Quasi-polynomial-time algorithm for Independent Set in $P_t$-free and $C_{>t}$-free graphs via shrinking the space of connecting subgraphs

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In a recent work, Gartland and Lokshtanov [FOCS 2020] gave a quasi-polynomial-time algorithm for MAXIMUM WEIGHT INDEPENDENT SET in $P_t$-free graphs, that is, graphs excluding a path on $t$ vertices as an induced subgraph. Their algorithm runs in time $n^{O((\log^2 n))}$, where $t$ is assumed to be a constant.

Inspired by their ideas, we present an arguably simpler algorithm with an improved running time bound of $n^{O((\log n))}$. Our main insight is that a connected $P_t$-free graph always contains a vertex $w$ whose neighborhood intersects, for a constant fraction of pairs $\{u, v\} \in (V(G))^2$, a constant fraction of induced $u-v$ paths. Since a $P_t$-free graph contains $O(n^{t-1})$ induced paths in total, branching on such a vertex and recursing independently on the connected components leads to a quasi-polynomial running time bound.

In a subsequent and very recent work, Gartland and Lokshtanov [arXiv:2007.11402] extended their ideas to $C_{>t}$-free graphs: graphs that do not contain a cycle on more than $t$ vertices as an induced subgraph. They obtained an algorithm for MAXIMUM WEIGHT INDEPENDENT SET in this graph class with running time $n^{O((\log^2 n))}$. We show that it is possible to combine their ideas with our new understanding, and thus obtain an algorithm that runs in time $n^{O((\log n))}$.

We also show how to use the same approach to obtain quasi-polynomial-time algorithms for related problems, including MAXIMUM WEIGHT INDUCED MATCHING and 3-COLORING, in $P_t$-free and $C_{>t}$-free graphs.

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1 Introduction

Understanding the boundary of tractability of fundamental graph problems, depending on restrictions put on the input graph, has been a very active area of research in the last three decades. A methodological way of studying restrictions on the input graph is to focus on hereditary graph classes, that is, graph classes closed under vertex deletion. From this point of view, one starts with studying $H$-free graphs: graphs excluding one fixed graph $H$ as an induced subgraph. The goal is to classify for which graphs $H$, the problem at hand admits an efficient algorithm in $H$-free graphs.

Arguably, one of the most intriguing cases is $H = P_t$, where $P_t$ denotes a path on $t$ vertices. Alekseev [1] observed already in 1982 that for Maximum Independent Set, the known NP-hardness reductions prove hardness for $H$-free graphs unless every component of $H$ is a tree with at most three leaves. Since then, we know no more hardness results; in particular, no NP-hardness reduction for Maximum Independent Set is known that would not produce long induced paths in the output graph. On the positive side, polynomial time algorithms are known for only a few small graphs $H$ [11, 18, 15, 23, 20, 2, 19].

Very recently, we have learned that there is a reason for that. First, two authors of this paper together with Chudnovsky and Thomassé [9] showed that Maximum Weight Independent Set (the weighted generalization of Maximum Independent Set) admits a quasi-polynomial-time approximation scheme (QPTAS) in $H$-free graphs in all cases left open by Alekseev. Second, in a recent breakthrough result, Gartland and Lokshtanov [13] showed a quasi-polynomial-time algorithm for Maximum Weight Independent Set in $P_t$-free graphs with running time bound $n^{O((\log n)^4)}$, providing the first decisive evidence against NP-hardness of the problem in these graph classes.

In this work, inspired by the combinatorial insights of [13], we present an arguably simpler algorithm for Maximum Weight Independent Set in $P_t$-free graphs with an improved running time bound.

**Theorem 1.** For every fixed $t \in \mathbb{N}$, the Maximum Weight Independent Set problem can be solved in time $n^{O((\log n)^4)}$ in $n$-vertex $P_t$-free graphs.

Apart from $P_t$-free graphs, we also consider the broader class of $C_{\geq t}$-free graphs: graphs that do not contain a cycle on more than $t$ vertices as an induced subgraph. Note that every $P_t$-free graph is in particular $C_{\geq t}$-free. In [9], a QPTAS for Maximum Weight Independent Set in $C_{\geq t}$-free graphs is also presented. Very recently, Gartland and Lokshtanov [12] announced a quasi-polynomial-time algorithm for Maximum Weight Independent Set in $C_{\geq t}$-free graphs, where the running time bound is $n^{O((\log n)^3)}$ for every fixed $t$. We show that our approach can also be combined with the technique of [12] so that, again, we improve the running time bound by one log-factor in the exponent.

**Theorem 2.** For every fixed $t \in \mathbb{N}$, the Maximum Weight Independent Set problem can be solved in time $n^{O((\log n)^4)}$ in $n$-vertex $C_{\geq t}$-free graphs.

Finally, we observe that both the approach of Gartland and Lokshtanov [13, 12] and our approach are quite robust and can be used to obtain quasi-polynomial-time algorithms for many other graph problems. A notable example is 3-Coloring, whose complexity in $P_t$-free graphs is a well-known open problem [6, 10].

**Theorem 3.** For every fixed $t \in \mathbb{N}$, the 3-Coloring problem can be solved in time $n^{O((\log n)^2)}$ in $n$-vertex $P_t$-free graphs and in time $n^{O((\log n)^4)}$ in $n$-vertex $C_{\geq t}$-free graphs.

Let us now discuss the main ideas behind Theorem 1. We consider the space of all induced paths in the graph. In a $P_t$-free graph, this space is small, of size $O(n^{t-1})$, and thus can be enumerated in polynomial time. The main idea is to use the size of this space to guide a branching algorithm.
For Maximum Weight Independent Set, a natural branching step is the following: take a vertex \( w \) and branch into two cases: either take \( w \) to the constructed independent set and delete \( N[w] \) (i.e., the set consisting of \( w \) and all its neighbors) from the graph (the successful branch) or do not take \( w \) and delete \( w \) from the graph (the failure branch). In the failure branch, we cannot hope for much progress, as we delete only one vertex. However, if the branching pivot \( w \) is chosen carefully, we can hope to guarantee large progress in the successful branch, leading to a good running time guarantee.

The crucial combinatorial property of \( P_t \)-free graphs, used also in \([9, 13]\), is the following corollary of the Gyárfás path argument. By \( N[X] \) we mean the set \( \bigcup_{w \in X} N[w] \).

**Theorem 4 (Gyárfás \([16]\), Chudnovsky et al. \([8]\)).** Let \( G \) be a \( C_{t+1} \)-free graph and let \( A \subseteq V(G) \). Then there is a set of vertices \( X \) of size at most \( t \) such that \( G[X] \) is connected and every connected component of \( G - N[X] \) contains at most \(|A|/2\) vertices of \( A \). Furthermore, such a set can be found in polynomial time.

Let \( G \) be a connected \( P_t \)-free graph. Take all the \( O(n^{t-1}) \) induced paths in \( G \) and partition them into buckets: \( \{B_{u,v} : \{u, v\} \in \binom{V(G)}{2}\} \) according to their endpoints: \( B_{u,v} \) comprises all induced paths with endpoints \( u \) and \( v \). Take a set \( X \) given by Theorem 4. By the separation property, \( N[X] \) intersects all paths from at least half of all the \( \binom{|V(G)|}{2} \) buckets. Hence, as \(|X| \leq t\), there exists a vertex \( w \in X \) such that \( N[w] \) intersects at least \( \frac{1}{t} \) paths from at least \( \frac{1}{2} \) buckets.

Such \( w \) is an excellent branching pivot: after \( O(\log |V(G)|) \) successful branches, a constant fraction of buckets become empty and this implies that \( G \) got disconnected into connected components of multiplicatively smaller size. Furthermore, since we can enumerate all buckets in polynomial time, such a vertex \( w \) can be identified in polynomial time.

In Section 2 we prove formally Theorem 1. In Section 3 we prove Theorem 2. In Section 4 we discuss the possible extensions of the algorithm for 3-COLORING and related problems. These extensions follow by suitably adapting the strategy outlined above.

2 \( P_t \)-free graphs

In this section we prove Theorem 1, restated below.

**Theorem 1.** For every fixed \( t \in \mathbb{N} \), the Maximum Weight Independent Set problem can be solved in time \( n^{O(\log^2 n)} \) in \( n \)-vertex \( P_t \)-free graphs.

Let \((G, w)\) be the instance of the Maximum Weight Independent Set problem, where \( G \) is \( P_t \)-free and \( w \) is the weight function. Without loss of generality, assume \( t \geq 5 \). To simplify the notation, we allow that the domain of \( w \) to be a superset of \( V(G) \).

Consider the set of all induced paths in \( G \). We partition them into buckets. For a pair of distinct vertices \( u, v \), the bucket \( B_{u,v} \) contains all induced paths with one endvertex \( u \) and the other \( v \). Since \( G \) is \( P_t \)-free, the total size of all the buckets is \(|V(G)|^{t-1}\).

Let \( \varepsilon > 0 \) be a constant. We say that a vertex \( w \) \( \varepsilon \)-hits a bucket \( B_{u,v} \) if \( N[w] \) intersects at least \( \varepsilon \cdot |B_{u,v}| \) paths in \( B_{u,v} \). A vertex \( w \) is \( \varepsilon \)-heavy if it \( \varepsilon \)-hits at least \( \varepsilon \cdot \binom{|V(G)|}{2} \) buckets (i.e., if \( N[w] \) intersects at least \( \varepsilon \)-fraction of paths in at least an \( \varepsilon \)-fraction of buckets). The crucial idea of our algorithm is encapsulated in the following claim, whose proof is inspired by the result of Gartland and Lokshtanov \([13]\).

**Lemma 5.** A connected \( P_t \)-free graph has a \( \frac{1}{2t} \)-heavy vertex.
Proof. Let \( n \) be the number of vertices of the considered graph \( G \). Let \( X \) be the set given by Theorem 4 for \( A = V(G) \). We claim that \( N[X] \) intersects all paths in at least \( \frac{1}{2} \binom{n}{2} \) buckets.

Observe that \( B_{u,v} \) is non-empty if and only if \( u \) and \( v \) are in the same connected component. So, as \( G \) is connected, all the buckets are non-empty. As each connected component in \( G - N[X] \) has at most \( n/2 \) vertices, the number of buckets that contain at least one path disjoint with \( N[X] \) is

\[
\sum_{C: \text{component of } G - N[X]} \binom{|V(C)|}{2} \leq 2 \cdot \frac{n(n - 1)}{2} \leq \frac{1}{2} \cdot \frac{n^2}{2}.
\]

Here, the first inequality follows from the fact that \( |V(C)| \leq n/2 \) and the convexity of the mapping \( x \mapsto \binom{x}{2} \). It follows that \( N[X] \) intersects all the paths in at least half of the buckets.

Recall that \( |X| \leq t \). Thus, by the pigeonhole principle, there is \( w \in X \) such that \( N[w] \) intersects at least \( \frac{1}{2} \)-fraction of paths in at least \( \frac{1}{2t} \binom{n}{2} \) buckets. \( \square \)

We now proceed to describing the algorithm. For simplicity, the algorithm returns the maximum weight of a solution, but it is straightforward to adapt it so that a solution witnessing this value is constructed. The key step is branching on heavy vertices. For a vertex \( w \), we will separately consider two instances: \( G - w \) (indicating that \( w \) is not chosen to the solution, we call this the failure branch), and \( G - N[w] \) (indicating that \( w \) is chosen to the solution, this branch is called successful). Clearly, the optimum weight of a solution is the maximum of the return value of the first call and the return value of the second call, plus \( w(w) \).

Now, the algorithm is very simple. If the vertex set is empty, then we return \( 0 \). If \( G \) has one vertex, then we return its weight. If \( G \) is disconnected, we call the algorithm recursively for every connected component of \( G \). Otherwise, we enumerate all induced paths in \( G \) and partition them into buckets, find a \( \frac{1}{2t} \)-heavy vertex, and we branch on it. The pseudo-code is given in Algorithm 1.

**Algorithm 1: FindMIS**

**Input:** \( P \)-free graph \( G \), weight function \( w \)

1. If \( |V(G)| \leq 1 \) then return the total weight of \( V(G) \)
2. If \( G \) is disconnected then return \( \sum_{C: \text{component of } G} \text{FindMIS}(C, w) \)
3. Initialize buckets \( B_{u,v} \) for all \( u, v \in V(G) \)
4. \( w \leftarrow \) a \( \frac{1}{2t} \)-heavy vertex in \( G \)
5. Return \( \max\{\text{FindMIS}(G - w, w), \text{FindMIS}(G - N[w], w) + w(w)\} \)

Lemma 5 asserts that in line 4 we can always find a \( \frac{1}{2t} \)-heavy vertex. Note that in each recursive call the number of vertices of the instance graph decreases. Thus it is clear that the algorithm terminates and returns the correct value. Furthermore, the local computation in each node of the recursion tree can be performed in time \( |V(G)|^{O(t)} \). It remains to show that the number of nodes in the recursion tree is bounded by \( |V(G)|^{O(\log^2 |V(G)|)} \).

To this end, consider the recursion tree \( T \) of the algorithm applied on a graph \( G \). For a call on a graph \( H \) (which is a subgraph of \( G \)), the local subtree of the call consists of all descendant calls that treat a graph with at least \( 0.99|V(H)| \) vertices. We greedily find a partition \( \mathcal{P} \) of \( T \) into local subtrees as follows: start with \( \mathcal{P} = \emptyset \) and, as long as there exists a call in \( T \) that is in neither of the local trees in \( \mathcal{P} \), take such a call closest to the root and add its local subtree to \( \mathcal{P} \).

Clearly, a root-to-leaf path in the recursion tree intersects \( O(\log |V(G)|) \) local subtrees of \( \mathcal{P} \). Thus, it suffices to prove that any local subtree, say for a call on a graph \( H \), contains at most \( |V(H)|^{O(\log |V(H)|)} \) leaves.
Let $S$ be the local subtree rooted at a call on a graph $H$. Mark the following edges of $S$:

1. For every call in $S$ on a disconnected graph, say on $H'$, observe that there is at most one child call of this call that also belongs to $S$. Indeed, this call must be on a connected component $H''$ of $H'$ satisfying $|V(H'')| \geq 0.99 |V(H')|$, and there is at most one such component. If there exists such a unique child call that belongs to $S$, mark the edge to it.

2. For every call in $S$ on a connected graph, mark the edge to the call in the failure branch (provided it belongs to $S$).

Thus, every call in $S$ has at most one marked edge to a child. Hence, the marked edges form a family of vertex-disjoint upwards paths in $S$. Let $S'$ be the tree obtained from $S$ by contracting all the marked edges; the parent-child relation is naturally inherited from $S$. Then, every node in $S'$ has $O(|V(H)|) = O(|V(G)|)$ children and every edge of $S'$ corresponds to a successful branch in some call in $S$. It suffices to show that $S'$ has depth $O(\log |V(H)|) = O(\log |V(G)|)$.

To this end, we introduce the potential of a call in $S$, say on a graph $H'$:

$$\mu(H') := - \sum_{\{u,v\} \in \binom{V(H')}{2}} \log_{\log(1-1/2t)}(1 + |\mathcal{B}_{u,v}|).$$

At the initial call on $H$, we have $\mu(H) = O(|V(H)|^2 \log |V(H)|)$, because the size of each bucket is at most $|V(H)|^{t-1}$. Since in a successful branch we remove the closed neighborhood of a $\frac{1}{2t}$-heavy vertex, a successful branch at a call in $S$ on a graph $H'$ results in decreasing the potential $\mu$ by at least

$$\frac{1}{2t} \binom{|V(H')|}{2} \geq \frac{1}{2t} \left( \left\lceil 0.99 |V(H)| \right\rceil \right) \geq \frac{0.9}{2t} \binom{|V(H)|}{2}.$$

Since $\mu$ is nonnegative, it follows that the depth of $S'$ is bounded by $O(\log |V(H)|)$, as desired.

### 3 $C_{>t}$-free graphs

In this section we extend the algorithm from Section 2 to the class of $C_{>t}$-free graphs and prove Theorem 2. The algorithm is based on the same high-level idea as the algorithm for $P_t$-free graphs, but is more technically involved. The main combinatorial insights heavily follow [12].

Let $(G, \omega)$ be an instance of MAXIMUM WEIGHT INDEPENDENT SET, where $G$ is $C_{>t}$-free. Without loss of generality we can assume that $t$ is even and at least 6. Again, we will measure the progress of our algorithm by keeping track of the number of some suitably defined objects in the graph.

A connector is a graph with three designated vertices, called tips, obtained in the following way. Take three induced paths $Q_1, Q_2, Q_3$; here we allow degenerated, one-vertex paths. The paths $Q_1, Q_2, Q_3$ will be called the legs of the connector. The endvertices of $Q_i$ are called $a_i$ and $b_i$. Now, join these paths in one of the following ways:

a) identify $a_1, a_2$, and $a_3$ into a single vertex, i.e., $a_1 = a_2 = a_3$, or

b) add edges $a_1a_2$, $a_2a_3$, and $a_1a_3$.

Furthermore, if the endvertices are identified, then at most one leg may be degenerated. There are no other edges between the legs of the connector. The vertices $b_1, b_2, b_3$ are the tips of the connector, and the
Figure 1: Two connectors with two long legs and one short leg; one connector with $a_i$s identified and one with $a_i$s forming a triangle. Vertices $b'_i$ are the tips of the core $T$ of the connector. The gray area depicts the set $T^*$. set \{a_1, a_2, a_3\} is called the center (this set can have either three or one element); see Figure 1. If one of the paths forming the connector has only one vertex, and the endvertices were identified, then the connector is just an induced path with the tips being the endpoints of the path plus one internal vertex of the path. Note that, given a connector as a graph and its tips, the legs and the center of the connector are defined uniquely.

We will need the following folklore observation.

**Lemma 6.** Let $G$ be a graph, $A \subseteq V(G)$ be a set consisting of exactly three vertices in the same connected component of $G$, and let $A \subseteq B \subseteq V(G)$ be an inclusion-wise minimal set such that $G[B]$ is connected. Then the graph $G[B]$ with the set $A$ as tips is a connector.

**Proof.** Let $A = \{u, v, w\}$, let $P_{uv}$ be a shortest path from $u$ to $v$ in $G[B]$ and let $P_w$ be a shortest path from $w$ to $V(P_{w})$ in $G[B]$. By minimality of $B$, we have $B = V(P_{uv}) \cup V(P_w)$.

If $w \in V(P_{uv})$ (equivalently, $|V(P_w)| = 1$), then $G[B]$ is a path and we are done. Otherwise, let $q \in V(P_w) \cap V(P_{uv})$ be the endpoint of $P_w$ distinct than $w$ and let $p$ be the unique neighbor of $q$ on $P_w$. By the mimality of $P_w$, $p$ and $q$ are the only vertices of $P_w$ that may have neighbors on $P_{uv}$. If $p$ has two neighbors $x, y \in V(P_{w})$ that are not consecutive on $P_{uv}$, then $G[B]$ remains connected after the deletion from $B$ of all vertices on $P_{uv}$ between $x$ and $y$ (exclusive), a contradiction to the choice of $B$. Thus, $N(p) \cap V(P_{uv})$ consists of $q$ and possibly one neighbor of $q$ on $P_{uv}$. We infer that $G[B]$ is a connector with tips $u, v, w$, as desired. □

A **tripod** is a connector where each of the paths $Q_1, Q_2, Q_3$ has at most $t/2 + 1$ vertices. A leg of a tripod is **long** if it contains exactly $t/2 + 1$ vertices, and **short** otherwise. The **core** of a connector $C$ with legs $Q_1, Q_2, Q_3$, is the tripod consisting of the first $t/2 + 1$ (or all of them, if the corresponding path $Q_i$ is shorter) vertices of each path $Q_i$, starting from $a_i$. A tripod **in** $G$ is a tripod that is an induced subgraph of $G$. Note that each tripod has at most $3t/2 + 3$ vertices, hence given $G$, we can enumerate all tripods in $G$ in time $n^{O(t)}$. 

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Let $T$ be a tripod in $G$ with legs $Q_1, Q_2, Q_3$. Let $L(T)$ denote the tips of $T$ which are endvertices of the long legs. We denote

$$T^* := N_G[V(T) - L(T)] - L(T).$$

In other words, $T^*$ is the closed neighborhood of the tripod $T$ in $G$, except that we exclude the neighbors of tips of long legs and these tips themselves. Note that tips of short legs are included in $T^*$.

We have the following simple observation.

**Lemma 7.** For every tripod $T$ in $G$ and for every connected component $C$ of $G - T^*$, the component $C$ contains at most one tip of $L(T)$ and no tip of a short leg.

**Proof.** First, note that tips of short legs are contained in $T^*$, hence they are not contained in $G - T^*$.

For contradiction, without loss of generality assume that $b_1, b_2 \in C \cap L(T)$. Then, $Q_1, Q_2$, and a shortest path from $b_1$ to $b_2$ in $C$ yield an induced cycle on more than $t$ vertices in $G$, a contradiction. \(\square\)

Suppose $T$ is a tripod in $G$. With each tip $b_i$ of $T$ we associate a bag $B_i$ defined as follows:

- if $b_i$ is the endpoint of a long leg, then $B_i$ is the vertex set of the connected component of $G - T^*$ that contains $b_i$, and
- otherwise, $B_i = \{b_i\}$.

Note that Lemma 7 implies that the bags $B_1, B_2, B_3$ are pairwise disjoint and nonadjacent in $G$, except for the corner case of two adjacent tips of short legs.

Recall that in the previous section, we were grouping induced paths in $G$ into buckets. Here, the objects that will populate buckets will be tripods in $G$. This time, each bucket will be indexed by an unordered triple of distinct vertices of $G$. Every tripod $T$ in $G$ with bags $B_1, B_2, B_3$ belongs to the bucket $B_{u,v,w}$ for all triples $u, v, w$ such that $u \in B_1$, $v \in B_2$, and $w \in B_3$. Note that thus, the buckets do not have to be pairwise disjoint. Observe the following.

**Lemma 8.** For every $\{u, v, w\} \in \binom{V(G)}{3}$ and every $T \in B_{u,v,w}$, there exists a connector $T'$ with tips $u, v, w$ whose core equals $T$. Consequently, the bucket $B_{u,v,w}$ is nonempty if and only if $u, v, w$ lie in the same connected component of $G$.

**Proof.** Let $Q_{u, v}, Q_{v, w}, Q_{w, u}$ be the legs of $T$ with tips $b_u, b_v, b_w$ and bags $B_u, B_v, B_w$, respectively, such that $u \in B_u$, $v \in B_v$, $w \in B_w$. Let $Q'_u$ be the concatenation of $Q_u$ and a shortest path from $b_u$ to $u$ in $B_u$. Similarly define $Q'_v$ and $Q'_w$.

Recall that the bags $B_u, B_v,$ and $B_w$ are pairwise distinct and nonadjacent (except for the case of two adjacent tips of short legs). Hence, $Q'_u, Q'_v,$ and $Q'_w$ form a connector $T'$ with tips $u, v,$ and $w$. Since $B_u \neq \{b_u\}$ only if the leg $Q_u$ is long, $T$ is the core of $T'$.

\(\square\)

The next combinatorial observation lies at the heart of the algorithm of Gartland and Lokshtanov [12].

**Lemma 9.** Let $G$ be a connected $C_{\geq t}$-free graph, and let $u, v, w$ be its three distinct vertices. Let $X \subseteq V(G)$ be such that $G[X]$ is connected and no two of $u, v, w$ are in the same connected component of $G - N[X]$. Then $N[X]$ intersects all tripods in $B_{u,v,w}$.

**Proof.** Let $T \in B_{u,v,w}$. Let $Q_u, Q_v, Q_w$ be the legs of $T$ with tips $b_u, b_v, b_w$ and bags $B_u, B_v, B_w$, respectively. Let $T'$ be the connector for $T$ given by Lemma 8 with legs $Q'_u, Q'_v,$ and $Q'_w$. Since no two of $u, v, w$ lie in the same connected component of $G - N[X]$, the set $N[X]$ intersects at least two legs of $Q'_w$. 6
Section 2

Theorem 4

The outcome of this strategy is encapsulated in the following lemma.

Lemma 11. Let $G$ be a connected $C_{2t}$-free graph and let $X \subseteq V(G)$ be a set of size at most $t$ such that $G[X]$ is connected and every connected component of $G - N[X]$ has at most $0.1 \cdot |V(G)|$ vertices. Then there exists a $1/(2t)$-heavy vertex in $G$.

Proof. Let $n = |V(G)|$. Assume $n \geq 5$, as otherwise $N[X] = V(G)$ and the statement is trivial. The number of buckets $B_{u,v,w}$ such that two of $u,v,w$ are in the same connected component of $G - N[X]$ is bounded by

$$\sum_{C: \text{component of } G - N[X]} \left( \frac{|V(C)|}{3} \right) + \left( \frac{|V(C)|}{2} \right) \cdot (n - |V(C)|)$$

$$\leq n \cdot \left( \frac{(0.1n - 1)(0.1n - 2)}{6} + \frac{(0.1n - 1) \cdot 0.9n}{2} \right)$$

$$\leq n \cdot 0.1 \cdot \frac{n - 1}{2} \cdot \frac{2.8n - 2}{3}$$

$$= \frac{1}{2} n \cdot \frac{n - 1}{2} \cdot \frac{0.56n - 0.4}{3}$$

$$\leq \frac{1}{2} \frac{n}{3}.$$ 

Here, the last inequality uses $n \geq 5$. Hence, by Lemma 9, the set $N[X]$ intersects all tripods in at least half of the buckets. Since $|X| \leq t$, there exists $w \in X$ such that $N[w]$ intersects at least $1/t$ fraction of tripods in at least $1/(2t)$ fraction of the buckets. \hfill \square

Unfortunately, Theorem 4 for $A = V(G)$ gives us only a connected set $X$ of size at most $t$ such that every component of $G - N[X]$ has at most $|V(G)|/2$ vertices. The example of a long path shows that the fraction $1/2$ cannot be improved while keeping $X$ both connected and of constant size.

Following the ideas of [12], in the absence of a heavy vertex, we shift to a secondary branching strategy. The outcome of this strategy is encapsulated in the following lemma.

Lemma 11. For every fixed $t$ there exists an algorithm that, given an $n$-vertex $C_{2t}$-free graph $G$ and a set $X \subseteq V(G)$ such that $G[X]$ is connected, runs in time $n^{O(\log^2 n)}$ and outputs a family $G$ of $n^{O(\log^2 n)}$ induced subgraphs of $G$ with the following properties:
Lemma 11

(a) For every $H \in \mathcal{G}$, we have $X \subseteq V(H)$ and either every connected component of $H$ has at most $0.99n$ vertices, or every connected component $C$ of $H - N_H[X]$ that has more than $0.01n$ vertices satisfies $N_H[C] \cap N_H[X] = \emptyset$.

(b) For every independent set $I$ in $G$, there exists $H \in \mathcal{G}$ with $I \subseteq V(H)$.

The proof of Lemma 11 is postponed to Section 3.1.

With Lemma 11 in hand, our recursive algorithm for MIS in $C_{> t}$-free graphs is now easy to describe. A recursive call, given a graph $G$, applies the following:

1. If $G$ is of constant size, solve MIS on $G$ by brute-force.
2. If $G$ is disconnected, recurse on each connected component independently.
3. If $G$ is connected and contains a $1/(10t)$-heavy vertex $w$, branch on $w$. That is, in the failure branch delete $w$ from the graph, and in the successful branch add $w$ to the constructed independent set and delete $N[w]$ from the graph.
4. If $G$ is connected, but does not contain a $1/(10t)$-heavy vertex, do the following:
   - Construct a set $X$ from Theorem 4 for $A = V(G)$.
   - Invoke the secondary branching algorithm of Lemma 11 on $G$ and $X$, thus obtaining a family $\mathcal{G}$.
   - For every $H \in \mathcal{G}$, recurse on $H$ to find a maximum weight independent set in $H$.
   - Output the maximum weight independent set among those found in the recursive calls.

The exhaustiveness of the branching step in Step 3 and the second property of Lemma 11 for Step 4 ensures that the algorithm indeed finds an independent set with maximum weight in $G$. It remains to establish the running time bound.

To this end, the crucial observation is the following.

Lemma 12. Assume $G$ is a connected $C_{> t}$-free graph that does not contain a $1/(10t)$-heavy vertex. Let $X$ be a result of application of Theorem 4 to $G$ and $A = V(G)$ and let $\mathcal{G}$ be a result of application of Lemma 11 to $G$ and $X$. Then, for every $H \in \mathcal{G}$, at least one of the following assertions hold:

(a) every connected component of $H$ is has at most $0.99|V(G)|$ vertices; or

(b) there is subset $F \subseteq \binom{V(H)}{3}$ of size at least $\frac{1}{1000} \cdot \binom{|V(G)|}{3}$ such that for every $\{u, v, w\} \in F$, the size of the bucket $B_{u,v,w}$ in $H$ is of size at most one-fifth of its size in $G$.

Proof. Let $H \in \mathcal{G}$ be such that there is a connected component $D$ of $H$ with more than $0.99|V(G)|$ vertices. For a tuple $\{u, v, w\} \in \binom{V(D)}{3}$, let $B^D_{u,v,w}$, $B^H_{u,v,w}$, and $B^G_{u,v,w}$ denote the buckets for $u, v, w$ in the graphs $D$, $H$, and $G$, respectively. Observe that since $D$ is a connected component of $H$ and $H$ is an induced subgraph of $G$, it follows that $B^D_{u,v,w} = B^H_{u,v,w} \subseteq B^G_{u,v,w}$.

Since every connected component of $G - N[X]$ has at most $|V(G)|/2$ vertices, $G[X]$ is connected, and $X \subseteq V(H)$, it follows that $X \subseteq V(D)$. Then, by the properties promised by Lemma 11, every connected component $C$ of $D - N_H[X]$ contains at most $0.01|V(G)|$ vertices. Since $0.01|V(G)| < 0.1|V(D)|$, Lemma 10 implies that there is a set $F_0 \subseteq \binom{V(D)}{3}$ of size at least $\frac{1}{22} \cdot \binom{|V(D)|}{3}$ and a vertex $x \in V(D)$ such that $N_D[x]$ intersects at least a fraction of $\frac{1}{22} \binom{|V(D)|}{3}$ tripods from every bucket $B^D_{u,v,w} = B^H_{u,v,w}$ for $\{u, v, w\} \in F_0$.

As $|V(D)| \geq 0.99|V(G)|$, we have that $|F_0| \geq \frac{1}{22} \binom{|V(G)|}{3}$.
Note that $N_D[x] = N_H[x] \subseteq N_G[x]$. Hence, for fixed $\{u, v, w\} \in F_0$, every tripod hit by $N_H[x]$ in $B_{u,v,w}^H$, is also hit by $N_G[x]$ in $B_{u,v,w}^G$. However, $x$ is not $1/(10t)$-heavy in $G$, because we assumed that $G$ has no $1/(10t)$-heavy vertices. This implies that there is a set $F \subseteq F_0$ of size at least

$$\left( \frac{1}{4t} - \frac{1}{10t} \right) \left( \frac{|V(G)|}{3} \right) \geq \frac{1}{10t} \left( \frac{|V(G)|}{3} \right)$$

such that for every $\{u, v, w\} \in F$, the neighborhood $N_G[x]$ hits less than a $\frac{1}{10t}$-fraction of tripods in $B_{u,v,w}^G$. However, for $\{u, v, w\} \in F$, $N_H[x]$ hits a $\frac{1}{2}$-fraction of tripods in $B_{u,v,w}^H$ and every tripod hit by $N_H[x]$ in $H$ is hit by $N_G[x]$ in $G$. Consequently, for every $\{u, v, w\} \in F$ we have $|B_{u,v,w}^G| \geq 5|B_{u,v,w}^H|$, as desired. \(\square\)

With Lemma 12 established, the analysis now is very similar to the one for $P_t$-free graphs. Consider the recursion tree $T$ of the algorithm applied on a graph $G$. As before, for a call on a graph $H$ (which is an induced subgraph of $G$), the local subtree of the call consists of all descendant calls that treat a graph with at least $0.99|V(H)|$ vertices. We greedily find a partition $\mathcal{P}$ of $T$ into local subtrees as follows: start with $\mathcal{P} = \emptyset$ and, as long as there exists a call in $T$ that is in none of the local trees in $\mathcal{P}$, take such a call closest to the root and add its local subtree to $\mathcal{P}$.

Clearly, any root-to-leaf path in the recursion tree intersects $O(\log |V(G)|)$ local subtrees of $\mathcal{P}$. Thus, it suffices to prove the following claim: any local subtree, say rooted at a call on a graph $H$, contains at most $|V(H)|^{O(\log^2 |V(H)|)} \leq |V(G)|^{O(\log^3 |V(G)|)}$ leaves.

Let $S_0$ be the local subtree rooted at a call on a graph $H$. First, we observe that $S_0$ may contain nodes corresponding to calls on a graph $H'$ that has at least $0.99|V(H)|$ vertices, but each connected component of $H'$ has fewer than $0.99|V(H)|$ vertices. However, every such call is a leaf in $S_0$. Remove all such nodes from $S_0$, thus obtaining a tree $S$. Since every node in $S_0$ has $|V(H)|^{O(\log^2 |V(H)|)}$ children, it suffices to prove that $S$ has $|V(H)|^{O(\log^3 |V(H)|)}$ leaves.

Now, for every node in $S$, say corresponding to a call on a graph $H'$, $H'$ contains a connected component with at least $0.99|V(H)|$ vertices. Mark the following edges of $S$:

1. For every call in $S$ on a disconnected graph, say on $H'$, observe that there is exactly one child call of this call that also belongs to $S$. Indeed, this is the call on the unique connected component $H''$ of $H'$ satisfying $|V(H'')| \geq 0.99|V(H')|$. Mark the edge to this child call.

2. For every call in $S$ on a connected graph with branching in Step 3, mark the edge to the call in the failure branch (provided it belongs to $S$).

Every call in $S$ has at most one marked edge to a child. Thus, the marked edges form a family of vertex-disjoint upwards paths in $S$. Let $S'$ be the tree obtained from $S$ by contracting all the marked edges; the parent-child relation is naturally inherited from $S$. Then, by Lemma 11, every node in $S'$ has at most $|V(G)|^{O(\log^2 |V(G)|)}$ children. It now suffices to show that $S'$ has depth $O(\log |V(G)|)$.

Every edge of $S'$ corresponds to an edge of $S$, and thus to one of the calls in the branching algorithm. Consider such an edge $e$ and assume that in $S$ the parent endpoint of $e$ is the call on a graph $H'$. Then, $e$ corresponds to either a successful branch in Step 3 on some vertex $x \in V(H')$, or to a choice of $H'' \in \mathcal{G}$ in Step 4, where $H''$ necessarily contains a connected component $D$ with at least $0.99|V(H')| \geq 0.99|V(H')|$ vertices. The definition of a heavy vertex and Lemma 12 imply that in both cases, there are at least $\frac{1}{10t} \left( \frac{|V(H')|}{3} \right)$ buckets of $H'$ that in the child call have size decreased by at least a multiplicative factor of 

$$(1 - \frac{3}{10t})$$.
On the other hand, in both cases mentioned in the previous paragraph the graph \( H' \) is connected. For such \( H' \), for all \( \{u,v,w\} \subseteq \binom{V(H')}{3} \) the bucket \( B_{u,v,w} \) is nonempty and \( |B_{u,v,w}| \geq 0.9 |V(H')| \). We introduce the potential

\[
\mu(H') := - \sum_{\{u,v,w\} \in \binom{V(H')}{3}} \log(1 - 1/(10t))(1 + |B_{u,v,w}|).
\]

Similarly as in the case of \( P_t \)-free graphs, at the initial call on \( H \) (the root of \( S \)) we have \( \mu(H) = O(|V(H)|^3 \log |V(H)|) \), because every bucket is of size \( |V(H)|^{O(1)} \). On the other hand, the conclusion of the previous paragraph shows that for every edge \( e \) of \( S' \), if \( H' \) and \( H'' \) are the graphs of the parent and child calls of \( e \) in \( S \), respectively, then \( \mu(H') - \mu(H'') = \Omega(|V(H)|^3) \). Hence, the depth of \( S' \) is \( O(\log |V(G)|) \), as desired.

3.1 Secondary branching

We are left with proving Lemma 11. Recall that the setting is as follows: we consider a \( C_{>t} \)-free graph \( G \) and a set \( X \subseteq V(G) \) such that \( G[X] \) is connected. Denote \( n = |V(G)| \).

For a graph \( G \), a set \( C \subseteq V(G) \) such that \( G[C] \) is connected, and distinct vertices \( u, v \in N_G(C) \), a \( C \)-link between \( u \) and \( v \) is a path \( P \) in \( G \) with the following properties:

- \( P \) has endpoints \( u \) and \( v \) and length at least 2;
- all internal vertices of \( P \) belong to \( C \); and
- \( P \) is an induced path in \( G - E(G[N_G(C)]) \) (i.e. \( P \) is an induced path in \( G \), except that we allow the existence of the edge \( uv \)).

The following lemma is an adaptation of of [12, Observation 4].

**Lemma 13.** Let \( G \) be a \( C_{>t} \)-free graph, let \( X \subseteq V(G) \) be such that \( G[X] \) is connected, and let \( C \) be a connected component of \( G - N[X] \). Then every \( C \)-link has at most \( t \) vertices.

**Proof.** Let \( P \) be a path with endpoints \( u, v \) and all internal vertices in \( C \) and let \( Q \) be a shortest path with endpoints \( u, v \) and internal vertices in \( X \). Then \( P \cup Q \) is an induced cycle in \( G \). Thus, both \( P \) and \( Q \) have at most \( t \) vertices. \( \square \)

The algorithm will again follow a branching strategy. A recursive call is given an induced subgraph \( H \) of \( G \) such that \( X \subseteq V(H) \). The initial call is made on \( H = G \). The induced subgraphs \( H \) considered in the leaves of the recursion will be exactly the subgraphs inserted into the constructed family \( G \).

We fix a threshold \( \tau := 0.01 |V(G)| \). A chip in \( H \) is the vertex set \( C \) of a connected component of \( H - N_H[X] \) satisfying the following properties: \( C \) contains more than \( \tau \) vertices and \( N_H[C] \cap N_H[X] \neq \emptyset \).

The algorithm works as follows. We consider a recursive call on \( H \). If every connected component of \( H \) has at most \( 0.99 |V(G)| \) vertices or \( H \) contains no chip, then we declare the call a leaf call: we insert \( H \) into \( G \) and do not invoke any recursive subcalls. Otherwise, we again look at buckets of some objects to choose a branching pivot \( x \in V(H) - X \); the exact choice of the pivot will be described later. The algorithm branches into two subcalls, one failure branch on \( H - x \) and one successful branch on \( H - (N_H(x) - X) \). Note that this definition ensures that \( X \) is contained in the subgraphs passed to recursive calls.

Observe that for every independent set \( I \) in \( H \), \( I \) is contained in \( H - x \) if \( x \notin I \) and \( I \) is contained in \( H - (N_H(x) - X) \) if \( x \in I \). Consequently, by a straightforward bottom-up induction on the recursion
Lemma 11: for every independent set \( I \) in \( G \), at least one enumerated graph \( H \in \mathcal{G} \) contains \( I \).

It remains to show a strategy of choosing branching pivots that ensures that each recursive subcall is invoked on a strictly smaller graph (i.e., the chosen branching pivot \( x \) has at least one neighbor not in \( X \), so that \( H - (N_H(x) - X) \) is a proper subgraph of \( H \)) and that the branching tree has \( n^{O(\log^2 n)} \) leaves. This immediately implies the desired bounds on the running time and on the size of the output family \( \mathcal{G} \).

Consider a recursive call on a graph \( H \). If \( H - N_H[X] \) contains a chip \( C \) with \( |N_H(C)| = 1 \), we choose the unique element of \( N_H(C) \) as the branching pivot. Note that after branching on such a pivot, both in the failure and in the successful branches (the remainder of) the chip \( C \) is in a different connected component of a graph than \( X \). As a result, the children subcalls are leaf subcalls: if \( |C| \leq 0.99|V(G)| \), then every connected component in a child subcall has at most \( 0.99 |V(G)| \) vertices due to \( |C| \geq \tau = 0.01 |V(G)| \), and if \( |C| > 0.99 |V(G)| \), then child subcalls contain no chips.

We are left with the following case: in \( H \), there is at least one chip and every chip \( C \) in \( H \) satisfies \( |N_H(C)| \geq 2 \). We define buckets as follows. The buckets are indexed by a pair consisting of a vertex \( w \in V(H) \) and an unordered pair \( \{u, v\} \in (N_H(X))^2 \). For such a choice of \( w, u, v \), the bucket \( \mathcal{L}_{w,u,v} \) contains all \( C \)-links with endpoints \( u \) and \( v \) where \( C \) is a chip in \( H \) and \( w \in V(C) \). Lemma 13 ensures that every path in a bucket has at most \( \log n \) vertices and, consequently, every bucket has size \( O(n^\epsilon) \) and can be enumerated in polynomial time. Note that \( \mathcal{L}_{w,u,v} \) is nonempty if and only if there exists a chip \( C \) with \( w \in V(C) \) and \( u, v \in N_H(C) \).

For \( \epsilon > 0 \), a vertex \( x \in V(H) - X \) is \( \epsilon \)-heavy if \( N_H(x) - X \) intersects at least an \( \epsilon \) fraction of links in at least an \( \epsilon \) fraction of nonempty buckets. Following the lines of the algorithm for \( P_t \)-free graphs, we prove the following.

Lemma 14. If there exists a nonempty bucket, then there exists a \( \frac{1}{200t} \)-heavy vertex.

**Proof.** Since there are at most \( n/\tau = 100 \) chips, pick a chip \( C \) such that for at least a fraction of 0.01 nonempty buckets \( \mathcal{L}_{w,u,v} \) we have \( w \in V(C) \). Let \( H_C := H[N_H[C]] \); note that \( C \subseteq N_H[C] \subseteq C \cup N_H(X) \). Apply Theorem 4 to \( H_C \) with \( A = N_H(C) \), obtaining a set \( Y_C \) of size at most \( t \) such that every connected component of \( H_C - N[Y_C] \) contains exactly \( |N_H(C)|/2 \) vertices of \( N_H(C) \). Consequently, \( N_H[C][Y_C] \) intersects all links in at least half of the bucket \( \mathcal{L}_{w,u,v} \) with \( w \in V(C) \). We infer that there is \( y \in Y_C \) such that \( N_{H_C}[y] \) intersects at least a \( \frac{1}{400} \) fraction of links in at least a \( \frac{1}{200t} \) fraction of all nonempty buckets. Since we exclude single-vertex paths from the link definition and all links are vertex-disjoint with \( X \), every link intersected by \( N_{H_C}[y] \) is also intersected by \( N_{H}(y) - X \). This completes the proof.

If there exists a chip \( C \) with \( |N_H(C)| \geq 2 \), then there exists a nonempty bucket, as every bucket \( \mathcal{L}_{w,u,v} \) for \( w \in V(C) \) and \( \{u, v\} \in (N_H(C))^2 \) is nonempty. Hence, Lemma 14 allows us to choose a \( \frac{1}{200t} \)-heavy vertex as a branching pivot. Note that for such a heavy vertex \( x \), \( N_H(x) - X \) is necessarily nonempty in order to intersect any link. It remains to show that with this choice of the branching pivot, the recursion tree has \( n^{O(\log^2 n)} \) leaves.

Let \( T \) be the recursion tree of the algorithm. As argued, for a call on a graph \( H \), if there exists a chip \( C \) with \( |N_H(C)| = 1 \), then we branch on the unique element of \( N_H(C) \) and both child subcalls are leaf subcalls. Remove from \( T \) both such child subcalls. The number of leaves decreased by at most a half and now, every internal node of \( T \) corresponds to a call where the algorithm branches on a \( \frac{1}{200t} \)-heavy vertex.

By \( H_e \) and \( H'_e \) we denote the graphs considered by the parent and the child node of an edge \( e \) in \( T \), respectively. To avoid confusion, by \( \mathcal{L}^H_{w,u,v} \) we denote the bucket for \( \{w, \{u, v\}\} \) in the graph \( H \).

Let \( T' \) be the tree obtained from \( T \) by contracting all edges corresponding to failure branches. The parent-child relation in \( T' \) is naturally inherited from \( T \). Note that since every node of \( T \) has at most
one child connected by an edge corresponding to a failure branch, it follows $T'$ is obtained from $T$ by contracting disjoint upward paths, so every node in $T'$ has $O(n)$ children. Hence, it suffices to show that the depth of $T'$ is $O(\log^2 n)$.

Every edge $e$ of $T'$ is also present in $T$ and corresponds to a successful branch. For an edge $e$ of $T'$, the level of $e$ is defined as

$$\lambda(e) := \left\lfloor \log_2 (1 + |\{(w, \{u, v\}) \mid \mathcal{L}_{u,v}^H \neq \emptyset\}|) \right\rfloor.$$ 

Note that the level is positive if and only if there is a nonempty bucket. Since at the beginning (for $H = G$) there are $O(n^3)$ nonempty buckets, there are $O(\log n)$ possible levels. Furthermore, during the recursion the buckets can only shrink, so the level of an edge $e$ is never higher than the level of any edge on the path from $e$ to the root of $T'$.

It suffices to prove that for every subtree $S$ of $T'$ containing only edges of the same level $\ell$, the depth of $S$ is $O(\log n)$. To this end, consider the following potential for a graph $H$.

$$\mu(H) := - \sum_{(w, \{u, v\}) \in \mathcal{L}_{u,v}^H \neq \emptyset} \log \left(1 - \frac{1}{200t}\right) \left(1 + |\mathcal{L}_{u,v}^H|\right).$$

For every edge $e$ in $S$, there are between $2^{\ell} - 1$ and $2^{\ell+1} - 2$ nonempty buckets of $H_e$. Hence, for every edge $e$ of $S$,

$$\mu(H_e) \leq (2^{\ell+1} - 2) \cdot O(\log n) = 2(2^\ell - 1) \cdot O(\log n).$$

Since every edge $e$ of $S$ corresponds to a successful branch, we have

$$\mu(H_e) - \mu(H_e') \geq \frac{1}{200t} \cdot (2^\ell - 1).$$

Since the potential $\mu$ is nonnegative, it can decrease only $O(\log n)$ times at a successful branch. Thus, $S$ is of depth $O(\log n)$, as desired.

This completes the proof of Lemma 11 and thus of Theorem 2.

4 Extensions of the algorithm

In this section we discuss possible extensions of the algorithm from Section 2. The crucial step of many subexponential-time algorithms for $P_t$-free graphs and $C_{>t}$-free graphs is branching on a high-degree vertex \cite{3, 14, 22, 8}. We observe that some of these algorithms can be turned into quasi-polynomial ones with the new approach.

4.1 Partitioning vertices: 3-COLORING

Let us consider the 3-COLORING problem, whose complexity in $P_t$-free graphs is a well-known open problem \cite{6, 10}. We aim to show Theorem 3, restated below.

**Theorem 3.** For every fixed $t \in \mathbb{N}$, the 3-COLORING problem can be solved in time $n^{O(\log^2 n)}$ in $n$-vertex $P_t$-free graphs and in time $n^{O(\log^4 n)}$ in $n$-vertex $C_{>t}$-free graphs.

Let us first focus on the case of $P_t$-free graphs. The algorithm and its analysis are very similar to the proof of Theorem 1, so we will only point out the differences. The adaptation is inspired by the known subexponential-time algorithms for 3-COLORING \cite{5, 14}. We remark that a quasi-polynomial-time algorithm
for $3$-COLORING in $P_1$-free graphs with running time $n^{O(\log^2 n)}$ can be also derived from the work of Gartland and Lokshntanov [13], by an analogous adaptation of their approach.

Actually, we will solve the more general LIST $3$-COLORING, where every vertex $v$ has a list $L(v) \subseteq \{1, 2, 3\}$, and we ask for a coloring respecting lists $L$ in the sense that the color of each vertex belongs to its list.

As the first step, we preprocess the instance as follows. If there exists a vertex with an empty list, then there is no way to properly color the graph with lists $L$ and thus we can immediately terminate the current call, as we deal with a no-instance. Further, if there is a vertex $v$ with a one-element list, say $L(v) = \{c\}$, then we can obtain an equivalent instance by removing $c$ from the lists of neighbors of $v$ and deleting $v$ from the graph. This corresponds to coloring $v$ with the color $c$. Finally, we enumerate all sets $S \subseteq V(G)$ of size at most $t - 1$, and all their proper colorings, respecting lists $L$. If for some set $S$, the graph $G[S]$ cannot be properly colored with lists $L$, then we terminate the call and report a no-instance. Moreover, if for some $S$, some $v \in S$, and some $c \in L(v)$, the vertex $v$ is not colored $c$ in any proper coloring of $G[S]$, respecting lists $L$, then we can safely remove $c$ from $L(v)$. We perform these steps exhaustively; this can clearly be done in polynomial time. Thus, after the preprocessing, the instance satisfies the following properties:

(P1) Each list has two or three elements.

(P2) For each $v \in V(G)$, each $c \in L(v)$, and each $S \subseteq V(G)$, such that $v \in S$ and $|S| \leq t - 1$, there is a proper coloring of $G[S]$, respecting lists $L$, in which the color of $v$ is $c$.

Similarly to Algorithm 1, our algorithm has two key steps. If the graph is disconnected, we call the algorithm for each connected component independently, and report a yes-instance if all these calls report yes-instances.

Otherwise, if the graph is connected, then we will branch on a vertex. Again, this vertex will be carefully chosen using buckets. This time the objects in a bucket $B_{u,v}$ will be colored induced $u-v$ paths, i.e., for each induced $u-v$ path we additionally enumerate all its proper colorings, respecting lists $L$. Observe that by property (P2) we know that every induced $u-v$ path $P$ appears at least once in $B_{u,v}$ as a colored path. Even stronger, if $w$ is a vertex of $P$ and $c \in L(w)$, then $P$ appears in $B_{u,v}$ as a colored path, where $w$ is colored $c$. Observe that thus, we still have the property that $B_{u,v}$ is non-empty if and only if $u$ and $v$ are in the same connected component of $G$. Note that the total size of all buckets is at most $|V(G)|^{t-1} \cdot 3^{t-1} = (3|V(G)|)^{t-1}$, and we can compute them in time $|V(G)|^{O(t)}$.

The crucial observation is that the analogue of Lemma 5 holds for these buckets too.

**Lemma 15.** Suppose $G$ is a connected $P_1$-free graph and $L : V(G) \to 2^{\{1,2,3\}}$ is a list assignment that satisfies properties (P1) and (P2). Then there is a vertex $w$ and a color $c \in L(w)$ such that for at least $\frac{1}{8t} \cdot \left(\binom{|V(G)|}{2}\right)$ pairs $\{u,v\} \in \binom{V(G)}{2}$, at least $\frac{1}{8t} \cdot |B_{u,v}|$ colored paths in $B_{u,v}$ contain a vertex colored $c$ that belongs to $N[w]$.

**Proof.** For all $\{u,v\} \in \binom{V(G)}{2}$, let $B'_{u,v}$ be the bucket from Algorithm 1, i.e., the collection of all induced $u-v$ paths. Let $w$ be a $\frac{1}{2t}$-heavy vertex given by Lemma 5 for the buckets $\{B'_{u,v} : \{u,v\} \in \binom{V(G)}{2}\}$.

Recall that by property (P1), each vertex in $G$ has one of four possible lists. Thus, by the pigeonhole principle, there exists a list $R \subseteq \{1, 2, 3\}$ and a subset $Q \subseteq \binom{V(G)}{2}$ of size at least $\frac{1}{8t} \cdot \left(\binom{|V(G)|}{2}\right)$ such that for all $\{u,v\} \in Q$, there exists $B'_{u,v} \subseteq B'_{u,v}$ of size at least $\frac{1}{8t} \cdot |B'_{u,v}|$ with the property that each path in $B'_{u,v}$ contains a vertex that belongs to $N[w]$ and whose list is $R$.

By property (P1) we know that $|L(w)| \geq 2$ and $|R| \geq 2$, so there is a color $c \in R \cap L(w)$. Furthermore, by property (P2), each path in $B'_{u,v}$ gives rise to at least one colored path in $B_{u,v}$ which contains a vertex colored $c$. Moreover, these colored paths are pairwise distinct.
Now the claim follows from the observation that $|B_{u,v}| \leq 3^{t-1} \cdot |B'_{u,v}|$, so for each $\{u, v\} \in Q$, we have selected at least $|B'_{u,v}| \geq \frac{1}{8t} |B_{u,v}| \geq \frac{1}{8t3^{t-1}} |B_{u,v}|$ paths in $B_{u,v}$.

Lemma 15 gives us an efficient way to perform branching in case of dealing with a connected graph. Let $w$ and $c$ be as in the statement of the lemma, note that they can be found in polynomial time, as the total size of all buckets is $|V(G)|^{O(t)}$. We branch on coloring $w$ with color $c$. In the first branch we remove $c$ from $L(w)$ (this corresponds to deciding not to color $w$ with $c$). In the second branch, we assign the color $c$ to $w$, i.e., we remove all other colors from $L(w)$. Note that the preprocessing step at the beginning of the subsequent recursive call will remove the vertex $w$ from the graph, and the color $c$ from the lists of all the neighbors of $w$. We report a yes-instance if at least one of the two recursive calls reports a yes-instance.

The algorithm clearly terminates, as in each call we reduce the total size of all lists, and returns the correct answer. Recall that the size of each bucket is polynomial in $|V(G)|$. Lemma 15 implies that in the preprocessing phase in the branch where we color $w$ with the color $c$, we remove a constant fraction of colored paths in a constant fraction of buckets. Note that a path may be removed in one of two ways: either it contains $w$, so it will be removed when we delete $w$, or it contains a neighbor of $w$ colored $c$, so it will be removed when we delete $c$ from the lists of neighbors of $w$.

Now the analysis of the running time of the algorithm is essentially the same as that in the proof of Theorem 1; we leave the details to the reader. This completes the proof of Theorem 3 in the case for $P_t$-free graphs.

The algorithm for $C_{>t}$-free graphs is a modification of the algorithm of Theorem 2 along the same lines as above, so we skip the details.

Let us point out that the above algorithm works also in the weighted setting, i.e., with each pair $(v, c)$, where $v \in V(G)$ and $c \in \{1, 2, 3\}$, we are given a cost $w(v, c)$ of coloring $v$ with $c$, and we ask for a proper coloring minimizing the total cost. A natural special case of this problem is INDEPENDENT ODD CYCLE TRANSVERSAL, where we ask for a minimum-sized independent set which intersects all odd cycles. The complexity of this problem in $P_t$-free graphs is another open problem in the area [4].

Furthermore, the algorithm from this section can be extended to some family of (weighted) graph homomorphism problems, which generalize both the MAXIMUM WEIGHT INDEPENDENT SET problem and the (LIST) 3-COLORING problem, similarly to the work of Groenland et al. [14], see also [22]. We skip the details, as they do not bring any new insight.

### 4.2 Packing fixed patterns: Maximum Weight Induced Matching

Another way to generalize MAXIMUM WEIGHT INDEPENDENT SET is to pack induced, non-adjacent copies of some fixed pattern in the host graph $G$. A natural example of such a problem is MAXIMUM WEIGHT INDUCED MATCHING, where the pattern is $K_2$. This problem can be equivalently formulated as the MAXIMUM WEIGHT INDEPENDENT SET problem on $L^2(G)$, i.e., the square of the line graph of $G$. The vertex set of $L^2(G)$ is $E(G)$, and the edges $e_1, e_2 \in E(G)$ are adjacent in $L^2(G)$ if and only if they do not form an induced matching in $G$, i.e., they either intersect, or some vertex of $e_1$ is adjacent to some vertex of $e_2$. The following structural property of $L^2(G)$ was shown by Cameron, Sritharan, and Tang [7] for $C_{>t}$-free graphs and by Kobler and Rotics [17] for $P_t$-free graphs.

**Lemma 16 ([7, 17]).** The following implications hold:

1. For every $t \geq 3$, if $G$ is $C_{>t}$-free, then $L^2(G)$ is $C_{>t}$-free.
2. For every \( t \geq 4 \), if \( G \) is \( P_t \)-free, then \( L^2(G) \) is \( P_t \)-free.

As the number of vertices of \( L^2(G) \) is at most \( |V(G)|^2 \), Lemma 16 combined with Theorem 1 and Theorem 2 immediately yields the following.

**Corollary 17.** For every fixed \( t \in \mathbb{N} \), the maximum weight induced matching problem can be solved in time \( n^{O((\log^2 n))} \) in \( n \)-vertex \( P_t \)-free graphs and in time \( n^{O((\log^4 n))} \) in \( n \)-vertex \( C_{> t} \)-free graphs.

Let us point out that we could also obtain an algorithm for maximum weight induced matching by a direct modification of Algorithm 1 in a spirit similar to Theorem 3, see also [5]. Moreover, the algorithm can be further generalized to solve the maximum \( H \)-packing problem for any fixed family \( H \) of graphs. In this problem we ask for a maximum-size (or, more generally, maximum-weight) set \( X \) such that every connected component of \( G[X] \) is isomorphic to some graph in \( H \). Again, we skip the technical details and refer the reader to [5].

### 4.3 Finding induced subgraphs of bounded treewidth: \textsc{Min Feedback Vertex Set}

Another way to look at maximum weight independent set is to find a maximum-weight induced subgraph of treewidth 0. A natural next step is to look for a maximum induced forest, i.e., a subgraph of treewidth 1. By complementation, this problem is equivalent to the \textsc{Min Feedback Vertex Set} problem, where we want to find a minimum-size (or minimum-weight) set which intersects all cycles.

A subexponential-time algorithm for \textsc{Min Feedback Vertex Set} in \( P_t \)-free graphs and in \( C_{> t} \)-free graphs is known [21] and also involves branching on a high-degree vertex. However, it has also one more step of exhaustive guessing the large-degree vertices that are not in the optimum feedback vertex set, and it is not clear how to avoid this. It would be interesting to investigate if the methods of Gartland of Lokshtanov [13, 12] or our approach could be used to obtain a quasi-polynomial-time algorithm for \textsc{Min Feedback Vertex Set} in \( P_t \)-free graphs and \( C_{> t} \)-free graphs.

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