Some Classical Solutions of Relativistic Membrane Equations in 4 Space-Time Dimensions

J Hoppe*
Isaac Newton Institute for Mathematical Sciences
20 Clarkson Road
Cambridge CB3 0EH
UK

Abstract

Various reductions, and some solutions of the classical equations of motion of a relativistic membrane are given.

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* Heisenberg Fellow
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As there now exists quite a variety of non-trivial reformulations, and simplifications, of the standard parametric ‘minimal hypersurface’ equations, different kinds of reductions, and insight, can be used to find explicit solutions. Static solutions of the hydrodynamic equations derived in [1], e.g., correspond to infinitely extended membranes of fixed shape, moving with the velocity of light. Alternatively, solutions may be derived by viewing the non parametric $z$-equation (cp [2]) as a consistency condition of a higher dimensional free (!) wave equation and a non-linear constraint. Using the orthonormal light cone gauge [3], on the other hand, for a non-compact membrane with one rotational symmetry, an algebraic ‘shape-changing’ solution is found. Also, while the closed membrane light cone solution of [3] contracts to a point, one can write down solutions of the same equation without the above point-singularity, by relaxing the $S^1$-symmetry. Two classes of symmetry reductions of the membrane equations to ordinary differential equations are then deduced, as well as the classical equivalence of axially-symmetric membranes with strings in a given curved 3-dimensional background. At the end an idea is sketched which indicates a possible ‘linearisation’ of membrane dynamics, when viewed as ‘moments of continuous mass in interaction’. Finally, a note is added which mentions two infinite classes of minimal hypersurfaces, one of which corresponds to self similarly expanding, or contracting, open membranes of very intricate shapes.

Let me start by showing how the usual parametric equations,

$$\partial_\alpha \sqrt{G} G^{\alpha\beta} \partial_\beta x^\mu = 0 \quad (1)$$

simplify in the orthonormal light cone gauge

$$\varphi^0 = \frac{1}{2} (x^0 + x^3) \quad (2)$$

$$\partial_r \zeta = \dot{x} \partial_r \dot{x}, \quad r = 1...M, \quad (3)$$

$$\dot{\zeta}^2 = \frac{\dot{x}^2}{x} + g \quad (4)$$

(cp. [3]), where $\zeta := x^0 - x^3$, $\dot{x} = (x^1, x^2, x^4, ..., x^{D-1})$, $\dot{x}$ denotes differentiation with respect to $\varphi^0 = \frac{x^0 + x^3}{2}$ (called ‘time’, $t$, in the following argument) and $g$ is the determinant of the $M \times M$ matrix formed by $g_{rs} := \partial_r \dot{x} \partial_s \dot{x}$.

The second order differential operator acting on $x^\mu$ (in (1)) then becomes

$$D = \partial_t^2 - \partial_r gg^{rs} \partial_s \quad (5),$$
\( g^{rs} g_{sr'} = \delta^r_{r'} \), and it is straightforward to show that

\[
D \mathbf{\dot{x}} = 0 \tag{6}
\]

implies \( D \zeta = 0 \), as well as the consistency of (4) with each of (3). As \( D t \) is automatically zero (cp. (4)), the only remaining condition is the consistency of (3), i.e.

\[
\partial_r \mathbf{\dot{x}} \delta_{s} \mathbf{\dot{x}} - \partial_s \mathbf{\dot{x}} \delta_r \mathbf{\dot{x}} = 0 \quad r, s = 1...M \tag{7}
\]

So far for minimal surfaces of arbitrary dimension and co-dimension.

For \( M = 2, D = 4 \), two simple ways of satisfying (7) are:

\[
\mathbf{\dot{x}} = \begin{pmatrix} x(t) \cdot X(\mu, \varphi) \\ y(t) \cdot Y(\mu, \varphi) \end{pmatrix} \tag{8}
\]

or, assuming rotational symmetry around the \( x_3 \)-axis,

\[
\mathbf{\dot{x}} = R(t, \mu) \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} \quad \varphi \in [0, 2\pi] \tag{9}
\]

(\( \mu \) and \( \varphi \) denoting the two spatial membrane parameters). The dynamical equations, \( D \mathbf{\dot{x}} = 0 \), will be satisfied provided \([3]\)

\[
2 \ddot{R} = R(R^2)'' \tag{10}
\]

\( ( = \frac{\partial}{\partial \mu} \), respectively

\[
\ddot{x} = -yx^2, \quad \ddot{y} = -yx^2 \tag{11}
\]

together with

\[
X = \{\{X, Y\}, Y\}, \quad Y = \{\{Y, X\}, X\} \tag{12}
\]

(\( \{X, Y\} \) denoting \( \partial_\mu X \partial_\varphi Y - \partial_\varphi X \partial_\mu Y \) - which basically has only one type of solution,

\[
X = \sqrt{1 - \mu^2} \cos \varphi, \quad \mu \in [-1, +1] \tag{13}
\]

\[
Y = \sqrt{1 - \mu^2} \sin \varphi
\]

(11) describes the motion of a unit mass point particle in 2 dimensions, moving under the influence of the potential \( V(x, y) = \frac{1}{2} x^2 y^2 \), a problem which has quite an interesting history (see e.g. [4]-[7]). Constructing \( \zeta \) from (3) and (4) one finds for (8) (with (11)/(13))

\[
\zeta(t, \mu, \varphi) = \frac{1 - \mu^2}{2} (x\dot{x} \cos^2 \varphi + y\dot{y} \sin^2 \varphi)
\]

\[ + \frac{1}{2} \int_0^t x^2(\tau)y^2(\tau)d\tau + \zeta_0 \tag{14} \]
which (upon \( \zeta = x^0 - x^3, t = \frac{x^0 + x^3}{2} \)) gives an implicit equation for \( x^3 \) as a function of \( x^0, \mu \) and \( \varphi \), respectively \( x^0, x^1 \) and \( x^2 \) (when using that the \( \mu, \varphi \) dependant term in (14) is equal to \( \frac{1}{2}(x_1 \frac{\dot{x}}{x} + x_2 \frac{\ddot{y}}{y}) \)).

However, even in the simple case

\[
x(t) = y(t) = (2E)^{\frac{t}{4}} cn((2E)^{\frac{t}{4}}(t - t_0); \frac{1}{\sqrt{2}})
\]

(15)

it seems difficult to explicitly solve for \( x_3 \). The only immediately tractable case, \( y(t) \equiv 0 \) and \( x(t) = at + b \), giving \( x_1^2 + (x_3 - \frac{b}{a})^2 = (x_0 + \frac{b}{a})^2 \) for \( a \neq 0 \) (a contracting, or expanding, circle), and \( x^3 = x^0 \) (a rigid piece of string, of length \( 2 \mid b \mid \), moving with the velocity of light) for \( a = 0 \), is of course a ‘fake’ solution, as \( G \equiv 0 \).

From (10), on the other hand, one may obtain a simple solution which is both 3 dimensional, and algebraic, by letting

\[
R(t, \mu) = \pm \sqrt{2} \frac{\sqrt{\mu^2 + e^2}}{t}
\]

(16).

From (3)/(4) one gets

\[
\zeta = -\frac{\left(\mu^2 + \frac{1}{3}e^2\right)}{t^3}
\]

(17),

which together with \( x_1^2 + x_2^2 = R^2 = 2\frac{\mu^2 + e^2}{t^2} \) gives

\[
(x^0 + x^3)^2(x_0^2 - x_3^2 + x_1^2 + x_2^2) = \frac{16}{3} e^2
\]

(18).

Note that for \( \mu \in (-\infty, +\infty) \), (10) has two independant scaling symmetries, which can be used to look for solutions

\[
R(t, \mu) = t^a f(t^b \mu), \quad a + b + 1 = 0,
\]

(19),

\[
a(a - 1)f(s) + sf'(s)(a + 1)(2 - a) + (a + 1)^2 s^2 f''(s)
\]

\[= (f'^2 + ff'')f(s)
\]

(20)

(the case of \( a = -1, b = 0 \), corresponding to (18)).

Let us now look at various other (cp. [1],[2]) reformulations, and reductions, of (1):

Choosing \( \varphi^0 = \frac{x^0 + x^3}{2} \) (called \( t \), as above), \( \varphi^1 = x^1, \varphi^2 = x^2 \) (called \( x \) and \( y \), respectively) one obtains a ‘non-parametric light-cone’ or ‘hydrodynamic’equation for \( x^0 - x^3 =: p(t, x, y) \),

\[
\ddot{p} + 2(\nabla p \nabla \dot{p} - \dot{p} \nabla^2 p) + \frac{1}{2} \nabla^2 p \nabla^2 (\nabla^2 p) - (\nabla p)^2 \nabla^2 p = 0
\]

(21),

while getting the ‘usual non-parametric’, or ‘Born-Infeld’ - equation,

\[
(1 - z^\alpha z_\alpha)\Box z + z^\alpha z^\beta z_{\alpha\beta} = 0
\]

(22)
\[ z_\alpha := \frac{\partial z}{\partial x^\alpha}, \quad z_{\alpha \beta} = \frac{\partial^2 z}{\partial x^\alpha \partial x^\beta}, \quad \eta_{\alpha \beta} = z_{\alpha \beta} \eta^{\alpha \beta} \]

\[ \eta^{\alpha \beta} = (1-1-1) \]

for \( x^3 = z(x^0, x^1, x^2) \) by choosing \( \varphi^0 = x^0, \varphi^1 = x^1, \varphi^2 = x^2 \) (which of course is appropriate only if the hypersurface can be represented as a graph). In the first gauge, \( G = 2\dot{p} + (\nabla \ddot{p})^2 \) while \( G = 1 - \dot{z}^2 + (\nabla \dot{z})^2 \) in the second (here, \( \dot{z} := \frac{\partial z}{\partial x^\alpha} \)).

Also, one could represent the hypersurface as the set of zeroes (more generally: a level set) of some scalar function \( u \); the dynamical equation for \( u \) that guarantees extremality of the volume \( \int \sqrt{G} \) is

\[ u^\rho u_\rho = -u^\mu u_\mu u_{\mu \nu} = 0 \quad (23) \]

\[ (u_\rho = \frac{\partial u}{\partial x^\rho}, \rho = 0...3; u_{\mu \nu} = \frac{\partial^2 u}{\partial x^\mu \partial x^\nu}, \eta_{\mu \nu} = u_{\mu \nu} \eta^{\mu \nu}). \]

Assuming the hypersurface to be rotationally symmetric around the \( x_3 \)-axis (cp. (9)) one obtains

\[ \ddot{p} + 2(p' \dddot{p} - \dddot{p}'') = \frac{1}{r}(p'^3 + 2\dot{p}p') \quad (24), \]

\[ \ddot{z}(1 + z'^2) - z''(1 - \dot{z}^2) - 2\dot{z}z' \dot{z}' = \frac{1}{r}(z'^3 + z'(1 - \dot{z}^2)) \quad (25) \]

from (21), respectively (22). Note that both forms look considerably more complicated than (10), which corresponds to the Lagrangian density \( L = \frac{1}{2}(\dddot{R}^2 - R^2 \dddot{R}'^2) \) or - with \( \phi = \frac{1}{2}R^2 \) - a Hamiltonian density \( H = \phi \pi^2 + \frac{1}{2} \phi'^2 \). The relation with strings in a curved 3 dimensional background is easily seen by noting that the action for (25), \( S = \int rdrdt\sqrt{1 - \dot{z}^2 + z'^2} \), is a non-parametric version of \( S = \int d\varphi^0 d\varphi^1 \sqrt{-det(\partial_\alpha x^\alpha \partial_\beta x^\beta g_{\alpha \beta})} \), with \( g_{\alpha \beta} = r \eta^{\alpha \beta}, ds^2 = r(dt^2 - dr^2 - dz^2) \) (implying a curvature singularity according to \( R = -\frac{3}{2r^3} \)).

Further reducing (24) by letting

\[ p(t, r = \sqrt{x_1^2 + x_2^2}) = t^a P(t^c r = Z) \quad (26) \]

\[ a + 2c = 1 = 0 \]

one gets

\[ a(a - 1)P - \frac{3}{4}(a^2 - 1)Z P' + \frac{1}{4}(a + 1)^2 Z^2 P'' - \frac{P'^3}{Z} + 2a(P'^2 - \frac{PP'}{Z} - PP'') = 0 \quad (27). \]

At least the case of \( a = 0 \) can be solved by elementary methods, yielding an elliptic integral, respectively

\[ x^3 - x^0 = \pm \frac{1}{2} \int_0^1 \frac{du}{\sqrt{1 - (\frac{u}{u_0})^4}} + \text{const.} \quad (28) \]
A rather large class of solutions can be obtained from the Ansatz

\[ z(x^0, x^1, x^2) = x^0 - p(x^1, x^2) \]  

(29),

respectively \( \dot{p} = 0 \) in (21), yielding the equation

\[ p_x^2 p_{yy} + p_y^2 p_{xx} - 2p_x p_y p_{xy} = 0 \]  

(30)

for the ‘shape of the surface that moves with the velocity of light in the \( x^3 \)-direction’. While the integrability of (30) must have been known for quite a long time, the above connection with extremal hypersurfaces in Minkowski space was noted only recently, in collaboration with M. Bordemann. Instead of resorting to the general method of linearisation by hodograph - transform (applicable to any 2-dimensional field equation that comes from a lagrangean which depends only on the first derivatives of the field) solutions of (30) may actually be obtained in rather more explicit form, which is quite useful for a qualitative discussion in the membrane context. It is e.g. easy to show that

\[ p(x, y) := \tilde{p}(x + v(x, y)y) \]  

(31),

with

\[ v(x, y) = \tilde{v}(x + v(x, y)y) \]  

(32),

solves (30), for any smooth \( \tilde{p} \) and \( \tilde{v} \). As (32) implies

\[ v_x = \frac{\tilde{v}'}{1 - y\tilde{v}'}, \quad v_y = \frac{\tilde{v}'}{1 - y\tilde{v}'} \]  

(33),

\( v \) and \( p \) will generically have ‘cusps’ in the \((xy)\) plane, as well as (when moved according to (29)) somewhere vanishing \( G \) (both, however, only on measure zero sets), according to

\[ G = (\tilde{p}')^2 \frac{1 + v^2}{(1 - y\tilde{v}')^2} \]  

(34).

Note that for \( \tilde{v} = \text{const.} \) any strictly montonic \( \tilde{p} \) determines a hypersurface which is everywhere regular. In a different context, Fairlie et al [8] (partly referring to work of Bateman and Garabedian) have discussed various aspects of (30), including the existence of solutions defined in terms of two functions, \( F \) and \( H \), via \( xF(p) + yH(p) \equiv 1 \). When moved according to (29), the corresponding hypersurfaces will have \( G = \frac{F^2 + H^2}{(xF' + yH')^2}. \) Similarly, \( p = p(\cot^{-1}(\frac{y}{x})) \) will give \( \int \sqrt{G} \sim \int_0^{2\pi} d\theta \mid p'(\theta) \mid. \)

Finally note that (22) may be viewed as a consistency condition of a free wave equation in 4 dimensions,

\[ h^{\mu\nu} \partial^2_{\mu\nu} z = 0, \quad h^{\mu\nu} = \text{diag}(1, -1, -1, +1) \]  

(35)

and a non-linear constraint,

\[ h^{\mu\nu} \partial_{\mu} z \partial_{\nu} z = 1 \]  

(36).
In light cone coordinates, \( r = \frac{x^0 + x^2}{2}, \ s = \frac{x^0 - x^2}{2}, \ u = \frac{x^1 + x^3}{2}, \ v = \frac{x^1 - x^3}{2} \) the two equations read

\[
\begin{align*}
    f_{rs} &= f_{uv}, \\
    f_r f_s - f_u f_v &= 1
\end{align*}
\] (37).

The simplest solutions obtained this way are, \( \pm f = r + s + g (u \text{ or } v) \), respectively \( u - v + g (r \text{ or } s) \).

Let me conclude by mentioning a kind of ‘linearization’ that occurs when looking at eq. (23) in the following way (I thank Martin Bordemann for many discussions on this direction): A field dependant change of variables from \( x^\mu \) to \( p^\mu := \frac{\partial u}{\partial x^\mu} \) transforms \( S = \int d^4 x \sqrt{\partial_\mu u \partial^\mu u} \) into \( S = \int d^4 p \sqrt{p^\mu p_\mu} \det M (p) \), where \( M \) is the matrix of second derivatives of a scalar field \( f(p_0, ..., p_3) \). This suggests taking \( \sqrt{p^\mu p_\mu} \) as one of the independent variables, ie. foliating the relevant part of flat Minkowski \( p \)-space according to \( p_\mu = \tilde{r} e_\mu (\vec{q}) \), \( e_\mu e^\mu = 1 \) \( (\vec{q} = (q^1, q^2, q^3) \) parametrizing the hyperboloid \( H^3 \)). Due to the specific nature of (23), terms explicitly depending on the independent coordinates cancel in the equation of motion for \( f = f(r, \tilde{q}) \), which simply becomes

\[
\epsilon^{i\mu_1 \mu_2 \mu_3} \epsilon_{i}^{\nu_1 \nu_2 \nu_3} f;_{\mu_1} f;_{\mu_2} f;_{\mu_3} = 0 \tag{38}
\]

; indicating covariant differentiation with respect to the induced metric, ie. \( f;_{00} = f_{rr}, f;_{0i} = f_{ri} - \frac{1}{r} f_i, f;_{ij} = f_{ij} - r g_{ij} f_r \) \( (g_{ij} \text{ being the metric on } H^3, \ eg. \frac{4}{(1 - \vec{q})^2} \delta_{ij} \text{ in the ‘Poincaré-ball’ model, } 0 \leq |\vec{q}| \leq 1) \).

For \( f(r, \tilde{q}) = \sum_{0}^{\infty} r^n Y^{(n)}(\tilde{q}) \), (38) becomes a system of non-linear differential equations for the set \( \{Y^{(n)}\} \) of scalar functions on \( H^3 \). However, the equation for \( Y^{(0)} \) is just the level-set version for minimal surfaces in \( H^3 \) (cp. (23), or [2]; and [9, 10, 12] for the integrable nature of this problem, and examples), while each \( Y^{(n)}, n > 0 \), may recursively be determined in terms of the \( Y^{(m < n)} \) as a solution of a linear(!) equation.

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Note added:

Let \( e : \Sigma_2 \to H^3 := \{ \epsilon \mu \epsilon R^{1,3} | \epsilon^\mu \epsilon_{\mu} = +1 \} \) \(^{(39)}\),
resp.
\( \tilde{e} : \tilde{\Sigma}_2 \to S^{2,1} := \{ \tilde{\epsilon}^\mu \epsilon R^{1,3} | \tilde{\epsilon}^\mu \tilde{\epsilon}_{\mu} = -1 \} \) \(^{(40)}\),
be a spacelike (resp. timelike) 2 dimensional surface with zero mean curvature in \( H^3 \) (resp. \( S^{2,1} \)). The cone \( C(\Sigma) \) (resp. \( \tilde{C}(\tilde{\Sigma}) \)), defined by
\[
X : (0, \infty) \times \Sigma_2 \to R^{1,3} \\
(\rho, (\varphi^1, \varphi^2) \mapsto \rho \cdot e(\varphi^1, \varphi^2)
\]
resp.
\[
\tilde{X} : \tilde{\Sigma}_2 \times (0, \infty) \to R^{1,3} \\
(\varphi^0, \varphi^1, \tilde{\rho} \mapsto \tilde{\rho} \cdot \tilde{e}(\varphi^0, \varphi^1)
\]
is then a 3 dimensional timelike hypersurface with zero mean curvature in \( R^{1,3} \) \(^{(11)}\).

This is easy to check, as eg. (1) for \( X \) (resp. \( \tilde{X} \)), with \( \varphi^0 := \rho \) (implying \( G_{00} = +1, G_{0r} = 0, G_{rs} = \rho^2 h_{rs} \), with \( h_{rs} := \partial_r e^\mu \partial_s e_{\mu} \) negative definite; \( r, s = 1, 2 \) - respectively \( \varphi^2 := \tilde{\rho} \) (implying \( G_{22} = -1, G_{a2} = 0, G_{ab} = \tilde{\rho}^2 \tilde{h}_{ab}, \sqrt{G} = \tilde{\rho}^2 \sqrt{-\tilde{h}} \), where \( \tilde{h} \) is the determinant of \( \tilde{h}_{ab} := \partial_a \tilde{\epsilon}^\mu \partial_b \tilde{\epsilon}_{\mu}; a, b = 0, 1 \) ) reads
\[
(2 + \frac{1}{\sqrt{\tilde{h}}} \partial_r \sqrt{\tilde{h}} h^{rs} \partial_s) e^\mu = 0
\]
resp.
\[
(-2 + \frac{1}{\sqrt{-\tilde{h}}} \partial^a \sqrt{-\tilde{h}} \tilde{h}^{ab} \partial_b) \tilde{e}^\mu = 0
\]
Correspondingly, one may eg. choose \( \varphi^0 := x^0 \), and consider the Ansatz
\[
\vec{x}(t) = t \cdot \vec{v}(\varphi^1, \varphi^2)
\]
\( | \vec{v} | \leq 1 \). The resulting equation for \( \vec{v} \) is the minimal surface equation in the ‘Klein-ball’ model of \( H^3 \). As much is known about minimal surfaces in hyperbolic 3-space \(^{[9, 10, 12]} \) (basically, the problem reduces to solving variants of the integrable Sinh-Gordon equation), infinitely many possible membrane shapes, expanding according to (45), may thus be determined. Finally, note again the relation with strings in curved 3 dimensional backgrounds (via (40), or (10)/(23), eg.), especially with regard to a quantum theory of relativistic surfaces.
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