Linearly representable games and pseudo-polynomial calculation of the Shapley value

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May 26, 2022

Abstract

We introduce the notion of linearly representable games. Broadly speaking, these are TU games that can be described by as many parameters as the number of players, like weighted voting games, airport games, or bankruptcy games. We show that the Shapley value calculation is pseudo-polynomial for linearly representable games. This is a generalization of many classical and recent results in the literature (Mann and Shapley [11], Aziz [2]). Our method naturally turns into a strictly polynomial algorithm when the parameters are polynomial in the number of players.

Keywords: Computer science, Pseudo-polynomial algorithm, Game theory, TU games, Operations research, Shapley value

1 Introduction

The calculation of the Shapley value for games with transferable utility (TU) is a challenging task since its introduction in Shapley’s famous paper [14]. Though most games can be described by a few parameters, the Shapley value is the mean of exponentially many numbers, which cannot be calculated in practice based on its definition if the number of players is large. In the recent 70 years many great algorithms have been developed to overcome the difficulties. It turned out that TU games are diverse in the complexity of the Shapley value calculation, it is polynomial for some game classes (see for example Megiddo [12], Granot et al. [9] and Castro et al. [3]), but it can also be NP-hard (see Deng and Papadimmitriou [6] or Faigle and Kern [8]). We have no good algorithms to solve NP-hard problems, and we probably never will. There is, however, a compromise. Some NP-complete problems can be solved in pseudo-polynomial time. An algorithm is pseudo-polynomial if its running time is a polynomial function of the numeric value of the input. This is different from a polynomial algorithm, where the running time is a polynomial function of the size of the input. The difference is crucial, because inputs of computational problems are usually numbers and numbers are represented by their digits and the number of digits of a number equals its logarithm.

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Therefore, a pseudo-polynomial algorithm can still be exponential in the worst case, but completely usable in practice if the parameters are moderate.

The first pseudo-polynomial algorithm for the Shapley value was introduced by Mann and Shapley [11] for weighted voting games, while Deng and Papadimitriou [6] showed that the Shapley value calculation is NP-hard for this game class. Since then, similar results have been proved for other game classes, like bankruptcy games by Aziz [2] or liability games by Csóka et al. [5]. On the other hand, Ando [1] shows that the Shapley value calculation for minimum spanning tree games is NP-hard even if the value of each coalition is 0 or 1. This obviously implies that there cannot be a pseudo-polynomial algorithm for such games as it would automatically be polynomial because of the bounded weights.

In this paper, we show that the crucial difference between the bankruptcy or liability games and the minimum spanning tree games is that the former ones can be described by assigning a number (a weight) to each player and the value of each coalition is a function of the sum of the weights of its members, while minimum spanning tree games can be described by a weighted graph, which is a “more complicated” representation. We abstract this common property of bankruptcy games, liability games, weighted voting games, and many other game classes by introducing the notion of linearly representable games. It turns out the above-mentioned positive result, the existence of a pseudo-polynomial algorithm for the Shapley value, holds for every linearly representable game. The algorithm can also be slightly modified to calculate the Shapley value for some special game classes for example airport games in polynomial time and space.

The rest of the paper is structured as follows. In Section 2 and 3 we introduce the basic notations and the concept of linearly representable games, respectively. In Section 4, we provide our main result, a pseudo-polynomial algorithm to calculate the Shapley value of one player in a linearly representable TU game, and in Section 5 we discuss how the algorithm can be modified to get the Shapley values of all of the players without repeating the whole calculation \( n \) times. In Section 6 we broadly discuss possible extensions, special cases, and generalizations.

2 Preliminaries

A cooperative game with transferable utility is a pair \((N, v)\), where \(N\) is a finite set of players and the characteristic function \(v : 2^N \to \mathbb{R}\) is a function over the subsets of players such that \(v(\emptyset) = 0\). The number of players is denoted by \(n = |N|\). Subsets of \(N\) are called coalitions.

The Shapley value assigns a unique value to each player as follows. Let \(i \in N\) be a player and consider a fix permutation \(\pi\) of all the players. The marginal contribution of player \(i\) is

\[
\varphi_i(\pi) = v(P_{\pi}(i) \cup \{i\}) - v(P_{\pi}(i)),
\]

where \(P_{\pi}(i)\) is the set of players preceding \(i\) in the sequence \(\pi\). The Shapley value of player \(i\) is the mean of the marginal contributions over all possible permutations.

\[
Sh(i) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \varphi_i(\pi)
\]  

1In other words this problem is strongly NP-hard

2It is known that the Shapley value for airport games is polynomial (Castro et al. [3]), but we generalize this result as well.
where $\Pi_N$ stands for all possible permutations of the player’s set $N$. It is clear that any two terms in Equation (1) are equal if the sets of players preceding player $i$ are the same: $\nu'_{\pi_1}(i) = \nu'_{\pi_2}(i)$ if $Pre_{\pi_1}(i) = Pre_{\pi_2}(i)$. If we make the summation over these equivalent terms, we get the formula

$$Sh(i) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} (v(S \cup \{i\}) - v(S)) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \cdot \nu'_i(S), \quad (2)$$

where $\nu'_i(S) = v(S \cup \{i\}) - v(S)$. The number of terms in expression (2) is $2^n - 1$, which is definitely much less than $n!$, the number of terms in Formula (1), but this does not help much to calculate it, since it is still exponential.

Let us consider the Euler integral $\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} \, dx = \frac{(a-1)! \cdot (b-1)!}{(a+b-1)!}$.

By Formula (2):

$$Sh_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \cdot \nu'_i(S).$$

For $a = |S| + 1$ and $b = n - |S|$ we get

$$Sh_i = \int_0^1 x^{a-1}(1-x)^{b-1} \, dx = \sum_{S \subseteq N \setminus \{i\}} \frac{(a-1)! \cdot (b-1)!}{(a+b-1)!} \cdot \nu'_i(S) = \sum_{S \subseteq N \setminus \{i\}} \int_0^1 x^{a-1}(1-x)^{b-1} \cdot \nu'_i(S) \, dx = \int_0^1 \sum_{S \subseteq N \setminus \{i\}} x^{|S|}(1-x)^{n-|S|-1} \cdot \nu'_i(S) \, dx.$$

The idea of expressing the Shapley value as an integral originally came from Owen [13] and is very well described and used for a simulation method for voting games by Leech [10].

For a fix $S \subseteq N$, the expression $x^{|S|}(1-x)^{n-|S|-1}$ can be interpreted as the probability of forming $S$ if all players in $N \setminus \{i\}$ decide independently about joining a coalition with probability $x$. In other words, let us say that each player but $i$ have to make a binary decision independently of one another wheather or not they want to cooperate with the others. The expression $x^{|S|}(1-x)^{n-|S|-1}$ is the probability that the coalition of players that decided to cooperate is exactly $S$. As it is summed

\footnote{With integration by parts:
$$\int_0^1 x^{a-1}(1-x)^{b-1} \, dx = \left[ \frac{x^a}{a} (1-x)^{b-1} \right]_{x=0}^{x=1} - \int_0^1 \frac{x^a}{a} (b-1)(1-x)^{b-2} \, dx$$

i.e. $\beta(a, b) = \frac{b-1}{a} \beta(a+1, b-1)$. By induction on $b$ we can get the well-known formula $\beta(a, b) = \frac{(a-1)! \cdot (b-1)!}{(a+b-1)!}$.}

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over all possible coalitions $S \subseteq N \setminus \{i\}$, the weighted sum under the integral is the expected value of the marginal contribution of player $i$ to this random coalition. Besides, the integral of the function from 0 to 1 (devided by 1) can be considered as the average value of the function. This is another interpretation of the Shapley value as the average marginal contribution of the given player. In a concise form we can write that

$$Sh(i) = \int_0^1 \mathbb{P}_x v_i'(S) \, dx,$$

where $\mathbb{P}_x$ is a probability measure defined by $\mathbb{P}_x(S) = x^{|S|}(1 - x)^{n-|S|-1}$ for all $S \subseteq N \setminus \{i\}$.

### 3 Linearly representable games

In this section we introduce the linear representation of games with some examples and basic properties.

**Definition 3.1.** A TU-game $G = (N, v)$ is linearly representable if there exists a set of positive weights $a_j \in \mathbb{N}$ : $j \in N$ and a function $f : \mathbb{N} \to \mathbb{N}$ such that the value of each coalition $S \subseteq N$ can be expressed in the form $v(S) = f\left(\sum_{i \in S} a_i\right)$. In this case, we say that this function $f$ and the weights $a_j$ are a linear representation of game $G$.

**Remark 3.1.** There can be multiple linear representations for the same game.

The following games are for example, “naturally” linearly representable, since they are initially defined by a set of weights.

**Example 3.1** (Weighted voting games). Let $w_1, w_2, \ldots, w_n \in \mathbb{Z}$ be the voting powers of the players and let $q \in \mathbb{Z}$ be the quota. The value of a coalition $S$ is

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q, \\ 0 & \text{if } \sum_{i \in S} w_i < q. \end{cases}$$

Weighted voting games are linearly representable, with $a_i = w_i$ and

$$f(x) = \begin{cases} 1 & \text{if } x \geq q, \\ 0 & \text{if } x < q. \end{cases}$$

**Example 3.2** (Bankruptcy games). Let $A, l_1, l_2, \ldots, l_n \in \mathbb{N}$ and let the characteristic function be

$$v(S) = \max \left\{0, A - \sum_{j \not\in S} l_j\right\}.$$ 

For the interpretation of the formula and more details on bankruptcy games, see Driessen [13] or Thomson [14]. Bankruptcy games are also linearly representable. Here $a_i = l_i$ and $f(x) = \max \{0, A - (L - x)\}$, where $L = \sum l_j$. 

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Example 3.3 (Liability games). Liability games have recently been introduced by Csóka and Herings [4] and are similar to bankruptcy games. Let \( N = \{0, 1, \ldots, k\} \) and let \( A, l_1, l_2, \ldots, l_k \in \mathbb{N} \) where \( A < \sum l_i \), and let the characteristic function be

\[
v(S) = \begin{cases} 
\min\{A, \sum_{j \in S \setminus \{0\}} l_j\}, & \text{if } 0 \in S, \\
\max\{0, A - \sum_{j \not\in S \cup \{0\}} l_j\}, & \text{if } 0 \not\in S.
\end{cases}
\]

Because of the two cases in the characteristic function, it is not completely trivial that liability games are linearly representable. The trick is to assign a sufficiently large number to player 0 (that represents a defaulting firm, which has no liabilities to itself), in a way that we can tell from the sum of the weights whether or not the firm is in the coalition. So let \( L = \sum l_i + 1 \), \( a_0 = L \), \( a_i = l_i \) for \( i \geq 1 \) and

\[
f(x) = \begin{cases} 
\min\{A, x - L\}, & \text{if } x \geq L, \\
\max\{0, A - (L - 1 - x)\}, & \text{if } x < L.
\end{cases}
\]

The Shapley value for all of these games are NP-hard to compute (see Deng and Papadimitriou [6], Aziz [2] and Csóka et al. [5]). On the other hand, we know that it is pseudo-polynomial for voting games and bankruptcy games (see Mann and Shapley [11] and Aziz [2]) and in this paper we put the icing on the cake and provide a pseudo-polynomial algorithm for every linearly representable game.

Remark 3.2. It is easy to see that every TU game is linearly representable with at most exponentially large weights. For example, number the players from 0 to \( n - 1 \) and assign \( 2^j \) to the \( j \)th player. For all \( 1 \leq k \leq 2^n \) let \( f(k) \) be the value \( v(S) \) of the coalition \( S \) for which \( k = \sum_{i \in S} a_i \). Such coalition exists and is unique, because every integer is the sum of powers of 2 and the characteristic vector of \( S \) is the binary representation of \( k \).

This observation does not make the concept of linearly representable games meaningless. As we will see, the linear representation with exponentially large weights is not useful, in this case the pseudo-polynomial algorithm has a worse complexity than the brute force one. It is absolutely not clear how to compute a linear representation with weights less than exponential or how to decide if such representation exists. This can be a question for further research. For now, the method we propose in the next section is only meaningful for games that are linearly representable with “small enough” weights.

4 An algorithm to calculate the Shapley value of one player

Here we present our main result.

Theorem 4.1. Let \( (f, \{a_j : j \in N\}) \) be a linear representation of game \( G = (N, v) \) and let \( i \in N \) be one of the players. Let us assume that \( a_j \leq 2^{\lvert N \rvert} \) and function \( f \) can be calculated in linear time and space. The Shapley value of player \( i \) can be calculated in \( O(n^3 K) \) time and \( O(n^2 K) \) space, where \( K = \sum_{j \neq i} a_j \).
Remark 4.1. The exponential upper bound for the weights is justified by Remark 3.2 and can be relaxed, just like the condition about the complexity of function $f$. The statement holds without those conditions with some technical modifications.

Proof. By Formula (3),

$$Sh(i) = \int_0^1 \mathbb{E}_x v'_i.$$

Let $S \subseteq N$ be a coalition such that $i \notin S$. Since $G$ is linearly representable, $v(S) = f \left( \sum_{j \in S} a_j \right)$, and we get that

$$v'_i(S) = f \left( \sum_{j \in S} a_j + a_i \right) - f \left( \sum_{j \in S} a_j \right).$$

Let $S_x \subseteq N \setminus \{i\}$ be a random coalition formed by players in $N \setminus i$, when they decide about the cooperation with probability $x$. With this notation,

$$\mathbb{E}_x v'_i = \sum_{S \subseteq N \setminus \{i\}} P(S_x = S) \cdot \left( f \left( \sum_{j \in S} a_j + a_i \right) - f \left( \sum_{j \in S} a_j \right) \right).$$

Let $K = \sum_{j \in N \setminus \{i\}} a_j$ and let us group the terms in the expression by the sum of the weights in $S$:

$$\mathbb{E}_x v'_i = \sum_{k=0}^K \sum_{S \subseteq N \setminus \{i\}} \frac{\sum_{j \in S} a_j = k}{P(S_x = S)} \cdot \left( f(k + a_i) - f(k) \right) =$$

$$= \sum_{k=0}^K \left( f(k + a_i) - f(k) \right) \cdot \sum_{S \subseteq N \setminus \{i\}} P(S_x = S) = \sum_{k=0}^K \left( f(k + a_i) - f(k) \right) \cdot P(G_x = k),$$

where $G_x = \sum_{j \in S_x} a_j$ the total weight of the random coalition $S_x$.

The probabilities $P(G_x = k)$ are $n-1$ degree polynomials of the indeterminate $x$ with integer coefficients. We can calculate them as follows.

1. Let $i_1, i_2, \ldots, i_{n-1}$ be an arbitrary order of all the players but $i$ and let $a_\ell$ be the weight of the $\ell^{th}$ player.

2. For $0 \leq j \leq n-1$ and $0 \leq k \leq K$, let $P[j, k]$ denote the probability that the first $j$ players form a coalition with total weight $k$.\footnote{The matrix $P[j, k]$ depends on the arbitrarily chosen order of the players, on which the Shapley value obviously cannot depend. We will address this issue later.} For each $j$ and $k$, $P[j, k]$ is a polynomial of $x$ with degree $j$ and with integer coefficients. For $j = 0$, $P[0, 0] = 1$ and $P[0, k] = 0$, for $k \geq 1$. 

$$\mathbb{E}_x v'_i = \sum_{k=0}^K \sum_{S \subseteq N \setminus \{i\}} P(G_x = k).$$
3. Let us assume recursively that $P[j-1,k]$ is already known for each $0 \leq k \leq K$. By the law of iterated expectations, we get that

$$
P[j,k] = \begin{cases} 
(1-x) \cdot P[j-1,k] & \text{if } k < a_j, \\
x \cdot P[j-1,k] + (1-x) \cdot P[j-1,k] & \text{if } k \geq a_j.
\end{cases}
$$

The polynomials $P[n-1,k]$ are exactly the probabilities $\mathbb{P}(G_x = k)$. Formula (9) is simply their weighted sum, $E_x v_i' = \sum_{k=0}^{K} \left( f(k + a_i) - f(k) \right) \cdot \mathbb{P}(G_x = k)$, which is also a polynomial of degree $n-1$, therefore its integral can be calculated analytically. This is the algorithm.

Let us examine its complexity. At first sight, it seems we have to fill an $N \times K$ matrix, and each entry can be calculated with one operation. But this is oversimplification. The matrix $P[j,k]$ consists of polynomials, i.e. a sequence of coefficients, that are exponentially large in $n$. To calculate it correctly, let us observe that the first row only contains 0 or 1 constant polynomials. Each polynomial is the combination of at most three other polynomials occurred in the previous row. By induction on $j$ each coefficient of $P[j,k]$ is at most $3^j$ that is at most $3^n$, which is an $O(n)$ bits long number. Therefore each polynomial $P[j,k]$ can be stored in $O(n^2)$ memory and can be calculated in $O(n^2)$ time, so the matrix $P[j,k]$ can be calculated in $O(n^3K)$ time. The algorithm does not have to store the whole matrix, it only needs the last 2 rows at any given point of the calculation, which only requires $O(n^2K)$ space. It is easy to see that the rest of the algorithm (evaluation of function $f$ at most $K$ times, calculation of the weighted sum of the polynomials, and the integral) is very fast under the simplifying conditions, and do not need extra space. Therefore, the complexity of the algorithm is $O(n^3K)$ time and $O(n^2K)$ space.

**Example 4.1.** As an illustration, consider a bankruptcy game with four players with liabilities $l_1 = 2$, $l_2 = 3$, $l_3 = 5$, $l_4 = 7$ and assets value $A = 9$. Let us calculate the Shapley value of the fourth player. The matrix $P[j,k]$ is shown in Table 4. The first row in the matrix shows that the empty coalition is worth 0 with probability 1 all other entries of the matrix are initialized to 0. Each row but the first one is associated to a liability $l_j$ in any fixed order. The algorithm fills the matrix row by row and left to right as follows. It takes the next nonzero element of the current row, multiplies it by $1-x$, adds the result to the element right under it (next row, same column), multiplies it by $x$ and adds the result to the element in the next row, $l_j$ columns to the right of it. In other words, each element of the matrix is the sum of $1-x$ times the element above it and $x$ times the element $l_j$ columns to the left of it in the previous row. We have to take the weighted sum of these polynomials and integrate it, so we should store them as the sum of monomials as Table 2 shows. The last row of the matrix shows the probabilities of the first three players forming a coalition with $k$ total claims for $k = 0, 1, \ldots, 10$. The marginal contributions of the fourth player to such a coalition is indicated in an extra row under Table 3. The expected marginal contribution of the fourth player is the sum of the polynomials (probabilities) times the marginal contributions

$$
x^3 - 2x^2 + x + 2(x^3 - 2x^2 + x) + 4(-x^2 + x) + 6(-x^3 + x^2) + 7(-x^3 + x^2) + 7x^3 = -3x^3 + 3x^2 + 7x.
$$

The Shapley value is the integral of this polynomial from 0 to 1 i.e. the sum of the coefficients of its primitive function, $\int \frac{-3}{4} + 1 + \frac{7}{2} = \frac{15}{4}$. 

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Table 1: Recursive calculation of the polynomials in Example 4.1.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|----|
| 0 | 1 | 0 | 0  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| $l_1 = 2$ | 1 - $x$ | 0 | $x$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| $l_2 = 3$ | $(1 - x)^2$ | 0 | $x(1 - x)$ | $x(1 - x)$ | 0 | $x^2$ | 0 | 0 | 0 | 0 | 0  |
| $l_3 = 5$ | $(1 - x)^3$ | 0 | $x(1 - x)^2$ | $x(1 - x)^2$ | 0 | $x^2(1 - x) + x(1 - x)^2$ | 0 | $x^2(1 - x)$ | $x^2(1 - x)$ | 0 | $x^3$ |

Table 2: Polynomials of Example 4.1 expanded to sum of monomials.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|----|
| 0 | 1 | 0 | 0  | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| $l_1 = 2$ | $-x + 1$ | 0 | $x$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0  |
| $l_2 = 3$ | $x^2 - 2x + 1$ | 0 | $-x^2 + x$ | $-x^2 + x$ | 0 | $x^2$ | 0 | 0 | 0 | 0 | 0  |
| $l_3 = 5$ | $-x^3 + 3x^2 - 3x + 1$ | 0 | $x^3 - 2x^2 + x$ | $x^3 - 2x^2 + x$ | 0 | $-x^2 + x$ | 0 | $-x^3 + x^2$ | $-x^3 + x^2$ | 0 | $x^3$ |

Table 2: Polynomials of Example 4.1 expanded to sum of monomials.
5 Shapley value of the game

Theorem 4.1 states that the Shapley value of one player in a linearly representable game can be calculated in pseudo-polynomial time and space. In this section, we show that it has the exact same complexity to calculate the Shapley value of all of the $n$ players.

Lemma 5.1. Let $G = (N, f, \{a_j : j \in N\})$ be a linearly representable game and consider the matrix $P[j, k]$ introduced in the proof of Theorem 4.1 which depends not only on the set of the players (weights), but also on the order of player we consider when constructing the matrix.

- If we swap players $i_\ell$ and $i_{\ell+1}$ only the $\ell^{th}$ row of the $P$ matrix is changed.
- The first $j$ rows of the matrix is independent of the order of the first $j$ (and the last $n-1-j$) players.
- The last row of the matrix is independent of the order of the players.

Remark 5.1. Based on the interpretation of the matrix $P$ ($P[j, k]$ is the probability that the first $j$ player form a coalition of total weight $k$) the lemma must be trivial. A formal proof shows that it really is and our notations are concise and reflect the natural properties of the studied objects.

Proof. Let us observe that the second and third statements of the lemma follow from the first one, we only need to prove that. Applying the recursive formula for $P[j, k]$ twice, we get that

$$P[\ell+1, k] = x \cdot P[\ell, k - a_{\ell+1}] + (1-x) \cdot P[\ell, k] =$$

$$= x \cdot \left( x \cdot P[\ell - 1, k - a_{\ell+1} - a_{\ell}] + (1-x) \cdot P[\ell - 1, k - a_{\ell+1}] \right) +$$

$$+ (1-x) \cdot \left( x \cdot P[\ell - 1, k - a_{\ell}] + (1-x) \cdot P[\ell - 1, k] \right) =$$

$$= x^2 \cdot P[\ell, k - (a_{\ell+1} + a_{\ell})] + x(1-x) \left( P[\ell - 1, k - a_{\ell+1}] + P[\ell - 1, k - a_{\ell}] \right) + (1-x)^2 \cdot P[\ell - 1, k],$$

where the non-existent elements are considered as 0. This formula is indeed symmetric in $a_{\ell}$ and $a_{\ell+1}$, which completes the proof. □

Lemma 5.2. Considering a fixed order of the players, based on any row of the $P$ matrix, we can calculate its previous row in $O(n^2 K)$ time.

Proof. Let $\ell$ be a positive integer $(1 \leq \ell \leq n - 1)$, and let $a_{\ell}$ be the weight to calculate row $\ell$ from the previous one. We have to invert the recursion

$$P[\ell, k] = x \cdot P[\ell - 1, k - a_{\ell}] + (1-x) \cdot P[\ell - 1, k]$$

where the left term is considered 0 for the unmeaning indices. We can calculate the polynomials $P[\ell - 1, k]$ from left to right, i.e. from $k = 0$ to $k = K$ (where $K = \sum_{j \neq i} a_j$ the rightmost column of $P$) as follows.

- For $k < a_{\ell}$: $P[\ell, k] = (1-x) \cdot P[\ell - 1, k]$ where both sides are polynomials with integer coefficients, so we can calculate $P[\ell - 1, k]$ with long division of polynomials.
For $k \geq a_i$: the left term $x \cdot P[\ell - 1, k - a_i]$ exists too, but as $k - a_i < k$, the value of $P[\ell - 1, k - a_i]$ has already been calculated in a previous step, so we can simply solve the equation for $P[\ell - 1, k]$ by subtracting the left term and dividing by $1 - x$.

To sum up

$$P_i[\ell - 1, k] = \begin{cases} \frac{P[\ell, k]}{1 - x}, & \text{if } k < a_i, \\ \frac{P[\ell, k] \cdot x \cdot P[\ell - 1, k - a_i]}{1 - x}, & \text{if } k \geq a_i, \end{cases}$$

including that the numerators are always divisible by $1 - x$ without remainders.

As for the complexity, the division of polynomials is a fast and easy task, but let us point out that in these special cases it is even more simple. We can assume that the polynomials are stored as a vector (or array) of coefficients from the lowest term to the highest one, so multiplication by $x$ is simply a shift of the numbers and putting an extra 0 for the constant. If we use a smarter data structure it can be faster, but in the worst case it is $O(n^2)$ copy operations. As for the division by $1 - x$, the task is to find coefficients $\mu_s$ such that:

$$\lambda_m \cdot x^m + \lambda_{m-1} \cdot x^{m-1} + \cdots + \lambda_s \cdot x^s + \cdots + \lambda_1 \cdot x + \lambda_0 =$$

$$= (1 - x) \cdot \left( \mu_{m-1} \cdot x^{m-1} + \mu_{m-2} \cdot x^{m-2} + \cdots + \mu_s x^s + \cdots + \mu_0 x \right),$$

where $\lambda_s$ are the known coefficients of the given polynomial. It is clear that the polynomial on the left side is divisible by $1 - x$ iff $\sum \lambda_s = 0$ and in this case $\mu_s = -(\lambda_m + \lambda_{m-1} + \cdots + \lambda_{s+1})$. Therefore, the division by $1 - x$ is simply the calculation of cumulated sums of the coefficients, which can obviously be done in $O(n^2)$ time too.

Now it is easy to see how to calculate the Shapley value for all of the players. First we calculate the matrix $P[j, k]$ for all of the players, we do not exclude anybody. This is an $(n + 1) \times K$ matrix of polynomials where $K = \sum a_j$. This is the expensive part of the algorithm with $O(n^3K)$ time complexity, but this is the part we only have to do once. We still do not need the whole matrix, we only need to store the current and the next row at a time. At the end we only need to keep the last row $P[n, k]$ ($1 \leq k \leq K$), which is $O(n^2K)$ space. Let us call these polynomials base polynomials, as these store all the information we need to calculate the Shapley values. Let $i \in N$ be any of the players and calculate the last but one row of the matrix $P$ based on Lemma 5.2 as if $i$ were the last player when we calculated the last row. It does not matter if $i$ was not the last player when calculating the matrix as the last row is independent of the order of players, we can still revert back to the previous step to get the $(n - 1)^{th}$ row as if it was calculated from a sequence of all the players but $i$. We can use the polynomials in this row to calculate the Shapley value of player $i$. According to Lemma 5.2 this can be done in $O(n^2K)$ time. Now we can replace player $i$ by another one, but we do not have to calculate the whole matrix from zero. We can use the last row to revert back to the previous step as if this new player were excluded, and again use the last but one row to calculate the Shapley value for this player. We can calculate the Shapley value for all of the players one by one, but each calculation takes only $O(n^2K)$ steps. So the algorithm runs in $O(n^3K)$ time. Let us summarize the result.

**Theorem 5.1.** Let $(f, \{a_j : j \in N\})$ be a linear representation of game $G = (N, v)$ and let us assume that $a_j \leq 2^{N}$ and function $f$ can be calculated in linear time and space. The sequence of the Shapley values of all of the players can be calculated in $O(n^3K)$ time and $O(n^3K)$ space, where $K = \sum a_j$. 


Remark 5.2. It is worth to point out that the base polynomials are independent of function $f$ in the representation of the game. If we want to calculate the Shapley value for a different game where the weights $a_j$ are the same, the most expensive part of the calculation is already done.

Table 3 illustrates the first step of the algorithm (for the same bankruptcy game we have in Example 4.1). It is exactly the same as Table 2 with one more row for the fourth player. The last row contains the base polynomials. Table 4 shows the second step. There are four possibilities for the third row of the matrix depending the excluded player. These four rows can be used to calculate the four Shapley values. The last case where the fourth player is excluded gives back the third row of Table 3. From this point the calculation is the same as in the previous version of the algorithm shown by Table 2. We calculate the sum of these polynomials weighted by the marginal contributions of the excluded player and integrate it from 0 to 1 to get that

\[
\begin{align*}
Sh_1 &= \int_0^1 (-x^3 + 4x^2 + x) \, dx = \left[ -\frac{3}{4}x^4 + \frac{4}{3}x^3 + \frac{1}{2}x^2 \right]_0^1 = -\frac{3}{4} + \frac{4}{3} + \frac{1}{2} = \frac{13}{12},
\end{align*}
\]

\[
\begin{align*}
Sh_2 &= \int_0^1 (-3x^3 + 4x^2 + 2x) \, dx = \left[ -\frac{3}{4}x^4 + \frac{4}{3}x^3 + x^2 \right]_0^1 = -\frac{3}{4} + \frac{4}{3} + 1 = \frac{19}{12},
\end{align*}
\]

\[
\begin{align*}
Sh_3 &= \int_0^1 (-3x^3 + 4x^2 + 4x) \, dx = \left[ -\frac{3}{4}x^4 + \frac{4}{3}x^3 + 2x^2 \right]_0^1 = -\frac{3}{4} + \frac{4}{3} + 2 = \frac{31}{12},
\end{align*}
\]

\[
\begin{align*}
Sh_4 &= \int_0^1 (-3x^3 + 3x^2 + 7x) \, dx = \left[ -\frac{3}{4}x^4 + x^3 + \frac{7}{2}x^2 \right]_0^1 = -\frac{3}{4} + 1 + \frac{7}{2} = \frac{15}{4}.
\end{align*}
\]
Table 3: Base polynomials for the bankruptcy game in Example 4.1.

| k  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| l_1 | 2 | x + 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| l_2 | 3 | x^2 + 2x + 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| l_3 | 5 | -x^3 + 3x^2 - 3x + 1 | 0 | x^3 - 2x^2 + x | 0 | -x^2 + x | 0 | -x^3 + x^2 | 0 | x^3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| l_4 | 7 | x^4 - 4x^3 + 6x^2 - 4x + 1 | 0 | -x^3 + 3x^2 - 3x^2 + x | -x^4 + 3x^3 - 3x^2 + x | 0 | x^3 - 2x^2 + x | 0 | x^4 - 2x^3 + x | 0 | x^4 - 2x^3 + x | 0 | x^4 - 2x^3 + x | 0 | x^4 - 2x^3 + x | 0 | x^4 - 2x^3 + x | 0 | x^4 |

Table 4: Reverting the last step of the algorithm.
6 Summary and possible extensions

In this section, we briefly discuss how the algorithm can be improved and used for other game classes.

6.1 Optimizing the algorithm

The algorithm in its current form has a drawback. The very first step is to create an exponentially large nullmatrix. This makes the algorithm slow even in cases where only a polynomial number of probabilities $P[j, k]$ are nonzero. We can fix this issue, if we only store the nonzero $P[j, k]$ values in an associative array. An associative array is a data structure that stores key-value pairs such that we can access the values “quickly” by their unique keys. This way, we do not have to initialize a large matrix of zeros, only a very small data structure that initially stores nothing but the polynomial $P[0, 0] = 1$, and then add only the nonzero new values when the algorithm calculates them.

In the second step of the algorithm we iterate the polynomials $P[n, k]$ in increasing order, so we have to keep the array sorted by keys, or we have to sort it at the end of the first step. This has some extra computational cost, which is only worth it, if we have a lot of zero polynomials $P[j, k]$. On the other hand, this modification makes the algorithm as fast as possible, by automatically skipping all meaningless operations with unnecessarily stored zeros without any further information requirement (we do not have to know anything about the number of nonzero probabilities in advance).

6.2 Application for airport games

In this section we demonstrate that the proposed algorithm can be used under more general conditions than linearly representable games. The simplest example is probably airport games, where each player is associated with a cost $c_i \in \mathbb{N}$ and the value of a coalition is

$$v(S) = \max\{c_i \mid i \in S\}.$$  

This is not a linear represented of the game, because the value of a coalition is not a function of the sum of the weights of the members, but the maximum of them. Mathematical intuition suggests that this nonlinearity makes the problem much more complicated. In fact, not just that we can use a slightly modified version of the algorithm, but it is actually polynomial in this case.

The similarity between games with a linear representation and this one is that we only need to know the probabilities

$$P[j, k] = \mathbb{P}\left(\text{The first } j \text{ players forms a coalition with total cost } k\right)$$

to calculate the Shapley value. However, in case of airport games we only have $n + 1$ possible $k$ values to consider, the costs $c_i$ themselves and 0. So the number of nonzero columns of the matrix $P$ is at most $n - 1$. The matrix is “small”, the algorithm is polynomial, we only need to find the recursive formula for $P[j, k]$, which is very easy. Let us consider the players in ascending order by their weights: $c_1 \leq c_2 \leq \cdots \leq c_n$. It is clear that

$$P[j, c_k] = \begin{cases} 
(1 - x) \cdot P[j - 1, c_k], & \text{if } c_j \neq c_k, \\
(1 - x) \cdot P[j - 1, c_k] + x, & \text{if } c_j = c_k.
\end{cases}$$
Thus we can calculate the matrix $P$ as follows. We have $n+1$ rows (where $n$ is the number of players) and $L+1$ columns, where $L$ is the number of distinct costs. Each row is labeled by a player (except the first one, which is labeled by the empty set or 0) in nondecreasing order by their costs and each column is labeled by a cost also in nondecreasing order (except the first column, which is labeled by 0 cost). Therefore, each column may correspond to a set of consecutive rows labeled by players with the same cost. The top leftmost element is initialized as $P[0, 0] = 1$. Now we can fill the matrix row by row as follows. Let the current row be labeled by player $j$.

- If $c_j$ is a larger cost than all of the previous ones, we locate the cell in column labeled by $c_j$ and set it to $x$. All cells in its left side will be $(1 - x)$ times the polynomial above them. All cells in the right remain 0.
- If $c_j$ is a repeated cost, do almost the same. Locate the column labeled by $c_j$. In the previous row there is a nonzero polynomial in it. Multiply it by $1 - x$ add $x$ and put the result in this cell. All other cells are calculated the same way as in the previous case (simply multiply them by $1 - x$).

Let us illustrate the process with an airport game with costs $c_1 = 1, c_2 = c_3 = 5, c_4 = c_5 = 7, c_6 = 10$. We calculate the Shapley value of player 5. We exclude player 5, sort the rest of the players in ascending order by their costs and fill the matrix in Table 5. First, we set $P[0, 0] = 0$ and then each row is simply $1 - x$ times the previous row except the only column that corresponds to the cost of the player in the current row, where we add one extra $x$. After calculating the last row, we get the Shapley value the exact same way we did in case of linearly representable games.

Though the algorithm is polynomial in this case, it is worth to point out that we do not really need to calculate the matrix $P$ row by row, because

$$P(A \text{ random coalition } S \text{ has total cost } k) = \left(1 - (1 - x)^{n_k}\right) \cdot (1 - x)^{n_{k+}} ,$$

where $n_k$ is the number of players with cost exactly $k$ and $n_{k+}$ is the number of players with cost strictly bigger than $k$. This practically leads to an algorithm to calculate and store binomial coefficients with alternating signs.

|   | 0       | 1       | 5       | 7       | 10      |
|---|---------|---------|---------|---------|---------|
| 0 | 1       |         |         |         |         |
| 1 | 1 - $x$ | $x$     |         |         |         |
| 2 | 1 - 2$x$ + $x^2$ | $x - x^2$ | $x$     |         |         |
| 3 | 1 - 3$x$ + 3$x^2$ - $x^3$ | $x - 2x^2 + x^3$ | 2$x$ - $x^2$ |         |         |
| 4 | 1 - 4$x$ + 6$x^2$ - 4$x^3$ + $x^4$ | $x - 3x^2 + 3x^3 - x^4$ | 2$x$ - 3$x^2$ + $x^3$ | $x$     |         |
| 5 | 1 - 5$x$ + 10$x^2$ - 10$x^3$ + 5$x^4$ - $x^5$ | $x - 4x^2 + 6x^3 - 4x^4 + x^5$ | 2$x$ - 5$x^2$ + 4$x^3$ - $x^4$ | $x - x^2$ | $x$     |

Table 5: matrix $P[j, k]$ for the Airport game
6.3 Generalized linearly representable games

The common feature of linearly representable games and airport games is that each player has a weight $w_i$ such that the characteristic functions can be expressed in the form

$$v(S) = g(\omega(S)),$$

where $\omega(S) = \{w_i \mid i \in S\}$ is the set of weights of the coalition and $g : 2^\mathbb{N} \mapsto \mathbb{N}$ is a function. This is of course no restriction (see Remark 3.2), but the algorithm can only be used if function $g$ satisfies the condition

$$\forall w \notin H_1 \cup H_2 \subseteq \mathbb{N} \quad g(H_1) = g(H_2) \implies g(H_1 \cup \{w\}) = g(H_2 \cup \{w\}).$$

Moreover, this relation must be computable, i.e. there must exist an easily computable function $h : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$ such that $g(H \cup \{w\}) = h(g(H), w)$. If two coalitions have the same value, each player outside of both coalitions has the same marginal contribution to both coalitions, and the value of the extended coalitions can be calculated based on the value of the coalitions and the weight of the player. In this case, we can use the algorithm in the exact same way we did in the special cases. Create an $(n+1) \times (K+1)$ matrix of polynomials, denoted by $P$ where $K = \max(g)$, initialize it with 0s except that $P[0, 0] = 1$, then fill the matrix row by row as follows

For $j = 0 \ldots (n-1)$:

For $k = 0 \ldots K$:

$$P[j + 1, h(k, w_i)] = P[j + 1, h(k, w_i)] + P[j, k] \cdot (1 - x) + x \cdot P[j, k],$$

where player $i$ is associated with row $(j + 1)$. A natural question arises. For a given game, how can we find a suitable function $g$ and weights, (or at least prove that they exist) such that the algorithm runs faster than the brute force one? If the weights are exponentially large, it is too slow. We want the upper bound $K = \max(g)$ or the number of nonzero columns in $P$ be as small as possible. This question is far beyond the scope of this paper, but it can be an interesting research topic for the future.
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