INDUCTIVE CONSTRUCTION OF THE $p$-ADIC ZETA FUNCTIONS FOR NON-COMMUTATIVE $p$-EXTENSIONS OF TOTALLY REAL FIELDS WITH EXPONENT $p$

TAKASHI HARA

Abstract. We construct the $p$-adic zeta function for a one-dimensional (as a $p$-adic Lie extension) non-commutative $p$-extension $F_\infty$ of a totally real number field $F$ such that the finite part of its Galois group $G$ is a $p$-group with exponent $p$. We first calculate the Whitehead groups of the Iwasawa algebra $\Lambda(G)$ and its canonical Ore localisation $\Lambda(G)_S$ by using Oliver-Taylor’s theory upon integral logarithms. This calculation reduces the existence of the non-commutative $p$-adic zeta function to certain congruence conditions among abelian $p$-adic zeta pseudomeasures. Then we finally verify these congruences by using Deligne-Ribet’s theory and certain inductive technique. As an application we shall prove a special case of (the $p$-part of) the non-commutative equivariant Tamagawa number conjecture for critical Tate motives. The main results of this paper give generalisation of those of the preceding paper of the author.

0. Introduction

One of the most important topics in non-commutative Iwasawa theory is to construct the $p$-adic zeta function and to verify the main conjecture, as well as in classical theory. Up to the present, there have been several successful examples upon this topic for $p$-adic Lie extensions of totally real number fields: the results of Jürgen Ritter and Alfred Weiss [RW7], Kazuya Kato [Kato2], Mahesh Kakde [Kakde1] and so on. In this article, we shall construct different type of example for certain non-commutative $p$-extensions of totally real number fields.

Let $p$ be a positive odd prime number and $F$ a totally real number field. Let $F_\infty$ be a totally real $p$-adic Lie extension of $F$ which contains the cyclotomic $\mathbb{Z}_p$-extension $F_{\text{cyc}}$ of $F$, and assume that all but finitely many primes of $F$ ramify in $F_\infty$. For a moment we admit Iwasawa’s $\mu = 0$ conjecture to simplify conditions (see Section 1.1 (F_\infty-3) for more general $\mu = 0$ condition). The aim of this article is to prove the following theorem under these conditions:

**Theorem 0.1** (=Theorem 3.1). Let $G$ denote the Galois group of $F_\infty/F$. Then for $F_\infty/F$ the $p$-adic zeta function $\xi_{F_\infty/F}$ exists and the Iwasawa main
conjecture is true if $G$ is isomorphic to the direct product of a finite $p$-group $G'$ with exponent $p$ and the abelian $p$-adic Lie group $\Gamma$ isomorphic to $\mathbb{Z}_p$.

We shall review the characterisation of the $p$-adic zeta function and the precise statement of the non-commutative Iwasawa main conjecture in Section 1.1. In the preceding paper [H], we constructed the $p$-adic zeta function and verified the main conjecture when the Galois group $\text{Gal}(F'_\infty/F)$ is isomorphic to the pro-$p$ group

$$
\begin{pmatrix}
1 & \mathbb{F}_p & \mathbb{F}_p \\
0 & 1 & \mathbb{F}_p \\
0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix} \times \Gamma
$$

and $p$ is not equal to either 2 or 3. Theorem 0.1 generalises this result.

Philosophically the Iwasawa main conjecture is closely related to the special values of $L$-functions (as implied by many people including Kazuya Kato, Annette Huber-Klawitter, Guido Kings, David Burns, Mathias Flach,.....); hence our main theorem (Theorem 0.1) should also suggest validity of conjectures upon these values in some sense even in non-commutative coefficient cases. In fact we may verify a special case of (the $p$-part of) the equivariant Tamagawa number conjecture for critical Tate motives with non-commutative coefficient combining Theorem 0.1 with descent theory established by David Burns and Otmar Venjakob [BurVen].

**Corollary 0.2 (=Corollary 3.7).** Let $F'_\infty$ be a $p$-adic Lie extension of a totally real number field $F$ as in Theorem 0.1 and $F'$ an arbitrary finite Galois subextension of $F'_\infty/F$. Then for an arbitrary natural number $r$ divisible by $p - 1$, the $p$-part of the equivariant Tamagawa number conjecture for $\mathbb{Q}(1-r)_{F'/F}$ is true (here $\mathbb{Q}(1-r)_{F'/F} = h^0(\text{Spec} F')(1-r)$ denotes the $(1-r)$-fold Tate motive regarded as defined over $F$).

This may also be regarded as an analogue of the cohomological Lichtenbaum conjecture (in special cases), which was proven by Barry Mazur and Andrew Wiles [MazWil, Wiles] when $F'$ is the same field as $F$—the Bloch-Kato conjecture case—as the direct consequence of the main conjecture (for commutative cases) which they verified. Later we shall give a brief review upon the formulation of the equivariant Tamagawa number conjecture for Tate motives (see Section 1.2).

Now let us summarise the main idea to prove Theorem 0.1. Consider the family $\mathfrak{F}_B$ of all pairs $(U, V)$ such that $U$ is an open subgroup of $G$ containing $\Gamma$ and $V$ is the commutator subgroup of $U$. By classical induction theorem of Richard Brauer [Serre1, Théorème 22], an arbitrary Artin representation of $G$ is isomorphic to a $\mathbb{Z}$-linear combination of representations induced by characters of abelian groups $U/V$ (as a virtual representation) where each $(U, V)$ is in $\mathfrak{F}_B$. Let $\theta_{U,V}$ and $\theta_{S,U,V}$ denote the composite maps

$$
K_1(\Lambda(G)) \xrightarrow{\text{norm}} K_1(\Lambda(U)) \xrightarrow{\text{canonical}} \Lambda(U/V)^\times, \\
K_1(\Lambda(G)_S) \xrightarrow{\text{norm}} K_1(\Lambda(U)_S) \xrightarrow{\text{canonical}} \Lambda(U/V)_S
$$

for each $(U, V)$ in $\mathfrak{F}_B$ where $\Lambda(G)_S$ (resp. $\Lambda(U)_S$, $\Lambda(U/V)_S$) is the canonical Ore localisation of the Iwasawa algebra $\Lambda(G)$ (resp. $\Lambda(U)$, $\Lambda(U/V)$)
introduced in [CFKSV Section 2] (see also Section 1 in this article). Set \( \theta = (\theta_U V)(U,V) \in \mathfrak{F}_B \) and \( \theta_S = (\theta_{S,U,V})(U,V) \in \mathfrak{F}_B \).

Let \( F_U \) (resp. \( F_V \)) be the maximal subfield of \( F_\infty \) fixed by \( U \) (resp. \( V \)). Then the \( p \)-adic zeta function exists for each abelian extension \( F_V/F_U \); Pierre Deligne and Kenneth Alan Ribet first constructed it [DR], and by using their results Jean-Pierre Serre reconstructed it as a unique element \( \xi_{U,V} \) in the total quotient ring of \( \Lambda(U') \) which satisfies certain interpolation formulae [Serre2]. Now suppose that there exists an element \( \xi \) in \( K_1(\Lambda(G)_S) \) which satisfies the equation

\[
\theta_S(\xi) = (\xi_U V)(U,V) \in \mathfrak{F}_B.
\]

Then we may verify by Brauer induction that \( \xi \) satisfies an interpolation formula which characterises \( \xi \) as the \( p \)-adic zeta function for \( F_\infty/F \). This observation motivates us to prove that \( (\xi_U V)(U,V) \in \mathfrak{F}_B \) to be contained in the image of \( \theta_S \). It seems, however, to be difficult in general to characterise the image of the theta map \( \theta_S \) completely for the localised Iwasawa algebra \( \Lambda(G)_S \). Therefore we shall first determine the image of the theta map \( \theta \) for the (integral) Iwasawa algebra \( \Lambda(G) \), and then construct an element \( \xi \) satisfying (0.1) by using this calculation and certain diagram chasing. The strategy which we introduced here was first proposed by David Burns (and hence we call this method Burns’ technique in this article). We shall discuss its details in Section 2.

Let \( (U,\{e\}) \) be an element in \( \mathfrak{F}_B \) such that the cardinality of the finite part of \( U \) is at most \( p^2 \) (and \( U \) is hence abelian). Let \( I_{S,U} \) denote the image of \( \theta_{S,U,\{e\}} \) for each of such \( (U,\{e\}) \)'s. By virtue of Burns’ technique, we may reduce the condition for \( (\xi_{U,V})(U,V) \in \mathfrak{F}_B \) to be contained in the image of \( \theta_S \) to the following type of congruence:

\[
\xi_{U,\{e\}} \equiv \varphi(\xi_{G,\Gamma})^{(G,U)}/p \mod I_{S,U}
\]

where \( \varphi \) is the Frobenius endomorphism \( \varphi: \Lambda(G_{ab})_S \to \Lambda(\Gamma)_p \) induced by the group homomorphism \( G_{ab} \to \Gamma; g \mapsto g^p \). Kato, Ritter, Weiss and Kakde verified such type of congruence when the index \( (G : U) \) exactly equals \( p \) [Kato2, RW6, Kakde1] by using the theory of Deligne and Ribet upon Hilbert modular forms [DR]. It seems, however, to be almost impossible to deduce such congruences only from Deligne-Ribet’s theory when the index \( (G : U) \) is strictly larger than \( p \). Nevertheless in Sections 8 and 9 we shall verify these congruences by combining Deligne-Ribet’s theory with certain inductive technique which was first introduced in [H].

In computation of the images of \( \theta \) and \( \theta_S \) we use theory upon \( p \)-adic logarithmic homomorphisms. This causes ambiguity upon \( p \)-power torsion elements in the whole calculation, and hence we have to eliminate this ambiguity as the final step of the proof. We shall complete this step by utilising the existence of the \( p \)-adic zeta functions for Ritter-Weiss-type extensions [RW7] and certain inductive arguments.

The detailed content of this article is as follows. We shall briefly review the basic formulations of the non-commutative Iwasawa main conjecture and the equivariant Tamagawa number conjecture in Section 1. Then we discuss David Burns’ outstanding strategy for construction of the \( p \)-adic zeta
function in Section 2. The precise statement of our main theorem and its application will be dealt with in Section 3. Sections 4, 5 and 6 are devoted to computation of the image of (a certain variant of) the theta map \( \tilde{\theta} \); we first construct “the additive theta isomorphism” \( \theta^+ \) in Section 4, and then translate it into the multiplicative morphism \( \tilde{\theta} \) by utilising logarithmic homomorphisms in Section 6. Section 5 is the preliminary section for Section 6. We study the image of the norm map \( \tilde{\theta}_S \) for the localised Iwasawa algebra \( \Lambda(G)_S \) in Section 7 and derive certain “weak congruences” upon abelian \( p \)-adic zeta pseudomeasures in Section 8 by applying Deligne-Ribet’s \( q \)-expansion principle [DR] and Ritter-Weiss’ approximation technique [RW6]. In Section 9 we refine the congruences obtained in the previous section by using induction, and construct the \( p \)-adic zeta function “modulo \( p \)-torsion.” We shall finally eliminate ambiguity of the \( p \)-power torsion part.

**Contents**

0. Introduction 1  
1. Reviews upon non-commutative arithmetic theory 5  
2. Burns’ technique 11  
3. The main theorem and its application 13  
4. Construction of the theta isomorphism I —additive theory— 18  
5. Preliminaries for logarithmic translation 23  
6. Construction of the theta isomorphism II —translation— 28  
7. Localized version 33  
8. Weak congruences upon abelian \( p \)-adic zeta functions 35  
9. Inductive construction of the \( p \)-adic zeta functions 41  
References 50

**Notation.** In this article, \( p \) always denotes a positive prime number. We denote by \( \mathbb{N} \) the set of natural numbers (the set of strictly positive integers). We also denote by \( \mathbb{Z} \) (resp. \( \mathbb{Z}_p \)) the ring of integers (resp. \( p \)-adic integers). The symbol \( \mathbb{Q} \) (resp. \( \mathbb{Q}_p \)) denotes the rational number field (resp. the \( p \)-adic number field). For an arbitrary group \( G \) let \( \text{Conj}(G) \) denote the set of all conjugacy classes of \( G \). For an arbitrary pro-finite group \( P \), we always denote by \( \Lambda(P) \) its Iwasawa algebra over \( \mathbb{Z}_p \) and by \( \Omega(P) \) its Iwasawa algebra over \( \mathbb{F}_p \). More specifically, \( \Lambda(P) \) and \( \Omega(P) \) are defined by

\[
\Lambda(P) = \lim_{\leftarrow U} \mathbb{Z}_p[P/U], \quad \Omega(P) = \lim_{\leftarrow U} \mathbb{F}_p[P/U]
\]

where \( U \) runs over all open normal subgroups of \( P \). Let \( \Gamma \) denote the commutative \( p \)-adic Lie group isomorphic to \( \mathbb{Z}_p \) (corresponding to the Galois group of the cyclotomic \( \mathbb{Z}_p \)-extension). Throughout this paper, we fix a topological generator \( \gamma \) of \( \Gamma \). In other words, we fix Iwasawa-Serre isomorphisms

\[
\Lambda(\Gamma) \xrightarrow{\sim} \mathbb{Z}_p[[T]], \quad \Omega(\Gamma) \xrightarrow{\sim} \mathbb{F}_p[[T]]
\]

\[
\gamma \mapsto 1 + T, \quad \gamma \mapsto 1 + T
\]

where \( \mathbb{Z}_p[[T]] \) (resp. \( \mathbb{F}_p[[T]] \)) is the formal power series ring over \( \mathbb{Z}_p \) (resp. \( \mathbb{F}_p \)). For an arbitrary \( p \)-adic Lie group \( W \) isomorphic to the direct product of a
finite $p$-group and $\Gamma$, $W^f$ denotes the finite part of $W$. We always assume that every associative ring has unity. The centre of an associative ring $R$ is denoted by $Z(R)$. For an associative ring $R$, we denote by $M_n(R)$ the ring of $n \times n$-matrices with entries in $R$ and by $GL_n(R)$ the multiplicative group of $M_n(R)$. We always consider that all Grothendieck groups are additive abelian groups, whereas all Whitehead groups are multiplicative abelian groups. For an arbitrary multiplicative abelian group $A$, let $A_{p\text{-tors}}$ (resp. $A_{\text{tors}}$) denote the $p$-power torsion part (resp. the torsion part) of $A$. We set $\tilde{\Lambda}(G) = \Lambda(G)/\Lambda(G)_{p\text{-tors}}$ for an arbitrary associative ring $G$. Similarly, we set $\tilde{\Lambda}(P) = \Lambda(P)/\Lambda(P)_{p\text{-tors}}$ for an arbitrary pro-finite group $P$.

Acknowledgements. The author would like to express his sincere gratitude to Professor Takeshi Tsuji for much fruitful discussion and many helpful comments (especially the suggestion that we use augmentation ideals in the translation of the additive theta isomorphism, see Section 5.1 for details). He is also grateful to Mahesh Kakde for useful comments upon the preliminary version of this article.

1. Reviews upon non-commutative arithmetic theory

In this section we shall review the formulations of the non-commutative Iwasawa main conjecture (only for the cases of totally real number fields) and the equivariant Tamagawa number conjecture (only for the Tate motives). Refer to [Bass, Swan] for basic results upon (low-dimensional) algebraic $K$-theory used in this section.

1.1. Non-commutative Iwasawa theory for totally real fields. In this subsection we review the formulation of the non-commutative Iwasawa main conjecture for totally real number fields following John Henry Coates, Takako Fukaya, Kazuya Kato, Ramdorai Sujatha and Otmar Venjakob [CFKSV, FukKat]. We remark that Jürgen Ritter and Alfred Weiss also formulated the non-commutative Iwasawa main conjecture — “the ‘main conjecture’ of equivariant Iwasawa theory” in their terminology — in somewhat different manner [RW1, RW2, RW3, RW4]. Let $F$ be a totally real number field and $p$ a positive odd prime number. Let $F_\infty$ be a totally real $p$-adic Lie extension of $F$ satisfying the following three conditions:

1. The cyclotomic $\mathbb{Z}_p$-extension $F_{\text{cyc}}$ of $F$ is contained in $F_\infty$;
2. Only finitely many primes of $F$ ramify in $F_\infty$;
3. There exists a finite subextension $F'$ of $F_\infty/F$ such that $F_\infty/F'$ is pro-$p$ and the Iwasawa $\mu$-invariant for its cyclotomic $\mathbb{Z}_p$-extension $F'_{\text{cyc}}/F'$ equals zero.

Fix a finite set $\Sigma$ of finite places of $F$ which contains all of those ramifying in $F_\infty$. For an arbitrary algebraic extension $E$ of $F$, we shall denote $\Sigma_E$ the union of the set $\Sigma_\infty$ of all infinite places and the set $\Sigma_F$ of all finite places above $\Sigma$. Set $G = \text{Gal}(F_\infty/F)$, $H = \text{Gal}(F_\infty/F_{\text{cyc}})$ and $\Gamma = \text{Gal}(F_{\text{cyc}}/F)$. The pro-$p$ group $\Gamma$ is isomorphic to $\mathbb{Z}_p$ by definition.

Let $S$ be the subset of $\Lambda(G)$ consisting of all elements $f$ such that the quotient module $\Lambda(G)/\Lambda(G)f$ is finitely generated as a left $\Lambda(H)$-module. The set $S$ is a left and right Ore set of $\Lambda(G)$ with no zero divisors [CFKSV, Theorem 2.4], which is called the canonical Ore set for $F_\infty/F$ (refer to
The Ore localisation $\Lambda(G) \to \Lambda(G)_S$ induces the following localisation exact sequence in algebraic $K$-theory (due to Charles Weibel, Dongyuan Yao, Alan Johnathan Berrick and Michael Keating [WeibYao, BerKeat]):

$$K_1(\Lambda(G)) \to K_1(\Lambda(G)_S) \xrightarrow{\partial} K_0(\Lambda(G), \Lambda(G)_S) \to 0.$$ 

Surjectivity of the connecting homomorphism $\partial$ was proven in [CFKSV, Proposition 3.4]. Now let $\mathcal{C}^{\text{perf}}(\Lambda(G))$ denote the category of perfect complexes of finitely generated left $\Lambda(G)$-modules (that is, the category of complexes of finitely generated left $\Lambda(G)$-modules which are quasi-isomorphic to bounded complexes of finitely generated projective left $\Lambda(G)$-modules), and let $\mathcal{C}_S^{\text{perf}}(\Lambda(G))$ denote the full subcategory of $\mathcal{C}^{\text{perf}}(\Lambda(G))$ generated by all objects whose cohomology groups are $S$-modules. The category $\mathcal{C}_S^{\text{perf}}(\Lambda(G))$ (resp. $\mathcal{C}_S^{\text{quad}}(\Lambda(G))$) is regarded as a Waldhausen category (resp. $\mathcal{C}_S^{\text{quad}}(\Lambda(G))$, $\text{qis}$) equipped with weak equivalences consisting of all quasi-isomorphisms. Then it is well known that the relative Grothendieck group $K_0(\Lambda(G), \Lambda(G)_S)$ is canonically identified with the Waldhausen-Grothendieck group $K_0(\mathcal{C}_S^{\text{perf}}(\Lambda(G)), \text{qis})$. Set

$$C_{F_{\infty}/F} = R\Gamma_{\text{et}}(\text{Spec} \mathcal{O}_{F_{\infty}}^{\Sigma'}, \mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p)$$

where $\Gamma_{\text{et}}$ denotes the global section functor for the étale topology and $\mathcal{O}_{F_{\infty}}^{\Sigma'}$ denotes the $\Sigma'$-integer ring of $F_{\infty}$. Its cohomology groups are calculated as follows:

$$(1.1) \quad H^i(C_{F_{\infty}/F}) = \begin{cases} 
\mathbb{Z}_p & \text{if } i = 0, \\
X_\Sigma & \text{if } i = -1, \\
0 & \text{otherwise}.
\end{cases}$$

Here $X_\Sigma = \text{Gal}(M_\Sigma/F_{\infty})$ is the Galois group of the maximal abelian pro-$p$ extension $M_\Sigma$ of $F_{\infty}$ unramified outside $\Sigma$. Note that $\mathbb{Z}_p$ is an $S$-torsion module since it is finitely generated as a left $\Lambda(H)$-module (see [CFKSV, Proposition 2.3] for details). The Galois group $X_\Sigma$ is also an $S$-torsion module by condition (1.1.3) (due to the lemma of Yoshitaka Hachimori and Romyar Sharifi [HachShar, Lemma 3.4], see also [H, Section 1.2]). Therefore we may regard $C_{F_{\infty}/F}$ as an object of $\mathcal{C}_S^{\text{perf}}(\Lambda(G))$, and by surjectivity of $\partial$ there exists an element $f_{F_{\infty}/F}$ in $K_1(\Lambda(G)_S)$ satisfying

$$(1.2) \quad \partial(f_{F_{\infty}/F}) = -[C_{F_{\infty}/F}],$$

which is called a characteristic element for $F_{\infty}/F$. Characteristic elements are determined uniquely up to multiplication by elements in the image of the canonical homomorphism $K_1(\Lambda(G)) \to K_1(\Lambda(G)_S)$ (due to the localisation exact sequence).

We next consider the $p$-adic zeta function for $F_{\infty}/F$. From now on we fix an algebraic closure of the $p$-adic number field $\overline{\mathbb{Q}}_p$, and we also fix embeddings of the algebraic closure $\overline{\mathbb{Q}}$ of the rational number field $\mathbb{Q}$ into $\mathbb{C}$ — the complex number field — and $\overline{\mathbb{Q}}_p$ till the end of this subsection. By the argument in [CFKSV, p. 172–173], we may define the evaluation map

$$K_1(\Lambda(G)_S) \to \overline{\mathbb{Q}}_p \cup \{\infty\}; f \mapsto f(\rho)$$
for an arbitrary continuous representation \( \rho: G \to \text{GL}_d(O) \) (where \( O \) is the ring of integers of a certain finite extension of \( \mathbb{Q}_p \)). Now let \( L_\Sigma(s; F_\infty/F, \rho) \) be the complex Artin \( L \)-function associated to an Artin representation \( \rho \) (recall that \( \rho \) is an Artin representation if its image is finite) whose local factors at places belonging to \( \Sigma \) are removed. If there exists an element \( \xi_{F_\infty/F} \) in \( K_1(\Lambda(G)_S) \) which satisfies the interpolation formula
\[
\xi_{F_\infty/F}(\rho^r) = L_\Sigma(1 - r; F_\infty/F, \rho)
\]
for an arbitrary Artin representation \( \rho \) of \( G \) and an arbitrary natural number \( r \) divisible by \( p - 1 \), we call \( \xi_{F_\infty/F} \) the \( p \)-adic zeta function for \( F_\infty/F \). The Iwasawa main conjecture for totally real number fields is then formulated as follows:

**Conjecture 1.1.** Let \( p, F \) and \( F_\infty/F \) be as above. Then

1. (existence and uniqueness of the \( p \)-adic zeta function) the \( p \)-adic zeta function \( \xi_{F_\infty/F} \) for \( F_\infty/F \) exists uniquely;
2. (the Iwasawa main conjecture) the equation \( \partial(\xi_{F_\infty/F}) = -[C_{F_\infty/F}] \) holds.

### 1.2. The equivariant Tamagawa number conjecture for Tate motives.

The Tamagawa number conjecture, which predicts the special values of the \( L \)-functions associated to motives in terms of “motivic Tamagawa numbers,” was first formulated by Spencer Bloch and Kazuya Kato [BlKat] following the earlier results of Pierre Deligne [Deligne1] and Alexander Beilinson [Beilinson]. Then Jean-Marc Fontaine, Berdenette Perrin-Riou [FPR] and Kazuya Kato [Kato1] gave its reformulation in a somewhat sophisticated way. The equivariant version of the Tamagawa number conjecture was finally formulated by David Burns and Mathias Flach [BurFl3] which included cases with non-commutative coefficient. In this subsection we shall give a brief review of the formulation of the equivariant Tamagawa number conjecture for Tate motives.

First recall the notion of the determinant functor
\[
\det_R: \mathcal{C}^{\text{Perf}}(R)_{\text{qis}} \to V(R)
\]
which was constructed by Pierre Deligne [Deligne2] for an arbitrary associative ring (and by Finn Faye Knudsen and David Mumford [KnudMum] when \( R \) is commutative) where \( \mathcal{C}^{\text{Perf}}(R)_{\text{qis}} \) denotes the subcategory of \( \mathcal{C}^{\text{Perf}}(R) \) —the category of perfect complexes of finitely generated left \( R \)-modules— whose objects are the same as \( \mathcal{C}^{\text{Perf}}(R) \) and morphisms are restricted to quasi-isomorphisms. The (small) Picard category \( V(R) \) constructed by Deligne is called the category of virtual objects associated to \( R \), which satisfies associative and commutative constraints. In fact Deligne has constructed the category of virtual objects for an arbitrary exact category in [Deligne2] by using Daniel Quillen’s \( S \)-construction [Quillen], but we shall omit the details. Here we just remark that Takako Fukaya and Kazuya Kato gave an alternative and direct construction of \( V(R) \) [FukKat]. In this article we shall never use the explicit description of \( V(R) \).

The determinant functor \( \det_R \) enjoys the following properties (here we denote the product structure of the Picard category \( V(R) \) by \( \cdot \)):

...
I) for an arbitrary exact sequence $0 \to C' \to C \to C'' \to 0$ in the category $\text{C}^{\text{Perf}}(R)$, the determinant functor $\det_R$ induces a canonical isomorphism

$$\det_R(C) \xrightarrow{\sim} \det_R(C') \cdot \det_R(C'')$$

in $V(R)$, which is functorial and satisfies so-called “the 9-terms relation;”

II) if an object $C$ of $\text{C}^{\text{Perf}}(R)$ is acyclic, the quasi-isomorphism $0 \to C$ induces a canonical isomorphism $1_R \xrightarrow{\sim} \det_R(C)$ where $1_R = \det_R(0)$ denotes the unit object of the Picard category $V(R)$;

III) the equation $\det_R(C[r]) = \det_R^{(-1)^r}(C)$ holds for an arbitrary object $C$ of $\text{C}^{\text{Perf}}(R)$ and an arbitrary integer $r$. Here $C[r]$ is the $r$-th translation of $C$ (that is, the cochain complex defined by $C[r]^i = C^{i+r}$ with an appropriate differential);

IV) the functor $\det_R$ factorises $\mathcal{D}^{\text{Perf}}(R)$, the image of $\text{C}^{\text{Perf}}(R)$ in the derived category $\mathcal{D}^{b}(R)$ of bounded complexes of finitely generated left $R$-modules. Moreover it extends to $\mathcal{D}^{\text{Perf}}(R)_{\text{isom}}$, the subcategory of $\mathcal{D}^{\text{Perf}}(R)$ whose morphisms are restricted to isomorphisms;

V) if an object $C$ of $\text{C}^{\text{Perf}}(R)$ is cohomologically perfect (that is, if all cohomology groups $H^i(C)[0]$ belongs to $\mathcal{D}^{\text{Perf}}(R)$), the equation

$$\det_R(C) = \prod_{i \in \mathbb{Z}} \det_R^{(-1)^i}(H^i(C))$$

holds;

VI) the determinant functor $\det_R$ is stable under arbitrary base change; that is, the diagram

$$
\begin{array}{ccc}
\mathcal{D}^{\text{Perf}}(R)_{\text{isom}} & \xrightarrow{\det_R} & V(R) \\
\downarrow \text{R'} \otimes^{\mathbb{L}}_{R} \downarrow & \text{~} & \downarrow \text{R'} \otimes^{\mathbb{L}}_{R} \\
\mathcal{D}^{\text{Perf}}(R')_{\text{isom}} & \xrightarrow{\det_{R'}} & V(R')
\end{array}
$$

commutes for an arbitrary $R$-algebra $R'$.

We denote the group of isomorphism classes of $V(R)$ by $\pi_0(V(R))$ and the group of isomorphisms of the unit object $1_R$ by $\pi_1(V(R))$ respectively. Then there exist canonical isomorphisms

$$K_0(R) \xrightarrow{\sim} \pi_0(V(R)); \quad [P] \mapsto [\det_R(P)],$$

$$K_1(R) \xrightarrow{\sim} \pi_1(V(R)); \quad [f : P \xrightarrow{\sim} P] \mapsto [1_R \xrightarrow{\det_R(f)} 1_R]$$

where we abbreviate to $\det_R(f)$ the isomorphism

$$1_R = \det_R(P) \cdot \det_R^{-1}(P) \xrightarrow{\det_R(f) \cdot \text{id} \cdot \det_R^{-1}(P)} \det_R(P) \cdot \det_R^{-1}(P) = 1_R$$

induced by $f$. For an arbitrary $R$-algebra $R'$, we define the category $V(R, R')$ as the fibre-product category $V(R) \times_{V(R')} V_0$ if we let $V_0$ denote the trivial Picard category consisting of a unique object 1 equipped with the trivial automorphism group $\{\text{id}_1\}$; in particular an object of $V(R, R')$ is a pair $(L, \lambda)$ with $L$ an object of $V(R)$ and $\lambda$ an isomorphism $R' \otimes_R L \xrightarrow{\sim} 1_{R'}$ in $V(R')$. 

(we call $\lambda$ a trivialisation of $R' \otimes_R L$). In fact the category $V(R, R')$ is a Picard category and there exists a canonical isomorphism between $K_0(R, R')$ and $\pi_0(V(R, R'))$. For details see [BrBu1, Section 5].

Next let $F$ be a number field and $F'$ its finite Galois extension with the Galois group $G_{F'/F}$. For an arbitrary strictly negative integer $m$, consider the $m$-fold Tate motive $Q(m)_{F'/F} = h^0(\text{Spec } F')(m)$ endowed with equivariant action of the Galois group $G_{F'/F}$ (regarded as defined over $F$). The Betti realisation of $Q(m)_{F'/F}$ is explicitly described as

$$Q(m)_{F'/F,B} = \prod_{\tau: F' \to \mathbb{C}} (2\pi \sqrt{-1})^m$$

where $\tau$ runs over all embeddings of $F'$ into $\mathbb{C}$. Note that the complex conjugate acts upon both $\tau$'s and $(2\pi \sqrt{-1})^m$ in a natural way. Let $Q(m)_{F'/F,B}$ denote the maximal submodule of $Q(m)_{F'/F,B}$ fixed by the action of the complex conjugate and set

$$\Xi(Q(m)_{F'/F}) = \det_{Q[G_{F'/F}]}(K_{1-2m}(F')_0^\bullet) \cdot \det_{Q[G_{F'/F}]}^{-1}(Q(m)_{F'/F,B}^+)$$

where we denote by $X^*$ the $\mathbb{Q}$-dual of a left $Q[G_{F'/F}]-$projective module $X$ endowed with contragredient $G_{F'/F}$-action (the virtual object $\Xi(Q(m)_{F'/F})$ is nothing but the fundamental line for the Tate motive in the terminology of [PPR]). As is well known, Beilinson’s regulator map (or Borel’s regulator map) induces an isomorphism between $K_{1-2m}(F')_0^\bullet \otimes \mathbb{Z} \mathbb{R}$ and the $\mathbb{R}$-dual of $Q(m)_{F'/F,B} \otimes \mathbb{Q} \mathbb{R}$; hence we obtain an isomorphism in $V(\mathbb{R}[G_{F'/F}])$ (the period-regulator isomorphism in the terminology of [FukKat])

$$\vartheta_\infty: \Xi(Q(m)_{F'/F}) \otimes \mathbb{Q} \mathbb{R} \cong 1_{\mathbb{R}[G_{F'/F}]}.$$

We may regard the pair $(\Xi(Q(m)_{F'/F}), \vartheta_\infty)$ as the object of the relative Picard category $V(\mathbb{Q}[G_{F'/F}], \mathbb{R}[G_{F'/F}]).$

Now let $R\Gamma_{c,\et}(\mathcal{O}_{F'}^{\Sigma'}, \mathbb{Q}_p(m))$ (or $R\Gamma_{c,\et}(\text{Spec } \mathcal{O}_{F'}^{\Sigma'}, \mathbb{Q}_p(m))$) be the compactly supported étale cohomology of $\text{Spec } \mathcal{O}_{F'}^{\Sigma'}$ with coefficient in $\mathbb{Q}_p(m)$ (regarded as an object of the derived category) characterised by the distinguished triangle

$$R\Gamma_{c,\et}(\mathcal{O}_{F'}^{\Sigma'}, \mathbb{Q}_p(m)) \to R\Gamma_{\et}(\mathcal{O}_{F'}^{\Sigma'}, \mathbb{Q}_p(m)) \to \bigoplus_{\varphi \in \Sigma'} R\Gamma(G_{F'\varphi}, \mathbb{Q}_p(m)) \to$$

where $R\Gamma(G_{F'\varphi}, \mathbb{Q}_p(m))$ denotes the $\mathbb{Q}_p(m)$-coefficient Galois cohomology of the decomposition group of $G_{F'\varphi}$ at $\varphi$. Then we may construct an isomorphism in $V(\mathbb{Q}_p[G_{F'/F}])$ (the $p$-adic period-regulator isomorphism in the terminology of [FukKat])

$$\vartheta_p: \Xi(Q(m)_{F'/F}) \otimes \mathbb{Q}_p \cong \det_{Q[G_{F'/F}]}(R\Gamma_{c,\et}(\mathcal{O}_{F'}^{\Sigma'}, \mathbb{Q}_p(m)))$$

which is essentially derived from the $p$-adic Chern class isomorphism

$$K_{1-2m}(\mathcal{O}_{F'}) \otimes \mathbb{Z} \mathbb{Q}_p \cong H^1(\mathcal{O}_{F'}/1/p), \mathbb{Q}_p(m)).$$

Refer to [BurFi1] Sections 1.2.2 and 1.6] or [BurFi2] Section 2] for details.

We finally consider the leading terms of the equivariant Artin $L$-functions. Let $\text{Irr}(G_{F'/F})$ denote the set of all isomorphism classes of irreducible finite
dimensional ($\mathbb{Q}$-valued) representations of $G_{F'/F}$. Then by Wedderburn’s decomposition theorem we obtain an isomorphism

$$\left(1.5\right) \quad Z(\mathbb{C}[G_{F'/F}]) \xrightarrow{\sim} \prod_{\rho \in \text{Irr}(G_{F'/F})} \mathbb{C} \cdot \text{id}_{V_{\rho}}; \quad x \mapsto (\rho(x))_{\rho \in \text{Irr}(G_{F'/F})}$$

where $V_{\rho}$ denotes the representation space of $\rho$. We identify the right-hand side of (1.5) with the direct product $\prod_{\rho \in \text{Irr}(G_{F'/F})} \mathbb{C}$. Then we may define the leading term of the equivariant Artin $L$-function $L^{*}(\mathbb{Q}(m)_{F'/F})$ associated to the Tate motive $\mathbb{Q}(m)_{F'/F}$ as the element in $Z(\mathbb{C}[G_{F'/F}])^{\times}$ corresponding to $(L^{*}(m, \rho))_{\rho \in \text{Irr}(G_{F'/F})}$ via the isomorphism (1.5) (we denote by $L^{*}(m, \rho)$ the leading term of the complex Artin $L$-function $L(s; F'/F, \rho)$ at $s = m$). In fact $L^{*}(\mathbb{Q}(m)_{F'/F})$ is contained in $Z(\mathbb{R}[G_{F'/F}])^{\times}$ and there exists an element $\lambda$ in $Z(\mathbb{Q}[G_{F'/F}])^{\times}$ such that $\lambda L^{*}(\mathbb{Q}(m)_{F'/F})$ is contained in the image of the reduced norm map

$$\text{nrd}_{\mathbb{R}[G_{F'/F}]}: K_{1}(\mathbb{R}[G_{F'/F}]) \to Z(\mathbb{R}[G_{F'/F}])^{\times}.$$

See [BurFî3, Lemma 8.9] for details.

**Conjecture 1.2** (Rationality conjecture, [BurFî3, Conjecture 5]). Let $\partial$ be the connecting homomorphism $K_{1}(\mathbb{R}[G_{F'/F}]) \to K_{0}(\mathbb{Q}[G_{F'/F}], \mathbb{R}[G_{F'/F}])$ appearing in the localisation exact sequence associated to the localisation $\mathbb{Q}[G_{F'/F}] \to \mathbb{R}[G_{F'/F}]$. Then in $K_{0}(\mathbb{Q}[G_{F'/F}], \mathbb{R}[G_{F'/F}])$ the equation

$$\partial(\text{nrd}_{\mathbb{R}[G_{F'/F}]}^{-1}(\lambda L^{*}(\mathbb{Q}(m)_{F'/F}))) + [\Xi(\mathbb{Q}(m)_{F'/F}), \vartheta_{\infty}] = 0$$

holds (via the canonical isomorphism as already explained, we identify the element $[\Xi(\mathbb{Q}(m)_{F'/F}), \vartheta_{\infty}]$ in $\pi_{0}(V(\mathbb{Q}[G_{F'/F}], \mathbb{R}[G_{F'/F}])$) with the corresponding one in the relative Grothendieck group $K_{0}(\mathbb{Q}[G_{F'/F}], \mathbb{R}[G_{F'/F}])$).

It is known that Conjecture 1.2 for the $m$-fold Tate motive $\mathbb{Q}(m)_{F'/F}$ with negative $m$ is equivalent to the central conjecture of Benedict Hyman Gross [Gross]. Conjecture 1.2 implies that there exists an isomorphism

$$\vartheta^{(\lambda)}: \Xi(\mathbb{Q}(m)_{F'/F}) \xrightarrow{\sim} 1_{\mathbb{Q}[G_{F'/F}]}$$

such that the scalar extension $\vartheta^{(\lambda)} = \vartheta^{(\lambda)} \otimes_{\mathbb{Q}} \mathbb{R}$ coincides with the trivialisation of $\Xi(\mathbb{Q}(m)_{F'/F}) \otimes_{\mathbb{Q}} \mathbb{R}$ described as $\text{nrd}_{\mathbb{R}[G_{F'/F}]}^{-1}(\lambda L^{*}(\mathbb{Q}(m)_{F'/F})) \circ \vartheta_{\infty}$.

**Conjecture 1.3** ($p$-part of the Tamagawa number conjecture). Let $\partial_{p}$ denote the connecting homomorphism appearing in the localisation exact sequence associated to the localisation $\mathbb{Z}_{p}[G_{F'/F}] \to \mathbb{Q}_{p}[G_{F'/F}]$. Then the element $T\Omega(\mathbb{Q}(m)_{F'/F})_{p}$ in $K_{0}(\mathbb{Z}_{p}[G_{F'/F}], \mathbb{Q}_{p}[G_{F'/F}])$ defined by

$$T\Omega(\mathbb{Q}(m)_{F'/F})_{p} = [\det_{\mathbb{Z}_{p}[G_{F'/F}]}(\mathcal{O}_{F'}^{\Sigma}, \mathbb{Z}_{p}(m)), \vartheta^{(\lambda)} \circ \vartheta_{p}^{-1}] - \partial_{p}(\text{nrd}_{\mathbb{Q}_{p}[G_{F'/F}]}^{-1}(\lambda L^{*}(\mathbb{Q}(m)_{F'/F})))$$

vanishes where $\vartheta^{(\lambda)}$ denotes the scalar extension $\vartheta^{(\lambda)} \otimes_{\mathbb{Q}} \mathbb{Q}_{p}$ (see also [BurFî3, Conjecture 6]).
Remark 1.4. If $F'/\mathbb{Q}$ is a finite abelian Galois extension and $F$ is a subfield of $F'$, the equivariant Tamagawa number conjecture for the Tate motives $\mathbb{Q}(m)_{F'/F}$ has been proven for an arbitrary prime number $p$ and an arbitrary integer $m$ (not necessarily negative) by David Burns, Cornelius Greither and Mathias Flach. Refer to [BurGr] (for negative $m$ and odd $p$), [Flach] (for negative $m$ and arbitrary $p$) and [BurFl4] (for arbitrary $m$ and $p$). Independently Annette Huber-Klawitter, Guido Kings and Kensuke Itakura have also proven the Bloch-Kato conjecture for Dirichlet motives—a somewhat weaker conjecture than the equivariant Tamagawa number conjecture—by using rather different technique. See [HubKin2] (for $p \neq 2$) and [Itakura] (for $p = 2$) for details. In spite of such great development upon commutative cases, very little seems to be known for non-commutative cases.

2. Burns’ technique

There exists a standard strategy to construct the $p$-adic zeta functions for non-commutative extensions by “patching” Serre’s $p$-adic zeta pseudomeasures for abelian extensions. It was first observed by David Burns and applied by Kazuya Kato to his pioneering work [Kato2]. Here we shall introduce this outstanding technique in a little generalised way.

Throughout this section we fix embeddings of $\mathbb{Q}$ into $\mathbb{C}$ and $\mathbb{Q}_p$. Let $p$ be a positive odd prime number, $F$ a totally real number field and $F_{\infty}/F$ a totally real $p$-adic Lie extension satisfying conditions $(F_{\infty}-1)$, $(F_{\infty}-2)$ and $(F_{\infty}-3)$ in the previous section. Let $G$, $H$ and $\Gamma$ be $p$-adic Lie groups defined as in Section 1.1.

**Definition 2.1** (Brauer families). Let $\mathfrak{F}_B$ be a family consisting of a pair $(U,V)$ where $U$ is an open subgroup of $G$ and $V$ is that of $H$ such that $V$ is normal in $U$ and the quotient group $U/V$ is abelian. We call $\mathfrak{F}_B$ a Brauer family for the group $G$ if it satisfies the following condition $(\sharp)_B$:

$$(\sharp)_B \text{ an arbitrary Artin representation of } G \text{ is isomorphic to a } \mathbb{Z}\text{-linear combination (as a virtual representation) of induced representations } \text{Ind}_{U}^{G}(\chi_{U/V}) \text{, where each } (U,V) \text{ is an element in } \mathfrak{F}_B \text{ and } \chi_{U/V} \text{ is a finite-order character of the abelian group } U/V.$$ 

Suppose that there exists a Brauer family $\mathfrak{F}_B$ for $G$. Let $\theta_{U,V}$ be the composition

$$K_1(\Lambda(G)) \xrightarrow{\text{Nr}_{\Lambda(G)/\Lambda(U)}} K_1(\Lambda(U)) \xrightarrow{\text{canonical}} K_1(\Lambda(U/V)) \xrightarrow{\gamma} \Lambda(U/V)^\times$$

for each $(U,V)$ in $\mathfrak{F}_B$ where $\text{Nr}_{\Lambda(G)/\Lambda(U)}$ is the norm map in algebraic $K$-theory. Set

$$\theta = (\theta_{U,V})_{(U,V) \in \mathfrak{F}_B} : K_1(\Lambda(G)) \rightarrow \prod_{(U,V) \in \mathfrak{F}_B} \Lambda(U/V)^\times.$$ 

Similarly we may construct the map

$$\theta_S = (\theta_{S,U,V})_{(U,V) \in \mathfrak{F}_B} : K_1(\Lambda(G)_S) \rightarrow \prod_{(U,V) \in \mathfrak{F}_B} \Lambda(U/V)^\times_S$$

\footnote{We use the same symbol $S$ for the canonical Ore set for $F_V/F_U$ by abuse of notation.}
for the localised Iwasawa algebra $\Lambda(G)_S$. Then we obtain the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
K_1(\Lambda(G)) & \overset{\theta}{\longrightarrow} & K_1(\Lambda(G)_S) & \overset{\partial}{\longrightarrow} & K_0(\Lambda(G), \Lambda(G)_S) & \overset{\text{norm}}{\longrightarrow} 0 \\
\prod_{\overline{S}} \Lambda(U/V)_\infty & \overset{\varphi_S}{\longrightarrow} & \prod_{\overline{S}} \Lambda(U/V)_S & \overset{\partial}{\longrightarrow} & \prod_{\overline{S}} K_0(\Lambda(U/V), \Lambda(U/V)_S) & \overset{0}{\longrightarrow} 0.
\end{array}
$$

Let $f$ be an arbitrary characteristic element for $F_\infty/F$ (that is, an element in $K_1(\Lambda(G)_S)$ satisfying the relation (1.2)) and $(f_{U,V})(U,V)\in \overline{S}_B$ its image under the map $\theta_S$. Then each $f_{U,V}$ satisfies the relation $\partial(f_{U,V}) = -[C_{U,V}]$ by the functoriality of the connecting homomorphism $\partial$. Now recall that for each pair $(U, V)$ in $\overline{S}_B$, the $p$-adic zeta pseudomeasure $\xi_{U,V}$ for the abelian extension $F_V/F_U$ exists as an element in $\Lambda(U/V)_\infty$ by the localisation exact sequence, and consider the following assumption:

**Assumption (b)** the element $(w_{U,V})(U,V)\in \overline{S}_B$ is contained in the image of $\theta$.

Then under Assumption (b) there exists an element $w$ in $K_1(\Lambda(G))$ which satisfies $\theta(w) = (w_{U,V})(U,V)\in \overline{S}_B$. Let $\xi$ be the element in $K_1(\Lambda(G)_S)$ defined as $fw$. Then $\xi$ readily satisfies the following two conditions by easy diagram chasing:

(ξ-1) the equation $\partial(\xi) = -[C_{F_\infty/F}]$ holds;
(ξ-2) the equation $\theta_S(\xi) = (\xi_{U,V})(U,V)\in \overline{S}_B$ holds.

By using condition (ξ)B, condition (ξ-2) and the interpolation formulae (2.1), we may verify that $\xi$ satisfies the interpolation formula (1.3) as follows:

$$
\xi(\rho \kappa^r) = \xi \left( \kappa^r \sum_{(U,V)\in \overline{S}_B} a_{U,V} \text{Ind}_U^G(\chi_{U/V}) \right) \quad \text{(by (ξ)B)}
$$

$$
= \prod_{(U,V)\in \overline{S}_B} \text{Nr}_{\Lambda(G)_S/\Lambda(U)_S}(\xi)(\chi_{U/V}\kappa^r)^{au,V}
$$

$$
= \prod_{(U,V)\in \overline{S}_B} \xi_{U,V}(\chi_{U/V}\kappa^r)^{au,V} \quad \text{(by (ξ-2))}
$$

$$
= \prod_{(U,V)\in \overline{S}_B} L_{\Sigma}(1 - r; F_V/F_U, \chi_{U/V})^{au,V} \quad \text{(by (2.1))}
$$

$$
= L_{\Sigma}(1 - r; F_\infty/F, \sum_{(U,V)\in \overline{S}_B} a_{U,V} \text{Ind}_U^G(\chi_{U/V})) = L_{\Sigma}(1 - r; F_\infty/F, \rho)
$$

where $\rho$ is an arbitrary Artin representation of $G$ and $r$ is an arbitrary natural number divisible by $p - 1$. Therefore $\xi$ is the desired $p$-adic zeta
function. Furthermore \((\zeta_1-1)\) implies that \(\zeta\) is also a characteristic element for \(F_\infty/F\); in other words the Iwasawa main conjecture holds for \(F_\infty/F\).

By virtue of Burns’ technique, both construction of the \(p\)-adic zeta function and verification of the Iwasawa main conjecture are reduced to the following two tasks:

- characterisation of the images of \(\theta\) and \(\theta_S\);
- verification of Assumption \((\flat)\).

In general, there are so many pairs in a Brauer family \(\mathfrak{B}_B\) that it is hard to compute and characterise the image of the norm maps \(\theta\) and \(\theta_S\). Therefore we shall use not only Brauer families but also Artinian families in arguments of the rest of this article.

**Definition 2.2 (Artinian families).** If a family \(\mathfrak{F}_A\) consisting of an abelian open subgroup of \(G\) satisfies the following condition \((\sharp)_A\), we call \(\mathfrak{F}_A\) an Artinian family for the group \(G\):

\[(\sharp)_A\text{ an arbitrary Artin representation of }G\text{ is isomorphic to a }\mathbb{Z}[1/p]-\text{linear combination (as a virtual representation)}\] of induced representations \(\text{Ind}^G_U(\chi_U)\), where each \(U\) is an element in \(\mathfrak{F}_A\) and \(\chi_U\) is a finite-order character of the abelian group \(U\).

Artinian families tend to contain much fewer elements than Brauer families, which often makes computation remarkably simpler.

### 3. The main theorem and its application

#### 3.1. The main theorem.** The precise statement of the main result in this article is as follows:

**Theorem 3.1 (Main theorem).** Let \(p\) be a positive odd prime number and \(F\) a totally real number field, and let \(F_\infty\) be a totally real \(p\)-adic Lie extension of \(F\) satisfying conditions \((F_\infty-1)\), \((F_\infty-2)\) and \((F_\infty-3)\) in Section 1.1. Suppose that the Galois group of \(F_\infty/F\) is isomorphic to the direct product of a finite \(p\)-group \(G_f\) with exponent \(p\) and the commutative \(p\)-adic Lie group \(\Gamma\). Then the \(p\)-adic zeta function \(\zeta_{F_\infty/F}\) for \(F_\infty/F\) exists uniquely up to multiplication by an element in \(SK_1(\mathbb{Z}_p[G_f])\). Moreover, the Iwasawa main conjecture (Conjecture 1.1(2)) is true for \(F_\infty/F\) (for arbitrary fixed embeddings \(\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}\) and \(\overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p\)).

**Remark 3.2.** If the image of \(SK_1(\mathbb{Z}_p[G_f])\) under the canonical localisation homomorphism \(K_1(\Lambda(G)) \to K_1(\Lambda(G)_S)\) vanishes, we may establish the uniqueness result upon the \(p\)-adic zeta function for \(F_\infty/F\) in Theorem 3.1. However the author does not have any ideas either \(SK_1(\mathbb{Z}_p[G_f])\) always vanishes or not in \(K_1(\Lambda(G)_S)\).

Now let us consider an easy but interesting application of our main theorem. Let \(B^N(\mathbb{F}_p)\) be a multiplicative \(p\)-group defined as the subgroup of the general linear group \(\text{GL}_{N+1}(\mathbb{F}_p)\) of degree \(N+1\) generated by all strongly upper-triangular matrices; that is,
Corollary 3.3. Let \( p \) be a positive odd prime number, \( F \) a totally real number field and \( F_B/F \) a totally real \( p \)-adic Lie extension satisfying conditions (\( F_\infty \)-1), (\( F_\infty \)-2) and (\( F_\infty \)-3) in Section 1. Assume also that

(i) there exists a certain non-negative integer \( N \) such that the Galois group of \( F_B/F \) is isomorphic to the direct product of \( B^N(F_p) \) and the commutative \( p \)-adic Lie group \( \Gamma \);

(ii) the prime number \( p \) is strictly larger than \( N \).

Then the \( p \)-adic zeta function \( \xi_{F_B/F} \) for \( F_B/F \) exists uniquely up to multiplication by an element in \( SK_1(Z_p[B^N(F_p)]) \). Moreover the Iwasawa main conjecture (Conjecture 1.1 (2)) is true for \( F_B/F \) (for arbitrary fixed embeddings \( \overline{\mathbb{Q}} \oplus \mathbb{C} \rightarrow \mathbb{C} \) and \( \overline{\mathbb{Q}} \rightarrow \mathbb{Q}_p \)).

Proof. For each \( p \) strictly larger than \( N \), the exponent of \( B^N(F_p) \) equals \( p \). Therefore the claim is directly deduced from Theorem 3.1. \( \square \)

Remark 3.4. The author is grateful to Peter Schneider and Otmar Venjakob for kindly informing him that they have verified triviality of the group \( SK_1(Z_p[B^N(F_p)]) \) for arbitrary \( N \) in their ongoing project upon the “non-commutative Coleman map” (the case where \( N \) equals 2 has been already known by the results of Robert Oliver [Oliver, Proposition 12.7]). By combining their results with Corollary 3.3, we may verify uniqueness of the \( p \)-adic zeta function for \( F_B/F \).

Remark 3.5. The case where \( N \) is equal to 2 is a special case of Kato’s Heisenberg-type extensions [Kato2]. The case where \( N \) is equal to 3 is nothing but the main results of the preceding paper [H]. Our original motivation upon this study was to generalise these results to the cases where \( N \) is larger than 4, and it was convenient to consider the problem under more general conditions as in Theorem 3.1.

In the rest of this article we mainly deal with cases under the conditions of our main theorem (Theorem 3.1).

3.2. Application to the equivariant Tamagawa number conjecture for critical Tate motives. The non-commutative Iwasawa main conjecture should be deeply related to the (non-commutative) equivariant Tamagawa number conjecture, as was pointed out in, for example, [HubKin1] and [FukKat]. In this subsection, we shall show that the \( p \)-part of the (non-commutative) equivariant Tamagawa number conjecture for critical Tate
motives follows from the Iwasawa main conjecture (Conjecture 1.4) by applying a standard descent argument. This can be regarded as evidence implying mystic relationship between the non-commutative Iwasawa main conjecture and the non-commutative Tamagawa number conjecture.

**Proposition 3.6.** Let $p$ be a positive odd prime number and $F$ a totally real number field. Let $F_\infty$ be a totally real $p$-adic Lie extension of $F$ satisfying conditions $(F_\infty \setminus 1)$, $(F_\infty \setminus 2)$ and $(F_\infty \setminus 3)$ in Section 1.4. Suppose also that Conjectures 1.3 (1), (2) are true for $F^\infty / F$. Then the $p$-part of the equivariant Tamagawa number conjecture (Conjecture 1.3) for $Q(1-r)_{F/F}$ is true for an arbitrary finite Galois subextension $F'$ of $F_\infty / F$ and an arbitrary natural number $r$ divisible by $p-1$.

Note that the Tate motive $Q(1-r)_{F'/F}$ is a critical motive in the sense of Pierre Deligne [Deligne1, Definition 1.3] since both $F$ and $F'$ are totally real and $r$ is even. Combining this proposition with Theorem 3.3 we obtain:

**Corollary 3.7.** Let $p, F$, and $F_\infty / F$ be as in Proposition 3.6. Suppose that the Galois group of $F_\infty / F$ is isomorphic to the direct product of a finite $p$-group $G^f$ with exponent $p$ and the commutative $p$-adic Lie group $\Gamma$. Then the $p$-part of the equivariant Tamagawa number conjecture for $Q(1-r)_{F/F}$ is true for an arbitrary finite Galois subextension $F'$ of $F_\infty / F$ and an arbitrary natural number $r$ divisible by $p-1$.

This corollary gives a simple but non-trivial example strongly suggesting validity of the equivariant Tamagawa number conjecture for motives with non-commutative coefficient. Proposition 3.6 is just the direct consequence of the Iwasawa main conjecture and descent theory established by David Burns and Otmar Venjakob [CurVen], and all materials used in the proof should be essentially contained in [CurVen]. There, however, does not seem to exist explicit suggestion upon critical Tate motives there, and thus we shall give the proof of Proposition 3.6 in the rest of this subsection.

First we may easily see that both $Q(1-r)_{F'/F, B}$ and $K_{2r-1}(F')_Q$ are trivial because $F$ and $F'$ are totally real fields and $r$ is an even natural number (triviality of $K_{2r-1}(F')_Q$ is due to Armand Borel [Borel1, Borel2]), and thus $\Xi(Q(1-r)_{F/F})$ is also trivial and the period-regulator map $\vartheta_\infty$ degenerates to the identity map on the unit object $1_{\mathbb{E}[G_{F'/F}]}$.

**Lemma 3.8.** The leading term $L^*(Q(1-r)_{F'/F})$ of the equivariant Artin $L$-function for $Q(1-r)_{F'/F}$ is contained in $Z(Q[G_{F'/F}])^\infty$.

**Proof.** We may verify the claim by an argument similar to that in [Deligne1, Proposition 6.7]: first note that $L(s; F'/F, \rho)$ does not vanish at $s = 1-r$ for each $\rho$ in $\text{Irr}(G_{F'/F})$ because $1-r$ is a critical value for $L(s; F'/F, \rho)$. Therefore $L^*(Q(1-r)_{F'/F})$ corresponds to $(L(1-r; F'/F, \rho))_{\rho \in \text{Irr}(G_{F'/F})}$. Since the action of an automorphism $\tau$ of $\mathbb{C}$ upon $\prod_{\rho \in \text{Irr}(G_{F'/F})} \mathbb{C}$ is given by

$$(x_\rho)_{\rho \in \text{Irr}(G_{F'/F})} \mapsto (\tau(x_{\tau^{-1} \rho}))_{\rho \in \text{Irr}(G_{F'/F})},$$

it suffices to prove that $L(1-r; F'/F, \tau \rho) = \tau L(1-r; F'/F, \rho)$ holds for an arbitrary automorphism $\tau$ of $\mathbb{C}$; this follows from the classical results of
Lemma 3.8 implies that the rationality conjecture [1.2] is true for the Tate motive $\mathbb{Q}(1 - r)^{F'/F}$. Choose an element $\lambda$ in $Z(\mathbb{Q}[G_{F'/F}]^\times$ such that $\lambda L^*(\mathbb{Q}(1 - r)^{F'/F})$ is contained in the image of the reduced norm map $\text{nr}_{\mathbb{Q}[G_{F'/F}]}(\lambda L^*(\mathbb{Q}(1 - r)^{F'/F}))$. Set $\Delta(1 - r)^{F'} = RT_{c,\ell}(O_L^{\Sigma^\vee}, \mathbb{Z}_p(1 - r))$. We can easily check that the natural homomorphism

$$K_0(\mathbb{Z}_p[G_{F'/F}], \mathbb{Q}_p[G_{F'/F}], \mathbb{Z}_p[G_{F'/F}]) \rightarrow K_0(\mathbb{Z}_p[G_{F'/F}], \mathbb{Q}_p[G_{F'/F}], \mathbb{Z}_p[G_{F'/F}])$$

induced by the canonical embedding $\mathbb{Q}_p \hookrightarrow \mathbb{Q}_p$ is injective and can calculate the image $T\Omega(\mathbb{Q}(1 - r)^{F'/F})_{p,\mathbb{Z}_p}$ of $T\Omega(\mathbb{Q}(1 - r)^{F'/F})_p$ under this map as

$$\partial_p(\text{nr}_{\mathbb{Q}_p[G_{F'/F}]}) = \partial_p^L(\mathbb{Q}(1 - r)^{F'/F})$$

where $\partial_p$ is the connecting homomorphism appearing in the localisation exact sequence associated to $\mathbb{Z}_p[G_{F'/F}] \rightarrow \mathbb{Q}_p[G_{F'/F}]$. Here we use the following relations:

$$[\Delta(1 - r)^{F'}, \vartheta^{(\lambda)} \circ \vartheta_p^{-1}] = [\Delta(1 - r)^{F'}, \vartheta_p^{-1}] + \partial_p(\vartheta^{(\lambda)}),$$

$$\partial_p(\vartheta^{(\lambda)}) = \partial_p(\text{nr}_{\mathbb{Q}_p[G_{F'/F}]}) = \partial_p(\text{nr}_{\mathbb{Q}_p[G_{F'/F}]})(\lambda L^*(\mathbb{Q}(1 - r)^{F'/F})).$$

Therefore in order to prove Proposition 3.6 it suffices to show that the element $T\Omega(\mathbb{Q}(1 - r)^{F'/F})_{p,\mathbb{Z}_p}$ vanishes.

**Proof of Proposition 3.2.** Let $\xi_{F_{\infty}/F}$ be the $p$-adic zeta function for $F_{\infty}/F$ and assume that the Iwasawa main conjecture $\partial(\xi_{F_{\infty}/F}) = -\{C_{F_{\infty}/F}\}$ is valid (for an arbitrary fixed embedding $j_p: \mathbb{Q} \hookrightarrow \mathbb{Q}_p$). Since $RT_{c,\ell}(O_{\mathbb{Q}_{\infty}^{\vee}}, \mathbb{Q}_p[Z_p])$ is identified with the injective limit of complexes $RT_{c,\ell}(O_{\mathbb{Q}_p}^{\vee}, \mathbb{Z}_p/Z_p)$ for all finite Galois subextensions $L/F$ of $F_{\infty}/F$, we may easily see that $C_{F_{\infty}/F}$ is isomorphic to the complex $\lim_{\leftarrow L} RT_{c,\ell}(\text{Spec } O_L^{\vee}, \mathbb{Z}_p(1))$ by virtue of the Poitou-Tate/Artin-Verdier duality theorem. Furthermore for each $L$ the complex $RT_{c,\ell}(O_{\mathbb{Q}_p}^{\vee}, \mathbb{Z}_p(1))$ is isomorphic to $RT_{c,\ell}(O_{\mathbb{Q}_p}^{\vee}, \mathbb{Z}_p[G_{L/F}]^\sharp(1))$ by Shapiro’s lemma (here $\mathbb{Z}_p[G_{L/F}]^\sharp$ denotes $\mathbb{Z}_p[G_{L/F}]$ regarded as a left $G_{L/F}$-module whose $G_{L/F}$-action is given by the right multiplication of the inverse element). Refer to, for example, [HorKim, Appendix B] for details. Hence the following equation holds in $K_0(\mathbb{Q}_p^{\text{Perf}}(G), \text{qis})$:

$$\partial(\xi_{F_{\infty}/F}) = \lim_{\leftarrow L} RT_{c,\ell}(\text{Spec } O_{\mathbb{Q}_p}^{\vee}, \mathbb{Z}_p[G_{L/F}]^\sharp(1)) = [RT_{c,\ell}(\text{Spec } O_{\mathbb{Q}_p}^{\vee}, L(\gamma)^\sharp(1))].$$

Now for each natural number $r$ divisible by $p - 1$ consider the $\mathbb{Z}_p$-linear map $\vartheta^r: \Lambda(G) \rightarrow \Lambda(G)$ induced by $\sigma \mapsto \kappa^r(\sigma)\sigma$ for $\sigma$ in $G$, which is in fact a ring automorphism because $\kappa^r(\sigma)$ is an element in the centre of $\Lambda(G)$.
This map also induces a ring automorphism \( g_{S,\kappa} \) on the canonical Ore localisation \( \Lambda(G)_{S} \) of \( \Lambda(G) \) and its composition with the homomorphism \( \Lambda(G)_{S} \to \text{Frac}(\Lambda(\Gamma)) \) induced by the projection \( G \to \Gamma \) coincides with the morphism \( \Phi_{\kappa} \) introduced in [CFKSV, Lemma 3.3]. Hence the definition of the evaluation map asserts that the interpolation formula

\[
g_{S,\kappa}(\xi_{F_{\infty}/F})(\rho) = L_{\Sigma}(1 - r; F_{\infty}/F, \rho)
\]

holds for an arbitrary Artin representation \( \rho \) of \( G \). On the other hand the Tate twist \( \Lambda(G)^{\sharp}(1) \to \Lambda(G)^{\sharp}(1 - r) \) defines a \( g_{S,\kappa} \)-semilinear isomorphism (due to the \( \Lambda(G) \)-module structure of \( \Lambda(G)^{\sharp} \)), and in \( G^{\text{Perf}}(\Lambda(G)) \) we therefore obtain the isomorphism

\[
\Lambda(G) \otimes_{\Lambda(G),g_{\kappa}} RT_{c,\ell}^{*}(\mathcal{O}_{F}^{\Sigma^{\vee}}, \Lambda(G)^{\sharp}(1)) \xrightarrow{\sim} RT_{c,\ell}^{*}(\mathcal{O}_{F}^{\Sigma^{\vee}}, \Lambda(G)^{\sharp}(1 - r))
\]

(we remark that this argument is a non-commutative variant of that in [Flach, Lemma 5.13 a]). Then we obtain

\[
(3.1) \quad \partial(g_{S,\kappa}^{*}(\xi_{F_{\infty}/F})) = [RT_{c,\ell}^{*}(\mathcal{O}_{F}^{\Sigma^{\vee}}, \Lambda(G)^{\sharp}(1 - r))]
\]

by the functoriality of the connecting homomorphism. Moreover

\[
Z_{p}[G_{F'/F}] \otimes_{\Lambda(G),g_{\kappa}} RT_{c,\ell}^{*}(\mathcal{O}_{F}^{\Sigma^{\vee}}, \Lambda(G)^{\sharp}(1 - r)) = RT_{c,\ell}^{*}(\mathcal{O}_{F}^{\Sigma^{\vee}}, Z_{p}[G_{F'/F}]^{\sharp}(1 - r)) = RT_{c,\ell}^{*}(\mathcal{O}_{F}^{\Sigma^{\vee}}, Z_{p}(1 - r)) = \Delta(1 - r)_{F'}
\]

holds for an arbitrary finite Galois subextension \( F'/F \) of \( F_{\infty}/F \). Since the localisation \( Z_{p} \otimes_{Z_{p}} \Delta(1 - r)_{F'} \) is acyclic (essentially due to the criticalness of \( \mathbb{Q}(1 - r)_{F'/F} \); refer to [BurFl2, (9),(10)]), the equation (3.1) descends to

\[
(3.2) \quad K_{1}(\mathbb{Q}_{p}[G_{F'/F}]) = \prod_{v \in \Sigma} \phi_{v}^{-1}
\]

by the results of Burns and Venjakob [BurVen, Theorem 2.2]. Here we remark that the element \( [\Delta(1 - r)_{F'}] \) in the relative Grothendieck group can be naturally identified with the element \( -[\det_{\mathbb{Q}_{p}[G_{F'/F}]}(\Delta(1 - r)_{F'})] \) in \( \pi_{0}(\mathbb{Q}_{p}[G_{F'/F}]) \) where “acyc” denotes the trivialisation induced by acyclicity of \( \mathbb{Q}_{p} \otimes_{Z_{p}} \Delta(1 - r)_{F'} \) (see Remark 3.11 for the problem upon sign convention). The difference between two trivialisations \( \partial_{v}^{-1} \frac{1}{p \mathbb{Q}_{p}} \) and acyc is calculated in [BurFl2] as

\[
\text{acyc} = \partial_{v}^{-1} \frac{1}{p \mathbb{Q}_{p}} \prod_{v \in \Sigma} \phi_{v}^{-1}
\]

where each \( \mathbb{Q}_{p} \)-isomorphism \( \phi_{v} : V \to V \) is defined as in [BurFl1, Section 1.2] or [FukKat, Section 2.4.2] which we regard as an element in \( K_{1}(\mathbb{Q}_{p}[G_{F'/F}]) \). Then by definition the image of \( \phi_{v}^{-1} \) under the Wedderburn decomposition

\[
K_{1}(\mathbb{Q}_{p}[G_{F'/F}]) \xrightarrow{\text{mrred}_{\mathbb{Q}_{p}[G_{F'/F}]}} Z(\mathbb{Q}_{p}[G_{F'/F}])^{\times} \xrightarrow{\prod_{\rho \in \text{Irr}(G_{F'/F})}} \mathbb{Q}_{p}^{\times}
\]

coincides with \( (L_{v}(1 - r; F'/F, \rho))_{\rho \in \text{Irr}(G_{F'/F})} \), the local factor of the equivariant Artin \( L \)-function at \( v \). Combining this fact with the relation (3.2), we can easily verify that \( T\Omega(\mathbb{Q}(1 - r)_{F'/F})_{p \mathbb{Q}_{p}} \) vanishes.
Remark 3.9. If we take $F' = F$, Proposition 3.6 is equivalent to the $p$-part of the cohomological Lichtenbaum conjecture
\[
|\zeta_F(1-r)|_{p}^{-1} = \frac{\|H^2_{c,\mathrm{et}}(\mathcal{O}^{\vee}_F, \mathbb{Z}_p(1-r))\|_{p}^{-1}}{\|H^1_{c,\mathrm{et}}(\mathcal{O}^{\vee}_F, \mathbb{Z}_p(1-r))\|_{p}^{-1}}
\]
via certain specialisation (here $\zeta_F(s)$ is the complex Dedekind zeta function for $F$ and $| \cdot |_p$ is the $p$-adic valuation normalised by $|p|_p = 1/p$). This is directly deduced from the (classical) Iwasawa main conjecture for totally real number fields verified by Andrew Wiles [Wiles]. Proposition 3.6 gives its certain generalisation for cases with non-commutative coefficient.

Remark 3.10 (sign convention). Let $R$ be an associative ring and $S$ a left Ore subset of $R$. We let $S^{-1}R$ denote the left Ore localisation of $R$ with respect to $S$. In [Swan], the relative Grothendieck group $K_0(R, S^{-1}R)$ is defined as a certain quotient of the free abelian group generated by all triples $[P, \lambda, Q]$ where each $P$ and $Q$ are finitely generated projective left $R$-modules and $\lambda$ is an $S^{-1}R$-isomorphism $\lambda : S^{-1}R \otimes_R P \to S^{-1}R \otimes_R Q$. Then we may identify a homomorphism $P \to Q$ induced by $\lambda$ with a cochain complex concentrated in terms of degree 0 and 1, and we use this identification as the normalisation of the isomorphism between $K_0(R, S^{-1}R)$ and $K_0(\phi_S^{\mathrm{Perf}}(R), \acute{\mathrm{qis}})$ (this normalisation is the same one as used in [FukKat]). In [BurVen], however, they identify $K_0(R, S^{-1}R)$ with $\pi_0(V(R, S^{-1}R))$ in the following manner: when both $\ker(\lambda)$ and $\coker(\lambda)$ are projective, the element $[P, \lambda, Q]$ in $K_0(R, S^{-1}R)$ is identified with the element in $\pi_0(V(R, S^{-1}R))$ defined as $[\det^{-1}(P) \cdot \det^R_Q(\lambda) \cdot \id_{\det^R_Q(\lambda)}]$; in other words they implicitly regard $P \to Q$ as a complex concentrated in terms of degree $-1$ and 0. Hence there appears difference in sign convention
\[
K_0(\phi_S^{\mathrm{Perf}}(R), \acute{\mathrm{qis}}) \xrightarrow{[C]} \pi_0(V(R, S^{-1}R))
\]
(on the other hand they used, in [BtBur], the different normalisation
\[
[P, \lambda, Q] \leftrightarrow [\det_R(P) \cdot \det^{-1}_R(Q) \cdot \det_R(\lambda) \cdot \id_{\det^{-1}_R(Q)}],
\]
and the element $[C]$ in $K_0(\phi_S^{\mathrm{Perf}}(R), \acute{\mathrm{qis}})$ therefore corresponds to the element $[\det(C), \acute{\mathrm{acyc}}]$ in $\pi_0(V(R, S^{-1}R)))$.

4. Construction of the theta isomorphism I —additive theory—

In this section we first define the Artinian families $\mathfrak{F}_A$, $\mathfrak{F}_B$, and the Brauer family $\mathfrak{F}_B$ (see Section 3.1 for definition), which will play important roles in the following arguments. We then construct the additive version of the theta isomorphism (see Section 4.3). Later we shall translate it into the multiplicative morphism in Section 3. We remark that Mahesh Kakde has recently established more general construction of the additive theta isomorphism [Kakde2] (his construction can be applied to case in which $G'$ is an arbitrary finite $p$-group not necessarily with exponent $p$).
4.1. Artinian families $\mathfrak{F}_A$, $\mathfrak{F}_A^c$ and Brauer family $\mathfrak{F}_B$. Let $p$, $F$, and $F_\infty/F$ be as in Theorem 3.1. Let $G$ be the Galois group of $F_\infty/F$ and $p^N$ the order of the finite part $G^f$ of $G$ (and $N$ is hence a non-negative integer). The finite $p$-group $G^f$ acts upon the set of all its cyclic subgroups by conjugation. Choose a set of representatives of the orbital decomposition under this action, and choose also a generator for each representative cyclic group. Let $\mathfrak{F}$ denote the set of these fixed generators, and for each $h$ in $\mathfrak{F}$ let $U_h^f$ be the cyclic subgroup of $G^f$ generated by $h$. Since the exponent of $G^f$ is $p$, the degree of each $U_h^f$ exactly equals $p$ except for $U_e^f = \{e\}$ (here we denote the unit of $G^f$ by $e$). Let $U_h$ be the open subgroup of $G$ isomorphic to the direct product of $U_h^f$ and $\Gamma$ for each $h$ in $\mathfrak{F}$, and consider the family of open subgroups of $G$ consisting of all such $U_h$ which we denote by $\mathfrak{F}_A$ (we always identify $U_e$ with $\Gamma$).

**Proposition 4.1.** The family $\mathfrak{F}_A$ satisfies condition $(z)_A$. In other words, the family $\mathfrak{F}_A$ is an Artinian family for the group $G$.

**Proof.** The claim is directly deduced from the classical induction theorem of Emil Artin (see, for example, [Serre1, Corollaire de Théorème 15]).

For the usage of induction in Section 3, we now introduce another Artinian family $\mathfrak{F}_A^c$. When $N$ equals zero, we set $\mathfrak{F}_A^c = \mathfrak{F}_A = \{\Gamma, \{e\}\}$. When $N$ is larger than 1, choose a central element $c \neq e$ in $\mathfrak{F}$ and fix it (there exists such an element $c$ because $G^f$ is a $p$-group). For each $h$ in $\mathfrak{F}$, let $U_{h,c}^f$ be the abelian subgroup of $G^f$ generated by $h$ and $c$, and let $U_{h,c}$ be the open subgroup of $G$ isomorphic to the direct product of $U_{h,c}^f$ and $\Gamma$. Let $\mathfrak{F}_A^c$ denote the family of open subgroups of $G$ consisting of all elements in $\mathfrak{F}_A$ and all $U_{h,c}$ (we identify both $U_e$ and $U_{e,c}$ with $U_e$). Then the family $\mathfrak{F}_A^c$ is also an Artinian family for $G$ because $\mathfrak{F}_A^c$ contains the Artinian family $\mathfrak{F}_A$.

We finally define $\mathfrak{F}_B$ as the family consisting of all pairs $(U, V)$ such that $U$ is an open subgroup of $G$ containing $\Gamma$ and $V$ is the commutator subgroup of $U$. Then the family $\mathfrak{F}_B$ satisfies condition $(z)_B$ by Richard Brauer’s induction theorem [Serre1, Théorème 22] (note that for an arbitrary finite $p$-group, the family of all its Brauer elementary subgroups coincides with that of all its subgroups by definition); hence $\mathfrak{F}_B$ is a Brauer family.

4.2. Calculation of the images of trace homomorphisms. First recall the definition of trace homomorphisms: for an arbitrary finite group $\Delta$, let $\mathbb{Z}_p[\text{Conj}(\Delta)]$ be the free $\mathbb{Z}_p$-module of finite rank with basis $\text{Conj}(\Delta)$, and for an arbitrary pro-finite group $P$, let $\mathbb{Z}_p[[\text{Conj}(P)]]$ be the projective limit of free $\mathbb{Z}_p$-modules $\mathbb{Z}_p[\text{Conj}(P_\lambda)]$ over finite quotient groups $P_\lambda$ of $P$.

**Definition 4.2** (trace homomorphisms). Let $P$ be an arbitrary pro-finite group and $U$ its arbitrary open subgroup. Let $\{a_1, a_2, \ldots, a_s\}$ be one of the representatives of the left coset decomposition $P/U$. For an arbitrary conjugacy class $[g]$ of $P$ and for each integer $1 \leq j \leq s$, define $\tau_j([g])$ as

$$
\tau_j([g]) = \begin{cases} 
[a_j^{-1}ga_j] & \text{if } a_j^{-1}ga_j \text{ is contained in } U, \\
0 & \text{otherwise.}
\end{cases}
$$
Then the element $\text{Tr}_{Z_p[[\text{Conj}(P)]]/Z_p[[\text{Conj}(U)]]}([g]) = \sum_{j=1}^{s} \tau_j([g])$ is determined independently of the choice of representatives $\{a_j\}_{j=1}^{s}$. We call the induced $Z_p$-module homomorphism

$$\text{Tr}_{Z_p[[\text{Conj}(P)]]/Z_p[[\text{Conj}(U)]]} : Z_p[[\text{Conj}(P)]] \to Z_p[[\text{Conj}(U)]]$$

the trace homomorphism from $Z_p[[\text{Conj}(P)]]$ to $Z_p[[\text{Conj}(U)]]$.

Let $c$ be the fixed central element of $G^f$ as in the previous subsection and let $\theta^+_U$ denote the trace homomorphism $\text{Tr}_{Z_p[[\text{Conj}(G)]]/Z_p[[U]]}$ for each $U$ in $\mathfrak{F}_A$. We now calculate each image $I_{U}$ of $\theta^+_U$. Let $NU^f$ denote the normaliser of $U^f$ in $G^f$ for each $U$ in $\mathfrak{F}_A$. We denote by $p^{n_h}$ the cardinality of $NU^f_h$ for each $h$ in $\mathfrak{F}_A$.

**Calculation of $I_{U} (= I_{U_h})$.** When $N$ is equal to zero, the $Z_p$-module $I_U$ obviously coincides with $\Lambda(G) = \Lambda(\Gamma)$. Now suppose that $N$ is larger than 1. In this case, $\theta^+_U([g])$ does not vanish if and only if $g$ is contained in $\Gamma$. We may regard the finite part $G^f$ as a set of representatives of the left coset decomposition $G/\Gamma$, and for each $\gamma$ in $\Gamma$ and $a$ in $G^f$, the element $a^{-1}\gamma a$ equals $\gamma$ (note that $\gamma$ and $a$ commute). Therefore we have

$$I_{U} = p^{N}Z_p[[\Gamma]]$$

(this equality is also valid for the case in which $N$ equals zero).

**Calculation of $I_{U_h}$ for $h$ in $\mathfrak{F}$ except for $e$ ($N \geq 1$).** When $N$ is equal to 1, the $Z_p$-module $I_{U_h}$ obviously coincides with $\Lambda(U_h)$. Hence suppose that $N$ is larger than 2. In this case $\theta^+_U([g])$ does not vanish if and only if $g$ is contained in one of the conjugates of $U_h$, and we may therefore assume that $g$ is contained in $U_h$ without loss of generality. The normaliser $NU^f_h$ acts upon $U^f_h$ by conjugation, which induces a group antihomomorphism $\text{inn}: (NU^f_h)^{op} \to \text{Aut}(U^f_h) \cong \mathbb{F}_p^\times$. Note that it is trivial since $NU^f_h$ is a $p$-group. Therefore for every $u$ in $U_h$ not contained in $\Gamma$, its conjugate $a^{-1}ua$ is equal to $u$ if $a$ is contained in $NU^f_h$ and is not contained in $U_h$ otherwise. For each $\gamma$ in $\Gamma$, its conjugate $a^{-1}\gamma a$ always equals $\gamma$ as in the previous case. Consequently we have

$$I_{U_h} = p^{N-1}Z_p[[\Gamma]] \oplus \bigoplus_{i=1}^{p-1} p^{n_h-1}h^iZ_p[[\Gamma]]$$

(this equality is also valid when $N$ equals 1).

**Calculation of $I_{U_{h,c}}$ for $h$ in $\mathfrak{F}$ except for $e$ and $c$ ($N \geq 2$).** We obtain a group antihomomorphism $\text{inn}: (NU^f_{h,c})^{op} \to \text{Aut}(U^f_{h,c})$ in the same argument as that in the previous case. Since the automorphism group $\text{Aut}(U^f_{h,c})$ is isomorphic to $\text{GL}_2(\mathbb{F}_p)$ and its cardinality is equal to $p(p - 1)^2(p + 1)/2$, we have to consider the following two cases:

**Case (a)** the image of $\text{inn}$ is trivial;

**Case (b)** the image of $\text{inn}$ is a cyclic group of degree $p$. 

Inductive Construction of $p$-adic Zeta Functions

Figure 1. Trace and norm relation for $\mathfrak{F}_A$.

In Case (a) it is easy to see that $NU_{h,c}^f$ coincides with $NU_h^f$ (in particular the cardinality of $NU_{h,c}^f$ is equal to $p^n$). Therefore we may calculate $I_{U_{h,c}}$ in the same way as $I_{U_h}$, and we obtain

$$I_{U_{h,c}} = p^{N-2}Z_p[[U_c]] \oplus \bigoplus_{i=1}^{p-1} p^{n_k-2} h^i Z_p[[U_c]].$$

In Case (b) we may readily show by easy computation that the image of the map $\text{inn}$ is generated by automorphisms induced by $h^i c^j \mapsto h^i c^{kj+i}$ for each $0 \leq k \leq p-1$. Its kernel obviously coincides with $NU_{h,c}^f$, and the cardinality of $NU_{h,c}^f$ is thus equal to $p^{n+k+1}$. This enables us to calculate $I_{U_{h,c}}$ as

$$I_{U_{h,c}} = p^{N-2}Z_p[[U_c]] \oplus \bigoplus_{i=1}^{p-1} p^{n_k-2} h^i (1 + c + \ldots + c^{p-1}) Z_p[[\Gamma]].$$

4.3. Additive theta isomorphisms. Now set

$$\theta^+_A = (\theta^+_U)_{U \in \mathfrak{F}_A}: Z_p[[\text{Conj}(G)]] \to \prod_{U \in \mathfrak{F}_A} Z_p[[U]]$$

and let $\Phi$ be the $Z_p$-submodule of $\prod_{U \in \mathfrak{F}_A} Z_p[[U]]$ consisting of all elements $y_*$ satisfying the following two conditions:

- (trace relation) the equation $\text{Tr}_{Z_p[[U_h]]/Z_p[[\Gamma]]} y_h = y_e$ holds for each $Z_p[[U_h]]$-component $y_h$ of $y_*$ (see Figure 1);
- each $Z_p[[U]]$-component $y_U$ of $y_*$ is contained in $I_U$.

Proposition 4.3. The map $\theta^+_A$ induces an isomorphism of $Z_p$-modules

$$\theta^+_A: Z_p[[\text{Conj}(G)]] \xrightarrow{\sim} \Phi.$$ We call the induced isomorphism $\theta^+_A$ the additive theta isomorphism for $\mathfrak{F}_A$.

Proof. It is easy to see that $\Phi$ contains the image of $\theta^+_A$ by construction.

Injectivity. Take an element $y$ from the kernel of $\theta^+_A$ and let $\rho$ be an arbitrary Artin representation of $G$. Note that $\rho$ is isomorphic to a $\mathbb{Z}[1/p]$-linear combination $\sum_{U \in \mathfrak{F}_A} a_U \text{Ind}^G_U \chi_U$ by condition $(\sharp)_A$ where each $\chi_U$ is a finite-order character of the abelian group $U$. If we let $\chi_\rho$ denote the character associated to the Artin representation $\rho$, we obtain the equation

$$\chi_\rho(y) = \sum_{U \in \mathfrak{F}_A} a_U \chi_U \circ \text{Tr}_{Z_p[[\text{Conj}(G)]]/Z_p[[U]]}(y).$$
Corollary 4.4. Every element \( y \) in \( \mathbb{Z}_p[[U/V]] \) is completely determined by its trace images \( \{\theta^+_U(y)\}_{U \in \mathfrak{F}_B} \).

Next we extend the notion of the additive theta isomorphism to the Brauer family \( \mathfrak{F}_B \); for each \((U,V)\) in \( \mathfrak{F}_B \) let \( \theta^+_{U,V} \) be the composite map

\[
\mathbb{Z}_p[[\text{Conj}(G)]] \xrightarrow{\text{Tr}} \mathbb{Z}_p[[\text{Conj}(U)]] \xrightarrow{\text{canonical}} \mathbb{Z}_p[[U/V]]
\]

and set \( \theta^+_B = \left( \theta^+_{U,V} \right)_{(U,V) \in \mathfrak{F}_B} \). We define the \( \mathbb{Z}_p \)-submodule \( \Phi_B \) of the direct product \( \prod_{(U,V) \in \mathfrak{F}_B} \mathbb{Z}_p[[U/V]] \) as the submodule consisting of all elements \( (y_{U,V})_{(U,V) \in \mathfrak{F}_B} \) satisfying the following trace compatibility condition (TCC, see Figure 2):

- the equation \( \text{Tr} \mathbb{Z}_p[[U/V]]/\mathbb{Z}_p[[U'/V']] (y_{U,V}) = \text{can}_{V'}(y_{U',V'}) \) holds for arbitrary pairs \((U,V)\) and \((U',V')\) in \( \mathfrak{F}_B \) such that \( U \) contains \( U' \) and \( U' \) contains \( V \) respectively (we denote by \( \text{can}_{V'} \) the canonical surjection \( \mathbb{Z}_p[[U'/V']] \to \mathbb{Z}_p[[U'/V]] \));

and the following conjugacy compatibility condition (CCC+):

- the equation \( y_{U',V'} = \psi_{a}(y_{U,V}) \) holds if \((U,V)\) and \((U',V')\) are elements in \( \mathfrak{F}_B \) such that \( U' = a^{-1}Ua \) and \( V' = a^{-1}Va \) hold for a certain element \( a \) in \( G \) (we denote by \( \psi_{a} \) the isomorphism \( \mathbb{Z}_p[[U/V]] \to \mathbb{Z}_p[[U'/V']] \) induced by the conjugation \( U/V \to U'/V'; u \to a^{-1}ua \)).

Note that we may naturally regard \( \mathfrak{F}_A \) as a subfamily of \( \mathfrak{F}_B \) (by identifying \( U \) in \( \mathfrak{F}_A \) with the pair \((U,\{e\})\) in \( \mathfrak{F}_B \)).

**Proposition 4.5.** Let \( (y_{U,V})_{(U,V) \in \mathfrak{F}_B} \) be an element in \( \Phi_B \) and assume that \( (y_{U,\{e\}})_{U \in \mathfrak{F}_A} \) is contained in \( \Phi \). Then there exists a unique element \( y \) in \( \mathbb{Z}_p[[\text{Conj}(G)]] \) which satisfies \( \theta^+_B(y) = (y_{U,V})_{(U,V) \in \mathfrak{F}_B} \).
**Proof.** Consider the following commutative diagram (we denote the canonical projection by proj):

\[
\begin{array}{ccc}
Z_p[[\text{Conj}(G)]] & \xrightarrow{\theta^+_B} & \prod_{(U,V) \in \mathfrak{B}_B} Z_p[[U/V]] \\
\downarrow & & \downarrow \text{proj} \\
Z_p[[\text{Conj}(G)]] & \xrightarrow{\sim} & \Phi \xrightarrow{\theta^-_A} \prod_{U \in \mathfrak{G}_A} Z_p[[U]].
\end{array}
\]

Then \(\text{proj}((y_{U,V})(U,V) \in \mathfrak{B}_B)\) is contained in \(\Phi\) by assumption, and thus there exists a unique element \(y\) in \(Z_p[[\text{Conj}(G)]]\) which corresponds to the element \(\text{proj}((y_{U,V})(U,V) \in \mathfrak{B}_B)\) via the additive theta isomorphism \(\theta^-_A\) for \(\mathfrak{G}_A\) (Proposition 4.3). We have to show that \(\theta^+_B(y)\) coincides with \((y_{U,V})(U,V) \in \mathfrak{B}_B\), and for this purpose it suffices to show that \(\text{proj}\) induces an injection on \(\Phi_B\) (note that the element \((\theta^+_B(y))(U,V) \in \mathfrak{B}_B\) obviously satisfies both \((\text{TCC})\) and \((\text{CCC+})\) by construction; hence \(\theta^+_B(y)\) is an element in \(\Phi_B\)). Let \((z_{U,V})(U,V) \in \mathfrak{B}_B\) be an element in \(\Phi_B\) satisfying the following equation:

\[
(4.1) \quad \text{proj}((z_{U,V})(U,V) \in \mathfrak{B}_B) = (z_{U,\{e\}})_{U \in \mathfrak{G}_A} = 0.
\]

We shall prove that \(z_{U,V} = 0\) holds for each \((U,V)\) in \(\mathfrak{B}_B\) by induction on the cardinality of \(U^f\). First note that \(z_{U,\{e\}} = 0\) holds for \((U,\{e\})\) if the cardinality of \(U^f\) is smaller than \(p\) (use (4.1) and the conjugacy compatibility condition \((\text{CCC+})\)). Now let \((U,V)\) be an element in \(\mathfrak{B}_B\) such that the degree of \(U^f\) is equal to \(p^k\) for certain \(k\) larger than 2 and set \(W = U^f/V^f\). Then the abelian group \(W\) is isomorphic to \((\mathbb{Z}/p\mathbb{Z})^{\leq d}\) for a certain natural number \(d\) smaller than \(k\) (the structure theorem of finite abelian groups).

Moreover we may assume that \(d\) is larger than \(2^2\). Since the element \(z_{U,V}\) is represented as a \(\Lambda(\Gamma)\)-linear combination \(\sum_{w \in W} a_w w\), it suffices to prove that \(a_w\) equals zero for every \(w\) in \(W\). Take an arbitrary element \(x\) of degree \(p\) in \(W\), and let \(U^f_x\) denote the inverse image of \(\langle x \rangle\)—the cyclic subgroup of \(W\) generated by \(x\)—under the canonical surjection \(U^f \to W\). Obviously the cardinality of \(U^f_x\) is strictly smaller than \(p^k\). If we set \(U_x = U^f_x \times \Gamma\), we may explicitly calculate the image of \(z_{U,V}\) under the trace map from \(Z_p[[U/V]]\) to \(Z_p[[U_x/V]]\) as \(\sum_{i=0}^{p^k-1} p^{k-i} a_x x^i\). On the other hand the element \(z_{U_x,V_x}\) is equal to zero by induction hypothesis (here \(V_x\) denotes the commutator subgroup of \(U_x\)). Therefore \(a_x = 0\) holds for each \(i\) by \((\text{TCC})\). Replacing \(x\) appropriately, we may verify that \(a_w = 0\) holds for every \(w\) in \(W\). \(\square\)

5. Preliminaries for logarithmic translation

This section is devoted to technical preliminaries for arguments in Section 4.

5.1. Augmentation theory. For each \((U,V)\) in \(\mathfrak{B}_B\), let \(\text{aug}_{U,V}\) denote the augmentation map from \(\Lambda(U/V)\) to \(\Lambda(\Gamma)\) (namely it is a ring homomorphism induced by the projection \(U/V \to \Gamma\)), and let \(\text{aug}_{U,V} : \Omega(U/V) \to \Omega(\Gamma)\)

\footnote{We may easily verify that the cardinality of \(V^f\) is always smaller than \(p^{k-2}\) by induction on the cardinality of \(U^f\).}
be its reduction modulo $p$. Let $\varphi: \Lambda(G) \to \Lambda(\Gamma)$ denote “the Frobenius endomorphism” on $\Lambda(G)$ defined as the composition

$$\Lambda(G) \xrightarrow{\text{aug}_G} \Lambda(\Gamma) \xrightarrow{\gamma^p} \Lambda(\Gamma)$$

where $\text{aug}_G$ is the canonical augmentation map and $\varphi_G$ is the Frobenius endomorphism on $\Lambda(\Gamma)$ induced by $\gamma \mapsto \gamma^p$. Let $\theta_{U,V}$ denote the composition of the norm map $\text{Nr}_{\Lambda(G)/\Lambda(U)}$ with the canonical map $K_1(\Lambda(U)) \to \Lambda(U/V)^\times$.

The author is grateful to Takeshi Tsuji for presenting the following useful proposition to him.

**Proposition 5.1.** Let $(U, V)$ be an element in $\mathfrak{S}_B$ and $J_{U,V}$ the kernel of the composite map

$$\Lambda(U/V) \xrightarrow{\text{aug}_{U,V}} \Lambda(\Gamma) \xrightarrow{\text{mod}_p} \Omega(\Gamma).$$

Then the element $\varphi(x)^{-(G:U)/p} \theta_{U,V}(x)$ is contained in $1 + J_{U,V}$ for each $x$ in $K_1(\Lambda(G))$ if $U$ is a proper subgroup of $G$. In other words, the congruence $\theta_{U,V}(x) \equiv \varphi(x)^{(G:U)/p} \mod J_{U,V}$ holds unless $U$ coincides with $G$.

Before the proof we remark that the image of an element $x$ in $K_1(\Lambda(G))$ under the map $\theta_{U,V}$ can be calculated as follows: since the Iwasawa algebra $\Lambda(G)$ is regarded as a left free $\Lambda(U)$-module of finite rank $r = (G : U)$, the right-multiplication-$x$ map is represented by an invertible matrix $A_x$ with entries in $\Lambda(U)$. The element $\theta_{U,V}(x)$ then coincides with the determinant of the image of $A_x$ under the canonical map $\text{GL}_r(\Lambda(U)) \to \text{GL}_r(\Lambda(U/V))$. 

**Proof.** The claim is equivalent to the following Proposition 5.2 since both $\Lambda(G)$ and $\Lambda(U/V)$ are $p$-adically complete. \qed

Let $\bar{\varphi}: \Omega(G) \to \Omega(\Gamma)$ denote the Frobenius endomorphism on $\Omega(G)$ defined as $\bar{\varphi} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and let $\bar{\theta}_{U,V}$ denote the composition of the norm map $\text{Nr}_{\Omega(G)/\Omega(U)}$ with the canonical map $K_1(\Omega(U)) \to \Omega(U/V)^\times$.

**Proposition 5.2.** Let $\bar{J}_{U,V}$ be the kernel of the augmentation map

$$\Omega(U/V) \xrightarrow{\text{mod}_{U,V}} \Omega(\Gamma).$$

Then the element defined as $\bar{\varphi}(x)^{-(G:U)/p} \bar{\theta}_{U,V}(x)$ is contained in $1 + \bar{J}_{U,V}$ for each $x$ in $K_1(\Omega(G))$.

**Remark 5.3.** Proposition 5.2 is valid even if $U$ coincides with $G$ (indeed $\bar{\varphi}(x)$ can be described as a $p$-th power of a certain element, see the proof of Proposition 5.2). However there exists an obstruction for taking the projective limit if the exponent $(G : U)/p$ of $\varphi(x)$ is not integral. Therefore the case where $U$ coincides with $G$ remains as an exception to Proposition 5.1.

Proposition 5.2 is deduced from the following elementary lemma in modular representation theory.

**Lemma 5.4.** Let $K$ be a field of positive characteristic $p$, $\Delta$ a finite $p$-group and $V$ a finite dimensional representation space of $\Delta$ over $K$. Let $\text{aug}$ denote the canonical augmentation map $K[\Delta] \to K$. Take a natural number $n$ such that $p^n$ is larger than the $K$-dimension of $V$. Then for each $x$ in $K[\Delta]$, 

\[ x \equiv x^{p^n} \mod p^n \]
the action of $x^p^n$ upon $V$ coincides with the multiplication by \( \text{aug}(x)^{p^n} \). In particular the equation \( x^{p\Delta} = \text{aug}(x)^{p\Delta} \) holds.

**Proof.** Let \( d \) be the \( K \)-dimension of \( V \). The group ring \( K[\Delta] \) is a local ring whose maximal ideal is the augmentation ideal since \( K \) is of characteristic \( p \) and \( \Delta \) is a \( p \)-group. Therefore the only simple \( K[\Delta] \)-module (up to isomorphisms) is \( K \) endowed with trivial \( \Delta \)-action, and moreover there exists a Jordan-Schreier composition series

\[
V = V_1 \supseteq V_2 \supseteq \cdots \supseteq V_d \supseteq V_{d+1} = \{0\}
\]

such that each quotient space \( V_i/V_{i+1} \) is isomorphic to \( K \). Take an arbitrary element \( e_i \) in \( V_i \) not contained in \( V_{i+1} \) for each \( 1 \leq i \leq d \). Then \( \{e_1, e_2, \ldots, e_d\} \) forms a basis of \( V \) over \( K \), with respect to which the action of \( x \) is represented by an upper triangular matrix all of whose diagonal components are equal to \( \text{aug}(x) \). This implies the first claim. The second claim is deduced from the first one (take \( n \) as the \( p \)-order of \( \Delta \) and apply the claim to the regular representation \( V = K[\Delta] \)). \( \square \)

**Proof of Proposition 5.2.** Identify the modulo \( p \) Iwasawa algebra \( \Omega(U/V) \) with the group ring \( \Omega(\Gamma)[U^J/V^J] \), and let \( K \) be the fractional field of \( \Omega(\Gamma) \). Then we may naturally regard each \( t \) in \( \Omega(U/V) \) as an element in \( K[U^J/V^J] \), and therefore the equation

\[
\varphi(U^J/V^J) = \text{aug}_{U,V}(t)\varphi(U^J/V^J)
\]

holds by Lemma 5.4. Now let \( x \) be an arbitrary element in \( \Omega(G)^{\times} \), and set \( z = \text{aug}_G(x) \) and \( y = xz^{-1} \). Then we obtain \( \text{aug}_G(y) = 1 \) and

\[
\bar{\theta}(x) = \bar{\theta}(y)\bar{\theta}(z)
\]

by definition (here we denote the map \( \bar{\theta}_{U,V} \) by \( \bar{\theta} \) to simplify the notation). Since \( z \) is an element in \( \Omega(\Gamma) \) (and hence \( z \) is contained in the centre of \( \Omega(G) \)), the image of \( z \) under the norm map \( \bar{\theta} \) coincides with \( z^{(G:U)} \) by direct calculation. On the other hand we may calculate \( \bar{\varphi}(x) \) as follows:

\[
\bar{\varphi}(x) = \bar{\varphi}(y)\bar{\varphi}(z) = \bar{\varphi}(\text{aug}_G(y))\bar{\varphi}(z) = z^p \quad \text{(use } \text{aug}_G(y) = 1) .
\]

Hence the equation \( \bar{\varphi}(x)^{-(G:U)/p}\bar{\theta}(x) = \bar{\theta}(y) \) holds by (5.2) and (5.3). Moreover (5.1) implies that \( y^{p^N} \) is equal to \( \text{aug}_G(y)^{p^N} = 1 \), and therefore \( \bar{\theta}(y)^{p^N} \) is also trivial. The same argument as above derives a similar equation

\[
\bar{\theta}(y)^{p(U^J/V^J)} = \text{aug}_{U,V}(\bar{\theta}(y))\bar{\theta}(y)^{p(U^J/V^J)},
\]

and consequently the equation

\[
\text{aug}_{U,V}(\bar{\theta}(y))^{p^N} = \bar{\theta}(y)^{p^N} = 1
\]

holds. Since \( \Omega(\Gamma) \) is a domain (recall that \( \Omega(\Gamma) \) is isomorphic to the formal power series ring \( \mathbb{F}_p[[T]] \) ), the last equation implies that \( \text{aug}_{U,V}(\bar{\theta}(y)) \) itself is trivial; in other words \( \bar{\theta}(y) \) is contained in \( 1 + J_{U,V} \). \( \square \)

The last paragraph of the proof above implies that \( \theta_{U,V}(y) \) is contained in \( 1 + J_{U,V} \) if \( y \) is an element in \( \Lambda(G) \) satisfying \( \text{aug}_G(y) \equiv 1 \mod p \). By replacing \( G \) and \( U \) appropriately, we obtain the following useful corollary:

**Corollary 5.5.** Let \( (U, V) \) be an element in \( \mathfrak{F}_B \) such that \( U \) does not contain a non-trivial central element \( c \). Then the norm map \( \text{Nr}_{\Lambda(U \times \langle c \rangle, V)}/\Lambda(U/V) \) induces a group homomorphism from \( 1 + J_{U \times \langle c \rangle, V} \) to \( 1 + J_{U,V} \).
Remark 5.6. Both \(1 + J_{U,V}(e).V\) and \(1 + J_{U,V}\) are actually multiplicative groups; see Proposition 5.7 for details.

5.2. Logarithmic theory. Let us study the \(p\)-adic logarithm on \(1 + J_{U,V}\) for each \((U,V)\) in \(S_B\), as well as those on \(1 + I_U\) for each \(U\) in \(S_A\).

Proposition 5.7. For each \((U,V)\) in \(S_B\), let \(J_{U,V}\) be as in Proposition 5.1. Then

1. the subset \(1 + J_{U,V}\) of \(\Lambda(U/V)\) is a multiplicative subgroup of \(\Lambda(U/V)^\times\); 
2. for each \(y\) in \(J_{U,V}\), the logarithm \(\log(1 + y) = \sum_{m=1}^{\infty}(-1)^{m-1}y^m/m\) converges \(p\)-adically in \(\Lambda(U/V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\); 
3. the kernel (resp. image) of the induced homomorphism

\[\log : 1 + J_{U,V} \to \Lambda(U/V) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\]

is \(\mu_p(\Lambda(U/V))\) (resp. is contained in \(\Lambda(U/V))\) where \(\mu_p(\Lambda(U/V))\) denotes the multiplicative subgroup of \(\Lambda(U/V)^\times\) consisting of all \(p\)-power roots of unity.

Proof. Define \(\bar{\Lambda}_{U,V} : \Omega(U/V) \to \Omega(\Gamma)\) similarly to the previous subsection, and let \(\bar{J}_{U,V}\) be the kernel of \(\bar{\Lambda}_{U,V}\). Since \(\Omega(U/V)\) is commutative, we have

\[\bar{y} = \sum_{u \in U/V} \bar{y}_u u = \left(\sum_{u \in U/V} \bar{y}_u\right)^p = (\bar{\Lambda}_{U,V}(\bar{y}))^p = 0\]

for an element \(\bar{y} = \sum_{u \in U/V} \bar{y}_u u\) in \(\bar{J}_{U,V}\) (each \(\bar{y}_u\) is an element in \(\Omega(\Gamma)\)). Therefore \(y^p\) is contained in \(p\Lambda(U/V)\) for each \(y\) in \(J_{U,V}\).

(1) By the remark above, \((1 + y)^{-1} = \sum_{m=0}^{\infty}(-y)^m\) converges \(p\)-adically in \(1 + J_{U,V}\) for each \(1 + y\) in \(1 + J_{U,V}\).

(2) In a similar way the element \(y^m\) is contained in \(p^{[m/p]}\Lambda(U/V)\) for each \(y\) in \(J_{U,V}\) (for a real number \(x\), we denote by \([x]\) the largest integer not greater than \(x\)), and hence the claim holds.

(3) If we take an element \(x = 1 + y\) from \(1 + J_{U,V}\), we may calculate as \(\bar{x}^p = 1 + \bar{y}^p = 1\) where \(\bar{x} = 1 + \bar{y}\) is the image of \(x\) in \(1 + \bar{J}_{U,V}\). This implies that \(x^p\) is an element in \(1 + p\Lambda(U/V)\) since the \(p\)-adic exponential map and the \(p\)-adic logarithmic map define an isomorphism between \(p\Lambda(U/V)\) and \(1 + p\Lambda(U/V)\) in general (recall that \(p\) is odd). Therefore \(p\log x = \log x^p\) is contained in \(p\Lambda(U/V)\), or equivalently \(\log(1 + J_{U,V})\) is contained in \(\Lambda(U/V)\). Furthermore if we assume that \(\log x = 0\) holds for \(x\) in \(1 + J_{U,V}\), we obtain \(x^p = 1\) by the calculation above, which implies that \(x\) is an element in \(\mu_p(\Lambda(U/V))\). Conversely \(\log x\) vanishes for an arbitrary element \(x\) in \(\mu_p(\Lambda(U/V))\) since \(\Lambda(U/V)\) is free of \(p\)-torsion.

\[\square\]

Lemma 5.8. For each \(U\) in \(S_A\), let \(I_U\) be the \(\mathbb{Z}_p\)-submodule of \(\Lambda(U)\) defined as in Section 3. Then \(I^2\) is contained in \(I_U\). Moreover,

1. when \(N\) is larger than \(1\), the \(\mathbb{Z}_p\)-module \(I_F\) is contained in \(J_F = p\Lambda(\Gamma)\);
2. when \(N\) is larger than \(2\), the \(\mathbb{Z}_p\)-module \(I_{U_h}\) is contained in \(p\Lambda(U_h)\) (hence also in \(J_{U_h}\)) for each \(h\) in \(S\setminus\{e\}\), and there exist canonical inclusions \(p^{k(N-1)}I_{U_h} \subseteq I^{k+1}_{U_h} \subseteq p^{k(N-1)}I_{U_h}\) for an arbitrary natural number \(k\);
Proposition 5.9. When $N$ is larger than 3, the $\mathbb{Z}_p$-module $I_{U_{h,c}}$ is contained in $p\Lambda(U_{h,c})$ (hence also in $J_{U_{h,c}}$) for each $h$ in $\mathcal{S} \setminus \{e,c\}$ satisfying the condition of Case (a), and $p^{k(N-2)}I_{U_{h,c}} \subseteq I_{U_{h,c}}^{k+1} \subseteq p^{k(n_h-2)}I_{U_{h,c}}$ holds for an arbitrary natural number $k$.

(3) Recall that $\# NU_{h,c}$ is contained in the centre of $G$, and therefore $n_h$ is at least 2. In both cases $I_{U_h}$ is contained in $p\Lambda(U_h)$. The last claim is obvious from the explicit description of $I_{U_h}$.

(4) First note that $2 \leq n_h \leq N - 1$ holds since the cardinality of $NU_{h,c}^I$ (which is smaller than $p^N$) is equal to $p^{n_h+1}$. By using this fact, we may exactly calculate as

$$I_{U_{h,c}}^k = (p^{k(N-2)}\mathbb{Z}_p[[U_c]] + p^{k(n_h-1)-1}(1 + c + \cdots + c^{p-1})\mathbb{Z}_p[[\Gamma]])$$

$$\oplus \bigoplus_{i=1}^{p-1} p^{k(n_h-1)-1}i(1 + c + \cdots + c^{p-1})\mathbb{Z}_p[[\Gamma]].$$

for each $k$ larger than 2. The claim holds by this calculation.

Proof. The first claim is easily checked by direct calculation.

(1) Obvious from the exact description of $I_\Gamma$ (see Section 4.3).

(2) If $h$ is contained in the centre of $G^I$, the equation $n_h = N$ clearly holds. Otherwise $NU_{h,c}^I$ has to contain the centre of $G^I$, and therefore $n_h$ is at least 2. In both cases $I_{U_h}$ is contained in $p\Lambda(U_h)$. The last claim is obvious from the explicit description of $I_{U_h}$.

(3) Recall that $\# NU_{h,c}^I = p^{n_h}$ holds in Case (a). Let $\bar{U}_h^I$ denote the quotient group $U_{h,c}^I/U_c^I$. If $\bar{h}$ (the image of $h$ in $\bar{U}_h^I$) is contained in the centre of $G^I = G^I/\bar{U}_c^I$, the normaliser of $\bar{U}_h^I$ obviously coincides with $G^I$, which implies that $n_h$ is equal to $N$. Otherwise there exists a non-trivial element $\bar{a}$ in the centre of $G^I$. Let $a$ be its lift to $G^I$, then the finite subgroup of $G^I$ generated by $c, h$ and $a$ is contained in $NU_{h,c}^I$ by construction. This implies that $n_h$ is at least 3. In both cases we may conclude that $I_{U_{h,c}}$ is contained in $p\Lambda(U_{h,c})$. The last claim is obvious from the explicit description of $I_{U_{h,c}}$.

(4) First note that $2 \leq n_h \leq N - 1$ holds since the cardinality of $NU_{h,c}^I$ (which is smaller than $p^N$) is equal to $p^{n_h+1}$. By using this fact, we may exactly calculate as

$$I_{U_{h,c}}^k = (p^{k(N-2)}\mathbb{Z}_p[[U_c]] + p^{k(n_h-1)-1}(1 + c + \cdots + c^{p-1})\mathbb{Z}_p[[\Gamma]])$$

$$\oplus \bigoplus_{i=1}^{p-1} p^{k(n_h-1)-1}i(1 + c + \cdots + c^{p-1})\mathbb{Z}_p[[\Gamma]].$$

for each $k$ larger than 2. The claim holds by this calculation.

\[\square\]

Proposition 5.9. Let $U$ be an element in $\mathfrak{B}_A$ and assume that $U$ does not coincide with $G$ if $N$ equals either 0, 1 or 2. Then

(1) the subset $1 + I_U$ of $\Lambda(U)$ is a multiplicative subgroup of $\Lambda(U)^\times$;

(2) for each $y$ in $I_U$, the logarithm $\log(1 + y) = \sum_{m=1}^{\infty} (-1)^{m-1}y^m/m$ converges $p$-adically in $I_U$;

(3) the $p$-adic logarithmic homomorphism induces an isomorphism between $1 + I_U$ and $I_U$.

Proof. The claims of (1) and (2) follow from Lemma 5.8 (use the fact that $y^{p^m}/p^m$ is contained in $I_U$ for each $y$ in $I_U$ if $p$ is odd). For (3), first note that $1 + I_U$ is a multiplicative subgroup of $1 + I_U$ and the $p$-adic logarithm induces a homomorphism from $1 + I_U$ to $I_U$ for each natural number $k$ (similarly to (1) and (2)). Moreover the $I_U$-adic topology on $I_U$ coincides with the
$p$-adic topology by Lemma 5.8. Therefore it suffices to show that the $p$-adic logarithm induces an isomorphism
\[
\log : 1 + I_U^k/1 + I_U^{k+1} \to I_U^k/I_U^{k+1}; \ 1 + y \mapsto y
\]
for each natural number $k$. Let $y$ be an element in $I_U^k$. We have only to show that $y^p^m/p^m$ is contained in $I_U^{k+1}$ for each $m \geq 1$, or equivalently, $p^{-m}I_U^kp^m$ is contained in $I_U^{k+1}$ for every $k$ and $m$. We may verify it by direct calculation.$^4$

Remark 5.10. Suppose that $N$ equals either 0, 1 or 2. Then the Artinian family $\mathfrak{F}_A$ contains the whole group $G$ by definition. The $\mathbb{Z}_p$-module $I_G$ obviously coincides with $\Lambda(G)$. Thus the $p$-adic logarithm never converges on $1 + I_G$. We remark that the $\mathbb{Z}_p$-module $I_G = \Lambda(G)$ is the only exception to our logarithmic theory discussed in this subsection.

6. CONSTRUCTION OF THE THETA ISOMORPHISM II —TRANSLATION—

In this section we shall construct the multiplicative theta isomorphism by using the facts studied in Section 5.

6.1. The multiplicative theta isomorphism. Let $(U, V)$ be an element in $\mathfrak{F}_B$. We use the notion “$x \equiv y \mod{J}$” for elements $x$ and $y$ in $\widetilde{\Lambda}(U/V)^\times$ such that $xy^{-1}$ is contained in $1 + J$ —the image of $1 + J$ under the canonical surjection $\Lambda(U/V)^\times \to \widetilde{\Lambda}(U/V)^\times$— if $J$ is a $\mathbb{Z}_p$-submodule of $\Lambda(U/V)$ such that $1 + J$ is a multiplicative subgroup of $\Lambda(U/V)^\times$. Let $\Psi'$ denote the subgroup of $\prod_{(U,V)\in\mathfrak{F}_B} \Lambda(U/V)^\times$ consisting of all elements $\eta^\ast = (\eta_U,V)(U,V)\in\mathfrak{F}_B$ satisfying the following three conditions:

- (norm compatibility condition, NCC)
  the equation $N_{\Lambda(U/V)/\Lambda(U'/V')}(\eta_U,V) = \text{can}_V'((\eta_{U',V'})(U',V')$ holds for $(U, V)$ and $(U', V')$ in $\mathfrak{F}_B$ such that $U$ contains $U'$ and $U'$ contains $V$ respectively (here $\text{can}_V'$ is the canonical map $\Lambda(U'/V') \to \Lambda(U'/V')$);

- (conjugacy compatibility condition, CCC)
  the equation $\eta_{U',V'} = \psi_a(\eta_{U,V})$ holds for $(U, V)$ and $(U', V')$ in $\mathfrak{F}_B$ such that $U' = a^{-1}U$ and $V' = a^{-1}V$ hold for a certain element $a$ in $G$ (we denote by $\psi_a$ the isomorphism $\Lambda(U/V)^\times \to \Lambda(U'/V')^\times$ induced by the conjugation $U/V \to U'/V'$; $u \mapsto a^{-1}ua$);

- (congruence condition)
  the congruence $\eta_U,V \equiv \varphi(\eta_{ab})^{(G,U)/p} \mod{J_{U,V}}$ holds for $(U, V)$ in $\mathfrak{F}_B$ except for $(G, [G, G])$ where $\eta_{ab}$ denotes the $\widetilde{\Lambda}(G^{ab})^\times$-component of $\eta^\ast$ (see the previous section for the definition of $J_{U,V}$).

Let $\Psi$ (resp. $\Psi_e$) be the subgroup of $\Psi'$ consisting of all elements $\eta^\ast$ satisfying the following additional congruence condition (see Section 2A for the definition of $I_U$):

- (additional congruence condition)
  the congruence $\eta_U \equiv \varphi(\eta_{ab})^{(G,U)/p} \mod{I_U}$ holds for each $U$ in $\mathfrak{F}_A$ (resp. $\mathfrak{F}_A^e$).

$^4$In this calculation we use the fact that $p$ is odd.
Remark 6.1. When $N$ equals either 0, 1 or 2, we regard the additional congruence condition for the total group $G$ as the trivial condition (in other words, we do not impose any congruence condition upon $G$). Therefore we have only to consider an element $(U, V)$ in $\mathfrak{S}_B$ (resp. $U$ in $\mathfrak{S}_A$) such that $U$ is a proper subgroup of $G$ in arguments concerning with congruence conditions.

Remark 6.2. For each $U$ in $\mathfrak{S}_A$, we may easily check that the ideal $J_U$ contains $I_U$ unless $U$ coincides with $G$ by using the explicit description of $I_U$ given in Section 4.2; in particular $\tilde{\Psi}$ is a subgroup of $\tilde{\Psi}$.

Let $\theta_{U,V}$ be as in Section 5.1 and set $\theta = (\theta_{U,V})_{(U,V)\in\mathfrak{S}_B}$, then the map $\theta$ induces a group homomorphism $\tilde{\theta}: \tilde{K}_1(\Lambda(G)) \to \prod_{(U,V)\in\mathfrak{S}_B} \tilde{\Lambda}(U/V)^\times$.

Proposition 6.3. The multiplicative group $\tilde{\Psi}$ coincides with $\tilde{\Psi}_c$. Moreover the map $\tilde{\theta}$ induces an isomorphism $\tilde{\theta}: \tilde{K}_1(\Lambda(G)) \sim \tilde{\Psi}$.\hspace{1cm}(= \tilde{\Psi}_c).

In order to prove Proposition 6.3, it suffices to verify surjectivity of $\tilde{K}_1(\Lambda(G)) \to \tilde{\Psi}$ and injectivity of $\tilde{K}_1(\Lambda(G)) \to \tilde{\Psi}_c$ (see Remark 6.2). The arguments to verify these two claims will occupy the rest of this section.

6.2. Integral logarithmic homomorphism. We now introduce the integral logarithmic homomorphisms; for an arbitrary finite $p$-group $\Delta$, Robert Oliver and Laurence Robert Taylor defined a homomorphism of abelian groups (called the integral logarithm)

$$\Gamma_\Delta: K_1(\mathbb{Z}_p[\Delta]) \to \mathbb{Z}_p[\text{Conj}(\Delta)]; \ x \mapsto \log(x) - p^{-1}\varphi(\log(x))$$

where $\varphi$ is “the Frobenius correspondence” on $\mathbb{Z}_p[\text{Conj}(\Delta)]$ characterised by

$$\varphi \left( \sum_{[d] \in \text{Conj}(\Delta)} a_{[d]}[d] \right) = \sum_{[d] \in \text{Conj}(\Delta)} a_{[d]}[d^p].$$

The integral logarithmic homomorphisms are compatible with group homomorphisms; that is, the diagram

$$
\begin{array}{ccc}
K_1(\mathbb{Z}_p[\Delta]) & \xrightarrow{\Gamma_\Delta} & \mathbb{Z}_p[\text{Conj}(\Delta)] \\
\downarrow f_* & & \downarrow f_* \\
K_1(\mathbb{Z}_p[\Delta']) & \xrightarrow{\Gamma_{\Delta'}} & \mathbb{Z}_p[\text{Conj}(\Delta')] \\
\end{array}
$$

commutes for an arbitrary homomorphism $f: \Delta \to \Delta'$ of finite $p$-groups (the symbol $f_*$ denotes the homomorphism of abelian groups induced by $f$). It is known that the sequence

$$1 \to K_1(\mathbb{Z}_p[\Delta])/K_1(\mathbb{Z}_p[\Delta])_{\text{tors}} \xrightarrow{\Gamma_\Delta} \mathbb{Z}_p[\text{Conj}(\Delta)] \xrightarrow{\omega_\Delta} \Delta^{ab} \to 1$$

is exact where $\omega_\Delta$ is the homomorphism of abelian groups defined by

$$\omega_\Delta \left( \sum_{[d] \in \text{Conj}(\Delta)} a_{[d]}[d] \right) = \prod_{[d] \in \text{Conj}(\Delta)} \tilde{d}^{\omega_\Delta}.\hspace{1cm}(6.2)$$
structure of the torsion part of details of the properties of integral logarithms.

Section 4.4; in fact, it is described as by the theorem of Charles Terence Clegg Wall [Wall, Theorem 4.1] where unity. By taking the projective limit, [\( \leftarrow \)] we obtain the following exact sequence (note that the projective limit \( \lim_{n} K_1(\mathbb{Z}_p[\mathbb{G}(n)]) \) actually coincides with \( K_1(\mathbb{G}(n)) \); see [FukKat, Proposition 1.5.1]):

\[
1 \to K_1(\mathbb{G}(n))/\lim_{n} K_1(\mathbb{Z}_p[\mathbb{G}(n)])_{\text{tors}} \xrightarrow{\Gamma_{\mathbb{G}}} \mathbb{Z}_p[[\text{Conj}(\mathbb{G})]] \xrightarrow{\omega_\mathbb{G}} \mathbb{G}_{ab} \to 1.
\]

Moreover \( (6.3) \) implies that the projective limit \( \lim_{n} K_1(\mathbb{Z}_p[\mathbb{G}(n)])_{\text{tors}} \) is isomorphic to the direct product \( \mu_{p-1}(\mathbb{Z}_p) \times \mathbb{G}_{ab} \times SK_1(\mathbb{Z}_p[\mathbb{G}(n)]) \). We may, therefore, identify the \( p \)-torsion part \( K_1(\mathbb{G}(n))_{p}\text{-tors} \) of the Whitehead group \( K_1(\mathbb{G}(n)) \) with \( \mathbb{G}_{ab} \times SK_1(\mathbb{Z}_p[\mathbb{G}(n)]) \) (recall that \( SK_1(\mathbb{Z}_p[\mathbb{G}(n)]) \) is a finite \( p \)-group [Wall, Theorem 2.5]).

We remark that the \( p \)-th power Frobenius endomorphism \( g \to g^p \) is well defined on \( G\) in our case since the exponent of \( G^f \) equals \( p \). We use the same symbol \( \varphi \) for the Frobenius endomorphism on \( G \), then it obviously induces the Frobenius correspondence on \( \mathbb{Z}_p[[\text{Conj}(\mathbb{G})]] \). The notion \( \varphi \) introduced here is compatible with the one defined in Section 5.

6.3. The group \( \tilde{\Psi}_c \) contains the image of \( \tilde{\theta} \). In this subsection we prove that \( \tilde{\Psi}_c \) contains the image of \( \tilde{\theta} \) (and hence \( \Psi \) also does by Remark 6.1).

Lemma 6.4. The multiplicative group \( \tilde{\Psi}_c \) contains the image of \( \tilde{\theta} \).

Proof. The element \( (\tilde{\theta}_{U,V}(\eta))(U,V) \in \tilde{\mathcal{A}}_c \) satisfies both (NCC) and (CCC) for each \( \eta \) in \( \tilde{K}_1(\mathbb{G}(n)) \) by the basic properties of norm maps in algebraic \( K \)-theory. Moreover the congruence \( \tilde{\theta}_{U,V}(\eta) \equiv \varphi(\tilde{\theta}_{ab}(\eta))(G^{U'})/p \mod J_{U,V} \) holds unless \( U \) coincides with \( G \) by Proposition 5.1 (we denote by \( \tilde{\theta}_{ab} \) the homomorphism \( \tilde{K}_1(\mathbb{G}(n)) \to \Lambda(\mathbb{G}_{ab})^x \) induced by the abelisation map; note that \( \varphi(\tilde{\theta}_{ab}(\eta)) \) obviously coincides with \( \varphi(\eta) \) by definition).

By virtue of Lemma 6.4 we have only to verify the following proposition to show that \( \tilde{\Psi}_c \) contains the image of \( \tilde{\theta} \).

Proposition 6.5. Let \( \eta \) be an element in \( \tilde{K}_1(\mathbb{G}(n)) \). Then the congruence \( \tilde{\theta}_{U}(\eta) \equiv \varphi(\tilde{\theta}_{ab}(\eta))(G^{U'})/p \mod J_U \) holds for each \( U \) in \( \tilde{\mathcal{A}}_c \).

\[\text{Since } K_1(\mathbb{Z}_p[\mathbb{G}(n+1)])/K_1(\mathbb{Z}_p[\mathbb{G}(n+1)])_{\text{tors}} \to K_1(\mathbb{Z}_p[\mathbb{G}(n)])/K_1(\mathbb{Z}_p[\mathbb{G}(n)])_{\text{tors}} \text{ is surjective, the exact sequence } (6.2) \text{ for the projective system with respect to } (\mathbb{G}(n))_{n \in \mathbb{N}} \text{ satisfies the Mittag-Leffler condition. Therefore we may take the projective limit.}\]
The following lemma relates norm maps in algebraic $K$-theory to trace homomorphisms defined in Section 4.2 via $p$-adic logarithms.

**Lemma 6.6** (compatibility lemma). Let $(U, V)$ and $(U', V')$ be elements in $\mathcal{F}_B$ such that $U$ contains $U'$. Then the following diagram commutes:

$$
\begin{array}{ccc}
K_1(\Lambda(U)) & \log & Q_p[[\text{Conj}(U)]] \\
\text{Nr}_{\Lambda(U)/\Lambda(U')} & & \downarrow\text{Tr}_{Q_p[[\text{Conj}(U)]]/Q_p[[\text{Conj}(U')]]} \\
K_1(\Lambda(U')) & \log & Q_p[[\text{Conj}(U')]].
\end{array}
$$

**Proof.** We may prove that the diagram commutes for each finite quotient $U^{(n)} = U^I \times \Gamma/\Gamma_p^n$ and $U'^{(n)} = U'^I \times \Gamma/\Gamma_p^n$ by the same argument as that in [H] Lemma 4.7. Hence the claim holds by taking the projective limit. □

**Proof of Proposition 6.7.** We may assume that $U$ does not coincide with $G$ without loss of generality (see Remark 6.1). Let $\theta_{ab}$ (resp. $\theta_{ab}^+$) be the homomorphism $K_1(\Lambda(G)) \to \Lambda(\mathcal{F}_{ab})^\times$ (resp. $\mathbb{Z}_p[[\text{Conj}(G)]] \to \mathbb{Z}_p[[\mathcal{F}_{ab}]]$) induced by the abelisation map $G \to \mathcal{F}_{ab}$. Then we may easily check that the following diagram commutes for each $(U, V)$ in $\mathcal{F}_B$:

$$
\begin{array}{ccc}
Q_p[[\text{Conj}(G)]] & \xrightarrow{Q_p \otimes_{\mathbb{Z}_p} \theta_{ab}^+} & Q_p[[\mathcal{F}_{ab}]] \\
\frac{1}{p} \phi & & \text{Tr}_{p \mathcal{F}_{ab}/p \mathcal{F}_G} \\
Q_p[[\text{Conj}(G)]] & \xrightarrow{Q_p \otimes_{\mathbb{Z}_p} \theta_{ab}^{+U,V}} & Q_p[[U/V]].
\end{array}
$$

(6.5)

Note that $\phi(\tilde{\theta}_{ab}(\eta))^{-(G;U)/p \tilde{\theta}_U(\eta)}$ is contained in $1 + \mathcal{F}_{U}$ for each $U$ in $\mathcal{F}_B$ because $(\tilde{\theta}_{U,V}(\eta))(U,V) \in \mathcal{F}_B$ is an element in $\mathcal{F}_B$ (Lemma 6.3). Then Proposition 5.7(3) asserts that the element $\log(\phi(\tilde{\theta}_{ab}(\eta))^{-(G;U)/p \tilde{\theta}_U(\eta)})$ is contained in $\Lambda(U)$. On the other hand, we may calculate as (6.6)

$$
\theta_{ab}^{+U,V} \circ \Gamma_G(\eta) = (Q_p \otimes_{\mathbb{Z}_p} \theta_{ab}^{+U,V})(\log(\eta)) = (Q_p \otimes_{\mathbb{Z}_p} \theta_{ab}^{U,V})(p^{-1}\phi(\log(\eta))) = \log(\theta_{U,V}(\eta)) - \frac{(G;U)}{p} \phi(\log(\theta_{ab}(\eta))) = \log(\theta_{U,V}(\eta)) - \frac{(G;U)}{p} \phi(\log(\theta_{ab}(\eta)))
$$

for each $(U, V)$ in $\mathcal{F}_B$ (the first equality is nothing but the definition of the integral logarithm and the second follows from Lemma 6.6 and (6.5)). In particular $\log(\phi(\tilde{\theta}_{ab}(\eta))^{-(G;U)/p \tilde{\theta}_U(\eta)})$ is contained in $I_U$ for each $U$ in $\mathcal{F}_B$ by definition. Recall that for each $U$ in $\mathcal{F}_B$ the $p$-adic logarithm is injective on $1 + J_U$ (Proposition 5.7) and it induces an isomorphism between $1 + I_U$ and $I_U$ (Proposition 5.7 (3)) unless $U$ coincides with $G$. Therefore we may conclude that $\phi(\tilde{\theta}_{ab}(\eta))^{-(G;U)/p \tilde{\theta}_U(\eta)}$ is contained in $1 + I_U$, which implies the desired additional congruence for $U$. □

---

\textsuperscript{6} For the abelisation $\theta_{ab} = \theta_{G,[G,G]}$, we use the notation $\log(\phi(\eta)^{-1/p \theta_{ab}(\eta)})$ for an element defined as $\Gamma_{G,ab}(\theta_{ab}(\eta)) = \log(\theta_{ab}(\eta)) - p^{-1}\log(\phi(\theta_{ab}(\eta)))$ by abuse of notation.
By Lemma 6.4 and Proposition 6.5, we may conclude that \( \widetilde{\Psi} \) (resp. \( \widetilde{\Psi}_c \)) contains the image of \( \tilde{\theta} \); in other words, \( \tilde{\theta} \) induces a homomorphism

\[ \tilde{\theta} : \widetilde{K}_1(\Lambda(G)) \to \widetilde{\Psi} \quad \text{(resp. } \widetilde{\Psi}_c). \]

6.4. **Proof of the isomorphy of \( \tilde{\theta} \).** We shall verify the isomorphy of \( \tilde{\theta} \) in this subsection.

**Proposition 6.7.** The homomorphism \( \tilde{\theta} : \widetilde{K}_1(\Lambda(G)) \to \widetilde{\Psi} \) is injective.

**Proof.** Take an arbitrary element from the kernel of \( \tilde{\theta} \) and let \( \eta \) denote its lift to \( K_1(\Lambda(G)) \). Then \( \theta_G^{[U,V]} \circ \Gamma_G(\eta) \) vanishes for each \((U,V)\) in \( \mathfrak{F}_B \) by (6.6). Hence \( \Gamma_G(\eta) \) coincides with zero since \( \theta_B^{[U,V]} \) is injective (Proposition 4.5); equivalently the element \( \eta \) is contained in the kernel of the integral logarithm \( \Gamma_G \). Combining this fact with Wall’s theorem (see [Wall Theorem 4.1] and (6.3)), we may regard \( \eta \) as an element in \( \mu_{p^{-1}}(\mathbb{Z}_p) \times G^{ab} \times SK_1(\mathbb{Z}_p[G^f]) \). Furthermore the abelisation map \( \theta_{ab} \) induces the canonical projection from \( \mu_{p^{-1}}(\mathbb{Z}_p) \times G^{ab} \times SK_1(\mathbb{Z}_p[G^f]) \) onto \( \mu_{p^{-1}}(\mathbb{Z}_p) \times G^{ab} \) when it is restricted to the kernel of \( \Gamma_G \). Since \( \theta_{ab}(\eta) \) vanishes by assumption, the element \( \eta \) is contained in \( G^{ab,f} \times SK_1(\mathbb{Z}_p[G^f]) \), and in particular \( \eta \) is a \( p \)-torsion element. This implies that the image of \( \eta \) in \( \widetilde{K}_1(\Lambda(G)) \) reduces to be trivial. \( \square \)

**Proposition 6.8.** The homomorphism \( \tilde{\theta} : \widetilde{K}_1(\Lambda(G)) \to \widetilde{\Psi} \) is surjective.

Let \( \eta \) be an element in \( \widetilde{\Psi} \). Since \( \eta \) is in particular contained in \( \widetilde{\Psi} \), the element \( \log(\varphi(\eta_{ab})^{-1}(G,U)/p_{U,V}) \) can be defined as an element in \( \Lambda(U/V) \) for each \((U,V)\) in \( \mathfrak{F}_B \) (Proposition 5.7 (2) and the definition of the integral logarithm for \( G^{ab} \)).

**Lemma 6.9.** The element \( \log(\varphi(\eta_{ab})^{-1}(G,U)/p_{U,V}) \) is contained in \( \Phi_B \). Moreover \( \log(\varphi(\eta_{ab})^{-1}(G,U)/p_{U,V}) \) is contained in \( \Phi \).

**Proof.** Set \( y_{U,V} = \log(\varphi(\eta_{ab})^{-1}(G,U)/p_{U,V}) \) for each \((U,V)\) in \( \mathfrak{F}_B \). Then we may easily verify that \( y_{U,V}(U,V) \in \mathfrak{F}_B \) satisfies both (TCC) and (CCC+) (due to (NCC), (CCC) and Lemma 6.6). Hence \( y_{U,V}(U,V) \) is contained in \( \Phi_B \). Moreover \( \varphi(\eta_{ab})^{-1}(G,U)/p_{U,V} \) is contained in \( 1 + I_U \) for each \( U \) in \( \mathfrak{F}_A \) by additional congruence condition, and thus \( y_{U} = \log(\varphi(\eta_{ab})^{-1}(G,U)/p_{U,V}) \) is contained in \( I_U \) by Proposition 5.9. This implies that \( y_{U,V} \) is an element in \( \Phi \). \( \square \)

**Proof of Proposition 6.8.** First note that there exists a unique element \( y \) in \( \mathbb{Z}_p[[\text{Conj}(G)]] \) which satisfies \( \theta_B^{[U,V]}(y) = (\log(\varphi(\eta_{ab})^{-1}(G,U)/p_{U,V}))(U,V) \in \mathfrak{F}_B \) by Proposition 4.5 and Lemma 6.9. In particular the equation

\[ (6.7) \quad \theta_{ab}^\gamma(y) = \log(\eta_{ab}) - \frac{1}{p}\varphi(\log(\eta_{ab})) = \Gamma_{G^{ab}}(\eta_{ab}) \]

holds. Then we may calculate as

\[ \omega_G(y) = \omega_{G^{ab}} \circ \theta_{ab}^\gamma(y) = \omega_{G^{ab}} \circ \Gamma_{G^{ab}}(\eta_{ab}) = 1 \]

where the first equality directly follows from the definition of \( \omega_G \) and \( \omega_{G^{ab}} \) (see Section 6.2), the second follows from (6.7) and the last follows from
The sequence (6.4) also asserts that there exists an element \( \eta' \) in \( K_1(\Lambda(G)) \) which satisfies \( \Gamma_G(\eta') = y \). Furthermore we obtain
\[
\Gamma_{G,ab}(\theta_{ab}(\eta')) = \theta_{ab}^+ \circ \Gamma_G(\eta') = \theta_{ab}^+(y) = \Gamma_{G,ab}(y_{ab})
\]
by using (6.7). Since the kernel of \( \Gamma_{G,ab} \) is identified with \( \mu_{p^{-1}}(\mathbb{Z}_p) \times G_{ab} \) by the theorem of Graham Higman [Higman], there exists an element \( \tau \) in \( \mu_{p^{-1}}(\mathbb{Z}_p) \times G_{ab} \) such that the equation \( \theta_{ab}(\eta') \tau = \eta_{ab} \) holds. Set \( \eta = \eta' \tau \).

By construction, the abelianisation \( \tilde{\eta}(\eta) \) of \( \eta \) coincides with \( \eta_{ab} \), and
\[
\log \frac{\eta_{UV}}{\varphi(\eta_{ab})(G:U)/p} = \theta_{UV}^+(y) = \theta_{UV}^+ \circ \Gamma_G(\eta) = \log \frac{\theta_{UV}(\eta)}{\varphi(\theta_{ab}(\eta))(G:U)/p}
\]
holds for each \((U, V)\) in \( \mathfrak{F}_B \) except for \((G, [G, G])\) (the first equality is due to the construction of \( y \) and the last is due to (6.6)). Then \( \tilde{\theta}_{UV}(\eta) \) coincides with \( \eta_{UV} \) because the \( p \)-adic logarithm induces an injection on \( 1 + \mathfrak{J}_{UV} \) (Proposition 5.7); in other words the image of \( \eta \) under the map \( \tilde{\eta} \) coincides with \( \eta_\bullet \), which asserts that \( \tilde{\eta} : K_1(\Lambda(G)) \to \tilde{\Psi} \) is surjective.

7. Localized version

In this section we study “the localised theta map;” more precisely, let \( \theta_{S,U,V} \) be the composition of the norm map \( \text{Nr}_{\Lambda(G)_S/\Lambda(U,V)_S} \) with the canonical homomorphism \( K_1(\Lambda(U,V)_S) \to \Lambda(U,V)_S^\times \) for each \((U, V)\) in \( \mathfrak{F}_B \) and set \( \theta_S = (\theta_{S,U,V})_{(U,V)\in \mathfrak{F}_B} \). It is obvious that \( \theta_S \) induces a group homomorphism \( \tilde{\theta}_S : \tilde{K}_1(\Lambda(G)_S) \to \prod_{(U,V)\in \mathfrak{F}_B} \Lambda(U,V)_S^\times \). We shall study the image of \( \tilde{\theta}_S \).

Let \( \Lambda(\Gamma)_{(p)} \) denote the localisation of the Iwasawa algebra \( \Lambda(\Gamma) \) with respect to the prime ideal \( p\Lambda(\Gamma) \), and let \( R \) denote its \( p \)-adic completion \( \Lambda(\Gamma)_{(p)} \hat{} \) for simplicity. We remark that for each finite \( p \)-group \( \Delta \), the localised Iwasawa algebra \( \Lambda(\Delta \times \Gamma)_{S} \) is identified with the group ring \( \Lambda(\Gamma)_{(p)}[\Delta] \) under the identification \( \Lambda(\Delta \times \Gamma) \cong \Lambda(\Gamma)[\Delta] \) (see [CFKS, Lemma 2.1]). Now for each \((U, V)\) in \( \mathfrak{F}_B \), let \( J_{S,U,V} \) (resp. \( J_{U,V} \)) be the kernel of the composition
\[
\Lambda(U/V)_S \xrightarrow{\text{augmentation}} \Lambda(\Gamma)_{(p)} / \Lambda(\Gamma)_{(p)}/p\Lambda(\Gamma)_{(p)} \xrightarrow{\text{augmentation}} R \xrightarrow{R[U/V]^f} R/pR.
\]

Then we may easily verify that the intersection of \( J_{U,V} \) and \( \Lambda(U/V)_S \) (resp. \( J_{S,U,V} \) and \( \Lambda(U/V) \)) coincides with \( J_{S,U,V} \) (resp. \( J_{U,V} \)) under the identification \( \Lambda(U/V)_S \cong \Lambda(\Gamma)_{(p)}[U/V]^f \). Since the group ring \( R[U/V]^f \) is \( p \)-adically complete, the \( p \)-adic logarithm converges on \( 1 + \mathfrak{J}_{UV} \) and induces an injection \( \log : (1 + \mathfrak{J}_{UV}) \to R[U/V]^f \) unless \( U \) coincides with \( G \) (similarly to Proposition 5.7). Let \( \tilde{\Psi} \) be the subgroup of the direct product \( \prod_{(U,V)\in \mathfrak{F}_B} \Lambda(U/V)_S^\times \) consisting of all elements \( \eta_{S,\bullet} \) satisfying norm compatibility condition \( (\text{NCC})_S \), conjugacy compatibility condition \( (\text{CCC})_S \) and the following congruence for each \((U, V)\) in \( \mathfrak{F}_B \) except for \((G, [G, G])\):
\[
\eta_{S,U,V} \equiv \varphi(\eta_{S,ab})(G:U)/p \mod J_{S,U,V}.
\]

\({}^7\)We may naturally extend both (NCC) and (CCC) to the localised versions \( (\text{NCC})_S \) and \( (\text{CCC})_S \) in an obvious manner.
Let $\tilde{\Psi}_S$ (resp. $\tilde{\Psi}_{S,c}$) be the subgroup of $\tilde{\Psi}'_S$ consisting of all elements $\eta_S$, satisfying the following additional congruence condition:

(additional congruence condition)

the congruence $\eta_{S,U} \equiv \varphi(\eta_{S,ab})^{(G,U)/p} \mod I_{S,U}$ holds for each $U$ in $A_\Lambda$ (resp. $A_\Lambda^c$) where $I_{S,U}$ is the $\Lambda(\Gamma)(p)$-module defined as $I_U \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)(p)$. The group $\tilde{\Psi}_{S,c}$ is a subgroup of $\tilde{\Psi}_S$ (as $\tilde{\Psi}_c$ is that of $\tilde{\Psi}$; see also Remark 6.1).

**Lemma 7.1.** The intersection of $I_U$ and $\Lambda(U)_S$ (resp. $I_{S,U}$ and $\Lambda(U)$) coincides with $I_{S,U}$ (resp. $I_U$).

**Proof.** We shall only prove the claim $I_{S,U} \cap \Lambda(U) = I_U$ (the other one is verified by much simpler calculation). The $\mathbb{Z}_p$-module $I_U$ is obviously contained in the intersection $I_{S,U} \cap \Lambda(U)$ by construction. Note that $I_{S,U}$ is a free $\Lambda(\Gamma)(p)$-submodule of $\Lambda(U)_S$ each of whose generators is obtained as finite sum of $\{p^iu\}_{0 \leq i \leq N_{ab}U}$ (see the explicit description of $I_U$ given in Section 4.2). Hence an arbitrary element in $I_{S,U} \cap \Lambda(U)$ is described as a $\Lambda(\Gamma)(p) \cap \Lambda(U)[p^{-1}]$-linear combination of generators of $I_{S,U}$, which implies that the intersection $I_{S,U} \cap \Lambda(U)$ is contained in $I_U$ (observe that $\Lambda(\Gamma)(p) \cap \Lambda(U)[p^{-1}]$ coincides with $\Lambda(U)$ and generators of $I_{S,U}$ over $\Lambda(U)$ coincide with those of $I_U$ over $\Lambda(U)$).

**Proposition 7.2.** Both $\tilde{\Psi}_S$ and $\tilde{\Psi}_{S,c}$ contain the image of $\hat{\theta}_S$.

**Sketch of the proof.** Let $\eta_S$ be an arbitrary element in $\tilde{K}_1(\Lambda(G)_S)$. By the same argument as that in the proof of Lemma 6.1, we may verify that $\tilde{\Psi}'_S$ contains the image of $\hat{\theta}_S$. Then the element $\varphi(\theta_{S,ab}(\eta_S))^{(G,U)/p} \theta_{S,U,V}(\eta_S)$ (which we denote by $\eta_{S,U,V}$ in the following) is contained in $1 + J_{S,U,V}$ for $(U,V)$ in $B$ except for $(G, [G,G])$ by congruence condition, and it is regarded as an element in $1 + J_{U,V}$ in a natural way. Hence we may define $\log \eta_{S,U,V}$ as an element in $R[U'/V']$. On the other hand we may easily show that for each $U$ in $A_\Lambda$ the image of the trace map $\text{Tr}_{R[\text{Conj}(G')]/[R[U']]}$ coincides with the $R$-module $I_U$ defined as $I_U \otimes_{\Lambda(\Gamma)} R$ (here we assume that $U$ does not coincide with $G$; see Remark 6.1). Moreover the image of $\eta_S$ under the composite map $\text{Tr}_{R[\text{Conj}(G')]/[R[U']]} \circ \Gamma_{R,G'}$ is calculated as $\log \eta_{S,U,V}$ by calculation similar to (6.6) where $\hat{\Gamma}_{R,G'}$ is the integral logarithm $\hat{K}_1(R[G']) \to R[\text{Conj}(G')]$ with coefficient in $R$ (see [11, Section 1.1 and Remark 5.2]). This implies that $\log \eta_{S,U,V}$ is contained in $I_U$ for each $U$ in $A_\Lambda^c$. Then we obtain the congruence $\tilde{\theta}_{S,U}(\eta_S) \equiv \varphi(\hat{\theta}_{S,ab}(\eta_S))^{(G,U)/p} \mod I_{S,U}$ by the logarithmic isomorphism $1 + I_U \rightiso I_U$ (readily verified in the same manner as Proposition 6.1) and the relation $I_U \cap \Lambda(U)_S = I_{S,U}$ (Lemma 7.1). Consequently $(\tilde{\theta}_{S,U,V}(\eta_S))_{U,V} \in B_\Lambda$ is contained in $\tilde{\Psi}_{S,c}$ (and hence in $\tilde{\Psi}_S$).

**Proposition 7.3.** The intersection of $\tilde{\Psi}_S$ (resp. $\tilde{\Psi}_{S,c}$) and the direct product $\prod_{U,V} \Lambda(U/V)^\times$ coincides with $\tilde{\Psi}$ (resp. $\tilde{\Psi}_c$).

**Proof.** Use the relations $I_{S,U} \cap \Lambda(U) = I_U$ for each $U$ in $A_\Lambda^c$ (Lemma 7.1) and $J_{S,U,V} \cap \Lambda(U/V) = J_{U,V}$ for each $(U,V)$ in $B_\Lambda$.
8. Weak congruences upon abelian \( p \)-adic \( \zeta \) functions

In this section we study properties of the \( p \)-adic \( \zeta \) pseudomeasures for extensions corresponding to certain abelian subquotients of \( G \), especially congruences which they satisfy. In the rest of this article, we fix embeddings \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \) and \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \).

8.1. Weak Congruences. For each \((U, V)\) in \( \mathfrak{F}_B \), let \( \xi_{U,V} \) denote Serre’s \( p \)-adic \( \zeta \) pseudomeasure for the abelian extension \( F_V/F_U \) (which is an element in \( \Lambda(U/V)_{\mathfrak{e}} \)).

Lemma 8.1. The element \( (\xi_{U,V})(U,V) \in \mathfrak{F}_B \) in \( \prod_{(U,V) \in \mathfrak{F}_B} \tilde{\Lambda}(U/V)^{\chi}_{\mathfrak{e}} \) satisfies both norm compatibility condition (NCC) and conjugacy compatibility condition (CCC).

Proof. Let \((U, V)\) and \((U', V')\) be elements in \( \mathfrak{F}_B \) such that \( U \) contains \( U' \) and \( U \) contains \( V \) respectively. Then we may easily verify that

\[
\text{Nr}_{\Lambda(U/V)_{\mathfrak{e}}/\Lambda(U'/V')_{\mathfrak{e}}} (f) (\rho) = f (\text{Ind}_{U'}^U (\rho))
\]

holds for an arbitrary element \( f \) in \( \Lambda(U/V)_{\mathfrak{e}} \) and an arbitrary continuous \( p \)-adic character \( \rho \) of the abelian group \( U'/V' \) (due to the definition of the evaluation map). Hence for an arbitrary finite-order character \( \chi \) of \( U'/V \) and an arbitrary natural number \( r \) divisible by \( p - 1 \), the following equation holds by the interpolation property (2.1) of \( \xi_{U,V} \):

\[
\text{Nr}_{\Lambda(U/V)_{\mathfrak{e}}/\Lambda(U'/V')_{\mathfrak{e}}} (\xi_{U,V}) (\chi \kappa^r) = \xi_{U',V'} (\text{Ind}_{U'}^U (\chi \kappa^r)) = L_{\Sigma} (1 - r; F_V/F_U, \text{Ind}_{U'}^U (\chi)) = L_{\Sigma} (1 - r; F_V/F_{U'}, \chi) = \xi_{U',V'} (\chi \kappa^r).
\]

Then uniqueness of the abelian \( p \)-adic \( \zeta \) pseudomeasures for \( F_V/F_{U'} \) asserts that the norm image \( \text{Nr}_{\Lambda(U/V)_{\mathfrak{e}}/\Lambda(U'/V')_{\mathfrak{e}}} (\xi_{U,V}) \) of \( \xi_{U,V} \) coincides with \( \xi_{U',V} \). The equation can also be verified straightforwardly, and therefore \( (\xi_{U,V})(U,V) \in \mathfrak{F}_B \) satisfies (NCC). By a similar formal argument we may also prove that \( (\xi_{U,V})(U,V) \in \mathfrak{F}_B \) satisfies (CCC), but we omit the details.

Therefore if \( (\xi_{U,V})(U,V) \in \mathfrak{F}_B \) satisfies both congruence condition and additional congruence condition, we may conclude that \( (\xi_{U,V})(U,V) \in \mathfrak{F}_B \) is contained in \( \tilde{\Psi}_{S,c} \) (hence also in \( \tilde{\Psi}_S \)). It is, however, difficult to prove the desired congruences for \( \{\xi_{U,V}\}(U,V) \in \mathfrak{F}_B \) directly. In the rest of this section we shall prove the following weak congruences by using Deligne-Ribet’s theory upon Hilbert modular forms [DR] (especially using the \( q \)-expansion principle).

Proposition 8.2 (weak congruences). Let \((U, V)\) be an element in \( \mathfrak{F}_B \) such that \( U \) does not coincide with \( G \), then there exists an element \( c_{U,V} \) in \( \tilde{\Lambda}(\Gamma)^{\mathfrak{e}}_{(p)} \) and the congruence

\[
(8.1) \quad \xi_{U,V} \equiv c_{U,V} \mod J_{S,U,V}
\]

holds. If \( U \) is an element in \( \mathfrak{F}_A \), the congruence

\[
(8.2) \quad \xi_{U} \equiv c_{U} \mod I_{S,U}
\]
also holds where $I'_U$ is the image of the trace map from $\mathbb{Z}_p[[\text{Conj}(NU)]]$ to $\mathbb{Z}_p[[U]]$ and $I'_{S,U}$ is its scalar extension $I'_U \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)(p)$.

Remark 8.3. We may obtain the explicit description of each $I'_U$ by calculation similar to that in Section 4.2 as follows:

\[ I'_1 = p^N \mathbb{Z}_p[[\Gamma]], \]
\[ I'_{U_h} = p^{n_h-1} \mathbb{Z}_p[[U_h]] \quad \text{for } h \in \mathfrak{H} \setminus \{e\}, \]
\[ I'_{U_h,c} = p^{n_h-2} \mathbb{Z}_p[[U_{h,c}]] \quad \text{for } h \in \mathfrak{H} \setminus \{e,c\} \text{ satisfying (Case-1),} \]
\[ I'_{U_h,c} = p^{n_h-1} \mathbb{Z}_p[[U_c]] \oplus \bigoplus_{i=1}^{p-1} p^{n_h-2} h^i (1 + c^2 + \cdots + c^{p-1}) \mathbb{Z}_p[[\Gamma]] \quad \text{for } h \in \mathfrak{H} \setminus \{e,c\} \text{ satisfying (Case-2).} \]

Each $I'_U$ (resp. $I'_{S,U}$) obviously contains $I_U$ (resp. $I_{S,U}$). Moreover the $p$-adic logarithm induces an isomorphism between $1 + I'_U$ and $I'_U$ (resp. between $1 + (I'_U)^c$ and $(I'_U)^c$ where $(I'_U)^c$ is defined as $I'_U \otimes_{\Lambda(\Gamma)} \mathbb{R}$) by the same argument as that in the proof of Proposition 5.9.

8.2. Ritter-Weiss’ approximation technique. In [RW6], Jürgen Ritter and Alfred Weiss approximated the $p$-adic zeta pseudomeasure by using special values of partial zeta functions, and derived certain congruences among $p$-adic zeta pseudomeasures. In this section we shall derive sufficient condition for Proposition 8.2 to hold by applying their approximation technique. Fix an element $(U, V)$ in $\mathfrak{H}$ such that $U$ does not coincide with $G$, and let $W$ denote the quotient group $U/V$ (which is abelian by definition). For an arbitrary open subgroup $U$ of $W$, we define the natural number $m(U)$ by $\kappa^{-1}(U) = 1 + m(U) \mathbb{Z}_p$ where $\kappa$ is the $p$-adic cyclotomic character. Then we obtain an isomorphism

\[ (8.3) \quad \mathbb{Z}_p[[W]] \xrightarrow{\sim} \lim_{\substack{U \subset W: \text{open}}} \mathbb{Z}_p[[W/U]]/p^{m(U)} \mathbb{Z}_p[[W/U]] \]

(see [RW6] Lemma 1 for details).

Definition 8.4 (partial zeta function). Let $\varepsilon$ be a $\mathbb{C}$-valued locally constant function on $W$. If $\varepsilon$ is constant on an open subgroup $U$ of $W$, we may identify $\varepsilon$ with a $\mathbb{C}$-linear combination $\sum_{x \in W/U} \varepsilon(x) \delta^{(x)}(s, \varepsilon)$ where $\delta^{(x)}$ is “the Dirac delta function at $x$” (that is, $\delta^{(x)}(w)$ equals 1 if $w$ is in the coset $x$ and 0 otherwise). Then we define the $(\Sigma$-truncated) partial zeta function $\zeta_{\Sigma}^{\sum} F_{V/F_{U}} (s, \varepsilon)$ for $F_{V/F_{U}}$ with respect to the locally constant function $\varepsilon$ as $\sum_{x \in W/U} \varepsilon(x) \zeta_{\Sigma}^{\sum} F_{V/F_{U}} (s, \delta^{(x)})$ where $\zeta_{\Sigma}^{\sum} F_{V/F_{U}} (s, \delta^{(x)})$ is defined as the Dirichlet series

\[ \zeta_{\Sigma}^{\sum} F_{V/F_{U}} (s, \delta^{(x)}) = \sum_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_{F_{U}}} \frac{\delta^{(x)}((F_{V/F_{U}}, \mathfrak{a}))}{(N\mathfrak{a})^s} \]

(the symbol $(F_{V/F_{U}}, -)$ denotes the Artin symbol for the abelian extension $F_{V/F_{U}}$ and $N\mathfrak{a}$ denotes the absolute norm of the ideal $\mathfrak{a}$). It is meromorphically continued to the whole complex plane $\mathbb{C}$. 
For an arbitrary natural number $k$ divisible by $p - 1$ and an arbitrary element $w$ in $W$, set
\[
\Delta_{F_{V}/F_U}^w(1 - k, \varepsilon) = \zeta^w_{F_{V}/F_U}(1 - k, \varepsilon) - \kappa(w)^k \zeta^w_{F_{V}/F_U}(1 - k, \varepsilon_w)
\]
which is a $p$-adic rational number due to the results of Helmut Klingen and Carl Ludwig Siegel [Klingen, Siegel] (we denote by $\varepsilon_w$ the function defined by $\varepsilon_w(w') = \varepsilon(ww')$).

**Proposition 8.5** (approximation lemma, Ritter-Weiss). Let $\mathcal{U}$ be an arbitrary open normal subgroup of $W$. Then for each $k$ divisible by $p - 1$ and each $w$ in $W$, the image of the element $(1 - w)\xi_{U,V}$ under the canonical surjection $\mathbb{Z}_p[[W]] \to \mathbb{Z}_p[W/\mathcal{U}]/p^{m(\mathcal{U})}\mathbb{Z}_p[W/\mathcal{U}]$ is described as
\[
\sum_{x \in W/\mathcal{U}} \Delta_{F_{V}/F_U}^w(1 - k, \delta(x))\kappa(x)^{-k}x \mod p^{m(\mathcal{U})}.
\]

**Proof.** See [RW6, Proposition 2]. \qed

Let $j$ be a natural number and $NU$ the normaliser of $U$. Then the quotient group $NU/U$ acts upon $W/\Gamma^p$ by conjugation (recall that $\Gamma^p$ is abelian). For each coset $y$ of $W/\Gamma^p$, let $(NU/U)_y$ denote the isotropy subgroup of $NU/U$ at $y$ under this action.

**Proposition 8.6** (sufficient condition). Let $(U, V)$ be an element in $\mathfrak{F}_B$ except for $(G, [G, G])$. Then the congruence \eqref{B.1} holds if the congruence \eqref{2.5} holds for an arbitrary element $w$ in $\Gamma$, an arbitrary natural number $k$ divisible by $p - 1$ and an arbitrary coset $y$ of $W/\Gamma^p$ not contained in $\Gamma$. If $U = (U, \{e\})$ is an element in $\mathfrak{F}_A$, the congruence \eqref{2.3} also gives sufficient condition for the congruence \eqref{2.2} to hold.

**Proof.** Apply the approximation lemma (Proposition 8.5) to the element $(1 - w)\xi_{U,V}$, and then its image under the canonical surjection from $\mathbb{Z}_p[[W]]$ onto $\mathbb{Z}_p[W/\Gamma^p]/p^{m(\Gamma^p)}\mathbb{Z}_p[W/\Gamma^p]$ is described as
\[
\sum_{y \in W/\Gamma^p} \Delta_{F_{V}/F_U}^w(1 - k, \delta(y))\kappa(y)^{-k}y \mod p^{m(\Gamma^p)}
\]
for an arbitrary natural number $k$ divisible by $p - 1$. Let $y$ be a coset of $W/\Gamma^p$ not contained in $\Gamma$, and consider the $NU/U$-orbital sum in \eqref{2.5} containing the term associated to $y$. We may calculate it by applying \eqref{2.4} as follows:
\[
\sum_{\sigma \in (NU/U)/(NU/U)_y} \Delta_{F_{V}/F_U}^w(1 - k, \delta(\sigma^{-1}y\sigma))\kappa(\sigma^{-1}y\sigma)^{-k}\sigma^{-1}y\sigma
\]
\[
= \Delta_{F_{V}/F_U}^w(1 - k, \delta(y))\kappa(y)^{-k} \sum_{\sigma \in (NU/U)/(NU/U)_y} \sigma^{-1}y\sigma
\]
\[
\equiv \sharp(NU/U)_y \sum_{\sigma \in (NU/U)/(NU/U)_y} \sigma^{-1}y\sigma \mod \sharp(NU/U)_y.
\]
The element \( \text{aug}_{U,V}(P_y) = \sharp(NU/U)\text{aug}_{U,V}(y) \) is obviously divisible by \( p \) if we set \( P_y \) as the last expression of the equation above (note that the normaliser of \( U \) is strictly larger than \( U \) since \( U \) is a proper open subgroup of the pro-\( p \) group \( G \)). This calculation implies that \( P_y \) is an element in \( J_{U,V} \otimes_{\Lambda(\Gamma)} \mathbb{Z}_p[\Gamma/\Gamma^p]/p^m(\Gamma^p) \). If \( U \) is an element in \( \mathfrak{F}^A \), the element \( P_y \) is no other than the image of \( y \) under the trace map from \( \mathbb{Z}_p[[\text{Conj}(NU)]] \) to \( \mathbb{Z}_p[[U]] \). Therefore \( P_y \) is contained in \( I_U \otimes_{\Lambda(\Gamma)} \mathbb{Z}_p[\Gamma/\Gamma^p]/p^m(\Gamma^p) \). Clearly \( \Delta_{F^U/F^V}(1-w)\xi_{U,V}(1-w)^{-k} \) is an element in \( \mathbb{Z}_p[\Gamma/\Gamma^p]/p^m(\Gamma^p) \) if \( y \) is a coset contained in \( \Gamma \), and hence we may show by taking the projective limit that the element \( (1-w)\xi_{U,V} \) (resp. \( (1-w)\xi_U \) for \( U \) in \( \mathfrak{F}^A \)) is contained in \( \Lambda(\Gamma) + J_{U,V} \) (resp. \( \Lambda(\Gamma) + I_U \)). Since \( 1-w \) is an invertible element in \( \Lambda(U/V)_S \), we obtain the desired congruences (8.1) and (8.2).

8.3. Deligne-Ribet’s theory upon Hilbert modular forms. We briefly summarise the theory of Pierre Deligne and Kenneth Alan Ribet upon Hilbert modular forms [DR] in this subsection, which we shall use in verification of sufficient condition (8.4).

Let \( K \) be a totally real number field of degree \( r \) and \( K_{\infty}/K \) an abelian totally real \( p \)-adic Lie extension. Let \( \mathcal{O} \) be the different of \( K \) and \( \Sigma \) a finite set of prime ideals of \( K \) and assume that \( \Sigma \) contains all primes which ramify in \( K_{\infty} \) (we fix such a finite set \( \Sigma \) throughout the following argument). We denote by \( \mathfrak{h}_K \) the Hilbert upper-half space associated to \( K \) defined as \( \{ \tau \in K \otimes \mathbb{C} \mid \text{Im}(\tau) > 0 \} \). For an even natural number \( k \), we define the action of \( \text{GL}_2(K)^+ \) — subgroup of \( \text{GL}_2(K) \) consisting of all matrices with totally positive determinants — upon the set of \( \mathbb{C} \)-valued functions on \( \mathfrak{h}_K \) by

\[
(F|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix})(\tau) = N(ad-bc)^{k/2}N(c\tau+d)^{-k}F(\frac{a\tau+b}{c\tau+d})
\]

where \( N : K \otimes \mathbb{C} \to \mathbb{C} \) denotes the usual norm map.

**Definition 8.7** (Hilbert modular forms). Let \( \mathfrak{f} \) be an integral ideal of \( \mathcal{O}_K \) all of whose prime factors are contained in \( \Sigma \) and set

\[
\Gamma_{00}(\mathfrak{f}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(K) \mid a, d \in 1 + \mathfrak{f}, b \in \mathcal{O}_K^{-1}, c \in \mathfrak{f}\mathcal{O} \right\}.
\]

Then a *Hilbert modular form* \( F \) of (parallel) weight \( k \) on \( \Gamma_{00}(\mathfrak{f}) \) is defined as a holomorphic function \( F : \mathfrak{h}_K \to \mathbb{C} \) which is fixed by the action of \( \Gamma_{00}(\mathfrak{f}) \) (namely \( F|_k M = F \) holds for an arbitrary element \( M \) in \( \Gamma_{00}(\mathfrak{f}) \)).

Let \( \mathcal{A}_K^{\text{fin}} \) denote the finite adèles ring of \( K \). Then \( \text{SL}_2(\mathcal{A}_K^{\text{fin}}) \) is decomposed as \( \hat{\Gamma}_{00}(\mathfrak{f}) \cdot \text{SL}_2(K) \) by the strong approximation theorem (we denote by \( \hat{\Gamma}_{00}(\mathfrak{f}) \) the topological closure of \( \Gamma_{00}(\mathfrak{f}) \) in \( \text{SL}_2(\mathcal{A}_K^{\text{fin}}) \)). We define the action of \( \text{SL}_2(\mathcal{A}_K^{\text{fin}}) \) upon the set of all \( \mathbb{C} \)-valued functions on \( \mathfrak{h}_K \) by \( F|_k M = F|_k M_{\text{SL}_2(K)} \) where \( M_{\text{SL}_2(K)} \) is the \( \text{SL}_2(K) \)-factor of \( M \) in \( \text{SL}_2(\mathcal{A}_K^{\text{fin}}) \). For a finite idèle \( \alpha \) of \( K \) and a Hilbert modular form \( F \) of weight \( k \) on \( \Gamma_{00}(\mathfrak{f}) \), set

\[
F_{\alpha} = F|_k \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.
\]

If \( K \) is the rational number field \( \mathbb{Q} \), we assume that \( F \) is holomorphic at the cusp \( \infty \).
Then $F_\alpha$ has a Fourier series expansion

$$F_\alpha = c(0, \alpha) + \sum_{\mu \in \mathcal{O}_K, \mu \gg 0} c(\mu, \alpha)q^K_\mu, \quad q^K_\mu = \exp(2\pi \sqrt{-1} \text{Tr}_{K/\mathbb{Q}}(\mu \tau))$$

which we call the \textit{q-expansion} of $F$ at the cusp determined by $\alpha$. Especially, the $q$-expansion of $F$ at the cusp $\infty$ (determined by 1) is called the standard $q$-expansion of $F$. Deligne and Ribet proved the following deep theorem.

**Theorem 8.8** ([DR Theorem (0.2)]). Let $F_k$ be a Hilbert modular form of weight $k$ on $\Gamma_0(\mathfrak{m})$. Assume that all coefficients of the $q$-expansion of $F_k$ at an arbitrary cusp are rational numbers, and assume also that $F_k$ is equal to zero for all but finitely many $k$. Set $\mathcal{F}(\alpha) = \sum_{k \geq 0} \mathcal{N}_p^{-k} F_k,\alpha$ for a finite idèle $\alpha$ of $K$ whose $p$-th component we denote by $\alpha_p$. Then if the $q$-expansion of $\mathcal{F}(\gamma)$ has all its coefficients in $p^j\mathbb{Z}_p$ for a certain finite idèle $\gamma$ and a certain integer $j$, the $q$-expansion of $\mathcal{F}(\alpha)$ for an arbitrary finite idèle $\alpha$ also has all its coefficients in $p^j\mathbb{Z}_p$.

The following corollary—so-called the \textit{q-expansion principle}—plays the most important role in verification of sufficient condition (8.4).

**Corollary 8.9** (q-expansion principle). Let $F_k$ and $\mathcal{F}(\alpha)$ be as in Theorem 8.8 and $j$ an integer. Suppose that the $q$-expansion of $\mathcal{F}(\gamma)$ has all its non-constant coefficients in $p^j\mathbb{Z}_p$ for a certain finite idèle $\gamma$. Then for arbitrary two distinct finite idèles $\alpha$ and $\beta$, the difference between the constant terms of the $q$-expansions of $\mathcal{F}(\alpha)$ and $\mathcal{F}(\beta)$ is also contained in $p^j\mathbb{Z}_p$.

**Proof.** Just apply Theorem 8.8 to $\mathcal{F}(\alpha) - c(0, \gamma)$ and $\mathcal{F}(\beta) - c(0, \gamma)$ where $c(0, \gamma)$ is the constant term of the $q$-expansion of $\mathcal{F}(\gamma)$. See also [DR Corollary (0.3)].

Finally we introduce the Hilbert-Eisenstein series attached to a locally constant $\mathbb{C}$-valued function $\varepsilon$ on $\text{Gal}(K_\infty/K)$.

**Theorem 8.10** (Hilbert-Eisenstein series). Let $\varepsilon$ be a locally constant function on $\text{Gal}(K_\infty/K)$ and $k$ an even natural number. Then there exists an integral ideal $\mathfrak{f}$ of $\mathcal{O}_K$ all of whose prime factors are contained in $\Sigma$, and there exists a Hilbert modular form $G_{k,\varepsilon}$ of weight $k$ on $\Gamma_0(\mathfrak{f})$ (which is called the Hilbert-Eisenstein series of weight $k$ attached to $\varepsilon$) whose standard $q$-expansion is given by

$$2^{-\rho} \zeta_{K_\infty/K}(1-k, \varepsilon) + \sum_{\mu \in \mathcal{O}_K, \mu \gg 0} \left( \sum_{\mu \in \mathfrak{a} \subseteq \mathcal{O}_K, \text{prime to } \Sigma} \varepsilon(\mathfrak{a}) \kappa(\mathfrak{a})^{k-1} \right) q^K_\mu$$

(we use the notation $\varepsilon(\mathfrak{a})$ and $\kappa(\mathfrak{a})$ for elements defined as $\varepsilon((K_\infty/K, \mathfrak{a}))$ and $\kappa((K_\infty/K, \mathfrak{a}))$ respectively where $(K_\infty/K, -)$ denotes the Artin symbol for the abelian extension $K_\infty/K$). The $q$-expansion of $G_{k,\varepsilon}$ at the cusp determined by a finite idèle $\alpha$ is given by

$$(8.6)$$

$$\mathcal{N}((\alpha))^k \left\{ 2^{-\rho} \zeta_{K_\infty/K}(1-k, \varepsilon_\alpha) + \sum_{\mu \in \mathcal{O}_K, \mu \gg 0} \left( \sum_{\mu \in \mathfrak{a} \subseteq \mathcal{O}_K, \text{prime to } \Sigma} \varepsilon_\alpha(\mathfrak{a}) \kappa(\mathfrak{a})^{k-1} \right) q^K_\mu \right\}$$
where \((\alpha)\) is the ideal generated by \(\alpha\) and \(a\) is an element in \(\text{Gal}(K_\infty/K)\) defined as \((K_\infty/K, (\alpha)a^{-1})\).

For details, see [DR, Theorem (6.1)].

8.4. Proof of sufficient conditions. In the rest of this section we shall verify sufficient condition (8.4). This part is a subtle generalisation of the argument in [H Section 6.6]. Let \(j\) be a sufficiently large integer and \(y\) a coset of \(W/T^p\) not contained in \(\Gamma\). Choose an integral ideal \(\mathfrak{f}\) of \(\mathcal{O}_{FU}\) such that the Hilbert-Eisenstein series \(G_{k,\mathfrak{f}}\) over \(\mathfrak{f}_{FU}\) is defined on \(\Gamma_{00}(\mathfrak{f}|\mathcal{O}_{FU})\).

Then it is easy to see that the restriction \(\mathcal{G} = G_{k,\mathfrak{f}|\mathfrak{f}_{NU}}\) of \(G_{k,\mathfrak{f}}\) to \(\mathfrak{f}_{NU}\) is also a Hilbert modular form of weight \(p_{nu}\) on \(\Gamma_{00}(\mathfrak{f})\) where \(p_{nu}\) is the cardinality of the quotient group \(NU/U\). The \(q\)-expansion of \(\mathcal{G}\) is directly calculated as

\[
2^{-[FU:Q]}q_{\Sigma_{F_{\nu}/F}}(1 - k, \delta(y)) + \sum_{\nu \in \mathcal{O}_{FU}, \nu > 0} \sum_{\nu \in \mathfrak{b} \subseteq \mathcal{O}_{FU}, \mathfrak{b} \text{ prime to } \Sigma} \delta(y)(\mathfrak{b})\kappa(\mathfrak{b})^{-k-1} q_{\mathfrak{f}_{NU}}^{(\nu)}
\]

where \(q_{\mathfrak{f}_{NU}}^{(\nu)}\) denotes \(\exp(2\pi \sqrt{-1} \text{Tr}_{\mathfrak{f}_{NU}/Q}(\text{Tr}_{F_{\nu}/F_{NU}}(\nu)\tau))\). Note that the quotient group \(NU/U\) naturally acts upon the set of all pairs \((\mathfrak{b}, \nu)\) such that \(\mathfrak{b}\) is a non-zero integral ideal of \(\mathcal{O}_{FU}\) prime to \(\Sigma\) and \(\nu\) is a totally positive element in \(\mathfrak{b}\). First suppose that the isotropy subgroup \((NU/U)_{(\mathfrak{b}, \nu)}\) is trivial. Then we can easily calculate the \(NU/U\)-orbital sum in the \(q\)-expansion of \(\mathcal{G}\) containing the term associated to \((\mathfrak{b}, \nu)\) as follows:

\[
\sum_{\sigma\in NU/U} \delta(y)(\sigma\mathfrak{b})\kappa(\sigma\mathfrak{b})^{-k-1} q_{\mathfrak{f}_{NU}}^{(\nu)} = \sharp(NU/U)_{y} \sum_{\sigma\in(NU/U)/(NU/U)_{y}} \delta^{(\nu\sigma^{-1})}(\mathfrak{b})\kappa(\mathfrak{b})^{-k-1} q_{\mathfrak{f}_{NU}}^{(\nu)}
\]

(we use the obvious formula \(\text{Tr}_{F_{\nu}/F_{NU}}(\nu) = \text{Tr}_{F_{\nu}/F_{NU}}(\nu)\)).

Next suppose that the isotropy subgroup \((NU/U)_{(\mathfrak{b}, \nu)}\) is not trivial. Let \(F_{(\mathfrak{b}, \nu)}\) be the fixed subfield of \(F_{\nu}\) by \((NU/U)_{(\mathfrak{b}, \nu)}\) and \(F_{\text{comm}}^{(\mathfrak{b}, \nu)}\) the fixed subfield of \(F_{\infty}\) by the commutator subgroup of \(NU_{(\mathfrak{b}, \nu)}\). Then \((\mathfrak{b}, \nu)\) is fixed by the action of \(\text{Gal}(F_{\nu}/F_{(\mathfrak{b}, \nu)})\), and hence \(\nu\) is an element in \(F_{(\mathfrak{b}, \nu)}\) and there exists a non-zero integral ideal \(\mathfrak{a}\) of \(\mathcal{O}_{F_{(\mathfrak{b}, \nu)}}\) such that \(\mathfrak{a}\mathcal{O}_{F_{\nu}}\) coincides with \(\mathfrak{b}\). For such \((\mathfrak{a}, \nu)\), the equation

\[
\delta(y)(\mathfrak{b}) = \delta(y)((F_{\nu}/F_{U}, \mathfrak{a}\mathcal{O}_{F_{\nu}})) = \delta(y) \circ \text{Ver}((F_{\text{comm}}^{(\mathfrak{b}, \nu)}/F_{(\mathfrak{b}, \nu)}, \mathfrak{a})) = 0
\]

holds because the image of the Verlagerung homomorphism is contained in \(\Gamma\) (indeed the Verlagerung coincides with the \(n_{U}\)-th power of the Frobenius endomorphism \(\varphi^{n_{U}}\) if the finite part of the Galois group is of exponent \(p\); see [H Lemma 4.3] for details) but \(y\) is not contained in \(\Gamma\).

The calculation above implies that \(\mathcal{G}\) has all non-constant coefficients in \(\sharp(NU/U)_{y}T_{(p)}\). Take a finite idele \(\gamma\) such that \((F_{\nu}/F_{U}, (\gamma)\gamma^{-1})\) coincides with \(w\). Then by Deligne-Ribet’s \(q\)-expansion principle (Corollary 8.9) the constant term of \(\mathcal{G} - \mathcal{G}(\gamma)\) is also contained in \(\sharp(NU/U)_{y}T_{(p)}\), which we may calculate as \(2^{-[FU:Q]}\Delta_{F_{\nu}/F_{U}}(1 - k, \delta(y))\) (use the explicit formula of \(8.6\) for the \(q\)-expansion of \(\mathcal{G}(\gamma)\)). Therefore sufficient condition (8.4) holds (recall that 2 is invertible since \(p\) is odd).
9. Inductive Construction of the $p$-adic Zeta Functions

We shall complete the proof of our main theorem (Theorem 3.1). We first construct the $p$-adic zeta function “modulo $p$-torsion” for $F_\infty/F$, and then eliminate ambiguity of the $p$-torsion part.

9.1. Choice of the central element $c$. In order to let induction work effectively, we have to choose a “good” central element $c$ which is used in the construction of the Artinian family $\mathfrak{F}_A$ (see Section 4.1). The following elementary lemma implies how to choose such a “good” central element.

**Lemma 9.1.** Let $\Delta$ be a finite $p$-group with exponent $p$ and $c$ a non-trivial central element in $\Delta$. If $c$ is not contained in the commutator subgroup of $\Delta$, the $p$-group $\Delta$ is isomorphic to the direct product of the cyclic group $\langle c \rangle$ generated by $c$ and the quotient group $\Delta/\langle c \rangle$.

**Proof.** By the structure theorem of finite abelian groups, the abelisation $\Delta^{ab}$ of $\Delta$ is decomposed as the direct product of the image of the cyclic group $\langle c \rangle$ and a certain finite abelian $p$-group $\bar{H}$ with exponent $p$. Let $H$ denote the inverse image of $\bar{H}$ under the abelisation map $\Delta \mapsto \Delta^{ab}$. Then one may easily verify that $H$ and $\langle c \rangle$ generate $\Delta$. The intersection of $H$ and $\langle c \rangle$ is obviously trivial, and $H$ commutes with elements in $\langle c \rangle$ since $c$ is central.

This lemma asserts that there exists a non-trivial central element $c$ which is contained in the commutator subgroup of $Gf$ if $G$ is not abelian. We may assume that $G$ is non-commutative without loss of generality (abelian cases are just the results of Deligne, Ribet and Wiles), and thus we may always find a non-trivial central element contained in $[G,G]$.

In the following argument we take a non-trivial central element $c$ from the commutator subgroup of $G$ and fix it.

9.2. Construction of the $p$-adic zeta function “modulo $p$-torsion”. In this subsection we construct the $p$-adic zeta function “modulo $p$-torsion” for $F_\infty/F$, by mimicking Burns’ technique (see Section 2). First consider the following commutative diagram with exact rows:

\[ K_1(\Lambda(G)) \longrightarrow K_1(\Lambda(G)_S) \longrightarrow 0 \]

\[ \prod_{\mathfrak{S}_B} \Lambda(U/V)^x \longrightarrow \prod_{\mathfrak{S}_B} \Lambda(U/V)_S^\times \longrightarrow \prod_{\mathfrak{S}_B} \Lambda(\Lambda(U/V)_S) \longrightarrow 0. \]

Let $f$ be an arbitrary characteristic element for $F_\infty/F$ (see Section 1.1) and set $\theta_S(f) = (f_{U,V})_{(U,V)\in\mathfrak{S}_B}$. For each $(U,V)$ in $\mathfrak{S}_B$ let $w_{U,V}$ be the element defined as $\xi_{U,V} f_{U,V}^{-1}$, which is contained in $\Lambda(U/V)^x$ by an argument similar to Burns’ technique. Let $\tilde{w}_{U,V}$ denote the image of $w_{U,V}$ in $\Lambda(U/V)^x$.

Since both $(f_{U,V})_{(U,V)\in\mathfrak{S}_B}$ and $(\xi_{U,V})_{(U,V)\in\mathfrak{S}_B}$ satisfy conditions (NCC) and (CCC) (see Proposition 7.2 and Lemma 8.1), the element $(\tilde{w}_{U,V})_{(U,V)\in\mathfrak{S}_B}$ also satisfies them. Moreover there exists an element $d_{U,V}$ (resp. $\tilde{d}_{U,V}$) in $\Lambda(\Gamma)^x_{(p)}$ such that the congruence

\[ \tilde{w}_{U,V} \equiv d_{U,V} \mod J_{S,U,V} \quad (\text{resp. } \tilde{w}_{U} \equiv \tilde{d}_{U} \mod I_{S,U}) \]
holds for each \((U, V)\) in \(\mathfrak{S}_B\) except for \((G, [G, G])\) (resp. for each \(U\) in \(\mathfrak{S}_A\)) by Proposition 7.2 and Proposition 8.2. We remark that these congruences are not sufficient to prove that \(\tilde{w}_{U,V}(U,V)\in\mathfrak{S}_B\) is contained in \(\Psi\) (or equivalently in \(\Psi_c\)).

**Remark 9.2.** Unfortunately the congruences (9.1) hold not in \(\Lambda(U/V)\) (resp. \(\Lambda(U)\)) but in \(\Lambda(U/V)_S\) (resp. \(\Lambda(U)_S\)), but we may obtain the integral congruences (9.5) later by “eliminating \(\hat{d}_{U,V}\) and \(\hat{d}_U\).” The author would like to appreciate Mahesh Kakde for pointing out the wrong arguments around Theorem 9.3.

**Proof.** Recall that the non-negative integer \(N\) is defined by \(\mathcal{Z}^N = p^N\). We shall prove the claim by induction on \(N\). We first assume that \(G\) is abelian. Then the element \((\xi_{U,V}(e))\in\mathfrak{S}_B\) is in fact contained in the image of \(\hat{g}\) (use the existence of the \(p\)-adic zeta pseudomeasure for \(F_{\infty}/F\)), and hence \((\xi_{U,V}(e))\in\mathfrak{S}_B\) satisfies desired congruence condition and additional congruence condition. This implies that \((\hat{w}_{U,V}(e))\in\mathfrak{S}_B\) also satisfies them. In particular the cases where \(N\) equals either 0, 1 or 2 are done. Therefore we assume that \(N\) is larger than 3 and \(G\) is non-commutative in the following argument.

Now let \((U, V)\) (resp. \(U\)) be an element in \(\mathfrak{S}_B\) (resp. \(\mathfrak{S}_A\)) such that \(U\) contains the fixed central element \(c\) chosen as in Section 9.4. Set \(G = G/(c)\), \(\tilde{U} = U/\langle c \rangle\) and \(\tilde{V} = V/\langle c \rangle\) respectively. Clearly the set of all such \((\tilde{U}, \tilde{V})\) is a Brauer family \(\mathfrak{S}_B\) for \(G\), and the set of \(\tilde{U}_h = U_{h,c}/\langle c \rangle\) for all \(h\) in \(\mathfrak{S}\) is an Artinian family \(\mathfrak{S}_A\) for \(G\). Note that the norm map \(\text{Nr}_{\Lambda(G)_S/\Lambda(U)_S}\) and the canonical homomorphism \(K_1(\Lambda(G)_S) \to K_1(\Lambda(G)_S)\) are compatible, and note also that the image of \(\xi_{U,V}\) under the canonical quotient map \(\Lambda(U/V)_S^\mathcal{Z} \to \Lambda(U/V)_S^\mathcal{Z}\) coincides with the \(p\)-adic zeta pseudomeasure \(\xi_{U,V}\) for \(F_{\infty}/F\) (easily follows from its interpolation property). Hence we may apply the induction hypothesis to the image \(\tilde{w}_{U,V}\) of \(\hat{w}_{U,V}\) in \(\tilde{\Lambda}(U/V)^\mathcal{Z}\); in other words, the congruences

\[
\tilde{w}_{U,V} \equiv \varphi(\tilde{w}_{ab})^{(G,U)/p} \mod J_{U,V} \quad \text{(resp.} \quad \tilde{w}_U \equiv \varphi(\tilde{w}_{ab})^{(G,U)/p} \mod I_U) \]

hold for each \((\tilde{U}, \tilde{V})\) in \(\mathfrak{S}_B\) except for \((G, [G, G])\) and for each \(\tilde{U}\) in \(\mathfrak{S}_A\) if we define \(J_{U,V}\) and \(I_U\) analogously to \(J_{U,V}\) and \(I_U\). On the other hand we may readily verify that the natural surjection \(\Lambda(U/V) \to \Lambda(U/V)^\mathcal{Z}\) maps \(J_{U,V}\) to \(J_{U,V}\) and \(I_U\) to \(I_U\) respectively (use the definition of \(J_{U,V}\) and the explicit description of \(I_U\)). Let \(I'_U\) denote the image of the trace map \(\text{Tr}_{Z_p([\text{Con}(\mathcal{N}(U))]/Z_p[U])}\) for each \(U\) in \(\mathfrak{S}_A\), and let \(J_{S,U,V}\) (resp. \(I_{S,U}\)) denote the scalar extension \(J_{U,V} \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)_{(p)}\) (resp. \(I'_U \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)_{(p)}\)). Then we obtain the congruences

\[
\tilde{w}_{U,V} \equiv \tilde{d}_{U,V} \mod J_{S,U,V}, \quad \tilde{w}_U \equiv \tilde{d}_U \mod I'_{S,U}.
\]
by applying the canonical surjection $\Lambda(U/V) \to \Lambda(\bar{U}/\bar{V})$ to \eqref{9.1} (recall that $I_S^t$ contains $I_S(U)$. The congruences \eqref{9.2} and \eqref{9.3} implies that for $(U, V)$ in $\mathfrak{S}_B$ except for $(G, [G, G])$ the element $\varphi(\bar{w}_{ab})^{-((G:U)/p)}d_{U,V}$ is contained in $1 + J_{S, U, V} \cap \widetilde{\Lambda}(\Gamma)_{(p)}^\times$ which coincides with $1 + p\Lambda(\Gamma)_{(p)}$ by definition. Furthermore for $U$ in $\mathfrak{S}_A$ the element $\varphi(\bar{w}_{ab})^{-((G:U)/p)}d_U$ is contained in $(1 + I_{S, U}^t)^\times \cap \widetilde{\Lambda}(\Gamma)_{(p)}^\times$, which coincides with $1 + p^{a_h - \epsilon}\Lambda(\Gamma)_{(p)}$ by the explicit description of $I_{S, U}$ (the integer $\epsilon$ is defined as 2 for (Case-1) and 1 for (Case-2)). Obviously the equations $\varphi(\bar{w}_{ab}) = \varphi(\bar{w}_{ab})$ and $(G : \bar{U}) = (G : U)$ hold by construction, and therefore the congruences

\begin{equation}
\tag{9.4}
d_{U,V} \equiv \varphi(\bar{w}_{ab})^{(G:U)/p} \mod p\Lambda(\Gamma)_{(p)},
d_U \equiv \varphi(\bar{w}_{ab})^{(G:U)/p} \mod p^{n_h - \epsilon}\Lambda(\Gamma)_{(p)}.
\end{equation}

hold. Combining \eqref{9.4} with \eqref{9.1}, we obtain the following congruences\footnote{Since both $\bar{w}_{U,V}$ (resp. $\bar{w}_{U}$) and $\varphi(\bar{w}_{ab})^{(G:U)/p}$ are contained in $\bar{\Lambda}(U/V)^\times$, the congruence \eqref{9.5} actually holds in $\Lambda(U/V)$ (resp. in $\Lambda(U)$) and we may remove the sub-index $S$ from the congruence. This is the “eliminating $d$” procedure mentioned in Remark 9.3.}

\begin{equation}
\tag{9.5}
\bar{w}_{U,V} \equiv \varphi(\bar{w}_{ab})^{(G:U)/p} \mod J_{U,V}, \quad \bar{w}_U \equiv \varphi(\bar{w}_{ab})^{(G:U)/p} \mod I_U^t.
\end{equation}

The former congruence is no other than the desired one. The latter one for $U_c$ is also the desired one because $I_{U_c}^t$ coincides with $I_{U_c}$ by definition. Now consider the congruence for $U_{h,c}$. It suffices to consider the case where $U_{h,c}$ is a proper subgroup of $G$ (see Remark 6.1). Since $\log(\varphi(\bar{w}_{ab})^{-p^{n_h-3}\bar{w}_{U_{h,c}}})$ is contained in $I_{U_{h,c}}^t$ by \eqref{9.5}, it is explicitly described as

$$
\log \frac{\bar{w}_{U_{h,c}}}{\varphi(\bar{w}_{ab})^{p^{n_h-3}}} = \sum_{i=0}^{p-1} p^{n_h - \epsilon}_i a_i c^i + \text{(terms containing } h)$$

where each $a_i$ is an element in $\Lambda(\Gamma)$. Furthermore the equation

$$
\log \frac{\bar{w}_{U_{h,c}}}{\varphi(\bar{w}_{ab})^{p^{n_h-3}}} = \text{Tr}_{\mathbb{Z}_p[[U_{h,c}]][/\mathbb{Z}_p[[U_c]]]} (\log \frac{\bar{w}_{U_{h,c}}}{\varphi(\bar{w}_{ab})^{p^{n_h-3}}}) = \sum_{i=0}^{p-1} p^{n_h - \epsilon + 1} a_i c^i
$$

holds by (TCC). The first expression of the equation above is contained in $I_{U_{h,c}} = p^{n_h - 1}\mathbb{Z}_p[[U_{h,c}]]$ as we have already remarked, and hence there exists an element $b_i$ in $\Lambda(\Gamma)$ such that $p^{n_h - \epsilon + 1} a_i$ coincides with $p^{n_h - 1} b_i$ for each $i$. Therefore we may conclude that the element $\log(\varphi(\bar{w}_{ab})^{-p^{n_h-3}\bar{w}_{U_{h,c}}})$ is contained in $I_{U_{h,c}}$. This implies the desired congruence for $U_{h,c}$ because the logarithm induces an injection on $1 + \bar{J}_{U_{h,c}}$ (Proposition 5.7) and an isomorphism between $1 + \bar{J}_{U_{h,c}}$ and $I_{U_{h,c}}$ (Proposition 5.9).

Next let $(U, V)$ be an element in $\mathfrak{S}_B$ such that $U$ does not contain the fixed central element $c$. We claim that $U \times \langle c \rangle$ does not coincide with $G$; indeed if it does, the commutator subgroup of $G$ automatically coincides with $V$ which does not contain $c$. This is contradiction since we choose such $c$ as contained in $[G, G]$ in Section 9.1. Now we apply the argument above to the pair $(U \times \langle c \rangle, V)$ and obtain the congruence

$$
\bar{w}_{U \times \langle c \rangle, V} \equiv \varphi(\bar{w}_{ab})^{(G:U)/p^2} \mod J_{U \times \langle c \rangle, V}
$$

This completes the proof of Theorem 9.2.
(use the obvious equation \((G : U \times (c)) = (G : U)/p\)). By using (NCC) and the fact that \(\varphi(\tilde{w}_{ab})\) is contained in the centre of \(\Lambda(U \times (c))\), we have

\[
\text{Nr}_{\Lambda(U \times (c))/\Lambda(U)}(\varphi(\tilde{w}_{ab}) - (G : U)/p^2 \tilde{w}_{U \times (c)}) = \varphi(\tilde{w}_{ab}) - (G : U)/p \tilde{w}_{U,V}.
\]

On the other hand the left hand side of the equation above is contained in \(1 + J_{U,V}\) by Corollary 5.3. The desired congruence thus holds for \((U,V)\).

Finally let \(U_h\) be an element in \(\mathfrak{S}_A\) and assume that \(h\) does not coincide with \(c\). By the same argument as above, we may conclude that \(\varphi(\tilde{w}_{ab})^{-p^{N-2}} \tilde{w}_{U_h}\) is contained in \(1 + J_{U_h}\). On the other hand the element \(\varphi(\tilde{w}_{ab})^{-p^{N-3}} \tilde{w}_{U_h,c}\) is contained in \(1 + I_{U_h,c}\) by the argument above. Now the compatibility lemma (Lemma 5.6) enables us to calculate as follows:

\[
\text{Tr}_{\mathbb{Z}_p[[U_h,c]]/\mathbb{Z}_p[[U_h]]}(\log \frac{\tilde{w}_{U_h,c}}{\varphi(\tilde{w}_{ab})^{p^{N-3}}} = \log \text{Nr}_{\Lambda(U_h,c)/\Lambda(U)}(\frac{\tilde{w}_{U_h,c}}{\varphi(\tilde{w}_{ab})^{p^{N-3}}})
\]

\[
= \log \frac{\tilde{w}_{U_h}}{\varphi(\tilde{w}_{ab})^{p^{N-2}}}.
\]

The \(\mathbb{Z}_p\)-module \(\text{Tr}_{\mathbb{Z}_p[[U_h,c]]/\mathbb{Z}_p[[U_h]]}(I_{U_h,c})\) is contained in \(I_{U_h}\) by definition, and thus \(\log(\varphi(\tilde{w}_{ab})^{-p^{N-2}} \tilde{w}_{U_h})\) is also contained in \(I_{U_h}\). The desired congruence now holds for \(U_h\) because the logarithm induces an injection on \(1 + J_{U_h}\) (Proposition 5.7) and an isomorphism between \(1 + I_{U_h}\) and \(I_{U_h}\) (Proposition 5.9).

By virtue of Theorem 5.3, we may conclude that \((\tilde{w}_{U_h,V})(U,V)\in\mathfrak{S}_B\) is an element in \(\mathfrak{S}_c\). Hence there exists a unique element \(\tilde{w}\) in \(K_1(\Lambda(G))\) such that \(\tilde{\theta}(\tilde{w}) = (\tilde{w}_{U_h,V})_{(U,V)\in\mathfrak{S}_B}\) holds (Proposition 6.7) and Proposition 6.8). Take an arbitrary lift of \(\tilde{w}\) to \(K_1(\Lambda(G))\) and set \(\xi = f \tilde{w}\). Then by construction, we may easily check that \(\tilde{\xi}\) satisfies the following two properties:

\((\xi-1)\) the equation \(\tilde{\theta}(\tilde{\xi}) = -[C_{F_\infty/F}]\) holds;

\((\xi-2)\) there exists an element \(\tau_{U,V}\) in \(\Lambda(U/V)_{p\text{-tors}}\) for each \((U,V)\) in \(\mathfrak{S}_B\) such that the equation \(\tilde{\theta}_B(\tilde{\xi}) = (\xi_{U,V}\tau_{U,V})(U,V)\in\mathfrak{S}_B\) holds.

By using \((\xi-1)\) and \((\xi-2)\), we may show that there exists a \(p\)-power root of unity \(\zeta_{p^r}\) such that the equation \(\xi(\rho_{p^r} = \zeta_{p^r}L_{\Sigma}(1 - r; C_{F_\infty/F}, \rho)\) holds for an arbitrary Artin representation \(\rho\) of \(G\) and an arbitrary natural number \(r\) divisible by \(p - 1\). Roughly speaking, the element \(\xi\) is the \(p\)-adic zeta function “modulo \(p\)-torsion” for \(F_\infty/F\) which interpolates special values of complex Artin L-functions up to multiplication by a \(p\)-power root of unity.

9.3. Refinement of the \(p\)-torsion part. We shall finally modify \(\tilde{\xi}\) and reconstruct the \(p\)-adic zeta function \(\xi\) for \(F_\infty/F\) without any ambiguity upon \(p\)-torsion elements. The author strongly believes that our argument to remove ambiguity of the \(p\)-torsion part is based upon essentially the same spirits as “the torsion congruence method” used by Jürgen Ritter and Alfred Weiss [RW5]. We shall, however, adopt somewhat different formalism from theirs.

Let \(\tilde{\xi}\) be the \(p\)-adic zeta function “modulo \(p\)-torsion” for \(F_\infty/F\) and set \(\tau_{U,V} = \xi_{U,V}\theta_{S_{U,V}}(\tilde{\xi})^{-1}\) for each \((U,V)\) in \(\mathfrak{S}_B\). Then \(\tau_{U,V}\) is a \(p\)-torsion element by definition. Moreover \(\tau_{U,V}\) is an element in \(\Lambda(U/V)^{\times}\) by the
same argument as Burns’ technique. Since the $p$-torsion part of $K_1(\Lambda(G))$ is identified with $G^{f,ab} \times SK_1(\mathbb{Z}_p[G^{f}])$ and that of $\Lambda(G^{ab})^\times$ is identified with $G^{f,ab}$ respectively (see Section 6.2), we may naturally regard $\tau_{ab}$ as an element in $K_1(\Lambda(G))_{p\operatorname{-tors}}$. Let $\xi = \tau_{ab} \bar{\xi}$. Then $\theta_{S,ab}(\xi) = \xi_{ab}$ obviously holds by construction.

**Theorem 9.4.** The equation $\theta_{S,U,V}(\xi) = \xi_{U,V}$ holds for each $(U,V)$ in $\mathfrak{F}_B$.

If the claim is verified, we may conclude that $\xi$ satisfies the interpolation formula (13) without any ambiguity by Brauer induction (see Section 2). Therefore $\xi$ is no other than the “true” $p$-adic zeta function for $F_{\infty}/F$.

We shall prove Theorem 9.4 by induction on $N$. First assume that $G$ is abelian. Then the obvious equation $\xi = \theta_{S,ab}(\xi) = \xi_{ab}$ implies that $\xi$ is actually the $p$-adic zeta function for $F_{\infty}/F$. In particular the cases in which $N$ equals either 0, 1 or 2 are done.

Now suppose that $N$ is larger than 3 and $G$ is non-commutative. Let $c$ be a non-trivial central element in $G$ chosen as in Section 9.1 and set $\bar{G} = G/\langle c \rangle$. Let $\bar{\xi}$ be the image of $\xi$ under the canonical map $K_1(\Lambda(G)_S) \rightarrow K_1(\Lambda(\bar{G})_S)$. Then the element $\bar{\xi}$ is the $p$-adic zeta function “modulo $p$-torsion” for $F_{\langle c \rangle}/F$ by construction. Furthermore the following diagram commutes since $c$ is contained in the commutator subgroup of $G$ (here $\bar{\theta}_{S,ab}$ denotes the abelisation map for $\Lambda(\bar{G})_S$):

$$
\begin{array}{ccc}
K_1(\Lambda(G)_S) & \xrightarrow{\theta_{S,ab}} & \Lambda(G^{ab})^\times_S \\
\downarrow \text{canonical} & & \downarrow \\
K_1(\Lambda(\bar{G})_S) & \xrightarrow{\bar{\theta}_{S,ab}} & \Lambda(\bar{G}^{ab})^\times_{\bar{S}}.
\end{array}
$$

This asserts that $\bar{\theta}_{S,ab}(\bar{\xi}) = \xi_{ab}$ holds, and we may thus apply the induction hypothesis to $\bar{\xi}$; in other words we may assume that $\bar{\xi}$ is the “true” $p$-adic zeta function for $F_{\langle c \rangle}/F$. Now take an arbitrary pair $(U,V)$ in $\mathfrak{F}_B$.

**(Case-1).** Suppose that $c$ is contained in $V$. Let $\bar{U}$ and $\bar{V}$ denote the quotient groups $U/\langle c \rangle$ and $V/\langle c \rangle$ respectively. Let $\bar{\theta}_{S,U,V}$ be the composition of the norm map $\text{Nr}_{\Lambda(\bar{G})_S/\Lambda(\bar{U})_S}$ with the canonical homomorphism $K_1(\Lambda(\bar{U})_S) \rightarrow \Lambda(\bar{U}/\bar{V})^\times_{\bar{S}}$. Then it is clear that $U/V$ coincides with $\bar{U}/\bar{V}$ and the theta maps $\theta_{S,U,V}$ and $\bar{\theta}_{S,U,V}$ are compatible. Hence we obtain

$$
\theta_{S,U,V}(\xi) = \bar{\theta}_{S,U,V}(\bar{\xi}) = \xi_{U,V} = \xi_{U,V}
$$

which is the desired result (the last equality follows from the fact that $F_V/F_{\bar{U}}$ is the completely same extension as $F_V/F_U$).

**(Case-2).** Suppose that $c$ is contained in $U$ but not contained in $V$. Let $U'$ be an open subgroup of $G$ which contains $U$ as a subgroup of index $p$ (and $U$ is hence normal in $U'$). Let $V'$ denote the commutator subgroup of $U'$. We claim that we may reduce to the case in which $\theta_{S,U',V'}(\xi) = \xi_{U',V'}$ holds; indeed the desired equation holds if $V'$ contains $c$ by (Case-1). Assume that $V'$ does not contain $c$. Then the pair $(U_1,V_1) = (U',V')$ also satisfies the condition of (Case-2), and recursively we may obtain a sequence of pairs $\{(U_i,V_i)\}_{i \in \mathbb{Z}_{\geq 0}}$ such that $(U_0,V_0)$ is equal to $(U,V)$ and $U_{i+1}$ contains $U_i$ as
its normal subgroup of index $p$ (each $V_i$ denotes the commutator subgroup of $U_i$). Therefore there exists a natural number $n$ such that $V_n$ contains $c$ because we now assume that the commutator subgroup of $G$ contains $c$.

We now apply the following theorem:

**Theorem 9.5.** The $p$-adic zeta function $\xi_{U', V'}$ for $F_{U'}/F_{U'}$ exists uniquely as an element in $K_1(\Lambda(U'/V')_S)$.

This is the special case of the deep results of Jürgen Ritter and Alfred Weiss [RW7]. For the convenience of the readers, we shall give the sketch of the proof in the following subsection.

Now assume that Theorem 9.5 is valid for a moment. Let $\text{can}_{U', V'}$ denote the canonical homomorphism $K_1(\Lambda(U')_S) \to K_1(\Lambda(U'/V)_S)$. Note that the element $\text{can}_{U', V'} \circ \text{Nr}_{\Lambda(G)_S/\Lambda(U')_S}(\xi)$ is the $p$-adic zeta function “modulo $p$-torsion” for $F_{U'}/F_{U'}$ by the interpolation property, and hence there exists an element $\tau$ in $U'f'/V'f'$ such that $\text{can}_{U', V'} \circ \text{Nr}_{\Lambda(G)_S/\Lambda(U')_S}(\xi)$ coincides with $\tau\xi_{U', V'}$ (here we remark that the $p$-torsion part of $K_1(\Lambda(U'/V))$ coincides with $(U'f'/V'f')^{ab} = U'f'/V'f'$ because $SK_1(\mathbb{Z}_p[U'f'/V'f'])$ is trivial by [Oliver Theorem 8.10]). Then easy calculation verifies that the equation

$$
\xi_{U', V'} = \theta_{S, U', V'}(\xi) = \text{can}_{V'}(\tau\xi_{U', V'}) = \tau\xi_{U', V'}
$$

holds where $\text{can}_{V'}: K_1(\Lambda(U'/V)_S) \to \Lambda(U'/V')_S^{\times}$ denotes the canonical homomorphism. This implies that $\tau$ is trivial. On the other hand, the norm relation $\text{Nr}_{\Lambda(U'/V)_S/\Lambda(U/V)_S}(\xi_{U', V'}) = \xi_{U, V}$ holds since $\xi_{U', V'}$ is the $p$-adic zeta function for $F_{U'}/F_{U'}$. Therefore we obtain the desired equation

$$
\theta_{S, U, V}(\xi) = \text{Nr}_{\Lambda(U'/V)_S/\Lambda(U/V)_S} \circ \text{can}_{U', V'} \circ \text{Nr}_{\Lambda(G)_S/\Lambda(U')_S}(\xi)
$$

equals $\text{Nr}_{\Lambda(U'/V)_S/\Lambda(U/V)_S}(\xi_{U, V}) = \xi_{U, V}$.

**Case-3.** Suppose that $c$ is contained in neither $U$ nor $V$. In this case the pair $(U \times \langle c \rangle, V)$ satisfies the condition of (Case-2), and thus the equation $\theta_{S, U \times \langle c \rangle, V}(\xi) = \xi_{U \times \langle c \rangle, V}$ holds. Then by using the commutative diagram

$\begin{array}{ccc}
K_1(\Lambda(G)_S) & \xrightarrow{\theta_{S, U \times \langle c \rangle, V}} & \Lambda(U \times \langle c \rangle/V')_S \\
\downarrow{\theta_{S, U, V}} & & \downarrow{\text{Nr}_{\Lambda(U \times \langle c \rangle/V)_S/\Lambda(U/V)_S}} \\
\Lambda(U/V)_S & & \end{array}
$

we obtain

$$
\theta_{S, U, V}(\xi) = \text{Nr}_{\Lambda(U \times \langle c \rangle/V)_S/\Lambda(U/V)_S} \circ \theta_{S, U \times \langle c \rangle, V}(\xi)
$$

equals $\text{Nr}_{\Lambda(U \times \langle c \rangle/V)_S/\Lambda(U/V)_S}(\xi_{U \times \langle c \rangle, V}) = \xi_{U, V}$,

which is the desired result.\footnote{We may derive the desired result for (Case-3) even if we only assume that $\theta_{S, U \times \langle c \rangle, V}(\xi) = c^j\xi_{U \times \langle c \rangle, V}$ holds for certain $j$ (which we may verify by the arguments similar to (Case-1)); hence the essentially difficult part is just (Case-2). Note that $\text{Nr}_{\Lambda(U \times \langle c \rangle/V)_S/\Lambda(U/V)_S}(c^j)$ coincides with $(c^j)^p = 1$ because $c^j$ is contained in the centre of $\Lambda(U \times \langle c \rangle/V)_S$.}
9.4. Outline of the proof of Theorem 9.5. In this subsection we shall give the rough sketch of the proof of the following theorem.

**Theorem 9.6** (Ritter-Weiss). Let $p$ be a positive odd prime number and $F$ a totally real number field. Let $F_\infty$ be a totally real $p$-adic Lie extension of $F$ satisfying conditions $(F_\infty\text{-}1)$, $(F_\infty\text{-}2)$ and $(F_\infty\text{-}3)$ in Section 7.7. Let $G$ denote the Galois group of $F_\infty/F$ and suppose that the following two conditions are satisfied:

(i) the Galois group $G$ is isomorphic to the direct product of a finite $p$-group $G^f$ and the commutative $p$-adic Lie group $\Gamma$;

(ii) the finite part $G^f$ has an abelian subgroup $W^f$ of index $p$ (which is automatically normal in $G$).

Then the $p$-adic zeta function $\xi_{F_\infty/F}$ for $F_\infty/F$ exists uniquely as an element in $K_1(\Lambda(G)_{S})$.

Theorem 9.5 is the direct consequence of the claim above. Here we shall give the proof of this theorem which is based upon the method of Kazuya Kato in [Kato2]. In [RW7] Jürgen Ritter and Alfred Weiss proved the more general claim in another manner (refer also to [RW5] [RW6]).

**Sketch of the proof.** Set $W = W^f \times \Gamma$ and choose a generator $\lambda$ of the quotient group $G^f/W^f$. Then we obtain the splitting exact sequence

$$1 \to W^f \to G^f \to \langle \lambda \rangle \to 1.$$ 

Recall that Serre’s $p$-adic zeta pseudomeasure $\xi_{ab}$ (resp. $\xi_W$) for $F_{[G,G]}/F$ (resp. $F_\infty/F_W$) exists as a unique element in $\Lambda(G^{ab})_S^\otimes$ (resp. $\Lambda(W)_S^\otimes$).

**Step 1.** Construction of the Brauer family $\bar{\mathfrak{X}}$.

By identifying $W^f$ with a $d$-dimensional $F_p$-vector space, the action of $\lambda$ upon $W^f$ (which we regard as a left action) is described as the Jordan normal form $J_\lambda = \bigoplus_{i=1}^d J_i$ where each $J_i$ is the Jordan block of rank $m_i$ with eigenvalue 1. Then the summation of $\{m_i\}_{i=1}^d$ equals $d$ by definition. Moreover each $m_i$ is less than $p$ since the order of $J_\lambda - \text{id}$ is just equal to $p$. Let $W^f = \bigoplus_{i=1}^d W_i^f$ be the corresponding generalised eigenspace decomposition of $W^f$. Fix a Jordan basis $\{e_{i,j}^{m_i}\}_{i,j=1}^t$ of each $W_i$. Note that the abelisation of $G^f$ is identified with $W \times \langle \lambda \rangle$ where $W$ is the quotient space of $W^f$ isomorphic to the subspace $W'$ of $W^f$ spanned by $\{e_{i,m_i}\}_{i=1}^t$. Now let $\mathfrak{X}(W^f)$ denote the space of characters on $W^f$. The cyclic group $\langle \lambda \rangle$ also acts upon $\mathfrak{X}(W^f)$ from the right by

$$\mathfrak{X}(W^f) \times \langle \lambda \rangle \to \mathfrak{X}(W^f); \quad (\chi, \lambda) \mapsto (\chi \ast \lambda: w \mapsto \chi(\lambda w)).$$

It is clear that the fixed subspace of $\mathfrak{X}(W^f)$ under this action is identified with $\mathfrak{X}(W)$ (use the Jordan basis above). By [Serre1] Théorème 17, every irreducible representation of $G^f = W^f \times \langle \lambda \rangle$ is isomorphic to one of the following types of induced representations:

- $\text{Ind}_{W^f}^{G^f}(\chi_1)$ where $\chi_1$ is an element in $\mathfrak{X}(W^f) \setminus \mathfrak{X}(W)$,
- $\chi_2 \otimes \chi'_2$ where $\chi_2$ is an element in $\mathfrak{X}(W)$ and $\chi'_2$ is a character of $\langle \lambda \rangle$.

We may, however, each $\chi_2 \otimes \chi'_2$ as a character of the abelisation $G^{f,ab}$ of $G^f$. Therefore an arbitrary irreducible representation of $G^f$ is obtained as an
induced representation of a character of either $W^f$ or $G^{f,\text{ab}}$; in other words, the family $\mathfrak{F} = \{(G, [G, G]), (W, \{e\})\}$ is a Brauer family for $G$.

**Step 2.** Additive theory.

Let $\theta^+_{ab}: \mathbb{Z}_p[[\text{Conj}(G)]] \to \mathbb{Z}_p[[G^{ab}]]$ denote the abelianisation homomorphism and $\theta^+_W$ the trace homomorphism from $\mathbb{Z}_p[[\text{Conj}(G)]]$ to $\mathbb{Z}_p[[W]]$. Let

$$I_W = \langle \theta^+_W(w) = \sum_{k=0}^{p-1} \lambda^k w \mid w \in W^f \rangle_{\mathbb{Z}_p[[\Gamma]]}$$

be the image of $\theta^+_W$ in $\mathbb{Z}_p[[W]]$ and $\Phi$ the $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p[[G^{ab}]] \times \mathbb{Z}_p[[W]]$ consisting of all pairs $(y_{ab}, y_W)$ satisfying the following two conditions:

- the equation $\text{Tr}_{\mathbb{Z}_p[[G^{ab}]]/\mathbb{Z}_p[[W^f]]}(y_{ab}) = \text{can}_{W,W}(y_W)$ holds;
- the element $y_W$ is contained in $I_W$.

Moreover we may also prove that $W$ is injective, where $\text{Conj}(\mathbb{Z}_p[[G^{ab}]])$.

**Step 3.** Logarithmic theory.

First note that $I_W$ is an ideal of the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[[W]]$ — the maximal subalgebra of $\mathbb{Z}_p[[W]]$ fixed by the action of $(\lambda)$ —, and thus $I_W$ obviously contains $I_W^f$. Moreover $I_W$ is contained in the augmentation kernel

$$J_W = \ker(\mathbb{Z}_p[[W]] \stackrel{\text{aug}_W}{\to} \mathbb{Z}_p[[\Gamma]] \to \mathbb{F}_p[[\Gamma]])$$

because $\text{aug}_W \circ \theta^+_W(w) = p$ holds for each $w$ in $W^f$. By the same argument as that in Proposition 5.7, we may prove that $1 + J_W$ is a multiplicative subgroup of $\Lambda(W)^\times$ and the $p$-adic logarithm induces a homomorphism

$$\log: 1 + J_W \to \Lambda(W).$$

Next we shall prove that the restriction of the map (9.6) to the multiplicative subgroup $1 + I_W$ is injective. The kernel of the homomorphism

$$\log: 1 + J_W \to \Lambda(W).$$

1\text{By the argument in the proof of Proposition 5.7, the element $x^p$ is especially contained in $p\Lambda(W) \cap I_W$ for $x$ in $I_W$. It implies that $(1 + x)^{-1} = \sum_{j=0}^{\infty} (-x)^j$ converges $p$-adically in $1 + I_W$, and hence $1 + I_W$ is a multiplicative subgroup of $\Lambda(W)^\times$.}
\[ \text{(9.6)} \text{ coincides with } \mu_p(\Lambda(W)) \text{ by Proposition 5.7 which is isomorphic to } \mu_p(\Lambda(\Gamma)) \times W^f \text{ by the theorem of Graham Higman [Higman] and Charles Terence Clegg Wall [Wall] Theorem 4.1. Note that } W^f \text{ is not contained in } 1 + I_W; \text{ if } x \text{ is an element in } I_W, \text{ its coefficient of the unit of } W^f \text{ is divisible by } p \text{ because } p \text{ is one of the free basis of } I_W \text{ over } \Lambda(\Gamma). \text{ Therefore } w - 1 \text{ is not contained in } I_W \text{ for each } w \text{ in } W^f \text{ except for the unit. Combining this fact with triviality of } \mu_p(\Lambda(\Gamma)), \text{ the homomorphism } 1 + I_W \to \Lambda(W) \text{ induced by the } p\text{-adic logarithm is injective.} \\

\textbf{Step 4. Translation.} \\
\text{Let } \theta_{ab}: K_1(\Lambda(\Gamma)) \to \Lambda(\Gamma)^{\times} \text{ denote the abelisation homomorphism and } \theta_W \text{ the norm homomorphism } \text{Nr}_{\Lambda(\Gamma)/\Lambda(W)}. \text{ Let } \Psi \text{ be the subgroup of } \Lambda(\Gamma)^{\times} \times \Lambda(W)^{\times} \text{ consisting of all pairs } (\eta_{ab}, \eta_W) \text{ satisfying the following two conditions:} \\
\begin{itemize}
  \item the equation } \text{Nr}_{\Lambda(\Gamma)/\Lambda(W)}(\eta_{ab}) = \text{can}_{W/W}(\eta_W) \text{ holds;}
  \item the congruence } \eta_W \equiv \varphi(\eta_{ab}) \text{ mod } I_W \text{ holds.}
\end{itemize}
\text{Then we may prove that } \theta = (\theta_{ab}, \theta_W) \text{ induces a surjective homomorphism } K_1(\Lambda(\Gamma)) \to \Psi \text{ with kernel } SK_1(\mathbb{Z}_p[G^f]) \text{ by the same arguments as those in Sections 6.3–6.4; more precisely,} \\
\begin{itemize}
  \item the congruence } \theta_W(\eta) \equiv \varphi(\theta_{ab}(\eta)) \text{ mod } I_W \text{ holds for } \eta \text{ in } K_1(\Lambda(\Gamma)) \text{ by direct calculation (refer to [Taylor] Section 5, Lemma 1.7); in other words, } \Psi \text{ contains the image of } \theta; \\
  \item for an arbitrary element } (\eta_{ab}, \eta_W) \text{ in } \Psi, \text{ there exists a unique element } y \text{ in } \mathbb{Z}_p[[\text{Conj}(G)]] \text{ which satisfies } \theta^+(y) = (\Gamma_{G^{ab}}(\eta_{ab}), \log(\varphi(\eta_{ab})^{-1}\eta_W)) \text{(recall that } \theta^+ \text{ is isomorphic). By applying theory upon integral logarithmic homomorphisms, we may prove that there exists a unique element } \eta \text{ in } K_1(\Lambda(\Gamma)) \text{satisfying } \Gamma_{G}(\eta) = y \text{ up to multiplication by an element in } SK_1(\mathbb{Z}_p[G^f]). \text{ The desired relation } \theta(\eta) = (\eta_{ab}, \eta_W) \text{ follows from the construction of } \eta \text{ (mimic the argument in the proof of Proposition 6.8).} \footnote{12} \\
\end{itemize}
\text{Note that } SK_1(\mathbb{Z}_p[G^f]) \text{ is trivial because } G^f \text{ has an abelian normal subgroup with cyclic quotient (see [Oliver] Theorem 8.10]), and therefore the induced homomorphism } \theta: K_1(\Lambda(\Gamma)) \to \Psi \text{ is an isomorphism.} \\

\textbf{Step 5. Localised version.} \\
\text{Let } \theta_{S,ab}: K_1(\Lambda(\Gamma)_S) \to \Lambda(\Gamma)^{\times}_S \text{ be the abelisation homomorphism and } \theta_{S,W} \text{ the norm homomorphism } \text{Nr}_{\Lambda(\Gamma)_S/\Lambda(W)_S}. \text{ Set } I_{S,W} = I_W \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)_S \text{ and let } \Psi_S \text{ be the subgroup of } \Lambda(\Gamma)^{\times}_S \times \Lambda(W)^{\times}_S \text{ consisting of all pairs } (\eta_{S,ab}, \eta_{S,W}) \text{ satisfying the following two conditions:} \\
\begin{itemize}
  \item the equation } \text{Nr}_{\Lambda(\Gamma)_S/\Lambda(W)_S}(\eta_{S,ab}) = \text{can}_{W/W}(\eta_{S,W}) \text{ holds;}
  \item the congruence } \eta_{S,W} \equiv \varphi(\eta_{S,ab}) \text{ mod } I_{S,W} \text{ holds.}
\end{itemize}
\text{Then we may prove that } \Psi_S \text{ contains the image of } \theta_S = (\theta_{S,ab}, \theta_{S,W}) \text{ by mimicking the argument in Step 4 (see also Section 7). One of the most important points is the following fact: since Higman-Wall’s theorem also}
holds for $R[W^f]$ the intersection of $1 + \hat{I}_W$ and $\mu_p(R[W^f]) = \mu_p(R) \times W^f$ is trivial if we set $\hat{I}_W = I_W \otimes_{\Lambda(\Gamma)} R$ (use triviality of $\mu_p(R)$ and mimic the argument in Step 3), and hence the induced homomorphism $log: 1 + \hat{I}_W \rightarrow R[W^f]$ is also injective.

**Step 6.** Verification of the congruence.

Under these settings, we may prove the claim if we verify the congruence $\xi_W \equiv \varphi(\xi_{ab}) \mod I_{\xi,W}$ by virtue of Burns’ technique (see Section 2). It is, however, no other than a special case of the Ritter-Weiss’ congruence [RW6, Theorem]. □

**References**

[Bass] Bass, H., *Algebraic K-theory*, Benjamin (1968).

[Bellinson] Bellinson, A., *Higher regulators and values of L-functions* (Russian), Current problems in mathematics, 24, I.ogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow (1984) 181–238.

[BerKean] Berwick, A. J., and Keating, M. E., *The localization sequence in K-theory*, K-Theory, 9 (1995) 577–589.

[BlKat] Bloch, S., and Kato, K., *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., 86 (1990) 333–400.

[Borel1] Borel, A., *Stable real cohomology of arithmetic groups*, Ann. Sci., ENS 7 (1974) 235–272.

[Borel2] Borel, A., *Cohomologie de $SL_n$ et valeurs de fonctions zêta aux points entiers*, Ann. Sc. Norm. Sup. Pisa 4 (1977) 613–636.

[BrBur] Breuning, M., and Burns, D., *Additivity of Euler characteristics in relative algebraic K-groups*, Homology, Homotopy Appl., 7 (2005) 11–36.

[BurFl1] Burns, D., and Flach, M., *Motivic L-functions and Galois module structures*, Math. Ann., 305 (1996) 65–102.

[BurFl2] Burns, D., and Flach, M., *On Galois structure invariants associated to Tate motives*, Amer. J. Math., 120 (1998) 1343–1397.

[CFKSV] Coates, J., Fukaya, T., Kato, K., Sujatha, R., and Venjakob, O., *The GL$_2$ main conjecture for elliptic curves without complex multiplication*, Publ. Math. Inst. Hautes Études Sci., 101 (2005) 163–208.

[CL] Coates, J., and Lichtenbaum, S., *On $\ell$-adic zeta functions*, Ann. of Math. (2) 98 (1973) 498–550.

[Deligne1] Deligne, P., *Valeurs de fonctions L et périodes d’intégrales*, in: *Automorphic forms, representations and L-functions*, Proc. Sympos. Pure. Math., XXXIII, Part 2 (1979) 313–346.

[Deligne2] Deligne, P., *Le déterminant de la cohomologie*, in: *Current trends in arithmetic algebraic geometry* (Arcata, Calif., 1985) Contemp., Math., 67, Amer. Math. Soc. Providence, RI (1987) 93–177.

13This is because $R$ is the $p$-adic completion of $\Lambda(\Gamma)_{(p)}$; refer to [Wall] the remark after Theorem 4.1.]
INDUCTIVE CONSTRUCTION OF p-ADIC ZETA FUNCTIONS

[DR] Deligne, P., and Ribet, K. A., Values of abelian L-functions at negative integers over totally real fields, Invent. Math., 59 (1980) 227–286.

[FERWash] Ferrero, B., and Washington, L. C., The Iwasawa invariant $\mu_p$ vanishes for abelian number fields, Ann. of Math., 109 (1979) 377–395.

[Flach] Flach, M., with an appendix by Greither, C., The equivariant Tamagawa number conjecture: a survey, in: Stark’s conjectures: recent work and new directions, Contemp. Math., 358, Amer. Math. Soc., Providence, RI (2004) 79–125.

[FPF] Fontaine, J.-M., and Perrin-Riou, B., Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L, in: Motives (Seattle 1991), Sympos. Pure Math., 55, Part 1 (1994) 599–706.

[FukKat] Fukaya, T., and Kato, K., A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory, Proceedings of the St. Petersburg Mathematical Society, Vol. XII, 1–85, Amer. Math. Soc. Transl. Ser. 2, 219, Amer. Math. Soc., Providence, RI (2006).

[Gross] Gross, B. H., On the values of Artin L-functions, preprint.

[HachShar] Hachimori, Y., and Sharifi, R. T., On the failure of pseudo-nullity of Iwasawa modules, J. Algebraic Geom., 14 (2005) 567–591.

[Hara] Hara, T., Iwasawa theory of totally real fields for certain non-commutative $p$-extensions, J. Number theory, 130, Issue 4 (2010) 1068–1097.

[Higman] Higman, G., The units of group rings, Proc. London Math. Soc. (2) 46 (1940) 231–248.

[Hornbostel] Hornbostel, J., and Kings, G., On non-commutative twisting in étale and motivic cohomology, Ann. Inst. Fourier (Grenoble) 56 (2006) 1257–1279.

[HubKin1] Huber, A., and Kings, G., Equivariant Bloch–Kato conjecture and non-abelian Iwasawa main conjecture, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing 2002) 149–162.

[HubKin2] Huber, A., and Kings, G., Bloch–Kato conjecture and Main Conjecture of Iwasawa theory for Dirichlet motives, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing 2002) 149–162.

[Itakura] Itakura, K., Tamagawa Number Conjecture of Bloch–Kato for Dirichlet motives at the prime 2, preprint.

[Kakde1] Kakde, M., Proof of the main conjecture of noncommutative Iwasawa theory for totally real number fields in certain cases, preprint, [arXiv:0902.2274v2 [math.NT]].

[Kakde2] Kakde, M., $K_1$ of some non-commutative p-adic group rings, in preparation.

[Kato1] Kato, K., Iwasawa theory and p-adic Hodge theory, Kodai Math. J., 16 (1993) no. 1, 1–31.

[Kato2] Kato, K., Iwasawa theory of totally real fields for Galois extensions of Heisenberg type, preprint.

[Klingen] Klingen, H., Über die Werte der Dedekindschen Zetafunktion, Math. Ann., 145 (1961/1962) 265–272.

[KnudMum] Knudsen, F. F., and Mumford, D., The projectivity of the moduli space of stable curves I: Preliminaries on “det” and “Div,” Math. Scand., 39 (1976) no. 1, 19–55.

[Kings] Kings, G., The Bloch–Kato conjecture on special values of L-functions. A survey of known results, J. Théor. Nombres Bordeaux, 15 (2003) 179–198.

[MazWil] Mazur, B., and Wiles, A., Class fields of abelian extensions of $\mathbb{Q}$, Invent. Math., 76, no. 2 (1984) 179–330.

[McRob] McConnell, J. C., and Robson, J. C., Noncommutative Noetherian Rings, Graduate Studies in Math., 30, American Mathematical Society (1987).

[Oliver] Oliver, R., Whitehead groups of finite groups, London Mathematical Society Lecture Note Series, 132 (1988) Cambridge Univ. Press.

[OT] Oliver, R., and Taylor, L. R., Logarithmic descriptions of Whitehead groups and class groups for p-groups, Mem. Amer. Math. Soc., 76 (1988) no. 392.

[Quillen] Quillen, D., Higher algebraic K-theory I, in: Algebraic K-theory I: Higher K-theories (Proc. Conf. Battelle Memorial Inst., Seattle, Wash., 1972) Lecture Notes in Math., 341, Springer, Berlin (1973) 85–147.

[RW] Ritter, J., and Weiss, A., Toward equivariant Iwasawa theory. I, Manuscripta Math., 100 (2002) 131–146.
[RW2] Ritter, J., and Weiss, A., Toward equivariant Iwasawa theory: II, Indag. Math. (N.S.) 15 (2004) 549–572.

[RW3] Ritter, J., and Weiss, A., Toward equivariant Iwasawa theory: III, Math. Ann., 336 (2006) 27–49.

[RW4] Ritter, J., and Weiss, A., Toward equivariant Iwasawa theory: IV, Homology, Homotopy Appl., 7 (2005) 155–171.

[RW5] Ritter, J., and Weiss, A., Non-abelian pseudomeasures and congruences between abelian Iwasawa L-functions, Pure Appl. Math. Q., 4 (2008) no. 4, part 1, 1085–1106.

[RW6] Ritter, J., and Weiss, A., Congruences between abelian pseudomeasures, Math. Res. Lett., 15 (2008) no. 4, 715–725.

[RW7] Ritter, J., and Weiss, A., Equivariant Iwasawa theory: an example, Doc. Math., 13 (2008) 117–129.

[Serre1] Serre, J.-P., Représentations linéaires des groupes finis, Hermann (1967).

[Serre2] Serre, J.-P., Sur le résidu de la fonction zêta p-adique d’un corps de nombres, C. R. Acad. Sci., Paris, 287 (1978) série A, 183–188.

[Siegel] Siegel, C. L., Über die Fourierschen Koeffizienten von Modulformen, Nachr. Akad. Wiss. Göttingen Math.–Phys. Kl. II, 3 (1970) 15–56.

[Stenström] Stenström, B., Rings of Quotients, Springer-Verlag, New York–Heidelberg (1975).

[Swan] Swan, R. G., Algebraic K-theory, Lecture Notes in Mathematics, 76, Springer-Verlag, Berlin–New York (1968).

[Taylor] Taylor, M. J., Classgroups of group rings, London Math. Soc. Lect. Note Ser., 91, Cambridge Univ. Press (1984).

[Wall] Wall, C. T. C., Norms of units in group rings, Proc. London Math. Soc. (3) 29 (1974) 593–632.

[WeibYao] Weibel, C., and Yao, D., Localization for the K-theory of noncommutative rings, in: Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989) Contemp. Math., 126, Amer. Math. Soc., Providence, RI (1992) 219–230.

[Wiles] Wiles, A., The Iwasawa conjecture for totally real fields, Ann. of Math. Second Ser., 131 (1990) no.3, 493–540.

Graduate School of Mathematical Sciences, The University of Tokyo, 8-1 Komaba 3-chome, Meguro-ku, Tokyo, 153-8914, Japan
E-mail address: thara@ms.u-tokyo.ac.jp