Electroweak interaction beyond the Standard Model and Dark Matter in the Tangent Bundle Quantum Field Theory

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A generalized theory of electroweak interaction is developed based on the underlying geometrical structure of the tangent bundle with symmetries arising from transformations of tangent vectors along the fiber axis at a fixed space-time point, leaving the scalar product invariant. Transformations with this property are given by the $SO(3,1)$ group with the little groups $SU(2), E^c(2)$ and $SU(1,1)$ where the group $E^c(2)$ is the central extended group of the Euclidian group $E(2)$. Electroweak interaction beyond the standard model (SM) is described by the transformation group $SU(2) \otimes E^c(2)$ without a priori introduction of a phenomenologically determined gauge group. The Laplacian on this group yields the known internal quantum numbers of isospin and hypercharge, but in addition the extra $E^c$-charge $\kappa$ and the family quantum number $n$ which explains the existence of families in the SM. The connection coefficients deliver the SM gauge potentials but also hypothetical gauge bosons and other hypothetical particles as well as candidate Dark Matter particles are predicted. It is shown that the interpretation of the $SO(3,1)$ connection coefficients as electroweak gauge potentials is compatible with teleparallel gauge gravity theory based on the translational group.

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1. Introduction

The formal equivalence of gauge theories with the geometry of fiber bundles has been recognized since the 1960s [1-5]. In the fiber bundle formalism, gauge potentials are understood as a geometrical entity - the connections on the principal bundles, and matter fields are described by associated fiber bundles. The geometrical interpretation of gauge theories by the mathematical fiber bundle theory is a beautiful and mathematically profound concept. However in earlier investigations the transformation groups of the fibers were taken from the phenomenologically determined internal gauge groups of the Standard Model (SM). Therefore, up to now the fiber bundle interpretation delivers mainly a re-interpretation of the gauge fields and it turned out to be mainly an issue for mathematical physics and less for elementary particle theory.

In this paper we consider as a general hypothesis that the fundamental physical interactions can be described within the geometrical structure of the most fundamental fiber bundle - the tangent bundle, and gauge transformations can be identified with transformations at a fixed spacetime point along the tangent vector axis. This means the gauge group is not assumed by phenomenological reasons but arises self-consistently from the invariance of the scalar product with respect to tangent fiber transformations described by the group $SO(3,1)$. Since the action of this group is not transitive, the vector space decomposes into different orbits and the most general (projective) irreducible representations of $SO(3,1)$ can be constructed by the little groups $SU(2), E^c(2)$ and $SU(1,1)$ where the group $E^c(2)$ is the central extended Euclidian group. Based on differential geometry on the tangent bundle with covariant derivatives determined by the generators of the transformation group $G = SU(2) \otimes E^c(2)$ and corresponding connection coefficients (gauge potentials) a generalized theory of the electroweak interaction is derived. In addition to the internal quantum numbers (IQN) of isospin and hypercharge, the $E^c$-charge $\kappa$ and the family quantum number $n$ arise which could elucidate the existence of families in the SM. In this approach the known $Z$ and $W^{\pm}$ gauge bosons can be found again but in addition new extra $E^c$ and $B^\pm$ gauge bosons and other hypothetical particles are predicted. A notable feature of the theory presented is the possibility of identifying candidate stable or unstable hypothetical Dark Matter (DM) vector bosons, DM scalars and DM fermions with zero hypercharge and zero isospin but non-zero $E^c$-charge $\kappa \neq 0$. Here we present only the basics, specific in-depth observable consequences are beyond the present paper. Note that the more general transformation group $SO(3,1) \rtimes T(3,1)$ (where $T(3,1)$ is the translational group and $\rtimes$ represents the semi-direct product) includes teleparallel gravity into the tangent bundle geometry based on translational transformations $T(3,1)$ of the tangent fibers. According to this approach the interpretation of the $SO(3,1)$ connection coefficients as electroweak gauge potentials is compatible with teleparallel gauge gravity theory which is fully equivalent to Einstein’s general relativity theory.

2. Differential geometry on the tangent bundle

At the beginning we start with a brief description of the geometry of the tangent bundle on a manifold (see e.g. [6]). The tangent space $T_x(M)$ at the point $x$ on the space-time manifold $M$ is the set of all tangent vectors spanned by frame vectors in the coordinate basis $e_\mu = \partial_\mu \ (\mu = 0,1,2,3)$. The tangent bundle is the union of all tangent spaces at all points $x$ of the manifold $M : TM = \bigcup_{x \in M} T_x(M)$. In coordinate description a point in $TM$ is described by the numbers of pairs...
$u = (x, v)$ with $x = \{x^0, x^1, x^2, x^3\}$ as the coordinates of the spacetime manifold and $v = \{v^0, v^1, v^2, v^3\}$ are the coordinates of the tangent vectors. Thus the tangent fiber bundle geometry introduces four additional variables $v$ for the description of the tangent fiber. To aid understanding, it is convenient to consider $M$ as a Pseudo-Riemannian spacetime manifold with indefinite metric $g^{\mu\nu}(x)$. Besides the frame vectors in the coordinate basis $e_\mu = \partial_\mu$ ($\mu = 0, 1, 2, 3$), one can introduce the tetrads as another geometric object on the tangent space:

$$ e_a = e_a^\mu(x)\partial_\mu. \quad (1) $$

Each vector described in the coordinate basis $e_\mu = \partial_\mu$ can be expressed by a vector with respect to the tetrad frame basis $e_a$ according to the rule

$$ v^\nu = e^\nu_a(x)e^a. \quad (2) $$

The subscript $a, b, ..$ numbers the vectors ($a, b = 0, 1, 2, 3$) and $\mu$ their components in the coordinate basis. The dual basis of the frame fields $e_a$ are cotangent frame 1-forms $e^\mu_a = e^\mu_a dx^\mu$ satisfying the orthogonality relation $e^\mu_a(x)e^\nu_b(x) = \delta^\mu_b$. By using the tetrads of the pseudo-Riemannian manifold, the scalar product of two vectors gets the form of a Lorentzian manifold:

$$ (v, u) = g_{\mu\nu}(x)v^\mu u^\nu = g_{\mu\nu}(x)e^\mu_a(x)\times e^\nu_b(x)v^a u^b = \eta_{ab}v^a u^b, \quad (3) $$

where $\eta_{ab} = diag(-1, 1, 1, 1)$ is the metric of the Minkowski space.

The geometric properties of manifolds are usually related to the invariance of certain geometrical structure relations under the action of certain transformation groups. The definition of the scalar product (3) is the governing structure relation defining the geometry of the tangent bundle. Tangent vectors manifest two kinds of transformations which do not change the scalar product in (3). Under general coordinate transformations of the spacetime manifold $x^\alpha \rightarrow y^\alpha = y^\alpha(x)$ vectors transform as $v^\alpha(x) = (\partial y^\alpha/\partial x^\nu)v^\nu(x)$. On the other hand, the vector components in the tetrad frame basis remain unchanged: $v^a(x) = v^a$. A second type of transformations exists that does not change the scalar product. These are transformations at a fixed point $x$ of the spacetime manifold $\mathcal{M}$ transforming the tangent vectors along the tangent fiber directions as follows:

$$ v^a = T^a_b(x)v^b, \quad e^\alpha_a = (T^a_b(x))e^\alpha_b, \quad (4a) $$

$$ e^\mu_b(x) = (T^a_b(x))^{-1}e^\mu_a(x), \quad (4b) $$

where $T^a_b(x)$ are matrices satisfying the conditions $\eta_{\alpha\beta}T^\alpha_cT^\beta_d = \eta_{cd}$. On the other hand, the tangent vectors which refer to the coordinate frame remain unchanged: $v^\mu = v^\mu$. The transformation group of tangent vectors along the tangent fiber is the $SO(3, 1)$ group of special linear transformations, with matrix elements $T^a_b(x) \in SO(3, 1)$ depending on the spacetime point $x$ as a parameter.

Note that the transformation of the tangent vectors by the group $SO(3, 1)$ is not the most general transformation. Actually, the fact that the group $SO(3, 1)$ leaves the scalar product of tangent vectors invariant is not sufficient because we need the infinitesimal tangent vector line elements to be invariant.

$$ (dv, du) = g_{\mu\nu}(x)dv^\mu du^\nu = \eta_{ab}dv^a du^b. \quad (5) $$

This allows us to add constant translations to the transformations in (4):

$$ v^a = T^a_b(x)v^b + a^a(x), \quad (6) $$

and leads to the more general transformation group $SO(3, 1) \times T(3, 1)$.

Poincare transformations and the transformation group (6) of tangent vectors $T^a_b(x)$ in the tetrad basis are described by the same group $SO(3, 1) \times T(3, 1)$ but both have principal different geometrical and physical meaning: the first transforms the coordinates of a flat spacetime manifold while the second describes transformations within the tangent fiber $F = T_x(M)$ leaving the spacetime point $x$ unchanged.

### 3. Connections on the Tangent Bundle and Teleparallel Gauge Gravity Theory

The geometric construction of tetrads is closely linked to the conceptional basis of gravity theories and its extensions to gravity gauge theories. To facilitate a proper understanding of the underlying geometric structure and a unified description including gravity we first consider the general inhomogenous transformation group $SO(3, 1) \times T(3, 1)$ and its relationship with gravity. Differential geometry on the tangent bundle can be obtained using the general rules for principal fibre bundles $P(M; G)$ requiring the definition of connections and covariant derivatives on the bundle. The definition of a covariant derivative demands to consider vectors which point from one fiber to the other at different points $x$ and $x'$ of the spacetime manifold. The generators $L_a$ of the group $G$ are vertical vectors pointing along the fiber and therefore belong to the vertical subspace $V_a(P)$. Vectors which point away from the fiber, i.e. elements of the tangent space of the fiber bundle $U_a(P)$ that complement the vertical vectors in $V_a(P)$ can be constructed by the definition of a connection as an assignment to each point in the principle fiber $\pi: \mathcal{X} \rightarrow \mathbb{X}$ such that

$$ U_a(P) = H_a(P) \oplus V_a(P). \quad (7) $$

The definition of a connection can be used for the definition of a covariant differentiation along the curves horizontally lifted to the principal bundle:

$$ \frac{d}{d\tau} = \frac{dx^\mu}{d\tau}D_\mu, \quad (8) $$
where
\[ D_\mu = \frac{\partial}{\partial x^\mu} + i \tilde{A}_\mu^a L_a, \] (9)
is the covariant derivative on the principal fiber bundle. \( L_a \) are the right-invariant fundamental vector fields (generators) on the group manifold \( G = \{g_{ij}\} \) and \( \tilde{A}_\mu^a \) the connection coefficient of the group \( G \). A connection on a principal bundle induces a connection on the associated bundle. The covariant derivative on the associated bundle is given by (9) substituting the generators \( L_a \) by the left-invariant fundamental vector fields on the section of the associated bundle which describe matter fields.

The geometric transformations of tangent vectors in a tangent bundle are described by the group \( G = SO(3,1) \times T(3,1) \). According to (9) the covariant derivative along the horizontal lifted curve on the principal bundle \( P(M;G) \) of this group is given by
\[ D_\mu = \frac{\partial}{\partial x^\mu} + i \omega_a^\mu P_a + \frac{i}{2} \Omega_{\mu}^{ab} M_{ab}, \] (10)
where \( M_{ab} \) are related with the 6 generators of the group \( SO(3,1) \) with \( J_a = e_{abc} M_{bc} \) and \( K_a = -M_{aa} \) are the generators of the translational group \( T(3,1) \). Here \( \omega_a^\mu \) and \( \Omega_{\mu}^{ab} \) are connection 1-forms of the \( T(3,1) \) and \( SO(3,1) \) group, respectively. The total field strength tensor can be defined as
\[ F_{\mu \nu} = [D_\mu, D_\nu] = T_\mu^a P_a + \frac{1}{2} R_{\mu \nu}^{ab} M_{ab}, \] (11)
with the torsion tensor
\[ T_{\nu \mu} = \partial_\nu \omega_\mu^a - \partial_\mu \omega_\nu^a + (\Omega_\nu^a \omega_\mu^b - \Omega_\mu^a \omega_\nu^b), \] (12)
and the curvature tensor
\[ R_{\nu \mu \rho}^a = \partial_\nu \Omega_\mu^a - \partial_\mu \Omega_\nu^a + (\Omega_\nu^b \omega_\mu^a - \Omega_\mu^b \omega_\nu^a). \] (13)
Through contraction with tetrads, tensors can be transformed to spacetime indexed forms as e.g., \( u^\mu = \epsilon^a_\mu v^a \), \( R^a_\nu T_{\rho \lambda} = \epsilon^a_\nu \epsilon^\rho_\lambda R^a_{\rho \lambda} \) and the lower frame index \( v_a \) can be raised by the Lorentz metric \( \mu = g^{\mu \nu} v_\nu \). The connections \( \omega_\mu^a \) and \( \Omega_{\mu}^{ab} \) are fundamental structure functions characterizing the specific tangent bundle.

Parallel transport of a tangent vector \( v^a \) from a point \( x \) in the spacetime manifold to a neighboring point \( x' \) is defined by the covariant derivative
\[ D_\mu v^a = \partial_\mu v^a + \Omega_\mu^{ab} v^b, \] (14)
where \( \Omega_\mu^{ab} \) is denoted as frame connection. On the other hand the covariant derivative of vectors which refer to the coordinate basis can be written as
\[ D_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu \rho} v^\rho, \] (15)
with the coordinate connection \( \Gamma^\nu_{\mu \rho} \). The coordinate connection \( \Gamma^a_{\mu \lambda} \) is connected with the frame connection \( \Omega_\mu^{ab} \) by requiring \( D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma_\nu^a \omega_{\mu b}^c + \Omega_\mu^{ab} e_c^b = 0 \) from which the following relation can be derived
\[ \Omega_\mu^{ab} = e_c^a \partial_\mu e_c^b + e_c^b \partial_\mu e_c^a. \] (16)
For the description of gravity tetrads \( e^a_\mu \) and a specific coordinate connection \( \Omega_\mu^{a} \) has to be defined. For a manifold with vanishing metricity described by the condition \( D_\lambda g_{\mu \nu} = \partial_\lambda g_{\mu \nu} - \Gamma^a_{\mu \lambda} g_{ab} - \Gamma^b_{\mu \lambda} g_{ba} = 0 \) one gets
\[ \Gamma^a_{\mu \nu} = \tilde{\Gamma}^a_{\mu \nu} + K^a_{\mu \nu}, \] (17)
where \( \tilde{\Gamma}^a_{\mu \nu} \) is the Levi-Civita connection and \( K^a_{\mu \nu} \) is the contortion tensor
\[ K^a_{\mu \nu} = \frac{1}{2} (T^\nu_{\mu a} - T^\mu_{\nu a} - T^a_{\nu \mu}), \] (18)
with the torsion tensor \( T^a_{\nu \mu} = \Gamma^a_{\nu \rho} - \Gamma^a_{\mu \rho} \). The underlying geometric structure with respect to translational and rotational transformations (6) of tangent vectors leads to a Riemann-Cartan spacetime endowed with frame connections \( \omega_\mu^a \) and \( \Omega_\mu^{ab} \), non-vanishing curvature \( R^a_{\rho \lambda \mu} \), non-vanishing torsion \( T^a_{\nu \mu} \) but vanishing metricity. The special case of a Riemann spacetime and the General Theory of Relativity (GTR) can be obtained from the above formulas by setting the torsion tensor to be identically vanishing. The coordinate connection \( \Gamma^a_{\nu \mu} \) then is given by the Levi-Civita connection \( \tilde{\Gamma}^a_{\nu \mu} \) related with the metric tensor \( g_{\mu \nu} \). The frame connection \( \Omega^a_{\nu \mu} \) in this case is usually denoted as spin connection \( \Omega^a_{\nu \mu} \) related with the Levi-Civita connection \( \tilde{\Gamma}^a_{\nu \mu} \) by the equation (16). On the other hand based on the symmetry with respect of translational transformations of tangent vectors a gauge theory of gravity has been developed by analogy with internal symmetries [9,12]. This gauge gravity theory corresponds to the teleparallel gravity theory, which is an alternative but equivalent formulation of the GTR describing the very same gravitational field. In teleparallel gravity, the torsion is non-vanishing, acting as a gravitational force while the curvature \( R^a_{\rho \lambda \mu} \) vanishes identically. The translational connections \( \omega_\mu^a \) and the tetrad coframe \( e^a_\mu \) turn out to be conceptional distinct entities [13], since it does not transform inhomogeneous with a gradient term under gauge transformation. The coframes depend on the translational connection in the form \( e^a_\mu = \omega^a_\mu + D_\mu \xi^b \), \( D_\mu = \partial_\mu \delta_{\alpha b} + \Omega^a_{\alpha b} \) where \( \xi^a = \xi^a(x) \) is a coset vector. Locally at a given point \( x \) on the spacetime manifold one can transform \( D_\mu \xi^a \) into \( \delta_{\alpha b} \) (where \( \delta \) is the Kronecker symbol). Note that the coframe \( e^a_\mu \) induce a metric as an independent dynamically quantity. In the geometry of teleparallel gravity the coordinate connection \( \Gamma^a_{\nu \mu} \) takes the form of the Weizenb"{o}ck connection \( \Gamma^a_{\nu \mu \lambda} = W \Gamma^a_{\nu \mu \lambda}(x) \), defined as
\[ W \Gamma^a_{\nu \mu \lambda}(x) = e^a_\alpha(x) \partial_\nu e^\alpha_{\mu \lambda}(x). \] (19)
If we consider only gravitational effects with the choice of the Weizenböck connection in teleparallel gravity a vector is parallel transported if its projections on the tetrads is proportional, regardless the path connecting both tangent spaces. This can we see from (15) with the substitution of $T^{a}_{\mu\nu}(x)$ by $^{W}\Gamma^{a}_{\mu\nu}T^{a}_{\mu\nu}(x)$ which yields for the covariant derivative $D^{\nu}_{\mu} = \partial^{\nu}_{\mu} + \omega^{a}_{\mu\nu}  a^{\nu}$. In teleparallel gravity the connection coefficients $\Omega^{a}_{\mu\nu}$ represents a pure inertial effect [12] where $\Omega^{a}_{\mu\nu}$ is the spin connection in teleparallel gravity. There exist a class of inertial frames in which $\Omega^{a}_{\mu\nu}$ vanishes: $\Omega^{a}_{\mu\nu} = 0$.

The Weisenböck torsion can be used to build up the Lagrangian of teleparallel gravity with quadratic Weizenböck scalars [9,12], given by

$$L = -\frac{h}{16\pi} \left[ T^{abc}_{\mu\nu} T_{abc} + \frac{1}{2} T^{abc}_{\mu\nu} T_{bac} - T^{a}_{\mu\nu} T_{a}\right],$$

(20)

where $h = \det(\omega^{a}_{\mu})$ and $T_{b} = T^{a}_{b\mu}$. The Lagrangian is up to a divergence equivalent to the Lagrangian of Einstein’s GTR, this means both theories are simply alternative formulations for the description of gravity but are based on different principles.

A principal bundle $P(M; G)$ encodes the essential data of gauge transformations and the frame connection $\Omega^{a}_{\mu\nu}$ and $\omega^{a}_{\mu\nu}$ are additional structure functions that are attached to it and are in general independent defined on the existence of a metric (7,8). If the structure group $G = SO(3,1) \times T(3,1)$ is restricted to the translational subgroup $T(3,1)$ equ.(12) and (13) can be taken with $\Omega^{a}_{\mu\nu}$ put every where equal to zero. The torsion tensor is now determined by $T^{a}_{\mu\nu} = \partial^{a}_{\mu} \omega^{a}_{\mu\nu} - \partial^{a}_{\mu} \omega^{a}_{\mu\nu}$. However the structure group of the tangent bundle is the larger group $G = SO(3,1) \times T(3,1)$. This raise the question of the physical meaning of the other subgroup $SO(3,1)$.

If $\Omega^{a}_{\mu\nu}$ do not vanishes we find from (13) a non-Abelian field strength tensor which in teleparallel gauge gravity theory based on the translational symmetry is not related to gravity. In this paper the main hypothesis is elaborated that non-Abelian fields in electroweak interaction can be identified with the connection coefficients $\Omega^{a}_{\mu\nu}$ arising from transformations along the tangent fiber axis described by the group $SO(3,1)$. This interpretation differs from Poincare gravity gauge theory based on the localization of the Poincare group as gauge group [13] (for a review see [14]). In this theory both the translational part and the rotational part of the local Poincare group is related with gravity leading to a hypothetical generalized gravity theory, denoted as the Einstein-Cartan gravity theory.

4. The generators on the little groups

From now on we neglect gravity arising from the translational part $a^{\nu}(x)$ of the transformation of tangent vectors in (6). Since the action of $SO(3,1)$ on a tangent vector is not transitive, the vector space decomposes into different orbits with the little groups $SO(3), E(2)$ and $SO(1,2)$. The unitary representations $T_{L}(g)$ of the little groups $SO(3)$ and $E(2)$ are well known. The composition law of these so-called vector representations $T_{L}(g)$ satisfy the functional equation $T_{L}(g_{1})T_{L}(g_{2}) = T_{L}(g_{1}g_{2})$ and encodes the law of group transformations on the set of vector states. However it is known that this composite law is too restrictive and leads in special cases to certain pathologies, as e.g. the Dirac equation is not invariant under the Poincare group, but under its universal covering group. In quantum theory the physical symmetry of a group of transformations on a set of vector states has to preserve the transition probability between two vector states $|\Phi, T_{L}(g)|\Psi>|^{2} = |\Phi, \Psi>|^{2}$. Therefore as shown by Wigner [16] and systematically studied by Bargman [17] the problem of pathologies can be solved if the above given composite law is replaced by a weaker one: $T_{L}(g_{1})T_{L}(g_{2}) = \varepsilon(g_{1}, g_{2})T_{L}(g_{1}g_{2})$ where $\varepsilon(g_{1}, g_{2})$ is a complex-valued antisymmetric function of the group elements with $|\varepsilon(g_{1}, g_{2})| = 1$. Such representations are called projective representations. For the case of simply connected groups like the rotation group $SO(3)$ projective representations are obtained by replacing the group $SO(3)$ by its universal cover $SU(2)$.

However in the case of the Euclidean group $E(2)$ the covering group is not enough, one has to substitute this group by a larger group: the universal central extension $E^{(2)}$ which includes in addition to the group elements of $E(2)$ the group $U(1)$ of phases factors $\varepsilon(g_{1}, g_{2})$ with $|\varepsilon(g_{1}, g_{2})| = 1$.

In general a central extension $E^{(G)}$ of a group $G$ with elements $(g, \varsigma) \in G^{\ast}$ and $g \in G, \varsigma \in U(1)$ satisfy the group law [17]

$$(g, \varsigma) = (g_{1}, \varsigma_{1}) \ast (g_{2}, \varsigma_{2}) = ((g_{1} \ast g_{2}, \varsigma_{1}\varsigma_{2}) \exp[i\xi(g_{1}, g_{2})]),$$

(21)

where $\xi(g_{1}, g_{2})$ is the 2-cocycle satisfying the relation $\xi(g_{1}, g_{2}) + \xi(g_{1} \ast g_{2}, g_{3}) = \xi(g_{1}, g_{2} \ast g_{3}) + \xi(g_{2}, g_{3}), \xi(\epsilon, g) = 0$.

The 2-dimensional Euclidian group $E(2) = T(2) \times SO(2)$ with $E(2) = \{ (\alpha, a) \ | \ \alpha \in R_{\ast}(\mod 2\pi), a = (a^{1}, a^{2})^{\ast} = E^{2} \}$ is a semi-direct product of translations and rotations of the two-dimensional Euclidian plane. Since the invariant subgroup $SO(2)$ is abelian this group is not semi-simple and not compact and the unitary representations are infinite dimensional. The representations of the group $E(2)$ constructed on the space of functions $f(\xi)$ are well known. The action of the group $E(2)$ with $g = (\alpha, a) = (a_{1}, a_{2})$ on a vector $Z = (\varphi, \xi_{1}, \xi_{2})$ is given by:

$$(\alpha, a)[(\varphi, \xi_{1}, \xi_{2})] = (\xi_{1}\cos \alpha + \xi_{2}\sin \alpha + a^{1},$$

$$\xi_{1}\sin \alpha - \xi_{2}\cos \alpha + a^{2}).$$

(22)

The most general (projective) representations of $E(2)$ can not be obtained from its universal covering group but by the central extended group $E^{(2)}$. This group has been studied previously as e.g. in [17,19]. The central extension embodies in addition a $U(1)$ subgroup characterized.
by a complex parameter $\zeta = \exp(i\omega)$. $E^c(2)$ consists of elements $(\alpha, a, \omega)$ with $(\alpha, a) \in E(2), \omega \in \mathbb{R}$. The action of the group $E^c(2)$ on a vector $Z = (\xi_1, \xi_2, \beta)$ is described by $\mathbf{17}$ $\mathbf{19}$.

$$(\alpha, a, \omega)(\xi_1, \xi_2, \beta) = (\xi_1 \cos \alpha + \xi_2 \sin \alpha + a^1, -\xi_1 \sin \alpha + \xi_2 \cos \alpha + a^2, \beta + \omega + \frac{i}{2} m(\alpha, a, \varphi, \xi)), $$

where $m(\alpha, a, \varphi, \xi)$ is the two-cocycle which gives the desired central extension parametrized as

$$m(\alpha, a, \varphi, \xi) = (a_1 \xi_1 + a_2 \xi_2) \sin \alpha - (a_1 \xi_2 - a_2 \xi_1) \cos \alpha.$$  

By using of (23) and (24) we can find the infinitesimal transformations and the generators of the Lie algebra given by

$$T^1 = -i \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T^2 = -i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, $$

$$T^3 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E = -i \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$  

which satisfy the communication rules

$$[T^1, T^2] = iE, [T^1, T^3] = -iT^2, [T^2, T^3] = iT^1, [T^0, E] = 0.$$  

Of particular interest are the generators on the group $E^c(2)$:

$$T^1 = -i \left( \frac{\partial}{\partial \xi_1} + \frac{i}{2} \xi_2 \frac{\partial}{\partial \beta} \right), T^2 = -i \left( \frac{\partial}{\partial \xi_2} - \xi_1 \frac{1}{2} \frac{\partial}{\partial \beta} \right), T^3 = -i \left( \frac{\partial}{\partial \xi_1} - \frac{\partial}{\partial \xi_2} \right), E = -i \frac{\partial}{\partial \beta}.$$  

The group Laplacian (Casimir operator) is determined by

$$\Delta = (T^1)^2 + (T^2)^2 + 2T^3E.$$  

To derive a canonical basis we use the eigenfunction of $T^3$ and the Laplace operator $\Delta$ of $E^c(2)$. Using polar coordinates $\xi_1 = \xi \cos \phi, \xi_2 = \xi \sin \phi$ and $h_{nm\varphi}(\xi, \beta, \phi) = \exp(i\beta)(\exp(im\phi)g_{nm\varphi}(\xi)$ we find with the Laplacian (28) the following equation for $g_{nm\varphi}(\xi)$:

$$\left[\left(-\frac{1}{\xi} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} + \frac{1}{\xi} m^2\right) + \xi^2 \cdot \frac{\partial}{\partial \xi} - 2\xi m\right]g_{nm\varphi}(\xi) = \epsilon_{nm\varphi}g_{nm\varphi}(\xi).$$  

(29)

With the solutions of (29) we find for $h_{nm\varphi}(\xi, \beta, \phi)$

$$h_{nm\varphi}(\xi, \beta, \phi) = N_{nm\varphi} |\varphi|^{\frac{1}{2}} \exp(i\beta)(\exp(im\phi)$$

$$\exp\left(-\frac{1}{2} |\varphi|^2\right)\xi^{|m|}L_n^{|m|}(\varphi |\xi|^2),$$  

(30)

with $N_{nm\varphi} = \sqrt{\frac{2}{\pi}}(\frac{2}{|m|+1})^\frac{1}{2}, \epsilon_{nm\varphi} = 4\pi(n+\frac{1}{2}+\frac{1}{2})(m+|m|)$, with $n = 0, 1, 2, ..., m = 0, \pm 1, \pm 2, ..., \varphi = \pm 1, \pm 2, ..$

Here $L_n^{|m|}(x)$ are the associated Legendre polynomials.

The solutions $h_{nm\varphi}$ form an orthonormalized set and have the analog form like the solutions of the Schrödinger equation in the symmetric gauge for electrons in a constant magnetic field $\mathbf{20}$. The group $E^c(2)$ is isomorphic to the quantum harmonic oscillator group studied e.g. in $\mathbf{21}$.

The special eigenvalue $\varphi = 0$ plays a particular role. The solution of (29) for $\varphi = 0$ is given by

$$h_{0000}(\varphi) = \frac{1}{\sqrt{2\pi}} \exp(im\phi).$$  

(32)

As described later the generator $T_3$ corresponds to the hypercharge generator in the SM. It is convenient to introduce for $T_1$ and $T_2$ the generators $T^\pm = \frac{1}{\sqrt{2}}(T^1 \pm iT^2)$. The action of the operator $T^+$ on the eigenfunctions $g_{nm\varphi}$ increases the hypercharge from $m$ to $m+1$, but simultaneously generates states with different family numbers $n$: $T^+g_{nm\varphi} = \sum_i A_i g_{i,m+1,\varphi}$. Corresponding the generator $T^-$ reduce the hypercharge: $T^-g_{nm\varphi} = \sum_i B_i g_{i,m-1,\varphi}$. $A_i$ and $B_i$ are coefficients, respectively. The operator product $(T^- T_+ + T_+ T_-)$ on the eigenfunctions is given by

$$(T^- T_+ + T_+ T_-)g_{nm\varphi}(\xi, \phi, \beta) = q_B g_{nm\varphi}(\xi, \phi, \beta),$$  

(33)

with $q_B(n, m, \varphi) = 4\pi[n + \frac{1}{2}(1 + |m|)]$.

The generators $J_1, J_2, J_3$ on the group $SU(2)$ are well-known and, by using the Laplacian $\Delta = (J_1^2 + J_2^2 + J_3^2)$ on this group, all finite-dimensional representations can be found (see e.g. $\mathbf{22}$ $\mathbf{23}$). The application of the operators $J_\pm = 2^{-\frac{1}{2}}(J_1 \pm J_2)$ and $J_3$ on the eigenfunctions of the Laplacian $\Delta$ leads to

$$J_\pm f_{j_3} = \frac{1}{2} [j(j+1) - j_3(j_3 \pm 1)]^{1/2} f_{j_3 \pm 1},$$  

(34)
\[ J_{3j} f_{j}^i = f^3 f_{j}^i, \] (35)

where \( j = (-j_3, -j_3+1, \ldots, j_3) \) are the isospin quantum numbers, \( j_3 \) is the projection on the third isospin axis. Besides we find

\[ (J_+ J_- + J_- J_+) f_{j}^i = q_w f_{j}^i, \] (36)

with \( q_w = [j(j+1) - j_3^2] \). Using the parametrization \( z_1 = \cos \frac{\theta}{2} \exp[i(\psi - \varphi)/2] \), \( z_2 = i \sin \frac{\theta}{2} \exp[i(\psi + \varphi)] \) for \( j = 1/2, j_3 = 1/2 \) one finds \( f_{j}^1 = z_1 \) and for \( j = 1/2, j_3 = -1/2 \) : \( f_{j}^1 = z_2 \). For the isospin numbers \( j = 1, j_3 = 1 \) we have \( f_{j}^1 = (z_1^2)/\sqrt{2} \), for \( j = 0 \) we have \( f_{j}^0 = z_1 z_2^2 \) and for \( j = -1 \) one finds \( f_{j}^1 = (z_2^2)/\sqrt{2} \). The eigenfunctions for the general case for the isospin \( j \) are given by [23]

\[ f_{j3} = \left( \frac{1}{(j+j_3)! (j-j_3)!} \right)^{1/2} z_1^{j_3} z_2^{j-j_3}, \] (37)

5. Quantization on the tangent bundle with the transformation group \( SU(2) \otimes E^c(2) \)

According to the above described orbit decomposition the covariant derivative (10) with \( \omega^a_{\mu} = 0 \) can be rewritten for the group \( SU(2) \otimes E^c(2) \) as

\[ D_\mu = \frac{\partial}{\partial x_\mu} + ig_1 A^a_{\mu}(x) J_a + ig_2 \frac{1}{2} B^a_{\mu}(x) T_a + ig_3 C_\mu(x) E, \] (38)

where \( J_a, T_a \) and \( E \) are the operators on the group \( SU(2) \otimes E^c(2) \). \( A^a_{\mu}(x), B^a_{\mu}(x) \) and \( C_\mu(x) \) are the frame connection coefficients (gauge potentials) depending on the point \( x \) of the spacetime manifold. The electromagnetic field should not couple to the neutrino, and the Lagrangian of scalar neutral particles \( L = (D_\mu \Phi S) (D^\nu \Phi S) \) (with \( Q = j_3 + m/2 = 0 \)) should be diagonalized. Therefore the fields \( A^a_3, B_3 \) and \( C_\mu \) have to be transformed to new fields expressed by the relations

\[ B_\mu = \cos \theta_W A_\mu - \sin \theta_W (\cos \theta_D Z_\mu - \sin \theta_D E^c_\mu), \]
\[ A^3_\mu = \sin \theta_W A_\mu + \cos \theta_W (\cos \theta_D Z_\mu - \sin \theta_D E^c_\mu), \]
\[ C^3_\mu = \sin \theta_D Z_\mu + \cos \theta_D E^c_\mu. \] (39)

Here \( \theta_W \) is the Weinberg angle, \( g_2 = g \sin \theta_W, g_1 = g \cos \theta_W, g = ((g_1)^2 + (g_2)^2)^{1/2}, e = g \cos \theta_W \tan \theta_D \) and the mixing angle \( \theta_D \) is defined by \( \tan 2 \theta_D = \frac{g_3 m \nu}{[(g_3)^2 - (\frac{\pi}{2} g)^2]} \). The covariant derivative can be rewritten as

\[ D_\mu = \frac{\partial}{\partial x_\mu} + i[W^+_{\mu} Q^- - W^-_{\mu} Q^+ + B^+_{\mu} Q^- - B^-_{\mu} Q^+] + Z_\mu Q^+ + A_\mu Q + E^c_\mu Q E], \] (40)

with \( W^\pm_\mu = 2^{-1/2}(W^\pm_1 \pm iW^\pm_2), B^\pm_\mu = 2^{-1/2}(B^1_\mu \pm iB^2_\mu) \). The following operators are introduced

\[ Q = e(J_3 + \frac{1}{2} T_3), \]
\[ Q_Z = g \cos \theta_D (\cos^2 \theta_W J_3 - \sin^2 \theta_W \frac{1}{2} T_3) \]
\[ + g_3 \sin \theta_D E, \]
\[ Q_{W\pm} = g_1 J^\pm, \]
\[ Q_{E\pm} = g (- \cos^2 \theta_W J_3 + \sin^2 \theta_W \frac{1}{2} T_3) \sin \theta_D \]
\[ + g_3 \cos \theta_D E. \] (41-44)

Note, that in standard gauge theory the matrix-valued generators on the Lie algebra are used instead of the generators on the group. For compact groups with finite element representations, both definitions lead to equivalent results.

The gauge field strength tensors can be obtained from the commutators \([ D_\mu, D_\nu ]\) which yields for the neutral fields \( A_{\mu \nu}, Z_{\mu \nu} \) and for the hypothetical gauge field \( E^c_{\mu \nu} \) the following relations

\[ A_{\mu \nu} = \frac{\partial}{\partial x_\mu} A_\nu - \frac{\partial}{\partial x_\nu} A_\mu + i e (W^+_{\mu \nu} W^- - W^-_{\mu \nu} W^+), \]
\[ Z_{\mu \nu} = \frac{\partial}{\partial x_\mu} Z_\nu - \frac{\partial}{\partial x_\nu} Z_\mu + \frac{i g \theta_D g_3^2}{g} (W^+_{\mu \nu} W^- - W^-_{\mu \nu} W^+) \]
\[ + \frac{i g_3^2 \cos \theta_D}{g_3} (B^+_\mu B^-_\nu - B^-_\mu B^+_\nu), \]
\[ E^c_{\mu \nu} = \frac{\partial}{\partial x_\mu} E^c_\nu - \frac{\partial}{\partial x_\nu} E^c_\mu - \frac{i g_2^2}{g} \sin \theta_D (W^+_{\mu \nu} W^- - W^-_{\mu \nu} W^+) \]
\[ + \frac{g_3^2 \cos \theta_D}{g_3} (B^+_\mu B^-_\nu - B^-_\mu B^+_\nu). \] (45-47)

For the \( W^\pm \) bosons carrying the iso-spin number \( j_3 = \pm 1 \) we get

\[ W^\pm_{\mu \nu} = \frac{\partial}{\partial x_\mu} W^\pm - \frac{\partial}{\partial x_\nu} W^\pm \pm i e (W^+_\mu A_\nu - W^-_\mu A_\nu) \]
\[ \pm i \cos \theta_D g \cos^2 \theta_W (W^+_{\mu} Z_\nu - W^-_{\mu} Z_\nu) \]
\[ \pm g \cos^2 \theta_W \sin \theta_D (W^+_{\mu} E^c_\nu - W^-_{\mu} E^c_\nu). \] (48)

Besides, additional hypothetical gauge bosons \( B^\pm \) with hypercharge \( m = \pm 1 \) are predicted with the gauge field strength

\[ B^\pm_{\mu \nu} = \frac{\partial}{\partial x_\mu} B^\pm - \frac{\partial}{\partial x_\nu} B^\pm \pm i e (B^+_\mu A_\nu - B^-_\mu A_\nu) \]
\[ \pm i \cos \theta_D g \sin^2 \theta_W (B^+_\mu Z_\nu - B^-_\mu Z_\nu) \]
\[ \pm i g \sin^2 \theta_W \sin \theta_D (B^+_\mu E^c_\nu - B^-_\mu E^c_\nu). \] (49)

Any scalar function \( \Phi(x, z) \) defined on the fiber bundle can be expanded into the form \( \Phi = \sum_M (\phi_M(x) \chi_M(u) + \sum_m \phi_m(x) \chi_m(u) ). \)
\( \phi_I^f(x) \chi_M^*(u) \) with the IQNs \( M = (n, m, \kappa, j, j_3) \) depending on the coordinates of the spacetime manifold \( x \) and the eigenfunctions \( \chi_M(u) = h_{mn}(\xi, \phi, \beta) f_{j j_3}(s_1, s_2) \) of the Laplacian of the group \( SU(2) \otimes E^c(2) \) defined by (30) and (37).

In the quantum-field theoretical approach one-particle states of the fermion Dirac field \( \Psi_f(x, u) = \sum_{M,s}(\psi_M^f(x)\chi_M^*(u) + \psi_M^f(x)\chi_M^*(u)) \) labeled by the three-momentum \( \mathbf{p} \) are described on the tangent bundle by

\[
\Psi_f(x, u) = \sum_K \frac{1}{\sqrt{2 E_M^f V}} \left[ a_K^f u_M^f(p) \chi_M^f(u) \exp(i \mathbf{p} \mathbf{x}) + b_K^f u_M^f(p) \chi_M^f(u) \exp(-i \mathbf{p} \mathbf{x}) \right],
\]

where the index \( s \) characterizes the helicity \( s = \{L, R\} \) and \( K = \{M, \mathbf{p}, s\} \). \( a_K^f(t) \) is the annihilation operator for a particle in the interaction representation, \( b_K^f(t) \), the antiparticle creation operator satisfying the anticommutation rules. \( E_M^f \) is the single particle energy and \( V \) the volume. \( u_M^f(p) \) and \( v_M^f(p) \) are the plane wave solutions of the Dirac equation for particles and antiparticles, respectively. The general construction of states in the TB suggests that not only leptons but also scalars and gauge bosons carry the IQN \( M_A = \{n, m, \kappa, j, j_3\} \). This means scalar fields and gauge fields can be expanded in the analog form as (50). For a compact representation we introduce for the gauge particles the notation: \( g = (g^0, g^c) \) with \( g^0 = (A, Z, E^c) \) and \( g^c = (W^\pm, B^\pm) \) and for the gauge potentials \( A_{g0} = A_{g0}^+ + A_{g0}^- \) with \( A_{g0}^\pm = a_{g0}^\pm(x) \chi_{M_0}^\pm(u) \). The general structure of the theory based on the TB suggest an identification of an elementary particle as a state with specific internal quantum numbers \( M \) analogous as in quantum mechanics of atoms discrete quantum states with different quantum numbers and energy levels exist. Therefore we do not fix the particle content from the beginning but let the existence of "exotic" particles open which do not appear in the SM and are not observed so far. The potential observation of such particles depends on its parameters as mass and lifetime, but also on selection rules as discussed later.

The dependence of the quantized field operators (50) on the eigenfunctions \( \chi_M(u) \) and on the coordinates \( u \) of the tangent vectors is a specific trait of the here presented approach based on the underlying geometrical structure of the TB. The internal symmetries arise here from the inherent geometrical symmetries of the TB in an analogous way as symmetries in quantum mechanics originate from spacetime symmetries in a given physical system. This differ in a principle way from standard QFT, therefore we denote the here presented theory as Tangent Bundle Quantum Field Theory (TB-QFT).

The total Lagrangian can be presented by

\[
L = L_f + L_S + L_g + L_{Yuk}.
\]

Here the Lagrangian of fermions \( L_f \) is defined in the chiral representation as

\[
L_f = i \sum_{s, g} \left[ (\bar{\Psi}_f^{1,f}(x, u)\sigma^\mu_\nu D_{\mu} \bar{\Psi}_f^{1,f}(x, u)) \Psi_f^s(x, u) \right],
\]

with \( s = \{L, R\} \) and \( \sigma^\mu_\nu = (\sigma^0, \sigma^c, \sigma^\mu, \sigma^c, \sigma^\mu, \sigma^c) \) and \( \Psi_f^s(x, u) = \sum_{M,s}(\psi_M^f(x)\chi_M^*(u) + \psi_M^f(x)\chi_M^*(u)) \). The Lagrangian of scalar fields \( \Phi^s \) becomes

\[
L_S = \frac{\partial}{\partial x^\mu}\bar{\Phi}_S^s \frac{\partial}{\partial x^\mu}\Phi_S^s + \sum_{g} A_{g0}^\mu A_{g0}^\nu (\bar{\Phi}_S^s(\Xi_g)\Phi_S^s - V(\Phi_S^s)),
\]

with \( \Xi_g = Q^g_{sR} \) and \( \Xi_g = Q^g_{sL} + Q^g_{sL} \) and charge operators as given in (41)-(44). In (53) the SM self-interaction potential \( V(\Phi_S^s) = -\mu^2 | \Phi_S^s |^2 + \lambda | \Phi_S^s |^4 \) is included. Besides \( L_{Yuk} \) denotes the SM Yukawa interaction term given by

\[
L_{Yuk} = -\sum_{M,N} \kappa_{M,N} s_{M} \bar{\Phi}_S^s \Phi_S^s + h c,
\]

with \( s = R \) and \( s' = L \). The self-interaction term \( V(\Phi_S^s) \) and \( L_{Yuk} \) do not arise in the tree level approximation but are included by phenomenological reasons in the same way as in the SM. A microscopic foundation of these phenomenological terms is an unsolved problem in the SM as well as in the here presented approach. With (53) we find for the SM Higgs particle \( \Phi_H \) with \( m = 1, j_3 = -1/2 \) and assuming \( \kappa = 0 \) the parameters \( \theta_0 = 0, q_B = 0 \) and \( Q_{E^c} = 0 \). Spontaneous symmetry breaking generates a vacuum expectation value \( < \Phi_S^s(\Xi) > = \Phi_0^s \) and leads to the generation of masses for the \( Z \) and \( W^\pm \) bosons identical with their values in the SM \( M_Z = 83 \text{ GeV} / \sqrt{2} \) and \( M_{W^\pm} = 80 \text{ GeV} / \sqrt{2} \) but vanishing masses for the extra gauge bosons \( E^c \) and \( B^\pm \). If the SM Higgs carry the \( E^c \) charge \( \kappa = 1 \) the \( B^\pm \) boson gets a mass \( M_{B^\pm} = \Phi_0 g_2 2 \sqrt{2} \simeq 227.5 \text{ GeV} \) (using \( \Phi_0 = 180 \text{ GeV} \) and \( n = 0 \)). The mass of the \( E^c \) boson is \( M_{E^c} \simeq \Phi_0 6 \sqrt{2} g_3 \) for \( g_3 \ll g \) which is much smaller than \( M_{B^\pm} \). However this does not exclude the possible existence of other mass generation mechanism for these extra gauge bosons. Note that with \( \kappa = 1 \) analogous as lepton families also Higgs families should exist.

The Lagrangian \( L_g \) of the gauge bosons is given by

\[
L_g = -\frac{1}{4} \left[ (A_{\mu\nu} A_{\mu\nu}^\ast + Z_{\mu\nu} Z_{\mu\nu}^\ast + E_{\mu\nu} E_{\mu\nu}^\ast + W_{\mu\nu}^\pm W_{\mu\nu}^\pm + B_{\mu\nu}^\pm B_{\mu\nu}^\pm + 2 W_{\mu\nu}^\pm W_{-\mu\nu}^\pm + 2 B_{\mu\nu}^\pm B_{-\mu\nu}^\pm \right],
\]

with field strength tensors as defined in (45)-(49).

Inserting the expansion (50) into the fermion Laplacian (52) the Hamiltonian is easy to build with an interaction.
term with gauge particles $g$. For the unperturbed fermion Hamiltonian we get
\[ H_0 = \frac{1}{2} \sum_p \varepsilon_p [a_p^f a_p^f - b_p^f b_p^f], \]  
(56)
with $P = \{M, p, s\}$ and with the one-particle energy $\varepsilon_p = |p|$. For the interaction Hamiltonian $H_I$ one gets
\[ H_I = \sum_{PRK} \left[ (c^g_P V^g_{PRK} + c^g_K V^g_{PRK}) a^f_{PRK} a^f_{PRK} + a^g_{PRK} V^g_{PRK} \right] + b^{g*}_{PRK} V^{g*}_{PRK} a^f_{PRK} a^f_{PRK} + (c^g_P U^{g*}_{PRK} + a^g_{PRK} U^{g*}_{PRK}) a^f_{PRK} \]
\[ + b^{g*}_{PRK} U^{g*}_{PRK} b^f_{PRK} b^f_{PRK}, \]  
(57)
with $R = \{N, r, s\}$, $K = \{M_g, k, \lambda\}$, $\tilde{K} = \{M_g, -k, \lambda\}$ and
\[ V^g_{PRK} = \frac{u^g_P \sigma^\mu_{PRK} \epsilon^*_{PRK} (k)}{2 \sqrt{2 E_p E_r E_k^* V}}, \]  
(58)
\[ U^g_{PRK} = \frac{\bar{u}^g_P \sigma^\mu_{PRK} \epsilon^*_{PRK} (k)}{2 \sqrt{2 E_p E_r E_k^* V}}. \]  
(59)
Here we introduced the matrix elements
\[ I_{M,N}^{f} = \int d\mu \chi^g_{M} (u) \chi^*_{N} (u) Q_{g}^{f} \]  
(60)
with the integration measure $d\mu (u) = d\mu_{SU(2)} d\mu_{E^c} \mu_{SU(2)} = (16\pi^2)^{-1} \sin \theta d\phi d\psi / \sin \phi$ and $d\mu_{E^c} = (4\pi^2)^{-1} \sin \theta d\phi d\beta$. The compact representation (48) includes both electromagnetic interaction with gauge boson $g = A$, weak interaction with $g = Z$ and $g = W^\pm$ as well as interaction with the gauge boson $g = E^c$ with the charge operator $Q_{E^c}$. In addition to isospin $I$ and hypercharge $m$ the $E^c$-charge $\chi$ and the family quantum number $n$ exist. In difference to the SM the three lepton families are differentiated not only by mass, but now it find a deeper explanation by the additional family quantum number $n$ for a non-zero $E^c$-charge $\chi$. From the matrix elements (60) selection rules can be derived for the interaction of a fermion with a certain IQN with an another via an intermediate gauge boson. If the value of the integral $I_{M,N}^{f}$ is zero the interaction is forbidden. These selection rules arise in a similar way as the selection rules in atomic systems if the transition moment integral is vanishing and constrains the possible transitions of a system from one quantum state to another. The operators $Q, Q_Z, Q_{W^\pm}$ and $Q_{E^c}$ do not change the family number $n$. From (60) we see that leptons couple only to gauge boson states with $\chi = 0$ and therefore we find for the matrix element $I_{M,N}^{f} = 1$. This means family universal electroweak coupling to leptons with gauge bosons exist also in this extension of the SM for the electroweak interaction and the interaction via the $E^c$-boson. The operator $Q_{B^\pm}$ shift the hypercharge by one units. Since right and left-handed leptons carry different isospin components $j_3$ the matrix element (60) vanishes for $g = B^\pm$, this means that despite that $B^\pm$ bosons carry hypercharge its coupling of leptons is not allowed. Note that a violation of lepton universality could arise via the interaction with $E^c$ bosons if different lepton families carry different $E^c$ charges $\chi$ as e.g. $\chi = +1$ and $\chi = -1$.

The interaction Hamiltonian of scalar particles in the unitary gauge is described by
\[ H_I^S = - \sum_{PQKR} \left[ (c^S_P \sigma^\mu_{PQ} + c^S_K \sigma^\mu_{QR}) M_{PQKR}^g \right] \]
\[ + (a^{g*}_K c^S_{PR} + b^{g*}_K c^S_{KR}) M_{PQKR}^g. \]  
(61)
Here we introduced the symbols $P = \{M_S, p, \tilde{P} = \{M_S, -p\}, Q = \{M_S, q\}, \tilde{Q} = \{M_S, -q\}, K = \{M_g, k, \lambda\}, \tilde{K} = \{M_g, -k, \lambda\}, R = \{M_g, r, \sigma\}, \tilde{R} = \{M_g, -r, \sigma\}$ and the matrix elements
\[ M_{PQKR}^g = \frac{1}{8} \frac{I_{M_S, M_g}^{f} \lambda \delta (p - \mathbf{q} + \mathbf{k} - r)}{V \sqrt{E_p E_q E_k^*}}, \]  
(62)
with $q_A = Q_A^2, q_2 = Q_2^2, q_{E^c} = Q_{E^c}^2, Q_A = e(j_3 + \frac{1}{2} m), Q_Z = g \cos \theta_D (\cos \omega j_3 - \sin \omega \theta_D) + g_3 \sin \theta_D, q_B = 4\chi |n + \frac{1}{2} (1 + | m |)|, q_W = | j(j + 1) - j_3^2 |, Q_{E^c} = (-g \cos \omega j_3 + g \sin \omega \theta_D) \sin \theta_D + g_3 \cos \theta_D$ and
\[ I_{M_S, M_g}^{f} = \int d\mu (u) \chi^{S^*}_{M_S} (u) \chi^{S}_{M_g} (u) \chi^{g*}_{M_S} (u) \chi^{g}_{M_g} (u). \]  
(63)
In contrast to the coupling to fermions the coupling of scalar particles to $B^\pm$ bosons is allowed. If exotic gauge bosons with non-zero $E^c$-charge $\chi \neq 0$ exist they can also couple to scalar particles, but they do not couple to fermions.

The possible existence of $E^c$-bosons leads to a fundamental fifth interaction. The parameter space of $E^c$ mass and mixing angle $\theta_D$ or the coupling coefficient $g_3$ is constrained by existing data from lepton anomalous magnetic momenta, beam dump experiments and others, and can be found analog to that for $U(1)$ extended models (see. e.g. [24, 25]). Much of these data hints to the assumption that the coupling coefficient $g_3$ and the mixing angle $\theta_D$ are very small with $g_3 \ll 10^{-3}, \theta_D \approx g_3 / q < 10^{-3}$. This means that the fifth fundamental interaction mediated by the $E^c$-boson is much weaker than the SM weak interaction. As seen in (57)-(63) the $E^c$-charge $\chi$ of leptons or scalars also affect the weak interaction with the SM Z bosons, but its influence is highly suppressed since $g_3 \theta_D \ll 10^{-6}$. The
very small coupling between SM particles and the extra gauge boson is the reason that their effects are too feeble to be observed so far.

Note that the existence of new vector bosons is a common feature of many extensions of the SM (for a review see [27]). In particular models with an extra gauge group $U(1)$ are studied in large number of papers (see e.g. [27-30]).

6. Dark Matter candidates

Astrophysical and cosmological observations show that the largest part of matter in our universe is constituted by unknown non-luminous particles denoted as Dark Matter (DM) that have a very weak interaction with the visible sector of the universe. Such particles do not exist in the SM, but there are many attempts for an extension of the SM with possible DM candidates such as Weakly Interacting Massive Particles (WIMPs), sterile neutrinos, the lightest neutralinos in super-symmetric models, axions (see e.g. [31, 32]) or in the Dark Sector Model [26]. One of the most notable feature of the generalization of the SM by the gauge group $SU(2) \otimes E^c(2)$ is the possibility that dark matter candidates lie within the new gauge sector. An obvious way for the assignment of the IQNs to left- and right-handed Dark Fermions and Dark Scalars can be made by the choice of zero hypercharge ($m = 0$) and isospin ($j = 0$) but non-zero $E^c$-charge $\kappa \neq 0$. As a result one can expect that similar as SM leptons DM fermions and DM scalars are grouped in families with the IQN $n = 1, 2, 3$. According to (30) the eigenfunctions $\chi_M$ of the Laplacian on the group $SU(2) \otimes E^c(2)$ with $j = m = 0$ take the form

$$\chi_M^D = h_{00\kappa}(\xi, \phi) = \sqrt{\frac{\kappa}{\pi}} \exp(i\kappa\beta) \exp(-\frac{|\kappa|^2}{2} L_n(|\kappa| \xi^2)).$$

6.1. Dark vector gauge bosons

In the present approach new vector bosons $E^c$ and $B^\pm$ arise naturally by the geometric TB symmetry described by the group $E^c(2)$. These particles can be interpreted as DM vector gauge bosons. If the SM Higgs carries the IQN $\kappa = 1$, the $E^c$ and $B^\pm$ gauge boson yields a mass as discussed in chapter 5. The $E^c$ gauge boson is not completely decoupled from the SM particles, according to (57) there is a coupling of the new $E^c$-boson to SM leptons without the assumption of kinetic mixing. Despite that $B^\pm$ bosons carry hypercharge due to the selection rule (60) the interaction with leptons is forbidden but as seen in (63) its coupling to scalar particles is allowed.

From (45) - (49) we can see that the DM vector bosons with the gauge potentials $E^c_\mu$ and $B^\pm_\mu$ interact with each other but also with the SM gauge bosons $A$, $Z$ and $W^\pm$. Note that the hypothesis of self-interacting DM (in contrast to collisionless cold Dark Matter) enables to resolve a number of conflicts between observations and predictions of collisionless DM simulations [33, 34] and has also been assumed as light thermal DM relics [35].

There exists an additional possibility to identify a new class of hidden DM vector bosons. As explained before coupling of all gauge bosons to fermions only arise if the gauge bosons carry the $E^c$ charge $\kappa = 0$. However in principle in addition to gauge boson states with $E^c$-charge $\kappa = 0$ also gauge boson states with $E^c$-charge $\kappa \neq 0$ could exist. Such hypothetical exotic gauge particles do not interact with fermions but the coupling to scalar particles is allowed.

6.2. Dark scalars

In contrast to SM Higgs particles with zero electric charge there do not exist fundamental DM scalars (with $m = j = 0$) with zero $E^c$-charge. Such scalar particles can only exist as bound particles with opposite $E^c$-charges. The Lagrangian of DM scalars with $j = m = 0$ but nonzero $\kappa \neq 0$ is given by

$$L_D^D = \partial_\mu \Phi_S^D \partial^\mu \Phi_S^D + |\Phi_S^D|^2 \left(-g_3^2 2 \kappa (n + \frac{1}{2}) B^+_\mu B^-\mu + Z_\mu Z^{\mu}(g_3 \sin \theta_D)^2 + E^{\mu \nu}(g_3 \cos \theta_D)^2 \right) \right).$$

The coupling of a DM scalar particle to gauge bosons $g = (E^c, Z, B^\pm)$ is described by the matrix elements (63) and with the eigenfunctions $\chi_M^D(u)$ as given in (64). As seen coupling of DM scalars to DM gauge bosons $E^c$ and $B^\pm$ is allowed. But albeit with a very weak coupling coefficient $g_3 \sin \theta_D$ there exists also a coupling to the SM $Z$ boson arising from the diagonalization of the Laplacian for neutral scalars.

Note that a much discussed possible scenario of DM particles is the Higgs portal model [36] based on the coupling of DM scalars to SM Higgs particles through a phenomenological quartic self-interaction term of the Higgs model specifically through a $\Phi_D \Phi_D^* \Phi_D \Phi_D^*$ term. This so-called Higgs -portal can be included by a phenomenological Higgs potential $V(\Phi_S^D, \Phi_H^D)$ in (65) which delivers an interaction term of the Dark Scalar and the SM Higgs but is based on a phenomenological model assumption. As explained above in TB-QFT a fundamental DM Higgs with zero $E^c$-charge and $m = j = 0$ do not exists.

6.3. Dark Fermions

In the same way we can assume that DM fermions with vanishing hyper-charge and isospin ($j = 0$, $m = 0$) but nonzero $E^c$-charge ($\kappa \neq 0$) could exist. The Lagrangian of these DM fermions is given by

$$L_D^F = i \sum_{M_+} \Psi_{M_+}^D \sigma_\mu [\partial_\mu + ig_3^2 \kappa (\sin \theta_D Z_\mu + \cos \theta_D E^{\mu}_c)] \Psi_{M_+}^D,$$

with $M = (n, 0, \kappa, 0, 0)$. As one can see different types of DM particles with different $E^c$-charges $\kappa = \pm 1, \pm 2, \ldots$ are predicted which couple to the $E^c_\mu$ gauge potential with the coupling coefficient $g_3 \kappa \cos \theta_D$. Besides, for every
with symmetries described by the group $SO(3, 1)$. Projective irreducible representations of this group can be constructed by using the little groups $SU(2)$, $E^c(2)$ and $SU(1, 1)$. Using the covariant derivative given by the operators on the transformation group $G = SU(2) \otimes E^c(2)$, and the corresponding connection coefficients (gauge potentials) a generalized theory of the electroweak interaction is derived. In this approach wave functions depend on the space-time coordinates $x$ as well as on the coordinates of the tangent fibers $v$. The known SM $Z$ and $W^\pm$ gauge bosons can be found again but in addition new extra gauge bosons $E^c$ and $B^\pm$ are predicted which constitute a fifth fundamental interaction. In addition to the SM quantum numbers of isospin and hypercharge there exist the $E^c$-charge $\kappa$ and the family quantum number $n$ which explains the existence of lepton families. The existence of families requires that leptons carry a non-zero $E^c$-charge $\kappa \neq 0$. In contrast, SM gauge bosons carry zero $E^c$-charge $\kappa = 0$. The derived selection rule reveals that family universal coupling of leptons persists for the interaction with $A$, $Z$ and $W^\pm$ gauge bosons but also for the allowed interaction with the $E^c$ boson. However a violation of lepton universality could arise via the $E^c$ boson interaction if different SM lepton families carry different $E^c$ charges $\kappa$. The interaction of leptons with the $B^\pm$ boson is forbidden but its coupling to scalar particles is allowed. If the SM Higgs carry the IQN $\kappa = 1$ the mass of the $B^\pm$ boson can be estimate to be $M_{B^\pm} \approx 227.5\text{GeV}$ and analogous as lepton families also Higgs families should exist. The mass of the $E^c$ boson is determined by the coupling coefficient $g_3$ with $M_{E^c} = \Phi_0 e\sqrt{2}g_3$. A notable feature of the theory presented is the possibility of identifying candidate stable or unstable hypothetical DM fermions and DM scalars with zero hypercharge and zero isospin but non-zero $E^c$-charges $\kappa \neq 0$ which should grouped in different families with different family numbers $n = 1, 2, 3$ analogous as leptons.

Finally, the approach presented could open-up an extended prospect and a step towards the solution of the long-standing problem of the unification of all fundamental interactions on a geometrical basis. The tangent bundle is the geometrical fundament for teleparallel gravity gauge theory based on translational transformations of tangent vectors along the fiber axis. That is, the more general symmetry group $SO(3, 1) \times T(3, 1)$ in the TB contains both gravity and the generalized electroweak interaction with the inclusion of candidate Dark Matter particles as predicted here. Therefore gravity and electroweak interaction are described by the same fundamental geometrical structure of the TB. Note that strong interaction is in this scenario still missing. However there exists a surprising analogy of fractional charged quarks with the anomalous Quantum Hall effect. In addition to the analog effect of fractional charge quantization the eigenfunctions of the group $E^c(2)$ in (25) have the same
form as electrons in a 2D quantum Hall system in an external magnetic field. This gives a hint that between both phenomena a close internal relationship could exist. In such analogy quarks appear as bound states of fermions and and two quantized vortices (composite fermions) via an emergent SU(3) gauge field. In such approach the SU(3) color symmetry in Quantum Chromodynamics could originate from emergent gauge fields similar as the emergent fields in the theory of the anomalous Quantum Hall effect arise from the existence of composite fermions bounded to topological vortices by Chern-Simons effective fields (see e.g. [20, 42]). The study of this problem is presently in elaboration.

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[1] E. Lubkin, Ann.Phys. (N.Y.) 23, 233 (1963),
[2] Trautman, Rep. Math. Phys. 1, 29 (1970),
[3] W. Drechsler W and M. E. Meyer, "Fibre Bundle Techniques in Gauge Theories", Lecture Notes in Physics, Springer1977,
[4] M. Daniel and C. M. Viallet, Rev. Mod. Phys. 52, 175 (1980),
[5] M. Nakahara, "Geometry, Topology and Physics", Institute of Physics, Bristo 1990,
[6] Bo-Yuan Hou, "Differential Geometry for Physicists", Singapore 1997,
[7] S. Kobayashi an K. Nomizu, "Foundation of Differential Geometry", Interscience Publishers (1963),
[8] Chris J Isham, "Modern Differential Geometry for Physicists", World Scientific (1999),
[9] K. Hayashi and T. Nakno, Prog Theor. Phys. 38, 491 (1967),
[10] Y. M. Cho, Phys. Rev.D 14, 2521 (1976),
[11] T. Dass, Pramana 23, 433 (1984),
[12] R. Aldrovandi, j. G. Pereira "Teleparallel Gravity; an Introduction" (Sprinder, Dordrecht, 2012),
[13] T. W. Kibble, J. Math. Phys. 2, 212, (1961),
[14] F. W. Hehl,J. D. McCrea, E. W. Mielke an Y. Ne`eman, Phys. Rep. 258, 1 (1995),
[15] F. W. Hehl,J. D. McCrea, E. W. Mielke an Y. Ne`eman, Phys.Rev. D 48, 48 (1993),
[16] E. P. Wigner, Ann. Math. 40, 149 (1939),
[17] V. Bargman, Ann. Math. 59, 1 (1954),
[18] H. Hoogland, J. Phys. A Math. Gen. 11, 1557 (1978),
[19] J. F. Carina, M. A. del Olmo and M Santander, J.Phys.A: Math.Gen. 17, 3091 (1984),
[20] J. J. Quinn, A. Wójs, K.-S. Yi, and G. Simion, Physics Reports 481.29 (2009),
[21] V. Aldaya, J. Navarro-Salas, J. Bisquert and R. Loll, J. Math. Phys. 33, 3087 (1992),
[22] I.M. Shapiro, R.A. Minlos and Z.Ya Shapiro, "Representations of the Rotation and Lorentz Groups and their Applications", Pergamon Press 1963,
[23] W. J. Holman and L. C. Biedenhans, Jr Annals of Physics 39, 1, (1966),
[24] R. Essig et al., JHEP 02, 009 (2011),
[25] J. D. Bjorken et al. Phys. Rev. D 80, 07501 (2009),
[26] J. Alexander et al.: Dark Sectors 2016 Workshop: Community Report, arXiv:1608.08632v1,
[27] P. Langacker, Rev. Mod. Physics 81, 1199 (2009),
[28] T. Appelquist, B.A. Dobrescu and A. R. Hopper, Phys. Rev. D 68, 035012 (2003),
[29] K. S. Babu, C. Kolda and J. March-Russell, Phys. Rev. D 57, 6788 (1998),
[30] F. Blas and M Perez-Victoria F. del Aguila, J. de Blas and M Perez-Victoria JHEP 1009,033 (2010),
[31] G. Bertone, D. Hooper and J. Silk, Particle dark matter: evidence, candidates and constraints. Phys. Rep. 405 (2005) 279,
[32] Stefan Profumo, "An Introduction to Particle Dark Matter", World Scientific (2017),
[33] D.N.Steinhardt, J.Paul, Phys. Rev. Lett. 84, 3760 (2000),
[34] S. Tulin and H.-B. Yu, Phys. Rep. 730, 1 (2018)
[35] Y. Hochberg et al. Phys. Rev. Lett. 113, 171301 (2014),
[36] V. Silveira and A. Zee, Phys. Lett. B 161, 136 (1985),
[37] Thomas Hambye, JHEP 01 (2009) 028,
[38] B. Holdom, Phys. Lett. B 166, 196 (1986),
[39] P-F Yin and S.-H. Zhu, Front. Phys. 11, 111403 (2016),
[40] N. Bernal et al. JCAP 1603, 028 (2016),
[41] S.-M. Choi, arXiv:1707.01433, (2017).