Static output feedback $H_2/H_\infty$ control with spectrum constraints for stochastic systems subject to multiplicative noises

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**ABSTRACT**

This paper investigates the problem of static output feedback $H_2/H_\infty$ control with spectrum constraints for linear continuous-time stochastic systems with state dependent noises. By using the spectrum technique, the notion of $D(-\beta, -\alpha)$ stability for continuous-time stochastic systems is proposed. Subsequently, the design method of the static output feedback controller is presented by solving a set of linear matrix inequalities. Under $D(-\beta, -\alpha)$ stability, a sufficient condition for the existence of stochastic $H_2/H_\infty$ control is derived, which transforms the problem of $H_2/H_\infty$ control into a convex optimization problem. Finally, a numerical example is given to show the effectiveness of the obtained results.

1. Introduction

Over the past decades, stochastic systems have been played more and more significant roles in control engineering practice. Especially, stochastic systems modelled by Itô differential equations have received an increasing attention because of their wide applications in various areas such as aerospace engineering, industrial process control, economics, synthetic biology, etc. (Ding, Wang, Ho, & Wei, 2017; Geng, Liang, Liu, & Alsaadi, 2018; Geng, Wang, Liang, Cheng, & Alsaadi, 2017; Khandani, Majd, & Tahmasebi, 2017; Verdejo, Vargas, & Kliemann, 2012; Wang & Zhu, 2015; Xie & Duan, 2010; Yuan, Wang, & Guo, 2018). As one of the most basic models, many issues have been studied for linear stochastic systems, such as observability and detectability (Shen, Sun, & Wu, 2013), stability and stabilization (Ren & Xiong, 2017; Song & Zhu, 2018), $H_\infty$ control (Kao, Xie, Wang, & Karimi, 2015; Yang, Wang, Shu, Alsaadi, & Hayat, 2016), and optimal control (Wang, Zhang, & Zhang, 2014; Yuan, Wang, Zhang, & Dong, 2018; Yuan, Yuan, Wang, Guo, & Yang, 2017). In addition, due to the powerful model ability, many complicated topics regarding stochastic systems have been investigated extensively. For instance, the problem of $H_\infty$ control for nonlinear stochastic Takagi–Sugeno (T–S) fuzzy systems with time-delay has been solved in Senthilkumar and Balasubramaniam (2011) by means of linear matrix inequalities (LMIs). By introducing a new Lyapunov-Krasovskii functional, the issue of stabilization for nonlinear stochastic Markovian jump systems with mixed mode-dependent time-delays has been studied in Wang, Liu, and Liu (2010).

It is well recognized that stability and stabilization are fundamental concepts in modern control theory. During the past several decades, various notions of stability for stochastic systems have been proposed by applying Lyapunov function approach, see e.g. (Bolzern, Colaneri, & Nicolaio, 2010; Fang, Loparo, & Feng, 1994; Liu, Zhang, & Zhang, 2010; Zhu, Xi, & Li, 2012) where the stochastic stability, asymptotic mean square stability, exponential stability, $\delta$-moment stability and global asymptotical stability have been studied. In actually, for a stochastic system, we would be interested not only steady-state performance but also transient response. Therefore, the spectrum technique and the relevant $H$-representation method have been put forwarded in Zhang and Chen (2012) for stochastic systems, and the spectrum technique can also be used to describe some novel concepts of stability: critical stability, $\alpha$-stability, interval stability, and essential instability (Hou, Zhang, & Chen, 2011; Sheng, Gao, & Zhang, 2013). It is worth mentioning that we can obtain the desired convergence rate of the system by means of the spectrum technique, which was interesting and useful in stochastic control theory.

In practical applications, the mixed $H_2/H_\infty$ control serves as a typical multiobjective control method, which requires one to design a controller not only to attenuate
the exogenous disturbance but also to minimize the desired output when the worst case disturbance is implemented, that is, the controlled system could satisfy both the pre-specified $H_\infty$ and $H_2$ performance indices at the same time. For linear deterministic systems, several methods have been put forward to solve the mixed $H_2/H_\infty$ control problems, such as the time-domain Nash game method (Sheng, Zhang, & Gao, 2014), convex optimization method (Khargonekar & Rotea, 1994) and linear matrix inequality method (Scherer, Gahient, & Chilali, 1997). Up to now, the problem of state feedback $H_2/H_\infty$ control has been extensively studied for stochastic systems (Ma, Wang, Bo, & Guo, 2011; Orihuela, Millan, Vivas, & Rubio, 2014; Qiu, Zhang, Xu, Pan, & Yao, 2016), while the related results for the corresponding static output feedback $H_2/H_\infty$ control have been scattered because of the difficulty in solving the linear matrix inequality. Moreover, most of the existing literatures have focused on the design of the output feedback controller with less constraints. For example, a new approach has been proposed in Chang, Park, and Zhou (2015) for robust static output feedback $H_\infty$ controller design, which was described by a set of linear matrix inequalities. It should be pointed out that, unfortunately, the static output feedback $H_2/H_\infty$ control has not been investigated so far. It is, therefore, the purpose of this paper to shorten such a gap.

Summarizing the above discussions, the objective of this paper is to deal with $H_2/H_\infty$ control issue with spectrum constraints for a stochastic system with multiplicative noises. The key of the issue is to design a static output feedback controller such that the closed-loop system satisfies the pre-specified $H_\infty$ disturbance attenuation level and the energy of system output is minimized, simultaneously. The main contribution of this paper can be summarized as follows. (1) The problem of the output feedback $H_2/H_\infty$ control with spectrum constraints is firstly proposed for continuous-time stochastic systems. (2) A sufficient condition is derived under which the closed-loop system is $D(-\beta,-\alpha)$ and satisfies the $H_2$ and $H_\infty$ performances at the same time. In conclusion, the control problem considered in this paper is significant in both the theoretical and practical senses.

The rest of the paper is organized as follows. In Section 2, the concept of interval stability and some properties are proposed based on the spectrum technique, and a sufficient condition for stochastic systems to be $D(-\beta,-\alpha)$-stable is given. In Section 3, the problem of $H_2/H_\infty$ control with spectrum constraints is solved, where the design method of static output feedback controller is presented. Section 4 gives a numerical example to show the effectiveness of the proposed method. Finally, some conclusions are put forward in Section 5.

Notations: The notation used here is fairly standard except where otherwise stated. $\mathbb{R}$ (respectively, $\mathbb{C}$, $\mathbb{C}^\times$) is the space of all real (respectively, complex, negative complex) numbers. $\mathbb{R}^n$ represents the $n$-dimensional Euclidean space, and $\mathbb{R}^{m \times n}$ represents the set of all $m \times n$ real matrices. $A > 0$ ($A \geq 0$) means that $A$ is a real symmetric positive definite (positive semidefinite) matrix. $I$ is the unit matrix with appropriate dimensions and $S_m$ represents the set of all $n \times n$ real symmetric matrices. $A^T$ denotes the transpose of a matrix $A$. $\theta(\mathcal{L})$ stands for the spectrum set of the operator or matrix $\mathcal{L}$. $\|x\|$ describes the Euclidean norm of a vector $x$. $\text{diag}\{\cdots\}$ represents a diagonal matrix and $E(x)$ refers to the mathematical expectation of $x$. The asterisk $*$ in a matrix is used to represent the term that is induced by symmetry. $L_2[0, \infty)$ stands for the space of square integrable vector functions over $[0, \infty)$. The notation $Het(A)$ denotes $A + A^T$ for simplicity, and the notation $\eta_{m \times n}$ represents that $\eta \in \mathbb{R}^{m \times n}$. The vertical strips region of the left-half plane is described as $D(-\beta,-\alpha) := \{ \kappa : \kappa \in \mathbb{C}^\times, \alpha < |\kappa| < \beta \}$ where $0 \leq \alpha < \beta$ are real constants.

2. Stability and stabilization

Consider the following continuous-time Itô linear stochastic system with $x$-dependent noise

$$\frac{dx(t)}{dt} = (Ax(t) + Bu(t))dt + Dx(t)d\omega(t), \quad y(t) = Ex(t),$$

where $x(t) \in \mathbb{R}^nx$, $u(t) \in \mathbb{R}^{nu}$ and $y(t) \in \mathbb{R}^ny$ respectively represent the system state, control and the measurement output. $\omega(t)$ is the one-dimensional standard Wiener process with $E(\omega(t)) = 0$ and $E(\omega^2(t)) = 1$. $A, B, D$ and $E$ are known, real matrices with appropriate dimensions. Without loss of generality, the matrix $E$ is assumed to be row full rank.

As for system (1), the static output feedback controller is described by:

$$u(t) = Ky(t) = KE\hat{x}(t).$$

where $K \in n_u \times n_y$ is called the controller gain. Substituting (2) into (1), one can obtain the following closed-loop system

$$\frac{dx(t)}{dt} = (A + BKE)x(t)dt + Dx(t)d\omega(t).$$

The system (1) is called unforced system when $u(t) = 0$, which can be described as

$$\frac{dx(t)}{dt} = Ax(t)dt + Dx(t)d\omega(t), \quad x(0) = x_0 \in \mathbb{R}^nx.$$
The spectrum of \( \text{stable of the system (4)} \), we introduce a linear operator from \( S_n \) to \( S_n \) as follows:

\[
\mathcal{L}_{A,D} : X \in S_n \mapsto AX + XA^T + DXD^T \in S_n.
\] (5)

By the operator theory, we define the spectrum of \( \mathcal{L}_{A,D} \) as

\[
\theta(\mathcal{L}_{A,D}) = \{ \lambda \in \mathbb{C} : \mathcal{L}_{A,D}(X) = \lambda X, X \in S_n, X \neq 0 \}.
\] (6)

With the concept of the spectrum, we shall introduce a spectral description for the asymptotically mean square stability

**Lemma 2.1 (Zhang & Chen, 2012):** The following three conditions are equivalent:

(a) The system (4) is asymptotically stable in mean square, i.e. \( \lim_{t \to \infty} E[\|x(t)\|^2] = 0 \).

(b) The spectrum of \( \mathcal{L}_{A,D} \) located in left-half plane, that is, \( \theta(\mathcal{L}_{A,D}) \subset \mathbb{C}^{-\infty} \).

(c) There exists a positive definite symmetric matrix \( X > 0 \) such that \( \mathcal{L}_{A,D}(X) < 0 \).

**Definition 2.2 (Zhang & Chen, 2012):** Given constants \( 0 \leq \alpha < \beta \), the system (4) is said to be \( D(-\beta, -\alpha) \)-stable, if the spectrum of \( \mathcal{L}_{A,D} \) belongs to \( D(-\beta, -\alpha) \), written as, \( \theta(\mathcal{L}_{A,D}) \subset \mathbb{C}^{-\alpha} \).

**Remark 2.1:** \( D(-\beta, -\alpha) \) stability is closely related to the convergence rate of the system state response. Assume that the system (4) is \( D(-\beta, -\alpha) \)-stable, then for any sufficiently small \( \epsilon > 0 \), there exist some real constants \( \delta_1, \delta_2 > 0 \) such that

\[
\delta_2 \|x_0\|^2 e^{-\beta \epsilon^T} \leq E[\|x(t)\|^2] \leq \delta_1 \|x_0\|^2 e^{-\alpha \epsilon^T}.
\] (7)

**Lemma 2.2 (Song & Zhu, 2018):** For a given symmetric matrix \( S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \), the following three conditions are equivalent:

\[
S < 0,
\]

\[
S_{11} < 0, \quad S_{22} - S_{12}^T S_{12}^{-1} S_{11} < 0,
\]

\[
S_{22} < 0, \quad S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0.
\] (8) (9) (10)

**Lemma 2.3 (Chang et al., 2015):** For matrices \( A, B, C, D \) with appropriate dimensions and a constant \( \xi > 0 \), the inequality

\[
B < 0, \quad B + D^T A^T + AD < 0
\] (11)

is satisfied if the following condition holds:

\[
\begin{bmatrix}
B & * \\
\xi A^T + CD & -\xi C - \xi C^T
\end{bmatrix} < 0.
\] (12)

In the following lemma, we start with a \( D(-\beta, -\alpha) \) stabilization analysis of the closed-loop system (3).

**Lemma 2.4:** Consider the closed-loop system (3). For known scalar parameters \( 0 \leq \alpha < \beta \), if there exist matrices \( M > 0 \) and \( K \) satisfying the following matrix inequalities:

\[
\begin{bmatrix}
He\left(AM + BKEM + \frac{\alpha}{2} M\right) & * \\
DM & -M
\end{bmatrix} < 0,
\] (13)

\[
\begin{bmatrix}
He\left(-AM - BKEM - \frac{\beta}{2} M\right) & * & * \\
0 & He\left(-AM - BKEM - \frac{\beta}{2} M\right) & * & * \\
0 & -DM & -M & * \\
DM & 0 & 0 & -M
\end{bmatrix} < 0
\] (14)

then the system is \( D(-\beta, -\alpha) \)-stabilizable.

**Proof:** From the Definition 2.2, we know that \( \theta(\mathcal{L}_{A,D,K}) \subset \mathbb{C}_{-\beta}^{-\alpha} \) iff \( \theta(\mathcal{L}_{A,D,K}) \subset \mathbb{C}_{-\beta}^{-\alpha} \cap \mathbb{C}_{-\infty}^{-\beta} \).

Above all, notice that \( \theta(\mathcal{L}_{A,D,K}) \subset \mathbb{C}_{-\beta}^{-\alpha} \) iff \( \theta(\mathcal{L}_{A,D,K}) \subset \mathbb{C}^{-\infty} \), where \( \mathbb{C}_{-\beta}^{-\alpha} := \mathcal{L}_{A,D,K} + \alpha I \). Based on the Lemma 2.1, the system is asymptotically stable in mean square, which is equivalent to that there exists a \( X > 0 \) satisfying the following inequality

\[
X \left( A + BKE + \frac{\alpha}{2} I \right)^T + \left( A + BKE + \frac{\alpha}{2} I \right) X + DXD^T < 0
\] (15)

By applying Schur’s complement in Lemma 2.2, (15) can be written as

\[
\begin{bmatrix}
He\left(AX + BKEX + \frac{\alpha}{2} X\right) & * \\
D & -X^{-1}
\end{bmatrix} < 0
\] (16)

Set \( M = X^{-1} \), pre- and post-multiplying (16) by \( \text{diag}(M^T, I) \) and its transpose, respectively, then we have the inequality (13).

Next, note that \( \theta(\mathcal{L}_{A,D,K}) \subset \mathbb{C}_{-\beta}^{-\alpha} \) iff \( \theta(-\mathcal{L}_{A,D,K}) \subset \mathbb{C}^{-\beta} \). In the following, a sufficient condition for \( \theta(-\mathcal{L}_{A,D,K}) \subset \mathbb{C}^{-\beta} \) is \( \theta(\mathcal{L}_{A,D}) \subset \mathbb{C}^{-\beta} \) is proved by contradiction, such that

\[
\hat{A} = \begin{bmatrix}
-A - BKE - \frac{\beta}{2} I & 0 \\
0 & -A - BKE - \frac{\beta}{2} I
\end{bmatrix},
\]

\[
\hat{D} = \begin{bmatrix}
0 & -D \\
D & 0
\end{bmatrix}.
\] (17)

Assume that \( \theta(\mathcal{L}_{A,D}) \subset \mathbb{C}^{-\beta} \) but \( \theta(-\mathcal{L}_{A,D,K}) \not\subset \mathbb{C}^{-\beta} \), then there exist corresponding \( \lambda \in \mathbb{C} \) with \( \text{Re}(\lambda) \geq 0 \) and \( P \neq 0 \).
0 ∈ ℝₙ, it follows that

\[ P \left( -A - BK - \frac{\beta}{2} I \right)^\top + \left( -A - BK - \frac{\beta}{2} I \right) P^{-1} + DPD^\top = \lambda P. \]  

(18)

It is easy to verify that \( \hat{P} = \left[ \rho \ p \ p \right] \) solves \( \hat{P} \tilde{A}^\top + \tilde{A} \hat{P} + \tilde{D} \hat{P} \hat{D} = \lambda \hat{P} \), which implies that \( \lambda \in \theta(\mathcal{L}_{\tilde{A}, \tilde{D}}) \) and \( \text{Re}(\lambda) \geq 0 \). Therefore, the assumption is invalid. According to the Lemma 2.1, the system is asymptotically stable in mean square if there exist a \( \hat{X} \) with \( \tilde{X} = \text{diag}(\hat{X}, \hat{X}) \) satisfying the following inequality

\[ \hat{X} \tilde{A}^\top + \hat{A} \hat{X} + \hat{D} \hat{X} \hat{D}^\top < 0. \]  

(19)

Repeating the similar procedure as above, (14) is obtained, which ends the proof. □

By combining Lemmas 2.3 and 2.4, we have the following theorem, which provides a novel sufficient condition for designing the static output feedback controller.

**Theorem 2.1:** Consider the closed-loop system (3). Given a scalar \( \xi > 0 \), for known two scalar parameters \( 0 \leq \alpha < \beta \), the system is \( D(-\beta, -\alpha) \)-stabilizable if there exist matrices \( M > 0, V \) and \( U \) such that the following LMIs hold

\[
\begin{bmatrix}
\Omega_1 & * & * & * \\
DM & -M & * & * \\
\Omega_2 & 0 & -\xi U - \xi U^T & * \\
0 & -DM & -M & * \\
0 & 0 & 0 & -\xi U - \xi U^T \\
\Omega_4 & 0 & 0 & 0 & 0 & -\xi U - \xi U^T
\end{bmatrix} < 0,
\]

(20)

where \( \rho_{ny \times nx} \) is the same as (Chang et al., 2015) and \( \rho_{ny \times nx} = (E^T)^{-1} E \),

\[
\Omega_1 = \text{He} \left( AM + BV \rho_{ny \times nx} + \frac{\alpha}{2} M \right), \\
\Omega_2 = \xi \bar{V}^T B^T + EM - U \rho_{ny \times nx}, \\
\Omega_3 = \text{He} \left( -AM - BV \rho_{ny \times nx} - \frac{\beta}{2} M \right), \\
\Omega_4 = -\xi \bar{V}^T B^T + EM - U \rho_{ny \times nx}.
\]

Furthermore, the static output feedback controller gain matrix is given by

\[ K = VU^{-1}. \]  

(22)

**Proof:** Suppose that inequalities (20) and (21) hold. The feasible solution of these inequalities satisfies \( -\xi U - \xi U^T < 0 \), which implies that matrix \( U \) is non-singular.

Firstly, let us verify whether the inequality (20) is satisfied or not. By Lemma (2.3) with

\[ A = \begin{bmatrix} BV \\ 0 \end{bmatrix}, \]

\[ B = \begin{bmatrix} \text{He} \left( AM + BV \rho_{ny \times nx} + \frac{\alpha}{2} M \right) \\ DM \end{bmatrix}, \]

\[ C = U, \text{ and } D = U^{-1} \left[ EM - U \rho_{ny \times nx} \right], \]

the inequality (20) can guarantee

\[ \text{He} \left[ \begin{bmatrix} BV \\ 0 \end{bmatrix} \times U^{-1} \left[ EM - U \rho_{ny \times nx} \right] \right] < 0. \]

(23)

Then, (23) can be rewritten as

\[ \text{He} \left[ \begin{bmatrix} AM + \frac{\alpha}{2} M \\ DM \end{bmatrix} \right] \]

\[ + \text{He} \left[ \begin{bmatrix} BV \\ 0 \end{bmatrix} \times U^{-1} \left[ U \rho_{ny \times nx} \right] \right] \]

\[ + \text{He} \left[ \begin{bmatrix} BV \\ 0 \end{bmatrix} \times U^{-1} \left[ EM - U \rho_{ny \times nx} \right] \right] < 0. \]

(24)

Considering (22), rewriting (24) in the following form

\[ \text{He} \left[ \begin{bmatrix} AM + BKEM + \frac{\alpha}{2} M \\ DM \end{bmatrix} \right] \]

\[ < 0, \]  

(25)

it implies that the inequality (15) in Lemma 2.4 holds.

Secondly, by similar proof of (20), it can be obtained (21). The proof is omitted. □

**Remark 2.2:** It is worth pointing out that the design method of the robust static output feedback controller in Chang et al. (2015) for general uncertain systems can be extended to solve the static output feedback control problems for stochastic systems. And in general, the \( \rho_{ny \times nx} = (EE^T)^{-1} E \) when the matrix \( E \) is of full row rank.
3. Infinite horizon $H_2/H_\infty$ control with spectrum constraints

In this section, we will discuss the static output feedback $H_2/H_\infty$ control with spectrum constraints.

Consider the following linear controlled stochastic system

$$
\begin{align*}
\dot{x}(t) &= [A + BKE]x(t) + (C + BKF)v(t) \, dt + D\dot{x}(t) \, dw(t), \\
y(t) &= Ex(t) + Fv(t), \\
z(t) &= Gx(t) + Hu(t),
\end{align*}
$$

(26)

where $z(t) \in \mathbb{R}^{\mathcal{O}}$ is the controlled output. $v(t) \in \mathbb{R}^{\mathcal{E}}$ denotes the exogenous disturbance input which belongs to $L_2[0, \infty)$. $C,F,G$ and $H$ are known, real matrices with appropriate dimensions.

Under the static output feedback control $u(t) = Ky(t)$, the closed-loop system is obtained as

$$
\begin{align*}
\dot{x}(t) &= [(A + BKE)x(t) + (C + BKF)v(t)] \, dt \\
&\quad + D\dot{x}(t) \, dw(t), \\
z(t) &= (G + HKF)x(t) + HKFv(t).
\end{align*}
$$

(27)

For a prescribed disturbance attenuation level $\gamma > 0$, define the following two performances

$$
J_2(u, v) = \mathbb{E} \int_{0}^{\infty} \|z(t)\|^2 \, dt - \gamma^2 \mathbb{E} \int_{0}^{\infty} \|v(t)\|^2 \, dt
$$

(28)

and

$$
J_2(u, 0) = \lim_{t \to \infty} \frac{1}{t} \mathbb{E} \int_{0}^{t} \|z(s)\|^2 \, ds.
$$

(29)

The infinite horizon $H_2/H_\infty$ control with spectrum constraints of linear stochastic system (26) is formulated as follows.

Consider the system (26) and given a scalar $\gamma > 0$, the objective of the $H_2/H_\infty$ control is to find a static output feedback controller $u(t)$ such that

(i) Given constants $0 \leq \alpha < \beta$, the closed-loop system (27) is $\mathcal{D}(-\beta, -\alpha)$-stabilizable when the external disturbance $v(t) = 0$.

(ii) Under the zero initial condition, the $H_\infty$ performance $J_\infty(u, v) < 0$ is satisfied for any nonzero $v(t) \in L_2[0, \infty)$.

(iii) Based on the constraints (i) and (ii), the upper bound of $H_2$ performance $J_2(u, 0)$ is minimized.

If the above $u(t)$ exists, then the infinite horizon $H_2/H_\infty$ control problem with spectrum constraints of linear stochastic system (26) is solvable. In the following, we shall design a static output feedback controller for system (26) such that the design objectives (i), (ii) and (iii) are satisfied simultaneously. It turns out that the solvability of the addressed $H_2/H_\infty$ control problem can be determined by the solvability of LMIs.

**Lemma 3.1:** Consider the closed-loop system (27). Given a scalar $\gamma > 0$, if there exist matrices $M > 0$ and $K$ satisfying the following matrix inequality:

$$
\begin{bmatrix}
He(AM + BKE) & * & * \\
(C + BKE)^T & -\gamma^2 I & * \\
DM & 0 & -M \\
GM + HKEM & HKF & 0 & -I
\end{bmatrix} < 0,
$$

(30)

then the $H_\infty$ performance of the system is guaranteed.

**Proof:** By analyzing the stochastic bounded real lemma, a well-known $H_\infty$ performance analysis condition for the closed-loop system (27) is that there exist matrices $K$ and $D > 0$ such that

$$
\begin{bmatrix}
He(XA + XBKE) & * & * & * \\
(C + BKE)^T X & -\gamma^2 I & * & * \\
D & 0 & -X^{-1} & * \\
G + HK & HKF & 0 & -I
\end{bmatrix} < 0.
$$

(31)

Letting $M = X^{-1}$, pre- and post-multiplying (31) by $\text{diag}(M^T, I, I, I)$ and its transpose, then the inequality (30) can be obtained. 

In existing literatures, the $H_2$ control problem is commonly regarded as an optimization problem, which the objective is to minimize the $H_2$ performance. According to the condition (i), we know that the closed-loop system (27) is $\mathcal{D}(-\beta, -\alpha)$-stabilizable in the absence of external disturbance $v(t)$. Based on this assumption, one can obtain

$$
J_2(u^*, 0) = \min_{u(t)} J_2(u, 0) = \min_{u(t)} \lim_{t \to \infty} \frac{1}{t} \mathbb{E} \int_{0}^{t} z^T(s)z(s) \, ds
$$

$$
= \min_{u(t)} \lim_{t \to \infty} \frac{1}{t} \mathbb{E} \int_{0}^{t} [x^T(s)(G + HK) + (G + HK)X(s)(G + HK)^T] \, ds
$$

$$
\leq \text{min Trace}(G + HK)X(G + HK)^T.
$$

(32)

By introducing a positive definite symmetric matrix $Z$ and applying Schur’s complement in Lemma 2.2, the $H_2$ control problem is transformed into the following optimization problem

$$
\min \text{Trace}(Z),
$$

s.t. $\begin{bmatrix}
-Z & * \\
(GM + HKEM)^T & -M
\end{bmatrix} < 0.
$$

(33)

The following theorem presents a sufficient condition for the closed-loop system (27) to guarantee among
The static output feedback $H_2/H_\infty$ controller gain matrix is given by $K = \sqrt{U}L^{-1}$, which assigns the spectrum of $\mathcal{L}_{A,D,K}$ into $D(-\beta, -\alpha)$, satisfies the $H_\infty$ performance, and lets the $H_2$ performance less than $\text{Trace}(Z)$.

**Proof:** The (34) and (35) are directly obtained from (20) and (21) in Theorem 2.1. The (30) and (33) can be respectively translated into (36) and (37) by applying Lemma (2.3), and the proof procedure is basically similar to the Theorem 2.1. Therefore, the proof is omitted here. □

### 4. A numerical example

Consider a stochastic continuous-time system with multiplicative noises described by (26) with the model parameters given as follows:

$$A = \begin{bmatrix} -3 & 1 \\ 2 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.1 \\ 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.8 & 0 \\ 0 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad F = 0.75, \quad G = \begin{bmatrix} 1.24 & 3.12 \end{bmatrix}, \quad H = 1.6.$$

According to the Definition 2.1, the spectra of the open-loop system (26) are $\theta(\mathcal{L}_{A,D}) = (-6.5817, -2.6895, 1.2112)$. From Lemma 2.1, it can be verified that the system (26) is not asymptotically mean square stable. By setting the initial condition $x_0 = [1.2 \ 2]^T$, the external disturbance $v(t) = \cos(t)e^{(-0.2t)}\sin(0.3t)$, and applying the Euler-Maruyama Method, we can obtain the state responses of the system (26) which are shown in Figure 1.

Setting $\alpha = 4.8, \beta = 7.2, \gamma = 3.25$, and solving the optimal problem in Theorem 3.1, the relationship between $\xi$ and min $\text{Trace}(Z)$ is shown in Figure 2, where we can acquire a suboptimal solution min $\text{Trace}(Z)$ when $\xi = 0.035$. For given a scalar parameter $\xi = 0.035$, we acquire...
Based on the Definition 2.1, the spectrums of the closed-loop system (27) are obtained as \( \theta(L_{\alpha,D,K}) = \{-4.9617 + 3.1631i, -4.9617 - 3.1631i, -4.9512\} \), which indicate the spectrums are located in \( D(-7.2,-4.8) \). The state responses of the closed-loop system (27) is asymptotically mean square stable with respect to \( D(-7.2,-4.8) \) under the static output feedback \( H_2/H_\infty \) controller.

5. Conclusions

In this paper, the issue of static output feedback \( H_2/H_\infty \) control with spectrum constraints has been studied for a linear stochastic continuous-time systems with state dependent noises. The concept of the \( D(-\beta,-\alpha) \) stability has been defined based on the spectrum technique. Furthermore, the relationship between the \( D(-\beta,-\alpha) \) stability and the decay rate of the system state response has been revealed. Under \( D(-\beta,-\alpha) \) stability, a sufficient condition for the existence of stochastic \( H_2/H_\infty \) control has been derived, which transforms the design of \( H_2/H_\infty \) control into a convex optimization problem. A numerical example has been presented to show the effectiveness and applicability of the proposed method. Further research topics include the extension of the main results to \( H_\infty \) filtering problem for network control systems with missing measurements or fading channels, and so on.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by National Natural Science Foundation of China [grant numbers 61773400, 61573377], Fundamental Research Fund for the Central Universities of China [grant numbers 15CX08014A, 17CX02059], and the Research Fund for the Taishan Scholar Project of Shandong Province of China.

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