Metric Invariants of Spherical Harmonics

Valentin Lychagin

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Spherical harmonics of degree $k$ are smooth solutions of the Euler

$$xu_x + yu_y + zu_z - ku = 0,$$

and the Laplace

$$u_{xx} + u_{yy} + u_{zz} = 0,$$

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Denote by $E(i)\supset J^i, i = 1, 2, .., k$ the corresponding equations and their prolongations.

Lie group $\text{SO}(3)$ is obvious symmetry group of these equations and all $E(i)$ are affine algebraic manifolds equipped with the algebraic $\text{SO}(3)$ -action.
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We say that a rational $\text{SO}(3)$-invariant function on $\mathbb{H}_k$ is an *algebraic metric invariant of spheric harmonics*, having degree $k$. 

The field of algebraic invariants we denote by $F_{a,k}$.

We say that a rational $\text{SO}(3)$-invariant function on an affine manifold $E^i$ is a *differential metric invariant of spheric harmonics*, having order $i$.

The field of differential invariants we denote by $F_{d,k}$. 

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3. The field of algebraic invariants we denote by $\mathcal{F}_k^a$. 

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How structure on Weyl algebra

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The Lie algebra $\mathfrak{sl}(2) \subset \mathbb{A}_3$, generated by the following operators

$$X_+ = \frac{r^2}{2}, \ H = \delta + \frac{3}{2}, \ X_- = \frac{\Delta}{2},$$

where

$$r^2 = x^2 + y^2 + z^2, \ \delta = x\partial_x + y\partial_y + z\partial_z, \ \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2,$$

and operators $(X_+, H, X_-)$ form the Weyl basis in $\mathfrak{sl}(2)$:

$$[H, X_+] = 2X_+, \ [H, X_-] = -2X_-, \ [X_-, X_+] = H.$$
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\]

3. The Lie algebra \( \mathfrak{so}(3) \subset \mathbb{A}_3 \) generated by the angular momentum operators

\[
L_z = x\partial_y - y\partial_x, \quad L_y = x\partial_z - z\partial_x, \quad L_x = y\partial_z - z\partial_y.
\]
These Lie algebras mutually commute and the \textit{universal enveloping algebra} $U(\mathfrak{sl}(2)) \subset A_3$ is the subalgebra of $\textit{so}(3)$-invariant operators in $A_3$. 

The Casimir operator in Lie algebra $\textit{so}(3)$ is the orbital angular momentum operator $M = L_x^2 + L_y^2 + L_z^2$ and it coincides with the Casimir operator in Lie algebra $\mathfrak{sl}(2)$: $M = r^2 \Delta \delta^2 \delta$. 

Operator $M$ is also the spherical Laplace operator.
These Lie algebras mutually commute and the universal enveloping algebra $U(\mathfrak{sl}(2)) \subset A_3$ is the subalgebra of $so\, (3)$-invariant operators in $A_3$.

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Harmonic polynomials

The following sequence

$$0 \rightarrow \mathbb{H}_k \rightarrow \mathbb{P}_k \xrightarrow{\Delta} \mathbb{P}_{k-2} \rightarrow 0$$

is exact, and $\dim \mathbb{H}_k = 2k + 1$. 

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2. Splitting \( \mathbb{P}_k \): for any homogeneous polynomial \( p_k \in \mathbb{P}_k \) there are (and unique) spheric harmonics \( h_{k-2i} \in \mathbb{H}_{k-2i}, 0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \), such that

\[ p = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} r^{2i} h_{k-2i}. \]
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for all \( h_k \in \mathbb{H}_k \).

4. The restriction of spheric harmonics on the unit sphere \( S^2 \subset \mathbb{R}^3 \) are eigenfunctions of the spherical laplacian \( \Delta_S \) with eigenvalues \(-k(k+1)\) and any continuous function on \( S^2 \) could be approximated (with any accuracy) by linear combination of spherical harmonics.
Harmonic projections $\eta_{k,2i} : \mathbb{P}_k \rightarrow \mathbb{H}_{k-2i}$ are the following

$$\eta_{k,2i} = r^{-2i} Q_{k,2i} (M),$$

where

$$Q_{k,2i} (\lambda) = \prod_{j \neq i}^{\left[ \frac{k}{2} \right]} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j}, \quad \lambda_i = -(k - 2i)(k - 2i + 1).$$
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$$0 \to \mathbb{P}_{k-2} \xrightarrow{r^2} \mathbb{P}_k \xrightarrow{\eta_{k,0}} \mathbb{H}_k \to 0$$

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$$\mathbf{0} \rightarrow \mathbb{P}_{k-2} \xrightarrow{r^2} \mathbb{P}_k \xrightarrow{\eta_{k,0}} \mathbb{H}_k \rightarrow \mathbf{0}$$

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3. Define product of spheric harmonics $h_k \in \mathbb{H}_k$, $h_l \in \mathbb{H}_l$ as follows

$$h_k \ast h_l = \eta_{k+l,0}(h_k h_l) \in \mathbb{H}_{k+l}.$$
Harmonic projections

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\]

4. Here

\[
\eta_{k+l,0} = \prod_{i} \frac{M + (k + l - 2j)(k + l - 2j + 1)}{2j(2j - 2k - 2l - 1)}.
\]
1. Spherical harmonics

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\[ x \ast x + y \ast y + z \ast z = 0. \]
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form a graded commutative algebra with respect to the product \(*\).

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3. The complixification \(H_* = H \otimes \mathbb{C}\) is the algebra of regular functions on the null cone \(\{x^2 + y^2 + z^2 = 0\}\) in \(\mathbb{C}^3\).
Algebra of spherical harmonics

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4. Example.

\[ x \star x = xx - \frac{r^2}{3}, x \star y = xy. \]
The space \( \mathbb{H}_k \) of spherical harmonics is a vector space of dimension \( 2k + 1 \). The Lie group \( \text{SO}(3) \) acts in algebraic way on \( \mathbb{H}_k \), and in \( \mathbb{H}_k \) are realized all irreducible representations of \( \text{SO}(3) \).
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3. Due to Rosenlicht theorem rational invariants of this action (i.e. rational invariants of spherical harmonics) form a field of transcendence degree equals the codimension of regular orbit.

Regular orbit has codimension $(2k^2)$, when $k \geq 2$, and codimension 1, when $k = 1$. Therefore, in order to define a regular orbit we need $2k^2$ algebraically independent rational invariants, for $k > 2$, and only one invariant, for $k = 1$. 

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Regular orbit has codimension $(2k - 2)$, when $k \geq 2$, and codimension 1, when $k = 1$. Therefore, in order to define a regular orbit we need $2k - 2$ algebraically independent rational invariants, for $k > 2$, and only one invariant, for $k = 1$. 

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Equations $\mathcal{E}^{(i)}$ are affine manifolds of dimension $2i + 4$, if $2 \leq i < k$. The regular $\text{SO}(3)$–orbits (that correspond to smooth points of quotient $\mathcal{E}^{(i)}/\text{SO}(3)$) Thus, due to Hilbert theorem, the quotients are affine manifolds of dimension $2i + 1$. Rational differential invariants of order $\leq i$ are rational functions on $\mathcal{E}^{(i)}/\text{SO}(3)$ and therefore the transcendency degree of field $\mathcal{F}_i^d$ equals to $2i + 1$. 
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As we have seen, the transcendence degree of field $\mathcal{F}^a_k$ equals $2(k - 1)$. 

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Equations \( \mathcal{E}^{(i)} \) are affine manifolds of dimension \( 2i + 4 \), if \( 2 \leq i < k \). The regular \( \text{SO}(3) \) -orbits (that correspond to smooth points of quotient \( \mathcal{E}^{(i)}/\text{SO}(3) \)) Thus, due to Hilbert theorem, the quotients are affine manifolds of dimension \( 2i + 1 \). Rational differential invariants of order \( \leq i \) are rational functions on \( \mathcal{E}^{(i)}/\text{SO}(3) \) and therefore the transcendence degree of field \( \mathcal{F}_i^d \) equals to \( 2i + 1 \).

As we have seen, the transcendence degree of field \( \mathcal{F}_k^a \) equals \( 2(k - 1) \).

Take a regular harmonic \( h \in \mathbb{H}_k \). Then it is easy to check that the \( \text{SO}(3) \) -orbit of the 2-jet \( j_2(h) \) into \( \mathcal{E}^{(2)} \) is a 6-dimensional submanifold into 8-dimensional manifold \( \mathcal{E}^{(2)} \) and therefore we need 2 differential invariants of order 2 to describe the orbit (compare with \( 2(k - 1) \) algebraic invariants).
Invariant coframe

Total differentials of the obvious invariants $J_{-1} = \frac{r^2}{2}$ and $J_0 = u$ give us two $\text{SO}(3)$-invariant horizontal 1–forms:

\[
\begin{align*}
\omega_1 &= xdx + ydy + zdz, \\
\omega_2 &= u_x dx + u_y dy + u_z dz.
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2. Their cross product gives us

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   \omega_3 = (yu_z - zu_y) \, dx + (zu_x - xu_z) \, dy + (xu_y - yu_x) \, dz.
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Invariant coframe

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3. Then coframe $(\omega_1, \omega_2, \omega_3)$ is $\text{SO}(3)$-invariant.
Invariant frame

\[ D_1 = x \frac{d}{dx} + y \frac{d}{dy} + z \frac{d}{dz}, \]

\[ D_2 = u_x \frac{d}{dx} + u_y \frac{d}{dy} + u_z \frac{d}{dz}, \]

\[ D_3 = (yu_z - zu_y) \frac{d}{dx} + (zu_x - xu_z) \frac{d}{dy} + (xu_y - yu_x) \frac{d}{dz}. \]
First invariants

\[ J_{-1} = \frac{r^2}{2}, \quad J_0 = u, \]
\[ J_1 = D_2(J_0) = u_x^2 + u_y^2 + u_z^2, \]
\[ J_{21} = \frac{D_2(J_1)}{2} = u_x^2 u_{xx} + u_y^2 u_{yy} + u_z^2 u_{zz} + 2(u_x u_y u_{xy} + u_x u_z u_{xz} + u_y u_z u_{yz}). \]
Invariant symmetric forms and operators

1 Symmetric differential $i$-forms

$$\Theta_i = \sum_{i_1+i_2+i_3=i} u_{i_1,i_2,i_3} \frac{dx^{i_1}}{i_1!} \cdot \frac{dy^{i_2}}{i_2!} \cdot \frac{dz^{i_3}}{i_3!}$$

are invariants with respect to Lie group of affine transformations in $\mathbb{R}^3$. 
Invariant symmetric forms and operators

1. Symmetric differential $i$-forms

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are invariants with respect to Lie group of affine transformations in $\mathbb{R}^3$.

2. Differential operators

$$\hat{\Theta}_i = \sum_{i_1+i_2+i_3=i} u_{i_1,i_2,i_3} \frac{d^k}{i_1!i_2!i_3!} \frac{dx^{i_1}}{i_1!} \frac{dy^{i_2}}{i_2!} \frac{dz^{i_3}}{i_3!}$$

are $SO(3)$-invariant.
Invariants

Let

\[
\begin{align*}
    dx &= t_{11} \omega_1 + t_{12} \omega_2 + t_{13} \omega_3, \\
    dy &= t_{21} \omega_1 + t_{22} \omega_2 + t_{23} \omega_3, \\
    dz &= t_{31} \omega_1 + t_{32} \omega_2 + t_{33} \omega_3,
\end{align*}
\]

where \( t_{ij} \) are rational functions on \( J^1(\mathbb{R}^3) \), and let

\[
\Theta_i = \sum_{i_1 + i_2 + i_3 = i} T_{i_1, i_2, i_3} \frac{\omega_1^{i_1}}{i_1!} \cdot \frac{\omega_2^{i_2}}{i_2!} \cdot \frac{\omega_3^{i_3}}{i_3!}.
\]

Theorem

Functions \( T_{i_1, i_2, i_3} \) are rational differential \( \text{SO}(3) \)-invariants of order \( i = i_1 + i_2 + i_3 \) and any rational differential \( \text{SO}(3) \)-invariants of order \( i \) is a rational function of them.
Remark that invariants

\[ G_i = \Theta_i (u) = \sum_{i_1 + i_2 + i_3 = i} \frac{u_{i_1, i_2, i_3}^2}{i_1! i_2! i_3!} \]

are squares of lengths of symmetric forms \( \Theta_i \).

Thus,

\[
\Theta_1 = u_x \frac{d}{dx} + u_y \frac{d}{dy} + u_z \frac{d}{dz},
\]

\[
\Theta_2 = \frac{1}{2} \left( u_{xx} \frac{d^2}{dx^2} + u_{yy} \frac{d^2}{dy^2} + u_{zz} \frac{d^2}{dz^2} \right) + u_{xy} \frac{d^2}{dxdy} + u_{xz} \frac{d^2}{dxdz} + u_{yz} \frac{d^2}{dydz}
\]

and

\[
\Theta_1 (u) = u_x^2 + u_y^2 + u_z^2,
\]

\[
\Theta_1 (u) = J_{22} = \frac{u_{xx}^2 + u_{yy}^2 + u_{zz}^2}{2} + u_{xy}^2 + u_{xz}^2 + u_{yz}^2.
\]
Theorem

The field of rational differential \( \text{SO}(3) \)-invariants of spherical harmonics is generated by invariants \( J_{-1} = \frac{r^2}{2}, J_0 = u, J_{22} \) and derivation \( \nabla = \hat{\Theta}_1 \).
Monoid of invariants

$\mathbf{SO}(3)$—Invariants $\iff \mathbf{SO}(3)$—invariant differential operators:

$$\phi \in C^\infty \left( J^k(\mathbb{R}^3) \right) \iff \Delta_\phi : C^\infty(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3),$$

$$\Delta_\phi(f) = j_k(f)^*(\phi).$$
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2. Monoid structure on \( \textbf{SO} (3) \) — invariants defines by the composition of invariant operators, and \( \text{id} = u \).
Monoid of invariants

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2. Monoid structure on $\text{SO}(3)$ — invariants defines by the composition of invariant operators, and $\text{id} = u$.

3. Thus, the field $\mathcal{F}_k^k$ is the monoid.
Let

\[ W = x \partial_x + y \partial_y + z \partial_z + ku \partial_u, \]

and let \( W^* \) be its \( \infty \)-prolongation.
Let
\[ W = x\partial_x + y\partial_y + z\partial_z + ku\partial_u, \]
and let \( W^* \) be its \( \infty \)-prolongation.

We say that a polynomial differential invariant \( I \) has weight \( w(I) \) if
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We say that a polynomial differential invariant \( I \) has weight \( w(I) \) if
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In other words, if \( h \) is a homogeneous polynomial of degree \( k \) then \( I(h) \) has degree \( w(I) \).
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2. Let $I$ be a polynomial differential invariant of weight $w$, and $h \in \mathbb{H}_k$. Then $I(h)$ is a scalar, i.e., invariant $G_w$ is an algebraic invariant.
1. Algebraic invariants on $\mathcal{H}_k$ are differential invariants of order $k$.

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3. Then $I(h) \in \mathbb{P}_w$, $(\eta_{w,2l} \circ \Delta_l)(h) \in \mathcal{H}_{w-2l}$ and its length $(\Delta_{G_{w-2l}} \circ \eta_{w,2l} \circ \Delta_l)(h)$ is a scalar, i.e invariant $G_{w-2l} \circ \eta_{w,2l} \circ I$ is an algebraic invariant.