $\xi$ depends on the chosen normalization of the generators $t^\alpha$’s, getting fixed by the explicit computation.

Remarkably, the $K$-tensor is related to the well-known invariant homogeneous polynomial $I_4$ of $\mathbb{R}$ ($G_4$), used to classify extremal black hole solutions in Maxwell–Einstein $D = 4$ supergravities with symmetric scalar manifolds (see e.g. [29] for a review and a list of references). Indeed, $I_4$ is defined as the contraction of the $K$-tensor with four copies of the charge vector of $\mathbb{R}$ ($A = 0, 1, \ldots, f/2 - 1$)

$$Q^M = (p, q)^M,$$ \hspace{1cm} (5.13)

namely

$$I_4 \equiv \bar{K}_{MNPQ}Q^M Q^N Q^P Q^Q = \xi t^\nu_{MN[t^\alpha_{(\omega)]PQ} Q^M Q^N Q^P Q^Q},$$ \hspace{1cm} (5.14)

resulting in a homogeneous polynomial of degree 4 in the black hole charges $Q$.

It is here worth remarking that in $D = 5$ the role of the $K$-tensor is played by the so-called $d$-tensors defined in (4.9), used to construct the $G_5$-invariant cubic homogeneous polynomials $I_{3,e}$ (electric) and $I_{3,m}$ (magnetic) (see e.g. [30, 54–56]). A key difference is that the $d$-tensor is primitive, namely it cannot be expressed in terms of other independent tensor structures in
any irreps of $G_5$, while the $K$-tensor is not primitive (from its very definition (5.5)). Moreover, at least in symmetric geometries, the $d$-tensor satisfies the fundamental identity

$$d_{M(R PQ(R S) d(5)} = \frac{2}{15} \delta_{(5} d_{(P Q(R S) d(5)}.$$  \hfill (5.15)

This identity can be derived from the identities obtained in section 4, by contracting (4.16) with $d_{5(R} d_{5)}$, and then by symmetrizing with respect to all the lower indices and using (4.20). Note the different normalization of (5.15) e.g. with respect to references [18, 48] (see footnote 11).

In analogy with the $D = 5$ case, the issue of deriving an identity analogue to (5.15) involving the $K$-tensor naturally arises out. By exploiting the definition (5.12) of the $K$-tensor and the decomposition (C.1), the following result can be achieved (recall (5.11)):

$$K_{MNPQ} K_{RSTU} C^{QR} = K_{(MN P)Q} K_{(RSTU)} C^{QR}$$

$$= \xi_1 \frac{1}{f} C_{(MN P)(STU)} + \xi_2 \frac{1}{f} C_{(MN P)(STU)} - \xi_2 \frac{1}{f} C_{(MN P)(STU)} - \xi_2 \frac{1}{f} C_{(MN P)(STU)}$$

$$= \xi_1 \frac{1}{f} C_{(MN P)(STU)} + \xi_2 \frac{1}{f} C_{(MN P)(STU)} - \xi_2 \frac{1}{f} C_{(MN P)(STU)},$$

where $C_{(MN P)(STU)}$ means symmetrization for the triplets of indices $(M, N, P)$ and $(S, T, U)$. In (5.16) and (5.17) we introduced the fundamental invariant tensor $C_{(MN P)}^{(STU)}$ (see appendix C) that does not arise from the reduction of the $E_3$-identities considered in this paper.

By some algebra, equation (5.10) yields (recall (5.11))

$$K_{MNPQ} K_{RSTU} C^{QR} = \xi_{[2(\tau - 1)]} K_{MNPQ} + \xi_{[2(\tau - 1)]} C_{MNPQ},$$

Identity (5.18) implies that arbitrary powers of the $K$-tensor, each having a couple of indices contracted, are always linear in the $K$-tensor and in $C_{MNPQ}$. By further contracting with the charges $Q^M Q^N Q^P Q^Q$ and recalling definition (5.14), one obtains

$$K_{MNPQ} K_{RSTU} C^{QR} = \xi_{[2(\tau - 1)]} I_4.$$  \hfill (5.19)

On the other hand, by suitably changing the order of the indices of the $K$-tensor and recalling equation (2.7), one can compute

$$K_{MNPQ} K_{RSTU} C^{QR} = \xi_{[2(\tau - 1)]} I_4.$$  \hfill (5.20)

From the complete symmetry of the $K$-tensor, the fact that the left-hand sides of (5.19) and (5.20) are equal implies the following relation:

$$g^\tau = 4(1 - 3\tau),$$  \hfill (5.21)

which, through (5.11), relates the dual Coxeter number $g^\tau$, the Dynkin index of $R$, and the dimensions of $R$ and $Adj$. Result (5.21) holds at least for all $G_\lambda$'s of supergravity theories reported in table 1. For these groups, equations (2.7) and (5.21) imply that the general result

$$C_{Adj} \over C_R = \frac{f}{d} g^\tau$$  \hfill (5.22)

can be further elaborated as

$$C_{Adj} \over C_R = \frac{4f}{d} (1 - 3\tau).$$  \hfill (5.23)
The ‘master’ identity (5.17) has potential application in the study of independent tensor structures in the 0-brane (black hole) charge irrep $\mathbf{R}$ of $G_4$’s of symmetric $D = 4$ supergravities. Due to recent advances in the investigation of UV finiteness properties [32], the case of the $\mathbf{56}$ of $G_4 = E_{7(7)}$, $U$-duality group of $\mathcal{N} = 8$, $D = 4$ supergravity, is especially relevant. In this case, the primitive tensors of the $\mathbf{56}$ of $E_{7(7)}$ are related to the classification of discrete $E_{7(7)}(\mathbb{Z})$-invariants; indeed, as discussed e.g. in [35, 36, 57, 58] (and, in particular, in sections 3 and 4 of the fourth reference of [32], and in appendix E of [37]), the discrete $E_{7(7)}(\mathbb{Z})$-invariants are given by the greatest common divisor (gcd) of certain sets of numbers which correspond to covariant tensors of $E_{7(7)}(\mathbb{R})$. Physically, $E_{7(7)}(\mathbb{Z})$-invariants would determine the algebraic classification of the charge orbits of extremal black holes in the presence of Dirac–Zwanziger–Schwinger quantization conditions. The currently known set of invariants of the $\mathbf{56}$ of $E_{7(7)}(\mathbb{Z})$ is given by the gcd of suitable projections of contractions of the $K$-tensor itself with some charge vectors $Q$’s [35–37] (a manifestly $(\text{SL}(2, \mathbb{R}) \times \text{SO}(6, 6))$-covariant formalism is worked out in [57, 58]). Unfortunately, with the exception of the so-called projective black holes [35], the known set of discrete invariants does not allow for a complete classification of black hole states. Thus, it is natural to ask if the missing invariants derived by taking the gcd of some independent tensors of the $\mathbf{56}$ of $E_{7(7)}(\mathbb{R})$, given by suitable tensor products of the $K$-tensor, suitably projected onto $E_7$-irreps and contracted with charge vectors $Q$’s. The ‘master’ identity (5.17), yielding to various relations constraining invariant structures of $E_7$, may actually provide a systematic way to figure out a complete set of independent invariant tensor structures.

Furthermore, the ‘master’ identity (5.17) is relevant to derive and study the algebraic independence of higher-order $U$-invariant polynomials appearing in the study of multi-center black hole solutions [33, 34]. We leave these interesting issues for future investigation.

Acknowledgments

AM and EO would like to thank the Department of Mathematics, King’s College, London, UK, where part of this work was done, for warm hospitality and inspiring environment. FR would like to thank the Politecnico di Torino and Turin University for hospitality. AM would like to thank Sergio Ferrara for enlightening discussions, and EO would like to thank Mario Trigiante for useful discussions. The work of AM has been supported in part by an INFN visiting Theoretical Fellowship at SITP, Stanford University, Stanford, CA, USA. The work of EO has been supported by the ERC Advanced Grant no 226455, ‘Supersymmetry, Quantum Gravity and Gauge Fields’ (SUPERFIELDS). The work of FR has been supported by the STFC rolling grant ST/G000/395/1.

Appendix A. Conventions for $SU(2)$

In this appendix we summarize our conventions for $SU(2)$. The generators are anti-Hermitian:

$$D_{i,a}^b = \frac{1}{2} \sigma_{i,a}^b ,$$  \hspace{1cm} (A.1)

where the $\sigma$’s denote the Pauli matrices. We are using a negative-definite Cartan–Killing metric

$$g_{ij} = -\delta_{ij} ,$$  \hspace{1cm} (A.2)

which means that $D^i = - \frac{1}{2} \sigma_i$. The symmetric part of the general Fierz identity reads

$$D_b^{ic} D_c^b + D_b^{ic} D_c^b = \frac{1}{2} g^{ij} g_{a}^i ,$$  \hspace{1cm} (A.3)
and its contraction with \(g_{ij}\) yields
\[
D^{ab}D_{ij}{}^{cd} = -\frac{1}{4}[\epsilon^{ae}\epsilon_{bd} + \epsilon^{ad}\epsilon_{be}]. \tag{A.4}
\]

From identities (A.3)–(A.4), by further contracting with \(g_{ij}\) and/or \(\epsilon_{ab}\), one can obtain the following identities:
\[
D_{\alpha i}D^i{}^b = \frac{1}{2}\delta^b_\alpha; \tag{A.5}
\]
\[
D^b_iD^i{}_{\alpha} = \frac{1}{2}g^{ij}; \tag{A.6}
\]
\[
[D_i, D_j] = D_{ij}{}^c D_{j|cb} - D_{j|i}{}^c D_{i|cb} = \epsilon_{ij}{}^k D_k. \tag{A.7}
\]

Finally, consistent with the negative definiteness of the metric, the following product of Levi-Civita symbols is used:
\[
\epsilon_{ijkl}\epsilon^{kl} = -2g_{ij}. \tag{A.8}
\]

**Appendix B. \(E_6 \supset SO(10) \times U(1)\)**

For completeness, in this appendix we consider the reduction of some \(E_6\)-identities derived in section 4, according to the group embedding (1.5). The fundamental and the adjoint irreps of \(E_6\) respectively decompose as follows:
\[
27 \rightarrow 10_{-2} + 16_1 + 1_4, \quad 78 \rightarrow 45_0 + 16_{-3} + 16_1 + 1_0, \tag{B.1}
\]
where subscripts denote the \(U(1)\)-charges. We denote here with indices \(A, B, \ldots\) the vector \(10\) of \(SO(10)\), while the spinor representations are denoted with \(a, b, \ldots\), where a lower index denotes the \(16\) and an upper index denotes the \(\bar{16}\). Thus, the decomposition (B.1) implies the indices split as follows:
\[
V^M \rightarrow (V_A, V_a, V_1); \tag{B.2}
\]
\[
\tilde{V}^M \rightarrow (V_A, V^a, V^\gamma); \tag{B.3}
\]
\[
\tilde{V}^{\dot{a}} \rightarrow (V_{AB}, V_a, V^a, V_1). \tag{B.4}
\]

Before proceeding with the reduction, let us summarize our \(SO(10)\) conventions. The charge conjugation matrix \(C\) converts an upper \(a\) index to a lower index \(\dot{a}\), and vice versa:
\[
C = \begin{pmatrix} 0 & C^{ab} \\ C^{\dot{a}b} & 0 \end{pmatrix}; \tag{B.5}
\]

it is antisymmetric and unitary, that is,
\[
C^{ab} = -C^{ba}, \tag{B.6}
\]

and \(\dagger\) denotes the Hermitian conjugation
\[
C_{ab}^\dagger C^{bc} = \delta^c_a, \quad C_{\dot{a}b}^\dagger C^{\dot{c}c} = \delta^c_{\dot{a}}. \tag{B.7}
\]

The \(\Gamma\)-matrices have the form
\[
\Gamma_A = \begin{pmatrix} 0 & \Gamma_{A,a}^a \\ \Gamma_{A,\dot{a}b} & 0 \end{pmatrix}, \tag{B.8}
\]

and they satisfy the Clifford algebra
\[
\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}. \tag{B.9}
\]
where $\eta_{AB}$ is the $D = 10$ Minkowski metric (in the compact case $\eta_{AB} = \delta_{AB}$), as well as the property
\[
C \Gamma_A C^\dagger = -\Gamma_A^\dagger.
\] (B.10)

Note also that the matrix
\[
(\Sigma \Gamma_A)^{ab} = C^{ab} \Gamma_a^b
\] (B.11)
is symmetric in the indices $ab$.

We now perform the reduction of some $E_6$-identities. The Cartan–Killing metric of $E_6$ decomposes according to
\[
\delta_{\vec{a}\vec{b}} \to \left( -\delta^{CD}_{AB} \delta_{\vec{a}\vec{b}}^{1(16)} \right),
\] (B.12)

whereas the invariant tensor $\delta^{AB}_{\vec{M}}$ branches as
\[
\delta^{\vec{N}}_{\vec{M}} \to \left( \delta^{AB}_{\vec{M}} \left( 10 10 \right), \delta^{ab}_{\vec{M}} \left( 16 16 \right), 1 \left( 1 1 \right) \right).
\] (B.13)

Here $\delta^{CD}_{AB} \equiv \frac{1}{2} \left( \delta^C_A \delta^D_B - \delta^C_B \delta^D_A \right)$, and the minus sign in the first term on the right-hand side of (B.12) has been chosen for convenience, so that all coefficients in the reductions under consideration are real. Note that all terms occurring in the reduction must be invariant tensors of $\text{SO}(10) \times U(1)$, and thus they trivially have vanishing $U(1)$-charge.

We consider the reduction of (4.10), (4.16), (4.17) and (4.20), as well as of the identities (2.5) and (2.6) for $E_6$. For simplicity’s sake, we do not consider here the reduction of $E_6$-identities involving more than three $E_6$-invariant tensors, as well as of identities involving the $E_6$ structure constants. The reduction of the invariant tensors $d_{\vec{M} \vec{N}}$ and $d_{\vec{M} \vec{N}} \vec{P}$ reads
\[
d_{\vec{M} \vec{N}} \to \left( \frac{1}{\sqrt{10}} \eta_{AB} \left( 10 10 \right), \frac{1}{2\sqrt{3}} (\Gamma_A C^\dagger)_{ab} \left( 10 16 16 \right) \right),
\] (B.14)

while the reduction of the generators $t_{\vec{a} \vec{b} \vec{N}}$ is
\[
t_{\vec{a} \vec{b} \vec{N}} \to \left( \begin{array}{c} \frac{1}{4\sqrt{3}} (\Gamma_A C^\dagger)_{ab} \left( 45 16 16 \right), \\ \frac{1}{\sqrt{10}} \delta^{AB}_{\vec{a} \vec{b}} \left( 45 16 10 \right), \\ \frac{1}{\sqrt{3}} \delta^{AB}_{\vec{a} \vec{b}} \left( 1 10 10 \right), \\ \frac{1}{\sqrt{6}} \delta^{ab}_{\vec{a} \vec{b}} \left( 16 16 16 \right), \\ -\frac{1}{\sqrt{6}} \delta^{ab}_{\vec{a} \vec{b}} \left( 16 1 16 \right), \\ -\frac{1}{\sqrt{6}} \delta^{ab}_{\vec{a} \vec{b}} \left( 1 1 1 \right), \\ \frac{1}{2\sqrt{3}} (\Gamma_A C^\dagger)_{ab} \left( 16 16 10 \right), \\ \frac{1}{2\sqrt{3}} (\Gamma_A C^\dagger)_{ab} \left( 16 16 10 \right), \\ -\frac{1}{\sqrt{6}} \delta^{ab}_{\vec{a} \vec{b}} \left( 16 1 16 \right), \\ -\frac{1}{\sqrt{6}} \delta^{ab}_{\vec{a} \vec{b}} \left( 1 1 1 \right), \\ \end{array} \right).
\] (B.15)

where $\Gamma_{AB} \equiv \Gamma_{A} \Gamma_{B}$.

One can then show that the reduction of $E_6$-identities (4.10), (2.5), (2.6) and (4.17) leads to the relations defining the Clifford algebra, together with trivial $\Gamma$-matrices identities. The reduction of (4.20) leads, among other more trivial identities, to the well-known $\text{SO}(10)$ Fierz identity
\[
(\Sigma \Gamma_A)^{ab} (\Sigma \Gamma_A)^{cd} = 0,
\] (B.16)

while the reduction of (4.16) leads, among the rest, to the Fierz identity
\[
(\Gamma_A C^\dagger)_{ac} (\Sigma \Gamma_A)^{bd} - 2\delta^c_d \delta^a_b - \frac{1}{2} \delta^a_d \delta^c_b = \frac{1}{4} (\Gamma_A C^\dagger)_{ac} (\Gamma_A B)_{bd} = 0.
\] (B.17)

Similarly, the reduction of the other $E_6$-identities derived in section 4 will give rise to additional $\text{SO}(10)$ $\Gamma$-matrices identities, including additional Fierz identities.
Appendix C. A useful decomposition

A useful decomposition used in section 5, holding at least for all $U$-duality Lie groups $G_4$ of $D = 4$ supergravities reported in table 1, reads

$$t_{a|M}N_{\beta|NQ} = -t_{a|M}P_{\beta|NQ} C^{MN} + \frac{1}{2} f_{ab|\gamma} t_{\gamma|MQ} + S_{a(\beta|M\gamma)}.$$  \hspace{1cm} (C.1)

where the $G_4$-invariant tensor $S_{a(\beta|M\gamma)}$ is such that

$$S_{a(\beta|M\gamma)k^{a\beta}} = 0; \quad S_{a(\beta|M\gamma)C^{MQ}} = 0.$$  \hspace{1cm} (C.2)

It is here worth pointing out that the left-hand side of (C.1), namely $t_{a|\gamma}^{\gamma} t_{\gamma|\beta}$, is a $G_4$-singlet (because $t_{a|\gamma}^{\gamma}$ is a $G_4$-singlet itself; see equation (5.1)). Thus, due to its symmetry properties, $t_{a|M}N_{\beta|NQ}$ enjoys a decomposition into irreducible $G_4$-invariants terms, antisymmetric under the simultaneous exchanges $M \leftrightarrow Q$ and $\alpha \leftrightarrow \beta$. In other words, the adjoint indices and symplectic indices of $t_{a|M}N_{\beta|NQ}$ must have opposite symmetry properties.

For simplicity’s sake, let us derive (C.1)–(C.2) in a particular case, namely for $G_4 = E_7$ (the generalization is straightforward). The case $G_4 = E_7$ pertains both to magic octonionic $\mathcal{N} = 2$ ($G_4 = E_7(-25)$, $J^G_8$-related) supergravity and to $\mathcal{N} = 8$ maximal theory ($G_4 = E_7(7)$, $J^D_8$-related). For this group, it holds that

$$g_{a\beta} = 1 \in 133 \times 133;$$  \hspace{1cm} (C.3)

$$f_{a\beta}^{\gamma} t_{\gamma} = 1 \in 133 \times 133;$$  \hspace{1cm} (C.4)

$$C_{MN} = 1 \in 56 \times 56.$$  \hspace{1cm} (C.5)

Starting from the tensor product of the $R$ ($E_7 = 56$ and $\text{Adj} (E_7) = 133$ irreps, it follows that

$$56 \times 56 = 133 + 1463;$$  \hspace{1cm} (C.6)

$$56 \times 56 = 1 + 1539;$$  \hspace{1cm} (C.7)

$$133 \times 133 = 1 + 1539 + 7371;$$  \hspace{1cm} (C.8)

$$133 \times 133 = 133 + 8645.$$  \hspace{1cm} (C.9)

These considerations lead to a decomposition of $t_{a|M}N_{\beta|NQ}$ that can contain only three terms. Namely

- two terms with symmetric adjoint indices $(a, b \in \mathbb{R})$:
  $$ag_{a\beta} C_{MN} = 1 \times 1 \in (133 \times_5 133) \times (56 \times 56);$$  \hspace{1cm} (C.10)

  $$bS_{a(\beta|M\gamma)} = 1 \in (1539 \times 1539) \in (133 \times 133) \times (56 \times 56).$$  \hspace{1cm} (C.11)

Note that no other possibilities with symmetric adjoint indices arise, because $1 \notin (7371 \times 1539)$;

- one term with antisymmetric adjoint indices $(c \in \mathbb{R})$:
  $$c f_{a\beta}^{\gamma} t_{\gamma|M\gamma} = 1 \in (133 \times 133) \in (133 \times 133) \times (56 \times 56).$$  \hspace{1cm} (C.12)

No other possibilities with antisymmetric adjoint indices arise, because $1 \notin (1463 \times 133)$, $1 \notin (1463 \times 8645)$ and $1 \notin (133 \times 8645)$. 

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Thus, \( t_{ij}^0 \) of \( G_4 \) can be \( E_7 \)-irreducibly decomposed as follows:

\[
t_{ij}^0 \equiv a g_{ij}\mathcal{C}_{MN} + b S_{ij}^{MN} + c f_{ij} f_{kl} (MN).
\]

(C.13)

In order to compute the constants \( a, b, c \in \mathbb{R} \), we recall that all terms of (C.13) are irreducible, as also implied by (C.2). Thus, by saturating (C.13) with \( \mathbb{C}^{MN} \), one obtains

\[
t_{ij}^0 \epsilon_{ij}^{MN} = a g_{ij}\mathcal{C}_{MN} \epsilon^{MN} = f a g_{ij} \Leftrightarrow a = \frac{1}{f}.
\]

(C.14)

On the other hand, by recalling definition (2.4) of the structure constants of the Lie algebra \( g_4 \) of \( G_4 \) and using (C.13), it follows that

\[
f_{ij}^{MN} \epsilon_{ij}^{MN} = 2 t_{ij}^0 \epsilon_{ij}^{MN} = t_{ij}^0 \epsilon_{ij}^{MN} - t_{ij}^0 \epsilon_{ij}^{MN} = 2 c f_{ij} f_{kl} (MN) \Leftrightarrow c = \frac{1}{2}.
\]

(C.15)

Obviously, the constant \( b \) can be reabsorbed in a re-definition of \( S_{ij}^{MN} \). Thus, the irreducible decomposition (C.1) has been proved to hold.

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