CURRENTS AND FLAT CHAINS
ASSOCIATED TO VARIFOLDS, WITH AN
APPLICATION TO MEAN CURVATURE FLOW

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ABSTRACT. We prove under suitable hypotheses that convergence of integral varifolds implies convergence of associated mod 2 flat chains and subsequential convergence of associated integer-multiplicity rectifiable currents. The convergence results imply restrictions on the kinds of singularities that can occur in mean curvature flow.

1. INTRODUCTION

Let $U$ be an open subset of $\mathbb{R}^N$. Let $\mathcal{L}_{m,\text{rec}}(U, \mathbb{Z}^+)$ denote the space of functions on $U$ that take values in nonnegative integers, that are locally $L^1$ with respect to Hausdorff $m$-dimensional measure on $U$, and that vanish except on a countable disjoint union of $m$-dimensional $C^1$ submanifolds of $U$. We identify functions that agree except on a set of Hausdorff $m$-dimensional measure zero. Let $\mathcal{L}_{m,\text{rec}}(U, \mathbb{Z}_2)$ be the corresponding space with the nonnegative integers $\mathbb{Z}^+$ replaced by $\mathbb{Z}_2$, the integers mod 2.

The space of $m$-dimensional integral varifolds in $U$ is naturally isomorphic to $\mathcal{L}_{m,\text{rec}}(U, \mathbb{Z}^+)$: given any such varifold $V$, the corresponding function is the density function $\Theta(V, \cdot)$ given by

$$\Theta(V, x) = \lim_{r \to 0} \frac{\mu_V(B(x, r))}{\omega_m r^m}$$

where $\mu_V$ is the radon measure on $U$ determined by $V$ and $\omega_m$ is the volume of the unit ball in $\mathbb{R}^m$. In particular, this limit exists and is a nonnegative integer for $\mathcal{H}^m$-almost every $x \in U$.

Similarly, the space of $m$-dimensional rectifiable mod 2 flat chains in $U$ is naturally isomorphic to $\mathcal{L}_{m,\text{rec}}(U, \mathbb{Z}_2)$: given any such flat chain $A$, the corresponding function is the density function $\Theta(A, \cdot)$ given by

$$\Theta(A, x) = \lim_{r \to 0} \frac{\mu_A(B(x, r))}{\omega_m r^m} = \lim_{r \to 0} \frac{M(A \cap B(x, r))}{\omega_m r^m}$$

where $\mu_A$ is the radon measure on $U$ determined by $A$. In particular, this limit exists and is 0 or 1 for $\mathcal{H}^m$-almost every $x \in U$.

The surjective homomorphism

$$[\cdot] : \mathbb{Z}^+ \to \mathbb{Z}_2$$

$$k \mapsto [k]$$

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determines a homomorphism from $L_{\text{m-rec}}(U, \mathbb{Z}^+)$ to $L_{\text{m-rec}}(U, \mathbb{Z}_2)$, and thus also a homomorphism from the additive semigroup of integral varifolds in $U$ to the additive group of rectifiable mod 2 flat chains in $U$. If $V$ is such a varifold, we let $[V]$ denote the corresponding rectifiable mod 2 flat chain. Thus $[V]$ is the unique rectifiable mod 2 flat chain in $U$ such that
\[ \Theta([V], x) = [\Theta(V, x)] \]
for $\mathcal{H}^m$-almost every $x \in U$.

Although in some ways integral varifolds and rectifiable mod 2 flat chains are similar, the notions of convergence are quite different. Typically (and throughout this paper) convergence of varifolds means weak convergence as radon measures on $U \times \mathcal{G}_m(\mathbb{R}^N)$ (where $\mathcal{G}_m(\mathbb{R}^N)$ is the set of $m$-dimensional linear subspaces of $\mathbb{R}^N$), and convergence of flat chains means convergence with respect to the flat topology (see Section 5). A sequence $V(i)$ of integral varifolds may converge even though the associated flat chains $[V(i)]$ do not converge. Similarly, the flat chains $[V(i)]$ may converge even though the varifolds $V(i)$ do not. Furthermore, the $V(i)$ and $[V(i)]$ may converge to limits $V$ and $A$, respectively, with $A \neq [V]$. See Section 2.4 for examples.

This paper identifies an important situation in which convergence of integral varifolds implies convergence of the corresponding mod 2 flat chains to the expected limit. In practice, one often proves existence of convergent sequences of integral varifolds by appealing to Allard’s compactness theorem (described in Section 3 below). Here we prove that if a sequence of integral varifolds with limit $V$ satisfies the hypotheses of Allard’s compactness theorem plus one additional hypothesis, then the corresponding mod 2 flat chains converge to $[V]$:

1.1. Theorem. Let $V(i)$ be a sequence of $m$-dimensional integral varifolds in an open set $U$ of $\mathbb{R}^N$ that converges to a limit $V$. Suppose that

1. The $V(i)$ satisfy the hypotheses of Allard’s compactness theorem for integral varifolds, and
2. The boundaries $\partial[V(i)]$ of the mod 2 flat chains $[V(i)]$ converge in the flat topology.

Then the chains $[V(i)]$ converge in the flat topology to $[V]$.

I do not know whether hypothesis (2) is really necessary.

There is an analogous theorem with rectifiable currents in place of mod 2 flat chains. Suppose $A$ is an $m$-dimensional integer-multiplicity rectifiable current in $U$ and that $V$ is an $m$-dimensional integral varifold in $U$. Recall that $A$ determines an integral varifold $v(A)$ by forgetting orientations [Sim83, §27]. We say that $A$ and $V$ are compatible provided
\[ V = v(A) + 2W \]
for some integral varifold $W$ in $U$. Thus $A$ and $V$ are compatible if and only if they determine the same mod 2 rectifiable chain. Equivalently, $A$ and $V$ are compatible provided
\[ \Theta(V, x) - \Theta(A, x) \]
is a nonnegative, even integer for $\mathcal{H}^m$-almost every $x \in U$.

The analog of Theorem 1.1 for integer-multiplicity currents is the following:
1.2. **Theorem.** Let $V(i)$ and $A(i)$ be sequences of $m$-dimensional integral varifolds and integer-multiplicity currents, respectively, in $U$, such that $V(i)$ and $A(i)$ are compatible for each $i$. Suppose the $V(i)$ satisfy the hypotheses of Allard’s compactness theorem for integral varifolds. Suppose also that the boundaries $\partial A(i)$ converge (in the integral flat topology) to a limit current. Then there is a subsequence $i(k)$ such that the $V(i(k))$ converge to an integral varifold $V$, the $A(i(k))$ converge to a limit integer-multiplicity current $A$, and such that $A$ and $V$ are compatible.

The existence of a subsequence for which the limits $V$ and $A$ exist follows immediately from Allard’s compactness theorem for integral varifolds and from the Federer-Fleming compactness theorem for integer-multiplicity currents. What is new here is the compatibility of the limits $A$ and $V$.

2. **Preliminaries**

2.1. **Terminology.** For mod 2 flat chains, see Fleming’s original paper [Fle66] or, for a different approach, Federer’s book [Fed69, §4.2.26]. Unfortunately (for the purposes of this paper), a multiplicity $[1]$ plane does not qualify as a mod 2 flat chain under either definition. By contrast, a multiplicity 1 plane does qualify as an integral varifold. Thus in order for the map $V \mapsto [V]$ (as described in the introduction) to be a homomorphism from integral varifolds to mod 2 flat chains, one must either restrict the class of varifolds or enlarge the class of flat chains.

If one prefers to restrict, then one should (throughout this paper) replace “varifold” by “compactly supported varifold” and “flat chain” by “compactly supported flat chain”. (Federer’s flat chains are automatically compactly supported, but Fleming’s need not be.) Likewise $L_{m,rec}(U, \mathbb{Z}^+)$ and $L_{m,rec}(U, \mathbb{Z}_2)$ should be replaced by the subsets consisting of compactly supported functions. In particular, the main theorem, Theorem 3.3, remains true if one makes those replacements.

However, in this paper we have chosen instead to enlarge the class of flat chains. Fortunately, only a slight modification in Fleming’s definition (or Federer’s) is required to produce the “correct” class of flat chains. (Flat chains so defined would, in the terminology of [Fed69], be called “locally flat chains”. However, although locally flat chains over the integers are briefly mentioned in [Fed69] (in Section 4.1.24), the mod 2 versions are not.)

See Section 5 for the required modification.

When the coefficient group is the integers (with the standard metric), the “correct” class of flat chains is defined in [Sim83], and the rectifiability and compactness theorems are proved there.

2.2. **Notation.** Suppose $M$ is a Borel subset of a properly embedded $m$-dimensional $C^1$ submanifold of $U$, or of a countable union of such manifolds. If $M$ has locally finite $\mathcal{H}^m$ measure, we let $[M]$ denote the mod 2 flat chain associated to $M$ and $\nu(M)$ denote the integral varifold associated to $M$. More generally, if $f : M \to \mathbb{Z}^+$ is a function such that the extension

$$F : U \to \mathbb{Z}^+$$

$$F(x) = \begin{cases} f(x) & \text{if } x \in M \\ 0 & \text{if } x \in U \setminus M \end{cases}$$

\(^1\)Federer’s definition requires that a flat chain have compact support, and Fleming’s definition requires that a flat chain have finite flat norm.
is in $L_{m\text{-rec}}(U,\mathbb{Z}^+)$, then we let $v(M,f)$ be the integral varifold in $U$ corresponding to $F$.

2.3. **Push-forwards.** Suppose that $V$ is an integral varifold in $U$ and that $\phi : U \to W$ is a $C^1$ map that is proper on $U \cap \text{spt}(\mu_V)$. Then the push-forward $\phi_# V$ is also an integral varifold in $W$ and it satisfies

$$\Theta(\phi_# V, y) = \sum_{\phi(x)=y} \Theta(V, x)$$

for $H^m$-almost every $y \in W$.

Similarly, if $A$ is a rectifiable mod 2 flat chain in $U$ and if $\phi : U \to W$ is locally lipschitz on $U \cap \text{spt} \mu_A$, then the image chain $\phi_# A$ satisfies

$$[\Theta(\phi_# A, y)] = \sum_{\phi(x)=y} [\Theta(A, x)]$$

for $H^m$-almost every $y \in W$.

Note that this determines $\Theta(\phi_# A, y)$ for $H^m$-almost every $y$ since its value is 0 or 1 almost everywhere. In other words, for $H^m$-almost every $y \in W$,

$$\Theta(\phi_# A, y) = \begin{cases} 1, & \text{if } \sum_{\phi(x)=y} \Theta(A, x) \text{ is odd,} \\ 0, & \text{if the sum is even.} \end{cases}$$

Together (1) and (3) imply that $\phi_# [V] = [\phi_# V]$.

We shall need push-forwards only in the special cases where $\phi$ is a dilation or an affine projection.

2.4. **Examples.** Although they are not needed in this paper, some examples illustrating the differences between flat chain convergence and varifold convergence may be instructive.

First, consider a sequence of smooth, simple closed curves $C_i$ lying in a compact region of $\mathbb{R}^2$ such that the lengths tend to infinity but the enclosed areas tend to 0. Let $V_i = v(C_i)$ be the corresponding one-dimensional integral varifolds. Then the varifolds $V_i$ do not converge, but the corresponding mod 2 flat chains $[V_i]$ converge to 0.

Next let

$$J_n = \cup \left\{ \left[ \frac{k}{2n}, \frac{k+1}{2n} \right] : k \text{ odd, } 0 < k < 2n \right\}$$

and let

$$S_n = J_n \times \{0, 1/(2n)\} \subset \mathbb{R}^2.$$ 

Thus $S_n$ consists of $2n$ horizontal intervals, each of length $1/(2n)$. Let $V_n = v(S_n)$ be the corresponding integral varifold. Then the $V_n$ converge to $v(I)$, where

$$I = \{(x,0) : 0 \leq x \leq 1\}.$$ 

However, the corresponding mod 2 flat chains $[V_n]$ do not converge. To see this, suppose to the contrary that the $[V_n]$ converge to a limit chain $T$. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be the projections given by $f(x,y) = x$ and $g(x,y) = x - y$. Then $f_# [V_n] = 0$ and $g_# [V_n] = [0,1]$. Passing to the limit, we get

$$f_# T = 0, \quad g_# T = [0,1].$$

However, $T$ is clearly supported in $I$ and $f|I = g|I$, so $f \neq T = g \neq T$ (by (2)), contradicting (3). This proves that the $[V_n]$ do not converge.

For a final example, let

$$Q_n = J_n \times [0, (1/n^2)]$$

where $J_n$ is given by (1). Thus $Q_n$ is the union of $n$ closed rectangles, each with base $1/(2n)$ and height $1/(n^2)$. Let $V_n$ be the one-dimensional varifold associated to the set-theoretic boundary of $Q_n$: $V_n = v(\partial Q_n)$. Then the $V_n$ converge to $V = v(I)$, where $I$ is given by (5), but the flat chains $[V_n]$ converge to 0 since the area of $Q_n$ tends to 0. Thus the varifolds $V_n$ converge to $V$ and the chains $[V_n]$ converge to 0, but $[V] \neq 0$.

3. PROOFS OF THE MAIN RESULTS

Let $V(i)$ be a sequence of $m$-dimensional varifolds in an open subset $U$ of $\mathbb{R}^N$. If the $V(i)$ converge to a varifold $V$, then of course

$$\limsup_{i} \mu_{V(i)} W < \infty \text{ for all } W \subset U. \tag{7}$$

Conversely, if (7) holds, then the $V(i)$ have a convergent subsequence (by the compactness theorem for radon measures.)

3.1. Definition. Suppose that $V(i)$, $i = 1, 2, 3, \ldots$, and $V$ are $m$-dimensional varifolds in an open subset $U$ of $\mathbb{R}^N$. In this paper, we will say that $V(i)$ converges with locally bounded first variation to $V$ provided $V(i) \to V$ as varifolds and

$$\limsup_{i} \| \delta V(i) \|(W) < \infty \tag{8}$$

for every $W \subset U$.

To understand the definition, the reader may find it helpful to recall that if $V$ is the multiplicity 1 varifold associated to a smooth, embedded manifold-with-boundary $M$, then

$$\| \delta V \|(W) = H^{m-1}(W \cap \partial M) + \int_{M \cap W} |H(x)| \, dH^m x$$

where $H(x)$ is the mean curvature vector of $M$ at $x$. Thus for a sequence $V(i)$ of such integral varifolds, the condition (8) means that the areas of the boundaries and the $L^1$ norms of the mean curvature are uniformly bounded on compact subsets of $U$.

(See [All72] or [Sim83, §39] for the general definition of $\| \delta V \|$.)

The following closure theorem of Allard ([All72 6.4] or [Sim83 §42.8]) is one of the key results in the theory of varifolds:

3.2. Theorem. If $V(i)$ is a sequence of integral varifolds that converges with locally bounded first variation to $V$, then $V$ is also an integral varifold.

Here we prove:

3.3. Theorem. Suppose $V(i)$ is a sequence of integral varifolds that converge with locally bounded first variation to an integral varifold $V$. If the boundaries $\partial [V(i)]$ converge (as mod 2 flat chains) to a limit chain $\Gamma$, then

$$[V(i)] \to [V]$$

and therefore $\partial [V] = \Gamma$. 
The last assertion (\( \partial[V] = \Gamma \)) follows because the boundary operator is continuous with respect to flat convergence.

The result is already interesting in the case where \( \partial[V(i)] = 0 \) for all \( i \).

**Proof.** Since \( V \) is rectifiable, there is a countable union \( \cup M \) of \( m \)-dimensional \( C^1 \) embedded manifolds such that

\[
\mu_V(U \setminus \cup M) = 0
\]

Without loss of generality, we may assume that the manifolds in \( M \) are disjoint.

By the compactness theorem for flat chains of locally finite mass (see Theorem 5.1), a subsequence of the \([V(i)]\) will converge to such a flat chain \( A \). (Here and throughout the proof, “flat chain” means “mod 2 flat chain”.) Using the rectifiability theorem (see Theorem 5.4), we can conclude that \( A \) is rectifiable.

We remark that here one may prove rectifiability of \( A \) directly (without invoking Theorem 5.4). One sees that as follows. By the lower-semicontinuity of mass with respect to flat convergence, the inequality

\[
\mu[V[i]] \leq \mu_V
\]

implies that

\[
(9) \quad \mu_A \leq \mu_V
\]

and therefore that

\[
(10) \quad \mu_A(R^N \setminus \cup M) \leq \mu_V(R^N \setminus \cup M) = 0.
\]

Hence \( A \) is rectifiable.

To show that \( A = [V] \), it suffices by (10) to show that

\[
(11) \quad \Theta(\mu_V, x) - \Theta(\mu_A, x) \text{ is an even integer}
\]

for \( \mu_V \)-almost every \( x \in U \). By (10), it suffices to show that (11) holds for \( \mu_V \)-almost every \( x \in \cup M \).

For \( H^m \)-almost \( x \in \cup M \) (and therefore in particular for \( \mu_V \)-almost every \( x \in \cup M \)) we have:

\[
(12) \quad \eta_{x,\lambda}[V] \to \Theta(V, x) v(P),
\eta_{x,\lambda}[A] \to \Theta(A, x)[P]
\]

as \( \lambda \to 0 \), where \( P \) is the tangent plane at \( x \) to the unique \( M \in M \) that contains \( x \). Here \( \eta_{x,\lambda} : R^N \to R^N \) is translation by \(-x\) followed by dilation by \(1/\lambda\):

\[
\eta_{x,\lambda}(y) = \frac{1}{\lambda}(y - x).
\]

The proof of Lemma 42.9 in [Sim83] shows that \( \mu_V \)-almost every \( x \) has an additional property, namely

\[
(13) \quad \liminf_i \| \delta V(i) \| B(x, r) \leq cr^m \text{ for all } r \in (0,1)
\]

where \( c = c(x) < \infty \).

We will complete the proof by showing that if \( x \) has properties (12) and (13), then \( \Theta(V, x) \) and \( \Theta(A, x) \) differ by an even integer.

For each fixed \( \lambda \),

\[
(14) \quad \eta_{x,\lambda}[V(i)] \to \eta_{x,\lambda}[V],
\eta_{x,\lambda}[A(i)] \to \eta_{x,\lambda}[A].
\]
Thus a standard diagonal argument (applied to (12) and (14)) shows that there is a sequence \( \lambda(i) \to 0 \) such that
\[
\tilde{V}(i) \to \Theta(V, x)v(P)
\]
\[
[\tilde{V}(i)] \to \Theta(A, x)[P]
\]
where
\[
\tilde{V}(i) = \eta_{\epsilon, \lambda(i)}#V(i).
\]
(One does not need to pass to a subsequence to achieve this. Rather, one simply chooses the \( \lambda_i \)'s to go to 0 sufficiently slowly.)

Note that \( \|\delta \tilde{V}(i)\|_B(0, r) \) scales like \( r^{m-1} \). Thus (13) implies that for each \( r \),
\[
\liminf_{i \to \infty} \|\delta \tilde{V}(i)\|_B(0, r) = 0.
\]
By passing to a further subsequence, we can assume that the liminf is in fact a limit, so that
\[
\|\delta \tilde{V}(i)\|_B(0, k) = 0
\]
for each \( k \).

Thus we will be done if we can show that (15) and (16) imply that \( \Theta(V, x) - \Theta(A, x) \) is an even integer. That is, we have reduced the theorem to the special case described in the following lemma.

3.4. Lemma. Suppose
(i) A sequence \( V(i) \) of integral varifolds converges to the varifold \( V = nv(P) \), where \( n \) is a nonnegative integer and \( P \) is an \( m \)-dimensional linear subspace of \( \mathbb{R}^N \).
(ii) The radon measures \( \|\delta V(i)\| \) converge to 0.
(iii) The associated mod 2 flat chains \( [V(i)] \) converge to \( A = a[P] \), where \( a \in \mathbb{Z}_2 \).

Then \( a = [n] \).

Proof. We may assume that \( P = \mathbb{R}^m \times (0)^{N-m} \subset \mathbb{R}^N \). Let
\[
\pi : \mathbb{R}^N \cong \mathbb{R}^m \times \mathbb{R}^{N-m} \to \mathbb{R}^m
\]
be the orthogonal projection map.

Hypothesis (iii) implies that for almost every \( R > 0 \),
\[
[V(i)], B^N(0, R) \to a[P] \cap B^N(0, R) = a[P \cap B^N(0, R)].
\]
We can assume that this is the case for \( R = 1 \). (Otherwise dilate by \( 1/R \).) We write \( B \) for \( B^N(0, 1) \).

Let \( W(i) = V(i) \cap B \). By (18),
\[
[\pi#W(i)] = [\pi#V(i)] \to a[B^m],
\]
where \( B^m = B^m(0, 1) \). Also,
\[
W(i) \to V \perp B = nv(P \cap B)
\]
and therefore
\[
\pi#W(i) \to nv(B^m).
\]
Note that
\[(\text{21}) \quad \pi_# W(i) = v(B^m, \theta_i) \]
where
\[
\theta_i(x) = \sum_{y \in B \cap \pi^{-1}(x)} \Theta(W(i), y).
\]

From hypotheses (i) and (ii), it follows that
\[(\text{22}) \quad \mathcal{L}^m Q_i \rightarrow 0\]
where
\[(\text{23}) \quad Q_i = \{x \in B^m : \theta_i(x) \neq n\}.
\]
(This is a very nontrivial fact. Indeed, it is a key part of the proof given in [Sim83] of the closure theorem for integral varifolds. See Remark 3.5 below for a more detailed discussion.)

Now
\[
\mathcal{L}^m (\pi_# W(i)) = \mathcal{L}^m (\{x \in B^m : \theta_i(x) \text{ is odd}\}).
\]
Thus
\[
[\pi_# W(i)] - [n v(B^m)] = \{x \in B^m : \theta_i(x) - n \text{ is odd}\},
\]
and so (by (22) and (23))
\[
M([\pi_# W(i)] - [n v(B^m)]) \leq \mathcal{L}^m (Q_i) \rightarrow 0.
\]
Consequently,
\[
[\pi_# W(i)] \rightarrow [n v(B^m)].
\]
This together with (19) implies that \(a[B^m] = [n v(B^m)]\) and thus that \(a = [n]\). □

3.5. Remark. Here we elaborate on statement (22) of the proof above, because it may not be immediately apparent to one who reads [Sim83] that the lemma we cite (Lemma 42.9) does actually justify that step. Note that
\[(\text{24}) \quad \int_{B^m} \theta_i \rightarrow n \mathcal{L}^m (B^m)\]
by (20) and (21). Let \(\epsilon > 0\). Write
\[(\text{25}) \quad \theta_i(x) = F_{i,\epsilon}(x) + G_{i,\epsilon}(x)\]
where
\[
F_{i,\epsilon}(x) = \sum \{\Theta(W(i), y) : y \in B \cap \pi^{-1}(x), |y| < \epsilon\}
\]
and
\[
G_{i,\epsilon}(x) = \sum \{\Theta(W(i), y) : y \in B \cap \pi^{-1}(x), |y| \geq \epsilon\}.
\]
Now
\[(\text{26}) \quad \int G_{i,\epsilon} \rightarrow 0\]
since \(W(i) \rightarrow n v(P)\). This together with (24) implies that
\[(\text{27}) \quad \int F_{i,\epsilon} \rightarrow n \mathcal{L}^m (B^m).
\]
According to [Sim83] Lemma 42.9,
\[(\text{28}) \quad \limsup_{i \rightarrow \infty} \int_{B^m} (F_{i,\epsilon} - n)^+ d\mathcal{L}^m \leq \omega(\epsilon)
\]
for some function \( \omega(\cdot) \) such that \( \omega(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

(Note: there is a mistake in the statement of \cite[Lemma 42.9]{Sim83}; instead of (28), it asserts the weaker inequality

\[
\limsup_{i \to \infty} L^m \{ x \in B^m : F_{i, \epsilon}(x) > n \} \leq \omega(\epsilon).
\]

However, the proof of \cite[Lemma 42.9]{Sim83} establishes the stronger statement (28). Indeed the stronger statement is essential in the proof of Allard’s integrality theorem \cite[§42.8]{Sim83}. In particular, the stronger statement is used in line (8) of that proof.)

From (27) and (28), we see that

\[
\limsup_{i \to \infty} \int_{B^m} |F_{i, \epsilon} - n| d\mathcal{L}^m \leq \omega(\epsilon)
\]

This together with (26) and (25) implies that

\[
(29) \quad \limsup_{i \to \infty} \int_{B^m} |\theta_i - n| d\mathcal{L}^m \leq \omega(\epsilon).
\]

Letting \( \epsilon \to 0 \) gives

\[
\limsup_{i \to \infty} \int_{B^m} |\theta_i - n| d\mathcal{L}^m = 0
\]

and thus (since \( \theta_i \) is integer-valued)

\[
\lim_{i \to \infty} \mathcal{L}^m \{ x \in B^m : \theta_i(x) \neq n \} = 0.
\]

3.6. **Theorem.** Suppose \( V(i) \) is a sequence of integral varifolds that converge with locally bounded first variation to an integral varifold \( V \). Suppose \( A(i) \) is a sequence of integer-multiplicity rectifiable currents such that \( V(i) \) and \( A(i) \) are compatible. If the boundaries \( \partial A(i) \) converge (in the integral flat topology) to a limit integral flat chain \( \Gamma \), then there is a subsequence \( i(k) \) such that the \( A(i(k)) \) converge to an integer-multiplicity rectifiable current \( A \). Furthermore, \( V \) and \( A \) must then be compatible, and \( \partial A \) must equal \( \Gamma \).

The proof is exactly analogous to the proof of Theorem 3.3. Alternatively, one can argue as follows. The existence of a subsequence \( A(i(k)) \) that converges to an integer-multiplicity rectifiable current \( A \) follows from the compactness theorem for such currents (see Theorems 5.1 and 5.4). The “furthermore” statement then follows immediately from Theorem 3.3, together with the observation that an integral varifold and an integer-multiplicity rectifiable current are compatible if and only if they determine the same mod 2 rectifiable flat chain.

4. **Application to Mean Curvature Flow**

Here we show how the results of this paper rule out certain kinds of singularities in mean curvature flows. In another paper, we will use similar arguments to prove, under mild hypotheses, boundary regularity at all times for hypersurfaces moving by mean curvature.

On both theoretical and experimental grounds, grain boundaries in certain annealing metals are believed to move by mean curvature flow \cite[Appendix A]{Bra78}. In such metals, one typically sees triple junctions where three smooth surfaces come together at equal angles along a smooth curve. Of course one also sees such triple junctions in soap films, which are equilibrium solutions to mean curvature flow.
Consider the following question: can an initially smooth surface evolve under mean curvature flow so as later to develop triple junction type singularities? More generally, can such a surface have as a blow-up flow (i.e., a limit of parabolic blow-ups) a static configuration of $k$ half-planes (counting multiplicity) meeting along a common edge? Using Theorem 3.3 we can (for a suitable formulation of mean curvature flow) prove that the answer is “no” if $k$ is odd.

(Suppose $\mathcal{M}$ is a Brakke flow, $X_i$ is a sequence of spacetime points converging to $X = (x, t)$ with $t > 0$, and $\lambda_i$ is a sequence of numbers tending to infinity. Translate $\mathcal{M}$ in spacetime by $-X_i$ and then dilate parabolically by $\lambda_i$ to get a flow $\mathcal{M}_i$. A blow-up flow of $\mathcal{M}$ is any Brakke flow that can be obtained as a subsequential limit of such a sequence.)

Let $I \subset \mathbb{R}$ be an interval, typically either $[0, \infty)$ or all of $\mathbb{R}$. Recall that a Brakke flow $t \mapsto V(t)$ of varifolds is called an integral Brakke flow provided $V(t)$ is an integral varifold for almost all $t \in I$. (See [Bra78, §3] or [Ilm94, §6] for the definition of Brakke flow.)

4.1. Definition. Let $t \mapsto V(t), t \in I$ be an integral Brakke flow in $U \subset \mathbb{R}^N$. We say that $V(\cdot)$ is cyclic mod 2 (or cyclic for short) provided $\partial[V(t)] = 0$ for almost every $t \in I$.

More generally, suppose $W$ is an open subset of $U$ and $J$ is a subinterval of $I$. We say that the Brakke flow $V(\cdot)$ is cyclic mod 2 in $W \times J$ if for almost all $t \in J$, $[V(t)]$ has no boundary in $W$.

We have:

4.2. Theorem. Suppose $t \mapsto V_i(t)$ is a sequence of integral Brakke flows that converge as Brakke flows to an integral Brakke flow $t \mapsto V(t)$. If the flows $V_i(\cdot)$ are cyclic mod 2, then so is the flow $V(\cdot)$. If the flows $V_i(\cdot)$ are cyclic mod 2 in $W \times J$, then so is the flow $V(\cdot)$.

Here convergence as Brakke flows means that for almost all $t$:

(30)  $\mu_{V_i(t)} \to \mu_{V(t)}$, and

(31)  there is a subsequence $i(k)$ (depending on $t$) such that $V_{i(k)}(t) \to V(t)$.

(This definition may seem peculiar, but this is precisely the convergence that occurs in Ilmanen’s compactness theorem for integral Brakke flows [Ilm94, §7].)

Theorem 4.2 follows immediately from Theorem 3.3 and the following lemma (which is implicit in [Ilm94], but is not actually stated there):

4.3. Lemma. Suppose $t \mapsto V_i(t)$ is a sequence of Brakke flows in $U \subset \mathbb{R}^N$ that converges to a Brakke flow $t \mapsto V(t)$. Then for almost every $t \in I$, there is a subsequence $i(k)$ such that $V_{i(k)}(t)$ converges with locally bounded first variation to $V(t)$. Indeed, we can choose the subsequence so that $\delta V_{i(k)}$ is absolutely continuous with respect to $\mu_{V_{i(k)}}$ and so that

$$\sup_{i(k)} \int_{x \in W} |H(V_{i(k)}(t), x)|^2 \, d\mu_{V_{i(k)}}(x) < \infty.$$

for every $W \subset \subset U$, where $H(V_{i(k)}(t), \cdot)$ is the generalized mean curvature of $V_{i(k)}(t)$.

Proof. For simplicity, let us assume that $I = [0, \infty)$. 
Recall that for almost all $t$, the varifold $V_i(t)$ has bounded first variation, and the singular part of the first variation measure is 0. Thus (for such $t$)

\begin{equation}
\|\delta V_i(t)\|_W = \int_W |H_{i,t}| \, d\mu_{i,t} \leq \left( \int_W |H_{i,t}|^2 \, d\mu_{i,t} \right)^{1/2} \left( \mu_{i,t}(W) \right)^{1/2}.
\end{equation}

where $H_{i,t}$ is the generalized mean curvature of $V_i(t)$ and $\mu_{i,t} = \mu_{V_i(t)}$.

Consider first the case that the varifolds $V_i(t)$ are all supported in some compact set. Then the initial total masses $M(V_i(0)) = \mu_{V_i(0)}(\mathbb{R}^N)$ are bounded above by some $C < \infty$. Since mass decreases under mean curvature flow, the same bound holds for all $t > 0$. By definition of Brakke flow,

$$\mathcal{T}_t M(V_i(t)) \leq -\int |H(V_i(t), \cdot)|^2 \, d\mu_{V_i(t)},$$

so

\begin{equation}
\int_{t \in I} \int |H(V_i(t), \cdot)|^2 \, d\mu_{V_i(t)} \, dt \leq C.
\end{equation}

Thus by Fatou’s theorem,

$$\liminf_{i} \int_{t \in I} |H(V_i(t), \cdot)|^2 \, d\mu_{V_i(t)} \, dt \leq C$$

In particular,

$$\liminf_{i} \int |H(V_i(t), \cdot)|^2 \, d\mu_{V_i(t)} < \infty$$

for almost every $t$. For each such $t$, there is a subsequence $i(k)$ such that

$$\sup_k \int |H(V_{i(k)}(t), \cdot)|^2 \, d\mu_{V_{i(k)}(t)} < \infty$$

This together with (32) implies that the $V_{i(k)}(t)$ converge with locally bounded first variation to $V(t)$ (in the sense of Definition 3.1). The general case (noncompactly supported varifolds) is essentially the same, except that instead of (33) one uses the local bound:

$$\sup_i \int_{t \in J} \int_{x \in W} |H(V_i(t), x)|^2 \, d\mu_{V_i(t)} \, dt < \infty$$

together with the mass bound

$$\sup_i \sup_{t \in J} \mu_{V_i(t)}(W) < \infty,$$

both of which bounds hold for all intervals $J \subset I$ and open subsets $W \subset U$ [Eck04, Proposition 4.9].

4.4. Remark. The lemma and Allard’s closure theorem (Theorem 3.2) imply that a limit of integral Brakke flows is also integral.

4.5. Corollary. Suppose $k$ is an odd integer. A static configuration of $k$-half planes (counting multiplicity) meeting along a common edge cannot occur as a blow-up flow to an integral Brakke flow that is cyclic mod 2.

Proof. Let $V$ be the varifold corresponding to $k$ halfplanes (counting multiplicity) meeting along an edge $E$. If the static flow $t \mapsto V$ is a limit flow to an integral Brakke flow that is cyclic mod 2, then this static flow is also cyclic mod 2 and thus $\partial[V] = 0$. But $\partial[V]$ is the common edge $E$ with multiplicity $[k]$, so $k$ must then be even. □
The following theorem shows that for rather arbitrary initial surfaces, there exist nontrivial integral Brakke flows that are cyclic mod 2.

4.6. **Theorem.** Let $A_0$ be any compactly supported rectifiable mod 2 cycle in $\mathbb{R}^N$. (For example, $A_0$ could be the mod 2 rectifiable flat chain associated to a $C^1$ compact, embedded submanifold.) Then there is an integral Brakke flow $t \in [0, \infty) \mapsto V(t)$ and a one-parameter family $t \in [0, \infty) \mapsto A(t)$ of rectifiable mod 2 flat chains with the following properties:

1. $A(0) = A_0$ and $\mu_{V(0)} = \mu_{A(0)}$.
2. $\partial A(t) = 0$ for all $t$.
3. $t \mapsto A(t)$ is continuous with respect to the flat topology.
4. $\mu_{A(t)} \leq \mu_{V(t)}$ for all $t$.
5. $A(t) = [V(t)]$ for almost every $t$.

In particular, the flow is cyclic mod 2, and thus triple (or more generally odd-multiplicity) junctions cannot occur in $V(\cdot)$ by Corollary 4.5.

(Remark about assertion 5: Since $V(\cdot)$ is an integral Brakke flow, $V(t)$ is an integral varifold for almost all $t$ and thus $[V(t)]$ is well-defined for almost all $t$.)

**Proof.** Except for assertion 5, this was proved by Ilmanen [Ilm94, 8.1 and 8.3]. He used integer-multiplicity currents rather than mod 2 flat chains, but his proof works equally well in either context. (The $A(t)$ here is the slice $T_t$ in Ilmanen’s notation.) The flat continuity (3) is not stated there, but it follows immediately from [Ilm94] 8.3.

Roughly speaking, Ilmanen constructs $V(\cdot)$ and $A(\cdot)$ as limits of “nice” examples $V_i(\cdot)$ and $A_i(\cdot)$ for which

$$\mu_i(t) = \mu_{A(i)}(t)$$

for all $t$.

Now his $A_i(t)$ are not quite cycles. However, $A_i(t)$ moves by translation, and it moves very fast if $i$ is large. In particular, if $U \subset \subset \mathbb{R}^N$ and $I \subset \subset (0, \infty)$, then for sufficiently large $i$ and for all $t \in I$, $\partial A_i(t)$ lies outside $U$.

Thus (exactly as in the proof of Theorem 4.2 or by Remark 4.4 and Theorem 3.6), we deduce (for almost every $t \in I$) that $A(t) \cdot U = [V(t)] \cdot U$ and that $\partial [V(t)]$ lies outside $U$.

Since $U$ is arbitrary, this gives 5. \qed

4.7. **Remark.** The description just given is a slightly simplified account of Ilmanen’s proof. Actually he does not quite get the pair $(V(\cdot), A(\cdot))$ as limits of nice examples. Rather he gets a pair of flows $(\mu^*(\cdot), A^*(\cdot))$ of one higher dimension as such a limit. The argument given above shows that $(\mu^*(\cdot), A^*(\cdot))$ has the property corresponding to property 5 above (and Ilmanen in his proof shows that it has properties 1–4). Now the pair $(\mu^*(\cdot), A^*(\cdot))$ is translation invariant in one spatial direction. By slicing, Ilmanen gets the desired pair $(\mu(\cdot), A(\cdot))$. Translational invariance implies (in a straightforward way) that properties 1–4 for $(\mu(\cdot), A(\cdot))$ are equivalent to the corresponding properties for $(\mu^*(\cdot), A^*(\cdot))$.

Theorem 4.6 has an analog for integer-multiplicity currents in place of mod 2 flat chains:

4.8. **Theorem.** Let $A_0$ be any compactly supported integer-multiplicity cycle (i.e., integer-multiplicity current with $\partial A_0 = 0$.) Then there is an integral Brakke flow
We define the flat seminorm $M_W$ we define the space $F$ the completion of $F$. It is straightforward to show that the resulting space is independent of the choice of $Q$. That among all sequences of compactly supported, finite-mass flat chains \((34) \lim \inf \mu \leq M_W(t) \leq \mu_V(t) \) for all $t$.

We let $G$ chains with coefficients in $\mathbb{Z}$ the distance from 0. The groups relevant for this paper are $U$ standard metrics. If $G$ thus has a quadruple junction. Ruling out even-multiplicity junctions is more subtle. In particular, limits of $G$ that $\mu$ and $G$ are compatible for almost every $G$.

We omit the proof since it is almost identical to the proof of the mod 2 case, Theorem 4.6. Note that if an integer-multiplicity current $A$ is compatible with an integral varifold $V$, then $[V]$ is the flat chain mod 2 corresponding to $A$. It follows that the Brakke flow $V(\cdot)$ in Theorem 4.8 is cyclic mod 2. In particular, triple (or more generally odd-multiplicity) junctions cannot occur in $V(\cdot)$ by Corollary 4.3.

5. Appendix: Flat Chains

Let $G$ be a metric abelian coefficient group, i.e., an abelian group with a translation invariant metric $d(\cdot, \cdot)$. The norm $|g|$ of a group element $g$ is defined to be its distance from 0. The groups relevant for this paper are $\mathbb{Z}_2$ and $\mathbb{Z}$, both with the standard metrics. If $U$ is an open subset of $\mathbb{R}^N$, let $\mathcal{F}_c(U; G)$ be the space of flat chains with coefficients in $G$ and with compact support in $U$, as defined in [Fle66]. We let $\mathcal{F}_{m,c}(U; G)$ denote the space of $m$-dimensional chains in $\mathcal{F}_c(U; G)$.

If $W$ is an open subset of $\mathbb{R}^N$ and $A \in \mathcal{F}_c(\mathbb{R}^N; G)$, we let $M_W(A)$ be the minimum of

\begin{equation}
(34) \; \lim \inf \mu_{A(t)}(W)
\end{equation}

among all sequences of compactly supported, finite-mass flat chains $A(i)$ such that $A(i)$ converges in the flat topology to $A$. By lower-semicontinuity of mass, $M_W(A) = \mu_A(W)$ for any chain $A$ of finite mass.

We define the flat seminorm $\mathcal{F}_W$ by

$$\mathcal{F}_W(A) = \inf \{M_W(A - \partial Q) + M_W(Q) \}$$

where the infimum is over all $Q \in \mathcal{F}_c(\mathbb{R}^N; G)$.

Let $U$ be an open subset of $\mathbb{R}^N$. Choose a countable collection $W$ of nested open sets whose union is $U$ and each of whose closures is a compact subset of $U$. We define the space $\mathcal{F}_{m,c}(U; G)$ of flat $m$-chains in $U$ with coefficients in $G$ to be the completion of $\mathcal{F}_{m,c}(U; G)$ with respect to the seminorms $\mathcal{F}_W$ for $W \in \mathcal{W}$. (It is straightforward to show that the resulting space is independent of the choice of $\mathcal{W}$.)

By continuity, the seminorms $\mathcal{F}_W$ extend to all of $\mathcal{F}_m(U; G)$. We also define the mass seminorms $M_W$ on all of $\mathcal{F}_m(U; G)$ exactly as above (34).
Convergence of flat chains means flat convergence, i.e., convergence with respect to the seminorms \( \mathcal{F}_W \) for all open \( W \subset U \) or, equivalently, for all \( W \in \mathcal{W} \) for a collection \( \mathcal{W} \) of nested open sets as above.

We define the support of a flat chain \( A \in \mathcal{F}_m(U; G) \) as follows: \( x \notin \text{spt } A \) if and only if there is a sequence \( A_i \in \mathcal{F}_{m,c}(U; G) \) and a ball \( B(x, r) \) such that \( A_i \to A \) and such that \( \text{spt } A_i \) is disjoint from \( B(x, r) \) for every \( i \).

In the proof of the main results, Theorems 3.3 and 3.6, we used the following version of the compactness theorem for flat chains. It is valid for any coefficient group \( G \) in which all sets of the form \( \{ g \in G : |g| \leq r \} \) are compact. In particular, it is valid for the integers with the usual norm and for the integers mod 2.

5.1. **Theorem** (Compactness Theorem). Let \( A_i \) be a sequence of flat \( m \)-chains in \( U \) such that the boundaries \( \partial A_i \) converge to a limit chain \( \Gamma \), and such that

\[
\limsup_i M_W(A_i) < \infty
\]

for every open \( W \subset U \). Then \( A_i \) has a convergent subsequence.

We first prove the version for compact supports:

5.2. **Lemma.** If \( A_i, \Gamma \in \mathcal{F}_c(\mathbb{R}^N; G) \) are supported in a fixed compact subset \( X \) of \( \mathbb{R}^N \), if \( \sup_i M(A_i) < \infty \), and if \( \mathcal{F}(\partial A_i - \Gamma) \to 0 \), then there is a subsequence \( A_{i(k)} \) and a chain \( A \) such that \( \mathcal{F}(A_{i(k)} - A) \to 0 \).

**Proof.** We may assume \( X \) is convex (otherwise replace it by its convex hull.) Since \( \partial A_i \to \Gamma \), we have \( \partial \Gamma = 0 \). It follows that \( \Gamma = \partial R \) for some chain \( R \) of finite mass.

By hypothesis,

\[
\mathcal{F}(\partial A_i - \partial R) \to 0.
\]

Thus there are chains \( Q_i \) such that

\[
M(Q_i) + M(\partial Q_i + \partial A_i - \partial R) \to 0.
\]

We may assume that \( R \) and the \( Q_i \) are supported in \( X \). (Otherwise map them into \( X \) by the nearest point retraction of \( \mathbb{R}^N \) to \( X \).)

Now let

\[
A_i^* = Q_i + A_i - R.
\]

Note that

\[
\limsup_i M(A_i^*) \leq \sup_i M(A_i) + M(R) < \infty
\]

since \( M(Q_i) \to 0 \) by (30). From (36) we also see that \( M(\partial A_i^*) \to 0 \), so in particular \( \sup_i M(\partial A_i^*) < \infty \).

Thus by the standard compactness theorem (see for example [Fle66, 7.4]), we may, by passing to a subsequence, assume that the \( A_i^* \) converge to a limit \( A^* \).

Hence

\[
\mathcal{F}(A_i - (A^* + R)) = \mathcal{F}(A_i^* - A^* - Q_i)
\]
\[
\leq \mathcal{F}(A_i^* - A^*) + \mathcal{F}(Q_i)
\]
\[
\leq \mathcal{F}(A_i^* - A^*) + M(Q_i)
\]
\[
\to 0
\]

since \( A_i^* \to A^* \) and \( M(Q_i) \to 0 \). Thus the \( A_i \) converge to \( A^* + R \).
Proof of Theorem 5.1. Let $W$ be an open set whose closure is a compact subset of $U$. Choose an open set $V$ whose interior contains the closure of $W$ and whose closure is a compact subset of $V$. The idea of the proof is to work in a one-point compactification of $V$ so that we can apply Lemma 5.2.

Let $u : \mathbb{R}^N \rightarrow [0, 1]$ be a smooth function that is 1 on $W$, that is strictly positive on $V$, and that vanishes on $\mathbb{R}^N \setminus V$.

Define $f : \mathbb{R}^N \rightarrow \mathbb{R}^{N+1}$ by

$$F(x) = u(x)(x, 1).$$

Note that $f$ is lipschitz and that $f$ maps the complement of $V$ to a point. (Indeed, $f(\mathbb{R}^N)$ may be regarded as a one-point compactification of $V$.) It follows that $M(f_\# S)$ and $F(f_\# S)$ can be bounded by a constant times $M_V(S)$ and $F_V(S)$, respectively.

Let $A_i^* = f_\# A_i$. Then the hypotheses of Lemma 5.2 are satisfied for the $A_i^*$. Thus by passing to a subsequence we may assume that the $A_i^*$ converge in the $F$ metric.

By passing to a further subsequence, we may assume that

$$\sum_i F(A_i^* - A_{i+1}^*) < \infty. \tag{37}$$

Let $H = H_\zeta$ be a halfspace of the form $\mathbb{R}^N \times [\zeta, \infty)$. From (37) it follows that

$$\sum_i F(A_i^* \cup H - A_{i+1}^* \cup H) < \infty \tag{38}$$

for almost every $\zeta$ (See [Fle66, Lemma 2.1]). Fix such a $\zeta \in (0, 1)$ and the corresponding $H$.

The radial projection map

$$\pi : H \rightarrow \mathbb{R}^N$$

$$\pi(x, y) = x/y$$

is lipschitz, so by (38) the chains $A_i^! := \pi_\# (A_i^* \cup H)$ are $F$-convergent.

It follows that the $A_i^!$ are also $F_W$ convergent (since $F_W \leq F$).

But $\pi \circ f$ is the identity on $W$. Hence $A_i^!$ and $A_i$ coincide in $W$. (In other words, $A_i - A_i^!$ is supported in $W^c_\zeta$.)

Thus the $A_i$ are also $F_W$ convergent.

We have shown that for every open $W \subset U$, there is an $F_W$-convergent subsequence of the $A_i$. Now apply the diagonal argument to a nested sequence of such $W$’s that exhaust $U$. □

5.3. Corollary. Suppose $A_i$ are flat chains in $U$ such that

$$\limsup_i (M_W(A_i) + M_W(\partial A_i)) < \infty$$

for every $W \subset U$. Then $A_i$ has a subsequence that converges in the flat topology.

Proof. By Theorem 5.1 applied to the $\partial A_i$, there is a subsequence $i(k)$ for which the boundaries $\partial A_{i(k)}$ converge. Consequently the $A_{i(k)}$ satisfy the hypotheses of Theorem 5.1. □

5.4. Theorem (Rectifiability Theorem). Suppose $A$ is a flat $m$-chain in $U$ with locally finite mass. Then $A$ is rectifiable.
Of course “A has locally finite mass” means “\( M_W(A) < \infty \) for every open \( W \subset \subset U \).”

The theorem was proved in the case \( G = \mathbb{Z} \) by Federer and Fleming [FF60]. The proof is also presented in [Fed69] and in [Sim83]. Rather different proofs are given in [Sol84] and [Whi89]. Fleming proved the rectifiability theorem for all finite coefficient groups [Fle66]. For the most general result, see [Whi99], which gives a simple necessary and sufficient condition on the coefficient group in order for the rectifiability theorem to hold.

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