1 Introduction

An $n$-fold Kuga variety, which we will refer to as a Kuga variety, is a variety over a Siegel modular variety such that each fibre is a product of $n$ copies of the abelian variety or Kummer variety to which it corresponds in the base. In this paper, we study the Kodaira dimension of $n$-fold Kuga varieties, $X^n_p$, over the moduli spaces $\mathcal{A}_p$ of $(1, p)$-polarised abelian surfaces with canonical level structure for prime $p \geq 3$.

There are results about the Kodaira dimensions of some special kinds of Kuga varieties: \cite{V}, \cite{FV} have proved the unirationality of Kuga families, i.e. the 1-fold Kuga varieties, over moduli spaces of principally polarised abelian varieties of dimension 4 and 5; in \cite{PSMS}, we computed the Kodaira dimension of any Kuga variety over moduli spaces of principally polarised abelian varieties of dimension $g \geq 2$. It is therefore natural to ask for the Kodaira dimension of Kuga varieties of other kinds, for example, the $X^n_p$ described above.

A connection between modular forms and differential forms on arbitrary Kuga varieties is established in \cite{M}. This gives us some information about the Kodaira dimensions, assuming a specific compactification for the Kuga varieties, which is referred to as a Namikawa compactification in \cite{PSMS}. Specifically, \cite{M} Theorem 1.3 translates to:

**Theorem 1.** Let $X$ be a Namikawa compactification of $X^n_p$. Then

$$\kappa(\overline{\mathcal{A}}_p, (n + 3)\mathcal{L} - \Delta_A) \leq \kappa(K_X) \leq 3$$

where $\overline{\mathcal{A}}_p$ is a toroidal compactification of $\mathcal{A}_p$, $\mathcal{L}$ is the $\mathbb{Q}$-line bundle of weight 1 modular forms of $\Gamma_p$ and $\Delta_A$ is the boundary divisor of $\overline{\mathcal{A}}_p$.

The Kodaira dimension is a birational invariant by definition, so $\kappa(X^n_p) = \kappa(X)$. Moreover, if $X$ has canonical singularities, then $\kappa(X) = \kappa(K_X)$, and hence $\kappa((n + 3)\mathcal{L} - \Delta_A)$ is a lower bound for $\kappa(X^n_p)$.

We say a Kuga variety $X^n_p$ is of relative general type if its Kodaira dimension equals the dimension of the base $\mathcal{A}_p$ of $X^n_p$, namely 3, which is also the maximum value $\kappa(X^n_p)$ can attain.
This paper is divided into two parts: in Section 2 we show that for \( n > 2 \) and any \( p \), the particular Namikawa compactification \( X \) of \( X^n_p \) constructed in [PSMS] has canonical singularities; in Section 3 we search for a lower bound of \((p, n)\) for which \( \kappa((n + 3)L - \Delta_A) = 3 \). We summarise our result in the following theorem:

**Theorem 2.** A Kuga variety \( X^n_p \) is of relative general type if

- \( p \geq 3 \) and \( n \geq 4 \);
- \( p \geq 5 \) and \( n \geq 3 \).

Combining the results of [GH] and [HS], which say \( X^0_p = A_p \) is of general type for \( p \geq 37 \), we can mark on the \((p, n)\)-plane a region for which the Kuga varieties are of relative general type as in Figure 1.

![Figure 1](image)

1.1 Construction of Kuga varieties

The moduli space \( A_p \) of \((1, p)\)-polarised abelian surfaces with canonical level structure, defined and studied in [HKW1 Chapter I.1], is given as the quotient of the Siegel upper half plane \( \mathbb{H}_2 \) by the action of a certain arithmetic subgroup \( \Gamma_p \) of \( \text{Sp}(4, \mathbb{Z}) \). Recall the Siegel upper half space of degree \( g \) is defined as

\[
\mathbb{H}_g = \{ \tau \in M_{g \times g}(\mathbb{C}) : \tau = \tau^t, \Im \tau > 0 \} .
\]

Note that for any integer \( k \geq 2 \), the Grassmannian \( \text{Gr}(2, \mathbb{C}^k) \) is isomorphic to the orbit space \( M_{k \times 2}(\mathbb{C})/\text{GL}(2, \mathbb{C}) \) of all \( k \times 2 \) matrices modulo right multiplication by the invertible matrices in \( \text{GL}(2, \mathbb{C}) \). So the Siegel upper
half plane $\mathbb{H}_2$ can be identified with a subset of $\text{Gr}(2, \mathbb{C}^4)$ by sending an element $\tau$ to the $\text{GL}(2, \mathbb{C})$-equivalence class of block matrices:

$$\tau \mapsto \begin{bmatrix} \tau \\ 1_2 \end{bmatrix}.$$ 

For any prime $p \geq 3$, we define

$$\Gamma_p = \left\{ \gamma \in \text{Sp}(4, \mathbb{Z}) : \gamma - 1 \in \begin{pmatrix} p \mathbb{Z} & p \mathbb{Z} & p \mathbb{Z} & p^2 \mathbb{Z} \\ p \mathbb{Z} & p \mathbb{Z} & p \mathbb{Z} & p \mathbb{Z} \\ p \mathbb{Z} & p \mathbb{Z} & p \mathbb{Z} & p \mathbb{Z} \\ p \mathbb{Z} & p \mathbb{Z} & p \mathbb{Z} & p \mathbb{Z} \end{pmatrix} \right\},$$

which is of finite index in $\text{Sp}(4, \mathbb{Z})$. This group $\Gamma_p$ acts on $H^2 \subset \text{Gr}(2, \mathbb{C}^4)$ by left multiplication on the $\text{GL}(2, \mathbb{C})$-equivalence classes of block matrices, which is an analogue of linear fractional transformations:

$$\gamma \cdot \tau = \begin{bmatrix} \gamma \cdot \begin{bmatrix} \tau \\ 1_2 \end{bmatrix} \end{bmatrix}, \quad \gamma \in \Gamma_p, \tau \in \mathbb{H}_2.$$

The quotient of $\mathbb{H}_2$ by this action of $\Gamma_p$ gives the moduli space $A_p$.

In [HKW1, Chapter I.2B], a 1-fold Kuga variety $X^1_p$ over $A_p$ is constructed. This method can be extended to construct an $n$-fold Kuga variety $X^n_p$ over $A_p$ for any positive integer $n$ by defining an extension $\tilde{\Gamma}_p^n$ of $\Gamma_p$ and a left action of it on $\mathbb{C}^{2n} \times \mathbb{H}_2$ which descends to that of $\Gamma_p$ on $\mathbb{H}_2$. First, by identifying $\mathbb{C}^{2n}$ with the set of $n \times 2$ complex matrices, we can identify $\mathbb{C}^{2n} \times \mathbb{H}_2$ with a subset of $\text{Gr}(2, \mathbb{C}^{n+4})$ by sending an element $(Z, \tau)$ to a $\text{GL}(2, \mathbb{C})$-equivalence class of block matrices:

$$(Z, \tau) \mapsto \begin{bmatrix} Z \\ \tau \\ 1_2 \end{bmatrix}.$$ 

We define the following group

$$\tilde{\Gamma}_p^n = \left\{ (l, \gamma) = \begin{pmatrix} 1_n & l \\ 0 & \gamma \end{pmatrix} \in M_{n \times 4}(\mathbb{Z}) \rtimes \Gamma_p : \gamma \in \Gamma_p \right\}.$$ 

The group $\tilde{\Gamma}_p^n$ acts on $\mathbb{C}^{n} \times \mathbb{H}_2$ by left multiplication on the $\text{GL}(2, \mathbb{C})$-equivalence classes of block matrices. Explicitly, for $\tilde{\gamma} = (l, \gamma) \in \tilde{\Gamma}_p^n$ and $\tilde{\tau} = (Z, \tau) \in \mathbb{C}^{2n} \times \mathbb{H}_2$, then

$$\tilde{\gamma} \cdot \tilde{\tau} = \begin{bmatrix} (1_n & l) \cdot \begin{bmatrix} Z \\ \tau \\ 1_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} Z + l \cdot \tau \\ \gamma \cdot \begin{bmatrix} \tau \\ 1_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} (Z + l \cdot \tau) \cdot N \\ \gamma \cdot \tau \\ 1_2 \end{bmatrix}$$

for some $N \in \text{GL}(2, \mathbb{C})$. 

3
The quotient of \((\mathbb{C}^{2n} \times \mathbb{H}_2)\) by this action of \(\tilde{\Gamma}_p^n\) gives the Kuga variety \(X_p^n\). The projection \(\mathbb{C}^{2n} \times \mathbb{H}_2 \to \mathbb{H}_2\) induces a map \(\pi: X_p^n \to A_p\). Indeed, [PSMS] [N], all fibres are the product of \(n\) copies of the torus parametrised by the corresponding point in the base \(A_p\), up to a change of basis of \(\mathbb{C}^2\) for each copy.

2 Canonical singularities

2.1 Namikawa compactification

The explicit construction of a Namikawa compactification \(X\) of the Kuga variety \(X_p^n\) is the main purpose of section 1 in [PSMS]. Here we briefly introduce its idea and some notations to be used in the latter sections.

**Definition 2.1.** A Namikawa compactification of \(X_p^n\) is an irreducible normal projective variety \(X\) containing \(X_p^n\) as an open subset, together with a projective toroidal compactification \(\overline{A}_p\) of \(A_p\) for which the following conditions hold.

1. \(\pi: X_p^n \to A_p\) extends to a projective morphism \(\overline{\pi}: X \to \overline{A}_p\);
2. every irreducible component of \(\Delta_X := X \setminus X_p^n\) dominates an irreducible component of \(\Delta_A := \overline{A}_p \setminus A_p\).

Therefore \(X\) sits inside the commutative diagram

\[
\begin{array}{ccc}
X_p^n & \longrightarrow & X \\
\downarrow \pi & & \downarrow \overline{\pi} \\
A_p & \longrightarrow & \overline{A}_p
\end{array}
\]

and \(\overline{\pi}\) does not contract any divisor.

Namikawa compactifications are constructed by toroidal methods in [N]. We now describe the partial compactification at an integral boundary component \(\tilde{F}\) of rank \(g' \leq 2\) of \(X_p^n\) which leads to a Namikawa compactification. The boundary component \(\tilde{F}\) can be written as \(\mathbb{C}^n \times F\), an extension of a rank \(g'\) boundary component \(F < \mathbb{H}_2\) of \(A_p\), where \(F\) corresponds to a rank \(g'' := g - g'\) isotropic sublattice of \(\mathbb{Z}^4\) defined up to the transitive action of \(\text{Sp}(4, \mathbb{Z})\). Let \(\tilde{\mathcal{P}}(\tilde{F})\) be the stabiliser subgroup of \(\tilde{F}\) in \(\mathbb{R}^{4n} \times \text{Sp}(4, \mathbb{R})\), which can be embedded in \(\text{GL}(n + 4, \mathbb{R})\). We also define the following subgroups of \(\tilde{\mathcal{P}}(\tilde{F})\):

\[
\begin{align*}
\tilde{\mathcal{P}}'(&\tilde{F}) := \text{Centre of the unipotent radical of } \tilde{\mathcal{P}}(\tilde{F}) \\
\tilde{\Gamma}^n := \tilde{\mathcal{P}}'(&\tilde{F}) \cap \tilde{\Gamma}_p^n \\
\tilde{\mathcal{P}}''(&\tilde{F}) := (\tilde{\mathcal{P}}(\tilde{F}) \cap \tilde{\Gamma}_p^n) / \tilde{\Gamma}^n
\end{align*}
\]
An explicit expression of the above matrix groups can be found in [N, Section 2] and [PSMS, Section 1]. Also note that both \( \tilde{\Upsilon}^n \) and \( P''(\tilde{F}) \) inherit the action of \( \Gamma_p^n \) on \( C^{2n} \times \mathbb{H}_2 \).

Consider the Siegel domain realisation of \( \mathbb{H}_2 \), which gives \( C^{2n} \times \mathbb{H}_2 \) as an open subset of

\[
(C^{g'\times n} \times C^{g''\times n}) \times (\mathbb{H}_{g'} \times M_{g'\times g''}(\mathbb{C}) \times M_{g''}^{sym}(\mathbb{C})).
\]

Then taking the partial quotient by \( \tilde{\Upsilon}^n \) near the boundary component \( \tilde{F} \) corresponds to translations in the imaginary directions of the factors \( C^{g'\times n} \) and \( M_{g''}^{sym}(\mathbb{C}) \) respectively. Therefore there is an open subset of

\[
C^{g'\times n} \times (\mathbb{C}^*)^{g''\times n} \times \mathbb{H}_{g'} \times C^{g'\times g''} \times (\mathbb{C}^*)^{g''\times g''}
\]

that uniformises \( X_p^n \).

Using a suitable cone decomposition \( \Sigma(\tilde{F}) \), we can extend the action of \( P''(\tilde{F}) \) to a smooth torus embedding \( \text{Temb}(\Sigma(\tilde{F})) \) for the torus part \( (\mathbb{C}^*)^{g''\times g''} \). The partial compactification of \( X_p^n \) at the boundary component \( \tilde{F} \) is then given as the quotient of (an open subset of) the torus bundle

\[
\tilde{X}(\tilde{F}) := \mathbb{H}_{g'} \times C^{g'\times g''} \times C^{g'\times n} \times \text{Temb}(\Sigma(\tilde{F}))
\]

by the action of \( P''(\tilde{F}) \). Note that this decomposition of \( \tilde{X}(\tilde{F}) \) into its factors is preserved by the quotient.

In practice, such a cone decomposition \( \Sigma(\tilde{F}) \) can be given by an extension of the perfect cone decomposition near the boundary component \( F \) of \( A_p \). Furthermore, this extension can be chosen carefully to satisfy more conditions as listed in [PSMS, proposition 1.4]. In particular, the set of cone decompositions is compatible at each cusp such that it results in a Namikawa compactification \( X \). Also, the local uniformising space \( \tilde{X}(\tilde{F}) \) of \( X \) has canonical singularities.

We will prove in the remaining subsections that for any \( p \) and \( n > 2 \), this Namikawa compactification \( X_p^n \) has canonical singularities.

### 2.2 The general strategy

We will separately examine the singularities in the interior and the boundary of \( X \), and check if they are canonical by applying the Reid–Shepherd-Barron–Tai (RST) criterion.

We will need the following set up to state the RST criterion [R]: Suppose \( G \) is a finite group acting on the complex vector space \( C^m \) linearly. For a non-trivial element \( \gamma \in G \) of order \( k \), the eigenvalues of its action on \( C^m \) can be expressed as an \( m \)-tuple \( (\xi^{a_1}, \cdots, \xi^{a_m}) \), with \( \xi \) being a primitive \( k \)-th
root of unity and $\alpha_j$ being a non-negative integer less than $k$ for any $j$. We define, with dependence on the choice of $\xi$, the type of $\gamma$ to be

$$\frac{1}{k}(\alpha_1, \cdots, \alpha_m)$$

and its associated RST sum to be

$$\text{RST}(\gamma) := \sum_{i=1}^{m} \frac{\alpha_i}{k}.$$ 

Furthermore, we say that $\gamma$ is a quasi-reflection if all but one $\alpha_j$’s are 0, or equivalently $\gamma$ preserves a divisor.

The RST criterion is then given by the following:

**Theorem 3** ([R, 4.11]). Let $G$ be a finite group which acts on $\mathbb{C}^m$ as above. Then $\mathbb{C}^m/G$ has a canonical singularity if $G$ contains no quasi-reflection, and if every non-trivial element $\gamma \in G$ satisfies the inequality

$$\text{RST}(\gamma) \geq 1.$$ 

Note that, since we need to check the above inequality involving the RST sum for every element in $G$, it does not matter which root of unity $\xi$ was chosen to give the type of a generator $\gamma$ of $G$.

2.2.1 Strategy in the interior

In the interior $X^n_p$ of $X$, a singularity corresponds to a point $\tilde{\tau} = (Z, \tau)$ in $\mathbb{C}^{2n} \times \mathbb{H}_2$ fixed by $\tilde{\Gamma}_p^n$. We are allowed to apply the RST criterion to check if $\tilde{\tau}$ corresponds to a canonical singularity: suppose $\tilde{\gamma}$ is an element in the isotropy group $\text{iso}(\tilde{\tau}) < \tilde{\Gamma}_p^n$ of $\tilde{\tau}$. By ([1]), it is clear that one can consider the action of $\tilde{\gamma}$ separately as the action of $\gamma$ on the $\mathbb{H}_2$ factor and that of $\tilde{\gamma}$ on the $\mathbb{C}^{2n}$ factor. Also, $\tilde{\gamma}$ fixes $\tilde{\tau}$ only if $\gamma$ fixes $\tau$. The isotropy group $\text{iso}(\tilde{\tau})$ of $\tilde{\tau}$ in $\tilde{\Gamma}_p^n$ is finite, so any nontrivial element $\tilde{\gamma} = (l, \gamma)$ in $\text{iso}(\tilde{\tau})$ is a torsion element and $l = 0$. As a result of ([1] Theorem 4.1), the induced action of any element $\gamma \in \text{iso}(\tau) \leq \Gamma_p$ of order $k$ on the tangent space $T_{\tau}(\mathbb{H}_2)$ can be diagonalised under suitable local coordinates. It will be shown that $\tilde{\gamma}$ also acts diagonally on $T_Z(\mathbb{C}^{2n})$. This gives us the finite dimensional representation of $\text{iso}(\tilde{\tau})$ required for the application of the RST criterion.

Note that it suffices to apply the RST criterion at a limited number of singularities in $X^n_p$:

**Lemma 1.** Let $\tilde{\tau} = (Z, \tau)$ be a point in $\mathbb{C}^{2n} \times \mathbb{H}_2$ that corresponds to a canonical singularity in $X^n_p$. Then either $\tau$ corresponds to a canonical singularity in $A_p$, or $\text{iso}(\tilde{\tau}) = \langle \tilde{\sigma} := (0, -1_4) \rangle < \tilde{\Gamma}_p^n$. In the latter case, $\tau$ corresponds to a smooth point.
Proof. The isotropy group of $\tilde{\tau}$, $\text{iso}(\tilde{\tau})$, cannot contain a quasi reflection: according to [M, Lemma 7.1], a non-trivial element $\tilde{\gamma} \in \text{iso}(\tilde{\tau})$ does not fix any divisor in $\mathcal{X}_p^n$.

Consider any nontrivial $\tilde{\gamma} := (0, \gamma) \in \text{iso}(\tilde{\tau})$. If $\gamma$ acts trivially on $H_2$, then $\gamma = -1_4$.

Moreover, by the definition of RST sums, we have

$$\text{RST}(\tilde{\gamma}) \geq \text{RST}(\gamma)$$

So $\tilde{\tau}$ corresponds to a canonical singularity in $\mathcal{X}_p^n$ if $\tau$ corresponds to a canonical singularity in $\mathcal{A}_p$.

2.2.2 Strategy in the boundary

A singularity in the boundary of $X$ correspond to a point $\tilde{\tau}$ in $\tilde{X}(\tilde{F})$ fixed by $\tilde{P}''(\tilde{F})$ near a boundary component $\tilde{F}$ of rank $g'$. Again, the RST criterion can be applied to check if $\tilde{\tau}$ corresponds to a canonical singularity: Let $\tilde{\tau} := (Z, \tau)$, where $Z \in C^{2n}$ and $\tau \in H_p \times \text{Temb}(\Sigma(\tilde{F}))$. As mentioned in section 2.1, $\tilde{P}''(\tilde{F})$ preserves the decomposition of $\tilde{X}(\tilde{F})$, so $\tilde{\gamma}$ acts on each factors of $\tilde{X}(\tilde{F})$ separately. A calculation similar to (*) shows that locally at $\tilde{\tau}$, $\tilde{\gamma} = (l, \gamma) \in \tilde{P}''(\tilde{F})$ fixes $\tilde{\tau}$ only if $\gamma$ fixes $\tau$. However, different from what we had in section 2.2.1, $\tilde{\gamma}$ may not be a torsion element, i.e. $l$ could be non-zero. Nevertheless, the action of $\tilde{\gamma}$ on the tangent space of a resolution of $\tilde{X}(\tilde{F})$ at $\tilde{\tau}$ at $\tilde{\tau}$ is of finite order, so the RST criterion can be applied there.

The following observations are useful for checking whether these singularities are canonical:

1. [PSMS] Lemma 1.3]: Let $(\tilde{X}(\tilde{F}))^*$ be a smooth $\tilde{P}''(\tilde{F})$-equivariant resolution of $\tilde{X}(\tilde{F})$. If $\tilde{P}''(\tilde{F})$ has no quasireflection, then the partial compactification $\tilde{P}''(\tilde{F}) \setminus \tilde{X}(\tilde{F})$ has canonical singularities if $\tilde{P}''(\tilde{F}) \setminus (\tilde{X}(\tilde{F}))^*$ has canonical singularities. In particular, this implies that we can apply the RST criterion at the singularities in $\tilde{P}''(\tilde{F}) \setminus (\tilde{X}(\tilde{F}))^*$ instead.

2. Let $\tilde{\tau} = (Z, \tau)$ correspond to a canonical singularity near $\tilde{F}$. Then either $\tau$ corresponds to a canonical singularity in the boundary of $\overline{\mathcal{A}_p}$, or $\text{iso}(\tilde{\tau}) = (\tilde{\sigma} := (l, -1_4)) < \tilde{P}''(\tilde{F})$ for some $l \in L$. In the latter case, $\tau$ corresponds to a smooth point. The proof is similar to that in Lemma 1. Again, this implies that we only need to apply the RST criterion at a limited number of singularities.

2.3 Singularities in the interior of compactification

In this section, we will identify the singularities in $\mathcal{X}_p^n$ and show that for $n > 2$, they are all canonical.
First we identify singularities that project to non-canonical singularities in $A_p$. It is given in the proof of [HKW2, Theorem 1.8] that for any odd prime $p$, the singular points in $A_p$ are exactly the points that lie on the two disjoint curves $C_1$ and $C_2$. Any point on one of these curves corresponds to a point $\tau$ in $H_2$, whose isotropy group in $\Gamma_p$ is generated by a single generator. Its induced action the tangent space of $H_2$ at $\tau$ is also given there: one can write any point in the tangent space $T_{\tau}(H_2)$ in the form 

$$
\begin{pmatrix}
\tau_1 + x & \tau_2 + y \\
\tau_2 + y & \tau_3 + z
\end{pmatrix}.
$$

So the tuple $(x, y, z)$ can be considered as the local coordinates for $T_{\tau}A_p$, and the respective action of a generator of $\text{iso}(\tau)$ on $T_{\tau}(H_2)$ with these coordinates is given by 

$$(x, y, z) \mapsto (-x, -iy, z) \text{ along } C_1;$$

$$(x, y, z) \mapsto (\rho^2 x, -\rho y, z) \text{ along } C_2,$$

where $\rho = e^{2\pi i/3}$.

Therefore, the chosen generators are of types $\frac{1}{4}(2, 3, 0)$ and $\frac{1}{6}(4, 5, 0)$ when the root of unity $\xi$ is chosen to be $i$ and $e^{2\pi i/6}$ respectively on each curve $C_1$ and $C_2$. By applying the RST criterion to the isotropy groups, it is clear that singularities on $C_1$ are canonical but those on $C_2$ are not.

Let $\tilde{\tau} := (Z, \tau) \in \mathbb{C}^{2n} \times \mathbb{H}_2$ such that $\tau$ corresponds to a point in $C_2$. Let $\tilde{\sigma} := (0, -1_4)$ and $\tilde{\gamma} := (0, \gamma)$, where $\gamma$ is the generator of $\text{iso}(\tau)$ with the action on $T_{\tau}(H_2)$ described above. Then either $\text{iso}(\tilde{\tau}) = \langle \tilde{\gamma}, \tilde{\sigma} \rangle$ or $\text{iso}(\tilde{\tau}) = \langle \tilde{\gamma} \rangle$.

We shall first compute the type of $\tilde{\gamma}$. We only need to understand the action of $\tilde{\gamma}$ at a point $\tilde{Y} = (Z + Y, \tau)$ on the tangent space $T_{\tau}(\mathbb{C}^{2n} \times \{\tau\}) \simeq T_Z(\mathbb{C}^{2n})$ to complete the type of $\tilde{\gamma}$. To do this, we need the explicit expressions of the set $C_2$ and its isotropy group $\text{iso}(\tau)$ from [HKW2, Definition 1.5]: 

$$C_2 = \left\{ \begin{pmatrix}
\rho & 0 \\
0 & \tau_3
\end{pmatrix} : \rho = e^{2\pi i/3}, \tau_3 \in \mathbb{H}_1 \right\},$$

$$\text{iso}(\tau) = \langle \gamma = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \rangle.$$

Following (8), the action of $\tilde{\gamma}$ at $\tilde{Y}$ in $T_{\tau}(\mathbb{C}^{2n} \times \{\tau\})$ is given by 

$$\tilde{\gamma} \cdot \tilde{Y} = \begin{pmatrix}
(Z + Y) \cdot N \\
\tau \\
1_2
\end{pmatrix},$$

where $N = \begin{pmatrix}
(\rho + 1)^{-1} & 0 \\
0 & 1
\end{pmatrix}$.

Since $\tilde{\gamma}$ fixes $(Z, \tau)$, $Z \cdot N = Z$ and $\tilde{\gamma}$ acts on $T_Z(\mathbb{C}^{2n})$ diagonally by sending the set of local coordinates $Y$ to $Y \cdot N$.
Note that \((\rho + 1)^{-1} = e^{2\pi i (5/6)}\). So by choosing the primitive root of unity to be \(e^{2\pi i /6}\), which is the same as that for the \(\mathbb{H}_2\) factor, we have an extra \(n\) copies of \(5/6\)'s and \(n\) copies of \(0\)'s in the RST sum of \(\tilde{\gamma}\). In other words, the type of \(\tilde{\gamma}\) is \(\frac{1}{6}(4,5,0,5,\cdots,5,0,\cdots,0)\).

As for the type of \(\tilde{\sigma}\), since \(\tilde{\sigma}\) acts trivially on \(T_\tau(\langle Z \rangle \times \mathbb{H}_2)\), the first entries in the type of \(\tilde{\sigma}\) which correspond to the \(\mathbb{H}_2\) factor are all \(0\)'s. On the other hand, the calculation in (2) shows that \(\tilde{\sigma}\) acts on the set of local coordinates in \(T_2(C^{2n})\) diagonally by \(X \mapsto -X\). So the type of \(\tilde{\sigma}\) is \(\frac{1}{6}(0,0,0,1,\ldots,1)\) when the primitive root of unity \(\xi\) is chosen to be \(-1\).

Since \(\tilde{\sigma}\) commutes with \(\tilde{\gamma}\), we can draw the following table which shows the type of a non-trivial element \(\tilde{\gamma}^{k_1} \tilde{\sigma}^{k_2} \in \text{iso}(\tilde{\tau})\), where \(0 \leq k_1 \leq 5\) and \(0 \leq k_2 \leq 1\).

| \(k_2\) | \(k_1\) | 0                      | 1                      |
|--------|--------|------------------------|------------------------|
| 0      |        | \(N/A\)                | \(\frac{1}{6}(0,0,0,1,\cdots,1,1,\cdots,1)\) |
| 1      | \(\frac{1}{6}\) | \(\frac{1}{6}(4,5,0,5,\cdots,5,0,\cdots,0)\) | \(\frac{1}{6}(4,5,0,2,\cdots,2,3,\cdots,3)\) |
| 2      | \(\frac{1}{6}\) | \(\frac{1}{6}(2,4,0,4,\cdots,4,0,\cdots,0)\) | \(\frac{1}{6}(2,4,0,1,\cdots,1,3,\cdots,3)\) |
| 3      | \(\frac{1}{6}\) | \(\frac{1}{6}(0,3,0,3,\cdots,3,0,\cdots,0)\) | \(\frac{1}{6}(0,3,0,0,\cdots,0,3,\cdots,3)\) |
| 4      | \(\frac{1}{6}\) | \(\frac{1}{6}(4,2,0,2,\cdots,2,0,\cdots,0)\) | \(\frac{1}{6}(4,2,0,5,\cdots,5,3,\cdots,3)\) |
| 5      | \(\frac{1}{6}\) | \(\frac{1}{6}(2,1,0,1,\cdots,1,0,\cdots,0)\) | \(\frac{1}{6}(2,1,0,4,\cdots,4,3,\cdots,3)\) |

The types of all non-trivial elements in \(\langle \tilde{\gamma} \rangle\) are given by the first column of the table, while that in \(\langle \tilde{\gamma}, \tilde{\sigma} \rangle\) are given by the entire table. Notice the RST criterion only fails when \(n \leq 2\):

\[
\text{RST}(\tilde{\gamma}^5) < 1.
\]

We conclude that for \(n > 2\), both \(\langle \tilde{\gamma} \rangle\) and \(\langle \tilde{\gamma}, \tilde{\sigma} \rangle\) satisfy the RST criterion, and therefore \(\tilde{\tau}\) is a canonical singularity in \(X^n_p\), no matter \(\text{iso}(\tilde{\tau}) = \langle \tilde{\gamma}, \tilde{\sigma} \rangle\) or \(\text{iso}(\tilde{\tau}) = \langle \tilde{\gamma} \rangle\).

Finally, for any singularity that corresponds to a point in \(C^{2n} \times \mathbb{H}_2\) whose isotropy group is \(\langle \tilde{\sigma} \rangle\), we only need study the first row of the table: there is no quasi-reflection and the RST inequality is satisfied for any \(n\). Therefore such singularity is always canonical.

### 2.4 Singularities in the boundary of compactification

In this section we will check that every singularity in the boundary of \(X\) is canonical.

First, we identify all the non-canonical singularities in \(\overline{\mathcal{X}}_p\). Consider the compact curves \(C^*_1\) and \(C^*_2\) containing \(C_1\) and \(C_2\) in \(\overline{\mathcal{X}}_p\). Then from [HKW2, Propositions 2.15 and 3.4], for any odd prime \(p\), the complement \(\overline{\mathcal{A}}_p \setminus (C^*_1 \cup C^*_2)\) contains only isolated singularities. The types of a generator
in the respective isotropy groups are given as $\frac{1}{2}(1, 1, 1)$ or $\frac{1}{3}(1, 2, 1)$. So both isotropy groups satisfy the RST criterion, and these singularities in $X$ are canonical. Therefore, any non-canonical singularity in $X$ has to project down to $C_1^* \setminus C_1$ or $C_2^* \setminus C_2$.

From the same source above, each set $C_1^* \setminus C_1$ and $C_2^* \setminus C_2$ consists of $(p^2 - 1)/2$ points, one in each of the rank 1 boundary components called peripheral components [HKW1 Definition I.3.105]. [HKW2 Proposition 2.8] further says that near one of these boundary component $F$, the singularities in $C_1^*$ and $C_2^*$ are represented by $Q_1 = (i, 0, 0)$ and $Q_2 = (p, 0, 0)$ as points in $\mathbb{H}_1 \times \mathbb{C} \times \mathbb{C}$, the Siegel domain realisation of $\mathbb{H}_2$, with $p = e^{2\pi i/3}$.

First consider the singularity in $X$ associated to $Q_2$: let $\tilde{\tau} := (Z, \tau) \in \tilde{X}(\tilde{F})$ such that $\tau = Q_2$. From [HKW2 Propositions 2.5 and 2.8], the stabiliser subgroup of $\tau$ in $P''(F) \cong (P(F) \cap \Gamma_p)/(P(F) \cap \Gamma_p)$ is generated by the order 6 element $\gamma = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Let $\tilde{\gamma} := (l, \gamma)$ be the corresponding generator in $\text{iso}(\tilde{\tau})$, and let $\tilde{\sigma} := (l, -1_4)$ for some $l \in L$. Then again $\text{iso}(\tilde{\tau}) = \langle \tilde{\gamma} \rangle$ or $\text{iso}(\tilde{\tau}) = \langle \tilde{\gamma}, \tilde{\sigma} \rangle$. To find the types of elements in $\text{iso}(\tilde{\tau})$, we consider their actions on each factor of $\tilde{X}(\tilde{F})$, a $P''(\tilde{F})$-equivariant resolution of $\tilde{X}(\tilde{F})$. [T Lemmas 5.1 and 5.2] describes such a resolution of singularities for the moduli space of polarised abelian $g$-folds, as well as a formula for the RST sum of a generator $\gamma$ in the isotropy group. Explicitly when $g = 2$, there are three factors in the resolution of $\mathbb{H}_2$: $\mathbb{H}_{g'}, \mathbb{C}^{g'g''}$ and a torus at infinity. The following submatrices are extracted from the entries $\gamma_{ij}$ of $\gamma$:

$$\gamma' = \begin{pmatrix} \gamma_{11} & \gamma_{13} \\ \gamma_{31} & \gamma_{33} \end{pmatrix}, \quad U = \begin{pmatrix} \gamma_{22} \end{pmatrix}.$$

Suppose $\gamma'$ has eigenvalues $\lambda^{\pm 1}$ and $U$ has eigenvalue $\mu$. Then the eigenvalues of the action of $\gamma$ on the tangent space of the $\mathbb{H}_{g'}$ factor, the $\mathbb{C}^{g'g''}$ factor and the torus at infinity in the resolution of $\mathbb{H}_2$ are $\lambda$, $\lambda \mu$ and 0 respectively.

In our case, $\gamma'$ has eigenvalues $e^{\pm 2\pi i/3}$ and $U$ has eigenvalue 1. Therefore, when $e^{2\pi i/6}$ is chosen to be the primitive root of unity, the $\mathbb{H}_{g'}$, $\mathbb{C}^{g'g''}$ and the torus at infinity factors contribute $\frac{1}{3}$, $\frac{1}{6}$ and $0$ to the RST sum respectively.

For the RST sum over the remaining $\mathbb{C}^n \times (\mathbb{C}^*)^n$ factor of $\tilde{\tau}$, follow [T] and consider the action of $\tilde{\gamma}$ on $\tilde{Y} := (Z + Y, \tau)$ in the tangent space at $Z$ of the resolved $\mathbb{C}^{2n}$ factor:

$$\tilde{\gamma} \cdot \tilde{Y} = \begin{pmatrix} (Z' + Y) \cdot N \\ \tau \\ 1_2 \end{pmatrix}$$

where $Z' = Z + l \cdot (\tau)$ and $N = \begin{pmatrix} \frac{1}{p+1} & 0 \\ 0 & 1 \end{pmatrix}$. 

10
Again \( \tilde{\gamma} \) fixes \( \tilde{\tau} \), so \( Z' \cdot N = Z \) and \( \tilde{\gamma} \) acts on the tangent space diagonally by sending the local coordinates \( Y \) to \( Y \cdot N \). The eigenvalues of the action are the eigenvalues of \( N \), which are \( e^{\pm \pi i/6} \) and 1. When we choose \( e^{\pi i/6} \) to be the primitive root of unity for \( \tilde{\gamma} \), which is the same choice as the other factors, they contribute \( n \) copies of \( \frac{e^{\pi i/6}}{6} \) and \( n \) copies of 0 to the RST sum.

Do the same for \( \tilde{\sigma} \) to find \( \text{RST}(\tilde{\sigma}) \): write \( \sigma = -1 \) and consider the submatrices \( \sigma' \) and \( U \) extracted from \( \sigma \) in the same way as above. Their eigenvalues are \( \{1, -1\} \) and \( -1 \) respectively, which contribute 0 to the RST sum for all 3 factors of the solution of \( \mathbb{H}_2 \) after resolving. Following (M), the action of \( \tilde{\sigma} \) on the tangent space of the resolved \( \mathbb{C}^n \times (\mathbb{C}^*)^n \) factor at \( Z \) is again multiplication by \( -1 \) to the local coordinates \( Y \).

Therefore, we can draw a similar table as in the previous subsection for each element \( \tilde{\sigma}^{k_1} \tilde{\sigma}^{k_2} \in \text{iso}(\tilde{\tau}) \), where \( 0 \leq k_1 \leq 5 \) and \( 0 \leq k_2 \leq 1 \):

\[
\begin{array}{c|cc}
  & 0 & 1 \\
\hline
k_2 & N/A & \frac{1}{2}(0, 0, 0, 1, \cdots, 1, 1, \cdots, 1) \\
0 & \frac{1}{2}(2, 1, 0, 5, \cdots, 5, 0, \cdots, 0) & \frac{1}{2}(2, 1, 0, 2, \cdots, 2, 3, \cdots, 3) \\
1 & \frac{1}{2}(4, 2, 0, 4, \cdots, 4, 0, \cdots, 0) & \frac{1}{2}(4, 2, 0, 1, \cdots, 1, 3, \cdots, 3) \\
2 & \frac{1}{2}(0, 3, 0, 3, \cdots, 3, 0, \cdots, 0) & \frac{1}{2}(0, 3, 0, 0, \cdots, 0, 3, \cdots, 3) \\
3 & \frac{1}{2}(2, 4, 0, 5, \cdots, 2, 0, \cdots, 0) & \frac{1}{2}(2, 4, 0, 5, \cdots, 5, 3, \cdots, 3) \\
4 & \frac{1}{2}(4, 5, 0, 1, \cdots, 1, 0, \cdots, 0) & \frac{1}{2}(4, 5, 0, 4, \cdots, 4, 3, \cdots, 3) \\
5 & \frac{1}{2}(4, 5, 0, 1, \cdots, 1, 0, \cdots, 0) & \frac{1}{2}(4, 5, 0, 4, \cdots, 4, 3, \cdots, 3) \\
\end{array}
\]

One can check that there is no quasi-reflection, and the RST sum is at least 1 everywhere on the table. So the RST criterion is satisfied for both \( \{\tilde{\gamma}\} \) and \( \{\tilde{\gamma}, \tilde{\sigma}\} \). Thus for all \( n \geq 1 \), the singularity in \( X \) that corresponds to \( (Z, Q_2) \) is canonical.

Now we replace \( Q_2 \) by \( Q_1 \) everywhere in the above to check whether the other singularity in the boundary component \( \tilde{F} \) is canonical or not. Again, let \( \tilde{\tau} = (Z, \tau) \) such that \( \tau = Q_1 \). The stabiliser subgroup of \( \tau = Q_1 \) in \( P^\mu(F) \) is generated by the order 4 element

\[
\gamma = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

First we calculate \( \text{RST}(\gamma) \). Extract the submatrices \( \gamma' \) and \( U \) as before. The eigenvalues of \( \gamma' \) are \( \pm \imath \) and the eigenvalue of \( U \) is 1. When \( \imath \) is the chosen primitive root of unity, the \( \mathbb{H}_{\imath} \) factor, the \( \mathbb{C}^n \) factor and the torus at infinity in the resolution of \( \mathbb{H}_2 \) contribute a \( \frac{2}{3} \), a \( \frac{1}{2} \) and a 0 to the RST sum respectively. Consider the action of \( \tilde{\gamma} \) at \( Y = (Z + Y, \tau) \). Then (M)
gives:
\[
\tilde{\gamma} \cdot \tilde{Y} = \begin{bmatrix} (Z' + Y) \cdot N \end{bmatrix} \quad \text{where} \quad Z' = Z + l \cdot \left( \begin{array}{c} \tau_2 \end{array} \right) \quad \text{and} \quad N = \left( \begin{array}{cc} -i & 0 \\ 0 & 1 \end{array} \right).
\]

Once more \(Z' \cdot N = Z\) and \(\tilde{\gamma}\) acts on the tangent space diagonally by sending the local coordinates \(Y\) to \(Y \cdot N\). This action has eigenvalues \(\frac{3 \pi i}{4}\) and 1, which contribute \(n\) copies of \(\frac{3}{4}\) and \(n\) copies of 0 to the RST sum over the resolved \(\mathbb{C}^n \times (\mathbb{C}^*)^n\) factor when the primitive root of unity chosen is \(i\).

The RST sums of \(\tilde{\sigma}\) restricted to each factor is the same as the case of \(Q_2\).

Therefore we can draw the table for the type of \(\tilde{\gamma}^{k_1} \tilde{\sigma}^{k_2} \in \text{iso}(\tau)\), where \(0 \leq k_1 \leq 3\) and \(0 \leq k_2 \leq 1:\)

| \(k_1\) | \(0\) | 1 |
|---|---|---|
| 0 | N/A | \(\frac{1}{2}(0, 0, 0, 1, \cdots, 1, 1, \cdots, 1)\) |
| 1 | \(\frac{1}{4}(2, 1, 0, 3, \cdots, 3, 0, \cdots, 0)\) | \(\frac{1}{2}(2, 1, 0, 1, \cdots, 1, 2, \cdots, 2)\) |
| 2 | \(\frac{1}{4}(0, 2, 0, 2, \cdots, 2, 0, \cdots, 0)\) | \(\frac{1}{2}(0, 2, 0, 0, \cdots, 0, 2, \cdots, 2)\) |
| 3 | \(\frac{1}{4}(2, 3, 0, 1, \cdots, 1, 0, \cdots, 0)\) | \(\frac{1}{2}(2, 3, 0, 3, \cdots, 3, 2, \cdots, 2)\) |

The RST criterion is satisfied for both \(\langle \tilde{\gamma} \rangle\) and \(\langle \tilde{\gamma}, \tilde{\sigma} \rangle\), so for all \(n \geq 1\), the singularity in \(X\) that corresponds to \((Z, Q_1)\) is canonical.

We summarise our findings in the following theorem:

**Theorem 4.** *Singularities in the Namikawa compactification \(X\) of \(X^n_p\) are canonical for \(n \geq 3\). For \(n = 1, 2\), the set of non-canonical singularities in \(X\) is exactly the preimage under \(\pi\) of the curve \(C_2\) in \(X^1_p\) and \(X^2_p\) respectively.*

### 3 Low weight cusp form trick

In this section, we will prove the following theorem:

**Theorem 5.** *The equality \(\kappa(\overline{A}_p, (n + 3)L - \Delta_A) = 3\) is satisfied for the following values of \(n\) and \(p\):

- \(p \geq 3\) and \(n \geq 4\);
- \(p \geq 5\) and \(n \geq 3\).*

To find a lower bound for \(\kappa((n + 3)L - \Delta_A)\), which is the rate of growth with respect to \(m\) of the dimension of the space of weight \(m(n + 3)\)-cusp forms of \(\Gamma_p\), we use the “low weight cusp form trick”, which has been used in this context in [GH] and [GS], and more widely thereafter.
Suppose $n > N$ and there exists a non-zero weight $3 + N$ cusp form $F$ of $\Gamma_p$, that is, $F \in H^0((3+N)L-\Delta_A)$. For any non-zero $F' \in H^0(m(n-N)L)$, 

$$F^m F' \in H^0(m(n+3)L-\Delta_A).$$

Fixing $F$, the space of cusp forms in the form of $F^m F'$ then grows at the same rate as $H^0(m(n-N)L)$ with respect to $m$, which is known to be $O(m^3)$. So $\kappa((n+3)L-\Delta_A) \geq 3$.

Therefore, $X^n_p$ is of relative general type if $h := \dim H^0((3+N)L-\Delta_A) > 0$.

To find a lower bound for $h$, we apply Gritsenko’s lifting of Jacobi cusp forms mentioned in [G, Theorem 3], which states the existence of an injective lifting $J_cusp(k, p) \hookrightarrow S_k(\Gamma[p])$ where $J_cusp(k, p)$ is the space of Jacobi cusp forms of weight $k$ and index $p \geq 1$, and $S_k(\Gamma[p])$ is the space of weight $k$ cusps forms of $\Gamma[p]$, with the paramodular group $\Gamma[p]$ defining the moduli space of $(1, p)$-polarised abelian surfaces without level structure as $\Gamma[p]\backslash \mathbb{H}_2$. But since $\Gamma_p \leq \Gamma[p]$, the image of the lifting is also contained in $S_k(\Gamma_p)$.

From [EZ], $\dim J_cusp(k, p) \geq j(k, p)$ (equality holds when $k \geq p$), where

$$j(k, p) := \begin{cases} \sum_{j=0}^{t} \left( \dim M_{k+2j} - \left( \left\lfloor \frac{k^2}{4p} \right\rfloor + 1 \right) \right), & \text{if } k \text{ is even} \\ \sum_{j=1}^{t} \left( \dim M_{k+2j-1} - \left( \left\lfloor \frac{k^2}{4p} \right\rfloor + 1 \right) \right), & \text{if } k \text{ is odd} \end{cases}$$

with $M_r$ being the space of modular forms of weight $r$ for $\text{SL}(2, \mathbb{Z})$.

It is a general fact that

$$\dim M_r = \begin{cases} \left\lfloor \frac{r^2}{12} \right\rfloor, & \text{if } r \equiv 2 \mod 12 \\ \left\lfloor \frac{r^2}{12} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

By a simple computation, it can be found that the first prime $p$ such that $j(k, p) > 0$ for $k = 5$ and 6 are $p = 5$ and 3 respectively. Note:

1. $\dim(S_k(\Gamma_p)) \geq j(k, p)$ for any $k, p$;
2. $j(k, p)$ increases with $p$;
3. the isomorphism in [M, Theorem 1.1]

$$\bigoplus_{m \geq 0} H^0(X^n_p, K^{\otimes m}_p) = \bigoplus_{m \geq 0} M_{(n+3)m}(\Gamma_p)$$

establishes that $\kappa(X^n_p)$ is non-decreasing with respect to $n$, because the same is true for $\dim(M_{(n+3)m}(\Gamma_p))$.

By letting $k = 3 + N = 2 + n$, this shows that for the values of $n$ and $p$ stated in Theorem 5, $\dim(S_k(\Gamma_p)) \geq j(k, p) \geq 1$. This concludes our proof for Theorem 5.
4 Possible improvements

By following [HS] and applying the Riemann-Roch theorem on the exceptional divisor $E$ of a blow-up at a non-canonical singularity in $X^1_{p^1}$, we may be able to improve our boundary at $n = 1$ by finding two consecutive primes $p'$ and $p''$ with $p' < p''$ such that $\kappa(X^1_{p'}) < \kappa(X^1_{p''})$. However, that would involve understanding the intersection behaviour of divisors on the 4-fold $E$, which is expected to be complicated. The low density of prime numbers near 37 makes the quest less promising: the estimate for $p'$ we find by this method may not be smaller than 31.

There are a few more questions that can be asked: for example, whether the boundary we have drawn can be improved for $p = 5$ and $p = 3$. The image of Gritsenko’s lift is not the entire $S_k(\Gamma_p)$ or even $S_k(\Gamma[p])$, so we might be able to find a weight 4 cusp form with respect to $\Gamma_p$ or $\Gamma[p]$ through other means which improves the bound at $p = 5$, and likewise for $p = 3$. Another question is to calculate $\kappa(X^n_{p^p})$ for other $X^n_{p^p}$ not of relative general type by considering the slope of Siegel cusp forms of $\Gamma_p$, which is the ratio between weight and vanishing order at $\infty$, and to draw a boundary on the $(p,n)$-plane separating the regions with $\kappa(X^n_{p^p}) = -\infty$ and $\kappa(X^n_{p^p}) \geq 0$. We can also extend the problem by considering $p = 2$, non-prime $p$, or abelian surfaces without level structure.

References

[EZ] M. Eichler, D. Zagier. The Theory of Jacobi Forms Birkhäuser, 1985.

[FV] G. Farkas, A. Verra. The universal abelian variety over $\mathcal{A}_5$. Ann. Sci. Éc. Norm. Supér. 49, 521-542, 2016.

[G] V. Gritsenko. Irrationality of the moduli spaces of polarized abelian surfaces Abelian Varieties: Proceedings of the International Conference held in Egloffstein, Germany, October 3-8, 1993, De Gruyter, 63–82, 2011.

[GH] V. Gritsenko, K. Hulek. Appendix to the paper “Irrationality of the moduli spaces of polarized abelian surfaces” Abelian Varieties: Proceedings of the International Conference held in Egloffstein, Germany, October 3-8, 1993, De Gruyter, 83–84, 2011.

[GS] V. A. Gritsenko, G. K. Sankaran. Moduli of Abelian surfaces with a $(1,p^2)$ polarisation Izv. RAN. Ser. Mat. 60(5), 19–26, 1996.

[HKW1] K. Hulek, C. Kahn, S. Weintraub. Moduli Spaces of Abelian Surfaces: Compactification, Degenerations and Theta Functions De Gruyter, 1993.
[HKW2] K. Hulek, C. Kahn, S. Weintraub. Singularities of the moduli spaces of certain abelian surfaces *Compositio Mathematica*. **79**, 231–253, 1991.

[HS] K. Hulek, G.K. Sankaran. The Kodaira dimension of certain moduli spaces of abelian surfaces *Compositio Mathematica*. **90**, 1–35, 1994.

[M] S. Ma. Universal abelian variety and Siegel modular forms *Algebra Number Theory*. **15**(8), 2089–2122, 2021.

[N] Y. Namikawa. Toroidal degeneration of abelian varieties II, *Math. Ann*. **245**, 117–150, 1979.

[PSMS] F. Poon, R. Salvati Manni, G.K. Sankaran. Slopes of Siegel cusp forms and geometry of compactified Kuga varieties, Preprint arXiv: 2109.06142v1, 2021.

[R] M. Reid. Young Persons Guide to Canonical Singularities *Proceedings of Symposia in Pure Mathematics*, **188**, 321–340, 1997.

[T] Y.-S. Tai. On the Kodaira dimension of the moduli space of abelian varieties. *Invent. Math.*, **68**, 425–439, 1982.

[V] A. Verra. On the universal principally polarized abelian variety of dimension 4. *Contemp. Math*. **465**, 253–274, AMS 2008.