Synthesis and Optimization of Reversible Circuits for Homogeneous Boolean Functions

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Abstract

Homogenous Boolean function is an essential part of any cryptographic system. The ability to construct an optimized reversible circuits for homogeneous Boolean functions might arise the possibility of building cryptographic system on novel computing paradigms such as quantum computers. This paper shows a factorization algorithm to synthesize such circuits.

1 Introduction

Reversible logic \cite{4,11} is one of the hot areas of research. It has many applications in quantum computation \cite{13,23}, low-power CMOS \cite{8,31} and many more. Synthesis and optimization of reversible circuits cannot be done using conventional ways \cite{29}.

The design and analysis of Boolean based devices has many applications in engineering and science, e.g. cryptography. Massive computation power is required as the complexity and so the strength of encryption algorithms

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increase. Secure encryption algorithms involves designing two elementary blocks: P-box and S-box \[25\]. The S-Box theory is the design and analysis of Boolean functions that have certain desirable cryptographic properties such as balance, symmetry and high nonlinearity \[26\]. Synthesis and optimization of Boolean systems on non-standard computers that promise to do computation more powerfully \[28\] than classical computers, such as quantum computers, is an essential aim in the exploration of the benefits that may be gain from such systems.

A lot of work has been done trying to find an efficient reversible circuit for an arbitrary reversible function. Reversible truth table can be seen as a permutation matrix of size \(2^n \times 2^n\). In one of the research directions, it was shown that the process of synthesizing linear reversible circuits can be reduced to a row reduction problem of \(n \times n\) non-singular matrix \[24\]. Standard row reduction methods such as Gaussian elimination and LU-decomposition have been proposed \[5\]. In another research direction, search algorithms and template matching tools using reversible gates libraries have been used \[10, 17, 18, 20, 21\]. These will work efficiently for small circuits. Benchmarks for reversible circuits have been established \[9, 7\].

Implementing Boolean functions with \(n\) inputs and a single output as reversible circuits using some of the above methods is not possible for certain classes of Boolean functions \[27\], where the inputs are required to stay unchanged as required by many quantum algorithms \[12, 33\], where further operations will be applied on the inputs based on the output of the Boolean function. For example, representing the truth table of a Boolean function as a non-singular \(n \times n\) matrix may not be possible in some cases. Implementing Boolean function as a reversible function using search algorithms could be unnecessarily exhaustive, since we can immediately drop half the reversible truth table and keep track only for the changes to the single output, since no inputs will be changed. Recently, there have been few efforts to find methods to create efficient Boolean reversible circuits. A method was proposed where it used a ROM-based model such that the inputs might not be changed even during an intermediate stage \[30\]. Using this model will require an exponential number of ROM calls. In \[32\], it was shown that there is a direct correspondence between reversible Boolean operations and certain forms of classical logic known as Reed-Muller expansions. This arises the possibility of handling the problem of synthesis and optimization of reversible Boolean logic within the field of Reed-Muller logic. In another research direction \[22\], it was suggested that fixed polarity Reed-Muller expansions (FPRM) \[2\] can
be used with binary decision diagrams (BDD) in an iterative algorithm to
generate reversible circuits for simple incompletely specified Boolean func-
tions with less than 10 variables. A method proposed in [10] used a modified
version of Karnaugh maps and depends on a clever choice of certain minterms
to be used in the minimization process. However, this algorithm may have
poor scalability because of the usage of Karnaugh maps. Another method is
given in [14], where a very useful set of transformations for Boolean quant-
num circuits was shown. In this method, extra auxiliary bits are used in the
construction that will increase the hardware cost.

In this paper, a simple algorithm for synthesis and optimization by fac-
torization of homogeneous Boolean functions of $n$ variables with degree $\leq 3$
will be presented. The algorithm highly optimizes the quantum cost and the
number of gates of the reversible circuit. The structure of the paper is as
follows: Section 2 reviews the necessary background. Section 3 presents two
methods for the reversible construction of homogeneous Boolean functions
where the experimental results will be presented. The paper ends up with a
conclusion in Section 4.

2 Background

2.1 Reversible Boolean Function

A Boolean function is a function that takes $n$ Boolean inputs and generates
a single Boolean output.

**Definition 2.1 (Boolean Function)**

Any Boolean function $f$ with $n$ variables $f : \{0,1\}^n \rightarrow \{0,1\}$ can be
represented as a Positive Polarity Reed-Muller (PPRM) expansion as follows [12],

$$f(x_{n-1},...,x_1,x_0) = \bigoplus_{i=0}^{2^n-1} b_i \varphi_i,$$  \hspace{1cm} (1)

where,

$$\varphi_i = \prod_{k=0}^{n-1} (x_k)^{i_k},$$  \hspace{1cm} (2)
where \(x_k, b_i \in \{0, 1\}\) and \(i_k\) represent the binary digits of \(i\). \(\varphi_i\) are known as product terms (minterms) and \(b_i\) determine whether a minterm is present or not. \(\oplus\) means that the arguments are subject to Boolean operation exclusive-OR (XOR) and multiplication is assumed to be the AND operation.

For example, the set of all 3-inputs Boolean functions can be represented as:

\[
f(x_2, x_1, x_0) = b_0 \oplus b_1 x_0 \oplus b_2 x_1 \oplus b_3 x_1 x_0 \oplus b_4 x_2 \oplus b_5 x_2 x_0 \oplus b_6 x_2 x_1 \oplus b_7 x_2 x_1 x_0. \tag{3}
\]

Setting \(b_i\)’s to 0 or 1 gives different 3-inputs Boolean functions.

### 2.2 Reversible Boolean Circuits

In building a reversible Boolean circuit for a given Boolean function with \(n\) variables, an \(n + 1 \times n + 1\) reversible circuit will be used, where the extra auxiliary bit will be initialize to zero, to hold the result of the Boolean function at the end of the computation. \(CNOT\) gate is the main primitive gate that will be used in building the circuit. \(CNOT\) gate is defined as follows:

**Definition 2.2 (CNOT gate)**

\(CNOT(x_{n-1}, x_{n-2}, \ldots, x_0; f)\) is a reversible gate with \(n + 1\) inputs \(x_{n-1}, x_{n-2}, \ldots, x_0\) (known as control bits) and \(f_{\text{in}}\) (known as target bit) and \(n + 1\) outputs \(y_{n-1}, y_{n-2}, \ldots, y_0\) and \(f_{\text{out}}\). The operation of the \(CNOT\) gate is defined as follows,

\[
y_i = x_i, \text{ for } 0 \leq i \leq n - 1,
\]

\[
f_{\text{out}} = f_{\text{in}} \oplus x_{n-1} x_{n-2} \ldots x_0, \tag{4}
\]

i.e. the target bit will be flipped if and only if all the control bits are set to 1. Some special cases of the general \(CNOT\) gate have their own names, \(CNOT\) gate with no control bits is called \(NOT\) gate (Figure 1b), where the bit which will be flipped unconditionally. \(CNOT\) gate with one control bit is called \(Feynman\) gate (Figure 1c). \(CNOT\) gate with two control bits is called \(Toffoli\) gate (Figure 1d).
Figure 1: CNOT gates. The back circle • indicates the control bits, and the symbol ⊕ indicates the target bit. (a) CNOT gate with n control bits. (b) CNOT gate with no control bits. (c) CNOT gate with one control bit. (d) CNOT gate with two control bits.

2.3 Quantum Cost

Quantum cost is a term appears in the literature to refer to the technological cost of building CNOT gates. The total quantum cost of a CNOT-based reversible circuit is subject to optimization as well as the number of CNOT gates used in the circuit. Quantum cost based primarily on the number of control bits per CNOT gates. Quantum cost refers to the number of elementary operations required to build the CNOT gate [3, 19]. The calculation of quantum cost for the circuits in this paper will be based on the cost table available in [7]. The state-of-art shows that both CNOT(x_i) and CNOT(x_i; f) have quantum cost = 1, CNOT(x_i, x_j; f) has a quantum cost = 5, and CNOT(x_i, x_j, x_k; f) has a quantum cost = 13. Interaction of such gates in different ways may give a total quantum cost of the circuit less than the sum of the quantum cost of each CNOT gate in the circuit.

2.4 Homogeneous Boolean Functions

Homogeneity and nonlinearity are important properties of Boolean functions when used in cryptographic algorithms. Boolean functions with the highest possible nonlinearity are called homogeneous bent functions [25, 26].

Definition 2.3 (Homogeneous Boolean Function)

A Boolean function f : {0, 1}^n \rightarrow {0, 1} is homogeneous of degree k if it can be represented as follows [12],

\[ f(x_{n-1}, \ldots, x_1, x_0) = \bigoplus_{0 \leq i_1 \leq \ldots \leq i_k \leq n-1} b_{i_1 \ldots i_k} x_{i_1} \ldots x_{i_k}, \quad (5) \]
where each term \( x_{i_1} \ldots x_{i_k} \), \( b_{i_1 \ldots i_k} \) is a product of precisely \( k \) variables.

For example, it was found that there are 20 distinct minterms of degree 3 in a Boolean function of six variables and hence there are \( 2^{20} \) possible homogeneous Boolean function of degree 3 on six variables. Within this space, there are 30 homogeneous bent functions with 16 minterms. For example, a representative of a homogeneous bent function of degree 3 on six variables is as follows [26],

\[
\begin{align*}
f &= x_0x_1x_2 \oplus x_0x_1x_3 \oplus x_0x_1x_4 \oplus x_0x_1x_5 \\
&\quad \oplus x_0x_2x_3 \oplus x_0x_2x_5 \oplus x_0x_3x_4 \oplus x_0x_4x_5 \\
&\quad \oplus x_1x_2x_3 \oplus x_1x_2x_4 \oplus x_1x_3x_5 \oplus x_1x_4x_5 \\
&\quad \oplus x_2x_3x_4 \oplus x_2x_3x_5 \oplus x_2x_4x_5 \oplus x_3x_4x_5.
\end{align*}
\]

(6)

3 Synthesis Algorithm

3.1 Direct Synthesis

In this section, the steps to implement any arbitrary Boolean function \( f \) using positive polarity RM expansions as reversible circuits will be presented. For example, consider the homogeneous function,

\[
f (x_4, x_3, x_2, x_1, x_0) = x_2x_1x_0 \oplus x_4x_3x_0 \oplus x_4x_3x_1 \oplus x_4x_3x_2 \oplus x_3x_1x_0.
\]

(7)

To find the reversible circuit implementation for this function, we may follow the following procedure:

1- For a Boolean function with \( n \) variables, prepare \( n \) bits and initialize an extra bit \( t \) to 0, which will hold the result of the Boolean function.

2- Add a CNOT gate for each product term in this expansion taking the Boolean variables in this product term as control bits and the extra bit as the target bit.

3- For the product term, which contains 1, add a CNOT(\( t \)), so the final circuit for \( f \) shown in Eqn. [7] will be as shown in Fig. [2].
3.2 Optimization by Factorization

The main idea of factorization is to find the common terms in a Boolean function that have high quantum cost and implement them in a way to decrease multi-calculations of such terms and so decrease the total quantum cost and/or the total number of \( CNOT \) gates. The following observations are essential in understanding the final construction:

**Observation 3.1** For a homogeneous Boolean function of degree 3 of the form,

\[
\begin{align*}
x_i x_j x_{k_1} & \oplus x_i x_j x_{k_2} \oplus \cdots \oplus x_i x_j x_{k_n}, \\
\end{align*}
\]

it can be factorized to take the form,

\[
\begin{align*}
x_i x_j (x_{k_1} \oplus x_{k_2} \oplus \cdots \oplus x_{k_n}).
\end{align*}
\]

Realization of that expression as a reversible circuit is as follows,

\[
\begin{align*}
& CNOT(x_{k_1}; x_{k_2}) CNOT(x_{k_2}; x_{k_3}) \cdots CNOT(x_{k_{n-1}}; x_{k_n}) \\
& CNOT(x_i; x_j; x_{k_n}; f) CNOT(x_{k_n-1}; x_{k_n}) \cdots CNOT(x_{k_2}; x_{k_3}) \\
& CNOT(x_{k_1}; x_{k_2}).
\end{align*}
\]

Synthesis of that circuit will decrease the quantum cost from \( 13n \) is constructed using the direct method to \( 13 + 2(n - 1) \) after factorization. If the reservation of the inputs is not important, then the quantum cost can be decreased to \( 12 + n \), where the last \( (n - 1) \) \( CNOT \) gates used to restore the
state of the inputs can be removed. For example, the following function with a circuit of quantum cost = 39,

\[ f = x_0 x_3 x_4 \oplus x_1 x_3 x_4 \oplus x_2 x_3 x_4, \]  

(11)
can be re-written as follows,

\[ f = x_3 x_4 (x_0 \oplus x_1 \oplus x_2), \]  

(12)
such function has a circuit of quantum cost = 17 as shown in Fig. 3. Note that, the last two CNOT gates can be removed of the values of the inputs is not important.

Figure 3: Reversible circuit realization for the Boolean function shown in Eqn. 11 based on Observation 3.1

**Observation 3.2** If a homogeneous Boolean function can be represented in the form,

\[ x_{i_1} x_{j_1} (x_{k_0} \oplus \cdots \oplus x_{k_n}) \oplus x_{i_2} x_{j_2} (x_{k_0} \oplus \cdots \oplus x_{k_n} \oplus x_{k_{n+1}}) \oplus \cdots \oplus x_{i_m} x_{j_m} (x_{k_0} \oplus \cdots \oplus x_{k_n} \oplus \cdots \oplus x_{k_{n+m-1}}), \]  

(13)
then the upper bound of the quantum cost will be decreased from 13m (n + \( \frac{m+1}{2} \)) = O (m^2) using the direct method to 13m + 2(n + m − 1) = O (m) and the number of CNOT gates will be m + 2(n + m − 1) instead of m (n + \( \frac{m+1}{2} \)). For example, the function,

\[ f = x_2 x_3 (x_0 \oplus x_1) \oplus x_3 x_4 (x_0 \oplus x_1 \oplus x_2), \]  

(14)
has a circuit with quantum cost = 30 instead of 65 \((m = 2, n = 1)\) if synthesized using the direct method as shown in Fig. 4. Note that, the last two CNOT gates can be removed if the values of the inputs is not important. Finding the maximum number of common factors \(m\) leads to a decrease in the number of CNOT gates as well as the total quantum cost of the reversible circuit.

Figure 4: Reversible circuit realization for the Boolean function shown in Eqn. 14 based on Observation 3.2

3.2.1 Factorization Algorithm

From Observation 3.2, we can see that finding the maximum number of common factors in the expression will optimize the quantum cost and the number of CNOT gates of the reversible circuit. Similar observations can be constructed for homogeneous Boolean functions of degrees 2 by omitting one of the variables and using 5 instead of 13. In this section, an algorithm that helps in factorizing the expression to find the maximum number of common factors will be presented. Experimental results for that algorithm will be shown in the next section. To factorize an expression maximizing the number of factors, we follow the following algorithm:

1- Group the Boolean expression to a collection of minterms of the same degree and apply the following steps separately on each collection.

2- For a homogeneous Boolean function on \(n\) variables and \(m\) minterms, draw a table of \(n\) columns and \(m\) rows such that each column represents a variable and each row represents a minterm.
3- For each minterm in the expression (row), if \( x_i \) exists then place 1 in the corresponding cell.

4- Count the number of 1’s in each column and choose the variable with maximum occurrence as a common factor, if more than one variable have the same count, then choose the first one. We will call such variable as a factor variable and the terms in the corresponding bracket as a factor group.

5- Repeat Step 3 recursively for the remaining rows of the table collecting a set of factor variables and the corresponding factor groups, where some minterms will be left over (remainder minterms).

6- Find the common terms within all the factor groups and take them as a common factor generating groups of factor variables. This step arise the form shown in Observation 3.1

7- Group the common terms with the same group of factor variables and repeat Step 3 recursively within each group.

8- Find the subset relationships between the groups of factor variables. This arises the form shown in Observation 3.2

9- For each set of related groups, construct the reversible circuit as shown in Observation 3.2. Taking into account to reset the used variables before constructing the next related group.

10- For the remainder minterms, construct the corresponding CNOT gates using the direct method.

For example, consider the Boolean function shown in Eqn. \( \text{6} \). The factorization table from steps 1-3 on the expression is shown in Table.1. In this case, all the variables occurs 8 times in the expression, so we choose \( x_0 \) as the first factor variable. Applying Step 4 recursively on the remaining rows, we can find that \( x_2 \) and \( x_5 \) are the remaining factor variables. Next thing to do is to find the common terms in the factor groups as follows,

\[
f = x_0(x_1x_2 \oplus x_1x_3 \oplus x_1x_1 \oplus x_1x_5 \oplus x_2x_3 \oplus x_2x_5 \oplus x_3x_4 \oplus x_4x_5) \\
\oplus x_2(x_1x_3 \oplus x_1x_4 \oplus x_3x_4 \oplus x_3x_5 \oplus x_4x_5) \\
\oplus x_5(x_1x_3 \oplus x_1x_4 \oplus x_3x_4).
\]  
\[
(15)
\]
3.3 Experimental Results

Even though the art of synthesis and optimization of reversible circuits has been there for sometimes \[16, 14, 27, 22\], there are none of the work has
been done focused on the construction of reversible circuits for homogeneous Boolean functions where some of the mentioned work couldn’t optimize such circuits because of the nonlinearity of that functions. The proposed algorithm has been examined on the known benchmarks $2of5$ and $4mod5$ \[7\] that can be represented as a set of homogeneous Boolean functions of degree at most 3, for example, the Boolean expression of the $4mod5$ is as follows,

$$f = 1 \oplus x_0 \oplus x_1 \oplus x_2 \oplus x_3$$
$$\oplus x_0x_1 \oplus x_1x_2 \oplus x_0x_3 \oplus x_2x_3,$$  

(17)

where it contains 5 minterms of degree 1 and 4 minterms of degree 2, applying the factorization algorithm, the synthesized circuit is shown in Fig. \[6\] where the last three $CNOT$ gates can be removed if the state of the inputs is not important.

The algorithm also has been tested on representatives of homogenous bent functions of degree 3 with 6 and 8 variables. It was shown in \[6\] a set of 30 homogeneous bent functions of degree 3 with 6 variables each with 16 minterms ($BFV6_{d3}$), and a set of 20 homogeneous bent functions of degree 3 with 8 variables, 4 functions with 24 minterms ($BFV8_{d3,24}$), 6 functions with 28 minterms ($BFV8_{d3,28}$), 4 functions with 32 minterms ($BFV8_{d3,32}$) and 6 functions with 35 minterms ($BFV8_{d3,35}$). The experimental results is shown in Table \[2\] The results are shown by omitting the $CNOT$ gates that restore the state of the inputs to follow the known benchmarks for $2of5$ and $4mod5$\[7\]. Even if the missing $CNOT$ gates are added to the results, there will be no increase in the number of $CNOT$ gates compared with the direct
synthesis method. The results show an improvement in the quantum cost compared with the best known results in the literature. It can be seen from the results of bent functions of 8 variables that the degree of improvement increases as the number of minterms increases.

![Figure 6: Reversible circuit realization of the 4 mod 5 function using the factorization algorithm.](image)

Table 2: Reversible circuits benchmarks, where − indicates that results are not available.

| Benchmark | Inputs | Benchmark [7] | Direct Synthesis | Factorization |
|-----------|--------|---------------|------------------|---------------|
| 4 mod 5   | 4      | 5             | 13               | 9             | 25            | 6              | 8              |
| 2 of 5    | 5      | 15            | 107              | 20            | 180           | 17             | 75             |
| BFV6.d3   | 6      | -             | -                | 16            | 208           | 13             | 85             |
| BFV8.d3.24| 8      | -             | -                | 24            | 312           | 24             | 196            |
| BFV8.d3.28| 8      | -             | -                | 28            | 364           | 28             | 184            |
| BFV8.d3.32| 8      | -             | -                | 32            | 416           | 32             | 188            |
| BFV8.d3.35| 8      | -             | -                | 35            | 455           | 32             | 176            |

4 Conclusion

This paper shown a simple algorithm based on factorization of the algebraic expression of a homogeneous Boolean function of degree at most 3. The algorithm finds the common parts of the circuit and decrease the number
of their computation, this will decrease the quantum cost of the generated circuit.

The algorithm can also perform on a Boolean function that can be divided into a group of homogeneous Boolean functions. The ability of the algorithm to minimize the quantum cost increases as the number of minterms of the same degree increases.

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