EXTENSION OF THE STRUCTURE THEOREM OF BORCHERS
AND ITS APPLICATION TO HALF-SIDED MODULAR
INCLUSIONS

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Dedicated to Professor D. Buchholz on his 60th birthday

Abstract. A result of H.-W. Wiesbrock is extended from the case of a common cyclic and separating vector for the half-sided modular inclusion \( N \subset M \) of von Neumann algebras to the case of a common faithful normal semifinite weight and at the same time a gap in Wiesbrock’s proof is filled in.

1. Introduction

J. Bisognano and E. Wichmann ([4]) made a discovery about the connection of the modular operator and the modular conjugation for the von Neumann algebra generated by quantum fields in a wedge region of the Minkowski space-time with kinematical transformations, namely pure Lorentz transformation and the TCP\(_1\) operator.

H. J. Borchers ([6]) formulates an important feature of this connection in the abstract setting of a pair of von Neumann algebras \( N \subset M \) with a common cyclic and separating vector \( \Omega \), and a one-parameter group of unitaries \( U(\lambda) \) having a positive generator, which induces a semigroup of endomorphisms of \( (M, \Omega) \), obtaining a commutation relation of \( U(\lambda) \) with the modular operator and the modular conjugation for \( (M, \Omega) \), which reproduces the kinematical commutation relations in the Bisognano-Wichmann situation.

A further development has been achieved by H.-W. Wiesbrock ([37], [38], [39], [40], [41], [42]), who introduces the notion of the half-sided modular inclusion and obtains an underlying group structure (cf. [41]), as well as an imbedding of the canonical endomorphisms of the subfactor theory into a one-parameter semigroup of endomorphisms in this specific situation. Thus he gets a correspondence between 2-dimensional chiral conformal field theories and a class of type III\(_1\) subfactors.

Unfortunately, there is a gap in Wiesbrock’s proof of his basic theorem ([37], Theorem 3, Corollary 6 and Corollary 7, [42]). We will fill in this gap in Wiesbrock’s proof and further generalize the result to the case of a common normal semifinite faithful weight.

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As a basic tool to prove general half-sided modular inclusion results, we generalize a structure theorem of H. J. Borchers ([7], Theorem B) considerably, making Borchers’ proof at the same time more transparent.

The extension from the state case to the weights turns out to be not straightforward. For this purpose we introduce as a basic tool the notion of a Hermitian map by using modular structure.

It seems that before the summer of 1995, when we independently noticed the gap in the proof of Wiesbrock’s half-sided modular inclusion theorem, this gap was generally overlooked. Thanks to Prof. Detlev Buchholz, who has been visiting the first named author in the fall of 1995, we learned about each other’s insights and started to collaborate on this paper. The first version of the paper, containing a complete proof of the General Half-sided Modular Inclusion Theorem, Theorem 2.1, was available already at the end of 1995. It had a restricted circulation, but it was presented at several conferences. Other topics, like Theorem 2.2 on the structure and type of the involved von Neumann algebras and Proposition 2.4 on pathologies of the analytic extension of orbits of one-parameter automorphism groups, are of more recent date.

We notice that since 1995 a number of papers appeared, containing proposals for a complete proof of the half-sided modular inclusion theorem (see, for example, [17], Section 3 and [10], pp. 608 and 609). None the less, till now we have no knowledge of a completely elaborated proof, even in the state case.

2. Main Results

(a) Notations and facts from the Modular Theory of von Neumann Algebras (see, for example, [35], Chapter 10).

For two Hilbert spaces $H$ and $K$ we denote by $B(K, H)$ the Banach space of all bounded linear maps from $K$ to $H$. $B(H, H)$ will be denoted simply by $B(H)$. If $T$ is a not necessarily everywhere defined linear operator from $K$ to $H$, then $\text{Dom}(T)$ will stand for the domain of $T$.

We denote the weak and the strong operator topology on $B(K, H)$ respectively by $\text{wo}$ and $\text{so}$. The weak topology defined on $B(K, H)$ by all linear functionals belonging to the norm-closure of the $\text{wo}$-continuous linear functionals in the dual of $B(K, H)$ will be denoted by $\text{w}$. Further, the locally convex vector space topology defined on $B(K, H)$ by the seminorms $B(K, H) \ni T \mapsto \varphi(T^*T)^{1/2}$, where $\varphi$ ranges over all $\text{w}$-continuous positive linear functionals on $B(K)$, will be denoted by $\text{s}$. We notice that on any bounded subset of $B(K, H)$ $\text{wo} = \text{w}$ and $\text{so} = \text{s}$.

For a weight $\varphi$ on a von Neumann algebra $M$ we use the standard notations:

$\mathfrak{N}_\varphi = \{x \in M : \varphi(x^*x) < +\infty\}$ (left ideal),

$\mathfrak{M}_\varphi = (\mathfrak{N}_\varphi)^*\mathfrak{N}_\varphi = \text{the linear span of } \{a \in M^+ : \varphi(a) < +\infty\}$ (hereditary $*$-subalgebra),

$\mathfrak{A}_\varphi = (\mathfrak{N}_\varphi)^* \cap \mathfrak{N}_\varphi \supset \mathfrak{M}_\varphi$ ($*$-subalgebra).
We notice that for \( a \in M^+ \) we have \( a \in \mathfrak{M}_\phi \iff \phi(a) < +\infty \).

A von Neumann algebra \( M \) on a Hilbert space \( H \) is in standard form with respect to a normal semifinite faithful weight \( \phi \) on \( M \) if there is a linear map with dense range

\[
\mathfrak{N}_\phi \ni x \mapsto x_\phi \in H
\]
such that

\[
\phi(x^*x) = \|x_\phi\|^2, \quad (ax)_\phi = ax_\phi, \quad x \in \mathfrak{N}_\phi, \quad a \in M.
\]

In particular, by the faithfulness of \( \phi \), the map \( x \mapsto x_\phi \) is injective. We notice also that the above map \( x \mapsto x_\phi \) is unique up to natural unitary equivalence. If \( \phi \) is bounded, then \( \xi_\phi = (1_H)_\phi \) is a cyclic and separating vector for \( M \) and \( x_\phi = x\xi_\phi \) for all \( x \in M = \mathfrak{M}_\phi \). In this case \( \phi \) is the vector form \( M \ni x \mapsto \omega_{\xi_\phi}(x) = (x\xi_\phi \mid \xi_\phi) \).

Let \( M \) be a von Neumann algebra on a Hilbert space \( H \), in standard form with respect to a normal semifinite faithful weight \( \phi \) on \( M \). Then the antilinear operator

\[
H \ni \{x_\phi; x \in \mathfrak{N}_\phi \} \ni x_\phi \mapsto \omega_{\phi^*}(x) = (x \xi_\phi \mid \xi_\phi)
\]

has closure \( S_\phi \) and the invertible positive selfadjoint operator \( \Delta_\phi = S_\phi^*S_\phi \) is called the modular operator of \( \phi \). If

\[
S_\phi = J_\phi \Delta_{\phi}^{1/2}
\]
is the polar decomposition of \( S_\phi \), then \( J_\phi \) is an involutive antiunitary operator (antilinear surjective isometry with \( J_\phi^2 = 1_H \)), called the modular conjugation of \( \phi \). The operators \( \Delta_\phi \) and \( J_\phi \) satisfy the commutation relation

\[
J_\phi \Delta_{\phi}^z = \Delta_{\phi}^{-z}J_\phi, \quad z \in \mathbb{C}, \tag{2.1}
\]
in particular,

\[
S_\phi = J_\phi \Delta_{\phi}^{1/2} = \Delta_{\phi}^{-1/2}J_\phi, \quad J_\phi \Delta_{\phi}^{it} = \Delta_{\phi}^{it}J_\phi, \quad t \in \mathbb{R}. \tag{2.2}
\]

If \( \phi \) is bounded and \( \xi_\phi = (1_H)_\phi \) is the associated cyclic and separating vector, then

\[
S_\phi \xi_\phi = \xi_\phi, \quad \Delta_\phi \xi_\phi = \xi_\phi, \quad J_\phi \xi_\phi = \xi_\phi.
\]

The fundamental result of the modular theory claims that

\[
x \in \mathfrak{N}_\phi, \quad t \in \mathbb{R} \implies \Delta_{\phi}^{it}x \Delta_{\phi}^{-it} \in \mathfrak{N}_\phi, \quad (\Delta_{\phi}^{it}x \Delta_{\phi}^{-it})_\phi = \Delta_{\phi}^{it}x_\phi, \tag{2.3}
\]

so that

\[
M \ni x \mapsto \sigma_t^\phi(x) = \Delta_{\phi}^{it}x \Delta_{\phi}^{-it} \in M, \quad t \in \mathbb{R} \tag{2.4}
\]
defines an so-continuous one-parameter group of automorphisms \( (\sigma_t^\phi)_{t \in \mathbb{R}} \) of \( M \), called the modular automorphism group of \( \phi \), and

\[
J_\phi M J_\phi = M', \quad x, y \in \mathfrak{N}_\phi \implies xJ_\phi y = J_\phi xy, \quad (J_\phi x)_\phi = J_\phi (x_\phi), \tag{2.5}
\]
so that \( M \ni x \mapsto J_\phi x^*J_\phi \in M' \) is a \(*\)-antiisomorphism. Moreover, the weight \( \phi \) is invariant under the action of the modular automorphism group:

\[
\sigma_t^\phi(a) = \phi(a), \quad a \in M^+, \quad t \in \mathbb{R}. \tag{2.6}
\]

The center \( Z(M) \) of \( M \) is contained in the fixed point von Neumann subalgebra \( \{x \in M; \sigma_t^\phi(x) = x, \quad t \in \mathbb{R}\} \subset M \), which is usually denoted by \( M^\phi \). On the other hand, \( J_\phi z J_\phi = z^* \) for all \( z \in Z(M) \). We recall also (see the proof of [23], Lemma 5.2 or [45], Corollary 1.2) :

\[
x \in \mathfrak{N}_\phi, \quad y \in M^\phi \implies xy \in \mathfrak{N}_\phi, \quad (xy)_\phi = J_\phi y^*J_\phi x_\phi. \tag{2.7}
\]

Let \( e \in M^\phi \) be a projection and let \( \varphi_e \) denote the restriction of \( \varphi \) to \( eMe \). By [23], Proposition 4.1 and Theorem 4.6 (see also [33], Propositions 4.5 and 4.7), \( \varphi_e \)
is a normal semifinite faithful weight and its modular group is the restriction of the modular group of \( \phi \) to \( eMe \). Thus, if \( \pi_e : eMe \rightarrow B(eH) \) is the faithful normal \(*\)-representation which associates to every \( x \in eMe \) the restriction \( x | eH \) considered as a linear operator \( eH \rightarrow eH \), then the modular group of the weight \( \varphi_e \circ \pi_e^{-1} \) on \( \pi_e(eMe) \), that is \( (\pi_e \circ \sigma_t^{it} \circ \pi_e^{-1}) \in \mathbb{R} \), is implemented by the unitary group \( (\Delta^u_t | eH)_{t \in \mathbb{R}} \) on \( eH \). Nevertheless, \( \pi_e(eMe) \) is not always in standard form with respect to \( \varphi_e \circ \pi_e^{-1} \) (indeed, if \( M \subset B(\mathbb{C}^4) \) is a type II \(_1\) factor, in standard form with respect to its trace, and \( e \in M \) is a minimal projection, then \( \pi_e(M) \) is one-dimensional, while its commutant \( \pi_e(M)' \) is four-dimensional, so \( \pi_e(M) \) and \( \pi_e(M)' \) are not antiisomorphic).

However, for any projection \( e \in M^\varphi \),
\[
\pi = \pi_{eJ_\varphi} : eMe \ni x \mapsto x | eJ_\varphi eH \in B(eJ_\varphi eJ_\varphi H) \quad (2.8)
\]
is a faithful normal \(*\)-representation, such that the von Neumann algebra \( \pi(eMe) \) is in standard form with respect to \( \varphi_e \circ \pi_e^{-1} \) (cf. [19], Lemma 2.6). Moreover, \( \Delta_\varphi \) and \( J_\varphi \) commute with \( eJ_\varphi eJ_\varphi \) and we have the identifications
\[
\Delta_{\varphi^{\circ \circ -1}}^{\circ \circ} = eJ_\varphi eJ_\varphi H, \quad J_{\varphi^{\circ \circ -1}} = eJ_\varphi eJ_\varphi H. \quad (2.9)
\]
For the convenience of the reader, let us outline the proof of (2.9).

For the faithfulness of \( \pi \), let \( x \in eMe \) be such that \( x | eJ_\varphi eJ_\varphi = 0 \). Then
\[
x J_\varphi eJ_\varphi eJ_\varphi = x eJ_\varphi M eJ_\varphi eJ_\varphi = J_\varphi M eJ_\varphi x eJ_\varphi eJ_\varphi = 0,
\]
so \( x J_\varphi \) vanishes on \( MeH \), hence on the range of the central support \( z(e) \in Z(M) \) of \( e \). Thus \( x = x z(e) = x J_\varphi z(e) J_\varphi = 0 \).

To see that \( \pi(eMe) \) is in standard form with respect to \( \psi = \varphi_e \circ \pi_e^{-1} \), first we notice that, according to (2.7), \( \mathfrak{N}_\psi = \pi(\mathfrak{N}_{\varphi_e}) = e \mathfrak{N}_{\varphi_e} e \). Next, the linear map
\[
\mathfrak{N}_\psi = \pi(\mathfrak{N}_{\varphi_e}) \ni \pi(x) \mapsto x_\varphi = (exe)_\varphi \in eJ_\varphi eJ_\varphi H
\]
has dense range. Indeed, every vector in \( eJ_\varphi eJ_\varphi H \) belongs to the closure of
\[
e J_\varphi eJ_\varphi \{x_\varphi : x \in \mathfrak{N}_{\varphi_e}\} = \{(exe)_\varphi : x \in \mathfrak{N}_{\varphi_e}\} = \{x_\varphi : x \in \mathfrak{N}_{\varphi_e}\}. \quad (2.10)
\]
Finally, for every \( \pi(x) \in \pi(\mathfrak{N}_{\varphi_e}) = \mathfrak{N}_\psi \) and \( \pi(a) \in \pi(eMe) \) hold true:
\[
\psi(\pi(x)^* \pi(x)) = \varphi_e(x^* x) = \varphi(x^* x) = \|x_\varphi\|^2, \quad (ax)_\varphi = ax_\varphi = \pi(a)x_\varphi.
\]

The commutation of \( J_\varphi \) with \( eJ_\varphi eJ_\varphi \) follows immediately from the commutation of \( e \) with \( J_\varphi eJ_\varphi \). Let \( J_\varphi \) denote the involutive antiumiti operator
\[
e J_\varphi eJ_\varphi H \ni \xi \mapsto J_\varphi \xi \in eJ_\varphi eJ_\varphi H.
\]
Further, using (2.2) and \( e \in M^\varphi \), we obtain for every \( t \in \mathbb{R} \):
\[
e J_\varphi eJ_\varphi \Delta^u_t = e J_\varphi e \Delta^u_t J_\varphi = e J_\varphi e J_\varphi e J_\varphi = e \Delta^u_t J_\varphi e J_\varphi e J_\varphi = \Delta^u_t e J_\varphi e J_\varphi .
\]
Thus also \( \Delta_\varphi \) commutes with \( eJ_\varphi eJ_\varphi \), so
\[
e J_\varphi eJ_\varphi H \ni \Delta_\varphi \xi \ni \Delta_\varphi \xi \ni eJ_\varphi eJ_\varphi H
\]
is an invertible positive selfadjoint operator \( \Delta_\varphi \), whose positive selfadjoint square root is
\[
e J_\varphi eJ_\varphi H \ni \Delta^{1/2}_\varphi \ni \Delta^{1/2}_\varphi \xi \ni eJ_\varphi eJ_\varphi H.
\]
Since, for every \( \pi(x) \in \pi(\mathfrak{N}_{\varphi_e}) = \mathfrak{N}_\psi \),
\[
S_\psi x_\varphi = (x^*)_\varphi = S_\varphi x_\varphi = J_\varphi \Delta^{1/2}_\varphi x_\varphi = J_e \Delta^{1/2}_\varphi x_\varphi = J_e \Delta^{1/2}_\varphi x_\varphi ,
\]
we deduce that \( S_\psi \subset J_\psi \Delta_2 \). For the equality \( S_\psi = J_\psi \Delta_2 \), which will imply \(2.3\), let \( \xi \in \text{Dom}(\Delta_2) = \text{Dom}(\Delta_2) \cap (eJ_\psi eJ_\psi H) = \text{Dom}(S_\psi) \cap (eJ_\psi eJ_\psi H) \)
be arbitrary. Then there is a sequence \( (x_n)_{n \geq 1} \) in \( \mathfrak{A}_\psi \) such that \( (x_n)_\varphi \to \xi \) and \( (x_n)_\varphi \to S_\varphi \xi \). By \(2.4\) the sequence \( (e_{x_n} e)_{n \geq 1} \) belongs to \( \mathfrak{A}_\varphi \) and we have
\[
(e_{x_n} e) = e_{x_n} eJ_\varphi (x_n)_{\varphi} \to e_{x_n} eJ_\varphi \xi = \xi, \\
S_\psi (e_{x_n} e)_{\varphi} = ((e_{x_n} e)^*)_{\varphi} = e_{x_n} eJ_\varphi (x_n)_{\varphi} \to e_{x_n} eJ_\varphi S_\varphi \xi.
\]
Now the closedness of the graph of \( S_\psi \) yields \( \xi \in \text{Dom}(S_\psi) \).

For a projection \( p \in Z(M) \subset M^\varphi \) we have \( J_\varphi pJ_\varphi = p \), so \(2.3\) yields
\[
\Delta_{\varphi, p \sigma_{\varphi}^{-1}} = \Delta_\varphi | pH, \\
J_{\varphi, p \sigma_{\varphi}^{-1}} = J_\varphi | pH.
\]

Let \( M \neq \{0\} \) be a von Neumann algebra, in standard form with respect to a normal semifinite faithful weight \( \varphi \) on \( M \). Then the Connes spectrum \( \Gamma(\sigma_\varphi) \) of the modular automorphism group \( \sigma_\varphi \) of \( \varphi \) is the intersection of the Arveson spectra of all modular automorphism groups \( \sigma_\varphi, \) where \( \epsilon \) ranges over all non-zero projections \( e \in M^\varphi \). By \(13\), Lemme 1.2.2 and Théorème 2.2.4 (see also \(33\), Theorem 3.1 and Proposition 16.3), \( \Gamma(\sigma_\varphi) \) is a closed additive subgroup of \( \mathbb{R} \) and it does not depend on the choice of \( \varphi \), so it can be denoted (like in \(27\), 8.15) by \( \Gamma(M) \). Furthermore, by \(13\), Lemme 3.2.2 (see also \(33\), Proposition 28.1), \( \lambda \in \Gamma(M) \) if and only if \( e_\lambda \) belongs to the spectrum \( \sigma(\Delta_{\varphi, \epsilon}) \) of \( \Delta_{\varphi, \epsilon} \) for all non-zero projections \( e \in M^\varphi \).

According to \(13\), page 28, the von Neumann algebra \( M \neq \{0\} \) is called to be of type \( \text{III}_1 \) if \( \Gamma(M) = \mathbb{R} \), or equivalently, if \( \sigma(\Delta_{\varphi, \epsilon}) = [0, +\infty) \) for every non-zero projection \( e \in M^\varphi \). By \(2.9\) we have also:
\[
M \text{ is of type } \text{III}_1 \iff \left\{ \begin{array}{l}
\sigma(\Delta_\varphi | eJ_\varphi eJ_\varphi H) = [0, +\infty),
\text{for every projection } 0 \neq e \in M^\varphi.
\end{array} \right.
\]

(b) THE GENERAL HALF-SIDED MODULAR INCLUSION THEOREM.

Let \( M \) be a von Neumann algebra on a Hilbert space \( H \), in standard form with respect to a normal semifinite faithful weight \( \varphi \) on \( M \). Let further \( N \subset M \) be a von Neumann subalgebra such that the restriction \( \psi \) of \( \varphi \) to \( N \) is semifinite. If \( \{y_\varphi : y \in \mathfrak{R}_\psi\} \) is dense in \( H \), then \( N \) is in standard form with respect to \( \psi \) such that \( y_\psi = y_\varphi \) for all \( y \in \mathfrak{R}_\psi \). This happens, for example, if \( N \subset M \subset B(H) \) are von Neumann algebras having a common cyclic and separating vector \( \xi_o \), and \( \varphi \) is the vector form \( M \ni x \to (x \xi_o | \xi_o) \).

In the above situation, owing to \(2.6\), we have
\[
J_\psi J_\varphi M J_\varphi J_\psi = J_\psi M' J_\psi \subset J_\psi N' J_\psi = N,
\]
so the unitary \( J_\psi J_\varphi \) implements a unital \(*\)-homomorphism
\[
M \ni x \to \text{Ad}(J_\psi J_\varphi)(x) = J_\psi J_\varphi x J_\varphi J_\psi \in N \subset M,
\]
considered by R. Longo \((24, 25)\) and called the canonical endomorphism of the inclusion \( N \subset M \). The canonical endomorphism \( \gamma \), in particular the tunnel
\[
M \ni N \ni \gamma(M) \ni \gamma(N) \ni \gamma^2(M) \ni \gamma^2(N) \ni \ldots,
\]
plays an important role in the Subfactor Theory (see \(24\) and \(22\)).
Let $\mathcal{P}_\mathbb{R}^1(1)$ denote the two-dimensional Lie group generated by the hyperbolic rotations

$$L_t : \mathbb{R}^2 \ni \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mapsto \begin{pmatrix} \cosh(2\pi t) & -\sinh(2\pi t) \\ -\sinh(2\pi t) & \cosh(2\pi t) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2, \quad t \in \mathbb{R}$$

and the lightlike translations

$$T_s : \mathbb{R}^2 \ni \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 + s \\ \xi_2 + s \end{pmatrix} \in \mathbb{R}^2, \quad s \in \mathbb{R}$$

(cf. [2], Ch. 17, § 2, A), which is the Poincaré group on the light-ray. Furthermore, the commutation relation

$$T_s L_t = L_t T_{e^{2\pi t}s}, \quad s, t \in \mathbb{R}$$

implies that $(T_{s_1}L_{t_1})(T_{s_2}L_{t_2}) = T_{s_1+e^{-2\pi t_1}s_2}(L_{t_1+t_2})$. Therefore, endowing $\mathbb{R}^2$ with the Lie group structure defined by the composition law

$$(s_1, t_1) \cdot (s_2, t_2) = (s_1 + e^{-2\pi t_1}s_2, t_1 + t_2),$$

the mapping $\mathbb{R}^2 \ni (s, t) \mapsto T_s L_t \in \mathcal{P}_\mathbb{R}^1(1)$ becomes a Lie group isomorphism. In particular, $\mathcal{P}_\mathbb{R}^1(1)$ is connected and simply connected. On the other hand, the map $\left(\begin{array}{cc} e^{-2\pi t} & s \\ 0 & 1 \end{array}\right) \mapsto T_s L_t$ is a Lie group isomorphism of the two-dimensional $2 \times 2$ matrix group $\mathcal{G} = \left\{ \left(\begin{array}{cc} e^{-2\pi t} & s \\ 0 & 1 \end{array}\right) \mid s, t \in \mathbb{R}\right\}$ onto $\mathcal{P}_\mathbb{R}^1(1)$. If we identify $\mathcal{P}_\mathbb{R}^1(1)$ with $\mathcal{G}$ along the above isomorphism, the Lie algebra $\mathfrak{p}_\mathbb{R}^1(1)$ of $\mathcal{P}_\mathbb{R}^1(1)$ will be identified with the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$, and the exponential map $\mathfrak{p}_\mathbb{R}^1(1) \to \mathcal{P}_\mathbb{R}^1(1)$ with the exponential map $\mathfrak{g} \to \mathcal{G}$, that is with the usual exponentiation of the matrices belonging to $\mathfrak{g}$. We notice that $\mathfrak{g}$ is the set of all $2 \times 2$ real matrices $X$ such that $\exp(tX) \in \mathcal{G}$, $t \in \mathbb{R}$, and $[X, Y] = XY - YX$ for all $X, Y \in \mathfrak{g}$. The elements

$$X_1 = \begin{pmatrix} -2\pi & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -2\pi & 2\pi \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(2.13)

of $\mathfrak{g} \equiv \mathfrak{p}_\mathbb{R}^1(1)$ are of particular interest: we have

$$X_3 = \frac{1}{2\pi} (X_2 - X_1), \quad [X_2, X_1] = 4\pi^2X_3,$$ (2.14)

any two of $X_1, X_2, X_3$ is a basis for $\mathfrak{g} \equiv \mathfrak{p}_\mathbb{R}^1(1)$ and

$$\exp(tX_j) = \begin{pmatrix} e^{-2\pi t} & 0 \\ 0 & 1 \end{pmatrix}, \quad \left(\begin{array}{cc} e^{-2\pi t} & 1 - e^{-2\pi t} \\ 0 & 1 \end{array}\right), \quad \left(\begin{array}{cc} 1 & t \\ 0 & 1 \end{array}\right)$$

(2.15)

for $j = 1, 2, 3$, respectively.

According to the general theory of unitary representations of Lie groups (see e.g. [2], Ch. 11, § 1, B or [3], Section 10.1), if $\pi$ is an so-continuous unitary representation of $\mathcal{G} \equiv \mathcal{P}_\mathbb{R}^1(1)$ on a Hilbert space $H$ and $\mathcal{D}_{\mathcal{G}}(\pi)$ denotes the Gårding subspace of $H$ for $\pi$, then the formula

$$d\pi(X)\xi = \frac{d}{dt}\pi(\exp(tX))\xi \bigg|_{t=0}, \quad X \in \mathfrak{p}_\mathbb{R}^1(1), \quad \xi \in \mathcal{D}_{\mathcal{G}}(\pi)$$

defines a representation of the Lie algebra $\mathfrak{p}_\mathbb{R}^1(1)$ into the Lie algebra of all skew-symmetric linear mappings $\mathcal{D}_{\mathcal{G}}(\pi) \to \mathcal{D}_{\mathcal{G}}(\pi)$. Moreover, for every $X \in \mathfrak{p}_\mathbb{R}^1(1)$, the
linear mapping \( \text{id} \pi (X) : H \supset D_G(\pi) \rightarrow D_G(\pi) \subset H \) is essentially selfadjoint (see e.g. [2], Ch. 11, §2, Corollary 4 or [30]. Corollary 10.2.11). Therefore, if \( X \in \mathcal{P}_+^1(1) \) and \( A \) is the selfadjoint linear operator in \( H \), then

\[
\pi(\exp(tX)) = \exp(itA) \quad \text{for all } t \in \mathbb{R} \quad \implies \quad \overline{d\pi(X)} = iA \quad (2.16)
\]

(the first exp is the exponential map of the Lie group \( \mathcal{P}_+^1(1) \), while the second one indicates functional calculus). Indeed, by the definition of \( d\pi(X) \) we have \( d\pi(X) \subset iA \), so the selfadjoint operator \( -i\overline{d\pi(X)} \) is contained in the selfadjoint operator \( A \), which implies their equality.

We notice for completeness that, according to [16], Theorem 3.3, the Gårding subspace \( D_G(\pi) \) is actually equal to the set of all \( C^\infty \)-vectors for \( \pi \).

We notice also the following simple fact concerning the essential selfadjointness of sums of symmetric operators: If \( H \) is a Hilbert space, \( \mathcal{D} \subset H \) is a dense linear subspace and \( A, B : \mathcal{D} \rightarrow \mathcal{D} \) are linear operators, then

\[
A, B \text{ symmetric, } A + B \text{ essentially selfadjoint} \implies \overline{A + B} \subset \overline{A + B}. \quad (2.17)
\]

Consequently also \( \overline{A + B} \) is essentially selfadjoint and \( \overline{A + B} = \overline{A + B} \).

To prove (2.14), let \( \eta \in \text{Dom} (A) \cap \text{Dom} (B) \) be arbitrary. Then

\[
(\eta \mid (A + B) \xi) = (\eta \mid A \xi) + (\eta \mid B \xi) = (\overline{A} \eta \mid \xi) + (\overline{B} \eta \mid \xi) = \left( (\overline{A + B}) \eta \mid \xi \right), \quad \xi \in \mathcal{D},
\]

so \( \eta \) is in the domain of \( (A + B)^* = A + B \) and \( (A + B) \eta = (A + B)^* \eta = (\overline{A + B}) \eta \).

(2.14) implies that, for any so-continuous unitary representation \( \pi \) of \( \mathcal{P}_+^1(1) \),

\[
\overline{d\pi(X)} + \overline{d\pi(Y)} = \overline{d\pi(X + Y)}, \quad X, Y \in \mathcal{P}_+^1(1). \quad (2.18)
\]

**Theorem 2.1.** (General Half-sided Modular Inclusion Theorem) Let \( M \) be a von Neumann algebra on a Hilbert space \( H \), in standard form with respect to a normal semifinite faithful weight \( \varphi \) on \( M \), and \( N \subset M \) a von Neumann subalgebra such that the restriction \( \psi \) of \( \varphi \) to \( N \) is semifinite and \( N \) is in standard form with respect to \( \psi \). Let us denote for convenience

\[\Delta_M = \Delta_\varphi, \quad J_M = J_\varphi \quad \text{and} \quad \Delta_N = \Delta_\psi, \quad J_N = J_\psi\]

and assume the following half-sided modular inclusion:

\[
\Delta_M^{it} N \Delta_N^{-it} \subset N, \quad t \leq 0. \quad (2.19)
\]

Then

\[
\frac{1}{2\pi} (\log \Delta_N - \log \Delta_M), \quad (2.20)
\]

defined on the intersection of the domains of \( \log \Delta_N \) and \( \log \Delta_M \), is an essentially selfadjoint operator with positive selfadjoint closure \( P \) and, letting

\[
U(s) = \exp(isP), \quad s \in \mathbb{R}, \quad (2.21)
\]

we have the following:

1. \( \Delta_M^{it} U(s) \Delta_M^{-it} = \Delta_N^{-it} U(s) \Delta_N^{it} = U(e^{2\pi t}s), \quad s, t \in \mathbb{R}; \)
2. \( J_M U(s) J_M = J_N U(s) J_N = U(-s), \quad s \in \mathbb{R}; \)
3. \( U(1 - e^{2\pi t}) = \Delta_N^{-it} \Delta_M^{it} \quad \text{and} \quad \Delta_M^{it} U(1) = \Delta_M^{it} U(1) \Delta_M^{it} U(1)^*; \quad t \in \mathbb{R}; \)
4. \( U(2) = J_N J_M \quad \text{and} \quad J_N = U(1) J_M U(1)^*; \)
5. \( N = U(1) M U(1)^*; \)
6. \( U(s) M U(s)^* \subset M, \quad s \geq 0; \)
Theorem 2.2. Under the assumptions and with the notation as in Theorem 2.1, the following hold:

1. \( \gamma_s(z) = z \) for \( s \geq 0 \) and \( z \in Z(M) \), so \( Z(\gamma_s(M)) = Z(M) \) for all \( s \geq 0 \).
2. There exists the greatest central projection \( p \) of \( M \) satisfying \( M^p = N p \).

For a projection \( e \) in \( M \) we have

\[ e \leq p \iff U(s) e = e \text{ for all } s \in \mathbb{R}, \]

while for a projection \( e \) in \( M^p \) with \( \gamma_s(e) = e, s \geq 0 \), we have even

\[ e \leq p \iff U(s) e J_M e J_M \text{ for all } s \in \mathbb{R}. \]

3. \( M^p \subset \bigcap_{s \geq 0} \gamma_s(M) \Rightarrow \bigcap_{s \geq 0} \gamma_s(M) = \{ x \in M ; \gamma_s(x) = x, s \geq 0 \} \).
(4) \( \mathcal{M}_x \cap M^x = M^x \implies M^x \subseteq \{ x \in M ; \gamma_s(x) = x, s \geq 0 \} \)

\[ \implies M(1_H - p) \text{ and } N(1_H - p) \text{ are of type III} \]

whenever \( p \neq 1_H \).

Remarks. (1) is proved in the case of bounded \( \varphi \) in [9], Theorem 2.4, and for the general case we shall essentially repeat the same proof.

If \( \varphi \) is bounded, then the equality \( \bigcap_{s \geq 0} \gamma_s(M) = \{ x \in M ; \gamma_s(x) = x, s \geq 0 \} \) in (3) follows from [24], Corollary 2.2, but for the proof of (3) in our setting we need a different method.

Finally, for bounded \( \varphi \), the inclusion \( M^\varphi \subseteq \{ x \in M ; \gamma_s(x) = x, s \geq 0 \} \) and the type of \( M(1_H - p) \) and \( N(1_H - p) \) were established in [37] if \( M \) is a factor, and in [8], [9] in the case of a general \( M \). However, there is a gap in the proof in [8], [9]: it is shown only that, for every \( \varepsilon \in M^\varphi \) majorized by \( 1_H - p \), the spectrum of \( \Delta_\varphi \mid \varepsilon H \) is \([0, +\infty)\), while the right proof requires that the spectrum of the modular operator of \( \varphi \), which is \( \Delta_\varphi \mid \varepsilon J_\varphi \varepsilon J_\varphi H \) (see [24]), be equal to \([0, +\infty)\).

We don’t know if the above inclusion still holds without assuming the strong operator density of \( \mathcal{M}_x \cap M^x \) in \( M^\varphi \). We notice that, by (3) and (4) in the above theorem, if \( M^\varphi \subseteq \bigcap_{s \geq 0} \gamma_s(M) \) would hold in general, then we would always have:

\[ M^\varphi \subseteq \{ x \in M ; \gamma_s(x) = x, s \geq 0 \} = \bigcap_{s \geq 0} \gamma_s(M), \]

\[ M(1_H - p), N(1_H - p) \text{ are of type III in the case } p \neq 1_H. \]

(c) The Analytic Extension Theorem.

Let \( \beta \in \mathbb{R}, \beta \neq 0 \). Set

\[ S_\beta = \{ z \in \mathbb{C}; 0 < \beta^{-1} \Im z < 1 \}. \]

\( H^\infty(S_\beta) \) will denote the Banach algebra of all bounded analytic complex functions on \( S_\beta \).

Most parts of the following Analytic Extension Theorem are known, but we shall give a proof for the convenience of the reader:

**Theorem 2.3.** (Analytic Extension Theorem) Let \( A \) and \( B \) be invertible positive selfadjoint linear operators on the Hilbert spaces \( H \) and \( K \) respectively, \( 0 \neq \beta \in \mathbb{R} \), and \( T \in B(K, H) \). Then the next statements (1) – (5) are equivalent:

1. \( \mathbb{R} \ni s \mapsto A^{is} T B^{-i\beta} \) is a uniformly bounded so-continuous extension

\[ \mathcal{S}_\beta \ni z \mapsto T(z) \in B(K, H) \] (2.22)

which is analytic in \( S_\beta \).

2. \( \mathbb{R} \ni s \mapsto A^{is} T B^{-i\beta} \) has a w-continuous extension \( (2.22 \text{ w}) \), which is analytic in \( S_\beta \).

3. There exists a Borel set \( \Xi \subseteq \mathbb{R} \) of non-zero Lebesgue measure such that, for every \( \xi \in K \) and \( \eta \in H \), there exists an \( f_{\xi, \eta} \in H^\infty(S_\beta) \) satisfying

\[ \lim_{0 < t / \beta \to 0} f_{\xi, \eta}(s + it) = (A^{is} T B^{-i\beta} \xi | \eta) \] (2.23)

for almost all \( s \in \Xi \).

4. \( A^{-\beta} T B^\beta \) is defined and bounded on a core of \( B^\beta \).
(5) \( \text{Dom}(A^{-\beta} TB^\beta) = \text{Dom}(B^\beta) \) and \( A^{-\beta} TB^\beta \) is bounded. Moreover, if the above conditions are satisfied, then, for every \( z \in S_\beta \),

\[
\text{Dom}(A^{iz} TB^{-iz}) = \text{Dom}(B^{-iz}), \tag{2.24}
\]

\[
A^{iz} TB^{-iz} \subset T(z), \tag{2.25}
\]

\[
A^{it} T(z) B^{-it} = T(z + t), \quad t \in \mathbb{R}. \tag{2.26}
\]

**Remark.** A somewhat novel feature is the non-null Borel set \( \Xi_0 \) in (3). We shall apply this Theorem in the case \( K = H, T = 1_H, A = \Delta_M, B = \Delta_N \) and \( \beta = -\frac{1}{2} \) in the proof of the Modular Extension Theorem (Theorem 2.12).

We notice that, for \( A \) and \( B \) as in Theorem 2.3, \( R \ni s \mapsto \alpha_s = A^{is} \cdot B^{-is} \) is an so-continuous one-parameter group of linear isometries on \( B(K, H) \). If, for some \( T \in B(K, H) \) and \( z \in \mathbb{C} \), the orbit \( R \ni s \mapsto A^{is} T B^{-is} \) has an so-continuous extension

\[
\overline{S_{2\pi}} \ni \zeta \mapsto T(\zeta) \in B(K, H),
\]

which is analytic in \( S_{2\pi} \), then we say that \( T \) belongs to the domain of \( \alpha_z \) and define \( \alpha_z(T) = T(z) \). According to Theorem 2.3, for \( T \in B(K, H) \) the conditions

- \( T \in \text{Dom}(\alpha_z) \),
- \( A^{iz} TB^{-iz} \) is defined and bounded on a core of \( B^{-iz} \),
- \( \text{Dom}(A^{iz} TB^{-iz}) = \text{Dom}(B^{-iz}) \) and \( A^{iz} TB^{-iz} \) is bounded

are equivalent and, if they are satisfied, then \( \alpha_z(T) = A^{iz} T B^{-iz} \). For a more detailed study of the analytic extension operator \( \alpha_z \), especially in the most relevant case \( z = -i \), when it is called the **analytic generator** of the group \( \alpha \), we refer to [12], [43] and [44].

Let us point out that, in general, the five equivalent statements in Theorem 2.3 are not equivalent with

\( (4') \) \( A^{-\beta} TB^\beta \) is densely defined and bounded.

Indeed:

**Proposition 2.4.** There exist an invertible positive selfadjoint linear operator \( A \) on a Hilbert space \( H \) and a unitary \( v \in B(H) \), such that

\( A^{-1} v A \) is densely defined and bounded, but

\( \text{Dom}(A^{-1} v A) \) is not a core for \( A \).

**Remark.** Actually the proof of Proposition 2.4, which will be given in Section 3, works to prove the next more general statement:

If \( A \) and \( B \) are invertible positive selfadjoint linear operators on a non-zero Hilbert space, such that

\[
A^{it} B^{is} A^{-it} = e^{-its} B^{is}, \quad t, s \in \mathbb{R},
\]

hence \( A^{it} \exp(-B^\pi) A^{-it} = \left( \exp(-B^\pi) \right) e^{-it}, \quad t \in \mathbb{R}, \) and further,

\[
A^{it} b^{is} A^{-it} = b^{is} e^{-it}, \quad t, s \in \mathbb{R},
\]
then, for every $s > 0$,
\[ \text{Dom} (A^{-1} b^{i s} A) = \text{Dom} (A) \text{ and } A^{-1} b^{i s} A \subset b^{-i s}, \]

while
\[ A^{-1} b^{-i s} A \text{ is densely defined and } A^{-1} b^{-i s} A \subset b^{i s}, \text{ but } \]
\[ \text{Dom} (A^{-1} b^{-i s} A) \text{ is not a core of } A. \]

Nevertheless, there are situations in which the above statement (4') is equivalent with the statements (1) – (5) in Theorem 2.3. One of such situations occurs in Lemma 15.15 and Theorem 15.3, namely:

Let $M \subset B(H)$ be a von Neumann algebra, in standard form with respect to a normal semifinite faithful weight $\varphi$ on $M$. Then, for any $a \in \mathfrak{M}_\varphi$, the following are equivalent:

(a) $\Delta_\varphi^{-1/2} a \Delta_\varphi^{1/2}$ is densely defined and $\| \Delta_\varphi^{-1/2} a \Delta_\varphi^{1/2} \| \leq 1$,
(b) $\| \Delta_\varphi^{-1/2} a^* \Delta_\varphi^{-1/2} \| \leq 1$,
(c) $\text{Dom} (\Delta_\varphi^{-1/2} a \Delta_\varphi^{1/2}) = \text{Dom} (\Delta_\varphi^{1/2})$ and $\| \Delta_\varphi^{-1/2} a \Delta_\varphi^{1/2} \| \leq 1$.

Indeed, if (a) holds and $\eta \in \text{Dom} (\Delta_\varphi^{1/2} a^* \Delta_\varphi^{-1/2})$, then
\[ \| (\Delta_\varphi^{-1/2} a^* \Delta_\varphi^{-1/2} / \| \eta \|) (\xi) \| = \| (\eta / \| \eta \|) (\Delta_\varphi^{-1/2} a \Delta_\varphi^{1/2} / \| \xi \|) (\xi) \|, \quad \xi \in \text{Dom} (\Delta_\varphi^{-1/2} a \Delta_\varphi^{1/2}). \]

Since $\text{Dom} (\Delta_\varphi^{-1/2} a \Delta_\varphi^{1/2})$ is dense in $H$, we get $\| \Delta_\varphi^{-1/2} a^* \Delta_\varphi^{-1/2} \| \leq \| \eta \|$.

Next, $\text{Dom} (\Delta_\varphi^{1/2} a^* \Delta_\varphi^{-1/2})$ always contains the core $\{ J_\varphi x \varphi : x \in \mathfrak{A}_\varphi \}$ of $\Delta_\varphi^{-1/2}$. Indeed, if $x \in \mathfrak{A}_\varphi$, then $x a \in \mathfrak{A}_\varphi$ and so $a^* \Delta_\varphi^{-1/2} J_\varphi x \varphi = a^* S_\varphi x \varphi = a^* (x^*) = ((xa)^*)$ belongs to $\text{Dom} S_\varphi = \text{Dom} \Delta_\varphi^{1/2}$.

Consequently, according to Theorem 2.3, (b) implies that $a^* \in \text{Dom} (\sigma_{\text{i/2}}^+(a))$ and $\| \sigma_{\text{i/2}}^+(a^*) \| \leq 1$. But then $a \in \text{Dom} (\sigma_{\text{i/2}}^-(a))$ and $\sigma_{\text{i/2}}^+(a)^* = \sigma_{\text{i/2}}^-(a^*)$, hence $\| \sigma_{\text{i/2}}^-(a) \| = \| \sigma_{\text{i/2}}^+(a^*) \| \leq 1$. Using again Theorem 2.3, we obtain that (c) holds.

Finally, the implication (c) $\Rightarrow$ (a) is trivial.

(d) **Lebesgue continuity, Tomita algebras.**

Let $M \subset B(H)$ be a von Neumann algebra, in standard form with respect to a normal semifinite faithful weight $\varphi$ on $M$.

The next lemma shows that $1_H$ can be approximated by particularly regular elements of $\mathfrak{M}_\varphi$ with respect to the so-topology:

**Lemma 2.5.** There is an increasing net $\{ a_\iota \}$, in $\mathfrak{M}_\varphi \cap M^+$ such that, for any $\iota$, the orbit $\mathbb{R} \ni s \mapsto \sigma_\varphi^z (a_\iota) \in M$ has an entire extension $\mathbb{C} \ni z \mapsto \sigma_\varphi^z (a_\iota)$ and
\[ \sigma_\varphi^z (a_\iota) \in \mathfrak{M}_\varphi, \sigma_\varphi^z (a_\iota)^* = \sigma_\varphi^z (a_\iota), \| \sigma_\varphi^z (a_\iota) \| \leq e^{(3\| \varphi \|)^2} \text{ for all } \iota \text{ and } z \in \mathbb{C}, \]

so $\lim \sigma_\varphi^z (a_\iota) = 1_H$ for all $z \in \mathbb{C}$.

Nets $\{ a_\iota \}$ as in Lemma 2.5 (called in §1, regularizing nets for $\varphi$) will be used to prove the following description of $\mathfrak{M}_\varphi$:

**Lemma 2.6.** (1) For $x \in M$ and $c \geq 0$,
\[ x \in \mathfrak{M}_\varphi \text{ and } \| x \| \leq c \iff \| x J_\varphi y \varphi \| \leq c \| y \| \text{ for all } y \in \mathfrak{M}_\varphi. \]

(2) For $x \in M$ and $\xi \in H$,
\[ x \in \mathfrak{M}_\varphi \text{ and } x \varphi = \xi \iff x J_\varphi y \varphi = J_\varphi y J_\varphi \xi \text{ for all } y \in \mathfrak{M}_\varphi. \]
Using the above lemma, we get immediately

\[ f \in L^1(\mathbb{R}), \quad x \in \mathfrak{N}_\varphi \Rightarrow \begin{cases} \begin{aligned} \left( \omega - \int_R f(t) \sigma^\varphi_t(x) dt \right) y_\varphi &= \int_R f(t) \left( \sigma^\varphi_t(x) J_\varphi y_\varphi \right) dt \\
&= \int_R f(t) \left( J_\varphi y_\varphi \sigma^\varphi_t(x) \right) dt \\
&= J_\varphi y_\varphi \int_R f(t) \Delta^it_\varphi x_\varphi dt, \end{aligned} \end{cases} \tag{2.27} \]

where \( \hat{f} \) is the inverse Fourier transform of \( f \):

\[ \hat{f}(\lambda) = \int_R f(t) e^{i\lambda t} dt, \quad \lambda \in \mathbb{R}. \tag{2.28} \]

Indeed, by \( \omega \) and \( \omega \) we have for every \( y \in \mathfrak{N}_\varphi \)

\[ \left( \omega - \int_R f(t) \sigma^\varphi_t(x) dt \right) J_\varphi y_\varphi = \int_R f(t) \left( \sigma^\varphi_t(x) J_\varphi y_\varphi \right) dt \\
= \int_R f(t) \left( J_\varphi y_\varphi \sigma^\varphi_t(x) \right) dt \\
= J_\varphi y_\varphi \int_R f(t) \Delta^it_\varphi x_\varphi dt, \]

so we can apply Lemma \( \omega \) to \( \omega - \int_R f(t) \sigma^\varphi_t(x) dt \) and \( \int_R f(t) \Delta^it_\varphi x_\varphi dt \).

If \( \varphi \) is bounded, then the linear mapping \( M \ni x \mapsto x_\varphi = x \xi_\varphi \in H \) is bounded, but its inverse is in general not bounded. For unbounded \( \varphi \) even \( x \mapsto x_\varphi \) is not bounded. Nevertheless, both \( \mathfrak{N}_\varphi \ni x \mapsto x_\varphi \in H \) and its inverse have a dominated continuity property with respect to the \( \omega \)-topology on \( M \) and the weak topology on \( H \), called in \[13\], \S 2, Lebesgue continuity. For the proof of Theorem \[2.1\] we need the following variant of \[13\], Section 4.6, Propositions 1 and 2, concerning the Lebesgue continuity of \( x \mapsto x_\varphi \) and \( x_\varphi \mapsto x \):

**Proposition 2.7.** Let \( M \subset B(H) \) be a von Neumann algebra, in standard form with respect to a normal semifinite faithful weight \( \varphi \) on \( M \), and \( \{x_i\}_i \subset \mathfrak{N}_\varphi \) a net.

1. If \( \omega - \lim_i x_i = x \in M \) and \( \sup_i \|x_i\| < \infty \), then \( x \in \mathfrak{N}_\varphi \) and \( (x_i)_\varphi \longrightarrow x_\varphi \) in the weak topology of \( H \).

2. If \( (x_i)_\varphi \longrightarrow \xi \in H \) in the weak topology of \( H \) and \( \sup_i \|x_i\| < \infty \), then there exists \( x \in \mathfrak{N}_\varphi \) such that \( \omega - \lim_i x_i = x \) and \( x_\varphi = \xi \).

Let \( \mathfrak{T}_\varphi \) denote the set of all \( x \in \mathfrak{A}_\varphi \) such that \( \mathbb{R} \ni s \mapsto \sigma^\varphi_s(x) \in M \) has an entire extension \( \mathbb{C} \ni z \mapsto \sigma^\varphi_z(x) \in M \) satisfying \( \sigma^\varphi_s(x) \in \mathfrak{A}_\varphi \) for all \( s \in \mathbb{C} \). Since

\[ x, y \in \mathfrak{T}_\varphi \quad \Rightarrow \quad x y \in \mathfrak{T}_\varphi \quad \text{and} \quad \sigma^\varphi_s(x y) = \sigma^\varphi_s(x) \sigma^\varphi_s(y), \quad z \in \mathbb{C}, \]

\[ x \in \mathfrak{T}_\varphi \quad \Rightarrow \quad x^* \in \mathfrak{T}_\varphi \quad \text{and} \quad \sigma^\varphi_z(x^*) = \sigma^\varphi_z(x)^*, \quad z \in \mathbb{C}, \]

\( \mathfrak{T}_\varphi \) is a \( * \)-subalgebra of \( \mathfrak{A}_\varphi \), called the (maximal) **Tomita algebra** of \( \varphi \).

In the next variant of \[25\], 10.21, Corollary 1, certain standard properties of the Tomita algebra \( \mathfrak{T}_\varphi \) are formulated.

**Proposition 2.8.** Let \( M \subset B(H) \) be a von Neumann algebra, in standard form with respect to a normal semifinite faithful weight \( \varphi \) on \( M \). Then

\[ x \in \mathfrak{T}_\varphi, \quad z \in \mathbb{C} \quad \Rightarrow \quad x_\varphi \in \text{Dom} (\Delta^iz_\varphi) \quad \text{and} \quad \sigma^\varphi_z(x)_\varphi = \Delta^iz_\varphi x_\varphi \tag{2.29} \]
and, for every $y \in \mathfrak{A}_\varphi$, there exists a sequence $\{y_n\}_{n \geq 1}$ in $\mathfrak{T}_\varphi$ such that

\begin{align*}
- & y_n \xrightarrow{s.o.} y \text{ and } y_n^* \xrightarrow{s.o.} y^*, \\
- & (y_n)_{\varphi} \rightarrow y_{\varphi} \text{ and } (y_n^*)_{\varphi} \rightarrow (y^*)_{\varphi} \text{ in the norm-topology of } H, \\
- & \|\sigma(x)(y_n)\| \leq e^{n(3z^2)}\|y\| \text{ for all } n \geq 1 \text{ and } z \in \mathbb{C}, \\
- & \|\Delta^i_{\varphi}(y_n)\| \leq e^{n(3z^2)}\|y\|, \quad \|\Delta^i_{\varphi}(y_n^*)\| \leq e^{n(3z^2)}\|(y^*)_{\varphi}\| \text{ for all } n \geq 1 \text{ and } z \in \mathbb{C}.
\end{align*}

We notice that the set $\mathfrak{S}_\varphi$ of all $x \in \mathfrak{T}_\varphi$, for which

\begin{align*}
\|\sigma(x)\| \leq e^{c(x)3z^2}\|x\|, \quad \|\Delta^i_{\varphi}x_{\varphi}\| \leq e^{c(x)3z^2}\|x_{\varphi}\|, \quad z \in \mathbb{C}
\end{align*}

with $c(x) \geq 0$ a constant depending only on $x$, is a $*$-subalgebra of $\mathfrak{T}_\varphi$ and for every $y \in \mathfrak{A}_\varphi$ there exists a sequence $\{y_n\}_{n \geq 1}$ in $\mathfrak{S}_\varphi$ such that

\begin{align*}
- & y_n \xrightarrow{s.o.} y \text{ and } y_n^* \xrightarrow{s.o.} y^*, \\
- & (y_n)_{\varphi} \rightarrow y_{\varphi} \text{ and } (y_n^*)_{\varphi} \rightarrow (y^*)_{\varphi} \text{ in the norm-topology of } H
\end{align*}

(see [35, 10.22]).

(e) **Hermitian maps.**

Let $H$, $K$ be Hilbert spaces and $M \subset B(H)$, $N \subset B(K)$ von Neumann algebras, in standard form with respect to the normal semifinite faithful weights $\varphi$ on $M$ and $\psi$ on $N$. An essential role will be played by the fixed point real linear subspaces of $K$ and $H$ under $S_\varphi$ and $S_\psi$, respectively:

\begin{align*}
K^{S_\varphi} = \{\xi \in \text{Dom} (S_\psi): S_\psi \xi = \xi\}, \quad H^{S_\varphi} = \{\eta \in \text{Dom} (S_\varphi): S_\varphi \eta = \eta\}.
\end{align*}

They have been used by various authors earlier: see, for example, [29] and [15].

Let us formulate the basic properties, for example, of $K^{S_\varphi}$:

**Lemma 2.9.** (1) $K^{S_\varphi} = \{x_{\psi}: x^* = x \in \mathfrak{M}_\psi\}$.

(2) $\xi \in K$ belongs to $K^{S_\varphi}$ if and only if $(\xi \mid J_\psi x_{\psi}) \in \mathbb{R}$ for all $x^* = x \in \mathfrak{M}_\psi$.

(3) $\xi \in K$ belongs to $K^{S_\psi}$ if and only if $(\xi \mid J_\psi x_{\psi}) = (J_\psi (x^*_{\psi}) \mid \xi)$ for all $x \in \mathfrak{M}_\psi$.

(4) $\text{Dom } S_\psi = K^{S_\varphi} + iK^{S_\varphi}$.

**Definition 2.10.** (1) $T \in B(K,H)$ is said to be Hermitian with respect to the weight pair $(\psi,\varphi)$ if

\begin{align*}
TK^{S_\varphi} \subset H^{S_\varphi}.
\end{align*}

(2) $T \in B(K,H)$ is said to implement $\psi$ in $\varphi$ if

\begin{align*}
x \in \mathfrak{M}_\psi \implies T x T^* \in \mathfrak{M}_\varphi, \quad (T x T^*)_{\varphi} = T x_{\psi}.
\end{align*}

Statement (3) in the next lemma explains why we call the fulfilment of the implication in Definition 2.10 (2) “implementation of $\psi$ in $\varphi$ by $T$”.

**Lemma 2.11.** (1) $T \in B(K,H)$ is Hermitian with respect to $(\psi,\varphi)$ whenever it implements $\psi$ in $\varphi$.

(2) If $T \in B(K,H)$ implements $\psi$ in $\varphi$, then $TN T^* \subset M$.

(3) If an isometric $T \in B(K,H)$ implements $\psi$ in $\varphi$, then $N \ni x \mapsto T x T^* \in M$ is an injective $*$-homomorphism and

\begin{align*}
\psi(a) = \varphi(Ta T^*), \quad 0 \leq a \in \mathfrak{M}_\psi.
\end{align*}
(4) For bounded \( \psi \) and \( \varphi \) and the corresponding cyclic and separating vectors \( \xi_\psi = (1_K)_\psi \) and \( \eta_\varphi = (1_H)_\varphi \), an injective \( T \in B(K, H) \) implements \( \psi \) in \( \varphi \) if and only if

\[
TNT^* \subset M \quad \text{and} \quad T^* \eta_\varphi = \xi_\psi.
\]

The following result provides important criteria for Hermiticity:

**Theorem 2.12.** (Modular Extension Theorem) Let \( M \subset B(H) \), \( N \subset B(K) \) be von Neumann algebras, in standard form with respect to the normal semifinite faithful weights \( \varphi \) on \( M \) and \( \psi \) on \( N \). Then for \( T \in B(K, H) \) the following conditions (1) – (8) are equivalent:

1. \( T \) is Hermitian with respect to \( (\psi, \varphi) \);
2. \( Tx_\psi \in H_{S_\varphi^*} \) for all \( x^* = x \in \mathfrak{M}_\psi \);
3. \( (Tx_\psi | J_\varphi y_\psi ) \in \mathbb{R} \) for all \( x^* = x \in \mathfrak{M}_\psi \) and \( y^* = y \in \mathfrak{M}_\varphi \);
4. For every \( x \in \mathfrak{M}_\psi \) and \( y \in \mathfrak{M}_\varphi \) we have

\[
(Tx_\psi | J_\varphi y_\psi ) = (J_\varphi (y^*)_\varphi | T(x^*)_\psi);
\]
5. \( TS_\psi \subset S_\varphi^* T \);
6. \( \Delta_\varphi^{1/2} T \Delta_\psi^{-1/2} \) is defined on \( \text{Dom } \Delta_\varphi^{-1/2} \) and coincides there with \( J_\varphi T J_\psi \);
7. \( J_\varphi T^* J_\varphi \) is Hermitian with respect to \( (\varphi, \psi) \);
8. (Modular Extension Condition) \( \mathbb{R} \ni s \longmapsto \Delta_\varphi^s T \Delta_\psi^{-i s} \in B(K, H) \) extends to a bounded so-continuous map

\[
\mathbb{S}_{-1/2} \ni z \longmapsto T(z) \in B(K, H),
\]

analytic in \( \mathbb{S}_{-1/2} \) and satisfying

\[
T \left( -\frac{i}{2} \right) = J_\varphi T J_\psi. \tag{2.30}
\]

Moreover, if the above equivalent conditions are satisfied, then, with the notation from the Modular Extension Condition (8), we have

\[
\|T(z)\| \leq \|T\|, \quad z \in \mathbb{S}_{-1/2}, \tag{2.31}
\]

\[
T(z + t) = \Delta_\varphi^t T(z) \Delta_\psi^{-it}, \quad z \in \mathbb{S}_{-1/2}, t \in \mathbb{R}, \tag{2.32}
\]

\[
T \left( s - \frac{i}{2} \right) = J_\varphi T(s) J_\psi, \quad s \in \mathbb{R} \tag{2.33}
\]

and \( T(s) \) is Hermitian with respect to \( (\psi, \varphi) \) for all \( s \in \mathbb{R} \).

(f) Generalization of the structure theorem of Borchers.

Let \( M, N, \varphi, \psi \) be as in the preceding subsection, and \( T \in B(K, H) \) Hermitian with respect to \( (\psi, \varphi) \). Then, by Theorem 2.12, the orbit \( \mathbb{R} \ni s \longmapsto \Delta_\varphi^s T \Delta_\psi^{-i s} \) of \( T \) has a bounded so-continuous extension \( T(\cdot) \) to \( \mathbb{S}_{-1/2} \), analytic in \( \mathbb{S}_{-1/2} \), which satisfies the boundary conditions

\[
T(s) \text{ is Hermitian with respect to } (\psi, \varphi) \text{ for all } s \in \mathbb{R},
\]

\[
J_\varphi T \left( s - \frac{i}{2} \right) J_\psi = T(s) \text{ is Hermitian with respect to } (\psi, \varphi) \text{ for all } s \in \mathbb{R}.
\]
The next extension of a structure theorem of H. J. Borchers ([7], Theorem B, see also [6], Theorem 11.9 and [37], Theorem 2) shows, in particular, that also the converse statement holds, that is any bounded so-continuous map \( \overline{S_{-1/2}} \rightarrow B(K, H) \), which is analytic in \( S_{-1/2} \) and satisfies the above boundary conditions, arises from a Hermitian \( T \in B(K, H) \) as above.

**Theorem 2.13.** (Generalized Structure Theorem) Let \( M \subset B(H) \) and \( N \subset B(K) \) be von Neumann algebras, in standard form with respect to the normal semifinite faithful weights \( \varphi \) on \( M \) and \( \psi \) on \( N \). Further let \( 0 \neq \beta \in \mathbb{R} \), \( \Xi_o \) and \( \Xi_1 \) be Lebesgue null sets in \( \mathbb{R} \), and

\[
\overline{S_{\beta}} \setminus (\Xi_o \cup (\Xi_1 + i\beta)) \ni z \mapsto T(z) \in B(K, H)
\]

be a bounded map which is analytic in \( S_{\beta} \) and satisfies the boundary conditions

(i) \( T(s) \) is Hermitian with respect to \( (\psi, \varphi) \) for all \( s \in \mathbb{R} \setminus \Xi_o \) and

\[
T(s) = wo - \lim_{0 < t/\beta \to 0} T(s + it), \quad s \in \mathbb{R} \setminus \Xi_o,
\]

(ii) \( J_\varphi T(s + i\beta) J_\psi \) is Hermitian with respect to \( (\psi, \varphi) \) for all \( s \in \mathbb{R} \setminus \Xi_1 \) and

\[
T(s + i\beta) = wo - \lim_{1 > t/\beta \to 1} T(s + it), \quad s \in \mathbb{R} \setminus \Xi_1.
\]

Then, for some \( T \in B(K, H) \) which is Hermitian with respect to \( (\psi, \varphi) \),

\[
T(s) = \Delta_{\varphi}^{-i\frac{s}{2\beta}} T \Delta_{\psi}^{i\frac{s}{2\beta}}, \quad s \in \mathbb{R} \setminus \Xi_o.
\]

Hence the given map \( z \mapsto T(z) \) extends to an so-continuous map on the whole \( \overline{S_{\beta}} \) and, with the same notation \( T(\cdot) \) for the extension, it satisfies

\[
T(z + 2\beta t) = \Delta_{\varphi}^{-it} T(z) \Delta_{\psi}^{it}, \quad z \in \overline{S_{\beta}}, \quad t \in \mathbb{R},
\]

\[
T(s + i\beta) = J_\varphi T(s) J_\psi, \quad s \in \mathbb{R}.
\]

**Remark.** Our theorem owes much to Borchers’ work, its proof being based on the main idea of the proof of Theorem B in [7]. Nevertheless, our approach has several features of generality:

(a) \( z \mapsto T(z) \) is not assumed to be so-continuous on the whole \( \overline{S_{\beta}} \), but only the existence of radial limits are assumed almost everywhere on the boundary. In our application to the proof of Theorem 2.13 we shall use Theorem 2.13 with \( \Xi_o = \{0\} \) and \( \Xi_1 = \emptyset \).

(b) We are considering the case of arbitrary normal semifinite faithful weights \( \varphi \) and \( \psi \), without assuming their boundedness.

(c) On the boundary we assume only the Hermiticity of \( T(s) \) and \( J_\varphi T(s + i\beta) J_\psi \) rather than the implementation of \( \psi \) in \( \varphi \) by these operators. The advantage of our assumption consists in its linearity, which allows “mollification”, while Borchers’ assumption is of quadratic nature, more difficult to handle.

(d) Our proof is made more elementary, avoiding most arguments of the two-dimensional complex analysis and using instead of the Malgrange-Zerner Theorem only the elementary Osgood Lemma (the Hartogs Theorem for continuous functions) along with the Morera Theorem (one-dimensional edge-of-the-wedge theorem).

(g) **Complements to the implementation theorem of Borchers.**
Based on the ideas from [1], an invariant subspace theory was developed in [44] for the “bounded analytic” elements associated to an $so$-continuous one-parameter group $(\alpha_t)_{t \in \mathbb{R}}$ of $*$-automorphisms of a von Neumann algebra $M \subset B(H)$. This theory allows, starting with an already existent one-parameter group of unitaries on $H$ which implements $\alpha$, to construct canonically a new implementing group of unitaries on $H$, which has a minimality property and inherits certain properties of the $*$-automorphism group $\alpha$ (see [1], Proposition in Section 3, where the idea is formulated in the realm of a particular situation, and [44], Theorem 5.3, Corollary 5.4, Lemma 5.11 for the general theory).

The above method yields a proof for the one-parameter version of the celebrated implementation theorem of Borchers [5], claiming the innerness of $\alpha$ whenever it is implemented by a one-parameter group of unitaries $(U(s))_{s \in \mathbb{R}}$ having positive generator (see [1], Theorem 3.1 and [44], Corollary 5.7). Moreover, as we shall see in the next theorem, the obtained canonical inner implementing group of unitaries inherits certain commutation properties of the $*$-automorphism group $\alpha$.

We recall that, if $M$ is a von Neumann algebra and $(\alpha_s)_{s \in \mathbb{R}}$ is an $so$-continuous one-parameter group of $*$-automorphisms of $M$, then the spectral subspace of $\alpha$ corresponding to a closed set $F \subset \mathbb{R}$ is defined by

$$M^\alpha(F) = \left\{ x \in M \; ; \; \omega a - \int_{\mathbb{R}} f(s) \alpha_s(x) \, ds = 0 \; \text{if} \; f \in L^1(\mathbb{R}), \; F \cap \text{supp}(\hat{f}) = \emptyset \right\},$$

where $\hat{f}$ denotes the inverse Fourier transform (2.28) of $f$ (see [1], Definition 2.1).

**Theorem 2.14.** Let $M \subset B(H)$ be a von Neumann algebra and $P$ a selfadjoint operator in $H$, such that $P$ is bounded below and $\text{Ad} \, \exp(itP)$ leaves $M$ invariant for all $t \in \mathbb{R}$, defining thus an $so$-continuous one-parameter group $(\alpha_s)_{s \in \mathbb{R}}$ of $*$-automorphisms of $M$. Then there exists a unique injective $b \in M$, $0 \leq b \leq 1_H$, such that

(i) $\alpha_s(x) = b^{-is}x b^{is}$, $s \in \mathbb{R}, x \in M$,

(ii) for any injective $d \in M$, $0 \leq d \leq 1_H$, such that the implementation relation $\alpha_s(x) = d^{-is}x d^{is}$, $s \in \mathbb{R}, x \in M$ holds, we have

$$\chi_{(0,e^\lambda)}(b) \leq \chi_{(0,e^\lambda)}(d), \quad \lambda \in \mathbb{R},$$

where $\chi_{(0,e^\lambda)}$ stands for the characteristic function of $(0,e^\lambda)$.

Moreover,

(iii) for every $\lambda \in \mathbb{R}$, $\chi_{(0,e^\lambda)}(b)$ is the orthogonal projection onto

$$\bigcap_{\mu > \lambda} \text{the closed linear span of } M^\alpha([-\mu, +\infty)) H,$$

(iv) for any $*$-automorphism $\sigma$ of $M$ and $\lambda_\sigma > 0$, such that $\sigma \circ \alpha_s = \alpha_{\lambda_\sigma s} \circ \sigma$ for all $s \in \mathbb{R}$, we have $\sigma(b) = b^{\lambda_\sigma}$.

The above theorem will be used in the proof of Theorem 2.2.

(h) **Summary of the remaining part of the paper.**

The remainder of this paper presents proofs for the above results:

- Theorem 2.13 and Proposition 2.4 in Section 3,
- Lemma 2.2, Lemma 2.6, Proposition 2.7 and Proposition 2.8 in Section 4,
The aim of this section is to prove Theorem 2.3 and Proposition 2.4.

Proof of Theorem 2.3

The equivalence of conditions (2), (4) and (5), as well as the three additional statements 2.24, 2.25, 2.26 were proved in [12], Theorem 6.2.

For the proof of the remaining part, we introduce the following notation: let $K_c(B)$ and $H_c(A)$ be the set of all vectors $\xi \in K$ and $\eta \in H$, respectively, with compact spectral support for $\log B$ and $\log A$, respectively. For such $\xi$ and $\eta$, $\mathbb{C} \ni z \mapsto B^{iz} \xi$ and $\mathbb{C} \ni z \mapsto A^{iz} \eta$ are analytic functions of exponential type with respect to $\text{Im} z$ and they are uniformly bounded in $\overline{S_\beta}$. Furthermore, $K_c(B) \subset K$ and $H_c(A) \subset H$ are dense linear subspaces and they are cores of $B^{iz}$ and $A^{iz}$ for every $z \in \mathbb{C}$, respectively.

Proof of (2) $\Rightarrow$ (1). The uniform boundedness of $\overline{S_\beta} \ni z \mapsto T(z)$ and its so-contiuity are to be proved. The latter is automatic on $\overline{S_\beta}$, where $T(\cdot)$ is analytic.

Let $\xi \in K_c(B)$ and $\eta \in H_c(A)$. Then

$$
(T(z) \xi | \eta) = (TB^{-iz} \xi | A^{-iz} \eta), \quad z \in \overline{S_\beta},
$$

because the analytic function $\mathbb{C} \ni z \mapsto (TB^{-iz} \xi | A^{-iz} \eta)$ and the continuous function $\overline{S_\beta} \ni z \mapsto (T(z) \xi | \eta)$, which is analytic in the interior, coincide on $\mathbb{R}$. By Lemma 2.24 and by the density of $H_c(A)$ in $H$, it follows that

$$
T(z) \xi = A^{iz} B^{-iz} \xi, \quad z \in \overline{S_\beta}, \quad \xi \in K_c(B).
$$

Since $\|A^{iz} B^{-iz}\| = \|T\|$ for $z \in \mathbb{R}$ and $\|A^{iz} T B^{-iz}\| = \|A^{-\beta} T A^\beta\|$ for $z \in \mathbb{R} + i \beta$ by (5), we have by the Three Line Theorem

$$
\|T(z) \xi | \eta\| \leq \max \left( \|T\|, \|A^{-\beta} T A^\beta\| \right) \|\xi\| \|\eta\|, \quad z \in \overline{S_\beta},
$$

obtaining thus the uniform boundedness of $\overline{S_\beta} \ni z \mapsto T(z)$.

Due to this uniform boundedness, it suffices to prove the convergences

$$
\lim_{\overline{S_\beta} \ni z \to s} \|T(z) \xi - A^{is} B^{-is} \xi\| = 0, \quad s \in \mathbb{R},
$$

$$
\lim_{\overline{S_\beta} \ni z \to s + i \beta} \|T(z) \xi - A^{is-\beta} B^{-is+\beta} \xi\| = 0, \quad s \in \mathbb{R}
$$

for $\xi \in K_c(B)$. We give a proof explicitly only for $\beta > 0$, the treatment of the case $\beta < 0$ being completely similar.

Let $E$ denote the spectral projection of $\log A$ corresponding to $(-\infty,0]$. Owing to (3.1) we can split $T(z) \xi$ as follows:

$$
T(z) = (A^{iz} (1_H - E)) T B^{-iz} \xi + (A^{iz+\beta} E)(A^{-\beta} T B^\beta) B^{-iz-\beta} \xi, \quad z \in \overline{S_\beta}.
$$

We note that

$$
A^{iz} (1_H - E) \text{ is defined on } H \text{ and } \|A^{iz} (1_H - E)\| \leq 1 \text{ for all } z \in \mathbb{C}, \text{ Im } z \geq 0,
$$
$A^i z + \beta E$ is defined on $H$ and $\|A^i z + \beta E\| \leq 1$ for all $z \in \mathbb{C}$, $\text{Im} \ z \leq \beta$.

Now the norm-continuity of $S_\beta \ni z \mapsto B^{-iz} \xi$ and $S_\beta \ni z \mapsto B^{-i z - \beta} \xi$, the so-
continuity of $\overline{S}_\beta \ni z \mapsto A^i z (1_H - E)$ and $\overline{S}_\beta \ni z \mapsto A^i z + \beta E$, and the boundedness
of $A^{-i} T B^\beta$ on $K_c(B)$ yield the convergences (3.2) and (3.3).

**Proof of (1) ⇒ (3):** obvious, with $\Xi_o = \mathbb{R}$.

**Proof of (3) ⇒ (4).** We proceed in three steps.

**Step 1.** First we quote some results from the theory of the Hardy spaces on the
disc. Let $H^\infty(\mathbb{D})$ be the Banach algebra of all bounded analytic complex functions
on the unit disc

$$\mathbb{D} = \{ z \in \mathbb{C}; |z| < 1 \}.$$

Any $g \in H^\infty(\mathbb{D})$ has a non-tangential limit

$$\tilde{g}(\zeta) = \lim_{\mathbb{D} \ni z \rightarrow \zeta} g(z)$$

for almost all $\zeta$ in the boundary $\partial \mathbb{D}$ of $\mathbb{D}$ (the unit circle) due to Fatou’s Theorem
(see, for example, the second Corollary on page 38 of [20] or the theorems on pages
5 and 14 of [21]). Furthermore, the map $H^\infty(\mathbb{D}) \ni g \mapsto \tilde{g} \in L^\infty(\partial \mathbb{D})$ obtained
this way is an isometric algebra homomorphism. On the other hand, the range
$\{ \tilde{g}; g \in H^\infty(\mathbb{D}) \}$ of the above homomorphism is equal to

$$\{ \psi \in L^\infty(\partial \mathbb{D}); \int_0^{2\pi} \psi(e^{i\theta}) e^{ik\theta} \, ds = 0 \text{ for all } k = 1, 2, \ldots \}$$

and hence it is weak* closed (see, for example, [21], §20.1). We notice also that,
according to a uniqueness theorem of the Riesz brothers (see, for example, the
second Corollary on page 52 of [20] or the Theorem on page 76 of [21]), if for some
$g \in H^\infty(\mathbb{D})$ the boundary function $\tilde{g}$ vanishes almost everywhere on a Borel subset
of $\partial \mathbb{D}$ with non-zero arc length measure, then $g = 0$.

We consider the one point compactification of the right half and the left half
of $S_\beta$ and denote each added point by $+\infty$ and $-\infty$, respectively. We extend the function

$$\mathbb{D} \setminus \{ +1, -1 \} \ni \zeta \mapsto \Phi_\beta(\zeta) = \frac{\beta}{\pi} \log \left( \frac{1 + \zeta}{1 - \zeta} \right) \in \overline{S}_\beta$$

to be $+\infty$ at $\zeta = +1$ and $-\infty$ at $\zeta = -1$. Then the extended function

$$\overline{\mathbb{D}} \ni \zeta \mapsto \Phi_\beta(\zeta) = \frac{\beta}{\pi} \log \left( \frac{1 + \zeta}{1 - \zeta} \right) \in \overline{S}_\beta \cup \{-\infty, +\infty\}$$

is a homeomorphism, mapping $\mathbb{D}$ onto $S_\beta$ conformally, and the boundary $\partial \mathbb{D}$ onto
$\partial S_\beta \cup \{-\infty, +\infty\}$ absolutely bicontinuously with respect to the arc length measures: if $\Xi$ is a Borel set in $\partial \mathbb{D}$, then $\Xi$ has arc length measure 0 if and only if
$\Phi_\beta(\Xi)$ has arc length measure 0. Moreover, $\Phi_\beta$ maps paths in $\mathbb{D}$ tending to a
$\zeta \in \partial \mathbb{D} \setminus \{ +1, -1 \}$ from within a sector of opening $< \pi$ having vertex at $\zeta$, and
symmetric about the inner normal to $\partial \mathbb{D}$ in $\zeta$, to paths tending to $\Phi_\beta(\zeta) \in \partial S_\beta$ in
a similar non-tangential way. Therefore, if $f \in H^\infty(S_\beta)$, the non-tangential limit

$$\tilde{f}(\zeta) = \lim_{\mathbb{D} \ni z \rightarrow \zeta} f(z)$$

eexists for almost all $\zeta$ in $\partial S_\beta$ by Fatou’s Theorem applied to $f \circ \Phi_\beta$. Similarly
we can transcribe the above quoted results concerning $H^\infty(\mathbb{D})$ in the setting of
$H^\infty(S_\beta): H^\infty(S_\beta) \ni g \mapsto \tilde{g} \in L^\infty(\partial S_\beta)$ is an isometric algebra homomorphism


with weak* closed range in $L^\infty(\partial S_\beta)$ and $f \in H^\infty(S_\beta)$ is equal to zero whenever $\tilde{f}$ vanishes almost everywhere on a Borel subset of $\partial S_\beta$ with non-zero arc length measure.

Step 2. We consider the map

$$F : K \times H \ni (\xi, \eta) \mapsto \tilde{f}_{\xi, \eta} \in \{\tilde{f} : f \in H^\infty(S_\beta)\} \subset L^\infty(\partial S_\beta),$$

where, as noticed in Step 1, $\{\tilde{f} : f \in H^\infty(S_\beta)\}$ is a weak* closed subalgebra of $L^\infty(\partial S_\beta)$. The function $f_{\xi, \eta}$ is uniquely determined by (2.23) due to the uniqueness result quoted in Step 1. Since the right hand side of (2.23) is sesquilinear in $\xi$ and $\eta$, the mapping $F$ is also sesquilinear. We shall prove in this step that $F$ is bounded.

We first prove that the graph of $F$ is closed. Suppose that $\xi_n \to \xi_o$, $\eta_n \to \eta_o$ and $\tilde{f}_{\xi_n, \eta_n} \to \tilde{f}_o$ with respect to the norm of $L^\infty(\partial S_\beta)$, hence also $f_{\xi_n, \eta_n} \to f_o$ uniformly. By the continuity of the right hand side of (2.23) in $\xi$ and $\eta$, $\tilde{f}_o$ has to satisfy (2.23) for $\xi = \xi_o$ and $\eta = \eta_o$ almost everywhere on the set $\Sigma_o$. Therefore $\tilde{f}_o = \tilde{f}_{\xi_o, \eta_o}$, again by the uniqueness theorem of the Riesz brothers, proving that the graph of $F$ is closed.

Let us consider $F(\xi, \cdot)$ for a fixed $\xi$. Denote by $\overline{H}$ the conjugate of the Hilbert space $H$ and by $\overline{\eta}$ the canonical image of $\eta \in H$ in $\overline{H}$. By the above proved closedness of the graph of $F$, the graph of the linear map $\overline{H} \ni \overline{\eta} \mapsto F(\xi, \overline{\eta}) \in L^\infty(\partial S_\beta)$ is closed and hence, by the Closed Graph Theorem,

$$\|F(\xi, \eta)\| \leq c_\xi \|\eta\|, \quad \eta \in H$$

for some constant $c_\xi \geq 0$ depending on $\xi$. Thus $F(\xi, \cdot) : \overline{H} \to L^\infty(\partial S_\beta)$ is bounded.

Now we prove that the graph of the linear map

$$K \ni \xi \mapsto F(\xi, \cdot) \in B(\overline{H}, L^\infty(\partial S_\beta))$$

is closed. Suppose that $\xi_n \to \xi_o$ and $F(\xi_n, \cdot) \to T_o$ with respect to the norm of $B(\overline{H}, L^\infty(\partial S_\beta))$. Then, for every $\eta \in H$, $F(\xi_n, \eta) \to T_o \eta$ and by the closedness of the graph of $F$ it follows that $T_o \eta = F(\xi_o, \eta)$. Thus $T_o = F(\xi_o, \cdot)$.

By the Closed Graph Theorem

$$\|F(\xi, \cdot)\| \leq c \|\xi\|, \quad \xi \in K$$

for some constant $c \geq 0$, so

$$\|F(\xi, \eta)\| = \text{ess sup} \|\tilde{f}_{\xi, \eta}(\xi)\| = \sup_{z \in S_\beta} |f_{\xi, \eta}(z)| \leq c \|\xi\| \|\eta\|, \quad \xi \in K, \eta \in H.$$

Step 3. We now take $\xi \in K_c(B)$, $\eta \in H_c(A)$. Then

$$\mathbb{C} \ni z \mapsto g_{\xi, \eta}(z) = (TB^{-i\xi} \xi | A^{-i\eta} \eta)$$

is an entire function satisfying the boundary condition (2.23), so that $g_{\xi, \eta} = f_{\xi, \eta}$ by the uniqueness theorem of the Riesz brothers. Therefore

$$|g_{\xi, \eta}(z)| \leq c \|\xi\| \|\eta\|, \quad z \in S_\beta$$

and hence the same estimate holds for all $z \in \overline{S_\beta}$ by continuity. This implies

$$TB^{-i\xi} \xi \in \text{Dom}(A^{iz})$$

because $H_c(A)$ is a core of $A^{iz}$, and

$$\|A^{iz} TB^{-i\xi} \xi\| \leq c \|\xi\|, \quad z \in \overline{S_\beta}.$$
Since $K_\nu(B)$ is a core of $B^{-iz}$, setting $z = i\beta$ we obtain (4).

\textbf{Proof of Proposition 2.4}

Let us denote:

by $\lambda_t$ the translation operator $\xi \mapsto \xi(\cdot - t)$ on $L^2(\mathbb{R})$, $t \in \mathbb{R}$,

by $\alpha_t$ the $*$-automorphism $\text{Ad}(\lambda_t)$ of $B(H)$, $t \in \mathbb{R}$, and

by $b$ the multiplication operator with $e^{-ct}$ on $L^2(\mathbb{R})$.

Clearly, $(\alpha_t)_{t \in \mathbb{R}}$ is an so-continuous one-parameter group of $*$-automorphisms of $B(H)$, $0 \leq b \leq 1_{L^2(\mathbb{R})}$ and $b$ is injective. Since

$$\alpha_t(b) = \lambda_t b \lambda^*_t = b e^{-\pi z}, \quad t \in \mathbb{R},$$

we have

$$\alpha_t(b^s) = \lambda_t b^s \lambda^*_t = b^{2\pi z}, \quad t, s \in \mathbb{R}. \tag{3.4}$$

By the Stone Representation Theorem there exists an invertible positive selfadjoint linear operator $A$ on $L^2(\mathbb{R})$ such that $\lambda_t = A^t$, $t \in \mathbb{R}$. Then

$$\alpha_t = \text{Ad}(A^t), \quad t \in \mathbb{R}. \tag{3.5}$$

Let $s > 0$ be arbitrary. Since $0 \leq b \leq 1_{L^2(\mathbb{R})}$ and $\Re(ise^{-\pi z}) = se^{-\pi z} \sin(\pi z)$, the complex power $b^{ise^{-\pi z}} \in B(L^2(\mathbb{R}))$ is defined and $\|b^{ise^{-\pi z}}\| \leq 1$ for every $z$ in the closed strip $\mathbb{S}_1$. Using (3.4), it is easily seen that

$$F_1: \mathbb{S}_1 \ni z \mapsto b^{ise^{-\pi z}} \in B(L^2(\mathbb{R}))$$

is an so-continuous extension of $\mathbb{R} \ni t \mapsto \alpha_t(b^s)$, which is analytic in $\mathbb{S}_1$ and whose value at $i$ is $b^{is}$. Taking into account (3.5), Theorem 2.3 yields that

$$\text{Dom}(A^{-1}b^{is}A) = \text{Dom}(A)$$

that is $A^{-1}b^{is}A = b^{is} | \text{Dom}(A)$. Therefore

$$\text{Dom}(A^{-1}b^{is}A) = b^{is} | \text{Dom}(A)$$

is dense in $H$ and $A^{-1}b^{is}A \subset b^{is}$. \tag{3.6}

But

$$\text{Dom}(A^{-1}b^{is}A)$$

is not a core of $A$. \tag{3.7}

Indeed, assuming that $\text{Dom}(A^{-1}b^{is}A)$ is a core of $A$, (3.6) and Theorem 2.3 imply that $\mathbb{R} \ni t \mapsto \alpha_t(b^{is})$ has a uniformly bounded so-continuous extension

$$F_2: \mathbb{S}_1 \ni z \mapsto F_2(\overline{z})^* \in B(L^2(\mathbb{R})),$$

which is analytic in $\mathbb{S}_1$ and whose value at $i$ is $b^{is}$. Then

$$\mathbb{S}_{-1} \ni z \mapsto F_2(\overline{z})^* \in B(L^2(\mathbb{R}))$$

is an so-continuous extension of $\mathbb{R} \ni t \mapsto \alpha_t(b^{is})$, which is analytic in $\mathbb{S}_{-1}$ and whose value at $-i$ is $b^{-is}$. Consequently, $\mathbb{R} \ni t \mapsto \alpha_t(b^{is})$ has a uniformly bounded so-continuous extension

$$F: \mathbb{S}_1 \cup \mathbb{S}_{-1} \ni z \mapsto \begin{cases} F_1(z) & \text{if } \Im z \geq 0 \\ F_2(\overline{z})^* & \text{if } \Im z \leq 0 \end{cases},$$

which is analytic in the interior and which takes the same value $b^{-is}$ at $i$ and $-i$. Then, by (2.20), $F$ is periodic of period $2i$, so it extends to a uniformly bounded entire mapping, which must be constant by the Liouville Theorem. Thus the orbit $\mathbb{R} \ni t \mapsto \alpha_t(b^{is})$ is constant, that is $b^{is}$ commutes with every $\lambda_t$. Since $b^{is}$ is
the multiplication operator with \( \mathbb{R} \ni r \mapsto e^{-isr^2} \) on \( L^2(\mathbb{R}) \), this means that the above function is constant, what is plainly not true.

By \( \text{(3.6)} \) and by \( \text{(3.7)} \) we conclude that, choosing \( v = b^{-is} \) with \( s > 0 \), \( A^{-1}vA \) is densely defined and bounded, but \( \text{Dom}(A^{-1}vA) \) is not a core for \( A \).

\[ \square \]

4. Lebesgue Continuity, Tomita Algebras

In this section we prove Lemmas 2.5 and 2.6, as well as Propositions 2.7 and 2.8. Throughout this section \( M \subset B(H) \) will stand for a von Neumann algebra, in standard form with respect to a normal semifinite faithful weight \( \varphi \) on \( M \).

Proof of Lemma 2.5

Since \( M_\varphi \) is a hereditary \(*\)-subalgebra of \( M \), there is an increasing approximate unit \( \{b_\iota\} \), for \( M_\varphi \) (for example, the upward directed set \( \{ b \in M_\varphi \cap M^+; \|b\| < 1 \} \), labeled by itself). Then, by the so-density of \( M_\varphi \) in \( M \), we have

\[ \text{so-lim}_\iota b_\iota = 1 \_H \]

Setting

\[ a_\iota = \frac{1}{\sqrt{\pi}} wo - \int_{-\infty}^{\infty} e^{-t^2} \sigma^\varphi_\iota (b_\iota) \, dt \]

\( \{a_\iota\}_\iota \) is an increasing net in \( M^+ \) such that every orbit

\[ \mathbb{R} \ni s \mapsto \sigma^\varphi_s (a_\iota) = \frac{1}{\sqrt{\pi}} wo - \int_{-\infty}^{\infty} e^{-(t-s)^2} \sigma^\varphi_\iota (b_\iota) \, dt \in M \]

has an entire extension

\[ \mathbb{C} \ni z \mapsto \sigma^\varphi_z (a_\iota) = \frac{1}{\sqrt{\pi}} wo - \int_{-\infty}^{\infty} e^{-(t-z)^2} \sigma^\varphi_\iota (b_\iota) \, dt \in M \]

Clearly, \( \sigma^\varphi_\iota (a_\iota)^* = \sigma^\varphi_{-\iota} (a_\iota) \) for all \( \iota \) and \( z \in \mathbb{C} \). Since, for every \( z \in \mathbb{C} \), the function

\[ \mathbb{R} \ni t \mapsto e^{-(t-z)^2} = e^{-(t-\Re z)^2 + (\Im z)^2} e^{2i(t-\Re z) \Im z} \]

is of the form \( f_1 - f_2 + i (f_3 - f_4) \) with \( 0 \leq f_j \in L^1(\mathbb{R}) \), \( 1 \leq j \leq 4 \), using \( \text{(2.6)} \) we deduce easily that

\[ \sigma^\varphi_\iota (a_\iota) \in \mathfrak{M}_\varphi \text{ and } \|\sigma^\varphi_\iota (a_\iota)\| \leq e^{(\Im z)^2} \text{ for all } \iota \text{ and } z \in \mathbb{C} \]

On the other hand, \( \text{so-lim}_\iota b_\iota = 1 \_H \) yields

\[ \text{so-lim}_\iota \sigma^\varphi_\iota (a_\iota) = 1 \_H \text{ for all } z \in \mathbb{C} \]

\[ \square \]

Proof of Lemma 2.6

First we prove that

\[ y \in F_\varphi \cap \text{Dom}(\sigma^\varphi_-) \text{, } \sigma^\varphi_- (y) \in F_\varphi \implies \sigma^\varphi_- (y)_\varphi = J_\varphi (y^*)_\varphi \]  \hspace{1cm} \text{(4.1)}

For let \( x \in F_\varphi \) be arbitrary. Then \( \text{(2.5)} \) yields

\[ \sigma^\varphi_- (y) J_\varphi x_\varphi = J_\varphi x J_\varphi \sigma^\varphi_- (y)_\varphi \]  \hspace{1cm} \text{(4.2)}
Theorem 2.3 we obtain

\[ \sigma_{-\frac{1}{2}}^\varphi(x) = \Delta_{-\frac{1}{2}}^\varphi(x) \in \text{Dom}(\Delta_{-\frac{1}{2}}^\varphi); \]

using Theorem 2.3 we obtain

\[ \sigma_{-\frac{1}{2}}^\varphi(y) J_\varphi x = \Delta_{-\frac{1}{2}}^\varphi y J_\varphi x = \Delta_{-\frac{1}{2}}^\varphi (y x^*) = J_\varphi S_\varphi(y x^*) = J_\varphi x(y^*) \] (4.3)

Now, (4.2) and (4.3) imply

\[ J_\varphi x J_\varphi \sigma_{-\frac{1}{2}}^\varphi(y) = J_\varphi x(y^*)_\varphi, \quad x \in \mathfrak{M}_\varphi \]

and by the so-density of \( \mathfrak{M}_\varphi \) in \( M \ni 1_H \) we conclude that \( \sigma_{-\frac{1}{2}}^\varphi(y) = J_\varphi (y^*)_\varphi \).

(1) If \( x \in \mathfrak{M}_\varphi \), then by (4.2)

\[ \| x J_\varphi y \varphi \| = \| J_\varphi y J_\varphi x \varphi \| \leq \| y \| \| x \varphi \|, \quad y \in \mathfrak{M}_\varphi \supset M_\varphi. \]

Conversely, assume that \( x \in M \) and \( c \geq 0 \) are such that

\[ \| x J_\varphi y \varphi \| \leq c \| y \|, \quad y \in \mathfrak{M}_\varphi. \] (4.4)

Let \( \{ a_i \} \) be a net as in Lemma 2.5. Then we have for every \( \iota \)

\[ \| \sigma_{-\frac{1}{2}}^\varphi(a_i)^* x^* x \sigma_{-\frac{1}{2}}^\varphi(a_i) \| \leq \| x \| \| \sigma_{-\frac{1}{2}}^\varphi(a_i) \|^2 \leq \| x \|^2 e^{1/2} \]

and, according to (4.1) and (4.2),

\[ \varphi(\sigma_{-\frac{1}{2}}^\varphi(a_i)^* x^* x \sigma_{-\frac{1}{2}}^\varphi(a_i)) = \| x \| \sigma_{-\frac{1}{2}}^\varphi(a_i) \|^2 = \| x \| J_\varphi(a_i)_\varphi \|^2 \leq c^2 \| a_i \|^2 \leq c^2. \]

Since \( \sigma_{-\frac{1}{2}}^\varphi(a_i)^* x^* x \sigma_{-\frac{1}{2}}^\varphi(a_i) = \sigma_{-\frac{1}{2}}^\varphi(a_i)^* x^* x \sigma_{-\frac{1}{2}}^\varphi(a_i) \to x^* x \) in the so-topology and \( \varphi \) is lower \( \omega \)-semi-continuous on the bounded subsets of \( M^+ \), it follows that

\[ \varphi(x^* x) \leq c^2, \text{ that is } x \in \mathfrak{M}_\varphi \text{ and } \| x \varphi \| \leq c. \]

(2) Since the implication \( \Longrightarrow \) is an immediate consequence of (4.5), we have to prove only the converse implication. For let \( x \in M \) and \( \xi \in H \) be such that

\[ x J_\varphi y \varphi = J_\varphi y J_\varphi \xi, \quad y \in \mathfrak{M}_\varphi. \] (4.5)

Then (4.1) holds with \( c = \| \xi \| \), so by the above part of the proof we have \( x \in \mathfrak{M}_\varphi \). But then (4.2) and (4.3) yield

\[ J_\varphi y J_\varphi x \varphi = x J_\varphi y \varphi = J_\varphi y J_\varphi \xi, \]

so by the so-density of \( \mathfrak{M}_\varphi \) in \( M \ni 1_H \) we conclude that \( x \varphi = \xi \).

\[ \square \]

Proof of Proposition 2.7

(1) Let \( \mathcal{D} \) be the linear span of \( \{ J_\varphi a^* J_\varphi b_\varphi : a, b \in \mathfrak{M}_\varphi \} \), which is dense in \( H \). Define the linear functional \( F : \mathcal{D} \to \mathbb{C} \) by

\[ F(\eta) = \lim_i (\eta \mid (x_i)_\varphi), \quad \eta \in \mathcal{D}, \]

where the limit exists due to the convergence

\[ (J_\varphi a^* J_\varphi b_\varphi \mid (x_i)_\varphi) = (b_\varphi \mid J_\varphi a J_\varphi (x_i)_\varphi) \xrightarrow{2.8} (b_\varphi \mid x J_\varphi a_\varphi). \] (4.6)

Since \( F \) is bounded by

\[ \| F \| \leq \sup_i \| (x_i)_\varphi \| < \infty, \]
it extends to a continuous linear functional on $H$, and hence there exists $\xi \in H$ satisfying $F(\eta) = (\eta | \xi)$ for all $\eta \in \mathcal{D}$. In particular, by (4.6),

\[(b_{\varphi} | x J_{\varphi} a_{\varphi}) = F(J_{\varphi} a_{\varphi}^* J_{\varphi} b_{\varphi}) = (J_{\varphi} a_{\varphi}^* J_{\varphi} b_{\varphi} | \xi) = (b_{\varphi} | J_{\varphi} a_{\varphi} \xi), \quad a, b \in \mathcal{N}_{\varphi}.
\]

This implies that $x J_{\varphi} a_{\varphi} = J_{\varphi} a_{\varphi} J_{\varphi} \xi$ for all $a \in \mathcal{N}_{\varphi}$ and by Lemma 2.6(2) we get

\[x \in \mathcal{N}_{\varphi} \text{ and } x_{\varphi} = \xi.
\]

Furthermore,

\[
\lim_i (\eta | (x_i)_{\varphi}) = F(\eta) = (\eta | \xi) = (\eta | x_{\varphi}), \quad \eta \in \mathcal{D},
\]

the density of $\mathcal{D}$ in $H$ and the boundedness of the net $\{(x_i)_{\varphi}\}_i$, yield that

\[(x_i)_{\varphi} \xrightarrow{\text{wo}} x_{\varphi} \text{ in the weak topology of } H.
\]

(2) Let $x \in M$ be any wo-limit point of the bounded net $\{x_i\}_i$. Then, for every $a \in \mathcal{N}_{\varphi}$, $x J_{\varphi} a_{\varphi}$ is a weak limit point of the net $\{x_i J_{\varphi} a_{\varphi}\}_i$ by Lemma 2.6(2). Set

\[x J_{\varphi} a_{\varphi} = J_{\varphi} a_{\varphi} J_{\varphi} \xi, \quad a \in \mathcal{N}_{\varphi}
\]

and, using Lemma 2.6(2), we obtain

\[x \in \mathcal{N}_{\varphi} \text{ and } x_{\varphi} = \xi.
\]

By the injectivity of the mapping $\mathcal{N}_{\varphi} \ni y \mapsto y_{\varphi}$, the uniqueness of the wo-limit point $x$ of $\{x_i\}_i$ follows and we conclude that $\text{wo-} \lim_i x_i = x$.

\[\square\]

**Proof of Proposition 2.8**

First we show that every $y \in \mathcal{N}_{\varphi}$ can be approximated by a sequence $\{y_n\}_{n \geq 1}$ in $\mathcal{N}_{\varphi}$ as required in the statement and such that (2.24) holds for $x = y_n$, $n \geq 1$.

Set

\[y_n = \sqrt{\frac{n}{\pi}} \text{wo} - \int_{-\infty}^{\infty} e^{-n t^2} \sigma_1^\varphi(y) \, dt, \quad n \geq 1.
\]

Then every orbit

\[\mathbb{R} \ni s \mapsto \sigma_1^\varphi(y_n) = \sqrt{\frac{n}{\pi}} \text{wo} - \int_{-\infty}^{\infty} e^{-n (t-s)^2} \sigma_1^\varphi(y) \, dt \in M
\]

has the entire extension

\[\mathbb{C} \ni z \mapsto \sigma_z^\varphi(y_n) = \sqrt{\frac{n}{\pi}} \text{wo} - \int_{-\infty}^{\infty} e^{-n (t-z)^2} \sigma_z^\varphi(y) \, dt \in M
\]

and by (2.24) we have $\sigma_z^\varphi(y_n) \in \mathcal{N}_{\varphi}$ for every $z \in \mathbb{C}$. Similarly,

\[\mathbb{R} \ni s \mapsto \sigma_z^\varphi(y_n^*) = \sqrt{\frac{n}{\pi}} \text{wo} - \int_{-\infty}^{\infty} e^{-n (t-z)^2} \sigma_z^\varphi(y^*) \, dt \in M
\]

has an entire extension $\mathbb{C} \ni z \mapsto \sigma_z^\varphi(y_n^*)$ and $\sigma_z^\varphi(y_n^*) \in \mathcal{N}_{\varphi}$ for every $z \in \mathbb{C}$. Since $\sigma_z^\varphi(y_n^*) = \sigma_z^\varphi(y_n^*)$, we have

\[\sigma_z^\varphi(y_n) \in (\mathcal{N}_{\varphi})^* \cap \mathcal{N}_{\varphi} = \mathcal{N}_{\varphi}, \quad n \geq 1, z \in \mathbb{C},
\]

that is $y_n \in \mathcal{N}_{\varphi}$ for all $n \geq 1$. By the so-continuity of $\mathbb{R} \ni t \mapsto \sigma_t^\varphi(y) \in M$ and $\mathbb{R} \ni t \mapsto \sigma_t^\varphi(y^*) \in M$ we get $y_n \xrightarrow{s_0} y$ and $y_n^* \xrightarrow{s_0} y^*$, while using

\[|e^{-n(t-z)^2}| = |e^{-n(t-\mathbb{R})^2} e^{n(\mathbb{Z})^2} e^{2 n i (t-\mathbb{R}) \mathbb{Z}}| = e^{-n(t-\mathbb{R})^2} e^{n(\mathbb{Z})^2}
\]

(4.8)
it is easily seen that \( \|\sigma^x_\varphi (y_n)\| \leq e^{n(3z)^2} \|y\| \) for all \( n \geq 1 \) and \( z \in \mathbb{C} \).

On the other hand, by (2.29) we have
\[
(y_n)\varphi = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \Delta^t_y \varphi_y \, dt, \quad (y^*_n)\varphi = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-nt^2} \Delta^t_y (\varphi^*) \, dt
\]
and by the norm-continuity of \( \mathbb{R} \ni t \mapsto \Delta^t_y \varphi_y \in H \) and \( \mathbb{R} \ni t \mapsto \Delta^t_y (\varphi^*)_\varphi \in H \) we get the convergences \((y_n)\varphi \to \varphi_y \) and \((y^*_n)\varphi \to (\varphi^*)_\varphi \) in the norm-topology. Furthermore, every orbit
\[
\mathbb{R} \ni s \mapsto \Delta^s_\varphi (y_n)\varphi = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-n(t-s)^2} \Delta^t_y \varphi_y \, dt \in H
\]
has the entire extension
\[
\mathbb{C} \ni z \mapsto \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-n(t-z)^2} \Delta^t_y \varphi_y \, dt \in H
\]
and thus (see [28], Lemma 3.2 and [12], Theorem 6.1) \((y_n)\varphi \in \bigcap_{z \in \mathbb{C}} \text{Dom} \Delta^z_\varphi \) and
\[
\Delta^z_\varphi (y_n)\varphi = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-n(t-z)^2} \Delta^t_y \varphi_y \, dt, \quad z \in \mathbb{C}.
\] (4.9)

Similarly, \((y^*_n)\varphi \in \bigcap_{z \in \mathbb{C}} \text{Dom} \Delta^z_\varphi \) and
\[
\Delta^z_\varphi (y^*_n)\varphi = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-n(t-z)^2} \Delta^t_y (\varphi^*) \, dt, \quad z \in \mathbb{C}.
\]

Moreover, using (4.8), we get for every \( n \geq 1 \) and \( z \in \mathbb{C} \)
\[
\|\Delta^z_\varphi (y_n)\varphi\| \leq e^{n(3z)^2} \|y\|, \quad \|\Delta^z_\varphi (y^*_n)\varphi\| \leq e^{n(3z)^2} \|(\varphi^*)_\varphi\|.
\]

Finally, for every \( n \geq 1 \), (4.7), (2.29) and (1.3) yield
\[
\sigma^x_\varphi (y_n)\varphi = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{-n(t-z)^2} \sigma^t_\varphi (y)\varphi \, dt = \Delta^z_\varphi (y_n)\varphi, \quad z \in \mathbb{C},
\] (4.10)

hence (2.29) holds for \( x = y_n \).

It remains to prove that (2.29) holds in full generality. First we show that
\[
x \in \mathfrak{I}_\varphi, \ z \in \mathbb{C}, \ x_\varphi \in \text{Dom} (\Delta^z_\varphi) \implies \sigma^x_\varphi (x_\varphi) = \Delta^z_\varphi x_\varphi.
\] (4.11)

By Lemma 2.40 this is equivalent to the implication
\[
x \in \mathfrak{I}_\varphi, \ z \in \mathbb{C}, \ x_\varphi \in \text{Dom} (\Delta^z_\varphi), \ y \in \mathfrak{A}_\varphi \implies \sigma^x_\varphi (x) J_\varphi y_\varphi = J_\varphi y_\varphi J_\varphi \Delta^z_\varphi x_\varphi,
\]
what we now are going to prove. Choose a sequence \( \{y_n\}_{n \geq 1} \) in \( \mathfrak{I}_\varphi \) as in the above part of the proof. For each \( n \geq 1 \), \((y_n)\varphi \in \text{Dom} (\Delta^{-1}_\varphi)\) implies by (2.1) that
\[
J_\varphi (y_n)\varphi \in \text{Dom} (\Delta^{-1}_\varphi) \quad \text{and} \quad \Delta^{-i\pi}_\varphi J_\varphi (y_n)\varphi = J_\varphi \Delta^{-i\pi}_\varphi (y_n)\varphi,
\]
so, according to Theorem 2.20
\[
x \Delta^{-i\pi}_\varphi J_\varphi (y_n)\varphi = x J_\varphi \Delta^{-i\pi}_\varphi (y_n)\varphi \in \text{Dom} (\Delta^{-i\pi}_\varphi) \quad \text{and} \quad \sigma^x_\varphi (x) J_\varphi (y_n)\varphi = \Delta^{i\pi}_\varphi x J_\varphi \Delta^{-i\pi}_\varphi (y_n)\varphi.
\] (4.12)

Since, by (1.11) and by (2.5),
\[
x J_\varphi \Delta^{-i\pi}_\varphi (y_n)\varphi = x J_\varphi \sigma^x_\varphi (y_n)\varphi = J_\varphi \sigma^x_\varphi (y_n) J_\varphi x_\varphi,
\]
Lemma 2.9, Lemma 2.11 and Theorem 2.12. This last inclusion together with (4.11) imply (2.29).

On the other hand, for every $x$ we have

$$k \quad \text{holds for every integer } k \geq 1 .$$

Indeed, $x \in \mathcal{T}_x \Rightarrow x_\varphi \in \text{Dom } (S_\varphi)$ holds for some $k \geq 1$ and $x \in \mathcal{T}_x$, then we have by (4.11)

$$\Delta_\varphi^k x_\varphi = \sigma_\varphi^{k+1}(x_\varphi) \in \text{Dom}(S_\varphi) = \text{Dom}(\Delta_\varphi^k) ,$$

that is $x_\varphi \in \text{Dom}(\Delta_\varphi^k)$.

On the other hand, for every $x \in \mathcal{T}_x$ and $k \geq 1$ we have $\sigma_\varphi^k(x) \in \mathcal{T}_x$, so (4.14) yields $\sigma_\varphi^k(x_\varphi) \in \text{Dom}(\Delta_\varphi^k)$ and, using (4.11), we deduce

$$x_\varphi = \sigma_\varphi^k (\sigma_\varphi^k(x)) = \Delta_\varphi^k \sigma_\varphi^k(x) \in \text{Dom}(\Delta_\varphi^k) .$$

Therefore (4.14) holds also for every integer $k \leq -1$ that is $\mathcal{T}_x \in \bigcap_{z \in \mathbb{C}} \text{Dom}(\Delta_\varphi^z)$.

This last inclusion together with (4.11) imply (2.29).

\section{5. Hermitian Maps}

In this section we analyse the notion introduced in Definition 2.10 by proving Lemma 2.11, Lemma 2.12 and Theorem 2.12.

Proof of Lemma 2.11

Let $\xi \in \text{Dom } S_\psi$ be arbitrary. Then there is a sequence $(x_n)_{n \geq 1}$ in $\mathfrak{A}_\psi$ such that

$$(x_n)_\psi \rightarrow \xi , \quad (x_n^*_\psi) \rightarrow S_\psi \xi .$$

Then, denoting

$$\xi_+ = \frac{1}{2} (\xi + S_\psi \xi) , \quad \xi_- = \frac{1}{2i} (\xi - S_\psi \xi) , \quad a_n = \frac{1}{2} (x_n + x_n^*) ,$$

we have

$$\xi = \xi_+ + i \xi_- , \quad S_\psi \xi_\pm = \xi_\pm, \text{ i.e. } \xi_\pm \in K^{S_\psi} , \quad a_n^* = a_n \in \mathfrak{M}_\psi , \quad (a_n)_\psi \rightarrow \xi_+ .$$

(5.1)
Since $\xi_+ = \xi$ if $\xi \in K^{S\psi}$, (6.2) implies that $K^{S\psi} \subset \{ x_\psi : x^* = x \in \mathfrak{N}_\psi \}$. The converse inclusion being trivial, the equality $K^{S\psi} = \{ x_\psi : x^* = x \in \mathfrak{N}_\psi \}$ follows. On the other hand, (5.1) implies that $\text{Dom } S_\psi = K^{S\psi} + iK^{S\psi}$. This proves (1) and (4) in Lemma 2.9.

For (2) and (3) we first notice that, for every $\xi \in K^{S\psi}$ and $x \in \mathfrak{A}_\psi$,

$$(\xi \mid J_\psi x_\psi) = (\xi \mid J_\psi S_\psi (x^*)_\psi) = (\xi \mid \Delta^{1/2}_\psi (x^*)_\psi) = (\Delta^{1/2}_\psi \xi \mid (x^*)_\psi) = (\psi (x^*) \mid \xi).$$

In particular, $(\xi \mid J_\psi x_\psi) \in \mathbb{R}$ whenever $x = x^*$.

Conversely, let us assume that $\xi \in K$ is such that $(\xi \mid J_\psi x_\psi) \in \mathbb{R}$ if $x^* = x \in \mathfrak{N}_\psi$. For every $x \in \mathfrak{A}_\psi$ we have

$$a = \frac{1}{2} (x + x^*) , \quad b = \frac{1}{2i} (x - x^*) \in \mathfrak{A}_\psi$$

are selfadjoint and $x = a + ib$,

hence, by our assumption on $\xi$,

$$(\xi \mid J_\psi x_\psi) = \frac{1}{2} ((\xi \mid J_\psi a_\psi) + i (\xi \mid J_\psi b_\psi)) = \frac{1}{2} ( (J_\psi a_\psi) \mid \xi) + i (J_\psi b_\psi) \mid \xi) = (J_\psi (x^*)_\psi) \mid \xi).$$

It follows that $(x_\psi \mid J_\psi \xi) = (\Delta^{1/2}_\psi x_\psi) \mid \xi)$ for all $x \in \mathfrak{A}_\psi$, hence, $\{x_\psi : x \in \mathfrak{A}_\psi\}$ being a core of $\Delta^{1/2}_\psi$, $\xi$ belongs to the domain of $(\Delta^{1/2}_\psi)^* = \Delta^{1/2}_\psi$ and $\Delta^{1/2}_\psi \xi = J_\psi \xi$. In other words, $\xi \in \text{Dom } S_\psi$ and $S_\psi \xi = J_\psi \Delta^{1/2}_\psi \xi = \xi$, i.e. $\xi \in K^{S\psi}$.

\[ \square \]

Proof of Lemma 2.11

If $T \in B(K, H)$ implements $\psi$ in $\varphi$ and $x^* = x \in \mathfrak{N}_\psi$, then

$$T x_\psi = (T x T^*)_\varphi \text{ with } (T x T^*)^* = T x T^* \in \mathfrak{N}_\psi$$

and hence Lemma 2.9(1) implies $T K^{S\psi} \subset H^{S\varphi}$, proving (1). In this case the inclusion $T \mathfrak{N}_\psi T^* \subset \mathfrak{N}_\varphi$ and the *-density of $\mathfrak{N}_\psi$ in $\mathfrak{N}$ imply $T N T^* \subset M$, proving (2).

If $T$ is isometric in addition, then

$$x = T^* (T x T^*) T, \quad x \in \mathfrak{N}$$

shows the injectivity of the map $N \ni x \mapsto T x T^* \in M$, which is clearly also a *-homomorphism. Furthermore, $0 \leq a \in \mathfrak{N}_\psi$ implies $a^{1/2} \in \mathfrak{N}_\psi$ and

$$\psi(a) = \|(a^{1/2})_\psi\|^2 = ||(T a^{1/2} T^*)_\varphi||^2 = ||(T a^{1/2} T^*)_\varphi||^2 = \varphi(T a T^*).$$

Therefore (3) holds.

Let us finally assume that $\psi$ and $\varphi$ are bounded and $\xi_\psi = (1_K)_\psi, \eta_\varphi = (1_H)_\varphi$. If $T \in B(K, H)$ is injective and implements $\psi$ in $\varphi$, then $T N T^* \subset M$ by the above proved (2) and

$$T T^* \eta_\varphi = (T 1_K T^*)_\varphi = T (1_K)_\psi = T \xi_\psi \implies T^* \eta_\varphi = \xi_\psi$$

by the injectivity of $T$. Conversely, if $T \in B(K, H)$ is injective and satisfies $T N T^* \subset M$ and $T^* \eta_\varphi = \xi_\psi$, then for $x \in N$

$$(T x T^*)_\varphi = T x T^* \eta_\varphi = T x \xi_\psi = T x_\psi.$$ 

Hence we have (4).
Proof of Theorem 2.12

(1), (2) and (3) in Lemma 2.9 imply the equivalences (1) ⇔ (2), (2) ⇔ (3) and (2) ⇔ (4), respectively.

Let us assume that (1) holds. By (4) in Lemma 2.9, every ξ ∈ Dom S_ϕ is of the form ξ = ξ_1 + i ξ_2 with ξ_1, ξ_2 ∈ K^{S_ϕ}. Hence we get

\[ T(ξ) = T(ξ_1) + i T(ξ_2) ∈ H^{S_ϕ} + i H^{S_ϕ} ⊂ Dom S_ϕ, \]

\[ S_ϕ(T(ξ)) = T(ξ_1) - i T(ξ_2) = T(ξ_1 - i ξ_2) = T(S_ϕ(ξ)), \]

proving (5). Conversely, if (5) holds, then we have for every ξ ∈ K^{S_ϕ} ⊂ Dom S_ϕ

\[ T(ξ) ∈ Dom S_ϕ \quad \text{and} \quad S_ϕ(T(ξ)) = T(S_ϕ(ξ)) = T(ξ), \]

so T(ξ) ∈ H^{S_ϕ}. Therefore (1) ⇔ (5).

Since J_ψ is involutive and, by (2.12), S_ϕ = Δ_0^{-1/2} J_ψ and S_ϕ = Δ_0^{-1/2} J_ψ, (5) is equivalent to

\[ T \Delta_0^{-1/2} ⊂ Δ_0^{-1/2} J_ψ T J_ψ. \]

This equation, in turn, is equivalent to the validity of

\[ Δ_0^{1/2} T Δ_0^{-1/2} ξ = J_ψ T J_ψ ξ, \quad ξ ∈ Dom Δ_0^{-1/2} \]

and thus (5)⇔(6).

We have already seen that (1)⇔(3). Applying this equivalence to J_ψ T^* J_ψ, it follows that J_ψ T^* J_ψ is Hermitian with respect to (ϕ, ψ) if and only if

\[ (J_ψ T^* J_ψ y_ϕ | J_ψ x_ϕ) = (x_ψ | T^* J_ψ y_ϕ) = (T x_ψ | J_ψ y_ϕ) \]

is real for all x^* = x ∈ Ξ and y^* = y ∈ Ξ. But this means exactly (3), so (3)⇔(7).

By the equivalence of statements (1) and (5) in Theorem 2.3 with A = Δ_0, B = Δ_0, β = -1/2, and taking into account that they imply (2.22) and (2.23), we obtain the equivalence (6)⇔(8).

Now let us assume that the equivalent conditions (1)-(8) are satisfied. Then, by (2.22) in Theorem 2.3 we get (2.21). Further, using (2.22), we obtain (2.23) immediately from (2.22) and (2.24). Since the map T(·) is bounded and

\[ \|T(s)\| = \|Δ_i s T Δ_i^{-s}\| = \|T\|, \quad \|T(s - i/2)^{-1}\| \leq \|J_ψ T(s) J_ψ\| = \|T\|, \quad s ∈ \mathbb{R}, \]

we get also (2.24) by the Three Line Theorem. Finally, since K^{S_ϕ} and H^{S_ϕ} are invariant under Δ_0^{-is} and Δ_0^{is}, respectively, for every s ∈ \mathbb{R}, due to (2.23) and Lemma 2.9 (1), we obtain the Hermiticity of T(s) = Δ_0^{is} T Δ_0^{-is} from (1).

\[ \square \]

6. Generalization of the Structure Theorem of Borchers

We prove Theorem 2.12 in two steps: first we prove it for the case where Ξ_0 and Ξ_1 are empty, and then we reduce the proof of the general case to the above special case.

Step 1. Proof in the case of Ξ_0 = Ξ_1 = ∅ and wo-continuous T(·).
By our assumptions in this step, 
\( \mathbb{S}_\beta \ni z \mapsto T(z) \in B(K, H) \)

is a bounded \( \omega \)-continuous map which is analytic in \( \mathbb{S}_\beta \) and satisfies the boundary conditions

(i) \( T(s) \) is Hermitian with respect to \( (\psi, \varphi) \) for all \( s \in \mathbb{R} \),
(ii) \( J_\varphi T(s + i\beta) J_\psi \) is Hermitian with respect to \( (\psi, \varphi) \) for all \( s \in \mathbb{R} \).

Let \( x \in \mathcal{F}_\psi \) and \( y \in \mathcal{F}_\varphi \) be arbitrary (for the Tomita algebras \( \mathcal{F}_\psi \) and \( \mathcal{F}_\varphi \) see the comments before Proposition 2.8) such that

\[
\| \Delta_\varphi^{iz} x_\varphi \| \leq e^{c|\Delta z|^2} \| x_\varphi \|, \quad \| \Delta_\varphi^{iz} y_\varphi \| \leq e^{c|\Delta z|^2} \| y_\varphi \|, \quad z \in \mathbb{C}
\]  

(6.1)

for some constant \( c \geq 0 \). Consider the functions

\[
f_1 : \mathbb{C} \times \mathbb{S}_\beta \ni (z_1, z_2) \mapsto (T(z_2) \Delta_\psi^{-iz_1} x_\psi | J_\varphi \Delta_\varphi^{-iz_1} y_\varphi),
\]

\[
f_2 : \mathbb{C} \times \mathbb{S}_{-\beta} \ni (z_1, z_2) \mapsto (\Delta_\varphi^{-iz_1 + \frac{i}{2}} y_\varphi | T(z_2 J_\psi \Delta_\psi^{-iz_1 + \frac{i}{2}} x_\psi)).
\]

They are continuous and, according to (6.1), bounded on any set of the form \( \{z_1 \in \mathbb{C} ; |\Im z_1| \leq \delta \} \times \mathbb{S}_\beta \) and \( \{z_1 \in \mathbb{C} ; |\Im z_1| \leq \delta \} \times \mathbb{S}_{-\beta} \), respectively, \( \delta > 0 \).

Furthermore, the partial functions

\[
\mathbb{C} \ni z_1 \mapsto f_1(z_1, z_2), \quad z_2 \in \mathbb{S}_\beta, \quad \mathbb{S}_\beta \ni z_2 \mapsto f_1(z_1, z_2), \quad z_1 \in \mathbb{C},
\]

\[
\mathbb{C} \ni z_1 \mapsto f_2(z_1, z_2), \quad z_2 \in \mathbb{S}_{-\beta}, \quad \mathbb{S}_{-\beta} \ni z_2 \mapsto f_2(z_1, z_2), \quad z_1 \in \mathbb{C}
\]

are analytic.

Now, by (2.29) and Theorem 2.12 (4), (i) implies, for every \( z_1 \in \mathbb{C} \) and \( s \in \mathbb{R} \),

\[
f_1(z_1, s) = (T(s) \sigma_{-z_1}(x_\psi | J_\varphi \sigma_{-z_1}(y_\varphi)
\]

\[
= (J_\varphi (\sigma_{-z_1}(y_\varphi)^*), T(s) (\sigma_{-z_1}(x_\psi)^*)_\psi
\]

\[
= (J_\varphi S_\psi \sigma_{-z_1}(y_\varphi | T(s) S_\psi \sigma_{-z_1}(x_\psi
\]

\[
= (\Delta_\psi^{-iz_1 + \frac{i}{2}} y_\varphi | T(s J_\psi \Delta_\psi^{-iz_1 + \frac{i}{2}} x_\psi) = f_2(z_1, s).
\]

Therefore

\[
f : \mathbb{C} \times \{z_2 \in \mathbb{C} ; |\Im z_2| \leq |\beta|\} \ni (z_1, z_2) \mapsto \begin{cases} 
  f_1(z_1, z_2) & \text{if } z_2 \in \mathbb{S}_\beta, \\
  f_2(z_1, z_2) & \text{if } z_2 \in \mathbb{S}_{-\beta}
\end{cases}
\]

is a well defined continuous function, bounded on every set of the form

\[
\{z_1 \in \mathbb{C} ; |\Im z_1| \leq \delta\} \times \{z_2 \in \mathbb{C} ; |\Im z_2| \leq |\beta|\}, \quad \delta > 0.
\]

For each fixed \( z_1 \in \mathbb{C} \), the function \( \mathbb{S}_\beta \cup \mathbb{S}_{-\beta} \ni z_2 \mapsto f(z_1, z_2) \) is analytic. Hence, by the Morera Theorem (the one-dimensional edge-of-the-wedge theorem, see for example [3], 2.1.9.(2) or [11], II.2.7), it can be analytically extended across \( \mathbb{R} \), that is the partial functions

\[
\{z_2 \in \mathbb{C} ; |\Im z_2| < |\beta|\} \ni z_2 \mapsto f(z_1, z_2), \quad z_1 \in \mathbb{C}
\]

are analytic. Thus we can apply to \( f \) the Osgood Lemma (the Hartogs Theorem for continuous functions, see for example [13], Theorem I.A.2) and deduce that it is analytic, as function of two complex variables, on \( \mathbb{C} \times \{z_2 \in \mathbb{C} ; |\Im z_2| < |\beta|\} \).
For every \( z_1 \in \mathbb{C} \) and \( s \in \mathbb{R} \), (ii) implies by \((2.28)\) and Theorem \((2.12)\),

\[
f(z_1 + \frac{i}{2}, s + i \beta) = f_1(z_1 + \frac{i}{2}, s + i \beta) = \\
= (T(s + i \beta) J_\psi S_\psi \Delta_{\psi}^{-iz_1} x_\psi | S_\psi \Delta_{\psi}^{-iz_1} y_\psi) = \\
= (T(s + i \beta) J_\psi (\sigma_{-z_1}^{\psi}(x)^*) | (\sigma_{-z_1}^{\psi}(y)^*)_\psi) = \\
= (J_\varphi (\sigma_{-z_1}^{\varphi}(y)^*)_\varphi | J_\varphi T(s + i \beta) J_\psi (\sigma_{-z_1}^{\psi}(x)^*)_\psi) = \\
= (J_\varphi T(s + i \beta) J_\psi \sigma_{-z_1}^{\psi}(x)_\psi | J_\varphi \sigma_{-z_1}^{\varphi}(y)_\varphi) = \\
= (\sigma_{-z_1}^{\varphi}(y)_\varphi | T(s + i \beta) J_\psi \sigma_{-z_1}^{\psi}(x)_\psi) = \\
= (\Delta_{\varphi}^{-i(z_1 \frac{\pi}{2} + \frac{\beta}{2})} y_\varphi | T(s - i \beta) \Delta_{\psi}^{-i(z_1 \frac{\pi}{2} + \frac{\beta}{2})} x_\psi) = \\
= f_2(z_1 - \frac{i}{2}, s - i \beta) = f(z_1 - \frac{i}{2}, s - i \beta).
\]

Therefore, for each \( s \in \mathbb{R} \), the bounded, continuous function

\[
g_s : \left\{ \zeta \in \mathbb{C} : |\Im \zeta| \leq \frac{1}{2} \right\} \ni \zeta \mapsto f(\zeta, s + 2 \beta \zeta),
\]

which is analytic in the interior, satisfies

\[
g_s \left( t + \frac{i}{2} \right) = g_s \left( t - \frac{i}{2} \right), \quad t \in \mathbb{R}.
\]

By the Morera Theorem, \( g_s \) extends to a periodic entire function with period \( i \), still denoted by \( g_s \), which is bounded. By the Liouville Theorem it follows that \( g_s \) is constant, hence we get successively

\[
f_1(0, s) = f(0, s) = g_s(0) = g_s \left( \frac{-s}{2 \beta} \right) = f \left( \frac{-s}{2 \beta}, 0 \right) = f_1 \left( \frac{-s}{2 \beta}, 0 \right),
\]

\[
(T(s) x_\psi | J_\varphi y_\varphi) = (T(0) \Delta_{\varphi}^{i\frac{s\pi}{2}} x_\psi | J_\varphi \Delta_{\psi}^{i\frac{s\pi}{2}} y_\varphi) = \\
\overset{(2.28)}{=} \Delta_{\varphi}^{-i\frac{s\pi}{2}} T(0) \Delta_{\psi}^{i\frac{s\pi}{2}} x_\psi | J_\varphi y_\varphi.
\]

By the density property of \( \mathcal{I}_\varphi \) stated in Proposition \((2.8)\) the above equalities imply that

\[
T(s) = \Delta_{\varphi}^{-i\frac{s\pi}{2}} T(0) \Delta_{\psi}^{i\frac{s\pi}{2}}, \quad s \in \mathbb{R},
\]

hence \((2.36)\) holds with \( T = T(0) \). From \((2.32)\) and \((2.33)\) in Theorem \((2.12)\) we obtain \((2.37)\) and \((2.38)\).

**Step 2. Proof in the general case.**

Let us consider, for any integer \( n \geq 1 \), the entire function

\[
\mathbb{C} \ni z \mapsto f_n(z) = \sqrt{\frac{n}{\pi}} e^{-nz^2}
\]

and the mollification of \( T(\cdot) \)

\[
\overline{S_\beta} \ni z \mapsto T_n(z) = wo - \int_{-\infty}^{\infty} f_n(t) T(t + z) dt \in B(K, H).
\]

(6.2)

We notice that the mapping \( \mathbb{R} \ni t \mapsto T(t + z) \in B(K, H) \) is norm-continuous for \( z \in S_\beta \) and, due to the continuity conditions \((2.34)\) and \((2.35)\), \( wo \)-measurable with
It follows that the integral in (6.6) is convergent and defines an entire mapping \( T(\cdot) \) where

\[
T_n(s) = wo - \lim_{0<\beta \to 0} T_n(s + it), \quad s \in \mathbb{R},
\]

\[
T_n(s + i\beta) = wo - \lim_{1>\beta \to 1} T_n(s + it), \quad s \in \mathbb{R}.
\]

We compare the operator valued function \( T_n(\cdot) \) with

\[
\mathbb{C} \ni z \mapsto T_{\zeta,n}(z) = wo - \int_{\mathbb{R} + \i\zeta} f_n(w - z) T(w) \, dw
\]

\[
= wo - \int_{-\infty}^{\infty} f_n(t + \zeta - z) T(t + \zeta) \, dt \in B(K, H),
\]

where \( \zeta \in S_\beta \). Due to

\[
f_n(t + \zeta - z) = \sqrt{\frac{n}{\pi}} e^{-nt^2} e^{-2nt(\zeta - z) - n(\zeta - z)^2},
\]

the integral in (6.6) is convergent and defines an entire mapping \( T_{\zeta,n}(\cdot) \).

For \( \zeta_1, \zeta_2 \in S_\beta \) and \( z \in \mathbb{C} \) we have by the Cauchy Integral Theorem

\[
T_{\zeta_1,n}(z) = wo - \int_{\mathbb{R} + \i\zeta_1} f_n(w - z) T(w) \, dw = wo - \int_{\mathbb{R} + \i\zeta_2} f_n(w - z) T(w) \, dw = T_{\zeta_2,n}(z),
\]

so \( T_{\zeta,n}(\cdot) \) does not depend on \( \zeta \in S_\beta \). Therefore, for any \( \zeta \in S_\beta \),

\[
T_{\zeta,n}(z) = T_{\zeta,n}(z) = T_n(z), \quad z \in S_\beta.
\]

Let \( \zeta \in S_\beta \) be arbitrary. Since \( T_{\zeta,n}(\cdot) \) is an entire mapping, by 6.4, 6.5 and 6.7 we get for every \( s \in \mathbb{R} \)

\[
T_n(s) = wo - \lim_{0<\beta \to 0} T_n(s + it) = wo - \lim_{0<\beta \to 0} T_{\zeta,n}(s + it) = T_{\zeta,n}(s),
\]

\[
T_n(s + i\beta) = wo - \lim_{1>\beta \to 1} T_n(s + it) = wo - \lim_{1>\beta \to 1} T_{\zeta,n}(s + it) = T_{\zeta,n}(s + i\beta).
\]

Consequently, the mapping \( S_\beta \ni z \mapsto T_n(z) \) defined in (6.2) is a restriction of the entire mapping \( T_{\zeta,n}(\cdot) \). In particular, it is \( s \)-continuous (the role of \( T_{\zeta,n}(\cdot) \) is just to prove this statement) and its restriction to \( S_\beta \) is analytic. We recall that its boundedness was already noticed in 6.3.

Since \( \mathbb{R} \ni t \mapsto f_n(t) \) is a real function, (1) \( \Leftrightarrow \) (3) in Theorem 2.12 implies that the Hermiticity of \( T(s) \) and \( J_{\psi} T(s + i\beta) J_{\psi} \) for \( s \in \mathbb{R} \setminus \Xi_0 \) respectively \( s \in \mathbb{R} \setminus \Xi_1 \) is inherited by \( T_n(s) \) and \( J_{\psi} T_n(s + i\beta) J_{\psi} \) for all \( s \in \mathbb{R} \).

Thus \( T_n(\cdot) \) fulfills all the assumptions made in Step 1. Consequently there exists \( T_n \in B(K, H) \) satisfying

\[
T_n(s) = \Delta_{\psi}^{i\frac{s}{\beta}} T_n \Delta_{\psi}^{-i\frac{s}{\beta}}, \quad s \in \mathbb{R}.
\]

It follows that

\[
T_n(s + t) = \Delta_{\psi}^{i\frac{s}{\beta}} T_n(t) \Delta_{\psi}^{-i\frac{s}{\beta}}, \quad t, s \in \mathbb{R},
\]
which yields by analytic extension
\[ T_n(z + s) = \Delta^{-i\frac{s}{|s|}}_\varphi T_n(z) \Delta^i\frac{s}{|s|}_\psi, \quad z \in \mathbb{S}_\beta, \ s \in \mathbb{R}. \] (6.8)

On the other hand, from (6.2) we obtain
\[ \text{norm} \lim_{n \to \infty} T_n(z) = T(z), \quad z \in \mathbb{S}_\beta, \]
due to the boundedness and norm-continuity of \( T(\cdot) \) in \( \mathbb{S}_\beta \). Thus we get by (6.8)
\[ T(z + s) = \Delta^{-i\frac{s}{|s|}}_\varphi T(z) \Delta^i\frac{s}{|s|}_\psi, \quad z \in \mathbb{S}_\beta, \ s \in \mathbb{R}. \]

Now choose some \( s_o \in \mathbb{R} \setminus \Xi_o \) and denote
\[ T = \Delta^i\frac{so}{|so|}_\psi T(s_o) \Delta^{-i\frac{so}{|so|}}_\varphi. \]

Then \( T \) is Hermitian with respect to \((\psi, \varphi)\) and we have for every \( s \in \mathbb{R} \setminus \Xi_o \)
\[ T(s) = wo - \lim_{0 < t/|s| \to 0} T(s + it) \]
\[ = wo - \lim_{0 < t/|s| \to 0} \Delta^{-i\frac{so}{|so|}}_\varphi T(s_o + it) \Delta^i\frac{so}{|so|}_\psi \]
\[ = \Delta^{-i\frac{so}{|so|}}_\varphi \left( wo - \lim_{0 < t/|s| \to 0} T(s_o + it) \right) \Delta^i\frac{so}{|so|}_\psi \]
\[ = \Delta^{-i\frac{so}{|so|}}_\varphi T(s_o) \Delta^i\frac{so}{|so|}_\psi \]
\[ = \Delta^{-i\frac{so}{|so|}}_\varphi T \Delta^i\frac{so}{|so|}_\psi, \]
that is \( 2.36 \) holds. Using (1) \( \Leftrightarrow \) (8) in Theorem 2.12, (2.32) and (2.33), as well as
the uniqueness theorem of the Riesz brothers (see, in Section 3, Step 1 of the proof
of (3) \( \Rightarrow \) (4) in Theorem 2.3), we obtain that \( T(\cdot) \) extends to an so-continuous map
on \( \mathbb{S}_\beta \), for which \( 2.37 \) and \( 2.38 \) hold.

\[ \square \]

7. Proof of Theorem 2.1

We recall the setting:
- \( N \subset M \subset B(H) \) are von Neumann algebras,
- \( \varphi \) is a normal semifinite faithful weight on \( M \) such that its restriction \( \psi \) to
  \( N \) is semifinite,
- we assume that \( M \) is in standard form with respect to \( \varphi \) and \( \{ y_\varphi; y \in \mathcal{N}_\psi \} \)
  is dense in \( H \), hence \( N \) is in standard form with respect to \( \psi \) and \( y_\psi = y_\varphi \)
  for all \( y \in \mathcal{N}_\psi \).

We shall use the notations \( \Delta_M = \Delta_\varphi \), \( J_M = J_\varphi \) and \( \Delta_N = \Delta_\psi \), \( J_N = J_\psi \).

The proof of Theorem 2.1 will be performed in nine steps.

Step 1. Application of the Modular Extension Theorem.

Since \( S_\psi \subset S_\varphi \), hence \( H^{S_\psi} \subset H^{S_\varphi} \), the identity map \( I \) on \( H \) is Hermitian with
respect to \((\psi, \varphi)\). By (1) \( \Leftrightarrow \) (8) in Theorem 2.12, (2.32) and (2.33), we obtain an
so-continuous map
\[ \overline{S}_{-1/2} \ni z \mapsto I(z) \in B(H), \]
which is analytic in $S_{-1/2}$ and satisfies the conditions
$$I(s) = \Delta^{-i} \Delta_M^{-i} \Delta_N^{-i} \quad \text{and} \quad I\left(s - \frac{i}{2}\right) = J_M I(s) J_N, \quad s \in \mathbb{R},$$
$$\|I(z)\| \leq 1, \quad z \in S_{-1/2}.$$  

Therefore the mapping
$$S_{-1/2} \ni \zeta \mapsto W(\zeta) = I(-\zeta^*) \in B(H)$$
is so-continuous, analytic in $S_{-1/2}$, and such that
$$W(s) = \Delta_N^{-i} \Delta_M^{-i} \quad \text{and} \quad W\left(s - \frac{i}{2}\right) = J_N W(s) J_M, \quad s \in \mathbb{R}, \quad (7.1)$$
$$\|W(\zeta)\| \leq 1, \quad \zeta \in S_{-1/2}. \quad (7.2)$$

**Step 2. Hermiticity on the boundary.**

First we show that
$$\left( W\left(s - \frac{i}{2}\right) x_{\psi} \mid J_N y_{\psi} \right) \in \mathbb{R}, \quad x^* = x \in \mathcal{N}_{\psi}, \ y^* = y \in \mathcal{N}_{\psi},$$
what is equivalent, according to (1) $\Leftrightarrow$ (3) in Theorem 2.12, to
$$W\left(s - \frac{i}{2}\right) \text{ is Hermitian with respect to } (\psi, \psi), \quad s \in \mathbb{R}. \quad (7.3)$$

Indeed,
$$\left( W\left(s - \frac{i}{2}\right) x_{\psi} \mid J_N y_{\psi} \right) = \left( J_N \Delta_N^{-i} \Delta_M^{-i} J_M x_{\psi} \mid J_N y_{\psi} \right) = \left( \Delta_N^{-i} y_{\psi} \mid \Delta_M^{-i} J_M x_{\psi} \right)$$
$$\quad = \left( \Delta_N^{-i} y_{\psi} \mid J_M \Delta_M^{-i} x_{\phi} \right) \quad (\sigma_{\phi}^\psi(y)_{\phi} \mid J_M \sigma_{\phi}^\psi(x)_{\phi})$$
is real because of Lemma 2.19 (2).

Next we show, using the negative half-sided modular inclusion assumption (2.19), that
$$W(s) \text{ is Hermitian with respect to } (\psi, \psi), \quad s \leq 0, \quad (7.4)$$

$$J_N W(s) J_N \text{ is Hermitian with respect to } (\psi, \psi), \quad s \geq 0. \quad (7.5)$$

For (7.4), let $s \leq 0$ and $x^* = x \in \mathcal{N}_{\psi}$ be arbitrary. By (2.19) we have $\sigma_{\phi}^\psi(x) \in \mathbb{N}_{\psi}$ and $\sigma_{\phi}^\psi(x)_{\phi} = \sigma_{\phi}^\psi(x)_{\psi}$. Thus
$$W(s) x_{\psi} = \Delta_N^{-i} \Delta_M^{-i} x_{\phi} = \Delta_N^{-i} \sigma_{\phi}^\psi(x)_{\phi} = \Delta_N^{-i} \sigma_{\phi}^\psi(x)_{\psi} = \sigma_{\phi}^\psi(W(s) x_{\psi})_{\psi}$$
and the Hermiticity of $W(s)$ with respect to $(\psi, \psi)$ follows by using (2) $\Rightarrow$ (1) in Theorem 2.12.

Now, for (7.5), let $s \geq 0$ and $x^* = x \in \mathcal{N}_{\psi}$ be arbitrary. Then, due to (2.19), we have
$$\sigma_{\phi}^\psi(x)_{\psi} \in \mathcal{N}_{\psi} \quad \text{and} \quad \sigma_{\phi}^\psi(x)_{\phi} = \sigma_{\phi}^\psi(x)_{\psi} \quad (\Delta_N^{-i} \Delta_M^{-i} x_{\phi})_{\psi} = W(s)^* x_{\psi},$$
so $W(s)^* x_{\psi} \in H^{S_{\psi}}$. Therefore, owing to (2) $\Rightarrow$ (7) in Theorem 2.12, $J_N W(s) J_N$ is Hermitian with respect to $(\psi, \psi)$.

To summarize,
- $W(\zeta)$ is Hermitian with respect to $(\psi, \psi)$ for $\zeta \in (-\infty, 0] \cup \left( \mathbb{R} - \frac{i}{2} \right)$ and
- $J_N W(\zeta) J_N$ is Hermitian with respect to $(\psi, \psi)$ for $\zeta \in [0, \infty)$.

**Step 3. Change of Variable.**

We shall use the analytic logarithm branches

$$
\log_+ : \left\{ r e^{i\theta} ; r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \right\} \ni r e^{i\theta} \mapsto \log r + i \theta,
$$

$$
\log_- : \left\{ r e^{i\theta} ; r > 0, -\frac{3\pi}{2} < \theta < \frac{\pi}{2} \right\} \ni r e^{i\theta} \mapsto \log r + i \theta.
$$

For any $\beta \in \mathbb{R}$, $\beta \neq 0$, we consider (like in Section 3, in the proof of (3) $\Rightarrow$ (4) in Theorem 2.13) the one point compactification of the right half and the left half of $S_\beta$ and denote each added point by $+\infty$ and $-\infty$, respectively.

In order to apply Theorem 2.13 to $W$, we have to map $S_{-1/2}$ conformally onto some $S_\beta$, $\beta \neq 0$, such that $(-\infty, 0) \cup \{-\infty\} \cup \left( \mathbb{R} - \frac{i}{2} \right)$ correspond to $\mathbb{R}$, and $(0, \infty)$ to $\mathbb{R} + i \beta$. This is done, for $\beta = \pi$, by

$$
\overline{S_{-1/2}} \setminus \{0\} \ni \zeta \mapsto \Psi(\zeta) = \log_+ \left( 1 - e^{2\pi i} \right) \in \overline{S_{\pi}} \setminus \{0\},
$$

which extends to a homeomorphism $\Psi : \overline{S_{-1/2}} \cup \{-\infty, +\infty\} \rightarrow \overline{S_{\pi}} \cup \{-\infty, +\infty\}$ satisfying $\Psi(0) = -\infty$, $\Psi(-\infty) = 0$ and $\Psi(+\infty) = +\infty$.

The inverse homeomorphism $\Psi^{-1} : \overline{S_{\pi}} \cup \{-\infty, +\infty\} \rightarrow \overline{S_{-1/2}} \cup \{-\infty, +\infty\}$ is given by

$$
\overline{S_{\pi}} \setminus \{0\} \ni z \mapsto \Psi^{-1}(z) = \frac{1}{2\pi} \log_-(1 - e^{2i\pi}) \in \overline{S_{-1/2}} \setminus \{0\},
$$

so it maps $S_{\pi}$ conformally onto $S_{-1/2}$ and

$$
(-\infty, 0) \text{ onto } (-\infty, 0) \quad \left(0 \mapsto -\infty, -\infty \mapsto 0 \right),
$$

$$(0, +\infty) \text{ onto } \mathbb{R} - \frac{i}{2} \quad \left(0 \mapsto -\infty - \frac{i}{2}, \infty \mapsto +\infty + \frac{i}{2} \right),
$$

$$(\mathbb{R} + i\beta \text{ onto } 0, +\infty) \quad \left(-\infty + i\pi \mapsto 0, i\pi \mapsto \frac{\log 2}{2\pi}, +\infty + i\pi \mapsto +\infty \right).
$$

Thus we can consider the so-continuous mapping

$$
\overline{S_{\pi}} \setminus \{0\} \ni z \mapsto V(z) = W(\Psi^{-1}(z)) \in B(H),
$$

which is analytic in $S_{\pi}$ and, according to (7.6), (7.7), and (7.8), satisfies

$$
\|V(z)\| \leq 1, \quad z \in \overline{S_{\pi}} \setminus \{0\},
$$

$V(s)$ is Hermitian with respect to $(\psi, \psi), \quad s \in \mathbb{R} \setminus \{0\},

$$
J_N V(s + i\pi) J_N \text{ is Hermitian with respect to } (\psi, \psi), \quad s \in \mathbb{R}.
$$

**Step 4. Application of the Generalized Structure Theorem.**

Since the so-continuous mapping considered in (7.6) is analytic in $S_{\pi}$ and (7.7), (7.8), (7.9) hold, it satisfies the assumptions for $T(\cdot)$ in Theorem 2.14 with
$M$ replaced by $N$, $\varphi$ replaced by $\psi$, $\beta = \pi$, $\Xi_0 = \{0\}$ and $\Xi_1 = \emptyset$.

By Theorem 2.13 it follows that, for some $V \in B(H)$ which is Hermitian with
respect to $(\psi, \psi)$,

$$V(s) = \Delta_N^{-i \frac{s}{2\pi}} V \Delta_N^{i \frac{s}{2\pi}}, \quad s \in \mathbb{R} \setminus \{0\}$$

(7.10)

and the mapping (7.6) has an so-continuous extension

$$\mathbb{S}_\pi \ni z \mapsto V(z) \in B(H)$$

with $V(0) = V$, satisfying

$$V(z + 2\pi t) = \Delta_N^{-it} V(z) \Delta_N^{it}, \quad z \in \mathbb{S}_\pi, \ t \in \mathbb{R},$$

(7.11)

$$V(s + i\pi) = J_N V(s) J_N, \quad s \in \mathbb{R}.$$  

(7.12)

In particular,

$$V(0) = so - \lim_{\not\sim \pi \to 0} V(s) = so - \lim_{t \to -\infty} W(t) = so - \lim_{t \to -\infty} \Delta_N^{-it} \Delta_M^{it}$$

is a wave operator.

We notice an additional continuity property of $V(\cdot)$: since $\Psi^{-1}(-\infty) = 0$, the limit

$$so - \lim_{\mathbb{S}_\pi \ni z \to -\infty} V(z) = so - \lim_{\not\sim \pi \to 0} W(\zeta) = W(0) = 1_H$$

exists. Thus the mapping (7.6) has actually an so-continuous extension

$$\mathbb{S}_\pi \cup \{-\infty\} \ni z \mapsto V(z) \in B(H)$$

(7.13)

with $V(0) = so - \lim_{t \to -\infty} \Delta_N^{-it} \Delta_M^{it}$ and $V(-\infty) = 1_H$.



Step 5. Further change of variable.

We recall that

$$\mathbb{S}_\pi \ni z \mapsto \Theta(z) = e^z \in \{\zeta \in \mathbb{C} ; \Im \zeta \geq 0\} \setminus \{0\}$$

is a homeomorphism mapping $\mathbb{S}_\pi$ conformally onto $\{\zeta \in \mathbb{C} ; \Im \zeta > 0\}$, which extends to a homeomorphism $\Theta : \mathbb{S}_\pi \cup \{\infty\} \to \{\zeta \in \mathbb{C} ; \Im \zeta \geq 0\} \cup \{\infty\}$ satisfying $\Theta(-\infty) = 0$ and $\Theta(+\infty) = \infty$.

The inverse homeomorphism $\Theta^{-1} : \{\zeta \in \mathbb{C} ; \Im \zeta \geq 0\} \cup \{\infty\} \to \mathbb{S}_\pi \cup \{-\infty, +\infty\}$, which maps $\{\zeta \in \mathbb{C} ; \Im \zeta > 0\}$ conformally onto $\mathbb{S}_\pi$, is given by

$$\{\zeta \in \mathbb{C} ; \Im \zeta \geq 0\} \setminus \{0\} \ni \zeta \mapsto \Theta^{-1}(\zeta) = \log_+ \zeta \in \mathbb{S}_\pi,$$

$$\Theta^{-1}(0) = -\infty, \quad \Theta^{-1}(\infty) = +\infty.$$

Since the mapping (7.13) is so-continuous, also the mapping

$$\{\zeta \in \mathbb{C} ; \Im \zeta \geq 0\} \ni \zeta \mapsto U(\zeta) = V(\Theta^{-1}(\zeta)) \in B(H)$$

is so-continuous. Moreover, since $\mathbb{S}_\pi \ni z \mapsto V(z)$ is analytic, the restriction of
the above mapping to $\{\zeta \in \mathbb{C} ; \Im \zeta > 0\}$ is analytic.

For $\zeta = 0$ and $\zeta = 1$ we have

$$U(0) = V(-\infty) = 1_H, \quad U(1) = V(0) = so - \lim_{t \to -\infty} \Delta_N^{-it} \Delta_M^{it}. $$
In particular, \( U(0) \) and \( U(1) \) are unitaries and \( \Delta^u_N U(1) = U(1) \Delta^u_M \) for all \( s \in \mathbb{R} \). On the other hand, for \( \zeta \in \mathbb{C}, \Im \zeta \geq 0, \zeta \neq 0, 1 \),

\[
U(\zeta) = V(\Theta^{-1}(\zeta)) = V(\log_+ \zeta) = W(\Psi^{-1}(\log_+ \zeta)) = W\left(\frac{1}{2\pi} \log_-(1-\zeta)\right)
\]

holds. In particular, according to (7.16), also the operators

\[
\{U(s) : s \in \mathbb{R}, s \neq 0, 1\} = \{V(z) : z \in \partial S_\pi, z \neq 0\} = \{W(\zeta) : \zeta \in \partial S_{-i/2}, \zeta \neq 0\}
\]

are unitaries.

We summarize:

\[
\begin{align*}
U(0) &= 1_H \quad \text{and} \quad U(s) \quad \text{is unitary for every} \quad s \in \mathbb{R}, \\
U(1) &= \lim_{t \to -\infty} \Delta^{-it}_N \Delta^u_M \quad \text{and} \quad \Delta^u_N U(1) = U(1) \Delta^u_M, \quad s \in \mathbb{R}, \quad (7.14) \\
U(\zeta) &= V(\log_+ \zeta), \quad 0 \neq \zeta \in \mathbb{C}, \Im \zeta \geq 0, \quad (7.15) \\
U(\zeta) &= W\left(\frac{1}{2\pi} \log_-(1-\zeta)\right), \quad 1 \neq \zeta \in \mathbb{C}, \Im \zeta \geq 0. \quad (7.16)
\end{align*}
\]

Using (7.11), we obtain from (7.15) and (7.12)

\[
\begin{align*}
&\|U(\zeta)\| \leq 1, \quad \zeta \in \mathbb{C}, \Im \zeta \geq 0, \quad (7.17) \\
&U(e^{2\pi i} \zeta) = \Delta^{-it}_N U(\zeta) \Delta^u_M, \quad t \in \mathbb{R}, \quad \zeta \in \mathbb{C}, \Im \zeta \geq 0, \quad (7.18) \\
&U(-s) = J_N U(s) J_N, \quad s \in \mathbb{R}. \quad (7.19)
\end{align*}
\]

Indeed, (7.14) implies that \( \|U(\zeta)\| \leq 1 \) for \( 0 \neq \zeta \in \mathbb{C}, \Im \zeta \geq 0 \), while the norm of \( U(0) = 1_H \) is \( \leq 1 \). Similarly, the equality in (7.15) is an immediate consequence of (7.11) for \( 0 \neq \zeta \in \mathbb{C}, \Im \zeta \geq 0 \), while it is trivial for \( \zeta = 0 \). Finally, for any \( s > 0 \), (7.12) implies

\[
U(-s) = V(\log s + i\pi) = J_N V(\log s) J_N = J_N U(s) J_N,
\]

hence also \( J_N U(-s) J_N = J_N^2 U(s) J_N^2 = U(s) \). Therefore the equality in (7.19) holds for every \( s \in \mathbb{R} \) (it is trivial for \( s = 0 \)).

Furthermore, by (7.10) and (7.1),

\[
U(2) = W\left(\frac{1}{2\pi} \log_-(1)\right) = W\left(-\frac{i}{2}\right) = J_N W(0) J_M = J_N J_M. \quad (7.20)
\]

On the other hand, (7.16) is equivalent to the equality

\[
W(z) = U\left(1 - e^{2\pi z}\right), \quad z \in S_{-i/2},
\]

which yields

\[
\Delta^{-it}_N \Delta^u_M = W(t) = U\left(1 - e^{2\pi t}\right), \quad t \in \mathbb{R}. \quad (7.21)
\]

**Step 6. Group property of** \( U(\cdot) \).

We now prove the group property

\[
U(s_1) U(s_2) = U(s_1 + s_2), \quad s_1, s_2 \in \mathbb{R}. \quad (7.22)
\]
Let $s > 0$ and $t \in \mathbb{R}$ be arbitrary. By (7.15), (7.16), (7.18), and (7.19), we obtain
\[
U(s) = V(\log s) = \Delta_N^{-\frac{1}{2\pi}} V(0) \Delta_N^{\frac{1}{2\pi}} = \Delta_N^{-\frac{1}{2\pi}} U(1) \Delta_N^{\frac{1}{2\pi}} = U(1) \Delta_M^{-\frac{1}{2\pi}} \Delta_N^{\frac{1}{2\pi}} = U(1) W \left( -\log s \right) = U(1) (1-s)^{+},
\]
hence $U(s)U(1-s) = U(1)$.

By sandwiching this equation by $\text{Ad} \, \Delta^{-it}_N$ and taking into account (7.18), we get
\[
U(e^{2\pi t} s) U(e^{2\pi t}(1-s)) = U(e^{2\pi t}). \tag{7.23}
\]

Next let $r_1, r_2 \in \mathbb{R}$ be such that $r_1 > 0$ and $r_1 + r_2 > 0$. Then $s = \frac{r_1}{r_1 + r_2} > 0$
and, with $t = \frac{1}{2\pi}$ \((\log r_1 + r_2) \in \mathbb{R})$, we have $e^{2\pi t} = r_1, e^{2\pi t}(1-s) = r_2$. Thus
(7.23) yields
\[
U(r_1) U(r_2) = U(r_1 + r_2). \tag{7.24}
\]

Finally let $s_1, s_2 \in \mathbb{R}$ be arbitrary and choose $s \in \mathbb{R}$ such that $s > 0$, $s + s_1 > 0$
and $s + s_1 + s_2 > 0$. Then, using (7.24) with $(r_1, r_2)$ equal to $(s, s_1), (s + s_1, s_2)$
and $(s, s_1 + s_2)$, respectively, we obtain
\[
U(s) U(s_1) U(s_2) = U(s + s_1) U(s_2) = U(s + s_1 + s_2) = U(s) U(s_1 + s_2).
\]

Since $U(s)$ is unitary, the above equality implies that $U(s_1) U(s_2) = U(s_1 + s_2)$.

Therefore (7.22) is proved. In particular,
\[
U(s) U(-s) = U(0) = 1_H, \text{ that is } U(s)^* = U(-s), \quad s \in \mathbb{R}. \tag{7.25}
\]

Thus $\mathbb{R} \ni s \mapsto U(s) \in B(H)$ is an so-continuous one-parameter group of unitaries, which allows an so-continuous extension $\{\zeta \in \mathbb{C}; \Im \zeta \geq 0\} \ni \zeta \mapsto U(\zeta)$,
analytic in $\{\zeta \in \mathbb{C}; \Im \zeta > 0\}$ and satisfying (7.17). Consequently
\[
U(s) = \exp(isP), \quad s \in \mathbb{R} \quad \tag{7.26}
\]
for some positive selfadjoint operator $P$ in $H$.

**Step 7. Further properties of $U(\cdot)$**.

Here we show that the above constructed group $\mathbb{R} \ni s \mapsto U(s) \in B(H)$ satisfies properties (1) - (7) in Theorem 2.1

By (7.18), (7.19), (7.21) and (7.22), we obtain for all $s, t \in \mathbb{R}$
\[
\Delta_M^{-it} U(s) \Delta_M^{it} = (\Delta_M^{-it} \Delta_N^{it}) U(s) \Delta_M^{it} (\Delta_N^{-it} \Delta_M^{it}) = U(1 - e^{2\pi t})^* U(e^{2\pi t} s) U(1 - e^{2\pi t}) = U(e^{2\pi t} s). \tag{7.27}
\]

This equality and (7.18) show that property (1) in Theorem 2.1 is satisfied.

Similarly, (7.18), (7.20), (7.22) and (7.23) yield for every $s \in \mathbb{R}$
\[
J_M U(s) J_M = (J_M J_N) U(s) J_N (J_N J_M) = U(2)^* U(-s) U(2) = U(-s). \tag{7.28}
\]

Now property (2) in Theorem 2.1 is (7.19) together with (7.25),
The validity of property (3) in Theorem 2.1 follows from (7.21) and (7.14).
The first equality in property (4) in Theorem 2.1 is (7.20), while the second one follows from (7.25), (7.28), (7.22) and (7.23):
\[
U(1) J_M U(1)^* = U(1) J_M U(-1) = U(1)^2 J_M = U(2) J_M = J_N. \tag{7.29}
\]
Next we prove property (5). Since
\[ \Delta_M^{-it} \Delta_N^{it} = \Delta_M^{-it} N \Delta_N^{it} = \Delta_M^{-it} M \Delta_N^{it} = M, \quad t \in \mathbb{R}, \]
(7.31) implies 
\[ U(1)^* N U(1) \subseteq M. \] (7.30)
On the other hand, by sandwiching this relation by \( \text{Ad} J_M \) and using (7.24), we obtain
\[ M' = J_M M J_M \supset J_M U(1)^* N U(1) J_M = U(1)^* J_N N J_N U(1) = U(1)^* N' U(1). \]
Passing to the commutants, this inclusion relation yields
\[ M \subset U(1)^* N U(1), \]
which together with (7.30) gives \( U(1)^* N U(1) = M \), that is
\[ N = U(1)^* M U(1)^*. \] (7.31)
For property (6) in Theorem 2.1 we notice that (7.21) and the negative half-sided modular inclusion assumption (7.19) imply that \( W(t) N W(t)^* \subset N \) for all \( t \leq 0 \), what is by (7.21) equivalent to
\[ U(s) N U(s)^* \subset N, \quad 0 \leq s \leq 1. \]
Using (7.30), (7.22) and (7.26), we get for every \( 0 \leq s \leq 1 \)
\[ U(s) M U(s)^* = U(s) U(1)^* N U(1) U(s)^* = U(1)^* U(s) N U(s)^* U(1) \]
\[ \subset U(1)^* N U(1) = M. \]
Using now induction on \( n \), it follows that
\[ U(s) M U(s)^* \subset M, \quad 0 \leq s \leq n \]
holds for every integer \( n \geq 1 \), that is \( U(s) M U(s)^* \subset M \) for all \( s \geq 0 \).
(7) is an immediate consequence of (6), (5) and (4).

**Step 8. Invariance properties of \( U(\cdot) \) with respect to \( \varphi \).**

We show in the following that \( \mathbb{R} \ni s \mapsto U(s) \in B(H) \) satisfies property (8) in Theorem 2.1, hence also property (9), which is an immediate consequence of (8).

For any \( y \in \mathcal{N}_\varphi \) and \( t \in \mathbb{R} \), (2.26) yields
\[ \Delta_M^{it} y \Delta_N^{-it} \in \mathcal{M}_\varphi \subset \mathcal{N}_\varphi \text{ and } \left( \Delta_N^{it} y \Delta_N^{-it} \right)_\varphi = \left( \Delta_M^{it} y \Delta_N^{-it} \right)_\varphi = \Delta_N^{it} y \varphi. \]
Setting \( s = 1 - e^{2\pi t} \) and using (7.21), we obtain
\[ U(s)^* y U(s) = \Delta_M^{-it} \Delta_N^{it} y \Delta_N^{-it} \Delta_M^{it} = \Delta_M^{-it} \mathcal{M}_\varphi \Delta_M^{it} \subset \Delta_M^{-it} \mathcal{N}_\varphi \Delta_M^{it} \subset \mathcal{N}_\varphi, \]
\[ \left( U(s)^* y U(s) \right)_\varphi = \Delta_M^{-it} \left( \Delta_N^{it} y \Delta_N^{-it} \right)_\varphi = \Delta_M^{-it} \left( \Delta_N^{it} y \Delta_N^{-it} \right)_\varphi = \Delta_M^{-it} \Delta_N^{it} y \varphi \]
\[ = U(s)^* y \varphi. \]
Therefore we have for all \( s \in \mathbb{R}, s < 1, \)
\[ U(s)^* y U(s) \in \mathcal{N}_\varphi \text{ and } \left( U(s)^* y U(s) \right)_\varphi = U(s)^* y \varphi. \] (7.32)
Moreover, according to the Lebesgue continuity result Proposition 2.1 (7.32) holds also for \( s = 1 \).
According to (7.19), \( \gamma_1 : M \ni x \mapsto U(1)^* x U(1)^* \in N \) is a \( \ast \)-isomorphism with inverse \( \gamma_1^{-1} : N \ni y \mapsto U(1)^* y U(1) \in M \). The modular automorphism groups of
the normal semifinite faithful weights ψ and φ ◦ γ₁⁻¹ are equal. Indeed, by (7.14) we have for every y ∈ N and t ∈ ℝ:

\[ σ_t^{φ ◦ γ_1^{-1}}(y) = γ_1 \left( σ_t^φ \left( γ_1^{-1}(y) \right) \right) = U(1) \Delta_M^t U(1)^* y U(1) \Delta_M^t U(1)^* = \Delta_N^t y \Delta_N^{−it} = σ_t^φ(y). \]

On the other hand, for every y ∈ Mφ, using (7.32) with s = 1, we obtain

\[ ψ(y^s y) = \| y \|_φ^2 = \| U(1)^* y \|_φ^2 = \| γ_1^{-1}(y) \|_φ^2 = φ ◦ γ_1^{-1}(y^s y), \]

so φ ◦ γ₁⁻¹ and ψ coincide on Mφ. Thus [28], Proposition 5.9 yields ψ = φ ◦ γ₁⁻¹, that is

\[ φ ◦ γ_1 = ψ ◦ γ_1 = φ. \] (7.33)

In particular, for every x ∈ M and n ≥ 0,

\[ x ∈ Mᵦ ⇐⇒ γ_n(x) ∈ Mᵦ. \] (7.34)

Let x ∈ Mφ be arbitrary. By (7.33) and (7.31) we have γ₁(x) ∈ Mφ, so holds with y = γ₁(x) and any 0 ≤ s ≤ 1. Using the group property of U(·), we deduce that, for every 0 ≤ s ≤ 1,

\[ γ_s(x) = U(s)xU(s)^* = U(1 − s)^*γ₁(x)U(1 − s) ∈ Mᵦ \quad \text{and} \quad γ_s(x)φ = U(1 − s)^*γ₁(x)φ. \] (7.35)

In particular, for s = 0 we get xφ = U(1)^*γ₁(x)φ, that is γ₁(x)φ = U(1)xφ. Thus (7.35) yields

\[ γ_s(x) ∈ Mᵦ \quad \text{and} \quad γ_s(x)φ = U(s)xφ, \quad 0 ≤ s ≤ 1. \] (7.36)

Iterating (7.35), we obtain

\[ x ∈ Mᵦ, \ s ≥ 0 \quad ⇒ \ γ_s(x) ∈ Mᵦ \quad \text{and} \quad γ_s(x)φ = U(s)xφ. \] (7.37)

On the other hand, for x ∈ M and s ≥ 0, denoting by n the integer part of s, that is the integer n ≥ 0 with n ≤ s < n + 1, we have

\[ γ_s(x) ∈ Mᵦ \quad \text{and} \quad γ_{n+1}(x) = γ_{n+1−s}(γ_s(x)) ∈ Mᵦ \quad x ∈ Mᵦ. \]

Consequently property (8) in Theorem 2.1 is satisfied.

**Step 9. Description of the generator P.**

First we verify that statement (10) in Theorem 2.1 holds with P defined by (7.20).

We recall from Subsection (b) of Section 2 that, if we endow ℝ² with the Lie group structure defined by the composition law

\[ (s_1, t_1) \cdot (s_2, t_2) = (s_1 + e^{-2\pi t_1}s_2 + t_1 + t_2), \]

then ℝ² ⊃ (s, t) → T_s L_t ∈ P⁺₁(1) is a Lie group isomorphism. Hence, by (7.27),

\[ π : P⁺₁(1) ⊃ T_s L_t → U(s)Δ_M^{it} \]

is an so-continuous unitary representation on H and, according to (7.21), the group π(P⁺₁(1)) contains \{Δ_M^{it}, Δ_N^{st} : t, s ∈ ℝ\} and is generated by this set.
Let us consider the elements \( X_1, X_2, X_3 \) of the Lie algebra \( p_+^1(1) \equiv g \) defined in (2.13). By (2.15) and by the definition of \( \pi \), taking into account how \( P_+^1(1) \) was identified in Subsection (b) of Section 2 with \( \mathcal{G} \), we obtain for every \( t \in \mathbb{R} \):

\[
\pi\left( \exp(t X_1) \right) = \pi(L_t) = \Delta_M^t = \exp \left( it \log \Delta_M \right),
\]

\[
\pi\left( \exp(t X_2) \right) = \pi(T_{1-e^{-2\pi t}}) = U \left( 1 - e^{-2\pi t} \right) \Delta_M^t = \exp \left( it \log \Delta_N \right),
\]

\[
\pi\left( \exp(t X_3) \right) = \pi(T_t) = \Delta_N^t = \exp \left( it \log \Delta_N \right),
\]

Therefore, according to (2.16),

\[
\frac{d}{dt} \pi(X_1) = i \log \Delta_M,
\]

\[
\frac{d}{dt} \pi(X_2) = i \log \Delta_N,
\]

\[
\frac{d}{dt} \pi(X_3) = i P.
\]

Since any two of \( X_1, X_2, X_3 \) is a basis for \( g^1_+ \) and \( P_+^1(1) \) is connected and simply connected, the representation \( \pi \) is uniquely determined by any two of the relations (7.38) (see e.g. [2], Ch. 11, §5 or [30], Proposition 10.5.2).

Now, by (2.14), (2.18) and (7.38), we conclude that

\[
iP = \frac{1}{2\pi} d\pi(X_3) = \frac{1}{2\pi} \frac{d\pi(X_2 - X_1)}{d\pi(X_1) - d\pi(X_2)}
\]

\[
= \frac{1}{2\pi} \left( \log \Delta_N - \log \Delta_M \right),
\]

hence \( P \) is the closure of \( \frac{1}{2\pi} \left( \log \Delta_N - \log \Delta_M \right) \).

\[\square\]

8. Complements to the implementation theorem of Borchers and the proof of Theorem 2.2

First we prove Theorem 2.14 which will then be used to prove Theorem 2.2.

**Proof of Theorem 2.14**

**Step 1. The existence and the uniqueness of \( \pi \)** (it is essentially the proof of [1], Theorem 3.1 and [41], Corollary 5.7).

By the lower boundedness of \( P \) we have \( d_o = \exp(-P) \in B(H) \). Moreover, \( d_o \) is clearly positive and injective. Denoting \( \beta_s = \alpha_{-s}, (\beta_s)_{s \in \mathbb{R}} \) is an \( \ast \)-continuous group of \( \ast \)-automorphisms of \( M \) such that

\[
\beta_s(x) = d_o^{is} x d_o^{-is}, \quad s \in \mathbb{R}, \ x \in M.
\]

We recall that, according to [41], Theorem 1.4, we have for every \( \lambda \in \mathbb{R} \):

\[
M^\alpha([-\lambda, +\infty)) = M^\beta((-\infty, \lambda]) =
\]

\[
\left\{ x \in \bigcap_{z \in \mathbb{C}} \text{Dom} (\alpha_z) : \|\alpha_z(x)\| \leq e^{\lambda z} \|x\| \right\} =
\]

\[
\left\{ x \in \bigcap_{z \in \mathbb{C}} \text{Dom} (\beta_z) : \|\beta_z(x)\| \leq e^{-\lambda z} \|x\| \right\}.
\]

(8.1)
Denoting

\[ H_\lambda = \text{the closed linear span of } M^\alpha([-\lambda, +\infty)) H = \text{the closed linear span of } M^\beta((-\infty, \lambda]) H, \]

we have clearly \( H_\lambda \supset M^\beta((-\infty, 0]) H \supset 1_H H = H \), hence \( H_\lambda = H \), for all \( \lambda \geq 0 \).

In particular, \( H \) is an invariant subspace of support \( 1_H \) relative to \( \beta \), as defined in Section 5 of [44]. Moreover, since the spectral subspace of \( d_\sigma \) corresponding to \( (0, \|d_\sigma\|] \) is \( H \), the second statement in [44, Theorem 5.3] implies that \( H \) is simply invariant, that is \( \bigcap_\lambda H_\lambda = \{0\} \). Furthermore, \( M^\beta H \subset H_\lambda \) implies that the orthogonal projection \( p_\lambda \) onto \( H_\lambda \) belongs to \( M \).

Using now the first statement in [44, Theorem 5.3], it follows that there exists an injective \( b \in B(H) \), \( 0 \leq b \leq 1_H \), such that

\[ \beta_\lambda(x) = b^s x b^{-is}, \quad s \in \mathbb{R}, \; x \in M \]  

(8.2)

and, for every \( \lambda \in \mathbb{R} \), the spectral projection \( \chi_{(0, e)}(b) \) is the orthogonal projection onto \( H_{\lambda+0} = \bigcap_\mu H_\mu \), hence it is equal to \( p_{\lambda+0} = \lim_{\lambda < \mu \to \lambda} p_\mu \in M \). In particular, \( b \in M \).

Property (i) in the statement of Theorem 2.4 holds by [44]. In order to verify property (ii), let \( d \) be an arbitrary injective operator in \( M \) such that \( 0 \leq d \leq 1_H \) and \( \alpha_s(x) = d^{is} x d^{-is}, s \in \mathbb{R}, x \in M \), that is \( \beta_\lambda(x) = d^{is} x d^{-is}, s \in \mathbb{R}, x \in M \).

Since the spectral subspace of the unitary group \( (d^s)_{s \in \mathbb{R}} \) corresponding to \( (-\infty, \lambda] \) is \( \chi_{(0, e)}(d) H \), [44, Corollary 2.6] yields

\[
M^\beta((-\infty, \lambda]) H = M^\beta((-\infty, \lambda]) \chi_{(0, e)}(d) H \subset \chi_{(0, e)}(d) H, \quad \lambda \in \mathbb{R}.
\]

Consequently, \( \chi_{(0, e)}(b) = p_{\lambda+0} \leq \chi_{(0, e)}(d) \) for all \( \lambda \in \mathbb{R} \).

The uniqueness of \( b \) is an immediate consequence of (ii).

**Step 2. Proof of (iii) and (iv).**

(iii) is clear from the construction of \( b \) in Step 1.

In order to verify (iv), let \( \sigma \) be a *-automorphism of \( M \) such that, for some \( \lambda_\sigma > 0 \),

\[ \sigma \circ \alpha_s = \alpha_{\lambda_\sigma s} \circ \sigma, \quad s \in \mathbb{R}. \]

Then holds clearly

\[ \sigma \circ \alpha_s = \alpha_{\lambda_\sigma s} \circ \sigma, \quad s \in \mathbb{C}. \]  

(8.3)

There exists a faithful unital normal *-representation \( \pi : M \to B(K) \), which is covariant with respect to \( \sigma \), that is \( \pi\{\sigma(x)\} = U \pi(x) U^*, x \in M \), where \( U \) is an appropriate unitary on \( K \): for example, we can choose

\[ K = l^2(\mathbb{Z}; H), \]

the space of all square-summable two-sided sequences in \( H \),

\[
\pi(x) (\xi_k)_{k \in \mathbb{Z}} = \left( \sigma^k(x) \xi_k \right)_{k \in \mathbb{Z}} \quad \text{for} \quad x \in M, \quad (\xi_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}; H),
\]

\[
U (\xi_k)_{k \in \mathbb{Z}} = (\xi_{k+1})_{k \in \mathbb{Z}} \quad \text{for} \quad (\xi_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z}; H).
\]

Then \( (\pi \circ \alpha_s \circ \pi^{-1})_{s \in \mathbb{R}} \) is an so-continuous one-parameter group of *-automorphisms of the von Neumann algebra \( \pi(M) \subset B(K) \), \( \pi(b) \) is an injective element of \( \pi(M) \).
with \(0 \leq \pi(b) \leq 1_K\) and
\[
\left(\pi \circ \alpha_s \circ \pi^{-1}\right)(\pi(x)) = \pi(b)^{-is}\pi(x)\pi(b)^{is}, \quad s \in \mathbb{R}, \ x \in M.
\]
Moreover, by the definition of \(b\), for any injective \(\pi(d) \in \pi(M)\), \(0 \leq \pi(d) \leq 1_K\), such that
\[
\left(\pi \circ \alpha_s \circ \pi^{-1}\right)(\pi(x)) = \pi(d)^{-is}\pi(x)\pi(d)^{is}, \quad s \in \mathbb{R}, \ x \in M,
\]
we have
\[
\chi_{\left(0,\varepsilon\lambda\right)}(\pi(b)) \leq \chi_{\left(0,\varepsilon\lambda\right)}(\pi(d)), \quad \lambda \in \mathbb{R}.
\]
Applying the above proved (iii) to \(\pi(M)\), \((\pi \circ \alpha_s \circ \pi^{-1})_{s \in \mathbb{R}}\), \(\pi(b)\) instead of \(M\), \(\alpha\), \(b\), we obtain that, for every \(\lambda \in \mathbb{R}\), \(\pi\left(\chi_{\left(0,\varepsilon\lambda\right)}(b)\right) = \chi_{\left(0,\varepsilon\lambda\right)}(\pi(b))\) is the orthogonal projection onto
\[
\bigcap_{\mu > \lambda} \text{the closed linear span of } \pi(M)^{\pi \circ \alpha_s \circ \pi^{-1}}([-\mu, +\infty)) K = \bigcap_{\mu > \lambda} \text{the closed linear span of } \pi\left(M^{\alpha}\left([-\mu, +\infty)\right)\right) K
\]
For every \(\lambda \in \mathbb{R}\) and \(x \in M\), by (8.1) and by (8.3), the following four conditions are equivalent:
\[
x \in M^{\alpha}\left([-\lambda, +\infty)\right),
\]
\[
x \in \bigcap_{z \in \mathbb{C}, \exists z \geq 0} \text{Dom} (\alpha_z) \quad \text{and} \quad \|\sigma(\alpha_z(x))\| = \|\alpha_z(x)\| \leq e^{\lambda|z|}\|x\| \quad \text{for all } z \in \mathbb{C}, \ \exists z \geq 0,
\]
\[
\sigma(x) \in \bigcap_{z \in \mathbb{C}, \exists z \geq 0} \text{Dom} (\alpha_z) \quad \text{and} \quad \|\alpha_{\lambda z}(\sigma(x))\| \leq e^{\lambda|z|}\|x\| \quad \text{for all } z \in \mathbb{C}, \ \exists z \geq 0,
\]
Therefore
\[
\sigma\left(M^{\alpha}\left([-\lambda, +\infty)\right)\right) = M^{\alpha}\left([-\lambda^{-1}\lambda, +\infty)\right), \quad \lambda \in \mathbb{R}.
\]
Let next \(\lambda \in \mathbb{R}\) be arbitrary. By the covariance property of \(\pi\) and by (8.3), we have for every \(\mu > \lambda\):
\[
U \pi\left(M^{\alpha}\left([-\mu, +\infty)\right)\right) K = U \pi\left(M^{\alpha}\left([-\mu, +\infty)\right)\right) U^* K = \pi\left(\sigma\left(M^{\alpha}\left([-\mu, +\infty)\right)\right)\right) K = \pi\left(M^{\alpha}\left([-\lambda^{-1}\mu, +\infty)\right)\right) K.
\]
Consequently
\[
U \pi\left(\chi_{\left(0,\varepsilon\lambda\right)}(b)\right) K = \pi\left(\chi_{\left(0,\varepsilon\lambda^{-1}\lambda\right)}(\pi(b))\right) K, \quad \text{and so}
\]
\[
\pi\left(\sigma\left(\chi_{\left(0,\varepsilon\lambda\right)}(b)\right)\right) = U \pi\left(\chi_{\left(0,\varepsilon\lambda\right)}(b)\right) U^* = \pi\left(\chi_{\left(0,\varepsilon\lambda^{-1}\lambda\right)}(b)\right),
\]
\[
\chi_{\left(0,\varepsilon\lambda\right)}(\sigma(b)) = \sigma\left(\chi_{\left(0,\varepsilon\lambda\right)}(b)\right) = \chi_{\left(0,\varepsilon\lambda^{-1}\lambda\right)}(b) = \chi_{\left(0,\varepsilon\lambda\right)}(b^{\lambda_\mu}). \quad (8.5)
\]
Now, by (8.5) we conclude that \(\sigma(b) = b^{\lambda_\mu}\). \qed

Proof of Theorem 2.22
Proof of (1). Let us consider \( \gamma_s = \text{Ad} U(s) \) for all \( s \in \mathbb{R} \) (not only for \( s \geq 0 \)) and let \( z \) be an arbitrary selfadjoint element of the center \( Z(M) \) of \( M \). By (6) in Theorem 2.1 we have \( \gamma_s(z) \in M \) for all \( s \geq 0 \), hence \( z \) and \( \gamma_s(z) \) commute for any \( s \geq 0 \). But then the elements of the set \( \{ \gamma_s(z) : s \in \mathbb{R} \} \) are mutually commuting, so the von Neumann algebra \( C \) generated by this set is commutative. Since \( \gamma_s = \text{Ad}(U(s)|M) \leq C \) invariant for every \( s \in \mathbb{R} \) and \( U(s) = \exp(isP), s \in \mathbb{R} \), for some positive selfadjoint operator \( P \) in \( H \), according to the implementation theorem of Borchers [5] (see also Theorem 2.14) there exists an element \( b \in C \), \( 0 \leq b \leq 1_H \), such that
\[
\gamma_s(x) = b^{-is}x b^{is} = x, \quad x \in C, \quad s \in \mathbb{R}.
\]
Consequently, \( \gamma_s(z) = z \) for all \( s \in \mathbb{R} \).

Proof of (2). By (5) in Theorem 2.1 and by the above proved (1), \( Z(N) = Z(\gamma_1(M)) = Z(M) \). Now it is easy to see that the projection family \( \{ q \in Z(M) : Mq = Nq \} \) is upward directed and its lowest upper bound is the greatest projection \( p \in Z(M) \) satisfying \( M \leq p \).

The implication \( e \leq p \implies U(s)e = e \) holds for any projection \( e \in M \), because \( M \leq p \implies \varphi_p = \psi_p \), \( \Delta^u_M p = \Delta^u_N p, t \in \mathbb{R} \implies U(s)p = p, s \in \mathbb{R} \).

Now let \( e \in M \) be an arbitrary projection such that
\[
U(s)e = e, \quad s \in \mathbb{R}.
\]
(8.6)

For every \( a \in \mathfrak{A}_\varphi \) and \( x, y \in \mathfrak{H}_\varphi \), using (8) in Theorem 2.1 we deduce that
\[
(U(s)e J_M x_\varphi | J_M y_\varphi) \overset{8.7}{=} (\gamma_s(a)e J_M x_\varphi | J_M y_\varphi) = (e J_M x_\varphi | \gamma_s(a^*) J_M y_\varphi)
\]
\[
\overset{8.8}{=} (e J_M x_\varphi | J_M y_\varphi \gamma_s(a^*)_{\varphi})
\]
\[
= (J_M y^* J_M e J_M x_\varphi | U(s)(a^*)_{\varphi})
\]
\[
= (e J_M y^* x_\varphi | U(s)(a^*)_{\varphi})
\]
\[
\overset{8.9}{=} (e J_M y^* x_\varphi | (a^*)_{\varphi})
\]
do not depend on \( s \geq 0 \), so \( \left( 1_H - U(s) \right) a e J_M x_\varphi | J_M y_\varphi = 0 \) for all \( s \geq 0 \).

By \( \{ J_M x_\varphi : x \in \mathfrak{H}_\varphi \} = H \) and \( \mathfrak{A}_\varphi^{\psi_0} = M \), we get
\[
(1_H - U(s)) Me H = \{ 0 \}, \quad s \in \mathbb{R}.
\]
Since the orthogonal projection onto the closed linear span of \( Me H \) is the central support \( z(e) \in Z(M) \) of \( e \), we obtain that \( (1_H - U(s)) z(e) = 0 \) for all \( s \geq 0 \), hence \( U(s)z(e) = z(e), s \in \mathbb{R} \).

Consequently, \( Nz(e) = U(1)M U(1)^* z(e) = Mz(e) \) and so \( e \leq z(e) \leq p \).

Finally, let \( e \in M^e \) be a projection such that
\[
U(s)e = e U(s), \quad s \in \mathbb{R}, \quad (8.7)
\]
\[
U(s) e J_M e J_M = e J_M e J_M, \quad s \in \mathbb{R}. \quad (8.8)
\]
If \( \pi = \pi_{eJ_a} : eMe \to B(eJ_a eJ_a H) \) is the \(*\)-representation defined in (8.7), then, for every \( s \geq 0 \), (8.7) and (8.8) yield that \( \gamma_s(eMe) \subset eMe \) and
\[
\pi(\gamma_s(a)) = U(s)a U(-s) | eJ_M eJ_M H = a | eJ_M eJ_M H = \pi(a), \quad a \in eMe.
\]
Since \( \pi \) is faithful, we obtain that \( U(s)a U(-s) = \gamma_s(a) = a \) for all \( s \geq 0 \) and all \( a \in eMe \). In other words, every \( U(s) \) commutes with every operator in \( eMe \).

Consequently, for every \( s \in \mathbb{R} \), the unitary \( U(s) | eH : eH \to eH \) belongs to the commutant of the reduced von Neumann algebra \( \{ x | eH : eH \to eH ; x \in eMe \} \), hence to the induced von Neumann algebra \( \{ x' | eH : eH \to eH ; x' \in M' \} \). Since the kernel of the induction \(*\)-homomorphism \( M' \ni x' \mapsto x' | eH \) is \( M'(1_H - z(e)) \), where \( z(e) \in Z(M) \) stands for the central support of \( e \), there exists a one-parameter group \((u'_s)_{s \in \mathbb{R}}\) of unitaries in \( M'z(e) \) such that
\[
U(s) | eH = u'_s | eH, \text{ that is } U(s)e = u'_se, \quad s \in \mathbb{R}.
\]
Setting \( u_s = J_M u'_s J_M \), \((u_s)_{s \in \mathbb{R}}\) is a one-parameter group of unitaries in \( Mz(e) \) such that
\[
U(s)J_M eJ_M = J_M U(-s) eJ_M = J_M u'_s J_M = u_s J_M eJ_M, \quad s \in \mathbb{R}.
\]
Therefore we have, for every \( s \geq 0 \) and \( a \in Mz(e) \),
\[
\gamma_s(a)J_M eJ_M = J_M eJ_M U(s) a U(-s) J_M eJ_M = J_M eJ_M u_s a u_{-s} J_M eJ_M
\]
\[
= (u_s a u_{-s}) J_M eJ_M.
\]
Since the kernel of the induction \(*\)-homomorphism \( M \ni x \mapsto x | J_M eJ_M H \) is equal to \( M(1_H - z(J_M eJ_M)) = M(1_H - z(e)) \), we obtain that
\[
\gamma_s(a) = u_s a u_{-s}, \quad s \geq 0, \quad a \in Mz(e).
\]
In particular, \( Nz(e) = \gamma_1(Mz(e)) = u_1 Mz(e) u_{-1} = Mz(e) \), and so \( e \leq z(e) \leq p \).

**Proof of (3).** Let us assume that \( M_{-\infty} = \bigcap_{s \geq 0} \gamma_s(M) \) contains \( M^p \).

Ad \( U(s) \) leaves \( M_{-\infty} \) invariant for every \( s \in \mathbb{R} \), defining thus an \( s \)-continuous one-parameter group \((\alpha_s)_{s \in \mathbb{R}}\) of \(*\)-automorphisms of \( M_{-\infty} \). Using Theorem 2.14 we get an injective \( b \in M_{-\infty} \), \( 0 \leq b \leq 1_H \), such that
\[
\alpha_s(x) = b^{-is} x b^{is}, \quad s \in \mathbb{R}, \quad x \in M_{-\infty} \tag{8.9}
\]
and
\[
\sigma \text{ *-automorphism of } M_{-\infty} \text{ and } \lambda_\sigma > 0 \quad \left\{ \begin{array}{l}
\sigma \circ \alpha_s = \alpha_{\lambda_\sigma \sigma} \circ \sigma \text{ for all } s \in \mathbb{R} \\
\end{array} \right. \implies \sigma(b) = b^{\lambda_\sigma}. \tag{8.10}
\]
Since \( b \in M_{-\infty} \), (8.10) yields that
\[
U(s)b U(s)^* = \alpha_s(b) = b, \text{ that is } b \text{ commutes with } U(s), \quad s \in \mathbb{R}. \tag{8.11}
\]
In particular, \( b \in U(1) M U(1)^* = N \). Furthermore, by (1) in Theorem 2.1 we have
\[
\sigma_t^p \circ \gamma_s = \gamma_{-2\pi t} \circ \sigma_t^p, \quad s, t \in \mathbb{R}, \tag{8.12}
\]
so \( M_{-\infty} \) is left invariant by all \( \sigma_t^p \). Therefore, applying (8.10) with \( \sigma = \sigma_t^p \), we get
\[
\sigma_t^p(b) = b^{e^{-2\pi t}}, \quad t \in \mathbb{R}. \tag{8.13}
\]
Let \( \chi(\lambda) \) be the characteristic function of \( \{ \lambda \} \subset \mathbb{R} \). For every \( t \in \mathbb{R} \) we have

\[
\sigma_t^\varphi(b) = b e^{-2\pi it} \implies \sigma_t^\varphi(\chi(\lambda)(b)) = \chi(\lambda)(b),
\]
so \( \chi(\lambda)(b) \in M^\varphi \). On the other hand, (8.11) implies that \( \chi(\lambda)(b) \) commutes with all \( U(s) \). Therefore \( \chi(\lambda)(b) \in N \) commutes with all \( \Delta_N^it = U(1)\Delta_M^itU(1)^* \), and so it belongs to \( N^\psi \).

Let us denote

\[
e_0 = 1_H - \chi(\lambda)(b) \in M^\varphi \cap N^\psi,
\varphi_0 = \text{the restriction of } \varphi \text{ to } e_0 Me_0,
\psi_0 = \text{the restriction of } \psi \text{ (hence also of } \varphi \text{) to } e_0 Ne_0,
b_o = e_0be_0 = e_0b \in e_0Me_0.
\]

By [28], Proposition 4.1 and Theorem 4.6 (see also [33], Propositions 4.5 and 4.7), \( \varphi_0 \) is a normal semifinite faithful weight and its modular group is the restriction of the modular group of \( \varphi \) to \( e_0Me_0 \). Similarly, \( \psi_0 \) is a normal semifinite faithful weight and its modular group is the restriction of the modular group of \( \psi \) to \( e_0Ne_0 \). In particular, by (8.13), we have

\[
\sigma_t^{\varphi_0}(b_o) = b_o e^{-2\pi it}, \quad t \in \mathbb{R}.
\]

Since \( 0 \leq b_o \leq e_o \) and the supports of both \( b_o \) and \( e_o - b_o \) are equal to the unit \( e_o Me_o \), \( -\log b_o \) is a positive selfadjoint linear operator, of support \( e_o \) and affiliated with \( e_0Me_0 \). Consequently, defining

\[
u_s = (-\log b_o)^i s, \quad s \in \mathbb{R},
\]

\((u_s)_{s \in \mathbb{R}}\) is a strongly continuous one-parameter group of unitaries in \( e_0Me_0 \) and (8.14) yields

\[
\sigma_t^{\varphi_0}(u_s) = e^{-2\pi i ts} u_s, \quad s, t \in \mathbb{R}.
\]

Now the characterization theorem of M. Landstad [33], Theorem 2 (see also [34], Theorems I.3.3 and I.3.4, or [33], Theorem 19.9) implies that the von Neumann algebra \( e_0Me_0 \) is generated by \( (e_0Me_0)^{\varphi_0} = e_oM^\varphi e_0 \) and by \( e_0 \), that is

\[
e_0M^\varphi e_o \text{ and } b_o \text{ generate the von Neumann algebra } e_0Me_0.
\]

Since \( M^\varphi \subset M_{-\infty} \) and \( b \in M_{-\infty} \), we get that \( e_0Me_0 \subset M_{-\infty} \subset \gamma_1(M) = N \), that is \( e_0Me_o = e_oNe_0 \). Consequently \( \varphi_0 = \psi_0 \), and so the modular groups \( \sigma^\varphi \) and \( \sigma^\psi \) have the same restriction \( \sigma^{\varphi_0} = \sigma^{\psi_0} \) on \( e_0Me_o = e_oNe_0 \). Using (3) in Theorem 2.1 we obtain for every \( x \in e_0M^\varphi e_o \subset M^\varphi \cap N^\psi \) and \( t \in \mathbb{R} \):

\[
U(1 - e^{2\pi it})xU(1 - e^{2\pi it})^* = \Delta_N^{-it} \Delta_M^{-it} x \Delta_M^{-it} \Delta_N^{-it} = \Delta_N^{-it} \sigma_t^{\varphi_0}(x) \Delta_N^{it} = \sigma^{\psi_0}(x) = x.
\]

Therefore \( e_0M^\varphi e_o \subset \{ x \in M ; U(s)x = xU(s), s \in \mathbb{R} \} \), which yields together with (8.11) and (8.15)

\[
e_0M e_o \subset \{ x \in M ; U(s)x = xU(s), s \in \mathbb{R} \}.
\]

In other words, every \( \alpha_s \) acts identically on \( e_0Me_o \subset M_{-\infty} \). By (8.9) we conclude that \( b_o \) belongs to the center of \( e_0Me_0 \).

Since \( b_o \in Z(e_0Me_o) \) is invariant under the modular automorphism group of \( \varphi_0 \), which coincides with the restriction of the modular automorphism group of \( \varphi \) to \( e_0Me_o \) as discussed above, we have \( b_o \in M^\varphi \). Taking into account (8.11), we obtain

\[
b e^{-2\pi it} e_o = b_o e^{-2\pi it} = \sigma_t^{\varphi_0}(b_o) = b_o = b e_o, \quad t \in \mathbb{R},
\]
which is possible only if $e_o = 0$. Consequently $\chi_{\{1\}}(b) = 1_H$, that is $b = 1_H$. But then every $\alpha_s = \Ad(U(s)|\M_{-\infty}$ acts identically on $\M_{-\infty}$, hence

$$
\M_{-\infty} \subset \{x \in \M; U(s)x = xU(s), s \in \R\} = \{x \in \M; \gamma_s(x) = x, s \geq 0\}.
$$

Proof of (4). Let us first assume that $\M_{\varphi} \cap \M^{p_{\varphi}} = \M^{\varphi}$. Taking into account (8) in Theorem 2.1, we have successively

$$
\text{once we show that}
$$

$$
s > 0, x \in \M_{\varphi} \cap \M^{\varphi} \implies U(s)x_{\varphi} = x_{\varphi}.
$$

For let $s > 0$ and $x \in \M_{\varphi} \cap \M^{\varphi}$ be arbitrary. Using (8) in Theorem 2.1 and (8.12), we get successively

$$
\Delta_M^H U(s)x_{\varphi} = \sigma^H(\gamma_s(x))_{\varphi} = \gamma_{-2\pi it_s}(x)_{\varphi} = U(e^{-2\pi it_s})x_{\varphi} \xrightarrow{t \to \pm \infty} x_{\varphi},
$$

$$
\|U(s)x_{\varphi} - x_{\varphi}\| = \|\Delta_M^H U(s)x_{\varphi} - x_{\varphi}\| = \|\Delta_M^H U(s)x_{\varphi} - \sigma^H(x)_{\varphi}\| \xrightarrow{t \to \pm \infty} 0.
$$

For the second implication we prove that if

$$
\M^{\varphi} \subset \{x \in \M; \gamma_s(x) = x, s \geq 0\} \quad (8.16)
$$

and $p \neq 1_H$, then $M(1_{H-p})$ is of type III$_1$. Then also $N(1_{H-p}) = \gamma_1(M(1_{H-p}))$ will be of type III$_1$.

Taking into account (2.21), we have to prove that

$$
e \in \M^{\varphi} \text{ projection}, 0 \neq e \leq 1_H - p \implies \sigma(\Delta_M | eJ_M eJ_M H) = [0, +\infty).
$$

For this purpose, let the projection $e \in \M^{\varphi}$, $0 \neq e \leq 1_H - p$, be arbitrary. By the assumption (8.10) we have $\gamma_s(e) = e$ for all $s \geq 0$, hence (8.9) holds. Since $0 \neq e \leq 1_H - p$, the above proved (2) entails that (8.8) does not hold, that is $U(s) eJ_M eJ_M \neq eJ_M eJ_M$ for some $s \in \R$. Nevertheless, by (8.7) and by (2) in Theorem 2.1 all $U(s)$ commute with $eJ_M eJ_M$. Since $e \in \M^{\varphi}$, also every $\Delta_M^H$ commutes with $eJ_M eJ_M$.

According to (2.21), $U(i) = \exp(-P) \in B(H)$ is injective and $0 \leq U(i) \leq 1_H$. By the commutation relation (1) in Theorem 2.1 we have

$$
\Delta_M^H U(i) \Delta_M^{it} = U(i)e^{-2\pi it}, \quad t \in \R. \quad (8.17)
$$

Consequently, the spectral projection $f_o = \chi_{\{1\}}(U(i))$ commutes with every $\Delta_M^H$. On the other hand, $f_o$ clearly commutes with every $U(s)$. Finally, the commutation of $eJ_M eJ_M$ with all $U(s)$ implies that the projections $eJ_M eJ_M$ and $f_o$ commute.

We have already seen that $U(s) eJ_M eJ_M \neq eJ_M eJ_M$ for some $s \in \R$. On the other hand, since $U(s) = U(i)^{-is}$, we have $U(s)f_o = f_o$ for every $s \in \R$. Therefore $eJ_M eJ_M \not\subset f_o$, and so the projection

$$
f_1 = eJ_M eJ_M - f_o eJ_M eJ_M \leq eJ_M eJ_M
$$

is not zero. Since all $\Delta_M^H$ and all $U(s)$ commute with $eJ_M eJ_M$ and with $f_o$, they commute also with $f_1$. Therefore, we can define an so-continuous one-parameter group $(v_t)_{t \in \R}$ of unitaries on $f_1H$ by setting

$$
v_t = \Delta_M^H f_1H = (\Delta_M f_1H)^{it}, \quad t \in \R, \quad (8.18)
$$

as well as the operator

$$
b_1 = U(i) f_1H : f_1H \to B(f_1H),
$$
for which $0 \leq b_1 \leq f_1$ and $b_1, f_1 - b_1$ are injective. From (5.14) we get successively

$$v_t b_1 v_t^* = b_1 e^{-2\pi t}, \quad t \in \mathbb{R},$$

$$v_t (- \log b_1)^i s v_t^* = e^{-2\pi t s i} (- \log b_1)^i s, \quad t, s \in \mathbb{R}.$$ 

Now the Stone-von Neumann Uniqueness Theorem for canonical commutation relations (see e.g. [2], Ch. 20, § 2 or [32]) entails that there exist a Hilbert space $K \neq \{0\}$ and a unitary operator $W : K \otimes L^2(\mathbb{R}) \rightarrow f_1 H$ such that

$$v_t = W \circ (1_K \otimes m_{-2\pi t}) \circ W^*, \quad (- \log b_1)^i s = W \circ (1_K \otimes \lambda_s) \circ W^*, \quad t, s \in \mathbb{R},$$

where $m_t$ is the multiplication operator with $e^{it \cdot}$ on $L^2(\mathbb{R})$, $\lambda_s$ is the translation operator $\xi \mapsto \xi(\cdot - s)$ on $L^2(\mathbb{R})$, $s \in \mathbb{R}$.

Using (5.15), we deduce that $\Delta_M f_1 H = W \circ (1_K \otimes m_{2\pi i}) \circ W^*$, where $m_{2\pi i}$ is the unbounded positive selfadjoint multiplication operator with $e^{-2\pi \cdot}$ in $L^2(\mathbb{R})$. Consequently, the spectrum of $\Delta_M | f_1 H$ is equal to the spectrum of $1_K \otimes m_{2\pi i}$, that is to $[0, +\infty)$. Since $f_1 \leq e^{J_M e^{J_M} H}$, we conclude that also the spectrum of $\Delta_M | e^{J_M e^{J_M} H}$ is equal to $[0, +\infty)$.

□

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