Abstract. Recently a new algorithm for model reduction of second order linear dynamical systems with proportional damping, the Adaptive Iterative Rational Global Arnoldi (AIRGA) algorithm \cite{8}, has been proposed. The main computational cost of the AIRGA algorithm is solving a sequence of linear systems. Usually, direct methods (e.g., LU) are used for solving these systems. As model sizes grow, direct methods become prohibitively expensive. Iterative methods (e.g., Krylov) scale well with size, and hence, are a good choice with an appropriate preconditioner.

Preconditioned iterative methods introduce errors in linear solves because they are not exact. They solve linear systems up to a certain tolerance. We prove that, under mild conditions, the AIRGA algorithm is backward stable with respect to the errors introduced by these inexact linear solves. Our first assumption is use of a Ritz-Galerkin based solver that satisfies few extra orthogonality conditions. Since Conjugate Gradient (CG) is the most popular method based upon the Ritz-Galerkin theory, we use it. We show how to modify CG to achieve these extra orthogonalities.

Modifying CG with the suggested changes is non-trivial. Hence, we demonstrate that using Recycling CG (RCG) helps us achieve these orthogonalities with no code changes. The extra cost of orthogonalizations is often offset by savings because of recycling. Our second and third assumptions involve existence, invertibility and boundedness of two matrices, which are easy to satisfy.

While satisfying the backward stability assumptions, by numerical experiments we show that as we iteratively solve the linear systems arising in the AIRGA algorithm more accurately, we obtain a more accurate reduced system. Experiments are done on an one dimensional beam model and a Gyroscope model, both of which give rise to Symmetric Positive Definite (SPD) linear systems (as needed because of use of CG/RCG). Since Sparse Approximate Inverse (SPAI) and Incomplete Cholesky Factorization (ICHOL) are the most general types of preconditioners, we use both here. By numerical experiments, we also show that recycling Krylov subspaces helps satisfy the backward stability assumptions (extra-orthogonalities) at almost no extra cost.

Key words. Model Order Reduction, Global Arnoldi Algorithm, Moment Matching, Iterative Methods, Preconditioners, Backward Stability Analysis, Recycling Krylov Subspaces, Recycling CG.

AMS subject classifications. 34C20, 41A05, 65F10, 93A15, 93C05, 65L20.

1. Introduction. A continuous time-invariant second order linear dynamical system is of the form

\begin{equation}
M\ddot{x}(t) = -D\dot{x}(t) - Kx(t) + Fu(t),
\end{equation}

\begin{equation}
y(t) = C_p x(t) + C_v \dot{x}(t),
\end{equation}

where $M, D, K \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{n \times m}, C_p, C_v \in \mathbb{R}^{q \times n}$ are constant matrices. In (1), $x(t): \mathbb{R} \rightarrow \mathbb{R}^n$ is the state, $u(t): \mathbb{R} \rightarrow \mathbb{R}^m$ is the input, and $y(t): \mathbb{R} \rightarrow \mathbb{R}^q$ is the output. The mass matrix $M$ and the stiffness matrix $K$ need not hold any specific property (e.g., symmetry, positive definiteness etc.), but only the special case of proportional damping is considered. That is, the damping matrix is chosen as $D = \alpha M + \beta K$ for some choice of real $\alpha$ and $\beta$.

In many cases, the original system dimension $n$ is too large to allow for an efficient simulation of (1). Therefore, the goal of model reduction is to generate a low dimensional system that has, as best as possible, the same characteristics as the original system but whose simulation requires significantly less computational effort. The
reduced system of (1) is described by

\[
\begin{align*}
\dot{M}\ddot{x}(t) & = -\dot{D}\dot{x}(t) - \ddot{K}\dot{x}(t) + \ddot{F}u(t), \\
\dot{y}(t) & = \dot{C}_p\dot{x}(t) + \dot{C}_v\dot{x}(t),
\end{align*}
\]

where \(\dot{M}, \dot{K}, \dot{D} \in \mathbb{R}^{r \times r}, \ddot{F} \in \mathbb{R}^{r \times m}, \dot{C}_p, \dot{C}_v \in \mathbb{R}^{q \times r}\) and \(r \ll n\). We want \(\ddot{y}(t)\) to be nearly equal to \(y(t)\) for all acceptable inputs. In order to capture the relevant features of the original system, the damping matrix \(\ddot{D}\) of the reduced order system is required to be \(\ddot{D} = \alpha\dot{M} + \beta\dot{K}\).

We focus on a moment-matching projection based approach in order to compute the reduced order system. The objective is to generate a reduced order system for which the first moments of the transfer function match those of the original system. The transfer function of (1) is given by

\[H(s) = (C_p + sC_v) \left( s^2M + sD + K \right)^{-1} F =: (C_p + sC_v) X(s).\]

The power series expansion of \(X(s)\) around an expansion point \(s_0 \in \mathbb{R}\) is given by (see, e.g., [22])

\[
X(s) = \sum_{j=0}^{\infty} X^{(j)}(s_0) (s - s_0)^j,
\]

where,

\[
X^{(0)}(s_0) = \left( s_0^2M + s_0D + K \right)^{-1} F,
\]

\[
X^{(1)}(s_0) = \left( s_0^2M + s_0D + K \right)^{-1} \left( -(2s_0M + D) \right) X^{(0)}(s_0), \quad \text{and}
\]

\[
X^{(j)}(s_0) = \left( s_0^2M + s_0D + K \right)^{-1} \left[ -(2s_0M + D) X^{(j-1)}(s_0) - MX^{(j-2)}(s_0) \right],
\]

for \(j = 2, 3, \ldots\). Here, \(X^{(j)}(s_0)\) is called the \(j\)th-order system moment at \(s_0\).

Similarly, the transfer function of the reduced system (2) is given by

\[\dot{X}(s) = \left( \dot{C}_p + s\dot{C}_v \right) \dot{X}(s),\]

where \(\dot{X}(s) = \left( s^2\dot{M} + s\dot{D} + \dot{K} \right)^{-1} \ddot{F}\). The power series expansion of \(\dot{X}(s)\) around an expansion point \(s_0 \in \mathbb{R}\) is given by

\[
\dot{X}(s) = \sum_{j=0}^{\infty} \dot{X}^{(j)}(s_0) (s - s_0)^j.
\]

The \(j\)th-order system moment \(\dot{X}^{(j)}(s_0)\) is defined analogously to \(X^{(j)}(s_0)\).

The goal of moment-matching approach is to find a reduced order system such that the first few moments of (3) and (5) are matched, that is, \(X^{(j)}(s_0) = \dot{X}^{(j)}(s_0)\) for \(j = 0, 1, 2, \ldots, t\) for some \(t\). This can be achieved by the observation below. With

\[
P_1 = -\left( s_0^2M + s_0D + K \right)^{-1} (2s_0M + D),
\]

\[
P_2 = -\left( s_0^2M + s_0D + K \right)^{-1} M,
\]

\[
Q = \left( s_0^2M + s_0D + K \right)^{-1} F.
\]
we have from (4)

\[ X^{(0)}(s_0) = Q, \]
\[ X^{(1)}(s_0) = P_1 X^{(0)}(s_0), \quad \text{and} \]
\[ X^{(j)}(s_0) = P_1 X^{(j-1)}(s_0) + P_2 X^{(j-2)}(s_0) \]

for \( j \geq 2 \). As already observed in [3], these moments are just the blocks of the second order Krylov subspace

\[ \mathcal{G}^j(P_1, P_2, Q) = \text{span}\{Q, \mathcal{S}_1(P_1, P_2)Q, \mathcal{S}_2(P_1, P_2)Q, \ldots, \mathcal{S}_j(P_1, P_2)Q\}, \]

where \( \mathcal{S}_j(P_1, P_2) = P_1 \mathcal{S}_{j-1}(P_1, P_2) + P_2 \mathcal{S}_{j-2}(P_1, P_2) \) for \( j > 2 \), with \( \mathcal{S}_1(P_1, P_2) = P_1 \) and \( \mathcal{S}_2(P_1, P_2) = P_1^2 + P_2 \). For the special case of proportionally damped second order systems, it has been observed in [4] that with \( L = (s_0^2 M + s_0 D + K) \)

\[ \begin{align*}
\mathcal{G}^j(P_1, P_2, Q) &= \mathcal{G}^j(-L^{-1}(2s_0 M + D), -L^{-1} M, -L^{-1} F), \\
&= \mathcal{G}^j(-L^{-1}((2s_0 + \alpha) M + \beta K), -L^{-1} M, -L^{-1} F), \\
&= \mathcal{K}^j(-L^{-1} M, -L^{-1} F) = \mathcal{K}(P_2, Q),
\end{align*} \]

where \( \mathcal{K}(P_2, Q) \) is the standard block Krylov subspace

\[ \mathcal{K}(P_2, Q) = \text{span}\{Q, P_2Q, P_2^2Q, \ldots, P_2^{j-1}Q\}. \]

The reduced order system (2), which matches the first \([r/m] \) moments of the original system (1) can be obtained by projecting (1) with \( \Pi = VV^H \) with an orthonormal matrix \( V \in \mathbb{R}^{n \times r} \) whose columns span \( \mathcal{K}(P_2, Q) \);

\[ V^T \left( M\dot{\bar{x}}(t) + D\bar{x}(t) + KV\dot{\bar{x}}(t) - Fu(t) \right) = 0, \]
\[ \dot{\bar{y}}(t) = C_p V\dot{\bar{x}}(t) + C_v V\dot{\bar{x}}(t). \]

Thus, we have

\[ \begin{align*}
\hat{M} &= V^T M V, \quad \hat{D} = V^T D V, \quad \hat{K} = V^T K V, \quad \hat{F} = V^T F, \quad \hat{C}_p = C_p V \quad \text{and} \quad \hat{C}_v = C_v V.
\end{align*} \]

Standard efficient methods to compute the desired orthogonal basis of \( \mathcal{K}(P_2, Q) \) are, e.g., the block or the global Arnoldi algorithm [16, 12, 18]. In this paper, we consider the Adaptive Iterative Rational Global Arnoldi (AIRGA) algorithm [8], which generates \( V \) by a global Arnoldi method. Its relevant parts are given in Algorithm 1. Unlike as discussed above, the AIRGA algorithm uses not just one expansion point, but a set of \( \ell \) expansion points. This ensures a better reduced system in the entire frequency domain of interest. The method is adaptive, i.e. it automatically chooses the number of moments to be matched at each expansion point \( s_i \). This is controlled by the inner while loop starting at line 8. The variable \( j \) stores the total number of moments matched. The upper bound on \( j \) is \([r_{\text{max}}/m] \), where \( r_{\text{max}} \) is the maximum dimension to which we want to reduce the state variable (input from the user), and \( m \) is the dimension of the input. For a thorough discussion on how to determine convergence, to choose the expansion points in the inner loop as well as a new set of expansion points in the outer loop see [8].
The AIRGA algorithm requires solving sequences of linear systems of equations (line 4 and line 13). As the $s_i$ change in each iteration step, the linear systems to be solved change in each iteration step. These linear systems are usually solved by direct methods. But for large systems, this might be infeasible. Thus, in this work, we focus on the use of preconditioned iterative methods which solve these systems only inexactly. Certainly, the choice of an appropriate iterative algorithm is problem dependent; that is, it depends on whether $s_i^2 M + s_i D + K$ is, e.g., non-symmetric, symmetric or symmetric positive definite. Our goal is to answer the question whether the AIRGA algorithm is stable with respect to these inexact linear solves or not. Can one even give a recommendation on which iterative method to use?

We discuss stability of the AIRGA algorithm with respect to these inexact linear solves in Section 2. To achieve a stable AIRGA algorithm, we need to implement few extra orthogonality conditions in the underlying iterative solver. These details are discussed in Section 3. Here, we also support the fact that such an implementation does not incur any substantial extra cost. Numerical experiments, which support our preconditioned iterative solver theory are given in Section 4. Finally, we give conclusions and future directions in Section 5.

For the rest of this paper, $\| \cdot \|_f$ denotes the Frobenius norm and $\| \cdot \|_2$ the Euclidean norm for vectors and the induced spectral norm for matrices.

2. Stability Analysis of the AIRGA Algorithm. In order to clearly state the goal of our analysis, recall the definition of a backward stable algorithm. Suppose that an approximation $\tilde{y} = f(x)$ to $y = f(x)$ is computed by a certain method. Then the method is called backward stable, if, for any $x$, it produces a computed $\tilde{y}$ with a small backward error, that is, $\tilde{y} = f(x + \Delta x)$ for some small $\Delta x$. To be more precise, following [20], an algorithm $\hat{f}$ is said to be backward stable if

$$\hat{f}(x) = f(\hat{x}) \text{ for some } \hat{x} \text{ with } \frac{\| \hat{x} - x \|}{\| x \|} = O(\epsilon_{\text{machine}}),$$

where $\epsilon_{\text{machine}}$ is the machine precision and we are looking at finite precision arithmetic errors.

Here, we study the stability of the AIRGA algorithm with respect to use of iterative methods. So far we have considered the AIRGA algorithm as an algorithm which maps the original system represented by $M, D, K, F, C_p, C_v$ to the reduced order one represented by $\hat{M}, \hat{D}, \hat{K}, \hat{F}, \hat{C}_p, \hat{C}_v$. In this section, we will consider the AIRGA algorithm as an algorithm which maps the transfer function $H(s)$ of the original system to the transfer function $\hat{H}(s)$ of the reduced order system. That is, for the stability analysis the function $\hat{f}$ maps $H(s)$ to $\hat{H}(s)$, $\hat{f}(H(s)) = \hat{H}(s)$. This is represented by the AIRGA algorithm (Algorithm 1) when a direct solver for solving the linear systems at lines 4 and 13 is employed. This is called the exact AIRGA algorithm. The function $\hat{f}$ maps the transfer function $H(s)$ of the original system to the transfer function of the reduced order system employing an iterative solver in order to solve the linear systems at lines 4 and 13 of the AIRGA algorithm instead of a direct solver. This is denoted by $\hat{f}(\hat{H}(s)) = \hat{H}(s)$ and is called the inexact AIRGA algorithm.
Then, the AIRGA algorithm is stable with respect to the iterative solves if

\[
\tilde{f}(H(s)) = f\left(\tilde{H}(s)\right) \quad \text{for some } \tilde{H}(s) \text{ with }
\]

\[
\frac{\|H(s) - \tilde{H}(s)\|_\diamond}{\|H(s)\|_\diamond} = O\left(\|Z\|_\diamond\right),
\]

for a suitable norm \( \| \cdot \|_\diamond \). Here, as earlier, \( H(s) \) denotes the transfer function of the unperturbed (original) full system, \( \tilde{H}(s) \) is the transfer function of a perturbed full system, and \( Z \) is the corresponding perturbation. Also, \( f\left(\tilde{H}(s)\right) \) implies the exact AIRGA algorithm applied to the perturbed full system.

Thus, \( f(H(s)) \) is computed by Algorithm 1 employing a direct solver in lines 4 and 13. For our discussion, we are only interested in one outer iteration step. The

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**Algorithm 1** Adaptive Iterative Rational Global Arnoldi Algorithm [8]

1: Input: \( \{M, D, K, F, C_p, C_v, r_{\text{max}}\}; \) initial set of expansion points \( S = \{s_1, \ldots, s_r\} \)
2: while no convergence do
3:     for each \( s_i \in S \) do
4:         \( X^{(0)}(s_i) = (s_i^2M + s_iD + K)^{-1}F \)
5:         Compute \( QR = qr\left(X^{(0)}(s_i)\right), \ X^{(0)}(s_i) = Q \)
6:     end for
7:     \( j = 1 \)
8:     while no convergence and \( j < \lceil r_{\text{max}}/m \rceil \) do
9:         Choose an expansion point \( \sigma_j \in S \)
10:        \( V_j = X^{(j-1)}(\sigma_j)/\|X^{(j-1)}(\sigma_j)\|_f \)
11:        for \( i = 1, \ldots, \ell \) do
12:            if \( s_i = \sigma_j \) then
13:                \( X^{(j)}(s_i) = -(s_i^2M + s_iD + K)^{-1}MV_j \)
14:            else \( X^{(j)}(s_i) = X^{(j-1)}(s_i) \)
15:        end if
16:        for \( t = 1, 2, \ldots, j \) do
17:            \( \gamma_{t,j}(s_i) = \text{trace}(V_t^{(j)}X^{(j)}(s_i)) \)
18:            \( X^{(j)}(s_i) = X^{(j)}(s_i) - \gamma_{t,j}(s_i)V_t \)
19:        end for
20:     end while
21:     \( j = j+1 \)
22: end while
23: Set \( J = j \) and pick \( \sigma_J \in S \)
24: \( V_J = X^{(J-1)}(\sigma_J)/\|X^{(J-1)}(\sigma_J)\|_f \) and \( \tilde{V} = [V_1, V_2, \ldots, V_J] \)
25: Compute \( VY = qr(\tilde{V}) \)
26: Compute the reduced order system matrices \( \hat{M}, \hat{D} \) and \( \hat{K} \) with \( V \) as in (7)
27: Choose new set of expansion points \( S = \{s_1, \ldots, s_{\ell}\} \) using eigenvalues of the reduced system
28: end while
29: Compute the reduced order system matrices \( \hat{F}, \hat{C}_p, \) and \( \hat{C}_v \) with \( V \) as in (7)

**Note:** Here \( qr(T) \) denotes the usual \( QR \) factorization \( T = QR \) with an orthogonal matrix \( Q \) and an upper triangular matrix \( R \).
matrix $\tilde{V} = [V_1, V_2, \ldots, V_J]$ is generated and the reduced order system is computed with $V$ as in (7). This immediately gives $f(H(s))$.

Now assume that in lines 4 and 13 of Algorithm 1 an iterative solver is used instead of a direct solver, that is, the linear systems are solved inexactly giving $\hat{f}(H(s))$. Further, we need to assume that the choice of the expansion points is the same no matter whether iterative solves or a direct solve is used.

2.1. Conditions for Stability. Consider the linear systems for $X^{(0)}(s_i) \in \mathbb{R}^{n \times m}$ at line 4

$$(s_i^2 M + s_i D + K) X^{(0)}(s_i) = F,$$

where $s_i \in S = \{s_1, s_2, \ldots, s_\ell\}$. We denote the inexactly computed solution for $X^{(0)}(s_i)$ by $\hat{X}^{(0)}(s_i)$. Let the associated residual be $\eta_{0i} \in \mathbb{R}^{n \times m}$ for $i = 1, \ldots, \ell$. Then, the above equation is equivalent to

$$(s_i^2 M + s_i D + K) \hat{X}^{(0)}(s_i) = F + \eta_{0i}. \tag{10}$$

Next, in Algorithm 1 at line 10, at the first iteration of the while loop (i.e. $j=1$), $\hat{V}_1$ is computed as

$$\hat{V}_1 = \hat{X}^{(0)}(s_{t_1}) / \| \hat{X}^{(0)}(s_{t_1}) \|_f,$$ \tag{11}

where $s_{t_1} = \sigma_1$ is the expansion point chosen for this step. For notational convenience, we also refer this expansion point as $s_{t_0}$. That is, $s_{t_0} = s_{t_1}$.

Then, at line 13 in Algorithm 1 the inexact solve gives

$$(s_{t_1}^2 M + s_{t_1} D + K) \hat{X}^{(1)}(s_{t_1}) = M \hat{V}_1 + \eta_{1t_1}. \tag{12}$$

Next, in Algorithm 1 at line 10 after one iteration of the while loop (i.e. $j=2$), $\hat{V}_2$ is computed as

$$\hat{V}_2 = \hat{X}^{(1)}(s_{t_2}) / \| \hat{X}^{(1)}(s_{t_2}) \|_f,$$ \tag{13}

where $s_{t_2} = \sigma_2$ is the expansion point chosen for this step.

Further, at line 13 in Algorithm 1 the inexact solve yields for $j = 2, \ldots, J - 1$

$$(s_{t_j}^2 M + s_{t_j} D + K) \hat{X}^{(j)}(s_{t_j}) = M \hat{V}_j + \eta_{jt_j}. \tag{14}$$

Thus, in Algorithm 1 at line 10 for $j = 3, \ldots, J - 1$ and at line 24 for $j = J$, $\hat{V}_j$ is computed as

$$\hat{V}_j = \hat{X}^{(j-1)}(s_{t_j}) / \| \hat{X}^{(j-1)}(s_{t_j}) \|_f,$$ \tag{15}

where $s_{t_j} = \sigma_j$ is the expansion point chosen for the $j^{th}$ step.

Finally, $\tilde{V} = [\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_J]$ is set up and used to generate the reduced system (obtained by the inexact AIRGA algorithm),

$$(\tilde{M} = \tilde{V}^T M \tilde{V}, \tilde{D} = \tilde{V}^T D \tilde{V}, \tilde{K} = \tilde{V}^T K \tilde{V}),$$

$$\tilde{F} = \tilde{V}^T F, \tilde{C}_p = C_p \tilde{V}, \text{ and } \tilde{C}_v = C_v \tilde{V}.$$
This reduced order system is equivalent to \( \tilde{f} (H(s)) \).

Now we have to find a perturbed original system \( \tilde{H}(s) \), such that the exact solution \( f(\tilde{H}(s)) \) will give the reduced system as obtained by the inexact solution of the original full system, \( \tilde{f}(H(s)) \). That is, find \( \tilde{H}(s) \) such that \( \tilde{f}(H(s)) = f(\tilde{H}(s)) \).

This will satisfy the first stability condition (8).

Among the many systems \( \tilde{H}(s) \) one can consider here, we concentrate on those that have a constant perturbation \( Z \in \mathbb{R}^{n \times n} \) in \( K \) only. That is,

\[
\dot{K} = K + Z, \quad \dot{M} = M, \quad \dot{D} = D, \quad \dot{F} = F, \quad \dot{C}_p = C_p, \quad \text{and} \quad \dot{C}_v = C_v.
\]

Recall in (10) we solve \( \ell \) linear systems corresponding to the \( \ell \) expansion points. However, only solution of one of these systems (identified by \( s_{t_0} = s_{t_1} \) expansion point) is chosen to build \( \tilde{V}_1 \) as shown in (11). Then, for \( \tilde{H} \) we have that instead of (10), \( \tilde{X}^{(0)}(s_{t_0}) \) is the exact solution of

\[
(s_{t_0}^2 M + s_{t_0} D + (K + Z)) \tilde{X}^{(0)}(s_{t_0}) = F.
\]

Similarly, it follows that the linear systems (12) and (14) are solved exactly as

\[
(s_{t_j}^2 M + s_{t_j} D + (K + Z)) \tilde{X}^{(j)}(s_{t_j}) = M \tilde{V}_j \quad \text{for} \quad j = 1, \ldots, J - 1.
\]

The final matrix \( \tilde{V} = [\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_J] \) is exactly the same as before as \( \tilde{V}_{j+1} \) for \( j = 0, \ldots, J - 1 \) are given by (11), (13) and (15). This is because, the matrices \( \tilde{X}^{(0)}(s_{t_0}) \) and \( \tilde{X}^{(j)}(s_{t_j}) \) for \( j = 1, \ldots, J - 1 \) are the same in (10) and (17) as well as in (12), (14) and (18). Thus, the reduced order system (obtained by the exact AIRGA algorithm applied to the perturbed system \( \tilde{H} \)) is given by

\[
\begin{align*}
\dot{M} &= \tilde{V}^T \dot{M} \tilde{V} = \tilde{V}^T M \tilde{V} = \dot{\tilde{M}}, \\
\dot{D} &= \tilde{V}^T \dot{D} \tilde{V} = \tilde{V}^T D \tilde{V} = \dot{\tilde{D}}, \\
\dot{K} &= \tilde{V}^T \dot{K} \tilde{V} = \tilde{V}^T (K + Z) \tilde{V} = \dot{\tilde{K}} + \tilde{V}^T Z \tilde{V}, \\
\dot{F} &= \tilde{V}^T F = \tilde{V}^T F = \dot{\tilde{F}}, \\
\dot{C}_p &= \tilde{C}_p \tilde{V} = C_p \tilde{V} = \dot{\tilde{C}}_p, \quad \text{and} \\
\dot{C}_v &= \tilde{C}_v \tilde{V} = C_v \tilde{V} = \dot{\tilde{C}}_v.
\end{align*}
\]

This reduced order system is equivalent to \( f(\tilde{H}(s)) \). Obviously, this is already almost the same as \( \tilde{f}(H(s)) \).

Recall that our goal is to find \( \tilde{H}(s) \) such that \( \tilde{f}(H(s)) = f(\tilde{H}(s)) \). That is, we need to find \( Z \) such that \( \dot{\tilde{K}} = \tilde{K} \) or \( \tilde{V}^T Z \tilde{V} = 0 \).

Comparing (10) with (17), (12) and (14) with (18) gives

\[
Z \tilde{X}^{(j)}(s_{t_j}) = \eta_{j t_j} \quad \text{for} \quad j = 0, \ldots, J - 1.
\]

We can rewrite \( Z \) as

\[
Z \mathbf{X} = \eta,
\]
where \( Z \in \mathbb{R}^{n \times n}, \ X = \begin{bmatrix} \hat{X}^{(0)}(s_{t_0}) & \hat{X}^{(1)}(s_{t_1}) & \cdots & \hat{X}^{(J-1)}(s_{t_{(J-1)}}) \end{bmatrix} \in \mathbb{R}^{n \times m_J}, \) and 
\[
\eta = \begin{bmatrix} \eta_{t_0}, \ldots, \eta_{t_{(J-1)t_{(J-1)}}} \end{bmatrix} \in \mathbb{R}^{n \times m_J}.
\]
As discussed in Section 1, the upper bound for \( J \) is \([r_{\text{max}}/m]\), and hence, \( mJ \leq r_{\text{max}} \). Using the fact that \( r_{\text{max}} \ll n \), we have \( mJ < n \). Thus, we have an under-determined system of equations. One solution of this is
\[
Z = \eta X^T (XX^T)^{-1},
\]
assuming the inverse of \((XX^T)\) exists. Multiplying both sides of (20) with \( \tilde{V} \), we get
\[
\tilde{V}^T Z \tilde{V} = \tilde{V}^T \eta X^T (XX^T)^{-1} \tilde{V}.
\]
Assume that we are using a Ritz-Galerkin based iterative solver. Here, the solution space of the linear systems is orthogonal to the corresponding residuals, i.e. \( \tilde{V}_1 \perp \eta_{t_0} \), \( \tilde{V}_2 \perp \eta_{t_1} \), \( \ldots \), and \( \tilde{V}_J \perp \eta_{(J-1)t_{(J-1)}} \) [21]. Hence,
\[
\tilde{V}^T \eta = \begin{bmatrix} \tilde{V}_1^T \eta_{t_0} & \tilde{V}_2^T \eta_{t_1} & \cdots & \tilde{V}_J^T \eta_{t_{(J-1)}} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{V}_1^T \eta_{t_1} & \cdots & \tilde{V}_J^T \eta_{(J-2)t_{(J-2)}} & \tilde{V}_1^T \eta_{(J-1)t_{(J-1)}} \\
\tilde{V}_2^T \eta_{t_0} & 0 & \cdots & \tilde{V}_J^T \eta_{(J-2)t_{(J-2)}} & \tilde{V}_2^T \eta_{(J-1)t_{(J-1)}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tilde{V}_{J-1}^T \eta_{t_0} & \tilde{V}_{J-1}^T \eta_{t_1} & \cdots & 0 & \tilde{V}_{J-1}^T \eta_{(J-1)t_{(J-1)}} \\
\tilde{V}_J^T \eta_{t_0} & \tilde{V}_J^T \eta_{t_1} & \cdots & \tilde{V}_J^T \eta_{(J-2)t_{(J-2)}} & 0 \end{bmatrix}.
\]

Our goal here is to make the right hand side of the above equation equal to zero. The upper triangular part of the above matrix is zero if we have the following orthogonalities:
\[
[\tilde{V}_1] \perp \eta_{t_1},
[\tilde{V}_1 \ \tilde{V}_2] \perp \eta_{t_2},
\vdots
[\tilde{V}_1 \ \tilde{V}_2 \ \tilde{V}_3 \ \ldots \ \tilde{V}_{J-2}] \perp \eta_{(J-2)t_{(J-2)}},
[\tilde{V}_1 \ \tilde{V}_2 \ \tilde{V}_3 \ \ldots \ \tilde{V}_{J-2} \ \tilde{V}_{J-1}] \perp \eta_{(J-1)t_{(J-1)}},
\]
Similarly, for the lower triangular part of the above matrix to be zero we need the following orthogonalities:
\[
\tilde{V}_2 \perp [\eta_{t_0}],
\tilde{V}_3 \perp [\eta_{t_0} \ \eta_{t_1}],
\vdots
\tilde{V}_{J-1} \perp [\eta_{t_0} \ \eta_{t_1} \ \cdots \ \eta_{(J-3)t_{(J-3)}}],
\tilde{V}_J \perp [\eta_{t_0} \ \eta_{t_1} \ \cdots \ \eta_{(J-3)t_{(J-3)}} \ \eta_{(J-2)t_{(J-2)}}].
\]
These orthogonalities can be easily achieved by changing the underlying iterative solver. We discuss this in the next section (Section 3.1).

At the first glance, there seem to be two problems in achieving the above discussed orthogonalities in an iterative solver. One is the amount of code changes to be done. The other is the extra cost associated at every iterative step of the solver, which may undermine the benefit of using an iterative solver itself.

In Section 3.2, we show that both these issues can be easily resolved by using a recycling variant of the underlying iterative solver. While solving a sequence of linear systems, if the consecutive systems do not change much, then some information can be reused from solving one linear system to solving the next. In the context of Krylov based iterative linear solvers, this information is in the form of the generated Krylov subspace. The process of reusing Krylov subspaces from one linear system to the next is termed as “recycling” [14, 23, 2, 1].

A subset of \( \hat{V} \)'s and \( \eta \)'s of (23) and (24) can be used to span a recycle space, leading to almost no code changes in the recycling variant of the underlying iterative solver. In some cases, this choice of the recycle space can actually accelerate the convergence of the next linear system in the sequence. In case when this recycle space deteriorates the convergence of the next linear system, this behaviour is bounded.

In the numerical experiments section (Section 4), we support both these conjectures (acceleration and deterioration of the convergence of iterative linear solvers) with multiple examples.

Therefore, after applying (22), (23) and (24) to (21), we get \( \hat{V}^T Z \hat{V} = 0 \). Thus, \( \hat{K} = \hat{K} \) or

\[
\hat{f} (H(s)) = f (\hat{H}(s)) = \hat{H}(s),
\]

where \( H(s) = (C_p + sC_v) (s^2 M + sD + K)^{-1} F \), \( \hat{H}(s) = (C_p + sC_v) (s^2 \hat{M} + s\hat{D} + \hat{K})^{-1} \hat{F} = (\hat{C}_p + s\hat{C}_v) (s^2 \hat{M} + s\hat{D} + \hat{K})^{-1} \hat{F} \). Thus, we satisfy the first condition of stability.

According to the second condition of stability, given in (9), the difference between the unperturbed (original) full system and the perturbed full system should be of the order of the perturbation [20]. These errors are measured in the commonly used norms as below.

\[
\begin{align*}
H_2 - \text{norm:} & \quad \|H - G\|_{H_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|H(\omega) - G(\omega)\|_f d\omega, \\
H_\infty - \text{norm:} & \quad \|H - G\|_{H_\infty} = \max_{\omega \in \mathbb{R}} \|H(\omega) - G(\omega)\|_2,
\end{align*}
\]

where the transfer functions \( H \) and \( G \) belong to systems with the same input and output dimension. Theorem 4.3 from [5] helps in giving the desired result.

**Theorem 2.1.** If \( \|Z\|_2 < \frac{1}{\|L(s)^{-1}\|_{H_\infty}} \) then

\[
H(s) - \hat{H}(s)\|_{H_2} \leq \frac{\|C(s)L(s)^{-1}\|_{H_2} \|L(s)^{-1}F\|_{H_\infty} \|Z\|_2}{1 - \|L(s)^{-1}\|_{H_\infty} Z\|_2} \|Z\|_2,
\]

where \( L(s) = (s^2 M + sD + K) \) and \( C(s) = (C_p + sC_v) \).

If \( \|Z\|_2 < 1 \) and \( \|L(s)^{-1}\|_{H_\infty} < 1 \), then we have \( \|L(s)^{-1}\|_{H_\infty} \|Z\|_2 < 1 \), and hence,

\[
\frac{1}{1 - \|L(s)^{-1}\|_{H_\infty} \|Z\|_2} < \frac{1}{1 - \|L(s)^{-1}\|_{H_\infty}}.
\]
Substituting (26) in (25) we get

\[
\frac{\|H(s) - \tilde{H}(s)\|_{H_2}}{\|H(s)\|_{H_2}} \leq \frac{\|C(s)L(s)^{-1}\|_{H_2}\|L(s)^{-1}F\|_{H_\infty}}{\|H(s)\|_{H_2}} \frac{1}{1 - \|L(s)^{-1}\|_{H_\infty}} \cdot \|Z\|_2 = \mathcal{O}(\|Z\|_2).
\]

This proves the stability of the AIRGA algorithm. The next theorem summarizes this.

**Theorem 2.2.** If the linear systems arising in the AIRGA algorithm are solved by a Ritz-Galerkin based solver (i.e. the residual is orthogonal to the generated Krylov subspace) and it also holds the following conditions:

(a) the extra orthogonalities given by (23) and (24) are satisfied by such a solver,
(b) \(L(s)\) as defined in Theorem 2.1 is invertible and \(\|L(s)^{-1}\|_{H_\infty} < 1\),
(c) \(Z\) is given by (20) exists and \(\|Z\|_2 < 1\),

then the AIRGA algorithm is backward stable with respect to the inexact linear solves.

**2.2. Accuracy of the systems.** Using Theorem 15.1 of [20] we know that if the AIRGA algorithm is backward stable, then the relative accuracy of the reduced system obtained by using the inexact AIRGA algorithm, as compared to using the exact AIRGA algorithm, is given as follows:

\[
\frac{\|\tilde{H}(s) - \hat{H}(s)\|_{H_2}}{\|\tilde{H}(s)\|_{H_2}} = \mathcal{O}(\kappa(H(s)) \cdot \|Z\|_2),
\]

where \(\kappa(H(s))\) is the condition number of \(H(s)\) (discussed below), and \(Z\) is the perturbation in \(H(s)\). As earlier, \(\tilde{H}(s)\) is the reduced system obtained by using the exact AIRGA algorithm and \(\hat{H}(s)\) is the reduced system obtained by using the inexact AIRGA algorithm. We are looking at reduced systems obtained at line 26 of Algorithm 1. That is, after each step of the outer while loop (line 2). Thus, accuracy of the reduced system is dependent on the conditioning of the problem as well as the perturbation. Next, we look at both these quantities separately.

First, we want to compute conditioning of our system with respect to performing the inexact linear solves on lines 4 and 13 of Algorithm 1. Since for backward stability we equate the reduced system obtained by performing the inexact AIRGA algorithm on the unperturbed (original) full system \((H(s))\) and performing the exact AIRGA algorithm on the perturbed full system \((\tilde{H}(s))\), these inexact linear solves are captured by \(\tilde{H}(s)\). Thus, the condition number of our system with respect to computing the \(H_2\)-norm of the error system \(H(s) - \tilde{H}(s)\) will give us a good approximation to the condition number that we want to compute (with respect to computing the \(H_2\) - norm of \(H(s) - \hat{H}(s)\)).

Similar behaviour has been observed for first order linear dynamical systems (see Theorem 3.1 and 3.3 in [5]) and first order bilinear dynamical systems [9].

Recall, the condition number by definition means the relative change in the output \(\|H(s) - \tilde{H}(s)\|_{H_2}/\|H(s)\|_{H_2}\) with respect to the relative change in the input (for us this is \(\|Z\|_2/\|K\|_2\) since we are perturbing the \(K\) matrix) [9]. Hence,
from (27) we have

\[ \frac{\|H(s) - \tilde{H}(s)\|_{H_2}}{\|H(s)\|_{H_2}} \leq \frac{\|C(s)L(s)^{-1}\|_{H_2} \|L(s)^{-1}F\|_{H_\infty}}{\|H(s)\|_{H_2}}, \quad \frac{\|K\|_2}{1 - \|L(s)^{-1}\|_{H_\infty}} \cdot \frac{\|Z\|_2}{\|K\|_2}, \]

where it is assumed that \( \|Z\|_2 < 1 \) and \( \|L(s)^{-1}\|_{H_\infty} < 1 \). Hence, the above inequality is equivalent to

\[ \frac{\|H(s) - \tilde{H}(s)\|_{H_2}}{\|H(s)\|_{H_2}} \leq \kappa(H(s)) \cdot \frac{\|Z\|_2}{\|K\|_2}, \]

where,

\[ \kappa(H(s)) = \frac{\|C(s)L(s)^{-1}\|_{H_2} \|L(s)^{-1}F\|_{H_\infty}}{\|H(s)\|_{H_2}}, \quad \frac{\|K\|_2}{1 - \|L(s)^{-1}\|_{H_\infty}}. \]

In the numerical experiments section, for the first problem, we show that this condition number is fairly small, whereas, for the second problem it is large. In other words, the first problem is well conditioned and the second problem is ill-conditioned with respect to the \( H_2 \)-norm of the error system \( H(s) - \tilde{H}(s) \). Note that \( \|Z\|_2 < 1 \) and \( \|L(s)^{-1}\|_{H_\infty} < 1 \), as assumed here, come from the assumptions for backward stability of the AIRGA algorithm (see Theorem 2.2), and hence, we do not need any extra assumptions.

Second, we relate the perturbation \( Z \) and the cumulative residual \( \eta \). From (20) we have

\[ \|Z\|_2 \leq \|Z\|_f \leq \|\eta \cdot X^T (XX^T)^{-1}\|_f \leq \|\eta\|_f \cdot \|X^T (XX^T)^{-1}\|_f, \]

\[ \leq \left( \|\eta_{t_a}\|_f + \cdots + \|\eta_{t_{(J-1)t_{(J-1)}}}\|_f \right) \cdot \left( \|X^T (XX^T)^{-1}\|_f \right). \]

In the above equation, \( \eta_{t_a}, \ldots, \eta_{t_{(J-1)t_{(J-1)}}} \) represent the residuals obtained while solving the linear systems arising in the model reduction process. These residuals will reduce if we solve such linear systems more accurately. The second term \( \|X^T (XX^T)^{-1}\|_f \) is usually more dependent on the selection of the expansion points \( (s_i) \), and less on the accuracy to which we solve the linear systems [5]. We support this argument with numerical experiments as well.

Thus, as we solve the arising linear systems more accurately, from (32) we get that \( \|Z\| \) reduces. Thus, using the above with (28), ideally we should get a more accurate reduced system. We support this with experiments in Section 4.

### 3. Satisfying Backward Stability Conditions.

As already discussed in the earlier section, for a backward stable AIRGA algorithm we need to use a Ritz-Galerkin based method for solving the underlying linear systems. The Conjugate Gradient (CG) method is one of the most popular solver of such a type. The CG method is mainly used for solving Symmetric Positive Definite (SPD) linear systems. For solving non-symmetric linear systems, Full Orthogonalization Method (FOM) [12, 24] is the one that is based upon the Ritz-Galerkin theory.

In this paper, we focus on the CG method and show how to achieve the extra orthogonalities needed for a backward stable AIRGA algorithm (see Theorem 2.2).

\[ ^1 \text{This ill-conditioning of the second problem does not effect our main conjecture. We discuss this aspect in detail later.} \]
Hence, in the results section, we take models that lead to SPD linear systems in the AIRGA algorithm. Similar changes can be done for the FOM method, and is part of our future work.

Next, we first discuss how to change the theory of the CG method such that the extra orthogonalities, (23)-(24), are satisfied (in Section 3.1). Further, we describe how the recommended changes can be easily implemented (in Section 3.2).

3.1. Achieving Extra Orthogonalities. The CG method consists of two components. One is the Lanczos algorithm that gives a good basis of the generated Krylov subspace. The other is the Ritz-Galerkin projection to obtain solution estimates from this subspace. The orthogonalities in (24) can be achieved by modifying the Lanczos algorithm (discussed in Section 3.1.1), and those in (23) can be achieved by changing the Ritz-Galerkin projection (discussed in Section 3.1.2).

3.1.1. Adapting the Lanczos Process. Assume we are trying to solve the linear system of the form

\[(33) \quad Ax = b,\]

where \(A \in \mathbb{C}^{n \times n}\) and \(b \in \mathbb{C}^{n}\). Let \(x_0\) be the initial solution vector with \(r_0 = b - Ax_0\) as the corresponding residual. The Lanczos algorithm computes a good basis of the generated Krylov subspace involving \(A\) and \(r_0\) as [16]

\[(34) \quad w_{k+1} \in \mathbb{K}^k(A, r_0) \equiv \text{span}\{r_0, Ar_0, A^2r_0, \ldots, A^{k-1}r_0\} \quad \text{s.t.} \quad w_{k+1} \perp \begin{bmatrix} w_1 & w_2 & \cdots & w_k \end{bmatrix},\]

where \(w_{k+1}\) is the Lanczos vector at the \((k+1)\)th iterative step and \(w_1 = r_0/\|r_0\|^2\). Now, assume we are carrying some residual vector \(\tilde{r}\) from another linear system, which we need to make orthogonal to the final solution of (33). Then, the Lanczos algorithm above would consist of the following procedure:

\[w_{k+1} \in \mathbb{K}^k(A, r_0) \quad \text{s.t.} \quad w_{k+1} \perp \begin{bmatrix} w_1 & w_2 & \cdots & w_k & \tilde{r} \end{bmatrix}.\]

Recall from the previous section that in the AIRGA algorithm the first linear system to be solved iteratively is given by (10) (recall from (11) that \(s_{t_0}\) is the only expansion point chosen). That is,

\[(35) \quad (s_{t_0}^2 M + s_{t_0} D + K) \tilde{X}^{(0)}(s_{t_0}) = F + \eta_0 u_0.\]

Next, we need to iteratively solve (12), i.e.

\[(36) \quad (s_{t_1}^2 M + s_{t_1} D + K) \tilde{X}^{(1)}(s_{t_1}) = M \tilde{V}_1 + \eta_{t1}.\]

Here, we need a good basis of the Krylov subspace involving \(K_{t_1} = (s_{t_1}^2 M + s_{t_1} D + K)\) and \((r_{t_1})_0\), which is the initial residual of (36). Hence, the Lanczos algorithm here would consist of the following procedure:

\[(37) \quad (w_{t_1})_{k+1} \in \mathbb{K}^k(K_{t_1}, (r_{t_1})_0) \quad \text{s.t.} \quad (w_{t_1})_{k+1} \perp \begin{bmatrix} (w_{t_1})_1 & (w_{t_1})_2 & \cdots & (w_{t_1})_k \end{bmatrix},\]

\[w_{k+1} = Aw_k - c_1w_1 - c_2w_2 - \ldots - c_{k-1}w_{k-1} - c_kw_k.\]

Finally, the second equation of (34) gives us \(c_1, c_2, \ldots, c_k\). For a complete derivation of this, please see chapter 5 of [21].
where \((w_{t_1})_{k+1}\) is the Lanczos vector at the \((k + 1)^{th}\) iterative step and \((w_{t_1})_1 = (r_{t_1})_0 / \| (r_{t_1})_0 \|\). Now, adapting the above Lanczos algorithm to achieve the first orthogonality of (24), i.e. \(\tilde{V}_2 \perp [\eta_{t_0}]\) we get

\[
(w_{t_1})_{k+1} \in \mathbb{R}^k (K_{t_1}, (r_{t_1})_0)
\]

\[
\text{s.t. } (w_{t_1})_{k+1} \perp \left[ (w_{t_1})_1, (w_{t_1})_2, \ldots, (w_{t_1})_k, \eta_{t_0} \right],
\]

where \(\eta_{t_0}\) is the final residual obtained after solving (35) iteratively.

Next, we need to iteratively solve (14) for \(j = 2\), i.e.

\[
(s^2_{t_2} M + s_{t_2} D + K) \tilde{X}^{(2)} (s_{t_2}) = M\tilde{V}_2 + \eta_{2t_2}.
\]

Here, we need a good basis of the Krylov subspace involving \(K_{t_2} = (s^2_{t_2} M + s_{t_2} D + K)\) and \((r_{t_2})_0\), which is the initial residual of (39). Hence, the Lanczos algorithm here would consist of the following procedure:

\[
(w_{t_2})_{k+1} \in \mathbb{R}^k (K_{t_2}, (r_{t_2})_0)
\]

\[
\text{s.t. } (w_{t_2})_{k+1} \perp \left[ (w_{t_2})_1, (w_{t_2})_2, \ldots, (w_{t_2})_k \right],
\]

where \((w_{t_2})_{k+1}\) is the Lanczos vector at the \((k + 1)^{th}\) iterative step and \((w_{t_2})_1 = (r_{t_2})_0 / \| (r_{t_2})_0 \|\). To achieve the second set of orthogonalities in (24), i.e. \(\tilde{V}_3 \perp [\eta_{t_0} \eta_{t_1}]\) we need

\[
(w_{t_2})_{k+1} \in \mathbb{R}^k (K_{t_2}, (r_{t_2})_0)
\]

\[
\text{s.t. } (w_{t_2})_{k+1} \perp \left[ (w_{t_2})_1, (w_{t_2})_2, \ldots, (w_{t_2})_k, \eta_{t_0}, \eta_{t_1} \right],
\]

where \(\eta_{t_1}\) is the final residual obtained after solving (36) iteratively.

We need to repeat a similar procedure for (14) for all \(j = 3, \ldots, J - 1\). For the sake of clarity, we describe the changes for the last \(j\) i.e. \(j = J - 1\). Hence, we need to iteratively solve

\[
(s^2_{t_{j-1}} M + s_{t_{j-1}} D + K) \tilde{X}^{(j-1)} (s_{t_{j-1}}) = M\tilde{V}_{j-1} + \eta_{(j-1)t_{j-1}}.
\]

We need a good basis of the Krylov subspace involving \(K_{t_{j-1}} = (s^2_{t_{j-1}} M + s_{t_{j-1}} D + K)\) and \((r_{t_{j-1}})_0\), which is the initial residual of (42). Hence, the Lanczos algorithm here would consist of the following procedure:

\[
(w_{t_{j-1}})_{k+1} \in \mathbb{R}^k (K_{t_{j-1}}, (r_{t_{j-1}})_0)
\]

\[
\text{s.t. } (w_{t_{j-1}})_{k+1} \perp \left[ (w_{t_{j-1}})_1, (w_{t_{j-1}})_2, \ldots, (w_{t_{j-1}})_k \right],
\]

where \((w_{t_{j-1}})_{k+1}\) is the Lanczos vector at the \((k + 1)^{th}\) iterative step and \((w_{t_{j-1}})_1 = (r_{t_{j-1}})_0 / \| (r_{t_{j-1}})_0 \|\). Finally, to achieve the last set of orthogonalities in (24), i.e. \(\tilde{V}_J \perp [\eta_{t_0} \eta_{t_1} \ldots \eta_{(j-3)t_{j-3}} \eta_{(j-2)t_{j-2}}]\) we need

\[
(w_{t_{j-1}})_{k+1} \in \mathbb{R}^k (K_{t_{j-1}}, (r_{t_{j-1}})_0)
\]

\[
\text{s.t. } (w_{t_{j-1}})_{k+1} \perp \left[ (w_{t_{j-1}})_1, (w_{t_{j-1}})_2, \ldots, (w_{t_{j-1}})_k, \eta_{t_0}, \ldots, \eta_{(j-2)t_{j-2}} \right],
\]

where \(\eta_{(j-2)t_{j-2}}\) is the final residual obtained by iteratively solving the linear system at \(j = J - 2\) (equivalent to (42)).
3.1.2. Adapting the Ritz-Galerkin Projection. Recall that if we were trying to solve the linear system given in (33) by the CG method, then (34) gives a good basis of the generated Krylov subspace. The solution updates are then given as [16]

\[ x_k = x_0 + \zeta_k, \]

where \( \zeta_k = W_k y_k \) and \( W_k = [w_1 \ w_2 \ldots \ w_k] \) with the columns of this matrix given by (34). In the CG method, this \( y_k \) is defined by a Ritz-Galerkin projection

\[ r_k \perp W_k, \]

where \( r_k = b - A(x_0 + \zeta_k) = b - A(x_0 + W_k y_k) = r_0 - AW_k y_k \). Now, assume we are carrying some solution vector \( \tilde{x} \) from another linear system, which we need to make orthogonal to the final residual of (33). Then, the Ritz-Galerkin projection as above would consist of the following procedure:

\[ r_k \perp \left[ W_k \tilde{x} \right]. \]

Let us now look at the second linear system to solve in the AIRGA algorithm, i.e. (36). For this, a good basis of the generated Krylov subspace is given by (37). To find the solution vector here, the Ritz-Galerkin projection is defined as

\[ (\eta_{t_1})_k \perp (W_{t_1})_k, \]

where \( (\eta_{t_1})_k \) is the residual of (36) at the \( k^{th} \) iterative step and \( (W_{t_1})_k = [(w_{t_1})_1 \ (w_{t_1})_2 \ldots \ (w_{t_1})_k] \) with columns of this matrix given by (37). Note that \( \eta_{t_1} \) is the final residual of (36) (at convergence of CG).

To achieve the first orthogonality of (23), i.e. \( \tilde{V}_1 \perp \eta_{t_1} \), instead of (48) we need to have the following projection:

\[ (\eta_{t_1})_k \perp \left[ (W_{t_1})_k \tilde{V}_1 \right], \]

where \( \tilde{V}_1 \) is given by (11).

Similarly, to achieve the second set of orthogonalities of (23), i.e. \( \tilde{V}_1 \tilde{V}_2 \perp \eta_{t_2} \), the Ritz-Galerkin projection for solving the third linear system given by (39) needs to be adapted as follows:

\[ (\eta_{t_2})_k \perp \left[ (W_{t_2})_k \tilde{V}_1 \tilde{V}_2 \right], \]

where \( (\eta_{t_2})_k \) is the residual of (39) at the \( k^{th} \) iterative step, \( (W_{t_2})_k = [(w_{t_2})_1 \ (w_{t_2})_2 \ldots \ (w_{t_2})_k] \) with columns of this matrix given by (40), \( \tilde{V}_1 \) as above given by (11) and \( \tilde{V}_2 \) as given by (13). Note that \( \eta_{t_2} \) is the final residual of (39) (at convergence of CG).

We need to repeat a similar procedure for all other linear systems as well, which would help achieve the corresponding orthogonalities of (23). To achieve the last set of orthogonalities of (23), the Ritz-Galerkin projection of the last linear system (42) needs to be adapted as below

\[ (\eta_{(J-1)t_{(J-1)}}) \perp \left[ (W_{t_{(J-1)}}) \tilde{V}_1 \tilde{V}_2 \ldots \tilde{V}_{J-1} \right]. \]
where \( \eta_{t(J-1)I(J-1)} \) is the residual of (42) at the \( k^{th} \) iterative step, \( (W_{t(I-1)})_k = \left[ (w_{t(I-1)})_1 \ (w_{t(I-1)})_2 \ \cdots \ (w_{t(I-1)})_k \right] \) with columns of this matrix given by (43), \( \tilde{V}_1 \) given by (11), \( \tilde{V}_2 \) given by (13) and \( V_{J-1} \) given by (15). Note that \( \eta_{t(J-1)I(J-1)} \) is the final residual of (42) (at convergence of CG).

3.2. Implementation. Developing the CG algorithm that is based upon the adapted Lanczos process and the adapted Ritz-Galerkin projection is doable. However, developing its efficient implementation involving standard two/three term recurrences is non-trivial. Also, as the sequence number of the linear system increase (i.e. \( j \) gets larger), the number of orthogonalizations to be done also increase linearly. As briefly discussed in Section 2.1, using a recycling CG (RCG) [11, 15] helps alleviate both these problems. Hence, in this subsection we first discuss the idea behind RCG. Second we describe how to use RCG so as to easily achieve the earlier described extra orthogonalities. We do this with no code changes to the existing algorithm. Here, we also discuss the extra computational cost of such an implementation.

Assume that we want to solve the linear system in (33). Also assume that the recycle space is in the form of \( \text{span} \{U\} \), where columns of \( U \in \mathbb{R}^{n \times k} \) are linearly independent. If \( x_{-1} \) is the initial guess for (33) and \( r_{-1} = b - Ax_{-1} \) is the corresponding residual, then the projected initial guess \( x_0 \) is defined as [11, 15]

\[
x_0 = x_{-1} + U (U^T AU)^{-1} U^T r_{-1},
\]

with the corresponding residual \( r_0 = b - Ax_0 \).

At the \( k^{th} \) iterative step, the Lanczos process involves [17]

\[
w_{k+1} \in \mathbb{K}^k(A, U, r_0) \equiv \text{span}\{U, r_0, Ar_0, A^2r_0, \ldots, A^{k-1}r_0\} \quad s.t. \quad w_{k+1} \perp [U \ w_1 \ w_2 \ \cdots \ w_k],
\]

where \( w_{k+1} \), as earlier, is the \( (k + 1)^{th} \) Lanczos vector and \( w_1 = r_0/\|r_0\| \). The Ritz-Galerkin projection here is as follows:

\[
r_k \perp \mathbb{K}^k(A, U, r_0).
\]

The final solution update and the residual recurrences take the following form:

\[
x_{k+1} = x_k + \alpha_k p_k,
\]
\[
r_{k+1} = r_k + \alpha_k A p_k,
\]

where

\[
p_k = \beta_{k-1} p_{k-1} + (I - U(U^T AU)^{-1}(AU)^T) r_k,
\]
\[
\alpha_k = \left( r_k^T r_k \right) / (p_k^T A p_k),
\]
\[
\beta_{k-1} = \left( r_{k-1}^T r_{k-1} \right) / (r_k^T r_{k-1}).
\]

Next, we discuss how to use the above machinery for our requirements. Consider solving the linear system given by (36), originally (12). For adapting the Lanczos process in Section 3.1.1, while solving this linear system, we need to achieve the extra orthogonality in (38). Similarly, for adapting the Ritz-Galerkin projection in Section 3.1.2, while solving this linear system, we need to achieve the extra orthogonality in (49). Both these orthogonalities can be achieved if we take

\[
U = \begin{bmatrix} \eta_{1t} & \tilde{V}_1 \\ \end{bmatrix}
\]

(52)
in RCG.

At the first glance, it seems we are doing extra work here since \( \eta_0 \) orthogonality is needed only for Lanczos (not for Ritz-Galerkin), and \( \tilde{V}_1 \) is needed for Ritz-Galerkin (not for Lanczos). Also, note that by defining \( U \) as above, \( \eta_0 \) and \( \tilde{V}_1 \) are added in the Krylov search space, which is not needed in the adapted Lanczos process. That is, instead of (38) given by

\[
(w_{t_1})_{k+1}^T \in \mathbb{K}^k (\mathcal{K}_{t_1}, (r_{t_1})_0)
\]

\[\text{s.t.} \quad (w_{t_1})_1 \perp (w_{t_1})_2 \cdots (w_{t_1})_k \eta_0 \eta_0 \hat{H}
\]

we achieve the following:

\[
(w_{t_1})_{k+1}^T \in \mathbb{K}^k (\eta_0 \hat{H}, \tilde{V}_1, \mathcal{K}_{t_1}, (r_{t_1})_0)
\]

\[\text{s.t.} \quad (w_{t_1})_1 \perp (w_{t_1})_2 \cdots (w_{t_1})_k \eta_0 \tilde{V}_1 \]

These facts are true but besides the benefit of ease of implementation, this choice of space often leads to acceleration of the system. We support this with experiments in the next section. A theoretical study of this choice of space is the part of future work.

Next to achieve (41) from Section 3.1.1 and (50) from Section 3.1.2, we take

\[
U = \begin{bmatrix} \eta_0 \eta_0 \tilde{V}_1 \hat{V}_2 \end{bmatrix}
\]

in RCG. A similar selection of \( U \) is done for all the other linear systems. For the last linear system, i.e. to achieve (44) from Section 3.1.1 and (51) from Section 3.1.2, we take

\[
U = \begin{bmatrix} \eta_0 \eta_0 \cdots \eta_{(J-2)t(J-2)} \tilde{V}_1 \hat{V}_2 \cdots \tilde{V}_{J-1} \end{bmatrix}
\]

in RCG.

To summarize, \( \| \hat{H}(s) - \tilde{H}(s) \|_{H_2} \) is proportional to \( \kappa(H(s)) \) and \( \| Z \|_2 \). The problem is usually well conditioned and \( \| Z \|_2 \) is directly proportional to the cumulative residual norm \( \| \eta_f \| \). Thus, assuming backward stability conditions hold, as we iteratively solve the linear systems arising in the AIRGA algorithm more accurately (i.e. reduce the stopping tolerance of the linear solver), we should get a more accurate reduced system. This is very useful in deciding when to stop the linear solver. If we need a very accurate reduced system, then we need to iterate more in the linear solver, else we can stop earlier. We support this with numerical experiments in the next section.

4. Numerical Experiments. As motivated in Section 3, for stability we focus on the CG method for solving the linear systems arising in the AIRGA algorithm. Also, as discussed earlier, CG is optimal for SPD linear systems. Thus, we need to ensure that the coefficient matrices of all the linear systems to be solved are SPD.

The coefficient matrices are of the form \( s_i^2 M + s_j D + K \). To achieve that these matrices are SPD at start we do as below.

(a) We take input models that have M, D and K matrices as SPD. We use the one dimensional beam model (size 10,000) [4] and the Gyroscope model (size 17,361) [7]
that have such matrices and are commonly used. These models are of the form \([4, 7]\)

\[
\begin{align*}
M \ddot{x}(t) + D \dot{x}(t) + Kx(t) &= Fu(t), \\
y(t) &= C_p x(t),
\end{align*}
\]

(55)

where \(M, D, K \in \mathbb{R}^{n \times n}\) are the mass, the damping and the stiffness matrices, respectively, \(F \in \mathbb{R}^{n \times 1}\) and \(C_p \in \mathbb{R}^{1 \times n}\). These models are Single Input Single Output (SISO), and have proportional damping, i.e. \(D = \alpha M + \beta K\), where the damping coefficients \(\alpha\) and \(\beta\) belong to \((0, 1)\).

(b) We take the input expansion points \(s_i\) to be real and positive. In fact, we take three expansion points linearly spaced between 1 and 500 (based upon experience).

Next, we discuss how to ensure that the linear system matrices are SPD after the first AIRGA iteration (i.e. after start). As briefly mentioned in Section 1, after the first AIRGA iteration, the expansion points are chosen from the eigenvalues of the quadratic eigenvalue problems of the form \(\lambda^2 M + \lambda D + K\). For both our models, these eigenvalues turn out to be complex (case 3.8 of Table 1.1 in [19]). Thus, we get complex expansion points. Execution of the AIRGA algorithm as well as the accuracy of the reduced system does not get affected if one uses real expansion points or complex expansion points. Since real expansion points here are positive too (again because of case 3.8 of Table 1.1 in [19]), using them ensures that our coefficient matrices, \(s_i^2 M + s_i D + K\), are SPD at all the AIRGA iterations. Hence, we use real expansion points.

In Algorithm 1, at line 2, the overall iteration (while-loop) terminates when the change in the reduced system (computed as the \(H_2\)-error between the reduced systems of two consecutive AIRGA iterations) is less than a certain tolerance. We take this tolerance to be \(10^{-04}\) based on values in [8]. There is one more stopping criteria in this algorithm at line 8. This checks the \(H_2\)-error between two temporary reduced systems. We take this tolerance to be \(10^{-06}\) based upon values in [8].

As motivated in Section 3, to ensure that the extra orthogonalities for a stable AIRGA algorithm are satisfied, we use RCG instead of CG. As earlier, we refer to this as the inexact AIRGA algorithm. Preconditioning has to be employed when iterative methods fail or have a very slow convergence. Here, for the first model, we observe that the unpreconditioned RCG method has slow convergence whereas in the second model it fails to converge. Thus, we use a preconditioner. Since Sparse Approximate Inverse (SPAI) [10] and Incomplete Cholesky Factorization (ICHOL) [13, 16] are the most general types of preconditioners, we can use any of these preconditioners with RCG. Here, we use SPAI for the first model and ICHOL for the second model. For comparison, we solve all linear systems by a direct method as well. As earlier, we refer to this as the exact AIRGA algorithm. For certain types of analyses, we compare CG and RCG behaviours too.

We implement our codes in MATLAB (2016b), and test on a machine with the following configuration: Intel Xeon(R) CPU E5-1620 V3 @ 3.50 GHz., frequency 1200 MHz., 8 CPU and 64 GB RAM.

4.1. One Dimensional Beam Model. As discussed earlier, we do experiments on a system of size 10,000. Damping coefficients \(\alpha\) and \(\beta\) both are taken as 0.05 [4]. The maximum dimension to which we want to reduce the system \(r_{\text{max}}\) is taken as 8 based upon similar values in [4]. Thus, in the AIRGA algorithm, we have to solve linear systems of size \(10,000 \times 10,000\). While using RCG for solving these linear systems, we use two different stopping tolerances \(10^{-10}\) and \(10^{-14}\). Ideally, as
discussed earlier, we should obtain a more accurate reduced system for the smaller stopping tolerance.

First, let us look at the remaining assumptions for backward stability of the AIRGA algorithm (see Theorem 2.2). For all expansion points, \( L(s) \) is invertible and \( \|L(s)^{-1}\|_{H_{\infty}} \) is less than one. E.g., for the initial set of expansion points, \( \|L(s)^{-1}\|_{H_{\infty}} \) is \( 2.68 \times 10^{-02} \). Finally, \( \|Z\|_2 \), at the end of the first AIRGA iteration, for the RCG stopping tolerance of \( 10^{-10} \) and \( 10^{-14} \) is \( 8.59 \times 10^{-01} \) and \( 1.27 \times 10^{-04} \), respectively, both of which are also less than one. These values are less than one at the end of all the other AIRGA iterations as well. The condition number for our problem, as defined in (31), is \( 8.63 \times 10^{-02} \). This shows that the one dimensional beam model is well-conditioned.

The accuracy results are given in Fig. 1 and Table 1. In Fig. 1, we have the accuracy of the reduced system \( \|\hat{H}(s) - \tilde{H}(s)\|_{H_2} \) on the y-axis and the AIRGA iterations on the x-axis. Here, the dotted line corresponds to the RCG stopping tolerance of \( 10^{-10} \) while the solid line corresponds to the RCG stopping tolerance of \( 10^{-14} \). From Fig. 1, it is evident that we get a more accurate reduced system as we solve the linear systems more accurately (dotted line is above the solid one at all the AIRGA iterations). Table 1 gives the corresponding data.

The AIRGA algorithm gets more consistent as it converges to ideal expansion points. Hence, from AIRGA iteration 3 to 5, the accuracy of the reduced system for the RCG stopping tolerance of \( 10^{-14} \) is visibly better than the accuracy of the reduced system for the RCG stopping tolerance of \( 10^{-10} \).

Next, we do some other analysis corresponding to (32), i.e. the relation between the perturbation and the stopping tolerance. In Table 2, we demonstrate that \( \|X^T(XX^T)^{-1}\|_f \) is less sensitive to the accuracy to which we solve the linear systems. We notice from this table that as we reduce the stopping tolerance of RCG from \( 10^{-10} \) to \( 10^{-14} \), \( \|X^T(XX^T)^{-1}\|_f \) stays almost the same.

Recall, \( \|\eta\|_f \) is the normed sum of residuals obtained by solving different linear systems at one outer iteration of the AIRGA algorithm. These values are lower for smaller stopping tolerance as expected. However, the values of \( \|\eta\|_f \) for the RCG stopping tolerance \( 10^{-14} \) seem higher than the tolerance itself. There are two reasons
Table 1: Accuracy of the reduced system at each AIRGA iteration for the two different stopping tolerances in RCG; one dimensional beam model.

| AIRGA Iteration | RCG stopping tolerance of $10^{-10}$ | RCG stopping tolerance of $10^{-14}$ |
|-----------------|--------------------------------------|--------------------------------------|
|                 | $||H - \hat{H}||_2$                  | $||\hat{H}||_2$                      |
| 1               | $2.9 \times 10^{-07}$                | $1.8 \times 10^{-07}$                |
| 2               | $2.6 \times 10^{-06}$                | $2.2 \times 10^{-06}$                |
| 3               | $3.1 \times 10^{-06}$                | $3.2 \times 10^{-07}$                |
| 4               | $1.4 \times 10^{-06}$                | $8.1 \times 10^{-07}$                |
| 5               | $8.3 \times 10^{-06}$                | $4.0 \times 10^{-07}$                |

Table 2: The perturbation expression quantities for RCG at two different stopping tolerances; one dimensional beam model.

| AIRGA Iteration | RCG stopping tolerance of $10^{-10}$ | RCG stopping tolerance of $10^{-14}$ |
|-----------------|--------------------------------------|--------------------------------------|
|                 | $||\eta||_f$                          | $||\eta||_f$                          |
|                 | $||X^T(XX^T)^{-1}||_f$                | $||X^T(XX^T)^{-1}||_f$                |
| 1               | $7.7 \times 10^{-11}$                | $8.1 \times 10^{-12}$                |
| 2               | $8.2 \times 10^{-11}$                | $2.2 \times 10^{-12}$                |
| 3               | $7.4 \times 10^{-11}$                | $2.5 \times 10^{-12}$                |
| 4               | $8.2 \times 10^{-11}$                | $2.8 \times 10^{-12}$                |
| 5               | $8.2 \times 10^{-11}$                | $2.7 \times 10^{-12}$                |

for this. First, we are reporting absolute residuals here while stopping tolerances are relative. Second, we are ensuring that the norm of the residual of every linear system is less than the stopping tolerance while $||\eta||_f$ is the sum of all such normed residuals.

Finally, we support our earlier claim that the way required orthogonalities are achieved (see Section 3.2), it does not deteriorate the convergence of our linear solver. Thus, we solve all linear systems arising in the AIRGA algorithm with CG as well besides RCG. Table 3, gives the average iteration count of the two solvers (CG and RCG) at every AIRGA outer iteration. It is evident that using the recycle space as formulated in Section 3.2, neither accelerates nor deteriorates convergence.

It is important to note that by using a recycle space we are doing extra work in terms of more inner products. For this example, as in Table 3, CG and RCG both take only few iterations for convergence. Hence, the computation time at every AIRGA iteration is too less to realistically compare the two cases (less than about one hundredth of second). In the next example, where the CG and the RCG iteration counts are substantial, we do a timing comparison as well. There, by recycling Krylov subspaces, we see a reduction in the iteration count as well as time of RCG as compared to CG. Thus, for many cases, recycling spaces offsets the cost of extra orthogonalizations and even improves convergence (savings both in the iteration count...
Table 3: Convergence analysis of CG and RCG at two different stopping tolerances; one dimensional beam model.

| AIRGA Iteration | Stopping tol. $10^{-10}$ | Stopping tol. $10^{-14}$ |
|------------------|---------------------------|---------------------------|
|                  | Avg. CG Itr. | Avg. RCG Itr. | Avg. CG Itr. | Avg. RCG Itr. |
| 1                | 6            | 6             | 7            | 7             |
| 2                | 4            | 4             | 5            | 5             |
| 3                | 4            | 4             | 5            | 5             |
| 4                | 4            | 4             | 5            | 5             |
| 5                | 4            | 4             | 5            | 5             |
| Total            | 22           | 22            | 27           | 27            |

and time).

4.2. Gyroscope Model. As mentioned earlier, we do another experiment on a system of size 17, 361. Damping coefficients $\alpha$ and $\beta$ are taken as 0.2 and $1.34 \times 10^{-04}$, respectively [7]. The dimension to which we want to reduce the system ($r_{max}$) is taken as 12 based upon similar values in [7]. Here, in the AIRGA algorithm we have to solve the linear systems of size $17,361 \times 17,361$. Again, we use RCG for solving these linear systems. To demonstrate our main result, we ideally want the stopping tolerances to be four orders of magnitude different from each other. E.g., $10^{-10}$ and $10^{-14}$ in the previous problem. Here, we are unable to solve the linear systems for tolerances less than $10^{-10}$. As for the higher tolerance, if we go beyond $10^{-08}$, then the AIRGA algorithm’s convergence varies (differing iteration counts for convergence). Thus, we cannot compare results of the two cases. Hence, we use stopping tolerances of $10^{-08}$ and $10^{-10}$. As discussed earlier, we should obtain a more accurate reduced system for the smaller stopping tolerance.

Similar to the previous experiment, here also we look at the remaining assumptions for backward stability of the AIRGA algorithm (see Theorem 2.2). For all expansion points, $L(s)$ is invertible and $\|L(s)^{-1}\|_{H_{\infty}}$ is less than one. E.g., for the initial set of expansion points, $\|L(s)^{-1}\|_{H_{\infty}}$ is $6.46 \times 10^{-01}$. Finally, $\|Z\|_2$, at the end of the first AIRGA iteration, for the RCG stopping tolerance of $10^{-08}$ and $10^{-10}$ is $8.6 \times 10^{-01}$ and $3.3 \times 10^{-01}$, respectively, both of which are also less than one. These values are less than one at the end of all the other AIRGA iterations as well. The condition number for this problem, as defined in (31), is $5.15 \times 10^{09}$. This shows that the Gyroscope model is ill-conditioned.

Accuracy of the reduced system is proportional to the condition number $\kappa (H(s))$ and the perturbation $\|Z\|$ (see (28)). Since, the condition number here is high, we get a less accurate reduced system. However, this is still a good problem for us since we want to demonstrate that the reduction in perturbation (linked to linear solver stopping tolerance) improves accuracy. High condition number spoils the accuracy equally for both the RCG stopping tolerances ($10^{-08}$ and $10^{-10}$). The accuracy results are given in Table 4. It is again evident that we get a more accurate reduced system as we solve the linear systems more accurately.

Here also, we do some other analysis corresponding to (32), i.e. relation between the perturbation and the stopping tolerance. From Table 5, we demonstrate that $\|X^T (XX^T)^{-1} \|_f$ is less sensitive to the accuracy to which we solve the linear systems.
Table 4: Accuracy of the reduced system at each AIRGA iteration for the two different stopping tolerances in RCG; Gyroscope Model.

| AIRGA Iteration | RCG stopping tolerance $10^{-08}$ | RCG stopping tolerance $10^{-10}$ |
|-----------------|----------------------------------|----------------------------------|
|                 | $||\hat{H} - \tilde{H}||_{H_2}$ | $||\hat{H} - \tilde{H}||_{H_2}$ |
| 1               | $1.55 \times 10^{-03}$           | $8.66 \times 10^{-04}$           |
| 2               | $3.63 \times 10^{-05}$           | $3.14 \times 10^{-05}$           |

Table 5: The perturbation expression quantities for RCG at two different stopping tolerances; Gyroscope Model.

| AIRGA Iteration | RCG Stopping tolerance $10^{-08}$ | RCG Stopping tolerance $10^{-10}$ |
|-----------------|----------------------------------|----------------------------------|
|                 | $||\eta||_f$ | $||X^T(XX^T)^{-1}||_f$ | $||\eta||_f$ | $||X^T(XX^T)^{-1}||_f$ |
| 1               | $2.5 \times 10^{-09}$ | $1.34 \times 10^{09}$ | $2.9 \times 10^{-10}$ | $1.33 \times 10^{09}$ |
| 2               | $2.6 \times 10^{-09}$ | $1.98 \times 10^{11}$ | $2.6 \times 10^{-10}$ | $1.98 \times 10^{11}$ |

Again, we notice from this table that as we reduce the stopping tolerance of RCG from $10^{-08}$ to $10^{-10}$, $||X^T(XX^T)^{-1}||_f$ stays almost the same. For this model also, behaviour of $||\eta||_f$ is same as for the previous problem.

As discussed in the previous subsection, for this model, we observe that the number of iterations required for convergence of RCG is less than that of CG, both of which are given in Table 6. We see a savings of about 10% in the average linear solver iterations. The corresponding computation times are given in Table 7. The savings in iterations translate to about 5% savings in time.

5. Conclusion. We discuss application of preconditioned iterative methods for solving the large linear systems in the AIRGA algorithm. These methods find solutions only up to a certain tolerance. Hence, we show that under some mild assumptions, the AIRGA algorithm is backward stable with respect to these inexact linear solves. We also analyze the accuracy of the resulting reduced system, and support all our results with multiple numerical experiments.

The first assumption is the use of a Ritz-Galerkin based linear solver, where the residual of a linear system is made orthogonal to the corresponding Krylov subspace. This is not enough for stability, and hence, we assume few other orthogonalities. Since the CG method is the most popular linear solver based upon the Ritz-Galerkin theory and is ideal for SPD linear systems, we focus on SPD systems only. We use Recycling CG (RCG) to achieve the extra orthogonalities. The future work here involves modifying other methods based upon the Ritz-Galerkin theory (to achieve extra orthogonalities), which can be used to solve general non-symmetric indefinite linear systems. For example, the Full Orthogonalization Method (FOM).

The second set of assumptions are that $L(s)$, which is a function of the frequency $s$, the mass matrix $M$, the damping matrix $D$, and the stiffness matrix $K$, is invertible and its norm is bounded by one. These assumptions are easily satisfied for our models, but they may not always hold. Future work here involves better
Table 6: Convergence analysis of CG and RCG at two different stopping tolerances; Gyroscope Model.

| Iteration | Stopping tolerance $10^{-08}$ | Stopping tolerance $10^{-10}$ |
|-----------|-------------------------------|-------------------------------|
|           | Avg. CG Itr. | Avg. RCG Itr. | Avg. CG Itr. | Avg. RCG Itr. |
| 1         | 216           | 207             | 244           | 224           |
| 2         | 202           | 180             | 228           | 206           |
| **Total** | **418**       | **387**         | **472**       | **430**       |

Table 7: Computation time of CG and RCG at two different stopping tolerances; Gyroscope Model.

| Iteration | Stopping tolerance $10^{-08}$ | Stopping tolerance $10^{-10}$ |
|-----------|-------------------------------|-------------------------------|
|           | CG time (secs.) | RCG time (secs.) | CG time (secs.) | RCG time (secs.) |
| 1         | 2.35           | 2.20             | 2.49           | 2.41           |
| 2         | 2.04           | 1.95             | 2.33           | 2.23           |
| **Total** | **4.39**       | **4.15**         | **4.82**       | **4.64**       |

The third set of assumptions involve being able to compute perturbation $Z$ from the given expression and bounding its 2-norm by one. As earlier, although for both our models these assumptions are easily satisfied, they may not always hold. $Z$ is dependent on the linear solver stopping tolerances. Hence, we need to study range of these tolerances when the 2-norm of this perturbation could be bounded by one.

The condition number of the dynamical system, which we use is an approximation to the ideal condition number. That is, condition number of the dynamical system with respect to computing the $H_2$-norm of the error between the inexactly computed reduced system and the exactly computed reduced system. This is also part of the future work.

We also plan to apply preconditioned iterative methods in other model reduction algorithms for second order linear dynamical systems. Balanced truncation is an alternative to the projection based methods for performing model reduction. Here, large scale Lyapunov equations have to be solved. Alternating Directions Implicit (ADI) based algorithms are widely used for solving such equations. ADI algorithms also require solving sequences of large sparse linear systems.

Often the matrices of the dynamical systems (i.e. M, D, K etc.) are dependent on multiple parameters (e.g., material property etc.). Algorithms have been proposed for model reduction of such parametric second order linear dynamical systems as well. One such algorithm is given in [6]. The model reduction technique here is similar to the moment matching done in the AIRGA algorithm. The sequence of linear systems obtained here are dependent on the parameters of the system matrices as compared to the dependency on the expansion points as in case of the AIRGA algorithm. We
also plan to apply preconditioned iterative methods for solving large linear systems arising in such parametric settings.

Acknowledgments. We would like to deeply thank Prof. Dr. Heike Faßbender (at Institut Computational Mathematics, AG Numerik, Technische Universität Braunschweig, Germany) for discussions and help regarding different aspects of this project.

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