Spectral dimension of quaternion spheres

Bipul Saurabh

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Abstract

Employing ideas of noncommutative geometry, certain dimensional invariant for quantum homogeneous spaces has been proposed and here we take up its computation for quaternion spheres.

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1 Introduction

In Connes’ (2) formulation, a geometric space is given by a triple \((A, \mathcal{H}, D)\) called a spectral triple, with \(A\) being an involutive algebra represented as bounded operators on a Hilbert space \(\mathcal{H}\), and \(D\) being a selfadjoint operator with compact resolvent and having bounded commutators with the algebra elements. Connes further defined the dimension of a spectral triple to be the quantity \(\inf \{\delta : \text{Tr}(|D|^{-\delta}) < \infty\}\). Utilizing this, Chakraborty and Pal (3) introduced an invariant called spectral dimension, for an ergodic \(C^*\)-dynamical system or equivalently for a homogeneous space of a compact quantum group. They considered all finitely summable equivariant spectral triples on the GNS space of the state invariant under the group action and defined the spectral dimension of the homogeneous space to be the infimum of the summability of the associated Dirac operators. Compact quantum groups and their quotient spaces are natural examples of homogeneous spaces. In the same paper (3), Chakraborty and Pal computed spectral dimension of many such homogeneous spaces, both in classical and quantum situations and it was conjectured that the spectral dimension of a homogeneous space of a (classical) compact Lie group is same as its dimension as a differentiable manifold. The spectral dimensions of \(SU(2)\), \(SU_q(n)\) and \(S^2q^n+1\) point towards this conjecture. All these examples are homogeneous spaces of type \(A\) quantum groups. To examine the conjecture, we need to explore more examples. In this article, we take up the case of quaternion sphere \(H^n\). These spaces are homogeneous spaces of type \(C\) quantum groups. Here we show that the spectral
dimension is equal to its dimension as a real manifold. Therefore it strengthens the conjecture of Chakraborty and Pal. The computation of the invariant for $H^n$ is the first instance of computation for homogeneous spaces of type $C$ quantum groups.

We will sometimes write a spectral triple $(A, \mathcal{H}, D)$ as $(\mathcal{H}, \pi, D)$ where $\pi$ is the representation of $A$ in the Hilbert space $\mathcal{H}$. For a subset $S$ of a $C^*$-algebra, $\overline{S}$ will denote the closed linear span of $S$ in $A$.

## 2 Spectral dimension

In this section, we recall from [1] the definition of spectral dimension of a $C^*$-dynamical system. Let us begin with the definition of a homogeneous space.

**Definition 2.1.** A compact quantum group $G$ acts on a $C^*$-algebra $A$ if there exists a $*$-homomorphism $\tau : A \to A \otimes C(G)$ such that

1. $(\tau \otimes id)\tau = (id \otimes \Delta)\tau$,
2. $\{(I \otimes b)\tau(a) : a \in A, b \in C(G)\} = A \otimes C(G)$.

where $\Delta$ is the comultiplication map of $G$. We call an action $\tau$ homogeneous or ergodic if the fixed point subalgebra $\{a \in A : \tau(a) = a \otimes I\}$ is $\mathbb{C} I$. In that case, the associated $C^*$-algebra $A$ is called an homogeneous space of $G$ and the triple $(A, G, \tau)$ is called an ergodic $C^*$-dynamical system.

A covariant representation of a $C^*$-dynamical system $(A, G, \tau)$ is a pair $(\pi, U)$ consisting of a representation $\pi : A \to \mathcal{L}(\mathcal{H})$ and a unitary representation of $G$ on $\mathcal{H}$ such that for all $a \in A$, one has

$$(\pi \otimes id)\tau(a) = U(\pi(a) \otimes I)U^*.$$  

**Definition 2.2.** Let $(\pi, U)$ be a covariant representation of a $C^*$-dynamical system $(A, G, \tau)$ and $(\mathcal{H}, \pi, D)$ be a spectral triple for a dense $*$-subalgebra $A$ of $A$. We call $(\mathcal{H}, \pi, D)$ equivariant with respect to $(\pi, U)$ if $D \otimes I$ commutes with $U$.

Associated with a homogeneous action $\tau$ of $G$ is a unique invariant state $\rho$ on the homogeneous space $A$ that obeys

$$(\rho \otimes id)\tau(a) = \rho(a)I, \quad a \in A.$$  

Consider the GNS representation $(\mathcal{H}_\rho, \pi_\rho, \eta_\rho)$ of $A$ associated with the state $\rho$. Using the invariance property of $\tau$, one can show that the action $\tau$ induces a unitary representation $U_\tau$ of $G$ on $\mathcal{H}_\rho$ and the pair $(\pi_\rho, U_\tau)$ is a covariant representation of the system $(A, G, \tau)$. Let $O(G)$ be the dense $*$-Hopf subalgebra of $C(G)$ generated by matrix entries of irreducible unitary representations of $G$. Define

$$\mathcal{A} := \{a \in A : \tau(a) \in A \otimes_{alg} O(G)\}.$$  

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It follows from part (1) of Theorem 1.5 in [1] that \( \mathcal{A} \) is a dense \(*\)-subalgebra of \( A \). Define \( \xi \) to be the class of spectral triples of \( \mathcal{A} \) equivariant with respect to the covariant representation \((\pi_\rho, U_\rho)\). The spectral dimension denoted by \( Sdim(A,G,\tau) \) of the \( C^*\)-dynamical system \((A,G,\tau)\) is defined to be the quantity

\[
\inf\{p > 0 : \exists D \text{ such that } (\mathcal{A},\mathcal{H}_\rho, D) \in \xi \text{ and } D \text{ is } p\text{-summable}\}.
\]

### 3 Main result

Here we briefly recall some notions related to quaternion spheres \( H^n \) (or \( SP(2n)/SP(2n - 2) \)) and then compute its spectral dimension. For \( 1 \leq i,j \leq 2n \), define a continuous map

\[
u_i^j : SP(2n) \to \mathbb{C}; \quad A \mapsto a_j^i
\]

where \( a_j^i \) is the \( ij^{th} \) entry of \( A \in SP(2n) \). The \( C^*\)-algebra \( C(SP(2n)) \) is generated by elements of the set \( \{u_j^i : 1 \leq i,j \leq 2n\} \). In the same way, define the generators \( \{v_j^i : 1 \leq i,j \leq 2n - 2\} \) of \( C(SP(2n - 2)) \). Define the map \( \Phi : C(SP(2n)) \to C(SP(2n - 2)) \) as follows.

\[
\Phi(u_j^i) = \begin{cases} 
u_j^{i-1}, & \text{if } i \neq 1 \text{ or } 2n, \text{ or } j \neq 1 \text{ or } 2n, \\ \delta_{ij}, & \text{otherwise.} \end{cases}
\]

Clearly \( \phi \) is a \( C^*\)-epimorphism obeying \( \Delta \phi = (\phi \otimes \phi) \Delta \) where \( \Delta \) is the co-multiplication map of \( C(SP(2n)) \). In such a case, one defines the quotient space \( C(SP(2n)/SP(2n - 2)) \) by,

\[
C(SP(2n)/SP(2n - 2)) = \{a \in C(SP(2n)) : (\phi \otimes id)\Delta(a) = I \otimes a\}.
\]

The quotient space \( SP(2n)/SP(2n - 2) \) can be realized as the \( n\)-dimensional quaternion sphere \( H^n \). Also, each of the generators \( \{u_j^1 : 1 \leq j \leq 2n\} \) can be viewed as projection on to a fixed complex coordinate of a point in \( H^n \subset \mathbb{C}^{2n} \) and for \( 1 \leq j \leq 2n \), the map \( u_j^{2n} \) is the complex conjugate of \( u_{2n+1-j}^1 \). This shows that \( C(SP(2n)/SP(2n - 2)) \) is generated by \( \{u_j^i : i = 1 \text{ or } 2n, 1 \leq j \leq 2n\} \). Restricting the co-multiplication map to \( C(SP(2n)/SP(2n - 2)) \) gives an action \( \tau \) of the compact quantum group \( SP(2n) \) on \( SP(2n)/SP(2n - 2) \).

\[
\tau : C(SP(2n)/SP(2n - 2)) \to C(SP(2n)/SP(2n - 2)) \otimes C(SP(2n)) \\
a \mapsto \Delta a.
\]

It is not difficult to verify that the system \((C(SP(2n)/SP(2n - 2)), SP(2n), \tau)\) is an ergodic \( C^*\)-dynamical system and the invariant state \( \rho \) of \( \tau \) is the faithful Haar state \( h \) of \( C(SP(2n)) \) restricted to \( C(SP(2n)/SP(2n - 2)) \). By Theorem 1.5 of [1], we get

\[
C(SP(2n)/SP(2n - 2)) = \bigoplus_{\lambda \in SP(2n)} \bigoplus_{i \in I_\lambda} W_{\lambda,i}(3.1)
\]
where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ represents the highest weight of a finite-dimensional irreducible co-representation $u_\lambda$ of $C(SP(2n))$, $I_\lambda$ is the multiplicity of $u_\lambda$ and $W_{\lambda,i}$ corresponds to $u_\lambda$ in the sense of Podles (see page 4, [4]) for all $i \in I_\lambda$. Using Zhelobenko branching rule (see page 79, [4] and Theorem 1.7 in [4], page 145-146 in [4]), we get

\[
I_\lambda = \begin{cases} 
\lambda_1 - \lambda_2 + 1, & \text{if } \lambda_i = 0 \text{ for all } i \geq 3, \\
0, & \text{otherwise.}
\end{cases}
\]

Define

\[
O(SP(2n)/SP(2n-2)) := \bigoplus_{\lambda \in SP(2n)} \bigoplus_{i \in I_\lambda} W_{\lambda,i}.
\]

It is not difficult to see that $O(SP(2n)/SP(2n-2))$ is the algebra generated by the elements of the set $\{u_i^j : i = 1 \text{ or } 2n, 1 \leq j \leq 2n\}$. Moreover, the algebra $O(SP(2n)/SP(2n-2))$ is a dense Hopf $*$-algebra consisting of all $a \in C(SP(2n)/SP(2n-2))$ such that $\tau(a) \in C(SP(2n)/SP(2n-2)) \otimes_{alg} O(SP(2n))$.

Let $U(\mathfrak{sp}(2n))$ be the universal enveloping algebra of the Lie algebra $\mathfrak{sp}(2n)$. We will view $\mathfrak{sp}(2n)$ as a subset of $U(\mathfrak{sp}(2n))$. Then $U(\mathfrak{sp}(2n))$ is generated by $H_i, E_i, F_i \in \mathfrak{sp}(2n)$, $i = 1, 2, \cdots, n$, satisfying the relations given in page 160, [3]. Hopf $*$-structure of $U(\mathfrak{sp}(2n))$ comes from the following maps (see page 18 and page 21 of [3]):

\[
\Delta(r) = r \otimes 1 + 1 \otimes r, \quad S(r) = -r, \quad \epsilon(r) = 0, \quad r = r^* \quad \forall r \in \mathfrak{sp}(2n).
\]

Denote by $T_1$ the finite dimensional irreducible representation of $U(\mathfrak{sp}(2n))$ with highest weight $(1, 0, \cdots, 0)$. There exists unique nondegenerate dual pairing $\langle \cdot, \cdot \rangle$ between the Hopf $*$-algebras $U(\mathfrak{sp}(2n))$ and $O(SP(2n)/SP(2n-2))$ such that

\[
\langle f, u_i^k \rangle = t_{ki}(f); \quad \text{for } k = 1 \text{ or } 2n \text{ and } 1 \leq l \leq 2n,
\]

where $t_{ki}$ is the matrix element of $T_1$. Using this, one can give the algebra $O(SP(2n)/SP(2n-2))$ a $U(\mathfrak{sp}(2n))$-module structure in the following way.

\[
f(a) = (1 \otimes \langle f, \cdot \rangle) \Delta a,
\]

where $f \in U(\mathfrak{sp}(2n))$ and $a \in O(SP(2n)/SP(2n-2))$. We call an element $b \in O(SP(2n)/SP(2n-2))$ a highest weight vector with highest weight $(\lambda_1, \lambda_2, 0, \cdots, 0)$ if

\[
H_1(b) = (\lambda_1 - \lambda_2)b, \quad H_2(b) = \lambda_2b, \quad H_i(b) = 0 \quad \text{for } i \geq 2,
\]

and

\[
E_i(b) = 0 \quad \text{for } 1 \leq i \leq 2n.
\]

We will write down $\lambda_1 - \lambda_2 + 1$ linearly independent highest weight vectors explicitly in terms of $\{u_m^1, u_m^2 : 1 \leq m \leq 2n\} \subset O(SP(2n)/SP(2n-2))$. Let $x = u_{2n-1}^1, y = u_{2n-1}^2, z = u_{2n}^1$ and $w = u_{2n}^2$. For $j \in \{0, 1, \cdots, \lambda_1 - \lambda_2\}$, define

\[
\delta^j_{\lambda_1, \lambda_2} := z^j w^{\lambda_1 - \lambda_2 - j} (xw - yz)^{\lambda_2}.
\]
Proposition 3.1. Let $\lambda_1, \lambda_2$ be two positive integers such that $\lambda_1 \geq \lambda_2$. Then the set $\{b^{(\lambda_1, \lambda_2, j)} : \gamma \leq j \leq \lambda_1 - \lambda_2\}$ is a linearly independent set of highest weight vectors in the algebra $O(SP(2n)/SP(2n - 2))$ with highest weight $(\lambda_1, \lambda_2, 0, \cdots, 0)$.

Proof: It is easy to see that

$$E_i(x) = E_i(y) = E_i(z) = E_i(w) = 0 \quad \text{for} \quad i > 1.$$ 

Also,

$$E_1(x) = -z, E_1(y) = -w, E_2(z) = E_2(w) = 0.$$ 

Further,

$$H_1(x) = -x, H_1(y) = -y, H_1(z) = z, H_1(w) = w,$$

$$H_2(x) = x, H_2(y) = y, H_2(z) = 0, H_2(w) = 0,$$

and for $i > 2$, $H_i$ maps these elements to 0. Now using properties of Hopf algebra pairing (see page 21 of [3]), one can check that $\{b^{(\lambda_1, \lambda_2, j)} : \gamma \leq j \leq \lambda_1 - \lambda_2\}$ are highest weight vectors with highest weight $(\lambda_1, \lambda_2, 0, \cdots, 0)$. The proof of linear independence follows from the fact that $x, y, z$ and $w$ represent projections or conjugate of projections on to different coordinates of a point in $H^n$. \qed

Let

$$\Gamma = \{(\gamma_1, \gamma_2, \gamma_3) : \gamma_1, \gamma_2, \gamma_3 \in \mathbb{N}, 0 \leq \gamma_2 \leq \gamma_1, 0 \leq \gamma_3 \leq \gamma_1 - \gamma_2\}.$$ 

Here first two co-ordinates represent the highest weight and last co-ordinate is for multiplicity. We denote by $W_\gamma$ the vector space corresponding to irreducible representation of highest weight vector $b^\gamma$ in the sense of Podles (see page 4, [4]), by $N_\gamma$ the dimension of $W_\gamma$ and by $\{u_\gamma^i : i \in \{1, 2, \cdots, N_\gamma\}\}$ a basis of $W_\gamma$ such that $u_\gamma^1 = b^\gamma$. Hence we can write equation (3.1) as

$$O(SP(2n)/SP(2n - 2)) = \oplus_{\gamma \in \Gamma} W_\gamma.$$ 

Therefore the set $\{e_\gamma^i := \frac{u_\gamma^i}{\|u_\gamma^i\|} : i \in \{1, 2, \cdots, N_\gamma\}, \gamma \in \Gamma\}$ is an orthonormal basis of $L^2(\rho)$. Let $D$ be an equivariant Dirac operator. Then following the arguments in propositions 5.1-5.3 leading to the statement (5.22) in [1], we can assume that $D$ must be of the form

$$D e_\gamma^i = d^e e_\gamma^i, \quad i \in \{1, 2, \cdots, N_\gamma\}, \gamma \in \Gamma.$$ 

Further assume that $(O(SP(2n)/SP(2n - 2)), L^2(\rho), D)$ is an equivariant spectral triple of the system $(C(SP(2n)/SP(2n - 2)), SP(2n), \tau)$. Define the set

$$\Theta = \{(x, y, z, w) \in \mathbb{R}^4 : 0 \leq x, y, z, w \leq 1, x^2 + y^2 + z^2 + w^2 = 1\}.$$ 

For $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \Lambda$, define the function

$$g^{(\gamma_1, \gamma_2, \gamma_3)} : \Theta \rightarrow \mathbb{R}.$$
Hence we have
\[ \| u_1^\gamma \| = \| b^{(\gamma_1, \gamma_2 \gamma_3)} \| = \sup_{(x, y, z, w) \in \Theta} g^{(\gamma_1, \gamma_2, \gamma_3)}(x, y, z, w). \]

**Proposition 3.2.** Let \( \Theta \) be a compact subset of \( \mathbb{R}^n \) and \( f \) and \( h \) are two real valued continuous functions defined on \( \Theta \). Let \( x_0 \in \Theta \) be a point such that \( |f(x_0)| = \| f \| = \sup_{x \in \Theta} |f(x)| \neq 0 \) and \( h(x_0) \neq 0 \). Then one has

\[ \frac{\| h^m f \|}{\| h^{m+1} f \|} \leq \frac{1}{\| h(x_0) \|}. \]

**Proof:** For \( m > 0 \), choose \( x_m \in \Theta \) such that \( \| h^m f(x_m) \| = \| h^m f \|. \) Then for \( m \geq 0 \), we have

\[ \| h^{m+1} f \| \geq \| h^{m+1} f(x_m) \| \geq |h^m f(x_m)h(x_m)| = |h(x_m)| \| h^m f \|. \]

Further

\[ \| h^{m+1} f \| = \| h^{m+1} f(x_{m+1}) \| = |h(x_{m+1})| \| h^m f(x_{m+1}) \| \leq |h(x_{m+1})| \| h^m f \|. \]

Comparing the two inequalities, we get

\[ |h(x_m)| \leq |h(x_{m+1})| \]

Hence we have

\[ \frac{\| h^m f \|}{\| h^{m+1} f \|} \leq \frac{1}{\| h(x_m) \|} \leq \frac{1}{\| h(x_0) \|}. \]

\[ \Box \]

**Lemma 3.3.** For \( (m, n) \in \mathbb{N}^2 - \{0\} \), define \( f_{(m, n)} : \Theta \to \mathbb{R} \) by \( f_{(m, n)}(x, y, z, w) = (zw)^n(xz + yw)^m \). Let \( \theta_{(m, n)} := (\sqrt[2m]{\frac{\sqrt[m]{m}}{2\sqrt[n]{n+m}}, \sqrt[m]{m}, \sqrt[2m+n+m]{\sqrt[2m]{2m+n+m}}, \sqrt[2m+n+m]{\sqrt[2m]{2m+n+m}}}) \). Then \( \theta_{(m, n)} \in \Theta \) and \( f_{(m, n)}(\theta_{(m, n)}) = \| f_{(m, n)} \| \).

**Proof:** By symmetry, we can assume without loss of generality that \( x = y \) and \( z = w \). Now by a straightforward calculation, one can prove the claim. \( \Box \)

**Lemma 3.4.** Let \( \epsilon_1 = (1, 0, 0), \epsilon_2 = (0, 1, 0) \) and \( \epsilon_3 = (0, 0, 1) \). Then one has

1. \( \sup_{\gamma \in \Gamma: \gamma_1 = \gamma_2, \gamma_3 = 0} \| u_1^\gamma \|_{\| u_1^{\gamma_1+\gamma_2+\gamma_3} \|} < \infty. \)
2. \( \sup_{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 = 0} \| u_1^\gamma \|_{\| u_1^{\gamma_1+\gamma_2+\gamma_3} \|} < \infty. \)
3. \( \sup_{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 \geq 0} \| u_1^\gamma \|_{\| u_1^{\gamma_1+\gamma_2+\gamma_3} \|} < \infty. \)
4. \( \sup_{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 \leq 0} \| u_1^\gamma \|_{\| u_1^{\gamma_1+\gamma_2+\gamma_3} \|} < \infty. \)
Proof: Observe that $\|g^\gamma\| \neq 0$ for all $\gamma \in \Gamma$.

1. For $\gamma$ with $\gamma_1 = \gamma_2$ and $\gamma_3 = 0$, we have $g^\gamma = (xz + yw)^{\gamma_2}$ and $g^{\gamma + \epsilon_1 + \epsilon_2} = (xz + yw)^{\gamma_2 + 1}$. Hence

$$\sup_{\{\gamma \in \Gamma: \gamma_1 = \gamma_2, \gamma_3 = 0\}} \frac{\|u_1^{\gamma + \epsilon_1 + \epsilon_2}\|}{\|u_1^{\gamma + 2\epsilon_1 + \epsilon_2}\|} = \sup_{\{\gamma \in \Gamma: \gamma_1 = \gamma_2, \gamma_3 = 0\}} \frac{\|g^\gamma\|}{\|g^{\gamma + \epsilon_1 + \epsilon_2}\|} = \frac{1}{\|g^{(1,1,0)}\|} < \infty.$$ 

2. For $\gamma$ with $\gamma_1 - \gamma_2 - 2\gamma_3 = 0$, we have $g^\gamma = (zw)^{\gamma_3}(xz + yw)^{\gamma_2}$ and $g^{\gamma + \epsilon_1 + \epsilon_3} = (zw)^{\gamma_3 + 1}(xz + yw)^{\gamma_2}$. Also, using Lemma 3.3, one can see that at $(1/2, 1/2, 1/2, 1/2)$, the function $f(\gamma, 0)(x, y, z, w) = (xz + yw)^{\gamma_2}$ takes its maximum. Hence by Proposition 3.2 we get

$$\sup_{\{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 = 0\}} \frac{\|u_1^\gamma\|}{\|u_1^{\gamma + 2\epsilon_1 + \epsilon_3}\|} = \frac{1}{zw(\theta(\gamma, 0))} \leq 4 < \infty.$$ 

3. For $\gamma$ with $\gamma_1 - \gamma_2 - 2\gamma_3 \geq 0$, we have $g^\gamma = w^{\gamma_1 - \gamma_2 - 2\gamma_3}(zw)^{\gamma_3}(xz + yw)^{\gamma_2}$. Hence by Lemma 3.3 and Proposition 3.2 we get

$$\sup_{\{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 \leq 0\}} \frac{\|u_1^\gamma\|}{\|u_1^{\gamma + \epsilon_1 + \epsilon_3}\|} = \sup_{\{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 \leq 0\}} \frac{\|g^\gamma\|}{\|g^{\gamma + \epsilon_1 + \epsilon_3}\|} \leq \frac{1}{\sup_{\{\gamma: \gamma_1 - \gamma_2 - 2\gamma_3 \in \mathbb{N}\}} w(\theta(\gamma, \gamma_3))} \leq 2 < \infty.$$ 

4. For $\gamma$ with $\gamma_1 - \gamma_2 - 2\gamma_3 \leq 0$, we have $g^\gamma = z^{\gamma_2 + 2\gamma_3 - \gamma_1}(zw)^{\gamma_1 - \gamma_2 - \gamma_3}(xz + yw)^{\gamma_2}$. Hence by Lemma 3.3 and Proposition 3.2 we get

$$\sup_{\{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 \geq 0\}} \frac{\|u_1^\gamma\|}{\|u_1^{\gamma + \epsilon_1 + \epsilon_3}\|} = \sup_{\{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 \geq 0\}} \frac{\|g^\gamma\|}{\|g^{\gamma + \epsilon_1 + \epsilon_3}\|} \leq \frac{1}{\sup_{\{\gamma: \gamma_1 - \gamma_2 - 2\gamma_3 \in \mathbb{N}\}} z(\theta(\gamma_2, \gamma_1 - \gamma_2 - \gamma_3))} \leq 2 < \infty.$$ 

In the following lemma, we establish some estimate on the growth of $d^\gamma$.

Lemma 3.5. Let $\epsilon_1 = (1, 0, 0)$, $\epsilon_2 = (0, 1, 0)$ and $\epsilon_3 = (0, 0, 1)$. Then one has

1. $\sup_{\{\gamma \in \Gamma: \gamma_1 = \gamma_2, \gamma_3 = 0\}} |d^{\gamma + \epsilon_1 + \epsilon_2} - d^\gamma| < \infty$.
2. $\sup_{\{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 = 0\}} |d^{\gamma + 2\epsilon_1 + \epsilon_3} - d^\gamma| < \infty$.
3. $\sup_{\{\gamma \in \Gamma: \gamma_1 - \gamma_2 - 2\gamma_3 \geq 0\}} |d^{\gamma + \epsilon_1} - d^\gamma| < \infty$. 

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4. \( \sup_{\gamma \in \Gamma; \gamma_1 - \gamma_2 - 2\gamma_3 \leq 0} |d^{\gamma + \epsilon_1 + \epsilon_2} - d^0| < \infty. \)

**Proof:** Let \( \gamma \in \Gamma \) such that \( \gamma_1 = \gamma_2 \) and \( \gamma_3 = 0 \). Note that

\[
\| [D, (xw - yz)] u_1^\gamma \| = \| [D, (xw - yz)] (xw - yz)^\gamma \|
\]

\[
= \| (d^\gamma - d^{\gamma + \epsilon_1 + \epsilon_2}) u_1^{\gamma + \epsilon_1 + \epsilon_2} \|.
\]

Since \([D, (xw - yz)]\) is a bounded operator, we have

\[
\sup_{\gamma \in \Gamma; \gamma_1 = \gamma_2, \gamma_3 = 0} |d^\gamma - d^{\gamma + \epsilon_1 + \epsilon_2}| = \sup_{\gamma \in \Gamma; \gamma_1 = \gamma_2, \gamma_3 = 0} \| [D, (xw - yz)] u_1^\gamma \| \frac{\| u_1^\gamma \|}{\| u_1^{\gamma + \epsilon_1 + \epsilon_2} \|} \leq \sup_{\gamma \in \Gamma; \gamma_1 = \gamma_2, \gamma_3 = 0} \| [D, (xw - yz)] u_1^\gamma \| \frac{\| u_1^\gamma \|}{\| u_1^{\gamma + \epsilon_1 + \epsilon_2} \|} < \infty \quad (\text{by part (1) of the Lemma 3.5})
\]

Using Lemma 3.4 and the fact that \([D, zw], [D, z]\) and \([D, w]\) are bounded operators, other parts of the claim follow similarly. \( \square \)

Let \( c > 0 \) be an upper bound in all the four inequalities of the Lemma 3.5. Let \( \mathcal{G} \) be a graph with vertex set \( \Gamma \) and edge set \( \{(\gamma, \gamma') : |d^\gamma - d^{\gamma'}| < c\} \). The following lemma says that \( \mathcal{G} \) is a connected graph.

**Lemma 3.6.** Let \( \gamma \in \Gamma \). Then there is a path in \( \mathcal{G} \) joining \((0, 0, 0)\) and \( \gamma \) and of length less than or equal to \( \gamma_1 \).

**Proof:** If \( \gamma_1 - \gamma_2 - 2\gamma_3 \geq 0 \), then one possible path would be as follows.

\[
(0, 0, 0) \rightarrow (1, 1, 0) \rightarrow (2, 2, 0) \rightarrow \cdots \rightarrow (\gamma_2, \gamma_2, 0)
\]

(by part(1) of the Lemma 3.5)

\[
(\gamma_2, \gamma_2, 0) \rightarrow (\gamma_2 + 2, \gamma_2, 1) \rightarrow \cdots \rightarrow (\gamma_2 + 2\gamma_3, \gamma_2, \gamma_3)
\]

(by part(2) of the Lemma 3.5)

\[
(\gamma_2 + 2\gamma_3, \gamma_2, \gamma_3) \rightarrow (\gamma_2 + 2\gamma_3 + 1, \gamma_2, \gamma_3) \rightarrow \cdots (\gamma_1, \gamma_2, \gamma_3)
\]

(by part(3) of the Lemma 3.5).

If \( \gamma_1 - \gamma_2 - 2\gamma_3 \leq 0 \), then one possible path would be as follows.

\[
(0, 0, 0) \rightarrow (1, 1, 0) \rightarrow (2, 2, 0) \rightarrow \cdots \rightarrow (\gamma_2, \gamma_2, 0)
\]

(by part(1) of the Lemma 3.5)

\[
(\gamma_2, \gamma_2, 0) \rightarrow (\gamma_2 + 2, \gamma_2, 1) \rightarrow \cdots \rightarrow (2\gamma_1 - \gamma_2 - 2\gamma_3, \gamma_2, \gamma_1 - \gamma_2 - \gamma_3)
\]

(by part(2) of the Lemma 3.5)

\[
(2\gamma_1 - \gamma_2 - 2\gamma_3, \gamma_2, \gamma_1 - \gamma_2 - \gamma_3) \rightarrow (2\gamma_1 - \gamma_2 - 2\gamma_3 + 1, \gamma_2, \gamma_1 - \gamma_2 - \gamma_3 + 1)
\]

\[ \rightarrow \cdots \rightarrow (\gamma_1, \gamma_2, \gamma_3) \quad \text{(by part (4) of the Lemma 3.5).} \]
Moreover, the length of the paths in both cases are less than $\gamma_1$ as in each step the increment in the first coordinate is at least one. This settles the claim. \[\square\]

**Lemma 3.7.** Let $D : e_i^\gamma \mapsto d^i e_i^\gamma$ be an operator acting on the Hilbert space $L^2(\rho)$ such that the triple $(L^2(\rho), \pi_\rho, D)$ is an equivariant spectral triple of the system $(\mathcal{C}(\text{SP}(2n)/\text{SP}(2n - 2)), \text{SP}(2n), \tau)$. Then we have

\[d^i = O(\gamma_1).\]

**Proof:** It follows from Lemma 3.6. \[\square\]

**Lemma 3.8.** For $1 \leq m \leq 2n$ and $l = 1$ or $2n$, one has

\[u_m^l u_i^\gamma \subset \text{span}\{u_i^\beta : \gamma_1 - 1 \leq \beta_1 \leq \gamma_1 + 1\}\]

**Proof:** Let $e_i = (0,0,\cdots,0,1\underset{i}{\uparrow},0,\cdots,0)$. Then from equation ((14), page 210, [3]), we get

\[u_{(\gamma_1,\gamma_2,0,0,\cdots,0)} \otimes u_{(1,0,\cdots,0)} = \bigoplus_{n=1}^\infty u_{(\gamma_1,\gamma_2,0,0,\cdots,0)+e_i} \bigoplus \bigoplus_{n=1}^\infty u_{(\gamma_1,\gamma_2,0,0,\cdots,0)-e_i} \]

Hence $u_m^l u_i^\gamma$ is in the span of matrix entries of the irreducible representations of highest weight $(\beta_1,\beta_2,\cdots,\beta_n)$ such that $\beta_1 = \gamma_1$ or $\gamma_1 \pm 1$. Since $u_m^l u_i^\gamma \in \mathcal{O}(\text{SP}(2n)/\text{SP}(2n - 2))$ and \{u_i^\gamma : \gamma \in \Gamma\} is a basis of $\mathcal{O}(\text{SP}(2n)/\text{SP}(2n - 2))$, we get the claim. \[\square\]

**Theorem 3.9.** Let $D_{eq}$ be the Dirac operator $e_i^\gamma \mapsto \gamma_1 e_i^\gamma$ acting on the Hilbert space $L^2(\rho)$. Then the triple $(\mathcal{O}(\text{SP}(2n)/\text{SP}(2n - 2)), L^2(\rho), D)$ is a $(4n - 1)$-summable equivariant spectral triple of the system $(\mathcal{C}(\text{SP}(2n)/\text{SP}(2n - 2)), \text{SP}(2n), \tau)$. The operator $D_{eq}$ is optimal, i.e. if $D$ is any equivariant Dirac operator of the $C^*$-dynamical system $(\mathcal{C}(\text{SP}(2n)/\text{SP}(2n - 2)), \text{SP}(2n), \tau)$ acting on $L^2(\rho)$ then there exist positive reals $a$ and $b$ such that

\[|D| \leq a|D_{eq}| + b.\]

**Proof:** Clearly $D_{eq}$ is a selfadjoint operator with compact resolvent. That $D_{eq}$ has bounded commutators with the generators $\{u_m^1, u_m^{2n} : m \in \{1,2,\cdots,2n\}\}$ of $\mathcal{O}(\text{SP}(2n)/\text{SP}(2n - 2))$ follows from Lemma 3.8. This proves that the triple $(\mathcal{O}(\text{SP}(2n)/\text{SP}(2n - 2)), L^2(\rho), D)$ is an equivariant spectral triple of the system $(\mathcal{C}(\text{SP}(2n)/\text{SP}(2n - 2)), \text{SP}(2n), \tau)$. From Weyl dimension formula, we have

\[N_\gamma = O(\gamma_1^{2n-1} \gamma_2^{2n-3}).\]

This along with the fact that $0 \leq \gamma_2 \leq \gamma_1$ and $0 \leq \gamma_3 \leq \gamma_1 - \gamma_2$ shows that $D$ is $(4n - 1)$-summable. Optimality follows from Lemma 3.7. \[\square\]

**Theorem 3.10.** Spectral dimension of the quaternion spheres $\text{SP}(2n)/\text{SP}(2n - 2)$ is $4n - 1$.

**Proof:** It is a direct consequence of Theorem 3.9. \[\square\]
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Bipul Saurabh (saurabhbipul2@gmail.com)
Harish Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211019, INDIA