CROSSED EXTENSIONS OF LIE ALGEBRAS

APURBA DAS

Abstract. It is known that Hochschild cohomology groups are represented by crossed extensions of associative algebras. In this paper, we introduce crossed $n$-fold extensions of a Lie algebra $g$ by a module $M$, for $n \geq 2$. The equivalence classes of such extensions are represented by the $(n+1)$-th Chevalley-Eilenberg cohomology group $H_{CE}^{n+1}(g, M)$.

1. Introduction

Group cohomology of a group $G$ with coefficients in a $G$-module $M$ is related to crossed extensions of $G$ by $M$. More precisely, equivalence classes of crossed $n$-fold extensions of $G$ by $M$ are classified by the $(n+1)$-th group cohomology $H^{n+1}(G, M)$ [5]. A similar result for Hochschild cohomology was considered by Baues and Minian [2]. Namely, they introduce crossed $n$-fold extensions of an associative algebra $A$ by an $A$-bimodule $M$ and prove that equivalence classes of such extensions are isomorphic to the $(n+1)$-th Hochschild cohomology $HH^{n+1}(A, M)$ as abelian groups. A further generalization of this result has been obtained in [3].

The cohomology of a Lie algebra $g$ with coefficients in a $g$-module $M$ is given by the Chevalley-Eilenberg cohomology. This cohomology theory controls the deformation of a given Lie algebra structure. The Chevalley-Eilenberg cohomology groups $H_{CE}^n(g, M)$ are also related to special type of $L_\infty$-algebras [1, Theorem 6.7]. In this paper, we give another interpretation of Chevalley-Eilenberg cohomology in terms of crossed extensions. The idea is exactly same as Baues and Minian for associative algebra case. We introduce crossed $n$-fold extensions of a Lie algebra $g$ by a $g$-module $M$. The equivalence classes of such extensions are classified by $H_{CE}^{n+1}(g, M)$. We also find a similar result for Leibniz algebras.

All vector spaces are over a field $K$.

2. Chevalley-Eilenberg cohomology

Let $(g, [\cdot, \cdot])$ a Lie algebra. A module over $g$ consists of a vector space $M$ together with a $K$-bilinear map $[\cdot, \cdot]: g \times M \to M$ satisfying

$$[[x, y], m] = [x, [y, m]] - [y, [x, m]],$$

for $x, y \in g$ and $m \in M$. It is clear that $g$ is a module over $g$ with respect to the Lie bracket.

Given a Lie algebra $(g, [\cdot, \cdot])$ and a module $M$, the corresponding Chevalley-Eilenberg cochain groups $C^n_{CE}(g, M)$ are given by $C^n_{CE}(g, M) = M$ and $C^n_{CE}(g, M) = \text{Hom}_K(\wedge^n g, M)$, for $n \geq 1$. The coboundary map $\delta: C^n_{CE}(g, M) \to C^{n+1}_{CE}(g, M)$ is given by

$$(\delta m)(x) = [x, m], \quad \text{for } m \in M \text{ and } x \in g,$$

and

$$(\delta f)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} [x_i, f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})] + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}),$$

for $f \in C^n_{CE}(g, M)$ and $x_1, \ldots, x_{n+1} \in g$. The corresponding cohomology groups are denoted by $H^n_{CE}(g, M)$ and called the Chevalley-Eilenberg cohomology of $g$ with coefficients in the module $M$.

2010 Mathematics Subject Classification. 17B56, 17B55, 17A32.

Key words and phrases. Lie algebras, Chevalley-Eilenberg cohomology, Crossed modules, Crossed extensions.
It is easy to see that \( H^0_{CE}(g, M) \) is the submodule of invariants of \( M \):
\[
H^0_{CE}(g, M) = \{m \in M \mid [x, m] = 0, \forall x \in g\}.
\]
The first cohomology group \( H^1_{CE}(g, M) \) can be seen as the space of outer derivations
\[
H^1_{CE}(g, M) = \text{OutDer}(g, M) := \frac{\text{Der}(g, M)}{\text{InnDer}(g, M)}.
\]
where a derivation is a map \( f : g \to M \) satisfying \( f[x, y] = [x, fy] - [y, fx] \), for all \( x, y \in g \) and is called inner if \( f(x) = [x, m] \), for some \( m \in M \).

The second cohomology group \( H^2_{CE}(g, M) \) can be described as the space of equivalence classes of Lie algebra extensions of \( g \) by the module \( M \).

**Pushout of \( g \)-modules.** Pushout in a category is an important tool to study some nice properties of the category.

2.1. **Definition.** Let \( C \) be a category. Given two morphisms \( f : A \to B \) and \( g : A \to C \), a pushout (or fibered sum) is a triple \((D, i, j)\) where \( D \in \text{Ob}(C) \) and \( i : B \to D \) and \( j : C \to D \) are morphisms in \( C \) such that \( jg = if \) that satisfies the following universal property: for any triple \((D', i', j')\) with \( j'g = i'f \), there is a unique morphism \( \theta : D \to D' \) making the following diagram commute

\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\makebox[.5in]{\(f\)} & & \makebox[.5in]{
} \\
B & & D \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\makebox[.5in]{\(f\)} & & \makebox[.5in]{
} \\
B & & D' \\
\end{array}
\]

Let \( g \) be a Lie algebra. Consider the category \( g\text{-mod} \) of \( g \)-modules and \( g \)-module morphisms. Then the pushout of two maps \( f : A \to B \) and \( g : A \to C \) in \( g\text{-mod} \) exists.

Let \( S = \{(f(a), g(a)) \mid a \in A\} \). Then it is easy to see that \( S \) is a \( g \)-submodule of \( B \oplus C \). Take \( D = \frac{B \oplus C}{S} \) and define \( i : B \to D \) by \( i(b) = (b, 0) + S \) and \( j : C \to D \) by \( j(c) = (0, c) + S \). It is easy to see that \( jg = if \). Moreover, if \((D', i', j')\) is another triple with \( j'g = i'f \), we define \( \theta : D \to D' \) by \( \theta((b, c) + S) = i'(b) + j'(c) \). It is also easy to check that \( \theta \) is unique. Hence the claim.

3. **Crossed modules of Lie algebras**

In this section we give an interpretation of \( H^3_{CE}(g, M) \) in terms of crossed module over Lie algebras. However our notion of crossed module is different from the traditional one (see for example [7,9]). Our definition is motivated from the one given by Baues and Minian for associative algebras [2].

3.1. **Definition.** A crossed module over a Lie algebra is a triple \((V, L, \partial)\) in which \( L \) is a Lie algebra, \( V \) is a \( L \)-module and \( \partial : V \to L \) is a map of \( L \)-modules such that
\[
[\partial v, w] = -[\partial w, v], \quad \text{for} \ v, w \in V.
\]

3.2. **Definition.** Let \((V, L, \partial)\) and \((V', L', \partial')\) be two crossed modules. A map between them consists of a linear map \( \alpha : V \to V' \) and a Lie algebra map \( \beta : L \to L' \) such that the following diagram commute

\[
\begin{array}{ccc}
V & \xrightarrow{\alpha} & L \\
\makebox[.5in]{\(\partial\)} & & \makebox[.5in]{
} \\
V' & \xrightarrow{\beta} & L' \\
\end{array}
\]

and satisfying \( \alpha[x, v] = [\beta(x), \alpha(v)] \), for \( x \in L, v \in V \).
Given a crossed module \((V, L, \partial)\), we consider \(g = \text{coker}(\partial)\) and \(M = \ker(\partial)\). The Lie algebra structure on \(L\) induces a Lie algebra structure on \(g\) by \([\pi(x), \pi(y)] = \pi[x, y]\), where \(\pi : L \to g\) is the projection map. Moreover, the action of \(L\) on \(V\) induces an action of \(g\) on \(M\) via \([\pi(x), m] = [x, m]\), for \(m \in M\). Hence a crossed module yields an exact sequence

\[
0 \to M \xrightarrow{i} V \xrightarrow{\partial} L \xrightarrow{\pi} g \to 0.
\]

We call \((V, L, \partial)\) a crossed module over the Lie algebra \(g\) with kernel a \(g\)-module \(M\). Two crossed modules \((V, L, \partial)\) and \((V', L', \partial')\) over \(g\) with kernel \(M\) are said to be equivalent if there is a morphism of crossed modules \((V, L, \partial) \to (V', L', \partial')\) which induces identity maps on \(g\) and \(M\). Let \(\text{Cross}(g, M)\) be the equivalence classes of such crossed modules. In the next theorem we see that equivalence classes of crossed modules are in one-to-one correspondence with the third Chevalley-Eilenberg cohomology.

### 3.3. Theorem

There is a bijection

\[
\psi : \text{Cross}(g, M) \to H^3_{CE}(g, M).
\]

**Proof.** Let

\[
\mathcal{E} : 0 \to M \xrightarrow{i} V \xrightarrow{\partial} L \xrightarrow{\pi} g \to 0
\]

be a crossed module. Choose \(K\)-linear sections \(s : g \to L\) with \(\pi s = \text{id}\) and \(q : \text{Im}(\partial) \to V\) with \(\partial q = \text{id}\). For any \(x, y \in g\), we have

\[
\pi([s(x), s(y)] - s[x, y]) = 0.
\]

This shows that \([s(x), s(y)] - s[x, y]\) is in \(\ker(\pi) = \text{Im}(\partial)\). Take \(g(x, y) = q([s(x), s(y)] - s[x, y]) \in V\).

Define a map \(\theta_{E,s,q} : \mathfrak{g}^\otimes 3 \to M\) by

\[
\theta_{E,s,q}(x, y, z) = [s(x), g(y, z)] - [s(y), g(x, z)] + [s(z), g(x, y)] - g([x, y], z) + g([x, z], y) - g([y, z], x).
\]

Since \(\partial\) is a map of \(L\)-modules, it follows that \(\partial(\theta_{E,s,q}(x, y, z)) = 0\). Therefore, \(\theta_{E,s,q}(x, y, z) \in \ker(\partial) = M\). The map \(\theta_{E,s,q} : \mathfrak{g}^\otimes 3 \to M\) is skew-symmetric in \(x, y, z\). Hence \(\theta_{E,s,q} : \wedge^3 g \to M\). The map \(\theta_{E,s,q}\) defines a 3-cocycle on the Chevalley-Eilenberg cohomology of \(g\) with coefficients in \(M\).

We first show that the class of \(\theta_{E,s,q}\) in \(H^3_{CE}(g, M)\) does not depend on the section \(s\). Suppose \(\mathfrak{s} : g \to L\) is another section of \(\pi\) and let \(\theta_{E,s,q}\) be the corresponding 3-cocycle defined by using \(\mathfrak{s}\) instead of \(s\). Then there exists a linear map \(h : g \to V\) with \(s = \mathfrak{s} = \partial h\). Observe that

\[
[s(x), g(y, z)] - [\mathfrak{s}(x), \mathfrak{s}(y, z)] = [s(x) - \mathfrak{s}(x), g(y, z)] + [\mathfrak{s}(x), (g - \mathfrak{g})(y, z)]
\]

\[
= \partial[h(x), q([s(y), s(z)] - s[y, z])] + [\mathfrak{s}(x), (g - \mathfrak{g})(y, z)]
\]

\[
= -[[[s(y), s(z)] - s[y, z]], h(x)] + [\mathfrak{s}(x), (g - \mathfrak{g})(y, z)].
\]

Therefore,

\[
(\theta_{E,s,q} - \theta_{E,s,q})(x, y, z) = -[[[s(y), s(z)] - s[y, z]], h(x)] + [[[s(x), s(z)] - s[x, z]], h(y)]
\]

\[
- [[[s(x), s(y)] - s[x, y]], h(z)] + [\mathfrak{s}(x), (g - \mathfrak{g})(y, z)]
\]

\[
- [\mathfrak{s}(y), (g - \mathfrak{g})(x, z)] + [\mathfrak{s}(z), (g - \mathfrak{g})(x, y)]
\]

\[
- (g - \mathfrak{g})([x, y], z) + (g - \mathfrak{g})([x, z], y) - (g - \mathfrak{g})([y, z], x).
\]

Define a map \(b : \wedge^3 g \to V\) by

\[
b(x, y) = [s(x), h(y)] - h([x, y]) - [s(y), h(x)] - [\partial h(x), h(y)].
\]
Then a easy calculation shows that \( \partial b = \partial (g - \overline{g}) \). Hence \((g - \overline{g} - b) : \wedge^2 \mathfrak{g} \to M \). It follows from (1) that
\[
(\theta_{E,s,q} - \theta_{E',s,q})(x, y, z) = -[[s(y), s(z)] - s[y, z]], h(x)] + [[(s(x), s(z)) - s[x, z]], h(y)]
- [[s(x), s(y)] - s[x, y]], h(z)] + (\delta(g - \overline{g} - b))(x, y, z)
+ [\overline{s}(x), b(y, z)] - [\overline{s}(y), b(x, z)] + [\overline{s}(z), b(x, y)]
- b([x, y], z) + b([x, z], y) - b([y, z], x).
\]

In the right hand side of the above equation, we substitute the definition of \( b \) in last six terms. After many cancellations on the right hand side (one has to use the fact that \((V, L, \partial)\) is a crossed module for some cancellations), we are only left with the term \((\delta(g - \overline{g} - b))(x, y, z)\). Hence the class of \( \theta_{E,s,q} \) does not depend on the section \( s \).

Next consider a map
\[
E := 0 \longrightarrow M \xrightarrow{i} V \xrightarrow{\partial} L \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0
\]
\[
E' := 0 \longrightarrow M \xrightarrow{i} V' \xrightarrow{\partial'} L' \xrightarrow{\pi'} \mathfrak{g} \longrightarrow 0
\]
of crossed modules. Let \( s' : \mathfrak{g} \to L' \) and \( q' : \text{Im} (\partial') \to V' \) be sections of \( \pi' \) and \( \partial' \), respectively. Note that \((\pi' \beta s)(x) = (\pi s)(x) = x \), for all \( x \in \mathfrak{g} \). Therefore, \( \beta s : \mathfrak{g} \to L' \) is another section of \( \pi' \). Thus, we have
\[
(\theta_{E,s,q} - \theta_{E',s,q})(x, y, z)
= [s(x), g(y, z)] - [s(y), g(x, z)] + [s(z), g(x, y)] - g([x, y], z) + g([x, z], y) - g([y, z], x)
- [\beta s(x), g'(y, z)] + [\beta s(y), g'(x, z)] - [\beta s(z), g'(x, y)] + g'([x, y], z) - g'([x, z], y) - g'([y, z], x),
\]
where \( g'(x, y) = q'([\beta s(x), \beta s(y)] - \beta s[x, y]) \). Here we have used the same notation \([ , , ]\) to denote the action of \( L \) on \( V \) and the action on \( L' \) on \( V' \). Hence, we have
\[
(\theta_{E,s,q} - \theta_{E',s,q})(x, y, z)
= [\beta s(x), (\alpha q - q' \beta)(s(y), s(z)) - s[y, z]]) - [\beta s(y), (\alpha q - q' \beta)(s(x), s(z)) - s[x, z]])
+ [\beta s(z), (\alpha q - q' \beta)(s(x), s(y)) - s[x, y]]) - (\alpha q - q' \beta)(s[x, y], s[z]) - s[x, y], z]])
+ (\alpha q - q' \beta)(s[x, z], s(y)) - s[x, z], y]]) - (\alpha q - q' \beta)(s[y, z], s[x]) - s[y, z], x]).
\]
It follows from the above expression that \( (\theta_{E,s,q} - \theta_{E',s,q})(x, y, z) = (\delta \phi)(x, y, z) \), for some \( \phi : \wedge^2 \mathfrak{g} \to M \). Hence, \([\theta_{E,s,q}] = [\theta_{E',s,q}] \) in \( H^3_{CE}(\mathfrak{g}, M) \). Moreover, from the first part, we have \([\theta_{E,s,q} - \theta_{E',s,q}] \) \([\theta_{E',s',q'}]\). Hence the class \([\theta_{E,s,q}] \) does not depend on the sections \( s \) and \( q \). We denote the corresponding class by \([\theta_E] \). Therefore, the map
\[
\psi : \text{Cross}(\mathfrak{g}, M) \to H^3_{CE}(\mathfrak{g}, M), \ E \to [\theta_E]
\]
is well-defined. The surjectivity of \( \psi \) follows from the next observation. \( \square \)

Let \( \mathfrak{g} \) be a Lie algebra and \( 0 \to M \to M' \xrightarrow{\pi} M'' \to 0 \) an exact sequence of \( \mathfrak{g} \)-modules. Given an abelian extension
\[
0 \to M'' \to \epsilon \to \mathfrak{g} \to 0
\]
of \( \mathfrak{g} \) by \( M'' \), we consider the Yoneda product
\[
0 \to M \to M' \xrightarrow{\mu} \epsilon \to \mathfrak{g} \to 0.
\]
Writing $e = M'' \oplus g$ as a vector space, we have $\mu(m') = (\pi(m'),0)$. An $e$-action on $M'$ is induced from the $g$-action on $M'$, namely,

$$[(m'',x),m'] = [x,m'], \text{ for } (m'',x) \in e \text{ and } m' \in M'.$$

It is easy to see that (3) defines a crossed module of $g$ by $M$. It has been shown in [9] that the corresponding third cohomology class in $H^3_{CE}(g,M)$ as constructed above is the image of the second cohomology class in $H^2_{CE}(g,M'')$ (defined by the abelian extension (2)) under the connecting homomorphism $\partial$ in the cohomology long exact sequence

$$\cdots \rightarrow H^2_{CE}(g,M) \rightarrow H^2_{CE}(g,M') \rightarrow H^2_{CE}(g,M'') \xrightarrow{\partial} H^3_{CE}(g,M) \rightarrow \cdots.$$

**Surjectivity of the map $\psi$ (of Theorem 3.3).** The surjectivity of $\psi$ can be shown as follows [9]. As the category $g$-mod possesses enough injectives, we can choose an injective $g$-module $I$ and a monomorphism $i : M \rightarrow I$. Consider the short exact sequence of $g$-modules

$$0 \rightarrow M \xrightarrow{i} I \rightarrow M'' \rightarrow 0 \tag{4}$$

where $M''$ is the cokernel of the map $i$. Since $I$ is injective, the cohomology long exact sequence yields $H^2_{CE}(g,M'') \cong H^3_{CE}(g,M)$. This isomorphism is given by the connecting homomorphism in the long exact sequence of cohomology. Thus, a cohomology class $[\gamma] \in H^2_{CE}(g,M)$ corresponds to a class $[\alpha] \in H^2_{CE}(g,M'')$. We now consider an abelian extension $0 \rightarrow M'' \rightarrow e \rightarrow g \rightarrow 0$ corresponding to the cohomology class $[\alpha] \in H^2_{CE}(g,M'')$. It follows from the above discussion that the Yoneda product of (4) and the above abelian extension gives rise to a crossed module whose associated third cohomology class is given by $[\gamma]$.

4. **Crossed $n$-fold extensions of Lie algebras**

Using the notion of crossed module of previous section, we introduce crossed $n$-fold extensions of a Lie algebra $g$ by a module $M$. We show that there is an abelian group structure on $\text{Opext}^n(g,M)$ of equivalence classes of crossed $n$-fold extensions of $g$ by $M$, for $n \geq 2$. The construction is similar to the case of associative algebra [2]. Finally, we show that such extensions represent cohomology classes in $H^n_{CE}(g,M)$.

Let $g$ be a Lie algebra and $M$ be a $g$-module. Let $n \geq 2$.

4.1. **Definition.** A crossed $n$-fold extension of $g$ by $M$ is an exact sequence

$$0 \rightarrow M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} g \rightarrow 0$$

of $K$-vector spaces with the following properties:

- $(M_1, L, \partial_1)$ is a crossed module with $\text{coker}(\partial_1) = g$,
- for $1 < i \leq n-1$, $M_i$ is a $g$-module and $\partial_i$, $f$ are morphisms of $g$-modules.

4.2. **Definition.** Let

$$\mathcal{E} := (0 \rightarrow M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} g \rightarrow 0)$$

and

$$\mathcal{E}' := (0 \rightarrow M' \xrightarrow{f'} M'_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} M'_1 \xrightarrow{\partial'_1} L' \xrightarrow{\pi'} g \rightarrow 0)$$

be two crossed $n$-fold extensions of $g$ by $M$ and $M'$, respectively. A morphism between them consists of maps $\alpha : M \rightarrow M'$, $\beta : L \rightarrow L'$ and $\delta_i : M_i \rightarrow M_i'$ for $1 \leq i \leq n-1$ such that

- all squares commute (see figure (5)),
- $\alpha$, $\delta_i$ $(2 \leq i \leq n-1)$ are morphism of $g$-modules,
- $(\delta_1, \beta) : (M_1, L, \partial_1) \rightarrow (M'_1, L', \partial'_1)$ is a map of crossed modules inducing the identity map on $g$. 
The projection is defined coordinatewise. It is easy to check that \((M, L)\) is a \(\alpha\)-module. Then there exists an \(n\)-fold extensions of \(g\) by \(M\). Thus, it follows from the above proposition that a \(\alpha\)-module map \(\alpha : M \to M'\) induces a well-defined map

\[
\alpha_* : \text{Opext}^n(g, M) \to \text{Opext}^n(g, M'), \quad \mathcal{E} \mapsto \alpha \mathcal{E}.
\]

Let \(\text{Opext}^n(g, M)\) be the equivalence classes of crossed \(n\)-fold extensions of \(g\) by \(M\). Then it follows that \(\text{Opext}^2(g, M) = \text{Cross}(g, M)\). In the next, we show that \(\text{Opext}^n(g, M)\) has a natural abelian group structure. To do that, we start with the following proposition.

4.4. Definition. let \(\mathcal{E} \in \text{Opext}^n(g, M)\) be an \(n\)-fold extension of \(g\) by \(M\) and \(\alpha : M \to M'\) be an \(g\)-module map. Then there exists an \(n\)-fold extension \(\alpha \mathcal{E} \in \text{Opext}^n(g, M')\) and a morphism of the form \((\alpha, \delta_{n-1}, \ldots, \delta_1, \beta)\) from \(\mathcal{E}\) to \(\alpha \mathcal{E}\). Moreover, \(\alpha \mathcal{E} \in \text{Opext}^n(g, M')\) is the unique \(n\)-fold extension with this property.

Proof. The proof is based on pushout of \(g\)-modules. Let \(\mathcal{E} := (0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} L \xrightarrow{\pi} g \to 0) \in \text{Opext}^n(g, M)\) be a crossed \(n\)-fold extension. We consider the \(n\)-fold extension \(\alpha \mathcal{E} \in \text{Opext}^n(g, M)\) as

\[
\alpha \mathcal{E} := (0 \to M' \xrightarrow{f'} M_{n-1}' \xrightarrow{\delta_{n-1}'} \cdots \xrightarrow{\delta_2'} M_1' \xrightarrow{\delta_1'} L' \xrightarrow{\pi'} g \to 0)
\]

where \(M' \to M_{n-1}' \to M_{n-2}'\) is obtained from the following pushout of \(g\)-modules

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & M_{n-1} \\
\downarrow & & \downarrow \\
M' & \xrightarrow{\alpha} & M_{n-1}' \\
0 & & 0 \\
\end{array}
\]

Moreover \((\alpha, i, id, \ldots, id)\) defines a morphism from \(\mathcal{E}\) to \(\alpha \mathcal{E}\).

Finally, let \(\mathcal{E}' \in \text{Opext}^n(g, M')\) be an \(n\)-fold extension and there is a morphism \((\alpha, \delta_{n-1}, \ldots, \delta_1, \beta) : \mathcal{E} \to \mathcal{E}'\). Then by properties of pushout, one obtains a map \((1, j, \delta_{n-2}, \ldots, \delta_1, \beta) : \alpha \mathcal{E} \to \mathcal{E}'\). This shows that the class of \(\alpha \mathcal{E}\) and \(\mathcal{E}'\) is same in \(\text{Opext}^n(g, M)\).

Thus, it follows from the above proposition that a \(g\)-module map \(\alpha : M \to M'\) induces a well-defined map

\[
\alpha_* : \text{Opext}^n(g, M) \to \text{Opext}^n(g, M'), \quad \mathcal{E} \mapsto \alpha \mathcal{E}.
\]
4.5. **Definition.** (Baer sum) Let $\mathcal{E}, \mathcal{E}' \in \text{Opext}^n(\mathfrak{g}, M)$ with $n \geq 3$. Then the Baer sum $\mathcal{E} + \mathcal{E}' \in \text{Opext}^n(\mathfrak{g}, M)$ is defined by

$$\mathcal{E} + \mathcal{E}' = \nabla_M(\mathcal{E} \oplus \mathcal{E}')$$

where $\nabla_M : M \oplus M \to M$, $(m_1, m_2) \mapsto m_1 + m_2$ is the codiagonal map.

4.6. **Definition.** (Zero extension) Let $n \geq 3$. Then

$$0 \to M = M \to 0 \to \cdots \to 0 \to \mathfrak{g} \to 0 \to 0$$

is a crossed $n$-fold extension of $\mathfrak{g}$ by $M$. We define $0 \in \text{Opext}^n(\mathfrak{g}, M)$ to be the class of this extension.

4.7. **Remark.** Let $\mathcal{E} := (0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0) \in \text{Opext}^n(\mathfrak{g}, M)$ be a crossed $n$-fold extension, for $n \geq 3$. Suppose there is a map $g : M_{n-1} \to M$ of $\mathfrak{g}$-modules satisfying $gf = \text{id}_M$. Then there is a morphism

$$\begin{array}{ccccccc}
0 & \to & M & \xrightarrow{f} & M_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_2} & M_1 & \xrightarrow{\partial_1} & L & \xrightarrow{\pi} & \mathfrak{g} & \to 0
\end{array}$$

of crossed $n$-fold extensions. This shows that the class of $\mathcal{E}$ defines $0 \in \text{Opext}^n(\mathfrak{g}, M)$.

4.8. **Remark.** Let $\mathcal{E} := (0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0) \in \text{Opext}^n(\mathfrak{g}, M)$ be a crossed $n$-fold extension, for $n \geq 3$. It follows from Proposition 4.3 that $f\mathcal{E} \in \text{Opext}^n(\mathfrak{g}, M_{n-1})$ and is given by the bottom row of the following diagram

$$\begin{array}{ccccccc}
0 & \to & M & \xrightarrow{f} & M_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_2} & M_1 & \xrightarrow{\partial_1} & L & \xrightarrow{\pi} & \mathfrak{g} & \to 0
\end{array}$$

Observe that, in $f\mathcal{E}$, there is a map $g = \text{pr}_1 : M_{n-1} \oplus M_{n-2} \to M_{n-1}$ which satisfies $g \circ (\text{id}, 0) = \text{id}_{M_{n-1}}$. Hence, it follows from Remark 4.7 that $f\mathcal{E} = 0 \in \text{Opext}^n(\mathfrak{g}, M_{n-1})$.

4.9. **Definition.** (Inverse of an extension) Let

$$\mathcal{E} := (0 \to M \xrightarrow{f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0) \in \text{Opext}^n(\mathfrak{g}, M)$$

be an extension. Then

$$-\mathcal{E} := (0 \to M \xrightarrow{-f} M_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} M_1 \xrightarrow{\partial_1} L \xrightarrow{\pi} \mathfrak{g} \to 0) \in \text{Opext}^n(\mathfrak{g}, M)$$

defines a new crossed $n$-extension of $\mathfrak{g}$ by $M$.

The proof of the following theorem is straightforward.

4.10. **Theorem.** Let $n \geq 3$. Then the set $\text{Opext}^n(\mathfrak{g}, M)$ equipped with the Baer sum is an abelian group. The zero element and inverse elements of this group are given by Definitions 4.6 and 4.6, respectively.

Moreover, if $\alpha : M \to M'$ is a morphism of $\mathfrak{g}$-modules, the map $\alpha_* : \text{Opext}^n(\mathfrak{g}, M) \to \text{Opext}^n(\mathfrak{g}, M')$ is a morphism of groups.

4.11. **Remark.** The set $\text{Opext}^2(\mathfrak{g}, M)$ also inherits the structure of an abelian group. However, one can easily feel that the structure of this abelian group must be different than the one defined in Theorem 4.10.

Note that an element in $\text{Opext}^2(\mathfrak{g}, M)$ is an equivalence class of crossed modules. We denote $0 \in \text{Opext}^2(\mathfrak{g}, M)$ to be the class of the crossed module $0 \to M = M \xrightarrow{0} \mathfrak{g} = \mathfrak{g} \to 0$. The
Then algebra case \[ \delta \] we define \[ V \] for \( (V) \). Here of \( g \) as follows. Given an extension \( \delta \) be a short exact sequence of \( g \) us to prove our main theorem. Let \( E \) sum quotient map \( \rho \) for \( (V) \). Then the Baer sum \( E + \mathcal{E} \) is the class of the extension \( 0 \to M \xrightarrow{\partial} V + V' \xrightarrow{\partial'} L \xrightarrow{\pi} g \to 0 \).

Here \( V + V' \) is the pushout of \( \mathbb{K} \)-vector spaces

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array{To solve this problem, we need to interpret the mathematical symbols and expressions in the text. The document contains a proof involving a short exact sequence of modules, a crossed module, and a long exact sequence. The proof involves defining an action on a vector space, establishing a homomorphism from one module to another, and constructing a long exact sequence. The goal is to prove a main theorem using these concepts. Here is a step-by-step breakdown of the proof:

1. Define the setup: We start with a short exact sequence of modules \( 0 \to M \to M' \to M'' \to 0 \).

2. Define the action: We define an action \( \delta \) on the vector space \( V + V' \) via a quotient map \( \rho \).

3. Construct the long exact sequence: We consider a short exact sequence of modules \( 0 \to M \to M' \to M'' \to 0 \), and then use the Baer sum to construct a long exact sequence.

4. Prove the main theorem: The proof involves showing that the constructed long exact sequence is exact at each stage.

5. Conclusion: The main theorem is proven by showing that the long exact sequence is exact, which is consistent with the properties of abelian groups.

The proof relies on the foundational concepts of module theory and abelian groups, which are essential for understanding the structure and properties of these mathematical objects. The detailed steps in the proof are crucial for verifying the correctness of the main theorem.
Moreover, it follows from the cohomology long exact exact sequence that $H^{n+1}_{CE}(\mathfrak{g}, M''') \cong H^{n+2}_{CE}(\mathfrak{g}, M)$. Therefore,

$$\text{Opext}^3(\mathfrak{g}, M) \cong \text{Opext}^2(\mathfrak{g}, M''') \cong H^2_{CE}(\mathfrak{g}, M''') \quad (\text{by Theorem 3.3})$$

$$\cong H^2_{CE}(\mathfrak{g}, M).$$

We conclude the result by using the induction on $n$. \hfill \Box

4.14. Remark. Lie-Rinehart algebras are algebraic analogue of Lie algebroids and closely related to Lie algebras. These algebras pay special attention due to its connection Poisson geometry [6]. In [4] the authors studied crossed modules for Lie-Rinehart algebras and classify them by the third cohomology of Lie-Rinehart algebras. However, their crossed modules are similar to the traditional one for Lie algebras. It would be interesting to study crossed extensions of Lie-Rinehart algebras and their classification in terms of higher cohomologies of Lie-Rinehart algebras.

5. The case of a Leibniz algebra

Leibniz algebra was introduced by Loday in connection with cyclic homology and Hochschild homology of matrix algebras [8]. The cohomology theory of Leibniz algebras was introduced by the same author. In this section, we outline that Leibniz cohomology groups are also represented by crossed extensions of Leibniz algebras.

A (right) Leibniz algebra is a $\mathbb{K}$-vector space $\mathfrak{h}$ together with a $\mathbb{K}$-bilinear map $[\cdot, \cdot]: \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}$ satisfying

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad \text{for all } x, y, z \in \mathfrak{h}.$$

Let $(\mathfrak{h}, [\cdot, \cdot])$ be a Leibniz algebra. An $\mathfrak{h}$-module is a vector space $M$ together with two bilinear maps (both of them denoted by $[\cdot, \cdot]$) $\mathfrak{h} \times M \rightarrow M$ and $M \times \mathfrak{h} \rightarrow M$ satisfying

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

whenever one of the variable is from $M$ and the others two are from $\mathfrak{h}$.

The cohomology of the Leibniz algebra $\mathfrak{h}$ with coefficients in $M$ is the cohomology of the complex $(C^n_{Leib}(\mathfrak{h}, M), \delta)_{n \geq 0}$ where $C^0_{Leib}(\mathfrak{h}, M) = M$ and $C^n_{Leib}(\mathfrak{h}, M) = \text{Hom}_\mathbb{K}(\mathfrak{h}^\otimes n, M)$, for $n \geq 1$. The differential $\delta$ is given by $(\delta m)(x) = [x, m]$, for $m \in M$, $x \in \mathfrak{h}$ and

$$(\delta f)(x_1, \ldots, x_{n+1}) = [x_1, f(x_2, \ldots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^{i-1} f(x_i, \ldots, \hat{x}_i, \ldots, x_{n+1}),$$

$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} f(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_{n+1}).$$

Similar to the case of Chevalley-Eilenberg cohomology, the lower degree Leibniz cohomology has the following interpretations. The zero-th cohomology group $H^0_{Leib}(\mathfrak{h}, M)$ is the submodule of invariants on $M$ and the first cohomology group $H^1_{Leib}(\mathfrak{h}, M)$ is the space of outer derivations. The second cohomology group $H^2_{Leib}(\mathfrak{h}, M)$ classifies the equivalence classes of extensions of the Leibniz algebra $\mathfrak{h}$ by $M$.

Crossed modules over a Leibniz algebra can be defined in a similar way. One only needs to careful about non skew-symmetric version of the identities used in the case of Lie algebra.

A crossed module over a Leibniz algebra is a triple $(V, L, \partial)$ in which $L$ is a Leibniz algebra, $V$ is a $L$-module and $\partial : V \rightarrow L$ is a map of $L$-modules satisfying

$$[\partial v, w] = [v, \partial w], \quad \text{for all } v, w \in L.$$
In a similar way, a crossed module yields a Leibniz algebra structure on \( \mathfrak{h} = \text{coker}(\partial) \) and there is an exact sequence

\[
0 \to M \to V \xrightarrow{\partial} L \xrightarrow{\pi} \mathfrak{h} \to 0
\]

where \( M = \ker(\partial) \). Moreover, there is a Leibniz algebra action of \( \mathfrak{h} \) on \( M \). A morphism of crossed modules over Leibniz algebras can be defined in a similar way.

In this case, one can also prove that equivalence classes of crossed modules with cokernel \( \mathfrak{h} \) and kernel \( M \) are in one-to-one correspondence with \( H^3_{\text{Leib}}(\mathfrak{h}, M) \). The proof is similar to the Lie algebra case (Theorem 3.3). However, in this case, the Leibniz 3-cocycle \( \theta_{E,s,q} : \mathfrak{h} \otimes^3 \to M \) is given by

\[
\theta_{E,s,q}(x,y,z) = [s(x), g(y,z)] + [g(x,z), s(y)] - [g(x,y), s(z)] - g([x,y], z) + g([x,z], y) + g(x, [y,z]).
\]

 Associated to any third cohomology class in \( H^3_{\text{Leib}}(\mathfrak{h}, M) \) the existence of the corresponding crossed module can be shown by the way that have been described in Theorem 3.3.

Moreover, one can define crossed \( n \)-fold extension of a Leibniz algebra \( \mathfrak{h} \) by a module \( M \). Along the lines of Section 4, we obtain the following theorem for Leibniz algebras.

5.1. **Theorem.** The set \( \text{Opext}^n(\mathfrak{h}, M) \) of equivalence classes of crossed \( n \)-fold extensions of \( \mathfrak{h} \) by \( M \) forms an abelian group, for \( n \geq 2 \). Moreover, there exists an isomorphism of abelian groups

\[
\text{Opext}^n(\mathfrak{h}, M) \cong H^{n+1}_{\text{Leib}}(\mathfrak{h}, M).
\]

**Acknowledgement.** The research is supported by the fellowship of Indian Institute of Technology, Kanpur (India). The author would like to thank the Institute for their support.

**References**

[1] J. Baez and A. S. Crans, Higher-dimensional algebra. VI. Lie 2-algebras, *Theory Appl. Categ.* 12 (2004) 492-538.

[2] H.-J. Baues and E. G. Minian, Crossed extensions of algebras and Hochschild cohomology, *Homology Homotopy Appl.*, vol. 4 (2002), no. 2, 63-82.

[3] H.-J. Baues, E. G. Minian and B. Richter, Crossed modules over operads and operadic cohomology, *K-Theory*, vol. 31 (2004), no. 1, 39-69.

[4] J. M. Casas, M. Ladra and T. Pirashvili, Crossed modules for Lie-Rinehart algebras, *J. Algebra* 274 (2004), no. 1, 192-201.

[5] J. Huebschmann, Crossed \( n \)-fold extensions of groups and cohomology, *Comment. Math. Helv.*, vol. 55 (1980), no. 2, 302-313.

[6] J. Huebschmann, Poisson cohomology and quantization, *J. Reine Angew. Math.* 408 (1990), 57-113.

[7] C. Kassel and J.-L. Loday, Extensions centrales d’algèbres de Lie, *Ann. Inst. Fourier (Grenoble)* 32 (4) (1982) 119-142.

[8] J.-L. Loday, Une version non commutative des algèbres de Lie : les algèbres de Leibniz, *Enseign. Math.* (2) 39 (1993), no. 3-4, 269-293.

[9] F. Wagemann, On Lie Algebra Crossed Modules, *Comm. Algebra* 34 (2006), no. 5, 1699-1722.

**Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur 208016, Uttar Pradesh, India**

E-mail address: apurbadas348@gmail.com