Introducing the stability catalyzer for relativistic soliton solutions

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Abstract

For a real nonlinear Klein-Gordon (RNKG) lagrangian density with a special solitary wave solution (SSWS) which is essentially unstable, it is shown how adding a proper additional massless term, without changing the dominant dynamical equation and other properties of the SSWS, guarantees the stability of the SSWS appreciably, i.e. it is a stability catalyzer.
**Keywords**: nonlinear, Klein-Gordon, energetically stability, catalyzer, non-topological, soliton.

I. **INTRODUCTION**

The classical relativistic field equations with stable solitary wave solutions (solions) have been interesting for decades [1–5]. In fact, soliton solutions behave like real particles because they have the non-disperse localized energy density functions and satisfy the standard relativistic energy momentum relations. For example, the real nonlinear Klein-Gordon (RNKG) systems in $1+1$ dimensions with kink (anti-kink) solutions [1 5 27], Skyrme model of baryons [5 29 30] and 't Hooft Polyakov model which yields magnetic monopole solutions [1 5 31 32] in $3+1$ dimensions are three well known systems which yield stable solitary wave solutions or solitons. Note that, all three systems listed above, yield topological solitons and the topological feature is the main reason for their stability. For topological solutions, to have a many particle-like solution, there are usually complicated conditions on the boundaries, but for the non-topological solutions each arbitrary multi-particle-like solution can be obtained easily just by adding distinct far enough solitary wave solutions together.

There are many works to find the relativistic non-topological solitary wave solutions, among which one can mention the complex nonlinear KG systems which lead to non-dispersive solitary wave packet solutions that are called Q-ball [33–48]. Although, many of the Q-ball are stable objects according to the Vakhitov-Kolokolov (or the classical) criterion of the stability [33 36 49 51], but none of them are energetically stable objects. In fact, a special solitary wave solution is energetically stable if any arbitrary variation above the background of that leads to an increase in the total energy [52]. Of course, the Q-ball solutions have the minimum energy among other solutions with the same electrical charge [36 38] which is a necessary, but not a sufficient, condition for the energetically stability criterion.

In this paper, we introduce a special real nonlinear Klein-Gordon (RNKG) model in $1+1$ dimensions with a well-formed non-topological special solitary wave solution (SSWS) which is essentially unstable [28 33]. But we will show how adding a proper term to the original RNKG Lagrangian density changes the SSWS to a stable non-topological particle-like solution. We call this additional term the stability catalyzer term, because it behaves
as a massless spook \textsuperscript{1} which surrounds the SSWS and guarantees the stability of that, i.e. it prevents any change in the internal structure of the SSWS, and leaves the dominant dynamical equations and other properties of the SSWS invariant. Note that, we consider the model in 1 + 1 dimension just for simplicity, but it can be extended to 3 + 1 dimensions with some modifications.

The organization of this paper is as follows: In the next section, for the RNKG systems with a single scalar field, we set up the basic equations and consider a special RNKG model with a special non-topological non-vibrational solitary wave solution. In section III, we introduce the stability catalyst term which yields to an extended RNKG system with a non-topological energetically stable SSWS. In section IV, we study the stability of the SSWS for any arbitrary small deformations. The last section is devoted to summary and conclusions.

II. SINGLE FIELD RNKG SYSTEMS IN 1 + 1 DIMENSIONS

The simplest form of the real nonlinear Klein Gordon (RNKG) systems in 1+1 dimensions can be introduced by the following lagrangian density:

$$\mathcal{L}_o = \partial^\mu \phi \partial_\mu \phi - U(\phi), \tag{1}$$

in which $\phi$ is the single real scalar field and $U(\phi)$ is called potential. Note that, through the paper, for simplicity, we take the speed of light equal to one ($c = 1$). Using the principle of least action, then the related equation of motion is

$$\Box \phi = \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = - \frac{1}{2} \frac{dU}{d\phi}. \tag{2}$$

Using the Noether’s theorem, one can simply obtain the energy-momentum tensor:

$$T^{\mu\nu} = 2 \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}_o, \tag{3}$$

in which $g^{\mu\nu}$ is the Minkowski metric. The $T^{00} (T^{10} = T^{01})$ component of this tensor is the same energy (momentum) density function which is simplified to

$$T^{00} = \varepsilon_o(x, t) = \dot{\varphi}^2 + \varphi'^2 + U(\varphi) \quad (T^{01} = 2\dot{\varphi}\varphi'), \tag{4}$$

\textsuperscript{1} We chose the word ”spook” in order not to confuse with words like ”ghost” and ”phantom”, which have meaning in the literature.
in which dot and prime are symbols for time and space derivatives respectively. The integration of \( T^{00} \) (or \( T^{10} \)) over the whole space yields the same total energy \( E \) (momentum \( P \)) of the system and always remains constant.

In general, it is easy to show that there is not a stable non-topological non-vibrational solitary wave solution for the RNKG systems in 1 + 1 dimensions [28, 33]. For example, if the following form of the potential is considered:

\[
U(\varphi) = \varphi^4(1 - \varphi^2),
\]  
(5)

the equation of motion (2), for a static (non-moving and non-vibrational) solution \( \varphi = \varphi_o(x) \), leads to

\[
\frac{d^2 \varphi_o}{dx^2} = 2\varphi_o^3 - 3\varphi_o^5,
\]  
(6)

which has a non-topological solitary wave solution

\[
\varphi_o(x) = \pm \frac{1}{\sqrt{1 + x^2}},
\]  
(7)

Applying the Lorentz transformations, the boosted version of this solution \( \varphi(x,t) \) can be obtained as well:

\[
\varphi_v(x,t) = \varphi_o(\tilde{x}) = \pm \frac{1}{\sqrt{1 + \tilde{x}^2}},
\]  
(8)

where \( v \) is the velocity, \( \gamma = \frac{1}{\sqrt{1 - v^2}} \) and \( \tilde{x} = \gamma(x - vt) \). However, since the potential (5) for \( |\varphi| > \frac{\sqrt{6}}{3} \) is decreasing and for \( |\varphi| > 1 \) takes negative values. Therefore, without violating the conservation energy law, the effect of any small perturbation, causes the profile of the localized solution (7) to blow up (see Fig. 1).

In general, since the theory is relativistic, therefore the same standard well-known relativistic relations between the moving and non-moving versions of any arbitrary solution with a finite energy and localized energy density function would be obtained, i.e.

\[
E = m \equiv \int_{-\infty}^{+\infty} T^{00} dx = \int_{-\infty}^{+\infty} [\dot{\varphi}_v^2 + \varphi_v^2 + U(\varphi_v)] dx = \gamma E_o = \gamma m_o,
\]  
(9)

\[
P = \int_{-\infty}^{+\infty} T^{01} dx = 2 \int_{-\infty}^{+\infty} \dot{\varphi}_v \varphi_v' dx = \gamma m_o v.
\]  
(10)

where \( E_o \) (or \( m_o \)) is the same rest energy (mass) of the solution. Moreover, the width of any arbitrary moving solutions is always smaller than the non-moving version of that exactly according to the Lorentz contraction.
FIG. 1. The non-topological static solitary wave solution (7) of the RNKG system (5) is essentially unstable and spontaneously blows up. This Fig is obtained from a finite difference method in Matlab for the SSWS (7) as the initial condition of the PDE (2).

III. AN EXTENDED RNKG SYSTEM WITH A STABLE NON-TOPOLOGICAL SOLITON SOLUTION

In the standard classical field theory, one of our goals is to find the proper nonlinear KG (-like) field equations which lead to soliton solutions as the stable particle-like objects. Unfortunately, there are few models with soliton solutions. In this paper, our main goal is to find a proper additional term for the original lagrangian density (1) which guarantees the stability of the special solution (7) properly, but like a catalyst, it does not play any role in the dominant equation of motion of the free special solution (7). In other word, we expect just for the free special solution (7), the dominant lagrangian density and the dominant dynamical equation would be same as the original ones (1) and (2), respectively. However, the new lagrangian density is introduced as follows:

$$\mathcal{L} = \mathcal{L}_o + F = \partial^\mu \varphi \partial^\nu \varphi - U(\varphi) + F,$$

(11)
where $F$ is the same unknown additional (catalyzer) term which must be recognized properly. In fact, we expect for the SSWS (7), the new extended lagrangian density (11) is reduced to the same original version (1), i.e. we expect the new additional term $F$ for the SSWS (7) to be zero. Note that, the new extended system (11) and the original RNKG system (1) are essentially different relativistic field systems with different solutions except the SSWS (7) which we assume to be a common solution. Similar to the standard relativistic lagrangian densities in physics, we expect the unknown functional scalar $F$ must a function of the allowed scalars $\varphi$ and $\partial_{\mu}\varphi\partial_{\mu}\varphi$. However, the new equation of motion is

$$
\left[\Box\varphi + \frac{1}{2\,d}\frac{dU}{d\varphi}\right] + \frac{1}{2}\left[\frac{\partial}{\partial x^\mu}\left(\frac{\partial F}{\partial (\partial_{\mu}\varphi)}\right) - \left(\frac{\partial F}{\partial \varphi}\right)\right] = 0.
$$

For the SSWS (7) to be a solution of the new equation of motion (12) (or the new equation of motion (12) is reduced to the same original version (2)), since the first part of this new equation is satisfied automatically and the functional $F$ is not essentially linear in $L_o$, we conclude that the two distinct terms $\frac{\partial}{\partial x^\mu}\left(\frac{\partial F}{\partial (\partial_{\mu}\varphi)}\right)$ and $\frac{\partial F}{\partial \varphi}$ must be zero independently for the SSWS (7).

For all the pervious requirements to be satisfied, one can conclude that $F$ must be a function of the powers of $K$ ($K^n$’s with $n \geq 3$), where $K$ is a special scalar functional

$$
K = \partial_{\mu}\varphi\partial_{\mu}\varphi + U(\varphi) = \dot{\varphi}^2 - \varphi'^2 + \varphi^4(1 - \varphi^2),
$$

which is defined to be zero when we have a SSWS (7). For example, a simple choice for the functional $F$ is

$$
F = BK^3,
$$

where $B$ is a real positive number. For this special choice (14), we obtain

$$
\frac{\partial}{\partial x^\mu}\left(\frac{\partial F}{\partial (\partial_{\mu}\varphi)}\right) = BK [6K\partial_{\mu}\partial_{\mu}\varphi + 12\partial_{\mu}K\partial_{\mu}\varphi],
$$

$$
\frac{\partial F}{\partial \varphi} = \left[3BK^2\frac{\partial K}{\partial \varphi}\right],
$$

which both are obviously zero for the SSWS (7). In fact, each term in the right hand side of the above equations contains a power of $K$ and hence all are zero for the SSWS (7). Therefore, with this special choice (14), we are sure that the pervious SSWS (7) is again a solution of the new extended system (11), and the new dynamical equation (12) is reduced to the same original one (2) as the dominant dynamical equation of motion for the SSWS (7) as well.
The energy density function of the new extended system \(\text{(11)}\) can be obtained easily
\[
T^{00} = \frac{\partial L}{\partial \dot{\varphi}} - \dot{L} = \left[ \dot{\varphi}^2 + \varphi^2 + U(\varphi) \right] + \left[ B\mathcal{K}^2(6\dot{\varphi}^2 - \mathcal{K}) \right] = \varepsilon_o + \varepsilon_1. \tag{15}
\]
If one applies the standard definition of the scalar \(\mathcal{K}\) \(\text{(13)}\) in the second part of the energy density \(\text{(15)}\), it turns to
\[
\varepsilon_1 = B\mathcal{K}^2(5\dot{\varphi}^2 + \varphi^2 + \varphi^4(\varphi^2 - 1)), \tag{16}
\]
which is zero for the SSWS \(\text{(7)}\) and the vacuum state \(\varphi = 0\). However, it is not a positive definite function, because function \(\varphi^4(\varphi^2 - 1)\) in the range \(0 < |\varphi| < 1\) is negative. Hence, we can not be sure about the energetically stability of the SSWS \(\text{(7)}\). Of course, we have previously provided an article \([28]\) to show the stability of a SSWS in motion, but mathematically it is not rigorous. Nevertheless, in this paper, we present a new extended RNKG system with the complete mathematical rigour to show how the SSWS \(\text{(7)}\) becomes an energetically stable object and resists against any arbitrary deformation.

To introduced a model which guarantees the energetically stability of the SSWS \(\text{(7)}\), we should introduce a new scalar field \(\theta\) which can be called the phase field or the catalyzer field. It is used to introduced a new additional (catalyzer) term \(F\) as follows:
\[
F = B \sum_{i=1}^{3} \mathcal{K}_i^3, \tag{17}
\]
where
\[
\mathcal{K}_1 = \varphi^4S_1, \tag{18}
\]
\[
\mathcal{K}_2 = \varphi^4S_1 + S_2, \tag{19}
\]
\[
\mathcal{K}_3 = \varphi^4S_1 + S_2 + 2\varphi^2S_3, \tag{20}
\]
and
\[
S_1 = \partial_\mu \theta \partial^\mu \theta - 1, \tag{21}
\]
\[
S_2 = \partial_\mu \varphi \partial^\mu \varphi + \varphi^4(1 - \varphi^2), \tag{22}
\]
\[
S_3 = \partial_\mu \varphi \partial^\mu \theta. \tag{23}
\]
In general, since \(S_i\)'s are three independent scalars, it is not possible to be zero simultaneously except for the non-trivial SSWS \(\text{(7)}\) with \(\theta = \omega_s t = \pm t\). In other words, \(S_1 = 0, S_2 = 0\) and
\( S_3 = 0 \) are three independent coupled nonlinear PDEs which do not have any non-trivial common solution except the SSWS (7) with \( \theta = \pm t \) (see the Appendix A). The same result goes for \( K_i \)'s, in fact since \( K_i \)'s are three independent linear combination of the scalars \( S_i \)'s, then they are not zero simultaneously except for the SSWS (7) with \( \theta = \pm t \). Note that, for a moving version of the SSWS (8) which moves at the velocity of \( v \), the proper phase function \( \theta \), for which all \( S_i \)'s would be zero simultaneously, is \( \theta = k \mu x^\mu \), i.e. the boosted version of the function \( \theta = \pm t \), provided

\[
k^\mu \equiv (k^0, k^1) = (\omega, k) = (\omega, \omega v),
\]

where \( \omega = \gamma \omega_s \) and \( \omega_s = \pm 1 \).

However, the new dynamical equations of motion of the extended lagrangian density (11) with the new additional term (14) can be obtained easily as follows:

\[
\square \varphi + \frac{1}{2} \frac{dU}{d\varphi} + \frac{1}{2} \left[ \frac{\partial}{\partial x^\mu} \left( \frac{\partial F}{\partial (\partial_\mu \varphi)} \right) - \left( \frac{\partial F}{\partial \varphi} \right) \right] = 0, \tag{25}
\]

\[
\frac{\partial}{\partial x^\mu} \left( \frac{\partial F}{\partial (\partial_\mu \theta)} \right) = 0. \tag{26}
\]

Again, it is easy to show that all different first and second derivatives of the catalyzer term \( F \) (17), which were seen in the Eqs. (25) and (26), for the SSWS (7) with \( \theta = \pm t \), would be zero simultaneously. In other words, for the SSWS (7) with \( \theta = \pm t \), Eq. (26) is satisfied automatically and Eq. (25) is reduced to the same standard original version (2) as the dominant dynamical equation of the SSWS (7). This means, if one asks about the right equation of motion of the free SSWS (7), our answer would be the same Eq. (2). Of course, the SSWS (7) in the new extended system (11) is now considered along with a scalar field \( \theta = \pm t \). However, from here to the end of the paper, we mean that the non-moving SSWS is as follows:

\[
\varphi_s(x) = \varphi_o(x) = \frac{\pm 1}{\sqrt{1 + x^2}}, \quad \theta_s(t) = \omega_s t = \pm t. \tag{27}
\]

Hence, the moving version of the SSWS (27) would be

\[
\varphi_v(x, t) = \varphi_s(\tilde{x}) = \frac{\pm 1}{\sqrt{1 + \tilde{x}^2}}, \quad \theta_v(x, t) = \theta_s(\tilde{t}) = k_\mu x^\mu = \omega t - k x. \tag{28}
\]

where \( \tilde{t} = \gamma (t - vx) \) and \( \tilde{x} = \gamma (x - vt) \).
The new energy density function of the extended RNKG system (11) with the new catalyster term (17) is

\[
\varepsilon(x, t) = T^{00} = \frac{\partial L}{\partial \dot{\varphi}} + \frac{\partial L}{\partial \dot{\theta}} - L = \varepsilon_o + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \\
\left[ \varphi^2 + \varphi^2 + U(\varphi) \right] + B \sum_{i=1}^{3} \left[ 3C_i K_i^2 - K_i^3 \right],
\]

(29)

which are divided into four distinct parts and

\[
C_i = \frac{\partial K_i}{\partial \dot{\theta}} + \frac{\partial K_i}{\partial \dot{\varphi}} \dot{\varphi} = \begin{cases} 
2\varphi^4 \dot{\varphi}^2 & i=1 \\
2(\varphi^2 + \varphi^4 \dot{\varphi}) & i=2 \\
2(\varphi + \varphi^2 \dot{\varphi})^2 & i=3.
\end{cases}
\]

(30)

After a straightforward calculation, one can obtain:

\[
\varepsilon_1 = B K_1^2 \left[ 5\varphi^4 \dot{\varphi}^2 + \varphi^4 \theta^2 + \varphi^4 \right] \geq 0,
\]

(31)

\[
\varepsilon_2 = B K_2^2 \left[ 5\varphi^4 \dot{\varphi}^2 + 5\dot{\varphi}^2 + \varphi^4 \theta^2 + \varphi^2 + \varphi^6 \right] \geq 0,
\]

(32)

\[
\varepsilon_3 = B K_3^2 \left[ 5(\varphi^2 \dot{\varphi} + \varphi^2)^2 + (\varphi^2 \theta' + \varphi')^2 + \varphi^6 \right] \geq 0.
\]

(33)

All terms in the above relations are now positive definite, therefore all \(\varepsilon_i\)'s \((i = 1, 2, 3)\) are positive definite functions and bounded from below by zero. All \(\varepsilon_i\)'s \((i = 1, 2, 3)\) are zero simultaneously just for the trivial vacuum state \(\varphi = 0\) and the non-trivial SSWS (27), as we expect from the catalyster role. Now, if parameter \(B\) is considered to be a large number, since for any other solution at least one of the \(K_i\)'s is a non-zero function, then at least one of the \(\varepsilon_i\)'s \((i = 1, 2, 3)\) would be a large positive function. It means that for other solutions (except the ones which are very close to the vacuum \(\varphi = 0\)), the related energies are always larger than the rest energy of the SSWS (27). Unlike \(\varepsilon_i\)'s \((i = 1, 2, 3)\), which are three absolute positive functions and are minimum for the SSWS, \(\varepsilon_o\) is not. But, in the next section, we will show that the role of \(\varepsilon_o\) in the stability considerations, if we take an extended system with a large parameter \(B\), would be completely ineffective.

IV. ENERGETICALLY STABILITY OF THE SSWS

Let us to study the stability of a non-moving SSWS (27) for the small variations. Any small deformed SSWS (27) can be introduced as follow:

\[
\varphi(x, t) = \varphi_s(x) + \delta \varphi(x, t) \quad \text{and} \quad \theta(x, t) = \theta_s(t) + \delta \theta(x, t),
\]

(34)
where $\delta \varphi$ and $\delta \theta$ (variations) are small functions of space-time. Now, if we insert (34) in $\varepsilon_o(x, t)$ and keep the terms to the second order of variations, then it yields

$$
\varepsilon_o(x, t) = \varepsilon_{os}(x) + \delta \varepsilon_o(x, t) \approx \varepsilon_{os}(x) + O1[\varphi_s, \delta \varphi, \delta \varphi'] + O2[\varphi_s, (\delta \varphi)^2, (\delta \varphi')^2, (\delta \varphi)^2] = (\varphi_s^2 + U(\varphi_s)) + 2 \left( \varphi'_s(\delta \varphi') + \frac{1}{2} \frac{dU(\varphi_s)}{d\varphi_s}(\delta \varphi') \right) + \left( (\delta \varphi)^2 + (\delta \varphi')^2 + \frac{1}{2} \frac{d^2U(\varphi_s)}{d\varphi_s^2}(\delta \varphi)^2 \right) (35)
$$

where $O1$ and $O2$ are two functionals that are defined in the right hand side of the above Equation. Note that, for a non-moving SSWS (27), $\dot{\varphi}_s = 0$, $\theta'_s = 0$ and $\dot{\theta}_s = \omega_s = \pm 1$. It is obvious that $\delta \varepsilon_o$ is not necessarily a positive definite function. Now, let do this for the additional terms $\varepsilon_i$’s $(i = 1, 2, 3)$. If we insert a variation like (34) into $\varepsilon_i$ $(i = 1, 2, 3)$, it yields

$$
\varepsilon_i(x, t) = \varepsilon_{is} + \delta \varepsilon_i = \delta \varepsilon_i = B[3(C_{is} + \delta C_i)(K_{is} + \delta K_i)^2 - (K_{is} + \delta K_i)^3] = B[3(C_{is} + \delta C_i)(\delta K_i)^2 - (\delta K_i)^3] \approx B[3C_{is}(\delta K_i)^2 - (\delta K_i)^3] \approx B[3C_{is}(\delta K_i)^2] > 0 \quad (36)
$$

in which $\varepsilon_{is} = 0$, $K_{is} = 0$ and $C_{is} = \omega^2 \varphi^4_s$ are referred to the SSWS (27). Therefore, since $C_i > 0$ (30), according to Eq. (36), $\delta \varepsilon_i$’s are positive definite as we expect from the Eqs. (31), (32) and (33) generally.

According to Eqs. (18)-(23), for the SSWS (27), to the first order of small variations, we have:

$$
\delta K_1 \approx 2\omega_s \varphi^4_s \delta \dot{\theta},
\delta K_2 \approx \delta K_3 - 2\varphi'_s(\delta \varphi') - 2(3\varphi^5_s - 2\varphi^3_s)\delta \varphi,
\delta K_3 \approx \delta K_2 + 2\varphi^2_s(\omega_s \delta \dot{\varphi} - \varphi'_s \delta \theta') . \quad (37)
$$

Accordingly, since $\delta K_i$’s are linear in the first order of small variations $\delta \varphi$, $\delta \theta$ and their derivatives, then according to Eq. (36), $\delta \varepsilon_i$’s are positive definite linear functions of the second order of small variations and their derivatives which all of them are multiplied by $B$.

For any arbitrary small deformation of a non-moving SSWS (27), the variation of the total energy can be calculated by the integration of $\delta \varepsilon$ over the whole space:

$$
\delta E = \int_{-\infty}^{+\infty} (\delta \varepsilon) \, dx = \int_{-\infty}^{+\infty} (\delta \varepsilon_o + \sum_{i=1}^{3} \delta \varepsilon_i) \, dx = \sum_{j=0}^{3} \delta E_j , \quad (38)
$$

In general, to show that the SSWS (27) is energetically a stable soliton solution, we have to prove that $\delta E$ is always positive for any arbitrary small deformation. In other words,
if any arbitrary deformation needs external energies to happen, then the SSWS (27) is an energetically stable solitary wave solution or a soliton solution. Since \( \delta \varepsilon_1, \delta \varepsilon_2 \) and \( \delta \varepsilon_3 \) are positive definite small functions, then the integration of them over the whole space, i.e. \( \delta \varepsilon_1, \delta \varepsilon_2 \) and \( \delta \varepsilon_3 \), are always positive definite values. Now, let us to concentrate on the \( \delta \varepsilon_o \):

\[
\delta \varepsilon_o = \delta \varepsilon_{o1} + \delta \varepsilon_{o2} = \int_{-\infty}^{+\infty} O1 \, dx + \int_{-\infty}^{+\infty} O2 \, dx, \tag{39}
\]

where, \( \delta \varepsilon_{o1} \) is the integral contribution of the first-order of variations which we will show that it would be zero in general. For a non-moving SSWS (27), according to Eq. (2), as its dominant dynamical equation, we can use \( \varphi'' = \frac{d^2 \varphi_s}{d\varphi_s^2} \) instead of \( \frac{1}{2} \frac{dU(\varphi_s)}{d\varphi_s} \) in Eq. (35) for \( O1 \).

Then, it is easy to show that

\[
O1 = 2 \left[ \varphi' (\delta \varphi') + \frac{1}{2} \frac{dU(\varphi_s)}{d\varphi_s} (\delta \varphi) \right] = 2 \left[ \varphi' (\delta \varphi') + (\delta \varphi') \varphi'' \right] = 2 \frac{d}{dx} \left( \delta \varphi \frac{d\varphi_s}{dx} \right) \tag{40}
\]

Hence, the integration of \( O1 \) over of whole space leads to

\[
\int_{-\infty}^{+\infty} O1 \, dx = 2(\delta \varphi) \left. \frac{d\varphi_s}{dx} \right|_{+\infty} - 2(\delta \varphi) \left. \frac{d\varphi_s}{dx} \right|_{-\infty} = 0, \tag{41}
\]

Note that, \( \delta \varphi \) and \( \frac{d\varphi_s}{dx} \) at \( \pm \infty \) would be zero. Therefore, in relation to the stability, the terms in the first-order of variations must be ignored. In other words, \( \delta \varepsilon_o = \delta \varepsilon_{o2} \) and then we must just compare the magnitude of the term \( O2 \) with the term \( \sum_{i=1}^{3} \delta \varepsilon_i \) for the stability considerations. In other words, we can consider the effective variation of the energy density function \( \delta \varepsilon_e \) instead of \( \delta \varepsilon \), which is defined as follows:

\[
\delta \varepsilon_e = \delta \varepsilon - O1 = O2 + \sum_{i=1}^{3} \delta \varepsilon_i, \tag{42}
\]

Since \( \frac{1}{2} \frac{d^2 U(\varphi_s)}{d\varphi_s^2} = -15 \varphi_s^4 + 6 \varphi_s^2 > 0 \) for \( |\varphi_s| < \sqrt{\frac{16}{5}} \), hence undoubtedly \( O2 = (\delta \dot{\varphi})^2 + (\delta \varphi')^2 + \frac{1}{2} \frac{d^2 U(\varphi_s)}{d\varphi_s^2} (\delta \varphi)^2 \) itself would be positive for the points \( x \) that \( |\varphi_s(x)| \) is less than \( \sqrt{\frac{16}{5}} \), and then \( \delta \varepsilon_e > 0 \) for such points. The function \( \frac{1}{2} \frac{d^2 U(\varphi_s)}{d\varphi_s^2} \) for the points \( x \) that \( |\varphi_s(x)| > \sqrt{\frac{16}{5}} \), would be negative and we can not be sure about the positivity of \( \delta \varepsilon_e \). Hence, to be sure that these points (for which \( |\varphi_s(x)| > \sqrt{\frac{16}{5}} \) ) lead to \( \delta \varepsilon_e > 0 \), we have to consider systems for which parameter \( B \) being a large number. In fact, \( |O2| \) is a function of the second order of \( \delta \varphi, \delta \varphi' \) and \( \delta \dot{\varphi} \) which does not contain parameter \( B \), but \( \delta \varepsilon_i \)'s \( (i = 1, 2, 3) \) are also functions of the second order of variations \( \delta \varphi, \delta \theta \) and their derivatives which are multiplied by \( B \). Hence, we are sure that always \( \sum_{i=1}^{3} \delta \varepsilon_i > |O2| \) or \( \delta \varepsilon_e > 0 \), provided \( B \) being a large number.
FIG. 2. Variations of the total energy $E$ versus small $\xi$ and different $B$ at $t = 0$ for the SSWS \[27\]. The Figs a-f are related to different variations \[43\]-\[48\] respectively. Note that for the case $\xi = 0$, in all figures, the total energy is the same rest energy of the SSWS \[27\], i.e. $E(\xi = 0) = E_o = \frac{\pi}{4}$.

Numerically, let us to study the stability of the SSWS \[27\] for many arbitrary small deformations. For example, six arbitrary small deformations of the non-moving SSWS \[27\] can be introduced as follows:

$$\varphi(x, t) = \varphi_s + \delta\varphi = \frac{\pm 1}{\sqrt{1 + x^2}} + \xi \exp(-x^2), \quad \theta(x, t) = \omega_s t, \quad (43)$$
\[ \varphi(x,t) = \varphi_s + \delta \varphi = \frac{\pm 1 + \xi}{\sqrt{1 + x^2}}, \quad \theta(x,t) = \omega_s t, \]  
\( (44) \)

\[ \varphi(x,t) = \varphi_s + \delta \varphi = \frac{\pm 1}{\sqrt{1 + \xi + x^2}}, \quad \theta(x,y,z,t) = \omega_s t, \]  
\( (45) \)

\[ \varphi(x,t) = \varphi_s + \delta \varphi = \frac{\pm 1}{\sqrt{1 + (1 + \xi)x^2}}, \quad \theta(x,t) = \omega_s t, \]  
\( (46) \)

\[ \varphi(x,t) = \frac{\pm 1}{\sqrt{1 + x^2}}, \quad \theta(x,t) = \theta_s + \delta \theta = \omega_s t + \xi t, \]  
\( (47) \)

\[ \varphi(x,t) = \frac{\pm 1}{\sqrt{1 + x^2}}, \quad \theta(x,t) = \theta_s + \delta \theta = \omega_s t + \xi \exp(-x^2), \]  
\( (48) \)

in which \( \xi \) is a small parameter which, for any kind of the small deformations (43)-(48), can be considered as an indication of the amount of deformations (variations). The all deformed functions (43)-(48) turn to the same free non-deformed SSWS (27) for \( \xi = 0 \). For all arbitrary deformations (43)-(48), Fig. 2(a-f) show how larger values of parameter \( B \) lead to more stability, i.e. the larger values of \( B \) lead to further increase in the total energy versus \( \xi \). Figure. 2(a-c) show that clearly why the case \( B = 0 \) leads to an energetically unstable SSWS. In other words, for the case \( B = 0 \), in Figure. 2(a-c), the rest energy of the SSWS, i.e. \( E_o = E(\xi = 0) = \frac{\pi}{4} \), is not a minimum. Note that, the case \( B = 0 \) is related to the same original RNKG system (1) with the same SSWS (7). In general, there are the same results for any arbitrary deformations, i.e. the total energy always increases (and increases more for larger values of \( B \)) versus the amount of any arbitrary variation above the background of the SSWS (27). Although, the parameter \( B \) can be taken a large value, but it would not affect the dynamics and the observable of the SSWS (27). It just make it stable and does not appear in any of the observable.

A multi lump (particle-like) solution with different velocities can be easily constructed just by adding distinct far enough SSWS (28) together. But, since the phase field \( \theta \) for each SSWS (28) depends on its velocity, hence it must change from one to other. Namely, if there are two SSWS (28) which one of them being at rest and the other is moving, then the phase field must change from \( \theta = \omega_s t \) at the position of the first SSWS to \( \theta = k \mu x^n \) at the position of the second SSWS. In the regions between two SSWSs, the scalar field \( \varphi \) is almost zero.
and then $\varepsilon$ is almost zero everywhere, hence, there is not any rigorous restriction on $\theta$ to be in the standard forms $\theta = \omega_s t$ and $\theta = k_\mu x^\mu$ as the special solutions of the PDE $S_1 = 0$. In other words, the phase field $\theta$, where the scalar field $\varphi$ is almost zero, is completely free and evolve without any rigorous restriction, i.e. it can evolve slowly from $\theta = \omega_s t$ to $\theta = k_\mu x^\mu$ in the spaces for which $\varphi \approx 0$. In fact, for the case $\varphi \approx 0$, it is not necessary to satisfy $S_i = 0$ or $K_i = 0$ ($i = 1, 2, 3$) simultaneously, because for such situations, all $\varepsilon_i$’s ($i = 1, 2, 3$) would automatically be almost zero simultaneously without any restrictive condition.

V. SUMMERY AND CONCLUSION

In this paper, as an example, we introduced a new extended real nonlinear Klein-Gordon (RNKG) field system which analytically leads to a special stable solitary wave solution (SSWS). In other words, it leads to a soliton solution. This new extended lagrangian density is composed of two distinct parts, first, the original part which is the same known standard RNKG system, and second, an additional part which can be called the stability catalyzer term. The original standard RNKG lagrangian density is introduced for a single scalar field $\varphi$. But, introducing the stability catalyzer term is needed to use another scalar field $\theta$ (phase filed) along with the original field $\varphi$, i.e. the catalyzer term is a functional of $\varphi$ and $\theta$ together. The role of the stability catalyzer term is seems as a massless spook which surrounds the SSWS which guarantees the stability of the SSWS exactly. This catalyzer term does not have any role in the dominant dynamical equation and other properties of the SSWS, i.e. the dominant dynamical equation just for the SSWS is reduced to the same known standard RNKG version. However, it has the main role in the stability of the SSWS, that is any arbitrary small deformation in the internal structure of the SSWS leads to increase total energy. In other words, the rest energy of the SSWS is minimum among the other solutions of the new extended RNKG system except the ones which are very close to the vacuum state $\varphi = 0$. 

14
Appendix A

Here, we are going to show that three PDEs

\[
S_1 = \dot{\theta}^2 - \theta'^2 - 1 = 0, \quad (A1)
\]

\[
S_2 = \dot{\varphi}^2 - \varphi'^2 + \varphi^4(1 - \varphi^2) = 0, \quad (A2)
\]

\[
S_3 = \dot{\varphi} \dot{\theta} - \varphi \theta' = 0. \quad (A3)
\]

do not have any non-trivial common solution except the SSWS \[27\]. Equation (A3) leads to obtain \( \dot{\theta} \) in terms of \( \theta' \), \( \varphi' \) and \( \dot{\varphi} \) as follows:

\[
\dot{\theta} = \frac{\varphi' \theta'}{\dot{\varphi}}. \quad (A4)
\]

If insert this into Eq. (A1), we can obtain \( \theta' \) in terms of \( \varphi' \) and \( \dot{\varphi} \) as follows:

\[
\theta' = \pm \frac{\dot{\varphi}}{\sqrt{\varphi'^2 - \dot{\varphi}^2}}. \quad (A5)
\]

Using Eqs. (A4) and (A5), \( \dot{\theta} \) can be obtain as well:

\[
\dot{\theta} = \pm \frac{\varphi'}{\sqrt{\varphi'^2 - \dot{\varphi}^2}}. \quad (A6)
\]

The obvious mathematical expectation \((\dot{\theta})' = \frac{d}{dx} \frac{d\theta}{dt} = \frac{d}{dt} \frac{d\theta}{dx} = (\dot{\theta}')\) leads to the following result:

\[
\ddot{\varphi} - \varphi'' + \frac{1}{\sqrt{\varphi'^2 - \dot{\varphi}^2}}(\varphi'^2 \dot{\varphi} + \varphi'^2 \varphi'' - 2 \varphi \varphi' \dot{\varphi}') = 0, \quad (A7)
\]

which simply can be written in a covariant form:

\[
\partial_\mu \partial^\mu \varphi + \frac{1}{\sqrt{-\partial_\mu \varphi \partial^\mu \varphi}}(\partial_\nu \varphi \partial_\sigma \varphi)(\partial_\rho \varphi \partial^\rho \varphi) = 0 \quad (A8)
\]

Therefore, to find the common solutions of three independent nonlinear PDEs (A1), (A2) and (A3), equally we can search for the common solutions of the two different PDEs (A2) and (A8). In general, it is easy to show that each non-vibrational function \( \varphi_v(x, t) = \varphi_v(\gamma(x - vt)) \), would be a solution of the PDE (A8) or (A7). Moreover, for any non-vibrational solitary wave solution, Eqs. (A5) and (A6) lead to \( \theta' = \pm \gamma v = \omega v \) and \( \dot{\theta} = \pm \gamma = \gamma \omega_s = \omega \) as we expected. On the other hand, we know that the SSWS \[28\] is the single non-vibrational solution of the PDE (A2). Hence, for PDEs (A2) and (A8), the single common non-vibrational solitary wave solution is the SSWS \[28\], as we
expected. Accordingly, for the scalar field $\varphi$, there are two completely different PDEs (A2) and (A8), therefore it is not seem to exist other common vibrational solutions along with the non-vibrational SSWS [28].

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