ON THE SECOND EIGENVALUE OF A CAYLEY GRAPH OF THE
SYMMETRIC GROUP

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Abstract. In 2020, Siemons and Zalesski [On the second eigenvalue of some Cayley
graphs of the symmetric group. arXiv preprint arXiv:2012.12460, 2020] determined
the second eigenvalue of the Cayley graph $\Gamma_{n,k} = \text{Cay}(\text{Sym}(n), C(n,k))$ for $k = 0$
and $k = 1$, where $C(n,k)$ is the conjugacy class of $(n-k)$-cycles. In this paper, it
is proved that for any $n \geq 3$ and $k \in \mathbb{N}$ relatively small compared to $n$, the second
eigenvalue of $\Gamma_{n,k}$ is the eigenvalue afforded by the irreducible character of $\text{Sym}(n)$
that corresponds to the partition $[n-1,1]$. As a byproduct of our method, the result
of Siemons and Zalesski when $k \in \{0,1\}$ is retrieved. Moreover, we prove that the
second eigenvalue of $\Gamma_{n,n-5}$ is also equal to the eigenvalue afforded by the irreducible
character of the partition $[n-1,1]$.

1. Introduction

The second eigenvalue of a graph is a well-studied parameter in spectral graph theory
(see [3, 10, 11] for instance). Given a graph $X = (V,E)$, the eigenvalues of $X$ (i.e., the
eigenvalues of its adjacency matrix) are real and can be arranged in the following way
$\lambda_1(X) \geq \lambda_2(X) \geq \ldots \geq \lambda_{|V(X)|}(X)$. When $X$ is a regular graph, the largest eigenvalue
$\lambda_1(X)$ is equal to the valency of $X$. This paper is concerned with the study of the
second (largest) eigenvalue, $\lambda_2(X)$, of a Cayley graph of the symmetric group.

Let $G$ be a finite group with identity $e$, and let $C$ be a subset of $G \setminus \{e\}$ with the
property that if $x \in C$ then $x^{-1} \in C$. The Cayley graph $\text{Cay}(G,C)$ is the simple and
undirected graph with vertex set $G$ and two group elements $g$ and $h$ are adjacent if
and only if $hg^{-1} \in C$. We say that the graph $\text{Cay}(G,C)$ is a normal Cayley graph if
$gCg^{-1} = C$, for any $g \in G$. In other words, $\text{Cay}(G,C)$ is a normal Cayley graph if
and only if $C$ is a union of conjugacy classes of $G$. The graph $\text{Cay}(G,C)$ is regular
with valency equal to $|C|$. Since the elements of the left-regular representation $\text{R}(G) = \{\rho_g: G \to G, \rho_g(x) = gx \ | \ g \in G\}$ of $G$ act as automorphisms of $\text{Cay}(G,C)$, the latter
is also vertex-transitive.

Let $n$ and $k$ be two integers such that $0 \leq k \leq n - 1$. We denote the conjugacy class
of the $(n-k)$-cycles of $\text{Sym}(n)$ by $C(n,k)$. That is,

$$C(n,k) = \{\sigma(1,2,3,\ldots,n-k)\sigma^{-1} \ | \ \sigma \in \text{Sym}(n)\}.$$
Let $\Gamma_{n,k}$ be the Cayley graph of $\text{Sym}(n)$ with connection set equal to $C(n,k)$. In other words, $\Gamma_{n,k} := \text{Cay}(\text{Sym}(n), C(n,k))$. The graph $\Gamma_{n,k}$ is vertex-transitive and regular of valency $|C(n,k)| = \binom{2n}{k(n-k-1)}!$. Since $C(n,k)$ is a conjugacy class, $\Gamma_{n,k}$ is also a normal Cayley graph.

A famous result of Babai [1] links the eigenvalues of a normal Cayley graph $\text{Cay}(G,C)$ to the irreducible characters of $G$. In particular, each irreducible character of $G$ determines an eigenvalue of $\text{Cay}(G,C)$ and the spectrum of $\text{Cay}(G,C)$ (the multiset consisting of all the eigenvalues) can be determined with the knowledge of the set of all irreducible characters $\text{Irr}(G)$ of $G$.

The following theorem was recently proved by Siemons and Zalesski [13].

**Theorem 1.1.** For $n \geq 5$, the second eigenvalue of $\Gamma_{n,k}$, for $k \in \{0,1\}$, is as follows.

- If $k = 0$, then $\lambda_2(\Gamma_{n,k}) = (n-2)!$ when $n$ is even and $\lambda_2(\Gamma_{n,k}) = 2(n-3)!$ when $n$ is odd.
- If $k = 1$, then $\lambda_2(\Gamma_{n,k}) = 3(n-3)(n-5)!$ when $n$ is even and $\lambda_2(\Gamma_{n,k}) = 2(n-2)(n-4)!$ when $n$ is odd.

Finding the second eigenvalue of $\Gamma_{n,k}$, for arbitrary $k \leq n$, is an open problem posed by Siemons and Zalesski in [13]. In this paper, we find $\lambda_2(\Gamma_{n,k})$ when $k$ is relatively small. Our main result is the following.

**Theorem 1.2.** For any $n \geq 3$ and $k \in \mathbb{N}$ such that $2 \leq k \leq \min\left(n, 2\log_\frac{k}{e}\left(\frac{n(n-2)}{2e}\right) - 1\right)$, the second eigenvalue of $\Gamma_{n,k}$ is

$$
\lambda_2(\Gamma_{n,k}) = \frac{k-1}{n-1}\binom{n}{k}(n-k-1)!. 
$$

Moreover, $\lambda_2(\Gamma_{n,k})$ is afforded by the irreducible character of $\text{Sym}(n)$ that corresponds to the partition $[n-1,1]$.

It is well-known that a Cayley graph $\text{Cay}(G,C)$ is connected if and only if $\langle C \rangle = G$. Note that if the graph $\Gamma_{n,k}$ is connected then, $C(n,k)$ has to contain odd permutations. Indeed, if $C(n,k)$ only contains even permutations, then due to the fact that the product of even permutations is an even permutation, we have $\langle C(n,k) \rangle \leq \text{Alt}(n)$, where $\text{Alt}(n)$ is the alternating group on $\{1,2,\ldots,n\}$. In fact, one can easily check that the set $\{1,2,x\mid x \in \{3,4,\ldots,n\}\}$ is contained in $\langle C(n,k) \rangle$, which implies that $\langle C(n,k) \rangle = \text{Alt}(n)$. In general, if $X = \text{Cay}(\text{Sym}(n),C)$ is a normal Cayley graph, then it is straightforward that $\langle C \rangle \triangleleft \text{Sym}(n)$. Since $\text{Alt}(n)$ is the only minimal normal subgroup of $\text{Sym}(n)$, for $n \geq 5$, we have $\langle C \rangle \in \{\text{Alt}(n), \text{Sym}(n)\}$.

Consequently, if $n-k$ is odd, then the graph $\Gamma_{n,k}$ is disconnected and has two components. Since $\Gamma_{n,k}$ is vertex-transitive, these two components are isomorphic. Hence, the second eigenvalue of $\Gamma_{n,k}$ is equal to the second eigenvalue of $\text{Cay}(\text{Alt}(n),C(n,k))$, when $n-k$ is odd. The following result was proved by Huang and Huang [7].

**Theorem 1.3.** For any $n \geq 5$, the second eigenvalue of $\text{Cay}(\text{Alt}(n),C(n,n-3))$ is

$$
\frac{n(n-2)(n-4)}{3}.
$$

Therefore, the second eigenvalue of $\Gamma_{n,n-3}$ is $\frac{n(n-2)(n-4)}{3}$. It can be verified that this eigenvalue is also the one afforded by irreducible character corresponding to the partition $[n-1,1]$. 
Other results on the second eigenvalue of $\Gamma_{n,k}$, when $k$ is large are also known. For instance, Diaconis and Shahshahani [4] proved that $\lambda_2(\Gamma_{n,n-2}) = \left(\frac{2}{n}\right)$, which coincides with our result in Theorem 1.2 when $k = n - 2$. This eigenvalue corresponds to the irreducible character of the partition $[n-1,1]$. Moreover, Huang et al. [8] proved that $\lambda_2(\Gamma_{n,n-4}) = \frac{n(n-2)(n-3)(n-5)}{4}$, which is again the eigenvalue corresponding to the partition $[n-1,1]$. We believe that our main result, Theorem 1.2, also holds in the general case when $2 \leq k \leq n - 2$. For $n \in \{3,4,5,6,\ldots,21\}$ and $2 \leq k \leq n - 2$, an exhaustive search on Sagemath [14] for the second eigenvalue of $\Gamma_{n,k}$, showed that $\lambda_2(\Gamma_{n,k}) = \frac{k-1}{n-1}(n-1)!$. Due to these observations, we make the following conjecture.

**Conjecture 1.4.** For any $n \in \mathbb{N}$ and $2 \leq k \leq n - 2$, the second eigenvalue of $\Gamma_{n,k}$ is given by the irreducible character corresponding to $[n-1,1]$, and its value is

$$\lambda_2(\Gamma_{n,k}) = \frac{k-1}{n-1}\binom{n}{k}(n-k-1)!.$$

Our next result is about Conjecture 1.4, for the 5-cycles.

**Theorem 1.5.** For any $n \geq 7$, $\lambda_2(\Gamma_{n,n-5}) = \frac{n(n-2)(n-3)(n-4)(n-6)}{2}$. Moreover, this eigenvalue is afforded by the irreducible character corresponding to the partition $[n-1,1]$.

This paper is organized as follows. In Section 2.1, we prove Theorem 1.2. In our proof of Theorem 1.2, we use the representation theory of the symmetric group. In particular, our proof relies on the recursive Murnaghan-Nakayama rule (see [12]). In Section 2.2, we show that our method can also be applied to retrieve the result of Siemons and Zalesski in Theorem 1.1. Finally, we prove Theorem 1.5 in Section 3.

2. **Proof of the main results**

2.1. **Proof of Theorem 1.2.** Throughout this section, we let $n,k \in \mathbb{N}$ such that $2 \leq k \leq \min\left(n, 2\log_2\left(\frac{n(n-2)}{2e}\right) - 1\right)$. It is also assumed that the reader is familiar with the notion of Specht modules [12, Section 2.3], the Hook Length Formula [12, Section 3.10] and the Murnaghan-Nakayama Rule [12, Section 4.10].

We note that since the connection set of $\Gamma_{n,k}$ is a conjugacy class, the graph $\Gamma_{n,k}$ is a normal Cayley graph. It is well-known that the spectrum of a normal Cayley graph on a group $G$ can be determined by the irreducible characters of $G$ (see [1] and [9, Theorem 2.5] for more details).

Recall that a partition of the integer $n$ is a non-increasing sequence of positive integers summing to $n$. If $\lambda = [n_1, n_2, \ldots, n_t]$ is a partition of $n$, then we write $\lambda \vdash n$. It is well-known that there is a bijective correspondence between the irreducible representations of $\text{Sym}(n)$ and the set of all partitions of $n$. Each $\lambda \vdash n$ corresponds to an irreducible $\mathbb{C}\text{Sym}(n)$-module $S^\lambda$ (i.e., a complex irreducible representation of $\text{Sym}(n)$) called the $\lambda$-Specht module. Let $\chi^\lambda$ be the character afforded by $S^\lambda$. For any $\lambda \vdash n$, we denote the dimension of $S^\lambda$ by $f^\lambda$ (i.e., $f^\lambda = \chi^\lambda(id)$). If $\lambda \vdash n$ and $\sigma \in \text{Sym}(n)$ has cycle type $\tau$, then we define $\chi^\lambda_\sigma := \chi^\lambda(\sigma)$. Note that since $C(n,k)$ is the conjugacy class of the $(n-k)$-cycles, the cycle type of a permutation of $C(n,k)$ is $(n-k,1^k)$. 


Using the famous result of Babai [1], the eigenvalues of $\Gamma_{n,k}$ are given in the following lemma.

**Lemma 2.1.** For any $n, k \in \mathbb{N}$ such that $1 \leq k \leq n$, the eigenvalues of $\Gamma_{n,k}$ are the numbers

$$\xi_{\lambda} = \frac{\lambda^{(n-k,1^k)}}{f^\lambda} \binom{n}{k} (n-k-1)!,$$

for all $\lambda \vdash n$.

Given two Young diagrams $\lambda$ and $\mu$ such that $\mu \subset \lambda$, the skew diagram $\lambda/\mu$ is the set of cells of $\lambda$ that are not in $\mu$. A rim hook $\rho$ of a Young diagram $\lambda \vdash n$ is a skew diagram whose cells are on a path with only upward and rightward steps (see [12] for details on this). The length of $\rho$ denoted by $|\rho|$ is the number of its cells, and the leg-length $\ell(\rho)$ is the number of rows that $\rho$ spans minus 1. For any $\lambda \vdash n$, we let $\text{RH}_{n-k}(\lambda)$ be the set of all rim hooks of length $n-k$ of the partition $\lambda$. As a rim hook $\rho \in \text{RH}_{n-k}(\lambda)$ is a skew diagram, removal of the cells of $\lambda$ which are in $\rho$ results in a Young diagram that corresponds to a partition of $k$. We denote the Young diagram obtained from such removal by $\lambda \setminus \rho$. Using the recursive Murnaghan-Nakayama rule, we obtain the following lemma.

**Lemma 2.2.** For any $\lambda \vdash n$, the character value of the irreducible character $\chi_{\lambda}$ is

$$\chi_{\lambda} = \sum_{\rho \in \text{RH}_{n-k}(\lambda)} (-1)^{\ell(\rho)} f^{\lambda \setminus \rho}.$$ 

Proof. Let $\rho_1, \rho_2, \ldots, \rho_\ell$ be all the distinct rim hooks of length $n-k$ in $\lambda$. By the Murnaghan-Nakayama rule, we have

$$\chi_{\lambda} = \sum_{i=1}^\ell (-1)^{\ell(\rho_i)} \chi_{\lambda \setminus \rho_i} = \sum_{i=1}^\ell (-1)^{\ell(\rho_i)} f^{\lambda \setminus \rho_i}.$$

Our strategy is to use the fact that $k$ is relatively small so that there is at most one rim hook of length $n-k$ in any Young diagram $\lambda$. We recall the following lemma which is proved in [2]. The proof is given for completeness.

**Lemma 2.3.** Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_q] \vdash n$. Assume that $\lambda$ has a rim hook $\rho$ of length $n-k$ and $\mu = [\mu_1, \mu_2, \ldots, \mu_t] \vdash k$ is the Young diagram obtained from $\lambda$ by removing $\rho$. That is, $\mu = \lambda \setminus \rho$.

If $\ell$ is a positive integer such that $2\ell - n \geq k + 1$, then $\lambda$ has at most one rim hook of length $\ell$. In particular, if $3k + 1 < n$, then there is at most one rim hook of length $n-k$ in any Young diagram $\lambda \vdash n$.

Proof. First, we claim that if $\zeta$ is a rim hook of $\lambda$ which is contained in the skew diagram $\lambda/\lambda_1[1^{q-1}]$, then the length of $\zeta$ is at most $k$. Indeed, since $\zeta$ is contained in $\lambda/\lambda_1[1^{q-1}]$ and $\mu = \lambda \setminus \rho$, the rim hook $\zeta$ has at most $\mu_1$ columns and $t$ rows. In total, $\zeta$ has at most $\mu_1 + t - 1$ cells. Therefore, the length of $\zeta$ is at most $\mu_1 + t - 1 \leq k$.

As a consequence of this result, if a rim hook $\zeta$ has length $k' > k$, then $\zeta$ is not contained in $\lambda/\lambda_1[1^{q-1}]$. That is, $\zeta$ contains a cell of $\lambda$ of coordinate $(1,1)$ or $(1, \cdot)$. If $\zeta$
contains a cell of coordinate (.1), then it must start from the cell of coordinate (q, 1).
In other words, ζ is the unique rim hook of length k' starting at the cell of coordinate
(q, 1). Similarly, if ζ contains a cell of coordinate (1, .), then ζ must be the unique rim
hook of length k' ending at the cell of coordinate (1, λ1).

Finally, we prove that there is a unique rim hook of length ℓ in λ. Let γ and γ' be
two rim hooks in RH_{ℓ}(λ). It is easy to see that γ and γ' have common cells, otherwise
we would have 2ℓ ≤ n, which contradicts the fact that 2ℓ − n ≥ k + 1. Consider the
set of common cells of γ ∩ γ' of γ and γ'. By the claim that we proved above, the part of
the rim hook γ ∩ γ' which is contained in λ/[λ1, 1^{n−1}] has length at most k. Since
the length of the union of γ and γ' is 2ℓ − |γ ∩ γ'| = |γ| + |γ' − |γ ∩ γ'| ≤ n, we conclude
that 2ℓ − n ≤ |γ ∩ γ'|. Combining this with the hypothesis that 2ℓ − n ≥ k + 1, we
have |γ ∩ γ'| ≥ k + 1. In other words, γ and γ' has a common cell of coordinate (.1) or
(1, .). It follows that γ = γ'. The second statement of Lemma 2.3 is obtained by taking
ℓ = n − k.

By Lemma 2.3, there is at most one rim hook of length k in λ | n, whenever 2 ≤ k ≤ \left[\frac{2n}{3}\right]. Since k belongs to this interval (by the hypothesis of Theorem 1.2), the
non-zero eigenvalues of Γ_{n,k} are of the form

\[ ξ_{χ^λ} = (-1)^{ℓ(ρ)} \frac{f^{λ}_ρ}{f^λ} \binom{n}{k} (n − k − 1)!, \]  

where λ | n and ρ is the unique rim hook of length n − k of λ. We prove that the
maximum of \{ξ_{χ^λ}\}_{λ | n, λ \neq [n,1^{n}]} is attained by the partition [n − 1, 1]. The transpose
[2,1^{n−2}] of [n − 1, 1] also gives the second eigenvalue of Γ_{n,k}, depending on the parity
of n and k.

Now, we divide the proof of Theorem 1.2 into two cases: the cases k = 2 and k ≥ 3.

a) When k = 2

By Lemma 2.3, λ | n has at most one rim hook of length n − 2. If λ | n does not
have a rim hook of length n − 2, then the corresponding eigenvalue is equal to 0. If ρ is
the unique rim hook of length n − 2 of λ | n, then λ \ ρ ∈ \{[2,1^{n−2}]\}. Hence, the
dimension of the (λ \ ρ)-Specht module is always equal to 1. Consequently, the non-zero
eigenvalues of Γ_{n,2} are

\[ ξ_{χ^λ} = (-1)^{ℓ(ρ)} \frac{1}{f^λ} \binom{n}{k} (n − k − 1)!, \]

where λ | n has a rim hook of length n − 2. It is easy to see that the second eigenvalue
of Γ_{n,2} is obtained from the irreducible characters of smallest dimension which is not
equal to 1. When n ≥ 7, it is well-known that the only irreducible characters of Sym(n)
of degree less than n are those corresponding to the partitions [n], [1^{n}], [n − 1, 1] and
[2,1^{n−2}]. It is obvious that the partition giving the second eigenvalue (independent of
k) is [n − 1, 1] since the leg-length of a rim hook of length n − 2 on [n − 1, 1] is equal to
0. When 3 ≤ n ≤ 6, we use Sagemath [14] to verify that the second eigenvalue is given
by [n − 1, 1]. Therefore, λ_{2}(Γ_{n,2}) = \frac{1}{n−1} \binom{n−3}{2}.
b) When \( k \geq 3 \)

First, we recall a basic result about the largest dimension of the irreducible characters of a finite group. If \( G \) is a finite group, then we let \( b(G) \) be the largest dimension of an irreducible character of \( G \). We recall the following well-known result (see [6] for example).

**Proposition 2.4** ([6]). If \( G \) is a finite group, then \( b(G) \leq \sqrt{|G|} \).

Next, we present a lemma on the low dimensional irreducible characters of \( \text{Sym}(n) \).

**Lemma 2.5.** Let \( n \geq 19 \). If \( \phi \) is a character of \( \text{Sym}(n) \) of degree less than \( 3^{(n)} \) and \( \chi^\lambda \) a constituent of \( \phi \), then \( \lambda \) is one of \( \binom{n}{3} \), \( \binom{n}{2} \), \( \binom{n}{1} \), \( \binom{n-2}{2} \), \( \binom{n-3}{2} \), \( \binom{n-2}{1} \), \( \binom{n-3}{1} \), \( \binom{n-4}{1} \), \( \binom{n-2}{3} \), \( \binom{n-3}{2} \), \( \binom{n-4}{2} \), \( \binom{n-3}{3} \), or \( \binom{n-2}{5} \).

A proof of Lemma 2.5 can be found in [2, Lemma 3.4 and Remark 3.5].

Suppose that \( n \geq 19 \). It is easy to see that \((1)\) depends only on \((-1)^{\ell(\rho)} \frac{f^\lambda}{f^\rho}\). Using Proposition 2.4, if \( f^\lambda > 3^{(n)} \), then

\[
\frac{f^\lambda \rho}{f^\lambda} \leq \frac{b(\text{Sym}(k))}{3^{(n)}} \leq \frac{\sqrt{k!}}{3^{(n)}}.
\]

Using the classical bound \( \frac{kk^{k-1}}{e^{k-1}} \leq k! \leq \frac{kk^{k+1}}{e^{k+1}} \), it is easy to verify that when

\[
3 \leq k \leq \min \left( n, 2 \log \frac{e}{k} \left( \frac{n(a-2)}{2e} \right) - 1 \right), \quad \text{we have} \quad \sqrt{k!} \leq \frac{3^{(n)}}{n}.
\]

Consequently, if \( f^\lambda > 3^{(n)} \), then

\[
0 < \frac{f^\lambda \rho}{f^\lambda} \leq \frac{1}{n-1}.
\]  

(2)

Now, we compute the eigenvalues that correspond to the irreducible characters of \( \text{Sym}(n) \) with dimension less than \( 3^{(n)} \). The trivial character always affords the valency of \( \Gamma_{n,k} \), which is \( c_{n,k} := \binom{n}{k}(n-k-1)! \). We note that if \( \lambda \vdash n \) and \( \lambda' \) is the transpose of \( \lambda \), then \( \chi^{\lambda'}(\sigma) = \chi^{[1^n]}(\sigma) \chi^{\lambda}(\sigma) \), for any \( \sigma \in \text{Sym}(n) \). Therefore, the eigenvalue afforded by the partition \( [1^n] \) is either equal to \( c_{n,k} \) or \(-c_{n,k} \), in which case \( \Gamma_{n,k} \) is bipartite. The eigenvalue corresponding to \( [1^n] \) is precisely \((-1)^{n-k-1}c_{n,k} \). The eigenvalues of \( \Gamma_{n,k} \) corresponding to the other irreducible characters of dimension less than \( 3^{(n)} \) are given in Table 1. These values were computed using the Murnaghan-Nakayama rule and the Hook Length Formula.
An analysis of the eigenvalues in Table 1 shows that the largest eigenvalue is \( \frac{k-1}{n-1}c_{n,k} = \frac{k-1}{n-1} \binom{n}{k} (n-k-1)! \), when \( n \geq 19 \) and \( 3 \leq k \leq n-2 \). Moreover, this eigenvalue is afforded by the irreducible character of Sym(\( n \)) corresponding to the partition \([n-1, 1]\).

We conclude that \( \lambda_2(\Gamma_{n,k}) = \frac{k-1}{n-1}c_{n,k} \), when \( n \geq 19 \) and \( 3 \leq k \leq \min \left( n, 2 \log_2 \frac{n(n-2)}{2e} \right) - 1 \).

For the cases where \( 3 \leq n \leq 18 \) and \( 3 \leq k \leq n-2 \), we use Sagemath [14] to verify that the second eigenvalue is given by the partition \([n-1, 1]\) and it is equal to \( \lambda_2(\Gamma_{n,k}) = \frac{k-1}{n-1}c_{n,k} \). This completes the proof of Theorem 1.2.

2.2. Proof of Theorem 1.1. In this section, we use Table 1 to retrieve the results of Siemons and Zalesski in Theorem 1.1.

The case \( k = 0 \). The eigenvalue afforded by the irreducible character corresponding to \([n-1, 1]\) is \( \frac{-c_{n,0}}{n-1} < 0 \). When \( n \) is odd, it is easy to see that the largest eigenvalue in Table 1 is \( \binom{n-1}{2}^{-1} c_{n,0} = 2(n-3)! \), which is afforded by \([n-2, 1^2]\). When \( n \) is even, the largest eigenvalue is \( \frac{c_{n,0}}{n-1} = (n-2)! \), which is afforded by \([2, 1^{n-2}]\).

The case \( k = 1 \). The eigenvalues afforded by \([n-1, 1]\) and \([2, 1^{n-2}]\) are both equal to 0 in this case. When \( n \) is odd, the second eigenvalue is afforded by \([2^2, 1^{n-4}]\) and is equal to \( (-1)^{n-1} \binom{n-1}{2}^{-1} c_{n,1} = 2(n-2)(n-4)! \). When \( n \) is even, the eigenvalue afforded by \([2^2, 1^{n-4}]\)
is negative and the second eigenvalue of $\Gamma_{n,k}$ is equal to $-\frac{(\begin{pmatrix}-1 & -3 \\
n-2 & n-4 \end{pmatrix})n_{1}}{n(n-2)(n-4)} = 3(n-3)(n-5)!$, which is afforded by $[n - 3, 2, 1]$.

3. SECOND EIGENVALUE OF $\Gamma_{n,n-5}$

In this section, we prove Theorem 1.5.

3.1. Equitable partitions. In this subsection, we show that the eigenvalue afforded by the irreducible character corresponding to the partition $[n - 1, 1]$ appears as an eigenvalue of an equitable partition of $\Gamma_{n,k}$.

The following lemma is straightforward and is given without a proof (see [8, Lemma 5]).

**Lemma 3.1.** Let $G$ be a finite group and $H \leq G$. If $X = \text{Cay}(G,C)$ is a Cayley graph of $G$, then the partition of $X$ into left cosets of $H$ is an equitable partition of $X$.

Let $G := \text{Sym}(n)$ and for any $i \in \{1, 2, \ldots, n\}$, let $G_i$ be the stabilizer of $i$ in the natural action of $G$ on the set $\{1, 2, \ldots, n\}$. For any $s, t \in \{1, 2, \ldots, n\}$, we define $G_{t,s} := \{\sigma \in G \mid \sigma(t) = s\}$. By Lemma 3.1, the partition $G/G_i$ is equitable for any $i \in \{1, 2, \ldots, n\}$. If $B_{\Pi_i} = (b_{s,t})_{s,t \in \{1, 2, \ldots, n\}}$ is the quotient matrix corresponding to the equitable partition $\Pi_i$ of $\Gamma_{n,k}$ given by $G/G_i$, then by [8]

$$b_{s,t} = |C(n, k) \cap G_{t,s}| = \begin{cases} \frac{n-1}{n-k}(n-k-1)! & \text{if } t = s \\ \frac{n-2}{n-k-2}(n-k-2)! & \text{if } t \neq s. \end{cases}$$

It is not hard to see that

$$B_{\Pi_i} = \binom{n-2}{n-k-2}(n-k-2)!(J - I) + \binom{n-1}{n-k}(n-k-1)!I,$$

where $J$ is the $n \times n$ all ones matrix and $I$ is the $n \times n$ identity matrix. The eigenvalues of $B_{\Pi_i}$ are

$$\binom{n}{k}(n-k-1)! \text{ and } \frac{k-1}{n-1}\binom{n}{k}(n-k-1)!,$$

which are the eigenvalues afforded by the irreducible characters corresponding to $[n]$ and $[n-1, 1]$, respectively.

3.2. A recursive method. In [8], Huang et al. gave a recursive method to compute the second eigenvalue of highly transitive groups.

Throughout this subsection, we let $G \leq \text{Sym}(\Omega)$ be a finite transitive group acting on $\Omega = \{1, 2, \ldots, n\}$. Let $G^{(0)} = G$ and for any $k \geq 1$, let

$$G^{(k)} = G_n \cap G_{n-1} \cap \ldots \cap G_{n-k+1}.$$ 

Let $T$ be a union of conjugacy classes of $G$. Define

$$\begin{cases} T_k = T_{k-1} \setminus (T_{k-1} \cap G_k), \text{ for any } k \geq 1, \\ T_0 = T. \end{cases}$$
In other words, if \( \text{Supp}(\sigma) = \{i \in \Omega \mid \sigma(i) \neq i\} \), then \( T_k = \{\sigma \in T \mid \{1, 2, \ldots, k\} \subset \text{supp}(\sigma)\} \). For any \( i \geq 0 \) and \( k \geq 0 \), let

\[
X_{k,i} = \text{Cay}(G^{(i)}, T_k \cap G^{(i)}).
\]

Let \( \Pi \) be the partition of \( G \) into left cosets of any stabilizer of a point of \( G \). From the results in the previous subsection, we know that this partition is equitable. Let \( B_\Pi \) be the quotient matrix corresponding to this equitable partition. By Lemma 3.1, the partition of \( G^{(i)} \) into left cosets of any of its point-stabilizers is an equitable partition of \( X_{k,i} \), for any \( k \geq 0 \). Let \( B_\Pi^{(k,i)} \) be the quotient matrix of this equitable partition. The second eigenvalue \( \lambda_2 \left( B_\Pi^{(k,i)} \right) \) was computed in \([8]\) and it is equal to

\[
\lambda_2 \left( B_\Pi^{(k,i)} \right) = |T_k \cap G^{(i)} \cap G_{k+1}| - |T_k \cap G^{(i)} \cap G_{k+2,k+1}|.
\]

When \( G \) is highly transitive, the second eigenvalue of the normal Cayley graph \( \text{Cay}(G, T) \) can be computed via a recursive method on the graphs defined in \((3)\).

**Lemma 3.2.** \([8, \text{Theorem 14}]\) Let \( m = \max_{\sigma \in T} |\text{supp}(\sigma)| \). If \( G \) is \((m + a)\)-transitive for some \( a \geq 1 \) and \( \lambda_2(X_{k,a-1}) = \lambda_2 \left( B_\Pi^{(k,a-1)} \right) \) for any \( 0 \leq k \leq m - 1 \), then

\[
\lambda_2(\text{Cay}(G, T)) = \lambda_2(X_{0,0}) = \lambda_2(B_\Pi).
\]

In the next subsection, we find the second eigenvalue of \( \Gamma_{n,n-5} \) using this recursive method.

### 3.3. Proof of Theorem 1.5

Let \( T = C(n, n - 5) \) be the conjugacy class of 5-cycles of \( \text{Sym}(n) \). Our main tool to prove Theorem 1.5 is Lemma 3.2. Since \( \langle T \rangle = \text{Alt}(n) \), the graph \( \Gamma_{n,n-5} \) is disconnected and is the disjoint union of two copies of \( \text{Cay} (\text{Alt}(n), T) \). Therefore, \( \lambda_2(\Gamma_{n,n-5}) = \lambda_2(\text{Cay}(\text{Alt}(n), T)) \). Let us apply Lemma 3.2 on \( X = \text{Cay}(\text{Alt}(n), T) \).

As the elements of \( T \) are 5-cycles, \( m = \max_{\sigma \in T} |\text{supp}(\sigma)| = 5 \). Since \( \text{Alt}(n) \) is \((n - 2)\)-transitive, it is easy to see that \( \text{Alt}(n) \) is \((m + a)\)-transitive, whenever \( a \leq n - 7 \). Let \( a = n - 7 \) and \( G = \text{Alt}(n) \). Now, it is enough to verify that

\[
\lambda_2(X_{k,n-8}) = |T_k \cap G^{(n-8)} \cap G_{k+1}| - |T_k \cap G^{(n-8)} \cap G_{k+2,k+1}|,
\]

for any \( 0 \leq k \leq 4 \).

Note that \( G^{(n-8)} = \text{Alt}(8) \) and for any \( 0 \leq k \leq 4 \), \( X_{k,n-8} = \text{Cay}(\text{Alt}(8), T_k \cap \text{Alt}(8)) \). Using Sagemath \([14]\) and SciPy \([15]\), we were able to verify \((4)\) for any \( 0 \leq k \leq 4 \). We have compiled in the following table the values of \( \lambda_2(X_{k,n-8}) \) and \( \lambda_1(X_{k,n-8}) \).

| \( k \) | \( \lambda_1(\text{Cay}(\text{Alt}(8), T_k \cap \text{Alt}(8))) \) | \( \lambda_2(\text{Cay}(\text{Alt}(8), T_k \cap \text{Alt}(8))) \) |
|---|---|---|
| 0 | 1344 | 384 |
| 1 | 840 | 300 |
| 2 | 480 | 216 |
| 3 | 240 | 138 |
| 4 | 96 | 72 |
By Lemma 3.2, we conclude that $\lambda_2(\Gamma_{n,n-5}) = \frac{n-6}{n-1} \binom{n}{5} 4! = \frac{n(n-2)(n-3)(n-4)(n-6)}{5}$. 

REFERENCES

[1] L. Babai. Spectra of Cayley graphs. *Journal of Combinatorial Theory, Series B*, 27(2):180–189, 1979.

[2] A. Behajaina, R. Maleki, A. T. Rasoamanana, and A. S. Razafimahatratra. 3-setwise intersecting families of the symmetric group. *Discrete Mathematics*, 344(8):112467, 2021.

[3] D. Cvetković and S. Simić. The second largest eigenvalue of a graph (a survey). *Filomat*, pages 449–472, 1995.

[4] P. Diaconis and M. Shahshahani. Generating a random permutation with random transpositions. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 57(2):159–179, 1981.

[5] C. Godsil and K. Meagher. *Erdős-Ko-Rado Theorems: Algebraic Approaches*. Cambridge University Press, 2016.

[6] Z. Halasi, C. Hannusch, and H. Nguyen. The largest character degrees of the symmetric and alternating groups. *Proceedings of the American Mathematical Society*, 144(5):1947–1960, 2016.

[7] X. Huang and Q. Huang. The second largest eigenvalues of some Cayley graphs on alternating groups. *Journal of Algebraic Combinatorics*, 50(1):99–111, 2019.

[8] X. Huang, Q. Huang, and S. M. Cioabă. The second eigenvalue of some normal Cayley graphs of highly transitive groups. *The Electronic Journal of Combinatorics*, 26(2), 2019.

[9] X. Liu and S. Zhou. Eigenvalues of Cayley graphs. *arXiv preprint arXiv:1809.09829*, 2018.

[10] A. Neumaier. The second largest eigenvalue of a tree. *Linear Algebra and its Applications*, 46:9–25, 1982.

[11] A. Nilli. On the second eigenvalue of a graph. *Discrete Mathematics*, 91(2):207–210, 1991.

[12] B. E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions (Graduate Texts in Mathematics)*. New York: Springer, 2001.

[13] J. Siemons and A. Zalesski. On the second largest eigenvalue of some Cayley graphs of the symmetric group. *arXiv preprint arXiv:2012.12460*, 2020.

[14] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.9)*, 2020. https://www.sagemath.org.

[15] P. Virtanen et al. SciPy 1.0: fundamental algorithms for scientific computing in Python. *Nature methods*, 17(3):261–272, 2020.

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