THE BOUNDARY OF THE MILNOR FIBRE OF COMPLEX AND REAL ANALYTIC NON-ISOLATED SINGULARITIES

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Abstract. Let $f$ and $g$ be holomorphic functions vanishing at the origin of the affine space of dimension three. Suppose that the singular set of $(fg)^{-1}(0)$ is 1-dimensional and that the real analytic function $f \bar{g}$ has an isolated critical value at 0. By results of A. Pichon and J. Seade the function $f \bar{g}$ has a Milnor fibration. We prove that the boundary of the Milnor fibre is a Waldhausen manifold. As an intermediate milestone we describe geometrically the Milnor fibre of functions of type $f \bar{g}$ defined in the complex plane, and prove an A’Campo-type formula for the zeta function of their monodromy.

1. Introduction

It is classically known that there is a rich interplay between 3-manifold theory and the topology of isolated singularities in complex surfaces. This goes back to the work of F. Klein by the end of the 19th century, and then made clearer by in the early 1960s, by the work of Grauert [4] and Mumford [11]. A closed oriented 3-manifold $M$ is the link of some isolated complex surface singularity $(V, p)$ if and only if $M$ is a Waldhausen manifold with negative definite intersection form.

This important theorem has played, on one hand, a key role for understanding the topology of surface singularities through the work of W. Neumann and many others. On the other hand, the links of isolated surface singularities provide a very interesting class of 3-manifolds which, thanks to their rich algebraic nature, have proved to be rather useful for 3-manifold theory, as for instance for the understanding of the Casson invariant, Seiberg-Witten invariants, Floer homology and other important invariants of 3-manifolds that have been discovered in the last decades.

In this sense, it is interesting to find new classes of 3-manifolds, besides the links of isolated complex surfaces singularities, that have a rich (possibly algebraic) geometric structure. And in that sense one has the interesting theorem of F. Michel, A. Pichon, A. Nemethi and A. Szilard, stating that if $f$ is a holomorphic map-germ $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with a 1-dimensional critical set, then the boundary of the Milnor fiber is a Waldhausen manifold. The theorem was announced first by F. Michel and A. Pichon [3], but its proof contained a gap. In [8] F. Michel, A. Pichon and C. Weber provided a proof valid for some classes of singularities. The first
complete proof of the general case of the theorem was provided by A. Nemethi and A. Szilard in a long paper, where even an algorithm to compute the graph describing the Waldhausen manifold is given. Shortly afterwards F. Michel and A. Pichon have provided another complete proof which is more in the spirit of the original method they proposed.

In this article we envisage a similar problem. Consider \( f, g \) holomorphic map-germs such that the germ \( f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) has an isolated critical value at 0. Notice that \( f\bar{g} \) has necessarily non-isolated critical points. Yet, we know from [15] that the germ \( f\bar{g} \) has the Thom a\text{f}-property, and therefore it has a Milnor fibration:

\[ f : N(\epsilon, \delta^*) \to \delta^*, \]

where \( \epsilon \) is a small closed ball around 0 in \( \mathbb{C}^3 \), \( \delta \) is a disc in \( \mathbb{C} \) of sufficiently small radius with respect to \( \epsilon \), and \( \delta^* := \delta \setminus \{0\} \).

Our main theorem (Theorem 5) in this article states that the boundary of the corresponding Milnor fibre is a Waldhausen manifold.

Although our proof has some inspiration from the method of Nemethi and Szilard, and has some points in common with that of Michel and Pichon, it provides a much shorter proof of the theorem for the holomorphic case which immediately generalises to non-holomorphic real analytic germs of the form \( f\bar{g} \). It is based in a detailed understanding of the Milnor fibre of a germ of the form \( f\bar{g} \) defined in the plane in terms of an embedded resolution of \( \{fg = 0\} \). Such a detailed understanding allows to generalise for real analytic germs of type \( f\bar{g} \) defined in the plane, the A’Campo formula for computing the zeta-function of the monodromy in terms of an embedded resolution of singularities. This is Theorem 9.

A natural next step, which is open by now, is to find an algorithm to determine the graph associated to the boundary of the Milnor fibre.

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2. A DESCRIPTION OF THE MILNOR FIBRE OF \( f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) IN TERMS OF ITS EMBEDDED RESOLUTION

Let \( f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) be two holomorphic functions such that \( (fg)^{-1}(0) \) has an isolated singularity at the origin and the real analytic germ given by \( f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) has an isolated critical value at \( 0 \in \mathbb{C} \).

In [14] Pichon and Seade prove that there exist sufficient small positive reals \( 0 < \delta << \epsilon \) such that the restriction

\[ f\bar{g} : (fg)^{-1}(\mathbb{D}_\delta \setminus \{0\}) \cap B^4_\epsilon \to \mathbb{D}_\delta \setminus \{0\} \]

is a locally trivial fibration, where \( \mathbb{D}_\delta \) denotes the disk of radius \( \delta \) in \( \mathbb{C} \) centered at 0 and \( B^4_\epsilon \) denotes the ball in \( \mathbb{C}^2 \) of radius \( \epsilon \) centered at 0.

Let \( \pi : \tilde{M} \to B^4_\epsilon \) be a good embedded resolution of \( (fg)^{-1}(0) \) and consider:

- \( \pi^{-1}(0) := E = \bigcup_{i=1}^p E_i \), the exceptional divisor with its decomposition in irreducible components;
- \( (fg \circ \pi)^{-1}(0) = (fg \circ \pi)^{-1}(0) = \sum_{i=1}^p k_i E_i + \tilde{C} \) the total transform, where \( \tilde{C} \) is the strict transform, which has a decomposition into connected components \( \tilde{C} = \bigcup_{p=1}^m \tilde{C}_p \);
- \( F := (fg \circ \pi)^{-1}(\delta) \), the Milnor fibre of \( f\bar{g} \).
For each $i$, $1 \leq i \leq s$, let $U_i$ be a tubular neighbourhood of $E_i$, and for each $p$, $1 \leq p \leq w$ let $\tilde{U}_p$ be a tubular neighbourhood of $\tilde{C}_p$. Then we define the sets

- $V_{ij} := U_i \cap U_j$;
- $\tilde{V}_{ip} := U_i \cap \tilde{U}_p$;
- $V_i := U_i \setminus \bigcup_{p \neq j} \bigcup_{p=1}^w \tilde{V}_{ip}$;
- $\tilde{V}_p := \tilde{U}_p \setminus \bigcup_{i=1}^s V_i$.

We decompose the Milnor fibre $F$ as follows:

$$F = \left( \bigcup_{i=1}^s (V_i \cap F) \right) \cup \left( \bigcup_{i,j=1}^s (V_{ij} \cap F) \right) \cup \left( \bigcup_{p=1}^w (\tilde{V}_p \cap F) \right) \cup \left( \bigcup_{1 \leq i \leq s} (\tilde{V}_{ip} \cap F) \right).$$

Note that doing some convenient change of coordinates, each part $V_{ij} \cap F$ or $\tilde{V}_{ip} \cap F$ of the Milnor fibre $F$ has equation of the form

$$x^{a_i} \bar{x}^{b_i} y^{a_j} \bar{y}^{b_j} \varphi_1 \varphi_2 = \delta,$$

and each part $V_i \cap F$ or $\tilde{V}_p \cap F$ of the Milnor fibre has equation of the form

$$x^{a_i} \bar{x}^{b_i} \varphi_1 \varphi_2 = \delta,$$

where $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are units in $\mathbb{C}\{x, y\}$, $a_i$ is the multiplicity of $f$ corresponding to $E_i$ and $b_i$ is the multiplicity of $g$ corresponding to $E_i$ (obviously either $a_p = 1$ and $b_p = 0$ or $a_p = 0$ and $b_p = 1$, for each $p = 1, \ldots, w$).

**Lemma 1.** The intersection of the Milnor fibre $F$ with each neighbourhood $V_i$, $V_{ij}$, $\tilde{V}_p$ or $\tilde{V}_{ip}$ is either a finite disjoint union of cylinders (cases (i), (iii) and (iv) in the proof) or it is a finite cover over a disk minus some disks (case (ii) in the proof).

**Proof.** There are four cases to consider:

(i) $F \cap V_{ij}$ with $a_i \neq b_i$ and $a_j \neq b_j$; and $F \cap \tilde{V}_{ip}$ with $a_i \neq b_i$.

By some change of coordinates we can locally consider

$$(fg \circ \pi) = x^{a_i} \bar{x}^{b_i} y^{a_j} \bar{y}^{b_j} \varphi_1 \varphi_2,$$

where

$$\begin{align*}
\varphi_1(x, y) &= \alpha + \psi_1(x, y) \\
\varphi_2(x, y) &= \beta + \psi_2(x, y),
\end{align*}$$

with $\alpha, \beta \in \mathbb{C}^*$ and $\psi_1(0) = \psi_2(0) = 0$. If we set

$$\begin{align*}
\varphi_{1,t}(x, y) &= \alpha + t\psi_1(x, y) \\
\varphi_{2,t}(x, y) &= \beta + t\psi_2(x, y),
\end{align*}$$

we can define the 1-parameter family

$$h_t = fg \circ \pi = x^{a_i} \bar{x}^{b_i} y^{a_j} \bar{y}^{b_j} \varphi_{1,t} \varphi_{2,t},$$

which gives a homotopy between $h_1 = fg \circ \pi$ and $h_0 = x^{a_i} \bar{x}^{b_i} y^{a_j} \bar{y}^{b_j} \alpha \beta$.

Consider the real analytic mapping

$$H : V_{ij} \times [0, 1] \to \mathbb{C} \times [0, 1]$$

$$(z, t) \mapsto (h_t(z), t)$$

and its restriction

$$H_1 : (V_{ij} \times [0, 1]) \cap H^{-1}(D_\delta \setminus \{0\} \times [0, 1]) \to D_\delta \setminus \{0\} \times [0, 1].$$
Then $H_i$ has three properties:

- it is proper.
- It is a submersion on $(V_{ij} \times [0, 1]) \cap H^{-1}(D_{ij}\{0\} \times [0, 1])$.

The Jacobian matrix of $H$ is given by

$$
\begin{pmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial \bar{t}} & f_1 \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial \bar{t}} & f_2 \\
\frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial \bar{t}} & f_3 \\
\frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial \bar{t}} & f_4
\end{pmatrix}
\begin{pmatrix}
\frac{\partial h_1}{\partial x} \\
\frac{\partial h_2}{\partial x} \\
\frac{\partial h_3}{\partial x} \\
\frac{\partial h_4}{\partial x}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial f_1}{\partial x} \phi_1 & \frac{\partial f_1}{\partial y} \phi_1 & \frac{\partial f_1}{\partial \bar{t}} \phi_1 & x^{a_i} \phi_1 \psi_1 g_1 \\
\frac{\partial f_2}{\partial x} \phi_2 & \frac{\partial f_2}{\partial y} \phi_2 & \frac{\partial f_2}{\partial \bar{t}} \phi_2 & x^{a_j} \phi_2 \psi_2 g_2 \\
\frac{\partial f_3}{\partial x} \phi_3 & \frac{\partial f_3}{\partial y} \phi_3 & \frac{\partial f_3}{\partial \bar{t}} \phi_3 & x^{b_i} \phi_3 \psi_3 g_3 \\
\frac{\partial f_4}{\partial x} \phi_4 & \frac{\partial f_4}{\partial y} \phi_4 & \frac{\partial f_4}{\partial \bar{t}} \phi_4 & x^{b_j} \phi_4 \psi_4 g_4
\end{pmatrix}
$$

Then $H_i$ is a submersion in a point $p$ if, and only if, it is not a solution of at least one of the four following equations:

\[
\begin{align*}
(1) & \quad |\frac{\partial f_i}{\partial x}|^2 |g_t|^2 - |\frac{\partial y_i}{\partial y}|^2 |f_t|^2 = 0 \\
(2) & \quad |\frac{\partial f_i}{\partial y}|^2 |g_t|^2 - |\frac{\partial y_i}{\partial y}|^2 |f_t|^2 = 0 \\
(3) & \quad |f_i|^2 |\frac{\partial f_i}{\partial x} g_t| - |g_t|^2 |\frac{\partial f_i}{\partial y} g_t| = 0 \\
(4) & \quad f g \left( \frac{\partial f_i}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f_i}{\partial y} \frac{\partial g}{\partial x} \right) = 0
\end{align*}
\]

Note that $f_t = x^{a_i} y^{a_j} \varphi_{1,t}$ and $g_t = x^{b_i} y^{b_j} \varphi_{2,t}$. Then setting

\[
\begin{align*}
\zeta_1 &= a_i \varphi_{1,t} + x^{a_i} \\
\zeta_2 &= a_j \varphi_{1,t} + y^{a_j} \\
\zeta_3 &= b_i \varphi_{2,t} + x^{b_i} \\
\zeta_4 &= b_j \varphi_{2,t} + y^{b_j}
\end{align*}
\]

we have that

\[
\begin{align*}
\frac{\partial f_1}{\partial x} &= x^{a_i} y^{a_j} \zeta_1 \\
\frac{\partial f_1}{\partial y} &= x^{a_i} y^{a_j} \zeta_2 \\
\frac{\partial g_1}{\partial x} &= x^{b_i} y^{b_j} \zeta_3 \\
\frac{\partial g_1}{\partial y} &= x^{b_i} y^{b_j} \zeta_4
\end{align*}
\]

Substituting on equations (1) to (4), we have the equations:

\[
\begin{align*}
(1) & \quad |x|^{2(a_i+b_j-1)} |y|^{2(a_j+b_i)} |\varphi_{1,t}|^2 |\varphi_{2,t}|^2 \frac{\zeta_1^2}{\zeta_4^2} = 0 \\
(2) & \quad |x|^{2(a_i+b_j)} |y|^{2(a_j+b_i-1)} |\varphi_{1,t}|^2 |\varphi_{2,t}|^2 \frac{\zeta_2^2}{\zeta_4^2} = 0 \\
(3) & \quad x y |x|^{2(a_i+b_j-1)} |y|^{2(a_j+b_i-1)} |\varphi_{1,t}|^2 |\varphi_{2,t}|^2 \frac{\zeta_1 \zeta_2}{\zeta_3 \zeta_4} = 0 \\
(4) & \quad x^{2(a_i+b_j-1)} y^{2(a_j+b_i-1)} \varphi_{1,t} \varphi_{2,t} |\zeta_1 \zeta_2 | \frac{\zeta_3 \zeta_4}{\zeta_1 \zeta_3} = 0
\end{align*}
\]

Since

\[
\begin{align*}
|\varphi_{1,t}(0)|^2 & \quad |\varphi_{2,t}(0)|^2 = |a_i^2 - b_j^2| |\varphi_{1,t}(0)\varphi_{2,t}(0)|^2 \\
\zeta_1(0)^2 & \quad \zeta_3(0)^2 = |a_i^2 - b_j^2| |\varphi_{1,t}(0)\varphi_{2,t}(0)|^2
\end{align*}
\]

and

\[
\begin{align*}
|\varphi_{1,t}(0)|^2 & \quad |\varphi_{2,t}(0)|^2 = |a_i^2 - b_j^2| |\varphi_{1,t}(0)\varphi_{2,t}(0)|^2 \\
\zeta_2(0)^2 & \quad \zeta_4(0)^2 = |a_i^2 - b_j^2| |\varphi_{1,t}(0)\varphi_{2,t}(0)|^2
\end{align*}
\]

the result follows.
• It is a submersion on the boundary
  \[ \partial(V_{ij} \times D_{\epsilon_2} \times [0,1]) \cap H^{-1}(D_\epsilon \setminus \{0\} \times [0,1]). \]

Indeed, the tubular neighbourhoods can be chosen so that \(V_{ij}\) is the polydisc \(D_{\epsilon_1} \times D_{\epsilon_2}\) for small \(\epsilon_i\). Note that \(\delta << \min\{\epsilon_1, \epsilon_2\}\). Then

\[ \partial(D_{\epsilon_1} \times D_{\epsilon_2}) = (\partial D_{\epsilon_1} \times D_{\epsilon_2}) \cup (D_{\epsilon_1} \times \partial D_{\epsilon_2}) \]

and therefore it is an \(n\)-cylinder. So we conclude that \(h^{-1}_1(\delta)\) intersects \(\partial D_{\epsilon_1} \times \partial D_{\epsilon_2}\) transversally and, by an analogous argument, that \(h^{-1}_2(\delta)\) intersects \(D_{\epsilon_1} \times \partial D_{\epsilon_2}\) transversally if \(a_j \neq b_j\).

Then if follows from Ehresmann’s Fibration Lemma that the Milnor fibre of \(h_1 = f \circ \pi\) is diffeomorphic to the Milnor fibre of \(h_0 = x^{a_1} \bar{x}^{b_1} y^{a_2} \bar{y}^{b_2} \alpha \beta\), which is diffeomorphic to

\[ \{ x^{a_1} \bar{x}^{b_1} = \frac{\delta}{y^{a_2} \bar{y}^{b_2}} \} \cap (D_{\epsilon_1} \times D_{\epsilon_2}), \]

which is a covering over an annulus of degree \(|a_i - b_i|\), and therefore it is a disjoint union of at most \(|a_i - b_i|\)-cylinders.

(ii) \(F \cap V_i\) with \(a_i \neq b_i\) and \(F \cap V_{ip}\):

We can apply exactly the same proof of case (i), considering \((f \circ \pi) = x^{a_1} \bar{x}^{b_1} \varphi_1 \varphi_2\). Then we get that the Milnor fibre of \(f \circ \pi\) inside \(V_i\) is diffeomorphic to the set

\[ \{ x^{a_1} \bar{x}^{b_1} = \delta \}, \]

and therefore it is an \(|a_i - b_i|\)-covering of \(V_i \cap E_i\), which is a disk minus some disks.

(iii) \(F \cap V_{ij}\) with \(a_i = b_i\) and \(a_j \neq b_j\) or with \(a_i \neq b_i\) and \(a_j = b_j\); and \(F \cap V_{ip}\) with \(a_i = b_i\):

Consider the mapping germ

\[ (f \circ \pi, g \circ \pi) : (\mathbb{C}_1^2, 0) \rightarrow (\mathbb{C}_2^2, 0) \]

\[ (x, y) \mapsto (x^{a_1} y^{a_2} \varphi_1, x^{b_1} y^{b_2} \varphi_2) \]

We want to find a change of coordinates \(\Theta = (\Theta_1, \Theta_2) : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)\) such that

\[ (f \circ \pi, g \circ \pi) = (x^{a_1} y^{a_2}, x^{b_1} y^{b_2}) \circ \Theta, \]

which happens if and only if \((f \circ \pi, g \circ \pi) = (\Theta_1^a, \Theta_2^a, \Theta_1^b, \Theta_2^b)\). If we set \(\Theta_1 = x \theta_1\) and \(\Theta_2 = y \theta_2\), with \(\theta_1(0) \neq 0\) and \(\theta_2(0) \neq 0\), then our problem is to find \(\theta_1, \theta_2 : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)\) such that

\[ (f \circ \pi, g \circ \pi) = (x^{a_1} y^{a_2} \theta_1^a \theta_2^a, x^{b_1} y^{b_2} \theta_1^b \theta_2^b). \]

This happens if and only if the system

\[ \begin{cases} 
  x^{a_1} y^{a_2} \theta_1^a \theta_2^a = x^{a_1} y^{a_2} \varphi_1 \\
  x^{b_1} y^{b_2} \theta_1^b \theta_2^b = x^{b_1} y^{b_2} \varphi_2 
\end{cases} \]
has solution \((\theta_1, \theta_2)\). This is equivalent to

\[
\begin{aligned}
\theta_1^a \theta_2^a &= \varphi_1 \\
\theta_1^b \theta_2^b &= \varphi_2,
\end{aligned}
\]

which has solution, if and only if the linear system with indeterminates \(\log \theta_1, \log \theta_2\)

\[
\begin{aligned}
a_i \log \theta_1 + a_j \log \theta_2 &= \log \varphi_1 \\
b_i \log \theta_1 + b_j \log \theta_2 &= \log \varphi_2,
\end{aligned}
\]

has solution. But we have the non-vanishing of the determinant

\[
\begin{vmatrix}
a_i & a_j \\
b_i & b_j
\end{vmatrix} \neq 0,
\]

and so the system has solutions. Then \((fg \circ \pi) = (x^{a_i} \bar{x}^{b_i} | y|^{2b_i}) \circ \Theta\) and therefore its Milnor fibre is given by the equation

\[
\{|y|^{2b_i} = \frac{\delta}{x^{a_i} \bar{x}^{b_i}} \} \cap V_{ij},
\]

which is a disjoint union of \(|a_i - b_i|\) cylinders (which intersect \(\partial V_{ij}\) transversally).

\((iv)\) \(F \cap V_i\) with \(a_i = b_i\) and \(F \cap V_{ij}\) with \(a_i = b_i\) and \(a_j = b_j\):

A. Pichon and J. Seade [14] proved that \(a_i = b_i\) implies that \(E_i\) does not represent a rupture vertex of the dual graph of the total transform of \((fg)^{-1}(0)\) by \(\pi\).

Let \(S\) be the union of all the exceptional divisors \(E_i\) such that \(a_i = b_i\) and consider its decomposition in connected components \(S = S_1 \cup \cdots \cup S_k\). For each \(S_i\), let \(\Omega_i\) be the union of all the \(V_i, V_{ij}\) and \(\bar{V}_{ip}\) that intersect \(S_i\), excluding the \(V_{ij}\)'s that intersect some \(E_i\) with \(a_i \neq b_i\) (there are at least one and at most two of them). There are only two cases to consider:

\((a)\) There is just one \(V_{ij}\) in \(S_i\) that intersects some \(E_i\) with \(a_i \neq b_i\).

\((b)\) There are exactly two \(V_{ij}\)'s in \(S_i\) that intersect some \(E_i\)'s with \(a_i \neq b_i\).

We claim that case \((a)\) does never occur. In fact, we know that if \(M\) denotes the \((s \times s)\)-intersection matrix of \(E\), that is,

\[
m_{ij} = \begin{cases} 
E_i^2, & \text{if } i = j; \\
1, & \text{if } i \neq j \text{ and } E_i \text{ intersects } E_j; \\
0, & \text{otherwise}
\end{cases}
\]

and if \(u_i\) denotes the number of intersection points between \(E_i\) and the strict transform of \(f\) minus the number of intersection points between \(E_i\) and the strict transform of \(g\), then (see [7], Theorem 18.2)

\[
M \cdot \begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_s - b_s \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_s \end{pmatrix} = 0
\]
Let \( f, g : (\mathbb{C}^2, 0) \to \mathbb{C} \), be families of holomorphic germs depending holomorphically on a parameter \( t \) which varies in a disc \( D \). Suppose that \( f_t g_t \) has an isolated singularity at the origin whose Milnor number is independent
of \( t \). For any \( t_0 \in D \) there exists a neighbourhood \( t_0 \subset U \subset D \), and a positive number \( \epsilon \) which is a Milnor radius for \( f\bar{g} \), for any \( t \in U \).

**Proof.** A Milnor radius for a real analytic germ of the form \( f\bar{g} \) is a positive radius \( \epsilon \) such that for any other radius \( \epsilon' \leq \epsilon \), the sphere of radius \( \epsilon' \) centered at the origin meets \( fg^{-1}(0) \) transversely. Thus \( \epsilon \) is a Milnor radius for the real analytic germ \( f\bar{g} \) if and only if it is a Milnor radius for the holomorphic germ \( f\bar{g} \). The assertion of the Remark follows from the corresponding one for \( \mu \)-constant families of holomorphic germs of functions in two variables: its well known that a \( \mu \)-constant family of plane curves is Whitney equisingular and admits a uniform Milnor radius.

\[ \square \]

### 3. The boundary of the Milnor fibre of non-isolated singularities

Let \( f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) be two holomorphic functions such that the real analytic map-germ \( f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) has an isolated critical value at 0 \( \in \mathbb{C} \). Seade and Pichon [15] prove that such real analytic function \( f\bar{g} \) as above always have Thom’s \( a_{F} \)-property. Hence there exist \( \epsilon, \delta \in \mathbb{R} \) sufficiently small, \( 0 < \delta << \epsilon \), such that \( f\bar{g} \) can be given a Milnor-Lê fibration in the tube as follows:

\[
(f\bar{g})^{-1}(D_\delta \setminus \{0\}) \cap B_\epsilon^3 \to D_\delta \setminus \{0\}.
\]

We would like to prove that the boundary of the Milnor fibre

\[
L_t := (f\bar{g})^{-1}(t) \cap S_\epsilon^2,
\]

for \( t \in D_\delta \setminus \{0\} \) is a Waldhausen manifold.

#### 3.1. Systems of neighbourhoods

In the holomorphic case it is well known that it is possible to define the Milnor fibration using different systems of neighbourhoods at the origin. Balls and polydisks are the most widely used systems of neighbourhoods. In the proof of our main result we need to work with a Milnor fibration defined using a polydisk instead of a ball. We shall prove now that the boundary of the Milnor fibre defined for polydisks is homeomorphic to the boundary of the Milnor fibre defined for balls.

Let \( \Sigma \) be the singular set of \( \{fg = 0\} \). It is the complex curve given by

\[
\Sigma(f\bar{g}) = \Sigma(f) \cup \Sigma(g) \cup (f^{-1}(0) \cap g^{-1}(0)) = \Sigma(fg).
\]

Choose a coordinate system \((x, y, z)\) of \( \mathbb{C}^3 \) such that there exists \( \epsilon \) such that for any \( \epsilon' \leq \epsilon \) the boundary of the polydisk

\[
\Delta_\epsilon = \{(x, y, z) \in \mathbb{C}^3 : \max\{|x|, |y|, |z|\} \leq \epsilon\}
\]

meets \( \Sigma \) transversely at the open face

\[
\{(x, y, z) \in \mathbb{C}^3 : \max\{|x|, |y|\} < \epsilon, |z| = \epsilon\}.
\]

Consider the family of norms in \( \mathbb{C}^3 \)

\[
||\langle x, y, z \rangle ||_s := (|x|^{1/s} + |y|^{1/s} + |z|^{1/s})^s,
\]

\[
||\langle x, y, z \rangle ||_0 := \max\{|x|, |y|, |z|\}
\]

for \( s \in [0, 1/2] \), which depends continuously on the parameter \( s \).

A positive number \( \epsilon \) is a Milnor radius for the function \( \{fg = 0\} \) with respect to the norm \( ||\cdot|| \), if for any positive \( \epsilon' \leq \epsilon \) the hypersurface \( fg = 0 \) is transverse in the stratified sense to the sphere \( ||\langle x, y, z \rangle ||_s = \epsilon' \). It is well known that for any \( s \in [0, 1/2] \) there is a Milnor radius for \( \{fg = 0\} \) with respect to the norm \( ||\cdot||_s \).

Moreover, by the continuity of the norms in the parameter \( s \), if \( \epsilon \) is a Milnor radius
for \( \{fg = 0\} \) with respect to the norm \(|| \cdot ||_s\), then there exists a neighbourhood \( U \) of \( s \in [0, 1/2] \) such that \( \epsilon \) is a Milnor radius for \( \{fg = 0\} \) with respect to the norm \(|| \cdot ||_{s'}\) for any \( s' \in U \). Using the compactness of \([0, 1/2]\) we find a radius \( \epsilon \) which is a Milnor radius for \( \{fg = 0\} \) with respect to the norm \(|| \cdot ||_s\) for any \( s \in [0, 1/2] \).

Given any \( U \subset [0, 1/2] \) we define the set
\[
B^U_\epsilon := \{(x, y, z, s) \in \mathbb{C}^3 \times U : ||(x, y, z)||_s \leq \epsilon\}.
\]
It must be viewed as a family of Milnor balls for varying norms.

Since the function \( fg \) satisfies the Thom \( a_F \)-property, for any \( s \in [0, 1/2] \) there exists a neighbourhood \( U \) of \( s \in [0, 1/2] \) and a positive \( \delta \) such that the mapping
\[
F^U : B^U_\delta \cap (fg)^{-1}(D_\delta \setminus \{0\}) \to D_\delta \setminus \{0\} \times U
\]
defined by \( F^U(x, y, z, s) := (f(x, y, z)\overline{g}(x, y, z), s) \) is a topological locally trivial fibration.

By compactness of \([0, 1/2]\) there exists a positive \( \delta \) such that the mapping
\[
F : B^{[0, 1/2]}_\delta \cap (fg)^{-1}(D_\delta \setminus \{0\}) \to D_\delta \setminus \{0\} \times [0, 1/2]
\]
is a topological locally trivial fibration.

Consequently, the boundaries of fibres \( F^{-1}(t, 0) \) and \( F^{-1}(t, 1/2) \) are homeomorphic for any \( t \in D_\delta \setminus \{0\} \), and hence the boundary of the Milnor fibre defined for polydiscs is homeomorphic to the boundary of the Milnor fibre defined for balls.

3.2. The Waldhausen structure. Our main goal is to prove that the boundary of the Milnor fibre
\[
L_t = (fg)^{-1}(t) \cap S^5_\epsilon,
\]
for \( t \in D_\delta \setminus \{0\} \) is a Waldhausen manifold. Since we have proved that the boundary of the Milnor fibre defined using balls is homeomorphic to the boundary of the Milnor fibre using polydisk, from this moment on we will assume that \( S^5_\epsilon \) denotes the boundary of the ball of Milnor radius \( \epsilon \) for the norm \(|| \cdot ||_0\) as in 3.1.

The singular set of
\[
L_0 = (fg)^{-1}(0) \cap S^5_\epsilon = (fg)^{-1}(0) \cap S^5_\epsilon
\]
is the intersection of the sphere \( S^5_\epsilon \) with the complex curve \( \Sigma \). We know that
\[
L(\Sigma) := \Sigma \cap S^5_\epsilon
\]
is a finite disjoint union of circles \( S^1 \) contained in the open face
\[
\{(x, y, z) \in \mathbb{C}^3 : \max\{|x|, |y|\} < \epsilon, |z| = \epsilon\}.
\]

Let
\[
n : \tilde{X} \to (fg)^{-1}(0)
\]
be the normalization of \((fg)^{-1}(0)\) and set \( \Sigma := n^{-1}(\Sigma) \).

Let \( W \) denote the vanishing zone of \( f\overline{g} \), which is nothing but a tubular neighbourhood of \( L(\Sigma) \) in \( S^5_\epsilon \), and we define \( W_0 := W \cap L_0 \) and \( W_t := W \cap L_t \). Then it is well known that \( \tilde{X} \setminus \tilde{W}_0 \) is a Waldhausen manifold, where \( \tilde{W}_0 := n^{-1}(W_0) \) is a tubular neighbourhood of \( \Sigma \) in \( \tilde{X} \) (see [5] for instance). Since the normalization is an isomorphism outside \( W_0 \), we find that \( L_0 \setminus W_0 \) is a Waldhausen manifold.

It is also easy to check that \( L_t \setminus W_t \) is diffeomorphic to \( L_0 \setminus W_0 \), and then it follows that \( L_t \setminus W_t \) is Waldhausen. Since \( \partial W_t \) is a finite disjoint union of tori, all we have to prove is that \( W_t \) is a Waldhausen manifold.

Now, if \( \Sigma = \Sigma_1 \cup \cdots \cup \Sigma_k \) is the decomposition of \( \Sigma \) into irreducible components, we get the decomposition into disjoint connected components \( W = W[1] \cup \cdots \cup W[k] \),
where $W[i]$ is a small tubular neighbourhood of the circle $\Sigma_i \cap S^5_i$ in $S^5_i$, for $i = 1, \ldots, k$.

Fix a component $\Sigma_l$. Given $p \in \Sigma_l \setminus \{0\}$ let $H_p$ be the 2-dimensional affine hyperplane of $\mathbb{C}^3$ passing through $p$ and parallel to $\{z = 0\}$. Choosing $\epsilon$ small enough we may assume that the Milnor number of the germ $(fg|_{H_p}, p)$ is independent of $p$. Therefore, by Remark 2 and the compactness of $\Sigma_l \cap S^5$ we deduce the existence of a positive $\eta$ such that for any $p \in \Sigma_l \cap S^5$ the ball in $H_p$ centered at $p$ and of radius $\eta$ is a Milnor ball for $(fg|_{H_p}, p)$. We may choose $W[l]$ to be the union of those balls when $p$ varies in $\Sigma_l \cap S^5$. With this definition there is an obvious fibration

$$\sigma_l : W[l] \rightarrow \Sigma_l \cap S^5$$

with fibre a complex 2-ball.

Since $\bar{f}\bar{g}$ satisfies Thom’s $\alpha_F$-property there exists a positive $\delta$ such that the mapping

$$\Psi_l : W[l] \cap (\bar{f}\bar{g})^{-1}(D\delta) \rightarrow D\delta \times (\Sigma_l \cap S^5)$$

defined by $\Psi_l := (\bar{f}\bar{g}, \sigma_l)$ has only the circle $\{0\} \times (\Sigma_l \cap S^5)$ as critical values. Therefore, for $t \in D\delta \setminus \{0\}$, the restriction

$$\sigma_{il} : L_t \cap W[l] : L_t \cap W[l] \rightarrow \Sigma_l \cap S^5$$

is a locally trivial fibration. Its fibre is called the Transversal Milnor fibre of $\bar{f}\bar{g}$ at $\Sigma_l$ and its monodromy $h$ the vertical monodromy along $\Sigma_l$, see [10].

To prove that $W_l$ is a Waldhausen manifold, we have to show that each connected component $L_t \cap W[l]$, for $i = 1, \ldots, k$, is a Waldhausen manifold. To do that, it is sufficient to give a decomposition of each transversal Milnor fibre which is invariant under the corresponding vertical monodromy such that the corresponding pieces of $L_t \cap W[l]$ are Seifert manifolds. We will prove that they are Seifert manifolds either by proving that they are fibrations with base a circle and fibre a cylinder, or by showing directly that the restriction of $h$ to the corresponding piece of transversal Milnor fibre is periodic.

Let us fix on an irreducible component $\Sigma_l$ of $\Sigma$, which by an easy argument can be assumed, without losing generality, to be the $z$-axis (see [7], Lemma 4.4). Let $D$ be the disk of radius $\epsilon$ around the origin of $\Sigma_l$. This coincides with the intersection of $\Sigma_l$ with the polydisc of size $\epsilon$. Let $S^1$ be its boundary circle. The region $W[l]$ coincides now with the product $S^1 \times B_\eta$, where $B_\eta$ is the ball of radius $\eta$ in the $(x, y)$-complex 2-plane, and the mapping $\sigma_l$ coincides with the projection to the first factor.

We can look at the restriction $(\bar{f}\bar{g})_l : D^* \times \mathbb{C}^2 \rightarrow \mathbb{C}$ as a family in the parameter $D^*$. We denote by $\bar{f}\bar{g}_s$ the restriction $\bar{f}\bar{g}_{(s)} : D^* \times \mathbb{C}^2$. The corresponding holomorphic family $\bar{f}\bar{g} : D^* \times \mathbb{C}^2 \rightarrow \mathbb{C}$ is $\mu$-constant over $D^*$. Then we can consider a minimal embedded resolution in family

$$\pi : \tilde{M} \rightarrow D^* \times \mathbb{C}^2$$

where

- $\pi^{-1}(D^* \times \{0\}) := E$ is the exceptional divisor, with a decomposition in irreducible components $E = \cup_{i=1}^r E_i$, where an irreducible component is defined as the closure of a connected component of $E \setminus Sing(E)$;
- for each $s \in S^1$ define $X_s := \pi^{-1}(\{s\} \times B_\eta)$. Then

$$\pi_s : X_s \rightarrow \mathbb{C}^2$$
is the minimal embedded resolution of the plane curve singularity that the restriction of \( f \) to \( H_s \) defines at the point \((s,0,0)\). We denote by \( E^s \) the exceptional divisor of \( \pi_s \), and by \( E^s_i \) the set of irreducible components of \( E^s \) contained in \( E_i \).

For each \( i \in \{1, \ldots, r\} \), let \( U_i \) be a tubular neighbourhood of \( E_i \) and define the boxes

\[
V_{ij} := U_i \cap U_j \quad \text{and} \quad V_i := U_i \setminus \cup_{j \neq i} V_{ij}.
\]

Then for each \( s \in S^1 \), the transversal Milnor fibre of \((f \bar{g})^{-1}(t) \cap \sigma^{-1}(s)\) is diffeomorphic to \((f \bar{g}_s \circ \pi_s)^{-1}(t)\), and this set can be decomposed as follows:

\[
(f \bar{g}_s \circ \pi_s)^{-1}(t) = \left( \bigcup_i \left(V_i \cap (f \bar{g}_s \circ \pi_s)^{-1}(t)\right) \right) \cup \left( \bigcup_{i,j} \left(V_{ij} \cap (f \bar{g}_s \circ \pi_s)^{-1}(t)\right) \right).
\]

This decomposition is preserved by the vertical monodromy.

In the holomorphic case (when \( g \) is constant), we know that each part of the Milnor fibre of type \( V_{ij} \cap (f \circ \pi_s)^{-1}(t) \) has equation of the form \( x^{k_i} y^{b_i} = t \), and therefore it is a finite union of cylinders. In Lemma \([\text{I}]\) we show that the same happens in the general case, that is, \( V_{ij} \cap (f \bar{g}_s \circ \pi_s)^{-1}(t) \) is also a finite disjoint union of cylinders. Hence \( V_{ij} \cap (f \bar{g})^{-1}(t) \cap \pi^{-1}(S^1 \times \mathbb{C}^2) \) is a fibre bundle over \( S^1 \) with fibre a finite disjoint union of cylinders. The classification of such kind of fibrations yields that all their total spaces are Seifert manifolds.

In the general case, the \( V_i \)'s satisfying \( a_i = b_i \) are grouped in a finite number of connected regions \( \Omega_k \) and by Lemma \([\text{I}]\) case (iv), we have that \( \Omega_k \cap (f \bar{g}_s \circ \pi_s)^{-1}(t) \) is a disjoint union of cylinders. As before we deduce that the corresponding piece is a Seifert manifold.

In the holomorphic case, each part of the Milnor fibre of type \( V_i \cap (f \circ \pi_s)^{-1}(t) \) has equation of the form \( x^{k_i} = t \), and therefore \( V_i \cap (f \circ \pi_s)^{-1}(t) \) is a finite covering over \( E^s_i \cap V_i \).

We showed in Lemma \([\text{I}]\) case (ii), that when \( a_i \neq b_i \) the very same happens in the general case, that is,

\[
V_i \cap (f \bar{g}_s \circ \pi_s)^{-1}(t) \quad \text{and} \quad E^s_i \cap V_i = \mathbb{P}^1 \setminus \text{disks}
\]

is a finite covering, where \( \rho_i \) is the projection of the tubular neighbourhood of \( E^s_i \) to \( E^s_i \).

If the connected components of \( E^s_i \) do not represent rupture vertices of the dual graph of the total transform of the transversal singularity by \( \pi_s \), then \( V_i \cap (f \bar{g})^{-1}(t) \cap \pi^{-1}(S^1 \times \mathbb{C}^2) \) is a fibre bundle over \( S^1 \) with fibre a finite disjoint union of either disks or cylinders, and therefore it is a Seifert manifold. So now we shall see what happens in \( V_i \) if the connected components of \( E^s_i \) represent rupture vertices.

**Proposition 3.**  
(i) A finite cover of a Seifert manifold is a Seifert manifold;  
(ii) A finite cover of a Waldhausen manifold is a Waldhausen manifold.

**Proof.** Let \( \pi : M' \to M \) be a finite cover of a Waldhausen manifold. Write \( M = \cup M_i \), where each Seifert piece \( M_i \) is a fibre bundle over a compact surface with boundary \( F_i \) and fibre \( S^1 \) (with finitely many multiple special fibres), and projection
$p_i : M_i \to F_i$. It is enough to prove that each piece $M_i' := \pi^{-1}(M_i)$ is a Seifert manifold. The composition

$$p_i \circ \pi : M_i' \to F_i$$

is a bundle (with finitely many multiple special fibres) with fibre a finite cover of $S^1$. If such fibre is connected, then we have expressed $M_i'$ as a bundle over a surface with fibre $S^1$, and therefore it is a Seifert manifold.

If the fibre is not connected, we define the following equivalence relation in $M_i'$: two points are identified if they belong to the same connected component of the same fibre of $p_i \circ \pi$. The quotient of $M_i'$ by this equivalence relation is a surface $F_i'$ that covers $F_i$, and the quotient application $q : M_i' \to F_i'$ expresses $M_i'$ as a bundle over $F_i'$ with fibre $S^1$.

Now fix $i \in \{1, \ldots, r\}$ and note that the component $E_i$ fibres over the punctured disk $D^*$ with projection

$$\pi_1 : E_i \to D^*,$$

where $\pi_1$ is the composition of the mapping $\pi$ restricted to $E_i$ and the projection $p(x, y, z) = z$ of $\mathbb{C}^3$ to the $z$-axis $\Sigma_i$. Its fibre $E_i^*$ is a disjoint union of $\mathbb{P}^1$'s. According to Proposition 3, all we have to prove is that $E_i \cap V_i \cap \pi_1^{-1}(S^1)$ is Waldhausen.

To do that, we would like $\pi_1$ to have connected fibre. It is easy to produce a finite cover

$$\tau : D^* \to D^*$$

and an analytic lift

$$\pi_2 : E_i \to D^*$$

which is a locally trivial bundle with fibre $\mathbb{P}^1$ such that $\tau \circ \pi_2 = \pi_1$.

**Lemma 4.** If $E_i$ represents a rupture vertex of the dual graph of $\pi$, then, after pullback by a finite covering of the base, the lift $\pi_2 : E_i \to D^*$ is the projection of a trivial bundle.

**Proof.** Let $\varsigma_1, \varsigma_2, \varsigma_3 : D^* \to E_i$ be three pairwise disjoint sections of $\pi_2$ given by the intersection of $E_i$ with three distinct components of the total transform of $(fg)^{-1}(0)$ by $\pi$ (either other exceptional divisors or components of the strict transform). Note that it may be necessary to make a base change of $D^*$ so the sections are uni-valued.

Consider the fibre bundle homeomorphism $\Psi : E_i \to \mathbb{P}^1 \times D^*$ defined by sending three pairwise different constant sections of the first bundle to the sections $\varsigma_1, \varsigma_2, \varsigma_3$ on the second and imposing that fibrewise $\Psi$ is a projective isomorphism.

From the previous Lemma it is immediate that the restriction of the vertical monodromy $h_i : E_i^s \to E_i^v$, for $s \in S^1$, is periodic (it is the identity after a finite base change). Therefore $E_i \cap V_i \cap \pi_1^{-1}(S^1)$ is a Seifert manifold. We have proved:

**Theorem 5.** Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that the real analytic germ given by $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ has an isolated critical value at $0 \in \mathbb{C}$. Then the boundary of the Milnor fibre of $f\bar{g}$ is a Waldhausen manifold.

4. The Zeta Function of the Monodromy

Now we want to give a formula to calculate zeta function of the monodromy $h$ of $f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with an isolated singularity, in terms of the combinatorics
of the embedded resolution of \(fg\), in the same way A’Campo \(\Pi\) did to calculate
the monodromy of holomorphic functions.

If we set \(F_\theta := (fg)^{-1}(\delta e^{\theta})\), for \(\delta\) sufficiently small, then for each \(q \geq 0\), the monodromy \(h : F_\theta \rightarrow F_\theta\) defines an isomorphism between vector spaces (the homology groups) given by \(h^* : H^q(F_\theta; \mathbb{C}) \rightarrow H^q(F_\theta; \mathbb{C})\), the so called algebraic monodromy.

The zeta function of such monodromy is defined by
\[
Z(t) = \prod_{q \geq 0} \left( \text{det}(Id^* - th^*) \right)^{(-1)^{q+1}}.
\]

There is a very classic way of calculating the zeta function of \(h\) in terms of the Lefschetz numbers of \(h\), which we describe below:

For each \(k \geq 1\), the Lefschetz number of the \(k\)-iteration of \(h\) is defined by
\[
\Lambda(h^k) = \sum_{q \geq 0} (-1)^q \text{trace}[(h^*)^k : H^q(F_\theta; \mathbb{C}) \rightarrow H^q(F_\theta; \mathbb{C})].
\]

If we define the integers \(s_1, s_2, \ldots\) by the relations
\[
\Lambda(h^k) = \sum_{i \mid k} s_i,
\]
then the zeta function of \(h\) is given by
\[
Z(t) = \prod_{i \geq 1} (1 - t^i)^{-s_i/i}.
\]

So all we have to do is to calculate the Lefschetz numbers of \(h\). First we recall the following lemma:

**Lemma 6.** Consider the following commutative chain map on an exact sequence:
\[
\begin{array}{cccccccc}
0 & \longrightarrow & G_0 & \longrightarrow & G_1 & \longrightarrow & \ldots & \longrightarrow & G_n & \longrightarrow & 0 \\
& & \downarrow \varphi_0 & & \downarrow \varphi_1 & & \ldots & & \downarrow \varphi_n & & \\
0 & \longrightarrow & G_0 & \longrightarrow & G_1 & \longrightarrow & \ldots & \longrightarrow & G_n & \longrightarrow & 0 \\
\end{array}
\]
Then
\[
\sum_{i=0}^n (-1)^i \text{trace}[\varphi_i] = 0
\]

Let \(\pi : \tilde{M} \rightarrow \mathbb{C}^2\) be an embedded resolution of the germ \((fg)^{-1}(0)\) at the origin. Let \(E = \bigcup_{i=1}^s\) be a decomposition of the exceptional divisor of \(\pi\) in irreducible components. Let \(a_i\) and \(b_i\) denote the multiplicity of \(E_i\) in the total transform of \(f^{-1}(0)\) and \(g^{-1}(0)\) respectively.

We apply the previous lemma to the Mayer-Vietoris sequence associated to the decomposition of the Milnor fibre of \(fg\) in the boxes \(V_i\) and \(V_{ij}\) and \(\Omega_i\) as in Lemma \(\Pi\) (here, in order to simplify notation, we do not make distinction between \(V_i\) and \(V_p\) nor between \(V_{ij}\) and \(V_{ip}\) we get:

\[
\Lambda(h^k) = \sum_{i=1}^s \Lambda(h^k_{V_i \cap F_p}) + \sum_{i,j=1}^s \Lambda(h^k_{\Omega_{ij} \cap F_p}) - \sum_{i=1}^s \Lambda(h^k_{\partial V_i \cap F_p}).
\]

Recall that we have grouped the components \(V_i\) with \(a_i = b_j\) and \(V_{ij}\) with \(a_i = b_i\) and \(a_j = b_j\) in larger domains \(\Omega_i\), and we have proved in Lemma \(\Pi\) that the part
of the Milnor number contained in these components is a finite union of cylinders. Moreover, we have seen that each of these domains \( \Omega_i \) contains only one \( V_i \), that is, \( \Omega_i = V_i \), with \( a_i = b_i \).

**Lemma 7.** We have:

(i) \( \Lambda(h^k_{V_i \cap F_0}) = 0 \), for any \( i \neq j \);
(ii) \( \Lambda(h^k_{\Omega_l \cap F_0}) = 0 \), for any \( l \);
(iii) \( \Lambda(h^k_{\partial V_i \cap F_0}) = 0 \), for any \( i \).

**Proof.** Each of the pieces of the Milnor fibre considered are either finite unions of cylinders (cases (i) and (ii)) or of circles (case (iii)), by the way the Milnor fibre have been splitted (see Lemma 1). The cylinders of cases (i) and (ii) have always a circle of case (iii) as a boundary component. Since a cylinder can be retracted to its boundary, it is enough to prove the result for case (iii). The finite union of circles \( \partial V_i \cap F_0 \) is a finite covering over a circle of \( E_i \) and the monodromy is compatible with the covering projection. Since the Euler characteristic of the circle vanishes, the result then follows from the following lemma: \( \square \)

**Lemma 8.** Let \( \pi : X \to B \) be a \( m \)-covering of a compact manifold with boundary and let \( h \) be an automorphism of \( X \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{h} & X \\
\downarrow{\pi} & & \downarrow{\pi} \\
B & & B
\end{array}
\]

commutes. For \( b \in B \), denote by \( h_b : \pi^{-1}(b) \to \pi^{-1}(b) \) the permutation induced by \( h \), and suppose this permutation is cyclic and transitive. Then

\[
\Lambda(h^k) = \chi(B) \cdot \Lambda(h_b^k) = \begin{cases} 
\chi(B) \cdot m, & \text{if } m \mid k; \\
0, & \text{if } m \nmid k
\end{cases}
\]

**Proof.** Suppose that \( B \) is contractible. Then \( X \simeq B \times \pi^{-1}(b) \) and then \( H^q(X) \simeq H^q(\pi^{-1}(b)) \) and therefore \( \Lambda(h^k) = \Lambda(h_b^k) \). If \( B \) is not contractible, we can write it as a (finite) union of contractible sets \( B_i \), \( i \in \{1, \ldots, c\} \), such that \( B_i \cap B_j \) is contractible, for any \( i, j \in \{1, \ldots, c\} \). Then we proceed by induction on \( c \).

If the result is true for \( c - 1 \), we define \( \hat{B} = \cup_{i=1}^{c-1} B_i \), \( X_{cup} = \pi^{-1}(\hat{B}) \) and \( X_c = \pi^{-1}(B_c) \). Then write \( B = \hat{B} \cup B_c \) and applying the Mayer-Vietoris sequence associated to this decomposition one gets

\[
\Lambda(h) = \Lambda(h|_{X_{cup}}) + \Lambda(h|_{X_c}) - \lambda(h|_{X_{cup} \cap X_c}) = \\
\chi(\hat{B}) \Lambda(h_b^k) + \chi(B_c) \Lambda(h_b^k) - \chi(\hat{B} \cap B_c) \Lambda(h_b^k) = \\
\chi(\hat{B}) \Lambda(h_b^k).
\]

Now observe that \( \Lambda(h_b^k) = \sum_{q \geq 0} (-1)^q \text{trace}((h_b^k)^* : H^q(\pi^{-1}(b)) \to H^q(\pi^{-1}(b))) \), which is the trace of the induced isomorphism \( (h_b^k)^* : \mathbb{Z} \times \cdots \times \mathbb{Z} \to \mathbb{Z} \times \cdots \times \mathbb{Z} \),

which is equal to \( m \) if \( m \mid k \), or zero otherwise. \( \blacksquare \)

Now, if \( a_i \neq b_i \), then \( V_i \cap F_0 \) is a degree \( d_i = |a_i - b_i| \)-covering over \( E_i \) minus \( r_i \)-disks, where \( r_i \) is the number of double points of the total transform of \( (fg)^{-1}(0) \)
on $E_i$. Moreover, the monodromy $h$ is compatible with the covering projection. Hence using the two previous lemmata we get the following formula:

$$\Lambda(h^k) = \sum_{i=1}^{s} \Lambda(h_{X_i}^k) = \sum_{i=1}^{s} d_i (2 - r_i).$$

Moreover, since $h^0$ denotes the identity, we also have

$$\Lambda(h^0) = \sum_{i=1}^{s} \Lambda(h_{V_i \cap F_{\theta}}^0) = \sum_{i=1}^{s} \chi(V_i \cap F_{\theta}) = \sum_{i=1}^{s} d_i (2 - r_i).$$

Now, since for each $k \geq 1$ we have

$$\sum_{d_i | k} d_i (2 - r_i) = \Lambda(h^k) = \sum_{d_i | k} s_{d_i},$$

it follows that

$$s_{d_i} = d_i (2 - r_i)$$

and then we have the following theorem:

**Theorem 9.** Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic functions such that the real analytic map-germ $fg : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ has an isolated singularity. Let $\pi : \hat{M} \to \mathbb{C}^2$ be an embedded resolution of the germ $(fg)^{-1}(0)$ at the origin. Let $E = \bigcup_{i=1}^{s} E_i$ be a decomposition of the exceptional divisor of $\pi$ in irreducible components. Let $a_i$ and $b_i$ denote the multiplicity of $E_i$ in the total transform of $f^{-1}(0)$ and $g^{-1}(0)$ respectively. Define $d_i := |a_i - b_i|$. Let $r_i$ be the number of double points of the total transform of $fg$ in $E_i$. The zeta function of the monodromy of the Milnor fibration of $fg$ is given by

$$Z(t) = \prod_{i=1}^{s} (1 - t^{d_i})^{r_i - 1}.$$  

**Example 10.** Consider the holomorphic functions $f(x, y) = x^2 + y^3$ and $g(x, y) = x^3 + y^2$. Then the real analytic germ $\hat{f} \hat{g} : (\mathbb{R}^4, 0) \to (\mathbb{R}^2, 0)$ has an isolated singularity at $0 \in \mathbb{C}^2$. The graph of the resolution of the complex curve $\{fg = 0\} = \{\hat{f} \hat{g} = 0\}$ is given below:

```
E_2 (3,2)  E_4 (2,3)
      |      |
      f     g
      |
E_3 (6,4)  E_5 (4,6)
```

In the holomorphic case $fg$, the part of the Milnor fibre inside each box $V_{ij}$ is a disjoint union of $\gcd(a_i + b_i, a_j + b_j)$ cylinders, and inside each box $V_i$ it is an $(a_i + b_i)$-covering of a sphere minus $r_i$ disks, with Euler characteristic $(a_i + b_i)(2 - r_i)$. 
Then the part of the Milnor fibre inside: $V_{13}$ and $V_{15}$ are two cylinders; $V_{23}$ and $V_{45}$ are five cylinders; $V_{1}$ is two cylinders; $V_{2}$ and $V_{4}$ are five disks; $V_{3}$ and $V_{5}$ are compact surfaces of genus 2 with boundaries eight circles. So the Milnor fibre of $fg$ is a twice-perforated surface of genus 5.

In the real analytic case $f\bar{g}$, according to Lemma 1, the part of the Milnor fibre inside: $V_{13}$ and $V_{15}$ are two cylinders; $V_{23}$ and $V_{45}$ are one cylinder; $V_{1}$ are two cylinders; $V_{2}$ and $V_{4}$ are one disk; $V_{3}$ and $V_{5}$ are compact surfaces of genus 0, that is, spheres, with boundaries four circles each one. Hence the Milnor fibre of $f\bar{g}$ is a twice-perforated surface of genus 1, that is, a torus with boundary two disjoint circles.

Moreover, by Theorem 9 the zeta function of $h^{-1/2}$ is given by
\[
Z(t) = (1 - t^5)^{-2}(1 - t^{10})^2
\]
and the zeta function of $h^{-1/2}$ is given by
\[
Z(t) = (1 - t)^{-2}(1 - t^2)^2.
\]

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