A Robust Beamforming Algorithm Based on Covariance Matrix Reconstruction and Steering Vector Estimation

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Abstract. Adaptive beamforming algorithms often suffer from performance degradation due to the presence of the desired signal component in the interference-plus-noise covariance matrix (IPNCM) and mismatch of steering vector. In order to improve the robustness of the beamforming algorithm performance, a new robust adaptive beamforming algorithm is proposed in this paper. In the proposed algorithm, Capon spatial spectrum is first used to reconstruct the IPNCM. Then, based on the robust Capon beamforming algorithm based on the steering vector uncertain set, new constraints are added so that the steering vector does not converge to the interference direction. The proposed beamforming algorithm is a non-convex quadratic constraint quadratic programming (QCQP) problem that is solved using semidefinite programming (SDP) relaxation. Simulation results show that the proposed algorithm is more robust than some other existing algorithms and is closest to the optimal solution.

1. Introduction

Adaptive beamforming technology has been widely used in radar, communication, sonar, microphone array speech processing, medical image processing and other fields\(^1\). When the IPNCM and the steering vector of the signal of interest (SOI) are very accurate, the traditional minimum variance distortionless response (MVDR) algorithm can minimize the output interference noise power and respond to the desired signal without distortion\(^2\). However, the fact that the actual training data contains SOI components or a small amount of data will make the IPNCM estimation inaccurate, resulting in a decrease in the performance of the beamformer\(^3\). In addition, the misalignment of the steering vector of SOI caused by many factors including array calibration errors, array geometric distortion, etc. will also significantly reduce the performance of the beamformer\(^4\). Therefore, researches\(^5-11\) on robust adaptive beamforming algorithms have become a research focus in recent years.

In general, the existing robust adaptive beamforming algorithms can be divided into two categories, which are essentially based on MVDR algorithms. The first type of algorithm uses various algorithms to estimate the IPNCM from the training data. The diagonal loading (DL) technique\(^5, 6\) belongs to this type of algorithm. The DL algorithm can artificially inject white noise into the diagonal elements of the sample covariance matrix (SCM), reduce the noise eigenvalue spread, and prevent beam main lobe distortion. Li W et al.\(^5\) improved the loading level of adaptive zero-stretched beams by adopting adaptive variable DL technology and covariance matrix taper method. The DL technology cannot determine the optimal loading amount under different conditions of error and interference\(^7\). The second type of algorithm is to estimate the expected signal steering vector. Both the eigenspace-based
(ESB) algorithm[8,9] and the uncertainty set constraint algorithm[10,11] belong to this type of algorithm. The ESB eigen decomposes SCM and uses the orthogonality of the signal plus interference subspace and noise subspace to project the steering vector of desired signal onto the signal plus interference subspace. This algorithm is generally only suitable for high signal-to-noise ratio (SNR). Yuan X et al.[8] used the intersection of the interference subspace and signal subspace obtained from the spectral estimation and the signal interference subspace obtained from the SCM eigen decomposition as the signal steering vector and the interference steering vector respectively, and reconstructed the INPCM, which led to a good beam performance with high SNR and low SNR, but the algorithm needed to know some prior information about signal and interference. The uncertainty set constraint algorithm models the uncertainty set of the steering vector and constrains it into the algorithm to eliminate the error of the steering vector mismatch. However, the algorithm generally does not give an analytical solution to the problem and needs to solve the QCQP problem[12]. The robust Capon beamforming algorithm proposed by Li J et al.[10] added a quadratic constraint to the steering vector, and the robustness of the algorithm was improved. But this research showed that when the interference-to-noise ratio (INR) was greater than the SNR, the estimation of the steering vector may converge to the interference direction, resulting in a dramatic decrease in the performance of the beamformer.

In this paper, we propose a new robust algorithm to improve the performance of the beamformer. In this algorithm, the Capon spatial spectrum[13] is first used to reconstruct the IPNCM. Second, based on the robust Capon algorithm, new constraints are added to form a new robust Capon beamforming algorithm formula. After analysis, the new beamforming algorithm can well prevent the steering vector from converging to the interference direction. We use the SDP relaxation[14] approximation to solve the optimal solution, and use Matlab CVX toolbox[15] for simulation since the new beamformer is a non-convex QCQP problem. The simulation results show that the new algorithm has better robustness, and the main lobe direction can well point to the direction of arrival (DOA) of the actual desired signal. At the same time, as the input SNR changes, the output signal-to-interference-to-noise ratios (SINR) of the proposed algorithm is always higher than some other existing algorithms and closest to the optimal value.

This paper contributes to the field of robust adaptive beamforming in the following two aspects.
1) The formation reason of beamformer pointing in the direction of interference in SNR < INR is analyzed.
2) A new robust beamforming algorithm is proposed, including adding new constraints to the robust Capon algorithm, solving the beamformer pointing problem and using Capon spatial spectrum to reconstruct IPNCM.

The remainder of this paper is organized as follows. In section 2, the problem model of robust adaptive beamforming is briefly described, and the defects of robust Capon algorithm are analyzed by simulation. In section 3, we propose a robust adaptive beamforming algorithm based on covariance matrix reconstruction and steering vector estimation. In section 4, the performance of the proposed algorithm is verified by simulation results. The conclusions are drawn in section 5.

2. The problem model
Consider a uniform linear array (ULA) with M array elements. The array output can be expressed as
\[ y(k) = w^H x(k) \] (1)
where \( w \) is the adaptive weight vector of \( M \times 1 \), and \( x(k) \) is the signal received by the antenna, which can be expressed as
\[ x(k) = x_s(k) + x_i(k) + x_n(k) \] (2)
where \( x_s(k) \) is the desired signal, \( x_i(k) \) is the interference signal, \( x_n(k) \) is the Gaussian noise with zero mean and variance of \( \sigma^2 \). Herein, it is assumed that the desired signal is not related to the interference signal. Use the MVDR algorithm to get the optimal weighting vector as
where $a$ is the steering vector of the desired signal, and $R$ is the INPCM. The SCM $\hat{R}$ is used instead of $R$ since $R$ cannot be accurately known in actual situations, which is called sample matrix inversion (SMI). The $\hat{R}$ expression is as follows

$$\hat{R} = \frac{1}{N} \sum_{n=1}^{N} x(k)x^H(k)$$

where $N$ is the number of snapshots.

In practice, except for IPNCM, the steering vector $a$ of the desired signal is often inaccurate. Usually, a presumed steering vector $\bar{a}$ is estimated based on the knowledge of antenna array geometry, parameters of the SOI and some additional assumptions about antenna array calibration\cite{16}. Due to the mismatch between the actual steering vector and the presumed steering vector, the performance of (3) degrades dramatically. Therefore, many robust steering vector estimation algorithms have been proposed. Among them, the robust Capon beamforming algorithm constrains the steering vector uncertainty set and can effectively deal with the steering vector mismatch problem. The expression of its beamformer is:

$$\min a^H \hat{R}^{-1} a \quad \text{s.t.} \quad \|a - \bar{a}\| \leq \varepsilon$$

where $\varepsilon$ is the presumed error parameter.

Here we set the direction of arrival (DOA) of the desired signal to $10^\circ$, assuming the DOA of the desired signal is $14^\circ$, there are two interference signals whose DOA are $-30^\circ$, $60^\circ$ respectively, the number of elements is 10, the INR is 10dB, and the SNR are 5dB and 15dB. We can get the Figure 1 by using Matlab CVX toolbox to simulate (5). It can be seen from the Figure 1 that when the SNR $> \text{INR}$, the beamformer (5) can correctly point to the actual DOA, but when the SNR $< \text{INR}$, it points to the interference direction. Still use the above simulation conditions to simulate the target function of (5) and get the Figure 2.

It can be seen from the Figure 2 that when the SNR $< \text{INR}$, the objective function of the desired signal will be larger than the interference signal. If $\varepsilon$ is not selected properly at this time, the beamformer will point in the interference direction. A proper $\varepsilon$ requires a lot of precise prior knowledge, which is often difficult to acquire.

3. The proposed algorithm
In this part, we propose a new robust adaptive beamforming algorithm, which is divided into two parts. The first part is the reconstruction of INPCM, and the second part is the steering vector estimation.
3.1. Reconstruction of IPNCM  
In general, INPCM is replaced by the SCM of equation (4), which works well in low SINR. However, the beamformer suppresses the desired signal as an interference signal as the SINR gradually increases, which leads to more SOI component contained in SCM. Herein, we use the integration of the Capon spatial spectrum over an angular sector separated from the SOI direction as an estimate of the IPNCM.

\[ \hat{R}_{i,w} = \int_{\Theta} a(\theta) a^H(\theta) R^{-1} a(\theta) \]

where \( a(\theta) \) is a steering vector in the direction \( \theta \), and \( \Theta \) represents an angular sector separated from the SOI direction. Suppose \( \Theta \) represents the angular sector in which SOI is located, then \( \bar{\Theta} \) is the complement of \( \Theta \), \( \Theta \cup \bar{\Theta} \) covers the entire space, and \( \Theta \cap \bar{\Theta} \) is the empty set. Herein, we sample the continuous angular sector \( \Theta \), get the discrete angular sector \( \Theta_d = \{ \theta_1, \theta_2, \ldots, \theta_p \} \), and find the corresponding steering vector set \( A = \{ a(\theta_1), a(\theta_2), \ldots, a(\theta_p) \} \) since the high complexity of the integration operation. The estimation of IPNCM is obtained by summing the discrete angular sectors instead of integrating.

\[ \hat{R}_{i,w} = \sum_{k=1}^{p} \frac{a(\theta_k) a^H(\theta_k)}{a^H(\theta_k) R^{-1} a(\theta_k)} \]  

3.2. Steering vector estimation  
Based on the robust Capon beamforming algorithm, we added new constraints and the new beamforming algorithm formula is

\[
\begin{aligned}
&\min_{\mathbf{a}} \quad a^H \hat{R}^{-1} \mathbf{a} \\
\text{s.t.} \quad &\| \mathbf{a} - \mathbf{a}_d \|^2 \leq \varepsilon \\
&\mathbf{a}^H \tilde{C} \mathbf{a} \leq \Delta_0 \\
&\| \mathbf{f} \|^2 = M
\end{aligned}
\]

where \( \varepsilon \), \( \Delta_0 \) are presumed parameters, \( \tilde{C} = \int_\Theta a(\theta) a^H(\theta) d\theta \). Assuming that the direction of the desired signal is \( \theta_1 \) and the corresponding steering vector \( a = a(\theta_1) \), then

\[ a^H \tilde{C} a = \int_\Theta (M + 2 \sum_{n=1}^{M-1} (M-n) \cos n(\theta - \theta_1)) \]

if \( \bar{\Theta} = [\theta_2, \theta_1] \), then

\[ f(\theta_1) = a^H \tilde{C} a = M(\theta_1 - \theta_2) + 2 \sum_{n=1}^{M-1} \frac{M-n}{n} [\sin n(\theta_3 - \theta_1) - \sin n(\theta_2 - \theta_1)] \]

In order to better explain the effect of \( a^H \tilde{C} a \leq \Delta_0 \), suppose the desired signal DOA is 10°, \( \bar{\Theta} = [-90°, 5°] \cup [15°, 90°] \), \( M = 10 \), and simulate the formula (10).
According to Figure 3, the steering vector will be constrained near the actual steering vector by appropriate $\Delta_0$. Combined with Figure 2, the final steering vector will not converge to the interference direction due to the constraint of $a^H \tilde{C}a \leq \Delta_0$ when INR > SNR. In addition, the role of constraint $\|a\|^2 = M$ is to prevent the program from converging to a trivial solution $a = 0$.

We can modify the solution to similar problems in [17] since the proposed algorithm is obviously a non-convex QCQP problem. Using SDP relaxation for equation (8) gives the following form:

$$
\begin{align*}
\min_{X,\gamma} & \quad \text{tr}(\hat{R}^{-1}X) \\
\text{s.t.} & \quad \text{tr}(\hat{C}X) \leq \Delta_0 \\
& \quad \text{tr}(X) = M \\
& \quad \text{tr}\left(\begin{bmatrix}
I & -a_0^H \\
-a_0 & M
\end{bmatrix}\begin{bmatrix}
X & y \\
y^H & 1
\end{bmatrix}\right) \leq \varepsilon \\
& \quad \begin{bmatrix}
X & y \\
y^H & 1
\end{bmatrix} \succeq 0
\end{align*}
$$

where $\text{tr}(\cdot)$ is the trace of matrix, $X$ is a semi-positive definite matrix of $M \times M$. Obviously, the optimal solution $X$ is $aa^H$. Using SDP relaxation to convert equation (8) to a relaxed convex problem, $X$ can be solved using Matlab CVX toolbox. Suppose $X^*$ is the optimal solution of the relaxation problem (11), and $X^*$ satisfies $X^* = YY^H$. If the rank of $X^*$ is $r$, then $Y$ is the full rank matrix of $M \times r$. If $r = 1$, the optimal solution of the original problem is $Y$, otherwise, the optimal solution is $Yv$, where $v$ is a vector of $r \times 1$ and satisfies the following formula

$$
\|Y\| = \sqrt{M}, \quad v^H Y^H \tilde{C}Yv = \text{tr}(Y^H \tilde{C}Y)
$$

A possible solution of vector $v$ is proportional to the sum of all eigenvectors of the matrix below

$$
D = \frac{1}{M} Y^H Y - \frac{Y^H \tilde{C}Y}{\text{tr}(Y^H \tilde{C}Y)}
$$

The steering vector of the desired signal obtained by solving the relaxed convex problem is $\hat{a}$, combined with $\hat{R}_{i,n}^{-1}$, and substituted into formula (3), the weight expression of the proposed algorithm is

$$
w_{\text{proposed}} = \frac{\hat{R}_{i,n}^{-1} \hat{a}}{\hat{a}^H \hat{R}_{i,n}^{-1} \hat{a}}
$$
4. Simulation results

In our simulation, the simulation object is a ULA with the number of array elements being 10 and the array element spacing to wavelength ratio being 0.5. Among them, we set two interference signals whose DOA are -30°, 60° and Gaussian noise with zero mean and variance of 1. The actual DOA of the desired signal which is not correlated with the interference signal is 10°, and the possible angular sector where the desired signal is located is \( \Theta = [5^\circ, 15^\circ] \). It is herein assumed that the DOA of the desired signal is 14°. The proposed method is compared with the representative robust Capon algorithm and the worst-case algorithm[18] among the robust adaptive beamforming algorithms.

Figure 4 and 5 are beam patterns of SNR = 15dB, INR = 10dB and SNR = 5dB, INR = 10dB, respectively. In Figure 4, all three methods have certain robustness, but if we observe the main lobe orientation carefully, we can find that the proposed method is closer to the actual DOA of the desired signal. In Figure 5, as our analysis in the second section is consistent, the robust Capon algorithm cannot handle the case of INR > SNR, and the main lobe points directly to the interference direction. The proposed method and the worst case algorithm can point to the actual direction of the desired signal, but the proposed method is better. In addition, as the IPNCM estimation in the proposed method is more accurate, a null is formed near the interference.

![Figure 4. Beam pattern of formula (14) when SNR = 15dB, INR = 10dB](image1)

![Figure 5. Beam pattern of formula (14) when SNR = 5dB, INR = 10dB](image2)

Figure 4 is the relationship between output SINR and input SNR. The input INR is set to 20dB. In order to eliminate contingency, the simulation result is the average of 200 Monte Carlo experiments. From the figure, we can see that the robust Capon algorithm has a sudden rise in the output SINR at the input SNR = 20dB. This is also because when the SNR > INR, the Capon algorithm can work...
normally and have good performance. However, the curve of the proposed algorithm is always higher than other curves, and is closest to the optimal curve.

5. Conclusion
In this paper, a new robust adaptive beamforming algorithm is proposed which can improve the robustness of beamforming from two aspects, namely the reconstruction of the IPNCM and the estimation of steering vector of the desired signal. It uses Capon spatial spectrum for reconstructing the IPNCM to eliminate the impact of SOI components on the algorithm. Second, new constraints are incorporated with the existing robust Capon algorithm to improve the accuracy of the steering vector estimation. Simulation is performed using one desired signal and two interference signals incident to a ULA. Simulation results show that this algorithm is more robust than some other existing algorithms, and has a higher output SINR that is closest to the optimal output SINR.

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