Constraints on Non-Singular Cosmological Models with Quadratic Lagrangians

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We consider the generalized set of theories of gravitation whose Lagrangians contain the term \( R^2 : L = \sqrt{-g}(R + \beta R^2) \). Inserting the RW metric with an imposed non-singular and inflationary behaviour of the scale factor \( a(t) \), and using an arbitrary perfect fluid, we study the properties of \( \rho \) and \( p \) in this context. By requiring the positivity of the energy density, as well as real and finite velocity of sound, we can obtain the range of values of \( \beta \) that ensure the inflationary behaviour and absence of singularity.

I. INTRODUCTION

There are many motivations to study quadratic-order curvature Lagrangians in cosmology: quantum corrections [1,2] and renormalization of divergences, [3–5] effective actions of superstring theory, [6] inflation without scalar fields, [7] etc.

The aim of the present work is to explore the consequences of adding quadratic-order curvature terms to the Einstein action. From the beginning, a specified non-singular and inflationary scale factor and the Robertson-Walker metric are imposed, and the unknown energy-momentum tensor is then analysed.

The only energy-momentum tensor \( T_{\mu\nu} \) choice in this work will be an arbitrary perfect fluid described by the energy density \( \rho \) and the isotropic pressure \( p \). Just a couple of simple energy conditions will be used here,

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\[ \rho > 0 \ , \quad (\rho + p) > 0 , \quad (1) \]

and additionally the adiabatic speed of sound, \( v_s^2 = (\partial p/\partial \rho) \), must obey

\[ 0 \leq v_s^2 \leq 1 . \quad (2) \]

For each value of the spatial curvature \( k \), the form of \( \rho \) and \( p \) are obtained as functions of the scale factor \( a(t) \) and other constants. The energy (1) and velocity of sound (2) conditions can give constraints on the constant \( \beta \) associated with the quadratic-order curvature terms. It is also possible to compare the violations of the conditions (1,2) with and without the quadratic-order curvature terms, which gives some insight on the consequences of using quadratic Lagrangians.

**II. THE MODIFIED FRIEDMANN EQUATION**

Considering the fact that the quantity \( \sqrt{-g}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2) \), known as Gauss-Bonnet term, is a total divergence in four-dimensional spacetime, the \( R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \) term can be expressed in terms of \( R^2 \) and \( R_{\mu\nu}R^{\mu\nu} \), so the Riemann tensor is not necessary to account for the curvature square corrections.

Representing a general quadratic order curvature Lagrangian (but without the cosmological constant), let us take the Lagrangian

\[ L = \sqrt{-g}(R + \beta_1 R^2 + \beta_2 R_{\mu\nu}R^{\mu\nu}) . \quad (3) \]

Calculating the variation of this action with respect to the metric \( g^{\mu\nu} \), the modified Einstein equations are

\[ G_{\mu\nu} + \beta_1 W^{(1)}_{\mu\nu} + \beta_2 W^{(2)}_{\mu\nu} = \kappa T_{\mu\nu} . \quad (4) \]

We will use the Robertson-Walker (RW) metric

\[ ds^2 = -dt^2 + a(t)^2 \left[ \left( \frac{1}{1 - kr^2} \right) dr^2 + r^2 \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right] , \quad (5) \]
where the spatial curvature $k$ takes the values 0, +1, −1 (corresponding to flat, closed and open spatial sections). In the case of this spherically symmetric homogeneous metric, the Weyl tensor has null components. So the identity $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$ is equal to zero, yielding another equation that now allows us to write $R_{\mu\nu}R^{\mu\nu}$ in terms of $R^2$

$$L = \sqrt{-g} \left[ R + \left( \beta_1 + \frac{\beta_2}{3} \right) R^2 \right] = \sqrt{-g} (R + \beta R^2) . \quad (6)$$

With the use of a perfect fluid, the modified Friedmann third-order differential equation is obtained:

$$\frac{a'(t)^2 + k}{a(t)^2} - \frac{\kappa \rho(t)}{3} - \frac{18 \beta}{a(t)^4} \left\{ k^2 - 3a'(t)^4 - a(t)^2 a''(t)^2 - 2a'(t)^2 [k - a(t) a''(t)] + 2a(t)^2 a'(t) a'''(t) \right\} = 0 , \quad (7)$$

where $\beta = (\beta_1 + \frac{\beta_2}{3})$ different from zero means that the quadratic-order curvature corrections are present.

With the RW metric (5), the energy-momentum conservation equations yield

$$\rho'(a) = \frac{-3(\rho + p)}{a} , \quad (8)$$

so that $p(a)$ can be calculated using $\rho(a)$.

\section*{III. NON-SINGULAR SCALE FACTORS WITH INFLATIONARY BEHAVIOURS}

Non-singular and inflationary scale factors for the three values of spatial curvature $k$ are studied. They show inflationary behaviour matching the classical radiation dominated behaviour.

\section*{1) The $k = 0$ case :}

The following equation
\[ t = \frac{a(t)^2}{a_0^2} + \alpha \log \left[ \frac{a(t)^2}{a(t)^2 + a_0^2} \right], \quad (9) \]

allows to obtain a scale factor \( a(t) \), which for \( \alpha = 0 \) (or \( a >> a_0 \)) yields \( a(t) = a_0 \sqrt{t} \) (the known solution for radiation dominated era in FRW cosmology with \( k = 0 \)), and for \( t \to -\infty \) (or \( a << a_0 \)) gives \( a(t) = a_0 \exp(t/2\alpha) \). Figure 1 shows this behaviour compared to \( a(t) = a_0 \sqrt{t} \).

![a(t) vs t](image)

**FIG. 1.** \( a(t) \times t \) for the case \( k = 0 \), with \( a_0 = \alpha = 1 \). The solid line is the non-singular scale factor with inflationary behaviour; the dashed line is the known scale factor, \( a(t) = a_0 \sqrt{t} \), for radiation era in FRW cosmology.

It is possible to derive Eq. (9) in respect to \( t \) and write

\[ a'(t) = \frac{a(t) a_0^2 [a(t)^2 + a_0^2]}{2 [a(t)^4 + a(t)^2 a_0^2 + a_0^4 \alpha]}, \quad (10) \]

deriving this Eq. (10), the second derivative can be obtained,

\[ a''(t) = \frac{a(t) a_0^8 [a(t)^2 + a_0^2][3 a(t)^2 + a_0^2] \alpha - a(t)^3 a_0^4 [a(t)^2 + a_0^2]^3}{4 [a(t)^4 + a(t)^2 a_0^2 + a_0^4 \alpha]^3}, \quad (11) \]

and deriving this Eq. (11), the third derivative is given by
\[ a''(t) = \frac{a(t) a_0^6 [a(t)^2 + a_0^2]}{8 [a(t)^4 + a(t)^2 a_0^2 + a_0^4 \alpha]} \left\{ 3 a(t)^4 [a(t)^2 + a_0^2]^4 - 2 a(t)^2 a_0^4 [a(t)^2 + a_0^2] [15 a(t)^4 + 15 a(t)^2 a_0^2 + 4 a_0^4] \alpha + a_0^8 [15 a(t)^4 + 12 a(t)^2 a_0^2 + a_0^4] \alpha^2 \right\}. \]  

(12)

The scale factor \( a(t) \) and its derivatives are smooth functions of \( t \).

II) The \( k = +1 \) case:

Generalizing the equation Eq. (9),

\[
    t = \left(1 - \sqrt{1 - \frac{a(t)^2}{a_0^2}}\right) t_0 + \alpha \log \left[ \frac{a(t)^2}{a(t)^2 + a_0^2} \right],
\]

(13)
allows to obtain \( a(t) \), which for \( \alpha = 0 \) (or \( a \approx a_0 \)) yields \( a(t) = a_0 \sqrt{1 - (1 - t/t_0)^2} \) (the known solution for radiation dominated era in FRW cosmology with \( k = +1 \)), and for \( t \to -\infty \) (or \( a << a_0 \)) gives \( a(t) = a_0 \exp(t/2\alpha) \).

As in the \( k = 0 \), it is possible to derive Eq. (13) in respect to \( t \) and write

\[
    a'(t) = \frac{a(t) a_0 \sqrt{a_0^2 - a(t)^2} [a(t)^2 + a_0^2]}{a(t)^4 t_0 + a(t)^2 a_0^2 t_0 + 2 a_0^3 \sqrt{a_0^2 - a(t)^2} \alpha},
\]

(14)
and the second and third derivatives can also be obtained without problems. All these functions are smooth.

III) The \( k = -1 \) case:

Also generalizing the equation Eq. (9),

\[
    t = \left(-1 + \sqrt{1 + \frac{a(t)^2}{a_0^2}}\right) t_0 + \alpha \log \left[ \frac{a(t)^2}{a(t)^2 + a_0^2} \right],
\]

(15)
allows to obtain \( a(t) \), which for \( \alpha = 0 \) (or \( a >> a_0 \)) yields \( a(t) = a_0 \sqrt{-1 + (1 + t/t_0)^2} \) (the known solution for radiation dominated era in FRW cosmology with \( k = -1 \)), and for \( t \to -\infty \) (or \( a << a_0 \)) gives \( a(t) = a_0 \exp(t/2\alpha) \).

As in the \( k = 0 \), it is possible to derive Eq. (13) in respect to \( t \) and write
\[ a'(t) = \frac{a(t)a_0 \left[ a(t)^2 + a_0^2 \right]^{3/2}}{a(t)^4 t_0 + a(t)^2 a_0^2 t_0 + 2a_0^3 \sqrt{a_0^2 + a(t)^2 \alpha}}, \] (16)

and the second and third derivatives can also be easily obtained. All these derivatives are smooth functions.

**IV. ANALYSIS OF \( \rho \) AND \( P \)**

Instead of trying to obtain solutions for \( a(t) \) of the modified Friedmann equation, Eq. (3), which can only be solved numerically, another way of analysis will be chosen. Replacing the derivatives of \( a(t) \), then the energy density \( \rho(t) \) can be written as \( \rho(a) \), also depending on the constants \( k \) and \( \beta \) and the constants of the definition of \( a(t) \).

**I) The \( k = 0 \) case :**

The energy density is given by

\[
\rho(a) = \frac{3a_0^4 (a^2 + a_0^2)^2}{4 (a^4 + a^2 a_0^2 + a_0^4 \alpha)^2 \kappa} + \frac{27 a^2 a_0^12 (a^2 + a_0^2)^2 \alpha \beta}{2 (a^4 + a^2 a_0^2 + a_0^4 \alpha)^6 \kappa} \times \left[ 14 a_6^6 + 29 a^4 a_0^2 + 2 a^2 a_0^4 (10 - 3 \alpha) - 5 a_0^6 (-1 + \alpha) \right],
\] (17)

and a thorough analysis shows that \( \rho \geq 0 \) if \( \alpha \) and \( \beta \) are inside the region

\[
Q_1 = \{ 0 \leq \alpha < 1, \beta > \beta_- \} \cup \{ \alpha > 1, \beta_- < \beta < \beta_+ \},
\] (18)

where \( \beta_- \) and \( \beta_+ \) are functions of \( \alpha \). The fact of adding the non-linear terms in the Lagrangian is represented by \( \beta \neq 0 \), and does not force \( \rho \) to be negative.

The Figure 2 shows the behaviour of \( \rho(a) \), which has no divergences if \( \alpha > 0 \).
FIG. 2. $\rho(a) \times a$ for the case $k = 0$, with $a_0 = \kappa = 1$, $\alpha = 0.5$, $\beta = 0.01$. The solid line uses the non-singular scale factor with inflationary behaviour; the dashed line is obtained using the known scale factor, $a(t) = a_0 \sqrt{t}$, for radiation era in FRW cosmology.

The pressure $p$ can be obtained from the Eq. (19),

$$p(a) = \frac{a_0^4 (a^2 + a_0^2) [a^6 + 2 a_0^4 (1 - 7 \alpha) - 3 a_0^6 \alpha]}{4 (a^4 + a_0^2 + a_0^4 \alpha)^3 \beta} +$$

$$+ \frac{9 a^2 a_0^{12} (a^2 + a_0^2) \alpha \beta}{2 (a^4 + a_0^2 + a_0^4 \alpha)^7 \beta} \left\{126 a^{12} + 459 a^{10} a_0^2 + 8 a^8 a_0^4 (85 - 36 \alpha) +
25 a_0^{12} (-1 + \alpha) \alpha - 6 a^6 a_0^6 (-87 + 119 \alpha) +
3 a^4 a_0^8 [70 + \alpha (-207 + 22 \alpha)] + a^2 a_0^{10} [35 + \alpha (-220 + 87 \alpha)]\right\},$$

If $\alpha > 0$, the pressure $p$ is always negative for some region in the $a(t)$ or $t$ domain, independent of $\beta$. For $\alpha > 0$, $p$ has no divergences.

Testing whether $\rho + p \geq 0$ is true just restricts $\alpha$ and $\beta$ inside the region

$$Q_2 = \{0 \leq \alpha < 1, \beta_- < \beta < \beta_+\},$$

where $\beta_-$ and $\beta_+$ are another functions of $\alpha$, different from the analysis of $\rho$.

The velocity of sound,
\[
v_s^2 = \frac{\partial p}{\partial \rho} = \frac{\partial p/\partial a}{\partial \rho/\partial a},
\]

(21)
does not diverge if \(\alpha\) and \(\beta\) are inside the same region \(Q_2\). But it can be imaginary for any value of \((\alpha, \beta) \neq (0, 0)\).

So the conclusion for the \(k = 0\) case is that the addition of high-order terms in the Einstein action does not lead to any other anomaly in the effective perfect fluid, i.e., with or without \(\beta\), the behaviour of \(\rho, p, \rho + p\) and \(v_s\) is the same (under the condition that \(\alpha\) and \(\beta\) are inside the region \(Q_2\)).

II) The \(k = +1\) case :

Analogous to the \(k = 0\) case, but with more difficult calculations, the energy density is given by

\[
\rho(a) = \frac{3 a_0^2}{(a^5 + a^3 a_0^2 + 2 a a_0^2 \sqrt{-a^2 + a_0^2} \alpha)^2} \left[ a^6 + 2 a^4 a_0^2 + a^2 a_0^4 + 4 a^2 \sqrt{-a^2 + a_0^2} \left( a^2 + a_0^2 \right) \alpha - 4 a^2 a_0^2 \alpha^2 + 4 a_0^4 \alpha^2 \right] + \frac{216 a_0^2 \alpha \beta}{a^4 \left( a^4 + a^2 a_0^2 + 2 a_0^2 \sqrt{-a^2 + a_0^2} \alpha \right)^6} \left\{ a^6 \left( a - a_0 \right) a_0^2 \left( a + a_0 \right) \times \left( a^2 + a_0^2 \right)^2 \left( 51 a^6 + 14 a^4 a_0^2 - 13 a^2 a_0^4 + 8 a_0^6 \right) \alpha + 4 a_0^6 \left( a^5 - a_0^5 \right)^2 \times \left( -17 a^2 + 2 a_0^2 \right) \alpha^3 - 16 a_0^{10} \left( -a^2 + a_0^2 \right)^3 \alpha^5 + a^2 \sqrt{-a^2 + a_0^2} \times \left( a^4 - a_0^4 \right) \left[ a^4 \left( a^2 + a_0^2 \right)^2 \left( 5 a^6 - 34 a^4 a_0^2 - 33 a^2 a_0^4 - 10 a_0^6 \right) + 8 a^2 a_0^4 \left( 7 a^2 - 2 a_0^2 \right) \left( a^2 + a_0^2 \right)^2 \alpha^2 + 48 a_0^8 \left( -a^2 + a_0^2 \right) \alpha^4 \right] \right\},
\]

and a long analysis shows that \(\rho \geq 0\) if \(\alpha\) and \(\beta\) are inside the region

\[
Q_1 = \{ 0 \leq \alpha < \alpha_{\text{max}}, \beta_- < \beta \leq 0 \} \cup \{ \alpha > \alpha_{\text{max}}, \beta < 0 \},
\]

(23)
where \(\beta_-\) is function of \(\alpha\) and \(a_0\), and \(\alpha_{\text{max}}\) is a function of \(a_0\), \(\alpha_{\text{max}} \approx 0.73 \ a_0\). Figure 3 shows the region \(Q_1\). The non-linear terms in the Lagrangian do not force \(\rho\) to be negative.
FIG. 3. The region $Q_1$ for the case $k = +1$, where $\rho > 0$, with $a_0 = 1$.

The pressure $p$ can be obtained from the Eq. (8) and will not be shown here. Even $p \geq 0$ is possible if $\alpha$ and $\beta$ are inside the region

$$Q_2 = \{\alpha > \alpha_{\text{min}}, \beta < \beta_- \cup \beta > \beta_+\},$$

(24)

where $\beta_-$ and $\beta_+$ are functions of $\alpha$ and $a_0$, and $\alpha_{\text{min}}$ is a function of $a_0$, $\alpha_{\text{min}} \approx 0.84 a_0$. As $\beta = 0$ is not included in the region $Q_2$, this result is a benefit of adding non-linear terms in the Lagrangian.

To obtain $\rho + p \geq 0$, $\alpha$ and $\beta$ must be inside the region

$$Q_3 = \{0 \leq \alpha < \alpha_{\text{max}}, \beta_- < \beta \leq 0\} \cup \{\alpha > \alpha_{\text{max}}, \beta < 0\},$$

(25)

where $\beta_-$ and $\beta_+$ are another functions of $\alpha$, different from the analysis of $\rho$, and $\alpha_{\text{max}}$ is a function of $a_0$, $\alpha_{\text{max}} \approx 0.80 a_0$. The regions $Q_1$ and $Q_3$ are similar.

The velocity of sound does not diverge if $\alpha$ and $\beta$ are inside the same region $Q_3$. But it can be imaginary for any value of $(\alpha, \beta) \neq (0, 0)$; maybe only in the $Q_2$ region $v_s$ has real values, this must be confirmed.

So the conclusion for the $k = +1$ case is that the addition of high-order terms in the Einstein action does not lead to any other anomaly in the effective perfect fluid, i.e., with
or without $\beta$, the behaviour of $\rho$, $p$, $\rho + p$ and $v_s$ is the same (under the condition that $\alpha$ and $\beta$ are inside the region $Q_3$).

There is even an advantage because $p$ can be always positive, with only the problem that the region $Q_2$ leads to almost suppression of the radiation era. The calculus of the region $Q_2$ is not easy, and it is a task for further work to verify if inflationary behaviour can exist with $\rho > 0$, $p > 0$ and $0 < v_s < 1$.

**III) The $k = -1$ case**:

Like the $k = +1$ case, the calculations are very long, and the energy density is given by

$$
\rho(a) = \frac{3 a_0^2 (a^2 + a_0^2) \left[ a^4 - 4 a_0^2 \alpha^2 + a^2 \left( a_0^2 - 4 \sqrt{a^2 + a_0^2} \alpha \right) \right]}{(a^5 + a^3 a_0^2 + 2 a a_0^2 \sqrt{a^2 + a_0^2} \alpha)^2 \kappa} + \frac{216 a_0^2 \alpha \beta}{a^4 \left( a^4 + a^2 a_0^2 + 2 a_0^2 \sqrt{a^2 + a_0^2} \alpha \right)^6 \kappa} \left\{ -a^6 a_0^2 \left( 51 a^2 + 32 a_0^2 \right) \right. \times \left. \left( a^2 + a_0^2 \right)^5 \alpha - 4 a^2 a_0^6 \left( a^2 + a_0^2 \right)^4 \left( 17 a^2 + 2 a_0^2 \right) \alpha^3 - 16 \left( a^2 + a_0^2 \right)^3 \times a_0^{10} \alpha^5 + a^2 \left( a^2 + a_0^2 \right)^{7/2} \left[ a^4 \left( a^2 + a_0^2 \right)^2 \left( 5 a^4 + 17 a^2 a_0^2 + 10 a_0^4 \right) - 8 a^2 a_0^4 \left( a^2 + a_0^2 \right) \left( 7 a^2 + 2 a_0^2 \right) \alpha^2 - 48 a_0^8 \alpha^4 \right] \right\},
$$

and $\rho \geq 0$ is impossible for $\alpha > 0$, independent of $\beta$.

The pressure $p$ will not be shown here. Like $\rho$, $p \geq 0$ is impossible for $\alpha > 0$ and any $\beta$.

Obviously, for $\alpha > 0$ and any $\beta$ it is not possible to obtain $\rho + p \geq 0$.

The velocity of sound is divergent and imaginary if $\alpha > 0$, independent of $\beta$.

The case $k = -1$ does not provide useful conclusions, because even without the high-order terms in the Einstein action there are a lot of anomalies in the effective perfect fluid. The hope that quadratic-order terms could avoid these anomalies was not achieved.

**V. CONCLUSION**

In all three cases of spatial curvature, it is possible to choose the values of $\alpha$ and $\beta$ so that the quadratic-order terms in the Lagrangian do not add more anomalies to the effective perfect fluid.
For closed spatial section, the opposite happens, i.e., anomalies are suppressed, and this case of non-singular inflationary scale factor with well-behaved perfect fluid deserves more investigation.

Other interesting improvements would be: use another solutions of the scale factor, more energy conditions, compare the $\beta$'s constraints with observational values and study many types of effective perfect fluids.

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