To logconcavity and beyond

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Abstract

In 1976 Brascamp and Lieb proved that the heat flow preserves logconcavity. In this paper, introducing a variation of concavity, we show that it preserves in fact a stronger property than logconcavity and we identify the strongest concavity preserved by the heat flow.

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1 Introduction

A nonnegative function \( u \) in \( \mathbb{R}^N \) is said \textit{logconcave} in \( \mathbb{R}^N \) if
\[
u((1-\mu)x + \mu x) \geq \nu(x)^{1-\mu} \nu(y)^\mu
\]
for \( \mu \in [0,1] \) and \( x, y \in \mathbb{R}^N \) such that \( u(x)u(y) > 0 \). This is equivalent to that the set
\[S_u := \{ x \in \mathbb{R}^N : u(x) > 0 \}\] is convex and \( \log u \) is concave in \( S_u \). Logconcavity is a very useful variation of concavity and plays an important role in various fields such as PDEs, geometry, probability, statics, optimization theory and so on (see e.g. [17]). Most of its relevance, especially for elliptic and parabolic equations, is due to the fact that the Gauss kernel
\[
G(x,t) := (4\pi t)^{-\frac{N}{2}} \exp \left( -\frac{|x|^2}{4t} \right)
\]
is logconcave in \( \mathbb{R}^N \) for any fixed \( t > 0 \). Indeed,
\[
\log G(x,t) = -\frac{|x|^2}{4t} + \log(4\pi t)^{-\frac{N}{2}}
\]
is concave in \( \mathbb{R}^N \) for any fixed \( t > 0 \). Exploiting the logconcavity of the Gauss kernel, Brascamp and Lieb [4] proved that logconcavity is preserved by the heat flow and they also obtained the logconcavity of the first positive Dirichlet eigenfunction for the Laplace operator \(-\Delta \) in a bounded convex domain. (See also [7, 13].) For later convenience, we state explicitly these two classical results below.

(a) Let \( u \) be a bounded nonnegative solution of
\[
\begin{cases}
\partial_t u = \Delta u & \text{in } \Omega \times (0, \infty), \\
u = 0 & \text{on } \partial \Omega \times (0, \infty) \text{ if } \partial \Omega \neq \emptyset, \\
u(x,0) = u_0(x) & \text{in } \Omega,
\end{cases}
\]
where \( \Omega \) is a convex domain in \( \mathbb{R}^N \) and \( u_0 \) is a bounded nonnegative function in \( \Omega \). Then \( u(\cdot, t) \) is logconcave in \( \Omega \) for any \( t > 0 \) if \( u_0 \) is logconcave in \( \Omega \).

(b) Let \( \Omega \) be a bounded convex domain in \( \mathbb{R}^N \) and \( \lambda_1 \) the first Dirichlet eigenvalue for the Laplace operator in \( \Omega \). If \( \phi \) solves
\[
\begin{cases}
-\Delta \phi = \lambda_1 \phi & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega, \\
\phi > 0 & \text{in } \Omega,
\end{cases}
\]
then \( \phi \) is logconcave in \( \Omega \).

We denote by \( e^{t\Delta} u_0 \) the (unique) solution to problem (1.3). In particular, in the case of \( \Omega = \mathbb{R}^N \), we write \( e^{t\Delta} u_0 = e^{t\Delta} R^N u_0 \), that is,
\[
[e^{t\Delta} u_0](x) = \int_{\mathbb{R}^N} G(x-y, t) u_0(y) \, dy, \quad x \in \mathbb{R}^N, \ t > 0.
\]
Logconcavity is so naturally and deeply linked to heat transfer that $e^{t\Delta}u_0$ spontaneously becomes logconcave in $\mathbb{R}^N$ even without the logconcavity of initial function $u_0$. Indeed, Lee and Vázquez [14] proved the following:

(c) Let $u_0$ be a bounded nonnegative function in $\mathbb{R}^N$ with compact support. Then there exists $T > 0$ such that $e^{t\Delta}u_0$ is logconcave in $\mathbb{R}^N$ for $t \geq T$. (See [14, Theorem 5.1].)

Due to the above reasons, logconcavity is commonly regarded as the optimal concavity for the heat flow and for the first positive Dirichlet eigenfunction for $-\Delta$.

In this paper we dare to ask the following question:

(Q1) Is logconcavity the strongest concavity preserved by the heat flow in a convex domain?

If not, what is the strongest concavity preserved by the heat flow?

We introduce a new variation of concavity and give answers to (Q1). More precisely, we introduce $\alpha$-logconcavity as a refinement of $p$-concavity at $p = 0$ (see Section 2) and show that the heat flow preserves 2-logconcavity (see Theorem 3.1). Here 2-logconcavity is stronger than usual logconcavity and we prove that 2-logconcavity is exactly the strongest concavity property preserved by the heat flow (see Theorem 3.2).

Another natural question which spontaneously arises after (Q1) is the following: is logconcavity the strongest concavity shared by the solution $\phi$ of (1.4) for any bounded convex domain $\Omega$? We are not able to give here an exhaustive answer to this question, but we conjecture that it is negative and that also the first positive Dirichlet eigenfunction for $-\Delta$ in every convex domain is 2-logconcave. See Remark 4.2 about this.

The rest of this paper is organized as follows. In Section 2 we introduce a new variation of concavity and prove some lemmas. In particular, we show that 2-logconcavity is the strongest concavity for the Gauss kernel $G(\cdot, t)$ to satisfy. In Section 3 we state the main results of this paper. The proofs of the main results are given in Sections 4 and 5.

2 Logarithmic power concavity

For $x \in \mathbb{R}^N$ and $R > 0$, set $B(x, R) := \{y \in \mathbb{R}^N : |x - y| < R\}$. For any measurable set $E$, we denote by $\chi_E$ the characteristic function of $E$. Furthermore, for any function $u$ in a set $\Omega$ in $\mathbb{R}^N$, we say that $U$ is the zero extension of $u$ if $U(x) = u(x)$ for $x \in \Omega$ and $U(x) = 0$ for $x \notin \Omega$. We often identify $u$ with its zero extension $U$. A function $f : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ is a (proper) concave function if $f((1 - \mu)x + \mu y) \geq (1 - \mu)f(x) + \mu f(y)$ for $x, y \in \mathbb{R}^N, \mu \in [0, 1]$ (and $f(x) > -\infty$ for at least one $x \in \mathbb{R}^N$). Here we deal with $-\infty$ in the obvious way, that is: $-\infty + a = -\infty$ for $a \in \mathbb{R}$ and $-\infty \geq -\infty$.

Let $G = G(x, t)$ be the Gauss kernel (see (1.1)). Similarly to (1.2), for any fixed $t > 0$, it follows that

$$-(-\log [\kappa G(x, t)])^{\frac{1}{2}} = -\left[\frac{|x|^2}{4t} - \log \left((4\pi t)^{-\frac{N}{2}} \kappa \right)\right]^{\frac{1}{2}}$$
is still concave in $\mathbb{R}^N$ for any sufficiently small $\kappa > 0$. Motivated by this, we formulate a definition of $\alpha$-logconcavity ($\alpha > 0$). Let $L_\alpha = L_\alpha(s)$ be a strictly increasing function on $[0,1]$ defined by

$$L_\alpha(s) := -(-\log s)^\frac{1}{\alpha} \quad \text{for} \quad s \in (0,1], \quad L_\alpha(s) := -\infty \quad \text{for} \quad s = 0.$$

**Definition 2.1** Let $\alpha > 0$.

(i) Let $u$ be a bounded nonnegative function in $\mathbb{R}^N$. We say that $u$ is $\alpha$-logconcave in $\mathbb{R}^N$ if

$$L_\alpha(\kappa u((1-\mu)x + \mu y)) \geq (1-\mu)L_\alpha(\kappa u(x)) + \mu L_\alpha(\kappa u(y)), \quad x, y \in \mathbb{R}^N, \; \mu \in [0,1],$$

for all sufficiently small $\kappa > 0$.

(ii) Let $u$ be a bounded nonnegative function in a convex set $\Omega$ in $\mathbb{R}^N$. We say that $u$ is $\alpha$-logconcave in $\Omega$ if the zero extension of $u$ is $\alpha$-logconcave in $\mathbb{R}^N$.

Definition 2.1 means that, for any bounded nonnegative function $u$ in a convex set $\Omega$, $u$ is $\alpha$-logconcave in $\Omega$ if and only if the function $L_\alpha(\kappa U)$ is concave in $\mathbb{R}^N$ for all sufficiently small $\kappa > 0$. Due to Definition 2.1, we easily see the following properties:

- Logconcavity corresponds to 1-logconcavity;
- If $u$ is $\alpha$-logconcave in $\Omega$ for some $\alpha > 0$, then $\kappa u$ is also $\alpha$-logconcave in $\Omega$ for any $\kappa > 0$;
- If $0 < \alpha \leq \beta$ and $u$ is $\beta$-logconcave in $\Omega$, then $u$ is $\alpha$-logconcave in $\Omega$;
- The Gauss kernel $G(\cdot, t)$ is $2$-logconcave in $\mathbb{R}^N$ for any $t > 0$.

Furthermore, we have:

**Lemma 2.1** Let $\alpha \geq 1$. Let $u$ be a function in a convex set $\Omega$ such that $0 \leq u \leq 1$ in $\Omega$. If $L_\alpha(u)$ is concave in $\Omega$, then $L_\alpha(\kappa u)$ is also concave in $\Omega$ for any $0 < \kappa \leq 1$.

**Proof.** Let $x, y \in \Omega$ and $\mu \in [0,1]$. Assume that $u(x)u(y) > 0$. Since $L_\alpha^{-1}(s) = \exp(-(-s)^\alpha)$ for $s \in (-\infty,0]$, we find

$$\psi(\kappa) := \kappa^{-1}L_\alpha^{-1}((1-\mu)L_\alpha(\kappa u(x)) + \mu L_\alpha(\kappa u(y)))$$

$$= \kappa^{-1}\exp\left\{ -\left[(1-\mu)(-\log \kappa u(x))^{\frac{1}{\alpha}} + \mu(-\log \kappa u(y))^{\frac{1}{\alpha}} \right]^\alpha \right\}$$

for $0 < \kappa \leq 1$. Since $L_\alpha(u)$ is concave in $\Omega$, it follows that

$$(1-\mu)u(x) + \mu u(y) \geq \psi(1).$$

(2.1)

For $a, b > 0$ and $\gamma \in (-\infty, \infty)$, set

$$M_\gamma(a,b;\mu) := [(1-\mu)a^\gamma + \mu b^\gamma]^{\frac{1}{\gamma}}.$$
Then
\[ \psi'(\kappa) = -\kappa^{-2} \exp \left\{ -M_{\frac{1}{\alpha}}(-\log \kappa u(x), -\log \kappa u(y); \mu) \right\} \\
+ \kappa^{-1} \exp \left\{ -M_{\frac{1}{\alpha}}(-\log \kappa u(x), -\log \kappa u(y); \mu) \right\} \\
\times \left[ (1 - \mu)(-\log \kappa u(x))^\frac{1}{\alpha} + \mu(-\log \kappa u(y))^\frac{1}{\alpha} \right]^{\alpha-1} \\
\times \kappa^{-1} \left[ (1 - \mu)(-\log \kappa u(x))^{\frac{1-\alpha}{\alpha}} + \mu(-\log \kappa u(y))^{\frac{1-\alpha}{\alpha}} \right] \]

(2.2)

\( = \kappa^{-2} \exp \left\{ -M_{\frac{1}{\alpha}}(-\log \kappa u(x), -\log \kappa u(y); \mu) \right\} \\
\times \left[ -1 + M_{\frac{1}{\alpha}}(-\log \kappa u(x), -\log \kappa u(y); \mu)^{\frac{\alpha-1}{\alpha}} M_{\frac{1}{\alpha}}(-\log \kappa u(x), -\log \kappa u(y); \mu^{\frac{1-\alpha}{\alpha}}) \right] \]

for 0 < \kappa \leq 1. On the other hand, since \( \alpha \geq 1 \), it follows that \( 1/\alpha \geq (1 - \alpha)/\alpha \). Then the Jensen inequality yields
\[ M_{\frac{1}{\alpha}}(a, b; \mu) \geq M_{\frac{1}{\alpha}}(a, b; \mu) \quad \text{for} \quad a, b > 0. \]

This together with (2.2) implies that \( \psi'(\kappa) \geq 0 \) for 0 < \kappa \leq 1. Therefore, by (2.1) we obtain
\[ (1 - \mu)u(x) + \mu u(y) \geq \psi(\kappa) = \kappa^{-1}L_{\alpha}^{-1}[(1 - \mu)L_{\alpha}(\kappa u(x)) + \mu L_{\alpha}(\kappa u(y))] \]

(2.3)

for 0 < \kappa \leq 1 in the case of \( u(x)u(y) > 0 \). In the case of \( u(x)u(y) = 0 \), by the definition of \( L_{\alpha} \) we easily obtain (2.3) for 0 < \kappa \leq 1. These mean that \( L_{\alpha}(\kappa u) \) is concave in \( \Omega \) for 0 < \kappa \leq 1. Thus Lemma 2.1 follows. \( \square \)

**Remark 2.1** (i) Let \( u \) be a bounded nonnegative function in a convex set \( \Omega \) and \( \alpha > 0 \). We say that \( u \) is weakly \( \alpha \)-logconcave in \( \Omega \) if
\[ L_{\alpha}(\kappa U((1 - \mu)x + \mu y)) \geq (1 - \mu)L_{\alpha}(\kappa U(x)) + \mu L_{\alpha}(\kappa U(y)), \quad x, y \in \mathbb{R}^N, \ \mu \in [0, 1], \]

for some \( \kappa > 0 \). Here \( U \) is the zero extension of \( u \). Lemma 2.1 implies that \( \alpha \)-logconcavity is equivalent to weak \( \alpha \)-logconcavity in the case of \( \alpha \geq 1 \).

(ii) For 0 < \alpha < 1, \( \alpha \)-logconcavity is not equivalent to weak \( \alpha \)-logconcavity. Indeed, set \( u(x) := \exp(-|x|^\alpha)\chi_{B(0, R)}, \) where \( 0 < R \leq \infty \). Then \( L_{\alpha}(u) = -|x| \) is concave in \( B(0, R) \).

On the other hand, for any 0 < \kappa < 1, we have
\[ \frac{\partial}{\partial r} L_{\alpha}(\kappa u(x)) = -(-\log \kappa + |x|^\alpha)^{-1+\frac{1}{\alpha}}|x|^{-1+\alpha}; \]
\[ \frac{\partial^2}{\partial r^2} L_{\alpha}(\kappa u(x)) = (\alpha - 1)(-\log \kappa + |x|^\alpha)^{-2+\frac{2}{\alpha}}|x|^{-2+2\alpha} + (1 - \alpha)(-\log \kappa + |x|^\alpha)^{-1+\frac{1}{\alpha}}|x|^{-2+\alpha}; \]
\[ = (1 - \alpha)(-\log \kappa + |x|^\alpha)^{-2+\frac{1}{\alpha}}|x|^{-2+\alpha}(-\log \kappa) > 0, \]

for \( x \in \overline{B(0, R) \setminus \{0\}}, \) where \( r := |x| > 0 \). This means that \( L_{\alpha}(\kappa u) \) is not concave in \( B(0, R) \) for any 0 < \kappa < 1.
Next we introduce the notion of $F$-concavity, which generalises and embraces all the notions of concavity we have already seen.

**Definition 2.2** Let $\Omega$ be a convex set in $\mathbb{R}^N$.

(i) A function $F : [0, 1] \to \mathbb{R} \cup \{-\infty\}$ is said admissible if $F$ is strictly increasing continuous in $(0, 1]$, $F(0) = -\infty$ and $F(s) \neq -\infty$ for $s > 0$.

(ii) Let $F$ be admissible. Let $u$ be a bounded nonnegative function in $\Omega$ and $U$ the zero extension of $u$. Then $u$ is said $F$-concave in $\Omega$ if $0 \leq kU(x) \leq 1$ in $\mathbb{R}^N$ and

$$F(kU((1 - \mu)x + \mu y)) \geq (1 - \mu)F(kU(x)) + \mu F(kU(y)),$$

for all sufficiently small $k > 0$. We denote by $C_{\Omega}[F]$ the set of $F$-concave functions in $\Omega$. Furthermore, in the case of $\Omega = \mathbb{R}^N$, we write $C[F] = C_{\Omega}[F]$ for simplicity.

(iii) Let $F_1$ and $F_2$ be admissible. We say that $F_1$-concavity is stronger than $F_2$-concavity in $\Omega$ if $C_{\Omega}[F_1] \subsetneq C_{\Omega}[F_2]$.

We recall that a bounded nonnegative function $u$ in a convex set $\Omega$ is said $p$-concave in $\Omega$, where $p \in \mathbb{R}$, if $u$ is $F$-concave with $F = F_p$ in $\Omega$, where

$$F_p(s) := \begin{cases} 
\frac{1}{p} & \text{for } s > 0 \text{ if } p \neq 0, \\
\log s & \text{for } s > 0 \text{ if } p = 0, \\
-\infty & \text{for } s = 0.
\end{cases}$$

Here 1-concavity corresponds to usual concavity while 0-concavity corresponds to usual logconcavity (in other words, 1-logconcavity). Furthermore, $u$ is said quasiconcave or $-\infty$-concave in $\Omega$ if all superlevel sets of $u$ are convex, while it is said $\infty$-concave in $\Omega$ if $u$ satisfies

$$u((1 - \mu)x + \mu y) \geq \max\{u(x), u(y)\}$$

for $x, y \in \Omega$ with $u(x)u(y) > 0$ and $\mu \in [0, 1]$. Then, by the Jensen inequality we have:

- Let $-\infty \leq p \leq q \leq \infty$. If $u$ is $q$-concave in a convex set $\Omega$, then $u$ is also $p$-concave in $\Omega$.

Among concavity properties, apart from usual concavity, of course logconcavity has been the most deeply investigated, especially for its importance in probability and convex geometry (see for instance [6] for an overview and the series of papers [1, 2, 3, 15], which recently broadened and structured the theory of log-concave functions). Clearly, if a function $u$ is $F$-concave in $\Omega$ for some admissible $F$, then it is quasiconcave in $\Omega$; vice versa, if $u$ is $\infty$-concave in $\Omega$, then it is $F$-concave in $\Omega$ for any admissible $F$. These mean that quasiconcavity (resp. $\infty$-concavity) is the weakest (resp. strongest) conceivable concavity. Notice that $\alpha$-logconcavity ($\alpha > 0$) corresponds to $F$-concavity with $F = L_\alpha$ and it is weaker (resp. stronger) than $p$-concavity for any $p > 0$ (resp. $p < 0$). Indeed, the following lemma holds.

**Lemma 2.2** Let $\Omega$ be a convex set in $\mathbb{R}^N$ and $u$ a nonnegative bounded function in $\Omega$.

(i) If $u$ is $p$-concave in $\Omega$ for some $p > 0$, then $u$ is $\alpha$-logconcave in $\Omega$ for any $\alpha > 0$.

(ii) If $u$ is $\alpha$-logconcave in $\Omega$ for some $\alpha > 0$, then $u$ is $p$-concave in $\Omega$ for any $p < 0$.  


Proof. We prove assertion (i). Let \( p > 0 \) and \( \alpha > 0 \). It suffices to prove that

\[
[(1 - \mu)a^p + \mu b^p]^\frac{1}{p} \geq \exp \left\{ - \left[ (1 - \mu)(- \log a)^\frac{1}{p} + \mu(- \log b)^\frac{1}{p} \right]^{\alpha} \right\}
\]  

holds for all sufficiently small \( a, b > 0 \) and all \( \mu \in [0, 1] \). This is equivalent to that the inequality

\[
\left( - \frac{1}{p} \log \left( (1 - \mu)\tilde{a} + \mu \tilde{b} \right) \right)^\frac{1}{\alpha} \leq (1 - \mu) \left( - \frac{1}{p} \log \tilde{a} \right)^\frac{1}{\alpha} + \mu \left( - \frac{1}{p} \log \tilde{b} \right)^\frac{1}{\alpha}
\]  

holds for all sufficiently small \( \tilde{a} := a^p, \tilde{b} := b^p > 0 \) and all \( \mu \in [0, 1] \). Inequality (2.5) follows from the fact that the function \( s \mapsto \left( - \frac{1}{p} \log s \right)^\frac{1}{\alpha} \) is convex for all sufficiently small \( s > 0 \). Thus (2.4) holds for all sufficiently small \( a, b > 0 \) and all \( \mu \in [0, 1] \) and assertion (i) follows. Similarly, we obtain assertion (ii) and the proof is complete. \( \square \)

Lemma 2.2 implies that \( \alpha \)-logconcavity is a refinement of \( p \)-concavity at \( p = 0 \).

At the end of this section we show that 2-logconcavity is the strongest concavity for the Gauss kernel \( G(\cdot, t) \) to satisfy. This plays a crucial role in giving an answer to the second part of (Q1).

Lemma 2.3 Let \( F \) be admissible such that \( G(\cdot, t) \) is \( F \)-concave in \( \mathbb{R}^N \) for some \( t > 0 \). Then a bounded nonnegative function \( u \) in \( \mathbb{R}^N \) is \( F \)-concave in \( \mathbb{R}^N \) if it is 2-logconcave in \( \mathbb{R}^N \). Furthermore,

\[
C[L_2] = \bigcap_{F \in \{H(\cdot, t) \in C[H]\}} C[F] \text{ for any } t > 0.
\]  

(2.6)

Proof. Assume that \( G(\cdot, t) \) is \( F \)-concave in \( \mathbb{R}^N \) for some \( t > 0 \). It follows from Definition 2.1 that the function \( e^{-|x|^2} \) is \( F \)-concave in \( \mathbb{R}^N \). Then we obtain the \( F \)-concavity of \( e^{-s^2} \). Let \( u \) be 2-logconcave in \( \mathbb{R}^N \). By Definition 2.1 we see that \( L_2(\kappa u) \) is concave in \( \mathbb{R}^N \) for all sufficiently small \( \kappa > 0 \). Set

\[
w(x) := -L_2(\kappa u(x)) = \begin{cases} \sqrt{-\log \kappa u(x)} & \text{if } u(x) > 0, \\ \infty & \text{if } u(x) = 0. \end{cases}
\]

Then \( w \) is nonnegative and convex in \( \mathbb{R}^N \), that is,

\[
0 \leq w((1 - \mu)x + \mu y) \leq (1 - \mu)w(x) + \mu w(y)
\]  

for \( x, y \in \mathbb{R}^N \) and \( \mu \in [0, 1] \). On the other hand, by \( F \)-concavity of \( e^{-s^2} \) we have

\[
F \left( \kappa e^{-[(1 - \mu)w(x) + \mu w(y)]^2} \right) \geq (1 - \mu)F(\kappa e^{-w(x)^2}) + \mu F(\kappa e^{-w(y)^2})
\]  

(2.8)
for all sufficiently small $\kappa > 0$. Since $F$ is an increasing function, by (2.7) and (2.8) we obtain
\[
F(\kappa^2 u((1 - \mu)x + \mu y)) = F(\kappa \exp(-w((1 - \mu)x + \mu y)^2)) \\
\geq F(\kappa e^{-[(1 - \mu)w(x) + \mu w(y)]^2}) \geq (1 - \mu)F(\kappa e^{-w(x)^2}) + \mu F(\kappa e^{-w(y)^2}) \\
= (1 - \mu)F(\kappa^2 u(x)) + \mu F(\kappa^2 u(y))
\]
for all sufficiently small $\kappa > 0$ if $u(x)u(y) > 0$. This inequality also holds in the case of $u(x)u(y) = 0$. These imply that $u$ is $F$-concave in $\mathbb{R}^N$ and
\[
C[L_2] \subset \bigcap_{F \in \{H : G(\cdot, t) \in \mathcal{C}[H]\}} C[F]. \quad (2.9)
\]
On the other hand, since $G(\cdot, t)$ is 2-logconcave, it turns out that
\[
\bigcap_{F \in \{H : G(\cdot, t) \in \mathcal{C}[H]\}} C[F] \subset C[L_2].
\]
This together with (2.9) implies (2.6). Thus Lemma 2.3 follows. $\square$

3 Main results

We are now ready to state the main results of this paper. The first one ensures that the heat flow preserves $\alpha$-logconcavity with $1 \leq \alpha \leq 2$.

**Theorem 3.1** Let $\Omega$ be a convex domain in $\mathbb{R}^N$ and $1 \leq \alpha \leq 2$. Let $u_0$ be a bounded nonnegative function in $\Omega$ and $u := e^{t\Delta_\Omega}u_0$. Assume that $0 \leq u_0 \leq 1$ and $L_\alpha(u_0)$ is concave in $\Omega$. Then $L_\alpha(u(\cdot, t))$ is concave in $\Omega$ for any $t > 0$.

Since $\alpha$-logconcavity with $\alpha > 1$ is stronger than usual logconcavity, Theorems 3.1 gives answer to the first part of (Q1). Furthermore, as a corollary of Theorem 3.1 we have the following.

**Corollary 3.1** Let $\Omega$ be a convex domain in $\mathbb{R}^N$. Let $u_0$ be a bounded nonnegative function in $\Omega$. If $1 \leq \alpha \leq 2$ and $u_0$ is $\alpha$-logconcave in $\Omega$, then $e^{t\Delta_\Omega}u_0$ is $\alpha$-logconcave in $\Omega$ for any $t > 0$.

Next we state a result which shows that 2-logconcavity is the strongest concavity preserved by the heat flow. This addresses the second part of (Q1).

**Theorem 3.2** Let $F$ be admissible and $\Omega$ a convex domain in $\mathbb{R}^N$. Assume that $F$-concavity is stronger than 2-logconcavity in $\Omega$, that is, $C_\Omega[F] \subset C_\Omega[L_2]$. Then there exists $u_0 \in C_\Omega[F]$ such that
\[
e^{T\Delta_\Omega}u_0 \notin C_\Omega[F] \quad \text{for some } T > 0.
\]

Here the following question naturally arises:

(Q2) What is the weakest concavity preserved by the heat flow?
Unfortunately we have no answers to (Q2) and it is open. Notice that the heat flow does not necessarily preserve $p$-concavity for some $p < 0$. See [10, II]. (See also [5].)

Finally we assure that $e^{t \Delta} u_0$ spontaneously becomes $\alpha$-logconcave for any $\alpha \in [1, 2]$ if $u_0$ has compact support. This improves assertion (c).

**Theorem 3.3** Let $u_0$ be a bounded nonnegative function in $\mathbb{R}^N$ with compact support. Then, for any given $1 < \alpha < 2$, there exists $T_\alpha > 0$ such that, for any $t \geq T_\alpha$, $L_\alpha(e^{t \Delta} u_0)$ is concave in $\mathbb{R}^N$, in particular, $e^{t \Delta} u_0$ is $\alpha$-logconcave in $\mathbb{R}^N$.

We conjecture that Theorem 3.3 holds true even for $\alpha = 2$, but we cannot prove it here. Indeed, in our proof of Theorem 3.3, $T_\alpha \to \infty$ as $\alpha \to 2$.

In Section 4 we prove Theorems 3.1 and 3.2. Theorem 3.1 is shown as an application of [9] however the proof is somewhat tricky (see Remark 4.1). Furthermore, we prove Theorem 3.2 by the use of Lemma 2.3. In Section 5 we study the large time behavior of the second order derivatives of $e^{t \Delta} u_0$. This proves Theorem 3.3.

### 4 Proofs of Theorems 3.1 and 3.2

Firstly we prove Theorem 3.1 and show the preservation of $\alpha$-logconcavity ($1 \leq \alpha \leq 2$) by the heat flow.

**Proof of Theorem 3.1.** Let $\Omega$ be a convex domain in $\mathbb{R}^N$. Let $u_0$ be a nontrivial function in $\Omega$ such that $0 \leq u_0(x) \leq 1$ in $\Omega$. Then it follows from the strong maximum principle that $0 < u < 1$ in $\Omega \times (0, \infty)$. Assume that $L_\alpha(u_0)$ is concave in $\Omega$ for some $\alpha \in [1, 2]$.

1st step: We consider the case where $\Omega$ is a bounded smooth convex domain, $u_0 \in C(\Omega)$ and $u_0 = 0$ on $\partial \Omega$. Set

$$w(x, t) := - L_\alpha(u(x, t)) \geq 0, \quad w_0(x) := - L_\alpha(u_0(x)) \geq 0.$$ 

Here $u_0$ is convex in $\Omega$. Then it follows that

$$\begin{cases}
  w_t - \Delta w + \frac{1}{\gamma} \frac{|\nabla w|^2}{w} + \frac{\gamma - 1}{\gamma} \frac{|\nabla w|^2}{w} = 0 & \text{in } \Omega \times (0, \infty), \\
  w(x, 0) = w_0(x) & \text{in } \Omega, \\
  w > 0 & \text{in } \Omega \times (0, \infty), \\
  w(x, t) \to +\infty & \text{as } \text{dist}(x, \partial \Omega) \to 0 \text{ for any } t > 0,
\end{cases}$$

(4.1)

where $\gamma := 1/\alpha \in [1/2, 1]$.

We prove that $w(\cdot, t)$ is convex in $\Omega$ for any $t > 0$. For this aim, we set $z := e^{-w}$ and show that $z(\cdot, t)$ is logconcave in $\Omega$ for any $t > 0$. (See Remark 4.1.) It follows from (4.1) that

$$\begin{cases}
  z_t - \Delta z + \frac{|\nabla z|^2}{z} \left[ \frac{1}{\gamma} (\log z)^{-1} \right] + \frac{\gamma - 1}{\gamma} (\log z)^{-1} + 1 = 0 & \text{in } \Omega \times (0, \infty), \\
  z(x, 0) = e^{-w_0(x)} & \text{in } \Omega, \\
  z(x, t) = 0 & \text{on } \partial \Omega \times (0, \infty).
\end{cases}$$

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Furthermore, thanks to the convexity of \( w_0 \), we see that \( z(\cdot,0) = e^{-w_0} \) is logconcave in \( \Omega \). Applying [9, Theorem 4.2, Corollary 4.2] (see also [8, Theorem 4.2]), we deduce that \( z(\cdot,t) \) is logconcave in \( \Omega \) for any \( t > 0 \) if

\[
h(s, A) := e^{-s} [ -e^{s} \text{trace}(A) + f(e^{s}, e^{s} \theta) ]
\]

is convex for \( (s, A) \in (-\infty,0) \times \text{Sym}_N \) for any fixed \( \theta \in \mathbb{R}^N \).

Here \( \text{Sym}_N \) denotes the space of real \( N \times N \) symmetric matrices and

\[
f(\zeta, \vartheta) := \frac{|\vartheta|^2}{\zeta} \left[ -\frac{1}{\gamma} (-\log \zeta)^{-\frac{\gamma^{-1}}{\gamma}} + \frac{\gamma - 1}{\gamma} (\log \zeta)^{-1} + 1 \right]
\]

for \( (\zeta, \vartheta) \in (0,1) \times \mathbb{R}^N \).

On the other hand, for any fixed \( \theta \in \mathbb{R}^N \),

\[
h(s, A) = -\text{trace}(A) + |\theta|^2 \left[ -\frac{1}{\gamma} (-s)^{-\frac{\gamma^{-1}}{\gamma}} + \frac{\gamma - 1}{\gamma} s^{-1} + 1 \right]
\]

is convex for \( (s, A) \in (-\infty,0) \times \text{Sym}_N \) if and only if \( 1/2 \leq \gamma \leq 1 \). Therefore \( z(\cdot,t) \) is logconcave in \( \Omega \) for any \( t > 0 \). This implies that \( u(\cdot,t) \) is \( \alpha \)-logconcave for any \( t > 0 \).

2nd step: We consider the case where \( \Omega \) is a bounded smooth convex domain. In this step we do not assume that \( u_0 = 0 \) on \( \partial \Omega \). Since \( u_0 \) is \( \alpha \)-logconcave in \( \Omega \), we see that \( L_\alpha(u_0) \) is concave in \( \Omega \). Set

\[v_0(x) := \begin{cases} \exp(L_\alpha(u_0(x))) & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega, \end{cases} \]

\[v(x,t) := [e^{t \Delta} v_0](x),\]

for \( x \in \mathbb{R}^N \) and \( t > 0 \). Here we let \( e^{-\infty} := 0 \). Then \( v_0 \) is logconcave in \( \mathbb{R}^N \). We deduce from assertion (a) that \( v(\cdot,t) \) is logconcave for any \( t > 0 \). Furthermore, we deduce from \( v_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) that

\[
v(\cdot,t) \text{ is a positive continuous function in } \mathbb{R}^N \text{ for any } t > 0,
\]

\[
\|v(t)\|_{L^\infty(\mathbb{R}^N)} < \|v_0\|_{L^\infty(\mathbb{R}^N)} \text{ for any } t > 0,
\]

\[
\lim_{t \to 0} \|v(t) - v_0\|_{L^1(\mathbb{R}^N)} = 0.
\]

By (4.4) we can find a sequence \( \{t_n\} \subset (0,\infty) \) with \( \lim_{n \to \infty} t_n = 0 \) such that

\[
\lim_{n \to \infty} v(x,t_n) = v_0(x)
\]

for almost all \( x \in \mathbb{R}^N \).

Let \( \eta \) solve

\[-\Delta \eta = 1 \text{ in } \Omega, \quad \eta > 0 \text{ in } \Omega, \quad \eta = 0 \text{ on } \partial \Omega.
\]

Then \( \eta \) is \( 1/2 \)-concave in \( \Omega \) (see e.g. [12, Theorem 4.1]), which implies that \( \log \eta \) is concave in \( \Omega \) and \( \log \eta \to -\infty \) as \( \text{dist}(x,\partial \Omega) \to 0 \). By (4.2) and (4.3) we can find a sequence \( \{m_n\} \subset (1,\infty) \) with \( \lim_{n \to \infty} m_n = \infty \) such that

\[V_\alpha(x) := \log v(x,t_n) + m_n^{-1} \log \eta(x)\]
is continuous and concave in $\Omega$ and
\[ \sup_{x \in \Omega} V_n(x) \leq \text{ess sup}_{x \in \Omega} \log v_0. \]

Furthermore, by (4.5) we have
\[ \lim_{n \to \infty} V_n(x) = \log v_0(x) = L_\alpha(u_0(x)) \text{ for almost all } x \in \Omega, \]
\[ V_n(x) \to -\infty \text{ as } \text{dist}(x, \partial \Omega) \to 0. \]

Then the function $u_{0,n}(x) := L_\alpha^{-1}(V_n(x))$ satisfies
\[ 0 \leq u_{0,n} \leq 1 \text{ in } \Omega, \]
\[ u_{0,n} = 0 \text{ on } \partial \Omega \text{ and } \lim_{n \to \infty} u_{0,n}(x) = u_0(x) \text{ for almost all } x \in \Omega. \] (4.6)

Furthermore, $u_{0,n}$ is continuous on $\overline{\Omega}$ and $L_\alpha(u_{0,n})$ is concave in $\Omega$. Let
\[ u_n(x,t) := [e^{t \Delta \Omega} u_{0,n}](x) = \int_{\Omega} G_\Omega(x,y,t) u_{0,n}(y) \, dy, \]
where $G_\Omega = G_\Omega(x,y,t)$ is the Dirichlet heat kernel in $\Omega$. Then, by (4.6) we apply the Lebesgue dominated convergence theorem to obtain
\[ \lim_{n \to \infty} u_n(x,t) = \int_{\Omega} G_\Omega(x,y,t) u_0(y) \, dy = u(x,t), \quad x \in \Omega, \ t > 0. \] (4.7)

On the other hand, by the argument in 1st step we see that $L_\alpha(u_n(\cdot, t))$ is concave in $\Omega$ for any $t > 0$. Then we deduce from (4.7) that $L_\alpha(u(\cdot, t))$ is also concave in $\Omega$ for any $t > 0$. Thus Theorem 3.1 follows in the case where $\Omega$ is a bounded smooth convex domain.

3rd step: We complete the proof of Theorem 3.1. There exists a sequence of bounded convex smooth domains $\{\Omega_n\}_{n=1}^\infty$ such that
\[ \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots, \quad \bigcup_{n=1}^\infty \Omega_n = \Omega. \]
(This is for instance a trivial consequence of [16, Theorem 2.7.1]).

For any $n = 1, 2, \ldots$, let $u_n := e^{t \Delta \Omega_n} (u_0 \chi_{\Omega_n})$. The argument in 2nd step implies that $L_\alpha(u_n(\cdot, t))$ is concave in $\Omega_n$ for any $t > 0$. Furthermore, by the comparison principle we see that
\[ u_n(x,t) \leq u_{n+1}(x,t) \leq u(x,t) \quad \text{in } \Omega_n \times (0, \infty), \]
\[ u(x,t) = \lim_{n \to \infty} u_n(x,t) \quad \text{in } \Omega \times (0, \infty). \]

Then we observe that $L_\alpha(u(\cdot, t))$ is concave in $\Omega$ for any $t > 0$. Thus Theorem 3.1 follows. \(\square\)

**Remark 4.1** Sufficient conditions for the concavity of solutions to parabolic equations were discussed in [9, Section 4.1]. However we cannot apply the arguments in [9, Section 4.1] to show the concavity of $-w(\cdot, t)$, because assumption (F3) with $p = 1$ in [9] is not satisfied for the equation satisfied by $-w$. 

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Remark 4.2 Theorem 3.1 implies that $e^{t \Delta \chi}{\Omega}(x)$ is $2$-logconcave (with respect to $x$) for every $t > 0$. As it is well known, by the eigenfunction expansion of solutions and the regularity theorems for the heat equation, we have $$\lim_{t \to \infty} e^{\lambda_1 t}[e^{t \Delta \chi}{\Omega}](x) = c\phi(x)/\|\phi\|_{L^2(\Omega)}$$ uniformly on $\overline{\Omega}$, where $\lambda_1$ and $\phi$ are as in assertion (b) of the Introduction and $c = \int_{\Omega} \phi(x) \, dx / \|\phi\|_{L^2(\Omega)} > 0$.

Then we may think to obtain the 2-logconcavity of $\phi$ just by letting $t \to +\infty$ and using the preservation of 2-logconcavity by pointwise convergence. Unfortunately this approach does not work, since the parameter $\kappa$ of Definition 2.1 for $e^{t \Delta \chi}{\Omega}(x)$ may tend to 0 as $t$ tends to $+\infty$, while 2-logconcavity is preserved only if $\kappa$ remains strictly positive.

Proof of Corollary 3.1. Corollary 3.1 directly follows from Theorem 3.1 Definition 2.1 and the linearity of the heat equation. □

At the end of this section we prove Theorem 3.2 with the aid of Lemma 2.3.

Proof of Theorem 3.2. Let us consider the case of $\Omega = \mathbb{R}^N$. Since $F$ is stronger than 2-logconcavity, by Lemma 2.3 we see that $G(\cdot, t) \notin C[F]$ for any $t > 0$. Then, for any $\epsilon > 0$, there exist $\kappa \in (0, \epsilon)$, $\mu \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}^N$ such that

$$F \left( \kappa e^{-[(1-\mu)x_1+\mu x_2]^2} \right) - (1-\mu)F \left( \kappa e^{-|x_1|^2} \right) - \mu F \left( \kappa e^{-|x_2|^2} \right) < 0.$$ (4.8)

Let $K$ be a bounded convex set in $\mathbb{R}^N$ such that $|K| > 0$ and set $u := e^{t \Delta \chi K}$. Since $\chi_K$ is $\infty$-concave, we see that $\chi_K$ is $F$-concave.

On the other hand, it follows from (1.5) that

$$\lim_{t \to \infty} \frac{N}{2} \|u(t) - |K|G(t)\|_{L^\infty(\mathbb{R}^N)} = 0.$$ This implies that

$$\lim_{t \to \infty} (4\pi t)^{\frac{N}{2}} |K|^{-1} u(2\sqrt{t}\xi, t) = e^{-|\xi|^2}, \quad \xi \in \mathbb{R}^N.$$ (4.9)

For any $t \geq 1$ and $i = 1, 2$, set $\xi_i^t := 2\sqrt{t}x_i$. Since $F$ is continuous in $(0, 1]$, by (4.8) and (4.9) we have

$$F \left( \left(4\pi t \right)^{\frac{N}{2}} |K|^{-1} \kappa u((1-\mu)\xi_1^t + \mu \xi_2^t, t) \right) - (1-\mu)F \left( \left(4\pi t \right)^{\frac{N}{2}} |K|^{-1} \kappa u(\xi_1^t, t) \right) - \mu F \left( \left(4\pi t \right)^{\frac{N}{2}} |K|^{-1} \kappa u(\xi_2^t, t) \right)$$

$$\to F \left( \kappa e^{-[(1-\mu)x_1+\mu x_2]^2} \right) - (1-\mu)F \left( \kappa e^{-|x_1|^2} \right) - \mu F \left( \kappa e^{-|x_2|^2} \right) < 0$$

as $t \to \infty$. Since $\epsilon$ is arbitrary, we see that

$$u(\cdot, T) = e^{T \Delta \chi K}$$ is not $F$-concave for all sufficiently large $T$. (4.10)
Thus $F$-concavity is not preserved by the heat flow in $\mathbb{R}^N$.

Next we consider the case of $\Omega \neq \mathbb{R}^N$. We can assume, without loss of generality, that $0 \in \Omega$ and $K \subset \Omega$. For $n = 1, 2, \ldots$, set $\Omega_n := n\Omega$. Then

$$
\lim_{n \to \infty} [e^{t\Delta_n} \chi_K](x) = [e^{t\Delta} \chi_K](x)
$$

(4.11)

for any $x \in \mathbb{R}^N$ and $t > 0$. Let $T'$ be a sufficiently large constant. By (4.10) and (4.11) we observe that $e^{T'\Delta_n} \chi_K$ is not $F$-concave for all sufficiently large $n$. Since

$$
[e^{n^2t\Delta_n} \chi_K](nx) = [e^{t\Delta} \chi_{n^{-1}K}](x), \quad x \in \Omega, \ t > 0,
$$

we see that $e^{n^{-2}T'\Delta_n} \chi_{n^{-1}K}$ is not $F$-concave for all sufficiently large $n$. Combining the fact that $\chi_{n^{-1}K}$ is $F$-concave, we see that $F$-concavity is not preserved by the heat flow in $\Omega$. Thus Theorem 3.2 follows. $\Box$

5 Proof of Theorem 3.3

We modify the arguments in the proof of [14, Theorem 5.1] and prove Theorem 3.3.

Proof of Theorem 3.3. Let $u_0$ be a nontrivial bounded nonnegative function in $\mathbb{R}^N$ such that $\text{supp } u_0 \subset B(0, R)$ for some $R > 0$. Without loss of generality, we can assume that

$$
\int_{\mathbb{R}^N} y_i u_0(y) \, dy = 0, \quad i = 1, \ldots, N.
$$

(5.1)

Let $u := e^{t\Delta} u_0$. It follows from (1.1) and (1.5) that

$$
0 < u \leq \min \left\{ \| u_0 \|_{L^\infty(\mathbb{R}^N)}, (4\pi t)^{-\frac{N}{2}} \| u_0 \|_{L^1(\mathbb{R}^N)} \right\}
$$

(5.2)

for $(x, t) \in \mathbb{R}^N \times (0, \infty)$. In particular, $0 < u(x, t) < 1$ in $\mathbb{R}^N \times (T, \infty)$ for some $T > 0$.

Let $\gamma := 1/\alpha \in (1/2, 1]$. For the proof of Theorem 3.3 it suffices to prove that

$$
v(x, t) := (-\log u(x, t))^\gamma
$$

is convex in $\mathbb{R}^N$ for all sufficiently large $t$. By (1.1), (1.5) and (5.1), for $i = 1, \ldots, N$, we have

$$
\frac{u_{x_i}(x, t)}{u(x, t)} = \left[ -\frac{x_i}{2t} + \frac{1}{2t} X_i \right] = \frac{x_i^2}{4t^2} - \frac{x_i}{2t^2} X_i + \frac{1}{4t^2} X_i^2,
$$

$$
\frac{u_{x_ix_i}(x, t)}{u(x, t)} = -\frac{1}{2t} + \frac{x_i^2}{4t^2} - \frac{x_i}{2t^2} X_i + \frac{1}{4t^2} Y_i,
$$

(5.3)

for $(x, t) \in \mathbb{R}^N \times (0, \infty)$, where

$$
X_i := \frac{1}{u} \int_{\mathbb{R}^N} y_i G(x - y, t) u_0(y) \, dy, \quad Y_i := \frac{1}{u} \int_{\mathbb{R}^N} y_i^2 G(x - y, t) u_0(y) \, dy.
$$

(5.4)

It follows from $\text{supp } u_0 \subset B(0, R)$ that

$$
|X_i| \leq R, \quad 0 \leq Y_i \leq R^2.
$$

(5.4)
By (5.4) and (5.6) we have
\[ v_{x_i} = -\gamma(-\log u)^{(1-\gamma)} \frac{u_{x_i}}{u}, \]
\[ v_{x,x_i} = -\gamma(1-\gamma)(-\log u)^{(2-\gamma)} \frac{(u_{x_i})^2}{u^2} + \gamma(-\log u)^{(1-\gamma)} \frac{(u_{x_i})^2}{u^2} - \gamma(-\log u)^{(1-\gamma)} \frac{u_{x,x_i}}{u}, \]
by (5.3) we obtain
\[ \frac{2t}{\gamma}(-\log u)^{1-\gamma} v_{x,x_i} = 2t \left[ -\frac{u_{x,x_i}}{u} + \frac{(u_{x_i})^2}{u^2} \right] - 2t(1-\gamma)(-\log u)^{1}(\frac{u_{x_i})^2}{u^2} = 1 + \frac{1}{2t} X_i^2 - \frac{1}{2t} Y_i - (1-\gamma)(-\log u)^{-1} \left[ \frac{x_i^2}{2t} - \frac{x_i}{t} X_i + \frac{1}{2t} X_i^2 \right] \]
for \((x,t) \in \mathbb{R}^N \times (T, \infty)\). Since \(\lim_{t \to \infty} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = 0\) (see (5.2)), taking a sufficiently large \(T\) if necessary, we have
\[ \frac{2t}{\gamma}(-\log u)^{1-\gamma} v_{x,x_i} \geq 1 + \frac{1}{4t} X_i^2 - \frac{1}{2t} Y_i - (1-\gamma)(-\log u)^{-1} \left[ \frac{x_i^2}{2t} - \frac{x_i}{t} X_i \right] \]
for \((x,t) \in \mathbb{R}^N \times (T, \infty)\).

Let \(0 < \epsilon < 1\). By (5.2) we take a sufficiently large \(T\) so that
\[ (-\log u)^{-1} \leq \left( -\log \left[ (4\pi t)^{-\frac{N}{2}} \|u_0\|_{L^1(\mathbb{R}^N)} \right] \right)^{-1} \leq (\frac{N}{4 \log t})^{-1} \]
for \((x,t) \in \mathbb{R}^N \times (T, \infty)\). We consider the case where \((x,t) \in \mathbb{R}^N \times (T, \infty)\) with \(|x|^2 \leq \epsilon t \log t\).

By (5.4) and (5.3) we have
\[ (-\log u)^{-1} \left[ \frac{x_i^2}{2t} - \frac{x_i}{t} X_i \right] \leq \left( \frac{N}{4 \log t} \right)^{-1} \left[ \frac{|x|^2}{2t} + R \frac{|x|}{t} \right] \]
\[ \leq \left( \frac{N}{4 \log t} \right)^{-1} \left[ \frac{\epsilon t \log t}{2} + R \epsilon t^{-\frac{1}{2}} (\log t)^{-\frac{1}{2}} \right] \]
\[ = \frac{2 \epsilon}{N} + \frac{4 \epsilon}{N} R \epsilon t^{-\frac{1}{2}} (\log t)^{-\frac{1}{2}}. \]

By (5.4), (5.5) and (5.7), taking a sufficiently small \(\epsilon > 0\) and a sufficiently large \(T\) if necessary, we obtain
\[ \frac{2t}{\gamma}(-\log u)^{1-\gamma} v_{x,x_i} \geq 1 - \frac{R^2}{2t} - (1-\gamma) \left[ \frac{2 \epsilon}{N} + \frac{4 \epsilon}{N} R \epsilon t^{-\frac{1}{2}} (\log t)^{-\frac{1}{2}} \right] \geq \frac{1}{2} \]
for \((x,t) \in \mathbb{R}^N \times (T, \infty)\) with \(|x|^2 \leq \epsilon t \log t\).

We consider the case where \((x,t) \in \mathbb{R}^N \times (T, \infty)\) with \(|x|^2 > \epsilon t \log t\). Let \(\delta\) be a positive constant to be chosen later. Since \(\text{supp } u_0 \subset B(0,R)\), by (1.5), taking a sufficiently large \(T\) if necessary, we have
\[ u(x,t) \leq (4\pi t)^{-\frac{N}{2}} \exp \left( -\frac{|x|^2}{4(1+\delta)t} \right) \|u_0\|_{L^1(\mathbb{R}^N)}. \]
This implies that
\[ (-\log u)^{-1} \leq \left( \frac{N}{2} \log(4\pi t) + \frac{|x|^2}{4(1+\delta)t} - \log \|u_0\|_{L^1(\mathbb{R}^N)} \right)^{-1} \leq \frac{4(1+\delta)t}{|x|^2}. \]

It follows from (5.4) and (5.5) that
\[ \frac{2t}{\gamma} (-\log u)^{1-\gamma} v_{x_i x_i} \geq 1 - \frac{R^2}{2t} - (1-\gamma) \frac{4(1+\delta)t}{|x|^2} \left( \frac{|x|^2}{2t} + R \frac{|x|}{t} \right) \]
\[ \geq 1 - 2(1-\gamma)(1+\delta) - \frac{R^2}{2t} - (1-\gamma) \frac{4(1+\delta)}{(et \log t)^{\frac{1}{2}}} \tag{5.9} \]

Since \(1/2 < \gamma \leq 1\), taking a sufficiently small \(\delta > 0\), we see that \(1 - 2(1-\gamma)(1+\delta) \geq \delta\). Then, by (5.9), taking a sufficiently large \(T\) if necessary, we obtain
\[ \frac{2t}{\gamma} (-\log u)^{1-\gamma} v_{x_i x_i} \geq \frac{\delta}{2} \tag{5.10} \]

for \((x, t) \in \mathbb{R}^N \times (T, \infty)\) with \(|x|^2 > et \log t\). Combining (5.8) and (5.10), we deduce that \(v(\cdot, t)\) is convex in \(\mathbb{R}^N\) for \(t \geq T\). Therefore we see that \(L_\alpha(u(\cdot, t))\) is concave in \(\mathbb{R}^N\) for \(t \geq T\). This together with Lemma 2.1 implies that \(u(\cdot, t)\) is \(\alpha\)-logconcave in \(\mathbb{R}^N\) for \(t \geq T\). Thus Theorem 3.3 follows. \(\square\)

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