New n-mode squeezing operator and squeezed states with standard squeezing

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received 10 February 2009; accepted 24 February 2009
published online 11 March 2009

PACS 03.65.-w - Quantum mechanics
PACS 42.50.-p - Quantum optics

Abstract - We find that the exponential operator $V = \exp[1\lambda(Q_1P_2 + Q_2P_1 + \cdots + Q_{n-1}P_n + Q_nP_1)]$, $Q_i, P_i$, are, respectively, the coordinate and momentum operators, is an n-mode squeezing operator which engenders standard squeezing. By virtue of the technique of integration within an ordered product of operators we derive $V$’s normally ordered expansion and obtain the n-mode squeezed vacuum states, its Wigner function is calculated by using the Weyl ordering invariance under similar transformations.

Introduction. – Quantum entanglement is a weird, remarkable feature of quantum mechanics though it implies intricacy. In recent years, various entangled states have attracted considerable attention and the interest of physicists because of their potential uses in quantum communication [1,2]. Among them the two-mode of physicists because of their potential uses in quantum communication [1,2],. Among them the two-mode squeezed state exhibits quantum entanglement between the idle mode and the signal mode in a frequency domain manifestly, and is a typical entangled state of continuous variables. Theoretically, the two-mode squeezed state is constructed by the two-mode squeezing operator $S = \exp[\lambda(a_1a_2 - a_1^\dagger a_2^\dagger)]$ [3–5] acting on the two-mode vacuum state $|00\rangle$:

$S|00\rangle = \operatorname{sech}\lambda \exp \left[-a_1^\dagger a_2 \tanh \lambda \right] |00\rangle$, (1)

where $\lambda$ is the squeezing parameter, the disentangling of $S$ can be obtained by using the SU(1,1) Lie algebra, $|a_1^\dagger a_2 a_1 a_2^\dagger = a_1^\dagger a_1 + a_2^\dagger a_2 + 1$, or by using the entangled state representation $|\eta\rangle = \eta_1 + \eta_2\rangle$ [6,7]:

$|\eta\rangle = \exp \left[-(a_1^\dagger a_2^\dagger + a_1^\dagger a_1 + a_2^\dagger a_2) \right]|00\rangle$, (2)

$|\eta\rangle$ is the common eigenvector of the two particles’ relative position $(Q_1 - Q_2)$ and the total momentum $(P_1 + P_2)$, and obeys the eigenvector equations, $(Q_1 - Q_2)|\eta\rangle = \sqrt{2}\eta_1 |\eta\rangle$, $(P_1 + P_2) = |\eta\rangle = \sqrt{2}\eta_2 |\eta\rangle$, and the orthonormal-complete relation

$$\int \frac{d^2\eta}{\pi} |\eta\rangle \langle \eta| = 1, \quad \langle \eta' | \eta\rangle = \pi \delta(\eta - \eta') (\eta^* - \eta'^*),$$

(3)

because the two-mode squeezing operator has its natural representation in the $|\eta\rangle$ basis:

$S = \exp \left[\lambda \left( a_1 a_2 - a_1^\dagger a_2^\dagger \right) \right]$ [3–5]

$S|\eta\rangle = \frac{1}{\sqrt{\mu}} \eta \langle \eta|, \quad \mu = e^{-\lambda}$. (4)

The proof of eq. (4) proceeds by virtue of the technique of integration within an ordered product (IWOP) of operators [8–10]:

$$\int \frac{d^2\eta}{\pi\mu} |\eta/\mu\rangle \langle \eta| = \int \frac{d^2\eta}{\pi\mu} \exp \left\{-\frac{\mu^2 + 1}{2\mu^2} |\eta|^2 + \eta \left( \frac{a_1^\dagger}{\mu} - a_2 \right) \right\} + \eta^* \left( \frac{a_2^\dagger}{\mu} + a_1^\dagger a_2 + a_1 a_2^\dagger - a_1^\dagger a_1 - a_2^\dagger a_2 \right) \right) = \frac{2\mu}{1 + \mu^2} \exp \left\{ \frac{\mu^2}{1 + \mu^2} \left( \frac{a_1^\dagger}{\mu} - a_2 \right) \left( a_1 - a_2^\dagger \right) \right\} - \left( a_1 - a_2^\dagger \right) \left( a_1 - a_2^\dagger \right) = e^{-a_1^\dagger a_2^\dagger \tanh \lambda} e^{(a_1 a_2 + a_2 a_1 + 1)} \ln \operatorname{sech} \lambda e^{a_1 a_2 \tanh \lambda} \equiv S; \quad (5)$

eq. (4) confirms that the two-mode squeezed state itself is an entangled state which entangles the idle mode.
and signal mode as the outcome of a parametric down-conversion process [11]. The $|\eta\rangle$ state was constructed in refs. [6, 7] according to the idea of Einstein, Podolsky and Rosen in their argument that quantum mechanics is incomplete [12].

Using the relation between bosonic operators and the coordinate $Q_i$, momentum $P_i$, $Q_i = (a_i + a_i^\dagger)/\sqrt{2}$, $P_i = (a_i - a_i^\dagger)/\sqrt{2}$, and introducing the two-mode quadrature operators of light field as in ref. [4], $x_1 = (Q_1 + P_2)/2$, $x_2 = (P_1 + P_2)/2$, the variances of $x_1$ and $x_2$ in the state $S(00)$ are in the standard form

$$(00)|S(00)\rangle = \frac{1}{4} e^{-2\lambda}, \quad (00)|S(02)S(00)\rangle = \frac{1}{4} e^{2\lambda},$$

(6)

thus we get the standard squeezing for the two quadratures: $x_1 \rightarrow \frac{1}{2} e^{-\lambda} x_1$, $x_2 \rightarrow \frac{1}{2} e^{\lambda} x_2$. On the other hand, the two-mode squeezing operator can also be recast into the form $S = \exp\{i\lambda (Q_1 P_2 + Q_2 P_1)\}$. Then an interesting question naturally rises: what is the property of the $n$-mode operator

$$V \equiv \exp\{i\lambda (Q_1 P_2 + Q_2 P_3 + \cdots + Q_{n-1} P_n + Q_n P_1)\},$$

(7)

and is it a squeezing operator which can engenders the standard squeezing for $n$-mode quadratures? What is the normally ordered expansion of $V$ and what is the state $V(0)$ ($0$ is the $n$-mode vacuum state)? In this work we shall study $V$ in detail. But how to disentangle the exponential of $V$? Since all terms of the set $Q_i P_{i+1}$ ($i = 1, \ldots, n$) do not make up a closed Lie algebra, the problem of what $V$’s normally ordered form is seems difficult. Thus we appeal to the IWOP technique to solve this problem. Our work is arranged in this way: firstly we use the IWOP technique to derive the normally ordered expansion of $V$ and obtain the explicit form of $V(0)$; then we examine the variances of the $n$-mode quadrature operators in the state $V(0)$, we find that $V$ just causes standard squeezing. Thus $V$ is a squeezing operator. The Wigner function of $V(0)$ is calculated by using the Weyl ordering invariance under similar transformations. Some examples are discussed in the last section.

**The normal product form of $V$.** – In order to disentangle operator $V$, let $A$ be

$$A = \begin{pmatrix}
    0 & 1 & 0 & \cdots & 0 \\
    0 & 0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & \cdots & \cdots & 1 \\
    1 & 0 & \cdots & \cdots & 0
\end{pmatrix},$$

(8)

then $V$ in (7) is compactly expressed as

$$V = \exp\left[\pmatrix{1} \lambda \sum_{i,j=1}^{n} Q_i A_{ij} P_j \right].$$

(9)

Using the Baker-Hausdorff formula, $e^A B e^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]] + \frac{1}{6} [A, [A, [A, B]]] + \cdots$, we have (here and henceforth the repeated indices represent the Einstein summation notation)

$$V^{-1} Q_k V = Q_k - \lambda Q_i A_{ik} + \frac{1}{2!} \lambda^2 [Q_i A_{ij} P_j, Q_i A_{ik}] + \cdots = Q_i (e^{-A})_{ik} + \lambda \lambda_{ik} Q_i,$$

$$V^{-1} P_k V = P_k + \lambda A_{ki} P_i + \frac{1}{2!} \lambda^2 [A_{ki} P_i, Q_i A_{im} P_m] + \cdots = (e^{A})_{ki} P_i.$$  

(10)

(11)

From eq. (10) we see that when $V$ acts on the $n$-mode coordinate eigenstate $|\eta\rangle$, where $\eta = (q_1, q_2, \ldots, q_n)$, it squeezes $|\eta\rangle$ in this way:

$$V|\eta\rangle = |\lambda|^{1/2}|\lambda \eta\rangle, \quad \Lambda = e^{-\lambda A}, \quad |\lambda| \equiv \det \Lambda.$$  

(12)

Thus $V$ has the representation, on the coordinate $|\eta\rangle$ basis,

$$V = \int d^n q V|\eta\rangle \langle \eta | = |\lambda|^{1/2} \int d^n q |\lambda \eta\rangle \langle \lambda \eta |, \quad V^\dagger = V^{-1},$$

(13)

since $\int d^n q |\lambda \eta\rangle \langle \lambda \eta | = 1$. Using the expression of eigenstate $|\lambda \eta\rangle$ in Fock space

$$|\lambda \eta\rangle = \pi^{-n/4} \exp \left[\frac{1}{2} \Sigma q_i \eta_i + \sqrt{2} \eta a^\dagger - \frac{1}{2} a^\dagger a\right] |0\rangle,$$

(14)

$$\overline{a}^\dagger = (a_1^\dagger, a_2^\dagger, \ldots, a_n^\dagger),$$

and $|0\rangle \langle 0| = \exp[-\overline{a}^\dagger a\dagger]$; we can put $V$ into the normal ordering form,

$$V = \pi^{-n/2}|\lambda|^{1/2} \int d^n \eta \exp \left[-\frac{1}{2} \overline{\eta} (1 + \overline{\Lambda} \Lambda) \eta + \sqrt{2} \overline{\eta} \Lambda \overline{a}^\dagger + a\right] \times \exp \left[\frac{1}{2} \overline{a}^\dagger (\Lambda N^{-1} - I) a\right],$$

(15)

To compute the integration in eq. (15) by virtue of the IWOP technique, we use the mathematical formula

$$\int d^n x \exp[-\overline{x} F x + \overline{x} v] = \pi^{n/2} \det F^{-1/2} \exp \left[\frac{1}{4} \overline{v} F^{-1} v\right],$$

(16)

then we derive

$$V = \left(\frac{\det \Lambda}{\det N}\right)^{1/2} \exp \left[\frac{1}{2} \overline{a}^\dagger \left(\Lambda N^{-1} - I\right) a\right] \times \exp \left[\frac{1}{2} \overline{a}^\dagger (\Lambda N^{-1} - I) a\right],$$

(17)

where $N = (\overline{1} + \overline{\Lambda} \Lambda)/2$. Equation (17) is just the normal product form of $V$.  

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The squeezing property of $V|0\rangle$. - Operating $V$ on the $n$-mode vacuum state $|0\rangle$, we obtain the squeezed vacuum state

$$V|0\rangle = \left( \frac{\det A}{\det N} \right)^{1/2} \exp \left[ \frac{1}{2} \overline{a} \left( \Lambda N^{-1} A - I \right) a \right] |0\rangle.$$  \hfill (18)

Now we evaluate the variances of the $n$-mode quadratures. The quadratures in the $n$-mode case are defined as

$$X_1 = \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} Q_i, \quad X_2 = \frac{1}{\sqrt{2n}} \sum_{i=1}^{n} P_i,$$  \hfill (19)

obeying $[X_1, X_2] = \frac{i}{2}$. Their variances are $(\Delta X_i)^2 = \langle X_i^2 \rangle - \langle X_i \rangle^2$, $i = 1, 2$. Noting the expectation values of $X_1$ and $X_2$ in the state $V|0\rangle$, $(X_1)(X_2) = 0$, and using eqs. (10) and (11) we see that the variances are

$$(\Delta X_1)^2 = \langle 0|V^{-1} X_1^2 V|0\rangle = \frac{1}{2n} \langle 0|V^{-1} \sum_{i=1}^{n} Q_i \sum_{j=1}^{n} Q_j V|0\rangle$$

$$= \frac{1}{2n} \sum_{i,j} \frac{\overline{Q}_k (e^{-\lambda A})_{ki} \overline{Q}_j (e^{-\lambda A})_{ji}}{Q_k Q_l} [Q_k Q_l]|0\rangle$$

$$= \frac{1}{2n} \sum_{i,j} \frac{(e^{-\lambda A})_{ki} (e^{-\lambda A})_{ji}}{Q_k Q_l} [Q_k Q_l]|0\rangle$$

$$= \frac{1}{2n} \sum_{i,j} \frac{(e^{-\lambda A})_{ki} (e^{-\lambda A})_{ji}}{Q_k Q_l} [Q_k Q_l]|0\rangle$$

$$(\Delta X_2)^2 = \langle 0|V^{-1} X_2^2 V|0\rangle = \frac{1}{4n} \sum_{i,j} \tilde{\Lambda} \Lambda_{ij},$$  \hfill (20)

similarly we have

$$(\Delta X_2)^2 = \langle 0|V^{-1} X_2^2 V|0\rangle = \frac{1}{4n} \sum_{i,j} \tilde{\Lambda} \Lambda_{ij}.$$  \hfill (21)

Equations (20), (21) are the quadrature variance formulae in the transformed vacuum state acted by the operator $\exp[i \lambda \sum_{i,j=1}^{n} Q_i A_i P_j]$. By observing that $A$ in (8) is a cyclic matrix, we see that

$$\sum_{i,j} \left[ (A + \tilde{A})^\dagger \right]_{ij} = 2n,$$  \hfill (22)

then using $\tilde{A} = A\Lambda$, so $\tilde{\Lambda} = e^{-\lambda (A + \tilde{A})}$, a symmetric matrix, we have

$$\sum_{i,j=1}^{n} \tilde{\Lambda} \Lambda_{ij} = \sum_{i,j=1}^{n} \left[ (A + \tilde{A})^\dagger \right]_{ij} = n \sum_{i,j=1}^{n} \left[ (A + \tilde{A})^\dagger \right]_{ij} = n \sum_{i,j=1}^{n} \left[ (A + \tilde{A})^\dagger \right]_{ij} = ne^{-2\lambda},$$  \hfill (23)

and

$$\sum_{i,j=1}^{n} \tilde{\Lambda} \Lambda_{ij} = n e^{2\lambda}.$$  \hfill (24)

It then follows that

$$(\Delta X_1)^2 = \frac{1}{4n} \sum_{i,j} \tilde{\Lambda} \Lambda_{ij} = \frac{e^{-2\lambda}}{4},$$  \hfill (25)

$$(\Delta X_2)^2 = \frac{1}{4n} \sum_{i,j} \tilde{\Lambda} \Lambda_{ij} = \frac{e^{2\lambda}}{4}.$$  \hfill (26)

This leads to $\Delta X_1 \cdot \Delta X_2 = \frac{1}{4}$, which shows that $V$ is a correct $n$-mode squeezing operator for the $n$-mode quadratures in eq. (19) and produces a standard squeezing similar to eq. (6).

The Wigner function of $V|0\rangle$. - Wigner distribution functions [13–15] of quantum states are widely studied in quantum statistics and quantum optics. Now we derive the expression of the Wigner function of $V|0\rangle$. Here we take a new method to do it. Recalling that in refs. [16–18] we have introduced the Weyl ordering form of the single-mode Wigner operator $\Delta_1(q_1, p_1)$,

$$\Delta_1(q_1, p_1) = \delta (q_1 - Q_1) \delta (p_1 - P_1)$$  \hfill (27)

its normal ordering form is

$$\Delta_1(q_1, p_1) = \frac{1}{\pi} : \exp \left[ -(q_1 - Q_1)^2 - (p_1 - P_1)^2 \right] :$$  \hfill (28)

where the symbols $::$ and $::$ denote the normal ordering and the Weyl ordering, respectively. Note that the order of Bose operators $a_1$ and $a_1^\dagger$ within a normally ordered product and a Weyl ordered product can be permuted. That is to say, even though $[a_1, a_1^\dagger] = 1$, we can have $a_1 a_1^\dagger := a_1^\dagger a_1$ and $a_1 a_1^\dagger := a_1^\dagger a_1$. The Weyl ordering has a remarkable property, i.e., the order invariance of Weyl-ordered operators under similar transformations [16–18], which means

$$U : \{ \circ \circ \} : U^{-1} = : U \{ \circ \circ \} U^{-1} :$$  \hfill (29)

as if the “fence” did not exist.

For $n$-mode case, the Weyl ordering form of the Wigner operator is

$$\Delta_n(q, p) = \delta (q - \tilde{Q}) \delta (p - \tilde{P}),$$  \hfill (30)

where $\tilde{Q} = (Q_1, Q_2, \ldots, Q_n)$ and $\tilde{P} = (P_1, P_2, \ldots, P_n)$. Then according to the Weyl ordering invariance under similar transformations and eqs. (10) and (11) we have

$$V^{-1} \Delta_n (q, p) V = V^{-1} : \delta (q - \tilde{Q}) \delta (p - \tilde{P}) : V$$

$$= \delta (q_k - (e^{-\lambda A})_{ki} Q_k) \delta (p_k - (e^{-\lambda A})_{ki} P_k)$$

$$= \delta (e^{\lambda A} \tilde{q} - \tilde{Q}) \delta (e^{-\lambda A} \tilde{p} - \tilde{P})$$

$$= \Delta (e^{\lambda A} \tilde{q}, e^{-\lambda A} \tilde{p}).$$  \hfill (31)
Thus using eqs. (27) and (31) the Wigner function of $V(0)$ is

$$\langle 0 | V^{-1} \Delta_n (q, p) | 0 \rangle = \frac{1}{\sinh^2 \lambda} \exp \left[ -2 \sum_{i=1}^{n} |\alpha_i|^2 \right]$$

which engenders standard squeezing for the four-modes squeezed states [19]. Then substituting eq. (38) into eq. (32) we obtain

$$\langle 0 | V^{-1} \Delta_n (q, p) | 0 \rangle = \frac{1}{\sinh^2 \lambda} \exp \left[ -2 \sum_{i=1}^{n} |\alpha_i|^2 \right]$$

from which one can see that the four-modes squeezed state is not the same as the direct product of the two two-mode squeezed states in eq. (1). In sum, by virtue of the IWOP technique, we have introduced a kind of an $n$-mode squeezing operator $V \equiv \exp \left[ \sum_{i=1}^{n} (Q_i P_2 + Q_2 P_3 + \cdots + Q_{n-1} P_n + Q_n P_1) \right]$, which engenders standard squeezing for the $n$-mode quadratures. We have derived $V$’s normally ordered expansion and obtained the expression of $n$-mode squeezed vacuum states and evaluated its Wigner function with the aid of the Weyl ordering invariance under similar transformations.

**This work was supported by the National Natural Science Foundation of China under grants 10775097 and 10874174.**
APPENDIX

Derivation of eq. (38). – For the completeness of this paper, here we analytically derive eq. (38). Noticing that for the case of $n = 4$, $A^4 = I$, $I$ is the $4 \times 4$ unit matrix, from the Cayley-Hamilton theorem we know that the expanding form of $\exp(-\lambda \hat{A})$ must be

$$\Lambda = \exp(-\lambda \hat{A}) = c_0(\lambda)I + c_1(\lambda)\hat{A} + c_2(\lambda)\hat{A}^2 + c_3(\lambda)\hat{A}^3.$$  

(A.1)

To determine $c_j(\lambda)$, we take $\hat{A}$ to be $e^{i(j/2)\pi}$ $(j = 0, 1, 2, 3)$ respectively, then we have

$$\begin{align*}
\exp(-\lambda) &= c_0(\lambda) + c_1(\lambda) + c_2(\lambda) + c_3(\lambda), \\
\exp(-\lambda e^{i(1/2)\pi}) &= c_0(\lambda) + c_1(\lambda)e^{i(1/2)\pi} + c_2(\lambda)e^{i(3/2)\pi} + c_3(\lambda)e^{i3\pi}, \\
\exp(-\lambda e^{i\pi}) &= c_0(\lambda) + c_1(\lambda)e^{i\pi} + c_2(\lambda)e^{i2\pi} + c_3(\lambda)e^{i3\pi}, \\
\exp(-\lambda e^{i3(1/2)\pi}) &= c_0(\lambda) + c_1(\lambda)e^{i3(1/2)\pi} + c_2(\lambda)e^{i(9/2)\pi}.
\end{align*}$$

(A.2)

Its solution is

$$\begin{align*}
c_0(\lambda) &= \frac{1}{2} (\cosh \lambda + \cos \lambda), \\
c_1(\lambda) &= \frac{1}{2} (\sinh \lambda - \sin \lambda), \\
c_2(\lambda) &= \frac{1}{2} (\cosh \lambda - \cos \lambda), \\
c_3(\lambda) &= \frac{1}{2} (\sinh \lambda + \sin \lambda).
\end{align*}$$

(A.3)

It follows that

$$\Lambda = \begin{pmatrix} c_0 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_3 \\ c_3 & c_2 & c_1 & c_0 \end{pmatrix}, \quad \det \Lambda = 1,$$

(A.4)

and

$$\hat{\Lambda} = \left[ c_0(\lambda)I + c_1(\lambda)\hat{A} + c_2(\lambda)\hat{A}^2 + c_3(\lambda)\hat{A}^3 \right] \cdot \left[ c_0(\lambda)I + c_1(\lambda)\hat{A} + c_2(\lambda)\hat{A}^2 + c_3(\lambda)\hat{A}^3 \right]$$

which is just eq. (38).

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