Nuclides as a liquid phase of $SU(2)_L \times SU(2)_R$ chiral perturbation theory
I: emergence of pion-less $SU(2)_\chi PT$

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Abstract

The Standard Model of particle physics (SM), augmented with neutrino mixing, is either the complete theory of interactions of known particles at energies accessible to Nature on Earth, or very nearly so. Candidate effective theories of nuclear structure must therefore reflect SM symmetries, especially the chiral global $SU(2)_L \times SU(2)_R$ symmetry of two-massless-quark QCD. For ground-state nuclei, $SU(2)_\chi PT$ enables perturbation/truncation in inverse powers of $\Lambda_{\chi SB} \approx 1\text{GeV}$, with analytic operators renormalized to all loop orders. We show that $SU(2)_\chi PT$ of protons, neutrons and 3 Nambu-Goldstone boson (NGB) pions admits a semi-classical “liquid” phase, with energy required to increase or decrease the density of constituents.

We show that “Pion-less" $SU(2)_\chi PT$ emerges in the chiral liquid: far-infrared NGB pions decouple from “Static Chiral Nucleon Liquids (Static$\chi NL$),” vastly simplifying the derivation of saturated nuclear matter (the infinite liquid phase) and of finite microscopic liquid drops (ground-state nuclides). Static$\chi NL$s explain the power of pion-less $SU(2)_\chi PT$ to capture experimental ground-state properties of certain nuclides, tracing that (no-longer-mysterious) empirical success directly to the global symmetries of two-massless-quark QCD. Static$\chi NL$ are made entirely of

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nucleons. They have even parity; total spin zero; even proton number $Z$, and neutron number $N$; and are arranged so local expectation values for spin and momenta vanish.

We derive the Static-$\chi$NL effective $SU(2)\chi PT$ Lagrangian, including all order $\Lambda_{\chi SB}^{-1}$ operators. These include: all 4-nucleon operators that survive Fierz rearrangement in the non-relativistic limit, including operators that vanish for the non-relativistic $SU(2)\chi PT$ deuteron; effective Lorentz-vector iso-vector neutral “$\rho$-exchange” operators crucial to $Z\neq N$ asymmetry effects. Static-$\chi$NL motivate nuclear matter, seen as non-topological solitons at zero internal and external pressure: the Nuclear Liquid Drop Model and Bethe-Weizsäcker Semi-Empirical Mass Formula emerge in an explicit Thomas-Fermi construction provided in the companion paper. For chosen nuclides, nuclear Density Functional and Skyrme models are justified to order $\Lambda_{\chi SB}^{-1}$. We conjecture that inclusion of $\Lambda_{\chi SB}^{-2}$ and $\Lambda_{\chi SB}^{-3}$ operators will result in accurate ”natural” Skyrme, No-Core-Shell, and ordinary neutron star models, with approximate liquid structure.

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1 Introduction

In the Standard Model (SM) of particle physics, Quantum Chromodynamics (QCD) describes the strong interactions among quarks and gluons. At low energies, quarks and gluons are confined inside hadrons, concealing their degrees of freedom in such a way that we must employ an effective field theory (EFT) of hadrons. In doing so, we acknowledge as a starting point a still-mysterious experimental fact: Nature first makes hadrons and then assembles nuclei from them [1, 2, 3, 4].

Since nuclei are made of hadrons, the fundamental challenge of nuclear physics is to identify the correct EFT of hadrons and use it to characterize all nuclear physics observations. Many such EFTs have been considered [5, 6, 7, 8, 9]. Ultimately, the correct choice will both match the observations and be derivable from the SM, i.e. QCD.

Chiral perturbation theory ($\chi PT$) [10, 11, 12, 13, 14, 15] is a low-energy perturbative approach to identifying the operators in the EFT that are allowed by the global symmetries of the SM. It builds on the observation that the up ($m_{up} \approx 6MeV$) and down ($m_{down} \approx 12MeV$) quarks, as well as the 3 pions ($\pi^+\pi^0\pi^-$)–which are pseudo Nambu-Goldstone bosons–are all nearly massless compared to the other energy scales ($\Lambda_{\chi SB} \approx 1GeV$) in low-energy hadronic physics.

The effective-Lagrangian power counting [16] of $SU(2)_L \times SU(2)_R$ chiral perturbation theory ($SU(2)\chi PT$) incorporates all analytic higher-order quantum-loop corrections into tree-level amplitudes. The resultant perturbation expansion in the inverse of the chiral-symmetry-breaking scale $\Lambda_{\chi SB}^{-1}$ renders $SU(2)\chi PT$’s strong-interaction predictions calculable in practice. Its
low-energy dynamics of a proton-neutron nucleon doublet and three pions as a pseudo-Nambu-Goldstone boson (pseudo-NGB) triplet are our best understanding, together with lattice QCD, of the experimentally observed low-energy dynamics of QCD strong interactions. This understanding encompasses: pseudo-Nambu-Goldstone-boson (NGB) masses, soft-pion scattering, the applicability of $SU(2)_{L+R} \times SU(2)_{L-R}$ current algebra, the conserved vector current (CVC) and partially conserved axial-vector current (PCAC) hypothesis, semi-leptonic $\pi$ decay, leptonic $\pi$ decay, semi-leptonic nucleon decay, second class currents, nucleon axial-vector couplings, the Goldberger-Treiman relation, nuclear beta decay (e.g. $^{14}O \rightarrow ^{14}Ne + \nu_e$), precise measurement of Cabibbo angle, et cetera.

$SU(2)\chi PT$'s effective-field-theoretic predictive power \cite{10, 12, 13, 14, 15, 16, 17, 18, 19} derives from its ability to control its analytic quantum loops by power counting in $\frac{1}{\Lambda_{\chi SB}}$, thus maintaining a well-ordered low-energy perturbation expansion that can be truncated. This predictive power stands in stark contrast with theories of strong interactions that lose their field-theoretic predictive power. These include any model of light or heavy nuclei not demonstrably derivable from the Standard Model \cite{20}, such as theories of quark bags and other confinement models of hadronic structure \cite{21, 22} strange quark matter (and strange quark stars) \cite{23, 24, 25, 26, 27} and multi-Skyrmions in chiral pseudo-Goldstone symmetry \cite{28, 29, 30}.

In contrast, QCD lattice-gauge-theory calculations of quarks and gluons \cite{31, 32, 33} control their quantum loops, and we may hope that the detailed properties of the deuteron, the alpha particle, and maybe even heavy nuclei, may someday be directly calculated in lattice QCD.

Triumphant in claiming a role in nuclear physics, $SU(2)\chi PT$ of dynamical nucleons and pions has been demonstrated \cite{34, 35} to explain, to high accuracy, the detailed structure of the deuteron.

B.W. Lynn \cite{36} first introduced the idea that $SU(2)\chi PT$ could also admit a liquid phase: “It is legitimate to inquire whether the effective (power-counting) Lagrangian (A.13) ... contains a liquid phase. An ‘$SU(2)_{L} \times SU(2)_{R}$ chiral liquid’ is defined as a statistically significant number of baryons interacting via chiral operators ... with an almost constant (saturated) density ... (which) can survive as localized ‘(liquid) drops’ at zero external pressure”. Lynn’s Lagrangian included $SU(2)\chi PT$ terms of $O(\Lambda_{\chi SB})$ and $O(\Lambda_{\chi SB}^0)$ ignoring electromagnetic breaking. Anticipating the Static Chiral Nucleon Liquids studied here, he argued that, in the exact chiral limit, nucleons in the liquid phase interact with each other only via the contact terms (20). He did not derive pion-less $SU(2)\chi PT$. The study of chiral liquids in \cite{36} focused on those explicit chiral symmetry breaking terms whose origin lies entirely in non-zero light quark masses $m_{up}, m_{down} \neq 0$. (The $m = 0, l = 1, n = 1$ contributions in (A.13).) Here, we focus our study of chiral liquids instead on the $n = 0$ chiral limit, and prove the emergence of Pionless $SU(2)\chi PT$ in that chiral limit.
1.1 The fatal flaw in nuclear liquid drop models not based on $SU(2) \chi PT$

T.D. Lee and G.C. Wick [37] first identified non-topological solitons with the ground state of heavy-nuclei, as well as possible super-heavy nuclei, thus making the crucial connection to nuclear liquids. Mathematically, such non-topological solitons emerge as a species of fermion Q-Ball [38, 39, 40, 41, 42, 43], or non-topological soliton [44, 37, 45, 46, 47, 48, 49, 50, 51, 52]. A practical goal would be to identify nuclear non-topological solitons with the ground state of ordinary even-even spin-zero spherically symmetric heavy nuclei, such as $^{20}_{40}$Ca, $^{50}_{90}$Zr, and $^{126}_{82}$Pb.

Nuclear non-topological solitons identified as nuclear liquids became popular with the ingenious work of Chin and Walecka [53] carried forward by [54]. Nuclear “Walecka models” [55, 56, 57] contain four dynamical particles: protons, neutrons, the Lorentz-scalar iso-scalar $\sigma$, and the Lorentz-vector iso-scalar $\omega_\mu$. Nucleons are treated as locally free-particles in Thomas-Fermi approximation. Finite-width nuclear surfaces are generated by dynamical attractive $\sigma$-particle exchange, allowing them to exist at zero external pressure. The empirical success of Walecka models is based on balancing $\sigma$ boson-exchange attraction against $\omega_\mu$-boson-exchange repulsion. That that balance must be fine-tuned remains a famous mystery of the structure of the Walecka ground state. In the absence of long-ranged electromagnetic forces, infinite symmetric ($Z = N$) nuclear matter, as well as finite microscopic ground-state ($Z = N$) nuclides, appear as symmetric nuclear liquid drops.

Both T.D. Lee’s and J.D. Walecka’s nuclear non-topological solitons are to be classified as “liquids” because:

- they have no crystalline or other “solid” structure;
- it costs energy to either increase or decrease the density of the constituent nucleons compared to an optimum value;
- they survive at zero external pressure, e.g. in the absence of gravity, so they are not a “gas.”

Despite their successes, there is a fatal flaw in all such current non-topological nuclear models, and in all nuclear models not based on $SU(2) \chi PT$. To see this,

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1 In practice, the $\omega_\mu$ is best treated as a very heavy non-dynamical auxiliary field and integrated out of the theory, but, in order to be able to properly discuss the renormalizability of the Walecka model, we won’t do so here.
examine the renormalizable\(^2\) tree-level Walecka Lagrangian:

\[
\begin{align*}
L_{\text{Walecka}} &= L_{\text{Nucleons}} + L_{\text{Walecka}}^\sigma + L_{\text{Walecka}}^\omega \\
L_{\text{Nucleons}} &= \bar{N} \left( i \gamma^\mu (\partial_\mu + \omega_\mu) - m_N + g_\sigma \sigma \right) N \\
L_{\text{Walecka}}^\sigma &= \frac{1}{2} (\partial_\sigma \sigma)^2 - V(\sigma); \quad V(\sigma) = \frac{1}{2} m_\sigma^2 \sigma^2 + \frac{1}{4} \lambda_\sigma \sigma^4 \\
L_{\text{Walecka}}^\omega &= -\frac{1}{4} \left[ \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \right] \left[ \partial^\mu \omega^\nu - \partial^\nu \omega^\mu \right] + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu
\end{align*}
\]

where \(\frac{g_\sigma}{m_\sigma} = 284.3 \text{GeV}^{-2}\) and \(\frac{g_\omega}{m_\omega} = 208.8 \text{GeV}^{-2}\) are fit to the experimentally inferred values of the number density \((k_{\text{Fermi}} \approx 1.42/fm)\) and saturated volume energy \((E_{\text{binding/nucleon}} \approx 16 \text{MeV})\) of infinite symmetric \(Z = N\) nuclear matter – taken to be the interior of \(^{208}\text{Pb}\) – neglecting Coulomb and isospin effects.

The fatal flaw manifests when treating (1) as a quantum field theory beyond tree level. Inclusion of 1-loop quantum corrections will strongly renormalize the values of \(m_N, g_\sigma, m_\sigma^2, \lambda_\sigma, g_\omega,\) and \(m_\omega^2\), and induce higher-order terms \(\sim \sigma^6, \sigma^{32}, \sigma^{784}, \ldots\) with coefficients that depend on those parameters. We can dutifully re-fit (e.g. via Coleman-Weinberg) the 1-loop parameters to symmetric nuclear matter, including nuclear surface terms and compressibility, which now also depend on those new higher-power \(\sigma\) interactions. Next include 2-loop strong-interactions and re-fit. Because these are strong hadronic interactions, 2-loop effects will be just as large as 1-loop effects, and cannot be truncated. Now include 3,4,5,..., 283 quantum loops (which are all required in any quantum field theory of strong hadronic interactions) and re-fit. Not only is such a program impossible in practice but, much worse, all the nuclear predictive power of the Walecka model has been completely lost!

This paper will cure those problems by strict compliance with the requirements of \(SU(2) \chi PT\) effective field theory of protons, neutrons and pions. The static chiral nucleon liquids (Static\(\chi\)NL) studied below are true solutions to \(SU(2) \chi PT\). They include renormalized all-loop-orders analytic quantum corrections, are dependent on just a few experimentally measurable chiral coefficients, and restore theoretical predictive power over nuclides.

## 2 The emergence of pion-less Static\(\chi\)NL

We recall the \(SU(2) \chi PT\) Lagrangian\(^2\) to order \(\Lambda_{\chi SB}\) and \((\Lambda_{\chi SB})^0\) in the chiral limit.

\(^2\) Imagine \(m_{\pi_0}^2\) arising from a spontaneously broken \(U(1)\) gauge theory.

\(^3\) Important Infra-Red non-analytic terms in the pion sector are included in Appendix A
\[ L_{\chi PT}^{\text{Symmetric}} = L_{\chi PT}^{\pi} + L_{\chi PT}^{N} + L_{\chi PT}^{A-N} \]

\[ L_{\chi PT}^{\pi} = \frac{f_{\pi}^2}{4} \text{Tr} \partial_{\mu} \Sigma \bar{\partial} \Sigma + L_{\chi PT}^{\pi,\text{Non-Analytic}} \]  

\[ L_{\chi PT}^{N} = \overline{N} \left( i \gamma^\mu (\partial_{\mu} V_{\mu}) - m_{N} 1 \right) N - g_{A} \gamma_{\mu} \gamma^{5} A_{\mu} N \]

\[ L_{\chi PT}^{A-N} = C_{A} \frac{1}{2 f_{\pi}^2} (\bar{N} \gamma_{\sigma} N) (\bar{N} \gamma_{\sigma} N) + ++, \]

with fermion bi-linear currents

\[ \bar{J}^\mu = \overline{N} \gamma^\mu N; \quad \bar{J}^{\mu 5} = \overline{N} \gamma^\mu \gamma^5 N \]

\[ V_{\mu} = \vec{V}_{\mu}; \quad \vec{V}_{\mu} = 2i \left( \frac{\sin(\frac{\pi}{f_{\pi}})}{\frac{\pi}{f_{\pi}}} \right)^2 \vec{r} \times (\partial_{\mu} \vec{r}) \]

\[ A_{\mu} = \vec{A}_{\mu}; \quad \vec{A}_{\mu} = -\frac{2}{\pi^2} \left[ \frac{\cos(\frac{\pi}{f_{\pi}}) \sin(\frac{\pi}{f_{\pi}})}{\frac{\pi}{f_{\pi}}} \vec{r} \times (\partial_{\mu} \vec{r}) + \frac{\cos(\frac{\pi}{f_{\pi}}) \sin(\frac{\pi}{f_{\pi}})}{\frac{\pi}{f_{\pi}}} \vec{r} \cdot (\partial_{\mu} \vec{r}) \right] \]

where \( \pi = |\vec{r}| = \sqrt{\vec{r}^2} \).

The parentheses in the four-nucleon Lagrangian indicate the order of SU(2) index contraction, and ++ indicates that one should include all possible combinations of such contractions. As usual, \( \gamma^{\sigma} \equiv (1, \gamma^\mu, i \sigma_{\mu\nu}, i \gamma^\mu \gamma^5, \gamma^5) \), for \( \sigma_{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu] \). These are commonly referred to as scalar (S), vector (V), tensor (T), axial-vector (A), and pseudo-scalar (P) respectively. \( C_{\sigma} \) are a set of chiral constants.

In the chiral limit, where \( \pi_{\sigma} \)s are massless, the presence of quantum nucleon sources could allow the massless NGB to build up, with tree-level interactions only, a non-linear quantum pion cloud. If we minimize the resultant action with respect to variations in the pion field, the equations of motion\(^4\) capture the part of the quantum cloud that is to be characterized as a classical soft-pion field, thus giving us the pion ground-state (and content/configuration/structure) in the presence of the ground-state “Chiral Nucleon Liquid” \( \chi NL \) with fixed baryon

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\(^4\) This is a chiral limit \( SU(2) \chi PT \) analogue of QED where, in the presence of quantum lepton sources, a specific superposition of massless Infra-Red photons builds up into a classical electromagnetic field. Important examples are the “exponentiation” of IR photons in \( e^+ e^- \rightarrow \mu^+ \mu^- \) asymmetries, and \( e^+ e^- \rightarrow e^+ e^- \) Bhabha scattering, at LEP1. Understanding the classical fields generated by initial-state and final-state soft photon radiation \([58, 59]\) is crucial to dis-entangling high precision electro-weak loop effects, such as the experimentally confirmed precise Standard Model predictions for the top-quark \([60]\) and Higgs’ \([60, 61]\) masses.
number $A = Z + N$

\[
0 = \left[ \frac{\partial}{\partial (\rho, \pi^m)} - \frac{\partial}{\partial \pi^m} \right] L^{\text{Symmetric}}_{\chi^{\text{PT}}} \]

\[
= \left[ \frac{\partial}{\partial (\rho, \pi^m)} - \frac{\partial}{\partial \pi^m} \right] L^{\text{Symmetric}}_{\chi^{\text{PT}}} \\
+ i \tilde{J}^\mu : \left[ \frac{\partial}{\partial (\rho, \pi^m)} - \frac{\partial}{\partial \pi^m} \right] \tilde{V}_\mu - g_A \tilde{J}^{\mu, 5} : \left[ \frac{\partial}{\partial (\rho, \pi^m)} - \frac{\partial}{\partial \pi^m} \right] \tilde{A}_\mu \\
- 2 \partial_\mu \tilde{J}^\mu \left( \frac{\sin(\frac{\pi}{2j\pi})}{\left( \frac{\pi}{2j\pi} \right)} \right)^2 (\vec{\pi} \times \vec{m}) \\
+ \frac{2}{\pi^2} g_A \partial_\mu \tilde{J}^{\mu, 5} : \tilde{\pi} (\vec{\pi} \cdot \vec{m}) + \frac{\cos(\frac{\pi}{2j\pi}) \sin(\frac{\pi}{2j\pi})}{\left( \frac{\pi}{2j\pi} \right)} \tilde{\pi} \times (\vec{m} \times \vec{\pi}) \right]
\]

We divide the classical pion field into “Infra-Red” and “Non-IR” parts. By definition, only “IR” pions survive the internal projection operators associated with taking expectation values of the classical NGB $\vec{\pi}$s in the $|\chi_{\text{NL}}\rangle$ quantum state

\[
\langle \chi_{\text{NL}} | \text{Function}(\partial_\mu \vec{\pi}, \vec{\pi}) | \chi_{\text{NL}} \rangle = \langle \chi_{\text{NL}} | \text{IRPartOf} \left[ \text{Function}(\partial_\mu \vec{\pi}, \vec{\pi}) \right] | \chi_{\text{NL}} \rangle \\
\equiv \langle \text{Function}(\partial_\mu \vec{\pi}, \vec{\pi}) \rangle_{\text{IR}} \\
0 = \langle \chi_{\text{NL}} | \text{Non-IRPartOf} \left[ \text{Function}(\partial_\mu \vec{\pi}, \vec{\pi}) \right] | \chi_{\text{NL}} \rangle
\]

The IR part does not change the $\chi_{\text{NL}}$. It could in principle be an important part of the $\chi_{\text{NL}}$: a $\vec{\pi}$ condensate, a giant resonance, a breathing mode, a time-dependent flashing-pion mode. To ignore such classical IR $\vec{\pi}$s would therefore be an incorrect definition of $\chi_{\text{NL}}$. For finite $\chi_{\text{NL}}$, it could be just a passing pion (of any frequency) which simply does not strike the $\chi_{\text{NL}}$.

We call these “IR pions” by keeping in mind a simple picture, where the $\vec{\pi}$ wavelength is “long”, i.e. longer than the scale within the $\chi_{\text{NL}}$ over which the local mean values of nucleon spin and momentum vanish. Only “IR” pions survive the internal projection operators associated with taking expectation values of the classical NGB $\vec{\pi}$s in the $|\chi_{\text{NL}}\rangle$ quantum state

We now take expectation values of the $\vec{\pi}$ equations of motion. In the presence
of the quantum $\chi_{NL}$ source, the classical NGB $\pi$ cloud obeys

$$
0 = \left( \chi_{NL} \right| \left[ \partial_\nu \frac{\partial}{\partial (\partial_\nu \pi^m)} - \frac{\partial}{\partial (\partial_\nu \pi^m)} \right] L^{\pi; Symmetric}_\chi | \chi_{NL} \rangle \right.
$$

$$
= \left[ \left[ \partial_\nu \frac{\partial}{\partial (\partial_\nu \pi^m)} - \frac{\partial}{\partial (\partial_\nu \pi^m)} \right] L^{\pi; Symmetric}_\chi \right]_{IR}
$$

$$
+ i \left( \chi_{NL} | \vec{p} | \chi_{NL} \right) \cdot \left[ \left[ \partial_\nu \frac{\partial}{\partial (\partial_\nu \pi^m)} - \frac{\partial}{\partial (\partial_\nu \pi^m)} \right] \vec{v}_\nu \right]_{IR}
$$

$$
- g_A \left( \chi_{NL} | \vec{p} \mu | \chi_{NL} \right) \cdot \left[ \left[ \partial_\nu \frac{\partial}{\partial (\partial_\nu \pi^m)} - \frac{\partial}{\partial (\partial_\nu \pi^m)} \right] \vec{A}_\mu \right]_{IR}
$$

$$
- 2 \left( \chi_{NL} | \partial_\mu \vec{p} | \chi_{NL} \right) \cdot \left\{ \left( \frac{\sin(\frac{\pi}{2\Gamma})}{(\frac{\pi}{2\Gamma})} \right)^2 \vec{\pi} \times \hat{m} \right\}_{IR}
$$

$$
+ \frac{2}{\pi} g_A \left( \chi_{NL} | \partial_\mu \vec{p} \mu | \chi_{NL} \right) \cdot \left\{ \vec{\pi} \cdot \hat{m} \right\}_{IR}
$$

Examining the ground-state expectation values of the nucleon currents and their divergences in $[13]$, we find that almost all of them vanish:

$$
\left( \chi_{NL} | J_\mu^\pm | \chi_{NL} \right) = 0, \quad \left( \chi_{NL} | J_\mu^{5,5} | \chi_{NL} \right) = 0,
$$

$$
\left( \chi_{NL} | \partial_\mu J_\mu^\pm | \chi_{NL} \right) = 0, \quad \left( \chi_{NL} | \partial_\mu J_\mu^{5,5} | \chi_{NL} \right) = 0,
$$

because $J_\mu^\pm$ and $J_\mu^{5,5}$ change neutron and proton number. Since the liquid ground state is homogeneous and isotropic, spatial components of vector currents vanish, in particular

$$
\left( \chi_{NL} | J_3^\pm | \chi_{NL} \right) = 0
$$

for Lorentz index $i = 1, 2, 3$. Because left-handed and right-handed protons and nucleons are equally represented in the nuclear ground state,

$$
\left( \chi_{NL} | J_5^{3,5} | \chi_{NL} \right) \approx 0
$$

for all $\mu$. Current conservation (see section 4) enforces

$$
\left( \chi_{NL} | \partial_\mu J_\mu^3 | \chi_{NL} \right) = 0, \quad \left( \chi_{NL} | \partial_\mu J_\mu^{5,5} | \chi_{NL} \right) = 0.
$$

This leaves only a single non-vanishing current expectation value.

$$
\left( \chi_{NL} | J_0^3 | \chi_{NL} \right) \neq 0.
$$

Equation (5), governing the classical pion cloud, is thus enormously simplified

$$
0 \approx \left[ \left[ \partial_\nu \frac{\partial}{\partial (\partial_\nu \pi^m)} - \frac{\partial}{\partial (\partial_\nu \pi^m)} \right] L^{\pi; Symmetric}_\chi \right]_{IR}
$$

$$
+ i \left( \chi_{NL} | J_0^3 | \chi_{NL} \right) \left[ \left[ \partial_\nu \frac{\partial}{\partial (\partial_\nu \pi^m)} - \frac{\partial}{\partial (\partial_\nu \pi^m)} \right] \mathcal{V}_0^3 \right]_{IR}
$$
with
\[
\left\{ \left[ \partial_{\mu} \frac{\partial}{\partial (\partial_{\nu} \pi^m)} - \frac{\partial}{\partial \pi^m} \right] V_0^3 \right\}_{IR} = 0 \tag{12}
\]
\[
2 \left\{ \left[ (\partial_0 \vec{\pi}) \times \vec{\pi} + \vec{\pi} \times \partial_0 \vec{\pi} - \partial_0 \vec{\pi} \right] \partial_0 \pi^m \right\} \left( \sin^2 \left( \frac{\pi f}{2} \right) \right) \left( \frac{\pi f}{2} \right)^2 \right\}_{IR}.
\]

A crucial observation is that (12) is linear in \(\partial_0 \pi\), i.e. in the energy of the classical NGB IR \(\pi\) field. Expecting the nuclear ground state, and thus its classical IR \(\pi\) field, to be static, we enforce
\[
\left\{ \partial_0 \vec{\pi} \right\}_{IR} = 0. \tag{13}
\]
It now follows that
\[
\left\{ \left[ \partial_{\mu} \frac{\partial}{\partial (\partial_{\nu} \pi^m)} - \frac{\partial}{\partial \pi^m} \right] V_0^3 \right\}_{IR} = 0. \tag{14}
\]

independent of \(\chi_{NL} \langle J^{3;0} \chi_{NL} \rangle\). The IR pion equation of motion
\[
\left\{ \left[ \partial_{\mu} \frac{\partial}{\partial (\partial_{\nu} \pi^m)} - \frac{\partial}{\partial \pi^m} \right] L_\pi^{Symmetric} \right\}_{IR} = 0 \tag{15}
\]
therefore has no nucleon source. \(L_\pi^{Symmetric} \) in (15) includes both its analytic and non-analytic contributions (cf. appendix equation (A.21)). The ground-state nucleons are not a source of any static IR NGB \(\vec{\pi}\) classical field.

The nuclear ground state in the chiral liquid is thus a static chiral nucleon liquid (Static\(\chi_{NL}\)), with no \(\vec{\pi}\) condensate \(\langle \vec{\pi} \rangle\) or time-dependent pion-flashing modes. We now write \(\langle \chi_{NL} \rangle_0\) for the ground state to emphasize that it is static.

We want to quantize the nucleons in the background field of the static \(\chi_{NL}\), and so consider the expectation value of the nucleon equation of motion in the chiral nucleon liquid ground state:
\[
0 = \langle \chi_{NL} \rangle_0 \left\{ \partial_{\mu} L_\pi^{Symmetric} \right\}_0 \tag{16}
\]
\[
= \langle \chi_{NL} \rangle_0 \left\{ i \gamma^\mu \partial_\mu - m^N \right\} \langle \chi_{NL} \rangle_0 + \frac{1}{f_A} \langle \chi_{NL} \rangle_0 \left\{ C_A (\bar{N} \gamma^{\mu} N) \langle \chi_{NL} \rangle_0 \right\} + \cdots
\]
\[+ \text{higher order terms} \]

\[\text{After explicit chiral symmetry breaking, with non-zero } u, d \text{ quark and resultant pion masses, and with Partially Conserved Axial Currents (PCAC), a static S-wave } \vec{\pi} \text{ condensate is a logical possibility} \text{.} \]
Since most of the nucleon $SU(2)_L \times SU(2)_R$ currents vanish in the Static $\chi$NL, and \( \{ \bar{\partial}_\nu \vec{r} \}_R \) = 0,

\[
0 \approx \left( \chi_{NL} \right| \bar{N} \left( i \gamma^\mu \partial_\mu - m^N \mathbb{1} \right) N \left| \chi_{NL} \right)_0
\]

\[
+ \frac{1}{f^2} \left( \chi_{NL} \right| C_{af} (\vec{N} \gamma^{af} N)(\vec{N} \gamma^{af} N) ++ + \left| \chi_{NL} \right)_0.
\]

Equations (15) and (17) show that, to order \( \Lambda_{\chi SB} \) and \( (\Lambda_{\chi SB})^0 \), Static $\chi$NL are composed entirely of nucleons. That is also the basic premise of many empirical models of the nuclear ground state: Pion-less $SU(2)$ $\chi$PT, Weizsächer’s Semi-empirical Mass Formula, the Nuclear Liquid Drop Model, Nuclear Density Functional Models, no-core Nuclear Shell Models, and Nuclear Skyrme Models. We have shown that that empirical nuclear premise can be (approximately) traced directly to the global $SU(2)_L \times SU(2)_R$ symmetries of 2-massless-quark Quantum Chromodynamics, i.e. directly to the Standard Model of elementary particles.

The effective Lagrangian derived from $SU(2)_L \times SU(2)_R$ $\chi$PT governing Static $\chi$NL can now be written

\[
L_{\text{Static} \chi NL} = L_{\text{Free Nucleons}} + L_{\text{Static} \chi NL}^{4-N}
\]

\[
L_{\text{Free Nucleons}}^{\text{Static} \chi NL} = \left( \chi_{NL} \right| \bar{N} \left( i \gamma^\mu \partial_\mu - m^N \mathbb{1} \right) N \left| \chi_{NL} \right)_0
\]

\[
L_{\text{Static} \chi NL}^{4-N} = \left( \chi_{NL} \right| \frac{1}{2f^2} C_{af} (\bar{N} \gamma^{af} N)(\bar{N} \gamma^{af} N) ++ + \left| \chi_{NL} \right)_0.
\]

Pion-less $SU(2)$ $\chi$PT thus emerges inside nuclear Static $\chi$NL. Within all-analytic-orders renormalized $SU(2)$ $\chi$PT, infrared NGB pions effectively decouple from Static $\chi$NL, vastly simplifying the derivation of the properties of saturated nuclear matter (the infinite liquid phase) and of finite microscopic liquid drops (the nuclides). Static $\chi$NL thus explain the (previously puzzling) power of pion-less $SU(2)$ $\chi$PT to capture experimental ground-state facts of certain nuclides, by tracing that (no-longer-mysterious) empirical success directly to the global symmetries of two-massless-quark QCD.

It will be shown below that static $\chi$NLs satisfy all relevant $SU(2)_L \times SU(2)_R$ vector and axial-vector current-conservation equations in the liquid phase. Static $\chi$NL are therefore solutions of the semi-classical liquid equations of motion; they are not just an ansatz.

3 Semi-classical Static $\chi$NL as the approximate ground state of certain nuclei

To further elucidate the properties of the Static $\chi$NL, we must address the four-nucleon interactions. This is best done by resolving $L_{\text{Static} \chi NL}^{4-N}$ into terms that
are the products of two ground-state current expectation values – usually described as boson-exchange contact interactions – and terms that are the products of transition matrix elements between the ground state and an excited nuclear state,

\[ L_{4-N}^{\text{Static}_{\chi N L}} = L_{4-N}^{\text{BosonExchange}} + L_{4-N}^{\text{ExcitedNucleon}}. \] (19)

A priori there are 10 possible contact interactions representing isosinglet and isotriplet channels for each of five “spatial” current types: scalar, vector, tensor, pseudo-scalar and axial-vector, and so 10 chiral coefficients parametrizing 4-nucleon contact terms:

\[
\begin{align*}
C_T^S &= 0_S, \\
C_T^V &= 1_S, \\
C_T^T &= 0_T, \\
C_T^P &= 1_T, \\
C_T^A &= 0_A, \\
C_S^S &= 1_S,
\end{align*}
\]

with only 4 independent chiral coefficients:

\[
\begin{align*}
C_S^{200} &= C_T^{T=0}, \\
-C_S^{200} &= \frac{1}{2}\left[ \frac{1}{2} C_T^{T=0} + \frac{5}{4} C_T^{T=1} \right] + 3 \left( C_T^{T=0} + \frac{1}{2} C_T^{T=1} \right) + \frac{1}{2} \left( C_T^{T=0} + \frac{1}{2} C_T^{T=1} \right), \\
C_V^{200} &= C_T^{T=0}, \\
-C_V^{200} &= \frac{1}{2} \left[ -C_T^{T=0} + C_T^{T=1} \right] + \frac{1}{2} C_T^{T=1}. 
\end{align*}
\] (21)

This is a vast improvement in the predictive power of the theory, while still providing sufficient free parameters to balance vector repulsive forces against scalar attractive forces, when fitting (to order \( (\chi_{SB})^0 \) ) Non-topological Soliton, Density Functional and Skyrme nuclear models to the experimental structure of ground-state nuclei.

* In this Section, we will adopt the shorthand \( \langle 0|\chi_{NL}\rangle \) as well as \( \langle 1|\chi_{NL}\rangle \).
There is yet another simplification for a sufficiently large number of nucleons: simple Hartree analysis of (20) is equivalent to far-more accurate Hartree-Fock analysis of the same Lagrangian without spinor-interchange terms. More coefficients would be required to parametrize the excited-nucleon interactions:

\[ -L^{4-N; \text{ExcitedNucleon}}_{\text{Static\chi NL}} \]

\[
= \sum_{\Psi} \sum_{\chi_{NL}} \left[ C_{\Psi_{\chi_{NL}}}^{T=0} \langle \chi_{NL} N_{c}^{\gamma_{\mu} \alpha_{\beta} N_{c}^{\beta}} | \Psi(N_{c}^{\gamma_{\mu} \alpha_{\beta} N_{c}^{\beta}} \chi_{NL}) \rangle \right] 
+ \sum_{B} C_{B}^{T=1} \langle \chi_{NL} N_{c}^{\gamma_{\mu} \alpha_{\beta} N_{c}^{\beta}} | \Psi(N_{c}^{\gamma_{\mu} \alpha_{\beta} N_{c}^{\beta}} \chi_{NL}) \rangle.
\]

However, excited-nuclear contributions, which will also include states that are not proton-and-neutron-even, are beyond the scope of this paper, and will be ignored. To the extent that such excited states are energetically well above the ground state, this should be a satisfactory approximation.

We now see that a nucleon living in the self-consistent field of the other nucleons inside the Static\chi NL obeys the Dirac equation

\[
0 = \left( i \gamma_{\mu} \partial_{\mu} + \Theta \right) \gamma^{\nu} N_{\chi_{NL}}^{\nu} = \gamma^{\mu} N_{\chi_{NL}}^{\mu} - \Theta \gamma^{0} N_{\chi_{NL}}^{0},
\]

where

\[
\Theta = -m^{N} + \frac{1}{f_{\pi}^{2}} C_{200}^{S} - \frac{1}{f_{\pi}^{2}} C_{200}^{V} \gamma^{0},
\]

with

\[
C_{200}^{S} = \left( C_{200}^{S} - \frac{1}{2} C_{200}^{S} \right) \left( N_{\chi_{NL}} \right) - \frac{1}{2} C_{200}^{S} \left( N_{\chi_{NL}} \right) t_{3},
\]

\[
C_{200}^{V} = \left( C_{200}^{V} - \frac{1}{2} C_{200}^{V} \right) \left( N_{\chi_{NL}} \right) - \frac{1}{2} C_{200}^{V} \left( N_{\chi_{NL}} \right) t_{3}.
\]

Ignoring \( L_{\text{Static\chi NL}}^{4-N; \text{ExcitedNucleon}} \), baryon-number and the third component of isospin are both conserved, i.e. the associated currents \( J_{\text{Baryon}}^{\mu} \equiv \overline{N} \gamma^{\mu} N \) and \( J_{3}^{\mu} \equiv \overline{N} \gamma^{\mu} t_{3} N \) are both divergence-free. The neutral axial-vector current \( J_{8}^{\mu} \equiv \frac{1}{\sqrt{3}} \overline{N} \gamma^{\mu} \gamma^{5} N \), corresponding to the projection onto SU(2) of the ‘eta’ NGB \( \eta \), part of the unbroken \( SU(3)_{L} \times SU(3)_{R} \) meson octet, is also divergence free,

\[
\frac{2}{\sqrt{3}} \left[ i \partial_{\mu} J_{8}^{\mu} \right] = \left( \overline{N} \left( \Theta, \gamma^{5} N \right) \right)
= 2 \left( \overline{N} \left( -m^{N} - \frac{1}{f_{\pi}^{2}} C_{200}^{S} \right) \gamma^{5} N \right)
= 0.
\]
This can be understood as a statement that the $\eta$ particle cannot survive in the parity-even interior of a Static\(\chi\)NL, since it is a NGB pseudo-scalar in the chiral limit.

Similarly, the axial-vector current of the 3rd component of $SU(2)_{L-R}$ isospin $J_3^{5,\mu} \equiv \bar{N} \gamma^\mu \gamma^5 t_3 N$ is divergence-free,

\[
\left\langle i \partial_\mu J_3^{5,\mu} \right\rangle = \left\langle \bar{N} \left\{ \Theta, \gamma^5 \right\} t_3 N \right\rangle \\
= 2 \left\langle \bar{N} \left( -m_N - \frac{1}{f_\pi^2} C \right) \gamma^5 t_3 N \right\rangle \\
\approx 0,
\]

because the $SU(2)\chi PT$ $\pi_3$ particle is also a NGB pseudo-scalar in the chiral limit, and cannot survive in the interior of a parity-even Static\(\chi\)NL.

Even though explicit pion and $\eta$ fields vanish in Static\(\chi\)NL, their quantum numbers reappear in its PCAC properties from nucleon bi-bilinears and four-nucleon terms in the divergences of axial vector currents. That these average to zero in Static\(\chi\)NL plays a crucial role in the conservation of axial-vector currents within the liquid.

It is now straightforward to see that, in the liquid approximation, a homogeneous $SU(2)\chi PT$ nucleon liquid drop with no meson condensate satisfies all relevant CVC and PCAC equations. In fact, of all the space-time components of the three $SU(2)_{L+R}$ vector currents $J_\mu^a$ and three $SU(2)_{L-R}$ axial vector currents $J_5^{5,\mu}$, only $J_0^3$ does not vanish in Static\(\chi\)NL.

The neutral $SU(3)_L \times SU(3)_R$ currents are conserved $\left\langle \partial_\mu J_8^\mu \right\rangle = \left\langle \partial_\mu J_8^{5,\mu} \right\rangle = 0$ in the Static\(\chi\)NL mean field. In addition, the neutral $SU(3)_{L+R}$ vector current’s spatial components $J_8^{a=1,2,3}$ and $SU(3)_{L-R}$ axial-vector currents $J_5^{5,\mu}$ all vanish. Only $J_0^3$, proportional to the baryon number density, survives in the Static\(\chi\)NL mean field.

Since Static\(\chi\)NL chiral nuclear liquids satisfy all relevant $\chi PT$ CVC and PCAC equations in the liquid phase, they are true solutions of the all-orders-renormalized tree level semi-classical liquid equations of motion truncated at $O(\Lambda_{\chi SB}^0)$.

4 Relation of Static\(\chi\)NL to standard nuclear models

In a companion paper, we apply the Thomas-Fermi approximation to construct explicit liquid solutions of $SU(2)\chi PT$ of protons, neutrons and three Nambu-Goldstone boson (NGB) pions. Constant-density non-topological solitons, i.e. liquids comprised entirely of nucleons, emerge as homogeneous and isotropic semi-classical static solutions at zero pressure. They thus serve as models of the ground state of both infinite nuclear matter and finite liquid drops. There is no need for an additional confining interaction to define the finite-drop surface.
By construction, these drops have total spin $\tilde{S} = 0$, even proton number $Z$, and even neutron number $N$. We first show symmetric $Z = N$ ground-state zero-pressure Hartree-Fock soliton solutions, fit to inferred experimental values for symmetric-nuclear-matter density and volume binding energy. Then, copying nuclides, we add $(Z - N)^2 \ll 1$, and derive asymmetric $Z \neq N$ nuclear matter, for which fermion-exchange terms are crucial. Finally, we show finite zero-pressure microscopic liquid drops closely resembling the Nuclear Liquid Drop Model. After crude inclusion of electromagnetic chiral symmetry breaking, our microscopic Static $\chi$NL solitons’ saturated nucleon density, as well as their volume, asymmetry and electromagnetic terms, fit the Bethe-Weizsäcker Semi-Empirical Mass Formula.

This empirical success, coupled with the fact that chiral perturbation theory is a direct consequence of the Standard Model of particle physics – as correct nuclear physics, atomic physics, etcetera must ultimately be – motivates us to consider the connection of certain mainstream nuclear-model frameworks to the Static $\chi$NL solutions we have identified.

### 4.1 Density Functionals

The basic building blocks of current relativistic nuclear density functionals [62] are the densities bilinear in the Dirac-spinor field $N$ of the nucleon doublet:

$$
\left[ (\tilde{N}O_r \gamma_{\mu} N) \left( \tilde{N}O_r \gamma^{\mu} N \right) \right].
$$

where $O_r = (1, 2\vec{I})$ and $\gamma_{\mu} = (S, V, T, A, P)$. The nuclear-ground-state density and energy are determined by the self-consistent solution of relativistic linear Kohn-Sham [63] equations. To derive those equations, Niksic et.al. [62] construct an interaction Lagrangian with four-fermion (contact) interaction terms in the various Lorentz-space isospace channels: scalar-isoscalar $\sigma$ exchange, vector-isoscalar $\omega_{\mu}$ exchange, vector-isovector $\rho_{\mu}$ exchange, and scalar-isovector $\delta$ exchange. Ignoring explicit electro-magnetic chiral symmetry breaking

$$
{\mathcal{L}}_{\text{Niksic}} = {\mathcal{L}}_{\text{Isoscalar Niksic}} + {\mathcal{L}}_{\text{Isovector Niksic}} + {\mathcal{L}}_{\text{Surface Niksic}}
$$

$$
{\mathcal{L}}_{\text{Isoscalar Niksic}} = \tilde{N}(i\gamma_{\mu}\partial^{\mu} - m_N)N - \frac{1}{2}a_S \left[ (\tilde{N}N)(\tilde{N}N) \right] - \frac{1}{2}a_V \left[ (\tilde{N}\gamma_{\mu}N)(\tilde{N}\gamma^{\mu}N) \right]
$$

$$
{\mathcal{L}}_{\text{Isovector Niksic}} = -\frac{1}{2}a_T S \left[ (N2\vec{I}N) \cdot (N2\vec{I}N) \right] - \frac{1}{2}a_T V \left[ (N2\vec{I}\gamma_{\mu}N) \cdot (N2\vec{I}\gamma^{\mu}N) \right]
$$

$$
{\mathcal{L}}_{\text{Surface Niksic}} = -\frac{1}{2} \delta S \left[ \partial_{\nu}(\tilde{N}N) \partial^{\nu}(\tilde{N}N) \right].
$$
where the coefficients are themselves functions of the nuclear number density normalized to that of nuclear matter:

\[
\alpha_S \left( \frac{N^\dagger N}{[N^\dagger N]_{\text{Nuclear\ Matter}}} \right), \quad \alpha_V \left( \frac{N^\dagger N}{[N^\dagger N]_{\text{Nuclear\ Matter}}} \right), \quad (29)
\]

\[
\alpha_{TS} \left( \frac{N^\dagger N}{[N^\dagger N]_{\text{Nuclear\ Matter}}} \right), \quad \alpha_{TV} \left( \frac{N^\dagger N}{[N^\dagger N]_{\text{Nuclear\ Matter}}} \right).
\]

In order to be consistent with, and thus legitimately employ, emergent Pionless\textit{SU(2)}\textit{\times}\textit{\chi} PT, density-functional models must be made to obey all-orders-renormalized power-counting to at least \(\Lambda_{\text{SB}}^{-1}\). A beginning would be to re-scale density functional coefficients to reflect \textit{SU(2)}\textit{\times}\textit{\chi} PT power counting, and Lorentz invariance, as

\[
\alpha_S \left( \frac{\bar{N}N}{f_\pi^2 \Lambda_{\text{SB}}} \right), \quad \alpha_V \left( \frac{\bar{N}N}{f_\pi^2 \Lambda_{\text{SB}}} \right), \quad \alpha_{TS} \left( \frac{\bar{N}N}{f_\pi^2 \Lambda_{\text{SB}}} \right), \quad \alpha_{TV} \left( \frac{\bar{N}N}{f_\pi^2 \Lambda_{\text{SB}}} \right) \quad (30)
\]

Current nuclear density-functional models contain non-analytic terms inside \(\alpha_S, \alpha_V, \alpha_{TS}, \alpha_{TV}\). These must be made to map onto any known non-analytic terms in \textit{SU(2)}\textit{\times}\textit{\chi} PT [17].

Exchange terms must be included for Hartree-Fock results.

Chhanda Samanta [64] claims that “density functional theory currently predicts long-lived super-heavy elements in a variety of shapes, including spherical, axial and triaxial configurations. Only when N=184 is approached one expects superheavy nuclei that are spherical in their ground states. Magic islands of extra-stability have been predicted to be around Z=114, 124 or, 126 with N=184, and Z=120, with N=172.” If Pionless \textit{SU(2)}\textit{\times}\textit{\chi} PT confirmed such statements, islands of nuclear stability would move from fantasy to probable fact.

In the end, the requirement that \textit{SU(2)}_L \times \textit{SU(2)}_R \textit{\chi} PT have high-accuracy experimental predictive power tied to two-massless-quark QCD, i.e. the requirement that all chirally invariant terms be included in the Lagrangian and that their coefficients be “natural”, will force nuclear density-functional theories to obey analytic power counting to at least \(O(\Lambda_{\text{SB}}^{-2})\).

### 4.2 Nuclear Skyrme models

A large preexisting class of high-accuracy Nuclear Skyrme Models [9] were first identified by Friar, Madland, and Lynn [5, 65] as (almost) derivable from \textit{SU(2)}_L \times \textit{SU(2)}_R \textit{\chi} PT liquid \(O\left(\Lambda_{\text{SB}}^n\right),\; n = 1, 0, -1, -2\) operators, thus introducing the \textit{\chi} PT power-counting concept of “Naturalness” to nuclear Skyrme models.

Careful and successful comparison of theory to experiment for the ground state of certain even-even spin-zero spherical closed-shell heavy nuclei is a major
triumph for Relativistic-Mean-Field Point-Coupling Hartree-Fock (RMF-PC-HF) "Skyrme" models of nuclear many-body forces [6, 7, 8, 9]. For such nuclides, nuclear Skyrme models (almost) obey a much-simplified $SU(2)\chi PT$, in which the set of liquid operators is much fewer than the total set of possible non-liquid operators.

Without prior consideration of chiral liquid $SU(2)\chi PT$, Nikolaus, Hoch, and Madland [9] fit nine coefficients, spanning the range of $10^{-4}\text{MeV}^{-2}$ to $10^{-18}\text{MeV}^{-8}$, to the properties of just three heavy nuclei. They then predicted the properties of another 57 heavy nuclei quite accurately. The observational success of their model, with the improvements of [6] is competitive with other nuclear models [20]: binding energies are fit to within $\pm 0.15\%$; charge radii are fit to $\pm 0.2\%$; diffraction radii are fit to $\pm 0.5\%$; surface thicknesses are fit to within $\pm 50\%$; spin-orbit splittings are fit to $\pm 5\%$; and pairing gaps are fit to $\pm 0.05\text{MeV}$. The observed isotonic chains, fission barriers, etc. are also fit to various high accuracies.

When these 9 coefficients were rescaled [5, 65], as appropriate to $SU(2)_{L} \times SU(2)_{R} \chi PT$ liquids, these (almost) obeyed $SU(2)\chi PT$ power-counting in $\Lambda_{\chi SB}^{-1}$ through order $\Lambda_{\chi SB}^{-2}$, with order-one chiral coefficients. A high-accuracy fit and its predictions for properties of such heavy nuclei [3] showed that two apparent exceptions have since improved. Burvenich, Madland, Marhun, and Reinhard used 11 coupling constants that, when appropriately rescaled with $\Lambda_{\chi SB}$, almost obey naturalness [5] for $SU(2)\chi PT$ chiral nuclear liquids. This obedience of nuclear-Skyrme-model coefficients to $\Lambda_{\chi SB}^{-1}$ power-counting in $SU(2)\chi PT$ is now commonly referred to as "$\chi PT$-naturalness" in the heavy-nuclear-structure literature [66]. Symmetric and asymmetric (finite) nuclear liquid drops and bulk nuclear matter in nuclear Skyrme models are therefore nearly obedient to $SU(2)\chi PT$.

In order that nuclear Skyrme models emerge with chiral-liquid-like properties within Pionless $SU(2)\chi PT$, one must carefully consider which operators emerge, and how their coefficients are related. But current Skyrme both over-count and omit certain liquid operators. In order to be consistent with, and thus legitimately employ, emergent Pionless $SU(2)\chi PT$, they must be made to strictly obey all-orders-renormalized power-counting to $O(\Lambda_{\chi SB}^{-2})$.

- First re-scale coefficients in (28) as insisted above to reflect $SU(2)\chi PT$ power counting, and Lorentz invariance, as

$$\alpha_S \left( \frac{\overline{NN}}{f^2_\pi \Lambda_{\chi SB}} \right), \alpha_V \left( \frac{\overline{NN}}{f^2_\pi \Lambda_{\chi SB}} \right), \alpha_{TS} \left( \frac{\overline{NN}}{f^2_\pi \Lambda_{\chi SB}} \right), \alpha_{TV} \left( \frac{\overline{NN}}{f^2_\pi \Lambda_{\chi SB}} \right)$$

(31)

- 2-Nucleon forces (4-N operators)

  - For constant $\alpha_V, \alpha_S, \alpha_{TV}, \alpha_{TS}, \delta_S$, the Lagrangian (28) is derivable from 2-massless-quark QCD, via a Static $\chi NL$ with its Pionless $SU(2)\chi PT$.

  - To begin, nuclear Skyrme models should test empirically whether excited nucleon contact terms can really be ignored.
If so, replace $L_{\text{Niksic}}^{\text{Isosvector}}$ with

\[
L_{\text{Static}N_\text{L}}^{\text{Isosvector}} = \frac{1}{2} \alpha_{TS} \left[ (\bar{N}2t_3N) \cdot (\bar{N}2t_3N) \right] - \frac{1}{2} \alpha_{TV} \left[ (\bar{N}2t_3\gamma\mu N) \cdot (\bar{N}2t_3\gamma\mu N) \right]
\] (32)

It remains to be seen whether $L_{\text{Static}N_\text{L}}^{\text{Isosvector}}$, which arises from fermion exchange-terms but vaguely resembles neutral $\rho_3^\mu$ and $\delta_3$ boson exchange, can account, with high accuracy, for the known isovector properties of nuclear ground-states in nuclear Skyrme models. In practice, Niksic et al. neglect the isovector-scalar $\vec{\delta}$ exchange (i.e. they set $\alpha_{TS} = 0$), arguing that, although the total isovector strength has a relatively well-defined value, the distribution between the scalar $\alpha_{TS}$ and vector $\alpha_{TV}$ channels is not determined by ground-state data.

- When $O\left(\Lambda_{\chi SB}^0\right)$ include spinor-interchange and boson-exchange terms, constant coefficients $\alpha_V, \alpha_S, \alpha_{TV}, \alpha_{TS}$ are linear combinations of $C_{200}^S, C_{200}^V, C_{200}^{SV}$.

• 3-nucleon (6-N operators) and 4-nucleon (8-N operators) contact forces

- 3-nucleon forces, of order $O\left(\Lambda_{\chi SB}^{-1}\right) + O\left(\Lambda_{\chi SB}^{-2}\right)$, are smaller than $O\left(\Lambda_{\chi SB}^0\right)$ 2-nucleon forces [2, 67, 68].

- 4-nucleon $O\left(\Lambda_{\chi SB}^{-2}\right)$ forces are smaller still than 3-nucleon $O\left(\Lambda_{\chi SB}^{-1}\right)$ forces [2, 67, 68]. Including separate $C_{400}^S$ and $C_{400}^V$ over-counts independent chiral coefficients.

- For example, to $O(\Lambda_{\chi SB}^{-2})$,

\[
\alpha_V \left(\frac{1}{f_{30}^2} \Lambda_{\chi SB}^{-1}\right) \approx \alpha_V(0) \left[ \frac{N^N}{f_{30}^2 \Lambda_{\chi SB}} \right] + \frac{1}{3} C_{300} \left[ \frac{N^N}{f_{30}^2 \Lambda_{\chi SB}} \right] + \frac{1}{4} C_{400} \left[ \frac{N^N}{f_{30}^2 \Lambda_{\chi SB}} \right]^2
\]

\[
\alpha_S \left(\frac{1}{f_{30}^2} \Lambda_{\chi SB}^{-1}\right) \approx \alpha_S(0)
\]

representing 2-nucleon, 3-nucleon and 4-nucleon contact terms respectively. Since non-relativistic $N^N$ and $NN$ differ by relativistic corrections of $O(\Lambda_{\chi SB}^{-2})$, $SU(2)\chi PT$ requires $\alpha_V(0), C_{300}, C_{400}$ to be $\approx O(1)$ natural.

- Spinor-interchange terms must also be added to consistently preserve Hartree-Fock quantum loop power counting.

• Incorporate $O\left(\Lambda_{\chi SB}^{-2}\right)$ nuclear-surface terms. Because they only involve differentials of the baryon number density, only certain surface terms are
invariant under local $SU(2)\chi PT$ transformations, and do not contribute to $SU(2)_{L+R}$ or $SU(2)_{L-R}$ currents affecting CVC or PCAC properties. These terms replace the scalar $\sigma$ particle in the Chin-Walecka model in describing the nuclear surface \[53, 57, 56\]. The surface term must be re-scaled

$$L_{\text{Niksic}}^{\text{Surface}} \rightarrow L_{\text{Static} \chi N L}^{\text{Surface}} = -\frac{1}{2} C_{220} \left[ \frac{\partial_\nu}{\Lambda_{\chi_{SB}}} (\bar{N} N) \frac{\partial^\nu}{\Lambda_{\chi_{SB}}} (\bar{N} N) \right].$$

with constant $C_{220} \approx O(1)$ in order to obey naturalness and absorb all-orders quantum loops. $L_{\text{Static} \chi N L}^{\text{Surface}}$ is invariant under $SU(2)_L \times SU(2)_R$ transformations, including pions, but is automatically pion-less, even without the liquid approximation. It contains no dangerous $\partial_0 \sim m_N$ nucleon mass terms, so non-relativistic re-ordering is un-necessary. Nucleon-exchange and spinor-interchange interactions must also be included.

- According to strict $O\left(\Lambda^{-2}_{\chi_{SB}}\right)$ power counting \[2, 67, 68\], current nuclear Skyrme models sometimes over-count chiral-liquid operators. For example, to $O\left(\Lambda^{-2}_{\chi_{SB}}\right)$, only eight of eleven coupling constants considered in \[6\] are truly independent.

- The authors of \[6\] are also missing chiral-liquid operators where non-relativistic re-ordering of the large nucleon mass term $\sim \partial_0$ is necessary. Such time-dependent operators may be important for high-accuracy nuclear structure, and will also affect ordinary heavy nuclear $SU(2)_{L+R}$ and $SU(2)_{L-R}$ currents, CVC and PCAC.

### 4.2.1 Ab Initio calculations

It is beyond the scope of this paper to construct a complete minimal $O\left(\Lambda^{-2}_{\chi_{SB}}\right)$ set of chiral-liquid operators for nuclear Skyrme models, but a systematic program of calculation of detailed properties of the ground state of even-even spin-zero spherical closed-shell heavy nuclei in RMF-PC-HF (and nuclear liquid drops) with that set is necessary in order to extract predictions for nuclear structure from Static$\chi NL$ emergent from $SU(2)\chi PT$.

Going forward, it is important to understand whether the contribution of

$$-L_{\text{Static} \chi N L:_{t,t_c}}^{4-N: \text{ExcitedNucleon}} = \frac{1}{2} \sum_{\Psi_{\text{Static} \chi N L}} \frac{1}{f_0^2} \left[ C_{S}^{T=1} \left( \chi NL \left| \sum_{c,d} \left( N^c_e(t_c)_{c,d} N^d_e(t_c) \right) \bar{\Psi}\Psi \right| N^c_e(t_c)_{c,d} N^d_e(t_c) \left| \chi NL \right) \right] 
+ C_{V}^{T=1} \left( \chi NL \left| \sum_{c,d} \gamma^{\mu,\alpha,\beta} N^d_e(t_c) \bar{\Psi}\Psi \right| N^c_e(t_c)_{c,d} \gamma^{\lambda,\sigma,\tau} N^d_e(t_c) \left| \chi NL \right) \right]$$

is numerically material to empirical models. Such terms involve proton-odd neutron-odd intermediate states, and may require explicit pion-exchange effects lying outside pion-less $SU(2)\chi PT$, thus significantly complicating Ab Initio calculations.
If Nuclear Skyrme Models properly incorporating strict $SU(2)_L \times SU(2)_R$ power-counting to $O\left(\frac{\Lambda^2_{\chi SB}}{\chi_{SB}}\right)$ in chiral liquids, and possibly including higher representations such as the $\Delta(1232)$, were to capture empirical reality, including no-core shell structure, to high accuracy for those nuclear-ground-states which are to be regarded as liquid drops, that success would have been traced directly to the global symmetries of 2-massless-quark QCD.

4.3 Neutron Stars

Putting aside exotica (i.e. quark condensates, strange-kaon condensates, etc.), we conjecture that much of the structure of neutron stars may be traced directly to 2-massless-quark QCD, and thus directly to the Standard Model.

The models of Harrison & Wheeler [69], Salpeter [70] and Baym, Pethic and Sutherland [71], all based on the Bethe-Weizsächer Semi-Empirical Mass Formula [72], would seem to be implied by our companion paper [73]. If Density Functional and Skyrme models can be modified to strictly obey $SU(2)_\chi PT$, highly credible and predictive Standard-Model neutron-star structure would follow.

Kim et. al. [74] have recently used gravitational wave observations [75], density functional technology, and reasonable constraints on Skyrme models (from stable nuclei, nuclear matter and the maximum mass of neutron stars), to constrain the tidal deformities in single neutron stars and binaries. If such constraints could be traced directly to QCD and the SM as conjectured here, a strong new connection between General Relativity and the Standard Model will have appeared.

4.4 2-light-mass-quark QCD’s $SU(2)_\chi PT$ symmetry-breaking terms

B.W. Lynn[36] first introduced the idea that $SU(2)_\chi PT$ could admit a liquid phase. His Lagrangian included only $SU(2)_\chi PT$ terms of $O(\Lambda_{\chi SB})$ and $O(\Lambda^0_{\chi SB})$ and ignored electro-magnetic breaking. These included strong-interaction terms which survive the chiral limit, as well as explicit chiral symmetry breaking terms which do not. But he was careful to include only/all those terms which survive the approximate Static $\chi NL$ dynamical symmetries discussed in this paper.

The symmetry-breaking terms have $m = 0, l = 1, n = 1$ in (A.13). Ignoring $\pi^* - \pi^0$ mass splitting, these are

$$L^{N;3SB}_{\chi PT} \cong \left[ m_{up} + m_{down} \right] \left[ a_1 + a_2 + a_3 \right] \left[ 1 - \cos \frac{2\pi}{f_{\pi}} \right]$$

$$(a_1, a_2, a_3; m_{up}, m_{down}) = (0.28, -0.56, 1.3 \pm 0.2; 6 MeV, 12 MeV),$$

with constants measured in $SU(3)_L \times SU(3)_R \chi PT$ processes [17] and [76].

Since $L^{N;3SB}_{\chi PT} > 0$, the symmetry-breaking terms have the effect of lowering the effective nucleon mass inside a static $\pi = |\vec{r}|$ condensate. [39] showed that an unphysical ‘pion-nucleon coupling’ $\beta \sigma_{\pi N} \geq 400 MeV$, with $\beta \geq 6.66$, causes
an S-wave $\vec{p}^2$ pion condensate to form. He also showed that the experimental values $\beta = 1; \sigma_{\pi N} \approx 60 MeV$ allow no such S-wave condensate to form in ordinary heavy nuclei, in agreement with observation [1], and thus avoided the disaster of “parity doubling" in the chart of the nuclides.

Note that (36) is further suppressed in the 2-light-quark sector because

$$\left(m_{\text{up}} + m_{\text{down}}\right) \sim \frac{m_\pi^2}{\Lambda_{\chi SB}} \sim 0.02 GeV$$

which is why un-physically large $\beta$ was necessary to form the un-physical $\pi$-condensate. Maybe this also explains why it is empirically successful to take certain nuclear structure to be independent of the pion mass [77].

We conjecture that Pion-less SU(2)$_L \times$ SU(2)$_R$ of Static $\chi NL$ for certain nuclides can be shown to effectively include all $O(\Lambda_{\chi SB})$ and $O(\Lambda_{\chi SB}^0)$ non-strange power-counting terms, both those from the chiral-limit and those from $m_{\text{up}}, m_{\text{down}} \neq 0$ chiral symmetry breaking.

5 Conclusions

The Standard Model of particle physics, augmented by neutrino mixing and General Relativity (i.e. Frank Wilczek’s “Core Theory" [81]) is the most powerful, accurate, predictive, successful and experimentally successful scientific theory known to humans. No experimental counter-example has ever been observed in the known universe. Its local SU(3)$_{\text{Color}}$ Quantum Chromo-Dynamic subset is, according to all experimental evidence, the complete and correct theory of the strong interactions of known fundamental particles at all energies accessible to current technology. It must therefore underlie the complete and correct theory of the structure and interactions of atomic nuclei.

In this paper, we have explored some of the implications of this inescapable connection for nuclear structure as directly derivable from Standard Model, especially from the global symmetries of QCD. In this we have been guided by two key observations: that nuclei are made of protons and neutrons, not quarks; and that the up and down quarks, which are the fermionic constituents of the protons and neutrons, are much lighter than the principal mass scales of QCD, such as the proton and neutron masses. Taken together, these strongly suggest that the full complexity of the Standard Model can largely be captured, for the purposes of nuclear physics, by an effective field theory (EFT) – SU(2)$_L \times$ SU(2)$_R$ chiral perturbation theory ($SU(2) \times \chi PT$) of protons and neutrons.

In writing down an EFT Lagrangian, one incorporates all analytic higher-order quantum-loop corrections into tree-level amplitudes. $SU(2) \times \chi PT$ enables

7 Strange Chiral Nuclear Liquids [76], a form of Strange Baryon Matter [78], consist of a Static $\chi NL$ immersed in a kaon condensate driven by large $m_{\text{strange}} \approx 0.24 GeV, \beta = 9$. These strange chiral liquids are identified [76], [79] as a possible SU(3)$_L \times$ SU(3)$_R \times \chi PT$ MACRO dark matter candidate [79] non-topological soliton [80] which is fully consistent with the dynamics of ordinary nuclides in this paper.

8 Neutrinos may have undiscovered interactions connected to their mass and to their flavor oscillations. These are unlikely to affect the conclusions of this paper.
the operators of that EFT Lagrangian (and the states) to be expressed as a perturbation expansion in inverse powers of the chiral-symmetry-breaking scale $\Lambda_{\chi SB} \approx 1\text{GeV}$.

Building on this longstanding insight, we have studied the chiral limit of $SU(2) \chi PT$ EFT, including only operators of order $\Lambda_{\chi SB}$ and $\Lambda_{\chi SB}^0$. We find that $SU(2) \chi PT$ of protons, neutrons and 3 Nambu-Goldstone boson (NGB) pions - admits a semi-classical liquid phase, a Static Chiral Nucleon Liquid (Static $\chi$NL).

Static $\chi$NLs are made entirely of nucleons, with zero anti-proton and anti-neutron content. They are parity even and time-independent. As we have studied them so far, not just the total nuclear spin $\vec{S} = 0$, but also the local expectation value for spin $\langle \vec{S} \rangle = 0$. Similarly, the nucleon momenta vanish locally in the Static $\chi$NL rest frame. For these reasons, our study of Static $\chi$NL is applicable to bulk ground-state spin-zero nuclear matter, and to the ground state of appropriate spin-zero parity-even nuclei with an even number $Z$ of protons and an even number $N$ of neutrons.

We classify these solutions of $SU(2) \chi PT$ as “liquid” because energy is required both to pull the constituent nucleons further apart and to push them closer together. This is analogous with the balancing of the attractive Lorentz-scalar $\sigma$-exchange force and the repulsive Lorentz-vector $\omega_{\mu}$-exchange force in the Walecka model. The nucleon number density therefore takes a saturated value even in zero external pressure (e.g. in the absence of gravity), so are not a “gas.” Meanwhile they are statistically homogeneous and isotropic, lacking the reduced symmetries of crystals or other solids.

We have shown that in this ground-state liquid phase, the expectation values of many of the allowed operators of the most general $SU(2) \chi PT$ EFT Lagrangian vanish or are small. We have further conjectured that, for studying (static) ground-state systems, many more operators are small because they involve transitions to excited intermediate states. Going forward, it is imperative to understand the contribution of $L^{4-N:ExcitedNucleon}_{\text{Static} \chi$NL} to empirical models of nuclear ground states.

We have also shown that this ground-state liquid phase does not support a classical pion field – infrared pions decouple from these solution. We expect that this emergence of “pion-less $SU(2) \chi PT$ is at the heart of the apparent theoretical independence of much successful nuclear physics from pion properties such as the pion mass.

In a companion paper, we will use the Thomas-Fermi approximation to provide an explicit “proof of principal” solution for a liquid phase of $SU(2) \chi PT$, i.e. Static $\chi$NL non-topological solitons. We demonstrate there the existence of zero-pressure non-topological soliton Static $\chi$NLs, with both macroscopic (infinite nuclear matter) and microscopic (nucleide ground states).

We conjecture that, for appropriate nuclides, proper inclusion of $1/\Lambda_{\chi SB}$ and $(1/\Lambda_{\chi SB})^2$ $SU(2) \chi PT$ operators will result in accurate ”natural” nuclear Skyrme models,

9 Note that in the chiral limit, electromagnetic interactions are ignored.
exhibiting “no-core” shell structure, with approximate StaticχNL structure.

We speculate that the extension of the line of thinking contained in this paper to $SU(3)_L \times SU(3)_R \chi PT$, will be instructive on the experimentally current question of strange nuclei, and on the astrophysically relevant question of strange nuclear matter.

The Standard Model (augmented by neutrino mixing) is, as a result of five decades of experimental and theoretical effort, a remarkably complete and correct description of all non-gravitational interactions of known fundamental particles, without experimentally identified exception. Nature has been kind, by building atoms out of electrons and nuclei, and nuclei out of protons and neutrons, and by making the up and down quark so much lighter than those, to afford us a possible pathway to relate the emergent physics of: atoms\(^{10}\); the deuteron \([34, 35]\); the heavy nuclides in this paper; and the structure of the proton \([82]\) in lattice gauge theory, directly to the fundamental interactions of the Standard Model. It is incumbent on us to avail ourselves of that kindness by striving to obediently connect our phenomenological/empirical models to Nature’s magnificent fundamental theory.

Acknowledgments

This paper is dedicated to BWL’s teacher, Gerald Feinberg (1933-1992) who predicted the muon neutrino; and teased emergent physics - positronium, muonium (i.e. renormalizable bound states), the periodic table (i.e. atomic physics), and the polarizability of di-hydrogen (i.e. chemistry) - directly from the Standard Model.

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Appendix A \( SU(2)_L \times SU(2)_R \) \( \chi PT \) of a nucleon doublet and a pion triplet in the chiral limit

The chiral symmetry of two light quark flavors in QCD, together with the symmetry-breaking and Goldstone’s theorem, makes it possible to obtain an approximate solution to QCD at low energies using a \( SU(2)_L \times SU(2)_R \) EFT, where the degrees of freedom are hadrons [10, 11, 12, 13, 14, 15, 16, 17, 18]. In particular, the non-linear \( SU(2) \) \( \chi PT \) effective Lagrangian has been shown to successfully model the interactions of pions with nucleons, where a perturbation expansion (e.g., in soft momentum \( \frac{k}{\Lambda_{\chi SB}} \ll 1 \), baryon number density \( \frac{N_f}{\pi} \ll 1 \), for chiral symmetry breaking scale \( \Lambda_{\chi SB} \approx 1 \) GeV) has demonstrated predictive power. Power-counting in \( \Lambda_{\chi SB}^{-1} \) includes all analytic quantum-loop effects into experimentally measurable coefficients of \( SU(2)_L \times SU(2)_R \) current-algebraic operators obedient to the global symmetries of QCD, with light-quark masses generating additional explicit chiral-symmetry-breaking terms. Therefore, \( SU(2) \) \( \chi PT \) tree-level calculations with a power-counting effective Lagrangian are to be regarded as true predictions of QCD and the Standard \( SU(3)_C \times SU(2)_L \times U(1)_Y \) Model of elementary particles.

A.1 Non-linear transformation properties

We present the Lagrangian of unbroken \( SU(2) \) \( \chi PT \) of a nucleon doublet and a pseudo-Nambu-Goldstone-Boson (pNGB) triplet. We employ the defining SU(2) strong-isospin representation of unitary 2×2 Pauli matrices \( \sigma_a \), with asymmetric structure constants \( f_{abc} = \epsilon_{abc} \)

\[
\begin{align*}
    t_a &= \sigma_a \frac{1}{2}, \quad a = 1, 3 \\
    \text{Tr}(t_{ab}) &= \frac{\delta_{ab}}{2} \\
    [t_{a}, t_{b}] &= i f_{abc} t_{c} \\
    \{t_{a}, t_{b}\} &= \frac{\delta_{ab}}{2}.
\end{align*}
\]

The \( SU(2)_L^+ \times SU(2)_R^+ \) vector and \( SU(2)_L^- \times SU(2)_R^- \) axial-vector charges obey the algebra

\[
\begin{align*}
    [Q_{a}^{+L+R}, Q_{b}^{+L+R}] &= i f_{abc} Q_{c}^{+L+R} \\
    [Q_{a}^{-L+R}, Q_{b}^{-L+R}] &= i f_{abc} Q_{c}^{+L+R} \\
    [Q_{a}^{-L+R}, Q_{b}^{-L-}] &= i f_{abc} Q_{c}^{+L-}. \quad (A.2)
\end{align*}
\]

We consider a triplet representation of NGBs,

\[
\pi_{a}^{a} = \frac{1}{\sqrt{2}} \left[ \begin{array}{c}
    \pi^{0} \\
    \pi^{-} \\
    -\frac{\pi^{0}}{\sqrt{2}}
\end{array} \right]. \quad (A.3)
\]
and a doublet of nucleons,

\[ N = \begin{bmatrix} p \\ n \end{bmatrix}. \]  

(A.4)

For pedagogical simplicity, representations of higher mass are neglected, even though the \( SU(3)_L \times SU(3)_R \) baryon decuplet (especially \( \Delta_{1232} \)) is known to have important nuclear structure [1] and scattering [34] effects. Since \( SU(2) \times SU(2) \) matrix elements are independent of representation [12, 13], we choose a representation [16, 83, 17] where the NGB octet has only derivative couplings,

\[ \Sigma \equiv \exp\left(\frac{2i\pi a_t}{f_\pi}\right). \]  

(A.5)

Under a unitary global \( SU(2)_L \times SU(2)_R \) transformation, given by \( L \equiv \exp(i\xi a_t) \) and \( R \equiv \exp(i\eta a_t) \),

\[ \Sigma \rightarrow \Sigma' = L \Sigma R^\dagger. \]  

(A.6)

It also proves useful to introduce the “square root” of \( \Sigma \)

\[ \xi \equiv \exp\left(i\pi a_t/f_\pi\right), \]  

(A.7)

which transforms as

\[ \xi \rightarrow \xi' = \exp\left(i\pi a_t/f_\pi\right). \]

(A.8)

We observe that

\[ \xi' = L\xi U^\dagger = U\xi R^\dagger, \]

for some unitary local transformation matrix \( U(L, R, \pi_a(t, x)) \).

The vector and axial-vector NGB currents

\[ V_\mu \equiv \frac{1}{2}(\xi\partial_\mu\xi + \xi\partial_\mu\xi^\dagger) \]

\[ A_\mu \equiv i\frac{1}{2}(\xi\partial_\mu\xi - \xi\partial_\mu\xi^\dagger) \]

transform straightforwardly as

\[ V_\mu \rightarrow V' = UV_\mu U^\dagger + U\partial_\mu U^\dagger \]

\[ A_\mu \rightarrow A' = UA_\mu U^\dagger. \]  

(A.10)

Meanwhile the nucleons transform as

\[ N \rightarrow N' = UN \]  

(A.11)

and

\[ D_\mu N \equiv \partial_\mu N + V_\mu N \rightarrow U(D_\mu N). \]  

(A.12)
A.2 $\Lambda_{\chi SB}$ power counting

The $SU(2)\chi PT$ Lagrangian, including all analytic quantum-loop effects for soft momenta ($\ll 1$ GeV) \cite{16, 83}, can now be written:

$$L_{\chi PT} = (A.13)$$

$$\sum_{l,m,n} C_{lmn} f_r^2 \Lambda_{\chi SB}^2 \left( \frac{\partial \mu}{\Lambda_{\chi SB}} \right)^m \left( \frac{N}{f_\pi \sqrt{\Lambda_{\chi SB}}} \right)^l \left( \frac{m_{\text{quark}}}{\Lambda_{\chi SB}} \right)^n f_{lmn} \left( \frac{f_a}{f_\pi} \right)$$

where $f_{lmn}$ is an analytic function, and the dimensionless constants $C_{lmn}$ are $O(\Lambda_{\chi SB}^0)$ and, presumably, $\sim 1$. As a power series in $\Lambda_{\chi SB}$,

$$L_{\chi PT} \sim \Lambda_{\chi SB} + (\Lambda_{\chi SB})^0 + \left( \frac{1}{\Lambda_{\chi SB}} \right)^2 + ...$$

We take, self-consistently, $\Lambda_{\chi SB} \approx 1$ GeV and, in higher orders, reorder the non-relativistic perturbation expansion in $\partial_0$ to converge with large nucleon mass $m_N \approx \Lambda_{\chi SB}$ \cite{2, 67, 68}. As the terms in (A.13) already include all loop corrections, we can perform tree-level calculations to arrive at strong-interaction predictions.

A.3 The Chiral Limit

For the purposes of this paper, we retain from (A.13) only terms of order $\Lambda_{\chi SB}$ and $\Lambda_{\chi SB}^0$, i.e. $1 \leq m + l + n \leq 2$. We can further divide $L_{\chi PT}$ into a symmetric piece (i.e., spontaneous $SU(2)_{L-R}$ breaking with massless Goldstones) and a symmetry-breaking piece (i.e., explicit $SU(2)_{L-R}$ breaking, traceable to quark masses) generating three massive pNGB:

$$L_{\chi PT} = L_{\chi PT}^{\text{Symmetric}} + L_{\chi PT}^{\text{Symmetry-Breaking}}.$$  

In this paper, we are interested only in unbroken $SU(2) \chi PT$ and so take $n = 0$ in (A.13)

$$L_{\chi PT}^{\text{Symmetry-Breaking}} = 0.$$  

We separate $L_{\chi PT}^{\text{Symmetric}}$ into pure-meson terms, terms quadratic in baryons (i.e. nucleons), and four-baryon terms:

$$L_{\chi PT}^{\text{Symmetric}} = L_{\chi PT}^{\pi \text{Symmetric}} + L_{\chi PT}^{N \text{Symmetric}} + L_{\chi PT}^{4-N \text{Symmetric}}$$

with (as in \ref{2})

$$L_{\chi PT}^{\pi \text{Symmetric}} = \frac{f_\pi^2}{4 \lambda} \text{Tr} \partial_\mu \Sigma \partial_\mu \Sigma^\dagger + L_{\chi PT}^{\pi \text{Symmetric \ non-Analytic}}$$

$$L_{\chi PT}^{N \text{Symmetric}} = \bar{N} \left( i\gamma_\mu D_\mu - m_N \right) N - g_A \bar{N} \gamma_\mu \gamma^5 A_\mu N$$

$$L_{\chi PT}^{4-N \text{Symmetric}} \sim \frac{1}{f_\pi^2} \left( \bar{N} \gamma_\mu \gamma_\nu N \right) \left( \bar{N} \gamma_\sigma \gamma_\tau N \right) + ...$$
As described below [2], the parentheses in the four-nucleon Lagrangian indicate the order of SU(2) index contraction, and + + + indicates that one should include all possible combinations of such contractions. As usual, \( \gamma^{ab} \equiv (1, \gamma^\mu, i\sigma^{\mu\nu}\gamma^5, \gamma^5) \), for \( a, b = 1, \ldots, 16 \) (with \( \sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] \)). These are commonly referred to as scalar (S), vector (V), tensor (T), axial-vector (A), and pseudo-scalar (P) respectively.

### A.4 SU(2)_L \times SU(2)_R invariant 4-nucleon contact interactions

Focus on the 4-fermion terms in (A.18).

Using the completeness relation for \( 2 \times 2 \) matrices (sum over \( A = 0, 3 \))

\[
\sigma^B = (1, \vec{\sigma}); \quad \delta_{cf}\delta_{ed} = \frac{1}{4} \sum_{B=0}^3 \sigma^B_{cd}\sigma^B_{ef}. \tag{A.19}
\]

(We use \( a...\sigma \) for relativistic spinor indices, while \( a...f \) are isospin indices.) Both iso-scalar and iso-vector 4-nucleon contact interactions appear in the \( SU(2)_L \times SU(2)_R \) invariant Lagrangian:

\[
L_{\text{chiral}}^{4-\text{symmetric}} = \frac{1}{f_\pi} C_{\cd}^{T=0} (N_c^a \gamma^{ab} \sigma_{ad}) (N_c^b \gamma^{ab} \sigma_{bd}) + \frac{1}{f_\pi^2} C_{\cd}^{T=1} (N_c^a \gamma^{ab} \sigma_{ad}) (N_c^b \gamma^{ab} \sigma_{bd})
\]

\[
\to \frac{1}{f_\pi^2} C_{\cd}^{T=0} (N_c^a \gamma^{ab} \sigma_{ad}) (N_c^b \gamma^{ab} \sigma_{bf} \gamma^{ab} \sigma_{cf}) + \frac{1}{f_\pi^2} C_{\cd}^{T=1} (N_c^a \gamma^{ab} \sigma_{ad}) (N_c^b \gamma^{ab} \sigma_{bf} \gamma^{ab} \sigma_{cf}) \tag{A.20}
\]

\[
= \frac{1}{f_\pi^2} C_{\cd}^{T=0} (N_c^a \gamma^{ab} \sigma_{ad}) (N_c^b \gamma^{ab} \sigma_{bf} \gamma^{ab} \sigma_{cf}) + \frac{1}{f_\pi^2} C_{\cd}^{T=1} (N_c^a \gamma^{ab} \sigma_{ad}) (N_c^b \gamma^{ab} \sigma_{bf} \gamma^{ab} \sigma_{cf})
\]

\[
= \frac{1}{f_\pi^2} C_{\cd}^{T=0} (N_c^a \gamma^{ab} \sigma_{ad}) (N_c^b \gamma^{ab} \sigma_{bf} \gamma^{ab} \sigma_{cf}) + \frac{1}{4} \sum_{B=0}^3 \sum_{B=0}^3 C_{\cd}^{T=1} (N_c^a \sigma^B_{cd} \gamma^{ab} \sigma^B_{ad}) (N_c^b \sigma^B_{ef} \gamma^{ab} \sigma^B_{bf} \gamma^{ab} \sigma^B_{cf} \gamma^{ab} \sigma^B_{cf}).
\]

### A.5 Non-analytic NGB pion interactions

Non-analytic interactions of pions are induced in quantum loops. There are situations where loop effects are important and can be qualitatively distinguished from tree-level interactions by their analytic structure. For example, the \( \pi_a + \pi_b \to \pi_c + \pi_d \) scattering amplitude contains a term [17] in the chiral
Here \( s = (p_a + p_b)^2 \), \( t = (p_a - p_c)^2 \), \( u = (p_a - p_d)^2 \) are Mandelstam variables and \( \kappa \) is an arbitrary renormalization scale.

This paper crucially concerns itself with the far-infrared region of NGB pion momenta. The imaginary part of \( \ln(-s\kappa) \) arises from the unitarity of the S-matrix and is related to a total cross-section. The real part of \( \ln(-s\kappa) \) diverges in the far-infrared, and might have been important to \( \chi_{NL} \). We show that it is not! 11

Following [85], we pack this non-analytic S-Matrix \( O (\Lambda_0 \chi_{SB}) \) term, and all other such non-analytic terms in the pure pion sector, into a non-analytic effective Lagrangian \( L_{\pi; \text{Symmetric}}^{\text{non-Analytic}} \), which is also to be analyzed at tree-level.

### Appendix B 4-nucleon contact interactions in Static\( \chi \)NLs

#### B.1 Boson-exchange-inspired vs. excited-nucleon-inspired 4-nucleon contact interactions

We wish to study the ground state expectation value of \( L_{\chi_{PT}}^{4-N; \text{Symmetric}} \). Using (A.20)

\[
\langle 0 | \chi_{NL} | - L_{\chi_{PT}}^{4-N; \text{Symmetric}} | \chi_{NL} \rangle_0 = \frac{1}{2 f_\pi^2} \sum_{\sigma \sigma'} \left\{ \left( \chi_{NL} \right| C_{\sigma \sigma'}^{T=0} \left( \bar{N}_{c}^{\sigma'} a^{B_{c}} N_{e}^{B_{c}} \right) \left( \bar{N}_{c}^{\sigma} a^{B_{c}} N_{e}^{B_{c}} \right) | \chi_{NL} \rangle \right\} + \frac{1}{4} \sum_{B} \left\{ \langle \chi_{NL} | C_{\sigma \sigma'}^{T=1} \left( \bar{N}_{c}^{\sigma'} a^{B_{c}} N_{e}^{B_{c}} \right) \left( \bar{N}_{c}^{\sigma} a^{B_{c}} N_{e}^{B_{c}} \right) | \chi_{NL} \rangle \right\}.
\]

Now introduce a complete set of states

\[
1 = \left| \chi_{NL} \right\rangle_0 \langle \chi_{NL} \right| + \sum_{\Psi \neq \text{Static} \chi_{NL}} \left| \Psi \right\rangle \langle \Psi \right|
\]

and classify 4-nucleon Static\( \chi \)NL interaction terms as either inspired by “boson

\[\text{limit.}\]

\[
L_{\pi; \text{Symmetric}}^{\text{non-Analytic}} \leftrightarrow \left[ -\delta_{ab}\delta_{cd} \frac{s^2}{32\pi^2} - \delta_{ac}\delta_{bd} \frac{3s^2 + u^2 - t^2}{196\pi^2} \right. \\
\left. - \delta_{ad}\delta_{bc} \frac{3s^2 + t^2 - u^2}{196\pi^2} \right] \ln \left( -\frac{s}{\kappa} \right) + \text{cross-terms}
\]

(\text{A.21})

It clarifies things to regularize \( \ln(-\frac{s}{\kappa}) \) with \( |s_{IR}| > 0 \) in the Infra-Red.
Taking expectation values inside the Static\chi NL exchange”

\[-L_{\text{BosonExchange}} \text{\ Static}\chi \text{NL} = \frac{1}{2f_{\pi}} \sum_{d}\chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \left( \chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \right)
\]

or “excited-nucleon” inspired

\[-L_{\text{ExcitedNucleon}} \text{\ Static}\chi \text{NL} = \frac{1}{2f_{\pi}} \sum_{d}\chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \left( \chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \right)
\]

A useful theorem is

\[
\frac{1}{4} \left\langle \chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \left( \chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \right) + \frac{1}{4} \left\langle \chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \left( \chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \right) \right. \\
= \frac{1}{2} \left\langle \chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \left( \chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \right)
\]

Going forward, we will use the notation \( \equiv \chi \text{\ NL} \bigg|_{0} \) and \( \equiv \left( \chi \text{\ NL} \bigg|_{0} \chi \text{\ NL} \right) \) in this Appendix.

B.2 Contact-interactions that mimic hadron-exchange

Taking expectation values inside the Static\chi NL,

\[-L_{\text{BosonExchange}} \text{\ Static}\chi \text{NL} = \frac{1}{2f_{\pi}} \sum_{d}\chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \left( \chi \text{\ NL} \left( N_{c}^{d} \gamma^{\alpha \beta} N_{c}^{d} \right) \chi \text{\ NL} \bigg|_{0} \right)
\]

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The factorization in $L_{\text{BosonExchange}}$, and its name, are inspired by a simple picture of forces carried by heavy hadronic-boson exchange; i.e. we have integrated out the auxiliary fields:

- Lorentz-scalar isoscalar $\sigma$, with chiral coefficient $C^{T=0}_S$;
- Lorentz-vector isoscalar $\omega_\mu$ with chiral coefficient $C^{T=0}_V$;
- Lorentz-scalar isovector $\delta$, with chiral coefficient $C^{T=1}_S$;
- Lorentz-vector isovector $\rho_\mu$, with chiral coefficient $C^{T=1}_V$.

To order $\Lambda^0_{SB}$, the only 4-nucleon contact terms allowed by local $SU(2)\chi PT$ symmetry are exhibited in (B.3) (i.e. (B.5)) and (B.4). Note that isospin operators $\bar{t} = \frac{1}{2}\sigma_{Pauli}$ have appeared. However, quantum-loop power counting requires inclusion of nucleon Lorentz-spinor-interchange interactions, in order to enforce anti-symmetrization of fermion wavefunctions. These are the same magnitude, $(\Lambda \chi SB)^0$, as direct interactions. The empirical nuclear models of Manakos and Mannel [86, 87] were specifically built to include such spinor-interchange terms.

Explicit inclusion of spinor-interchange terms yields a great technical advantage for the liquid approximation: it allows us to treat $\chi NL$ in Hartree-Fock approximation, i.e. including fermion wavefunction anti-symmetrization, rather than in less-accurate Hartree approximation.

Because of normal-ordering, such point-coupling contact spinor-interchange terms don’t appear in the analysis of the deuteron [34, 35], which has only 1 proton and 1 neutron.

### B.3 Contact-interactions, including spinor-interchange terms enforcing effective anti-symmetrization of fermion wavefunctions in the Hartree-Fock approximation

In this section, we write an effective $\chi NL$ Lagrangian in terms of the 10 independent chiral coefficients $C^{T=0}_S, C^{T=0}_V, C^{T=0}_A, C^{T=0}_P, C^{T=1}_S, C^{T=1}_V, C^{T=1}_A, C^{T=1}_P$ governing 4-nucleon contact interactions.

For pedagogical simplicity, we first focus on the “boson-exchange-inspired” terms, with power-counting contact-interactions of order $(\Lambda \chi SB)^0$. “Direct” terms depend only on $C^{T=0}_S, C^{T=0}_V, C^{T=1}_S, C^{T=1}_V$; because isoscalar $(C^{T=0}_T, C^{T=0}_A, C^{T=0}_P$ and $C^{T=0}_T$) and isovector $(C^{T=1}_T, C^{T=1}_A, C^{T=1}_P$) vanish when evaluated in the liquid. “Spinor-interchange” terms depend all 10 coefficients after Fierz rearrangement. Such terms do not appear in the $SU(2)\chi PT$ analysis of the deuteron ground state, because it only has 1 proton and 1 neutron. The combination of direct and spinor-interchange terms (which we refer to below as “Total”) depend on all 10 coefficients.
Because of the inclusion of spinor-interchange terms, Hartree treatment of the resultant Static\textsubscript{NL} Lagrangian is equivalent to Hartree-Fock treatment of the liquid. When building the semi-classical liquid quantum state, this enforces the antisymmetrization of the fermion wavefunctions. A crucial observation is that the resultant liquid depends on only four independent chiral coefficients, $C^{S}_{200}$, $C^{V}_{200}$, $\overline{C^{S}_{200}}$, and $\overline{C^{V}_{200}}$. These provide sufficient free parameters to balance the scalar repulsive force carried by $C^{S}_{200}$ and $\overline{C^{S}_{200}}$ against the vector repulsive force carried by $C^{V}_{200}$ and $\overline{C^{V}_{200}}$ when fitting to the experimentally observed structure of ground-state nuclei. This is the case for our Non-topological Soliton nuclear model, where $C^{S}_{200} - \frac{1}{2} C^{V}_{200} < 0$ and $C^{V}_{200} - \frac{1}{2} C^{S}_{200} > 0$, and we conjecture it to persist in Density Functional and Skyrme nuclear models.

Motivated by the empirical success of Non-topological Soliton, Density Functional and Skyrme nuclear models, we also conjecture that excited-nucleon-inspired contact-interaction terms are small, and that the simple picture of scalar attraction balanced against vector repulsion persists when including them. But such analysis is beyond the scope of this paper.

### B.3.1 Lorentz Vector (V) and Axial-vector (A) forces

$$\left\{ L^{-N.V.A} \right\} = L^{V.A}_{\text{StaticNL}}$$

$$-L^{V.A}_{\text{StaticNL}} = \frac{1}{2 f^2} \sum_{s,f=V,A} \left[ C^{T=0}_{sf} \left< (\overline{C^{S}_{200}} \gamma^s \alpha \beta N^{B}_{c}) \left< (\overline{C^{V}_{200}} \gamma^s \alpha \beta N^{A}_{c}) \right> \right> \right] + \frac{1}{4} \sum_{B} C^{T=1}_{sf} \left< \left( \overline{C^{S}_{200}} \sigma^{B}_{cd} \gamma^{s} \alpha \beta N^{d}_{c} \right) \left< (\overline{C^{V}_{200}} \sigma^{B}_{cd} \gamma^{s} \alpha \beta N^{d}_{f}) \right> \right>$$

$$-L^{V.A}_{\text{StaticNL,ExcitedNucleon}} = \frac{1}{2 f^2} \sum_{s,f=V,A} \left[ C^{T=0}_{sf} \left< \left( \overline{C^{S}_{200}} \gamma^s \alpha \beta N^{B}_{c} \right) \left< \overline{C^{V}_{200}} \gamma^s \alpha \beta N^{A}_{f} \right> \right> \right] + \frac{1}{4} \sum_{B} C^{T=1}_{sf} \left< \left( \overline{C^{S}_{200}} \sigma^{B}_{cd} \gamma^{s} \alpha \beta N^{d}_{c} \right) \left< \overline{C^{V}_{200}} \sigma^{B}_{cd} \gamma^{s} \alpha \beta N^{d}_{f} \right> \right>$$

We have

$$-L^{V.A}_{\text{StaticNL}} = \frac{1}{2 f^2} \sum_{s,f=V,A} \left[ C^{T=0}_{sf} \left< \left( \overline{p^{2}_{\gamma^{s} \alpha \beta N^{B}_{c}}} \right) \left< (n^{2}_{\gamma^{s} \alpha \beta N^{A}_{f}}) \right> \right> \right]$$

$$+ \left[ C^{T=0}_{sf} + \frac{1}{2} C^{T=1}_{sf} \right] \left< \left( \overline{p^{2}_{\gamma^{s} \alpha \beta N^{B}_{c}}} \right) \left< (n^{2}_{\gamma^{s} \alpha \beta N^{A}_{f}}) \right> \right>$$

$$- L^{V.A}_{\text{StaticNL,ExcitedNucleon}}$$

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Direct terms: The properties of Static NLs vastly simplify this expression

\[-L^{V,A}_{\text{Static NL Direct}} = \frac{1}{2f^2} C_{V}^{T=0} \left\{ 2 \langle p^\dagger p \rangle \langle n^\dagger n \rangle \right\} + \frac{1}{2f^2} \left[ C_{V}^{T=0} + \frac{1}{2} C_{V}^{T=1} \right] \left\{ \langle p^\dagger p \rangle \langle p^\dagger p \rangle + \langle n^\dagger n \rangle \langle n^\dagger n \rangle \right\} - L^{V,A}_{\text{Static NL Excited Nucleon Direct}} \]

with simplified notation \( \langle p^\dagger p \rangle \langle n^\dagger n \rangle \equiv \langle p^\dagger p \rangle \langle n^\dagger n \rangle \).

Spinor-interchange terms: After interchanging the appropriate spinors, normal ordering creation and annihilation operators, and Fierz re-arrangement, spinor-interchange contributions depend on \( C_{V}^{T=0}, C_{A}^{T=0}, C_{V}^{T=1}, C_{A}^{T=1} \).

\[-L^{V,A}_{\text{Static NL: Spinor Interchange}} = \frac{1}{2f^2} \left[ - \left( C_{V}^{T=0} + \frac{1}{2} C_{V}^{T=1} \right) + \left( C_{A}^{T=0} + \frac{1}{2} C_{A}^{T=1} \right) \right] \left\{ \langle p^\dagger p \rangle \langle p^\dagger p \rangle + \langle n^\dagger n \rangle \langle n^\dagger n \rangle \right\} \]

where we have divided \( p = p_L + p_R \) and \( n = n_L + n_R \) into left-handed and right-handed spinors.

Total direct and spinor-interchange terms:

\[-L^{V,A}_{\text{Static NL Total}} = \frac{1}{2f^2} C_{V}^{T=0} \left\{ 2 \langle p^\dagger p \rangle \langle n^\dagger n \rangle \right\} + \frac{1}{2f^2} \left[ C_{V}^{T=0} + \frac{1}{2} C_{V}^{T=1} \right] \left\{ 2 \langle p^\dagger p \rangle \langle p^\dagger p \rangle + 2 \langle n^\dagger n \rangle \langle n^\dagger n \rangle \right\} \]

The reader should note the cancellation of the term

\[ \frac{1}{2f^2} \left[ C_{V}^{T=0} + \frac{1}{2} C_{V}^{T=1} \right] \left\{ \langle p^\dagger p \rangle \langle p^\dagger p \rangle + \langle n^\dagger n \rangle \langle n^\dagger n \rangle \right\} , \]
showing that vector-boson exchange cannot carry forces between same-handed protons, or between same-handed neutrons.

Significant simplification follows because Static $\chi$NLs are defined to have equal left-handed and right-handed densities

$$\langle p_L^\dagger p_L \rangle = \langle p_R^\dagger p_R \rangle = \frac{1}{2} \langle p^\dagger p \rangle$$  \hspace{1cm} (B.11)

$$\langle n_L^\dagger n_L \rangle = \langle n_R^\dagger n_R \rangle = \frac{1}{2} \langle n^\dagger n \rangle.$$  

so that the contribution of (B.10) to the Lorentz-spinor-interchange Lagrangian is

$$-L_{V,A}^{\text{Static} \chi \text{NL}; \text{Total}} = \frac{1}{2 f_{\pi}^2} C_V^{(200)} \left\{ N^\dagger N \right\} \left\{ N^\dagger N \right\}$$

$$\left\{ N^\dagger N \right\} \left\{ N^\dagger N \right\} + 4 \left\{ N^\dagger \gamma_5 N \right\} \left\{ N^\dagger \gamma_5 N \right\}$$

$$-L_{V,A}^{\text{Static} \chi \text{NL}; \text{ExcitedNucleon}; \text{Total}}$$

with

$$C_V^{(200)} = C_V^{T=0}$$

$$C_V^{(200)} = \frac{1}{2} \left[ - C_V^{T=0} + C_A^{T=0} + \frac{1}{2} C_V^{T=1} + \frac{1}{2} C_A^{T=1} \right]$$  \hspace{1cm} (B.13)

The crucial observation is that (B.12, B.13) depend on just two independent chiral coefficients, $C_V^{(200)}$ and $C_V^{(200)}$, instead of four, while still providing sufficient free parameters to fit the vector repulsive force (i.e. within Non-topological Soliton, Density Functional and Skyrme nuclear models) up to power-counting order $(\Lambda_{\chi SB})^0$, to the experimentally observed structure of ground-state nuclei.

### B.3.2 Lorentz Scalar (S), Tensor (T) and Pseudo-scalar (P) forces

$$\left\{ L_{A-H: \text{ScalarTensor Pseudoscalar}} \right\} = L_{\text{ScalarTensor Pseudoscalar}}^{\text{Static} \chi \text{NL}}$$

$$-L_{\text{ScalarTensor Pseudoscalar}}^{\text{Static} \chi \text{NL}} = \frac{1}{2 f_{\pi}^2} \sum_{id=ST, P} \left\{ C_V^{T=0} \left( N^\dagger e_c \gamma^a d c B N^\beta_c \right) \right\} \left\{ N^\dagger e_c \gamma^a d c B N^\beta_c \right\}$$

$$+ \frac{1}{4} \sum_B C_V^{T=1} \left( N^\dagger e_c \gamma^a d c B N^\beta_c \right) \left( N^\dagger e_c \gamma^a d c B N^\beta_c \right)$$

$$-L_{\text{ScalarTensor Pseudoscalar}}^{\text{Static} \chi \text{NL}; \text{ExcitedNucleon}}$$

$$-L_{\text{ScalarTensor Pseudoscalar}}^{\text{Static} \chi \text{NL}; \text{ExcitedNucleon}} = \frac{1}{2 f_{\pi}^2} \sum_{id=ST, P} \left\{ C_V^{T=0} \left( N^\dagger e_c \gamma^a d c B N^\beta_c \right) \right\} \left\{ N^\dagger e_c \gamma^a d c B N^\beta_c \right\}$$

$$+ \frac{1}{4} \sum_B C_V^{T=1} \left( N^\dagger e_c \gamma^a d c B N^\beta_c \right) \left( N^\dagger e_c \gamma^a d c B N^\beta_c \right).$$
We have

$$-L^{\text{Scalar Tensor Pseudoscalar}}_{\text{Static } \chi NL} = \frac{1}{2 f_R^2} \sum_{s = \pm S, T, P} \left[ C_{T=0}^{s=1} \left\langle \frac{2}{(p_c^2 + m_c^2)} \left\langle \left( n_T^2 + m_T^2 \right) \gamma_\mu n_\nu \right\rangle \right] + \left[ C_{T=0}^{s=1} + \frac{1}{2} C_{T=1}^{s=1} \right] \left\langle \left( p_c^2 + m_c^2 \right) \gamma_\mu p_\nu \right\rangle \right] + \left( \frac{n_T^2 + m_T^2}{n_T^2 + m_T^2} \right) \gamma_\mu n_\nu$$

Direct terms: The properties of Static \( \chi NLs \) give

$$-L^{\text{Scalar Tensor Pseudoscalar}}_{\text{Static } \chi NL; \text{Direct}} = \frac{1}{2 f_R^2} C_S^{T=0} \left\langle \left( n \right) \left\langle n \right\rangle \right\rangle$$

Spinor-interchange terms: Spinor-interchange contributions depend on 6 chiral coefficients: isoscalars \( C_{T=0}^{s=0}, C_{T=0}^{s=1}, C_{T=0}^{s=1} \) and isovectors \( C_{T=1}^{s=0}, C_{T=1}^{s=1}, C_{T=1}^{s=1} \).

$$-L^{\text{Scalar Tensor Pseudoscalar}}_{\text{Static } \chi NL; \text{Spinor Interchange}} = \frac{1}{2 f_R^2} C_S^{T=0} \left( \left\langle \left( p \right) \left\langle p \right\rangle \right\rangle + \left\langle \left( n \right) \left\langle n \right\rangle \right\rangle \right) - L^{\text{Scalar Tensor Pseudoscalar}}_{\text{Static } \chi NL; \text{Excited Nucleon; Direct}}$$

Total direct and spinor-interchange terms: As above, the fact that Static \( \chi NLs \) are defined to have equal left-handed and right-handed scalar densities simplifies the total direct and spinor-interchange contribution:

$$-L^{\text{Scalar Tensor Pseudoscalar}}_{\text{Static } \chi NL; \text{Total}} = \frac{1}{2 f_R^2} C_S^{T=0} \left\langle \left( \left( n \right) \left\langle n \right\rangle \right\rangle \right\rangle$$

with

$$C_{S}^{T=0} = C_{T=0}^{s=0}$$

$$-C_{S}^{T=0} = \frac{1}{2} \left( C_{T=0}^{s=1} + \frac{5}{4} C_{T=1}^{s=1} + 3 \left( C_{T=0}^{s=1} + \frac{1}{2} C_{T=1}^{s=1} \right) \right)$$
Once again we find that (B.18, B.19) depend on just two independent chiral coefficients, $C_{200}^S$ and $C_{200}^S$, instead of six, while still providing sufficient free parameters to fit the scalar attractive force (i.e. within Non-topological Soliton, Density Functional and Skyrme nuclear models) up to power-counting order $(\Lambda_{\chi B} S^0)$, to the experimentally observed structure of ground-state nuclei.

**Appendix C  Nucleon bi-linears and nuclear currents in Static $\chi$NL**

The structure of Static $\chi$NL suppresses various nucleon bi-linears:

- **Vectors’ space-components**: because it is a 3-vector, parity odd and stationary

\[
0 \langle \chi_{NL} (N_L \gamma^\alpha \sigma^\beta N_R^c) | \chi_{NL} \rangle_0 \sim 0 \quad (C.1)
\]

- **Tensors**: because the local expectation value of the nuclear spin $\vec{s} = \frac{1}{2} \hat{\vec{s}} \approx 0$

1. $\sigma^{0j}$:

\[
0 \langle \chi_{NL} (\overline{N_L} \sigma^{0j} \sigma^\beta N_R^c) | \chi_{NL} \rangle_0 \\
= 0 \langle \chi_{NL} (\overline{N_L} \sigma^{0j} N_R) | \chi_{NL} \rangle_0 + 0 \langle \chi_{NL} (\overline{N_R} \sigma^{0j} N_L) | \chi_{NL} \rangle_0 \\
= 2 \langle \chi_{NL} (\overline{N_L} \begin{pmatrix} 0 & \vec{s}_j \\ \vec{s}_j & 0 \end{pmatrix} N_R) | \chi_{NL} \rangle_0 \\
+ 2 \langle \chi_{NL} (\overline{N_R} \begin{pmatrix} 0 & \vec{s}_j \\ \vec{s}_j & 0 \end{pmatrix} N_L) | \chi_{NL} \rangle_0 \approx 0 \quad (C.2)
\]

2. $\sigma^{ij}$:

\[
0 \langle \chi_{NL} (\overline{N_L} \sigma^{ij} \sigma^\beta N_R^c) | \chi_{NL} \rangle_0 \\
= 0 \langle \chi_{NL} (\overline{N_L} \sigma^{ij} N_R) | \chi_{NL} \rangle_0 + 0 \langle \chi_{NL} (\overline{N_R} \sigma^{ij} N_L) | \chi_{NL} \rangle_0 \\
= -2i \epsilon_{ijk} \langle \chi_{NL} (\overline{N_L} \vec{s}_k N_R) | \chi_{NL} \rangle_0 \\
- 2i \epsilon_{ijk} \langle \chi_{NL} (\overline{N_R} \vec{s}_k N_L) | \chi_{NL} \rangle_0 \approx 0 \quad (C.3)
\]

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• Axial-vectors: because $p_L, p_R$ are equally represented in Static $\chi_{NL}$, as are $n_L, n_R$

\[
\begin{align*}
0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^{\alpha\beta} N_R^\beta) \right| \chi_{NL} \right\rangle \\
= 0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^5 N_L) \right| \chi_{NL} \right\rangle + 0\left\langle \chi_{NL} \left| (N_R^\alpha \gamma^5 N_R) \right| \chi_{NL} \right\rangle \\
= -0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^5 N_L) \right| \chi_{NL} \right\rangle + 0\left\langle \chi_{NL} \left| (N_R^\alpha \gamma^5 N_R) \right| \chi_{NL} \right\rangle \\
\simeq 0
\end{align*}
\]

(C.4)

• Pseudo-scalars: because Static $\chi_{NL}$ are of even parity

\[
\begin{align*}
0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^{\alpha\beta} N_R^\beta) \right| \chi_{NL} \right\rangle \\
= 0\left\langle \chi_{NL} \left| (N_R^\alpha \gamma^5 N_L) \right| \chi_{NL} \right\rangle + 0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^5 N_R) \right| \chi_{NL} \right\rangle \\
= -0\left\langle \chi_{NL} \left| (N_R^\alpha \gamma^5 N_L) \right| \chi_{NL} \right\rangle + 0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^5 N_R) \right| \chi_{NL} \right\rangle \\
\simeq 0
\end{align*}
\]

(C.5)

Therefore, various Lorentz and isospin representations are suppressed in Static $\chi_{NL}$. In summary: Isoscalars

\[
\begin{align*}
0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^5 N_L) \right| \chi_{NL} \right\rangle &\neq 0 \\
0\left\langle \chi_{NL} \left| (N_R^\alpha \gamma^5 N_R) \right| \chi_{NL} \right\rangle &\neq 0 \\
0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^{\alpha\beta} N_R^\beta) \right| \chi_{NL} \right\rangle &\simeq 0 \\
0\left\langle \chi_{NL} \left| (N_R^\alpha \gamma^{\alpha\beta} N_L^\beta) \right| \chi_{NL} \right\rangle &\simeq 0 \\
0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^{T,\alpha\beta} N_R^\beta) \right| \chi_{NL} \right\rangle &\simeq 0 \\
0\left\langle \chi_{NL} \left| (N_R^\alpha \gamma^{T,\alpha\beta} N_L^\beta) \right| \chi_{NL} \right\rangle &\simeq 0 \\
0\left\langle \chi_{NL} \left| (N_L^\alpha \gamma^{P,\alpha\beta} N_R^\beta) \right| \chi_{NL} \right\rangle &\simeq 0 \\
0\left\langle \chi_{NL} \left| (N_R^\alpha \gamma^{P,\alpha\beta} N_L^\beta) \right| \chi_{NL} \right\rangle &\simeq 0
\end{align*}
\]

(C.6)
and Isovectors

\[
\langle 0 | \chi_{NL} \left( N_c^3 \gamma^{\alpha \beta} N_d^\beta \right) | \chi_{NL} \rangle_0 = 0 \\
\langle 0 | \chi_{NL} \left( N_c^3 N_d^\alpha N_d^\beta \right) | \chi_{NL} \rangle_0 \neq 0 \\
\langle 0 | \chi_{NL} \left( N_c^3 \gamma^{0 \alpha \beta} N_d^\beta \right) | \chi_{NL} \rangle_0 \neq 0 \\
\langle 0 | \chi_{NL} \left( N_c^3 \gamma^{\alpha \beta} N_d^\beta \right) | \chi_{NL} \rangle_0 = 0 \\
\langle 0 | \chi_{NL} \left( N_c^3 \gamma^{T \alpha \beta} N_d^\beta \right) | \chi_{NL} \rangle_0 \approx 0 \\
\langle 0 | \chi_{NL} \left( N_c^3 \gamma^{A \alpha \beta} N_d^\beta \right) | \chi_{NL} \rangle_0 \approx 0 \\
\langle 0 | \chi_{NL} \left( N_c^3 \gamma^{P \alpha \beta} N_d^\beta \right) | \chi_{NL} \rangle_0 \approx 0
\]

\[ (C.7) \]

Now form the nuclear currents

\[
J^\mu_k = \gamma^\mu_{1k} N \quad k = 1, 2, 3 \\
J^\mu_\pm = J^\mu_1 \pm i J^\mu_2 = \left\{ \begin{array}{c} \bar{\nu} \gamma^\mu n \\ \bar{\nu} \gamma^\mu p \end{array} \right\} \\
J_5^\mu = \frac{1}{2} (\bar{\nu} \gamma^\mu p - \bar{\nu} \gamma^\mu n) \\
J_8^\mu = \sqrt{\frac{3}{2}} (\bar{\nu} \gamma^\mu p + \bar{\nu} \gamma^\mu n) \\
J_{QED}^\mu = \frac{1}{\sqrt{3}} J_8^\mu + J_5^\mu = \bar{\nu} \gamma^\mu p \\
J_{Baryon}^\mu = \frac{2}{\sqrt{3}} J_8^\mu = \bar{\nu} \gamma^\mu p + \bar{\nu} \gamma^\mu n \\
J_5^\mu_k = \gamma^\mu_{5k} N \quad k = 1, 2, 3 \\
J_5^\mu_\pm = J_5^\mu_1 \pm i J_5^\mu_2 = \left\{ \begin{array}{c} \bar{\nu} \gamma^\mu 5^\nu n \\ \bar{\nu} \gamma^\mu 5^\nu p \end{array} \right\} \\
J_3^\mu = \frac{1}{2} (\bar{\nu} \gamma^\mu 5^\nu p - \bar{\nu} \gamma^\mu 5^\nu n) \\
J_8^\mu = \sqrt{\frac{3}{2}} (\bar{\nu} \gamma^\mu 5^\nu p + \bar{\nu} \gamma^\mu 5^\nu n)
\]

\[ (C.8) \]

\[ SU(2)_L \times SU(2)_R \] nucleon currents within Static $\chi_{NL}$ are obedient to its
symmetries

\[ \langle \chi_{NL} | J_0^\mu | \chi_{NL} \rangle \neq 0; \quad \langle \chi_{NL} | \partial_\mu J_3^\mu | \chi_{NL} \rangle \approx 0 \]

\[ \langle \chi_{NL} | J_3^\mu | \chi_{NL} \rangle = 0; \quad \langle \chi_{NL} | \partial_\mu J_3^\mu | \chi_{NL} \rangle = 0 \]

\[ \langle \chi_{NL} | \partial_\mu J_5^\mu | \chi_{NL} \rangle \approx 0; \quad \langle \chi_{NL} | \partial_\mu J_5^{\mu+} | \chi_{NL} \rangle = 0 \]

\[ \langle \chi_{NL} | J_\mu | \chi_{NL} \rangle \neq 0; \quad \langle \chi_{NL} | \partial_\mu J_8^\mu | \chi_{NL} \rangle \approx 0 \]

\[ \langle \chi_{NL} | J_{QED}^\mu | \chi_{NL} \rangle \neq 0; \quad \langle \chi_{NL} | \partial_\mu J_{QED}^\mu | \chi_{NL} \rangle \approx 0 \]

\[ \langle \chi_{NL} | J_{Baryon}^\mu | \chi_{NL} \rangle \neq 0; \quad \langle \chi_{NL} | \partial_\mu J_{Baryon}^\mu | \chi_{NL} \rangle \approx 0 \]

\[ \langle \chi_{NL} | J_{QED}^{\mu=1,2,3} | \chi_{NL} \rangle \approx 0; \quad \langle \chi_{NL} | J_{Baryon}^{\mu=1,2,3} | \chi_{NL} \rangle \approx 0 \]

\[ \langle \chi_{NL} | J_{QED}^{\mu=1,2,3} | \chi_{NL} \rangle \approx 0; \quad \langle \chi_{NL} | J_{Baryon}^{\mu=1,2,3} | \chi_{NL} \rangle \approx 0 \] (C.9)