Decoding $q$-ary lattices in the Lee metric

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Abstract—$q$-ary lattices can be obtained from $q$-ary codes using the so-called Construction A. We investigate these lattices in the Lee metric and show how their decoding process can be related to the associated codes. For prime $q$ we derive a Lee sphere decoding algorithm for $q$-ary lattices, present a brief discussion on its complexity and some comparisons with the classic sphere decoding.

I. INTRODUCTION

A $q$-ary lattice [15], [19] is an integer lattice in the Euclidean space $\mathbb{R}^n$ which contains $q\mathbb{Z}^n$ as a sublattice. It can be obtained via Construction A [4] from a linear code in the module $\mathbb{Z}_q^n$. Those lattices have deserved special attention in recent years due to their use in cryptographic schemes based on lattices, one of the so-called “post-quantum” methods [15], [16]. One important problem concerning general lattices (thus particularly $q$-ary lattices) is the CVP (Closest Vector Problem) which asks for the closest lattice point to a received point in $\mathbb{R}^n$. A method largely used to solve this problem in the Euclidean metric is the sphere decoding [7], [20], [21], which has exponential expected complexity [11]. Other methods include basis reductions such as LLL [14] and BKZ [18], and trellis algorithm [3].

Codes in the Lee metric, on the other hand, were introduced in [13] and since then have been the object of study of many works from both theoretical (e.g. [1], [6], [8], [12]) and practical (e.g. [5], [17]) points of view. The Lee metric has a close relation with the $l_1$ metric, also called Manhattan or Taxi Cab metric, explored for example in [1], [8], concerning the existence of perfect codes and more recently [6], where the authors show how to construct dense error-correcting codes in the Lee metric from dense lattice packings of $n$-dimensional cross-polytopes.

The contributions in this paper are organized as follows. In Section III we derive connections between a $q$-ary lattice and its associated code decoding processes in the Lee metric through Propositions 1 and 2. This illustrates the fact that the Lee metric seems to have a “natural” geometry when dealing with $q$-ary codes (and lattices) for $q \in \mathbb{N}$, $q > 3$. In Section IV we propose an adaptation of the traditional sphere decoding ideas for $q$-ary lattices ($q$ prime) in the Lee metric and discuss its expected complexity through arguments which are similar to the ones presented in [11]. Using geometric arguments, we also make some comparisons with the classic sphere decoding and perform some low dimensional simulations.

II. PRELIMINARIES

In this section we summarize some concepts and properties related to $q$-ary lattices, Lee metric and establish the notation to be used from now on.

A. $q$-ary lattices

Given $q \in \mathbb{N}$, a $q$-ary linear code $C$ is a $\mathbb{Z}_q$-submodule of $\mathbb{Z}_q^n$. For prime $q$, there is always a generator matrix for $C$ in the systematic form, i.e., $A_{n \times k} \sim \begin{bmatrix} I_{k \times k} & B_{k \times (n-k)}^t \end{bmatrix}$. A lattice $\Lambda$ is a discrete additive subgroup of $\mathbb{R}^n$. Equivalently, $\Lambda \subseteq \mathbb{R}^n$ is a lattice iff there are linearly independent vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, such that any $y \in \Lambda$ can be written as $y = \sum_{i=1}^m \alpha_i v_i$, $\alpha_i \in \mathbb{Z}$. The set $\{v_1, \ldots, v_m\}$ is called a basis for $\Lambda$. A matrix $M$ whose columns are these vectors is said to be a generator matrix for $\Lambda$. Given a metric $d$ in $\mathbb{R}^n$, the Voronoi region of $x \in \Lambda$ is the set $V(x) = \{y \in \mathbb{R}^n : d(y, x) \leq d(y, x')$, for all $x' \in \Lambda\}$. To decode $y \in \mathbb{R}^n$ is to find the closest lattice point to $y$.

The so-called Construction A extended for $q$-ary codes [19], can be described by the surjective map $\phi : \mathbb{Z}_q^n \rightarrow \mathbb{Z}_q^n$, $\phi(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. Given a linear code $C \subseteq \mathbb{Z}_q^n$, $\Lambda_q(C) = \phi^{-1}(C)$ is said to be the $q$-ary lattice associated to $C$ and $\Lambda_q(C)/q\mathbb{Z}^n \approx C$. The code $C$ can be viewed as the set of representatives of the above quotient inside the hypercube $[0, q)^n$ and $\Lambda_q(C)$ is given by translations of this set by multiples of $q$ in each direction. For $q$ prime, $\Lambda_q(C)$ is generated by the matrix:

$$M = \begin{bmatrix} I_{k \times k} & 0_{k \times (n-k)} \\ B_{(n-k) \times k} & qI_{(n-k) \times (n-k)} \end{bmatrix}$$

provided that $[I_{k \times k} | B_{k \times (n-k)}^t]^t$ is the associated generator matrix for $C$.

B. Lee metric

Instead of the usual Hamming metric for codes and Euclidean for lattices we consider here the Lee metric for both spaces which seems to be more natural when dealing with $q$-ary lattices and codes.
For \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), the \( l_1 \) or sum distance is defined as 
\[
d_{l_1}(x, y) = \sum_{i=1}^{n} |x_i - y_i|.
\]
The Lee distance in either \( \mathbb{Z}^n / q \mathbb{Z}^n \) or \( \mathbb{R}^n / q \mathbb{Z}^n \) is the distance induced by \( d_{l_1} \) through the quotient map:
\[
d_{\text{Lee}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \min\{y_i - x_i \pmod{q}, y_i - x_i + q \pmod{q}\}.
\]
We will denote here either \( d_{l_1} \) or \( d_{\text{Lee}} \) by \( d \) and call both Lee distance. The minimum norm \( \mu \) of a lattice \( \Lambda \) is 
\[
\mu = \min_{\mathbf{0} \neq \mathbf{x} \in \Lambda} d(\mathbf{x}, \mathbf{0}) \text{ and for a } q\text{-ary lattice } \Lambda_q(C) \text{ we have } \mu = \min\{q, d(C)\} \quad [19].
\]

III. Decoding \( q \)-ary lattices via Construction A

A decoding process for lattices constructed from binary codes via Construction A is presented in [4]. It is shown that decoding a binary code \( C \subseteq \mathbb{Z}_q^n \) corresponds to decoding in the binary lattice \( \Lambda_2(C) \subseteq \mathbb{R}^2 \) in the Euclidean metric. In this section we obtain the same kind of relation between code and lattice decoding with the Lee metric.

Let \( C \subseteq \mathbb{Z}_q^n \) be a \( q \)-ary code. Due to the isomorphism \( \Lambda_q(C)/q \mathbb{Z}^n \cong C \), we will not distinguish the elements of \( \Lambda_q(C)/q \mathbb{Z}^n \) from codewords of \( C \) and we will denote by \( | \cdot | \) the rounding to the nearest integer. Given a received vector \( r \in \mathbb{R}^n \), let \( z \) be its closest point in \( \Lambda_q(C) \) considering the Lee metric. In the next Propositions (1) and (2), we show how to find via Construction A a representative of \( z \) which is given by a codeword in \( C \).

**Proposition 1:** Let \( \Lambda_q(C) \) be a \( q \)-ary lattice and \( r = (r_1, \ldots, r_n)^t \in \mathbb{R}^n \) a received vector. Given an element \( \mathbf{x} \in \Lambda_q(C)/q \mathbb{Z}^n \), \( x = (x_1, \ldots, x_n)^t \), the representative \( z = (z_1, \ldots, z_n)^t \in \Lambda_q(C) \) of \( \mathbf{x} \) which is closest to \( r \) in \( \Lambda_q(C) \) considering the Lee metric is given by \( z_i = x_i + q w_i \) where 
\[
w_i = \left\lfloor \frac{r_i - x_i}{q} \right\rfloor, \text{ for each } i = 1, \ldots, n.
\]

**Proof:** The proof is straightforward. A representative of the class of \( r \) is given by \( z = x + qw \), where \( w \in \mathbb{Z}^n \) and the Lee distance \( d(r, z) = \sum_{i=1}^{n} |r_i - x_i - qw_i| \) is minimum when \( w_i = \left\lfloor \frac{r_i - x_i}{q} \right\rfloor \).

**Proposition 2:** Let \( \Lambda_q(C) \) be a \( q \)-ary lattice. Given \( r = (r_1, \ldots, r_n)^t \in \mathbb{R}^n \) a received vector let \( r \pmod{q} \in \{0, q]\)^n, obtained from \( r \) by reductions modulo \( q \) in each entry. If \( \mathbf{x} \in C \) is an element of \( C \) closest to \( r \pmod{q} \) considering the Lee metric, \( z \in \Lambda_q(C) \), \( \mathbf{z} = \mathbf{x} \), given by Proposition (1) is a lattice point nearest to \( r \).

**Proof:** Let \( r = (r_1, \ldots, r_n)^t = (r_1^*, \ldots, r_n^*)^t + q(t_1, \ldots, t_n)^t \), with \( 0 \leq r_i^* \leq q, t_i \in \mathbb{Z}, \text{ for } i = 1, \ldots, n \), that is \( r \pmod{q} = (r_1^*, \ldots, r_n^*)^t \). Let \( \mathbf{x} \in C, x = (x_1, \ldots, x_n)^t \), \( 0 \leq x_i \leq q - 1 \) for \( i = 1, \ldots, n \), be a closest point to \( r \pmod{q} \) considering the Lee metric. We will show that a closest point to \( r \) in \( A \) is in the same class that \( x \) in \( \Lambda_q(C)/q \mathbb{Z}^n \). For each class \( \mathbf{a} \in C, a = (a_1, \ldots, a_n)^t \), by Proposition (1) we find the representative \( a^* \) closest to \( r \) considering the Lee metric. We will show that \( d(r, a^*) = d(r \pmod{q}, \mathbf{a}) \). For the Lee distance we have
\[
d(r, a^*) = \sum_{i=1}^{n} |r_i^* - a_i - \alpha_i q|,
\]
where \( \alpha_i = \left\lfloor \frac{r_i^* - a_i + t_i}{q} \right\rfloor - t_i \). Since \( 1 \leq \alpha_i \leq 0 \leq q \), then \( \alpha_i \in \{-1,0,1\} \), and \( d(r, a^*) = \sum_{i=1}^{n} |r_i^* - a_i| = d(r \pmod{q}, \mathbf{a}) \).

Figure (1) shows the lattice \( \Lambda_{13}(C) \) and its Voronoi regions.

**Example 1:** Consider the cyclic 13-ary code in \( \mathbb{Z}_{13}^3, C \simeq \langle (1, 5)^t \rangle \). It has minimum Lee distance \( d(C) = 5 \) and error correction capacity \( t = 2 \). For the received vector \( r = (0, -6)^t \) the Lee-closest codeword to \( r \pmod{q} = (0, 7)^t \) is \( x = (12, 7)^t \). Hence in Proposition (1), \( w_1 = w_2 = -1 \) and by Proposition (2) the closest lattice point to \( r \) is \( z = (-5, -5, 5)^t \).
Let \( \mathbf{r} = (0,7,4,8,0,12,0)^t \) be a received vector. The closest code point to \( \mathbf{r} \) (mod 9) = (0,7,4,8,0,12,0)^t is \( \mathbf{x} = (0,0,0,0,0,0,0)^t \). Hence in Proposition (1), \( w_1 = 0, w_2 = \left\lceil \frac{7}{9} \right\rceil = 2, w_3 = \left\lceil \frac{4}{9} \right\rceil = 1, w_4 = \left\lceil \frac{8}{9} \right\rceil = 2, w_5 = 0, w_6 = \left\lceil \frac{0}{9} \right\rceil = 0 \) and \( w_7 = 0 \). Then, \( z = (0,0,0,0,0,0,0)^t + 4(0,2,1,2,3,0)^t = (0,8,4,8,0,12,0)^t \) is the closest point to \( \mathbf{r} \).

Proposition (2) provides a decoding process for \( q \)-ary lattices with the Lee metric via its generator code. This can be specially interesting for associated codes with an efficient Lee decoding algorithm. Decoding algorithms for some \( q \)-ary codes in the Lee metric assuming integer coordinates for the received point \( \mathbf{r} \) can be found in [1], [12], [17]. The algorithm derived in the next section is based only on the lattice structure and allows real coordinates for \( \mathbf{r} \).

IV. LEE SPHERE DECODING

The algorithm proposed here is analogous to the classic sphere decoding in the Euclidean metric and follows the same basic ideas. Nevertheless, we show that the structure of \( q \)-ary lattices in the Lee metric yields some important simplifications in comparison to the traditional algorithm. From now on, \( \| \cdot \| \) will always stand for the Lee norm.

Let \( \Lambda_q(C) \) be a \( q \)-ary lattice with generator matrix \( M \) in the special form given in (1) and \( \mathbf{r} \) a received point. We remark that \( \Lambda_q(C) \) always have a generator matrix in that form for \( q \) prime and in some cases for \( q \) not prime, as can be seen in Example (2). Given \( R > 0 \) we want to enumerate all \( y \in \Lambda_q(C) \) such that \( \| y - \mathbf{r} \| = \| Mx - \mathbf{r} \| \leq R \) and then find the closest lattice point to \( \mathbf{r} \).

Fixing the vector \( x^t = (x_1,\ldots,x_k)^t \), the minimum of \( \| Mx - \mathbf{r} \| \) is obtained by simply taking

\[
x_j = \left\lfloor (r_j - (Bx^t)_{j-k})/q \right\rfloor \quad (j = k+1,\ldots,n),
\]

what can be seen as a consequence of proposition (1). Hence, in order to decode the received vector \( \mathbf{r} \), it is not necessary to enumerate all lattice points inside the Lee sphere above-cited, which allows us to discard many points during the enumeration step by choosing the exact path of the sphere decoding tree (Figure 2) that leads to the minimum norm value, given the first \( k \) nodes (Figure 2).

The lattice points tested by the algorithm are those whose coordinate vector \( \mathbf{x} \) satisfies \( l_j \leq x_j \leq u_j \), \( (j = 1,\ldots,k) \), where

\[
\begin{align*}
  l_j & = \left\lceil R + r_j - \sum_{i=1}^{j-1} |r_i - x_i| \right\rceil \quad \text{and} \\
  u_j & = \left\lfloor -R + r_j + \sum_{i=1}^{j-1} |r_i - x_i| \right\rfloor
\end{align*}
\]

i.e., the points satisfying \( \| x^t - (r_1,\ldots,r_k) \| \leq R \). In this case, the number of feasible points corresponds to the number of \( \mathbb{Z}^k \) points inside a Lee sphere of radius \( R \) centered at \( \mathbf{r} \), which we estimate by the volume of the sphere, i.e., \( \pi^k R^k/k! \). There is a subtle difference between feasible points and nodes visited by the Lee sphere decoding algorithm which will become clear later. If we continue the search until depth \( n \) we will get an estimated number of feasible points as \( R^n 2^n/n! \), which, for \( n \) much larger than \( k \) represents a drastic reduction. We can now describe the algorithm of the search done in a node at depth \( j \leq k \) as the enumeration of all \( \mathbb{Z}^k \) in the Lee sphere centered at \( (e_1,\ldots,e_k) \) with radius \( R \). For \( j = k+1 \) we choose \( x_j \) according to Equation (2) and check if \( \| Mx - \mathbf{r} \| \leq R \). In order to speed up the search some backtracking strategies for updating the decoding radius are also possible, but we will not consider those in our discussion on the complexity of the algorithm.

Remark 1: We remind that in the classic sphere decoding there are no restrictions on the generator matrix \( M \) in order to perform enumerations, since it is possible to triangularize \( M \), for example, via QR factorization where \( Q \) is an orthogonal matrix. Unfortunately this approach cannot be employed here since rotations are not isometries in the Lee metric. Thus, the “systematic” form (1) of the generator matrix for \( \Lambda_q(C) \) is crucial in the process above-described.

A. Choosing the decoding radius

The radius choice is a critical part of sphere decoding. For the Euclidean case, Viterbo and Biglieri [20] first proposed the covering radius of a lattice, which can be estimated by Roger’s bound. Hassib and Vikalo [11] suggested that the radius could be chosen accordingly to the signal-to-noise ratio (SNR) of the channel. Another possible strategy is the so-called Babai’s estimate which can be easily adapted to the Lee norm. The stategy is to take:

\[
\hat{R} = \| M |x| - \mathbf{r} \|
\]

where \( x \) is the (real) solution of \( Mx = \mathbf{r} \) and \( |x| \) is the vector whose coordinates are \( x \)'s entries rounded off to the closest integer. This estimate guarantees at least one lattice point inside the Lee sphere of radius \( \hat{R} \) and allows us to take advantage of the interesting structure of \( q \)-ary lattices. For this estimate and matrix \( M \) as in (1), we have:

\[
\hat{R} \leq \frac{k}{2} + q(n-k)/2.
\]

Therefore, if \( \hat{n}_j \) is the number of visited nodes at depth \( j \) (corresponding to the number of \( \mathbb{Z}^k \) points inside the ball \( \|(x_1-r_1,\ldots,x_j-r_j)\| \leq \hat{R} \)), we have the following
equation as an upper bound for the expected number of nodes visited by the algorithm until depth $k$:

$$E[\# \text{ of nodes}] = \sum_{j=0}^{k} \hat{n}_j \leq \sum_{j=0}^{k} \left( j + q(n - j) \right)^j / j!.$$  \hspace{1cm} (6)

In fact, reasoning in the same way as [11], we can argue that $\hat{R} \approx k^{1+1/k}/2e$ for large $k$ and hence the expected complexity of the algorithm is exponential, which is inherent to problem itself.

In the special case that the received vector is in $\mathbb{Z}^n$ (or at least the first $k$ coordinates of $r$ are integers) we have the following:

**Proposition 3:** Suppose the received vector $r$ is such that $(r_1, \ldots, r_k) \in \mathbb{Z}^k$. Then the number of nodes of the Lee sphere decoding tree until depth $k$ is exactly

$$\sum_{j=0}^{k} \sum_{i=0}^{\min(j,R)} 2^i \binom{j}{i} \binom{R}{i}.$$  \hspace{1cm} (7)

**Proof:** The proof comes from previous arguments of this section and the fact that the number of points of $\mathbb{Z}^j$ inside a Lee sphere of radius $R$ is [3]:

$$\sum_{i=0}^{\min(j,R)} 2^i \binom{j}{i} \binom{R}{i}.$$  \hspace{1cm} (8)

B. Comparisons

There are several efficient algorithms to solve (exactly or approximately) the Euclidean version of CVP. A first approach to approximately solve the Lee sphere decoding problem could be through the so-called Nearest Plane Algorithm [10] which essentially projects the target vector on a LLL reduced basis for the lattice. This approach yields a polynomial time algorithm with an exponential approximate factor. If used to approximate CVP in the Lee metric, the Nearest Plane Algorithm outputs a vector which satisfies:

$$\|y - r\| \leq \frac{2}{\sqrt{n}} \left( \frac{2}{\sqrt{3}} \right)^n \|y - r\|$$  \hspace{1cm} (9)

where $y \in \Lambda$ is the closest point to $r$ in the Lee norm and the $1/\sqrt{n}$ factor is, of course, explained by the equivalence relation between the $l_1$ and $l_2$ norms.

Concerning the comparison with the classic sphere decoding, we will not go into detail on the number of arithmetic operations performed by the algorithms, and let this more careful analysis for a further work. However, since the performance of the algorithm is closely related to the volume of the spheres involved in the process, it is worth to study whether the Lee sphere has a smaller volume than the Euclidean one, given a received point and its Babai estimate (in both norms). In what follows we show that when the dimension $(k)$ increases, the Euclidean spheres have greater volume than the Lee spheres, in average.

Stating the problem more formally, let $r = Mx + e$ be a received point and $\hat{R}_1$ and $\hat{R}_2$ the Babai’s estimate to the decoding radius in Lee and Euclidean norm, respectively. Clearly $\hat{R}_1 \geq \hat{R}_2$. We want to know whether $\text{Vol}(B_{\text{Lee}}(\hat{R}_1)) \geq \text{Vol}(B_{\text{Euclid}}(\hat{R}_2))$ in average or not, where $\text{Vol}(S)$ stands for the Euclidean volume of a set $S$. Without lost of generality we assume that the transmitted point is the origin. If we fix the value $\hat{R}_2$ and take the average volume of all Lee spheres centered at the origin and containing a point of the surface of the Euclidean sphere of radius $\hat{R}_2$, we have:

$$\text{Vol}(B_{\text{Lee}}) = \int \cdots \int_{S} V_{\text{Lee}}(\phi_1, \ldots, \phi_{n-1}) d\phi_1 \cdots d\phi_{n-1} \frac{1}{(\pi/2)^{n-1}}$$

$$= \frac{\hat{R}_2^{2n}}{n!(\pi/2)^{n-1}} \int \cdots \int_{S} \sum_{i=0}^{\min(j,R)} 2^i \binom{j}{i} \binom{R}{i}$$  \hspace{1cm} (10)

where $(x_1, \ldots, x_n)$ is in the surface $S$ of the Euclidean sphere and the angles $\phi_1, \ldots, \phi_n$ are the hyperspherical coordinates. If we define

$$I(n, j) := \int \cdots \int_{S} \sum_{i=0}^{\min(j,R)} 2^i \binom{j}{i} \binom{R}{i}$$

the following expressions can be derived:

$$I(n, n) = \sum_{j=0}^{n} \binom{n}{j} \frac{\Gamma(j+1)\Gamma(n-j+1)}{2\Gamma(n+1)} I(n-1, j)$$  \hspace{1cm} (11)

and

$$\text{Vol}(B_{\text{Lee}}) = \frac{\hat{R}_2^{2n}}{n!(\pi/2)^{n-1}} I(n, n).$$  \hspace{1cm} (12)

We can then show that

$$\lim_{n \to \infty} \frac{I(n, n)2^n}{n!(\pi/2)^{n-1}} = 0, \hspace{1cm} (13)$$

what means there is a value $n_0$ such that for all $n \geq n_0$, we have $\text{Vol}(B_{\text{Lee}}) < \text{Vol}(B_{\text{Euclid}}(\hat{R}_2))$. We illustrate this fact in Figure [3].

**Remark 2:** It is a well-known fact that the ratio between the volume of a sphere in the $l_1$ norm and a Euclidean sphere of the same radius vanishes while increasing the dimension. This fact, however, does not imply Equation [13], since the spheres considered here have different radius.

C. Simulations

To simulate what was proposed in the previous sections we considerer received vectors of the form:

$$r = Mx + e$$  \hspace{1cm} (14)

where $M$ is in the form [1], the entries of its submatrix $B$ are uniform on $\mathbb{Z}_q^n$ and the entries of $e$ are i.i.d. zero mean random variables with Laplace distribution. Our choice of this noise instead of the usual Gaussian noise is explained by the relation of Laplace distribution with the $l_1$ norm. Indeed, while Gaussian noise samples are Euclidean spherical distributed...
Connections between the decoding process on codes and lattices may provide tools for error correcting codes and cryptographic schemes. We discuss here this connection in the case of \( q \)-ary lattices, which are obtained from \( q \)-ary linear block codes through Construction A, considered with the Lee distance, and present a Lee sphere decoding algorithm for lattices. Extensions of the presented approach here to other constructions of lattices will be considered in a future work.

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