COTORSION TORSION TRIPLES AND THE REPRESENTATION
THEORY OF FILTERED HIERARCHICAL CLUSTERING

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Abstract. We give a full classification of representation types of the subcate-
gories of representations of an \( m \times n \) rectangular grid with monomorphisms
(dually, epimorphisms) in one or both directions, which appear naturally in
the context of clustering as two-parameter persistent homology in degree zero.
We show that these subcategories are equivalent to the category of all rep-
resentations of a smaller grid, modulo a finite number of indecomposables.
This equivalence is constructed from a certain cotorsion torsion triple, which is
obtained from a tilting subcategory generated by said indecomposables.

1. Introduction

Clustering analysis encompasses a wide range of statistical methods for inferring
structure in data. Loosely speaking, a clustering method aims to partition the
data into clusters such that data of different clusters are significantly more different
than data belonging to the same cluster. Such methods play an important role in
unsupervised data analysis.

In order to provide context for our results, we first discuss why clustering methods
are commonly based on the choice of two parameters, explaining the relevance for
studying two parameter grid representations, and also explaining why the special
case of epimorphisms in one parameter direction arises from this scenario.

Define a clustering function to be a map \( \mathcal{C} \) which associates to every finite metric
space \( (M,d) \) a surjective set map \( \mathcal{C}(M,d): M \to \mathcal{C} \) from the points of \( M \) to a set
of clusters \( \mathcal{C} \). We say that two elements \( m \) and \( m' \) in \( M \) are
clustered (with respect to \( \mathcal{C} \)) if \( \mathcal{C}(M,d)(m_1) = \mathcal{C}(M,d)(m_2) \). We write \( \mathcal{C}(M,d) \cong \mathcal{C}(M,d') \) if
\[
\mathcal{C}(M,d)(m_1) = \mathcal{C}(M,d)(m_2) \iff \mathcal{C}(M,d')(m_1) = \mathcal{C}(M,d')(m_2).
\]

Example 1.1. Fix an \( \varepsilon \geq 0 \) and define the geometric graph at scale \( \varepsilon \), \( \mathcal{G}_\varepsilon(M,d) \), to be the graph on \( M \) with an edge connecting \( m \) and \( m' \) if and only if \( d(m,m') \leq \varepsilon \).
The canonical epimorphism \( M \to \pi_0(\mathcal{G}_\varepsilon(M,d)) \) to the connected components
of the geometric graph defines a clustering function. If \( M \) is a subspace of some
Euclidean space (or, more generally, of a length metric space), then an equivalent
clustering function is given by considering the connected components of a union of
closed balls, \( M \to \pi_0(\bigcup_{x \in M} B_{\varepsilon(x)/2}) \), as illustrated in Figure 1.

Given the abundance of different clustering techniques, it is natural to ask what
kind of properties a clustering method may satisfy. Consider the following two
desirable properties of a clustering function:

- **Scale invariance**: For all \( \alpha > 0 \), \( \mathcal{C}(M,d) \cong \mathcal{C}(M,\alpha \cdot d) \).
- **Consistency**: For any two metrics \( d \) and \( d' \) on \( M \) satisfying
  - \( d'(x,y) \geq d(x,y) \) if \( \mathcal{C}(M,d)(x) \neq \mathcal{C}(M,d')(y) \), and
  - \( d'(x,y) \leq d(x,y) \) if \( \mathcal{C}(M,d)(x) = \mathcal{C}(M,d')(y) \),
we have \( \mathcal{C}(M,d) \cong \mathcal{C}(M,d') \).

2010 Mathematics Subject Classification. 16G20, 16S90 (primary); 55N99 (secondary).
Figure 1. Example illustrating the multiscale nature of the clustering problem. For single linkage clustering, the clusters correspond to the connected components of a unions of balls. At a coarse scale, two clusters (dark shading) are apparent. At a finer scale, each of these two clusters appear to decompose further into two subclusters (light shading).

Figure 2. Dendrogram for the point set shown in Figure 1.

It is not hard to see that the clustering method \( S_\varepsilon \) is consistent but not scale invariant, whereas a normalized version of \( S_\varepsilon \) would be scale invariant but not consistent. An immediate consequence of a theorem by Kleinberg [12, Theorem 3.1] is that an isometry invariant clustering function simultaneously satisfying scale invariance and consistency must be trivial, either always returning a single cluster, or always returning a separate cluster for each point.

An implication of this is that (non-trivial) clustering methods are inherently unstable; a small perturbation of the input data may produce a vastly different graph. Furthermore, there might be no unique correct scale at which the data should be considered. Indeed, Figure 1 illustrates that what appears to be well-defined clusters at one scale, may reveal a finer structure upon inspection at a smaller scale.

One may attempt to rectify these issues by considering hierarchical clustering methods. Such methods do not assign a single set map to the input metric space, but rather a one-parameter family of set maps \( \{ C_\varepsilon(M,d) : M \rightarrow C \}_{\varepsilon \geq 0} \) such that

\[
C_\varepsilon(M,d)(m_1) = C_\varepsilon(M,d)(m_2) \implies C_{\varepsilon'}(M,d)(m_1) = C_{\varepsilon'}(M,d)(m_2)
\]

for all \( \varepsilon \leq \varepsilon' \). The output of a hierarchical clustering method applied to a metric space can be visualized by means of a rooted tree called a dendrogram, see Figure 2 for an example.

**Example 1.2.** The clustering method from Example 1.1 based on geometric graphs \( \mathcal{G}_\varepsilon(M,d) \) extends naturally to a hierarchical clustering method. Indeed, observe that there is an inclusion of graphs \( \mathcal{G}_\varepsilon(M,d) \hookrightarrow \mathcal{G}_{\varepsilon'}(M,d) \) for all \( \varepsilon \leq \varepsilon' \). By post-composing with \( \pi_0 \), this yields a hierarchical clustering method \( S_- = \pi_0 \circ \mathcal{G}_- \)
commonly referred to as **single linkage hierarchical clustering**. For other hierarchical methods such as **average** and **complete linkage hierarchical clustering**, see, e.g., [16].

Carlsson and Memoli [6] give examples showing that average and complete linkage clustering are unstable with respect to perturbation of the input data. Intuitively, this means that a small distortion to the input data may produce a vastly different rooted tree. In contrast, they show that single linkage clustering is stable with respect to the Gromov–Hausdorff distance. Furthermore, they go on to classify single linkage clustering as the *unique functorial* hierarchical clustering method (under mild additional assumptions; see [6, Theorem 7.1] for the precise statement). In practice, however, other hierarchical methods are preferred over the single linkage, as it suffers from the so-called *chaining effect*. Intuitively, this means that single linkage clustering may fail to detect two dense regions connected by regions of low density.

One may attempt to rectify this by considering the hierarchical clustering at a specific density threshold. However, similar to fixing a geometric scale above, such choices would lead to instability in the method. Therefore we will consider clustering methods which offer a multi-scale view of both density and scale. This approach to clustering was first considered in [5].

Define a *filtration* of a metric space \((M, d)\) to be a family of subspaces \(\{(M_\delta, d)\}_{\delta \geq 0}\) such that \(M_\delta \subseteq M_{\delta'}\) for \(\delta \leq \delta'\). For example, let \(\sigma: M \rightarrow [0, \infty)\) be any function (e.g., a density estimate) and define a filtration of \(M\) by letting \(M_\delta = \sigma^{-1}[\delta, \infty)\).

A **filtered hierarchical clustering method** is a map which associates to every \(M_\delta\) in a filtration of \(M\) a one-parameter family of set maps \(\{C_\varepsilon(M_\delta, d): M_\delta \rightarrow C\}_{\varepsilon \geq 0}\) such that

\[
C_\varepsilon(M_\delta, d)(m_1) = C_\varepsilon(M_\delta, d)(m_2) \implies C_\varepsilon(M_{\delta'}, d)(m_1) = C_\varepsilon(M_{\delta'}, d)(m_2)
\]
for all $\delta \leq \delta'$ and $\epsilon \leq \epsilon'$. As an example, the clustering method of Example 1.1 naturally extends to a hierarchical clustering method for filtered spaces by defining $S_{\epsilon, \delta}(M) = \pi_0(S_\epsilon(M_\delta))$.

For a filtration of $M$ defined by a density estimate $\sigma: M \to [0, \infty)$, the collection of sets $\{S_{\epsilon, \delta}(M)\}_{\epsilon, \delta \geq 0}$ contains plentiful information. For a fixed $\delta$, the family $\{S_{\epsilon, \delta}(M)\}_{\epsilon \geq 0}$ simply yields single linkage clustering of all points in $m \in M$ with $\sigma(m) \geq \delta$. Similarly, fixing a scale $\epsilon$ and by considering the subset $\{S_{\epsilon, \delta}(M)\}_{\delta \geq 0}$ gives a density-based clustering method at a fixed geometric scale, akin to the Topological Mode Analysis method proposed in [7]. Unfortunately, whereas hierarchical clustering methods enjoy simple graphical representations, it is not clear how to visualize hierarchical clustering methods for filtered spaces in any reasonable way.

### 1.1. From sets to vector spaces

We will now rephrase the above constructions in terms of functors. For a finite metric space $(M, d)$, and a hierarchical clustering method $C$, the assignment $\epsilon \mapsto C_\epsilon(M, d)(M) \subseteq C$ defines a functor $C(M, d): [0, \infty) \to \text{set}$. Observe that all the internal morphisms of this functor are surjective. By further post-composing the functor $C(M, d)$ with the free functor $F_k: \text{set} \to \text{mod } k$ to the category of finite dimensional $k$-vector spaces, we get a functor $F_k \circ C(M, d): [0, \infty) \to \text{mod } k$. Furthermore, it is easy to see that all the internal morphisms are epimorphisms. Note that for hierarchical clustering methods defined using the path-component functor $\pi_0$, such as single linkage clustering $\pi_0 \circ S_\epsilon(M, d)$, this can also be interpreted as applying the 0-th homology functor $H_0(\cdot; k)$ with coefficients in $k$, since $H_0(\cdot; k) \simeq F_k \circ \pi_0$. In this case, the resulting functor $H_0(S_\epsilon(M, d); k): [0, \infty) \to \text{mod } k$ is the persistent homology in degree zero of the geometric graphs.

In general, functors of the type $[0, \infty) \to \text{mod } k$ (not necessarily assuming epimorphisms) are commonly referred to as persistence modules, and they are the main objects of study in the field of topological data analysis. What makes such functors so useful is that they decompose as a direct sum of persistence modules $k_I$ (see the beginning Section 2 for the precise definition), where $I \subseteq [0, \infty)$ is an interval [8]. This collection of intervals, typically referred to as a persistence barcode, is then used to extract topological information from the data at hand; a “long” interval corresponds to a topological feature that persists over a significant range.

The process of linearizing through post-composition with the 0-th homology functor allows for the decomposition of the rooted tree associated to the functor $C(M, d)$ into a collection of intervals describing the evolution of the clusters as we increase the scale parameter. The reduction in complexity comes at the expense that we no longer have precise information of which points belong to each cluster. Even though having this precise information of the clusters may be helpful for visualization, the persistence barcode is better suited for statistical analysis and machine learning.

A natural question to ask is whether filtered hierarchical clustering methods can be linearized in a similar fashion such that they decompose into simple components which can be used to interpret the evolution of connected components across multiple scales.

Following the arguments above, we see that a filtered hierarchical clustering method transforms a filtered finite metric space $M = (M_\delta)$ into a functor $C(M, d): [0, \infty)^2 \to \text{set}$, where $C(M, d)_{(\epsilon, \delta)} := C_\epsilon(M_\delta, d)$.

Similarly to above, $C(M, d)_{(\epsilon, \delta)} \to C(M, d)_{(\epsilon', \delta)}$ is a surjection for all $\epsilon \leq \epsilon'$, and post-composing with $F_k$ yields a functor $F_k \circ C(M, d): [0, \infty)^2 \to \text{mod } k$ where the morphisms $F_k \circ C(M, d)_{(\epsilon, \delta)} \to F_k \circ C(M, d)_{(\epsilon', \delta)}$ are all epimorphisms.
It is well known (see Theorem 1.3 below) that the representation theory of functors $[0, \infty)^2 \to \text{mod } k$ is very complicated. But what about the subcategory of functors carrying the additional property that all morphisms are epimorphic along one or both parameters? Or the subcategory of functors whose morphisms are injective along one parameter and surjective along the other? In what follows we shall precisely capture the representation type of such functors under the assumption that they can be re-indexed over a finite regular grid.

To make this precise, denote by $\vec{A}^n$ the quiver $1 \to 2 \to \cdots \to n$. Alternatively, the reader may think of $\vec{A}^n$ as the poset $\{1 < 2 < \cdots < n\}$, or its poset category. We denote by $\vec{A}_m \otimes \vec{A}_n$ the quiver with relations given by the fully commutative grid

$$
(1, 1) \to (2, 1) \to (3, 1) \to \cdots \to (m, 1) \\
(1, 2) \to (2, 2) \to (3, 2) \to \cdots \to (m, 2) \\
(1, n) \to (2, n) \to (3, n) \to \cdots \to (m, n)
$$

Equivalently, we may think of $\vec{A}_m \otimes \vec{A}_n$ as the product of $\vec{A}_m$ and $\vec{A}_n$ in the category of posets or in the category of small categories.

Since our metric spaces are assumed to be finite, we may replace $[0, \infty)^2$ in the above discussion by $\vec{A}_m \otimes \vec{A}_n$. Specifically, we denote by $\text{rep}_k(\vec{A}_m \otimes \vec{A}_n)$ the category of fully commutative grids of finite dimensional $k$-vector spaces. There is a complete classification of when these categories are finite or tame, but unfortunately these are very few cases beyond the one-parameter situation:

**Theorem 1.3 ([15, Theorem 5], [14, Theorem 2.5]).** The category of representations $\text{rep}_k(\vec{A}_m \otimes \vec{A}_n)$ contains finitely many indecomposables precisely in the cases

- $m = 1$ or $n = 1$,
- $(m, n) \in \{(2, 2), (2, 3), (2, 4), (3, 2), (4, 2)\}$.

It is of tame representation type precisely in the cases

- $(m, n) \in \{(2, 5), (3, 3), (5, 2)\}$.

In all other cases it is of wild representation type.

It is not hard to see that the remaining cases are of wild representation type. As an example, the following diagrams show that any representation of a particular quiver of wild representation type lifts to a representation of $\vec{A}_4 \otimes \vec{A}_5$:

As discussed above, the representations coming from hierarchical clustering on a filtered space will contain epimorphisms on all horizontal arrows. We denote by $\text{rep}_k^e(\vec{A}_m \otimes \vec{A}_n)$ and $\text{rep}_k^{**}(\vec{A}_m \otimes \vec{A}_n)$ the full subcategories of $\text{rep}_k(\vec{A}_m \otimes \vec{A}_n)$ consisting of all grid-shaped diagrams of vector spaces such that the horizontal morphisms, respectively all morphisms, are epimorphisms. Given that these are
proper subcategories, one may hope that their indecomposables are classifiable for a wider range of values for $m$ and $n$.

One starting point for this project was the observation that the subcategory of representations $\text{rep}_{\mathbf{k}}^{e,\ast}(\widetilde{A}_m \otimes \widetilde{A}_n)$ can be studied using Auslander–Reiten-theory. To begin with, one observes that this subcategory is closed under extensions and quotients. By [1, Corollaries 3.7 and 3.8], it has almost split sequences, and these are induced by almost split sequences in $\text{rep}_{\mathbf{k}}(\widetilde{A}_m \otimes \widetilde{A}_n)$. In the cases that $\text{rep}_{\mathbf{k}}^{e,\ast}(\widetilde{A}_m \otimes \widetilde{A}_n)$ contains only finitely many indecomposables, these can be obtained by constructing the Auslander–Reiten quiver, starting from the injectives.

One then observes that these finite Auslander–Reiten quivers look very similar to Auslander–Reiten quivers for categories $\text{rep}_{\mathbf{k}}(\widetilde{A}_m \otimes \widetilde{A}_{n-1})$ — see Figure 4. Only the thin modules with rectangular support “in the lower left corner”, modules which we will denote by $k\{1,\ldots,i\} \times \{j,\ldots,m\}$ — see Figure 5a — do not seem to appear in this correspondence.

This observation led us to suspect that the following natural construction, transforming the horizontal epimorphisms into general morphisms using kernels, might be useful in this context — a hope that was ultimately justified by the theorem below.

**Construction 1.4.** Let $X = (X_{i,j}) \in \text{rep}_{\mathbf{k}}^{e,\ast}(\widetilde{A}_m \otimes \widetilde{A}_n)$. We denote by $tX$ the object in $\text{rep}_{\mathbf{k}}(\widetilde{A}_m \otimes \widetilde{A}_n)$ given by

$$(tX)_{i,j} = \ker[X_{i,j} \rightarrow X_{i,n}].$$
One easily observes that \( t \) defines a functor \( \text{rep} \rightarrow \text{rep} \), and that the objects sent to 0 by \( t \) are precisely those where all vertical maps are monomorphisms, that is the objects in \( \text{rep}, \text{m} \).

**Theorem 1.5 (See Corollary 3.17).** The functor \( t \) induces an equivalence

\[
\frac{\text{rep}^e \text{e}(A_m \otimes A_n)}{\text{rep}^m \text{e}(A_m \otimes A_n)} \rightarrow \text{rep} \text{e}(A_m \otimes A_{n-1}).
\]

Moreover, \( \text{rep}^m \text{e}(A_m \otimes A_n) \) consists precisely of all finite direct sums of thin modules of the form \( k \{(x, y) \mid x < i \text{ or } y < j \} \), as depicted in Figure 5a.

Spelled out, this means that the representations in \( \text{rep} \text{e}(A_m \otimes A_n) \) that are surjective in one direction correspond (up to a finitely classified, completely explicit list of direct summands) to the representations in \( \text{rep} \text{e}(A_m \otimes A_{n-1}) \).

We will also see, in Corollary 2.37, that this correspondence preserves the Auslander–Reiten structure.

As an immediate application we can combine Theorem 1.5 with Theorem 1.3 and obtain the following classification.

**Corollary 1.6.** The category \( \text{rep}^e \text{e}(A_m \otimes A_n) \) contains finitely many indecomposables precisely in the cases

- \( n \leq 2 \) or \( m = 1 \);
- \( (m, n) \in \{(2, 3), (3, 3), (4, 3)\} \).

It is of tame representation type precisely in the cases

- \( (m, n) \in \{(3, 5), (4, 3)\} \).

In all other cases it is of wild representation type.

In fact, for \( n \leq 2 \), all the indecomposables are constant modules on connected subsets. In the context of clustering above, this means that \( F_k \circ \mathcal{C}(M, d) \) can be understood from a collection of simple regions if and only if the filtration of \( M \) is trivial or essentially a two-step filtration \( M_{\delta_1} \subseteq M_{\delta_2} \).

For the categories of the form \( \text{rep}^e \text{e}(A_m \otimes A_n) \) we obtain the following variant of Theorem 1.5.

**Theorem 1.7 (Dual of Corollary 3.26).** There is an equivalence

\[
\frac{\text{rep}^e \text{e}(A_m \otimes A_n)}{\mathcal{C}_{e,e}} \cong \text{rep} \text{e}(A_{m-1} \otimes A_{n-1}),
\]

where \( \mathcal{C}_{e,e} \) is the subcategory whose indecomposables are of the form \( k \{(x, y) \mid x < i \text{ or } y < j \} \) as depicted in Figure 5b.

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**Figure 5.** Certain special modules

(a) Diagrammatic depiction of the thin module \( k_{(1, \ldots, i) \times (j, \ldots, n)} \)

(b) Diagrammatic depiction of the module \( k_{|(x, y)|x < i \text{ or } y < j} \) in \( \mathcal{C}_{e,e} \).
Although the representation types of both \( \text{rep}^e(\vec{A}_m \otimes \vec{A}_n) \) and \( \text{rep}^e(\vec{A}_m \otimes \vec{A}_n) \) are wild already for relatively small values of \( m \) and \( n \), it is not clear what indecomposables appear as summands of, say, \( H_k \circ \delta(M, d) \) for some metric space \((M, d)\) whose filtration is given by a “density” function \( \sigma: M \rightarrow [0, \infty) \). Perhaps more interestingly, which indecomposables do we see in relevant data sets? These are questions which hopefully will be answered in future work.

1.2. Torsion and cotorsion. While a direct approach to the above theorem is possible, we chose here to prove it by considering the concepts of torsion and cotorsion pairs. The advantage of this approach is two-fold: Firstly, it is easier to get a feeling for why the proof should work, rather than just a technical verification. Secondly, the result in this language is actually much more general than the above application, and is therefore of independent interest both from a purely representation theoretic point of view and with respect to applicability to other specific instances.

For a formal definition of torsion and cotorsion pairs we refer to Section 2.1. For this introduction, the main point is that both torsion and cotorsion pairs are a way of “orthogonally” decomposing an abelian category into two parts, where “orthogonal” refers to Hom vanishing for torsion and to Ext\(^1\)-vanishing for cotorsion pairs. If a subcategory appears in both a torsion and a cotorsion pair, it is natural to wonder if its two complements are related. The most abstract version of the main result gives a positive answer to this suspicion:

**Theorem 1.8** (See Theorem 2.33). Let \( \mathcal{T}, \mathcal{F}, \mathcal{D} \) be three subcategories of an abelian category, such that \((\mathcal{T}, \mathcal{F})\) is a torsion pair and \((\mathcal{F}, \mathcal{D})\) is a cotorsion pair. Then

\[
\mathcal{T} \cong \frac{\mathcal{D}}{\mathcal{F} \cap \mathcal{D}}.
\]

Closely related results, in particular, equivalences coming from torsion and cotorsion pairs, are discussed in [4]. In particular their Proposition V.5.2 seems to be a special case of our theorem, with a number of additional technical conditions.

From this general result, we will deduce Theorem 1.5 by observing that the triple of subcategories of \( \text{rep}_k(\vec{A}_m \otimes \vec{A}_n) \) given by

\[
(\text{rep}_k(\vec{A}_m \otimes \vec{A}_{n-1}), \text{rep}^e_k(\vec{A}_m \otimes \vec{A}_n), \text{rep}^e(\vec{A}_m \otimes \vec{A}_n))
\]

satisfies the conditions of Theorem 1.8.

2. Background

In this section we establish the general theory that will later be applied to examples coming from multi-parameter persistent homology. This includes recalling known results from tilting theory and the theory of (co)torsion pairs while illustrating the results using an example from ordinary (one-parameter) persistent homology.

Let \((X, \leq)\) be a poset and \( k \) a field. By considering \((X, \leq)\) as a category, we form the abelian category of pointwise finite dimensional (covariant) representations, \( \text{Rep}^{\text{fd}}_k X \), consisting of functors

\[
M: X \rightarrow \text{mod} k
\]

into the category of finite dimensional vector spaces and natural transformations. A subset \( C \) of \( X \) is convex if for every pair \( x, y \in C \), \( x \leq z \leq y \) implies \( z \in C \). For a convex subset \( C \subseteq X \), the constant representation \( k_C \) is defined as

\[
k_C(x) = \begin{cases} k, & x \in C \\ 0, & x \notin C \end{cases}
\]
Remark 2.4. This is seen by considering the short exact sequence associated to \( A \). Putting \( P_x = k_{\{y \in X \mid y \geq x\}} \), we observe the following standard fact: For \( x \in X \) and \( M \in \text{Rep}^{\text{fpd}}_k X \) there is an isomorphism

\[
\text{Hom}_{\text{Rep}_k X}(P_x, M) \cong M(x),
\]

natural in both \( x \) and \( M \). This is an application of the Yoneda lemma; a natural transformation \( \eta: P_x \rightarrow M \) is mapped to \( \eta_x(1) \in M(x) \), and an element \( m \in M(x) \) is mapped to the uniquely determined natural transformation sending \( 1 \in P_x(x) = k \) to \( m \in M(x) \). In particular, \( \text{Hom}_{\text{Rep}_k X}(P_x, -) \) is right exact, so \( P_x \) is a projective object of \( \text{Rep}^{\text{fpd}}_k X \).

Based on this, a representation \( M \in \text{Rep}^{\text{fpd}}_k X \) is \textit{finitely generated} if there exists a finite indexing set \( I \), a set of elements \( \{x_i \in X\}_{i \in I} \) and an epimorphism of functors \( \bigoplus_{i \in I} P_{x_i} \rightarrow M \). It is \textit{finitely presented} if, additionally, the kernel of every such epimorphism is again finitely generated. The full subcategory of finitely presented representations is denoted \( \text{Rep}^{\text{fp}}_k X \subseteq \text{Rep}^{\text{fpd}}_k X \).

The following will be our running example.

Example 2.1. Fix a field \( k \) and let \( \mathbb{R}_{\geq 0} \) be the totally ordered set of non-negative real numbers considered as a category. For each \( x \geq 0 \), the projective \( P_x \) in \( \text{Rep}^{\text{fpd}}_k \mathbb{R}_{\geq 0} \) is concretely given as \( k_{[x, \infty)} \).

The indecomposable objects in this category are completely classified by the interval representations \( k_I \), where \( I \) is an (open, half-open or closed) interval; see [8, Theorem 1.1].

The abelian subcategory \( \text{Rep}^{\text{fp}}_k \mathbb{R}_{\geq 0} \) of finitely presented representations therefore has as indecomposable objects the \( k_{[x,y)} \), where we allow \( y = \infty \), as they have presentations

\[
(P_y \hookrightarrow P_x \twoheadrightarrow k_{[x,y)}).
\]

Conversely, any finitely presented interval module clearly must be of the form \( k_{[x,y)} \).

2.1. Cotorsion torsion triples.

Definition 2.3. Let \( \mathcal{A} \) be an abelian category. A \textit{torsion pair} is a pair \((\mathcal{T}, \mathcal{F})\) of full subcategories of \( \mathcal{A} \), the \textit{torsion} and \textit{torsion-free} parts, respectively, satisfying

1. \( \text{Hom}_\mathcal{A}(\mathcal{T}, \mathcal{F}) = 0 \), and
2. for any object \( A \in \mathcal{A} \) there is a short exact sequence \( tA \hookrightarrow A \twoheadrightarrow fA \) with \( tA \in \mathcal{T} \) and \( fA \in \mathcal{F} \).

Remark 2.4. The subcategories in a torsion pair determine each other, namely

\[
\mathcal{T} = \mathcal{T}^\perp := \{ A \in \mathcal{A} \mid \text{Hom}_\mathcal{A}(\mathcal{T}, A) = 0 \}
\]

\[
\mathcal{F} = \mathcal{F}^\perp := \{ A \in \mathcal{A} \mid \text{Hom}_\mathcal{A}(A, \mathcal{F}) = 0 \}.
\]

This is seen by considering the short exact sequence associated to \( A \): If \( A \in \mathcal{T}^\perp \), then \( A \twoheadrightarrow fA \) necessarily becomes an isomorphism.

The subcategory \( \mathcal{T} \) is extension-closed, as can be seen by applying the left-exact functor \( \text{Hom}_\mathcal{A}(-, F) \) with \( F \in \mathcal{T} \) to any short exact sequence \( T \hookrightarrow X \twoheadrightarrow T' \) with \( T, T' \in \mathcal{T} \). Similarly, \( \mathcal{F} \) is extension-closed as well. Moreover, left exactness of the Hom-functors also implies that \( \mathcal{T} \) is closed under factors, while \( \mathcal{F} \) is closed under subobjects.

The short exact sequence in the definition of torsion pair is functorial. More precisely, for each object of \( \mathcal{A} \), fix a short exact sequence as in the definition. For any morphism \( g: A \rightarrow A' \) the composite \( tA \hookrightarrow A \twoheadrightarrow fA' \) is zero, since \( \text{Hom}_\mathcal{A}(\mathcal{T}, \mathcal{F}) = 0 \). By universality of kernels and cokernels, we can complete the
following diagram uniquely.

\[
\begin{array}{ccc}
tA & \to & A \\
\downarrow t_g & & \downarrow g \\
tA' & \to & A'
\end{array}
\]

This defines the functors \( t: \mathcal{A} \to \mathcal{T} \) and \( f: \mathcal{A} \to \mathcal{F} \).

Applying \( \text{Hom}_{\mathcal{A}}(-, F) \), where \( F \in \mathcal{F} \), to \( tA \to A \to fA \) yields an isomorphism \( \text{Hom}_{\mathcal{A}}(A, F) \cong \text{Hom}_{\mathcal{A}}(fA, F) \), using left-exactness and \( \text{Hom}_{\mathcal{A}}(tA, F) = 0 \). Thus \( f \) is left adjoint to the inclusion \( \mathcal{T} \hookrightarrow \mathcal{A} \), and similarly \( t \) is right adjoint to \( \mathcal{F} \hookrightarrow \mathcal{A} \).

**Example 2.5.** Consider \( \mathcal{A} = \text{Rep}_{k}^{\text{fp}} \mathbb{R}_{\geq 0} \), and put

\[
\mathcal{T} = \text{add} \left\{ \{k_{[0,y]} \mid 0 < y \leq \infty \} \cup \{k_{[x,y]} \mid 1 \leq x < y \leq \infty \} \right\}
\]

\[
\mathcal{F} = \text{add} \left\{ \{k_{[x,y]} \mid 0 < x < y \leq 1 \} \right\}.
\]

Then \((\mathcal{T}, \mathcal{F})\) forms a torsion pair, as can be verified directly. In Example 2.23 we arrive at this conclusion by constructing the pair from a tilting subcategory.

The “interesting” indecomposables, i.e., the ones not lying in either \( \mathcal{T} \) or \( \mathcal{F} \) already, are the \( k_{[x,y]} \) with \( 0 < x < 1 < y \leq \infty \). For these, the functorial short exact sequence is given as

\[
\begin{array}{ccc}
k_{[1,y]} & \hookrightarrow & k_{[x,y]} \\
\in \mathcal{T} & & \in \mathcal{F} \end{array}
\]

For \( X \subseteq \mathcal{A} \) define full subcategories by

\[
X^{\perp c} := \left\{ A \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^{1}(X, A) = 0 \right\}
\]

\[
X^{\perp c} := \left\{ A \in \mathcal{A} \mid \text{Ext}_{\mathcal{A}}^{1}(A, X) = 0 \right\}.
\]

**Definition 2.6** (See [4, §V.3.3]). A cotorsion pair is a pair \((\mathcal{C}, \mathcal{D})\) of full subcategories of an abelian category \( \mathcal{A} \), the cotorsion and cotorsion-free parts, respectively, such that

1. \( \mathcal{C} = \mathcal{C}^{\perp c} \) and \( \mathcal{D} = \mathcal{D}^{\perp c} \), and
2. for any object \( A \in \mathcal{A} \) there are two short exact sequences

\[
\begin{array}{ccc}
dA & \hookrightarrow & cA \\
\tilde{d}A & \twoheadrightarrow & \tilde{c}A \end{array}
\]

with \( cA \) and \( \tilde{c}A \) in \( \mathcal{C} \) and \( dA \) and \( \tilde{d}A \) in \( \mathcal{D} \).

**Remark 2.7.** Other sources define cotorsion pairs as pairs only satisfying the first condition, and rather call pairs additionally providing the two exact sequences complete. The existence of these short exact sequences will be crucial for our results, so we have chosen to follow Beligiannis–Reiten’s convention.

In contrast to the case with torsion pairs, the constructions \( c, \tilde{c}, d, \tilde{d} \) are not functorial. They do, however, define functors to certain additive quotients; see Lemma 2.11.

In the presence of condition (2), we may replace condition (1) by the following

\[(1') \text{ Ext}^{1}(\mathcal{C}, \mathcal{D}) = 0 \text{ and } \mathcal{C} \text{ and } \mathcal{D} \text{ are closed under direct summands.}\]

To see this, consider a short exact sequence \( dA \hookrightarrow cA \twoheadrightarrow A \), where \( A \in \mathcal{C}^{\perp c} \).

Then this sequence splits and therefore \( A \) is a summand in \( cA \). Since \( \mathcal{C} \) is closed under summands, this implies \( A \in \mathcal{C} \). Analogously, we also obtain the second inclusion \( \mathcal{C}^{\perp c} \subseteq \mathcal{D} \).

Finally, we note that the condition \( \mathcal{C} = \mathcal{C}^{\perp c} \) readily implies that \( \mathcal{C} \) is closed under taking extensions, by an argument similar to that of Remark 2.4. Furthermore,
\( \mathcal{C} \) must contain every projective object in \( \mathcal{A} \). Likewise, \( \mathcal{D} \) is extension-closed and contains every injective in \( \mathcal{A} \).

**Example 2.8.** Consider \( \mathcal{A} = \text{Rep}_{kp}^{fp} \mathbb{R}_{\geq 0} \), and put
\[
\mathcal{C} = \text{add} \left( \{ k_{[x, \infty)} | 0 \leq x < \infty \} \cup \{ k_{[x, y)} | 0 \leq x < y < 1 \} \right) \\
\mathcal{D} = \text{add} \left( \{ k_{[0, y)} | 0 < y \leq \infty \} \cup \{ k_{[x, y)} | 1 \leq x < y \leq \infty \} \right).
\]

Now let \( k_{[a, b)} \) be an indecomposable in \( \mathcal{A} \). The first short exact sequence is given by
\[
\begin{array}{c}
\mathcal{D} \\
\xymatrix{
k_{[0, \infty)} & k_{[a, \infty)} \\
\ar[u] \ar[r] & k_{[a, b)} \ar[u]}
\end{array}
\]
for \( b \geq 1 \). (If \( b < 1 \) then the indecomposable is already in \( \mathcal{C} \).) The second short exact sequence is non-trivial for \( k_{[a, b)} \) with \( a < 1 \), and in that case it is given as
\[
\begin{array}{c}
\mathcal{D} \\
\xymatrix{
k_{[a, b)} & k_{[0, b)} \\
\ar[u] \ar[r] & k_{[0, a)} \ar[u]}
\end{array}
\]

The following definition introduces the main object of study in this section.

**Definition 2.9.** A cotorsion torsion triple is a triple of subcategories \(( \mathcal{C}, \mathcal{T}, \mathcal{F} )\) in an abelian category such that \(( \mathcal{C}, \mathcal{T} )\) is a cotorsion pair and \(( \mathcal{T}, \mathcal{F} )\) is a torsion pair.

**Example 2.10.** Comparing Examples 2.5 and 2.8, we thus obtain an example of a cotorsion torsion triple in \( \mathcal{A} = \text{Rep}_{kp}^{fp} \mathbb{R}_{\geq 0} \):
\[
\begin{array}{c}
\mathcal{C} = \text{add} \left( \{ k_{[x, \infty)} | 0 \leq x < \infty \} \cup \{ k_{[x, y)} | 0 \leq x < y < 1 \} \right) \\
\mathcal{T} = \text{add} \left( \{ k_{[0, y)} | 0 < y \leq \infty \} \cup \{ k_{[x, y)} | 1 \leq x < y \leq \infty \} \right) \\
\mathcal{F} = \text{add} \left( \{ k_{[x, y)} | 0 < x < y \leq 1 \} \right).
\end{array}
\]

In preparation for the next lemma, we recall the construction of additive quotients. Let \( \mathcal{E} \) be an additive category, and \( \mathcal{X} \subseteq \mathcal{E} \) a full subcategory which is closed under finite direct sums. Define the category \( \mathcal{E}/\mathcal{X} \) as having the same objects as \( \mathcal{E} \), and with
\[
\text{Hom}_{\mathcal{E}/\mathcal{X}}(E, E') := \text{Hom}_{\mathcal{E}}(E, E')/\sim,
\]
where \( f \sim g \) if and only if \( f - g \) factors through an object of \( \mathcal{X} \). In particular, all objects of \( \mathcal{X} \) become zero in \( \mathcal{E}/\mathcal{X} \), and the canonical quotient functor \( \mathcal{E} \rightarrow \mathcal{E}/\mathcal{X} \) enjoys the following universal property: If \( F : \mathcal{E} \rightarrow \mathcal{E}' \) is an additive functor such that \( F(\mathcal{X}) = 0 \), then there exists a unique additive functor \( \tilde{F} : \mathcal{E}/\mathcal{X} \rightarrow \mathcal{E}' \) such that
\[
\begin{array}{c}
\mathcal{E} \\
\xymatrix{\mathcal{E} \ar[r]^F & \mathcal{E}'}
\end{array}
\]

commutes.

**Lemma 2.11.** Let \(( \mathcal{C}, \mathcal{D} )\) be a cotorsion pair in \( \mathcal{A} \). Then any map from \( \mathcal{C} \) to \( \mathcal{D} \) factors through some object in \( \mathcal{C} \cap \mathcal{D} \).

Moreover, the constructions \( \overline{c} \), \( \overline{d} \) and \( \overline{e} \) define functors from \( \mathcal{A} \) to \( \overline{\mathcal{D}} \) and \( \overline{\mathcal{C}} \), respectively.

**Proof.** For the first claim, let \( f : C \rightarrow D \), with \( C \in \mathcal{C} \) and \( D \in \mathcal{D} \). Consider the short exact sequence \( dD \rightarrow C \rightarrow D \). Since \( \text{Ext}_A^1(C, dD) = 0 \), the map \( \text{Hom}_A(C, cD) \rightarrow \text{Hom}_A(C, D) \) is an epimorphism, so \( f \) factors through \( cD \). But \( cD \) is in \( \mathcal{C} \) by construction and in \( \mathcal{D} \) since \( \mathcal{D} \) is closed under extensions.
For the second part, we only show that $c$ and $d$ are functors, the statement for $\tilde{c}$ and $\tilde{d}$ being dual. Note that any map $g: A \to B$ in $A$ may be lifted to a map of short exact sequences

$\begin{align*}
\text{d}A & \hookrightarrow cA \longrightarrow A \\
\text{d}B & \hookrightarrow cB \longrightarrow B.
\end{align*}$

first to $\tilde{g}: cA \to cB$ since $\text{Ext}^1_A(cA, dB) = 0$, and then to $\hat{g}: \text{d}A \to \text{d}B$ by the universal property of kernels.

We need to check that the choice of lifts to $cA \to cB$ and $\text{d}A \to \text{d}B$ are unique in $C \cap T$ and $D$. Equivalently, we show that any lifts of the zero map $A \to B$ are zero in $C \cap D$ and $D \cap C$. Indeed, any choice of lift $\tilde{g}$ of the zero map $A \to B$ factors through some $h: cA \to \text{d}B$; and $\hat{g}$ factors through this $h$ as well. By the first part of this lemma, $h$, and thus $\tilde{g}$ and $\hat{g}$, factor through an object of $C \cap D$. Thus $\tilde{g}$ is zero in $\frac{c}{c \cap T}$ and $\hat{g}$ is zero in $\frac{d}{D \cap T}$. □

The cotorsion pairs we study will usually come from a cotorsion torsion triple. The following lemma guarantees that in this case, the cotorsion part consists of objects whose projective dimension is at most one.

**Lemma 2.12.** Let $(C, D)$ be a cotorsion pair in an abelian category $A$. Then $D$ is closed under factor modules if and only if all objects in $C$ have projective dimension at most one.

In particular, if $(C, T, F)$ is a cotorsion torsion triple, then all objects in $C$ have projective dimension at most one.

**Proof.** Assume first that $D$ is closed under factor modules. Let $C$ in $C$, and consider any 2-extension from $C$ to any object $A \in A$, as in the upper line of the following diagram. By the second property of cotorsion pairs there is a monomorphism $E \hookrightarrow \tilde{d}E$ with $\tilde{d}E \in D$. We construct the lower half of the diagram as the pushout along this map.

$\begin{align*}
A & \to E \to F \to C \\
\text{d}A & \to \text{d}E \overset{\tau}{\to} F' \to C
\end{align*}$

The image of the map $\text{d}E \to F'$ lies in $D$, since $D$ is closed under factor modules. Thus, by the first property of cotorsion pairs, the 1-extension from $C$ to this image splits at $F'$, whence our original 2-extension is trivial. Thus we have shown that $\text{Ext}^1_A(C, -) = 0$, that is, the projective dimension of $C$ is at most one.

Assume now that all objects $C$ in $C$ have projective dimension at most one. It follows that $\text{Ext}^1_A(C, -)$ is right exact, and thus that $D$ is closed under quotients.

Finally, assume $(C, T, F)$ is a cotorsion torsion triple. Since $(T, F)$ is a torsion pair, $F$ is automatically closed under factors. It follows from the first part that all objects of $C$ have projective dimension at most one. □

The following result will be of central significance in the sequel, providing the foundation for the main equivalences established in our work.

**Theorem 2.13.** Let $(C, T, F)$ be a cotorsion torsion triple in an abelian category $A$. Then $c: A \to C \cap T$ and $f: A \to F$ induce mutually inverse equivalences $F \simeq \frac{C}{C \cap F}$. 
Proof. The functor \( f : A \rightarrow \mathcal{F} \) satisfies \( f(\mathcal{T}) = 0 \), and so its restriction to \( \mathcal{C} \) induces a unique functor
\[
\text{слова} \cdot \mathcal{C} \xrightarrow{\text{слова}} \mathcal{F}
\]
making the top triangle in the following diagram commute.

We claim that the restriction of \( \text{слова} \) in Lemma 2.11 to \( \mathcal{F} \),
\[
\text{слова}|_{\mathcal{F}} : \mathcal{F} \rightarrow \text{слова} \cap \mathcal{F},
\]
is a quasi-inverse.

By the proof of the previous lemma, for any \( g : A \rightarrow B \) in \( \mathcal{F} \), we obtain a commutative diagram in \( A \)
\[
d A \leftarrow c A \rightarrow A
\]
where the lifting \( \text{слова}(g) = \hat{g} : c A \rightarrow c B \) is unique in \( \text{слова} \cap \mathcal{F} \). Since \( \hat{g} : d A \rightarrow d B \) is in \( \mathcal{T} \) and \( g : A \rightarrow B \) is in \( \mathcal{F} \), the functoriality of \( \text{слова} \), pointed out in Remark 2.4, implies that \( \text{слова} \circ \text{слова}|_{\mathcal{T}} \cong \text{id}_{\mathcal{F}} \).

For the other composition, let \( h : C \rightarrow C' \) be an arbitrary morphism in \( \text{слова} \cap \mathcal{F} \), and choose a preimage \( k \) in \( \mathcal{C} \). In \( A \) we obtain the commutative diagram
\[
t C \leftarrow C \rightarrow t C
\]
from the torsion pair \((\mathcal{T}, \mathcal{F})\). Since \( C, C' \in \mathcal{C} \), this tells us that \( k \) is a lift of \( f k \). Because this lift is unique in \( \text{слова} \cap \mathcal{F} \), we conclude that
\[
h = \text{слова} \circ f(k) = \text{слова} \circ f|_{\mathcal{C}}(h),
\]
and so \( \text{слова}|_{\mathcal{C}} \circ \text{слова}|_{\mathcal{F}} = \text{id}_{\text{слова}} \).

Example 2.14. Let \( A = \text{Rep}_{k}^{fp} R_{\geq 0} \), and let \((\mathcal{C}, \mathcal{T}, \mathcal{F})\) be the cotorsion torsion triple of Example 2.10. Note that
\[
\mathcal{C} \cap \mathcal{T} = \text{add} \left( \{ k_{[x, \infty)} \mid x \geq 1 \} \cup \{ k_{[0, y)} \mid y < 1 \} \cup \{ k_{[0, \infty)} \} \right).
\]
We obtain the indecomposable objects in \( \text{слова} \cap \mathcal{F} \) by removing those in \( \mathcal{C} \cap \mathcal{T} \), so
\[
\text{Ob} \left( \frac{\mathcal{C}}{\mathcal{C} \cap \mathcal{T}} \right) = \text{add} \{ k_{[x, \infty)} \mid 0 < x < 1 \} \cup \{ k_{[x, y)} \mid 0 < x < y < 1 \}.
\]
Recall that
\[
\mathcal{F} = \text{add} \{ k_{[x, y)} \mid 0 < x < y \leq 1 \}.
\]
One may verify that the bijection on objects given by the equivalence $\mathcal{C} \simeq \mathcal{F}$ of Theorem 2.13 is

$$
\begin{align*}
\text{Ob} \left( \frac{\mathcal{C}}{\mathcal{C} \cap \mathcal{T}} \right) & \leftrightarrow \text{Ob}(\mathcal{F}) \\
\kappa_{[x,y)} & \leftrightarrow \kappa_{[x,y)} \\
\kappa_{[x,\infty)} & \leftrightarrow \kappa_{[x,1)}
\end{align*}
$$

for all $0 < x < y < 1$.

2.2. Torsion and cotorsion pairs and tilting. In this section we discuss how to produce (co)torsion pairs in a class of very general abelian categories. To this end we will introduce the notion of (weak) tilting subcategories. When applying this to our running example, we notice, in Example 2.21, that only one out of two very similar subcategories give rise to a good tilting theory. The deciding factor turns out to be whether or not the subcategory approximates the abelian category well enough. The following definition makes this condition precise.

**Definition 2.15.** Let $\mathcal{E}$ be an additive category, with $\mathcal{X} \subseteq \mathcal{E}$ a full subcategory and $E$ an object of $\mathcal{E}$. A morphism $\varphi: X \rightarrow E$, with $X \in \mathcal{X}$, is said to be a right $\mathcal{X}$-approximation of $E$ if for every $X' \longrightarrow E$, with $X' \in \mathcal{X}$, there exists a morphism $X' \longrightarrow X$ making the following diagram commute

$$
\begin{array}{ccc}
X' & \longrightarrow & E, \\
\varphi & \downarrow & \\
X & \longrightarrow & E,
\end{array}
$$

or, in equivalent terms, $\varphi$ induces an epimorphism of functors

$$
\text{Hom}(-,X)|_\mathcal{X} \xrightarrow{\varphi^*} \text{Hom}(-,E)|_\mathcal{X}.
$$

If every $E \in \mathcal{E}$ admits a right $\mathcal{X}$-approximation, then $\mathcal{X} \subseteq \mathcal{E}$ is said to be contravariantly finite.

Dually, the morphism $\psi: E \longrightarrow X$ is a left $\mathcal{X}$-approximation of $E$ if $\psi$ induces an epimorphism of functors

$$
\text{Hom}(X,-)|_\mathcal{X} \xrightarrow{\psi^*} \text{Hom}(E,-)|_\mathcal{X},
$$

and if every $E$ admits such a left $\mathcal{X}$-approximation, $\mathcal{X} \subseteq \mathcal{E}$ is covariantly finite.

The subcategory $\mathcal{X} \subseteq \mathcal{E}$ is simply called functorially finite, provided it is both covariantly and contravariantly finite.

The use of the word finite stems from the fact that the existence of a right approximation of an object $E$ implies that the functor $\text{Hom}(-,E)|_\mathcal{X}: \mathcal{X}^{\text{op}} \longrightarrow \text{Ab}$ is finitely generated, in the following sense.

**Definition 2.16.** Let $\mathcal{X}$ be an additive category. An additive functor $F: \mathcal{X}^{\text{op}} \longrightarrow \text{Ab}$ is finitely generated provided there is an object $X \in \mathcal{X}$ and an epimorphism of functors $\text{Hom}(\mathcal{X},-)|_\mathcal{X} \longrightarrow F$.

**Definition 2.17** (Compare [17, Proposition 4.3]). Let $\mathcal{A}$ be an abelian category with enough projectives. An additively closed full subcategory $\mathcal{T}$ of $\mathcal{A}$, i.e., one which is closed under taking finite direct sums and summands, is a weak tilting subcategory if

1. $\text{Ext}^1_{\mathcal{A}}(T_1, T_2) = 0$ for all $T_1, T_2 \in \mathcal{T}$. 

(2) Any object \( T \in \mathcal{T} \) has projective dimension at most 1, that is, it appears in a short exact sequence
\[
P_1 \hookrightarrow P_0 \twoheadrightarrow T
\]
with \( P_1 \) projective in \( A \).

(3) For any \( P \) projective in \( A \), there is a short exact sequence
\[
P \hookrightarrow T^0 \twoheadrightarrow T^1
\]
with \( T^1 \in \mathcal{T} \).

A weak tilting subcategory \( \mathcal{T} \subseteq A \) is tilting provided \( \mathcal{T} \) is contravariantly finite in \( A \).

Remark 2.18. Clearly we do not need to verify (2) for all objects in \( \mathcal{T} \): It suffices to do so for a subcollection \( \mathcal{T}' \subseteq \mathcal{T} \) such that \( \text{add} \mathcal{T}' = \mathcal{T} \).

Similarly we do not need to check (3) for all projectives, it suffices to check it for a subcategory \( \mathcal{P} \subseteq \text{Proj} A \) such that \( \text{add} \mathcal{P} = \text{Proj} A \). Indeed, the collection of objects having two-term \( \mathcal{T} \) coresolutions is clearly closed under direct sums. Thus it remains to show that it is also closed under summands. So assume \( P \oplus Q \) has a two-term \( \mathcal{T} \) coresolutions as in the top row of the following diagram.

\[
\begin{array}{ccc}
P \oplus Q & \xrightarrow{(f \ 0)} & T^0 \\
p \oplus q & \xrightarrow{(f \ 0)} & (T^0)^2 & \xrightarrow{\text{coker } f \oplus \text{coker } g} & T^1 \\
\end{array}
\]

We can clearly draw the remaining solid part of this diagram, and it follows that there is the dashed short exact sequence in the right column. Since \( \text{Ext}_A^1(T^1, T_0) \subseteq \text{Ext}_A^1(T, T) = 0 \) this sequence splits, and thus both \( \text{coker } f \) and \( \text{coker } g \) are in \( \mathcal{T} \).

Remark 2.19. The existence of enough projectives is not essential for tilting theory. However we keep this assumption here since it simplifies the formulation of the definition, and it will be satisfied in the examples we are interested in.

Conditions (1) to (3) are the standard conditions for tilting, while assuming contravariant finiteness is less commonly made explicit. In fact, we proceed to show that this is automatically satisfied if the abelian category \( A \) is noetherian, that is, any ascending chain of subobjects of a given object eventually becomes stationary. We will see below, however, that it is not automatically satisfied in general, even for fairly reasonable categories.

Proposition 2.20. Let \( A \) be a noetherian abelian category and assume that \( \mathcal{T} \) is a weak tilting subcategory of \( A \). Then \( \mathcal{T} \) is automatically contravariantly finite, and therefore already tilting.

A related result can be found in the proof of [18, Theorem 2.5(ii)].

Proof. We need to show that \( \mathcal{T} \subseteq A \) is contravariantly finite, i.e., for any given object \( M \in \mathcal{A} \), the functor
\[
\text{Hom}_A(-, M)|_{\mathcal{T}} : \mathcal{T}^\text{op} \rightarrow \text{Ab}
\]
is finitely generated. To this end, first observe that by noetherianity, \( M \) has a (unique) maximal subobject \( tM \) that is an epimorphic image of an object \( T \in \mathcal{T} \).
This means that for $X \in \mathcal{A}$ and $f : X \to M$ the factorization through $tM$ in

$$\begin{array}{ccc}
X & \xrightarrow{f} & M \\
\downarrow & & \downarrow \\
tM & \xleftarrow{\text{im } f} & tM
\end{array}$$

exists and is unique. From this follows that $\text{Hom}_A(-,M)|_{\mathcal{A}} \cong \text{Hom}_A(-,tM)|_{\mathcal{A}}$: Any morphism $f : X \to M$ is sent to the composition $X \to \text{im } f \to tM$. The other direction is then clearly given by sending $X \to tM$ to $X \to tM \xrightarrow{f} M$.

Denote by $K$ the kernel of the epimorphism $T \twoheadrightarrow tM$. Then we obtain an exact sequence

$$\text{Hom}_A(-,T)|_{\mathcal{A}} \xrightarrow{\varphi_*} \text{Hom}_A(-,tM)|_{\mathcal{A}} \to \text{Ext}^1_A(-,K)|_{\mathcal{A}} \to \text{Ext}^1_A(-,T)|_{\mathcal{A}}.$$ 

Since $\text{Hom}_A(-,M)|_{\mathcal{A}} \cong \text{Hom}_A(-,tM)|_{\mathcal{A}}$ and the last term vanishes by (1), we obtain a short exact sequence

$$\text{im } \varphi_* \hookrightarrow \text{Hom}_A(-,M)|_{\mathcal{A}} \to \text{Ext}^1_A(-,K)|_{\mathcal{A}},$$

where the image of $\varphi_*$ is finitely generated via $\text{Hom}_A(-,T)|_{\mathcal{A}} \to \text{im } \varphi_*$. Thus by the horseshoe lemma, $\text{Hom}_A(-,M)|_{\mathcal{A}}$ is finitely generated provided $\text{Ext}^1_A(-,K)|_{\mathcal{A}}$ is.

Let $P \to K$ be an epimorphism from a projective in $\mathcal{A}$. Since the projective dimension of $\mathcal{T}$ is at most 1 by (2), we obtain an epimorphism $\text{Ext}^1_A(-,P)|_{\mathcal{A}} \to \text{Ext}^1_A(-,K)|_{\mathcal{A}}$. Thus it suffices to check that $\text{Ext}^1_A(-,P)|_{\mathcal{A}}$ is finitely generated for $P$ projective. So let $P \to T^0 \to T^1$ be the sequence from (3). This sequence induces a long exact sequence, whose relevant part is

$$\text{Hom}_A(-,T^1)|_{\mathcal{A}} \to \text{Ext}^1_A(-,P)|_{\mathcal{A}} \to \text{Ext}^1_A(-,T^0)|_{\mathcal{A}}.$$ 

Since the last term is 0 by (1), $\text{Ext}^1_A(-,P)|_{\mathcal{A}}$ is finitely generated and the claim follows. □

**Example 2.21.** Consider the abelian category $\text{Rep}^p_k \mathbb{R}_{\geq 0}$ of finitely presented (covariant) representations of $\mathbb{R}_{\geq 0}$ described in Example 2.1. Recall that the functors $k_{(x,\infty)}$ are projective, while the $k_{[0,x)}$ are injective, for any $x \in \mathbb{R}_{\geq 0}$.

Define two subcategories by

$$T_{>1} = \{k_{[x,\infty)} \mid x > 1\} \cup \{k_{[0,y)} \mid y \leq 1\} \cup \{k_{[0,\infty)}\}. \quad \text{and} \quad T_{\geq 1} = \{k_{[x,\infty)} \mid x \geq 1\} \cup \{k_{[0,y)} \mid y < 1\} \cup \{k_{[0,\infty)}\}.$$ 

Thus both subcategories consist of a projective object $k_{(x,\infty)}$ for each $x > 1$, an injective object $k_{[0,y)}$ for each $y < 1$ and the projective-injective $k_{[0,\infty)}$. The only difference is that we in addition include $k_{[0,1]}$ in the former subcategory and $k_{[1,\infty)}$ in the latter.

We claim that $T_{>1}$ is weakly tilting: Every indecomposable object of $T_{>1}$ is either projective or injective, so that $\text{Ext}^1$ vanishes, and each indecomposable object has projective dimension at most 1 by (2.2). Lastly, every indecomposable projective in $\text{Rep}^p_k \mathbb{R}_{\geq 0}$ is of form $k_{[a,\infty)}$, which appear in short exact sequences

$$k_{[a,\infty)} \cong k_{[a,\infty)} \to 0, \quad a > 1$$

$$k_{[a,\infty)} \hookrightarrow k_{[0,\infty)} \to k_{[0,a)}, \quad a \leq 1,$$

showing that (3) also holds. Similarly, $T_{\geq 1}$ is weakly tilting.

Moreover, $T_{\geq 1}$ is contravariantly finite, i.e., is tilting. To see this, we first note that $\text{Hom}(k_{(c,d)},k_{[a,b)}) = k$ if and only if $a \leq c < b \leq d$, and zero otherwise. With
this in mind, it is straight forward to check that any indecomposable representation \( k_{(a,b)} \) has a right \( T_{\geq 1} \)-approximation, namely

\[
\begin{align*}
\kappa_{(0,\infty)} & \longrightarrow \kappa_{(0,b)}, \quad 0 < b \leq \infty \\
\kappa_{(a,1)} & \longrightarrow \kappa_{(a,b)}, \quad 0 < a < b \leq 1 \\
\kappa_{(1,\infty)} & \longrightarrow \kappa_{(a,b)}, \quad 0 < a \leq 1 < b \leq \infty \\
\kappa_{(a,\infty)} & \longrightarrow \kappa_{(0,\infty)}, \quad 1 \leq a < b \leq \infty.
\end{align*}
\]

On the other hand, \( T_{>1} \) is not contravariantly finite, and therefore not tilting: Indeed one observes that \( \kappa_{(1,\infty)} \) has no right \( T_{>1} \)-approximation. A right approximation would have to contain summands of form \( \kappa_{(x,\infty)} \to \kappa_{(1,\infty)} \), for some \( x > 1 \), but then any \( k_{[y,\infty)} \) with \( x > y > 1 \) would violate the lifting property.

Recall that the factor category \( \text{Fac}X \subset A \) is the full subcategory consisting of factor objects of finite direct sums of objects in \( X \). The following result is well known in classical tilting theory; see for instance [9, Lemma 4.5]. We include a proof for later reference.

**Proposition 2.22.** Let \( A \) be an abelian category with enough projectives. If \( T \subset A \) is weakly tilting, then \( \text{Fac}T = T^{\perp 1} \). If \( T \) is additionally contravariantly finite, i.e., tilting, then

\[ (\text{Fac}T, T^{\perp}) \]

is a torsion pair.

**Proof.** Let \( X \in \text{Fac}T \), so there is an epimorphism \( T \to X \), where \( T \) is an object of \( T \). Since each object of \( T \) has projective dimension at most 1 by (2), \( \text{Ext}_A^1(T,-) \) is a right exact functor. Thus \( \text{Ext}_A^1(T,T) \to \text{Ext}_A^1(T,X) \) is an epimorphism, so \( X \in T_{\geq 1} \) by (1). For the converse, let \( P \to X \) be an epimorphism from a projective object in \( T \). By (3) we can form the following diagram, with \( T^i \in T \), by taking a pushout:

\[
\begin{array}{c}
P \\
| \\
\leftarrow \downarrow \gamma \\
\leftarrow \downarrow \gamma \\
X \\
\rightarrow \downarrow \gamma \\
\rightarrow \downarrow \gamma \\
Y \\
\rightarrow \downarrow \gamma \\
\rightarrow \downarrow \gamma \\
T^1
\end{array}
\]

Thus \( Y \in \text{Fac}T \), so since every extension of \( T^1 \) by \( X \) splits, \( X \) is a summand of \( Y \), and thus in \( T \). This completes the proof of \( \text{Fac}T = T^{\perp 1} \).

Now we show that \( (\text{Fac}T, T^{\perp}) \) is a torsion pair. Let \( X \in \text{Fac}T \) and choose an epimorphism \( T \to X \) with \( T \in T \). This gives rise to a monomorphism \( \text{Hom}_A(X,Y) \to \text{Hom}_A(T,Y) \), showing that \( \text{Hom}_A(X,Y) = 0 \) whenever \( Y \in T^{\perp} \).

For an arbitrary object \( A \in A \), contravariant finiteness guarantees the existence of a right \( T \)-approximation \( \varphi: T \to A \). The short exact sequence

\[
\text{im} \varphi \to A \to \text{coker} \varphi,
\]

where \( \text{im} \varphi \in \text{Fac}T \), yields a long exact sequence

\[
0 \to \text{Hom}_A(-, \text{im} \varphi)|_T \xrightarrow{i_*} \text{Hom}_A(-, A)|_T \to \text{Hom}_A(-, \text{coker} \varphi)|_T \to \text{Ext}_A^1(-, \text{im} \varphi)|_T.
\]

Now \( i_* \) is an epimorphism since \( \varphi \) is an approximation, and the rightmost term is 0 since \( \text{im} \varphi \in \text{Fac}T = T^{\perp 1} \). Thus \( \text{coker} \varphi \in T^\perp \).

**Example 2.23.** Consider \( A = \text{Rep}_{k}^{fp} \mathbb{R}_{\geq 0} \), and

\[ T_{\geq 1} = \text{add} \left( \{k_{[x,\infty)} \mid x \geq 1 \} \cup \{k_{(0,x)} \mid x < 1 \} \cup \{k_{(0,\infty)} \} \right) \]
as in Example 2.21. Then the associated torsion pair is given by
\[ T = \text{Fac}\, T_{\geq 1} = \text{add}\left( \{k_{[0,x)} | 0 < x \leq \infty \} \cup \{k_{[x,y)} | 1 \leq x < y \leq \infty \} \right) \]
\[ \mathcal{T} = T_{\geq 1}^\perp = \text{add}\left( \{k_{[x,y)} | 0 < x < y \leq 1 \} \right) \].
In other words, we recover the torsion pair in Example 2.5.

Remark 2.24. We used contravariant finiteness of \( T \) to show existence of the short exact sequence \( tA \to A \to fA \). The following example shows that a weak tilting subcategory is, in general, not sufficient to produce a torsion pair.

Example 2.25. If one were to try the same construction with the subcategory
\[ T_{> 1} = \text{add}\left( \{k_{[x,\infty)} | x > 1 \} \cup \{k_{[0,x)} | x \leq 1 \} \cup \{k_{[0,\infty)} \} \right) \],
also from Example 2.21, but which is not contravariantly finite, we observe that
\[ \text{Fac}\, T_{> 1} = \text{add}\left( \{k_{[x,\infty)} | x > 1 \} \cup \{k_{[x,y)} | 1 < x < y \leq \infty \} \right) \]
\[ T_{> 1}^\perp = \text{add}\left( \{k_{[x,y)} | 0 < x < y \leq 1 \} \right) \].
This pair of subcategories does not form a torsion pair: Either abstractly, because \( T_{> 1}^\perp \cap \text{Fac}\, T_{> 1} \neq T \), or concretely, because \( k_{[1,\infty)} \) does not appear as an extension between these two subcategories.

To contrast the construction of torsion pairs from tilting subcategories as in Proposition 2.22, we now show that weak tilting subcategories are already sufficient to produce cotorsion pairs; cf. [2, Theorem 5.5]

Proposition 2.26. Let \( T \) be weakly tilting in an abelian category \( A \) with enough projectives. Then
\[ (T^\perp(\text{Fac}\, T), \text{Fac}\, T) \]
is a cotorsion pair, and moreover
\[ T^\perp(\text{Fac}\, T) = \{ X \in T^\perp | \text{pdim } X \leq 1 \} \].

If \( T \) is additionally contravariantly finite, i.e., tilting, then the intersection of the cotorsion and cotorsion-free part is
\[ T^\perp(\text{Fac}\, T) \cap \text{Fac}\, T = T \].

Proof. Showing that the first condition of a cotorsion pair holds amounts to showing
\[ \text{Fac}\, T = \left( T^\perp(\text{Fac}\, T) \right)^{\perp 1} \].
The first part of the proof of Proposition 2.22 shows that \( T \subseteq (T^\perp(\text{Fac}\, T))^{\perp 1} \). Thus \( T \subseteq \left( T^\perp(\text{Fac}\, T) \right)^{\perp 1} \). Clearly, for any subcategory \( S \), we have \( S \subseteq \left( T^\perp(\text{Fac}\, T) \right)^{\perp 1} \), so that in particular
\[ \text{Fac}\, T \subseteq \left( T^\perp(\text{Fac}\, T) \right)^{\perp 1} \subseteq T^{\perp 1} \].
Then the maximal Ext\(^1\)-orthogonality follows from \( \text{Fac}\, T = T^{\perp 1} \), by Proposition 2.22.

To conclude that this is a cotorsion pair, it thus remains to construct the two short exact sequences, as above, for a given object \( A \in A \). First consider a projective presentation
\[ P_1 \to P_0 \to A \]
with the short exact sequence
\[ P_1 \to T^0 \to T^1 \]
from the definition of tilting. Forming a pushout we obtain a diagram

\[
\begin{array}{ccc}
P_1 & \rightarrow & T^0 & \rightarrow & T^1 \\
\downarrow & & \downarrow f & & \downarrow \\
P_0 & \rightarrow & V & \rightarrow & T^1
\end{array}
\]

Since \(P_0\) and \(T^1\) are in \(^\perp T\), and this subcategory is extension closed, so is the pushout, denoted \(V\). The image of \(f: T^0 \rightarrow V\) is a quotient of \(T^0\), hence in \(\text{Fac } T\). Since \(\text{coker } f \cong A\), we thus get the first desired short exact sequence as

\[
\text{im } f \hookrightarrow V \rightarrow A.
\]

For the second short exact sequence, we consider \(P_0 \rightarrow T^0 \rightarrow T^1\), again from the definition of tilting, and the pushout diagram

\[
\begin{array}{ccc}
P_0 & \rightarrow & T^0 & \rightarrow & T^1 \\
\downarrow & & \downarrow \gamma & & \downarrow \\
A & \rightarrow & V & \rightarrow & T^1
\end{array}
\]

Here the second horizontal sequence is the desired one, since the pushout is in \(\text{Fac } T\) and \(T^1 \in T \subseteq ^\perp T\).

For the equality \(^\perp T = \{X \in ^\perp T \mid \text{pdim } X \leq 1\}\), first note that the inclusion \(\subseteq\) holds by Lemma 2.12. For the converse inclusion, let \(X \in ^\perp T\) be of projective dimension at most one. Let \(F \in \text{Fac } T\), and choose \(\varphi: T \rightarrow F\) be any epimorphism with \(T \in T\). Then the exact sequence

\[
\text{Ext}^1_A(X,T) \rightarrow \text{Ext}^1_A(X,F) \rightarrow \text{Ext}^2_A(X,\ker \varphi)
\]

shows that \(\text{Ext}^1_A(X,F) = 0\) since the two outer terms are zero. It follows that \(C \in ^\perp T\).

We now assume that \(T\) is tilting and let \(X \in ^\perp T \cap \text{Fac } T\). Since \(X \in \text{Fac } T\), i.e., there exists an epimorphism from an object in \(T\), and since \(T\) is contravariantly finite, there is a right-approximation \(T \rightarrow X\), which necessarily is an epimorphism. By definition, the kernel \(K\) of this approximation satisfies \(\text{Ext}^1_T(-,K)\) is 0, or, in other words, \(K \in T^<\perp = \text{Fac } T\). But then the assumption \(X \in ^\perp T\) forces this short exact sequence to split, so \(K \oplus X \cong T\), whence \(X \in \text{Fac } T\).

**Example 2.27.** Consider \(\mathcal{A} = \text{Rep}_k \mathbb{R}_{\geq 0}\), and

\[
T_{\geq 1} = \text{add } \{\mathbf{k}_{[x,\infty)} \mid x \geq 1\} \cup \{\mathbf{k}_{[0,x)} \mid x < 1\} \cup \{\mathbf{k}_{[0,\infty)}\}
\]

as in Example 2.21. Then the associated cotorsion pair is given by

\[
\mathcal{C} = ^\perp (\text{Fac } T_{\geq 1}) = \text{add } \{\mathbf{k}_{[x,\infty)} \mid 0 \leq x < \infty\} \cup \{\mathbf{k}_{[x,y)} \mid 0 \leq x < y < 1\}
\]

\[
\mathcal{D} = \text{Fac } T_{\geq 1} = \text{add } \{\mathbf{k}_{[0,x)} \mid 0 < x \leq \infty\} \cup \{\mathbf{k}_{[x,y)} \mid 1 \leq x < y \leq \infty\},
\]

and so we recover the cotorsion pair of Example 2.8.

As \(T_{\geq 1}\) of Example 2.21 is weakly tilting, the previous proposition applies and one obtains a very similar example of a cotorsion pair.

So far, we have seen that tilting subcategories give rise to both a torsion and a cotorsion pair. The next proposition gives a converse, at least in the presence of both.

**Proposition 2.28.** Let \(\mathcal{A}\) be an abelian category with enough projectives, and \((\mathcal{C}, \mathcal{T}, \mathcal{F})\) be a cotorsion torsion triple in \(\mathcal{A}\).

Then \(\mathcal{C} \cap \mathcal{T}\) is a tilting subcategory.
Proof. First we observe that, by definition of cotorsion pairs, we have
\[ \text{Ext}_A^1(\mathcal{C} \cap \mathcal{T}, \mathcal{C} \cap \mathcal{T}) \subseteq \text{Ext}_A^1(\mathcal{C}, \mathcal{T}) = 0. \]
Next, by Lemma 2.12, the projective dimension of any object in \( \mathcal{C} \) is at most one, so in particular the same holds for any object in \( \mathcal{C} \cap \mathcal{T} \).

To verify the third point of the definition of tilting, let \( P \) be projective, and consider a short exact sequence
\[ P \hookrightarrow \tilde{d}P \twoheadrightarrow \tilde{c}P \]
with \( \tilde{d}P \in \mathcal{T} \) and \( \tilde{c}P \in \mathcal{C} \) as in the definition of a cotorsion pair. Since \( \mathcal{T} \) is closed under factor modules, we observe that \( \tilde{c}P \in \mathcal{T} \). Since \( P \in \mathcal{C} \) and \( \mathcal{C} \) is closed under extensions it follows that also \( \tilde{d}P \in \mathcal{C} \). Thus both \( \tilde{d}P \) and \( \tilde{c}P \) are in \( \mathcal{C} \cap \mathcal{T} \).

Finally we show that \( \mathcal{C} \cap \mathcal{T} \) is contravariantly finite in \( A \). Let \( A \in A \). Any map from \( \mathcal{T} \) to \( A \) factors through \( tA \). Consider the short exact sequence
\[ \text{dt}A \hookrightarrow \text{ct}A \twoheadrightarrow tA \]
with \( \text{dt}A \in \mathcal{T} \) and \( \text{ct}A \in \mathcal{C} \) from the definition of a cotorsion pair. Observe that both outer terms lie in \( \mathcal{T} \), whence so does the middle, i.e., \( \text{ct}A \in \mathcal{C} \cap \mathcal{T} \).

We complete the proof by showing that the composition \( \text{ct}A \twoheadrightarrow tA \hookrightarrow A \) is a right \( \mathcal{C} \cap \mathcal{T} \)-approximation. Let \( X \twoheadrightarrow A \) be any morphism, where \( X \in \mathcal{C} \cap \mathcal{T} \). Since \( X \in \mathcal{T} \) we first observe that \( X \twoheadrightarrow A \) factors through \( tA \hookrightarrow A \). Next, since \( X \in \mathcal{C} \) we have \( \text{Ext}_A^1(X, \text{dt}A) = 0 \), whence the map further factors through \( \text{ct}A \twoheadrightarrow tA \). \( \square \)

Combining our previous results, we obtain the following correspondence.

**Theorem 2.29.** Let \( A \) be an abelian category with enough projectives. Then the two constructions
\[ \{\text{tilting subcategories}\} \leftrightarrow \{\text{cotorsion torsion triples}\} \]
\[ \mathcal{T} \mapsto \{\{X \in \mathcal{T}^\perp | \text{pdim} X \leq 1\}, \text{Fac}(\mathcal{T}), \mathcal{T}^\perp\} \]
\[ \mathcal{C} \cap \mathcal{T} \leftrightarrow (\mathcal{C}, \mathcal{T}, \mathcal{F}) \]
give mutually inverse bijections between the collection of cotorsion torsion triples and the collection of tilting subcategories.

**Proof.** We have seen in Proposition 2.28 that the map from left to right is well-defined, and in Propositions 2.22 and 2.26 that the map from right to left is. Thus it only remains to check that they are mutually inverse.

Starting with a cotorsion torsion triple \( (\mathcal{C}, \mathcal{T}, \mathcal{F}) \) it suffices to verify that \( \mathcal{T} = \text{Fac}(\mathcal{C} \cap \mathcal{T}) \), since cotorsion torsion triples are determined by any of their parts. The inclusion ‘\( \mathcal{C} \)’ is immediate since \( \mathcal{T} \) is closed under quotients. For ‘\( \subseteq \)’, let \( T \in \mathcal{T} \) and consider the short exact sequence \( dt \hookrightarrow ct \twoheadrightarrow T \), with \( dt \in \mathcal{T} \) and \( ct \in \mathcal{C} \). Since \( \mathcal{T} \) is closed under extensions, we infer that \( ct \in \mathcal{C} \cap \mathcal{T} \), whence \( T \in \text{Fac}(\mathcal{C} \cap \mathcal{T}) \).

Conversely, starting with a tilting subcategory \( \mathcal{T} \), Proposition 2.26 tells us that \( \mathcal{T} \) is recovered as the intersection of cotorsion and cotorsion-free part of the associated cotorsion pair. \( \square \)

Combining this with Theorem 2.13, we get:

**Corollary 2.30.** Let \( \mathcal{T} \) be a tilting subcategory in an abelian category \( A \) with enough projective objects. Then there is an equivalence
\[ \{X \in \mathcal{T}^\perp | \text{pdim} X \leq 1\} \simeq \mathcal{T}^\perp. \]
Example 2.31. Let $\mathcal{A} = \text{Rep}_k \mathbb{R}_{\geq 0}$, and $\mathcal{T}_{\geq 1}$ as in Example 2.21. We have already described $\mathcal{C}$, $\mathcal{T}$, and $\mathcal{F}$ in Examples 2.23 and 2.27 above. We obtain the indecomposables for $\mathcal{C}/\mathcal{T}_{\geq 1}$ by removing those in $\mathcal{T}_{\geq 1}$, so

$$\text{Ob}(\mathcal{C}/\mathcal{T}_{\geq 1}) = \text{add}\{k_{[x,\infty)} \mid 0 < x < 1\} \cup \{k_{[x,y)} \mid 0 < x < y < 1\}.$$ 

Recall that $\mathcal{F} = \text{add}\{k_{[x,y)} \mid 0 < x < y \leq 1\}$. One may verify that the bijection on objects given by the equivalence $\mathcal{C}/\mathcal{T}_{\geq 1} \simeq \mathcal{F}$ of Theorem 2.13 is

$$\text{Ob}(\mathcal{C}/\mathcal{T}_{\geq 1}) \leftrightarrow \text{Ob}(\mathcal{F})$$

$$k_{[x,y)} \leftrightarrow k_{[x,y)}$$

$$k_{[x,\infty)} \leftrightarrow k_{[x,1)}$$

for all $0 < x < 1$.

2.3. Summary of dual statements. Each result discussed so far in this section has dual results (with dual proofs). We summarize the results below, keeping in mind that torsion pairs and cotorsion pairs are self-dual structures.

Suppose $\mathcal{A}$ is an abelian category with enough injectives. An additively closed full subcategory $\mathcal{C}$ of $\mathcal{A}$ is a weak cotilting subcategory if and only if it is weakly tilting in $\mathcal{A}^{\text{op}}$. In other words, it needs to satisfy the dual requirements

1. $\text{Ext}^1_\mathcal{A}(C_1, C_2) = 0$ for all $C_1, C_2 \in \mathcal{C}$.
2. Any object $C \in \mathcal{C}$ has injective dimension at most 1.
3. For any $I$ projective in $\mathcal{A}$, there is a short exact sequence

$$C^1 \hookrightarrow C^0 \twoheadrightarrow I$$

with $C^i \in \mathcal{C}$.

A weak cotilting subcategory $\mathcal{C} \subseteq \mathcal{A}$ is cotilting if it is additionally covariantly finite in $\mathcal{A}$.

The subobject category $\text{Sub} \mathcal{X} \subseteq \mathcal{A}$ is the full subcategory determined by subobjects of finite direct sums of objects in $\mathcal{X}$.

Proposition 2.32 (Dual of Propositions 2.22 and 2.26). Let $\mathcal{A}$ be an abelian category with enough injectives. If $\mathcal{C} \subseteq \mathcal{A}$ is weakly cotilting, then $\text{Sub} \mathcal{C} = \perp^1 \mathcal{C}$, $(\text{Sub} \mathcal{C})^{\perp 1} = \{X \in \mathcal{C}^{\perp 1} \mid \text{idim} X \leq 1\}$, and

$$(\text{Sub} \mathcal{C}, (\text{Sub} \mathcal{C})^{\perp 1})$$

is a cotorsion pair. If $\mathcal{C}$ is additionally cotilting, then

$$(\perp^1 \mathcal{C}, \text{Sub} \mathcal{C})$$

is a torsion pair and

$$(\text{Sub} \mathcal{C})^{\perp 1} \cap \text{Sub} \mathcal{C} = \mathcal{C}.$$ 

As expected, by torsion cotorsion triple in an abelian category $\mathcal{A}$ we mean a triple of subcategories $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ where $(\mathcal{T}, \mathcal{F})$ is a torsion pair and $(\mathcal{F}, \mathcal{D})$ is a cotorsion pair.

Theorem 2.33 (Dual of Theorem 2.13). Let $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ be a torsion cotorsion triple in an abelian category. Then $t: \mathcal{A} \rightarrow \mathcal{T}$ and $\mathcal{d}: \mathcal{A} \rightarrow \mathcal{D}/\mathcal{F}$ induce mutually inverse equivalences

$$t: \mathcal{T} \simeq \mathcal{D}/\mathcal{D} \cap \mathcal{F}.$$
Theorem 2.34 (Dual of Theorem 2.29 and Corollary 2.30). Let $A$ be an abelian category with enough injectives. Cotilting subcategories $C$ correspond bijectively to torsion cotorsion triples $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ by

$$C \mapsto (\perp C, \text{Sub} C, \{X \in C^{-1} | \text{idim} X \leq 1\})$$

$$\mathcal{F} \cap \mathcal{D} \leftrightarrow (\mathcal{T}, \mathcal{F}, \mathcal{D})$$

and for any cotilting subcategory $C$, there is an equivalence

$$\{X \in C^{-1} | \text{idim} X \leq 1\} \cong \perp C.$$

2.4. Artin algebras. In this subsection, we specialize to the case that our abelian category $A$ is the category $\text{mod} \Lambda$ of finitely generated modules over some Artin $R$-algebra $\Lambda$. The main advantage over the more general setup discussed previously is the presence of Auslander–Reiten theory. None of the results of this section will be used in the remaining parts of the paper, but they give a nice complement to Theorem 2.13 and explain why the AR-quivers in Figure 4 agree. Throughout this section we assume familiarity with the basic concepts of Auslander–Reiten theory; see [3] for an introduction.

Recall that the duality $D: \text{mod} \Lambda \rightarrow \text{mod} \Lambda^{op}$ is defined as $D = \text{Hom}_R(-, E)$, where $E$ is an injective envelope of $R/\text{rad} R$.

Using Auslander–Reiten theory in $\text{mod} \Lambda$ we obtain a version of Theorem 2.13:

**Theorem 2.35.** Let $\Lambda$ be an Artin algebra, and $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ a cotorsion torsion triple in $\text{mod} \Lambda$. Then the Auslander–Reiten translation $\tau$ defines an equivalence

$$\frac{\mathcal{C}}{\text{proj} \Lambda} \cong \mathcal{F}.$$

**Proof.** By general Auslander–Reiten theory it is known that $\tau$ defines an equivalence

$$\frac{\text{mod} \Lambda}{\text{proj} \Lambda} \cong \frac{\text{mod} \Lambda}{\text{inj} \Lambda}.$$

Clearly $\frac{\mathcal{C}}{\text{proj} \Lambda}$ is a full subcategory of the left hand side, and we observe that $\mathcal{F}$ is a full subcategory of the right hand side: Since $\text{inj} \Lambda \subseteq \text{Fac} T$ we have $\text{Hom}_\Lambda(\text{inj} \Lambda, \mathcal{F}) = 0$, and thus no non-zero maps between objects in $\mathcal{F}$ factor through injective modules.

Therefore it suffices to check that a module $M$ lies in $\mathcal{C}$ if and only if $\tau M$ lies in $\mathcal{F}$. We observe that

$$M \in \mathcal{C} \iff \text{Ext}_\Lambda^1(M, \mathcal{T}) = 0 \iff \text{Hom}_\Lambda(\mathcal{T}, \tau M) = 0 \iff \tau M \in \mathcal{F},$$

where the middle equivalence follows from Auslander–Reiten duality: A priori Auslander–Reiten duality says that $\text{Ext}_\Lambda^1(M, \mathcal{T}) = D \text{Hom}_{\text{mod} \Lambda}(\mathcal{T}, \tau M)$, but since $\mathcal{T}$ is closed under quotients one sees that if there is a non-zero map from $\mathcal{T}$ to $\tau M$ then the inclusion of its image is also a map from $\mathcal{T}$ to $\tau M$ and is non-zero even in the quotient category modulo injectives. \qed

**Theorem 2.36.** Let $\Lambda$ be an Artin algebra, and $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ a cotorsion torsion triple in $\text{mod} \Lambda$. Then $\mathcal{C}$ has almost split sequences, and the triangle of functors

$$\xymatrix{ \frac{\mathcal{C}}{\text{proj} \Lambda} \ar[r] \ar[d]_{\tau_\mathcal{C}} & \mathcal{F} \ar[d]_{\tau \mathcal{F}} \\ \frac{\mathcal{C}}{\mathcal{F} \cap \mathcal{D}} \ar[r] & \mathcal{F} \ar[r] \ar[l] }$$

commutes, where the functors to the right are the equivalences of Theorems 2.13 and 2.35, and the vertical functor is induced by the internal Auslander–Reiten translation in $\mathcal{C}$. 
In particular, $\tau_{\mathcal{C}} : \frac{e}{\text{proj } \Lambda} \rightarrow \frac{e}{\text{cyl } \mathcal{D}}$ is an equivalence.

Proof. The category $\mathcal{C}$ has almost split sequences by [13, Corollary 2.9]. (Note that their notion of cotorsion pairs is slightly stronger than ours: They require all extensions from the first to the second category to vanish, not just first extensions. However, given a cotorsion torsion triple, the cotorsion pair is actually a cotorsion pair in their sense, since $\mathcal{F}$ is closed under cosyzygies.)

The following construction of almost split sequences in $\mathcal{C}$ is very similar to [1, Corollary 3.5]: Let $C \in \mathcal{C}$ indecomposable and not projective. Consider the almost split sequence in mod $\Lambda$ ending in $C$, as in the top row of the following diagram.

$$
\begin{array}{cccc}
\tau C & \rightarrow & E & \rightarrow & C \\
\uparrow & & \uparrow & & \uparrow \\
\c(c(\tau C)) & \leftarrow & F & \rightarrow & C \\
\uparrow & & \uparrow & & \uparrow \\
d(\tau C) & \rightarrow & \text{d}(\tau C)
\end{array}
$$

The left vertical short exact sequence comes from the definition of cotorsion pair, see Definition 2.6. Since $\text{Ext}^2(\mathcal{C}, \mathcal{D}) \subseteq \text{Ext}^2(\mathcal{E}, \mathcal{F}) = 0$, we can complete the diagram as indicated with the dashed arrows above.

Since $E \rightarrow C$ is right almost split in mod $\Lambda$, it follows that $F \rightarrow C$ is right almost split in $\mathcal{C}$. Moreover, note that $c(\tau C)$ is indecomposable up to summands in $\mathcal{C} \cap \mathcal{D}$ by the two equivalences to $\mathcal{F}$ that we already have established. Thus the middle horizontal sequence is the almost split sequence in $\mathcal{C}$ ending in $C$, up to possible superfluous summands in $\mathcal{C} \cap \mathcal{D}$. In particular $\tau_{\mathcal{C}} = \c_{\mathcal{C} \Lambda}$ as functors to $\frac{C}{\text{cyl } \mathcal{D}}$, since $\c = \text{id}_T$. We have already established the commutativity of the triangle. \hfill $\Box$

Corollary 2.37. The equivalences of Theorem 2.36 commute with the Auslander–Reiten translation in the respective subcategories of mod $\Lambda$.

Proof. Firstly, $\mathcal{F}$ has almost split sequences by [1, Corollary 3.8]. Since the diagram in Theorem 2.36 commutes and $\tau_{\mathcal{C}}$ clearly commutes with itself, it is sufficient to show that $f \circ \tau_{\mathcal{C}} = \tau_{\mathcal{F}} \circ f$ commutes with the Auslander–Reiten translation, that is, $f \tau_{\mathcal{C}} = \tau_{\mathcal{F}} f$. We have already established that $\tau_{\mathcal{A}} = \tau_{\mathcal{C}} c$. Fix an indecomposable $F \in \mathcal{F}$. By [1, Proposition 3.3], $\tau_{\mathcal{F}} F = 0$ if and only if $f(\tau_{\mathcal{A}} F) = 0$, which is true if and only if $f(\tau_{\mathcal{C}}(cF)) = 0$. If it is not relative injective, $\tau_{\mathcal{F}} F$ is a direct summand of $f(\tau_{\mathcal{C}}(cF)) = f(\tau_{\mathcal{A}} F)$, by [1, Corollary 3.5]. Thus, if $\tau_{\mathcal{F}} F \neq 0$, we have $\tau_{\mathcal{F}} F = f(\tau_{\mathcal{C}}(cF))$, since the latter is indecomposable. It follows that $\tau_{\mathcal{F}} = f \tau_{\mathcal{C}} c$. \hfill $\Box$

3. Application to quivers

Throughout this section, we consider the following special case, where $\mathcal{A}$ is an abelian category with enough projectives $\text{Proj } \mathcal{A} \subseteq \mathcal{A}$, and $Q$ a finite acyclic quiver. (For the dual results we instead assume that $\mathcal{A}$ has enough injectives $\text{Inj } \mathcal{A} \subseteq \mathcal{A}$.) We denote by $[Q, \mathcal{A}]$ the category of (covariant) representations of $Q$ in $\mathcal{A}$, or, equivalently, the functor category from the free category generated by $Q$ to $\mathcal{A}$. This category inherits a lot of structure from $\mathcal{A}$; in particular it is abelian. In this context, we can produce particular families of objects of $[Q, \mathcal{A}]$:

Definition 3.1. For each vertex $i$ of $Q$ we have the evaluation functor

$$-i : [Q, \mathcal{A}] \rightarrow \mathcal{A}, \quad X \mapsto X_i$$

sending each representation to its value on this vertex. The functor $-i$ is clearly exact, and moreover has an exact left adjoint

$$P_i : \mathcal{A} \rightarrow [Q, \mathcal{A}]$$
where, for each vertex \( j \) we have
\[
P_i(A)_j = \bigoplus_{\rho: i \to j} A^{(\rho)}.
\]
Here the superscript indices are just used to distinguish the otherwise identical summands \( A^{(\rho)} \), in order to refer to the individual summands by their index later. The structure maps of \( P_i(A) \) are as follows: For any arrow \( \alpha: j \to k \) in the quiver,
\[
P_i(A)_j \to P_i(A)_k
\]
is uniquely determined by sending any summand \( A^{(\rho)} \) corresponding to a path \( \rho: i \to j \) identically to the summand \( A^{(\rho \circ \alpha)} \).

Any arrow \( \alpha: i \to j \) in \( Q \) gives rise to a natural monomorphism of functors
\[
P_\alpha: P_j \to P_i,
\]
where \( P_\alpha(A)_k: P_j(A)_k \to P_i(A)_k \) is given by sending any summand \( A^{(\rho)} \) corresponding to a path \( \rho: j \to k \) identically to \( A^{(\rho \circ \alpha)} \).

Dually to the construction of \( P_i \), the evaluation functor \( -_i \) has an exact right adjoint denoted by \( I_i \), which is explicitly given on vertices by
\[
I_i(A)_j = \bigoplus_{\rho: j \to i} A^{(\rho)}.
\]

For a more detailed discussion of these functors, along with a proof that \( P_i \) and \( I_i \) are adjoint to \( -_i \), see [11, Section 3].

**Remark 3.2.** As \( I_i \) and \( P_i \) are adjoint to the exact functor \( -_i \), it is a standard fact that \( I_i \) preserves injectives and \( P_i \) preserves projectives; that is, for \( I \) injective and \( P \) projective in \( \mathcal{A} \), \( I_i(I) \) is injective and \( P_i(P) \) is projective in \([Q, \mathcal{A}]\).

This can be seen by considering \( P_i(P) \), where \( P \) is projective in \( \mathcal{A} \), and letting \( f: X \to Y \) be an arbitrary epimorphism in \([Q, \mathcal{A}]\). Via the adjunction, \( f_* \) equals the composition
\[
\text{Hom}_{[Q, \mathcal{A}]}(P_i(P), X) \cong \text{Hom}_{\mathcal{A}}(P, X_i) \xrightarrow{f_i} \text{Hom}_{\mathcal{A}}(P, Y_i) \cong \text{Hom}_{[Q, \mathcal{A}]}(P_i(P), Y_i),
\]
and is therefore an epimorphism. Thus \( P_i(P) \) is a projective. Dually, \( I_i(I) \) is injective.

**Remark 3.3.** As soon as \( \mathcal{A} \) has enough projectives, the same is true for \([Q, \mathcal{A}]\). This is shown in [11, Corollary 3.10] for an arbitrary quiver \( Q \) and \( \mathcal{A} \) admitting small coproducts. We give a short argument for our more restrictive setting.

Let \( X \in [Q, \mathcal{A}] \) be any object. For any vertex \( i \) in \( Q \), the identity on \( X_i \) is adjoint to some morphism \( P_i(X_i) \to X \) in \([Q, \mathcal{A}]\). This morphism is moreover the identity at the given vertex \( i \), since the unit \( \eta: \text{id}_{\mathcal{A}} \to -_i \circ P_i \) is the identity, as there is a unique path from \( i \) to \( i \). Thus we have a canonical epimorphism
\[
p: \bigoplus_{i \in Q_0} P_i(X_i) \to X.
\]

For each \( X_i \), choose an epimorphism from a projective \( P^i \to X_i \). Since each \( P_i \) preserves projectives and is (right) exact, we get
\[
\bigoplus_{i \in Q_0} P_i(P^i) \to \bigoplus_{i \in Q_0} P_i(X_i) \to X,
\]
which is an epimorphism from a projective object onto \( X \). Thus \([Q, \mathcal{A}]\) has enough projectives.

We use this notation in the proof of the following lemma, which shows that the “standard” projectives give rise to all projectives.
Lemma 3.4. The subcategory of projective objects in \([Q,A]\) is
\[ \text{Proj}[Q,A] = \text{add}\{P_i(P) \mid i \in Q_0, P \in \text{Proj}A\}. \]

Proof. Each object \(P_i(P)\) is projective in \([Q,A]\) by Remark 3.2. This establishes \(\supseteq\). Conversely, for a projective object \(X\) in \([Q,A]\), the epimorphism
\[ \bigoplus_{i \in Q_0} P_i(P^i) \twoheadrightarrow X \]
of Remark 3.3 splits. Thus \(X \in \text{add}\{P_i(P) \mid i \in Q_0, P \in \text{Proj}A\}\), establishing \(\subseteq\). \(\square\)

Lemma 3.5. Let \(X \in [Q,A]\) be any representation. Then there is a short exact sequence
\[ \bigoplus_{\alpha: j \rightarrow k} P_k(X_j) \xrightarrow{m} \bigoplus_{j \in Q_0} P_j(X_j) \xrightarrow{p} X, \]
which is natural in \(X\).
In particular, if each component \(X_i\) is projective, then \(X\) has projective dimension at most 1.

Proof. The morphism \(p\) is the canonical epimorphism of Remark 3.3. Given an arrow \(\alpha: j \rightarrow k\) in the quiver, we define \(m\) at the corresponding summand \(P_k(X_j)\) as the morphism
\[ \begin{pmatrix} P_{\alpha}(X_j) \\ -P_k(X_{\alpha}) \end{pmatrix} : P_k(X_j) \rightarrow P_j(X_j) \oplus P_k(X_k). \]

We show that this sequence is component-wise split exact, establishing the first claim. The restriction of the two morphisms above to an arbitrary vertex \(i\) is
\[ (3.6) \quad \bigoplus_{\alpha: j \rightarrow k \atop \rho: k \rightarrow i} X_j^{(\alpha, \rho)} \xrightarrow{m} \bigoplus_{j \in Q_0} X_j^{(\pi)} \xrightarrow{p} X_i, \]
Here, again, the superscript indices are just used to distinguish the otherwise identical summands \(X_j^{(\alpha, \rho)} = X_j\) and \(X_j^{(\pi)} = X_j\). At the summand \(X_j^{(\alpha, \rho)}\) corresponding to an arrow \(\alpha: j \rightarrow k\) and a path \(\rho: k \rightarrow i\), the morphism \(m\) takes the form
\[ \begin{pmatrix} \text{id}_{X_j} \\ -X_{\alpha} \end{pmatrix} : X_j^{(\alpha, \rho)} \rightarrow X_j^{(\rho a)} \oplus X_k^{(\rho)}. \]
Similarly, at the summand \(X_j^{(\pi)}\), the morphism \(p\) takes the form
\[ X_\pi : X_j^{(\pi)} \rightarrow X_i. \]
In particular one immediately verifies that \(p \circ m = 0\).

Note that the only maps from objects of the form \(X_j\) to \(X_j\) are induced by the structure maps of \(X\), and in particular in the direction of the quiver \(Q\) (recall that \(Q\) has no oriented cycles). We can thus extend the partial order on the vertices of \(Q\) to a total order \(\preceq\), which induces a filtration of the complex (3.6) by restricting only to summands of the form \(X_j\) with \(j \preceq \ell\) for a given \(\ell\). The consecutive steps in this filtration yield subquotient complexes involving only terms \(X_j\) for a given \(j\). These are of the form
\[ \bigoplus_{\alpha: j \rightarrow k \atop \rho: k \rightarrow i} X_j^{(\alpha, \rho)} \xrightarrow{m_{j}} \bigoplus_{j \in Q_0} X_j^{(\pi)} \xrightarrow{p_{j}} X_i, \]
Note that now at the summand \(X_j^{(\alpha, \rho)}\), the morphism \(m_{j}\) takes the form
\[ \text{id}_{X_j} : X_j^{(\alpha, \rho)} \rightarrow X_j^{(\rho a)}, \]
and for \( j = i \) the morphism \( p_j \) projects to the summand corresponding to the trivial path. In particular, all these subquotients are trivially split exact sequences, whence the complexes \((3.6)\) also are split exact sequences. \( \square \)

We recall the following useful trick for constructing adjoint functors:

**Lemma 3.7.** Let

\[
F \xrightarrow{f} G \longrightarrow H
\]

be an exact sequence of functors between abelian categories \( \mathcal{A} \) and \( \mathcal{B} \). If both \( F, G: \mathcal{A} \rightarrow \mathcal{B} \) admit right adjoints, then so does \( H \).

**Proof.** Letting \( F', G': \mathcal{B} \rightarrow \mathcal{A} \) be the adjoint functors to \( F, G \), we obtain a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{B} (H(-), -) & \xrightarrow{f^*} & \text{Hom}_\mathcal{B} (G(-), -) \\
\downarrow & & \downarrow \\
\text{Hom}_\mathcal{A} (-, \ker f') & \xrightarrow{f'^*} & \text{Hom}_\mathcal{A} (-, G'(-))
\end{array}
\]

where \( f^* \) is the unique morphism making the rightmost square commute, yielding a natural transformation \( f': G' \rightarrow F' \) by the Yoneda lemma. Thus, by universality of the kernels in the left column, we obtain a natural isomorphism

\[
\text{Hom}_\mathcal{B} (H(-), -) \cong \text{Hom}_\mathcal{A} (-, (\ker f')(\cdot)),
\]

making \( H' := \ker f' \) a right adjoint to \( H \). \( \square \)

**Lemma 3.8.** The functor \( I_i: \mathcal{A} \rightarrow [Q, \mathcal{A}] \) has a right adjoint \( RI_i: [Q, \mathcal{A}] \rightarrow \mathcal{A} \).

**Proof.** Applying Lemma 3.5, and recalling that \( I_i(A_j) = A^{[Q(j,i)]} \), where \( [Q(j,i)] \) is the number of paths \( i \rightarrow j \) in \( Q \), we have the following short exact sequence of functors for all \( i \),

\[
\bigoplus_{\alpha: j \rightarrow k} P_{k}^{[Q(j,i)]} \xrightarrow{m} \bigoplus_{j \in Q_0} P_{j}^{[Q(j,i)]} \longrightarrow I_i.
\]

The first two terms have right adjoints as they are direct sums of the functors \( P_j \). Thus, by Lemma 3.7, \( RI_i \) exists and is a right adjoint to \( I_i \), for all \( i \). \( \square \)

With these lemmas at hand, we produce tilting subcategories of \([Q, \mathcal{A}]\).

**Proposition 3.9.** Let \( \mathcal{A} \) be an abelian category with enough projectives and put

\[
\mathbb{T} = \text{add}\{I_i(P) \mid i \in Q_0 \text{ and } P \in \text{Proj} \mathcal{A}\}.
\]

Then \( \mathbb{T} \) is tilting in \([Q, \mathcal{A}]\).

**Proof.** We need to check the definition of tilting; see Definition 2.17. The first point follows from the fact that exact adjoint functors of abelian categories extend to Ext-groups. In particular, the adjunction between \( -j \) and \( I_j \) extends to \( \text{Ext}^1 \), namely

\[
\text{Ext}^1_{[Q, \mathcal{A}]} (I_i(P), I_j(P')) \cong \text{Ext}^1_{\mathcal{A}} (I_i(P)_j, P').
\]

This particular case is straightforward to verify. Indeed, let \( X \in [Q, \mathcal{A}] \) have projective resolution \( P^* \rightarrow X \). Then \( P_j^* \rightarrow X_j \) is a projective resolution in \( \mathcal{A} \), as \( -j \) both is exact and preserves projectives. The adjunction gives an isomorphism of complexes

\[
\text{Hom}_{[Q, \mathcal{A}]} (P^*, I_j(A)) \cong \text{Hom}_{\mathcal{A}} (P_j^*, A),
\]

which compute \( \text{Ext}^1_{[Q, \mathcal{A}]} (X, I_j(A)) \) and \( \text{Ext}^1_{\mathcal{A}} (X_j, A) \), respectively.

Now \( I_i(P)_j \) is a (possibly empty) sum of copies of \( P \), hence projective. It follows that this \( \text{Ext}^1_{[Q, \mathcal{A}]} (I_i(P), I_j(P')) \) is zero.
For the second point, we need to check that every object in $\mathcal{T}$ has projective dimension at most 1. This follows from Lemma 3.5, since every component of $l_i(P)$ is projective in $\mathcal{A}$.

The third point follows in a similar way: By Lemma 3.4 and Remark 2.18 it suffices to consider projectives of the form $P_j(P)$. By the dual of Lemma 3.5, applied to the particular representation $P_j(P)$, there is a short exact sequence

$$
P_j(P) \hookrightarrow \bigoplus_{j \in Q_0} l_j(P_j(P)_{\alpha}) \longrightarrow \bigoplus_{\alpha : k \rightarrow j} l_k(P_j(P)_j),$$

whose two right terms are in $\mathcal{T}$.

In order to conclude that $\mathcal{T}$ is tilting, we need to show that $\mathcal{T}$ is contravariantly finite in $[Q, \mathcal{A}]$. Let $X \in [Q, \mathcal{A}]$ be any representation. Let $\mathcal{R}_i$ be the right adjoint of $l_i$, as given in Lemma 3.8. For each $i$, choose an epimorphism $p^i : P^i \longrightarrow \mathcal{R}_i(X)$ from a projective object $P^i \in \text{Proj} \mathcal{A}$. We claim that the sum of the adjoints

$$\bigoplus_{i \in Q_0} l_i(P^i) \xrightarrow{\varphi} X$$

is a right $\mathcal{T}$-approximation, i.e., any morphism $T \longrightarrow X$ with $T \in \mathcal{T}$ factors through $\varphi$. It is sufficient to check the claim on the generators of $\mathcal{T}$, namely the objects $l_i(P)$, where $P$ is projective and $i$ is a vertex. To this end, choose any morphism $f : l_i(P) \longrightarrow X$. It is adjunct to a morphism $P \longrightarrow \mathcal{R}_i(X)$, which factors through $p^i$ by projectivity of $P^i$. Thus $f$ factors through the $i$th component of $\varphi$, $l_i(P) \longrightarrow X$, and therefore through $\varphi$ itself. □

**Corollary 3.10.** Let $\mathcal{T}$ be as in Proposition 3.9. There is a cotorsion torsion triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ in $[Q, \mathcal{A}]$ given by

$$\mathcal{C} = [Q, \text{Proj} \mathcal{A}],$$

$$\mathcal{F} = \text{Fac} \mathcal{T},$$

$$\mathcal{T} = \mathcal{T}^\perp.$$

Moreover, we have $\mathcal{T} \cap \mathcal{C} = \mathcal{T}$.

**Proof.** Except for the description of $\mathcal{C}$, this is one direction of Theorem 2.29. Thus the only thing to show is that

$$\mathcal{T}^\perp \cap \text{Fac} \mathcal{T} = \{X \in \mathcal{T}^\perp \mid \text{pdim } X \leq 1\}$$

coincides with $[Q, \text{Proj} \mathcal{A}]$.

Note first that since $l_i$ is exact and $\mathcal{A}$ has enough projectives, $l_i(A)$ is in $\text{Fac} \mathcal{T}$, for all $A \in \mathcal{A}$. Thus, for $X \in \mathcal{T}^\perp \cap \text{Fac} \mathcal{T}$ one has

$$\text{Ext}_A^1(X, A) \cong \text{Ext}_{[Q, \mathcal{A}]}^1(X, l_i(A)) = 0.$$ 

It follows that $X_i$ is projective.

Going the other way, let $X \in [Q, \mathcal{A}]$ such that all $X_i$ are projective. By Lemma 3.5, $\text{pdim } X \leq 1$, so we need only show that $X \in \mathcal{T}^\perp$. This follows from

$$\text{Ext}_{[Q, \mathcal{A}]}^1(X, l_i(P)) \cong \text{Ext}_A^1(X_i, P) = 0.$$ □

**Theorem 3.11.** Let $\mathcal{A}$ be abelian with enough projectives and $Q$ a finite acyclic quiver. With $\mathcal{T}$ as in Proposition 3.9, we have

$$[Q, \text{Proj} \mathcal{A}] \cong \mathcal{T}^\perp$$

induced by $f$ and $c$ coming from the (co)torsion pairs.

**Proof.** This is now just a combination of Theorem 2.13 and Corollary 3.10 □
We explicitly state the dual version of this theorem, which additionally summarizes the dual of previous results.

**Theorem 3.12.** Let $\mathcal{A}$ be an abelian category with enough injective objects, $Q$ a finite acyclic quiver, and put

$$\mathcal{C} = \text{add}\{P_i(I) \mid i \in Q_0 \text{ and } I \in \text{Inj }\mathcal{A}\}.$$ 

Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair and $(\mathcal{F}, \mathcal{D})$ is cotorsion pair in $\mathcal{A}$, where

$$\mathcal{T} = \perp \mathcal{C}, \quad \mathcal{F} = \text{Sub }\mathcal{C}, \quad \mathcal{D} = [Q, \text{Inj }\mathcal{A}],$$

and there is an equivalence

$$[Q, \text{Inj }\mathcal{A}] \simeq \perp \mathcal{C}$$

induced by $t: \mathcal{A} \to \mathcal{T}$ and $\tilde{d}: \mathcal{A} \to \mathcal{F} \cap \mathcal{D} = \mathcal{C}$.

3.1. **Special cases.** In general, the right hand sides of the equivalences are fairly non-explicit. We therefore further interpret these theorems in the case where $Q$ is a Dynkin quiver. We start with the easiest case, which is also the most relevant to our applications, namely the linearly oriented Dynkin quiver of type $A$:

$$Q = \vec{A}_n: 1 \to 2 \to \cdots \to n.$$ 

**Lemma 3.13.** Let $\mathcal{A}$ be abelian with enough projectives and put

$$\mathcal{T} = \text{add}\{I_i(P) \mid 1 \leq i \leq n \text{ and } P \in \text{Proj }\mathcal{A}\} \subseteq [\vec{A}_n, \mathcal{A}].$$

Then

1. $X \in \mathcal{T}^\perp$ if and only if $X_1 = 0$; and
2. $X \in \text{Fac }\mathcal{T}$ if and only if all structure maps of $X$ are epimorphisms.

**Proof.** Recall first that for this quiver, $I_i(P)$ is just

$$P \overset{i \text{ copies}}{\longrightarrow} P \overset{id}{\longrightarrow} \cdots \overset{id}{\longrightarrow} P \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0.$$ 

For (1) we note that $X \in \mathcal{T}^\perp$ if and only if $\text{Hom}_{[\vec{A}_n, \mathcal{A}]}(I_i(P), X) = 0$ for all $1 \leq i \leq n$ and $P \in \text{Proj }\mathcal{A}$. Any morphism $P \to X_1$ lifts uniquely to $I_n(P) \to X$, so if $X \in \mathcal{T}^\perp$, then $P \to X_1$ is zero. Since $\mathcal{A}$ has enough projectives, $X_1 = 0$. On the other hand, if $X_1 = 0$, then $\text{Hom}_{[\mathcal{Q}, \mathcal{A}]}(I_i(P), X) = 0$ for all $1 \leq i \leq n$ and $P \in \text{Proj }\mathcal{A}$.

For (2), let first $X \in \text{Fac }\mathcal{T}$, i.e., there is an epimorphism $T \to X$ with $T \in \mathcal{T}$. Since the structure maps for each $I_i(P)$ is an epimorphism, the same is true for $T$. Because $T \to X$, each structure map of $X$ is an epimorphism. Conversely, assume that the structure maps of $X$ are all epimorphisms and choose an epimorphism $P \to X_1$. It extends to a unique epimorphism $I_n(P) \to X$, that is to say, $X \in \text{Fac }\mathcal{T}$. \hfill $\square$

Noting that in this case $\mathcal{T}^\perp \simeq [\vec{A}_{n-1}, \mathcal{A}]$, we arrive at the following corollary for **Theorem 3.11**.

**Corollary 3.14.** Let $\mathcal{A}$ be abelian with enough projectives. Then

$$[\vec{A}_n, \text{Proj }\mathcal{A}]_\mathcal{T} \simeq [\vec{A}_{n-1}, \mathcal{A}].$$
where $\mathcal{T}$ is the additive subcategory generated by representations of the form $\mathcal{Y}(P)$ with $P \in \text{Proj} \mathcal{A}$. In particular, $\mathcal{T}$ is the subcategory of epimorphic representations with projectives at every vertex.

**Proof.** It only remains to establish the last claim. By Corollary 3.10 we have

$$\mathcal{T} = \text{Fac} \mathcal{I} \cap [Q, \text{Proj} \mathcal{A}].$$

The second point of Lemma 3.13 completes the proof. □

Dually, from Theorem 3.12 one immediately obtains the following result.

**Corollary 3.15.** Let $\mathcal{A}$ be abelian with enough injectives. Then

$$[\mathcal{A}_{n-1}, \mathcal{A}]_{\mathcal{C}} \simeq [\mathcal{A}_{n-1}, \mathcal{A}],$$

where $\mathcal{C}$ is the additive subcategory generated by representations of the form $P_i(I)$ with $I \in \text{Inj} \mathcal{A}$. In particular, $\mathcal{C}$ is the subcategory of monomorphic representations with injectives at every vertex.

**Remark 3.16.** For $n = 2$, these results just state that passing between modules and their projective (or injective) presentations preserves most information.

We now restrict to the case where $\mathcal{A} = \text{rep}_k \mathcal{A}_m$ to recover Theorem 1.5 of the introduction.

**Corollary 3.17.** The functor $t$ induces an equivalence

$$\text{rep}_k^e(\mathcal{A}_m \otimes \mathcal{A}_n) \rightarrow \text{rep}_k(\mathcal{A}_m \otimes \mathcal{A}_{n-1}).$$

Moreover, $\text{rep}_k^m(\mathcal{A}_m \otimes \mathcal{A}_n)$ consists precisely of all direct sums of thin modules of the form $k_{\{1, \ldots, i\} \times \{j, \ldots, m\}}$.

**Proof.** Letting $\mathcal{A} = \text{rep}_k \mathcal{A}_m$ in Corollary 3.15, we have

$$[\mathcal{A}_n, \mathcal{A}] \simeq \text{rep}_k(\mathcal{A}_m \otimes \mathcal{A}_n).$$

Since the injective representations of $\mathcal{A}_m$ are precisely the representations where every structure map is an epimorphism, we have

$$[\mathcal{A}_n, \text{Inj} \mathcal{A}] \simeq \text{rep}_k^e(\mathcal{A}_m \otimes \mathcal{A}_n).$$

Moreover, $\mathcal{C} \subseteq \text{rep}_k(\mathcal{A}_m \otimes \mathcal{A}_n)$ is the subcategory additively generated by the representations

$$P_j(I_i) = I_i \otimes P_j = k_{\{1, \ldots, i\} \times \{j, \ldots, m\}},$$

and $\mathcal{C} = \text{rep}_k^m(\mathcal{A}_m \otimes \mathcal{A}_n)$, since it is the subcategory of representations with monomorphisms in the $\mathcal{A}_n$-direction between injective $\mathcal{A}_m$-representations. □

**Construction 3.18.** The equivalence of Theorem 3.11 has only been given rather abstractly, but in the special case of Corollary 3.14 we can explicitly describe the functors $f : [\mathcal{A}_n, \mathcal{A}] \rightarrow \mathcal{F}$ and $c : [\mathcal{A}_n, \mathcal{A}] \rightarrow \mathcal{C}/\mathcal{T}$ giving rise to the equivalence.

Recall that the cotorsion torsion triple is given by

$$\mathcal{C} = \text{Fac} \mathcal{I} = [\mathcal{A}_n, \text{Proj} \mathcal{A}];$$

$$\mathcal{F} = \text{Fac} \mathcal{I} = \{X \in [\mathcal{A}_n, \mathcal{A}] \mid \text{all structure maps of } X \text{ are epimorphisms}\};$$

$$\mathcal{C}/\mathcal{T} = \{X \in [\mathcal{A}_n, \mathcal{A}] \mid X_1 = 0\} \simeq [\mathcal{A}_{n-1}, \mathcal{A}],$$

by Corollary 3.10 and Lemma 3.13.
The functor \( f : [\tilde{A}_n, \mathcal{A}] \to \mathcal{F} \) is explicitly given by sending
\[
X := [X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n]
\]
to
\[
fX := \left[ 0 \to \text{coker } f_1 \to \text{coker } f_2 f_1 \to \cdots \to \text{coker } f_{n-1} \cdots f_1 \right];
\]
the reason being that the canonical epimorphism \( X \to fX \) has kernel
\[
tX := [X_1 \to \text{im } f_1 \to \text{im } f_2 f_1 \to \cdots \to \text{im } f_{n-1} \cdots f_2 f_1]
\]
which lies in \( \mathcal{T} \), as every structure map is an epimorphism.

Conversely, the construction from \([\tilde{A}_n, \mathcal{A}]\) to \( \mathcal{C} \), which gives rise to the functor
\[
[\tilde{A}_{n-1}, \mathcal{A}] \to \left[ \tilde{A}_n \mathcal{Proj} \right],
\]
by Lemma 2.11, is given by constructing a short exact sequence
\[
dX \hookrightarrow \mathfrak{c}X \twoheadrightarrow X,
\]
where \( dX \in \mathcal{T} \) and \( \mathfrak{c}X \in \mathcal{C} \). This is done in the following manner:

Starting with a representation \( X \in [\tilde{A}_n, \mathcal{A}] \), one constructs the following diagram from right to left.

\[
\begin{array}{cccccccccc}
X_1 & \to & X_2 & \to & \cdots & \to & X_{n-2} & \to & X_{n-1} & \to & X_n \\
\downarrow & & & & & & & & & & \\
P_1 & \to & R_2 & \to & P_2 & \to & \cdots & \to & P_{n-2} & \to & R \to P_n \\
\downarrow & & & & & & & & & & \\
\Omega X_1 & \to & \Omega X_2 & \to & \cdots & \to & \Omega X_{n-2} & \to & \Omega X_{n-1} & \to & \Omega X_n
\end{array}
\]

That is, start with an epimorphism from a projective \( P_n \) to \( X_n \) and take the pullback along \( X_{n-1} \to X_n \), denoted by \( R_n \). These two epimorphisms have the same kernel, which we denote by \( \Omega X_n \). Next, take an epimorphism from a projective \( P_{n-1} \) to the pullback, which also gives an epimorphism \( P_{n-1} \to X_{n-1} \). Note that the map of kernels \( \Omega X_{n-1} \to \Omega X_n \) is an epimorphism, as can be seen by an easy diagram chase. Take the pullback along \( X_{n-2} \to X_{n-1} \) and iterate this procedure down to \( X_1 \).

Denoting the middle row of projective objects by \( \mathfrak{c}X \) and the lower row by \( dX \), we have constructed the desired short exact sequence, as \( \mathfrak{c}X \in \mathcal{C} \) and \( dX \in \mathcal{T} \). Thus the image of \( X \) under \( \mathfrak{c} : [\tilde{A}_n, \mathcal{A}] \to \left[ \tilde{A}_n \mathcal{Proj} \right] \) is given by \( [P_1 \to P_2 \to \cdots \to P_n] \).

The constructions of \( f \) and \( \mathfrak{c} \) can be formally dualized to produce the equivalence of Corollary 3.15. In particular, we recover Construction 1.4.

**Example 3.19.** We take \( \mathcal{A} = \text{Rep}_k^P \mathbb{R}_{\geq 0} \), and denote by \( \text{proj}_k \mathbb{R}_{\geq 0} = \text{add}\{k_{[x,\infty)}\} \) its subcategory of projectives. Equivalently, \( \text{proj}_k \mathbb{R}_{\geq 0} \) is the subcategory given by requesting the property that all structure maps are monomorphic.

We are interested in determining all indecomposables in \([\tilde{A}_2, \text{proj}_k \mathbb{R}_{\geq 0}]\). By Corollary 3.14, we have
\[
[\tilde{A}_2, \text{proj}_k \mathbb{R}_{\geq 0}] \simeq \mathcal{A}.
\]
Thus the indecomposables in \([\tilde{A}_2, \text{proj}_k \mathbb{R}_{\geq 0}]\) are those in \( \mathcal{T} \), along the ones coming from the indecomposables in \( \mathcal{A} \). In this setup we know both collections:
\[
\mathcal{T} = \text{add}\{k_{[x,\infty)} \to k_{[x,\infty)}\} \cup \{k_{[x,\infty)} \to 0\}
\]
\[
\mathcal{A} = \text{add}\{k_{[x,y)}\}
\]
Figure 6. The four types of indecomposable representations of \([\vec{A}_2, \text{proj}_k \mathbb{R} \geq 0]\). From left to right: \([k_{[x, \infty)} \to k_{[x, \infty)}]\), \([k_{[y, \infty)} \to 0]\), \([k_{[y, \infty)} \to k_{[x, \infty)}]\), and \([0 \to k_{[x, \infty)}]\). The first two objects are contained in \(T\), and thus sent to 0 in \(\text{Rep}^{fp}_{k \mathbb{R} \geq 0}\), whereas the latter two are sent to \(k_{[x, y)}\) and \(k_{[x, \infty)}\).

Under the construction above, the representations of \(T\) are sent to 0 and the indecomposable \(k_{[x, y)}\) in \(\vec{A}\) corresponds to \([k_{[y, \infty)} \to k_{[x, \infty)}] \in [\vec{A}_2, \text{proj}_k \mathbb{R} \geq 0]\).

Thus our complete list of indecomposables in \([\vec{A}_2, \text{proj}_k \mathbb{R} \geq 0]\) is

- \([k_{[x, \infty)} \to k_{[x, \infty)}]\), for \(0 \leq x < \infty\),
- \([k_{[y, \infty)} \to 0]\), for \(0 \leq y < \infty\), and
- \([k_{[y, \infty)} \to k_{[x, \infty)}]\), for \(0 \leq x < y \leq \infty\).

Respectively, we may picture these as in Figure 6.

Dually, and more relevant to clustering, we may study the subcategory \(\text{inj} \mathbb{R} \geq 0\) of \(\text{Rep}^{fp}_{k \mathbb{R} \geq 0}\) where all structure maps are surjections. These are the injectives in \(\text{Rep}^{fp}_{k \mathbb{R} \geq 0}\). We obtain a complete list of indecomposables in \([\vec{A}_2, \text{inj} \mathbb{R} \geq 0]\) as

- \(k_{[0, x)} \to k_{[0, x)}\) with \(0 < x \leq \infty\),
- \(0 \to k_{[0, x)}\) with \(0 < x \leq \infty\), and
- \(k_{[0, y)} \to k_{[0, x)}\) with \(0 < y < x \leq \infty\).

Example 3.20. Now we consider filtered hierarchical clustering for 3-step filtration. That is, we would like to find all indecomposables in \([\vec{A}_3, \text{inj} \mathbb{R} \geq 0]\). This turns out to be a lot harder. Let us illustrate this by restricting to 5 fixed critical values, that is, we replace \(\text{rep}^{fp}_{k \mathbb{R}}\mathbb{R} \geq 0\) with \(\text{rep} k \vec{A}_5\). By Corollary 3.15 we have

\[
\frac{[\vec{A}_3, \text{inj} k \vec{A}_5]}{C} \simeq [\vec{A}_2, \text{rep} k \vec{A}_5],
\]

where \(C\) is generated by the representations \(P(I)\). On the right hand side, one easily finds a one-parameter family of indecomposables. (The one below comes from embedding an infinite family for \(\vec{D}_4\).)

Abstractly, it follows immediately from the equivalence that also \([\vec{A}_3, \text{inj} k \vec{A}_5]\) has a one-parameter family of indecomposables.
Concretely, one may follow the (dual) instructions in Construction 3.18 above and obtain the following 1-parameter family of indecomposables in $[\bar{A}_3, \text{inj} \bar{A}_5]$:

\[
\begin{array}{cccc}
\kappa^2 & \kappa^2 & \kappa^2 & \kappa^2 \\
(0, 1, 0) & (0, 1, 0) & (1, 0, 1) & (1, 0) \\
\kappa^3 & \kappa^3 & \kappa^2 & \kappa \\
(0, 0, 1) & (0, 0, 1) & (1, 0) & 0 \\
\kappa^2 & \kappa^2 & 0 & 0 \\
(1, 0, 0) & (0, 1) & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}
\]

As a next special case of Theorem 3.11, we consider the case that $Q$ is arbitrary while the abelian category $A$ is the category of finite dimensional modules over some finite dimensional $k$-algebra $\Lambda$. In particular in this case we have $[Q, A] = \text{mod} \Lambda$, and we will present the following results in this more familiar language.

Before we can rephrase Theorem 3.11 in this setup, note that there are natural forgetful functors $\text{mod} \Lambda \rightarrow \text{mod} \Lambda$ and $\text{mod} \Lambda \rightarrow \text{mod} kQ$. The image of a module $M$ under these functors will be denoted by $M_\Lambda$ and $M_kQ$, respectively.

With this notation, we obtain the following version of Theorem 3.11.

Corollary 3.21. Let $\Lambda$ be a finite dimensional $k$-algebra, and $Q$ a finite acyclic quiver. Then

\[
\left\{ M \in \text{mod} \Lambda \mid M_\Lambda \in \text{proj} \Lambda \right\} \simeq \left\{ M \in \text{mod} \Lambda \mid M_kQ \in \text{mod}_{\text{non-inj}} kQ \right\},
\]

where

\[
T = \text{add}\{l_i(P) \mid i \in Q_0 \text{ and } P \in \text{proj} \Lambda\}
\]

and $\text{mod}_{\text{non-inj}} kQ$ denotes the subcategory of $\text{mod} kQ$ containing all modules without non-zero injective summands.

Proof. We apply Theorem 3.11. The left hand side of the equivalence is literally the same as in that theorem, so it only remains to simplify the right hand side. Here we have

\[
T^\perp = \left\{ M \in \text{mod} \Lambda \mid \text{Hom}_{\text{mod} \Lambda} (l_i(\Lambda), M) = 0 \text{ for all } i \in Q_0 \right\}
\]

\[
= \left\{ M \in \text{mod} \Lambda \mid \text{Hom}_{\text{mod} kQ} (l_i(k), M) = 0 \text{ for all } i \in Q_0 \right\}
\]

\[
= \left\{ M \in \text{mod} \Lambda \mid M_kQ \in \text{mod}_{\text{non-inj}} kQ \right\}.
\]

Here, the first equality follows from the fact that $l(\Lambda) = \Lambda \otimes l(k)$, together with the fact that tensoring with $\Lambda$ is left adjoint to the forgetful functor $\text{mod} \Lambda \rightarrow \text{mod} kQ$. The second equality holds since for $\text{mod} kQ$, only injective modules permit non-zero homomorphisms from the indecomposable injectives $l_i(k)$.

In the next application, we combine Corollaries 3.14 and 3.21. To set the stage, recall that we denote by $\text{rep}_k^m(Q \otimes A_n)$ the subcategory of $\text{rep}_k(Q \otimes A_n)$ where all structure maps in the $A_n$-direction are monomorphisms.

Definition 3.22. Let $Q$ be a Dynkin quiver. The costable Auslander algebra of $kQ$ is given as

\[
\Gamma = \text{End}_{kQ} \left( \bigoplus_M M \right),
\]

where the sum runs over all isomorphism classes of non-injective indecomposables. (Note that this is indeed a finite sum, since we assumed $Q$ to be Dynkin.)
Corollary 3.23. Let $Q$ be a Dynkin quiver, and $\Gamma$ its costable Auslander algebra. Then
\[
\operatorname{rep}_k^*(Q \otimes \vec{A}_n) \simeq \Mod \Gamma \vec{A}_{n-1},
\]
where $X$ be the subcategory whose indecomposables are the representations
- $X = \cdots = X \hookrightarrow I = \cdots = I$,
  for $X \hookrightarrow I$ an injective envelope of an indecomposable $kQ$-module $X$,
- $0 = \cdots = 0 \rightarrow I = \cdots = I$,
  where $I$ is an indecomposable injective $kQ$-module.

In particular, for $n = 2$ above, we have
\[
\operatorname{rep}_k^*(Q \otimes \vec{A}_2) \simeq \Mod \Gamma.
\]

Proof. Since $k\vec{A}_n$-modules are projective if and only if all structure maps are monomorphisms, we may reformulate the denominator of the left hand side as
\[
\operatorname{rep}_k^*(Q \otimes \vec{A}_n) = \{M \in \Mod kQ \otimes k\vec{A}_n \mid M_{k\vec{A}_n} \in \proj k\vec{A}_n\}.
\]
Thus, by Corollary 3.21, we have
\[
\frac{\operatorname{rep}_k^*(Q \otimes \vec{A}_n)}{\mathbb{T}} \simeq \{M \in \Mod kQ \otimes k\vec{A}_n \mid M_{kQ} \in \mod_{\text{non-inj}} kQ\},
\]
where $\mathbb{T}$ consists precisely of (sums of) representations of the form $I \otimes P$, where $I$ is injective over $kQ$ and $P$ is projective over $k\vec{A}_n$. For indecomposable $P$ we may depict this representation as
\[
0 = \cdots = 0 \rightarrow I = \cdots = I.
\]

The next step is completely formal: Since $\mod_{\text{non-inj}} kQ \simeq \proj \Gamma$, we obtain an equivalence
\[
\{M \in \Mod kQ \otimes k\vec{A}_n \mid M_{kQ} \in \mod_{\text{non-inj}} kQ\} \simeq \{M \in \Mod \Gamma \vec{A}_n \mid M_{\Gamma} \in \proj \Gamma\}.
\]
Thus we are in the setup of Corollary 3.14, and obtain
\[
\frac{\{M \in \Mod \Gamma \vec{A}_n \mid M_{\Gamma} \in \proj \Gamma\}}{\mathbb{T}'} \simeq \Mod \Gamma \vec{A}_{n-1},
\]
where $\mathbb{T}'$ consists of direct sums of representations
\[
P = \cdots = P \rightarrow 0 = \cdots = 0
\]
with $P \in \proj \Gamma$.

Thus we complete the proof by observing that the representations in $\mathbb{T}'$ correspond to representations of the form
\[
[X = \cdots = X \hookrightarrow 0 = \cdots = 0] \in \operatorname{rep}_k(Q \otimes \vec{A}_n)
\]
with $X \in \mod_{\text{non-inj}} kQ$ under the second equivalence in this proof, and finally to
\[
[X = \cdots = X \hookrightarrow I = \cdots = I] \in \operatorname{rep}_k^*(Q \otimes \vec{A}_n)
\]
under the first equivalence above. \hfill \Box

Example 3.24. Let $Q = [1 \rightarrow 2 \leftarrow 3]$. We are interested in the subcategory $\operatorname{rep}_k^*(Q \otimes \vec{A}_n)$ of $\operatorname{rep}_k(Q \otimes \vec{A}_n)$. By the above corollary, up to finitely many indecomposables, $\operatorname{rep}_k^*(Q \otimes \vec{A}_n)$ is equivalent to $\operatorname{rep}_k(\vec{A}_{n-1} \otimes Q)$. (Here we use that for this specific quiver $Q$, the costable Auslander algebra is equivalent to $kQ$ again.)
In particular, $\text{rep}_k^*(Q \otimes \tilde{\mathcal{A}}_n)$ has finitely many indecomposables if and only if $kQ \otimes k\tilde{\mathcal{A}}_{n-1}$ is representation finite. This is known to be the case if and only if $n \leq 3$ by \cite[Theorems 2.4 and 2.5]{14}.

**Theorem 3.25.** The category $\text{rep}_k^*(Q \otimes \tilde{\mathcal{A}}_2)$ has finitely many indecomposables if and only if

- $Q$ is of Dynkin type $A_1$, $A_2$, $A_3$, or $A_4$, or
- $Q$ is of Dynkin type $A_5$ and the Loewy length of $kQ$ is at least 4.

**Proof.** First consider the case that $Q$ is not Dynkin. Then $\text{mod} kQ$ already contains infinitely many indecomposables, and we obtain infinitely many indecomposables in $\text{rep}_k^*(Q \otimes k\tilde{\mathcal{A}}_2)$ by considering isomorphisms in the $\tilde{\mathcal{A}}_2$-direction.

Thus we may assume that $Q$ is Dynkin, and hence the “In particular”-part of Corollary 3.23 applies. Recall that this asserts that $\text{rep}_k^*(Q \otimes k\tilde{\mathcal{A}}_2)/\mathcal{X} \simeq \text{mod} \Gamma$, where $\Gamma$ is the costable Auslander algebra of $kQ$, that is the algebra given by the Auslander–Reiten quiver of $kQ$ without the vertices corresponding to injective modules, and subject to mesh relations.

Since the category $\mathcal{X}$ only contains finitely many indecomposables, it follows that $\text{rep}_k^*(Q \otimes k\tilde{\mathcal{A}}_2)$ has finitely many indecomposables if and only if $\text{mod} \Gamma$ does, i.e. if $\Gamma$ is representation finite. So we only need to investigate for which Dynkin quivers $Q$ the costable Auslander algebra is representation finite.

Note that for a subquiver $Q'$ of $Q$, the corresponding (costable) Auslander algebra $\Gamma_{Q'}$ is an idempotent subalgebra of $\Gamma_Q$. In particular, if the costable Auslander algebra of $Q'$ is representation infinite then so is the costable Auslander algebra of $Q$. Therefore, the “only if” part of the theorem is proven provided we can show that the costable Auslander algebras of

- type $D_4$, with any orientation,
- $1 \longleftrightarrow 2 \rightarrow 3 \rightarrow 4 \longleftrightarrow 5$,
- $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \longleftrightarrow 5$,
- $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \longleftrightarrow 6$

are representation infinite. (Here the double arrows indicate that we have to consider all possible orientations. A priori there are additional cases to consider, but these are the opposites of the ones in the latter three points.)

We investigate them one by one, but using the same strategy in all cases. The Auslander–Reiten quivers of $\text{mod} kQ$ and $\text{mod} kQ = \text{mod} k\tilde{\mathcal{A}}_2$ are depicted in Table 1 for all 4 cases. In all cases, we marked a subset of the vertices in the costable Auslander–Reiten quiver by solid dots. The idempotent subalgebras of the costable Auslander algebras given by this subset of vertices are, in the four cases:

- The Kronecker algebra $\circ \xrightarrow{\gamma} \circ$.
- The algebra of type $\tilde{D}_4$, with any orientation.
- The algebra given by the quiver with relations as on the left below. This algebra has the “frame” as depicted on the right in the terminology of Happel and Vossieck’s \cite{10}. In particular it appears in their list of tame concealed algebras of type $\tilde{E}_7$.

```
\circ \xrightarrow{\gamma} \circ \xrightarrow{\gamma} \circ \xrightarrow{\gamma} \circ \xrightarrow{\gamma} \circ \xrightarrow{\gamma} \circ
```

- The algebra of type $\tilde{D}_4$, as above.

In particular these algebras are all representation infinite, and hence so are the costable Auslander algebras from Table 1.
For the “if” part of the statement, it suffices to show that all maximal quivers in the list actually give rise to finite categories $\text{rep}_{\mathbf{k}}^*\mathbf{m}(Q \otimes \bar{A}_2)$. These are $1 \rightarrow 2 \leftarrow 3 \rightarrow 4$, $1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5$, and $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$. The finiteness of $\text{rep}_{\mathbf{k}}^*\mathbf{m}(Q \otimes \bar{A}_2)$ for these three cases can be checked by calculating explicitly the Auslander–Reiten quivers.

Our final explicit quiver application concerns the case that we have monomorphisms in all directions. It differs slightly from all the above, in that it does not build on Theorem 3.11, but rather on the underlying Corollary 2.30.

**Corollary 3.26.** In the subcategory $\text{rep}_{\mathbf{k}}^m(\bar{A}_m \otimes \bar{A}_n)$ of $\text{rep}_{\mathbf{k}}(\bar{A}_m \otimes \bar{A}_n)$, let $T$ the subcategory formed by finite direct sums of modules of the form

$$T_{i,j} = \mathbf{k}\{ (x,y) | x > i \text{ or } y > j \}.$$

Then

$$\frac{\text{rep}_{\mathbf{k}}^m(\bar{A}_m \otimes \bar{A}_n)}{T} \simeq \text{rep}_{\mathbf{k}}(\bar{A}_{m-1} \otimes \bar{A}_{n-1}).$$
In particular, $\text{rep}_{k}^{m,n}(\tilde{A}_m \otimes \tilde{A}_n)$ contains finitely many indecomposables precisely in the cases

- $m \leq 2$ or $n \leq 2$,
- $(m,n) \in \{(3,3),(3,4),(3,5),(4,3),(5,3)\}$.

It is of tame representation type precisely in the cases

- $(m,n) \in \{(3,6),(4,4),(6,3)\}$.

In all other cases it is of wild representation type.

\textbf{Proof.} One easily verifies that $T$ is tilting in $\text{rep}_{k}({\vec{A}}_m \otimes {\vec{A}}_n)$. Thus Corollary 2.30 applies, and gives

$$\{ M \in \mathcal{T} \mid \text{pdim\ }M \leq 1 \} \simeq \mathcal{T}^\perp.$$ 

The category $\mathcal{A} = \text{rep}_{k}(\tilde{A}_m \otimes \tilde{A}_n)$ has projective and injective modules

$$P_{i,j} = k_{\{i,...,m\} \times \{j,...,n\}},$$

$$I_{i,j} = k_{\{1,...,i\} \times \{1,...,j\}},$$

so $T_{i,j}$ has the injective resolution $T_{i,j} \hookrightarrow I_{m,n} \rightarrow I_{i,j}$. It follows that

$$\text{Ext}^1_{\mathcal{A}}(M,T_{i,j}) = \text{coker}[\text{Hom}_{\mathcal{A}}(M,I_{m,n}) \rightarrow \text{Hom}_{\mathcal{A}}(M,I_{i,j})]$$

$$= D\ker[M_{i,j} \rightarrow M_{m,n}],$$

where $D$ denotes the vector space dual. Thus $\mathcal{T}^\perp = \text{rep}_{k}^{m,n}(\tilde{A}_m \otimes \tilde{A}_n)$. Since any module in $\text{rep}_{k}^{m,n}(\tilde{A}_m \otimes \tilde{A}_n)$ is a submodule of (copies of) the projective module $P_{1,1}$, and the global dimension of $k\tilde{A}_m \otimes k\tilde{A}_n$ is 2, these modules automatically have projective dimension at most 1. Thus we have shown that

$$\{ M \in \mathcal{T} \mid \text{pdim\ }M \leq 1 \} = \text{rep}_{k}^{m,n}(\tilde{A}_m \otimes \tilde{A}_n).$$

Now we explicitly describe $\mathcal{T}^\perp$. First observe that $P_{i,1} = T_{i-1,n}$ and $P_{i,j} = T_{m,j-1}$, whence all modules in $\mathcal{T}^\perp$ vanish in the vertices of the forms $(i,1)$ and $(1,j)$. On the other hand, all modules in $\mathcal{T}$ are generated in these vertices, so there cannot be any nonzero maps from $\mathcal{T}$ to modules vanishing in these vertices. In other words, we have shown that

$$\mathcal{T}^\perp = \{ M \in \text{rep}_{k}(\tilde{A}_m \otimes \tilde{A}_n) \mid \forall i: M_{i,1} = 0 \text{ and } \forall j: M_{1,j} = 0 \}.$$ 

Clearly this latter subcategory may be identified with $\text{rep}_{k}(\tilde{A}_{m-1} \otimes \tilde{A}_{n-1})$. \qed

\textbf{Acknowledgements}

UB and MB have been supported by the DFG Collaborative Research Center TRR109 Discretization in Geometry and Dynamics. SO has been supported by Norwegian Research Council project 250056, “Representation theory via subcategories”. JS has been partially supported by Norwegian Research Council project 231000, “Clusters, combinatorics and computations in algebra”.
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