Relaxation-based revision operators in description logics

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Abstract
As ontologies and description logics (DLs) reach out to a broader audience, several reasoning services are developed in this context. Belief revision is one of them, of prime importance when knowledge is prone to change and inconsistency. In this paper we address both the generalization of the well-known AGM postulates, and the definition of concrete and well-founded revision operators in different DL families. We introduce a model-theoretic version of the AGM postulates with a general definition of inconsistency, hence enlarging their scope to a wide family of non-classical logics, in particular negation-free DL families. We propose a general framework for defining revision operators based on the notion of relaxation, introduced recently for defining dissimilarity measures between DL concepts. A revision operator in this framework amounts to relax the set of models of the old belief until it reaches the sets of models of the new piece of knowledge. We demonstrate that such a relaxation-based revision operator defines a faithful assignment and satisfies the generalized AGM postulates. Another important contribution concerns the definition of several concrete relaxation operators suited to the syntax of some DLs (\textit{ALC} and its fragments \textit{EL} and \textit{ELU}).

Keywords: Revision in DL, AGM theory, relaxation, retraction.

1 Introduction
Belief revision is at the core of artificial intelligence and philosophy questionings. It is defined as the process of changing an agent belief with a new acquired knowledge. Three change operations are usually considered: expansion, contraction and revision. We focus here on the revision, i.e. the process of adding \textit{consistently} the new belief sets. Belief revision has been intensively studied in classical logics (e.g. propositional logic) mostly under the prism of AGM theory [Alchourrón et al., 1985]. With the growing interest in non-classical logics, such as Horn Logics and Description Logics [Baader, 2003], several attempts to generalize AGM theory, making it compliant to the meta-logical flavor of these logics, have been introduced recently [Flouris et al., 2005], [Delgrande and Peppas, 2015], [Ribeiro et al., 2013], [Ribeiro and Wassermann, 2014].

In this paper, we are interested in defining concrete revision operators in Description Logics (DLs). DLs are now pervasive in many knowledge-based representation systems, such as ontological reasoning, semantic web, scene understanding, cognitive robotics, to mention a few. In all these domains, the expert knowledge is rather a flux evolving through time, requiring hence the definition of rational revision operators. Revision is then a cornerstone in ontology engineering life-cycle where the expert knowledge is prone to change and inconsistency. This paper contributes to the effort of defining such rational revision operators, in line with the recent art of the domain [Qi et al., 2006a], [Qi et al., 2006b], [Flouris et al., 2005], [Flouris et al., 2006]. In Section 2 we discuss the adaptation of AGM theory to non-classical logics, including DLs, and introduce, as a first contribution, a model-theoretic rewriting of AGM postulates. In Section 3 we introduce our general framework of relaxation-based revision operators. As a second contribution, we demonstrate that they satisfy the AGM postulates and lead to a faithful assignment. Our third contribution is detailed in Section 4 by providing concrete theory relaxation operators in different DLs (namely \textit{ALC} and its fragments \textit{EL} and \textit{ELU}). Section 5 positions our contributions with regards to the literature and finally Section 6 draws some conclusions and perspectives.

2 Preliminaries
2.1 Description Logics
Description logics are a family of knowledge representation formalisms (see e.g. [Baader, 2003] for more details). We consider, in this paper, the logic \textit{ALC} and its fragments \textit{EL} and \textit{ELU}. In the following we provide the syntax and semantics of \textit{ALC}, from which those of \textit{EL} and \textit{ELU} are easily deducible. Signatures in DLs are triplets \( (N_C, N_R, I) \) where \( N_C, N_R \) and \( I \) are nonempty pairwise disjoint sets such that elements in \( N_C \), \( N_R \) and \( I \) are concept names, role names and individuals, respectively. Given a signature \( \Sigma \in \text{Sign} \), \text{Sen} \( (\Sigma) \) contains all the sentences of the form \( C \sqsubseteq D \), \( x : C \) and \( (x,y) : r \) where \( x,y \in I \), \( r \in N_R \) and \( C \) is an \textit{ALC}-concept inductively defined from \( N_C \) and binary and unary operators in \( \{\sqcap, \sqcup, \top, \bot\} \) and in \( \{\forall, \forall r, \forall \}, \exists r \), respectively. The set of concept descriptions provided by \( \Sigma \) is denoted by \( C(\Sigma) \). The semantics of concept descriptions is defined using interpretations. An interpretation \( \mathcal{I} \) is a pair \( \mathcal{I} = (\Delta, \cdot) \)
consisting of an interpretation domain $\Delta_T$ and an interpretation function $I$ which maps concepts to subsets of the domain $\Delta_T$ and role names to binary relations on the domain. A concept description $C$ is said to subsume a concept description $D$ (denoted by $C \sqsubseteq D$) if $C^I \subseteq D^I$ holds for every interpretation $I$. Two concepts $C$ and $D$ are equivalent (denoted by $C \equiv D$) if both $C \sqsubseteq D$ and $D \sqsubseteq C$ hold. An interpretation $I$ is a model of a $\Sigma$-sentence (TBox or ABox axiom) if it satisfies this sentence (e.g. $I \models (C \sqsubseteq D)$ iff $C^I \subseteq D^I$).

A DL knowledge base $T$ is a set of $\Sigma$-sentences (i.e. $I \subseteq \text{Sen}(\Sigma)$). An interpretation $I$ is a model of a DL knowledge base $T$ if it satisfies every sentence in $T$. In the following, we use $\text{Mod}(\phi)$ (or $\text{Mod}(T)$) to denote the set of all the models of a $\Sigma$-sentence $\phi$ (or DL knowledge base $T$). A knowledge base is said to be a theory if and only if $T = \text{Cn}(T)$, where $\text{Cn}()$ is the consequence operator defined as: $\text{Cn}(T) = \{ \phi \in \text{Sen}(\Sigma) \mid \forall I \in \text{Mod}(T), I \models_{\Sigma} \phi \}$ and satisfying monotonicity, inclusion and idempotence. Hence DLs can be considered as Tarski logics, i.e. pairs $\langle \Sigma, \text{Cn}() \rangle$.

Classically, consistency of a theory $T$ in DLs is defined as $\text{Mod}(T) \neq \emptyset$. Such a definition raised several issues in adapting revision postulates to Description Logics (see [Flouris et al., 2006b], [Ribeiro et al., 2013b], [Ribeiro and Wassermann, 2014]). We consider in this paper a more general definition of consistency the meaning of which is that there is at least a sentence which is not a semantic consequence: $T \subseteq \text{Sen}(\Sigma)$ is consistent if $\text{Cn}(T) \neq \text{Sen}(\Sigma)$.

### 2.2 AGM theory and Description Logics

AGM theory [Alchourrón et al., 1985] is probably the most influential paradigm in belief revision theory [Gärdenfors, 2003]. It provides, at an abstract logical level, a set of postulates that a revision operator should satisfy so that the old belief is changed minimally and rationally to become consistent with the new one. These postulates require the logic to be closed under negation and usual propositional connectives ($\land, \lor, \implies, \neg$) which prevents its use in many non-classical logics, including DLs. Indeed, many DLs do not allow for negation of concepts (e.g. $\mathcal{EL}$) and a fortiori disjunction between TBox and ABox sentences is not defined in all DLs. Recently, many papers have addressed the adaptation of AGM theory to non-classical logics, e.g. [Flouris et al., 2005], [Ribeiro et al., 2013], [Delgrande and Peppas, 2015], [Ribeiro and Wassermann, 2014]. The first efforts concentrated on the adaptation of contraction postulates, but more recently, [Ribeiro and Wassermann, 2014] discussed the adaptation of revision postulates and introduced new minimality criteria, not necessarily related to the contraction operator, throwing out the need for negation. However, one can find in [Qi et al., 2006a] an attempt to adapt the AGM revision postulates to DL in a model-theoretic way, following the seminal work of [Katsuno and Mendelzon, 1991] that translated the AGM postulates in propositional logic semantics. The translation in [Qi et al., 2006a] is provided with the classical notion of consistency (a theory $T$ is consistent if $\text{Mod}(T) \neq 0$) which is not adequate to revision purposes (see [Flouris et al., 2006] for a discussion). In this paper we consider a model-theoretic translation of AGM postulates, similar to the ones in [Qi et al., 2006b], with the notable difference that consistency is defined through the consequence operator as introduced in the previous section. This translation is in accordance with the postulates as introduced in [Ribeiro and Wassermann, 2014].

Given two knowledge bases $T, T' \subseteq \text{Sen}(\Sigma)$, $T \circ T'$ denotes the revision of the old belief $T$ by the new one $T'$. The model-theoretic translation of AGM postulates writes:

(G1) $\text{Mod}(T \circ T') \subseteq \text{Mod}(T')$.

(G2) If $T \cup T'$ is consistent, then $T \circ T' = T \cup T'$.

(G3) If $T'$ is consistent, then so is $T \circ T'$.

(G4) If $\text{Mod}(T_1) = \text{Mod}(T_1')$ and $\text{Mod}(T_2) = \text{Mod}(T_2')$, then $\text{Mod}(T_1 \circ T_2) = \text{Mod}(T_1' \circ T_2')$.

(G5) $\text{Mod}(T \circ T') \cap \text{Mod}(T'') \subseteq \text{Mod}(T \circ (T' \cup T''))$.

(G6) If $T \cup T' \cup T''$ is consistent, then $T \circ (T' \cup T'') = (T \circ T') \cup T''$.

Besides these postulates, we consider a minimality criterion introduced in [Ribeiro and Wassermann, 2014].

**Relevance** If $\phi \in T \setminus (T \circ T')$, then there exists $X, T \cap (T \setminus T') \subseteq X \subseteq T$, such that $\text{Cn}(X \cup \{\phi\} \neq \text{Sen}(\Sigma)$ and $\text{Cn}(X \cup \{\phi\} \setminus \text{Cn}(\Sigma)) = \text{Cn}(\Sigma)$.

A classical construction in belief theory is to characterize the revision operator in terms of faithful assignments [Katsuno and Mendelzon, 1991] [Grove, 1988]. We provide here a similar representation theorem for the postulates defined above. The proof can be found in [Aguier et al., 2015].

**Definition 1** (Faithful assignment) Let $T \subseteq \text{Sen}(\Sigma)$ be a knowledge base. Let $\preceq_T \subseteq \text{Mod}(\Sigma) \times \text{Mod}(\Sigma)$ be a total pre-order. $\preceq_T$ is a faithful assignment (FA) if the following three conditions are satisfied:

1. If $I, I' \in \text{Mod}(T)$, $I \not\prec_T I'$.

2. For every $I \in \text{Mod}(T)$ and every $I' \in \text{Mod}(\Sigma) \setminus \text{Mod}(T)$, $I \not\prec_T I'$.

3. For every $T' \subseteq \text{Sen}(\Sigma)$, if $\text{Mod}(T) = \text{Mod}(T')$, then $\preceq_T = \preceq_{T'}$.

**Theorem 1** A revision operator $\circ$ satisfies the postulates (G1)-(G6) if and only if for any DL knowledge base $T$, there exists a well-founded (i.e. the min is well defined) FA $\preceq_T$ such that $\text{Mod}(T \circ T') = \text{min}(\text{Mod}(T') \setminus M^*, \preceq_T)$, with $M^* = \{ I \in \text{Mod}(\Sigma) \mid \phi \in \text{Sen}(\Sigma) \mid I \models_{\Sigma} \phi \subseteq \text{Sen}(\Sigma)\}$.

### 3 Relaxation of theories and associated revision operator

The notion of relaxation has been introduced in [Distel et al., 2014a] [Distel et al., 2014b] to define dissimilarity measures between DL concept descriptions. In this paper we generalize this notion to formula relaxation and subsequently to theory relaxation in order to define revision operators.
Definition 2 (Concept Relaxation [Distel et al., 2014c]) A (concept) relaxation is an operator $\rho : C(\Sigma) \to C(\Sigma)$ that satisfies the following two properties for all $C \in C(\Sigma)$:

1. $\rho$ is extensive, i.e., $C \subseteq \rho(C)$,
2. $\rho$ is exhaustive, i.e., $\exists k \in \mathbb{N} : \top \subseteq \rho^k(C)$,

where $\rho^k$ denotes $\rho$ applied $k$ times, and $\rho^0$ is the identity mapping.

Our idea to define revision operators is to relax the set of models of the old belief until it becomes consistent with the new pieces of knowledge. This is illustrated in Figure 1 where theories are represented as sets of their models. Intermediate steps to define the revision operators are then the definition of formula and theory relaxations. The whole scheme of our framework is provided in Figure 2.

Definition 3 (Formula Relaxation) Given a signature $\Sigma \in \text{Sign}$, a $\Sigma$-formula relaxation is a mapping $\rho_\Sigma : \text{Sen}(\Sigma) \to \text{Sen}(\Sigma)$ satisfying the following properties:

- Extensivity: $\forall \phi \in \text{Sen}(\Sigma), \text{Mod}(\phi) \subseteq \text{Mod}(\rho_\Sigma(\phi))$,
- Exhaustivity: $\exists k \in \mathbb{N}, \text{Mod}(\rho^k_\Sigma(\phi)) = \text{Mod}(\Sigma)$, where $\rho^k_\Sigma$ denotes $\rho_\Sigma$ applied $k$ times, and $\rho^0_\Sigma$ is the identity mapping.

Definition 4 ($\Sigma$-theory relaxation) Let $T$ be a theory, $T \subseteq \mathcal{P}(\text{Sen}(\Sigma))$, $\rho : \mathcal{P}(\text{Sen}(\Sigma)) \to \mathcal{P}(\text{Sen}(\Sigma))$ defined as:

$$\rho^K(T) = \bigcup_{\varphi \in T} \rho^K(\varphi).$$

Proposition 1 $\rho^K$ is extensive ($\forall T \subseteq \text{Sen}(\Sigma), \text{Mod}(T) \subseteq \text{Mod}(\rho^K(T))$), and exhaustive ($\exists K \subseteq \mathbb{N}, \text{Mod}(\rho^K(T)) = \text{Mod}(\Sigma)$).

Definition 5 (Definition-based revision) Let $\rho^K$ be a $\Sigma$-theory relaxation. We define the revision operator $\circ : \mathcal{P}(\text{Sen}(\Sigma)) \times \mathcal{P}(\text{Sen}(\Sigma)) \to \mathcal{P}(\text{Sen}(\Sigma))$ as follows:

$$T_1 \circ T_2 = \rho^K(T_1) \cup T_2$$

for a set $K$ such that $\rho^K(T_1) \cup T_1 \cup T_2$ is consistent, and $\forall K'$ s.t. $\rho^K(T_1') \cup T_1' \cup T_2$ is consistent, $\bigcup_{k \in K'} k \leq \sum_{k \in K'} k$, and $T_1 = T_1' \cup T_1'$ (disjoint union) such that:

1. $\text{Ch}(T_1 \cup T_2) = \text{Sen}(\Sigma)$,
2. $\text{Ch}(T_1'' \cup T_2) \neq \text{Sen}(\Sigma)$,
3. $\forall T, T_1' \subseteq T \subseteq T_1, \text{Ch}(T \cup T_2) = \text{Sen}(\Sigma)$.

Partitioning $T_1$ into $T_1'$ and $T_1''$ is not unique and the only constraint is that $T_1''$ is of maximal size. The set $K$ may not be unique either.

Theorem 2 From any $\Sigma$-theory relaxation $\rho^K$ and every knowledge base $T \subseteq \text{Sen}(\Sigma)$, the binary relation $\succeq_T \subseteq \text{Mod}(\Sigma) \times \text{Mod}(\Sigma)$ defined by $I \succeq_T I'$ if:

$$\min_{K \in \mathcal{I} \subseteq \text{Mod}(\rho^K(T))} \sum_{k \in K} k \leq \min_{K' \in \mathcal{I}' \subseteq \text{Mod}(\rho^K'(T))} \sum_{k \in K'} k,$$

is a well-founded faithful assignment such that for every $T' \subseteq \text{Sen}(\Sigma), \text{Mod}(T \circ T') = \text{Mod}(\text{Mod}(T') \setminus M^*, \succeq_T)$.

Proof: By construction, $\succeq_T$ is obviously a total pre-order. Well-foundedness follows from exhaustivity. The two first conditions follow from the fact that $I \in \text{Mod}(T') \iff \min_{K \in \mathcal{I} \subseteq \text{Mod}(\rho^K(T))} \sum_{k \in K} k = 0$. The third one is obvious.

It remains to show that $\text{Mod}(\rho^K(T)) \setminus M^*, \succeq_T = \text{Mod}(\text{Mod}(T') \setminus M^*, \succeq_T)$. To simplify the proof, we suppose that $T \circ T' = \rho^K(T') \cup T'$ (i.e. if $T = T_1 \bigcup T_2$, then $T_2 = \emptyset$), the more general case where $T_2 \neq \emptyset$ being easily obtained from this more simple case.

(i) Let $I \in \text{Mod}(\rho^K(T'))$. By definition of $\rho$, there exists a set $K \subseteq \mathbb{N}$ such that $I \in \text{Mod}(\rho^K(T') \cup T')$, and then $I \in \text{Mod}(T')$. Let $I' \in \text{Mod}(T')$. If $I' \in \text{Mod}(\rho^K(T))$, then $I' \not\succeq_T I$ and $I' \not\succeq_T I$. Otherwise $I \not\in \text{Mod}(\rho^K(T))$ and $\forall K'$ such that $I' \in \text{Mod}(\rho^K(T'))$ we have $\sum_{k \in K'} k \geq \sum_{k \in K} k$, which implies $I \succeq_T I'$. We can hence conclude that $I \in \min(\text{Mod}(T') \setminus M^*, \succeq_T)$.

(ii) Conversely, let $I \in \min(\text{Mod}(T') \setminus M^*, \succeq_T)$. By definition of $\rho'$, there exists a set $K$ of minimal sum such that $\rho^K(T') \cup T'$ is consistent and $T \circ T' = \rho^K(T') \cup T'$. As $I \in \min(\text{Mod}(T') \setminus M^*, \succeq_T)$, this means that, for every $I' \in \text{Mod}(\rho^K(T') \cup T')$, $I' \succeq_T I'$, and then $I \in \text{Mod}(\rho^K(T) \cup T') = I \in \text{Mod}(T \circ T')$.

Proposition 2 The revision in Definition 5 satisfies the Relevance minimalisation criterion.

The proof is direct by setting $X = T''_1$.

4 Concrete theory relaxations in different $ALC$ fragments

In this section, we introduce concrete relaxation operators suited to the syntax of the logic $ALC$, as defined in Section 2.1 and its fragments $EL$ and $ELU$. $EL$-concept description constructors are existential restriction ($\exists$), conjunction ($\land$), and parallel ($\parallel$), while $ELU$-concept constructors are those of $EL$ enriched with conjunction ($\land$).

Formulas in DL are of the GCI form: $C \subseteq D$, where $C$ and $D$ are any two complex concepts, or Abox assertions: $(a : C, (a, b) : \tau)$, with $\tau$ a role. We propose to define a $\Sigma$-formula relaxation in two ways (other definitions may also...
exist). For GCIs, a first approach consists in relaxing the set of models of D while another one amounts to “retract” the set of models of C.

**Definition 6** Let C and D be any two complex concepts defined over the signature Σ. The concept relaxation based Σ-formula relaxation, denoted \( r_{Σ} \), is defined as follows:

\[
\begin{align*}
\tau_{Σ}(C \sqsubseteq D) &= C \sqsubseteq ρ(D) \\
\tau_{Σ}(a : C) &= a : ρ(C), \quad \tau_{Σ}((a, b) : r) = (a, b) : r_T
\end{align*}
\]

where \( r_T = Δ^2 × Δ^2 \) and ρ is a concept relaxation as in Definition 2.

**Proposition 3** \( r_{Σ} \) is a Σ-formula relaxation, that is extensive and exhaustive.

The proof directly follows from the extensivity and exhaustivity of ρ.

**Definition 7** (Concept Retraction) A (concept) retraction is an operator \( κ : C(Σ) \to C(Σ) \) that satisfies the following three properties for all \( C, D ∈ C(Σ) \):

1. \( κ \) is anti-extensive, i.e., \( κ(C) \sqsubseteq C \), and
2. \( κ \) is exhaustive, i.e., \( ∀D ∈ C(Σ), ∃k ∈ N \mid κ^k(C) ⊆ D \), where \( κ^k \) denotes \( k \) applied \( k \) times, and \( k^0 \) is the identity mapping.

**Definition 8** Let C and D be any two complex concepts defined over the signature Σ. The concept retraction based Σ-formula relaxation, denoted \( r_{Σ} \), is defined as follows:

\[
\tau_{Σ}(C \sqsubseteq D) ≡ κ(C) \sqsubseteq D
\]

where \( κ \) is a concept retraction.

The definition for Abox assertions is similar as in Definition 6.

**Proposition 4** \( r_{Σ} \) is a Σ-formula relaxation.

For coming up with revision operators, it remains to define concrete relaxation and retraction operators at the concept level (cf. Figure 2). Some examples of retraction and relaxation operators are given below.

### 4.1 Relaxation and retraction in \( EL \)

**EL-Concept Relaxations**. A trivial relaxation is the operator \( ρ_T \) that maps every concept to \( ⊤ \). Other non-trivial \( EL \)-concept description relaxations have been introduced in [Distel et al., 2014b]. We summarize here some of these operators.

\( EL \) concept descriptions can appropriately be represented as labeled trees, often called \( EL \) description trees [Baader et al., 1999]. An \( EL \) description tree is a tree whose nodes are labeled with sets of concept names and whose edges are labeled with role names. An \( EL \) concept description

\[
C ≡ P_1 ⊓ \ldots ⊓ P_n ⊓ ∃r_1.C_1 ⊓ \ldots ⊓ ∃r_m.C_m
\]

with \( P_i ∈ N_C ∪ \{ ⊤ \} \), can be translated into a description tree by labeling the root node \( v_0 \) with \( \{ P_1, \ldots, P_n \} \), creating an \( r_j \)-successor, and then proceeding inductively by expanding \( C_j \) for the \( r_j \)-successor node for all \( j ∈ \{ 1, \ldots, m \} \).

An \( EL \)-concept description relaxation then amounts to apply simple tree operations. Two relaxations can hence be defined [Distel et al., 2014b]: (i) \( ρ_{\text{depth}} \) that reduces the role depth of each concept by 1, simply by pruning the description tree, and (ii) \( ρ_{\text{leaves}} \) that removes all leaves from a description tree.

### 4.2 Relaxations in \( EL_U \)

The relaxation defined above exploits the strong property that an \( EL \) concept description is isomorphic to a description tree. This is arguably not true for more expressive DLs. Let us try to go a one step further in expressivity and consider the logic \( EL_U \). A relaxation operator as introduced in [Distel et al., 2014b] requires a concept description to be in a special normal form, called normal form with grouping of existentials, defined recursively as follows.

**Definition 9** We say that an \( EL \)-concept D is written in normal form with grouping of existential restrictions if it is of the form

\[
D = \bigcap_{A ∈ N_D} A \sqcap \bigcap_{r ∈ N_R} D_r
\]

where \( N_D ⊆ N_C \) is a set of concept names and the concepts \( D_r \) are of the form

\[
D_r = \bigcap_{E ∈ C_{D_r}} \exists r.E,
\]

where no subsumption relation holds between two distinct conjuncts and \( C_{D_r} \) is a set of complex \( EL \)-concepts that are themselves in normal form with grouping of existential restrictions.

The purpose of \( D_r \) terms is simply to group existential restrictions that share the same role name. For an \( EL_U \)-concept C we say that C is in normal form if it is of the form (\( \overline{C} ≡ C_1∪C_2∪\ldots∪C_k \)) and each of the \( C_i \) is an \( EL \)-concept in normal form with grouping of existential restrictions.

**Definition 10** [Distel et al., 2014b] Given an \( EL_U \)-concept description C we define an operator \( ρ_e \) recursively as follows. For \( C = A ∈ N_C \) and for \( C = ⊤ \) we define \( ρ_e(A) = ρ_e(⊤) = ⊤ \). For \( C = D_r \), where \( D_r \) is a group of existential restrictions as in (3), we need to distinguish two cases:
Def. 4
The new agent's belief, up to a rewriting, becomes

\[
\exists \rho : \text{Sen}(\Sigma) \rightarrow \text{Sen}(\Sigma)
\]

Then \(\exists \rho: \text{Sen}(\Sigma) \rightarrow \text{Sen}(\Sigma)\).

\[
\exists \rho : \text{Sen}(\Sigma) \rightarrow \text{Sen}(\Sigma)
\]

Def. 4
Def. 4

\[
\exists \rho : \text{Sen}(\Sigma) \rightarrow \text{Sen}(\Sigma)
\]

1. if \(D_r \equiv \exists r. T\) we define \(\rho_r(D_r) = T\) and
2. if \(D_r \neq \exists r. T\) then we define \(\rho_r(D_r) = \bigcup_{S \subseteq C_{D_r}} (\rho_e(G) \cap \bigcap_{H \in C_{D_r} \setminus G} H)\), where \(C_D = N_D \cup \{D_r \mid r \in N_R\}\). Finally for \(C = \{C_1 \cup \cdots \cup C_k\}\) we set \(\rho_e(C) = \rho_e(C_1) \cup \rho_e(C_2) \cup \cdots \cup \rho_e(C_k)\).

The proof of \(\rho_r\) being a relaxation, i.e. satisfying exhaustivity and extensivity is detailed in [Distel et al., 2014a]. Let us illustrate this operator on an example.

**Example 2** Suppose an agent believes that a person Bob is married to a female judge: \(T = \{\text{Bob} \subseteq \text{Male} \cap \exists \text{MarriedTo. (Female} \cap \text{Judge}}\}\). Suppose now that due to some obscurantist law, it happens that females are not allowed to be judges. This new belief is captured as \(T' = \{\text{Judge} \subseteq \text{Female} \subseteq \bot\}\). By applying \(\rho_r\) one can resolve the conflict between the two belief sets. To ease the reading, let us rewrite the concepts as follows: \(A \equiv \text{Male}, B \equiv \text{Female}, C \equiv \text{Judge}, m \equiv \text{MarriedTo}, D \equiv \exists \text{MarriedTo. (Female} \cap \text{Judge}}\). Hence \(\rho_e(A \cap D) \equiv (\rho_e(A) \cap D) \cup (A \cap \rho_e(D))\), with \(\rho_e(A) \equiv T\) and

\[
\rho_e(D) \equiv \exists m. \rho_e(B) \cap C \cup (\exists m. B \cap \exists m. \rho_e(C)) \cup (\exists m. \rho_e(B) \cap C)
\]

\[
\equiv \exists m. (B \cap C) \cup (\exists m. B \cap \exists m. C) \cup (\exists m. B \cap \exists m. C)
\]

\[
\equiv \exists m. B \cap \exists m. C \cup \exists m. B \cap \exists m. C
\]

Then \(\rho_e(A \cap D) \equiv (\rho_e(A) \cap D) \cup (A \cap \rho_e(D))\).

The new agent’s belief, up to a rewriting, becomes \(\{\text{Bob} \subseteq \exists \text{MarriedTo. (Female} \cap \text{Judge}}\} \cup \{\text{Male} \cap \exists \text{MarriedTo. (Female} \cap \text{Judge}}\} \cup \{\text{Judge} \subseteq \bot\}\).

Another possibility for defining a relaxation in \(\mathcal{ELU}\) is obtained by exploiting the disjunction constructor by augmenting a concept description with a set of exceptions.

**Definition 5** Given an exception set \(E = \{E_1, \ldots, E_n\}\), we define a relaxation of degree \(k\) of an \(\mathcal{ELU}\)-concept description \(C\) as follows: for a finite set \(E^k \subseteq E\) with \(|E^k| = k\)

\[
\rho^k_E(C) = C \cup E_{i_1} \cup \cdots \cup E_{i_k}, \forall j, E_{i_j} \in E^k \text{ and } E_{i_j} \cap C \subseteq \bot
\]

Extensivity of this operator follows directly from the definition. However, exhaustivity is not necessarily satisfied unless

- if \(D_r \equiv \exists r. T\) we define \(\rho_r(D_r) = T\) and
- if \(D_r \neq \exists r. T\) then we define

\[
\rho_r(D_r) = \bigcup_{S \subseteq C_{D_r}} (\rho_e(G) \cap \bigcap_{H \in C_{D_r} \setminus G} H)
\]

Notice that in the latter case \(T \not\subset C_{D_r}\), since \(D_r\) is in normal form. For \(C = D\) as in (2) we define \(\rho_e(D) = \bigcup_{G \subset C_D} (\rho_e(G) \cap \bigcap_{H \in C_{D_r} \setminus G} H)\), where \(C_D = N_D \cup \{D_r \mid r \in N_R\}\). Then we define \(\rho_e(C) = \rho_e(C_1) \cup \rho_e(C_2) \cup \cdots \cup \rho_e(C_k)\).

4.3 Relaxation and retraction in \(\mathcal{ALC}\)

We consider here operators suited to \(\mathcal{ALC}\) language. Of course, all the operators defined for \(\mathcal{EL}\) and \(\mathcal{ELU}\) remain valid.

**\(\mathcal{ALC}\)-Concept Retractions.** A first possibility for defining retraction is to remove iteratively from an \(\mathcal{ALC}\)-concept description one or a set of its subconcepts. A similar construction has been introduced in [Qi et al., 2006b] by transforming ABox assertions to nominals and conjuncting their negations from the concept they belong to. Interestingly enough, almost all the operators defined in [Qi et al., 2006b] [Gorogiannis and Hunter, 2008] are relaxations.

**Definition 12** Let \(C\) be any \(\mathcal{ALC}\)-concept description, we define \(\kappa^2_C(C) = C \cap E_1 \cap \cdots \cap E_n\) s.t. \(E_1 \subseteq C, \ldots, E_n \subseteq C\).

Consider again Example 7 \(\kappa^2_C(\text{Bird}) = \text{Bird} \cap \text{Tweety}\). The resulting revised knowledge base is then \{\text{Tweety} \cap \text{Bird}, \text{Bird} \cap \text{Tweety}, \text{Tweety} \cap \text{Flies}, \text{Tweety} \cap \text{Flies} \subseteq \bot\} which is consistent.

Another possibility, suggested in [Gorogiannis and Hunter, 2008] and related to operators defined in propositional logic as introduced in [Bloch and Lang, 2002], consists in applying the retraction at the atomic level. This captures somehow the Dalal’s idea of revision operators in propositional logic [Dalal, 1988].

**Definition 13** Let \(C\) be an \(\mathcal{ALC}\)-concept description of the form \(Q_1 r_1 \cdots Q_m r_m D\), where \(Q_i\) is a quantifier and \(D\) is quantifier-free and in CNF form, i.e. \(D = E_1 \cap E_2 \cdots \cap E_n\) with \(E_i\) being disjunctions of possibly negated atomic concepts. Define, as in the propositional case [Bloch and Lang, 2002], \(\kappa_p(D) = \bigcap_{i=1}^n (\bigcup_{j \neq i} E_i)\). Then \(\kappa^\mathcal{ALC}_{\text{Dalal}}(C) = Q_1 r_1 \cdots Q_m r_m \kappa_p(D)\).

This idea can be generalized to consider any retraction defined in \(\mathcal{ELU}\).

**Definition 14** Let \(C\) be an \(\mathcal{ALC}\)-concept description of the form \(Q_1 r_1 \cdots Q_m r_m D\), where \(Q_i\) is a quantifier and \(D\) is quantifier-free. Then \(\kappa^\mathcal{ALC}_{\text{Dalal}}(C) = Q_1 r_1 \cdots Q_m r_m \kappa_p(D)\).
Another possible \(\mathcal{ALC}\)-concept description retraction is obtained by substituting the existential restriction by an universal one. This idea has been sketched in [Gorogiannis and Hunter, 2008] for defining dilation operators (then by transforming ' into \(\exists\)), i.e., special relaxation operators enjoying additional properties [Distel et al., 2014b; Distel et al., 2014b]. We adapt it here to define retraction in DL syntax.

**Definition 15** Let \(C\) be an \(\mathcal{ALC}\)-concept description of the form \(Q_1 r_1 \cdots Q_n r_n \cdot D\), where \(Q_i\) is a quantifier and \(D\) is quantifier-free, then

\[
\kappa_q(C) = \bigcap\{Q'_1 r_1 \cdots Q'_n r_n \cdot D \mid \exists j \leq n \text{ s.t. } Q_j = \exists \text{ and } Q'_j = \forall, \text{ and for all } i \leq n \text{ s.t. } i \neq j, Q'_i = Q_i\}
\]

**\(\mathcal{ALC}\)-Concept Relaxations.** Let us now introduce some relaxation operators suited to \(\mathcal{ALC}\) language.

**Definition 16** Let \(C\) be an \(\mathcal{ALC}\)-concept description of the form \(Q_1 r_1 \cdots Q_m r_m \cdot D\), where \(Q_i\) is a quantifier and \(D\) is quantifier-free and in DNF form, i.e. \(D = E_1 \sqcup E_2 \sqcup \cdots E_s\), with \(E_i\) being conjunction of possibly negated atomic concepts. Define, as in the propositional case [Bloch and Lang, 2002], \(\rho_p(D) = \bigcup_{i=1}^s (\bigcap_{i \neq j} E_i)\), then \(\rho_p^{\text{Dalal}}(C) = Q_1 r_1 \cdots Q_m r_m \cdot \rho_p^p(D)\).

As for retraction, this idea can be generalized to consider any relaxation defined in \(ECU\).

**Definition 17** Let \(C\) be an \(\mathcal{ALC}\)-concept description of the form \(Q_1 r_1 \cdots Q_n r_n \cdot D\), where \(Q_i\) is a quantifier and \(D\) is quantifier-free, then \(\rho_q^{\text{Dalal}}(C) = Q_1 r_1 \cdots Q_n r_n \cdot \rho^\rho(D)\).

Let us consider another example adapted from the literature to illustrate these operators [Qi et al., 2006b].

**Example 3** Let us consider the following knowledge bases: \(T = \{\text{BOB} \sqsubseteq \exists \text{HASCHILD.RICH}, \text{BOB} \sqsubseteq \exists \text{HASCHILD.MARY}, \text{MARY} \sqsubseteq \text{RICH}\}\) and \(T' = \{\text{BOB} \sqsubseteq \exists \text{HASCHILD.JOHN}, \text{JOHN} \sqsubseteq \text{RICH}\}\). Relaxing the formula \(\text{BOB} \sqsubseteq \exists \text{HASCHILD.RICH}\) by applying \(\rho^\rho\) to the concept on the right hand side results in the following formula \(\text{BOB} \sqsubseteq \exists \text{HASCHILD} \sqcap \text{RICH} \sqcup \text{JOHN}\) which resolves the conflict between the two knowledge bases.

A last possibility, dual to the retraction operator given in Definition [15] consists in transforming universal quantifiers to existential ones.

**Definition 18** Let \(C\) be an \(\mathcal{ALC}\)-concept description of the form \(Q_1 r_1 \cdots Q_n r_n \cdot D\), where \(Q_i\) is a quantifier and \(D\) is quantifier-free, then

\[
\rho_q(C) = \bigcup\{Q'_1 r_1 \cdots Q'_n r_n \cdot D \mid \exists j \leq n \text{ s.t. } Q_j = \exists \text{ and } Q'_j = \forall, \text{ and for all } i \leq n \text{ s.t. } i \neq j, Q'_i = Q_i\}
\]

If we consider again Example [3] relaxing the formula \(\text{BOB} \sqsubseteq \exists \text{HASCHILD.RICH}\) by applying \(\rho_q\) to the concept on the right hand side results in the following formula \(\text{BOB} \sqsubseteq \exists \text{HASCHILD.RICH}\), which resolves the conflict between the two knowledge bases.

The following proposition summarizes the properties of the introduced operators.

**Proposition 5** The operators \(\rho^\top, \rho^\text{depth}, \rho^\text{leaves}, \rho^e, \rho^\text{Dalal}, \rho^q\) are extensive and exhaustive. The operators \(\rho^e, \rho^p\) are extensive but not exhaustive. The operators \(\kappa^\top, \kappa^e, \kappa^\text{Dalal}, \kappa^q\) are anti-extensive and exhaustive. The operators \(\kappa_q\) are anti-extensive but not exhaustive.

These properties are directly derived from the definitions and from properties of \(\rho_p\) and \(\kappa_p\) detailed in [Bloch and Lang, 2002]. Note that for \(\kappa_q\) exhaustivity can be obtained by further removing recursively the remaining universal quantifiers and apply at the final step any retraction defined above on the concept \(D\).

5 Related works

In the last decade, several works have studied revision operators in Description Logics. While most of them concentrated on the adaptation of AGM theory, few works have concerned the definition of concrete operators [Meyer et al., 2005; Qi et al., 2006a; Qi et al., 2006b]. A closely related field is inconsistency handling in ontologies (e.g. [Schlobach and Cornet, 2003; Schlobach et al., 2007]), with the main difference that the rationality of inconsistency repairing operators is not investigated, as suggested by AGM theory.

Some of our relaxation operators are closely related to the ones introduced in [Qi et al., 2006b] for knowledge bases revision and in [Gorogiannis and Hunter, 2008] for merging first-order theories. Our relaxation-based revision framework, being abstract enough (i.e. defined through easily satisfied properties), encompasses these operators. Moreover, the revision operator defined in [Qi et al., 2006b] considers only inconsistencies due to Abox assertions. Our operators are general in the sense that Abox assertions are handled as any formula of the language.

The relaxation idea originates from the work on MorphoLogics, initially introduced in [Bloch and Lang, 2002; Bloch et al., 2004]. In this seminal work, revision operators (and explanatory relations) were defined through dilation and erosion operators. These operators share some similarities with relaxation and retraction as defined in this paper. Dilation is a sup-preserving operator and erosion is inf-preserving, hence both are increasing. Some particular dilations and erosions are exhaustive and extensive while relaxation and retraction operators are defined to be exhaustive and extensive but not necessarily sup- and inf-preserving.

Another contribution in this paper concerns the generalization of AGM postulates and their translation in a model-theoretic writing with a definition of inconsistency, allowing using them in a wide class of non-classical logics. This follows recent works on the adaptation of AGM theory (e.g. [Ribeiro et al., 2013; Ribeiro and Wassermann, 2014; Delgrande and Peppas, 2015; Flouris et al., 2005]). Our generalization is closely related to the one recently introduced in [Ribeiro and Wassermann, 2014] and could be seen as its
counterpart in a model-theoretic setting. It also extends the one introduced in [Qi et al., 2006b].

6 Conclusion

The contribution of this paper is threefold. First, we provided a generalization of AGM postulates so as they become applicable to a wide class of non-classical logics. Secondly we proposed a general framework for defining revision operators based on the notion of relaxation. We demonstrated that such a relaxation-based framework for belief revision satisfies the AGM postulates and leads to a faithful assignment. Thirdly, we introduced a bunch of concrete relaxations, discussed their properties and illustrated them through simple examples. Future work will concern the study of the complexity of these operators, the comparison of their induced ordering, and their generalization to other non-classical logics such as Horn logic.

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