ON THE CAUCHY PROBLEM FOR THE HOMOGENEOUS BOLTZMANN-NORDHEIM EQUATION FOR BOSONS

Marc Briant, joint work with Amit Einav

Division of Applied Mathematics, Brown University

GAPDE seminar
University of Cambridge, February 16th 2015
1. General presentation of the Boltzmann collisional model

2. A quantic version of the Boltzmann equation

3. Local existence and uniqueness of solutions

4. Futur developments
General presentation of the Boltzmann collisional model

Marc Briant

THE HOMOGENEOUS BOLTZMANN-NORDHEIM EQUATION
A probabilistic approach of systems of particles

- **System**: \( N \) bodies moving in a domain \( \Omega \subseteq \mathbb{R}^d \)

- Particles are subject to external forces and interaction between particles

- Newton’s laws yield the reknown \( N \)-body problem (system of \( 2Nd \) equations)
A probabilistic approach of systems of particles

- **SYSTEM**: $N$ bodies moving in a domain $\Omega \subset \mathbb{R}^d$

- Particles are subject to external forces and interaction between particles

- Newton’s laws yield the reknown $N$-body problem (system of $2Nd$ equations)

$\Rightarrow$ Already problematic when $N \geq 3$!
A probabilistic approach of systems of particles

**Idea**: focus on the average behaviour of the system

establish the equation for the density function

\[ f : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^+ \\
(t, x, v) \mapsto f(t, x, v) \]

\[ f(t, x, v) dx dv \text{ is the probability of having a particle in } B(x, dx) \text{ with a velocity in } B(v, dv) \text{ at time } t \]
A probabilistic approach of systems of particles

- **Idea**: focus on the average behaviour of the system
- establish the equation for the density function

\[ f : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^+ \]

\[ (t, x, v) \rightarrow f(t, x, v) \]

- \( f(t, x, v)dx dv \) is the probability of having a particle in \( B(x, dx) \) with a velocity in \( B(v, dv) \) at time \( t \)

\[ \Rightarrow \text{Minimal assumption : } \]
\[ \forall t \in [0, T], \quad f(t, \cdot, \cdot) \in L^1_{loc} (\Omega, L^1_v (\mathbb{R}^d)) \]
1. **Binary collisions**: particles close enough to each other are deviated (Boltzmann-Grad limit)
The collisional process

1. **Binary collisions**: particles close enough to each other are deviated (Boltzmann-Grad limit)

2. **Localised collisions**: deviation of trajectories happens very quickly
The collisional process

1. **Binary collisions**: particles close enough to each other are deviated (Boltzmann-Grad limit)

2. **Localised collisions**: deviation of trajectories happens very quickly

3. **Elastic collisions**:

\[
\mathbf{v}' + \mathbf{v}_* = \mathbf{v} + \mathbf{v}_*
\]

\[
|\mathbf{v}'|^2 + |\mathbf{v}_*|^2 = |\mathbf{v}|^2 + |\mathbf{v}_*|^2
\]
The collisional process

1. **Binary collisions**: particles close enough to each other are deviated (Boltzmann-Grad limit)

2. **Localised collisions**: deviation of trajectories happens very quickly

3. **Elastic collisions**:

   \[ v' + v_*' = v + v_* \]

   \[ |v'|^2 + |v_*'|^2 = |v|^2 + |v_*|^2 \]

4. **Microreversible process**: microscopic dynamics are reversible in time
The collisional process

1. **Binary collisions**: particles close enough to each other are deviated (Boltzmann-Grad limit)

2. **Localised collisions**: deviation of trajectories happens very quickly

3. **Elastic collisions**:

   \[ v' + v_*' = v + v_* \]
   \[ |v'|^2 + |v_*'|^2 = |v|^2 + |v_*|^2 \]

4. **Microreversible process**: microscopic dynamics are reversible in time

5. **Molecular chaos**: particles evolve independently
The Boltzmann Equation

\[ \forall t \geq 0, \forall (x, v) \in \Omega \times \mathbb{R}^d, \quad \partial_t f + v \cdot \nabla_x f = Q(f, f) \]
The Boltzmann Equation

**The collision operator**:

\[
Q(f, f) = \int_{\mathbb{R}^d \times S^{d-1}} |v - v_*|^{\gamma} b(\cos \theta) \left[ f' f'_* - f f_* \right] dv_* d\sigma
\]

\[
\begin{align*}
    v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \\
    v_*' &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma
\end{align*}
\]

and \( \cos \theta = \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle \).
The Boltzmann Equation

• **The collision operator**: 

\[
Q(f, f) = \int_{\mathbb{R}^d \times S^{d-1}} |v - v_*|^\gamma b(\cos \theta) \left[ f' f_*' - f f_* \right] dv_* d\sigma
\]

\[
\begin{aligned}
    v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \\
    v_*' &= \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma
\end{aligned}
\]

• \(\gamma\) belongs to \((-d, 1]\)
• \((b \circ \cos)\) is continuous in \((0, \pi]\), strictly positive near \(\theta \sim \pi/2\),

\[
b(\cos \theta) \sin^{d-2} \theta \xrightarrow[\theta \to 0^+]{} b_0 \theta^{-(1+\nu)},
\]

for \(b_0 > 0\) and \(\nu\) in \((-\infty, 2)\)
Physical Observables

- **THE LOCAL DENSITY**:
  \[
  \rho(t, x) = \int_{\mathbb{R}^d} f(t, x, v) \, dv
  \]

- **THE LOCAL VELOCITY**:
  \[
  u(t, x) = \frac{1}{\rho(t, x)} \int_{\mathbb{R}^d} vf(t, x, v) \, dv
  \]

- **THE LOCAL ENERGY**:
  \[
  E(t, x) = \int_{\mathbb{R}^d} \frac{|v|^2}{2} f(t, x, v) \, dv = \rho(t, x) \frac{|u|^2}{2} + d \frac{\rho(t, x) \theta(t, x)}{2}
  \]
Conservation laws

- the preservation of the total mass

\[
\frac{d}{dt} \int_{\Omega} \rho(t, x) \, dx = 0,
\]
Conservation laws

- the preservation of the total mass
  \[
  \frac{d}{dt} \int_{\Omega} \rho(t,x) \, dx = 0,
  \]

- the preservation of total energy if \( \Omega \) has no boundary or if boundary conditions are bounce-back or specular reflections
  \[
  \frac{d}{dt} \int_{\Omega} E(t,x) \, dx = 0,
  \]
Conservation laws

- the preservation of the total mass
  \[ \frac{d}{dt} \int_{\Omega} \rho(t, x) \, dx = 0, \]

- the preservation of total energy if \( \Omega \) has no boundary or if boundary conditions are bounce-back or specular reflections
  \[ \frac{d}{dt} \int_{\Omega} E(t, x) \, dx = 0, \]

- the preservation of total momentum if \( \Omega \) has no boundary
  \[ \frac{d}{dt} \int_{\Omega} \rho(t, x)u(t, x) \, dx = 0, \]
H-Theorem and entropy dissipation

- **THE ENTROPY**: 
  \[ S(f) = \int_{\mathbb{R}^d} f \log f \, dx \, dv \]

- **THE ENTROPY DISSIPATION**: 
  \[ D(f) = - \int_{\mathbb{R}^d} Q(f, f) \log f \, dx \, dv \]

- **H-THEOREM**: 
  \[ \frac{d}{dt} S(f) = - \int_{\Omega} D(f) \, dx \leq 0. \]
H-Theorem and entropy dissipation

- **THE ENTROPY:**

  \[ S(f) = \int_{\mathbb{R}^d} f \log f \, dx dv \]

- **THE ENTROPY DISSIPATION:**

  \[ D(f) = - \int_{\mathbb{R}^d} Q(f, f) \log f \, dx dv \]

- **H-THEOREM:**

  \[ \frac{d}{dt} S(f) = - \int_{\Omega} D(f) \, dx \leq 0. \]

  \[ \Rightarrow \text{Irreversibility of the Boltzmann equation!} \]
Local and global equilibria

- The H-theorem implies that local equilibria are **THE LOCAL MAXWELLIANS**:

\[
M(\rho(t,x),u(t,x),\theta(t,x))(v) = \frac{\rho}{(2\pi \theta)^{d/2}} e^{-\frac{|v-u|^2}{2\theta}}.
\]
Local and global equilibria

- **The H-theorem implies that local equilibria are THE LOCAL MAXWELLIANS:**

  \[ M(\rho(t,x),u(t,x),\theta(t,x))(v) = \rho \frac{e^{-\frac{|v-u|^2}{2\theta}}}{(2\pi\theta)^{d/2}}. \]

- **Global equilibria:**

  \[ \forall (x,v) \in \Omega \times \mathbb{R}^d, \quad v \cdot \nabla_x M(\rho,u,\theta) = 0. \]
Local and global equilibria

- The H-theorem implies that local equilibria are **THE LOCAL MAXWELLIANS**:

  \[ M_{\rho(t,x),u(t,x),\theta(t,x)}(v) = \frac{\rho}{(2\pi\theta)^{d/2}} e^{-\frac{|v-u|^2}{2\theta}}. \]

- **GLOBAL EQUILIBRIA**:

  \[ \forall (x,v) \in \Omega \times \mathbb{R}^d, \quad v \cdot \nabla_x M_{\rho,u,\theta} = 0. \]

- in the case of the torus or non-axis symmetric bounded domains with bounce-back or specular reflection boundary conditions, the only global equilibrium is

  \[ \mu(v) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}. \]
A quantic version of the Boltzmann equation

The homogeneous Boltzmann-Nordheim equation

Marc Briant
Quantum effects in gases of bosons

- **Quantic collision properties**: The probability of two particles colliding also depends on the number of particles already in the outcoming velocity.
Quantum effects in gases of bosons

- **Quantic collision properties**: The probability of two particles colliding also depends on the number of particles already in the outcoming velocity.

- **The Boltzmann-Nordheim operator**:

\[
\int_{\mathbb{R}^{N} \times \mathbb{S}^{d-1}} |v - v^*|^{\gamma} b(\cos \theta) \left[ f'(1 + f)f^*(1 + f^*) - f(1 + f')f^*(1 + f^*) \right] dv^* d\sigma.
\]
Two types of equilibrium

- **Entropy of the solution**: always increasing in time

\[
S(f) = \int_{\mathbb{R}^d} \left[ (1 + f) \log(1 + f) - f \log(f) \right] dv
\]
Two types of equilibrium

- **Entropy of the solution**: always increasing in time

\[ S(f) = \int_{\mathbb{R}^d} [(1 + f)\log(1 + f) - f\log(f)] \, dv \]

\[ \Rightarrow \text{No control over concentration phenomena} \]
Two types of equilibrium

- **Entropy of the solution**: always increasing in time

\[ S(f) = \int_{\mathbb{R}^d} [(1 + f)\log(1 + f) - f\log(f)] \, dv \]

\[ \Rightarrow \text{No control over concentration phenomena} \]

- **Equilibria**:

\[ F_{BE}(v) = m_0\delta(v - v_0) + \frac{1}{e^{\frac{\beta}{2}(|v-v_0|^2-\mu)} - 1} \]

- \( m_0 \geq 0, \beta \) in \((0, +\infty]\), \(-\infty < \mu \leq 0\)
- \( \mu.m_0 = 0 \)
Two types of equilibrium

- **Entropy of the solution**: always increasing in time

\[ S(f) = \int_{\mathbb{R}^d} [(1 + f) \log(1 + f) - f \log(f)] \, dv \]

\[ \Rightarrow \text{No control over concentration phenomena} \]

- **Equilibria**:

\[ F_{BE}(v) = m_0 \delta(v - v_0) + \frac{1}{e^{\frac{\beta}{2} (|v - v_0|^2 - \mu)} - 1} \]

- \( m_0 \geq 0, \beta \in (0, +\infty], -\infty < \mu \leq 0 \)
- \( \mu. m_0 = 0 \)

\[ \Rightarrow \text{Unique for given } (M_0, v_0, T_0) \text{ and } m_0 = 0 \text{ iff } T_0 \geq T_c(M_0) \]
Local existence and uniqueness of solutions
Plan
- General presentation of the Boltzmann collisional model
- A quantic version of the Boltzmann equation
- Local existence and uniqueness of solutions
- Futur developments

The homogeneous setting
- Local Cauchy theory and main ingredients
- A quick glance at the proof of existence
- A longer stare at the proof of uniqueness

Our framework

- **Spatially homogeneous equation**: \( \partial_t f = Q(f) \)
- solutions in \( L^\infty \cap L^1 \left( 1 + |v|^2 \right) \)
Our framework

- **Spatially homogeneous equation:**
  \[ \partial_t f = Q(f) \]
  
  solutions in \( L^\infty \cap L^1 \left( 1 + |v|^2 \right) \)

- **The collision operator:**
  - Hard and Maxwellian potentials \( 0 \leq \gamma \leq 1 \)
  - Grad’s angular cutoff \( b \circ \cos \) integrable on the sphere
Our framework

\[ \partial_t f = Q^+(f) - fQ^-(f) \]

where we defined

\[ Q^+(f) = C_\phi \int_{\mathbb{R}^N \times S^{d-1}} |v - v_*|^\gamma b(\cos \theta) f' f_*'(1 + f + f_*) dv_* d\sigma \]

\[ Q^-(f) = C_\phi \int_{\mathbb{R}^N \times S^{d-1}} |v - v_*|^\gamma b(\cos \theta) f_*(1 + f' + f_*') dv_* d\sigma \]
Previous studies

- **Isotropic Cauchy theory**: radially symmetric solutions
  - Lu (2000 – 2005): global solutions in $L^1 \left(1 + |v|^2\right)$ and weak form for distributions
  - Escobedo-Velázquez (preprint): locally in time in $L^\infty(1 + |v|^{6+0})$
Previous studies

- **Isotropic Cauchy theory**: radially symmetric solutions
  - Lu (2000 – 2005): global solutions in $L^1 \left(1 + |v|^2\right)$ and weak form for distributions
  - Escobedo-Velázquez (preprint): locally in time in $L^\infty(1 + |v|^{6+0})$

- **Bose-Einstein condensate**:
  - Lu: long-time convergence towards equilibrium
  - Escobedo-Velázquez: blow-up in finite time in isotropic setting
Main arguments

**From classical Boltzmann works:**
- Regularity methods from Arkeryd (1972)
- Gain of regularity thanks to regularising properties of $Q^+$

Marc Briant

THE HOMOGENEOUS BOLTZMANN-NORDHEIM EQUATION
Main arguments

- **From classical Boltzmann works:**
  - Regularity methods from Arkeryd (1972)
    - Gain of regularity thanks to regularising properties of $Q^+$
  - Strategy of Mischler-Wennberg (1999)
    - Existence: Explicit Euler scheme and truncation of $Q$
    - Uniqueness: quantification of the blow-up of the $(2 + \gamma)^{th}$ moment and Nagumo’s uniqueness argument
Main arguments

**From classical Boltzmann works:**
- Regularity methods from Arkeryd (1972)
  - Gain of regularity thanks to regularising properties of $Q^+$
- Strategy of Mischler-Wennberg (1999)
  - Existence: Explicit Euler scheme and truncation of $Q$
  - Uniqueness: quantification of the blow-up of the $(2 + \gamma)^{th}$ moment and Nagumo’s uniqueness argument

**Our main contributions:**
- Control of the trilinear term and of the lack of control of $Q^-$ by entropy
- New $L^\infty$ estimates for the gain operator via integrability on Carleman’s hyperplane
- Refinement of a Povzner-type inequality (evolution of convex/concave functions through a collision)
The key role of the $L^\infty$-norm: properties of the gain operator

- **Carleman’s hyperplanes**:
  - $E_{vv'}$ is the hyperplane orthogonal to $v - v'$ going through $v$
  - With a Carleman representation we show

\[
\|Q^+(f)\|_{L^\infty_v} \leq C_+ \left( \|f\|_{L^\infty_v}, \|f\|_{L^1_v} \right) \sup_{v, v' \in \mathbb{R}^d} \left[ \int_{E_{vv'}} f'_* \, dE(v'_*) \right]
\]
The key role of the $L^\infty$-norm: properties of the gain operator

- **Carleman’s hyperplanes:**
  - $E_{vv'}$ is the hyperplane orthogonal to $v - v'$ going through $v$
  - With a Carleman representation we show

  \[
  \| Q^+(f) \|_{L^\infty_v} \leq C_+ \left( \| f \|_{L^\infty_v}, \| f \|_{L^1_v} \right) \sup_{v,v' \in \mathbb{R}^d} \left[ \int_{E_{vv'}} f'_* \, dE(v'_*) \right]
  \]

- **Regularising effect of $Q^+$**

  \[
  \int_{E_{vv'}} Q^+(f)(v'_*) \, dE(v'_*) \leq C_+E \left( \| f \|_{L^\infty_v}, \| f \|_{L^1_v} \right)
  \]
The key role of the $L^\infty$-norm: control of the condensate

- **Propagation of integrability on hyperplanes**: if $f$ is a mass and energy preserving solution that belongs to $L^\infty_v$ on $[0, T]$ then

\[
\int_{E_{v'}} f'_*(t) \, dE(v'_*) \leq C_E \left( \| f \|_{L^\infty_v} \right) \left[ 1 + e^{-C_0 t} \int_{E_{v'}} f_0(v'_*) \, dE(v'_*) \right]
\]
The key role of the $L^\infty$-norm: control of the condensate

- **Propagation of integrability on hyperplanes**: if $f$ is a mass and energy preserving solution that belongs to $L^\infty_v$ on $[0, T]$ then

$$
\int_{E_{v'}} f'_*(t) \, dE(v'_*) \leq C_E \left( \|f\|_{L^\infty_v} \right) \left[ 1 + e^{-C_0 t} \int_{E_{v'}} f_0(v'_*) \, dE(v'_*) \right]
$$

- **Gain of regularity at infinity**: if $f_0 \in L^1_{2,v} \cap L^\infty_{s,v}$ with $s > d - 1 \geq 2$ and if $f$ preserves mass and energy and belongs to $L^\infty_v$ on $[0, T]$ then there exists $2 < \bar{s} \leq s$ such that

$$
\forall 0 < s' < \bar{s}, \forall t \in [0, T], \quad \|f(t, \cdot)\|_{L^\infty_{s',v}} \leq C_T \left( \sup_{[0,T]} \|f\|_{L^\infty_v} \right).
$$
Our result

**Theorem**

Let $f_0$ be in $L_{2,v}^1 \cap L_{s,v}^\infty$ when $s > d - 1$. Then there exists $T_0 > 0$ such that there exists a unique mass and energy preserving solution $f$ in $L_{loc}^\infty \left([0, T_0), L_{2,v}^1 \cap L_v^\infty\right)$. This solution moreover satisfies

1. $\forall s' < s, \quad f \in L_{loc}^\infty \left([0, T_0), L_{2,v}^1 \cap L_{s',v}^\infty\right)$,
2. $T_0 = +\infty$ or $\lim_{T \to T_0^-} \|f\|_{L_{loc}^\infty([0,T] \times \mathbb{R}^d)} = +\infty$,
3. $f$ preserves the momentum $f_0$,
4. For all $\alpha > 0$ and for all $0 < T < T_0$, the $\alpha^{th}$ moment of $f$ belongs to $L_{loc}^\infty([T, T_0))$.
The explicit Euler scheme

- **Truncated operators**: we consider $Q_n$ which collision kernel is $(n \wedge |v - v_*|)^\gamma$
The explicit Euler scheme

- **Truncated operators**: we consider $Q_n$ which collision kernel is $(n \wedge |v - v_*|)^\gamma$

- **Plugged into a Euler scheme**: solving

$$\partial_t f_n = Q_n(f_n)$$

with a time-discrete scheme on $[0, T_0]$ with a time-step $\Delta_n$:

$$\begin{cases}
    f_n^{(0)}(v) = f_0(v) \\
    f_n^{(k+1)}(v) = f_n^{(k)}(v) \left(1 - \Delta_n Q_n^- \left(f_n^{(k)}\right)\right) + \Delta_n Q_n^+ \left(f_n^{(k)}\right)
\end{cases}$$
THE explicit Euler scheme : convergence

- **Propagation of bounds**: \( T_0 \) and \( \Delta_n \) chosen such that
  - \( f_n^{(k)} \geq 0 \)
  - \( f_n^{(k)} \) has same mass, momentum and energy as \( f_0 \)
  - \( \int_{E_{v'}} f_n^{(k)}(v')dE(v') \leq E_\infty \)
  - \( \left\| f_n^{(k)} \right\|_{L^\infty_v} \leq N_\infty \)
The explicit Euler scheme : convergence

- **Propagat**ion of bounds : \( T_0 \) and \( \Delta_n \) chosen such that
  - \( f_n^{(k)} \geq 0 \)
  - \( f_n^{(k)} \) has same mass, momentum and energy as \( f_0 \)
  - \( \int_{E_{v*}} f_n^{(k)}(v')dE(v') \leq E_\infty \)
  - \( \left\| f_n^{(k)} \right\|_{L_\infty^v} \leq N_\infty \)

- **Weak convergence** : Dunford-Pettis theorem and the slightly better estimate

\[
\sup_{v \in \mathbb{R}^d} \left[ f_n^{(k)}(v) + \Delta_n \sum_{j=0}^{k-1} (n^\gamma \wedge (1 + |v|^{\gamma})) f_n^{(j)}(v) \right] \leq K_\infty
\]
The explicit Euler scheme: convergence

- **Propagation of bounds:** $T_0$ and $\Delta_n$ chosen such that
  - $f_n^{(k)} \geq 0$
  - $f_n^{(k)}$ has same mass, momentum and energy as $f_0$
  - $\int_{E_{v'}} f_n^{(k)}(v')dE(v') \leq E_\infty$
  - $\|f_n^{(k)}\|_{L_\infty^v} \leq N_\infty$

- **Weak convergence:** Dunford-Pettis theorem and the slightly better estimate
  \[
  \sup_{v \in \mathbb{R}^d} \left[ f_n^{(k)}(v) + \Delta_n \sum_{j=0}^{k-1} (n^\gamma \wedge (1 + |v|\gamma)) f_n^{(j)}(v) \right] \leq K_\infty
  \]

- **Globally defined solutions?** $T_0$ depends on $\|f_0\|_{L_\infty^v}$ at $T_0$ we can only prove $\|f_\infty\|_{L_\infty^v} \leq 2 \|f_0\|_{L_\infty^v}$...
The necessity of controlling moments
The necessity of controlling moments

- **Evolution of** $\|f - g\|_{L^1_v}$: for all $0 \leq t \leq T < T_0$,

$$\frac{d}{dt} \|f - g\|_{L^1_v} \leq C_T \left[ \|f - g\|_{L^1_{2,v}} + \|f - g\|_{L^\infty_v} \right]$$
The necessity of controlling moments

- **Evolution of** $\|f - g\|_{L^1_v}$: for all $0 \leq t \leq T < T_0$,
  \[
  \frac{d}{dt} \|f - g\|_{L^1_v} \leq C_T \left[ \|f - g\|_{L^1_{2,v}} + \|f - g\|_{L^\infty_v} \right]
  \]

- **Evolution of** $\|f - g\|_{L^1_{2,v}}$: for all $0 \leq t \leq T < T_0$,
  \[
  \frac{d}{dt} \|f - g\|_{L^1_{2,v}} \leq C_T \left[ M_{2+\gamma}(t) \|f - g\|_{L^1_v} + \|f - g\|_{L^1_{2,v}} \right.
  
  \left. + (1 + M_{2+\gamma}(t)) \|f - g\|_{L^\infty_v} \right]
  \]
The necessity of controlling moments

- **Evolution of** $\|f - g\|_{L^1_v}$: for all $0 \leq t \leq T < T_0$,
  $$\frac{d}{dt} \|f - g\|_{L^1_v} \leq C_T \left[ \|f - g\|_{L^2_v} + \|f - g\|_{L^\infty_v} \right]$$

- **Evolution of** $\|f - g\|_{L^1_{2,v}}$: for all $0 \leq t \leq T < T_0$,
  $$\frac{d}{dt} \|f - g\|_{L^1_{2,v}} \leq C_T [M_{2+\gamma}(t) \|f - g\|_{L^1_v} + \|f - g\|_{L^1_{2,v}}$$
  $$+ (1 + M_{2+\gamma}(t)) \|f - g\|_{L^\infty_v}]$$

- **Control of** $\|f - g\|_{[0, T], v}$: for all $0 \leq t \leq T < T_0$,
  $$\|f - g\|_{L^\infty_v} \leq C_T \int_0^t \left[ \|f - g\|_{L^1_{2,v}}(u) + \|f - g\|_{L^\infty_v}(u) \right] du.$$
Existence and blow-up of moments

**Evolution of quantities during a collision:**

\[
\int_{\mathbb{R}^d} Q(f) \psi \, dv = \int_{\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}} q(f) \left[ \psi_* + \psi' - \psi_* - \psi \right] \, d\sigma \, dv \, dv_*
\]

where

\[
q(f)(v, v_*) = |v - v_*| \gamma b(\cos \theta) ff_* (1 + f' + f'_*)
\]
Existence and blow-up of moments

- **Evolution of quantities during a collision:**

  \[
  \int_{\mathbb{R}^d} Q(f) \psi \, dv = \int_{\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}} q(f) \left[ \psi_\ast + \psi' - \psi_\ast - \psi \right] d\sigma dv dv_\ast
  \]

  where

  \[
  q(f)(v, v_\ast) = |v - v_\ast| \gamma b(\cos \theta) ff_\ast (1 + f' + f'_\ast)
  \]

- **Povzner-type inequalities:** as an example, consider

  \[
  1 \leq F \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d \times S^{d-1})
  \]
Existence and blow-up of moments

\[ \int_{\mathbb{S}^{d-1}} F(v, v_*, \sigma) b(\theta) \left( \psi(|v'|^2) + \psi(|v|^2) - \psi(|v_*|^2) - \psi(|v|^2) \right) d\sigma = G_\psi(v, v_*) - H_\psi(v, v_*) \]

In the case \( \psi(x) = x^{1+\alpha} \) with \( \alpha > 0 \),

1. \[ |G(v, v_*)| \leq C_G \alpha (|v| |v_*|)^{1+\alpha} \]
2. \[ H(v, v_*) \geq C_H \alpha \left( |v|^{2+2\alpha} + |v_*|^{2+2\alpha} \right) \left[ 1 - 1_{\{|v|<|v_*|<2|v|\}} \right] \]
Existence and blow-up of moments

- **Creation of moments**: for $\alpha > 0$, as soon as $T > 0$,

\[
\int_{\mathbb{R}^d} |v|^\alpha f(t, v) \, dv \in L^\infty_{\text{loc}} ([T, T_0])
\]
Existence and blow-up of moments

- **Creation of moments**: for $\alpha > 0$, as soon as $T > 0$,

  $$\int_{\mathbb{R}^d} |v|^\alpha f(t, v) \, dv \in L^\infty_{\text{loc}}([T, T_0])$$

- **Behaviour at $t = 0$**: for all $0 < T < T_0$ there exists $C_T > 0$ such that

  $$\forall t \in (0, T], \quad M_{2+\gamma}(t) \leq \frac{C_T}{t}$$
Nagumo’s uniqueness argument

For all $0 < t \leq T < T_0$, 

\[
\frac{d}{dt} \| f - g \|_{L^1_v} \leq C_T \left[ \| f - g \|_{L^1_v} + \| f - g \|_{L^1_{2,v}} \right]
\]

\[
\frac{d}{dt} \| f - g \|_{L^1_{2,v}} \leq C_T \left[ M_{2+\gamma}(t) \| f - g \|_{L^1_v} + \| f - g \|_{L^1_{2,v}} + (1 + M_{2+\gamma}(t)) \| f - g \|_{L^\infty_v} \right]
\]

\[
\| f - g \|_{L^\infty_v} \leq \int_0^t \left[ \| f - g \|_{L^1_{2,v}} (u) + \| f - g \|_{L^\infty_v} (u) \right] du
\]
Nagumo’s uniqueness argument

For all $0 < t \leq T < T_0$, 

$$\|f - g\|_{L^1_v} \leq C_T t$$

$$\frac{d}{dt} \|f - g\|_{L^1_{2,v}} \leq C_T \left[ \frac{1}{t} \|f - g\|_{L^1_v} + \|f - g\|_{L^1_{2,v}} + \left(1 + \frac{1}{t}\right) \|f - g\|_{L^\infty_v} \right]$$

$$\|f - g\|_{L^\infty_v} \leq C_T t$$
For all $0 < t \leq T < T_0$,

$$\frac{d}{dt} \| f - g \|_{L^1_v} \leq C_T \left[ \| f - g \|_{L^1_v} + \| f - g \|_{L^1_{2,v}} \right]$$

$$\frac{d}{dt} \| f - g \|_{L^1_{2,v}} \leq C_T t$$

$$\| f - g \|_{L^\infty_v} \leq \int_0^t \left[ \| f - g \|_{L^1_{2,v}} (u) + \| f - g \|_{L^\infty_v} (u) \right] du$$
Nagumo’s uniqueness argument

For all $0 < t \leq T < T_0$,

$$\left\| f - g \right\|_{L^1_v} \leq C_T t^2$$

$$\frac{d}{dt} \left\| f - g \right\|_{L^1_{2,v}} \leq C_T \left[ \frac{1}{t} \left\| f - g \right\|_{L^1_v} + \left\| f - g \right\|_{L^1_{2,v}} \right] + \left( 1 + \frac{1}{t} \right) \left\| f - g \right\|_{L^\infty_v}$$

$$\left\| f - g \right\|_{L^\infty_v} \leq C_T t^2$$
Nagumo’s uniqueness argument

For all $0 < t \leq T < T_0$, 

$$
\| f - g \|_{L^1_v} \leq C_T t^n \\
\| f - g \|_{L^1_{2,v}} \leq C_T t^n \\
\| f - g \|_{L^\infty_{[0,t],v}} \leq C_T t^n
$$
Nagumo’s uniqueness argument

For all $0 < t \leq T < T_0$, 

\[ \| f - g \|_{L^1_v} \leq C_T t^n \]

\[ \| f - g \|_{L^1_{2,v}} \leq C_T t^n \]

\[ \| f - g \|_{L^\infty_{[0,t],v}} \leq C_T t^n \]

$\Rightarrow C_T$ depends on $n$!
∀t ∈ [0, T], \[ \frac{d}{dt} \| f - g \|_{L^1_{2,v}} \leq \frac{K_1}{t} \| f - g \|_{L^1_{2,v}} + K_2 \sup_{[0,t],v} \| f - g \|_{L^1_{2,v}} \]

Look at \( X(t) = \| f - g \|_{L^1_{2,v}} / t^n : X(0) = 0, X'(0) = 0 \)
∀ \( t \in [0, T] \), \( \frac{d}{dt} \| f - g \|_{L^1_{2,v}} \leq \frac{K_1}{t} \| f - g \|_{L^1_{2,v}} + K_2 \sup_{[0,t],v} \| f - g \|_{L^1_{2,v}} \)

1. Look at \( X(t) = \| f - g \|_{L^1_{2,v}} / t^n : X(0) = 0, X'(0) = 0 \)

2. Choose \( n \) large enough such that

\[
X(t) \leq tK_2 \sup_{[0,t],v} X(s)
\]
Nagumo’s uniqueness argument : conclusion

\[ \forall t \in [0, T], \quad \frac{d}{dt} \| f - g \|_{L^1_{2,v}} \leq K_1 \| f - g \|_{L^1_{2,v}} + K_2 \sup_{[0,t],v} \| f - g \|_{L^1_{2,v}} \]

1. Look at \( X(t) = \frac{\| f - g \|_{L^1_{2,v}}}{t^n} : X(0) = 0, X'(0) = 0 \)
2. Choose \( n \) large enough such that \( X(t) \leq tK_2 \sup_{[0,t],v} X(s) \)
3. By induction
4. On \([0, \tau]\), find \( X(t) = 0. \) Extend to \([\tau, T]\) by Grönwall’s lemma
In an ideal future:

- Showing the creation of the Bose-Einstein condensate in non-isotropic setting (we do not even know how to obtain global existence...)
- Obtaining a constructive proof of the creation of the condensate (we do not even know if it is unique...)
I thank you for your attention

1 In an ideal future:
   - Showing the creation of the Bose-Einstein condensate in non-isotropic setting (we do not even know how to obtain global existence...)
   - Obtaining a constructive proof of the creation of the condensate (we do not even know if it is unique...)

2 In the real world:
   - Understand how to obtain solutions globally defined in time
   - Understand if blow-up can occur away from the equilibrium momentum
   - Understand the collisional process of the trilinear term that generates a condensate
**1. In an ideal future:**
- Showing the creation of the Bose-Einstein condensate in non-isotropic setting (we do not even know how to obtain global existence...)
- Obtaining a constructive proof of the creation of the condensate (we do not even know if it is unique...)

**2. In the real world:**
- Understand how to obtain solutions globally defined in time
- Understand if blow-up can occur away from the equilibrium momentum
- Understand the collisional process of the trilinear term that generates a condensate