TRANSLATION INVARIANCE OF FOCK SPACES

KEHE ZHU

ABSTRACT. We show that there is only one Hilbert space of entire functions that is invariant under the action of naturally defined weighted translations.

1. INTRODUCTION

For $\alpha > 0$ and $0 < p \leq \infty$ the Fock space $F_p^\alpha$ consists of entire functions $f$ in $\mathbb{C}^n$ such that the function $f(z)e^{-\alpha|z|^2/2}$ belongs to $L^p(\mathbb{C}^n, dv)$, where $dv$ is the Lebesgue volume measure on $\mathbb{C}^n$. The spaces $F_p^\alpha$ are sometimes called Bargmann or Segal-Bargmann spaces as well.

When $0 < p < \infty$ and $f \in F_p^\alpha$, we write

$$\|f\|_{p,\alpha}^p = \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbb{C}^n} \left|f(z)e^{-\alpha|z|^2/2}\right|^p dv(z).$$

For $f \in F_\infty^\alpha$ we write

$$\|f\|_{\infty,\alpha} = \sup_{z \in \mathbb{C}^n} |f(z)|e^{-\alpha|z|^2/2}.$$

If we define

$$d\lambda_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} dv(z),$$

then $F_\alpha^2$ is a closed subspace of $L^2(\mathbb{C}^n, d\lambda_\alpha)$, and hence is a Hilbert space itself with the following inherited inner product from $L^2(\mathbb{C}^n, d\lambda_\alpha)$:

$$\langle f, g \rangle_\alpha = \int_{\mathbb{C}^n} f(z)\overline{g(z)} d\lambda_\alpha(z).$$

For any point $a \in \mathbb{C}^n$ we define a linear operator $T_a$ as follows.

$$T_a f(z) = e^{\alpha\overline{\mathbf{a}} \cdot \mathbf{z} - \alpha|\mathbf{a}|^2} f(z - a),$$

where

$$z\overline{a} = z_1\overline{a}_1 + \cdots + z_n\overline{a}_n$$

for $z = (z_1, \ldots, z_n)$ and $a = (a_1, \ldots, a_n)$ in $\mathbb{C}^n$. These operators will be called (weighted) translation operators. A direct calculation shows that

2000 Mathematics Subject Classification. 30H20.

Key words and phrases. Fock spaces, Gaussian measures, translation invariance, atomic decomposition, reproducing kernel, Heisenberg group.
they are surjective isometries on all the Fock spaces $F^p_\alpha$, $0 < p \leq \infty$. In particular, each $T_a$ is a unitary operator on the Fock space $F^2_\alpha$.

The purpose of this note is to show that $F^2_\alpha$ is the only Hilbert space of entire functions in $\mathbb{C}^n$ that is invariant under the action of the weighted translations $T_a$:

$$\langle T_a f, T_a g \rangle_\alpha = \langle f, g \rangle_\alpha$$

for all $f$ and $g$ in $F^2_\alpha$. In other words, if $H$ is any other Hilbert space of entire functions in $\mathbb{C}^n$ such that (1) holds for all $f$ and $g$ in $H$, then $H = F^2_\alpha$ and there exists a positive constant $c$ with $\langle f, g \rangle_H = c \langle f, g \rangle_\alpha$ for all $f$ and $g$ in $H$.

Along the way, we will also demonstrate that, in some sense, $F^1_\alpha$ is minimal among translation invariant Banach spaces of entire functions in $\mathbb{C}^n$, and $F^\infty_\alpha$ is maximal among translation invariant Banach spaces of entire functions in $\mathbb{C}^n$.

The uniqueness of $F^2_\alpha$ here is not a surprise. The corresponding result for Bergman and Besov spaces has been known for a long time. See [2, 3, 10]. In the setting of Bergman and Besov spaces, the rotation group plays a key role. In our setting here, the translations do not include any rotations, so a new approach is needed. We present two new approaches here: one that is specific for the Fock space setting and another that can be used in the Bergman space setting as well.

2. Preliminaries

Since $F^2_\alpha$ is a closed subspace of the Hilbert space $L^2(\mathbb{C}^n, d\lambda_\alpha)$, there exists an orthogonal projection $P_\alpha : L^2(\mathbb{C}^n, d\lambda_\alpha) \to F^2_\alpha$.

It turns out that $P_\alpha$ is an integral operator,

$$P_\alpha f(z) = \int_{\mathbb{C}^n} f(w) e^{\alpha z \bar{w}} d\lambda_\alpha(w), \quad f \in L^2(\mathbb{C}^n, d\lambda_\alpha),$$

where $K_w(z) = K(z, w) = e^{\alpha z \bar{w}}$ is the reproducing kernel of $F^2_\alpha$. In terms of this reproducing kernel we can rewrite

$$T_a f(z) = f(z - a) k_a(z),$$

where

$$k_a(z) = K_a(z)/\|K_a\|_{2, \alpha} = e^{\alpha z \bar{a} - \frac{1}{2} |a|^2}$$

is a unit vector in $F^2_\alpha$ and is called the normalized reproducing kernel of $F^2_\alpha$ at the point $a \in \mathbb{C}^n$. 

**Lemma 1.** For any two points $a$ and $b$ in $\mathbb{C}^n$ we have
\[ T_a T_b = e^{-\alpha \text{Im}(\overline{a}b)} T_{a+b} = e^{\alpha \text{Im}(\overline{ab})} T_{a+b}. \] (2)

**Proof.** This follows from a direct calculation and we leave the routine details to the interested reader. $\square$

**Corollary 2.** Each $T_a$ is invertible on $F^p_\alpha$ with $T_a^{-1} = T_{-a}$. In particular, each $T_a$ is a unitary operator on $F^2_\alpha$ with $T_a^{-1} = T_a = T_{-a}$.

Recall that the Heisenberg group is $\mathbb{H} = \mathbb{C}^n \times \mathbb{R}$ with the group operation defined by
\[ (z, t) \oplus (w, s) = (z + w, t + s - \text{Im}(\overline{zw})). \]
It follows that the weighted translations $T_a$, $a \in \mathbb{C}^n$, can be thought of as elements of the Heisenberg group. They do not form a subgroup though. Furthermore, the mapping $(z, t) \mapsto e^{\alpha it} T_z$ is a unitary representation of the Heisenberg group on $F^2_\alpha$. Although it is nice to know this connection, we will not need the full action of the Heisenberg group. We only need the action of these weighted translations.

The following result, often referred to as the atomic decomposition for Fock spaces, will play a key role in our analysis.

**Theorem 3.** Let $0 < p \leq \infty$. There exists a sequence $\{z_j\}$ in $\mathbb{C}^n$ with the following property: an entire function $f$ in $\mathbb{C}^n$ belongs to the Fock space $F^p_\alpha$ if and only if it can be represented as
\[ f(z) = \sum_{j=1}^{\infty} c_j k_{z_j}(z), \] (3)
where $\{c_j\} \in l^p$ and
\[ \|f\|_{p,\alpha} \sim \inf \{\|c_j\|_{l^p}\}. \] (4)
Here the infimum is taken over all sequences $\{c_j\}$ satisfying (3).

**Proof.** See [6] and [9]. $\square$

### 3. Two Extreme Cases

In this section we show that the Fock spaces $F^1_\alpha$ and $F^\infty_\alpha$ are extremal among translation invariant Banach spaces of entire functions in $\mathbb{C}^n$. The next two propositions were stated as Corollary 8.1 in [6], derived from a general framework of Banach spaces invariant under the action of the Heisenberg group. We include these results here for completeness and note that only the weighted translations from the Heisenberg group are needed.
Proposition 4. The Fock space $F_\alpha^\infty$ is maximal in the following sense. If $X$ is any Banach space of entire functions in $\mathbb{C}^n$ satisfying

(a) $\|T_a f\|_X = \|f\|_X$ for all $a \in \mathbb{C}^n$ and $f \in X$.

(b) $f \mapsto f(0)$ is a bounded linear functional on $X$.

Then $X \subset F_\alpha^\infty$ and the inclusion is continuous.

Proof. Condition (a) implies that $T_a f \in X$ for every $f \in X$ and every $a \in \mathbb{C}^n$. Combining this with condition (b) we see that for every $a \in \mathbb{C}^n$ the point evaluation $f \mapsto f(a)$ is also a bounded linear functional on $X$. Furthermore,

$$e^{-\frac{1}{2}|a|^2 |f(a)|} = |T_{-a} f(0)| \leq C \|T_{-a} f\|_X = C \|f\|_X,$$

where $C$ is a positive constant that is independent of $a \in \mathbb{C}^n$ and $f \in X$. Since $a$ is arbitrary, we conclude that $f \in F_\alpha^\infty$ with $\|f\|_{\infty, \alpha} \leq C \|f\|_X$ for all $f \in X$. $\square$

Proposition 5. The Fock space $F_\alpha^1$ is minimal in the following sense. If $X$ is a Banach space of entire functions in $\mathbb{C}^n$ satisfying

(a) $\|T_a f\|_X = \|f\|_X$ for all $a \in \mathbb{C}^n$ and $f \in X$.

(b) $X$ contains all constant functions.

Then $F_\alpha^1 \subset X$ and the inclusion is continuous.

Proof. Since $X$ contains all constant functions, applying $T_a$ to the constant function 1 shows that for each $a \in \mathbb{C}^n$ the function $k_a(z) = e^{\alpha z - \frac{1}{2}|a|^2}$ belongs to $X$. Furthermore, $\|k_a\|_X = \|T_a 1\|_X = \|1\|_X$ for all $a \in \mathbb{C}^n$.

Let $\{z_j\}$ denote a sequence in $\mathbb{C}^n$ on which we have atomic decomposition for $F_\alpha^1$. If $f \in F_\alpha^1$, there exists a sequence $\{c_j\} \in l^1$ such that

$$f = \sum_{j=1}^{\infty} c_j k_{z_j}. \quad (5)$$

Since each $k_{z_j}$ belongs to $X$ and $\sum |c_j| < \infty$, we conclude that $f \in X$ with

$$\|f\|_X \leq \sum_{j=1}^{\infty} |c_j| \|k_{z_j}\|_X = C \sum_{j=1}^{\infty} |c_j|,$$

where $C = \|1\|_X > 0$. Taking the infimum over all sequences $\{c_j\}$ satisfying (5) and applying (4), we obtain another constant $C > 0$ such that

$$\|f\|_X \leq C \|f\|_{F_\alpha^1}, \quad f \in F_\alpha^1.$$

This proves the desired result. $\square$
4. Uniqueness of $F^2_\alpha$

In this section we show that there is only one Hilbert space of entire functions in $\mathbb{C}^n$ that is invariant under the action of the weighted translations.

There have been several similar results in the literature concerning the uniqueness of certain Hilbert spaces of analytic functions. The first such result appeared in [2], where it was shown that the Dirichlet space is unique among Möbius invariant (pre-)Hilbert spaces of analytic functions in the unit disk. This result was generalized in [10] to the case of the unit ball in $\mathbb{C}^n$. Then a more systematic study was made in [3] concerning Möbius invariant Hilbert spaces of analytic functions in bounded symmetric domains. In each of these papers, a key idea was to average over a certain subgroup of the full group of automorphisms of the underlying domain. For example, in the case of the unit disk, the average was taken over the rotation group.

In the Fock space case, the translation operators do not involve any rotation, and there does not seem to be any natural average that one can use. We introduce two different approaches here. One uses the group operation in the Heisenberg group in a critical way and so is specific to the Fock space setting. The other is based on reproducing kernel techniques and so can be used in the Bergman space setting as well.

**Theorem 6.** The Fock space $F^2_\alpha$ is unique in the following sense. If $H$ is any separable Hilbert space of entire functions in $\mathbb{C}^n$ satisfying

(a) $H$ contains all constant functions.
(b) $\|T_\alpha f\|_H = \|f\|_H$ for all $\alpha \in \mathbb{C}^n$ and $f \in H$.
(c) $f \mapsto f(0)$ is a bounded linear functional on $H$.

Then $H = F^2_\alpha$ and there exists a positive constant $c$ such that $\langle f, g \rangle_H = c\langle f, g \rangle_\alpha$ for all $f$ and $g$ in $H$.

**Proof.** For the duration of this proof we use $\langle f, g \rangle$ to denote the inner product in $H$. The inner product in $F^2_\alpha$ will be denoted by $\langle f, g \rangle_\alpha$.

By Proposition [5], $H$ contains all the functions

$$k_\alpha(z) = e^{\alpha z \bar{\alpha} - \frac{1}{2} |\alpha|^2}, \quad z \in \mathbb{C}^n.$$ 

Let $K_\alpha(z) = e^{\alpha z \bar{\alpha}}$ denote the reproducing kernel of $F^2_\alpha$ and define a function $F$ on $\mathbb{C}^n$ by

$$F(a) = \langle 1, K_\alpha \rangle, \quad a \in \mathbb{C}^n.$$ 

We claim that $F$ is an entire function. To see this, assume $n = 1$ (the higher dimensional case is proved in the same way) and observe that for any fixed $a \in \mathbb{C}$ we have

$$K_\alpha(z) = e^{\alpha z \bar{\alpha}} = \sum_{k=0}^{\infty} \frac{\alpha^k a^k}{k!} z^k. \quad (6)$$
By Proposition 5, we have \( F^1_α \subset H \) and the inclusion is continuous. In particular, \( H \) contains all polynomials and

\[
\|z^k\|_H \leq C \|z^k\|_{F^1_α} \sim \left( \sqrt{\frac{2}{\alpha}} \right)^k \Gamma \left( \frac{k}{2} + 1 \right).
\] (7)

It follows that the series in (6) converges in \( H \) for any fixed \( a \). Therefore,

\[
F(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!} \langle 1, z^k \rangle a^k.
\]

From this, the estimate in (7), and Stirling’s formula, we see that \( F \) is entire.

On the other hand, it follows from condition (b) that \( T_a \) is a unitary operator on \( H \). Since \( T_a T_{-a} = I \) by direct calculations, we have \( T_a^* = T_{-a} \) on \( H \), just as in the case of \( F^2_α \). Furthermore,

\[
e^{-\frac{4}{a^2}} F(a) = \langle 1, k_a \rangle = \langle 1, T_a 1 \rangle = \langle T_{-a} 1, 1 \rangle = \langle k_{-a}, 1 \rangle
\]

\[
= e^{-\frac{4}{a^2}} \langle K_{-a}, 1 \rangle = e^{-\frac{4}{a^2}} \langle 1, K_{-a} \rangle
\]

\[
= e^{-\frac{4}{a^2}} F(-a).
\]

This shows that \( F(z) = F(-z) \) for all \( z \in \mathbb{C}^n \) and hence \( F \) must be constant. Let \( c = \langle 1, 1 \rangle = F(0) > 0 \). Then \( F(z) = c \) for all \( z \in \mathbb{C}^n \).

Let \( a \) and \( b \) be any two points in \( \mathbb{C}^n \). It follows from the observation \( T_a^* = T_{-a} \) and Lemma 1 that

\[
\langle k_a, k_b \rangle = \langle T_a 1, T_b 1 \rangle = \langle 1, T_{-a} T_b 1 \rangle = e^{-\alpha \Im (\overline{a} b)} \langle 1, T_{-a+b} 1 \rangle
\]

\[
= e^{\alpha \Im (\overline{a} b)} e^{-\frac{4}{a^2} - |a-b|^2} F(-a + b)
\]

\[
= ce^{-\frac{4}{a^2} - \frac{4}{b^2} |a-b|^2 + \alpha \overline{a} b}
\]

\[
= c \langle k_a, k_b \rangle \alpha.
\]

This along with the atomic decomposition for \( F^2_α \) gives \( \langle f, g \rangle = c \langle f, g \rangle_α \) for all \( f \) and \( g \) in \( F^2_α \). This shows that \( F^2_α \subset H \) and the inner products \( \langle f, g \rangle_α \) and \( \langle f, g \rangle_H \) only differ by a positive scalar.

We now introduce another approach which will actually give the equality \( F^2_α = H \).

It follows from conditions (b) and (c) that every point evaluation is a bounded linear functional on \( H \). Furthermore, for every compact set \( A \subset \mathbb{C}^n \) there exists a positive constant \( C > 0 \) such that \( |f(z)| \leq C \|f\|_H \) for all \( f \in H \) and all \( z \in A \). This implies that \( H \) possesses a reproducing kernel \( K_H(z, w) \).

It is well known that for any orthonormal basis \( \{e_k\} \) of \( H \) we have

\[
K_H(z, w) = \sum_{k=1}^{\infty} e_k(z) \overline{e_k(w)}
\]
for all \( z \) and \( w \) in \( \mathbb{C}^n \). See [5, 8, 11] for basic information about reproducing Hilbert spaces of analytic functions.

Let \( \{ e_k \} \) be an orthonormal basis for \( H \). Then for any fixed \( a \in \mathbb{C}^n \), the functions
\[
\sigma_k(z) = T_a e_k(z) = e^{\alpha z - \frac{\alpha}{2}|a|^2} e_k(z - a)
\]
also form an orthonormal basis for \( H \). Therefore,
\[
K_H(z, w) = \sum_{k=1}^{\infty} \sigma_k(z) \overline{\sigma_k(w)}
= k_a(z) k_a(w) \sum_{k=1}^{\infty} e_n(z - a) \overline{e_n(w - a)}
= k_a(z) k_a(w) K_H(z - a, w - a).
\]
Let \( z = w = a \). We obtain
\[
K_H(z, z) = e^{\alpha |z|^2} K_H(0, 0), \quad z \in \mathbb{C}^n.
\]
By a well-known result in the function theory of several complex variables, any reproducing kernel is uniquely determined by its values on the diagonal. See [5, 7, 8]. Therefore, if we write \( K(z, w) = e^{\alpha z \overline{w}} \) for the reproducing kernel of \( F_\alpha^2 \), we must have \( K_H(z, w) = c K(z, w) \) for all \( z \) and \( w \), where \( c = K_H(0, 0) > 0 \) as \( H \) contains the constant function 1. This shows that, after an adjustment of the inner product by a positive scalar, the two spaces \( H \) and \( F_\alpha^2 \) have the same reproducing kernel, from which it follows that \( H = F_\alpha^2 \). This completes the proof of the theorem.

The Fock space \( F_\alpha^2 \) is obviously invariant under the action of the full Heisenberg group. Consequently, \( F_\alpha^2 \) is also the unique Hilbert space of entire functions in \( \mathbb{C}^n \) that is invariant under the action of the Heisenberg group, or more precisely, under the action of the previously mentioned unitary representation of the Heisenberg group.

It is not really necessary for us to assume that \( H \) contains all constant functions in Theorem 6. According to the reproducing kernel approach, to ensure that \( K_H(0, 0) > 0 \), all we need to assume is \( H \neq (0) \). In fact, if \( f \) is a function in \( H \) that is not identically zero, then \( f(a) \neq 0 \) for some \( a \). Combining this with translation invariance, we see that \( f(0) \neq 0 \) for some \( f \in H \), so \( K_H(0, 0) > 0 \).

5. ANOTHER LOOK AT BERGMAN AND HARDY SPACES

The reproducing kernel approach used in the proof of Theorem 6 also works for several other situations. We illustrate this using Bergman and Hardy spaces on the unit disk. The generalization to bounded symmetric domains is obvious.
In this section we consider the following weighted area measures on the unit disk $D$,

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z),$$

where $\alpha > -1$ and $dA$ is area measure normalized so that the unit disk has area 1. The Bergman space $A^2_\alpha$ is the closed subspace of $L^2(D, dA_\alpha)$ consisting of analytic functions. Thus $A^2_\alpha$ is a separable Hilbert space with the inherited inner product $\langle f, g \rangle_\alpha$ from $L^2(D, dA_\alpha)$. It is clear that every point evaluation at $z \in D$ is a bounded linear functional on $A^2_\alpha$. The reproducing kernel of $A^2_\alpha$ is given by the function

$$K_\alpha(z, w) = \frac{1}{(1 - zw)^{2+\alpha}}.$$

For any point $a \in \mathbb{D}$ define an operator $U_a$ by

$$U_a f(z) = k_a(z)f(\varphi_a(z)), \quad z \in \mathbb{D},$$

where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ is an involutive Möbius map and

$$k_a(z) = \frac{K_\alpha(z, a)}{\sqrt{K_\alpha(a, a)}} = \frac{(1 - |a|^2)^{(\alpha+2)/2}}{(1 - \overline{a}a)^{\alpha+2}}$$

is the normalized reproducing kernel of $A^2_\alpha$ at the point $a$.

**Theorem 7.** For each $a \in \mathbb{D}$ the operator $U_a$ is unitary on $A^2_\alpha$ with $U_a^{-1} = U_a$. Furthermore, if $H$ is any separable Hilbert space of analytic functions in $\mathbb{D}$ satisfying

(i) $\|U_a f\|_H = \|f\|_H$ for all $f$ in $H$ and $a \in \mathbb{D}$.

(ii) $f \mapsto f(0)$ is a nonzero bounded linear functional on $H$.

Then $H = A^2_\alpha$ and there exists a positive constant $c$ such that $\langle f, g \rangle_H = c\langle f, g \rangle_\alpha$ for all $f$ and $g$ in $H$.

**Proof.** This result was proved in [3]. We now show that it follows from the reproducing kernel approach introduced earlier, without appealing to averaging operations on any subgroup of the Möbius group. In fact, we are not even assuming that $H$ is invariant under the action of rotations.

First observe that condition (i) together with the identity $U_a^2 = I$, which can be checked easily, shows that each $U_a$ is a unitary operator on $H$. Next observe that conditions (i) and (ii) imply that for every $z \in \mathbb{D}$, the point evaluation $f \mapsto f(z)$ is a bounded linear functional on $H$, and if $z$ is restricted to any compact subset of $\mathbb{D}$, then the norm of these bounded linear functionals is uniformly bounded. Therefore, $H$ possesses a reproducing kernel $K_H(z, w)$. Since $f \mapsto f(0)$ is nontrivial, we have $c = K_H(0, 0) > 0$.

Let $\{e_j\}$ be any orthonormal basis for $H$. Then the functions

$$\sigma_j(z) = U_a e_j(z) = k_a(z)e_j(\varphi_a(z))$$
form an orthonormal basis for $H$ as well. Thus
\[
K_H(z, w) = \sum_{j=1}^{\infty} \sigma_j(z) \overline{\sigma_j(w)}
\]
\[
= k_a(z) k_a(w) \sum_{j=1}^{\infty} e_j(\varphi_a(z)) e_j(\varphi_a(w))
\]
\[
= k_a(z) k_a(w) K_H(\varphi_a(z), \varphi_a(w)).
\]
Let $z = w = a$. Then
\[
K_H(z, z) = |k_z(z)|^2 K_H(0, 0) = c K_\alpha(z, z), \quad z \in \mathbb{D}.
\]
Since any reproducing kernel is determined by its values on the diagonal, we have $K_H(z, w) = c K_\alpha(z, w)$ for all $z$ and $w$ in $\mathbb{D}$. This shows that $H = A_\alpha^2$ and $(f, g)_H = c (f, g)_\alpha$ for all $f$ and $g$ in $H$. □

Recall that the Hardy space $H^2$ consists of analytic functions in the unit disk $\mathbb{D}$ such that
\[
\|f\|^2 = \sum_{k=0}^{\infty} |a_k|^2 < \infty, \quad f(z) = \sum_{k=0}^{\infty} a_k z^k.
\]
It is well known that every function in $H^2$ has nontangential limits at almost every point of the unit circle $\mathbb{T}$. Furthermore, when $f$ is identified with its boundary function, we can think of $H^2$ as a closed subspace of $L^2(\mathbb{T}, d\sigma)$, where
\[
d\sigma(\zeta) = \frac{1}{2\pi} d\theta, \quad \zeta = e^{i\theta}.
\]
Thus $H^2$ is a Hilbert space with the inner product inherited from $L^2(\mathbb{T}, d\sigma)$.

**Theorem 8.** For any $a \in \mathbb{D}$ the operator $U_a$ defined by
\[
U_a f(z) = \sqrt{1 - |a|^2} \frac{1}{1 - za} f(\varphi_a(z))
\]
is unitary on $H^2$ with $U_a^* = U_a^{-1} = U_a$. Furthermore, if $H$ is any separable Hilbert space of analytic functions in $\mathbb{D}$ satisfying
(i) $\|U_a f\|_H = \|f\|_H$ for all $f$ in $H$ and $a \in \mathbb{D}$.
(ii) $f \mapsto f(0)$ is a nontrivial bounded linear functional on $H$.
Then $H = H^2$ and there exists a positive constant $c$ such that $(f, g)_H = c (f, g)_{H^2}$ for all $f$ and $g$ in $H$.

**Proof.** This is proved in exactly the same way as Theorem 7 was proved. □
Finally we mention that the reproducing kernel approach here does not seem to work in cases like the Dirichlet space when the following Möbius invariant semi-inner product is used:

$$\langle f, g \rangle = \int_D f'(z)\overline{g'(z)} \, dA(z).$$

In this case, we have $\langle 1, 1 \rangle = \|1\|^2 = 0$. If we somehow make $\|1\| > 0$, then the resulting inner product will no longer be Möbius invariant.

REFERENCES

[1] J. Arazy and S. Fisher, Some aspects of the minimal, Möbius invariant space of analytic functions on the unit disk, *Springer Lecture Notes in Math.* **1070** (1984), Springer, New York, 24-44.
[2] J. Arazy and S. Fisher, The uniqueness of the Dirichlet space among Möbius invariant Hilbert spaces, *Illinois J. Math.* **29** (1985), 449-462.
[3] J. Arazy and S. Fisher, Invariant Hilbert spaces of analytic functions on bounded symmetric domains, in *Topics in Operator Theory*, edited by L. de Branges, I. Gohberg, and J. Rovnyak, *Operator Theory: Advances and Applications* **48**, 1990, 67-91.
[4] J. Arazy, S. Fisher, and J. Peetre, Möbius invariant function spaces, *J. Reine Angew. Math.* **363** (1985), 110-145.
[5] N. Aronszajn, Theory of reproducing kernels, *Trans. Amer. Math. Soc.* **68** (1950), 337-404.
[6] S. Janson, J. Peetre, and R. Rochberg, Hankel forms and the Fock space, *Revista Mat. Ibero-Amer.* **3** (1987), 61-138.
[7] S. Krantz, *Function Theory of Several Complex Variables* (2nd edition), Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, California, 1992.
[8] S. Saitoh, *Theory of Reproducing Kernels and Its Applications*, Pitman Research Notes in Mathematics **189**, 1988.
[9] R. Wallstén, The $S^p$-criterion for Hankel forms on the Fock space, $0 < p < 1$, *Math. Scand.* **64** (1989), 123-132.
[10] K. Zhu, Möbius invariant Hilbert spaces of holomorphic functions on the unit ball of $\mathbb{C}^n$, *Trans. Amer. Math. Soc.* **323** (1991), 823-842.
[11] K. Zhu, *Operator Theory in Function Spaces* (2nd edition), American Mathematical Society, 2007.