Note on the stability of planar stationary flows in an exterior domain without symmetry

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Abstract The asymptotic stability of two-dimensional stationary flows in a non-symmetric exterior domain is considered. Under the smallness condition on initial perturbations, we show the stability of the small stationary flow whose leading profile at spatial infinity is given by the rotating flow decaying in the scale-critical order $O(|x|^{-1})$. Especially, we prove the $L^p$-$L^q$ estimates to the semigroup associated with the linearized equations.

Keywords Navier-Stokes equations · two-dimensional exterior flows · scale-criticality · stability of stationary solutions

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1 Introduction

In this paper we consider the perturbed Stokes equations for viscous incompressible flows in a two-dimensional exterior domain.

\[
\begin{aligned}
\begin{cases}
\partial_t v - \Delta v + V \cdot \nabla v + v \cdot \nabla V + \nabla q = 0, & t > 0, \ x \in \Omega, \\
\text{div} \ v = 0, & t \geq 0, \ x \in \Omega, \\
v |_{\partial \Omega} = 0, & t > 0, \\
v |_{t=0} = v_0, & x \in \Omega.
\end{cases}
\end{aligned}
\]

(PS)

Here the unknown functions $v = v(t,x) = (v_1(t,x), v_2(t,x))^T$ and $q = q(t,x)$ are respectively the velocity field and the pressure field of the fluid, and $v_0 = v_0(x) = (v_{0,1}(x), v_{0,2}(x))^T$ is a given initial velocity field. The given vector field $V = V(x) = (V_1(x), V_2(x))^T$ is assumed to be time-independent and decay in the scale-critical order $V(x) = O(|x|^{-1})$ at spatial infinity. We use the standard notations for differential operators with respect to the variable $t$ and $x = (x_1, x_2)^T$: $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\Delta = \partial_1^2 + \partial_2^2$, $V \cdot \nabla v + v \cdot \nabla V = \sum_{j=1}^2 V_j \partial_j v + v_j \partial_j V$, $\text{div} \ v = \partial_1 v_1 + \partial_2 v_2$. The exterior domain $\Omega$ is assumed to be contained by the domain exterior to the radius-$\frac{1}{2}$ disk $\{ x \in \mathbb{R}^2 | |x| > \frac{1}{2} \}$.

The aim of this paper is to investigate the time-decay estimates to the equations (PS), under a suitable condition on the vector field $V$. The equations (PS) has been studied as the linearization of the Navier-Stokes equations around a stationary solution $V$. In the three-dimensional case, Borchers and Miyakawa [2] establishes the $L^p$-$L^q$ estimates to (PS) for the small stationary Navier-Stokes flow $V$ satisfying $V(x) = O(|x|^{-1})$ as $|x| \to \infty$. This result is extended to the case when $V$ belongs to the Lorenz space $L^{3,\infty}(\Omega)$ by Kozono and

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Yamazaki [10]. We also refer to the whole-space result by Hishida and Schonbek [9] considering the time-dependent $V = V(t, x)$ in the scale-critical space $L^\infty(0, \infty; L^{6,\infty}(\mathbb{R}^3))$, where the $L^p-L^q$ estimates are obtained for the evolution operator associated with the linearized equations around $V(t, x)$.

For the two-dimensional problem as in (PS), the analysis becomes quite complicated and there is no general result especially for the time-decay estimate so far. The difficulty arises from the unavailability of the Hardy inequality. Maekawa [11] studies the stability of the flow $\alpha U$ where the inequality (3) essentially depends on the potential property of $V$. Hence, if we consider the problem (PS) with $V = \alpha U$, $\alpha \in \mathbb{R} \setminus \{0\}$, the linearized term $\alpha(U \cdot \nabla v + v \cdot \nabla U)$ can no more be regarded as a perturbation from the Laplacian, and we cannot avoid the difficulty coming from the lack of the Hardy inequality. Maekawa [11] obtains the $L^p-L^q$ estimates to (PS) with $V = \alpha U$ for small $\alpha$, and shows the asymptotic $L^2$-stability.
of $\alpha U$ if $\alpha$ and initial perturbations are sufficiently small. This result is extended by the same author in [12] for the more general class of $V$ in (PS) including the flow of the form $V = \alpha U + \delta W$ with small $\alpha$ and $\delta$; see [12] for details.

Our first motivation is to generalize the result in [11] to the case when the domain loses symmetry (and the second one is explained in Remark 1.2 (3) below). Let us prepare the assumptions on the domain $\Omega$ and the stationary vector field $V$ in (PS) considered in this paper. We denote by $B_\rho(0)$ the two-dimensional disk of radius $\rho > 0$ centered at the origin.

**Assumption 1.1**

1. There is a positive constant $d \in (0, \frac{1}{4})$ such that the complement of the domain $\Omega$ satisfies

$$B_{1-2d}(0) \subset \Omega^c \subset B_{1-d}(0).$$

2. Let the constants $\alpha \in (0, 1)$ and $d \in (0, \frac{1}{4})$ be sufficiently small. Then the vector field $V$ in (PS) satisfies $\text{div} V = 0$ in $\Omega$ and the asymptotic behavior

$$V(x) = \beta U(x) + R(x), \quad x \in \Omega,$$

where $U(x)$ is the rotating flow in [4]. The constant $\beta$ and the remainder $R(x)$ are assumed to satisfy the following conditions with some $\gamma \in (\frac{1}{2}, 1)$ and $\kappa \in (0, 1)$:

$$\beta = \alpha + \tilde{\alpha}_d, \quad |\tilde{\alpha}_d| \leq Cd, \quad \beta \in (0, 1),$$

$$\sup_{x \in \Omega} |x|^{1+\gamma}|R(x)| \leq C\beta^{\kappa}d,$$

where the constant $C$ depends only on $\Omega$ and $\gamma$.

**Remark 1.2**

1. Formally taking $d = 0$ in [5]–[8] we obtain the flow $V = \alpha U$ in the exterior disk $\Omega = \mathbb{R}^2 \setminus B_1(0)$, which solves the following two-dimensional stationary Navier-Stokes equations (SNS): $-\Delta u + u \cdot \nabla u + \nabla p = f$, $\text{div} u = 0$ in $\Omega$, $u = b$ on $\partial \Omega$, and $u \to 0$ as $|x| \to \infty$ with $f = 0$ and $b = \alpha x^\perp$. The vector field $V$ in [6]–[8] describes the flow around $\alpha U$ created from a small perturbation to the exterior disk, and hence, one can naturally expect the existence of such solution $V$ to the nonlinear problem (SNS) if $f$ and $b - \alpha x^\perp$ are sufficiently small with respect to $0 < d \ll 1$. Indeed, imposing the symmetry on the domain perturbation in [5], we can construct the Navier-Stokes flow $V$ satisfying at least [6] and [7] for small symmetric given data, based on the energy method and the recovered Hardy-inequality [11] thanks to the symmetry of the domain $\Omega$ and the remainder $R$. We refer to Galdi [3], Russo [14], Yamazaki [17], and Pileckas and Russo [13] for the solvability of (SNS) under the symmetry condition. The reader is also referred to Hillairet and Wittwer [17] proving the existence of solutions to (SNS) in the exterior disk with $f = 0$ and $b = \alpha x^\perp + \tilde{b}$ when $\alpha$ is large enough and $\tilde{b}$ is sufficiently small.

2. The novelty of our assumption is that we do not impose the symmetry either on the domain $\Omega$ and the flow $V$, and it is a crucial assumption for the stability analysis in [4, 18] to resolve the difficulty related to the lack of the Hardy inequality. While one can realize the exterior disk case in [11] by putting $d = 0$ to [5]–[8] formally. In this sense, the assumption above gives a generalization of the setting in [11] to non-symmetric domain cases.

3. Another motivation for the assumption on $V$ is explained as follows. Let us consider the situation where the obstacle $\Omega^c$ rotates around the origin with a constant speed $\alpha \in \mathbb{R} \setminus \{0\}$. Then the time-periodic Navier-Stokes flow moving with the rotating obstacle gives a stationary solution to the problem (RNS): $\partial_t u - \Delta u - \alpha(x^\perp \cdot \nabla u - u^\perp) + u \cdot \nabla u + \nabla p = f$. 


\[ \text{div} \, u = 0 \text{ in } \Omega, \ u = \alpha x^+ \text{ on } \partial \Omega, \text{ and } u \to 0 \text{ as } |x| \to \infty. \] The reference frame attached to the obstacle; see Hishida [8] for details. The stationary problem of (RNS) is analyzed by Higaki, Maekawa, and Nakahara [6], where the existence and uniqueness of stationary solutions decaying in the order \( O(|x|^{-1}) \) is proved when \( \alpha \) is sufficiently small and \( f \) is of a divergence form \( f = \text{div} \, F \) for some \( F \) which is small in a scale-critical norm. Moreover, the leading profile at spatial infinity is shown to be \( C \frac{1}{|x|} + O(|x|^{-1-\gamma}) \) for some constant \( C \) if \( F \) satisfies a decay condition \( F = O(|x|^{-2-\gamma}) \) with \( \gamma \in (0, 1) \).

The motivation comes from the stability analysis of the stationary solutions to (RNS). Indeed, one can construct the solutions \( V \) to (RNS) satisfying the estimates (6), (7), and (8) with \( \kappa = \frac{1-\gamma}{2} \) under the condition on the domain \( \Omega \) (this result can be shown by extending the proof in [6] but we omit the details). Obviously, letting us denote the linearization to (RNS) by \( \sigma \) we denote by \( (\text{PRS}) \) and \( (\text{PS}) \) are different from each other due to the additional term \(-\alpha (x^+ \cdot \nabla v - v^+) \) in (PRS). However, if we consider the *resolvent problems* of each equation, there are some common features thanks to the property of the term \( \alpha (x^+ \cdot \nabla v - v^+) = \sum_{n \in \mathbb{Z}} i \alpha n P_n v \), which is derived from the Fourier expansion of \( v \{ |x| > 1 \} \); see (20) and (21) in Subsection 2.4. In particular, we can reproduce a similar calculation performed in this paper to the resolvent problem of (PRS), by observing that the appearance of \( \sum_{n \in \mathbb{Z}} i \alpha n P_n v \) in the resolvent equation (restricted on \( |x| > 1 \)) leads to the shifting of the resolvent parameter from \( \lambda \in \mathbb{C} \) to \( \lambda + i \alpha \) in the \( n \)-Fourier mode. Although the stability of the stationary solutions \( V \) to (RNS) still remains open, our analysis in this paper will contribute to the resolvent estimate of the linearized problem (PRS).

Before stating the main result, let us introduce some notations and basic facts related to the problem \( (\text{PS}) \). We denote by \( L^2_0(\Omega) \) the \( L^2 \)-closure of \( C_0^\infty(\Omega) \). The orthogonal projection \( \mathbb{P} : L^2(\Omega)^2 \to L^2_0(\Omega) \) is called the Helmholtz projection. Then the Stokes operator \( \mathbb{A} \) with the domain \( D_{L^2}(\mathbb{A}) = L^2_0(\Omega) \cap W^{1,2}_0(\Omega)^2 \cap W^{2,2}(\Omega)^2 \) is defined as \( \mathbb{A} = -\mathbb{P} \Delta \), and it is well known that the Stokes operator is nonnegative and self-adjoint in \( L^2_0(\Omega) \). Finally we define the perturbed Stokes operator \( \mathbb{A}_V \) as

\[
\begin{align*}
D_{L^2}(\mathbb{A}_V) & = D_{L^2}(\mathbb{A}), \\
\mathbb{A}_V v & = \mathbb{A} v + \mathbb{P}(V \cdot \nabla v + v \cdot \nabla v).
\end{align*}
\]

The perturbation theory for sectorial operators implies that \( -\mathbb{A}_V \) generates a \( C_0 \)-analytic semigroup in \( L^2_0(\Omega) \). We denote this semigroup by \( e^{-t \mathbb{A}_V} \). Then our main result is stated as follows. Let \( d, \beta, \alpha \) be the constants in Assumption 1.1.

**Theorem 1.3** There are positive constants \( \beta_* \) and \( \mu_* \) such that if \( \beta \in (0, \beta_*) \) and \( d \in (0, \mu_* \beta_*^2 - \kappa) \) then the following statement holds. Let \( q \in (1, 2] \). Then we have

\[
\begin{align*}
\|e^{-t \mathbb{A}_V} f \|_{L^2_0(\Omega)} & \leq C \beta^{-\frac{1}{q}+\frac{1}{2}} \| f \|_{L^q(\Omega)}, & t > 0, \\
\|\nabla e^{-t \mathbb{A}_V} f \|_{L^2_0(\Omega)} & \leq C \beta^{-\frac{1}{q}+\frac{1}{2}} \| f \|_{L^q(\Omega)}, & t > 0,
\end{align*}
\]

for \( f \in L^q_0(\Omega) \cap L^q(\Omega)^2 \). Here the constant \( C \) is independent of \( \beta \) and depends on \( q \).

As an application of Theorem 1.3, we can prove the nonlinear stability of \( V \) for the Navier-Stokes equations, whose integral form is given by

\[
v(t) = e^{-t \mathbb{A}_V} v_0 - \int_0^t e^{-(t-s) \mathbb{A}_V} \mathbb{P}(v \cdot \nabla v)(s) \, ds, \quad t > 0.
\]

\[\text{{INS}}\]
The proof of the following result is omitted in this paper, since the argument is quite straightforward using the Banach fixed point theorem.

**Theorem 1.4** Let $\beta_*$ and $\mu_*$ be the constants in Theorem 1.3. Then there is a positive constant $\nu_*$ such that if $\beta \in (0, \beta_*),$ $d \in (0, \mu_* \beta_*^{\frac{1}{2}-\gamma}),$ and $\|v_0\|_{L^2(\Omega)} \in (0, \nu_\beta \beta^2)$ then there exists a unique solution $v \in C([0, \infty); L^2(\Omega)) \cap C((0, \infty); W^{1,2}_0(\Omega)^2)$ to (INS) satisfying

$$\lim_{t \to \infty} t^{\frac{k}{2}} \|\nabla^k v\|_{L^2(\Omega)} = 0, \quad k = 0, 1. \quad (12)$$

The proof of Theorem 1.4 relies on the resolvent estimate to the perturbed Stokes operator $A_V.$ Since the difference $A_V - A$ is a compact operator, one can show that the spectrum of $-A_V$ has the structure $\sigma(-A_V) = [0, \infty) \cup \sigma_{disc}(-A_V)$ in $L^2_2(\Omega),$ where $\sigma_{disc}(-A_V)$ denotes the set of discrete spectrum of $-A_V$; see [11, Lemma 2.11 and Proposition 2.12].

By using the identity $v \cdot \nabla v = \frac{1}{2} \nabla |v|^2 + v^\perp \cdot \nabla v$ with $rot v = \partial_1 v_2 - \partial_2 v_1$ and $rot U = 0$ in $x \in \Omega,$ we can write the resolvent problem associated with (PS) as

$$\begin{cases}
\lambda v - \Delta v + \beta U^\perp \cdot \nabla v + div (R \otimes v + v \otimes R) + \nabla q = f, & x \in \Omega, \\
div v = 0, & x \in \Omega, \\
v|_{\partial \Omega} = 0.
\end{cases} \quad (RS)$$

Here $\lambda \in \mathbb{C}$ is the resolvent parameter and we have used the conditions $\text{div} v = \text{div} R = 0$ to derive $R \cdot \nabla v + v \cdot \nabla R = \text{div} (R \otimes v + v \otimes R).$ Hence, the proof of Theorem 1.3 is complete as soon as we show that there is a sector $\Sigma$ included in the resolvent set $\rho(-A_V),$ and that the following estimates to (RS) hold for $q \in (1, 2)$ and $f \in L^2_2(\Omega) \cap L^q(\Omega)^2.$

$$\begin{align*}
&\|(\lambda + A_V)^{-1} f\|_{L^2(\Omega)} \leq C\beta^\frac{1}{2} |\lambda|^{-\frac{n}{2} + \frac{1}{q}} \|f\|_{L^q(\Omega)}, & \lambda \in \Sigma \cap \{\text{Re}(z) < 0\}, \\
&\|\nabla(\lambda + A_V)^{-1} f\|_{L^2(\Omega)} \leq C\beta^{-\frac{1}{2}} |\lambda|^{-\frac{n}{2} + \frac{1}{q}} \|f\|_{L^q(\Omega)}, & \lambda \in \Sigma \cap \{\text{Re}(z) < 0\}. \quad (13)
\end{align*}$$

Let us prepare the ingredients for the proof of the resolvent estimates (13). Our approach is based on the energy method to (RS), and thus one of the most important steps is to obtain the estimate for the term $\|\langle \beta U^\perp \cdot \nabla v, v \rangle_{L^2(\Omega)}\|$ which enables us to close the energy computation. Again we note that the bound $\|\langle \beta U^\perp \cdot \nabla v, v \rangle_{L^2(\Omega)}\| \leq C\beta^2 \|\nabla v\|_{L^2(\Omega)}^2$ is no longer available contrary to the three-dimensional cases.

Firstly let us examine the next inequality containing the parameter $T \gg 1$:

$$\|\langle \beta U^\perp \cdot \nabla v, v \rangle_{L^2(\Omega)}\| \leq \frac{\beta}{T} \|\nabla v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + C\beta \Theta(T) \|\nabla v\|_{L^2(\Omega)}^2, \quad (14)$$

where the function $\Theta(T)$ satisfies $\Theta(T) \approx \log T$ if $T \gg 1.$ This inequality leads to the closed energy computation for (RS), as long as the coefficient $C \beta \Theta(T)$ is small enough so that the second term in the right-hand side of (14) can be controlled by the dissipation from the Laplacian in (RS). However, this observation does not give the information about the spectrum of $-A_V$ near the origin. More precisely, we cannot close the energy computation when the resolvent parameter $\lambda$ is exponentially small with respect to $\beta,$ that is, when $0 < |\lambda| \leq O(e^{-\frac{1}{\beta}}).$ We emphasize that this difficulty is essentially due to the unavailability of the Hardy inequality (1) in two-dimensional exterior domains.
To overcome the difficulty for the case $0 < |\lambda| \leq O(e^{-\frac{1}{d\beta}})$, we rely on the representation formula to the resolvent problem in the exterior unit disk established in \[11\]. Since the restriction $\{v_1|\{x|>1\}, q_1|\{x|>1\}\}$ gives a unique solution to the next problem for $(w, r)$:
\[
\begin{align*}
\lambda w - \Delta w + \beta U^{-1} \text{rot } w + \nabla r &= -\text{div } (R \otimes v + v \otimes R) + f, \quad |x| > 1, \\
d\text{div } w &= 0, \quad |x| > 1, \\
w_1|_{\{x|=1\}} &= v_1|_{\{x|=1\}},
\end{align*}
\]
we can study the a priori estimates of $w = v_1|_{\{x|>1\}}$ based on the solution formula to $\text{(RS)}^\text{ed}$. Then a detailed calculation shows that $|\langle \beta U^{-1} \text{rot } v, v \rangle_{L^2(\{|x|>1\})}|$ satisfies
\[
|\langle \beta U^{-1} \text{rot } v, v \rangle_{L^2(\{|x|>1\})}| \leq \frac{C}{\beta^4} \left( \| R \otimes v + v \otimes R \|_{L^2(\Omega)} + \beta \| v \|_{L^2(\{|x|=1\})} \right)^2 + \frac{C}{\beta^4} |\lambda|^{-\frac{2}{d} + \frac{2}{\beta}} \| f \|_{L^2(\Omega)}^2 + C \beta \| \nabla v \|_{L^2(\Omega)}^2,
\]
and once we obtain (15) then the estimate of $|\langle \beta U^{-1} \text{rot } v, v \rangle_{L^2(\Omega)}|$ is derived by using the Poincaré inequality on the bounded domain $\Omega \setminus \{ |x| \geq 1 \}$. However, in closing the energy computation, we need to be careful about the $\beta$-singularity in the coefficients in (15). In fact, the first term in the right-hand side of (15) have to be controlled by the dissipation as
\[
\frac{C}{\beta^4} \left( \| R \otimes v + v \otimes R \|_{L^2(\Omega)} + \beta \| v \|_{L^2(\{|x|=1\})} \right)^2 \leq \frac{Cd^2}{\beta^4 - 2d} \| \nabla v \|_{L^2(\Omega)}^2,
\]
and then the smallness of the coefficient $Cd^2 \beta^{-4 + 2d} \ll 1$ is required in order to close the energy computation. This condition is achieved by imposing the smallness on the distance $d$ between the domain $\Omega$ and the exterior unit disk, which is introduced in Assumption \[11\].

This paper is organized as follows. In Section 2 we recall some basic facts from vector calculus in polar coordinates, and derive the resolvent estimate to $\text{(RS)}$ when $|\lambda| \geq O(\beta^2 e^{-\frac{1}{d\beta}})$ by a standard energy method. In Section 3 the resolvent problem is discussed for the case $0 < |\lambda| < e^{-\frac{1}{d\beta}}$. In Subsections 3.1 and 3.2 we derive the estimates to the problem $\text{(RS)}^\text{ed}$ by using the representation formula. The results in Subsections 3.1 and 3.3 are applied in Subsection 3.4, where the resolvent estimate to $\text{(RS)}$ is established in the expected region $0 < |\lambda| < e^{-\frac{1}{d\beta}}$. Section 4 is devoted to the proof of Theorem 1.3.

2 Preliminaries

This section is devoted to the preliminary analysis on the resolvent problem $\text{(RS)}$ and $\text{(RS)}^\text{ed}$ in the introduction. In Subsections 2.1 and 2.2 we recall some basic facts from vector calculus in polar coordinates. In Subsection 2.3 we show that the resolvent estimates in (13) are valid if the resolvent parameter $\lambda$ satisfies $|\lambda| \geq O(\beta^2 e^{-\frac{1}{d\beta}})$. Throughout this section, let us denote by $D$ the exterior unit disk $\mathbb{R}^2 \setminus B_1(0) = \{ x \in \mathbb{R}^2 \mid |x| > 1 \}$.

2.1 Vector calculus in polar coordinates and Fourier series

We introduce the usual polar coordinates on $D$. Set
\[
x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad r = |x| \geq 1, \quad \theta \in [0, 2\pi),
\]
\( e_r = \frac{x}{|x|}, \quad e_\theta = \frac{x^\perp}{|x|} = \partial_\theta e_r. \)

Let \( v = (v_1, v_2)^T \) be a vector field defined on \( D \). Then we set

\[
v = v_r e_r + v_\theta e_\theta, \quad v_r = v \cdot e_r, \quad v_\theta = v \cdot e_\theta.
\]

The following formulas will be used:

\[
\text{div} \, v = \partial_1 v_1 + \partial_2 v_2 = \frac{1}{r} \partial_r (rv_r) + \frac{1}{r} \partial_\theta v_\theta, \tag{16}
\]

\[
\text{rot} \, v = \partial_1 v_2 - \partial_2 v_1 = \frac{1}{r} \partial_r (rv_\theta) - \frac{1}{r} \partial_\theta v_r, \tag{17}
\]

\[|\nabla v|^2 = |\partial_r v_r|^2 + |\partial_r v_\theta|^2 + \frac{1}{r^2} (|\partial_\theta v_r - v_\theta|^2 + |v_r + \partial_\theta v_\theta|^2), \tag{18}
\]

and

\[
-\Delta v = \left( -\partial_r \left( \frac{1}{r} \partial_r (rv_r) \right) - \frac{1}{r^2} \partial_\theta^2 v_r + \frac{2}{r^2} \partial_\theta v_\theta \right) e_r \\
+ \left( -\partial_r \left( \frac{1}{r} \partial_r (rv_\theta) \right) - \frac{1}{r^2} \partial_\theta^2 v_\theta - \frac{2}{r^2} \partial_\theta v_r \right) e_\theta. \tag{19}
\]

The formulas

\[
e_r \cdot \nabla v = (\partial_r v_r) e_r + (\partial_\theta v_\theta) e_\theta, \quad e_\theta \cdot \nabla v = \frac{\partial_\theta v_r - v_\theta}{r} e_r + \frac{\partial_\theta v_\theta + v_r}{r} e_\theta
\]

imply the following equality:

\[
x^\perp \cdot \nabla v - v^\perp = |x|(e_\theta \cdot \nabla v) - (v_r e_\perp^r + v_\theta e_\perp^\theta) \\
= (\partial_\theta v_r - v_\theta) e_r + (\partial_\theta v_\theta + v_r) e_\theta - (v_r e_\perp^r + v_\theta e_\perp^\theta) \\
= \partial_\theta v_r e_r + \partial_\theta v_\theta e_\theta. \tag{20}
\]

For each \( n \in \mathbb{Z} \), we denote by \( P_n \) the projection on the Fourier mode \( n \) with respect to the angular variable \( \theta \):

\[
P_n v = v_{r,n}(r) e^{in\theta} e_r + v_{\theta,n}(r) e^{in\theta} e_\theta, \tag{21}
\]

where

\[
v_{r,n}(r) = \frac{1}{2\pi} \int_0^{2\pi} v_r(r \cos \theta, r \sin \theta) e^{-in\theta} \, d\theta,
\]

\[
v_{\theta,n}(r) = \frac{1}{2\pi} \int_0^{2\pi} v_\theta(r \cos \theta, r \sin \theta) e^{-in\theta} \, d\theta.
\]

We also set for \( m \in \mathbb{N} \cup \{0\},

\[
Q_m v = \sum_{|n|=m+1} \infty P_n v. \tag{22}
\]
For notational simplicity we often write \( v_n \) instead of \( \mathcal{P}_n v \). Each \( \mathcal{P}_n \) defines an orthogonal projection in \( L^2(D) \). From (18) and (21), for \( n \in \mathbb{N} \cup \{0\} \) and \( v \) in \( W^{1,2}(D)^2 \) we see that
\[
\|\nabla v\|_{L^2(D)}^2 = \sum_{n \in \mathbb{Z}} \|\nabla \mathcal{P}_n v\|_{L^2(D)}^2 ,
\]
\[
|\nabla \mathcal{P}_n v|^2 = |\partial_r v_{r,n}|^2 + \frac{1 + n^2}{r^2} |v_{r,n}|^2 + |\partial_{\theta} v_{\theta,n}|^2 + \frac{1 + n^2}{r^2} |v_{\theta,n}|^2 - \frac{4n}{r^2} \text{Im}(v_{\theta,n} v_{r,n}) .
\]
In particular, we have
\[
|\nabla \mathcal{P}_n v|^2 \geq |\partial_r v_{r,n}|^2 + \frac{(|n| - 1)^2}{r^2} |v_{r,n}|^2 + |\partial_{\theta} v_{\theta,n}|^2 + \frac{(|n| - 1)^2}{r^2} |v_{\theta,n}|^2 ,
\]
and thus, from the definition of \( Q_m \) in (22), we have for \( m \in \mathbb{N} \cup \{0\} \),
\[
\|\nabla Q_m v\|_{L^2(D)}^2 \geq |\partial_r (Q_m v)_r|^2 + |\partial_{\theta} (Q_m v)_{\theta}|^2 + m^2 \|v\|_{W^{1,2}(D)}^2 .
\]

2.2 The Biot-Savart law in polar coordinates

For a given scalar field \( \omega \) in \( D \), the streamfunction \( \psi \) is formally defined as the solution to the Poisson equation: \(-\Delta \psi = \omega \) in \( D \) and \( \psi = 0 \) on \( \partial D \). For \( n \in \mathbb{Z} \) and \( \omega \in L^2(D) \) we set \( \mathcal{P}_n \omega = \mathcal{P}_n \omega(r, \theta) \) and \( \omega_n = \omega_n(r) \) as
\[
\mathcal{P}_n \omega = \left( \frac{1}{2\pi} \int_0^{2\pi} \omega(r \cos s, r \sin s) e^{-in s} \, ds \right) e^{in\theta} , \quad \omega_n = (\mathcal{P}_n \omega) e^{-in\theta} .
\]
From the Poisson equation in polar coordinates, we see that each \( n \)-Fourier mode of \( \psi \) satisfies the following ODE:
\[
-\frac{d^2 \psi_n}{dr^2} - \frac{1}{r} \frac{d \psi_n}{dr} + \frac{n^2}{r^2} \psi_n = \omega_n , \quad r > 1 , \quad \psi_n(1) = 0 .
\]
Let \( |n| \geq 1 \). Then the solution \( \psi_n = \psi_n[\omega_n] \) to (26) decaying at spatial infinity is given by
\[
\psi_n[\omega_n](r) = \frac{1}{2n} \left( - \frac{d \omega_n}{r |n|} + \frac{n}{r |n|} \int_1^r s^{1+|n|} \omega_n(s) \, ds + r |n| \int_r^\infty s^{-1-|n|} \omega_n(s) \, ds \right) ,
\]
\[
d_n[\omega_n] = \int_1^\infty s^{-1-|n|} \omega_n(s) \, ds .
\]
The formula \( V_n[\omega_n] \) in the next is called the Biot-Savart law for \( \mathcal{P}_n \omega \):
\[
V_n[\omega_n] = V_{r,n}[\omega_n](r)e^{in\theta} e_r + V_{\theta,n}[\omega_n](r)e^{in\theta} e_\theta ,
\]
\[
V_{r,n}[\omega_n] = \frac{im}{r} \psi_n[\omega_n] , \quad V_{\theta,n}[\omega_n] = -\frac{1}{r} \frac{d}{dr} \psi_n[\omega_n] .
\]
The velocity \( V_n[\omega_n] \) is well defined at least when \( r^{1-|n|} \omega_n \in L^1((1, \infty)) \), and it is straightforward to see that
\[
\text{div} V_n[\omega_n] = 0 , \quad \text{rot} V_n[\omega_n] = \mathcal{P}_n \omega \quad \text{in} \quad D ,
\]
\[
e_r \cdot V_n[\omega_n] = 0 \quad \text{on} \quad \partial D .
\]
The condition \( r^{1-n}|n| \omega_n \in L^1((1, \infty)) \) is automatically satisfied when \( \omega \in L^2(D) \) and \( |n| \geq 2 \). When \( |n| = 1 \), however, the integral in the definition of \( \psi_n \omega_n \) does not converge absolutely for general \( \omega \in L^2(\Omega) \). We can justify this integral for \( |n| = 1 \) if \( \omega \) is given in a rotation form \( \omega = \text{rot } u \) with some \( u \in W^{1,2}(D)^2 \), since the integration by parts leads to the convergence of \( \lim_{N \to \infty} \int_{\Gamma}^N \omega_n \, dv \). Hence, for any \( v \in L^2_n(\Omega) \cap W^{1,2}(\Omega)^2 \), the \( n \)-mode \( v_n = \mathcal{P}_n v \) can be expressed in terms of its vorticity \( \omega_n \) by the formula (27) when \( |n| \geq 1 \).

### 2.3 A priori resolvent estimate by energy method

In this subsection we study the energy estimate to the resolvent problem \( (RS) \):

\[
\begin{align*}
\lambda v - \Delta v + \beta U^{-1} \text{rot } v + \text{div } (R \otimes v + v \otimes R) + \nabla q &= f, \quad x \in \Omega, \\
\text{div } v &= 0, \quad x \in \Omega, \\
v|_{\partial \Omega} &= 0.
\end{align*}
\]

(RS)

Here \( \lambda \in \mathbb{C} \) is the resolvent parameter, the vector field \( U \) is the rotating flow of \( \Psi \) in the introduction, and \( \beta \) and \( R \) are defined in Assumption \( 1.1 \). The first result of this subsection is the a priori estimates to \( (RS) \) obtained by the energy method. We recall that \( D \) denotes the exterior disk \( \{ x \in \mathbb{R}^2 \mid |x| > 1 \} \), and that \( \gamma \) and \( \kappa \) are the constants in Assumption \( 1.1 \).

**Proposition 2.1** Let \( q \in (1, 2], f \in L^2(\Omega)^2 \), and \( \lambda \in \mathbb{C} \). Suppose that \( v \in D(\mathcal{A}_V) \) is a solution to \( (RS) \). Then there is a constant \( \beta_1 \in (0, 1) \) depending only on \( \Omega, \gamma, \) and \( \kappa \) such that the following estimates hold.

\[
\begin{align*}
\text{Re}(\lambda) \| v \|_{L^2(\Omega)}^2 + \frac{3}{4} \| \nabla v \|_{L^2(\Omega)}^2 &\leq \beta \sum_{|n|=1} \langle \text{rot } v, \frac{v_n}{|x|} \rangle_{L^2(D)} + C \| f \|_{L^2(\Omega)}^\frac{2\gamma}{\gamma - 2} \| v \|_{L^2(\Omega)}^\frac{4(\gamma - 1)}{\gamma - 2}, \\
|\text{Im}(\lambda)\| v \|_{L^2(\Omega)}^2 &\leq \frac{1}{4} \| \nabla v \|_{L^2(\Omega)}^2 + \beta \sum_{|n|=1} \langle \text{rot } v, \frac{v_n}{|x|} \rangle_{L^2(D)} + C \| f \|_{L^2(\Omega)}^\frac{2\gamma}{\gamma - 2} \| v \|_{L^2(\Omega)}^\frac{4(\gamma - 1)}{\gamma - 2},
\end{align*}
\]

(29)

as long as \( \beta \in (0, \beta_1) \). The constant \( C \) is independent of \( \beta \).

**Proof:** Taking the inner product with \( v \) to the first equation of \( (RS) \), we find

\[
\begin{align*}
\text{Re}(\lambda) \| v \|_{L^2(\Omega)}^2 + \| \nabla v \|_{L^2(\Omega)}^2 &= -\beta \text{Re}(U^{-1} \text{rot } v, v)_{L^2(\Omega)} + \text{Re}(R \otimes v + v \otimes R, \nabla v)_{L^2(\Omega)} + \text{Re}(f, v)_{L^2(\Omega)}, \\
\text{Im}(\lambda) \| v \|_{L^2(\Omega)}^2 &= -\beta \text{Im}(U^{-1} \text{rot } v, v)_{L^2(\Omega)} + \text{Im}(R \otimes v + v \otimes R, \nabla v)_{L^2(\Omega)} + \text{Im}(f, v)_{L^2(\Omega)}.
\end{align*}
\]

(30)

After decomposing the domain \( \Omega = (\Omega \setminus D) \cup D \), from \( U^+ = -\frac{v_n}{r} \) on \( D \) we have

\[
\beta |U^+ \text{rot } v, v|_{L^2(\Omega)} \leq \beta |U^+ \text{rot } v, v|_{L^2(\Omega \setminus D)} + \beta |\langle \text{rot } v, \frac{v_n}{|x|} \rangle_{L^2(D)}|.
\]

(33)
Then the Poincare inequality on $\Omega \setminus D$ implies that
\[
\beta \| (U^\perp \rot v, v)_{L^2(\Omega, \partial D)} \| \leq C \beta \| \nabla v \|_{L^2(\Omega)}^2,
\] (34)
and by applying the Fourier series expansion on $D$, we see from (21) and (25) that
\[
\left| \langle \rot v, \frac{v_{r,n}}{|x|} \rangle_{L^2(D)} \right| = \left| \left( \sum_{|n| = 1} + \sum_{n \in Z\setminus\{\pm 1\}} \right) \langle \rot v, \frac{v_{r,n}}{|x|} \rangle_{L^2(D)} \right|
\leq \left| \sum_{|n| = 1} \langle \rot v, \frac{v_{r,n}}{|x|} \rangle_{L^2(D)} \right| + \sum_{n \in Z\setminus\{\pm 1\}} \| \rot v_n \|_{L^2(D)} \| \frac{v_{r,n}}{|x|} \|_{L^2(D)}. \tag{35}
\]
Then the inequality (24) ensures that
\[
\sum_{n \in Z\setminus\{\pm 1\}} \| \rot v_n \|_{L^2(D)} \| \frac{v_{r,n}}{|x|} \|_{L^2(D)} \leq C \sum_{n \in Z\setminus\{\pm 1\}} \| \nabla v_n \|_{L^2(\Omega)}^2 \leq C \| \nabla v \|_{L^2(\Omega)}^2. \tag{36}
\]
Inserting (34)–(36) into (33) we obtain
\[
\beta \| (U^\perp \rot v, v)_{L^2(\Omega)} \| \leq C_1 \beta \| \nabla v \|_{L^2(\Omega)}^2 + \beta \left| \sum_{|n| = 1} \langle \rot v, \frac{v_{r,n}}{|x|} \rangle_{L^2(D)} \right|. \tag{37}
\]
Next by (8) in Assumption 1.1 we have
\[
\| (R \otimes v + v \otimes R, \nabla v)_{L^2(\Omega)} \| \leq C_2 \beta \| \nabla v \|_{L^2(\Omega)}^2, \tag{38}
\]
where the inequality $\| |x|^{-(1+\gamma)} v \|_{L^2} \leq C \| \nabla v \|_{L^2}$ is applied. The constant $C_2$ depends on $\gamma \in (0, 1)$. By the Gagliardo-Nirenberg inequality we see that for $q \in (1, 2]$ and $q' = \frac{q}{q-1}$,
\[
\| f, v \|_{L^2(\Omega)} \| \leq C \| f \|_{L^q(\Omega)} \| v \|_{L^{q'}(\Omega)} \tag{39}
\]
where the Young inequality is applied in the last line. Now we take $\beta_1$ small enough so that
\[
C_1 \beta_1 + C_2 \beta_1 d \leq 2 \max\{C_1, C_2\} \beta_1^\kappa \leq \frac{1}{8}, \tag{40}
\]
holds for $\kappa \in (0, 1)$. Then the assertions (29) and (30) are proved by inserting (37)–(39) into (31) and (32), and using the condition (40). This completes the proof. □

As can be seen from Proposition 2.1 the key object in closing the energy computation is to derive the estimate for the next term appearing in the right-hand sides of (29) and (30):
\[
\left| \sum_{|n| = 1} \langle \rot v, \frac{v_{r,n}}{|x|} \rangle_{L^2(D)} \right|.
\]
Note that the Hardy inequality in polar coordinates (23) cannot be applied to this term. The next proposition shows that this term can be handled if $\lambda$ in (RS) satisfies $|\lambda| \geq O(\beta^2 e^{-\eta d/2})$. 

10
Proposition 2.2 Let $\beta_1$ be the constant in Proposition 2.1. Then the following statements hold.

(1) Fix a positive number $\beta_2 \in (0, \min\{\frac{1}{12}, \beta_1\})$. Then the set

$$S_{\beta} = \{ z \in \mathbb{C} \mid |\text{Im}(\lambda)| > -\text{Re}(\lambda) + 12e^{\frac{1}{2}}\beta_2e^{-\frac{1}{2} \beta_2} \}$$

(41)
is included in the resolvent $\rho(-\mathcal{A}_v)$ for any $\beta \in (0, \beta_2)$.

(2) Let $q \in (1, 2]$ and $f \in L^q(\Omega) \cap L^q(\Omega)^2$. Then we have

$$\| (\lambda + \mathcal{A}_v)^{-1} f \|_{L^2(\Omega)} \leq C|\lambda|^{-\frac{3}{2} + \frac{1}{q}} \| f \|_{L^q(\Omega)}, \quad \lambda \in S_{\beta} \cap \{\text{Re}(z) < 0\},$$

(42)

and using the nondegenerate condition \( \{x \in \mathbb{R}^2 \mid |x| \leq \frac{1}{2}\} \subset \Omega^c \). Then the Young inequality yields

$$\frac{\beta}{T} \| v \|_{L^2} \| \nabla v \|_{L^2} \leq \frac{\beta \Theta(T)}{2} \| \nabla v \|_{L^2}^2 + \frac{\beta}{2T^2 \Theta(T)} \| v \|_{L^2}^2.$$

(43)

Inserting (43) into (29) and (30) in Proposition 2.1 we see that

$$\left(\frac{\text{Re}(\lambda)}{2T^2 \Theta(T)} \right) \| v \|_{L^2}^2 + \left(\frac{1}{4} - 3\frac{\beta \Theta(T)}{2} \right) \| \nabla v \|_{L^2}^2 \leq C \| f \|_T \frac{2q}{3q-2} \| v \|_{L^2}^2 \frac{4(q-1)}{2q-2},$$

(47)

$$\left( \frac{\| |\text{Im}(\lambda)| - \frac{\beta}{T^2 \Theta(T)} \| v \|_{L^2}^2}{2} \right) \leq \left(\frac{1}{4} + 3\frac{\beta \Theta(T)}{2} \right) \| \nabla v \|_{L^2}^2 + C \| f \|_T \frac{2q}{3q-2} \| v \|_{L^2}^2 \frac{4(q-1)}{2q-2}. \leq C \| f \|_T \frac{2q}{3q-2} \| v \|_{L^2}^2 \frac{4(q-1)}{2q-2},$$

(48)

Then (47) and (48) lead to

$$\left( |\text{Im}(\lambda)| + \text{Re}(\lambda) - \frac{\beta}{T^2 \Theta(T)} \right) \| v \|_{L^2}^2 + \left(\frac{1}{2} - 3\beta \Theta(T) \right) \| \nabla v \|_{L^2}^2 \leq C \| f \|_T \frac{2q}{3q-2} \| v \|_{L^2}^2 \frac{4(q-1)}{2q-2}. \leq C \| f \|_T \frac{2q}{3q-2} \| v \|_{L^2}^2 \frac{4(q-1)}{2q-2},$$

(49)
Now let us take \( T = e^{\frac{1}{2}\beta \rho} \). Since \( T > e \) by the condition \( \beta \in (0, \frac{1}{12}) \), from \((44)\) we have

\[
3\beta \Theta(T) \leq 3\beta \log T = \frac{1}{4} \quad \text{and} \quad \frac{\beta}{T^2 \Theta(T)} \leq \frac{e^{\frac{1}{4}\beta}}{T^2 \log T} = 12e^{\frac{1}{4}\beta}2e^{-\frac{3}{8}}.
\]

(50)

By inserting \((50)\) into \((49)\) we obtain the assertion \( S_\beta \subset \rho(-A_V) \).

(2) Let \( \lambda \in S_\beta \cap \{ z \in \mathbb{C} \mid \Re(z) < 0 \} \). Note that this condition ensures that

\[
|\Im(\lambda)| \geq \frac{\beta}{T^2 \Theta(T)} \quad \text{and} \quad |\Im(\lambda)| \leq \sqrt{2}|\Im(\lambda)|.
\]

Then we see from \((48)\) and \((49)\) that,

\[
|\lambda||v||^2_{L^2} \leq \frac{6\sqrt{2}}{8} \|\nabla v\|^2_{L^2} + 2\sqrt{2}C||f||_{L^2}^2 \|v\|^2_{L^2} \quad \text{and} \quad \|\nabla v\|^2_{L^2} \leq 4C||f||_{L^2}^2 \|v\|^2_{L^2},
\]

(51)

(52)

where the constant \( C \) is independent of \( \beta \). The estimates in \((42)\) follow from \((51)\) and \((52)\). This completes the proof of Proposition 2.2. \( \square \)

3 Resolvent analysis in region exponentially close to the origin

The resolvent analysis in Proposition 2.2 is applicable to the problem \((RS)\) only when the resolvent parameter \( \lambda \in \mathbb{C} \) satisfies \( |\lambda| \geq e^{-\frac{1}{a}} \) for some \( a \in (1, \infty) \), and we have taken \( a = 6 \) in the proof for simplicity. This restriction is essentially due to the unavailability of the Hardy inequality in two-dimensional exterior domains. In fact, in the proof of Proposition 2.2 we rely on the following inequality singular in \( T \gg 1 \):

\[
\left| \sum_{|n|=1} \langle (\text{rot} \ v)_n, \frac{v_n}{|x|} \rangle_{L^2(D)} \right| \leq \frac{1}{T} ||v||_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \log T \|\nabla v\|^2_{L^2(\Omega)},
\]

as a substitute for the Hardy inequality, and this leads to the lack of information about the spectrum of \(-A_V\) in the region \( 0 < |\lambda| \leq O(e^{-\frac{1}{a}}) \). Here we set \( D = \{ x \in \mathbb{R}^2 \mid |x| > 1 \} \).

To perform the resolvent analysis in the region exponentially close to the origin, we firstly observe that a solution \((v, q)\) to \((RS)\) satisfies the next problem in the exterior disk \( D \):

\[
\begin{cases}
\lambda w - \Delta w + \beta U^\perp \text{rot} \ w + \nabla r = (-\text{div} (R \otimes v + v \otimes R) + f)|_D, & x \in D, \\
\text{div} \ w = 0, & x \in D, \\
w|_{\partial D} = v|_{\partial D}.
\end{cases}
\]

\((RS^{ed})\)

Then thanks to the symmetry, we can use a solution formula to \((RS^{ed})\) by using polar coordinates, and study the a priori estimate for \( w = v|_D \). To make calculation simple, we decompose the linear problem \((RS^{ed})\) into three parts \((RS^{ed/F}), (RS^{ed/F}), \text{ and } (RS^{ed}),\) which are respectively introduced in Subsections 3.1, 3.2, and 3.3. Then we derive the estimates to each problem in the corresponding subsections, and finally we collect them in Subsection 3.4 in order to establish the resolvent estimate to \((RS)\) when \( 0 < |\lambda| < e^{-\frac{1}{a}} \).
3.1 Problem I: External force \( f \) and Dirichlet condition

In this subsection we study the following resolvent problem for \((w,r) = (w^\text{ed},r^\text{ed})\):

\[
\begin{cases}
\lambda w - \Delta w + \beta U^\perp \text{rot} w + \nabla r = f, & x \in D, \\
\text{div} w = 0, & x \in D, \\
w|_{\partial D} = 0.
\end{cases}
\]  

\((\text{RS}^f_{\text{ed}})\)

Especially, we are interested in the estimates for the \(\pm 1\)-Fourier mode of \(w^\text{ed}\). Although the \(L^p-L^q\) estimates to \((\text{RS}^f_{\text{ed}})\) are already proved in [11], we revisit this problem here in order to study the \(\beta\)-dependence in these estimates, and it is one of the most important steps for the energy computation when \(0 < |\lambda| < e^{-\frac{1}{6}}\).

Let us recall the representation formula established in [11] for the solution to \((\text{RS}^f_{\text{ed}})\) in each Fourier mode. Fix \(n \in \mathbb{Z} \setminus \{0\}\) and \(\lambda \in \mathbb{C} \setminus \mathbb{R}^-, \mathbb{R}^- = (-\infty, 0]\). Then, by applying the Fourier mode projection \(P_n\) to \((\text{RS}^f_{\text{ed}})\) and using the invariant property \(P_n(U^\perp \text{rot} w) = U^\perp \text{rot} P_n w\) in [11, Lemma 2.9], we observe that the \(n\)-mode \(w_n = P_n w\) solves

\[
\begin{cases}
\lambda w_n - \Delta w_n + \beta U^\perp \text{rot} w_n + P_n \nabla r = f_n, & x \in D, \\
\text{div} w_n = 0, & x \in D, \\
w_n|_{\partial D} = 0.
\end{cases}
\]  

\((\text{RS}^f_{\text{n,ed}})\)

Since the formula in [11] is written in terms of some special functions, we introduce these definitions here. The modified Bessel function of first kind \(I_\mu(z)\) of order \(\mu\) is defined as

\[
I_\mu(z) = (\frac{z}{2})^\mu \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(\mu + m + 1)} \left(\frac{z}{2}\right)^{2m}, \quad z \in \mathbb{C} \setminus \mathbb{R}^-, \tag{53}
\]

where \(z^\mu = e^{\mu \text{Log} z}\) and \(\text{Log} z\) denotes the principal branch to the logarithm of \(z \in \mathbb{C} \setminus \mathbb{R}^-,\) and the function \(\Gamma(z)\) in (53) denotes the Gamma function. Next we define the modified Bessel function of second kind \(K_\mu(z)\) of order \(\mu \notin \mathbb{Z}\) in the following manner:

\[
K_\mu(z) = \frac{\pi}{2} \frac{I_{-\mu}(z) - I_\mu(z)}{\sin \mu \pi}, \quad z \in \mathbb{C} \setminus \mathbb{R}^-.
\]  

\((\text{54})\)

It is classical that \(K_\mu(z)\) and \(I_\mu(z)\) are linearly independent solutions to the ODE

\[
-\frac{d^2 \omega}{dz^2} - \frac{1}{z} \frac{d \omega}{dz} + \left(1 + \frac{\mu^2}{z^2}\right) \omega = 0,
\]  

\((\text{55})\)

and that their the Wronskian is \(z^{-1}\). Applying the rotation operator \(\text{rot}\) to the first equation of \((\text{RS}^f_{\text{n,ed}})\), we find that \(\omega = (\text{rot} w)_n = (\text{rot} w_n)e^{-in\theta}\) satisfies the ODE

\[
-\frac{d^2 \omega}{dr^2} - \frac{1}{r} \frac{d \omega}{dr} + \left(\lambda \frac{r^2 + in\beta}{r^2}\right) \omega = (\text{rot} f)_n, \quad r > 1.
\]  

\((\text{56})\)

Hence, if we set

\[
\mu_n = \mu_n(\beta) = \left(n^2 + in\beta\right)^{\frac{1}{2}}, \quad \text{Re}(\mu_n) > 0,
\]  

\((\text{57})\)
then $K_{\mu_n}(\sqrt{r})$ and $I_{\mu_n}(\sqrt{r})$ give linearly independent solutions to the homogeneous equation of (56) and their Wronskian is $r^{-1}$. Here and in the following we always take the square root $\sqrt{z}$ so that $\text{Re}(\sqrt{z}) > 0$ for $z \in \mathbb{C} \setminus \mathbb{R}^-$. Furthermore, we set
\[
F_n(\sqrt{\lambda} ; \beta) = \int_1^\infty s^{1-|n|} K_{\mu_n}(\sqrt{\lambda}s) \, ds, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}^-, \tag{58}
\]
and denote by $\mathcal{Z}(F_n)$ the set of the zeros of $F_n(\sqrt{\lambda} ; \beta)$ lying in $\mathbb{C} \setminus \mathbb{R}^-$;
\[
\mathcal{Z}(F_n) = \{ z \in \mathbb{C} \setminus \mathbb{R}^- \mid F_n(z; \beta) = 0 \}. \tag{59}
\]
Let $\lambda \in \mathbb{C} \setminus (\mathbb{R}^- \cup \mathcal{Z}(F_n))$. Then, from the argument in [11, Section 3], we have the following representation formula for $w_{f,n}^{ed}$ solving (RS$_{f,n}$):
\[
w_{f,n}^{ed} = -\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda} ; \beta)} V_n[K_{\mu_n}(\sqrt{\lambda})] + V_n[\Phi_{n,\lambda}[f_n]]. \tag{60}
\]
Here $V_n[\cdot]$ is the Biot-Savart law in (27) and the function $\Phi_{n,\lambda}[f_n]$ is defined as
\[
\Phi_{n,\lambda}[f_n](r) = -K_{\mu_n}(\sqrt{r})\left(\int_1^r I_{\mu_n}(\sqrt{\lambda}s) \left(\mu_n f_{\theta,n}(s) + i n f_{r,n}(s)\right) \, ds + \sqrt{\lambda} \int_1^r s I_{\mu_n + 1}(\sqrt{\lambda}s) f_{\theta,n}(s) \, ds\right) + I_{\mu_n}(\sqrt{r})\left(\int_r^\infty K_{\mu_n}(\sqrt{\lambda}s) \left(\mu_n f_{\theta,n}(s) - i n f_{r,n}(s)\right) \, ds + \sqrt{\lambda} \int_r^\infty s K_{\mu_n - 1}(\sqrt{\lambda}s) f_{\theta,n}(s) \, ds\right), \tag{61}
\]
while the constant $c_{n,\lambda}[f_n]$ is defined as
\[
c_{n,\lambda}[f_n] = \int_1^\infty s^{1-|n|} \Phi_{n,\lambda}[f_n](s) \, ds. \tag{62}
\]
Moreover, the vorticity $\text{rot } w_{f,n}^{ed}$ is represented as
\[
\text{rot } w_{f,n}^{ed} = -\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda} ; \beta)} K_{\mu_n}(\sqrt{\lambda}) e^{i \theta} + \Phi_{n,\lambda}[f_n](r) e^{i \theta}. \tag{63}
\]
We shall estimate $w_{f,n}^{ed}$ and $\text{rot } w_{f,n}^{ed}$, represented respectively as in (60) and (63), when $|n| = 1$ in the following two subsections. Our main tools for the proof are the asymptotic analysis of $\mu_n = \mu_n(\beta)$ for small $\beta$ in Appendix A and the detailed estimates to the modified Bessel functions in Appendix B. Before going into details, let us state the estimate of $F_n(\sqrt{\lambda} ; \beta)$ in a region exponentially close to the origin with respect to $\beta$. We denote by $\Sigma_{\phi}$ the sector $\{ z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \phi \}$, $\phi \in (0, \pi)$, in the complex plane $\mathbb{C}$, and by $B_\rho(0) \subset \mathbb{C}$ the disk centered at the origin with radius $\rho > 0$.

**Proposition 3.1** Let $|n| = 1$. Then for any $\epsilon \in (0, \frac{\pi}{2})$ there is a positive constant $\beta_0$ depending only on $\epsilon$ such that as long as $\beta \in (0, \beta_0)$ and $\lambda \in \Sigma_{\pi-\epsilon} \cap B_{\rho}(0)$ we have
\[
\frac{1}{|F_n(\sqrt{\lambda}; \beta)|} \leq C|\lambda|^{-\frac{\text{Re}(\mu_n)}{2}}, \tag{64}
\]
where the constant $C$ depends only on $\epsilon$. In particular, we have $\mathcal{Z}(F_n) \cap B_{\rho}(0) = \emptyset$.  

14
Proof: The assertion follows from Lemma C.1 in Appendix C since we have $e^{-\frac{1}{D}} \beta^2 < \beta^4$ for any $\beta \in (0, 1)$. See Appendix C for the proof of Lemma C.1.

3.1.1 Estimates of the velocity solving $\{R_{f,n}^{ed}\}$ with $|n| = 1$

In this subsection we derive the estimates for the solution $w_{f,n}^{ed}$ to $\{R_{f,n}^{ed}\}$ which is now represented as (60). The novelty of the following result is the investigation on the $\beta$-singularity appearing in each estimate. Let $\beta_0$ be the constant in Proposition 3.1.

**Theorem 3.2** Let $|n| = 1$ and $1 \leq q < p \leq \infty$ or $1 < q \leq p < \infty$. Fix $\epsilon \in (0, \frac{1}{2})$. Then there is a positive constant $C = C(q, p, \epsilon)$ independent of $\beta$ such that the following statement holds. Let $f \in C^0_{0}(D)^2$ and $\beta \in (0, \beta_0)$. Then for $\lambda \in \Sigma_{\pi-\epsilon} \cap B_{e^{-\frac{1}{D}}}(0)$ we have

$$
\|w_{f,n}^{ed}\|_{L^p(D)} \leq \frac{C}{\beta^2} |\lambda|^{1+\frac{1}{q}-\frac{1}{p}} \|f\|_{L^q(D)},
$$

(65)

$$
\frac{\|w_{f,n}^{ed}(1)|x|\|_{L^2(D)}}{L^2(D)} \leq \frac{C}{\beta^2} \left(1 + |\log \text{Re}(\sqrt{\lambda})|^{\frac{1}{2}}\right) |\lambda|^{1+\frac{1}{q}} \|f\|_{L^q(D)}.
$$

(66)

Moreover, (65) and (66) hold all for $f \in L^q(D)^2$.

**Remark 3.3** The logarithmic factor $|\log \text{Re}(\sqrt{\lambda})|$ in (66) cannot be removed in our analysis. This singularity might prevent us from closing the energy computation in view of the scaling, however, we observe that it is resolved by considering the following products:

$$
|\langle \omega_{f,n}^{ed}(1), \omega_{\div f,n}^{ed}(1) \rangle_{L^2(D)}|, \quad |\langle \omega_{\div f,n}^{ed}(1), \frac{w_{f,n}^{ed}(1)}{|x|} \rangle_{L^2(D)}|, \quad |\langle \omega_{\div f,n}^{ed}(1), \frac{w_{f,n}^{ed}(1)}{|x|} \rangle_{L^2(D)}|.
$$

Here the vorticities $\omega_{f,n}^{ed}(1), \omega_{\div f,n}^{ed}(1), \omega_{\div f,n}^{ed}$ will be introduced respectively in Subsections 3.1.2, 3.2.2 and 3.3. This is indeed a key observation in proving Proposition 3.23 in Subsection 3.4 where the estimate for $\langle (\text{rot } v)_n \frac{w_{f,n}}{|x|} \rangle_{L^2(D)}$ is established when $0 < |\lambda| < e^{-\frac{1}{D}}$.

We postpone the proof of Theorem 3.2 at the end of this subsection, and focus on the term $V_n(\Phi_{n,\lambda}[f_n])$ in (60) for the time being. In order to estimate $V_n(\Phi_{n,\lambda}[f_n])$, taking into account the definition of $V_n[\cdot]$ in (27), firstly we study the following two integrals

$$
\frac{1}{r^{n}} \int_{1}^{r} s^{1+|n|} \Phi_{n,\lambda}[f_n](s)(s) \, ds, \quad r^{n} \int_{r}^{\infty} s^{1-|n|} \Phi_{n,\lambda}[f_n](s) \, ds.
$$

Let us recall the decompositions for them used in (11) which are useful in calculations. To state the result we define the functions $g_n^{(1)}(r)$ and $g_n^{(2)}(r)$ by

$$
g_n^{(1)}(r) = \mu_n f_{\theta,n}(r) + in f_{r,n}(r), \quad g_n^{(2)}(r) = \mu_n f_{\theta,n}(r) - in f_{r,n}(r),
$$

and fix a resolvent parameter $\lambda \in \mathbb{C} \setminus \mathbb{R}_{-}$. 

**Lemma 3.4** ([11] Lemmas 3.6 and 3.9)] Let $n \in \mathbb{Z} \setminus \{0\}$ and $f \in C^0_{0}(D)^2$. Then we have

$$
\frac{1}{r^{n}} \int_{1}^{r} s^{1+|n|} \Phi_{n,\lambda}[f_n](s)(s) \, ds = \sum_{l=1}^{9} J_{l}^{(1)}[f_n](r),
$$

(67)
where

\[ J_1^{(1)}[f_n](r) = -\frac{1}{r^{[n]}} \int_1^r I_{\mu_n}(\sqrt{\lambda r}) g_n^{(1)}(\tau) \int_\tau^r s^{1+[n]} K_{\mu_n}(\sqrt{\lambda s}) \, ds \, d\tau, \]

\[ J_2^{(1)}[f_n](r) = -\frac{\mu_n + [n]}{r^{[n]}} \int_1^r \tau I_{\mu_n+1}(\sqrt{\lambda r}) f_{\theta,n}(\tau) \int_\tau^r s^{[n]} K_{\mu_n-1}(\sqrt{\lambda s}) \, ds \, d\tau, \]

\[ J_3^{(1)}[f_n](r) = \frac{1}{r^{[n]}} \int_1^r K_{\mu_n}(\sqrt{\lambda r}) g_n^{(2)}(\tau) \int_\tau^r s^{1+[n]} I_{\mu_n}(\sqrt{\lambda s}) \, ds, \]

\[ J_4^{(1)}[f_n](r) = \frac{\mu_n - [n]}{r^{[n]}} \int_1^r \tau K_{\mu_n-1}(\sqrt{\lambda r}) f_{\theta,n}(\tau) \int_\tau^r s^{[n]} I_{\mu_n+1}(\sqrt{\lambda s}) \, ds, \]

\[ J_5^{(1)}[f_n](r) = \frac{1}{r^{[n]}} \left( \int_1^\infty K_{\mu_n}(\sqrt{\lambda s}) g_n^{(2)}(s) \, ds \right) \left( \int_1^r s^{1+[n]} I_{\mu_n}(\sqrt{\lambda s}) \, ds \right), \]

\[ J_6^{(1)}[f_n](r) = \frac{\mu_n - [n]}{r^{[n]}} \left( \int_1^\infty s K_{\mu_n-1}(\sqrt{\lambda s}) f_{\theta,n}(s) \, ds \right) \left( \int_1^r s^{[n]} I_{\mu_n+1}(\sqrt{\lambda s}) \, ds \right), \]

\[ J_7^{(1)}[f_n](r) = r K_{\mu_n-1}(\sqrt{\lambda r}) \int_1^r s I_{\mu_n+1}(\sqrt{\lambda s}) f_{\theta,n}(s) \, ds, \]

\[ J_8^{(1)}[f_n](r) = r I_{\mu_n+1}(\sqrt{\lambda r}) \int_1^\infty s K_{\mu_n-1}(\sqrt{\lambda s}) f_{\theta,n}(s) \, ds, \]

\[ J_9^{(1)}[f_n](r) = -\frac{I_{\mu_n+1}(\sqrt{\lambda r})}{r^{[n]}} \int_1^\infty s K_{\mu_n-1}(\sqrt{\lambda s}) f_{\theta,n}(s) \, ds. \]

and

\[ r^{[n]} \int_r^\infty s^{1-[n]} \Phi_{n,\lambda}[f_n](s) \, ds = \sum_{i=10}^{17} J_i^{(1)}[f_n](r), \]

where

\[ J_{10}^{(1)}[f_n](r) = -r^{[n]} \left( \int_1^r I_{\mu_n}(\sqrt{\lambda s}) g_n^{(1)}(s) \, ds \right) \left( \int_1^\infty s^{1-[n]} K_{\mu_n}(\sqrt{\lambda s}) \, ds \right), \]

\[ J_{11}^{(1)}[f_n](r) = -r^{[n]} \int_1^r I_{\mu_n}(\sqrt{\lambda r}) g_n^{(1)}(\tau) \int_\tau^\infty s^{1-[n]} K_{\mu_n}(\sqrt{\lambda s}) \, ds \, d\tau, \]

\[ J_{12}^{(1)}[f_n](r) = -(\mu_n - [n]) r^{[n]} \left( \int_1^r s I_{\mu_n+1}(\sqrt{\lambda s}) f_{\theta,n}(s) \, ds \right) \left( \int_1^\infty s^{-[n]} K_{\mu_n-1}(\sqrt{\lambda s}) \, ds \right), \]

\[ J_{13}^{(1)}[f_n](r) = -(\mu_n - [n]) r^{[n]} \int_1^r \tau I_{\mu_n+1}(\sqrt{\lambda r}) f_{\theta,n}(\tau) \int_\tau^\infty s^{-[n]} K_{\mu_n-1}(\sqrt{\lambda s}) \, ds \, d\tau, \]

\[ J_{14}^{(1)}[f_n](r) = r^{[n]} \int_1^\infty K_{\mu_n}(\sqrt{\lambda r}) g_n^{(2)}(\tau) \int_\tau^\infty s^{1-[n]} I_{\mu_n}(\sqrt{\lambda s}) \, ds \, d\tau, \]

\[ J_{15}^{(1)}[f_n](r) = (\mu_n + [n]) r^{[n]} \int_1^\infty \tau K_{\mu_n-1}(\sqrt{\lambda r}) f_{\theta,n}(\tau) \int_\tau^\infty s^{-[n]} I_{\mu_n+1}(\sqrt{\lambda s}) \, ds \, d\tau, \]

\[ J_{16}^{(1)}[f_n](r) = -r K_{\mu_n-1}(\sqrt{\lambda r}) \int_1^r s I_{\mu_n+1}(\sqrt{\lambda s}) f_{\theta,n}(s) \, ds, \]

\[ J_{17}^{(1)}[f_n](r) = -r I_{\mu_n+1}(\sqrt{\lambda r}) \int_1^\infty s K_{\mu_n-1}(\sqrt{\lambda s}) f_{\theta,n}(s) \, ds. \]

**Remark 3.5** (1) The estimate to the term \( J_9^{(1)}[f_n] \) is not needed in the following analysis thanks to the cancellation \( J_9^{(1)}[f_n](r) - r^{-[n]} J_7^{(1)}[f_n](1) = 0 \) in the Biot-Savart law.
\[ V_n[\Phi_n, \lambda[f_n]]. \] This fact will be used in the proof of Proposition 3.10.

(2) Note that \( J_{l}^{(1)}[f_n] = -J_{l}^{(1)}[f_n] \) and \( J_{l}^{(2)}[f_n] = -J_{l}^{(2)}[f_n] \) hold. Therefore we will skip the derivation of the estimates for \( J_{l}^{(1)}[f_n] \) and \( J_{l}^{(2)}[f_n] \) in Lemma 3.7.

(3) We can express the constant \( c_{n, \lambda}[f_n] \) in (62) in terms of \( J_{l}^{(1)}[f_n](r) \) as
\[ c_{n, \lambda}[f_n] = \sum_{l=1,13,14,15,17} J_{l}^{(1)}[f_n](1). \]

The estimates to \( J_{l}^{(1)}[f_n], l \in \{1, \ldots, 8\} \), in Lemma 3.4 are given as follows.

**Lemma 3.6** Let \(|n| = 1 \) and \(|q| \in [1, \infty) \), and let \( \lambda \in \Sigma_{\pi-\epsilon} \cap B_{1}(0) \) for some \( \epsilon \in (0, \frac{\pi}{2}) \).

Then there is a positive constant \( C = C(q, \epsilon) \) independent of \( \beta \) such that the following statements hold.

1. Let \( f \in C_{0}^{\infty}(D)^{2} \). Then for \( l \in \{1, \ldots, 8\} \) we have
\[
|J_{l}^{(1)}[f_n](r)| \leq \frac{C}{\beta} r^{3-\frac{2}{q}} \|f\|_{L^{q}(D)}, \quad 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}. \tag{67}
\]

On the other hand, for \( l \in \{1, \ldots, 6\} \) we have
\[
|J_{l}^{(1)}[f_n](r)| \leq \frac{C}{\beta} |\lambda|^{-1} r^{1-\frac{2}{q}} \|f\|_{L^{q}(D)}, \quad r \geq \text{Re}(\sqrt{\lambda})^{-1}, \tag{68}
\]

while for \( l \in \{7, 8\} \) we have
\[
|J_{l}^{(1)}[f_n](r)| \leq C|\lambda|^{-1+\frac{1}{\beta}} r^{1-\frac{4}{q}} \|f\|_{L^{q}(D)}, \quad r \geq \text{Re}(\sqrt{\lambda})^{-1}. \tag{69}
\]

2. Let \( f \in C_{0}^{\infty}(D)^{2} \). Then for \( l \in \{7, 8\} \) we have
\[
\|r^{-1}J_{l}^{(1)}[f_n]\|_{L^{\infty}(D)} \leq \frac{C}{\beta} |\lambda|^{-1} \|f\|_{L^{\infty}(D)}, \tag{70}
\]
\[
\|r^{-1}J_{l}^{(1)}[f_n]\|_{L^{1}(D)} \leq \frac{C}{\beta} |\lambda|^{-1} \|f\|_{L^{1}(D)}. \tag{71}
\]

**Proof:** (1) (i) Estimate of \( J_{1}^{(1)}[f_n] \): For \( 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1} \), by (153) for \( k = 0 \) in Lemma 3.1 and (156) for \( k = 0 \) in Lemma 3.2 in Appendix B we find
\[
|J_{1}^{(1)}[f_n](r)| \leq r^{-1} \left( \int_{1}^{r} I_{\mu_{n}}(\sqrt{\lambda} \tau) g_{n}^{(1)}(\tau) \right) \left( \int_{\tau}^{r} s^{2} K_{\mu_{n}}(\sqrt{\lambda} s) \, ds \right) d\tau \leq Cr \int_{1}^{r} |f_{n}(\tau)| \tau \, d\tau,
\]
which leads to the estimate (67). For \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \), by (153) and (155) for \( k = 0 \) in Lemma 3.1 and (157) and (158) for \( k = 0 \) in Lemma 3.2 we have
\[
|J_{1}^{(1)}[f_n](r)| \leq r^{-1} \left( \int_{1}^{r} \frac{1}{\text{Re}(\sqrt{\lambda})} + \int_{\text{Re}(\sqrt{\lambda})}^{r} \right) |I_{\mu_{n}}(\sqrt{\lambda} \tau) g_{n}^{(1)}(\tau)| \left( \int_{\tau}^{r} s^{2} K_{\mu_{n}}(\sqrt{\lambda} s) \, ds \right) d\tau \leq C|\lambda|^{-1} r^{-1} \int_{1}^{r} |f_{n}(\tau)| \tau \, d\tau + C|\lambda|^{-1} r^{-1} \int_{1}^{r} |f_{n}(\tau)| \tau \, d\tau,
\]
which implies the estimate (68).

(ii) Estimate of $J_2^{(1)}[f_n]$: The proof is parallel to that for $J_1^{(1)}[f_n]$ using the results in Lemmas B.1 and B.2 for $k = 1$. We omit the details here.

(iii) Estimate of $J_3^{(1)}[f_n]$: For $1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}$, by (149) and (151) in Lemma B.1 and (161) for $k = 0$ in Lemma B.3, we see that

\[
|J_3^{(1)}[f_n](r)| \leq r^{-1} \int_1^r |H_{\mu_n}(\sqrt{\lambda}\tau) g_\tau^{(2)}(\tau)| \int_1^\tau |s^2 I_{\mu_n}(\sqrt{\lambda}s)| \, ds \, d\tau \\
\leq C r \int_1^r |f_n(\tau)| \tau \, d\tau.
\]

Thus we have (67). For $r \geq \text{Re}(\sqrt{\lambda})^{-1}$, by (149), (151), and (154) for $k = 0$ in Lemma B.1 and (161) and (162) for $k = 0$ in Lemma B.3, we have

\[
|J_3^{(1)}[f_n](r)| \leq r^{-1} \left( \int_1^{\text{Re}(\sqrt{\lambda})} + \int_1^{\infty} \right) |H_{\mu_n}(\sqrt{\lambda}\tau) g_\tau^{(2)}(\tau)| \int_1^\tau |s^2 I_{\mu_n}(\sqrt{\lambda}s)| \, ds \, d\tau \\
\leq C |\lambda|^{-1} r^{-1} \int_1^{\text{Re}(\sqrt{\lambda})} |f_n(\tau)| \tau \, d\tau + C |\lambda|^{-1} r^{-1} \int_1^{\infty} |f_n(\tau)| \tau \, d\tau,
\]

which leads to (68).

(iv) Estimate of $J_4^{(1)}[f_n]$: The proof is parallel to that for $J_3^{(1)}[f_n]$ using the results in Lemmas B.1 and B.3 for $k = 1$, and we omit here.

(v) Estimates of $J_5^{(1)}[f_n]$ and $J_6^{(1)}[f_n]$: We give a proof only for $J_5^{(1)}[f_n]$ since the proof for $J_6^{(1)}[f_n]$ is similar. For $1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}$, we observe that

\[
|J_5^{(1)}[f_n](r)| \leq r^{-1} \int_1^r |s^2 I_{\mu_n}(\sqrt{\lambda}s)| \, ds \left( \int_1^{\text{Re}(\sqrt{\lambda})} + \int_1^{\infty} \right) |H_{\mu_n}(\sqrt{\lambda}s) g_\tau^{(2)}(s)| \, ds \\
\leq C_1 r^{\text{Re}(\mu)+2} \int_1^{\text{Re}(\sqrt{\lambda})} s^{-(\text{Re}(\mu)-1)} |f_n(s)| \, ds \\
+ C |\lambda|^{-\frac{\text{Re}(\mu)}{2}} \int_1^{\infty} s^{-\frac{\text{Re}(\sqrt{\lambda})}{2}} e^{-\text{Re}(\sqrt{\lambda})s} |f_n(s)| \, ds.
\]

Then a direct calculation shows (67). For $r \geq \text{Re}(\sqrt{\lambda})^{-1}$ we have

\[
|J_5^{(1)}[f_n](r)| \leq r^{-1} \int_1^r |s^2 I_{\mu_n}(\sqrt{\lambda}s)| \, ds \int_r^{\infty} |H_{\mu_n}(\sqrt{\lambda}s) g_\tau^{(2)}(s)| \, ds \\
\leq C |\lambda|^{-1} r^{-\frac{1}{2}} e^{\text{Re}(\sqrt{\lambda})r} \int_r^{\infty} s^{-\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda})s} |f_n(s)| \, ds,
\]

which implies (68).

(vi) Estimate of $J_7^{(1)}[f_n]$: For $1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}$, by (150), (152), and (153) for $k = 1$ in Lemma B.1 we find

\[
|J_7^{(1)}[f_n](r)| \leq r |K_{\mu_n-1}(\sqrt{\lambda}r)| \int_1^r |I_{\mu_n+1}(\sqrt{\lambda}s) f_{\theta,n}(s)| \, ds \\
\leq C_1 |\lambda|^{-1} r \int_1^r |f_n(s)| \, ds.
\]

(72)
Thus we have (67). For \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \), by (153)–(155) for \( k = 1 \) in Lemma B.1, we have
\[
|J_l^{(1)}[f_n](r)| \leq |r K_{\mu_n-1}(\sqrt{\lambda}r)| \left( \int_{1}^{\text{Re}(\sqrt{\lambda})} + \int_{\text{Re}(\sqrt{\lambda})}^{r} \right) |I_{\mu_n+1}(\sqrt{\lambda}s) f_{\theta,n}(s)s| \, ds
\]
\[
\leq C |\lambda|^{-\frac{1}{2}} r^{\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda})r} \int_{1}^{r} |f_n(s)| s \, ds + C |\lambda|^{-\frac{1}{2}} r^{3} e^{-\text{Re}(\sqrt{\lambda})s} \int_{1}^{r} s^{-\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda})s} |f_n(s)| s \, ds,
\]
which leads to (69).

(vii) Estimate of \( J_8^{(1)}[f_n] \): For \( 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1} \) we find
\[
|J_8^{(1)}[f_n](r)| \leq |r I_{\mu_n+1}(\sqrt{\lambda}r)| \left( \int_{1}^{\text{Re}(\sqrt{\lambda})} + \int_{\text{Re}(\sqrt{\lambda})}^{r} \right) |K_{\mu_n-1}(\sqrt{\lambda}s) f_{\theta,n}(s)s| \, ds
\]
\[
\leq C \beta^{-1} |\lambda| r^{3} \int_{1}^{r} |f_n(s)| s \, ds + C |\lambda|^{-\frac{1}{2}} r^{3} e^{-\text{Re}(\sqrt{\lambda})s} \int_{1}^{r} s^{-\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda})s} |f_n(s)| s \, ds,
\]
which implies (67). For \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \) we have
\[
|J_8^{(1)}[f_n](r)| \leq |r I_{\mu_n+1}(\sqrt{\lambda}r)| \int_{r}^{\infty} |K_{\mu_n-1}(\sqrt{\lambda}s) f_{\theta,n}(s)s| \, ds
\]
\[
\leq C |\lambda|^{-\frac{1}{2}} r^{\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda})r} \int_{r}^{\infty} s^{-\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda})s} |f_n(s)| s \, ds,
\]
which leads to (69). Hence we obtain the assertion (1) of Lemma 3.6.

(2) The estimate (70) follows from (72)–(75) in the above. For the proof of (71), one can reproduce the calculation performed in [11, Lemma 3.7] using the results in Lemma B.1, and hence we omit the details here. This completes the proof of Lemma 3.6.

The next lemma summarizes the estimates to \( J_l^{(1)}[f_n](r) \), \( l \in \{10, \ldots, 17\} \), in Lemma 3.4. We skip the proofs for \( J_{16}^{(1)}[f_n] \) and \( J_{17}^{(1)}[f_n] \) as is already mentioned in Remark 3.5 (2).

**Lemma 3.7** Let \( |n| = 1 \) and \( q \in [1, \infty) \), and let \( \lambda \in \Sigma_{\sigma-\epsilon} \cap B_{1}(0) \) for some \( \epsilon \in (0, \frac{\pi}{\lambda}) \). Then there is a positive constant \( C = C(q, \epsilon) \) independent of \( \beta \) such that the following statements hold.

(1) Let \( f \in C_{0}^{\infty}(D)^{2} \). Then for \( l \in \{10, \ldots, 17\} \) we have
\[
|J_l^{(1)}[f_n](r)| \leq C |\lambda|^{-1+\frac{1}{2}l} r^{1-q} \|f\|_{L^{q}(D)}, \quad 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}.
\]

On the other hand, for \( l \in \{10, \ldots, 15\} \) we have
\[
|J_l^{(1)}[f_n](r)| \leq C |\lambda|^{-1+\frac{1}{2}l} r^{1-q} \|f\|_{L^{q}(D)}, \quad r \geq \text{Re}(\sqrt{\lambda})^{-1},
\]
while for \( l \in \{16, 17\} \) we have
\[
|J_l^{(1)}[f_n](r)| \leq C |\lambda|^{-1+\frac{1}{2}l} r^{1-q} \|f\|_{L^{q}(D)}, \quad r \geq \text{Re}(\sqrt{\lambda})^{-1}.
\]
(2) Let $f \in C_0^\infty (D)^2$. Then for $l \in \{ 16, 17 \}$ we have
\[
\| r^{-1} J_l^{(1)} [f_n] \|_{L^\infty (D)} \leq \frac{C}{\beta} |\lambda|^{-1} \| f \|_{L^1 (D)}, \tag{79}
\]
\[
\| r^{-1} J_l^{(1)} [f_n] \|_{L^1 (D)} \leq \frac{C}{\beta} |\lambda|^{-1} \| f \|_{L^1 (D)}. \tag{80}
\]

**Proof:** (1) (i) Estimate of $J_{10}^{(1)} [f_n]$: For $1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}$, by (153) for $k = 0$ in Lemma B.1 and (159) for $k = 0$ in Lemma B.2 in Appendix B we find
\[
| J_{10}^{(1)} [f_n] (r) | \leq r \int_1^\infty K_{\mu_n} (\sqrt{\lambda} s) \, ds \left( \int_1^{\text{Re}(\sqrt{\lambda})} + \int_{\text{Re}(\sqrt{\lambda})}^r \right) | I_{\mu_n} (\sqrt{\lambda} s) g_n^{(1)} (s) | \, ds 
\]
\[
\leq C \beta^{-1} r \int_1^r | f_n (s) | s \, ds,
\]
which implies (76). For $r \geq \text{Re}(\sqrt{\lambda})^{-1}$, by (153) and (155) for $k = 0$ in Lemma B.1 and (160) for $k = 0$ in Lemma B.2 we have
\[
| J_{10}^{(1)} [f_n] (r) | \leq r \int_1^\infty K_{\mu_n} (\sqrt{\lambda} s) \, ds \left( \int_1^{\text{Re}(\sqrt{\lambda})} + \int_{\text{Re}(\sqrt{\lambda})}^r \right) | I_{\mu_n} (\sqrt{\lambda} s) g_n^{(1)} (s) | \, ds 
\]
\[
\leq C |\lambda|^{-1} r \frac{\beta}{4} e^{-\text{Re}(\sqrt{\lambda}) r} \int_1^{\text{Re}(\sqrt{\lambda})} | f_n (s) | s \, ds 
\]
\[
+ C |\lambda|^{-1} r \frac{\beta}{4} e^{-\text{Re}(\sqrt{\lambda}) r} \int_{\text{Re}(\sqrt{\lambda})}^r s^{-\frac{\beta}{2}} e^{\text{Re}(\sqrt{\lambda}) s} | f_n (s) | s \, ds,
\]
which leads to (77).

(ii) Estimate of $J_{11}^{(1)} [f_n]$: For $1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}$, by (153) and (155) for $k = 0$ in Lemma B.1 and (159) and (160) for $k = 0$ in Lemma B.2 we see that
\[
| J_{11}^{(1)} [f_n] (r) | \leq r \left( \int_r^{\text{Re}(\sqrt{\lambda})} + \int_{\text{Re}(\sqrt{\lambda})}^\infty \right) | I_{\mu_n} (\sqrt{\lambda} \tau) g_n^{(1)} (\tau) | \left( \int_1^\infty K_{\mu_n} (\sqrt{\lambda} s) \, ds \right) \, d\tau 
\]
\[
\leq C \beta^{-1} r \int_r^{\text{Re}(\sqrt{\lambda})} | f_n (\tau) | \tau \, d\tau + C |\lambda|^{-1} r \int_{\text{Re}(\sqrt{\lambda})}^\infty \tau^{-2} | f_n (\tau) | \tau \, d\tau,
\]
which implies (76). For $r \geq \text{Re}(\sqrt{\lambda})^{-1}$, by (155) for $k = 0$ in Lemma B.1 and (160) for $k = 0$ in Lemma B.2 we have
\[
| J_{11}^{(1)} [f_n] (r) | \leq r \int_r^\infty | I_{\mu_n} (\sqrt{\lambda} \tau) g_n^{(1)} (\tau) | \int_\tau^\infty | K_{\mu_n} (\sqrt{\lambda} s) | \, ds \, d\tau 
\]
\[
\leq C |\lambda|^{-1} r \int_r^\infty \tau^{-2} | f_n (\tau) | \tau \, d\tau,
\]
which leads to (77).

(iii) Estimates of $J_{12}^{(1)} [f_n]$ and $J_{13}^{(1)} [f_n]$: The proof for $J_{12}^{(1)} [f_n]$ is parallel to that for $J_{10}^{(1)} [f_n]$ using the bound $|\mu_n - 1| \leq C \beta$ and the results in Lemmas B.1 and B.2 for $k = 1$. The proof for $J_{13}^{(1)} [f_n]$ is similar to that for $J_{11}^{(1)} [f_n]$. Thus we omit the details here.
Lemmas 3.6 and 3.7 lead to the next important estimates that we shall need in the proof of B.3 for $k = 0$ in Lemma B.1 and (163) and (164) for $k = 0$ in Lemma B.3 we observe that

$$|J_{14}^{(1)}[f_n](r)| \leq r \left( \int_{Re(\sqrt{\lambda})}^{1} + \int_{1}^{\infty} \right) |K_{\mu_n}(\sqrt{\lambda}r) g_{\mu_n}(2)(\tau)| \int_{r}^{\tau} |I_{\mu_n}(\sqrt{\lambda}s)| \, ds \, d\tau$$

$$\leq C r \int_{Re(\sqrt{\lambda})}^{1} |f_n(\tau)| \, d\tau + C|\lambda|^{-1} r \int_{1}^{\infty} \tau^{-2} |f_n(\tau)| \, d\tau,$$

which implies (76). For $r \geq Re(\sqrt{\lambda})^{-1}$, by (154) in Lemma B.1 and (165) in Lemma B.3 for $k = 0$ we have

$$|J_{14}^{(1)}[f_n](r)| \leq r \int_{r}^{\infty} |K_{\mu_n}(\sqrt{\lambda}r) g_{\mu_n}(2)(\tau)| \int_{r}^{\tau} |I_{\mu_n}(\sqrt{\lambda}s)| \, ds \, d\tau$$

$$\leq C|\lambda|^{-1} r \int_{r}^{\infty} \tau^{-2} |f_n(\tau)| \, d\tau,$$

which leads to (77).

(v) Estimate of $J_{15}^{(1)}[f_n]$: The proof is parallel to that for $J_{14}^{(1)}[f_n]$ using Lemmas B.1 and B.3 for $k = 1$, and thus we omit here. This completes the proof of Lemma 3.7. $\square$

Lemmas 3.6 and 3.7 lead to the next important estimates that we shall need in the proof of Proposition 3.10 below. Let $c_{n,\lambda}[f_n]$ be the constant in (62).

**Corollary 3.8** Let $|n| = 1$ and $1 \leq q < p \leq \infty$ or $1 < q \leq p \leq \infty$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap B_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(q, p, \epsilon)$ independent of $\beta$ such that the following statement holds. Let $f \in C_{0}^{\infty}(D)^2$. Then for $l \in \{1, \ldots, 17\} \setminus \{9\}$ we have

$$|c_{n,\lambda}[f_n]| \leq \frac{C|\beta|^{-1+\frac{1}{q}}}{|\lambda|^{1+\frac{1}{q}}} \|f\|_{L^q(D)}, \quad (81)$$

$$\|r^{-1}J_{l}^{(1)}[f_n]\|_{L^p(D)} \leq \frac{C|\beta|^{-1+\frac{1}{q}}}{|\lambda|^{1+\frac{1}{q}}} \|f\|_{L^q(D)}, \quad (82)$$

$$\|r^{-2}J_{l}^{(1)}[f_n]\|_{L^2(D)} \leq \frac{C|\beta|^{-1+\frac{1}{q}}}{|\lambda|^{1+\frac{1}{q}}} \log \text{Re}((\sqrt{\lambda}))^{\frac{1}{2}} \|f\|_{L^q(D)}.$$

**Proof:** (i) Estimate of $c_{n,\lambda}[f_n]$: Remark 3.5 (3) ensures that

$$|c_{n,\lambda}[f_n]| \leq \sum_{l=11,13,14,15,17} |J_{l}^{(1)}[f_n](1)|.$$

Then the estimate (81) follows from putting $r = 1$ to (76) in Lemma 3.7.

(ii) Estimate of $r^{-1}J_{l}^{(1)}[f_n]$: If $l \in \{1, \ldots, 17\} \setminus \{7, 8, 9, 16, 17\}$, then it is easy to see from the pointwise estimates in Lemmas 3.6 and 3.7 that

$$\sup_{r \geq 1} r^{\frac{2}{p-q}} |r^{-1}J_{l}^{(1)}[f_n](r)| \leq C \beta^{-1} |\lambda|^{-1} \|f\|_{L^q(D)}, \quad 1 \leq q < \infty.$$
Thus by the Marcinkiewicz interpolation theorem we have (82) for the case $1 < p = q < \infty$. Moreover, again from Lemmas 3.6 and 3.7 one can see that
\[
\sup_{r \geq 1} |r^{-1} J_{\frac{1}{2}}^{(1)} [f_{n}] (r)| \leq C \beta^{-1} \|f\|_{L^1 (D)},
\]
which leads to (82) for the case $1 < p \leq \infty$ and $q = 1$. Hence finally we have (82) for $1 \leq q < p \leq \infty$ and $1 < q \leq p < \infty$ by the Marcinkiewicz interpolation theorem again.

If $l \in \{7, 8, 16, 17\}$, from (70), (71), (79), and (80) we have (82) for the case $1 \leq p = q \leq \infty$ by the interpolation argument. Moreover, (67), (69), (76), and (78) lead to the estimate in the form (84) for $l \in \{7, 8, 16, 17\}$. Thus we obtain (82) for the case $1 \leq p \leq \infty$ and $q = 1$, and hence (82) for $1 \leq q \leq p \leq \infty$ by the Marcinkiewicz interpolation theorem.

(iii) Estimate of $r^{-2} J_{\frac{1}{2}}^{(1)} [f_{n}]$: The assertion (83) can be checked easily by a direct calculation using Lemmas 3.6 and 3.7. We note that the logarithmic factor in (83) is due to the estimate (70). The proof of Corollary 3.8 is complete.

Now we are in position to prove the main theorem of this subsection. Let us start with the simple proposition about the estimate for the term $V_{n} [K_{\mu_{n}} (\sqrt{\lambda} \cdot)]$ in (60).

**Proposition 3.9** Let $|n| = 1$, $p \in (1, \infty)$, and let $\lambda \in \Sigma_{\nu - \epsilon} \cap B_{1} (0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C (p, \epsilon)$ independent of $\beta$ such that we have
\[
\|V_{n} [K_{\mu_{n}} (\sqrt{\lambda} \cdot)]\|_{L^p (D)} \leq C \beta^{-1} |\lambda|^{-\frac{\text{Re}(\mu_{n}) - 1}{2}} \|f\|_{L^p (D)},
\]
\[
\frac{\|V_{n} [K_{\mu_{n}} (\sqrt{\lambda} \cdot)]\|_{L^2 (D)}}{|x|} \leq C \beta |\lambda|^{-\frac{\text{Re}(\mu_{n})}{2}}.
\]

**Proof:** It is easy to see from the definition of $V_{n} [\cdot]$ in (27) that
\[
|V_{n} [K_{\mu_{n}} (\sqrt{\lambda} \cdot)]| \leq C r^{-2} \left( |F_{n} (\sqrt{\lambda}; \beta)| + \left| \int_{0}^{r} s^{2} K_{\mu_{n}} (\sqrt{\lambda} s) \, ds \right| \right) + C \left| \int_{r}^{\infty} K_{\mu_{n}} (\sqrt{\lambda} s) \, ds \right|.
\]

By the results in Lemma B.2 for $k = 0$ in Appendix B we have
\[
|V_{n} [K_{\mu_{n}} (\sqrt{\lambda} \cdot)] (r)| \leq C \beta^{-1} |\lambda|^{-\frac{\text{Re}(\mu_{n})}{2}} r^{-\text{Re}(\mu_{n}) - 1}, \quad 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1},
\]
\[
|V_{n} [K_{\mu_{n}} (\sqrt{\lambda} \cdot)] (r)| \leq C \beta^{-1} |\lambda|^{-\frac{1}{2} + \frac{1}{2}} r^{-2}, \quad r \geq \text{Re}(\sqrt{\lambda})^{-1}.
\]

Then for $p \in [1, \infty]$ we find
\[
\sup_{r \geq 1} r^{|p/2 - 1|} |V_{n} [K_{\mu_{n}} (\sqrt{\lambda} \cdot)] (r)| \leq C \beta^{-1} |\lambda|^{-\frac{\text{Re}(\mu_{n})}{2}} \|f\|_{L^p (D)}.
\]
Hence by an interpolation argument (85) follows. Moreover, a direct calculation combined with (87), (88), and $(\text{Re}(\mu_{n} (\beta)) - 1)^{\frac{1}{2}} \approx O(\beta)$ yield (86). This completes the proof.

The next proposition gives the estimate for the term $V_{n} [\Phi_{n, \lambda} [f_{n}]]$ in (60).
Proof: The definition of the Biot-Savart law \( V_r \) in (21) leads to the next representations for the radial part \( V_{r,n} \Phi_{r,n}(f_n) \) and the angular part \( V_{\theta,n} \Phi_{r,n}(f_n) \) of \( V_n \Phi_{r,n}(f_n) \):

\[
V_{r,n}(\Phi_{r,n}(f_n)) = \frac{-in}{2r} \left( \frac{c_{n,\lambda}[f_n]}{r} - \frac{1}{r} \int_{r}^{\infty} s^{2} \Phi_{r,n}(f_n)(s) \, ds \right),
\]

\[
V_{\theta,n}(\Phi_{r,n}(f_n)) = \frac{1}{2r} \left( \frac{c_{n,\lambda}[f_n]}{r} - \frac{1}{r} \int_{r}^{\infty} s^{2} \Phi_{r,n}(f_n)(s) \, ds + \int_{r}^{\infty} \Phi_{r,n}(f_n)(s) \, ds \right),
\]

where \( c_{n,\lambda}[f_n] \) is defined in (62). From Lemma 3.4 and Remark 3.5 (1) and (3) we see that

\[
\frac{c_{n,\lambda}[f_n]}{r} - \frac{1}{r} \int_{r}^{\infty} s^{2} \Phi_{r,n}(f_n)(s) \, ds \leq r^{-1} \sum_{l=1,11,13,14,15} J_{l}^{(1)}[f_n](1) - \sum_{l=1}^{8} r^{-1} J_{l}^{(1)}[f_n](r). \tag{91}
\]

Then, by (91) and the decomposition of the integral \( \int_{r}^{\infty} \Phi_{r,n}(f_n)(s) \, ds \) in Lemma 3.4 we find the following pointwise estimate of \( V_{n}(\Phi_{r,n}(f_n))(r) \):

\[
|V_{n}(\Phi_{r,n}(f_n))(r)| \leq C \left( r^{-2} \sum_{l=1,11,13,14,15} |J_{l}^{(1)}[f_n](1)| + \sum_{l \in \{1, ..., 17\} \setminus \{9\}} |r^{-1} J_{l}^{(1)}[f_n](r)| \right). \tag{92}
\]

Thus the assertions (89) and (90) follow from Corollary 3.8. This completes the proof. \( \square \)

Finally we give a proof of Theorem 3.2 which is a direct consequence of Corollary 3.8 and Propositions 3.9 and 3.10.

Proof of Theorem 3.2: In view of Proposition 3.10 it suffices to show that the first term in the right-hand side of (60) satisfies the estimates (65) and (66). By using Proposition 3.1 and (81) in Corollary 3.8 one can see that (65) and (66) respectively follow from (65) and (66) in Proposition 3.9. This completes the proof of Theorem 3.2. \( \square \)

3.1.2 Estimates of the vorticity for \( \text{RS}_{f,n}^{\text{ed}} \) with \( |n| = 1 \)

This subsection is devoted to the estimate of the vorticity \( \omega_{f,n}^{\text{ed}}(r) = (\text{rot} \, w_{f,n}^{\text{ed}}) e^{-in\theta} \) with \( |n| = 1 \), where \( w_{f,n}^{\text{ed}} \) solves \( \text{RS}_{f,n}^{\text{ed}} \) in Subsection 3.1. We recall that \( \omega_{f,n}^{\text{ed}} \) is represented as

\[
\omega_{f,n}^{\text{ed}}(r) = -\frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda}; \beta)} K_{\mu_n}(\sqrt{\lambda}r) + \Phi_{r,n}(f_n)(r)
\]

by (63). The main result is stated as follows. Let \( \beta_0 \) be the constant in Proposition 3.1.
Theorem 3.11 Let \(|n| = 1, q \in (1, \infty)\), and \(\tilde{q} \in (\max\{1, \frac{q}{2}\}, q]\). Fix \(\varepsilon \in (0, \frac{q}{2})\). Then there is a positive constant \(C = C(q, \tilde{q}, \varepsilon)\) independent of \(\beta\) such that the following statement holds. Let \(f \in C_0^\infty(D)^2\) and \(\beta \in (0, \beta_0)\). Set

\[
\omega_{f,n}^{\text{ed}(1)}(r) = \frac{c_{n,\lambda}[f_n]}{F_n(\sqrt{\lambda}/\beta)} K_{\mu_n}(\sqrt{\lambda}r), \quad \omega_{f,n}^{\text{ed}(2)}(r) = \Phi_{n,\lambda}[f_n](r) .
\]

Then for \(\lambda \in \Sigma_{\varepsilon} \cap \mathcal{B} = \mathcal{B}(0, \frac{1}{\varepsilon})\) we have

\[
\|\omega_{f,n}^{\text{ed}(1)}\|_{L^2(D)} \leq \frac{C}{\beta^2} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(D)} ,
\]

\[
\|\omega_{f,n}^{\text{ed}(2)}\|_{L^2(D)} \leq \frac{C}{\beta^2} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(D)} ,
\]

\[
\left|\omega_{f,n}^{\text{ed}(1)}(r), \omega_{f,n}^{\text{ed}(2)}(r)\right|_{L^2(D)} \leq \frac{C}{\beta^2} |\lambda|^{-2+\frac{2}{q}} \|f\|_{L^q(D)}^2 .
\]

Moreover, (94), (95), and (96) hold all for \(f \in L^q(D)^2\).

Proof: (i) Estimate of \(\omega_{f,n}^{\text{ed}(1)}\): The estimate (94) is a direct consequence of Proposition 3.1 (81) in Corollary 3.8 and (166) with \(p = 2\) in Lemma B.4 in Appendix B.

(ii) Estimate of \(|x|^{-1} \omega_{f,n}^{\text{ed}(2)}\): We decompose \(\omega_{f,n}^{\text{ed}(2)}\) into \(\omega_{f,n}^{\text{ed}(2)} = \sum_{l=1}^4 \Phi_{n,\lambda}^{(l)}[f_n]\) by setting

\[
\Phi_{n,\lambda}^{(1)}[f_n] = -K_{\mu_n}(\sqrt{\lambda}r) \int_1^r I_{\mu_n}(\sqrt{\lambda}s) g_n^{(1)}(s) ds ,
\]

\[
\Phi_{n,\lambda}^{(2)}[f_n] = -\sqrt{\lambda} K_{\mu_n}(\sqrt{\lambda}r) \int_1^r s I_{\mu_{n+1}}(\sqrt{\lambda}s) f_{\theta,n}(s) ds ,
\]

\[
\Phi_{n,\lambda}^{(3)}[f_n] = I_{\mu_n}(\sqrt{\lambda}r) \int_r^\infty K_{\mu_n}(\sqrt{\lambda}s) g_n^{(2)}(s) ds ,
\]

\[
\Phi_{n,\lambda}^{(4)}[f_n] = \sqrt{\lambda} I_{\mu_n}(\sqrt{\lambda}r) \int_r^\infty s K_{\mu_{n-1}}(\sqrt{\lambda}s) f_{\theta,n}(s) ds .
\]

Then the assertion (95) follows from the estimates of each term \(|x|^{-1} \Phi_{n,\lambda}^{(l)}[f_n], l \in \{1,2,3,4\}\). (I) Estimates of \(|x|^{-1} \Phi_{n,\lambda}^{(1)}[f_n]\) and \(|x|^{-1} \Phi_{n,\lambda}^{(2)}[f_n]\): We give a proof only for \(|x|^{-1} \Phi_{n,\lambda}^{(2)}[f_n]\) since the proof for \(|x|^{-1} \Phi_{n,\lambda}^{(1)}[f_n]\) is similar. The Minkowski inequality leads to

\[
\|\Phi_{n,\lambda}^{(2)}[f_n]\|_{L^q(D)} = |\lambda|^{-\frac{1}{2}} \left(\int_1^\infty \left(\int_r^\infty r^{-1} K_{\mu_n}(\sqrt{\lambda}s) s I_{\mu_{n+1}}(\sqrt{\lambda}s) f_{\theta,n}(s) ds\right)^{\tilde{q}} r dr\right)^{\frac{1}{\tilde{q}}}
\]

\[
\leq |\lambda|^{-\frac{1}{2}} \int_1^\infty \left(\int_s^\infty |r^{-1} K_{\mu_n}(\sqrt{\lambda}s)|^{\tilde{q}} r dr\right)^{\frac{1}{\tilde{q}}} s I_{\mu_{n+1}}(\sqrt{\lambda}s) f_{\theta,n}(s) ds .
\]

By (153) and (155) for \(k = 1\) in Lemma B.1 and (167) and (168) in Lemma B.3 we have

\[
\|\Phi_{n,\lambda}^{(2)}[f_n]\|_{L^q(D)} \leq C|\lambda| \int_1^{\text{Re}(\sqrt{\lambda})} s^{-\frac{2}{\tilde{q}}} |f_n(s)| s ds + C |\lambda|^{-\frac{1}{2}} \int_{\text{Re}(\sqrt{\lambda})}^\infty s^{-2+\frac{2}{\tilde{q}}} |f_n(s)| s ds ,
\]

24
which implies (95) since $\frac{q}{q-1}(-2 + \frac{1}{q}) + 2 < 0$ holds if $\tilde{q} \in \{1, \frac{q}{q-1}\}$.

(II) Estimates of $|x|^{-1} \Phi_{n,\lambda}^{(1)}[f_n]$ and $|x|^{-1} \Phi_{n,\lambda}^{(4)}[f_n]$: We give a proof only for $|x|^{-1} \Phi_{n,\lambda}^{(4)}[f_n]$. After using the Minkowski inequality in the same way as above, from (150), (152), and (154) with $k = 1$ in Lemma B.1 and (169) and (170) in Lemma B.4, we have
\[
\| \Phi_{n,\lambda}^{(4)}[f_n] \|_{L^p(D)} \leq C|\lambda|^{\frac{1}{2}} \int_{\frac{1}{\Re(\sqrt{\lambda})}}^{\infty} s^2 K_{\mu_n-1}(\sqrt{\lambda} s) f_{\theta_n}(s) \left( \int_{1}^{s} |r^{-1} I_{\mu_n}(\sqrt{\lambda} r)|^q r dr \right)^{\frac{1}{q}} ds 
\leq C\beta^{-1}|\lambda| \int_{1}^{\frac{1}{\Re(\sqrt{\lambda})}} s^2 |f_n(s)| s ds + C|\lambda|^{-\frac{1}{2}} \int_{\frac{1}{\Re(\sqrt{\lambda})}}^{\infty} s^{-2+\frac{1}{q}} |f_n(s)| s ds,
\]
which leads to (95). Hence we obtain the assertion (95).

(iii) Estimate of $\left| \left\langle \omega_{f,\lambda}^{(1)}(x), |x|^{-1}(w_{f,\lambda}^{ed})_n \right\rangle \right|_{L^2(D)}$: From (90) and (91) we see that
\[
\left| \left\langle \omega_{f,\lambda}^{(1)}(x), \frac{|x|}{f_n} \right\rangle \left\langle f_n, \frac{|x|}{f_n} \right\rangle \right|_{L^2(D)} \leq \left| \frac{c_{n,\lambda} f_n}{F_n(\sqrt{\lambda}; \beta)} \right|^2 \left| \left\langle K_{\mu_n}(\sqrt{\lambda}), \frac{V_{f,\lambda}[K_{\mu_n}(\sqrt{\lambda})]}{|x|} \right\rangle \right|_{L^2(D)} + \left| \frac{c_{n,\lambda} f_n}{F_n(\sqrt{\lambda}; \beta)} \right|^2 \left| \left\langle K_{\mu_n}(\sqrt{\lambda}), \frac{V_{f,\lambda}[\Phi_{n,\lambda}]}{|x|} \right\rangle \right|_{L^2(D)}.
\]

Then, by Proposition 3.1 and (81) in Corollary 3.8 combined with the results in Lemma B.1 for $k = 0$ and (87) and (88) in the proof Proposition 3.9, we have
\[
\left| \left\langle \omega_{f,\lambda}^{(1)}(x), \frac{|x|}{f_n} \right\rangle \left\langle f_n, \frac{|x|}{f_n} \right\rangle \right|_{L^2(D)} \leq C\beta^{-3}|\lambda|^{-2+\frac{2}{q}} \| f \|_{L^q(D)}^2 \left( \int_{1}^{\frac{1}{\Re(\sqrt{\lambda})}} \frac{1}{r} |r^{-\Re(\mu_n)}| dr + |\lambda|^{-\frac{1}{2}} \int_{\frac{1}{\Re(\sqrt{\lambda})}}^{\infty} e^{-\Re(\sqrt{\lambda}) r} dr \right).
\]

By (92) in the proof of Proposition 3.10 combined with Lemmas 3.6 and 3.7, we have
\[
\left| \left\langle \omega_{f,\lambda}^{(1)}(x), \frac{|x|}{f_n} \right\rangle \left\langle f_n, \frac{|x|}{f_n} \right\rangle \right|_{L^2(D)} \leq C\beta^{-2}|\lambda|^{-2+\frac{2}{q}} \| f \|_{L^2(D)}^2 \left( \int_{1}^{\frac{1}{\Re(\sqrt{\lambda})}} \frac{1}{r} |r^{-\Re(\mu_n)}| dr + |\lambda|^{-\frac{1}{2}} \int_{\frac{1}{\Re(\sqrt{\lambda})}}^{\infty} r^{\frac{1}{2}} e^{-\Re(\sqrt{\lambda}) r} dr \right).
\]

Hence, by inserting the above estimates into (97), one can check that the assertion (96) holds. This completes the proof of Theorem 3.11.

\[
\square
\]

3.2 Problem II: External force $\text{div} \, F$ and Dirichlet condition

In this subsection we consider the following resolvent problem for $(w, r) = (w_{\text{div}F}^{ed}, r_{\text{div}F}^{ed})$:
\[
\begin{cases}
\lambda w - \Delta w + \beta U^\perp \text{rot} \, w + \nabla r = \text{div} \, F, & x \in D, \\
div w = 0, & x \in D, \\
w|_{\partial D} = 0.
\end{cases}
\]

In particular, the estimates for the $\pm 1$-Fourier mode of $w_{\text{div}F}^{ed}$ are our interest. Here $F = (F_{ij}(x))_{1 \leq i, j \leq 2}$ is a $2 \times 2$ matrix. We recall that the operator $\text{div}$ on matrices $G = \]
$$(G_{ij}(x))_{1 \leq i,j \leq 2}$$ is defined as $\text{div} \, G = (\partial_1 G_{11} + \partial_2 G_{12}, \partial_1 G_{21} + \partial_2 G_{22})^T$. The assumption on $F$ is as follows: let us take the constant $\gamma \in (\frac{1}{2}, 1)$ of Assumption 1.1 in the introduction. Fix $\gamma' \in (\frac{1}{2}, \gamma)$. Then we assume that $F$ belongs to the function space $X_{\gamma'}(D)$ defined as

$$X_{\gamma'}(D) = \{ F \in L^2(D)^{2\times 2} \mid \| F \|_{\gamma'} \in L^2(D)^{2\times 2} \}. \quad (98)$$

This definition is motivated from the property of the matrix $R \otimes v + R \otimes v$ appearing in $\text{RS}_{\text{ed}}$, where $R$ is the function in Assumption 1.1 and $v \in D(H^1)$. The weak solution to (RS) in view of the regularity of the solution to (RS), we can represent the weak solution to (RS) as

$$w_n \in P_n(L^0(D) \cap W^{1,p}(D)^2)$$

is said to be a weak solution to $\text{RS}_{\text{ed},F}$ replacing $\text{div} \, F$ by $(\text{div} \, F)_n = P_n \text{div} \, F$ if

$$\lambda < w_n \varphi >_{L^2(D)} + \langle \nabla w_n, \nabla \varphi \rangle_{L^2(D)} + \beta (U^\perp \text{rot} \, w_n \varphi)_{L^2(D)}$$

holds for all $\varphi \in C_0^\infty(D)^2$. Then the pressure $r \in W^{1,p}_{\text{loc}}(D)$ is recovered by a standard functional analytic argument; see [40] page 73, Lemma 2.21 for instance. The uniqueness of weak solutions is trivial thanks to the representation formula (99) below. In the following we consider the solutions to $\text{RS}_{\text{div},F,n}$ for given $F \in X_{\gamma'}(D)$.

Let $n \in \mathbb{Z} \setminus \{0\}$. By the solution formula (60) in Subsection 3.1, at least when $F \in C_0^\infty(D)^{2\times 2}$, we can represent the $n$-Fourier mode of the solution $w_{\text{ed},F,n}$ to $\text{RS}_{\text{div},F}$ as

$$w_{\text{ed},F,n} = \frac{c_{n,\lambda}[\text{div} \, F]_n}{F_n(\sqrt{X}; \beta)} V_n[K_{\mu,\lambda}(\sqrt{x}; \beta)] + V_n[\Phi_{n,\lambda}(\text{div} \, F)_n], \quad (99)$$

if $\lambda \in \mathbb{C} \setminus \mathbb{R}$ satisfies $F_n(\sqrt{x}; \beta) \neq 0$. Here $c_{n,\lambda}[\cdot], F_n(\sqrt{x}; \beta), V_n[\cdot], \Phi_{n,\lambda}[\cdot]$ are respectively defined in (62), (58), (27), and (61). Then the vorticity of $w_{\text{ed},F,n}$ is given by

$$\text{rot} \, w_{\text{ed},F,n} = -\frac{c_{n,\lambda}[\text{div} \, F]_n}{F_n(\sqrt{x}; \beta)} K_{\mu,\lambda}(\sqrt{x}; \beta) e^{i \theta} + \Phi_{n,\lambda}(\text{div} \, F)_n (r) e^{i \theta}. \quad (100)$$

We prove the estimates of (99) and (100) in the next two subsections. Before concluding this subsection, we prepare a useful lemma for the calculation concerning $\Phi_{n,\lambda}(\text{div} \, F)_n$.

**Lemma 3.12** Let $n \in \mathbb{Z} \setminus \{0\}$ and $F \in C_0^\infty(D)^{2\times 2}$. Then there are functions $\tilde{F}_{n,k} = \tilde{F}_{n,k}(r), k \in \{1, \ldots, 7\}$, each of which is a linear combination containing the $n$-Fourier mode of the components of $F = (F_{ij})_{1 \leq i,j \leq 2}$, such that $\Phi_{n,\lambda}(\text{div} \, F)_n$ is represented as

$$\Phi_{n,\lambda}(\text{div} \, F)_n (r) = -K_{\mu,\lambda}(\sqrt{x}; \beta) \left( \int_r^\infty s^{-1} I_{\mu,\lambda}(\sqrt{x}; s) \tilde{F}_{n,1}^{(1)}(s) \, ds ight.$$

$$+ \sqrt{\lambda} \int_r^\infty I_{\mu+1,\lambda}(\sqrt{x}; s) \tilde{F}_{n,2}^{(2)}(s) \, ds - \lambda \int_1^r s I_{\mu,\lambda}(\sqrt{x}; s) \tilde{F}_{n,3}^{(3)}(s) \, ds \bigg)$$

$$+ I_{\mu,\lambda}(\sqrt{x}; \beta) \left( \int_r^\infty s^{-1} K_{\mu,\lambda}(\sqrt{x}; s) \tilde{F}_{n,4}^{(4)}(s) \, ds ight.$$

$$+ \sqrt{\lambda} \int_r^\infty K_{\mu,\lambda}(\sqrt{x}; s) \tilde{F}_{n,5}^{(5)}(s) \, ds + \lambda \int_r^\infty s K_{\mu,\lambda}(\sqrt{x}; s) \tilde{F}_{n,6}^{(6)}(s) \, ds \bigg)$$

$$- \sqrt{\lambda r}(K_{\mu,\lambda}(\sqrt{x}; \beta) I_{\mu+1,\lambda}(\sqrt{x}; s) + K_{\mu-1,\lambda}(\sqrt{x}; s) I_{\mu,\lambda}(\sqrt{x}; s)) \tilde{F}_{n,7}^{(7)}(r). \quad (101)$$

26
Proof: Let \( n \in \mathbb{Z} \setminus \{0\} \). By the definition of \( \text{div} \ F \), there are functions \( G_n^{(l)} \in C_0^\infty((1, \infty)) \), \( l \in \{1, \ldots, 4\} \), such that the \( n \)-Fourier mode \( (\text{div} \ F)_n \) has a representation

\[
(\text{div} \ F)_n = (\text{div} \ F)_{n \cdot r} e^{i n \theta} e_r + (\text{div} \ F)_{\theta \cdot r} e^{i n \theta} e_\theta
\]

\[=
(\partial_r G_n^{(1)}(r) + \frac{1}{r} G_n^{(2)}(r)) e^{i n \theta} e_r + (\partial_r G_n^{(3)}(r) + \frac{1}{r} G_n^{(4)}(r)) e^{i n \theta} e_\theta.
\]

(102)

Then there are functions \( H_n^{(m)} \in C_0^\infty((1, \infty)) \), \( m \in \{1, \ldots, 4\} \), each of which is a linear combination containing the \( n \)-mode of the components of \( F = (F_{ij})_{1 \leq i,j \leq 2} \), such that

\[
\mu_n(\text{div} \ F)_{\theta \cdot r} (r) + \text{im}(\text{div} \ F)_{r \cdot n} (r) = \partial_r H_n^{(1)}(r) + \frac{1}{r} H_n^{(2)}(r),
\]

(103)

\[
\mu_n(\text{div} \ F)_{\theta \cdot r} (r) - \text{im}(\text{div} \ F)_{r \cdot n} (r) = \partial_r H_n^{(3)}(r) + \frac{1}{r} H_n^{(4)}(r).
\]

(104)

By inserting (102)–(104) into the representation of \( \Phi[f_n] \) in (61) replacing \( f_n \) by \( (\text{div} \ F)_n \), and using the next relations of Bessel functions \( I_\mu(z) \) and \( K_\mu(z) \) (see [1] page 376):

\[
\frac{dI_\mu(z)}{dz} = \frac{\mu}{z} I_\mu(z) + I_{\mu+1}(z), \quad \frac{dK_\mu(z)}{dz} = -\frac{\mu}{z} K_\mu(z) - K_{\mu-1}(z),
\]

we can obtain the assertion (101). We omit the details since the calculations are straightforward using integration by parts. The proof is complete. \( \square \)

3.2.1 Estimates of the velocity solving \( \text{RS}^{\text{ed}}_{\text{div} F,n} \) with \( |n| = 1 \)

The main result of this subsection is the estimates of \( u_{\text{div} F,n}^{\text{ed}} \) represented as in (99). Let us recall that \( \beta_0 \) is the constant in Proposition 3.1.

Theorem 3.13 Let \( |n| = 1, \gamma' \in (\frac{1}{2}, \gamma) \), and \( p \in (\frac{2}{\gamma'}, \infty) \). Fix \( \epsilon \in (0, \frac{\gamma}{2}) \). Then there is a positive constant \( C = C(\gamma', p, \epsilon) \) independent of \( \beta \) such that the following statement holds. Let \( F \in C_0^\infty(D)^{2 \times 2} \) and \( \beta \in (0, \beta_0) \). Then for \( \lambda \in \Sigma_{\pi-\epsilon} \cap B_{\epsilon^\frac{1}{\beta \gamma}}(0) \) we have

\[
\|u_{\text{div} F,n}^{\text{ed}}\|_{L^p(D)} \leq \frac{C}{\beta^2 \gamma 4^{\gamma-1} \gamma'} \|x|^{\gamma'} F\|_{L^2(D)}, \quad (105)
\]

\[
\|u_{\text{div} F,n}^{\text{ed}}\|_{L^2(D)} \leq \frac{C}{\beta^2} \|x|^{\gamma'} F\|_{L^2(D)}. \quad (106)
\]

Moreover, (105) and (106) hold all for \( F \in X_{\gamma'}(D) \) defined in (98).

By following a similar procedure as in Subsection 3.1.1, we give the proof of Theorem 3.13 at the end of this subsection. We firstly focus on the term \( V_n[\Phi_n, \lambda][(\text{div} \ F)_n] \) in (99). By using Lemma 3.12 one can see that the next decomposition holds. Let \( \tilde{F}_n^{(k)}(r) \), \( k \in \{1, \ldots, 7\} \), be the functions in Lemma 3.12.

Lemma 3.14 Let \( n \in \mathbb{Z} \setminus \{0\} \) and \( F \in C_0^\infty(D)^{2 \times 2} \). Then we have

\[
\int_{-1}^{1} \int_{1}^{r} s^{1+|n|} \Phi_n, \lambda, [(\text{div} \ F)_n](s) \, ds = \sum_{l=1}^{10} J_l^{(2)}([\text{div} \ F)_n](r),
\]

(107)
where

\[
J_1^{(2)}[(\text{div } F)_n](r) = -\frac{1}{r^n} \int_1^r \tau^{-1} I_{\mu_n}(\sqrt{\lambda} \tau) \tilde{F}_n^{(1)}(\tau) \int_\tau^r s^{1+n} |\mu_n(\sqrt{\lambda} s) d\tau ,
\]

\[
J_2^{(2)}[(\text{div } F)_n](r) = -\frac{\sqrt{\lambda}}{r^n} \int_1^r I_{\mu_{n+1}}(\sqrt{\lambda} \tau) \tilde{F}_n^{(2)}(\tau) \int_\tau^r s^{1+n} |\mu_n(\sqrt{\lambda} s) d\tau ,
\]

\[
J_3^{(2)}[(\text{div } F)_n](r) = -\frac{\lambda}{r^{n+1}} \int_1^r \tau I_{\mu_n}(\sqrt{\lambda} \tau) \tilde{F}_n^{(3)}(\tau) \int_\tau^r s^{1+n} |\mu_n(\sqrt{\lambda} s) d\tau ,
\]

\[
J_4^{(2)}[(\text{div } F)_n](r) = \frac{1}{r^n} \int_1^r \tau^{-1} I_{\mu_n}(\sqrt{\lambda} \tau) \tilde{F}_n^{(4)}(\tau) \int_\tau^r s^{1+n} |\mu_n(\sqrt{\lambda} s) d\tau ,
\]

\[
J_5^{(2)}[(\text{div } F)_n](r) = \frac{1}{r^n} \left( \int_1^r \tau^{-1} I_{\mu_n}(\sqrt{\lambda} \tau) \tilde{F}_n^{(5)}(\tau) d\tau \right) \left( \int_\tau^r s^{1+n} |\mu_n(\sqrt{\lambda} s) d\tau ,
\]

\[
J_6^{(2)}[(\text{div } F)_n](r) = \frac{\sqrt{\lambda}}{r^n} \left( \int_1^r K_{\mu_{n-1}}(\sqrt{\lambda} \tau) \tilde{F}_n^{(6)}(\tau) d\tau \right) \left( \int_\tau^r s^{1+n} |\mu_n(\sqrt{\lambda} s) d\tau ,
\]

\[
J_7^{(2)}[(\text{div } F)_n](r) = \frac{\lambda}{r^{n+1}} \left( \int_1^r \tau K_{\mu_n}(\sqrt{\lambda} \tau) \tilde{F}_n^{(7)}(\tau) d\tau \right) \left( \int_\tau^r s^{1+n} |\mu_n(\sqrt{\lambda} s) d\tau ,
\]

\[
J_8^{(2)}[(\text{div } F)_n](r) = -\frac{\sqrt{\lambda}}{r^n} \int_1^r s \left( K_{\mu_n}(\sqrt{\lambda} s) I_{\mu_{n+1}}(\sqrt{\lambda} s) + K_{\mu_{n-1}}(\sqrt{\lambda} s) I_{\mu_n}(\sqrt{\lambda} s) \right) \tilde{F}_n^{(8)}(s) d\tau .
\]

Here \(\tilde{F}_n^{(k)}(r), k \in \{1, \ldots, 7\},\) are the functions in Lemma 3.12

**Proof:** The assertion follows by inserting (101) in Lemma 3.12 into the left-hand side of (107), and changing order of integration as \(\int_1^r f_1^s d\tau ds = \int_1^r \int_1^s f_1^s d\tau ds \) and \(\int_1^r \int_1^\infty f_1^\infty d\tau ds = \int_1^r \int_1^\infty d\tau ds + \int_1^\infty \int_1^\infty d\tau ds .\) This completes the proof.

The next lemma gives the estimates to \(J_1^{(2)}[(\text{div } F)_n], l = \{1, \ldots, 10\},\) in Lemma 3.14

**Lemma 3.15** Let \(|\lambda| = 1\) and \(\gamma' \in (\frac{1}{2}, \gamma),\) and let \(\lambda \in \Sigma_{\pi, -\epsilon} \cap B_1(0)\) for some \(\epsilon \in (0, \frac{\pi}{2}).\) Then there is a positive constant \(C = C(\gamma', \epsilon)\) independent of \(\beta\) such that the following statement holds. Let \(F \in C_0^\infty(D)^{2 \times 2} .\) Then for \(l = \{1, \ldots, 10\}\) we have

\[
|J_1^{(2)}[(\text{div } F)_n](r)| \leq \frac{C}{\beta}(|\lambda|^{\frac{1}{2}} r^2 + r^{2-\text{Re}(\mu_n)} + r^{1-\gamma'}) ||x|^{\gamma'} F||_{L^2(D)} , \quad 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1} - , \quad (108)
\]

\[
|J_1^{(2)}[(\text{div } F)_n](r)| \leq \frac{C}{\beta}(|\lambda|^{-\frac{1}{2}} r + r^{1-\gamma'}) ||x|^{\gamma'} F||_{L^2(D)} , \quad r \geq \text{Re}(\sqrt{\lambda})^{-1} - . \quad (109)
\]

**Proof:** (i) Estimate of \(J_1^{(2)}[(\text{div } F)_n]:\) For \(1 \leq r < \text{Re}(\sqrt{\lambda})^{-1},\) by (153) for \(k = 0\) in

28
Lemma B.1 and (156) for \( k = 0 \) in Lemma B.2 in Appendix B we find
\[
|J_1^{(2)}[(\text{div } F_n)(r)]| \leq C r^{-1} \int_1^r |r^{-\frac{1}{2}} I_{\mu_n} (\sqrt{\lambda} r) \tilde{F}_n^{(1)}(\tau)| \int_\tau^r s^2 K_{\mu_n} (\sqrt{s} \lambda) \, ds \, d\tau
\]
\[
\leq C r^{2 - \text{Re}(\mu_n)} \int_1^r r^{\text{Re}(\mu_n) - 2} |F_n(\tau)| r \, d\tau,
\]
which implies \( |J_1^{(2)}[(\text{div } F_n)(r)]| \leq C r^{2 - \text{Re}(\mu_n)} \| x^\gamma F \|_{L^2} \). For \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \), by (153) and (155) for \( k = 0 \) in Lemma B.1 and (157) and (158) for \( k = 0 \) in Lemma B.2 we have
\[
|J_1^{(2)}[(\text{div } F_n)(r)]| \leq C |\lambda|^{-\frac{1}{2}} \| x^\gamma F \|_{L^2}.
\]
(ii) Estimate of \( J_2^{(2)}[(\text{div } F_n)(r)] \): In the similar manner as the proof of \( J_1^{(2)}[(\text{div } F_n)(r)] \), for \( 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1} \) we have \( |J_2^{(2)}[(\text{div } F_n)(r)]| \leq C |\lambda|^{-\frac{3}{2}} \| F \|_{L^2} \), and for \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \) we have \( |J_2^{(2)}[(\text{div } F_n)(r)]| \leq C |\lambda|^{-\frac{3}{2}} \| F \|_{L^2} \). We omit since the proof is straightforward.
(iii) Estimate of \( J_3^{(2)}[(\text{div } F_n)(r)] \): For \( 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1} \), we have \( |J_3^{(2)}[(\text{div } F_n)(r)]| \leq C |\lambda|^{-\frac{3}{2}} \| F \|_{L^2} \) by same way as the proof of \( J_1^{(2)}[f_n] \). For \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \), we observe that
\[
|J_3^{(2)}[(\text{div } F_n)(r)]| \leq C |\lambda|^{-\frac{1}{2}} \| x^\gamma F \|_{L^2}.
\]
thus we have \( |J_3^{(2)}[(\text{div } F_n)(r)]| \leq C |\lambda|^{-\frac{3}{2}} \| F \|_{L^2} + r^{1 - \gamma} \| x^\gamma F \|_{L^2} \).
(iv) Estimate of \( J_4^{(2)}[(\text{div } F_n)] \): For \( 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1} \), by (139) and (151) in Lemma B.1 and (161) for \( k = 0 \) in Lemma B.3 we find
\[
|J_4^{(2)}[(\text{div } F_n)(r)]| \leq r^{-1} \int_1^r |r^{-\frac{1}{2}} K_{\mu_n} (\sqrt{\lambda} r) \tilde{F}_n^{(4)}(\tau)| \int_\tau^r s^2 I_{\mu_n} (\sqrt{s} \lambda) \, ds \, d\tau
\]
\[
\leq C \int_1^r r^{\frac{1}{2}} |F_n(\tau)| r \, d\tau,
\]
which implies \( |J_4^{(2)}[(\text{div } F_n)(r)]| \leq C r^{1 - \gamma} \| x^\gamma F \|_{L^2} \). For \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \), (149), (151), and (154) for \( k = 0 \) in Lemma B.1 and (161) and (162) for \( k = 0 \) in Lemma B.3 yield
\[
|J_4^{(2)}[(\text{div } F_n)(r)]| \leq C r^{-1} \left( \int_1^r \frac{r^{-\frac{1}{2}} K_{\mu_n} (\sqrt{\lambda} r) \tilde{F}_n^{(4)}(\tau)}{r^{\frac{1}{2}} |F_n(\tau)| r} \right) \int_1^r \frac{s^2 I_{\mu_n} (\sqrt{s} \lambda)}{r^{\frac{1}{2}} |F_n(\tau)| r} \, ds \, d\tau
\]
\[
\leq C |\lambda|^{-\frac{3}{2}} \| F \|_{L^2} + C |\lambda|^{-\frac{3}{2}} \int_1^r |F_n(\tau)| r \, d\tau,
\]
which leads to \( |J_4^{(2)}[(\text{div } F)_n](r)| \leq C|\lambda|^{-\frac{1}{2}}\|F\|_{L^2} \).

(v) Estimate of \( J_{05}^{(2)}[(\text{div } F)_n] \): For \( 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1} \), we find
\[
|J_{05}^{(2)}[(\text{div } F)_n](r)| \
\leq r^{-1} \int_1^r |s^2 I_{\mu_n}(\sqrt{\lambda}s)| \ ds \left( \int_1^{\text{Re}(\sqrt{\lambda})^{-1}} |\tau^{-1} K_{\mu_n} (\sqrt{\lambda} \tau) \tilde{F}_n^{(4)}(\tau)| \ d\tau \right) \
\leq C r^{\text{Re}(\mu_n)+2} \int_1^{\text{Re}(\sqrt{\lambda})^{-1}} \tau^{-\text{Re}(\mu_n)-2}|F_n(\tau)| \ d\tau \
+ C |\lambda|^{\text{Re}(\mu_n)} \frac{1}{4} r^{\text{Re}(\mu_n)+2} \int_1^{\text{Re}(\sqrt{\lambda})^{-1}} \tau^{-\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda}) \tau}|F_n(\tau)| \ d\tau ,
\]

and thus we see that \( |J_{05}^{(2)}[(\text{div } F)_n](r)| \leq C(r^{1-\gamma'}||x|\gamma'F||_{L^2} + |\lambda|^{\frac{1}{2}} r^2 ||F||_{L^2}) \) holds. For \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \), we have
\[
|J_{05}^{(2)}[(\text{div } F)_n](r)| \
\leq r^{-1} \int_1^r |s^2 I_{\mu_n}(\sqrt{\lambda}s)| \ ds \int_1^{\text{Re}(\sqrt{\lambda})^{-1}} |\tau^{-1} K_{\mu_n} (\sqrt{\lambda} \tau) \tilde{F}_n^{(4)}(\tau)| \ d\tau \
\leq C |\lambda|^{-1} r^{\frac{1}{2}} e^{\text{Re}(\sqrt{\lambda}) r} \int_1^{\text{Re}(\sqrt{\lambda})^{-1}} \tau^{-\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda}) \tau}|F_n(\tau)| \ d\tau ,
\]

which implies \( |J_{05}^{(2)}[(\text{div } F)_n](r)| \leq C|\lambda|^{-\frac{1}{2}}\|F\|_{L^2} \).

(vi) Estimates of \( J_l^{(2)}[(\text{div } F)_n] \), \( l \in \{6, 7, 8, 9\} \): In the similar manner as the proofs for \( J_4^{(2)}[f_n] \) and \( J_5^{(2)}[(\text{div } F)_n] \), we see that
\[
|J_l^{(2)}[(\text{div } F)_n](r)| \leq C\beta^{-1}|\lambda|^{\frac{1}{2}} r^2 \|F\|_{L^2} , \quad 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1} ,
\]
\[
|J_l^{(2)}[(\text{div } F)_n](r)| \leq C(\beta^{-1}|\lambda|^{-\frac{1}{2}} \|F\|_{L^2} + r^{1-\gamma'}||x|\gamma'F||_{L^2}) , \quad r \geq \text{Re}(\sqrt{\lambda})^{-1} ,
\]
for \( l \in \{6, 7, 8, 9\} \). We omit the details since the calculations are straightforward.

(vii) Estimate of \( J_{10}^{(2)}[(\text{div } F)_n] \): For \( 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1} \), we have
\[
|J_{10}^{(2)}[(\text{div } F)_n](r)| \leq C\beta^{-1}|\lambda| \int_1^r |F_n(s)| s \ ds ,
\]
which implies \( |J_{10}^{(2)}[(\text{div } F)_n](r)| \leq C\beta^{-1}|\lambda|^{\frac{1}{2}} r^2 \|F\|_{L^2} \). For \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \), we have
\[
|J_{10}^{(2)}[(\text{div } F)_n](r)| \leq C\beta^{-1}|\lambda|^{\frac{1}{2}} r^{-1} \int_1^{\text{Re}(\sqrt{\lambda})^{-1}} s |F_n(s)| s \ ds + C r^{-1} \int_1^{\text{Re}(\sqrt{\lambda})^{-1}} s^{-1} |F_n(s)| s \ ds ,
\]

which leads to \( |J_{10}^{(2)}[(\text{div } F)_n](r)| \leq C(\beta^{-1}|\lambda|^{-\frac{1}{2}} \|F\|_{L^2} + r^{1-\gamma'}||x|\gamma'F||_{L^2}) \). This completes the proof of Lemma 3.15.

We continue the analysis on \( V_{n1}[\Phi_{n,\lambda}[(\text{div } F)_n]] \) in (99). The next decomposition is also useful in calculation as is Lemma 3.14.

**Lemma 3.16** Let \( n \in \mathbb{Z} \setminus \{0\} \) and \( F \in C_0^\infty(D)^{2 \times 2} \). Then we have
\[
r^{[n]} \int_r^\infty s^{1-[n]} |\Phi_{n,\lambda}[(\text{div } F)_n]|(s) \ ds = \sum_{l=11}^{20} J_l^{(2)}[(\text{div } F)_n](r) , \quad (110)
\]
where

\[ J_{11}^{(2)}[(\text{div } F)_n](r) = -r^n \int_1^\infty \tau^{-1} I_{\mu_n}(\sqrt{\lambda \tau}) \tilde{F}_n^{(1)}(\tau) \int_r^\infty s^{1-|n|} K_{\mu_n}(\sqrt{s}) \, ds \, d\tau, \]

\[ J_{12}^{(2)}[(\text{div } F)_n](r) = -r^n \int_1^\infty \tau^{-1} I_{\mu_n}(\sqrt{\lambda \tau}) \tilde{F}_n^{(1)}(\tau) \int_r^\infty s^{1-|n|} K_{\mu_n}(\sqrt{s}) \, ds \, d\tau, \]

\[ J_{13}^{(2)}[(\text{div } F)_n](r) = -\sqrt{\lambda} r^n \int_1^\infty \tau^{-1} I_{\mu_{n+1}}(\sqrt{\lambda \tau}) \tilde{F}_n^{(2)}(\tau) \int_r^\infty s^{1-|n|} K_{\mu_n}(\sqrt{s}) \, ds \, d\tau, \]

\[ J_{14}^{(2)}[(\text{div } F)_n](r) = -\sqrt{\lambda} r^n \int_1^\infty \tau^{-1} I_{\mu_{n+1}}(\sqrt{\lambda \tau}) \tilde{F}_n^{(2)}(\tau) \int_r^\infty s^{1-|n|} K_{\mu_n}(\sqrt{s}) \, ds \, d\tau, \]

\[ J_{15}^{(2)}[(\text{div } F)_n](r) = \lambda r^n \int_1^\infty \tau I_{\mu_n}(\sqrt{\lambda \tau}) \tilde{F}_n^{(3)}(\tau) \int_r^\infty s^{1-|n|} K_{\mu_n}(\sqrt{s}) \, ds \, d\tau, \]

\[ J_{16}^{(2)}[(\text{div } F)_n](r) = \lambda r^n \int_1^\infty \tau I_{\mu_n}(\sqrt{\lambda \tau}) \tilde{F}_n^{(3)}(\tau) \int_r^\infty s^{1-|n|} K_{\mu_n}(\sqrt{s}) \, ds \, d\tau, \]

\[ J_{17}^{(2)}[(\text{div } F)_n](r) = r^n \int_1^\infty \tau^{-1} K_{\mu_n}(\sqrt{\lambda \tau}) \tilde{F}_n^{(4)}(\tau) \int_r^\infty s^{1-|n|} I_{\mu_n}(\sqrt{s}) \, ds \, d\tau, \]

\[ J_{18}^{(2)}[(\text{div } F)_n](r) = \sqrt{\lambda} r^n \int_1^\infty K_{\mu_{n-1}}(\sqrt{\lambda \tau}) \tilde{F}_n^{(5)}(\tau) \int_r^\infty s^{1-|n|} I_{\mu_n}(\sqrt{s}) \, ds \, d\tau, \]

\[ J_{19}^{(2)}[(\text{div } F)_n](r) = \lambda r^n \int_1^\infty s K_{\mu_n}(\sqrt{\lambda \tau}) \tilde{F}_n^{(6)}(\tau) \int_r^\infty s^{1-|n|} I_{\mu_n}(\sqrt{s}) \, ds \, d\tau, \]

\[ J_{20}^{(2)}[(\text{div } F)_n](r) = -\sqrt{\lambda} r^n \int_1^\infty s (K_{\mu_n}(\sqrt{\lambda \tau}) I_{\mu_n+1}(\sqrt{s}) + K_{\mu_{n-1}}(\sqrt{s}) I_{\mu_n}(\sqrt{s})) \tilde{F}_n^{(7)}(s) \, ds. \]

Here \( \tilde{F}_n^{(k)}(s) \), \( k \in \{1, \ldots, 7\} \), are the functions in Lemma 3.12.

Proof: The assertion is a consequence of inserting (101) in Lemma 3.12 into the left-hand side of (110), and changing order of integration as \( \int_r^\infty \int_s^\infty ds \, d\tau = \int_r^\infty \int_s^\infty ds \, d\tau \) and \( \int_r^\infty \int_s^\infty ds \, d\tau = \int_r^\infty \int_s^\infty ds \, d\tau \). This completes the proof. \( \square \)

The next lemma summarizes the estimates to \( J_l^{(2)}[F_n], l \in \{11, \ldots, 20\} \), in Lemma 3.16.

Lemma 3.17 Let \( |n| = 1 \) and \( \gamma' \in (\frac{1}{2}, \gamma) \), and let \( \lambda \in \Sigma_{\gamma'/2} \cap B_1(0) \) for some \( \epsilon \in (0, \frac{\pi}{2}) \). Then there is a positive constant \( C = C(\gamma', \epsilon) \) independent of \( \beta \) such that the following statement holds. Let \( F \in C_0^\infty(D)^{2 \times 2} \). Then for \( l \in \{11, \ldots, 20\} \) we have

\[ |J_l^{(2)}[(\text{div } F)_n](r)| \leq C \left| \frac{1}{\beta} \right| \left| |\lambda|^{-\frac{3}{2}} + r^{2-\Re(\mu_n)} + r^{1-\gamma'} \right| \|x|^\gamma F\|_{L^2(D)}, \]

(111) \( 1 \leq r < \Re(\sqrt{\lambda})^{-1} \),

\[ |J_l^{(2)}[(\text{div } F)_n](r)| \leq C \left| |\lambda|^{-\frac{3}{2}} + r^{1-\gamma'} \right| \|x|^\gamma F\|_{L^2(D)}, \quad r \geq \Re(\sqrt{\lambda})^{-1}. \]

(112)

Proof: (i) Estimate of \( J_{11}^{(2)}[(\text{div } F)_n] \): For \( 1 \leq r < \Re(\sqrt{\lambda})^{-1} \), by (153) for \( k = 0 \) in Lemma B.1 and (159) for \( k = 0 \) in Lemma B.2 in Appendix B we find

\[ |J_{11}^{(2)}[(\text{div } F)_n](r)| \leq r \left| \int_r^\infty K_{\mu_n}(\sqrt{s}) \, ds \right| \int_1^\infty \tau^{-1} I_{\mu_n}(\sqrt{\lambda \tau}) \tilde{F}_n^{(1)}(\tau) \, d\tau \]

\[ \leq C \beta^{-1} r^{2-\Re(\mu_n)} \int_1^\infty r^{\Re(\mu_n)-2} |F_n(\tau)| \, d\tau, \]

(113)
which implies \(|J_{11}^{(2)}[(\text{div } F)_n](r)| \leq C\beta^{-1}r^{2-\text{Re}(\mu_n)}||x|^{\gamma'} F||_{L^2}^r\). For \(r \geq \text{Re}(\sqrt{\lambda})^{-1}\), by (153) and (155) for \(k = 0\) in Lemma B.1 and (160) for \(k = 0\) in Lemma B.2 we see that
\[
|J_{11}^{(2)}[(\text{div } F)_n](r)| \leq r \int_r^\infty |K_{\mu_n}(\sqrt{\lambda}s)| \, ds \left( \int_1^{\text{Re}(\sqrt{\lambda})} + \int_{\text{Re}(\sqrt{\lambda})}^r \right) |\tau^{-1}I_{\mu_n}(\sqrt{\lambda}\tau)\tilde{F}_n^{(1)}(\tau)| \, d\tau \\
\leq C|\lambda|^{\frac{3}{2}}r^{\frac{1}{2}}e^{-\text{Re}(\sqrt{\lambda})r} \int_1^{\text{Re}(\sqrt{\lambda})} \tau^{-\frac{3}{2}}e^{\text{Re}(\sqrt{\lambda})\tau} |F_n(\tau)| \, d\tau + C|\lambda|^{-1}r \int_1^{\text{Re}(\sqrt{\lambda})} \tau^{-\frac{5}{2}}e^{\text{Re}(\sqrt{\lambda})\tau} |F_n(\tau)| \, d\tau.
\]
Thus we have \(|J_{11}^{(2)}[(\text{div } F)_n](r)| \leq C|\lambda|^{-\frac{3}{2}}||x|^{\gamma'} F||_{L^2}^r\).

(ii) Estimate of \(J_{12}^{(2)}[(\text{div } F)_n]:\) For \(1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}\), by (153) and (155) for \(k = 0\) in Lemma B.1 and (159) and (160) for \(k = 0\) in Lemma B.2 we observe that
\[
|J_{12}^{(2)}[(\text{div } F)_n](r)| \leq r \left( \int_r^\infty + \int_{\text{Re}(\sqrt{\lambda})}^\infty \right) |\tau^{-1}I_{\mu_n}(\sqrt{\lambda}\tau)\tilde{F}_n^{(1)}(\tau)| \int_\tau^\infty |K_{\mu_n}(\sqrt{\lambda}s)| \, ds \, d\tau \\
\leq C\beta^{-1}r \int_r^{\text{Re}(\sqrt{\lambda})} \tau^{-1}|F_n(\tau)| \, d\tau + C|\lambda|^{-1}r \int_r^{\text{Re}(\sqrt{\lambda})} \tau^{-3}|F_n(\tau)| \, d\tau,
\]
which implies \(|J_{12}^{(2)}[(\text{div } F)_n](r)| \leq C\beta^{-1}r^{-\gamma'}||x|^{\gamma'} F||_{L^2}^r\). For \(r \geq \text{Re}(\sqrt{\lambda})^{-1}\), by (155) for \(k = 0\) in Lemma B.1 and (160) for \(k = 0\) in Lemma B.2 we find
\[
|J_{12}^{(2)}[(\text{div } F)_n](r)| \leq Cr \int_r^\infty |\tau^{-1}I_{\mu_n}(\sqrt{\lambda}\tau)\tilde{F}_n^{(1)}(\tau)| \int_\tau^\infty |K_{\mu_n}(\sqrt{\lambda}s)| \, ds \, d\tau \\
\leq C|\lambda|^{-1}r \int_r^\infty \tau^{-3}|F_n(\tau)| \, d\tau,
\]
which leads to \(|J_{12}^{(2)}[(\text{div } F)_n](r)| \leq C|\lambda|^{-\frac{5}{2}}||F||_{L^2}^r\).

(iii) Estimates of \(J_{1}^{(2)}[(\text{div } F)_n], l \in \{13, 14, 15, 16\}:\) In the similar manner as the proofs of \(J_{11}^{(2)}[f_n]\) and \(J_{12}^{(2)}[(\text{div } F)_n]\), we have
\[
|J_{l}^{(2)}[(\text{div } F)_n](r)| \leq C\beta^{-1}(|\lambda|^{\frac{3}{2}}r^2||F||_{L^2}^r + r^{-1-\gamma'}||x|^{\gamma'} F||_{L^2}^r), \quad 1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}, \\
|J_{l}^{(2)}[(\text{div } F)_n](r)| \leq C(|\lambda|^{-\frac{5}{2}}||F||_{L^2}^r + r^{-1-\gamma'}||x|^{\gamma'} F||_{L^2}^r), \quad r \geq \text{Re}(\sqrt{\lambda})^{-1},
\]
for \(l \in \{13, 14, 15, 16\}\). We omit the details since the calculations are straightforward.

(iv) Estimates of \(J_{l}^{(2)}[(\text{div } F)_n], l \in \{17, 18, 19\}:\) We give a proof only for \(J_{19}^{(2)}[(\text{div } F)_n]\) since the proofs for \(J_{17}^{(2)}[(\text{div } F)_n]\) and \(J_{18}^{(2)}[(\text{div } F)_n]\) are similar. For \(1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}\), from (149), (151), and (154) for \(k = 0\) in Lemma B.1 and (163) and (164) for \(k = 0\) in Lemma B.3 we observe that
\[
|J_{19}^{(2)}[(\text{div } F)_n](r)| \leq |\lambda|r \left( \int_r^{\text{Re}(\sqrt{\lambda})} + \int_{\text{Re}(\sqrt{\lambda})}^r \right) |\tau K_{\mu_n}(\sqrt{\lambda}\tau)\tilde{F}_n^{(6)}(\tau)| \int_\tau^r |I_{\mu_n}(\sqrt{\lambda}s)| \, ds \, d\tau \\
\leq C|\lambda|r \int_r^{\text{Re}(\sqrt{\lambda})} \tau|F_n(\tau)| \, d\tau + Cr \int_r^\infty \tau^{-1}|F_n(\tau)| \, d\tau,
\]
which completes the proof of the estimates. 

32
which implies $|J_{19}^{(2)}[(\text{div } F)_n](r)| \leq C r^{1-\gamma'}||x|\gamma' F||_{L^2}$. For $r \geq \text{Re}(\sqrt{\lambda})^{-1}$, by (154) for $k = 0$ in Lemma B.1 and (165) in Lemma B.3 for $k = 0$, we have

$$|J_{19}^{(2)}[(\text{div } F)_n](r)| \leq |\lambda| r \int_{r}^{\infty} |\tau K_{\mu_n}(\sqrt{\lambda} \tau) \tilde{F}_n^{(6)}(\tau)| \int_{r}^{\tau} |I_{\mu_n}(\sqrt{\lambda} s)| \, ds \, d\tau \leq C r \int_{r}^{\infty} \tau^{-1} |F_n(\tau)| \, d\tau,$$

which leads to $|J_{19}^{(2)}[(\text{div } F)_n](r)| \leq C r^{1-\gamma'}||x|\gamma' F||_{L^2}$.

(v) Estimate of $J_{20}^{(2)}[(\text{div } F)_n]$ for $1 \leq r < \text{Re}(\sqrt{\lambda})^{-1}$, we have

$$|J_{20}^{(2)}[(\text{div } F)_n](r)| \leq C \beta^{-1} |\lambda| r \int_{r}^{\text{Re}(\sqrt{\lambda})} s |F_n(s)| \, ds + C r \int_{r}^{\infty} s^{-1} |F_n(s)| \, ds.$$

Thus we have $|J_{20}^{(2)}[(\text{div } F)_n](r)| \leq C \beta^{-1} r^{1-\gamma'}||x|\gamma' F||_{L^2}$. For $r \geq \text{Re}(\sqrt{\lambda})^{-1}$, we have

$$|J_{20}^{(2)}[(\text{div } F)_n](r)| \leq C r \int_{r}^{\infty} \tau^{-1} |F_n(\tau)| \, d\tau,$$

which implies $|J_{20}^{(2)}[(\text{div } F)_n](r)| \leq C r^{1-\gamma'}||x|\gamma' F||_{L^2}$. This completes the proof of Lemma 3.17.

From Lemmas 3.15 and 3.17 we see that the following estimates hold.

**Corollary 3.18** Let $|n| = 1$, $\gamma' \in \left(\frac{1}{2}, \gamma\right)$, and $p \in \left(\frac{1}{2}, \infty\right)$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap B_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(\gamma', p, \epsilon)$ independent of $\beta$ such that the following statement holds. Let $F \in C^{\alpha}(D)^{2\times 2}$. Then for $l \in \{1, \ldots, 20\}$ we have

$$|c_{n,\lambda}[(\text{div } F)_n]| \leq \frac{C}{\beta} ||x|\gamma' F||_{L^2(D)}, \quad (113)$$

$$||r^{-1} J_l^{(2)}[(\text{div } F)_n]|_{L^p(D)}| \leq \frac{C}{\beta} |\lambda|^{-\frac{1}{2}} ||x|\gamma' F||_{L^2(D)}, \quad (114)$$

$$||r^{-2} J_l^{(2)}[(\text{div } F)_n]|_{L^2(D)}| \leq \frac{C}{\beta^2} ||x|\gamma' F||_{L^2(D)}. \quad (115)$$

Here $c_{n,\lambda}[(\text{div } F)_n]$ is the constant in (62) replacing $f_n$ by $(\text{div } F)_n$.

**Proof:** (i) Estimate of $c_{n,\lambda}[(\text{div } F)_n]$: By the definitions of $J_l^{(2)}[f_n]$ for $l \in \{11, \ldots, 20\}$ in Lemma 3.16 we see that $|c_{n,\lambda}[(\text{div } F)_n]| \leq \sum_{l=12,14,16,17,18,19,20} |J_{l}^{(2)}[f_n](1)|$. Hence we obtain the estimate (113) by putting $r = 1$ to (111) in Lemma 3.17.

(ii) Estimate of $r^{-1} J_l^{(2)}[(\text{div } F)_n]$: By Lemmas 3.15 and 3.17 for $p \in \left(\frac{1}{2}, \infty\right)$ we have

$$\sup_{r \geq 1} r^{\frac{5}{4}} |r^{-1} J_l^{(2)}[(\text{div } F)_n](r)| \leq C \beta^{-1} |\lambda|^{-\frac{1}{2}} ||x|\gamma' F||_{L^2(D)}.$$

Thus by the Marcinkiewicz interpolation theorem we obtain (114) for $p \in \left(\frac{1}{2}, \infty\right)$.

(iii) Estimate of $r^{-2} J_l^{(2)}[(\text{div } F)_n]$: The assertion (115) can be checked easily by using Lemmas 3.15 and 3.17 and $(\text{Re}(\mu_n) - 1)^{\frac{3}{2}} \approx O(\beta)$. This completes the proof.

The next proposition gives the estimate for the term $V_n[\Phi_{n,\lambda}[(\text{div } F)_n]]$ in (99).
Proposition 3.19  Let \( |n| = 1, \gamma' \in (\frac{1}{2}, \gamma), \) and \( p \in (\frac{2}{\gamma'}, \infty), \) and let \( \lambda \in \Sigma_{\pi - \epsilon} \cap B_1(0) \) for some \( \epsilon \in (0, \frac{\pi}{2}) \). Then there is a positive constant \( C = C(\gamma', p, \epsilon) \) independent of \( \beta \) such that for \( \Phi \in C_0^\infty(D)^{2 \times 2} \) we have

\[
\| V_n[\Phi_{n,\lambda}[(\text{div} F)]]_{n} \|_{L^p(D)} \leq \frac{C}{\beta} |\lambda| - \frac{1}{2} \| |x| \gamma' \|_{L^2(D)},
\]

\[
\| V_n[\Phi_{n,\lambda}[(\text{div} F)]]_{n} \|_{L^2(D)} \leq \frac{C}{\beta^2} \| |x| \gamma' \|_{L^2(D)}.
\]

Proof: In the similar manner as the proof of Proposition 3.10, we find

\[
|V_n[\Phi_{n,\lambda}[(\text{div} F)]]_{n}(r)| \leq C \left( r^{-2} \sum_{l=1}^{20} |J_1^{(1)}[(\text{div} F)]]_{n}(1)| + \sum_{l=1}^{20} |r^{-1} J_1^{(1)}[(\text{div} F)]]_{n}(r)| \right).
\]

Thus the assertions (116) and (117) follow from Corollary 3.18. The proof is complete. \( \square \)

From Corollary 3.18 and Proposition 3.19, Theorem 3.13 follows.

Proof of Theorem 3.13: (i) Estimate for the case \( F \in C_0^\infty(D)^{2 \times 2} \): It suffices to prove that the first term in the right-hand side of (99) has the estimates (105) and (106) in view of Proposition 3.10. By using Proposition 3.19 and (113) in Corollary 3.18, we see that (105) and (106) respectively follow from (85) and (86) in Proposition 3.9.

(ii) Estimate for the case \( F \in X_\gamma(D) \): Let us take sequences \( \{G^{(m)}\}_{m=1}^{\infty} \subset C_0^\infty(D)^{2 \times 2} \) and \( \{w_n^{(m)}\}_{n=1}^{\infty} \subset \mathcal{P}_n(L^p_0(D) \cap W^{1,p}_0(D)^2) \) such that \( \lim_{m \to \infty} \| |x| \gamma' (F - G^{(m)}) \|_{L^2(D)} = 0 \) and \( w_n^{(m)} \) is a (unique) solution to \( \text{RS}^{ed}_\text{div} \) replacing \( F \) by \( G^{(m)} \). Then, since \( w_n^{(m)} \) satisfies (105) and the estimates in Theorem 3.20 below replacing \( F \) by \( G^{(m)} \), by using \( \| \nabla h \|_{L^2(D)} \leq C \| \text{rot} h \|_{L^2(D)} \) for \( h \in L^p_0(D) \cap W^{1,p}_0(D)^2 \), we observe that the limit \( w_n = \lim_{m \to \infty} w_n^{(m)} \in \mathcal{P}_n(L^p_0(D) \cap W^{1,p}_0(D)^2) \) exists and also satisfies (105) and the estimates in Theorem 3.20. Moreover, by taking the limit \( m \to \infty \) in \( \text{RS}^{ed}_\text{div} \) replacing \( F \) by \( G^{(m)} \), we see that \( w_n \) gives a weak solution to \( \text{RS}^{ed}_\text{div} \). This completes the proof. \( \square \)

3.2.2 Estimates of the vorticity for \( \text{RS}^{ed}_\text{div} \) with \( |n| = 1 \)

In this subsection we estimate the vorticity \( \omega^{ed}_{\text{div},n}(r) = (\text{rot} w^{ed}_{\text{div},n}) e^{-in\theta} \) with \( |n| = 1 \), where \( \text{rot} w^{ed}_{\text{div},n} \) is represented as (100). We take the constant \( \beta_0 \) in Proposition 3.1.

Theorem 3.20  Let \( |n| = 1, \gamma' \in (\frac{1}{2}, \gamma), \) and \( p \in [2, \infty) \), and \( q \in (1, \infty) \). Fix \( \epsilon \in (0, \frac{\pi}{2}) \). Then there is a positive constant \( C = C(\gamma', p, q, \epsilon) \) independent of \( \beta \) such that the following statement holds. Let \( F \in C_0^\infty(D)^{2 \times 2}, f \in L^q(D)^2, \) and \( \beta \in (0, \beta_0) \). Set

\[
\omega^{ed}_{\text{div},n}(r) = \frac{c_n, \lambda}{F_n(\sqrt{\lambda}; \beta)} K_{\mu n}(\sqrt{\lambda} r), \quad \omega^{ed}_{\text{div},n}(r) = \Phi_{n,\lambda}[(\text{div} F_n)](r).
\]

34
Then for $\lambda \in \Sigma_{x-\epsilon} \cap \mathcal{B}_{e^{-\frac{1}{e^\epsilon}}}(0)$ we have
\begin{align*}
\|\omega_{\text{div}F,n}^{(1)}\|_{L^p(D)} & \leq \frac{C}{\beta(p\text{Re}(\mu_n) - 2)^\frac{1}{p}}\||x|^\gamma F\|_{L^2(D)}; \quad (119) \\
\|\omega_{\text{div}F,n}^{(2)}\|_{L^p(D)} + \beta\|\omega_{\text{div}F,n}^{(2)}\|_{L^1(D)} & \leq \frac{C}{\beta}\||x|^\gamma F\|_{L^2(D)}; \quad (120) \\
\|\omega_{\text{div}F,n}^{(1)}\|_{L^p(D)} & \leq \frac{C}{\beta}\||x|^\gamma F\|_{L^2(D)}; \quad (121)
\end{align*}

Moreover, (119), (120), and (121) hold all for $F \in X_{x'}(D)$ defined in (98) by a density argument as in the proof of Theorem 3.73 above.

**Proof:** (i) Estimate of $\omega_{\text{div}F,n}^{(1)}$: The estimate (119) is a direct consequence of Proposition 3.1.11 and (113) in Corollary 3.18 and (166) in Lemma B.4 in Appendix B.

(ii) Estimates of $\omega_{\text{div}F,n}^{(2)}$ and $|x|^{-1}\omega_{\text{div}F,n}^{(2)}$: Firstly we decompose $\omega_{\text{div}F,n}^{(2)} = \sum_{n=1}^7 \Phi_{n,\lambda}^{(l)}[(\text{div} F)_n]$ as in Lemma 3.12. Then the assertion (120) follows from the estimates of each term $\Phi_{n,\lambda}^{(l)}[(\text{div} F)_n], l \in \{1, \ldots, 7\}$.

(I) Estimates of $\Phi_{n,\lambda}^{(l)}[(\text{div} F)_n], l \in \{1, 2, 3\}$: We give a proof only for $\Phi_{n,\lambda}^{(3)}[(\text{div} F)_n]$ since the proofs for $\Phi_{n,\lambda}^{(1)}[(\text{div} F)_n]$ and $\Phi_{n,\lambda}^{(2)}[(\text{div} F)_n]$ are similar. The Minkowski inequality and the Fubini theorem lead to
\begin{align*}
\|\Phi_{n,\lambda}^{(3)}[(\text{div} F)_n]\|_{L^p(D)} + \beta\|\Phi_{n,\lambda}^{(3)}[(\text{div} F)_n]\|_{L^1(D)} & \leq |\lambda| \int_1^\infty |s I_{\mu_n}(\sqrt{\lambda s}) F_{n}^{(3)}(s)| \left( \int_s^\infty |K_{\mu_n}(\sqrt{x r})|^p r dr \right)^{\frac{1}{p}} + \beta \int_s^\infty |K_{\mu_n}(\sqrt{x r})| dr \right) ds.
\end{align*}

By (153) and (155) for $k = 0$ in Lemma B.1 and (171) and (172) in Lemma B.4 we have
\begin{align*}
\|\Phi_{n,\lambda}^{(3)}[(\text{div} F)_n]\|_{L^p(D)} + \beta\|\Phi_{n,\lambda}^{(3)}[(\text{div} F)_n]\|_{L^1(D)} & \leq C \beta^{-1}|\lambda|^\frac{1}{p} \int_{\text{Re}(\sqrt{\lambda s})}^1 |s F_n(s)| s ds + C|\lambda|^\frac{1}{p} \int_{\text{Re}(\sqrt{\lambda s})}^\infty s^{-\frac{2}{p}} |s F_n(s)| s ds,
\end{align*}

which implies (120) since the condition $\gamma' \in (\frac{1}{2}, 1)$ is assumed.

(II) Estimates of $\Phi_{n,\lambda}^{(l)}[(\text{div} F)_n], l \in \{4, 5, 6\}$: We give a proof only for $\Phi_{n,\lambda}^{(6)}[(\text{div} F)_n]$ since the proofs for $\Phi_{n,\lambda}^{(4)}[(\text{div} F)_n]$ and $\Phi_{n,\lambda}^{(5)}[(\text{div} F)_n]$ are similar. After using the Minkowski inequality and the Fubini theorem, by (149), (151), and (154) for $k = 0$ in Lemma B.1 and (173) and (174) in Lemma B.4 we observe that
\begin{align*}
\|\Phi_{n,\lambda}^{(6)}[(\text{div} F)_n]\|_{L^p(D)} + \beta\|\Phi_{n,\lambda}^{(6)}[(\text{div} F)_n]\|_{L^1(D)} & \leq |\lambda| \int_1^\infty |s K_{\mu_n}(\sqrt{\lambda s}) F_{n}^{(6)}(s)| \left( \int_s^\infty |I_{\mu_n}(\sqrt{x r})|^p r dr \right)^{\frac{1}{p}} + \int_1^s |I_{\mu_n}(\sqrt{x r})| dr ds.
\end{align*}
Lemma 3.21

Let \( \lambda \) be a constant in Proposition 3.1. Recall that \( \beta \) is the constant in Proposition 3.1. Then, we have

\[
\frac{1}{R_0(\sqrt{\lambda})} \int_1^{\infty} s^{-\frac{1}{2}} \int_{R_0(\sqrt{\lambda})} s^\gamma |s^{\frac{1}{3}} f_n(s)| s \, ds,
\]

which leads to \((120)\) by the condition \( \gamma' \in (\frac{1}{2}, 1) \).

(iii) Estimate of \( \Phi_n^{(7)}[\text{div}(F_n)] \): The proof is straightforward using the results in Lemma 3.11 and thus we omit the details.

(ii) Estimate of \( |\omega_n^{(1)}| |x|^{-1} (\omega_n^{(1)} w_n) \|_{L^2(D)} \): We omit since the proof is parallel to that for \((96)\) in Theorem 3.11 using \((113)\) in Corollary 3.18. The proof is complete.

3.3 Problem III: No external force and boundary data \( b \)

In this subsection we give the estimates for \( (b_n, r_n) \) solving the next problem:

\[
\begin{aligned}
\lambda w - \Delta w + \beta U \cdot \text{rot} w + \nabla r &= 0, \quad x \in D, \\
\text{div} w &= 0, \quad x \in D, \\
w|\partial D &= b.
\end{aligned}
\]

\( w \) is a solution to \((\text{RS}_b^{ed}) \) and \( \text{div} \) is the operator \( \text{div} \) with some pressure \( \gamma \) solves \((120)\) by the condition \( \gamma' \in (\frac{1}{2}, 1) \). The proof is complete.

Lemma 3.21

Let \( |n| = 1 \) and \( b \in L^\infty(\partial D)^2 \), and let \( \lambda \in \mathbb{C} \setminus (\mathbb{R}_- \cup \mathcal{Z}(F_n)) \). Suppose that \( w_b^{ed} \) is a solution to \((\text{RS}_b^{ed}) \). Then the \( n \)-Fourier modes \( w_{b,n}^{ed} \) and \( \omega_{b,n}^{ed} \) satisfy the following representations:

\[
\begin{aligned}
w_{b,n}^{ed} &= \frac{T_n(b)}{F_n(\sqrt{\lambda}; \beta)} V_n[K_{\mu_n}(\sqrt{\lambda}) + \frac{V_n[b](\theta)}{r^2}], \\
\omega_{b,n}^{ed}(r) &= \frac{T_n(b)}{F_n(\sqrt{\lambda}; \beta)} K_{\mu_n}(\sqrt{\lambda}r),
\end{aligned}
\]

where the operator \( T_n(b) \) and the vector field \( V_n[b](\theta) \) are defined as

\[
\begin{aligned}
T_n(b) &= \frac{b_{r,n}}{\bar{m}} - b_{\theta,n}, \\
V_n[b](\theta) &= b_{r,n} e^{i\theta} e_r + \frac{b_{r,n}}{\bar{m}} e_{\theta}.
\end{aligned}
\]

Here \( \mathcal{Z}(F_n) \) is the set in \((59)\) and \( V_n[\cdot] \) is the Biot-Savart law in \((27)\).

Proof: It is easy to see that \( u = \frac{T_n(b)}{F_n(\sqrt{\lambda}; \beta)} V_n[K_{\mu_n}(\sqrt{\lambda})] \) solves

\[
\begin{aligned}
\lambda u - \Delta u + \beta U \cdot \text{rot} u + \nabla p &= 0, \quad x \in D, \\
\text{div} u &= 0, \quad x \in D, \\
u_r|\partial D &= 0, \quad u_{\theta}|\partial D = -T_n(b),
\end{aligned}
\]

with some pressure \( p \in W_{L^1}^{1,1}(\Omega) \). The vector field \( \frac{V_n[b](\theta)}{r^2} \) corrects the boundary condition in \((125)\) so that \( u + \frac{V_n[b](\theta)}{r^2} \) solves \((\text{RS}_b^{ed})\) replacing \( b \) by \( b_n \). The proof is complete.

The estimates for \( w_{b,n}^{ed} \) and \( \omega_{b,n}^{ed} \) in Lemma 3.21 are the main results of this subsection. We recall that \( \beta_0 \) is the constant in Proposition 3.1.
Theorem 3.22 Let $|n| = 1$, $p \in (1, \infty)$, and $q \in (1, \infty)$. Fix $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(p, q, \epsilon)$ independent of $\beta$ such that the following statement holds. Let $b \in L^\infty(\partial D)^2$, $f \in L^3(D)^2$, and $\beta \in (0, \beta_0)$. Then for $\lambda \in \Sigma_{\pi-\epsilon} \cap B_{\epsilon^{-\frac{1}{\beta}}} (0)$ we have

\[
\|u_{b,n}^{\text{ed}}\|_{L^p(D)} \leq \frac{C}{\beta} |\lambda|^{-\frac{1}{q}} \|b\|_{L^\infty(\partial D)},
\]

(126) \[
\|u_{b,n}^{\text{ed}}\|_{L^2(\mathbb{R}^2)} \leq \frac{C}{\beta^2} |\lambda|^{-1} \|b\|_{L^\infty(\partial D)},
\]

(127) \[
\|\omega_{b,n}^{\text{ed}}\|_{L^2(\mathbb{R}^2)} \leq \frac{C}{\beta} |\lambda|^{-1} \|f\|_{L^3(D)} \|b\|_{L^\infty(\partial D)},
\]

(128) \[
\left| \langle \omega_{b,n}^{ed}, \frac{(w_{\theta})^n}{|x|} \rangle_{L^2(D)} \right| \leq \frac{C}{\beta^2} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^3(D)} \|b\|_{L^\infty(\partial D)}.
\]

(129)

Proof: The estimates (126) and (127) follow by Propositions 3.1 and 3.9 while (128) follows by Proposition 3.1 and (166) with $p = 2$ in Lemma B.3 in Appendix B. The proof for (129) is parallel to that for (96) in Theorem 3.11. The proof is complete.

3.4 Resolvent estimate in region exponentially close to the origin

In this subsection we treat the problem (RS) when the resolvent parameter $\lambda$ is exponentially close to the origin. We start with the a priori estimate of the term $\langle (\text{rot } v)_n, \frac{v_{\theta,n}}{|x|} \rangle_{L^2(D)}$, where $|n| = 1$, $0 < |\lambda| < \epsilon^{-\frac{1}{\beta}}$, which is needed in closing the energy computation. We recall that $D$ denotes the exterior disk $\{x \in \mathbb{R}^2 \mid |x| > 1\}$, and that $R$, $\gamma$, $\kappa$ are defined in Assumption I.1. Let $\beta_0$ be the constant in Proposition 3.1.

Proposition 3.23 Let $|n| = 1$, $q \in (1, 2]$, and $f \in L^q(\Omega)^2$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap B_{\epsilon^{-\frac{1}{2\beta}}} (0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Suppose that $v \in D(\mathcal{A}_{\mathcal{V}})$ is a solution to (RS). Then we have

\[
\left| \langle (\text{rot } v)_n, \frac{v_{\theta,n}}{|x|} \rangle_{L^2(D)} \right| \leq \frac{C}{\beta^2} |\lambda|^{-2+\frac{2}{q}} \|f\|_{L^q(\Omega)}^2 + \frac{Kd^2}{\beta^2-2\kappa} \|\nabla v\|_{L^2(\Omega)}^2,
\]

(130) \[
\text{as long as } \beta \in (0, \beta_0). \quad \text{The constant } C \text{ is independent of } \beta \text{ and depends on } \gamma, q, \text{ and } \epsilon, \text{ while } K \text{ is independent of } \beta \text{ and } q, \text{ and depends on } \gamma \text{ and } \epsilon.
\]

Proof: In this proof we denote the function space $L^q(\Omega)$ by $L^q$ to simplify notation. Firstly we fix a positive number $\gamma' \in (\frac{1}{2}, \gamma)$, and set $F = -(R \otimes v + v \otimes R)|_D$ and $b = v|_{\partial D}$. It is easy to see that $F$ belongs to the function space $X_{\gamma'}(D)$ defined in (98), and that $b \in L^\infty(\partial D)^2$. Moreover, a direct calculation and Assumption I.1 imply that

\[
\|x|^{\gamma} F\|_{L^2} \leq C \beta^\gamma d \|\nabla v\|_{L^2(\Omega)}, \quad \|b\|_{L^\infty(\partial D)} \leq C d \|\nabla v\|_{L^2(\Omega)}.
\]

(131)

In the following we use the notations in Subsections 3.1, 3.2. Since $v|_D$ is a solution to the problem (RS$^{\text{ed}}$), by the solution formula we have the decompositions for $v_n$, $|n| = 1$:

\[
v_n = u_{f,n}^{ed} + w_{\text{div } F,n}^{ed} + w_{b,n}^{ed} \quad \text{in } D,
\]

(132) \[
(\text{rot } v)_n = \omega_{f,n}^{ed} + \omega_{\text{div } F,n}^{ed} + \omega_{b,n}^{ed} \quad \text{in } D.
\]

(133)
Then, in view of \((133)\), the assertion \((130)\) follows from estimating the next three terms:

\[
\|\langle \omega_{f,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^2} + \|\langle \omega_{v,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^2} + \|\langle \omega_{b,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^2}.
\]

(i) Estimate of \(\|\langle \omega_{f,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^2}\): We fix a number \(p \in (\frac{q}{p'}, \infty)\). Note that \(p \in (q, \infty)\) holds since \(\frac{q}{p'} > 2\). Then setting \(p' = \frac{q}{p-1} \in (1, q)\) and using the Hölder inequality, we see that

\[
\|\langle \omega_{f,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^2} \leq \|\langle \omega_{f,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^{p'}} \leq \|\langle \omega_{f,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^p} \|v_{n}\|_{L^p}.
\]

From \((94)\) and \((96)\) in Theorem 3.11, \((106)\) in Theorem 3.13, and \((127)\) in Theorem 3.22 we observe that

\[
\|\langle \omega_{f,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^p} \leq \|\langle \omega_{f,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^p} + \|\langle \omega_{v,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^p} + \|\langle \omega_{b,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^p}.
\]

Then by the condition \(\kappa \in (0, 1)\) and \((131)\) we find

\[
\|\langle \omega_{f,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^p} \leq \frac{C}{\beta^3} |\lambda|^{-1 + \frac{1}{q}} \|f\|_{L^q} (|\lambda|^{-1 + \frac{1}{q}} \|f\|_{L^q} + \beta^\gamma |b| \|\nabla v\|_{L^2(\Omega)}).
\]

On the other hand, since \(\frac{1}{q} + \frac{1}{p'} = 1\) holds, by using \((95)\) replacing \(\tilde{q}\) by \(p'\) in Theorem 3.11 \((65)\) in Theorem 3.2, \((105)\) in Theorem 3.13 and \((126)\) in Theorem 3.22 we have

\[
\|\langle \omega_{f,n}^{ed} \frac{v_{r,n}}{|x|} \rangle_{L^{p'}} \|_{L^{p'}} \leq \frac{C}{\beta^3} |\lambda|^{-1 + \frac{1}{q}} \|\nabla f\|_{L^q} (|\lambda|^{-1 + \frac{1}{q}} \|\nabla f\|_{L^q} + \beta^\gamma \|\nabla v\|_{L^2(\Omega)}).
\]

Then inserting \((135)\) and \((136)\) into \((134)\) we obtain

\[
\|\langle \omega_{f,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^p} \leq \frac{C}{\beta^3} |\lambda|^{-1 + \frac{1}{q}} \|\nabla f\|_{L^q} (|\lambda|^{-1 + \frac{1}{q}} \|\nabla f\|_{L^q} + \beta^\gamma |b| \|\nabla v\|_{L^2(\Omega)}).
\]

(ii) Estimate of \(\|\langle \omega_{v,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^2}\): By using the Hölder inequality we find

\[
\|\langle \omega_{v,n}^{ed}, \frac{v_{r,n}}{|x|} \rangle_{L^2} \|_{L^2} \leq \|\omega_{v,n}^{ed} \|_{L^2} \left(\|\frac{u_{r,n}^{ed}}{|x|} \|_{L^2} + \|\frac{u_{b,n}^{ed}}{|x|} \|_{L^2} \right)
\]

By Theorem 3.20 \((106)\) in Theorem 3.13 and \((127)\) in Theorem 3.22 we see that

\[
\|\omega_{v,n}^{ed} \|_{L^2} \left(\|\frac{u_{r,n}^{ed}}{|x|} \|_{L^2} + \|\frac{u_{b,n}^{ed}}{|x|} \|_{L^2} \right) \leq \frac{K}{\beta^2} \|\nabla f\|_{L^2} (|\lambda|^{-1 + \frac{1}{q}} \|\nabla f\|_{L^2} + \beta |b| \|\nabla v\|_{L^2(\Omega)}).
\]
where we note that the constant $K$ depends only on $\epsilon$ and $\gamma$, and is independent of $\beta$ and, in particular, of $q \in (1, 2)$. Theorem 3.2 and (65) with $p = \infty$ in Theorem 3.2 lead to

$$\left| \langle \omega^{ed}_{\text{div}}(1), \frac{(u^{ed}_{\text{div}F,n})}{|x|} \rangle \right|_{L^2} \leq C \beta^2 \|f\|_{L^p(\Omega)}\|\omega^{ed}_{\text{div}F,n}\|_{L^1} \|u^{ed}_{F,n}\|_{L^\infty} \leq \frac{C d}{\beta^5} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(\Omega)} \|\nabla v\|_{L^2(\Omega)}. \tag{140}$$

Inserting (139) and (140) into (138) we have

$$\left| \langle \omega^{ed}_{\text{div}F,n}, \frac{v_r n}{|x|^2} \rangle \right|_{L^2} \leq \frac{C d}{\beta^5} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \frac{K d^2}{\beta^{3-2\kappa}} \|\nabla v\|_{L^2(\Omega)}^2. \tag{141}$$

(iii) Estimate of $|\langle \omega^{ed}_{b,n}, \frac{v_r n}{|x|^2} \rangle|_{L^2}$: Using the Schwartz inequality and Theorem 3.22 we find

$$\left| \langle \omega^{ed}_{b,n}, \frac{v_r n}{|x|^2} \rangle \right|_{L^2(D)} \leq \left| \langle \omega^{ed}_{b,n}, \frac{(u^{ed}_{\text{div}F,n})}{|x|^2} \rangle \right|_{L^2} \leq \frac{1}{\beta^3} \left( C |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(\Omega)}\|\nabla v\|_{L^2} + \|\omega^{ed}_{b,n}\|_{L^2} \left( \|\omega^{ed}_{\text{div}F,n}\|_{L^2} + \|u^{ed}_{\text{div}}\|_{L^2} \right) \right) \leq \frac{C d}{\beta^5} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \frac{K d^2}{\beta^{3-2\kappa}} \|\nabla v\|_{L^2(\Omega)}^2. \tag{142}$$

where we have applied the condition $\kappa \in (0, 1)$. Finally we obtain the assertion (130) by collecting (137), (141), and (142), and using the Young inequality in the form

$$\frac{C d}{\beta^5} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(\Omega)}\|\nabla v\|_{L^2(\Omega)} \leq \frac{C d}{\beta^5} |\lambda|^{-2+\frac{2}{q}} \|f\|_{L^q(\Omega)}^2 + \frac{K d^2}{\beta^{3-2\kappa}} \|\nabla v\|_{L^2(\Omega)}^2. \tag{143}$$

The proof is complete.

Now we shall establish the resolvent estimate to (RS) when $0 < |\lambda| < e^{-\frac{1}{2\pi}}$, by closing the energy computation starting from Proposition 2.1 in Subsection 2.3.

**Proposition 3.24** Let $\epsilon \in (0, \frac{\pi}{2})$, and let $\beta_1$, $\beta_0$, and $K$ be the constants respectively in Propositions 2.1, 2.1, and 3.23. Then the following statements hold.

1. Fix positive numbers $\beta_3 \in (0, \min\{\beta_1, \beta_0\})$ and $\mu_\ast \in (0, (2\sqrt{2K\pi})^{-1})$. Then the set

$$\Sigma_{\frac{1}{2\pi}} \epsilon \cap B_{e^{-\frac{1}{2\pi}}} = (0) \tag{143}$$

is included in the resolvent $\rho(-A_V)$ for any $\beta \in (0, \beta_3)$ and $d \in (0, \mu_\ast^2 - \kappa)$.

2. Let $q \in (1, 2)$ and $f \in L^2(\Omega) \cap L^q(\Omega)^2$. Then we have

$$\|\lambda + A_V\|^{-1} \|f\|_{L^2(\Omega)} \leq \frac{C}{\beta^2} |\lambda|^{-\frac{1}{2} + \frac{1}{q}} \|f\|_{L^q(\Omega)}, \quad \lambda \in \Sigma_{\frac{1}{2\pi}} \epsilon \cap B_{e^{-\frac{1}{2\pi}}} = (0), \tag{144}$$

$$\|\nabla (\lambda + A_V)^{-1} f\|_{L^2(\Omega)} \leq \frac{C}{\beta^2} |\lambda|^{-1+\frac{1}{q}} \|f\|_{L^q(\Omega)}, \quad \lambda \in \Sigma_{\frac{1}{2\pi}} \epsilon \cap B_{e^{-\frac{1}{2\pi}}} = (0),$$

as long as $\beta \in (0, \beta_3)$ and $d \in (0, \mu_\ast^2 - \kappa)$. The constant $C$ is independent of $\beta$. 39
Proof: (1) Let \( \lambda \in \Sigma_{\frac{3\pi}{4} - \epsilon} \cap B_{\frac{1}{2\max(4\beta,1)}}(0) \) and suppose that \( v \in D(\mathcal{A}_V) \) is a solution to (RS). Since the condition \( d \in (0, \mu_0 \beta^{2-\epsilon}) \) ensures \( K d^2 \beta^{-4+2\epsilon} < \frac{1}{2} \), by inserting (130) in Proposition 3.23 into (29) and (30) in Subsections 2.3 and 3.4. In this section we prove Theorem 1.3. The proof is an easy consequence of Propositions 2.2 and 3.24 respectively.

(2) The estimate (144) can be easily checked by using (145). The proof is complete. \( \square \)

4 Proof of Theorem 1.3

In this section we prove Theorem 1.3. The proof is an easy consequence of Propositions 2.2 and 3.24 respectively in Subsections 2.3 and 3.4.

Proof of Theorem 1.3: Let \( \beta_2 \) be the constant in Proposition 2.2. Choose a number \( \beta_4 \in (0, \beta_2) \) small enough so that \( S_{\beta_4} \cap B_{\frac{1}{2\max(4\beta,1)}}(0) \neq \emptyset \). Then there is a constant \( \epsilon_0 \in (\frac{\pi}{4}, \frac{T}{4}) \) such that the sector \( \Sigma_{\pi-\epsilon_0} \) is included in the set \( S_{\beta} \cup B_{\frac{1}{2\max(4\beta,1)}}(0) \) for any \( \beta \in (0, \beta_4) \).

Let \( \beta_3 \) be the constant in Proposition 3.24. Fix a number \( \beta_* \in (0, \min\{\beta_3, \beta_4\}) \). Then by Propositions 2.2 and 3.24 there is a positive constant \( \mu_\ast \) such that the sector \( \Sigma_{\pi-\epsilon_0} \) is included in the resolvent \( \rho(-\mathcal{A}_V) \) as long as \( \beta \in (0, \beta_*) \) and \( d \in (0, \mu_\ast, \beta^{2-\epsilon}) \). Moreover, from the same propositions, for \( q \in (1, 2] \) and \( f \in L^q_\sigma(\Omega) \cap L^q(\Omega)^2 \) we have

\[
\| (\lambda + \mathcal{A}_V)^{-1} f \|_{L^q(\Omega)} \leq \frac{C}{\beta^2} |\lambda|^{-\frac{q}{2} + \frac{1}{4}} \| f \|_{L^q(\Omega)}, \quad \lambda \in \Sigma_{\pi-\epsilon_0} \cap \{ \Re(z) < 0 \},
\]

\[
\|\nabla (\lambda + \mathcal{A}_V)^{-1} f \|_{L^q(\Omega)} \leq \frac{C}{\beta^2} |\lambda|^{-\frac{q}{2} + \frac{1}{4}} \| f \|_{L^q(\Omega)}, \quad \lambda \in \Sigma_{\pi-\epsilon_0} \cap \{ \Re(z) < 0 \}.
\]

(146)

Next we fix a number \( \phi \in (\frac{\pi}{4}, \pi - \epsilon_0) \) and take a curve \( \gamma(b) = \{ z \in \mathbb{C} \mid |\arg z| = \phi, |z| \geq b \} \cup \{ z \in \mathbb{C} \mid |\arg z| \leq \phi, |z| = b \}, b \in (0, 1) \), oriented counterclockwise. For sufficiently small \( b \), the semigroup \( \{ e^{-t\mathcal{A}_V} \}_{t \geq 0} \) admits a Dunford integral representation

\[
e^{-t\mathcal{A}_V} f = \frac{1}{2\pi i} \int_{\gamma(b)} e^{\lambda} (\lambda + \mathcal{A}_V)^{-1} f d\lambda, \quad t > 0,
\]

for \( f \in L^q_\sigma(\Omega) \cap L^q(\Omega)^2 \). Then by taking the limit \( b \to 0 \) we observe from (146) that

\[
\| e^{-t\mathcal{A}_V} f \|_{L^q(\Omega)} \leq \frac{C}{\beta^2} \| f \|_{L^q(\Omega)} \int_{0}^{\infty} s^{-\frac{q}{2} + \frac{1}{4}} e^{-(\cos \phi)ts} ds \leq \frac{C}{\beta^2} t^{-\frac{q}{2} + \frac{1}{4}} \| f \|_{L^q(\Omega)}, \quad t > 0,
\]

which shows that (11) holds. The estimate (11) can be obtained in a similar manner using the Dunford integral. This completes the proof of Theorem 1.3. \( \square \)
A Asymptotics of the order $\mu_n(\beta)$ for small $\beta$

This appendix is devoted to the statement of the asymptotic behavior for $\mu_n(\beta) = (n^2 + \imath n \beta)^\frac{1}{2}, \; \Re(\mu_n) > 0,$ with $|n| = 1$ when the constant $\beta \in (0, 1)$ in Assumption 1.1 reaches to zero. The following result is essentially proved in \[11\].

**Lemma A.1 (\[11\] Lemma B.1)** Let $|n| = 1$. Then $\mu_n(\beta)$ satisfies the expansion

\[\begin{align*}
\Re(\mu_n(\beta)) &= 1 + \frac{\beta^2}{8} + O(\beta^4), \quad 0 < \beta \ll 1, \\
\Im(\mu_n(\beta)) &= \frac{\beta}{2} + O(\beta^3), \quad 0 < \beta \ll 1.
\end{align*}\]

B Estimates of the Modified Bessel Function

In this appendix we collect the basic estimates for the modified Bessel functions $K_{\mu_n}(z)$ and $I_{\mu_n}(z)$ of the order $\mu_n = (n^2 + \imath n \beta)^\frac{1}{2}, \; \Re(\mu_n) > 0,$ with $|n| = 1$ and $\beta \in (0, 1)$. We are especially interested in the $\beta$-dependence in each estimate, since our analysis in Section 3 requires the smallness of $\beta$. We denote by $B_\rho(0)$ the disk in the complex plane $\mathbb{C}$ centered at the origin with radius $\rho > 0$.

**Lemma B.1** Let $|n| = 1, k = 0, 1, \text{ and } R \in [1, \infty)$. Fix $\epsilon \in (0, \frac{\pi}{2})$. Then there is a positive constant $C = C(R, \epsilon)$ independent of $\beta$ such that the following statements hold.

1. Let $z \in \Sigma_\epsilon \cap B_R(0)$. Then $K_{\mu_n}(z)$ and $K_{\mu_n-1}(z)$ satisfy the expansions

\[\begin{align*}
K_{\mu_n}(z) &= \frac{\Gamma(\mu_n)}{2\pi} \left(\frac{z}{2}\right)^{-\mu_n} + R_n^{(1)}(z), \\
K_{\mu_n-1}(z) &= \frac{1}{2\sin((\mu_n - 1)\pi)} \left(\frac{z}{2}\right)^{-\mu_n} - \frac{1}{\Gamma(\mu_n)} \left(\frac{z}{2}\right)^{\mu_n} + R_n^{(2)}(z).
\end{align*}\]

Here $\Gamma(z)$ denotes the Gamma function and the remainders $R_n^{(1)}(z)$ and $R_n^{(2)}(z)$ satisfy

\[\begin{align*}
|R_n^{(1)}(z)| &\leq C|z|^{2 - \Re(\mu_n)}(1 + |\log |z||), \quad z \in \Sigma_\epsilon \cap B_R(0), \\
|R_n^{(2)}(z)| &\leq C|z|^{3 - \Re(\mu_n)}(1 + |\log |z||), \quad z \in \Sigma_\epsilon \cap B_R(0).
\end{align*}\]

2. The following estimates hold.

\[\begin{align*}
|I_{\mu_n+k}(z)| &\leq C|z|^{|\Re(\mu_n)|+k}, \quad z \in \Sigma_\epsilon \cap B_R(0), \\
|K_{\mu_n-k}(z)| &\leq C|z|^{-\frac{2}{\pi}e^{-\Re(z)}}, \quad z \in \Sigma_\epsilon \cap B_R(0)^c, \\
|I_{\mu_n+k}(z)| &\leq C|z|^{-\frac{2}{\pi}e^\Re(z)}, \quad z \in \Sigma_\epsilon \cap B_R(0)^c.
\end{align*}\]

**Proof:** (1) The expansions (149) and (150) follow from the definition of $K_{\mu}(z)$ in Subsection 3.1 combined with the well-known Euler reflection formula for the Gamma function. The estimates of the remainder terms (151) and (152) are also consequences of the same definition, and we omit the calculations which are easily checked.
Lemma B.2 Let $|n| = 1$ and $k = 0, 1$, and let $\lambda \in \Sigma_{n-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a constant $C > 0$ independent of $\beta$ such that the following statements hold.

1. If $1 \leq \tau \leq r \leq \Re(\sqrt{\lambda})^{-1}$, then
   \[
   \left| \int_{\tau}^{r} s^{-k} K_{\mu_n - k}(\sqrt{\lambda} s) \, ds \right| \leq C |\lambda|^{-\frac{3}{2}} e^{\frac{1}{2} \tau \, k - \Re(\mu_n) + 3}. \tag{156}
   \]

2. If $1 \leq \tau \leq \Re(\sqrt{\lambda})^{-1} \leq r$, then
   \[
   \left| \int_{\tau}^{r} s^{-k} K_{\mu_n - k}(\sqrt{\lambda} s) \, ds \right| \leq C |\lambda|^{-\frac{3}{2}} e^{\frac{1}{2} \tau \, k}. \tag{157}
   \]

3. If $\Re(\sqrt{\lambda})^{-1} \leq \tau \leq r$, then
   \[
   \int_{\tau}^{r} |s^{-k} K_{\mu_n - k}(\sqrt{\lambda} s)| \, ds \leq C |\lambda|^{-\frac{3}{2}} e^{\frac{1}{2} \tau \, k - \Re(\sqrt{\lambda}) \tau}. \tag{158}
   \]

4. If $1 \leq \tau \leq \Re(\sqrt{\lambda})^{-1}$, then
   \[
   \left| \int_{\tau}^{\infty} s^{-k} K_{\mu_n - k}(\sqrt{\lambda} s) \, ds \right| \leq C |\lambda|^{-\frac{3}{2}} e^{\frac{1}{2} \tau \, k \, k - \Re(\mu_n) + 1}. \tag{159}
   \]

5. If $\tau \geq \Re(\sqrt{\lambda})^{-1}$, then
   \[
   \int_{\tau}^{\infty} |s^{-k} K_{\mu_n - k}(\sqrt{\lambda} s)| \, ds \leq C |\lambda|^{-\frac{3}{2}} e^{\frac{1}{2} \tau \, k - \Re(\sqrt{\lambda}) \tau}. \tag{160}
   \]

Lemma B.3 Let $|n| = 1$ and $k = 0, 1$, and let $\lambda \in \Sigma_{n-\epsilon} \cap \mathcal{B}_1(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a constant $C > 0$ independent of $\beta$ such that the following statements hold.

1. If $1 \leq \tau \leq \Re(\sqrt{\lambda})^{-1}$, then
   \[
   \int_{1}^{\tau} |s^{-k} I_{\mu_n + k}(\sqrt{\lambda} s)| \, ds \leq C |\lambda|^{-\frac{3}{2}} e^{\frac{1}{2} \tau \, k \, k - \Re(\mu_n) + 3}. \tag{161}
   \]

2. If $\tau \geq \Re(\sqrt{\lambda})^{-1}$, then
   \[
   \int_{1}^{\tau} |s^{-k} I_{\mu_n + k}(\sqrt{\lambda} s)| \, ds \leq C |\lambda|^{-\frac{3}{2}} e^{\frac{1}{2} \tau \, k \, k \, k - \Re(\sqrt{\lambda}) \tau}. \tag{162}
   \]
(3) If $1 \leq r \leq \text{Re}(\sqrt{\lambda})^{-1}$, then
\[
\int_{r}^{\tau} |s^{-k}I_{\mu_n+k}(\sqrt{\lambda}s)| \, ds \leq C|\lambda|^{\frac{\text{Re}(\mu_n)}{2}} + \frac{1}{2} \tau \text{Re}(\mu_n)+1. \tag{163}
\]

(4) If $1 \leq r \leq \text{Re}(\sqrt{\lambda})^{-1} \leq \tau$, then
\[
\int_{r}^{\tau} |s^{-k}I_{\mu_n+k}(\sqrt{\lambda}s)| \, ds \leq C|\lambda|^{-\frac{3}{4}} \tau^{-\frac{1}{2}} - \frac{1}{2} e^{\text{Re}(\sqrt{\lambda})r}. \tag{164}
\]

(5) If $\text{Re}(\sqrt{\lambda})^{-1} \leq r \leq \tau$, then
\[
\int_{r}^{\tau} |s^{-k}I_{\mu_n+k}(\sqrt{\lambda}s)| \, ds \leq C|\lambda|^{-\frac{3}{4}} \tau^{-\frac{1}{2}} - \frac{1}{2} e^{\text{Re}(\sqrt{\lambda})r}. \tag{165}
\]

Lemma B.4 Let $|n| = 1$ and $p \in (1, \infty)$, and let $\lambda \in \Sigma_{\pi-\epsilon} \cap B_{1}(0)$ for some $\epsilon \in (0, \frac{\pi}{2})$. Then there is a constant $C > 0$ independent of $\beta$ such that the following statements hold.

(1) If additionally $p \in [2, \infty)$, then
\[
||K_{\mu_n}(\sqrt{\lambda})\|_{L^p([1,\infty);r \, dr)} \leq \frac{C}{(p \text{Re}(\mu_n) - 2) \frac{1}{p}} |\lambda|^{\frac{\text{Re}(\mu_n)}{2}} \tag{166}
\]

(2) If $1 \leq r \leq \text{Re}(\sqrt{\lambda})^{-1}$, then
\[
\left( \int_{r}^{\infty} |s^{-1}K_{\mu_n}(\sqrt{\lambda}s)|^p s \, ds \right)^\frac{1}{p} \leq C|\lambda|^{-\frac{1}{2}} \frac{1}{2} p^{-\frac{1}{2}} \frac{1}{2} + \frac{1}{2} e^{-\text{Re}(\sqrt{\lambda})r}. \tag{167}
\]

(3) If $\tau \geq \text{Re}(\sqrt{\lambda})^{-1}$, then
\[
\left( \int_{r}^{\infty} |s^{-1}K_{\mu_n}(\sqrt{\lambda}s)|^p s \, ds \right)^\frac{1}{p} \leq C|\lambda|^{-\frac{1}{2}} \frac{1}{2} p^{-\frac{1}{2}} \frac{1}{2} + \frac{1}{2} e^{-\text{Re}(\sqrt{\lambda})r}. \tag{168}
\]

(4) If $1 \leq r \leq \text{Re}(\sqrt{\lambda})^{-1}$, then
\[
\left( \int_{r}^{\infty} |s^{-1}I_{\mu_n}(\sqrt{\lambda}s)|^p s \, ds \right)^\frac{1}{p} \leq C|\lambda|^{-\frac{1}{2}} \frac{1}{2} p + \frac{1}{2} e^{-\text{Re}(\sqrt{\lambda})r}. \tag{169}
\]

(5) If $\tau \geq \text{Re}(\sqrt{\lambda})^{-1}$, then
\[
\left( \int_{r}^{\infty} |s^{-1}I_{\mu_n}(\sqrt{\lambda}s)|^p s \, ds \right)^\frac{1}{p} \leq C|\lambda|^{-\frac{1}{2}} \frac{1}{2} p + \frac{1}{2} e^{-\text{Re}(\sqrt{\lambda})r}. \tag{170}
\]

(6) If additionally $p \in [2, \infty)$ and if $1 \leq r \leq \text{Re}(\sqrt{\lambda})^{-1}$, then
\[
\left( \int_{r}^{\infty} |K_{\mu_n}(\sqrt{\lambda}s)|^p s \, ds \right)^\frac{1}{p} + \beta \int_{r}^{\infty} |K_{\mu_n}(\sqrt{\lambda}s)| \, ds \leq C|\lambda|^{-\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda})r}. \tag{171}
\]

(7) If additionally $p \in [2, \infty)$ and if $\tau \geq \text{Re}(\sqrt{\lambda})^{-1}$, then
\[
\left( \int_{r}^{\infty} |K_{\mu_n}(\sqrt{\lambda}s)|^p s \, ds \right)^\frac{1}{p} + \int_{r}^{\infty} |K_{\mu_n}(\sqrt{\lambda}s)| \, ds \leq C|\lambda|^{-\frac{1}{2}} e^{-\text{Re}(\sqrt{\lambda})r}. \tag{172}
\]
Lemma C.1
Let \( \varepsilon \) depending only on sections 3.31–3.33 and Corollary A.8 in [11], we observe that the next expansion holds:

\[
(8) \quad \left( \int_1^r |I_{\mu_n}(\sqrt{\lambda}s)|^p \, ds \right)^\frac{1}{p} + \int_1^r |I_{\mu_n}(\sqrt{\lambda}s)| \, ds \leq C|\lambda|^{-\frac{1}{2}} e^{Re(\sqrt{\lambda})} r. \tag{173}
\]

(9) If additionally \( p \in [2, \infty) \) and if \( r \geq \text{Re}(\sqrt{\lambda})^{-1} \), then

\[
\left( \int_1^r |I_{\mu_n}(\sqrt{\lambda}s)|^p \, ds \right)^\frac{1}{p} + \int_1^r |I_{\mu_n}(\sqrt{\lambda}s)| \, ds \leq C|\lambda|^{-\frac{1}{2}} e^{\text{Re}(\sqrt{\lambda})} r. \tag{174}
\]

\[ \text{C Proof of Proposition 3.1} \]

Proposition 3.1 is a direct consequence of the next lemma. Let us recall that \( B_\rho(0) \) denotes the disk in the complex plane \( \mathbb{C} \) centered at the origin with radius \( \rho > 0 \).

**Lemma C.1** Let \( |n| = 1 \). Then for any \( \epsilon \in (0, \frac{\pi}{2}) \) there is a positive constant \( \beta_0 = \beta_0(\epsilon) \) depending only on \( \epsilon \) such that as long as \( \beta \in (0, \beta_0) \) and \( \lambda \in \Sigma_{\pi-\epsilon} \cap B_{\beta}(0) \) we have

\[
|F_n(\sqrt{\lambda}; \beta)| \geq \frac{C}{\beta} |\lambda|^{-\frac{\text{Re}(\mu_n)}{2}} \min\{1, -\beta^2 \text{log}|z|\}, \tag{175}
\]

where \( F_n(\sqrt{\lambda}; \beta) \) is the function in (58) and the constant \( C \) depends only on \( \epsilon \).

**Proof:** The proof is carried out with the similar spirit as in [11] Proposition 3.34, where the nonexistence of zeros of \( F_n(\sqrt{\lambda}; \beta) \) in \( \lambda \in B_{\beta}(0) \) is proved. However, its proof is based on a contradiction argument, and quantitative estimates are not explicitly stated. Hence here we give the lower bound estimate of \( |F_n(\sqrt{\lambda}; \beta)| \) for completeness.

Let \( \lambda \in \Sigma_{\pi-\epsilon} \cap B_{\frac{\beta}{2}}(0) \) and set \( \zeta_n = \zeta_n(\beta) = \mu_n(\beta) - 1 \). Then, by combining Lemmas 3.31–3.33 and Corollary A.8 in [11], we observe that the next expansion holds:

\[
\zeta_n F_n(\sqrt{\lambda}; \beta) = \frac{\Gamma(1 + \zeta_n)(\sqrt{\lambda})^{-\zeta_n}}{\sqrt{\lambda}} \left( 1 - (e^{\gamma(\zeta_n)} \sqrt{\lambda})^{-\zeta_n} + R_n(\lambda) \right), \tag{176}
\]

for sufficiently small \( \beta \) depending on \( \epsilon \in (0, \frac{\pi}{2}) \). Here the function \( \gamma(\zeta_n) \) have the expansion

\[
\gamma(\zeta_n) = \gamma + O(|\zeta_n|) \quad \text{as} \quad |\zeta_n| \to 0, \tag{177}
\]

where \( \gamma \) denotes the Euler constant \( \gamma = 0.5772 \cdots \). The remainder \( R_n \) in (176) satisfies

\[
|R_n(\lambda)| \leq C_1 |\lambda|^{-\frac{\text{Re}(\mu_n)}{2}}, \quad \lambda \in \Sigma_{\pi-\epsilon} \cap B_{\frac{\beta}{2}}(0), \tag{178}
\]

with a constant \( C_1 = C_1(\epsilon) \) independent of small \( \beta \). To simplify notation we set

\[
z = \sqrt{\lambda}, \quad \tilde{\zeta} = e^{\gamma(\zeta_n)} \sqrt{\frac{\lambda}{2}}, \quad \theta(\tilde{\zeta}) = \text{arg} \tilde{\zeta}. \tag{179}
\]

If \( \beta \) is sufficiently small, then we see from (177) and (179) that

\[
\frac{1}{2} \leq \left| \frac{\tilde{\zeta}}{z} \right| \leq 1, \quad |\theta(\tilde{\zeta})| \leq \frac{\pi}{2} - \frac{\epsilon}{4}, \quad \lambda \in \Sigma_{\pi-\epsilon} \cap B_{\frac{\beta}{2}}(0). \tag{180}
\]
Now we set
\[ h(\tilde{z}, \zeta_n) = \text{Re}(\zeta_n) \log |\tilde{z}| - \text{Im}(\zeta_n) \theta(\tilde{z}), \quad (181) \]
\[ \Omega(\tilde{z}, \zeta_n) = \text{Re}(\zeta_n) \theta(\tilde{z}) + \text{Im}(\zeta_n) \log |\tilde{z}| \]
\[ = \left( \text{Re}(\zeta_n) + \frac{\text{Im}(\zeta_n)^2}{\text{Re}(\zeta_n)} \right) \theta(\tilde{z}) + \frac{\text{Im}(\zeta_n)}{\text{Re}(\zeta_n)} h(\tilde{z}, \zeta_n). \quad (182) \]

Then it is easy to see that
\[ 1 - \tilde{z}^\zeta_n = 1 - e^{h(\tilde{z}, \zeta_n)} e^{i\Omega(\tilde{z}, \zeta_n)}. \quad (183) \]

In the following we show the lower bound estimate of $|1 - \tilde{z}^\zeta_n|$. Firstly let us take a small positive constant $\kappa = \kappa(\epsilon) \ll 1$ so that
\[ \left( \text{Re}(\zeta_n) + (1 + \kappa) \frac{\text{Im}(\zeta_n)^2}{\text{Re}(\zeta_n)} \right) \left( \frac{\pi}{2} - \frac{\epsilon}{4} \right) < \pi \quad (184) \]
holds. The existence of such $\kappa$ is verified by using Lemma A.1 in Appendix A if $\beta$ is sufficiently small depending on $\epsilon$. Note that the smallness of $\kappa$ depends only on $\epsilon$.

(i) Case $|h(\tilde{z}, \zeta_n)| \leq \kappa |\text{Im}(\zeta_n)||\theta(\tilde{z})|$: In this case, (180), (182), and (184) ensure that
\[ |\Omega(\tilde{z}, \zeta_n)| < \pi, \quad (185) \]
and thus that $e^{i\Omega(\tilde{z}, \zeta_n)}$ is close to 1 if and only if $\Omega(\tilde{z}, \zeta_n)$ is close to 0. From (181) we have
\[ -\text{Re}(\zeta_n) \log |\tilde{z}| \leq (1 + \kappa) |\text{Im}(\zeta_n)||\theta(\tilde{z})|, \]
which leads to, for sufficiently small $\beta$,
\[ |\theta(\tilde{z})| \geq -\frac{1}{1 + \kappa} \frac{\text{Re}(\zeta_n)}{|\text{Im}(\zeta_n)|} \log |\tilde{z}| \geq -\frac{\beta}{2} \log |\tilde{z}|, \]
where $\frac{\text{Re}(\zeta_n)}{|\text{Im}(\zeta_n)|} = \frac{\beta}{2} + O(\beta^3)$ is applied in Lemma A.1. Then from (182) we have
\[ |\Omega(\tilde{z}, \zeta_n)| \geq (\text{Re}(\zeta_n) + (1 - \kappa) \frac{\text{Im}(\zeta_n)^2}{\text{Re}(\zeta_n)}) |\theta(\tilde{z})| \geq -\beta \log |\tilde{z}|, \]
if $\beta$ is small enough. On the other hand, it is straightforward to see that
\[ |1 - \tilde{z}^\zeta_n| \geq \max \{|1 - e^{h(\tilde{z}, \zeta_n)} \cos \Omega(\tilde{z}, \zeta_n)|, e^{h(\tilde{z}, \zeta_n)} |\sin \Omega(\tilde{z}, \zeta_n)|\}. \]
Since $e^{h(\tilde{z}, \zeta_n)} \in [\frac{1}{2}, 1]$, $|\sin x| \geq \frac{2|x|}{\pi}$ on $|x| \in [0, \frac{\pi}{2}]$, and $1 > \frac{|\Omega(\tilde{z}, \zeta_n)|}{\pi}$ by (185), we have
\[ |1 - \tilde{z}^\zeta_n| \geq \min \left\{ 1, \frac{|\Omega(\tilde{z}, \zeta_n)|}{\pi} \right\} \geq -\frac{\beta}{\pi} \log |\tilde{z}|. \quad (186) \]

(ii) Case $|h(\tilde{z}, \zeta_n)| > \kappa |\text{Im}(\zeta_n)||\theta(\tilde{z})|$: When $|\theta(\tilde{z})| > \frac{1}{2} \frac{\text{Re}(\zeta_n)}{|\text{Im}(\zeta_n)|} \log |\tilde{z}|$, we have
\[ |h(\tilde{z}, \zeta_n)| \geq \frac{\kappa \beta^2}{2} \log |\tilde{z}|. \]
On the other hand, when \(|\theta(\tilde{z})| \leq \frac{1}{2} \frac{\text{Re}(\zeta_n)}{\text{Im}(\zeta_n)} \log |\tilde{z}|\), (181) implies that
\[
|h(\tilde{z}, \zeta_n)| \geq \frac{1}{2} \text{Re}(\zeta_n) \log |\tilde{z}| \geq -\frac{\beta^2}{2} \log |\tilde{z}|.
\]
Thus in the case (ii), since \(|1 - \tilde{z}^n| \geq |1 - |\tilde{z}^n|| = |1 - e^{h(\tilde{z}, \zeta_n)}|\), we observe that
\[
|1 - \tilde{z}^n| \geq \min\{1, |h(\tilde{z}, \zeta_n)|\} \geq \min\{1, -\frac{\kappa \beta^2}{2} \log |\tilde{z}|\}.
\]
Hence, by collecting (180), (186), and (187), we have the next lower estimate of \(|1 - \tilde{z}^n|\):
\[
|1 - \tilde{z}^n| \geq \frac{\kappa}{4} \min\{1, -\beta^2 \log |z|\}.
\]
Finally by inserting (178) and (188) into (176) we obtain
\[
|\zeta_n F_n(\sqrt{\lambda}; \beta)| \geq C|\lambda|^{-\frac{\text{Re}(\mu_n)}{2}} (\kappa \min\{1, -\beta^2 \log |z|\} - C_1 |\lambda|^{-\frac{\text{Re}(\mu_n)}{2}}),
\]
which implies the assertion (175) if \(\lambda \in \Sigma_{\pi - \epsilon} \cap E\beta(0)\) and \(\beta\) is sufficiently small depending on \(\epsilon\). The proof is complete. \(\square\)

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