Paradoxical decompositions and finitary colouring rules

Robert Samuel Simon and Grzegorz Tomkowicz

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Robert Simon
London School of Economics
Department of Mathematics
Houghton Street
London WC2A 2AE
e-mail: R.S.Simon@lse.ac.uk

Grzegorz Tomkowicz
Centrum Edukacji $G^2$
ul.Moniuszki 9
41-902 Bytom
Poland
e-mail: gtomko@vp.pl
Abstract

We colour every point $x$ of a probability space $X$ according to the colours of a finite list $x_1, x_2, \ldots, x_k$ of points such that each of the $x_i$, as a function of $x$, is a measure preserving transformation. We ask two questions about a colouring rule: (1) does there exist a finitely additive extension of the probability measure for which the $x_i$ remain measure preserving and also a colouring obeying the rule almost everywhere that is measurable with respect to this extension?, and (2) does there exist some colouring obeying the rule almost everywhere? If the answer to the first question is no and to the second question yes, we say that the colouring rule is paradoxical. A paradoxical colouring rule not only allows for a paradoxical partition of the space, it requires one. If a colouring rule is paradoxical, an axiom of choice is used to prove (2) but not typically used to prove (1). We show that a form of paradoxical decomposition can be created from the colour classes of any such colouring. We show that proper vertex colouring can be paradoxical. We conclude with several new topics and open questions.

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1 Introduction

F. Hausdorff showed in [7] that there is no finitely additive rotation-invariant measure \( \mu \) defined on all subsets of the unit sphere \( \mathbb{S}^2 \) and such that \( \mu(\mathbb{S}^2) = 1 \). He started with some subsets and showed that they obeyed certain rules that prevented them from being measurable with respect to any finitely additive rotation invariant measure.

In the present paper we reverse the above perspective. We start with rules concerning how to colour points in a probability space according to measure preserving transformations. We call such rules colouring rules and we are interested in colouring rules where the only sets that satisfy these rules cannot be measurable with respect to any finitely additive \( G \)-invariant measure \( \mu \) extending the initial probability measure. Furthermore we are not really interested in such rules that are merely inconsistent, but rather in those for which, using AC, the rule can be satisfied in some way. We call colouring rules paradoxical when they can be satisfied almost everywhere but never in a finitely additive measurable way as described above. The name for our rules is motivated by our Theorem 1, which is closely related to a theorem of Tarski (see [14], Thm. 11.1 and Cor. 11.2 for a proof), that shows that the existence of a paradoxical colouring rule defined on a standard probability space using a topologically transitive group action implies a paradoxical decomposition of the whole space or some of its subsets. It is worthwhile to observe that paradoxical colouring rules concern non-amenable group or semigroup actions and non-amenable groups may act in an amenable way (see [5] or [15]). Therefore it is natural to link the existence of paradoxical colouring rules with the following famous problem of Greenleaf: What are sufficient and necessary conditions for the lack of a finitely additive \( G \)-invariant probability measure on \( X \), where \( G \) is a non-amenable group acting transitively on the space \( X \)? We will discuss this further in the third section.

Let us analyse closer the Hausdorff’s construction in [7] that the unit sphere \( \mathbb{S}^2 \) can be decomposed, modulo a countable set \( D \) (the set of all fixed points of non-identity rotations from the appropriate subgroup of rotations) into the sets \( A, B \) and \( C \) satisfying the relation

\[(*) \ A \cong B \cong C \cong B \cup C,\]

where \( \cong \) denotes the congruence of sets. Here the congruences are witnessed by a subgroup of \( SO_3(\mathbb{R}) \) isomorphic to \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) (and for convenience we will
call this subgroup $\mathbb{Z}_2 \ast \mathbb{Z}_3$). With $\sigma$ generating $\mathbb{Z}_2$ and $\tau$ generating $\mathbb{Z}_3$, we have the requirements $\sigma(A) = B \cup C$, $\sigma(B \cup C) = A$, $\tau(A) = B$, $\tau(B) = C$, and $\tau(C) = A$. We call these requirements the Hausdorff colouring rule. Measurability with respect to any measure where both $\sigma$ and $\tau$ are still measure preserving would require that both half and one-third of the space be taken by the set $A$. And from these three sets, we can also create six sets that partition the whole space, namely $A \cap \sigma(B)$, $A \cap \sigma(C)$, $\tau(A \cap \sigma(B))$, $\tau(A \cap \sigma(C))$, $\tau^2(A \cap \sigma(B))$ and $\tau^2(A \cap \sigma(C))$, such that by choosing the appropriate rotations these six sets recreate two copies of the whole sphere, modulo the countable set $D$.

Now we perceive the above congruences, using the actions $\sigma$ and $\tau$ and the colours $A$, $B$ and $C$, as the composite of two colouring rules. We colour any point $x$ according to the colours of $\tau^{-1}(x)$ and $\sigma(x)$. Suppose that $\tau^{-1}(x)$ is coloured already by $B$ and $\sigma(x)$ is coloured already by $C$; the $\sigma$ part of the rule say that $x$ should be coloured $A$ and the $\tau$ part of the rule say that $x$ should be coloured $C$. How could this conflict be reconciled? One way would be to move further afield to determine the colour of $x$. For example, one could specify a new colour $E$ that can exist at only one point in every orbit of $\mathbb{Z}_2 \ast \mathbb{Z}_3$, and then colour the rest of the orbit by $A$, $B$, and $C$ according to the group element needed to move to the point in question from the representative coloured $E$ (as was done in the original AC construction of the Hausdorff paradox). However for our purposes such a set of rules based on the colour $E$ is not satisfactory, because the determination of membership in $A$, $B$, $C$, or $E$ would involve an unbounded number of group elements.

We are interested only in colouring rules that are finitary in character, meaning that the colour of a point is determined by finitely many group or semigroup elements. Of special interest are rules that always allow for some assigned colour no matter how descendants are coloured (unlike the above Hausdorff rules); we say that such colouring rules have rank one. Rank one rules relate measure theoretic paradoxes to problems of optimisation. A determination of one best colour (or a subset of equivalently optimal colours) according to finite many parameters is within the grasp of an automaton.

Furthermore we allow for colouring rules that depend on the position in the space $X$. For example, let $X$ be $\{0, 1\}^G$ where the group or semigroup $G$ acts on $X$. There could be two distinct set of rules, depending on whether the coordinate $x^e$ is equal to 0 or 1. We identify a special class of colouring rules: those that do not depend on position, called stationary rules, with the others called non-stationary rules. The distinction of stationary vs non-stationary
plays an important role in this paper. As we prefer colouring rules that can be preformed by an automaton, it should be easy to determine which part of the rule should apply. We require that the change in rules should be measurable with respect to the original probability distribution (and with all our examples it is determined by membership in clopen sets).

There is a further distinction of whether the colours are discrete or continuous. Paradoxical colouring rules with finitely colours are easy to find; the challenges lie with the special properties these rules must obey. The existence of paradoxical colouring rules when the colours belong to a continuum and continuity of the colouring rule is required (with respect to location and neighboring colours) is done in [11] and [12].

The primary concern with Borel colouring (the questions raised in [8]); is what is the Borel chromatic number, the least number of colours needed for a proper vertex colouring such that each colour class defines a Borel measurable set. There are two differences between our approach and Borel colouring. Proper colouring is only one kind of colouring rule; colouring rules may or may not imply that the colouring is proper. And measurable with our approach means with respect to some finitely additive measure.

With a one dimensional compact continuum of colours (a probability simplex determined by two extremal colours) and a Cantor set, Simon and Tomkowicz [10] demonstrated a colouring rule using a semigroup action for which there is no $\epsilon$-Borel colouring for sufficiently small $\epsilon > 0$ (meaning a colouring function that is Borel measurable and obeys the rule in all but a set of measure $\epsilon$), though there is a colouring that uses almost everywhere the two extremal colours of the continuum.

Ramsey Theory is a weakly related topic. A non-amenable group $G$ could act invariantly on two probability spaces $X$ and $Y$ such that there is a measurable surjection from $X$ to $Y$ that commutes with the actions of the group. Assume there is a paradoxical colouring rule for the space $X$ using the group $G$ however $Y$ possesses a finitely additive $G$-invariant extension measure defined on all subsets. The consequence would be that this rule cannot be realised in any way on the space $Y$, though it would not be contradictory in a logical way independent of the space. The failure of the colouring rule on part of such a space $Y$ is analogous to the existence of a special structure with Ramsey Theory.

The rest of this paper is organised as follows. The second section introduces the basic definitions. The third section shows that a kind of a
paradoxical decomposition follows from a paradoxical rule and links this relationship to the Greenleaf problem. The fourth section demonstrates some simple examples of paradoxical colouring rules. The fifth section shows that proper vertex colouring can be paradoxical. The concluding section explores open questions and directions for further study.

2 The Basics

Let $X$ be a probability space with $\mathcal{F}$ the sigma algebra on which a probability measure $m$ is defined. Usually $X$ will have a topology and $\mathcal{F}$ will be the induced Borel sets.

We say that the relation $R \subseteq X \times X$ is admissible if:

(i) for any $x \in X$ there are finitely many elements $x_1, \ldots, x_k$ of $X$ such that $(x, x_i) \in R$ for $i = 1, \ldots, k$.

(ii) for almost all $x \in X$, $R$ is irreflexive, meaning that $(x, x)$ is not in $R$. We call the elements $x_1, \ldots, x_k$ appearing in (i) the descendants of $x$.

Let $R \subseteq X \times X$ be an admissible relation and let $A$ be a set of colours. Most of our colouring rules have a finite $A$. If $A$ is not finite we require a measurable structure to $A$, namely a sigma algebra of subsets. We say that $F : X \times A^k \to 2^A$ is a colouring rule on $X$ if

(1) the graph of $F$ is a measurable subset of $A \times (X \times A^k)$ (with respect to the $\sigma$-algebra generated by products of measurable sets using the measurability structure on $A$ and the completion of the probability measure on $X$), and

(2) all descendants $x_1, \ldots, x_k$ are measure preserving transformations as functions of $x$, with respect to $m$.

It is a colouring rule of rank one if the correspondence $F$ is non-empty, meaning that for every $x$ and for every choice of an element $b$ in $A^k$ the $F(x, b)$ is non-empty.

We will say that a colouring rule is deterministic if for any $x \in X$ and $b \in A^k$ the set $F(x, b)$ is a single point. The colouring rule is stationary if $F(x, b)$ is determined only by $b$. The colouring rule is continuous if the probability space $X$ and $A$ have a topology, $\mathcal{F}$ are the Borel sets of $X$, and
the $F(x, b)$ is a continuous function of $x$ and $b$. If there are finitely many colours, a colouring rule of rank 2 is one that is not of rank one and is equivalent to two colouring rules of rank one using the same descendents. We mean that for all $x$ and $b$ the set $F(x, b)$ is equal to the intersection of the $F_1(x, b) \cap F_2(x, b)$ for the two correspondences $F_1, F_2$ of rank one rules. With at least two different colours $c_1, c_2$ and no additional requirements on the correspondences we can always define $F_1(x, b) = F(x, b)$ if $F(x, b) \neq \emptyset$ and $F_1(x, b) = \{c_1\}$, $F_2(x, b) = \{c_2\}$ if $F(x, b) = \emptyset$. For the present purposes of finitely many colours we don’t need ranks higher than two, however they may be some contexts where it makes sense to have rules of rank higher than two. It is possible that a rank two colouring rule may be implied by a rank one colouring rule when applied to some space, but we exclude this consideration from the definition of rank, which has to do with whether the correspondence as defined is somewhere empty.

A colouring $c : X \to A$ satisfies a colouring rule $F$ if almost everywhere (with respect to $m$) it follows that $c(x) \subseteq F(x, c(x_1), \ldots, c(x_n))$. We say that a colouring rule on $X$ is non-contradictory if it is satisfied by some colouring of the space. It is paradoxical if it is non-contradictory and for every finitely additive extension $\mu$ of $m$ for which the descendants are still measure preserving there is no colouring $c$ of the space satisfying the colouring rule almost everywhere such that $c$ is a $\mu$ measurable function (meaning that for all such $\mu$ there is a measurable subset $D$ of $A$ such that $c^{-1}(D)$ is not measurable with respect to $\mu$).

3 Paradoxical rules and paradoxical decompositions

We begin this section with a definition of a semigroup, introduced by A. Tarski [13] (see also [14], Chap. 10), that allows one to describe the paradoxical decompositions in an arithmetic way.

Let $G$ be a group acting on a set $X$. Recall that $A \subseteq X$ is $G$-equidecomposable to a subset $B$ of $X$ if there exist a partition of $A$ into the sets $A_1, \ldots, A_k$ and elements $g_1, \ldots, g_k$ of $G$ such that $g_1(A_1), \ldots, g_k(A_k)$ is a partition of $B$.

Define the set $X^* = X \times \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers and define the group $G^* = \{(g, \pi) : g \in G \text{ and } \pi \text{ is a permutation of } \mathbb{N}\}$. Further, we define the action $G^*$ on $X^*$ as follows: $g^*(x, n) = (g(x), \pi(n))$. 
Let $A$ be a subset of $X^*$. We will call those $n \in \mathbb{N}$ such that $A$ has at least one element with second coordinate $n$ the *levels* of $A$. And we will say that $A \subseteq X^*$ is *bounded* if it has finitely many levels.

Now we are ready to define the *semigroup of types* $\mathcal{S}$ as follows: the set $\mathcal{S}$ is the set of equivalence classes defined by the $G^*$-equidecomposability of bounded subsets of $X^*$. Let $A \subseteq X^*$ be a bounded set, we will denote the element of $\mathcal{S}$ corresponding to $A$ by $[A]$ and call it the *type* of $A$.

We define the addition $+$ in $\mathcal{S}$ as $[A] + [B] = [A \cup B']$, where $B'$ is a bounded set such that the levels of $B'$ and the levels of $A$ are disjoint. It is easy to check that $+$ is well-defined and the element $[\emptyset]$ is the identity. Moreover, we can define an order in $\mathcal{S}$ given by $[A] \leq [B]$ if and only if there exists a $[C]$ such that $[A] + [C] = [B]$. Thus $[A] \leq [B]$ if and only if $A$ is $G^*$-equidecomposable with a subset of $B$.

Now we can express the fact that $E \subseteq X$ is $G$-paradoxical as $[E] = 2[E]$ in the semigroup of types. See [14] for more informations about the semigroup.

The following theorem proved by D. König (see [14], Thm. 10.20) is essential for our considerations:

**Theorem A.** (Cancellation Law) If $\alpha, \beta \in \mathcal{S}$ and $n$ is a positive integer then $n\alpha \leq n\beta$ implies $\alpha \leq \beta$.

The proof of the Cancellation Law uses a form of the uncountable Axiom of Choice and it may happen that when applied to the algebra of Borel sets in some topological space it produces a non-Borel set. Therefore the problem of whether the Cancellation Law is true for an algebra of Borel sets (even in $\mathbb{R}$) remains open.

Let $\mathcal{A}$ be an algebra of subsets of $X$. We can define an algebra $\mathcal{A}^*$ of bounded subsets in $X^*$ as the algebra with the property that each level of $A \in \mathcal{A}^*$ is a subset of $\mathcal{A}$. Now we can define the semigroup $\mathcal{S}(\mathcal{A})$ as the set of equivalence classes restricted to the sets in $\mathcal{A}^*$.

We will need the following fundamental theorem of Tarski (see [14], Thm. 11.1):

**Theorem B.** Let $\mathcal{T}$ be one of $\mathcal{S}$ or $\mathcal{S}(\mathcal{A})$ and let $\epsilon$ be a specified element. Then the following are equivalent:

(i) For all $n \in \mathbb{N}$ it is not the case that $(n + 1)\epsilon \leq n\epsilon$ in $\mathcal{T}$;
There exists a semigroup homomorphism $\mu : \mathcal{T} \to [0, \infty]$ such that $\mu(\epsilon) = 1$.

The proof of Theorem B involves the following Extension Theorem proved by Tarski [13] (Satz 1.55.):

**Theorem C (Extension Theorem).** Let $\mathcal{U}$ be a subsemigroup of the semigroup of types $\mathcal{T}$ and let $\epsilon \in \mathcal{U}$ be an element that satisfies (i) from Theorem B. If $\mu$ is a measure on $\mathcal{U}$ with $\mu(\epsilon) = 1$ (a semigroup homomorphism $\mu : \mathcal{U} \to [0, \infty]$ such that $\mu(\epsilon) = 1$), then there is an extension of $\mu$ to $\mathcal{T}$.

In what follows we will use the fact that in the case of $\sigma$-algebras, the inequality of Theorem B can be substituted by equality. This follows from the Banach-Schröder-Bernstein theorem (see [14], Thm. 3.6).

Let $(X, \mu)$ be a standard space and let $G$ be a group acting on $X$ such that $\mu$ is $G$-invariant. We will say that a set $A \subset X$ is **absolutely non-measurable** if it is not measurable with respect to any finitely additive $G$-invariant measure that extends $\mu$. We will say also that a set $E \subseteq X$ is **weakly paradoxical** with respect to some $G$-invariant algebra $\mathcal{A}$ if $(n+1)[E] = n[E]$ in the semigroup $S(\mathcal{A})$ for some positive integer $n$, meaning that one can pack $n+1$ copies of $E$ into $n$ copies of $E$. Notice that $(n+1)[E] = n[E]$ implies that $m[E] = n[E]$ for all $m > n$. What may be problematic is walking this back to $2[E] = [E]$. When $2[E] = [E]$, then we will say that $E$ is **strongly paradoxical**.

In many situations the proof of existence of weak paradoxical decomposition needs some additional assumptions. But we can introduce the following notion that leads to a theorem that justifies the term “paradoxical colouring rule”. Let $\mathcal{A}$ be the $\sigma$-algebra generated by the Borel sets and the colour classes of some colouring determined by a paradoxical rule. We say that a Borel subset $A$ of a standard Borel space $(X, \mu)$ is **measurably $G$-paradoxical** if there exists a Borel set $B$ with $\mu(A) \neq \mu(B)$ and $A$ is $G$-equidecomposable to $B$ using pieces in $\mathcal{A}$.

We have also the following notion that sharpen the definition of paradoxical rule; we say that a paradoxical finitary rule is **superparadoxical** if any colouring satisfying the rule is not measurable with respect to any finitely additive $G$-invariant measure defined on the $\sigma$-algebra $\mathcal{A}$. The motivation for this notion is that it harmonizes with the Tarski’s Theorem B and as we conjecture it relates paradoxical rules to the Greenleaf problem mentioned
in the introduction.

We mentioned above that to get more paradoxical decompositions we need some additional assumptions. Here we show that under some additional assumptions the space exhibits the strongest form of paradoxical decompositions.

Let $X$ be a metric space. Consider now the type $[U]$ of a bounded open set $U \subset X$, defined for the semigroup of types $S(A)$, where $A$ is a $\sigma$-algebra of subsets of $X$. We will say that bounded open sets in $X$ admit sequences of similar sets if for any bounded open set $U$ there exists a positive integer $n$ such that $(n+1)[U] = n[U]$ and there exists a sequence $V_i$ of open sets with diameters decreasing monotonically to 0 and such that $(n+1)[V_i] = n[V_i]$. An example of a metric space where open sets admit sequences of similar sets provides Euclidean space. To observe it we use some conjugations by similarities.

Since the finitely additive measures appearing in the definition of paradoxical rules are not a priori unique, we need a kind of notion that assures the uniqueness. These topics is related to some properties of groups acting on the space in question.

Let $S^n$ be the $n$-dimensional Euclidean sphere. Recall that for $F \in L^2(S^n, \lambda)$ with $\|F\|_2 = (\int_{S^n} |F|^2 d\lambda)^{1/2}$ the action of a countable group of isometries $G$ on $S^n$ has spectral gap if there exist a finite set $S \subset G$ and a constant $\kappa > 0$ such that $\|F\|_2 \leq \kappa \sum_{g \in S} |g \cdot F - F|_2$ for any $F \in L^2(S^n, \lambda)$ with $\int_{S^n} F d\lambda = 0$. Clearly, the notion of spectral gap can be extended to probability spaces.

Recently R. Boutonnet, A. Ioana and A. S. Salehi Golsefidy [3] generalized the notion of spectral gap as follows: Let $G$ be a countable group acting on a standard measure space $X$ then the action has local spectral gap with respect to a measurable set $B \subset X$ of finite measure if there exist a finite set $S \subset G$ and a constant $\kappa > 0$ such that $\|F\|_{2,B} \leq \kappa \sum_{g \in S} |g \cdot F - F|_{2,B}$ for any $F \in L^2(S^n, \lambda)$ with $\int_B F d\lambda = 0$. Here $\|F\|_{2,B}$ denotes $(\int_B |F|^2 d\lambda)^{1/2}$.

The notion of local spectral gap appears implicitly, in the case of action of group of isometries on $\mathbb{R}^n$, in Margulis [9] and is used to get a positive answer to Ruziewicz problem for $\mathbb{R}^n$ ($n \geq 3$).

Let $X$ be a metric space and let $G$ be a group acting on $X$, we will say that the action of $G$ is paradoxical if any two bounded subsets of $X$ with compact closures and non-empty interiors are $G$-equidecomposable with respect to the algebra of all sets.
It is well-known that if the action of a group $G$ is paradoxical and has spectral gap than the finitely additive extension of the completed Borel measure is unique (see [3],[6]). Therefore we have a broad class of spaces where the uniqueness of our extension holds.

**Theorem 1:** Let $F$ be a paradoxical rule with finitely many colours defined by a topologically transitive action of a group $G$ on a standard Borel space $(X, \mu)$ with $G$-invariant measure $\mu$. Then:

(i) For any colouring $c$ satisfying the rule $F$ there exists a Borel set that is measurably $G$-paradoxical with respect to the $G$-invariant $\sigma$-algebra $A$ generated by the Borel sets and all colour classes of $c$;

(ii) Assuming additionally that the extension of $\mu$ to $A$ is unique, for any colouring $c$ that satisfies $F$, any bounded open subset of $X$ is weakly $G$-paradoxical with respect to $A$;

(iii) Moreover, suppose that $X$ is metrizable and open subsets of $X$ admit sequences of similar sets. Then any compact set of $A$ with non-empty interior is strongly paradoxical using pieces in $A$;

(iv) If the rule $F$ is superparadoxical, then any compact subset of $X$ with non-empty interior is weakly $G$-paradoxical in $A$.

Proof of part (i). It is easy to see that any measure on the semigroup of types $S(A)$ induces a finitely additive $G$-invariant measure $m$ on the corresponding algebra $A$. Indeed, for any subset $A \in A$ we put $m(A) = \mu([A])$, where $\mu$ is the measure defined on $S(A)$ and $[A]$ is the type of $A$.

On the other hand, any finitely $G$-invariant measure $m$ defined on $A$ induces a measure on $S(A)$. To get such a measure it is enough to sum up the $m$-measures of finitely many levels defining any type of $S(A)$.

Suppose to the contrary that no Borel set is measurably $G$-paradoxical with respect to $A$. Then the Borel measure $\mu$ induces a measure on the subsemigroup $U$ of $S(A)$ that corresponds to the types of Borel sets. Clearly, the measure is well-defined by the assumption that no two Borel sets are measurably $G$-paradoxical with respect to $A$. Thus, by the Extension Theorem, there exists a measure on $S(A)$ that extends the measure on $U$. Clearly, the measure induces a $G$-invariant measure on $A$ that extends $\mu$. Therefore the colouring $c$ is measurable with respect to $m$. But this contradicts the fact that $F$ is a paradoxical rule and finishes the proof.

Proof of part (ii). Suppose to the contrary that there exists a bounded
open subset of $X$ that is not weakly $G$-paradoxical with respect to $\mathcal{A}$. Then Theorem B and the uniqueness of the extension of $\mu$ as the Borel measure imply the existence of a finitely additive, $G$-invariant measure $\nu$ on $\mathcal{A}$ with $\nu(U) = 1$. And this in turn implies that the colouring $c$ is $\nu$-measurable, in contradiction to the paradoxicality of the rule $F$. $\square$

**Proof of part (iii).** First, we observe that since bounded open sets are weakly paradoxical and that $\mathcal{A}$ is a $\sigma$-algebra, there exists a positive integer $m$ such that $(m + 1)[U] = m[U]$ in $\mathcal{S}(\mathcal{A})$ for any open set $U \subseteq X$. And this implies:

\[(1) \quad k[U] = m[U] \text{ in } \mathcal{S}(\mathcal{A}) \text{ for any integer } k \geq m.\]

Now take, a compact subset $K \subset X$ with nonempty interior and let $V \subset K$ be an open set. Since the action of $G$ is topologically transitive, $K$ can be covered by the union of $l$ congruent copies of $V$. Therefore we have:

\[(2) \quad [K] \leq l[V] \text{ in } \mathcal{S}(\mathcal{A}) \text{ for some integer } l \geq 2.\]

Clearly, (2) implies:

\[(3) \quad m[K] \leq ml[V] \text{ in } \mathcal{S}(\mathcal{A}) \text{ for some integer } l \geq 2,\]

and by (1) and (3) we get

\[(4) \quad m[K] \leq ml[V] = m[V] \text{ in } \mathcal{S}(\mathcal{A}).\]

On the other hand, since $V \subseteq K$, we get $m[V] \leq m[K]$ which by the Banach-Schröder-Bernstein theorem and by (4) implies $m[K] = m[V]$ in $\mathcal{S}(\mathcal{A})$.

Since open sets admit sequences of similar sets, there are is an open set $W$ such that that $(m + 1)[W] = m[W]$ and the union of $m$, $G$-congruent disjoint copies of $W$ is contained in $K$.

Hence, by the Banach-Schröder-Bernstein theorem and by (4) we get $m[K] = m[W]$ and also $m[K] \leq [K]$ in $\mathcal{S}(\mathcal{A})$. Clearly, $[K] \leq m[K]$ is also true. Again, the Banach-Schröder-Bernstein theorem implies $[K] = m[K]$ in $\mathcal{S}(\mathcal{A})$.

Finally, to get the strong paradox we observe that $[K] \leq 2[K] \leq m[K]$ in
$S(\mathcal{A})$ and so the Banach-Schröder-Bernstein theorem imples $[K] = 2[K]$ in $S(\mathcal{A})$. □

*Proof of part (iv).* This follows in exactly the same manner like in the cases (ii) and (iii) taking into an account that any colouring $c$ is not measurable with respect to any finitely additive $G$-invariant measure defined on $\mathcal{A}$. □

In the context of the above we may ask the following question such that a positive answer would extend the results of Tarski:

**Question 1:** Let $G$ be a group acting in a measure preserving way on a standard Borel probability space $(X, \mu)$. Is the existence of a $G$-paradoxical decomposition of a subset $E$ of $X$ with $\mu(E) > 0$ equivalent to the existence of a paradoxical colouring rule using the group $G$ to define the descendents?

M. Bounds [2] obtained recently a partial solution to this question. Whenever there is a partition of the space $X$ witnessing $n[X] = (n + 1)[X]$ he showed that there is a paradoxical colouring rule for which that partition is a solution. We note that a positive answer to Question 1 may shed some light on the following problem of Greenleaf: Let $G$ be a group acting faithfully and transitively on a space $X$. What are necessary and sufficient conditions for the action such that there is no finitely additive $G$-invariant probabilistic measure on all subsets of $X$?

**Question 2:** Let $F$ be a paradoxical rule and let $c$ be a colouring satisfying the rule. Are some of the colour classes of $c$ absolutely non-measurable?

A positive answer to this question would lend relevance to the above mentioned result of M. Bounds. The following construction that shows that the pieces witnessing the Banach-Tarski paradoxical decomposition are absolutely non-measurable with respect to any $\sigma$-algebra that contains the pieces. It indicates the answer to Question 2 might be positive:

Let $F$ be a free non-abelian group of rank two generated by $s$ and $t$. Consider the sets $w(s), w(s^{-1})$ and $w(t), w(t^{-1})$ of reduced words of $F$ begining on the left on $s, s^{-1}$ and $t, t^{-1}$, respectively. Then we have:
\[ sw(s^{-1}) = w(s^{-1}) \cup w(t) \cup w(t^{-1}) \text{ and also } \]
\[ tw(t^{-1}) = w(t^{-1}) \cup w(s) \cup w(s^{-1}). \]

**Proposition 1** Let \( \mathcal{A} \) be the \( G \)-invariant \( \sigma \)-algebra generated by the set \( w(s^{-1}) \). Then \( w(s), w(t), w(t^{-1}) \in \mathcal{A} \).

**Proof.** By \( G \)-invariance of \( \mathcal{A} \) we obtain that \( sw(s^{-1}) \in \mathcal{A} \) and thus \( w(s) \in \mathcal{A} \). Since \( \mathcal{A} \) is closed under complementation we get that \( w(t^{-1}) \cup w(t) \cup \{1\} \in \mathcal{A} \). Now, by \( G \)-invariance we get that \( t(w(t^{-1}) \cup w(t) \cup \{1\}) = w(s^{-1}) \cup w(s) \cup w(t^{-1}) \cup tw(t) \cup \{t\} \in \mathcal{A} \) too. Since \( w(s^{-1}) \cup w(s) \in \mathcal{A} \), then subtracting it from \( w(s^{-1}) \cup w(s) \cup w(t^{-1}) \cup tw(t) \cup \{t\} \) still leaves us in \( \mathcal{A} \). So \( w(t^{-1}) \cup tw(t) \cup \{t\} \in \mathcal{A} \). Iterating, the process of moving by \( t \) and subtracting then by \( w(s^{-1}) \cup w(s) \) we obtain that the set \( w(t^{-1}) \cup t^n w(t) \cup \{t^n\} \in \mathcal{A} \) for any positive integer \( n \).

Now, since \( \mathcal{A} \) is a \( \sigma \)-algebra and the group \( F \) is free we have that
\[
\bigcap_{n=1}^{\infty} (w(t^{-1}) \cup t^n w(t) \cup \{t^n\}) = w(t^{-1}) \in \mathcal{A}.
\]

Now, by the \( G \)-invariance we infer that \( tw(t^{-1}) \in \mathcal{A} \) and so \( w(t) \in \mathcal{A} \) too.

\[ \Box \]

## 4 Easy Examples

The Hausdorff rule is of rank two. We need to establish that there are paradoxical rules of rank one. The first example is a rank one stationary rule with three colours whose satisfaction is equivalent to the satisfaction of the Hausdorff rule. Example 1 was given to us by M. Bounds [2]. The second and third examples are rank one rules that mimick rank two rules. We are not so interested in the second and third examples because they represent an oversimplification, a cutting of a Gordian knot using an enlarged space or an enlarged colour set.

It is worthwile to note that the colour classes of any colouring satisfying the rule from Example 1 form a paradoxical decomposition of \( X \). In particular, we can show that the unit sphere \( S^2 \) up to a countable set of fixed points is paradoxical with sets defined by any of the colourings satisfying our rule. This in turn implies (by the arguments in Chapter 3 in [14]) a paradoxical
decomposition of Banach and Tarski of the unit ball in $\mathbb{R}^3$.

**Example 1** Let $G$ be any group generated freely by $\tau, \sigma_1, \ldots, \sigma_{k-1}$, where the order of $\tau$ is either infinite or divisible by 3, and the order of all the $\sigma_i$ are either infinite or divisible by 2. Let $X$ be a probability space acted upon freely by $G$ almost everywhere such that every element of $G$ is measure preserving. There are three colours $A_1, A_2, A_3$ and the arithmetic is modulo 3. Assume that $\tau^{-1}(x)$ is coloured $A_i$. If $\tau(x)$ is coloured either $A_i$ or $A_{i+1}$ then colour $x$ with $A_{i+1}$. If $\tau(x)$ is coloured $A_{i-1}$ then count how many points among $\tau(x), \tau^{-1}(x), \sigma_1(x), \ldots, \sigma_{k-1}(x)$ are coloured $A_1$ (which cannot be all of them because $\tau(x)$ and $\tau^{-1}(x)$ are coloured differently). If the number is strictly between 0 and $k$, colour $x$ with the same colour as that of $\tau(x)$. Otherwise $x$ is coloured with $A_{i+1}$. \hfill \Box

**Proposition 2** The colouring rule of Example 1 is paradoxical.

**Proof:** If the colouring $c$ satisfies the colouring rule, we claim that for almost all $x$ if $\tau^{-1}(x)$ is coloured $A_i$ then $x$ is coloured $A_{i+1}$. The colour given to $x$ by the rule is never the same as the colour given to $\tau^{-1}(x)$. The only way for $x$ to be not coloured with $A_{i+1}$ is for $\tau(x)$ to be coloured $A_{i-1}$. But then $x$ is also assigned the colour $A_{i-1}$, which is not possible because then $\tau(x)$ would not have been assigned the colour $A_{i-1}$. Now that we know that the colours cycle forward in the $\tau$ direction, by the rule it must follow that either none of the $\tau(x), \tau^{-1}(x), \sigma_1(x), \ldots, \sigma_{k-1}(x)$ are coloured $A_1$ or all but one are coloured $A_1$. If $x$ is coloured $A_1$ then because both $\tau^{-1}(x)$ and $\tau(x)$ cannot be coloured $A_1$ it must follow that none of the $\sigma_1(x), \ldots, \sigma_{k-1}(x)$ are coloured $A_1$. If $x$ is not coloured $A_1$ then at least one of $\tau^{-1}(x)$ and $\tau(x)$ is coloured $A_1$, hence all of the $\sigma_1(x), \ldots, \sigma_{k-1}(x)$ are coloured $A_1$.

Assuming that the colouring is measurable with respect to some finitely additive $G$-invariant extension measure, it follows from the measure preserving properties of all the generators that the global probability for $A_1$ must be simultaneously at most $\frac{1}{3}$ and at least $\frac{2}{3}$, a contradiction. Next we show that the colouring rule can be satisfied.

Choosing an $x$ in any orbit of $G$ (using some axiom of choice) for which $g_1(x) = g_2(x)$ implies that $g_1 = g_2$, colour $x$ with $A_1$, then colour the $\tau$ orbit of $x$ accordingly. Then continue with the rule that if $y$ is coloured $A_1$ then all of the $\sigma_i(y)$ and $\sigma^{-1}(y)$ are not coloured $A_1$ and if $y$ is not coloured $A_1$ then
all of the $\sigma_i(y)$ and $\sigma_i^{-1}$ are coloured $A_1$. Continue the process indefinitely on the whole orbit.

Example 2 Assume that $R$ is a paradoxical rule using a semigroup $G$ acting on $X$. Let $\hat{X}$ be $X \times \{a, b\}$, with the probability given to both $\{a\} \times A$ and $\{b\} \times A$ half of that given to $A$ in the space $X$. We define $\rho$ to be the measure preserving involution that switches between $(a, x)$ and $(b, x)$ for every $x \in X$. We use the same colours, and assume that there are at least two colours. The new rule assigns to every $(b, y)$ the same colour as that of $(a, y)$. The rule for colouring $(a, y)$ is easy to describe. If the colour of $(b, y)$ follows the rule $R$ with respect to the $(a, z_i)$ for all the descendents $z_i$ of $y$, as it should be after dropping the $a$ and $b$ coordinates, then colour $(a, y)$ the same colour as that of $(b, y)$. Otherwise colour $(a, y)$ differently than $(b, y)$. In any colouring satisfying this new rule the points $(a, y)$ and $(b, y)$ are given the same colour, implying that the $R$ rule is being followed on both copies of $X$.

Example 3 Assume that $R$ is a paradoxical rule using a semigroup $G$ acting on $X$ where one of the descendents is defined by an invertible $g \in G$. Let $C$ be the colouring set of the rule $R$ and instead colour $X$ with $C^2$ colours. The new rules requires that the second colour of $x$ is the copy of first colour of $g(x)$, and the first colour of $x$ agrees with the second colour of $g^{-1}(x)$ if and only if the second colour of $g^{-1}(x)$ is the correct colour for $x$ when following the rule $R$ with respect to the first colours of descendents of $x$. The process is really the same as that of Example 2, with the second colour of $g^{-1}(x)$ playing the same role as the colour of $(b, x)$ in Example 2.

5 Proper Colouring

In this section we present a context in which the proper vertex colouring rule is paradoxical. The distinction between stationary and non-stationary rules is especially important with proper colouring. With proper colouring, we make no requirement of the colouring other than that the colour of $x$ and the colour of any descendent of $x$ must be different. With other colouring rules, we could keep the finite set of descendents fixed and allow some of them to be irrelevant, dependent on location. But with proper colouring all descendents must be relevant and in the same way. The best way to resolve
this is to define our proper colouring rule to be non-stationary if the subset of descendants relevant to proper colouring changes with location. For one, this allows the degree of the relevant graph to vary by location. If all descendants are relevant everywhere to the proper colouring then it is stationary, and free action almost everywhere implies that almost everywhere the degree of the vertices is a constant. If the rule is non-stationary, we do require the Borel property, meaning in the context of finitely many colours and finitely many descendants that the subset of relevant descendants is determined by membership in finitely many Borel sets that partition the space.

Below we present an example where non-stationary proper colouring is paradoxical. Before we do this, we first present a paradoxical colouring rule where some colour is always allowed, meaning rank one. We move from this to a proper list colouring that is paradoxical, and finally to a non-stationary proper colouring that is paradoxical.

5.1 The fundamental example:

Example 4 With \( X = \{-1, 1\}^2 \), let \( T_1 \) and \( T_2 \) be the two generators of \( \mathbb{F}_2 \). We create four colours \( a_{1,u}, a_{1,c}, a_{2,u} \) and \( a_{2,c} \), the \( c \) or \( u \) standing for “crowded” or “uncrowded”. The colour is broken into two parts, its active part, the \( 1 \) or \( 2 \), and its passive part, the \( c \) or \( u \). From every \( x \in X \) we must direct an arrow from \( x \) to one of two of its neighbours. If \( x^e = -1 \), our choice is between \( T_{1}^{-1}x \) or \( T_{2}^{-1}x \). If \( x^e = 1 \) our choice is between \( T_{1}x \) or \( T_{2}x \). If both of our choices for such neighboring points are coloured with the passive colour \( c \) (meaning either \( a_{1,c} \) or \( a_{2,c} \)) or both of our choices are coloured with \( u \), the colouring rule allows us to place the arrow in either direction. But if one of these two points is coloured with \( c \) and the other is coloured with \( u \) then the colouring rule demands that the arrow must be placed toward the one coloured with \( u \). If two or more arrows are directed toward a point \( x \), then the colouring rule requires that its passive colour is \( c \). Otherwise its passive colour must be \( u \). The active colour \( 1 \) or \( 2 \) refers to the direction of the arrow, the active colour \( i \) if the arrow is directed to either \( T_{i} \) or \( T_{i}^{-1} \). Notice that every point receives an active and a passive colour and that the colouring scheme pertains to two independent systems on \( X \) that are separated by alternating applications of the generators.

We define the p-degree of a vertex as the number of potential arrows that could be pointed toward this vertex (the passive degree, to distinguish from
the usual definition of degree in a graph).

**Proposition 3** The colouring rule of Example 4 is paradoxical.

**Proof**
The vertices with p-degree zero take up $\frac{1}{16}$ of the space. Our claim is that with a colouring satisfying the rule almost all vertices are uncrowded, meaning they have the passive colouring $u$. This implies that the colouring cannot be measurable with respect to any finite extension for which the group $\mathbb{F}_2$ is measure preserving, because at least $\frac{1}{16}$ of the vertices cannot have any arrows pointed toward them. By one accounting there is an arrow exiting every vertex but by another accounting the average number of arrows coming in to vertices must be no more than $\frac{15}{16}$. Another way to see the paradox is that the arrows define an almost everywhere injective mapping via group elements from the whole space to a cylinder set of measure no more than $\frac{15}{16}$.

Now let us assume that $x$ is a crowded vertex and see what is necessary to maintain this situation in a colouring satisfying the rule. There must be two distinct vertices $y$ and $z$ such that there is an arrow from $y$ to $x$ and an arrow from $z$ to $x$. Let's focus on just one of them, without loss of generality the $y$. As the colouring rule is satisfied, the existence of an arrow from $y$ to $x$ implies that the other vertex toward which $y$ could direct an arrow is another crowded vertex, call it $w$. As the arrow is already defined from $y$ to $x$ it means that there are at least two arrows pointed inward to $w$ that do not start at $y$. Letting $v_1$ and $v_2$ be two of those vertices, let $u_1$ and $u_2$ be the vertices for which there could have been an arrow from $v_i$ to $u_i$, however instead the arrow was from the $v_i$ to $w$. We recognise by induction the existence of a chain of backwardly directed arrows, starting at $w$, moving to $u_1$ and $u_2$ (from vertices so far un-named) and beyond, such that at alternating stages the induced graph branches into either two or three directions. If there are three such branches in some places, we could reduce the problem to the existence of a binary tree. Each of these vertices of the binary tree has p-degree at least three. Now let $p$ be the probability of there existing such an infinite chain, the probability relative to the start at a vertex like $y$ moving in the direction away from $x$. As the space is defined homogeneously (that the probabilities for $-1$ or $1$ are independent regardless of shift distances) we can calculate $p$ recursively. From the start, at $x$ and $y$, there are two possibilities: the next vertex $w$ could be of p-degree three and the chain of backward arrows
continues with these two adjacent vertices on the other side of \( y \), or \( w \) is of p-degree four and the chain continues with at least two of these three vertices on the other side of \( y \). In the first case the conditional probability that \( w \) is of p-degree three is \( \frac{3}{8} \) (conditioned on the move from \( y \) to \( w \)) and then of continuation of the chain indefinitely happens with probability \( p^2 \). In the second case, the conditional probability of p-degree four is \( \frac{1}{8} \) and the probability of continuation 3\( p^2 - 2p^3 \) (three choices for the two next vertices minus the possibility, counted twice, that continuation is possible in all three directions). We have the formula \( p = \frac{3}{8}p^2 + \frac{1}{8}(3p^2 - 2p^3) = \frac{3p^2 - p^3}{4} \). Factoring out the \( p = 0 \) solution, we are left with \( p^2 - 3p + 4 \), which has no real solutions. We conclude that the stochastic structure of p-degrees does not allow for crowded points to exist in more than a set of measure zero.

Now we show (using AC) that there does exist a colouring satisfying the rule. For every orbit choose a representative \( x \) and label every other vertex in this same orbit as \( gx \) by the group element \( g \) used to travel from \( x \) to \( gx \). Every group element \( g \) has a length, the minimal number of uses of \( T_1, T_1^{-1}, T_2 \) and \( T_2^{-1} \) used to construct \( g \) (where the length of the identity is zero). Let the length of a vertex \( gx \) in the orbit be the length of \( g \). Colour \( x \) with either colour 1 or 2. Colour all vertices of length 1 next, then all vertices of length 2, and so on in the following way. At every stage of the process, a vertex of length \( l \) is adjacent to one vertex of length \( l - 1 \) and three of length \( l + 1 \). Therefore from any vertex \( y \) of length \( l \) one can always point the arrow toward a vertex of length \( l + 1 \), regardless of the value of \( y^e \). There can be no other arrow toward \( y \), as the three other points adjacent to \( y \) are all of length \( l + 2 \). Seeing that there are no vertices receiving two arrows, all vertices can be coloured with either \( a_1, u \) or \( a_2, u \).

\( \square \)

5.2 Proper list colouring:

Example 5 Consider the four group elements \( T_1T_2^{-1}, T_1^{-1}T_2, T_2T_1^{-1}, T_2^{-1}T_1 \), and the twelve length four group elements created by multiplying two of them together. Let \( g_1, \ldots, g_{16} \) be these distinct 16 group elements, with \( g_1, \ldots, g_4 \) the ones of length 2 and the \( g_5, \ldots, g_{16} \) the ones of length 4. Let \( X' \) be the subset of \( X \) such that \( g \) acts freely on \( X' \). Notice that these 16 group elements are paired up by inverses. By (see [1] or [4]) there are 17 Borel sets, partitioning the space \( X' \), such that \( g, x \) and \( x \) belong to different sets for every choice of \( i \) and \( x \in X' \). To each of these 17 sets associate a colour. For
every \( x \), we create a list of colours of cardinality 2 from the list of 17 colours. If \( x^e = 1 \), then the colours allowed at \( x \) are the colours given to \( T_1 x \) and \( T_2 x \) (by membership in the Borel sets). If \( x^e = -1 \), then the colours allowed at \( x \) are the colours given to \( T_1^{-1} x \) and \( T_2^{-1} x \). This structure follows that of Example 4 with the same concept of two potential arrows directed in two directions as determined by the \( e \) coordinate.

Now we define the adjacencies for the purpose of the list colouring. The point \( x \) is adjacent to \( y \) if there is a \( z \) such that an arrow could be pointed from \( x \) to \( z \) and an arrow could be pointed from \( y \) to \( z \) as with Example 4. Notice that these points \( x \) and \( y \) differ by one of the \( g_1, g_2, g_3, g_4 \). In this way, a graph is created by cliques such that for every \( z \) there is a clique of size equal to the p-degree of \( z \), consisting of the vertices from which an arrow could be directed to \( z \). We call this clique the clique centered at \( z \). If \( x \) is in \( X \setminus X' \) then we assume it has no adjacencies, so it is irrelevant which two colours are in the list. This graph of cliques we call the secondary graph. \( \square \)

Next we attempt to colour the vertices of this secondary graph (of cliques) using the same set of 17 colours, forgetting how each point was coloured originally, but retaining the lists of colours and the above defined graph adjacencies from cliques centered about points.

**Proposition 4** Proper list colouring of the secondary graph of Example 7 is paradoxical.

**Proof**
First assume we have a proper list colouring. Because the original colours are distinct on either side of the \( g_1, \ldots, g_4 \), there are two distinct colours in every list, and they correspond to the two directions an arrow can be directed, in the same manner as with Example 4. That the colouring is proper implies that there are no two arrows directed to the same point. If a list colouring is proper and measurable (finitely additive \( G \)-invariant), there is one arrow coming out of every point in \( X' \) however coming toward the centres of cliques the global average is no more than \( \frac{15}{16} \). As with Example 4 each point can be shifted according to its colour in the direction of its arrow to define a measure preserving injective map from all of the space to a finite collection of cylinders of measure \( \frac{15}{16} \).

Now we show that there are proper list colourings of the secondary graph. Because the colours are different on either side of the length four elements
$g_5, \ldots, g_{16}$, if it is possible to point an arrow from two distinct $x, y \in X'$ toward some $z$ but the choice is made instead to point both of these arrows in the other directions, $x$ and $y$ cannot get coloured with the same colour. (This is important, because they share a clique, the one centered at $z$.) It follows from this that any orientation of arrows satisfying the colouring rule in Example 4 will also define a proper list colouring of the secondary graph. \hfill \Box

5.3 Proper colouring:

**Example 6** We use the same structure of Borel sets and colours of Example 5 and assume that the number of sets/colours of that example is minimal, meaning that each set/colour is given positive probability and, due to the ergodic property of each of the $g_1, \ldots, g_{16}$, for every positive $\epsilon$ there is an odd $N$ such that from all but a Borel subset $Q$ of measure no more than $\epsilon$ for all $x$ in $X \setminus Q$ all the colours appear within a radius $N$ of $x$ when applying words of odd length. (By ergodicity, this is true when restricted to just one of the 16 group elements $g_i$.) We call this the original colouring of $X$. We create two copies of the space $X$, with the map $\rho$ defining the “identity” map from the first copy to the second copy. If it helps to understand, we could say that each element of $G$ commutes with $\rho$ and $\rho$ is a measure preserving involution between the two halves of the new space, each half given probability $\frac{1}{2}$. We keep the previous adjacency relations of the secondary graph of Example 5 within the first copy of $X$, meaning adjacencies defined by cliques centred about every $z$ of size equal to the p-degree of $z$. In the second copy of $X$, for every $x$ we connect with edges the point $\rho(x)$ with all the $\rho(y)$ such that $y$ is within a distance of $2N + 10$ from $x$ applying words of even length and $y$ coloured differently from $x$ in the original colouring. Furthermore any $x \in X \setminus Q$ in the first copy is connected to any $\rho(z)$ in the second copy if one moves from $x$ to $z$ with a word of odd length of distance less than or equal to $N$ and $z$ is not coloured the same originally as either $z_1$ or $z_2$ such that there is a potential arrow from $x$ to the $z_i$ (according to the construction of Example 4, which continues to define the structure of Example 5 and this example). If $x$ belongs to $Q$, then from $x$ in the first copy there are no adjacencies.

Next we assume that $\epsilon$ is strictly less than $\frac{1}{512}$. $N$ is so defined according to $\epsilon = \frac{1}{512}$, we drop the original colouring but keep the adjacencies that then define a graph with finite degree. \hfill \Box
Proposition 5  Proper colouring of Example 6 is paradoxical.

Proof  
We assume that we have a proper colouring of the two copies of $X$, using the same colours of the original colouring. If $z$ defines the centre of a clique in the first copy, these colours in the clique are distinct. Furthermore, with probability at least $1 - \frac{1}{512}$ (membership in $X \setminus Q$) for an $x \in X$ the proper colouring at $x$ defines an arrow from $x$ to one of two potential points $z_1$ or $z_2$ according to the way the two points $\rho(z_1)$ and $\rho(z_2)$ are now coloured. Assuming a finitely additive $G$-invariant measure, the arrows defined outward have a global weight totaling at least $1 - \frac{1}{512}$. On the other hand, given a $z$ defining a clique in the first copy of $X$, the probability that at least one point in the clique centered at $z$ is in $Q$ is no more than $\frac{1}{128}$. That means the weight of arrows toward such points $z$, on the average throughout $X$, can be no more than $1 + \frac{4}{128} - \frac{1}{128} = \frac{31}{32}$. 

To show that there is some colouring according to the rule, we start by colouring the second copy of $X$ according to the original colouring of the first copy into Borel sets (via the map $\rho$). Because the set $Q$ has no adjacencies, any previous solution from Example 5 on the first copy of $X$ will define a proper colouring. Hence the colouring is paradoxical.

6  Further study

Question 3: Given any probability space $X$ and a finitely generated measure preserving group $G$ acting on $X$, let $P$ be the stationary colouring rule that requires only that the colouring must be proper, where adjacency is defined through the Cayley graph using the finitely many generators. Is there such an $X$ and $G$ such that the stationary rule $P$ with finitely many colours is paradoxical?

Our first instinct is to believe that such a colouring rule $P$ cannot be paradoxical, because the rule $P$ is not complex enough. Through increasing the number of colours, so one could think, one jumps from impossible in any way to possible with respect to some finitely additive and invariant way. The situation is likely very different to that with Borel colouring [7], where the Borel property separates what can be accomplished with different numbers
of colours.

A colouring rule with \( k \) colours is (stationarily) essential if it is paradoxical of rank two and there exists no (stationary) paradoxical rank one colouring rule with the same colours whose satisfaction implies the satisfaction of the original rule. A space \( X \) is (stationarily) essential with \( k \) colours if there is no (stationary) paradoxical rank one colouring rule with \( k \) colours defined on \( X \), and yet there is a paradoxical colouring rule with \( k \) colours defined on \( X \) of rank two. Intriguing is the difference between what can be accomplished by stationary and non-stationary colouring rules. Example 1 shows that the Hausdorff rule is not stationarily essential.

**Question 4:** Is there a colouring rule that is stationarily essential?

We conjecture that the answer to Question 4 is yes, though before communication with M. Bounds we thought that the Hausdorff rule was stationarily essential.

**Question 5:** Does there exist a space and a semigroup action that is essential with respect to any number of colours?

When none of the descendants are invertible Example 3 doesn’t apply, and it is plausible that some paradoxical colouring rules are not rank one in character no matter how many colours are allowed.

There is something satisfying about Example 4 and unsatisfying about Examples 1, 2, and 3. Example 4 employs a stochastic process that seems to push colourings toward the satisfaction of the rule. On the other hand, the existence of colourings witnessing the rules of the first three examples seem to be either accidental or contrived. One could perceive satisfaction of a colouring rule to be a kind of fixed colouring, with the colouring rule defining some kind of iterative process that does or doesn’t bring the colourings closer to satisfaction of the rule. Of course if a colouring rule forced almost everywhere (with respect to the original measure \( m \)) an eventually stable colour in finitely many colouring stages with respect to some initial measurable colouring then there would be measurable colouring solutions with respect to the completion and the rule could not be paradoxical (given that the descendants remain measure preserving with respect to the completion). Although Example 4 does seem to contain a force moving toward its
satisfaction, iterations of the rule would likely involve long periods of stability followed by solutions from one area of the space colliding occasionally with solutions from another. A desired property may concern the relative stability of the colourings in the limit, and inspires the following question.

**Question 6:** Is there a refinement to the definition of a rank one paradoxical colouring rule that identifies a credible force toward its satisfaction?

Suppose one had a colouring rule for $S^G$ where $S$ is a finite set, $G$ is a group, and $C$ are the finite set of colours. Another structure to consider is $\{S \times C\}^G$, where we assume a random colouring start to $S^G$ inherited from the $C$ coordinate. We could consider how the colouring rule generates iterations of colourings on $\{S \times C\}^G$.

Like in Section 2 we will denote by $\mathcal{F}$ the sigma algebra on which a probability measure $m$ is defined, in most cases $\mathcal{F}$ will denote Borel sets. The *deficiency* of a colouring rule on $X$ is the infimum of all probabilities $\rho$ such that there is a $\mathcal{F}$ measurable subset of size $1 - \rho$ where the rule can be satisfied in some finitely additive and invariant way.

**Question 7:** Do all paradoxical colouring rules have positive deficiency?

Assume a sequence of colouring functions $c_1, c_2, \ldots$, corresponding to finitely additive measures $\mu_1, \mu_2, \ldots$ extending the original measure (with the descendants measure preserving) and subsets $X_1, X_2, \ldots$ such that for each $i = 1, 2, \ldots$ the measure of $X_i$ is greater than $1 - \frac{1}{7}$, the $c_i$ are $\mu_i$ measurable, and the colouring rule is satisfied by $c_i$ on $X_i$. Will there exist a colouring $c$ and a corresponding finitely additive $\mu$ (with the descendants measure preserving) that satisfies the rule almost everywhere and is $\mu$ measurable? Initially it seems that the answer should be no, because there is no guarantee that any colouring in a pointwise limit should have any measurable properties. However we suspect that the colouring rule will allow for finitely additive measurability in the limit, meaning that the answer to Question 7 is yes and this is another difference between paradoxical colouring rules and the topic of Borel colouring.

The problem of extending the Cancellation Law inspires the following question.

**Question 8:** Can one extend the conclusion of Theorem 1 to $2[X] = |X|$?
In the definition of a colouring rule, we use that the descendants are measure preserving. This was a natural way to connect colouring rules to measure theoretic paradoxes. We could be more general in the application of an admissible relation.

**Question 9:** Is there a reasonable and more general definition for paradoxical colouring rules that does not require that the descendants be measure preserving?

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