Entanglement and statistics in Hong-Ou-Mandel interferometry

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Hong-Ou-Mandel interferometry allows one to detect the presence of entanglement in two-photon input states. The same result holds for two-particles input states which obey to Fermionic statistics. In the latter case however anti-bouncing introduces qualitative differences in the interferometer response. This effect is analyzed in a Gedankenexperiment where the particles entering the interferometer are assumed to belong to a one-parameter family of quons which continuously interpolate between the Bosonic and Fermionic statistics.

II. USING HONG-OU-MANDEL INTERFEROMETER TO DETECT ENTANGLEMENT IN MULTI-MODE TWO-PHOTON STATES

The prototypical HOM set-up is sketched in Fig. 1. Here two multi-mode photons enter the interferometer following, respectively, the optical paths associated with the input ports $A_1$ and $A_2$. Their (possibly entangled) state is described by the vector

$$|\Psi\rangle = \sum_{k_1,k_2} \Phi(k_1,k_2) a_1^+(k_1)a_2^+(k_2)|\emptyset\rangle,$$

where $|\emptyset\rangle$ is the vacuum and where $a_j(k)$ is the photonic annihilation operator associated with $k$-th mode of the port $A_j$ which obeys standard Bosonic commutation relations, i.e.

$$[a_j(k), a_{j'}(k')] = 0$$

$$[a_j(k), a_{j}^+(k')] = \delta_{jj'} \delta_{kk'}.$$
FIG. 1: Schematic of the Hong-Ou-Mandel interferometer: the two-particle state $|\Psi\rangle$ of Eq. (1) enters the set-up from the multi-mode input ports $A_1$ and $A_2$ and combines at the 50/50 beam-splitter BS. Coincidence counts (5) are measured at the multi-mode output ports $B_1$ and $B_2$ for different values of the controllable phase-shift delay introduced in white-box area.

In these expressions $k$ labels the different frequencies of the modes propagating along the optical paths $A_{1,2}$ (for the sake of simplicity the entering signals are supposed to have definite polarization) and $\Phi(k_1, k_2)$ is the two-photon spectral amplitude which satisfies the normalization condition

$$\sum_{k_1, k_2} |\Phi(k_1, k_2)|^2 = 1.$$  (4)

Within the interferometer the modes of port $A_2$ undergo to phase-shifts transformations introduced through controllable delays (represented by the white box of Fig. 1) which transform $a_2(k)$ into $a_2(k)e^{i\phi_k}$. The two optical paths then interfere at the 50/50 beam-splitter BS where photo-coincidence $C_{12}$ are measured at the output ports $B_1$ and $B_2$. We will see that the presence of entanglement in the photon input state $|\Psi\rangle$ can be revealed by studying $C_{12}$ as a function of the controllable delays.

Assuming perfect detection efficiency and considering the limit of long detection times $T$ the average coincidence counts of the photo-detectors operating on $B_1$ and $B_2$ can be expressed as

$$C_{12} = \lim_{T \to \infty} \int_0^T dt_1 \int_0^T dt_2 \frac{\langle \Psi | I_{B_1}(t_1) I_{B_2}(t_2) | \Psi \rangle}{T^2},$$  (5)

where $I_{B_{1,2}}(t)$ are the normalized output intensity operators defined by

$$I_{B_j}(t) = \sum_{k_1} \sum_{k_2} e^{i(\omega_k_1 - \omega_k_2)t} b_j^\dagger(k_1) b_j(k_2),$$  (6)

with $b_j(k)$ being the Bosonic annihilation operator associated with the $k$-th outgoing mode of port $B_j$. Evaluating the integrals and the limit of Eq. (5) gives

$$C_{12} = \langle \Psi | N_1 N_2 | \Psi \rangle,$$  (7)

where for $j = 1, 2$, $N_j$ is the total photon number operator of the $B_j$ output port, i.e.

$$N_j = \sum_k n_j(k),$$  (8)

with $n_j(k) = b_j^\dagger(k)b_j(k)$ being the number operator of the $k$-th mode. Analogously the average output photo-counts at $B_1$ and $B_2$ yield

$$i_j = \lim_{T \to \infty} \int_0^T dt \frac{\langle \Psi | I_{B_j}(t) | \Psi \rangle}{T} = \langle \Psi | N_j | \Psi \rangle.$$  (9)

The operators $b_j(k)$ and $a_j(k)$ are connected through the input-output relations of the HOM interferometer, i.e.

$$b_1(k) = \left[ a_1(k) + e^{i\phi_k} a_2(k) \right] / \sqrt{2}$$
$$b_2(k) = \left[ a_1(k) - e^{i\phi_k} a_2(k) \right] / \sqrt{2},$$  (10)
FIG. 2: Pictorial representation of the coincidence counts dependence upon the interferometric delay \( \varphi_k \) in the HOM interferometer of Fig. 1. Left: Bosonic case. Among the two-particle states \(|\Psi\rangle\) of Eq. (1) only the entangled one (represented by the continuous curve) can have \( C_{12} > 1/2 \); separable states (dotted curve) stay always below the 1/2 threshold: the gray area above 1/2 is not accessible to them. Right: Fermionic case. Here the area which is not accessible to separable states is the region below 1/2 (see Ref. [5] for details). The continuous lines represent the average value \( \langle C_{12} \rangle \) of \( C_{12} \).

with \( \varphi_k \) being the controllable phase-shift and with the coefficient \( 1/\sqrt{2} \) coming from the 50/50 beam-splitter transformation. Notice that Eq. (10) couples only those modes of \( A_1 \) and \( A_2 \) which share the same value of \( k \) and that \( \varphi_k \) depends explicitly upon such index. [In Refs. [2, 3] for instance it is \( \varphi_k = \omega_k \tau \) with \( \omega_k \) being the frequency of the \( k \)-th mode and with \( \tau \) being a tunable delay.] With the help of the above definitions one can show that the average photon number \( \langle n \rangle \) at each of the two output ports equals 1. On the other hand Eq. (7) yields

\[
C_{12} = 1 - \frac{w}{2},
\]  

with

\[
w = \sum_{k_1, k_2} \Phi^*(k_1, k_2) \Phi(k_2, k_1) e^{i(\varphi_{k_1} - \varphi_{k_2})}.
\]

It is possible to verify that for a suitable choice of the delays \( \varphi_k \) and the spectral function \( \Phi(k_1, k_2) \), the real quantity \( w \) can take any values in the interval \([-1, 1]\). Correspondingly \( C_{12} \) can take values over the interval \([0, 1]\). In the twin-beam state case of Ref. [2] for instance, the plot of \( w \) with respect to the controllable delay \( \tau \) shows the so called Mandel dip, i.e. it nullifies for \( \tau = 0 \) and approaches 1 for sufficiently large \( \tau \). This implies that, by defining the visibility \( V \) of \( C_{12} \) as the difference between the maximum and minimum values it assumes when varying \( \varphi_k \), the maximum achievable value of \( V \) for the two-particles state \(|\Psi\rangle\) is 1.

On the other hand, consider what happens when \(|\Psi\rangle\) factorizes with respect to \( A_1 \) and \( A_2 \). In this case the spectral amplitude has the form \( \Phi(k_1, k_2) = \phi_1(k_1) \phi_2(k_2) \) and Eq. (12) becomes

\[
w = w_{\text{sep}} \equiv |\sum_k \phi_1^*(k) \phi_2(k) e^{i\varphi_k}|^2 \geqslant 0.
\]

Consequently, the coincidence counts \( C_{12} \) of factorizable input states \(|\Psi\rangle\) are limited into the interval \([0, 1/2]\) and their maximum allowed visibility (i.e. 1/2) is only one half of the maximum achievable value (see Fig. 2). As discussed in Ref. [5] the above result generalize to any (non necessarily pure) two-photon input density matrices which are separable with respect to \( A_1 \) and \( A_2 \). Therefore we can use \( C_{12} \) as an entanglement witness for two-photon input states. Indeed it is sufficient to repeat the coincidence measurements for different values of the controllable delay: by realizing a value of \( C_{12} \) or a visibility \( V \) which are strictly greater than 1/2 one can conclude that the input state of the system was entangled.

A. Bosons vs. Fermions

In Ref. [5] a Fermionic implementation of the HOM interferometer has been proposed. There, effective two-electron states analog to the two-photon input states of Eq. (1) originate from a solid-state entangler [4] and propagate along
metallic leads which take the place of the optical ports $A_1$, $A_2$, $B_1$ and $B_2$ of Fig. 1. Moreover, the coincidence counts $C_{12}$ of Eq. (5) is replaced by the current-current correlator of the outgoing leads $B_1$ and $B_2$. We refer the reader to Ref. 5 for a detailed discussion of the physical assumptions underlying such a scheme. For the purposes of the present manuscript it is sufficient to observe that an effective but rigorous description of such interferometer is obtained from the Eqs. (11), (7) and (9) by simply replacing the bosonic commutation rules (2) and (3) with their fermionic counterpart, i.e.

$$\{a_j(k), a_{j'}(k')\} = 0 \quad \text{Eq. (14)}$$

$$\{a_j(k), a_{j'}(k')\} = \delta_{jj'} \delta_{kk'}, \quad \text{Eq. (15)}$$

with $\{r, s\} = rs + sr$ being the anti-commutator of the operators $r$ and $s$. With this substitution the average output electron-numbers $i_{1,2}$ of Eq. (6) remain equal to 1. However the expression (11) for the coincidence-counts $C_{12}$ is replaced by

$$C_{12} = \frac{1 + w}{2} \quad \text{Eq. (16)}$$

with $w$ as in Eq. (12). Since the dependence of $w$ upon the factorizable properties of the input states $|\Psi\rangle$ is the same as in the bosonic case, one can still use the visibility $V$ as a signature of entanglement. Indeed also in the fermionic case, entangled inputs can have visibility $V$ greater than 1/2, while separable have always visibility $V$ smaller than or equal to 1/2. In the case where the label $k$ refers to the spin degree of freedom of the incoming electron this effect was first noted in Ref. 11.

The sign difference between Eqs. (11) and (16) implies that separable states of fermions are forced to have $C_{12}$ greater than or equal to 1/2 while separable states of bosons have $C_{12}$ smaller than or equal to 1/2 (see Fig. 2). This phenomenon can be interpreted in terms of the different bouncing and anti-bouncing attitudes of boson and fermion statistics. Indeed due to the exclusion principle one expects the coincidence counts $C_{12}$ of fermions to be typically higher than the corresponding bosonic coincidence counts. To make this a quantitative statement let us consider the average value of $C_{12}$ over all possible two-particle states of Eq. (11), i.e. over all the normalized two-particles spectral amplitudes $\Phi(k_1, k_2)$. A simple symmetric argument can be used to show

$$\bar{w} = \int d\mu(\Phi) \left[ \sum_{k_1, k_2} \Phi(k_1, k_2)\Phi^*(k_2, k_1)e^{i(\phi_{k_1} - \bar{\phi}_{k_2})} \right] = 1/M, \quad \text{Eq. (17)}$$

with $M$ being the number of the $k$-th modes. Therefore the average coincidence counts is

$$\bar{C}_{12} = \begin{cases} 1/2 - 1/(2M) & \text{Bosons} \\ 1/2 + 1/(2M) & \text{Fermions} \end{cases} \quad \text{Eq. (18)}$$

It is worth noticing that entangled input states are, to some extent, insensitive to the complementarity behavior of Eq. (18): indeed, as shown in Fig. 2 there are no regions which are strictly forbidden to them by the particle statistics.

**III. HOM INTERFEROMETRY WITH QUONS**

In this section a Gedankenexperiment is analyzed where the particles entering the HOM interferometer are assumed to be quons 12, 13. These have been introduced by Greenberg as an example of continuous interpolation between Bose and Fermi algebras (see Refs. 8, 14 for alternative definitions). We will use them to describe the sharp transition from the left part to the right part of Fig. 2 in terms of a continuous deformation of the particle statistics.

For $q \in [-1, 1]$ the quon algebra is obtained by replacing the relations (3) and (15) with the identity

$$a_j(k)a_{j'}(k') = q a_{j'}(k')a_j(k) = \delta_{jj'} \delta_{kk'}, \quad \text{Eq. (19)}$$

and by neglecting (12) the relations (2) and (14). For $|q| < 1$, Eq. (19) can be used to define a valid non-relativistic field theory but poses serious problems in deriving a reasonable relativistic quantum field theory 12, 13. A proper Fock-like representation can be derived upon enforcing the vacuum condition

$$a_j(k)|0\rangle = 0, \quad \text{Eq. (20)}$$
FIG. 3: Plot of the maximum and minimum values of $C_{12}$ of Eq. (22) allowed for a two-particle quon state as a function of the parameter $q$ of the quon algebra. The shaded region represents the allowed values (21) of $C_{12}$. The gray region is accessible only to non-factorizable input configurations – see Eq. (25). The continuous line represents the average value (26) of $C_{12}$ over all possible input two-particles states $|\Psi\rangle$. For $q = 1$ and $q = -1$ the boundaries coincide with those of Bosonic and Fermionic case respectively. The Boltzmann statistics $q = 0$ admits only the value 1/2.

for all $j$ and $k$ [12, 13]. In particular, it can be shown that for $|q| < 1$ the squared norms of all vectors made by polynomials of $a_j^\dagger(k)$ acting on the vacuum state $|\Omega\rangle$ are strictly positive. For $q = 1$ and $q = -1$ instead the squared norms are never negative and nullify for those configurations which are, respectively, totally symmetric and totally antisymmetric under permutations. This allows us to recover the Bosonic and Fermionic statistics as limiting cases of Eq. (19) without explicitly imposing the conditions (2) and (14), respectively. Of particular interest is also the $q = 0$ case whose statistics can be interpreted [12] as quantum version of the Boltzmann statistics, based on a system of identical particles having infinite number of internal degree of freedom.

The possibility of defining a proper Fock-like structure for the quon algebra (19) implies the existence of number operators $n_j(k)$ which satisfy the standard commutation relations

$$[n_j(k), a_{j'}^\dagger(k')] = -\delta_{jj'}\delta_{kk'} a_j(k),$$

and which reduce to $a_j^\dagger(k)a_j(k)$ in the Bosonic and Fermionic limits [12, 13]. In the following we will adopt a pragmatic point of view assuming that the equations of the previous section yield a legitimate description of our quon HOM interferometer. This is in part justified by the fact that the input-output relations (10) map the quon annihilation operators $a_j(k)$ into annihilation operators $b_j(k)$ which still satisfy the quon algebra (19) and the vacuum condition (20). The only technical problem of our approach comes from the fact that for $|q| < 1$ Eq. (7) is not necessarily a real quantity (it is a consequence of the fact that no commutation relation between $a_1(k_1)$ and $a_2(k_2)$ is defined [15]). Consequently we replace the product $N_1 N_2$ with its Hermitian part, i.e.

$$C_{12} \equiv \langle \Psi | \frac{N_1 N_2 + N_2 N_1}{2} | \Psi \rangle = \Re \langle \Psi | N_1 N_2 | \Psi \rangle .$$

(22)

[In this expression $N_{1,2}$ is defined through Eq. (8) in terms of the number operators $n_j(k)$ of Eq. (21)]. The right hand side term of Eq. (22) can now be computed by inverting the transformation (10) and expressing the input state $|\Psi\rangle$ of Eq. (11) in terms of the operators $b_j(k)$. With the help of the relations (21) and (14) and using the vacuum condition (20) one can then verify that the quon “coincidence counts” of the $q$-algebra are [16],

$$C_{12}(q) = \frac{1 - qw}{2}$$

(23)

with $w$ as in Eq. (12). Analogously one can verify that for normalized input states Eq. (23) still holds and that the average number $i_{1,2}$ is equal to 1 for all $q$. As expected, for $q = \pm 1$ Eq. (23) reduces to the Bosonic and Fermionic case. More interestingly for $q = 0$, $C_{12}(q)$ does not depend upon the two-particle input state $|\Psi\rangle$ and has constant value 1/2. This is exactly what one would expect from a classical model of the interferometer confirming the Boltzmann interpretation [12] of the $q = 0$ algebra.
Taking into account that for generic input state one has \( w \in [-1,1] \) the following bounds can be derived:

\[
\frac{1-\nu}{2} \leq C_{12}(q) \leq \frac{1+\nu}{2} \quad \text{for } q \in [0,1]
\]

\[
\frac{1+\nu}{2} \leq C_{12}(q) \leq \frac{1-\nu}{2} \quad \text{for } q \in [-1,0].
\]

(24)

On the other hand for factorizable amplitudes \( \Phi(k_1, k_2) \), the function \( w \) obeys Eq. (13). Therefore for these states one has

\[
\frac{1-\nu}{2} \leq C_{12}(q) \leq \frac{1}{2} \quad \text{for } q \in [0,1]
\]

\[
\frac{1}{2} \leq C_{12}(q) \leq \frac{1-\nu}{2} \quad \text{for } q \in [-1,0].
\]

(25)

Finally, we can use Eq. (17) to compute the average value of value of \( C_{12} \) with respect all possible two-particle input states \( |\Psi\rangle \), i.e.

\[
\overline{C}_{12}(q) = \frac{1}{2} - \frac{q}{2M}(2M).
\]

(26)

The above constraints and Eq. (26) have been plotted in Fig. 3; they indicate that the transition from the Bosonic regime to the Fermionic regime is characterized by a critical point at \( q = 0 \). Here the values \( C_{12} > 1/2 \) which for \( q > 0 \) were pertinent to non factorizable input states \( |\Psi\rangle \), become accessible to factorizable configurations. At the same time, however the values \( C_{12} < 1/2 \) become inaccessible to them. Following the discussion of the previous section this effect can be interpreted as a continuous transition from a bouncing behavior to an anti-bouncing behavior with \( q = 0 \) corresponding to the classical “neutral” point. Moreover, Eq. (26) provides an indirect confirmation of the interpretation [12] of the \( q = 0 \) case in terms of the statistics a system of identical particles having infinite number of internal degree of freedom. In effect according to such expression the limit \( M \to \infty \) and \( q \to 0 \) of the average coincidence counts \( \overline{C}_{12}(q) \) are equivalent.

IV. CONCLUSIONS

We analyzed how different quantum statistics affect HOM interferometry. In particular we focused on the interferometer response to initially separable/entangled inputs. By introducing a family of \( q \)-deformed algebras we interpolated between the Bosonic and Fermionic regimes which were previously discussed in Ref. [5]. Our results indicate a progressive attenuation of the bouncing behavior when moving from the Bose statistics to the Fermi statistics a progressive attenuation of the bouncing behavior.

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[15] For $q \neq \pm 1$, no (deformed) commutation rules can be imposed among $a_j(k)$ and $a^\dagger_j(k')$ (see Ref. [12] for details).
[16] It should be stressed that to derive Eq. (23) the exact expression of the $n_j(k)$ in term of the annihilation operators is not required. Only the commutation relation (21) is needed.