Abstract

The aim of this paper is to study the local and the global thermodynamic properties of the 3-dimensional rotating Gauss-Bonnet BTZ black hole. We are interested in deriving the conditions for local and global thermodynamic stability of the solution in a given ensemble of state quantities. It is well-known that determining the local thermodynamic stability requires the calculation of the specific heats of the system. We identify the regions of stability for every physical specific heat together with the existing Davies curves. In addition to the local analysis, we generalize the notion of global thermodynamic stability from the standard thermodynamics to describe the global equilibrium of black holes. Here, we apply a proper Legendre transformation of the energy or the entropy to find the natural thermodynamic potential for the given ensemble of macro parameters. The global stability analysis is now conducted on the properties of the new thermodynamic potential. The advantage of this method is that it allows one to chose different potentials, corresponding to different constraints to which the system may be subjected. Finally, we find it natural to impose global thermodynamic stability only where local one exists for the given black hole solution.
1 Introduction

In the past few decades investigating lower dimensional gravity theories has become very attractive area of research. This is mainly due to the remarkable gauge/gravity correspondence [1] which states a duality between particular gravitational and quantum systems. A modern review of the correspondence can be found in [2]. Within this framework, a number of $D = 3$ black hole solutions are shown to be dual to two-dimensional quantum field theories at a finite temperature. The most famous of these solutions is the Banados-Teitelboim-Zanelli (BTZ) black hole [3] and its generalizations.

Until recently there were two ways of constructing three-dimensional models of gravity. In the first approach one adds topological Chern-Simons terms to the standard Einstein-Hilbert action [4–6]. In the second approach the Einstein-Hilbert action is modified by higher-derivative correction terms [7,8]. Lately a third approach, involving higher-curvature corrections$^1$ to the Einstein-Hilbert action, found its way down to three-dimensional gravity. It was shown that the $D > 5$ Einstein-Gauss-Bonnet theory$^2$ possesses a non-trivial limit to four [10,11] and lower spacetime dimensions [12,13]. The latter has been suggested to circumvent the Lovelock theorem and allows the contribution of the higher-curvature Gauss-Bonnet term to the local dynamics. While the proposed regularization procedure is not consistent for general gravitational fields [14–17], it leads to correct predictions in a number of cases with high symmetries.

$^1$Higher-curvature corrections are known to occur often in models of quantum gravity. There they arise precisely as quantum corrections to the Einstein-Hilbert action.

$^2$Gauss-Bonnet gravity is the simplest representative of the Lovelock class of theories of gravity in higher than four spacetime dimensions. Lovelock gravity [9] maintains the property of having second-order field equations for all backgrounds.
Although there are many gravitational solutions in three dimensions, to our knowledge there exist only two novel $D = 3$ Gauss-bonnet black holes found in [12,13]. The solution given in [12] is a Gauss-Bonnet generalization of the static BTZ black hole with non-trivial scalar field profile. The other solution is the rotating Gauss-Bonnet BTZ black hole [13]. As a generalization of the standard BTZ solution, which is known to be dual to a 2-dimensional conformal field theory, we expect the rotating Gauss-Bonnet BTZ (RGB-BTZ) black hole also to be dual to a certain 2d CFT. The goal of our paper is to study the thermodynamic properties of the RGB-BTZ black hole from local and global perspective. Our analysis can later be transferred to the dual quantum system. Similar analysis has already been conducted in [18], where one can constrain the dual left and the right central CFT charges using the bulk thermodynamics of the warped AdS$_3$ black hole.

In standard black hole thermodynamics, one is interested only in the proper state quantities – those being the energy, the entropy, the charges and the angular momentum of the black hole. That being said, black holes are thermal systems – this means that they might not necessarily be in thermodynamic equilibrium with their environment. Thus, further considerations have to be taken into account. Specifically, one discerns two types of equilibrium – local and global.

If a given system is in global thermodynamic equilibrium then, by definition, it has the same temperature, the same pressure, the same chemical potentials etc, everywhere in space. In this case, one can study the global thermodynamic stability (GTDS) in a given ensemble by considering the properties of the corresponding thermodynamic potentials.

The system is said to be in local thermodynamic equilibrium if one can divide it into smaller constituents, which are individually in thermodynamic equilibrium, at least approximately. These partial systems can also be described by thermodynamic state quantities. However, it is of crucial importance that the partial systems can be chosen large enough for a statistical description to be reasonable. Nevertheless, in each partial system the intensive thermodynamic state quantities assume definite constant values and do not vary too strongly from one partial system to another, i.e. only small gradients are allowed.

If the system is in local equilibrium, the local thermodynamic stability (LTDS) does not imply a global one. On the other hand, it is natural to assume that a system in global thermodynamic equilibrium is also locally stable. Thus, it is evident that GTDS always imply LTDS, but not vice versa. This notion will be of topmost importance in our considerations. Whilst analysing the numerous specific heats definable in our extended thermodynamic picture we will look for regions of intersection between LTDS and GTDS. Only in such regions can one define true global thermodynamic equilibrium with respect to the corresponding specific heat.

The structure of the paper is as follows. In Section 2 we present the RGB-BTZ black hole solution and its thermodynamics. Although the local thermodynamic stability of black holes has been fully developed [19], to our knowledge the global thermodynamic stability analysis has not been stated properly or in full for black holes. For that reason in Section 4 we are going to generalize the notion of global thermodynamic stability from standard thermodynamics to describe the global equilibrium of black holes. We will achieve this by applying a proper Legendre transformation of the energy or the entropy of the RGB-BTZ solution to find the natural thermodynamic potential for the given ensemble of macro parameters. The global stability analysis is now conducted on the properties of the new thermodynamic potential. The advantage of this method is that it allows one to chose different potentials, corresponding to different constraints to which the system may be subjected. In Section 3 we fix our equilibrium ensemble. In Section 5 we investigate the local thermodynamic stability of the RGB-BTZ black hole by analyzing the proper specific heats. As stated previously, we find it natural to impose global thermodynamic stability only where local one exists. Finally, in Section 6 we give our concluding remarks.
2 The Rotating Gauss–Bonnet BTZ Black Hole

The Einstein-Gauss-Bonnet action in $D = d + 1$ space-time dimensions is given by

$$I = \frac{1}{16\pi} \int d^Dx \sqrt{|g|} \left[ R - 2\Lambda + \alpha \left( \phi G + 4G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 4(\partial\phi)^2\Box\phi + 2(\partial\phi)^4 \right) \right],$$  \hspace{1cm} (2.1)

where $\phi(t, \vec{x})$ is a scalar field, $\Lambda = -d(d-1)/(2\ell^2)$ is the cosmological parameter, $\alpha$ is the Gauss–Bonnet coupling and $G = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss–Bonnet term, which identically vanishes for $D < 4$.

In $D = 2 + 1$ dimensions the metric for the static BTZ black hole solution in Gauss-Bonnet theory is [12]:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\varphi^2, \quad \phi = \log \frac{r}{l},$$  \hspace{1cm} (2.2)

where $l$ is an integration constant and

$$f^\pm = \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + \frac{4\alpha}{r^2} f_E} \right)$$  \hspace{1cm} (2.3)

is the Gauss–Bonnet generalization of the (static) Einstein theory BTZ metric [3]:

$$f_E = \frac{r^2}{\ell^2} - m.$$  \hspace{1cm} (2.4)

In this case, only $f^-$ gives a black hole solution, which at $\alpha \to 0$ reduces to the standard Einstein BTZ with $R = -6/\ell^2$. Here, one also has the cosmological length scale $\ell > 0$ and two arbitrary integration constants $l, m$.

The $D = 2 + 1$ rotating Gauss-Bonnet BTZ (RGB-BTZ) black hole solution can be obtained from (2.2) after performing the following boost transformation on the coordinates [13]:

$$t \to \Xi t - a\varphi, \quad \varphi \to \frac{at}{\ell^2} - \Xi \varphi, \quad \Xi^2 = 1 + \frac{a^2}{L^2}, \quad L = \sqrt{\frac{2\alpha}{\sqrt{1 + \frac{4\alpha}{r^2} - 1}}, \quad a \geq 0.}$$  \hspace{1cm} (2.5)

Hence, the metric of the RGB-BTZ black hole yields

$$ds^2 = -f^-(\Xi dt - ad\varphi)^2 + \frac{r^2}{L^4} (adt - \Xi L^2 d\varphi)^2 + \frac{dr^2}{f^-}, \quad \phi = \log \frac{r}{l}.$$  \hspace{1cm} (2.6)

As in the static case, one has a black hole solution for $f^-$, which has a proper limit at $\alpha \to 0$. The inner Cauchy horizon is located at $r = 0$ and the outer event horizon resides at $r_h = \ell \sqrt{m}$, where $m > 0$. Furthermore, the scalar curvature is

$$R = \frac{3r^2(4\alpha + \ell^2) \left( r\ell\sqrt{X} - (X - 2am\ell^2) \right) - 12amr\ell^3\sqrt{X} - 16a^2m^2\ell^4}{\alpha r\ell^3 X^{3/2}},$$  \hspace{1cm} (2.7)

where $X = r^2(4\alpha + \ell^2) - 4am\ell^2$. The curvature $R$ indicates physical singularities at $r \to 0$ and

$$r_{cs} = \frac{2\sqrt{ma\ell}}{\sqrt{4\alpha + \ell^2}}.$$  \hspace{1cm} (2.8)

When $\alpha > 0$ the curvature singularity is $r_{cs} > 0$ and the metric function cannot be extended all the way to $r = 0$. When $-\ell^2/4 < \alpha < 0$ the curvature singularity $r_{cs}$ is not real, thus the metric function can be extended down to $r = 0$. Finally, when $\alpha < -\ell^2/4$, the singularity at $r_{cs}$ reappears positive and real. This case, however, corresponds to a naked singularity $r_{cs} > r_h$ and it will not be considered.

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3We also have the notations $\sqrt{|g|} \Box \phi = \partial_\mu(\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi)$, $(\partial \phi)^2 = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ and $(\partial \phi)^4 = (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi)^2$. 
3 Extended thermodynamics and equilibrium space

3.1 Extended thermodynamics

The relevant thermodynamic state quantities of the RGB-BTZ black hole have already been obtained in [13]. They can be written in the following way

\[ T = \frac{\sqrt{m}}{2\pi\ell \sqrt{\frac{a^2 Y}{2\alpha} + 1}}, \quad S = \frac{\sqrt{m\pi\ell}}{2} \sqrt{\frac{a^2 Y}{2\alpha} + 1}, \quad (3.1) \]

\[ \Omega = \frac{aY}{2\alpha \sqrt{\frac{a^2 Y}{2\alpha} + 1}}, \quad J = \frac{am}{4} \sqrt{\frac{a^2 Y}{2\alpha} + 1}, \quad (3.2) \]

\[ V = \pi m \left( \frac{a^2}{1 + Y} + \ell^2 \right), \quad P = \frac{1}{8\pi \ell^2}, \quad (3.3) \]

\[ M = \frac{m}{8} \left( \frac{a^2 Y}{\alpha} + 1 \right), \quad \Psi = \frac{a^2 m}{16a^2(1 + Y)} \left( Y - \frac{2\alpha}{\ell^2} \right), \quad (3.4) \]

where for convenience we have defined the parameter

\[ Y = \sqrt{\frac{4\alpha}{\ell^2}} + 1 - 1. \quad (3.5) \]

Furthermore, the parameter \( P = -\Lambda/(8\pi) \) is proportional to the cosmological constant and is interpreted as pressure. Its conjugate thermodynamic variable \( V \) is the thermodynamic volume of the black hole. The state quantity \( \Psi \) is the chemical potential for the Gauss-Bonnet parameter \( \alpha \). The first law of thermodynamics yields

\[ \delta M = T\delta S + \Omega \delta J + V\delta P + \Psi \delta \alpha. \quad (3.6) \]

Additionally, one has the Smarr relation

\[ 0 = TS - 2PV + \Omega J + 2\Psi \alpha. \quad (3.7) \]

In equilibrium, the standard relations between the intensive and the extensive parameters hold:

\[ T = \left. \frac{\partial M}{\partial S} \right|_{aJ\alpha}, \quad \Omega = \left. \frac{\partial M}{\partial J} \right|_{SP\alpha}, \quad V = \left. \frac{\partial M}{\partial P} \right|_{SJ\alpha}, \quad \Psi = \left. \frac{\partial M}{\partial \alpha} \right|_{SPJ}. \quad (3.8) \]

In the limit \( a \to 0 \) we recover the thermodynamics of the static Gauss-Bonnet BTZ solution, which is identical to the Einstein BTZ black hole as pointed out by [13]:

\[ M = \frac{m}{8}, \quad T = \frac{\sqrt{m}}{2\pi\ell}, \quad S = \frac{\pi\ell\sqrt{m}}{2}, \quad P = \frac{1}{8\pi \ell^2}, \quad V = m\pi\ell^2, \quad \Psi = 0, \quad (3.9) \]

with the first law and the Smarr relation reducing to

\[ \delta M = T\delta S + V\delta P + \Psi \delta \alpha \quad \text{and} \quad 0 = TS - 2PV + 2\Psi \alpha, \quad (3.10) \]

respectively.

It is natural to impose that certain physical parameters are always positive, namely the mass, entropy, temperature and volume. It can be shown that \( M, S, T, V > 0 \) leads to

\[ \alpha > 0 \quad \text{or} \quad -\ell^2/4 < \alpha < 0, \quad (3.11) \]

thus confirming the two sectors of the solution. One can also consider \( \alpha \to 0^\pm \) and \( \alpha \to -\ell^2/4 \) as special cases, when it is possible. Note that the naked singularity case \( \alpha < -\ell^2/4 \) has been discarded.
3.2 The space of equilibrium states

In order to mitigate some computational complexity in the study of the thermodynamics of the system we express \((M, S, J, V, \Psi)\) in terms of the parameters \((T, \Omega, P, \alpha)\). The later set will span our equilibrium space\(^4\). To do so one can solve \(T\) and \(\Omega\) (3.1,3.2) for \(m\) and \(a\):

\[
m = \frac{2\pi^2 T^2 \ell^2 (a^2 Y + 2\alpha)}{\alpha}, \quad a_\pm = \pm \frac{2\alpha \Omega}{\sqrt{Y} (Y - 2\alpha \ell^2)}.
\]  

(3.12)

where we take \(a_+ > 0\) for \(\alpha > 0\) and \(a_- > 0\) for \(-\ell^2 / 4 < \alpha < 0\). This follows from the sign of \(Y\), i.e.

\[
Y = \sqrt{\frac{4\alpha}{\ell^2}} + 1 - 1 = \sqrt{32\pi \alpha P + 1} - 1 \Rightarrow \begin{cases} Y > 0, & \alpha > 0, \\ -1 < Y < 0, & -\frac{\ell^2}{T} < \alpha < 0. \end{cases}
\]

(3.13)

One has to be careful with the condition \(-1 < Y < 0\), because \(Y\) depends on \(\alpha\). The right and left bounds are \(Y_1(\alpha, P)\) and \(Y_2(\alpha, P)\), which satisfy \(Y_1(\alpha \to -\ell^2 / 4) \to -1\) and \(Y_2(\alpha \to 0^-) \to 0\). Therefore this condition actually looks like

\[-1 < Y_1 \leq Y \leq Y_2 < 0.\]

(3.14)

In both sectors for \(\alpha\) the thermodynamics in \((T, \Omega, P, \alpha)\) equilibrium space takes the form:

\[
\begin{align*}
M &= \frac{2\pi^2 \alpha T^2 (2\alpha \Omega^2 + Y)}{(Y + 2) (Y - 2\alpha \Omega^2)^2}, \\
S &= \frac{4\pi^2 \alpha T}{Y + 2} (Y - 2\alpha \Omega^2), \\
J &= \frac{8\pi^2 \alpha^2 T^2 \Omega}{(Y + 2) (Y - 2\alpha \Omega^2)^2}, \\
V &= \frac{64\pi^3 \alpha^3 T^2 (-\alpha \Omega^2 + Y + 1)}{(Y + 1) (Y + 2) (Y - 2\alpha \Omega^2)^2}, \\
\Psi &= -\frac{2\pi^2 \alpha^2 T^2 Y \Omega^2}{(Y^2 + 3Y + 2) (Y - 2\alpha \Omega^2)^2}.
\end{align*}
\]

(3.15)

Assuming \(T, P, \Omega > 0\), all parameters in (3.15) have a common divergence in \((T, \Omega, P, \alpha)\) space given by the following temperature independent spinodal curve \(Y - 2\alpha \Omega^2 = 0\). More explicitly one has

\[
s = \sqrt{32\pi P \alpha + 1} - 1 - 2\omega,
\]

(3.16)

where \(\omega = \Omega^2 > 0\). From now on it is convenient to work with \(\omega\) instead of \(\Omega\), where latter will appear only when necessarily. Solving \(s = 0\) with respect to \(\omega\), one finds the following critical value:

\[
\omega_c = \frac{\sqrt{32\pi P + 1} - 1}{2\alpha} = \frac{Y}{2\alpha}.
\]

(3.17)

One notes that \(\omega_c > 0\) in both sectors for \(\alpha\). Furthermore, since the entropy from (3.15) has to be positive, \(S > 0\), it is evident that the following restriction on \(\omega\) must hold

\[
\frac{\alpha}{Y - 2\alpha \omega} > 0,
\]

(3.18)

which reduces to

\[
\omega < \omega_c.
\]

(3.19)

Thus, all physically meaningful states occur for values of \(\omega\) less than the critical value. No physical states exist for \(\omega > \omega_c\). The analysis of the thermodynamic properties of the system, close to the spinodal curve \(s = 0\) (\(\omega = \omega_c\)) requires the methods of non-equilibrium thermodynamics. We leave this path of inquiry for future work.

\(^4\)It becomes an equilibrium manifold after defining a proper Riemannian metric on it, which is a case of study by the framework of thermodynamic information geometry.
The critical squared angular velocity $\omega_c$ is a decreasing function of $\alpha$. For positive values of $\alpha$ this parameter is bounded from above and below, i.e.

$$0 < \omega_c < 8\pi P, \quad \alpha > 0,$$

which follows from the limits $\lim_{\alpha \to 0^+} \omega_c = 8\pi P$ and $\lim_{\alpha \to \infty} \omega_c = 0$. The negative values of $\alpha$ are bounded from below $\alpha_p < \alpha$, where $\alpha_p = -\ell^2/4 = -1/(32\pi P)$ is the physical lower bound. In this case, the parameter $\omega_c$ is bounded from both sides,

$$8\pi P < \omega_c < 16\pi P, \quad \alpha_p < \alpha < 0,$$

where $\lim_{\alpha \to \alpha_p} \omega_c = 16\pi P$.

In what follows we will investigate the local and global thermodynamic properties of the rotating Gauss-Bonnet BTZ solution in both sectors for $\alpha$ in $(T, \Omega, P, \alpha)$ equilibrium space.

## 4 Global thermodynamic stability

### 4.1 Positive Gauss–Bonnet parameter, $\alpha > 0$

One can study the global thermodynamic stability in a given ensemble by considering the properties of the corresponding thermodynamic potentials. In thermodynamics it is conventional to begin with the energy potential. In the extended black hole thermodynamics the mass does not coincide with the energy of the black hole, but it is interpreted as the enthalpy of spacetime:

$$M = E + PV = H.$$  

(4.1)

If we take the differential from both sides of this equation and solve for $dE$ and then compare with the first law (4.2) we find

$$\delta E = T\delta S + \Omega\delta J - P\delta V + \Psi \delta \alpha.$$  

(4.2)

The first law in this form fixes the natural parameters for the energy potential. In this case one has $E = E(S, J, V, \alpha)$. On the other hand, the energy potential is globally convex in its natural parameters$^5$, which means that its Hessian should be positive definite$^6$:

$$\frac{\partial^2 E}{\partial S^2} \bigg|_{J,V,\alpha} \geq 0, \quad \frac{\partial^2 E}{\partial J^2} \bigg|_{S,V,\alpha} \geq 0, \quad \frac{\partial^2 E}{\partial V^2} \bigg|_{S,J,\alpha} \geq 0, \quad \frac{\partial^2 E}{\partial \alpha^2} \bigg|_{S,J,V} \geq 0.$$  

(4.3)

However, we have a solution for the energy potential in $(T, \Omega, P, \alpha)$ space,

$$E(T, \Omega, P, \alpha) = M - PV = \frac{\pi \alpha T^2 Y (3Y + 2)\Omega^2}{16P(Y + 1)(Y - 2\alpha\Omega^2)^2},$$  

(4.4)

thus a change of variables has to be performed in the above conditions. This can actually be achieved if one finds the appropriate thermodynamic potential whose natural state parameters are $(T, \Omega, P, \alpha)$. In this case, this is given by the potential

$$\Phi = \mathcal{L}_{S,J,V} E = E - TS - \Omega J + PV = -\frac{\pi T^2 Y}{16PY - 32\alpha P\Omega^2};$$  

(4.5)

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$^5$In equilibrium the energy of the black hole should be minimal.

$^6$If the Hessian of a function is positive definite at some point, then the function has a local minimum at that point. The Hessian is positive definite if its diagonal elements, its eigenvalues and its determinant are all positive. In this case, one can also use the Sylvester criterion: all principal minors of the Hessian matrix should be positive.
which is obtained by a Legendre transformation from \((S, J, V, \alpha)\) space to \((T, \Omega, P, \alpha)\). The first law of thermodynamics now reads
\[
\delta \Phi = -S \delta T - J \delta \Omega + V \delta P + \Psi \delta \alpha.
\] (4.6)

Therefore, the natural state parameters for \(\Phi\) are exactly \((T, \Omega, P, \alpha)\), which leads to the following global thermodynamic stability conditions for the RGB-Black hole:
\[
\frac{\partial^2 \Phi}{\partial T^2} \bigg|_{T, \Omega, \alpha} \leq 0, \quad \frac{\partial^2 \Phi}{\partial \Omega^2} \bigg|_{T, P, \alpha} \leq 0, \quad \frac{\partial^2 \Phi}{\partial P^2} \bigg|_{T, \Omega, \alpha} \leq 0, \quad \frac{\partial^2 \Phi}{\partial \alpha^2} \bigg|_{T, \Omega, P} \geq 0.
\] (4.7)

The flip of the signs of the first three conditions in (4.7), as compared to the global minimization conditions of the energy (4.3), is due to the fact that the product of the corresponding conjugate thermodynamic variables in the Legendre transformation is always taken as minus. We can explicitly express these conditions with respect to the relevant parameters. For example, we choose to solve simultaneously (4.7) with respect to \(\omega\). In this case, the GTDS conditions reduce to\(^7\)
\[
0 < \omega \leq \omega_g,
\] (4.8)
where
\[
\omega_g = \frac{Y + 2}{3Y + 4} \omega_c = \frac{2\pi P \left(-1 + 3 \sqrt{32\pi \alpha P + 1}\right)}{36\pi \alpha P + 1}.
\] (4.9)

Noticing that \((Y + 2)/(3Y + 4) < 1\), it follows that \(\omega \leq \omega_g < \omega_c\) for all values of \(P\) and \(\alpha\) in this sector. Therefore, the global spinodal \(\omega_c\) is a boundary of the region of global thermodynamic stability. The hierarchy between different \(\omega\) in this sector is
\[
0 < \omega \leq \omega_g < \omega_c < 8\pi P.
\] (4.10)

Furthermore, the upper global bound \(\omega_g\) never intersects with \(\omega_c\), unless \(P \to 0\), where \(\omega_g = \omega_c = 0\). The GTDS in this sector is depicted in Fig. 1a. We also note that \(\omega\) can become greater than \(\omega_g\) for LTDS.

### 4.2 Negative Gauss–Bonnet parameter, \(\alpha_p < \alpha < 0\)

In this sector the Gauss-Bonnet parameter is negative and bounded from below \(\alpha_p < \alpha < 0\). The condition for GTDS from Eq. (4.7) lead to couple of distinct cases. We keep in mind that in all cases \(\omega_g < \omega_c\) holds.

- The simplest GTDS case corresponds to:
  \[
  0 < \omega \leq \omega_g, \quad -\frac{1}{36\pi P} \leq \alpha < 0.
  \] (4.11)
  This situation is shown on Fig. 1b.

- The second GTDS case is valid for \(\alpha_p < \alpha < -\frac{1}{36\pi P}\). It divides in two disjoint cases, where a more strict condition for \(\alpha\) emerges:
  \[
  0 < \omega \leq \omega_+, \quad \omega_- \leq \omega \leq \omega_g, \quad \text{where} \quad \alpha_p < \alpha \leq -\frac{3}{100\pi P},
  \] (4.12)
  with \(\omega_\pm\) are given by
  \[
  \omega_\pm = \frac{9(Y + 2) + 8 \pm (Y + 2)\sqrt{-(3Y + 4)(5Y + 4)}}{4\alpha(3Y + 4)}.
  \] (4.13)
  This case is pictured on Fig. 1c.

\(^7\)The index \(g\) in \(\omega_g\) stands for “global”.
Let us shortly discuss the results for the global thermodynamic stability of the RGB-BTZ black hole. In the $\alpha > 0$ sector there is only one condition for GTDS given in Eq. (4.8). The situation in the $\alpha_p < \alpha < 0$ sector is more complicated. Here one has two distinct cases for GTDS of the black hole. However, when considering the global thermodynamic stability together with the local one, we have to take into account only the intersections between both types of stabilities in order to have true GTDS. The reason for this follows from the fact that while LTDS does not require GTDS, the GTDS always imply LTDS. This analysis will be conducted in Section 5.

Figure 1: Global thermodynamic stability a) GTDS for $\alpha > 0$ occurs at $0 < \omega \leq \omega_g$; b) GTDS for $-\frac{1}{36\pi P} \leq \alpha < 0$ occurs at $0 < \omega \leq \omega_g$. c) GTDS for $\alpha_p < \alpha \leq -\frac{3}{100\pi P}$ or $\omega_\pm \leq \omega \leq \omega_g$. The point $\omega_+$ coincides with $\omega_-$ when $\alpha = -\frac{3}{100\pi P}$. In this case the left and right GTDS merge together.

### 4.3 Legendre transformation and other thermodynamic potentials

In order to make our global analysis more complete, let us say few words about some of the other thermodynamic potentials. Due to the fact that different potentials correspond to different constraints to which the system may be subjected one can study GTDS by constructing other energy derived thermodynamic potentials (see Appendix A). The latter can be obtained by the proper Legendre transformation of the energy potential along given natural state quantities. For example the enthalpy of spacetime, the Gibbs free energy and the Helmholtz free energy are given by

$$M = H = L_V E = E - (-PV) = E + PV, \quad (4.14)$$
$$G = L_{S,V} E = E - TS + PV, \quad (4.15)$$
$$F = L_{S,E} E = E - TS. \quad (4.16)$$

The natural parameters of these potentials can be obtained from the corresponding form of the first law:

$$\frac{\partial^2 M}{\partial S^2} \bigg|_{J,P,\alpha} \geq 0, \quad \frac{\partial^2 M}{\partial J^2} \bigg|_{S,P,\alpha} \geq 0, \quad \frac{\partial^2 M}{\partial P^2} \bigg|_{S,J,\alpha} \leq 0, \quad \frac{\partial^2 M}{\partial \alpha^2} \bigg|_{S,J,P} \geq 0. \quad (4.20)$$

Therefore the mass potential is convex function of $S, J$ and $\alpha$, but concave of $P$. Similar reasoning holds for $G$ and $F$:

$$\frac{\partial^2 G}{\partial T^2} \bigg|_{J,P,\alpha} \leq 0, \quad \frac{\partial^2 G}{\partial J^2} \bigg|_{T,P,\alpha} \geq 0, \quad \frac{\partial^2 G}{\partial P^2} \bigg|_{T,J,\alpha} \leq 0, \quad \frac{\partial^2 G}{\partial \alpha^2} \bigg|_{T,J,P} \geq 0. \quad (4.21)$$
\[
\frac{\partial^2 F}{\partial T^2} \bigg|_{J,V,\alpha} \leq 0, \quad \frac{\partial^2 F}{\partial J^2} \bigg|_{T,V,\alpha} \geq 0, \quad \frac{\partial^2 F}{\partial V^2} \bigg|_{T,J,\alpha} \geq 0, \quad \frac{\partial^2 F}{\partial \alpha^2} \bigg|_{T,J,V} \geq 0.
\] (4.22)

Therefore, the Gibbs potential is convex function of \( J \) and \( \alpha \), but concave of \( T \) and \( P \). The Helmholtz potential is convex in \( J, V \) and \( \alpha \), but concave function of \( T \).

Using the Legendre transformation of the energy \( E = E(S,J,V,\alpha) \) one can construct more energy derived thermodynamic potentials. We give the full list in Appendix A.

The energy derived thermodynamic potentials are not the only possibility. For example, if one starts with the entropy potential one can use the Legendre transformation of the entropy to construct new thermodynamic potentials called Massieu–Planck or free entropies\(^8\)(see Appendix A). To see how to do that, one rewrites the first law with respect to the entropy

\[
\delta S = \frac{1}{T} \delta E - \frac{\Omega}{T} \delta J + \frac{P}{T} \delta V - \frac{\Psi}{T} \delta \alpha,
\] (4.23)

where the parameter \( \beta = 1/T \) is the conjugate variable of \( E \), the parameter \( \Omega/T \) is conjugate to \( J \) and so on. Now it is obvious that the natural parameters for the entropy are \( S = S(E,J,V,\alpha) \).

In equilibrium the entropy is maximal thus it is globally concave in its natural parameters, which means that its Hessian should be negative definite:\(^9\)

\[
\frac{\partial^2 S}{\partial E^2} \bigg|_{J,V,\alpha} \leq 0, \quad \frac{\partial^2 S}{\partial J^2} \bigg|_{E,V,\alpha} \leq 0, \quad \frac{\partial^2 S}{\partial V^2} \bigg|_{E,J,\alpha} \leq 0, \quad \frac{\partial^2 S}{\partial \alpha^2} \bigg|_{E,J,V} \leq 0.
\] (4.24)

The relevant Massieu–Planck potential in \((T,\Omega,P,\alpha)\) space is

\[
\Sigma = \mathcal{L}_{E,J,V} S = S - \frac{1}{T} E + \frac{\Omega}{T} J - \frac{P}{T} V = \frac{\pi TY}{16PY - 32\alpha P\Omega^2}.
\] (4.25)

The first law now changes to\(^10\)

\[
\delta \Sigma = -(E - \Omega J + PV)\delta \frac{1}{T} + \frac{J}{T} \delta \Omega - \frac{V}{T} \delta P - \frac{\Psi}{T} \delta \alpha.
\] (4.26)

The conditions for global thermodynamic stability change sign along \( T, \Omega \) and \( P \) with respect to the inequalities along their conjugate variables \( E, J \) and \( V \) from (4.24):

\[
\frac{\partial^2 \Sigma}{\partial T^2} \bigg|_{\Omega,P,\alpha} \geq 0, \quad \frac{\partial^2 \Sigma}{\partial \Omega^2} \bigg|_{T,P,\alpha} \geq 0, \quad \frac{\partial^2 \Sigma}{\partial P^2} \bigg|_{T,\Omega,\alpha} \geq 0, \quad \frac{\partial^2 \Sigma}{\partial \alpha^2} \bigg|_{T,J,P} \leq 0.
\] (4.27)

These conditions lead to the same regions of global thermodynamic stability as discussed in the previous subsection. This confirms the correctness of our global thermodynamic analysis.

## 5 Local thermodynamic stability

The local thermodynamic stability (LTDS) of the RGB-BTZ black hole can be determined by investigating the properties of the corresponding specific heats. The direct way could be to

\(^8\)Sometimes they are called free information.

\(^9\)If the Hessian of a function is negative definite at some point, then the function has a local maximum at that point. The Hessian is negative definite if its diagonal elements and its eigenvalues are all negative. However, the determinant is not necessarily negative, because it is equal to the product of the eigenvalues: the determinant is positive if \( n \) is even, and negative if \( n \) is odd, where \( n \times n \) is the size of the Hessian matrix.

\(^10\)The natural parameters for the \( \Sigma \) potential are \((\beta,\Omega,P,\alpha)\), where \( \beta = 1/T \).
just take the derivative of the entropy with respect to the temperature, which will result in the following heat capacity:

\[ C = T \left( \frac{\partial S}{\partial T} \right) = \frac{4\pi^2 \alpha T}{(Y + 2)(Y - 2\alpha \Omega^2)}, \]  

(5.1)

which is just the entropy itself. The problem in multi-parameter thermodynamic systems, such as the RGB-BTZ black hole, is that there are multiple specific heats to choose from, because one has to keep track of which state quantities are fixed when calculating the specific heats. In this case, we can refer to the Nambu bracket formalism developed by [19]. Let's start by an ensemble with parameters \((A, B, C, D)\). In this case, for specific heat with constant parameters \(E, F, G\) the following relation holds:

\[ C_{E,F,G} = T \left( \frac{\partial S}{\partial T} \right)_{E,F,G} = T \frac{\{S, E, F, G\}_{A,B,C,D}}{\{T, E, F, G\}_{A,B,C,D}}. \]  

(5.2)

In our case \((A, B, C, D) = (T, \Omega, P, \alpha)\) define the parameter of the ensemble, and \((E, F, G)\) span all the other parameters \((\Omega, J, V, P, \Psi, \alpha)\), with \(\Phi\) being the thermodynamic potential. Therefore, the relevant heat capacities for the RGB-BTZ solutions are (see Appendix B):

\[ C_{\Omega,P,\alpha} = \frac{4\pi^2 \alpha T}{(Y + 2)(Y - 2\alpha \omega)}, \]  

(5.3)

\[ C_{J,P,\alpha} = \frac{4\pi^2 \alpha T}{(Y + 2)(Y + 6\alpha \omega)}, \]  

(5.4)

\[ C_{\Omega,P,\Psi} = \frac{4\pi^2 \alpha T}{Y(Y + 2) - 2\alpha(3Y + 4)\omega}, \]  

(5.5)

\[ C_{J,P,\Psi} = \frac{4\pi^2 \alpha T(3Y + 4)}{(Y + 2)(2\alpha \omega(7Y + 10) + Y(3Y + 4))}, \]  

(5.6)

\[ C_{\Omega,V,\alpha} = \frac{4\pi^2 \alpha^2 \omega}{\alpha \omega(3Y + 4)(2 + 3Y - 2\alpha \omega) - 4(Y + 1)^3}, \]  

(5.7)

\[ C_{\Omega,V,\Psi} = \frac{4\pi^2 \alpha^2 T \omega}{(Y + 2)(Y - 2\alpha \omega)(\alpha \omega + Y(Y + 1)(2Y - 2\alpha \omega))}, \]  

(5.8)

\[ C_{J,V,\alpha} = \frac{4\pi^2 \alpha^2 T(3Y + 4)\omega}{(Y + 2)(\alpha \omega(-5Y^2 + 2\alpha(7Y + 10)\omega - 12Y - 8) - 4(Y + 1)^3)}, \]  

(5.9)

\[ C_{J,V,\Psi} = \frac{4\pi^2 \alpha^2 T(1 - 2Y(Y + 1))\omega}{(Y + 2)(\alpha \omega(Y(2Y^2 + 6Y + 5) - 2\alpha(2Y - 1)(2Y + 3)\omega) + (Y + 1)(3Y + 4)Y^2)}. \]  

(5.10)

One notes that the heat capacity from Eq. (5.1) now corresponds to \(C_{\Omega,P,\alpha}\), which has only one Davies curve \(Y = 2\alpha \omega\), matching the spinodal (3.16) of the natural global thermodynamic potential \(\Phi\) in \((T, \Omega, P, \alpha)\) space.

In general, if a given specific heat is positive then the system is thermodynamically stable from local standpoint with respect to this specific heat. If the corresponding specific heat is negative – the system is not in a local equilibrium. Finally, if the specific heat changes sign or diverges it indicates phase transitions of the system.

Let us now study the local thermodynamic stability of the RGB-BTZ black hole.
5.1 Positive Gauss–Bonnet parameter, \( \alpha > 0 \)

5.1.1 Specific heat \( C_{\Omega,P,\alpha} \)

The first and natural heat capacity in \((T, \Omega, P, \alpha)\) space is \( C_{\Omega,P,\alpha} \). It has one divergence at \( Y = 2\pi \omega \), which corresponds to the global spinodal curve \( \omega_c \) for the potential \( \Phi \). The heat capacity \( C_{\Omega,P,\alpha} \) is positive for \( \omega < \omega_c \). The latter defines the region of local thermodynamic stability for the RGB-BTZ black hole with respect to fixed \((\Omega, P, \alpha)\). One notes that the global stability from Eq. (4.8) falls within \( \omega < \omega_c \), due to the fact that \( \omega_g < \omega_c \), as shown on Fig. 2.

![Figure 2: Intersection of GTDS (red curve) and LTDS (blue curve) for fixed \((\Omega, P, \alpha)\). The same situation occurs also for fixed \((J, P, \alpha)\), \((J, P, \Psi)\) and \((\Omega, V, \Psi)\).](image)

Several limiting cases occur at \( \alpha \to 0^+ \), \( \Omega \to 0 \) and \( P \to 0 \):

\[
C_{\Omega,P,\alpha \to 0^+} = \frac{\pi^2 T}{8\pi P - \Omega^2}, \quad C_{\Omega \to 0, P, \alpha} = \frac{\pi T}{8P}, \quad C_{\Omega, P \to 0, \alpha} = -\frac{\pi^2 T}{\Omega^2} \quad (5.11)
\]

This suggest that in the rotating BTZ case \( C_{\Omega,P,\alpha \to 0^+} > 0 \), if only \( \omega < 8\pi P \), and negative \( C_{\Omega,P,\alpha \to 0^+} < 0 \) for \( \omega > 8\pi P \). In the static case \( C_{\Omega \to 0, P, \alpha} > 0 \) is always positive. In the non-extended case \( C_{\Omega, P \to 0, \alpha} < 0 \) is always negative.

5.1.2 Specific heat \( C_{J,P,\alpha} \)

The next specific heat \( C_{J,P,\alpha} \) has no occurring divergences and is always positive in this sector. Thus the RGB-BTZ black hole is locally stable at constant \((J, P, \alpha)\) in the physical region \( 0 < \omega < \omega_c \) (see Fig. 2). The three limiting cases here are:

\[
C_{J,P,\alpha \to 0^+} = \frac{\pi^2 T}{8\pi P + 3\Omega^2}, \quad C_{J,P,\alpha} = \frac{\pi T}{8P}, \quad C_{J,P \to 0, \alpha} = \frac{\pi^2 T}{3\Omega^2} \quad (5.12)
\]

5.1.3 Specific heat \( C_{\Omega,\Psi} \)

For constant \((\Omega, P, \Psi)\) the specific heat \( C_{\Omega,\Psi} \) is positive if \( 0 < \omega < \omega_g \), where \( \omega_g \) is defined in Eq. (4.8) and corresponds to a divergence in the heat capacity. In this case, it is obvious that the GTDS coincides with the LTDS, with \( \omega = \omega_g \) not included in GTDS. Furthermore, the singular curve \( \omega_g \) and the global spinodal \( \omega_c \) never intersect with one another in this sector.

![Figure 3: Intersection of GTDS (red curve) and LTDS (blue curve) for fixed \((\Omega, P, \Psi)\).](image)

The three limiting cases are:

\[
C_{\Omega,\Psi \to 0^+} = \frac{\pi^2 T}{2(4\pi P - \Omega^2)}, \quad C_{\Omega \to 0, \Psi} = \frac{\pi T}{8P}, \quad C_{\Omega, P \to 0, \Psi} = -\frac{\pi^2 T}{2\Omega^2} \quad (5.13)
\]
5.1.4 Specific heat $C_{J,P,\Psi}$

For constant $(J, P, \Psi)$ the relevant specific heat is $C_{J,P,\Psi}$. It has no apparent divergences and is always positive in this sector (see Fig. 2). The three limiting cases for $C_{J,P,\Psi}$ are:

$$
C_{J,P,\Psi} \rightarrow 0^+ = \frac{2\pi^2 T}{16\pi P + 5\Omega^2}, \quad C_{J,P,\Psi} = \frac{\pi T}{8P}, \quad C_{J,P,\Psi} \rightarrow 0, \quad C_{J,P,\Psi} = \frac{2\pi^2 T}{5\Omega^2}.
$$

\hspace{1cm} (5.14)

5.1.5 Specific heat $C_{\Omega,V,\alpha}$

The specific heat $C_{\Omega,V,\alpha} < 0$ is always negative in this sector, thus the RGB-BTZ black hole is locally unstable from thermodynamic standpoint with respect to fixed $(\Omega, V, \alpha)$. There are no physical divergences occurring for this specific heat. The three limiting cases are:

$$
C_{\Omega,V,\alpha} \rightarrow 0^+ = 0, \quad C_{\Omega,0,V,\alpha} = 0, \quad C_{\Omega,V,\alpha} \rightarrow 0 = -\frac{\pi^2 \alpha^2 T \Omega^2}{2\alpha^2 \Omega^4 - 2\alpha \Omega^2 + 1}.
$$

\hspace{1cm} (5.15)

We note that the limiting cases at $\alpha \rightarrow 0^+$ and $\omega \rightarrow 0$ the specific heat $C_{\Omega,V,\alpha}$ vanishes. This is not unexpected situation, because zero specific heat corresponds to a phase transition of the system, which can occur when some of the parameters are taken to their limits. This type of situations correspond to different gravitational solutions than the RGB-BTZ black hole and their thermodynamics is also different.

5.1.6 Specific heat $C_{\Omega,V,\Psi}$

The denominator of $C_{\Omega,V,\Psi}$ is a quadratic function of $\omega$ with roots $\omega_- = \omega_c$ and $\omega_+$ given by

$$
\omega_+ = \frac{2(Y + 1)(3Y + 4)}{2Y^2 + 2Y - 1} \omega_g.
$$

\hspace{1cm} (5.16)

Inspecting the coefficient in front of $\omega^2$ one notices three cases, namely:

- $0 < \alpha < \sqrt{\frac{3}{64\pi P}}$. In this sector, the specific heat is positive for

  $$
  0 < \omega < \omega_c,
  $$

  which defines the LTDS for this case. Since $\omega_g < \omega_c$ there is an intersection between the local and global thermodynamic stability (see Fig. 2).

- $\alpha = \sqrt{\frac{3}{64\pi P}}$. In this case, the LTDS region is

  $$
  0 < \omega < \omega_c, \quad \text{where} \quad \omega_c = \left(1 - \frac{\sqrt{3}}{3}\right) 16\pi P.
  $$

  Substituting the value for alpha in $\omega_g$, we find that $\omega < \omega_g < \omega_c$ and thus LTDS includes the GTDS. This is the same situation depicted on Fig. 2.

- $\alpha > \sqrt{\frac{3}{64\pi P}}$. In this case the LTDS is

  $$
  0 < \omega < \omega_c.
  $$

  Here $\omega_g < \omega_c$ and the situation coincides again with Fig. 2.

Finally, the three limiting cases for $C_{\Omega,V,\Psi}$ are:

$$
C_{\Omega,V,\Psi} \rightarrow 0^+ = \frac{\pi^2 T \Omega^2}{(8\pi P - \Omega^2)(32\pi P + \Omega^2)}, \quad C_{\Omega,0,V,\Psi} = 0, \quad C_{\Omega,V,\Psi} \rightarrow 0 = -\frac{\pi^2 T}{\Omega^2}.
$$

\hspace{1cm} (5.20)
5.1.7 Specific heat $C_{J,V,\alpha}$

The denominator of $C_{J,V,\alpha}$ is again a quadratic polynomial $f(\omega)$ with respect to $\omega$. The only positive root is

$$\omega_+ = \frac{Y(5Y + 12) + 8 + \sqrt{(3Y + 4)(83Y^3 + 260Y^2 + 272Y + 96)}}{4\alpha(7Y + 10)}.$$  \hfill (5.21)

One can check that the LTDS condition $C_{J,V,\alpha} > 0$ requires $f(\omega) > 0$. The latter leads to $\omega_c < \omega_+ < \omega$, thus the scope of LTDS is beyond the physical interval $0 < \omega < \omega_c$. Therefore the RGB-BTZ black hole can not be locally nor globally stable for fixed $(J, V, \alpha)$.

The three limiting cases for $C_{J,V,\alpha}$ are:

$$C_{J,V,\alpha \rightarrow 0^+} = 0, \quad C_{J,V,\alpha \rightarrow \Omega_0} = 0, \quad C_{J,V,\alpha \rightarrow P_0} = \frac{2\pi^2 \alpha^2 T^4 \Omega^2}{5\alpha^2 \Omega^4 - 2\alpha \Omega^2 - 1}. \hfill (5.22)$$

5.1.8 Specific heat $C_{J,V,\Psi}$

The final relevant specific heat in this sector is $C_{J,V,\Psi}$. It is always negative. Thus no LTDS or GTDS exist in this case. The three limiting cases for $C_{J,V,\Psi}$ are:

$$C_{J,V,\Psi \rightarrow 0^+} = \frac{2\pi^2 T}{16\pi P + 5\Omega^2}, \quad C_{J,V,\Psi \rightarrow \Omega_0} = \frac{\pi T}{8P}, \quad C_{J,V,\Psi \rightarrow P_0} = \frac{2\pi^2 T}{5\Omega^2}. \hfill (5.23)$$

However one notes that in the limiting cases shown above the heat capacity $C_{J,V,\Psi}$ is positive. Once again, this is due to the fact that these cases correspond to different gravitational systems than RGB-BTZ black hole.

Let us make a short summary of the result from this section. We have analyzed the behavior of all the relevant specific heats of the RGB-BTZ black hole in the $\alpha > 0$ sector. Six of the specific heats can be positive in some regions of the equilibrium space. In two cases, namely for fixed $(\Omega, V, \alpha)$ and fixed $(J, V, \alpha)$, we do not have a local thermodynamic stability. This leads to the conclusion that there doesn’t exist a sector in the phase space, where the system is in local equilibrium with respect to all of its parameters.

5.2 Negative Gauss–Bonnet parameter, $\alpha < 0$

5.2.1 Specific heat $C_{\Omega,P,\alpha}$

In this case the condition for LTDS $C_{\Omega,P,\alpha} > 0$ reduces to

$$0 < \omega < \omega_c.$$  \hfill (5.24)

Now let us check if the global and the local thermodynamic stability intersect or not. There are three relevant cases as depicted on Fig. 4.

![Figure 4: Thermodynamic stability](image)

- (a) $-\frac{1}{36\pi P} \leq \alpha < 0$.
- (b) $\alpha_p < \alpha \leq -\frac{3}{100\pi P}$.

Figure 4: Thermodynamic stability: a) GTDS for $-1/(36\pi P) \leq \alpha < 0$ occurs at $0 < \omega \leq \omega_g$ (red curve) and LTDS occurs at $0 < \omega < \omega_c$ (blue curve). b) GTDS for $\alpha_p < \alpha \leq -3/(100\pi P)$ occurs at $0 < \omega \leq \omega_+$ or $\omega_- \leq \omega \leq \omega_g$ (red curves), and the LTDS occurs at $0 < \omega < \omega_c$ (blue curve).
5.2.2 Specific heat $C_{J,P,\alpha}$

In this case $C_{J,P,\alpha}$ is always positive and the black hole is in LTDS for all $\omega < \omega_c$. This situation corresponds to Fig. 4.

5.2.3 Specific heat $C_{\Omega,P,\psi}$

The LTDS in this case is given by

$$0 < \omega < \omega_g.$$  \hfill (5.25)

The comparison with GTDS is shown on Fig. 5.

\begin{figure}[h]
\centering
\includegraphics{figure5.png}
\caption{Thermodynamic stability: a) GTDS for $-1/(36\pi P) \leq \alpha < 0$ occurs at $0 < \omega < \omega_g$ (red curve) and LTDS occurs at $0 < \omega < \omega_g$ (blue curve). b) GTDS for $\alpha_p < \alpha \leq -3/(100\pi P)$ occurs at $0 < \omega \leq \omega_+$ or $\omega_- \leq \omega < \omega_g$ (red curves), and the LTDS is defined by $0 < \omega < \omega_g$ (blue curve).}
\end{figure}

5.2.4 Specific heat $C_{J,P,\psi}$

In this case $C_{J,P,\psi}$ is always positive and the black hole is in LTDS for all $0 < \omega < \omega_c$. This situation corresponds to Fig. 4.

5.2.5 Specific heat $C_{\Omega,V,\alpha}$

In the positive $\alpha$ case this specific heat was always negative. It turns out that this is not the case for $\alpha < 0$. For $\alpha \leq -3/(100\pi P)$ both roots of the denominator are positive and so $C_{\Omega,V,\alpha} > 0$, thus the LTDS is given by

$$\omega_+ < \omega < \omega_-, \quad \alpha \leq -\frac{3}{100\pi P},$$  \hfill (5.26)

where $\omega_{\pm}$ are defined in (4.13). This case is illustrated in Fig. 6.

\begin{figure}[h]
\centering
\includegraphics{figure6.png}
\caption{Thermodynamic stability: a) GTDS for $-1/(36\pi P) \leq \alpha < 0$ occurs at $0 < \omega \leq \omega_g$ (red curve) and LTDS occurs at $\omega_+ < \omega < \omega_-$ (blue curve). b) GTDS for $\alpha_p < \alpha \leq -3/(100\pi P)$ occurs at $0 < \omega \leq \omega_+$ or $\omega_- \leq \omega \leq \omega_g$ (red curves), and the LTDS occurs at $\omega_+ < \omega < \omega_-$ (blue curve).}
\end{figure}

5.2.6 Specific heat $C_{\Omega,V,\psi}$

For this specific heat one can show that the LTDS condition is trivial, $\omega < \omega_c$. Furthermore the GTDS is again given by Fig. 4.
5.2.7 Specific heat $C_{J,V,\alpha}$

The denominator has only one positive root,

$$\tilde{\omega} = \frac{Y(5Y+12)+8-\sqrt{(3Y+4)(83Y^3+260Y^2+272Y+96)}}{4\alpha(7Y+10)}$$  

The LTDS condition in this case is satisfied by $\tilde{\omega} < \omega < \omega_c$ and $\alpha < -\frac{5}{288\pi P}$.

The first GTDS case $(-1/36\pi P < \alpha < 0)$ is further bounded by

$$\alpha < -\frac{1-x_2}{32\pi P} \approx -\frac{0.773}{36\pi P},$$  

where $x_2 \approx 0.313$, which comes from the intersection of the curves $\tilde{\omega} = \omega_g$, resulting in the polynomial equation

$$4 + 9x - 2x^2 - 24x^3 - 2x^4 + 7x^5 = 0.$$  

Therefore, below the intersection point $x_2$ one has $\tilde{\omega} < \omega \leq \omega_g$ and LTDS and GTDS have a common region as shown on Fig. 7a.

For the second GTDS case (4.12) more complex situation is realized (Fig. 7b),

$$\tilde{\omega} < \omega \leq \omega_+, \quad \omega_- \leq \omega \leq \omega_g$$

Figure 7: Thermodynamic stability: a) GTDS for $-1/(36\pi P) \leq \alpha < -\frac{0.773}{36\pi P}$ occurs at $\tilde{\omega} < \omega \leq \omega_g$ (red curve) and LTDS occurs at $\tilde{\omega} < \omega < \omega_c$ (blue curve). b) GTDS for $\alpha_p < \alpha \leq -3/(100\pi P)$ occurs at $\tilde{\omega} < \omega \leq \omega_+$ or $\omega_- \leq \omega \leq \omega_g$ (red curves), and the LTDS occurs at $\tilde{\omega} < \omega < \omega_c$ (blue curve).

5.2.8 Specific heat $C_{J,V,\Psi}$

The specific heat $C_{J,V,\Psi}$ is always positive, thus the RGB-BTZ black hole is locally stable for $0 < \omega < \omega_c$. The comparison with GTDS is depicted in Fig. 4.

As a short summary: we found that in this sector all heat capacities posses regions of local thermodynamic stability. Contrary to the situation in the previous sector, now one could find a region, where the black hole is thermodynamically stable in all of its parameters.

6 Conclusion

In the present paper we have analysed the conditions for local and global thermodynamic equilibrium of the 3-dimensional rotating Gauss-Bonnet BTZ black holes. We have presented a full analysis of the global thermodynamic stability utilising the most natural thermodynamic potential for the given ensemble of macro parameters. Since all of the state quantities in this ensemble share a common divergence and the entropy has to be positive, it turns out that physical states occur for values of the angular velocity $\Omega^2 < (\sqrt{32\pi \alpha P} + T - 1)/(2\alpha)$. Our study included both sectors of the Gauss–Bonnet parameter $\alpha$. In both of them one can impose global thermodynamic stability.
Due to the fact that global thermodynamic stability implies local one, we have performed an exhaustive analysis of the local thermodynamic picture. This was done via the Nambu bracket formalism, developed in [19]. All of the 8 possible specific heats have been analysed for both sectors for $\alpha$. Interestingly, in the $\alpha > 0$ case, not all heat capacities posse a region of local thermodynamic stability. Namely, the specific heats $C_{\Omega, V, \alpha}$ and $C_{J, V, \Psi}$ are always negative. The missing underlying local stability also implies that for fixed $(\Omega, V, \alpha)$ and $(J, V, \Psi)$ a global one can not be established. For $\alpha_p < \alpha < 0$, this is not the case as all specific heats have a region of positivity and thus the black hole can be locally thermodynamically stable.

It is natural to assume that true thermodynamic equilibrium can only be properly established in regions where local and global thermodynamic stability occur at the same time. For this reason, we have looked for intersections between LTDS and GTDS in both sectors for the Gauss-Bonnet parameter. We have discovered that proper equilibrium exists for all specific heats in the case $\alpha_p < \alpha < 0$. In the $\alpha > 0$ this is true only with respect to some of the specific heats. When LTDS and GTDS intersect non-trivial conditions on $\alpha$ as a function of $P$ emerge, which highly restricts the physics in this regions.

To our surprise, the global thermodynamic analysis and its relations to the local one till now has not been presented in full for black holes. For this reason we felt compelled to state it clearly for the first time. Although we presented it on a three dimensional system, it holds valid in any dimensions, when there is a well-defined first law of thermodynamics.

This paper is intended to be the first of series of papers, where different aspects of the RGB-BTZ black hole will be investigated. One direction is to consider the thermodynamic geometry, where one can study the proper thermodynamic metrics on the space of the equilibrium states of the black hole. Investigating the holographic complexity of the RGB-BTZ black hole is another interesting problem. Studying the role of non-extensive thermodynamics over the extensive one presents yet another challenge. Finally, one can extend this work by including non-perturbative correction to the entropy, where the new coupling parameters in the correction terms can be constrained in a highly non-trivial way. It must be noted that the type of analysis presented in the current work can also be applied to a broad class of multi-parameter thermal systems besides black holes.

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**A Energy and entropy derived thermodynamic potentials**

**A.1 Energy derived thermodynamic potentials**

Using the Legendre transformation of the energy $E = E(S, J, V, \alpha)$ one can derive the following thermodynamic potentials for the RGB-BTZ black hole:

\[ \mathcal{L}_S E = E - TS, \]
\[ \mathcal{L}_J E = E - \Omega J, \]
\[ \mathcal{L}_V E = E + PV, \]
\[ \mathcal{L}_a E = E - \Psi \alpha, \]  
\[ \mathcal{L}_{S,J} E = E - TS - \Omega J, \]  
\[ \mathcal{L}_{S,V} E = E - TS + PV, \]  
\[ \mathcal{L}_{S,\alpha} E = E - TS - \Psi \alpha, \]  
\[ \mathcal{L}_{J,V} E = E - \Omega J + PV, \]  
\[ \mathcal{L}_{J,\alpha} E = E - \Omega J - \Psi \alpha, \]  
\[ \mathcal{L}_{V,\alpha} E = E + PV - \Psi \alpha, \]  
\[ \mathcal{L}_{S,J,V} E = E - TS - \Omega J + PV, \]  
\[ \mathcal{L}_{S,J,\alpha} E = E - TS - \Omega J - \Psi \alpha, \]  
\[ \mathcal{L}_{S,V,\alpha} E = E - TS + PV - \Psi \alpha, \]

This list of thermodynamic potentials include all the standard ones (Gibbs free energy, Helmholtz free energy, enthalpy etc.).

**A.2 Entropy derived thermodynamic potentials**

Using the Legendre transformation of the entropy \( S = S(E, J, V, \alpha) \) one can derive the following Massieu–Planck thermodynamic potentials for the RGB-BTZ black hole:

\[ \mathcal{L}_E S = S - \frac{E}{T}, \]  
\[ \mathcal{L}_J S = S + \frac{\Omega J}{T}, \]  
\[ \mathcal{L}_V S = S - \frac{PV}{T}, \]  
\[ \mathcal{L}_\alpha S = S + \frac{\Psi \alpha}{T}, \]  
\[ \mathcal{L}_{E,J} S = S - \frac{E}{T} + \frac{\Omega J}{T}, \]  
\[ \mathcal{L}_{E,V} S = S - \frac{E}{T} - \frac{PV}{T}, \]  
\[ \mathcal{L}_{E,\alpha} S = S - \frac{E}{T} + \frac{\Psi \alpha}{T}, \]  
\[ \mathcal{L}_{J,V} S = S + \frac{\Omega J}{T} - \frac{PV}{T}, \]  
\[ \mathcal{L}_{J,\alpha} S = S + \frac{\Omega J}{T} + \frac{\Psi \alpha}{T}, \]  
\[ \mathcal{L}_{V,\alpha} S = S - \frac{PV}{T} + \frac{\Psi \alpha}{T}, \]  
\[ \mathcal{L}_{E,J,V} S = S - \frac{E}{T} + \frac{\Omega J}{T} - \frac{PV}{T}, \]  
\[ \mathcal{L}_{E,J,\alpha} S = S - \frac{E}{T} + \frac{\Omega J}{T} + \frac{\Psi \alpha}{T}, \]  
\[ \mathcal{L}_{E,V,\alpha} S = S - \frac{E}{T} - \frac{PV}{T} + \frac{\Psi \alpha}{T}, \]  
\[ \mathcal{L}_{J,V,\alpha} S = S + \frac{\Omega J}{T} - \frac{PV}{T} + \frac{\Psi \alpha}{T}, \]
\[
\mathcal{L}_{S,J,V,\alpha} = S - \frac{E}{T} + \frac{\Omega J}{T} - \frac{PV}{T} + \frac{\Psi\alpha}{T}. 
\] (A.30)

This list includes all the standard free entropy potentials (Gibbs free entropy, Helmholtz free entropy, Planck potential, etc.).

## B Nambu brackets and specific heats

The local heat capacities in \((T, \Omega, P, \alpha)\) space of the RGB-BTZ black hole are given by

\[
C_{J,P,\alpha} = T \left( \frac{\partial S}{\partial T} \right)_{J,P,\alpha} = T \left\{ \frac{\{S, J, P, \alpha\}}{\{T, J, P, \alpha\}} \right\}_{T,\Omega,P,\alpha}, 
\] (B.1)

\[
C_{J,V,\alpha} = T \left( \frac{\partial S}{\partial T} \right)_{J,V,\alpha} = T \left\{ \frac{\{S, J, V, \alpha\}}{\{T, J, V, \alpha\}} \right\}_{T,\Omega,P,\alpha}, 
\] (B.2)

\[
C_{J,P,\Psi} = T \left( \frac{\partial S}{\partial T} \right)_{J,P,\Psi} = T \left\{ \frac{\{S, J, P, \Psi\}}{\{T, J, P, \Psi\}} \right\}_{T,\Omega,P,\alpha}, 
\] (B.3)

\[
C_{J,V,\Psi} = T \left( \frac{\partial S}{\partial T} \right)_{J,V,\Psi} = T \left\{ \frac{\{S, J, V, \Psi\}}{\{T, J, V, \Psi\}} \right\}_{T,\Omega,P,\alpha}, 
\] (B.4)

\[
C_{\Omega,P,\alpha} = T \left( \frac{\partial S}{\partial T} \right)_{\Omega,P,\alpha} = T \left\{ \frac{\{S, \Omega, P, \alpha\}}{\{T, \Omega, P, \alpha\}} \right\}_{T,\Omega,P,\alpha}, 
\] (B.5)

\[
C_{\Omega,V,\alpha} = T \left( \frac{\partial S}{\partial T} \right)_{\Omega,V,\alpha} = T \left\{ \frac{\{S, \Omega, V, \alpha\}}{\{T, \Omega, V, \alpha\}} \right\}_{T,\Omega,P,\alpha}, 
\] (B.6)

\[
C_{\Omega,P,\Psi} = T \left( \frac{\partial S}{\partial T} \right)_{\Omega,P,\Psi} = T \left\{ \frac{\{S, \Omega, P, \Psi\}}{\{T, \Omega, P, \Psi\}} \right\}_{T,\Omega,P,\alpha}, 
\] (B.7)

\[
C_{\Omega,V,\Psi} = T \left( \frac{\partial S}{\partial T} \right)_{\Omega,V,\Psi} = T \left\{ \frac{\{S, \Omega, V, \Psi\}}{\{T, \Omega, V, \Psi\}} \right\}_{T,\Omega,P,\alpha}. 
\] (B.8)

For example, the explicit calculation for \(C_{J,P,\alpha}\) in \((T, \Omega, P, \alpha)\) equilibrium space looks like

\[
C_{J,P,\alpha} = T \left( \frac{\partial S}{\partial T} \right)_{J,P,\alpha} = T \left\{ \frac{\{S, J, P, \alpha\}}{\{T, J, P, \alpha\}} \right\}_{T,\Omega,P,\alpha} = T \left| \begin{array}{cccc}
\frac{\partial S}{\partial T} & \frac{\partial S}{\partial J} & \frac{\partial S}{\partial P} & \frac{\partial S}{\partial \alpha} \\
\frac{\partial J}{\partial T} & \frac{\partial J}{\partial J} & \frac{\partial J}{\partial P} & \frac{\partial J}{\partial \alpha} \\
\frac{\partial P}{\partial T} & \frac{\partial P}{\partial J} & \frac{\partial P}{\partial P} & \frac{\partial P}{\partial \alpha} \\
\frac{\partial \alpha}{\partial T} & \frac{\partial \alpha}{\partial J} & \frac{\partial \alpha}{\partial P} & \frac{\partial \alpha}{\partial \alpha} \\
\end{array} \right| = T \left| \begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right|,
\] (B.9)

where we note that all derivatives of our parameters \((T, \Omega, P, \alpha)\) are equal to zero or one.

The expressions for the specific heats from the list above follow from the Nambu bracket formalism introduced by [19].
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