Symmetry Analysis of Telegraph Equation

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Abstract
Lie symmetry group method is applied to study the Telegraph equation. The symmetry group and its optimal system are given, and group invariant solutions associated to the symmetries are obtained. Finally the structure of the Lie algebra symmetries is determined.

Key words: Lie group analysis, Symmetry group, Optimal system, Invariant solution.

1 Introduction

The telegrapher’s equations (or just telegraph equations) are a pair of linear differential equations which describe the voltage and current on an electrical transmission line with distance and time. The equations come from Oliver Heaviside who developed the transmission line model. Oliver Heaviside (May 18, 1850 – February 3, 1925) was a self-taught English electrical engineer, mathematician, and physicist who adapted complex numbers to the study of electrical circuits, invented mathematical techniques to the solution of differential equations (later found to be equivalent to Laplace transforms), reformulated Maxwell’s field equations in terms of electric and magnetic forces and energy flux, and independently co-formulated vector analysis. Although at odds with the scientific establishment for most of his life, Heaviside changed the face of mathematics and science for years to come the theory applies to high-frequency transmission lines (such as telegraph wires and radio frequency conductors) but is also important for designing high-voltage energy transmission lines. The model demonstrates that the electromagnetic waves can be reflected on the wire, and that wave patterns can appear along the line. The telegrapher’s equations can be understood as a simplified case of Maxwell’s equations. In a more practical approach, one assumes that the conductors are composed of an infinite series of two-port elementary components, each representing an infinitesimally short segment of the transmission line.

2 Lie Symmetries of the Equation

A PDE with $p$—independent and $q$—dependent variables has a Lie point transformations

$$
\tilde{x}_i = x_i + \varepsilon \xi_i(x, u) + \mathcal{O}(\varepsilon^2), \quad \tilde{u}_\alpha = u_\alpha + \varepsilon \varphi_\alpha(x, u) + \mathcal{O}(\varepsilon^2)
$$

where $\xi_i = \left. \frac{\partial \tilde{x}_i}{\partial \varepsilon} \right|_{\varepsilon=0}$ for $i = 1, ..., p$ and $\varphi_\alpha = \left. \frac{\partial \tilde{u}_\alpha}{\partial \varepsilon} \right|_{\varepsilon=0}$ for $\alpha = 1, ..., q$. The action of the Lie group can be considered by its associated infinitesimal generator

$$
v = \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x, u) \frac{\partial}{\partial u_\alpha}
$$

References:

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on the total space of PDE (the space containing independent and dependent variables). Furthermore, the characteristic of the vector field (1) is given by

\[ Q^\alpha(x, u^{(1)}) = \varphi_\alpha(x, u) - \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial u^\alpha}{\partial x_i}, \]

and its \( n \)-th prolongation is determined by

\[ v^{(n)} = \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial}{\partial x_i} + \sum_{a=1}^{q} \sum_{j=-n}^{n} \varphi^j_a(x, u) \frac{\partial}{\partial u^a_j}, \]

where \( \varphi^j_a = D_jQ^a + \sum_i \xi_i u^a_{t,i}. \) (\( D_j \) is the total derivative operator describes in (3)).

The aim is to analysis the point symmetry structure of the Telegraph equation, which is

\[ u_{tt} + ku_t = a^2 \left[ \frac{1}{r} \left( r \frac{\partial u}{\partial r} \right)_r + \frac{1}{r^2} u_{xx} + u_{yy} \right], \]  

(2)

where \( u \) is a smooth function of \((r, x, y, t)\).

Let us consider a one-parameter Lie group of infinitesimal transformations \((x, t, u)\) given by

\[ \tilde{r} = r + \varepsilon_1(r, x, y, t, u) + O(\varepsilon^2), \quad \tilde{x} = x + \varepsilon_2(r, x, y, t, u) + O(\varepsilon^2), \]

\[ \tilde{y} = y + \varepsilon_3(r, x, y, t, u) + O(\varepsilon^2), \quad \tilde{u} = u + \varepsilon \eta(r, x, y, t, u) + O(\varepsilon^2), \]

where \( \varepsilon \) is the group parameter. Then one requires that this transformations leaves invariant the set of solutions of the Eq. (2). This yields to the linear system of equations for the infinitesimals \( \xi_1(r, x, y, t, u), \ \xi_2(r, x, y, t, u), \)

\( \xi_3(r, x, y, t, u), \ \xi_4(r, x, y, t, u) \) and \( \eta(r, x, y, t, u) \). The Lie algebra of infinitesimal symmetries is the set of vector fields in the form of \( \mathbf{v} = \xi_1 \partial_r + \xi_2 \partial_x + \xi_3 \partial_y + \xi_4 \partial_t + \eta \partial_u. \) This vector field has the second prolongation

\[ \mathbf{v}^{(2)} = \mathbf{v} + \varphi^r \partial_r + \varphi^x \partial_x + \varphi^y \partial_y + \varphi^t \partial_t + \varphi^{rt} \partial_{ru} + \varphi^{rx} \partial_{ru} + \cdots + \varphi^{yy} \partial_{uy} + \varphi^{yt} \partial_{ut} + \varphi^{tt} \partial_{tt} \]

with the coefficients

\[ \varphi^r = D_rQ + \xi_1 u_{rr} + \xi_2 u_{rx} + \xi_3 u_{ry} + \xi_4 u_{rt}, \]

\[ \varphi^x = D_xQ + \xi_1 u_{rx} + \xi_2 u_{xx} + \xi_3 u_{xy} + \xi_4 u_{xt}, \]

\[ \varphi^y = D_yQ + \xi_1 u_{ry} + \xi_2 u_{xy} + \xi_3 u_{yy} + \xi_4 u_{yt}, \]

\[ \varphi^t = D_tQ + \xi_1 u_{rt} + \xi_2 u_{xt} + \xi_3 u_{yt} + \xi_4 u_{tt}, \]

where the operators \( D_r, D_x, D_y \) and \( D_t \) denote the total derivative with respect to \( r, x, y \) and \( t \):

\[ D_r = \partial_r + u_r \partial_u + u_{rr} \partial_{u_r} + \cdots, \]

\[ D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{rx} \partial_{u_r} + \cdots, \]

\[ D_y = \partial_y + u_y \partial_u + u_{yy} \partial_{u_y} + u_{ry} \partial_{u_r} + \cdots, \]

\[ D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{rt} \partial_{u_r} + \cdots, \]  

(3)

Using the invariance condition, i.e., applying the second prolongation \( \mathbf{v}^{(2)} \) to Eq. (2), the following system of 27
Table 1

Commutation relations of \( \mathfrak{g} \)

| . | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( v_5 \) | \( v_6 \) | \( v_7 \) | \( v_8 \) | \( v_9 \) | \( v_{10} \) | \( v_{11} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( v_1 \) | 0 | 0 | 0 | 0 | 0 | \( -v_7 \) | \( v_6 \) | \( -v_9 \) | \( v_8 \) | \( -v_{11} \) | \( v_{10} \) |
| \( v_2 \) | 0 | 0 | 0 | 0 | \( 2a^2v_3 - kv_4 \) | 0 | 0 | \( v_7 \) | \( -v_6 \) | 0 | 0 |
| \( v_3 \) | 0 | 0 | 0 | 0 | \( v_2 \) | 0 | 0 | 0 | 0 | \( v_5 \) | \( v_7 \) |
| \( v_4 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( v_5 \) | 0 | \( -2a^2v_3 + v_4 \) | \( -v_2 \) | 0 | 0 | 0 | 0 | \( v_{11} \) | \( -v_{10} \) | \( -v_9 \) | \( v_8 \) |
| \( v_6 \) | \( v_7 \) | 0 | 0 | 0 | 0 | 0 | 0 | \( v_7 \) | \( -v_6 \) | 0 | 0 |
| \( v_7 \) | \( -v_6 \) | 0 | 0 | 0 | 0 | 0 | 0 | \( -v_2 \) | 0 | 0 | \( 2a^2v_3 - kv_4 \) |
| \( v_8 \) | \( v_9 \) | \( -v_7 \) | 0 | 0 | \( -v_{11} \) | 0 | \( -v_2 \) | 0 | \( -v_1 \) | 0 | \( v_5 \) |
| \( v_9 \) | \( -v_8 \) | \( v_6 \) | 0 | 0 | \( v_{10} \) | \( -v_2 \) | 0 | \( v_1 \) | 0 | \( -v_5 \) | 0 |
| \( v_{10} \) | \( v_{11} \) | 0 | \( -v_6 \) | 0 | \( -v_9 \) | \( -2a^2v_3 + kv_4 \) | 0 | 0 | \( v_7 \) | 0 | \( a^2v_1 \) |
| \( v_{11} \) | \( -v_{10} \) | 0 | \( -v_7 \) | 0 | \( -v_8 \) | 0 | \( -2a^2v_3 + kv_4 \) | \( -v_5 \) | 0 | \( -a^2v_1 \) | 0 |

determining equations yields:

\[
\begin{align*}
\xi_{2u} &= 0, & \xi_{2yy} &= 0, & \xi_{3y} &= 0, & \xi_{3u} &= 0, & \xi_{3u} &= 0, \\
\xi_{4t} &= 0, & \xi_{4y} &= 0, & \xi_{4rr} &= 0, & \xi_{4xy} &= 0, \\
\xi_{4yy} &= 0, & \xi_{4u} &= 0, & \eta_{tu} &= 0, & \eta_{uu} &= 0, \\
k\xi_{4y} + \eta_{ru} &= 0, & \xi_1 + r\xi_{2x} &= 0, & \xi_2 + r\xi_{2x} &= 0, & \xi_2 + r\xi_{2xy} &= 0, & \xi_2 + r\xi_{2xy} &= 0, \\
\xi_{2xx} - r\xi_{2x} &= 0, & k\xi_{4y} + 2\eta_{yu} &= 0, & 2\xi_{2r} + r\xi_{2rr} &= 0, & \xi_{3r} - r\xi_{2xy} &= 0, \\
\xi_{3x} &+ r^2\xi_{2y} &= 0, \quad \xi_{3t} - a^2\xi_{4y} &= 0, & \xi_{4x} - r\xi_{4xx} &= 0, & \xi_{4xx} + r\xi_{4r} &= 0, \\
r^2\xi_{2t} - a^2\xi_{4x} &= 0, & k\xi_{4x} + 2\eta_{ax} &= 0, & a^2r^2\eta_{rr} - kr^2\eta_r + a^2\eta_{rr} + a^2\eta_{yy} - r^2\eta_{tt} + a^2\eta_{xx} &= 0.
\end{align*}
\]

The solution of the above system gives the following coefficients of the vector field \( v \):

\[
\begin{align*}
\xi_1 &= c_6 \sin x - c_7 \cos x - c_8 y \cos x - c_9 y \sin x + 2c_{10}a^2t \sin x - 2c_{11}a^2t \cos x, \\
\xi_2 &= c_1 + c_6r^{-1} \cos x + c_7r^{-1} \sin x + c_8 y r^{-1} \sin x - c_9 y r^{-1} \sin x + 2c_{10}a^2t^{-1} \cos x - 2c_{11}a^2t^{-1} \sin x, \\
\xi_3 &= c_2 + 2c_5a^2t + c_6r \cos x + c_9 r \sin x, \\
\xi_4 &= c_3 + 2c_5at + 2c_{10}r \sin x - 2c_{11}r \cos x, & \eta &= c_4 u - c_5 k y u - c_{10} k r u \sin x + c_{11} k r u \cos x,
\end{align*}
\]

where \( c_1, \ldots, c_{11} \) are arbitrary constants, thus the Lie algebra \( \mathfrak{g} \) of the telegraph equation is spanned by the seven vector fields

\[
\begin{align*}
v_1 &= \partial_x, & v_2 &= \partial_y, & v_3 &= \partial_t, & v_4 &= u \partial_u, \\
v_5 &= 2a^2t \partial_y + 2y \partial_t - k y u \partial_u, & v_6 &= \sin x \partial_r + r^{-1} \cos x \partial_x, \\
v_7 &= - \cos x \partial_r + r^{-1} \sin x \partial_x, & v_8 &= -y \cos x \partial_r + r^{-1} y \sin x \partial_x + r \cos x \partial_y, \\
v_9 &= -y \sin x \partial_r - r^{-1} y \cos x \partial_x + r \sin x \partial_y, & v_{10} &= 2a^2t \sin x \partial_r + 2a^2t^{-1} \cos x \partial_x + 2r \sin x \partial_t - k r u \sin x \partial_u, \\
v_{11} &= -2a^2t \cos x \partial_r + 2a^2t^{-1} \sin x \partial_x - 2r \cos x \partial_t + k r u \cos x \partial_u,
\end{align*}
\]

which \( v_1, v_2 \) and \( v_3 \) are translation on \( x, t \) and \( u \), \( v_4 \) is rotation on \( u \) and \( x \) and \( v_7 \) is scaling on \( x, t \) and \( u \). The commutation relations between these vector fields is given by the (1), where entry in row \( i \) and column \( j \) representing \( [v_i, v_j] \). The one-parameter groups \( G_i \) generated by the base of \( \mathfrak{g} \) are given in the following table.

\[
\begin{align*}
g_1 : (r, x, y, t, u) &\rightarrow (r, x + s, y, t, u), & g_2 : (r, x, y, t, u) &\rightarrow (r, x, y + s, t, u),
\end{align*}
\]

\[3\]
g_3 : (r, x, y, t, u) \mapsto (r, x, y, t + s, u), \quad g_4 : (r, x, y, t, u) \mapsto (r, x, y, t, u e^s),

\begin{align*}
g_5 : (r, x, y, t, u) &\mapsto (r, x, y + st, \frac{1}{a^2} s y + t, -\frac{1}{2a^2} s k y + u), \\
g_6 : (r, x, y, t, u) &\mapsto (s \sin x + r, s \frac{y}{r} \cos x + x, y, t, u), \\
g_7 : (r, x, y, t, u) &\mapsto (-s \cos x + r, s \frac{y}{r} \sin x + y, t, u), \\
g_8 : (r, x, y, t, u) &\mapsto (-s y \cos x + r, -\frac{y}{r} \cos x + x, s r \cos x + y, t, u), \\
g_9 : (r, x, y, t, u) &\mapsto (-s y \sin x + r, -\frac{y}{r} \cos x + x, s r \sin x + y, t, u), \\
g_{10} : (r, x, y, t, u) &\mapsto (s t \sin x + r, s \frac{t}{r} \cos x + x, y, -\frac{s}{a^2} r \sin x + t - \frac{s}{2a^2} k r u \sin x + u), \\
g_{11} : (r, x, y, t, u) &\mapsto (-s t \cos x + r, s \frac{t}{r} \sin x + x, y, -\frac{s}{a^2} r \cos x + t, \frac{s}{2a^2} k r u \cos x + u).
\end{align*}

Since each group \(G_i\) is a symmetry group and if \(u = f(r, x, y, t)\) is a solution of the Telegraph equation, so are the functions

\begin{align*}
u^1 &= U(r, x + \epsilon, y, t), \\
&\quad u^2 = U(r, x, y + \epsilon, t), \\
&\quad u^3 = U(r, x, y, t + \epsilon), \\
&\quad u^4 = e^{-\epsilon} U(r, x, y, t), \\
&\quad u^5 = (2a^2 + \epsilon k y) U(r, x, y + \epsilon t, \frac{1}{a^2} s y + t), \\
&\quad u^6 = U\left(\epsilon \sin x + r, \epsilon \frac{x}{r} \cos x + x, y, t\right), \\
&\quad u^7 = U\left(-\epsilon \cos x + r, \epsilon \frac{x}{r} \sin x + y, t\right), \\
&\quad u^8 = U\left(-\epsilon \sin x + r, \epsilon \frac{x}{r} \cos x + x, \epsilon r \cos x + y, t\right), \\
&\quad u^9 = U\left(-\epsilon y \sin x + r, -\frac{\epsilon}{r} y \cos x + x, \epsilon r \sin x + y, t\right), \\
&\quad u^{10} = (2a^2 + \epsilon k r x) U\left(\epsilon t \sin x + r, -\frac{\epsilon}{r} t \cos x + x, y, -\frac{\epsilon}{a^2} r \sin x + t\right), \\
&\quad u^{11} = (2a^2 - \epsilon k r x) U\left(-\epsilon t \cos x + r, -\frac{\epsilon}{r} t \sin x + x, y, -\frac{\epsilon}{a^2} r \cos x + t\right).
\end{align*}

where \(\epsilon\) is a real number. Here we can find the general group of the symmetries by considering a general linear combination \(c_1 \mathbf{v}_1 + \cdots + c_1 \mathbf{v}_{11}\) of the given vector fields. In particular if \(g\) is the action of the symmetry group near the identity, it can be represented in the form \(g = \exp(\epsilon \mathbf{v}_{11}) \cdots \exp(\epsilon \mathbf{v}_1)\).

### 3 Optimal system of Telegraph equation

As is well known, the theoretical Lie group method plays an important role in finding exact solutions and performing symmetry reductions of differential equations. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the differential equation. So, a mean of determining which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system \([8]\). The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. The idea of using the adjoint representation for classifying group-invariant solutions is due to \([2,4,7,8]\).

The adjoint action is given by the Lie series

\[
\text{Ad}(\exp(\epsilon \mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \epsilon [\mathbf{v}_i, \mathbf{v}_j] + \frac{\epsilon^2}{2} [\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]] - \cdots,
\]

where \([\mathbf{v}_i, \mathbf{v}_j]\) is the commutator for the Lie algebra, \(\epsilon\) is a parameter, and \(i, j = 1, \cdots, 11\). Let \(F_\epsilon^i : \mathfrak{g} \to \mathfrak{g}\) defined by \(\mathbf{v} \mapsto \text{Ad}(\exp(\epsilon \mathbf{v}_i)\mathbf{v})\) is a linear map, for \(i = 1, \cdots, 11\). The matrices \(M_\epsilon^i\) of \(F_\epsilon^i\), \(i = 1, \cdots, 11\), with respect to basis
\[ \{v_1, \ldots, v_{11}\} \text{ are} \]

\[
M_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

7 matrices \[ \cdots M_1 = \begin{pmatrix}
\cosh as & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cosh \sqrt{7}ss & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{7}ss \sinh \sqrt{7}ss & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sinh \sqrt{7}ss & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad \sinh as = 0. 
\]

by acting these matrices on a vector field \( v \) alternatively we can show that a one-dimensional optimal system of \( g \) is given by

\[
X_1 = a_1v_1 + a_2v_3 + a_3v_4 + a_4v_8 \\
X_2 = a_1v_1 + a_2v_3 + a_3v_4 + a_4v_5 + a_6v_9 \\
X_3 = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_6v_8 \\
X_4 = a_1v_1 + a_2v_3 + a_3v_4 + a_4v_5 + a_6v_8 \\
X_5 = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_6v_8 \\
X_6 = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_4(v_6 - v_{10}) \\
X_7 = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + v_7 - v_{11}, \\
X_8 = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6, \\
X_9 = a_1v_1 + a_2v_2 + v_3 + v_4 - (2a^2 - k)v_5 + v_7, \\
X_{10} = a_1v_1 + a_2v_2 + v_3 + a_4v_4 - (2a^2 - k)v_5 + v_6, \\
X_{11} = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 + a_7v_{11}, \\
X_{12} = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 + a_7v_7 + a_8v_{11}, \\
X_{13} = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6 - a_7v_7 - a_8v_9, \\
X_{14} = a_1v_1 + a_2v_2 + a_4v_3 + a_4v_4 + a_5v_5 + a_6v_6 + v_7 - (2a + k)v_{11}, \\
X_{15} = \frac{1}{2a^2 - k}(v_1 + v_8) + a_1v_2 + a_2v_3 + a_3v_4 + a_5v_5 + v_6 + v_7, \\
X_{16} = a_1v_1 + a_2v_2 + v_3 + v_4 - (2a^2 - k - 1)v_5 + a_3v_6 + a_4v_7 + a_5v_8 + a_6v_9.
\]

\section{Lie Algebra Structure}

In this part, we determine the structure of symmetry Lie algebra of the telegraph equation. \( g \) has a \textit{Levi decomposition} in the form of \( g = \mathfrak{r} \ltimes \mathfrak{h} \), where \( \mathfrak{r} = \langle v_2, v_3, v_4, v_5, v_6 \rangle \) is the radical (the large solvable ideal) of \( g \) which is a nilpotent nilradical of \( g \) and \( \mathfrak{h} = \langle v_1, v_7, v_8, v_9, v_{10}, v_{11} \rangle \) is a solvable and non-semisimple subalgebra of \( g \) with centralizer \( \langle v_4 \rangle \) containing in the minimal ideal \( \langle v_1, v_2, 2a^2v_3 - kv_4, v_5, \ldots, v_{11} \rangle \).

Here we can find the quotient algebra generated from \( g \) such as

\[
g_1 = g/\mathfrak{r}, \quad \text{(5)}
\]

with commutators table (2), where \( w_i = v_i + \mathfrak{r} \) for \( i = 1, \ldots, 11 \) are members of \( g_1 \).
The (5) helps us to reduction differential equations. If we want to integration an involutive distribution, the process decomposes into two steps:

- integration of the involutive distribution with symmetry Lie algebra \( g/\tau \), and
- integration on integral manifolds with symmetry algebra \( \tau \).

First, applying this procedure to the radical \( \tau \) we decompose the integration problem into two parts: the integration of the distribution with semisimple algebra \( g/\tau \), then the integration of the restriction of distribution to the integral manifold with the solvable symmetry algebra \( \tau \).

The last step can be performed by quadratures. Moreover, every semisimple Lie algebra \( g/\tau \) is a direct sum of simple ones which are ideal in \( g/\tau \). Thus, the Lie-Bianchi theorem reduces the integration problem to involutive distributions equipped with simple algebras of symmetries.

Both \( g \) and \( g_1 \) are non-solvable, because if \( g^{(1)} = \langle v_i, [v_i, v_j] \rangle = [g, g] \), and \( g_1^{(1)} = \langle w_i, [w_i, w_j] \rangle = [g_1, g_1] \), be the derived subalgebra of \( g \) and \( g_1 \) we have

\[
g^{(1)} = [g, g] = \langle v_1, v_2, 2a^2 v_3 - kv_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} \rangle = [g^{(1)}, g^{(1)}] = g^{(2)},
\]

and

\[
g_1^{(1)} = [g_1, g_1] = \langle w_1, w_2, w_3, w_4, w_5, w_6 \rangle = g_1.
\]

Thus, we have a chain of ideals \( g \supset g^{(1)} = g^{(2)} \neq 0 \), \( g_1 \supset g_1^{(1)} = g_1 \neq 0 \), which sows the non-solvability of \( g \) and \( g_1 \).

5 Conclusion

In this article group classification of telegraph equation and the algebraic structure of the symmetry group is considered. Classification of one-dimensional subalgebra is determined by constructing one-dimensional optimal system. The structure of Lie algebra symmetries is analyzed.

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