On the Schrödinger equations with time-dependent potentials growing polynomially in the spatial direction

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Abstract

The Cauchy problem for the Schrödinger equations is studied with time-dependent potentials growing polynomially in the spatial direction. First the existence and the uniqueness of solutions are shown in the weighted Sobolev spaces. In addition, we suppose that our potentials are depending on a parameter. Secondly it is shown that if potentials depend continuously and differentiably on the parameter, the solutions to the Schrödinger equations respectively become continuous and differentiable with respect to its parameter.

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**Keywords** Schrödinger equations; time-dependent potentials; polynomially growing potentials in the spatial direction; continuity and differentiability with respect to a parameter.

**AMS Subject Classification (2010)** 35Q41; 35Q40.

1 Introduction

Let $T > 0$ be an arbitrary constant. We will study the Schrödinger equations

\[ i\hbar \frac{\partial u}{\partial t}(t) = H(t)u(t) \]

\[ := \left[ \frac{1}{2m} \sum_{j=1}^{d} \left( \frac{\hbar}{i} \frac{\partial}{\partial x_j} - qA_j(t, x) \right)^2 + qV(t, x) \right] u(t), \]

(1.1)

where $t \in [0, T], x = (x_1, \ldots, x_d) \in \mathbb{R}^d, (V(t, x), A(t, x)) = (V, A_1, A_2, \ldots, A_d) \in \mathbb{R}^{d+1}$ are electromagnetic potentials, $\hbar$ is the Planck constant, $m > 0$ is the mass of a particle and $q \in \mathbb{R}$ is its charge. For the sake of simplicity we suppose $\hbar = 1$ and $q = 1$ hereafter.

In the present paper we will consider scalar potentials $V(t, x)$ that are time-dependent and growing polynomially in $\mathbb{R}^d$. That is,

\[ C_0 < x >^{2(M+1)} - C_1 \leq V(t, x) \leq C_2 < x >^{2(M+1)} \]

(1.2)

in $[0, T] \times \mathbb{R}^d$ with constants $M \geq 0, C_0 > 0, C_1 \geq 0$ and $C_2 > 0$, where $|x| = \left( \sum_{j=1}^{d} x_j^2 \right)^{1/2}$ and $< x > = (1 + |x|^2)^{1/2}$. We denote by $L^2 = L^2(\mathbb{R}^d)$ the space of all square integrable functions on $\mathbb{R}^d$ with inner product $(f, g) := \int f(x)g(x)^*dx$ and norm $\|f\|$, where $g^*$ is the complex conjugate of $g$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ we write $|\alpha| = \sum_{j=1}^{d} \alpha_j, \partial_{x_j} = \partial/\partial x_j$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$. For $M \geq 0$ in (1.2) let us introduce the weighted Sobolev
spaces

\[ B^a(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d) ; \| f \|_a := \| f \| + \sum_{|\alpha| \leq 2a} \| \partial_\alpha^2 f \| + \| \cdot^{2a(M+1)} \| f \| < \infty \} \]  \quad (1.3)

for \( a = 1, 2, \ldots \). We denote the dual space of \( B^a \) (\( a = 1, 2, \ldots \)) by \( B^{-a} \) and the \( L^2 \) space by \( B^0 \).

The main aim in the present paper is to prove that for any \( u_0 \in B^a \) (\( a = 0, \pm 1, \pm 2, \ldots \)) there exists the unique solution \( u(t) \in {\mathcal E}_t^0([0,T]; B^a) \cap {\mathcal E}_t^1([0,T]; B^{a-1}) \) with \( u_0 \) at \( t = 0 \) to (1.1), where \( {\mathcal E}_t^j([0,T]; B^a) \) (\( j = 0, 1, \ldots \)) denotes the space of all \( B^a \)-valued \( j \)-times continuously differentiable functions on \([0,T] \).

Our results above will be applied in [8] to the proof of the convergence of the Feynman path integrals for the Schrödinger equations (1.1) with potentials growing polynomially in \( \mathbb{R}^d_x \), i.e. satisfying (1.2). The proof of the convergence of the Feynman path integrals for such Schrödinger equations has been hardly obtained (cf. §10.2 in [1]). It should be noted that the existence and the uniqueness of solutions not only in \( L^2 \) but also in \( B^1 \) have been necessary to prove the convergence in the \( L^2 \) space of the Feynman path integrals as seen in [5, 7].

The results of the existence and the uniqueness of solutions to (1.1) will be extended for the multi-particle systems. For the sake of simplicity we will
consider the 4-particle systems
\[ i \frac{\partial u}{\partial t}(t) = H(t)u(t) := \left[ \sum_{k=1}^{4} \left\{ \frac{1}{2m_k} \sum_{j=1}^{d} \left( \frac{1}{i} \frac{\partial}{\partial x_j^{(k)}} - A_j^{(k)}(t, x^{(k)}) \right) \right\}^2 + V_k(t, x^{(k)}) \right\} + \left[ \sum_{1 \leq i < j \leq 4} W_{ij}(t, x^{(i)} - x^{(j)}) \right] u(t) \]
\[ \equiv \left[ \sum_{k=1}^{4} H_k(t) + \sum_{1 \leq i < j \leq 4} W_{ij}(t, x^{(i)} - x^{(j)}) \right] u(t), \] (1.4)
where \( x^{(k)} \in \mathbb{R}^d \) (\( k = 1, 2, 3, 4 \)). Our results in the present paper for the 4-particle systems will be easily extended for the general multi-particle systems.

In addition, we suppose that the potentials of (1.1) and (1.4) are depending on a parameter. The second aim in the present paper is to prove that if potentials depend continuously and differentiably on the parameter, the solutions to (1.1) and (1.4) respectively become continuous and differentiable with respect to its parameter in \( \mathcal{E}_0^0([0, T]; B^a) \) (\( a = 0, \pm 1, \pm 2, \ldots \)). Such results have been well known in the theory of ordinary differential equations as the fundamental ones.

When the Hamiltonian \( H(t) \) is independent of \( t \in [0, T] \), i.e. \( H(t) = H \), the existence and the uniqueness of solutions in the \( L^2 \) space to (1.1) and (1.4) are equivalent to the self-adjointness of \( H \) (cf. §8.4 in [11]). The self-adjointness of \( H \) in \( L^2 \) has almost been settled now (cf. [3, 10, 12]), as stated in the introductions of [13, 14, 15].

It should also be noted that the Hamiltonian \( H_0 = -\sum_{j=1}^{d} \partial_{x_j}^2 - a|x|^b \) with constants \( a > 0 \) and \( b > 2 \) on \( C_0^\infty(\mathbb{R}^d) \) is not essentially self-adjoint in \( L^2 \) (cf. pp. 157-159 in [2]), \( H_0 \) has equal deficiency (cf. Theorem X.3 in [12]) and so \( H_0 \) has an infinite number of different self-adjoint extensions in \( L^2 \) from Theorem X.2 and its corollary in [12], where \( C_0^\infty(\mathbb{R}^d) \) denotes the space of all
infinitely differentiable functions with compact support on \( \mathbb{R}^d \). Consequently, as well known (cf. Theorem VIII.7 in [11]), the Stone theorem shows that the uniqueness of solutions to (1.1) with \( H(t) = H_0 \) doesn’t always hold in \( L^2 \).

If \( H(t) \) is not independent of \( t \in [0, T] \), the problem is never simple. In [13] Yajima has proved the existence and the uniqueness of solutions to (1.1) in \( B^a \) \( (a = 0, \pm 1, \pm 2, \ldots) \) with \( M = 1 \) under the assumptions

\[
|\partial_x^\alpha V(t, x)| \leq C_\alpha, \ |\alpha| \geq 2,
\]

\[
\sum_{j=1}^{d} (|\partial_x^\alpha A_j(t, x)| + |\partial_x^\alpha \partial_t A_j(t, x)|) \leq C_\alpha, \ |\alpha| \geq 1,
\]

\[
\sum_{1 \leq j < k \leq d} |\partial_x^\alpha B_{jk}(t, x)| \leq C_\alpha < x >^{-\langle 1 + \delta_\alpha \rangle}, \ |\alpha| \geq 1
\]

with constants \( \delta_\alpha > 0 \) and \( C_\alpha \geq 0 \) by the theory of Fourier integral operators, where \( B_{jk} = \partial A_k/\partial x_j - \partial A_j/\partial x_k \). In the present paper we often use symbols \( C, C_\alpha, C_{\alpha, \beta}, C_{\alpha, \delta} \) and \( \delta_\alpha \) to write down constants, though these values are different in general. In [4] the author has proved the existence and the uniqueness of solutions in \( B^a \) \( (a = 0, \pm 1, \pm 2, \ldots) \) with \( M = 1 \) under the assumptions

\[
|\partial_x^\alpha V(t, x)| \leq C_\alpha < x >, \ |\alpha| \geq 1,
\]

\[
\sum_{j=1}^{d} |\partial_x^\alpha A_j(t, x)| \leq C_\alpha, \ |\alpha| \geq 1
\]

by the energy method. Recently, general results about the existence and the uniqueness of solutions to (1.1) in the \( L^2 \) space have been obtained in [14] by the semi-group method. In [15] results in \( B^a \) \( (a = 0, 1) \) with \( M = 1 \) for the multi-particle systems, e.g. (1.4) have been obtained with singular potentials \( W_{ij} \) under the assumptions (1.5) for \( (V_k, A^{(k)}) \) by the theories of semi-groups and Fourier integral operators.

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It should be noted that our results in the present paper and in addition even the results in [14] are not enough to study the equations (1.1) in a general way. For example, both of these results can not be applied to the simple equations (1.1) with \( V = a(t)|x|^4 + |x|^2 \) and \( A = 0 \) where \( a(0) = 0 \) and \( a(t) > 0 \) \((t \in (0, T])\), as mentioned in Remark 2.1 of the present paper.

Next let us consider the Schrödinger equations with potentials dependent on a parameter. When the Hamiltonian \( H(t) \) is independent of \( t \in [0, T] \), it follows from Theorems VIII. 21 and VIII. 25 in [11] that if potentials are continuous with respect to its parameter, in the \( L^2 \) space so are the solutions to (1.1) and (1.4).

If \( H(t) \) is not independent of \( t \in [0, T] \), the problem is never simple like the existence and the uniqueness of solutions. In [6] the author has proved that if potentials depend continuously and differentiably on a parameter under the assumptions (1.6) and (1.7), the solutions to (1.1) respectively become continuous and differentiable with respect to its parameter in \( \mathcal{E}_0^0([0, T]; B^a) \) \((a = 0, 1, 2, \ldots)\) with \( M = 1 \).

All proofs of our results will be given by the energy method as in [4] and [6]. The crucial point in the proofs of our results for (1.1) is to introduce a family of bounded operators \( \{H_\varepsilon(t)\}_{0 < \varepsilon \leq 1} \) in \( B^a \) \((a = 0, \pm 1, \pm 2, \ldots)\) by (4.1) as an approximation of \( H(t) \) in (1.1). Then we can prove Proposition 4.2, by which we can complete the proofs of our results for (1.1) as in [4] and [6]. In the same way the crucial point in the proofs of our results for (1.4) is to introduce \( \{H_\varepsilon(t)\}_{0 < \varepsilon \leq 1} \) by (5.31) as an approximation of \( H(t) \) in (1.4). As in the proofs for (1.1) we can complete the proofs for (1.4). We note that the results in the present paper for the 4-particle systems (1.4) give generalizations of those for (1.1) in the present paper and [4, 6].
The plan of the present paper is as follows. In §2 we will state all theorems. §3 is devoted to preparing for the proofs of the theorems for (1.1). In §4 we will prove all theorems for (1.1). In §5 we will prove all theorems for (1.4).

2 Theorems

Assumption 2.1. We assume for all $\alpha$ and $k = 0, 1$ that $\partial_x^\alpha \partial_t^k V(t, x)$ and $\partial_x^\alpha \partial t^k A_j(t, x)$ ($j = 1, 2, \ldots, d$) are continuous in $[0, T] \times \mathbb{R}^d$. In $[0, T] \times \mathbb{R}^d$ we assume (1.2) with constants $M \geq 0, C_0 > 0, C_1 \geq 0$ and $C_2 > 0$, and the following for $j = 1, 2, \ldots, d$. We have

\[ |\partial_x^\alpha V(t, x)| \leq C_\alpha < x >^{2(M+1)}, |\alpha| \geq 1, \quad (2.1) \]
\[ |\partial_x^\alpha \partial_t V(t, x)| \leq C_\alpha < x >^{2(M+1)} \quad (2.2) \]

for all $\alpha$,

\[ |A_j(t, x)| \leq C < x >^{M+1-\delta} \quad (2.3) \]

with a constant $\delta > 0$,

\[ |\partial_x^\alpha A_j(t, x)| \leq C_\alpha < x >^{M+1}, |\alpha| \geq 1 \quad (2.4) \]

and

\[ |\partial_x^\alpha \partial_t A_j(t, x)| \leq C_\alpha < x >^{M+1} \quad (2.5) \]

for all $\alpha$.

Let $B^a$ be the weighted Sobolev spaces introduced in §1.

Theorem 2.1. Under Assumption 2.1 for any $u_0 \in B^a$ ($a = 0, \pm 1, \pm 2, \ldots$) there exists the unique solution $u(t) \in \mathcal{E}^0_t([0, T]; B^a) \cap \mathcal{E}^1_t([0, T]; B^{a-1})$ with $u(0) = u_0$ to (1.1). This solution $u(t)$ satisfies

\[ \|u(t)\|_a \leq C_a \|u_0\|_a \quad (0 \leq t \leq T) \quad (2.6) \]
and in particular

\[ \|u(t)\| = \|u_0\| \quad (0 \leq t \leq T). \] (2.7)

**Remark 2.1.** Let \(a(t)\) be a continuous function on \([0, T]\) such that \(a(0) = 0\) and \(a(t) > 0\) \((0 < t \leq T)\). Since \(V := a(t)|x|^4 + |x|^2\) does not satisfy (1.2) for any \(M \geq 0\), Theorem 2.1 cannot be applied to (1.1) with \(H(t) := (1/2m) \sum_{j=1}^{d} (-i\partial_{x_j})^2 + a(t)|x|^4 + |x|^2\). In addition, Theorems 1.2 and 1.4 in [14] cannot be applied either, because these self-adjoint operators \(H(t) (0 \leq t \leq T)\) in \(L^2(\mathbb{R}^d)\) don’t have a common domain.

Next, let us consider the Schrödinger equations (1.1) with potentials \((V(t,x;\rho), A(t,x;\rho))\) dependent on a parameter \(\rho \in \mathcal{O}\), where \(\mathcal{O}\) is an open set in \(\mathbb{R}\).

**Theorem 2.2.** We suppose that \((V(t,x;\rho), A(t,x;\rho))\) for all \(\rho \in \mathcal{O}\) satisfy Assumption 2.1 and have the uniform estimates (1.2) and (2.1) - (2.5) with respect to \(\rho \in \mathcal{O}\). In addition, we assume that \(\partial_{\alpha}^\alpha V(t,x;\rho)\) and \(\partial_{\alpha}^\alpha A_j(t,x;\rho)\) \((j = 1,2,\ldots,d)\) for all \(\alpha\) are continuous in \([0,T] \times \mathbb{R}^d \times \mathcal{O}\). Let \(u_0 \in B^a\) \((a = 0, \pm 1, \pm 2, \ldots)\) be independent of \(\rho\) and \(u(t;\rho)\) the solutions to (1.1) with \(u(0;\rho) = u_0\) determined in Theorem 2.1. Then, the mapping : \(\mathcal{O} \ni \rho \rightarrow u(t;\rho) \in \mathcal{E}_t^0([0,T];B^a)\) is continuous, where the norm in \(\mathcal{E}_t^0([0,T];B^a)\) is \(\max_{0 \leq t \leq T} \|f(t)\|_a\).

We set

\[ h(t,x,\xi) := \frac{1}{2m} |\xi - A(t,x)|^2 + V(t,x). \] (2.8)

Let \(\mathcal{S}(\mathbb{R}^d)\) denote the Schwartz space of all rapidly decreasing functions on \(\mathbb{R}^d\) and \(\chi \in \mathcal{S}(\mathbb{R}^d)\) such that \(\chi(0) = 1\). Then, using the oscillatory integral, we can write \(H(t)f\) in (1.1) for \(f \in \mathcal{S}(\mathbb{R}^d)\) as

\[
H(t)f = H \left( t, \frac{X + X'}{2}, D_x \right) f = Os - \int \int e^{i(x-y) \cdot \xi} h \left( t, \frac{x + y}{2}, \xi \right) f(y) dy d\xi \\
:= \lim_{\epsilon \to 0} \int \int e^{i(x-y) \cdot \xi} \chi(\epsilon \xi) h \left( t, \frac{x + y}{2}, \xi \right) f(y) dy d\xi, \quad d\xi = (2\pi)^{-d} d\xi \] (2.9)
in the form of the pseudo-differential operator with the double symbol $h(t, (x + x')/2, \xi)$ (cf. \[9\]).

**Theorem 2.3.** Besides the assumptions of Theorem 2.2 we suppose for all $\alpha$ and $j = 1, 2, \ldots, d$ that $\partial_\rho \partial_x^\alpha V(t, x; \rho)$ and $\partial_\rho \partial_x^\alpha A_j(t, x; \rho)$ are continuous in $[0, T] \times \mathbb{R}^d \times \mathcal{O}$ and satisfy

$$
\sup_{\rho \in \mathcal{O}} |\partial_\rho \partial_x^\alpha V(t, x; \rho)| \leq C_\alpha < x >^{2(M+1)} \quad (2.10)
$$

$$
\sup_{\rho \in \mathcal{O}} |\partial_\rho \partial_x^\alpha A_j(t, x; \rho)| \leq C_\alpha < x >^{M+1} \quad (2.11)
$$

in $[0, T] \times \mathbb{R}^d \times \mathcal{O}$. Let $u_0 \in B^{a+1}$ ($a = 0, \pm 1, \pm 2, \ldots$) be independent of $\rho$ and $u(t; \rho)$ the solutions to (1.1) with $u(0) = u_0$. Then, the mapping : $\mathcal{O} \ni \rho \mapsto u(t; \rho) \in \mathcal{E}_i^0([0, T]; B^a)$ is continuously differentiable with respect to $\rho$, we have

$$
\sup_{\rho \in \mathcal{O}} \|\partial_\rho u(t; \rho)\|_a \leq C_a \|u_0\|_{a+1} \quad (0 \leq t \leq T) \quad (2.12)
$$

and $\partial_\rho u(t; \rho)$ is the solution to

$$
i \frac{\partial}{\partial t} w(t; \rho) = H(t; \rho) w(t; \rho) + \frac{\partial H(t; \rho)}{\partial \rho} u(t; \rho) \quad (2.13)
$$

with $w(0) = 0$, where $\partial_\rho H(t; \rho)$ is the pseudo-differential operator with the double symbol $\partial_\rho h(t, (x + x')/2, \xi; \rho)$.

Now, we consider the 4-particle systems (1.4).

**Assumption 2.2.** We assume the following in $[0, T] \times \mathbb{R}^d$ : (1) Each $(V_k(t, x), A^{(k)}(t, x))$ $(k = 1, 2)$ satisfies Assumption 2.1 with $M = M_k \geq 0$. (2) Each $(V_k, A^{(k)})$ $(k = 3, 4)$ satisfies (1.6) and (1.7). (3) For $M_0 := \min(M_1, M_2)$ $W_{12}$ satisfies

$$
|W_{12}(t, x)| \leq C < x >^{2(M_0+1)-\delta} \quad (2.14)
$$
with a constant \( \delta > 0 \) and
\[
|\partial_x^\alpha W_{12}(t, x)| \leq C_\alpha < x >^{2(M_0+1)}, \quad |\alpha| \geq 1.
\] (2.15)

(4) Each \( W_{ij}(t, x) \) except \( W_{12} \) satisfies (1.6).

We introduce the weighted Sobolev spaces \( B^a(\mathbb{R}^d) := \{ f \in L^2(\mathbb{R}^d); \| f \|_a := \| f \| + \sum_{|\alpha| \leq 2a} \| \partial_x^\alpha f \| + \sum_{k=1}^4 < x^{(k)} >^{2a(M_k+1)} f \| < \infty \} \) with \( M_3 = M_4 = 0 \), and denote the dual space of \( B^a \) by \( B^{-a} \) and \( L^2 \) by \( B^0 \).

**Theorem 2.4.** Under Assumption 2.2 for any \( u_0 \in B^a(\mathbb{R}^d) \) \( (a = 0, \pm 1, \pm 2, \ldots) \) there exists the unique solution \( u(t) \in \mathcal{E}_l^0([0, T]; B^a) \cap \mathcal{E}_l^1([0, T]; B^{-a}) \) with \( u(0) = u_0 \) to (1.4). This solution \( u(t) \) satisfies
\[
\| u(t) \|_a \leq C_a \| u_0 \|_a \quad (0 \leq t \leq T)
\] (2.16)
and in particular
\[
\| u(t) \| = \| u_0 \| \quad (0 \leq t \leq T).
\] (2.17)

We will consider the 4-particle systems (1.4) with potentials dependent on a parameter \( \rho \in \mathcal{O} \).

**Theorem 2.5.** We suppose that \( (V_k(t, x; \rho), A^{(k)}(t, x; \rho)) \) \( (k = 1, 2, 3, 4) \) and \( W_{ij}(t, x; \rho) \) \( (1 \leq i < j \leq 4) \) for all \( \rho \in \mathcal{O} \) satisfy Assumption 2.2 and have the uniform estimates with respect to \( \rho \in \mathcal{O} \) (1.2) and (2.1) for \( V_k, A^{(k)} \) \( (k = 1, 2) \) with \( M = M_k \) \( (1.6) \cdot (1.7) \) for \( V_k, A^{(k)} \) \( (k = 3, 4) \) \( (2.14) \) \( - \) (2.15) for \( W_{12} \) and \( (1.6) \) for \( W_{ij} \) except \( W_{12} \). In addition, we assume that \( \partial_x^\alpha V_k(t, x; \rho), \partial_x^\alpha A^{(k)}_j(t, x; \rho) \) \( (k = 1, 2, 3, 4, j = 1, 2, \ldots, d) \) and \( \partial_x^\alpha W_{ij}(t, x; \rho) \) \( (1 \leq i < j \leq 4) \) for all \( \alpha \) are continuous in \( [0, T] \times \mathbb{R}^d \times \mathcal{O} \).

Let \( u_0 \in B^a \) \( (a = 0, \pm 1, \pm 2, \ldots) \) be independent of \( \rho \) and \( u(t; \rho) \) the solutions to (1.4) with \( u(0; \rho) = u_0 \) determined in Theorem 2.4. Then, the mapping \( \mathcal{O} \ni \rho \rightarrow u(t; \rho) \in \mathcal{E}_l^0([0, T]; B^a) \) is continuous.
Theorem 2.6. Besides the assumptions of Theorem 2.5 we suppose for all \( \alpha \) that all functions \( \partial_{\rho} \partial_{x}^{\alpha} V_{k}(t, x; \rho), \partial_{\rho} \partial_{x}^{\alpha} A_{j}^{(k)}(t, x; \rho) \) and \( \partial_{\rho} \partial_{x}^{\alpha} W_{ij}(t, x; \rho) \) are continuous in \([0, T] \times \mathbb{R}^{d} \times \mathcal{O}\). In addition, we assume (2.10)-(2.11) for \((V_{k}, A^{(k)}) \) \((k = 1, 2)\) with \( M = M_{k} \),

\[
\sup_{\rho \in \mathcal{O}} |\partial_{\rho} \partial_{x}^{\alpha} V_{k}(t, x; \rho)| \leq C_{\alpha} < x >, \ |\alpha| \geq 1, \tag{2.18}
\]

\[
\sup_{\rho \in \mathcal{O}} |\partial_{\rho} \partial_{x}^{\alpha} A_{j}^{(k)}(t, x; \rho)| \leq C_{\alpha}, \ |\alpha| \geq 1 \tag{2.19}
\]

in \([0, T] \times \mathbb{R}^{d}\) for \( k = 3, 4 \),

\[
\sup_{\rho \in \mathcal{O}} |\partial_{\rho} \partial_{x}^{\alpha} W_{12}(t, x; \rho)| \leq C_{\alpha} < x >^{2(M_{0}+1)} \tag{2.20}
\]

for all \( \alpha \) and

\[
\sup_{\rho \in \mathcal{O}} |\partial_{\rho} \partial_{x}^{\alpha} W_{ij}(t, x; \rho)| \leq C_{\alpha} < x >, \ |\alpha| \geq 1 \tag{2.21}
\]

for \((i, j) \neq (1, 2)\).

Let \( u_{0} \in B^{a+1} \) \((a = 0, \pm 1, \pm 2, \ldots)\) be independent of \( \rho \) and \( u(t; \rho) \) the solutions to (1.4) with \( u(0; \rho) = u_{0} \). Then we have the same assertion as in Theorem 2.3.

Remark 2.2. Theorems 2.4 - 2.6 in the present paper give generalizations of Theorems 2.1 - 2.3 in the present paper, Theorem in [4] and Theorems 2.1 - 2.4 in [6].

3 Preliminaries

Let \( h(t, x, \xi) \) be the function defined by (2.8).
Lemma 3.1. Assume (1.2) and (2.3). Then, there exist constant $C_0^* > 0$ and $C_1^* \geq 0$ such that

$$C_0^* (\xi >^2 + < x >^{2(M+1)}) - C_1^* \leq h(t, x, \xi) \leq C_0^{*-1} (\xi >^2 + < x >^{2(M+1)})$$

(3.1)
in $[0, T] \times \mathbb{R}^d$.

Proof. From (2.8) we have $h(t, x, \xi) \leq (|\xi|^2 + |A(t, x)|^2)/m + V(t, x)$ and hence by (1.2) and (2.3)

$$h(t, x, \xi) \leq C(< \xi >^2 + < x >^{2(M+1)})$$
in $[0, T] \times \mathbb{R}^d$ with a constant $C \geq 0$.

We may assume $0 < \delta \leq M + 1$ in (2.3). Take $p > 1$ and $q > 1$ so that

$$\frac{1}{p} = \frac{1}{2} \left( 1 - \frac{1}{2} \cdot \frac{\delta}{M+1} \right), \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Then we have

$$p(M + 1 - \delta) = 2(M + 1) \cdot \frac{1 - \frac{\delta}{M+1}}{1 - \frac{1}{2} \cdot \frac{\delta}{M+1}} \equiv 2(M + 1)\delta_1,$$

$$q = \frac{2}{1 + \frac{1}{2} \cdot \frac{\delta}{M+1}} \equiv 2\delta_2$$

with $0 < \delta_j < 1$ $(j = 1, 2)$. Hence, Young’s inequality and (2.3) show

$$|A(t, x)| \cdot |\xi| \leq \frac{1}{p} |A|^p + \frac{1}{q} |\xi|^q \leq \frac{1}{p} < x >^{p(M+1-\delta)} + \frac{1}{q} |\xi|^q$$

$$= \frac{1}{p} < x >^{2(M+1)\delta_1} + \frac{1}{q} |\xi|^{2\delta_2}.$$

Applying this, (1.2) and (2.3) to (2.8), we have

$$h(t, x, \xi) \geq \frac{1}{2m} \left( |\xi|^2 - 2|A| \cdot |\xi| \right) + V$$

$$\geq C_0(< \xi >^2 - < x >^{2(M+1)\delta_1} - < \xi >^{2\delta_2} + < x >^{2(M+1)}) - C_1$$

with constants $C_0 > 0$ and $C_1 \geq 0$. Therefore, we obtain (3.1).
We fix $C_0^*$ and $C_1^*$ in Lemma 3.1 hereafter. We set

$$h_s(t, x, \xi) = h(t, x, \xi) + \frac{i}{2m} \nabla \cdot A(t, x),$$  \hspace{1cm} (3.2)

where $\nabla \cdot A(t, x) = \sum_{j=1}^d \partial_{x_j} A_j(t, x)$. Since the real part $\text{Re} h_s(t, x, \xi)$ of $h_s(t, x, \xi)$ is equal to $h(t, x, \xi)$, we can determine

$$p_\mu(t, x, \xi) := \frac{1}{\mu + h_s(t, x, \xi)}$$  \hspace{1cm} (3.3)

for $\mu \geq C_1^*$ under the assumptions of Lemma 3.1. We denote by $H_s(t, X, D_x)f$ the pseudo-differential operator

$$\int e^{ix\cdot\xi} h_s(t, x, \xi) \bar{d}\xi \int e^{-iy\cdot\xi} f(y) dy$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ with the symbol $h_s(t, x, \xi)$. As is well known (cf. Theorem 2.5 in Chapter 2 of [9]), $H_s(t, X, D_x) = H(t)$ holds, where $H(t)$ is the operator defined by (1.1) or (2.9).

**Lemma 3.2.** Assume (1.2), (2.1) and (2.3) - (2.4). Then we have

$$[\mu + H(t)] P_\mu(t, X, D_x) = I + R_\mu(t, X, D_x),$$  \hspace{1cm} (3.4)

$$\left| r_{(\alpha) \mu (\beta)}^{(x)}(t, x, \xi) \right| \leq C_{\alpha\beta} (\mu - C_1^*)^{-1/2}$$  \hspace{1cm} (3.5)

in $[0, T] \times \mathbb{R}^{2d}$ for $\mu \geq C_0^*/2 + C_1^*$ with constants $C_{\alpha\beta}$ independent of $\mu$, where $r_{(\alpha) \mu (\beta)}^{(x)} = \partial_\xi^\alpha (-i\partial_x)^\beta r_\mu$.

**Proof.** Let $\mu \geq C_1^*$. By Lemma 3.1 and (3.2) we have

$$C_0^* \left( \xi > 2 + \left< x \right>^{2(M+1)} \right) + \mu - C_1^* \leq \mu + \text{Re} h_s(t, x, \xi).$$  \hspace{1cm} (3.6)
Since $H(t) = H_s(t, X, D_x)$, from (2.13) in [4] we have

$$r_\mu(t, x, \xi) = \sum_{|\alpha|=1} \int_0^1 d\theta \, O_s - \iint e^{-iy\cdot\eta} h_\alpha^{(\alpha)}(t, x, \xi + \theta\eta) p_{\mu(\alpha)}(t, x + y, \xi) dy d\eta$$

$$= \sum_{|\alpha|=1} \int_0^1 d\theta \, O_s - \iint e^{-iy\cdot\eta} < y >^{-2l_0} < D_\eta >^{2l_0} < \eta >^{-2l_1} < D_y >^{2l_1}$$

$$\cdot h_\alpha^{(\alpha)}(t, x, \xi + \theta\eta)p_{\mu(\alpha)}(t, x + y, \xi) dy d\eta$$

(3.7)

for large integers $l_0$ and $l_1$, where $< D_\eta >^{2l_0} = 1 - \sum_{j=1}^d \partial^2_{\eta j}$.

Now, using (2.1) and (2.3) - (2.4), from (3.2) we have

$$|\partial_x h_s(t, x + y, \xi)| \leq C \left( < \xi > + < x + y >^{M+1} + < x + y >^{2(M+1)} \right)$$

$$\leq C' \left( < \xi >^2 + < x + y >^{2(M+1)} \right).$$

In the same way we can prove

$$|h_\alpha^{(\alpha)}(t, x + y, \xi)| \leq C \left( < \xi >^2 + < x + y >^{2(M+1)} \right)$$

(3.8)

for all $\alpha$ and $|\beta| \geq 1$, and

$$|h_\alpha^{(\alpha)}(t, x, \xi + \theta\eta)| \leq C \left( < \xi > + < x >^{M+1} \right) < \eta >$$

(3.9)

for $|\alpha| \geq 1$ and all $\beta$. We also note

$$\frac{1}{< \xi >^2 + < x + y >^{2(M+1)}} \leq \frac{1}{< \xi >^2 + < x >^{2(M+1)} / (\sqrt{2} < y >)^{2(M+1)}}$$

$$\leq \frac{1}{< \xi >^2 + < x >^{2(M+1)}}.$$

Apply (3.6) and (3.8) - (3.9) to (3.7). Then, taking integers $l_0$ and $l_1$ so that

$$2l_0 - 2(M + 1) > d$$

and $2l_1 - 1 > d$, we have

$$|r_\mu(t, x, \xi)| \leq C \iint < y >^{-2l_0} < \eta >^{-2l_1} < \eta ~> < y >^{2(M+1)} dy d\eta$$

$$\times \frac{\Theta^{1/2}}{\mu - C_1^* + C_0^* \Theta} \leq C' \max_{1 \leq \theta} \frac{\theta^{1/2}}{\mu - C_1^* + C_0^* \theta}$$

(3.10)
with constants $C$ and $C'$ independent of $\mu \geq C_1^*$, where $\Theta = < \xi >^2 + < x >^{2(M+1)}$. Applying (2.9) in [4] with $\kappa = 1$ and $\tau = 2$ to (3.10), we have

$$|r_\mu(t, x, \xi)| \leq C_0(\mu - C_1^*)^{-1/2}$$

for $\mu \geq C_0^*/2 + C_1^*$. In the same way we can prove (3.5) from (3.7) - (3.9). □

**Proposition 3.3.** Under the assumptions of Lemma 3.2 there exist a constant $\mu \geq C_0^*/2 + C_1^*$ and a function $w(t, x, \xi)$ in $[0, T] \times \mathbb{R}^d$ satisfying

$$|w^{(\alpha)}_{(\beta)}(t, x, \xi)| \leq C_{\alpha\beta} \left(< \xi >^2 + < x >^{2(M+1)}\right)^{-1}$$

for all $\alpha, \beta$ and

$$W(t, X, D_x) = (\mu + H(t))^{-1}. \quad (3.12)$$

**Proof.** Let $\mu \geq C_0^*/2 + C_1^*$. From (3.3), (3.6) and (3.8) - (3.9) we see

$$|p^{(\alpha)}_{(\beta)}(t, x, \xi)| \leq C_{\alpha\beta} \left(< \xi >^2 + < x >^{2(M+1)}\right)^{-1}$$

for all $\alpha$ and $\beta$. Therefore, we can complete the proof of Proposition 3.3 as in the proof of (2.16) of [4] by using Lemma 3.2. □

We take a constant $\mu \geq C_0^*/2 + C_1^*$ in Proposition 3.3 and fix it hereafter throughout §3 and §4. Set

$$\lambda(t, x, \xi) := \mu + h_s(t, x, \xi). \quad (3.13)$$

Then, from (3.2) we have

$$\Lambda(t, X, D_x) = \mu + H_s(t, X, D_x) = \mu + H(t). \quad (3.14)$$

We take a $\chi \in S(\mathbb{R}^d)$ such that $\chi(0) = 1$ and set

$$\chi_\epsilon(t, x, \xi) := \chi(\epsilon(\mu + h(t, x, \xi))) \quad (3.15)$$
for constants $0 < \epsilon \leq 1$. We note that $h(t, x, \xi)$ defined by (2.8) is a real-valued function.

The following is crucial in the present paper.

**Lemma 3.4.** Under the assumptions of Lemma 3.2 there exist functions $k_\epsilon(t, x, \xi) (0 < \epsilon \leq 1)$ in $[0, T] \times \mathbb{R}^d$ satisfying

$$
\sup_{0 < \epsilon \leq 1} \sup_{t, x, \xi} |k_\epsilon^{(\alpha)}(t, x, \xi)| \leq C_{\alpha \beta} < \infty
$$

(3.16)

for all $\alpha, \beta$ and

$$
K_\epsilon(t, X, D_x) = \left[ X_\epsilon(t, X, D_x), \Lambda(t, X, D_x) \right],
$$

(3.17)

where $[\cdot, \cdot]$ denotes the commutator of operators.

**Proof.** Apply Theorem 3.1 in Chapter 2 of [9] to the right-hand side of (3.17). Then we have

$$
k_\epsilon(t, x, \xi) = \sum_{|\alpha| = 1} \left\{ \chi^{(\alpha)}_\epsilon(t, x, \xi)\lambda_{(\alpha)}(t, x, \xi) - \lambda^{(\alpha)}(t, x, \xi)\chi_{\epsilon(\alpha)}(t, x, \xi) \right\}
$$

$$
+ 2 \sum_{|\gamma| = 2} \frac{1}{\gamma!} \int_0^1 (1 - \theta)d\theta \operatorname{Os} - \int \int e^{-iy \cdot \eta} \left\{ \chi^{(\gamma)}_\epsilon(t, x, \xi + \theta\eta)\lambda_{(\gamma)}(t, x + y, \xi)

- \lambda^{(\gamma)}(t, x, \xi + \theta\eta)\chi_{\epsilon(\gamma)}(t, x + y, \xi) \right\} dy d\eta \equiv I_{1\epsilon} + I_{2\epsilon}.
$$

(3.18)

By (3.2), (3.13) and (3.15) we can write

$$
I_{1\epsilon}(t, x, \xi) = \epsilon\chi'(\epsilon(\mu + h)) \sum_{|\alpha| = 1} \left\{ h^{(\alpha)}(h^{(\alpha)} - h^{(\alpha)}) \right\}
$$

$$
= \epsilon\chi'(\epsilon(\mu + h(t, x, \xi))) \sum_{|\alpha| = 1} \frac{i}{2m^2}(\xi_{\alpha} - A_{\alpha}(t, x))(-i\partial_x)\alpha \nabla \cdot A(t, x).
$$

(3.19)

Hence, using $\epsilon\chi'(\epsilon(\mu + h)) = (\mu + h)^{-1}\epsilon(\mu + h)\chi'(\epsilon(\mu + h))$ and Lemma 3.1, from (2.3) - (2.4) we can prove $\sup_{0 < \epsilon \leq 1} \sup_{t, x, \xi} |I_{1\epsilon}| < \infty$. In the same way
from (3.19) we can prove
\[ \sup_{0 < \epsilon \leq 1} \sup_{t, x, \xi} |I_{1\epsilon}^{(\alpha)}(t, x, \xi)| \leq C_{\alpha \beta} < \infty \] (3.20)
for all \( \alpha \) and \( \beta \).

Next we will consider \( I_2\epsilon \). Let \( \gamma = 2 \). Since from (3.15) we have
\[ \partial_{\xi_j} \chi_\epsilon(t, x, \xi) = \frac{1}{m} \epsilon \chi'(\epsilon(\mu + h))(\xi_j - A_j) \]
and
\[ \epsilon(\xi_j - A_j) \partial_{\xi_k} \chi'(\epsilon(\mu + h)) = \frac{1}{m} \epsilon^2 (\xi_j - A_j)(\xi_k - A_k) \chi''(\epsilon(\mu + h)), \]
as in the proof of (3.20) we can easily prove
\[ \sup_{0 < \epsilon \leq 1} |\chi_\epsilon^{(\alpha + \gamma)}(t, x, \xi)| \leq C_{\alpha \beta} \left( < \xi >^2 + < x >^{2(M+1)} \right)^{-1} \] (3.21)
for all \( \alpha \) and \( \beta \). In the same way we can also prove
\[ \sup_{0 < \epsilon \leq 1} |\chi_\epsilon^{(\gamma)}(t, x, \xi)| \leq C_{\alpha \beta} < \infty \] (3.22)
for all \( \alpha \) and \( \beta \). We also note from (3.13) that each of \( \lambda^{(\gamma)} = h^{(\gamma)} \) for \( |\gamma| = 2 \) is equal to 0 or \( 1/m \). Hence, applying (3.8) and (3.21) - (3.22) to \( I_2\epsilon \) in (3.18), as in the proof of (3.10) we have \( \sup_{0 < \epsilon \leq 1} \sup_{t, x, \xi} |I_2\epsilon| < \infty \). In the same way we can prove
\[ \sup_{0 < \epsilon \leq 1} \sup_{t, x, \xi} |I_2^{(\alpha)}(t, x, \xi)| \leq C_{\alpha \beta} < \infty \]
for all \( \alpha \) and \( \beta \), which completes the proof of Lemma 3.4 together with (3.20).

Let
\[ \lambda_M(x, \xi) := \mu' + \frac{1}{2m} |\xi|^2 + < x >^{2(M+1)}, \] (3.23)
which is equal to \( \lambda(t, x, \xi) \) defined by (3.13) with \( V = < x >^{2(M+1)} \) and \( A = 0 \).
Let \( B^a (a = 0, \pm 1, \pm 2, \ldots) \) be the weighted Sobolev spaces introduced in §1.

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Proposition 3.5. (1) There exist a constant $\mu' \geq 0$ and a function $w_M(x, \xi)$ in $[0, T] \times \mathbb{R}^{2d}$ satisfying (3.11) for all $\alpha, \beta$ and

$$W_M(X, D_x) = \Lambda_M(X, D_x)^{-1}. \quad (3.24)$$

(2) We take a $\mu' \geq 0$ satisfying (1). Let $f$ be in the dual space $S'(\mathbb{R}^d)$ of $S(\mathbb{R}^d)$. Then, $B^\alpha \ni f (a = 0, \pm 1, \pm 2, \ldots)$ is equivalent to $(\Lambda_M)^\alpha f \in L^2$.

Proof. The assertion (1) follows from Proposition 3.3. The assertion (2) follows from Lemma 2.4 of [4] with $s = a, a = 2(M + 1)$ and $b = 2$.

Let $\mu' \geq 0$ be a constant in Proposition 3.5 and fix it hereafter throughout the present paper. Under the assumptions of Lemma 3.2 from (3.1), (3.8) and (3.9) we have

$$|h_{s(\beta)}^{(\alpha)}(t, x, \xi)| \leq C_{\alpha\beta} \left( <\xi>^2 + <x>^{2(M+1)} \right) \quad (3.25)$$

in $[0, T] \times \mathbb{R}^{2d}$ for all $\alpha$ and $\beta$.

4 Proofs of Theorems 2.1 - 2.3

Let $\lambda(t, x, \xi)$ and $\chi_\epsilon(t, x, \xi)$ ($0 < \epsilon \leq 1$) be the functions defined by (3.13) and (3.15), respectively. We define the approximation of $H(t)$ by the product of operators

$$H_\epsilon(t) := X_\epsilon(t, X, D_x)^\dagger H(t) X_\epsilon(t, X, D_x), \quad (4.1)$$

where $X_\epsilon(t, X, D_x)^\dagger$ denotes the formally adjoint operator of $X_\epsilon(t, X, D_x)$.

Lemma 4.1. Under Assumption 2.1 there exist functions $q_\epsilon(t, x, \xi)$ ($0 < \epsilon \leq 1$) satisfying

$$\sup_{0<\epsilon\leq1} \sup_{t,x,\xi} |q_\epsilon^{(\alpha)}(t, x, \xi)| \leq C_{\alpha\beta} < \infty \quad (4.2)$$
for all $\alpha, \beta$ and

$$Q_\epsilon(t, X, D_x) = \left[ \Lambda(t, X, D_x), H_\epsilon(t) \right] \Lambda(t, X, D_x)^{-1}$$

$$+ i \frac{\partial \Lambda}{\partial t}(t, X, D_x) \Lambda(t, X, D_x)^{-1}. \quad (4.3)$$

**Proof.** We first note

$$\left[ \Lambda(t, X, D_x), H_\epsilon(t) \right] = \left[ \Lambda(t), X_\epsilon(t) \right]^\dagger H(t) X_\epsilon(t)$$

$$+ X_\epsilon(t)^\dagger \left[ \Lambda(t), H(t) \right] X_\epsilon(t) + X_\epsilon(t)^\dagger H(t) \left[ \Lambda(t), X_\epsilon(t) \right].$$

Since $\Lambda(t) = \mu + H(t)$ from (3.14), we have $[\Lambda(t), H(t)] = 0$ and $\Lambda(t)^\dagger = \Lambda(t)$.

Hence

$$\left[ \Lambda(t), H_\epsilon(t) \right] = - \left[ \Lambda(t), X_\epsilon(t) \right]^\dagger H(t) X_\epsilon(t) + X_\epsilon(t)^\dagger H(t) \left[ \Lambda(t), X_\epsilon(t) \right]. \quad (4.4)$$

Thereby, applying Lemma 3.4 to (4.4), we see from Proposition 3.3 and (3.25) that there exist functions $q_\epsilon(t, x, \xi) \ (0 < \epsilon \leq 1)$ satisfying (4.2) and

$$Q_\epsilon(t, X, D_x) = \left[ \Lambda(t, X, D_x), H_\epsilon(t) \right] \Lambda(t, X, D_x)^{-1}. \quad (4.5)$$

It is easy to study the second term in the right-hand side of (4.3) by Proposition 3.3. Thus, our proof is complete. $\square$

**Proposition 4.2.** Under Assumption 2.1 there exist functions $q_a_\epsilon(t, x, \xi) \ (a = 0, \pm 1, \pm 2, \ldots, 0 < \epsilon \leq 1)$ satisfying (4.2) and

$$Q_a_\epsilon(t, X, D_x) = \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^a \right] \Lambda(t)^{-a}. \quad (4.5)$$

**Proof.** For $a = 0$ the assertion is clear. For $a = 1$ the assertion follows from Lemma 4.1. Let us consider the case $a = 2$. We note

$$[P, QR] = [P, Q] R + Q [P, R] \quad (4.6)$$
and thereby
\[
\left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^2 \right] \Lambda(t)^{-2} = \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t) \right] \Lambda(t)^{-1} + \Lambda(t) \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t) \right] \Lambda(t)^{-2} = Q_\epsilon(t) + \Lambda(t) Q_\epsilon(t) \Lambda(t)^{-1}.
\]

Hence, it follows from Lemma 4.1 and Proposition 3.3 that the assertion holds.

We consider the case \(a = 3\). From (4.6) we have
\[
\left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^3 \right] \Lambda(t)^{-3} = \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t) \right] \Lambda(t)^{-1} + \Lambda(t) \left\{ \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^2 \right] \Lambda(t)^{-2} \right\} \Lambda(t)^{-1}
\]
\[
= Q_\epsilon(t) + \Lambda(t) Q_\epsilon(t) \Lambda(t)^{-1}.
\]

Consequently, using the results for \(a = 1, 2\), we see that the assertion holds.

In the same way we can prove the assertion for \(a = 0, 1, 2, \ldots\) by induction.

Next we consider the case \(a = -1, -2, \ldots\). From (4.6) we have
\[
0 = \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^{-1} \Lambda(t) \right] = \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^{-1} \right] \Lambda(t)
\]
\[
+ \Lambda(t)^{-1} \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t) \right],
\]
which shows
\[
\left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^{-1} \right] \Lambda(t) = -\Lambda(t)^{-1} \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t) \right] \Lambda(t)^{-1} \Lambda(t)
\]
\[
= -\Lambda(t)^{-1} Q_\epsilon(t) \Lambda(t).
\]

Hence the assertion for \(a = -1\) holds. We consider the case \(a = -2\). From
we have
\[
\left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^{-1} \right] \Lambda(t)^2 = \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^{-1} \right] \Lambda(t) + \Lambda(t)^{-1}\left\{ \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^{-1} \right] \Lambda(t) \right\} \Lambda(t)
\]
\[
= Q_{-1\epsilon}(t) + \Lambda(t)^{-1}Q_{-1\epsilon}(t)\Lambda(t),
\]
which shows the assertion. In the same way we can prove the assertion for \( a = -1, -2, \ldots \) by induction. Therefore, our proof is complete.

We consider the equation
\[
i \frac{\partial u}{\partial t}(t) = H_\epsilon(t)u(t) + f(t).
\]

Proposition 4.3. Let \( u_0 \in B^a \) (\( a = 0, \pm 1, \pm 2, \ldots \)) and \( f(t) \in \mathcal{E}_t^0([0, T]; B^a) \). Then, under Assumption 2.1 there exist solutions \( u_\epsilon(t) \in \mathcal{E}_t^1([0, T]; B^a) \) (\( 0 < \epsilon \leq 1 \)) with \( u_\epsilon(0) = u_0 \) to (4.7) satisfying
\[
\sup_{0 < \epsilon \leq 1} \|u_\epsilon(t)\|_a \leq C_a \left( \|u_0\|_a + \int_0^t \|f(\theta)\|_a d\theta \right).
\]

In particular, if \( u_0 \in L^2 \) and \( f = 0 \), we have \( \|u_\epsilon(t)\| = \|u_0\| \).

Proof. Applying Theorem 2.5 in Chapter 2 of [9] to (4.1), we see that each of \( H_\epsilon(t) \) (\( 0 < \epsilon \leq 1 \)) is written as the pseudo-differential operator with the symbol \( p_\epsilon(t, x, \xi) \) satisfying
\[
\sup_{t, x, \xi} \left| p_\epsilon^{(\alpha)}(\beta)(t, x, \xi) \right| \leq C_{\alpha\beta} < \infty
\]
for all \( \alpha \) and \( \beta \), where \( C_{\alpha\beta} \) may depend on \( 0 < \epsilon \leq 1 \). Consequently, it follows from Lemma 2.5 of [4] with \( s = a, a = 2(M + 1) \) and \( b = 2 \) and (1.3) in the present paper that we have
\[
\sup_{0 \leq t \leq T} \|H_\epsilon(t)f\|_a \leq C_{ae}\|f\|_a
\]
for $a = 0, \pm 1, \pm 2, \ldots$ with constants $C_{ac} \geq 0$ dependent on $0 < \epsilon \leq 1$. Hence, noting that the equation (4.7) is equivalent to

$$iu(t) = iu_0 + \int_{0}^{t} \left\{ H_\epsilon(\theta)u(\theta) + f(\theta) \right\} d\theta,$$

we can find a solution $u_\epsilon(t) \in \mathcal{E}_t^1([0, T]; B^a)$ by the successive iteration for each $0 < \epsilon \leq 1$. From (4.5) and (4.7) we have

$$i \frac{\partial}{\partial t} \Lambda(t)^a u_\epsilon(t) = H_\epsilon(t)\Lambda(t)^a u_\epsilon(t) + Q_{ae}(t)\Lambda(t)^a u_\epsilon(t) + \Lambda(t)^a f(t). \quad (4.9)$$

Applying the Calderón-Vaillancourt theorem (cf. p.224 in [9]), from (3.25), Propositions 3.3 and 3.5 we have $\Lambda(t)^a u_\epsilon(t) \in \mathcal{E}_t^1([0, T]; L^2)$ because of $u_\epsilon(t) \in \mathcal{E}_t^1([0, T]; B^a)$. Noting that $H_\epsilon(t)$ defined by (4.1) is symmetric on $L^2$, from (4.9) we have

$$\frac{d}{dt} \|\Lambda(t)^a u_\epsilon(t)\|^2 = 2 \text{Re} \left( \frac{\partial}{\partial t} \Lambda(t)^a u_\epsilon(t), \Lambda(t)^a u_\epsilon(t) \right)$$

$$= -2 \text{Re} (iQ_{ae}(t)\Lambda(t)^a u_\epsilon(t), \Lambda(t)^a u_\epsilon(t)) - 2 \text{Re} (i\Lambda(t)^a f(t), \Lambda(t)^a u_\epsilon(t)).$$

Hence, using Proposition 4.2 and the Calderón-Vaillancourt theorem, we have

$$\frac{d}{dt} \|\Lambda(t)^a u_\epsilon(t)\|^2 \leq 2C_a \left( \|\Lambda(t)^a u_\epsilon(t)\|^2 + \|\Lambda(t)^a f(t)\| \cdot \|\Lambda(t)^a u_\epsilon(t)\| \right) \quad (4.10)$$

with a constant $C_a$ independent of $0 < \epsilon \leq 1$.

For a moment take a constant $\eta > 0$ and set $v(t) := (\|\Lambda(t)^a u_\epsilon(t)\|^2 + \eta)^{1/2}$, which is a positive and continuously differentiable function with respect to $t$. From (4.10) we have

$$\frac{d}{dt} v(t)^2 \leq 2C_a \left( v(t)^2 + \|\Lambda(t)^a f(t)\| v(t) \right)$$

and so $v'(t) \leq C_a \left( v(t) + \|\Lambda(t)^a f(t)\| \right)$. Hence we see

$$v(t) \leq e^{C_at} v(0) + C_a \int_{0}^{t} e^{C_a(t-\theta)} \|\Lambda(\theta)^a f(\theta)\| d\theta.$$
Letting $\eta$ to 0, we get

$$\|\Lambda(t)^a u_\epsilon(t)\| \leq e^{C_a t} \|\Lambda(0)^a u_0\| + C_a \int_0^t e^{C_a (t-\theta)} \|\Lambda(\theta)^a f(\theta)\| d\theta.$$  

(4.11)

Therefore, noting (3.25), Propositions 3.3 and 3.5, we can prove (4.8) with another constants $C_a \geq 0$.

Finally, we consider the case where $a = 0$ and $f = 0$. We have $d \|u_\epsilon(t)\|^2 / dt = 0$ instead of (4.10), which shows $\|u_\epsilon(t)\| = \|u_0\|$.

The following has been proved in Lemma 3.1 of [6].

**Lemma 4.4.** Let $a = 0, \pm 1, \pm 2, \ldots$. Then, the embedding map from $B^{a+1}$ into $B^a$ is compact.

Now we will prove Theorem 2.1. Our proof is similar to that of Theorem in [4].

**1st step.** Throughout 1st step we suppose $u_0 \in B^{a+1}$ and $f(t) \in E^0_1([0, T]; B^{a+1})$. Let $u_\epsilon(t) \in E^1_1([0, T]; B^{a+1})$ ($0 < \epsilon \leq 1$) be the solutions to (4.7) with $u(0) = u_0$ found in Proposition 4.3. We see from (3.25), Propositions 3.5 and 4.3 that the family $\{u_\epsilon(t)\}_{0 < \epsilon \leq 1}$ is bounded in $E^0_1([0, T]; B^{a+1})$ and equi-continuous in $E^0_1([0, T]; B^a)$ because

$$i\{u_\epsilon(t) - u_\epsilon(t')\} = \int_{t'}^t H_\epsilon(\theta) u_\epsilon(\theta) d\theta + \int_{t'}^t f(\theta) d\theta$$

and

$$\sup_{0 < \epsilon \leq 1} \max_{0 \leq t \leq T} \|H_\epsilon(t) u_\epsilon(\theta)\| \leq C_a \sup_{0 < \epsilon \leq 1} \max_{0 \leq t \leq T} \|u_\epsilon(t)\|_{a+1} \leq C'_a \|u_0\|_{a+1}.$$  

Consequently, it follows from Lemma 4.4 that we can apply the Ascoli-Arzelà theorem to $\{u_\epsilon(t)\}_{0 < \epsilon \leq 1}$ in $E^0_1([0, T]; B^a)$. Hence, there exist a sequence $\{\epsilon_j\}_{j=1}^\infty$ tending to zero and a function $u(t) \in E^0_1([0, T]; B^a)$ such that

$$\lim_{j \to \infty} u_{\epsilon_j}(t) = u(t)$$  

in $E^0_1([0, T]; B^a)$.  

(4.12)
Then, since from (4.7) we have
\[ u_{\epsilon_j}(t) = u_0 - i \int_0^t H_{\epsilon_j}(\theta) u_{\epsilon_j}(\theta) d\theta - i \int_0^t f(\theta) d\theta \]
we get
\[ u_{\epsilon_j}(t) = u_0 - i \int_0^t H_{\epsilon_j}(\theta) u(\theta) d\theta - i \int_0^t H_{\epsilon_j}(\theta) \{ u_{\epsilon_j}(\theta) - u(\theta) \} d\theta - i \int_0^t f(\theta) d\theta, \]
as in the proof of (3.14) in [4] we have
\[ u(t) = u_0 - i \int_0^t H(\theta) u(\theta) d\theta - i \int_0^t f(\theta) d\theta \]
in \( \mathcal{E}_0^0([0,T];B^{a-1}) \) by using Lemma 2.2 in [4]. Therefore, we see that \( u(t) \) belongs to \( \mathcal{E}_t^0([0,T];B^a) \cap \mathcal{E}_t^1([0,T];B^{a-1}) \) and satisfies
\[ i \frac{\partial u}{\partial t}(t) = H(t)u(t) + f(t) \]  \hspace{2em} (4.13)
with \( u(0) = u_0 \). From (4.8) and (4.12) we also have
\[ \|u(t)\|_a \leq C_a \left( \|u_0\|_a + \int_0^T \|f(\theta)\|_a d\theta \right). \]  \hspace{2em} (4.14)

2nd step. In this step we will prove that a solution to (1.1) or (4.13) with a given initial data at \( t = 0 \) is uniquely determined in \( \mathcal{E}_t^0([0,T];B^a) \cap \mathcal{E}_t^1([0,T];B^{a-1}) \) for any \( a = 0, \pm 1, \pm 2, \ldots \).

Let \( u(t) \in \mathcal{E}_t^0([0,T];B^a) \cap \mathcal{E}_t^1([0,T];B^{a-1}) \) be a solution to (1.1), i.e.
\[ i \frac{\partial u}{\partial t}(t) = H(t)u(t) \]
with \( u(0) = 0 \). We may assume \( a \leq 0 \) because of \( B^{a+1} \subset B^a \). Let \( g(t) \in \mathcal{E}_t^0([0,T];B^{-a+2}) \) be an arbitrary function and consider the equation
\[ i \frac{\partial v}{\partial t}(t) = H(t)v(t) + g(t) \]
with \( v(T) = 0 \). From the 1st step we can get a solution \( v(t) \in \mathcal{E}_t^0([0,T];B^{-a+1}) \cap \mathcal{E}_t^1([0,T];B^{-a}) \). Then we have
\[
0 = \int_0^T \left( i \frac{\partial u}{\partial t}(t) - H(t)u(t), v(t) \right) dt \\
= \int_0^T \left( u(t), i \frac{\partial v}{\partial t}(t) - H(t)v(t) \right) dt = \int_0^T (u(t), g(t)) dt,
\]
which shows $u(t) = 0$.

**3rd step.** Let $u_0 \in B^a$. We take $\{u_{0j}\}_{j=1}^{\infty}$ in $B^{a+1}$ such that $\lim_{j \to \infty} u_{0j} = u_0$ in $B^a$. Let $u_j(t) \in \mathcal{E}_t^0([0, T]; B^a) \cap \mathcal{E}_t^1([0, T]; B^{a-1})$ be the solution to (1.1) with $u(0) = u_{0j}$, uniquely determined in the above 2 steps. Since $u_j(t) - u_k(t) \in \mathcal{E}_t^0([0, T]; B^a) \cap \mathcal{E}_t^1([0, T]; B^{a-1})$ is the solution to (1.1) with $u(0) = u_{0j} - u_{0k} \in B^{a+1}$, from (4.14) we have

$$\|u_j(t) - u_k(t)\|_a \leq C_a \|u_{0j} - u_{0k}\|_a. \quad (4.15)$$

Consequently, there exists a $u(t) \in \mathcal{E}_t^0([0, T]; B^a)$ such that $\lim_{j \to \infty} u_j(t) = u(t)$ in $\mathcal{E}_t^0([0, T]; B^a)$. Since $u(t)$ belongs to $\mathcal{E}_t^0([0, T]; B^a) \cap \mathcal{E}_t^1([0, T]; B^{a-1})$ and satisfies (1.1) with $u(0) = u_0$. We can also prove (2.6) because $\|u_j(t)\|_a \leq C_a \|u_{0j}\|_a$ holds from (4.14).

Finally, we will prove (2.7). Let $u_0 \in B^1$ and $u_\epsilon(t) (0 < \epsilon \leq 1)$ the solutions found in Proposition 4.3 to (4.7) with $u(0) = u_0$ and $f(t) = 0$. Then as in the proof of (4.14) we have $\|u(t)\| = \|u_0\|$ from $\|u_\epsilon(t)\| = \|u_0\|$ and (4.12). Let $u_0 \in L^2$. Take $\{u_{0j}\}_{j=1}^{\infty}$ in $B^1$ such that $\lim_{j \to \infty} u_{0j} = u_0$ in $L^2$ and let $u_j(t)$ be the solutions to (1.1) with $u(0) = u_{0j}$. Then we have $\|u_j(t)\| = \|u_{0j}\|$. Since we have proved $\lim_{j \to \infty} u_j(t) = u(t)$ in $\mathcal{E}_t^0([0, T]; L^2)$ from (4.15), we see (2.7).

Thus, our proof of Theorem 2.1 is complete.

Next, we will prove Theorem 2.2. Our proof below is similar to that of Theorem 4.1 in [6].

Let $u(t; \rho) (\rho \in \mathcal{O})$ be the solutions to (1.1) with $u(0; \rho) = u_0 \in B^a \ (a = 0, \pm 1, \pm 2, \ldots)$. Then, following the proof of Theorem 2.1, under the assump-
In Theorem 2.2, we have
\[ \sup_{\rho \in \mathcal{O}} \|u(t; \rho)\|_a \leq C_a \|u_0\|_a \quad (0 \leq t \leq T). \] (4.16)

We first assume \( u_0 \in B^{a+1} \). Then from (4.16) we have
\[ \sup_{\rho \in \mathcal{O}} \|u(t; \rho)\|_{a+1} \leq C_{a+1} \|u_0\|_{a+1} \]
and hence as in the 1st step of the proof of Theorem 2.1
\[ \|u(t; \rho) - u(t'; \rho)\|_a \leq C'_a |t - t'| \|u_0\|_{a+1} \]
with a constant \( C'_a \) independent of \( \rho \). Consequently, we see that the family \( \{u(t; \rho)\}_{\rho \in \mathcal{O}} \) is bounded in \( \mathcal{E}^0_t([0, T]; B^{a+1}) \) and equi-continuous in \( \mathcal{E}^1_t([0, T]; B^{a-1}) \).

Let \( \rho_j \to \rho \) in \( \mathcal{O} \) as \( j \to \infty \). Noting Lemma 4.4, we can apply the Ascoli-Arzelà theorem to \( \{u(t; \rho_j)\}_{j=1}^\infty \) in \( \mathcal{E}^0_t([0, T]; B^a) \). Then, there exist a subsequence \( \{j_k\}_{k=1}^\infty \) and a function \( v(t) \in \mathcal{E}^0_t([0, T]; B^a) \) such that \( \lim_{k \to \infty} u(t; \rho_{j_k}) = v(t) \) in \( \mathcal{E}^0_t([0, T]; B^a) \). As in the proof of (4.13) we see that \( v(t) \) belongs to \( \mathcal{E}^1_t([0, T]; B^{a-1}) \) and satisfies (1.1) with \( u(0) = u_0 \). The uniqueness of solutions to (1.1) gives \( v(t) = u(t; \rho) \), which shows \( \lim_{k \to \infty} u(t; \rho_{j_k}) = u(t; \rho) \). Using the uniqueness of solutions to (1.1) again, we have
\[ \lim_{j \to \infty} u(t; \rho_j) = u(t; \rho) \] in \( \mathcal{E}^0_t([0, T]; B^a) \).

Therefore, we see that the mapping \( \mathcal{O} \ni \rho \to u(t; \rho) \in \mathcal{E}^0_t([0, T]; B^a) \) is continuous.

Now let \( u_0 \in B^a \) and \( u(t; \rho) \) (\( \rho \in \mathcal{O} \)) the solutions to (1.1) with \( u(0) = u_0 \). We take \( \{u_{0k}\}_{k=1}^\infty \) in \( B^{a+1} \) such that \( \lim_{k \to \infty} u_{0k} = u_0 \) in \( B^a \) and let \( u_k(t; \rho) \in \mathcal{E}^0_t([0, T]; B^a) \cap \mathcal{E}^1_t([0, T]; B^{a-1}) \) be the solutions to (1.1) with \( u(0) = u_{0k} \). Then, from (4.16) we have
\[ \sup_{\rho} \max_t \|u_k(t; \rho) - u(t; \rho)\|_a \leq C_a \|u_{0k} - u_0\|_a, \]
which shows that $u(t; \rho)$ is continuous in $\mathcal{E}_t^0([0, T]; B^a)$ with respect to $\rho \in \mathcal{O}$, because so is $u_k(t; \rho)$. Thus, our proof of Theorem 2.2 is complete.

In the end of this section we will prove Theorem 2.3. Our proof below is similar to that of Theorem 2.3 in [6].

Let $u_0 \in B^a$ ($a = 0, \pm 1, \pm 2, \ldots$) and $f(t) \in \mathcal{E}_t^0([0, T]; B^a)$. Then, we see that under Assumption 2.1 there exists the unique solution $u(t) \in \mathcal{E}_t^0([0, T]; B^a) \cap \mathcal{E}_t^1([0, T]; B^{a-1})$ to (4.13) with $u(0) = u_0$, which satisfies

$$\|u(t)\|_a \leq C_a \left(\|u_0\|_a + \int_0^t \|f(\theta)\|_a d\theta\right).$$

(4.17)

Its proof can be completed by using (4.14) as in the 3rd step of the proof of Theorem 2.1.

Let $u_0 \in B^{a+1}$ and $u(t; \rho) \in \mathcal{E}_t^0([0, T]; B^{a+1}) \cap \mathcal{E}_t^1([0, T]; B^a)$ ($\rho \in \mathcal{O}$) the solutions to (1.1) with $u(0) = u_0$. Let $\rho \in \mathcal{O}$ be fixed and $\tau \neq 0$ a small constant such that $\rho + \tau \in \mathcal{O}$. We set

$$w_\tau(t; \rho) := \frac{u(t; \rho + \tau) - u(t; \rho)}{\tau},$$

(4.18)

which belongs to $\mathcal{E}_t^0([0, T]; B^a) \cap \mathcal{E}_t^1([0, T]; B^{a-1})$. Then, we have $w_\tau(0; \rho) = 0$ and from (1.1)

$$i \frac{\partial}{\partial t} w_\tau(t; \rho) = H(t; \rho) w_\tau(t; \rho) + \int_0^1 \frac{\partial H}{\partial \rho}(t; \rho + \theta \tau) d\theta u(t; \rho + \tau).$$

(4.19)

Hence from (4.16) and (4.17) we get

$$\|w_\tau(t; \rho)\|_a \leq C_a \int_0^t \int_0^1 \left\| \frac{\partial H}{\partial \rho}(t'; \rho + \theta \tau) u(t'; \rho + \tau) \right\|_a d\theta dt' \leq C_a' \int_0^t \|u(t'; \rho + \tau)\|_{a+1} dt' \leq C_a'' \|u_0\|_{a+1}.$$ 

Consequently,

$$\sup_{\tau} \|w_\tau(t; \rho)\|_a \leq C_a \|u_0\|_{a+1}$$

(4.20)
with another constant $C_a$.

We first assume $u_0 \in B^{a+2}$. From (4.20) we have

$$\sup_\tau \|w_\tau(t; \rho)\|_{a+1} \leq C_{a+1} \|u_0\|_{a+2}.$$ 

Thereby from (4.16) and (4.19) we have

$$\sup_\tau \|w_\tau(t; \rho) - w_\tau(t'; \rho)\|_a \leq C_a|t - t'| \|u_0\|_{a+2}$$

as in the 1st step of the proof of Theorem 2.1. Hence, we can apply the Ascoli-Arzelà theorem to $\{w_\tau(t; \rho)\}_\tau$ in $\mathcal{E}_t^0([0, T]; B^a)$. In addition, we can use the uniqueness of solutions to (2.13) or (4.13). Then, using Theorem 2.2, as in the 3rd step of the proof of Theorem 2.1 we can prove from (4.19) that there exists a function $w(t; \rho) \in E_0^0(t([0, T]; B^a)) \cap E_1^1(t([0, T]; B^{a-1})$ satisfying (2.13) with $w(0) = 0$ and

$$\lim_{\tau \to 0} w_\tau(t; \rho) = w(t; \rho) \text{ in } E_0^0([0, T]; B^a).$$

(4.21)

Now let $u_0 \in B^{a+1}$. Let $u(t; \rho)$ be the solution to (1.1) with $u(0) = u_0$ and $w_\tau(t; \rho)$ be defined by (4.13). We take $\{u_{0k}\}_{k=1}^\infty$ in $B^{a+2}$ such that $\lim_{k \to \infty} u_{0k} = u_0$ in $B^{a+1}$. Let $u_k(t; \rho) \in \mathcal{E}_t^0([0, T]; B^{a+1}) \cap \mathcal{E}_t^1([0, T]; B^a)$ be the solution to (1.1) with $u(0) = u_{0k}$. We define $w_{k\tau}$ and $w_\tau$ by (4.18) and (4.21) for $u = u_k$, respectively. From (4.19) we have

$$i \frac{\partial}{\partial t} \{w_{k\tau}(t; \rho) - w_\tau(t; \rho)\} = H(t; \rho)\{w_{k\tau}(t; \rho) - w_\tau(t; \rho)\}$$

$$+ \int_0^1 \frac{\partial H}{\partial \rho}(t; \rho + \theta \tau) d\theta \{u_k(t; \rho + \tau) - u(t; \rho + \tau)\}$$

and $w_{k\tau} - w_\tau \in \mathcal{E}_t^0([0, T]; B^a) \cap \mathcal{E}_t^1([0, T]; B^{a-1})$. Hence, using (4.17), from (4.16) we have

$$\sup_\tau \|w_{k\tau}(t; \rho) - w_\tau(t; \rho)\|_a \leq C_a \|u_{0k} - u_0\|_{a+1}.$$ 

(4.22)
As noted in the early part of this proof of Theorem 2.3, there exists the solution 
\( \mathbf{w}(t; \rho) \in \mathcal{E}_t^0([0, T]; B^a) \cap \mathcal{E}_t^1([0, T]; B^{a-1}) \) to \((2.13)\) with \(w(0) = 0\) because of \(\partial \rho H(t; \rho) u(t; \rho) \in \mathcal{E}_t^0([0, T]; B^a)\). Then, as in the proof of \((4.22)\) from \((2.13)\) we have
\[
\| \mathbf{w}_k(t; \rho) - \mathbf{w}(t; \rho) \|_a \leq C_a \| \mathbf{u}_0 - \mathbf{u}_0 \|_{a+1}
\] (4.23)

with constants \(C_a \geq 0\) independent of \(\rho \in \mathcal{O}\) because both of \(\mathbf{w}_k\) and \(\mathbf{w}\) are the solutions to \((2.13)\). Consequently, we have
\[
\| \mathbf{w}_\tau(t; \rho) - \mathbf{w}(t; \rho) \|_a \leq \| \mathbf{w}_\tau - \mathbf{w}_{k\tau} \|_a + \| \mathbf{w}_{k\tau} - \mathbf{w}_k \|_a + \| \mathbf{w}_k - \mathbf{w} \|_a
\]
\[
\leq 2C_a \| \mathbf{u}_0 - \mathbf{u}_0 \|_{a+1} + \| \mathbf{w}_{k\tau} - \mathbf{w}_k \|_a.
\]

Therefore, we see from \((4.21)\) that for any \(\epsilon > 0\) we get \(\lim_{\tau \to 0} \max_{0 \leq t \leq T} \| \mathbf{w}_\tau(t; \rho) - \mathbf{w}(t; \rho) \|_a < \epsilon\), which shows
\[
\lim_{\tau \to 0} \max_{0 \leq t \leq T} \| \mathbf{w}_\tau(t; \rho) - \mathbf{w}(t; \rho) \|_a = 0.
\] (4.24)

We also have \((2.12)\) from \((4.20)\) and \((4.24)\).

In the end of this proof we will prove that \(w(t; \rho) = \partial \rho u(t; \rho)\) for \(u_0 \in B^{a+1}\) is continuous in \(\mathcal{E}_t^0([0, T]; B^a)\) with respect to \(\rho \in \mathcal{O}\). We first assume \(u_0 \in B^{a+2}\). Then we have \((4.16)\) where \(a\) is replaced by \(a + 2\). Since \(w(t; \rho)\) is the solution to \((2.13)\) with \(w(0) = 0\), we see from \((4.17)\) as in the proof of \((4.21)\) that the family \(\{ w(t; \rho) \}_{\rho \in \mathcal{O}} \) is bounded in \(\mathcal{E}_t^0([0, T]; B^{a+1})\) and equi-continuous in \(\mathcal{E}_t^0([0, T]; B^a)\). Hence, noting that \(u(t; \rho)\) is continuous in \(\mathcal{E}_t^0([0, T]; B^{a+2})\) with respect to \(\rho\), we see that so is \(w(t; \rho)\) in \(\mathcal{E}_t^0([0, T]; B^a)\) as in the proof of Theorem 2.2. Now let \(u_0 \in B^{a+1}\). We take \(\{ u_{0k} \}_{k=1}^{\infty} \) in \(B^{a+2}\) such that \(\lim_{k \to \infty} u_{0k} = u_0\) in \(B^{a+1}\) and write as \(w_k(t, \rho)\) the solutions to \((2.13)\) with \(u(t; \rho) = u_k(t; \rho)\) and \(w(0) = 0\). Then we have \((4.23)\), which shows that \(w(t; \rho)\) is continuous with respect to \(\rho \in \mathcal{O}\) in \(\mathcal{E}_t^0([0, T]; B^a)\). Therefore, our proof of Theorem 2.3 is complete.
5 Proofs of Theorems 2.4 - 2.6

In this section we will study the 4-particle systems (1.4). Let \((x, \xi) \in \mathbb{R}^{2d}\) and write

\[
h_k(t, x, \xi) := \frac{1}{2m_k} |\xi - A^{(k)}(t, x)|^2 + V_k(t, x) \quad (k = 1, 2, 3, 4)
\]

and

\[
l_k(x, \xi) := \frac{1}{2m_k} |\xi|^2 + <x>^2 \quad (k = 3, 4).
\]

We set

\[
\tilde{h}(t, z, \zeta) := \sum_{k=1}^{2} h_k(t, x^{(k)}, \xi^{(k)}) + W_{12}(t, x^{(1)} - x^{(2)}) + \sum_{k=3}^{4} l_k(x^{(k)}, \xi^{(k)})
\]

and write

\[
\tilde{H}(t) := \tilde{H} \left( t, \frac{Z + Z'}{2}, D_z \right),
\]

where \(z = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)})\) and \(\zeta = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)})\) in \(\mathbb{R}^{4d}\). We also set

\[
\tilde{h}_s(t, z, \zeta) := \tilde{h}(t, z, \zeta) + i \sum_{k=1}^{2} \frac{1}{2m_k} \nabla \cdot A^{(k)}(t, x^{(k)})
\]

and

\[
p_\mu(t, z, \zeta) := \frac{1}{\mu + \tilde{h}_s(t, z, \zeta)}
\]

for large \(\mu\) as in (3.2) and (3.3), respectively.

**Lemma 5.1.** Assume (1.2), (2.1) and (2.3 - 2.4) for \((V_k, A^{(k)}) (k = 1, 2)\) with \(M = M_k\) and (2.14 - 2.15) for \(W_{12}\). Then, there exist a constant \(\mu^* \geq 0\) and functions \(r_\mu(t, z, \zeta) (\mu \geq \mu^*)\) such that

\[
\mu^* + \text{Re} \, \tilde{h}_s(t, z, \zeta) \geq C_0^* (<\zeta>^2 + \Phi(z)^2),
\]

\[
[\mu + \tilde{H}(t)] P_\mu(t, Z, D_z) = I + R_\mu(t, Z, D_z),
\]

for large \(\mu\) as in (3.2) and (3.3), respectively.
\[
| r^{(\alpha)}_{\mu, (\beta)}(t, z, \zeta) | \leq C_{\alpha\beta} \mu^{-1/2}
\]
for all \( \alpha, \beta \) and \( \mu \geq \mu^* \) with constants \( C_0^* > 0 \) and \( C_{\alpha\beta} \) independent of \( \mu \), where

\[
\Phi(z) = \sum_{k=1}^{2} < x^{(k)} >_{M_{k+1}} + \sum_{k=3}^{4} < x^{(k)} >.
\]

**Proof.** As in the proof of (3.6) we see

\[
\text{Re } \tilde{h}_s(t, z, \zeta) = \tilde{h}(t, z, \zeta) \geq C_0 \left( < \zeta >^2 + \Phi(z)^2 \right) - |W_{12}(t, x^{(1)} - x^{(2)})| - C_1
\]

with constants \( C_0 > 0 \) and \( C_1 \geq 0 \). Hence, using the assumption (2.14), we can determine constants \( \mu^* \geq 0 \) and \( C_0^* > 0 \) satisfying (5.6). Then, using (5.6), as in the proof of (3.7) for \( \mu \geq \mu^* \) we have

\[
r_{\mu}(t, z, \zeta) = \sum_{|\alpha|=1}^{1} \int_{0}^{1} d\theta \text{ Os} - \int \int e^{-iy\cdot\eta} < y >^{-2l_0} < D_\eta >^{2l_0} < \eta >^{-2l_1} < D_y >^{2l_1}
\]

\[
\cdot \tilde{h}^{(\alpha)}_{s}(t, z, \zeta + \theta\eta)p_{\mu(\alpha)}(t, z + y, \xi)dyd\eta
\]

for large integers \( l_0 \) and \( l_1 \). In addition, as in the proofs of (3.8) - (3.9) we can show

\[
| \tilde{h}^{(\alpha)}_{s}(t, z + y, \zeta) | \leq C_{\alpha\beta} \left( < \zeta >^2 + \Phi(z + y)^2 \right)
\]

for all \( \alpha \) and \( |\beta| \geq 1 \), and

\[
| \tilde{h}^{(\alpha)}_{s}(t, z, \zeta + \theta\eta) | \leq C_{\alpha\beta} \left( < \zeta > + \Phi(z) \right) < \eta >
\]

for \( |\alpha| \geq 1 \) and all \( \beta \). Therefore, we can complete the proof of Lemma 5.1 from (5.10) - (5.12) as in the proof of Lemma 3.2.

We can easily see from (5.11) and (5.12) as in the proof of (3.25) that under the assumptions of Lemma 5.1 we have

\[
| \tilde{h}^{(\alpha)}_{s}(t, z, \zeta) | \leq C_{\alpha\beta} \left( < \zeta >^2 + \Phi(z)^2 \right)
\]

for all \( \alpha \) and \( \beta \).
Proposition 5.2. Under the assumptions of Lemma 5.1 there exist a constant $\mu \geq \mu^*$ and a function $w(t, z, \zeta)$ satisfying

$$|w^{(\alpha)}_{(\beta)}(t, z, \zeta)| \leq C_{\alpha\beta} \left( <\zeta>^2 + \Phi(z)^2 \right)^{-1}$$

(5.14)

for all $\alpha, \beta$ and

$$W(t, Z, D_z) = (\mu + \tilde{H}(t))^{-1}.$$  

(5.15)

Proof. If $\mu \geq \mu^*$, from (5.6) and (5.11) - (5.12) we see

$$|p^{(\alpha)}_{(\beta)}(t, z, \zeta)| \leq C_{\alpha\beta} \left( <\zeta>^2 + \Phi(z)^2 \right)^{-1}$$

for all $\alpha$ and $\beta$ as in the proof of Proposition 3.3. Hence, using Lemma 5.1, we can prove Proposition 5.2 as in the proof of Proposition 3.3. \square

We take a $\mu$ in Proposition 5.2 and fix it hereafter. We set

$$\lambda(t, z, \zeta) := \mu + \tilde{h_s}(t, z, \zeta)$$

(5.16)

as in (3.13). Then, from (5.11) - (5.5) we have

$$\Lambda(t) = \Lambda(t, Z, D_z) = \mu + \tilde{H}_s(t, Z, D_z) = \mu + \tilde{H}(t)$$

$$= \mu + H_1(t) + H_2(t) + W_{12}(t) + L_3(t) + L_4(t),$$

(5.17)

where $H_k(t)$ are the operators defined by (1.4) and $L_k(t)$ the pseudo-differential operators with symbols $l_k(x^{(k)}, \xi^{(k)})$ defined by (5.2). Using the real-valued function $\tilde{h}(t, z, \zeta)$ defined by (5.3), we determine

$$\chi_\epsilon(t, z, \zeta) = \chi(\epsilon(\mu + \tilde{h}(t, z, \zeta))$$

(5.18)

with constants $0 < \epsilon \leq 1$ and $\chi \in \mathcal{S}(\mathbb{R})$ such that $\chi(0) = 1$ as in (3.15).

Lemmas 5.3 and 5.4 below are crucial in this section.
Lemma 5.3. Under the assumptions of Lemma 5.1 there exist functions
\( k_\varepsilon(t, z, \zeta) \) \((0 < \varepsilon \leq 1)\) in \([0, T] \times \mathbb{R}^d\) satisfying
\[
\sup_{0 < \varepsilon \leq 1} \sup_{t, z, \zeta} |k_\varepsilon^{(\alpha)}(t, z, \zeta)| \leq C_{\alpha\beta} < \infty \tag{5.19}
\]
for all \(\alpha, \beta\) and
\[
K_\varepsilon(t, Z, D_z) = \left[ X_\varepsilon(t, Z, D_z), \Lambda(t, Z, D_z) \right]. \tag{5.20}
\]

Proof. As in the proof of (3.18) we see
\[
k_\varepsilon(t, z, \zeta) = \sum_{|\alpha|=1} \left\{ \chi_\varepsilon^{(\alpha)}(t, z, \zeta) \lambda_{(\alpha)}(t, z, \zeta) - \lambda^{(\alpha)}(t, z, \zeta) \chi_{\varepsilon(\alpha)}(t, z, \zeta) \right\}
\]
\[
+ 2 \sum_{|\gamma|=2} \frac{1}{\gamma!} \int_0^1 (1 - \theta)d\theta \text{Os} - \iint e^{-iy\eta} \left\{ \chi_\varepsilon^{(\gamma)}(t, z, \zeta + \theta\eta) \lambda^{(\gamma)}(t, z + y, \zeta)
\]
\[
- \lambda^{(\gamma)}(t, z, \zeta + \theta\eta) \chi_\varepsilon^{(\gamma)}(t, z + y, \xi) \right\} dyd\eta \equiv I_{1\varepsilon} + I_{2\varepsilon}. \tag{5.21}
\]
From (5.5), (5.16) and (5.18) we can write
\[
I_{1\varepsilon}(t, z, \zeta) = \varepsilon \chi'(\varepsilon(\mu + \tilde{h})) \sum_{|\alpha|=1} \left\{ \tilde{h}_s^{(\alpha)}(t, z, \zeta) - \tilde{h}_s^{(\alpha)}(t, z, \zeta) \right\}
\]
\[
= i\varepsilon \chi'(\varepsilon(\mu + \tilde{h})) \sum_{|\alpha|=1} \tilde{h}_s^{(\alpha)}(t, z, \zeta) \sum_{k=1}^2 \frac{1}{2m_k} (-i\partial_z)\alpha \nabla \cdot A^{(k)}(t, x^{(k)}, \zeta^{(k)}). \tag{5.22}
\]
From (5.6) we have
\[
(\mu + \tilde{h}(t, z, \zeta))^{-1} \leq C_0(\zeta^2 + \Phi(z)^2)^{-1} \tag{5.23}
\]
because of \( \tilde{h} = \text{Re}\tilde{h}_s \). Hence, from (2.4), (5.12) and (5.22) - (5.23) we can prove \(\sup_{t, z, \zeta}|I_{1\varepsilon}| < \infty\) as in the proof of (3.20). In the same way we can prove
\[
\sup_{0 < \varepsilon \leq 1} \sup_{t, z, \zeta} |I_{\varepsilon(\beta)}^{(\alpha)}(t, z, \zeta)| \leq C_{\alpha\beta} < \infty \tag{5.24}
\]
for all \( \alpha \) and \( \beta \).

Let \( |\gamma| = 2 \). Then, from (5.5) and (5.11) - (5.12) we have the similar inequalities

\[
\sup_{0 < \varepsilon \leq 1} |\chi^{(\alpha+\gamma)}_\varepsilon(t, z, \zeta)| \leq C_{\alpha\beta} \left( < \zeta^2 + \Phi(z)^2 \right)^{-1}
\]

and

\[
\sup_{0 < \varepsilon \leq 1} |\chi^{(\alpha)}_{\varepsilon(\beta+\gamma)}(t, z, \zeta)| \leq C_{\alpha\beta} \leq \infty
\]

to (3.21) and (3.22) for all \( \alpha \) and \( \beta \), respectively. Consequently, noting that \( \lambda^{(\gamma)}(t, z, \zeta) = \tilde{h}_{\gamma}^{(\gamma)}(t, z, \zeta) \) are constants, from (5.21) we can prove

\[
\sup_{0 < \varepsilon \leq 1} \sup_{t, z, \zeta} |I^{(\alpha)}_{\varepsilon(\beta)}(t, z, \zeta)| \leq C_{\alpha\beta} < \infty
\]

for all \( \alpha \) and \( \beta \) as in the proof of Lemma 3.4, which completes the proof together with (5.21) and (5.24). \( \square \)

Let \( H(t) \) be the operator defined by (1.4).

\textbf{Lemma 5.4.} Besides the assumptions of Lemma 5.1 we suppose that each \((V_k, A^{(k)}) \ (k = 3, 4)\) satisfies (1.6) - (1.7) and each \( W_{ij}(t, x) \) (1 \( \leq i < j \leq 4 \)) except \( W_{12} \) satisfies (1.6). Then, there exists a function \( \tilde{q}(t, z, \zeta) \) satisfying

\[
\sup_{t, z, \zeta} |\tilde{q}^{(\alpha)}_{\varepsilon(\beta)}(t, z, \zeta)| \leq C_{\alpha\beta} < \infty \quad (5.25)
\]

for all \( \alpha, \beta \) and

\[
\tilde{Q}(t, Z, D_z) = \left[ \Lambda(t), H(t) \right] \Lambda(t)^{-1}. \quad (5.26)
\]

\textbf{Proof.} We write \( H(t) \) as

\[
H(t) = \sum_{k=1}^{4} H_k(t) + W_{12}(t) + \sum \ 'W_{ij}(t). \quad (5.27)
\]
Then from (5.17) we see
\[
[H(t), \Lambda(t)] = [(H_1 + H_2 + W_{12}) + H_3 + H_4 + \sum 'W_{ij},
\]
\[
(H_1 + H_2 + W_{12}) + L_3 + L_4] = [H_3, L_3] + [H_4, L_4] +
\]
\[
\sum 'W_{ij}, H_1 + H_2 + L_3 + L_4].
\] (5.28)

Lemma 3.1 in [4] has showed that each of \([H_k, L_k] = \Lambda^{-1}(k = 3, 4)\) is written as the pseudo-differential operator with the symbol satisfying (5.25).

We can easily see that \(m_1 [W_{13}(t), H_1(t)]\) is written as the pseudo-differential operator with the symbol
\[
\tilde{q}_1(t, z, \zeta) = i \frac{\partial W_{13}}{\partial x}(t, x^{(1)} - x^{(3)}) \cdot \xi^{(1)} + \frac{1}{2} \Delta x W_{13}(t, x^{(1)} - x^{(3)})
\]
\[
- i A^{(1)}(t, x^{(1)}) \cdot \frac{\partial W_{13}}{\partial x}(t, x^{(1)} - x^{(3)}).
\] (5.29)

Hence, from the assumptions we have
\[
|\tilde{q}_1(t, z, \zeta)| \leq C_1(\xi^{(1)} > 2 + < x^{(1)} >^{M_1 + 1} < x^{(1)} - x^{(3)} >)
\]
\[
\leq C_2(\xi^{(1)} > 2 + < x^{(1)} >^{M_1 + 2} + < x^{(1)} >^{2(M_1 + 1)} + < x^{(3)} >^2)
\]
\[
\leq C_3(\xi^{(1)} > 2 + < x^{(1)} >^{2(M_1 + 1)} + < x^{(3)} >^2).
\]

In the same way we have
\[
|\tilde{q}_1^{(\alpha)}(t, z, \zeta)| \leq C_{\alpha\beta}(\xi < \zeta >^2 + \Phi(z)^2)
\] (5.30)

for all \(\alpha\) and \(\beta\). Consequently, by Proposition 5.2 we see that \([W_{13}(t), H_1(t)] = \Lambda(t)^{-1}\) is written as the pseudo-differential operator with the symbol satisfying (5.25).

In the same way we can complete the proof of Proposition 5.4.

Using the function \(\chi_\epsilon(t, z, \zeta)\) defined by (5.18), we define
\[
H_\epsilon(t) := X_\epsilon(t, Z, D_z)^\dagger H(t) X_\epsilon(t, Z, D_z)
\] (5.31)
as in (4.11).
Lemma 5.5. Under Assumption 2.2 there exist functions $q_\epsilon(t, z, \zeta) (0 < \epsilon \leq 1)$ satisfying (5.19) and
\[
Q_\epsilon(t, Z, D_z) = \left[ \Lambda(t, Z, D_z), H_\epsilon(t) \right] \Lambda(t, Z, D_z)^{-1}
+ i \frac{\partial \Lambda}{\partial t}(t, Z, D_z) \Lambda(t, Z, D_z)^{-1}.
\] (5.32)

Proof. From (5.31) and $\Lambda(t)^\dagger = \Lambda(t)$ we have
\[
\left[ \Lambda(t), H_\epsilon(t) \right] = - \left[ \Lambda(t), X_\epsilon(t) \right]^\dagger H(t) X_\epsilon(t)
+ X_\epsilon(t)^\dagger \left[ \Lambda(t), H(t) \right] X_\epsilon(t) + X_\epsilon(t)^\dagger H(t) \left[ \Lambda(t), X_\epsilon(t) \right]
\]
as in the proof of Lemma 4.1. Let’s apply Proposition 5.2 and Lemmas 5.3-5.4 to the above and apply Proposition 5.2 to $(i\partial \Lambda(t)/\partial t)\Lambda(t)^{-1}$. Then, we can prove Lemma 5.5 as in the proof of Lemma 4.1.

Using Lemma 5.5, we can prove the following as in the proof of Proposition 4.2.

Proposition 5.6. Under Assumption 2.2 there exist functions $q_{a\epsilon}(t, z, \zeta) (a = 0, \pm 1, \pm 2, \ldots, 0 < \epsilon \leq 1)$ satisfying (5.19) and
\[
Q_{a\epsilon}(t, Z, D_z) = \left[ i \frac{\partial}{\partial t} - H_\epsilon(t), \Lambda(t)^a \right] \Lambda(t)^{-a}.
\] (5.33)

Let $B^a(\mathbb{R}^d) (a = 0, \pm 1, \pm 2, \ldots)$ be the weighted Sobolev spaces introduced in §2. Then we see as in Lemma 4.4 that the embedding map from $B^{a+1}$ into $B^a$ is compact. We also get the similar result to Proposition 3.5 from Proposition 5.2. Therefore, using Proposition 5.6, we can prove Theorems 2.4 - 2.6 as in the proofs of Theorems 2.1 - 2.3, respectively.
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