Probing the quark condensate by means of $\pi\pi$ scattering

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The available experimental information is consistent with the Weinberg predictions for the threshold parameters of $\pi\pi$-scattering. The data, however, only test those relations that are insensitive to the quark masses $m_u$ and $m_d$. Recent theoretical progress leads to remarkably sharp predictions for the behaviour of the S- and P-wave phase shifts in the elastic region – one of the very few low energy phenomena in QCD, where theory is ahead of experiment and where new precision data would subject the theory to a crucial test.

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1. Introduction

My talk at the DAΦCE meeting consisted of two parts: an introductory discussion of chiral perturbation theory and a report concerning recent work on the application of this method to $\pi\pi$ scattering. In the following, I omit the first part, referring the reader to one of the published reviews \cite{1}. In particular, the handbook \cite{2} offers an excellent overview of the applications of interest in connection with the experiments planned at DAΦNE. This reference also describes the status of our current knowledge of low energy $\pi\pi$-scattering, both from the phenomenological point of view \cite{3} and within the framework of chiral perturbation theory \cite{4}, \cite{5}. For a comprehensive discussion of the general properties of the $\pi\pi$ scattering amplitude, see refs. \cite{4}, \cite{5}.

Since the mass difference $m_u - m_d$ is small, the strong interaction approximately conserves isospin. Neglecting the electromagnetic interaction, the various elastic reactions among two pions may be represented by a single scattering amplitude $A(s,t,u)$. Only two of the Mandelstam variables are independent, $s+t+u = 4M^2_\pi$ and, as a consequence of Bose statistics, the amplitude is invariant under an interchange of $t$ and $u$.

Equivalently, the amplitude may be described in terms of the corresponding partial waves $t^I_\ell(q)$, where $q$ is the momentum in the centre-of-mass-system, $\ell = 0,1,2,\ldots$ is the angular momentum and the isospin index $I$ takes the values $I = 0,1,2$. Bose symmetry entails that only even values of $\ell+I$ occur. The first two terms in the threshold expansion of the partial waves,

\[ \text{Re} t^I_\ell(q) = q^\ell (a^I_\ell + q^2 b^I_\ell + \ldots) , \]

are referred to as scattering length ($a^I_\ell$) and effective range ($b^I_\ell$).

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The pions are the Goldstone bosons of a spontaneously broken approximate symmetry. As shown by Weinberg [8], chiral symmetry fully determines the leading term in the low energy expansion of the $\pi\pi$ scattering amplitude, in terms of the pion decay constant:

$$A(s, t, u) = \frac{s - M^2_\pi}{F^2_\pi} + O(p^4) \ .$$

(2)

The formula implies that (i) Re $A(s, t, u)$ contains an Adler zero in the vicinity of $s = M^2_\pi$, (ii) the scattering amplitude rapidly grows with the energy and (iii) in the channels with $I = 0$ and $I = 1$, the interaction is attractive, while for $I = 2$ it is repulsive.

The present paper concerns the behaviour of the scattering amplitude in the region $4M^2_\pi < s < 4M^2_K$, where the Adler zero and the pole due to $\rho$-exchange are the main features, which determine the properties of the amplitude to a large extent [9]. The $\rho$ may be incorporated into effective field theory models by introducing a corresponding field. Indeed, the tree graphs of a suitable model of this type provide a simple, successful parametrization of the scattering amplitude [10]. The following discussion does not deal with the main features, however, but addresses the symmetry breaking effects due to the quark masses $m_u, m_d$. These are very small and cannot reliably be analyzed on the basis of such simplified models. Instead, I am using the Roy equations, which follow from general principles and are perfectly suited for the low energy analysis.

2. Roy equations

As was shown by Roy [11], analyticity and crossing symmetry imply that the partial wave amplitudes satisfy a system of coupled dispersion relations of the form

$$\text{Re } t^\ell(q) = c^\ell(q) + \sum_{\ell' = 0}^2 \sum_{\ell''} \int_4^\infty dq' Q^{II'\ell'\ell}(q, q') \text{Im } t^\ell(q') \ .$$

(3)

The kernels $Q^{II'\ell'\ell}(q, q')$ are explicitly known kinematic quantities that may be expressed in terms of associated Legendre functions and $c^\ell(q)$ represents a subtraction polynomial. In principle, a single subtraction constant suffices for the above dispersion relations to converge, but the convergence is then rather slow, so that the behaviour of the imaginary parts at high energies play a significant role. For the purpose of the low energy precision analysis which I wish to discuss, it is more appropriate to work with two subtraction constants, which may be identified with the S-wave scattering lengths $a^0_0, a^0_2$. The subtraction polynomials then take the form [11]

$$c^0_0(q) = a^0_0 + \frac{1}{3}(2a^0_0 - 5a^2_0)q^2 \ , \quad c^2_0(q) = a^2_0 - \frac{1}{6}(2a^0_0 - 5a^2_0)q^2 \ ,$$

$$c^1_1(q) = \frac{1}{18}(2a^0_0 - 5a^2_0)q^2 \ , \quad c^\ell_\ell(q) = 0 \text{ for } \ell \geq 2 .$$

(4)

(5)

The Roy equations express the real part of $t^\ell(q)$ in terms of the imaginary part. We may represent the partial waves in terms of the phase shift $\delta^\ell(q)$ and the elasticity $\eta^\ell(q)$:

$$t^\ell(q) = \frac{\sqrt{M^2_\pi + q^2}}{2i q} \left\{ \eta^\ell(q) e^{2i\delta^\ell(q)} - 1 \right\} .$$

(6)
Unitarity implies that in the elastic region \( \eta_L(q) \) is equal to 1, so that the real and imaginary parts of \( t^I_L(q) \) are determined by a single real function. For \( s > 16M^2 \), on the other hand, unitarity only leads to the bound \( 0 \leq \eta_L(q) \leq 1 \). Hence two real functions are required to specify the behaviour of the partial waves there. If the subtraction constants \( a^0_0, a^2_0 \) and the elasticities are taken as known, the Roy equations amount to a set of coupled integral equations for the phase shifts, which may be solved iteratively. It is clear, however, that we cannot extract two real quantities from one real equation. When using the Roy equations to determine the partial waves, we thus need two categories of input:

1. data in the inelastic region \( \sqrt{s} > 4M \simeq 550 \text{ MeV} \)

2. values of the subtraction constants \( a^0_0, a^2_0 \)

The early literature on the problem \[12\], \[13\], \[14\] shows that the subtraction constants represent the main source of uncertainty here – the data available for \( \sqrt{s} > 550 \text{ MeV} \) determine the values of the dispersion integrals in the region below this point to within rather narrow limits. The present experimental situation is described in \[3\]. For a thorough recent analysis of the data in the range \( 600 \text{ MeV} < \sqrt{s} < 1900 \text{ MeV} \), see \[15\].

The available experimental information does not allow us to determine the S-wave scattering lengths very accurately. Various results are quoted in the literature, for instance \[16\]

\[
a^0_0 = 0.26 \pm 0.05, \quad a^2_0 = -0.028 \pm 0.012, \quad 2a^0_0 - 5a^2_0 = 0.66 \pm 0.05. \tag{7}
\]

The value for \( a^0_0 \) mainly relies on the \( K_{e4} \)-data of the Geneva-Saclay collaboration \[17\] (the final state interaction among the two pions allows a measurement of the phase difference \( \delta^0_0 - \delta^1_1 \) in the region \( \sqrt{s} < M_K \)). The value inferred for \( a^2_0 \) then emerges from the correlation between the two S-wave scattering lengths that was first noted by Morgan and Shaw \[18\] (“universal curve”). The correlation is due to the fact that, as mentioned earlier, one of the two subtraction constants in the Roy equations is superfluous: The combination \( 2a^0_0 - 5a^2_0 \) may be represented as a convergent dispersion integral over the imaginary part of the amplitude. The number given above indicates that the available experimental information about the imaginary part allows us to evaluate the dispersion integral to within about 10 %.

3. Wanders sum rules

As an illustration of the statement that, if the two S-wave scattering lengths are taken as known, the available data allow a remarkably precise determination of the scattering amplitude in the low energy region, I briefly discuss one of the Wanders sum rules \[19\]. Evaluating the second derivative of the Roy equation for \( t^0_0(q) \) at \( q = 0 \), we obtain a sum rule for the effective range:

\[
b^0_0 = \frac{1}{3}(2a^0_0 - 5a^2_0) + \text{dispersion integral} \tag{8}
\]

The relation represents a variant of the sum rule underlying the universal curve. The main difference is that the dispersion integral occurring here converges more rapidly, so that
the result is less sensitive to the experimental uncertainties in the high energy region. Moreover, the contribution from the scattering lengths dominates the result – the one from the dispersion integral amounts to about 20\%. If $a_0$ and $a_2$ are assumed known, the sum rule determines the effective range to within a few percent. Similar sum rules exist for the other threshold parameters, $a_1$, $b_1$, $a_0^2$, $a_2^2$, \ldots [20], [21], [22].

Note that the threshold region generates significant contributions to the dispersion integrals, so that an iterative procedure must be used to obtain reliable values. The sum rules for the threshold parameters may be viewed as limiting cases of the Roy equations. A more systematic analysis of the low energy behaviour of the partial waves calls for an iterative solution of these equations. The main point here is that $a_0$, $a_2$ represent the essential low energy parameters. Once these are known, the available data suffice to pin down the phase shifts quite accurately, despite the fact that, in the elastic region, the experimental error bars are large.

4. Chiral perturbation theory

The representation [2] implies that the expansion of the S-wave scattering lengths in powers of the quark masses $m_u$, $m_d$ starts with [3]

$$a_0 = \frac{7 M^2}{32 \pi F^2_\pi} + O(M^4_\pi), \quad a_2 = -\frac{M^2}{16 \pi F^2_\pi} + O(M^4_\pi). \quad (9)$$

These expressions are proportional to $M^2_\pi$, i.e. the S-wave scattering lengths vanish in the chiral limit, $m_u, m_d \to 0$. Indeed, chiral symmetry prevents Goldstone bosons of zero momentum from interacting with one another, so that, in the symmetry limit, the effective Lagrangian exclusively contains derivative couplings. The S-wave scattering lengths therefore offer direct probes of the chiral symmetry breaking generated by the quark masses. Since $m_u$ and $m_d$ are very small, these effects are difficult to measure. In the case of the $I = 0$ S-wave, for instance, where the threshold expansion starts with $\text{Re} \, \ell_0(q) = a_0^0 + q^2 b_0^0 + \ldots$, the scattering length only dominates in the immediate vicinity of threshold: For $q > M_\pi$, the effective range term is more important, because this term does not represent a symmetry breaking effect (the low energy expansion starts with $b_0 = 1/(4 \pi F^2_\pi) + \ldots$).

Weinberg’s low energy theorem only accounts for the leading term in the expansion of the scattering lengths in powers of $m_u, m_d$. The higher order corrections may be worked out by means of chiral perturbation theory, which yields a series of the form

$$a_0 = \frac{7 M^2}{32 \pi F^2_\pi} \left\{ 1 + \epsilon_1 + \epsilon_2 + \ldots \right\}, \quad \epsilon_n = O(M^{2n}_\pi). \quad (10)$$

The explicit expression for the one loop correction reads [23]

$$\epsilon_1 = \frac{5 M^2}{84 \pi^2 F^2_\pi} \left( \bar{l}_1 + 2 \bar{l}_2 - \frac{3}{8} \bar{l}_3 + \frac{21}{10} \bar{l}_4 + \frac{21}{8} \right), \quad (11)$$

where $\bar{l}_1, \ldots, \bar{l}_4$ are effective coupling constants of the chiral Lagrangian, normalized at running scale $\mu = M_\pi$. 


5. Effective coupling constants

The coupling constants $\bar{l}_1$ and $\bar{l}_2$ may be determined in several independent ways. In particular, an analysis of the $K_{e4}$-data yields \[ \bar{l}_1 = -1.7 \pm 1.0, \bar{l}_2 = 6.1 \pm 0.5. \] Alternatively, the values of these coupling constants may be determined by comparing the chiral representation of the $\pi\pi$ scattering amplitude with the data \[ [23], [25]. \] Recent work in this direction \[ [20], [21], [22] \] has clarified the role of the higher order contributions considerably. It leads to a coherent low energy representation of the scattering amplitude that confirms the results obtained from $K_{e4}$-decay. Moreover, the physics underneath the phenomenological values of these couplings is well understood: The main contribution stems from the singularity generated by the exchange of a $\rho$-meson \[ [26], [27], [9], \] while the exchange of particles of spin 0 or 2 only gives rise to comparatively small corrections \[ [28], [29]. \]

The coupling constant $\bar{l}_4$ may be determined on the basis of SU(3)×SU(3), using the observed value of the ratio $F_K/F_\pi$ of decay constants, or, alternatively, the slope of the scalar form factor occurring in $K_\ell_3$-decay \[ [23]. \] The result, $\bar{l}_4 = 4.3 \pm 0.9$, predicts a value for the combination $2a_0^0 - 5a_2^0$ that is in good agreement with the universal curve of Morgan and Shaw. Moreover, there is an independent determination of $\bar{l}_4$, based on a dispersive analysis of the scalar pion form factor \[ [30]. \] The value obtained in that reference for the scalar “charge” radius of the pion, $\langle r^2 \rangle \pi = 0.61$ fm$^2$, corresponds to $\bar{l}_4 = 4.6$ and thus also confirms the above estimate.

Concerning $\bar{l}_3$, however, the direct experimental information is very meagre. The physical significance of this coupling constant is best seen in the formula for the mass of the pion. The Gell-Mann-Oakes-Renner relation, \[ F_\pi^2 M_\pi^2 = (m_u + m_d) |\langle 0|\bar{u}u|0\rangle| + O(m^2), \] states that, at leading order of the expansion in powers of $m_u$ and $m_d$, the square of the pion mass is linear in the quark masses. The constant $\bar{l}_3$ is the coefficient of the correction occurring at second order: \[ M_\pi^2 = M^2 - \frac{\bar{l}_3 M^4}{32 \pi^2 F_\pi^2} + O(M^6), \]

\[ M^2 \equiv (m_u + m_d) B_0, \quad B_0 = \lim_{m \to 0} \frac{1}{F_\pi^2} |\langle 0|\bar{u}u|0\rangle|. \]

In principle, the dependence of the pion mass on $m_u$ and $m_d$ can be determined on the lattice \[ [31]. \] The available lattice results do indicate that $M_\pi^2$ grows linearly with the quark mass, but it is difficult to estimate the uncertainties arising from the fact that, for the time being, dynamical quarks with realistic masses are out of reach.

In the standard picture, the correction term is expected to be very small. In particular, none of the singularities generated by low lying resonances yields a large contribution to $\bar{l}_3$. Crude estimates, for instance the one based on $\eta'$-dominance for $L_\tau$ and on the Zweig rule, lead to $\bar{l}_3 \simeq 3$, indicating that the departure from the linear mass formula is of order 2%. The expression \[ (10) \] shows that in the case of the scattering length, the contribution due to $\bar{l}_3$ is smaller than the one in the mass formula by the factor $\frac{5}{7}$. Unless the theoretical estimates for $\bar{l}_3$ are entirely incorrect, the precise value of this constant does not significantly affect the prediction for $a_0^0$. 
6. Value of $a_0^0$

Weinberg’s formula leads to $a_0^0 = 0.16$. With the above estimates for the coupling constants, the corrections increase this number to 0.20, lower than the experimental value (7) by about one standard deviation.

At first sight, the corrections appear to be unreasonably large. After all, these are of the type $\epsilon_1 = (m_u + m_d)/\Lambda_0$. A familiar rule of thumb states that the scale $\Lambda_0$ should be of order 500 MeV or 1 GeV, indicating that the corrections should be of the order of a few percent. The same bookkeeping also applies within chiral perturbation theory, where the correction is given by an expression of the type $\epsilon_1 = c \frac{M^2}{(4\pi F)}$ and $\frac{M^2}{(4\pi F)^2} = 0.014$ is a very small number.

The resolution of the paradox is that the coefficient contains an infrared singularity. The explicit expression is of the form $c = 9 \ln(\Lambda_1/M_\pi)$, where $\Lambda_1$ is independent of the quark masses. In the limit $m_u, m_d \to 0$, the coefficient $c$ therefore diverges logarithmically. Infrared singularities of this type are a common occurrence in chiral perturbation theory. In the present case, the logarithm has an unusually large coefficient, partly because we are considering the value of the partial wave amplitude at threshold, i.e. at the branch point singularity associated with two-pion intermediate states. The estimates for the coupling constants given above yield $c \simeq 20$, which corresponds $\Lambda_1 \simeq 1.3$ GeV, a perfectly reasonable scale.

In the meantime, the $\pi\pi$ scattering amplitude has been worked out to two loops of chiral perturbation theory [32]. Using theoretical estimates for the additional coupling constants occurring at that order, the authors obtain $a_0^0 = 0.217$. The infrared singularities again yield the dominating contribution. A systematic analysis of the effective couplings occurring in the two loop result is under way [33]. Two types of coupling constants occur: those which survive in the chiral limit and which manifest themselves in the momentum dependence of the amplitude and those which account for the symmetry breaking due to $m_u, m_d$. As demonstrated in ref. [21], the former may be determined phenomenologically, on the basis of the available data. In the standard picture, the latter only generate very small effects, dominated by the contribution from $\tilde{l}_4$. A rough estimate of the uncertainties in the various elements of the calculation indicates that the two loop representation of the scattering amplitude will allow us to predict the value of $a_0^0$ to an accuracy of 2 or 3 %.

It should be noted that, at this level of precision, electromagnetic contributions are not negligible. In particular, the above numbers rely on $M_\pi = M_{\pi^+}$, the scale conventionally used to express dimensionful quantities in pion mass units. If the electromagnetic interaction is turned off, the mass of the charged pion decreases by about 4 MeV. The correction reduces the prediction for the S-wave scattering lengths by 6 %. Also, the pion decay constant must be corrected for radiative effects, which act in the opposite direction. Taken together, these modifications reduce the two loop result to $a_0^0 = 0.208$. For a discussion of isospin breaking in the $\pi\pi$ scattering amplitude, I refer to [14].

7. Need for low energy precision experiments

As discussed above, the prediction for $a_0^0$ relies on theoretical estimates for the coupling constant $\tilde{l}_3$. Stern and collaborators have emphasized that the direct experimental
information on the magnitude of this coupling constant leaves much to be desired \cite{footnote1,footnote2}. Expressed in terms of $\bar{l}_3$, the experimental uncertainty in the value of $a_0^0$ roughly corresponds to
\begin{equation}
    a_0^0 = 0.26 \pm 0.05 \quad \iff \quad \bar{l}_3 = -60 \pm 60 \text{.}
\end{equation}

If it should turn out that the data confirm the central value, the immediate conclusion to draw would be that $\bar{l}_3$ is negative and very large. This in turn would imply that the higher order "corrections" which the Gell-Mann-Oakes-Renner relation neglects are as big as the "leading" term – the standard theoretical picture underlying our current understanding of the low energy structure of QCD would be proven wrong. In particular, the standard pattern of the light quark masses, where $(m_u + m_d)/m_s \simeq M_\pi^2/M_K^2$, would then be incorrect.

As discussed in detail in refs. \cite{footnote1,footnote2}, the standard picture relies on the hypothesis that the quark condensate $\langle 0 | \bar{u}u | 0 \rangle$ is the leading order parameter of the spontaneously broken symmetry. The condensate plays a role analogous to the one of the spontaneous magnetization of a magnet. In that case, it is well-known that spontaneous symmetry breakdown occurs in two quite different modes: ferromagnets and antiferromagnets. For the former, the magnetization develops a non-zero expectation value, while for the latter, this does not happen. In either case, the symmetry is spontaneously broken (for a discussion of the phenomenon within the effective Lagrangian framework, see \cite{footnote33}). The example illustrates that operators which are allowed by the symmetry to pick up an expectation value may but need not do so. In particular, general principles do not exclude the possibility that the quark condensate vanishes in the chiral limit.

Since the form of the effective chiral Lagrangian is determined by symmetry, the standard expression, characterized by the coupling constants $F_\pi$, $B$, $\bar{l}_1$, $\bar{l}_2$, $\bar{l}_3$, \ldots also applies if $\bar{l}_3$ is taken to be large, but it then becomes inconsistent to treat the interaction generated by this coupling constant as a perturbation. Instead, the term proportional to $\bar{l}_3$ must then be included among the leading contributions of order $p^2$: The effective Lagrangian remains the same, but the chiral perturbation series must be reordered ("generalized chiral perturbation theory"). The implications for some of the low energy observables of interest, in particular also for $\pi\pi$ scattering have been worked out in detail \cite{footnote1,footnote2}.

If the picture underlying the standard framework should turn out to be incorrect, much of the predictive power of the effective theory would be lost. In the case of $a_0^0$, for instance, the generalized scenario does not yield a prediction, because $\bar{l}_3$ is treated as a free parameter. Another example is the Gell-Mann-Okubo formula for the masses of the pseudoscalar octet, which represents a very neat check of the standard framework – the generalized scheme does not lead to such a formula, but can accommodate the observed mass pattern if the corresponding effective coupling constants are properly tuned.

Quite irrespective, however, of whether or not the alternative is theoretically attractive, it is important to subject the issue to experimental test. Low energy precision measurements of the $\pi\pi$ scattering amplitude would allow us to settle the matter. In particular, an analysis of the momentum distribution in the decay $K \to \pi\pi e^+\nu$ does provide a test of those predictions that concern the breaking of chiral symmetry due to $m_u$ and $m_d$. At DAΦNE, this can be done, with significantly better statistics than in the Geneva-Saclay experiment. Moreover, the properties of the transition matrix elements are now under
better control, so that the data analysis can be performed on a more solid basis. There is a beautiful alternative proposal due to Nemenov, based on the observation that $\pi^+\pi^-$ atoms decay into a pair of neutral pions, through the strong transition $\pi^+\pi^- \rightarrow \pi^0\pi^0$. Since the momentum transfer nearly vanishes, the decay rate is proportional to the square of the combination $a_0^2 - a_2^0$ of S-wave $\pi\pi$ scattering lengths. A measurement of the lifetime of a $\pi^+\pi^-$ atom would thus also allow us to decide whether or not the quark condensate represents the leading order parameter. An experiment with this goal is under way at CERN.

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