Rigid $G_2$-Representations and motives of Type $G_2$

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1 Introduction

We consider a family of motives $(M_s)_{s \in \mathbb{A}^1 \setminus \{0,1\}}$ whose generic member has motivic Galois group isomorphic to a simple algebraic group of type $G_2$. Such families, attached to certain rigid local systems with $G_2$-monodromy, are introduced and studied in [DRK10]. There are exactly 5 local systems, whose associated Galois representations possibly can be defined over $\mathbb{Q}$. In this paper we study the family of motives attached to the local system on $\mathbb{A}^1 \setminus \{0,1\}$ whose local monodromy at $0, 1, \infty$ is given by

$\text{diag}(-1, -1, -1, 1, 1, 1), \quad J(2) \oplus J(2) \oplus J(3), \quad \text{diag}(\zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5, \zeta_7^6, 1),$

(respectively) where $J(k)$ denotes a Jordan block of length $k$. The family of motives arises from the family of varieties (with respect to the coordinate $X_7$), defined by

$Y^{14} = \prod_{1 \leq i \leq 7; j=1,2} (X_i - T_j)^{e(i,j)} \prod_{1 \leq j \leq 6} (X_{j+1} - X_j)^{f(j)},$

where the $e(i,j), f(j)$ are certain natural numbers between 1 and 13 and where $T_1 = 0, T_2 = 1$ (cf. Section 2).

It is a consequence of the rigidity criterion in inverse Galois theory that, although the underlying motives are a priori only defined over $\mathbb{Q}(\zeta_7)$ ($\zeta_7$ denoting a primitive 7-th root of unity), the monodromy representation

$\rho_\ell = \rho_{\mathcal{Y}_{\mathbb{Q}(\zeta_7)}} : \pi_1(\mathbb{A}^1_{\mathbb{Q}(\zeta_7)} \setminus \{0,1\}) \to \text{GL}_7(\mathbb{Q}_\ell)$

of the étale realization $\mathcal{Y}_{\mathbb{Q}(\zeta_7)}$ of the family $(M_s)_{s \in \mathbb{A}^1 \setminus \{0,1\}}$ has the property that the residual representation can be defined over $\mathbb{Q}$: There exists a lisse sheaf $\overline{\mathcal{Y}}_{\mathbb{Q}}$ on $\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}$ with monodromy representation

$\overline{\rho}_\ell = \rho_{\overline{\mathcal{Y}}_{\mathbb{Q}}} : \pi_1(\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}) \to \text{GL}_7(\mathbb{F}_\ell)$

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such that \( \tilde{\rho}_\ell \) restricted to \( \pi_1(\mathbb{A}_{\mathbb{Q}(\zeta)}^1 \setminus \{0,1\}) \) is isomorphic to the residual representation of \( \rho_\ell \).

For a point \( s \in S(K) \), where \( K \) is any field and \( S \) a scheme, we have a map 
\( i_s : G_K \to \pi_1(S) \) by the functoriality of the functor \( \pi_1 \). The composition \( \rho_\ell \circ i_s : G_K \to \GL_7(\mathbb{Q}_\ell) \) (resp. \( \tilde{\rho}_\ell \circ i_s : G_K \to \GL_7(\mathbb{F}_\ell) \)) is called the specialization of \( \rho_\ell \) (resp. \( \tilde{\rho}_\ell \)) with respect to \( s \) and is denoted \( \rho_\ell^{(s)} \) (resp. \( \tilde{\rho}_\ell^{(s)} \)). We prove the following effective Hilbert irreducibility result for the specializations of \( \tilde{\rho}_\ell \) (Thm. 3.4):

1.1 Theorem. Let \( \ell \neq 2, 3, 7, 11 \) and 13. Let \( s \in \mathbb{Q} \setminus \{0,1\} \) and suppose there exist prime numbers \( p,q \neq 2,7,\ell \) with \( p \nmid G_2(\ell) \), \( p \equiv 3 \) or 5 mod 7, \( \nu_p(s) < 0 < \nu_q(s-1) \), 7 \( \nmid \nu_q(s-1) \) and \( \ell \nmid \nu_q(s-1) \). Then \( \tilde{\rho}_\ell^{(s)} : G_\mathbb{Q} \to G_2(\mathbb{F}_\ell) \) is surjective.

Since the reduction modulo-\( \ell \) map \( G_2(\mathbb{Z}_\ell) \to G_2(\mathbb{F}_\ell) \) has the Frattini property (cf. [Wei95]), Thm. 1.1 implies that for \( s \) as in the theorem, the twist by a certain character of order \( \leq 2 \) of the Galois representation
\[
\rho_\ell^{(s)} : G_\mathbb{Q}(\zeta) \to \GL_7(\mathbb{Q}_\ell)
\]
has an image equal to \( G_2(\mathbb{Z}_\ell) \). Under the Tate conjecture, this implies that the motive \( M_s \), where \( s \) is as in the theorem, has motivic Galois group of type \( G_2 \) (Thm. 4.5).

2 Construction of sheaves with monodromy \( G_2 \)

2.1 Introduction of the basic objects. For \( N \) a natural number denote by \( R_N \) the ring \( \mathbb{Z}[\zeta, \frac{1}{13N}] \). As in [DRK10] we want to construct in a nice geometric way a lisse sheaf on \( \mathbb{A}_{R_N}^1 \setminus \{0,1\} \) having monodromy (Zariski dense inside) \( G_2(\mathbb{Q}_\ell) \) and whose mod \( \ell \) reduction has monodromy contained in \( G_2(\mathbb{F}_\ell) \).

Denote by \( S_N \) the ring \( R_N[T_1,T_2,\frac{1}{13N}] \). Set
\[
\Delta_{T_1,T_2} := \prod_{i,j} (X_i - T_j) \prod_k (X_{k+1} - X_k),
\]
where \( i \) runs through \( \{1,\ldots,7\} \), \( j \) through \( \{0,1\} \) and \( k \) runs through \( \{1,\ldots,6\} \). Denote by \( \text{Hyp}_N \) the hypersurface in \( \mathbb{G}_{m,R_N} \otimes_{R_N} (\mathbb{A}_{S_N}^7 \setminus v(\Delta_{T_1,T_2})) \) defined by
\[
Y^{14} = \prod_{1 \leq i \leq 7} \prod_{j=1,2} (X_i - T_j)^{e(i,j)} \prod_{1 \leq j \leq 6} (X_{j+1} - X_j)^{f(j)},
\]
where \( Y \) denotes the coordinate function of \( \mathbb{G}_{m,R_N} \), \( X_1,\ldots,X_7 \) denotes the coordinate functions of \( \mathbb{A}_{S_N}^7 \) and the exponents are as follows:

| \( e(1,1) \) | \( e(2,1) \) | \( e(3,1) \) | \( e(4,1) \) | \( e(5,1) \) | \( e(6,1) \) | \( e(7,1) \) |
|---|---|---|---|---|---|---|
| 7 | 0 | 7 | 0 | 7 | 0 | 7 |
| \( e(1,2) \) | \( e(2,2) \) | \( e(3,2) \) | \( e(4,2) \) | \( e(5,2) \) | \( e(6,2) \) | \( e(7,2) \) |
| 13 | 5 | 0 | 5 | 0 | 5 | 0 |
and
\[
\begin{array}{cccccc}
 f(1) & f(2) & f(3) & f(4) & f(5) & f(6) \\
 11 & 3 & 1 & 13 & 5 & 9 \\
\end{array}
\]

Let
\[
\pi_N : \text{Hyp}_N \longrightarrow \mathbb{A}^1_{SN} \setminus \{T_1, T_2\}
\]
be the morphism induced by the inclusion
\[
S_N[X_7, (X_7-T_1)(X_7-T_2)][X_1, \ldots, X_7, Y, \Theta_{T_1, T_2}].
\]

We think of \( \bm{14} \) as the group of 14th roots of unity in \( R_N \). It acts on \( \text{Hyp}_N \) via \( Y \mapsto \omega Y \) (for \( \omega \in \bm{14} \)), inducing an \( \bm{14} \)-action on \( R^q(\pi_N)|/(\mathbb{Q}_\ell) \). Think of \( R_N \) being embedded in \( \mathbb{Q}_\ell \). Let \( \chi \) be the \( \mathbb{Q}_\ell \)-valued character of \( \bm{14} \) given by this embedding. By \( (R^q(\pi_N)|/(\mathbb{Q}_\ell))^\chi \) we mean the \( \chi \) eigenspace of the \( \bm{14} \)-action. Then by [DRK10] Thm. 2.3.1 and Thm. 2.4.1, \( (R^q(\pi_N)|/(\mathbb{Q}_\ell))^\chi \) is a lisse \( \mathbb{Q}_\ell \)-sheaf on \( \mathbb{A}^1_{SN} \setminus \{T_1, T_2\} \) of mixed weights \( \leq q \) and
\[
\mathcal{V} := W^6[(R^6(\pi_N)|/(\mathbb{Q}_\ell))^\chi]|_{\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}}
\]
is cohomological rigid (see [Kat96] (5.0.1)) with induced monodromy representation \( \rho_\mathcal{V} \) Zariski dense in \( G_2(\mathbb{Q}_\ell) \). Further, its monodromy tuple \( (g_0, g_1, g_\infty) \) has the following Jordan canonical forms:
\[
\begin{align*}
\text{JCF}(g_0) &= \text{diag}(-1,-1,-1,-1,1,1), \\
\text{JCF}(g_1) &= J(2) \oplus J(2) \oplus J(3) \\
\text{JCF}(g_\infty) &= \text{diag}(\zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5, \zeta_7^6, 1).
\end{align*}
\]

Throughout the paper we use the following notation: If \( \mathcal{W} \) is a lisse sheaf on a connected scheme \( X \), then \( \rho_{\mathcal{W}} : \pi_1(X, \bar{s}) \rightarrow \text{Aut}(\mathcal{W}_{\bar{s}}) \) denotes the monodromy representation of \( \mathcal{W} \) with base point \( \bar{s} \). For \( k \) be a number field containing \( \zeta_7 \) set \( \mathcal{V}_k := W^6[(R^6(\pi_N)|/(\mathbb{Q}_\ell))^\chi]|_{\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\}} \) (similar define \( \mathcal{V}_{SN}, \mathcal{V}_{R_\infty}, \ldots \)). By definition, this is an extension of \( \mathcal{V} \) to \( \mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\} \). Since the geometric fundamental group is a normal subgroup of \( \pi_1^{\text{et}}(\mathbb{A}^n_{\mathbb{Q}} \setminus \{0,1\}) \), the image of the induced representation \( \rho_{\mathcal{V}_k} \) normalizes the image of \( \rho_\mathcal{V} \).

By [DRK10] Thm. 2.3.1, \( \rho_{\mathcal{V}_{SN}} \) is already defined over \( \mathbb{Q}_\ell(\zeta_7) \) and standard arguments give us that it is \( \text{GL}_7(\mathbb{Q}_\ell(\zeta_7)) \)-conjugate to a representation with values in \( \text{GL}_7(\mathbb{Z}_\ell(\zeta_7)) \). Thus we may assume, that \( \rho_\mathcal{V}, \rho_{\mathcal{V}_{SN}}, \rho_{\mathcal{V}_{R_\infty}}, \rho_{\mathcal{V}_k}, \ldots \) have values in \( \text{GL}_7(\mathbb{Z}_\ell(\zeta_7)) \). Define the mod \( \ell \) reductions \( \mathcal{V}, \mathcal{V}_{SN}, \mathcal{V}_{R_\infty}, \mathcal{V}_k, \ldots \) as the étale locally constant sheaves given by the naive mod \( \ell \) reductions of the above representations.

### 2.2 Computation of the monodromy of \( \mathcal{V} \)

If \( G \) is an algebraic group defined over \( \mathbb{Z} \) (e.g. \( G = \text{GL}_n \) or \( G_2 \leq \text{GL}_7 \)) and if \( \ell^m \) is a power of a prime number \( \ell \), we write \( G(\mathbb{F}_{\ell^m}) \) for the group \( G(\mathbb{F}_{\ell^m}) \). From now on let \( \ell \neq 2,3,7 \). Let \( \bm{\sigma} := (g_0, g_1, g_\infty) \) be the monodromy tuple of \( \mathcal{V} \). It is not hard to see that the Jordan canonical forms remain untouched: \( g_0 \) is an involution so its Jordan
canonical form is determined by its trace, which clearly is the mod \( \ell \) reduction of \( \text{tr}(g_0) \). By our assumption on \( \ell \), \( \text{JCF}(g_0) = \overline{\text{JCF}(g_0)} \) follows easily. By looking at the minimal polynomial it is clear that \( g_1 \) is unipotent with Jordan blocks of length less than three. If it has more than three blocks, \( \text{dim}(\text{Eig}(\overline{g_1}, 1)) \geq 4 \). But \( \text{dim}(\text{Eig}(g_0, -1)) = 4 \). In particular the intersection of both eigenspaces is non trivial and thus the product relation \( g_0 g_1 \overline{g_\infty} = 1 \) implies that \( g_\infty \) has \(-1\) as eigenvalue which is clearly a contradiction. The only remaining possibility except \( \text{JCF}(g_1) = \overline{\text{JCF}(g_1)} \) is two blocks of length three and one of length one. But then \( \text{rk}((\overline{g_1} - 1)^2) \) would be greater then \( \text{rk}((g_1 - 1)^2) \) which is impossible. Finally, by our assumptions on \( \ell \) the characteristic polynomial of \( g_\infty \) is still separable, hence \( \text{JCF}(\overline{g_\infty}) = \overline{\text{JCF}(g_\infty)} \) holds as well.

As a result, the \( \text{GL}_7(\overline{\mathbb{F}_\ell}) \)-conjugacy classes of the \( \overline{g}_i \)‘s are \( \phi_\ell \)-stable, where \( \phi_\ell \) denotes the component wise Frobenius map. Moreover, we still have the product relation \( \phi_\ell(g_0) \phi_\ell(g_1) \phi_\ell(g_\infty) = 1 \). But the Katz algorithm for rigid \cite{Kat96} Chapter 6 resp. its analogue to arbitrary algebraically closed fields of \cite{DR00} yields that \( \sigma \) is an absolutely irreducible linearly rigid tuple. Here linearly rigid means that whenever a triple of elements in the \( \text{GL}_7(\overline{\mathbb{F}_\ell}) \)-conjugacy classes of the \( \overline{g}_i \)‘s satisfies the above product relation, then they are even simultaneously conjugate to our original \( \overline{g}_i \)‘s. In particular we get a single element \( g \in \text{GL}_7(\overline{\mathbb{F}_\ell}) \) s.t. \( \phi_\ell(g_i) \) equals \( \overline{g}_i^b \) for \( i = 0, 1 \) resp. \( \infty \). Now \( \text{GL}_7(\overline{\mathbb{F}_\ell}) \) is connected so there is an element \( h \) with \( g = h \phi_\ell(h)^{-1} \) by Lang-Steinberg (see e.g. \cite{Car85} 1.17), i.e. \( \phi_\ell \) fixes \( \overline{g}_i^b \). Thus we have shown:

\begin{figure}[h]
\centering
\begin{tabular}{|c|l|}
\hline
Group: & Remarks: \\
\hline
\( P_{\alpha}, P_{\beta} \) & maximal parabolic subgroups \\
\hline
\( C_{G_2(\ell^n)}(\iota) \) & centralizer of the involution \( \iota \) \\
\hline
\( K_\varepsilon \cong \text{L}_\varepsilon \rtimes \mathbb{Z}/2 \) & \( \varepsilon = \pm, \text{L}_+ \cong \text{SL}_3(\ell^n), \text{L}_- \cong \text{SU}_3(\ell^n) \) \\
\hline
\( \text{PGL}_2(\ell^n) \) & for \( \ell \geq 7, \ell^n \geq 11 \) \\
\hline
\((\mathbb{Z}/2)^3: \text{L}_3(2)\) & for \( n = 1 \) \\
\hline
\( L_2(8) \) & for \( \ell \geq 5, \mathbb{F}_{\ell^n} = \mathbb{F}_\ell(\omega) \) with \( \omega^3 - 3\omega + 1 = 0 \) \\
\hline
\( L_2(13) \) & for \( \ell \neq 13, \mathbb{F}_{\ell^n} = \mathbb{F}_\ell(\sqrt{13}) \) \\
\hline
\( G_2(2) \) & for \( \ell^n = \ell \geq 5 \) \\
\hline
\( J_1 \) & for \( \ell^n = 11 \) \\
\hline
\( C_{G_2(\ell^n)}(\phi_{\ell^m}) = G_2(\ell^m) \) & for \( \alpha = \frac{m}{n} \) prime \\
\hline
\end{tabular}
\end{figure}

Here we denote by \((\mathbb{Z}/2)^3: \text{L}_3(2)\) a suitable non-split group extension of \((\mathbb{Z}/2)^3\).
with $L_3(2)$. We do not need an exact definition of $P_\alpha$ and $P_\beta$ at this point, but it can be found latter in section 3.3. We will use this list to prove the following proposition:

2.2 Proposition. Let $\ell \neq 2, 3, 7, 11$ and 13. Suppose $\text{im}(\rho_\psi) \leq G_2(\ell^n)$. Then $\text{im}(\rho_\psi) = G_2(\ell^n)$ or $\text{im}(\rho_\psi)$ is contained in a subgroup $G_2(\ell^n)$-conjugate to $C_{G_2(\ell^n)}(\phi_{\ell m}) = G_2(\ell^n)$ for $\alpha = \frac{a_1}{m}$ a prime number.

Proof: We simply have to check for every type of group in the list, whether the group contains $\text{im}(\rho_\psi)$.

(i) We have seen above that $\text{im}(\rho_\psi)$ acts irreducible on $V(\ell^n) := \mathbb{F}_{\ell^n}$ but the first three groups in the table act reducible on $V(\ell^n)$. Hence $\text{im}(\rho_\psi)$ is not contained in any subgroup $G_2(\ell^n)$-conjugate to one of these subgroups.

(ii) As in the proof of [Kle88] Lem. 4.2.1 $V(\ell^n)$ as PGL$_2(\ell^n)$ module is isomorphic to the $\mathbb{F}_{\ell^n}$-vector space of homogeneous polynomials in $X$ and $Y$ of degree 6 as $\langle \rho(SL_2(\ell^n), \mu^{-3}\rho(\text{diag}(\mu, 1)))\rangle$-module, where $\mu$ is a non square in $\mathbb{F}_{\ell^n}$ and $\rho$ is the GL$_2(\ell^n)$ representation given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : X^i Y^j \mapsto (a X + b Y)^i (c X + d Y)^j.$$

The non trivial unipotent elements in GL$_2(\ell^n)$ clearly form a single conjugacy class. Hence an easy computation shows, that the Jordan canonical form of each $\ell$ as $F$ is the GL$_2(\ell^n)$ as $F$ algebra.

Finally, we can exclude all other maximal subgroups but $G_2(\ell^n)$-conjugate to PGL$_2(\ell^n)$ contains $\text{im}(\rho_\psi)$.

(iii) Finally, we can exclude all other maximal subgroups but $C_{G_2(\ell^n)}(\phi_{\alpha})$ in the above table by comparing their orders with our assumptions on $\ell$. □

Iterative application of this together with Prop. 2.1 give us the following

2.3 Corollary. Let $\ell \neq 2, 3, 7, 11$ and 13. Then the image of $\rho_\psi$ is $G_2(\ell)$.

2.3 An extension of $\tilde{V}$ to $\mathbb{A}^1_\mathbb{Q} \setminus \{0, 1\}$. Let $S$ be the set of places $\{t, t-1, t^{-1}\}$ of $\tilde{Q}(t)$. As in [MM99] I Thm. 2.2 denote by $\tilde{M}_S$ the maximal algebraic extension field of $\tilde{Q}(t)$ inside $\tilde{Q}(t)$ unramified outside $S$. It is well known that $\text{Gal}(\tilde{M}_S/k(t))$ is canonically isomorphic to $\pi^1(\mathbb{A}^1_\mathbb{K} \setminus \{0, 1\}, 0)$ for $\tilde{K}$ an algebraic extension field of $\mathbb{Q}$, where $0$ denotes the geometric point corresponding to the choice of an embedding of $k(t)$ to $\overline{\tilde{Q}(t)}$. In particular $\rho_\psi$ gives rise to an intermediate field of $\tilde{M}_S/\tilde{Q}(t)$, Galois over $\tilde{Q}(t)$ with group isomorphic to $G_2(\ell)$. As in the Hurwitz classification of such intermediate fields in [MM99] I.4.1 denote this field by $\tilde{N}_\sigma$ (i.e. $\rho_\psi$ corresponds to $\psi_{\sigma}$ in loc. cit.). Now, if $\tilde{Q}(t)$ is a field of definition of $\tilde{N}_\sigma/G_2(\ell)\tilde{Q}(t)$ (see [MM99] I.3.1), we get an epimorphism $\rho_\tilde{V}_\tilde{Q} : \text{Gal}(\tilde{M}_S/\tilde{Q}(t)) \rightarrow G_2(\ell)$ which, after a possible modification by an automorphism of $G_2(\ell)$, restricts to $\rho_\psi$. As a result the locally constant sheaf $\tilde{V}_\tilde{Q}$ on $\mathbb{A}^1_\mathbb{Q} \setminus \{0, 1\}$ defined by $\rho_\tilde{V}_\tilde{Q}$ is an extension of $\tilde{V}$.  

5
Our tuple $\sigma$ is rigid in the sense of [MM99] if for each triple $\{h_i\}$ in $G_2(\ell)^3$ satisfying the product relation $\overline{g_0^h g_i^h g_\infty^h} = 1$, there is even an element $h$ of $G_2(\ell)$ with $\overline{g_i^h} = \overline{g_i^h}$ simultaneously for all $i$. Further, $\sigma$ is rational in the sense of [MM99] if for all primes $p$ not dividing the order of $G_2(\ell)$ the $p$ power map preserves the $G_2(\ell)$-conjugacy class of each $\overline{g_i}$. Now if $\sigma$ is both rigid and rational $\mathbb{Q}(\ell)$ is indeed a field of definition of $\overline{N_\sigma} / G_2(\ell) \mathbb{Q}(\ell)$ by the Rigidity Theorem (see [MM99] I Thm. 4.8).

We start with checking the rationality of $\sigma$. By [MM99] I Prop. 4.4 this is equivalent to the statement that every complex irreducible character of $G_2(\ell)$ has rational values for $\overline{g_0}, \overline{g_1}$ and $\overline{g_\infty}$. But this is straightforward, using the character table in [CR74]. Thus $\sigma$ is indeed rational.

Let $p \equiv 3$ or $5 \mod 7$ be a prime number not dividing the order of $G_2(\ell)$. Then the rationality of $\sigma$ implies that there exists an element $h_{\infty,p}$ in $G_2(\ell)$ with $\overline{g_\infty^h} = \overline{g_0^p}$. It is not hard to see that relative to the Jordan canonical form of $\overline{g_\infty}$, $h_{\infty,p}$ is of the form:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \pm 1 \times \text{an element of } D
$$

(2.1)

where $D$ denotes the diagonal torus of $GL_7(\mathbb{F}_\ell(\zeta_7))$. With the aid of $h_{\infty,\ell}$ we can show that

$$
N_{GL_7(\mathbb{F}_\ell)}(G_2(\ell)) = G_2(\ell) \times \mathbb{F}_\ell^x :
$$

First note that $N_K(G_2(K)) = G_2(K) \times K^\times$ holds for $K$ algebraically closed. Indeed, if $h$ is an element of the normalizer and $f(\cdot, \cdot, \cdot)$ denotes the Dickson alternating trilinear form then $G_2(K)$ still fixes the form $f(h(\cdot), h(\cdot), h(\cdot))$. Hence by [Asc87] Thm. 5 (2) this form is a scalar multiple of the original Dickson form. Since $K$ is assumed to be algebraically closed, a scalar multiple of $h$ fixes the Dickson form. By [Asc87] (2.11) and (3.4) this means $h \in G_2(K) \times K^\times$ ($G_2$ is a simple group!), as claimed. The other inclusion is clear.

Now for $g \in G_2(\ell)$ and $h \in N_{GL_7(\mathbb{F}_\ell)}(G_2(\ell))$ arbitrary we have $g^{\phi_\ell(h)} = \phi_\ell(g^h) = g^h$, i.e. $h^{-1} \phi_\ell(h) \in C_{GL_7(\mathbb{F}_\ell)}(G_2(\ell))$. Let $h'$ be arbitrary in this centralizer. We may assume, that $\overline{g_\infty}$ is in its Jordan canonical form. In particular $h'$ is contained in $D$ (since this is just the centralizer of $\overline{g_\infty}$). Thus it follows from (2.1) that $h'$ is of the form $\text{diag}(a, \ldots, a, b)$. But e.g. from $\overline{g_1^h} = \overline{g_1}$ we can deduce that $a = b$. In particular $h^{-1} \phi_\ell(h)$ lies in the connected center of $GL_7(\mathbb{F}_\ell)$. So by Lang-Steinberg there is an element $\lambda$ in this center s.t. $h^{-1} \phi_\ell(h)$ equals $\lambda \phi_\ell(\lambda)^{-1}$, i.e. the product
$h\lambda$ is fixed by $\phi_\ell$. But our above computation of $N_{\text{GL}_7}(\bar{k}_\ell)(G_2(\bar{k}_\ell))$ shows, that
the $\phi_\ell$ fixed points are just $G_2(\ell) \times \mathbb{F}_\ell^\times$, which completes the proof of (2.2).

Combining this with the linear rigidity of $\sigma$ we finally get the rigidity in the
device sense of [MM99] and thus our desired extension of $\bar{\mathcal{V}}$ to $\mathbb{A}_Q^1 \setminus \{0,1\}$.

**2.4 Proposition.** Let $\ell \neq 2, 3, 7, 11$ and 13. Then there exists an extension $\bar{\mathcal{V}}_\mathbb{Q}$
of $\mathcal{V}$ to $\mathbb{A}_\mathbb{Q}^1 \setminus \{0,1\}$, satisfying
\[ \text{im}(\rho_{\mathcal{V}_\mathbb{Q}}) = \text{im}(\rho_{\mathcal{V}}) (= G_2(\ell)). \]

**3 Specializations of $\rho_{\mathcal{V}_\mathbb{Q}}$**

From now on always assume $\ell \neq 2, 3, 7, 11$ and 13. Denote the $\mathbb{Q}$-rational point of
$\mathbb{A}_\mathbb{Q}^1 \setminus \{0,1\}$ induced by $s \in \mathbb{Q} \setminus \{0,1\}$ by $s$ as well. Composition with the induced
map on fundamental groups with $\rho_{\mathcal{V}_\mathbb{Q}}$ gives us the specialization at $s$
\[ \rho_{\mathcal{V}_\mathbb{Q}}^{(s)} : G_\mathbb{Q} \longrightarrow G_2(\ell) \]
(similar for $\mathcal{V}_k, \bar{\mathcal{V}}_k, \ldots$). In this section we want to proof that for suitable choices
of $s$, the Galois representation $\rho_{\mathcal{V}_\mathbb{Q}}^{(s)}$ is onto.

**3.1 Restriction of $\rho_{\mathcal{V}_\mathbb{Q}}^{(s)}$ to inertia subgroups.** First we have to modify
$\bar{\mathcal{V}}_{R_N}, \bar{\mathcal{V}}_k$ s.t. the resulting sheaves are compatible with $\bar{\mathcal{V}}_\mathbb{Q}$. Since $\pi_\ell^{(s)}(\mathbb{A}_Q^1 \setminus \{0,1\})$
is a normal subgroup of $\pi_\ell^{(s)}(\mathbb{A}_Q^1(\zeta) \setminus \{0,1\})$ and $\pi_\ell^{(s)}(\mathbb{A}_R^1 \setminus \{0,1\})$ is a quotient of
the latter group, the image of $\rho_{\mathcal{V}_{R_N}}$ is contained in $N_{\text{GL}_7(\mathbb{F}(\zeta))}(G_2(\ell)) = G_2(\ell) \times
\mathbb{F}_\ell(\zeta)^\times$. As a result we get a continuous character
\[ \pi_\ell^{(s)}(\mathbb{A}_R^1 \setminus \{0,1\}) \xrightarrow{\rho_{\mathcal{V}_{R_N}}} G_2(\ell) \times \mathbb{F}_\ell(\zeta)^\times \xrightarrow{\text{pr}_{\mathcal{V}_\mathbb{Q}}^{\mathcal{V}_\mathbb{Q}}} \mathbb{F}_\ell(\zeta)^\times \xrightarrow{(\cdot)^{-1}} \mathbb{F}_\ell(\zeta)^\times. \]

Define $\bar{\mathcal{V}}_{R_N}^\varepsilon, \bar{\mathcal{V}}_k^\varepsilon, \ldots$ as the twisted of $\bar{\mathcal{V}}_{R_N}, \bar{\mathcal{V}}_k, \ldots$ by the restricted characters.
Clearly $\mathcal{V}_k^\varepsilon = \mathcal{V}$. In particular these sheaves are still extensions of $\bar{\mathcal{V}}$, with
\[ \text{im}(\rho_{\mathcal{V}_k^\varepsilon}) = G_2(\ell). \]
Set $N := M_{\text{S}_k}^{\text{ker}(\rho_{\mathcal{V}_k})}$. Using basic arguments involving the profi-
finite Galois correspondence we derive that $N \cdot \mathbb{Q}(t) = \bar{N}_{\sigma}$ and $N \cap \mathbb{Q}(t) = \mathbb{Q}(\zeta)(t)$.
We can deduce the analog statements for $N' := \mathbb{Q}(\zeta)(t) \cdot M_{\text{S}_k}^{\text{ker}(\rho_{\mathcal{V}_k})}$: Indeed, the
desired properties hold for $M_{\text{S}_k}^{\text{ker}(\rho_{\mathcal{V}_k})}$ over $\mathbb{Q}(t)$. Thus, using arguments from the
proof of [MM99] I Prop. 3.1, we see that $N$ and $N'$ coincides. In other words
(after a possible modification by an automorphism of the monodromy) we get that
$\mathcal{V}_{k}^\varepsilon$ is the restriction of $\mathcal{V}_{\mathbb{Q}}$.

**3.1 Lemma.** Let $p \neq 2, 7$ and $\ell$ be a prime number. Further, let $s \in \mathbb{Q} \setminus \{0,1\}$
with $\nu_\ell(s) > 0$ and $2 \nmid \nu_\ell(s)$ (for $i = 0$), $\nu_\ell(s-1) > 0$ and $\ell \nmid \nu_\ell(s-1)$ (for $i = 1$)
resp. $\nu_\ell(s) < 0$ and $7 \nmid \nu_\ell(s)$ (for $i = \infty$). Then the image of an inertia subgroup
$I_p \leq G_\mathbb{Q}$ under the specialization $\rho_{\mathcal{V}_\mathbb{Q}}^{(s)}$ at $s$ contains an element conjugate to $\bar{g}_i$. 
Recall that the absolute Galois groups resp. étale fundamental group of \( \hat{\mathbb{Q}}_{p}^{nr} \), \( \mathbb{C}_{p}((z)) \) resp. \( \Delta_{p} := \text{Spec}W_{p}((z)) \) can be computed using the well known diagram

\[
\begin{array}{ccc}
G_{\mathbb{C}_{p}((z))} & \xrightarrow{\sim} & \lim_{\mathbb{n}} \mu_{n} \\
\downarrow \text{pr} & & \\
\pi_{1}^{\text{ét}}(\Delta_{p}) & \xrightarrow{\sim} & \lim_{\mathbb{p} \nmid n} \mu_{n} \\
\downarrow \text{pr} & & \\
G_{\hat{\mathbb{Q}}_{p}^{nr}} & \xrightarrow{\text{res}} & \lim_{\mathbb{p} \nmid n} \mu_{n}
\end{array}
\]

(3.1)

(here \( W_{p} \) denotes the Wittring of \( \overline{\mathbb{F}}_{p} \) with field of fractions \( \hat{\mathbb{Q}}_{p}^{nr} \) and \( G_{\hat{\mathbb{Q}}_{p}^{nr}} \rightarrow \pi_{1}^{\text{ét}}(\Delta_{p}) \) is induced by the evaluation \( z \mapsto a \) for \( a \in W_{p} \)). Note that the upper and lower commutative squares are taken with resp. to different base points of \( \Delta_{p} \). But since the claim of the lemma is only about conjugacy classes, we will often omit the base points in our notation. We will apply (3.1) for \( z = t \) (for \( i = 0 \)), \( z = t^{-1} \) (for \( i = 1 \)) resp. \( z = \frac{1}{t} \) (for \( i = \infty \)) and \( a = z(s) \).

**Proof** (of the lemma): Suppose \( p \nmid N \). The canonical inclusions of the resp. coordinate rings gives us the commutative diagram

\[
\begin{array}{ccc}
G_{2}(\ell) & \xrightarrow{\rho_{V,R_{N}}} & \pi_{1}^{\text{ét}}(A_{R_{N}}^{1} \setminus \{0,1\}) \\
\rho_{\overline{V}} & & \pi_{1}^{\text{ét}}(A_{\overline{Q}}^{1} \setminus \{0,1\}) \\
\pi_{1}^{\text{ét}}(\Delta_{p}) & \xrightarrow{\text{res}} & G_{\mathbb{C}_{p}((z))} \xrightarrow{\sim} G_{\hat{\mathbb{Q}}((z))}.
\end{array}
\]

The restriction map \( G_{\hat{\mathbb{Q}}((z))} \rightarrow \text{Gal}(|\overline{M}_{S}/\overline{Q}(t)) = \pi_{1}^{\text{ét}}(A_{\overline{Q}}^{1} \setminus \{0,1\}) \) identifies \( G_{\hat{\mathbb{Q}}((z))} \) with one of the conjugated inertia groups corresponding to \( z \) in \( S \). Since one of these inertia groups is topological generated by \( \gamma_{i} \) with \( \rho_{V}(\gamma_{i}) = \bar{g}_{i} \) (see e.g. [MM99] I Thm. 1.4), the image of the resulting representation \( G_{\hat{\mathbb{Q}}((z))} \rightarrow G_{2}(\ell) \) is conjugate to the subgroup generated by \( \bar{g}_{i} \). The same is true for the representation \( \pi_{1}^{\text{ét}}(\Delta_{p}) \rightarrow G_{2}(\ell) \) induced by \( (V_{R})_{\Delta_{p}} \): Indeed, by (3.1) \( G_{\mathbb{C}_{p}((z))} \rightarrow \pi_{1}^{\text{ét}}(\Delta_{p}) \) is an epimorphism.

The diagram consisting of canonical inclusions and evaluations \( t \mapsto s \)

\[
\begin{array}{ccc}
R_{N}[t, \frac{1}{t(t-1)}] & \xrightarrow{\mathbb{Q}(\zeta_{7})[t, \frac{1}{t(t-1)}]} & \mathbb{Q}[t, \frac{1}{t(t-1)}] \\
\downarrow \text{pr} & & \\
W_{p}((z)) & \xrightarrow{\hat{\mathbb{Q}}_{p}^{nr}} & \mathbb{Q}_{p}^{nr} \xrightarrow{\mathbb{Q}_{p}^{nr}} \mathbb{Q}.
\end{array}
\]

8
Since $\rho$ gives us the commutative diagram

\[
\begin{array}{ccc}
\pi^1_1(A_{RN}^N \setminus \{0, 1, \bar{s}\}) & \xrightarrow{\rho_{\bar{v}Q}} & \pi_1^1(A_{QQ}^Q \setminus \{0, 1, \bar{s}\}) \\
\pi^1_1(\Delta_p, \bar{s}) & \xrightarrow{G_{\bar{Q}}^Q} & I_p^G \xrightarrow{G_{\bar{Q}}^Q} G_{\bar{Q}}.
\end{array}
\]

In particular, $(\rho_{\bar{v}Q}^{(s)})_{G_{\bar{Q}}^Q}$ factors through the above representation induced by $(\bar{V}_{Rn})_{|_{\Delta_p}}$. But by (3.1) $G_{\bar{Q}}^Q \to \pi^1_1(\Delta_p)$ is the $\nu_p(a)$ power map and $\nu_p(a)$ is coprime to $\text{ord}(\bar{g}_i)$ by our assumptions. It follows that the image of $(\rho_{\bar{v}Q}^{(s)})_{G_{\bar{Q}}^Q}$ is still conjugate to the subgroup generated by $\bar{g}_i$, which completes the proof. \(\square\)

Consider the maximal tame algebraic extension $Q_p^{\text{tame}}/Q_p$. The tame inertia group $I_p^{\text{tame}}$ is given by the exact sequence

\[
1 \longrightarrow I_p^{\text{tame}} \longrightarrow G_{Q_p}^{\text{tame}} \longrightarrow G_{\bar{Q}} \longrightarrow 1.
\]

Since $I_p^{\text{tame}}$ is abelian, $G_{\bar{Q}}$ acts on $I_p^{\text{tame}}$. It is not hard to see that the Frobenius acts as $p$ power map.

Let $p$ be as in the last lemma for $i = \infty$. Moreover suppose that $p$ does not divide the order of $G_2(l)$. Since $I_p^{\text{tame}}$ is the quotient of $I_p$ by its unique $p$-Sylow group (see e.g. [Neu92] II Satz 9.12), $(\rho_{\bar{v}Q}^{(s)})_{|_{I_p}}$ factors through $I_p^{\text{tame}}$. In particular, a lift of the Frobenius to $G_{Q_p}$ acts on the image of $(\rho_{\bar{v}Q}^{(s)})_{|_{I_p}}$ as the $p$ power map. Thus for $p \equiv 3$ or $5 \mod 7$ and our $\bar{g}_{\infty}$ conjugate in the image of $(\rho_{\bar{v}Q}^{(s)})_{|_{I_p}}$ we get an element in the image of $\rho_{\bar{v}Q}^{(s)}$ with analogue properties as $h_{\infty,p}$ has for $\bar{g}_{\infty}$.

For notational convenience we assume that our $\bar{g}_i$ conjugates given by the lemma are just $\bar{g}_i$ themselves and that the image of the Frobenius is just $h_{\infty,p}$.

3.2 The structure of $s^*\bar{V}_Q$ as $F_\ell[H]$-module. For $p$ as above let $H$ be the subgroup of the image of $\rho_{\bar{v}Q}^{(s)}$ generated by $\bar{g}_1, \bar{g}_{\infty}$ and $h_{\infty,p}$. For a field extension $K/F_\ell$ set $V(K) := s^*\bar{V}_Q \otimes_{F_\ell} K$.

3.2 Lemma. The decomposition of $V(F_\ell)$ into $V_1(F_\ell) \oplus V_2(F_\ell)$, where the direct factors are $V_1(F_\ell) := \text{Eig}(g_{\infty}, 1)$ and $V_2(F_\ell) := \bigoplus_{i=1}^6 \text{Eig}(g_{\infty}, \bar{2}_i)$, is defined over $\bar{F}_\ell$. If $H$ acts reducible on $s^*\bar{V}_Q$, the only possible non trivial $F_\ell[H]$-submodules are $V_1(F_\ell)$ and $V_2(F_\ell)$. Further, if they exist, they are irreducible submodules.

Proof: Denote by $H'$ the subgroup generated only by $\bar{g}_{\infty}$ and $h_{\infty,p}$. First we prove that $V(F_\ell) = V_1(F_\ell) \oplus V_2(F_\ell)$ is an irreducible decomposition of $V(F_\ell)$ as a $\bar{F}_\ell[H']$-module. Let $v_i$ be a $\zeta_i^2$-eigenvector of $\bar{g}_{\infty}$. In particular (2.1) gives us a description for $h_{\infty,p}$ with resp. to the basis $\{v_1, \ldots, v_7\}$. Thus $V(F_\ell) =$
$V_1(\mathbb{F}_ℓ) \oplus V_2(\mathbb{F}_ℓ)$ is clearly a decomposition of $V(\mathbb{F}_ℓ)$ as a $\mathbb{F}_ℓ[H']$-module. It remains to prove the $H'$-irreducibility of $V_2(\mathbb{F}_ℓ)$. For a non trivial $v \in V_2(\mathbb{F}_ℓ)$ define $M_v$ as the set of all $i$ s.t. the projection of $v$ to $\mathbb{F}_ℓ.v_i$ is nontrivial and $m_v$ as $|M_v|$. Let $P \in \mathbb{F}_ℓ[X]$ be non trivial of minimal degree with $P(\bar{g}_∞)v = 0$. It follows that $P(\zeta_ℓ^i) = 0$ for $i \in M_v$. But $\prod_{i \in M_v} (X - \zeta_ℓ^i)$ gives another such non trivial relation, i.e. $\deg(P) = m_v$. Clearly $\sum_{i \in \mathbb{Z}} \bar{g}_ℓ.i.\bar{g}_ℓ^i(v)$ is contained in the $m_v$-dim. space $\bigoplus_{i \in M_v} \mathbb{F}_ℓ.v_i$. As a result these spaces coincide. In particular, there exists an $i ≤ 6$ with $v_i$ in the $\mathbb{F}_ℓ$-span of $H'v$. But by (2.1) this span is already $V_2(\mathbb{F}_ℓ)$.

Now the rest of the proof is quite easy: The minimal polynomial of $\bar{g}_∞$ is separable. So we see that the decomposition is already defined over $\mathbb{F}_ℓ$ by looking at the rational canonical form of $\bar{g}_∞$. Let $U ≤ V(\mathbb{F}_ℓ)$ be a non trivial $\mathbb{F}_ℓ[H']$-hence also $\mathbb{F}_ℓ[H']$-submodule. Using the $H'$-irreducibility of $V_1(\mathbb{F}_ℓ)$ and $V_2(\mathbb{F}_ℓ)$ we get that $U$ equals either $V_1(\mathbb{F}_ℓ)$ or $V_2(\mathbb{F}_ℓ)$ (compute the possible dimensions of their intersections with $U$).

It follows that $s^V_\mathbb{Q} = V_1(\mathbb{F}_ℓ) \oplus V_2(\mathbb{F}_ℓ)$ is the only possible non trivial decomposition as an $\mathbb{F}_ℓ[H]$-module. Suppose that this is indeed a decomposition. In particular both $V_1(\mathbb{F}_ℓ)$ and $V_2(\mathbb{F}_ℓ)$ are $\bar{g}_1$-invariant, which contradicts that the Jordan canonical form of $\bar{g}_1$ contains no Jordan block of length 1. Thus we get:

3.3 Corollary. $s^V_\mathbb{Q}$ is indecomposable as an $\mathbb{F}_ℓ[H]$-module.

3.3 Computation of the image of $\rho_{V_\mathbb{Q}}^{(s)}$. First we recall the following fact about our representation of $G_2(\ell)$ which can be found for example in [Mal03]: There is a basis $\{v_1, \ldots, v_7\}$ of $s^V_\mathbb{Q}$ s.t. every maximal parabolic subgroup of $G_2(\ell)$ is given either by $P_\alpha := \langle \mathcal{T}, x_\alpha \rangle$ or $P_\beta := \langle \mathcal{T}, x_\beta \rangle$, where $x_\alpha$ and $x_\beta$ are generators of the root subgroup of the simple root $\alpha$ respectively $\beta$ given as

$$
x_\alpha := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad x_\beta := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathcal{T} := \{ \text{diag}(t_1, t_2, t_1t_2^{-1}, 1, t_1^{-1}t_2, t_2, t_1^{-1}) \mid t_1, t_2 \in \mathbb{F}_ℓ^\times \} \text{ is a maximal split torus of } G_2(\ell) \text{ in } \text{GL}_7(\mathbb{F}_ℓ).
$$

We will use the above list of types of maximal subgroups of $G_2(\ell)$ to prove the following:

3.4 Theorem. Let $\ell ≠ 2, 3, 7, 11$ and 13. Let $s \in \mathbb{Q} \setminus \{0, 1\}$ and suppose there exist prime numbers $p, q ≠ 2, 7, \ell$ with $p ∤ |G_2(\ell)|$, $p ≡ 3$ or 5 mod 7, $\nu_p(s) < 0 <$
\[ \nu_q(s - 1), \ 7 \nmid \nu_p(s) \ \text{and} \ \ell \mid \nu_q(s - 1). \] Then \( \rho^{(s)}_{\mathbb{Q}} : G_{\mathbb{Q}} \to G_2(\ell) \) is surjective.

**Proof:** It suffices to prove that no maximal subgroup \( G_2(\ell) \)-conjugate to one of the subgroups in the above list contains \( H \) (hence \( H = G_2(\ell) \)).

By the same arguments as in the proof of Prop. 2.3, \( H \) is not contained in any maximal subgroup of \( G_2(\ell) \) (conjugate to a subgroup) in the above list except the first three ones.

(i) Suppose \( H \leq M \) with \( M \) a maximal parabolic subgroup of \( G_2(\ell) \). If \( M = P_\alpha \), then \( v_1 \) and \( v_2 \) span a 2-dimensional \( \mathbb{F}_\ell[P_\alpha] \)-submodule of \( s^*\tilde{V}_Q \). If \( M = P_\beta \), then \( v_1, v_2 \) and \( v_3 \) span a 3-dimensional \( \mathbb{F}_\ell[P_\beta] \)-submodule of \( \mathbb{F}_\ell[H] \)-submodule of \( s^*\tilde{V}_Q \), which both contradicts Lemma 3.2.

(ii) Suppose \( H \leq C_{G_2(\ell)}(\iota) \) for an involution \( \iota \) as in the above list. Now \( \iota \) is semisimple \( \neq -1 \) (otherwise \( C_{G_2(\ell)}(\iota) \) would be \( G_2(\ell) \)). Thus \( \iota \) has Jordan canonical form

\[
\text{JCF}(\iota) = \begin{pmatrix}
\text{id}_{r \times r} & 0 \\
0 & -\text{id}_{s \times s}
\end{pmatrix}
\]

for suitable \( r, s > 0 \) and \( r + s = 7 \). But then \( C_{\text{GL}(s^*\tilde{V}_Q)}(\iota) \) is isomorphic to \( \text{GL}(s^*\tilde{V}_Q) \oplus \text{GL}(s_2) \). In particular even \( C_{\text{GL}(s^*\tilde{V}_Q)}(\iota) \) acts decomposable on \( s^*\tilde{V}_Q \), which contradicts Cor. 3.3.

(iii) Suppose \( H \leq M \) with \( M \) \( G_2(\ell) \)-conjugate to \( K_\varepsilon = L_\varepsilon \rtimes \mathbb{Z}/2 \). Without loss of generality we may assume that \( M \) equals \( K_\varepsilon \). According to the proof of [Kle88] Prop. 2.2 we may choose \( L_\varepsilon \) in our representation to be generated by three long root subgroups of \( G_2(\ell) \), all of which act trivial on a 5-dimensional subspace \( W_i \) of \( s^*\tilde{V}_Q \) (resp.). In particular \( L_\varepsilon \) acts trivial on the non trivial subspace \( W_1 \cap W_2 \cap W_3 \). Say \( v \) is a non trivial Element of this space and let \( \sigma \) be the generator of \( \mathbb{Z}/2 \) in \( K_\varepsilon \). Since \( \sigma \) normalizes \( L_\varepsilon \), \( L_\varepsilon \) acts trivial on \( \sigma v \) as well. Thus the space \( W \) spanned by \( v \) and \( \sigma v \) is a 1- or 2-dimensional \( \mathbb{F}_\ell[K_\varepsilon] \)-submodule of \( s^*\tilde{V}_Q \). Further, [Kle88] Prop. 2.2 (iii) tells us that \( K_\varepsilon \) stabilizes a 6-dimensional subspace \( U \) of \( s^*\tilde{V}_Q \). From Lem. 3.2 we get that \( U = V_2(\ell) \) is an irreducible \( \mathbb{F}_\ell[H] \)-submodule. In particular \( U \) and \( W \) intersects trivially. It follows that \( s^*\tilde{V}_Q = U \oplus W \) is a decomposition as an \( \mathbb{F}_\ell[H] \)-module, which again contradicts Cor. 3.3.

By the above list \( H \) is not contained in any maximal subgroup of \( G_2(\ell) \), so the theorem follows.

\[ \Box \]

**3.5 Remark.** Note that this theorem is in particular an explicit version of Hilbert’s irreducibility theorem for \( G_2(\ell) \). A similar result was proved by S. Beckmann in [Bec91] (see loc. cit. Cor. 1.3), using a different approach. She needs a generating set of certain \( \{ \sigma_\iota \mid \iota \in T \} \) in the image of certain inertia groups with the property that for all \( \tau_\iota \) in the \( G_2(\ell) \)-conjugacy class of \( \sigma_\iota \), \( \{ \tau_\iota \mid \iota \in T \} \) is still a generating set. But restriction of our specialization to the inertia groups \( I_p \) and \( I_q \) alone only yields two elements, which not necessary generates \( G_2(\ell) \). Thus her result does not apply in our case. Further, we have to exclude less primes: \( q \) might as well divide the order of \( G_2(\ell) \), hence might lie inside the exceptional set \( S_{\text{bad}} \) in [Bec91].
4 Motives with motivic Galois group $G_2$

Assuming the Tate conjecture, we will construct a family of pure motives with motivic Galois group (with resp. to the $\ell$-adic realization) $G_2(\overline{\mathbb{Q}}_\ell)$.

4.1 Interpretation of $\bar{s}^*V_k$ as $\ell$-adic realization. By the construction in section 2.1 our way is clear: we have to interpret suitable stalks of $V_k$ in terms of $\ell$-adic cohomology of suitable smooth projective varieties.

For a $k$-valued point $s : \text{Spec}(k) \to \mathbb{A}^1_{\mathbb{R}}$ denote by $\pi_s : \text{Hyp}_s \to \text{Spec}(k)$ the basechange of $\pi_N$ along $s$.

4.1 Proposition. There is a 6-dimensional smooth projective $\mu_{14}$-equivariant compactification

$$\xymatrix{ \text{Hyp}_s \ar[r]^-j & X \ar@{-}[d]_-{\pi_s} \ar@{-}[ld]^-{\pi_D} & D \ar[l]_-i \\ \text{Spec}(k) }$$

of $\pi_s$ with $X = \text{Hyp}_s \amalg D$ and $D = \bigcup_i D_i$ a closed subscheme with smooth projective $D_i$.

Proof: Recall from the construction in Sec. 2.1 that we have a canonical embedding $\text{Hyp}_s \hookrightarrow \mathbb{P}^1_k \otimes_k \mathbb{P}^6_k$. The $\mu_{14}$-action clearly extents to $\mathbb{P}^1_k \otimes_k \mathbb{P}^6_k$. Thus we get a $\mu_{14}$-action on the closed subvariety $\overline{\text{Hyp}_s}$ given by the closure of $\text{Hyp}_s$ in $\mathbb{P}^1_k \otimes_k \mathbb{P}^6_k$. Clearly $\text{Hyp}_s$ lies in the regular locus of $\overline{\text{Hyp}_s}$. Apply [BEV05] Thm. 2.4 to get an iterated blow up of $\mathbb{P}^1_k \otimes_k \mathbb{P}^6_k$ with regular $\mu_{14}$-equivariant strict transform $\overline{\text{Hyp}_s} \to \text{Hyp}_s$ extending the $\mu_{14}$-action on $\text{Hyp}_s$.

Let $D'$ be the closed subvariety of $\overline{\text{Hyp}_s}$ given by the complement of $\text{Hyp}_s$. The $\mu_{14}$-action restricts to $D'$. We may assume that $D'$ is equidimensional (otherwise iterate the next step for every $\mu_{14}$-orbit of the induced action on the connected components of $D'$). Again by [BEV05] Thm. 2.4 we get a $\mu_{14}$-equivariant iterated blow up $X \to \overline{\text{Hyp}_s}$ in $\text{SmProj}$ with regular $\mu_{14}$-equivariant strict transform $D_0 \to D'$. In particular we still have a $\mu_{14}$-equivariant open embedding $\text{Hyp}_s \hookrightarrow X$. Let $\bigcup_{i=1}^r D_i \hookrightarrow X$ be the exceptional locus (denoted $\bigcup_i H_i$ in [BEV05]). But then $D_i$ is smooth projective by definition of a pair in loc. cit. (see Def. 2.2). Set $D := \bigcup_{i=0}^r D_i$. Then $X$ equals $\text{Hyp}_s \amalg D$, which completes the proof.

For $s \in \mathbb{Q} \setminus \{0, 1\}$ there is an $N \gg 0$ s.t. $s$ induces an $R_N$-valued point of $\mathbb{A}^1_{R_N} \setminus \{0, 1\}$ (also denoted by $s$) and s.t. the above compactification has a model

$$\xymatrix{ s^*\text{Hyp}_N \ar[r]^-j & X_N \ar@{-}[d]_-{\pi_N} \ar@{-}[ld]^-{\pi_{D_N}} & D_N = \bigcup_i D_{N,i} \ar[l]_-i \\ \text{Spec}(R_N) }$$

over $R_N$ with good reduction at every prime of $R_N$. For such $N$ we get the following.

\[\text{(4.1)}\]
4.2 Lemma. Let $\Pi$ be the indempotent endomorphism of $H^6_{l}(X)$ given in the group ring $\mathbb{Q}[\mu_{14}]$ as $\frac{1}{14} \sum_{i=0}^{13} \zeta_{14}^{i} \omega^{i}$ for a generator $\omega$ of $\mu_{14}$ with $\chi(\omega) = \zeta_{14}$. Then the canonical morphism $\alpha : \prod D_{i} \rightarrow X$ induces a canonical isomorphism of $G_{k}$-modules

$$\tilde{s}^{*}W^{6}[(\mathbb{R}^{6}(\pi_{N}^{*})(\widetilde{\mathbb{Q}_{l}}))^{\chi}] \xrightarrow{\sim} \Pi[\ker(H^{6}_{l}(\alpha))]$$

Proof: We argue as in the proof of [DRK10] Cor. 2.4.2: First, using Weil II we see

$$W^{6}[(\mathbb{R}^{6}(\pi_{N}^{*})(\widetilde{\mathbb{Q}_{l}}))^{\chi}] \equiv \ker(\mathbb{R}^{6}(\pi_{N}^{*})_{*}(\widetilde{\mathbb{Q}_{l}}) \rightarrow \mathbb{R}^{6}(\pi_{D_{N}}^{*})_{*}(\widetilde{\mathbb{Q}_{l}}))^{\chi}.$$

Let $\alpha_{N}$ be the canonical morphism $\prod_{i} D_{N,i} \rightarrow X_{N}$. Suppose the canonical morphism

$$\mathbb{R}^{6}(\pi_{D_{N}}^{*})_{*}(\widetilde{\mathbb{Q}_{l}}) = \mathbb{R}^{6}(\pi_{N}^{*})_{*}(i_{*}i^{*}\widetilde{\mathbb{Q}_{l}}) \longrightarrow \mathbb{R}^{6}(\pi_{N}^{*})_{*}((\alpha_{N})_{*}\alpha_{N}^{*}\widetilde{\mathbb{Q}_{l}})$$

is a monomorphism. It follows that we can express $W^{6}[(\mathbb{R}^{6}(\pi_{N}^{*})(\widetilde{\mathbb{Q}_{l}}))^{\chi}]$ as $\ker(\mathbb{R}^{6}(\pi_{N}^{*})_{*}(\widetilde{\mathbb{Q}_{l}}) \rightarrow \mathbb{R}^{6}(\pi_{N}^{*})_{*}(a_{*}a^{*}\widetilde{\mathbb{Q}_{l}}))^{\chi}$. Now taking stalks at $\tilde{s}$ completes the proof.

Thus it remains to show that $\mathbb{R}^{3}(\pi_{N}^{*})_{*}(i_{*}i^{*}\widetilde{\mathbb{Q}_{l}}) \rightarrow \mathbb{R}^{3}(\pi_{N}^{*})_{*}((\alpha_{N})_{*}\alpha_{N}^{*}\widetilde{\mathbb{Q}_{l}})$ is a monomorphism. Clearly $\alpha_{N}$ factors through $i$. The corresponding morphism $a : \prod_{i} D_{N,i} \rightarrow D_{N}$ is finite, hence $\mathbb{Q}_{l} \rightarrow a_{*}a^{*}\mathbb{Q}_{l}$ is a monomorphism. From this we get that $i_{*}i^{*}\widetilde{\mathbb{Q}_{l}} \rightarrow (\alpha_{N})_{*}\alpha_{N}^{*}\widetilde{\mathbb{Q}_{l}}$ is a monomorphism as well. The induced short exact sequence gives us a long exact sequence of $\mathbb{R}^{q}(\pi_{N}^{*})_{*}$-terms. Finally, by good reduction of $\pi_{N}^{*}$ we can apply Weil II to show that all the connecting morphisms vanish (compare the weights of the domain and codomain), which completes the proof.

From now on assume the Tate conjecture holds. In particular the standard conjectures holds for $H^{\bullet}_{l}$. We work in the (therefore) Tannakian category $\mathbf{NM}(k)_{\mathbb{Q}_{l}}$ of numerical motives over $k$ with coefficients $\mathbb{Q}_{l}$ and fibre functor $H^{\bullet}_{l}$. Note that $\alpha$ is $\mu_{14}$-equivariant: indeed by our construction of $D$, $D_{0}$ is fixed by the $\mu_{14}$-action while the action permutes the $D_{i}$’s for $i > 0$ (see the equivariance part in the proof of [BEV05] Thm. 2.4). It follows that $\Pi$ commutes with the projection $p_{\ker(\alpha)} : h(X) \rightarrow \ker(\alpha)$. It is clear that the sixth Künneth projector $\pi_{X}^{6}$ commutes with both idempotents. As a result $p_{s} := \Pi \circ \pi_{X}^{6} \circ p_{\ker(\alpha)}$ is idempotent as well and $p_{s}h(X)$ a well defined numerical motive. The following corollary is then clear:

4.3 Corollary. The $l$-adic realization $H^{\bullet}_{l}(p_{s}h(X))$ of $p_{s}h(X)$ isomorphic to $\tilde{s}^{*}N_{k}$ as a $G_{k}$-module.

4.2 Motives of type $G_{2}$. In this final section we will proof that for suitable choices of $s$ the motive $p_{s}h(X)(3)$ has motivic Galois group $G_{2}(\mathbb{Q}_{l})$.

As above let $s \in \mathbb{Q} \setminus \{0,1\}$ and choose $N \gg 0$ as in the last subsection. Let $l \neq 2, 3, 7, 11$ and 13. Furthermore, let $p, q \neq 2, 7$ be prime numbers with $p \equiv 3$ or 5 mod $7$, $p \nmid N|G_{2}(\ell)|$ and the additional properties $\nu_{p}(s) < 0 < \nu_{q}(s-1)$, $7 \nmid \nu_{p}(s)$ and $\ell \nmid \nu_{q}(s-1)$.
Again \( \rho_{\mathcal{V}_N} \) normalizes \( G_2(\mathbb{Q}_\ell) \). In abuse of notation we denote by \( \varepsilon \) the resulting continuous character

\[
\pi_1^{\mathbb{C}}(A_{N/R_N}) \xrightarrow{\rho_{\mathcal{V}_N}} G_2(\mathbb{Q}_\ell) \times \mathbb{Q}_\ell^\times \xrightarrow{\text{pr}_{\mathbb{Q}_\ell^\times}^{-1}} \mathbb{Q}_\ell^\times
\]

as well. As above this gives us extensions \( \mathcal{V}_N^\varepsilon, \mathcal{V}_R_N, \ldots \) of \( \mathcal{V} \). These are still defined over \( \mathbb{Q}_\ell(\zeta_7) \). Hence we get back \( \mathcal{V}_N^\varepsilon, \mathcal{V}_R_N, \ldots \) by mod \( \ell \) reduction analogue to the above construction. Let \( k/\mathbb{Q} \) be a Galois extension containing \( \zeta_7 \) s.t. the \( G_k \)-representation \( \rho_{\mathcal{V}_N}^\varepsilon \) is non trivial (e.g. \( k = \mathbb{Q}(\zeta_7) \)). Since this representation is a restriction of the absolutely irreducible \( G_2 \)-representation \( \rho_{\mathcal{V}_0}^\varepsilon \) (see Thm. 1.1) with image the simple group \( G_2(\ell) \), \( G_k \) acts on \( s^*\mathcal{V}_N^\varepsilon \) absolutely irreducible as well. But then the \( G_k \)-module \( s^*\mathcal{V}_N^\varepsilon \) hence \( s^*\mathcal{V}_N^\varepsilon \cong H_\varepsilon^s(p_kh(X)) \) is irreducible, too.

Recall that \( X \) has dimension 6. Poincaré duality gives us a non degenerate symmetric bilinear form \( H_\varepsilon^s(X)(3)^{\otimes 2} \to \mathbb{Q}_\ell \). By the projection formula (see e.g. [Ando04] 3.3.2) the adjoint of a morphism given by an element of \( G_k \) with resp. to this form is just its inverse, i.e. the Galois action is orthogonal.

Now \( H_\varepsilon^s(p_k) \) is a morphism of Galois representations. Further, the kernel of the canonical map \( H_\varepsilon^s(p_kh(X)(3)) \to H_\varepsilon^s(p_kh(X)(3))^\vee \) induced by the above bilinear form is \( G_k \)-invariant. But \( H_\varepsilon^s(p_kh(X)(3)) \) is an irreducible \( G_k \)-module, i.e. the restriction of the above bilinear form gives us a non degenerate symmetric bilinear form \( b \) on \( H_\varepsilon^s(p_kh(X)(3)) \) resp. \( s^*\mathcal{V}_k^\varepsilon(3) \). For \( G_k \) acts orthogonal with resp. to this form.

As a result the image of \( \rho_{\mathcal{V}_k}^{(s)}(3) \) is even contained inside \( N_{D_s\mathcal{V}_k(3),b}(G_2(\mathbb{Q}_\ell)) = G_2(\mathbb{Q}_\ell) \times \langle \pm 1 \rangle \). In particular \( \varepsilon^2 \) is trivial, hence we can choose a \( k \) as above with the additional properties \( \mathcal{V}_k^\varepsilon(3) = \mathcal{V}_k(3) \) and \( [k : \mathbb{Q}(\zeta_7)] \leq 2 \). For this \( k \) we get the following

4.4 Proposition. The image of the specialization \( \rho_{\mathcal{V}_k}^{(s)}(3) \) at \( s \) is a Zariski dense subgroup of \( G_2(\mathbb{Q}_\ell) \).

Proof: We have seen above that \( G_\mathbb{Q}(\mathbb{Q}(\zeta_7)) \) acts as an irreducible subgroup of \( G_2(\mathbb{Q}_\ell) \) on \( s^*\mathcal{V}_\mathbb{Q}(\mathbb{Q}(\zeta_7))(3) \). Using [Asc87] Cor. 12 we get that the only possible proper maximal Zariski closed subgroups of \( G_2(\mathbb{Q}_\ell) \) containing the image of \( \rho_{\mathcal{V}_k}^{(s)}(3) \) are of fifth type in loc. cit.. We will show that this is not the case:

Note that the image of \( \rho_{\mathcal{V}_k(\mathbb{Q}(\zeta_7))}^{(s)}(3) \) contains an element conjugate to \( g_1^{\mu_{\mathbb{Q}(\zeta_7)}(s-1)} \):

Indeed, since \( p \nmid N \) we can argue as in the proof of Lem. 3.2. The Jordan canonical form of \( g_1 \) consists of three Jordan blocks. As a result \( g_1 \) has a 3-dimensional 1-eigenspace. It follows that \( g_1^{\mu_{\mathbb{Q}(\zeta_7)}(s-1)} \) has at least a 3-dimensional 1-eigenspace, i.e. its Jordan canonical form consists of at least three blocks. But \( g_1^{\mu_{\mathbb{Q}(\zeta_7)}(s-1)} \) is clearly non trivial. Thus we can argue as in the second part of the proof of Prop. 2.2 to get that the image of \( \rho_{\mathcal{V}_k(\mathbb{Q}(\zeta_7))}^{(s)}(3) \) is a Zariski dense subgroup of \( G_2(\mathbb{Q}_\ell) \). In particular the closure of the image of \( \rho_{\mathcal{V}_k(3)}^{(s)} = \rho_{\mathcal{V}_k(3)}^{(s)} \) is a non trivial normal subgroup of the simple group \( G_2(\mathbb{Q}_\ell) \), which finishes the proof. \( \square \)
All in all we have seen that the $G_k$-action on objects in the $\otimes$-category generated by $H^*_\ell(p_s \mathfrak{h}(X)(3))$ in $\text{Vec}_{\bar{\mathbb{Q}}}$ fixes $b \in S^2H^*_\ell(p_s \mathfrak{h}(X)(3))$ and the Dickson form $f \in \Lambda^3H^*_\ell(p_s \mathfrak{h}(X)(3))$, i.e. $b$ and $f$ are Tate cycles. Granting the Tate conjecture this gives us that the motivic Galois group $G_{\text{mot}}(p_s \mathfrak{h}(X)(3))$ fixes these cycles, hence is contained inside $G_2(\bar{\mathbb{Q}_\ell})$ by [Asc87] (2.11) and (3.4).

On the other hand every algebraic cycle is a Tate cycle, so $\text{im}(\rho_{\mathcal{V}_k(3)})$ fixes all algebraic cycles in $\mathfrak{s}^*V_k(3) = H^*_\ell(p_s \mathfrak{h}(X)(3))$ and hence on all $\otimes$-constructions of this space. Further, it is a Zariski dense subgroup of $G_2(\bar{\mathbb{Q}_\ell})$ as a Zariski closed subgroup of $\text{GL}(H^*_\ell(p_s \mathfrak{h}(X)(3)))$. So the same is true for $G_2(\bar{\mathbb{Q}_\ell})$. But $G_2(\bar{\mathbb{Q}_\ell})$ is reductive, so it is contained in $G_{\text{mot}}(p_s \mathfrak{h}(X)(3))$ by [And04] 6.3.1.

This gives us our final result:

4.5 Theorem. Suppose the Tate conjecture holds. Let $s \in \mathbb{Q} \setminus \{0, 1\}$ and choose $N \gg 0$ s.t. $X$ and the $D_i$’s are defined over $\mathbb{Z}_N$ with good reduction at every prime. Let $\ell \neq 2, 3, 7, 11$ and 13. Furthermore, let $p, q \neq 2, 7, \ell$ be prime numbers with $p \equiv 3 \text{ or}\ 5 \text{ mod } 7$, $p \nmid N \cdot |G_2(\ell)|$ and the additional properties $\nu_p(s) < 0 < \nu_q(s - 1)$, $7 \nmid \nu_p(s)$ and $\ell \nmid \nu_q(s - 1)$. Then the motive $p_s \mathfrak{h}(X)(3)$ has motivic Galois group of type $G_2$.

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