TIME ANALYTICITY FOR HEAT EQUATION ON GRADIENT SHRINKING RICCI SOLITONS

JIA-YONG WU

Abstract. On a complete non-compact gradient shrinking Ricci soliton, we prove the analyticity in time for smooth solutions of the heat equation with quadratic exponential growth in the space variable. This growth condition is sharp. As an application, we give a necessary and sufficient condition on the solvability of the backward heat equation in a class of functions with quadratic exponential growth on shrinkers.

1. Introduction

As is well known, generic solutions of the heat equation are usually analytic in space but not analytic in time. In Euclidean space, it is not difficult to construct non-time-analytic solutions of the heat equation in a finite space-time cylinder. It is therefore an interesting task to seek suitable conditions for ensuring the time analyticity of solutions of the heat equation (see [1]). In a recent paper [2], Zhang proved that the ancient solutions with exponential growth in the space variable are time-analytic on a complete non-compact Riemannian manifold with the Ricci curvature bounded below. He also gave a necessary and sufficient condition on the solvability of the backward heat equation in a class of functions with exponential growth. Later, by choosing suitable space-time cutoff functions, Dong and Zhang [3] extended Zhang’s results to the solutions with quadratic exponential growth in the space variable. Meanwhile, they provided an example to indicate that the growth condition is sharp. For more results about time analyticity for parabolic equations, see [1, 5, 6, 7, 8, 9] and references therein.

In this paper, we will study the time analyticity for smooth solutions of the heat equation on a complete non-compact gradient shrinking Ricci soliton (see the definition below). We will prove that the analyticity in time always holds on a fixed gradient shrinking Ricci soliton without any curvature assumption, provided that the solutions have quadratic exponential growth in the space variable. This result may be useful for understanding the function theory of gradient shrinking Ricci solitons.

Date: August 22, 2022.

2010 Mathematics Subject Classification. Primary 53C21; Secondary 35C10, 35K05.

Key words and phrases. gradient shrinking Ricci soliton; heat equation; time analyticity.
Recall that an \( n \)-dimensional Riemannian manifold \((M, g)\) is called a gradient shrinking Ricci soliton \((M, g, f)\) (also called a shrinker for short) (see [10]) if there exists a smooth potential function \( f \) on \( M \) such that

\[
\text{Ric} + \text{Hess} f = \frac{1}{2} g, \tag{1.1}
\]

where Ric is the Ricci curvature of \((M, g)\) and Hess \( f \) is the Hessian of \( f \). Obviously, the flat Euclidean space \((\mathbb{R}^n, \delta_{ij})\) is a gradient shrinking Ricci soliton with potential function \( f = \frac{|x|^2}{4} \), which is called the Gaussian shrinking Ricci soliton \((\mathbb{R}^n, \delta_{ij}, |x|^2/4)\). Shrinkers play an important role in the Ricci flow [10] and in Perelman’s resolutions of the Poincaré Conjecture [11, 12, 13], as they are self-similar solutions and arise as limits of dilations of Type I singularities in the Ricci flow. Over the past two decades, the geometric and analytic properties of shrinkers have become an active issue, as this is useful for understanding the structure of manifolds (see [14] for an excellent survey).

From (1.1), Hamilton [10] observed that

\[
C(g) := R + |\nabla f|^2 - f
\]

is a finite constant, where \( R \) is the scalar curvature of \((M, g)\). Adding \( f \) by a constant, without loss of generality, we assume that

\[
R + |\nabla f|^2 = f. \tag{1.2}
\]

Under this normalization, it is not hard to see that (see also [15])

\[
\int_M (4\pi)^{-\frac{n}{2}} e^{-f} dv = e^\mu, \tag{1.3}
\]

where \( dv \) is the Riemannian volume element on \((M, g)\) and \( \mu = \mu(g, 1) \) is the entropy functional of Perelman [11]. For the Ricci flow, Perelman’s entropy functional is time-dependent, but for a fixed gradient shrinking Ricci soliton it is constant and finite.

Now we give the analyticity in time for smooth solutions of the heat equation on a complete non-compact gradient shrinking Ricci soliton.

**Theorem 1.1.** Let \((M, g, f)\) be an \( n \)-dimensional complete non-compact gradient shrinking Ricci soliton satisfying (1.1), (1.2) and (1.3). Let \( u(x, t) \) be a smooth solution of the heat equation \((\Delta - \partial_t)u = 0\) on \( M \times [-2, 0] \). For a fixed point \( p \in M \), if \( u \) satisfies quadratic exponential growth in the space variable, i.e.,

\[
|u(x, t)| \leq A_1 e^{A_2 d^2(x, p)} \tag{1.4}
\]

for all \((x, t) \in M \times [-2, 0]\), where \( A_1 \) and \( A_2 \) are some positive constants, and \( d(x, p) \) is the distance function from \( p \) to \( x \), then \( u(x, t) \) is analytic in time \( t \in [-1, 0] \) with radius \( \delta > 0 \) depending only on \( n, A_2, \mu \) and \( f \). Moreover, we have that

\[
u(x, t) = \sum_{j=0}^\infty a_j(x) t^j, \quad j!=-1;
\]
with $\Delta a_j(x) = a_{j+1}(x)$ and

$$|a_j(x)| \leq A_1 e^{-\mu/2} e^{f(x)/2} (f(x) + 1)^{n/4} A_3^{j+1} j! e^{2A_2 \delta^2(x,p)}, \quad j = 0, 1, 2, \ldots,$$

where $A_3$ is some constant depending only on $n$ and $A_2$, and $0^0 = 1$. Here $\mu = \mu(g, 1)$ denotes Perelman’s entropy functional.

Remark 1.2. The growth condition is necessary. As in [3], let $v(x, t)$ be Tychonov’s solution of the heat equation in $(\mathbb{R}^n, \delta_{ij}, |x|^2/4) \times \mathbb{R}$ such that $v = 0$ if $t \leq 0$ and $v$ is nontrivial for $t > 0$. Then $u := v(x, t + 1)$ is a nontrivial ancient solution of the heat equation and is not analytic in time. Note that $|u(x, t)|$ grows faster than $e^{c|x|^2}$ for any $c > 0$, but for any $\epsilon > 0$, $|u(x, t)|$ is bounded by $c_1 e^{c_2 |x|^{2+\epsilon}}$ for some constants $c_1$ and $c_2$. This implies that growth condition (1.4) is sharp.

In general, the Cauchy problem to the backward heat equation is not solvable. However, on a complete non-compact shrinker, we can obtain a solvable result by a simple application of Theorem 1.1.

Corollary 1.3. Let $(M, g, f)$ be an $n$-dimensional complete non-compact gradient shrinking Ricci soliton satisfying (1.1), (1.2) and (1.3). For a fixed point $p \in M$, the Cauchy problem for the backward heat equation

$$\begin{cases}
(\Delta + \partial_t)u = 0, \\
u(x, 0) = a(x)
\end{cases}$$

has a smooth solution with quadratic exponential growth in $M \times (0, \delta)$ for some $\delta > 0$ if and only if

$$|\Delta^j a(x)| \leq e^{-\mu/2} e^{f(x)/2} (f(x) + 1)^{n/4} A_3^{j+1} j! e^{A_4 \delta^2(x,p)}, \quad j = 0, 1, 2, \ldots,$$

where $A_3$ and $A_4$ are some positive constants.

The rest of this paper is organized as follows: in Section 2, we recall some properties of gradient shrinking Ricci solitons. In particular, we give a local mean value type inequality on gradient shrinking Ricci solitons. In Section 3, adapting Dong-Zhang’s proof strategy [3], we apply the mean value type inequality of Section 2 to prove Theorem 1.1 and Corollary 1.3.

Acknowledgement. The author sincerely thanks Professor Qi S. Zhang for answering several questions about the work [3]. This work was partially supported by NSFC (11671141) and NSFS (17ZR1412800).

2. Some properties of shrinkers

In this section, we will present some basic propositions about complete non-compact gradient shrinking Ricci solitons; these will be used in the proofs of our main results.

On an $n$-dimensional complete non-compact gradient shrinking Ricci soliton $(M, g, f)$ satisfying (1.1), (1.2) and (1.3), from Chen’s work (Proposition 2.2 in [16]), we know that the scalar curvature is $R \geq 0$. Moreover, by
we know that the scalar curvature $R$ must be strictly positive, unless $(M, g, f)$ is the Gaussian shrinking Ricci soliton $\left(\mathbb{R}^n, \delta_{ij}, |x|^2/4\right)$.

For any fixed point $p \in M$, by Theorem 1.1 of Cao-Zhou [18] (later refined by Chow et al. [19]), we have

$$\frac{1}{4} \left[ \left( d(x, p) - 2\sqrt{f(p)} - 4n + \frac{4}{3} \right)_+^2 \right] \leq f(x) \leq \frac{1}{4} \left( d(x, p) + 2\sqrt{f(p)} \right)^2$$

for any $x \in M$, where $d(x, p)$ is the distance function from $p$ to $x$. From this, $2\sqrt{f(x)}$ could be regarded as a distance-like function. By the Gaussian shrinking Ricci soliton $\left(\mathbb{R}^n, \delta_{ij}, |x|^2/4\right)$, the growth estimate of $f$ is sharp.

Combining this estimate and (1.2), for any point $p \in M$, we get that

$$(2.1) \quad 0 \leq R(x) \leq \frac{1}{4} \left( d(x, p) + 2\sqrt{f(p)} \right)^2$$

on $(M, g, f)$. It remains an interesting open question as to whether or not the scalar curvature $R$ is bounded from above by a uniform constant.

By Cao-Zhou [18] and Munteanu [20], the volume growth of a gradient shrinking Ricci soliton can be regarded as an analogue of Bishop’s theorem for manifolds with non-negative Ricci curvature (see [21]). That is, there exists a constant $c(n)$ depending only on $n$ such that

$$(2.2) \quad V_p(r) \leq c(n)e^{f(p)r^n}$$

for any $r > 0$ and $p \in M$, where $V_p(r)$ denotes the volume of geodesic ball $B_p(r)$ with radius $r$ and center $p \in M$. From Haslhofer-Müller [22], there exists a point $p_0 \in M$ where $f$ attains its infimum such that $f(p_0) \leq n/2$.

In [23], Li and Wang proved a local Sobolev inequality on complete non-compact gradient shrinking Ricci solitons without any assumption.

**Lemma 2.1.** Let $(M, g, f)$ be an $n$-dimensional complete gradient shrinking Ricci soliton satisfying (1.1), (1.2) and (1.3). Then, for each compactly supported locally Lipschitz function $u(x)$ with support in $B_p(r)$, where $p \in M$ and $r > 0$, we have that

$$(2.3) \quad \left( \int_{B_p(r)} u^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{n}} \leq C(n)e^{-\frac{2\mu}{n}} \int_{B_p(r)} \left( 4|\nabla u|^2 + Ru^2 \right) \, dv$$

for some constant $C(n)$ depending only on $n$, where $\mu := \mu(g, 1)$ is the entropy functional of Perelman, and $R$ is the scalar curvature of $(M, g, f)$.

The proof of Lemma 2.1 mainly depends on Perelman’s entropy functional and the Markov semigroup technique; see [23]. If the scalar curvature $R$ is bounded, (2.3) is similar to a classical Sobolev inequality on compact manifolds [24]. In this paper, we will apply the Sobolev inequality (2.3) to obtain a local mean value type inequality on shrinkers.

**Proposition 2.2.** Let $(M, g, f)$ be an $n$-dimensional complete gradient shrinking Ricci soliton satisfying (1.1), (1.2) and (1.3). Fix $0 < m < \infty$. Then there exists a positive constant $C(n, m)$ depending on $n$ and $m$, such that,
for any \( s \in \mathbb{R} \), and for any \( 0 < \delta < 1 \), and for any smooth nonnegative solution \( v \) of
\[
(\Delta - \partial_t) v(x, t) \geq 0
\]
in the parabolic cylinder \( Q_r(p, s) := B_p(r) \times [s - r^2, s] \), where \( p \in M \) and \( 0 < r < 2 \), we have
\[
\sup_{Q_{dr}(p, s)} \{ v^m \} \leq \frac{C(n, m)(R_M + 1)^{n/2}}{(1 - \delta)^{2+n} e^{n} r^{2+n}} \int_{Q_r(p, s)} v^m \, dx \, dt,
\]
where \( R_M := \sup_{x \in B_p(r)} R(x) \) and \( \mu := \mu(g, 1) \) is Perelman’s entropy functional.

Remark 2.3. It should be noted that our local mean value type inequality only holds for a local geodesic ball (here we choose the radius \( 0 < r < 2 \)), and does not hold for a geodesic ball of any radius.

The point \((p, s)\) and the radius \(r\) in Proposition 2.2 is customarily called the vertex and the size of parabolic cylinder \(Q_r(p, s)\), respectively. Compared with the classical mean value type inequality of manifolds, there seems to be a lack of a volume factor \(V_p(r)\) in (2.4). However if the factor \(r^n\) is regarded as \(V_p(r)\), this inequality is very similar to the classical case.

Proof of Proposition 2.2. Following the argument of [25], the proof technique used here is the delicate Moser iteration applied to local Sobolev inequality (2.3). It is emphasized that the explicit coefficients of the mean value inequality in terms of the Sobolev constants in (2.3) should be carefully examined.

We first prove (2.4) for \( m = 2 \). Given a nonnegative smooth solution \( u \) of \((\Delta - \partial_t) v \geq 0\) for any nonnegative function \( \phi \in C^\infty_0(B) \), where \( B := B_p(r) \), \( p \in M \) and \( r > 0 \), we have that
\[
\int_B (\phi u_t + \nabla \phi \nabla u) \, dv \leq 0.
\]
Set \( \phi = \psi^2 u, \psi \in C^\infty_0(B) \). Then
\[
\int_B (\psi^2 u u_t + \psi^2 |\nabla u|^2) \, dv \leq 2 \int_B w \psi \nabla u \nabla \psi \, dv
\]
\[
\leq 2 \int_B |\nabla \psi|^2 u^2 \, dv + \frac{1}{2} \int_B \psi^2 |\nabla u|^2 \, dv,
\]
so we get that
\[
\int_B (2\psi^2 u u_t + |\nabla (\psi u)|^2) \, dv \leq 4 \| \nabla \psi \|^2 \int_{\text{supp}(\psi)} u^2 \, dv.
\]
Multiplying a smooth function $\lambda(t)$, which will be determined later, from the above inequality, we have that

$$
\partial_t \left( \int_B (\lambda \psi u)^2 dv \right) + \lambda^2 \int_B |\nabla (\psi u)|^2 dv
$$

(2.5)

$$
\leq C \lambda \left( \lambda \|\nabla \psi\|_\infty^2 + \|\lambda\| \sup \psi^2 \right) \int_{\text{supp}(\psi)} u^2 dv,
$$

where $C$ is finitely constant, though this may change from line to line in the ensuing computations.

We choose $\psi$ and $\lambda$ such that, for $0 < \sigma' < \sigma < 1$, $\kappa = \sigma - \sigma'$,

1. $0 \leq \psi \leq 1$, $\text{supp}(\psi) \subset \sigma B$, $\psi = 1$ in $\sigma' B$ and $|\nabla \psi| \leq 2(\kappa r)^{-1}$, where $\sigma B := B_p(\sigma r)$;

2. $0 \leq \lambda \leq 1$, $\lambda = 0$ in $(-\infty, s - \sigma r^2)$, $\lambda = 1$ in $(s - \sigma' r^2, +\infty)$, and $|\lambda'(t)| \leq 2(\kappa r)^{-2}$.

Set $I_o := [s - \sigma r^2, s]$. For any $t \in I_o$, integrating (2.5) over $[s - r^2, t]$,

(2.6)

$$
\sup_{I_o} \left\{ \int_B \psi u^2 dv \right\} + \int_{B \times I_o} |\nabla (\psi u)|^2 dv dt \leq C (\kappa r)^{-2} \int_{\sigma B \times I_o} u^2 dv dt.
$$

On the other hand, by Hölder’s inequality and Lemma 2.1, we have that

(2.7)

$$
\int_B \varphi^{2\left(1+2/n\right)} dv \leq \left( \int_B \varphi^{2/n} dv \right)^{2/n} \left( \int_B |\varphi|^{2n} dv \right)^{n/2}
$$

$$
\leq \left( \int_B \varphi^{2/n} dv \right)^{2/n} \left[ C(n) e^{-\frac{2a}{n}} \int_B \left( 4|\nabla \varphi|^2 + R \varphi^2 \right) dv \right]
$$

for all $\varphi \in C_0^\infty(B)$. Combining (2.6) and (2.7), we finally get that

$$
\int_{\sigma' B \times I_o} u^{2\theta} dv dt \leq C(B) \left( C(\kappa r)^{-2} \int_{\sigma B \times I_o} u^2 dv dt \right)^{\theta},
$$

with $\theta = 1 + 2/n$, where $E(B) := C(n) e^{-\frac{2a}{n}} (R_M + 1)$ and $R_M := \sup_{x \in B(\sigma r)} R(x)$.

We would like to point out that we have used the condition $0 < r < 2$ in the above inequality.

For any $m \geq 1$, $u^m$ is also a nonnegative solution of $(\Delta - \partial_t) u \geq 0$. Hence the above inequality indeed implies that

(2.8)

$$
\int_{\sigma' B \times I_o} u^{2m\theta} dv dt \leq C(B) \left( C(\kappa r)^{-2} \int_{\sigma B \times I_o} u^{2m} dv dt \right)^{\theta}
$$

for $m \geq 1$.

Let $\kappa_i = (1 - \delta)2^{-i}$, which satisfies $\sum_1^\infty \kappa_i = 1 - \delta$. Let $\sigma_0 = 1$ and $\sigma_{i+1} = \sigma_i - \kappa_i = 1 - \sum_1^i \kappa_j$. Applying (2.8) for $m = \theta^i$, $\sigma = \sigma_i$, $\sigma' = \sigma_{i+1}$, we have that

$$
\int_{\sigma_{i+1} B \times I_{\sigma_{i+1}}} u^{2\theta^{i+1}} dv dt \leq E(B) \left\{ \frac{C^{i+1}}{([1 - \delta] r)^2} \int_{\sigma_i B \times I_{\sigma_i}} u^{2\theta^i} dv dt \right\}^{\theta}.
$$
Therefore,
\[
\left( \int_{\sigma_{i+1} B \times I_{\sigma_{i+1}}} u^{2^i+1} dvdt \right)^{\theta^{-i-1}} \leq \frac{C \Sigma j \theta^{-j} E(B) \Sigma \theta^{-j}}{[(1-\delta)r]^{2\Sigma \theta^{-j}}} \int_{Q_r(p,s)} u^2 dvdt,
\]
where \( \Sigma \) denotes the summations from 1 to \( i+1 \). Letting \( i \to \infty \),
\[
(2.9) \quad \sup_{\delta B \times I_{\delta}} \{ u^2 \} \leq \frac{C E(B)^{n/2}}{[(1-\delta)r]^{2+n}} \| u \|_{2,Q_r(p,s)}^2,
\]
which clearly implies (2.4) when \( m = 2 \), since \( I_{\delta^2} \subset I_{\delta} \).

The case \( m > 2 \) then follows by the case \( m = 2 \), because if \( u \) is a nonnegative solution of \((\Delta - \partial_t)v \geq 0\), then \( u^m \), \( m \geq 1 \), is also a nonnegative solution of \((\Delta - \partial_t)v \geq 0\). All in all, we do, in fact, prove (2.4) when \( m > 2 \).

When \( 0 < m < 2 \), we will apply (2.9) to prove (2.4) by a different iterative argument. Let \( \sigma \in (0,1) \) and \( \rho = \sigma + (1-\sigma)/3 \). Then (2.9) implies that
\[
\sup_{\sigma B \times I_{\sigma}} \{ u \} \leq \frac{F(B)}{(1-\sigma)^{1+n/2}} \| u \|_{2,\rho B \times I_{\rho}},
\]
where \( F(B) := C(n)e^{-\mu/2} (R_M + 1)^{n/4} r^{-1-n/2} \).

Using
\[
\| u \|_{2,Q} \leq \| u \|_{\infty,Q}^{1-m/2} \cdot \| u \|_{m,Q}^{m/2}, \quad 0 < m < 2
\]
for any parabolic cylinder \( Q \), we then have that
\[
(2.10) \quad \| u \|_{\infty,\sigma B \times I_{\sigma}} \leq \frac{G(B)}{(1-\sigma)^{1+n/2}} \| u \|_{\infty,\sigma B \times I_{\sigma}}^{1-m/2},
\]
where \( G(B) := F(B) \| u \|_{m,Q_r(p,s)}^{m/2} \).

Now fix \( \delta \in (0,1) \) and let \( \sigma_0 = \delta \) and \( \sigma_{i+1} = \sigma_i + (1-\sigma_i)/4 \), which satisfies that \( 1-\sigma_i = (3/4)^i (1-\delta) \). Applying (2.10) to \( \sigma = \sigma_i \) and \( \rho = \sigma_{i+1} \) for each \( i \), we have that
\[
\| u \|_{\infty,\sigma_i B \times I_{\sigma_i}} \leq \frac{(4/3)^i (1+n/2)}{(1-\delta)^{1+n/2}} \frac{G(B)}{(1-\sigma_i)^{1+n/2}} \| u \|_{\infty,\sigma_{i+1} B \times I_{\sigma_{i+1}}}^{1-m/2}.
\]

Therefore, for any \( i \),
\[
\| u \|_{\infty,\delta B \times I_{\delta}} \leq \frac{(4/3)^i (1+n/2) \Sigma j (1-m/2)^j}{(1-\delta)^{1+n/2}} \frac{G(B)}{(1-\sigma_i)^{1+n/2}} \| u \|_{\infty,\sigma_i B \times I_{\sigma_i}}^{1-m/2} \| u \|_{\infty,\sigma_{i+1} B \times I_{\sigma_{i+1}}}^{1-m/2},
\]
where \( \Sigma \) denotes the summations from 0 to \( i-1 \). Letting \( i \to \infty \),
\[
\| u \|_{\infty,\delta B \times I_{\delta}} \leq \frac{(4/3)^{2/n} (2+n)}{\Sigma j (1-\delta)^{1+n/2}} \frac{G(B)}{(1-\sigma_i)^{1+n/2}} \| u \|_{\infty,\sigma_i B \times I_{\sigma_i}}^{2/n} \| u \|_{\infty,\sigma_{i+1} B \times I_{\sigma_{i+1}}}^{2/n},
\]
which implies (2.4) for \( 0 < m < 2 \), by \( I_{\delta^2} \subset I_{\delta} \) and the definition of \( G(B) \).\( \square \)
3. Proof of results

In this section, adapting the argument of Dong-Zhang [3], we will apply the preceding propositions of shrinkers in Section 2 to prove Theorem 1.1 and Corollary 1.3. We first prove Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality, we may assume that $A_1 = 1$, because the heat equation is linear. To prove the theorem, it suffices to confirm the result at space-time point $(x, 0)$ for any $x \in M$.

Since $u$ is a given smooth solution to the heat equation, $u^2$ is a nonnegative subsolution to the heat equation. Given a point $x_0 \in M$ and a positive integer $k$, by letting $s = 0$, $r = 1/\sqrt{k}$, $m = 1$, $\delta = 1/2$ in the mean value type inequality of Proposition 2.2, we have that

$$
\sup_{Q_{1/(2\sqrt{k})}(x_0,0)} u^2 \leq C_1(n) \left[ \frac{1}{4} \left( \frac{1}{\sqrt{k}} + 2\sqrt{f(x_0)} \right)^2 + 1 \right]^{n/2} \int_{Q_{1/\sqrt{k}}(x_0,0)} u^2(x,t) \, dvdt
$$

$$
\leq C_2(n) e^{-\mu k^{n/2} + (f(x_0) + 1)^{n/2}} \int_{Q_{1/\sqrt{k}}(x_0,0)} u^2(x,t) \, dvdt,
$$

where constants $C_1(n)$ and $C_2(n)$ both depend only on $n$. Here, in the first inequality above, we used the estimate

$$
\sup_{B_{x_0}(1/\sqrt{k})} R \leq \frac{1}{4} \left( \frac{1}{\sqrt{k}} + 2\sqrt{f(x_0)} \right)^2,
$$

and in the second inequality we only used the simple fact that the size of the cubes is less than one.

Since $\partial_t^k u$ is also a solution to the heat equation, then substituting this into (3.1) gives

$$
\sup_{Q_{1/(2\sqrt{k})}(x_0,0)} (\partial_t^k u)^2 \leq C_2(n) e^{-\mu k^{n/2} + (f(x_0) + 1)^{n/2}} \int_{Q_{1/\sqrt{k}}(x_0,0)} (\partial_t^k u)^2(x,t) \, dvdt.
$$

In that follows, following we will bound the right hand side of (3.2). For integers $j = 1, 2, \ldots, k$, consider the domains

$$
\Omega_j^1 = B_{x_0} \left( \frac{j}{\sqrt{k}} \right) \times \left[ -\frac{j}{k}, 0 \right], \quad \Omega_j^2 = B_{x_0} \left( \frac{j+0.5}{\sqrt{k}} \right) \times \left[ -\frac{j+0.5}{k}, 0 \right].
$$

Obviously, $\Omega_j^1 \subset \Omega_j^2 \subset \Omega_{j+1}$. Let $\psi_j^{(1)}$ be a standard Lipschitz cutoff function supported in

$$
B_{x_0} \left( \frac{j+0.5}{\sqrt{k}} \right) \times \left[ -\frac{j+0.5}{k}, \frac{j+0.5}{k} \right]
$$

such that

$$
\psi_j^{(1)} = 1 \text{ in } \Omega_j^1 \quad \text{and} \quad \left| \nabla \psi_j^{(1)} \right|^2 + \left| \partial_t \psi_j^{(1)} \right| \leq Ck,
$$
where $C$ is a universal constant (this includes the following constants $C$) that may be changed line by line.

For the above cutoff function $\psi = \psi^{(1)}_j$, multiplying $(u_t)^2 = u_t \Delta u$ by $\psi^2$, integrating it over $\Omega^2_j$, and using integration by parts, we compute that

$$
\int_{\Omega^2_j} (u_t)^2 \psi^2 \, dx \, dt = \int_{\Omega^2_j} u_t \Delta u \psi^2 \, dv \, dt
$$

$$
= - \int_{\Omega^2_j} ((\nabla u)_t \nabla u) \psi^2 \, dv \, dt - \int_{\Omega^2_j} u_t \nabla u \nabla \psi^2 \, dv \, dt
$$

$$
= \frac{1}{2} \int_{\Omega^2_j} |(\nabla u)^2|_t \psi^2 \, dv \, dt - 2 \int_{\Omega^2_j} u_t \psi \nabla u \nabla \psi \, dv \, dt
$$

$$
\leq \frac{1}{2} \int_{\Omega^2_j} |\nabla u|_t^2 (\psi^2) \, dv \, dt + \frac{1}{2} \int_{\Omega^2_j} (u_t)^2 \psi^2 \, dv \, dt + 2 \int_{\Omega^2_j} |\nabla u| |\nabla \psi|^2 \, dv \, dt
$$

$$
\leq C k \int_{\Omega^2_j} |\nabla u| \, dv \, dt + \frac{1}{2} \int_{\Omega^2_j} (u_t)^2 \psi^2 \, dv \, dt,
$$

where we used the property of cutoff function $\psi$ in the fifth line. Using $\Omega^1_j \subset \Omega^2_j$ and the property of $\psi$, the above inequality implies that

$$
(3.3) \quad \int_{\Omega^1_j} (u_t)^2 \, dv \, dt \leq C k \int_{\Omega^2_j} |\nabla u|^2 \, dv \, dt.
$$

Let $\psi^{(2)}_j$ also be a standard Lipschitz cutoff function supported in

$$
B_{x_0} \left( \frac{j+1}{\sqrt{k}} \right) \times \left[ - \frac{j+1}{k}, \frac{j+1}{k} \right]
$$

such that $\psi^{(2)}_j = 1$ in $\Omega^2_j$ and $|\nabla \psi^{(2)}_j|^2 + |\partial_t \psi^{(2)}_j| \leq C k$. Then we can apply the standard Caccioppoli inequality (energy estimate) between the cubes $\Omega^2_j$ and $\Omega^1_{j+1}$ to obtain that

$$
(3.4) \quad \int_{\Omega^2_j} |\nabla u|^2 \, dv \, dt \leq C k \int_{\Omega^1_{j+1}} u^2 \, dv \, dt.
$$
Indeed, for the cutoff function \( \varphi = \psi_j^{(2)} \), multiplying \( uu_t = u\Delta u \) by \( \varphi^2 \), integrating it over \( \Omega_{j+1}^1 \), and using integration by parts, we have that

\[
\frac{1}{2} \int_{\Omega_{j+1}^1} \partial_t (u^2 \varphi^2) \, dv \, dt - \int_{\Omega_{j+1}^1} \varphi \varphi_t u^2 \, dv \, dt
= \int_{\Omega_{j+1}^1} uu_t \varphi^2 \, dv \, dt
= \int_{\Omega_{j+1}^1} u\Delta u \varphi^2 \, dv \, dt
= -\int_{\Omega_{j+1}^1} |\nabla u|^2 \varphi^2 \, dv \, dt - 2 \int_{\Omega_{j+1}^1} u\nabla u \varphi \nabla \varphi \, dv \, dt.
\]

Observing that the first term of left hand side is

\[
\frac{1}{2} \int_{\Omega_{j+1}^1} \partial_t (u^2 \varphi^2) \, dv \, dt = \frac{1}{2} \int_{B(x_0, 2/\sqrt{k})} u^2(x, 0) \, dv \geq 0,
\]

we conclude that

\[
\int_{\Omega_{j+1}^1} |\nabla u|^2 \varphi^2 \, dv \, dt \leq \int_{\Omega_{j+1}^1} \varphi \varphi_t u^2 \, dv \, dt - 2 \int_{\Omega_{j+1}^1} u\nabla u \varphi \nabla \varphi \, dv \, dt
\leq \int_{\Omega_{j+1}^1} \varphi \varphi_t u^2 \, dv \, dt + \frac{1}{2} \int_{\Omega_{j+1}^1} |\nabla u|^2 \varphi^2 \, dv \, dt + 2 \int_{\Omega_{j+1}^1} u^2 |\nabla \varphi|^2 \, dv \, dt,
\]

where we have used Young’s inequality in the second inequality above.

Therefore,

\[
\int_{\Omega_{j+1}^1} |\nabla u|^2 \varphi^2 \, dv \, dt \leq 2 \int_{\Omega_{j+1}^1} \varphi \varphi_t u^2 \, dv \, dt + 4 \int_{\Omega_{j+1}^1} u^2 |\nabla \varphi|^2 \, dv \, dt.
\]

Then (3.4) follows by \( \Omega_2 \subset \Omega_{j+1}^1 \) and the property of cut-off function \( \varphi \).

Combining (3.4) and (3.3) yields that

\[
\int_{\Omega_{j}^1} (u_t)^2 \, dv \, dt \leq C \int_{\Omega_{j+1}^1} u^2 \, dv \, dt.
\]

Since \( \partial_t u \) is also a solution of the heat equation, we can replace \( u \) in the above inequality by \( \partial_t u \) to deduce, by induction, that

\[
\int_{\Omega_{j}^1} (\partial_t^k u)^2 \, dv \, dt \leq C^k \int_{\Omega_k} u^2 \, dv \, dt.
\]

Noticing that \( \Omega_1^1 = Q_{1/(2\sqrt{k})}(x_0, 0) \) and \( \Omega_{1/2}^1 = B_{x_0}(\sqrt{k}) \times [-1, 0] \), we substitute the above inequality into (3.2) to get that

(3.5)

\[
\sup_{Q_{1/(2\sqrt{k})}(x_0, 0)} (\partial_t^k u)^2(x, t) \leq C_2(n) e^{-n/2} (f(x_0) + 1)^{n/2} C^k \int_{B_{x_0}(\sqrt{k}) \times [-1, 0]} u^2 \, dv \, dt.
\]
Using quadratic exponential growth condition (1.4) and the volume growth of shrinker (2.2), (3.5) can be further simplified as
\[
\sup_{Q_{1/(2\sqrt{k})}(x_0,0)} |\partial_t^k u(x,t)| \leq C_2(n) e^{-\mu/2} k^{n/2+1/2} (f(x_0) + 1)^{n/4} C_{k/2} k^{k} e^{f(x_0)/2} \sqrt{n/4} e^{A_2 d^2(\xi,p)}
\]
for some point \(\xi \in B_{x_0}(\sqrt{k})\) and for all integers \(k \geq 1\). By the triangle inequality,
\[
d(\xi,p) \leq d(x_0,p) + d(x_0,\xi) \leq d(x_0,p) + k^{1/2},
\]
so
\[
d^2(\xi,p) \leq (d(x_0,p) + k^{1/2})^2 \leq 2d^2(x_0,p) + 2k.
\]
Therefore,
\[
\left|\partial_t^k u(x,t)\right| \leq C_2(n) e^{-\mu/2} e^{f(x_0)/2} (f(x_0) + 1)^{n/4} k^{n/2+1/2} C_{k/2} k^{k} e^{2A_2 k} e^{2A_2 d^2(x_0,p)}
\]
\[
\leq e^{-\mu/2} e^{f(x_0)/2} (f(x_0) + 1)^{n/4} A_3^{k+1} k^{k} e^{2A_2 d^2(x_0,p)}
\]
for all \((x,t) \in Q_{1/(2\sqrt{k})}(x_0,0)\), and for all integers \(k \geq 1\), where \(A_3\) is a positive constant depending only on \(n\), \(C\) and \(A_2\).

Now fix a real number \(R \geq 1\). For any point \(x \in B_p(R)\), we choose a positive integer \(j\) and \(t \in [-\delta,0]\) for some small \(\delta > 0\). By Taylor’s theorem,
\[
u(x,t) - \sum_{i=0}^{j-1} \partial_t^i u(x,0) \frac{t^i}{i!} = \frac{t^j}{j!} \partial_s^j u(x,s),
\]
where \(s = s(x,t,j) \in [t,0]\). Using (3.6), we know that, for sufficiently small \(\delta > 0\), which depends on \(n\), \(A_2\), \(A_3\), \(\mu\) and \(f\), the right hand side of (3.7) converges to 0 uniformly for \(x \in B_p(R)\) as \(j \to \infty\). Hence
\[
u(x,t) = \sum_{j=0}^{\infty} \partial_t^j u(x,0) \frac{t^j}{j!}.
\]
Thus \(u(x,t)\) is analytic in time \(t\) with radius \(\delta\). Set \(a_j = a_j(x) = \partial_t^j u(x,0)\). By (3.6) again, we have that
\[
\partial_t u(x,t) = \sum_{j=0}^{\infty} a_{j+1}(x) \frac{t^j}{j!} \quad \text{and} \quad \Delta u(x,t) = \sum_{j=0}^{\infty} \Delta a_j(x) \frac{t^j}{j!},
\]
where both series converge uniformly for \((x,t) \in B_p(R) \times [-\delta,0]\) for any fixed \(R > 0\). Since \(u(x,t)\) is a solution of the heat equation, this implies that
\[
\Delta a_j(x) = a_{j+1}(x),
\]
with
\[
|a_j(x)| \leq A_3^{j+1} e^{-\mu/2} e^{f(x)/2} (f(x) + 1)^{n/4} j^j e^{2A_2 d^2(x,p)},
\]
where \(A_3\) is a positive constant depending only on \(n\) and \(A_2\).

In the end, we apply Theorem 1.3 to prove Corollary 1.3.
Proof of Corollary 1.3. Assume that $u(x,t)$ is a smooth solution to the Cauchy problem of the backward heat equation (1.5) with quadratic exponential growth. Then $u(x,-t)$ is also a smooth solution of the heat equation with quadratic exponential growth. By Theorem 1.1, we have that

$$u(x,-t) = \sum_{j=0}^{\infty} \Delta^j a(x) \frac{(-t)^j}{j!}.$$ 

Then (1.6) follows by letting $\Delta^j a(x) = a_j(x)$ in the theorem.

On the other hand, we assume that (1.6) holds. We then claim that

$$u(x,t) = \sum_{j=0}^{\infty} \Delta^j a(x) \frac{t^j}{j!}$$

is a smooth solution to the heat equation for $t \in [-\delta,0]$ with some constant $\delta > 0$ sufficiently small. Indeed, (1.6) guarantees that the above series and the two series

$$\sum_{j=0}^{\infty} \Delta^{j+1} a(x) \frac{t^j}{j!} \quad \text{and} \quad \sum_{j=0}^{\infty} \Delta^j a(x) \frac{\partial^j t^j}{j!}$$

all converge absolutely and uniformly in $[-\delta,0] \times B_p(R)$ for any fixed $R > 0$. Hence $(\Delta - \partial_t)u(x,t) = 0$, and the claim follows. Moreover, we observe that

$$|u(x,t)| \leq \sum_{j=0}^{\infty} |\Delta^j a(x)| \frac{|t|^j}{j!} \leq e^{-\mu/2} e^{f(x)/2} (f(x) + 1)^{n/4} A_3 e^{A_4 d^2(x,p)} \sum_{j=0}^{\infty} \frac{(A_3 j |t|^j)}{j!} \leq e^{-\mu/2} e^{f(x)/2} (f(x) + 1)^{n/4} A_3 A_5 e^{A_4 d^2(x,p)},$$

provided that $t \in [-\delta,0]$ with some sufficiently small constant $\delta > 0$, where we used the fact that the series $\sum_{j=0}^{\infty} \frac{(A_3 j |t|^j)}{j!}$ converges in $[-\delta,0]$ and its summation is no more than some constant $A_5 > 0$; that is, $u(x,t)$ has quadratic exponential growth. Hence, $u(x,-t)$ is a solution to the Cauchy problem of the backward heat equation (1.5) of quadratic exponential growth. □

References

1. Widder D V. Analytic solutions of the heat equation, Duke Math J, 1962, 29: 497-503
2. Zhang Q S. A note on time analyticity for ancient solutions of the heat equation, Proc Amer Math Soc, 2020, 148(4): 1665-1670
3. Dong H J, Zhang Q S. Time analyticity for the heat equation and Navier-Stokes equations, J Funct Anal, 2020, 279(4): 108563, 15pp
4. Masuda K. On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation, Proc Japan Acad, 1967, 43: 827-832
5. Kinderlehrer D, Nirenberg L. Analyticity at the boundary of solutions of nonlinear second-order parabolic equations, Comm Pure Appl Math, 1978, 31(3): 283-338
6. Komatsu G. Global analyticity up to the boundary of solutions of the Navier-Stokes equation, Comm Pure Appl Math, 1980, 33(4): 545-566
7. Giga Y. Time and spatial analyticity of solutions of the Navier-Stokes equations, Comm Partial Differential Equations, 1983, 8(8): 929-948
8. Escariaiza L, Montaner S, Zhang C. Analyticity of solutions to parabolic evolutions and applications, SIAM J Math Anal, 2017, 49(5): 4064-4092
9. Han F W, Hua B B, Wang L L. Time analyticity of solutions to the heat equation on graphs, Proc Amer Math Soc, 2021, 149(6): 2279-2290
10. Hamilton R. The formation of singularities in the Ricci flow. Surveys in Differential Geometry, International Press, Boston, vol. 2, 1995, 7-136
11. Perelman G. The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159
12. Perelman G. Ricci flow with surgery on three-manifolds, arXiv:math.DG/0303109
13. Perelman G. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, arXiv:math.DG/0307245
14. Cao H D. Recent progress on Ricci solitons, Recent Advances in Geometric Analysis, In: Y.-I. Lee, C.-S. Lin, M.-P. Tsui (eds.) Advanced Lectures in Mathematics (ALM), International Press, Somerville, 2010, 11: 1-38
15. Carrillo J, Ni L. Sharp logarithmic Sobolev inequalities on gradient solitons and applications, Comm Anal Geom, 2009, 17(4): 721-753
16. Chen B L. Strong uniqueness of the Ricci flow, J Diff Geom, 2009, 82(2): 363-382
17. Pigola S, Rimoldi M, Setti A G. Remarks on non-compact gradient Ricci solitons, Math Z, 2011, 268(3-4): 777-790
18. Cao H D, Zhou D T. On complete gradient shrinking Ricci solitons, J Diff Geom, 2010, 85(2): 175-186
19. Chow B, Chu S C, Glickenstein D, Guenther C, Isenberg J, Ivey T, Knopf D, Lu P, Luo F, Ni L. The Ricci flow: techniques and applications, part IV: long-time solutions and related topics, Mathematical Surveys and Monographs, vol. 206, American Mathematical Society, Providence, RI, 2015
20. Munteanu O. The volume growth of complete gradient shrinking Ricci solitons, arXiv:0904.0798v2
21. Munteanu O, Wang J P. Geometry of manifolds with densities, Adv Math, 2014, 259: 269-305
22. Haslhofer R, Müller R. A compactness theorem for complete Ricci shrinkers, Geom Funct Anal, 2011, 21(5): 1091-1116
23. Li Y, Wang B. Heat kernel on Ricci shrinkers, Calc. Var. Partial Differential Equations, 2020, 59(6): Art. 194
24. Zhang Q S. Sobolev inequalities, heat kernels under Ricci flow, and the Poincaré conjecture, CRC Press, Boca Raton, FL, 2011
25. Wu J Y, Wu P. Heat kernels on smooth metric measure spaces with nonnegative curvature, Math Ann, 2015, 362(3-4): 717-742

Department of Mathematics, Shanghai University, Shanghai 200444, China
Email address: jyw81@yahoo.com