FRANK FEATHERSTONE BONSALL
31 March 1920 — 22 February 2011
FRANK FEATHERSTONE BONSALL

31 March 1920 — 22 February 2011

Elected FRS 1970

BY T. A. GILLESPIE FRSE*

School of Mathematics, University of Edinburgh, James Clerk Maxwell Building,
Peter Guthrie Tate Road, Edinburgh EH9 3FD, UK

Frank Bonsall played a significant role in the mathematical life of the United Kingdom in the decades following the Second World War. He had a particular impact in Scotland and the north of England, especially in research and graduate education. His research interests focused primarily on functional analysis, the area of mathematics that brings together various strands of analysis under a single abstract framework, and on the related theory of linear operators on Banach spaces. He influenced a generation of young mathematicians with the elegance of his written and oral expositions, both of his own research and that of others. The quality of his caring and thorough research supervision was reflected in his many PhD students who would continue in research and go on to successful academic careers in their own right, both in the United Kingdom and beyond.

1. FAMILY BACKGROUND AND EARLY YEARS

Frank Featherstone Bonsall, the son of Wilfred Cook Bonsall and Sarah (née Frank), was born at Crouch End in North London. His father’s forebears came from Derbyshire, where the surname Bonsall is relatively common, although Wilfred himself was brought up in Middlesbrough. Widely self-educated and well read, Wilfred qualified as an accountant by correspondence course and was to become the company secretary of an import-export firm. Sarah’s family hailed originally from North Yorkshire, where the Franks had farmed for generations. She studied at University of Leeds, where she was in the first cohort of graduates when the university received its charter. Frank was the younger of Wilfred and Sarah’s two

* E-mail: t.a.gillespie@ed.ac.uk

© 2020 The Author(s)
children and both were to achieve distinction in their chosen careers. His elder brother, Arthur Wilfred (1917–2014), was to serve as Director of GCHQ from 1973 to 1978 and was knighted in 1977.

In 1923 the family moved to Welwyn Garden City, then a new town that had been founded only three years earlier. As Frank recalled later, it was a good place for a child to be brought up, with plenty of fields and woods and all the interests of both farms and building sites. The house was home to many books, with the *Children's encyclopaedia* and *Webster's dictionary* making the greatest impression on Frank. There were also frequent visits from A. F. Hallimond ScD, the geologist and designer of microscopes, who was a close family friend.

From 1926 to 1933, Frank attended Fretherne House Preparatory School, where he received an excellent grounding in mathematics from the headmaster. Although this involved the usual drill in routines, sight was never lost of the underlying problem in hand. When he was 13 years old, his parents sent him to Bishop’s Stortford College. This independent school, which at that time took only boys, had both day pupils and boarders; Frank attended as a boarder. Although he did not particularly enjoy boarding school life, he valued the excellent staff and excelled academically, winning most of the available prizes, including the school leaving scholarship. In 1938, he went up to Merton College, Oxford, to read mathematics. He described that first year 1938–1939 at Oxford as one of his happiest, despite the shadow of the coming war. In contrast to boarding school, he was able to enjoy the freedom and relative privacy of university life, as well as his first encounters with the power of rigorous analysis, the area of mathematics that was to become his speciality. A course by W. L. Ferrar on convergence made a particularly strong impact on him. Although Frank had failed to win an Entrance Scholarship to Merton, he was high among the Firsts in the Mathematics Moderations at the end of his first year and, as a result, the college elected him to an Honorary Postmastership (scholarship) and a Commoner’s Exhibition. Following the outbreak of war, Merton College was taken over by a government department, and the undergraduates were evacuated to University College. As he waited to receive his call-up papers, Frank continued to make good progress in mathematics throughout the academic year 1939–1940, but the year was inevitably less idyllic than the previous one.

### 2. War Service

Frank joined the Royal Engineers in July 1940. Well aware of the need for an effective army and with a strong sense of duty, he was an enthusiastic officer cadet, winning a prize at the passing out parade. Initially stationed at Colchester, he saw something of the great air battles of that summer. Indeed, a German bomber flew over the barracks on one occasion, firing its maching guns. Fortunately, nobody was hurt, and Frank would later comment that, in his six years of service in the army, this was the closest he ever came to the enemy!

In June 1941, he was posted to a field company in the 53rd Division, at that time stationed in the pleasant seaside village of Dundrum in Northern Ireland. However, his company was to remain in Northern Ireland for only a few months before being transferred to Liverpool, to replace the 18th Division, which had recently sailed for Singapore. Liverpool had been heavily bombed not long before and, with the darkness and wet of winter, was a depressing place, particularly compared with the relative tranquility of County Down. After attending a course at the School of Military Engineering at Ripon, he was appointed as an instructor there, with the temporary rank of captain. He returned later to his field company, by then
stationed in Maidstone in advance of the invasion of France. However, with the pressing need for mathematicians, he was appointed to Operational Research shortly before D-day and, rather than going to France with the 53rd Division, was sent to India with a small unit set up to test equipment under jungle conditions. This was a bit more interesting, involving some application of the scientific method and statistics. He was demobbed in 1946, finishing with the rank of major and in time to return to Oxford in October of that year to complete his undergraduate studies.

3. Return to academic life

Although Frank had had little time for mathematics while in the army, he did manage to take a few books with him to India, notable among which was a copy of Titchmarsh’s *Theory of functions*. This took on a rather peculiar appearance due to exposure to monsoon conditions. Despite the break in his university education, he found on his return to Oxford that he had developed an improved understanding of the subject; this he put down to the fact that, although he had forgotten a certain amount from his pre-war time at Oxford, long term memory is seldom important in mathematics. During that third year at Oxford, he particularly enjoyed Titchmarsh’s lectures on complex function theory, although the lecturing style was rather eccentric, with formulae written in an apparently random fashion on a small blackboard. What impressed him most was J. H. C. Whitehead’s postgraduate seminar on topological groups, based on Pontrjagin’s book; this gave Frank his first introduction to abstract analysis. He graduated with first class honours in the summer of 1947.

That third year at Oxford was also significant at a personal level in that Frank met his future wife, Gillian (Jill) Patrick, a fellow mathematician in her final year at Somerville College. She and Frank married in July 1947 and were to enjoy a very long and happy marriage. They had no children, but for many years provided a home for two faithful cats whom they called A and B, as only mathematicians would do. Jill was to become a very successful secondary school teacher of mathematics. She survived Frank by two years.

It was intended that, following graduation, Frank would start studying for a doctorate under Whitehead. However, now a married man, he decided instead to accept the offer of a temporary lectureship at The University of Edinburgh. This position was caused by the temporary absence of A. Erdélyi, then a lecturer at Edinburgh, who was visiting Caltech for a year. During that year, Frank taught the analysis course to a fourth year class that included James Mackay (later Lord Mackay of Clashfern, the Lord Chancellor) and George Mackie (later appointed to the Chair of Applied Mathematics at University of Edinburgh). Frank’s move to Edinburgh was to have a significant impact on his subsequent academic career (and by happenchance he, Erdélyi and Mackie would become colleagues many years later).

Although Frank made no real start with research during that year in Edinburgh, a comment by a colleague prompted him to seek out *Théorie des Opérations Linéaires* (Banach 1932), the seminal work on linear abstract analysis. Further, in the spring of 1948, W. W. Rogosinski, who had recently been appointed to a chair at what was then King’s College, Newcastle, visited Edinburgh to give a lecture on the Lebesgue integral. Rogosinski so impressed Frank, both as a mathematician and as a charming man, that he invited him back to the basement flat that he and Jill were renting. Despite the strictures of post-war rationing and a guest suddenly appearing without any warning, Jill managed to rustle up a meal. This was the beginning of a friendship that would last for the rest of Rogosinski’s life.
Following the year’s temporary appointment in Edinburgh, Frank was very pleased to be appointed by Rogosinski to a lectureship at Newcastle in 1948. As a self-taught research mathematician, Frank benefited greatly from Rogosinski’s influence and received every encouragement to undertake research. During the next two years, he made a start on research in analysis along fairly classical lines, with papers on inequalities, generalized convex functions and, in postal collaboration with Morris Marden at the University of Milwaukee, on complex polynomials. His interest in abstract analysis arose initially from reading Banach’s book and was reinforced when he attended what he later described as a brilliant lecture by F. Smithies on commutative Banach algebras at the 1950 British Mathematical Colloquium. However, it was during the year 1950–1951, spent as a visiting associate professor with N. Aronszajn’s research group at Oklahoma State University, that he had his first real opportunity for the serious study of functional analysis, a subject that unifies different parts of mathematical analysis within a single abstract framework. This was when Senator Joseph McCarthy was beginning his witch hunts against those whom he regarded as having left-wing or liberal sympathies. University staff, as state employees, were required to sign documents avowing loyalty to the USA. Frank refused on principle, his salary was duly cut off, and he and Jill had to live on savings for the remainder of their stay. Apart from the year in Oklahoma, Frank was to remain in Newcastle from 1948 to 1965, first as a lecturer, then as a reader (1956–1959) and finally as professor from 1959 after Rogosinski retired.

Frank left Newcastle in 1965 when he was appointed to the newly established McLaurin Chair of Mathematics at The University of Edinburgh (figure 1). This move has some resonance with his earlier time at Edinburgh. Erdélyi had gone back to Caltech in 1948 to a permanent position, but returned to Edinburgh in 1964 to the historic chair of mathematics. At the same time, the university decided to establish a second chair of mathematics, the McLaurin Chair, and it was to this that Frank was appointed in 1965. The committee making the appointment was chaired by the principal, Sir Edward Appleton, who immediately wrote offering Frank the position. This was to be one of Appleton’s very last acts as he died suddenly that night. Frank remained in Edinburgh until he retired in 1984, spending the final three years on full-time research under the Special Replacement Scheme (the ‘New Blood’ scheme) set up at the time by the Science and Engineering Research Council (SERC). As he had by-passed PhD study after his undergraduate years, this represented a neat reversal of the usual pattern of academic life. Following retirement, he and Jill moved to Harrogate and Frank was appointed Honorary Visiting Fellow at the University of Leeds. He remained active in research for many more years, travelling weekly to a small seminar at the University of York. His last research publication appeared in 2000, just two years before he and Jill moved into a retirement home.

Frank was a very lucid lecturer and took his undergraduate teaching responsibilities seriously. However, he will be remembered more for his impact on graduate education. As a leading figure in functional analysis, he attracted many research students over the years, among whom he was affectionately known as FFB. He was an extremely conscientious supervisor, seeing each student for one hour every week and always able to suggest suitable areas for study. During his time at Newcastle and Edinburgh, 25 of his students were awarded PhDs; almost all were to go on to successful academic careers in their own right. Interestingly, his very first student, A. Olubummo (PhD in 1955), was to become the first Nigerian professor of mathematics. During the academic year 1962–1963, Frank was supervising no less than nine PhD students, a remarkable number for a pure mathematician. At the same time as this commitment to his students, he was managing to maintain a steady flow of
high quality research papers. Along with John Ringrose and Barry Johnson at Newcastle, Frank founded the North British Functional Analysis Seminar (NBFAS), one of the first inter-university seminars in mathematics and a model for many others. NBFAS brought numerous distinguished functional analysts to the UK, thereby enriching the experiences of the many research students of the time. Its inaugural meeting was held in Edinburgh in February 1968; in recognition of the key role that Frank played in its foundation, a special meeting in Frank’s memory was held a few months after his death. NBFAS celebrated its fiftieth anniversary, again in Edinburgh, at a meeting in April 2018; both of these later meetings were attended by many of Frank’s former students and colleagues.

Throughout his career, Frank maintained extensive correspondence with mathematicians around the world and collaborated with no fewer than 17 different mathematicians. In addition to the year at Oklahoma State University, he spent four months as a visiting professor at the Tata Institute of Fundamental Research, Bombay (December 1960–March 1961), and the academic year 1966–1967 as a visiting professor at Yale University.

Although he never sought committee work, Frank’s strong sense of duty made him take such work seriously when it came to him. Over the years, he served on the Council of the London Mathematical Society, committees of the Royal Society and the Science Research Council, and on numerous editorial boards. His achievements were recognized with various honours, including the award of the Senior Berwick Prize of the London Mathematical Society and election to the Royal Society of Edinburgh in 1966, election to the Royal Society in 1970, presidency of the Edinburgh Mathematical Society in 1976–1977 and an honorary doctorate from the University of York in 1990. Mention has already been made of the British Mathematical Colloquium, the annual meeting of the British mathematical community. This was established in 1949 and has been meeting every year since then. Frank attended almost
every meeting from 1949 until he retired, both speaking at it and serving on its organizing committee on various occasions.

Frank had well-formed views of the issues confronting a working research mathematician, and he articulated these in ‘A down-to-earth view of mathematics’ (22). In this article, he discussed both the issue of quality versus applicability in any piece of mathematics and the nature of proof. He developed the thesis that, despite the fact that mathematicians are human and all humans make mistakes, nevertheless the discipline has developed a self-correcting mechanism—to use someone else’s results, a mathematician should be satisfied that the earlier result is both correct and applicable to the situation in which it is being applied. This relies on individual mathematicians taking full responsibility for the results they publish and requires absolute integrity; don’t use someone else’s result unless you are convinced that it is correct.

He viewed this as the mathematical equivalent of the scientific method of the experimental sciences that requires the experimenter to give sufficient detail of experimental techniques to allow repeatability. This was something that, many years earlier, he had discussed in his inaugural address after his appointment to the McLaurin Chair, when he pointed out the dangers of using the results of others without being sure of their truth. In this 1982 article, he also took something of a swipe at proofs that required numerous verifications of special cases when the number of cases involved needed the assistance of a computer. For Frank, such so-called proofs did not constitute genuine proofs. In a similar vein, in his acceptance speech for the York honorary doctorate, he spoke of the dangers of over-dependence on computer models in science.

Frank had many interests beyond mathematics: he enjoyed cricket at school and maintained an interest in the sport throughout his life; when snooker became a major sport on television in the 1970s, he became an avid follower; he recorded extensive weather data for Harrogate; and he and Jill were enthusiastic solvers of the Ximenes crossword puzzle in The Observer, their names being listed as winners on more than one occasion. They also shared a love of gardening; the fruits of their large garden in Edinburgh were widely distributed to friends, and the garden in Harrogate was spectacular. It consisted essentially of three different gardens: flowering shrubs and trees, roses and herbaceous borders; exotic flowers in the sun parlour and patio (Jill’s responsibility); and (Frank’s passion) fruit and vegetables. They also shared an enjoyment of climbing and walking on hills, with holidays in the Austrian Alps and memorable visits to the mountains of Norway and Iceland. However, Frank’s chief love was for the Scottish hills. Before moving to Edinburgh, he had already ascended about 40 Munros, the Scottish mountains at least 3000 feet high. These were originally listed by Sir Hector Munro in 1891 and, after moving to Edinburgh, Frank was to complete the ascent of the full set in 1997, an achievement he recorded in his CV. Munro’s list has always been the subject of some controversy (when do two close tops count as separate Munros, for example). In two articles for the Scottish Mountaineering Club Journal (31, 32), Frank developed a rule for determining this. Application of this rule yields a list very close to Munro’s original one and influenced some subsequent revisions of the club’s definitive list. After the move to Harrogate, Frank continued to enjoy walking the Yorkshire hills and dales for many years.

4. SCIENTIFIC CONTRIBUTIONS

As has already been explained, Frank took up a temporary lectureship at Edinburgh immediately following his final undergraduate year at Oxford and so had no period of formal
postgraduate study or guidance in research. To that extent, therefore, he was a self-taught research mathematician.

His earliest papers, written during his first few years at Newcastle, reflected the classical analysis that he had studied as an undergraduate. Inter alia, these involved generalizing the notions of convexity and subharmonicity. Thus, while a convex function is a real-valued function of a real variable, the generalized convex functions that he introduced, calling them sub-(L) functions, are dominated by the solutions of a given second order linear differential equation $L_y = 0$. In higher dimensions, a similar idea generalizes subharmonic functions, with local domination by harmonic functions being replaced by local domination by solutions of a given elliptic second order partial differential equation. Particular mention should be made of his early paper (1) involving extensions and analogues of inequalities appearing in the classic work by Hardy, Littlewood and Pólya. Some of the ideas in this paper, such as seeking best constants in inequalities, are still topical over 60 years later.

However, it was to abstract analysis and, more specifically, to functional analysis and operator theory that Frank would focus his research for the rest of his scientific life. His interests in this broad area were wide, encompassing seven main themes: (i) Banach algebra theory; (ii) operators on partial ordered Banach spaces; (iii) semi-algebras; (iv) the numerical ranges of operators; (v) compact operators; (vi) Hankel operators on Hilbert space; and (vii) atomic decompositions for Banach spaces.

If there is one feature that marks out his research, it is its elegance and aesthetic simplicity. He had the ability to take any topic that he was currently interested in, strip out extraneous complications, get to the heart of the matter and then convey this, either in writing or in a lecture, to a wider audience. This aesthetic simplicity is well illustrated with one anecdote, recalled by John Duncan, one of his PhD students. In the early 1960s, the graduate seminar at Newcastle was struggling through a long complicated proof of a theorem of Choquet about extreme points of compact convex sets in general spaces. Frank reduced the whole proof to one simple application of the Hahn–Banach theorem. When he expounded the proof at the British Mathematical Colloquium, the lecture hall was filled to overflowing—and even included many statisticians!

### 4.1. Banach algebras

A Banach algebra is a linear associative algebra $A$ equipped with a norm $\| \cdot \|$ that satisfies $\|xy\| \leq \|x\|\|y\|$ for all $x,y \in A$ and with respect to which $A$ is a Banach space. The underlying scalar field is usually taken to be the complexes, although some parts of the theory can be developed with real scalars. There were a number of early papers in the 1930s in which the notion of multiplication in a Banach space was studied, but the systematic development of a general theory of Banach algebras was initiated by I. M. Gelfand in his seminal paper (Gelfand 1941a) in which he used elementary ideal theory to great effect. In particular, he showed that a semi-simple commutative Banach algebra with an identity element (assumed without any real loss of generality to have norm 1) is isomorphic to an algebra of continuous functions on a compact Hausdorff space. In a subsequent paper (Gelfand 1941b), Gelfand used his theory to give an elegant proof of Wiener’s result that, if $f$ is a non-vanishing function on the unit circle with absolutely convergent Fourier series, then $f^{-1}$ also has an absolutely convergent Fourier series. This proof attracted considerable attention at the time and it was the lecture by F. Smithies on Gelfand’s work, mentioned above, that cemented Frank’s interest in abstract analysis.
Frank’s earliest work on Banach algebras was undertaken with the algebraist A. W. Goldie, who had been appointed to a lectureship at Newcastle on the same day as Frank. They studied algebras with a trace through which all continuous linear functionals can be represented (2), and showed that such algebras are dual algebras in the sense of I. Kaplansky (that is, each closed one-sided ideal is its own second annihilator). However, unaware at the time of Kaplansky’s work, they developed a structure theory for dual algebras, which they found applied to a wider class of algebras and which they called annihilator algebras. These are algebras in which every proper closed one-sided ideal has a non-zero annihilator. The standard model for the building blocks of an annihilator algebra is the algebra $F(X)$ of all uniform limits of finite rank operators on a reflexive Banach space $X$. Each such algebra $F(X)$ is an annihilator algebra but can fail to be a dual algebra, this latter fact being established only much later by A. M. Davie (Davie 1973). Another strand to Frank’s work on Banach algebras, this time with John Duncan, involved the concept of a dual representation of a Banach algebra. This used linear functionals on the algebra to construct representations and resulted in interesting connections between the algebraic and geometric structures of the algebra (10, 11). One other feature of Frank’s work on Banach algebras worth highlighting involves the Jordan product, that is the mapping $(x, y) \rightarrow xy + yx$. He developed an interest in this in the latter half of the 1970s; of particular note are the papers concerning the connection between order structure and the Jordan product (19, 20).

4.2. Operators on partially ordered Banach spaces

During the late fifties, Frank became particularly interested in the Krein–Rutman theorem. This, a generalization of the classical Perron–Frobenius theorem of linear algebra, asserts that, if $X^+$ is a closed cone in a Banach space such that $X^+ - X^+$ is dense in $X$ and $T$ is a compact linear operator on $X$ that maps $X^+$ into itself, then the spectral radius of $T$ is an eigenvalue of $T$ with an eigenvector in $X^+$. As so often was the case, Frank found an elementary proof of this result; it rests on the fact that a boundary point of the spectrum of an element of a Banach algebra gives rise to a topological divisor of zero. This led to a series of papers appearing in the latter half of the 1950s giving variants and generalizations of the Krein–Rutman theorem. Of particular note here is where the context is of a real normed space $X$ with a (non-zero) cone $X^+$ that is complete with respect to the norm and a linear operator $T$ mapping $X^+$ continuously into itself and acting compactly on $X^+$ in a natural sense (3). There is then a natural definition of the partial spectral radius $\mu$ of $T$, given by $\mu = \lim_{n \to \infty} \left( p(T^n) \right)^{1/n}$, where $p(T^n) = \sup \{ \| T^n x \| : x \in X^+ \text{ and } \| x \| \leq 1 \}$, and it is shown that $\mu$ is an eigenvalue of $T$ with a corresponding eigenvector in $X^+$.

4.3. Semi-algebras

Frank’s work on semi-algebras had two components: semi-algebras of continuous functions and locally compact semi-algebras.

Semi-algebras of continuous functions: Let $C(E)$ be the algebra of all continuous real-valued functions on a compact Hausdorff space $E$. When Frank started to consider this algebra, it was well known that its closed subalgebras containing the constant function 1 are determined by the sets of pairs of points that the subalgebra fails to separate (i.e. the pairs $x, y$ in $E$ such that $f(x) = f(y)$ for all $f$ in the subalgebra); this is the content of the Stone–Weierstrass theorem and its descendants. Instead of considering subalgebras, Frank was interested in exploring
what could be said about closed semi-algebras in $C(E)$, non-void closed subsets $A$ of $C(E)$ with the property that

$$f, g \in A \text{ and } \alpha \geq 0 \Rightarrow fg, f + g \text{ and } \alpha f \in A.$$  

As they stand, these axioms are too weak for anything substantial to be proved, so, in order to make progress, he introduced the notion of type. A semi-algebra $A$ of $C(E)$ is of type $n$ for $n$ a non-negative integer if

$$f \in A \Rightarrow f^n(1 + f)^{-1} \in A.$$  

The motivation here was consideration of the semi-algebras of $C[0, 1]$ consisting of the functions with successive non-negative $k^{th}$ differences up to $k = n$. He was able to give complete descriptions of the type $n$ semi-algebras of $C(E)$ for $n = 0$ and $1$ (the case $n = 1$ involving the introduction of a partial order on $E$) and some partial results when $n = 2$ (see (4), (5), (6)). Beyond that, no progress was made until, many years later, he returned to the topic in his final paper (28).

Locally compact semi-algebras: Here, Frank considered the semi-algebras $A$ in a Banach algebra $B$ with the property that the intersection of $A$ with the unit ball of $B$ is compact in the norm topology. He developed a satisfactory Wedderburn theory for such semi-algebras (7), and followed this with a paper where the theory was applied to the semi-algebra generated by a compact linear operator on a Banach space (8). These ideas were taken up later by M. A. Kaashoek and T. T. West (Kaashoek & West 1974).

4.4. The numerical ranges of an operator

For an operator $T$ on a Hilbert space $H$, its numerical range $W(T)$ is given by

$$W(T) = \{(Tx, x) : x \in H, \quad (x, x) = 1\}.$$  

The concept was introduced by Toeplitz in 1918 and, a year later, Hausdorff proved that $W(T)$ is convex. Given the rich theory of operators on Hilbert space, the concept has played only a minor role in Hilbert space operator theory. However, with few tools available for the study of operators on Banach spaces, the concept was generalized in the Banach setting independently by G. Lumer (1961) and H. Bauer (1962). Lumer’s definition involved the notion of a semi-inner-product on a Banach space (such always exist but are not unique), while Bauer’s relies purely on duality between the space and its dual. This latter definition has proved more effective. More precisely, the spatial numerical range $V(T)$ of an operator $T$ on a Banach space $X$ is defined as

$$V(T) = \{f(Tx) : x \in X, f \in X^* \text{ and } \|x\| = \|f\| = f(x) = 1\},$$  

where, as usual, $X^*$ denotes the dual space of $X$.

The introduction of numerical range concepts for Banach spaces was followed by a decade of hectic activity. Frank published a number of papers in the area (12, 13, 15, 17, 18), as well as two expository articles (16, 21). Further, together with John Duncan, he organized a highly successful conference in Aberdeen in the summer of 1971 that brought all the main players in the area together and he co-authored two sets of lecture notes (29, 30).
4.5. Compact operators

The notion of a compact operator is a natural generalization of a finite rank operator to an infinite dimensional setting. It goes back to the origins of functional analysis at the beginning of the twentieth century with the work of F. Riesz and J. Schauder, inter alia. Frank wrote three papers involving compact operators, in which he used ideas from Banach algebra theory to give a more algebraic approach to their study. To illustrate, he observed that, if $T$ is a compact operator on a Banach space $X$ and $Y$ is the algebra of all operators on $X$ that commute with $T$ (endowed with the operator norm), then the mapping $S \rightarrow TS$ ($S \in Y$) is a compact operator on $Y$ (9). He then used the theory developed earlier (8) to obtain the classical Riesz–Schauder theory. This approach to compact operators was further developed later in (14).

4.6. Hankel operators on Hilbert space

A Hankel matrix is an infinite matrix with $i,j$th entry of the form $a_{i+j}$ ($i,j \geq 0$) for some complex sequence $\{a_n\}$ and a Hankel operator on the Hilbert space $H^2$ of square integrable functions on the unit circle $\mathbb{T}$ with vanishing negative Fourier coefficients is an operator whose matrix relative to the natural orthonormal basis $\{\chi_n\}_{n \geq 0}$ of $H^2$, where $\chi_n(\xi) = \xi^n$, is a Hankel matrix. The fundamental result on Hankel operators, Nehari’s theorem, asserts that $\{a_n\}$ determines a bounded operator on $H^2$ in this way if and only if there is a function $\varphi$ in $L^\infty(\mathbb{T})$ with Fourier coefficients satisfying $\hat{\varphi}(n) = a_n$ for $n \geq 0$ and that there exists such a $\varphi$ with $\|\varphi\|_\infty$ equal to the norm of the operator. Such a function is called a symbol of the operator, which is often denoted by $S_\varphi$. The relationship between a Hankel operator $S_\varphi$ and its symbol $\varphi$ provides a fruitful connection between operator theory and classical function theory. Frank’s interest in Hankel operators dates back to the 1970s, but his work in this area intensified some 10 years later. By then, he had lost some of his enthusiasm for abstraction and was drawn more to the borderland of functional and classical analysis. In his proposal to SERC under the Special Replacement Scheme, he gave as his main aim ‘to seek new results and new proofs in function theory by using operator theory and Banach algebra techniques’. Hankel operators provided a good starting point for this programme.

The first problem that he tackled was of finding criteria for a Hankel matrix $(a_{i+j})_{i,j \geq 0}$ to give a bounded operator on Hilbert space. In a sense, Nehari’s theorem provides a solution, but this is unsatisfactory as a test since there is no description of the non-negative Fourier coefficients of a bounded function on $\mathbb{T}$. To establish such a criterion, he considered the functions $v_z$ for $z$ in $D$, the open unit disc in the complex plane, defined by

$$v_z(\xi) = (1 - |z|^2)^{1/2}(1 - \bar{z}\xi)^{-1} \quad (\xi \in \mathbb{T}).$$

He showed (23) that $(a_{i+j})_{i,j \geq 0}$ is bounded if and only if

$$\sup_{z \in D} \|R_\varphi v_z\|_2 < \infty.$$

Here $R_\varphi$ is the mapping of $H^\infty$ (the bounded functions in $H^2$) to $H^2$ given by $R_\varphi = PM_\varphi J$, where $J$ is the reversion mapping sending $f(\xi)$ to $f(\bar{\xi})$, $M_\varphi$ is multiplication by $\varphi$ on $L^2$ and $P$ is the orthogonal projection of $L^2$ onto $H^2$. A second sufficient condition for boundedness was given in the same paper and this led to a proof of C. Fefferman’s unpublished criterion when the entries of $(a_{i+j})_{i,j \geq 0}$ are non-negative: such a matrix is bounded if and only if the sequence
\{c_n\}_{n \geq 1} \text{ is bounded, where}

\begin{equation*}
    c_n = \sum_{j=1}^{\infty} \left( \sum_{r=0}^{n-1} a_{j+r} \right)^2.
\end{equation*}

This second condition for boundedness was improved by F. Holland and D. Walsh (Holland & Walsh 1984). The criterion for boundedness (23) was strengthened (24), where it was shown that the condition \(\sup_{n \geq 1} \| R_{\phi} v_{z_n} \|_2 < \infty\) suffices for boundedness, where \(\{z_n\}\) is an appropriately chosen sequence of points in \(D\).

4.7. Atomic decompositions for Banach spaces

An atomic decomposition theorem for a Banach space \(X\) establishes that every element of \(X\) is of the form \(\sum_{n=1}^{\infty} \lambda_n u_n\), where the \(\lambda_n\)'s are scalars with \(\sum_{n=1}^{\infty} |\lambda_n| < \infty\) and the \(u_n\)'s belong to some given bounded subset of \(X\). An early example of such a result was a decomposition theorem of Coifman and Rochberg for the Bergman space \(L^1_a\) of functions that are analytic and integrable on \(D\). Motivated by an approach to this result due to D. H. Luecking (Luecking 1985), Frank established a general atomic decomposition theorem for Banach spaces (25). This takes the following form. Let \(X\) be a Banach space with dual space \(X^*\) and let \(E\) be a given subset of \(X\). For \(f\) in \(X\), let \(\Lambda(E,f)\) denote the set (possibly empty) of absolutely summable complex sequences \(\{\lambda_k\}\) such that \(f = \sum_{k=1}^{\infty} \lambda_k u_k\) for some \(u_k\) in \(E\) and let \(m, M\) be positive constants. Then the following statements are equivalent.

(i) For each \(\phi\) in \(X^*\),

\[m\|\phi\| \leq \sup\{|\phi(u)| : u \in E\} \leq M\|\phi\|.

(ii) For each \(f\) in \(X\), \(\Lambda(E,f)\) is non-empty and

\[M^{-1}\|f\| \leq \inf\{\|\lambda\|_1 : \lambda \in \Lambda(E,f)\} \leq m^{-1}\|f\|.

Later, he was able to add a further equivalent condition:

(iii) \(mX_1 \subseteq \text{abco } E \subseteq MX_1\).

Here, \(X_1\) is the closed unit ball of \(X\) and \(\text{abco } E\) denotes the closed absolutely convex hull of \(E\); these ideas were subsequently developed further (26, 27).

5. ACKNOWLEDGEMENTS

This memoir draws heavily on extensive autobiographical notes that Frank left, as well as on an obituary published by the Royal Society of Edinburgh and co-authored by John Duncan and myself. I am also grateful to Frank’s nephew, Professor Peter Bonsall, Emeritus Professor of Transport Planning at the University of Leeds, for background family information.

The portrait photograph was taken by Godfrey Argent and is Copyright © Godfrey Argent Studio.

REFERENCES TO OTHER AUTHORS

Banach, S. 1932 Théorie des Opérations Linéaires. In Monografje Matematyczne Warsaw; 2nd edn reprinted by Chelsea Publ. Co., New York, 1963.
Bibliography

The following publications are those referred to directly in the text. A full bibliography is available as electronic supplementary material at http://dx.doi.org/10.1098/rsbm.2020.0007.

(1) 1951 Inequalities with non-conjugate parameters. Q. J. Math. (Oxford Ser. (2)) 2, 135–150.
(2) 1953 (With A. W. Goldie) Algebras which represent their linear functionals. Proc. Cambr. Phil. Soc. 49, 1–14. (doi:10.1017/S0305004100027973)
(3) 1958 Linear operators in complete positive cones. Proc. Lond. Math. Soc. S3–8, 53–75. (doi:10.1112/plms/s3-8.1.53)
(4) 1960 Semi-algebras of continuous functions. Proc. Lond. Math. Soc. S3–10, 122–140. (doi:10.1112/plms/s3-10.1.122)
(5) 1960 Semi-algebras of continuous functions. Proc. Int. Symp. Linear Spaces, Jerusalem, pp. 101–114.
(6) 1962 On type 2 semi-algebras of continuous functions. Proc. Lond. Math. Soc. S3–12, 133–143. (doi:10.1112/plms/s3-12.1.133)
(7) 1963 Locally compact semi-algebras. Proc. Lond. Math. Soc. S3–13, 51–70. (doi:10.1112/plms/s3-13.1.51)
(8) 1965 (With B. J. Tomiuk) The semi-algebra generated by a compact linear operator. Proc. Edinb. Math. Soc. 14(3), 179–196.
(9) 1967 Compact linear operators from an algebraic standpoint. Glas. Math. J. 8, 41–49. (doi:10.1017/S0017089500000069)
(10) 1967 (With J. Duncan) Dual representations of Banach algebras. Acta Math. 117, 79–102. (doi:10.1007/BF02395041)
(11) 1968 (With J. Duncan) Dually irreducible representations of Banach algebras. Q. J. Math. (Oxford Ser. (2)) 19, 67–111. (doi:10.1093/qmath/19.1.67)
(12) 1968 (With B. E. Cain & H. Scheider) The numerical range of a continuous mapping of a normed space. Aequationes mathematicae 2, 86–93.
(13) 1969 The numerical range of an element of a normed algebra. Proc. Glas. Math. Assoc. 10, 68–72.
(14) 1969 Operators that act compactly on an algebra of operators. Bull. Lond. Math. Soc. 1, 163–170. (doi:10.1112/blms/1.2.163)
(15) 1970 (With M. J. Crabb) The spectral radius of a Hermitian element of a Banach algebra. Bull. Lond. Math. Soc. 2, 178–180. (doi:10.1112/blms/2.2.178)
(16) 1970 The numerical range of an operator. Eureka 33, 24–27.
(17) 1972 Hermitian operators on Banach spaces. Colloquia Math. Soc. János Bolyai 5, 65–75.
(18) 1973 An inclusion theorem for the matrix and essential ranges of operators. J. Lond. Math. Soc. 6, 329–332.
(19) 1975 Locally multiplicative wedges in Banach algebras. Proc. Lond. Math. Soc. S3–30, 239–256. (doi:10.1112/plms/s3-30.2.239)
(20) 1978 Jordan subalgebras of Banach algebras. Proc. Edinb. Math. Soc. 21, 103–110. (doi:10.1017/S0013091500016060)
(21) 1980 (With J. Duncan) Numerical ranges. In Studies in functional analysis ed. 1–49. Mathematical Association of America.
(22) 1982 A down-to-earth view of mathematics. Am. Math. Month. 89, 8–15. (doi:10.1080/00029890.1982.11995374)
(23) 1984 Boundedness of Hankel matrices. *J. Lond. Math. Soc.* **29** (2), 289–300.
(24) 1984 Criteria for boundedness and compactness of Hankel operators. *Contemp. Math.* **32**, 83–95. (doi:10.1090/conm/032/769499)
(25) 1986 Decompositions of functions as sums of elementary functions. *Q. J. Math. (Oxford Ser. (2))* **37**, 129–136. (doi:10.1093/qmath/37.2.129)
(26) 1987 Domination of the supremum of a bounded harmonic function by its supremum over a countable set. *Proc. Edinb. Math. Soc.* **30**, 471–477. (doi:10.1017/S0013091500026869)
(27) 1991 A general atomic decomposition theorem and Banach’s closed range theorem. *Q. J. Math. (Oxford Ser. (2))* **42**, 9–14. (doi:10.1093/qmath/42.1.9)
(28) 2000 Type 2 semi-algebras of continuous functions. *Proc. Lond. Math. Soc.* **S3–81**, 725–746.

**Books and Lecture Notes**

(29) 1971 (With J. Duncan) *Numerical ranges of operators on normed spaces and of elements of normed algebras*, Lond. Math. Soc. Lect. Notes Series 2, Cambridge: London Mathematical Society.
(30) 1973 (With J. Duncan) *Numerical ranges II*, Lond. Math. Soc. Lect. Notes Series 10. Cambridge: London Mathematical Society.
(31) 1973 (with J. Duncan) *Complete Normed Algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete 80*, Springer-Verlag: New York.

**Non-mathematical Articles**

(32) 1973 The separation of mountains. *Scottish Mountaineering Club J.* **30**, 153–156.
(33) 1974 The separation of Munros. *Scottish Mountaineering Club J.* **30**, 254–256.