Why *Explicit* Strangeness Is Not Relevant In Compact Stars

Mannque Rho

1 *Institut de Physique Théorique, CEA Saclay, 91191 Gif-sur-Yvette cédex, France*

(Dated: October 13, 2018)

Drawing largely from my work with co–workers, I present arguments that strangeness does not play an *explicit* role in compact-star matter. They are based on the skyrmion description of dense matter combined with Wilsoninan renormalization group approach to kaon nuclear interactions. The key idea is that quark degrees of freedom, carrying strangeness, can be traded in by topology for hadron degrees of freedom.

I. INTRODUCTION

In a recent study of the structure of massive compact stars [1], there was found to be no need to introduce strangeness degrees of freedom, either hadronic or quarkish, in satisfactorily describing the EoS of compact-star matter at high density. The framework resorted to an effective field theory approach anchored on certain symmetries “hidden” in QCD, namely, hidden local symmetry associated with the vector mesons $\rho$ and $\omega$ and hidden scale symmetry associated with the scalar dilaton $\sigma$. No symmetry associated with the $s$ quark was taken into account. The question naturally arises as to whether it is justified to simply ignore strangeness degrees of freedom such as kaon condensation and/or hyperon presence at the density relevant to the interior of massive compact stars in confronting the recently observed $\sim 2$-solar mass stars [2, 3].

In standard chiral perturbation approaches in flavor $SU(3)$ as first considered by Kapan and Nelson [4] and studied extensively as recounted in numerous reviews, e.g., [5], the $SU(3)$ chiral Lagrangian would predict, at the mean field level, s-wave anti-kaons (referred to simply as kaons from here on) condensing at a relatively low density, typically at $\sim 3n_0$. Such a low-density kaon condensation would have a large impact on a variety of astrophysical phenomena including possible low-mass black holes [5]. The recent discovery of $\sim 2$ solar-mass compact stars [2, 3] raised the poignant issue as to whether such kaon condensation – and equally hyperon presence, triggering softness in the EoS as naively suspected, are not unequivocally ruled out by the observed massive compact stars. Indeed, the verdict pronounced in [2] was that one of the manifestation of strangeness, namely, the kaon condensation, is ruled out by the observation.

So the question raised is: Is the discovery of $\sim 2$-solar-mass stars the smoking-gun signal for the death of kaon condensation or equivalently of hyperons in compact star matter?

There are a variety of scenarios in the literature, either invoking or not – up-to-date unestablished – repulsive interactions in strange-flavor channels, that claim that the so-called “strangeness problem,” both hyperon problem and kaon condensation problem, can be avoided. It is fair however to say that up to date, given the paucity of experimental information and consistent theoretical tools, there are no generally acceptable scenarios. Thus this issue remains a totally open problem to be resolved.

I would claim that the verdict pronounced in [2], though it cannot be ruled out, is however too premature at present given the large uncertainty in the theoretical treatment of strangeness in dense matter. The theoretical results the conclusion made in [2] is drawn from is based on naively attaching tree-order chiral Lagrangian treatments of kaon-nucleon interactions to correlated nuclear dynamics, both treated separately at a very different level of sophistication and with no feedbacks between the two, which could play a crucial role. Given the preeminent role of the scalar- and vector-meson degrees of freedom in the EoS for matter at density $n \gtrsim 2n_0$ as described in [1], which is not taken into account in the conventional treatment of kaon condensation, such a hasty verdict should not be taken seriously. As I will argue below, the situation can be a lot subtler than one would summarily think. In fact, there is an indication known since a long time that a unified treatment of both meson and baryon sectors treated on the same footing is necessary for a reliable description. For example, when strongly repulsive short-range correlations among the nucleons are suitably taken into account, the naive chiral perturbative estimate can be qualitatively wrong [6]. Indeed the repulsion operative in the nuclear sector, well recognized in nuclear dynamics, is of the same origin as the attraction between kaons and nucleons responsible for boson condensation and is found to push the condensation threshold density way beyond the density relevant for compact stars. This indicates both the baryon sector and meson sector and their interconnections, i.e., back-reactions, must be treated consistently. To the best of my knowledge, such a consistent treatment has not been done up to date.

Within the formalism adopted in [1], it should be in principle feasible to approach the problem with the same $V_{\text{lowk}}$ RG employed for two flavors, generalized to three flavors, say, $V_{\text{lowk}}^{SU(3)}$ RG. It should rely on $V_{\text{lowk}}$’s in coupled channels involving hyperons and kaons. The procedure, although complicated and perhaps even daunting, should however be straightforward. The proper treatment will require $V_{\text{lowk}}$’s in the octet baryon sector, octet pseudo-scalar sector and their couplings, suitably taking into account what’s known as intrinsic density dependence (IDD) in the effective Lagrangian inherited from
QCD [1]. Working this out to confront nature is however beyond the present capability, mainly due to the paucity of experimental information that does not allow the large number of “bare” parameters (with IDDs endowed properly) of three-flavor scale-invariant HLS Lagrangian to be adequately controlled. In what follows, I will work with a drastically simplified version of the approach and try to arrive at a qualitative confirmation that strangeness via hyperons and kaons does not figure in compact-star equation of state. For this I will first argue that kaon condensation and hyperon presence, treated on the same footing, appear at the same density in the large $N_c$ limit where the in-medium kaon mass is of $\mathcal{O}(N_c^{-1})$ term and also in the chiral limit. In either or both of these limits, $c^* \to 1$. In the matter-free space, it is found to be $c^* \approx 0.5$. Although presently there is no proof, it seems likely that $c^* < 1$ in medium, approaching 1 from below near chiral restoration. If this is the case, that would suggest that hyperons appear before kaons condense and they ultimately join in the vicinity of chiral restoration. It is however difficult to be more precise on this point since the effect is at $\mathcal{O}(1/N_c)$ hyperfine corrections, namely, when the skyrmion-kaon system is rotationally quantized. A simple quasi-particle approximation leads to

$$ E_N^* - E_{N^*}^* = \omega_K^* + \frac{3}{8\Omega^*}(c^{*2} - 1) $$(4)

where $\Omega > 0$ is the moment of inertia of skyrmion rotator and $c^*$ is the in-medium hyperfine coefficient multiplying the effective spin operator of strangeness -1. The coefficient $c$ is highly model-dependent even in the matter-free space [3], so it is unknown in dense matter except in the large $N_c$ limit. In either or both of these limits, $c^* \to 1$. In the matter-free space, it is found to be $c^* \approx 0.5$. Although presently there is no proof, it seems likely that $c^* < 1$ in medium, approaching 1 from below near chiral restoration. If this is the case, that would suggest that hyperons appear before kaons condense and they ultimately join in the vicinity of chiral restoration. It is however difficult to be more precise on this point since the effect is at $\mathcal{O}(1/N_c)$ and at that order there are many other corrections, such as higher-order nuclear correlations, that go beyond the mean-field order, something that can be done in the three-flavor $V_{lowk}$ approach. In the absence of a realistic calculation with kaon mass and $1/N_c$ corrections, it seems reasonable to assume that kaon condensation and hyperons appear at about the same density.

III. HYPERON PROBLEM

It will be shown below that kaon condensation will inevitably take place at some high density above $n_0$. The precise value for the density at which kaons start condensing cannot be pinned down theoretically unless a full 3-flavor $V_{lowk}$RG analysis is performed. Here I develop an argument based on a mean-field treatment of $bs$HLS that hyperons may not destabilize massive stars. With the possibility that kaons condense at about the same density as discussed above, the same will apply to this process.

It is easy to see by simple energetic considerations that hyperons could be present at high density in compact-star matter. Specifically, the lowest-lying hyperon $\Lambda$, with its attractive interaction, is estimated to appear at matter density $\sim 2n_0$ with the others possibly appearing at higher density. This suggests that the hyperons could appear at about the same density as the one at which

$$ \mu_e \geq E_\Lambda^* - E_N^* $$
the half-skrymon phase appears in the skyrmion matter [1]. If this were the case, then the prediction made in [1] would make no sense without the hyperonic degree of freedom duly taken into account.

In an admittedly phenomenological, but sophisticated, study using a Monte Carlo simulation over parameters that enter in the EoS for symmetric and asymmetric nuclear matter such as the compression moduli $K$ and $L$, symmetry energies $E_{sym}$ and $E^{\Lambda}_{sym} = S_{\Lambda}$, Bedaque and Steiner obtain the range of density $\Delta$ constrained by hydrodynamic stability of the system that ensures that stars with $M > 2M_{\odot}$ could be supported [11]. The $\Delta$ is then the range of density within which the in-medium $\Lambda$ mass should become greater than the vacuum value. One expects – and it is confirmed experimentally – that the $\Lambda$-nucleon interaction is attractive at normal nuclear density, so $\Lambda$s could be bound in nuclear matter. In compact star matter, as density increases, the chemical potential difference between neutron and proton increases, and it can become energetically favored to have the non-strange quarks in $\Lambda$ vs. 3 in nucleon. Then $\mu_{\Lambda} - \mu_3$ factor accounting for the two non-strange quarks in $\Lambda$ vs. 3 in nucleon. Then

$$\mu_{\Lambda} = m_{\Lambda}^s - \frac{g_{\sigma \Lambda}^s g_{\omega N}^s}{m_{\Lambda}^s} n_s + \frac{g_{\omega \Lambda}^s g_{\omega N}^s}{m_{\Lambda}^s} n$$  (5)

where $n_s$ and $n$ are, respectively, nucleon scalar density and nucleon number density as defined in Fig. 1. The notations for $(\sigma, \omega)$ coupling to $\Lambda$ and $N$ are self-evident. The asterisk stands for IDD-scaling parameters.

Apart from the $\Lambda$ coupling to the mesons, the large cancelation between the dilaton $(\sigma)$ attraction and the vector $(\omega)$ repulsion responsible for small binding energy for nuclear matter must take place also in this case. In fact, using the standard constituent quark (equivalent to quasi-quark) counting, we may take $g_{\sigma \Lambda} \approx \frac{2}{3} g_{\sigma N}$ and $g_{\omega \Lambda} \approx \frac{5}{3} g_{\omega N}$, the $2/3$ factor accounting for the two non-strange quarks in $\Lambda$ vs. 3 in nucleon. Then

$$\mu_{\Lambda} = m_{\Lambda}^s + 2 \frac{2}{3} \left( - \frac{g_{\sigma N}^2}{m_{\sigma}^2} n_s + \frac{g_{\omega N}^2}{m_{\omega}^2} n \right).$$  (6)

To make a numerical estimate of the $\Lambda$ mass shift in dense medium, we take into account the IDD-scaling – assuming flavor $SU(3)$ symmetry – of the parameters in the sHLS Lagrangian with baryons incorporated explicitly (call it bsHLS) in the mean-field calculation which corresponds to the “single-decimation procedure” [12]. In doing this, one should recognize that the scaling parameter $c_I$ in this procedure could well be different, i.e., renormalized, from the IDD coefficient entering into the double-decimation procedure with $V_{\text{trunc}}$ employed in [11] addressed to compact stars. This is because in the single-decimation procedure, the scaling function $\Phi_I$ is related to the Fermi-liquid fixed-point parameters as shown in [13] and encodes certain nonperturbative quasi-particle interactions on top of the IDD effects, termed “induced

FIG. 1. Tadpole diagram for self-energies for the nucleon $N$ and the hyperon $\Lambda$ in medium with the IDDs. The loop corresponds to the nucleon scalar density $n_s = \langle N N \rangle$ for coupling to $\sigma$ and the nucleon number density $n = \langle N^1 N \rangle$ for coupling to $\omega$.
density dependence\(^3\) or DD\(_{\text{induced}}\) in [13]—manifesting scale-chiral symmetry.

The result is plotted in Fig. 2. We see that \(\mu_A - m_A\) crosses zero at a density 1.5 \(< n/n_0 < 2.0\). The result is insensitive to the demarcation density for the regions. In fact what comes out in the mean field in \(bS\text{HLS}\)\(^4\) is quite easy to understand. Since \(m_A^2\) stops dropping with \(f_\sigma^2\) stabilizing at \(2n_0\), what matters is the interplay of the ratio \((f_\sigma^2/D_s)^2\) for the scalar and vector mesons—with an opposite sign—multiplied, respectively, by the scalar density \(n_s\) and by the baryon number density \(n\). The vector repulsion wins over the scalar attraction as density increases in the same way as it does in nuclear matter. Although the estimate is admittedly approximate—and it could be done much more realistically in the \(V_{\text{lowk}}\) approach used above, that the BS constraint [11] is met is most likely robust. We conclude from this that within the formalism developed in [16] and with the prescription given in [11], the hyperon problem could be avoided. A more definitive answer awaits a full \(V_{\text{lowk}}^{SU(3)}\) RG calculation.

An intriguing—and perhaps important—point to note here is that the density involved corresponds more or less to the skyrmion-half-skyrmion transition at \(n_{1/2} \sim 2n_0\) [11] [13] which corresponds roughly to the smooth hadron-quark (or quarkyonic) transition in the model where quark degrees of freedom (including strange quark) are explicitly incorporated [17].

**IV. KAON CONDENSATION PROBLEM**

The mechanism that the discussion of hyperons vs. kaons given above relied on was the binding of \(K^-\)'s to dense baryonic matter. There in the presence of beta equilibrium, phase transitions were not involved. Unlike hyperons, however, kaons can bose-condense at high enough a density. In discussing the binding, what’s involved in chiral Lagrangians is the leading chiral order (LO) term known as Weinberg-Tomozawa (WT) term. In the literature discussing on \(\Lambda(1405)\) with respect to KN interactions, both free and bound in matter, it is this leading-order (LO) term that plays the key role. This term however turns out to be marginally irrelevant from the RG point of view. More on this matter below. What is relevant involves a scalar meson, i.e., the dilaton \(\sigma\), the pseudo-NG boson from spontaneously broken scale symmetry discussed in [1]. It is found that kaons condense ultimately after the appearance of hyperons when the dilaton condensate \(\langle \chi \rangle\) goes to zero, i.e., at the dilaton limit fixed point [1].

---

\(^3\) We recall that here “mean field” endowed with IDDs goes beyond what is used in the standard mean-field theory.

---

### A. Kaons in skyrmion crystal

In the Callan-Klebanov model with which the issue of hyperons vs. kaons was discussed, the topological Wess-Zumino term played the dominant role. Viewed in terms of the \(bS\text{HLS}\) Lagrangian, it is roughly equivalent to the \(\omega\) exchange term which when localized, corresponds to the Weinberg-Tomozawa term in \(S\chi\text{PT}\). But this interaction is not the relevant interaction at high density. It is rather the scalar dilaton that plays the crucial role in dense matter at \(n > n_{1/2} \sim 2n_0\).

Since we are primarily interested in qualitative rather than numerically accurate features, one can use the Skyrme model (skyrmion- \(\pi\) with Nambu-Goldstone bosons only). Strangeness is involved, so we will deal with three flavors with the topological Wess-Zumino term included. The dilaton will be introduced in the way described in [11] [13] [18]. Other heavy-meson fields, while quantitatively relevant, are not expected to modify the general features, so they will be ignored. The price to pay for this simplification is that the density at which the skyrmion-half-skyrmion transition, which here again will figure importantly, takes place will not be given reliably. I believe this does not affect the main argument.

Take the scale-chiral Lagrangian valid in the leading order in scale symmetry (“LOSS”) as derived in [13] [19] and given a support in [1] [14],

\[
\mathcal{L}_{sk} = \frac{f^2}{4} \left( \frac{\chi}{f_\sigma^2} \right)^2 \text{Tr}(L_\mu L^\mu) + \frac{1}{32\pi^2} \text{Tr}[L_\mu, L_\nu]^2 + \frac{f^2}{4} \left( \frac{\chi}{f_\sigma^2} \right)^3 \text{Tr}\mathcal{M}(U + U^\dagger - 2) + \frac{1}{2} \partial_\mu \chi \partial_\mu \chi + V(\chi)
\] (7)

where \(V(\chi)\) is the dilaton potential, \(L_\mu = U^\dagger \partial_\mu U\), with \(U\) the chiral field taking values in \(SU(3)\). One ignores the pion mass for simplicity, so the explicit chiral symmetry-breaking mass term is given by the mass matrix \(M = \text{diag}(0, 0, 2m_0^2)\). In \(SU(3)\), the anomaly term, i.e., the Wess-Zumino term, \(S_{WZ} = -i\frac{N_C}{240\pi^2} \int d^4x e^{\mu\nu\lambda\sigma} \text{Tr}(\partial_\mu L_\mu L_\nu L_\lambda L_\sigma)\), turns out to play a crucial role in our approach:

Now consider the fluctuation of kaons in the background of the skyrmion matter \(u_0\)

\[
U(\vec{x}, t) = \sqrt{U_K(\vec{x}, t)}U_0(\vec{x})\sqrt{U_K(\vec{x}, t)},
\] (8)

\[
U_K(\vec{x}, t) = e^{\frac{-i}{\sqrt{2}f_\sigma^2} \left( \begin{array}{cc} 0 & K \\ K^\dagger & 0 \end{array} \right)}, \quad U_0(\vec{x}) =\left( \begin{array}{cc} u_0(\vec{x}) & 0 \\ 0 & 1 \end{array} \right).
\] (9)

Substituting (8) into (7) and the Wess-Zumino term, one gets the kaon Lagrangian in the background matter field...
In the spirit of mean field approximation, the space aver-
ation is the background effective kaon mass. This equation will be used for evaluating the in-medium behaviour of the kaon. For the model, we find $m_{K+}^* = 720$ MeV and dilaton mass $m_\sigma = 720$ MeV.

In the spirit of mean field approximation, the space average is taken on the background matter fields $u_0$ and $\chi_0$, obtaining

$$L_K = \alpha (\partial_\mu K^{\dagger} K) + i \beta (K^{\dagger} \dot{K} - \dot{K} K^{\dagger}) - \gamma K^{\dagger} K$$

where $\alpha = \langle \kappa^2 G \rangle$, $\beta = \frac{N_c}{4 f_\pi^2} \langle B^0 G \rangle$, $\gamma = \langle \kappa^3 G \rangle m_K^2$.

with $\kappa = \chi_0 / f_\chi$. Lagrangian (14) yields a dispersion relation for the kaon in the skyrmion matter as

$$\alpha (\omega_K^2 - p_K^2) + 2 \beta \omega_K + \gamma = 0.$$ (16)

Solving this for $\omega_K$ and taking the limit of $p_K \to 0$, one gets

$$m_K^* = \lim_{p_K \to 0} \omega_K = \frac{-\beta + \sqrt{\beta^2 + \alpha \gamma}}{\alpha}.$$ (17)

This equation will be used for evaluating the in-medium effective kaon mass.

The key element in Eq. (11) for the kaonic fluctuation is the background $u_0$ which reflects the “vacuum” modified by the dense skyrmion matter. The classical dilaton field tracks the quark condensate affected by this skyrmion background $u_0$ that carries information on chiral symmetry of dense medium. Our approach here is to describe this background $u_0$ in terms of crystal configuration. The pertinent $u_0$ has been worked out in detail in [20], from which we shall simply import the results for this work. As shown there, skyrmions put on an FCC – which is the favored crystal configuration – make a transition at a density $n_{1/2}$ to a matter consisting of half-skyrmions. With the parameters of the Lagrangian picked for the model, we find $n_{1/2} \sim 1.3 n_0$ but this is not significant. As noted before, the reasonable range of density range is $n_{1/2} \sim (2 - 3) n_0$.

We now discuss the results given in Fig. 3 in some detail.

The results are given for the dilaton mass $m_\sigma \approx 720$ MeV. The dilaton mass is currently known neither experimentally nor theoretically. It is taken as $m_\sigma \approx 720$ so as to give $\sim 600$ MeV in nuclear matter consistent with the Fermi-liquid description of nuclear matter implementing IDDs [21]. The density at which the half-skyrmion matter appears turns out to be more or less independent of the dilaton mass [20]. In fact, the $n_{1/2}$ is entirely dictated by the parameters that give the background $u_0$ (for which the hidden local symmetric mesons cannot be ignored).

For the skyrmion background, $\sqrt{2} e f_\pi$ – which is the vacuum pion mass at tree order– can be taken to be $\sim 780$ MeV which fixes $e$ for the given $f_\pi \approx 93$ MeV. What is controlled by the dilaton mass are the density at which the dilaton condensate vanishes together with the pion decay constant, i.e., the dilaton limit fixed point and more significantly, the rate at which the kaon mass drops as density exceeds $n_{1/2}$.

What is novel in this model is that the behavior of the kaon is characterized by three different phases, not two as in conventional approaches. In the low density regime up to $\sim n_{1/2}$, the kaon interaction can be described by standard chiral symmetry treatments. For instance, the binding energy of the kaon at nuclear matter density comes out to be $\sim (80 - 100)$ MeV for $m_\sigma = 720$ MeV. Go-

---

2 Recall that this background-dependence is supposed to effectively mock up the IDDs.
ing above $n_{1/2}$, however, as the matter enters the half-skyrmion phase with vanishing quark condensate (on average) and non-zero pion decay constant, the kaon mass starts dropping more steeply. This form of matter arising from the skyrmion-half-skyrmion transition involving the dilaton condensate — which is undoubtedly highly non-perturbative missing in $S$PT — is not captured by the Weinberg-Tomozawa term. Finally at $n_{c}$ at which both the quark condensate and the pion decay constant vanish, the kaon mass vanishes. This corresponds to the dilaton limit fixed point mentioned above. This happens in this model at density $\approx 4.0 n_{0}$ — which is clearly much too low — for the dilaton mass $m_{\sigma} = 720$ MeV. The reason for this is that the dilaton condensate, $(\chi)$, vanishes at the dilaton limit fixed point. This means that “kaon condensation” in symmetric nuclear matter takes place at the point at which the scale symmetry associated with the soft dilaton is restored. In the realistic treatment given in [1], it was found that the vector manifestation fixed point and the dilaton limit fixed point are to arrive at $n \gg 10 n_{0}$. Therefore this anomalous behavior of the kaon mass at the skyrmion-1/2-skyrmion transition must take place at a much higher density than what’s estimated in the simple model given above. In fact this observation will be supported in the renormalization-group treatment given below.

### B. Kaons in Fermi liquid

The crucial role of the dilaton in the strangeness problem in dense compact stars could be unravelled in a $V_{\text{SU}(3)}^\text{lowk}$RG treatment. In the absence of such a treatment, we look at a greatly simplified model that captures essential ingredients involved in the process.

The model is formulated as follows. Given that the $V_{\text{lowk}}$RG in two flavors employed in [1] is equivalent to doing Landau-Fermi liquid theory with a bsHLS Lagrangian endowed with IDDs as argued in [24], the formalism is extended to three flavors. The key requirement is to account for both nuclear interactions in the Fermi liquid and kaonic fluctuations on top in Wilsonian RG, taking into account kaon properties warped by the baryonic background in dense medium as indicated by the result of Pandharipande et al. [3]. There are in principle many terms in the action. The simplification consists of localizing $V_{\text{lowk}}$s in all three channels, namely baryon-baryon, baryon-kaon and kaon-kaon. Assuming that the process takes place in the half-skyrmion phase from which kaons are to condense is in Fermi liquid, we then look at the kaons coupling to the Fermi liquid via exchange of a scalar dilaton that can be tuned to criticality (i.e., zero mass). We ignore the effect of the electron chemical potential [22] in this discussion. We write the simplified Euclidean action for kaon-Fermi-liquid nuclear systems in momentum space in the form [23]

$$Z = \int [d\phi][d\phi^*][d\psi][d\bar{\psi}] e^{-S^E}$$ \hspace{1cm} (18)

with the actions

$$\tilde{S}^E = \tilde{S}^E_{\psi} + \tilde{S}^E_{\phi} + \tilde{S}^E_{\psi\phi} \hspace{1cm} (19)$$

$$\tilde{S}^E_{\psi} = \int \frac{d\omega d^3k}{(2\pi)^4} \left\{ \frac{-\omega - e(k) - e_F}{\omega - e(k) - e_F} \right\} \bar{\psi}_\sigma \psi_\sigma + \int \frac{d\omega d^3k}{(2\pi)^4} \lambda \bar{\psi}_\sigma \psi_\sigma \psi_\omega \delta^4(\omega, k) \hspace{1cm} (20)$$

$$\tilde{S}^E_{\phi} = \int \frac{d\omega d^3q}{(2\pi)^4} \left\{ \phi^*(\omega^2 + q^2) + m^2 \phi \phi^* + \cdots \right\} \hspace{1cm} (21)$$

$$\tilde{S}^E_{\psi\phi} = \int \frac{d\omega d^3k}{(2\pi)^4} \left\{ \phi^*(\omega^2 + q^2) + m^2 \phi \phi^* \right\} \int \frac{d\omega d^3q}{(2\pi)^4} \lambda \phi^* \bar{\psi}_\sigma \psi_\omega \delta^4(\omega, \epsilon, q, k) \hspace{1cm} (22)$$

where $\psi$ and $\bar{\psi}$ are the eigenvalues of $\Psi$ and $\Psi^\dagger$ acting on $|\psi\rangle$ and $\langle \bar{\psi}|$, fermion coherent state,

$$\Psi |\psi\rangle = \psi |\psi\rangle \hspace{0.5cm} \text{and} \hspace{0.5cm} \langle \bar{\psi}| \Psi^\dagger = \langle \bar{\psi}| \bar{\psi} \rangle \hspace{1cm} (23)$$

Although our notations for the fields are general as they can apply to other systems like pions/nucleons and electrons/phonons, we should keep in mind that we are specializing to the $K^-$ field for the boson and the non-strange baryons, proton and neutron for the fermion.

It is interesting that a same-type action figures in condensed matter physics. In fact, [19] is quite similar to the action studied for quantum critical metals in [20] from which we shall borrow various scaling properties for the renormalization group.

We have taken here a local KN interaction which in terms of our bsHLS Lagrangian can be interpreted as arising from the exchanges of the dilaton and $\omega$ meson with the mesons integrated out. Similarly we have taken only one generic four-Fermi interaction. Hyperons could figure in the Fermion sector but we will focus on nucleons. Therefore hyperon problem is not addressed directly. More realistically it would have various channels, including the channels corresponding to the dilaton exchange and the $\omega$ exchange. The intricate interplay between the various channels, in particular the scalar and vector channels, must play a crucial role in giving the mechanism leading to kaon condensation. One notable observation is that in the nucleon-nucleon interaction, the attractive scalar exchange and repulsive vector exchange compete whereas in the KN interaction (where $N$ is a neutron) both are attractive and add. We will not go into these intricacies here but just give a generic feature

---

3 An action similar to [19] was is studied in [25] except for the bosonic action. The bosonic action used there is different from [21] in that the system considered there was assumed to be a compact-star matter in weak and chemical equilibrium (with the boson approximated to satisfy a quadratic dispersion relation is most likely incorrect for kaons) whereas here we are dealing with relativistic bosons which could be tuned to criticality by attractive KN interactions.
of what a consistent treatment might mean in a simple situation. Needless to say, this simple treatment cannot possibly be truly realistic.

It is instructive to compare the constant $h$ in (22) with what is given in the standard chiral Lagrangian. In terms of the chiral counting, the leading term ($O(p)$) is “Weinberg-Tomozawa-type” term, coming from the exchange of the $\omega$ meson and gives, for the $s$-wave kaon,

$$h_{WT} \propto \frac{q_0}{f_\pi^2}$$

(24)

where $q_0$ is the fourth component of the kaon 4-momentum and the next chiral order ($O(p^2)$) term is the “KN sigma-type” term

$$h_{\Sigma} \propto \frac{\Sigma_K N}{f_\pi^2}$$

(25)

which can be thought of arising from a scalar meson exchange. In our approach, both terms can be considered as resulting from integrating out the “heavy” mesons from the $bs$HLS Lagrangian.

We should stress that the fermion $\psi$ is a quasiparticle field in Fermi-liquid near its fixed point, so the coupling $h$ encodes more than just the “bare” couplings in the $bs$HLS Lagrangian. This point was made in [27]. For convenience, we shall use these chiral Lagrangian terms for identifying the two terms contributing to $h$. It should be remembered that they are not the same.

As written, the action for the fermion system (20) is for a Fermi liquid with marginal four-Fermi interactions [28]. So the energy-momentum of the quasiparticle (fermion) is measured with respect to the Fermi energy $\epsilon_{\text{F}}$ and Fermi momentum $k_F$. The bosonic action is for massive Klein-Gordon field, which later will be associated with a pseudo-Goldstone field, with higher-field terms ignored. In contrast to that of the quasi-particle, the boson energy-momentum will be measured from zero. We would like to look at meson-field fluctuations on the background of a fermionic matter given by the Fermi-liquid theory. The Fermi-liquid structure is expected to be valid as long as $N \equiv k_F/\bar{\Lambda} \gg 1$ where $\bar{\Lambda} = \Lambda - k_F$ where $\Lambda$ is the (momentum) cutoff scale from which mode decimation – in the sense of Wilsonian – is made. For nucleon systems, the action (20) gives the nuclear matter stabilized at the fixed point, Landau Fermi-liquid fixed point, with corrections suppressed by $1/N$ [28]. The four-Fermi interactions and the effective mass of the quasiparticle are the fixed-point parameters. We expect that as long as $N \gg 1$, the Fermi-liquid action can be trusted even at higher densities than $n_0$. Recall that the skyrmion-half-skyrmion “transition” which figures importantly is not a phase transition in the sense of Ginzburg-Landau-Wilson paradigm. This is because the half-skyrmions are confined although their presence makes a dramatic impact on the EoS.

For simplicity, we shall assume a spherically symmetric Fermi surface in which case we can set for the quasiparticle momentum

$$\vec{k} = \vec{k}_F + \vec{q} \approx \hat{\Omega}(k_F + \vec{q}).$$

(26)

Then for $\bar{\Lambda} \ll k_F$, we have

$$e(k) - e_F \approx \vec{v}_F \cdot \vec{q} + O(l^2)$$

(27)

where $v_F$ is the Fermi velocity $v_F = k_F/m^*_L$ with $m^*_L$ the effective quasiparticle (Landau) mass.

1. Scaling

To do the Wilsonian decimation, we need to set the cutoff scale $\Lambda$ at which the classical action has bare parameters. Quantum effects are taken into account by doing the mode elimination by lowering the cutoff from $\Lambda$ to $s\Lambda$ with $s < 1$. The important point to note here is that the boson and fermion fields have different kinematics. While the boson momentum is measured from the origin – and hence the momentum cutoff is $\Lambda$ , the fermion momentum is measured from the Fermi momentum $k_F$. Therefore the mode elimination for the fermion involves lowering the fermion momentum from $\Lambda = \Lambda - k_F$ to $s\Lambda$. As noted above, the strategy in the fermion sector is to take the large $N$ limit where $N \equiv k_F/\bar{\Lambda}^6$.

We define the scaling laws of fields and other quantities like the delta functions by requiring that the kinetic energy terms of the fermion and the boson be invariant under the scaling

$$\epsilon \rightarrow s\epsilon, \quad \omega \rightarrow s\omega, \quad l \rightarrow sl, \quad \vec{q} \rightarrow s\vec{q}.$$  

(28)

Only the fermion momentum orthogonal to the Fermi surface scales in the way the fermion energy and all components of the boson momentum do. Since the quasiparticle mass $m^*_L$ is a fixed-point quantity, the Fermi velocity does not scale. Therefore we have the fields scaling as

$$\phi \rightarrow s^{-3}\phi, \quad \psi \rightarrow s^{-3/2}\psi,$$

(29)

for which we denote the scaling dimensions as $[\phi] = -3$ and $[\psi] = -3/2$ and the meson mass term scales

$$\int d\omega d^3q \, m^2_{\phi} \phi^* \phi \rightarrow \int d\omega d^3q \, m^2_{\phi} s^{-2} \phi^* \phi = \int d\omega d^3q \, m^2_{\phi} s^{2} \phi^* \phi,$$

(30)

5 We remind the readers that the larger the $k_F$ or density, the more $\bar{\Lambda}$ shrinks, which would mean that the large $N$ argument would hold better in the fermion sector as the density increases. This suggests that the mean field approximation with effective Lagrangians – chiral Lagrangian or hidden local symmetry Lagrangian – would get better the higher the density.

6 This notation will be used in what follows.
with \( m_\phi' \equiv s^{-1} m_\phi \). This shows the well-known fact that the meson mass term is “relevant.” Using the procedure of scaling toward the Fermi surface, we have

\[
[d\omega d^3k] = 2, \quad [\delta(\epsilon, k)] = -2.
\]

This confirms that the four-Fermi interaction term in (20) is marginal.7

The scaling of the coupling term (22) is more subtle. Using the scaling dimensions

\[
[\phi] = -3, \quad [\psi] = -3/2, \quad [d\omega d^3q] = 4, \quad (32)
\]

we find the scaling of the integrand \( I_{\psi\phi} \) of the action (22) written as \( \int I_{\psi\phi} \) is

\[
[I_{\psi\phi}] = 3 + [h] + [\delta(\omega, \epsilon, q, k)], \quad (33)
\]

where the bare coupling constant \( h \) will have the scaling dimension \( [h_{WT}] = [g_0] = 1 \) and \( [h_\Sigma] = 0 \). The \( \psi\phi \) coupling will be “relevant” if \( [I_{\psi\phi}] < 0 \), marginal if \( = 0 \) and “irrelevant” if \( > 0 \). It is thus the scaling of the delta function that determines the scaling of the coupling \( h \).

It will follow from arguments based on bSHLS developed so far that the \( \delta \) function scales as

\[
[\delta(\omega, \epsilon, q, k)] = -4. \quad (34)
\]

This then leads to

\[
[I_{\psi\phi}]_{h_{WT}} = 0, \quad [I_{\psi\phi}]_{h_{\Sigma}} = -1. \quad (35)
\]

Thus the Weinberg-Tomozawa (WT)-type coupling is marginal, while the sigma-term-type coupling is relevant. The one-loop contribution to the former turns out to be irrelevant \( [23] \), so the WT-type term is marginally irrelevant. Therefore one can focus on the latter.

Briefly the scaling (34) is arrived at as follows. (For details, we refer to \( [23] \).)

First note that the KN coupling giving rise to (22) differs from that leading to the four-Fermi interaction (20) by that one of the two \( \psi\phi \) vertices in the former is replaced by a \( KK\phi \) vertex. Within the scaling rule we adopt, the \( \psi\phi \) vertex is marginal but the \( KK\phi \) vertex is relevant. All other quantities are the same. Since in our approach, we want the four-Fermi interaction to remain marginal so that the baryonic matter remains in Fermi liquid, we need to enforce that the presence of the \( \phi \) field leave unaffected the marginal four-Fermi interaction. This is achieved by assigning a definite scaling property to the \( \phi \) propagator. By imposing this scaling condition on the \( \phi \) propagator that figures in the KN-KN interaction, one sees immediately that the \( \Sigma \)-term type KN coupling in (22) must be relevant. This is simply because the \( KK\phi \) vertex is relevant. One can verify this result by an explicit counting of the scalings involved in the Feynman diagrams \( [23] \).

The same simple counting rule shows that the WT term should be marginal. For this, simply replace the \( \psi\phi \) vertex of the scalar-exchange term by a \( \psi\omega \) coupling and the \( KK\phi \) vertex by a \( q_0KK\omega \) vertex, which amounts to replacing a marginal coupling by a marginal coupling and a relevant coupling by a marginal coupling. The net result is marginal because \( q_0KK \omega \) vertex is marginal.

2. Wilsonian renormalization group

We are interested in the flows of the kaon mass and the kaon-nucleon coupling and the effect of kaon condensation on the Fermi liquid structure of the baryonic system. For this we need the scaling properties of the diagrams.

At tree order, we have

\[
h' = s^{-1} h, \quad (36)
\]

i.e., relevant as found above. The kaon mass scaling is of course relevant,

\[
m_{\phi}'^2 = s^{-2} m_\phi^2. \quad (37)
\]

FIG. 4. One-loop graphs contributing to \( h \). The process corresponds to \( f(\text{fermion}) (\epsilon, k) + m(\text{meson}) (\omega, q) \rightarrow f(\text{fermion}) (\epsilon', k') + m(\text{meson}) (\omega', q') \).

FIG. 5. One-loop graph contributing to \( m_\phi \).

At one-loop order, there are two diagrams, Figs. 4, contributing to \( h \) and one diagram, Fig. 5, to \( m_\phi \). The one-loop graph Fig. 5, contributing to the \( \phi \) mass is easy to evaluate. By decimating from \( \Lambda \) to \( s\Lambda \) in the nucleon loop, we get

\[
\int \frac{d\omega d^3q}{(2\pi)^3} \frac{1}{\phi^*\phi s^2} \left[ -m_\phi^2 + \frac{\gamma h k_F^2}{2\pi^2} \int_{\Lambda < |l'| < s\Lambda} dl' |\text{sgn} (l')| \right], \quad (38)
\]
where $\gamma$ is the degeneracy factor for flavor (=2) and spin (=2): $\gamma = 4$ for nuclear matter and $\gamma = 2$ for neutron matter. It follows from above that

$$
\delta [m_{\phi}^2] = s^{-2} \left[ m_{\phi}^2 - \frac{\gamma h_{\phi}^2}{\pi^2} (1 - s) \right] - m_{\phi}^2. \tag{39}
$$

As for the loop contribution to the $h_\phi \phi \psi \psi$ coupling, it turns out that the two diagrams (a) and (b) of Fig. 4 are related to each other. And they are down by $1/N$ relative to the tree term. Furthermore both are found to vanish. Therefore to one-loop order the matter becomes very simple.

3. Flow analysis

From what’s discussed above, it is straightforward to write down the RGEs. Setting $t \equiv -\ln s$,

$$
\frac{dm_{\phi}^2}{dt} = 2m_{\phi}^2 - Ah, \tag{40}
$$

$$
\frac{dh}{dt} = h - Bh^2, \tag{41}
$$

where

$$
A = \frac{\gamma k_F^2 \bar{\Lambda}}{\pi^2} = \frac{\gamma k_F^2}{N \pi^2}, \tag{42}
$$

$$
B = 0. \tag{43}
$$

It is easy to get the analytic solutions for $m_{\phi}$ and $h$

$$
m_{\phi}^2(t) = \left( m_{\phi}^2(0) - h(0) A \right) e^{2t} + h(0) A e^t, \tag{44}
$$

$$
h(t) = h(0) e^t. \tag{45}
$$

Given the analytic solution, we can work out how $m_{\phi}^2$ and $h$ flow as $t$ increases ($s$ decreases) for given values of $A$, $h(0)$ and $m_{\phi}^2(0)$. From Eqs. (44) and (45), we have the relation,

$$
\frac{m_{\phi}^2(t) - Ah(t)}{[h(t)]^2} = \frac{m_{\phi}^2(0) - Ah(0)}{[h(0)]^2}, \tag{46}
$$

which is satisfied for any value of $t$. It is convenient to introduce the new parameter $c_{\phi}$, which is independent of $t$,

$$
c \equiv \frac{m_{\phi}^2(0) - Ah(0)}{[h(0)]^2}. \tag{47}
$$

The quantities at $t = 0$, i.e., $m_{\phi}^2(0)$ and $h(0)$, are given at $s = 1$, that is, at the scale from which the decimation starts for a given $k_F$. These parameters in the bare action and the parameter $c$ depend only on density. Since the flow depends on $c$, the RG properties of dense medium depend on the density dependence of the parameters of the bare chiral Lagrangian at the scale $\Lambda$. This is equivalent to the IDD that is obtained in relativistic mean field treatment of effective Lagrangians of the HLS-type which is again equivalent to Landau Fermi-liquid theory.

Now using Eqs. (46) and (47), we readily obtain

$$
m_{\phi}^2(t) = ch(t)^2 + Ah(t). \tag{48}
$$

The flows $m_{\phi}(t)$ vs. $h(t)$ are plotted in Fig. 6 for various values of $c_{\phi}$ that depend on $k_F/\Lambda$. We note that $m_{\phi}^2(t)$ flows toward zero and becomes negative for $c < 0$. This signals the instability toward kaon condensation. What happens after the condensation is a matter that goes beyond the model. It could be stabilized by terms higher order in the kaon field not taken into account in our analysis.

The conclusion is that only the $m_{\phi}(0)$ and $h(0)$ that satisfy

$$
m_{\phi}^2(0) < Ah(0) \tag{49}
$$

will be in the parameter space for the condensation to take place. The value $c = 0$ defines the critical line in the $h(t)$ vs. $m_{\phi}^2(t)$ plane, any point on which the parameters flow from the origin satisfying

$$
m_{\phi}^2(t) = Ah(t). \tag{50}
$$

Note that in the RG flows of $m_{\phi}(t)$ and $h(t)$, as $t$ goes up, the attractive interaction gets stronger.

4. Instability toward the condensation

Although one cannot determine the critical density at which kaons condense from the RG flow analysis, it is nonetheless feasible to indicate whether and how the condensation can take place. This can be done by looking at the “parameter window” that signals the instability toward condensation.
At some density, according to the values of $c$, the RG flows of $m_{\phi}^2$ and $h(t)$ will follow certain of the flow lines in Fig. 6. If the value of $c$ is changed by changing $k_F$ to $k_F'$, the RG flow of $m_{\phi}^2$ and $h(t)$ follows from the line (in $k_F$) to the line (in $k_F'$). By locating the sign change of

$$m_{\phi}^2(0) - Ah(0),$$

(51)

one can then locate the density at which the phase is moved to the region that signals the instability toward the kaon condensation. While this argument does not allow to pinpoint what the condensation density can be, it nevertheless can give the lower bound of the critical density. Let’s use Eq. (49) to make this estimate. The sign change of $c$ takes place when

$$n = \gamma \frac{k_F^3}{6\pi^2} = \frac{N m_{\phi}^2(0)}{6 h(0)}.$$  

(52)

What this says is that since the better the quasiparticle picture is in the baryon sector which would be the case the larger $N$ is, the higher the critical density $n_K$. This indicates that since the half-skyrmion phase, as argued, is very likely in Fermi liquid at its fixed point [1], if kaons do condense from the half-skyrmion phase which sets in at $n \gtrsim n_{1/2} \sim 2n_0$, $n_K$ will inevitably be high. In compact stars in beta equilibrium, the electron chemical potential $\mu_e$ that increases with density, would help reduce $n_K$ but the qualitative result obtained is not expected to be modified seriously.

Just to have an idea of what sort of density range is involved, let me take $h(0) = \Sigma_{KN}/f$ from chiral Lagrangian. This is just to get a rough idea since the parameter $h$ should in our formulation represent coupling of a kaon on top of the Fermi sea to a quasiparticle warped by the background, both IDD and $DD_{\text{induced}}$, which of course could be quite different from $\Sigma_{KN}/f^2$. From (52), replacing $\phi$ by $K$ and $f$ by $f_\pi$, we find the condensation density $n_K$ bound by

$$n_K > \frac{N m_{\phi}^2 f_\pi^2}{6 \Sigma_{KN}}.$$  

(53)

For numerical estimation, take $m_K \approx 500$ MeV and $\Sigma_{KN} \approx 250$ MeV using a recent lattice measurement for $\Sigma_{KN}$ [31]. This gives at the Fermi-liquid fixed point

$$n_K > \frac{Nm_{\phi}^2}{3} / 3 \approx Nn_0 \to \infty.$$  

(54)

In order to have an idea what $N$ is, let’s take the cutoff used in the $V_{\text{lowk}}$RG approach applied in [1]. The optimal cutoff turns out to be $\Lambda \sim (2 - 3) \text{fm}^{-1}$ [30]. If one takes $\Lambda \sim 2 \text{fm}^{-1}$, then in the half-skyrmion phase that sets in at $n \gtrsim 2n_0$, $N = \frac{k_F}{\Lambda - k_F} \gtrsim 5n_0$. This means once the half-skyrmion state sets in, the system rapidly moves towards the Fermi-liquid fixed point (with zero beta function). In the density range $(6 - 7)n_0$ relevant to the interior of massive compact stars, we will have $1/N \sim 0$. This implies that the condensation density $n_K$ will be pushed way beyond the maximum interior density of massive stars.

V. CONCLUSION

Combining the skyrmion description of dense matter with possible smooth hadron matter-quark matter transition traded in for topology change at a density $n \sim (2 - 3)n_0$ and an Wilsonian RG analysis of coupled baryon-meson systems in dense medium, I argued that (1) hyperons and condensed kaons must appear simultaneously in the large $N_c$ limit and (2) kaon condensation is banished to very high density beyond the density relevant to the interior of massive compact stars. The conclusion is that strangeness is unlikely to figure explicitly and cause the strangeness (hyperons and condensed kaons) problem in compact stars.

I should mention a caveat here. The reasoning that $N$ is large in (54) relies on that the matter treated is a good Fermi liquid. However the four-Fermi interaction corresponding to a scalar exchange in (22), though a fairly good approximation near normal nuclear matter density [32], cannot be valid when the dilaton limit with the vanishing dlaton mass is approached. In other words, the Fermi-liquid structure must break down. What happens when this breakdown occurs is unknown. It is however most likely that the breakdown takes place at a density much higher than is relevant for compact stars. A $V_{\text{lowk}}^{SU(3)}$ calculation could settle this issue.

Acknowledgments

I am grateful for discussions with Hyun Kyu Lee and Won-Gi Paeng. This note is largely based on the work done with them in the World Class Program at Hanyang University, Seoul, Korea.

[1] H. Dong, T. T. S. Kuo, H. K. Lee, R. Machleidt and M. Rho, “Half-Skyrmions and the Equation of State for Compact-Star Matter,” Phys. Rev. C 87, 054332 (2013); W. G. Paeng, T. T. S. Kuo, H. K. Lee and M. Rho, “Scale-Invariant Hidden Local Symmetry, Topology Change and Dense Baryonic Matter,” Phys. Rev. C 93, no. 5, 055203 (2016); W. G. Paeng, T. T. S. Kuo, H. K. Lee, Y. L. Ma and M. Rho, “Scale-invariant hidden local symmetry, topology change, and dense baryonic matter. II.,” Phys. Rev. D 96, no. 1, 014031 (2017);

[2] P. Demorest et al. “Shapiro delay measurement of a two solar mass neutron star,” Nature 467, 1081 (2010).

[3] J. Antoniadis et al., “A massive pulsar in a compact relativistic binary,” Science 340, 6131 (2013)
[4] D.B Kaplan and A.E. Nelson, “Kaon condensation in dense matter,” Nucl. Phys. A 479, 273c. (1988).
[5] G.E. Brown, C.H. Lee, and M. Rho, “Recent developments on kaon condensation and its astrophysical implications,” Phys. Rept. 462, 1 (2008).
[6] V.R. Pandharipande, C.J. Pethick, and V. Thorsson, “Kaon energies in dense matter,” Phys. Rev. Lett. 75, 4567 (1995).
[7] H.K. Lee and M. Rho, “Hyperons and condensed kaons in compact stars,” arXiv:1301.0067 [nucl-th].
[8] G.E. Brown, “Recent developments on kaon condensation and its astrophysical implications,” Phys. Rept. 462, 1 (2008).
[9] V.R. Pandharipande, C.J. Pethick, and V. Thorsson, “Kaon energies in dense matter,” Phys. Rev. Lett. 75, 4567 (1995).
[10] H.K. Lee and M. Rho, “Hyperons and condensed kaons in compact stars,” arXiv:1301.0067 [nucl-th].
[11] C.G. Callan Jr. and I.R. Klebanov, “Bound state approach to strangeness in the Skyrme model,” Nucl. Phys. B 262, 365 (1985).
[12] I.R. Klebanov, “Strangeness in the Skyrme model,” Cargese Lecture on Hadron and Hadronic Matter, Cargese, France, 1989.
[13] I.R. Klebanov, “Nuclear matter in the Skyrme model,” Nucl. Phys. B 262, 133 (1985).
[14] P. Bedaque and A.W. Steiner, “Hypernuclei and the hyperon problem in neutron stars,” Phys. Rev. C 92, 025803 (2015).
[15] G.E. Brown and M. Rho, “Double decimation and sliding vacua in the nuclear many body system,” Phys. Rept. 396, 1 (2004).
[16] Y.L. Ma and M. Rho, Effective Field Theories, Dense Matter and Compact Stars (World Scientific, Singapore, 2015).
[17] Y. L. Li and Y. L. Ma, “Derivation of Brown-Rho scaling from scale-chiral perturbation theory,” arXiv:1710.02839 [hep-ph].
[18] H.J. Lee, B.Y. Park, D.P. Min, M. Rho and V. Vento, “A unified approach to high density: Pion fluctuations in skyrmion matter,” Nucl. Phys. A 723, 427 (2003).
[19] C. Song, G.E. Brown, D.P. Min, and M. Rho, “Fluctuations in ‘BR scaled’ chiral Lagrangians,” Phys. Rev. C 56, 2244 (1997).
[20] J.W. Holt, G.E. Brown, J.D. Holt and T.T.S. Kuo, “Nuclear matter with Brown-Rho-scaled Fermi liquid interactions,” Nucl. Phys. A 785, 322 (2007).
[21] H.K. Lee, M. Rho and S.J. Sin, “Renormalization group flow analysis of meson condensations in dense matter,” Phys. Lett. B 348, 290 (1995).
[22] A.L. Fitzpatrick, S. Kachru, J. Kaplan and S. Raghu, “Non-Fermi liquid fixed point in a Wilsonian theory of quantum critical metals,” Phys. Rev. B 88, 125116 (2013).
[23] M. Rho, “Exchange currents from chiral Lagrangians,” Phys. Rept. 66, 1275 (1991).
[24] R. Shankar, “Renormalization group approach to interacting fermions,” Rev. Mod. Phys. 66, 129 (1994).
[25] S. B. Gudnason and M. Nitta, “Fractional Skyrmions and their molecules,” Phys. Rev. D 91, 085040 (2015); P. Zhang, K. Kimm, L. Zou and Y. M. Cho, “Reinterpretation of Skyrme theory: New topological structures,” arXiv:1704.05975 [hep-th].