Rigidity of $K$-theory under deformation quantization

Jonathan Rosenberg*

Dedicated to Calvin C. Moore on his 60th birthday

Abstract

Quantization, at least in some formulations, involves replacing some algebra of observables by a (more non-commutative) deformed algebra. In view of the fundamental role played by $K$-theory in non-commutative geometry and topology, it is of interest to ask to what extent $K$-theory remains "rigid" under this process. We show that some positive results can be obtained using ideas of Gabber, Gillet-Thomason, and Suslin. From this we derive that the algebraic $K$-theory with finite coefficients of a deformation quantization of the functions on a compact symplectic manifold, forgetting the topology, recovers the topological $K$-theory of the manifold.

Key words: deformation quantization, star-product, algebraic $K$-theory, $K$-theory with finite coefficients, power series ring.

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Notation. If $A$ is a ring, $K(A)$ will denote its (connective) $K$-theory spectrum, the spectrum associated to the infinite loop space $K_0(A) \times BGL(A)^+$, where $BGL(A)^+$ is the result of applying the Quillen $+$-construction to the classifying space of the infinite general linear group over $A$. By definition, the (algebraic) $K$-groups $K_i(A)$ of $A$ are (at least in positive degrees) the homotopy groups of $K_0(A)$, and the $K$-groups of $A$ with finite coefficients $\mathbb{Z}/(m)$, $K_i(A; \mathbb{Z}/(m))$, are defined (at least in positive degrees) to be the homotopy groups of $S(\mathbb{Z}/(m)) \wedge K(A)$, where $S(\mathbb{Z}/(m))$ is the $\mathbb{Z}/(m)$ Moore spectrum. These come with universal coefficient short exact sequences

$$0 \to K_i(A) \otimes \mathbb{Z}/(m) \to K_i(A; \mathbb{Z}/(m)) \to \text{Tor}_2(K_{i-1}(A), \mathbb{Z}/(m)) \to 0.$$

(This is almost, but not quite, the definition of Browder in [1]; for an explanation of the difference between the two definitions, see [1], pp. 285–286.)

In the one case below where confusion might be possible between algebraic and topological $K$-groups, we denote these by $K^\text{alg}_j$ and $K^\text{top}_j$, respectively.

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Now we begin with a very general definition of (formal) deformation quantization. Intuitively, this is a formal deformation of the multiplication on an “algebra of observables,” the deformation parameter being identified with “Planck’s constant” $\hbar$.

**Definition 1** Let $\mathcal{A}_0 = (A_0, \cdot)$ be an algebra over a commutative ring $k$ (with unit), where $A_0$ is the underlying $k$-module of $\mathcal{A}_0$ and $\cdot$ is the multiplication in $A$. A (formal) deformation quantization of $\mathcal{A}_0$ will mean an (associative) algebra $A = (A_0[[\hbar]], \star)$ over $k[[\hbar]]$ (the commutative ring of formal power series over $k$ in a variable $\hbar$) with underlying $k[[\hbar]]$-module $A_0[[\hbar]]$, where the multiplication $\star$ in $A$ is defined by perturbing the multiplication $\cdot$ in $A_0$ to a new product $\star$ via

$$a \star b = a \cdot b + \hbar \phi_1(a, b) + \hbar^2 \phi_2(a, b) + \cdots, \quad a, b \in A_0,$$

and then extending to series in the obvious way:

$$\left( \sum_{j=0}^{\infty} a_j \hbar^j \right) \star \left( \sum_{l=0}^{\infty} b_l \hbar^l \right) = \sum_{j, l=0}^{\infty} \hbar^{j+l} \left( a_j \cdot b_l + \sum_{p=1}^{\infty} \hbar^p \phi_p(a_j, b_l) \right).$$

Here $\phi_j$, $j = 1, 2, \ldots$ are $k$-bilinear maps $A_0 \times A_0 \to A_0$. Note that $\mathcal{A}_0 \cong A/(\hbar)$ as algebras, so that one has a natural algebra map $e_0 : A \to \mathcal{A}_0$ (“setting $\hbar$ to 0”). We call the map $e_0$ the classical limit map.

**Example 2** A trivial but still important example is the case where the multiplication on $\mathcal{A}_0$ is undeformed. In this case $A = \mathcal{A}_0[[\hbar]]$ is simply a ring of formal power series in one variable over the ring $\mathcal{A}_0$. $\square$

**Example 3** In one of the most important examples, $k = \mathbb{C}$ and $\mathcal{A}_0 = C^\infty(M)$, where $M$ is a symplectic manifold. Then there exist non-commutative algebras $A$ satisfying Definition 1 for which

$$\phi_1(f, g) - \phi_1(g, f) = \{f, g\},$$

where $\{\ , \ \}$ is the Poisson bracket on $M$. This was shown in [14] and [3]. The $\star$-product is obtained by a patching procedure using the Weyl quantization of $C^\infty(\mathbb{R}^{2n})$. $\square$

In this generality, it turns out that $K_0$ is preserved under deformation quantization.

**Theorem 4** Let $k$ be a commutative ring with unit, let $\mathcal{A}_0$ be an algebra (with unit) over $k$, let $A$ be a deformation quantization of $\mathcal{A}_0$ in the sense of Definition 1, and let $e_0 : A \to \mathcal{A}_0$ be the associated classical limit map. Then the map $(e_0)_* : K_0(A) \to K_0(\mathcal{A}_0)$ induced by $e_0$ is an isomorphism.

We will need the following simple lemma.
Lemma 5 Let $k$ be a commutative ring with unit, let $A_0$ be a $k$-algebra, and let $A$ be a deformation quantization of $\overline{A}_0$ in the sense of Definition 4. Then an element $a = \sum_{j=0}^{\infty} a_j h^j$ of $A$ ($a_j \in A_0$) is invertible if and only if $c_0(a) = a_0$ is invertible in $\overline{A}_0$. Similarly, an element $a = \sum_{j=0}^{n-1} a_j h^j$ of $A/\,(h^n)$ ($a_j \in A_0$) is invertible if and only if $c_0(a) = a_0$ is invertible in $\overline{A}_0$.

Proof. The “only if” direction is trivial, and the “if” direction in the case of $A/(h^n)$ follows from the result for $A$. The proof (in the case of $A$) for the “if” direction is the usual algorithm for inversion of power series. More specifically, suppose $a_0$ is invertible for the multiplication · in $A_0$, and let $a = \sum_{j=0}^{\infty} a_j h^j \in A$ ($a_j \in A_0$). We can construct an inverse $b = \sum_{l=0}^{\infty} b_l h^l$ for $a$ with respect to the product $\ast$ in $A$ by letting $b_0 = a_0^{-1}$ (the inverse of $a_0$ in $\overline{A}_0$) and then solving for the coefficients $b_l$ by iteration in the equation

$$1 = a \ast b = \left( \sum_{j=0}^{\infty} a_j h^j \right) \ast \left( \sum_{l=0}^{\infty} b_l h^l \right) = \sum_{j,l=0}^{\infty} h^{j+l} \left( a_j \cdot b_l + \sum_{p=1}^{\infty} h^p \phi_p(a_j,b_l) \right).$$

(2)

Equating coefficients of powers of $h$ on the two sides of (2) gives for each $q \geq 1$ an equation (in $A_0$)

$$\sum_{j+l+p=q} \phi_p(a_j,b_l) = 0.$$  

(3)

where for convenience we let $\phi_0(a_j,b_l) = a_j \cdot b_l$. To show these equations are (uniquely) solvable, note that assuming we have solved for $b_0, \ldots, b_{q-1}$, $q \geq 1$, (3) reduces to

$$a_0 \cdot b_q + \sum_{j+l+p=q, l \leq q-1} \phi_p(a_j,b_l) = 0,$$

or

$$b_q = - \sum_{j+l+p=q, l \leq q-1} a_0^{-1} \cdot \phi_p(a_j,b_l).$$

Thus, by induction on $q$, (3) has a unique solution which is a right $\ast$-inverse to $a$. Similarly, $a$ has a unique left $\ast$-inverse. By the usual argument, these must be equal, so $a$ is invertible in $A$. □

Proof of Theorem 4 For the injectivity, it is enough to show that if $M$ and $N$ are (left) $A$-modules with $M \oplus N = A^n$ for some $n$, and if $\overline{A}_0 \otimes_{c_0} M$ and $\overline{A}_0 \otimes_{c_0} N$ are free $\overline{A}_0$-modules, then $M$ and $N$ are free $A$-modules. Since the kernel $(h)$ of $c_0$ is contained in the radical of $A$ (this follows immediately from Lemma 3), the proof of Theorem 1.3.11 in [4] applies without change.

The proof of surjectivity is based on a version of Hensel’s Lemma. Since $A = \varprojlim A/(h^n)$ and we can replace $A$ by $M_n(A)$, the $n \times n$ matrices over $A$, if necessary, it is enough to show that for $j \geq 1$, any idempotent $\overline{\pi}$ in $A/(h^j)$ can be lifted to an idempotent in $A/(h^{j+1})$. (Then an idempotent in $\overline{A}_0 = A/\,(h)$
Theorem 6  Let $k$ be a field of characteristic zero, let $\overline{A}_0$ be an algebra (with unit) over $k$, let $A$ be a deformation quantization of $\overline{A}_0$ in the sense of Definition 1, and let $e_0 : A \rightarrow \overline{A}_0$ be the associated classical limit map. Then $e_0$ induces isomorphisms $K_j(A; \mathbb{Z}/(m)) \cong K_j(\overline{A}_0; \mathbb{Z}/(m))$ on K-theory with finite coefficients for any $m > 1$, $j > 0$.

The motto of the theorem is: passage to the classical limit preserves K-theory with finite coefficients. But perhaps a few words of explanation for the peculiar formulation are in order.

1. We certainly cannot expect $e_0$ to induce isomorphisms of K-groups integrally, since this is false in the case of Example 2. If $\overline{A}_0 = k$, $*=\cdot$, and $A=k[[\hbar]]$, then $A$ is a commutative local ring and thus (see for instance [11], Corollary 2.2.6) $K_1(A) = A^\times$, which is vastly bigger than $K_1(\overline{A}_0) = k^\times$, and in fact the kernel of the map induced by $(e_0)_*$ on $\pi_1$ may be identified with a $k$-vector space of uncountable dimension.

2. There is some subtlety in the result since $A$ is as a $k$-vector space an infinite product of copies of $A_0$, but the K-theory groups of an infinite product of rings are in general not the products of the K-groups of the factors. For a simple counterexample, let $R_j = C(S^{2j})$ (the continuous complex-valued functions on a sphere), $j = 1, 2, \ldots$. By Bott periodicity, $K_0(R_j) \cong \mathbb{Z}$. Let $b_j \in K_0(R_j)$ have non-trivial projection into $K_0(R_j)$. Then the element $(b_1, b_2, \ldots)$ of $\prod_j K_0(R_j)$ does not lie in the image of $K_0(\prod_j R_j)$, since realizing $b_j$ as a formal difference of idempotent matrices requires matrices of increasing size as $j \rightarrow \infty$, so that $(b_1, b_2, \ldots)$ cannot
come from matrices of finite size over $\prod_j R_j$. The $K$-theory of categories does commute with infinite products [8], but for quite non-trivial reasons. However, if $\mathcal{P}(R)$ denotes the category of finitely generated projective $R$-modules for a ring $R$ (the relevant category for $K$-theory of rings), then $\mathcal{P}(\prod_j R_j)$ is not generally equivalent to $\prod_j \mathcal{P}(R_j)$.

Before giving the proof, we need two preliminaries.

**Lemma 7** Let $k$ be a field, let $A_0$ be a $k$-algebra, and let $A$ be a deformation quantization of $\overline{A}_0$ in the sense of Definition 8. Then for any $n \geq 1$, the natural maps $GL(n, A/(h^{j+1})) \to GL(n, A/(h^j))$ ($j = 1, 2, \ldots$) are all surjective, and $GL(n, A) = \varprojlim GL(n, A/(h^j))$.

**Proof.** This follows immediately from Lemma 8, applied not to $A$ but to $M_n(A)$, the $n \times n$ matrices over $A$. □

**Proposition 8** Let $k$, $\overline{A}_0$, and $A$ be as in Theorem 9. Then for any integers $n$, $j \geq 1$, $m > 1$, the natural map $GL(n, A/(h^j)) \to GL(n, \overline{A}_0)$ induces an isomorphism on homology with $\mathbb{Z}/(m)$ coefficients.

**Proof.** We fix $n$ and prove this by induction on $j$. The statement is trivially true when $j = 1$. So assume $j \geq 1$ and the statement is true for $j$; we’ll prove it for $j + 1$. Consider the exact sequence of $k$-algebras

$$0 \to I \to A/(h^{j+1}) \to A/(h^j) \to 0,$$

where as a vector space, $I = h^j A_0$, but the multiplication on $I$ vanishes since $2j \geq j + 1$. By the previous lemma, the induced map $GL(n, A/(h^{j+1})) \to GL(n, A/(h^j))$ is surjective, and the kernel $K$ consists of matrices of the form $1 + x$, $x \in M_n(I)$. Since $I^2 = 0$, multiplication in $K$ is given by $(1 + x)(1 + y) = 1 + x + y$, i.e., $K \cong M_n(I)$ with its additive group structure. Since $k$ is of characteristic zero, $K$ is therefore isomorphic to the underlying additive group of a $\mathbb{Q}$-vector space, which is uniquely divisible. Hence $K$ is $\mathbb{Z}/(m)$-acyclic, and the Hochschild-Serre spectral sequence for

$$1 \to K \to GL(n, A/(h^{j+1})) \to GL(n, A/(h^j)) \to 1$$

collapses to give $H_\bullet(GL(n, A/(h^{j+1})); \mathbb{Z}/(m)) \cong H_\bullet(GL(n, A/(h^j)); \mathbb{Z}/(m))$. This gives the inductive step. □

**Proof of Theorem 9.** By Lemma 8, $GL(n, A) = \varprojlim GL(n, A/(h^j))$ (for any $n$). Hence the $\mathbb{Z}/(m)$-homology of $GL(n, A)$ can be computed from that of the $GL(n, A/(h^j))$ by the Milnor limit sequence. But by Proposition 9, the maps $GL(n, A/(h^{j+1})) \to GL(n, A/(h^j))$ are all $\mathbb{Z}/(m)$-homology isomorphisms. Hence the inverse system $H_\bullet(GL(n, A/(h^j)); \mathbb{Z}/(m))$ (for fixed $n$) satisfies the Mittag-Leffler criterion, and

$$H_\bullet(GL(n, A); \mathbb{Z}/(m)) \cong H_\bullet(GL(n, \overline{A}_0); \mathbb{Z}/(m)).$$
Now pass the to the limit as \( n \to \infty \). We deduce that the map of groups \( GL(A) \to GL(\mathcal{A}_0) \) induces a \( \mathbb{Z}/(m) \)-homology isomorphism. Applying the classifying space functor and the Quillen +-construction yields that \( BGL(A)^+ \to BGL(\mathcal{A}_0)^+ \) is a \( \mathbb{Z}/(m) \)-homology equivalence (and of course also an infinite loop map). Now the usual connective \( K \)-theory spectrum of \( A, \mathbb{K}(A) \), is just the spectrum associated to the infinite loop structure on \( K_0(A) \times BGL(A)^+ \), and \( K \)-theory with finite coefficients (in positive degrees, at least) is computed by taking the homotopy groups of \( K \) and \( \mathbb{K}(A) \). Combining the fact that \( BGL(A)^+ \to BGL(\mathcal{A}_0)^+ \) is a \( \mathbb{Z}/(m) \)-homology equivalence with the fact that \( K_0(A) \to K_0(\mathcal{A}_0) \) is an isomorphism (Theorem \( \mathbb{H} \), we see \( \mathbb{K}(A; \mathbb{Z}/(m)) \to \mathbb{K}(\mathcal{A}_0; \mathbb{Z}/(m)) \) is a homology equivalence, hence a homotopy equivalence by the Hurewicz Theorem (which applies to connective spectra). (This argument bypassed the sort of reasoning used in \( \mathbb{K} \), Proposition 1.5, but one could use that here instead.) So \( \pi_j(\mathbb{K}(A; \mathbb{Z}/(m))) \to \pi_j(\mathbb{K}(\mathcal{A}_0; \mathbb{Z}/(m))) \), i.e., \( K_j(A; \mathbb{Z}/(m)) \to K_j(\mathcal{A}_0; \mathbb{Z}/(m)) \), for \( j > 0 \). \( \square \)

**Corollary 9** (Cf. \( \mathbb{H} \), Theorem 1, for the commutative case.) If \( k \) is a field of characteristic zero and if \( B \) is a \( k \)-algebra, then for \( j > 0 \) and any \( m > 1 \), \( K_j(B[[t]]; \mathbb{Z}/(m)) \to K_j(B; \mathbb{Z}/(m)) \).

**Proof.** Apply Theorem \( \mathbb{H} \) to Example \( \mathbb{H} \). \( \square \)

**Corollary 10** Let \( M \) be a compact symplectic manifold, let \( \mathcal{A}_0 = C^\infty(M) \) with its usual Poisson structure, and let \( A \) be a deformation quantization of \( \mathcal{A}_0 \). Then for \( j > 0 \) and any \( m > 1 \), \( R_j^{alg}(A; \mathbb{Z}/(m)) \cong K_{top}^{-j}(M; \mathbb{Z}/(m)) \), the topological \( K \)-theory of \( M \) with finite coefficients.

**Proof.** We apply our results to Example \( \mathbb{H} \). By Theorem \( \mathbb{H} \), \( K_0(A) \times BGL(A)^+ \to K_0(\mathcal{A}_0) \times BGL(\mathcal{A}_0)^+ \) is a \( \mathbb{Z}/(m) \)-homotopy equivalence, so for \( j > 0 \),

\[
K_j^{alg}(A; \mathbb{Z}/(m)) \cong K_j^{alg}(C^\infty(M); \mathbb{Z}/(m)).
\]

The group on the right is known to coincide with \( K_{top}^{-j}(M; \mathbb{Z}/(m)) \) by \( \mathbb{H} \). This requires comment: Fischer’s theorem is stated for the algebra of continuous functions on a compact space \( X \), but since the proof is sheaf-theoretic, when \( X \) is a manifold \( M \), one can replace the sheaf of germs of continuous functions by the sheaf of germs of \( C^\infty \) functions, and all the arguments go through. The essential facts needed to make everything work are:

1. the local ring of germs of \( C^\infty \) functions at a point of a smooth manifold is Henselian;

2. for \( G \) a Lie group (in particular, for \( G = GL(n, \mathbb{C}) \)), the group \( C^\infty(M, G) \) is a “locally convex” topological group in the sense of \( \mathbb{H} \), that is, that it is a topological group in the \( C^\infty \) topology, and that functions \( M \to G \) which are close in the \( C^\infty \) topology can be joined by a smooth path; and
3. The topological K-theory of \( C^\infty(M) \) coincides with that of \( C(M) \) (a well-known consequence of \( C^\infty \) approximation).

□

**Remark 11** Exactly the same statement as in Theorem 6 works when the ground ring \( k \) is a field of characteristic \( p \), except that in this case one has to assume \( (m, p) = 1 \). The only difference in the proof is that in the proof of Proposition 8 one should substitute the fact that if \( (m, p) = 1 \), then a \( \mathbb{Z}/(p) \)-vector space (regarded as a group under addition) is \( \mathbb{Z}/(m) \)-acyclic. In fact one can even take \( k = \mathbb{Z}[\frac{1}{m}] \) and the argument still works (see [13], Lemma 1.1). □

**Remark 12** In fact the connective K-theory spectrum is the connective cover of a non-connective K-theory spectrum \( K^\text{non-conn}(A) \), whose homotopy groups in non-negative degrees are the same as those of \( K_0(A) \times BGL(A)^+ \) (in other words, the Quillen K-groups), and whose negative homotopy groups are the negative K-groups of Bass. (One of the many constructions of this spectrum may be found in [9], and a proof that it is equivalent to all the other standard definitions of this spectrum may be found in [10], §§5–6.) An optimal statement along the lines of Theorem 6—I am not sure whether this is correct or not—would thus be that

\[ (e_0)_* : K^\text{non-conn}(A; \mathbb{Z}/(m)) \xrightarrow{\cong} K^\text{non-conn}(A_0; \mathbb{Z}/(m)), \]

so that one gets isomorphisms similar to those of Theorem 6 for negative K-theory as well, but we have been unable to prove this. The difficulty is that the natural way to deloop the equivalence of Theorem 6 would be to replace \( \mathcal{A}_0 \) by \( \mathcal{B}_0 = \mathcal{A}_0[t, t^{-1}] \) and define \( B \) from \( B_0 \) by the obvious formula derived from (1), keeping \( t \) central in \( B \). The problem is that the resulting \( B \) is not just \( A[t, t^{-1}] \) (which is not (\( h \))-adically complete), but rather its (\( h \))-adic completion, and it’s not clear what effect the completion process has on K-groups. Other delooping techniques run into similar problems having to do with the failure of products and coproducts to commute. □

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JONATHAN ROSENBERG
Department of Mathematics
University of Maryland
College Park, MD 20742
email: jmr@math.umd.edu