COMPARISON OF NOVEL INDEX WITH GEOMETRIC−ARITHMETIC AND SUM−CONNECTIVITY INDICES

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Abstract. This work focusses on the minimal, second minimal, maximal and second maximal values of three indices for various unicyclic, bicyclic graphs. Motivated by the works of Ghorbani, this work compares the extremal values among three indices viz., geometric−arithmetic, sum−connectivity and proposed novel index (geometric−harmonic) and these indices are computed for star, cycle and path. Relations are established among the indices for star, cycle, path, tree, complete, unicyclic and bicyclic graphs.

Keywords: geometric−arithmetic index; sum−connectivity index; geometric−harmonic index.

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1. INTRODUCTION

In this paper, the graphs considered are simple and loop free. A graph is a collection of vertices and edges, where edge is a link between two vertices. The degree of a vertex is the number of edges incident to that vertex. The path is a sequence of vertices placed adjacent to
each other and joined using edges. A connected graph is one which has a path from any point to any other point in the graph. A graph which exactly depicts like a star in which a central node $p$ is connected to $p - 1$ pendant edges is a star graph. The degree of the central node will be $p - 1$ and the degree of the $p - 1$ edges will be unity.

A complete graph is a graph in which each pair of graph vertices is connected by an edge. The complete graph with $n$ vertices has $n(n - 1)/2$ number of edges. A cycle is a connected graph with all vertices of degree 2. If $n$ vertices are considered in a cycle $C_n$, then there are $n$ edges in the cycle $C_n$ [4].

The graph theory has its use in almost every fields [2, 5, 6, 7, 8, 10, 14] and it plays a significant role in mathematical chemistry for drawing the information of a chemical compound and also in the design of drugs. Robust methods and a large database are available for designing drugs. Survey relating the molecular structure to a particular property using tools of statistics are very significant. This is often referred to as QSPR and QSAR studies [1, 11, 12, 15, 16, 18]. The structure in which the atoms are bonded to each other are very important in order to carry out the study on it. The structure of the compound is a treasure of information of the respective compound. A two-dimensional descriptor that considers the arrangement of compounds, size, shape, branching etc, conceals the information in numerical form. Topological indices play a key role in the applications of the compounds in QSAR and QSPR studies [3, 9, 13, 17, 19, 21].

In this work, we discuss the extremal values of three indices and establish relations among them.

## 2. Preliminaries

Vukicevic et al., [20] designed a topological index called geometric–arithmetic index is the ratio of geometric mean of end vertex degrees of an edge $uv$ to arithmetic mean of end vertex degrees of the edge $uv$ and is defined as

$$GA(G) = \sum \frac{2 \sqrt{d_u d_v}}{d_u + d_v}$$

where $(u, v)$ is an element of $E(G)$. 

Zhou et al., [22] introduced the sum–connectivity index and is defined as

\[ \chi(G) = \sum \frac{1}{\sqrt{d_u + d_v}} \]

where \((u, v)\) is an element of \(E(G)\).

Motivated by Vukicevic in designing GA index, an attempt to design a new degree based index geometric–harmonic index is made and introduced as the ratio of geometric mean of end vertex degrees of an edge \(uv\) to harmonic mean of end vertex degrees of the edge \(uv\). It is defined as

\[ GH(G) = \sum \frac{(d_u + d_v)\sqrt{d_u d_v}}{2} \]

where \((u, v)\) is an element of \(E(G)\).

This index is computed and comparison of all the above defined three indices for unicyclic and bicyclic graphs are carried out.

3. MAIN RESULTS

**Theorem 3.1.** Suppose \(S_n\) be the star graph on \(n\) vertices. The degree of the central vertex is \((n - 1)\) and others are pendant vertices. Then the above discussed three indices are given by

\[ GA(S_n) = \frac{2(n - 1)^{\frac{3}{2}}}{n} \]
\[ \chi(S_n) = \frac{(n - 1)}{\sqrt{n}} \]
\[ GH(S_n) = \frac{n(n - 1)^{\frac{3}{2}}}{2} \]

**Proof.** The proof is trivial. \(\Box\)

**Theorem 3.2.** Let \(K_n\) be a complete graph with \(n\) vertices. Then the indices are respectively given by

\[ GA(K_n) = \frac{n(n - 1)}{2} \]
\[ \chi(K_n) = \frac{n\sqrt{(n - 1)}}{2\sqrt{2}} \]
\[ GH(K_n) = \frac{n(n - 1)^{\frac{3}{2}}}{2} \]
Proof. The proof is trivial. □

**Theorem 3.3.** Consider a path $P_n$ with $n$ vertices. Then the indices are respectively given by

\[
GA(P_n) = \frac{4\sqrt{2}}{3} + (n-3),
\]

\[
\chi(P_n) = \frac{(n-2)}{2} + \frac{2}{\sqrt{3}},
\]

\[
GH(P_n) = 4n + (3\sqrt{2} - 12).
\]

Proof. The proof is trivial. □

**Theorem 3.4.** Let $C_n$ be a cycle with $n$ vertices and each vertex degree is 2. Then the indices are respectively given by

\[
GA(C_n) = n.
\]

\[
\chi(C_n) = \frac{n}{2},
\]

\[
GH(C_n) = 4n.
\]

Proof. The proof is trivial. However, an interesting point to note in this particular case is that, the three indices for a cycle $C_n$ are related as

\[
\chi(C_n) = \frac{GA(C_n)}{2}.
\]

\[
GH(C_n) = 4GA(C_n).
\]

\[
GH(C_n) = 8\chi(C_n).
\]

□

**Theorem 3.5.** Let $G$ be any connected graph with $n$ vertices. Then the indices take the expression

\[
GA(G) \leq GA(K_n).
\]

\[
\chi(G) \leq \chi(K_n).
\]

\[
GH(G) \leq GH(K_n).
\]
Proof. As the complete graph has \(n(n-1)/2\) number of edges of degree \((n-1,n-1)\), the topological index will be larger for complete graph compared to any simple graph \(G\).

The proof is trivial. \(\Box\)

Theorem 3.6. A tree with \(n\) vertices with minimal GA index is a star \(S_n\). The GA index of \(S_n\) is given by

\[
\text{GA}(S_n) = \frac{2(n-1)^{3/2}}{n}.
\]

where \(n \geq 4\)

Proof. Consider a star \(n \geq 4\) for which one vertex is of degree \((n-1)\) and the rest \((n-1)\) vertices are pendant vertices. A star \(S_n\) has \((n-1)\) edges and end degree vertices are \((n-1)\) and 1 respectively. Then using GA index of star graph is

\[
\text{GA}(S_n) = \frac{2(n-1)^{3/2}}{n}.
\]

Thus, it is clear that any tree of vertices \(n \geq 4\) has GA index more than that of a star graph. \(\Box\)

Theorem 3.7. The minimal GA index and minimal sum−connectivity index of a \(n\)−vertex connected unicyclic graph is \((S_n + e)\) and is given by

\[
\text{GA}(S_n + e) = \frac{2(n-3)(n+1)\sqrt{n-1}+n(n+1)+4n\sqrt{2n-2}}{n(n+1)}.
\]

\[
\chi(S_n + e) = \frac{(n-3)}{\sqrt{n}} + \frac{1}{2} + \frac{2}{\sqrt{(n+1)}}.
\]

\textbf{Figure 1. The Graph } S_n + e.

Proof. Any \(n\)−vertex connected unicyclic graph has GA index and sum−connectivity index more than that of a graph in Fig.1. The graph as shown in Fig.1 has \((n-3)\) edges of \((1,n-1)\),
1 edge of \((2, 2)\) and 2 edges of \((2, n - 1)\) types. Using all these vertices and edges and definition of \(GA\) index and sum–connectivity index, we arrive at the results.

\[
GA(S_n + e) = \frac{2(n - 3)(n + 1)\sqrt{n - 1} + n(n + 1) + 4n\sqrt{2n - 2}}{n(n + 1)}.
\]

\[
\chi(S_n + e) = \frac{(n - 3)}{\sqrt{n}} + \frac{1}{2} + \frac{2}{\sqrt{(n + 1)}}.
\]

\[\square\]

**Theorem 3.8.** Among all the unicyclic graphs on \(n\) vertices, the graph \(S_n + e\) has the maximal geometric–harmonic index. The geometric–harmonic index for this graph is

\[
GH(S_n + e) = \frac{n(n - 3)\sqrt{n - 1}}{2} + 4 + (n + 1)\sqrt{2n - 2}.
\]

**Proof.** From the Fig.1 considering the total number of edges with its respective end vertices, the \(GH\) index results in

\[
GH(S_n + e) = \frac{n(n - 3)\sqrt{n - 1}}{2} + 4 + (n + 1)\sqrt{2n - 2}.
\]

\[\square\]

**Theorem 3.9.** The minimal \(GA\) index among all bicyclic graphs is for the graph depicted in Fig.2. This is given by

\[
GA(G) = 2\frac{(n - 4)\sqrt{(n - 1)}}{n} + 4\frac{\sqrt{(2n - 2)}}{(n + 1)} + 2\frac{\sqrt{(3n - 3)}}{(n + 2)} + \frac{4\sqrt{6}}{5}.
\]

**Figure 2.** The Graph G.
Proof. The graph $G$ depicted in Fig. 2 has $(n - 4)$ edges of $(1, n - 1)$, $2$ edges of $(2, n - 1)$, $1$ edge of $(3, n - 1)$ and $2$ edges of $(2, 3)$ types. Considering the total number of edges with its respective end vertices, the $GA$ index results in

$$GA(G) = \frac{2(n - 4)\sqrt{(n - 1)}}{n} + 4\sqrt{\frac{2(n - 2)}{n + 1}} + \frac{2\sqrt{3n - 3}}{n + 2} + \frac{4\sqrt{6}}{5}.$$ 

It is found that $GA$ index is the minimal for the graph $G$ as shown in Fig. 2 among $n$–vertex connected bicyclic graphs. 

**Theorem 3.10.** The minimal sum–connectivity index (only for $n \leq 10$) among all bicyclic graphs, is for the graph in Fig. 2. It is given by

$$\chi(G) = \frac{n - 4}{\sqrt{n}} + \frac{2}{\sqrt{n + 1}} + \frac{1}{\sqrt{n + 2}} + \frac{2}{\sqrt{5}}.$$ 

Proof. From the Fig. 2 considering the total number of edges with its respective end vertices, the sum–connectivity index results in

$$\chi(G) = \frac{n - 4}{\sqrt{n}} + \frac{2}{\sqrt{n + 1}} + \frac{1}{\sqrt{n + 2}} + \frac{2}{\sqrt{5}}.$$ 

It is found that the sum–connectivity index is the minimal for a graph $G$ as shown in Fig. 2 among $n$–vertex connected bicyclic graphs. 

**Theorem 3.11.** The maximal geometric–harmonic index in all bicyclic graphs is for the graph $G$ depicted in the graph Fig. 2. The value of the index for the graph $G$ is

$$GH(G) = \frac{n(n - 4)\sqrt{n - 1} + 2(n + 1)\sqrt{2n - 2} + (n + 2)\sqrt{3n - 3} + 10\sqrt{6}}{2}.$$ 

Proof. From the Fig. 2 considering the total number of edges with its respective end vertices, the $GH$ index results in

$$GH(G) = \frac{n(n - 4)\sqrt{n - 1} + 2(n + 1)\sqrt{2n - 2} + (n + 2)\sqrt{3n - 3} + 10\sqrt{6}}{2}.$$ 

**Theorem 3.12.** The second minimal $GA$ index among all unicyclic graphs is for the graph in Fig. 3. It is given by

$$GA(S_n + e + e') = \frac{2(n - 4)\sqrt{n - 2}}{n - 1} + \frac{4\sqrt{2n - 2}}{n} + 2.$$
Figure 3. The Graph $S_n + e + e'$.

Proof. The graph $S_n + e + e'$ depicted in the Fig.3 has $(n - 4)$ edges of $(n - 2, 1)$, 2 edges of $(2, n - 2)$ and 2 edges of $(2, 2)$ types. Considering the total number of edges with its respective end vertices, the $GA$ index results in

$$GA(S_n + e + e') = \frac{2(n-4)\sqrt{n-2}}{n-1} + \frac{4\sqrt{2n-2}}{n} + 2.$$ 

It is found that the $GA$ index is the second minimal for a graph $S_n + e + e'$ as shown in Fig.3 among $n$–vertex connected unicyclic graphs.

\[ \square \]

Theorem 3.13. The second minimal sum–connectivity index among all unicyclic graphs is for the graph in Fig.3. It is given by

$$\chi(S_n + e + e') = \frac{n-4}{\sqrt{n-1}} + \frac{2}{\sqrt{n}} + 1.$$ 

Proof. From the Fig.3 considering the total number of edges with its respective end vertices, the sum–connectivity index results in

$$\chi(S_n + e + e') = \frac{n-4}{\sqrt{n-1}} + \frac{2}{\sqrt{n}} + 1.$$ 

It is found that the sum–connectivity index is the second minimal for a graph $S_n + e + e'$ as shown in Fig.3 among $n$–vertex connected unicyclic graphs.

\[ \square \]

Theorem 3.14. The second maximal $GH$ index among all unicyclic graphs is for the graph in Fig.3. It is given by

$$GH(S_n + e + e') = \frac{(n-4)(n-1)\sqrt{n-2} + 2n\sqrt{2n-4} + 16}{2}.$$
Proof. From the Fig.3 considering the total number of edges with its respective end vertices, the $GH$ index results in

$$GH(S_n + e + e') = \frac{(n-4)(n-1)\sqrt{n-2} + 2n\sqrt{2n-4} + 16}{2}.$$ 

It is found that the $GH$ index is the second maximal for a graph $S_n + e + e'$ as shown in Fig.3 among $n$–vertex connected unicyclic graphs.

Theorem 3.15. The $n$–vertex tree with maximal GA index is the path $P_n$ in which $GA(T) < GA(P_n)$. The GA index of a path $P_n$ is given in the previous results.

Proof. The graph $H$ is a $n$–vertex bicyclic and connected graph. It has $n$ vertices and $n+1$ edges. The graph in the Fig.4 has 4 edges of $(2,3)$, 1 edge of $(3,3)$ and $n-4$ edges of $(2,2)$ types. GA index is the maximal for the graph $H$ among bicyclic graphs. The GA index results in

$$GA(H) = n - 3 + \frac{8\sqrt{6}}{5}.$$ 

Theorem 3.16. Among all bicyclic graphs on $n$ vertices, the graph $H$ in Fig.4 has the second maximal sum–connectivity index and it is given by

$$\chi(H) = \frac{4}{\sqrt{5}} + \frac{1}{3} + \frac{n-4}{2}.$$ 

Proof. From the Fig.4 considering the total number of edges with its respective end vertices, the sum–connectivity index results in

$$\chi(H) = \frac{4}{\sqrt{5}} + \frac{1}{3} + \frac{n-4}{2}.$$
It is found that sum–connectivity index is the second maximal for the graph $H$ as shown in Fig.4 among connected bicyclic graphs.

**Theorem 3.17.** Among all bicyclic graphs on $n$ vertices, the graph $H$ in Fig.4 has minimal $GH$ index and it is given by

$$GH(H) = 4n + 10\sqrt{6} - 12.$$ 

*Proof.* From the Fig.4 considering the total number of edges with its respective end vertices, the $GH$ index results in

$$GH(H) = 4n + 10\sqrt{6} - 12.$$ 

It is found that $GH$ index is the minimal for the graph $H$ as shown in Fig.4 among connected bicyclic graphs.

The $n$–vertex connected tree with second minimal GA index is the graph $(S_n)'$. The GA index of $(S_n)'$ is

$$GA(S_n)' = \frac{2(n-3)\sqrt{n-2}}{n-1} + \frac{2\sqrt{2n-4}}{n} + \frac{2\sqrt{2}}{3}.$$ 

*Figure 5. The Graph $S_n'$.*

*Proof.* The graph $(S_n)'$ has $n$ vertices and $n - 1$ edges. The $(S_n)'$ has $n - 3$ edges of $(n - 2, 1)$, 1 edge of $(n - 2, 2)$ and 1 edge of $(1, 2)$ types. Using GA index for $(S_n)'$, we get the required result and found to be the second minimal among the trees.

**Theorem 3.19.** The $n$–vertex connected tree with minimal $GH$ index is the graph $(S_n)'$. The $GH$ index of $(S_n)'$ is

$$GH(S_n)' = \frac{(n^2 - 4n + 3)\sqrt{n-2} + n\sqrt{2n-4} + 3\sqrt{2}}{2}.$$
Proof. From the Fig.5 considering the total number of edges with its respective end vertices, the GH index results in

\[ GH(S_n)' = \frac{(n^2 - 4n + 3)\sqrt{n-2} + n\sqrt{2n-4} + 3\sqrt{2}}{2}. \]

\[ \square \]

**Theorem 3.20.** Among all bicyclic graphs on \( n \) vertices, the graph \( F \) in Fig.6 has the second minimal GA index and is given by

\[ GA(F) = \frac{2(n-5)\sqrt{n-2}}{n-1} + \frac{4\sqrt{2n-4}}{n} + \frac{2\sqrt{3n-6}}{n+1} + \frac{4\sqrt{6}}{5} + 1. \]

![Figure 6. The Graph F.](image)

Proof. In graph \( F \), \( n \)-vertex connected bicyclic graph, there are \( n+1 \) edges. There are \( n-5 \) edges of \((1, n-2)\), 2 edges of \((2, n-2)\), 1 edge of \((3, n-2)\), 2 edges of \((2, 3)\) and 1 edge of \((2, 2)\) types in graph \( F \). \( GA(F) \) results in the following formula by using above edges and found to be second minimal among all bicyclic graphs. Hence follows the result.

\[ GA(F) = \frac{2(n-5)\sqrt{n-2}}{n-1} + \frac{4\sqrt{2n-4}}{n} + \frac{2\sqrt{3n-6}}{n+1} + \frac{4\sqrt{6}}{5} + 1. \]

\[ \square \]

**Theorem 3.21.** Among all bicyclic graphs for \( n \) vertices, the graph \( F \) in Fig.6 has the second minimal sum-connectivity index and is given by

\[ \chi(F) = \frac{n-5}{\sqrt{n-1}} + \frac{2}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \frac{2}{\sqrt{5}} + \frac{1}{2}. \]
Proof. From the Fig.6 considering the total number of edges with its respective end vertices, the sum–connectivity index results in

\[ \chi(F) = \frac{n - 5}{\sqrt{n - 1}} + \frac{2}{\sqrt{n}} + \frac{1}{\sqrt{n + 1}} + \frac{2}{\sqrt{5}} + \frac{1}{2}. \]

\[ \square \]

**Theorem 3.22.** Among all bicyclic graphs for \( n \leq 10 \) vertices, the graph \( F \) in Fig.6 has the second maximal \( GH \) index and is given by

\[ GH(F) = \frac{(n^2 - 6n + 5)\sqrt{n - 2} + 2n\sqrt{2n - 4} + (n + 1)\sqrt{3n - 6} + 10\sqrt{6} + 8}{2}. \]

Proof. From the Fig.6 considering the total number of edges with its respective end vertices, the \( GH \) index results in

\[ GH(F) = \frac{(n^2 - 6n + 5)\sqrt{n - 2} + 2n\sqrt{2n - 4} + (n + 1)\sqrt{3n - 6} + 10\sqrt{6} + 8}{2}. \]

It is found that the \( GH \) index is the second maximal for a graph \( F \) as shown in Fig.6 among all bicyclic graphs.

\[ \square \]

**Theorem 3.23.** The \( n \)–vertex tree with maximal \( GA \) index, sum–connectivity index and \( GH \) index is the graph \( Q \) in Fig.7. It is given by

\[ GA(Q) = n - 6 + \frac{4}{\sqrt{3}} + \frac{4\sqrt{6}}{5} + \frac{\sqrt{3}}{2}. \]

\[ \chi(Q) = \frac{n - 6}{2} + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{5}} + \frac{1}{2}. \]

\[ GH(Q) = \frac{8(n - 6) + 6\sqrt{2} + 10\sqrt{6} + 4\sqrt{3}}{2}. \]

**Figure 7.** The Graph Q.
Proof. The graph $Q$ has $n$ vertices and $n - 1$ edges. There are $n - 6$ edges of $(2,2)$, 2 edges of $(1,2)$, 2 edges of $(2,3)$ and 1 edge of $(1,3)$ types in graph $Q$. The number of edges are considered and $GA$ index, sum−connectivity index and $GH$ index are found to be maximal for the graph $Q$. It is obtained as

$$GA(Q) = n - 6 + \frac{4}{\sqrt{3}} + \frac{4\sqrt{6}}{5} + \frac{\sqrt{3}}{2}.$$  

$$\chi(Q) = \frac{n - 6}{2} + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{5}} + \frac{1}{2}.$$  

$$GH(Q) = \frac{8(n - 6) + 6\sqrt{2} + 10\sqrt{6} + 4\sqrt{3}}{2}.$$  

\[\square\]

**Theorem 3.24.** The $n$−vertex connected unicyclic graph with second maximal $GA$ and sum−connectivity index is graph $R$, depicted in Fig.8. It is given by

$$GA(R) = n - 4 + \frac{2\sqrt{2}}{3} + \frac{6\sqrt{6}}{5}.$$  

$$\chi(R) = \frac{n - 4}{2} + \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{5}}.$$  

The $GA$ index of graph $R$ and $C_n$ are related by

$$GA(R) = GA(C_n) - 0.11779.$$  

The sum−connectivity index of graph $R$ and $C_n$ are related by

$$\chi(R) = \chi(C_n) - 0.0810.$$  

**Figure 8.** The Graph R.

Proof. The graph $R$ in Fig.8 has $n$ vertices and $n$ edges. The edges found in graph $R$ are $n - 4$ of $(2,2)$, 1 of $(1,2)$ and 3 of $(2,3)$ types. Using this data, the results are as follows. It is found that
among the $n$–vertex connected unicyclic graphs, graph $R$ has the second maximal $GA$ index and sum–connectivity index. It is clear by the relation that $GA$ index of graph $R$ reduces by a quantity of $0.11779$ compared to the $GA$ index of $C_n$ among all unicyclic graphs. It is clear by the relation that sum–connectivity index of graph $R$ reduces by a quantity of $0.0810$ compared to the sum–connectivity index of $C_n$ among all unicyclic graphs.

$$GA(R) = GA(C_n) - 0.11779.$$  
$$\chi(R) = \chi(C_n) - 0.0810.$$  

□

**Theorem 3.25.** The $n$–vertex connected unicyclic graph with second minimal $GH$ index is graph $R$, depicted in Fig.8. It is given by

$$GH(R) = \frac{8(n - 4) + 3\sqrt{2} + 15\sqrt{6}}{2}.$$  
The $GH$ index of graph $R$ and $C_n$ are related by

$$GH(R) = GH(C_n) + 4.4923.$$  

**Proof.** The graph $R$ in Fig.8 has $n$ vertices and $n$ edges. The edges found in graph $R$ are $n - 4$ of $(2, 2)$, $1$ of $(1, 2)$ and $3$ of $(2, 3)$ types. Using this data, the results are as follows. It is found that among the $n$–vertex connected unicyclic graphs, graph $R$ has the second minimal $GH$ index and it is clear by the relation that $GH$ index of graph $R$ increases by a quantity $4.4923$ compared to the $GH$ index of $C_n$ among all unicyclic graphs. □

**Theorem 3.26.** Among the bicyclic graphs on $n$ vertices, the graph $S$ in Fig.9 has the second maximal $GA$ index and the sum–connectivity index is maximal and the $GH$ index is second minimal ($n \geq 12$) for the bicyclic graph $S$ and are given by

$$GA(S) = n - 5 + \frac{12\sqrt{6}}{5}.$$  
$$\chi(S) = \frac{n - 5}{2} + \frac{6}{\sqrt{5}}.$$  
$$GH(S) = \frac{8(n - 5) + 30\sqrt{6}}{2}.$$
The relation of \( GA \) index for the graphs \( S \) and \( H \) is given by

\[
GA(S) = GA(H) - 0.040408.
\]

**Proof.** The graph \( S \) is a bicyclic \( n \)—vertex connected graph with \( n + 1 \) edges. There are \( n - 5 \) edges of \((2,2)\) and 6 edges of \((2,3)\) types. The \( GA \) index follows as above. It is found to be the second maximal among bicyclic graphs. The \( GA \) index of the graph \( S \) is second maximal when compared to the \( GA \) index of the graph \( H \) as \( GA \) index of the graph \( S \) reduces by a quantity 0.040408 to that of \( GA \) index of the graph \( H \). The sum—connectivity index is found to be maximal among all bicyclic graphs considered in this study. The \( GH \) index is minimal for the bicyclic graph \( S \) up to \( n \leq 9 \) and it is second minimal as \( n \geq 12 \). Hence,

\[
GA(S) = n - 5 + \frac{12\sqrt{6}}{5}.
\]

\[
\chi(S) = \frac{n - 5}{2} + \frac{6}{\sqrt{5}}.
\]

\[
GH(S) = \frac{8(n - 5) + 30\sqrt{6}}{2}.
\]

\[
GA(S) = GA(H) - 0.040408.
\]

**□**

4. **Conclusion**

In this study the extremal values of unicyclic and bicyclic graphs using three indices viz., geometric-arithmetic(\( GA \)), sum-connectivity(\( \chi \)), geometric-harmonic(\( GH \)) are computed. A novel topological index known as geometric-harmonic index is introduced and relations are established among the indices for star, cycle, path, tree, complete, unicyclic and bicyclic graphs using other two indices considered under the study.
CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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