LOCAL UNIFORMIZATION THROUGH MONOMIALIZATION OF KEY ELEMENTS.

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Abstract. We give a new proof of the simultaneous embedded local uniformization Theorem in zero characteristic for essentially of finite type rings and for quasi excellent rings. The results are a consequence of the simultaneous monomialization presented here. The methods develop the key elements theory that is a more subtle notion than the notion of key polynomials.

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Part 1. Introduction.

The resolution of singularities can be formulated in the following way.

Let $V$ be a singular variety. The variety $V$ admits a resolution of singularities if there exists a smooth variety $W$ and a proper birational morphism $W \to V$.

This problem has been solved in many cases but remains an open problem in others. In characteristic zero Hironaka proved resolution of singularities in all dimensions ([13]) in 1964. So the problem remains open in positive characteristic.

The two-dimensional case has been solved by Abhyankar in 1956 ([1]) and the 3-dimensional case by Cossart and Piltant in 2014 ([7]). In higher dimensions some results were obtained, but not for every characteristic.

To try to solve this problem numerous methods were introduced, in particular Zariski and Abhyankar used the local uniformization. But it does not allow at the moment to solve completely the problem.

We are interested in a stronger problem than the local uniformization: the monomialization problem. In this work we solve the monomialization problem in characteristic zero. We hope that these methods, applicable in positive characteristic, may help to attack the global problem of resolution of singularities on a different point of view.

One of the essential tools to handle the monomialization or the local uniformization is a valuation. Let us look on an example how valuations naturally fit into the problem.

Let $V$ be a singular variety and $Z$ be an irreducible closed set of $V$. If we knew how to resolve the singularities of $V$, we would have a smooth variety $W$ and a proper birational morphism $W \to V$. In $W$, we can consider a irreducible set $Z'$ whose image is $Z$. And so the regular local ring $O_{W,Z'}$ dominates the non regular local ring $O_{V,Z}$. It means that we have an inclusion $O_{V,Z} \subseteq O_{W,Z'}$ and the maximal ideal of $O_{V,Z}$ is the intersection of those of $O_{W,Z'}$ with $O_{V,Z}$. Up to a blow-up $Z'$ is a hypersurface and so $O_{W,Z'}$ is dominated by a discrete valuation ring. In this case the valuation is the order of vanishing along the hypersurface.

Before stating the local uniformization Theorem, we need a classical notion that will be very important: the center of a valuation. For details, we can read ([29]) or ([22, sections 2 and 3]).

Let $K$ be a field and $\nu$ be a valuation defined over $K$. We set

$$R_\nu := \{ x \in K \text{ such that } \nu(x) \geq 0 \},$$

the valuation ring of $\nu$, and $m_\nu$ its maximal ideal.

We consider a subring $A$ of $K$ such that $A \subset R_\nu$. Then the center of $\nu$ in $A$ is the ideal $p$ of $A$ such that $p = A \cap m_\nu$.

Now we consider an algebraic variety $V$ over a field $k$ and $K$ its fractions field. Assume $V$ is an affine variety. Then $V = \text{Spec} (A)$ where $A$ is a finite type integral $k$-algebra with $A \subseteq K$. If $A \subseteq R_\nu$, then the center of $\nu$ over $V$ is the point $\zeta$ of $V$ which corresponds to the prime ideal $A \cap m_\nu$ of $A$.

The irreducible closed sub-scheme $Z$ of $V$ defined by $A \cap m_\nu$ (it means the image of the morphism $\text{Spec} (A/m_\nu) \to \text{Spec} (A)$) has a generic point $\xi$. Equivalently $\xi$ is the point associated to the zero ideal. We say that $Z$ is the center of $\nu$ over $V$.

Now let us state the local uniformization Theorem. It has been proved in characteristic zero but it is always a conjecture in positive characteristic.
Theorem (Zariski [29]). Let $X = \text{Spec}(A)$ be an affine variety of fractions field $K$ over a field $k$. We consider $\nu$ a valuation over $K$ of valuation ring $R_\nu$.

Then $A$ can be embedded in a regular local sub-ring $A'$ essentially of finite type over $k$ and dominated by $R_\nu$.

In this work we prove a stronger result: the simultaneous monomialization Theorem. We are going to explain what is the monomialization and what are the objects that we handle.

Let $k$ be a field of characteristic zero and $f \in k[u_1, \ldots, u_n]$ be a polynomial in $n$ variables, irreducible over $k$. We denote by $V(f)$ the hypersurface defined by $f$ and we assume that it has a singularity at the origin. Then we set $R := k[u_1, \ldots, u_n]_{(u_1, \ldots, u_n)}$. This is a regular local ring that is essentially of finite type over the field $k$. The vector $u = (u_1, \ldots, u_n)$ is a regular system of parameters of $R$. We use the notation $(R, u)$ to express the fact that $u$ is a regular system of parameters of the regular local ring $R$.

Definition (4.9). The element $f$ is monomializable if there exists a map $(R, u) \rightarrow (R', u' = (u'_1, \ldots, u'_n))$ that is a sequence of blow-ups such that the total transform of $f$ is a monomial. It means that in $R'$, the total transform of $f$ is $v \prod_{i=1}^n (u'_i)^{\alpha_i}$, with $v$ a unit of $R'$.

Now we can give a simplified version of one of the main theorems of this work.

Theorem (7.1). Let $(R, u)$ be a regular local ring that is essentially of finite type over a field $k$ of characteristic zero.

Then there exists a countable sequence of blow-ups

$$(R, u) \rightarrow \cdots \rightarrow (R_i, u^{(i)}) \rightarrow \cdots$$

that monomializes simultaneously all the elements of $R$.

Equivalently, it means that for each element $f$ in $R$, there exists an index $i$ such that in $R_i$, $f$ is one monomial.

If $f$ is an irreducible polynomial of $k[u_1, \ldots, u_n]$, then $A := \frac{k}{(f)}$ is a local domain. We can find a valuation $\nu$ over Frac $(A)$ centered in $R$. One consequence of Theorem (7.1) is that the total transform of $f$ in one of the $R_i$ is $v \prod_{j=1}^n (u_j^{(i)})^{\alpha_j}$. By the irreducibility of $f$ its strict transform is exactly $u_n^{(i)}$.

Hence there exists an embedding of $A$ into the ring $A' = \frac{R_i}{(u_n^{(i)})}$ which is dominated by $R_\nu$. So a consequence of Theorem (7.1) is the Local Uniformization Theorem as announced.

And we obtain a stronger result here: the total transform is a normal crossing divisor. We call this result the embedded local uniformization. We will give a new proof of this theorem in this work.

Let us explain why simultaneous monomialization is a stronger result than the embedded local uniformization Theorem. First we monomialize all the elements of $R$ with the same sequence of blow-ups. Secondly, this sequence is effective and at each step of the process we can express the $u^{(i+1)}$ in terms of the $u^{(i)}$. Indeed, we consider an essentially of finite type regular local ring $R$, and a valuation centered
in $R$. Thanks to this valuation we construct an effective sequence of blow-ups that monomializes all the elements of $R$. One more advantage of the proof we give here is that in the essentially of finite type case, we prove the simultaneous embedded local uniformization whatever is the valuation. In particular we do not need any hypothesis on the rank of the valuation.

One of the most important ingredient in the proof of this theorem is the notion of key polynomial. We give here a new definition of key polynomial, introduced by Spivakovsky and appearing for the first time in ([9] and [19]). Let $K$ be a field, $\nu$ be a valuation over $K$ and we denote by $\partial_{b} := \frac{\partial}{\partial x^{b}}$ the formal derivative of the order $b$ on $K[X]$. For every polynomial $P \in K[X]$, we set

$$\epsilon_{\nu}(P) := \max_{b \in \mathbb{N}^{*}} \left\{ \frac{\nu(P) - \nu(\partial_{b}P)}{b} \right\}.$$

**Definition (1.7).** Let $Q \in K[X]$ be a monic polynomial. The polynomial $Q$ is a key polynomial for $\nu$ if for every polynomial $P \in K[X]$:

$$\epsilon_{\nu}(P) \geq \epsilon_{\nu}(Q) \Rightarrow \deg_{X}(P) \geq \deg_{X}(Q).$$

One of the interests of this new definition is the following notion:

**Definition (2.1).** Let $Q_{1}$ and $Q_{2}$ be two key polynomials. We say that $Q_{2}$ is an immediate successor of $Q_{1}$ if $\epsilon(Q_{1}) < \epsilon(Q_{2})$ and if $Q_{2}$ is of minimal degree for this property. We denote this by $Q_{1} < Q_{2}$.

We denote by $M_{Q_{1}}$ the set of immediate successors of $Q_{1}$. We assume that they all have the same degree as $Q_{1}$ and that $\epsilon(M_{Q_{1}})$ does not have any maximal element.

**Definition (2.10).** We assume that there exists a key polynomial $Q'$ such that $\epsilon(Q') > \epsilon(M_{Q_{1}})$. We call immediate limit successor of $Q_{1}$ every polynomial $Q_{2}$ of minimal degree satisfying $\epsilon(Q_{2}) > \epsilon(M_{Q_{1}})$, and we denote this by $Q_{1} <_{\lim} Q_{2}$.

Let $Q_{1}$ and $Q_{2}$ be two key polynomials. Let us write $Q_{2}$ according to the powers of $Q_{1}$, $Q_{2} = \sum_{i=0}^{\infty} q_{i}Q_{1}^{i}$ where the $q_{i}$ are polynomials of degree strictly less than $Q_{1}$. We call this expression the $Q_{1}$-expansion of $Q_{2}$.

An important result in this work, and the only one for which we need the characteristic zero hypothesis, is the following Theorem.

**Theorem (2.17).** Let $Q_{2}$ be an immediate limit successor of $Q_{1}$. Then the terms of the $Q_{1}$-expansion of $Q_{2}$ that minimize the valuation are exactly those of degrees 0 and 1.

Then the hypothesis of characteristic zero is necessary also for the results that follow from this theorem.

Here we give an idea of our proof of Theorem 7.1. Let us consider a regular local ring $R$ essentially of finite type over a field $k$ of characteristic zero. We fix $u = (u_{1}, \ldots, u_{n})$ a regular system of parameters of $R$.

The first ingredient in the proof is the notion of non degeneration.

**Definition (5.1).** We say that an element $f$ of $R$ is non degenerated with respect to $u$ if there exists an ideal $N$ of $R$, generated by monomials in $u$, such that $\nu(f) = \min_{x \in N} \{\nu(x)\}$. 


The first step is to monomialize all the elements that are non degenerated with respect to a regular system of parameters of \( R \). So let \( f \) be an element of \( R \) that is non degenerated with respect to \( u \). We construct a sequence of blow-ups
\[
(R, u) \to \cdots \to (R', u')
\]
such that the strict transform of \( f \) in \( R' \) is a monomial in \( u' \).

There exist elements \( f \) of \( R \) that are not non degenerated with respect to \( u \). So we wonder if we could find a sequence of blow-ups
\[
(R, u) \to \cdots \to (T, t)
\]
such that \( f \) is non degenerated with respect to \( t \). If we can, after a new sequence of blow-ups, we monomialize \( f \). Doing this for all the elements of \( R \) would be too complicated. So we would want to find a sequence of blow-ups
\[
(R, u) \to \cdots \to (R', u')
\]
such that all the elements of \( R \) are non degenerated with respect to \( u' \). It is a little optimistic and we need to do something more subtle. We will find an infinite sequence of blow-ups
\[
(R, u) \to \left(R_1, u^{(1)}\right) \to \cdots \to \left(R_i, u^{(i)}\right) \to \cdots
\]
such that for each element \( f \) of \( R \), there exists \( i \) such that \( f \) is non degenerated with respect to \( u^{(i)} \).

For this, we need the second main ingredient: the key polynomials.

We construct a sequence of key polynomials \( (Q_i) \) such that each element \( f \) of \( R \) is non degenerated with respect to some \( Q_i \). It means that:
\[
\forall f \in R, \exists i \text{ such that } \nu(f) = \nu_{Q_i}(f).
\]

We construct the sequence \( (Q_i) \) step by step. We require the following properties for this sequence: for every index \( i \), the polynomial \( Q_{i+1} \) is an (eventually limit) immediate successor of \( Q_i \). Furthermore the sequence \( (\epsilon(Q_i)) \) is cofinal in \( \epsilon(\Lambda) \) where \( \Lambda \) is the set of key polynomials of the extension \( k(u_1, \ldots, u_{n-1}) (u_n) \).

Equivalently it means:
\[
\begin{cases}
\forall i, Q_i < Q_{i+1} \text{ or } Q_i <_{\text{lim}} Q_{i+1}, \\
\forall Q \in \Lambda \exists i \text{ such that } \epsilon(Q_i) \geq \epsilon(Q).
\end{cases}
\]

Assume now that we can construct a sequence of blow-ups
\[
(R, u) \to \cdots \to \left(R_j, u^{(j)}\right) \to \cdots
\]
such that all the \( Q_i \) belong to a regular system of parameters. It means that
\[
\forall i, \exists j, k \text{ such that } Q_i^{\text{strict}, j} = u_k^{(j)},
\]
where \( Q_i^{\text{strict}, j} \) is the strict transform of \( Q_i \) in \( R_j \). Then every element \( f \) of \( R \) which is non degenerated with respect to \( Q_i \) is non degenerated with respect to \( u^{(j)} \). Thus it is monomializable. So the next step is to monomialize all the \( Q_i \).

In order to do this once again we have to be subtle. The notion of key polynomial is not stable by blow-up, so we need a better notion: the notion of key element. Let \( (Q_i, Q_{i+1}) \) a couple of (eventually limit) immediate successors of our sequence. We consider \( Q_{i+1} = \sum_{j=0} q_j Q_i^j \) the \( Q_i \)-expansion of \( Q_{i+1} \). Then we associate to \( Q_{i+1} \) a key element \( Q'_{i+1} \) defined as follows.

\[
Q'_{i+1} = \sum_{j=0} q_j Q_i^j.
\]
Definition (3.11). An element \( Q'_{i+1} = \sum_{j=0}^{a_j Q'_i} a_j q_j Q^j_i \) where the \( a_j \) are units is called a key element associated to \( Q_{i+1} \).

In fact we also have a notion of (eventually limit) immediate successors in this case.

Definition (3.13 and 3.14). Let \( P'_1 \) and \( P'_2 \) be two key elements. We say that \( P'_1 \) and \( P'_2 \) are (eventually limit) immediate successors key elements if their respective associated key polynomials \( P_1 \) and \( P_2 \) are such that \( P_1 < P_2 \) (eventually \( P_1 <_{\text{lim}} P_2 \)).

After some blow ups we prove that (eventually limit) immediate successors become (eventually limit) immediate successors key elements. So we monomialize these key elements. For this we construct a sequence of blow-ups

\[
(R, u) \rightarrow \cdots \rightarrow \left( R_{s_i}, u^{(s_i)} \right) \rightarrow \cdots
\]

that monomializes all the key polynomials \( Q_i \). More precisely, for every index \( i \) there exists an index \( s_i \) such that in \( R_{s_i}, Q_i \) is a monomial in \( u^{(s_i)} \) up to a unit of \( R_{s_i} \).

So in the case of essentially of finite type regular local rings, no matter the rank of the valuation is, we prove the embedded local uniformization Theorem. And we do this using only a sequence of blow-ups for all the elements of the ring, and in an effective way. It means that every blow-up is effective and we know how to express all the systems of coordinates.

Then we want to prove the same kind of result over more general rings, even if it means adding conditions on the valuation. We work with quasi excellent rings. Indeed, Grothendieck and Nagata showed that there is no resolution of singularities for rings that are not quasi excellent.

The second main result of this paper can be express in the following simplified form.

Theorem (12.3). Let \( R \) be a noetherian quasi excellent complete regular local ring and \( \nu \) be a valuation centered in \( R \).

Assume that \( \nu \) is of rank 1, or of rank 2 but composed with a discrete valuation, and that \( \text{car} (k_{\nu}) = 0 \).

There exists a countable sequence of blow-ups

\[
(R, u) \rightarrow \cdots \rightarrow \left( R_{s}, u^{(s)} \right) \rightarrow \cdots
\]

that monomializes all the element of \( R \).

So let \( R \) be a quasi excellent local domain. This time \( R \) is not assume to be of finite type, so we cannot repeat what we did before. We need to introduce one more ingredient: the implicit prime ideal.

Let \( \nu \) be a valuation of the fractions field of \( R \) centered in \( R \). We call implicit prime ideal of \( R \) associated to \( \nu \) the ideal of the completion \( \hat{R} \) of \( R \) defined by:

\[
H := \bigcap_{\beta \in \nu(R, \{0\})} P_{\beta} \hat{R}
\]

where \( P_{\beta} := \{ f \in R \text{ such that } \nu(f) \geq \beta \} \).
One can show that in this case desingularizing $R$ means desingularizing $\hat{R}$. In the last part of this work we also prove that to desingularize $R$, we only need to desingularize $\hat{R}_H$ and (up to one more sequence of blow-ups) $\hat{R}_H$. We prove that the implicit prime ideal satisfies the property that $\hat{R}_H$ is regular. So we only have to desingularize $\hat{R}_H$ and this is done by Theorem 11.2.

**Acknowledgments.** The author is really grateful to her PHD advisor Mark Spivakovsky for all the helpful discussions.
Part 2. Key polynomials.

The notion of key polynomials has been first introduced by Saunders Mac Lane in 1936, in the case of discrete valuations of rank 1. The first motivation to introduce this notion was to describe all the extensions of a valuation to a field extension. Let \( K \rightarrow L \) be an extension of field and \( \nu \) a valuation over \( K \). We consider \( \mu \) a valuation which extends \( \nu \) to \( L \). In the case where \( \nu \) is of rank 1 and where \( L \) is a simple algebraic extension of \( K \), Mac Lane created the notion of key polynomial for \( \mu \). He also created the notion of augmented valuations. Given a valuation \( \mu \) and \( Q \) a key polynomial of Mac Lane, we write

\[
    f = \sum_{i=0}^{r} f_j Q^j
\]

the \( Q \)-expansion of an element \( f \in K[[X]] \). An augmented valuation \( \mu' \) of \( \mu \) is one defined by

\[
    \mu'(f) = \min_{0 \leq j \leq r} \{ \mu(f_j) + j\delta \}
\]

where \( \delta > \mu(Q) \). He proved that \( \mu \) is the limit of a family of augmented valuations over the ring \( K[x] \). Michel Vaquié extended this definition to arbitrary valued field \( K \), it means without assuming that \( \nu \) is discrete. The most important difference between these notions is the fact that those of Vaquié involves limit key polynomials while those of Mac Lane not.

More recently, the notion of key polynomials has been used by Spivakovsky to study the local uniformization problem, and to do this he created a new notion of key polynomials. It is the one we use here.

1. **Key polynomials of Spivakovsky and al.**

For some results of this part, we send the reader to [9], but we recall the definitions and properties used in this work to have a self-contained manuscript.

First, recall the definition of a valuation.

**Definition 1.1.** Let \( R \) be a monic commutative domain, \( K \) be a commutative field and \( \Gamma \) be a totally ordered abelian group. We set \( \Gamma_{\infty} := \Gamma \cup \{+\infty\} \).

A valuation of \( R \) is a map

\[
    \nu: R \rightarrow \Gamma_{\infty}
\]

such that:

1. \( \forall x \in R, \nu(x) = +\infty \Leftrightarrow x = 0 \),
2. \( \forall (x, y) \in R^2, \nu(xy) = \nu(x) + \nu(y) \),
3. \( \forall (x, y) \in R^2, \nu(x + y) \geq \min \{\nu(x), \nu(y)\} \).

Let us give three examples of valuations.

**Example 1.2.** The map \( \nu_1 : \mathbb{C}[x] \rightarrow \mathbb{Z} \cup \{+\infty\} \) which sends a polynomial \( P = \sum_{i=0}^{d} p_i x^i \) to \( \min \{i \text{ such that } p_i \neq 0\} \) is a valuation.

**Example 1.3.** We want to define a valuation \( \nu_2 \) on \( \mathbb{C}(x, y, z) \). The value of a quotient \( \frac{P}{Q} \) is \( \nu_2(P) - \nu_2(Q) \).

And we define the value of a polynomial \( P = \sum_{i} p_i x^{i1} y^{i2} z^{i3} \) as the minimal of the values of \( p_i x^{i1} y^{i2} z^{i3} \).

Then we only have to define the values of the generators \( x, y \) and \( z \).

Hence the map \( \nu_2 : \mathbb{C}(x, y, z) \rightarrow \mathbb{R}_{\infty} \) which sends \( x \) to 1, \( y \) to 2\( \pi \) and \( z \) to \( 1 + \pi \) is a valuation.
Example 1.6. Let us set $Q = z^2 - x^2y$. Every polynomial $P \in \mathbb{C}[x, y, z]$ can be written according to the powers of $Q$. We write $P = \sum p_i Q^i$ with the $p_i \in \mathbb{C}[x, y][z]$ of degree in $z$ strictly less than $\deg_z(Q) = 2$. Assume that the first non-zero $p_i$ is $p_n$. Then the map $\nu_3 : \mathbb{C}(x, y, z) \to (\mathbb{R}^2, \text{lex})$ that sends $P$ on $(n, \nu_2(p_n))$ defines a valuation, with $\nu_2$ the valuation defined in Example 1.3.

We consider $K$ a field with a valuation $\nu$ and we consider a simple transcendental extension

$$K \hookrightarrow K(X)$$

with a valuation $\nu$ that extends $\mu$ to $K(X)$. We still denote by $\nu$ the restriction of $\nu$ to $K[X]$. For every non-zero integer $b$, we set $\partial_b := \frac{\partial b}{\partial x}$. This is called the formal derivative at the order $b$.

For every polynomial $P \in K[X]$, we set

$$\epsilon_\nu(P) := \max_{b \in \mathbb{N}^*} \left\{ \frac{\nu(P) - \nu(\partial_b P)}{b} \right\}.$$

Remark 1.5. Most of the time we will note $\epsilon(P) := \epsilon_\nu(P)$.

Example 1.6. We consider $\mathbb{C}(x, y)[z]$ and the valuation $\nu := \nu_3$ defined in 1.3.

We have $\nu(z) = (0, 1 + \pi)$ and $\nu(\partial z) = \nu(1) = (0, 0)$. So

$$\epsilon(z) = \max_{b \in \mathbb{N}^*} \left\{ \frac{\nu(z) - \nu(\partial_b z)}{b} \right\} = \frac{\nu(z) - \nu(\partial z)}{1} = \nu(z) = (0, 1 + \pi).$$

Also we have $\nu(x) = (0, 1)$ and $\nu(\partial x) = \nu(0) = (+\infty, +\infty)$ so $\epsilon(x) = (-\infty, -\infty)$.

And also $\epsilon(y) = (-\infty, -\infty)$.

Finally, let us compute $\epsilon(Q = z^2 - x^2y)$. We have $\nu(Q) = (1, 0)$, $\nu(\partial Q) = \nu(2z) = (0, 1 + \pi)$ and $\nu(\partial_2 Q) = \nu(2) = (0, 0)$.

So $\epsilon(Q) = \max \left\{ \frac{\nu(Q) - \nu(\partial Q)}{1}, \frac{\nu(Q) - \nu(\partial_2 Q)}{2} \right\} = \max \left\{ \frac{(1, 0) - (0, 1 + \pi)}{1}, \frac{(1, 0) - (0, 0)}{2} \right\} = (1, -1 - \pi)$.

Definition 1.7. Let $Q \in K[X]$ be a monic polynomial. We say that $Q$ is a key polynomial for $\nu$ if for every polynomial $P \in K[X]$, we have:

$$\epsilon_\nu(P) \geq \epsilon_\nu(Q) \Rightarrow \deg_X(P) \geq \deg_X(Q).$$

Example 1.8. We consider the same example as in example 1.6.

Let us show that $z$ is a key polynomial. We do a proof by contrapositive. Let $P$ be a polynomial of degree in $z$ strictly less than $\deg_z(z) = 1$. So $P$ does not depend on $z$. Then we saw that $\epsilon(P) = (-\infty, -\infty)$. So $\epsilon(P) < \epsilon(z)$ and $z$ is a key polynomial.

Now, let us show that $Q = z^2 - x^2z$ is a key polynomial. We consider a polynomial $P$ such that $\epsilon(P) \geq \epsilon(Q) = (1, -1 - \pi)$.

Then $\epsilon(P) = (n, *)$ where $n\geq 1$ and $*$ is a scalar. So $\nu(P) = (m, *)$ where $m \geq 1$. Hence $Q^m | P$ and so $\deg_z(P) \geq \deg_z(Q)$.

We proved that $Q$ is a key polynomial.
We have two key polynomials \( z \) and \( Q \) and we have \( \epsilon (z) < \epsilon (Q) \). One can show that \( Q \) is of minimal degree for this property. We will see further that \( Q \) is an immediate successor of \( z \).

For every polynomial \( P \in K[X] \), we set
\[
b_{\nu} (P) := \min I (P)
\]
where
\[
I (P) := \left\{ b \in \mathbb{N}^* \text{ such that } \frac{\nu (P) - \nu (\partial_b P)}{b} = \epsilon_\nu (P) \right\}.
\]
Again, if there is no confusion, we will omit the index \( \nu \).

Let \( P \) and \( Q \) two polynomials such that \( Q \) is monic. Then \( P \) can be written
\[
P = \sum_{j=1}^{n} p_j Q^j \quad \text{with} \quad p_j \text{ polynomials of degree strictly less than the degree of } Q.
\]
This expression is unique and it is called the \( Q \)-expansion of \( P \).

**Definition 1.9.** Let \((P, Q) \in K[X]^2 \) such that \( Q \) is monic, and we consider \( P = \sum_{j=1}^{n} p_j Q^j \) the \( Q \)-expansion of the polynomial \( P \). Then we set \( \nu_Q (P) := \min_{0 \leq j \leq n} \nu (p_j Q^j) \). The map \( \nu_Q \) is called the \( Q \)-truncation of \( \nu \).

Also we set
\[
S_Q (P) := \{ j \in \{0, \ldots, n\} \text{ such that } \nu (p_j Q^j) = \nu_Q (P) \}
\]
and
\[
\delta_Q (P) := \max \{ S_Q (P) \}.
\]
Now, we set
\[
\tilde{P}_{\nu,Q} := \sum_{j \in S_Q (P)} p_j Q^j.
\]

**Remark 1.10.** In the general case, \( \nu_Q \) is not a valuation. But if \( Q \) is a key polynomial, we are going to show that \( \nu_Q \) is a valuation.

In order to do that, we need the next result, which will also be needed for a proof of the fundamental theorem \[2.17\]

**Lemma 1.11.** Let \( t \in \mathbb{N}_{>1} \) and \( Q \) be a key polynomial. We consider \( P_1, \ldots, P_t \) some polynomials of \( K[X] \) all of degree strictly less than \( \deg (Q) \) and we set \( \prod_{i=1}^{t} P_i := qQ + r \) the euclidian division of \( \prod_{i=1}^{t} P_i \) by \( Q \) in \( K[X] \). Then:
\[
\nu (r) = \nu \left( \prod_{i=1}^{t} P_i \right) < \nu (qQ).
\]

**Proof.** We do an induction on \( t \).

Initialisation: \( t = 2 \). So we want to show that \( \nu (P_1 P_2) < \nu (qQ) \).

Indeed, if \( \nu (P_1 P_2) < \nu (qQ) \), then
\[
\nu (r) = \nu (P_1 P_2 - qQ) = \nu (P_1 P_2) < \nu (qQ)
\]
and we have the result.
Assume by contradiction that $\nu(P_1 P_2) \geq \nu(q Q)$ and so $\nu(r) \geq \nu(q Q)$. Since $Q$ is a key polynomial, every polynomial $P$ of degree strictly less than $\deg(Q)$ satisfies $\epsilon(P) < \epsilon(Q)$. In particular, for every non zero integer $j$, we have $\nu(P) - \nu(\partial_j P) < j \epsilon(Q)$. So it is the case for $P_1$, $P_2$ and $r$. Since $P_1$ and $P_2$ of degree strictly less than $\deg(Q)$, we have
\[
\deg_X(P_1 P_2) = \deg_X(P_1) + \deg_X(P_2) < 2 \deg_X(Q).
\]
However, $\deg_X(P_1 P_2) = \deg_X(q Q) = \deg_X(q) + \deg_X(Q)$. So $q$ is of degree strictly less than $\deg(Q)$ too, and then $q$ satisfies, for every non zero integer $j:
\nu(q) - \nu(\partial_j q) < j \epsilon(Q)$. We are going to compute $\nu(\partial_b(q Q))$ by two distinct ways to get the contradiction.

First,
\[
\nu(\partial_b(q Q)) = \nu\left(\sum_{j=0}^{b(Q)} (\partial_b(q Q) - j \partial_j q)\right).
\]

Look at the first term of the sum: $q \partial_b(q Q)$, and compute its value $\nu(q \partial_b(q Q))$. We are going to show that its value is the smallest of the sum.

We have
\[
\nu(q \partial_b(q Q)) = \nu(q) + \nu(\partial_b(q Q)) = \nu(q) + \nu(Q) - b(Q) \epsilon(Q)
\]
by definition of $b(Q)$. But we know that for every non zero integer $j$, we have $\nu(q) < j \epsilon(Q) + \nu(\partial_j q)$, so
\[
\nu(q \partial_b(q Q)) < (j - b(Q)) \epsilon(Q) + \nu(Q) + \nu(\partial_j q) \leq \nu(\partial_j q) + \nu(\partial_b(q Q)) = \nu(q \partial_b(q Q))
\]
so
\[
\nu(q \partial_b(q Q)) = \nu(\partial_b(q Q)) = \nu(q) + \nu(\partial_b(q Q)) = \nu(q) + b(Q) \epsilon(Q).
\]

Then $q \partial_b(q Q)$ is the term of smallest value in the sum. In particular,
\[
\nu(\partial_b(q Q)) = \nu(q \partial_b(q Q)) = \nu(q) + \nu(\partial_b(q Q)) = \nu(q) + b(Q) \epsilon(Q).
\]

Now we compute this value of a distinct way. We have:
\[
\nu(\partial_b(q Q)) = \nu(\partial_b(q Q) (P_1 P_2 - r)) = \nu(\partial_b(q Q) (P_1 P_2) - \partial_b(q Q) (r)) \geq \min\{\nu(\partial_b(q Q) (P_1 P_2)), \nu(\partial_b(q Q) (r))\}.
\]

But also:
\[
\nu(\partial_b(q Q) (P_1 P_2)) = \nu\left(\sum_{j=0}^{b(Q)} \partial_j (P_1) \partial_b(q Q) - j (P_2)\right) \geq \min_{0 \leq j \leq b(Q)} \{\nu(\partial_j P_1) + \nu(\partial_b(q Q) - j (P_2))\}.
\]

If $j \neq 0$, we have $\nu(P_1) < j \epsilon(Q) + \nu(\partial_j (P_1))$ and so
\[
\nu(\partial_j (P_1)) > \nu(P_1) - j \epsilon(Q)
\]
because $\deg_X(P_1) < \deg_X(Q)$. If $0 \leq j < b(Q)$, we also have
\[
\nu(\partial_b(q Q) - j (P_2)) \geq \nu(P_2) - (b(Q) - j) \epsilon(Q).
\]
So if $0 < j < b(Q)$, we have
\[ \nu(\partial_j P_1) + \nu(\partial_{b(Q)-j}(P_2)) > \nu(P_1P_2) - b(Q)\epsilon(Q). \]
This inequality stays true if $j = 0$ and $j = b(Q)$, so:
\[ \nu(\partial_{b(Q)}(P_1P_2)) > \nu(P_1P_2) - b(Q)\epsilon(Q). \]
By hypothesis, $\nu(P_1P_2) \geq \nu(qQ)$, so
\[ \nu(\partial_{b(Q)}(P_1P_2)) > \nu(qQ) - b(Q)\epsilon(Q). \]
But since $r$ is of degree strictly less than $\deg(Q)$, we know that $\nu(\partial_{b(Q)}(r)) > \nu(r) - b(Q)\epsilon(Q)$, and by hypothesis $\nu(r) \geq \nu(qQ)$. Then $\nu(\partial_{b(Q)}(r)) > \nu(qQ) - b(Q)\epsilon(Q)$.

So
\[ \nu(\partial_{b(Q)}(qQ)) \geq \min \{ \nu(\partial_{b(Q)}(P_1P_2)) \nu(\partial_{b(Q)}(r)) \} > \nu(qQ) - b(Q)\epsilon(Q) \]
which contradicts \eqref{eq:contradiction}. So we do have $\nu(r) = \nu(P_1P_2) < \nu(qQ)$, and this concludes the initialisation.

We now assume the result true at the rank $t - 1 \geq 2$ and we are going to show it at the rank $t$. We set $P := \prod_{i=1}^{t-1} P_i$.

Let $P = q_1Q + r_1$
be the euclidian division of $P$ by $Q$ and
\[ r_1P_t = q_2Q + r_2 \]
be those of $r_1P_t$ by $Q$. Since $PP_t = qQ + r$, we have $r = r_2$ and $q = q_1P_t + q_2$.

By induction hypothesis, $\nu(r_1) = \nu(P) < \nu(q_1Q)$. In particular,
\[ \nu(r_1P_t) = \nu\left(\prod_{i=1}^{t} P_i\right) < \nu(q_1P_tQ). \]
Since the polynomials $r_1$ and $P_t$ are both of degree strictly less than $\deg(Q)$, we can apply the initialisation case and so
\[ \nu(r_1P_t) = \nu(r_2) < \nu(q_2Q). \]
So $\nu(r) = \nu(r_2) = \nu(r_1P_t) = \nu\left(\prod_{i=1}^{t} P_i\right)$ and furthermore this value is strictly less than $\nu(q_1P_tQ)$ and than $\nu(q_2Q)$. So it is strictly less than the minimum, which is less or equal than $\nu(q_1P_tQ + q_2Q)$ by definition of a valuation. So
\[
\nu(r) = \nu\left(\prod_{i=1}^{t} P_i\right) < \nu((q_1P_t + q_2)Q) = \nu(qQ)
\]
which concludes the proof. \hfill $\square$

We can now show the next theorem.

**Theorem 1.12.** Let $Q$ be a key polynomial. The map $\nu_Q$ is a valuation.
Proof. The only thing we have to prove is that for every \((P_1, P_2) \in K[X]^2\), we have
\[
\nu_Q(P_1P_2) = \nu_Q(P_1) + \nu_Q(P_2).
\]
First case: \(P_1\) and \(P_2\) are both of degree strictly less than \(\deg(Q)\). Then \(\nu_Q(P_1) = \nu(P_1)\) and \(\nu_Q(P_2) = \nu(P_2)\). Since \(\nu\) is a valuation, we have \(\nu(P_1P_2) = \nu(P_1) + \nu(P_2)\).

Then, \(\nu(P_1P_2) = \nu_Q(P_1) + \nu_Q(P_2)\). Since \(P_1\) and \(P_2\) are both of degree strictly less than \(\deg(Q)\), by previous Lemma, we have \(\nu_Q(P_1P_2) = \nu(P_1P_2)\) and we are done.

Second case: \(P_1 = p_i^{(1)}Q^j\) and \(P_2 = p_j^{(2)}Q^i\), with \(p_i^{(1)}\) and \(p_j^{(2)}\) both of degree strictly less than \(\deg(Q)\).

Let \(p_i^{(1)}p_j^{(2)} = qQ + r\) be the euclidian division of \(p_i^{(1)}p_j^{(2)}\) by \(Q\). Since \(\deg_X(p_i^{(1)}p_j^{(2)}) < 2\deg_X(Q)\), we know that \(\deg_X(q) < \deg_X(Q)\), and by definition of the euclidian division, we have \(\deg_X(r) < \deg_X(Q)\). So \(P_1P_2 = qQ^{i+j+1} + rQ^{i+j}\) is the \(Q\)-expansion of \(P_1P_2\).

We are going to prove that in this case we still have
\[
\nu_Q(P_1P_2) = \nu(P_1P_2),
\]
and since \(\nu\) is a valuation, we will still have the result. We have:
\[
\nu_Q(P_1P_2) = \nu_Q(qQ^{i+j+1} + rQ^{i+j}) = \min \{ \nu(qQ^{i+j+1}), \nu(rQ^{i+j}) \} = \min \{ \nu(qQ) + \nu(Q^{i+j}), \nu(r) + \nu(Q^{i+j}) \}.
\]
However, we can apply the previous Lemma to the product
\[
p_i^{(1)}p_j^{(2)} = qQ + r
\]
and conclude that \(\nu(r) = \nu\left(p_i^{(1)}p_j^{(2)}\right) < \nu(qQ)\).

Then
\[
\nu_Q(P_1P_2) = \nu(r) + \nu\left(Q^{i+j}\right) = \nu\left(p_i^{(1)}p_j^{(2)}\right) + \nu\left(Q^{i+j}\right) = \nu(P_1P_2)
\]
and we still have the result.

Last case: general case. Since we only look at the terms of smallest value, we can replace \(P_1\) by
\[
\left(P_1\right)_{\nu,Q} = \sum_{j \in S_Q(P_1)} p_j^{(1)}Q^j
\]
and \(P_2\) by
\[
\left(P_2\right)_{\nu,Q} = \sum_{i \in S_Q(P_2)} p_i^{(2)}Q^i.
\]
We know that
\[
\nu_Q(P_1 + P_2) \geq \min \{ \nu_Q(P_1), \nu_Q(P_2) \}
\]
and
\[
\nu_Q\left(p_j^{(1)}Q^j p_i^{(2)}Q^i\right) = \nu_Q\left(p_j^{(1)}Q^j\right) + \nu_Q\left(p_i^{(2)}Q^i\right).
\]
So
\[ \nu_Q (P_1 P_2) = \nu_Q \left( \sum p_j^{(1)} p_i^{(2)} Q^{j+i} \right) \geq \min \left\{ \nu_Q \left( p_j^{(1)} Q^j \right), \nu_Q \left( p_i^{(2)} Q^i \right) \right\}. \]

However
\[ \nu_Q \left( p_j^{(1)} Q^j \right) = \nu \left( p_j^{(1)} Q^j \right) = \nu_Q (P_1) \]
and
\[ \nu_Q \left( p_i^{(2)} Q^i \right) = \nu \left( p_i^{(2)} Q^i \right) = \nu_Q (P_2). \]

So \( \nu_Q (P_1 P_2) \geq \nu_Q (P_1) + \nu_Q (P_2) \), and we only have to show that it is an equality. In order to do that, it is enough to find a term in the \( Q \)-expansion of \( P_1 P_2 \) which value is exactly \( \nu_Q (P_1) + \nu_Q (P_2) \). Let us consider the term of smallest value in each \( Q \)-expansion, so let us consider \( p_n^{(1)} Q^{n_1} \) and \( p_m^{(2)} Q^{m_2} \), where \( n_1 = \min S_Q (P_1) \) and \( m_2 = \min S_Q (P_2) \).

Let \( p_n^{(1)} Q^{m_2} = q Q + r \) be the euclidian division of \( p_n^{(1)} p_m^{(2)} \) by \( Q \), which is its \( Q \)-expansion too.

By Lemma \textbf{1.11} we have \( \nu(r) = \nu \left( p_n^{(1)} p_m^{(2)} \right) \). In fact, in the \( Q \)-expansion of \( P_1 P_2 \), there is the term \( r Q^{n_1+m_2} \), and we have:

\[ \nu_Q \left( r Q^{n_1+m_2} \right) = \nu \left( r Q^{n_1+m_2} \right) = \nu \left( p_n^{(1)} p_m^{(2)} Q^{n_1+m_2} \right) = \nu_Q (P_1) + \nu_Q (P_2) \]

which concludes the proof. \( \square \)

**Remark 1.13.** For every polynomial \( P \in \mathbb{K}[X] \), we have
\[ \nu_Q (P) \leq \nu (P). \]

It will be very important to be able to determine when this inequality is an equality.

A key polynomial \( P \) which satisfies the strict inequality and which is of minimal degree for this property will be called an immediate successor of \( Q \) (Definition \textbf{2.11}). We will study these polynomials with more details in this work. First, let us concentrate on the equality case.

**Definition 1.14.** Let \( Q \) be a key polynomial and \( P \) be a polynomial such that \( \nu_Q (P) = \nu (P) \). We say that \( P \) is non degenerated with respect to \( Q \).

Another thing very important is to be able to compare the \( \epsilon \) of key polynomials. Indeed, if I have two key polynomials \( Q_1 \) and \( Q_2 \), do I have \( \epsilon (Q_1) < \epsilon (Q_2) \), or do I have \( \epsilon (Q_1) = \epsilon (Q_2) \) ? Being able to answer will be crucial. The next four results can be found in [9] but we recall them for more clarity.

**Lemma 1.15.** For every polynomial \( P \in \mathbb{K}[X] \) and every strictly positive integer \( d \), we have:
\[ \nu_Q (\partial_d P) \geq \nu_Q (P) - d \epsilon (Q) \]

**Proof.** We consider \( P = \sum_{i=0}^n p_i Q^i \) the \( Q \)-expansion of \( P \).

Assume we have the result for \( p_i Q^i \). It means that
\[ \nu_Q \left( \partial_d \left( p_i Q^i \right) \right) \geq \nu_Q \left( p_i Q^i \right) - d \epsilon (Q) \]
for every index $i$. Then:

\[
\nu_Q(\partial_d P) = \nu_Q \left( \partial_d \left( \sum_{i=0}^{n} p_i Q^i \right) \right) \\
= \nu_Q \left( \sum_{i=0}^{n} \partial_d (p_i Q^i) \right) \\
\geq \min_{0 \leq i \leq n} \nu_Q(\partial_d (p_i Q^i)) \\
\geq \min_{0 \leq i \leq n} \{ \nu_Q(p_i Q^i) - d \epsilon(Q) \} \\
\geq \min_{0 \leq i \leq n} \{ \nu_Q(p_i Q^i) \} - d \epsilon(Q) \\
\geq \nu_Q(P) - d \epsilon(Q)
\]

and the proof is done.

So we just have to prove the result for $P = p_i Q^i$.

First, by Lemma 1.15, we can replace the result for two polynomials, we have the result for two polynomials, we have the result for the product.

So let us prove the result for $P = p_i$.

Since $\deg_X(p_i) < \deg_X(Q)$ and since $Q$ is a key polynomial, we have $\epsilon(p_i) < \epsilon(Q)$. So, for every strictly positive integer $d$, we have:

\[
\nu_Q(\partial_d p_i) = \nu(\partial_d p_i) \\
\geq \nu(p_i) - d \epsilon(p_i) \\
= \nu_Q(p_i) - d \epsilon(p_i) \\
> \nu_Q(p_i) - d \epsilon(Q).
\]

Now, it just remains to prove that if we have the result for two polynomials $P$ and $S$, then we have it for $PS$. Assume the result proven for $P$ and $S$. Then:

\[
\nu_Q(\partial_{d}(PS)) = \nu_Q \left( \sum_{r=0}^{d} \partial_r(P) \partial_{d-r}(S) \right) \\
\geq \min_{0 \leq r \leq d} \{ \nu(\partial_r(P)) + \nu(\partial_{d-r}(S)) \} \\
\geq \min_{0 \leq r \leq d} \{ \nu_Q(P) - r \epsilon(Q) + \nu_Q(S) - (d-r) \epsilon(Q) \} \\
\geq \nu_Q(PS) - d \epsilon(Q)
\]

This concludes the proof. \hfill \Box

**Proposition 1.16.** Let $Q$ be a key polynomial and $P \in K[X]$ a polynomial such that $S_Q(P) \neq \{0\}$.

Then there exists a strictly positive integer $b$ such that

\[
\frac{\nu_Q(P) - \nu_Q(\partial_d P)}{b} = \epsilon(Q).
\]

**Proof.** First, by Lemma 1.15 we can replace $P$ by $\tilde{P}_r, Q = \sum_{i \in S_Q(P)} p_i Q^i$.

We want to show the existence of a strictly positive integer $b$ such that $\nu_Q(P) - \nu_Q(\partial_b P) = b \epsilon(Q)$.

Since $S_Q(P) \neq \{0\}$, we can consider the strictly less integer non zero $l$ of $S_Q(P)$. We write $l = p^r u$, with $p \nmid u$. 

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Lemma 1.17. We have \( \partial_b (P) = urQ^{l-p} + Q^{l-p+1}R + S \), where:

1. The polynomial \( r \) is the rest of the euclidian division of \( p_l (\partial_b(Q))^{p_r} \) by \( Q \),
2. The polynomials \( R \) and \( S \) satisfy

\[
\nu_Q(S) > \nu_Q(P) - b e(Q).
\]

Proof. First let us show that the Lemma is true for \( P = p_lQ^l \) and that for every \( j \in S_Q(P) \setminus \{ l \} \), we have

\[
\partial_b (p_jQ^j) = Q^{l-p^*+1}R_j + S_j,
\]

where \( R_j \) and \( S_j \) are two polynomials, and where \( \nu_Q(S_j) > \nu_Q(P) - b e(Q) \).

So we consider \( j \in S_Q(P) \). We set

\[
M_j := \left\{ B_s = (b_0, \ldots, b_s) \in \mathbb{N}_s+1 \text{ such that } \sum_{i=0}^s b_i = b \text{ and } s \leq j \right\}.
\]

The generalized Leibniz rule tells us that:

\[
\partial_b (p_jQ^j) = \sum_{B_s \in M_j} (T(B_s))
\]

where

\[
T(B_s) = T((b_0, \ldots, b_s)) = C(B_s) \partial_{b_0}(p_j) \left( \prod_{i=1}^s \partial_{b_i}(Q) \right) Q^{j-s}
\]

with \( C(B_s) \) some elements of \( K \) whose exact value can be found in \( \mathbb{I}4 \). We set

\[
\alpha := (0, b(Q), \ldots, b(Q)) \in \mathbb{N}^{p+1}.
\]

Recall that \( I(Q) = \left\{ d \in \mathbb{N}^s \text{ such that } \frac{\nu(Q)-\nu(\partial_{b_0}Q)}{d} = e(Q) \right\} \). We set

\[
N_j := \{ B_s = (b_0, \ldots, b_s) \in M_j \text{ such that } b_0 > 0 \text{ or } \{b_1, \ldots, b_s\} \not\subseteq I(Q) \},
\]

\[
S_j := \sum_{B_s \in N_j} T(B_s)
\]

and finally we set

\[
Q^{l-p^*+1}R_j := \begin{cases} 
\sum_{B_s \in M_j \setminus N_j} T(B_s) & \text{if } j \neq l \\
\sum_{B_s \in M_j \setminus (N_j \cup \{\alpha\})} T(B_s) & \text{if } j = l 
\end{cases}
\]

If \( j = l \), the term \( T(\alpha) \) appears \( \left( \begin{array}{c} l \\ p_r \end{array} \right) \) times in \( \partial_b (p_lQ^l) \). Equivalently, \( C(\alpha) = u \) and so

\[
T(\alpha) = up_l (\partial_b(Q))^p_r Q^{l-p^*} = u (qQ + r) Q^{l-p^*}
\]

where \( qQ + r \) is the euclidian division of \( p_l (\partial_b(Q))^p_r \) by \( Q \).
It means that
\[ T(\alpha) = \nu Q^{l-p^*+1} + urQ^{l-p^*}. \]

So if \( j \neq l \), then \( \partial_b (p_j Q^j) = Q^{l-p^*+1} R_j + S_j \). It remains to prove that \( \nu_Q (S_j) > \nu_Q (p_j Q^j) - b e (Q) \).

But:

\[ \nu_Q (S_j) = \nu_Q \left( \sum_{B_s \in N_j} T(B_s) \right) \]
\[ = \nu_Q \left( \sum_{B_s \in N_j} C(B_s) \partial_{b_0} (p_j) \left( \prod_{i=1}^s \partial_{b_i} (Q) \right) Q^{j-s} \right) \]
\[ \geq \min_{B_s \in N_j} \left\{ \nu (\partial_{b_0} (p_j)) + \sum_{i=1}^s \nu (\partial_{b_i} (Q)) + (j-s) \nu (Q) \right\} \cdot \]

Since \( B_s \in N_j \), we have two options. Or \( b_0 = 0 \) and \( \{b_1, \ldots, b_s\} \not\subseteq I(Q) \). It means that for every \( i \in \{1, \ldots, s\} \) we have \( \nu (\partial_{b_i} (Q)) \geq \nu (Q) - b_i \epsilon (Q) \). And then the inequality is strict for at least an index \( i \in \{1, \ldots, s\} \). Either \( b_0 > 0 \) and then

\[ \frac{\nu (p_j) - \nu (\partial_{b_0} (p_j))}{b_0} \leq \epsilon (p_j) < \epsilon (Q) \]

because \( \text{deg}_X (p_j) < \text{deg}_X (Q) \) and \( Q \) is a key polynomial. Equivalently,

\[ \nu (\partial_{b_0} (p_j)) > \nu (p_j) - b_0 \epsilon (Q). \]

So if \( b_0 = 0 \) and \( \{b_1, \ldots, b_s\} \not\subseteq I(Q) \), we have

\[ \nu (\partial_{b_0} (p_j)) + \sum_{i=1}^s \nu (\partial_{b_i} (Q)) + (j-s) \nu (Q) > \nu (p_j) + s \nu (Q) - b e (Q) + (j-s) \nu (Q). \]

And if \( b_0 > 0 \), then

\[ \nu (\partial_{b_0} (p_j)) + \sum_{i=1}^s \nu (\partial_{b_i} (Q)) + (j-s) \nu (Q) > \nu (p_j) - b_0 \epsilon (Q) + s \nu (Q) - \sum_{i=1}^s b_i \epsilon (Q) + (j-s) \nu (Q). \]

So:

\[ \nu_Q (S_j) > \min_{B_s \in N_j} \{ \nu (p_j Q^j) - b e (Q) \} \]
\[ > \nu_Q (P) - b e (Q) \]

as wanted.

If \( j = l \), then

\[ \partial_b (p_l Q^j) = (R_0 + R_l) Q^{l-p^*+1} + S_l + urQ^{l-p^*} \]
and with the same argument than before, $\nu_Q(S_i) > \nu_Q(P) - b\epsilon(Q)$.

It remains to show the general case. We have:

$$\partial_b(P) = \partial_b\left(\sum_{i \in S_Q(P)} p_iQ^i\right) = \partial_b(p_iQ^i) + \sum_{j \in S_Q(P) \setminus \{l\}} \partial_b(p_jQ^j).$$

Then:

$$\partial_b(P) = (R_0 + R_l)Q^{l-p^*+1} + S_l + urQ^{l-p^*} + \sum_{j \in S_Q(P) \setminus \{l\}} (Q^{l-p^*} + 1R_j + S_j)$$

$$= urQ^{l-p^*} + Q^{l-p^*+1}R + S$$

where

$$R := R_0 + \sum_{j \in S_Q(P)} R_j$$

and

$$S := \sum_{j \in S_Q(P)} S_j.$$ 

We have

$$\nu_Q(S) \geq \min_{j \in S_Q(P)} \{\nu_Q(S_j)\} > \nu_Q(P) - b\epsilon(Q).$$

This concludes the proof of the Lemma. □

Recall that we want to show that

$$\nu_Q(\partial_bP) = \nu_Q(P) - b\epsilon(Q).$$

We just saw that the $Q$-expansion of $\partial_bP$ contains the term $urQ^{l-p^*}$, some terms divisible by $Q^{l-p^*+1}$ and others of value strictly higher than $\nu_Q(P) - b\epsilon(Q)$. It is sufficient now to show that

$$\nu_Q(\partial_bP) \geq \nu_Q(P) - b\epsilon(Q)$$

and that

$$\nu_Q\left(urQ^{l-p^*}\right) = \nu_Q(P) - b\epsilon(Q).$$

Let us compute $\nu_Q\left(urQ^{l-p^*}\right)$. Recall that $p_i(\partial_{b(Q)}Q)^{p^*} = qQ + r$. By Lemma 1.11 we have $\nu(r) = \nu(p_i(\partial_{b(Q)}Q)^{p^*})$. So:

$$\nu_Q\left(urQ^{l-p^*}\right) = \nu_Q\left(rQ^{l-p^*}\right) = \nu\left(rQ^{l-p^*}\right) = \nu\left(p_i(\partial_{b(Q)}Q)^{p^*}\right) + \nu(Q^{l-p^*}) = \nu(p_iQ^i) + p^r\nu(\partial_{b(Q)}Q) - p^r\nu(Q) = \nu_Q(P) + p^r\nu(\partial_{b(Q)}Q) - \nu(Q) = \nu_Q(P) + p^r(-b\epsilon(Q)) = \nu_Q(P) - b\epsilon(Q).$$

To conclude, we just have to use Lemma 1.16. □

Remark 1.18. One can show that the proposition is in fact an equivalence.
Proposition 1.19. Let $Q$ be a key polynomial and $P$ a polynomial such that there exists a strictly positive integer $b$ such that

$$\nu_Q (P) - \nu_Q (\partial_b P) = b \epsilon (Q)$$

and

$$\nu_Q (\partial_b P) = \nu (\partial_b P).$$

Then $\epsilon (P) \geq \epsilon (Q)$.

If in addition $\nu (P) > \nu_Q (P)$, then $\epsilon (P) > \epsilon (Q)$.

Proof. We have

$$\epsilon (P) \geq \frac{\nu (P) - \nu (\partial_b P)}{b} = \frac{\nu (P) - \nu_Q (\partial_b P)}{b} = \epsilon (Q) + \frac{\nu (P) - \nu_Q (P)}{b}.$$

We know that for every polynomial $P$, we have $\nu (P) \geq \nu_Q (P)$, so $\epsilon (P) \geq \epsilon (Q)$. And if $\nu (P) > \nu_Q (P)$, we do have the strict inequality $\epsilon (P) > \epsilon (Q)$. \qed

Proposition 1.20. Let $Q_1$ and $Q_2$ be two key polynomials such that

$$\epsilon (Q_1) \leq \epsilon (Q_2)$$

and let $P \in K[X]$ be a polynomial.

Then $\nu_{Q_1} (P) \leq \nu_{Q_2} (P)$.

Furthermore, if $\nu_{Q_1} (P) = \nu (P)$, then $\nu_{Q_2} (P) = \nu (P)$.

Proof. First, we show that $\nu_{Q_2} (Q_1) = \nu (Q_1)$. If $\deg_X (Q_1) < \deg_X (Q_2)$, we do have this equality. Otherwise we have $\deg_X (Q_1) = \deg_X (Q_2)$ since $\epsilon (Q_1) \leq \epsilon (Q_2)$ and since $Q_1$ is a key polynomial.

Assume by contradiction that $\nu_{Q_2} (Q_1) < \nu (Q_1)$.

So $S_{Q_2} (Q_1) \neq \{0\}$ and by Proposition 1.19 there exists a non zero integer $b$ such that $\nu_{Q_2} (Q_1) - \nu_{Q_2} (\partial_b Q_1) = b \epsilon (Q_2)$. However $\deg_X (\partial_b Q_1) < \deg_X (Q_2)$, so $\nu_{Q_2} (\partial_b Q_1) = \nu (\partial_b Q_1)$ and by Proposition 1.19 we have $\epsilon (Q_1) > \epsilon (Q_2)$. This is a contradiction by hypothesis. So we do have $\nu_{Q_2} (Q_1) = \nu (Q_1)$.

Let $P = \sum_{i=0}^n p_i Q_1^i$ the $Q_1$-expansion of $P$.

For every $i \in \{0, \ldots, n\}$, we have:

$$\nu_{Q_2} (p_i Q_1^i) = \nu_{Q_2} (p_i) + i \nu_{Q_2} (Q_1) = \nu_{Q_2} (p_i) + i \epsilon (Q_1).$$

But $\deg_X (p_i) < \deg_X (Q_1) \leq \deg_X (Q_2)$, so $\nu_{Q_2} (p_i) = \nu (p_i)$ and $\nu_{Q_2} (p_i Q_1^i) = \nu (p_i Q_1^i)$.

Then

$$\nu_{Q_2} (P) \geq \min_{0 \leq i \leq n} \{ \nu_{Q_2} (p_i Q_1^i) \} = \min_{0 \leq i \leq n} \{ \nu (p_i Q_1^i) \} = \nu_{Q_1} (P).$$

Assume that in addition, $\nu_{Q_1} (P) = \nu (P)$. Then $\nu (P) \leq \nu_{Q_2} (P)$. By definition of $\nu_{Q_2}$, we have $\nu_{Q_2} (P) \leq \nu (P)$, and so the equality. \qed
Proposition 1.21. Let $P_1, \ldots, P_n \in K[X]$ be some polynomials and we set $d := \max_{1 \leq i \leq n} \{ \deg_X (P_i) \}$.

Then there exists a key polynomial $Q$ of degree less or equal to $d$ such that all the $P_i$ are non degenerated with respect to $Q$. Equivalently, there exists a key polynomial $Q$ such that for every $i$, we have $\nu Q (P_i) = \nu (P_i)$.

Proof. Assume the result for only one polynomial and assume $n > 1$.

So we have $Q_1, \ldots, Q_n$ some key polynomials of degrees less or equal to $d$ such that for every $i \in \{1, \ldots, n\}$, the polynomial $P_i$ is non degenerated with respect to $Q_i$. It means that $\nu Q_i (P_i) = \nu (P_i)$.

We can assume

$$\epsilon (Q_n) = \max_{1 \leq i \leq n} \{ \epsilon (Q_i) \}.$$ 

By Proposition 1.20 for every $i \in \{1, \ldots, n\}$, we have $\nu Q_i (P_i) = \nu (P_i)$.

So all the $P_i$ are non degenerated with respect to $Q_n$. This concludes the proof.

It remains to show the result for $n = 1$. We give a proof by contradiction. Assume the existence of a polynomial $P$ such that for every key polynomial $Q$ of degree less or equal to $d$, we have $\nu Q (P) < \nu (P)$. We choose $P$ of minimal degree for this property.

Let us show that there exists a key polynomial $Q$, of degree less or equal to $d = \deg_X (P)$, such that for every $b > 0$, we have $\nu Q (\partial_b P) = \nu (\partial_b P)$.

First, for every $b > d$, we have $\partial_b P = 0$. Then, by minimality of the degree of $P$, for every $b \in \{1, \ldots, d\}$, there exists a key polynomial $Q_b$ such that $\nu Q_b (\partial_b P) = \nu (\partial_b P)$.

We consider $Q$ among the $Q_b$ such that $\epsilon (Q) = \max_{1 \leq b \leq d} \{ \epsilon (Q_b) \}$. By Proposition 1.20 we have $\nu Q (\partial_b P) = \nu (\partial_b P)$, for every $b > 0$.

So we have $\nu Q (P) < \nu (P)$. In particular, $S_Q (P) \neq \{0\}$, and $\nu Q (\partial_b P) = \nu (\partial_b P)$ for every $b > 0$. By Proposition 1.19 and Corollary 1.19 we conclude that $\epsilon (P) > \epsilon (Q)$.

Let us show that this last inequality is true for every key polynomial of degree less or equal than $\deg (P)$. Let $Q_0$ be such a key polynomial.

First case: $\epsilon (Q_0) \leq \epsilon (Q)$. Then $\epsilon (Q_0) < \epsilon (P)$ since $\epsilon (Q) < \epsilon (P)$.

Last case: $\epsilon (Q_0) > \epsilon (Q)$. By Proposition 1.20 we have $\nu (\partial_b P) = \nu Q_0 (\partial_b P)$ for every $b > 0$. By hypothesis we know that $\nu Q_0 (P) < \nu (P)$. So by Proposition 1.19 and Corollary 1.19 we have $\epsilon (P) > \epsilon (Q_0)$ as wanted.

So we know that for every key polynomial of degree less or equan than those of $P$, we have $\epsilon (P) > \epsilon (Q)$. But by definition of the key polynomials, there exists a key polynomial $\bar{Q}$ of degree less or equal than those of $P$ and such that $\epsilon (P) \leq \epsilon (\bar{Q})$.

Contradiction. This concludes the proof. □
2. Immediate successors key polynomials.

**Definition 2.1.** Let $Q_1$ and $Q_2$ be two key polynomials. We say that $Q_2$ is an *immediate successor* of $Q_1$ and we note $Q_1 < Q_2$ if $\epsilon(Q_1) < \epsilon(Q_2)$ and if $Q_2$ is of minimal degree for this property.

**Remark 2.2.** We consider the same hypothesis than in example 1.8. Then we have $z < z^2 - x^2y$.

**Definition 2.3.** It will be useful to have more simple ways to check if a key polynomial is an immediate successor of another key polynomial. This is why we give these two results.

**Proposition 2.4.** Let $Q_1$ and $Q_2$ be two key polynomials. The following are equivalent.

1. The polynomials $Q_1$ and $Q_2$ satisfy $Q_1 < Q_2$.
2. We have $\nu(Q_1) < \nu(Q_2)$ and $Q_2$ is of minimal degree for this property.

**Proof.** First let us show that $\epsilon(Q_1) < \epsilon(Q_2) \Rightarrow \nu(Q_1) < \nu(Q_2)$.

We set $b := b(Q_2) = \min \{ b \in \mathbb{N}^* \text{ such that } \nu(Q_2) - \nu(\partial_b Q_2) = \epsilon(Q_2) \}$. We have $\epsilon(Q_1) < \epsilon(Q_2) \iff be(Q_1) < \nu(Q_2) - \nu(\partial_b Q_2) \Rightarrow be(Q_1) < \nu(Q_2) - \nu_Q, (\partial_b Q_2)$

for every polynomial $g$, we have $\nu_Q, (g) \leq \nu(g)$.

But by Lemma 2.15, $\nu_Q, (Q_2) - \nu_Q, (\partial_b Q_2) \leq be(Q_1)$, so

$\nu_Q, (Q_2) - \nu_Q, (\partial_b Q_2) < \nu(Q_2) - \nu_Q, (\partial_b Q_2)$.

Then $\nu_Q, (Q_2) < \nu(Q_2)$.

Now let us show that $\nu_Q, (Q_2) < \nu(Q_2) \Rightarrow \epsilon(Q_1) < \epsilon(Q_2)$. Assume by contradiction that $\epsilon(Q_1) \geq \epsilon(Q_2)$. Then $\deg(Q_1) \geq \deg(Q_2)$.

If we have $\deg(Q_1) > \deg(Q_2)$, then $\nu_Q, (Q_2) = \nu(Q_2)$ and this is a contradiction.

So assume $Q_1$ and $Q_2$ have same degree.

Let $Q_2 = Q_1 + (Q_2 - Q_1)$ the $Q_1$-expansion of $Q_2$.

If $\nu(Q_1) \neq \nu(Q_2 - Q_1)$, then $\nu(Q_2) = \min \{ \nu(Q_1), \nu(Q_2 - Q_1) \} = \nu_Q, (Q_2)$

and again it is a contradiction.

So $\nu(Q_1) = \nu(Q_2 - Q_1) = \nu_Q, (Q_2) < \nu(Q_2)$.

But $\nu(Q_2) = \nu_Q, (Q_2) \leq \nu_Q, (Q_2)$ by Proposition 1.20. This is still a contradiction.

So we showed that $\epsilon(Q_1) < \epsilon(Q_2) \iff \nu_Q, (Q_2) < \nu(Q_2)$.

Let $Q_2$ be of minimal degree for the first property.

Assume the existence of $Q_3$ of degree strictly less than $Q_2$ such that $\nu_Q, (Q_3) < \nu(Q_3)$. So $\epsilon(Q_1) < \epsilon(Q_3)$, which is in contradiction with the minimality of the degree of $Q_2$ for this property.

So we have $Q_1 < Q_2 \Rightarrow \nu_Q, (Q_2) < \nu(Q_2)$ and $Q_2$ is of minimal degree for this property.

Then let us set $Q_2$ such that $\nu_Q, (Q_2) < \nu(Q_2)$ and $Q_2$ is of minimal degree for this property. Assume the existence of $Q_3$ of degree strictly less than $Q_2$ and such
that $\epsilon(Q_1) < \epsilon(Q_3)$. By this last property, we have that $\nu_{Q_1}(Q_3) < \nu(Q_3)$, which is in contradiction with the minimality of the degree of $Q_2$ for this property.

This concludes the proof. □

**Proposition 2.5.** Let $Q_1$ and $Q_2$ be two key polynomials, and let

$$Q_2 = \sum_{j \in \Theta} q_j Q_1^j$$

be the $Q_1$-expansion of $Q_2$.

The following are equivalent:

1. The polynomials $Q_1$ and $Q_2$ satisfy $Q_1 < Q_2$.
2. We have that $\sum_{j \in S_{Q_1}(Q_2)} \text{in}_\nu(q_j Q_1^j) = 0$ with $Q_2$ of minimal degree for this property.

**Proof.** First, let us show that $Q_1 < Q_2 \Rightarrow \sum_{j \in S_{Q_1}(Q_2)} \text{in}_\nu(q_j Q_1^j) = 0$.

Assume $Q_1 < Q_2$. By Proposition 2.4, we know that $\nu_{Q_1}(Q_2) < \nu(Q_2)$. So by definition

$$\sum_{j \in S_{Q_1}(Q_2)} \text{in}_\nu(q_j Q_1^j) = 0.$$

Furthermore, if $Q_1 < Q_2$, we do have that $Q_2$ is of minimal degree for this property by definition of successor immediate.

Now let us show that if $\sum_{j \in S_{Q_1}(Q_2)} \text{in}_\nu(q_j Q_1^j) = 0$ with $Q_2$ of minimal degree for this property, then $Q_1 < Q_2$.

So let us assume $\sum_{j \in S_{Q_1}(Q_2)} \text{in}_\nu(q_j Q_1^j) = 0$. Then

$$\nu(Q_2) > \min_{j \in \Theta} \nu(q_j Q_1^j) = \nu_{Q_1}(Q_2),$$

and so $Q_2 > Q_1$ by Proposition 2.4. □

**Remark 2.6.** Let $Q_1 < Q_2$ be two immediate successors and let $Q_2 = \sum_{j \in \Theta} q_j Q_1^j$ be the $Q_1$-expansion of $Q_2$. We set

$$\tilde{Q}_2 = \sum_{j \in S_{Q_1}(Q_2)} q_j Q_1^j.$$

We will show that $\tilde{Q}_2$ is an immediate successor of $Q_1$. Then we will always consider “optimal” immediate successors key polynomials, it means that all the terms in their expansion according to the powers of the previous key polynomial are of same value.

**Proposition 2.7.** Let $Q_1 < Q_2$ be two immediate successors and let $Q_2 = \sum_{j \in \Theta} q_j Q_1^j$ be the $Q_1$-expansion of $Q_2$. We set

$$\tilde{Q}_2 = \sum_{j \in S_{Q_1}(Q_2)} q_j Q_1^j.$$

Then $\tilde{Q}_2$ is an immediate successor of $Q_1$. 
Proof. First, by Definition of $\bar{Q}_2$, we have $\deg (\bar{Q}_2) \leq \deg (Q_2)$. We are going to show that this inequality is in fact an equality.

We have $\sum_{j \in S_{Q_1}(Q_2)} \nu_{Q_1} (q_j Q_1^j) = \sum_{j \in S_{Q_1}(Q_2)} \nu_{Q_1} (q_j Q_1^j) = 0$. Since $Q_2$ is of minimal degree for this property, we know that its term of greatest degree appears in this sum. So $\deg_X (\bar{Q}_2) = \deg_X (Q_2)$.

Now let us show that $\epsilon (\bar{Q}_2) > \epsilon (Q_1)$.

Since $\sum_{j \in S_{Q_1}(Q_2)} \nu_{Q_1} (q_j Q_1^j) = 0$, we have $\nu_{Q_1} (\bar{Q}_2) < \nu (\bar{Q}_2)$, and $\bar{Q}_2$ is still of minimal degree for this property. Then $S_{Q_1} (\bar{Q}_2) \neq \{0\}$ and for every non zero integer $b$, we have $\nu_{Q_1} (\partial_b \bar{Q}_2) = \nu (\partial_b \bar{Q}_2)$. By Proposition 1.16 there exists a strictly positive integer $b$ such that $\nu_{Q_1} (P) - \nu_{Q_1} (\partial_b P) = b \epsilon (Q)$. So we can use Corollary 1.19 to conclude that $\epsilon (\bar{Q}_2) > \epsilon (Q_1)$.

Assume that we already know that $\bar{Q}_2$ is a key polynomial. Since $\deg (\bar{Q}_2) = \deg (Q_2)$, we have that $\bar{Q}_2$ is of minimal degree for the property $\epsilon (\bar{Q}_2) > \epsilon (Q_1)$, and so $Q_1 < \bar{Q}_2$.

So we now just have to prove that $\bar{Q}_2$ is a key polynomial.

Assume by contradiction that $\bar{Q}_2$ is not a key polynomial. So there exists a polynomial $P \in K[X]$ such that $\epsilon (P) \geq \epsilon (\bar{Q}_2)$

and

$\deg_X (P) < \deg_X (\bar{Q}_2)$.

We consider $P$ of minimal degree for this property. We can also assume that $P$ is monic. Then let us show that $P$ is a key polynomial.

Let $S \in K[X]$ be a polynomial such that $\epsilon (S) \geq \epsilon (P)$. So $\epsilon (S) \geq \epsilon (\bar{Q}_2)$. If $\deg_X (S) \geq \deg_X (\bar{Q}_2)$, then $\deg_X (S) > \deg_X (P)$ and it is over. So let us assume that $\deg_X (S) < \deg_X (\bar{Q}_2)$.

So we have $\epsilon (S) \geq \epsilon (\bar{Q}_2)$ and $\deg_X (S) < \deg_X (\bar{Q}_2)$. By minimality of the degree of $P$ for this property, we have $\deg_X (S) \geq \deg_X (P)$, and then $P$ is a key polynomial.

So there exists a key polynomial $P$ such that $\epsilon (P) \geq \epsilon (\bar{Q}_2)$

and

$\deg_X (P) < \deg_X (\bar{Q}_2)$.

Since $\epsilon (\bar{Q}_2) > \epsilon (Q_1)$, we also have $\epsilon (P) > \epsilon (Q_1)$. By minimality of the degree of $Q_2$ among the key polynomials which satisfy this inequality, we have
Since \( Q_2 \) does not have a maximal element and that for every element \( Q \in \Lambda \), we have \( \deg(Q) < \epsilon(Q') \).

We also assume that there exists a key polynomial \( Q' \in \Lambda \) such that \( \epsilon(Q') > \epsilon(M_Q) \).

We call \textit{limit immediate successor} of \( Q \) every polynomial \( Q' \) of minimal degree which satisfies this property, and we note \( Q <_{\text{lim}} Q' \).

**Proposition 2.11.** Let \( Q \) and \( Q' \) be two key polynomials such that \( \epsilon(Q) < \epsilon(Q') \). Then there exists a sequence \( Q_1 = Q, \ldots, Q_k = Q' \) where for every index \( i \), the polynomial \( Q_{i+1} \) is either an immediate successor of \( Q_i \) or a limit immediate successor of \( Q_i \).

**Proof.** If \( Q' \) is an immediate successor of \( Q \), we are done, so we assume that \( Q' \) is not an immediate successor of \( Q \), and we note this \( Q \not< Q' \).

Let us first look at \( M_Q = M_{Q_1} \). If this set has a maximum, we denote this maximum by \( Q_2 \). So we have:

\[
\begin{cases}
Q < Q_2 \\
\epsilon(Q) < \epsilon(Q') \\
Q \not< Q'
\end{cases}
\]

and by minimality of the degree of \( Q_2 \) we know that \( \deg_X(Q_2) < \deg_X(Q') \). But \( Q' \) is a key polynomial, so \( \epsilon(Q_2) < \epsilon(Q') \).

Then we have

\[
\begin{cases}
Q = Q_1 < Q_2 \\
\epsilon(Q) < \epsilon(Q_2) < \epsilon(Q')
\end{cases}
\]

and since \( Q < Q_2 \), we know that \( \deg_X(Q) \leq \deg_Q(Q_2) \).

We iterate the process as long as \( M_Q \) has a maximum.

Assume that there exists an index \( i \) such that \( M_{Q_i} \) does not have any maximum. Assume \( \epsilon(M_{Q_i}) \not< \epsilon(Q') \). So there exists \( g_i \in M_{Q_i} \) such that \( \epsilon(g_i) \geq \epsilon(Q') \).

Since \( Q' \) is a key polynomial, we know that \( \deg_X(g_i) \geq \deg_X(Q') \).

So we have:

\[
\begin{cases}
\epsilon(Q_i) < \epsilon(Q') \\
Q_i < g_i \\
\deg_X(Q') \leq \deg_X(g_i)
\end{cases}
\]
By definition of immediate successors, we have \( Q_i < Q' \) and we set \( Q_{i+1} = Q' \). This concludes the proof.

So now assume that \( \epsilon \left( Q' \right) > \epsilon \left( M_{Q_i} \right) \).

Since \( \deg_X Q_i \leq \deg_X Q \), for every index \( i \), there exists a rank \( N \) from which for every index \( j \geq N \), we have

\[
\deg_X Q_j = \deg_X Q_{j+1} < \deg_X Q'.
\]

Let \( P \in M_{Q_N} \). By construction, \( \epsilon \left( P \right) \leq \epsilon \left( Q_{N+1} \right) < \epsilon \left( Q' \right) \). If \( Q' \) is not of minimal degree for this property, then there exists a key polynomial \( P' \) limit immediat successor of \( Q_N \), of degree strictly less than the degree of \( Q' \). So

\[
\deg_X Q_{N+1} < \deg_X P' < \deg_X Q'.
\]

Then we exchange \( Q_{N+1} \) with \( P' \) and we iterate the processus, which ends because the sequence of the degrees increase strictly.

Otherwise, \( Q' \) is of minimal degree among all the key polynomials such that \( \epsilon \left( M_{Q_N} \right) < \epsilon \left( Q' \right) \), so \( Q' \) is a limit immediat successor of \( Q_N \) and the processus ends at \( Q_{N+1} = Q' \).

In each case, we construct a family of key polynomials which begins at \( Q \), ends at \( Q' \) and such that for every index \( i \), the polynomial \( Q_{i+1} \) is either an immediate successor of \( Q_i \), or a limit immediate successor of \( Q_i \), this ends the proof. \( \square \)

**Proposition 2.12.** Let \( Q \) and \( Q' \) be two key polynomials such that \( \epsilon \left( Q \right) < \epsilon \left( Q' \right) \). Then there exists a sequence \( Q_1 = Q, \ldots, Q_h = Q' \) where for every index \( i \), the polynomial \( Q_{i+1} \) is either an optimal immediate successor of \( Q_i \) or a limit immediate successor of \( Q_i \).

**Proof.** Let \( Q_2 \) be an optimal immediate successor of \( Q \). We look at \( M_Q = M_{Q_1} \). If this set has a maximum, we denote this maximum by \( P \).

If \( \epsilon \left( Q_2 \right) = \epsilon \left( P \right) \), we set \( P = Q_2 \). Otherwise, \( \epsilon \left( Q_2 \right) < \epsilon \left( P \right) \). Since \( P \) and \( Q_2 \) are both immediate successors of \( Q \), they have same degree.

Hence \( P \) is an immediate successor of \( Q_2 \), of the same degree than \( Q_2 \). The polynomial \( P \) is then an optimal immediate successor of \( Q_2 \).

So we set \( Q_3 = P \).

In fact, we have a finite sequence of optimal immediate successors which begins at \( Q \) and ends at \( P = \max \{ M_Q \} \).

We iterate the processus as long as \( M_Q \) has a maximum. Assume that there exists an index \( i \) such that \( M_{Q_i} \) does not have any maximum.

Then we do exactly the same thing that we did in the proof of Lemma 2.13 and this ends the proof. \( \square \)

**Lemma 2.13.** Let \( Q \) and \( Q' \) be two key polynomials such that \( Q < Q' \) and we note \( Q' = \sum_{j=0}^{m} q_j Q^j \) the \( Q \)-expansion of \( Q' \). Then \( q_m = 1 \).

**Proof.** Since \( \epsilon \left( Q \right) < \epsilon \left( Q' \right) \), we know by Proposition 2.10 that \( \sum_{j=0}^{m} \text{in}_\nu \left( q_j Q^j \right) = 0 \).

In fact we have

\[
\text{in}_\nu \left( q_m \right) \text{in}_\nu \left( Q \right)^m + \cdots + \text{in}_\nu \left( q_1 \right) \text{in}_\nu \left( Q \right) + \text{in}_\nu \left( q_0 \right) = 0.
\]
Then, since \( \text{in}_\nu(q_m) \neq 0 \), we have
\[
Q_{\nu}(Q)^m + \cdots + \frac{\text{in}_\nu(q_1)}{\text{in}_\nu(q_m)} \text{in}_\nu(Q) + \frac{\text{in}_\nu(q_0)}{\text{in}_\nu(q_m)} = 0.
\]

We set \( a := \text{deg}_X(Q) \) and we consider \( G_{\leq a} \) the subalgebra of \( \text{gr}_\nu(K[X]) \) generated by the initial forms of all the polynomials of degree strictly less than \( a \).

Hence \( G_{\leq a} \) is a saturated algebra, and then all the coefficients of the form \( \frac{\text{in}_\nu(q_i)}{\text{in}_\nu(q_m)} \) of the equation (2.1) can be represented by polynomials and we denote by \( h_i \) some lifts, of degrees strictly less than \( a \).

The element \( \text{in}_\nu(Q) \) is hence a solution of an homogeneous integer equation with coefficients in \( G_{\leq a} \) and which coefficient of greatest degree is \( 1 \).

We consider the polynomial \( \tilde{Q} = Q^m + \sum_{j=0}^{m-1} h_j Q^j \), with, by hypothesis, \( \text{deg}_X(Q) \leq \text{deg}_X(Q') \). By construction we have
\[
\text{in}_\nu(Q)^m + \sum_{j=0}^{m-1} \text{in}_\nu(h_j) \text{in}_\nu(Q)^j = 0
\]
and by the proof of the proposition 2.5, we have \( \epsilon(Q) > \epsilon(Q) \).

By minimality of the degree of \( Q' \) for the property, if we can show that \( \tilde{Q} \) is a key polynomial, then we would have \( \text{deg}_X(Q') = \text{deg}_X(Q) \) and so \( q_m = 1 \).

Hence let us show that \( \tilde{Q} \) is a key polynomial.

Assume by contradiction that it is not. Then there exists a polynomial \( P \) such that \( \epsilon(P) \geq \epsilon(Q) \) and \( \text{deg}_X(P) < \text{deg}_X(Q) \). We choose \( P \) monic and of minimal degree for this property. Let us show that \( P \) is a key polynomial.

Let \( S \) be a polynomial such that \( \epsilon(S) \geq \epsilon(P) \). Then \( \epsilon(S) \geq \epsilon(Q) \).

If \( \text{deg}_X(S) \geq \text{deg}_X(Q) \), then, since \( \text{deg}_X(P) < \text{deg}_X(Q) \), it is done.

So let us assume that \( \text{deg}_X(S) < \text{deg}_X(Q) \). Then \( \epsilon(S) \geq \epsilon(Q) \) and \( \text{deg}_X(S) < \text{deg}_X(Q) \). By minimality of the degree of \( P \) for that property, \( \text{deg}_X(S) \geq \text{deg}_X(P) \) and it is over.

So there exists a key polynomial \( P \) such that \( \epsilon(P) \geq \epsilon(Q) \) and \( \text{deg}_X(P) < \text{deg}_X(Q) \).

Since \( \epsilon(Q) > \epsilon(Q) \), we have \( \epsilon(P) > \epsilon(Q) \).

So we have a key polynomial \( P \) such that \( \epsilon(P) > \epsilon(Q) \). By minimality of the degree of \( Q' \) for this property, we know that \( \text{deg}_X(Q') \leq \text{deg}_X(P) \). But \( \text{deg}_X(P) < \text{deg}_X(Q) \), and this implies that \( \text{deg}_X(Q') < \text{deg}_X(Q) \), which is a contradiction.

Then \( Q \) is a key polynomial.

\[ \square \]

**Proposition 2.14.** Let \( Q \) and \( Q' \) be two key polynomials such that
\[ \epsilon(Q) < \epsilon(Q'). \]

Let \( c \) and \( c' \) be two polynomials of degrees strictly less than \( \text{deg}_X Q' \) and let \( j \) and \( j' \) be two integers such that :
Proof. Then:

\[
\begin{align*}
\nu_Q(c) &= \nu(c), \\
\nu_Q(c') &= \nu(c'), \\
\nu_Q(c(Q')^j) &= \nu(c'(Q')^j).
\end{align*}
\]

Then:

\[
\nu(c(Q')^j) \leq \nu(c'(Q')^j).
\]

Furthermore, if in addition either \(j < j'\) or \(\nu_Q(c(Q')^j) < \nu_Q(c'(Q')^j)\), then

\[
\nu(c(Q')^j) < \nu(c'(Q')^j).
\]

Proof. We know that \(\nu_Q(Q') \leq \nu(Q')\), then

\[
\nu(Q') - \nu_Q(Q') \geq 0.
\]

Since we assumed that \(j \leq j'\), we have

\[
j(\nu(Q') - \nu_Q(Q')) \leq j'(\nu(Q') - \nu_Q(Q')).
\]

Furthermore, we know that \(\nu_Q(c(Q')^j) \leq \nu_Q(c'(Q')^j)\), hence

\[
\nu_Q(c(Q')^j) + j(\nu(Q') - \nu_Q(Q')) \leq \nu_Q(c'(Q')^j) + j'(\nu(Q') - \nu_Q(Q')).
\]

So we have the inequality

\[
\nu_Q(c) + j\nu_Q(Q') + j\nu(Q') - j
\]

Equivalently \(\nu_Q(c) + j\nu(Q') \leq \nu_Q(c') + j'\nu_Q(Q')\).

But \(\nu_Q(c) = \nu(c)\) and \(\nu_Q(c') = \nu(c')\), so \(\nu(c(Q')^j) \leq \nu(c'(Q')^j)\).

If in addition either \(j < j'\) or \(\nu_Q(c(Q')^j) < \nu_Q(c'(Q')^j)\), then we have

\[
\nu(c(Q')^j) < \nu(c'(Q')^j).
\]

Lemma 2.15. Let \(Q\) and \(Q'\) be two polynomials such that

\[
\epsilon(Q) < \epsilon(Q')
\]

and let \(f \in K[X]\) be a polynomial which \(Q'\)-expansion is \(f = \sum_{j=0}^{r} f_j (Q')^j\). Then

\[
\nu_Q(f) = \min_{0 \leq j \leq r} \left\{ \nu_Q\left(f_j (Q')^j\right) \right\}.
\]

If we set

\[
T_{Q,Q'}(f) := \{ j \in \{0, \ldots, r\} \text{ such that } \nu_Q\left(f_j (Q')^j\right) = \nu_Q(f) \},
\]

then we have

\[
in_{\nu_Q}(f) = \sum_{j \in T_{Q,Q'}(f)} \nu_{\nu_Q}\left(f_j (Q')^j\right).
\]
Proof. Uniquely in this proof, we will note

$$\nu' (f) := \min_{0 \leq j \leq r} \{ \nu_Q \left( f_j (Q')^i \right) \}$$

and

$$T'^{t} (f) := \{ j \in \{0, \ldots, r \} \text{ such that } \nu_Q \left( f_j (Q')^i \right) = \nu' (f) \}.$$ 

Let us show that $\nu_Q (f) = \nu' (f)$.

First, we have

$$\nu_Q \left( \sum_{j \in T'^{t} (f)} f_j (Q')^i \right) \geq \min_{j \in T'^{t} (f)} \nu_Q \left( f_j (Q')^i \right) = \min_{j \in T'^{t} (f)} \nu' (f) = \nu' (f).$$

Set now $b' = \max T'^{t} (f)$ and $b = \delta_Q (f_{b'})$. It means that $b = \max \{ j \in \{0, \ldots, n \} \text{ such that } \nu (a_j Q^i) = \nu_Q (f_{b'}) \}$ where $f_{b'} = \sum_{j=0}^{n} a_j Q^i$. Hence, the element $\sum_{j \in T'^{t} (f)} f_j (Q')^i$ contains the term

$$a_{b Q} Q^b Q^{b + b'_Q} Q'.$$

Then for every $j \in \{0, \ldots, r \}$ such that $f_j \neq 0$, we have:

$$\nu_Q \left( f_j (Q')^i \right) \geq \min_{0 \leq j \leq r} \{ \nu_Q \left( f_j (Q')^i \right) \} = \nu' (f) = \nu_Q \left( f_i (Q')^i \right)$$

for every index $i \in T'^{t} (f)$. So in particular,

$$\nu_Q \left( f_j (Q')^i \right) \geq \nu_Q \left( f_{b'} (Q')^{b'} \right) = \nu_Q \left( f_{b'} \right) + \nu_Q \left( (Q')^{b'} \right) \geq \nu (a_{b Q} Q^b) + \nu_Q \left( (Q')^{b'} \right) \geq \nu (a_{b Q} Q^b) + \nu \left( c_{Q Q} Q^b Q^{b + b'_Q} Q' \right) \geq \nu \left( a_{b c Q} Q^b Q^{b + b'_Q} Q' \right)$$

with strict inequality if $j \notin T'^{t} (f)$.

So

$$\nu \left( a_{b c Q} Q^b Q^{b + b'_Q} Q' \right) = \nu' (f)$$

and

$$\nu_Q \left( \sum_{j \notin T'^{t} (f)} f_j (Q')^i \right) > \nu' (f).$$

By maximality of $b$ and $b'$, the term $a_{b c Q} Q^b Q^{b + b'_Q} Q'$ cannot be compensated and so $\nu_Q (f) = \nu \left( a_{b c Q} Q^b Q^{b + b'_Q} Q' \right) = \nu' (f)$. It means that $\nu_Q (f) = \min_{0 \leq j \leq r} \{ \nu_Q \left( f_j (Q')^i \right) \}$. So we also have

$$T'^{t} (f) = T_{Q Q'} (f).$$
Then $\sum_{j \in T(f)} \text{in}_{\nu_Q} (f_j (Q')^j)$ is a non zero element of $G_{\nu_Q}$, equal to $\text{in}_{\nu_Q} (f)$. This concludes the proof.

\[\square\]

**Corollary 2.16.** Let $Q$ and $Q'$ be two key polynomials such that $\epsilon (Q) < \epsilon (Q')$

and let

$$f = \sum_{j=0}^{r} f_j (Q')^j = \sum_{j=0}^{n} a_j Q^j$$

be the $Q'$ and $Q$-expansions of an element $f \in K[X]$. We set

$$\theta := \min_{Q \leq Q'} \{Q \in K[X] \text{ such that } \nu_Q \left( f_j (Q')^j \right) = \nu_Q (f) \}$$

and we assume that $\nu_Q \left( f_{\delta_Q} (f) \right) = \nu \left( f_{\delta_Q} \right)$ and that $\nu_Q (f_\theta) = \nu (f_\theta)$.

Then:

1. $\delta_{Q'} (f) \deg_Q Q' \leq \delta_Q (f)$, and so $\delta_{Q'} (f) \leq \delta_Q (f)$.
2. If $\delta_Q (f) = \delta_{Q'} (f)$, we set $\delta := \delta_Q (f)$ and then

$$\deg_Q Q' = 1,$$

and

$$T_{Q,Q'} (f) = \{\delta\}$$

$$\text{in}_{\nu_Q} (f) = (\text{in}_{\nu_Q} a_\delta) \left( \text{in}_{\nu_Q} Q' \right)^\delta.$$

**Proof.** First let us show the point 1.

By the proof of the previous Lemma, we know that

$$\theta \deg_Q Q' \leq \delta_Q (f).$$

Furthermore,

$$\nu_Q \left( f_{\delta_Q} (f) \right) = \nu \left( f_{\delta_Q} \right),$$

$$\nu_Q (f_\theta) = \nu (f_\theta).$$

By definition of $\delta = \delta_{Q'} (f)$, we have $\nu \left( f_\delta (Q')^\delta \right) \leq \nu \left( f_\theta (Q')^\theta \right)$. We know by Lemma 2.15 that $\nu_Q (f) = \min_{0 \leq j \leq r} \left\{ \nu_Q \left( f_j (Q')^j \right) \right\}$. Since $\theta = \min_{Q \leq Q'} \{Q \in K[X] \text{ such that } \nu_Q \left( f_j (Q')^j \right) \leq \nu_Q (f) \}$, we have

$$\nu_Q \left( f_\theta (Q')^\theta \right) = \nu_Q (f) = \min_{0 \leq j \leq r} \left\{ \nu_Q \left( f_j (Q')^j \right) \right\}.$$

Hence $\nu_Q \left( f_\theta (Q')^\theta \right) \leq \nu_Q \left( f_\delta (Q')^\delta \right)$.

Then, since $\nu_Q (f_\theta) = \nu (f_\theta)$ and $\nu_Q (f_\delta) = \nu (f_\delta)$:

$$\nu_Q \left( f_\theta (Q')^\theta \right) \leq \nu_Q \left( f_\delta (Q')^\delta \right) \iff \nu_Q (f_\theta) + \theta \nu_Q (Q') \leq \nu_Q (f_\delta) + \delta \nu_Q (Q') \iff \nu (f_\theta) + \theta \nu (Q') \leq \nu (f_\delta) + \delta \nu (Q').$$

Assume we have equality on $\nu$, it means that $\nu \left( f_\theta (Q')^\theta \right) = \nu \left( f_\delta (Q')^\delta \right)$. So

$$\nu (f_\theta) = \nu (f_\delta) + \delta \nu (Q') - \theta \nu (Q')$$

and
\[ \nu_{Q} \left( f_{\delta} (Q')^{\theta} \right) \leq \nu_{Q} \left( f_{\delta} (Q)^{\delta} \right) \]

\[ \Leftrightarrow \nu (f_{\delta}) + \delta \nu (Q') - \theta \nu (Q') + \theta \nu_{Q} (Q') \leq \nu (f_{\delta}) + \delta \nu_{Q} (Q') \]

\[ \Leftrightarrow (\delta - \theta) \nu (Q') \leq (\delta - \theta) \nu_{Q} (Q'). \]

Since we know that \( \epsilon (Q) < \epsilon (Q') \), by proof of Proposition \( 2.14 \), we know that \( \nu_{Q} (Q') < \nu (Q') \) and then \( \delta - \theta \leq 0 \), it means that \( \delta \leq \theta \).

Otherwise we have \( \nu \left( f_{\delta} (Q')^{\delta} \right) < \nu \left( f_{\theta} (Q')^{\theta} \right) \).

Then we have the four hypothesis:

\[
\begin{align*}
\nu_{Q} (f_{\delta}) &= \nu (f_{\delta}) \\
\nu_{Q} (f_{\delta}) &= \nu (f_{\delta}) \\
\nu_{Q} \left( f_{\delta} (Q')^{\delta} \right) &= \nu_{Q} \left( f_{\delta} (Q')^{\delta} \right) \\
\nu \left( f_{\delta} (Q')^{\delta} \right) &= \nu \left( f_{\theta} (Q')^{\theta} \right).
\end{align*}
\]

By contraposition of Proposition \( 2.14 \), we deduce that \( \delta < \theta \).

In each case, we have \( \delta \leq \theta \). Then since \( \theta \deg_{Q} Q' \leq \delta_{Q} (f) \), we know that \( \delta \deg_{Q} Q' \leq \delta_{Q} (f) \). So in particular \( \delta_{Q} (f) \leq \delta_{Q} (f) \).

Now let us show the point 2.

Assume \( \delta_{Q} (f) = \delta_{Q} (f) = \delta \). We just saw that \( \delta_{Q} (f) \deg_{Q} Q' \leq \delta_{Q} (f) \), so we have that \( \deg_{Q} Q' = 1 \). Then \( Q' = Q + b \) with \( b \) a polynomial of degree strictly less than the degree of \( Q \).

We know by the proof of the first point that \( \delta \leq \theta \). Furthermore, we know that \( \theta \deg_{Q} Q' \leq \delta_{Q} (f) = \delta \), it means that \( \theta \leq \delta \) since \( \deg_{Q} Q' = 1 \).

Hence \( \delta \leq \theta \leq \delta \), it means that \( \theta = \delta = \min T_{Q,Q'} (f) \). We now have to prove that for every index \( j > \delta \), we have \( j \not\in T_{Q,Q'} (f) \). Equivalently that:

\[ \nu_{Q} \left( f_{j} (Q')^{j} \right) > \nu_{Q} (f) = \min_{0 \leq i \leq r} \left\{ \nu_{Q} \left( f_{i} (Q')^{i} \right) \right\}. \]

And then we will have \( T_{Q,Q'} (f) = \{ \delta \} \).

So let \( j > \delta \). By definition of \( \delta_{Q} (f) \) and \( \delta_{Q'} (f) \), we know that \( \nu \left( f_{j} (Q')^{j} \right) > \nu_{Q'} (f) \) and \( \nu \left( a_{Q} Q' \right) > \nu_{Q} (f) \).

Furthermore, since \( \delta \in T_{Q,Q'} (f) \), we have \( \nu_{Q} \left( f_{\delta} (Q')^{\delta} \right) = \nu_{Q} (f) \). Then we want to show that \( \nu_{Q} \left( f_{j} (Q')^{j} \right) > \nu_{Q} \left( f_{\delta} (Q')^{\delta} \right) \) for every index \( j \in \{ \delta + 1, \ldots, r \} \).

We know that:

\[
\begin{align*}
\nu \left( f_{\delta} (Q')^{\delta} \right) &= \nu_{Q'} (f) < \nu \left( f_{j} (Q')^{j} \right) \\
\nu_{Q} (f_{\delta}) &= \nu (f_{\delta}) \\
\nu_{Q} (f_{j}) &= \nu (f_{j}) \\
\delta < j
\end{align*}
\]

because \( \deg_{X} (f_{\delta}) < \deg_{X} (Q') = \deg_{X} (Q) \)

because \( \deg_{X} (f_{j}) < \deg_{X} (Q') = \deg_{X} (Q) \)

By contraposition of Proposition \( 2.14 \), we have:

\[ \nu_{Q} \left( f_{\delta} (Q')^{\delta} \right) < \nu_{Q} \left( f_{j} (Q')^{j} \right). \]

So we do have \( T_{Q,Q'} (f) = \{ \delta \} \).

By Lemma \( 2.15 \), we have
\[
\text{in}_{\nu_Q} (f) = \sum_{j \in T_{\bar{Q}, Q'} (f)} \text{in}_{\nu_Q} \left( f_j \left( Q' \right)^j \right) \\
= \text{in}_{\nu_Q} \left( f_\delta \left( Q' \right)^\delta \right) \\
= \text{in}_{\nu_Q} \left( f_\delta \right) \left( \text{in}_{\nu_Q} \left( Q' \right) \right)^\delta.
\]

**Theorem 2.17.** Let \( Q \) and \( Q' \) be two key polynomials such that
\[
\epsilon \left( Q \right) < \epsilon \left( Q' \right).
\]
We recall that \( \text{car} \left( k_\nu \right) = 0 \). If \( Q' \) is a limit immediate successor of \( Q \), then \( \delta_Q \left( Q' \right) = 1 \).

**Proof.** We do a proof by contradiction and so assume that \( \delta_Q \left( Q' \right) > 1 \). Among all the couples \( \left( Q, Q' \right) \) such that \( Q' \) is a limit immediate successor of \( Q \) and such that \( \delta_Q \left( Q' \right) > 1 \), we choose \( Q \) and \( Q' \) such that \( \text{deg} \left( Q' \right) - \text{deg} \left( Q \right) \) is minimal.

By definition of a limit immediate successor, for every sequence of immediate successors \( Q \) such that \( \left( Q, Q' \right) \) and we set
\[
\begin{align*}
\text{in}_{\nu_Q} \left( f_\delta \right) \left( \text{in}_{\nu_Q} \left( Q' \right) \right)^\delta.
\end{align*}
\]

Then, by minimality of \( \text{deg} \left( Q' \right) - \text{deg} \left( Q \right) \), we know that there exists a finite sequence of immediate successors between \( Q \) and \( \bar{Q} \) and that there exists a finite sequence of immediate successors between \( \bar{Q} \) and \( Q' \). Then we have a finite sequence of immediate successors between \( Q \) and \( Q' \), which is a contradiction.

Then there exists a key polynomial \( \bar{Q} \) such that
\[
\epsilon \left( Q \right) < \epsilon \left( \bar{Q} \right) < \epsilon \left( Q' \right)
\]
and \( \text{deg} \left( Q \right) < \text{deg} \left( \bar{Q} \right) < \text{deg} \left( Q' \right) \).

Then, by minimality of \( \text{deg} \left( Q' \right) - \text{deg} \left( Q \right) \), we know that there exists a finite sequence of immediate successors between \( Q \) and \( \bar{Q} \) and that there exists a finite sequence of immediate successors between \( \bar{Q} \) and \( Q' \). Then we have a finite sequence of immediate successors between \( Q \) and \( Q' \), which is a contradiction.

Then there exists a key polynomial \( \bar{Q} \) such that
\[
\epsilon \left( Q \right) < \epsilon \left( \bar{Q} \right) < \epsilon \left( Q' \right)
\]
and \( \text{deg} \left( \bar{Q} \right) < \text{deg} \left( Q' \right) \), and so \( \text{deg} \left( Q \right) = \text{deg} \left( \bar{Q} \right) \).

So let us set \( \bar{Q} \) a such key polynomial. We have \( \bar{Q} := Q - a \) where \( a \) is a polynomial of degree strictly less than the degree of \( Q \).

Since \( \epsilon \left( Q \right) < \epsilon \left( \bar{Q} \right) \), by Proposition 2.12 we know that \( \text{in}_{\nu} \left( Q \right) = \text{in}_{\nu} \left( a \right) \).

We also consider \( \sum_{j=0} \delta_j Q^j \) the \( Q \)-expansion of \( Q' \). We can assume that \( \delta_Q \left( Q' \right) = \delta_Q \left( Q^j \right) \) and we set \( \delta := \delta_Q \left( Q' \right) \).

By Corollary 2.16 we know that \( \text{in}_{\nu_Q} \left( Q' \right) = \text{in}_{\nu_Q} \left( a \right) \text{in}_{\nu_Q} \left( Q - a \right) \). It means that
\[
\text{in}_{\nu_Q} \left( Q' \right) = \text{in}_{\nu_Q} \left( a \right) \text{in}_{\nu_Q} \left( Q - a \right) \delta.
\]

Furthermore, \( \partial Q' = \sum_{j=0} \partial_j a_j Q^j + a_j jQ^{j-1} \partial Q \).

We first show that the terms \( \partial_j a_j Q^j \) don’t intervene in \( \text{in}_{\nu} \left( \partial Q' \right) \). So let \( j \in \{0, \ldots, n\} \).
We have
\[ \nu_Q (\partial a_j) = \nu (\partial a_j) \geq \nu (a_j) - \epsilon (a_j). \]

But \( Q \) is a key polynomial and \( a_j \) is of degree strictly less than the degree of \( Q \) since it is a coefficient of a \( Q \)-expansion. Then \( \epsilon (a_j) < \epsilon (Q) \).

So
\[ \nu_Q (\partial a_j) > \nu (a_j) - \epsilon (Q) = \nu_Q (a_j) - \epsilon (Q). \]

By proof of Proposition \[1.16\] we know that, since we are in characteristic zero,
\[ \nu_Q (Q) - \nu_Q (\partial Q) = \epsilon (Q). \]

Then \( \nu_Q (\partial a_j) > \nu_Q (a_j) - \nu_Q (Q) + \nu_Q (\partial Q) \). In fact,
\[ \nu_Q (\partial a_j) + \nu_Q (Q) > \nu_Q (a_j) + \nu_Q (\partial Q). \]

It means that \( \nu_Q (Q \partial a_j) > \nu_Q (a_j \partial Q) \), and adding \( \nu_Q (Q^{-1}) \) in each side, we obtain:
\[ \nu_Q (Q^2 \partial a_j) > \nu_Q (a_j Q^{-1} \partial Q) = \nu_Q (ja_j Q^{-1} \partial Q). \]

So
\[ \text{in}_{\nu_Q} (\partial Q') = \text{in}_{\nu_Q} \left( \sum_{j=1}^{n} [ja_j Q^{-1} \partial Q] \right). \]

Even if the expression \( \sum_{j=1}^{n} [ja_j Q^{-1} \partial Q] \) is not a \( Q \)-expansion, since \( a_j \) and \( \partial Q \) are of degrees strictly less than the degree of \( Q \) in characteristic zero, by Lemma \[1.11\] the \( \nu_Q \)-initial form of \( a_j \partial Q \) is equal to the initial form of its rest of the euclidian division by \( Q \). So we conserv this expression and consider it like a \( Q \)-expansion.

Now let us show that \( \delta_Q (\partial Q') = \delta - 1 \).

Exchange \( Q \) by \( \tilde{Q} \) in the computation of the initial form of \( Q' \) with respect to \( Q \) (respectively \( \tilde{Q} \)) does not change anything, and we assume that \( \delta \) stabilies from \( Q \). Then, if \( \delta_Q (\partial Q') = \delta - 1 \), we would also have \( \delta_{\tilde{Q}} (\partial Q') = \delta - 1 \).

Let \( j > \delta \). Let us first show that
\[ \nu_Q (ja_j Q^{-1} \partial Q) > \nu_Q (\delta a_j Q^{-1} \partial Q). \]

It means that
\[ \nu_Q (ja_j Q^{-1}) > \nu_Q (\delta a_j Q^{-1}). \]

But by definition of \( \delta \), we have \( \nu_Q (a_j Q^j) > \nu_Q (a_j Q^\delta) \). So
\[ \nu_Q (a_j Q^j) > \nu_Q (a_j Q^\delta), \]
then \( \nu_Q (ja_j Q^{-1}) > \nu_Q (\delta a_j Q^\delta). \)

We now have to prove that the value of the term \( \delta - 1 \) is minimal.

Let \( j < \delta \). We know that \( \nu_Q (a_j Q^j) = \nu_Q (a_j Q^\delta) \), and then
\[ \nu_Q (a_j Q^{-1} \partial Q) = \nu_Q (a_j Q^{\delta-1} \partial Q). \]

So \( \nu_Q (ja_j Q^{-1} \partial Q) = \nu_Q (\delta a_j Q^\delta \partial Q) \) since we are in characteristic zero.

So we do have \( \delta_Q (\partial Q') = \delta_{\tilde{Q}} (\partial Q') = \delta - 1 \). By Corollary \[2.16\] we have:
\[ \text{in}_{\nu_Q} (\partial Q') = \text{in}_{\nu_Q} (\delta a_j Q) \text{in}_{\nu_Q} (\tilde{Q})^{\delta-1}. \]
It means that
\[ \text{in}_{\nu_Q} (\partial Q') = \delta \text{in}_{\nu_Q} (a_\delta \partial Q) \text{in}_{\nu_Q} (Q - a)^{\delta - 1}. \]
We know that \( \nu_Q (Q - a) < \nu (Q - a) \). Then, since \( \delta > 1 \),
\[ \nu_Q \left( \delta a_\delta \partial Q (Q - a)^{\delta - 1} \right) < \nu \left( \delta a_\delta \partial Q (Q - a)^{\delta - 1} \right). \]
It means that the image by \( \varphi : \text{gr}_{\nu_Q} K [x] \rightarrow \text{gr}_{\nu} K [x] \) of
\[ \text{in}_{\nu_Q} \left( \delta a_\delta \partial Q (Q - a)^{\delta - 1} \right) \]
is zero. Then, the image by \( \varphi \) of \( \text{in}_{\nu_Q} (\partial Q') \) is zero, and so
\[ \nu_Q (\partial Q') < \nu (\partial Q'). \]

By proof of Proposition 2.4, we have \( \epsilon (Q) < \epsilon (\partial Q') \). But we know that \( \deg (\partial Q') < \deg (Q') \), and since \( Q' \) is a key polynomial, we have \( \epsilon (\partial Q') < \epsilon (Q') \).

More generally, the previous argumentation is true exchanging \( Q \) by any key polynomial \( \tilde{Q} \) of the same degree than \( Q \).
So for every key polynomial \( \tilde{Q} \) of the same degree than \( \deg (Q) \), we have \( \epsilon (\tilde{Q}) < \epsilon (\partial Q') \).
In fact, \( \epsilon (Q) < \epsilon (\partial Q') < \epsilon (Q') \) and \( \deg (\partial Q') < \deg (Q') \). So if we show that \( \partial Q' \) is a key polynomial, we will have
\[ \deg (Q) = \deg (\partial Q'). \]
So let us show that \( \partial Q' \) is a key polynomial. We assume by contradiction that it is not. So there exists a polynomial \( P \) such that \( \epsilon (P) \geq \epsilon (\partial Q') \) and \( \deg (P) < \deg (\partial Q') \). We choose \( P \) of minimal degree for this property. With the same idea we did before, we can show that \( P \) is a key polynomial.
We have \( \deg (P) < \deg (\partial Q') \), then \( \deg (P) < \deg (Q') \) and since \( Q' \) is a key polynomial, we have \( \epsilon (P) < \epsilon (Q') \).
But since \( \epsilon (P) \geq \epsilon (\partial Q') \), we have \( \epsilon (P) > \epsilon (Q) \) too.
So we have another key polynomial \( P \) such that \( \epsilon (Q) < \epsilon (P) < \epsilon (Q') \) and \( \deg (P) < \deg (Q') \). Then we know that \( \deg (P) = \deg (Q) \). Hence the polynomial \( P \) is a key polynomial of same degree than \( Q \), and so \( \epsilon (P) < \epsilon (\partial Q') \), which is a contradiction.
So we proved that \( \partial Q' \) was a key polynomial. Then \( \deg (Q) = \deg (\partial Q') \). But then \( \epsilon (\partial Q') < \epsilon (\partial Q') \) and this in a contradiction. This concludes the proof. \( \square \)
Part 3. **Simultaneous local uniformization in the case of rings essentially of finite type over a field without restriction on the valuation’s rank.**

The objective of this part is to give a proof of the local uniformization in the case of rings essentially of finite type over a field of zero characteristic without any restriction on the valuation’s rank. The proof of the local uniformization is well known in zero characteristic. It has been proved for the first time by Zariski in 1940 ([29]) in every dimension and in zero characteristic. The benefit of this new proof is to present an universal construction which works for all the elements of the regular ring we start with, and in which we know by advance all the coordinates step by step. Thus we will have an infinite sequence of blow-ups given explicitely, in which every coordinate are described, and which monomialize any element of our algebra essentially of finite type, whatever the valuation’s rank.

To do this, we will proceed by steps. Let us give the idea.

Let $k$ be a characteristic zero field, $R$ a regular local $k$-algebra essentially of finite type over $k$, of residual field $k$. We consider $u = (u_1, \ldots, u_n)$ a regular system of parameters of $R$, $\nu$ a valuation centered in $R$ of value group $\Gamma$ and $K = k(u_1, \ldots, u_{n-1})$. We assume that $k = k_{\nu}$. This property is stable by blow-ups. Thus every ring we will have in the local blow-ups along the valuation $\nu$ will have the same residual field: $k$.

We will construct a unique sequence of blow-ups which monomialize every element of $R$ provided we look far enough in the sequence. To do this, we will construct a particular sequence of immediate successors (eventually limit) key polynomials. Indeed, every element $f$ of $R$ will be non degenerated with respect to a key polynomial $Q$ of this sequence, it means that we will have $\nu_Q(f) = \nu(f)$. Furthermore, all the polynomials of this sequence will be monomializable. At this point we would have proved that every element of $R$ is non degenerated with respect to a regular system of parameters of a regular ring. Then we will just have to see that every element non degenerated with respect to a regular system of parameters is monomializable by our sequence of blow-ups.

We will begin this part by some preliminaries, where we define the non degenerescence, the framed blow-ups and the monomial blow-ups.

Then, we will see that every element non degenerated with respect to a regular system of paramaters is monomializable. And then it will be sufficient to prove that it is the case of all the elements we are interested in.

So, after that, we construct a sequence of (eventually limit) immediate successors key polynomials which satisfies that every element $f$ of $R$ is non degenerated with respect to one of these key polynomials.

In the last parts, we will see that all the key polynomials of this sequence are monomializable, and that we have proven the simultaneous local uniformization. To do this we will need a new notion: the one of key element. Indeed, modified by the blow-ups, the key polynomials of the above mentioned sequence have no reason to still be polynomials. So we will give a new definition, this one of key element. This notion has the benefit to be conserved by blow-ups. So we will monomialize the key elements and not the key polynomials, and we will conclude by induction.
3. Preliminaries.

Let $k$ be a field of characteristic zero and $R$ a regular local $k$-algebra which is essentially of finite type over $k$. We consider $u = (u_1, \ldots, u_n)$ a regular system of parameters of $R$ and $\nu$ a valuation centered on $R$ which group of values is denoted by $\Gamma$. We write $\beta_i = \nu(u_i)$ for every integer $i \in \{1, \ldots, n\}$, and $K = k(u_1, \ldots, u_{n-1})$.

3.1. Non degenerated elements.

Definition 3.1. Let $f \in R$. We say that $f$ is non degenerated with respect to $\nu$ and $u$ if we have $\nu_u(f) = \nu(f)$, where $\nu_u$ is the monomial valuation with respect to $u$.

We need a more convenient way to know if an element is non degenerated with respect to a regular system of parameters. It is the objective of the following Proposition.

Proposition 3.2. Let $f \in R$. The element $f$ is non degenerated with respect to $\nu$ and $u$ if and only if there exists an ideal $N$ of $R$ which contains $f$, monomial with respect to $u$ and such that

$$\nu(f) = \nu(N) = \min_{x \in N} \{\nu(x)\}.$$  

Proof. Let us show that if there exists an ideal $N$ of $R$ which contains $f$, monomial with respect to $u$ and such that

$$\nu(f) = \nu(N) = \min_{x \in N} \{\nu(x)\},$$

then $\nu_u(f) = \nu(f)$. So let $N$ be a such ideal. As $N$ is monomial with respect to $u$, we have $\nu_u(N) = \nu(N)$ and $\nu_u(N) \leq \nu_u(f)$ since $f \in N$.

So $\nu(f) = \nu(N) \leq \nu_u(f)$, which give us equality.

Now let us show that if $\nu_u(f) = \nu(f)$, then there exists an ideal $N$ of $R$ which contains $f$, monomial with respect to $u$ and such that $\nu(f) = \nu(N) = \min_{x \in N} \{\nu(x)\}$.

So let suppose that $\nu_u(f) = \nu(f)$. Let $N$ be the smallest ideal of $R$ generated by monomials in $u$ containing $f$. So $\nu(N) = \nu_u(N) = \nu_u(f)$ and since $\nu_u(f) = \nu(f)$, we have $\nu(N) = \nu(f)$. \hfill \square

3.2. Framed and monomial blow-up. Let $J_1 \subset \{1, \ldots, n\}$, $A_1 = \{1, \ldots, n\} \setminus J_1$ and $j_1 \in J_1$.

We write

$$u'_q = \begin{cases} \frac{u_q}{u_{j_1}} & \text{if } q \in J_1 \setminus \{j_1\} \\ u_q & \text{otherwise} \end{cases}$$

and we consider $R_1$ a localisation of $R' = R\left[u'_1, \ldots, u'_{n-1}\right]$ by a prime ideal, let say $R_1 = R_{m^r}$ of maximal ideal $m_1 = mR_1$. Since $R$ is regular, $R'$ and $R_1$ are regular. Let $u^{(1)} = (u_1^{(1)}, \ldots, u_{n-1}^{(1)})$ be a regular system of parameters of $m_1$.

We note

$$B_1 := \{q \in J_1 \setminus \{j_1\} \text{ such that } u'_q \notin R_1^\times\}$$

and

$$C_1 := J_1 \setminus (B_1 \cup \{j_1\}).$$

Since $u$ is a regular system of parameters of $R$, we have the disjoint union

$$u' = u'A_1 \cup u'_B_1 \cup u'_C_1 \cup \{u'_{j_1}\}.$$
Let \( \pi : R \to R_1 \) be the natural map. We suppose that 
\[
J_1 = \{1, \ldots, h\}.
\]

**Definition 3.3.** We say that \( \pi : (R, u) \to (R_1, u^{(1)}) \) is a framed blow-up of \( (R, u) \) along \((u, j_1)\) with respect to \( \nu \) if it exists \( D_1 \subset \{1, \ldots, n_1\} \) such that 
\[
u'|_{A_1 \cup B_1 \cup \{j_1\}} = \nu^{(1)}_D
\]
and if \( m' = \{x \in R' \text{ such that } \nu(x) > 0\} \).

**Remark 3.4.** A blow-up \( \pi \) is framed if in the generators of the maximal ideal \( m_1 \) of \( R_1 \), we have all the elements of \( u' \), except eventually those who are invertibles in \( R_1 \). It means except eventually those who are invertibles in \( R_1 \).

It is framed with respect to \( \nu \) if we localised in the center of \( \nu \).

Let \( \pi \) be such a blow-up.

**Definition 3.5.** We say that \( \pi \) is monomial if \( B_1 = J_1 \setminus \{j_1\} \).

**Remark 3.6.** Let \( \pi \) be a monomial blow-up.

Then \( n_1 = n \) and \( D_1 = \{1, \ldots, n\} \).

**Definition 3.7.** Let \( \pi : (R, u) \to (R_1, u^{(1)}) \) be a framed blow-up and \( T \subset \{1, \ldots, n\} \).

We say that \( \pi \) is independant of \( u_T \) if \( T \cap J_1 = \emptyset \), so if \( T \subset A_1 \).

**Remark 3.8.** Since we look at blow-ups with respect to a valuation \( \nu \), we have blow-ups such that \( \nu(R_1) \geq 0 \). Since \( u'_q \in R_1 \) for every \( q \in J_1 \), we want \( \nu \left( \frac{u_q}{u_q^{(1)}} \right) \geq 0 \), so \( \nu(u_q) \geq \nu(u_{j_1}) \) for every \( q \in J_1 \setminus \{j_1\} \). So we can set \( j_1 \) as an element of \( J_1 \) such that \( \beta_{j_1} = \min_{q \in J_1} \{\beta_q\} \).

We have:
\[
B_1 := \{ q \in J_1 \setminus \{j_1\} \text{ such that } u'_q \notin R_1^\times \}
= \{ q \in J_1 \setminus \{j_1\} \text{ such that } \nu \left( \frac{u_q}{u_q^{(1)}} \right) > 0 \}
= \{ q \in J_1 \setminus \{j_1\} \text{ such that } \beta_q > \beta_{j_1} \}.
\]

And \( C_1 = \{ q \in J_1 \setminus \{j_1\} \text{ such that } \beta_q = \beta_{j_1} \} \).

Let \( k_1 \) be the residue field of \( R_1 \) and \( t_{k_1} \) be the transcendence degree of \( k \to k_1 \).

Let us show that \( t_{k_1} \leq zC \).

We note \( \tilde{R} = \frac{R}{mR'} \) and also \( \tilde{u}_q \) the image of \( u'_q \) in \( \tilde{R} \) for every \( q \in J_1 \setminus \{j_1\} \). So \( \tilde{R} = k \left[ \tilde{u}_{B_1}, \tilde{u}_{C_1}^{1} \right] \). We have \( R \to R' \to R_1 \to k_1 \), so \( k \to \tilde{R} \to \frac{R_1}{m_1R_1} \to k_1 \).

We have \( m = m_1 \cap R = m'R_1 \cap R = m' \cap R \). Let \( m' = \frac{m'}{m' \cap R} \). We have 
\[
\frac{R_1}{m_1R_1} = \frac{R'}{mR'} \frac{m'R}{m' \cap R} = \frac{R'}{mR'} \frac{m'}{m' \cap R}
= \frac{\tilde{R}}{\tilde{m}}
\]

it means
(3.1)
\[
k \to \tilde{R} \to \tilde{R}_m \to k_1.
\]

Since \( u'_A \cup B_1 \cup \{j_1\} \subset m' \), then for every \( q \in A_1 \cup B_1 \cup \{j_1\} \), the image of \( u'_q \) in \( k_1 \) is zero. So \( k_1 \) is generated over \( k \) by the images of the \( u'_q \) with \( q \in C_1 \). So \( t_{k_1} \leq zC_1 \).
But we have \( C_1 := J_1 \setminus (B_1 \cup \{ j_1 \}) \). So \( \sharp C_1 + \sharp B_1 + 1 = \sharp J_1 = h \), and:

\[
(3.2) \quad \sharp B_1 + 1 \leq t_{k_1} + \sharp B_1 + 1 \leq \sharp C_1 + \sharp B_1 + 1 = h \leq n.
\]

We will often set \( J_1 \subset \{ 1, \ldots, r, n \} \) where \( r \) is the dimension of \( \sum_{i=1}^{n} u_i \mathbb{Q} \) in \( \Gamma \otimes \mathbb{Q} \). If \( J_1 \subset \{ 1, \ldots, r \} \), the family \( \beta_j \) is a family of elements \( \mathbb{Q} \)-linearly independent, and so \( B_1 = J_1 \setminus \{ j_1 \} \).

Otherwise \( n \in J_1 \). Then we have \( B_1 = J_1 \setminus \{ j_1 \} \) or \( B_1 = J_1 \setminus \{ j_1, q_1 \} \) where \( q_1 \in J_1 \setminus \{ j_1 \} \). The interesting cases are those where \( h - 2 \leq \sharp B_1 \), so those where \( h - 1 \leq \sharp B_1 + 1 \).

Since (3.2), we have \( h - 1 + t_{k_1} \leq \sharp B_1 + 1 + t_{k_1} \leq h \).

Then we have three cases.

The first one, \( \sharp B_1 + 1 = h \) and \( t_{k_1} = 0 \), it occurs when the blow-up is monomial.

The second one, \( \sharp B_1 + 1 = h - 1 \) and \( t_{k_1} = 1 \).

The last one, \( \sharp B_1 + 1 = h - 1 \) and \( t_{k_1} = 0 \).

\[\text{Fact 3.9. In the cases 1 and 3, we have } n_1 = n \text{ and in the case 2 we have } n_1 = n - 1.\]

\[\text{Remark 3.10. In the rest of the chapter, we will assume that the valuation ring has } k \text{ as residue field. So } k_1 = k \text{ and } t_{k_1} = 0. \text{ So we will have } n_1 = n.\]

Since \( k_1 \simeq k^{[k]}(z) \), we know that \( \lambda(Z) \) is a polynomial of degree 1 over \( k \).

### 3.3. Key elements

We need a more general notion than the one of key polynomials. Indeed, after several blow-ups, a key polynomial might not be a polynomial anymore.

For example, we can have \( \frac{1}{u_{n-1}} u_{n-1} \), which is not a polynomial.

\[\text{Definition 3.11. Let } P_1 < P_2 \text{ be two immediate successors key polynomials of the extension } k \left( u_1^{(l)}, \ldots, u_n^{(l)} \right) \left( u_1^{(l)}, \ldots, u_n^{(l)} \right). \text{ We consider } P_2 = \sum_{j \in \mathcal{P}_1(P_2)} a_j P_1^j \text{ the } P_1 \text{-expansion of } P_2.\]

We call key element every element \( P'_2 \) of the form

\[
P'_2 = \sum_{j \in \mathcal{P}_1(P_2)} a_j b_j P_1^j
\]

where \( b_j \) are units of \( R_l = k \left( u_1^{(l)}, \ldots, u_n^{(l)} \right) \left( u_1^{(l)}, \ldots, u_n^{(l)} \right) \). The polynomial \( P_2 \) is the key polynomial associated to the key element \( P'_2 \).

\[\text{Remark 3.12. A key element is not necessarily a polynomial. Indeed, for example, } \frac{1}{1+u_{n-1}} \text{ is a unit of } R_l.\]

\[\text{Definition 3.13. Let } P'_1 \text{ and } P'_2 \text{ two key elements. We say that } P'_1 \text{ and } P'_2 \text{ are immediate successors key elements, and we note } P'_1 \ll P'_2, \text{ if their key polynomials associated are immediate successors.}\]

Now we define limit immediate successors key elements.

\[\text{Definition 3.14. Let } P'_1 \text{ and } P'_2 \text{ be two key elements. We say that } P'_1 \text{ and } P'_2 \text{ are limit immediate successors key elements}, \text{ and we note } P'_1 \ll_{\lim} P'_2, \text{ if their key polynomials associated } P_1 \text{ and } P_2 \text{ are such that } P_2 \text{ is a limit immediate successor key polynomial of } P_1.\]
4. Monomialization in the non degenerated case.

In this section, we will monomialize all the elements which are non degenerated with respect to a system of parameters.

Let $\alpha$ and $\gamma$ two elements of $\mathbb{Z}^n$, and $\delta = (\min \{\alpha_j, \gamma_j\})_{1 \leq j \leq n}$. We say that $u^\alpha | u^\gamma$ if for every integer $i$, $\alpha_i$ is less or equal to $\gamma_i$, it means that $\alpha$ is less or equal to $\beta$ one by one component.

Let set
\[
\tilde{\alpha} = \alpha - \delta = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_a, 0, \ldots, 0) \in \mathbb{N}^n.
\]
The objective is to build a sequence of blow-ups $(R, u) \to \cdots \to (R', u')$ such that in $R'$, we have $u^\alpha | u^\gamma$.

**Definition 4.1.** We say that $\alpha \preceq \gamma$ if for every index $i$, we have $\alpha_i \leq \gamma_i$.

We assume that $\gamma \not\preceq \alpha$ and that $\alpha \not\preceq \gamma$. So we can assume that $|\tilde{\alpha}| \neq 0$, and $\tilde{\alpha}_i > 0$ for every integer $i \in \{1, \ldots, a\}$.

With the same idea, we set
\[
\tilde{\gamma} = \gamma - \delta = (0, \ldots, 0, \tilde{\gamma}_{a+1}, \ldots, \tilde{\gamma}_n) \in \mathbb{N}^n.
\]
Even if it means to exchange $\alpha$ and $\gamma$, we assume $0 < |\tilde{\alpha}| \leq |\tilde{\gamma}|$.

4.1. Construction of a strictly decreasing numerical character.

**Definition 4.2.** Let $\tau: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{N}^2$ be the map such that
\[
\tau(\alpha, \gamma) = (|\tilde{\alpha}|, |\tilde{\gamma}|).
\]

Let $J$ be a minimal subset of $\{1, \ldots, n\}$ such that $\{1, \ldots, a\} \subset J$ and $\sum_{q \in J} \tilde{\gamma}_q \geq |\tilde{\alpha}|$.

Let $\pi: (R, u) \to (R_1, u^{(1)})$ be a framed blow-up along $(u, j)$. Let $j \in J$ such that $R_1$ is a localization of $R \left[ \frac{u_j}{u_j} \right]$.

If $q \in J \setminus \{j\}$, we recall that $u^{(q)} = \frac{u_q}{u_j}$, and $u^{(q)} = u_q$ otherwise.

We now define $\alpha^{(q)} = \tilde{\alpha}_q$ for $q \not\in J$, and $\alpha^{(q)} = 0$ otherwise. We set also $\gamma^{(q)} = \tilde{\gamma}_q$ if $q \not\in J$, $\gamma^{(q)} = \sum_{q \in J} \gamma_q - |\tilde{\alpha}|$ otherwise.

And finally we define
\[
d' = (\delta_1, \ldots, \delta_{j-1}, \sum_{q \in J} \delta_q + |\tilde{\alpha}|, \delta_{j+1}, \ldots, \delta_n).
\]

So we have:
\[
u^\alpha = \prod_{l = 1}^n u_l^{\alpha_l} = \prod_{l \in J \setminus \{j\}} u_l^{\alpha_l} \times \prod_{l \in J \setminus \{j\}} u_l^{\alpha_l}.
\]

But for every $l \in J \setminus \{j\}$, we have $u_l = u_l' \times u_j$ and for $l \notin J \setminus \{j\}$, we have $u_l = u_l'$. So
\[
u^\alpha = \prod_{l \in J \setminus \{j\}} (u_l' \times u_j)^{\alpha_l} \times \prod_{l \notin J \setminus \{j\}} (u_l')^{\alpha_l}.
\]
Let us isolate the term $u_j$, we obtain:

$$u^\alpha = u_j^{\sum_{i \in J \setminus \{j\}} \alpha_i} \times \prod_{l = 1}^n (u'_l)^{\alpha_l}$$

and since $\tilde{\alpha} = \alpha - \delta$, we have $\alpha = \tilde{\alpha} + \delta$ and then

$$u^\alpha = u_j^{\sum_{i \in J \setminus \{j\}} \alpha_i} \times \prod_{l = 1}^n (u'_l)^{\tilde{\alpha}_i + \delta_i} \times (u'_j)^{\tilde{\alpha}_j + \delta_j}$$

But $\tilde{\alpha}'_q = \tilde{\alpha}_q$ for $q \neq j$ and $\delta' = (\delta_1, \ldots, \delta_j - 1, \sum_{q \in J} \delta_q + |\tilde{\alpha}|, \delta_{j+1}, \ldots, \delta_n)$, so

$$u^\alpha = u_j^{\sum_{i \in J \setminus \{j\}} \alpha_i} \times \prod_{l = 1}^n (u'_l)^{\tilde{\alpha}'_i + \delta'_i} \times (u'_j)^{\tilde{\alpha}_j + \delta_j}$$

We include another time the term $l = j$ in the product, and then:

$$u^\alpha = u_j^{\sum_{i \in J \setminus \{j\}} \alpha_i + \tilde{\alpha}_j + \delta_j - \tilde{\alpha}'_j + \delta'_j} \times \prod_{l = 1}^n (u'_l)^{\tilde{\alpha}'_l + \delta'_l}$$

But we have

$$\sum_{l \in J \setminus \{j\}} \alpha_l + \tilde{\alpha}_j + \delta_j - \tilde{\alpha}'_j + \delta'_j = \sum_{l \in J \setminus \{j\}} \alpha_l + \tilde{\alpha}_j + \delta_j - \sum_{q \in J} \delta_q - |\tilde{\alpha}|$$

$$= \sum_{l \in J \setminus \{j\}} (\bar{\alpha}_l + \delta_l) + \tilde{\alpha}_j - \sum_{q \in J \setminus \{j\}} \delta_q - |\tilde{\alpha}|$$

$$= \sum_{l \in J \setminus \{j\}} \bar{\alpha}_l - |\tilde{\alpha}|$$

$$= 0.$$

So $u^\alpha = (u')^{\tilde{\alpha}' + \delta'}$, and with the same idea $u^\gamma = (u')^{\tilde{\alpha}' + \gamma'}$.

We set $\alpha' = \delta' + \tilde{\alpha}'$ and $\gamma' = \delta' + \tilde{\gamma}'$.

**Proposition 4.3.** We have $\tau(\alpha', \gamma') < \tau(\alpha, \gamma)$.

**Proof.** First case: $j \in \{1, \ldots, a\}$. Then

$$|\tilde{\alpha}'| = |\tilde{\alpha}| - \tilde{\alpha}_j < |\tilde{\alpha}|.$$
Second case: $j \in \{a + 1, \ldots, n\}$. Then $|\tilde{\alpha}'| = |\tilde{\alpha}|$. Let us show that $|\tilde{\gamma}'| < |\tilde{\gamma}|$.

But

$$|\tilde{\gamma}'| = \sum_{q = a + 1}^{n} \tilde{\gamma}_q + \sum_{q \in J} \tilde{\gamma}_q - |\tilde{\alpha}|$$

By minimality of $J$, we have

$$\sum_{q \in J \setminus \{j\}} \tilde{\gamma}_q - |\tilde{\alpha}| < 0,$$

and so

$$|\tilde{\gamma}'| < \sum_{q = a + 1}^{n} \tilde{\gamma}_q = |\tilde{\gamma}|.$$

In every case, we have $(|\tilde{\alpha}'|, |\tilde{\gamma}'|) < (|\tilde{\alpha}|, |\tilde{\gamma}|) = \tau(\alpha, \gamma)$.

If $|\tilde{\alpha}'| \leq |\tilde{\gamma}'|$, then $\tau(\alpha', \gamma') = (|\tilde{\alpha}'|, |\tilde{\gamma}'|)$ and this concludes the proof.

Otherwise, $|\tilde{\alpha}'| > |\tilde{\gamma}'|$, so

$$\tau(\alpha', \gamma') = (|\tilde{\gamma}'|, |\tilde{\alpha}'|) < (|\tilde{\alpha}'|, |\tilde{\gamma}'|),$$

and we have done. $\square$

Even if it means to renumber the $u'_q$, we can assume that $u'_q \notin R^*_l$ for every $q \in \{1, \ldots, s\}$ and $u'_q \in R^*_l$ otherwise. Since $\pi$ is a framed blow-up, we have $\{u'_1, \ldots, u'_s\} \subset u^{(1)}$, so even if it means to renumber again, we can assume $u'_q = u_q^{(1)}$ for every $q \in \{1, \ldots, s\}$. We set

$$\alpha^{(1)} = (\alpha'_1, \ldots, \alpha'_s, 0, \ldots, 0) \in \mathbb{Z}^{n_1}$$

and

$$\gamma^{(1)} = (\gamma'_1, \ldots, \gamma'_s, 0, \ldots, 0) \in \mathbb{Z}^{n_1}.$$

We have $\tau(\alpha^{(1)}, \gamma^{(1)}) \leq \tau(\alpha', \gamma')$. By Proposition 4.3, we have

$$\tau(\alpha^{(1)}, \gamma^{(1)}) < \tau(\alpha, \gamma).$$

4.2. Divisibility and change of variables. Let $s \in \{1, \ldots, n\}$ be an integer.

We write $u = (w, v)$ where

$$w = (w_1, \ldots, w_s) = (u_1, \ldots, u_s)$$

and

$$v = (v_1, \ldots, v_{n-s}).$$

We consider $\alpha$ and $\gamma$ two elements of $\mathbb{Z}^s$.

**Proposition 4.4.** There exists a framed local sequence

$$(R, u) \to \left( R_l, u^{(l)} \right),$$

with respect to $\nu$, independent of $v$, such that in $R_l$, we have $w^{\alpha} | w^{\gamma}$ or $w^{\gamma} | w^{\alpha}$. 
Proof. Unless that $\gamma \preceq \alpha$, or that $\alpha \preceq \gamma$, we can iterate the previous construction, choosing blow-up with respect to $\nu$ and independent of $\nu$. Since $\tau$ is a vector in $\mathbb{N}^2$ and is strictly decreasing, after a finite number of steps, the process stops. After these steps, we have $w^{\alpha} = U \times (u^{(0)})^{\alpha(i)}$, $w^{\gamma} = U \times (u^{(0)})^{\gamma(i)}$, with $U \in R_i^x$ and with $\gamma(i) \preceq \alpha(i)$, or $\alpha(i) \preceq \gamma(i)$. So we do have $w^{\alpha} | w^{\gamma}$ or $w^{\gamma} | w^{\alpha}$ in $R_i$. \hfill $\square$

Let us now study the change of variables we do at each blow-up. We consider $i$ and $i'$ some indexes of the framed local sequence, it means that we consider

$$ (4.1) \quad (R, u) \rightarrow \cdots \rightarrow (R_i, u^{(i)}) \rightarrow \cdots \rightarrow (R_{i'}, u^{(i')} \rightarrow \cdots \rightarrow (R_i, u^{(0)}). $$

**Proposition 4.5.** Let us consider $0 \leq i < i' \leq l$. We consider $m$ an element of $\{1, \ldots, n_i\}$ and $m'$ one of $\{1, \ldots, n_i\}$. So:

1. There exists a vector $\delta_{m'}^{(i',j)}$ of $\mathbb{N}^{D_i}$ such that

   $$ u_m^{(i)} \in \left( u_{{D_i}',j}^{(i',j)} \right)_{m} R_{i'}^\n. $$

2. If, in addition, the local sequence is independent of $u_T$, with $T \subset \{1, \ldots, n_i\}$; and if we assume that $u_m^{(i)} \notin u_T$, then $\left( u_{{D_i}',j}^{(i',j)} \right)_{m} \delta_{m'}^{(i',j)}$ is monomial in $u_{{D_i}',j}^{(i',j)} \setminus u_T$.

3. We assume that $i'' > 0$ such that $i \leq i'' < i'$. We have $D_{i'} = \{1, \ldots, n_{i''}\}$, and we assume that $m' \in D_{i'}$. Then there exists a vector $\gamma_{m',i'}^{(i,i')} \in \mathbb{Z}^{n_i}$ such that

   $$ u_m^{(i')} = \left( u^{(i)} \right)_{m'}^{\gamma_{m',i'}^{(i,i')}}. $$

4. If, in addition, the local sequence is independent of $u_T$ and if we assume that $u_m^{(i)} \notin u_T$, then $u_m^{(i)}$ is monomial in $u^{(i)} \setminus u_T$. 

**Proof.** We only consider the case $i' = i + 1$, the general case can be proved by induction. We can also assume that $i = 0$.

Let us show (1). By Definition 3.3 we have $u'_{A_i \cup B_i \cup \{j_i\}} = u_{D_i}^{(1)}$.

We denote by $D_1 = D_1^{A_1} \cup D_1^{B_1}$ where

$$ u_{A_1}^{(1)} = u_{D_1}^{A_1}, $$

and

$$ u_{B_1 \cup \{j_i\}}^{(1)} = u_{D_1}^{B_1}. $$

If $m \in A_1 \cup \{j_i\}$, so $u_m = u_m'$ and we are done. If $m \in B_1$ then $u_m = u_j, u_m' = u_m'$ and we are done.

If $m \in C_1$, so $u_m = u_m'$ and by definition, $u_m' \in R_1^x$, which give us the result.

Let us show (3). We have $m' \in D_1 = D_1^{A_1} \cup D_1^{B_1}$ and $u_{A_1 \cup B_1 \cup \{j_i\}}^{(1)} = u_{D_1}^{(1)}$. If $m' \in D_1^{A_1}$ then by definition $u_m^{(1)} = u_{A_1}^{(1)} = u_{A_1}$ and we have the result. Otherwise $m' \in D_1^{B_1}$. So

$$ u_m^{(1)} \in u_{B_1 \cup \{j_i\}}^{(1)} = \left\{ u_j, \frac{u_q}{u_j}, q \in B_1 \right\}. $$

This concludes the proof of (3).
Now let us assume that the sequence is independent of $u_T$. By definition we have $u_{J_i} \cap u_T = \emptyset$ and also

$$u_{D_1}^{(1)} \cap u_T = \emptyset.$$  

Let us show (2) and so assume that $u_m \notin u_T$.

If $m \in A_1$, then $u_m = u_m' \in u_{D_1}^{(1)}$ and $u_m \notin u_T$ and we are done. Otherwise $m \in J_1$. We saw in the proof of (1) that $m$ was monomial in $u_{D_1}^{(1)}$ uniquely, and since $u_{D_1}^{(1)} \cap u_T = \emptyset$, this concludes the proof of (2).

We only have now to prove (4). Then we assume that $u_m' \notin u_T$, with $m' \in D_1 = D_1^A \cup D_1^{B_1}$.

If $m' \in D_1^A$, then $u_{m'}^{(1)} \in u_{A_i} = u_{A_i}$. Since $u_{m'}^{(1)} \notin u_T$, we have $u_m^{(1)} \in u \setminus u_T$.

Otherwise $m' \in D_1^{B_1}$ and we saw that $u_m^{(1)}$ is monomial in $u_{B_1 \cup J_1} \subset u_J$. Since $u_J \cap u_T = \emptyset$, we are done.

\[\square\]

**Remark 4.6.** Let $T \subset A$, be a set of cardinal $t$, and $s := n - t$. We set

$$v = (v_1, \ldots, v_t) = u_T$$

and

$$w = (w_1, \ldots, w_s) = u_{\{1, \ldots, n\} \setminus T}.$$  

We assume here that $w$ do monomial blow-ups.

We have $u' = (v, w')$ where $w' = (w_1', \ldots, w_s') = (w^\gamma(1), \ldots, w^\gamma(s))$ with $\gamma(i) \in \mathbb{Z}^*$, by Proposition 4.5. By the proof of this Proposition, the matrix $F_s = [\gamma(1) \ldots \gamma(s)]$ is an unimodular matrix. For every $\delta \in \mathbb{Z}^*$, we have $w^\delta = w^F \delta$. In the same vein $w_i = w^{\delta(i)}$ and the $s$-vectors $\delta(1), \ldots, \delta(s)$ form an unimodular matrix equal to the inverse of $F_s$. Then we have $w^\gamma = w^\gamma k_i^{-1}$, for every $\gamma \in \mathbb{Z}^*$.

**Proposition 4.7.** We have:

$$w^\alpha | w^\gamma \text{ in } R_l \iff \nu(w^\alpha) \leq \nu(w^\gamma).$$  

**Proof.** We have $u_l^{(1)} = \left( w_l^{(1)}, \ldots, w_r^{(1)}, v \right)$.

By Proposition 4.5 there exists $\alpha^{(l)}, \gamma^{(l)} \in \mathbb{N}^r$ and $y, z \in R_l^{x}$ such that $w^\alpha = y (w^{(l)})^{\alpha^{(l)}}$ and $w^\gamma = z (w^{(l)})^{\gamma^{(l)}}$.

For every $i \in \{1, \ldots, r\}$, we have $\nu(w_i^{(l)}) \geq 0$ since the blow-up is with respect to $\nu$, so centered in $R_l$. By construction of $R_l$, we have that $\gamma^{(l)} \leq \alpha^{(l)}$ or $\alpha^{(l)} \leq \gamma^{(l)}$.

So

$$\left( w^{(l)} \right)^{\alpha^{(l)}} | \left( w^{(l)} \right)^{\gamma^{(l)}} \iff \nu \left( \left( w^{(l)} \right)^{\alpha^{(l)}} \right) \leq \nu \left( \left( w^{(l)} \right)^{\gamma^{(l)}} \right),$$  

it means that

$$w^\alpha | w^\gamma \iff \nu(w^\alpha) \leq \nu(w^\gamma).$$  

\[\square\]
4.3. Monomialization of the non degenerate elements. Let $N$ be an ideal of $R$ generated by monomials in $w$. We choose $w^{\alpha_0}, \ldots, w^{\alpha_k}$ some minimal generators of $N$, with $\nu(w^{\alpha_0}) \leq \nu(w^{\alpha_i})$ for every $i$.

**Proposition 4.8.** There exists a local framed sequence 

$$\phi: (R, u) \to (R_l, u^{(l)})$$

with respect to $\nu$, independent of $v$ and such that $NR_l = (w^{\alpha_0})R_l$.

**Proof.** Let $N$ be an ideal of $R$ generated by monomials in $w$. We choose some minimal generators $w^{\alpha_0}, \ldots, w^{\alpha_k}$ of $N$, with $\nu(w^{\alpha_0}) \leq \nu(w^{\alpha_i})$ for every $i$.

Assume $b \neq 0$. We consider $(w^{\alpha_{i_0}}, w^{\alpha_{j_0}})$ a couple which attains the minimum 

$$\min_{0 \leq i < j \leq b} \nu(w^{\alpha_i}, w^{\alpha_j}).$$

By Proposition 4.3, $\tau(N, w)$ is strictly decreasing at each blow-up.

Since the processus stops, $NR_l$ is generated by an unique element as an ideal of $R_l$. By Proposition 4.7, this element is $w^{\alpha_0}$ (which has the minimal value), which divides the others. Then $NR_l = (w^{\alpha_0})R_l$. 

**Definition 4.9.** An element $f$ of $R$ is monomializable if there exists a sequence of blow-ups 

$$(R, u) \to (R', u')$$

such that the total transform of $f$ is a monomial. It means that in $R'$, the total transform of $f$ is $v \prod_{i=1}^n (u'_i)^{\alpha_i}$, with $v$ a unit of $R'$.

**Theorem 4.10.** Let $f$ be a non degenerated element with respect to $u = (w, v)$, and let $N$ be the ideal which satisfies the conclusion of the Proposition 3.2, generated by monomials in $w$.

Then there exists a local framed sequence, independent of $v$, 

$$(R, u) \to (R', u')$$

such that $f$ is a monomial in $u'$ multiplicated by a unit of $R'$. Equivalently, $f$ is monomializable.

**Proof.** Let $(R, u) \to (R', u')$ be the local framed sequence of the Proposition 4.8. We have $NR' = w^{\alpha_0}R'$. Since $f \in N$ by the proof of the Proposition 3.2, we have the existence of an element $z \in R'$ such that $f = w^{\alpha_0}z$. Since $\nu$ is centered in $R'$, to show that $z$ is a unit of $R'$, we will show that $\nu(z) = 0$.

But $\nu(z) = \nu(f) - \nu(w^{\alpha_0}) = \nu(N) - \nu(w^{\alpha_0})$ by Proposition 3.2.

Since $NR' = w^{\alpha_0}R'$, we have $\nu(N) = \nu(w^{\alpha_0})$, and so $\nu(z) = 0$, and this concludes the proof. 

$\Box$
5. Non degenerescence and key polynomials.

Now that we monomialized every non degenerated element with respect to the
generators of the local ring, we are going to show that every element is non
degenerated with respect to a particular sequence of immediate successors. We denote
by $\Lambda$ the set of key polynomials and

$$
M_\alpha := \{ Q \in \Lambda \text{ such that } \deg(Q) = \alpha \}.
$$

**Proposition 5.1.** We consider $\nu$ an archimedian valuation centered in a noetherian
local domain $(R, m, k)$. We denote by $\Gamma$ the value group of $\nu$ and we set $\Phi := \nu(R \setminus \{0\})$.

The set $\Phi$ does not have any infinite bounded sequence.

**Proof.** Assume by contradiction that we have an infinite sequence $\alpha_1 < \alpha_2 < \ldots$
of elements of $\Phi$ bounded by an element $\beta \in \Phi$.

Then we have an infinite decreasing sequence $\cdots \subseteq P_{\alpha_2} \subseteq P_{\alpha_1}$ such that for
every index $i$, we have $P_{\beta} \subseteq P_{\alpha_i}$. And so we have an infinite decreasing sequence
of ideals of $R_{P_{\beta}}$.

We set $\delta = \nu(m) = \min_{x \in \Phi \setminus \{0\}} \{ \nu(x) \}$.

Since $\nu$ is archimedian, we know that there exists a non zero integer $n$ such that
$\beta \leq n\delta$, and so such that $m^n \subseteq P_{\beta}$. This way, we construct a epimorphism of
rings $R_{m^n} \rightarrow R_{P_{\beta}}$. Since the ring $R$ is noetherian, $R_{m^n}$ is artinian, and so is $R_{P_{\beta}}$. This
contradicts the existence of the infinite decreasing sequence of ideals of $R_{P_{\beta}}$.  \[\square\]

**Definition 5.2.** Assume that the set $M_\alpha$ is non empty and does not have any
maximum. Assume also that there exists a key polynomial $Q \in \Lambda$ such that $\epsilon(Q) > \epsilon(M_\alpha)$. We call a limit key polynomial every polynomial of minimal degree which satisfies this property.

**Definition 5.3.** Let $(Q_i)_{i \in \mathbb{N}}$ be a sequence of key polynomials. We say that it is
a sequence of immediate successors if for every integer $i$, we have $Q_i < Q_{i+1}$.

**Proposition 5.4.** If there is not any limit key polynomial, then there exists a
finite or infinite sequence of immediate successors $Q_1 < \ldots < Q_i < \ldots$ such that
the sequence $\{\epsilon(Q_i)\}$ is cofinal in $\epsilon(\Lambda)$. Equivalently, such that

$$
\forall Q \in \Lambda \exists i \text{ such that } \epsilon(Q_i) \geq \epsilon(Q).
$$

**Proof.** We do the proof by contrapositive.

Assume that for every finite or infinite sequence of immediate successors key
polynomials $(Q_i)$, the sequence $\{\epsilon(Q_i)\}$ is not cofinal in $\epsilon(\Lambda)$. Let us show that
there exists a limit key polynomial.

First let assume that for every $\alpha \in \Omega = \{ \beta \text{ such that } M_\beta \neq \emptyset \}$, $M_\alpha$ has a maxi-
mal element. It means that

$$
\forall \alpha \in \Omega \exists R_\alpha \in M_\alpha \text{ such that } \forall Q \in M_\alpha, \epsilon(R_\alpha) \geq \epsilon(Q).
$$

We set $M := \{ R_\alpha \}_{\alpha \in \Omega}$. All elements in $M$ are of distinct degree, so they are
strictly ordered by their degrees. So if $\alpha < \alpha'$, then $\deg(R_\alpha) < \deg(R_{\alpha'})$. Since
$R_{\alpha'}$ is a key polynomial, by definition, we have $\epsilon(R_{\alpha}) < \epsilon(R_{\alpha'})$ as soon as $\alpha < \alpha'$. Then in $M$ the elements are strictly ordered by their values of $\epsilon$.

Let us show that they are immediate successors. Let $R_{\alpha}$ and $R_{\alpha'}$ be two consecutive elements of $M$. We know that

$$\alpha = \deg(R_{\alpha}) < \deg(R_{\alpha'}) = \alpha'$$

and $\epsilon(R_{\alpha}) < \epsilon(R_{\alpha'})$. We want to show that $R_{\alpha'}$ is of minimal degree for the property. So let us set $R \in \Lambda$ such that $\epsilon(R_{\alpha}) < \epsilon(R)$ and $\deg(R) \leq \deg(R_{\alpha'})$. Let us show that $\deg(R) = \deg(R_{\alpha'}) = \alpha'$. Since $\epsilon(R_{\alpha}) < \epsilon(R)$ and since $R_{\alpha}$ is a key polynomial, by definition,

$$\deg(R_{\alpha}) = \alpha \leq \deg(R) \leq \alpha'.$$

Since $R$ is a key polynomial, if we had $\deg(R) = \deg(R_{\alpha})$, then we should have $\epsilon(R_{\alpha}) \geq \epsilon(R)$, which is a contradiction. Let us set $\lambda := \deg(R)$, so we have $\alpha < \lambda \leq \alpha'$, $R \in M_\lambda$ and $R_\lambda \in M$. Since the polynomials in $M$ are strictly ordered by their degrees and that $R_{\alpha}$ and $R_{\alpha'}$ are consecutive, then we have $\lambda = \alpha'$, and so $R_{\alpha} < R_{\alpha'}$.

So the set $M$ is a sequence of immediate successors. By hypothesis, the sequence $\epsilon(M)$ is not cofinal, so there exists $R \in \Lambda$ such that $\epsilon(R) > \epsilon(M)$. But then there exists $\alpha$ such that $R \in M_{\alpha}$ and then $\epsilon(R_{\alpha}) > \epsilon(R) > \epsilon(R_{\alpha})$. It is a contradiction.

So there exists $\alpha \in \Omega$ such that $M_{\alpha}$ does not have any maximal ideal. Then we have a sequence:

$$\epsilon(Q_1) < \epsilon(Q_2) < \ldots < \epsilon(Q_i) < \ldots$$

where $Q_i$ is an element of $M_{\alpha}$ for every integer $i$.

Let us show that the $Q_i$ are immediate successors. Let $R \in \Lambda$ such that $\epsilon(Q_i) < \epsilon(R)$ and $\deg(R) \leq \deg(Q_{i+1}) = \alpha$. Since $Q_i$ is a key polynomial, by definition, $\deg(R) \geq \deg(Q_i) = \alpha$. So $\deg(R) = \deg(Q_{i+1}) = \alpha$, and $Q_{i+1}$ is of minimal degree for the property. Then for every integer $i$, we have $Q_i < Q_{i+1}$.

By hypothesis, the sequence of the $Q_i$ is a sequence of immediate successors, so the sequence $\{\epsilon(Q_i)\}$, is not cofinal. So there exists a key polynomial $Q \in \Lambda$ such that $\epsilon(Q) > \epsilon(Q_i)$ for every integer $i$. Let $R \in M_{\alpha}$, since $M_{\alpha}$ does not have a maximal element, there exists $i$ such that $\epsilon(R) < \epsilon(Q_i) < \epsilon(Q)$. So there exists a key polynomial $Q \in \Lambda$ such that $\epsilon(Q) > \epsilon(M_{\alpha})$. Then the polynomial $Q$ is a limit key polynomial.

\begin{theorem}
There exists a finite or infinite sequence of optimal (eventually limit) immediate successors $(Q_i)_{i \geq 1}$ such that the sequence $\{\epsilon(Q_i)\}$ is cofinal in $\epsilon(\Lambda)$ where $\Lambda$ is the set of key polynomials.
\end{theorem}

\begin{proof}
We know that $x$ is a key polynomial. If for every key polynomial $Q \in \Lambda$, we have $\epsilon(x) \geq \epsilon(Q)$, then the sequence $\{\epsilon(x)\}$ is cofinal in $\epsilon(\Lambda)$ and it is done. Otherwise, it exists a key polynomial $Q \in \Lambda$ such that $\epsilon(x) < \epsilon(Q)$. If it exists a maximal element among the key polynomials of same degree than $Q$, then we exchange $Q$ by this element. By Proposition 2.12 it exists a finite sequence $Q_1 = x < \cdots < Q_p = Q$ of optimal (eventually limit) immediate successors which begins at $x$ and ends at $Q$.

If for every key polynomial $Q' \in \Lambda$, there exists a key polynomial of this sequence $Q_i$ such that $\epsilon(Q_i) \geq \epsilon(Q')$, then the sequence $\{\epsilon(Q_i)\}$ is cofinal in $\epsilon(\Lambda)$ and it is over.

\end{proof}
Otherwise there exists a polynomial $Q' \in \Lambda$ such that for every integer $i \in \{1, \ldots, p\}$, we have $\epsilon(Q_i) < \epsilon(Q')$. So $\epsilon(Q_p) < \epsilon(Q')$ and we use Proposition 2.12 again to construct a sequence of optimal (eventually limit) immediate successors which begins at $Q_p$ and ends at $Q'$. So we have a sequence $Q_1 = x, \ldots, Q_r = Q'$ of optimal (eventually limit) immediate successors which begins at $x$ and ends at $Q'$. We iterate the process until the sequence $\{\epsilon(Q_i)\}$ is cofinal in $\epsilon(\Lambda)$. If $Q_i$ is maximal among the set of key polynomials of degree $\deg_X(Q_i)$, then $\deg_X(Q_i) < \deg_X(Q_{i+1})$. If $Q_i < \lim Q_{i+1}$, we have again $\deg_X(Q_i) < \deg_X(Q_{i+1})$. In fact, the degree of the polynomials of the sequence strictly increase at least each two steps, so the process stops. □

Proposition 5.6. Assume that $k = k_\nu$. There exists a finite or infinite sequence of optimal (eventually limit) immediate successors $(Q_i)_i$ such that the sequence $\{\epsilon(Q_i)\}$ is cofinal in $\epsilon(\Lambda)$ where $\Lambda$ is the set of key polynomials.

And this sequence is such that: if $Q_i < Q_{i+1}$, then the $Q_i$-expansion of $Q_{i+1}$ has exactly two terms.

Proof. We have $Q_1 = x$, and we assume that $Q_1, Q_2, \ldots, Q_i$ have been constructed. We note $a := \deg_x(Q_i)$ and recall that

$$G_{\leq a} = \sum_{\deg_x(P) < a} \text{in}_{\nu Q_i}(P)G_\nu.$$ 

If $Q_i$ is maximal in $\Lambda$, we stop. Otherwise, $Q_i$ is not maximal and so it has an immediate successor.

We set $\alpha := \min \{h \in \mathbb{N}^* \text{ such that } h\nu(Q_i) \in \Delta_{\leq a}\}$ where $\Delta_{\leq a}$ is the subgroup of $\Gamma$ generated by the values of the elements of $G_{\leq a}$.

In fact, there exists a polynomial $f$ of degree strictly less than $a$ such that $\alpha \nu(Q_i) = \nu(Q_i^*) = \nu(f) \neq 0$.

Then, since $k_\nu = k$, there exists $c \in k^*$ such that $\text{in}_{\nu}(Q_i^*) = \text{in}_{\nu}(cf)$.

We set $Q = Q_i^* - cf$. By the proof of Proposition 2.5, we have $\epsilon(Q_i) < \epsilon(Q)$. Let us show that $Q_i < Q$. We only have to show that $Q$ is of minimal degree. So let us set $P$ a key polynomial such that $\epsilon(Q_i) < \epsilon(P)$.

Assume by contradiction that $\deg(P) < a\alpha$. We set $P = \sum_{j=0}^{a-1} p_j Q_i^j$ the $Q_i$-expansion of $P$. Then by the proof of Proposition 2.5, we have $\sum_{j=0}^{a-1} \text{in}_{\nu}(p_j) \text{in}_{\nu}(Q_i)^j = 0$, which contradicts the minimality of $\alpha$.

Then $Q$ is of minimal degree and $Q_i < Q$. Since it has just two terms in his $Q_i$-expansion, it is an optimal immediate successor of $Q_i$.

First case: $\alpha > 1$. Then we set $Q_{i+1} := Q$ and we iterate.

Second case: $\alpha = 1$. Then all the elements of $M_{Q_i}$ have same degree than $Q_i$. If $M_{Q_i}$ does not have any maximal element, then we do the same thing than in the proof of Proposition 2.12 and we set $Q_{i+1}$ a limit immediate successor of $Q_i$.

Otherwise, $M_{Q_i}$ has a maximal element $Q_{i+1}$. This element has same degree than $Q_i$, so we have $Q_{i+1} = Q_i - h$ with $h$ of degree strictly less than the degree of $Q_i$. Then it is an immediate successor of $Q_i$ which $Q_i$-expansion admits uniquely two terms. So it is optimal, and this concludes the proof. □
We now assume $k = k_\nu$ and consider $\mathcal{Q} := (Q_i)_i$ a sequence of optimal (eventually limit) immediate successors such that $(\epsilon(Q_i))_i$ is cofinal in $\epsilon(\Lambda)$ and such that if $Q_i < Q_{i+1}$, then the $Q_i$-expansion of $Q_{i+1}$ admits exactly two terms.

**Remark 5.7.** We consider the same hypothesis than in example 1.8. Then $\mathcal{Q} = \{z, Q\}$.

**Corollary 5.8.** For every polynomial $f$, there exists an index $i$ such that $\nu_{Q_i}(f) = \nu(f)$.

**Proof.** By Proposition 1.21 there exists a key polynomial $Q$ such that $\nu_Q(f) = \nu(f)$.

The sequence $\{\epsilon(Q_i)\}$ being cofinal, there exists an index $i$ such that

$$\epsilon(Q_i) \geq \epsilon(Q).$$

By Proposition 1.20 $\nu_Q(f) \leq \nu_{Q_i}(f)$ and since $\nu_Q(f) = \nu(f)$, we have $\nu_{Q_i}(f) = \nu(f)$. $\square$

**Remark 5.9.** So, for every polynomial $f$, there exists a key polynomial $Q_i$ of the sequence $\mathcal{Q}$ such that $f$ is non degenerated with respect to $Q_i$.

**Remark 5.10.** Let $Q_i \in \mathcal{Q}$. We don’t assume here $k = k_\nu$.

We set $a_i := \deg_x(Q_i)$ and $\Gamma_{<a_i}$ the group $\nu(G_{<a_i} \setminus \{0\})$.

If $\nu(Q_i) \notin \Gamma_{<a_i} \otimes_{\mathbb{Z}} \mathbb{Q}$, then $\epsilon(Q_i)$ is maximal in $\epsilon(\Lambda)$ and the sequence $\mathcal{Q}$ stops at $Q_i$. 
6. Monomialization of the key polynomials.

We set $K := k(u_1, \ldots, u_{n-1})$ and we consider the extension $K(u_n)$. We consider also a sequence of key polynomials $Q$ as in the section 5.

It means, $Q = (Q_i)_i$ is a sequence of optimal (eventually limit) immediate successors such that $(\epsilon(Q_i))_i$ is cofinal in $\epsilon(\Lambda)$.

Let $f$ be an element of $R$. We know that this element is non degenerated with respect to a key polynomial of the sequence $Q$. Also we know that every element non degenerated with respect to a regular system of parameters is monomializable.

Then, to monomialize $f$, it is enough to monomialize the set of key polynomials of this sequence. We assume in this part that the residual field is $k$.

6.1. Generalities. Let $r := r(R, u, \nu)$ be the dimension of

$$\sum_{i=1}^{n} \nu(u_i)Q$$

in $\Gamma \otimes \mathbb{Q}$. Even if it means to renumber, we can assume that $\nu(u_1), \ldots, \nu(u_r)$ are rationally independent and we consider $\Delta$ the subgroup of $\Gamma$ generated by $\nu(u_1), \ldots, \nu(u_r)$.

**Remark 6.1.** Let $(R, u) \to (R_1, u^{(1)})$ be a framed blow-up. Then $r \leq r_1 := r(R_1, u^{(1)}, \nu)$.

**Remark 6.2.** We will consider the framed local blow-ups

$$(R, u) \to \ldots \to (R_t, u^{(t)}) \to \ldots$$

Then we note $r_i := r(R_i, u^{(i)}, \nu)$.

We set $E := \{1, \ldots, r, n\}$ and $\alpha^{(0)} := \min \{h \in \mathbb{N}^* \text{ s.t. } h\nu(u_n) \in \Delta\}$.

So $\alpha^{(0)} \nu(u_n) = \sum_{j=1}^{r} \alpha^{(0)}_j \nu(u_j)$ with, even if it means to renumber the $\alpha^{(0)}_i$,

$$\alpha^{(0)}_1, \ldots, \alpha^{(0)}_r \geq 0$$

and

$$\alpha^{(0)}_{r+1}, \ldots, \alpha^{(0)}_n < 0.$$

We set

$$w = (w_1, \ldots, w_r, w_n) = (u_1, \ldots, u_r, u_n)$$

and

$$v = (v_1, \ldots, v_t) = (u_{r+1}, \ldots, u_{n-1}),$$

with $t = n - r - 1$.

We set $x_i = \nu_i u_i$, and then we have that $x_1, \ldots, x_r$ are algebraically independent over $k$ in $G\nu$. Let $\lambda_0$ be the minimal polynomial of $x_n$ over $k(x_1, \ldots, x_r)$, of degree $\alpha$.

We set:

$$y = \prod_{j=1}^{r} x_j^{\alpha^{(0)}_j},$$
\[ y = \prod_{j=1}^{r} w_{j}^{\alpha_{j}(0)}, \]
\[ z = \frac{x_{n}^{\alpha_{n}(0)}}{y}, \]

and
\[ \bar{z} = \frac{u_{n}^{\alpha_{n}(0)}}{y}. \]

We have
\[ \lambda_{0} = X^{\alpha} + c_{0}y, \]
where \( c_{0} \in k \), and \( Z + c_{0} \) is the minimal polynomial \( \lambda_{z} \) of \( z \) over \( \text{gr}_{\nu} k(x_{1}, \ldots, x_{r}). \)

Indeed, \( k_{\nu} \simeq k \simeq k[\lambda_{z}] \) so \( \lambda_{z} \) is of degree 1 in \( Z \). Then \( \lambda_{0} \) is of degree \( \alpha(0) \), and so \( \alpha = \alpha(0) \).

**Definition 6.3.** We say that \( Q_{i} \) is monomializable if there exists a sequence of blow-ups \((R, u) \to (R_{l}, u_{l})\) such that in \( R_{l}, Q_{i} \) can be written as \( u_{l}^{(l)} \) multiplicated by a monomial in \( \left(u_{1}^{(l)}, \ldots, u_{r_{l}}^{(l)}\right) \) up to a unit of \( R_{l} \), where \( r_{l} := r \left(R_{l}, u_{l}^{(l)}, \nu\right) \).

We are going to show that there exists a local framed sequence which monomialize all the \( Q_{i} \).

We have \( Q_{1} = u_{n} \), it is a monomial. By the blow-ups, \( Q_{1} \) stays a monomial. So we have to begin monomializing \( Q_{2} \).

Since we want to monomialize the key polynomials \( Q_{i} \) of the sequence \( Q \) constructed before by induction on \( i \), we are going to do something more general here: we consider \( Q_{2} \) an immediate successors (eventually limit) key element of \( Q_{1} \) instead of immediate successor (eventually limit) key polynomial of \( Q_{1} \).

Let us first consider
\[ Q = u_{n}^{\alpha} + a_{0}b_{0}y, \]
where \( b_{0} \in R \) such that \( b_{0} \equiv c_{0} \) modulo \( \mathfrak{m} \) and \( a_{0} \in R^{\times} \).

A priori, \( Q \) is not a key polynomial but we are going to see that we can bring this case to the case \( Q_{2} = Q \) by a local framed sequence independent of \( u_{n} \).

6.2. Puiseux packages.

Let
\[ \gamma = (\gamma_{1}, \ldots, \gamma_{r}, \gamma_{n}) = (\alpha_{1}(0), \ldots, \alpha_{s}(0), 0, \ldots, 0) \]
and
\[ \delta = (\delta_{1}, \ldots, \delta_{r}, \delta_{n}) = (0, \ldots, 0, -\alpha_{s+1}(0), \ldots, -\alpha_{r}(0), \alpha). \]

We have
\[ w^{\delta} = u_{n}^{\delta_{n}} \prod_{j=1}^{r} w_{j}^{\delta_{j}} = \frac{w_{n}^{\alpha_{n}(0)}}{\prod_{j=s+1}^{r} w_{j}^{\alpha_{j}(0)}} \]
and
\[ w^{\gamma} = \prod_{j=1}^{s} w_{j}^{\alpha_{j}(0)}. \]

So
\[ \frac{w^{\delta}}{w^{\gamma}} = \frac{w_{n}^{\alpha_{n}(0)}}{\prod_{j=1}^{r} w_{j}^{\alpha_{j}(0)}} = \bar{z}. \]
Let us compute the value of $w^\delta$.

$$
\nu(w^\delta) = \alpha \nu(w_n) - \sum_{j=s+1}^r \alpha_j^{(0)} \nu(u_j)
$$

$$
= \alpha \nu(u_n) - \sum_{j=s+1}^r \alpha_j^{(0)} \nu(u_j)
$$

$$
= \sum_{j=1}^s \alpha_j^{(0)} \nu(u_j) - \sum_{j=s+1}^r \alpha_j^{(0)} \nu(u_j)
$$

$$
= \sum_{j=1}^s \alpha_j^{(0)} \nu(w_j)
$$

$$
= \nu(\sum_{j=1}^s w_j^{(0)}).
$$

Theorem 6.4. There exists a local framed sequence

$$(R, w) \xrightarrow{\pi_0} (R_1, u^{(1)}) \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_{l-1}} (R_l, u^{(l)})$$

with respect to $\nu$, independent of $v$, and which satisfies the next properties:

For every integer $i \in \{1, \ldots, l\}$, we denote $u^{(i)} := (u_1^{(i)}, \ldots, u_n^{(i)})$ and we recall that $k$ is the residual field of $R_i$.

1. The blow-ups $\pi_0, \ldots, \pi_{l-2}$ are monomials.
2. We have $\pi \in R_i^\times$.
3. We set $u^{(l)} := (w_1^{(l)}, \ldots, w_r^{(l)}, v, w_n^{(l)})$. So for every integer $j \in \{1, \ldots, r, n\}$, $w_j$ is a monomial in $w_1^{(l)}, \ldots, w_r^{(l)}$ multiplied by an element of $R_i^\times$. And for every integer $j \in \{1, \ldots, r\}$, $w_j^{(l)} = w_\eta$ where $\eta \in \mathbb{Z}_{\geq 1}$.
4. We have $Q = w_n^{(l)} \times \pi$.

Proof. We apply Proposition 6.3 to $(w^\delta, w^\gamma)$ and so we obtain a local framed sequence for $\nu$, independent of $v$ and such that $w^\gamma \mid w^\delta$ in $R_i$.

By Proposition 6.7 and the fact that $w^\delta$ and $w^\gamma$ have same value, we have that $w^\delta \mid w^\gamma$ in $R_i$. In fact, $\pi, \pi^{-1} \in R_i^\times$. So we have (2).

We choose the local sequence to be minimal, it means that the sequence made by $\pi_0, \ldots, \pi_{l-2}$ does not satisfy the conclusion of the Proposition 6.3 for $(w^\delta, w^\gamma)$. Now we are going to show that this sequence satisfies the five properties of Theorem 6.4.

Let $i \in \{0, \ldots, l\}$. We write $w^{(i)} = (w_1^{(i)}, \ldots, w_r^{(i)}, w_n^{(i)})$, with $r = n - t - 1$ and define $J_i, A_i, B_i, j_i$ and $D_i$ the same way that we defined $J, A, B, j$ and $D_1$, considering the $i$-th blow-up.

Since $D_i \subset \{1, \ldots, n\}$, we have $\sharp D_i \leq n$. It means that $\sharp (A_i \cup (B_i \cup \{j_i\})) \leq n$, so $\sharp A_i + \sharp B_i + 1 \leq n$. As the sequence is independent of $v$, this implies that $T \subset A_i$, and so that $\sharp T \leq \sharp A_i$. Then $\sharp T + 1 + \sharp B_i \leq n$, so $t + 1 \leq n$, and so $r \geq 0$. By minimality of the sequence, we know that if $i < l$, $w^\delta \not| w^\gamma$ in $R_i$, and so $\sharp B_i \neq 0$, and so $r > 0$.

For every integers $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, n\}$, we set $\beta_j^{(i)} = \nu(\frac{u_j^{(i)}}{w_j^{(i)}})$. For each $i < l$, $\pi_i$ is a blow-up along an ideal of the form $\left(\frac{u_j^{(i)}}{w_j^{(i)}}\right)$. Even if it means to
renumber, we can assume that 1 ∈ Ji and that R_{i+1} is a localisation of R_i \left[ \frac{u_i^{(i)}}{u_1^{(i)}} \right].

So we have β_1^{(i)} = \min_{j \in J_i} \left\{ \beta_j^{(i)} \right\}.

**Fact 6.5.** Let X = (x_1, \ldots, x_n) ∈ \mathbb{Z}^n be a vector which elements are relatively prime. Then there exists a matrix A ∈ SL_n(\mathbb{Z}) of determinant 1 such that X is the first line of A.

**Proof.** This proof is made by induction on n and using Bezout theorem. □

**Lemma 6.6.** Let i ∈ \{0, \ldots, l - 1\}. We assume that the sequence π_0, \ldots, π_{i-1} of \([6.1]\) is monomial.

We set w^γ = (w^{(i)})^{γ^{(i)}} and w^δ = (w^{(i)})^{δ^{(i)}}. Then:

\begin{equation}
\sum_{q \in E} \left( \frac{\gamma_q^{(i)}}{} - \frac{\delta_q^{(i)}}{} \right) \beta_q^{(i)} = 0, \tag{6.2}
\end{equation}

(2) \text{pgcd} \left( \frac{\gamma_1^{(i)}}{}, \ldots, \frac{\gamma_r^{(i)}}{}, \frac{\gamma_n^{(i)}}{}, \frac{\delta_1^{(i)}}{}, \ldots, \frac{\delta_r^{(i)}}{}, \frac{\delta_n^{(i)}}{} \right) = 1,

(3) Every \mathbb{Z}\text{-linear dependence relation between } \beta_1^{(i)}, \ldots, \beta_r^{(i)}, \beta_n^{(i)} \text{ is an integer multiple of } \left\{ \frac{6.2}{} \right\}.

**Proof:**

(1) We have ν(w^γ) = ν(w^δ), it means ν \left( (w^{(i)})^{γ^{(i)}} \right) = ν \left( (w^{(i)})^{δ^{(i)}} \right). So,

\[ \nu \left( \prod_{j=1}^r \left( w_j^{(i)} \right)^{\gamma_j^{(i)}} \times \left( w_n^{(i)} \right)^{\gamma_n^{(i)}} \right) = \nu \left( \prod_{j=1}^r \left( w_j^{(i)} \right)^{\delta_j^{(i)}} \times \left( w_n^{(i)} \right)^{\delta_n^{(i)}} \right) \]

it means

\[ \sum_{j=1}^r \gamma_j^{(i)} \nu \left( w_j^{(i)} \right) + \gamma_n^{(i)} \nu \left( w_n^{(i)} \right) = \sum_{j=1}^r \delta_j^{(i)} \nu \left( w_j^{(i)} \right) + \delta_n^{(i)} \nu \left( w_n^{(i)} \right). \]

By the definition of w^{(i)}, for every integer j ∈ \{1, \ldots, r, n\}, we have w_j^{(i)} = w_j^{(i)}, so \nu(w_j^{(i)}) = β_j^{(i)}. Then:

\[ \sum_{j=1}^r \gamma_j^{(i)} β_j^{(i)} + γ_n^{(i)} β_n^{(i)} = \sum_{j=1}^r δ_j^{(i)} β_j^{(i)} + δ_n^{(i)} β_n^{(i)} \]

Then

\[ \sum_{j=1}^r \left( \frac{\gamma_j^{(i)}}{} - \frac{\delta_j^{(i)}}{} \right) β_j^{(i)} = 0. \]

(2) We do an induction. Case i = 0.

We have

\[ \text{pgcd} \left( \frac{\gamma_1^{(i)}}{0}, \ldots, \frac{\gamma_r^{(i)}}{0}, \frac{\gamma_n^{(i)}}{0}, \frac{\delta_1^{(i)}}{0}, \ldots, \frac{\delta_r^{(i)}}{0}, \frac{\delta_n^{(i)}}{0} \right) = \text{pgcd} \left( \frac{\gamma_1}{0}, \ldots, \frac{\gamma_r}{0}, \frac{\gamma_n}{0}, \frac{\delta_1}{0}, \ldots, \frac{\delta_r}{0}, \frac{\delta_n}{0} \right) = \text{pgcd} \left( α_1, \ldots, α_s, α_{s+1}, \ldots, α_r, -α_0 \right), \]
By definition
\[ \alpha = \alpha^{(0)} = \min_{h \in \mathbb{N}_0} \{ h \text{ such that } h \beta_n \in \Delta \} \]
and
\[ \alpha \beta_n = \sum_{j=1}^{r} a_j^{(0)} \beta_j. \]

So \( \text{pgcd} \left( \alpha_1^{(0)}, \ldots, \alpha_s^{(0)}, \alpha_{s+1}^{(0)}, \ldots, \alpha_r^{(0)}, -\alpha \right) = 1. \)

Case \( i > 0. \) We assume the result shown at the previous rank. We have \( \gamma^{(i)} = \gamma^{(i-1)} G^{(i)} \), \( \delta^{(i)} = \delta^{(i-1)} G^{(i)} \) and \( \beta^{(i)} = \beta^{(i-1)} F^{(i)} \) where \( F^{(i)} = (G^{(i)})^{-1} \) and \( G^{(i)} \in \text{SL}_{r+1}(\mathbb{Z}) \) such that
\[ G_{s q}^{(i)} = \begin{cases} 
1 & \text{if } s = q \\
1 & \text{if } q = j \text{ and } s \in J \\
0 & \text{otherwise.} 
\end{cases} \]

So \( (\gamma^{(i)} - \delta^{(i)}) = (\gamma^{(i-1)} - \delta^{(i-1)}) G^{(i)} = (\gamma - \delta) G \) where \( G \) is a product of unimodular matrices, and so \( G \) is unimodular.

By the case \( i = 0 \), \((\gamma - \delta)\) is a vector whose elements are relatively prime.

By \( \text{Lemma 6.5} \) this vector can be complete as a base of \( \mathbb{Z}^{r+1} \), which, by a unimodular matrix, stay a base of \( \mathbb{Z}^{r+1} \). The vector \((\gamma^{(i)} - \delta^{(i)})\) is then a vector of this base, so its elements are relatively prime.

(3) Case \( i = 0 \) is the fact that \( \beta_1, \ldots, \beta_r, \beta_n \) generate a vector space of dimension \( r \).

Let
\[ Z := \left\{ (x_1, \ldots, x_{r+1}) \in \mathbb{Z}^{r+1} \text{ such that } \sum_{j=1}^{r} x_j \beta_j + x_{r+1} \beta_n = 0 \right\}. \]

But \( \alpha \beta_n = \sum_{j=1}^{r} a_j^{(0)} \beta_j \), so:
\[ Z = \left\{ (x_1, \ldots, x_{r+1}) \in \mathbb{Z}^{r+1} \text{ such that } \sum_{j=1}^{r} (\alpha x_j + x_{r+1} a_j^{(0)}) \beta_j = 0 \right\}. \]

Since \( \beta_1, \ldots, \beta_r \) are \( \mathbb{Q} \)-linearly independents, we have that \( Z \) is free \( \mathbb{Z} \)-module of rank 1, so it is generated by a unique vector. By point (1), the vector \((\gamma - \delta)\) is in \( Z \), and by point (2), it is composed of elements relatively prime. This vector generates the free \( \mathbb{Z} \)-module of rank 1.

Let \( i > 0. \) We already know that \( \beta^{(i)} = \beta^{(i-1)} F^{(i)} = \beta F \) where \( F \) is a unimodular matrix, so an automorphism of \( \mathbb{Z}^r \).

Let
\[ Z^{(i)} := \left\{ (x_1, \ldots, x_{r+1}) \in \mathbb{Z}^{r+1} \text{ such that } \sum_{j=1}^{r} x_j \beta_j^{(i)} + x_{r+1} \beta_n^{(i)} = 0 \right\}. \]

So
\[ Z^{(i)} = \left\{ (x_1, \ldots, x_{r+1}) \in \mathbb{Z}^{r+1} \text{ such that } \sum_{j=1}^{r} x_j \beta_j F + x_{r+1} \beta_n F = 0 \right\}. \]
then
\[ Z^{(i)} = \left\{ (x_1, \ldots, x_{r+1}) \in \mathbb{Z}^{r+1} \text{ such that } \sum_{j=1}^{r} x_j \beta_j + x_{r+1} \beta_n = 0 \right\} . \]

Then the set \( Z^{(i)} \) is a free \( \mathbb{Z} \)-module of rank 1 by the case \( i = 0 \). And we know by (3) that the vector \( (\gamma^{(i)} - \delta^{(i)}) \) is a vector of \( Z^{(i)} \) composed of elements relatively prime, so it generates \( Z^{(i)} \), which ends the proof.

\[ \square \]

**Lemma 6.7.** The sequence \( (6.7) \) is not monomial.

*Proof.* Assume by contradiction that it is. By induction on \( i \), we have \( r_i = r \) for every \( i \in \{0, \ldots, l \} \). We know that \( w^{(i)} \) is a regular system of parameters of \( R_l \) and that \( w^\delta \) and \( w^\gamma \) divide each other in \( R_l \).

We saw that
\[ \gamma^{(i)} = \gamma^{(l-1)} G^{(i)} \]
and
\[ \delta^{(i)} = \delta^{(l-1)} G^{(i)} \]

So \( \delta^{(l)} = \gamma^{(l)} \).

But \( (\gamma^{(i)} - \delta^{(i)}) = (\gamma - \delta) G \) where \( G \) is a unimodular matrix, it means that \( \gamma = \delta \), which is a contradiction by definition of \( \gamma \) and \( \delta \).

\[ \square \]

**Lemma 6.8.** Let \( i \in \{0, \ldots, l-1\} \) and assume \( \pi_0, \ldots, \pi_{i-1} \) are all monomials. Then the following assertions are equivalent:

1. The blow-up \( \pi_i \) is not monomial.
2. There exists a unique index \( q \in J_i \setminus \{1\} \) such that \( \beta^{(i)}_q = \beta^{(i)}_1 \).
3. We have \( i = l - 1 \).

*Proof.* (3) \( \Rightarrow \) (1) by Lemma 6.7

(1) \( \Rightarrow \) (2) First, the existence. We have \( \beta^{(i)}_1 = \min_{j \in J_i} \{ \beta^{(i)}_j \} \). So \( \pi_i \) monomial

\[ \iff B_i = J_i \setminus \{1\} \iff \beta^{(i)}_q > \beta^{(i)}_1 \text{ for every } q \in J_i \setminus \{1\} . \]

Since the blow-up is not monomial by hypothesis, there exists \( q \in J_i \setminus \{1\} \) such that \( \beta^{(i)}_q = \beta^{(i)}_1 \).

Now let us show the unicity by contradiction. Assume there exist two different indexes \( q \) and \( q' \) in \( J_i \setminus \{1\} \) such that \( \beta^{(i)}_q - \beta^{(i)}_1 = 0 \) and \( \beta^{(i)}_{q'} - \beta^{(i)}_1 = 0 \).

Then we have two linear dependence relations between \( \beta^{(i)}_1, \ldots, \beta^{(i)}_n \) and the element \( \beta^{(i)}_n \), which are not linearly dependents. It is a contradiction by point (4) of Lemma 6.6

(2) \( \Rightarrow \) (3)

By Remark 6.6 we write \( w^{(i)}_1 = w^\epsilon \) and \( w^{(i)}_q = w^\mu \) where \( \epsilon \) and \( \mu \) are two colons of an unimodular matrix. Then \( \epsilon - \mu \) is unimodular, so its total gcd is one.

So
\[ \nu(w^\mu) = \sum_{s \in E} \mu_s \beta_s = \nu(w^{(i)}_q) = \beta^{(i)}_q \]
and
\[ \nu(w^i) = \sum_{s \in E} \epsilon_s \beta_s = \nu(w^{(i)}_1) = \beta^{(i)}_1. \]

But by hypothesis, \( \beta^{(i)}_q = \beta^{(i)}_1 \). Then \( \sum (\mu_s - \epsilon_s) \beta_s = 0 \) and by points (3) and (4) of Lemma 6.6 and the fact that the total pgcd of \( \mu - \epsilon \) is one, we have
\[ \mu - \epsilon = \pm (\gamma - \delta). \]
So \( w^{(i)}_q = w^{\pm(\gamma - \delta)} = \pm 1 \), then \( v \in R_{i+1} \) or \( v^{-1} \in R_{i+1} \).

To show that \( i = l - 1 \), we are going to show that \( i + 1 = l \). And to do this, we are going to use the fact that \( l \) has been chosen minimal such that \( v \in R_l^\times \). So let us show that \( v \in R_{l+1}^\times \).

Since \( v \in R_{i+1} \) or \( v^{-1} \in R_{i+1} \), we know that \( w^\delta \mid w^\gamma \) in \( R_{i+1} \) or the inverse. But by Proposition 4.7 and the fact that \( w^\delta \) and \( w^\gamma \) have same value, then \( w^\delta \mid w^\gamma \) in \( R_{i+1} \) if and only if the inverse is true. So \( v \in R_{i+1}^\times \), and we are done.

Doing an induction on \( i \) and using Lemma 6.8 we conclude that \( \pi_0, \ldots, \pi_{l-2} \) are monomials. So we have the first point of Theorem 6.3

Then we have to show the points (3) and (4).

By Lemma 6.8 we know the existence of a unique element \( q \in J_{l-1} \setminus \{j_{l-1}\} \) such that \( \beta^{(l-1)}_q = \beta^{(l-1)}_1 \), so we are in the case \( zB_{l-1} = zj_{l-1} - 1 \). Now we have to see if we are in the case \( t_{k_{l-1}} = 0 \) or in the case \( t_{k_{l-1}} = 1 \).

We recall that \( w^{(l-1)}_1 = w^\epsilon \) and \( w^{(l-1)}_q = w^\mu \) where \( \epsilon \) and \( \mu \) are two colons of a unimodular matrix \( l \) such that \( \mu - \epsilon = \pm (\gamma - \delta) \). So we have \( x^{(l-1)}_1 = x^\epsilon \) and \( x^{(l-1)}_q = x^\mu \), then
\[ \frac{x^{(l-1)}_q}{x^{(l-1)}_1} = x^{\mu - \epsilon} = x^{\pm(\gamma - \delta)} = x^{\pm \{\alpha^{(0)}_1, \ldots, \alpha^{(0)}_r, -\alpha^\epsilon\}}. \]

It means that
\[ \frac{x^{(l-1)}_q}{x^{(l-1)}_1} = \left( \prod_{j=1}^r \frac{x^{(0)}_j}{x^{(0)}_{n_j}} \right)^{\pm 1} = (z^{-1})^{\pm 1} = z^{\pm 1}. \]

Even if it means to exchange \( x^{(l-1)}_1 \) and \( x^{(l-1)}_q \), we can assume \( \frac{x^{(l-1)}_q}{x^{(l-1)}_1} = z \).

Since \( \beta^{(l-1)}_1, \ldots, \beta^{(l-1)}_r \) are linearly independents, we have \( q = n \).

We recall that \( \lambda_0 = X^\alpha + c_0 y \) where \( c_0 \in k \), and \( Z + c_0 \) is the minimal polynomial \( \lambda_z \) of \( z \) on \( \text{gr}_{r,k} (x_1, \ldots, x_r) \). By 3.9 we have
\[ w^{(l)}_n = u^{(l)}_n = \lambda_0(u^{(l)}_n) = \lambda_0 \left( \frac{u^{(l-1)}_n}{u^{(l-1)}_1} \right) = \frac{w^{(l-1)}_n}{w^{(l-1)}_1} = \frac{w^{(l)}_n}{w^{(l)}_1} = \lambda_0^{-1}(\tau) = \tau + a_0 b_0. \]

Remark 6.9. We know that \( \lambda_0(\tau) = \tau + b_0 g_0 \) where \( g_0 \) is a unit and \( b_0 \in R \) such that \( b_0 \equiv c_0 \) modulo \( m \). Then we choose \( g_0 = a_0 \).

But \( \tau = \frac{w^n}{y} \), so
\[ w^{(l)}_n = \frac{w^n}{y} + a_0 b_0 = \frac{w^n + a_0 b_0 y}{y} = \frac{Q}{y} \]
as wanted in point (4).

Let us show the point (3). We apply Proposition 4.5 at \( i = 0 \) and \( i' = l \). By monomiality of \( \pi_0, \ldots, \pi_{l-2} \), we know that \( D_i = \{1, \ldots, n\} \) for each \( i \in \{1, \ldots, l-1\} \), and we know that \( D_l = \{1, \ldots, n\} \). Here we set \( u_T = v \).

For every \( j \in \{1, \ldots, r, n\} \), the fact that \( w_j = u_j \) is a monomial in \( w_1^{(l)}, \ldots, w_r^{(l)} \), it means in \( u_1^{(l)}, \ldots, u_r^{(l)} \), multiplicated by an element of \( R_I \) is a consequence of Proposition 4.5.

And the fact that for every integer \( j \in \{1, \ldots, r\} \), we have \( w_j^{(l)} = w^{n} \) is a consequence of the same Proposition. This ends the proof. \( \square \)

Remark 6.10. In the case \( Q_2 = Q \), we monomialized \( Q_2 \) as wanted.

Definition 6.11. A local framed sequence which satisfies Theorem 6.4 is called a \( n \)-Puiseux package.

Let \( j \in \{r+1, \ldots, n\} \). A \( j \)-Puiseux package is a \( n \)-Puiseux package changing \( n \) by \( j \) in Theorem 6.4.

Lemma 6.12. Let \( P = u_n^\alpha + c_0 \) the \( u_n \)-expansion of an immediate successor key element of \( u_n \).

There exists a local framed sequence \( (R, u) \rightarrow (R_1, u^{(l)}) \), independant of \( u_n \), which transforms \( c_0 \) in a monomial in \( u_1^{(l)}, \ldots, u_r^{(l)} \), multiplicated by a unit of \( R_I \).

In particular, after this local framed sequence, the element \( P \) is of the form \( u_n^\alpha + a_0 b_0 \).

Proof. We will prove this Lemma in a more general version in Lemma 6.16. \( \square \)

Corollary 6.13. Let \( P \) an immediate successor key element of \( u_n \). Then \( P \) is monomializable.

Proof. If \( u_n \ll P \), we use Lemma 6.12 to bring us to the case \( P = u_n^\alpha + a_0 b_0 \). Then, by Theorem 6.4, we can monomialize \( P \). \( \square \)

Let \( G \) be a local ring essentially of finite type over \( k \) of dimension strictly less than \( n \) which has a valuation centered on \( G \).

Theorem 6.14. We assume that for every ring \( G \) as above, every element of \( G \) is monomializable.

We recall that \( \text{car} \ (k_n) = 0 \). If \( u_n \ll \lim P \), then, \( P \) is monomializable.

Proof. We write \( P = \sum_{j=0}^N b_j a_j u_n^j \) the \( u_n \)-expansion of \( P \), with \( a_j \in R^\times \) and \( Q = \sum_{j=0}^N b_j u_n^j \), limite immediate successor of \( u_n \).

By Theorem 2.17 we know that \( \delta_{u_n} (Q) = 1 \). Then:

\[ \nu(b_0) = \nu(b_1 u_n) < \nu(b_j u_n^j) \]

for every \( j > 1 \).

The elements \( a_i \) are units of \( R \), so for every \( j > 1 \) we have:

\[ \nu(a_0 b_0) = \nu(a_1 b_1 u_n) < \nu(a_j b_j u_n^j) \].
In fact, \( \nu(a_1 b_1) < \nu(a_0 b_0) \) and by hypothesis, after a sequence of blow-ups independent of \( u_n \), we can monomialize \( a_j b_j \) for every index \( j \), and assume that \( a_1 b_1 \mid a_0 b_0 \) by Proposition 4.7.

Then
\[
\nu(b_0) = \nu(b_1 u_n) < \nu(b_j) + j \nu(u_n) = \nu(b_j) + j (\nu(b_0) - \nu(b_1)).
\]

So \( \nu(b_0) < (b_j) + j (\nu(b_0) - \nu(b_1)) \).

In fact, \( \nu(b_1) \neq \nu(b_0) \).

Remark 6.15. Since \( Q_2 \) is an immediate successor (eventually limit) of \( u_n \), it is in particular an immediate successor (eventually limit) key element of \( u_n \). By Corollary 6.13, or Theorem 6.14, it is monomializable modulo Lemma 6.12.

6.3. Generalisation. Now we monomialized \( Q_2 \), but we want to monomialize every key polynomial of the sequence \( Q \). There the key elements will be useful. Indeed, modified by the blow-ups which monomialized \( Q_2 \), we cannot know if \( Q_3 \) is still a key polynomial.

To be more general, we will show that if \( Q_i \in \mathcal{Q} \) is monomializable, then \( Q_{i+1} \) is too.

Assume that the polynomial \( Q_i \) is monomializable after a sequence of blow-ups \( (R, u) \rightarrow (R_i, u^{(i)}) \).

Let \( \Delta_l \) be the group \( \nu(k \left\{ u_1^{(l)}, \ldots, u_n^{(l)} \right\} \setminus \{0\}) \). We set
\[
\alpha_l := \min \left\{ h \mid h_i b_n^{(l)} \in \Delta_l \right\}.
\]

We set \( X_j = \text{in}_\nu(u_j^{(l)}) \), \( W_j = w_j^{(l)} \) and \( \lambda_l \) the minimal polynomial of \( X_n \) over \( \text{gr}_\nu k \left( u_1^{(l)}, \ldots, u_n^{(l)} \right) \) of degree \( \alpha_l \).

Since \( k = k_\nu \), there exists \( c_0 \in \text{gr}_\nu k \left( u_1^{(l)}, \ldots, u_{n-1}^{(l)} \right) \) such that
\[
\lambda_l(X) = X^{\alpha_l} + c_0.
\]

Furthermore, we know that \( Q_i = \omega w_n^{(l)} \) with \( \omega \) a monomial in \( W_1, \ldots, W_{r_l} \) multiplicatied by a unit. We set \( \omega := \text{in}_\nu(\omega) \).

We know that \( Q_{i+1} \) is an optimal immediate successor of \( Q_i \), so we denote by
\[
Q_{i+1} = Q_i^{\omega} + b_0
\]
the \( Q_i \)-expansion of \( Q_{i+1} \) in \( k(u_1, \ldots, u_{n-1}) \) by Proposition 5.6 with \( c_0 = \text{in}_\nu(b_0) \).
Since $Q_i = \overline{\omega} W_n$ and $Q_{i+1} = Q_i^{\alpha_i} + b_0$, we have

$$Q_{i+1}^{\omega_i} = \left( u_n^{(i)} \right)^{\alpha_i} + \frac{b_0}{\omega_i}.$$  

We know that all the terms of the $Q_i$-expansion of $Q_{i+1}$ have same value. So these two terms are divisible by the same power of $\overline{\omega}$ after a suitable sequence of blow-ups $(*)_i$ independent of $u_n^{(i)}$.

We denote by $\tilde{Q}_{i+1}$ the strict transform of $Q_{i+1}$ by the composition of $(*)_i$ and of the sequence of blow-ups $(*)'_i$ which monomialize $Q_i$. We denote this composition by $(c_i)$. We consider $(R, u) \rightarrow (R_1, u^{(l)})$. We know that all the terms of the $Q_i$-expansion of $Q_{i+1}$ have same value. So these two terms are divisible by the same power of $\overline{\omega}$ after a suitable sequence of blow-ups $(*)_i$ independent of $u_n^{(i)}$.

We know that $\tilde{Q}_i$, the strict transform of $Q_i$ by $(c_i)$, is a regular parameter of the maximal ideal of $R_i$. Indeed, by Proposition 1.5 we know that every $u_j$ of $R$ can be written as a monomial in $u_i^{(l)}$, . . . , $u_n^{(l)}$. In fact, the reducptional divisor of this sequence of blow-ups is exactly $V(\overline{\omega})_{red}$. Then, since $Q_i = W_n \overline{\omega}$, we have that the strict transform of $Q_i$ is $\tilde{Q}_i = W_n = u_n^{(l)} = u_n^{(l)}$. So it is a key polynomial in the extension $k \left( u_1^{(l)}, \ldots, u_{n-1}^{(l)} \right) \left( u_n^{(l)} \right)$.

Let us show that $Q_{i+1} = \tilde{Q}_{i+1}$.

We have

$$Q_i^{\omega_i} = \overline{\omega_i} \left( u_n^{(l)} \right)^{\alpha_i}$$

and also $u_n^{(l)} \not| \overline{\omega}$. So $\overline{\omega_i}$ divides $Q_i^{\omega_i}$ and all the non zero terms of the $Q_i$-expansion of $Q_{i+1}$. Furthermore, it is the greatest power of $\overline{\omega}$ which divides each term, so $\tilde{Q}_{i+1}$ is $\tilde{Q}_{i+1}$, the strict transform of $Q_{i+1}$ by the sequence of blow-ups.

Let $G$ be a local ring essentially of finite type over $k$ of dimension strictly less than $n$ which has a valuation centered in $G$ which residual field is $k$.

**Lemma 6.16.** We assume that for every ring $G$ as above, every element of $G$ is monomializable.

Assume that $Q_i < Q_{i+1}$ in $Q$.

There exists a local framed sequence $(R_i, u^{(l)}) \rightarrow (R_e, u^{(e)})$ such that in $R_e$, the strict transform of $Q_{i+1}$ is of the form $\left( u_n^{(e)} \right)^{\alpha_i} + \tau_0 \eta$, where $\tau_0 \in R_e^\times$ and $\eta$ is a monomial in $u_1^{(e)}, \ldots, u_r^{(e)}$.

**Proof.** By hypothesis, after a sequence of blow-ups independent of $u_n^{(l)}$, we can monomialize $b_0$ and assume that it is a monomial in $u_1^{(l)}, \ldots, u_{n-1}^{(l)}$ multplicated by a unit of $R_i$.

For every $g \in \{ r_1 + 1, \ldots, n - 1 \}$, we do a $g$-Puiseux package, and then we have a sequence

$$(R_i, u^{(l)}) \rightarrow (R_t, u^{(t)})$$

such that every $u_g^{(t)}$ is a monomial in $u_1^{(l)}, \ldots, u_t^{(l)}$.

In fact, we can assume that $b_0$ is a monomial in $u_1^{(l)}, \ldots, u_t^{(l)}$ multplicated by a unit of $R_t$.

Since the strict transform $\tilde{Q}_{i+1} = \left( u_n^{(l)} \right)^{\alpha_i} + \frac{b_0}{\omega_i}$ is an immediate successor key element of $\tilde{Q}_i$, this ends the proof. \(\square\)
Remark 6.17. Lemma 6.12 is a particular case of Lemma 6.10.

Let $G$ be a local ring essentially of finite type over $k$ of dimension strictly less than $n$ which has a valuation centered in $G$ which residual field is $k$.

**Theorem 6.18.** Assume that for every ring $G$ as above, every element of $G$ is monomializable.

We recall that $\text{car}(k_v) = 0$. If $Q_i$ is monomializable, there exists a local framed sequence

\[
(R, u) \xrightarrow{\pi_0} (R_1, u^{(1)}) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{i-1}} (R_i, u^{(i)}) \xrightarrow{\pi_i} \cdots \xrightarrow{\pi_{n-1}} (R_n, u^{(n)})
\]

which monomialize $Q_{i+1}$.

**Proof.** There are two cases.

First: $Q_i < Q_{i+1}$. Then we just saw that the strict transform $\tilde{Q}_{i+1}$ of $Q_{i+1}$ by the sequence $(R, u) \rightarrow (R_i, u^{(i)})$ which monomialize $Q_i$ is an immediate successor key element of $\tilde{Q}_i = u^{(i)}_n$, and that we can bring us to the hypothesis of Theorem 6.4 by Lemma 6.16. So we use Theorem 6.4 exchanging $(R, u)$ for $(R_i, u^{(i)})$.

Then we constructed a local framed sequence (6.3) which monomialize $\tilde{Q}_{i+1}$.

Last case: $Q_i < \lim Q_{i+1}$. Then we saw that the strict transform $\tilde{Q}_{i+1}$ of $Q_{i+1}$ by the sequence $(R, u) \rightarrow (R_i, u^{(i)})$ which monomialize $Q_i$ is a limit immediate successor key element of $\tilde{Q}_i = u^{(i)}_n$. Then we apply Theorem 6.14 exchanging $Q_1$ by $\tilde{Q}_i$ and $Q_2$ by $\tilde{Q}_{i+1}$.

So we constructed a local framed sequence (6.3) which monomialize $\tilde{Q}_{i+1}$. \hfill $\square$

**Theorem 6.19.** There exists a local sequence

\[
(R, u) \xrightarrow{\pi_0} \cdots \xrightarrow{\pi_{s-1}} (R_s, u^{(s)}) \xrightarrow{\pi_s} \cdots
\]

which monomialize all the key polynomials of $Q$.

More precisely, for every index $i$, there exists an index $s_i$ such that in $R_{s_i}$, $Q_i$ is a monomial in $u^{(s_i)}$ multiplied by a unit of $R_{s_i}$.

**Proof.** Induction on the dimension $n$ et on the index $i$ and we iterate the previous processus. \hfill $\square$

### 6.4. Divisibility

We consider, for every integer $j$, the countable sets

\[
\mathcal{J}_j := \left\{ \prod_{i=1}^{n} (u_i^{(j)})^{\alpha_i^{(j)}}, \text{ with } \alpha_i^{(j)} \in \mathbb{Z} \right\}
\]

and

\[
\tilde{\mathcal{J}}_j := \{ (s_1, s_2) \in \mathcal{J}_j \times \mathcal{J}_j, \text{ with } \nu(s_1) \leq \nu(s_2) \}
\]

with the convention that for every $i \in \{1, \ldots, n\}$, $u_i^{(0)} = u_i$.

The set $\tilde{\mathcal{J}}_j$ being countable for every integer $j$, we can number its elements, and then we write $\tilde{\mathcal{J}}_j := \left\{ s^{(j)}_m \right\}_{m \in \mathbb{N}}$. We consider now the finite set

\[
\mathcal{J}'_j := \left\{ s^{(j)}_m, m \leq j \right\} \cup \left\{ s^{(m)}_j, m \leq j \right\}.
\]

Then $\bigcup_{j \in \mathbb{N}} (\mathcal{J}_j \times \mathcal{J}_j) = \bigcup_{j \in \mathbb{N}} \tilde{\mathcal{J}}_j = \bigcup_{j \in \mathbb{N}} \mathcal{J}'_j$ is a countable union of finite sets.
Now we fix a local framed sequence
\[ (R, u) \to \cdots \to (R_i, u^{(i)}) \].

**Theorem 6.20.** There exists a local framed finite sequence
\[ p_i : (R, u^{(i)}) \to \cdots \to (R_{i+q}, u^{(i+q,i)}) \]
such that for every integer \( j \leq i \) and for every element \( s \) of \( \mathcal{J}_j \), the first coordinate of \( s \) divides its second coordinate in \( R_{i+q_1} \).

**Proof.** Consider an integer \( j \leq i \) and an element \( s = (s_1, s_2) \in \mathcal{J}_j \). We want to construct a sequence of blow-ups such that at the end we have \( s_1 \mid s_2 \).

We know that \( s \in \mathcal{J}_m \) with \( m \leq j \). All cases being similar, we can assume \( s \in \mathcal{J}_j \) and then have
\[ s_1 = \prod_{i=1}^{n} (u_i^{(j)})^{a_{i,1}}, \]
and
\[ s_2 = \prod_{i=1}^{n} (u_i^{(j)})^{a_{i,2}}. \]

By Proposition 4.4 applied to \( R_i \) instead of \( R \), we have the existence of a sequence
\[ (R_i, u^{(i)}) \to \cdots \to (R_{i+l}, u^{(i+l)}) \]
such that in \( R_{i+l} \), \( s_1 \mid s_2 \) or \( s_2 \mid s_1 \). By definition \( \nu(s_1) \leq \nu(s_2) \), so we really have \( s_1 \mid s_2 \) by Proposition 4.7.

By point 4 of Theorem 6.3, we know that \( \mathcal{J}_j \subseteq R_{i+l}^{\times} \mathcal{J}_{i+l} \). It means that every element of \( \mathcal{J}_j \) can be written \( z_{i+l} s_{i+l} \) with \( z_{i+l} \in R_{i+l}^{\times} \) and \( s_{i+l} \in \mathcal{J}_{i+l} \).

Let \( (s_3, s_4) \in \mathcal{J}_j \), be another couple of \( \mathcal{J}_j \), let say that it is still in \( \mathcal{J}_j \). We just saw that \( s_3, s_4 \in R_{i+l}^{\times} \mathcal{J}_{i+l} \). Unless don’t have an effect on divisibility, so we can only consider the part of \( s_3 \) and \( s_4 \) which is in \( \mathcal{J}_{i+l} \). Hence we can iterate the Proposition 4.3 by applying it to \( (R_{i+l}, u^{(i+l)}) \). So we constructed an other sequence of blow-ups
\[ (R_{i+l}, u^{(i+l)}) \to \cdots \to (R_{i+h}, u^{(i+h)}) \]
such that \( R_{i+h} \) we have \( s_3 \mid s_4 \) or \( s_4 \mid s_3 \). Since \( \nu(s_3) \leq \nu(s_4) \), we know that \( s_3 \) divides \( s_4 \).

We iterate the process for all the couples of \( \mathcal{J}_j \), and for every \( j \leq i \) . This is a finite number of times since \( \mathcal{J}_j \) has a finite number of elements for every \( j \) and since we consider a finite number of such sets. Then we obtain a finite sequence of blow-ups
\[ (R_i, u^{(i)}) \to \cdots \to (R_{i+q}, u^{(i+q,i)}) \]
such that for every integer \( j \leq i \) and every \( s \) in \( \mathcal{J}_j \), the first coordinate of \( s \) divides the second coordinate in \( R_{i+q} \). \( \square \)

Since we consider all the key elements according to the variable \( u_n \), and more generally during the sequence of blow-ups according to the variable \( u^{(i)}_n \), and since we do an induction on the dimension, we have to monomialize the elements of \( B_i := k \left[ u_1^{(i)}, \ldots, u^{(i)}_{n-1} \right] \).
We are going to monomialize the elements of the
and which satisfies the next property:
\[ (6.5) \ \\
\text{which monomialize all the key polynomials, all the elements of } B_i \text{ for every index } i \text{ and which satisfies the next property:} \]
\[ \forall j \in \mathbb{N} \ \forall s = (s_1, s_2) \in \mathcal{S}_j \ \exists i \in \mathbb{N}_{\geq j} \text{ such that in } R_i \text{ we have } s_1 \mid s_2. \]

**Proof.** We are going to monomialize the elements of the \( B_i \) step by step. First, the elements of \( B_0 \). Then those of \( B_1 \) and so on. Restricting the valuation \( \nu \) to \( B_i \), we know that the \( \nu \)-ideals of \( B_i \) are countables and generated by a finite number of generators. We denote by \( (P_i^{(1)}_{\nu})_{i \in \mathbb{N}} \) the sequence of all the generators of all the \( \nu \)-ideals of \( B_i \). To monomialize all the elements of \( B_i \), we just have to monomialize all the generators.

In the infinite sequence \( (6.3) \) which monomialize all the key polynomials in the Theorem 6.19, we call \( \pi^{(i)} \) the part which monomialize the \( i \)-th key polynomial, which give the divisibility of the terms of the next key polynomial and which monomialize all the \( P_j^{(l)} \), where \( l, j \leq i - 1 \), which are not monomials. It means, \( \pi^{(1)} \) is the identity since the first key polynomial is a monomial in \( R \), composed with the morphism which monomialize \( P_0^{(0)} \). The morphism \( \pi^{(2)} \) is given by \( (6.1) \) in the Theorem 6.4 composed with \( (s_2') \) and with the morphism which monomialize \( P_0^{(1)} \), \( P_1^{(0)} \) and \( P_1^{(1)} \). The morphism \( \pi^{(3)} \) is the part \( (R_i, u^{(l)}) \rightarrow \cdots (R_m, u^{(m)}) \) of \( (6.3) \) given in the Theorem 6.18 composed with \( (s_3') \) and with the morphism which monomialize \( P_0^{(2)} \), \( P_1^{(2)} \), \( P_2^{(0)} \), \( P_2^{(1)} \) and \( P_2^{(2)} \) and so on.

First, we want that the monomials in the generators of \( R \) divide each others. So we apply Theorem 6.20 to \( i = 0 \). So we construct a sequence \( p_0 \). We now monomialize the second key polynomial. Our sequence \( \pi^{(2)} \) begins at \( R \) and not at \( R_{q_0} \), but it does not matter. Indeed, we can apply exactly the same argumentation as in the Theorem 6.4 beginning at \( R_{q_0} \). We still call \( \pi^{(2)} \) the sequence which monomialize \( Q_2 \) and which begins at \( R_{q_0} \). So we have a sequence
\[ \pi^{(2)} \circ p_0: (R, u) \rightarrow (R_i, u^{(l)}) \]
which monomialize the first two key polynomials and such that the coordinates of the couples of monomials in \( \mathcal{S}_j \) divide each other in \( R_i \). This sequence also allows the monomialization of all the \( P_j^{(i)} \) for \( i, j \leq 1 \).

Now we apply Theorem 6.20 at \( i = l \). Hence we construct a sequence \( p_l: (R_l, u^{(l)}) \rightarrow (R_{q_l}, u^{(q_l)}) \) such that in \( R_{q_l} \) we have the divisibility for every element of \( \mathcal{S}_j' \), for all \( j \leq l \).

Now we have a sequence \( p_1 \circ \pi^{(2)} \circ p_0 \) and we iterate the processus considering the sequence \( \pi^{(3)} \) which monomialize \( Q_3 \) and begins at \( R_{q_l} \). This sequence also allows the monomialization of all the \( P_j^{(3)} \) for \( i, j \leq 2 \).

We iterate the processus an infinite but countable number of times until monomializing all the key polynomials. Hence we obtain an infinite sequence \( (R, u) \rightarrow \cdots \rightarrow (R_m, u^{(m)}) \rightarrow \cdots \) which monomialize all the key polynomials and which satisfies the wanted property. □
7. Conclusion.

We can now prove the principal results of this chapter, and so the simultaneous embedded local uniformization for the local rings essentially of finite type over a field of zero characteristic.

A local algebra $K$ essentially of finite type over a field $k$ which has $k$ as residual field is an étale extension of

$$K' = k[u_1, \ldots, u_n]_{(u_1, \ldots, u_n)}.$$ 

Let $f \in K$ be an element irreducible over $k$ and

$$I := (f) \cap k[u_1, \ldots, u_n].$$

The ideal $I$ is a prime ideal of height 1, so it is principal. We consider $\tilde{f}$ a generator of $I$. Then $\frac{K'}{(f)} \rightarrow \frac{K'}{(f)}$ and each local sequence in $\frac{K'}{(f)}$ induced a local sequence in $\frac{K}{(f)}$.

So it is enough to prove local uniformization in the case of the rings $k[u_1, \ldots, u_n]_{(u_1, \ldots, u_n)}$ to have it in the general case of algebra essentially of finite type over a field $k$.

**Theorem 7.1.** Let us consider the sequence

$$(R, u) \rightarrow \cdots \rightarrow (R_m, u^{(m)}) \rightarrow \cdots$$

of the Theorem 6.21.

Then for every element $f$ of $R$, there exists $i$ such that in $R_i$, $f$ is a monomial multiplied by a unit.

**Proof.** Let $f \in R$. By Theorem 6.3, there exists a finite or infinite sequence $(Q_i)_i$ of key polynomials of the extension $K(u_n)$, optimal (eventually limit) immediate successors, such that $(\epsilon(Q_i))_i$ is cofinal in $\epsilon(\Lambda)$ where $\Lambda$ is the set of key polynomials.

Then by Remark 5.9 $f$ is non degenerated with respect to one of these polynomials $Q_j$. But we saw in Theorem 6.21 that there exists an index $l$ such that in $R_l$, all the $Q_j$ with $j \leq i$ are monomials, hence $f$ is non degenerated with respect to a regular system of parameters of $R_l$.

Let $N = (w_1, \ldots, w_s)$ be a monomial ideal in $u^{(l)}$ such that $\nu(N) = \nu(f)$ with $w_j$ monomials in $u^{(l)}$ such that $\nu(w_j) = \min \{\nu(w_j)\}$. By construction of the local framed sequence, there exists $l' \geq l$ such that in $R_{l'}$, $w_1 | w_j$ for all $j$. So in $R_{l'}$, $f$ is equal to $w_1$ multiplied by a unit of $R_{l'}$.

**Theorem 7.2** (Embedded local uniformization). Let $k$ be a zero characteristic field and $f = (f_1, \ldots, f_l) \in k[u_1, \ldots, u_n]^{l}$ a set of $l$ polynomials in $n$ variables, irreducible over $k$. We set $R := k[u_1, \ldots, u_n]_{(u_1, \ldots, u_n)}$ and $\nu$ a valuation centered in $R$ such that $k = k_\nu$.

We consider the sequence $(R, u) \rightarrow \cdots \rightarrow (R_m, u^{(m)}) \rightarrow \cdots$ of Theorem 6.21.

Then there exists an index $j$ such that the subscheme of $\text{Spec}(R_j)$ defined by the ideal $(f_1, \ldots, f_l)$ is a normal crossing divisor.

**Proof.** Even if it means to renumber, we assume $\nu(f_1) = \min \{\nu(f_j)\}$.

By Theorem 6.21 there exists an index $j_1$ such that in $R_{j_1}$, the total transform of $f_1$ is a monomial in $u^{(j_1)}$, and so is a normal crossing divisor.
Now we look the equation \( f_2 \) in \( R_j \). By Theorem 6.21, there exists an index \( j_2 \) such that in \( R_{j_2} \), the total transform of \( f_2 \) is a normal crossing divisor.

In \( R_2 \), the total transforms of \( f_1 \) and \( f_2 \) are normal crossing divisors.

We iterate the processus until the total transforms of \( f_1, \ldots, f_l \) are normal crossing divisors in \( R_{j_l} \).

By construction of the local framed sequence \((R, u) \to \cdots \to (R_m, u^{(m)}) \to \cdots\), there exists \( j \geq j_l \) such that in \( R_j \), we have \( f_1 \mid f_i \) for every index \( i \). \( \square \)

**Corollary 7.3.** We keep the same notations and hypothesis than in the previous Theorem.

Then \( R_v = \lim_{\to} R_i \).
Part 4. Simultaneous local uniformization in the case of quasi excellent rings for valuations of rank less or equal to 2.

8. Preliminaries.

Let $R$ be a local noetherian domain equicharacteristic of characteristic zero and $\nu$ be a valuation of rank 1 over $\text{Frac}(R)$, centered in $R$ and of value group $\Gamma$. We are going to define the implicit prime ideal $H$ of $R$ for the valuation $\nu$, which is a key object in local uniformization. Indeed, this ideal will be the ideal we have to desingularise. We are going to see in this part that to regularise, we only have to regularise $\hat{R}_H$ and $\hat{\mathcal{F}}$. At this point, the hypothesis of quasi excellence is very important: if $R$ is quasi excellent, the ring $\hat{R}_H$ is regular. So we will only have to monomialize the elements of $\hat{\mathcal{F}}$.

8.1. Quasi excellent rings and implicit prime ideal.

Definition 8.1. Let $R$ be a domain. We say that $R$ is a G-ring if for every prime ideal $p$ of $R$, the completion morphism $R_p \to \hat{R}_p$ is a regular homomorphism.

Definition 8.2. Let $R$ be a local ring. Then $R$ is quasi excellent if $R$ is a G-ring.

More generally, if $A$ is a ring, then $A$ is quasi excellent if $A$ is a local G-ring which regular locus is open.

Proposition 8.3. [16] A local noetherian ring $R$ is quasi excellent if the completion morphism $R \to \hat{R}$ is regular.

Remark 8.4. Let $R$ be a local ring. If $R$ is a G-ring, then its regular locus is open.

Definition 8.5. We call the implicit prime ideal $H$ of $R$ the ideal $H = \bigcap_{\beta \in \nu(R \setminus \{0\})} P_{\beta \hat{R}}$. The ideal $H$ is composed of the elements of $\hat{R}$ which are of infinite value.

Furthermore, the valuation $\nu$ extends uniquely to a valuation $\hat{\nu}$ centered in $\hat{R}$.

Proposition 8.6. Let $R$ be a quasi excellent local ring. Then $\hat{R}_H$ is regular.

Proof. The ring $R$ is a G-ring. Then for every prime ideal $p$ of $R$, we have the injective map $\kappa(p) \hookrightarrow \kappa(p) \otimes_R \hat{R}$ such that the fiber $\kappa(p) \otimes_R \hat{R}$ is geometrically regular over $\kappa(p)$, where $\kappa(p) := \frac{R_p}{\overline{\mathfrak{p}R_p}}$. Since $R$ is a domain, $(0)$ is a prime ideal of $R$.

Then we have the injective map $\text{Frac}(R) \hookrightarrow \text{Frac}(R) \otimes_R \hat{R}$ such that the fiber $\text{Frac}(R) \otimes_R \hat{R}$ is geometrically regular over $K := \text{Frac}(R)$. It means that the morphism $K \to K \otimes_R \hat{R}$ is regular.

But $R \setminus \{0\}$ and $\hat{R} \setminus H$ are two multiplicative parts of $\hat{R}$ such that $R \setminus \{0\} \subseteq \hat{R} \setminus H$, since $R \cap H = \{0\}$. Then, $\hat{R}_H$ is a localisation of $\hat{R}_{R \setminus \{0\}}$. If we show that $\hat{R}_{R \setminus \{0\}}$ is regular, then $\hat{R}_H$ will be also regular as localisation of a regular ring. But, by the universal property of tensor product, the ring $\hat{R}_{R \setminus \{0\}}$ is isomorphic to $K \otimes_R \hat{R}$, which is regular by hypothesis. This concludes the proof. 

8.2. Associated integers. Let $(S, q, L)$ be a local noetherian ring and $\mu$ a valuation centered in $S$. We write $\mu = \mu_2 \circ \mu_1$ with $\mu_1$ of rank 1. The valuation $\mu_2$ is trivial if and only if $\mu$ is also of rank 1. We note $G$ the value group of $\mu$ and $G_1$
the one of \( \mu_1 \). In fact we have that \( G_1 \) is the smallest isolated subgroup non trivial of \( G \). We set \( I := \{ x \in S \mid \mu(x) \notin G_1 \} \), and then \( \mu_1 \) induces a valuation of rank 1 over \( \frac{\hat{S}}{J} \). Let \( \hat{J} \) be the implicit prime ideal of \( \frac{\hat{S}}{J} \) for the valuation \( \mu_1 \) and \( J \) its preimage in \( \hat{S} \).

**Definition 8.7.** We set

\[
e(S, \mu) := \text{emb.dim} \left( \frac{\hat{S}}{J} \right).
\]

We assume that \( I \subseteq q^2 \). Let \( v = (v_1, \ldots, v_n) \) be a minimal set of generators of \( q \). We have \( \mu(v_j) \in G_1 \) for every index \( j \).

**Definition 8.8.** We have \( \sum_{j=1}^{n} \mathbb{Q} \mu(v_j) \subseteq G_1 \otimes \mathbb{Q} \), and we set

\[
r(S, v, \mu) := \dim_{\mathbb{Q}} \left( \sum_{j=1}^{n} \mathbb{Q} \mu(v_j) \right).
\]

**Remark 8.9.** We have \( r(S, v, \mu) \leq e(S, \mu) \).

Now we consider \( M \subset \{1, \ldots, n\} \) and

\[
(S, v) \mapsto \left( S_1, v^{(1)} = (v^{(1)}_1, \ldots, v^{(1)}_{n_1}) \right)
\]
a framed blow-up along \( (v_M) \). We set \( C' = \{1, \ldots, n_1\} \setminus D_1 \), where \( D_1 \) is as in 3.3.

If the elements of \( v_M \) are \( L \)-linearly independents in \( \frac{\hat{S}}{J+q^2 \hat{S}} \), then there exists a partition of \( A \) which we denote by \( A' \cup A'' \). This partition is such that \( v_M \cup v_{A'} \) are \( L \)-linearly independents modulo \( J + q^2 \hat{S} \) and \( v_{A''} \) is in the space generated by \( v_J \cup v_{A'} \) over \( L \) modulo \( J + q^2 \hat{S} \). As we know that \( v_{A' \cup B \cup \{j\}} = v_{D_1}^{(1)} \), we can identify \( A' \cup B \cup \{j\} \) with a subset of \( D_1 \).

Now we set \( I_1 := \{ x \in S_1 \mid \mu(x) \notin G_1 \} \) and we consider \( \hat{J} \) the implicit prime ideal of \( \frac{\hat{S}_1}{J_1 \hat{S}_1} \) with respect to \( \mu_1 \) and \( J_1 \) its preimage in \( \hat{S}_1 \). We call \( q_1 \) the maximal ideal of \( S_1 \) and \( L_1 \) its residual field.

**Remark 8.10.** We have \( e(S, \mu) = n \) if and only if the elements of \( v \) are \( L \)-linearly independents in \( \frac{\hat{S}}{J+q^2 \hat{S}} \).

**Theorem 8.11.** If \( e(S, \mu) = n \), then:

\[
e(S_1, \mu) \leq e(S, \mu).
\]

This inequality is strict once the elements of \( v_{A' \cup B \cup \{j\} \cup C}^{(1)} \) are \( L_1 \)-linearly dependent in \( \frac{q_1 \hat{S}_1}{J_1 + q_1^2 \hat{S}_1} \).

**Proof.** By definition, \( v^{(1)} \) generates the maximal ideal \( q_1 \) of \( S_1 \), and so induces a set of generators of \( \hat{q}_1 \hat{I}_1 \). Since \( n_1 \leq n \), by definition of a framed blow-up, we know that \( \hat{A}' \leq \hat{A} \).

Furthermore, we have \( e(S, \mu) = \hat{A}' + \hat{A} \). We also know that \( v_{D_1 \setminus (A' \cup B \cup \{j\})}^{(1)} \) is in the \( L \)-vector space of \( v_{A' \cup B \cup \{j\} \cup C}^{(1)} \) modulo \( J_1 + q_1^2 \hat{S}_1 \).
So:
\[
e(S_1, \mu) \leq \sharp A' + \sharp B + \sharp \{j\} + \sharp C' \\
\leq \sharp A' + \sharp B + 1 + \sharp C \\
= \sharp A' + \sharp M \\
= e(S, \mu).
\]

If in addition the elements of \(v^{(1)}_{A' \cup B \cup \{j\} \cup C} \) are \(L_1\)-linearly dependents in \(\mathfrak{a}_1 \hat{S}_1 \mathfrak{J}_1 + \mathfrak{a}_1 \hat{S}_1 \mathfrak{J}_2 \), then we have \(e(S_1, \mu) < \sharp A' + \sharp B + \sharp \{j\} + \sharp C' \) and so \(e(S_1, \mu) < e(S, \mu)\). □

**Theorem 8.12.** We have \(r(S_1, v^{(1)}, \mu) \geq r(S, v, \mu)\).

**Proof.** This is induced by the two last points of Proposition 4.5. □

**Corollary 8.13.** Once \(e(S, \mu) = n\), we have
\[
\left( e(S_1, \mu), e(S_1, \mu) - r(S_1, v^{(1)}, \mu) \right) \leq (e(S, \mu), e(S, \mu) - r(S, v, \mu)).
\]
The inequality is strict if \(e(S_1, \mu) < n\).

**Remark 8.14.** We are doing an induction on the dimension \(n\). We saw that this dimension decreases by the sequence of blow-ups.

If it decreases strictly, then it will happen a finite number of time and the proof is done.

Then, after now, we assume this dimension is constant by blow-up. It means that for all framed sequence \(S \rightarrow S_1\), we assume that \(e(S, \mu) = e(S_1, \mu) = n\).

With the same idea, we can assume that \(r(S, v, \mu) = r(S_1, v^{(1)}, \mu)\).
9. Implicit ideal.

Let \((R, \mathfrak{m}, k)\) be a local quasi excellent ring equicharacteristic and let \(\nu\) be a valuation of rank \(1\) of its field of fractions, centered in \(R\) and of value group \(\Gamma_1\). We denote by \(H\) the implicit prime ideal of \(R\) for the valuation \(\nu\).

By Cohen structure Theorem, there exists an epimorphism \(\Phi\) from a complete regular local ring \(A \simeq k[[u_1, \ldots, u_n]]\) of field of fractions \(K\) into \(\frac{R}{I}\). Its kernel \(I\) is a prime ideal of \(A\).

We consider \(\mu\) a monomial valuation with respect to a regular system of parameters of \(A_i\). It is a valuation on \(A\) centered in \(I\). Then we set \(\hat{\nu} := \nu \circ \mu\), hence we define a valuation on \(A\).

Let \(\Gamma\) be the value group of \(\hat{\nu}\).

Then, \(\Gamma_1\) is the smallest non trivial isolated subgroup of \(\Gamma\) and we have:

\[ I = \{ f \in A \text{ such that } \hat{\nu}(f) \notin \Gamma_1 \}. \]

**Definition 9.1.** Let \(\pi\): \((A, u) \rightarrow (A', u')\) be a framed blow-up and \(\sigma\): \(A' \rightarrow \hat{A}'\) be the formal completion of \(A'\). The composition \(\sigma \circ \pi\) is called formal framed blow-up.

A composition of such blow-ups is called a formal framed sequence.

Let \((A, u) \rightarrow (A_1, u^{(1)}) \rightarrow \cdots \rightarrow (A_l, u^{(l)})\) a formal sequence, which we denote by \((\ast)\).

**Definition 9.2.** The formal sequence \((A, u) \rightarrow (A_1, u^{(1)}) \rightarrow \cdots \rightarrow (A_l, u^{(l)})\) is said defined on \(\Gamma_1\) if for every integers \(i \in \{0, \ldots, l-1\}\) and \(q \in J_i\), we have \(\nu\left(\frac{u^{(i)}}{\ell-1}\right) \in \Gamma_1\).

Now we consider \(A_i \simeq k_i[[u_1^{(i)}, \ldots, u_n^{(i)}]]\) and we denote by \(I_i^{\text{strict}}\) the strict transform of \(I\) in \(A_i\).

**Definition 9.3.** We call formal transformed of \(I\) in \(A_i\), which we denote by \(I_i\), the preimage in \(A_i\) of the implicit ideal of \(\frac{A_i}{I_i^{\text{strict}}}\).

Let \(v_i\) the greatest integer of \(\{r, \ldots, n\}\) such that

\[ I_i \cap k_i[[u_1^{(i)}, \ldots, u_n^{(i)}]] = (0) \]

and we set

\[ B_i := k_i[[u_1^{(i)}, \ldots, u_v^{(i)}]]. \]

**Definition 9.4.** Let \(P\) a prime ideal of \(A\). We call \(\ell\)-th symbolic power of \(P\) the ideal \(P^{(\ell)} := (P^\ell A_P) \cap A\).

Equivalently, we have \(P^{(\ell)} = \{ x \in A \text{ such that } \exists y \in A \setminus P \text{ such that } xy \in P^{\ell}\}\).

It is the set composed by the elements which vanish with order at least \(\ell\) in the generic point of \(V(P)\).

Let \(G\) be a complete ring of dimension strictly less than \(n\) and let \(\theta\) be a valuation centered in \(G\), of value group \(\hat{\Gamma}\).

We consider \(\hat{\Gamma}_1\) the first non trivial isolated subgroup of \(\hat{\Gamma}\) and \(\mathfrak{g} := \{ g \in G \text{ such that } \theta(g) \notin \hat{\Gamma}_1\}\).

The next result will help us to prove the simultaneous local uniformization by induction.

**Proposition 9.5.** Assume that:
(1) In the formal sequence \((A, u) \rightarrow (A_1, u^{(1)}) \rightarrow \cdots \rightarrow (A_l, u^{(l)})\), there exists a formal framed sequence

\[ \pi: (A, u) \rightarrow (A_i, u^{(i)}) \]

such that \(v_i < n - 1\).

(2) For every ring \(G\) as above, every element in \(G \setminus g^{(2)}\) is monomializable by a formal framed sequence defined on \(\tilde{\Gamma}_1\).

Then for every element \(f\) of \(A \setminus I^{(2)}\), there exists a formal sequence

\[ (A, u) \rightarrow \cdots \rightarrow (A_l, u^{(l)}) \]

defined on \(\Gamma_1\) such that \(f\) can be written as a monomial in \(u^{(1)}_1, \ldots, u^{(l)}_n\) multiplicated by an element of \(A^\times_i\).

Proof. We assume that there exists a formal framed sequence

\[ \pi: (A, u) \rightarrow (A_i, u^{(i)}) \]

such that \(v_i < n - 1\). It means that \(v_i + 1 < n\). By definition of \(v_i\), we know that \(g_i := I_i \cap k_i\left[u^{(i)}_1, \ldots, u^{(i)}_{v_i+1}\right] \neq (0)\). So we consider an element \(g\) in \(g_i \setminus g^{(2)} \subseteq C_i \setminus g^{(2)}\), where \(C_i := k_i\left[u^{(i)}_1, \ldots, u^{(i)}_{v_i+1}\right]\). Since \(v_i + 1 < n\), the ring \(C_i\) is of dimension strictly less than \(n\). So we can use the second hypothesis on the element \(g\) in the ring \(C_i\).

Hence there exists a formal sequence defined on \(\Gamma_1\)

\[ (C_i, \left[u^{(i)}_1, \ldots, u^{(i)}_{v_i+1}\right]) \rightarrow \cdots \rightarrow (S', \left[u^{(i)}_1, \ldots, u^{(i)}_{v'}\right]) \]

where \(v' \leq v_i + 1\), and such that \(g\) can be written as a monomial in \(u^{(i)}_1, \ldots, u^{(i)}_{v'}\) multiplicated by an element of \(S'^\times\).

Since \(g \in g_i\), there exists a regular parameter of \(S'\), for example \(u^{(i)}_{v'}\), such that \(\nu(u^{(i)}_{v'}) \notin \Gamma_1\). Indeed, \(g \in g_i = I_i \cap C_i\), so \(g \in I_i\) and then it satisfies the hypothesis of \(I\). Equivalently, it satisfies \(\tilde{\nu}(g) \notin \Gamma_1\). Since \(g\) can be written as a monomial in the generators of the maximal ideal of \(S'\), one of these generators which appears in the decomposition of \(g\) must be in \(I\). Hence \(e(S', \tilde{\nu}) < v_i + 1\).

Exchanging every ring \(O\) which appears in

\[ (C_i, \left[u^{(i)}_1, \ldots, u^{(i)}_{v_i+1}\right]) \rightarrow \cdots \rightarrow (S', \left[u^{(i)}_1, \ldots, u^{(i)}_{v'}\right]) \]

by \(O\left[u^{(i)}_{v_i+2}, \ldots, u^{(i)}_n\right]\), we obtain a formal sequence

\[ \pi': \left(A_i, u^{(i)}\right) \rightarrow \cdots \rightarrow \left(A_l, u^{(l)}\right) \]

independent of \(u^{(i)}_{v_i+2}, \ldots, u^{(i)}_n\), with \(A_l = S'\left[u^{(i)}_{v_i+2}, \ldots, u^{(i)}_n\right]\). But we know that \(e(S', \tilde{\nu}) < v_i + 1\), and so \(e(A_l, \tilde{\nu}) < n\).

Let \(f\) be an element of \(A \setminus I^{(2)}\). Its image by \(\pi' \circ \pi\) is an element of \(A_l\), which dimension is strictly less than \(n\). Since all the \(A_l\) are quasi excellent, we have \(f \notin A_l \setminus I^{(2)}\) and we can use again the second hypothesis. Hence we constructed a formal sequence \(\pi' \circ \pi\) such that \(f\) can be written as a monomial in the generators of \(A_l\) multiplicated by a unit of \(A_l\). This concludes the proof. \(\square\)
Now, we assume that for every formal sequence \((A, u) \rightarrow (A_1, u^{(1)}) \rightarrow \cdots \rightarrow (A_i, u^{(i)})\) and for every integer \(i\), we have \(v_i \in \{n - 1, n\}\).

So for every integer \(i\), we have \(I_i \cap k_i \left[ \left[ u_1^{(i)}, \ldots, u_{n-1}^{(i)} \right] \right] = (0)\).

We consider \(G\) a complete local ring of dimension strictly less than \(n\) and a valuation \(\theta\) of rank 1 centered in \(G\).

**Lemma 9.6.** Assume that for every ring \(G\) as above, there exists a formal framed sequence which monomialize every element of \(G\).

Then \(I\) is at most of height 1.

**Proof.** If \(I = (0)\), it is done. So we assume \(I ≠ (0)\) and we consider \(f \in I \setminus \{0\}\). We write

\[ f = \sum_{j=0}^\infty a_j u_n^j \]

with \(a_j \in k[[u_1, \ldots, u_{n-1}]]\). We consider an integer \(N\) big enough such that every \(a_j\) with \(j > N\) is in the ideal generated by \((a_0, \ldots, a_N)\). Now let us consider

\[ \delta := \min \left\{ j \in \{0, \ldots, N\} \text{ such that } \nu(a_j) = \min_{0 ≤ s ≤ N} \{\nu(a_s)\} \right\}. \]

We set \(\overline{\nu} := (u_1, \ldots, u_{n-1})\) and \(B := k[[\overline{\nu}]]\). Since \(B\) is a complete ring of dimension strictly less than \(n\), by hypothesis we can construct a formal sequence \((B, \overline{\nu}) \rightarrow (B', \overline{\nu}')\) such that for every \(j \in \{0, \ldots, N\}\), the element \(a_j\) is a monomial in \(\overline{\nu}'\). By Propositions 4.4 and 4.7, we can construct a formal sequence \((B', \overline{\nu}') \rightarrow (B'', \overline{\nu}'')\) such that \(a_\delta | a_j\) for every \(j \in \{0, \ldots, N\}\) in \(B''\), since \(a_\delta\) has minimal value. So we have a sequence

\[ (B, \overline{\nu}) \rightarrow (B', \overline{\nu}') \rightarrow (B'', \overline{\nu}''). \]

We compose with the formal completion and obtain

\[ (B, \overline{\nu}) \rightarrow \left( \hat{B'}', \overline{\nu}' \right) \]

in which we still have \(a_\delta | a_j\) for every \(j \in \{0, \ldots, N\}\).

We exchange again all the rings \(O\) of the sequence \((B, \overline{\nu}) \rightarrow \left( \hat{B'}', \overline{\nu}' \right)\) by \(O[[u_n]]\), and obtain a sequence \((A, u) \rightarrow (A', u')\) independent of \(u_n\) and in which we still have \(a_\delta | a_j\) for every \(j \in \{0, \ldots, N\}\).

We recall that for every index \(i\), we have

\[ I_i \cap k_i \left[ \left[ u_1^{(i)}, \ldots, u_{n-1}^{(i)} \right] \right] = (0). \]

If we denote by \(I'\) the formal transformed of \(I\) in \(A'\), we obtain \(I' \cap \hat{B''} = (0)\). We know that \(\frac{d}{a_\delta} \in I'\), and by Weierstrass preparation Theorem, \(\frac{d}{a_\delta} = xy\) where \(x\) is an invertible formal series in \(k[[u_1, \ldots, u_{n-1}]][[u_n]]\). It means that \(x\) is a unit of \(A'\), and \(y\) is a monic polynomial in \(u_n\) of degree \(\delta\). Then the morphism \(\hat{B''} \rightarrow \frac{A'}{I'}\) is injective and finite.

Hence \(\dim \left( \frac{A'}{I'} \right) = \dim \left( \hat{B''} \right) = n - 1\). Since \(\dim(A') = n\), we have \(\text{ht}(I') ≤ \text{ht}(I') = \dim(A') - \dim \left( \frac{A'}{I'} \right) = n - (n - 1) = 1\), and we are done. \(\square\)
Corollary 9.7 (of Lemma 9.6). We keep the same hypothesis that in Lemma 9.6. Let $I = (h)$. There exists a formal framed sequence $(A, u) \to (A', u')$ such that in $A'$, the strict transform of $h$ is a monic polynomial of degree $\delta$.

Since now, we assume that $h$ is a monic polynomial of degree $\delta$.

Proposition 9.8. We keep the same hypothesis that in Lemma 9.6. Let $I = (h)$. The polynomial $h$ is a key polynomial.

Proof. By definition, $I = \{ f \in A$ such that $\hat{\nu} (f) \notin \Gamma_1 \}$, so $\hat{\nu} (h) \notin \Gamma_1$. Furthermore, for every non zero integer $b$, we have $\hat{\nu} (\partial_b h) \in \Gamma_1$ since $h$ is of minimal degree to generate $I$ and so $\partial_b h \notin I$.

Then $\epsilon (h) \notin \Gamma_1$.

Let $P$ be a polynomial such that $\deg (P) < \deg (h)$. To show that $h$ is a key polynomial, we only have to prove that $\epsilon (P) < \epsilon (h)$.

By minimality of $\deg (h)$, we still have $P \notin I$ and so $\hat{\nu} (P) \in \Gamma_1$. So for every non zero integer $b$, we also have $\hat{\nu} (\partial_b P) \in \Gamma_1$. Then $\epsilon (P) \in \Gamma_1$.

Assume by contradiction that $\epsilon (P) \geq \epsilon (h)$.

Then $- \epsilon (P) \leq \epsilon (h) \leq \epsilon (P)$ and since $\Gamma_1$ is an isolated subgroup, $\Gamma_1$ is a segment and so $\epsilon (h) \in \Gamma_1$. Contradiction.

Hence, $\epsilon (P) < \epsilon (h)$ and $h$ is a key polynomial.

Now we are going to monomialize the key polynomial $h$.

As in the previous part, we construct a sequence of optimal (eventually limit) immediate successors which begins at $x$ and ends at $h$. So since $\epsilon (h)$ is maximal in $\epsilon (\Lambda)$, we stop. Then we have a finite sequence $Q = (Q_i)$ of optimal (eventually limit) immediate successors which ends at $h$.

In the case $I = (0)$, we construct again a sequence $Q = (Q_i)$ of optimal (eventually limit) immediate successors such that $\epsilon (Q)$ is cofinal in $\epsilon (\Lambda)$.

Since we don’t assume $k = k_{\nu}$ in this part, we need a generalisation of the monomialization Theorems of the previous part.
10. Monomialization of the key polynomials.

Here we consider the ring $A \cong k[[u_1, \ldots, u_n]]$ and a valuation $\nu$ centered in $A$ of value group $\Gamma$. For more clarity, we recall some previous notations.

Let $r$ be the dimension of $\sum_{i=1}^{n} \nu(u_i)/\mathbb{Q}$ in $\Gamma \otimes \mathbb{Z} \mathbb{Q}$. Even if it means to renumber, we assume that $\nu(u_1), \ldots, \nu(u_r)$ are rationally independents and we consider $\Delta$ the subgroup of $\Gamma$ generated by $\nu(u_1), \ldots, \nu(u_r)$.

We set $E := \{1, \ldots, r, n \}$ and

$$\alpha^{(0)} := \min_{\alpha \in \mathbb{N}^*} \{ \alpha \text{ such that } \alpha \nu(u_n) \in \Delta \}.$$

So $\alpha^{(0)} \nu(u_n) = \sum_{j=1}^{r} \alpha_j^{(0)} \nu(u_j)$ with

$$\alpha_1^{(0)}, \ldots, \alpha_s^{(0)} \geq 0$$

and

$$\alpha_{s+1}^{(0)}, \ldots, \alpha_r^{(0)} < 0.$$

We set

$$w = (w_1, \ldots, w_r, w_n) = (u_1, \ldots, u_r, u_n)$$

and

$$v = (v_1, \ldots, v_t) = (u_{r+1}, \ldots, u_{n-1}),$$

with $t = n - r - 1$.

We note $x_i = \ln_k u_i$, and so $x_1, \ldots, x_r$ are algebraically independents over $k$ in $G_\nu$. Let $\lambda_0$ be the minimal polynomial of $x_n$ over $k[x_1, \ldots, x_r]$, of degree $\alpha$. If $x_n$ is transcendental, we set $\lambda_0 := 0$.

We consider

$$y = \prod_{j=1}^{r} x_j^{\alpha_j^{(0)}},$$

$$\overline{y} = \prod_{j=1}^{r} w_j^{\alpha_j^{(0)}},$$

$$z = \frac{x_n^{\alpha_n^{(0)}}}{y},$$

and

$$\overline{z} = \frac{u_n^{\alpha_n^{(0)}}}{\overline{y}}.$$

Let $d_0 := \frac{\alpha_0^{(0)}}{\alpha_n^{(0)}} \in \mathbb{N}$.

If $\lambda_0 \neq 0$, we have

$$\lambda_0 = \sum_{q=0}^{d_0} c_q y^{d_0-q} X^{q^{\alpha_n^{(0)}}}$$

where $c_q \in k$, $c_d = 1$ and $\sum_{q=0}^{d_0} c_q z^q$ is the minimal polynomial of $z$ over $G_\nu$.

We are going to show that there exists a formal framed sequence which monomialize all the $Q_i$. We have $Q_1 = u_n$ so we have to begin monomializing $Q_2$. 
First, let us consider
\[ Q = \sum_{q=0}^{d_0} a_q b_q y_{d_0-q} w_n^{a_q(0)} \]
where \( b_q \in \mathbb{R} \) such that \( b_q \equiv c_q \) modulo \( m \) and \( a_q \in A^\times \).

Then we will show that we can bring us to this case.

Let
\[ \gamma = (\gamma_1, \ldots, \gamma_r, \gamma_n) = (\alpha_1^{(0)}, \ldots, \alpha_s^{(0)}, 0, \ldots, 0) \]
and
\[ \delta = (\delta_1, \ldots, \delta_r, \delta_n) = (0, \ldots, 0, -\alpha_{s+1}^{(0)}, \ldots, -\alpha_r^{(0)}, \alpha_0^{(0)}). \]

We have
\[ w_\delta = w_\delta^\gamma \prod_{j=1}^{r} w_j^{\delta_j} = \frac{w_n^{\alpha(n)}}{\prod_{j=s+1}^{r} w_j^{\alpha_j(n)}} \]
and
\[ w_\gamma = \prod_{j=1}^{s} w_j^{\alpha_j(n)}. \]

So \( \frac{w_\delta}{w_\gamma} = \frac{w_n^{\alpha(n)}}{\prod_{j=s+1}^{r} w_j^{\alpha_j(n)}} = \pi. \)

Let us compute the value of \( w_\delta. \)

\[ \nu(w_\delta) = \alpha_0^{(0)} \nu(w_n) - \sum_{j=s+1}^{r} \alpha_j^{(0)} \nu(w_j) \]
\[ = \alpha_0^{(0)} \nu(u_n) - \sum_{j=s+1}^{r} \alpha_j^{(0)} \nu(u_j) \]
\[ = \sum_{j=1}^{r} \alpha_j^{(0)} \nu(u_j) - \sum_{j=s+1}^{r} \alpha_j^{(0)} \nu(u_j) \]
\[ = \sum_{j=1}^{s} \alpha_j^{(0)} \nu(u_j) \]
\[ = \nu(w_\gamma). \]

**Theorem 10.1.** There exists a local framed sequence

\[ (A, w) \xrightarrow{\pi_0} (A_1, u^{(1)}) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{l-1}} (A_l, u^{(l)}) \]

with respect to \( \nu \), independent of \( v \), which satisfies next properties:
For every integer \( i \in \{1, \ldots, l\} \), we note \( u^{(i)} = (u_1^{(i)}, \ldots, u_n^{(i)}) \) and \( k_i \) the residual field of \( A_i \).

1. The blow-ups \( \pi_0, \ldots, \pi_{l-2} \) are monomials.
2. We have \( \pi \in A_l^\times. \)
3. We have

\[ n_i = \begin{cases} 
  n & \text{if } \lambda \neq 0 \\
  n-1 & \text{otherwise.}
\end{cases} \]
(4) We set
\[ u^{(l)} = \begin{cases} 
(w_1^{(l)}, \ldots, w_r^{(l)}, v, w_n^{(l)}) & \text{if } \lambda \neq 0 \\
(w_1^{(l)}, \ldots, w_r^{(l)}, v) & \text{otherwise.}
\end{cases} \]

For every integer \( j \in \{1, \ldots, r, n\} \), \( w_j \) is a monomial in \( w_1^{(l)}, \ldots, w_r^{(l)} \) multiplied by an element of \( A_i^\times \). And for every integer \( j \in \{1, \ldots, r\} \), \( w_j^{(l)} = w^\eta \) where \( \eta \in \mathbb{Z}^{r+1} \).

(5) If \( \lambda_0 \neq 0 \), then \( Q = w_n^{(l)} \times \bar{y}^{\lambda_0} \).

**Proof.** We apply Proposition 4.4 to \( (w^\delta, w^\gamma) \) and obtain a local framed sequence for \( \nu \), independent of \( v \), such that \( w^\gamma | w^\delta \) in \( A_i \).

By Proposition 4.4 and by the fact that \( w^\delta \) and \( w^\gamma \) have same value, we have \( w^\delta | w^\gamma \) in \( R_i \). In fact \( \bar{z}, \bar{z}^{-1} \in A_i^\times \). So we have the point (2).

We choose the sequence to be minimal, it means that the sequence composed by \( \pi_0, \ldots, \pi_{i-2} \) does not satisfy the conclusion of Proposition 4.4 for \( (w^\delta, w^\gamma) \). We are now going to show that this sequence satisfies the properties of Theorem 10.1.

Let \( i \in \{0, \ldots, l\} \). We write \( w^{(i)} = (w_1^{(i)}, \ldots, w_r^{(i)}, w_n^{(i)}) \), with \( r_i = n_i - 1 > 0 \). For every integers \( i \in \{1, \ldots, l\} \) and \( j \in \{1, \ldots, n_i\} \), we write \( \beta_j^{(i)} = \nu \left( u_j^{(i)} \right) \).

For all \( i < l \), \( \pi_i \) is a blow-up along an ideal of the form \( \left( u_j^{(i)} \right) \). Even if it means to renumber, we can assume that \( 1 \in J_i \) and that \( A_{i+1} \) is a localisation of \( A_i \left[ \frac{u_i^{(i)}}{u_i^{(i)}} \right] \). Hence, \( \beta_1^{(i)} = \min \left\{ \beta_j^{(i)} : j \in J_i \right\} \).

**Lemma 10.2.** Let \( i \in \{0, \ldots, l-1\} \). We assume that the sequence \( \pi_0, \ldots, \pi_{i-1} \) of Theorem 10.1 is monomial.

We write \( w^\gamma = (w^{(i)})^{\gamma^{(i)}} \) and \( w^\delta = (w^{(i)})^{\delta^{(i)}} \). Then:

1. \( r_i = r \),
2. \[ \sum_{q \in \mathcal{A}} (\gamma_q^{(i)} - \delta_q^{(i)}) \beta_q^{(i)} = 0 \]
3. \( \gcd (\gamma_1^{(i)} - \delta_1^{(i)}, \ldots, \gamma_r^{(i)} - \delta_r^{(i)}, \gamma_n^{(i)} - \delta_n^{(i)}) = 1 \),
4. Every \( \mathbb{Z} \)-linear dependence relation between \( \beta_1^{(i)}, \ldots, \beta_r^{(i)}, \beta_n^{(i)} \) is an integer multiple of (10.2).

**Proof.**

1. It is enough to do an induction on \( i \) and use Remark 3.6.
2. We have \( \nu (w^\gamma) = \nu (w^\delta) \), it means that \( \nu ((w^{(i)})^{\gamma^{(i)}}) = \nu ((w^{(i)})^{\delta^{(i)}}) \).

Since \( w^{(i)} = (w_1^{(i)}, \ldots, w_r^{(i)}, w_n^{(i)}) \), we have:

\[ \nu \left( \prod_{j=1}^{r_i} (w_j^{(i)})^{\gamma_j^{(i)}} \times (w_n^{(i)})^{\gamma_n^{(i)}} \right) = \nu \left( \prod_{j=1}^{r_i} (w_j^{(i)})^{\delta_j^{(i)}} \times (w_n^{(i)})^{\delta_n^{(i)}} \right). \]
So we have
\[
\sum_{j=1}^{r_i} \gamma_j(w_j^{(i)}) = \sum_{j=1}^{r_i} \gamma_j(w_j^{(i)}) + \delta_j^{(i)}(w_{n_i}^{(i)}) = \sum_{j=1}^{r_i} \gamma_j^{(i)}(w_j^{(i)}) + \delta_j^{(i)}(w_{n_i}^{(i)}).
\]

By definition of \( w^{(i)} \), for every integer \( j \in \{1, \ldots, r_i, n_i\} \), we have \( w_j^{(i)} = u_j^{(i)} \). So \( \nu(w_j^{(i)}) = \beta_j^{(i)} \). Then:
\[
\sum_{j=1}^{r_i} \gamma_j^{(i)} \beta_j^{(i)} + \gamma_{n_i}^{(i)} \beta_{n_i}^{(i)} = \sum_{j=1}^{r_i} \delta_j^{(i)} \beta_j^{(i)} + \delta_{n_i}^{(i)} \beta_{n_i}^{(i)}.
\]

Hence
\[
\sum_{j=1}^{r_i} \left( \gamma_j^{(i)} - \delta_j^{(i)} \right) \beta_j^{(i)} = 0.
\]

But \( r_i = n_i - t - 1 = r \), so \( n_i = r + t + 1 = n \), and:
\[
\sum_{j=1}^{r_i} \left( \gamma_j^{(i)} - \delta_j^{(i)} \right) \beta_j^{(i)} = \sum_{j \in \{1, \ldots, r, n_i\}} \left( \gamma_j^{(i)} - \delta_j^{(i)} \right) \beta_j^{(i)} = \sum_{j \in \{1, \ldots, r, n_i\}} \left( \gamma_j^{(i)} - \delta_j^{(i)} \right) \beta_j^{(i)} = 0.
\]

(3) Same proof than in Theorem 6.4.
(4) Same proof than in Theorem 6.4.

\[\square\]

Lemma 10.3. The sequence \((A, u) \xrightarrow{\pi_0} (A_1, u^{(1)}) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{l-1}} (A_l, u^{(l)})\) of Theorem 10.1 is not monomial.

Proof. Same proof than Lemma 6.7.

Lemma 10.4. Let \( i \in \{0, \ldots, l-1\} \) and we assume that \( \pi_0, \ldots, \pi_{i-1} \) are all monomials. Then following properties are equivalent:

1. The blow-up \( \pi_i \) is not monomial.
2. There exists an unique index \( q \in J_i \setminus \{1\} \) such that \( \beta_q^{(i)} = \beta_1^{(i)} \).
3. We have \( i = l - 1 \).

Proof. Same proof than Lemma 10.8.

By doing an induction on \( i \) and using Lemma 10.3 we conclude that \( \pi_0, \ldots, \pi_{l-2} \) are monomials. So we do have the first point of the Theorem.

We now have to prove the three last points.

By Lemma 10.4 we know that there exists a unique element \( q \in J_{l-1} \setminus \{j_{l-1}\} \) such that \( \beta_q^{(l-1)} = \beta_{j_{l-1}}^{(l-1)} \), hence we are in the case \( x B_{l-1} + 1 = x J_{l-1} - 1 \). We know have to see if \( t_{k_{l-1}} = 0 \) or \( 1 \).

We recall that \( u_1^{(l-1)} = u^\epsilon \) and \( u_q^{(l-1)} = w^\mu \) where \( \epsilon \) and \( \mu \) are two colons of an unimodular matrix such that \( \mu - \epsilon = \pm(\gamma - \delta) \). So \( x_1^{(l-1)} = x^\epsilon \) and \( x_q^{(l-1)} = x^\mu \), then
\[
\frac{x_q^{(l-1)}}{x_1^{(l-1)}} = x^{\mu - \epsilon} = x^{\pm(\gamma - \delta)} = x^{\pm(\alpha_0^{(0)}, \ldots, \alpha_l^{(0)}, \alpha_{l+1}^{(0)})}.
\]
It means that
\[
x_q^{(l-1)} \frac{x_j^{(l-1)}}{x_i^{(l-1)}} = \left( \prod_{j=1}^{r} x_j^{(0)} \right)^{\pm 1} = (z^{-1})^{\pm 1} = z^{\pm 1}.
\]
So we can assume \( x_q^{(l-1)} \frac{x_j^{(l-1)}}{x_i^{(l-1)}} = z \).

The case \( t_{k^{-1}} = 1 \) corresponds to the fact that \( z \) is transcendental over \( k \), so once \( \lambda = 0 \). The case \( t_{k^{-1}} = 0 \) corresponds to the fact that \( z \) is algebraic over \( k \), and so once \( \lambda_0 \neq 0 \). The third point of the Theorem is then a consequence of 3.9.

Since \( \beta_1^{(l-1)}, \ldots, \beta_r^{(l-1)} \) are linearly independents, we have \( q = n \). By 3.9 if \( \lambda_0 \neq 0 \), we have
\[
w_n^{(l)} = u_n^{(l)} = \lambda_0(u'_n) = \lambda_0 \left( \frac{u_n^{(l-1)}}{u_1^{(l-1)}} \right) = \lambda_0 \left( \frac{w_n^{(l-1)}}{w_1^{(l-1)}} \right) = \lambda_0(z) = \sum_{i=0}^{d} a_i b_i z^i.
\]

Remark 10.5. We have \( \lambda_0(z) = \sum_{i=0}^{d} c_i b_i z^i \) where \( c_i \) are some units. Then we choose to set \( c_i = a_i \) for each index \( i \).

But since \( z = \frac{w_0^{(0)} y}{y_0} \), we have
\[
w_n^{(l)} = \sum_{i=0}^{d} a_i b_i \left( \frac{w_0^{(0)}}{y} \right)^i = \sum_{i=0}^{d} a_i b_i y^{d_0-i} \left( \frac{w_0^{(0)}}{y} \right)^i \frac{Q}{y^{d_0}}
\]
and the last point is proven.

So now we just have to prove the point (4).

We apply Proposition 1.5 to \( i = 0 \) and \( i = l \). By monomiality of \( \pi_0, \ldots, \pi_{l-2} \), we know that \( D_i = \{1, \ldots, n\} \) for every \( i \in \{1, \ldots, l-1\} \).

We know that \( D_i = \{1, \ldots, n\} \) if \( \lambda \neq 0 \) and \( D_i = \{1, \ldots, n-1\} \) otherwise. Here we set again \( u_T = v \).

For every \( j \in \{1, \ldots, r, n\} \), the fact that \( w_j = u_j \) is a monomial in \( w_1^{(l)}, \ldots, w_r^{(l)} \), it means that it is one in \( u_1^{(l)}, \ldots, u_r^{(l)} \), multiplicated by an element of \( A_1^{(l)} \) is a consequence of Proposition 1.5.

Same thing for the fact that for every integer \( j \in \{1, \ldots, r\} \), we have \( w_j^{(l)} = w^n \). This concludes the proof. \( \square \)

Remark 10.6. In the case \( Q_2 = Q \), the we constructed a local framed sequence such that the total transform of \( Q_2 \) is a monomial. We will bring us to this case.

Definition 10.7. \([8]\) A local framed sequence which satisfies Theorem [10.1] is called a \( n \)-generalized Puiseux package.

Let \( j \in \{r+1, \ldots, n\} \). A \( j \)-generalized Puiseux package is a \( n \)-generalized Puiseux package exchanging \( n \) by \( j \) in Theorem [10.1].

Remark 10.8. We consider \( (A, u) \to \cdots \to (A_i, u^{(i)}) \to \cdots \) a \( j \)-generalized Puiseux package, with \( j \in \{r+1, \ldots, n\} \). We exchange each ring of this sequence by its formal completion, hence we obtain a formal framed sequence that we call a formal
j-Puiseux package. So Theorem 10.1 induces a formal n-Puiseux package which satisfies the same conclusion than in Theorem 10.1.

Since we want to to an induction, now we will assume until the end of Theorem 10.14 that we know how to monomialize every complete local equicharacteristic quasi excellent ring $G$ of dimension strictly less than $n$ which has a valuation of rank 1 centered in $G$ by a formal framed sequence. This hypothesis is called $H_n$.

**Lemma 10.9.** Let $P = \sum_{j \in S_n(P)} c_j u_n^j$ the $u_n$-expansion of an optimal immediate successor key element of $u_n$.

There exists a formal framed sequence $(A, u) \to (A_1, u^{(l)})$ which transform each coefficient $c_j$ in a monomial in $(u_1^{(l)}, \ldots, u_n^{(l)})$, multiplied by a unit of $A_1$.

Hence, after this sequence, $P$ can be written like $\sum_{i=0}^{\infty} a_i b_i \tilde{y}^{i-1} \left( w_n^{(l)} \right)^i$.

**Proof.** We will prove a more general result in 10.12. \hfill \box

**Theorem 10.10.** If $u_n \ll_{\text{lim}} P$, then $P$ is monomializable.

**Proof.** Same proof than Theorem 6.14. \hfill \box

**Lemma 10.11.** There exists a formal framed sequence $(A, u) \to (A_1, u^{(l)})$ such that in $A_1$, the strict transform of the polynomial $Q_2$ is a monomial.

**Proof.** If $u_n < Q_2$, we use Lemma 10.9 and Theorem 10.1 to conclude. Otherwise, $u_n \ll_{\text{lim}} Q_2$ and so we use Theorem 6.14. \hfill \box

We constructed a formal framed sequence which monomialize $Q_2$. But we want one which monomialize all the key polynomials of $Q$.

Now we are going to show that if we constructed a formal framed sequence $(A, u) \to (A_1, u^{(l)})$ which monomialize $Q_i$, then we can associate another $(A_i, u^{(l)}) \to (A_s, u^{(s)})$ such that in $A_s$, the strict transform of $Q_{i+1}$ is also a monomial.

Let $\Delta_i$ be the group $\nu \left( k_1 \left( u_1^{(l)}, \ldots, u_{n-1}^{(l)} \right) \setminus \{0\} \right)$ and

$$\alpha_i := \min \left\{ h \text{ such that } h \tilde{y}_n^{(l)} \in \Delta_i \right\}.$$  

We set $X_j = \text{in}_\nu \left( u_j^{(l)} \right)$, $W_j = u_j^{(l)}$ and $\lambda_i$ the minimal polynomial of $X_n$ over $\text{gr}_\nu k_1 \left( u_1^{(l)}, \ldots, u_{n-1}^{(l)} \right)$ of degree $\alpha_i$.

We know that $Q_i = \tilde{w} w_n^{(l)}$ with $\tilde{w}$ a monomial in $W_1, \ldots, W_r$ multiplied by a unit. We set $\omega := \text{in}_\nu (\tilde{w})$.

If $Q_i \ll_{\text{lim}} Q_{i+1}$, we use Theorem 10.14 and it is over. So we assume that $Q_{i+1}$ is an optimal immediate successor of $Q_i$.

We write $Q_{i+1} = \sum_{j \in S_{Q_i(Q_{i+1})}} a_j Q_i^j = \sum_{j=0}^{s} a_j Q_i^j$ the $Q_i$-expansion of $Q_{i+1}$ in $k_1 \left( u_1^{(l)}, \ldots, u_{n-1}^{(l)} \right) \left( w_n^{(l)} \right)$.
Lemma 10.12. We assume that for every ring $Q$ and so all the non zero terms of the monomializable.

The maximal ideal of $n$ with a valuation centered in $u$ such that each $\nu \left( a_j Q_i \right) = \nu Q_i \left( Q_{i+1} \right)$.

So all non zero terms of the $Q_i$-expansion of $Q_{i+1}$ have same value. Then, by hypothesis $H_n$, all these terms are divisible by the same power of $\mathfrak{m}$ after an appropriate sequence of blow-ups $(*)_i$ independent of $u_n(\ell)$.

We denote by $Q_{i+1}$ the strict transform of $Q_{i+1}$ by the composition of $(*)_i$ with the sequence $(*)_i^0$ monomialize $Q_i$. We denote this composition by $(c_i)$.

We know that $Q_i$, the strict transform of $Q_i$ by $(c_i)$, is a regular parameter of the maximal ideal of $A_i$. Indeed, by Proposition 4.5, we know that each $u_j$ of $A$ can be written as a monomial on $w_1(\ell), \ldots, w_l(\ell)$. In fact, the reduced exceptional divisor of this sequence is exactly $V(\mathfrak{m})_{\text{red}}$. Hence, as we know that $Q_i = w_n(\ell)\mathfrak{m}$, we do have that the strict transform of $Q_i$ is $Q_i = w_n(\ell) = u_n(\ell)$. So it is a key polynomial in the extension $k_i \left( u_1(\ell), \ldots, u_n(\ell) \right)$.

Let us show that $Q_{i+1} = \frac{Q_{i+1}}{Q_i}$.

We have $a_1 = 1$ and $Q_i = \mathfrak{m} \left( u_n(\ell) \right)$ and also $u_n(\ell) \mathfrak{m}$, so $\mathfrak{m}$ divides the term $a_1 Q_i^s$ and so all the non zero terms of the $Q_i$-expansion of $Q_{i+1}$. Furthermore, it is the biggest power of $\mathfrak{m}$ which divides each term, hence $Q_{i+1} = \frac{Q_{i+1}}{Q_i} \left( u_n(\ell) \right) + \frac{a_1 \mathfrak{m}}{Q_i} \left( u_n(\ell) \right)^{s-1} + \cdots + \frac{a_0 \mathfrak{m}}{Q_i}$ is $Q_{i+1}$ the strict transform of $Q_{i+1}$ by the sequence of blow-ups, which satisfies $Q_i \ll Q_{i+1}$ by hypothesis.

Let $G$ be a complete local equicharacteristic ring of dimension strictly less than $n$ with a valuation centered in $G$.

Lemma 10.12. We assume that for every ring $G$ as above, every element of $G$ is monomializable.

Assume that $Q_i < Q_{i+1}$ in $Q$.

Then there exists a local framed sequence $(A_i, u(\ell)) \rightarrow (A_e, u(\ell))$ such that in $A_e$, the strict transform of $Q_{i+1}$ is of the form $\sum_{q=0}^{\infty} \tau_q \eta_q X_n^q$

where $\tau_q \in R_e^x$ and $\eta_q$ are monomials in $u_1(\ell), \ldots, u_r(\ell)$.

Proof. By hypothesis, after a sequence of blow-ups independent of $u_n(\ell)$, we can monomialize the $a_i$ and assume that they are monomials in $\left( u_1(\ell), \ldots, u_{n-1}(\ell) \right)$ multiplated by units of $A_i$.

For every $g \in \{ r+1, \ldots, n-1 \}$, we do a generalized $g$-Puiseux package as in Theorem 10.1 hence we have a sequence $(A_i, u(\ell)) \rightarrow (A_{e}, u(\ell))$ such that each $u_g(\ell)$ is a monomial in $\left( u_1(\ell), \ldots, u_r(\ell) \right)$.

In fact we can assume that the $a_j$ are monomials in $\left( u_1(\ell), \ldots, u_r(\ell) \right)$ multiplated by units of $A_i$. 
Theorem 10.14. We still assume 10.13
Remark ˜ is an immediate successor key element of \( Q \) more generally according to (10.3).

Proof. There are two cases.

Then the strict transform \( Q_{i+1} \) of \( Q_i \) by the sequence \( (A, u) \to (A_i, u^{(i)}) \) which monomialize \( Q_i \) is an immediate successor key element of \( \tilde{Q}_i = u_{(i)}^n \), and by Lemma 10.12 we just saw that we can bring us to the hypothesis of Theorem 10.1. So we use Theorem 10.1 exchanging \( Q_i \) by \( \tilde{Q}_i \) and \( Q_2 \) by \( Q_{i+1} \).

The last one: \( Q_i \leq \lim \tilde{Q}_{i+1} \).

We apply Theorem 10.10 exchanging \( u_i \) by \( \tilde{Q}_i \) and \( P \) by \( \tilde{Q}_{i+1} \).

As in the previous part, we consider, for every integer \( j \), the countable sets

\[
\mathcal{S}_j := \left\{ \prod_{i=1}^{n} \left( u_i^{(j)} \right)^{\alpha_i^{(j)}}, \text{ with } \alpha_i^{(j)} \in \mathbb{Z} \right\}
\]

and

\[
\tilde{\mathcal{S}}_j := \{(s_1, s_2) \in \mathcal{S}_j \times \tilde{\mathcal{S}}_j, \text{ with } \nu(s_1) \leq \nu(s_2)\}
\]

assuming that for every \( i \in \{1, \ldots, n\} \), \( u_i^{(0)} = u_i \).

The set \( \mathcal{S}_j \) is countable for every \( j \), so we can number its elements, and set \( \tilde{\mathcal{S}}_j := \left\{ s_m^{(j)} \right\}_{m \in \mathbb{N}} \). Now we consider the finite set

\[
\tilde{\mathcal{S}}_j' := \left\{ s_m^{(j)}, m \leq j \right\} \cup \left\{ s_m^{(j)}, m \leq j \right\}.
\]

Hence \( \bigcup_{j \in \mathbb{N}} (\mathcal{S}_j \times \tilde{\mathcal{S}}_j) = \bigcup_{j \in \mathbb{N}} \tilde{\mathcal{S}}_j = \bigcup_{j \in \mathbb{N}} \tilde{\mathcal{S}}_j' \) is a countable union of finite sets.

Since we consider all the elements according uniquely to the variable \( u_i \), and more generally according to \( u_i^{(j)} \), and since we do an induction on the dimension, we have to know how to monomialize the elements of \( B_i := k \left[ u_i^{(1)}, \ldots, u_i^{(n-1)} \right] \).

Theorem 10.15. Let \( A \simeq k \left[ [u_1, \ldots, u_n] \right] \) with a valuation \( \nu \) centered in \( A \).

We recall that \( \text{car}(k_i) = 0 \). There exists a formal sequence

\[
(A, u) \xrightarrow{\pi_0} \cdots \xrightarrow{\pi_{i-1}} (A_i, u^{(i)}) \xrightarrow{\pi_i} \cdots
\]

which monomialize all the key polynomials of \( Q \) and all the elements of the \( B_i \) for all \( i \). Furthermore, the sequence satisfies the property:

\[
\forall j \in \mathbb{N} \forall s = (s_1, s_2) \in \mathcal{S}_j \exists i \in \mathbb{N}_{\geq j} \text{ such that } s_1 | s_2 \text{ in } A_i.
\]
It means that for every index \( l \), there exists an index \( p_l \) such that in \( A_{p_l} \), \( Q_l \) is a monomial in \( u^{(p_l)} \) multiplicated by a unit of \( A_{p_l} \).

**Proof.** To show that we can choose the sequence \((10.4)\) such that

\[
\forall j \in \mathbb{N} \forall s = (s_1, s_2) \in \mathcal{L}' \exists i \in \mathbb{N}_{\geq j} \text{ such that } s_1 \mid s_2 \text{ in } A_i,
\]

and that all the elements of the \( B_i \) are monomialized, we do the same thing than in Theorem 6.21.

Then we do an induction on the dimension \( n \) and on the index \( i \) and we iterate the previous processus.

**Corollary 10.16.** Let \( A \simeq k[[u_1, \ldots, u_n]] \) with a valuation \( \hat{\nu} \) centered in \( A \), of value group \( \Gamma \). We assume

\[
I = \{ a \in A \text{ such that } \hat{\nu}(a) \notin \Gamma_1 \} = (h) \neq (0),
\]

where \( \Gamma_1 \) is the smallest isolated subgroup of \( \Gamma \). We recall that \( \text{car}(k_{\nu}) = 0 \).

There exists a formal framed sequence

\[
(A, u) \rightarrow \cdots \rightarrow (A_l, u^{(l)}) \rightarrow \cdots
\]

such that in \( A_l \), the polynomial \( h \) can be written as a monomial multiplicated by a unit.

**Proof.** The sequence \( Q \) has been constructed to contain \( h \), so we just have to use Theorem 10.15. \( \square \)
11. Reduction.

Let \((R, \mathfrak{m}, k)\) be a local quasi excellent equicharacteristic ring and let \(\nu\) be a valuation of its field of fractions, of rank 1, centered in \(R\) and of value group \(\Gamma_1\). We denote by \(\overline{H}\) the implicit ideal of \(R\).

We are going to see that in this case, we just have to regularise \(R/\overline{H}\).

We consider \(\mathcal{F} := \{f_1, \ldots, f_s\} \subseteq \mathfrak{m}\), and assume that \(f_1\) has minimal value.

**Remark 11.1.** We consider \(R \to \hat{R} \to R_1 \to \hat{R}_1\) a formal framed blow-up and we denote by \(H'\) the strict transformed of \(\overline{H}\) in \(R_1\).

Then we define \(\mathcal{T}_1\) as the preimage in \(\hat{R}_1\) of the implicit ideal of \(H'/R_1\).

We iterate this construction for every formal framed sequence.

**Theorem 11.2.** We recall that \(\text{car}(k_\nu) = 0\). There exists a formal framed sequence

\[
(R, u, k) = \left( R_0, u^{(0)}, k_0 \right) \to \cdots \to \left( R_i, u^{(i)} = \left( u_1^{(i)}, \ldots, u_n^{(i)} \right), k_i \right)
\]

such that:

1. The ring \(\hat{R}/\mathcal{T}_i\) is regular,
2. For every index \(j\), we have that \(f_j \mod (\mathcal{T}_i)\) is a monomial in \(u^{(i)}\) multiplied by a unit of \(\hat{R}/\mathcal{T}_i\),
3. For every index \(j\), we have \(f_1 \mod (\mathcal{T}_i) \mid f_j \mod (\mathcal{T}_i)\) in \(\hat{R}/\mathcal{T}_i\).

**Proof.** Set \(n := e(R, \nu)\) and \(u := (y, x)\) with

\[
y := (y_1, \ldots, y_{n-1})
\]

\[
x := (x_1, \ldots, x_n)
\]

such that the images of the \(x_j\) in \(\hat{R}/\mathcal{T}_i\) induce a minimal set of generators of \(\mathcal{T}_i\) and such that \(y\) generates \(\mathcal{T}_i\).

We do an induction on \((n_i, n_i - r_i, v_i)\).

We saw the existence of the surjection \(\Phi\) from \(A \simeq k[[x_1, \ldots, x_n]]\) to \(\hat{R}/\mathcal{T}_i\). of kernel \(I = \{f \in A\ s.t. \\tilde{\nu}(f) \notin \Gamma_1\} \subseteq \text{Spec}(A)\) where \(\tilde{\nu}\) is defined as in section 9. We denote by \(L\) the field of fractions of \(A\).

If \(v_0 < n - 1\), then we do the same thing than in Proposition 9.5 and we strictly decrease \(e(A, \tilde{\nu})\).

The we can assume \(v_0 \in \{n - 1, n\}\).

Assume \(v_0 = n - 1\).

Then we know that \(I = (h)\) and that there exists a formal framed sequence \((A, x) \to (A\ell, x^{(\ell)})\) which monomialize \(h\) by Corollary 10.10. So one of the generators which appears in its decomposition must be in \(I\). Hence there exists \(x_p^{(\ell)}\) such that \(\tilde{\nu}(x_p^{(\ell)}) \notin \Gamma_1\). So by Theorems 9.5 and 8.11 there exists a local framed sequence which decreases strictly \(e(A, \tilde{\nu})\), so this case can happen a finite number of time, and we bring us at the case \(I = (0)\). It means the case where \(\tilde{\nu}\) is regular.

Case \(I = (0)\). For every \(f_j\), we have \(\tilde{\nu}(f_j) \in \Gamma_1\). So the element \(f_j\) is a non zero formal series and by Weierstrass preparation Theorem, we know that we can see it like a polynomial in \(x_j\) with coefficients in \(k[[x_1, \ldots, x_{n-1}]]\). We construct a sequence of key polynomials in the extension \(k((x_1, \ldots, x_{n-1}))(x_n)\) as...
in previous section. It means that this sequence is a sequence of optimal (eventually limit) immediate successors which is cofinal in $\epsilon (\Lambda)$, where $\Lambda$ is the set of key polynomials. So the element $f_j$ is non degenerated with respect of one of these polynomials which all are monomializable by the previous part. Hence there exists a local framed sequence $(A, x) \rightarrow (A_i, x^{(i)})$ such that in $A_i$, the strict transform of $f_j$ is a monomial in $x^{(i)}$ multiplicated by a unit of $A_i$.

If there exists a formal framed sequence such that $v_i < n - 1$, then by Proposition 9.5 we can conclude by induction.

Iterating the case $I = (0)$, we assure the existence of a local framed sequence such that all the strict transforms of the $f_j$ are monomializable by units. Even if it means to do another blow-up, we assume the existence of a local framed sequence $(A, x) \rightarrow (A', x')$ such that all the strict transforms of the $f_j$ are monomials only in $x_1', \ldots, x'_r$.

By Proposition 11.3, we can assume that for every $j$ and every $p$, we have either $f_j \mid f_p$ or $f_p \mid f_j$.

So we have a local framed sequence

$$(A, x, k) \xrightarrow{\rho_0} (A_1, x^{(1)}, k_1) \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_s} (A_i, x^{(i)}, k_i)$$

which monomialize the $f_j$ and such that for all $j$ and $q$, we have $f_j \mid f_q$ or the inverse.

By minimality of $\nu (f_1)$, in $A_i$, we have $f_1 \mid f_j$ for every $j$.

We have also two maps

$$(R, u, k) \rightarrow \left( \hat{R}, x, k \right) \leftarrow (A, x, k),$$

and we know that $\hat{R} \simeq \hat{R}$ since $I = \ker (\Phi)$. Hence, looking at the strict transform of $\hat{R}$ at each step of the sequence $\{\rho_j\}_{0 \leq j \leq i}$, we obtain a local framed sequence

$$(\hat{R}, x, k) \xrightarrow{\rho_0} \left( \hat{R}_1, x^{(1)}, k_1 \right) \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_s} \left( \hat{R}_i, x^{(i)}, k_i \right).$$

So we have the diagram:

$$\begin{array}{ccc}
(A, x, k) & \xrightarrow{\rho_0} & (A_1, x^{(1)}, k_1) \\
\uparrow & & \uparrow \\
(\hat{R}, x, k) & \xrightarrow{\rho_0} & (\hat{R}_1, x^{(1)}, k_1) \\
\uparrow & & \uparrow \\
& \xrightarrow{\rho_1} & \cdots \xrightarrow{\rho_s} (\hat{R}_i, x^{(i)}, k_i) \\
& \uparrow & \uparrow \\
& (A, x, k) & \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_s} (A_i, x^{(i)}, k_i)
\end{array}$$

With the same previous argument, either $\hat{R}$ is regular, or the sequence $\{\rho_j\}$ can be chosen such that $e (R, \mu)$ strictly decreases.

So after a finite sequence of blow-ups, we bring us to the case where $\hat{R}$ is regular. Hence we can assume $\hat{R}$ regular and consider $f_1, \ldots, f_s$ elements of $R \setminus \{0\}$ such that $\nu (f_1) = \min_{1 \leq j \leq s} \nu (f_j)$. We know that the $f_j$ are all monomials in the $u^{(i)}$ and that $f_j \mod (\hat{R}) \mid f_j \mod (\hat{R})$. This concludes the proof.

**Theorem 11.3.** Let $R$ be a local quasi excellent domain and $H$ be his implicit prime ideal. We assume that $\hat{R}$ is regular.
We recall that \( \text{car}(k_v) = 0 \). There exists a sequence of blow-ups defined over \( R \) which resolve the singularities of \( R \).

**Proof.** The ring \( \hat{R}_H \) is regular by Proposition 8.6. So we know that there exist elements \( (\tilde{y}_1, \ldots, \tilde{y}_g) \) of \( H\hat{R}_H \) which form a regular system of parameters of \( \hat{R}_H \).

By definition of \( H\hat{R}_H \), it means that there exist \( y_1, \ldots, y_g \) elements of \( H \) and \( b_1, \ldots, b_g \) elements of \( \hat{R} \setminus H \) such that for every index \( i \), we have \( \tilde{y}_i = y_i b_i \).

But the \( b_i \) are elements of \( \hat{R} \times H \), so \( (\tilde{y}_1, \ldots, \tilde{y}_g) \hat{R}_H = (y_1, \ldots, y_g) \hat{R}_H \).

Then we have some elements \( (y_1, \ldots, y_g) \) of \( H \) which form a regular system of parameters of \( \hat{R}_H \).

Now we consider \( (x_1, \ldots, x_t) \) some elements of \( \hat{R} \setminus H \) which images \( (\tilde{x}_1, \ldots, \tilde{x}_t) \) modulo \( H \) form a regular system of parameters of \( \hat{R}_H \).

If \( (y_1, \ldots, y_g) \) generate \( H \), then \( \hat{R} \) is regular. Indeed, in this case, \( (y_1, \ldots, y_g, x_1, \ldots, x_t) \) generate \( \hat{m} = m \otimes_R \hat{R} \), which is the maximal ideal of \( \hat{R} \).

So \( \dim(\hat{R}) \leq g + t \).

But we know that \( g = \dim(\hat{R}_H) = \text{ht}(H) \) and \( t = \dim(\frac{\hat{R}}{H}) = \text{ht}(\frac{\hat{m}}{H}) \).

Then \( \dim(\hat{R}) = \text{ht}(\hat{m}) \geq \text{ht}(H) + \text{ht}(\frac{\hat{m}}{H}) = g + t \geq \dim(\hat{R}) \).

Then \( \dim(\hat{R}) = g + t \) and \( (y_1, \ldots, y_g, x_1, \ldots, x_t) \) is a minimal set of generators of \( \hat{m} \), and so \( \hat{R} \) is regular.

So now we assume that \( (y_1, \ldots, y_g) \) don’t generate \( H \) in \( \hat{R} \). So let us set \( (y_1, \ldots, y_g, y_{g+1}, \ldots, y_{g+s}) \) some elements which generate \( H \) in \( \hat{R} \).

We consider \( V := \frac{\hat{R}_H}{H^2 \hat{R}_H} \) which is a vector space of dimension \( g = \text{ht}(H) \) over the residual field of \( H \) since \( \hat{R}_H \) is regular.

We know that \( y_1, \ldots, y_{g+s} \) generate \( V \) and that \( g + s > \dim(V) = g \),

so there exist elements \( a_1, \ldots, a_{g+s} \) of \( \hat{R} \) such that \( a_1 y_1 + \cdots + a_{g+s} y_{g+s} \in H^2 \hat{R}_H \).
Hence we have an isomorphism $a$ and also that for every $z$ is of the form $z \in \mathbb{R}$.

So we transformed $z$ on $x$ in $\mathbb{R}$ can assume $\nu$ we have $\nu(a_1) = \min \{ \nu(a_i) \}$ and also that for every $i$, the element $a_i$ is not in $H$ or is zero.

Since the $a_i$ are in $\widehat{R}$, we look at them modulo $H$. By Theorem [6.21], we know that the classes $a_i$ of $a_i$ modulo $H$ are monomialisable in $\widehat{R}$ and that for every $i$, we have $a_i | a_i$.

Hence after a sequence of blow-ups, we have that $a$ is generated in $\widehat{R}$ by $(y_2, \ldots, y_{g+s})$.

Iterating, we could generate $H$ in $\widehat{R}$ by $g$ elements, and it would be over.

So let us show that we can do a sequence of blow-ups such that at the end $a_1$ divides all the $b_{i,j}$.

For every index $i \in \{1, \ldots, g+s\}$, there exists $n_i \in \mathbb{N}_{>1}$ such that $y_i \in \widehat{m}^{n_i-1} \setminus \widehat{m}^{n_i}$. We set $N := \max \{ n_i \}$, and then for every $i \in \{1, \ldots, g+s\}$, $y_i \notin \widehat{m}^N$.

We have a map $R \to \widehat{R}$ and we know that for every integer $c$, we have $\widehat{m}^c \cap R = m^c$. Hence we have an isomorphism $\frac{R}{m^c} \to \frac{\widehat{R}}{\widehat{m}^c}$.

So for all $i \in \{1, \ldots, g+s\}$, there exists $z_i \in R$ which class modulo $m^{N+2}$ is sent on $y_i$ by this map. Hence $z_i \mod (m^{N+2}) = y_i$. Even if it means to increase $N$, we can assume $\nu(\widehat{m}^N) > \nu(a_1)$.

More precisely $y_i = z_i + h_i + \zeta_i$ where $h_i \in (z_1, \ldots, z_{g+s})^2$ and $\zeta_i \in (x_1, \ldots, x_t)^N$.

After a sequence of blow-ups independent of $(z_1, \ldots, z_{g+s})$, we can assume that $w$, and so $a_1$, divides all the $\zeta_i$.

We do now $c_1$ blow-ups of $(z_1, \ldots, z_{g+s}, x_1)$. Each $z_i$ is transformed in a $z_i'$ which is of the form $\frac{z_i}{\tilde{x}_i}$.

We do now $c_2$ blow-ups of $(z_1', \ldots, z_{g+s}', x_2)$. Each $z_i'$ is transformed in a $z_i''$ which is of the form $\frac{z_i'}{\tilde{x}_2}$.

We iterate until doing $c_t$ blow-ups of

$$\left( z_1^{(t-1)}, \ldots, z_{g+s}^{(t-1)}, x_t \right).$$

So we transformed $z_i$ in $z_i^{(t)}$ which is of the form $\frac{z_i^{(t)}}{\tilde{x}_i}$.

Then $a_1$ divides all the $z_i^{(t)}$, and so all the $h_i^{(t)}$ and the $y_i^{(t)}$. The $b_{i,j}$ are elements of $\widehat{R}_H$, so after this sequence of blow-ups, since the strict transform of $H$ is generated by the $y_i^{(t)}$, we have that $a_1$ divides all the $b_{i,j}$, and it is over. \qed
12. Conclusion.

We know are going to give the principal results of this part. First we recall a fundamental result of Novacoski and Spivakovsky ([20]).

Theorem 12.1. Let $S$ be a noetherian local ring. If the local uniformization Theorem is true for every valuation of rank 1 centered in $S$, then it is true for any valuation centered in $S$.

So we just have to consider valuations of rank 1.

Theorem 12.2. Let $S$ be a noetherian equicharacteristic quasi excellent singular local ring of characteristic zero. We consider a valuation $\mu$ of rank 1 centered in $S$.

There exists a formal framed sequence

$$(S, u) \to \cdots \to (S_i, u^{(i)}) \to \cdots$$

such that for $j$ big enough, $S_j$ is regular and for every element $s$ of $S$, there exists $i$ such that in $S_i$, $s$ is a monomial.

Proof. We consider $\hat{S}$ the formal completion of $S$ and $H$ its implicit prime ideal. By Cohen structure Theorem, there exists an epimorphism $\Phi$ from a complete regular local ring $R$ in $\hat{S}$. We consider $\overline{H}$ the preimage of $H$ in $R$. We extend now $\mu$ to a valuation $\nu$ centered in $R$ by composition with a valuation centered in $H$.

By Proposition 8.6 we know that $\hat{S}_H$ is regular, and by Theorem 11.3 it is enough to show that $\hat{S}_{\overline{H}}$ is also regular.

We know that $\hat{S}_{\overline{H}} \simeq \frac{R}{H}$, so we just have to regularise $\frac{R}{H}$. We conclude with Theorem 10.15.

Now we prove the principal result of this part: the simultaneous embedded local uniformization for local noetherian quasi excellent equicharacteristic rings.

Theorem 12.3. Let $R$ be a local noetherian quasi excellent complete regular ring and $\nu$ be a valuation centered in $R$.

Assume that $\nu$ is of rank 1 or 2 but composed of a valuation $(f)$-adic where $f$ is an irreducible element of $R$. We assume $\text{car } (k_{\nu}) = 0$.

There exists a formal framed sequence

$$(R, u) \to \cdots \to (R_i, u^{(i)}) \to \cdots$$

such that for every element $g$ of $R$, there exists $i$ such that in $R_i$, $g$ is a monomial.

Proof. We consider the ring $A = \frac{R}{(f)}$. The valuation $\nu$ is of rank 2 composed of valuation $(f)$-adic, so $\nu$ can be written $\mu \circ \theta$ where $\theta$ is the valuation $(f)$-adic.

So we have a valuation $\mu$ centered in $A$ of rank 1. By Theorem 12.2 we can regularise $A$, and so there exists a local framed sequence $(R, u) \to \cdots \to (R_i, u^{(i)})$ such that in $R_i$, $f$ is a monomial. In $R_i$, we also have that every element $g$ of $\overline{R}$ can be written $g = \left(u_{h}^{(i)}\right)^{a} h$ where $u_{h}^{(i)}$ is the strict transform of $f$ and $h$ is not divisible by $u_{h}^{(i)}$. We apply another time Theorem 12.2 to construct a local framed sequence which monomialize $h$, and we are done.

Corollary 12.4. We keep the same notations and hypothesis than in the previous Theorem.

Then $\lim_{\rightarrow} R_i$ is a valuation ring.
Remark 12.5. The restriction on the rank of the valuation was settled to give an autosufficient proof. Otherwise, there exists a countable sequence of polynomials $\chi_i$ such that every $\nu$-ideal $P_\beta$ is generated by a subset of the $\chi_i$. Assume the embedded local uniformization Theorem.

Then there exists a local (respectively formal) framed sequence $(R, u) \to \cdots \to (R_i, u^{(i)}) \to \cdots$ which satisfies following properties:

1. For $i$ big enough, $R_i$ is regular.
2. For every finite set $\{f_1, \ldots, f_s\} \subseteq m$ there exists $i$ such that in $R_i$, every $f_j$ is a monomial and $f_1 \mid f_j$.

Then for every element $g$ in $R$, there exists $i$ such that in $R_i$, $g$ is a monomial.

References

1. Shreeram Abhyankar, *Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0* Annals of Mathematics, Second Series, 63 (3): 491-526
2. Shreeram Abhyankar, *Resolution of Singularities of Embedded Algebraic Surfaces* Academic Press, New York and London, 1966.
3. Edward Bierstone et Pierre D. Milman, *Local resolution of singularities*. In Real analytic and algebraic geometry (Trento, 1988), volume 1420 of Lecture Notes in Math., pages 42-64. Springer, Berlin, 1990.
4. Nicolas Bourbaki, *Eléments de mathématiques : Algèbre commutative*, Masson, 1985.
5. Vincent Cossart et Olivier Piltant, *Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings*. Journal of Algebra, 320 (3): 1051-1082
6. Vincent Cossart et Olivier Piltant, *Resolution of singularities of threefolds in positive characteristic. II* Journal of Algebra, 321 (7): 1836-1976
7. Vincent Cossart et Olivier Piltant, *Resolution of Singularities of Arithmetical Threefolds II* arXiv:1412.0868
8. Felipe Cano, Claude Roche et Mark Spivakovsky, *Reduction of singularities of three-dimensional line foliations* Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas108, Issue 1(2014), pages 221-258.
9. Julie Decaup, Wael Mahboub et Mark Spivakovsky, *Abstract key polynomials and comparison theorems with the key polynomials of Mac Lane-Vaquié* arXiv:611.06392.
10. Santiago Encinas et Herwig Hauser, *Strong resolution of singularities in characteristic zero*. Comment. Math. Helv., 77(4) :821-845, 2002.
11. David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer, 1995.
12. Santiago Encinas et Orlando Villamayor, *A new proof of desingularization over fields of characteristic zero*. In Proceedings of the International Conference on Algebraic Geometry and Singularities (Sevilla, 2001), volume 19, pages 339-353, 2003.
13. Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*. Ann. of Math. (2) 79 (1964), 109-203 ; ibid. (2), 79:205-326, 1964
14. Herrera Govantes, Mahboub, Olalla Acosta et Spivakovsky, *Key polynomials for simple extensions of valued fields* arXiv:1406.0657v3
15. Herrera Govantes, Olalla Acosta, Spivakovsky et Teissier, *Extending a valuation centered in a local domain to the formal completion* Proc. LMS105, Issue 3, 571-621, 2012.
16. Matsumura, *Commutative Algebra* Benjamin/Cummings Publishing Co., Reading, Mass., 1970.
17. Saunders Mac Lane, *A construction for absolute values in polynomial rings*, Transactions of the AMS, vol 40 (1936), pages 363-395.
18. Wael Mahboub, *Une construction explicite de polynômes-clefs pour des valuations de rang fini*, These, Université Toulouse 3, 2013.
19. Novacoski et Spivakovsky, *Key polynomials and pseudo-convergent sequences* Journal of Algebra 495 pages 199-219, 2018.
20. Novacoski et Spivakovsky, Reduction of local uniformization to the case of rank one valuations for rings with zero divisors. Valuation Theory in Interaction, EMS Series of Congress Reports, European Mathematical Society, 2014, pp. 404-431.

21. Teissier, Valuations, deformations and toric geometry, Proceedings of the Saskatoon Conference and Workshop on valuation theory, Vol II, F-V. Kuhlmann, S. Kuhlmann, M. Marshall, editors, Fields Institute Communications, 33 (2003), 361-459.

22. Michel Vaquié, Valuations and local uniformization, Advanced Studies in Pure Mathematics 43, 2006. Singularity Theory and Its Applications, p477-527

23. Michel Vaquié, Extension d’une valuation, Trans. Amer. Math. Soc. 359 (2007), no.7, 3439-3481.

24. Michel Vaquié, Valuations, dans Resolution of Singularities, Progr. in math. 181, 2000. MR1748635 (2001i:13005).

25. Orlando Villamayor, Constructiveness of Hironaka’s resolution. Ann. Sci. École Norm. Sup. (4), 22(1) :1-32, 1989.

26. Robert J. Walker, Reduction of the Singularities of an Algebraic Surface. Annals of Mathematics, Second Series, 36 (2): 336-365

27. Jaroslaw Wlodarczyk, Simple Hironaka resolution in characteristic zero. J. Amer. Math. Soc., 18(4) :779-822, 2005.

28. Oscar Zariski, The reduction of the singularities of an algebraic surface. Ann. of Math. (2), 40 :639-689, 1939.

29. Oscar Zariski, Local uniformization theorem on algebraic varieties. Ann. of Math. 41 :852-896, 1940.

30. Oscar Zariski, Reduction of singularities of algebraic three dimensional varieties Ann. of Math. 45, 472-542, 1944.

31. Oscar Zariski et Pierre Samuel, Commutative Algebra, Volume 2, Graduate Texts in Mathematics, 1960.

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