Hypercontractivity of spherical averages in Hamming space

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Abstract

Consider a linear space of functions on the binary hypercube and a linear operator $T_\delta$ acting by averaging a function over a Hamming sphere of radius $\delta n$. It is shown that such an operator has a dimension independent bound on the norm $L_p \to L_2$ with $p = 1 + (1 - 2\delta)^2$. This result evidently parallels a classical estimate of Bonami and Gross for $L_p \to L_q$ norms for the operator of convolution with a Bernoulli noise. The estimate for $T_\delta$ is harder to obtain since the latter is neither a part of a semigroup, nor a tensor power. The result is shown by a detailed study of the eigenvalues of $T_\delta$ and $L_p \to L_2$ norms of the Fourier multiplier operators $\Pi_a$ with symbol equal to a characteristic function of the Hamming sphere of radius $a$.

An application of the result to additive combinatorics is given: Any set $A \subset \mathbb{F}_2^n$ with the property that $A + A$ contains a large portion of some Hamming sphere (counted with multiplicity) must have cardinality a constant multiple of $2^n$. It is also demonstrated that this result does not follow from standard spectral gap and semi-definite (Lovász-Delsarte) methods.

I. MAIN RESULT AND DISCUSSION

Consider a linear space $\mathcal{L}$ of functions on $n$-dimensional Hamming cube $f : \mathbb{F}_2^n \to \mathbb{C}$. We endow $\mathcal{L}$ with the following norms and an inner product:

\begin{align}
\|f\|_p &\triangleq \mathbb{E} \frac{1}{p} |f(X)|^p, \quad 1 \leq p \leq \infty, \\
(f,g) &\triangleq \mathbb{E} [f(X)\overline{g(X)}],
\end{align}

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where $X$ is uniform on $\mathbb{F}_2^n$. For any linear operator $T : \mathcal{L} \to \mathcal{L}$ we define
\[
\|T\|_{p \to q} \triangleq \sup_{f \in \mathcal{L}} \frac{\|Tf\|_q}{\|f\|_p}.
\]

For the following operator
\[
N_\delta f(x) \triangleq \mathbb{E} [f(x + Z)], \quad Z \sim \text{i.i.d. Bern}(\delta), x, Z \in \mathbb{F}_2^n, 0 \leq \delta \leq 1
\]
the so-called “hypercontractive” inequality was established by Bonami and Gross [1], [2]:
\[
\|N_\delta\|_{p \to q} = 1, \quad p - 1 \geq (q - 1)(1 - 2\delta)^2, p, q \geq 1.
\]

In this paper we analyze $L_p \to L_2$ norm for an operator $T_\delta$ of averaging over a Hamming sphere $S_{\delta n}$. Specifically, for $x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ denote the Hamming weight of $x$ and the Hamming sphere centered at zero as
\[
|x| \triangleq |\{j : x_j = 1\}|
\]
\[
S_j \triangleq \{x : |x| = j\}.
\]

The operator $T_\delta$ is defined as follows: For $\delta < 1/2$
\[
T_\delta f(x) \triangleq \left( \frac{n}{[\delta n]} \right)^{-1} \sum_{y \in \mathbb{F}_2^n, |y| = [\delta n]} f(x + y)
\]
and for $\delta \geq 1/2$
\[
T_\delta f(x) \triangleq \left( \frac{n}{[\delta n]} \right)^{-1} \sum_{y \in \mathbb{F}_2^n, |y| = [\delta n]} f(x + y).
\]

Apart from rounding issues we may write
\[
T_\delta f \triangleq \frac{f \ast 1_{S_{\delta n}}}{|S_{\delta n}|},
\]
where $\ast$ denotes the convolution
\[
f \ast g(x) \triangleq \sum_{y \in \mathbb{F}_2^n} f(x - y)g(y).
\]

Our main result:

**Theorem 1:** Consider the set $F \subset [0, 1] \times [1, 2]$
\[
F = \{(\delta, p) : p \geq 1 + (1 - 2\delta)^2, 0 \leq \delta \leq 1, 1 < p \leq 2\}.
\]
For every compact subset $K$ of $F$ there exists a constant $C = C(K)$ such that for all $(\delta, p) \in K$, $n \geq 1$ and $f : \mathbb{F}_2^n \to \mathbb{C}$ we have

$$\|T_\delta f\|_2 \leq C \|f\|_p.$$  

(7)

Conversely, for any $(\delta, p) \notin F$ there is $E > 0$ such that

$$\sup_f \frac{\|T_\delta f\|_2}{\|f\|_p} \geq e^{nE + o(n)}, \quad n \to \infty$$

(8)

with the exception of $\delta = 1/2, p = 1$ for which we have

$$\sup_f \frac{\|T_{1/2} f\|_2}{\|f\|_1} = 2^{n/2} \left( \frac{n}{n/2} \right)^{-\frac{1}{2}} \sim \left( \frac{\pi n}{2} \right)^{\frac{1}{2}}.$$  

(9)

**Remark:** The constants that our proof yields are as follows: for $\delta \leq 0.174$ we have $C = \sqrt{2}$, while for larger $\delta$ we can take $C$ to be arbitrary close to $\sqrt{2}$ for sufficiently large $n$. Note also that the constant cannot be tightened to 1. Indeed, taking $f = 1_{\text{even}}$ to be the characteristic function of the set of all even-weight vectors we get

$$\|T_\delta\|_{p \to 2} \geq 2^{1-\frac{1}{p}}, \quad 1 \leq p \leq 2, 0 < \delta < 1,$$

regardless of dimension $n$.

There are a number of applications of hypercontractive inequalities in information theory [3], theoretical computer science [4] and probability [5]. In particular, a very simple argument, cf. [5, Theorem 3.7], shows that a discrete time finite Markov chain with state space $X$ which satisfies hypercontractive inequality mixes in time of order $O(\log \log |X|)$. For $T_\delta$, this Markov chain is a non-standard random walk on a hypercube $\mathbb{F}_2^n$ which jumps by a distance exactly $\delta n$ at each step. A simple coupling argument shows that indeed such a random walk must mix in time $O(\log n)$. This gives a probabilistic intuition to Theorem [1].

Our original interest in hypercontractivity was motivated by a remarkably simple solution it yields to a problem that the author attempted to solve using more conventional semi-definite programming (SDP), compare Sections IV in [6] and [7]. Here is an application of the new result (Theorem [1]) similar in spirit:

**Corollary 2:** For every $\epsilon \in (0, 1)$ there are constants $C_1, C_2 > 0$ such that for any dimension $n$ and any set $A \subset \mathbb{F}_2^n$ with the property that the average multiplicity of $A + A$ on at least some
Hamming sphere $S_j$ with $\epsilon n \leq j \leq (1-\epsilon)n$ exceeds $\lambda |A|$ implies that cardinality $|A| \geq C_1 \lambda^{C_2} 2^n$.

Formally:
\[
\sup_{j \in [\epsilon n, (1-\epsilon)n]} \frac{2^n (1_A * 1_A, 1_{S_j})}{|S_j| |A|} \geq \lambda \implies |A| \geq C_1 \lambda^{C_2} 2^n
\]

**Remark:** It is known that any linear subspace $V \subset \mathbb{F}_2^n$ which contains a $\Omega(1)$-fraction of any $S_{\delta n}$ must have codimension $O(1)$ (in $n \to \infty$). This corollary is a generalization: if a sumset $A + A$ contains a $\lambda$-fraction of any Hamming sphere $S_j$ (counted with multiplicity normalized by $|A|$) then a set must be of cardinality $\Omega(2^n)$.

**Proof:** For later reference we prove a stronger statement:
\[
\left( \phi * \phi, \frac{1_{S_j}}{|S_j|} \right) \geq \lambda \|\phi\|_2^2 \implies \|\phi\|_2^2 \leq \frac{1}{C_1} \lambda^{-C_2},
\]
from which the result follows by taking $\phi = 1_A$. To show (10) denote $\delta = \frac{j}{n}$ and consider the chain
\[
\lambda \|\phi\|_2^2 \leq \left( \phi * \phi, \frac{1_{S_j}}{|S_j|} \right) \leq (\phi, T_\delta \phi) \leq \|\phi\|_2 \|T_\delta \phi\|_2 \leq C \|\phi\|_2 \|\phi\|_p, \quad p = 1 + (1 - 2\epsilon)^2 < 2
\]
\[
\leq C \|\phi\|_2 \|\phi\|_2^{\frac{2}{p} - 1} \|\phi\|_2^{\frac{2}{p} - 2}
\]
where (13) is Cauchy-Schwartz, (14) is from Theorem 1 and (15) is from log-convexity of $\frac{1}{p} \mapsto \|\phi\|_p$. Rearranging terms yields (10).

Intuitively, a much more natural approach to proving the Corollary would be to apply the harmonic-analytic method (or linear programming, or SDP) of Delsarte, cf. [8]. Somewhat surprisingly, such a method works but only for sufficiently large values of $\lambda$. We illustrate this issue briefly below.

For $\lambda \in (0, 1]$ let us say that a distribution $\phi$ $\lambda$-approximates deconvolution of an i.i.d. Bern($\delta$) random variable $Z$ if
\[
\mathbb{P}[X + X' = Z] \geq \lambda \mathbb{P}[X + X' = 0],
\]
where $X, X'$ and $Z$ are independent and $X$ and $X'$ are distributed according to $\phi$. The idea here is that when $\lambda \to 1$ then the distribution of $X + X'$, whose highest peak necessarily occurs at 0,
is almost flat on the set where the $Z$ concentrates. The goal is to find maximally non-uniform \( \phi \) which becomes a \( \lambda \)-approximation to $Z$ after self-convolution. Formally:

\[
V_n(\lambda) = \max_{\phi \text{ sat. } \|\phi\|_1} \frac{\|\phi\|_2^2}{\|\phi\|_1^4}.
\]

Note that for \( \phi = \frac{1}{|A|} \) this corresponds to minimizing cardinality \(|A|\).

It turns out that regardless of dimension $n$ the value of $V_n(\lambda)$ is bounded by a constant. That is, every set $A$ deconvolving Bern($\delta$) is of cardinality $c_\lambda \cdot 2^n$. Obtaining this result from Bonami-Gross hypercontractivity (4) is very simple. To that end define $B_\delta(x) = \delta^{\|x\|} (1 - \delta)^{n-\|x\|}$ to be a distribution function of iid Bernoulli noise. Then, we have

\[
V_n(\lambda) = \max_{\phi \geq 0} \frac{(\phi, \phi)}{(\phi, 1)^2} \geq \lambda \|\phi\|_2^2
\]

(17)

The argument entirely similar to (13)-(15) invoking Bonami-Gross (4) instead of Theorem 1 demonstrates

\[
V_n(\lambda) \leq \lambda^{-s}
\]

for some \( s > 0 \) and all dimensions $n$.

Note that the problem in (17) is completely “$L_2$” and thus escaping to $L_p$ space in order to solve it looks somewhat artificial. Indeed, a much more natural approach would be to apply Fourier analysis or SDP relaxation. The Fourier-analytic approach (or the “spectral gap”) yields the following bound on $V_\lambda$: Since the second-largest eigenvalue of $N_\delta$ equals \((1 - 2\delta)\) we get

\[
(\phi_0, N_\delta \phi_0) \leq (1 - 2\delta) \|\phi_0\|,
\]

where $\phi_0 = \phi - (\phi, 1)$. Simple manipulations then imply

\[
V_n(\lambda) \leq \frac{2\delta}{\lambda - (1 - 2\delta)}, \quad \text{if } \lambda > (1 - 2\delta).
\]

This proves a correct estimate of $O(1)$ but only for large values of $\lambda$.

An improvement of this method comes with the use of an SDP relaxation. The latter is obtained by considering $\psi = \phi \ast \phi$ and retaining only the non-negative definiteness property of $\psi$. I.e. we
have the following upper bound:

\[ V_n(\lambda) \leq SDP(n, \lambda) \triangleq \max_{\psi \geq 0} \frac{2^n (\psi, B_0)}{(\psi, 1)}, \]

where \( B_0(x) = 1\{x = 0\} \) and \( \psi \geq 0 \) denotes that \( f \mapsto f \ast \psi \) is a non-negative definite operator. It can be shown that

\[ SDP(n, \lambda) = O(1), \quad \lambda > (1 - 2\delta)^2, \]

while for smaller values of \( \lambda \) \( SDP(n, \lambda) \) grows polynomially in \( n \). Thus SDP is unable to yield the correct estimate of \( V_n(\lambda) \) for the entire range of \( \lambda \). This example demonstrates that hypercontractivity may prove useful (in fact more powerful than SDP) even for questions that are entirely “\( L_2 \)”. 

Before delving into the proof of Theorem 1 we mention that traditional comparison techniques, cf. [9], for proving hypercontractive and log-Sobolev inequalities are not effective here. One problem is that our operators \( T_\delta \) do not form a semigroup. This may potentially be worked around by applying the discrete-time version of log-Sobolev inequalities developed by Miclo [10]. However, the natural comparison to \( N_\delta \) via Miclo’s method is unfortunately not useful: the primary reason is that the log-Sobolev constant of \( N_\delta \) is of order \( \frac{1}{n} \) which implies tight hypercontractive estimates when \( \delta \sim \frac{1}{n} \) and is very loose otherwise. 

Nevertheless, a direct comparison of \( T_\delta \) and \( N_\delta \) can still yield useful results

**Theorem 3:** For any \( \delta \) and \( p \geq 1 + (q - 1)(1 - 2\delta)^2 \) we have

\[ \|T_\delta\|_{p \rightarrow q} = O(\sqrt{n}). \]

**Proof:** Assuming without loss of generality that \( f \geq 0 \) it is easy to see from Stirling’s formula that

\[ \frac{1}{n} \sum_{|y| = \delta n} f(x + y) \leq O(\sqrt{n}) \sum_{|y| = \delta n} f(x + y)\delta^{|y|}(1 - \delta)^{n-|y|}. \]

Then extending summation to all of \( y \) we get

\[ T_\delta f(x) \leq O(\sqrt{n})N_\delta f(x) \quad \forall x \in \mathbb{F}_2^n. \]

\(^1\)These observations were made in collaboration with Prof. A. Megretski.
The result then follows from (4).

The main part of Theorem 1 is thus in establishing a constant estimate on \( \| T_\delta \|_{p \to q} \). Our proof compares the eigenvalues of \( T_\delta \) and \( N_\delta \) but also crucially depends on peculiar relation between norms of certain Fourier-multiplier operators on \( \mathbb{F}_2^n \) and eigenvalues of \( T_\delta \). Those estimates perhaps are of independent interest as they bound energies in the degree-a components of functions on the hypercube.

Finally, we close our discussion with mentioning the result of Semenov and Shneiberg [11]. Note that one of the most fascinating properties of (4) is that it shows the following “stickiness at 1” of \( \| \cdot \|_{p \to q} \) norms: as operator \( N_\delta \) starts to depart from the \( N_{\frac{1}{2}} \) the norm \( \| N_\delta \|_{p \to q} \) remains stuck at 1 for some range of values \( \delta \in [\delta_0, \frac{1}{2}] \) before starting to grow as \( \delta < \delta_0 \). This distinguishes the measure of dependence \( \| \cdot \|_{p \to q} \) from other measures (such as mutual information, or correlation coefficients). Interestingly, a similar effect was observed for Fourier multiplier operators and norms \( \| \cdot \|_{p \to p} \) and \( \| \cdot \|_{2 \to q} \) in [12], [13]. Semenov and Shneiberg showed this in general: If \( T \) is any operator with \( \| T \|_{p \to q} < \infty \) then in some neighborhood of \( \epsilon = 0 \) we have

\[
\|(1 - \epsilon)E + \epsilon T\|_{p \to q} = 1,
\]

provided that \( E \circ T = T \circ E \), \( T1 = 1 \) and \( (E,f)(x) \triangleq \mathbb{E} [f(X)] \) maps functions to constants. Paired with our Theorem 1 this allows to establish that many permutation-invariant (or \( S_n \)-equivariant) operators in Hamming space have \( L_p \to L_q \) norm equal to 1.

II. PROOF

A. Auxiliary results: Notation

For \( x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n \) define \( \bar{x} \triangleq (1 - x_1, \ldots, 1 - x_n) \). For each \( j = 1, \ldots, n \) let

\[
\chi_j(x_1, \ldots, x_n) \triangleq 1 \{ x_j = 0 \} - 1 \{ x_j = 1 \}.
\]

Define the characters, indexed by \( v \in \mathbb{F}_2^n \),

\[
\chi_v(x) \triangleq \prod_{j: v_j = 1} \chi_j(x) = (-1)^{<v,x>},
\]

where \( <v,x> = \sum_{j=1}^n v_j x_j \) is a non-degenerate bilinear form on \( \mathbb{F}_2^n \). The Fourier transform of \( f : \mathbb{F}_2^n \to \mathbb{C} \) is

\[
\hat{f}(\omega) \triangleq \sum_{x \in \mathbb{F}_2^n} \chi_\omega(x) f(x) = 2^n (f, \chi_\omega), \quad \omega \in \mathbb{F}_2^n.
\]
Lp norms are monotonic
\[ \|f\|_p \leq \|f\|_{p_1}, \quad p \leq p_1. \] (18)
and satisfy the Young inequality:
\[ \|f \ast g\|_p \leq 2^n \|f\|_q \|g\|_r \quad \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}, \quad 1 \leq p, q, r \leq \infty \] (19)
For the size of Hamming spheres we have
\[ |S_{\delta n}| = \binom{n}{\lfloor \delta n \rfloor} = e^{nh(\delta) - \frac{1}{2} \ln n + O(1)}, \quad n \to \infty \] (20)
where the estimate is a consequence of Stirling’s formula, \(O(1)\) is uniform in \(\delta\) on compact subsets of \((0, 1)\) and
\[ h(\delta) = -\delta \ln \delta - (1 - \delta) \ln(1 - \delta). \] (21)
Furthermore, for all \(0 \leq j \leq n\)
\[ e^{nh(\frac{j}{n})} \sqrt{\frac{1}{2n}} \leq |S_j| < e^{nh(\frac{j}{n})} \] (22)
and for \(1 \leq j \leq n - 1\), cf. \([14\text{, Exc. 5.8}]\),
\[ e^{nh(\frac{j}{n})} \sqrt{\frac{n}{8jn(n - j)}} \leq |S_j| \leq e^{nh(\frac{j}{n})} \sqrt{\frac{n}{2\pi j(n - j)}} \] (23)

**B. Auxiliary results: Asymptotics of Krawtchouk polynomials**

Krawtchouk polynomials are defined as Fourier transforms of Hamming spheres:
\[ K_j(x) \triangleq \hat{1}_{S_j}(x) = \sum_{k=0}^{n} (-1)^j \binom{|x|}{k} \binom{n - |x|}{j - k} \] (24)
Since \(K_j(x)\) only depends on \(x\) through its Hamming weight \(|x|\), we will abuse notation and write \(K_j(2)\) to mean value of \(K_j\) at a point with weight 2, etc.

Some useful properties of \(K_j\), cf. \([15]\):
\[ K_j(x) = (-1)^j K_j(n - x) \] (25)
\[ K_j(x) = (-1)^x K_{n-j}(x) \] (26)
\[ \frac{K_j(x)}{K_j(0)} = \frac{K_x(j)}{K_x(0)} \] (27)
\[ K_j(0) = \|K_j\|_2^2 = |S_j| = \binom{n}{j}, \] (28)
\[ K_j(x) = \sum_{|v|=j} \chi_v(x) \] (29)
It is also well-known that $K_j(x)$ has $j$ simple real roots. For $j \leq n/2$ all of them are in the interval \[ \frac{n}{2} - \sqrt{j(n-j)} \leq x \leq \frac{n}{2} + \sqrt{j(n-j)}. \]

Thus for $j = \delta n$ the location of the first root is at roughly
\[ \xi_{\text{crit}}(\delta) \triangleq \frac{1}{2} - \sqrt{\delta(1-\delta)}. \]

The following gives a convenient non-asymptotic estimate of the magnitude of $K_j(x)$:

**Lemma 4:** For all $x, j = 0, \ldots, n$ we have
\[ |K_j(x)| \leq e^{nE_{j/n}(x/n)}, \] (30)

where the function $E_{\delta}(\xi) = E_{1-\delta}(\xi)$ and for $\delta \in [0, 1/2]$:
\[ E_{\delta}(\xi) = \begin{cases} \frac{1}{2} (h(\delta) + \ln 2 - h(\xi)), & \xi_{\text{crit}}(\delta) \leq \xi \leq 1 - \xi_{\text{crit}}(\delta) \\ \phi(\xi, \omega), & \xi = \frac{1}{2} (1 - (1 - \delta)\omega - \delta\omega^{-1}) \end{cases} \] (31)

where in the second case $\omega$ ranges in
\[ \omega \in \left[-\sqrt{\frac{\delta}{1-\delta}}, -\frac{\delta}{1-\delta}\right] \cup \left[\frac{\delta}{1-\delta}, \sqrt{\frac{\delta}{1-\delta}}\right] \]
and
\[ \phi(\xi, \omega) \triangleq \xi \ln|1-\omega| + (1-\xi)\ln|1+\omega| - \delta \ln|\omega|. \] (32)

**Remark:** Exponent $E_{\xi}(\delta)$ was derived in [16] for $\xi \leq \xi_{\text{crit}}(\delta)$. Subsequently, a refined asymptotic expansion for all $\xi \in [0, 1]$ was found in [17]:
\[ K_{\delta n}(\xi n) = O(1) e^{nE_{\xi}(\delta)}, \] (33)
where the $O(1)$ term is $\theta(1)$ for $\xi \leq \xi_{\text{crit}}$, while for $\xi \in [\xi_{\text{crit}}, 1/2]$ the factor $O(1)$ is oscillating and may reduce the exponent for a few integer points $x \in [\xi_{\text{crit}} n, (1 - \xi_{\text{crit}})n]$, which are close to one of the roots of $K_j(\cdot)$.

**Proof:** Following [17] we have
\[ K_j(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \left\{ (1-z)^x (1+z)^{n-x} z^{-j} \right\} \frac{dz}{z}, \] (34)

Note that $K_j(\cdot)$ in [17] corresponds to $(-1)^j K_j(\cdot)$ in this paper.
where integration is over an arbitrary circle $C$ with center at $z = 0$. The derivative of the function in braces is zero when

$$n - 2x = (n - j)z + jz^{-1}. \quad (35)$$

Due to (26) it is sufficient to consider $j \leq n/2$. Among the two solutions of (35) denote by $\omega$ the unique one with smallest $|z|$ and $\Im(z) \geq 0$. Set, for convenience

$$\xi = x/n, \quad \delta = j/n \in [0, 1/2]$$

and note that we have the following relation between $\omega$ and $\xi$

$$\omega = \frac{1}{2(1 - \delta)} \left( 1 - 2\xi - \text{sgn}(1 - 2\xi) \cdot \sqrt{(1 - 2\xi)^2 - 1 + (1 - 2\delta)^2} \right) \quad (36)$$

$$1 - 2\xi = (1 - \delta)\omega + \frac{\delta}{\omega}. \quad (37)$$

As $\xi$ ranges from 0 to 1 the saddle point $\omega$ traverses the path

$$\omega : \frac{\delta}{1 - \delta} \rightarrow \sqrt{\frac{\delta}{1 - \delta}} \rightarrow -\sqrt{\frac{\delta}{1 - \delta}} \rightarrow -\frac{\delta}{1 - \delta},$$

where the middle segment is along the arc $e^{i\phi} \sqrt{\frac{\delta}{1 - \delta}}, \phi \in [0, \pi]$; Corresponding to these corner points the $\xi$ ranges as follows

$$\xi : 0 \rightarrow \xi_{\text{crit}} \rightarrow 1 - \xi_{\text{crit}} \rightarrow 1.$$

It is more convenient to reparameterize the answer in terms of location of the saddle-point $\omega$. If we take $C$ to be the circle passing through $\omega$, then as shown in [17] (3.4) and paragraph after (3.19) the maximum

$$\max_{z \in C} \left| (1 - z)^x(1 + z)^{n - x}z^{-j} \right|$$

is attained at $z = \omega$ and is equal to $e^{nE_\delta(\xi)}$, where

$$E_\delta(\xi) = \phi(\xi, \omega), \quad (38)$$

and $\xi$ is a function of $\omega$ defined via (37). Thus, upper-bounding the integrand $\{\cdot\}$ in (34) by the maximal value and noting that for any circle

$$\oint_C \left| \frac{dz}{z} \right| \leq 2\pi$$

we conclude that (30) holds.
Fig. 1. The exponent of $\frac{K_{\delta n}(\xi)}{K_j(0)}$ is equal $E_\delta(\xi) - h(\delta)$. The figure compares these exponents for two values of $\delta$. Asterisks mark the interval $[\xi_{crit}, 1 - \xi_{crit}]$ containing all the roots of $K_{\delta n}(\cdot)$. In this interval $K_{\delta n}(\cdot)$ is oscillatory.

It remains to show the simplified expression in (31) for $\xi \in [\xi_{crit}, 1 - \xi_{crit}]$. To that end, notice that such $\xi$ corresponds to

$$\omega = e^{i\phi} \sqrt{\frac{\delta}{1 - \delta}}, \quad \phi \in [0, \pi].$$

Substituting this $\omega$ into (38) we see that (31) is equivalent to

$$\xi \ln \frac{|1 - \omega|}{\sqrt{\xi}} + (1 - \xi) \ln \frac{|1 + \omega|}{\sqrt{1 - \xi}} = \frac{1}{2} \ln \frac{2}{1 - \delta}. \quad (39)$$

But for $\omega$ on the arc we have

$$\frac{|1 - \omega|}{\sqrt{\xi}} = \frac{|1 + \omega|}{\sqrt{1 - \xi}} = \sqrt{\frac{2}{1 - \delta}},$$

thus verifying (39) and (31).

For visual illustration of some properties summarized in the next lemma see Fig. 1.

**Lemma 5:** Properties of $E_\delta(\xi)$:

1) $(\delta, \xi) \mapsto E_\delta(\xi)$ is continuous on $[0, 1] \times [0, 1]$ and possesses two symmetries: $E_\delta(\xi) = E_{1-\delta}(\xi)$, $E_\delta(\xi) = E_\delta(1 - \xi)$.
2) $E_\delta(0) = E_\delta(1) = h(\delta)$, $E_\delta(1/2) = h(\delta)/2$
3) $E_{1/2}(\xi) = \ln 2 - h(\xi)/2$
4) $E_\delta(\xi) = h(\delta) - h(\xi) + E_\xi(\delta)$
5) $\xi \mapsto E_\delta(\xi)$ is monotonically decreasing on $[0, 1/2]$ and possesses continuous derivative on $[0, 1]$. 

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6) \( \delta \mapsto E_\delta(\xi) \) is monotonically increasing on \([0, 1/2]\).
7) \( \delta \mapsto E_\delta(\xi) - h(\delta) \) is monotonically decreasing on \([0, 1/2]\).
8) For fixed \( \delta \) and all \( \xi \leq \xi_{\text{crit}}(\delta) \) we have

\[
E_\delta(\xi) \leq \xi \ln(1 - 2\delta) + h(\delta). \tag{40}
\]

**Proof:** None of these properties are used below, so we omit fairly trivial details. \( \blacksquare \)

We will also need a more refined estimate for \( K_j(x) \) when \( x \) is small:

**Lemma 6:** For \( j \leq n/2 \) and \( 0 \leq x \leq n\xi_{\text{crit}}(j/n) = n/2 - \sqrt{n(n-j)} \) we have

\[
\frac{K_j(x)}{K_j(0)} \leq \left( 1 - \frac{2j}{n} \right)^x. \tag{41}
\]

**Remark:** With the additional factor \( O(\sqrt{n}) \) the estimate (41) follows from (40). Lemma 6 establishes the crucial relation between spectra of operators \( N_\delta \) and \( T_\delta \) powering Theorem 1.

**Proof:** In the mentioned range of \( x \) the polynomial \( K_j(x) \) is monotonically decreasing since \( K_j(0) > 0 \) and all roots are to the right of \( x \). Hence,

\[
0 < \frac{K_j(x+1)}{K_j(x)} < 1. \tag{42}
\]

On the other hand, e.g. \([15, (15)]\), \( K_j(\cdot) \) satisfies a three-term recurrence

\[
(n - x)K_j(x+1) - (n - 2j)K_j(x) + xK_j(x-1) = 0. \tag{43}
\]

Dividing by \( nK_j(x) \) we get

\[
\frac{K_j(x+1)}{K_j(x)} = \left( 1 - \frac{2j}{n} \right) - x \left( \frac{K_j(x-1)}{K_j(x)} - \frac{K_j(x+1)}{K_j(x)} \right) \tag{44}
\]

\[
\leq \left( 1 - \frac{2j}{n} \right), \tag{45}
\]

where (45) is from (42). The (41) then follows by iterating (45). \( \blacksquare \)

Note that for \( j \approx \frac{n}{2} \) conditions of Lemma 6 are not satisfied for any \( x \). For such \( j \) we prove another (somewhat loose) estimate below.

**Lemma 7:** Fix arbitrary \( \theta_1 \in (0, 1/2) \). Then for all \( x, j \) such that

\[
n - 2j \leq n\theta_1, \tag{46}
\]

\[
0 \leq x \leq 1 + \frac{\theta_1}{1 + \theta_1^2} (n\theta_1 - (n - 2j)) \tag{47}
\]
we have
\[ \left| \frac{K_j(x)}{K_j(0)} \right| \leq \theta_1^x. \] (48)

**Proof:** Denote \( \theta = 1 - 2\frac{j}{n} \leq \theta_1 \). Clearly (48) holds for \( x = 0 \). From (44) and (46) it also holds for \( x = 1 \). Let the induction hypothesis be that (48) holds for \( x \leq x_0 \). Then
\[ \left| \frac{K_j(x_0 + 1)}{K_j(0)} \right| = \frac{n\theta}{n - x_0} \frac{K_j(x_0)}{K_j(0)} - \frac{x_0}{n - x_0} \frac{K_j(x_0 - 1)}{K_j(0)} \leq \frac{n\theta}{n - x_0} \theta_1^{x_0} + \frac{x_0}{n - x_0} \theta_1^{x_0 - 1}, \] (49)
where (49) is from (43) and (50) is by induction hypothesis. Finally, it is easy to see that whenever \( n - x_0 > 0 \) it holds that
\[ x_0 \leq n \frac{\theta_1}{1 + \theta_1^2} (\theta_1 - \theta) \iff \frac{n\theta}{n - x_0} \theta_1^{x_0} + \frac{x_0}{n - x_0} \theta_1^{x_0 - 1} \leq \theta_1^{x_0 + 1}, \]
which concludes the proof of (48) for \( x = x_0 + 1 \). \( \square \)

On the other extreme, for small values of \( j \) we can extend Lemma 6 to the whole range \( 0 \leq x \leq \frac{n}{2} \):

**Lemma 8:** There exist \( C_1 \geq 1 \) and \( \delta_0 \in (0, 1) \) such that for all \( 0 \leq j \leq \delta_0 n \) we have
\[ \left| \frac{K_j(x)}{K_j(0)} \right| \leq C_1 \cdot \left( 1 - \frac{2j}{n} \right)^x , \quad 0 \leq x \leq \frac{n}{2}. \]

**Remark:** The proof given here yields \( C_1 = 1 \) and \( \delta_0 = 0.174 \).

**Proof:** For \( j = 0 \) the inequality is trivial. For \( x \leq \xi_{\text{crit}}(j/n) \) it follows from Lemma 6. Thus, it is sufficient to consider \( x \geq \xi_{\text{crit}}(j/n) \), \( j \geq 1 \). Denote \( \delta = j/n \). Then from Lemma 4 and (23) we have for all \( n \geq 1 \):
\[ \left| \frac{K_{\delta n}(x)}{K_j(0)(1 - 2\delta)^x} \right| \leq \sqrt{8(1 - \delta)} \cdot e^{x f(\delta) - \frac{1}{2} h(\delta)} \sqrt{n\delta}, \]
where
\[ f(\delta) = \max_{\xi \in [\xi_{\text{crit}}(\delta),1/2]} \frac{1}{2} (ln 2 - h(\xi)) - \xi \ln(1 - 2\delta). \]

From convexity of the function under maximization, we conclude
\[ f(\delta) = \frac{\ln 2}{2} - \frac{1}{2} \min (h(\xi_{\text{crit}}(\delta)) + 2\xi_{\text{crit}}(\delta) \ln(1 - 2\delta), \ln 2(1 - 2\delta)) . \]
Taking derivative at \( \delta = 0 \) we conclude that for some \( \delta_0' > 0 \) we have
\[ h(\xi_{\text{crit}}(\delta)) + 2\xi_{\text{crit}}(\delta) \ln(1 - 2\delta) \leq \ln 2(1 - 2\delta) , \quad \forall \delta \in [0, \delta_0']. \]
Consequently, for such \( \delta \)

\[
f(\delta) = \frac{1}{2} \left( \ln 2 - h(\xi_{\text{crit}}(\delta)) \right) - \xi_{\text{crit}}(\delta) \ln(1 - 2\delta). \]

Evidently, \( f \) is continuously differentiable and

\[
f(\delta) = 2\delta + o(\delta), \quad \delta \to 0.
\]

Therefore for some \( \delta_0 \in (0, \delta'_0] \) we must have

\[
f(\delta) - \frac{1}{2} h(\delta) < 0, \quad \forall \delta \in (0, \delta_0).
\]

The statement of the Lemma then follows with \( C_1 = \max(1, \sqrt{8(1 - \delta_0)C'_1}) \), where \( C'_1 \) is the finite supremum found in the following Lemma.

**Lemma 9:** Let \( \alpha, \delta_0, C > 0 \) and \( f \) — a continuous function on \([0, \delta_0] \) with \( f(0) = 0 \), derivative (one-sided at 0) bounded by \( C \) and satisfying

\[
f(\delta) - \alpha h(\delta) < 0, \quad \forall \delta \in (0, \delta_0].
\]

(51)

where \( h \) is a binary entropy function \((21)\). Then

\[
\sup_n \max_{\delta \in [0, \delta_0]} e^{n(f(\delta) - \alpha h(\delta))} \sqrt{n} \delta < \infty.
\]

**Proof:** Under conditions of the theorem there exists \( 0 < \delta_1 < \delta_0 \) such that

\[
f(\delta) \leq \frac{\alpha}{2} h(\delta), \quad \forall \delta \in [0, \delta_1].
\]

Thus we have

\[
\max_{\delta \in [0, \delta_1]} n(f(\delta) - \alpha h(\delta)) + \frac{1}{2} \ln(\delta n) \leq \frac{1}{2} \max_{\delta \in [0, \delta_1]} -\alpha nh(\delta) + \log \delta n \\
\leq \frac{1}{2} \max_{\delta \in [0, \delta_1]} \alpha n\delta \ln \delta + \ln(\delta n).
\]

(52)

(53)

Without loss of generality we may assume \( \delta_1 < \frac{1}{e} \) and \( n > \frac{e^2}{\alpha} \). In this case, maximization in (53) is attained at \( \delta^* \in (0, \frac{1}{e\alpha}) \). Consequently, upper-bounding the first term by zero and second by \( \ln(\frac{1}{\alpha \cdot n}) \) we get

\[
\frac{1}{2} \max_{\delta \in [0, \delta_1]} \alpha n\delta \ln \delta + \ln(\delta n) \leq -\frac{\ln \alpha}{2}.
\]

(54)

On the other hand, from (51) and continuity we get

\[
\max_{\delta \in [\delta_1, \delta_0]} f(\delta) - \alpha h(\delta) = -C_2 < 0.
\]
Therefore, putting both bounds together
\[
\max_{n \geq 1, \delta \in [0, \delta_0]} e^{n(f(\delta) - \alpha h(\delta))} \leq \max \left( \frac{1}{\sqrt{\alpha}}, \sup_n \sqrt{\delta_0 n e^{-C_2 n}} \right) < \infty.
\]

Finally, for illustration purposes we will need the following

Lemma 10: \( L_p \) norms of Krawtchouk polynomials are given asymptotically by the following parametric formula: Let \( \omega \in [0, 1] \) then for \( p \geq 2 \)
\[
\|K_{[\delta n]}\|_p = \exp \left\{ n \left( \frac{h(\xi) - \ln 2}{p} + \phi(\xi, \omega) \right) + O(\log n) \right\}, \quad n \to \infty
\]

\[
c = \frac{(1 + \omega)^p - (1 - \omega)^p}{(1 + \omega)^p + (1 - \omega)^p}
\]
\[
\xi = \frac{1 - c}{2} = \frac{1}{2}(1 - (1 - \delta)\omega - \delta\omega^{-1})
\]
\[
\delta = \frac{c\omega - \omega^2}{1 - \omega^2}
\]
and \( \phi(\xi, \omega) \) is given by (32). For \( p \leq 2 \) we have
\[
\|K_{[\delta n]}\|_p = \exp \left\{ \frac{n}{2} h(\delta) + O(\log n) \right\}, \quad n \to \infty.
\]

Proof: The lemma is shown by analyzing with exponential precision the expression
\[
\|K_{j}\|_p^p = \sum_{a=0}^{n} 2^{-n} \binom{n}{a} |K_j(a)|^p.
\]
From Lemma 4 it may be shown that for \( p \geq 2 \) the term exponentially dominating this sum occurs at \( a \leq n \xi_{crit}(j/n) \). Then, for such \( a = \xi n \) we know from [16] that \( K_{\delta n}(\xi n) = \exp\{nE_\delta(\xi) + O(\log n)\} \). We omit further details as this Lemma is not used in the proof of Theorem I

C. Auxiliary results: Norms of Fourier projection operators

The Fourier projection operators \( \Pi_a \) are defined as
\[
\hat{\Pi}_a f \overset{\triangle}{=} \hat{f} \cdot 1_{S_a}, \quad a = 0, 1, \ldots, n,
\]
or, equivalently,
\[
\Pi_a f \overset{\triangle}{=} 2^{-n} f * K_a.
\]
On the other hand from Young’s inequality [19] we have for any convolution operator:
\[
\|\phi * (\cdot)\|_{1 \to 2} = 2^n \|\phi\|_2.
\]
Thus we have
\[ \|\Pi_a\|_{1\to 2} = \sqrt{\frac{n}{a}}. \] (59)

Also, we note that
\[ \|\Pi_a\|_{p\to q} = \|\Pi_{n-a}\|_{p\to q}, \]
and thus we only consider \( a \leq \frac{n}{2} \) below. Estimates for other \( L_p \to L_2 \) follow from Bonami-Gross inequality (4) and complex interpolation:

**Lemma 11:** For any \( 1 \leq p \leq 2 \) and \( 0 \leq a \leq \frac{n}{2} \) we have
\[
\|\Pi_a\|_{p\to 2} \leq \begin{cases} 
(p-1)^{-\frac{\delta}{2}}, & p > p^*, \\
(p^* - 1) \left( \frac{(1-s)a}{2} \right)^{\frac{s-\delta}{2}}, & \frac{1-s}{p^*} + s, 0 \leq s \leq 1
\end{cases}
\] (60)

where \( p^* = p^*(a) = 2 \) if \( \frac{h(\delta)}{\delta} \leq 2 \), and otherwise \( p^* \in (1, 2) \) is a solution of
\[ p^* - \ln(p^* - 1) = \delta^{-1} h(\delta), \quad \delta = \frac{a}{n}. \]

We also have two weaker bounds
\[
\|\Pi_a\|_{p\to 2} \leq (p-1)^{-\frac{\delta}{2}} \quad (61)
\]
\[
\|\Pi_a\|_{p\to 2} \leq \left( \frac{n}{a} \right)^{\frac{\delta}{2}} \quad (62)
\]

**Remark:** The estimate (61) has been the basis of Kahn-Kalai-Linial results [4], so we refer to (61) as KKL bound. Note that \( p^*(a) = 2 \) corresponds to \( a > 0.3093n \), and then bound (60) coincides with (62).

**Proof:** From Riesz-Thorin interpolation, we know that the map \( \frac{1}{p} \mapsto \|\Pi_a\|_{p\to 2} \) is log-convex. Thus (60) follows from (61) and (62) by convexification (the value of \( p^* \) is chosen to minimize the resulting exponent when \( a = \delta n \)). Thus, it is sufficient to prove (61) and (62). The second one again follows from interpolating between (59) and \( \|\Pi_a\|_{2\to 2} = 1 \). For the first one notice that for any \( \tau \) we have
\[ N_\tau \Pi_a = \Pi_a N_\tau = (1 - 2\tau)^a \Pi_a. \]

And thus from (4) with \( (1 - 2\tau)^2 = p - 1 \) we get
\[ \|\Pi_a f\|_2 = |1 - 2\tau|^{-a} \|\Pi_a N_\tau f\|_2 \leq |1 - 2\tau|^{-a} \|N_\tau f\|_2 \leq |1 - 2\tau|^{-a} \|f\|_p. \]
Fig. 2. Exponent of $\|\Pi_a\|_{p \to 2}$ as a function of $a$ for two values of $p$. Two upper bounds correspond to Kahn-Kalai-Linial \cite{61} and the interpolated one \cite{60}. The lower bound is given by considering only permutation invariant functions (cf. Lemmas \cite{10} and \cite{12}).

To verify the tightness of our bounds we derive a simple lower bound by considering permutation invariant functions:

**Lemma 12:** For any $a \in \{0, \ldots, n\}$ and any $q, p \geq 1$ we have

$$\|\Pi_a\|_{p \to q} \geq \frac{\|K_a\|_q \|K_a\|_{p'}}{\|K_a\|_2},$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate.

**Proof:** The lower bound is shown by optimizing over a class of permutation invariant functions

$$f(x) = K_a(x) + \sum_{j \neq a} c_j K_j(x) \overset{\triangle}{=} K_a(x) + \Phi(x),$$

where $\Phi \perp K_a$. Note that

$$\inf_{\Phi \perp K_a} \|f\|_p = \inf_{\Phi \perp K_a} \sup_{g: \|g\|_{p'} \leq 1} (K_a + \Phi, g)$$

$$= \inf_{\Phi \perp K_a} \sup_{g - \text{sym.}: \|g\|_{p'} \leq 1} (K_a + \Phi, g) \quad \text{(64)}$$

$$= \sup_{g - \text{sym.}: \|g\|_{p'} \leq 1} \inf_{\Phi \perp K_a} (K_a + \Phi, g)$$

$$= \left( K_a, \frac{K_a}{\|K_a\|_{p'}} \right) = \frac{\|K_a\|_2^2}{\|K_a\|_{p'}},$$

where (63) is by duality $(L_p)^* = L_{p'}$, (64) states the obvious fact that supremization can be restricted to permutation-symmetric $g$, (65) is by von Neumann minimax theorem and (66) is
because the inner inf can only be finite if \( g \) belongs to the one-dimensional subspace spanned by \( K_a \), i.e. \( g = cK_a \) for a suitable \( c \).

Since \( \Pi_a(K_a + \Phi) = K_a \) we conclude that

\[
\|\Pi_a\|_{p\to q} \geq \frac{\|K_a\|_q}{\inf_{\Phi \perp K_a} \|K_a + \Phi\|_p} = \frac{\|K_a\|_q \|K_a\|_p'}{\|K_a\|_2^2}
\]

as claimed.

On Fig. 2 we compare the upper and lower bounds on \( \|\Pi_a\|_{p\to 2} \) as \( a \) ranges from 0 to \( n/2 \) for two values of \( p \). We note that KKL bound (61) is significantly suboptimal for small \( p \) and large \( a \). For example, for \( a > 0.3093n \) the bound (62) is strictly better than KKL.

D. Proof of Theorem 1

Denote the boundary of \( F \) as

\[
p(\delta) \triangleq 1 + (1 - 2\delta)^2.
\]

Note that every compact subset \( K' \) of \( F \) is contained in \( F \cap \{p \geq p_0\} \) for sufficiently small \( p_0 \) and in turn in some

\[
K = (F \cap \{\delta : |1 - 2\delta| \geq \theta\}) \cup \{(\delta, p) : |1 - 2\delta| \leq \theta, p \geq p_0\}
\]

for sufficiently small \( \theta \). In particular, we may choose \( \theta \) so small that \( p_0 > 1 + \theta^2 \). Next note that

\[
(f * 1_{S_{n-a}})(x) = (f * 1_{S_a})(\bar{x})
\]

and thus estimates for \( T_\delta \) and \( T_{1-\delta} \) coincide asymptotically. Due to this symmetry and thanks to the monotonicity (18) of norms, to prove a theorem it is sufficient to prove the following pair of statements, corresponding to the boundary of \( K' \):

S1. (critical estimate for \( \delta < 1/2 \)) For each \( \delta \) there is \( C_\delta \) such that for all \( n \geq 1 \) and all functions \( f \) we have

\[
\|T_\delta f\|_2 \leq C_\delta \|f\|_{p(\delta)}, \tag{68}
\]

and function \( \delta \mapsto C_\delta \) is bounded on each \([0, \Delta], \Delta < 1/2\).

S2. (subcritical estimate around \( \delta = 1/2 \)) For any \( p > 1 \) and sufficiently small \( \theta \) (in particular, \( p > 1 + \theta^2 \)) there is \( C \) such that for all \( \delta \in [(1 - \theta)/2, 1/2], n \geq 1 \) and functions \( f \) we have

\[
\|T_\delta f\|_2 \leq C \|f\|_p \tag{69}
\]
Asymptotic spectra of $T_δ$ and $N_δ$: $δ = 0.1$

Asymptotic spectra of $T_δ$ and $N_δ$: $δ = 0.25$

Fig. 3. Comparison of exponents of $a$-th eigenvalue of $T_δ$ and $N_δ$. For larger $δ$ we also show the negative of the exponent of $\|\Pi_a\|_{p(δ)} → 2$. $p(δ) = 1 + (1 - 2δ)^2$. As before asterisks denote the critical value $ξ_{crit}(δ)$, i.e. the smallest root of Krawtchouk polynomial $K_{δn}(·)$.

First we show S1. In accordance with (24)

$$\|T_δ f\|_2^2 = \sum_{a=0}^{n} \left| \frac{K_{δn}(a)}{K_{δn}(0)} \right|^2 \|f_a\|_2^2,$$

(70)

where we denoted

$$f_a \triangleq \Pi_a f .$$

The scheme of our proof is illustrated by Fig. 3:

1) First, we show that summation in (70) can be truncated to $a \leq \frac{n}{2}$.

2) Second, we show that for small values of $δ$ eigenvalues of $T_δ$ are upper-bounded by a constant multiple of eigenvalues of $N_δ$ defined in (3). This is the content of Lemma 8.

3) Third, for larger values of $δ$ we show that although eigenvalues of $T_δ$ can be exponentially larger than those of $N_δ$, such eigenvalues correspond to large $a$ for which $\frac{\|f_a\|_2^2}{\|f\|_p}$ is exponentially smaller.

For the first step note that any $f$ can be written as

$$f = f_{even} + f_{odd},$$

where each of the summands is supported on vectors $x \in F_2^n$ of even/odd weight. Note that $T_δ f_{even}$ and $T_δ f_{odd}$ are also of opposite parity. Thus,

$$\|T_δ f\|_2^2 = \|T_δ f_{even}\|_2^2 + \|T_δ f_{odd}\|_2^2 .$$
On the other hand, we have
\[
\left(\|f_{\text{even}}\|_p^2 + \|f_{\text{odd}}\|_p^2\right)^{\frac{1}{2}} \leq \left\|\sqrt{f_{\text{even}}^2 + f_{\text{odd}}^2}\right\|_p
\]
(71)
\[
= \|f\|_p ,
\]
(72)
where (71) is from Minkowski’s inequality and (72) is because the supports of \(f_{\text{even}}\) and \(f_{\text{odd}}\) are disjoint (we also assume, without loss of generality that \(f \geq 0\)). Thus, if (68) is established for both odd and even functions then (68) follows for all functions with the same constant \(C\).

Note that for both odd and even functions we have
\[
|\hat{f}(\omega)| = |\pm \hat{f}(\bar{\omega})| = |\hat{f}(\bar{\omega})| .
\]
and for any such \(f\) from (70) and (25) we get
\[
\|T_\delta f\|_2^2 \leq 2 \sum_{0 \leq a \leq n/2} \left|\frac{K_{\delta n}(a)}{K_{\delta n}(0)}\right|^2 \|f_a\|_2^2 .
\]
(73)

In the remaining we show that (73) is upper-bounded by \(C\|f\|_{p(\delta)}\) uniformly in \(f\) and \(\delta \leq \Delta < 1/2\). For all \(\delta \in [0, \delta_0]\) from Lemma 8 we have
\[
\|T_\delta f\|_2^2 \leq 2C_1^2 \sum_{0 \leq a \leq n/2} (1 - 2\delta)^{2a} \|f_a\|_2^2
\]
(74)
\[
\leq 2C_1^2 \sum_{0 \leq a \leq n/2} (1 - 2\delta)^{2a} \|f_a\|_2^2
\]
(75)
\[
= 2C_1^2 \|N_\delta f\|_2^2
\]
(76)
\[
\leq 2C_1^2 \|f\|_{p(\delta)}^2 ,
\]
(77)
where the last step follows from Bonami-Gross (4). For \(\delta \in [\delta_0, \Delta]\) we have from Lemma 6
\[
\left|\frac{K_{\delta n}(a)}{K_{\delta n}(0)}\right| \leq (1 - 2\delta)^a , \quad 0 \leq a \leq n\xi_{\text{crit}}(\delta) .
\]
(78)

On the other hand, for \(a \in [n\xi_{\text{crit}}(\delta), n/2]\) we have the following estimate:

**Lemma 13:** Fix arbitrary \(0 < \delta_0 < \Delta < 1/2\). Then there exist constants \(C_1, C_2 > 0\) such that for all \(n \geq 1\), all \(j \in [\delta_0 n, \Delta n]\) and all
\[
\frac{n}{2} - \sqrt{n(n - j)} \leq x \leq \frac{n}{2} + \sqrt{n(n - j)}
\]
we have
\[
\left|\frac{K_j(x)}{K_j(0)}\right| \cdot \|\Pi_j\|_p(\Delta n) \leq C_1 \sqrt{n} e^{-C_2n}
\]
(79)
where \( p(\delta) = 1 + (1 - 2\delta)^2 \).

Lemma is proved at the end of the section.

Putting together (77) and (79) we get similar to (77):

\[
\|T_\delta f\|_2^2 \leq 2\|N_\delta f\|_2^2 + 2\|f\|_p^2 \sum_{a \in [n \epsilon_{\text{crit}}(\delta), n/2]} (C_1)^2 ne^{-2C_2n} 
\]

where in the last step we applied (4). Since constants \( C_1 \) and \( C_2 \) only depend on \( \delta_0 \) and \( \Delta \) we finish the proof of (68) and of statement S1.

We proceed to statement S2. Showing (69) is significantly simpler since \( p > p(\delta) \) this time. Take \( \theta_1 = \sqrt{p - 1} > \theta \) and \( \delta_1 = \frac{1 - \theta_1}{2} \). Then, for all \( 0 \leq a \leq n\xi_1 \triangleq n\frac{\theta_1}{1 + \theta_1^2}(\theta_1 - \theta) \)
and all \( \delta \in [\frac{1 - \theta}{2}, \frac{1}{2}] \) we have from Lemma 7:

\[
\left| \frac{K_j(a)}{K_j(0)} \right| \leq (1 - 2\delta_1)^a.
\]

Thus, from (4) we get

\[
\sum_{a \in [0, n\xi_1]} \left| \frac{K_j(a)}{K_j(0)} \right|^2 \|f_a\|_2^2 \leq \|N_{\delta_1} f\|_2^2 \leq \|f\|_p^2.
\]

On the other hand, for \( a > n\xi_1 \) we have for some \( C_1, E > 0 \):

\[
\left| \frac{K_j(a)}{K_j(0)} \right| \cdot \frac{\|f_a\|_2}{\|f\|_p} \leq C_1 \sqrt{ne^{-nE}}, \quad \forall a \in [n\xi_1, n/2]
\]

Indeed, from Lemma 4 and (62) the exponent of the left-hand side of (84) is upper-bounded by

\[
\frac{1}{2} (\ln 2 - h(\delta)) + \left( \frac{1}{p} - 1 \right) h(\xi), \quad \xi \triangleq \frac{a}{n}.
\]

The largest value is attained when \( \delta = \frac{1 - \theta}{2} \) and \( \xi = \xi_1 \), yielding

\[
\frac{1}{2} (\ln 2 - h(\delta)) + \left( \frac{1}{p} - 1 \right) h(\xi) \leq \frac{1}{2} \left( \ln 2 - h\left( \frac{1 - \theta}{2} \right) \right) + \left( \frac{1}{p} - 1 \right) h\left( \frac{\theta_1(1 - \theta)}{1 + \theta_1^2} \right).
\]

Since \( p > 1 \) as \( \theta \to 0 \) the function on the right-hand side becomes negative. Thus the exponent of left-hand side in (84) is negative for sufficiently small \( \theta \).
Estimating the sum in (73) via (83) and (84) we get similar to (82) that
\[ \|T_\delta f\|_2^2 \leq 2(1 + (C_1)^2 n^2 e^{-2E_0}) \|f\|_p^2 \quad \forall \delta \in \left[ \frac{1 - \theta}{2}, \frac{1}{2} \right]. \]
This completes the proof of (69) and statement S2.

We proceed to lower bounds on \( \|T_\delta\|_{p \to 2} \). To show (8) consider function
\[ f(x) = \prod_{j=1}^{n} (1 + \epsilon \chi_j) = \sum_{t=0}^{n} (1 + \epsilon)^{n-t} (1 - \epsilon)^t 1_{S_t} = \sum_{k=0}^{n} \epsilon^k K_k(x). \]
On one hand,
\[ \|f\|_p = \left( \frac{(1 + \epsilon)^p}{2} + \frac{(1 - \epsilon)^p}{2} \right)^{\frac{1}{p}} = e^{\frac{n}{p-1} \epsilon^2 + o(\epsilon^2)}, \quad \epsilon \to 0 \]  
(85)
On the other hand, from Lemma 4 and (33) we have
\[ \|T_\delta f\|_2^2 = \sum_{a=0}^{n} e^{2n \left( E_\delta(\frac{\xi}{n}) - h(\delta) + \frac{\xi}{n} \ln \epsilon + \frac{1}{2} h(\frac{\xi}{n}) + o(n) \right)}, \]  
(87)
where we also used
\[ \|f_a\|_2 = \epsilon^a \left( \frac{n}{a} \right)^{\frac{1}{4}} = e^{a \ln \epsilon + nh(\frac{\xi}{n}) + o(n)}. \]
For convenience, set \( \xi = \frac{a}{n} \). Then it is not hard to show from (31) that
\[ E_\delta(\xi) - h(\delta) = \xi \ln(1 - 2\delta) + o(\xi). \]
Then setting \( \xi = \epsilon^2 (1 - 2\delta)^2 \) we find that
\[ E_\delta(\xi) - h(\delta) + \xi \ln \epsilon + \frac{1}{2} h(\xi) = \frac{(1 - 2\delta)^2}{2} \epsilon^2 + o(\epsilon^2), \quad \epsilon \to 0 \]
Thus from (87) and (86) we get
\[ \liminf_{n \to \infty} \frac{1}{n} \ln \frac{\|T_\delta f\|_2}{\|f\|_p} \geq \frac{(1 - 2\delta)^2 - (p - 1)}{2} \epsilon^2 + o(\epsilon^2). \]
Evidently, for \( p < 1 + (1 - 2\delta)^2 \) the norm \( \|T_\delta\|_{p \to 2} \) grows exponentially in dimension.

Finally, estimate (9) follows from Young’s inequality (19):
\[ \|T_{1/2} f\|_2 \leq 2^n \|f\|_1 \left( \left\| \frac{1_{S_{n/2}}}{|S_{n/2}|} \right\| \right) \]  
(88)
\[ = 2^n \cdot \left( 2^{-n/2} \left( \frac{n}{\lfloor n/2 \rfloor} \right)^{-1/2} \right) \|f\|_1 \]  
(89)
\[ = (1 + o(1)) \left( \frac{\pi n}{2} \right)^{\frac{1}{4}} \|f\|_1 \]  
(90)
This upper-bound is tight as \( f(x) = 1\{x = 0\} \) shows.

**Proof of Lemma 13**: Let \( \xi = \frac{a}{n} \) and \( \delta = j/n \). Since \( \xi \) is restricted to critical strip of Krawtchouk polynomial \( K_{\delta_0} \) from Lemma 4, bound (23) and Lemma 11 it is sufficient to show

\[
\max_{\delta_0 \leq \delta \leq \Delta} \max_{\xi:(1-2\xi)^2+\delta^2\leq 1} \frac{1}{2}(\ln 2 - h(\xi) - h(\delta)) + \pi(p(\delta), \xi) \leq -C_2 < 0, \tag{91}
\]

where \( p(\delta) = 1 + (1 - 2\delta)^2 \) and

\[
\frac{1}{p} \mapsto \pi(p, \xi)
\]

is the convexification of the function (cf. Lemma 11)

\[
\frac{1}{p} \mapsto \min \left\{ -\frac{\xi}{2} \ln(p - 1), \frac{1}{p} - \frac{1}{2}h(\xi) \right\}. \tag{92}
\]

To show (91) we first reparameterize the problem in terms of \( p \). Set

\[
p_0 = 1 + (1 - 2\Delta)^2, \tag{93}
\]

\[
p_1 = 1 + (1 - 2\Delta)^2. \tag{94}
\]

Then (91) is equivalent to (we also interchange the maxima in \( \xi \) and \( \delta \)):

\[
\max_{\xi:(1-2\xi)^2\leq 2-p_0} \max_{p:p_0\leq p\leq \min(p_1, 2-(1-2\xi)^2)} \eta(\xi, p) + \frac{\ln 2 - h(\xi)}{2} \leq -C_2 < 0 \tag{95}
\]

where

\[
\eta(\xi, p) \triangleq \pi(p, \xi) - \frac{1}{2}h \left( \frac{1 - \sqrt{p-1}}{2} \right).
\]

By construction, \( \frac{1}{p} \mapsto \pi(p, \xi) \) is convex. A simple verification shows that the second term \( h(\cdots) \) is concave in \( \frac{1}{p} \). Thus, the maximization over \( p \) in (95) is applied to a convex function and therefore must be achieved at one of the boundaries. Consequently, verifying (95) is equivalent to showing the following three strict inequalities, the maximum of which is taken to be \(-C_2\):

\[
\max_{\xi:(1-2\xi)^2\leq 2-p_0} \eta(\xi, p_0) + \frac{\ln 2 - h(\xi)}{2} < 0 \tag{96}
\]

\[
\max_{\xi:(1-2\xi)^2\leq 2-p_1} \eta(\xi, p_1) + \frac{\ln 2 - h(\xi)}{2} < 0 \tag{97}
\]

\[
\max_{\xi:2-p_1\leq (1-2\xi)^2\leq 2-p_0} \eta(\xi, 2 - (1 - 2\xi)^2) + \frac{\ln 2 - h(\xi)}{2} < 0 \tag{98}
\]

The first two are verified as follows: From (92) we have

\[
\pi(\xi, p) \leq -\frac{\xi}{2} \ln(p - 1).
\]
Thus optimization in (96) requires solving
\[
\max_{\xi: (1 - 2\xi)^2 \leq 2 - p} \left\{ -\frac{\xi}{2} \ln(p - 1) - \frac{1}{2} h(\xi) \right\}.
\]
Taking derivatives one finds that optimal $\xi^*(p) = 1 - \frac{1}{p}$ and it satisfies the constraint whenever $p > 1$. Consequently, substituting $\xi = \xi^*(p)$ we get
\[
\max_{(1 - 2\xi)^2 \leq 2 - p} \eta(\xi, p) + \frac{\ln 2 - h(\xi)}{2} \leq -\frac{\xi^*(p)}{2} \ln(p - 1) \leq \ln 2 - h(\xi^*(p)) - h \left( \frac{1 - \sqrt{p - 1}}{2} \right)
\]
Explicit function of a single variable $p$ on the right is continuous and can be verified to be non-positive and attain zero only at the edges of $p \in [1, 2]$. Since both $p_0$ and $p_1$ belong to the interior of $[1, 2]$, this completes the proof of (96) and (97).

To show (98) we apply the bound in (92) (without convexification):
\[
\max_{\xi} \eta(\xi, 2 - (1 - 2\xi)^2) + \frac{\ln 2 - h(\xi)}{2} \leq \max_{\xi} \frac{1}{2} f(\xi) \tag{99}
\]
where maximization is over
\[
2 - p_1 \leq (1 - 2\xi)^2 \leq 2 - p_0 \tag{100}
\]
and $f(\xi)$ is defined as
\[
f(\xi) \triangleq \min \left\{ \left( \frac{(1 - 2\xi)^2}{2 - (1 - 2\xi)^2} \right) h(\xi), -\xi \ln(4\xi(1 - \xi)) \right\} + \ln 2 - h(\xi) - h \left( \frac{1}{2} - \sqrt{\xi(1 - \xi)} \right) \tag{101}
\]
It may be shown that the minimum in this expression selects the first term for $\xi \in [\xi^*, 1 - \xi^*]$ and second term otherwise, where $\xi^*$ is the solution of
\[
8\xi(1 - \xi) \ln \xi + (2\xi - (1 - 2\xi)^2) \ln(1 - \xi) + 2\xi(2 - (1 - 2\xi)^2) \ln 2 = 0.
\]
Furthermore, function in (101) is non-positive, continuous and attains zero only at $\xi = 0, \frac{1}{2}, 1$ all of which are excluded by the constraints (100). Thus (98) holds.

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