We recently discovered that figure 3 in our paper (New J. Phys. 11 055017) contains a small error. The error occurred when we numerically evaluated the quantities plotted in the figure. The analytical expressions of these quantities that appear in the paper are correct. The corrected figure is given here.

\[ C_{dd} = 2.4. \]

Essentially, the vertical line in the figure has been moved from \( C_{dd} = 3.23 \) to \( C_{dd} = 2.4 \). Accordingly, the last sentence of section 3 should be modified to say ‘Our calculation indicates that the critical dipolar strength is about \( C_{dd} = 2.4 \).’ This error does not otherwise affect the contents or conclusions of the paper.
Dynamical properties of dipolar Fermi gases

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\textbf{Abstract.} In this paper, we investigate the dynamical properties of a one-component Fermi gas with dipole–dipole interaction between particles. Using a variational function based on the Thomas–Fermi density distribution in phase space representation, the total energy is described as a function of deformation parameters in both real and momentum spaces. Various thermodynamic quantities of a uniform dipolar Fermi gas are derived, and then instability of this system is discussed. For a trapped dipolar Fermi gas, the collective oscillation frequencies are derived with the energy-weighted sum rule method. The frequencies for the monopole, quadrupole, radial and axial modes are calculated, and softening against collapse is shown as the dipolar strength approaches the critical value. Finally, we investigate the expansion dynamics of the Fermi gas and show how the dipolar interaction manifests itself in the shape of an expanded cloud.

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1. Introduction

In recent years, atomic quantum dipolar gases have received much attention, for the simple reason that the anisotropic and long-range nature of the dipole–dipole interaction gives rise to a rich spectrum of novel properties in such systems. The theoretical study of dipolar Bose–Einstein condensates started in 2000. The properties of ground state [1, 2], collective oscillations [3, 4] and topological defects such as spin textures and vortex states [5, 6] have been studied. Moreover, when confined in optical lattice potentials, various quantum phases, such as ferromagnetism [7], supersolid state [8, 9], etc are predicted. The ground state [10], excitations [11], the BCS superfluidity [12] and rotating properties [13] of dipolar Fermi gases have also been theoretically explored. For a recent review of dipolar quantum gases, see [14].

In experiments, Bose–Einstein condensations of chromium atoms, which possess a magnetic dipole moment six times larger than that of alkali atoms, have been realized [15, 16]. The effect of dipole–dipole interaction in $^{52}$Cr condensate is observed in its expansion dynamics [17]. Besides chromium, heteronuclear molecules [18]–[23] and Rydberg atoms [24]–[26] are also expected to interact via a strong dipole–dipole force due to their large electric dipole moment, and their experimental realization is under way in a number of groups.

In [27], three of us studied the ground state properties of a dipolar Fermi gas by employing a variational Wigner function based on the Thomas–Fermi density of identical fermions. We showed that the dipole–dipole interaction induces a deformation of the momentum space distribution and identified that such a deformation arises from the Fock exchange term, which had not been paid particular attention in previous studies. The purpose of this paper is to extend the work of [27] and investigate the collective excitations and expansion dynamics of the dipolar Fermi gas. We want to emphasize that, due to the Pauli exclusion principle, the energy scales of a fermionic system are much larger than those of a Bose condensate. Consequently, the dipolar effects in Fermi gases only become significant when the dipole moment is very large. Our calculations show that for heteronuclear molecules with typical electric dipole moment of the order of 1 Debye, dipolar effects can be easily detected, while dipolar effects are usually negligible in atomic Fermi gases$^6$.

The content of the paper is organized as follows. In section 2, we present the model Hamiltonian and the total energy of the one-component dipolar Fermi gas under the

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$^6$ As pointed out in [14], in terms of the dipolar interaction strength, the magnetic dipole moment of chromium is equivalent to an electric dipole moment of 0.056 Debye.
Hartree–Fock approximation. In section 3, we derive the total energy function in a uniform system with a variational ansatz of Fermi surface and compute various thermodynamic quantities of the system. Here, we show how the Fock exchange interaction leads to Fermi surface deformation as well as instability of the system. In section 4, we turn our attention to a trapped system and investigate various modes of collective excitations using the sum-rule method, and show the softening of the excitation frequency as the interaction strength is increased toward a critical value. In section 5, we study the expansion dynamics of an initially trapped Fermi gas and show how the expanded cloud bears the signature of the underlying dipolar interaction. Finally, a summary and a discussion on the validity of the adopted variational method are presented in section 6.

2. Total energy functional in phase space representation

We consider a single-component Fermi gas of atoms or molecules with dipole moment aligned along the axial axis of a cylindrical harmonic trap at zero temperature. The Hamiltonian of this system is described by

\[ \hat{H} = \sum_i \left[ -\frac{\hbar^2}{2m} \nabla_i^2 + \frac{1}{2} \sum_{i \neq j} \left\{ \omega_p^2 (x_i^2 + y_i^2) + \omega_z^2 z_i^2 \right\} \right] + \sum_i V_{dd}(\mathbf{r}_i - \mathbf{r}_j), \]

where \( m \) is the mass of fermions, and \( \omega_p \) and \( \omega_z \) are the oscillation frequencies along the radial and axial axes, respectively. The dipole–dipole interaction of the last term in equation (1) is described by \( V_{dd}(\mathbf{r}) = d^2 (1 - 3 \cos^2 \theta)/r^3 \), where \( \theta \) is the angle between \( \mathbf{r} \) and the dipole moment \( \mathbf{d} \).

In the Hartree–Fock approximation, the total energy derived from Hamiltonian (1) can be written as the sum of the kinetic, trapping potential, Hartree direct and Fock exchange energies:

\[ E = E_{\text{kin}} + E_{\text{ho}} + E_d + E_{\text{ex}}, \]

\[ E_{\text{kin}} = \int d^3r \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} f(\mathbf{r}, \mathbf{k}), \]

\[ E_{\text{ho}} = \int d^3r \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} m [\omega_p^2 (x^2 + y^2) + \omega_z^2 z^2] f(\mathbf{r}, \mathbf{k}), \]

\[ E_d = \frac{1}{2} \int d^3r \int d^3r' \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} V_{dd}(\mathbf{r} - \mathbf{r}') f(\mathbf{r}, \mathbf{k}) f(\mathbf{r}', \mathbf{k}), \]

\[ E_{\text{ex}} = -\frac{1}{2} \int d^3R \int d^3s \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} V_{dd}(\mathbf{s}) e^{i(k-k')s} f(\mathbf{R}, \mathbf{k}) f(\mathbf{R}, \mathbf{k}'), \]

where we have introduced the Wigner function \( f(\mathbf{r}, \mathbf{k}) \) defined by the following transformation:

\[ n(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^3} \int d^3k e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} f \left( \frac{\mathbf{r} + \mathbf{r}'}{2}, \mathbf{k} \right), \]

where the one-body density matrix \( n(\mathbf{r}, \mathbf{r}') = \sum_i \psi_i(\mathbf{r}) \psi_i^*(\mathbf{r}') \) is defined in terms of a complete set of single-particle wave functions \( \{ \psi_i(\mathbf{r}) \} \). In equation (6), we have introduced the center
of mass coordinate \( \mathbf{R} = (\mathbf{r} + \mathbf{r}')/2 \) and relative coordinate \( \mathbf{s} = \mathbf{r} - \mathbf{r}' \). For the ground state, the summation over single-particle states \( i \) goes from the lowest one to the Fermi energy.

In our work, we do not calculate the Hartree–Fock energy represented by equation (2) in a fully self-consistent manner, which will be quite a complicated task. Instead, we adopt a much simpler semiclassical approach and calculate the total energy by employing a variational ansatz for the Wigner distribution function based on the Thomas–Fermi approximation, which assumes that the local Fermi surface at each spatial point has the same form as that in the homogeneous case. The ground state is then obtained by optimizing the Wigner function that minimizes the total energy. The details of this calculation can be found in \[27\]. In the present paper, we will focus on the dynamical properties such as the low-lying collective excitations and the expansion dynamics of the system.

3. Equilibrium properties of a homogeneous dipolar Fermi gas

It is instructive to first consider a homogeneous system \((\omega_z = \omega_x = 0)\) in a large box of volume \(V = \int d^3r\) with number density \(n_f\), as this will provide important insights into the trapped system to be studied later.

We introduce the following number-conserving variational ansatz for the Wigner function

\[
f(k) = \Theta \left( k_F^2 - \frac{k_p^2}{\alpha} - \alpha^2 k_s^2 \right),
\]

where \(\Theta(\cdot)\) is Heaviside’s step function, \(k_p^2 = k_x^2 + k_y^2\), and \(k_F = (6\pi^2 n_f)^{1/3}\) corresponds to the Fermi momentum. The parameter \(\alpha\) characterizes the deformation of the Fermi surface: \(\alpha > 1\) \((< 1)\) corresponds to an oblate disc-shaped (prolate cigar-shaped) Fermi surface. The physical origin of the Fermi surface deformation can be attributed to the anisotropic nature of the dipolar interaction.

Given the ansatz equation (8), the total energy of the homogeneous system can be derived as

\[
\varepsilon(\alpha) = E/V = \frac{\hbar^2}{m n_f^{5/3}} C_1 \left( \frac{2\alpha}{3} + \frac{1}{3\alpha^2} \right) - \frac{\pi}{3} C_{dd} I(\alpha),
\]

where \(C_1 = 3(6\pi^2)^{2/3}/10\), \(C_{dd} = m d^2 n_f^{1/3}/\hbar^2\) is the dimensionless dipolar interaction strength, and

\[
I(\alpha) = \int_0^\pi d\theta \sin \theta \left( \frac{3 \cos^2 \theta}{\alpha^3 \sin^2 \theta + \cos^2 \theta} - 1 \right)
\]

\[
= \begin{cases} 
6 & \frac{1}{1 - \alpha^3} \left[ 1 - \sqrt{\frac{\alpha^3}{1 - \alpha^3}} \arctan \left( \sqrt{\frac{1 - \alpha^3}{\alpha^3}} \right) \right] - 2 \quad (\alpha < 1), \\
0 & (\alpha = 1), \\
6 & \frac{1}{1 - \alpha^3} \left[ 1 + \frac{1}{2} \sqrt{\frac{\alpha^3}{\alpha^3 - 1}} \log \left( \frac{\sqrt{\alpha^3} + \sqrt{\alpha^3 - 1}}{\sqrt{\alpha^3} - \sqrt{\alpha^3 - 1}} \right) \right] - 2 \quad (\alpha > 1)
\end{cases}
\]
Figure 1. Deformation function $I(\alpha)$ as a function of $\alpha$.

Figure 2. Optimal deformation parameter $\alpha_0$ as a function of $C_{dd}$. For a molecular Fermi gas with electrical dipole moment $d = 1$ Debye, molecular mass $m = 100$ amu and density $n_f = 10^{13}$ cm$^{-3}$, we have $C_{dd} \approx 3.2$.

is the ‘deformation function’ [27] and is illustrated in figure 1. $I(\alpha)$ decreases monotonically from 4 to $-2$ as $\alpha$ increases from 0 to $\infty$, and passes through zero at $\alpha = 1$. The first and second terms in the square bracket of equation (9) represent the kinetic and the Fock exchange energy, respectively. For the homogeneous system, the Hartree direct energy vanishes.

Under this variational approach, the ground state is determined by the stationary condition for the total energy of equation (9) with respect to parameter $\alpha$: $[d\varepsilon/d\alpha]_{\alpha=\alpha_0} = 0$. The optimal value $\alpha_0$ is shown in figure 2 as a function of the dipolar strength $C_{dd}$. For free fermion systems, momentum density distribution is spherical, i.e. $\alpha_0 = 1$ at $C_{dd} = 0$. As the interaction strength increases, $\alpha_0$ decreases, which means that the momentum density distribution becomes cigar shaped. In other words, the Fermi surface is stretched along the direction of the dipoles.

Once we have the energy of the system as represented by equation (9), we can easily obtain other important thermodynamic quantities. Here, we provide our calculation for the pressure $P$, compressibility $K$ and chemical potential $\mu$, which are given by

$$P = -\frac{\partial E}{\partial V} \bigg|_N = n_f \frac{\partial E}{\partial n_f} \bigg|_N = \frac{C_{dd}}{3V} \frac{\partial E}{\partial C_{dd}} \bigg|_N,$$

$$\frac{1}{K} = n_f \frac{\partial P}{\partial n_f} = \frac{C_{dd}}{3} \frac{\partial P}{\partial C_{dd}}.$$
Figure 3. Chemical potential \( \mu \), pressure \( P \) and inverse compressibility or bulk modulus \( 1/K \) as functions of \( C_{dd} \). All quantities are normalized to their corresponding values in the non-interacting limit: \( \mu_0 = (5C_1/3)\hbar^2n_f^{2/3}/m = E_F \), \( P_0 = (2C_1/3)\hbar^2n_f^{5/3}/m \) and \( 1/K_0 = (10C_1/9)\hbar^2n_f^{5/3}/m \). The vertical line indicates the critical dipolar strength beyond which the system becomes unstable against collapse.

These quantities are illustrated in figure 3. One can see that \( P \), \( 1/K \) and \( \mu \) all monotonically decrease as the dipolar interaction strength increases. In particular, when the inverse compressibility (i.e. the bulk modulus) becomes negative, the system is no longer stable against collapse. Our calculation indicates that the critical dipolar strength is about \( C_{dd} = 3.23 \).

4. Collective oscillations of trapped dipolar Fermi gas

Let us now turn our attention to the trapped dipolar Fermi gas.

First, to obtain the total energy of equation (2), we introduce the following ansatz for the Wigner function:

\[
f(r,k) = \Theta \left( \frac{k_p^2}{\alpha} - \frac{\lambda^2}{a_{ho}^4} \left( \beta \rho^2 + \frac{z^2}{\beta^2} \right) \right),
\]

where \( \rho^2 = x^2 + y^2 \), and \( a_{ho} = \sqrt{\hbar/(m\omega)} \) with \( \omega = (\omega_\rho\omega_z)^{1/3} \). The variables \( \beta \) and \( \lambda \) represent the deformation and compression of the spatial density distribution of the system, respectively. When we take \( \alpha = 1 \), \( \beta = (\omega_\rho/\omega_z)^{2/3} \), and \( \lambda = 1 \), this trial function is consistent with the Thomas–Fermi density of a free Fermi gas in the harmonic trap. The Fermi wave number \( k_F \) is related to the number of fermions as

\[
N = \int d^3r \, n(r) = \int d^3r \int \frac{d^3k}{(2\pi)^3} f(r,k) = \frac{a_{ho}^6 k_F^6}{48\lambda^3}.
\]

Substituting equation (10) into equations (3)–(6), we obtain the total energy in units of \( N^{4/3}\hbar\omega \) as [27]

\[
\epsilon(\alpha, \beta, \gamma) = \frac{E}{N^{4/3}\hbar\omega} = \epsilon_{kin}(\alpha, \lambda) + \epsilon_{ho}(\beta, \lambda) + \epsilon_f(\beta, \lambda) + \epsilon_{ex}(\alpha, \lambda),
\]

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Figure 4. (a) Stability phase diagram in the space of the trap aspect ratio and the dipolar interaction strength. (b) Different energy terms, in units of $N^{4/3} \hbar \omega$, as functions of the trap aspect ratio $\beta_0$ at $N^{1/6} c_{dd} = 1.5$.

\[ \epsilon_{\text{kin}} = c_1 \lambda \left( 2 \alpha + \frac{1}{\alpha^2} \right), \quad \epsilon_{\text{ho}} = c_1 \left( \frac{2 \beta_0}{\beta} + \frac{\beta^2}{\beta_0^2} \right), \]

\[ \epsilon_d = N^{1/6} c_{dd} c_2 I(\beta) \lambda^{3/2}, \quad \epsilon_{\text{ex}} = -N^{1/6} c_{dd} c_2 I(\alpha) \lambda^{3/2}, \]  

where $\beta_0 = (\omega_0/\omega_z)^{2/3}$ represents the trap aspect ratio, $c_1 = 3^{1/3}/2^{8/3} \approx 0.2271$, $c_2 = 2^{10}/(3^{7/2} \times 5 \times 7 \pi^2) \approx 0.0634$, and $c_{dd} = d^2/\hbar \omega a_0^3$ is the dimensionless dipolar interaction strength for the trapped system. The momentum space deformation parameter $\alpha$, as in the homogeneous case, appears only in the kinetic and the exchange energy terms, both of which are independent of the spatial deformation parameter $\beta$. This indicates that the momentum space distribution of the trapped system will also be elongated along the direction of the dipoles, regardless of the geometry of the trapping potential. By contrast, $\beta$ appears only in the potential energy and the Hartree direct energy terms.

The ground state is determined by the stationary condition for equation (12) with respect to the three variables $\alpha, \beta$ and $\lambda$: $\partial \epsilon/\partial \alpha = \partial \epsilon/\partial \beta = \partial \epsilon/\partial \lambda = 0$. From the last condition, we can see that the energies of the dipolar Fermi gas satisfy the virial theorem:

\[ 2 \epsilon_{\text{kin}} - 2 \epsilon_{\text{ho}} + 3(\epsilon_d + \epsilon_{\text{ex}}) = 0. \]

In addition, the ground state has to satisfy the stability condition: the energy surface in the coordinates ($\alpha, \beta, \lambda$) has to be convex downward at the stationary point. If no values of ($\alpha, \beta, \lambda$) can be found to satisfy both the stationary and the stability conditions, the system is considered to be unstable against collapse [27]. This procedure leads to the stability phase diagram as shown in figure 4(a). Just as in the homogeneous case, the trapped dipolar gas is only stable for dipolar interaction strength below a critical value. In figure 4(b), we show the different energy terms (equations (13) and (14)) as functions of $\beta_0$ at $N^{1/6} c_{dd} = 1.5$. Several features are worth pointing out: (i) the exchange energy is always negative, as in the homogeneous case, regardless of the trap geometry, whereas the sign of the direct energy $\epsilon_d$ depends on trap geometry: $\epsilon_d > 0$ for $\beta_0 \lesssim 1$ (disc-shaped trap) and $\epsilon_d < 0$ for $\beta_0 \gtrsim 1$ (cigar-shaped trap). (ii) Both the kinetic and the trapping energies depend on the trap aspect ratio. By contrast, for the non-interacting system, when expressed in the same units, we have $\epsilon_{\text{kin}} = \epsilon_{\text{ho}} = 3c_1 \approx 0.68$ independent of $\beta_0$.

Next, we derive the collective oscillation frequencies for several low-lying excitation modes of the system using the sum rules in the present formulation [28]–[30]. In this approach,
we represent the excitation frequency \( \Omega \) using the first and third energy-weighted moments of the strength function for a given transition operator \( \hat{F} \):

\[
h\Omega = \sqrt{\frac{S_3}{S_1}},
\]

\[
S_1 = \sum_{v \neq 0} (E_v - E_0) |\langle v | \hat{F} | 0 \rangle|^2 = \frac{1}{2} \langle 0 | [\hat{F}^+, [\hat{H}, \hat{F}]] | 0 \rangle,
\]

\[
S_3 = \sum_{v \neq 0} (E_v - E_0)^3 |\langle v | \hat{F} | 0 \rangle|^2 = -\frac{1}{2} \langle 0 | [[H, F^+], [H, [H, F]]] | 0 \rangle,
\]

where \(|v\rangle\) denotes the \( v \)th eigenstate of the Hamiltonian with eigenenergy \( E_v \).

For our purpose, we choose the one-body operator as

\[
\hat{F} = \sum_{i=1}^N F(\vec{r}_i) = \sum_{i=1}^N \left[ \xi (x_i^2 + y_i^2) + \zeta z_i^2 \right],
\]

where \( \xi \) and \( \zeta \) are certain parameters that determine the nature of the excitation. A collective oscillation is compressive when \( \xi / \zeta = 1 \) and \( -1/2 \), respectively, whereas the axial (radial) mode is obtained by taking \( \xi = 0 \) and \( \zeta = 1 \) (\( \xi = 1 \) and \( \zeta = 0 \)).

Using equation (18), the collective excitation frequency \( \Omega \) in equation (15) in the present formulation can be shown to be

\[
\frac{\Omega}{\omega} = \sqrt{\frac{4(\epsilon_{\text{ho}}^2 + \epsilon_{\text{hz}}^2) + A(4\xi - \zeta) + B(4\xi - \zeta)(\xi - \zeta) + C(\xi - \zeta)^2}{(\epsilon_{\text{ho}}^2 + \beta_0^2)\epsilon_{\text{hz}}^2}},
\]

\[
A = \frac{1}{2}(\epsilon_d + \epsilon_{\text{ex}}), \quad B = \frac{2}{9} \left( \frac{\partial \epsilon_d}{\partial \beta} + \alpha \frac{\partial \epsilon_{\text{ex}}}{\partial \alpha} \right), \quad C = \frac{2}{9} \left( \beta_0^2 \frac{\partial^2 \epsilon_d}{\partial \beta^2} + \alpha^2 \frac{\partial^2 \epsilon_{\text{ex}}}{\partial \alpha^2} \right),
\]

where \( \epsilon_{\text{ho}} = 2c_1\beta_0/(\lambda \beta) \) and \( \epsilon_{\text{hz}} = c_1\beta_0^2/(\lambda \beta_0^2) \) are the radial and axial components of the trapping energy, respectively (see equation (13)). The excitation frequency \( \Omega \) is calculated by substituting the variational parameters (\( \alpha, \beta, \lambda \)) at the stationary point of the total energy (12).

From equation (19) we can easily find the excitation frequencies of the monopole, quadrupole, radial and axial modes, which have the following expressions:

\[
\Omega_M = \omega \sqrt{\frac{4\epsilon_{\text{ho}} + 3A}{(\epsilon_{\text{ho}}/\beta_0) + \beta_0^2\epsilon_{\text{ho}}}}, \quad \Omega_Q = \omega \sqrt{\frac{4\epsilon_{\text{ho}} + 16\epsilon_{\text{ho}} - 12A + 6B + 9C}{(\epsilon_{\text{ho}}/\beta_0) + 4\beta_0^2\epsilon_{\text{ho}}}},
\]

\[
\Omega_\rho = \omega \sqrt{\frac{4\epsilon_{\text{ho}} + 4B + C}{\epsilon_{\text{ho}}/\beta_0}}, \quad \Omega_\zeta = \omega \sqrt{\frac{4\epsilon_{\text{ho}} - A + B + C}{\beta_0^2\epsilon_{\text{ho}}}}.
\]

The corresponding frequencies for the non-interacting system can be recovered from equations (21) and (22) by taking \( A = B = C = 0 \) and \( \alpha = 1, \beta = \beta_0 \) as \( \Omega_M = \sqrt{12\omega_\rho^2\omega_\zeta^2/(\omega_\rho^2 + 2\omega_\zeta^2)} \),
Figure 5. Excitation frequencies of the monopole (a), quadrupole (b), radial (c) and axial (d) modes as functions of $\beta_0$. The frequencies are normalized to the corresponding values of the non-interacting system.

$$\Omega_0 = \sqrt{\frac{12\omega_\rho^2\omega_z^2}{(2\omega_\rho^2 + \omega_z^2)}}, \quad \Omega_{\rho 0} = 2\omega_\rho \quad \text{and} \quad \Omega_{z 0} = 2\omega_z,$$ which are in agreement with the results obtained in [30].

Figure 5 shows the excitation frequency of the four modes represented by equations (21) and (22). As can be seen in figure 4(b), the total interaction energy ($\epsilon_\text{ex} + \epsilon_\text{d}$) is positive (in other words, the overall dipolar interaction is repulsive) for disc-shaped traps ($\beta_0 < 1$), which makes the atomic cloud more compressible, hence $\Omega_M$ is increased compared with its non-interacting values. For cigar-shaped traps ($\beta_0 > 1$), the opposite will be true. This is consistent with the result shown in figure 5(a). The quadrupole and axial mode frequencies in figures 5(b) and (d), on the other hand, exhibit a roughly opposite trend.

To account for the hybridization of different modes, we parameterize $\xi$ and $\zeta$ in equation (18) as $\xi = \sin \theta$ and $\zeta = \cos \theta$, with $0 \leq \theta < \pi$. We then investigate the minimum value of the excitation frequency $\Omega(\theta)$ given by equation (19). The collective oscillation will be dominated by the compression mode for $0 < \theta < \pi/2$ and by the quadrupolar mode for $\pi/2 < \theta < \pi$. Moreover, $\theta = \pi/2$ represents the radial mode and $\theta = 0$ the axial mode. The natural monopole and quadrupole operators correspond to $\theta = \pi/4$ and $\theta = \pi - \arctan(1/2) \approx 0.85\pi$, respectively.

Figure 6(a) shows the minimum excitation frequency $\Omega_{\text{min}}$ as a function of the interaction strength $N^{1/6}_{\text{dd}}$ up to the critical value, and figure 6(b) shows the angle $\theta$ that minimizes $\Omega(\theta)$. For the spherical trap with $\beta_0 = 1.0$, the excitation frequency decreases monotonically as the interaction strength increases, and the minimum-energy mode is the monopole mode. For the cigar-shaped trap with $\beta_0 = 10.0$, the minimum-energy mode is dominated by the axial mode with $\theta \approx 0$ as the axial axis represents the direction of the soft confinement. Similarly, for the disc-shaped trap with $\beta_0 = 0.8$, the minimum-energy mode is dominated by the radial mode.
Figure 6. The minimum excitation frequency $\Omega_{\text{min}}$ (a) and the angle $\theta$ that minimizes $\Omega(\theta)$ (b) as functions of the interaction strength $N^{1/6}c_{\text{dd}}$ up to the critical values against instability. $\Omega_{\text{min}}$ is normalized to $\Omega_0$, the corresponding frequency for the non-interacting system at each $\beta_0$: $\Omega_0/\omega = 1.7$, 2.0 and 0.2 for $\beta_0 = 0.8$, 1.0 and 10, respectively. The critical values are $N^{1/6}c_{\text{dd}} = 2.433$, 2.166 and 1.603 for $\beta_0 = 0.8$, 1.0 and 10, respectively, see figure 4.

with $\theta \approx \pi/2$ as the radial direction now becomes the soft axis. However, as the interaction strength increases toward the critical value, in both of these cases, the minimum-energy mode shifts toward the monopole mode, and we clearly see the tendency of the softening of the collective mode, indicating the approaching of the collapse instability. We note that, in particular for the case of $\beta_0 = 0.8$, $\Omega_{\text{min}}$ does not completely decrease to zero at the critical value. This could be due to the calculation of the average frequency of the collective oscillation by the sum-rule method. Deeper insights into collective excitations may be obtained from microscopic approaches such as the random-phase approximation [31, 32].

5. Expansion dynamics

We now turn to the expansion dynamics of an initially trapped dipolar Fermi gas. This study is important as in most cold atom experiments, the atomic cloud is imaged after a period of free expansion, and such images can reveal many interesting properties of the system. Furthermore, the expansion dynamics may bear the signature of the underlying interaction. The dipolar effects in chromium condensate are first observed in the expansion dynamics [16, 17, 33].

Our starting point is the Boltzmann–Vlasov equation:

$$\frac{\partial f(\mathbf{r}, \mathbf{k}, t)}{\partial t} + \left( \frac{\hbar \mathbf{k}}{m} + \frac{1}{\hbar} \frac{\partial}{\partial \mathbf{k}} U(\mathbf{r}, \mathbf{k}, t) \right) \cdot \frac{\partial}{\partial \mathbf{r}} f(\mathbf{r}, \mathbf{k}, t) - \frac{1}{\hbar} \frac{\partial}{\partial \mathbf{r}} U(\mathbf{r}, \mathbf{k}, t) \cdot \frac{\partial}{\partial \mathbf{k}} f(\mathbf{r}, \mathbf{k}, t) = 0,$$

(23)

where the effective potential $U$ includes both the external harmonic trap potential $U_{\text{ho}}$ and the mean-field potential due to the dipole–dipole interaction:

$$U(\mathbf{r}, \mathbf{k}, t) = U_{\text{ho}}(\mathbf{r}) + \int d^3r' n(\mathbf{r}', t) V_{\text{dd}}(\mathbf{r} - \mathbf{r}') - \int \frac{d^3k'}{(2\pi)^3} \tilde{V}_{\text{dd}}(\mathbf{k} - \mathbf{k}') f(\mathbf{r}, \mathbf{k}', t),$$

(24)

where $\tilde{V}_{\text{dd}}(\mathbf{k}) = (4\pi d^2/3)(3k_z^2/k^2 - 1)$ is the Fourier transform of $V_{\text{dd}}(\mathbf{r})$. Note that the $\mathbf{k}$-dependence of the effective potential $U$ originates exclusively from the contribution of the exchange interaction, i.e. the last term on the rhs of equation (24).
To study the dynamics, we shall make use of the scaling transformation

\[ f(\mathbf{r}, \mathbf{k}, t) = f_0(\mathbf{R}(t), \mathbf{K}(t)) \],

\[ R_i(t) = r_i/b_i(t), \quad K_i(t) = b_i(t)k_i - mb_i(t)r_i/h \quad (i = x, y, z), \]

where \( f_0 \) represents the equilibrium Wigner distribution function obtained in the previous section, whose form is given by equation (10), and \( b_i \) are the dimensionless scaling parameters. This scaling approach has been used previously to study the expansion of Fermi gases [34, 35] and Bose–Fermi mixtures [36].

From the Boltzmann–Vlasov equation, we can derive the equations governing the scaling parameters [34, 36]

\[
\begin{align*}
\dot{b}_j + \gamma_j^2 b_j - \frac{\gamma_j^2}{(R_j^2)} \left[ \frac{T_j(b, b)}{b_j} - \frac{T_j(1, 0)}{b_j^3} \right] &= 0, \quad (25)
\end{align*}
\]

with \( \gamma_j = \omega_j/\omega, \quad \epsilon_{dd} = N^{1/6}c_{dd} \) and \( \langle R_j^2 \rangle = \int d^3R R_j^2 n_0(\mathbf{R}) \) with \( n_0 \) being the equilibrium density. The second and third terms of equation (25) represent, respectively, the restoring force and the kinetic energy. Collecting all contributions from the interaction we have

\[
\begin{align*}
T_j(b, b) &= \int d^3R d^3R' R_j \mathcal{W}(b; \mathbf{R} - \mathbf{R}')n_0(\mathbf{R}) \frac{\partial n_0(\mathbf{R})}{\partial R_j'} + \int d^3R d^3R d^3R' R_j K_j f_0(\mathbf{R}, \mathbf{K}) \\
&\quad \times \sum_i \frac{\partial f_0(\mathbf{R}, \mathbf{K})}{\partial K_i} \left[ \frac{\partial f_0(\mathbf{R}, \mathbf{K}')}{\partial R_i} - \frac{b_i b_i}{2\pi} \frac{\partial f_0(\mathbf{R}, \mathbf{K}')}}{\partial K_j'} \right] + \int d^3R d^3R d^3R' R_j K_j f_0(\mathbf{R}, \mathbf{K}) \\
&\quad \times \sum_i \frac{\partial \mathcal{W}(b; \mathbf{K} - \mathbf{K}')}{\partial K_i} \left[ - \frac{\partial f_0(\mathbf{R}, \mathbf{K})}{\partial R_i} + \frac{b_i b_i}{2\pi} \frac{\partial f_0(\mathbf{R}, \mathbf{K})}{\partial K_j} \right], \quad (26)
\end{align*}
\]

where

\[
\mathcal{W}(b; \mathbf{R}) = \frac{b_x^2 X^2 + b_y^2 Y^2 - 2b_z^2 Z^2}{(b_x^2 X^2 + b_y^2 Y^2 + b_z^2 Z^2)^{3/2}}
\]

is the dipole–dipole interaction potential under the scaling transformation and \( \mathcal{W}(b; \mathbf{K}) \) represents its Fourier transform. Given the Wigner function in equation (10), we obviously have \( b_x = b_y = b_z \) as the free expansion will not affect the cylindrical symmetry of the system. Moreover, the integrations for terms involving \( b_i \) in equation (26) vanish, so that \( T_j \) reduces to a function of \( \mathbf{b} \) only. The analytical expressions for \( T_j(\mathbf{b}) \) can be found as

\[
\begin{align*}
T_\rho(b) &= q b_{\rho}^{-2} b_{z}^{-1} \left[ d_\rho(\beta^{-3/2} b_{\rho}/b_z) - d_\rho(\alpha^{-3/2} b_{\rho}/b_z) \right], \\
T_\zeta(b) &= q b_{\rho}^{-2} b_{z}^{-1} \left[ d_\zeta(\beta^{-3/2} b_{\rho}/b_z) - d_\zeta(\alpha^{-3/2} b_{\rho}/b_z) \right], \quad (27)
\end{align*}
\]

where \( q = 1024(3\lambda)^{1/2}/(2835\pi^2) \) and the functions \( d_{\rho,\zeta} \) are defined as

\[
\begin{align*}
d_\rho(t) &= (1 - t^2)^{-2} \left[ 2 - 7t^2 + 4t^4 + 9t^4 \left( g(t) \right) \right], \\
d_\zeta(t) &= 2(1 - t^2)^{-2} \left[ 1 + 10t^2 - 2t^4 - 9t^4 \left( g(t) \right) \right],
\end{align*}
\]

with \( g(t) = \tanh^{-1} \sqrt{1 - t^2}/\sqrt{1 - t^2} \). We note that \( d_{\rho,\zeta}(t) \) are all monotonically decreasing functions of \( t \) and bounded between \( 2 \) and \( -4 \) for \( t \in [0, \infty) \).
Here, we focus on the time evolution of the atomic cloud aspect ratios in real and momentum spaces, which are defined, respectively, as

$$\kappa_r(t) = \sqrt{\frac{\langle r_x(t)^2 \rangle}{\langle r_z(t)^2 \rangle}} \quad \text{and} \quad \kappa_p(t) = \sqrt{\frac{\langle p_x(t)^2 \rangle}{\langle p_z(t)^2 \rangle}},$$

where initially the system is prepared in the ground state inside the external trap. Straightforward calculations yield

$$\kappa_r(t) = \beta^{-3/2} b_p(t)/b_z(t),$$

$$\kappa_p(t) = \left[ \frac{\beta \lambda^2 \alpha^3 b_p^{-2} + \alpha^2 b_z^2}{\beta \lambda^2 b_p^{-2} + \alpha^2 \beta^3 b_z^2} \right]^{1/2}.$$

The initial cloud aspect ratios are determined by the ground state Wigner function and can be easily shown to be $\kappa_r(0) = \beta^{-3/2}$ and $\kappa_p(0) = \alpha^{3/2}$. To study the expansion dynamics, we turn off the trapping potential at $t = 0$ and the cloud starts to expand. We then solve for $b_p(t)$ and $b_z(t)$ using equation (25) with the restoring force term $\gamma_j b_j$ removed and with the initial conditions $b_p(0) = b_z(0) = 1$. Before presenting our results, we recall that when the exchange interaction is ignored, the direct dipolar interaction always tends to stretch the cloud along the direction of dipole moments in both real and momentum spaces [35].

Figure 7 displays several examples of the cloud aspect ratio during time of flight for different trap geometries. As expected, asymptotically, the aspect ratios in momentum and real spaces become equal to each other, i.e. $\kappa_r(\infty) = \kappa_p(\infty) = \kappa_\infty$. A notable feature is that, regardless of the initial trap geometry, the shape of the expanding cloud eventually becomes cigar shaped as $\kappa_\infty < 1$. This result is in stark contrast to the expansion dynamics of a dipolar condensate whose asymptotic aspect ratio is sensitive to the initial trap geometry [16, 17, 33]. Furthermore, the interaction effects during the time of flight are also evident from figure 7: had interaction been ignored, the expansion would have become ballistic with $\kappa_p$ a constant in time. Figure 7(b) indicates that the expansion is essentially ballistic for an initial spherical trapping potential, as for such traps, the interaction energy is rather weak as shown in figure 4(b).
Figure 8. (a) The dipolar interaction strength dependences of asymptotic aspect ratios $\kappa_\infty$ (solid lines) and the initial momentum space aspect ratio $\kappa_p(0)$ (dashed lines) for various values of the trap aspect ratio $\beta_0$. (b) The difference between the asymptotic aspect ratio and the initial momentum space aspect ratio for $\beta_0 = 1$.

That the expanded cloud eventually becomes cigar shaped is also obtained in [35] when the exchange dipolar interaction is ignored, which indicates that the effect of the exchange interaction during the expansion is not very important. This is consistent with figure 4(b), which shows that, except for nearly spherical traps, the magnitude of the direct energy is, in general, much larger than that of the exchange energy. However, we want to emphasize that the exchange term is crucial for the equilibrium momentum distribution inside the trap: without the exchange term, the momentum distribution would be isotropic for any trap geometry. To get a closer look, we compare, in figure 8, $\kappa_\infty$ with the initial momentum space aspect ratio $\kappa_p(0)$, which characterizes the momentum distribution for the ground state in the trap. The initial momentum distribution is always cigar shaped as $\kappa_p(0) < 1$. In general, the effect of the interaction during the expansion, with the dominant contribution from the direct term, is to further enhance this anisotropy such that $\kappa_\infty < \kappa_p(0)$. Exceptions may occur for nearly spherical traps, for which one may have $\kappa_\infty > \kappa_p(0)$ as shown in figure 8(b). However, this effect is very small since, as we have already mentioned earlier, the total dipolar interaction is weak for such traps.

6. Summary

In summary, we have studied the properties of dipolar Fermi gases both in the homogeneous system and in the presence of a cylindrical harmonic trap with the dipole moments oriented along the symmetry axis. The total energy functional of this system is derived under the Hartree–Fock approximation. The one-body density matrix in the energy functional is obtained from a variational ansatz based on the Thomas–Fermi density distribution in phase-space representation, which accounts for the interaction-induced deformation in both real and momentum spaces. Our calculations show that deformation of the spatial density distribution comes from the Hartree direct energy term, whereas deformation of the momentum density distribution arises from the Fock exchange energy term. Note that the exchange term, a consequence of the anti-symmetry of the many-body fermionic wave function, does not appear in the Bose–Einstein condensate.

We have calculated several thermodynamic quantities such as the pressure, the compressibility and the chemical potential of the homogeneous system and investigated the
low-lying collective excitations of a trapped dipolar Fermi gas using the sum-rule method for various trap geometry and interaction strengths. We observe the softening of the collective excitations as the interaction strength approaches the critical value for collapse.

We have also studied the expansion dynamics of the initially trapped system. We show that, in stark contrast to the dipolar condensate [16, 17, 33], the atomic Fermi gas will eventually become elongated along the direction of the dipoles regardless of the initial trap geometry. This feature makes it convenient to detect the dipolar effects in Fermi gases.

Finally, we want to comment on the validity of the variational approach we have adopted here. As we have mentioned, our method is not self-consistent, which may potentially be a drawback. However, the crucial advantage of this method is its simplicity, which in turn allows one to capture the essential physics in a straightforward manner. Similar approaches have been used widely and successfully in various areas of physics in general, and in the area of ultracold atomic physics in particular. For the dipolar Fermi gas we have studied here, a natural improvement is to construct a fully self-consistent Hartree–Fock theory, a task we are undertaking at the moment. Our preliminary results for the homogeneous system show excellent agreement with the variational results reported here.

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