Littlewood–Paley characterization of BMO and Triebel–Lizorkin spaces

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Funding information
Russian Science Foundation, Grant/Award Number: 18-11-00053

Abstract
We prove a generalization of the Littlewood–Paley characterisation of the BMO space where the shifts of a Schwartz function are replaced by a family of functions with suitable conditions imposed on them. We also prove that a certain family of Triebel–Lizorkin spaces can be characterized in a similar way.

KEYWORDS
BMO, Littlewood–Paley decomposition, Triebel–Lizorkin spaces

MSC (2010)
42B25

1 | INTRODUCTION

One of the main results of this article is a generalization of the following statement, proved in [11].

Theorem A. Let \( \{\phi_n\}_{n \in \mathbb{Z}} \) be a uniformly bounded system of functions defined on \( \mathbb{R}^d \) and having weak derivatives up to the order \( d + 1 \). Assume that the following conditions hold:

1) \( \sum_{n \in \mathbb{Z}} \phi_n(x) \equiv 1 \) for all \( x \neq 0 \),
2) \( \text{supp} \phi_n \subseteq \{ x \in \mathbb{R}^d : 2^{n-1} \leq |x| < 2^{n+1} \} \),
3) \( 2^{-nd} \int |D^\alpha \phi_n(\xi)| \, d\xi \leq K 2^{-n|\alpha|} \) for \( 0 \leq |\alpha| \leq d + 1 \).

Here \( K \) is a constant which does not depend on \( n \). Define the operator \( \hat{\Lambda}_n f := \phi_n \hat{f} \) and the norm

\[ \|f\|_B := \sup_Q \left( \frac{1}{|Q|} \int_Q \sum_{Q \subseteq Q} |\Lambda_n f(x)|^2 \, dx \right)^{1/2}, \]

where we take the supremum over all cubes \( Q \), and \( l(Q) \) is the length of the edge of \( Q \). Then there exists a positive constant \( C_1 \) such that for all integrable functions \( f \in \text{BMO}(\mathbb{R}^d) \) the following inequality holds \( \|f\|_B \leq C_1 \|f\|_{\text{BMO}} \). Conversely, if an integrable function \( f \) satisfies \( \|f\|_B < \infty \), then \( \|f\|_{\text{BMO}} \leq C_2 \|f\|_B \), where \( C_2 > 0 \) is some universal constant.

Remark 1.1. Everywhere in this article \( \hat{f} \) stands for the Fourier transform of the function \( f \) and \( \tilde{f} \) stands for the inverse Fourier transform of \( f \).
This theorem is in turn a generalization of Bochkarev’s inequality (see [1]). Indeed, in [11] the authors replaced the de la Vallée–Poussin kernels used by Bochkarev with a more general system of functions. We draw the reader’s attention to the fact that this inequality was applied by S. V. Bochkarev to various problems of the theory of trigonometric sums, see [2] for further details.

In this paper we prove a generalization of Theorem A obtained by substituting \( \{ \psi_n \} \) for an even more general system of functions. In more detail, we completely dispense with the “\( d + 1 \) derivatives condition”, demanding bounds on the derivatives up to the order \( \lceil d/2 \rceil + 1 \) instead. The price to pay is the norm utilized, which we consider to be \( L^2 \) and not \( L^1 \). Let us state the main result of this paper.

**Theorem 1.2.** Let \( \{ \psi_n \} \) be a uniformly bounded system of functions defined on \( \mathbb{R}^d \) and having weak derivatives up to the order \( a = \lceil d/2 \rceil + 1 \). Assume that the following conditions hold:

1) \( \sum_{n \in \mathbb{Z}} \psi_n(x) \equiv 1 \) for all \( x \neq 0 \),
2) \( \text{supp} \psi_n \subseteq \{ x \in \mathbb{R}^d : 2^{n-1} \leq |x| < 2^{n+1} \} \),
3) \( 2^{-nd} \int |D^\alpha \psi_n(\xi)|^2 d\xi \leq K 2^{-n|\alpha|} \) for all \( 0 \leq |\alpha| \leq a \).

Here \( K \) is a constant which does not depend on \( n \). Define the operator \( \widehat{\Delta_n f} := \psi_n \widehat{f} \) and the norm

\[
\| f \|_D := \sup_Q \left( \frac{1}{|Q|^2} \int_{2^{-n} \leq |x| \leq 2^n} |\Delta_n f(x)|^2 \, dx \right)^{1/2},
\]

where we take the supremum over all cubes \( Q \), and \( l(Q) \) is the length of the edge of \( Q \). Then there exists a positive constant \( C_1 \) such that for all integrable functions \( f \in \text{BMO}(\mathbb{R}^d) \) the following inequality holds \( \| f \|_D \leq C_1 \| f \|_{\text{BMO}} \). Conversely, if an integrable function \( f \) satisfies \( \| f \|_D < \infty \), then \( \| f \|_{\text{BMO}} \leq C_2 \| f \|_D \), where \( C_2 > 0 \) is some universal constant.

**Remark 1.3.** The integrability of \( f \) is not an important issue in Theorem 1.2. Namely, this is a technical condition which we use only in the estimate (3.4). Moreover, the constants \( C_1 \) and \( C_2 \) will not depend on the \( L^1 \) norm of \( f \) as we shall see.

Several remarks are in order. First, Theorem 1.2 is indeed a generalization of Theorem A; this can easily be proved using the Sobolev–Gagliardo–Nirenberg inequality. Second, we emphasize that the conditions imposed on the system \( \{ \psi_n \} \) in Theorem 1.2 are exactly those of the Hörmander–Mikhlin multiplier theorem (consult [6] or [5] for the proof). Finally, we remark that it would not be difficult to obtain a variant of the Littlewood–Paley decomposition for the product space \( \text{BMO}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \) in the spirit of Theorem 1.2 along the lines of this result. However, we do not attack this problem here.

Theorem 1.2 provides a Littlewood–Paley characterization of the space \( \text{BMO}(\mathbb{R}^d) \). In the second main result of this article we establish such a characterization for the scale of Triebel–Lizorkin spaces. We introduce those in the following

**Definition 1.** Let \( \varphi \) be a collection of functions on \( \mathbb{R}^d \), \( \varphi = \{ \varphi_n \} \), such that

1) \( \text{supp} \varphi_n \subseteq \{ x \in \mathbb{R}^d : 2^{n-1} \leq |x| < 2^{n+1} \} \),
2) \( \sum_{n \in \mathbb{Z}} \varphi_n(x) = 1 \) for all \( x \neq 0 \),
3) \( 2^{-nd} \int_{\mathbb{R}^d} |D^\alpha \varphi_n(\xi)| \, d\xi \leq K_{\varphi} 2^{-n|\alpha|} \) for all \( 0 \leq |\alpha| \leq d + 1 \).

The Triebel–Lizorkin space \( F_{\infty,p}^{\varphi} \), for \( p \in (1, +\infty) \) is defined as follows: \( f \in F_{\infty,p}^{\varphi} \) if and only if

\[
\| f \|_p^\varphi := \sup_Q \left( \frac{1}{|Q|^2} \int_{2^{-n} \leq |x| \leq 2^n} |\Delta_n \varphi f(x)|^p \, dx \right)^{1/2} < \infty,
\]
where the sup is taken over all cubes $Q \subset \mathbb{R}^d$, $l(Q)$ is the length of an edge of $Q$, and the operators $\Delta_{n,\varphi}$ are defined by the following formula

$$\Delta_{n,\varphi} f(x) = \varphi_n(x) \cdot \hat{f}(x).$$

The classical Triebel–Lizorkin spaces arise in a particular case of this definition, namely when the functions $\varphi_n$ are dilatation of some function from the Schwartz class, see [7]. Our second main result here is following theorem.

**Theorem 1.4.** Let $\varphi = \{\varphi_n\}_{n \in \mathbb{Z}}$ and $\psi = \{\psi_n\}_{n \in \mathbb{Z}}$ be two collections of functions satisfying conditions 1), 2) and 3) from the definition. Then

$$\|f\|_{\varphi} \leq C \|f\|_{\psi},$$

where the constant $C$ depends only on $p$ and $K_{\psi}$.

Despite of the fact that it would be possible to strengthen Theorem 1.4 via imposing a condition demanding only $[d/2] + 1$ derivatives as we did in Theorem 1.2, we intentionally keep the stronger condition with $d + 1$ derivatives (the one used in [11]) and the norm $L^1$ here to enable the reader to compare the proofs of Theorems 1.4 and 1.2.

Let us also point out that obtaining a Littlewood–Paley decomposition for the scale of Triebel–Lizorkin spaces is a more difficult problem in comparison with the scale of Besov spaces (see [7, pp. 73–76]). One more remark is that it would be interesting (at least in our opinion) to compare the results of the present paper with those of [10], since similar characterizations were used there in order to describe some important properties of Triebel–Lizorkin and Besov spaces.

The Littlewood–Paley decompositions is a rapidly developing branch of the modern harmonic analysis. To demonstrate this we cite the paper [3] by A. El Baraka, where he characterised the scale of Morrey–Campanato spaces by means of Littlewood–Paley decompositions and a very recent preprint [9] by B. J. Park which discloses connections between Littlewood–Paley decompositions and vector valued multipliers. Both papers [3] and [9] have obvious connections with our article.

In what concerns the applications of the main results of this paper, we believe that our theorems can be used in the theory of trigonometric series as was done in the works by S. V. Bochkarev. However, at the present moment, the authors are unaware of any concrete examples of such applications.

We begin with the characterization of the scale of Triebel–Lizorkin spaces.

## 2 THE TRIEBEL–LIZORKIN SPACES

**Proof.** Denote $S_{n,\varphi} := \varphi_n$, $S_{n,\psi} := \varphi_n$ and $P_{n,\psi} := S_{n-1,\psi} + S_n,\psi + S_{n+1,\psi}$. Then

$$\Delta_{n,\varphi} f(x) = S_{n,\varphi} \ast f(x) = S_{n,\varphi} \ast P_{n,\psi} \ast f(x).$$

Indeed, this follows from the facts that $\sum_n \psi_n = 1$ and that $\text{supp} \psi_n \subseteq \{x \in \mathbb{R}^d : 2^{n-1} \leq |x| < 2^{n+1}\}$. We fix a cube $Q \subset \mathbb{R}^d$ and write

$$\frac{1}{|Q|} \int_Q \sum_{n \geq -\log_2 l(Q)} |\Delta_{n,\varphi} f(x)|^p \, dx = \frac{1}{|Q|} \int_Q \sum_{n \geq -\log_2 l(Q)} \left| f \ast S_{n,\varphi} \ast P_{n,\psi}(x) \right|^p \, dx$$

$$\leq \frac{1}{|Q|} \int_Q \sum_{n \geq -\log_2 l(Q)} \left| \int_{2Q} f \ast P_{n,\psi}(y) \cdot S_{n,\varphi}(x-y) \, dy \right|^p \, dx$$

$$+ \frac{1}{|Q|} \int_Q \sum_{n \geq -\log_2 l(Q)} \left| \int_{\mathbb{R}^d \setminus 2Q} f \ast P_{n,\psi}(y) \cdot S_{n,\varphi}(x-y) \, dy \right|^p \, dx,$$  \hspace{2cm} (2.1)
where $2Q$ stands for the cube whose center coincides with that of $Q$ and whose edge is two times longer than the edge of $Q$.

We first estimate the first integral:

$$I := \frac{1}{|Q|} \int_{Q} \sum_{n \geq -\log_2|Q|} \int_{2Q} |f \ast P_{n,\psi}(y) \cdot S_{n,\varphi}(x - y) dy|^p dx$$

$$\lesssim \frac{1}{|Q|} \sum_{n \geq -\log_2|Q|} \left\| \Delta_{n,\varphi} ((f \ast P_{n,\psi}) \cdot \chi_{2Q}) \right\|^p_{L^p(\mathbb{R}^d)}, (2.2)$$

where $\chi_A$ is the characteristic function of a set $A$. Using the Young inequality we conclude that for all $g \in L^p(\mathbb{R}^d)$ the following inequality holds:

$$\left\| \Delta_{n,\varphi} g \right\|_{L^p(\mathbb{R}^d)} = \left\| S_{n,\varphi} * g \right\|_{L^p(\mathbb{R}^d)} \leq \left\| S_{n,\varphi} \right\|_{L^1(\mathbb{R}^d)} \cdot \left\| g \right\|_{L^p(\mathbb{R}^d)}.$$

Next, for all $x \in \mathbb{R}^d$ we have the inequalities

$$|S_{n,\varphi}(x)| \lesssim 2^{nd}$$

and

$$|S_{n,\varphi}(x)| \lesssim 2^{-n} \cdot |x|^{-(d+1)}.$$

Indeed, the first estimate is a piece of cake:

$$|S_{n,\varphi}(x)| = \left| \int_{\mathbb{R}^d} \varphi_n(\xi) \cdot e^{2\pi i x \cdot \xi} d\xi \right| \leq \left\| \varphi_n \right\|_{L^1(\mathbb{R}^d)} \lesssim 2^{nd}.$$ 

In order to prove the second one we first infer that it suffices to show that

$$|x_{d+1} \cdot S_{n,\varphi}(x)| \lesssim 2^{-n}$$

for all $j \in [1, ..., d]$. We are going to prove this in the following way: we majorize the left-hand side of the last inequality using the properties of the Fourier transformation and condition 3 with the corresponding multi–index $d_j$:

$$|x_{d+1} \cdot S_{n,\varphi}(x)| = \left| \int_{\mathbb{R}^d} D_{d_j}^\varphi \varphi_n(x) \left\| D_{d_j}^{\varphi} \varphi_n(x) \right\|_{L^1(\mathbb{R}^d)} \leq 2^{nd-n \cdot d_j} = 2^{nd-n(d+1)} = 2^{-n},$$

and the second estimate follows as well. Note that these two inequalities yield

$$\left\| S_{n,\varphi} \right\|_{L^1(\mathbb{R}^d)} \leq \int_{\{x : |x| \leq 2^{-n}\}} |S_{n,\varphi}(x)| dx + \int_{\{x : |x| > 2^{-n}\}} |S_{n,\varphi}(x)| dx$$

$$\lesssim 2^{nd} \cdot 2^{-nd} + \int_{\{x : |x| > 2^{-n}\}} 2^{-n} \cdot |x|^{-(d+1)} dx \lesssim 1.$$

Now, we continue the estimate of the term $I$:

$$I \lesssim \frac{1}{|Q|} \sum_{n \geq -\log_2|2Q|} \int_{2Q} |f \ast P_{n,\psi}(\xi)|^p d\xi \lesssim \frac{1}{|Q|} \sum_{n \geq -\log_2|2Q|} \int_{2Q} |f \ast S_{n,\varphi}(\xi)|^p d\xi \lesssim \|f\|^p_{\psi}.$$
Let us proceed to the second term from (2.1), which we denote $J$:

$$J := \frac{1}{|Q|} \sum_{n \geq -\log_2 l(Q)} \left| \int_{\mathbb{R}^d \setminus 2Q} f \ast P_n \psi(y) \cdot S_n \varphi(x-y) \, dy \right|^p \, dx.$$ 

We first estimate the expression that is inside of the integral over the set $Q$:

$$\left| \int_{\mathbb{R}^d \setminus 2Q} |f \ast P_n \psi(y) \cdot S_n \varphi(x-y) \, dy \right| \leq \sum_{i=2}^{\infty} \int_{\Omega_i} |f \ast P_n \psi(y)| \cdot |S_n \varphi(x-y)| \, dy \leq \ldots,$$

where $\Omega_i = (i+1)Q \setminus iQ$. We use the bound $|S_n \varphi(x)| \leq 2^{-n} \cdot |x|^{-(d+1)}$ and the fact that if $x \in Q$ and $y \in \Omega_i$, then $|x-y| \approx i \cdot l(Q)$ and write

$$... \leq \sum_{i=2}^{\infty} \int_{\Omega_i} 2^{-n} \cdot |f \ast P_n \psi(y)| \, dy \leq \frac{2^{-n}}{l(Q)^{d+1}} \sum_{i=2}^{\infty} \frac{\|f\| \cdot |\Omega_i|}{i^{d+1}} \leq \frac{2^{-n} \cdot l(Q)^d}{l(Q)^{d+1}} \cdot \|f\| \cdot \sum_{i=2}^{\infty} \frac{1}{i^2} \leq \frac{2^{-n}}{l(Q)^{d+1}} \cdot \|f\| \psi,$$

where in the last inequality we have used the lemma that follows.

**Lemma 2.1.** Let $\Omega_i$ be as above. Then

$$\frac{1}{|\Omega_i|} \int_{\Omega_i} |f \ast P_n \psi(y)| \, dy \lesssim \|f\| \psi.$$

**Proof.** Split the set $\Omega_i$ into pairwise disjoint cubes $\{Q_j\}_{j=1}^N$ in a way that each of those is the image of $Q$ under a translation. Then the number $N$ of these cubes satisfies $N \lesssim i^{d-1}$. Note that for each $j \in [1,\ldots,N]$,

$$\left( \frac{1}{|Q_j|} \int_{Q_j} |f \ast P_n \psi(y)|^p \, dy \right)^{\frac{1}{p}} \lesssim \left( \frac{1}{|Q_j|} \int_{Q_j} |f \ast S_{n-1} \psi(y)|^p + |f \ast S_n \psi(y)|^p + |f \ast S_{n+1} \psi(y)|^p \, dy \right)^{\frac{1}{p}} \lesssim \|f\| \psi.$$

Using this bound together with the Hölder inequality we deduce that

$$\int_{\Omega_i} |f \ast P_n \psi(y)| \, dy = \sum_{j=1}^{N} \int_{Q_j} |f \ast P_n \psi(y)| \, dy$$

$$\leq \sum_{j=1}^{N} \left( \frac{1}{|Q_j|} \int_{Q_j} |f \ast P_n \psi(y)|^p \, dy \right)^{\frac{1}{p}} \cdot |Q_j|$$

$$\leq \|f\| \psi \cdot N \cdot |Q_j| \lesssim \|f\| \psi \cdot i^{d-1} \cdot |Q| \lesssim \|f\| \psi \cdot |\Omega_i|,$$

and the lemma follows.

We are ready now to finish the estimate of the term $J$:

$$J \lesssim \frac{1}{|Q|} \sum_{n \geq -\log_2 l(Q)} \frac{\|f\|_p^p 2^{-np}}{l(Q)^p} \, dx \lesssim \|f\|_p^p \cdot \frac{l(Q)^p}{l(Q)^p} \lesssim \|f\|_p^p,$$

and the theorem follows.
3 | THE BMO INEQUALITY

Proof of Theorem 1.2. We write \( f_Q \) for \( \frac{1}{|Q|} \int_Q f(x) \, dx \). We consider the following norm in the space BMO:

\[
\sup_Q \left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2}.
\]

We need to show that \( \|f\|_{\text{BMO}} \lesssim \|f\|_D \) and \( \|f\|_D \lesssim \|f\|_{\text{BMO}} \).

3.1 The inequality \( \|f\|_D \lesssim \|f\|_{\text{BMO}} \)

We begin with first part of the proof of the theorem. First of all we fix a cube \( Q \). We are going to estimate the integral

\[
\left( \frac{1}{|Q|} \int_Q \sum_{2^{-n} \leq l(Q)} |\Delta_n f(x)|^2 \, dx \right)^{1/2}.
\]

We decompose \( f \) into a sum of there functions. In more detail, we write

\[
(f - f_Q) \chi_{2Q} + (f - f_Q) \chi_{\mathbb{R}^d \setminus 2Q} + f_Q =: f_1 + f_2 + f_3.
\]

Note that \( f_3 \) is a constant, which means that \( \Delta_n f_3 = 0 \). Hence we infer the inequality

\[
\|f\|_D^2 \lesssim \left( \frac{1}{|Q|} \int_{\mathbb{R}^d} \sum_{2^{-n} \leq l(Q)} |\Delta_n f_1(x)|^2 \, dx \right)^{1/2} + \left( \frac{1}{|Q|} \int_Q \sum_{2^{-n} \leq l(Q)} |\Delta_n f_2(x)|^2 \, dx \right)^{1/2}.
\]

The first term here is a piece of cake. Indeed, using the fact that the \( \{\psi_n\} \) are uniformly bounded, we can conclude that \( \|Sg\|_{L^2} \lesssim \|g\|_{L^2} \) where \( Sg = \left( \sum_{n \in \mathbb{Z}} |\Delta_n g|^2 \right)^{1/2} \). So, the first term is less than or equal to

\[
\left( \frac{1}{|Q|} \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |\Delta_n ((f - f_Q) \chi_{2Q})|^2 \, dx \right)^{1/2} \lesssim \left( \frac{1}{|2Q|} \int_{2Q} |f(x) - f_Q|^2 \, dx \right)^{1/2}
\]

\[
\lesssim \left( \frac{1}{|2Q|} \int_{2Q} |f(x) - f_{2Q}|^2 \, dx \right)^{1/2} + |f_{2Q} - f_Q|.
\]

Both expressions here are less than \( C \|f\|_{\text{BMO}} \) for some universal constant \( C \). For the first term this is a consequence of the definition of the space BMO, and for the second one we must add the inequalities

\[
\int_Q |f(x) - f_Q| \, dx \leq |Q| \|f\|_{\text{BMO}} \quad \text{and} \quad \int_Q |f(x) - f_{2Q}| \, dx \leq \int_{2Q} |f(x) - f_{2Q}| \, dx \leq |Q| \|f\|_{\text{BMO}}.
\]

Hence it is left to estimate

\[
\left( \frac{1}{|Q|} \int_Q \sum_{2^{-n} \leq l(Q)} |\Delta_n ((f - f_Q) \chi_{\mathbb{R}^d \setminus 2Q})|^2 \, dx \right)^{1/2}.
\]

Let \( x \in Q \). Let us rewrite each term of the integrand in (3.1) (we denote \( S_n := \tilde{\psi}_n \)):
\[
\left| \Delta_n \left( (f - f_Q) \chi_{\mathbb{R}^d \setminus 2Q} \right)(x) \right| = \left| \int_{\mathbb{R}^d \setminus 2Q} (f(y) - f_Q) S_n(x - y) dy \right|
\leq \sum_{k=2}^{\infty} \int_{2^k Q \setminus 2^{k-1} Q} |f(y) - f_Q| |S_n(x - y)| dy
\leq \sum_{k=2}^{\infty} \left[ \left( \int_{2^k Q \setminus 2^{k-1} Q} |f(y) - f_Q|^2 dy \right)^{1/2} \left( \int_{2^k Q \setminus 2^{k-1} Q} |S_n(x - y)|^2 dy \right)^{1/2} \right]^{1/2}.
\] (3.2)

Next, we infer the estimate
\[
\left( \int_{2^k Q \setminus 2^{k-1} Q} |f(y) - f_Q|^2 dy \right)^{1/2} \leq \left( \int_{2^k Q} |f(y) - f_Q|^2 dy \right)^{1/2}
\leq \left( \int_{2^k Q} |f(y) - f_{2^k Q}|^2 dy \right)^{1/2} + |2^k Q|^{1/2} \cdot |f_{2^k Q} - f_Q|.
\]

The first term here does not exceed $|2^k Q|^{1/2} \| f \|_{\text{BMO}}$ whereas the second one is less than or equal to $k|2^k Q|^{1/2} \| f \|_{\text{BMO}}$, which can easily be proved by induction on $k$. Next, we estimate the second factor in (3.2). Note that once $x \in Q$ and $y \in 2^k Q \setminus 2^{k-1} Q$, we have $|x - y| \approx 2^k l(Q)$.

We continue our estimates, now using the Plancherel theorem and condition 3 with multi–indices $\alpha$ satisfying $|\alpha| = a$
\[
\ldots \lesssim \frac{1}{2^{2ak l(Q)^{2a}}} 2^{nd - 2na}.
\]

The inequalities above yield
\[
\left| \Delta_n \left( (f - f_Q) \chi_{\mathbb{R}^d \setminus 2Q} \right)(x) \right| \lesssim \sum_{k=2}^{\infty} k \left( \frac{2^k |Q|}{l(Q)} \right)^{1/2} \cdot 2^{\frac{nd - na}{2} - 2ak l(Q)^{-a}} \| f \|_{\text{BMO}}
= l(Q)^{d/2 - a} 2^{n(d/2 - a)} \| f \|_{\text{BMO}} \sum_{k=2}^{\infty} k \cdot 2^{k(d/2 - a)}
\lesssim l(Q)^{d/2 - a} 2^{n(d/2 - a)} \| f \|_{\text{BMO}}.
\]

Finally, we are able to write the following estimate for the expression (3.1):
\[
\left( \frac{1}{|Q|} \int_{Q} \sum_{2^{-n} \leq l(Q)} \left| \Delta_n \left( (f - f_Q) \chi_{\mathbb{R}^d \setminus 2Q} \right)(x) \right|^2 dx \right)^{1/2} \lesssim \| f \|_{\text{BMO}}^2 l(Q)^{d - 2a} \sum_{2^{-n} \leq l(Q)} 2^{n(d - 2a)}
\lesssim \| f \|_{\text{BMO}}^2,
\]

and the inequality $\| f \|_D \lesssim \| f \|_{\text{BMO}}$ follows.

Next, we proceed to the reverse inequality.
3.2 | The reverse inequality

So, we need to prove the inequality $\|f\|_{BMO} \lesssim \|f\|_D$. We first notice that for any constant $c$ the following inequality holds:

$$\left( \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx \right)^{1/2} \lesssim \left( \frac{1}{|Q|} \int_Q |f(x) - c|^2 \, dx \right)^{1/2}. \quad (3.3)$$

We remind the reader that we denote $S_n := \tilde{\psi}_n$ and $P_n := S_{n-1} + S_n + S_{n+1}$. Hence $\Delta_n f = S_n \ast f$. Let us choose $c := \sum_{2^{n-10} \leq |Q|} (f \ast S_n)(y)$ where $y$ is the center of $Q$. We infer the estimate

$$\left( \frac{1}{|Q|} \int_Q \left| \sum_{2^{n-10} \leq |Q|} \int_{\mathbb{R}^d} f(t) S_n(y-t) \, dt \right|^2 \, dx \right)^{1/2} \lesssim \frac{1}{|Q|^{1/2}} \left( \int_Q \left| \sum_{2^{n-10} \leq |Q|} \int_{\mathbb{R}^d} f(t) S_n(y-t) \, dt \right|^2 \, dx \right)^{1/2}. \quad (3.4)$$

Since $f$ is integrable, expressing $f(x)$ as $\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}^d} f(t) S_n(x-t) \, dt$ for almost every $x \in \mathbb{R}^d$, one readily sees that the right-hand side in (3.4) is less than or equal to

$$\frac{1}{|Q|^{1/2}} \left( \int_Q \left| \sum_{2^{n-10} \leq |Q|} \int_{\mathbb{R}^d} f(t) S_n(x-t) \, dt \right|^2 \, dx \right)^{1/2} \quad (3.5)$$

$$+ \frac{1}{|Q|^{1/2}} \left( \int_Q \left| \sum_{2^{n-10} \leq |Q|} \int_{\mathbb{R}^d} \left| f(t)(S_n(x-t) - S_n(y-t)) \right| \, dt \right|^2 \, dx \right)^{1/2} =: B_1 + B_2. \quad (3.6)$$

We estimate $B_1$ and $B_2$ separately. In both estimates, the essential ingredient is a suitable decomposition of $\mathbb{R}^d$ into a union of cubes. While estimating $B_2$ we shall also use the cancellation property of the kernels $S_n$ in a way similar to a detail in the proof of the Hörmander–Mihlin multiplier theorem.

3.2.1 | Estimate of term $B_1$

We need to prove that

$$\left( \int_Q \left| \sum_{2^{n-10} \leq |Q|} \int_{\mathbb{R}^d} f(t) S_n(x-t) \, dt \right|^2 \, dx \right)^{1/2} \lesssim |Q|^{1/2} \|f\|_D. \quad (3.7)$$

We estimate the square of the left-hand side in (3.7) with the help of the fact that $S_n = P_n \ast S_n$:

$$\int_Q \left| \sum_{2^{n-10} \leq |Q|} \int_{\mathbb{R}^d} (f \ast P_n)(u) S_n(x-u) \, du \right|^2 \, dx$$

$$\lesssim \int_Q \left| \sum_{2^{n-10} \leq |Q|} \int_{\mathbb{R}^d \setminus Q} (f \ast P_n)(u) S_n(x-u) \, du \right|^2 \, dx + \int_{\mathbb{R}^d} \sum_{2^{n-10} \leq |Q|} \int_Q (f \ast P_n)(u) S_n(x-u) \, du \, dx.$$
Hence we are done once we prove the following two inequalities:

\[
B_3 := \int_Q \left| \sum_{2^{-n} \leq l(Q)} \int_{\mathbb{R}^d \setminus Q} (f * P_n)(u) S_n(x-u) \, du \right|^2 \, dx \lesssim |Q| \|f\|_D^2, \tag{3.8}
\]

\[
B_4 := \int_{\mathbb{R}^d} \left| \sum_{2^{-n} \leq l(Q)} \int_Q (f * P_n)(u) S_n(x-u) \, du \right|^2 \, dx \lesssim |Q| \|f\|_D^2. \tag{3.9}
\]

Thanks to the triangle inequality in the space \(L^2\), the square root of the left-hand side in (3.8) is less than or equal to

\[
\sum_{2^{-n} \leq l(Q)} \left( \int_Q \left( \int_{\mathbb{R}^d \setminus Q} |S_n(x-u)| \cdot |(f * P_n)(u)| \, du \right)^2 \right)^{1/2}.
\]

In order to estimate this expression, we denote the summands there by \(G_n\) and we are going to estimate the quantity

\[
G_n^2 = \left( \int_Q \left( \int_{\mathbb{R}^d \setminus Q} |S_n(x-u)| \cdot |f * P_n(u)| \, du \right)^2 \right)^1.
\]

First, we prove the following simple statement.

**Lemma 3.1.** If \(\sigma\) is a cube with side length at least \(2^{-(n-1)}\), then the following inequality holds:

\[
\left( \frac{1}{|\sigma|} \int_{\sigma} |f * P_n|^2 \right)^{1/2} \lesssim \|f\|_D.
\]

**Proof.** We use the definition of \(P_n\) and write:

\[
\left( \frac{1}{|\sigma|} \int_{\sigma} |f * P_n|^2 \right)^{1/2} \leq \left( \frac{1}{|\sigma|} \int_{\sigma} |\Delta_{n-1} f + \Delta_n f + \Delta_{n+1} f|^2 \right)^{1/2} \lesssim \|f\|_D,
\]

where the last inequality is obvious due to the definition of the norm \(\|f\|_D\).

Denote by \(Q^-\) the cube with the same center as \(Q\) and with edge length equal to the length of the edge of \(Q\) minus \(2^{-(n-5)}\). It will be convenient for us to fix a decomposition of the whole space \(\mathbb{R}^d\) into a family of cubes \(\{\sigma_k\}_{k \in \mathbb{Z}^d}\) with edges of the order \(2^{-(n-4)}\) such that \(Q^- = \bigcup_{j \in J} \sigma_{k_j}\). Denote by \(G_1\) those cubes from \(\{\sigma_k\}\) which are contained in \(Q^-\), and by \(G_2\) those cubes from the family which have empty intersection with the cubes in \(G_1\). Note that in this case \(\mathbb{R}^d \setminus Q\) is contained in the union of cubes from \(G_2\). We see that

\[
I := \int_{Q^-} \left( \int_{\mathbb{R}^d \setminus Q} |S_n(x-u)| \cdot |f * P_n(u)| \, du \right)^2 \, dx
\]

\[
\leq \sum_{\sigma \in G_1} \int_{\sigma} \left( \sum_{\sigma_k \in G_2} \int_{\sigma_k} |S_n(x-u)| \cdot |f * P_n(u)| \, du \right)^2 \, dx
\]

\[
\leq \sum_{\sigma \in G_1} \int_{\sigma} \left( \sum_{\sigma_k \in G_2} \left( \int_{\sigma_k} \frac{|f * P_n(u)|^2}{|x-u|^{2a}} \, du \right)^{1/2} \left( \int_{\sigma_k} |S_n(x-u)||x-u|^{2a} \, du \right)^{1/2} \right)^2 \, dx
\]

\[
\leq \sum_{\sigma \in G_1} \int_{\sigma} \left( \sum_{\sigma_k \in G_2} \int_{\sigma_k} \frac{|f * P_n(u)|^2}{|x-u|^{2a}} \, du \right)^2 \left( \sum_{\sigma_k \in G_2} \int_{\sigma_k} |S_n(x-u)||x-u|^{2a} \, du \right) \, dx \leq \ldots .
\]
Note that the second sum over \( \sigma_k \in G_2 \) here is less than or equal to \( \int_{\mathbb{R}^d} |x|^{2a} |S_n(x)|^2 \mathrm{d}x \), which in turn does not exceed \( 2^{nd-2na} \), thanks to the Plancherel theorem. We continue the estimate, now referring to the fact that \( |x - u| \asymp 2^n |k - l| \), once \( x \in \sigma_k, y \in \sigma_l \):

\[
... \lesssim \sum_{\sigma_l \in G_1} \int_{\sigma_l} \left( \sum_{\sigma_k \in G_2} \int_{\sigma_k} |f * P_n(u)|^2 \mathrm{d}u \cdot 2^{2na} |k - l|^{-2a} \right) \cdot 2^{nd-2na} \mathrm{d}x \lesssim ... .
\]

Using the lemma, we first write \( \int_{\sigma_k} |f * P_n|^2 \lesssim 2^{-nd} \|f\|_D^2 \) and then continue the chain of the inequalities:

\[
... \lesssim \sum_{\sigma_l \in G_1} \int_{\sigma_l} \left( \sum_{\sigma_k \in G_2} \int_{\sigma_k} |f|^2 2^{-nd} 2^{2na} |k - l|^{-2a} \mathrm{d}x \lesssim \|f\|_D^2 2^{-nd} \sum_{\sigma_l \in G_1} \sum_{\sigma_k \in G_2} |k - l|^{-2a}.
\]

As we have already noticed, \( |k - l| \asymp 2^n |x - u| \) once \( x \in \sigma_k, y \in \sigma_l \), and hence

\[
2^{-2nd} |k - l|^{-2a} \asymp \int_{\sigma_l} \int_{\sigma_k} |k - l|^{-2a} \mathrm{d}x \asymp 2^{-2an} \int_{\sigma_l} \int_{\sigma_k} |x - u|^{-2a} \mathrm{d}u \mathrm{d}x,
\]

and consequently \( |k - l|^{-2a} \asymp 2^{2nd-2an} \int_{\sigma_l} \int_{\sigma_k} |x - u|^{-2a} \mathrm{d}u \mathrm{d}x \). We use this inequality in the estimate of the term \( I \):

\[
I \lesssim \|f\|_D^2 2^{-nd-2an} \sum_{\sigma_l \in G_1} \sum_{\sigma_k \in G_2} \int_{\sigma_l} \int_{\sigma_k} |x - u|^{-2a} \mathrm{d}u \mathrm{d}x = \|f\|_D^2 2^{-nd-2an} \int_{Q^-} \int_{\mathbb{R}^d \setminus Q'} |x - u|^{-2a} \mathrm{d}u \mathrm{d}x,
\]

where by \( Q' \) we denote the cube which is the union of all cubes that do not lie in \( G_2 \). In fact, \( Q' \) is the cube with the same center as \( Q^- \) and with the length of the edge equal to the length of the edge of \( Q^- \) plus \( 2^{-2(n-3)} \).

Let us now concentrate on the integral

\[
\int_{Q^-} \int_{\mathbb{R}^d \setminus Q'} |x - u|^{-2a} \mathrm{d}u \mathrm{d}x.
\]

For the sake of convenience we shift the cube \( Q^- \) in the way that \( Q^- = [0, l(Q^-)]^d \). With no loss of generality we take the inner integral here over the set

\[
\{u = (u_1, \ldots, u_d) \in \mathbb{R}^d : u_1, \ldots, u_k \in \left[-2^{-(n-3)}, l(Q^-) + 2^{-(n-3)}\right], u_{k+1}, \ldots, u_d \in \left(-\infty, -2^{-(n-3)}\right)\},
\]

where \( 0 \leq k \leq d - 1 \). It is obvious that \( |x - u| \) can be replaced by \( |x_1 - u_1| + \cdots + |x_d - u_d| \). Direct calculation of the corresponding integral shows that

\[
\int_{-2^{-(n-3)}}^{l(Q) + 2^{-(n-3)}} (|x_1 - u_1| + \cdots + |x_d - u_d|)^{-2a} \mathrm{d}u_1 \lesssim (|x_2 - u_2| + \cdots + |x_d - u_d|)^{-2a+1}.
\]

After that integration of the previous line \( k - 1 \) times with respect to the variables \( u_1, \ldots, u_k \), the left-hand side becomes \( (|x_{k+1} - u_{k+1}| + \cdots + |x_d - u_d|)^{-2a+k} \). The remaining integrals can be calculated explicitly:

\[
\int_{-\infty}^{-2^{-(n-3)}} (|x_{k+1} - u_{k+1}| + \cdots + |x_d - u_d|)^{-2a+k} \mathrm{d}u_{k+1}
\]

\[
\lesssim (|x_{k+1} + 2^{-(n-3)}| + |x_{k+2} - u_{k+2}| + \cdots + |x_d - u_d|)^{-2a+k+1}.
\]

After integrating this inequality with respect to the remaining variables, we see that it remains to estimate the integral

\[
\int_0^{l(Q^-)} \cdots \int_0^{l(Q^-)} (|x_{k+1} + 2^{-(n-3)}| + \cdots + |x_d + 2^{-(n-3)}|)^{-2a+d} \mathrm{d}x_1 \cdots \mathrm{d}x_d.
\]
It is obvious that this integral is maximal when \( k = d - 1 \). In the last case it is less than or equal to

\[
H := l(Q)^{d-1} \int_0^{l(Q)} (x_d + 2^{-(n-3)})^{-2a+d} \, dx_d.
\]

In order to bound \( H \), we treat two different cases separately. First, if \( d \) is even, then \( a = (d + 2)/2 \) and hence \( H \leq l(Q)^{d-1} 2^n \). In this case we readily conclude that

\[
I \lesssim ||f||_2^2 l(Q)^{d-1} 2^{-n}.
\]

If \( d \) is odd, then \( d = (d+1)/2 \) and

\[
H \leq l(Q)^{d-1} \log(l(Q) + 2^{-n}) - \log(2^{-n}) = l(Q)^{d-1} \log(1 + 2^n l(Q)).
\]

These inequalities yield

\[
I \lesssim ||f||_2^2 l(Q)^{d-1} 2^{-n} \log(1 + 2^n l(Q)).
\]

This implies that in either case

\[
\int_{Q_\perp} \left( \int_{\mathbb{R}^d \setminus Q} |f \ast P_n(u)| \cdot |S_n(x-u)| \, du \right)^2 \, dx \lesssim ||f||_2^2 l(Q)^{d-1} 2^{-n} \log(1 + 2^n l(Q)).
\]

In order to finish the estimate of the term \( G_n^2 \) it remains to bound

\[
\int_{Q_\perp} \left( \int_{\mathbb{R}^d \setminus Q} |f \ast P_n(u)| \cdot |S_n(x-u)| \, du \right)^2 \, dx
\]

from above. To this end, we split \( Q \setminus Q_\perp \) into the cubes \( \{ q_i \} \) with edges of length of order \( 2^{-(n-4)} \). There are about \( l(Q)^d 2^n (d-1) \) of these. Denote by \( q_i^+ \) the cube with the same center as \( q_i \) and with the length of the edge equal to \( 2^{-(n-5)} \) plus the length of the edge of \( q_i \). By analogy with what we have done above, we obtain the following

\[
\int_{q_i} \left( \int_{\mathbb{R}^d \setminus q_i^+} |f \ast P_n(u)| \cdot |S_n(x-u)| \, du \right)^2 \, dx \lesssim ||f||_2^2 l(q_i)^{d-1} 2^{-n} 2 \log(1 + 2^n l(q_i)) \lesssim ||f||_D^2 2^{-nd}.
\]

In order to estimate the integral that remains, we first use the Hölder inequality, and then the facts that

\[
\int_{q_i^+} |f \ast P_n(u)|^2 \, du \lesssim ||f||_D^2 2^{-nd} \quad \text{and} \quad \int_{q_i^+} |S_n(x-u)|^2 \, du \lesssim \| S_n \|_{L^2}^2 = \| \psi_n \|_{L^2}^2 \lesssim 2^{nd}:
\]

\[
\int_{q_i} \left( \int_{q_i^+} |f \ast P_n(u)| \cdot |S_n(x-u)| \, du \right)^2 \, dx \lesssim \int_{q_i} \left( \int_{q_i^+} |f \ast P_n(u)|^2 \, du \cdot \int_{q_i^+} |S_n(x-u)|^2 \, du \right) \, dx \lesssim ||f||_D^2 2^{-nd}.
\]

Taking into account the fact that there are about \( l(Q)^{d-1} 2^n (d-1) \) cubes in the collection \( \{ q_i \} \), we see that

\[
\int_{Q \setminus Q_\perp} \left( \int_{\mathbb{R}^d \setminus Q} |f \ast P_n(u)| \cdot |S_n(x-u)| \, du \right)^2 \, dx \lesssim \int_{q_i} \left( \int_{q_i^+} |f \ast P_n(u)|^2 \, du \cdot \int_{q_i^+} |S_n(x-u)|^2 \, du \right) \, dx
\]

\[
\lesssim ||f||_D^2 2^{-nd} \lesssim ||f||_D^2 l(Q)^{d-1} 2^{-n}.
\]
Finally the estimates yield
\[ G_n^2 \leq \|f\|_D^2 l(Q)^{d-1} 2^{-n} \log(1 + 2^n l(Q)). \]

Now we have all the ingredients to finish the estimate of the term \( B_3 \):
\[
\frac{B_3^{1/2}}{|Q|^{1/2}} \leq \frac{1}{|Q|^{1/2}} \sum_{n \geq -\log l(Q)} \|f\|_D l(Q)^{(d-1)/2} 2^{-n/2} \log_2(1 + 2^n l(Q))^{1/2}
\]
\[
\approx \|f\|_D l(Q)^{-1/2} \sum_{m \geq 0} 2^{-(m-\log_2 l(Q))/2} \log_2(1 + 2^m l(Q))^{1/2}
\]
\[
= \|f\|_D \sum_{m \geq 0} 2^{-m/2} \log_2(1 + 2^m)^{1/2} \lesssim \|f\|_D,
\]
and \( B_3 \) is estimated.

We proceed to inequality (3.9). Note that the functions
\[ x \mapsto \int_Q (f * P_n)(u)S_{n_1}(x - u) \, du, \]
\[ x \mapsto \int_Q (f * P_n)(u)S_{n_2}(x - u) \, du \]
are orthogonal in \( L^2(\mathbb{R}^d) \) once we suppose that \( |n_1 - n_2| \geq 2 \). This is a direct consequence of the fact that the functions \( \psi_{n_1} \) and \( \psi_{n_2} \) have nonintersecting supports. Thanks to this orthogonality, now we can estimate the expression \( B_4 \) on the left-hand side of (3.9):
\[
B_4 = \sum_{2^{-(n-10)} \leq l(Q)} \int_{\mathbb{R}^d} \left| \int_Q (f * P_n)(u)S_n(x - u) \, du \right|^2 \, dx
\]
\[
= \sum_{2^{-(n-10)} \leq l(Q)} \int_{\mathbb{R}^d \setminus Q} \left| \int_Q (f * P_n)(u)S_n(x - u) \, du \right|^2 \, dx + \sum_{2^{-(n-10)} \leq l(Q)} \int_{Q} \left| \int_Q (f * P_n)(u)S_n(x - u) \, du \right|^2 \, dx.
\]
The estimate of the first term here can be performed in the same way as the estimate of the term \( B_3 \). The second term is less than or equal to two times the sum
\[
\sum_{2^{-(n-10)} \leq l(Q)} \int_{\mathbb{R}^d} \left| \int_Q (f * P_n)(u)S_n(x - u) \, du \right|^2 \, dx + \sum_{2^{-(n-10)} \leq l(Q)} \int_{\mathbb{R}^d \setminus Q} \left| \int_Q (f * P_n)(u)S_n(x - u) \, du \right|^2 \, dx.
\]
The second sum here has been already estimated. Taking into account the fact that \( P_n * S_n = S_n \), we infer that the first one equals
\[
\sum_{2^{-(n-10)} \leq l(Q)} \int_Q \left| \int_{\mathbb{R}^d} f(t)S_n(x - t) \, dt \right|^2 \, dx,
\]
which in turn does not exceed \( \|f\|_D^2 |Q| \) by the definition of the norm \( \|f\|_D \). Hence the term \( B_1 \) is estimated.

### 3.2.2 Estimate of the term \( B_2 \)

In order to finish the proof of the theorem, we need to prove the following inequality:
\[
B_2 = \frac{1}{|Q|^{1/2}} \left( \int_Q \left| \sum_{2^{-(n-10)} > l(Q)} \int_{\mathbb{R}^d} f(t)(S_n(x - t) - S_n(y - t)) \, dt \right|^2 \, dx \right)^{1/2} \lesssim \|f\|_D.
\]
We remind the reader that $y$ here is the center of the (fixed) cube $Q$. Once again using the fact that $S_n = S_n \ast P_n$, we infer that the following inequality holds:

$$B_2 \leq \frac{1}{|Q|^{1/2}} \left( \int_Q \left| \sum_{j \neq k} \int_{2^{-(n-10)}}^{2^{-(n-10)+1}} \left( f \ast P_n \right)(u) \left| S_n(x-u) - S_n(y-u) \right| \, du \right|^2 \, dx \right)^{1/2}.$$ 

Without loss of generality, we assume that $Q$ is centered at 0 so that $y = 0$. We are now going to estimate the terms of this sum. In more detail, we shall consider the following expression:

$$\int_{\mathbb{R}^d} \left| \left( f \ast P_n \right)(-u) \right| S_n(u-x) - S_n(u) \, du. \tag{3.10}$$

We decompose $\mathbb{R}^d$ into the cubes $\{ \delta_k \}_{k \in \mathbb{Z}^d}$ with edge length $2^{-(n-10)}$ (so that the center of $\delta_k$ is $2^{-(n-10)}k$). We denote by $G_m, m \in \mathbb{N}$, the union of the cubes from $\{ \delta_k \}$ such that the maximal coordinates of their centers lie between $2^{-(n-10)+m}$ and $2^{-(n-10)+m+1}$ and by $G_0$ the union of the remaining cubes. Hence we can write the following inequality:

$$\sum_{m=1}^{\infty} \int_{G_m} \left| \left( f \ast P_n \right)(-u) \right| S_n(u-x) - S_n(u) \, du$$

$$\leq \sum_{m=1}^{\infty} \left( \int_{G_m} \left| \left( f \ast P_n \right)(u) \right|^2 \, du \right)^{1/2} \left( \int_{G_m} \left| S_n(u-x) - S_n(u) \right|^2 \, du \right)^{1/2}. \tag{3.11}$$

Lemma 3.1 yields

$$\left( \int_{G_m} \left| \left( f \ast P_n \right)(u) \right|^2 \, du \right)^{1/2} \lesssim |G_m|^{1/2} \| f \|_D \lesssim 2^{md/2} \cdot 2^{-nd/2} \cdot \| f \|_D.$$ 

Now, we concentrate on the second factor in (3.11):

$$\left( \int_{G_m} \left| S_n(u-x) - S_n(u) \right|^2 \, du \right)^{1/2} \leq \left( \int_{|u| \geq 2^{-n+m}} \left| S_n(u-x) - S_n(u) \right|^2 \, du \right)^{1/2}.$$

In order to estimate this integral, the cancellation of $\tilde{S}_n(u) := S_n(u-x) - S_n(u)$ must be involved. For any $r > 1$ we use the notation $r' = r/(r-1)$. We fix a number $q$, $1 < q < \infty$, which is greater than $d/(2a-d)$. The Hölder and the Hausdorff–Young inequalities yield

$$\int_{|u| \geq 2^{-n+m}} \left| \tilde{S}_n(u) \right|^2 \, du \lesssim \sum_{j=1}^{d} \left( \int_{|u| \geq 2^{-n+m}} u_j^{2q} \left| \tilde{S}_n(u) \right|^2 \, du \right)^{1/q'} \left( \int_{|u| \geq 2^{-n+m}} \left| u_j \right|^{-2aq} \, du \right)^{1/q}$$

$$\lesssim \sum_{j=1}^{d} 2^{(-n+m)(-2aq+d)/q} \left\| u_j^{2q} \tilde{S}_n(u) \right\|_{2q'}^2$$

$$\lesssim \sum_{j=1}^{d} 2^{(-n+m)(-2aq+d)/q} \left\| D_{\xi_j}^a \tilde{S}_n(\xi) \right\|_{(2q')'}^2.$$ 

Note that $(2q')' = 2q/(q + 1)$. Hence the expression (3.11) does not exceed

$$\sum_{m=1}^{\infty} 2^{md/2} \cdot 2^{-nd/2} \cdot \| f \|_D \cdot 2^{(-n+m)(-2aq+d)/2q} \cdot \max_j \left\| D_{\xi_j}^a \tilde{S}_n(\xi) \right\|_{2q/(q+1)}$$

$$\leq \| f \|_D \cdot \max_j \left\| D_{\xi_j}^a \tilde{S}_n(\xi) \right\|_{2q/(q+1)} \cdot 2^{-nd/2} \cdot 2^{na} \cdot 2^{-nd/2q} \cdot \sum_{m=1}^{\infty} 2^{m \left( \frac{d}{2} - a + \frac{d}{2q} \right)}.$$
Since $q$ is greater than $d/(2a - d)$, the sum here is equal to some finite constant and we are left with the expression

$$
\| f \|_D \cdot \max_j \left\| D_{\frac{a}{\frac{2d}{q+1}}} S_n(\xi) \right\| 2q \cdot 2^{-nd/2} \cdot 2^{na} \cdot 2^{-nd/2q}.
$$

In order to estimate $\left\| D_{\frac{a}{\frac{2d}{q+1}}} S_n(\xi) \right\|_2$, we observe that $S_n(\xi) = \psi_n(\xi)(e^{2\pi i x \cdot \xi} - 1)$ and hence

$$\text{supp} S_n \subset \{ 2^{n-1} \leq |\xi| < 2^{n+1} \}.$$ 

It is easy to see that if $\gamma \leq a$, then for $x \in Q$, $\xi \in \text{supp} \hat{S}_n$ we have $\left| D_{\frac{a}{\frac{2d}{q+1}}} e^{2\pi i x \cdot \xi} - 1 \right| \lesssim l(Q)^{2(n-1)}$. Indeed, for $\gamma = 0$ this is true because $|e^{2\pi i x \cdot \xi} - 1| \lesssim |x \cdot \xi| \lesssim 2^n l(Q)$ and for $\gamma > 0$ this is a consequence of the inequality $D_{\frac{a}{\frac{2d}{q+1}}} e^{2\pi i x \cdot \xi} \lesssim |x|^\gamma \lesssim l(Q)(2^{-n})^{\gamma - 1}$.

Using this estimate and the Leibniz rule, we deduce that

$$\left| D_{\frac{a}{\frac{2d}{q+1}}} S_n(\xi) \right| \lesssim l(Q) \left| \sum_{r=0}^{a} D_{\frac{a}{\frac{2d}{q+1}}} \psi_n(\xi) \right| 2^{-n(a-r-1)}.$$ 

This allows us to write

$$\left\| D_{\frac{a}{\frac{2d}{q+1}}} S_n(\xi) \right\|_2 \lesssim l(Q) \cdot \max_{0 \leq \gamma \leq a} \left\| D_{\frac{a}{\frac{2d}{q+1}}} \psi_n(\xi) \right\|_2.$$ 

The norm here can be estimated by means of the Hölder inequality:

$$\left\| D_{\frac{a}{\frac{2d}{q+1}}} \psi_n(\xi) \right\|_2 \frac{2d}{q+1} \frac{2q}{2q} = \int_{2^{n-1} \leq |\xi| < 2^{n+1}} \left| D_{\frac{a}{\frac{2d}{q+1}}} \psi_n(\xi) \right|_2^{\frac{2q}{q+1}} d\xi \leq \left\{ 2^{n-1} \leq |\xi| < 2^{n+1} \right\}^{1/(q+1)} \left( \int \left| D_{\frac{a}{\frac{2d}{q+1}}} \psi_n(\xi) \right|_2^2 d\xi \right)^{q/(q+1)}.$$ 

The first factor in the last line is less than or equal to $2^{nd/(q+1)}$ whereas the second one does not exceed $\left( 2^{nd-2np} \right)^{q/(q+1)}$ thanks to the third assumption of the theorem. So, we arrive at the following estimate:

$$\left\| D_{\frac{a}{\frac{2d}{q+1}}} \psi_n(\xi) \right\|_2 \lesssim 2^{\frac{nd}{2q}} \cdot 2^{-\frac{nd}{2} - np}.$$ 

Finally, collecting all the estimates, we see that the expression (3.11) is bounded from above by $2^n l(Q) \| f \|_D$.

In order to finish the estimate of the expression (3.10) it is left to deduce a similar bound for the integral

$$\int_{G_0} |f \ast P_n(-u)||S_n(u - x) - S_n(u)| du.$$ 

Note that $G_0$ is composed of the fixed number of cubes: this number depends on the dimension $d$ only. Hence we are done once we estimate integrals over $\delta$ for $\delta \in G_0$, namely

$$\int_{\xi} |f \ast P_n(-u)||S_n(u - x) - S_n(u)| du.$$
Using basic properties of the Fourier transformation along with the Hölder inequality, we see that

\[ |S_n(u - x) - S_n(u)| \leq l(Q) \| \nabla S_n \|_{\infty} \leq l(Q) \| \xi | \psi_n(\xi) \|_1 \]

\[ \leq l(Q) 2^n \| \psi_n \|_1 \leq l(Q) 2^n 2^{nd/2} \| \psi_n \|_2 \leq 2^{nd} l(Q) 2^n. \]

This implies the following bound

\[ \int_{\delta} |f \ast P_n(-u)||S_n(u - x) - S_n(u)| \, du \leq 2^{nd} l(Q) 2^n \int_{\delta} |f \ast P_n(-u)| \, du \]

\[ \leq 2^{nd} l(Q) 2^n |\delta|^{1/2} \left( \int_{\delta} |f \ast P_n(-u)|^2 \, du \right)^{1/2} \]

\[ \leq 2^{nd} |\delta| l(Q) 2^n \| f \|_D \leq \| f \|_D l(Q) 2^n. \]

Note that here we have used Lemma 3.1 once again.

Finally, we are ready to finish off the estimate of term \( B_2 \) and thus the proof of the theorem:

\[ B_2 \lesssim \sum_{2^{-n-10} > l(Q)} \| f \|_D \cdot l(Q) \cdot 2^n \lesssim \| f \|_D. \]

\[ \square \]

**ACKNOWLEDGEMENT**

This work was supported by the Russian Science Foundation (grant number 18–11–00053). The authors are kindly grateful to their scientific adviser Sergei V. Kislyakov for having posed the problem and for the continuous support during the process of its solution. We would also like to thank the anonymous referees for their careful reading of our paper, for a number of helpful suggestions, in particular for pointing out the importance of the integrability of the function \( f \) in Theorem 1.2.

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**How to cite this article:** Tselishchev A, Vasilyev I. Littlewood–Paley characterization of \( BMO \) and \( \text{Triebel}–\text{Lizorkin} \) spaces. Mathematische Nachrichten. 2020;293:2029–2043.

https://doi.org/10.1002/mana.201900059