EXISTENCE, UNIQUENESS AND REGULARITY OF THE SOLUTION OF THE TIME-FRACTIONAL FOKKER–PLANCK EQUATION WITH GENERAL FORCING

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Abstract. A time-fractional Fokker–Planck initial-boundary value problem is considered, with differential operator \( u_t - \nabla \cdot (\partial_t^{1-\alpha} \kappa u - F \partial_t^{1-\alpha} u) \), where \( 0 < \alpha < 1 \). The forcing function \( F = F(t,x) \), which is more difficult to analyse than the case \( F = F(x) \) investigated previously by other authors. The spatial domain \( \Omega \subset \mathbb{R}^d \), where \( d \geq 1 \), has a smooth boundary. Existence, uniqueness and regularity of a mild solution \( u \) is proved under the hypothesis that the initial data \( u_0 \) lies in \( L^2(\Omega) \). For \( 1/2 < \alpha < 1 \) and \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), it is shown that \( u \) becomes a classical solution of the problem. Estimates of time derivatives of the classical solution are derived—these are known to be needed in numerical analyses of this problem.

1. Introduction. In this paper, we study the existence, uniqueness and regularity of solutions to the following inhomogeneous, time-fractional Fokker–Planck initial-boundary value problem:

\[
\begin{align*}
&u_t(t,x) - \nabla \cdot (\partial_t^{1-\alpha} \kappa u - F \partial_t^{1-\alpha} u)(t,x) = g(t,x) \text{ for } (t,x) \in (0,T) \times \Omega, \quad (1a) \\
&u(0,x) = u_0(x) \text{ for } x \in \Omega, \quad (1b) \\
&u(t,x) = 0 \text{ for } x \in \partial \Omega \text{ and } 0 < t < T, \quad (1c)
\end{align*}
\]

where \( \kappa > 0 \) is constant and \( \Omega \) is an open bounded domain with \( C^2 \) boundary in \( \mathbb{R}^d \) for some \( d \geq 1 \). In (1a), one has \( 0 < \alpha < 1 \) and \( \partial_t^{1-\alpha} \) is the standard

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Riemann–Liouville fractional derivative operator defined by \( \partial_{1}^{1-\alpha}u = (J^{\alpha}u)_{t} \), where \( J^{\beta} \) denotes the Riemann–Liouville fractional integral operator of order \( \beta \), viz.,

\[
J^{\beta}u = \int_{0}^{t} \omega_{\beta}(t-s)u(s)ds \quad \text{where} \quad \omega_{\beta}(t) := t^{\beta-1} \frac{\Gamma(\beta)}{\Gamma(\beta)} \quad \text{for} \quad \beta > 0.
\]

Regularity hypotheses on \( F, g \) and \( u_{0} \) will be imposed later.

The problem (1) was considered in [5, 6, 15]. We describe it as “general forcing” since \( F = F(t,x) \); this is a more difficult problem than the special case where \( F = F(x) \), which can be reduced to a problem already studied by several authors (see, e.g., [3, 7, 8, 13, 16]). More precisely, when the force \( F \) may depend on \( t \) as well as \( x \), equation (1) cannot be rewritten in the form of the fractional evolution equation

\[
J^{1-\alpha}(u_{t}) + Au = h(t,u,\nabla u,g,F),
\]

in which the first term is a Caputo fractional derivative, the operator \( A = -\kappa_{\alpha}\Delta \), and the function \( h \) does not depend explicitly on \( \partial_{1}^{1-\alpha}u \).

The regularity of the solution to the Cauchy problem for (2) was studied in [3]; there a fundamental solution of that problem was constructed and investigated for a more general evolution equation where the operator \( A \) in (2) is a uniformly elliptic operator with variable coefficients that acts on the spatial variables. The Cauchy problem was also considered in [13] where \( h = h(t,u,g,F) \) lies in a space of weighted Hölder continuous functions, and in [16] for the case where \( A \) is almost sectorial. Existence and uniqueness of a solution to the initial-boundary value problem where (1a) is replaced by (2) is shown in [7, 8].

To the best of our knowledge, the well-posedness and regularity properties of solutions to (1) are open questions at present, apart from a pair of recent preprints [11, 12] which treat a wider class of problems that includes (1) as a special case. The analysis in [11, 12] proceeds along broadly similar lines to here—relying on Galerkin approximation, a fractional Gronwall inequality and compactness arguments—but employs a different sequence of a priori estimates and uses neither the weighted \( L^{2} \)-norm of Definition 2.2 nor the Aubin–Lions–Simon lemma (Lemma 3.8). An interesting consequence of the approach taken here is that the constants in our estimates remain bounded as \( \alpha \to 1 \), which one expects since in this limit (1) becomes the classical Fokker–Planck equation. However, the estimates in sections 6 and 7 are valid only for \( 1/2 < \alpha < 1 \), with constants that blow up as \( \alpha \to 1/2 \) (cf. the comment following Assumption 6.1). By contrast, the results in [11, 12] hold for the full range of values \( 0 < \alpha < 1 \), but with constants that blow up as \( \alpha \to 1 \). Also, the analysis is significantly longer than the one presented here.

The main contributions of our work are:

- A proof in Theorem 5.3 of existence and uniqueness of the mild solution of (1) for the case \( \alpha \in (0,1) \) and \( u_{0} \in L^{2}(\Omega) \);
- By imposing a further condition on \( u_{0} \) and restricting \( \alpha \) to lie in \((1/2,1)\), the mild solution becomes the classical solution of (1) described in Theorem 6.7;
- Estimates of time derivatives of the classical solution in Theorem 7.3.

The paper is organized as follows. Section 2 introduces our basic notation and the definitions of mild and classical solutions of (1). Various technical properties of fractional integral operators that will be used in our analysis are provided in Section 3. In Section 4, we introduce the Galerkin approximation of the solution of (1) and prove existence and uniqueness of approximate solutions. Properties of the mild and classical solutions are derived in Sections 5 and 6, respectively.
Finally, in Section 7, we provide estimates of the time derivatives of the classical solution in $L^2(\Omega)$ and $H^2(\Omega)$, needed for the error analysis of numerical methods for solving (1); see, e.g., [5, 6, 15].

2. Notation and definitions. Throughout the paper, we often suppress the spatial variables and write $v$ or $v(t)$ instead of $v(t, \cdot)$ for various functions $v$. We also use the notation $v'$ for the time derivative. Let $\| \cdot \|$ denote the $L^2(\Omega)$ norm defined by $\|v\|^2 = \langle v, v \rangle$, where $\langle \cdot, \cdot \rangle$ is the $L^2(\Omega)$ inner product. Let $\| \cdot \|_{H^r(\Omega)}$ and $| \cdot |_{H^r(\Omega)}$ be the standard Sobolev norm and seminorm on the Hilbert space of functions whose $r$th-order derivatives lie in $L^2(\Omega)$. We borrow some standard notation from parabolic partial differential equations, e.g., $C([0, T]; L^2(\Omega))$.

Assume throughout the paper that the forcing function $\mathbf{F} = (F_1, \ldots, F_d)^T \in W^{1, \infty}((0, T) \times \Omega)$ and that its divergence $\nabla \cdot \mathbf{F}$ is continuous on $[0, T] \times \Omega$. Then $\mathbf{F}$ is continuous on $[0, T] \times \overline{\Omega}$ and we set

$$\|\mathbf{F}\|_\infty := \max_{1 \leq i \leq d} \max_{(t, x) \in [0, T] \times \overline{\Omega}} |F_i(t, x)|$$

and

$$\|\mathbf{F}\|_{1, \infty} := \|\mathbf{F}\|_\infty + \max_{(x, t) \in [0, T] \times \overline{\Omega}} |\nabla \cdot \mathbf{F}(t, x)|.$$

Stronger assumptions on the regularity of $\mathbf{F}$ will be made in some sections.

We use $C$ to denote a constant that depends on the data $\Omega, \kappa, \mathbf{F}$ and $T$ of the problem (1) but is independent of any dimension of finite-dimensional spaces to be used in our Galerkin approximations. Here the unsubscripted constants $C$ are generic and can take different values in different places throughout the paper.

We now recall the definitions of some Banach spaces from [4, p.301]:

**Definition 2.1.** Let $X$ be a real Banach space with norm $\| \cdot \|_X$. The space $C([0, T]; X)$ comprises all continuous functions $v : [0, T] \to X$ with

$$\|v\|_{C([0, T]; X)} := \max_{0 \leq t \leq T} \|v(t)\|_X.$$

Let $p \in [1, \infty]$. The space $L^p(0, T; X)$ comprises all measurable functions $v : [0, T] \to X$ for which

$$\|v\|_{L^p(0, T; X)} := \begin{cases} \left( \int_0^T \|v(t)\|_X^p \, dt \right)^{1/p} < \infty & \text{when } 1 \leq p < \infty, \\ \text{ess sup}_{0 \leq t \leq T} \|v(t)\|_X < \infty & \text{when } p = \infty. \end{cases}$$

The space $W^{1, p}(0, T; X))$ comprises all measurable functions $v : [0, T] \to X$ for which

$$\|v\|_{W^{1, p}(0, T; X)} := \|v\|_{L^p(0, T; X)} + \|v'\|_{L^p(0, T; X)}$$

is finite.

Recall that $0 < \alpha < 1$.

**Definition 2.2.** Given a Banach space $X$ with a norm (or seminorm) $\| \cdot \|_X$, define $L^p_\alpha(0, T; X)$ to be the space of functions $v : [0, T] \to X$ for which the following norm (or seminorm) is finite:

$$\|v\|_{L^p_\alpha(0, T; X)} := \max_{0 \leq t \leq T} \left[ J^\alpha \left( \|v\|_X^2(t) \right) \right]^{1/2} = \max_{0 \leq t \leq T} \left[ \frac{1}{\Gamma(\alpha)} \int_{s=0}^t (t-s)^{\alpha-1} \|v(s)\|_X^2 \, ds \right]^{1/2}.$$
For any Banach space $X$, clearly $L^2_0(0,T;X) \subset L^2(0,T;X)$ for $0 < \alpha < 1$ and $\| \cdot \|_{L^2_0(0,T;X)} = \| \cdot \|_{L^2(0,T;X)}$ if we formally put $\alpha = 1$ in Definition 2.2. For brevity, when $X = L^2(\Omega)$ we write
\[ \|v\|_{L^2_0} = \|v\|_{L^2(0,T;L^2(\Omega))} \quad \text{and} \quad \|v\|_{L^2} = \|v\|_{L^2(0,T;L^2(\Omega))}. \]

The Mittag-Leffler function $E_\alpha(z)$ that is used in the fractional Gronwall inequality of Lemma 3.1 is defined by
\[ E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \]
for $z \in \mathbb{R}$. Its properties can be found in, e.g., [2].

We now introduce the definitions of mild solutions and classical solutions to problem (1). Set
\[ G(t) := u_0 + \int_0^t g(s) \, ds \quad \text{for} \quad 0 \leq t \leq T. \]

**Definition 2.3 (Mild solutions).** A mild solution of problem (1) is a function $u \in L^2(0,T;L^2(\Omega))$ such that $J^\alpha u \in L^2(0,T;H^2(\Omega) \cap H^1_0(\Omega))$ and $u$ satisfies
\[ u - \kappa_\alpha \Delta(J^\alpha u) + \nabla \cdot (F J^\alpha u) - \nabla \cdot \left( \int_0^t F'(s) J^\alpha u(s) \, ds \right) = G(t) \quad \text{a.e. on} \quad (0,T) \times \Omega. \]

**Definition 2.4 (Classical solutions).** A classical solution of problem (1) is a function $u$ belonging to the space $C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^2(\Omega))$ such that $u' \in L^2(0,T;L^2(\Omega))$ and $\partial_t^{1-\alpha} u \in L^2(0,T;H^2(\Omega))$, with $u$ satisfying (1a) a.e. on $(0,T) \times \Omega$, and (1b) a.e. on $\Omega$.

3. Technical preliminaries. This section provides some properties of fractional integrals that will be needed in our analysis.

**Lemma 3.1.** [17, Corollary 2] Let $\beta > 0$. Assume that $a$ and $b$ are non-negative and non-decreasing functions on the interval $[0,T]$, with $a \in L^1(0,T)$ and $b \in C[0,T]$. If $y \in L^1(0,T)$ satisfies
\[ 0 \leq y(t) \leq a(t) + b(t) \int_0^t \omega_\beta(t-s) y(s) \, ds \quad \text{for} \quad 0 \leq t \leq T, \]
then
\[ y(t) \leq a(t) E_\beta(b(t)t^\beta) \quad \text{for} \quad 0 \leq t \leq T. \]

The following lemmas will be used several times in our analysis.

**Lemma 3.2.** [5, Lemma 2.2] If $\alpha \in (1/2,1)$ and $v(x,\cdot) \in L^2(0,T)$ for each $x \in \Omega$, then for $t \in [0,T]$,
\[ J^{\alpha} \langle J^{\alpha} v, v \rangle(t) \geq \frac{1}{2} \| J^{\alpha} v(t) \|^2 \quad \text{and} \quad \int_0^t \langle J^{\alpha} v, v \rangle(s) \, ds \geq \frac{1}{2} J^{1-\alpha}(\| J^{\alpha} v \|^2)(t). \]

**Lemma 3.3.** [14, Lemma 3.1 (ii)] If $\alpha \in (0,1)$ and $v(x,\cdot) \in L^2(0,T)$ for each $x \in \Omega$, then for $t \in [0,T]$,
\[ \int_0^t \langle J^{\alpha} v, v \rangle(s) \, ds \geq \cos(\alpha \pi/2) \int_0^t \| J^{\alpha/2} v \|^2(s) \, ds. \]
Lemma 3.4. [5, Lemma 2.1] Let $\beta \in (0, 1)$. If $\phi(\cdot, t) \in L^2(\Omega)$ for $t \in [0, T]$, then
\[
\|J^\beta \phi(t)\|^2 \leq \omega_{\beta+1}(t) J^\beta(\|\phi\|^2)(t) \quad \text{for } 0 \leq t \leq T.
\]
Proof. As the proof is short, we give it here for completeness. The Cauchy–Schwarz inequality yields
\[
\|J^\beta \phi(t)\|^2 = \int_\Omega \left( \int_0^t \omega_\beta(t-s) \phi(x,s) ds \right)^2 dx \\
\leq \int_\Omega \left( \int_0^t \omega_\beta(t-s) ds \right) \left( \int_0^t \omega_\beta(t-s) \phi^2(x,s) ds \right) dx \\
= \omega_{\beta+1}(t) \int_0^t \omega_\beta(t-s) \int_\Omega \phi^2(x,s) dx ds = \omega_{\beta+1}(t) J^\beta(\|\phi\|^2)(t).
\]

Lemma 3.5. For any $t > 0$ and $\beta > 0$,
\[
\|J^\beta \phi(t)\| \leq \frac{t^\beta}{\Gamma(\beta+1)} \|\phi\|_{L^\infty(0,t;L^2)}, \quad \text{for all } \phi \in L^\infty(0,t;L^2).
\]
If $\beta > 1/2$, then
\[
\|J^\beta \phi(t)\| \leq \frac{t^{\beta-1/2}}{\Gamma(\beta)\sqrt{2\beta-1}} \|\phi\|_{L^2(0,t;L^2)}, \quad \text{for all } \phi \in L^2(0,t;L^2).
\]
Proof. Minkowski’s integral inequality gives
\[
\|J^\beta \phi(t)\| = \left( \int_\Omega \left( \int_0^t \omega_\beta(t-s) \phi(s) ds \right)^2 dx \right)^{1/2} \leq \int_0^t \omega_\beta(t-s) \|\phi(s)\| ds \; \quad (4)
\]
\[
\leq \|\phi\|_{L^\infty(0,t;L^2)} \int_0^t \omega_\beta(t-s) ds = \frac{t^\beta}{\Gamma(\beta+1)} \|\phi\|_{L^\infty(0,t;L^2)}.
\]
To prove the second inequality, apply Hölder’s inequality to (4) to obtain
\[
\|J^\beta \phi(t)\| \leq \int_0^t \omega_\beta(t-s) \|\phi(s)\| ds \leq \left( \int_0^t \omega_\beta^2(t-s) ds \right)^{1/2} \|\phi\|_{L^2(0,t;L^2)}
\]
\[
= \frac{t^{\beta-1/2}}{\sqrt{2\beta-1} \Gamma(\beta)} \|\phi\|_{L^2(0,t;L^2)}
\]
for any $\beta > 1/2$, which completes the proof of this lemma. 

Lemma 3.6. [10, Theorem A.1] For $t > 0$,
\[
\int_0^t \langle \beta_s^{1-\alpha} v, v \rangle ds \geq \rho_\alpha t^{\alpha-1} \int_0^t \|v(s)\|^2 ds \quad \text{where} \; \rho_\alpha = \pi^{1-\alpha} \frac{(1-\alpha)(1-2\alpha)}{2(2-\alpha)^2-\alpha} \sin(\frac{1}{2}\pi\alpha).
\]

The following estimate involving the force $F$ is used several times in our analysis.

Lemma 3.7. If $\phi : [0, T] \to H^1(\Omega)$, then
\[
\|\nabla \cdot (F(t)\phi(t))\| \leq \|F\|_{1,\infty}\|\phi(t)\|_{H^1(\Omega)} \quad \text{for } 0 \leq t \leq T.
\]
Proof. The vector field identity \( \nabla \cdot (F \phi) = (\nabla \cdot F) \phi + F \cdot (\nabla \phi) \) implies that
\[
\| \nabla \cdot (F(t) \phi(t)) \|^2 \leq \left( \| \nabla \cdot F(t) \|_{L^\infty(\Omega)} + \| F(t) \|_{L^\infty(\Omega, \mathbb{R}^d)} \right) \left( \| \phi(t) \|^2 + \| \nabla \phi(t) \|^2 \right)
\leq \left( \| \nabla \cdot F(t) \|_{L^\infty(\Omega)} + \| F(t) \|_{L^\infty(\Omega, \mathbb{R}^d)} \right)^2 \| \phi(t) \|^2_{H^1(\Omega)}
\leq \left( \| F \|_{1, \infty} \| \phi(t) \|_{H^1(\Omega)} \right)^2,
\]
which gives the desired estimate. \( \square \)

We now recall a fundamental compactness result that will be used several times in the proofs of our main results.

**Lemma 3.8** (Aubin–Lions–Simon). Let \( B_0 \subset B_1 \subset B_2 \) be three Banach spaces. Assume that the embedding of \( B_1 \) in \( B_2 \) is continuous and that the embedding of \( B_0 \) in \( B_1 \) is compact. Let \( p \) and \( r \) satisfy \( 1 \leq p, r \leq +\infty \). For \( T > 0 \), define the Banach space
\[
E_{p,r} := \{ v \in L^p((0, T); B_0) : \partial_t v \in L^r((0, T); B_2) \}
\]
with norm
\[
\| v \|_{E_{p,r}} := \| v \|_{L^p((0, T); B_0)} + \| v' \|_{L^r((0, T); B_2)}.
\]
Then,
- the embedding \( E_{p,r} \subset L^p((0, T), B_1) \) is compact when \( p < +\infty \), and
- the embedding \( E_{p,r} \subset C([0, T], B_1) \) is compact when \( p = +\infty \) and \( r > 1 \).

**Proof.** See, e.g., [1, Theorem II.5.16]. \( \square \)

4. **Galerkin approximation of the solution.** In this section we prove existence and uniqueness of a finite-dimensional Galerkin approximation of the solution of (1). This is a standard classical tool for deriving existence and regularity results for parabolic initial-boundary value problems; see, e.g., [4, Section 7.1.2].

Let \( \{ w_k \}_{k=1}^\infty \) be a complete set of eigenfunctions for the operator \(-\Delta\) in \( H_0^1(\Omega)\), with \( \{ w_k \} \) an orthonormal basis of \( L^2(\Omega) \) and an orthogonal basis of \( H_0^1(\Omega) \); see [4, Section 6.5.1]. For each positive integer \( m \), set \( W_m = \text{span}\{ w_1, w_2, \ldots, w_m \} \) and consider \( u_m : [0, T] \to W_m \) given by
\[
u_m(t) := \sum_{k=1}^m d_m^k(t) w_k(x).
\]
Let \( \Pi_m \) be the orthogonal projector from \( L^2(\Omega) \) onto \( W_m \) defined by: for each \( v \in L^2(\Omega) \), one has
\[
\Pi_m v \in W_m \quad \text{and} \quad \langle \Pi_m v, w \rangle = \langle v, w \rangle \quad \text{for all } w \in W_m.
\]
The projections of the source term and initial data are denoted by
\[
g_m(t) := \Pi_m g(t) \quad \text{and} \quad u_{0m} := \Pi_m u_0.
\]
We aim to choose the functions \( d_m^k \) so that for \( k = 1, 2, \ldots, m \) and \( t \in (0, T) \) one has
\[
u_m' - \kappa_0 \partial_t^{1-\alpha} \Delta u_m + \Pi_m \left( \nabla \cdot (F(t) \partial_t^{1-\alpha} u_m) \right) = g_m(t) \quad (5a)
\]
and
\[
d_m^k(0) = \langle u_0, w_k \rangle. \quad (5b)
\]
Existence and uniqueness of a solution to (5) are guaranteed by the following lemma.

**Lemma 4.1.** [6, Theorem 3.1] Let $F \in W^{1,\infty}(0,T; L^{\infty}(\Omega))$ and $g \in L^{1}(0,T; L^{2}(\Omega))$. Then for each positive integer $m$, the system of equations (5) has a solution $\{d_k^m\}_{k=1}^m$ with $u_m : [0,T] \rightarrow H^2(\Omega) \cap H^1_0(\Omega)$ absolutely continuous. This solution is unique among the space of absolutely continuous functions mapping $[0,T]$ to $H^1_0(\Omega)$.

**Proof.** Our argument is based mainly on the proof of [6, Theorem 3.1], but we fill a gap in that argument by verifying that $u_m$ is absolutely continuous. Define the linear operator $B_m(t) : W \rightarrow W$ by

$$
\langle B_m(t)v, w \rangle := -\kappa_a \langle \Delta v, w \rangle + \langle \Pi_m(\nabla \cdot (F(t,\cdot)v)), w \rangle
$$

for all $v, w \in W_m$, and rewrite (5a) as

$$
u_m'(t) + B_m(t)\partial^{1-\alpha}_t u_m(t) = g_m(t).$$

Formally integrating this equation in time we obtain the Volterra integral equation [6, p.1768]:

$$u_m(t) + \int_{0}^{t} K_m(t,s)u_m(s) \, ds = G_m(t) \text{ for } 0 \leq t \leq T, \tag{6}$$

where

$$K_m(t,s) = B_m(t)\omega_\alpha(t-s) - \int_{s}^{t} B_m'\tau)\omega_\alpha(t-\tau-s) \, d\tau \text{ and } G_m(t) := u_{0m} + \int_{0}^{t} g_m(s) \, ds.$$ 

It is shown in [6] that (6) has a unique solution $u_m \in C([0,T]; H^1_0(\Omega))$.

Now, $g \in L^{1}(0,T; L^{2}(\Omega))$ implies that $g_m \in L^{1}(0,T; L^{2}(\Omega))$, and it follows that $G_m : [0,T] \rightarrow L^2(\Omega)$ is absolutely continuous. Furthermore, Theorem 2.5 of [2] implies (using the continuity of $u_m$) that $t \mapsto \int_{s=0}^{t} K_m(t,s)u_m(s) \, ds$ is absolutely continuous. Hence, (6) shows that $u_m : [0,T] \rightarrow L^2(\Omega)$ is absolutely continuous.

We are now able to differentiate (6) (to differentiate the integral term, imitate the calculation in the proof of [2, Lemma 2.12]), obtaining

$$u_m'(t) + \int_{s=0}^{t} B_m(t)\omega_\alpha(t-s)u_m'(s) \, ds = g_m(t) \text{ for almost all } t \in [0,T].$$

The absolute continuity of $u_m(t)$ implies that $\partial^{1-\alpha}_t u_m(t)$ exists for almost all $t \in [0,T]$ by [2, Lemma 2.12]. Hence from the above equation, $u_m$ satisfies (5a). From (6), one sees immediately that $u_m$ satisfies (5b), so we have demonstrated the existence of a solution to (5).

To see that this solution of (5) is unique among the space of absolutely continuous functions, one can use the proof of [6, Theorem 3.1] since the absolute continuity of the solution is now known a priori. \qed

5. **Existence and uniqueness of the mild solution.** In this section, we assume that $\alpha \in (0,1)$, $F \in W^{1,\infty}((0,T) \times \Omega)$ and that the initial data $u_0 \in L^2(\Omega)$.

5.1. **A priori estimates.** In order to prove a priori estimates, we consider the integrated form of equation (5a):

$$u_m(t) - \kappa_a J^\alpha \Delta u_m(t) + \int_{0}^{t} \Pi_m(\nabla \cdot (F(s)\partial^{1-\alpha}_t u_m(s))) \, ds = G_m(t), \tag{7}$$

where $G_m(t) = \Pi_m G(t)$ as in (6).

Let $C_P$ denote the Poincaré constant for $\Omega$, viz., $\|v\|^2 \leq C_P \|\nabla v\|^2$ for $v \in H^1_0(\Omega)$. 
Lemma 5.1. Let $m$ be a positive integer. Let $u_m(t)$ be the absolutely continuous solution of (5a) that is guaranteed by Lemma 4.1. Then for any $t \in [0, T]$ one has

$$\cos(\alpha \pi/2) \int_0^t \|J^{\alpha/2} u_m(s)\|^2 ds + \kappa_\alpha \int_0^t \|J^{\alpha} u_m(s)\|^2_{H^1(\Omega)} ds \leq C_1 \int_0^t \|G_m(s)\|^2 ds \tag{8}$$

and

$$\int_0^t \|u_m(s)\|^2 ds \leq C_3 \int_0^t \|G_m(s)\|^2 ds, \tag{9}$$

where

$$C_1 := \frac{1 + C_0}{2} \left[1 + \frac{C_2 \omega_2^{1+\alpha/2}(T)}{\cos(\alpha \pi/2)} E_{\alpha/2} \left(\frac{C_2 \omega_2^{1+\alpha/2}(T)}{\cos(\alpha \pi/2)} T^\alpha\right)\right],$$

$$C_2 := 2 \left(1 + \frac{||F||^2_{\infty}}{\kappa_\alpha} + \frac{T^2 ||F'||^2_{\infty}}{\kappa_\alpha}\right),$$

$$C_3 := 2 + \frac{C_1}{\kappa_\alpha} \left(4||F||^2_{1, \infty} + 2T^2 ||F'||^2_{1, \infty}\right).$$

Proof. Taking the inner product of both sides of (7) with $J^{\alpha} u_m(t) \in W_m$ then integrating by parts with respect to $x$, we obtain

$$\langle J^{\alpha} u_m(t), u_m(t) \rangle + \kappa_\alpha \|J^{\alpha} \nabla u_m(t)\|^2$$

$$= \left\langle \int_0^t F(s) \partial_t^{1-\alpha} u_m(s) ds, J^{\alpha} \nabla u_m(t) \right\rangle + \langle G_m(t), J^{\alpha} u_m(t) \rangle$$

$$\leq \frac{\kappa_\alpha}{2} \|J^{\alpha} \nabla u_m(t)\|^2 + \frac{1}{2\kappa_\alpha} \left\|\int_0^t F(s) \partial_t^{1-\alpha} u_m(s) ds\right\|^2 + \frac{1}{4} \|G_m(t)\|^2 + \|J^{\alpha} u_m(t)\|^2. \tag{10}$$

Integrating by parts with respect to the time variable, and using Minkowski’s integral inequality and Hölder’s inequality, we have

$$\left\|\int_0^t F(s) \partial_t^{1-\alpha} u_m(s) ds\right\|^2 = \left\|F(t) J^{\alpha} u_m(t) - \int_0^t F'(s) J^{\alpha} u_m(s) ds\right\|^2$$

$$\leq 2 \|F\|_{\infty}^2 \|J^{\alpha} u_m(t)\|^2 + 2 \|F'\|_{\infty}^2 \left(\int_0^t \|J^{\alpha} u_m(s)\|^2 ds\right)^2$$

$$\leq 2 \|F\|_{\infty}^2 \|J^{\alpha} u_m(t)\|^2 + 2t \|F'\|_{\infty}^2 \int_0^t \|J^{\alpha} u_m(s)\|^2 ds. \tag{11}$$

It follows from (10) and (11) that

$$\langle J^{\alpha} u_m(t), u_m(t) \rangle + \frac{\kappa_\alpha}{2} \|J^{\alpha} \nabla u_m(t)\|^2 \leq \frac{1}{4} \|G_m(t)\|^2 + \left(1 + \frac{||F||^2_{\infty}}{\kappa_\alpha}\right) \|J^{\alpha} u_m(t)\|^2$$

$$+ \frac{t ||F'||^2_{\infty}}{\kappa_\alpha} \int_0^t \|J^{\alpha} u_m(s)\|^2 ds.$$
Integrating in time and invoking Lemma 3.3, we deduce that
\[
\cos(\alpha \pi / 2) \int_0^t \| J^{\alpha/2} u_m(s) \|^2 \, ds + \kappa_\alpha \int_0^t \| J^{\alpha} \nabla u_m(s) \|^2 \, ds \leq \frac{1}{2} \int_0^t \| G_m(s) \|^2 \, ds
\]
\[
+ 2 \left( 1 + \frac{\| F \|_\infty^2}{\kappa_\alpha} \right) \int_0^t \| J^{\alpha} u_m(s) \|^2 \, ds + \frac{2 \| F \|_\infty^2}{\kappa_\alpha} \int_0^t \int_0^s \| J^{\alpha} u_m(\tau) \|^2 \, d\tau \, ds
\]
\[
\leq \frac{1}{2} \int_0^t \| G_m(s) \|^2 \, ds + C_2 \int_0^t \| J^{\alpha} u_m(s) \|^2 \, ds.
\]
(12)

But Lemma 3.4 gives us
\[
\| J^{\alpha} u_m(s) \|^2 = \| J^{\alpha/2}(J^{\alpha/2} u_m)(s) \|^2 \leq \omega_{1+\alpha/2}(s) J^{\alpha/2}(\| J^{\alpha/2} u_m \|^2)(s).
\]
(13)

Thus, setting \( \psi_m(t) := J^1(\| J^{\alpha/2} u_m \|^2)(t) \), we deduce from (12) that
\[
\psi_m(t) \leq \frac{1}{2 \cos(\alpha \pi / 2)} \int_0^t \| G_m(s) \|^2 \, ds + \frac{C_2 \omega_{1+\alpha/2}(t)}{\cos(\alpha \pi / 2)} J^{1+\alpha/2}(\| J^{\alpha/2} u_m \|^2)(t)
\]
\[
= \frac{1}{2 \cos(\alpha \pi / 2)} \int_0^t \| G_m(s) \|^2 \, ds + \frac{C_2 \omega_{1+\alpha/2}(t)}{\cos(\alpha \pi / 2)} J^{\alpha/2} \psi_m(t).
\]

Applying Lemma 3.1, one obtains
\[
\psi_m(t) \leq E_{\alpha/2} \left( \frac{C_2 \omega_{1+\alpha/2}(t)}{\cos(\alpha \pi / 2)} t^{\alpha} \right) \frac{1}{2 \cos(\alpha \pi / 2)} \int_0^t \| G_m(s) \|^2 \, ds \quad \text{for } 0 \leq t \leq T.
\]
(14)

This inequality and (13) together yield
\[
\int_0^t \| J^{\alpha} u_m(s) \|^2 \, ds \leq \omega_{1+\alpha/2}(t) J^{\alpha/2} \psi_m(t)
\]
\[
\leq \frac{\omega_{1+\alpha/2}(t)}{2 \cos(\alpha \pi / 2)} \int_0^t \omega_{\alpha/2}(t - s) E_{\alpha/2} \left( \frac{C_2 \omega_{1+\alpha/2}(s)}{\cos(\alpha \pi / 2)} s^{\alpha} \right) \int_0^s \| G_m(z) \|^2 \, dz \, ds
\]
\[
\leq \frac{\omega_{1+\alpha/2}(t)}{2 \cos(\alpha \pi / 2)} E_{\alpha/2} \left( \frac{C_2 \omega_{1+\alpha/2}(t)}{\cos(\alpha \pi / 2)} t^{\alpha} \right) \int_0^t \| G_m(s) \|^2 \, ds.
\]

Now (8) follows immediately on recalling (12)–(14) and the Poincaré inequality.

In a similar fashion, we take the inner product of both sides of (7) with \( u_m(t) \in W_m \) and then integrate by parts with respect to \( x \), to obtain
\[
\| u_m(t) \|^2 + \kappa_\alpha \langle J^{\alpha} \nabla u_m(t), \nabla u_m(t) \rangle
\]
\[
= -\left\langle \int_0^t \nabla \cdot (F(s) \partial_{t^{1-\alpha}} u_m(s)) \, ds, u_m(t) \right\rangle + \langle G_m, u_m \rangle
\]
\[
\leq \| G_m(t) \|^2 + \frac{1}{2} \| u_m(t) \|^2 + \left\| \int_0^t \nabla \cdot (F(s) \partial_{t^{1-\alpha}} u_m(s)) \, ds \right\|^2.
\]
(15)

Using Lemma 3.7 and the same arguments as in the proof of (11), we also have
\[
\left\| \int_0^t \nabla \cdot (F(s) \partial_{t^{1-\alpha}} u_m(s)) \, ds \right\|^2
\]
\[
\leq 2 \| F \|_{1, \infty}^2 \| J^{\alpha} u_m(t) \|^2_{H^1(\Omega)} + 2 t \| F \|_{1, \infty}^2 \int_0^t \| J^{\alpha} u_m(s) \|^2_{H^1(\Omega)} \, ds.
\]
(16)
This estimate and (15) together imply
\[
\frac{1}{2} \|u_m(t)\|^2 + \kappa_\alpha (J^\alpha \nabla u_m(t), \nabla u_m(t)) \\
\leq \|G_m(t)\|^2 + 2\|F\|_{1,\infty}^2 \|J^\alpha u_m(t)\|_{H^1(\Omega)}^2 + 2t\|F\|_{1,\infty}^2 \int_0^t \|J^\alpha u_m(s)\|_{H^1(\Omega)}^2 ds.
\]
Integrating in time, we get
\[
\int_0^t \|u_m(s)\|^2 ds \\
\leq 2 \int_0^t \|G_m(s)\|^2 ds + \left(4\|F\|_{1,\infty}^2 + 2t^2\|F\|_{1,\infty}^2\right) \int_0^t \|J^\alpha u_m(s)\|_{H^1(\Omega)}^2 ds.
\]
Now apply the inequality (8) to complete the proof.

Lemma 5.2. Let \(m\) be a positive integer, and let \(u_m(t)\) be the absolutely continuous solution of (5a) that is guaranteed by Lemma 4.1. Then, for any \(t \in [0,T]\),
\[
\cos(\alpha \pi/2) \int_0^t \|J^{\alpha/2} \nabla u_m(s)\|^2 ds + \kappa_\alpha \int_0^t \|J^\alpha \Delta u_m(s)\|^2 ds \leq C_4 \int_0^t \|G_m(s)\|^2 ds
\]
and
\[
\|J^1 u_m(t)\|_{H^1(\Omega)}^2 \leq C_5 \int_0^t \|G_m(s)\|^2 ds,
\]
where
\[
C_4 := \frac{2}{\kappa_\alpha} + \frac{2C_1}{\kappa_\alpha^2} \left(2\|F\|_{1,\infty}^2 + T^2\|F\|_{1,\infty}^2\right) \quad \text{and} \quad C_5 := \frac{C_4 T^{1-\alpha} (1+C_T)}{(1-\alpha) \cos(\alpha \pi/2) \Gamma(1-\alpha/2)^2}.
\]

Proof. Taking the inner product of both sides of (7) with \(-J^\alpha \Delta u_m(t) \in W_m\) and then integrating by parts with respect to \(x\), we obtain
\[
\langle J^\alpha \nabla u_m(t), \nabla u_m(t) \rangle + \kappa_\alpha \langle J^\alpha \Delta u_m(t) \rangle
\]
\[
= \left( \int_0^t \nabla \cdot (F(s) \partial_t^{-\alpha} u_m(s)) ds, J^\alpha \Delta u_m(t) \right) - \langle G_m(t), J^\alpha \Delta u_m(t) \rangle
\]
\[
\leq \frac{\kappa_\alpha}{2} \|J^\alpha \Delta u_m(t)\|^2 + \frac{1}{\kappa_\alpha} \|G_m(t)\|^2 + \frac{1}{\kappa_\alpha} \left( \int_0^t \nabla \cdot (F(s) \partial_t^{-\alpha} u_m(s)) ds \right)^2.
\]
This inequality and (16) together imply
\[
\langle J^\alpha \nabla u_m(t), \nabla u_m(t) \rangle + \frac{\kappa_\alpha}{2} \|J^\alpha \Delta u_m(t)\|^2
\]
\[
\leq \frac{1}{\kappa_\alpha} \|G_m(t)\|^2 + \frac{2}{\kappa_\alpha} \|F\|_{1,\infty}^2 \|J^\alpha u_m(t)\|_{H^1(\Omega)}^2 + \frac{2t}{\kappa_\alpha} \|F\|_{1,\infty}^2 \left( \int_0^t \|J^\alpha u_m(s)\|_{H^1(\Omega)}^2 ds \right).
\]
Integrating in time and invoking Lemma 3.3, we deduce that
\[
2 \cos(\alpha \pi/2) J^1 (\|J^{\alpha/2} \nabla u_m\|^2) + \kappa_\alpha \int_0^t \|J^\alpha \Delta u_m(s)\|^2 ds
\]
\[
\leq \frac{2}{\kappa_\alpha} \int_0^t \|G_m(s)\|^2 ds + \frac{2}{\kappa_\alpha} \left(2\|F\|_{1,\infty}^2 + t^2\|F\|_{1,\infty}^2\right) \int_0^t \|J^\alpha u_m(s)\|_{H^1(\Omega)}^2 ds,
\]
which, after applying inequality (8) of Lemma 5.1, completes the proof of (17).
Applying (4) with \( \phi = J^{\alpha/2}u_m \) and \( \beta = 1 - \alpha/2 \) gives

\[
\| J^1 u_m(t) \|_{H^1(\Omega)} = \| J^{1-\alpha/2} J^{\alpha/2} u_m(t) \|_{H^1(\Omega)} \\
\leq t^{1-\alpha/2} \| J^{\alpha/2} u_m(t) \|_{H^1(\Omega)} = (\omega_{1-\alpha/2} \ast z)(t),
\]

where \( z(t) = \| J^{\alpha/2} u_m(t) \|_{H^1(\Omega)} \). Using Young’s convolution inequality we get

\[
\| J^1 u_m(t) \|_{H^1(\Omega)}^2 \leq \| \omega_{1-\alpha/2} \ast z \|_{L^\infty(\Omega)} \leq \| \omega_{1-\alpha/2} \|_{L^2(\Omega)}^2 \| z \|_{L^2(\Omega)}^2 = (1-\alpha)\Gamma(1-\alpha/2) \int_0^t \| J^{\alpha/2} u_m(s) \|_{H^1(\Omega)}^2 ds.
\]

The inequality (18) now follows immediately from (17).

5.2. The mild solution. Our assumption that \( \Omega \) has a \( C^2 \) boundary ensures that if \( v \in H^1_0(\Omega) \) satisfies \( \Delta v \in L^2(\Omega) \), then \( v \in H^2(\Omega) \). Moreover, there is a regularity constant \( C_R \), depending only on \( \Omega \), such that

\[
\| v \|_{H^2(\Omega)} \leq C_R \| \Delta v \| \quad \text{for} \quad v \in H^1_0(\Omega). \tag{19}
\]

Our next result requires a strengthening of the regularity hypothesis on \( F \).

**Theorem 5.3.** Assume that \( u_0 \in L^2(\Omega) \), \( F \in W^{2,\infty}((0,T) \times \Omega) \) and \( g \in L^2(0,T;L^2(\Omega)) \). Then there exists a unique mild solution \( u \) of (1) (in the sense of Definition 2.3) such that

\[
\| u \|_{L^2(0,T;L^2)} + \| J^s u \|_{L^2(0,T;H^2)} \leq \left( C_4 + \kappa_\alpha^{-1} C_3 C_R \right) \| G \|_{L^2} \tag{20}
\]

Proof. In order to prove the existence of a mild solution, we first prove the convergence of the approximate solutions \( u_m \), and then find the limit of equation (7) as \( m \) tends to infinity.

Note first that \( \| G_m(s) \| \leq \| G(s) \| \) because

\[
| \langle G_m(s), w \rangle | = | \langle G(s), \Pi_m w \rangle | \leq \| G(s) \| \| \Pi_m w \| \leq \| G(s) \| \| w \| \quad \text{for all} \quad w \in L^2(\Omega).
\]

Hence Lemma 5.1 shows that the sequence \( \{ \partial_t(J^1 u_m) \}_{m=1}^\infty = \{ u_m \}_{m=1}^\infty \) is bounded in \( L^2(0,T;L^2(\Omega)) \), and Lemma 5.2 shows that the sequence \( \{ J^1 u_m \}_{m=1}^\infty \) is bounded in \( L^\infty (0,T;H^1_0(\Omega)) \). Applying Lemma 3.8 with \( B_0 = H^1_0(\Omega) \), \( B_1 = B_2 = L^2(\Omega) \), \( p = +\infty \) and \( r = 2 \), it follows that there exists a subsequence of \( \{ J^1 u_m \}_{m=1}^\infty \), again denoted by \( \{ J^1 u_m \}_{m=1}^\infty \), and a \( v \in C([0,T];L^2(\Omega)) \), such that

\[
J^1 u_m \to v \quad \text{strongly in} \quad C([0,T];L^2(\Omega)). \tag{21}
\]

Furthermore, from the above bounds on \( \{ J^1 u_m \}_{m=1}^\infty \) and well-known results [1, Theorem II.2.7] for weak and weak-* compactness, by choosing sub-subsequences we get

\[
J^1 u_m \to v \quad \text{weak-* in} \quad L^\infty (0,T;H^1_0(\Omega)), \quad u_m = \partial_t (J^1 u_m) \to \partial_t v \quad \text{weakly in} \quad L^2 (0,T;L^2(\Omega)). \tag{22}
\]

By letting \( u := \partial_t v \in L^2 (0,T;L^2(\Omega)) \), we have \( u = J^1 u \). It remains to prove that \( J^\alpha u_m \) converges weakly to \( J^\alpha u \) in \( L^2 (0,T;H^2(\Omega)) \). Applying Lemma 3.5 with \( \phi = J^1 u_m \) and \( \beta = \alpha \), for any \( t \in [0,T] \) we deduce that

\[
\| J^{1+\alpha} u_m(t) \|_{H^1(\Omega)} \leq \frac{t^\alpha}{\Gamma(\alpha+1)} \| J^1 u_m \|_{L^\infty(0,t;H^1(\Omega))}.
\]

This inequality, together with Lemma 5.2, implies that the sequence \( \{ J^{1+\alpha} u_m \}_{m=1}^\infty \) is bounded in \( L^\infty (0,T;H^1_0(\Omega)) \). Also, Lemma 5.1 shows that the sequence...
\{\partial_t(J^{1+\alpha}u_m)\}_{m=1}^\infty = \{J^{\alpha}u_m\}_{m=1}^\infty \text{ is bounded in } L^2(0,T;H^0_1(\Omega)). \text{ It now follows from Lemma 3.8, again with } B_0 = H^0_1(\Omega), B_1 = B_2 = L^2(\Omega), p = +\infty \text{ and } r = 2, \text{ that there exists a subsequence of } \{J^{1+\alpha}u_m\}_{m=1}^\infty \text{ (still denoted by } \{J^{1+\alpha}u_m\}_{m=1}^\infty \text{) and } \bar{u} \in C([0,T];L^2(\Omega)) \text{ such that}
\begin{equation}
J^{1+\alpha}u_m \to \bar{u} \text{ strongly in } C([0,T];L^2(\Omega)).
\end{equation}
Furthermore, from the upper bound (17) of \{\partial_t(J^{1+\alpha}u_m)\}_{m=1}^\infty \text{ in } L^2(0,T;H^2(\Omega)), by choosing a subsequence one gets
\begin{equation}
J^{\alpha}u_m = \partial_t(J^{1+\alpha}u_m) \to \partial_t\bar{u} \text{ weakly in } L^2(0,T;H^2(\Omega)).
\end{equation}
On the other hand, by applying Lemma 3.5 with \phi = J^1(u_m - u) \text{ and } \beta = \alpha, \text{ we deduce that for any } t \in [0,T] \text{ one has}
\[\|J^{1+\alpha}(u_m - u)(t)\|_{L^2(\Omega)} \leq \frac{t^\alpha}{\Gamma(\alpha + 1)} \|J^1(u_m - u)\|_{L^\infty(0,t;L^2(\Omega))}.\]
Hence, (21) implies that \lim_{m \to \infty} \|J^{1+\alpha}(u_m - u)\|_{L^\infty(0,T;L^2(\Omega))} = 0. \text{ Recalling (23), we have } \bar{u} = J^{1+\alpha}u. \text{ By choosing subsequences, we obtain}
\begin{align}
J^{1+\alpha}u_m &\to J^{1+\alpha}u \text{ strongly in } C([0,T];L^2(\Omega)), \\
J^{1+\alpha}u_m &\to J^{1+\alpha}u \text{ weakly in } L^\infty(0,T;H^0_1(\Omega)), \\
J^{\alpha}u_m &\to J^{\alpha}u \text{ weakly in } L^2(0,T;H^2(\Omega)),
\end{align}
where we used the boundedness of \{J^{1+\alpha}u_m\}_{m=1}^\infty \text{ in } L^\infty(0,T;H^0_1(\Omega)) \text{ that was already mentioned, and (24).}

Multiplying both sides of (7) by a test function \xi \in C^\infty((0,T) \times \Omega), \text{ integrating over } (0,T) \times \Omega \text{ and noting that } \Pi_m \text{ is a self-adjoint operator on } L^2(\Omega) \text{ gives}
\begin{equation}
\langle u_m,\xi\rangle_{L^2(0,T;L^2)} - \kappa_\alpha \langle J^{\alpha}\Delta u_m,\xi\rangle_{L^2(0,T;L^2)} + \langle h_m,\Pi_m\xi\rangle_{L^2(0,T;L^2)} = \langle G_m,\xi\rangle_{L^2(0,T;L^2)},
\end{equation}
where \(h_m(t) := \int_0^t \nabla \cdot (F(s)\partial_t^{1-\alpha}u_m(s)) \, ds\). \text{ Using (22) and (27), as } m \to \infty \text{ one has}
\begin{align}
\langle u_m,\xi\rangle_{L^2(0,T;L^2)} &\to \langle u,\xi\rangle_{L^2(0,T;L^2)}, \\
\langle G_m,\xi\rangle_{L^2(0,T;L^2)} &\to \langle G,\xi\rangle_{L^2(0,T;L^2)}, \\
\langle J^{\alpha}\Delta u_m,\xi\rangle_{L^2(0,T;L^2)} &\to \langle J^{\alpha}\Delta u,\xi\rangle_{L^2(0,T;L^2)}, \\
\langle J^{\alpha}u_m,\xi\rangle_{L^2(0,T;L^2)} &\to \langle J^{\alpha}u,\xi\rangle_{L^2(0,T;L^2)}.
\end{align}
To find the limit of the most complicated term \langle h_m,\Pi_m\xi - \xi\rangle_{L^2(0,T;L^2)} \text{ in (28), we first integrate by parts twice with respect to the time variable:}
\begin{align}
h_m(t) &= \int_0^t \nabla \cdot (F(s)\partial_t^{1-\alpha}u_m(s)) \, ds \\
&= \nabla \cdot (F(t)J^{\alpha}u_m(t)) - \int_0^t \nabla \cdot (F'(s)J^{\alpha}u_m(s)) \, ds \\
&= \nabla \cdot (F(t)J^{\alpha}u_m(t)) - \nabla \cdot (F'(t)J^{1+\alpha}u_m(t)) + \int_0^t \nabla \cdot (F''(s)J^{1+\alpha}u_m(s)) \, ds.
\end{align}
It now follows from the boundedness of \{J^{\alpha}u_m\}_{m=1}^\infty \text{ and } \{J^{1+\alpha}u_m\}_{m=1}^\infty \text{ in } L^2(0,T;H^2(\Omega)) \text{ and } L^\infty(0,T;H^0_1(\Omega)), \text{ respectively, that } h_m \text{ is bounded in } L^2(0,T;L^2(\Omega)). \text{ Hence,}
\begin{equation}
\lim_{m \to \infty} \langle h_m,\Pi_m\xi - \xi\rangle_{L^2(0,T;L^2)} = 0.
\end{equation}
On the other hand, by using (30) and integration by parts with respect to \( x \), we have
\[
\langle h_m, \xi \rangle_{L^2(0,T;L^2)} = \langle \nabla \cdot (F J^{\alpha}u_m), \xi \rangle_{L^2(0,T;L^2)} - \langle \nabla \cdot (F' J^{1+\alpha}u_m), \xi \rangle_{L^2(0,T;L^2)} - \int_0^T \int_\Omega \left( \int_0^t F''(s) J^{1+\alpha}u_m(s) \, ds \right) \cdot \nabla \xi(t) \, dx \, dt.
\]

Combining this identity with (25)–(27) gives
\[
\lim_{m \to \infty} \langle h_m, \xi \rangle_{L^2(0,T;L^2)} = \langle \nabla \cdot (F J^{\alpha}u), \xi \rangle_{L^2(0,T;L^2)} - \langle \nabla \cdot (F' J^{1+\alpha}u), \xi \rangle_{L^2(0,T;L^2)} - \int_0^T \int_\Omega \nabla \cdot \left( \int_0^t F'(s) J^{\alpha}u(s) \, ds \right) \xi(t) \, dx \, dt.
\]

Now invoking (31) yields
\[
\lim_{m \to \infty} \langle h_m, \Pi_m \xi \rangle_{L^2(0,T;L^2)} = \langle \nabla \cdot (F J^{\alpha}u), \xi \rangle_{L^2(0,T;L^2)} - \int_0^T \int_\Omega \nabla \cdot \left( \int_0^t F'(s) J^{\alpha}u(s) \, ds \right) \xi(t) \, dx \, dt. \tag{32}
\]

Let \( m \to \infty \) in (28). Using (29) and (32), we deduce that for any \( \xi \in C_c^\infty((0,T) \times \Omega) \) one has
\[
\left\langle u - \kappa_d J^\alpha \Delta u + \nabla \cdot (F J^{\alpha}u) - \nabla \cdot \left( \int_0^t F'(s) J^{\alpha}u(s) \, ds \right), \xi \right\rangle_{L^2(0,T;L^2)} = \langle g, \xi \rangle_{L^2(0,T;L^2)}.
\]

Since \( C_c^\infty((0,T) \times \Omega) \) is dense in \( L^2((0,T) \times \Omega) \), the above equation also holds true for any test function \( \xi \in L^2((0,T) \times \Omega) \). Hence, \( u \) satisfies (3) a.e. on \( (0,T) \times \Omega \).

The weak convergence of \( u_m \) described in (22), and [1, Corollary II.2.8] with (9) together yield \( \|u\|_{L^2(0,T;L^2)}^2 \leq C_3 \|G\|_{L^2(0,T;L^2)}^2 \). Similarly, (27) and (17) imply that \( \kappa_d \|J^\alpha \Delta u\|_{L^2(0,T;H^2)}^2 \leq C_4 \|G\|_{L^2(0,T;L^2)}^2 \). Thus, (20) is proved.

The uniqueness of the solution \( u \) follows from linearity and (20), because if \( u_0 = 0 \) and \( g = 0 \) then \( G = 0 \) and hence \( u = 0 \).

6. Existence and uniqueness of the classical solution.

**Assumption 6.1.** In the rest of this paper, we assume that
\[
\frac{1}{2} < \alpha < 1.
\]

Assumption 6.1 is not overly restrictive because (1) is usually considered as a variant of the case \( \alpha = 1 \). We cannot avoid this restriction on \( \alpha \) in Sections 6 and 7 since our analysis makes heavy use of \( \partial_t 1^{-\alpha} u \), and for typical solutions \( u \) of (1), it will turn out that \( \|\partial_t 1^{-\alpha} u\|_{L^2(0,T;L^2(\Omega))} < \infty \) only for \( 1/2 < \alpha < 1 \). To see this heuristically, assume that \( u(x,t) = \phi(x) + v(x,t) \), where \( v \) vanishes as \( t \to 0 \) so \( \phi(x) \) is the dominant component near \( t = 0 \); then \( \partial_t 1^{-\alpha} u(x,t) = \phi(x) \omega_\alpha(t) + \partial_t 1^{-\alpha} v(x,t) \approx \phi(x) \omega_\alpha(t) \) near \( t = 0 \), and \( \int_0^T \omega_\alpha^2(t) \, dt \) is finite only if \( \alpha > 1/2 \).
6.1. **A priori estimates.** Since $\partial_t^{1-\alpha} v = \omega(\cdot, t)$, we can rewrite (5a) in terms of $v_m(t) := u_m(t) - u_{0m}$ as

$$
\begin{align*}
\nu' - \kappa_\alpha \partial_t^{1-\alpha} \Delta V_m + \Pi_m \left( \nabla \cdot \left( F(t) \partial_t^{1-\alpha} v_m \right) \right) \\
= \Pi_m g(t) - \omega(t) \left[ \Pi_m \left( \nabla \cdot \left( F(t) u_{0m} \right) - \kappa_\alpha \Delta u_{0m} \right) \right].
\end{align*}
\tag{33}
$$

We will require the following bound for $u_{0m}$.

**Lemma 6.2.** If $u_0 \in H^2(\Omega)$, then $\|u_{0m}\|_{H^2(\Omega)} \leq C_R \|u_0\|_{H^2(\Omega)}$ for all $m$, where the constant $C_R$ was defined in (19).

**Proof.** Write $d_{0m}^k = d_m^k(0) = \langle u_0, w_k \rangle$ for $1 \leq k \leq m$, and let $\lambda_k > 0$ denote the $k$th Dirichlet eigenvalue of the Laplacian so that $-\Delta w_k = \lambda_k w_k$ for all $k$. In this way,

$$
u_{0m}(x) = \sum_{k=1}^m d_{0m}^k w_k(x) \quad \text{and} \quad -\Delta u_{0m}(x) = \sum_{k=1}^m d_{0m}^k \lambda_k w_k(x).$$

If $u_0 \in H^2(\Omega)$ then $\Delta u_0 \in L_2(\Omega)$ so, using Parseval’s identity,

$$
\begin{align*}
\|\Delta u_{0m}\|^2 &= \sum_{k=1}^m |\langle \Delta u_{0m}, w_k \rangle|^2 = \sum_{k=1}^m \lambda_k^2 |\langle u_0, w_k \rangle|^2 \\
&\leq \sum_{k=1}^m \lambda_k^2 |\langle u_0, w_k \rangle|^2 = \sum_{k=1}^m |\langle u_0, \Delta w_k \rangle|^2 = \|\Delta u_0\|^2.
\end{align*}
$$

Thus, $\|u_{0m}\|_{H^2(\Omega)} \leq C_R \|u_{0m}\| \leq C_R \|u_0\| \leq C_R \|u_0\|_{H^2(\Omega)}$. \hfill \Box

We now prove upper bounds for $\|\partial_t^{1-\alpha} v_m(t)\|$ and $J^\alpha(\|\partial_t^{1-\alpha} \nabla v_m\|)(t)$ for any $t \in [0, T]$. The argument used in the following lemma is based on the proof of [5, Theorem 3.1].

**Lemma 6.3.** Let $m$ be a positive integer. Let $v_m(t)$ be the absolutely continuous solution of (33) that is guaranteed by Lemma 4.1. Then, for almost all $t \in [0, T]$,

$$
\|\partial_t^{1-\alpha} v_m(t)\|^2 \leq C_7 t^{2\alpha-1} \left( C_6 \|u_0\|_{H^2(\Omega)}^2 + \int_0^t \|g(s)\|^2 \, ds \right) \tag{34}
$$

and

$$
J^\alpha(\|\partial_t^{1-\alpha} \nabla v_m\|)(t) \leq 1 + C_7 E_{2\alpha - 1}(C_7 t^{2\alpha-1}) \omega_{2\alpha}(t) \left( C_6 \|u_0\|_{H^2(\Omega)}^2 + \int_0^t \|g(s)\|^2 \, ds \right), \tag{35}
$$

where

$$
C_6 := C_R^2 (\kappa_\alpha + \|F\|_{1, \infty})^2, \quad C_7 := \frac{\Gamma(2\alpha - 1)}{\Gamma(\alpha)} \left( 1 + \frac{T^{2\alpha-1}}{(2\alpha - 1)\Gamma(\alpha)} + \frac{\|F\|_{L_\infty}^2 \Gamma(\alpha) T^{1-\alpha}}{\kappa_\alpha \Gamma(2\alpha - 1)} \right).
$$

**Proof.** For notational convenience, set $z_m(t) := \partial_t^{1-\alpha} v_m(t) \in W_m$. Taking the inner product of both sides of (33) with $z_m(t)$ and integrating by parts with respect to $x$, we obtain

$$
\langle v'_m, z_m \rangle + \kappa_\alpha \|\nabla z_m\|^2 = \langle g(t), z_m \rangle + \langle F(t) z_m, \nabla z_m \rangle - \langle \nabla \cdot \left( F(t) u_{0m} \right), z_m \rangle \omega_\alpha(t). \tag{36}
$$
By the Cauchy–Schwarz and arithmetic–geometric mean inequalities, one has
\[
| \langle \mathbf{F}(t) z_m, \nabla z_m \rangle | \leq \| \mathbf{F} \|_\infty \| z_m \| \| \nabla z_m \| \leq \frac{\kappa_\alpha}{2} \| \nabla z_m \|^2 + \frac{\| \mathbf{F} \|^2_\infty}{2\kappa_\alpha} \| z_m \|^2
\]
and, using Lemma 3.7,
\[
\langle \nabla \cdot (\mathbf{F}(t) u_{0m}) - \kappa_\alpha \Delta u_{0m}, z_m \rangle \leq \| \mathbf{F} \|_{1,\infty} \| u_{0m} \|_{\dot{H}^1(\Omega)} + \kappa_\alpha \| \Delta u_{0m} \| \| z_m \|.
\]
Substituting these bounds into (36) and then applying Lemma 6.2, we obtain
\[
\langle \nu'_m, z_m \rangle + \frac{\kappa_\alpha}{2} \| \nabla z_m \|^2 \leq \| g(t) \| \| z_m \| + \| \mathbf{F} \|^2_\infty \| z_m \| + \frac{\| \mathbf{F} \|^2_\infty}{2\kappa_\alpha} \| z_m \|^2 + \sqrt{C_6} \| u_0 \|_{\dot{H}^2(\Omega)} \| z_m \| \omega_\alpha(t).
\]
But \( \nu_m(0) = 0 \), so \( z_m = \partial_t^{1-\alpha} v_m = C \partial_t^{1-\alpha} v_m = J^\alpha(v'_m) \) and thus \( \langle \nu'_m, z_m \rangle = \langle v'_m, J^\alpha(v'_m) \rangle \). Applying \( J^\alpha \) to both sides of (37) and invoking Lemma 3.2 to handle the first term, we get
\[
\| z_m \|^2 + \kappa_\alpha J^\alpha (\| \nabla z_m \|^2) \leq 2 J^\alpha (\| g \| \| z_m \|) + \left( \frac{\| \mathbf{F} \|^2_\infty}{\kappa_\alpha} J^\alpha (\| z_m \|^2) \right)
+ 2 \sqrt{C_6} \| u_0 \|_{\dot{H}^2(\Omega)} J^\alpha(\| z_m \| \omega_\alpha) \quad \text{for } 0 < t \leq T.
\]
By the Cauchy–Schwarz and arithmetic–geometric mean inequalities,
\[
2 J^\alpha (\| g \| \| z_m \|)(t) = 2 \int_{s=0}^{t} (t-s)^{\alpha-1} \frac{\Gamma(\alpha)}{t} \| g(s) \| \| z_m(s) \| ds
\leq 2 \left( \int_{s=0}^{t} |g(s)|^2 ds \right)^{1/2} \left( \int_{s=0}^{t} \frac{(t-s)^{2\alpha-2}}{\Gamma(\alpha)^2} \| z_m(s) \|^2 ds \right)^{1/2}
\leq \int_{s=0}^{t} \| g(s) \|^2 ds + \frac{\Gamma(2\alpha-1)}{\Gamma(\alpha)^2} \int_{s=0}^{t} \omega_{2\alpha-1}(t-s) \| z_m(s) \|^2 ds,
\]
and if \( 0 \leq s < t \), then \( (t-s)^{\alpha-1} = (t-s)^{1-\alpha} (t-s)^{2\alpha-2} \leq t^{1-\alpha} (t-s)^{2\alpha-2} \), so
\[
\frac{\| \mathbf{F} \|^2_\infty}{\kappa_\alpha} J^\alpha (\| z_m \|^2)(t) \leq \left( \frac{\| \mathbf{F} \|^2_\infty}{\kappa_\alpha \Gamma(\alpha)} \right)^{1-\alpha} \int_{s=0}^{t} (t-s)^{2\alpha-2} \| z_m(s) \|^2 ds.
\]
For the final term in (38), we have
\[
2 \sqrt{C_6} \| u_0 \|_{\dot{H}^2(\Omega)} J^\alpha(\| z_m \| \omega_\alpha) \leq C_6 \| u_0 \|_{\dot{H}^2(\Omega)} + \left[ J^\alpha(\| z_m \| \omega_\alpha) \right]^2
\]
with
\[
[J^\alpha(\| z_m \| \omega_\alpha)]^2 = \left( \int_{0}^{t} \omega_\alpha(t-s) \| z_m(s) \| \omega_\alpha(s) ds \right)^2
\leq \left( \int_{0}^{t} \omega_\alpha(s)^2 ds \right) \left( \int_{0}^{t} (t-s)^{2\alpha-2} \| z_m(s) \|^2 ds \right)
\leq \frac{\Gamma(2\alpha-1)t^{2\alpha-1}}{(2\alpha-1)^2} \int_{0}^{t} \omega_{2\alpha-1}(t-s) \| z_m(s) \|^2 ds.
\]
Hence, (38) yields
\[
\| z_m(t) \|^2 + \kappa_\alpha J^\alpha (\| \nabla z_m \|^2) (t) \leq C_6 \| u_0 \|_{\dot{H}^2(\Omega)} + \int_{0}^{t} \| g(s) \|^2 ds
+ C_7 \int_{0}^{t} \omega_{2\alpha-1}(t-s) \| z_m(s) \|^2 ds \quad \text{for } 0 < t \leq T.
\]
Discard the $\kappa_n$ term and then apply the fractional Gronwall inequality (Lemma 3.1) to get (34). Finally, after substituting the bound (34) into the right-hand side of (39), it is straightforward to deduce (35).

The next corollary follows easily from Lemma 6.3.

**Corollary 6.4.**

\[
\max_{0 \leq t \leq T} \| \partial_t^{1-\alpha} v_m(t) \|^2 \leq C_8 \left[ \| u_0 \|^2_{H^2(\Omega)} + \| g \|^2_{L^2} \right],
\]

\[
\| \partial_t^{1-\alpha} \nabla v_m \|^2_{L^2} \leq C_9 \left[ \| u_0 \|^2_{H^2(\Omega)} + \| g \|^2_{L^2} \right].
\]

Here, for $i = 8, 9$, the constants $C_i = C_i(\alpha, T, \kappa, \| F \|_{1, \infty})$ blow up as $\alpha \to (1/2)^+$ but are bounded as $\alpha \to 1^-$.  

Corollary 6.4 implies an $L^2(\Omega)$ bound on $v_m(t)$, which we give in Corollary 6.5.

**Corollary 6.5.** With $C_8$ as in Corollary 6.4, one has

\[
\| v_m(t) \|^2 \leq C_8 \omega_2^{-1/2} \left( \| u_0 \|^2_{H^2(\Omega)} + \| g \|^2_{L^2} \right) \quad \text{for } 0 \leq t \leq T, 
\]

and

\[
\| v_m \|^2_{L^2} \leq \frac{C_8 T^{2-\alpha}}{\Gamma(2-\alpha)} \left[ \| u_0 \|^2_{H^2(\Omega)} + \| g \|^2_{L^2} \right].
\]

**Proof.** As $u_m(t)$ is absolutely continuous, we have

\[
v_m(t) = (J^{1/2} u_m'(t)) = J^{1-\alpha}(J^{\alpha} u'_m(t)),
\]

where we used [2, Theorem 2.2]. Thus [2, Theorem 2.22] can be invoked, which yields

\[
v_m(t) = J^{1-\alpha} \partial_t^{1-\alpha} (u_m(t) - u_m(0)) \quad \text{for almost all } t.
\]

Set $z_m(t) = \partial_t^{1-\alpha} v_m(t)$, so $v_m(t) = J^{1-\alpha} z_m(t)$. Now Lemma 3.4 and Corollary 6.4 give

\[
\| v_m(t) \|^2 = \| J^{1-\alpha} z_m(t) \|^2 \leq \omega_2^{-1/2} \left( \int_0^t \omega_1^{-\alpha}(t-s) \| z_m(s) \|^2 \, ds \right)
\]

\[
\leq \omega_2^{-1/2} \omega_1^{-\alpha} C_8 \left[ \| u_0 \|^2_{H^2(\Omega)} + \| g \|^2_{L^2(0,T;L^2)} \right] \int_0^t \omega_1^{-\alpha}(t-s) \, ds
\]

\[
= C_8 \left( \| u_0 \|^2_{H^2(\Omega)} + \| g \|^2_{L^2(0,T;L^2)} \right) \omega_2^{-1/2}. \omega_1^{-\alpha}(t).
\]

As $u_m(t)$ is continuous, the inequality (40a) is valid for all $t$.

Next, using (40a) and the semigroup property $\omega_\alpha \ast \omega_\beta = \omega_{\alpha+\beta}$, we get

\[
\| v_m \|^2_{L^2} = \max_{0 \leq t \leq T} \int_0^t \omega_\alpha(t-s) \| v_m(s) \|^2 \, ds
\]

\[
\leq C_8 \left( \| u_0 \|^2_{H^2(\Omega)} + \| g \|^2_{L^2(0,T;L^2)} \right) \max_{0 \leq t \leq T} \int_0^t \omega_\alpha(t-s) \omega_2^{-1/2}(s) \, ds
\]

\[
\leq C_8 \left( \| u_0 \|^2_{H^2(\Omega)} + \| g \|^2_{L^2(0,T;L^2)} \right) \left( \max_{0 \leq t \leq T} \omega_2^{-1/2}(t) \right),
\]

which gives (40b).

In the next lemma, we also provide upper bounds for $\{v_m\}_m$ in $W^{1,2}(0,T;L^2) \cap L^2(0,T;H^2)$ and $\{\partial_t^{1-\alpha} \Delta v_m\}_m$ in $L^2(0,T;L^2)$. Recall that the constant $\rho_\alpha > 0$ was defined in Lemma 3.6.
Lemma 6.6. Let \( m \) be a positive integer. Let \( v_m(t) \) be the absolutely continuous solution of (33) that is guaranteed by Lemma 4.1. Then for almost all \( t \in [0,T] \), one has

\[
\|\nabla v_m(t)\|^2 + \kappa_\alpha \rho_\alpha t^{\alpha-1} \int_0^t \|\Delta v_m\|^2 \, ds \leq \frac{1^{1-\alpha}}{\kappa_\alpha \rho_\alpha} \left[ (C_{10} + C_{11}) \|u_0\|_{H^2(\Omega)}^2 + C_{10} \|g\|_{L^2}^2 \right],
\]

(41)

and

\[
\int_0^t \|v_m'(t)\|^2 \, dt \leq (C_{10} + C_{11}) C_R^2 \|u_0\|_{H^2(\Omega)}^2 + C_{10} \|g\|_{L^2}^2
\]

(42)

where

\[
C_{10} := 3 \left[ 1 + \|F\|^2_{L^\infty(\Omega)} (C_S T + C_S \Gamma(\alpha) T^{1-\alpha}) \right], \quad C_{11} := \frac{6 C_R^2 (\kappa_\alpha^2 + \|F\|^2_{L^\infty(\Omega)})}{(2\alpha - 1) \Gamma(\alpha)^2} T^{2\alpha-1}.
\]

Proof. Take the inner product of both sides of (33) with \(-\Delta v_m \in W_m\) and integrate by parts with respect to \( x \) to get

\[
\frac{1}{2} \frac{d}{dt} \|v_m\|^2 + \kappa_\alpha (\partial_t^{1-\alpha} \Delta v_m, \Delta v_m) = \langle \nabla \cdot (F(t) \partial_t^{1-\alpha} v_m), \Delta v_m \rangle - \langle g(t), \Delta v_m \rangle - \langle \kappa_\alpha \Delta u_{0m} - \nabla \cdot (F(t) u_{0m}), \Delta v_m \rangle \omega_\alpha(t).
\]

Integrating in time and noting that, by Lemma 3.6,

\[
\rho_\alpha t^{\alpha-1} \int_0^t \|\Delta v_m\|^2 \, ds \leq \int_0^t (\partial_t^{1-\alpha} \Delta v_m, \Delta v_m) \, ds,
\]

we obtain

\[
\frac{1}{2} \|v_m(t)\|^2 + \kappa_\alpha \rho_\alpha t^{\alpha-1} \int_0^t \|\Delta v_m\|^2 \, ds \leq 3 \epsilon \int_0^t \|\Delta v_m\|^2 \, ds
\]

\[
+ \frac{1}{4 \epsilon} \int_0^t \left[ \|\nabla \cdot (F(s) \partial_s^{1-\alpha} v_m)\|^2 + \|g(s)\|^2 + \|\kappa_\alpha \Delta u_{0m} - \nabla \cdot (F(s) u_{0m})\|^2 \omega_\alpha(s)^2 \right] \, ds,
\]

with a free parameter \( \epsilon > 0 \). Choosing \( \epsilon = \kappa_\alpha \rho_\alpha t^{\alpha-1}/6 \) and recalling Lemma 3.7 yields

\[
\|v_m(t)\|^2 + \kappa_\alpha \rho_\alpha t^{\alpha-1} \int_0^t \|\Delta v_m\|^2 \, ds
\]

\[
\leq \frac{3}{\kappa_\alpha \rho_\alpha t^{\alpha-1}} \left[ \int_0^t \|g(s)\|^2 + 2 \kappa_\alpha \|\Delta u_{0m}\|^2 \omega_\alpha(s)^2 \right] \, ds
\]

\[
+ \frac{3 \|F\|^2_{L^\infty(\Omega)}}{\kappa_\alpha \rho_\alpha t^{\alpha-1}} \left[ \int_0^t \left( \|\partial_s^{1-\alpha} v_m\|^2_{H^1(\Omega)} + 2 \|u_{0m}\|^2_{H^1(\Omega)} \omega_\alpha(s)^2 \right) \, ds \right]
\]

\[
\leq \frac{t^{1-\alpha}}{\kappa_\alpha \rho_\alpha} \left( C_{11} \|u_0\|_{H^2(\Omega)}^2 + 3 \|g\|^2_{L^2} + 3 \|F\|^2_{L^\infty(\Omega)} \int_0^t \|\partial_s^{1-\alpha} v_m\|^2_{H^1(\Omega)} \, ds \right),
\]
by Lemma 6.2. Invoking Corollary 6.4, we have
\[ \int_0^t ||\partial_t^{1-\alpha} v_m||^2_{H^1(\Omega)} \leq \int_0^t \left( ||\partial_s^{1-\alpha} v_m||^2 + \frac{\omega_\alpha(t-s)}{\omega_\alpha(t)} ||\partial_s^{1-\alpha} \nabla v_m||^2 \right) \, ds \]
\[ \leq \left( C_8 t + C_9 \Gamma(\alpha) t^{1-\alpha} \right) \left( ||u_0||_{H^2(\Omega)}^2 + ||g||_{L^2}^2 \right), \]  
(44)
and the bound (41) follows.

In a similar fashion, we next take the inner product of both sides of (33) with \( v_m' \in W_n \) and integrate by parts with respect to \( x \) to obtain
\[ \|v_m'(t)\|^2 + \kappa_\alpha \langle J^\alpha \nabla v_m', \nabla v_m' \rangle \]
\[ \leq -\langle \nabla \cdot (F(t)\partial_t^{1-\alpha} v_m), v_m' \rangle + \langle g(t), v_m' \rangle + \langle \nabla \cdot (F(t)u_0), \partial_t^{1-\alpha} v_m \rangle \omega_\alpha(t) \]
\[ \leq 3\epsilon \|v_m'(t)\|^2 + \frac{1}{4\epsilon} \left( \|g(t)\|^2 + \|\nabla \cdot (F(t)\partial_t^{1-\alpha} v_m)\|^2 \right) \]
\[ + \|\nabla \cdot (F(t)u_0) - \kappa_\alpha \Delta u_0 \|^2 \omega_\alpha^2(t). \]
Choosing \( \epsilon = 1/6 \) and invoking Lemma 3.7 gives
\[ \|v_m'(t)\|^2 + 2\kappa_\alpha \langle J^\alpha \nabla v_m', \nabla v_m' \rangle \leq 3\|g(t)\|^2 + 3\|F\|_{L^\infty} \|\partial_t^{1-\alpha} v_m\|_{L^2(\Omega)}^2 \]
\[ + 6(\kappa_\alpha^2 + \|F\|_{L^\infty}^2) \|u_0\|_{L^2(\Omega)} \|u_0\|_{H^2(\Omega)} \omega_\alpha^2(t). \]
Integrating both sides of the inequality in time and invoking Lemma 3.2, we deduce that
\[ \int_0^t ||v_m'||^2 \, ds \leq 3\|g\|_{L^2(0,T;L^2)}^2 + 3\|F\|_{L^\infty} \|\partial_t^{1-\alpha} v_m\|_{L^2(0,T;H^1(\Omega))^2}^2 \]
\[ + C_{11} \|u_0\|_{H^2(\Omega)}^2. \]  
(45)
The second result (42) now follows from (44), (45) and Lemma 6.2.

Using similar arguments, take the inner product of both sides of (33) with
\[ -\partial_t^{1-\alpha} \Delta v_m \in W_m, \]
integrate by parts with respect to \( x \) and note that \( \partial_t^{1-\alpha} \Delta v_m = J^\alpha \Delta v_m' \) to obtain
\[ \langle \nabla v_m', J^\alpha \nabla v_m' \rangle + \kappa_\alpha \|\partial_t^{1-\alpha} \Delta v_m\|^2 \]
\[ = \langle \nabla \cdot (F(t)\partial_t^{1-\alpha} v_m), \partial_t^{1-\alpha} \Delta v_m \rangle - \langle g(t), \partial_t^{1-\alpha} \Delta v_m \rangle \]
\[ - \langle \kappa_\alpha \Delta u_0 - \nabla \cdot (F(t)u_0), \partial_t^{1-\alpha} \Delta v_m \rangle \omega_\alpha(t) \]
\[ \leq \frac{\kappa_\alpha}{2} \|\partial_t^{1-\alpha} \Delta v_m\|^2 + \frac{3}{2\kappa_\alpha} \left( \|\nabla \cdot (F(t)\partial_t^{1-\alpha} v_m)\|^2 + \|g(t)\|^2 \right) \]
\[ + \|\kappa_\alpha \Delta u_0 - \nabla \cdot (F u_0) \|^2 \omega_\alpha^2(t). \]
Now integrate in time, invoking Lemma 3.2 and using (44), to deduce that
\[ \int_0^t ||\partial_t^{1-\alpha} \Delta v_m(s)||^2 \, ds \leq \frac{3}{\kappa_\alpha^2} \int_0^t \left( \|\nabla \cdot (F(s)\partial_s^{1-\alpha} v_m)\|^2 + \|g(s)\|^2 \right) \]
\[ + \|\kappa_\alpha \Delta u_0 - \nabla \cdot (F u_0) \|^2 \omega_\alpha^2(s) \, ds \]
\[ \leq \frac{3}{\kappa_\alpha^2} \int_0^t \left( \|g(s)\|^2 + 2\kappa_\alpha^2 \|\Delta u_0\|^2 \omega_\alpha^2(s) \right) \, ds \]
\[ + \frac{3\|F\|_{L^\infty}^2}{\kappa_\alpha^2} \int_0^t \left( \|\partial_s^{1-\alpha} v_m\|_{H^2(\Omega)}^2 + 2\|u_0\|_{H^2(\Omega)}^2 \omega_\alpha^2(s) \right) \, ds, \]
and (43) now follows by (44) and Lemma 6.2, which completes the proof of the lemma. \(\square\)

Inequality (44) may also be derived (with a different constant factor) by applying (19) to (43).

We remark that the function \(\alpha \mapsto \rho_\alpha\) is monotone increasing for \(\alpha \in (0, 1)\), with \(\rho_\alpha \to 1\) as \(\alpha \to 1\). Thus, \(\rho_{1/2} < \rho_\alpha < 1\) for \(1/2 < \alpha < 1\), with \(\rho_{1/2} = \sqrt{2\pi/27} = 0.48240\ldots\)

6.2. The classical solution. In this section, by using the method of compactness, we show that there is a subsequence of \(\{v_m\}_m\) such that the sum of its limit and the initial data satisfies equation (1) almost everywhere.

**Theorem 6.7.** Assume that \(\alpha \in (1/2, 1)\), \(u_0 \in H^2(\Omega) \cap H_0^1(\Omega)\), \(F \in W^{1,\infty}(0, T) \times \Omega\) and \(g \in L^2(0, T; L^2)\). Then there exists a unique classical solution of (1), in the sense of Definition 2.4, such that

\[
\sup_{0 \leq t \leq T} \|u(t)\|_{H^2(\Omega)}^2 + \|u'(t)\|_{L^2}^2 + \|\partial_t^{1-\alpha} u\|_{L^2(0, T; H^2)}^2 \leq C_{12} \left[\|u_0\|_{H^2(\Omega)} + \|g\|_{L^2}^2\right],
\]

where

\[C_{12} := \frac{C_S T^{2-\alpha}}{\Gamma(2-\alpha)} + (C_{10} + C_{11}) \left(1 + C_R^2 + \frac{T^{1-\alpha}}{\kappa_\alpha \rho_\alpha} + 1 \frac{1}{\kappa_\alpha^2}\right).
\]

**Proof.** From Corollary 6.5 and Lemma 6.6 we obtain

\[
\sup_{0 \leq t \leq T} \|v_m(t)\|_{H^2(\Omega)}^2 + \|v'_m(t)\|_{L^2}^2 + \|\partial_t^{1-\alpha} v_m\|_{L^2(0, T; H^2)}^2 \leq C_{12} \left[\|u_0\|_{H^2(\Omega)} + \|g\|_{L^2}^2\right],
\]

which shows that the sequence \(\{v_m\}_m\) is bounded in \(L^\infty(0, T; H^2) \cap L^2(0, T; H^2 \cap H_0^1)\) and that the sequence \(\{v'_m\}_m\) is bounded in \(L^2(0, T; L^2)\). Since the embeddings \(H^2 \hookrightarrow H^1 \hookrightarrow L^2\) are compact, it follows from Lemma 3.8 that there exists a subsequence of \(\{v_m\}_m\) (still denoted by \(\{v_m\}_m\)) such that

\[v_m \to v \text{ strongly in } C([0, T]; L^2) \cap L^2(0, T; H^1).
\]

Furthermore, from the upper bounds of \(\{v_m\}_m\) we have

\[v_m \to v \text{ weakly in } L^\infty(0, T; H^1) \cap L^2(0, T; H^2 \cap H_0^1),
\]

and \(v'_m \to v'\) weakly in \(L^2(0, T; L^2)\). \(\square\)

By virtue of Lemma 3.5, the strong convergence in (48) implies that

\[J^\alpha v_m \to J^\alpha v \text{ strongly in } C([0, T]; L^2) \cap L^2(0, T; H^1).
\]

This, together with Corollary 6.4 and (43), yields

\[\partial_t J^\alpha v_m \to \partial_t J^\alpha v \text{ weakly in } L^\infty(0, T; L^2) \cap L^2(0, T; H^2).
\]

Multiplying both sides of (33) by a test function \(\xi \in L^2(0, T; L^2)\), integrating over \((0, T) \times \Omega\) and noting that \(\Pi_m\) is a self-adjoint operator on \(L^2(\Omega)\), we deduce that

\[
\langle v'_m, \xi \rangle_{L^2(0, T; L^2)} - \kappa_\alpha \langle \partial_t^{1-\alpha} \Delta v_m, \xi \rangle_{L^2(0, T; L^2)} = \langle \nabla \cdot (F \partial_t^{1-\alpha} u_m), \Pi_m \xi \rangle_{L^2(0, T; L^2)}.
\]

Now let \(m \to \infty\) in this equation and recall (49) and (50). We get

\[
\langle v'_m, \xi \rangle_{L^2(0, T; L^2)} - \kappa_\alpha \langle \partial_t^{1-\alpha} \Delta u, \xi \rangle_{L^2(0, T; L^2)} = \langle g, \xi \rangle_{L^2(0, T; L^2)}
\]

and

\[
\langle v'_m, \xi \rangle_{L^2(0, T; L^2)} = \langle g, \xi \rangle_{L^2(0, T; L^2)}
\]

which is the desired result.
for all $\xi \in L^2(0,T;L^2)$, where $u := v + u_0$. From (48)–(50), we have
\[ u' \in L^2(0,T;L^2) \quad \text{and} \quad \partial_t^{1-\alpha} u \in L^2(0,T;H^2). \]
Hence, it follows from (51) that $u$ satisfies (1) a.e. in $(0,T) \times \Omega$.

Taking the limit as $m \to \infty$ in (47), we obtain (46). The uniqueness of the solution $u$ follows from (46), which completes the proof of the theorem. \qed

**Remark 6.8.** It follows from the uniqueness in Theorems 5.3 and 6.7 that the mild solution will become the classical solution when $\alpha \in (1/2,1)$ and $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Furthermore, the continuous dependence of both the mild and classical solutions on the initial data $u_0$ follows from (20) and (46).

7. **Regularity of the classical solution.** Recall that $1/2 < \alpha < 1$ and that in general $C=C(\Omega,\kappa,\mathbf{F},T)$. From Lemma 7.2 onwards, we allow $C=C(\Omega,\kappa,\mathbf{F},T,q)$, where $q$ appears in the statements of our results below.

From Theorem 6.7, for almost every $(t,x) \in (0,T) \times \Omega$ the solution $u(t,x)$ satisfies (1). Using the identity $\partial_t^{1-\alpha} u = (J^\alpha u)'(t) = (J^\alpha u)(t) + u(0)\omega_\alpha(t)$, we rewrite (1) as
\[ u' - \nabla \cdot (\kappa \nabla J^\alpha u - \mathbf{F} J^\alpha u) = g(t) + \nabla \cdot [\kappa \nabla u_0 - \mathbf{F}(t)u_0] \omega_\alpha(t). \quad (52) \]
From this equation and the fact that $J^\alpha \phi(0) = 0$ for any function $\phi \in L^2(0,T;L^2)$, we deduce that $u'(t) = O(t^{\alpha-1})$ when $t$ is close to 0. By letting $z(t,x) := tu'(t,x)$, we have $z(0) = 0$. The regularity of $z$ is examined in the following lemma.

**Lemma 7.1.** Assume that $\int_0^T \|tg'(t)\|^2 \, dt$ is finite. Then, the function $z$ defined above satisfies
\[ \sup_{0 \leq t \leq T} \|z(t)\|_{H^1(\Omega)}^2 + \|z'\|_{L^2}^2 + \|\partial_t^{1-\alpha} z\|_{L^2(0,T;H^2)} \leq C_{13}(\|u_0\|_{H^2(\Omega)}^2 + \|g\|_{L^2}^2 + \|g_1\|_{L^2}^2) \]
for some constant $C_{13}$.

**Proof.** For any $t > 0$, multiplying both sides of (52) by $t$ and using the elementary identity
\[ t(J^\alpha u')(t) = (J^\alpha z)(t) + \alpha (J^{\alpha+1} u')(t) = (J^\alpha z)(t) + \alpha ((J^\alpha u)(t) - u_0 \omega_{\alpha+1}(t)) \]
\[ = (J^\alpha z)(t) + \alpha (J^\alpha u)(t) - u_0 t \omega_\alpha(t), \]
we obtain a differential equation for $z$:
\[ z - \nabla \cdot (J^\alpha \kappa \nabla z - \mathbf{F} J^\alpha z) = tg(t) + \alpha \nabla \cdot (\kappa \nabla J^\alpha u - \mathbf{F} J^\alpha u). \]
Differentiating both sides of this equation with respect to $t$ and noting that $J^\alpha z' = \partial_t^{1-\alpha} z$, we have
\[ z' - \nabla \cdot (\partial_t^{1-\alpha} \kappa \nabla z - \mathbf{F} \partial_t^{1-\alpha} z) = G(t,x), \]
where
\[ G := g + tg' + \alpha \nabla \cdot (\kappa \nabla \partial_t^{1-\alpha} u - \mathbf{F}'(t) J^\alpha u - \mathbf{F}(t) \partial_t^{1-\alpha} u). \]
Applying Lemma 3.7 and letting $g_1(t) = tg'(t)$, we find that
\[ \|G\|^2_{L^2} \leq 4 \left( \|g\|^2_{L^2} + \|g_1\|^2_{L^2} + \alpha^2 (\kappa + \|\mathbf{F}\|_{1,\infty})^2 \|\partial_t^{1-\alpha} u\|^2_{L^2(0,T;H^2)} + \alpha^2 \|\mathbf{F}'\|^2_{1,\infty} \|J^\alpha u\|^2_{L^2(0,T,H^1)} \right), \]
with Lemma 5.1 and Theorem 6.7 implying that
\[ \|G\|^2_{L^2} \leq C(\|u_0\|^2_{H^2(\Omega)} + \|g\|^2_{L^2} + \|g_1\|^2_{L^2}) \quad \text{for some constant } C. \]
Thus, applying Theorem 6.7 to equation (55) with initial data \( z(0) = 0 \), we deduce the bound (53).

From Theorem 6.7, for almost every \((t, x) \in (0, T) \times \Omega\) we have the identity
\[
    u_t - \nabla \cdot (\partial_t^{1-\alpha} \kappa_\alpha \nabla u) = f,
\]
where \( f := g - \nabla \cdot (F \partial_t^{1-\alpha} u) \in L^2(0, T; H^1) \). The regularity of solutions to problem (56) subject to the initial condition \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) was studied in [9]. In order to apply [9, Theorem 5.7], we need at least an upper bound for \( q \) of \( \alpha \). Thus, applying Theorem 6.7 to equation (55) with initial data \( z \) where we used the identities (54), we note first that
\[
    \int_0^t s^2 \| f(q)(s) \|^2 \, ds \leq C \left( \| u_0 \|^2_{H^2(\Omega)} + \sum_{j=0}^q \int_0^t s^{2j} \| g^{(j)}(s) \|^2 \, ds \right).
\]

**Lemma 7.2.** Let \( u \) be the solution of (1) and \( f := g - \nabla \cdot (F \partial_t^{1-\alpha} u) \). Then, for \( q \in \{0, 1, 2, \ldots, \} \), \( F \in W^{q, \infty}(0, T; L^2(\Omega)) \), and for any \( t \in (0, T) \), there is a constant \( C = C(\Omega, \kappa_\alpha, F, q) \) such that
\[
    \int_0^t s^2 \| f(q)(s) \|^2 \, ds \leq C \left( \| u_0 \|^2_{H^2(\Omega)} + \sum_{j=0}^q \int_0^t s^{2j} \| g^{(j)}(s) \|^2 \, ds \right).
\]

**Proof.** Inequality (57) holds for \( q = 0 \) by virtue of (46) (with \( t \) playing the role of \( T \)) because
\[
    \| f(t) \|^2 \leq C(\| g(t) \|^2 + \| \partial_t^{1-\alpha} u \|^2_{H^1(\Omega)}).
\]

For the case \( q = 1 \), we note first that
\[
    t^2 \| f'(t) \|^2 = t^2 \| g'(t) - \nabla \cdot (F'(t) \partial_t^{1-\alpha} u + F(t) \partial_t^{2-\alpha} u) \|^2
\]
\[
    \leq C t^2 \left( \| g'(t) \|^2 + \| \partial_t^{1-\alpha} u \|^2_{H^1(\Omega)} + \| \partial_t^{2-\alpha} u \|^2_{H^2(\Omega)} + \| \partial_t^{3-\alpha} u \|^2_{H^1(\Omega)} \right).
\]

By (54) we have \((J^\alpha z)(t) = t(J^\alpha u')(t) - \alpha(J^{\alpha+1} u')(t), \) and differentiating with respect to \( t \) gives
\[
    \partial_t^{1-\alpha} z = t(J^\alpha u')'(t) - (\alpha - 1)(J^{\alpha+1} u')(t) = t\partial_t^{2-\alpha} u - (\alpha - 1)\partial_t^{1-\alpha} u,
\]
where we used the identities \((J^\alpha u')(t) = \partial_t^{1-\alpha} u - u_0 \omega_\alpha(t) \) and \((\alpha - 1)\omega_\alpha(t) = t\omega_{\alpha-1}(t) \). Thus,
\[
    t\partial_t^{2-\alpha} u = \partial_t^{1-\alpha} z + (\alpha - 1)\partial_t^{1-\alpha} u.
\]

Hence, by Theorem 6.7 and Lemma 7.1 (with \( t \) again playing the role of \( T \)),
\[
    \int_0^t s^2 \| \partial_t^{2-\alpha} u \|^2_{H^2(\Omega)} \, ds \leq C \left( \| u_0 \|^2_{H^2(\Omega)} + \int_0^t \left[ \| g(s) \|^2 + s^2 \| g'(s) \|^2 \right] \, ds \right),
\]

implying that the desired inequality (57) holds for \( q = 1 \).

Multiply both sides of (58) by \( t \) and then differentiate with respect to \( t \), obtaining
\[
    t^2 \partial_t^{3-\alpha} u = \partial_t^{1-\alpha} z + t\partial_t^{2-\alpha} z + (\alpha - 1)\partial_t^{1-\alpha} u + (\alpha - 3) t \partial_t^{2-\alpha} u.
\]

Since \( z \) satisfies (55) — an equation similar to (1a) but with a different source \( \tilde{G} \) and with \( \omega(0) = 0 \) — we get an estimate for \( z \) corresponding to (59):
\[
    \int_0^t s^2 \| \partial_t^{2-\alpha} z \|^2_{H^2(\Omega)} \, ds \leq C \int_0^t \left[ \| \tilde{G}(s) \|^2 + s^2 \| \tilde{G}'(s) \|^2 \right] \, ds.
\]

This inequality, together with (46), (53), (59) and (60), yields
\[
    \int_0^t s^4 \| \partial_t^{3-\alpha} u \|^2_{H^2(\Omega)} \, ds \leq C \left( \| u_0 \|^2_{H^2(\Omega)} + \int_0^t \left[ \| g(s) \|^2 + s^2 \| g'(s) \|^2 + s^4 \| g''(s) \|^2 \right] \, ds \right),
\]

(61)
which implies the desired inequality (57) for \( q = 2 \).

The general case follows by iterating the arguments above; cf. [12].

We can now prove regularity estimates for the classical solution \( u \).

**Theorem 7.3.** Let \( g_j(t) := t^j g^{(j)}(t) \) for \( j = 1, 2, 3, \ldots \) For \( q \in \{1, 2, 3, \ldots \} \), \( F \in W^{q, \infty}(0, T; L^2(\Omega)) \) and for any \( t \in (0, T) \),

\[
t^q \| \Delta u^{(q)}(t) \| \leq C t^{-\alpha (\alpha - 1)/2} \left( \| u_0 \|_{H^2(\Omega)} + \sum_{j=0}^{q+1} \| g_j \|_{L^2} \right)
\]

and

\[
t^q \| u^{(q)}(t) \| \leq C t^{1/2} \left( \| u_0 \|_{H^2(\Omega)} + \sum_{j=0}^{q} \| g_j \|_{L^2} \right).
\]

**Proof.** By (56), it follows from [9, Theorem 4.4] with \( r = 2 \) and \( \nu = \alpha \), and from [9, Theorem 5.6] with \( r = 0 \), \( \mu = 2 \) and \( \nu = \alpha \), that

\[
t^q \| \Delta u^{(q)}(t) \| \leq C \left( \| u_0 \|_{H^2(\Omega)} + t^{-\alpha} \sum_{j=0}^{q+1} \int_0^t s^j \| f^{(j)}(s) \| ds \right).
\]

Similarly, from [9, Theorem 4.4] with \( r = 2 \) and \( \nu = \alpha \), and from [9, Theorem 5.4] with \( r = \mu = 0 \), and

\[
t^q \| u^{(q)}(t) \| \leq C \left( t^\alpha \| u_0 \|_{H^2(\Omega)} + \sum_{j=0}^{q} \int_0^t s^j \| f^{(j)}(s) \| ds \right).
\]

The theorem follows by Lemma 7.2 since

\[
\int_0^t s^j \| f^{(j)}(s) \| ds \leq t^{1/2} \left( \int_0^t s^{2j} \| f^{(j)}(s) \|^2 ds \right)^{1/2}
\]

**Corollary 7.4.** Let \( \eta > 1/2 \). If \( \| g^{(j)}(t) \| \leq Mt^{\eta-1-j} \) for \( 0 \leq j \leq q + 1 \), \( F \in W^{q, \infty}(0, T; L^2(\Omega)) \) and \( t \in (0, T) \), then

\[
t^q \| \Delta u^{(q)}(t) \| \leq C (t^{-\alpha (\alpha - 1)/2} \| u_0 \|_{H^2(\Omega)} + Mt^{\eta-\alpha})
\]

and

\[
t^q \| u^{(q)}(t) \| \leq C (t^{1/2} \| u_0 \|_{H^2(\Omega)} + Mt^\eta).
\]

**Proof.** The assumption on \( g \) ensures that \( \| g_j \| \leq Mt^{\eta-1/2} \).

The alternative and longer analysis in [12, Theorems 12 and 13] shows that these bounds can be improved to

\[
t^q \| \Delta u^{(q)}(t) \| \leq C (\| u_0 \|_{H^2(\Omega)} + Mt^{\eta-\alpha}) \quad \text{and} \quad t^q \| u^{(q)}(t) \| \leq C (t^\alpha \| u_0 \|_{H^2(\Omega)} + Mt^\eta),
\]

for any \( \alpha \in (0, 1) \) and \( \eta > 0 \).
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