Inductive Construction of the Loop Transform for Abelian Gauge Theories

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Abstract

We construct the loop transform in the case of Abelian gauge theories as a unitary operator given by the inductive limit of Fourier transforms on tori. We also show that its range, i.e. the space of kinematical states of the quantum loop representation, is the Hilbert space of square integrable complex valued functions on the group of hoops.

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1 Introduction

The history of the loop transform begins in 1990 when Rovelli and Smolin [14] proposed, in the context of canonical quantum gravity, a formal transform to pass from functions on the space of connections to functions of loops. In a gauge theory where the gauge group \( G \) is assumed to be a closed subgroup of \( U(N) \), the loop transform \( \ell_\psi \) of a function \( \psi \) of connections is given by

\[
\ell_\psi(\alpha) := \int_\mathcal{A} T_\alpha(A)\psi(A)dA
\]

where \( \mathcal{A} \) is the space of smooth principal connections of a fixed principal fiber bundle \( P(M, G) \), \( dA \) is a formal measure on \( \mathcal{A} \), \( \alpha \) is a loop in \( M \) at least piecewise \( C^1 \) and \( T_\alpha \) is the associated Wilson function, which relates a connection \( A \in \mathcal{A} \) to the normalized trace of the holonomy of \( A \) around \( \alpha \), usually written

\[
T_\alpha(A) = \frac{1}{N} Tr(P \exp \oint_\alpha A),
\]

where \( P \) denotes the parallel transport.
The role of this transform is analogous to that of the Fourier transform in quantum mechanics, which enables to pass from the position representation to the momentum representation. In the canonical quantization of gauge theories, the loop transform should relate the connection representation, in which states are functions on the configuration space $\mathcal{A}$, to the loop representation in which states are functions on the loop space. Formally, the Wilson functions $T_\alpha$ play the role of the phase factors in the Fourier transforms and $\mathcal{A}$ the role of the finite dimensional configuration space $\mathbb{R}^n$ in quantum mechanics.

The loop transform is relevant for gauge theories invariant under diffeomorphisms as the Euclidean formulation of General Relativity or the Chern-Simon theory in three dimensions. In fact, if $\psi$ and $DA$ are invariant under diffeomorphisms and gauge transformations, then the function $\ell_\psi$ is a topological invariant of the manifold $M$ and it naturally satisfies the gauge and diffeomorphism constraints in the quantum theory [8].

The connection and the loop representations are equivalent if and only if the loop transform is an unitary operator. The first problem to get a loop transform was the construction of the measure $DA$. Measures invariant under gauge transformations and diffeomorphisms were obtained by Ashtekar and Lewandowski [2] and by Baez [4] on a suitable compact space $\overline{A/G}$ containing densely $A/G$, the space of connections modulo gauge transformations. Actually, $\overline{A/G}$ is the spectrum of the $C^*$-algebra generated by the Wilson functions $T_\alpha$ in the case the loops $\alpha$ are piecewise analytic and the gauge group is assumed to be $U(N)$ or $SU(N)$.

However the assumption of analyticity has the unpleasant consequence that the constructed measures are invariant only under analytic diffeomorphisms. For the gauge group $U(1)$ a measure was constructed in [2] starting from continuous piecewise $C^k$ loops, for any order $k \geq 1$ of differentiability. In [4] the case of more general gauge group was studied starting on piecewise smoothly immersed paths.

The loop transform amounts to the construction of a suitable basis on the Hilbert space of functions on $\overline{A/G}$ [10]. Other bases are given in [3] for the analytic case and in [12] for the smooth case. In all these settings the Hilbert spaces under consideration are inductive limits of Hilbert spaces. Here we treat the Abelian case, $G = U(1)$, where the theory of Abelian groups can be invoked to obtain the loop transform as inductive limit of Fourier transformations on tori.

2 Preliminaries

We fix a principal fiber bundle $P(M,G)$, where $M$ is an ordinary manifold and the group $G$ is $U(N)$ or $SU(N)$. The manifold $M$ admits a unique compatible real analytic structure up to $C^\infty$ diffeomorphisms (this can be worked out from §4.7 in [11]). In the following we shall use a fixed analytic structure.

We will consider continuous paths and loops on $M$ which are piecewise analytic, i.e. continuous maps $\gamma$ defined on a closed interval $[a,b]$ with a partition $t_0 = a \leq t_1 \leq ... \leq t_i \leq b$.
t_{i+1} \leq \ldots \leq t_n = b$ such that $\gamma$ agrees with an analytic map on each $[t_i, t_{i+1}]$. We call $\gamma$ an immediate retracing if $\gamma = \prod_{i=1}^{k} \gamma_i \gamma_i^{-1}$ for a finite collection $\gamma_1, \ldots, \gamma_k$ of paths in $M$. We call a loop $\alpha$ in $M$ thin if it is homotopic to the base point $\star$ with an homotopy whose image is entirely contained in the image of $\alpha$.

Two loops $\alpha$ and $\beta$ are said to be thin-equivalent if $\alpha \beta^{-1}$ is a thin loop. The thin equivalence agrees with the elementary equivalence, i.e. equivalence up to (order preserving) reparametrizations and up to immediate retracings.

The composition of loops based on $\star$ defines a group structure on the equivalence classes. We call this group the group of loops and we denote it by $\text{Loop}_\star(M)$. For sake of simplicity we will denote by $\alpha, \beta, \ldots$ parametrized loops as well as their equivalence classes.

Let $A$ be the space of smooth connections $A$ on the principal bundle $P(M, G)$ and $\mathcal{P}_\alpha$ the parallel transport defined by $A$ along $\alpha$. If we fix a point $u_0$ in the fiber on $\star$, the relation $u_0 = \mathcal{P}_\alpha(u_0)$ defines a homomorphism $H_A : \text{Loop}_\star(M) \to G$ called holonomy map of $A$.

Let $\phi$ be a gauge transformation of $P(M, G)$ (i.e. a $G$-equivariant automorphism of $P$ inducing the identity map on $M$) and $\phi^* A$ be the pull-back of $A$ by $\phi$. We have $H_{\phi^* A}(\alpha) = g_0 H_A(\alpha) g_0^{-1}$, where $g_0$ is the element of $G$ such that $\phi(u_0) = u_0 g_0$.

The Wilson function associated to a loop $\alpha$ is the gauge invariant function defined by $T_\alpha(A) = \frac{1}{N} \text{Tr} H_A(\alpha)$, which can be thought as a bounded complex valued function on $A/\mathcal{G}$, where $\mathcal{G}$ denotes the group of gauge transformations. Taking all the complex linear combinations of finite products of Wilson functions we get a *-algebra denoted with $\text{hol}(M, G)$. If we complete $\text{hol}(M, G)$ in the $\| \cdot \|_\infty$ norm we get an Abelian $\mathcal{C}^*$-algebra called holonomy $\mathcal{C}^*$-algebra, denoted by $\text{Hol}(M, G)$. If $\star$ is the constant loop, $T_\star(A) = 1$ for every $A$; thus $T_\star$ is the unit $\mathbf{1}$ in $\text{Hol}(M, G)$.

We quote here the main results on $\text{Hol}(M, G)$ given in several papers, f.i. [2] [3]; a short review can be also found in [4].

- The spectrum of $\text{Hol}(M, G)$ is a compact Hausdorff space in which $A/\mathcal{G}$ is densely embedded; for this reason it is usually indicated with $\overline{A/\mathcal{G}}$; its elements are called generalized connections and indicated by $\hat{A}$.

- $\overline{A/\mathcal{G}}$ agrees with the space $\text{Hom}(\text{Loop}_\star(M), G)/\text{Ad}_G$ of all homomorphisms $H : \text{Loop}_\star(M) \to G$ up to conjugation.

- $\text{Hol}(M, G)$ and $\text{hol}(M, G)$ do not depend on the principal bundle but only on $M$ and $G$.

3 A generalization of Bochner Theorem

In this section the gauge group $G$ is assumed to be $U(1)$ or $SU(2)$ and we denote by $\mathcal{W}$ the set of Wilson functions: $\mathcal{W} \equiv \{ T_\alpha, \alpha \in \text{Loop}_\star(M) \}$. 

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Definition 3.1 A complex valued function $\ell$ on $\mathcal{W}$ is positive definite if $\sum_{i=1}^{n} c_i \ell(T_\alpha_i) \geq 0$ whenever $\sum_{i=1}^{n} c_i T_\alpha_i(\bar{A}) \geq 0$ for every $\bar{A}$.

We denote by $\mathcal{B}(\mathcal{W})$ the set of bounded functions and by $\mathcal{P}(\mathcal{W})$ the set of positive definite functions on $\mathcal{W}$; we also identify $\text{Hol}(M, G)$ with $C(\bar{A}/\bar{G})$ by means of the Gelfand isomorphism.

Let $M(\bar{A}/\bar{G})$ denote the space of complex regular measures on $\bar{A}/\bar{G}$. To every $\mu \in M(\bar{A}/\bar{G})$, we associate the map $\ell_\mu$ on $\mathcal{W}$ defined by

$$\ell_\mu(T_\alpha) := \int_{\bar{A}/\bar{G}} T_\alpha(\bar{A}) d\mu(\bar{A}).$$

The function $\ell_\mu$ is the restriction to $\mathcal{W}$ of the bounded linear functional $I_\mu$ on $C(\bar{A}/\bar{G})$ defined by $I_\mu(f) = \int_{\bar{A}/\bar{G}} f d\mu$ for every $f \in C(\bar{A}/\bar{G})$. We have the following results.

Theorem 3.1 1. The map $\mathcal{L} : M(\bar{A}/\bar{G}) \to \mathcal{B}(\mathcal{W})$, $\mathcal{L}(\mu) = \ell_\mu$ is a continuous injective linear map.

2. The restriction of $\mathcal{L}$ to the cone $M_+(\bar{A}/\bar{G})$ of the positive measures gives rise to a one-to-one correspondence with $\mathcal{P}(\mathcal{W})$.

Proof. First we recall that the algebra $\text{hol}(M, G)$ agrees with the linear span of $\mathcal{W}$. This property is obvious for $U(1)$; for $SU(2)$ it follows from the Mandelstam identity: $2T_\alpha T_\beta = T_{\alpha\beta} + T_{\alpha\beta}^{-1}$.

The map $\mathcal{L}$ is injective as a consequence of the density of $\text{hol}(M, G)$ in $\text{Hol}(M, G)$. The inequality $|\ell_\mu(T_\alpha)| \leq \|\mu\|$ implies boundeness of $\ell_\mu$ and continuity of $\mathcal{L}$. Moreover, if $\mu$ is a positive measure, $\ell_\mu$ is positive definite.

Let now $\ell \in \mathcal{P}(\mathcal{W})$. We can associate to $\ell$ a functional $I$ on $\text{hol}(M, G)$ defined by $I(\sum_{i=1}^{n} c_i T_{\alpha_i}) := \sum_{i=1}^{n} c_i \ell(T_{\alpha_i})$. As $\ell$ is positive definite, $I$ is positive and well defined: actually $\sum_{i=1}^{n} c_i T_{\alpha_i} = 0$ implies $\sum_{i=1}^{n} c_i \ell(T_{\alpha_i}) = 0$. Now we prove that $I$ is bounded: for a real valued function $f$ in $\text{hol}(M, G)$ we have $-\|f\|_\infty \mathbf{I} \leq f \leq \|f\|_\infty \mathbf{I}$; positivity of $I$ implies $|I(f)| \leq \|f\|_\infty \ell(\mathbf{I})$. If $f$ is complex valued we obtain the same result by applying the above argument to the real and imaginary part of $f$.

As $\text{hol}(M, G)$ is dense in $\text{Hol}(M, G)$ we can extend $I$ to a positive continuous functional $\mathcal{I}$ on $\text{Hol}(M, G)$ obtaining, by Riesz-Markov Theorem, a regular measure $\mu$ on $\bar{A}/\bar{G}$. The positivity of $\mathcal{I}$ follows from these considerations: if $f \in \text{Hol}(M, G)$, $f \geq 0$, then $\sqrt{f} \in \text{Hol}(M, G)$. We can approximate $\sqrt{f}$ with a sequence $p_n$ of elements in $\text{hol}(M, G)$. Then $p_n p_n^* \geq 0$ and $p_n p_n^* \to f$ so that $\mathcal{I}(f) = \lim_n I(p_n p_n^*) \geq 0$. 

The above results were given in [2] with a different definition of positive definite function.

If $\mu \in M_+(\bar{A}/\bar{G})$, then every element $\psi \in L^2(\bar{A}/\bar{G}, \mu)$ defines a measure $\mu_\psi$ in $M(\bar{A}/\bar{G})$ such that $\int_{\bar{A}/\bar{G}} T_\alpha d\mu_\psi = \int_{\bar{A}/\bar{G}} T_\alpha \psi d\mu$. Hence the loop transform can be obtained simply by
restriction of $L$ to $L^2(\mathcal{A}/\mathcal{G}, \mu)$ for a suitable $\mu$. This map is still an injective linear operator and it is also continuous w.r.t. the $L^2$ norm because $|\ell_{\mu}(T_\alpha)| \leq \|\psi\|_2 \|\mu\|^{1/2}$. The range of the loop transform constructed in this way is not characterized in an explicit fashion. Our subsequent inductive construction will show that, in the Abelian case, an unitary loop transform with explicitly characterized range is actually available.

We stress that Theorem 3.1 does not depend on piecewise analyticity of the loops. Analogous results can be stated starting on piecewise $C^k$ loops, where $1 \leq k \leq \infty$. In this case $\mathcal{A}/\mathcal{G}$ will denote the spectrum of the holonomy algebra generated by the related Wilson functions.

For other gauge groups $G$ one can try to generalize Theorem 3.1 using some subsets $\mathcal{W}$ of $hol(M, G)$ containing, besides the Wilson functions, also products of Wilson functions to guarantee that their linear span is $hol(M, G)$. Mandelstam identities can be used to reduce the order of products to be considered. The results would be less appealing.

4 The loop transform in the Abelian case

In the following we will fix the gauge group to be $U(1)$. We first note that, in this case, $\text{Hoop}_*(M)$ is a group under pointwise multiplication, called the group of hoops in [2] and denoted by $\text{Hoop}_*(M)$. We want to prove that $\mathcal{W}$ is exactly $\text{Loop}_*(M)$ quotiented by its commutator subgroup, but to do this we have to introduce the notion of independent loops.

We call edge in $M$ an analytic path $e : [0, 1] \to M$ such that the restriction $e_{|(0,1)}$ is an embedding. We call vertex of an edge the starting or the ending point. A finite embedded graph $\Gamma$ is the image of a finite collection of edges which intersect themselves only at their vertices. For any finite collection $\alpha_1, \ldots, \alpha_n$ of $n$ (parametrized) loops based on $*$, the union of their images $\alpha_1^* \cup \ldots \cup \alpha_n^*$ is a finite embedded connected graph $\Gamma(\alpha_1, \ldots, \alpha_n)$. We stress that this result depends on the piecewise analyticity of the loops.

**Definition 4.1** The loops $\beta_1, \ldots, \beta_n$ are called independent if every $\beta_i$ contains an edge $e_i$ of the generated graph whose image $e_i^*$ is covered only once by $\beta_i$ and whose image intersects the other loops only in its vertices.

One can prove that $\beta_1, \ldots, \beta_n$ are independent if and only if they are the generators of $\pi_1(\Gamma(\beta_1, \ldots, \beta_n))$. We recall that $\pi_1(\Gamma(\beta_1, \ldots, \beta_n))$ is free and it can be considered as a subgroup of $\text{Loop}_*(M)$, so that an independent family of loops generates a free subgroup of $\text{Loop}_*(M)$ with $n$ generators.

The main property of the independent families of loops are stated in the next proposition (see [2]).

**Proposition 4.1** The following assertions hold:

1. every finite family of loops depends on an independent family;
2. let $\beta_1, \ldots, \beta_n$ an independent family and $(g_1, \ldots, g_n) \in G^n$, then there exists a connection $A \in \mathcal{A}$ such that $H_A(\alpha_i) = g_i$, $i = 1, \ldots, n$.

We say that a family of loops depends on another family if every loop of the first family can be written as product of elements of the second one and of their inverses. Property 2. is called the interpolation property.

We introduce the map $\tau : \text{Loop}_*(M) \to \text{Hoop}_*(M)$, $\tau(\alpha) = T_\alpha$; this map is certainly not injective (owing to the cyclic property of trace, one has $\tau(\alpha) = \tau(\beta\alpha\beta^{-1})$ for any other loop $\beta$). In the Abelian case $\tau$ is a group homomorphism whose kernel contains the commutator subgroup of $\text{Loop}_*(M)$, but one can say more.

**Proposition 4.2** Let $G = U(1)$; $\tau$ is a homomorphism whose kernel is the commutator subgroup of $\text{Loop}_*(M)$.

**Proof.** We recall that the commutator subgroup is the normal subgroup generated by elements of the form $\alpha\beta\alpha^{-1}\beta^{-1}$. By standard algebraic arguments it follows that the elements of the commutator subgroup are of the form:

$$\alpha = \beta_1^{k_{1,1}}\beta_2^{k_{2,1}}\cdots\beta_n^{k_{n,1}}\beta_1^{k_{1,2}}\cdots\beta_1^{k_{1,m}}\beta_n^{k_{n,m}}$$

where $\beta_1, \ldots, \beta_n$ are arbitrary elements of the group and $k_{i,j} \in \mathbb{Z}$ satisfy $Q_i \equiv \sum_{j=1}^m k_{i,j} = 0$ for every $i = 1, \ldots, n$. Therefore the commutator subgroup of $\text{Loop}_*(M)$ is contained in the kernel of $\tau$.

Let $\alpha \in \ker \tau$. We can write $\alpha$ by means of an independent family of loops $\beta_1, \ldots, \beta_n$ as in formula (1). Using the interpolation property we can find, for every $\theta \in \mathbb{R}$ and every $k = 1, \ldots, n$, a connection $A_{\theta,k}$ such that: $H_{A_{\theta,k}}(\beta_i) = 1$, for $i \neq k$ and $H_{A_{\theta,k}}(\beta_k) = e^{i\theta}$. As $\alpha$ satisfies $T_\alpha(A_{\theta,k}) = 1$ for $\theta \in \mathbb{R}$ and $k = 1, \ldots, n$, we get $Q_k = 0$ for every $k$. \qed

As a consequence of the identification $\mathcal{A}/\mathcal{G} \equiv \text{Hom}(\text{Loop}_*(M), G)/\text{Ad}_G$, we have that, in the Abelian case, $\mathcal{A}/\mathcal{G}$ agrees with the compact Abelian group $\text{Hom}(\text{Loop}_*(M), U(1))$ and $\text{Hoop}_*(M)$ with its dual group. In fact the Wilson functions are normalized characters of $\mathcal{A}/\mathcal{G}$ so they form an orthonormal set in $L^2(\mathcal{A}/\mathcal{G}, \mu_0)$, where by $\mu_0$ we denote the Haar measure. This set is complete because the span of the Wilson functions, i.e. $\text{hol}(M, U(1))$, is dense in $L^2(\mathcal{A}/\mathcal{G}, \mu_0)$. By Peter-Weyl Theorem we know that the normalized characters form an orthonormal basis in $L^2(\mathcal{A}/\mathcal{G}, \mu_0)$. We conclude that $\mathcal{W} \equiv \mathcal{A}/\mathcal{G}$.

By a classical result due to A. Weil [13], every compact group $G$ is the limit of a projective family of compact Lie groups. Furthermore, if $G$ is Abelian, it is the limit of a family $G_\mu$ of compact Abelian Lie groups. Instead of following Weil’s construction, we will use for the compact group $\text{Hom}(\text{Loop}_*(M), U(1))$ a more suitable projective family given by Marolf and Mourão in [13]. We will specialize their results for $G = U(1)$ to construct $\mathcal{A}/\mathcal{G}$ as projective limit of tori and to obtain the loop transform by means of the usual Fourier transforms on tori.
5 Projective and inductive limits

We recall the formal definitions and properties of projective and inductive limits of groups and Hilbert spaces. The definitions we will give here are not the general ones but adapted to our situation.

By a projective family of topological groups we mean a collection \((G_\mu, \pi_{\mu\nu}, J)\) where \(J\) is a directed set, \(G_\mu\) is a topological group for every \(\mu\in J\) and the maps \(\pi_{\mu\nu} : G_\nu \to G_\mu\), defined for every \(\mu \leq \nu\), are continuous surjective homomorphisms (called projections) satisfying the consistency conditions:

1. \(\pi_{\mu\mu} = id_{G_\mu}\);
2. \(\pi_{\mu\nu} \circ \pi_{\nu\lambda} = \pi_{\mu\lambda}\) for \(\mu \leq \nu \leq \lambda\).

We call projective limit of the family any topological group \(G\) such that for every \(\mu\) there exists a continuous surjective homomorphism \(p_\mu : G \to G_\mu\) satisfying

1. \(\pi_{\mu\nu} \circ p_\nu = p_\mu\) for \(\mu \leq \nu\);
2. \(p_\mu(g) = e_\mu\) for every \(\mu\) implies \(g = e\),

where by \(e\) and \(e_\mu\) we denote the units of \(G\) and \(G_\mu\), respectively. It is customary to indicate briefly \(G \equiv \lim_\leftarrow \mu G_\mu\). All such topological groups are isomorphic.

If \(G_\mu\) is a compact group for every \(\mu \in J\), the projective limit exists and it is a compact group. If \(G_\mu\) is a connected for every \(\mu\), then \(G\) is connected.

An inductive family of topological groups \((G_\mu, i_{\nu\mu}, J)\) is a collection of topological groups \(G_\mu\), where \(J\) is a directed set of indices and \(i_{\nu\mu} : G_\mu \to G_\nu\) are continuous injective homomorphisms (called inclusions) defined for every \(\nu \geq \mu\) and satisfying the consistency conditions:

1. \(i_{\mu\mu} = id_{G_\mu}\);
2. \(i_{\lambda\nu} \circ i_{\nu\mu} = i_{\lambda\mu}\) for \(\mu \leq \nu \leq \lambda\).

We call inductive limit of the family any topological group \(G\) such that for every \(\mu\) there exists a continuous injective homomorphism \(i_\mu : G_\mu \to G\) satisfying:

1. \(i_\nu \circ i_{\nu\mu} = i_\mu\) \(\mu \leq \nu\);
2. the entire \(G\) is covered by the union of the images of the inclusions \(i_\mu\).

We indicate briefly \(G = \lim_\rightarrow \mu G_\mu\). All such topological groups are isomorphic.

The dual group \(\hat{G}\) of a locally compact Abelian group \(G\) which is a projective limit of a family of locally compact Abelian groups \(G_\mu\) is the inductive limit of the dual groups \(\hat{G}_\mu\)
with the transposed maps \( t_i \) and \( t_{i\nu} \) as projections; this result can easily be worked out by §5 of A. Weil’s book [18].

We specialize the definition of inductive family and inductive limit to the category of Hilbert spaces by requiring the inclusions to be isometric linear maps. In this case we define the inductive limit to be any Hilbert space \( \mathcal{H} \) such that the inclusions \( i_\mu \) cover a dense linear subspace of \( \mathcal{H} \). All such Hilbert spaces are isomorphic.

Our definition of inductive family of Hilbert spaces is quite restrictive. In fact every inductive family \((\mathcal{H}_\mu, i_{\nu\mu}, J)\) generates a projective family \((\mathcal{H}_\mu, \pi_{\mu\nu}, J)\), taking as projections \( \pi_{\mu\nu} \) the adjoint maps \( t_{i\mu} \). The inductive limit \( \mathcal{H} \) with projections \( \pi_\mu = t_{i\mu} \) is also the projective limit of the family \((\mathcal{H}_\mu, \pi_{\mu\nu}, J)\).

Finally we give the definition of inductive and projective limit of unitary maps. Let \( \{\mathcal{H}_\mu, i_{\nu\mu}, J\} \) and \( \{\mathcal{H}_\mu', i_{\nu\mu}', J\} \) be inductive families of Hilbert spaces, with inductive limits \( \mathcal{H} \) and \( \mathcal{H}' \), respectively. A family of unitary maps \( \{\phi_\mu : \mathcal{H}_\mu \to \mathcal{H}_\mu'\}_{\mu \in J} \) is said to be inductive if it satisfies

\[
i_{\nu\mu}' \circ \phi_\mu = \phi_\nu \circ i_{\nu\mu} \quad \text{for} \quad \mu \leq \nu.
\]

The inductive limit is the unique unitary map \( \phi : \mathcal{H} \to \mathcal{H}' \) satisfying

\[
\phi \circ i_\mu = i_{\mu}' \circ \phi_\mu \quad \text{for every} \quad \mu.
\]

Analogously, in the projective case, a family of unitary maps \( \{\phi_\mu\}_{\mu \in J} \) is a projective family if it satisfies

\[
\pi_{\mu\nu}' \circ \phi_\mu = \phi_\mu \circ \pi_{\mu\nu} \quad \text{for} \quad \mu \leq \nu.
\]

Their projective limit is the unique unitary map \( \phi : \mathcal{H} \to \mathcal{H}' \) such that

\[
\pi_{\mu}' \circ \phi = \phi_\mu \circ \pi_\mu \quad \text{for every} \quad \mu.
\]

6 Inductive construction of the loop transform in the Abelian case

We first construct the group \( \overline{A/G} \) as a projective limit of tori following [13]. Let us consider the set \( J \) of the subgroups \( L \) of \( \text{Loop}_*(M) \) generated by a finite independent family of loops. By \( L \leq L' \) we mean that \( L \) is a subgroup of \( L' \); \( J \) is directed w.r.t. this ordering.

The projective family associated to \( \overline{A/G} \) is defined as follows:

- we take as index set the directed set \( J \);
- to every \( L \in J \) we associate the group \( \text{Hom}(L, U(1)) \);
- if \( L \leq L' \) we define the projection \( \pi_{LL'} : \text{Hom}(L', U(1)) \to \text{Hom}(L, U(1)) \) which restricts the homomorphisms \( H \in \text{Hom}(L', U(1)) \) to the subgroup \( L \).
To simplify the notation we denote $\text{Hom}(L, U(1))$ by $\overline{A/G}_L$ and its dual group by $\mathcal{W}_L$. For a given independent family of loops $(\alpha_1, \ldots, \alpha_n)$ the evaluation map

$$ev_{(\alpha_1, \ldots, \alpha_n)}(H) = (H(\alpha_1), \ldots, H(\alpha_n))$$  \hspace{1cm} (2)

is an isomorphism of $\overline{A/G}_L$ with the n-dimensional torus.

The group $\overline{A/G} \equiv \text{Hom}(\text{Loop}_* (M), U(1))$ is the projective limit of this family; actually the projection $\pi_l : \overline{A/G} \to \overline{A/G}_L$, which restricts the homomorphisms $H \in \text{Hom}(\text{Loop}_* (M), U(1))$ to $L$, is continuous and surjective owing to the interpolation property of independent loops.

The loop transform $L$ will be constructed as the inductive limit of the Fourier transforms $F_L$ between the Hilbert spaces $L^2(\overline{A/G}_L)$ and $L^2(\mathcal{W}_L)$ (for shortness we have omitted the relative Haar measures). The scheme of the work is visualized in this diagram:

$$
\begin{array}{ccc}
L^2(\overline{A/G}_L) & \xrightarrow{F_L} & L^2(\mathcal{W}_L) \\
\downarrow i_{L'L} & & \downarrow j_{L'L'} \\
L^2(\overline{A/G}_{L'}) & \xrightarrow{F_{L'}} & L^2(\mathcal{W}_{L'}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
L^2(\overline{A/G}) & \xrightarrow{\mathcal{L}} & L^2(\mathcal{W}) \equiv L^2(\text{Hoop}_* (M))
\end{array}
$$

To make the family $\{L^2(\overline{A/G}_L)\}$ an inductive family of Hilbert spaces we define the inclusions $i_{L'L}$ for $L \leq L'$ as follows: for every $\psi \in L^2(\overline{A/G}_L)$ we put

$$(i_{L'L}\psi)(H') := \psi(\pi_{LL'}(H')) \quad H' \in \overline{A/G}_{L'}.$$

These inclusions are linear and satisfy the consistency conditions, so we have only to prove that they are isometric maps. Suppose that $L$ and $L'$ are the free groups generated by the independent families $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_{n'}\}$, respectively, and that $L \leq L'$. We have:

$$
\begin{cases}
\alpha_1 = \beta_1^{k_{1,1}} \ldots \beta_{n'}^{k_{n',1}} \\
\vdots \\
\alpha_n = \beta_1^{k_{1,n}} \ldots \beta_{n'}^{k_{n',n}}
\end{cases}
$$

for some $k_{r,s} \in \mathbb{Z}$ for $r = 1, \ldots, n'$ and $s = 1, \ldots, n$.

For $ev_{(\beta_1, \ldots, \beta_{n'})}(H) = (e^{i\vartheta_1}, \ldots, e^{i\vartheta_{n'}})$, it follows

$$ev_{(\alpha_1, \ldots, \alpha_n)}(\pi_{LL'}(H)) = (e^{ik_{1,1}\vartheta_1} \ldots e^{ik_{n',1}\vartheta_{n'}}, \ldots, e^{ik_{1,n}\vartheta_1} \ldots e^{ik_{n',n}\vartheta_{n'}}).$$
By composition of the evaluation maps with \( i_{L'/L} \) one obtains the inclusions \( i_{n'n} : L^2(U(1)^n) \to L^2(U(1)^{n'}) \), defined by
\[
(i_{n'n}\psi)(e^{i\theta_1}, \ldots, e^{i\theta_{n'}}) = \psi(e^{i(k_1,1\vartheta_1+\ldots+k_{n',1}\vartheta_{n'})}, \ldots, e^{i(k_{1,n}\vartheta_1+\ldots+k_{n',n}\vartheta_{n'})}).
\]

From the normalization and the bi-invariance of the Haar measure it follows that the inclusions \( i_{n'n} \), and hence also the inclusions \( i_{L'/L} \), are isometric.

The inclusions \( j_{L'/L} \) are defined by the following commutative diagram:
\[
\begin{array}{ccc}
L^2(\overline{A/\mathcal{G}}_L) & \xrightarrow{F_L} & L^2(\mathcal{W}_L) \\
\uparrow_{i_{L'/L}} & & \uparrow_{j_{L'/L}} \\
L^2(\overline{A/\mathcal{G}}_{L'}) & \xrightarrow{F_{L'}} & L^2(\mathcal{W}_{L'})
\end{array}
\]

They are isometries as compositions of isometric maps. The diagram shows that the consistency conditions hold both for the inclusions \( j_{L'/L} \) and the Fourier transforms \( F_L \). So we have well defined inductive families.

**Theorem 6.1**

1. \( L^2(\overline{A/\mathcal{G}}) \) is the inductive limit of \( \{L^2(\overline{A/\mathcal{G}}_L)\}_L \);
2. \( L^2(\mathcal{W}) \) is the inductive limit of \( \{L^2(\mathcal{W}_L)\}_L \);
3. the loop transform \( \mathcal{L} \) on \( L^2(\overline{A/\mathcal{G}}) \) is the inductive limit of \( \{F_L\}_L \).

**Proof.** Let us define the inclusions \( i_L : L^2(\overline{A/\mathcal{G}}_L) \to L^2(\overline{A/\mathcal{G}}) \) by \( (i_L\psi_L)(H) = \psi_L(\pi_L(H)), \psi_L \in L^2(\overline{A/\mathcal{G}}_L) \).

Denoting by \( \mu_L \) the Haar measure on \( \overline{A/\mathcal{G}}_L \) and by \( \mu_0 \) the normalized Haar measure on \( \overline{A/\mathcal{G}} \), we have that \( (\pi_L)_*\mu_0 = \mu_L \). Therefore
\[
\|i_L\psi_L\| = \int_{\overline{A/\mathcal{G}}} |\psi_L \circ \pi_L|^2 \, d\mu_0(A) = \int_{\overline{A/\mathcal{G}}_L} |\psi_L|^2 \, d\mu_L = \|\psi_L\|_2
\]
so that the inclusions \( i_L \) are isometric. Moreover their images contain the Wilson functions, hence they cover a dense linear subspace of \( L^2(\overline{A/\mathcal{G}}) \). We conclude that \( L^2(\overline{A/\mathcal{G}}) \) is the inductive limit of the family \( \{L^2(\overline{A/\mathcal{G}}_L)\}_L \).

Now we define the inclusions \( j_L : L^2(\mathcal{W}_L) \to L^2(\text{Hoop}_*(M)) \) by the following commutative diagram:
\[
\begin{array}{ccc}
L^2(\overline{A/\mathcal{G}}) & \xrightarrow{\mathcal{L}} & L^2(\text{Hoop}_*(M)) \\
\uparrow_{i_L} & & \uparrow_{j_L} \\
L^2(\overline{A/\mathcal{G}}_L) & \xrightarrow{F_L} & L^2(\mathcal{W}_L)
\end{array}
\]

Repeating the same arguments on the inclusions \( j_L \) we get that \( L^2(\text{Hoop}_*(M)) \) is the inductive limit of the family \( \{L^2(\mathcal{W}_L)\}_L \) and that the inductive limit of \( \{F_L\}_L \) is the loop transform on \( L^2(\overline{A/\mathcal{G}}) \). \( \Box \)
From $\mathcal{L}\circ i_L = j_L \circ F_L$ we get the transposed equality $p_L \circ F^{-1} = F_L^{-1} \circ \hat{p}_L$, where $p_L := {}^t i_L$ and $\hat{p}_L := {}^t j_L$. This shows that the inverse Fourier transforms $F_L^{-1}$ form a projective family of unitary maps whose limit is the inverse loop transform. The inverse loop transform of a "loop state" $\ell : \text{Hoop}_\star(M) \to \mathbb{C}$ is given by $\psi = \sum_{\alpha \in \text{Hoop}_\star(M)} \ell(\alpha) T_{\alpha}$.

7 The smooth case

A piecewise smooth path is a continuous map $p$ on a closed interval $[a, b]$ in $M$ with a finite partition $t_0 = a \leq t_1 \leq ... \leq t_i \leq t_{i+1} \leq ... \leq t_n = b$ such that $p$ agrees with a $C^\infty$ curve in every interval $[t_i, t_{i+1}]$. Again we can construct the group of equivalence classes of piecewise smooth loops based on $\star$ up to reparametrizations and retracings; we will denote it by $\text{Loop}_\star(M)$. We recall that $\text{Loop}_\star(M)$ is a topological group endowed with the topology generated by piecewise smooth homotopies, i.e. curves in $\text{Loop}_\star(M)$ defined by continuous maps $\Phi : \mathbb{R} \times [0, 1] \to M$ which are smooth on $\mathbb{R} \times [t_k, t_{k+1}]$ for some partition $t_0 = 0 < t_1 < ... < t_n = 1$. This topology was introduced by Barrett in [7].

For a given $U(1)$-bundle $P$ on $M$, the group of hoops, denoted by $\text{Hoop}_\star(M)$, is the quotient of $\text{Loop}_\star(M)$ with respect to the subgroup $\{ \alpha \in \text{Loop}_\star(M) \mid H_A(\alpha) = 1 \ \forall A \}$, where $A$ denotes a connection on $P$. We will call this subgroup holonomy kernel. The following proposition follows from the characterization of the hoops given in [15].

Proposition 7.1 The holonomy kernel is the closure of the commutator subgroup in the Barrett topology.

Proof: The holonomy kernel contains the closure of the commutator subgroup: in fact the holonomy maps $H_A$ associated to connections $A$ are continuous in the Barrett topology and so the holonomy kernel is closed in this topology. Following Proposition 5.8 in [13] every loop in the holonomy kernel can be approximated by a loop in the commutator subgroup. $\square$

As a consequence of this characterization the hoop group is an Abelian group not depending on the bundle $P$ and can be constructed using the trivial bundle. Moreover $\text{Hoop}_\star(M)$ is torsion free (see Lemma A.2 in [4]).

In a torsion free $\mathbb{Z}$-modulus every finitely generated submodulus is freely generated; then in every finite generated subgroup $\text{Hoop}_\star(L)$ of $\text{Hoop}_\star(M)$ we can choose a finite family $\tilde{\alpha}_1, ..., \tilde{\alpha}_n$ of free generators and give an isomorphism of the group $\text{Hom}(L, U(1))$ with $U(1)^n$ as in §5.

In the case of trivial bundles and of bundles arising as pullback of the Hopf bundle $S^1 \to S^3 \to S^2$ the following weak form of the interpolation property holds, which assures that the spectrum of the holonomy algebra in the smooth case agrees with the compact Abelian group $\text{Hom}(\text{Hoop}_\star(M), U(1))$ (see [4]).

Proposition 7.2 For a family $\tilde{\alpha}_1, ..., \tilde{\alpha}_n$ of free generators the evaluation map $\text{ev}_{\alpha_1, ..., \alpha_n}(A) = (H_A(\alpha_1), ..., H_A(\alpha_n))$ has range dense in $U(1)^n$. 
Then a situation arises analogous to the analytic case.

**Proposition 7.3** \( \text{Hom}(\text{Hoop}_*(M), U(1)) \) is the projective limit of the family of Abelian compact groups \( \{ \text{Hom}(L, U(1)) \}_L \) where the index \( L \) runs over the finitely generated subgroups of \( \text{Hoop}_*(M) \).

**Proof:** The only non-trivial point is to show that the continuous projections \( p_L : \text{Hom}(\text{Hoop}_*(M), U(1)) \to \text{Hom}(L, U(1)) \) are surjective. This follows by proposition 7.2 using the fact that the image of \( p_L \) must be compact. \( \square \)

As in the analytic case we get that the group \( \text{Hoop}_*(M) \) is the dual group of \( \text{Hom}(\text{Hoop}_*(M), U(1)) \) and that a construction of the loop transform as an inductive limit of Fourier transforms of tori can be performed also in the smooth case. The remarkable difference is that a set of independent hoops is not easy characterized as in the analytic case.

### 8 The path transform

Let \( G \) be a compact group, \( A \) denote a connection on the trivial bundle \( M \times G \) and \( F : G^k \to \mathbb{C} \) a continuous function and \( p_1, ..., p_k \) piecewise analytic paths in \( M \); we can define the function \( f \) on \( A \) by

\[
f(A) = F(H_A(p_1), ..., H_A(p_k)),
\]

where \( H_A(p) \) denotes the parallel transport along \( p \) defined by the connection \( A \), identified with an element of \( G \). Functions of this form are called **cylinder functions** and are contained in \( \mathcal{B}(A) \). They generate a \( C^* \)-algebra with unit, called briefly the **cylinder \( C^* \)-algebra**, whose spectrum \( \overline{A} \) contains \( A \) densely.

The space \( \text{Path}(M) \) of the equivalence classes of piecewise analytic paths \( p \) in \( M \) up to reparametrizations and retracings is a groupoid where the composition \( p_1 p_2 \) is defined if the end point of \( p_1 \) agrees with the starting point of \( p_2 \) and the inverse \( p^{-1} \) is obtained by reversing the parametrization. Every parallel transport \( H_A : \text{Path}(M) \to G \) is a groupoid homomorphism: \( H_A(p_1 p_2) = H_A(p_1)H_A(p_2) \).

One can define families of independent paths as in Definition 4.1. Every family of paths depends on a family of independent paths and the interpolation property holds for independent paths as stated in Proposition 4.1 for independent loops. It follows that \( \overline{A} \) agrees with the space \( \text{Hom}(\text{Path}(M), G) \) of all homomorphisms from \( \text{Path}(M) \) to \( G \). The proof of this result is similar to the one used in [2] to prove that \( \overline{A}/G = \text{Hom}(\text{Loop}_*(M), G)/\text{Ad}G \) (see also [17]). \( \text{Hom}(\text{Path}(M), G) \) is a closed subset of the compact group \( G^{\text{Path}(M)} \).

In the Abelian case \( \overline{A} = \text{Hom}(\text{Path}(M), U(1)) \) is an Abelian compact group and it is the projective limit of the family of the compact groups \( \text{Hom}(L, U(1)) \) where \( L \) is the subgroupoid generated by a finite family of independent paths. Actually the interpolation property assures that the projections \( \pi_L : \text{Hom}(\text{Path}(M), U(1)) \to \text{Hom}(L, U(1)) \), defined by restrictions, are surjective homomorphisms.
Proposition 8.1 1) The dual group \( \hat{\text{Hom}}(\text{Path}(M), U(1)) \) is generated by the maps \( \chi_p : \text{Hom}(\text{Path}(M), U(1)) \rightarrow U(1), \chi_p(H) = H(p) \).

2) The kernel of the homomorphism \( \tau : \text{Path}(M) \rightarrow \hat{\text{Hom}}(\text{Path}(M), U(1)), \tau(p) = \chi_p \) is the subgroupoid generated by elements of the form
\[
p_1^{k_{1,1}}p_2^{k_{2,1}}...p_n^{k_{n,1}}p_1^{k_{1,2}}...p_1^{k_{1,m}}...p_n^{k_{n,m}}
\]
where \( k_{i,j} \in \mathbb{Z} \) and \( Q_i = \sum_{j=1}^m k_{i,j} = 0 \)

Proof. Every \( \chi_p \) is a continuous character: it is multiplicative and continuous as restriction to \( \text{Hom}(\text{Path}(M), U(1)) \) of the projection \( \pi_p : U(1)\text{Path}(M) \rightarrow U(1) \) on the \( p \) component. The group \( X \) generated by the characters \( \chi_p \) is separating on \( \text{Hom}(\text{Path}(M), U(1)) \). Then \( X \) is the entire dual group (apply Theorem 23.20 in [10]).

2) It follows by the interpolation property as in Proposition 4.2. \( \square \)

In the Abelian case one can introduce the path transform which is simply the Fourier transform on \( \mathcal{A} \). An inductive construction of the path transform can be obtained, as in §6 in the case of the loop transform, using as index set the family of subgroupoids generated by independent paths.

References

[1] Abbati, M.C. and Manià, A.: Holonomy algebras and their representations, UTM LNS, Trento, December 2000

[2] Ashtekar, A. and Lewandowski, J.: Representation theory of analytic holonomy \( C^* \)-algebras, in: J.C. Baez (ed.), Knots and Quantum Gravity, Oxford University Press, 1994

[3] Ashtekar, A. and Lewandowski, J.: Projective techniques and functional integration for gauge theories, J. Math. Phys. 36 (1995), 2170-2191

[4] Baez, J. C.: Diffeomorphism invariant generalized measure on the space of connections modulo gauge transformations, in: L. Crane and D. Yetter (eds.) Proceeding of the Conference on Quantum Topology, World Scientific, Singapore, 1994, pp. 213-223

[5] Baez, J. C.: Spin Network States in Gauge Theories, Adv.Math. 117 (1996) 253-272

[6] Baez, J.C. and Sawin S.: Functional Integration on Spaces of Connections, J. Funct. Anal. 150 (1997), 1-26

[7] Barrett, J. W.: Holonomy and Path Structures in General Relativity and Yang-Mills Theory, Int. Jour. Theor. Phys. 30 (1991) 1171-1215
[8] Brügmann, B.: *Loop representations*, available as arXiv:gr-qc/9312001.

[9] Gambini, R. and Pullin, J.: *Loops, Knots, Gauge Theories and Quantum Gravity*, Cambridge University Press, 1996

[10] Hewitt, E. and Ross, K.: *Abstract harmonic analysis*, vol 1, Springer, Berlin, 1963

[11] Hirsch, M. W.: *Differential Topology*, G.T.M. 33 Springer, Berlin, 1976

[12] Lewandowski, J. and Thiemann, T.: Diffeomorphism invariant Quantum Field Theories of Connections in terms of webs, *Class. Quant. Grav.* 16 (1999), 2299-2322

[13] Marolf, D. and Mourão, J. M.: On the Support of the Ashtekar-Lewandowski Measure, *Comm. Math. Phys.* 170 (1995), 583-605

[14] Rovelli, C. and Smolin, L.: Loop space representation of quantum general relativity, *Nucl. Phys. B* 331 (1990), 80-152

[15] Spallanzani, P.: Groups of Loops and Hoops, *Comm. Math. Phys.* 216 (2001), 243-253

[16] Thiemann, T.: The inverse loop transform, *J. Math. Phys.* 39 (1998), 1236-1248

[17] Velhinho, J.M.: A groupoid approach to spaces of generalized connections, available as arXiv:hep-th/0011200

[18] Weil, A.: *L’integration dans les groups topologiques et ses applications*, Hermann, Paris, 1940.