An analytic model for interacting dark energy and its observational constraints

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ABSTRACT

The paper deals with a theoretical model for interacting dark energy (DE). The interaction between the cold dark matter (dust) and the DE has been assumed to be non-gravitational in nature. Exact analytic cosmological solutions are obtained both for constant and variable EoS for DE. It is found that, for very small value of the coupling parameter (in the interaction term), the model asymptotically extends up to \( \Lambda \) cold dark matter, while the model can enter into the phantom domain asymptotically, if the coupling parameter is not so small. Both the solutions are then analysed with 194 Supernovae Type Ia data. The best-fitting parameters are shown with 1\( \sigma \) and 2\( \sigma \) confidence intervals. Finally, we have discussed the cosmographic parameters for both the cases.

Key words: cosmological parameters – dark energy – dark matter.

1 INTRODUCTION

Various observational data have confirmed that our Universe is accelerating at present (Riess et al. 1998; Perlmutter et al. 1999; Spergel et al. 2003; Tegmark et al. 2004; Eisenstein et al. 2005; Jain & Taylor 2003). To explain this acceleration within the framework of Einstein’s general relativity, dark energy (DE) having negative pressure was introduced. The cosmological constant \( \Lambda \) was proposed as the simplest candidate for DE, which was combined with cold dark matter (CDM) in the form of \( \Lambda \)CDM model. The model was good with most of the experimental data, however, it suffered from the cosmological constant, or, the fine tuning problem (Weinberg 1989; Carroll 2001), and, the cosmic coincidence problem (Copeland, Sami & Tsujikawa 2006).

Latest observations predict that 73 per cent of our Universe is filled with DE while 23 per cent with dark matter (DM) and remaining 4 percent being the usual baryonic matter and radiation. Thus, most of the recent research is aimed towards finding suitable DE candidates that are free from the above mentioned anomalies. Several DE models have already been proposed, such as, Quintessence (Caldwell, Dave & Steinhardt 1998), K-essence (Armendariz-Picon, Mukhanov & Steinhardt 2001), Tachyon (Padmanabhan 2002; Sen 2005), Phantom (Caldwell 2002; Caldwell, Kamionkowski & Weinberg 2003; Sami & Toporensky 2004), Quintom (Elizalde, Nojiri & Odintsov 2004; Feng, Wang & Zhang 2005; Cai et al. 2010), Chaplygin gas (Kamenshchik, Moschella & Pasquier 2001; M. Bento, Bertolami & Sen 2002), Holographic DE (Cohen, Kaplan & Nelson 1999; Li 2004), and, so on. However, the nature of the DE is still elusive. Attempts were made to explain DE through several models which were based on interactions between DE and DM (Wetterich 1995; Amendola 2000a; Amendola 2000b; Billyard & Coley 2000; Zimdahl, Pavon & Chimento 2001; Amendola & Quercellini 2003; Hoffman 2003; Chimento et al. 2003; Amendola 2004; Herrera, Pavon & Zimdahl 2004; Paliathanasis & Tsamparlis 2014). These also had the attribute that, it could alleviate the cosmic coincidence problem of the \( \Lambda \)CDM model. Although, there is no strong reason to exclude interaction dynamics, yet there is debate as to the exact form of the interaction between these two components. Sometimes, the choice of the interaction term is driven purely from mathematical aspect, while at other times, it is done to suit the latest data.

In the current work, we have considered DE interacting with CDM by some phenomenological interaction term between them. Two cases, one corresponding to DE having variable equation of state (EoS) parameter (\( \omega_d \)) and the other having constant EoS parameter are considered. It is interesting to note that, the case for variable EoS for DE with very small coupling parameter in the interaction can lead to the \( \Lambda \)CDM model for late time, while for large values of the coupling parameter (though ‘\(<1\)’ in the interaction term, we can get \( \omega(z) \) to cross the phantom divide line, i.e. \( \omega_d < -1 \). This being the familiar characteristic of the quintom models (Elizalde et al. 2004; Feng et al. 2005; Cai et al. 2010), and, some other models (Cai, Li, Piao & Zhang 2007; Pan & Chakraborty 2014). We can thus say that restriction in the variable EoS contains some noteworthy properties in the cosmic history.

Both our models are analysed with 194 Supernovae Type Ia data by Tonry et al. (2003) and Barris et al. (2004), where a maximum likelihood technique is used to determine the best-fitting values of the parameters with 1\( \sigma \) and 2\( \sigma \) errors. Also we have done a cosmographic analysis of the models and have graphically shown the...
variations of the cosmographic parameters throughout the expansion history of the universe.

The paper is organized in the following way. In Section 2, we have presented the interaction dynamics between the dark sectors: DM and DE. We have found the analytic solutions both for constant and variable EoS for DE. In Section 3, we have analysed our model using 194 Supernovae Type Ia data. Section 4 contains the cosmographic analysis for both the models. Finally, in Section 5 we have presented a brief summary of the work.

2 INTERACTING DARK SECTORS: TRACING THE COSMIC HISTORY

Consider that our Universe is well described by a flat Friedmann–Lemaître–Robertson–Walker (FLRW) line element

$$dx^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$

and, the matter distribution obeys the perfect fluid distribution with the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu},$$

where $u_\mu$ is the four velocity vector of the perfect fluid, $\rho$, $p$ are respectively the energy density and the thermodynamic pressure of the perfect fluid. Thus, the explicit form of the Einstein’s field equations (assuming $c = 1$)

$$G_{\mu\nu} = 8\pi GT_{\mu\nu},$$

are the Friedmann equations

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_d),$$

$$2H + 3H^2 = -8\pi G(p_m + p_d),$$

where $H = \dot{a}/a$, is the Hubble parameter, an overdot represents the differentiation with respect to the cosmic time $t$; $\rho_m$, $\rho_d$ are respectively the energy densities of DM and DE, and, $p_m$, $p_d$ are the corresponding thermodynamic pressures of the two dark components.

Further, we assume that, the interaction should be in the form of a pressure-less dust (i.e. $p_m = 0$), and, the DE satisfies the barotropic EoS, $p_d = \omega_d \rho_d$, where $\omega_d$ is the EoS for DE. Thus, considering the interaction between these two components, we can write the conservation equations both for DM and DE in the following coupled forms:

$$\dot{\rho}_m + 3H\rho_m = \dot{Q},$$

$$\dot{\rho}_d + 3H(1 + \omega_d)\rho_d = -\dot{Q}.$$  

Here, $\dot{Q}$ is the rate of energy density exchange between DM and DE, where

(i) $\dot{Q} > 0 \implies$ Energy goes from DE to DM,
(ii) $\dot{Q} < 0 \implies$ Energy goes from DM to DE.

We shall assume $\dot{Q}$ to be positive for the validity of the second law of thermodynamics. If we see the continuity equations (6) and (7), the interaction between DE and DM must be a function of the energy densities multiplied by a quantity having units of the inverse of time which has the natural choice as the Hubble parameter. Thus, the interaction between DE and DM can be expressed phenomenologically in the forms, such as, (i) $\dot{Q} = Q(H\rho_m)$, (ii) $\dot{Q} = Q(H\rho_d)$, (iii) $\dot{Q} = Q[H(\rho_d + \rho_m)]$, or, more generally, (iv) $\dot{Q} = Q(H\rho_d, H\rho_m)$. In the analytic model, the nature of DE has been studied considering different type of interactions (for details, see (He & Wang 2008)).

We consider for simplicity that, the interaction is in linear combinations of the dark sector densities as (Quartin et al. 2008; Boehmer et al. 2008; Caldera-Cabral, Maartens & Urena-Lopez 2009)

$$Q = 3\lambda_m H\rho_m + 3\lambda_d H\rho_d,$$  

where $\lambda_m$ and $\lambda_d$ are dimensionless constants. As from observational point of view, the interaction should be subdominant today (Chimento 2010, 2012), so, $|\lambda_m|$ and $|\lambda_d|$ are very small (i.e. $|\lambda_m| \ll 1$ and $|\lambda_d| \ll 1$). The factor ‘3’ in the above expression (8) for interaction is motivated purely from mathematical ground. This general form of interaction has been studied recently by several authors (Quartin et al. 2008; Boehmer et al. 2008; Caldera-Cabral et al. 2009), and, the particular cases $\lambda_m = \lambda_d$, in (Chimento et al. 2003), and, $\lambda_d = 0$, in (Billyard & Coley 2000).

Inserting equation (8) in the energy conservation equations (6) and (7) we have

$$\dot{\rho}_m + 3H \left(1 - \lambda_m - \frac{\lambda_d}{u}\right)\rho_m = 0,$$  

$$\dot{\rho}_d + 3H \left(1 + \omega_d + \lambda_d + \lambda_m u\right)\rho_d = 0,$$

where $u = \rho_m/\rho_d$. Equations (9) and (10) show that, we have effectively non-interacting two fluid system, where both the components have the energy densities as before, only pressure changes. If we define, $\rho_i = \rho_m + \rho_d$, as the total energy density of the combined fluid, then its evolution equation can be obtained from the conservation relations (either equations (6) and (7), or, equations (9) and (10)) as

$$\dot{\rho}_i = -3H\rho_m - 3H (1 + \omega_d)\rho_d$$

$$\implies \dot{\rho}_i + 3H (1 + \omega_i)\rho_i = 0,$$  

with the effective EoS ($\omega_i$) of the combined fluid as

$$\omega_i = \frac{\omega_d \rho_d}{\rho_i} = \omega_d \Omega_i,$$

where $\Omega_i = \rho_i/\rho_c$, ($\rho_c = 3H^2/8\pi G$, the critical energy density) is the density parameter of the DE which is related to the density parameter for DM ($\Omega_m = \rho_m/\rho_c$) by the following relation (a different look of the equation (4))

$$\Omega_i \equiv \Omega_d + \Omega_m = 1.$$  

It should be noted that, if $\omega_d$ is chosen to be a constant, $\omega_i$ still be a variable, i.e. the effective one fluid model has always varying EoS. According to present observations, ‘$\omega_d < -1$’ (Komatsu et al. 2011; Planck Collaboration XVI 2014). So, from equation (12), we see that, $\omega_i < -\Omega_d$, which shows that, for, $1/3 < \Omega_d < 1$, the EoS for combined fluid describes a DE universe.

Using equation (11), we can solve for $\rho_d$ and $\rho_m$ in the following way:

$$\rho_d = -\left(\frac{\rho_i + \rho_{i'}}{\omega_d}\right),$$  

$$\rho_m = \left(\frac{\rho_i' + (1 + \omega_d)\rho_i}{\omega_d}\right),$$

where ‘$'$ represents the differentiation with respect to $x = 3\ln a$. 

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Now, eliminating $\rho_3$ from equations (10) and (14), we obtain a second order differential equation for $\rho_1$ as 
\[
\rho_1'' + \left(2 + \omega_3 + \lambda_3 - \lambda_m - \frac{\omega_2}{\omega_3}\right) \rho_1' + \left[(1 + \omega_3)(1 - \lambda_m) + \lambda_3 - \frac{\omega_2}{\omega_3}\right] \rho_1 = 0.
\]  
(16)

We shall now solve the total energy density $\rho_1$ for both constant and variable DE EoS $\omega_3$.

(i) When $\omega_3$ is assumed to be a constant

For constant EoS for DE, the explicit form of $\rho_1$ is given by 
\[
\rho_1 = \rho_0 (1 + z)^{-3\omega_3} + \rho_1 (1 + z)^{3\mu_1},
\]  
where $\mu_0, \mu_1$ are the constants of integration, and, 
\[
\mu_0 = \frac{1}{2} \left[-2 + \omega_3 + \lambda_3 - \lambda_m + \sqrt{(\lambda_m + \omega_3 + \lambda_3)^2 - 4\lambda_m \lambda_3}\right],
\]
\[
\mu_1 = \frac{1}{2} \left[2 + \omega_3 + \lambda_3 - \lambda_m + \sqrt{(\lambda_m + \omega_3 + \lambda_3)^2 - 4\lambda_m \lambda_3}\right].
\]

The solution for $\rho_1$ in equation (17) is contained in the works by Chimento (Chimento 2010, 2012) in the context of interacting DE. As $|\lambda_m| \ll 1$ and $|\lambda_3| \ll 1$, so we neglect the product term $\lambda_m \lambda_3$, within the square root compared to the first term, then $\mu_0$ and $\mu_1$ are simplified to $\mu_0 \approx -(1 - \lambda_m)$, and, $\mu_1 \approx (1 + \omega_3 + \lambda_3)$. Thus, using the above approximations on $\mu_0$ and $\mu_1$ in (17), we have 
\[
\rho_1 = 3H^2 = \rho_0 (1 + z)^{3(\omega_3 - \mu_0)} + \rho_1 (1 + z)^{3(\omega_3 + \omega_2 + \mu_1)}.
\]  
(18)

The above equation (18) shows that, the present interacting DM and DE model is equivalent to a non-interacting two fluid model with constant EoS parameters $\omega_3$ and $\omega_3 + \omega_2$. Also, the integration constants $\mu_0$ and $\mu_1$ can be interpreted as the present energy densities of the two equivalent fluid components. Now, using (14) and (15), the energy densities of the two dark species respectively take the form 
\[
\rho_3 = \rho_1 \left(\frac{\lambda_3 + \omega_3}{\omega_3}\right) (1 + z)^{3(\omega_3 + \omega_2 + \mu_2)},
\]
\[- \rho_0 \left(\frac{\lambda_m}{\omega_3}\right)(1 + z)^{3(\omega_3 - \mu_3)},
\]  
(19)

\[
\rho_m = \rho_0 \left(\frac{\lambda_m + \omega_3}{\omega_3}\right) (1 + z)^{3(\omega_3 - \mu_3)},
\]
\[- \rho_1 \left(\frac{\lambda_3 + \omega_3}{\omega_3}\right) (1 + z)^{3(\omega_3 + \omega_2 + \mu_2)},
\]  
(20)

It should be noted that, the solutions for DE [equation (19)] and DM [equation (20)] were exactly found by Chimento (Chimento 2010, 2012). In connection with the analytic solutions, one may notice the analytic solutions for scalar field models in (Espichan Carrillo, Silva & Lima 2008; Paliathanasis et al. 2015). Now, using the equations (19) and (20), we can find the present (i.e. at $z = 0$) energy densities for DE ($\rho_3$) and DM ($\rho_m$). Further, we introduce the present values of the density parameters for DE ($\Omega_0$) and DM ($\Omega_0$) respectively as 
\[
\Omega_0 = \frac{\rho_0}{3H^2_0} = \Omega_1 \left(\frac{\lambda_3 + \omega_3}{\omega_3}\right) - \Omega_0 \left(\frac{\lambda_m}{\omega_3}\right),
\]  
(21)

\[
\Omega_0 = \frac{\rho_0}{3H^2_0} = \Omega_0 \left(\frac{\lambda_m + \omega_3}{\omega_3}\right) - \Omega_1 \left(\frac{\lambda_3}{\omega_3}\right).
\]  
(22)

where $\Omega_0 = \rho_0/3H^2_0$, $\Omega_1 = \rho_1/3H^2_0$, and, also, we see that, $\Omega_0 + \Omega_0 = \Omega_0 + \Omega_1 = 1$. Specifically, the quantities $\Omega_0$ and $\Omega_1$ can be expressed as 
\[
\Omega_0 = \frac{\lambda_3 + \omega_3}{\lambda_m + \lambda_3 + \omega_3},
\]
\[
\Omega_1 = \frac{\lambda_m + \omega_3}{\lambda_m + \lambda_3 + \omega_3}.
\]  
(23)

Furthermore, from equation (12), we have 
\[
\omega_3 = \frac{(\lambda_3 + \omega_3)(1 + z)^{3(\omega_3 + \omega_2 + \lambda_3)} - \lambda_m}{(1 + z)^{3(\omega_3 + \omega_2 + \lambda_3 + \omega_3)} + \frac{\omega_3}{\mu_0}}.
\]  
(25)

As the energy densities for both the dark components will be positive throughout the evolution, so, from equations (19) and (20), we must have, $0 \leq (\lambda_m, \lambda_3) < |\omega_3|$. From equation (25), we can identify the behaviour of the combined EoS of the dark sector as follows:

I: $\omega_3 + \lambda_m + \lambda_3 > 0$
As $z \to \infty$, $\omega_3 \to (\omega_3 + \lambda_m)$,
(26)

II: $\lambda_3 + \lambda_m + \lambda_3 < 0$
As $z \to 0$, $\omega_3 \to \omega_3 - \rho_0 \mu_0/\rho_1$, 
(27)

As $z \to -1$, $\omega_3$ is undefined.
(28)

As $z \to \infty$, $\omega_3$ is undefined,
(29)

As $z \to 0$, $\omega_3 \to \omega_3 - \rho_0 \mu_0/\rho_1$, 
(30)

As $z \to -1$, $\omega_3 \to (\omega_3 + \lambda_m)$.
(31)

The deceleration parameter $q$ is given by (Pan & Chakraborty 2013) 
\[
q = \frac{1}{2} + \frac{3}{2} \Omega_0 \omega_3 = \frac{1}{2} \left(1 + 3\omega_3\right)
\]
\[= \frac{1}{2} \left[rac{\left(1 - 3\lambda_m\right)\omega_3^2}{\mu_0} + N(1 + z)^{3P}\right],
\]  
(32)

where $N = (1 + 3\lambda_3 + 3\omega_3), P = (\omega_3 + \lambda_m + \lambda_3)$, and, consequently, in both cases I and II, the deceleration parameter $q_{I}$ (for case I) and $q_{II}$ (for case II) in the limit can be viewed as follows:

As $z \to \infty : q_I \to \frac{1}{2} (1 + 3\omega_3 + 3\lambda_3), & q_{II}$ is undefined
(33)

As $z \to 0 : q_{I} = q_{II} = \frac{1}{2} \left[1 + 3\left(\frac{\lambda_3 + \omega_3 - \rho_0 \mu_0}{1 + \frac{\omega_3}{\mu_0}}\right)\right].$
(34)

As $z \to -1 : q_{II}$ is undefined, but, $q_{II} = \frac{1}{2} (1 + 3\omega_3 + 3\lambda_3)$.
(35)

(ii) When $\omega_3$ is a variable

The differential equation (16) $\rho_1$ cannot be solved for arbitrary variation of the EoS $\omega_3$ for DE. We consider the case when $\lambda_m = 0$. 

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The figure shows the behaviour of the variable $\omega_d$ throughout the entire evolution of the universe.

Thus the equation (16) becomes

$$
\rho_t' + \left( 2 + \omega_d + \lambda_d - \frac{\omega_d}{\omega_{d0}} \right) \rho_t' + \left( 1 + \omega_d + \lambda_d - \frac{\omega_d}{\omega_{d0}} \right) \rho_t = 0,
$$

(33)

Further, we assume that the variation of $\omega_d$ to be restricted by the relation

$$
\omega_d' = n + \omega_d + \lambda_d,
$$

(34)

where $n$ is any real number. Thus, the solution for $\rho_t$ becomes

$$
\rho_t = \rho_0^0 (1 + z)^3 + \rho_1 (1 + z)^3 (1 - n),
$$

(35)

where $\rho_0^0$ and $\rho_1$ are integration constants. Now, depending on the values of $n$, $\rho_t$ behaves in the following way:

* $n > 1$:
  - $\rho_t \to \infty$, as, $z \to \infty$,
  - $\rho_t \to \rho_0^0 + \rho_1$, as, $z \to 0$,
  - $\rho_t \to \infty$, as, $z \to -1$.

* $n < 1$:
  - $\rho_t \to \infty$, as, $z \to \infty$,
  - $\rho_t \to \rho_0^0 + \rho_1$, as, $z \to 0$,
  - $\rho_t \to \rho_1$, as, $z \to -1$.

* $n = 1$:
  - $\rho_t \to \infty$, as, $z \to \infty$,
  - $\rho_t \to \rho_0^0 + \rho_1$, as, $z \to 0$,
  - $\rho_t \to \rho_1$, as, $z \to -1$.

As before, from the equation (35), we see that, the present interacting DE (with variable EoS) and DM is equivalent to a non-interacting two fluid system with constant equations of state ‘0’ (i.e., dust) and ‘−n’, respectively. The integration constants $\rho_0^0$ and $\rho_1$ are nothing but the energy densities (at the present epoch) of the equivalent two fluid components. Also, under the condition (34), the solution for $\omega_d$ looks

$$
\omega_d = \frac{n + \lambda_d}{\omega_{d0} (1 + z)^{3(n+\lambda_d)}} - 1,
$$

(36)

where $\omega_{d0}$ is the constant of integration, and, it has been taken to be negative to remove the singularity in $\omega_d$. Further, the graphical representations of $\omega_d$ for different values of the interaction parameter ($\lambda_d$) have been presented in Fig. 1, whereas Fig. 2 shows that in the high-redshift era, $\omega_d$ was negative, but very close to zero, thus, not dominating in nature, but it tracks $\rho_m$ after a certain redshift to start a DE era. Further, at present, it is very close to the $\Lambda$CDM ($\omega_d = -1$) for very small interaction parameter between the dark sectors, and, also it matches with the very latest Planck data (Panck Collaboration XIII, 2015).

Moreover, we can give the explicit solutions for $\rho_m$ and $\rho_d$ as follows:

$$
\rho_m = \rho_0^0 (1 + z)^3 + \left( \frac{\rho_0^1 \lambda_d}{n + \lambda_d} \right) (1 + z)^3 (1 - n),
$$

$$
+ \left( \frac{n \rho_0^1}{n + \lambda_d} \right) (1 + z)^3 (1 + \lambda_d),
$$

(37)

$$
\rho_d = \left( \frac{n \rho_0^1}{n + \lambda_d} \right) (1 + z)^3 (1 - n) - \omega_{d0} (1 + z)^3 (1 + \lambda_d). \quad (38)
$$

Similarly, for variable DE EoS, the present day density parameters for DM ($\Omega_{d0}$) and DE ($\Omega_{d0}$) can respectively be expressed as

$$
\Omega_{d0} = \Omega^0 + \left( \frac{\lambda_d + n \omega_{d0}}{n + \lambda_d} \right) \Omega^1,
$$

(39)

$$
\Omega_{d0} = (1 - \omega_{d0}) \left( \frac{n \Omega^1}{n + \lambda_d} \right),
$$

(40)

where $\Omega^0 = \rho_0^0 / 3 H_0^2$, $\Omega^1 = \rho_1 / 3 H_0^2$. Also, the explicit forms of $\Omega^0$ and $\Omega^1$ are

$$
\Omega^0 = \left( \frac{n + \lambda_d}{n(1 - \omega_{d0})} \right) \Omega_{d0},
$$

(41)

$$
\Omega^1 = \Omega_{d0} - \left( \frac{n \omega_{d0} + \lambda_d}{n(1 - \omega_{d0})} \right) \Omega_{d0},
$$

(42)

which immediately shows that, $\Omega_{d0} + \Omega_{d0} = \Omega^0 + \Omega^1 = 1$.

Now, we can give a comparative behaviour of the energy densities for both DE and DM in different eras throughout the entire evolution of the universe as follows:

* $n > 1$:
  - as $z \to \infty$: $\rho_d \to \infty$, $\rho_m$ is undefined,
  - as $z \to 0$: $\rho_d \to \frac{n \rho_0^1}{n + \lambda_d} (1 - \omega_{d0})$, $\rho_m \to \rho_1 + \frac{n \rho_0^1}{n + \lambda_d} + \frac{n \rho_0^1}{n + \lambda_d}$,
  - as $z \to -1$: $\rho_d \to \infty$, $\rho_m \to \infty$.

* $n < 1$:
Table 1. The table shows different cosmic phases depending on the parameters.

| Types  | Quintessence  | ΛCDM       | Phantom |
|--------|--------------|------------|---------|
| A      | \(1 - 3(n + \lambda_d) < \omega_0 \) < \(1 - (n + \lambda_d)\) | \(\omega_0 = 1 - (n + \lambda_d)\) | \(\omega_0 > 1 - (n + \lambda_d)\) |
| B      | \(1 - 3(n + \lambda_d) < \omega_0 \) < \(1 - (n + \lambda_d)\) | \(\omega_0 = 1 - (n + \lambda_d)\) | \(\omega_0 > 1 - (n + \lambda_d)\) |
| C      | \(1/e < a_0 < \exp(-1/3)\) | \(a_0 = \exp(-1/3)\) | \(a_0 \rightarrow 1\) |

as \(z \rightarrow \infty\): \(\rho_d \rightarrow \infty\), \(\rho_m\) is undefined,
as \(z \rightarrow 0\): \(\rho_d \rightarrow \frac{n + \lambda_d}{\omega_0 - 1}\), \(\rho_m \rightarrow \rho^\dagger + \frac{n + \lambda_d}{\omega_0 - 1}\),
as \(z \rightarrow -1\): \(\rho_d \rightarrow 0\), \(\rho_m \rightarrow 0\).

\(\ast n = 1:\)
as \(z \rightarrow \infty\): \(\rho_d \rightarrow \infty\), \(\rho_m\) is undefined,
as \(z \rightarrow 0\): \(\rho_d \rightarrow \frac{n + \lambda_d}{\omega_0 - 1}\), \(\rho_m \rightarrow \rho^\dagger + \frac{n + \lambda_d}{\omega_0 - 1}\),
As \(z \rightarrow -1\): \(\rho_d \rightarrow 0\), \(\rho_m \rightarrow \frac{n + \lambda_d}{\omega_0 - 1}\).

Further, the deceleration parameter can be given as (Pan & Chakraborty 2013)
\[q = \frac{1}{3} + \frac{3}{2} \Omega_d a_0,\]
where \(\Omega_d\) can be found from equation (38). Now, looking at equation (36) we see

\[(A): \text{for } n + \lambda_d > 0\]

\[\text{for } z \rightarrow \infty, \ \omega_d \rightarrow 0,\]

\[\text{for } z \rightarrow 0, \ \omega_d \rightarrow \frac{n + \lambda_d}{\omega_0 - 1},\]

\[\text{for } z \rightarrow -1, \ \omega_d \rightarrow -(n + \lambda_d).\]

\[(B): \text{for } n + \lambda_d < 0\]

\[\omega_d \rightarrow -(n + \lambda_d), \ \text{as } z \rightarrow \infty,\]

\[\omega_d \rightarrow \frac{n + \lambda_d}{\omega_0 - 1}, \ \text{as } z \rightarrow 0,\]

\[\omega_d \rightarrow 0, \ \text{as } z \rightarrow -1.\]

\[(C): \text{for } n + \lambda_d = 0\]

\[\omega_d = -\frac{1}{3 \ln(a/a_0)}, \ a_0 = \text{integration constant}\]

Also, \(\omega_d \rightarrow \frac{1}{3 \ln(a_0)}\), as \(z \rightarrow 0.\)

The cases A and B and C result Table 1, describing the different phases of the universe restricted by the model parameters.

### 3 Model Comparison With Observational Data

Here, we compare the models for both constant and variable \(\omega_d\) up to the redshift \(z = 1.75\) using the available 194 Supernovae Ia data (Tonry et al. 2003; Barris et al. 2004). The data is a compilation of the redshift \(z\) and the corresponding logarithm of the Hubble free luminosity distance \(\log(c D_L(z))\) with its 1σ error \(\sigma_{\log(D_L(z))}\). The Hubble constant free luminosity distance \(D_L(z)\) is related to the luminosity distance \(d_L(z)\) by the relation

\[D_L(z) = \frac{H_0}{c} d_L(z).\]

In terms of the comoving distance \(r(z)\) and the redshift \(z\),

\[D_L(z) = \frac{H_0}{c} r(z)(1 + z).\]

Again \(D_L(z)\) can be related to the theoretical model obtained using the relation

\[D_L^0(z) = \frac{(1 + z)}{H_0} \int_0^z \frac{dz'}{E(z')},\]

where \(E(z) = H(z)/H_0\). In order to determine the model parameters using the observational constraints, we use a maximum likelihood technique on the theoretical parameters whereby we minimize the function \(X^2\) given by

\[X^2 = \sum_{n=1}^{N} \left( \frac{\log_{10}(D_{obs}(z_n)) - \log_{10}(D_L^0(z_n))}{\sigma_{\log_{10}(D_{obs}(z_n))}} \right)^2 + \left( \frac{\log_{10}(D_{th}(z_n)) - \log_{10}(D_L(z_n))}{\sigma_{\log_{10}(D_{th}(z_n))}} \right)^2,\]

where \(N = 194\), and, \(\sigma_{z_n}\) is the 1σ error of the data corresponding to the redshift \(z_n\). A table of the data and the numerical program we used in this study can be downloaded in electronic form (Perivolaropoulos & Nesseris 2004).

#### 3.1 Observational constraints for constant \(\omega_d\)

From the equation (18), we can write

\[E^2 = \frac{1}{\mathbf{\Omega}_0} (1 + z)^{3(1 - \omega_m)} + \mathbf{\Omega}_1 (1 + z)^{3(1 + \omega_d + \lambda_d)}.\]

The parameters \(\mathbf{\Omega}_0\) and \(\mathbf{\Omega}_1\) are related with \(\omega_m\) and \(\omega_d\) given in (23) and (24), respectively. Further, \(\mathbf{\Omega}_0\) and \(\mathbf{\Omega}_1\) can be interpreted as the density parameters of the equivalent two fluids with corresponding values equivalent to that of \(\omega_m\) and \(\omega_d\), respectively.

Fig. 3 is a representation of the observed luminosity distance for the 194 Supernovae Ia data and the corresponding model predicted theoretical value for \(d_L(z)\). For evaluating theoretical \(d_L(z)\), we use \(\omega\)CDM model parameters for \(\mathbf{\Omega}_m = 0.34\) and \(\omega_d = -1.01\), i.e. we consider a small deviation of the \(\Lambda\)CDM model, and, thus, we get
Analytic model with observations

3.2 Observational constraints for variable $\omega_d$

For variable $\omega_d$ restricted by equation (34), we have

$$E^2 = \Omega^0(1 + z)^3 + \Omega^1(1 + z)^{3(1 - n)},$$

where $\Omega^0$ and $\Omega^1$ can be found in (41) and (42), respectively. In this case also, the parameters $\Omega^0$ and $\Omega^1$ are nothing but the density parameters for the hypothetical non-interacting two fluids with values similar to $\Omega_m$ and $\Omega_{de}$, respectively. The new parameter $n$ (any real number) arises due to the choice of $\omega_d$ that makes it variable, and, as a result, we can realize the different cosmic stages with the restrictions on the model parameters shown in Table 1. Fig. 5 shows the 1 and 2$\sigma$ contours in $\Omega_{m0}$, $\Omega_{de0}$ plane for three different values of $n$.

4 COSMOGRAPHY OF INTERACTING DE

The idea of cosmography in cosmology was motivated after the introduction of the statefinder parameters by Sahni et al. (2003). The interesting fact behind the statefinder parameters are that, they are dimensionless, geometrical, and, model independent in nature. As a result, they were widely used to filter the observationally sound DE models among the various theoretical DE models in the literature. The statefinder parameters $\{r, s\}$ are defined as

$$r = \frac{1}{aH^2} \frac{d^2a}{dt^2}, \quad \text{and,} \quad s = \frac{r - \frac{1}{3} (q - \frac{1}{2})}{\Omega_\gamma};$$  \hspace{1cm} (57)

Subsequently, this geometric investigation was extended by considering the Taylor series expansion of the scale factor about the present time in the following manner:

$$a(t) = a(t_0) + \frac{1}{H(t_0)^2} (t - t_0)^2 + \frac{1}{3!} \dot{H}(t_0) (t - t_0)^3 + \frac{1}{4!} H^2(t_0) (t - t_0)^4 + O((t - t_0)^5),$$

where we have some model independent and dimensionless parameters $j, s, l, m$ known as the cosmographic parameters (Visser 2004; Visser 2005) defined in the following way:

$$j = \frac{1}{aH^2} \frac{d^2a}{dt^2} \quad \text{and,} \quad m = \frac{1}{aH^6} \frac{d^6a}{dt^6};$$

The suffix ‘0’ stands for the value of the corresponding variable at the present epoch ($t_0$). The cosmographic parameters (from now we shall call these CP) are individually named as jerk ($j$) (this ‘$j$’ is same as ‘$r$’ defined by Sahni et al. (2003), snap ($s$) (this ‘$s$’ is different from the one defined by Sahni et al. (Sahni et al. 2003), lerk, and $m$ parameter (Visser 2004, 2005). Further, the above CP can be expressed in terms of the deceleration parameter ($q$), and, its higher derivatives

$$j = (1 + z) \frac{dq}{dz} + q(1 + 2q),$$  \hspace{1cm} (60)

$$s = -(1 + z) \frac{dj}{dz} + j - 3(1 + q)j,$$  \hspace{1cm} (61)

$$l = -(1 + z) \frac{ds}{dz} + s - 4(1 + q)s,$$  \hspace{1cm} (62)

$\Omega_{de0} = 0.66$. We choose the interaction parameters, $\lambda_m = 0.001$ and $\lambda_d = 0.002$ throughout all estimations as per our assumption of very low interaction. From the figure it is clear that for these values of the parameter, our model gives a good fit to the data.

Fig. 4 shows the 1 and 2$\sigma$ contours in $\Omega_{m0}$, $\Omega_{de0}$ plane, $\Omega_{m0}$, $\omega_m$ and $\Omega_{de0}$, $\omega_d$ plane, respectively. In all these results, we have found that, the best-fitting values of the free parameters $\Omega_{m0}/\Omega_{de0}$ and $\omega_d$ are consistent with the data. Thus, from the observational constraints, we can conclude that, the long term expansion history of the universe is overall in harmony with the existing models like $\Lambda$CDM.

Figure 4. 68 percent (1$\sigma$) and 95 percent (2$\sigma$) confidence level contours in ($\Omega_{m0}$, $\Omega_{de0}$), ($\Omega_{m0}$, $\omega_m$) and ($\Omega_{de0}$, $\omega_d$) plane have been shown with the best-fitting parameter indicated by the black dot in each plot.

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$$l = -(1 + z) \frac{ds}{dz} + s - 4(1 + q)s,$$
Figure 5. The 1σ and 2σ confidence levels in the \((\Omega_{\text{m}0}, \Omega_{\text{d}0})\) plane have been shown for \(n = 0.8, n = 0.9,\) and \(n = 1.1,\) respectively in the clockwise direction. The best-fitting values of \((\Omega_{\text{m}0}, \Omega_{\text{d}0})\) for different \(n\) have also been indicated by the black dot in each figure.

Thus, the evolution of the CP can be traced out only if \(q\) and its higher derivatives are differentiable throughout the entire cosmic history. In Figs 6 and 7, we have shown the variations of the CP with the

\[ m = -(1 + z) \frac{dl}{dz} + l - 5(1 + q)l. \]  

(63)

Figure 6. The figures show the variation of the four cosmographic parameters against the redshift \((z)\) for constant EoS \((\omega_d)\) for DE.
For variable EoS of DE ($\omega_d$) in equation (36), we have presented the four cosmographic parameters against the redshift ($z$). From the figures, one can notice that during the entire evolution of the universe, the nature of $j$ and $l$ are almost same in sign, similar behaviour is found for the parameters $s$ and $m$.

5 SUMMARY OF THE WORK

This work proposes a theoretical model of DE to match with recent observational pieces of evidence. Here, the DE is chosen in the form of perfect fluid with barotropic EoS, and, it interacts non-gravitationally with DM chosen as dust. Analytic solutions are obtained both for constant and variable EoS for DE with phenomenological choice for the interaction term. The asymptotic behaviour of the relevant physical parameters are discussed and their variations have been shown graphically. These theoretical models are then compared with 194 available Supernovae data by Tonry et al. (2003) and Barris et al. (2004). Using $\chi^2$ test for goodness of fit, the best-fitting values for the parameters are estimated, and, they are well accord with observed $1\sigma$ (or, $2\sigma$) level. For variable $\omega_d$, the effect of DE is negligible at early universe, and, it starts dominating in recent past. Also, the estimated value of $\omega_d$ nicely matches with the very recently released Planck data set (Planck Collaboration XIII 2015). Therefore, this particular interacting DE model can be considered as an alternative for $\Lambda$CDM model. Also, the variation of the cosmographic parameters are presented graphically for both the theoretical models for different choices of the parameters involved. Finally, as the present model includes only CDM and DE as the two main ingredients, one can include the baryonic matter and radiation to the picture (Erdem 2014) in future such that the era of the creation of baryonic matter and radiation is built into the model.

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Figure 7. For variable EoS of DE ($\omega_d$) in equation (36), we have presented the four cosmographic parameters against the redshift ($z$).
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