Inducing information stability and applications thereof to obtaining information theoretic necessary conditions directly from operational requirements

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Abstract

This work constructs a discrete random variable that, when conditioned upon, ensures information stability of quasi-images. Using this construction, a new methodology is derived to obtain information theoretic necessary conditions directly from operational requirements. In particular, this methodology is used to derive new necessary conditions for keyed authentication over discrete memoryless channels and to establish the capacity region of the wiretap channel, subject to finite leakage and finite error, under two different secrecy metrics. These examples establish the usefulness of the proposed methodology.

I. INTRODUCTION

Consider arbitrary discrete random variables (DRVs) \((M, X, Y)\), which form a Markov chain in that order, where \(X = (X_1, X_2, \ldots, X_n)\), \(Y = (Y_1, Y_2, \ldots, Y_n)\), and

\[
\Pr(Y = y | X = x) = \prod_{i=1}^{n} \Pr(Y_i = y_i | X_i = x_i) = \prod_{i=1}^{n} p_{Y_i|X}(y_i | x_i)
\]

for some conditional distribution \(p_{Y|X}\). DRVs of this nature are often found in the literature related to the information theory, starting when Shannon \([1]\) considered one way communication over a discrete memoryless channel (DMC) where \(M\) represents the message, \(X\) the output of the channel encoder, and \(Y\) the input to the channel decoder. The purpose of this work is to provide a DRV \(U\) such that

- the cardinality of its alphabet, \(|U|\), grows sub-exponentially with \(n\),
- \(U, X, Y\) form a Markov chain in that order, and
- \(U\) induces information stability (see \([2], [3]\)).

The last property above means that for probability that converges to unity with \(n\), a \(u \in U\) chosen randomly according to \(U\) will exhibit convergence in probability of

\[
- \log_2 p_{Y|U}(Y|u) \rightarrow H(Y|U = u)
\]

\[
- \log_2 p_{Y|M,U}(Y|M, u) \rightarrow H(Y|M, U = u)
\]

where

\[
H(Y|U = u) = - \sum_y p_{Y|U}(y|u) \log_2 p_{Y|U}(y|u)
\]

\[
H(Y|M, U = u) = - \sum_{m,y} p_{Y,M|U}(y, m|u) \log_2 p_{Y,M|U}(y, m, u)
\]

are conditional entropies. The construct of \(U\) is similar to that of a DRV that determines the empirical distribution, or type as defined in \([4]\) Chap. 2), of \(X\).

Such a property proves to be extremely useful in establishing information-theoretic necessary conditions directly from operational requirements. To understand the importance of such a capability, consider Fano’s inequality \([5]\), which states that given DRVs \(M, \hat{M}\) and \(\epsilon \in (0, 1)\), if \(\Pr(M = \hat{M}) < \epsilon\) then

\[
H(M | \hat{M} \leq \epsilon \log_2 |M| + H(B_\epsilon),
\]

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This research was presented in part at the 2014, 2016, and 2017 IEEE International Symposium on Information Theory. T. F. Wong was supported by the National Science Foundation under Grant CCF-1320086. Eric Graves was supported by something at Army Research Lab.
where $B_\epsilon$ is a Bernoulli random variable with parameter $\epsilon$. A typical application of Fano’s inequality generally views the DRV $M$ and $\hat{M}$ as respectively being a message sent at the transmitter and an estimate of the message made at the receiver of a communication system. Thus Fano’s inequality gives us an upper bound on $\mathbb{H}(M|\hat{M})$ from the operation requirement of maintaining a small transmission error probability, i.e., $\Pr(\hat{M} = M) < \epsilon$. While no one would argue against the utility of Fano’s inequality, it is clear that it can only provide a bound for one specific operational requirement, namely a small transmission error probability. In contrast, inducing information stability can allow for a replacement of stochastic terms with information theoretic averages, directly in the operational quantities. Thus, such a method is applicable to all operational requirements that can be written as functions of distributions of DRVs involved. To demonstrate this methodology we have provided three different examples: one way communication over a discrete memoryless channel; tighter bounds on the probability of intrusion in a generalization of a problem introduced by Lai et al. [6]; and establishing the capacity of the wire-tap channel with finite error and leakage under two different secrecy metrics. These problems are chosen as to present a wide-range of operational requirements for which this new methodology can extract information theoretic necessary conditions. Furthermore, while our first example is chosen simply to present the reader with a well-studied problem in information theory, the second and third examples establish new results.

The rest of the paper is organized as follows. First we conclude the introduction by describing the notation used through the rest of the paper in Section I-A particular attention should be given to the definition of a regular collection of DRVs (Definition 2) which defines where the theorems can be applied. Following this, we highlight relevant work in Section II. We then present our main results in Section III and applications thereof in Section IV. The proofs of each main theorem is given (Definition 2) which defines where the theorems can be applied. Following this, we highlight relevant work in Section II. We

\section{A. Notation}

Constants, random variables (R Vs), and sets will be denoted by lower case, upper case and script letters respectively. Function $Pr(\cdot)$ returns the probability of the event in the predicate. We will always employ the corresponding script form of a letter to denote the support set of any DRV. That is, if $X$ is a DRV, then $\mathcal{X}$ is the set of all $x$ for which $Pr(X = x) > 0$. Functions will be lower case or random variables if they are random or not. Conditional DRVs and events will be denoted by $\cdot|\cdot$, for example the DRV $X$ given the event $\{Y = y\}$ is written $X|\{Y = y\}$.

The set of positive integers is written as $\mathbb{N}_+$, and the set of positive real numbers is written as $\mathbb{R}_+$. Furthermore, $[i:j]$ denotes the set of integers starting at $i$ and ending at $j$, inclusively. We use $\otimes$ to denote collections of constants, DRVs, etc. For instance the collection of three DRVs $(X_1, X_2, X_3) = \otimes_{i=1}^3 X_i$. Throughout this paper $X \triangleq \otimes_{i=1}^n X_i$. That is, $X$ denotes a sequence of $n$ possibly mutually dependent DRVs, and $x$ denotes a sequence $n$ constants all from $\mathcal{X}$. Note that we have omitted the dependence on $n$ for simpler notation, and will continue to do so for the rest of the paper unless when it is necessary to highlight the dependence. The support set of $X$ is clearly a subset of $\mathcal{X}^n \triangleq \otimes_{i=1}^n \mathcal{X}$. Also when $X = 0$, by convention this defines $X$ as some unspecified constant. From here-forth, we will only rarely need to refer back to the individual elements in the $n$-length sequences of $x$ and $y$. As such, the subscripts of DRVs will be primarily used to denote a collection of multiple $n$-length DRV sequences, such as $Y_{[1:i]} \triangleq \otimes_{j=1}^i Y_j$, for some $i \in \mathbb{N}_+$.

Probability distributions, being deterministic functions over their support sets, will be denoted with lower case letters. Of particular importance will be $p$, which will always denote a probability distribution, and when written with the subscript of DRVs, specifically denotes the associated probability distribution over said random variables. For instance, $p_{X|Y}(x|y) \triangleq Pr(X = x|Y = y)$. With this notation $p_{X|Y}(X|Y)$ is itself a RV, while $p_{X|Y}(x|y)$ is a fixed value. When the context is clear, we may drop the subscript entirely. Furthermore, $p_X(A) = \sum_{x \in A} p_X(x)$ for any $A \subseteq \mathcal{X}$. The set of all possible conditional distributions of the form $p_{Y|X}(y|x)$, where $y \in \mathcal{Y}$ and $x \in \mathcal{X}$, is denoted $\mathcal{P}(\mathcal{Y}|\mathcal{X})$. For DRVs $Y$ and $X$ if $p_{Y|X}(y|x) = \prod_{i=1}^n p_{Y_i|X_i}(y_i|x_i)$, we will write $p_{Y|X}(y|x) = p_{X|Y}(y|x)$ or when clear $p_{X|Y}(y|x) = p^n(y|x)$. The empirical conditional distribution of $y|x$ is defined as $p_{y|x}(a|b) \triangleq \frac{\sum_{i \in [1:n]} |y_i = a, x_i = b|}{|\{i \in [1:n]|x_i = b\}|}$. The set of all valid empirical distributions for an $n$-length sequence will be denoted $\mathcal{P}_n$. For empirical conditional distributions we shall use $\mathcal{P}_n(\mathcal{Y}|p)$ where $p \in \mathcal{P}_n(\mathcal{X})$, to denote the set of conditional empirical distributions $w$ for which $w(y|x)p(x)$ is a valid distribution in $\mathcal{P}(\mathcal{Y}|\mathcal{X})$.

Many of the results to be presented in later sections involve DRVs that satisfy specific sets of relationship and/or properties. For relationships between DRVs in particular, we will use the following two operators. First if $X \leftrightarrow Y \leftrightarrow Z$, then DRVs $X, Y, Z$ form a Markov chain in that order. In other words $p_{X,Y,Z}(x,y,z) = p_X(x)p_{Y|X}(y|x)p_{Z|Y}(z|y)$ for all $(x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$. On the other hand, if $X \Rightarrow Y$, then $Y$ can be written as a deterministic function of $X$. For any DRVs $X, Y, Z$, if $X \Rightarrow Y$ then $Y \Rightarrow X \Rightarrow Z$. To simplify the statements of our results, we will adopt the standard set notation when describing DRVs satisfying a specific set of properties. For instance, the DRVs $U, X, Y$ that satisfy the conditions that $|U| \leq n$ and that $U \leftrightarrow X \leftrightarrow Y$ will be denoted by $(U, X, Y) : \{|U| \leq n, U \leftrightarrow X \leftrightarrow Y\}$. 

\section{B. Information Stability}

\subsection{Definition}
Information-theoretic quantities which are averages over probability distributions of DRVs will be denoted by blackboard bold letters. In specific, the following quantities will receive heavy use:

For DRVs $U, X, Y, Z$, and probability distributions $w, \hat{w}, \tilde{w} \in \mathcal{P}(Y|X)$ and $p \in \mathcal{P}(X)$,

\[
\mathbb{H}_w(X|Z) = - \sum_{(x,z) \in X \times Z} p_{X,Z|U}(x,z|u) \log_2 p_{X|Z,U}(x|z,u)
\]

\[
I_u(X;Y|Z) = \mathbb{H}_w(X|Z) - \mathbb{H}_u(X|Y,Z)
\]

\[
\mathbb{D}(w||\hat{w}|p) = \sum_{x,y} w(y|x)p(x) \log_2 \frac{w(y|x)}{\hat{w}(y|x)}
\]

\[
\mathbb{D}_w(w||\hat{w}|p) = \sum_{y,x} \hat{w}(y|x)p(x) \log_2 \frac{w(y|x)}{\hat{w}(y|x)}
\]

\[
= \mathbb{D}(\hat{w}||w|p) - \mathbb{D}(\hat{w}||\tilde{w}|p).
\]

It should be noted that while $\mathbb{H}(X|Z,U)$ is a constant, $\mathbb{H}_U(X|Z)$ is a RV. More specifically $\mathbb{H}(X|Z,U)$ is the expected value of $\mathbb{H}_U(X|Z)$ over $U$. Moreover we will employ the concept of entropy spectrum \[7\] in the development of some of our results. More specifically, we will mostly consider the \textit{entropy spectrum frequency} of $y|x$, which is defined as

\[
h_{Y|X}(y|x) \triangleq - \log_2 p_{Y|X}(y|x).
\]

The conditioning notation will be omitted in the special cases where $X = \emptyset$. Furthermore, the subscript may be omitted when the context is clear. Finally, we note that the exact bounds obtained in this paper quickly become unwieldy. This is unfortunate because this detracts from the elegance of the stated results. As a compromise, we introduce the following order terminology which is similar in spirit to Bachmann-Landau notation, but has a formal definition which has to be context sensitive.

**Definition 1.** For any $\epsilon \in \mathbb{R}_+$, we say $f(\epsilon) = O(g(\epsilon))$ if there exists a constant $c \in \mathbb{R}_+$ (that is possibly a function of the cardinalities of the alphabets involved) such that

\[
|f(\epsilon)| \leq c|g(\epsilon)|.
\]

Throughout the paper our results will be expressed in terms of $O(g(\epsilon))$, for some $g : \epsilon \to \mathbb{R}$, with the value of acceptable $\epsilon$ being itself a function of $n$. The exact calculations of the order terms are cumbersome and trivial, and we will skip most of such calculations except a few particularly important ones.

Now, we restrict the DRVs that our main theorems are applicable to.

**Definition 2.** (Regular collection of DRVs) For any arbitrary index set $W$ and any $l \in \mathbb{N}_+$, DRVs $(M_{[1:l]},X,Y,W)$ form a regular collection if

- $|X|$ and $\sup_{w \in W} |Y_w|$ are finite,
- $|X| \geq 2$ and $|Y_w| \geq 2$ for all $w \in W$,
- $M_{[1:l]} \xrightarrow{\text{random}} X \xrightarrow{\text{random}} Y_W$,
- $Y_w|X$ is distributed $p_{Y_w|X}$, where $p_{Y_w|X} \in \mathcal{P}(Y_w|X)$ for all $w \in W$, and
- $n \geq 27$.

Furthermore, to simplify notation we assume that $Y_w = Y$ for all $w \in W$, and when $W \subset \mathcal{P}(Y|X)$ we will assume that

\[
p_{Y_w|X}(y|x) = w^n(y|x).
\]

Note neither of these assumptions are in the least restrictive given the first requirement of the definition.

**II. BACKGROUND**

**A. Images and Quasi-images**

The manipulation of images and quasi-images will play an important role in establishing our theorems. Let us define these concepts. For all discussions and results in this section, it is assumed that $(X, Y)$ is a regular collection of DRVs.

**Definition 3.** ([4] Ch. 15) Let $p_{Y|X} \in \mathcal{P}(Y|X)$. For any $\eta \in (0, 1)$, a set $\mathcal{B} \subseteq \mathcal{Y}^n$ is called an $\eta$-image of $\mathcal{A} \subseteq \mathcal{X}^n$ (generated) by $p_{Y|X}$ if

\[
p_{Y|X}(\mathcal{B}|x) \geq \eta, \forall x \in \mathcal{A}.
\]

Furthermore $g_{Y|X}(\mathcal{A}, \eta)$ denotes the minimum cardinality (size) of $\eta$-images of $\mathcal{A}$ by $p_{Y|X} \in \mathcal{P}(Y|X)$. That is,

\[
g_{Y|X}(\mathcal{A}, \eta) = \min_{\mathcal{B} \subseteq \mathcal{Y}^n: \mathcal{B} \text{ is an } \eta\text{-image of } \mathcal{A} \text{ by } p_{Y|X}} |\mathcal{B}|.
\]
Lemma 5. (\cite[Problem 15.13]{13}) Let $p_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$. For any $\eta \in (0,1)$, a set $B \subseteq \mathcal{Y}^n$ is called an $\eta$-quasi image of $\mathbf{X}$ by $p_{Y|X}$ if
\[
\sum_x p_{Y|X}^n(B|x)p(x) \geq \eta.
\]
Furthermore $g_{Y|X}(\mathbf{X},\eta)$ denotes the minimum cardinality (size) of $\eta$-images of $\mathbf{X}$ by $p_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$.

Image sizes were originally introduced in Gács and Körner \cite{8} and Ahlswede et al. \cite{9}, and found use in proving strong converses due to the blowing up lemma, which the authors of \cite{8} and \cite{9} credit to Margulis \cite{10}. In our paper’s context, the blowing up lemma will play an important role because of how it relates image sizes. Before pointing out the lemmas which will find use in this paper, we refer readers to \cite[Chap. 5]{4}, \cite{11} and \cite[Chapter 3]{12} for an information theoretic context of the blowing up lemma.

Lemma 5. (\cite[Lemma 6.6]{4}) Given $\mathcal{X}$, $\mathcal{Y}$, $\alpha \in (0,1)$, and $\beta \in (0,1-\alpha]$, there exists $\tau_n : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, where
\[
\lim_{n \to \infty} \tau_n(\alpha,\beta) = 0
\]
for every $A \subseteq \mathcal{X}^n$, and every distribution $p_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$.

While it is possible to derive the theorems in Section \[III\] directly from Lemma \[8\] we take the further step now of providing an upper bound on $\tau_n(\alpha,\beta)$. This can be done from combining a lemma which discusses the change in probability given a blow up (see Liu et al. \cite{13} or Raginsky and Sason \cite[Lemma 3.6.2]{12}), with an upper bound on the increase in the image size due to the blow up (see Ahlswede et al. \cite[Lemma 3]{9} or Csiszár and Körner \cite[Lemma 5.1]{4}).

Lemma 6. For any $\alpha \in (0,1)$ and $\beta \in (0,1-\alpha]$, we have
\[
\tau_n(\alpha,\beta) \leq \mathbb{H}(B) + \frac{\sqrt{-\ln \beta} + \sqrt{-\ln \alpha}}{\sqrt{2n}} \log_2 |\mathcal{Y}|
\]
where $B$ is a Bernoulli DRV with parameter $\frac{\sqrt{-\ln \beta} + \sqrt{-\ln \alpha}}{\sqrt{2n}}$.

Remark 7. The value of $\tau_n$ will play a pivotal role in the bounds to come. In fact a tighter bound on the value of $\tau_n$ would directly lead to tighter bounds for multiple theorems in this paper. Because of this, we feel it necessary to bring forth recent work by Liu et al. \cite{13}, \cite{14} who endeavor to provide an alternative to the blowing-up lemma which offers tighter bounds for certain information theoretic problems. By using functional inequalities and the reverse hypercontractivity of particular Markov semigroups instead of the blowing up lemma, they have been able to obtain order tight bounds on the hypothesis testing problem. While hypothesis testing does not directly extend to determining minimum image and quasi-image sizes, it is clear that two problems are closely related. In specific the geometrical interpretations of their work may lead to further insight which allow for an improvement in the $\tau_n$ term.

In terms of applications Ahlswede \cite{13} used the blowing up lemma to prove a local strong converse for maximal error codes over a two-terminal DMC, showing that all bad codes have a good subcode of almost the same rate. Using the same lemma, Körner and Marton \cite{16} developed a general framework for determining the achievable rates of a number of source and channel networks. On the other hand, many of the strong converses for some of the most fundamental multi-terminal DMCs studied in literature were proven using image size characterization techniques. Körner and Martín \cite{17} employed such a technique to prove the strong converse of a discrete memoryless asymmetric broadcast channel. Later Dueck \cite{18} used these methods, combined with an ingenious “wringing technique” to prove the strong converse of the discrete memoryless multiple access channel with independent messages.

B. Other works of interest

Here we wish to briefly highlight a few of the methods by which information theoretic necessary conditions are generally obtained, first and foremost being Fano’s inequality \cite{5}. Fano’s inequality and generalizations (for instance, Han and Verdú \cite{19}), directly provide information theoretic necessary conditions from probability of error requirements. One significant problem is that it requires that the error probability go to zero with $n$ in order to obtain tight bounds in certain scenarios. One such scenario is establishing bounds on the number of message that can be reliably distinguished in one-way communication over a DMC, which we discuss in more detail in Section \[IV\]. While, as first claimed by Shannon and proven by Wolfowitz \cite{20}, this value does not change when allowing a finite error probability, the bound obtained from Fano’s inequality does increase with the error term.

\footnote{The lemma from Raginsky and Sason provides the same order for $\tau_n$ as can be obtained via \cite[Lemma 5.3.5.4]{4}, but is a little sharper, and much simpler to present.}
Actually, this allowed Wolfowitz to introduce the concept of a capacity dependent upon error, usually denoted by \( c(\epsilon) \). Because of this there exists a demarcation between converses which are primarily independent of the error rate, and those which are tight only if the probability of error vanishes. Following the terminology of Csiszár and Körner \([4, \text{Pg. 93}]\), a converse result showing \( c(\epsilon) = \lim_{n \to \infty} c(\epsilon^n) \) for all \( \epsilon \in (0, 1) \) is called a strong converse. Verdú and Han \([21]\) showed the stronger assertions that this is true for all finite \( n \), and that all rates larger must have error probability approaching unity hold for all two-terminal DMCs. More recently techniques such as the meta-converse by Polyanskiy et al. \([22]\) have been able to establish tight necessary condition as function of error probability up to the second order. The meta-converse leverages the idea that any decoder can be considered as a binary hypothesis test between the correct codeword set and the incorrect codeword set. Bounding the decoding error by the best binary hypothesis test, new bounds, which are relatively tight even for small values of \( n \), can be established.

Thus for the single operational requirement of transmission error probability, multiple different methodologies have been derived in order to obtain increasingly strong results. While each of these methodologies can be applied to different channels, they all still require the probability-of-error operational requirement as a starting point. The only general methodology that transcends this limitation are those related to the information spectrum as first defined by Verdú and Han \([21]\). For an in depth treatment of information spectrum methods, we point the reader to Han’s book \([7]\). The information-spectrum methods, in general, link operational quantities directly to information/entropy spectrum frequencies. Hence solving extremal problems of the information spectrum in turn determines the fundamental limits of these operational quantities. These methods are incredibly strong and universally applicable, but generally can not easily relate back to the more traditional information theoretic quantities like entropy and mutual information. Our work takes this further step, but at the cost of having to restrict our attention to DRVs that form a regular collection. Since such DRVs are of most common use, we feel this trade-off is one worth pursuing, because many operational requirements, in addition to the transmission error probability, are of recent interest.

### III. Main Results

Given a regular collection \((M_j, X, Y_j)\), our primary goal is to “stabilize” \( Y_w \), when conditioned on \( M_j \), where \( j \in [1 : l] \), in the sense that the entropy spectrum of \( Y_w | M_j \) is concentrated around a single frequency. More precisely, we want

\[
\Pr \left( \left| h_{Y_w|M_j}(Y_w|M_j) - c \right| > n \epsilon_n \right) < \delta_n
\]

for some \( c \in \mathbb{R}_+ \), and \( \epsilon_n \) and \( \delta_n \) that vanish with increasing \( n \). It is easy to see that a statement such as the above is not true in general. But, as we will show it can be achieved by introducing a particular stabilizing DRV. This stabilization will allow for the direct exchange of probabilities and entropy terms, thanks to the following lemma.

**Lemma 8.** Given DRVs \((Y, U)\), \( \mu \in \left(0, \frac{1}{2}\right) \) and \( \epsilon \in \mathbb{R}_+ \), if

\[
\Pr \left( \left| h(Y|U) - c \right| > \epsilon \right) < \mu,
\]

for some \( c \in \mathbb{R}_+ \), then

\[
|h(Y|U) - \epsilon| < \epsilon + \mu \log_2 \frac{|Y|}{\mu^2}.
\]

**Corollary 9.** Furthermore

\[
\Pr \left( \left| h(Y|U) - \epsilon|Y|U \right| > 2 \epsilon + \mu \log_2 \frac{|Y|}{\mu^2} \right) < \mu.
\]

The lemma’s proof can be found in Appendix \(\text{[A]}\). Thus stabilizing \( Y_w | M_j \) has the added benefit that \( p_{Y_w|M_j}(Y_w|M_j) \) converges to \( 2^{-H(Y_w|M_j)} \) in probability for large \( n \). From this exchange, we can easily create new necessary conditions for different information theoretic problems, as we demonstrate in Section \(\text{[IV]}\).

In order to construct the information stabilizing random variable, first for a given regular collection \((\emptyset, X, Y)\), we find a subset \( A^l \subseteq X^n \) for which the quasi-image of \( X \) \{ \( X \in A^l \) \} by a specific \( p_{Y|X} \in \mathcal{P}(Y|X) \) is stable.

**Theorem 10.** Given any regular collection \((\emptyset, X, Y)\) and any \( \alpha \in \left(\frac{\log_2 n}{n}, \frac{1}{8 \log_2 n}\right) \), there exists both a set \( A^l \subseteq X^n \), where

\[
p_{X}(A^l) \geq \frac{1}{n} \log_2 \frac{n}{8},
\]

and positive real numbers \( \delta = O(-\sqrt{\alpha} \log_2 \alpha) \) and \( r < 2n \log_2 |Y| \) such that

\[
\Pr (|h(Y|U) - r| > n\delta + h(U)|U = u) < 3 \cdot 2^{-n\alpha}
\]

for all \( U : U \rightarrow X \rightarrow Y \) and \( u \in U : \Pr \{ X \in A^l | U = u \} = 1 \).

The proof of Theorem \(\text{[10]}\) can be found in Section \(\text{[V]}\). Next we repeatedly use Theorem \(\text{[10]}\) to continually carve out different sets which induce stability. Thus directly building upon Theorem \(\text{[10]}\) we construct the following theorem.
Theorem 11. (Information stabilizing partitions) For any regular collection \((M_{[1:l]}, X, Y_{[1:k]})\) and real number \(\alpha \in \left(\frac{\log_2 n - \frac{1}{2}}{n} \right)\), we have

- a DRV \(V\) : \[ |V| \leq (2n^3 \log_2 |Y|)^{\frac{1}{k}} \]
- a positive real number \(\delta = O(-\sqrt{\alpha \log_2 \alpha})\), and
- for each DRV \(U\) : \[ (U, M_{[1:l]}) \rightarrow X \rightarrow Y_{[1:k]} \]

\(i \in [1:k], j \in [1:l]\), there exists a set \(U_{i|j} \subseteq U\) such that
\[ p_U(U_{i|j}) \geq 1 - 2^{-\frac{\alpha}{2 \log_2 \alpha}} \]

and
\[ \Pr(h(Y_i|M_j, U) - H_U(Y_i|M_j)) > n\delta + 3h(U)|U = u) < 4 \cdot 2^{-n\alpha}, \]

for all \(U \in U_{i|j}\).

The proof can be found in Section VI. In and of itself, Theorem 11 allows us a new methodology of providing information theoretic necessary conditions for certain problems. Still, the applicability of this methodology can be improved by also stabilizing \(M_{[1:l]}\).

Theorem 12. For any DRVs \(M_{[1:l]}\), positive integer \(\psi\), and positive real numbers \(\rho \in [1, \infty)\), we have:

- a DRV \(Q\) : \[ |Q| \leq (\psi + 1)^l \]
- a real number \(\beta = O(\rho + 2^{-\rho})\), and
- for each DRV \(U\) : \[ U \succ Q \]

\(j \in [1:l]\), there exists sets \(U_{j,(stable)} \subseteq U\) and \(U_{j,(answer)} \subseteq U\) such that
\[ p_U(U_{j,(stable)} \cup U_{j,(answer)}) \geq 1 - 2^{-\rho}, \]

\[ \Pr(h(M_j|U) - H_U(M_j)) > \beta + 3 \log_2 |U||U = u) < 2^{-\rho} \]

for all \(U \in U_{j,(stable)}\), and
\[ \Pr(h(M_j|U) < \psi - \beta - 3 \log_2 |U||U = u) < 2^{-\rho} \]

for all \(U \in U_{j,(answer)}\).

Furthermore, if \(M_j\) is uniform over \(M_j\), then
\[ p_U(U_{j,(stable)}) \geq 1 - 2^{-\rho} \]

and
\[ \Pr(h(M_j|U) - \log_2 |M_j|) > \beta + 3 \log_2 |U||U = u) < 2^{-\rho} \]

for all \(j \in [1:l]\) and \(U \in U_{j,(stable)}\).

Notice that providing stability to \(Y_i|M_j, Y_i\) and \(M_j\), also would then provide stability to \((Y_i, M_j)\) and \(M_j|Y_i\). Providing stability to \(M_j|Y_i\) may be instantly recognizable to the reader as stabilizing a message given an observation.

The need of our second augmentation theorem arises from the fact that Theorem 11 cannot in and of itself simultaneously provide stable quasi images for all product distributions in \(\mathcal{P}(Y|X)\). Indeed, the reason for this being that there are an infinite number of such distributions, with even the number of conditional empirical distributions possible for \(n\) symbols growing polynomial with \(n\). In turn then, the support set of \(Q\) would have to grow exponentially with \(n\), which is something which we are trying to avoid as it would make our results trivial. The following augmentation theorem rectifies this problem by providing a set \(\hat{\mathcal{P}} \subseteq \mathcal{P}(Y|X)\) which if stabilized then guarantee stability for all \(\mathcal{P}(Y|X)\).

First, a quick point of emphasis. For the upcoming theorem, we begin to adopt the notation outlined previously where \(Y_{\mathcal{P}(Y|X)} \triangleq \bigotimes_{w \in \mathcal{P}(Y|X)} Y_w\) and \(Y_w|X\) is distributed \(w^n(y|x)\) for \(w \in \mathcal{P}(Y|X)\).

Theorem 13. For any real number \(\epsilon \in \left(\frac{4|Y||Y|}{n} \log_2 n, 1\right)\), there exists a subset
\[ \hat{\mathcal{P}} \subseteq \mathcal{P}(Y|X) : \left\{|\hat{\mathcal{P}}| \leq \left|Y\right| \left(1 + \left|\frac{4|Y|^2}{\epsilon}\right|\right)^{|X||Y|}\right\} \]
with the following property: 

Given a regular collection \((M, X, Y, \mathcal{P}(Y | X))\), for each \(w \in \mathcal{P}(Y | X)\) there exists a \(\tilde{w}_w \in \tilde{\mathcal{P}}\) such that if 

\[
\Pr \left( |h(Y_{\tilde{w}_w} | M) - H(Y_{\tilde{w}_w} | M) | > n\delta \right) < 2^{-n\alpha}
\]

for some \(\delta, \alpha \in \mathbb{R}_+\), then 

\[
\Pr \left( |h(Y_w | M) - H(Y_w | M) | > n\delta \right) < 2^{-n\alpha} + 2^{-n(\alpha - \epsilon)}
\]

where 

\[
\delta = (2 + 2^{-n\alpha} + 2^{-n(\alpha - \epsilon)})(\delta + \epsilon) + (2^{-n\alpha} + 2^{-n(\alpha - \epsilon)}) \left( \log_2 |Y| - \frac{n}{2} \log_2 (2^{-n\alpha} + 2^{-n(\alpha - \epsilon)}) \right).
\]

At this point Theorems 11, 12, and 13 represent the main breadth of our contribution. But, it is clear that these Theorems are somewhat unwieldy. To simplify this procedure we will essentially combine Theorems 11, 12 and 13 into a single corollary which simultaneously stabilizes \(Y_w | M_i\) and \(M_i\) for all \(w \in \mathcal{P}(Y | X)\) and \(i \in [1 : l]\). Because of the tension between the accuracy of the stabilization, and the support set of the stabilizing random variable, we will construct the following corollary to only stabilize \(m_i\) such that \(-\log_2 p_m(m_i) < n^2\), with the remaining \(m_i\) being contained in their own set. Similar corollaries can be obtained with less accuracy on the stabilization, but with a much larger range of stabilized values (e.g., stabilize all \(m_i\) such that \(-\log_2 p_m(m_i) < 2n\). While such a trade off would be useful for scenarios such as ID coding, they would not be appropriate for the examples presented here.

In order to simplify analysis we introduce the following definition.

**Definition 14.** For any regular collection \((M_{[1:l]}, X, Y, \mathcal{P}(Y | X))\) and DRV \(U : \{(U, M_{[1:l]}) \rightarrow X \rightarrow Y, \mathcal{P}(Y | X)\}\), the \(\nu\)-stable sets for \(u \in U, w \in \mathcal{P}(Y | X)\), and \(M_j\) are 

\[
\mathcal{D}_{(stable),(M_j)}(u, w; \nu) \triangleq \left\{ (y_w, m_{[1:l]}) \in Y^n \times M_{[1:l]} : \begin{array}{l}
|h(y_w | m_j, u) - H_u(Y_w | M_j)| \leq n\nu \\
|h(m_j | u) - H_u(M_j)| \leq n\nu \\
|h(m_j | u) - h(m_j)| \leq n\nu
\end{array} \right\},
\]

while the the \(\nu\)-saturated sets for \(u \in U, w \in \mathcal{P}(Y | X)\), and \(M_j\) are 

\[
\mathcal{D}_{(saturated),(M_j)}(u, w; \nu) \triangleq \left\{ (y_w, m_{[1:l]}) \in Y^n \times M_{[1:l]} : \begin{array}{l}
|h(y_w | m_j, u) - H_u(Y_w | M_j)| \leq n\nu \\
h(m_j | u) - n^2 \geq -n\nu \\
|h(m_j | u) - h(m_j)| \leq n\nu
\end{array} \right\}.
\]

If \(M_j\) is uniform over \(M_j\), replace \(H_u(M_j)\) in the above with \(\log_2 |M_j|\).

Now if \((y_w, m_{[1:l]}) \in \mathcal{D}_{(stable),(M_j)}(u, w; \delta) \cup \mathcal{D}_{(saturated),(M_j)}(u, w; \delta)\) then \(p(y_w | m_j, u) \approx 2^{-H_u(Y_w | M_j) \pm n\delta}\). In addition, if \(h(m_j | u) < n^2 - 2n\delta\), then \(p(m_j | u) \approx p(m_j) \approx 2^{-H_u(M_j) \pm n\delta}\). In that sense \(\mathcal{D}_{(stable),(M_j)}(u, w; \delta)\) and \(\mathcal{D}_{(saturated),(M_j)}(u, w; \delta)\) consists of the probability terms which are well described by information theoretic quantities. Combining Theorems 11, 12, and Lemma 30 allow us to establish the following result.

**Corollary 15.** Let \(\varepsilon_n \triangleq n^{-1} \log_2 \log_2 n\). For any regular collection \((M_{[1:l]}, X, Y, \mathcal{P}(Y | X))\) and any DRV \(T : \{(T, M_{[1:l]}) \rightarrow X \rightarrow Y, \mathcal{P}(Y | X)\}\), there exists:

- a DRV \(U : \left\{ \log_2 |U| = O(|\log_2 |T| - l n \varepsilon_n \log_2 \varepsilon_n) \right\}, (U, M_{[1:l]} \rightarrow X \rightarrow Y, \mathcal{P}(Y | X)\),

- positive real number \(\nu_n = O(n^{-1} \log_2 |T| - l \sqrt{\varepsilon_n \log_2 \varepsilon_n})\), and

- set \(\tilde{U} \subseteq U\) such that 

\[
\Pr[\tilde{U}] \geq 1 - O(l^{2^{-n\varepsilon_n}})
\]

and for each \(j \in [1 : l]\) and \(u \in \tilde{U}\) either 

\[
\inf_{w \in \mathcal{P}(Y | X)} \Pr \left( (Y_w, M_{[1:l]}) \in \mathcal{D}_{(stable),(M_j)}(U, w; \nu_n) | U = u \right) \geq 1 - 8 \cdot 2^{-n\varepsilon_n},
\]

(12)
These bounds can be further simplified using the data processing inequality, and single-letterization techniques to show

\[
\inf_{w \in \mathcal{P}(Y|X)} \Pr \left( (Y_w, M_{[1:n]}) \in \mathcal{D}_{\text{uniform}}(M_j)(U, w; \nu_n) | U = u \right) \geq 1 - 8 \cdot 2^{-n\varepsilon n}.
\]

**Furthermore if** \(M_j\) **is uniform over** \(\mathcal{M}_j\), **then** \((12)\) **holds.**

The proof of which is in Appendix III. Note the error term is primarily due to the result holding simultaneously for all distributions in \(\mathcal{P}(Y|X)\). If this term is of importance in a potential application, and if only a finite and fixed number of quasi-images need to be stabilized, then the order term can be improved by simply combining Theorem 11 and 12.

### IV. Applications

In this section we will highlight a new methodology by which to obtain information theoretic necessary conditions. First we will apply this new methodology to a classical problem to highlight how it works, and how it differs from conventional approaches. In doing so we will provide extra commentary at each step in order that we make general application of the methodology plain. Next, we apply this methodology to establish new results for channels which require authentication, and wire-tap channels. These examples were chosen in order to demonstrate how this new methodology obtains information theoretic necessary conditions from a wide range of operational requirements.

#### A. One way communications over a DMC

Here we consider a classical problem in information theory, channel coding over a DMC \(p_{Y|X}\). In this model a source wants to send a message \(M\), which will be chosen at random according to some arbitrary distribution over \(\mathcal{M}\), to the destination. Connecting the source and destination is a DMC characterized by the conditional probability distribution \(p_{Y|X} \in \mathcal{P}(Y|X)\). To facilitate communications, the source and destination ahead of time agree upon a “code” consisting of an encoder \(F : \mathcal{M} \rightarrow \mathcal{X}^n\) and a decoder \(\Phi : \mathcal{Y}^n \rightarrow \mathcal{M}\), both of which may be stochastic.

For the code to be considered operational it must satisfy the following error probability criterion for some pre-arranged \(\delta \in (0, 1)\):

\[
\Pr (\Phi(Y) \neq M) < \delta.
\]

We note that the distribution of \(Y\) is induced by \(p^n_{Y|X}\) with \(X = F(M)\). Since \((\mathcal{M}, \emptyset, X, Y)\) form a regular collection of DRVVs, we can apply Corollary 13. Doing so, will allow us to directly transform \((14)\) into a set of information theoretic necessary conditions. Before a demonstration of this, we shall describe how one would apply Fano’s inequality to attempt to achieve the same task, and what the shortcomings of doing so are.

1) **Fano’s inequality:** Without a uniform distribution over \(\mathcal{M}\), Fano’s inequality can only (essentially) provide

\[
\mathbb{H}(M) < \mathbb{H}(\Phi(Y); M) + \Pr (\Phi(Y) \neq M) \log_2 |\mathcal{M}|.
\]

Now, if it were the case that \(M\) was uniform over \(\mathcal{M}\), then \((15)\) reduces to

\[
\log_2 |\mathcal{M}| < \frac{1}{1 - \Pr (\Phi(Y) \neq M)} \mathbb{H}(\Phi(Y); M),
\]

and if we were further to assume that \(\Pr (\Phi(Y) \neq M) \leq \delta_n\), for some \(\delta_n \rightarrow 0\), then asymptotically we could say

\[
\log_2 |\mathcal{M}| \leq \mathbb{I}(\Phi(Y); M) + O(n\delta_n).
\]

These bounds can be further simplified using the data processing inequality, and single-letterization techniques to show

\[
\mathbb{I}(\Phi(Y); M) \leq \mathbb{I}(Y; X) \leq n \max_{p(x)} \mathbb{I}(Y; X),
\]

which when substituted back into \((17)\) yields

\[
\lim_{n \rightarrow \infty} n^{-1} \log_2 |\mathcal{M}| \leq \max_{p(x)} \mathbb{I}(Y; X).
\]

But notice the assumptions that had to be made to obtain this result. First we had to assume \(M\) was uniform, and second we had to assume that the probability of error decay to zero with \(n\). The second assumption has already been the subject of much study, leading to the eventual demarcation between the strong and weak converse. With this in mind we instead consider what happens when you void the first assumption. In fact, repeating these steps with a non-uniform \(M\), gives

\[
\lim_{n \rightarrow \infty} n^{-1} H(M) \leq \max_{p(x)} \mathbb{I}(Y; X).
\]
Notice that Equation (20) looks like a sufficient condition as well, and actually is if \( M \) is information stable. But, this condition is actually not sufficient for general \( M \). Consider the following example to convince yourself of this fact. Let \( M, M = [0 : 2^{2n\max_p I(Y;X)}] \), have the following distribution

\[
p(m) = \begin{cases} \frac{3}{4} & \text{if } m = 0 \\ \frac{1}{2} - 2^{n\max_p I(Y;X)} & \text{if } m \in M \setminus \{0\} \end{cases}
\]  

(21)

This \( M \), on average, cannot be reliably transmitted over the channel. To see this consider a case where any potential decoder is given the side information that determines whether \( M = 0 \) or \( M \neq 0 \). When the decoder is informed that \( M = 0 \), then clearly the probability of error of the decoder can be eliminated. On the other hand when \( M \neq 0 \), then the number of potential messages greatly exceeds capacity and as a result the probability of error must be close to 1. This later fact being a by-product of the strong converse for the DMC. Thus, even with this side information, the best possible decoder could only obtain a minimum probability of error of just below 1/4. At the same time though, it is easy to calculate

\[
\mathbb{H}(M) = \frac{3}{4} \log_2 4 + \frac{1}{2} + \frac{n}{2} \max_p I(Y;X),
\]

(22)

which is less than \( n \max_p I(Y;X) \) for large enough \( n \) as long as \( \max_p I(Y;X) > 0 \). As a consequence Equation (20) can not also provide a matching sufficient condition, or in other words, Equation (20) only provides a loose necessary condition.

2) Information stable partitions: Now we move onto our methodology, which even without the assumption that \( M \) is information stable, nor that \( \Pr (\Phi(Y) \neq M) \rightarrow 0 \) as a function of \( n \), yields

\[
\Pr \left(n^{-1}h(M) > \max_p I(Y;X) + \zeta_n \right) < \delta + 2^{-n\zeta_n},
\]

(23)

for some \( \zeta_n : \zeta_n \rightarrow 0 \), as necessary to ensure (14). First shown by Han [7, Theorem 3.8.5 & 3.8.6], Equation (23) is not only necessary but also has a matching sufficient condition. We briefly discuss the sufficiency before completing the example. Observe that there can only be \( 2^{n\max_p I(Y;X)} \) values of \( m \in M \) such that \( h(m) \leq n \max_p I(Y;X) \). We will refer to this set of messages as the transmissible set. It would be simple to construct a reliable channel code for the transmissible set, while mapping all messages not from the transmissible set to some fixed codeword. As a result, if \( M \) is chosen from the transmissible set, the probability of error would be near 0, and otherwise 1. Hence there exists a coding scheme for which the error probability converges to the probability \( M \) is chosen from the transmissible set. This previous statement is essentially Equation (23).

Returning to establishing the necessary conditions. In general, our methodology looks to directly replace the operational requirement’s probability terms with information theoretic quantities. Here the operational requirement (Equation (14)) can be written as

\[
\Pr (\Phi(Y) = M) = \sum_{y,m} p_{\Phi|Y}(m|y)p_{Y,M}(y, m) > 1 - \delta.
\]

(24)

Next, because \((M, \emptyset, X, Y)\) constitute a regular collection of DRVs, there exists

- a DRV \( U : \{ \log_2|U| = O(-n\varepsilon_n \log_2 \varepsilon_n) \} \),
- positive real number \( \nu_n = O(-\sqrt{\varepsilon_n \log_2 \varepsilon_n}) \), and
- \( \bar{U} \subseteq U \) such that

\[
p_U(\bar{U}) \geq 1 - O(2^{-n\varepsilon_n})
\]

(25)

such that

\[
\Pr ((Y, M) \notin D_{(stable), \emptyset}(U, p_{Y|X}; \nu_n) | U = u) < 8 \cdot 2^{-n\varepsilon_n},
\]

(26)

and either

\[
\Pr ((Y, M) \notin D_{(stable), \emptyset,M}(U, p_{Y|X}; \nu_n) | U = u) < 8 \cdot 2^{-n\varepsilon_n},
\]

(27)

or

\[
\Pr ((Y, M) \notin D_{(stable), \emptyset,M}(U, p_{Y|X}; \nu_n) | U = u) < 8 \cdot 2^{-n\varepsilon_n},
\]

(28)

\footnote{With the usual asymmetry in the sign of the negligible terms.}
for all \( u \in \tilde{U} \), where \( \varepsilon_n = n^{-1/3} \) by Corollary 15. The set \( D_{(\text{sttue})}(U, p_Y|X; \nu_n) \) is not considered because the random variable \( \emptyset \) is trivially uniform by convention. Introducing \( U \) into the LHS of (24) via the law of total probability yields

\[
\Pr(\Phi(Y) = M) = \sum_u \sum_{y,m} p_Y(m|y)p_{Y,M,U}(y,m,u) > 1 - \delta.
\]  

(29)

Now, let \( \tilde{U}_{(\text{sttue})} \subseteq U \) be the set of \( u \) such that (27) holds. For each \( u \in \tilde{U}_{(\text{sttue})} \), let

\[
D_+(u, p_Y|X; \nu_n) = D_{(\text{sttue})}(u, p_Y|X; \nu_n) \cap D_{(\text{sttue})}(M)(u, p_Y|X; \nu_n).
\]

We are to establish that the terms for \( u \in \tilde{U}_{(\text{sttue})} \) and \((y,m) \in D_+(u, p_Y|X; \nu_n) \) dominate the sum in (29), and the contributions of all other terms to the sum become negligible as \( n \) increases. More specifically, we will to show

\[
\sum_{u \in \tilde{U}_{(\text{sttue})}} \sum_{(y,m) \in D_+(u, p_Y|X; \nu_n)} p_Y(m|y)p_{Y,M,U}(y,m,u) \leq \Pr(\Phi(Y) = M) \leq \sum_{u \in \tilde{U}_{(\text{sttue})}} \sum_{(y,m) \in D_+(u, p_Y|X; \nu_n)} p_Y(m|y)p_{Y,M,U}(y,m,u) + O(2^{-n\varepsilon_n})
\]

(30)

holds.

For the time being, let us first assume (30) does hold. Then it suffices to consider only \( u \in \tilde{U}_{(\text{sttue})} \) and \((y,m) \in D_+(u, p_Y|X; \nu_n) \), for which we have stability by Corollary 15 as discussed above. That is,

\[
2^{-H_u(Y|M) - n\nu_n} \leq p(y|m,u) \leq 2^{-H_u(Y|M) + n\nu_n}
\]

\[
2^{-H_u(Y) - n\nu_n} \leq p(y|u) \leq 2^{-H_u(Y) + n\nu_n}
\]

\[
2^{-H_u(M) - n\nu_n} \leq p(m|u) \leq 2^{-H_u(M) + n\nu_n}
\]

\[
2^{-H_u(M) - 2n\nu_n} \leq p(m) \leq 2^{-H_u(M) + 2n\nu_n},
\]

which imply

\[
p_{Y,M,U}(y,m,u) \leq p_{Y,U}(y,u)2^{-|\mathbb{H}_u(M) - \mathbb{H}_u(Y|M) - 3n\nu_n|} \]

(31)

for all \( u \in \tilde{U}_{(\text{sttue})} \) and \((y,m) \in D_+(u, p_Y|X; \nu_n) \). It is (31) that allows us to substitute the distribution terms in the necessary condition (29) with the corresponding information theoretic terms. In particular, putting (31) into (30) and combining with the necessary condition in (27)

\[
\sum_{u \in \tilde{U}_{(\text{sttue})}} p_U(u) 2^{-|\mathbb{I}_U(M) - \mathbb{I}_U(Y|M) - 3n\varepsilon_n|} \geq 1 - \delta - O(2^{-n\varepsilon_n})
\]

(32)

Continuing on, (32) also directly implies

\[
\Pr \left( \mathbb{I}_U(M) \leq \mathbb{I}_U(Y|M) + 3n\varepsilon_n \quad U \in \tilde{U}_{(\text{sttue})} \right) > 1 - \delta - O(2^{-n\varepsilon_n})
\]

(33)

via Markov’s inequality. But also note that

\[
\Pr \left( \mathbb{I}_U(M) \leq \mathbb{I}_U(Y|M) + 3n\varepsilon_n \quad U \in \tilde{U}_{(\text{sttue})} \right) \leq O(2^{-n\varepsilon_n})
\]

\[
+ \Pr \left( h(M) \leq \mathbb{I}_U(Y|M) + 5n\varepsilon_n \quad U \in \tilde{U}_{(\text{sttue})}, (Y,M) \in D_+(U, p_Y|X; \nu_n) \right) \leq O(2^{-n\varepsilon_n}) + \Pr (h(M) \leq \mathbb{I}_U(Y|M) + 5n\varepsilon_n + n\varepsilon_n)
\]

(34)

\[
\Pr (h(M) \leq \mathbb{I}_U(Y|M) + 5n\varepsilon_n + n\varepsilon_n) \leq O(2^{-n\varepsilon_n})
\]

(35)

\[\text{With } T = \emptyset\]

\[\text{The lower bound in (30) will not be used in establishing the necessary condition. It is provided mainly to show convergence.}\]
where (34) is because \( h(m) \geq h(m|u) - n\nu_n \geq \mathbb{I}(M|u) - 2n\nu_n \) for each \( u \in \tilde{U}_{(stable)} \) and \((y, m) \in D_\gamma(u, p_Y|X; \nu_n)\). In conclusion then, combining (33), (35), and the fact that \( I_u(Y; M) \leq n \max_{p(x)} I(Y; X) \) yields (23), since \((U, M) \to X \to Y\).

It remains to establish (30). The lower bound in (30) is trivial. The upper bound in (30) can be obtained by bounding the sum over three different sets of terms. First,

\[
\sum_{(y, m, u): \ u \in \tilde{U}} p_{\Phi}(y|m)p_{Y,M,U}(y, m, u) \leq 1 - p_U(\tilde{U}) < O(2^{-n\varepsilon_n}),
\]

(36)
directly follows from Equation (25), which bounds the sum over all terms not relating to a \( u \in \tilde{U} \). Given that \( u \notin \tilde{U} \), we bound the sum of all terms for which \( u \notin \tilde{U} \cup \tilde{U}_{(stable)} \) by leveraging that \((y, m) \in D_{(unstable), (M)}(u, p_Y|X; \nu_n)\) with high probability for such \( u \). In specific,

\[
\sum_{(y, m, u): \ u \in \tilde{U} \cup \tilde{U}_{(stable)}} p_{\Phi}(y|m)p_{Y,M,U}(y, m, u) \\
\leq O(2^{-n\varepsilon_n}) + \sum_{(y, m, u): \ u \in \tilde{U} \cup \tilde{U}_{(stable)}} p_{\Phi}(y|m)p_{Y,M,U}(y, m, u) \\
\leq O(2^{-n\varepsilon_n}) + \sum_{(y, m, u): \ u \in \tilde{U} \cup \tilde{U}_{(stable)}} p_{\Phi}(y|m)p_{Y,M,U}(y, m, u)
\]

(37)
where (37) is because Equation (28) must hold for all \( u \in \tilde{U} \cup \tilde{U}_{(stable)} \), while (38) is because \( p_{Y,M,U}(y, m, u) \leq p_M(m) \leq 2^{-n^2 + 2n\varepsilon_n} \) for all \((y, m) \in D_{(unstable), (M)}(u, p_Y|X; \nu_n)\). Thus all terms other than those with \( u \in \tilde{U}_{(stable)} \) do not contribute to the sum, and for the remaining terms it follows that

\[
\sum_{(y, m, u): \ u \in \tilde{U}_{(stable)}} p_{\Phi}(y|m)p_{Y,M,U}(y, m, u) \\
\leq O(2^{-n\varepsilon_n}) + \sum_{(y, m, u): \ u \in \tilde{U}_{(stable)}} p_{\Phi}(y|m)p_{Y,M,U}(y, m, u)
\]

(39)
since Equation (27) holds for all \( u \in \tilde{U}_{(stable)} \).

The process of deriving a necessary condition in the form of (32) will henceforth be referred to as our new methodology.

In essence, our new methodology is to introduce the variable \( U \) through Corollary (15) then prune out the unstable terms, and finally switch the distribution terms for information theoretic ones. Since our new methodology is relatively straightforward, in the future these steps will be described in less detail.

B. \((\delta, I)-capacity of the wiretap channel\)

In the wiretap channel, a source wants to reliably send a message \( M \) chosen uniformly from \( \{1, \ldots, 2^m\} \), to a given destination while ensuring a certain level of secrecy from an eavesdropper. The source is connected to the destination through a DMC \( p_{Y|X} \in \mathcal{P}(Y|X) \) and to the eavesdropper through a DMC \( p_{Z|X} \in \mathcal{P}(Y|X) \). Once again a code, consisting of an encoder \( F : M \to X \), and decoder \( \Phi : Y \to M \), must be designed to satisfy

\[
\Pr (\Phi(Y) = M) < \delta,
\]

(40)
for some fixed \( \delta \in (0, 1) \), but additionally there exists some secrecy requirement parameterized by \( \ell \). Since Wyner [23] first considered\(^5\) the wiretap channel model, their have been a large number of secrecy metrics proposed. Here we shall restrict our focus to the original metric, and a metric more commonly found in modern literature.

The original metric, the weak information leakage rate, is formally defined for all positive real numbers \( \ell \) as

\[
\mathbb{I}(Z; M) < n\ell.
\]

(41)
\(^5\)Wyner only presented the case where the channel to the eavesdropper was degraded. The general case was introduced by Csizsár and Körner in [24].
On the other hand, the more common modern metric is the variational distance metric parameterized by \( \ell \in (0, 1] \),

\[
\frac{1}{2} \sum_{z, m} |p_{Z,M}(z, m) - p_{Z}(z)p_M(m)|^+ < \ell.
\]  

(42)

We will say a code is a \((\delta, \ell)\)-code subject to weak information leakage if it satisfies Equation (40) and (41), and a \((\delta, \ell)\)-code subject to bounded variational distance if it satisfies Equation (40) and (42).

Regardless of secrecy metric, our primary concern will be to establish information theoretic necessary conditions for the existence of a \((\delta, \ell)\)-code. Specifically we shall establish this necessary conditions upon the value of \( r \). As a first point of order, note that for any \((\delta, \ell)\)-code subject to weak information leakage, where \( \delta \to 0 \) and \( \ell \to 0 \) as a function of \( n \), using Fano’s inequality it is possible to obtain

\[
nr \leq \mathbb{I}(Y; M) + n\delta_1
\]  

(43)

\[
n\ell > \mathbb{I}(Z; M) - n\delta_2,
\]  

(44)

where \( \delta_1, \delta_2 \to 0 \) with \( n \), which then leads to the two following inequalities

\[
nr < \mathbb{I}(Y; M) + n\delta_1
\]  

(45)

\[
nr < \mathbb{I}(Y; M) - \mathbb{I}(Z; M) + \ell + n\delta_1 + n\delta_2.
\]  

(46)

Equations (45) and (46) also imply

\[
r < c(\ell) + n(\delta_1 + \delta_2)
\]  

(47)

where

\[
c(\ell) = c(\ell, p_Y, z|x)
\]

\[\triangleq \max_{p(\nu)p(v)p(x|v)} \min (I(Y; V), I(Y; V|T) - I(Z; V|T) + \ell),\]  

(48)

\(|T| < |X| + 3 \text{ and } |Y| < |X|^2 + 4|X| + 3\). Although the derivation of Equation (47) from (45) and (46) is outside the scope of this paper, we point those interested to [24, Section V] or [4, Lemma 17.12] for the key identity in reducing Equation (46), and to [4, Lemma 15.4] or [25, Appendix C] for establishing the cardinality bounds. For our purposes, we will directly assume Equation (47) as an implication of (43) and (44).

Observe that \(((M, \emptyset), X, Y, p_{Y|X}[X])\) is a regular collection, and therefore there exists

- a DRV \( U : \{ (U, M) \xRightarrow{\mathcal{X}} Y_{p(|X|)} \} \),
- a positive real number \( \nu_n = O(-\sqrt{\varepsilon_n \log_2 \varepsilon_n}) \), and
- a set \( \tilde{U} \subseteq U \) such that

\[
p_U(U) \geq 1 - O(2^{-n\varepsilon_n})
\]  

(49)

and

\[
\inf_{w \in P_{Y|X}[X]} \Pr ((Y_w, M) \notin \mathcal{D}_+(U, w; \nu_n)| U = u) \leq 16 \cdot 2^{-n\varepsilon_n},
\]  

(50)

for all \( u \in \tilde{U} \),

where \( \varepsilon_n = n^{-\frac{3}{2(\log 2)^2}} \) and \( \varepsilon_n \leq \varepsilon_n \) and

\[
\mathcal{D}_+(u, w; \nu_n) = \mathcal{D}_{(\text{stable}, (M))}(u, w; \nu_n) \cap \mathcal{D}_{(\text{stable}, (\emptyset))}(u, w; \nu_n),
\]

by Corollary [15]. Using this new methodology we will establish necessary conditions under both secrecy metrics. Proof these conditions are sufficient can be found in our earlier work [26].

First, let us consider the weak information leakage.

**Theorem 16.**

\[
r \leq c \left( \frac{\ell}{1 - \delta} \right) + O(-\sqrt{\varepsilon_n \log_2 \varepsilon_n})
\]  

(51)

for any \((\delta, \ell)\) code subject to weak information leakage.

**Proof:** First, repeating the derivation of (35) from (24), we obtain

\[
\Pr (nr \leq \mathbb{I}_U(Y; M) + 5n\nu_n + n\varepsilon_n) \geq 1 - \delta - O(2^{-n\varepsilon_n})
\]  

(52)
Theorem 17. The set of $u \in \mathcal{U}^+ \triangleq \{ u : nr \leq I_U(Y; M) + 5n\nu_n + n\varepsilon_n \}$ will play a critical role in how error alters the leakage condition. In fact, starting from Equation (41) and using basic information inequalities we have

$$n\ell > I(Z; M) + \log |\mathcal{U}| - \sum_u \sum_{m: p_{Z,M}(z,m) \geq p_Z(z)} p_{Z,M}(z,m) - p_Z(z)p_M(m)$$

$$\geq -2n\log |\mathcal{U}| + \sum_{u \in \mathcal{U}^+} \sum_{m: p_{Z,M}(z,m) \geq p_Z(z)} p_{Z,M}(z,m) \geq \sum_{u \in \mathcal{U}^+} \sum_{m: p_{Z,M}(z,m) \geq 2^{-n\nu_n} p_Z(z)} p_{Z,M}(z,m) \geq \sum_{\{z,m,u\} \in \mathcal{D}_+} \sum_{u \in \mathcal{U}} \sum_{p_{Z,M}(z,m) \geq 2^{-n\nu_n} p_Z(z)} p_{Z,M}(z,m, u) \geq (1 - 2^{-n\nu_n}) \Pr( I_U(Z; M) \geq 3n\nu_n ) - O(2^{-n\varepsilon_n}) \tag{65}$$

Hence, from Equations (52) and (56) we obtain

$$nr \leq I(Y; M|U \in \mathcal{U}^+) + O(-n\sqrt{\varepsilon_n} \log_2 \varepsilon_n) \tag{57}$$

$$\frac{n\ell}{1 - \delta} \geq I(Z; M|U \in \mathcal{U}^+) - \frac{1}{1 - \delta} O(-n\varepsilon_n \log_2 \varepsilon_n + n2^{-n\varepsilon_n}) \tag{58}$$

Furthermore the preceding equations take on the form of Equations (43) and (44) since $(U, M) \xrightarrow{\ell} X \xrightarrow{\delta} Y \xrightarrow{c} Y|X$, and therefore

$$r \leq c \left( \frac{\ell}{1 - \delta} \right) + O(-\sqrt{\varepsilon_n} \log_2 \varepsilon_n) \tag{59}$$

Thus the capacity increases if the tolerable leakage increases or tolerable error increases. Only in the special case where the leakage is restricted to be 0 is there a strong converse. The necessary proof also hints at a simple direct proof where one code is constructed which has a probability of error near 1, but also does not have any information leakage, while a second code is constructed with a low error rate and is there a strong converse. The necessary proof also hints at a simple direct proof where one code will play a critical role in how error alters the leakage condition. In fact, starting from Equation (41) and using basic information inequalities we have

$$n\ell > I(Z; M) + \log |\mathcal{U}| - \sum_u \sum_{m: p_{Z,M}(z,m) \geq p_Z(z)} p_{Z,M}(z,m) - p_Z(z)p_M(m)$$

$$\geq -2n\log |\mathcal{U}| + \sum_{u \in \mathcal{U}^+} \sum_{m: p_{Z,M}(z,m) \geq p_Z(z)} p_{Z,M}(z,m) \geq \sum_{u \in \mathcal{U}^+} \sum_{m: p_{Z,M}(z,m) \geq 2^{-n\nu_n} p_Z(z)} p_{Z,M}(z,m) \geq \sum_{\{z,m,u\} \in \mathcal{D}_+} \sum_{u \in \mathcal{U}} \sum_{p_{Z,M}(z,m) \geq 2^{-n\nu_n} p_Z(z)} p_{Z,M}(z,m, u) \geq (1 - 2^{-n\nu_n}) \Pr( I_U(Z; M) \geq 3n\nu_n ) - O(2^{-n\varepsilon_n}) \tag{65}$$

for any $(\delta, \ell)$ code subject to bounded variational distance.

**Proof:** As in the proof for weak information leakage, meeting the reliability criterion requires

$$\Pr( nr \leq I_U(Y; M) + 5n\nu_n + n\varepsilon_n ) \geq 1 - \delta - O(2^{-n\varepsilon_n}) \tag{61}$$

Now, starting from Equation (42) we can derive new bounds as follows

$$\ell > \sum_{p_{Z,M}(z,m) \geq p_Z(z)} p_{Z,M}(z,m) - p_Z(z)p_M(m) \tag{62}$$

$$\geq \sum_{p_{Z,M}(z,m) \geq 2^{-n\nu_n} p_Z(z)} p_{Z,M}(z,m) \geq \sum_{\{z,m,u\} \in \mathcal{D}_+} \sum_{u \in \mathcal{U}} \sum_{p_{Z,M}(z,m) \geq 2^{-n\nu_n} p_Z(z)} p_{Z,M}(z,m, u) \geq (1 - 2^{-n\nu_n}) \Pr( I_U(Z; M) \geq 3n\nu_n ) - O(2^{-n\varepsilon_n}) \tag{65}$$
Combining Equation (61) and (65) yields

\[
\Pr \left( \frac{(Z, M) \in D_+(U, p_Z|X; \nu_n)}{\|U\|(Z, M) \geq 3\nu_n} \right) \\
\geq \Pr \left( \|U\|(Z, M) \geq 3\nu_n \right) - \Pr \left( (Z, M) \notin D_+(U, p_Z|X; \nu_n) \right) \\
\geq \Pr \left( \|U\|(Z, M) \geq 3\nu_n \right) - \Pr \left( U \notin \hat{U} \right) \\
- \Pr \left( (Z, M) \notin D_+(U, p_Z|X; \nu_n) \right) \left| U \in \hat{U} \right) \\
= \Pr \left( \|U\|(Z, M) \geq 3\nu_n \right) - O(2^{-n\epsilon_n}).
\]

These equations take the form of Equations (43) and (44) and therefore

\[
\Pr \left( nr \leq \|Y; M\|U = u + 5n\nu_n + n\epsilon_n \right) \\
0 > \|U\|(Z, M) - 3n\nu_n. 
\]

Therefore if \(1 - \delta - \ell > 0\) then for large enough \(n\) there exists a \(u \in \mathcal{U}\) such that

\[
r \leq c(0) + 8n\nu_n + \epsilon_n = c(0) + O(-\sqrt{n\log_2 \epsilon_n}).
\]

On the other hand, if \(1 - \delta - \ell > 0\) and \(\delta < 1\) then Equation (61) still directly implies

\[
r \leq \max_{p(x)} \|Y; X\| + 5n\nu_n + n\epsilon_n
\]

as discussed in the first example.

\[\Box\]

C. Converse for error exponents: keyed authentication

For this next example we consider a communication model recently employed by Lai et al. [6], and Gungor and Koksal [27]. Here the source and destination must now maintain reliable communications in the presence of an interloper who has the ability to modify any transmitted information. In order for communications to be considered reliable, the destination must be able to detect when the interloper has modified the transmission.

More formally, this model has two different modes of operation depending on if the interloper intercedes or not. If the interloper does not intercede then the source is connected to the destination through a DMC \(w_s \in \mathcal{P}(Y|X)\). However, if the interloper does intercede, then the source is connected first to the interloper through a DMC \(w_i \in \mathcal{P}(Y|X)\), and the interloper is then connected to the destination through a noiseless channel. In this case, the interloper may observe the entire \(n\)-length transmission sequence before arbitrarily choosing the value of \(Y\) that the destination will observe. In order to aid the source and destination in the detection of this interloper, they pre-share a secret key, \(K\), which is chosen uniformly from the set of all possible keys.

Our goal will be to establish information theoretic necessary conditions on the existence of a reliable encoder \(F : \mathcal{M} \times \mathcal{K} \rightarrow \mathcal{X}^n\), and decoder \(\Phi : \mathcal{Y}^n \times \mathcal{K} \rightarrow \mathcal{M} \cup \mathcal{I}\), where \(\mathcal{I}\) is a declaration of intrusion attempt. The probability of error of the message \(M\) will be completely ignored, and instead we focus solely on the necessary conditions which arise due to the need to detect intrusion. To that end let \(S(k) = \{ y : \Phi(y, k) \neq 1 \}\) denote the set of sequences of \(y\) which will be deemed authentic for given key \(k \in \mathcal{K}\). With this the probability of intrusion given intercession (that is, the false authentication probability) can be written as

\[
2^{-\beta} \triangleq \sup_{\psi \in \mathcal{P}(Y^n|X^n)} \sum_{y,y_{w_i},k} \psi(y|y_{w_i})p_{X_{w_i}K}(y_{w_i}, k).
\]

Indeed, in order for the decoder to falsely authenticate a transmission from the interloper, the interloper must choose a value \(y\) for which the decoder does not declare intrusion, in other words the interloper must choose a \(y \in S(k)\). Thus, \(\psi\) represents

\[\text{To be more precise, this model is a special case of the model found in [27].}\]

\[\text{[6] is only concerned with keys for which } n^{-1}\mathbb{H}(K) \text{ vanished asymptotically as } n \rightarrow \infty. \text{ We, nor [27], will make such a restrictions in our derivations.}\]
the attacking strategy of the interloper, in which the interloper receives the value \( y_w \), and then decides to transmit \( y \) with probability \( \psi(y|y_w) \).

The best previously established results were by Simmons [23] and Maurer [29] who demonstrated that

\[
\beta < \mathbb{I}(K; Y_w) \tag{73}
\]

\[
\beta < \mathbb{H}(K|Y_w). \tag{74}
\]

We will improve on these bounds by applying our new methodology to Equation (72). In particular our new bounds will show how the variation in the channel limits the value of \( \beta \).

To do this we assume some code exists with a \( 2^{-\beta} \) probability of intrusion. Next we set \( X = F(M, K) \) and observe that \(((K, X, \emptyset), X, Y, P(Y|X))\) is a regular collection, and DRV

\[ T \triangleq \sum_{t \in \mathcal{P}_n(X)} t \cdot 1 \{ X \in \{ x : p_x = t \} \}, \]

has the properties \( \log_2 |T| = O(\log_2 n) \) and \( T \ll X \). Therefore there exists:

- a DRV \( U : \begin{cases} \log_2 |U| = O(\log_2 n - n \varepsilon_n \log_2 \varepsilon_n) \end{cases} \)
- a real number \( \nu_n = O(n^{-1} \log_2 n - \sqrt{\varepsilon_n \log_2 \varepsilon_n}) \), and
- a set \( \tilde{U} \) such that

\[ p_U(\tilde{U}) \geq 1 - O(2^{-n \varepsilon_n}) \tag{75} \]

and

\[ \Pr((Y_w, K, X) \notin D_+(U, w; \nu_n)|U = u) < 16 \cdot 2^{-n \varepsilon_n} \tag{76} \]

for all \( u \in \tilde{U} \) and \( w \in \mathcal{P}(Y|X) \), where

\[ D_+(u, w; \nu_n) \triangleq \mathcal{D}_{(obs), (y)}(u, w; \nu_n) \cap \mathcal{D}_{(obs), (K)}(u, w; \nu_n) \]

and \( \varepsilon_n \triangleq n^{-|X||Y|+1}, \) by Corollary [15]

Because of this we will obtain the following theorem.

**Theorem 18.** Let \( n \) be large enough so that \((n + 1)^{-|X||Y|} \geq 17 \cdot 2^{-n \varepsilon_n}\). Then

\[ \beta \leq \inf_{(u, w) \in \mathcal{U} \times \mathcal{P}(Y|X)} \mathbb{I}(K; Y_w|u) + h(u) - \mathbb{H}(w|y_i|u) > 17 \cdot 2^{-n \varepsilon_n} \]

\[ \beta \leq \min_{(u, w) \in \mathcal{U} \times \mathcal{P}(Y|f_u)} \mathbb{I}(K; Y_w) + n \mathbb{H}(w|y_i|u) + O(-n \sqrt{\varepsilon_n \log_2 \varepsilon_n}), \tag{78} \]

where \( t_u \) denotes the \( t \) such that \( p|U(t)|u = 1 \) (that is the type of the distribution of \( X \)).

Before proof, we take a moment to discuss some of the implications of Theorem [18]. Note that restricting the code to a \( u \in \mathcal{U} \) is akin to restricting all of the codewords to be a particular type. So much like constant composition codes allow us to consider this type fixed, we will for the sake of discussion assume that the code is restricted to a single value of \( u \in \mathcal{U} \). In this case Theorem [18] simplifies to

\[ \beta \leq \inf_{w \in \mathcal{P}(Y|X)} \mathbb{I}(K; Y_w) + O(-n \sqrt{\varepsilon_n \log_2 \varepsilon_n}) \tag{79} \]

\[ \beta \leq \min_{w \in \mathcal{P}(Y|f_u)} \mathbb{I}(K; Y_w) + n \mathbb{H}(w|y_i|u) + O(-n \sqrt{\varepsilon_n \log_2 \varepsilon_n}). \tag{80} \]

From (79) we see that the mutual information between the key and the observation for any empirical channels (that is \( p_y|y \)) over which the correct key could be identified with non-zero probability will upper bound the probability of false authentication. This is compounded by the fact that \( \Pr(Y_w \in S(K)) > 17 \cdot 2^{-n \varepsilon_n} \) does not imply \( \mathbb{I}(K; Y_w) = \mathbb{H}(K) \) since the sets \( S(K) \) are not necessarily disjoint for different values of \( k \). This is unfortunate since choosing a code more robust to channel deviations may disproportionately increase the probability of false authentication.

Next, from (80) we see that the probability of false authentication will be constrained by the entropy of the key given the adversaries observation over a number of different empirical channels to the adversary. Moreover, the KL divergence term
can be thought of as relating to a correction term to account for the probability of the empirical channel being \( w \) while the actual channel is \( w_i \). Intuitively, one may think of the probability the empirical channel \( w_i \) occurs as \( 2^{-nD(w_i||w_i)} \), while the probability of false authentication given empirical channel \( w_i \) occurs as \( 2^{-n\mathbb{H}(K|Y_{w_i})} \). The stated bound then clearly follows.

Both of these equations are clearly less than the existing equations, which follows simply because the infimum of \( \mathcal{S}(\mathcal{K},w) \) generally includes \( w_s \), while the infimum of the set for \( \mathcal{S}(\mathcal{K},w) \) contains \( w_i \). We now prove Theorem 18.

**Proof:** To prove both Equation (77) and (78), it will be important to first note that Equation (72) directly implies

\[
2^{-\beta} \geq \sum_{y, y_{w_i}, k: y \in S(k)} \psi^*(y|y_{w_i}) p(y_{w_i}, k|u) p(u),
\]

for all \( \psi^* \in \mathcal{P}(\mathcal{Y}^n|\mathcal{Y}^n) \) and \( u \in \mathcal{U} \). To prove both Equation (77) and (78), specific values of \( \psi^* \) and \( u \) will be chosen.

To derive Equation (77), first fix a \( w \in \mathcal{P}(\mathcal{Y}|\mathcal{X}) \), and a \( u \in \mathcal{U} \) such that \( \Pr(Y_w \in S(K)|U = u) > 17 \cdot 2^{-n\varepsilon_n} \) and set \( \psi^*(y|y_{w_i}) \) equal to \( p_{Y_w|U}(y|u) \) in Equation (81). Doing so, we may derive Equation (77) as follows:

\[
2^{-\beta} \geq \sum_{y, y_{w_i}, k: y \in S(k)} p_{Y_w|U}(y|u) p(y_{w_i}, k, u) \geq \sum_{y, x, k: y \in S(k)} p_{Y_w, X, K, U}(y, x, k, u) \frac{p_{Y_w|U}(y|u)}{p_{Y_w|K, U}(y|k, u)} \geq \sum_{y, x, k: y \in S(k), (y, x, k) \in D_+(u, w; \delta)} p_U(u) 2^{-\mathbb{H}(Y_w|K|U) - 2n\epsilon_n} \geq p_U(u) 2^{-\mathbb{H}(Y_w|K|U) - 2n\epsilon_n - n\epsilon_n}
\]

where (84) is by our new methodology since \( u \in \mathcal{U} \), and (85) is because

\[
\Pr(Y_w \in S(K) \cap D_+(U, w; \nu_n)) \geq \Pr(Y_w \in S(K)|U = u) - \Pr((Y_w, X, K) \notin D_+(U, w; \nu_n)|U = u) \geq \Pr(Y_w \in S(K)|U = u) - 16 \cdot 2^{-n\varepsilon_n} \geq 2^{-n\varepsilon_n}.
\]

Moving on to proving Equation (78). First understand that if \( S(k) \) are not pairwise disjoint (i.e., \( S(k) \cap S(k') \neq \emptyset \) for some \( k \in \mathcal{K} \) and \( k' \in \mathcal{K} \setminus \{k\} \)), then the RHS of Equation (81) is

\[
\sum_{y, y_{w_i}, x, k: y \in S(k)} \psi^*(y|y_{w_i}) p(y_{w_i}, x, k, u),
\]

for any collection of sets \( \{S(k)\}_{k \in \mathcal{K}} \) that are pairwise disjoint and for which \( \hat{S}(k) \subseteq S(k) \) for all \( k \). As a point to note later, changing the decoder in such a way would not change the values of \( \mathbb{H}(K|Y_{w_i}, u) \) or \( D(w||w_i|t_u) \). As such we will proceed with the assumption that \( \{S(k)\}_{k \in \mathcal{K}} \) are pairwise disjoint from the outset, which is valid since we are looking for a lower bound on the probability of false authentication.

Now once again fix a \( u \in \mathcal{U} \) and \( w \in \mathcal{P}_n(\mathcal{Y}|t_u) \), but this time set \( \psi^* \) to be any distribution such that

\[
\psi^*(S(k)|y_{w_i}) = \frac{p_{Y_w|K, U}(y_{w_i}|k, u) p_K|U(k|u)}{p_{Y_w|U}(y_{w_i}|u)},
\]

for all \( (y_{w_i}, k), \{p_{Y_w|U}(y_{w_i}|u) \neq 0\} \). Since we assume the sets \( S(k) \) are disjoint we may perform the summation over \( y \) to obtain that

\[
2^{-\beta} \geq \sum_{y_{w_i}, x, k} p_K|U(k|y_{w_i}, u) w^n \psi^*(y_{w_i}|x)p(x, k, u)
\]

8Readers familiar with authentication problems should recognize these chosen values as relating to “substitution” and “impostor” attacks. Both of these attacks are encompassed by the general framework presented here.
is always necessary. Furthermore since all summands are positive, we may restrict the summation to only consider \( y_w \), such that \( p_{y_w} = w \), hence giving

\[
2^{-\beta} \geq \sum_{y_w, x, k; \mathcal{D}(u, w, v_n)} p(K|y_w, u)w_n^u(y_w|x)p(x, k, u).
\]

(88)

Now applying our new methodology yields that the RHS of Equation (88) is

\[
\geq p(u)2^{-E(K|y_w, u) - 3n\nu_n} \sum_{y_w, x, k; \mathcal{D}(u, w, v_n)} w_n^u(y_w|x)p(x, k|u)
\]

which may be continued to derive Equation (78) as

\[
= p(u)2^{-E(K|y_w, u) - D(w|v_n)} - 3n\nu_n
\]

where (90) is because \( w_n^u(y|x) = 2^{-nD(w|v_n)}y_n^u(y|x) \) for all \( y, x; p_{y|x} = w \) (see [30, Lemma 2.3]), and (91) is because

\[
\Pr(Y_u \in \{ y : p_{y|x} = w \} | X = x) \geq n^{-|X| |Y|} \geq 17 \cdot 2^{-n\varepsilon_n}
\]

for all \( w \in \mathcal{U} \).

\[
(92)
\]

\[
(93)
\]

V. PROOF OF THEOREM [10]

To prove Theorem [10] we construct here the subset \( \mathcal{A}^1 \subseteq \mathcal{X}^n \) with non-negligible probability for which the quasi-image of \( \mathcal{X} \) is stable. Our construction is based on information spectrum slicing [7]. In particular, the set \( \mathcal{A}^1 \) will be the pre-image of the union of a few entropy spectrum slices of \( \mathcal{Y} \). Therefore, before proving Theorem [10] we will first need to build up a few definitions related to the entropy spectrum. From there we will derive a few lemmas which will help to streamline the proof of Theorem [10].

A. Information Spectrum Slicing

Here we build a few results that relate to information (or entropy) spectrum introduced by Han [7]. Keep in mind that the overarching goal of this work is to create quasi-images with nearly uniform distribution. The entropy spectrum provides a language with which we can succinctly discuss these variations.

**Definition 19.** For DRV \( Y \) arbitrarily distributed over \( \mathcal{Y} \), the set

\[
S_Y(s; \lambda, t) \triangleq \left\{ y \in \mathcal{Y} : \min \left( \frac{h_Y(y)}{\lambda}, t \right) = s \right\},
\]

for a given \( s \in \mathbb{N} \), \( \lambda \in (0, \infty) \), and \( t \in \mathbb{N}_+ \), is the \((s; \lambda, t)\)-slice of \( Y \).

**Remark 20.** \( S_Y(s; \lambda, t) = \emptyset \) for all \( s \not\in [0 : t] \). Moreover as we care only about the support set \( \{ y \in \mathcal{Y} : p_Y(y) > 0 \} \) of \( Y \), we will interpret \( \bigcup_{t=0} S_Y(s; \lambda, t) \) as the support set.

The terminology of \( S_Y(s; \lambda, t) \) being a \((s; \lambda, t)\)-slice is supported by the following lemma.

**Lemma 21.** For every \( s \in [0 : t-1] \) and \( y \in S_Y(s; \lambda, t) \),

\[
s\lambda \leq h_Y(y) < (s + 1)\lambda.
\]

(94)

For \( s = t \), the lower bound in (94) still holds.
Proof: The double inequality is simply a restatement of Definition 19 for \( s \in [0 : t - 1] \), and as such the lower bound still clearly holds for \( s = t \).

Proposition 22. For every \( s \in [0 : t - 1] \),
\[
\begin{align*}
\lambda s - \varrho(s) & \leq \log_2 |\mathcal{S}_Y(s; \lambda, t)| < (s + 1)\lambda - \varrho(s), \\
\varrho(s) & \triangleq -\log_2 p_Y(\mathcal{S}_Y(s; \lambda, t)).
\end{align*}
\] (95)

In addition,
\[
\begin{align*}
\lambda t - \varrho(t) & \leq \log_2 |\mathcal{S}_Y(t; \lambda, t)| < (t + 1)\lambda \\
\text{if } t > \lambda^{-1} \log_2 |\mathcal{Y}|.
\end{align*}
\] (96)

Proof: For \( s \in [0 : t - 1] \), (94) implies
\[
2^{-(s+1)\lambda}|\mathcal{S}_Y(s; \lambda, t)| < \sum_{y \in \mathcal{S}_Y(s; \lambda, t)} p_Y(y) \leq 2^{-s\lambda}|\mathcal{S}_Y(s; \lambda, t)|,
\]
which in turn yields (95) since \( \sum_{y \in \mathcal{S}_Y(s; \lambda, t)} p_Y(y) = p_Y(\mathcal{S}_Y(s; \lambda, t)) \).

For the case of \( s = t \), the lower bound in (96) is still valid by the same argument above. The upper bound, on the other hand, is because \( \log_2 |\mathcal{S}_Y(t; \lambda, t)| \leq \log_2 |\mathcal{Y}| < t\lambda \).

B. Supporting Lemmas

Next, we begin to employ information spectrum slicing to derive a number of lemmas which help to simplify and streamline the proof of Theorem 10. All lemmas here are under the assumption that \((\emptyset, X, Y)\) form a regular collection of DRVs. Furthermore, throughout the proofs we let \(\mathcal{A}\) denote the support set of \(X\) (i.e., \(\mathcal{A} \triangleq \{ x \in \mathcal{A}^n : p_X(x) > 0 \}\)). This is important since an image is independent of \(X\)'s distribution, and instead is solely a function of the support set. To this end, as we will see, it will be helpful to let
\[
\eta_s \triangleq p_Y \left( \bigcup_{i=0}^{s} \mathcal{S}_Y(i) \right)
\]

for \( s \in [0 : t] \), and \( \eta_{s'} = 0 \) for all integers \( s' < 0 \) and \( \eta_{s'} = 1 \) for all integers \( s' \geq t \). Note then that \( 0 = \eta_{-1} \leq \eta_0 \leq \eta_1 \leq \cdots \leq \eta_t = 1 \). In addition \( \eta_{s-1} = \eta_s \) implies \( \mathcal{S}_Y(s) = \emptyset \). Of direct importance to the following lemmas is that \( \bigcup_{s=0}^t \mathcal{S}_Y(i) \) is an \( \eta_s \)-quasi image of \(X\) by \( p_{Y|X}\). As we will show later, it is in fact the unique minimum such \( \eta_s \)-quasi image.

First, we determine bounds on the image size, and the probability of, the set \(\mathcal{A}' \subseteq \mathcal{A}\) for which a minimum \(\alpha\)-quasi image of \(X\) by \( p_{Y|X}\) is also an \(\epsilon\)-image of \(\mathcal{A}'\) by \( p_{Y|X}\).

Lemma 23. Given \(\alpha, \epsilon \in (0, 1)\) and \( p_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X}) \), let \(\mathcal{B} \subseteq \mathcal{Y}^n\) be a minimum \(\alpha\)-quasi image of \(X\) by \( p_{Y|X}\). If
\[
\mathcal{A}' \triangleq \left\{ x \in \mathcal{A} : p_{Y|X}(\mathcal{B}|x) \geq \epsilon \right\},
\]
then
\[
p_X(\mathcal{A}') \geq \frac{\alpha - \epsilon}{1 - \epsilon},
\]
and
\[
\log_2 g_{Y|X}^{\alpha}(\mathcal{A}', 1 - \beta) \leq \log_2 g_{Y|X}^{\alpha}(X, \alpha) + n \tau_n(\epsilon, \beta),
\]
for each \( \beta \in (0, 1) \).

Proof: Not only is \(\mathcal{B}\) a minimum \(\alpha\)-quasi image of \(X\) by \( p_{Y|X}\), but clearly \(\mathcal{B}\) is an \(\epsilon\)-image of \(\mathcal{A}'\) by \( p_{Y|X}\). Hence
\[
\log_2 g_{Y|X}^{\alpha}(X, \alpha) = \log_2 |\mathcal{B}| \geq \log_2 g_{Y|X}^{\alpha}(\mathcal{A}', \epsilon)
\]
\[
\geq \log_2 g_{Y|X}^{\alpha}(\mathcal{A}', 1 - \beta) - n \tau_n(\epsilon, \beta)
\]
by Lemma 5. Furthermore as \(\mathcal{B}\) is a minimum \(\alpha\)-quasi image of \(X\) by \( p_{Y|X}\), we have
\[
\alpha \leq p_Y(\mathcal{B}) \leq \sum_{x \in \mathcal{A}} p_{Y|X}(\mathcal{B}|x)p_X(x)
\]
\[
= \sum_{x \in \mathcal{A}'} p_{Y|X}(\mathcal{B}|x)p_X(x) + \sum_{x \in \mathcal{A} \setminus \mathcal{A}'} p_{Y|X}(\mathcal{B}|x)p_X(x)
\]
\[
\leq p_X(\mathcal{A}') + (1 - p_X(\mathcal{A}'))\epsilon
\] (97)
Lemma 24. For each $s \in [0 : t]$, $\bigcap_{i=0}^{s} S_Y(i)$ is the unique minimum $\eta_s$-quasi image of $X$ by $p_{Y|X}$. Furthermore

$$\log_2 \tilde{g}_{Y|X}^n(X, \eta_s) < s\lambda + \lambda + \log_2(t + 1),$$

and

$$\log_2 \tilde{g}_{Y|X}^n(X, \eta_s) \geq s\lambda + \log_2 p_Y(S_Y(s)).$$

Proof: Assume first that $\bigcup_{i=0}^{s} S_Y(i)$ is the unique minimum $\eta_s$-quasi image of $X$ by $p_{Y|X}$. Under this assumption, it follows that

$$|S_Y(s)| \leq \tilde{g}_{Y|X}^n(X, \eta_s) = \sum_{i=0}^{s} |S_Y(i)|,$$

and thus (98) and (99) follow by Corollary 22.

Therefore what remains to be proven is that $\bigcup_{i=0}^{s} S_Y(i)$ is the unique minimum $\eta_s$-quasi image of $X$ by $p_{Y|X}$. Clearly $\bigcup_{i=0}^{s} S_Y(i)$ is an $\eta_s$-quasi image by definition. Hence assume, in hopes of a contradiction, that there exists a

$$B \subseteq Y^n : \left\{ \begin{array}{l} B \neq \bigcup_{i=0}^{s} S_Y(i) \\ p_Y(B) \geq \eta_s \\ |B| \leq |\bigcup_{i=0}^{s} S_Y(i)| \end{array} \right\}.$$ 

For $s = t$, $B$ cannot be a minimum $\eta_t$-quasi image of $X$ by $p_{Y|X}$, since $\bigcup_{i=0}^{s} S_Y(i)$ contains the entire support set of $Y$, which is also the minimum 1-quasi image by definition. On the other hand, for $s \in [0 : t - 1]$,

$$p_Y(B) = p_Y\left(\bigcup_{i=0}^{s} S_Y(i) \setminus B\right) = p_Y\left(\bigcup_{i=0}^{s} S_Y(i) \setminus B\right) + p_Y(B \setminus \bigcup_{i=0}^{s} S_Y(i))$$

$$= \eta_s - \sum_{i=0}^{s} p_Y(y) + \sum_{i=s+1}^{t} \sum_{y \in S_Y(i) \cap B} p_Y(y)$$

$$\leq \eta_s - \left(\bigcup_{i=0}^{s} S_Y(i) \setminus B\right) - \left(\bigcup_{i=0}^{s} S_Y(i) \setminus B\right) 2^{-s\lambda}$$

$$+ \left(\bigcup_{i=0}^{s} S_Y(i) \setminus B\right) 2^{-(s+1)\lambda}$$

$$< \eta_s,$$

where (100) is because the support set of $Y$ is $\bigcup_{i=0}^{t} S_Y(i)$, and (101) is because $|B| \leq |\bigcup_{i=0}^{s} S_Y(i)|$ by assumption. But this is a contradiction since $B$ is an $\eta_s$-quasi image of $X$ by $p_{Y|X}$.

Next we need to introduce an entropy spectrum equivalent to the inequality $|H(Y) - H(Y|U)| \leq \log_2 |\mathcal{U}|$. This is done in two parts.

Lemma 25. Fix any $U : \{U \rightarrow X \rightarrow Y\}$ and any $u \in \mathcal{U}$. Let $A_u$ be the support set of $X \{U = u\}$. If $\log_2 \tilde{g}_{Y|X}^n(A_u, 1 - \beta) \leq c$ for some $c \in \mathbb{R}_+$ and $\beta \in (0, 1)$, then

$$\Pr(h_{Y|U}(Y|U) \leq \mu + c|U = u) \geq 1 - 2^{-\mu} - \beta$$

for each $\mu \in \mathbb{R}_+$.

Proof: First observe that an upper bound on the minimum $(1 - \beta)$-image of $A_u$ by $p_{Y|X}$ also yields an upper bound on the minimum $(1 - \beta)$-quasi image of $X_u \triangleq X \{U = u\}$ by $p_{Y|X}$. In other words,

$$\tilde{g}_{Y|X}^n(X_u, 1 - \beta) \leq \log_2 \tilde{g}_{Y|X}^n(A_u, 1 - \beta) \leq 2^c,$$

which follows from the assumption that $U \rightarrow X \rightarrow Y$ and from the definitions of images and quasi-images.
Now letting $\mathcal{B}$ denote a minimum $(1 - \beta)$-quasi image of $X_u$ by $p_{Y|X}$, the result follows since
\[
\Pr \left( h_{Y|U}(Y|U) > \mu + c | U = u \right) \\
\leq 1 - \Pr \left( Y \in \mathcal{B} | U = u \right) \\
+ \Pr \left( h_{Y|U}(Y|U) > \mu + c, Y \in \mathcal{B} | U = u \right) \\
\leq \beta + \sum_{y \in \mathcal{B} : h_{Y|U}(y|u) < 2^{-\mu - c}} p_{Y|U}(y|u) \\
< \beta + 2^{-\mu},
\]
where the last inequality results from applying (103). ■

Lemma 26.

\[
h_{Y|U}(y|u) \geq h_Y(y) + \log_2 p_U(u)
\]
for any $u \in U$ and $U : \{U \leftrightarrow X \leftrightarrow Y\}$.

Proof: We can upper bound $p_{Y|U}(y|u)$ in terms of $p_Y(y)$, for any $u \in U$ and $U : \{U \leftrightarrow X \leftrightarrow Y\}$, as follows
\[
p_{Y|U}(y|u) = \sum_{x \in A} p^n_{Y|X}(y|x)p_{X|U}(x|u) \\
\leq \frac{1}{p_U(u)} \sum_{x \in A} p^n_{Y|X}(y|x)p_X(x) \\
= \frac{p_Y(y)}{p_U(u)}, \tag{104}
\]
Taking $-\log_2$ on both sides of (104) proves the lemma. ■

C. Proof of Theorem 10

Recall from the statement of Theorem 10 that $n \geq 27$, \(|\mathcal{Y}| \geq 2\), and $\alpha \in \left( \frac{\log_2 n}{n}, \frac{1}{8 \ln 2} \right)$. For the proof, we set
\[
\lambda = 2(\log_2 n)(1 - n^{-1} \log_2 n)^{-1}(\log_2 |\mathcal{Y}|) \leq 4|\mathcal{Y}| \log_2 n
\]
and
\[
t = 2n^{-1} \log_2 |\mathcal{Y}| = \frac{n}{\log_2 n} - 1,
\]
for which $t \lambda = 2n \log_2 |\mathcal{Y}| > \log_2 (|\mathcal{Y}|^n)$. Once again, we write $S_Y(i) = S_Y(i; \lambda, t)$ in order to simply notation.

Proof: First we will identify the set $\mathcal{A}^1$, and then prove (2) before proving (1). In the act of proving (2), we will by necessity prove the existence of the positive real numbers $\delta = O(-\sqrt{\alpha \log_2 \alpha})$ and $r$ described in the statement of the theorem.

To establish $\mathcal{A}^1$, observe there exists at least one $s^* \in [0 : t]$ such that $p_Y(S_Y(s^*)) \geq (t + 1)^{-1} = n^{-1} \log_2 n$ since $\sum_{i=0}^t p_Y(S_Y(i)) = 1$. Furthermore $s^* \neq t$, since $p_Y(y) \leq |\mathcal{Y}|^{-2n}$ for each $y \in S_Y(t)$, and thus $p_Y(S_Y(t)) \leq |\mathcal{Y}|^{-n} < n^{-1} \log_2 n$. The theorem follows by setting
\[
\mathcal{A}^1 = \mathcal{A}^+ \setminus \mathcal{A}^-
\]
where
\[
\mathcal{A}^+ \triangleq \left\{ x \in \mathcal{A} : p^n_{Y|X} \left( \bigcup_{i=0}^{s^*} S_Y(i) \right) x \geq 2^{-n}\alpha \right\}, \tag{106}
\]
\[
\mathcal{A}^- \triangleq \left\{ x \in \mathcal{A} : p^n_{Y|X} \left( \bigcup_{i=0}^{s^*} S_Y(i) \right) x \geq 2^{-n}\alpha \right\}, \tag{107}
\]
\[
s^- \triangleq \lfloor s^* - n\lambda^{-1}\delta \rfloor,
\]
\[
\tilde{\delta} \triangleq \tau_n(2^{-n\alpha}, 2^{-n\alpha}) + \alpha + 7.19|\mathcal{Y}|^{\log_2 n}/n.
\]

Note that if $s^- < 0$ in the above definition, we have $\bigcup_{i=0}^{s} S_Y(i) = \emptyset$ and hence $\mathcal{A}^- = \emptyset$. 

Having identified the set $\mathcal{A}^\dagger$, we move on to proving (1), and to that end consider any DRV $U : \{U \arrow X \arrow Y\}$ and $u \in \mathcal{U} : \{\Pr(X \in \mathcal{A}^\dagger|U = u) = 1\}$. Notice that such a $U$ always exists, e.g., consider $U$ as the indicator of $\mathcal{A}^\dagger$. Assume for the moment that

$$
\Pr \left( h(Y|U) > s^*\lambda + n\delta \right| U = u \right) \leq 2 \cdot 2^{-n\alpha} 
$$

(108)

$$
\Pr \left( h(Y|U) \leq s^-\lambda - h(U) \right| U = u \right) < 2^{-n\alpha}.
$$

(109)

Clearly, applying the union bound with (108), (109) gives

$$
\Pr \left( |h(Y|U) - s^*\lambda| < n\delta + h(U) \right| U = u \right) < 3 \cdot 2^{-n\alpha}.
$$

(110)

for a positive real number $\delta = O(-\sqrt{\alpha \log_2 \alpha})$. This therefore validates (1), since $s^*\lambda$ is a constant less than $\lambda t = 2n \log_2 |Y|$. A detailed verification of the order term of $\delta$ is provided in Appendix C.

Let us now turn to confirming (108) and (109). To verify (108), first observe

$$
\log_2 g^n_{Y|X}(\mathcal{A}_+^+, 1 - 2^{-n\alpha})
\leq \log_2 g^n_{Y|X}(X, \eta_{s^*}) + n\tau_n(2^{-n\alpha}, 2^{-n\alpha})
< s^*\lambda + \lambda + \log_2(t + 1) + n\tau_n(2^{-n\alpha}, 2^{-n\alpha})
$$

(111)

$$
\leq s^*\lambda + 3|Y| \log_2 n + \log_2 \left( \frac{n}{\log_2 n} \right) + n\tau_n(2^{-n\alpha}, 2^{-n\alpha})
\leq s^*\lambda + 7.19|Y| \log_2 n + n\tau_n(2^{-n\alpha}, 2^{-n\alpha})
= s^*\lambda + n\delta - n\alpha.
$$

(112)

(113)

where (111) results by applying Lemma 22 because of the definition of $\mathcal{A}^+$ and the fact that $\bigcup_{i=0}^{s^*} Y(i)$ is a minimum $\eta_{s^*}$-quasi image of $X$ by $p_{Y|X}$, while (112) is by Lemma 24. Equation (108) now directly follows from (113) and Lemma 25, since the support set of $X|\{U = u\}$ is a subset of $\mathcal{A}^+$.

On the other hand (109) can be derived as follows

$$
\Pr \left( h(Y|U) \leq s^-\lambda - h(U) \right| U = u \right)
\leq \Pr \left( h(Y) \leq s^-\lambda \right| U = u \right)
$$

(114)

$$
\leq p_{Y|U} \left( \bigcup_{i=0}^{s^-} Y(i) \right| U = u \right)
= \sum_{x \in \mathcal{A}^-} p^n_{Y|X} \left( \bigcup_{i=0}^{s^-} Y(i) \bigg| X = x \right) p_{X|U}(X|U)
\leq 2^{-n\alpha}
$$

(115)

where (114) is by Lemma 26, and (115) is because $\mathcal{A}^-$ contains all $x \in \mathcal{A}$ such that $p^n_{Y|X}(\bigcup_{i=0}^{s^-} Y(i)|x) \geq 2^{-n\alpha}$ yet $\mathcal{A}^\dagger \cap \mathcal{A}^- = \emptyset$.

Having identified the set $\mathcal{A}^\dagger$, the positive real numbers $\delta$ and $r$, and proven (1), we now move on to prove (2). To do so, we start by noting that if $\mathcal{A}^- \neq \emptyset$,

$$
\log_2 g^n_{Y|X}(\mathcal{A}^-, 1 - 2^{-n\alpha})
\leq \log_2 g^n_{Y|X}(X, \eta_{s^-}) + n\tau_n(2^{-n\alpha}, 2^{-n\alpha})
< s^-\lambda + \lambda + \log_2(t + 1) + n\tau_n(2^{-n\alpha}, 2^{-n\alpha})
$$

(116)

(117)

(118)

where (116) is due to Lemma 23 and the definition of $\mathcal{A}^-$ and $\bigcup_{i=0}^{s^-} Y(i)$ being a minimum $\eta_{s^-}$-quasi image of $X$ by $p_{Y|X}$; while (117) is due to Lemma 24. Finally (118) follows by applying the inequality that $s^*\lambda \geq s^-\lambda + n\delta$ and then the same chain of inequalities from (112) through (113). Also, we have $g^n_{Y|X}(\mathcal{A}^-, 1 - 2^{-n\alpha}) = 0$ trivially if $\mathcal{A}^- = \emptyset$. 


Now, let $B^−$ be a minimum $(1 − 2^{−\alpha})$-image of $A^−$ by $p_{Y|X}$. A lower bound on the probability of $S_Y(s^*) \setminus B^−$ can be constructed as follows:

$$p_Y(S_Y(s^*) \setminus B^−) = p_Y(S_Y(s^*)) - p_Y(S_Y(s^*) \cap B^−) \geq \frac{\log_2 n}{n} - 2^{−s^*\lambda} |S_Y(s^*) \cap B^−| \geq \frac{\log_2 n}{n} - 2^{−s^*\lambda} g^n_{Y|X}(A^−, 1 − 2^{−\alpha}) \geq \frac{\log_2 n}{n} - 2^{−2\alpha} \tag{119}$$

where (119) is because $p_Y(y) \leq 2^{−s^*\lambda}$ for all $y \in S_Y(s^*)$, and (120) is due to (118) when $A^− \neq \emptyset$ and due to $g^n_{Y|X}(A^−, 1 − 2^{−\alpha}) = 0$ when $A^− = \emptyset$. But this implies a lower bound on the probability of $A^|$ since now

$$\frac{\log_2 n}{n} - 2^{−\alpha} \leq p_Y(S_Y(s^*) \setminus B^−) = \sum_{x \in A^−} p^n_{Y|X}(S_Y(s^*) \setminus B^−|x)p_X(x) + \sum_{x \in A^\uparrow} p^n_{Y|X}(S_Y(s^*) \setminus B^−|x)p_X(x) + \sum_{x \in A\setminus(A^\uparrow \cup A^−)} p^n_{Y|X}(S_Y(s^*) \setminus B^−|x)p_X(x) \leq 2^{−2\alpha} + p_X(A^\uparrow) + 2^{−2\alpha} \tag{121}$$

where each term in (122) bounds the corresponding term in (121). In particular, the bound on the first term in (121) is due to the fact that each $x \in A^−$ satisfies $1 - p^n_{Y|X}(B^−|x) \leq 2^{−2\alpha}$. On the other hand, the bound on the third term in (121) is because $A^\uparrow \cup A^− = A^\uparrow \cup A^−$ contains all $x \in A$ such that $p^n_{Y|X}(S_Y(s^*)|x) \geq 2^{−2\alpha}$. Solving (122) for $p_X(A^\uparrow)$ and simplifying, we have

$$p_X(A^\uparrow) \geq \frac{\log_2 n}{n} - 3 \cdot 2^{−2\alpha} \geq \frac{\log_2 n}{n} - 3 \frac{n}{n} = \frac{1}{n} \log_2 n$$

since $\alpha \geq n^{−1} \log_2 n$.

\section{VI. Proof of Theorem 11: Information Stable Partitioning}

Throughout this section we will once again assume $(\emptyset, X, Y)$ are a regular collection of DRVs, and we will let $A$ denote the support set of $X$. Also, as before, we first present a few key lemmas.

\subsection{A. Supporting lemmas}

The first lemma repeatedly applies Theorem 10 in order to construct a DRV $V$ which provides stability when conditioned upon. That is, we apply Theorem 10 to $A$ obtaining a subset which gives stability. This subset is then removed from $A$, and Theorem 10 is applied to the remaining set again to obtain a new subset. This process is then repeated a number of times. The random variable $V$ is then an index to the stable subset that $X$ belongs.

\textbf{Lemma 27.} Given any regular collection $(\emptyset, X, Y)$, positive real number $\alpha \in \left(\frac{\log_2 n}{n}, \frac{1}{8n^2}\right)$, and $\zeta \in \mathbb{N}_+$, there exists:

\begin{itemize}
  \item a DRV $V : \{ \begin{array}{l}
  V = \{0 : \zeta - 1\} \\
  V \ll X \\
  h_V(0) > \frac{\zeta - 1}{\chi n} \log_2 \frac{n}{16} \end{array} \}$, \item positive real number $\delta = O(\sqrt{\alpha} \log_2 \alpha)$, and \item function $r : \mathcal{V} \to \mathbb{R}_+$ such that
\end{itemize}

$$\max_v r(v) < 2n \log_2 |\mathcal{V}|$$

and

$$\Pr(|h(Y|U) - r|V)| > n\delta + h(U)|U = u) < 3 \cdot 2^{−n\alpha}, \tag{123}$$
for all $DR\ V \ni U \overset{U \gg V}{\longrightarrow} \overset{U \overset{X \sim Y}{\longrightarrow}}{X \sim Y}$ and $u \in U : \{p_{V|U}(0|u) = 0\}$. 

Proof: By Theorem $[10]$ there exists a set $A^1_i \subseteq A$ where

$$p_X(A^1_i) \geq \frac{1}{n} \log_2 \frac{n}{8}, \quad (124)$$

and positive real numbers $\delta_i = O(-\sqrt{\alpha \log_2 \alpha})$ and $r_i < 2n \log_2 \{|Y|\}$ such that

$$\Pr \left(\left|\sum_{i} V \log_2 (1/r_i) - n\delta_i + h(U)|U = u\right| < 3 \cdot 2^{-n\alpha}\right)$$

for all $DR\ V : \{U \overset{U \sgtr V}{\longrightarrow} \overset{U \overset{X \sim Y}{\longrightarrow}}{X \sim Y}\}$ and $u \in U : \{\Pr \left(V \in A_i \mid U = u\right) = 1\}$. Now, for each $i \in [2 : \zeta - 1]$, given the recursively defined $X_i \triangleq X \setminus \{X \notin \bigcup_{j=1}^i A^1_j\}$ there exists a $A^1_i \subseteq A \setminus \bigcup_{j=1}^i A^1_j$ such that

$$p_X(A^1_i) \geq \frac{1}{n} \log_2 \frac{n}{8}, \quad (125)$$

and positive real numbers $\delta_i = O(-\sqrt{\alpha \log_2 \alpha})$ and $r_i < 2n \log_2 \{|Y|\}$ where

$$\Pr \left(\left|\sum_{i} V \log_2 (1/r_i) - n\delta_i + h(U)|U = u\right| < 3 \cdot 2^{-n\alpha}\right)$$

for all $DR\ V : \{U \overset{U \sgtr V}{\longrightarrow} \overset{U \overset{X \sim Y}{\longrightarrow}}{X \sim Y}\}$ and $u \in U : \{\Pr \left(V \in A_i \mid U = u\right) = 1\}$.

Now, define the following:

- **DRV**

$$V = \sum_{i=1}^{\zeta-1} i \cdot 1 \left(V \in A^1_i\right)$$

- positive real number $\delta = \max_{i \in [1 : \zeta - 1]} \delta_i$ (for which clearly $\delta = O(-\sqrt{\alpha \log_2 \alpha})$), and

- function $r(i) = r_i$ for $i \in [1 : \zeta - 1]$ and $r(0) = 3$ (for which clearly $\max_i r(v) < 2n \log_2 \{|Y|\}$).

Thus $V = [0 : \zeta - 1]$ and $X \gg V$ by construction. Before proving the upper bound on $h_Y(0)$, note $V = 0$ implies $X \in A_i^1$ for $i \in [1 : \zeta - 1]$, and if $V = 0$ then $X \in A \setminus \bigcup_{i=1}^{\zeta-1} A_i^1$. Hence,

$$p_V(0) = \Pr \left(V \in A \setminus \bigcup_{i=1}^{\zeta-1} A_i^1\right)$$

$$= \prod_{j=1}^{\zeta-1} \Pr \left(V \in A \setminus \bigcup_{i=1}^{j} A_i^1 \mid X \in A \setminus \bigcup_{i=1}^{j-1} A_i^1\right)$$

$$= \prod_{j=1}^{\zeta-1} \left(1 - p_X(A^1_j)\right)$$

$$\leq \prod_{j=1}^{\zeta-1} \left(1 - \frac{1}{n} \log_2 \frac{n}{8}\right)$$

$$\leq 2^{(\zeta-1) \log_2 (1 - n^{-1} \log_2 (n/8))} < 2^{\frac{\zeta-1}{n} \log_2 \frac{n}{8}} \quad (126)$$

where (126) is due to (124) and (125).

Finally we prove (123). If given a $DR\ V : \{U \gg V \overset{U \sim X \sim Y}{\longrightarrow}\}$, then for all $u \in U : \{p_{V|U}(0|u) = 0\}$ there must exist an $i \in [1 : \zeta - 1]$ for which $p_{V|U}(i|u) = 1$ since $V$ is a deterministic function of $U$. In turn then $\Pr \left(V \in A_i^1 \mid U = u\right) = 1$ since $V = i$ implies $X \in A_i^1$. Hence it also follows that

$$3 \cdot 2^{-n\alpha} \cdot \Pr \left(\left|\sum_{i} V \log_2 (1/r_i) - n\delta_i + h(U)|U = u\right| < 3 \cdot 2^{-n\alpha}\right)$$

$$\geq \Pr \left(\left|\sum_{i} V \log_2 (1/r_i) - n\delta_i + h(U)|U = u\right| < 3 \cdot 2^{-n\alpha}\right),$$

establishing (123).

Notice that Lemma 27 only applies to regular collections of the form $(\emptyset, X, Y)$. The next corollary is the first step in generalizing the result to regular collections of the form $(M, X, Y)$. This generalization is achieved by leveraging Lemma 27 with the fact that $(M, X, Y)|\{M = m\}$ is a regular collection for all $m \in M$. 


Corollary 28. Given any regular collection \((M, X, Y)\), positive real number \(\alpha \in \left(\frac{\log_2 n}{n}, \frac{1}{8\log_2 n}\right)\), and \(\zeta \in \mathbb{N}_+\), there exists:

- a DRV \(V\) : 
  \[
  \begin{cases}
  \mathcal{V} = \{0 : \zeta - 1\} \\
  V \ll (X, M) \\
  h_V(0) > \frac{\zeta - 1}{n} \log_2 \frac{\pi}{8}
  \end{cases}
  \]
- positive real number \(\delta = O(-\sqrt{\alpha \log_2 \alpha})\), and
- function \(r : \mathcal{V} \times M \to \mathbb{R}_+\) such that
  \[
  \sup_{(v, m) \in \mathcal{V} \times M} r(v, m) < 2n \log_2 |\mathcal{Y}|
  \]
  and
  \[
  \Pr \left( |h(Y|U, M) - r(V, M)| > n\delta + h(U|M)(U, M) = u, m \right) < 3 \cdot 2^{-n\alpha},
  \]
  for all DRV \(U\) : 
  \[
  \begin{cases}
  U \gg V \\
  (U, M) \to X \to Y
  \end{cases}
  \]
  and \(u \in \mathcal{U} : \{p_{V|M}(0|u) = 0\}\).

Proof: Let us be given a regular collection \((M, X, Y)\). For each \(m \in M\), let \((X_m, Y_m)\) be DRVVs defined by setting their distributions according to

\[
p_{X_m, Y_m}(x, y) = p_{Y|X|M}(x, y|m).
\]

It is clear then that the set \((\emptyset, X_m, Y_m)\) is also a regular collection. Thus for each \(m \in M\), there exists:

- a DRV \(V_m\) : 
  \[
  \begin{cases}
  \mathcal{V}_m = \{0 : \zeta - 1\} \\
  V_m \ll X_m \\
  h_{V_m}(0) > \frac{\zeta - 1}{n} \log_2 \frac{\pi}{8}
  \end{cases}
  \]
- a positive real number \(\delta_m = O(-\sqrt{\alpha \log_2 \alpha})\), and
- function \(r_m : \{0 : \zeta - 1\} \to \mathbb{R}_+\) such that
  \[
  \max_{v \in \{0 : \zeta - 1\}} r_m(v) < 2n \log_2 |\mathcal{Y}|
  \]
  and
  \[
  \Pr \left( |h(Y_m|U) - r_m(V_m)| > n\delta_m + h(U|M)(U, M = u) \right) < 3 \cdot 2^{-n\alpha},
  \]
  for all \(U\) : 
  \[
  \begin{cases}
  U \gg V_m \\
  U \to X_m \to Y_m
  \end{cases}
  \]
  and \(u \in \mathcal{U} : \{p_{V_m|M}(0|u) = 0\}\).

By Lemma 27, The DRV \(V\) in the corollary statement is constructed from the set of \(V_m\), \(m \in M\) by setting

\[
p_{Y, X, V|M}(y, x, v|m) = p_{Y_m, X_m, V_m}(y, x, v)
\]

for each \(m \in M\). Let us now verify the properties of \(V\) so constructed. Clearly \(\mathcal{V} = \bigcup_{m \in M} \mathcal{V}_m = \{0 : \zeta - 1\}\). Next, that \(V\) is a deterministic function of \((X, M)\) follows from (129) and \(V_m\) being a deterministic function of \(X_m\) for each \(m \in M\). Finally,

\[
p_{V}(0) = \sum_{m \in M} p_{M}(m)p_{V|M}(0|m) = \sum_{m \in M} p_{M}(m)p_{V_m}(0)
\]

\[
< \sum_{m \in M} p_{M}(m)2^{-\frac{\zeta - 1}{n} \log_2 \frac{\pi}{8}} = 2^{-\frac{\zeta - 1}{n} \log_2 \frac{\pi}{8}}.
\]

Further, let \(\delta = \sup_{m \in M, v \in \mathcal{V}} \delta_m\), for which clearly \(\delta = O(-\sqrt{\alpha \log_2 \alpha})\) since \(\delta_m = O(-\sqrt{\alpha \log_2 \alpha})\) for all \(m \in M\). Also let function \(r(v, m) = r_m(v)\) for each \(v \in \mathcal{V}\) and \(m \in M\). Clearly \(\sup_{v, m} r(v, m) < 2n \log_2 |\mathcal{Y}|\), since \(r_m(v) < 2n \log_2 |\mathcal{Y}|\) for all \(m \in M\) and \(v \in \mathcal{V}\).

To confirm (127), fix any DRV \(U\) : 
\[
\begin{cases}
  U \gg V \\
  (U, M) \to X \to Y
  \end{cases}
\]

and any \(u \in \mathcal{U} : \{p_{V|M}(0|u) = 0\}\). Define DRV \(U_m\) by setting

\[
p_{Y_m, X_m, V_m, U_m}(Y, X, v, u|m) = p_{Y, X, V, U|M}(y, x, v, u|m)
\]
for each $m \in \mathcal{M}$. Notice that $U_{(m)} \gg V_{(m)}$ and $U_{(m)} \xrightarrow{\mathcal{M}} X_{(m)} \xrightarrow{\mathcal{M}} Y_{(m)}$ and $p_{V_{(m)}|U_{(m)}}(0|u) = 0$ clearly follow from \[(130)\] since $U \gg V$ and $(U, M) \xrightarrow{\mathcal{M}} X \xrightarrow{\mathcal{M}} Y$ and $p_{V|U}(0|u) = 0$. Furthermore \[(128)\] must hold because $U_{(m)}$ satisfies these requirements. That is,

$$\Pr \left( |h(Y_{(m)}|U_{(m)}) - r_{(m)}(V_{(m)})| > n\delta_{(m)} + h(U_{(m)}) | U = u \right) < 3 \cdot 2^{-n\alpha}. \quad (131)$$

But, at the same time

$$h_{Y_{(m)}|U_{(m)}}(y|u) = h_{Y|U,M}(y|u, m)$$
$$h_{U_{(m)}}(u) = h_{U|M}(u|m)$$

for each $y, u, m \in Y^m \times \mathcal{U} \times \mathcal{M}$, by \[(130).\] Hence it follows that

$$\Pr \left( |h(Y|U, M) - r(V, M)| > n\delta + h(U|M)(U, M) = (u, m) \right) = \sum_{y : v} \sum_{u : m} \Pr_{Y, V|U, M}(y, v | u, m)$$
$$= \sum_{Y : V} \sum_{u : m} \Pr_{Y_{(m)}, V_{(m)}|U_{(m)}}(y, v | u)$$
$$= \Pr \left( |h(Y_{(m)}|U_{(m)}) - r_{(m)}(V) - n\delta + h(U_{(m)}) | U_{(m)} = u \right). \quad (132)$$

Equation \[(127)\] is therefore confirmed from combining \[(132)\] and \[(131)\] since $\delta \geq \delta_{(m)}$ for all $m \in \mathcal{M}$.

Next, we remove the dependence of the function $r$ upon $M$ since this will allow for $h(Y|M)$ to stabilize to $\mathbb{H}(Y|M)$.

**Lemma 29.** Given any regular collection $(M, X, Y)$, positive real number $\alpha \in \left( \frac{\log_2 \alpha}{n}, \frac{1}{\alpha \ln 2} \right)$, and $\zeta \in \mathbb{N}_+$, there exists:

- a DRV $V : \left\{ \begin{array}{l}
|V| \leq 2\zeta n \log_2 |Y| \\
V \ll (X, M)
\end{array} \right\}$,
- positive real number $\delta = O(-\sqrt{\alpha} \log_2 \alpha)$, and
- function $r : V \to \mathbb{R}_+$ such that

$$\Pr \left( |h(Y|U, M) - r(V) - n\delta + h(U)|U = u \right) < 4 \cdot 2^{-n\alpha}, \quad (133)$$

for all DRV $U : \left\{ \begin{array}{l}
U \gg V \\
(U, M) \xrightarrow{\mathcal{M}} X \xrightarrow{\mathcal{M}} Y
\end{array} \right\}$ and $u \in \mathcal{U} : \{p_{V|U}(v_0|u) = 0\}$.

**Proof:** For any regular collection $(M, X, Y)$ let $\tilde{V}$, $\tilde{v}$, and $\tilde{r}$ be the DRV, constant, and function respectively guaranteed by Corollary \[(28)\] Now define the following:

- DRV $\tilde{V} = \left\{ \tilde{r}(\tilde{V}, M) \right\}$.
- DRV $V = \left\{ \begin{array}{l}
(\tilde{V}, \tilde{V}) \quad \text{if} \quad \tilde{V} \neq 0 \\
v_0 \quad \text{o.w.}
\end{array} \right\}$.
- constant $\delta = \tilde{\delta} + \alpha + \frac{1}{n} = O(-\sqrt{\alpha} \log_2 \alpha)$

(see Appendix \[(22)\] for verification of the order term), and
- function $r(V) = \tilde{V}$.

First, let us verify the properties of $V$. Clearly $|V| \leq |\tilde{V}| |\tilde{V}| + 1$, where the additional term is to account for $v_0$. The upper bound on $|V|$ now follows since $|\tilde{V}| = 2n \log_2 |Y|$ and $|\tilde{V}| = \zeta$. Next, $(X, M) \gg V$ clearly follows from $(\tilde{V}, M) \gg \tilde{V}$ and $(X, M) \gg \tilde{V}$. Finally, $p_{V}(v_0) = p_{\tilde{V}}(0) < 2^{-\frac{\alpha}{\alpha \ln 2}}$. 


Now, to prove (133) fix any DRV $U : \left\{ \begin{array}{c} U \ni V \\ (U, M) \xrightarrow{\psi} X \xrightarrow{\phi} Y \end{array} \right\}$, and observe that

\[
\Pr \left( \left| r(V) - h(Y | M, U) \right| > n\delta + n\alpha + 1 + h(U) \right| \left| U = u \right) \\
\leq \Pr \left( \left| h(Y | M, U) - \tilde{r}(\tilde{V}, M) \right| > n\delta + h(U | M) \right| \left| U = u \right) \\
+ \Pr \left( \left| r(V) - \tilde{r}(\tilde{V}, M) \right| > 1 \right| \left| U = u \right) \\
+ \Pr \left( h(U | M) > h(U) + n\alpha \right| \left| U = u \right)
\]

(135)

Indeed, assume the predicates of all three probability terms on the right hand side (RHS) of (135) fail, then it would also follow that

\[
\left| r(V) - h(Y | M, U) \right| \\
\leq \left| r(V) - \tilde{r}(\tilde{V}, M) \right| + \left| h(Y | M, U) - \tilde{r}(\tilde{V}, M) \right| \\
\leq n\delta + h(U | M) + 1 \\
\leq n\delta + n\alpha + h(U) + 1 = n\delta + h(U).
\]

Equation (133) therefore follows from combining (135) with the following three to-be-proven inequalities:

\[
\Pr \left( \left| h(Y | M, U) - \tilde{r}(\tilde{V}, M) \right| > n\delta + h(U | M) \right| \left| U = u \right) < 2^{-n\alpha},
\]

(136)

\[
\Pr \left( h(U | M) > h(U) + n\alpha \right| \left| U = u \right) \leq 2^{-n\alpha},
\]

(137)

\[
\Pr \left( \left| r(V) - \tilde{r}(\tilde{V}, M) \right| > 1 \right| \left| U = u \right) = 0.
\]

(138)

We finish the proof by confirming (136) – (138). First, (136) is a property directly obtained from (129) of Corollary 28. Next, (137) can be derived as follows:

\[
\Pr (h(U | M) > h(U) + n\alpha | U = u) \\
= \sum_{m : p(u | m) < 2^{-n\alpha} p(u)} p(m | u) \\
= \sum_{m : p(u | m) < 2^{-n\alpha} p(u)} \frac{p(u | m) p(m)}{p(u)} \\
< \sum_m 2^{-n\alpha} p(m) = 2^{-n\alpha}.
\]

Finally, for proving (138),

\[
\left| r(V) - \tilde{r}(\tilde{V}, M) \right| \leq 1,
\]

follows by the definitions of $r$ and $\tilde{r}$ for all $v \neq v_0$.

$\blacksquare$

B. Proof of Theorem 77

Proof: Let us be given a regular collection $(M_{i : [1 : j]}, X, \{Y_{i : [1 : l]}\})$. For each $i \in [1 : k]$ and $j \in [1 : l]$, there exists

- a DRV $V_{i,j} : \begin{array}{c} |Y_{i,j}| \leq 2n^3 \log_2 |Y| \\ V_{i,j} \ll (X, M_j) \\ h_{V_{i,j}}(v_{0,i,j}) > \frac{n^2}{n^2} \log_2 \frac{\delta}{8} \end{array}$,

- positive real number $\delta_{i,j} = O(\sqrt{\alpha \log_2 \alpha})$, and

- function $r_{i,j} : V_{i,j} \to \mathbb{R}_+$ such that

\[
\Pr \left( \left| h(Y_i | U, M_j) - r_{i,j}(V_{i,j}) \right| > n\delta_{i,j} + h(U) \right| \left| U = u \right) < 4 \cdot 2^{-n\alpha},
\]

(139)

for all $U : \begin{array}{c} U \ni V_{i,j} \\ (U, M_j) \xrightarrow{\psi} X \xrightarrow{\phi} Y_{[1 : k]} \end{array}$, and $u \in U_{i,j} \triangleq \{ u \in U : p_{V_{i,j}}(v_{0,i,j} | u) = 0 \}$ by Lemma 29.

$^9$Set $\zeta = n^2$ in the lemma.
The DRV $V$ in the theorem statement can now be defined as
\[ V = \bigotimes_{i \in [1:k], j \in [1:l]} V_{i,j}. \]

Indeed, let us quickly verify the properties of $V$. First, $(X, M_{[1:l]}) \Rightarrow V$ since $(X, M_{[1:l]}) \Rightarrow V_{i,j}$ for each $i, j$. Second,
\[ |V| \leq (2n^3 \log_2 |Y|)^k \]
since $|V_{i,j}| \leq 2n^3 \log_2 |Y|$, while $|V|$ is at most $\prod_{i\in [1:k], j \in [1:l]} |V_{i,j}|$.

Going forth, it will be important to note that $V \Rightarrow V_{i,j}$, thus to every $v \in V$, $i \in [1:k]$ and $j \in [1:l]$ there exists a $v_{i,j}$ such that $p_{V_{i,j}|V}(v_{i,j}|v) = 1$. Furthermore if $U \Rightarrow V$, then similarly for each $i \in [1:k]$ and $j \in [1:l]$ there exists a $v_{i,j}$ such that $p_{V_{i,j}|U}(v_{i,j}|u) = 1$.

To prove the remaining properties, fix a DRV $U : \left\{ \begin{array}{l} U \Rightarrow V \\ (U, M_{[1:l]}) \xrightarrow{\to} X \xrightarrow{\to} Y_{[1:k]} \end{array} \right\}$, and let $u \in \mathcal{U}_{i,j}$. The probability of the event $\{U \in \mathcal{U}_{i,j}\}$ is determined by the probability that $V_{i,j} = v_{0,i,j}$; in specific
\[ p_U(U \setminus \mathcal{U}_{i,j}) = \sum_{u : p_{V_{i,j}|U}(v_{0,i,j}|u) = 1} p_U(u) \]
\[ = \sum_{u : p_{V_{i,j}|U}(v_{0,i,j}|u) = 1} p_{V_{i,j},U}(v_{0,i,j}, u) \]
\[ \leq p_{V_{i,j}}(v_{0,i,j}) \]
\[ < 2^{ \left( \frac{n^3 - 1}{n^3} \right) \log_2 \frac{n}{n}} \]
\[ \leq 2^{- \frac{n}{n} \log_2 \frac{n}{n}} \]  \hspace{1cm} (141)

where (140) is because $U \Rightarrow V \Rightarrow V_{i,j}$, while (141) is because $n \geq 27$.

Finally, note that $r_{i,j}(V_{i,j})$ is a constant given $U = u$ since $U \Rightarrow V \Rightarrow V_{i,j}$. Hence there exists a $\delta = O(-\sqrt{\alpha \log_2 \alpha})$ such that
\[ \Pr (|h(Y_i|U, M_j) - \mathbb{H}(Y_i|U, M_j)| > n\delta + 3h(U)|U = u) \]
\[ < 4 \cdot 2^{-n\alpha}, \]  \hspace{1cm} (142)

for all $i \in [1:k], j \in [1:l]$ and $u \in \mathcal{U}_{i,j}$ due to the combination of (139) and Corollary 9. A more detailed verification of which can be found in Appendix C.

\section{VII. Proof of Theorem 12}

The proof works by identifying two distinct subsets of elements in $M_j$ for each $j \in [1:l]$. The first subset contains those $m_j \in M_j$ such that $h(m_j)$ is small and the subset contains those $m_j \in M_j$ such that $h(m_j)$ is large. $M_j$ can be stabilized conditioned on the first subset, but not conditioned on the second subset. Luckily we may ignore the second subset since its probability of occurrence is small.

\subsection{A. Supporting Lemma}

We streamline the arguments of the proof by first considering the following lemma:

\textbf{Lemma 30.} For any DRVs $U$, $V$, and $\alpha \in \mathbb{R}_+$,
\[ \Pr (|h(V|U) - h(V)| > \alpha) < (|U| + 1)2^{-\alpha}. \]

\textbf{Proof:} By the union bound, it is clear that
\[ \Pr (|h(V|U)(V)| - h(V)| > \alpha) \leq p_{V,U}(A^+) + p_{V,U}(A^-) \]  \hspace{1cm} (143)

where
\[ A^- \triangleq \{(v, u) \in V \times U : h(v|u) < h(v) - \alpha\} \]
\[ A^+ \triangleq \{(v, u) \in V \times U : h(v|u) > h(v) + \alpha\}. \]

Thus the lemma is verified by (143) if we can show that $p_{V,U}(A^+) < |U|2^{-\alpha}$ and $p_{V,U}(A^-) \leq 2^{-\alpha}$.

First, observe that if there exists $v \in V$ such that $(v, u) \in A^-$, then
\[ p_U(u) < 2^{-\alpha}. \]  \hspace{1cm} (144)
Indeed, for all \((v, u) \in \mathcal{A}^-\) it must hold that
\[
 p_V(v) \geq p_{V|U}(v|u) p_U(u) > 2^\alpha p_V(v) p_U(u).
\]
The upper bound on \(p_{V,U}(\mathcal{A}^-)\) now follows from \(143\) as below:
\[
 p_{V,U}(\mathcal{A}^-) = \sum_{(v, u) \in \mathcal{A}^-} p_{V|U}(v|u) p_U(u) < \sum_{(v, u) \in \mathcal{A}^-} p_{V|U}(v|u) 2^{-\alpha} \leq |\mathcal{U}| 2^{-\alpha}.
\]
The upper bound on \(p_{V,U}(\mathcal{A}^+)\) follows similarly in that
\[
 p_{V,U}(\mathcal{A}^+) = \sum_{(v, u) \in \mathcal{A}^+} p_{V|U}(v|u) p_U(u) < \sum_{(v, u) \in \mathcal{A}^+} p_{V|U}(v|u) 2^{-\alpha} \leq 2^{-\alpha}.
\]

\[\text{B. Proof of Theorem 12}\]

\begin{proof}
Let \(Q_j = \lceil \min (h(M_j), \psi) \rceil\) for each \(j \in [1 : l]\). We first identify the DRV \(Q\) in the theorem statement as
\[
 Q = \prod_{j=1}^l Q_j.
\]
We start by verifying the properties of \(Q\). First,
\[
 |Q| \leq \prod_{j=1}^l |Q_j| \leq (\psi + 1)^l.
\]
Second, \(Q \ll (M_{[1:l]}\) since \(Q_j \ll M_j\) for each \(j \in [1 : l]\).

Next, choose any DRV \(U : \{U \gg Q\}\), and let \(\tilde{\rho} = 2\rho + \log_2(|\mathcal{U}| + 1)\). Fix \(j \in [1 : l]\), let \(q_j(u)\) be the unique element in \(Q_j\) that \(p_{Q_j|U}(q_j|u) = 1\) for each \(u \in \mathcal{U}\). Then identify the following sets:
\[
 \mathcal{U}_{j,(\text{stable})} = \mathcal{U}_{j,*} \cap \{u \in \mathcal{U} : q_j(u) < \psi\}
\]
\[
 \mathcal{U}_{j,(\text{same})} = \mathcal{U}_{j,*} \cap \{u \in \mathcal{U} : q_j(u) = \psi\}
\]
(145)
where
\[
 \mathcal{U}_{j,*} = \{u \in \mathcal{U} : \Pr (|h(M_j|U) - h(M_j)| > \tilde{\rho}|U = u) < 2^{-\rho}\}.
\]
We proceed to verify (5), (6), and (7) stated in the theorem.

First, observe that \(\mathcal{U}_{j,(\text{stable})} \cup \mathcal{U}_{j,(\text{same})} = \mathcal{U}_{j,*}\). By Lemma 30 we have
\[
 2^{-2\rho} > \Pr (|h(M_j) - h(M_j|U)| > 2\rho + \log_2(|\mathcal{U}| + 1))
\]
\[
 = \sum_{u} \Pr (|h(M_j|U) - h(M_j)| > \tilde{\rho}|U = u) p_U(u)
\]
\[
 \geq \sum_{u \notin \mathcal{U}_{j,*}} 2^{-\rho} p_U(u)
\]
(146)
where (146) is because
\[
 \Pr (|h(M_j|U) - h(M_j)| > \tilde{\rho}|U = u) \geq 2^{-\rho}
\]
for all \(u \notin \mathcal{U}_{j,*}\) by definition. Thus (5) is established, i.e.,
\[
 p_U(\mathcal{U}_{j,*}) > 1 - 2^{-\rho}.
\]
Next, to prove (6), we must show that there exists a \(\beta = O(\rho + 2^{-\rho}\psi)\) such that
\[
 \Pr (|h(M_j|U) - H_U(M_j)| > \beta + 3 \log_2 |\mathcal{U}||U = u) < 2^{-\rho}
\]
(147)
for all $u \in \mathcal{U}_{j,(\text{same})}$. To that end, observe first that
\[
\Pr \left( |h(M_j|U) - Q_j| > \hat{\rho} + 1|U = u \right) \\
\leq \Pr \left( |h(M_j|U) - h(M_j)| > \hat{\rho}|U = u \right) \\
+ \Pr \left( |h(M_j) - Q_j| > 1|U = u \right) \tag{148}
\]
and thus (149) results. Equation (147) follows directly from (149) and Corollary 9 by the fact that the support set, $M_{j,u}$, of $M_j|\{U = u\}$ satisfies $\log_2 |M_{j,u}| < \psi$ for each $u \in \mathcal{U}_{j,(\text{same})}$. A detailed calculation of the order term of $\beta$ can be found in Appendix C.

On the other hand, for each $u \in \mathcal{U}_{j,(\text{same})}$, note that $h(m_j) \geq \psi$ for all $m_j \in M_{j,u}$. Then since $u \in \mathcal{U}_{j,(\text{same})} \subseteq \mathcal{U}_{j,*}$, we must have
\[
\Pr \left( h(M_j|U) < \psi |U = u \right) < 2^{-\rho} \tag{150}
\]
for all $u \in \mathcal{U}_{j,(\text{same})}$ by the definition of $\mathcal{U}_{j,*}$. Replacing $\hat{\rho}$ in (150) with $\beta + 3 \log_2 |\mathcal{U}|$ gives (7) because $\beta + 3 \log_2 |\mathcal{U}| > \hat{\rho}$ (see Appendix C).

Finally, if $M_j$ is uniform over $M_j$ then $h(M_j) = \log_2 M_j$. Thus, by re-defining $\mathcal{U}_{j,(\text{same})} = \mathcal{U}_{j,*}$ and $\mathcal{U}_{j,*} = \emptyset$, $p_U(\mathcal{U}_{j,(\text{same})}) \geq 1 - 2^{-\rho}$ and
\[
\Pr \left( |h(M_j|U) - \log_2 |M_j|| > \hat{\rho}|U = u \right) < 2^{-\rho} \tag{151}
\]
for all $u \in \mathcal{U}_{j,*}$ by definition. Equation (8) can then be obtained from (151) by replacing $\hat{\rho}$ with $\beta + 3 \log_2 |\mathcal{U}|$ as above.

VIII. Proof of Theorem 13

To prove Theorem 13 we construct a finite but “dense” subset of conditional distributions in $\mathcal{P}(Y|X)$ that stability for all conditional distributions in the subset implies stability for the whole $\mathcal{P}(Y|X)$. It is clear that we will need to consider all possible product distributions from $\mathcal{P}_{Y|X}$. To simplify the notation required, given any two conditional distributions $w, \tilde{w} \in \mathcal{P}_{Y|X}$, we will write
\[
w_n(y|u) \triangleq \sum_x w^n(y|x)p_{X|U}(x|u) \\
\tilde{w}_n(y|u) \triangleq \sum_x \tilde{w}^n(y|x)p_{X|U}(x|u)
\]
throughout this section.

A. Supporting lemmas

**Lemma 31.** Let $(\emptyset, X, Y)_{\mathcal{P}(Y|X)}$ be a regular collection of DRVs, then
\[
\log_2 \frac{\tilde{w}_n(y|u)}{w_n(y|u)} \leq n \sup_{\tilde{w} \in \mathcal{P}(Y|X)} \mathbb{D}_{\tilde{w}}(w||\tilde{w}p) + 2|\mathcal{X}||Y| \log_2 n,
\]
for all $U : \{U \rightarrow X \rightarrow Y\}_{\mathcal{P}(Y|X)}$, $u \in U$, and $y \in Y^n : \{ \tilde{w}_n(y|u) > 0 \}$.

**Proof:** Let
\[
\zeta(\tilde{w}, \hat{\rho}, y) \triangleq \sum_{x \in \mathcal{P}_{Y|X} = \tilde{w}} \tilde{w}^n(y|x)p_{X|U}(x|u) \\
(\tilde{w}(y), \hat{\rho}(y)) \triangleq \arg \max_{(\tilde{w}, \hat{\rho}) \in \mathcal{P}_{Y|X}} \zeta(\tilde{w}, \hat{\rho}, y) 2^{-n \mathbb{D}(\tilde{w}(y)||\hat{\rho}(y))}.
\]
We will prove
\[
\log_2 \frac{w_n(y|u)}{\tilde{w}_n(y|u)} \leq n \mathbb{D}(\tilde{w}(y)||\hat{\rho}(y)) + 2|\mathcal{X}||Y| \log_2 n, \tag{152}
\]
of which the lemma is a clear consequence.
Towards proving (152), recognize

\[ w_n(y|u) = \sum_{(\hat{w}, \hat{\rho}) \in P_n(\mathcal{Y}')} \zeta(\hat{w}, \hat{\rho}, y) 2^{-nD(\hat{w}|\hat{\rho})} \]  

(153)

and (156).

\[ w_n(y|u) = \sum_{(\hat{w}, \hat{\rho}) \in P_n(\mathcal{Y}')} \zeta(\hat{w}, \hat{\rho}, y) 2^{-nD(\hat{w}|\hat{\rho})}, \]  

(154)

since

\[ w^n(y|x) = \hat{w}^n(y|x) 2^{-nD(\hat{w}|\hat{\rho})} \]

for any \((y, x) : \{p_{y,x}(a, b) = \hat{w}(a|b)\hat{\rho}(b) \quad \forall (a, b) \in \mathcal{Y} \times \mathcal{X} \} \) by [30] Lemma 2.3. Further notice that (153) and (154) imply

\[ w_n(y|u) \leq |P_n(\mathcal{Y}, \mathcal{X})| \zeta(\hat{w}(y), \hat{\rho}(y), y) 2^{-nD(\hat{w}(y)|\hat{\rho}(y))} \]

(155)

and

\[ \hat{w}_n(y|u) \geq \zeta(\hat{w}(y), \hat{\rho}(y), y) 2^{-nD(\hat{w}(y)|\hat{\rho}(y))}, \]

(156)

respectively. Using the fact [4] Lemma 2.1 that \(|P_n(\mathcal{Y}, \mathcal{X})| \leq (n + 1)^{|\mathcal{X}|} |\mathcal{Y}| \leq n^2 |\mathcal{X}| |\mathcal{Y}|\), we can arrive at (152) from (155) and (156).

**Lemma 32.** Let \((M, X, Y, P(\mathcal{Y}|\mathcal{X}))\) be a regular collection of DRVs, and \(\delta, \alpha \in \mathbb{R}_+, \) and real number \(\epsilon > \frac{2 |\mathcal{X}| |\mathcal{Y}|}{n} \log_2 n.\) If

\[ \Pr (|h(Y_{\hat{w}}|M) - \mathbb{H}(Y_{\hat{w}}|M|) > n\delta) < 2^{-na}, \]

(157)

for \(\hat{w} \in P(\mathcal{Y}|\mathcal{X})\), then

\[ \Pr \left( |h(Y_{\hat{w}}|M) - \mathbb{H}(Y_{\hat{w}}|M) - \hat{\delta} \right) < 2^{-na} + 2^{-n(\alpha - \epsilon)}, \]

(158)

where

\[ \hat{\delta} = (2 + 2^{-na} + 2^{-n(\alpha - \epsilon)})(\delta + \epsilon) + (2^{-na} + 2^{-n(\alpha - \epsilon)}) \left[ \log_2 |\mathcal{Y}| - \frac{2}{n} \log_2 (2^{-na} + 2^{-n(\alpha - \epsilon)}) \right], \]

for all \(w \in P(\mathcal{Y}|\mathcal{X})\) such that

\[ \sup_{\hat{w} \in P(\mathcal{Y}|\mathcal{X})} D(\hat{w}|w) \leq \epsilon - \frac{2 |\mathcal{X}| |\mathcal{Y}|}{n} \log_2 n. \]

\[ \left(159\right) \]

**Proof:** Given (157), consider any \(w \in P(\mathcal{Y}|\mathcal{X})\) that satisfies (159). In this case,

\[ h_{Y_{\hat{w}}|M}(y|m) - h_{Y_{\hat{w}}|M}(y|m) = \log_2 \frac{w_n(y|m)}{w_n(y|m)} \leq ne \]

(160)

for all \(m \in M\) and \(y \in \mathcal{Y}^n : \{w^n(y|m) > 0\}\) by Lemma 31. Because of (160), it must also follow that

\[ |h_{Y_{\hat{w}}|M}(y|m) - \mathbb{H}(Y_{\hat{w}}|M)| \leq n\delta + ne, \]

(161)

for all \(y \notin B^+(m) \cup B^-(m),\) where

\[ B^+(m) \triangleq \{ y : |h_{Y_{\hat{w}}|M}(y|m) - \mathbb{H}(Y_{\hat{w}}|M)| > n\delta \} \]

\[ B^-(m) \triangleq \{ y : \log_2 \frac{w_n(y|m)}{w_n(y|m)} < -ne \}. \]

Thus, we will have

\[ \Pr (|h_{Y_{\hat{w}}|M}(y|m) - \mathbb{H}(Y_{\hat{w}}|M)| > n(\delta + \epsilon)) < 2^{-n(\alpha - \epsilon)} + 2^{-ne} \]

(162)

if we can show that

\[ \Pr (Y_{\hat{w}} \in B^+(M)) < 2^{-n(\alpha - \epsilon)} \]

(163)
and
\[
\Pr \left( Y_w \in B^-(M) \right) < 2^{-n\epsilon}.
\] (164)

Equation (158) is then a direct result of combining (163) and Corollary 9 since \(\mathbb{H}(Y_w|M)\) is a constant positive real number.

The derivations of (163) and (164) though are straightforward. First, for (163),
\[
\Pr \left( Y_w \in B^*(M) \right)
= \sum_m P_M(m) \sum_{y \in B^*(m)} w_n(y|m)
\leq \sum_m P_M(m) \sum_{y \in B^*(m)} \tilde{w}_n(y|m) 2^{n\epsilon}
\leq 2^{n\epsilon} \Pr \left( Y_w \in B^*(M) \right) < 2^{-n(\alpha - \epsilon)}
\] (165)

where (165) is because of (160), and (166) is because \(\Pr \left( Y_w \in B^*(M) \right) < 2^{-n\alpha}\), which in turn is consequence of the hypothesis in (157). Second for (164), similarly,
\[
\Pr \left( Y_w \in B^-(M) \right) = \sum_m P_M(m) \sum_{y \in B^-(m)} w_n(y|m)
< \sum_m P_M(m) \sum_{y \in B^-(m)} \tilde{w}_n(y|m) 2^{-n\epsilon}
\leq 2^{-n\epsilon}
\] (167)

where (167) is because \(y \in B^-(m)\).

\[\text{□}\]

B. Proof of Theorem 13

Proof: Given an \(\epsilon \in \left( \frac{4|X| |Y|}{n} \log_2 n, 1 \right)\), we will create a finite set of distributions \(\mathcal{P}^{(\epsilon)}(Y|X)\) with cardinality
\[
|\mathcal{P}^{(\epsilon)}(Y|X)| \leq \left( |Y| \left( 1 + \frac{4|Y|^2}{\epsilon} \right) \right)^{|X| |Y|}
\] (168)

that can approximates \(\mathcal{P}(Y|X)\) in the following sense. For every \(w \in \mathcal{P}(Y|X)\), there exists a \(\tilde{w} \in \mathcal{P}^{(\epsilon)}(Y|X)\) such that
\[
\sup_{\tilde{p} \in \mathcal{P}^{(\epsilon)}(Y|X)} \mathcal{D}(w||\tilde{p}) \leq \epsilon - \frac{2|X||Y|}{n} \log_2 n.
\] (169)

Achieving this approximation, the theorem then follows directly from Lemma 12 with \(\hat{\mathcal{P}} = \mathcal{P}^{(\epsilon)}(Y|X)\).

To form \(\mathcal{P}^{(\epsilon)}(Y|X)\), we will first create a parameterized set of distributions over \(Y\), and then consider the union of these distributions over all possible parameters. We will use the union to approximate \(\mathcal{P}(Y)\). Finally this is extended to a set conditional distributions by observing that any \(w \in \mathcal{P}(Y|X)\) is simply an order collection of \(w_x \in \mathcal{P}(Y)\). To begin, let \(\tilde{\epsilon} = \frac{\epsilon}{4|Y|^2}\), and for each \(y \in Y\) construct the set of distributions \(\mathcal{P}^{(\epsilon)}(Y; y)\) such that contains all distributions in \(\mathcal{P}(Y)\) in the form:
\[
p(y) = \begin{cases} j_y \tilde{\epsilon} \text{ for some } j_y \in [0 : \lfloor 1/\tilde{\epsilon} \rfloor] & \text{if } \hat{y} \in Y \setminus \{y\} \\ 1 - \sum_{\hat{y} \notin Y \setminus \{y\}} p(\hat{y}) & \text{if } \hat{y} = y. \end{cases}
\]

Next, form
\[
\mathcal{P}^{(\epsilon)}(Y) \triangleq \bigcup_{y \in Y} \mathcal{P}^{(\epsilon)}(Y; y).
\]

Finally, we can define the approximating set as
\[
\mathcal{P}^{(\epsilon)}(Y|X) \triangleq \{ w \in \mathcal{P}(Y|X) : w(\cdot|x) \in \mathcal{P}^{(\epsilon)}(Y) \text{ for each } x \in X \}.
\]

Note that these definitions yield (168) since we have in progression
\[
|\mathcal{P}^{(\epsilon)}(Y; y)| \leq (1 + \lfloor 1/\tilde{\epsilon} \rfloor)^{|Y|}
|\mathcal{P}^{(\epsilon)}(Y)| \leq \sum_{y \in Y} |\mathcal{P}^{(\epsilon)}(Y; y)|
|\mathcal{P}^{(\epsilon)}(Y|X)| \leq |\mathcal{P}^{(\epsilon)}(Y)|^{|X|}.
\]
What remains is to establish (169), which can be done by providing upper bounds on the maximum of \( \log_2 \frac{w(y|x)}{w(y|x)} \), since clearly
\[
\sup_{\hat{w}, \hat{p}} \mathbb{D}_{\hat{w}}(w \| \hat{p}) \leq \max_{(y, x) : w(y|x) > 0} \log_2 \frac{w(y|x)}{w(y|x)}.
\] (170)

For every \( x \in \mathcal{X} \), let \( y_x \) denote a maximal element in \( \mathcal{Y} \) that \( w(y|x) = \max_{y \in \mathcal{Y}} w(y|x) \). Observe that for every \( w \in \mathcal{P}(\mathcal{Y} | \mathcal{X}) \) there exists a \( \hat{w} \in \mathcal{P}(\mathcal{Y} | \mathcal{X}) \) that has the following properties:
- \( \hat{w}(y|x) \geq w(y|x) \) for all \( (y, x) \neq (y_x, x) \), and
- \( \hat{w}(y_x|x) \geq w(y_x|x) - |\mathcal{Y}| \epsilon \) for all \( x \in \mathcal{X} \).

Hence
\[
\sup_{\hat{w}, \hat{p}} \mathbb{D}_{\hat{w}}(w \| \hat{p}) \leq \max_{(y, x)} \log_2 \frac{w(y_x|x)}{w(y_x|x) - |\mathcal{Y}| \epsilon} \leq \log_2 \frac{|\mathcal{Y}|-1}{|\mathcal{Y}|-1 - (4|\mathcal{Y}|)^{-1}\epsilon} \leq \frac{\epsilon}{2}
\] (172)
\[
\leq \epsilon - \frac{2|\mathcal{X}|^2}{n} \log_2 n
\] (173)
where (171) is obtained by combining the two aforementioned properties of \( \hat{w} \) and (170), while (172) is because \( w(y_x|x) \geq |\mathcal{Y}|^{-1} \), and (173) is because \( - \log_2 (1 - x) > 2x \) for \( x \in [0 : 1/2] \).

IX. CONCLUDING REMARKS

Our contribution is simply the construction of a DRV that provides information stability, and the examples demonstrating how such a result can be applied. This work is a self contained collection and refinement of our previous works [31], [32], [33], [26]. Perhaps more accurately, our current work is to our past work as Theseus’ final ship is to his starting ship. Every theorem and proof has been drastically changed as to provide results that could be more immediately applicable. In this regards we must thank all reviewers of our previous work, without whose adroit criticisms this work would not be possible.

There are multiple possible future directions this work may proceed in. As mentioned previous the error bounds can most likely be improved by simply finding an alternative to the blowing up lemma. They would be drastically improved if a replacement for Theorem 13 were to be found. Next, the work needs to be extended to consider continuous distributions. This seems as if it would be a natural extension since our methods are built similarly to those of the information spectrum. There are also some concerns regarding independent sources that must be addressed. In particular the provided DRV introduces correlation between independent \( M_{[1, l]} \), and thus care must be taken when applying these methods to models where achieving such correlation is impossible.

APPENDIX

A. Proof of Lemma \[3\]

**Proof:** Let \( \mathcal{B} = \{(y, u) \in \mathcal{Y} \times \mathcal{U} : |h_{Y|U}(y|u) - c| > \epsilon\} \) and \( Q = 1 \{ (Y, U) \in B \} \). From the hypothesis of the lemma, we have \( p_Q(1) < \mu < \frac{1}{2} \). The conditional entropy \( \mathbb{H}(Y|U) \) can always be expanded in the following manner:
\[
\mathbb{H}(Y|U) = \zeta_{Q=0} + \zeta_{Q=1}
\] (174)
where
\[
\zeta_{Q=0} = \sum_{y, u} p_{Y, U, Q}(y, u, 0) h_{Y|U}(y, 0|u)
\] (175)
\[
\zeta_{Q=1} = \sum_{y, u} p_{Y, U, Q}(y, u, 1) h_{Y|U}(y, 1|u).
\] (176)

Note that (175) and (176) result from that \( p_{Y, Q|U}(y, 0|u) = \begin{cases} p_{Y|U}(y|u) & \text{if } (y, u) \notin \mathcal{B} \\ 0 & \text{if } (y, u) \in \mathcal{B} \end{cases} \) and \( p_{Y, Q|U}(y, 1|u) = \begin{cases} p_{Y|U}(y|u) & \text{if } (y, u) \notin \mathcal{B} \\ 0 & \text{if } (y, u) \in \mathcal{B} \end{cases} \), respectively. Clearly
\[
(1 - \mu)(c - \epsilon) < (1 - p_Q(1)) (c - \epsilon) \leq \zeta_{Q=0} \leq c + \epsilon
\] (177)
follows by directly inserting the condition for \((y, u)\)’s inclusion in \(B\) into (175) and performing the summation. On the other hand,

\[
0 \leq \zeta_{Q=1} = p_Q(1) \mathbb{H}(Y|U, Q = 1) - \sum_u p_{Q|U}(1|u) p_U(u) \log_2 p_{Q|U}(1|u) \\
\leq p_Q(1) \mathbb{H}(Y|U, Q = 1) + \mathbb{H}(Q|U) \\
\leq \mu \log_2 \frac{|Y|}{\mu^2}
\]

(178)
due to \(\mu < \frac{1}{2}\).

Substituting (177), (178), and \(p_Q(1) < \mu\) into (174) yields

\[
(1 - \mu)(c - \epsilon) < \mathbb{H}(Y|U) < c + \epsilon + \mu \log_2 \frac{|Y|}{\mu^2}
\]

(179)
The proof can therefore be concluded from (179) by demonstrating that

\[
c \leq \epsilon - \log_2(1 - \mu) + \log_2 |Y| \leq \epsilon + \log_2 \frac{|Y|}{\mu}.
\]

Equation (180) can be verified by observing

\[
1 - \mu < p_Q(0) = \sum_{(y, u) \notin B} p_U(u) 2^{h(y|u)} \leq 2^{-c + \epsilon + \log_2 |Y|}
\]

(181)
where the last inequality is made by substituting in \(2^{-h(y|u)} \leq 2^{-c + \epsilon}\), since \((y, u) \notin B\), and then summing over all possible \((y, u)\). Solving for \(c\) in (181) gives the first inequality in (180), while the second inequality is because \(\mu < \frac{1}{2}\), which gives rise to \(- \log_2(1 - \mu) < - \log_2 \mu\).

**B. Proof of Corollary 13**

*Proof: Let us be given a regular collection \((M_{[1:l]}, X, Y_{P(Y|X)})\) and DRV \(T: (T, M_{[1:l]} \rightarrow X \rightarrow Y_{P(Y|X)})\). First we will use Theorem 13 to obtain a set \(\tilde{P} \subseteq \mathcal{P}(Y|X)\) which extends stability. Next we apply Theorems 11 and 12 to \((M_{[1:l]}, X, Y_{P})\) obtaining DRVs \(V\) and \(Q\) respectively. The DRV \(U\) described in the corollary is then constructed by setting \(U = (V, Q, T)\). From there we shall construct the set \(\tilde{U}\) and derive the properties of both \(U\) and \(\tilde{U}\).

To begin, let \(\varepsilon_n \triangleq n^{-4|Y|2/\varepsilon}\). By Theorem 13 there exists a set \(\tilde{P} \subseteq \mathcal{P}(Y|X)\), where

\[
|\tilde{P}| \leq \tilde{k} \triangleq \left(\frac{|Y|}{1 + \frac{4|Y|^2}{\varepsilon_n}}\right)^{|X|}\frac{|Y|}{|Y|}
\]

(182)
which has the following property; for all \(w \in \mathcal{P}(Y|X)\) there exists a corresponding \(\tilde{w}_w \in \tilde{P}\) such that if

\[
\Pr\left(|h(Y_{\tilde{w}_w}|M) - \mathbb{H}(Y_{\tilde{w}_w}|M) > n\tilde{\delta}\right) < 2^{-n\tilde{\alpha}}
\]

(183)
for some DRV \(M : (M \rightarrow X \rightarrow Y_{P(Y|X)})\) and positive real numbers \(\tilde{\delta}\) and \(\tilde{\alpha}\), then

\[
\Pr\left(|h(Y_w|M) - \mathbb{H}(Y_w|M) > n\tilde{\delta}\right) < 2^{-n\varepsilon_n + 2^{-n(\tilde{\alpha} - \varepsilon_n)}},
\]

(184)
where

\[
\tilde{\delta} = (2 + 2^{-n\varepsilon_n + 2^{-n(\tilde{\alpha} - \varepsilon_n)}})(\tilde{\delta} + \varepsilon_n)
\]

\[
+ (2^{-n\varepsilon_n + 2^{-n(\tilde{\alpha} - \varepsilon_n)}})(\log_2 |Y| - \frac{2}{n} \log_2 (2^{-n\varepsilon_n + 2^{-n(\tilde{\alpha} - \varepsilon_n)}})).
\]

Now for \((M_{[1:l]}, X, Y_{\tilde{P}})\), there exists:

- a DRV \(V: \{ |Y| \leq (2n^2 \log_2 |Y|)^{1/k} \ \ \ V \ll (X, M_{[1:l]}) \}, \)
- a real number \(\tilde{\delta} = O(-\sqrt{2\varepsilon_n \log_2 (2\varepsilon_n)})\), and
for each DRV $U : \{ U \gg V \}$, $\hat{w} \in \hat{P}$, and $j \in [1 : l]$ there exists a set $U_{\hat{w}|j} \subseteq U$ such that

$$p_U(U_{\hat{w}|j}) \geq 1 - 2^{-\frac{n}{2} \log_2 \hat{\tau}},$$

(185)

and

$$\Pr \left( |h(Y_{\hat{w}}|j,M_j,U) - H_U(Y_{\hat{w}}|M_j)| > n\hat{\delta} + 3h(U)|U = u \right) < 4 \cdot 2^{-2n\hat{\epsilon}_n}$$

(186)

for all $u \in U_{\hat{w}|j}$.

by Theorem[1][11] The existence of a $\delta = O(-\sqrt{\hat{\epsilon}_n \log_2 \hat{\epsilon}_n})$ so that

$$\Pr (|h(Y_w|j,M_j,U) - H_U(Y_w|M_j)| > n\hat{\delta} + 7h(U)|U = u) < 5 \cdot 2^{-2n\hat{\epsilon}_n}$$

(187)

for each $w \in P(Y|X)$ and $u \in U_{\hat{w}|j}$ follows since Equation (186) takes the form of Equation (183), and Equation (183) implies Equation (184). A verification of Equation (187) can be found in Appendix C.

Similarly for any DRVs $M_{[1:l]}$ there exists:

- DRV $Q : \{ |Q| \leq (n^2 + 1)^l \}$,
- a positive real number $\beta = O(n\hat{\epsilon}_n + n^22^{-n\hat{\epsilon}_n})$, and
- for each DRV $U : \{ U \gg Q \}$ and $j \in [1 : l]$, there exists sets $U_{j,(stable)} \subseteq U$ and $U_{j,(unstable)} \subseteq U$, such that

$$p_U(U_{j,(stable)} \cup U_{j,(unstable)}) \geq 1 - 2^{-n\hat{\epsilon}_n},$$

(188)

and

$$\Pr (|h(M_j|U) - H_U(M_j)| > \beta + 3\log_2 |U||U = u) < 2^{-n\hat{\epsilon}_n}$$

(189)

for all $u \in U_{j,(stable)}$, and

$$\Pr (h(M_j|U) - n^2 < -\beta - 3\log_2 |U||U = u) < 2^{-n\hat{\epsilon}_n}$$

(190)

for all $u \in U_{j,(unstable)}$.

by Theorem[1][12] Now set $U = (V,Q,T)$, and let us confirm the properties of $U$. First, $U \gg T$ is a direct consequence of the definition of $U$. Second,

$$\log_2 |U| = \log_2 |Y||Q||T|$$

(191)

$$\leq \log_2 |T| + 3l\log_2 n + l\log_2 (n^2 + 1) + (l\log_2 2\log_2 |Y|) \leq \log_2 |T| + l(3\log_2 + 4)\log_2 n + (l\log_2 2\log_2 |Y|)$$

$$= O(\log_2 |T| + l\log_2 n\log_2 \hat{\epsilon}_n),$$

(192)

with a detailed analysis of the order term appearing in Appendix C. Finally $(U,M_{[1:l]}) \rightsquigarrow X \rightsquigarrow Y_{P(Y|X)}$ since $Q \ll (X,M_{[1:l]}), V \ll (X,M_{[1:l]})$ and $(T,M_{[1:l]}) \rightsquigarrow X \rightsquigarrow Y_{P(Y|X)}$.

It is important to observe that Equation (187) applies since $U$ satisfies the properties just listed. Still Equation (187) depends on $h(U)$, and neither Equation (187) nor Equation (189) and (190) can bound the distance between $h(M_j)$ and $h(M_j|U)$. For this reason let $U$ be the set of all $u \in U$ such that

$$h(u) < n\hat{\epsilon}_n + \log_2 |U|$$

(193)

and for each $j \in [1 : l]$ let $U_j$ be the set of all $u \in U$ such that

$$\Pr (|h(M_j|U) - h(M_j)| > 2n\hat{\epsilon}_n + \log_2 (|U| + 1)|U = u) < 2^{-n\hat{\epsilon}_n}.$$
For these sets we have

\[ p_U(\mathcal{U}) \geq 1 - 2^{-n\varepsilon_n} \tag{195} \]

since

\[ 1 = \sum_u p_U(u) = p_U(\mathcal{U}) + \sum_{u \notin \mathcal{U}} 2^{-n\varepsilon_n} |\mathcal{U}|^{-1} \leq p_U(\mathcal{U}) + 2^{-n\varepsilon_n}. \tag{196} \]

Likewise

\[ p_U(\mathcal{U}_j) \geq 1 - 2^{-n\varepsilon_n} \tag{198} \]

which follows by Lemma 30 and Markov’s inequality.

Now we are in a position to construct the set \( \mathcal{U} \) from the corollary statement and prove the associated properties. In particular

\[ \mathcal{U} \triangleq \bigcap_{w \in \mathcal{P}, j \in [1:l]} \left( \mathcal{U}_{\tilde{w}, i} \cap \left( \mathcal{U}_j \cap \mathcal{U}_{\text{stable}} \right) \cap \mathcal{U}_j \cap \mathcal{U} \right), \tag{199} \]

and note that for each \( j \in [1:l] \) and each \( w \in \mathcal{P}(Y|X) \) one of the following two cases must occur; either \( u \in \mathcal{U}_{\tilde{w}, i} \cap \mathcal{U}_j \cap \mathcal{U} \) or \( u \in \mathcal{U}_{\tilde{w}, i} \cap \mathcal{U}_j \cap \mathcal{U} \). In the first case

\[
\Pr \left( \begin{array}{l}
|h(Y_w|M_j, U) - \mathbb{E}_U(Y_w|M_j)| > n \nu_n \\
|h(M_j|U) - \mathbb{E}_U(M_j)| > n \nu_n \\
|h(M_j) - h(M_j|U)| > n \nu_n
\end{array} \bigg| U = u \right) < 8 \cdot 2^{-n\varepsilon_n}, \tag{200}
\]

where

\[
\nu_n = \max \left( \delta + \frac{7h(U)}{n}, \frac{\beta + 3 \log_2|\mathcal{U}|}{n} \log_2(|\mathcal{U}| + 1) \right)
\]

\[
\leq \max \left( \delta + \frac{7 \varepsilon_n}{n}, \frac{\beta + 3 \log_2|\mathcal{U}|}{n} \log_2(|\mathcal{U}| + 1) \right) \tag{201}
\]

by Equations (187), (189), (193) and (194). Similarly in the second case

\[
\Pr \left. \begin{array}{l}
|h(Y_w|M_j, U) - \mathbb{E}_U(Y_w|M_j)| > n \nu_n \\
h(M_j|U) - n^2 < -n \nu_n
\end{array} \bigg| U = u \right) < 8 \cdot 2^{-n\varepsilon_n} \tag{202}
\]

by Equations (187), (190), (193) and (194). Furthermore

\[
\nu_n = O(n^{-1} \log_2 |\mathcal{T}| - l \sqrt{n} \log_2 \varepsilon_n) \tag{203}
\]

as detailed in Appendix C7 and

\[
\Pr \left( (Y_w, M_{[1:l]}) \notin \mathcal{D}(\text{stable}, (M_j)(U, w; \nu_n)) | U = u \right) < 8 \cdot 2^{-n\varepsilon_n} \tag{204}
\]

in the first case, while

\[
\Pr \left( (Y_w, M_{[1:l]}) \notin \mathcal{D}(\text{saturate}, (M_j)(U, w; \nu_n)) | U = u \right) < 8 \cdot 2^{-n\varepsilon_n}, \tag{205}
\]

in the second case.

Finally

\[
p_U(\mathcal{U}) \geq 1 - t \log_2 \frac{8}{t} - (2l + 1)2^{-n\varepsilon_n} \tag{206}
\]

follows by combining Equations (185), (188), (195), (198) and the union bound. A more detailed verification of the order term in Equation (206) can be found in Appendix C8.
C. Order terms

Before deriving the order terms explicitly, a few inequalities useful inequalities need to be derived for \( \alpha \in \left( \frac{\log_2 n}{n}, \frac{1}{8 \ln 2} \right) \), and \( n \geq 27 \). Letting \( \alpha_- = \frac{\log_2 n}{n} \) and \( \alpha_+ = \frac{1}{8 \ln 2} \) these inequalities are as follows:

\[
2^{-n\alpha} \leq 2^{-n\alpha_-} \leq \frac{1}{n} \\
\frac{1}{n} \leq -\sqrt{\alpha \log_2(\alpha)} \cdot \frac{1}{-n\sqrt{\alpha \log_2(\alpha_-)}} \\
\leq -\sqrt{\alpha \log_2(\alpha)} \cdot \frac{1}{\sqrt{n \log_2 n}} \\
\leq -\sqrt{\alpha \log_2(\alpha)} \cdot \frac{1}{2\sqrt{2} \ln 2 \log_2(8 \ln 2)} \\
\leq -\sqrt{\alpha \log_2(\alpha)} \cdot \frac{1}{\log_2(8 \ln 2)}. 
\]

Likewise if \( \varepsilon_n = n^{-\frac{1}{2\log_2(\alpha)}} \), \( |\mathcal{X}| \geq 2 \), \( |\mathcal{Y}| \geq 2 \) and \( n \geq 27 \) then

\[
\tilde{k} = \left( |\mathcal{Y}| \left( 1 + \frac{|\mathcal{Y}|^2}{\varepsilon_n} \right) \right)^{|\mathcal{X}| |\mathcal{Y}|} \\
\leq (|\mathcal{Y}| + 4|\mathcal{Y}|^3)^{|\mathcal{X}| |\mathcal{Y}|} \varepsilon_n^{-1} |\mathcal{X}| |\mathcal{Y}| n^{-1} \\
= (|\mathcal{Y}| + 4|\mathcal{Y}|^3)^{|\mathcal{X}| |\mathcal{Y}|} n^{-1} \varepsilon_n^{-1} \\
\leq (\sqrt{n \log_2 n} \varepsilon_n) \frac{2(|\mathcal{X}| |\mathcal{Y}| + 1)}{\log_2 n} \\
2^{-n\varepsilon} \leq (\sqrt{n \log_2 n} \varepsilon_n) \frac{(|\mathcal{X}| |\mathcal{Y}| + 1)}{2^n \log_2 n} \\
-\varepsilon_n \log_2 \varepsilon_n \leq -\varepsilon_n \log_2 \varepsilon_n \\
n2^{-n\varepsilon} \leq (\sqrt{n \log_2 n} \varepsilon_n) n^{-1} \frac{(|\mathcal{X}| |\mathcal{Y}| + 1)}{\log_2 n} \\
\leq (\sqrt{n \log_2 n} \varepsilon_n) (|\mathcal{X}| |\mathcal{Y}| + 1) n^{-1} 2^{-n} \\
< (\sqrt{n \log_2 n} \varepsilon_n) (|\mathcal{X}| |\mathcal{Y}| + 1) 
\]

1) Equation (110): In specific the combination of Equations (108) and (109) provides

\[
\Pr \left( h(Y|U) - s^* \lambda_n < n\delta + \lambda + h(U) \right) < 3 \cdot 2^{-n\alpha}. 
\]

Hence we need to show that \( \tilde{\delta} + n^{-1} \lambda = O(\sqrt{\alpha \log_2(\alpha)}). \) Towards this goal, let \( B \) be a Bernoulli random variable with parameter \( \sqrt{\alpha 2 \ln 2} \), and recall that

\[
\tilde{\delta} + \frac{\lambda}{n} = \tau_n (2^{-n\alpha}, 2^{-n\alpha}) + \alpha + 7.19|\mathcal{Y}| \frac{\log_2 n}{n} + \frac{\lambda}{n} \\
\leq \tau_n (2^{-n\alpha}, 2^{-n\alpha}) + \alpha + 11.19|\mathcal{Y}| \frac{\log_2 n}{n}, 
\]

and

\[
\tau_n (2^{-n\alpha}, 2^{-n\alpha}) \leq \mathcal{H}(B) + \sqrt{\alpha 2 \ln 2 \log_2 |\mathcal{Y}|} \\
\leq -\sqrt{\alpha 2 \ln 2 \log_2 \alpha} + \sqrt{\alpha 2 \ln 2 \log_2 |\mathcal{Y}|} 
\]

since the entropy of a Bernoulli random variable with parameter \( x \) is less than \(-2x \log_2 x\) for all \( x < \frac{1}{2} \) (also recall \( \alpha < (8 \ln 2)^{-1} \)). Combining Equations (220) and (222) with Equations (210)-(212) gives

\[
\tilde{\delta} + n^{-1} \lambda \leq -\mu \sqrt{\alpha \log_2(\alpha)} 
\]
follows by Corollary 9. But \( \rho \) where 

\[
\beta \quad \text{where} \quad \tilde{\delta} = 14
\]

\[
\mu \triangleq \sqrt{2 \ln 2} + \frac{\sqrt{2 \ln 2 \log |Y|}}{\log_2 (8 \ln 2)} + \frac{1}{2 \sqrt{2 \ln 2 \log_2 (8 \ln 2)}} + \frac{11.19|Y|}{\sqrt{n \log_2 n}}.
\]  

(224)

Clearly \( \mu \) has a maximum upper bound in terms of \(|Y|\) and thus \( \tilde{\delta} + n^{-1} \lambda = O(\sqrt{\alpha \log_2 \alpha}) \).

2) Equation (134): Observe that there exists a positive number \( \mu \) such that

\[
\delta = -\mu \sqrt{\alpha \log_2 \alpha} + n + 1 \quad \text{and} \quad \tilde{\delta} = O(\sqrt{\alpha \log_2 \alpha}).
\]

(225)

since \( \tilde{\delta} = \sqrt{\alpha \log_2 \alpha} \). It therefore follows that \( \delta = O(\sqrt{\alpha \log_2 \alpha}) \).

3) Equation (142): In particular the combination gives

\[
\Pr \left( \left| h(Y_i|U, M_j) - \mathbb{H}(Y_i|U, M_j) \right| U = u \right) < 4 \cdot 2^{-\alpha},
\]

(226)

where

\[
\delta = (2 + 4 \cdot 2^{-\alpha})(\max_{i,j} \delta, i, j) + 4 \cdot 2^{-\alpha} (\log_2 |Y|) - 4 + 2 \alpha.
\]

(227)

In turn then we have

\[
\Pr \left( \left| h(Y_i|U, M_j) - \mathbb{H}(Y_i|U, M_j) \right| U = u \right) < 4 \cdot 2^{-\alpha},
\]

(228)

and

\[
\delta \leq 3 (\max_{i,j} \delta, i, j) + \frac{4}{n} (\log_2 |Y|).
\]

(229)

since \( \alpha \in (n^{-1} \log_2 n, (8 \ln 2)^{-1}) \) and \( n \geq 27 \) imply \( 2^{-\alpha} \leq \frac{1}{n} \leq \frac{1}{24} \). To finish the proof note that \( \delta = O(\sqrt{\alpha \log_2 \alpha}) \).

4) Equation (147): Given Equation (149),

\[
\Pr \left( \left| \mathbb{H}(M_j|U) - h(M_j|U) \right| > \tilde{\beta} \left| U = u \right) < 2^{-\tau}
\]

(230)

where

\[
\tilde{\beta} = (2 + 2^{-\rho})(2 \rho + \log_2 (|U| + 1) + 1) + 2^{-\rho} \psi + 2^{-\rho} 2 \rho.
\]

(231)

follows by Corollary 9. But \( \rho \geq 1 \) and \( |U| \geq 1 \), and hence

\[
\tilde{\beta} < 8 \rho + 3 \log_2 |U| + 6 + 2^{-\rho} \psi \leq 14 \rho + 2^{-\rho} \psi + 3 \log_2 |U|.
\]

(232)

Thus

\[
\Pr \left( \left| \mathbb{H}(M_j|U) - h(M_j|U) \right| > \beta + 3 \log_2 |U| \left| U = u \right) < 2^{-\rho}
\]

(233)

where \( \beta = 14 \rho + 2^{-\rho} \psi = O(\rho + 2^{-\rho} \psi) \).

5) Equation (187): The specific bound guaranteed is

\[
\Pr \left( \left| h(Y_w|M_j, U) - \mathbb{H}_{U}(Y_w|M_j) \right| > n \tilde{\delta} \left| U = u \right) \leq 5 \cdot 2^{-n \varepsilon_n},
\]

(234)

where

\[
\tilde{\delta} = (2 + 5 \cdot 2^{-n \varepsilon_n}) \left( \tilde{\delta} + \frac{3 \log_2 |Y|}{n} + \varepsilon_n \right)
\]

(235)

\[
+ 5 \cdot 2^{-n \varepsilon_n} \left( \log_2 |Y| - \frac{2 \log_2 5}{n} + 2 \varepsilon_n \right).
\]

Notice that

\[
5 \cdot 2^{-n \varepsilon_n} = 5 \cdot 2^{-n \left( 1 - \frac{\log 2}{\log 5} \right)} \leq 5 \cdot 2^{-n \frac{4}{5}} < \frac{1}{3}.
\]

(236)
and hence
\[
\hat{\delta} \leq \frac{7}{3} \left( \delta + \varepsilon_n + \frac{3h(U)}{n} \right) + \frac{2\varepsilon_n}{3} + 5 \cdot 2^{-n\varepsilon_n} \log_2 |Y|,
\] (237)
since \( n \geq 27 \) and \(|\mathcal{Y}|, |\mathcal{X}| \geq 2 \). Thus
\[
\Pr \left( |h(Y_w|M_j,U) - \mathbb{E}_U(Y_w|M_j)| > n\delta + 7\log_2 |U| \right) = u \leq 5 \cdot 2^{-n\varepsilon_n},
\] (238)
for
\[
\delta = \frac{7}{3} \hat{\delta} + 3\varepsilon_n + 5 \cdot 2^{-n\varepsilon_n} \log_2 |Y|.
\]
Furthermore \( \delta = O(-\sqrt{n \log_2 \varepsilon_n}) \) because of Equations (214) and (215) and because \( \hat{\delta} = O(-\sqrt{n \log_2 \varepsilon_n}) \).

6) Equation (192): Begin by observing there exists a positive real number \( \mu \) such that
\[
\log_2 |U| \leq \log_2 |T| + l(3\mu n\varepsilon_n + 4) \log_2 n + l\mu n\varepsilon_n \log_2 (2 \log_2 |Y|),
\] (239)
since \( \hat{k} = O(n\varepsilon_n) \). Factoring out \(-ln\varepsilon_n \log_2 \varepsilon_n\) results in
\[
\log_2 |U| \leq \log_2 |T| - \hat{k} n\varepsilon_n \log_2 \varepsilon_n,
\] (240)
where
\[
\hat{k} = 3\mu \frac{-\log_2 n}{-\log_2 \varepsilon_n} + 4 \frac{\log_2 n}{-n\varepsilon_n \log_2 \varepsilon_n} + \mu \frac{\log_2 (2 \log_2 |Y|)}{-\log_2 \varepsilon_n}
= (|\mathcal{X}| |\mathcal{Y}| + 1) \left( 3\mu + 4n^{-1/2} |X||Y|^{-1} + \mu \frac{\log_2 (2 \log_2 |Y|)}{\log_2 n} \right).
\] (241)
Clearly then
\[
\log_2 |U| = O(\log_2 |T| - ln\varepsilon_n \log_2 \varepsilon_n).
\] (242)

7) Equation (204): First note that \( \nu_n \) is the maximum of three different terms. If we show that each of these terms is \( O(n^{-1} \log_2 |T| - l\sqrt{\varepsilon_n \log_2 \varepsilon_n}) \) then it must also follow that \( \nu_n = O(n^{-1} \log_2 |T| - l\sqrt{\varepsilon_n \log_2 \varepsilon_n}) \). First
\[
\delta + 7\varepsilon_n + \frac{7}{n} \log_2 |U| = O(-\sqrt{\varepsilon_n \log_2 \varepsilon_n}) + O(n^{-1} \log_2 |T| - ln\varepsilon_n \log_2 \varepsilon_n)
\leq O(n^{-1} \log_2 |T| - l\sqrt{\varepsilon_n \log_2 \varepsilon_n}),
\] (243)
by Equations (214) and (216), and because \( \delta = O(-\sqrt{\varepsilon_n \log_2 \varepsilon_n}) \) and \( \log_2 |U| = O(\log_2 |T| - ln\varepsilon_n \log_2 \varepsilon_n) \). Next
\[
\beta + 3 \log_2 |U| = O(\varepsilon_n + n2^{-n\varepsilon_n}) + O(n^{-2} \log_2 |T| - ln^{-1}\varepsilon_n \log_2 \varepsilon_n)
\leq O(n^{-1} \log_2 |T| - l\sqrt{\varepsilon_n \log_2 \varepsilon_n})
\] (244)
by Equations (214) and (213), and because \( \log_2 |U| = O(\log_2 |T| - ln\varepsilon_n \log_2 \varepsilon_n) \). Finally
\[
2\varepsilon_n + \frac{1}{n} \log_2 (|U| + 1) \leq 2\varepsilon_n + 1 + \frac{1}{n} \log_2 |U|
\leq O(n^{-1} \log_2 |T| - l\sqrt{\varepsilon_n \log_2 \varepsilon_n})
\] (245)
by Equation (218) and because \( \log_2 |U| = O(\log_2 |T| - ln\varepsilon_n \log_2 \varepsilon_n) \). Since all three terms are \( O(n^{-1} \log_2 |T| - l\sqrt{\varepsilon_n \log_2 \varepsilon_n}) \) it also follows that
\[
\nu_n = O(n^{-1} \log_2 |T| - l\sqrt{\varepsilon_n \log_2 \varepsilon_n}),
\] (246)
8) Equation (208): First, clearly,

$$(2l + 1)2^{-n\epsilon_n} = O(2^{-n\epsilon_n}),$$  \hspace{1cm} (247)

and on the other hand

$$\tilde{k}2^{-\frac{1}{2}\log_2 \frac{1}{n}} \leq (|\mathcal{Y}| + 4|\mathcal{Y}|^3)^{|X||\mathcal{Y}|n^{\epsilon_n}}2^{-\frac{1}{2}\log_2 \frac{1}{n}}$$  \hspace{1cm} (248)

$$\leq (|\mathcal{Y}| + 4|\mathcal{Y}|^3)^{|X||\mathcal{Y}|n^{\epsilon_n}2^{-\frac{1}{2}\log_2 \frac{1}{n}}}$$  \hspace{1cm} (249)

$$= 8(|\mathcal{Y}| + 4|\mathcal{Y}|^3)^{|X||\mathcal{Y}|2^{-\frac{1}{2}(\frac{1}{n} - 1)\log_2 \frac{1}{n}}}$$  \hspace{1cm} (250)

$$\leq 8(|\mathcal{Y}| + 4|\mathcal{Y}|^3)^{|X||\mathcal{Y}|2^{-n\epsilon_n}}.$$  \hspace{1cm} (251)

The summation of the two terms is therefore $O(2^{-n\epsilon_n})$.

REFERENCES

[1] C. E. Shannon, “A mathematical theory of communication,” Bell system technical journal, vol. 27, no. 3, pp. 379–423, 1948.
[2] R. Dobrushin, “A general formulation of shannons main theorem in information theory,” Amer. Math. Soc. Trans., vol. 33, pp. 323–348, 1963.
[3] M. S. Pinsker, “Information and information stability of random variables and processes,” 1960.
[4] I. Csiszár and J. Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems. Cambridge University Press, 2nd ed., 2011.
[5] R. Fano, Class notes for transmission of information, course 6.574, MIT, Cambridge, MA, 1952.
[6] L. Lai, H. El Gamal, and H. V. Poor, “Authentication over noisy channels,” Trans. Info. Theory, vol. 55, no. 2, pp. 906–916, 2009.
[7] T. S. Han and S. Verdú, “Beyond the blowing-up lemma: Optimal second-order converses via reverse hypercontractivity,” 2010.
[8] P. Gács and J. Körner, “Common information is far less than mutual information,” Problems of Control and Information Theory, vol. 2, no. 2, pp. 149–162, 1973.
[9] R. Ahlswede, P. Gács, and J. Körner, “Bounds on conditional probabilities with applications in multi-user communication,” Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, vol. 34, no. 2, pp. 157–177, 1976.
[10] G. Margulis, “Probabilistic characteristics of graphs with large connectivity,” Problemy Peredači Informacii, vol. 10, no. 2, pp. 101–108, 1974.
[11] K. Marton, “A simple proof of the blowing-up lemma,” IEEE Transactions on Information Theory, vol. 32, pp. 445–446, May 1986.
[12] M. Raginsky and I. Sason, “Concentration of measure inequalities in information theory, communications and coding,” CoRR, vol. abs/1212.4663, 2012.
[13] J. Liu, Information Theory from A Functional Viewpoint. PhD thesis, 2017.
[14] J. Liu, R. van Handel, and S. Verdú, “Beyond the blowing-up lemma: Optimal second-order converses via reverse hypercontractivity,” 2017.
[15] R. Ahlswede and G. Dueck, “Every bad code has a good subcode: A local converse to the coding theorem,” Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, vol. 34, no. 2, pp. 179–182, 1976.
[16] J. Körner and K. Marton, “Images of a set via two channels and their role in multi-user communication,” IEEE Trans. Info. Theory, vol. 23, no. 6, pp. 751–761, 1977.
[17] J. Körner and K. Marton, “General broadcast channels with degraded message sets,” IEEE Trans. Info. Theory, vol. 23, pp. 60–64, Jan. 1977.
[18] G. Dueck, “The strong converse to the coding theorem for the multiple-access channel,” J. Comb. Inform. Syst. Sci., vol. 6, no. 3, pp. 187–196, 1981.
[19] T. S. Han and S. Verdú, “Generalizing the Fano inequality,” Trans. Info. Theory, vol. 40, no. 4, pp. 1247–1251, 1994.
[20] J. Wolfowitz, “The coding of messages subject to chance errors,” Illinois Journal of Mathematics, vol. 1, pp. 591–606, Dec. 1957.
[21] S. Verdú and T. S. Han, “A general formula for channel capacity,” IEEE Trans. Info. Theory, vol. 40, pp. 1147–1157, July 1994.
[22] Y. Polyanskiy, H. V. Poor, and S. Verdú, “Channel coding rate in the finite blocklength regime,” IEEE Trans. Info. Theory, vol. 56, no. 5, pp. 2307–2359, 2010.
[23] A. D. Wyner, “The wire-tap channel,” The Bell System Technical Journal, vol. 54, pp. 1355–1387, Oct 1975.
[24] I. Csiszár and J. Körner, “Broadcast channels with confidential messages,” IEEE Transactions on Information Theory, vol. 24, pp. 339–348, May 1978.
[25] A. Gamal and Y. Kim, Network Information Theory. Cambridge University Press, 2011.
[26] E. Graves and T. F. Wong, “Wiretap channel capacity: Secrecy criteria, strong converse, and phase change,” in Int. Sym. Info. Theory, pp. 744–748, June 2017.
[27] O. Gungor and C. E. Koksal, “On the basic limits of rf-fingerprint-based authentication,” Trans. Info. Theory, vol. 62, no. 8, pp. 4523–4543, 2016.
[28] G. J. Simmons, “Authentication theory/coding theory,” in Advances in Cryptology, Proceedings of CRYPTO ’84, Santa Barbara, California, USA, August 19-22, 1984, Proceedings, pp. 411–431, 1984.
[29] U. Maurer, “Authentication theory and hypothesis testing,” IEEE Transactions on Information Theory, vol. 46, pp. 1350–1356, July 2000.
[30] I. Csiszár, P. C. Shields, et al., “Information theory and statistics: A tutorial,” Foundations and Trends® in Communications and Information Theory, vol. 1, no. 4, pp. 417–528, 2004.
[31] E. Graves and T. F. Wong, “Equalizing the achievable exponent region to the achievable entropy region by partitioning the source,” in Proc. 2014 IEEE Int. Symp. Info. Theory (ISIT), pp. 1346–1350, IEEE, 2014.
[32] E. Graves and T. F. Wong, “Equal-image-size source partitioning: Creating strong fano’s inequalities for multi-terminal discrete memoryless channels,” arXiv preprint arXiv:1512.00824, 2015.
[33] E. Graves and T. F. Wong, “Information stabilization of images over discrete memoryless channels,” in IEEE Int. Symp. Info. Theory, pp. 2619–2623, IEEE, 2016.