I. INTRODUCTION: QNMS AND (IN)STABILITY

A. The black hole QNM stability problem and the pseudospectrum

Structural stability is essential in the modelling and understanding of physical phenomena. In the context of spectral problems pervading physics, often related to wave phenomena in both classical and quantum theories, this concerns in particular the basic question about the stability of the physical spectrum of the system. Thereupon, one needs to assess the following questions: how does the spectrum react to small changes of the underlying system? Is the spectrum stable, i.e., do small perturbations lead to tiny deviations? Or is it unstable, with small changes in the system leading to drastic modifications of the spectrum? In the present work, we study such kind of spectral stability question in the setting of black hole (BH) spacetimes. Specifically, the problem we address here is the spectral robustness of BH QNMs, namely the stability of the resonant frequencies of BHs under perturbations. From a methodological perspective, our spectral (in)stability analysis is built upon the notion of the so-called pseudospectrum.

1. Spectral instability and pseudospectrum

The physical status of spectral stability depends crucially on whether the underlying system is conservative or not. In particular, conservative systems do have stable spectra and therefore the spectral instability question, being solved from scratch, is not relevant. Such spectral stability is familiar in (standard) quantum mechanics, where (time-independent) perturbation theory precisely relies on it. It is the selfadjoint nature of the relevant operators (namely “Hermitian matrices” in the finite dimensional case) that accounts for such spectral stability. More systematically, this is a consequence of the so-called ‘spectral theorem’ for selfadjoint operators: eigenvectors form an orthogonal and complete set, whereas eigenvalues are real and stable. This provides the mathematical background for the key physical notion of normal mode, associated with the characteristic (real) vibrating frequencies of a conservative system and its natural oscillation modes.

The situation is more delicate for non-conservative systems, modelled in terms of non-selfadjoint operators (non-Hermitian matrices). Such systems occur naturally whenever there exist flows (e.g. energy, particle, information) into external degrees of freedom that are out of the (Hilbert) space under consideration (see [1] for a recent and extensive review on non-Hermitian physics; cf. e.g. its Table 1 for a list of several classical and quantum systems described by non-Hermitian operators). In this setting the ‘spectral theorem’ is lost: eigenvectors are in general neither complete nor orthogonal, and eigenvalues (now in general complex) are potentially unstable. We focus here on this latter point, namely the potential spectral instability of a class of non-selfadjoint operators associated with the non-conservative system defined by the scattering of fields by BHs where, critically, the field leaks away from the system at far distances and through the BH horizon.

The notion of pseudospectrum provides a powerful tool for the analysis of the properties of non-selfadjoint operators. In particular, its use is well spread whenever stability issues of non-conservative systems are addressed, from pioneering applications in hydrodynamics to recent advances covering a wide range in physics. Broadly speaking in order to gain some first intuition, the pseudospectrum provides a visualization (and actually a characterization) of the spectral instability of our operator in terms of a kind of ‘topographic map’ on the complex plane, where the ‘peaks’ (actually end points of infinitely-high throats) lay at the complex eigenvalues of the non-perturbed operator. With this picture in mind, spectral stability is assessed in terms of the “thickness” of the throats: very thin throats decreasing fast from the eigenvalues correspond to spectral stability, whereas broad slowly decreasing throats indicate spectral instability. Expressing this in terms of ‘level-sets’, contour lines corresponding to ‘heights’ 1/ε define a family of nested sets around eigenvalues, referred to
as $\epsilon$-pseudospectra, that determine the regions in which eigenvalues can potentially ‘migrate’ under a system perturbation of size $\epsilon$. The (non-perturbed) spectrum corresponds to the set defined by $\epsilon \to 0$. Therefore, tightly packed contour lines around eigenvalues corresponding to strong gradients indicate spectral stability, whereas contour lines with low gradients extending far from the eigenvalues signal spectral instability.

2. Black hole QNMs in gravitational physics

BH QNMs encode the resonant response to (linear) perturbations of the BH spacetime. In spite of being triggered by perturbations, QNMs constitute an intrinsic property of the background and, therefore, QNM frequencies encode crucial geometric information about BHs and their environment. Thus, they have become a fundamental tool in astrophysics, fundamental gravitational physics, and mathematical relativity in their attempts to probe spacetime geometry through perturbation theory and scattering methods (see e.g. [14],[18] for systematic presentations and reviews).

Upon perturbation, and after an initial transient, the perturbative field propagating on the background spacetime shows an exponentially-damped, oscillatory behavior. QNM frequencies are the set of complex numbers encoding the oscillatory frequencies and decaying time scales of the propagating linear (scattered) field. To fix ideas, this is illustrated by the BH formed after the merger of a compact binary, in the emerging setting of gravitational wave (GW) astronomy. After the transient merger phase, the resulting perturbed BH evolves towards stationarity in a linear ringdown phase dominated by QNMs. In particular, the late time behaviour of the GW signal is controlled by the fundamental or slowest decaying QNM mode, namely the QNM frequency with smallest (in absolute value) imaginary part and therefore closest to the real axis. Nonetheless, QNMs with larger imaginary parts and referred to as overtones — with different oscillatory frequencies and faster decaying time scales than the fundamental QNM — are also present in the GW signal, its analysis being at the basis of the BH spectroscopy research program [19],[25]. Beyond GW physics, QNMs play a key role in gravitational physics as a crossroads among different limits and regimes of the theory, encompassing problems in the evoked GW astrophysical setting, in semiclassical gravity (e.g. [26]) and gravity-fluid (AdS/CFT) dualities (e.g. [27]), in analogue gravity [28] or in foundational questions in mathematical relativity (e.g. [29],[31]), among other problems ranging from the classical to the quantum regime.

To be more specific, the discussion of BH QNMs is set in the framework of BH perturbation theory. QNMs are obtained from the spectral problem associated with the wave equation under the step-like approximation for the potential (perturbed Schwarzschild curvature potential). More specifically, these works considered a family of step-like approximations to the Schwarzschild curvature potential. In a first step, the authors calculated the QNMs corresponding to the step-like approximation for the potential (perturbed QNMs) finding a strong deviation from the original values (non-perturbed QNMs), with a clear and systematic pattern: perturbed QNMs distribute along new QNM branches with a qualitative structure dramatically distinct from that of non-perturbed QNMs. In a second step, they performed time evolutions of the wave equation under the step-like approximated potential in a bid to identify and extract the perturbed QNMs from the wave signal. In contrast with the spectral problem, time evolutions presented an overall stable behaviour under perturbations of the potential. Specifically, Nollert & Price’s work demonstrates that, for the studied class of perturbations:

i) QNM overtones are strongly unstable, their instability increasing with their damping.

ii) The fundamental, slowest decaying, QNM is unstable.

iii) The black hole ringdowns, at intermediate late times, according to the non-perturbed fundamental mode. Only at very late times the ringdown frequency is controlled by the perturbed fundamental QNM mode.

These results have been confirmed and expanded in [38],[39] to perturbations of the scattering potential extending the step-like approximation, but still sharing the feature of presenting a discontinuity at the potential or some of its derivatives.
Beyond Nollert & Price’s works, research in BH QNM spectral (in)stability has been further pursued in various gravitational physics settings. In astrophysics, the understanding of possible environmental observational signatures in "dirty" BH scenarios has prompted a research line [40, 41] that has been significantly intensified recently [38, 39, 42–44]. On the other hand, regarding investigations on the fundamental structure of spacetime, the perspective of accessing quantum scales through high-frequency instabilities of QNM overtones has also tantalized a systematic research [32–37].

In spite of these efforts, a comprehensive picture of BH QNM (in)stability seems to be lacking. At this point it is worthwhile to explicitly distinguish between the instability in QNM frequencies and the instability in late-ringdown frequencies. The former refers to the spectral instability in the 'frequency domain' approach, when solving the spectral problem associated with the wave equation. The latter would refer to a dynamical instability in the 'time domain' approach, when solving the initial data dynamical problem. Both problems are intimately related, but are indeed different. In particular, it is known that the two sets of frequencies can indeed decouple (e.g. [1, 2, 38, 39, 45–48] in the gravitational context). Still, the separation between QNM and ringdown frequencies signals an 'anomaly' and, therefore, pinpoints a structural feature in the physical system requiring specific study. In the present work, we focus on QNM (in)stability in the spectral sense.

In this context, the stability status of the slowest decaying QNM — presenting precisely the tension described above between calculated spectral instability and observed robustness in the ringdown signal — remains unclear, whereas the elucidation of the lowest overtone subject to high-frequency instability is an open problem. Under the light of the discussion above on the fundamental role of BH QNMs in different settings of gravitational physics, the clarification of these two points is a first-order problem from a strictly physical perspective. Moreover, if establishing the stability status of BH QNMs is key in general BH physics, the problem is actually urgent in gravitational wave astrophysics. Indeed, in the era of gravitational-wave astronomy, the stability of the fundamental QNM and the overtones is paramount for BH spectroscopy.

The implementation of an analysis based on the pseudospectrum permits to address systematically these questions and to provide sound answers to points i) and ii) above. In short, and anticipating the results later discussed in detail, such analysis confirms the instability behaviour of QNM overtones — point i) — and provides a framework for its systematic study, whereas it disproves the instability of the fundamental QNM — point ii) — if asymptotic properties of the spacetime are respected, its unstable behaviour in [11] resulting an artifact consequence of “cutting” the effective potential at a finite distance. Regarding point iii), from the stability of the fundamental QNM we conclude that the late time ringdown is indeed dominated by the unperturbed slowest decaying QNM, (without any very late transition to a ‘perturbed ringdown’ frequency), but the systematic analysis of the detailed relation between QNM frequencies and BH ringdown frequencies lays beyond the scope of the present work and will be the subject of a specifically targeted research focused on the potential implications of BH QNM instability on GW astrophysics [49].

C. The present approach

1. The basic ingredients: hyperboloidal approach and pseudospectrum

The calculation of BH QNMs has been the subject of systematic study in gravitational physics and there exists a variety of standard approaches to address this problem (cf. e.g. [15–18]). From a methodological perspective, our discussion relies on two main ingredients at a conceptual level:

i) A hyperboloidal approach to QNMs: this geometric approach casts the QNM calculation as a proper eigenvalue problem of a particular non-selfadjoint operator.

ii) Pseudospectrum: together with related spectral tools, it provides the key instrument to study the potential spectral instability of the relevant non-selfadjoint operator.

The combination of these two elements permits to develop a systematic treatment of the problem. To the best of our knowledge, no systematic treatment of BH QNM (in)stability based on the pseudospectrum exists in the literature. At a first exploratory stage, prior to a full analytical study, the present work addresses pseudospectra in a numerical approach. This sets a challenging numerical problem demanding high accuracy, which is here addressed by introducing a third key ingredient in our approach: the use of spectral numerical methods.

2. Beyond gravity: QNMs, pseudospectrum and interdisciplinary physics

Before entering into the detailed discussion of BH QNM stability, let us stress that both QNMs and the pseudospectrum provide independent, but indeed complementary, arenas for interdisciplinary research in physics and related disciplines.

Regarding QNMs, beyond the present gravitational context, the notion of QNM spreads in physics, e.g. in electromagnetism and optics, acoustics, or — under the related notion of resonance in quantum mechanics — in atomic, nuclear and molecular physics. Beyond physics, QNMs enter in the discussion of scattering problems in geometry [50] and chaotic dynamics (see [51] for a systematic review of scattering resonances or QNMs from a mathematical perspective). Together with the extent of the applicability of the QNM notion, an important aspect concerns timing. Indeed, the synergy observed in this sense in recent years among different subdisciplines in the gravitational setting (namely GW astrophysics, AdS/CFT dualities and mathematical relativity) extends remarkably to other fields in physics, as perfectly illustrated by recent breakthroughs in optical nanoresonator QNMs, namely photonic and plasmonic resonances [52, 53].

Regarding the pseudospectrum, its use in physics naturally extends over the study of stability and spectral problems in...
non-conservative systems, from which we highlight its applications in hydrodynamics [4] and in non-Hermitian quantum mechanics [10]. Beyond physics, systematic applications are found in numerical analysis, the original context where the notion was formulated. This wide range of applications become intertwined methodologically by the pseudospectrum. The present approach to BH QNM stability, that introduces (to the best of our knowledge) the pseudospectrum into gravity, incorporates gravitational physics to this multifaceted research scheme. When combined with the large range of applicability of the QNM notion in physics, it outlines a robust and potentially rich frame for interdisciplinary research in physics.

The article is structured as follows. Section II presents a qualitative description of the hyperboloidal approach to scattering problems and reviews the literature on this geometrical framework. Beyond reviewing the main concepts, with a focus on QNMs, this section identifies and constructs the appropriate scalar product in the problem. Section III introduces the basic elements to study spectral instability of non-selfadjoint operators, in particular the notion of pseudospectrum. Section IV presents the numerical spectral tools to be employed in the present approach. Then, section V illustrates all the previous elements in the toy-model provided by the Pöschl-Teller potential, that also anticipates some of the main results in the BH setting. Section VI contains the main contribution in the present work, namely the construction of the Schwarzschild BH setting. Section VII presents a series of four appendices completing some points in the technical discussion of the main text. Throughout the work, we adopt units in which the speed of light and the gravitational constant are unit ($c = G = 1$).

II. HYPERBOLOIDAL APPROACH TO QNMS

A. Hyperboloidal approach: a heuristic introduction

Our approach to QNMs strongly relies on casting the discussion in terms of the spectral problem of a (non-selfadjoint) operator. In our scheme, this is achieved by means of a so-called hyperboloidal approach to wave propagation, that provides a systematic framework exploiting the geometric asymptotics of the spacetime, in particular enforcing the relevant outgoing boundary conditions in a geometric way. We start with a heuristic discussion of the basics, aiming at providing an intuitive picture and explicitly sacrificing rigor.

The notion of wave zone is a familiar concept in physics. It describes a region far away from a source where the degrees of a freedom of a given field (non-necessarily linear) propagate as a free wave, independently of their interior sources and obeying the superposition principle. Roughly speaking, this region is characterised by $r/R \gg 1$, where $r$ is the location of a distant observer and $R$ is a typical length scale of the source. This concept is addressed formally by taking appropriate limits $r \to \infty$ or $1/r \to 0$. From a spacetime perspective, however, such a limit must be carefully understood. To fix ideas, let us consider a physical scenario in spherical symmetry, where a wave propagating at finite speed is described in a standard spherical coordinate system $(t, r, \theta, \varphi)$ (for simplicity, let us consider momentarily a flat spacetime where we ignore gravity effects). The retarded time coordinate $u = t - r$ corresponds to the time at which an outgoing wave, passing by the observer at $r$ at time $t$, was emitted by a source located at the origin. Crucially, “light rays” propagate along (characteristic) curves satisfying $u = \text{const}$. In this setting, and as illustrated in Fig. 1 taking the limit $r \to \infty$ corresponds to completely different geometric statements depending on whether one stays at the hypersurface $t = \text{const}$ or rather on $u = \text{const}$. The limit attained by ‘space-like’ (geodesic) curves satisfying the former condition ($t = \text{const}$) is referred to as ‘space-like infinity’ and denoted $i^0$, whereas lightlike or null (future geodesic) curves satisfying the latter condition ($u = \text{const}$) attain a limit referred to as future null infinity, denoted as $\mathcal{I}^+$. It is future null infinity $\mathcal{I}^+$ that formally captures the intuitive notion of outgoing wave zone.

Other alternatives to the $t = \text{const}$ and $u = \text{const}$ hypersurfaces are possible, something natural in a general relativistic context implementing coordinate choice freedom. A particularly convenient possibility in our present problem consists in choosing a third alternative: to keep space-like hypersurfaces defined as level sets of an appropriate time function $\tau$, while reaching future null infinity as $r \to \infty$ so as to enforce the outgoing character of the radiation. Such a third option is displayed in Fig. 1 as a $\tau = \text{const}$ hypersurface. The asymptotic geometry of such hypersurfaces is that of a hyperboloid, a feature giving name to the resulting hyperboloidal approach.

The previous heuristic picture of spacetime asymptotics is formalised in the geometric notion of conformal infinity [55–61], that provides a rigorous and geometrically well-defined strategy to deal with radiation problems of compact isolated bodies. A conformal compactification maps the infinities of the physical spacetime into a finite region delimited by the boundaries of a conformal manifold. Specifically, $\mathcal{I}^+$ corresponds to the future endpoints of null geodesics, whereas a time function $\tau$ will be referred to as hyperboloidal if hypersurfaces $\tau = \text{const}$ intersect $\mathcal{I}^+$, being therefore adapted to the geometrical structure at the infinitely far away wave zone.

The hyperboloidal formulation has proved to be a powerful tool in mathematical and numerical relativity, permitting to obtain existence results in the non-linear treatment of Einstein equations, as illustrated in the semiglobal result in [52], or providing a natural framework for the extraction of the GW waveform in numerical dynamical evolutions of GW sources. Together with those fully non-linear studies, over the last decade the hyperboloidal approach has been successfully applied to problems defined on fixed spacetime backgrounds (see e.g. [63] and references therein). In particular, [64] proposes a hyperboloidal approach to BH perturbation theory. This is our setting for QNMs, where the hyperboloidal framework permits to implement geometrically the outgoing boundaries conditions at $\mathcal{I}^+$, in a strategy first proposed by Schmidt in [65]. The adopted (compactified) hyperboloidal approach provides a geometric framework to study QNMs, that characterizes resonant frequencies in terms of an eigen-
value problem. As explained above, the scheme geometrically imposes QNM outgoing boundary conditions by adopting a spacetime slicing that intersects future null infinity $\mathcal{I}^+$ and, in the BH setting, penetrates the horizon. Since light cones point outwards at the boundary of the domain, outgoing boundary conditions are automatically imposed for propagating physical degrees of freedom. Along these lines, our scheme to address the BH QNM (in)stability problem strongly relies on the hyperboloidal approach, since it provides the rationale to define the non-selfadjoint operator on which a pseudospectrum analysis is then performed.

![FIG. 1. Schematic representation of the different “$r \to \infty$” limits along curves within different types of spacetime hypersurfaces. Cauchy hypersurfaces, of spacelike character and represented by the “$t = \text{const.}$” condition in the figure, are such that this limit attains the so-called spatial infinity $i^0$, whereas in null hypersurfaces satisfying rather “$u = \text{const.}$” (with $u = t - r$ a null ‘retarded time’) the limit attains the outgoing wave zone modelled by future null infinity $\mathcal{I}^+$.

The hyperboloidal approach offers an intermediate possibility, where the limit is taken along spacelike hypersurfaces, formally represented by the “$\tau = \text{const.}$”, but still reaching $\mathcal{I}^+$ asymptotics.]

\[ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)\]  

and spatial infinity $i^0$. If we consider the rescaling
\[\Phi = \frac{1}{r} \phi,\]  

then Eq. (1) rewrites, expanding $\phi$ in spherical harmonics with $\phi_{\ell m}$ modes and using the tortoise coordinate defined by \[dr_* = f(r)\] (with the appropriate integration constant), as
\[\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0,\]  

where now $r_* \in \left[ -\infty, \infty \right]$. Remarkably, when considering electromagnetic and (linearized) gravitational fields, the respective geometric wave equations corresponding to Eq. (1) can be cast in the form of Eq. (4) for appropriate effective scalar potentials. Specifically, two scalar fields with different parity can be introduced, satisfying Eq. (4) with suitable potentials $V_\ell$. In the gravitational case, the axial parity is subject to the so-called Regge-Wheeler potential, whereas the polar one is controlled by the Zerilli potential (cf. e.g. [12]-[15]).

The BH event horizon and (spatial) infinity correspond, respectively, to $r_* \to -\infty$ and $r_* \to +\infty$. We extend the domain of $r_*$ to $[ -\infty, \infty ]$ and introduce the dimensionless quantities
\[\tilde{t} = \frac{t}{\lambda}, \quad \tilde{x} = \frac{r_*}{\lambda}, \quad \tilde{V}_\ell = \lambda^2 V_\ell,\]  

for an appropriate length scale $\lambda$ to be chosen in each specific setup. More importantly, we consider coordinates $(\tau, x)$ that implement the compactified hyperboloidal approach
\[\begin{align*}
\tilde{t} &= \tau - h(x) \\
\tilde{x} &= g(x)
\end{align*}\]  

Specifically (see Fig. 2):

i) The height function $h(x)$ implements the hyperboloidal slicing, i.e. $\tau = \text{const.}$ is a horizon-penetrating hyperboloidal slice $\Sigma_\tau$ intersecting future $\mathcal{I}^+$.

ii) The function $g(x)$ introduces a spatial compactification from $x \in [-\infty, \infty]$ to a compact interval $[a, b]$.

We note that the compactification is performed only in the spatial direction along the hyperboloidal slice, and not in time, so that the latter can be Fourier transformed in an unbounded domain. The relevant compactification here is a partial one, and not the total spacetime compactification leading to Carter-Penrose diagrams. The choice of $h(x)$ and $g(x)$ is, as we comment below, subject to certain restrictions. Under transformation (6), the wave equation (4) writes
\[\begin{align*}
\left[1 - \left(\frac{h'}{g'}\right)^2\right] \partial_\tau^2 - \frac{2}{g'} \left(\frac{h'}{g'}\right) \partial_\tau \partial_x &= \frac{1}{g'} \left(\frac{h'}{g'}\right)' \partial_x \\
+ \frac{1}{g'} \partial_\tau \left(\frac{1}{g'} \partial_x + \tilde{V}_\ell\right)\Phi_{\ell m} &= 0,
\end{align*}\]  

where the prime denotes derivative with respect to $x$. Admittedly, expression (7) appears more intricate than Eq. (4). However, this change encodes a neat geometric structure and, as we shall argue, it plays a crucial role in our construction and discussion of the relevant spectral problem.
The structure of $L_1$ is that of a Sturm-Liouville operator. In particular, functions $h(x)$ and $g(x)$ are chosen such that they guarantee the positivity of the weight function $w(x)$, namely $w(x) > 0$. The operator $L_2$ has also a neat geometric/analytic structure adapted to the integration by parts, being symmetric in the following form: $L_2 = \frac{1}{w(x)} \left( \gamma(x) \partial_x + \partial_x \left( \gamma(x) \partial_x \right) \right)$. A key property of coordinate transformation (6) is that it preserves, up to the overall constant $\lambda$, the timelike Killing vector $t^\alpha$ controlling stationarity

$$t^\alpha = \partial_t = \frac{1}{\lambda} \partial_t = \frac{1}{\lambda} \partial_x .$$

In this sense functions $t$ and $\lambda \tau$ “tick” at the same pace, namely they are natural parameters of $t^\alpha$, i.e. $t^\alpha(t) = t^\alpha(\lambda \tau) = 1$ (the role of the constant $\lambda$ being just that of keeping proper dimensions). This is crucial for the consistent definition of QNM frequencies by Fourier (or Laplace) transformation from Eqs. (4) and (7), since variables $\omega$ respectively conjugate to $t$ and $\tau$ then coincide (up to the constant $1/\lambda$). In other words: the change of time coordinate in Eq. (6) does not affect the values of the obtained QNM frequencies.

Performing then the Fourier transform in $\tau$ in the first-order (in time) form (9) of the wave equation (with standard sign convention for the Fourier modes, $u_{n,\ell m}(\tau, x) \sim u_{n,\ell m}(x) e^{i \omega \tau}$) we arrive at the spectral problem for the operator $L$

$$L u_{n,\ell m} = \omega_{n,\ell m} u_{n,\ell m} ,$$

or, more explicitly

$$\left( \begin{array}{cc} 0 & \frac{1}{L_1} \\ L_1 & L_2 \end{array} \right) \begin{pmatrix} \phi_{n,\ell m} \\ \psi_{n,\ell m} \end{pmatrix} = i \omega_{n,\ell m} \begin{pmatrix} \phi_{n,\ell m} \\ \psi_{n,\ell m} \end{pmatrix} .$$

1. Regularity and outgoing boundary conditions

As emphasized at the beginning of this section, a major motivation for the adopted hyperboloidal approach is the geometric imposition of outgoing boundary conditions at future null infinity and at the event horizon: being null hypersurfaces with light cones pointing outwards from the integration domain, the physical causally propagating degrees of freedom (as the scalar fields we consider here) should not admit boundary conditions, as long as they satisfy the appropriate regularity conditions. How does this translate into the analytic scheme resulting from the change of variables (6)?

The key point is that transformation (6) must be such that $p(x)$ in the Sturm-Liouville operator $L_1$ in Eq. (11) vanishes at the boundaries of the compactified spatial domain $[a, b]$

$$p(a) = p(b) = 0 .$$

This will be illustrated explicitly in the study cases discussed later. Then the elliptic operator $L_1$ is a “singular” Sturm-Liouville operator, this impacting directly on the boundary conditions it admits. Specifically, if (appropriate) regularity is enforced on eigenfunctions, then $L_1$ does not admit boundary conditions. Moreover, such absence of boundary conditions extends to the full operator $L$ in the hyperbolic problem.
In brief: if sufficient regularity is imposed on the space of functions \( u_{\alpha,\ell,m} \), then wave equations (7, 8) and the spectral problem (14) do not admit boundary conditions, as a consequence of the vanishing of \( p(x) \) at the boundaries of \([a, b]\).

This is the analytic counterpart of the geometric structure implemented in the compactified hyperboloidal approach. QNM boundary conditions are in-built, as regularity conditions, in the ‘bulk’ of the operator \( L \) in Eqs. (14) and (15).

### D. Scalar product: QNMs as a non-selfadjoint spectral problem

The outgoing boundary conditions in the present setting define a leaky system, with a loss of energy through the boundaries — null infinity and the black hole horizon — so that the system is not conservative. This suggests that the infinitesimal generator of the evolution in Eq. (7), namely the operator \( L \), should be non-selfadjoint. This requires the introduction of an appropriate scalar product in the problem. Moreover, such identification of the appropriate Hilbert space for solutions is also key for the regularity conditions evoked above.

Eq. (7) describes the evolution of each mode \( \phi_{\ell,m} \) in a background 1+1-Minkowski spacetime with a scattering potential \( V_\ell \). A natural scalar product in this reduced problem (cf. [72] for an extended discussion in terms of the full problem), both from the physical and the analytical point of view, is given in terms of the energy associated with such scalar field mode. In the context of the spectral problem (14), we must consider generically a complex scalar field \( \phi_{\ell,m} \), for which the associated stress-energy tensor writes (dropping \( \ell, m \) indices) is

\[
T_{ab} = \frac{1}{2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} \eta_{ab} \left( \nabla^c \phi \nabla_c \phi + V_\ell \phi \phi \right) + \text{c.c.} \right)
\]

where \( \eta_{ab} \) denotes the Minkowski metric in arbitrary coordinates and “c.c.” indicates “complex conjugate”. In a stationary situation, the “total energy” contained in the spatial slice \( \Sigma_\tau \) and associated with the mode \( \phi \) is given [59] by

\[
E = \int_{\Sigma_\tau} T_{ab} \phi \nabla^a \phi n^b d\Sigma_\tau,
\]

where \( t^a \) is again the timelike Killing vector associated with stationarity and \( n^a \) denotes the unit timelike normal to the spacelike slice \( \Sigma_\tau \). Writing explicitly the energy in the compactified hyperboloidal coordinates \((\tau, x)\) in (6), we get

\[
E(\phi, \partial_\tau \phi) = \int_{\Sigma_\tau} T_{ab} \phi \nabla^a \phi n^b d\Sigma_\tau
\]

\[
= \frac{1}{2} \int_a^b \left[ (g^{\tau^2} - h^{\tau^2}) \partial_\tau \bar{\phi} \partial_\tau \phi + \partial_x \bar{\phi} \partial_x \phi + g^{\tau^2} V_\ell \bar{\phi} \phi \right] \frac{1}{|g|} dx,
\]

where we identify the functions appearing in the definition of the \( L_I \) operator in (11) and (12). In particular, if \( g^{\tau^2} - h^{\tau^2} > 0 \) (as we have required above) and \( V_\ell > 0 \) (this is required for positivity of the norm) then, identifying \( \partial_\tau \phi = \psi \) as in (8), we can write the following norm for the vector \( u \) in (8)

\[
||u||_2^2 = ||\left( \begin{array}{c} \phi \\ \psi \end{array} \right) ||_2^2 := E(\phi, \psi)
\]

\[
= \frac{1}{2} \int_a^b \left( u(x) \bar{\phi}^2 + p(x) |\partial_x \phi|^2 + q(x) |\phi|^2 \right) dx.
\]

We refer in the following to this norm as the “energy norm”. We notice that \( \gamma(x) \) in Eq. (12), associated with \( L_2 \) does not enter in the norm, that is, in the energy. This norm comes indeed from a scalar product. Rewriting, for making its role more apparent, the \( q_\ell(x) \) function as the rescaled potential \( \tilde{V}_\ell \)

\[
\tilde{V}_\ell := q_\ell(x) = |g'(x)| \tilde{V}_\ell = \frac{\tilde{V}_\ell}{p(x)},
\]

and under the assumption above \( \tilde{V}_\ell > 0 \), we can introduce the “energy scalar product” for vector functions \( u \) in Eq. (8), as

\[
\langle u_1, u_2 \rangle_\tilde{V} = \left\langle \left( \begin{array}{c} \phi_1 \\ \psi_1 \end{array} \right), \left( \begin{array}{c} \phi_2 \\ \psi_2 \end{array} \right) \right\rangle_\tilde{V}
\]

\[
= \frac{1}{2} \int_a^b \left( u(x) \bar{\psi}_1 \psi_2 + p(x) \bar{\phi}_1 \phi_2 + \tilde{V}_\ell \bar{\phi}_1 \phi_2 \right) dx,
\]

and, by construction, it holds \( ||u||_2^2 = \langle u, u \rangle_\tilde{V} \). This will be the relevant scalar product in our discussion.

The full operator \( L \) in (14) is not selfadjoint in the scalar product (22). In fact, the first-order operator \( L_2 \) stands for a dissipative term encoding the energy leaking at \( \mathcal{I}^+ \) and the BH horizon [75]. One could, at a first look, consider that this is related to the first-order character of \( L_2 \), which makes it antisymmetric when integrating by parts with a \( L^2([a, b], w(x) dx) \) scalar product on \( \psi \), in contrast with the selfadjoint character of the Sturm-Liouville operator \( L_1 \) in \( L^2([a, b], w(x) dx) \) for \( \phi \) functions. However, this is misleading and actually would suggest a wrong ‘bulk’ dissipation mechanism. When calculating the formal adjoint \( L^\dagger \) of the full operator \( L \) with the scalar product (22), one gets

\[
L^\dagger = L + L^\dagger,
\]

where \( L^\dagger \) is an operator with support only on the boundaries of the interval \([a, b]\), that we can formally write as

\[
L^\dagger = \frac{1}{2} \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & L_2^\dagger \end{array} \right),
\]

with \( L_2^\dagger \) given by the expression

\[
L_2^\dagger = \frac{\gamma(x)}{w(x)} \left( \delta(x - a) - \delta(x - b) \right),
\]

where \( \delta(x) \) formally denotes a Dirac-delta distribution. This is just a formal expression, that underlines precisely the need of a more careful treatment on the involved functional spaces, but it has the virtue of making apparent that the obstruction to selfadjointness lays at the boundaries, as one expects in our QNM problem, and not in the bulk, as one could naively conclude from the presence of a first-order operator \( L_2 \) (cf. discussion above): \( L_2^\dagger \) explicitly entails a boundary dissipation.
mechanism. In particular, we note that $L$ is self-adjoint in the non-dissipative $L^2$ case, as expected, but that this has required the introduction of quite a non-trivial scalar product.

As a bottomline, in this section we have cast the QNM problem as the eigenvalue problem of a non-selfadjoint operator. In the following section we discuss the implications of this.

### III. SPECTRAL STABILITY AND PSEUDOSPECTRUM

The spectrum of a non-selfadjoint operator is potentially unstable under small perturbations of the operator. Let us consider a linear operator $A$ on a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, and denote its adjoint by $A^\dagger$, satisfying $\langle A^\dagger u, v \rangle = \langle u, Av \rangle$. The operator $A$ is called normal if and only if $[A, A^\dagger] = 0$. In particular, a selfadjoint operator $A^\dagger = A$ is normal. In this setting, the ‘spectral theorem’ (under the appropriate functional space assumptions) states that a normal operator is characterized as being unitarily diagonalizable. The eigenfunctions of $A$ form an orthonormal basis and, crucially in the present discussion, the eigenvalues are stable under perturbations of $A$. The lack of such a ‘spectral theorem’ for non-normal operators entails a severe loss of control on eigenfunction completeness and the potential instability of the spectrum of the operator $A$. Here, we focus on this second aspect.

#### A. Spectral instability: the eigenvalue condition number

Let us consider an operator $A$ and an eigenvalue $\lambda_i$. Left $u_i$ and right $v_i$ eigenvectors are characterised as [76]

$$ A^\dagger u_i = \bar{\lambda}_i u_i, \quad Av_i = \lambda_i v_i, \quad (26) $$

with $\bar{\lambda}_i$ the complex conjugate of $\lambda_i$. Let us consider, for $\epsilon > 0$, the perturbation of $A$ by a (bounded) operator $\delta A$

$$ A(\epsilon) = A + \epsilon \delta A, \quad ||\delta A|| = 1. \quad (27) $$

The eigenvalues [77] in the perturbed spectral problem

$$ A(\epsilon)v_i(\epsilon) = \lambda_i(\epsilon)v_i(\epsilon), \quad (28) $$

satisfy

$$ |\lambda_i(\epsilon) - \lambda_i| = \epsilon \left[ \frac{|u_i, \delta A v_i(\epsilon)|}{|u_i, v_i|} \right] = \epsilon \left[ \frac{|u_i, \delta A v_i|}{|u_i, v_i|} \right] + O(\epsilon^2) \quad (29) $$

$$ \leq \epsilon \left[ \frac{|u_i||\delta A v_i|}{|u_i, v_i|} \right] + O(\epsilon^2) \leq \epsilon \left[ \frac{|u_i||\delta A v_i|}{|v_i|} \right] + O(\epsilon^2), $$

where the first line generalizes [8] [78] the expression employed (for self-adjoint operators, where $u_i = v_i$) in quantum mechanics first-order perturbation theory, the first inequality in the second line is the Cauchy-Schwartz inequality and in the second inequality we make explicit use of an operator norm $||\cdot||$ induced from that of the vector Hilbert space, so that $||\delta A v_i|| \leq ||\delta A||v_i||$, and $||\delta A|| = 1$ in (27). Then, defining the condition number $\kappa_i$ associated with the eigenvalue $\lambda_i$, we can write the bound for the perturbation of the eigenvalue $\lambda_i$

$$ |\lambda_i(\epsilon) - \lambda_i| \leq \epsilon \kappa_i, \quad \kappa_i = \kappa(\lambda_i) := \frac{|u_i||v_i|}{|u_i, v_i|^2}. \quad (30) $$

In the normal operator case, $u_i$ and $v_i$ are proportional (namely, since $A$ and $A^\dagger$ commute they can be diagonalized in the same basis). Then, again by Cauchy-Schwartz, $\kappa_i = 1$ and we encounter spectral stability: a small perturbation of order $\epsilon$ of the operator $A$ entails a perturbation of the same order $\epsilon$ in the spectrum. In contrast, in the non-normal case, $u_i$ and $v_i$ are not necessarily collinear. In the absence of a spectral theorem nothing prevents $u_i$ and $v_i$ to become close to orthogonality and $\kappa_i$ can become very large: small perturbations of $A$ can produce large deviations in the eigenvalues. The relative values of $\kappa_i$ control the corresponding instability sensitivity of different $\lambda_i$’s to an operator perturbation [79].

#### B. Pseudospectrum

A complementary approach to the study of the spectral (in)stability of the operator $A$ under perturbations consists in considering the following questions:

*Given the operator $A$ and its spectrum $\sigma(A)$, which is the set of complex numbers $\lambda \in \mathbb{C}$ that are actual eigenvalues of “some” small perturbation $A + \delta A$, with $||\delta A|| < \epsilon$? Does this set extend in $\mathbb{C}$ far from the spectrum of $A$?*

In this setting, if we are dealing with an operator that is spectrally stable, we expect that the spectrum of $A + \delta A$ will not change strongly with respect to that of $A$, so that the set of $\lambda \in \mathbb{C}$ corresponding to the first question above will not be far from $\sigma(A)$, staying in its vicinity at a maximum distance of order $\epsilon$. On the contrary, if we find a tiny perturbation $\delta A$ of order $||\delta A|| < \epsilon$ such that the corresponding eigenvalues of $A + \delta A$ actually reach regions in $\mathbb{C}$ at distances far apart from $\sigma(A)$, namely orders of magnitude above $\epsilon$, we will conclude that our operator suffers of an actual spectral instability.

##### 1. Pseudospectrum and operator perturbations

The previous discussion is formalized in the notion of pseudospectrum, leading to its following (first) definition [80].

**Definition 1** (Pseudospectrum: perturbative approach). Given $A \in M_n(\mathbb{C})$ and $\epsilon > 0$, the $\epsilon$-pseudospectrum $\sigma^\epsilon(A)$ of $A$ is

$$ \sigma^\epsilon(A) = \{ \lambda \in \mathbb{C}, \exists \delta A \in M_n(\mathbb{C}), ||\delta A|| < \epsilon : \lambda \in \sigma(A + \delta A) \}. $$

This notion of $\epsilon$-pseudospectrum $\sigma^\epsilon(A)$ is a crucial one in our study of eigenvalue instability since it implies that points in $\sigma^\epsilon(A)$ are actual eigenvalues of some $\epsilon$-perturbation of $A$: if $\sigma^\epsilon(A)$ extends far from the spectrum $\sigma(A)$ for a small $\epsilon$, then a small physical perturbation $\delta A$ of $A$ can produce large actual deviations in the perturbed physical spectrum. The pseudospectrum becomes a systematic tool to assess spectral (in)stability, as illustrated in the hydrodynamics context [4].

Although the characterization [31] of $\sigma^\epsilon(A)$ neatly captures the notion of (in)stability of $A$, from a pragmatic perspective it suffers from the drawback of not providing a constructive
approach to build such sets $\sigma^\epsilon(A)$ for different $\epsilon$’s (see however subsection [HFC] below, for a further qualification of this question in terms of random perturbation probes).

2. Pseudospectrum and operator resolvent

To address the construction of pseudospectra, another characterization of the set $\sigma(A)$ in (31) of Definition 1 is very useful. Such second characterization is based on the notion of the resolvent $R_A(\lambda) = (\lambda\text{Id} - A)^{-1}$ of the operator $A$.

An eigenvalue $\lambda$ of $A$ is a complex number that makes singular the operator $(\lambda\text{Id} - A)$. More generally, the spectrum $\sigma(A)$ of $A$ is the set $\{ \lambda \in \mathbb{C} \}$ for which the resolvent $R_A(\lambda)$ does not exist as a bounded operator (cf. details and subtleties on this notion in e.g. [11, 78]). This spectrum concept is a key notion for normal operators but, due to spectral instabilities discussed above, $\sigma(A)$ is not necessarily the good object to consider for non-normal operators, in our context. Specifically, an equivalent characterization of the $\epsilon$-pseudospectrum set $\sigma^\epsilon(A)$ in Definition 1 is given by the following definition [8, 11].

**Definition 2** (Pseudospectrum: resolvent norm approach). Given $A \in \mathcal{B}(\mathbb{C})$, its resolvent $R_A(\lambda) = (\lambda\text{Id} - A)^{-1}$ and $\epsilon > 0$, the $\epsilon$-pseudospectrum $\sigma^\epsilon(A)$ of $A$ is characterised as

$$\sigma^\epsilon(A) = \{ \lambda \in \mathbb{C} : \| R_A(\lambda) \| = \| (\lambda\text{Id} - A)^{-1} \| > 1/\epsilon \} . \quad (32)$$

This characterization captures that, for non-normal operators, the norm of the resolvent $R_A(\lambda)$ can be very large far from the spectrum $\sigma(A)$. This is in contrast with the normal-operator case, where (in the $\| \cdot \|_2$ norm)

$$\| R_A(\lambda) \|_2 \leq \frac{1}{\text{dist}(\lambda, \sigma(A))} . \quad (33)$$

In the non-normal case, one can only guarantee (e.g. [8])

$$\| R_A(\lambda) \|_2 \leq \frac{\kappa}{\text{dist}(\lambda, \sigma(A))} , \quad (34)$$

where $\kappa$ is also a condition number, different but related to the eigenvalue condition numbers $\kappa_i$ in (39) (cf., associated with the matrix diagonalising $A$, provides an upper bound to the individual $\kappa_i$; see [8] for details). In the non-normal case, $\kappa$ can become very large and $\epsilon$-pseudospectra sets can extend far from the spectrum of $A$ for small values of $\epsilon$. The extension of $\sigma^\epsilon(A)$ far from $\sigma(A)$ is therefore a signature of strong non-normality and indicates a poor analytic behavior of $R_A(\lambda)$.

The important point here is that the characterization of the $\epsilon$-pseudospectrum in Definition 2, namely Eq. (32), provides a practical way of calculating $\sigma^\epsilon(A)$. If we calculate the norm of the resolvent $\| R_A(\lambda) \|$ as a function of $\lambda = \Re(\lambda) + i\text{Im}(\lambda) \in \mathbb{C}$, this provides a real function of two real variables $(\Re(\lambda), \text{Im}(\lambda))$, the boundaries of the $\sigma^\epsilon(A)$ sets are just the ‘contour lines’ of the plot of this function $\| R_A(\lambda) \|$. In particular, $\epsilon$-pseudospectra are nested sets in $\mathbb{C}$ around the spectrum $\sigma(A)$, with $\epsilon$ decreasing towards the ‘interior’ of such sets and such that $\lim_{\epsilon \to 0} \sigma^\epsilon(A) = \sigma(A)$.

3. Pseudospectrum and quasimodes

For completeness, we provide a third equivalent characterization of the pseudospectrum in the spirit of characterising $\lambda$’s in the $\epsilon$-pseudospectrum set $\sigma^\epsilon(A)$ as ‘approximate eigenvalues’ of $A$, ‘up-to an error’ $\epsilon$, with corresponding ‘approximate (right) eigenvectors’ $v$. Specifically, it holds [8, 11] that $\sigma^\epsilon(A)$ can be characterised also by the following (third) definition.

**Definition 3** (Pseudospectrum: quasimode approach). Given $A \in \mathcal{B}(\mathbb{C})$ and $\epsilon > 0$, the $\epsilon$-pseudospectrum $\sigma^\epsilon(A)$ of $A$ and its associated $\epsilon$-quasimodes $v \in \mathbb{C}^n$ are characterised by

$$\sigma^\epsilon(A) = \{ \lambda \in \mathbb{C}, \exists v \in \mathbb{C}^n : \| A v - \lambda v \| < \epsilon \} . \quad (35)$$

This characterisation introduces the notion of “$\epsilon$-quasimode” $v$ (referred to as “pseudo-mode” in [8]), a key notion in the semiclassical analysis approach to the spectral study of $A$ [11]. On the other hand, this third characterization also clearly indicates the numerical difficulty that may occur when trying to determine the actual eigenvalues of $A$, since round-off errors are unavoidable. This signals the need of a careful treating, whenever addressing numerically the spectral problem of a non-normal operator $A$.

4. Pseudospectrum and choice of the norm

In this subsection we have presented the $\epsilon$-pseudospectrum as a notion that may be more adapted to the analysis of non-normal operators than that of the spectrum. We must emphasize however, that the notion of spectrum $\sigma(A)$ is intrinsic to the operator $A$, whereas the $\epsilon$-pseudospectrum $\sigma^\epsilon(A)$ is not, since it also depends on the choice of an operator norm. This is crucial, since it determines what we mean by “big/small” when referring to the perturbation $\delta A$, and therefore critically impacts the assessment of stability: a small operator perturbation $\delta A$ in a given norm, can be a large one when considering another norm. In the first case, from a large variation $\delta \lambda$ in the eigenvalues we would conclude instability, whereas in the second case such variation could be consistent with stability.

In this sense, from a mathematical perspective, the study of spectral (in)stability through pseudospectra amounts, in a good measure, to the identification of the proper scalar product determining the norm, that is, to the identification of the proper Hilbert space in which the operator $A$ acts. However, from a physical perspective we might not have such a freedom to choose a mathematically conveniently rescaled norm, since what we mean by large and small may be fixed by the physics of the problem, e.g. by the size of involved amplitudes, intensities or the energy contained in the perturbations. Then, the choice of an appropriate norm, both from a mathematical and physical perspective, is a fundamental step in the analysis (cf. discussion in [25]). This is the rationale behind the choice of the energy norm $\| \cdot \|_E$ in (20). Once the norm is chosen, the equivalent characterizations in Definitions 1, 2 and 3, respectively Eqs. (31), (32) and (35), emphasize complementary aspects of the $\epsilon$-pseudospectrum notion and the $\sigma^\epsilon(A)$ sets.
C. Pseudospectrum and random perturbations

When considering the construction of pseudospectra, we have presented the characterization of $\sigma^\epsilon(A)$ in terms of the resolvent $R_A(\lambda)$ in Definition 2, Eq. (32), as better suited than the one in terms of spectra of perturbed operators in Definition 1, Eq. (31). The reason is that the former involves only the unperturbed operator $A$, whereas the latter demands a study of the spectral problem for any perturbed operator $A + \delta A$ with small $\delta A$: a priori, the difficulty to explicitly control such space of possible $\delta A$ perturbations hinders an approach based on such characterisation in Definition 1.

But the very nature of the obstacle suggests a possible solution, namely to consider the systematic study of the perturbed spectral problem under random perturbations $\delta A$ as an avenue to explore $\epsilon$-pseudospectra sets. This heuristic expectation actually withstands a more careful analysis and constitutes the basis of a rigorous approach to the analysis of pseudospectra [11]. From a practical perspective, the systematic study of the spectral problem of $A + \delta A$ with (bounded) random $\delta A$ with $||\delta A||_2 < \epsilon$, has proven to be an efficient tool to explore the ‘migration’ of eigenvalues through the complex plane (inside the $\epsilon$-pseudospectra) [8]. This is complementary to (and ‘technically’ independent from) the evaluation of $\sigma^\epsilon(A)$ from the contour-lines of the norm $||R_A(\lambda)||$ of the resolvent. Such complementarity of approaches will prove key later in our analysis of Nollert & Price’s high-frequency perturbations of the Schwarzschild’s potential and the related QNMs.

Two important by-products of this random perturbation approach to the pseudospectrum are the following:

i) Random perturbations help identifying instability-triggering perturbations: $\epsilon$-pseudospectra and condition numbers $\kappa_i$ are efficient in identifying the instability of the spectrum and/or a particular eigenvalue $\lambda_i$, respectively. However, they do not inform on the specific kind of perturbation actually triggering the instability. This can be crucial to assess the physical nature of the found instability. The use of families of random operators adapted to specific types of perturbations sheds light on this precise point. We will make critical use of this in our assessment of Schwarzschild’s (in)stability.

ii) Random perturbations improve analyticity: a remarkable and apparently counter-intuitive effect of random perturbations is the improvement of the analytic behaviour of $R_A(\lambda)$ in $\lambda \in \mathbb{C}$ [11]. In particular, the norm $||R_A(\lambda)||$ gets reduced away from $\sigma(A)$, as for normal operators [cf. Eq. (33)], so that the $\epsilon$-pseudospectra sets pattern becomes “flattened” (a signature of good analytical behaviour) below the random-perturbation scale $\epsilon$.

To complement this perspective on the relation between the two given approaches to spectral (in)stability, namely perturbation theory and $\epsilon$-pseudospectra, respectively subsections IIIA and IIIB, let us connect eigenvalue condition numbers $\kappa_i(\lambda_i)$ with $\epsilon$-pseudospectra $\sigma^\epsilon(A)$. The question we want to address is: how far can the $\epsilon$-pseudospectrum $\sigma^\epsilon(A)$ get away from the spectrum $\sigma(A)$? The $\kappa_i$’s provide the answer.

Let us define the ‘tubular neighbourhood’ $\Delta_\epsilon(A)$ of radius $\epsilon$ around the spectrum $\sigma(A)$ as

$$\Delta_\epsilon(A) = \{\lambda \in \mathbb{C} : \text{dist}(\lambda, \sigma(A)) < \epsilon\},$$

which is always contained in the $\epsilon$-pseudospectrum $\sigma^\epsilon(A)$ [8]

$$\Delta_\epsilon(A) \subseteq \sigma^\epsilon(A).$$

The key question is about the inclusion in the other direction. Normal operators indeed satisfy [8]

$$\sigma^\epsilon(A) = \Delta_\epsilon(A),$$

where $\sigma^\epsilon(A)$ indicates the use of a $|| \cdot ||_2$ norm. That is, a $((||\delta A|| < \epsilon)$ perturbed eigenvalue of a normal operator can move up to a distance $\epsilon$ from $\sigma(A)$. This is what we mean by spectral stability: an operator perturbation of order $\epsilon$ induces an eigenvalue perturbation also of order $\epsilon$. However, in the non-normal case, where $\kappa(\lambda_i) > 1$, it holds (for small $\epsilon$) [8]

$$\sigma^\epsilon(A) \subseteq \Delta_{\kappa(\lambda_i)(1+O(\epsilon^2))}(\lambda_i),$$

so that $\sigma^\epsilon(A)$ can extend into a much larger tubular neighbourhood of radius $\sim \kappa(\lambda_i)$ around each eigenvalue, signaling spectral instability if $\kappa(\lambda_i) \approx 1$. This bound is the essential content of the Bauer-Fike theorem relating pseudospectra and eigenvalue perturbations (cf. [8] for a precise formulation).

IV. NUMERICAL APPROACH: CHERYSHEV’S SPECTRAL METHODS

The present work is meant as a first assessment of BH QNM (in)stability by using pseudospectra. At this exploratory stage, we address the construction of pseudospectra in a numerical approach. As indicated in section III B 3, the study of the spectral stability of non-normal operators is a challenging problem that demands high accuracy. Spectral methods provide well-adapted tools for these calculations [8, 81, 82].

We discretize the differential operator $L$ in [9, 14] via Chebyshev differentiation matrices, built on Chebyshev-Lobatto $n$-point grids, producing $L^n$ matrix approximates (we note systematically $n = N + 1$ in spectral grids, cf. appendix C. Once the operator is discretized, the construction of the pseudospectrum requires the evaluation of matrix norms. A standard practical choice [8, 81] involves the matrix norm induced from the Euclidean $L^2$ norm in the vector space $\mathbb{C}^n$ that, starting from Eq. (32) in the Definition 2 of the pseudospectrum, leads to the following rewriting [8, 81]

$$\sigma^\epsilon(A) = \{\lambda \in \mathbb{C} : \sigma_{\text{min}}(\lambda id - A) < \epsilon\},$$

where $\sigma_{\text{min}}(M)$ denotes the smallest singular value of $M$, that is, $\sigma_{\text{min}}(M) = \min\{\sqrt{\lambda} : \lambda \in \sigma(M^\star M)\}$, with $M \in M_n(\mathbb{C})$ and $M^\star$ its conjugate-transpose $M^\star = M^t$.

Although Eq. (40) captures the spectral instability structure of $A$, the involved $L^2$ scalar product in $\mathbb{C}^n$ is neither faithful to the structure of the operator $L$ in Eq. (9), nor to the physics of the BH QNM problem (cf. discussion in section III B 4).
Instead, we rather use the natural norm in the problem, specifically the Chebyshev-discretised version of the ‘energy norm’ but following from the Chebyshev-discretised version of the scalar product. Specifically, we write the discretised scalar product in an appropriate basis as (we abuse the notation, since we use $\langle \cdot, \cdot \rangle_E$ as in (22), although this is now a scalar product in a finite-dimensional space $\mathbb{C}^n$)

$$
(u, v)_E = (u^*)^T G_E^{00} v = u^* \cdot G^E \cdot v , \quad u, v \in \mathbb{C}^n ,
$$

where $G_E^{00}$ is the Gram matrix corresponding to (22) (cf. appendix B for its construction) and we note $u^* = \bar{u}^T$. The adjoint $A^\dagger$ of $A$ with respect to $\langle \cdot, \cdot \rangle_E$ writes then

$$
A^\dagger = (G^E)^{-1} \cdot A^* \cdot G^E .
$$

The vector norm $\| \cdot \|_E$ in $\mathbb{C}^n$ associated with $\langle \cdot, \cdot \rangle_E$ in Eq. (41) induces a matrix norm $\| \cdot \|_M$ in $\mathbb{M}_n(\mathbb{C})$ (again, we abuse notation by using the same symbol for the norm in $\mathbb{C}^n$ and in $\mathbb{M}_n(\mathbb{C})$). Then, cf. appendix B the $\epsilon$-pseudospectrum $\sigma^\epsilon_E(A)$ of $A \in \mathbb{M}_n(\mathbb{C})$ in the norm $\| \cdot \|_E$ writes

$$
\sigma^\epsilon_E(A) = \{ \lambda \in \mathbb{C} : s^\epsilon_E(\lambda^2 I - A) < \epsilon \} ,
$$

where $s_E$ is the smallest of the “generalized singular values”

$$
s^\epsilon_E(M) = \min \{ \sqrt{\lambda} : \lambda \in \sigma(M^\dagger M) \} ,
$$

with $M \in \mathbb{M}_n(\mathbb{C})$ and its adjoint $M^\dagger$ given by Eq. (42).

V. A TOY MODEL: PÖSCHL-TELLER POTENTIAL

As presented in the previous sections, in our study of BH QNMs and their (in)stabilities, we exploit the geometrical framework of the hyperboloidal approach to analytically impose the physical boundary conditions at the BH horizon and at the radiation zone (future null infinity). As discussed in section II a crucial feature of such a strategy is that it allows us to cast the calculation of the QNM spectrum explicitly as the spectral problem of a non-selfadjoint differential operator, which is then the starting point for the tools assessing spectral instabilities presented in section III namely the construction of the pseudospectrum and the analysis of random perturbations. Finally, spectral methods discussed in section IV are employed to study these spectral issues through a discretisation for the derivative operators. Prior to the study of the BH case, the goal of this section is to illustrate this strategy in a toy model, namely the one given by the Pöschl-Teller potential.

A. Hyperboloidal approach in Pöschl-Teller

The Pöschl-Teller potential [83], given by the expression

$$
V(\bar{x}) = \frac{V_o }{ \cosh^2(\bar{x}) } = V_o \sech^2(\bar{x}) , \quad \bar{x} \in -\infty, \infty ,
$$

has been widely used as a benchmark for the study of QNMs in the context of BH perturbation theory (e.g. [84,85]). Interestingly, QNMs of this potential have been very recently revisited to illustrate, on the one hand, the hyperboloidal approach to QNMs in a discussion much akin to the present one (cf. [73], cast in the setting of de Sitter spacetime) or, on the other hand, functional analysis key issues related to the selfadjointness of the relevant operator [87]. Our interest in Pöschl-Teller stems from the fact that it shares the fundamental behavior regarding QNM (in)stability to be encountered later in the BH context, but in a mathematically much simpler setting. In particular, Pöschl-Teller presents weaker singularities than the Regge-Wheeler and Zerilli potentials in Schwarzschild, that translates in the absence of a continuous part of the spectrum of the relevant operator $L$ (corresponding to the “branch cut” in standard approaches to QNMs).

Let us consider the compactified hyperboloids given by Bizoń-Mach coordinates [88,89] mapping $\mathbb{R}$ to $[-1,1]$

$$
\begin{align*}
\tau &= \bar{t} - \ln(\cosh \bar{x}) , \\
x &= \tanh \bar{x} ,
\end{align*}
$$

or, equivalently

$$
\begin{align*}
\bar{t} &= \tau - \frac{1}{2} \ln(1 - x^2) , \\
\bar{x} &= \text{arctanh}(x) .
\end{align*}
$$

In the spirit of the conformal compactification along the hyperboloids described in section III we add the two points at (null) infinity (no BH horizon here), namely $x = \pm 1$, so that we work with the compact interval $[a, b] = [-1,1]$. Under this transformation the wave equation (41) reads

$$
(1 - x^2) \left( \partial_x^2 + 2x \partial_x \partial_x + \partial_x + 2x \partial_x - (1 - x^2) \partial_x^2 \right) + V(\bar{x}) \phi = 0 ,
$$

namely the version of Eq. (7) corresponding to the transformation (46). We notice that angular labels $(l, m)$ are not relevant in the one-dimensional Pöschl-Teller problem. If $x \neq 1$ we can divide by $(1 - x^2)$ and, defining

$$
\bar{V}(x) = \frac{V}{(1 - x^2)} ,
$$

we can write

$$
(\partial_x^2 + 2x \partial_x \partial_x + \partial_x + 2x \partial_x - (1 - x^2) \partial_x^2 + \bar{V}) \phi = 0 .
$$

This expression is formally valid for any given potential $V(x)$ (although analyticity issues may appear if the asymptotic decay is not sufficiently fast, as it is indeed the case for Schwarzschild potentials at $\mathcal{I}^+$). If we now insert the Pöschl-Teller expression (45) and notice $\sech^2(\bar{x}) = 1 - x^2$, we get a remarkably simple effective potential $\bar{V}$, actually a constant

$$
\bar{V}(x) = V_o .
$$

In particular, the Pöschl-Teller wave equation (50) exactly corresponds to Eq. (4) in [73], so that the Pöschl-Teller problem is equivalent to the Klein-Gordon equation in de Sitter spacetime with mass $m^2 = V_o$. In the following, we choose $\lambda = 1/\sqrt{V_o}$ in the rescaling (45), so that we can set

$$
\bar{V} = 1 .
$$
Performing now the first-order reduction in time (53-59) we get for \( w(x), p(x), q(x) \) and \( \gamma(x) \) in Eq. (12) the values
\[
w(x) = 1, \quad p(x) = (1 - x^2), \quad q(x) = \tilde{V} = 1
\]
\[
\gamma(x) = -x,
\]
and therefore the operators \( L_1 \) and \( L_2 \) building the operator \( L \) in Eq. (10) write, in the Pöschl-Teller case, as
\[
L_1 = \partial_x ((1 - x^2) \partial_x) - 1
\]
\[
L_2 = -(2x \partial_x + 1).
\]
As discussed in section II C 1, the function \( p(x) = 1 - x^2 \) vanishes at the boundaries of the interval \([a, b] = [-1, 1]\), defining a singular Sturm-Liouville operator. This is at the basis of the absence of boundary conditions, if sufficient regularity is enforced on the eigenfunctions of the spectral problem. Regularity therefore encodes the outgoing boundary conditions (see below). Finally, the scalar product (22) writes in this case as
\[
\langle u_1, u_2 \rangle_\epsilon = \left( \begin{array}{c} \phi_1 \\ \psi_1 \\ \phi_2 \\ \psi_2 \end{array} \right) = \frac{1}{2} \int_{-1}^{1} \left( \bar{\psi}_1 \phi_2 + (1 - x^2) \partial_x \bar{\phi}_1 \partial_x \phi_2 + \bar{\phi}_1 \phi_2 \right) dx.
\]

B. Pöschl-Teller QNM spectrum

1. Exact Pöschl-Teller QNM spectrum

Pöschl-Teller QNM spectrum can be obtained by solving the eigenvalue problem (14-15) with operators \( L_1 \) and \( L_2 \) given by Eq. (54). As discussed above, no boundary conditions need to be added, if we enforce the appropriate regularity. In this particular case, this eigenvalue problem can be solved exactly. The resolution itself is informative, since it illustrates this regularity issue concerning boundary conditions.

If we substitute the first component of (15) into the second or, simply, if we take the Fourier transform in \( \tau \) in Eq. (50) (with \( \tilde{V} = 1 \) from the chosen \( \lambda \) leading to Eq. (52)), we get
\[
\left[ (1 - x^2) \frac{d^2}{dx^2} - 2(i\omega + 1)x \frac{d}{dx} - i\omega(i\omega + 1) - 1 \right] \phi = (56)
\]
This equation can be solved in terms of the hypergeometric function \( 2F_1(a, b; c; z) \), with \( z = \frac{1 - x}{2} \) (see details in appendix D). In particular, for each value of the spectral parameter \( \omega \) we have a solution that can be written as a linear combination of linearly independent solutions obtained from \( 2F_1(a, b; c; z) \). Discrete QNMs are obtained only when we enforce the appropriate regularity, that encodes the outgoing boundary conditions. In this case, this is obtained by enforcing the solution to be analytic in \( x \in [-1, 1] \) (corresponding in \( z \) to analyticity in the full closed interval \([0, 1]\)), which amounts to truncate the hypergeometric series to a polynomial. We emphasize that such a need of truncating the infinite series to a polynomial, a familiar requirement encountered in many different physical settings, embodies here the enforcement of outgoing boundary conditions. In sum, this strategy leads to the Pöschl-Teller QNM frequencies (cf. e.g. (55, 56))
\[
\omega_n^\pm = \pm \sqrt{\frac{3}{2}} + i \left( n + \frac{1}{2} \right),
\]
with corresponding QNM eigenfunctions in this setting
\[
\phi_n^\pm(x) = P_n^{(i\omega_n^\pm, i\omega_n^\pm)}(x), \quad x \in [-1, 1],
\]
where \( P_n^{(\alpha, \beta)} \) are the Jacobi polynomials (see appendix D).

Two comments are in order here:

i) QNMs are normalizable: QNM eigenfunctions \( \phi_n^\pm(x) \) are finite and regular when making \( x \rightarrow \pm \infty \), corresponding to \( x = \pm 1 \). This is in contrast with the exponential divergence of QNM eigenfunctions in Cauchy approaches, where the time slices reach spatial infinity \( \tau \). This is a direct consequence of the hyperboloidal approach with slices reaching \( J^+ \). The resulting normalizability of the QNM eigenfunctions can be relevant in e.g. resonant expansions (cf. e.g. discussion in [54]).

ii) QNM regularity and outgoing conditions: In the present case, namely Pöschl-Teller in Bizoo-Mach coordinates, analyticity (actually polynomial structure) implements the regularity enforcing outgoing boundary conditions. Analyticity is too strong in the general case. But asking for smoothness is not enough (see e.g. [67]). In Refs. [70,72] this problem is approached in terms of Gevrey classes, that interpolate between analytic and (smooth) \( C^\infty \) functions, identifying the space of \( (\sigma, 2) \)-Gevrey functions as the proper regularity notion. The elucidation of the general adequate functional space for QNMs, tantamount of the consistent implementation of outgoing boundary conditions, is crucial for the characterization of QNMs in the hyperboloidal approach.

2. Numerical Pöschl-Teller QNM spectrum

Fig. 3 shows the result of the numerical counterpart of the Pöschl-Teller eigenvalue calculation, whose exact discussion has been presented above, by using the discretised operators \( L, L_1 \) and \( L_2 \) described in section [14] and appendix D.

\[
L^N v_n^{(N)} = \omega_n^{(N)} v_n^{(N)}.
\]
This indeed recovers numerically the analytical result in Eq. (57) (we drop the “\( \pm \)” label, focusing on one of the branches symmetric with respect the vertical axis).

We stress that the remarkable agreement between the numerical values from the bottom panel of Fig. 3 (see also Fig. 7 later) and the exact expression (57) is far from being a trivial result, as already illustrated in existing systematic numerical studies. This is in particular the case of Ref. [21] (where Pöschl-Teller is referred to as the Eckart barrier potential), where the fundamental mode \( \omega_0^+ \) in (57) is stable and accurately recovered, whereas all overtones \( \omega_n^{\pm}_{n \geq 1} \) suffer from
a strong instability (triggered, according to the discussion in [91, 92], by the $C^1$-regularity of the approximation modelling the Pöschl-Teller potential) and could not be recovered.

In our setting, a convergence study of the numerical values shows that the relative error

$$
\mathcal{E}_n^{(N)} = 1 - \frac{\omega_n^{(N)}}{\omega_n},
$$

between the exact QNM $\omega_n$ and the corresponding numerical approximation $\omega_n^{(N)}$ (obtained at a given truncation $N$ of the differential operator) actually increases with the resolution. This is a first hint of the instabilities to be discussed later. Indeed, the top panel of Fig. 4 displays the error for the fundamental mode $n = 0$ and the first overtones $n = 1, \ldots, 4$ when the eigenvalue problem for the discretised operator is naively solved with the standard machine roundoff error for floating point operations (typically, $\sim 10^{-16}$ for double precision).

It is astonishing how, despite the simplicity of the exact solution, the relative error grows significantly already for the first overtones and, crucially, more strongly as the damping grows with higher overtones. To mitigate such a drawback, one needs to modify the numerical treatment in order to allow for a smaller roundoff error in floating point operations. The bottom panel Fig. 4 shows the error $\mathcal{E}_n^{(N)}$ when the calculations are performed with an internal roundoff error according to $5 \times$ Machine Precision, i.e., $\sim 10^{-5 \times 16}$. In this case, the fundamental QNM $n = 0$ is “exactly” calculated at the numerical level (i.e. the difference between its exact value and the numerical approximation vanishes in the employed precision). The error for the overtones still grows, but in a safe range for all practical purposes. The values displayed in the bottom panel of Fig. 3 were obtained with internal roundoff error set to $10 \times$ Machine Precision and we can assure that the errors of all overtones are smaller than $10^{-100}$.

3. Condition numbers of QNM frequencies

The growth in the relative error as we move to higher overtones in Fig. 4 suggests an increasing spectral instability in $n$ of eigenvalues $\omega_n^{(N)}$, triggered by numerical errors related to machine precision, so that this instability can be reduced (but not eliminated) by improving the internal roundoff error.

At the level of the non-perturbed spectral problem (59), and in order to assess more systematically such spectral instability, we can apply the discussion in section IIIA to the Pöschl-Teller approximates $L_N$. Namely, solving the right-eigenvector problem (59) together with left-eigenvector one

$$
(L_N)^+ u_n^{(N)} = \bar{\omega}_n^{(N)} u_n^{(N)},
$$

FIG. 3. Pöschl-Teller QNM problem. Bottom panel: QNM spectrum for the Pöschl-Teller potential, calculated in the hyperboloidal approach described in section V A, with Chebyshev spectral methods and enhanced machine precision. Top panel: ratios of condition numbers $\kappa_n$ of the first QNMs over the condition number $\kappa_0$ of the fundamental QNM, indicating a growing spectral instability compatible with the need of using enhanced machine precision.
we can compute the condition numbers \( \kappa_n(N) = \kappa(\omega_n(N)) = \|v_n(N)\|_E / \|u_n(N)\|_E \) introduced in Eq. (30). Notice that this is quite a non-trivial calculation, since it involves: first, the construction of the adjoint operator \( (L^N)^\dagger = (G^E)^{-1} \cdot (L^N)^* \cdot G^E \) and, second, the calculation of scalar products \( \langle \cdot, \cdot \rangle_E \) and (vector) energy norms \( \| \cdot \|_E \). These calculations involve the determination of the Gram matrix \( G^E \) associated with the energy scalar product (55) by implementing expression (C23) in appendix C. These expressions are quite non-trivial and in the following section we provide a strong test to the associated analytical and numerical construction.

The result is shown in the top panel of Fig. 3. The ratio of the condition numbers \( \kappa_n \), relative to the condition number of the fundamental mode \( \kappa_0 \), grows strongly with \( n \). This indicates a strong and increasing spectral instability consistent with the error convergence displayed in Fig. 4. The rest of this section is devoted to address this spectral stability issue.

### C. Pöschl-Teller pseudospectrum

#### 1. Motivating the pseudospectrum

As the previous discussion makes apparent, a crucial question that arises after obtaining the QNM spectrum of the operator \( L \) in Eq. (10), with \( L_1 \) and \( L_2 \) in (54) is whether such QNM eigenvalues are stable under small perturbations of \( L \). More specifically for QNM physics, and in the context of the wave equation (4), whether the QNM spectrum is stable under small perturbations of the potential \( V \). The latter is the specific type of perturbation we are assessing in this work.

In the numerical approach we have adopted, perturbations in the spectrum under small perturbations in \( L \) may arise either from numerical noise resulting from the chosen discretisation strategy, or they can originate from “real-world sources”, namely small physical perturbations of the considered potential \( V \). Ultimately, in the BH setting for which Pöschl-Teller provides a toy model, such physical perturbations could stem from a “dirty” environment surrounding a black hole, and/or emergent fluctuations from quantum-gravity effects. Therefore, the question of whether QNM spectrum instability is a structural feature of the operator \( L \) — i.e. not just an artifact of a given numerical algorithm — is paramount for our understanding of the fundamental physics underlying the problem.

A pragmatic approach to address this question consists in explicitly introducing families of perturbations \( \delta L \) and study their effect on the QNM spectra \( N \), and evaluate the pseudospectrum of the unperturbed operator \( L \) — i.e. not just an artifact of a given numerical algorithm — is paramount for our understanding of the fundamental physics underlying the problem. We will make contact with this approach later in section V D but before that, we apply the pseudospectrum approach described in section III B to the Pöschl-Teller problem. Indeed, one of the main goals of our present work is to bring attention to and emphasise the fact that the unperturbed operator already contains crucial information to assess such (in)stability features. We have already encountered this fact in the evaluation of the condition numbers \( \kappa_n \) in Fig. 3, that only depends on the unperturbed operator \( L \), but we develop this theme further with the help of the pseudospectrum notion. Indeed, pseudospectrum analysis provides a framework to identify the (potential) spectral instability, which is oblivious to the particular perturbation employed. Then, in a second stage, actual perturbations of the operators with a particular emphasis on random perturbations along the lines in section III C can be used to complement and refine such pseudospectrum analysis.

Fig. 5 shows the pseudospectrum for the Pöschl-Teller potential in the energy norm of Eq. (20) associated with the scalar product (55). Let us explain the content of such a figure. According to the characterization in the Definition 1, namely Eq. (11), of the \( \epsilon \)-pseudospectrum of the operator \( L \), the set \( \sigma'(L) \) is the collection of all complex numbers \( \omega \in \mathbb{C} \) that are actual eigenvalues for some operator \( L + \delta L \), where \( \delta L \) is a small perturbation of “size” smaller than a given \( \epsilon \). Consequently and crucially, adding a perturbation \( \delta L \) with \( \|\delta L\|_E < \epsilon \) entails an actual (“physical”) change in the eigenvalues \( \omega_n \), that can reach up the boundary of the \( \sigma'(L) \) set, marked in white lines in Fig. 5. The key question is to assess if \( \epsilon \)-pseudospectra for small \( \epsilon \) can extend in large areas of \( \mathbb{C} \) or not. This is tightly related to condition numbers \( \kappa_n \), controlling eigenvalue spectral (in)stabilities, as explicitly estimated by the Bauer-Fike relation (39) between \( \epsilon \)-pseudospectra sets and ‘tubular neighbourhoods’ \( \Delta_{\epsilon \kappa} \) of radii \( \epsilon \kappa_n \) around the spectrum. Let us first discuss a selfadjoint test case and, in a second stage, the actual non-selfadjoint case [94].

#### 2. Pseudospectrum: selfadjoint case

As discussed in section II D setting \( L_2 = 0 \) in Eq. (10) —while keeping \( L_1 \) as in Eq. (54)— leads to a selfadjoint
Let us describe Fig. 6 in more detail. Boundaries of $\epsilon$-pseudospectra $\sigma^\epsilon(A)$ are marked in white lines, with $\epsilon$’s corresponding to the values in the color log-scale. Pseudospectra $\sigma^\epsilon(L)$ are, by construction, “nested sets” around the spectrum (red points in Fig. 6), the latter corresponding to the “innermost set” $\sigma^0(L)$ when $\epsilon \to 0$. In this selfadjoint case, condition numbers in (30) must satisfy $\kappa_n = 1$ for $\omega_n$ ($0 \leq n \leq 20$). Middle panel: Pseudospectrum: “flat” pattern typical of a spectrally stable (normal) operator. Bottom panel: Zoom near the spectrum, with concentric circles (“radius $\epsilon$” tubular regions around eigenvalues) characteristic of stability.

Horizonal boundaries of $\epsilon$-pseudospectra, when far from the spectrum, is a consequence (in this particular problem) of the use of the energy norm. If another norm is used, e.g. the standard one induced from the $L^2$ norm in $\mathbb{C}^n$, the global “flatness” of the pseudospectrum is still recovered, especially when comparing with the corresponding scales in Fig. 5 indicating already a much more stable situation than the general $L_2 \neq 0$ case. But when refining the scale, one would observe that pseudospectra contour lines far from the spectrum are not horizontal but present a slope growing with the frequency. This indicates that, under perturbations of the same size in that $L^2$ norm, higher frequencies can move further that low frequencies, this being in tension with the equal stability of all the eigenvalues. What is going on is the effect commented in section III B 4 concerning the impact of the norm choice on the notions of “big/small”: when using the $L^2$ norm, we would be marking with the same “small” $\epsilon$ different perturbations among which there exist $\delta L$ instances that actually excite strongly the high frequencies, but such a feature is blind to the $L^2$ norm. If using however a norm sensitive to high-frequency effects, as it is the case of the energy norm that has a $H^1$ character incorporating derivative terms, those same perturbations $\delta L$ would have a norm much larger than $\epsilon$, the derivative terms in the energy norm indeed weighting more as the frequency grows. What in the $L^2$ norm was a small perturbation $\delta L$, turns out to be a big one in the energy one, so stronger modifications in the eigenvalues are indeed consistent with stability. In practice, in order to construct a given $\epsilon$-pseudospectrum set, such “high-energy” perturbations $\delta L$ need to be renormalized to keep $\epsilon$ fixed, something that the energy norm does automatically. This is a neat example of how the choice of the norm affects the assessment of spectral stability and, in particular, of the importance of the energy norm in the present work, namely for high-frequency issues.

Fig. 6 may appear as a boring figure, but it is actually a tight and constraining test of our construction, both at the analytical and numerical level. First of all, panels in Fig. 6 correspond to different calculations: the top panel results from an eigenvalue calculation (actually two, one for $L$ and another for $L^1$), whereas the “map” in the middle and bottom panels is the result of calculating the energy norm of the resolvent $R_L(\omega) = (\omega |L| - L)^{-1}$ at each point $\omega \in \mathbb{C}$. Both calculations depend on the construction of the Gram matrix $G|L^2|G$, but are indeed different implementations. The $\kappa_n = 1$ values in the top panel constitute a most stringent test, since modifications in either the analytical structure of the scalar product (55) or the slightest mistake in the discrete counterpart (C23) spoil the result. As discussed at the end of section IV B 3, this provides a strong test both of the analytical treatment and the numerical discretization of the differential operator and scalar product. On the other hand, the plain flatness of the pseudospectrum in the middle panel is a strong test of the selfadjoint character of $L$ when $L_2 = 0$ that, given the subtleties of the spectral discretization explained in appendix C provides a reassuring
non-trivial to the whole numerical scheme.

3. Non-selfadjoint case: Pöschl-Teller pseudospectrum

In contrast with the selfadjoint case, when considering the actual $L_2 \neq 0$ of the Pöschl-Teller case, pseudospectra sets $\sigma(\epsilon)(L)$ with small $\epsilon$ extend in Fig. 5 into large regions of $\mathbb{C}$ (with typical sizes much larger than $\epsilon$) and therefore the operator $L$ is spectrally unstable: very small (physical) perturbations $\delta L$, with $||\delta L||_\epsilon < \epsilon$, can produce large variations in the eigenvalues up to the boundary of the now largely extended region $\sigma(\epsilon)(L)$. Such strong variations of the spectrum are not a numerical artifact, related e.g. to machine precision, but they rather correspond to an actual structural property of the non-perturbed operator. Indeed, large values of the condition numbers $\kappa_\epsilon$ in the top panel of Fig. 3 entail that the tubular sets $\Delta_\kappa(\epsilon)(L)$ in Eq. (34) extend now into large areas in $\mathbb{C}$. This fact on $\kappa_\epsilon$’s is consistent with the large regions in Fig. 5 corresponding to $\sigma(\epsilon)(L)$ sets with very small $\epsilon$’s. Such an non-trivial pattern of $\epsilon$-pseudospectra is a strong indication of spectral instability, although without a neat identification of the actual nature of the perturbations triggering instabilities.

4. Reading pseudospectra: “topographic maps” of the resolvent

In practice, if one wants to read from pseudospectra —such as those in Fig. 5 or Fig. 6— the possible effect on QNMs of a physical perturbation of (energy) norm of order $\epsilon$, one must first determine the “white-line” corresponding to that $\epsilon$ (using the log-scale). Then, eigenvalues can move potentially in all the region bounded by that line (namely, the $\epsilon$-pseudospectrum set for the non-perturbed operator $L$) that, in Fig. 5 corresponds to the region “above” white lines.

Pseudospectra can actually be seen as a “map” of the analytical structure of the resolvent $R_L(\omega) = (\omega I - L)^{-1}$ of the operator $L$, taken as a function of $\omega$. This corresponds to the characterization in Definition 2 of the pseudospectrum, Eq. (19), which is indeed the one used to effectively construct the pseudospectrum (specifically, its realisation (43) in the energy norm; cf. section 2.3 for details). In this view, the boundaries of the $\epsilon$-pseudospectra (white lines in Figs. 5 and 6) can be seen as “contour lines” of the “height function” $||R_L(\omega)||_\epsilon$, namely the norm of the resolvent $R_L(\omega)$. In quite a literal sense, the pseudospectrum can be read then as a topographic map, with stability characterised by very steep throats around eigenvalues fastly reaching flat zones away from the spectrum, whereas instability corresponds to non-trivial “topographic patterns” extending in large regions of the map far away from the eigenvalues.

In sum, this “topographic perspective” makes apparent the stark contrast between the flat pattern of the selfadjoint case of Fig. 6 corresponding to stability, and the non-trivial pattern of the (non-selfadjoint) Pöschl-Teller pseudospectrum in Fig. 5 in particular indicating a (strong) QNM sensitivity to perturbations that increases as damping grows.

D. Pöschl-Teller perturbed QNM spectra

Pseudospectra inform about the spectral stability and instability of an operator, but do not identify the specific type of perturbation triggering instabilities. Therefore, in a second stage, it is illuminating to complement the pseudospectrum information with the exploration of spectral instability with “perturbative probes” into the operator, always under the perspective acquired with the pseudospectrum. A link between both pseudospectra and perturbation strategies is provided by the Bauer-Fike theorem [8], as expressed in Eq. (39).

1. Physical instabilities: perturbations in the potential $V$

Not all possible perturbations of the $L$ operator are physically meaningful. An instance of this, in the setting of our numerical approach, are machine precision error perturbations $\delta L^N$ to the $L^N$ matrix. As discussed in section 3, machine precision errors indeed trigger large deviations in the spectrum, consistently with the non-trivial pattern of the pseudospectrum in Fig. 5 but clearly we should not consider such effects as physical. They are a genuine numerical artifact, since the structure of the perturbation $\delta L^N$ does not correspond to any physical or geometrical element in the problem.

The methodology we follow to address this issue is: i) given a grid resolution $N$, we first set the machine precision to a value sufficiently high so as to guarantee that all non-perturbed eigenvalues are correctly recovered, and ii) we then add a prescriped perturbation with the specific structure corresponding to the physical aspect we aim at studying.

In the present work we focus on a particular kind of perturbation, namely perturbations to the potential $V$ and more specifically, perturbations $\delta V$ to the rescaled potential $V$ in (21). This is in the spirit of studying the problem in [11]. That is, we consider perturbations $\delta L$ to the $L$ operator of the form

$$\delta L = \begin{pmatrix} 0 & 0 \\ \delta V & 0 \end{pmatrix}.$$ (62)

We note that, at the matrix level, the $\delta V$ submatrix is just a diagonal matrix. Therefore, the structure of $\delta L$ in Eq. (62) is a very particular one. The pseudospectrum in Fig. 5 tells us that $L$ is spectrally unstable, and we know that machine precision perturbations trigger such instabilities, but nothing guarantees that $L$ is actually unstable under a perturbation of the particular form (62). It is a remarkable fact, crucial for our physical discussion, that $L$ is indeed unstable under such perturbations and, therefore, under perturbations of the potential $V$.

2. Random and high-frequency perturbations in the potential $V$

We have considered two types of generic, but representative, perturbations $\delta L$ of the form given in Eq. (62):

i) Random perturbations $\delta \tilde{V}$: we set the perturbation according to a normal Gaussian distribution on the collocation points of the grid. This is, by construction, a
FIG. 7. Left column: Sequence of QNM spectra for the Pöschl-Teller potential subject to a random perturbation $\delta \tilde{V}_r$ of increasing “size” (in energy norm). The sequence shows how “switching on” a perturbation makes the QNMs migrate to a new branch (that actually follows closely a pseudospectrum contour line, compare with Fig. 5), in such a way that the instability starts appearing at highly-damped QNMs and descends in the spectrum as the perturbation grows (unperturbed values, in red, are kept along the sequence for comparison). The top panel corresponds to the non-perturbed potential shown in Fig. 3, the second panel shows how a random perturbation with (energy) norm $||\delta \tilde{V}_d||_E = 10^{-8}$ already reaches the 6th QNM overtone, whereas in the third panel a perturbation with $||\delta \tilde{V}_d||_E = 10^{-16}$ already reaches the 3rd overtone. This confirms the instability already detected in the pseudospectrum, indicating its high-frequency nature. Crucially, to reach the fundamental mode, $\delta \tilde{V}_d \approx \cos(2\pi k x)$, the first panel shows again the unperturbed potential, whereas the second one shows that a “low frequency” ($k = 1$) perturbation leaves the spectrum unperturbed, in spite of the $||\delta \tilde{V}_d||_E = 10^{-8}$ norm (compare with the random case with the same norm); this illustrates the harmless character of “low frequency” perturbations. The third panel shows how keeping the norm of the perturbation but increasing its frequency indeed “switches on” the instability, confirming the “high frequency” insight gained from random perturbations. The fourth panel shows how the instability increases with the frequency but less efficiently than with random perturbations of the same norm.

Right panel: Sequence of QNM spectra for Pöschl-Teller subject to a deterministic perturbation $\delta \tilde{V}_d \sim \cos(2\pi k x)$. The first panel shows again the unperturbed potential, whereas the second one shows that a “low frequency” ($k = 1$) perturbation leaves the spectrum unperturbed, in spite of the $||\delta \tilde{V}_d||_E = 10^{-8}$ norm (compare with the random case with the same norm). Crucially, to reach the fundamental mode, $\delta \tilde{V}_d \sim \cos(2\pi k x)$. The first panel shows again the unperturbed potential, whereas the second one shows that a “low frequency” ($k = 1$) perturbation leaves the spectrum unperturbed, in spite of the $||\delta \tilde{V}_d||_E = 10^{-8}$ norm (compare with the random case with the same norm). Crucially, to reach the fundamental mode, $\delta \tilde{V}_d \approx \cos(2\pi k x)$. The first panel shows again the unperturbed potential, whereas the second one shows that a “low frequency” ($k = 1$) perturbation leaves the spectrum unperturbed, in spite of the $||\delta \tilde{V}_d||_E = 10^{-8}$ norm (compare with the random case with the same norm). Crucially, to reach the fundamental mode, $\delta \tilde{V}_d \approx \cos(2\pi k x)$.
high-frequency perturbation. Random perturbations are a standard tool [3] to explore generic properties of spectral instability and there exists indeed a rich interplay between pseudospectra and random perturbations [11].

ii) Deterministic perturbations $\delta \hat{V}_i$: we have chosen

$$
\delta \hat{V}_i \sim \cos(2\pi k x),
$$

(63)

in order to address the specific impact of high and low frequency perturbations in QNM spectral stability, by exploring the effect of changing the wave number $k$.

Perturbations $\delta \hat{V}$ are then rescaled so as to guarantee $||\delta L||_\infty = \epsilon$. The impact on QNM frequencies resulting from adding these perturbations is shown in Fig. 7. In both random and deterministic cases, the sequence of images in Fig. 7 shows a high-frequency instability of QNM overtones, that “migrate” towards new QNM branches. The fundamental (slowest decaying) QNM is however stable under these perturbations. More generally, such QNM instability is sensitive with respect to both perturbations’ “size” and frequency.

One observes a systematic convergence, with the relative error dropping circa $10^8$ from $N = 150$ to $N = 400$. This result confirms that the spectrum corresponds indeed to the new, perturbed operator, and is not a numerical artifact. This neatly shows the unstable nature of the QNM spectrum of the unperturbed Pöschl-Teller potential.

3. Perturbed QNM branches and pseudospectrum

High-frequency perturbations trigger the migration of QNM overtone frequencies to new perturbed QNM branches. Fig. 9 displays the perturbed QNM spectra on the top of the pseudospectra for the unperturbed operator. The remarkable “predictive power” of the pseudospectrum becomes apparent: perturbed QNMs “follow” the boundaries of pseudospectrum sets. That is, QNM overtones “migrate” to new branches closely tracking the $\epsilon$-pseudospectra contour lines. This happens for both random and deterministic high-frequency perturbations. Crucially, no such instability is observed for low-frequency deterministic perturbations, with small wave number $k$. Consequently, we shall refer in the following to this effect as an ultraviolet instability of QNM overtones.

Remarkably, such high-frequency QNM instability is not limited to highly damped QNMs but indeed reaches the lowest overtones, the random perturbations being more effective in reaching the slowest decaying overtones for a given norm $||\delta V||_\infty = \epsilon$. This result is qualitatively consistent with analyses in [2, 44] for Dirac-delta potentials (cf compare e.g., perturbed QNM branches in Fig. 9 here with Fig. 1 in Ref. [44]). These findings advocate the use of pseudospectra to probe QNM instability, demonstrating its capability to capture it already at the level of the non-perturbed operator. At the same time, pseudospectra are oblivious to the nature of the perturbation triggering instabilities. A complementary perturbation analysis, in particular through random perturbations, has been then necessary to identify the high-frequency nature of the instability, confirming its physicality in the sense of being associated with actual perturbations of the potential $V$. 

Before we further discuss the details of the QNM instability, namely the nature of the new QNM branches, an important point must be addressed: whether the values obtained correspond to the actual eigenvalues of the new, perturbed operator $L + \delta L$, or whether they are an artifact of some numerical noise. As in the non-perturbed case discussed in section B 2 and as explained above when introducing the employed methodology, results are obtained with a high internal accuracy (10×Machine Precision), so that any numerical noise is below the range of showed values. Proceeding systematically, Fig. 8 presents the convergence tests for a few eigenvalues resulting from the deterministic perturbation (random perturbations do not admit this kind of test) with norm $||\delta V||_\infty = 10^{-5}$ and frequency $k = 20$ (bottom right panel of Fig. 7). The relative error is calculated as

$$
\mathcal{E}^{(N)} = \frac{1}{\omega_{(N=400)}} \left| \frac{\omega_{(N)}}{\omega_{(N=400)}} - 1 \right|,
$$

(64)

i.e., in the absence of exact results, we take as reference the values with a high resolution $N = 400$. As representative QNMs, we have chosen:

a) The last “unperturbed” overtone, whose value is actually very close to the (truly) unperturbed QNM $\omega_4$.

b) The first new QNM on the imaginary axis.

c) Three QNMs along the new branch with values spread in $1 \lesssim \text{Re}(\omega_n) \lesssim 10$ and $5 \lesssim \text{Im}(\omega_n) \lesssim 8$.

One observes a systematic convergence, with the relative error dropping circa 10 orders of magnitudes when the numerical resolution increases [95] from $N = 150$ to $N = 400$. This result confirms that the spectrum corresponds indeed to the new, perturbed operator, and is not a numerical artifact. This neatly shows the unstable nature of the QNM spectrum of the unperturbed Pöschl-Teller operator: eigenvalues indeed migrate to new branches under very small perturbations.
4. High-frequency stability of the slowest decaying QNM

The high-frequency instability observed for QNM overtones is absent in the fundamental QNM. The slowest decaying QNM is therefore ultraviolet stable. Such stability is already apparent in the pseudospectrum in Fig. 5 where the order of the \( \epsilon \)'s corresponding to \( \epsilon \)-pseudospectra sets around the fundamental QNM reaches the values in the stable self-adjoint case in Fig. 6. This high-frequency stability is then confirmed in the perturbation analysis. Indeed, Fig. 9 demonstrates the need of large perturbations in the operator in order to reach the fundamental QNM, namely (random) perturbations with a 'size' \( ||\delta V||_E \) of the same order as the induced variation in \( \omega_0^+ \). This behaviour is a tantamount of spectral stability.

The contrast between the high stability of \( \omega_0^+ \) and the instability of overtone resonances \( \omega_k^+ \geq 1 \) has already been evoked in VB 2, when referring to the large condition numbers \( \kappa_k/\kappa_0 \) in particular referring to Bindel & Zworski’s discussion in [21, 22]. This high-frequency stability of the fundamental mode is in tension with the instability found by Nollert in [11] for the slowest decaying mode for Schwarzschild. We will revisit this point in section VIB 3. For the time being, we simply emphasize that the observed stability relies critically on the faithfulness of the asymptotic structure of the potential, that is in-built in the adopted hyperboloidal approach permitting to capture the long-range structure of the potential up to null infinity \( S^+ \). It is only when we enforce a modification of the potential at “large distances” that the “low frequency” fundamental QNM is affected. This is illustrated in Fig. 10 (see also [29, 77]), corresponding to a Pöschl-Teller potential set to zero beyond a compact interval \( [x_{\text{min}}, x_{\text{max}}] \): such “cut” introduces high frequencies that make migrate the overtones to the new branches and, crucially, alters the asymptotic structure so that the fundamental QNM is also modified. Such “infrared” effect is however compatible with the spectral stability of the fundamental QNM, since such “cut” of the potential does not correspond to a small perturbation in \( \delta L \).

5. Regularization effect of random perturbations

Before proceeding to discuss the BH case, let us briefly comment on an apparently paradoxical phenomenon resulting from the interplay between random perturbations and the pseudospectrum. In contrast with what one might expect, the addition of a random perturbation to a spectrally unstable operator \( L \) does not worsen the regularity properties of \( L \), but, on the contrary, it improves the analytical behaviour of its resolvent \( R_L(\omega) \). This is illustrated in Fig. 11 that shows a series of pseudospectra corresponding to random perturbations of the Pöschl-Teller potential with increasing \( ||\delta V||_E \). In addition to the commented migration of QNM overtones towards pseudospectra contour lines, we observe two phenomena: i) \( \epsilon \)-pseudospectra sets with \( \epsilon > ||\delta V||_E \) are not affected by the perturbation, whereas ii) the pseudospectrum structure for \( \epsilon < ||\delta V||_E \) is smoothed into a “flat pattern”. As we have discussed in Fig. 6, such flat pseudospectra patterns are the signature of spectral stability, a tantamount of regularity of the resolvent \( R_L(\omega) \). The resulting improvement in the spectral stability of \( L + \delta L \), as compared to \( L \), is indeed consistent with the convergence properties of the respective QNM spectra, as illustrated by the contrast between the corresponding convergence tests in Figs. 8 and 9. In sum, random perturbations improve regularity, an intriguing effect seem-
the perspective of assessing the pioneering work in [11, 2].

A. Hyperboloidal approach in Schwarzschild

The attempt to implement the QNM stability analysis in the coordinate system employed for Pöschl-Teller, namely the Bizó-Mach chart [36], is unsuccessful. The reason is the bad analytic behaviour at null infinity of Schwarzschild potential(s) in the corresponding coordinate $x$. Instead of this, we resort to the ‘minimal gauge’ slicing [67, 68, 109], devised to improve regularity in the Schwarzschild(-like) case.

We start by considering standard Schwarzschild $(t, r)$ coordinates in the line element [2], with $f(r) = (1 - 2M/r)$ and BH horizon at $r = 2M$. “Axial” and “polar” Schwarzschild gravitational parities are described by the wave equation (4) with, respectively, Regge-Wheeler $V^{RW}_{\ell}(r)$ and Zerilli $V^{Z}_{\ell}(r)$ potentials. Specifically, we have

$$V^{RW}_{\ell, s}(r) = \left( 1 - \frac{2M}{r} \right) \left( \frac{\ell(\ell + 1)}{r^2} + \left( 1 - s^2 \right) \frac{2M}{r^3} \right)$$  (65)

for the axial case, where $s = 0, 1, 2$ correspond to the scalar, electromagnetic and (linearized) gravitational cases, and

$$V^{Z}_{\ell}(r) = \left( 1 - \frac{2M}{r} \right) \left( \frac{2\ell^2(n + 1)r^3}{r^3(\ell^2 + 3M^2)} \right)$$  (66)

with

$$n = \frac{\ell - 1}{2} + \frac{\ell + 2}{2} \quad (67)$$

for the polar case.

To construct horizon-penetrating coordinates reaching null infinity, one defines a height function $h$ in (6) by first considering an advanced time coordinate built on the rescaled tortoise coordinate $\bar{x} = r^*/\lambda$, with $r_* = r + 2M \ln(r/2M - 1)$, so that the BH horizon is at $\bar{x} \to -\infty$, and then enforcing a deformation of the Cauchy slicing into a hyperboloidal one through the choice of a ‘minimal gauge’, prescribed under the guideline of preserving a good analytic behavior at $\mathcal{J}^+$. In a second stage, the function $\theta$ in (6) implementing the compactification along hyperboloidal slices is implicitly determined by (note that instead of $\bar{x}$ in (6), we rather use $\tau$ for the spatial coordinate, so as to keep the standard usage in [67, 68, 109])

$$r = \frac{2M}{\sigma} \quad (68)$$

Choosing $\lambda = 4M$ in the rescaling $\bar{x} = r^*/\lambda$ of Eq. (5), the steps above result (see details in [67, 68, 109]) in the ‘minimal gauge’ hyperboloidal coordinates for the transformation (6)

$$\begin{cases}
\bar{\ell} = \tau - \frac{i}{2} \left( \ln \sigma + \ln(1 - \sigma) - \frac{1}{2} \right) \\
\bar{x} = \frac{1}{2} \left( \frac{i}{2} + \ln(1 - \sigma) - \ln \sigma \right)
\end{cases} \quad (69)$$

that, upon addition of the BH horizon and $\mathcal{J}^+$ points, maps $\bar{x} \in [-\infty, \infty]$ to the compact interval $\sigma \in [a, b] = [0, 1]$, with the BH horizon at $\sigma = 1$ and future null infinity at $\sigma = 0$.

VI. SCHWARZSCHILD QNM (IN)STABILITY

We address now the physical BH case, namely the stability of QNMs in Schwarzschild spacetime. Whereas the previous section has been devoted, to a large extent, to discuss some of the technical issues in QNM stability, the spirit in this section is to focus more on the physical implications, in particular

FIG. 11. Pseudospectra of Pöschl-Teller under random perturbations $\delta V_r$ of increasing norm, demonstrating the “regularizing” effect of random perturbations: pseudospectra sets $\sigma^*$ bounded by that “contour line” reached by perturbed QNMs become “flat”, a signature of improved analytic behaviour of the resolvent, as illustrated in Fig. 3. Pseudospectra sets not attained by the perturbation remain unchanged. Regularization of $R_{L+1L}(\omega)$ increases as $||\delta V_r||_E$ grows.

| $\ell$ | $r^*$ | $\sigma^*$ |
|-------|-------|-----------|
| 0     | 10    | 0.2       |
| 1     | 20    | 0.3       |
| 2     | 30    | 0.4       |
| 3     | 40    | 0.5       |

Implementing transformation (69) in the first-order reduction in time in Eqs. (8)-(9), we get for \( w(\sigma), p(\sigma), q_\ell(\sigma) \) (now explicitly depending on \( \ell \)) and \( \gamma(\sigma) \) in Eq. (12)

\[
\begin{align*}
w(\sigma) &= 2(1 + \sigma), & p(\sigma) &= 2\sigma^2(1 - \sigma), \\
q_\ell(\sigma) &= \frac{(4\lambda^2 V_\ell)}{2\sigma^2(1 - \sigma)}, & \gamma(\sigma) &= 1 - 2\sigma^2,
\end{align*}
\]

leading to the \( L_1 \) and \( L_2 \) operators building \( L \) in Eq. (10)

\[
\begin{align*}
L_1 &= \frac{1}{2(1 + \sigma)} \left( \partial_\sigma \left( 2\sigma^2(1 - \sigma)\partial_\sigma - \tilde{V}_\ell \right) \right), \\
L_2 &= \frac{1}{2(1 + \sigma)} \left( 2(1 - 2\sigma^2)\partial_\sigma - 4\sigma \right),
\end{align*}
\]

where the rescaled potential \( \tilde{V}_\ell(\sigma) := q_\ell(\sigma) \) results, in the respective axial and polar cases, in the explicit expressions

\[
\tilde{V}_\ell^{\text{RW,s}} = 2\left( \ell(\ell + 1) + (1 - s^2)\sigma \right),
\]

\[
\tilde{V}_\ell^{\text{Z}} = 2\left( \sigma + \frac{2n}{3} \left( 1 + 4n \right) \right) \left( \frac{3 + 2n}{2n + 3\sigma^2} \right).
\]

Finally, from Eqs. (70) and (22), the energy scalar product is

\[
(u_1, u_2)_E = \left\langle \frac{\phi_1}{\psi_1}, \frac{\phi_2}{\psi_2} \right\rangle_E
\]

\[
= \int_0^1 \left( (1 + \sigma)\dot{\psi}_1\psi_2 + \sigma^2(1 - \sigma)\dot{\phi}_2\phi_2 + \frac{\tilde{V}_\ell^{\text{Z}}}{2}\phi_1\dot{\phi}_2 \right) d\sigma,
\]

where the weight \( \tilde{V}_\ell \) is fixed by Eq. (72) for each polarization.

B. Schwarzschild QNM spectrum

As discussed in section [6C1], outgoing boundary conditions have been translated into regularity conditions on eigenfunctions. Specifically, as we have seen in the Pöschl-Teller case, the operator \( L_1 \) in (71) is a singular Sturm-Liouville operator, namely the function \( p(\sigma) = \sigma^2(1 - \sigma) \) vanishes at the boundaries of the interval \( [a, b] = [0, 1] \) consistently with Eq. (10). This translates into the fact that no boundary conditions can be imposed if enough regularity is required.

But there is a key difference between the Pöschl-Teller and the BH case: whereas in Pöschl-Teller the function \( p(x) = (1 - x)(1 + x) \) vanishes linearly at the boundaries, and therefore \( x = \pm 1 \) are regular singular points, in Schwarzschild this is true for \( \sigma = 1 \) (BH horizon) but not for \( \sigma = 0 \) (\( \mathcal{I}^+ \)), due to the quadratic \( \sigma^2 \) term. Null infinity is then an irregular singular point. This is the counterpart, in our compactified hyperboloidal formulation, of the power-law decay of Schwarzschild potentials responsible for the branch cut in the Green function of Eq. (1), with its associated “tails” in late decays of scattered fields. In the context of our spectral problem for the operator \( L \), this translates into the appearance of a (“branch cut”) continuous part in the spectrum. This has an important impact on the numerical approach, since the continuous branch cut is realized in terms of actual eigenvalues of the discretised approximates \( L^N \). Such eigenvalues are not QNMs and can indeed be unambiguously identified, but their presence has to be taken into account when performing the spectral stability analysis, that becomes a more delicate problem than in Pöschl-Teller. In this context, the latter becomes a crucial benchmark to guide the analysis in the BH case.

The Schwarzschild (gravitational) QNM spectrum for \( \ell = 2 \) is shown in Fig. 12 that presents the result of the numerical calculation of the spectrum of the \( L \) operator defined by (70). This is obtained either for the Regge-Wheeler or the Zerilli rescaled potentials in (72), corresponding respectively to potentials (65) and (66). This provides a crucial internal consistency check for the analytical and numerical construction, since both potentials are known to be QNM-isospectral (see below in section [VD2]). The branch cut structure is apparent in the eigenvalues along the upper imaginary axis. Such “branch cut” points can be easily distinguished from the special QNM corresponding to \( \omega_{\ell=2} \), also in the imaginary axis, simply by changing the resolution: branch points move “randomly” along the vertical axis, whereas \( \omega_{\ell=2} \) stays at the same frequency (see later [VID1] for a more systematic approach to establish the “non-branch” nature of \( \omega_{\ell=2} \), when we will consider high-frequency perturbations to QNMs). Moreover, eigenfunctions associated with algebraically special modes are polynomials, as shown in the detailed studies of these modes for Schwarzschild and Kerr in [67][112].

Due to the lack of an exact expression for the Schwarzschild QNMs, one must compare the obtained values against those...
available in the literature via alternative approaches — see, for instance [15,18,113,116]. An estimative for the errors when QNMs are calculated with the methods from this work is found in Ref. [109]. From the practical perspective, and regardless of the numerical methods, it is well known that the difficulty to accurately calculate numerically a given QNM overtone \( \omega_n^\pm \) increases significantly with \( n \). For instance, convergence and machine precision issues similar to the ones commented above are reported in Refs. [117,119], a control of the internal roundoff accuracy being required. Alternatively, iterative algorithms such as Leaver’s continued fraction method [120] require an initial seed relatively near a given QNM, which must be carefully adapted when dealing with the overtones [121]. The bottomline is that the calculation of BH QNM overtones is a challenging and very delicate issue.

In our understanding, the latter challenge is not a numerical hindrance but the consequence of a structural feature of the underlying analytical problem, namely the spectral instability of the Schwarzschild QNM problem. This is manifested already at the present stage of analysis, namely the calculation of QNM frequencies of non-perturbed Schwarzschild, in the eigenvalue condition numbers \( \kappa_{\omega_n} \)’s shown in the top panel of Fig. [12] we encounter again the pattern found in the Pöschl-Teller case, cf. Fig. [3] with a growth of the spectral instability as the damping increases, with the notably anomaly of an enhanced stability for the algebraically special QNM frequency, with \( n = 8 \). We devote the rest of the section to explore this spectral instability with the tools employed for Pöschl-Teller.

C. Schwarzschild pseudospectrum

The pseudospectrum of Schwarzschild is presented in Fig. [13]. As illustrated in Pöschl-Teller, the pseudospectrum provides a systematic and global tool to address QNM spectral instability, already at the level of the unperturbed potential. A “topographic map” of the analytic structure of the resolvent, where regions associated with small \( \epsilon \)-pseudospectra (light green) correspond to strong spectral instability, whereas regions with large \( \epsilon \) (namely \( O(\epsilon) \sim 1 \), dark blue) indicate spectral stability. The superposition of the QNM spectrum shows the respective spectral stability of QNM frequencies.

We can draw the following conclusions from Fig. [13]:

i) The Schwarzschild pseudospectrum indicates a strong instability of QNM overtones, an instability that grows fast with the damping. White-line boundaries corresponding to \( \epsilon \)-pseudospectra with very small \( \epsilon \)’s extend in large regions of the complex plane. This is compatible with the results in [11], providing a rationale — already at the level of the unperturbed potential— for the QNM overtone instability discovered by Nollert.

ii) The slowest decaying QNM is spectrally stable. Fig. [13] tells us that changing the fundamental QNM frequency requires perturbations in the operator of order \( \| \delta L \|_\infty \sim 1 \). This corresponds to spectral stability and is in tension with the results in [11], where the fundamental QNM is found to be unstable. We will address this point below.

iii) Schwarzschild and Pöschl-Teller potentials show qualitatively the same pseudospectrum pattern, with large “green regions” producing patterns in stark contrast with the flat selfadjoint case. On the one hand, this reinforces the usage of Pöschl-Teller as a convenient guideline for understanding the stability structure of BH QNMs and, on the other hand, it points towards an instability mechanism independent, at least in a certain measure, on some of the details of the potential.

We can conclude that Fig. [13] demonstrates —at the level of the unperturbed operator— the main features of the stability structure of the BH QNM spectrum, namely the QNM overtone instability and the stability of the fundamental QNM. However, the pseudospectrum does not inform us about the particular type of the perturbations that trigger the instabilities. This is addressed in the following subsection.

D. Perturbations of Schwarzschild potential

Once the Schwarzschild pseudospectrum, together with the condition numbers \( \kappa_{\omega_n} \), have presented evidence of QNM spectral instability at the level of the unperturbed operator, in this section we address the question about the actual physical character of perturbations triggering such instabilities.
1. Ultraviolet instability of BH QNM overtones

The qualitative agreement between Pöschl-Teller and Schwarzschild pseudospectra, cf. Figs. 3 and 13, together with the experience gained in the study of Pöschl-Teller perturbations regarding the high-frequency instability of all QNM overtones and the stability of the fundamental QNM, guide our steps in the analysis of the BH setting.

a. Random perturbations: spoils from the “branch cut”. The presence of a “branch cut” in the Schwarzschild spectrum, discussed in section VT18B, translates into a methodological subtlety when considering random perturbations in the BH case, as compared with the Pöschl-Teller one. The difficulty stems from the fact that not only the QNM eigenvalues, but also the eigenvalues associated with the discretized version of the branch cut, are sensitive to random perturbations \( \delta V_r \) of the potential. As a consequence, the possible contamination from eigenvalues from the branch cut complicates the analysis of the impact of random perturbations on QNM frequencies.

This does not mean that random perturbations have no use in our BH discussion. An illustrative example is the study of the stability of the algebraically special Schwarzschild QNM \( \omega_{n=8} \). Whereas random perturbations move “branch cut” eigenvalues away from the imaginary axis, the algebraically special QNM stays stable. This methodology provides a powerful and efficient tool to probe the “physicality” of specific eigenvalues in very general settings (cf. e.g. Fig. 4 in [122]).

b. Deterministic perturbations. Given the limitations for random \( V_d \)'s, in the present study we have focused on the class of deterministic perturbations to the potential \( \delta V_d \) provided by Eq. (63). Crucially, such perturbations do not perturb the “branch eigenvalues” as (much as) random \( V_d \) do, by-passing then the associated spectral instability contamination. Despite their simplicity, they provide a good toy-model to explore the effects of astrophysically motivated perturbations (assessment of “long range/low frequency” versus “small scale/high frequency” perturbations), as well as those arising from generic approaches to quantum gravity (“small scale/high frequency” effective fluctuations). They are, therefore, conveniently suited to address these instability issues.

The left column in Fig. 14 depicts (with \( ||\delta V_d||_k \sim 10^{-5} \)) the stability of the first overtones against low frequency perturbations \( k = 1 \), top-left panel) in contrast with the instability resulting from high-frequency perturbations \( k = 20 \), bottom-left panel). Pushing along this line, the right column in Fig. 14 zooms in to study the very first overtones, which are paramount for the incipient field of black-hole spectroscopy. Assessing the (in)stability of the very first overtones is therefore crucial for current research programs in gravitational astronomy. It becomes apparent that the first overtones, including the very first overtone, are indeed affected without any extraordinary or fine tuned perturbations \( \delta V_d \). In particular, and taking the left column as a reference, the first overtone is reached: i) either by considering a “slightly” more intense perturbation \( ||\delta V_d||_k \sim 10^{-4} \), \( k = 20 \), or ii) perturbations with sufficiently high frequency \( ||\delta V_d||_k \sim 10^{-8} \), \( k = 60 \).

From this perturbation analysis of the BH potential we conclude: i) all QNM overtones are ultraviolet unstable, as in Pöschl-Teller, the instability reaching the first overtone for sufficiently high frequency; ii) QNMs are stable under low frequency perturbations, this illustrating that spectral instability does not mean instability under “any” perturbation, in particular long-wave perturbations not affecting the QNM spectrum; iii) the slowest decaying QNM is ultraviolet stable, a result in tension with the instability of the fundamental QNM found in [11]. We revisit this point in section VI D 3 below.

2. Isospectrality loss: axial versus polar spectral instability

Regge-Wheeler and Zerilli potentials for axial and polar perturbations are known to be isospectrual in the QNM spectrum (cf. [14] [126] [125]; see also [74]). In particular, Chandrasekhar identified (cf. point 28 in [14]) a necessary condition for two (one-dimensional) potentials \( V_1(\bar{x}) \) and \( V_2(\bar{x}) \), with \( \bar{x} \in (-\infty, \infty) \) as the rescaled tortoise coordinate, to have the same transmission amplitude and present the same QNM spectrum. Specifically, both potentials must render the same values when evaluating an infinite hierarchy of integrals

\[
C_n = \int_{-\infty}^{\infty} v_n(\bar{x}) d\bar{x}, \tag{74}
\]

with

\[
v_1 = V, \quad v_3 = 2\sqrt{3} + V' \quad v_5 = 5V^4 + 10VV'^2 + V''^2, \quad v_{2n+1} = \ldots \tag{75}
\]

These quantities turn out to be the conserved quantities of the Korteweg-de Vries equation and connect the Schwarzschild QNM isospectrality problem to integrability theory through the inverse scattering transform of Gelfand-Schwarzid QNM isospectrality problem to integrability theory through the inverse scattering transform of Gelfand-Levitan-Marchenko (GLM) theory (cf. [126]; see [125] for an alternative approach in terms of Darboux transformations).

The key point for our spectral stability analysis of \( L \) is that axial and polar QNM isospectrality is the consequence of a subtle and "delicate" integrability property of stationary BH solutions, so we do not expect it to be robust under generic perturbations of \( V \). In particular, given the non-linear dependence in \( V \) of the conserved quantities \( C_n \) in (74), we would expect either random \( \delta V_r \) or deterministic \( \delta V_d \) perturbations to render different values of \( C_n \), therefore resulting in a loss of QNM isospectrality. Fig. 15 confirms this expectation: whereas the fundamental QNM mode remains stable under high-frequency perturbations, isospectrality is broken for the overtones with a slight, but systematic, enhanced damping in the axial case. Other mechanisms for BH isospectrality loss have been envisaged, e.g. in the study of the imprints of modified gravity theories [42], ultraviolet QNM overtone instability providing a possible mechanism inside general relativity.
In sum, isospectrality loss provides an interesting probe into QNM instability, with potential observable consequences and will be the subject of a specifically devoted study elsewhere.

3. “Infrared instability” of the fundamental QNM

Both the pseudospectrum and the explicit perturbations of the potential indicate a strong spectral stability of the slowest decaying Schwarzschild QNM. This is tension with the results in [12], where the instability affects the whole QNM spectrum, this including the slowest decaying QNM. This is a fundamental point to establish, since it directly impacts the dominating frequency in the late BH ringdown signal.

In our understanding, and as it was the case of the Pöschl-Teller potential discussed in section [VDT4], the instability of the fundamental QNM frequency found by [1] is an artifact of the implemented perturbations, namely step-like approximations to the Schwarzschild potential (in particular Regge-Wheeler, but the same applies for Zerilli) that modify the potential at large distances. Specifically, $V_T$ is set to zero beyond $[x_{\text{min}}, x_{\text{max}}]$, fundamentally altering the long-range nature of Schwarzschild potential that becomes of compact support. What we observe in Fig. 14 is that keeping a faithful treatment of the asymptotic structure at infinity through the compactified hyperboloidal approach keeps spectral stability.

To test this idea (cf. also the recent [39], as well as [97]), and as we did in Pöschl-Teller, we have implemented a “cut Schwarzschild” potential in our hyperboloidal approach, setting the potential to zero from a given distance (both towards null infinity and the BH horizon). The result is shown in Fig. 16 showing a similar qualitative behaviour to Pöschl-Teller in Fig. 10. Overtones are strongly perturbed into the QNM branches already observed in Fig. 14, consistently with the high-frequencies introduced by the Heaviside cut. But, crucially, now the fundamental QNM is indeed also modified, in contrast with its stable behaviour under high-frequency perturbations. This reinforces the understanding of this effect as a consequence of the “suppression” of the large-scale asymptotics of the potential [127]. However, the observed modific
FIG. 15. Loss of isospectrality in Schwarzschild, under high-frequency perturbations. The sequence of figures shows a zoom into the perturbation of lowest $\ell = 2$ axial and polar QNM overtones (the branch cut has been removed), with $\delta \tilde{V}_d$ fixed to a value reaching the first overtone, and then increasing the frequency. The breaking of axial and polar isospectrality is demonstrated, with perturbed axial overtones slightly more damped than polar perturbed counterparts, though both laying over the same perturbed QNM branches (actually tracking the pseudospectra contour lines, cf. Fig. 17 below). The fundamental QNM remains unchanged, consistently with its stability, so the dominating ringdown frequency remains “isospectral”.

FIG. 16. “Infrared” modification of the Schwarzschild fundamental QNM. As in the Pöschl-Teller case, cutting the Schwarzschild potential ($\ell = 2$, either Regge-Wheeler or Zerilli) outside a compact interval $[x_{\min}, x_{\max}]$ modifies the fundamental QNM, this accounting for its “instability” found in [1]. All QNM overtones are strongly perturbed due to the high-frequencies in the Heaviside cut, whereas (only) the fundamental QNM is recovered as $x_{\min}, x_{\max} \to \pm\infty$.}

with Eq. (4), in particular in the setting of a Cauchy slicing getting to spatial infinity $\partial^0$. Such asymptotic framework may be more sensitive to the modification of the potential that the hyperboloidal one, related to null infinity $\mathcal{I}^+$. In this setting, and lacking a better expression, we refer to this effect as an “infrared instability” of the fundamental QNM.

Enforcing the compact support nature of $V$ is naturally motivated in physical contexts such as optical cavities, and will be studied systematically in such settings [27]. In gravitation the physicality of such an effect is more difficult to assess, since gravity is a long-range interaction that, in contrast to the electromagnetic one, is not screened. In any case, insofar as a pertinent gravitational scenario may be envisaged for such “cut potential”, then the “infrared instability” shown for the first time in [11] would constitute a physical effect.

E. Nollert-Price BH QNM branches: instability and universality

We revisit the results in [112] (see also [38,39]), under the light of the elements introduced for the study of QNM spectral stability. Fig. 2 in [11] presents the migration of Schwarzschild QNMs to new branches, as the result of perturbing the (Regge-Wheeler) Schwarzschild potential with a step-like approximation with an increasing number “$N_{\text{cut}}$” of steps (cf. Fig. 1 in [11]). A salient feature of Nollert’s Fig. 2, further analysed with Price in [2], is that the new QNM branches distribute in a perfectly structured family of lines in the complex plane, unbounded in the real part of the frequency, that “move down” in the complex plane as $N_{\text{cut}}$ (i.e. the frequency in the pertur-
bution) increases [128]. A comparison with Schwarzschild’s pseudospectrum in our Fig. 13 shows two remarkable features: i) the pattern of the new branches found and studied by Nollert and Price is qualitatively similar to the contour lines of $\epsilon$-pseudospectra, ii) the effect of increasing the frequency perturbation indeed corresponds to an increment in the $\epsilon$ of the corresponding contour line (namely the “energy size” of the perturbation that, as a $H^1$ norm, includes the frequency). In other words, Nollert and Price’s BH QNMs branches seem indeed to be closely related to $\epsilon$-pseudospectrum contour lines.

In order to test this picture, we bring our perturbation analysis in section VTD into scene. Fig. 17 presents the superposition of perturbed QNM spectra in Fig. 14 onto the Schwarzschild pseudospectrum in Fig. 13. As in the Pöschl-Teller case, perturbed QNMs closely track $\epsilon$-pseudospectra lines, demonstrating the insight gained above on Nollert’s QNM instability by using the pseudospectrum: Nollert-Price QNM branches are identified as actual probes into the analytical structure of the non-perturbed wave operator. Moreover, the correlation of $\epsilon$-contour lines with the “size/frequency” of the perturbations, endows the pseudospectrum not only with an explicative but also with a predictive power, as a tool to calibrate the relation between spacetime perturbations and QNM frequency changes. The conceptual frame encoded in Fig. 17 is, in our understanding, the main contribution in this work.

1. QNM structural stability, universality and asymptotic analysis

Building on Nollert and Price’s work, our analysis strongly suggests that BH QNM overtones are indeed structurally unstable under high-frequency perturbations: BH QNM branches migrate to a qualitatively different class of QNM by Nollert and Price [2] provides an excellent illustration, with the identification of the large-$n$ asymptotic form of perturbed QNM branches, according to the logarithm dependence

$$\text{Im}(\omega_n) \sim C_1 + C_2 \ln \left(\text{Re}(\omega_n) + C_3\right), \quad n \gg 1,$$  \hspace{1cm} (76)

with $C_1$, $C_2$ and $C_3$ appropriate constants (note that $C_3$ can be put to zero for sufficiently high $n$, as in [2], since $\text{Re}(\omega_n) \to \infty$ as $n \to \infty$; we prefer to keep it to account for intermediate asymptotics [49]). It is suggestive that this makes direct contact with the possible universality of perturbed BH QNMs and (non-perturbed) QNMs of compact objects evoked above. Indeed, as shown in Ref. [129], $w$-modes of BH QNMs stars present exactly this logarithmic pattern [131]. Even more, this makes (an unexpected) contact with Pöschl-Teller, where the spectral instability discussed in section VTD is explained [31] in terms of so-called broad “Regge resonances” (not to confuse with “Regge poles”), precisely described by such a logarithmic dependence [132].

b. “Universality” in the high-frequency perturbations.

The QNM migration pattern seems independent of the detailed nature of the high-frequency perturbation in the Schwarzschild potential. First, such universality is manifested by the similar QNM perturbation pattern produced by very different perturbations: step-like perturbations in [1], the sinusoidal deterministic ones showed in Fig. 17 and also random perturbations (not presented here due to “blurring” issues, consequence of the “branch cut” contamination). Second, the new branches follow closely the pseudospectra contour lines, a key point in this universality discussion, since it is completely prior to and independent of perturbations.

c. Asymptotic analysis and universality.

How to address systematically a possible universality in the qualitative pattern of the perturbed QNM branches? Asymptotic analysis provides a sound approach. The study of the spiked TDP QNMs by Nollert and Price [2] provides an excellent illustration, with the identification of the large-$n$ asymptotic form of perturbed QNMs branches, according to the logarithm dependence

$$\text{Im}(\omega_n) \sim C_1 + C_2 \ln \left(\text{Re}(\omega_n) + C_3\right), \quad n \gg 1,$$  \hspace{1cm} (76)

with $C_1$, $C_2$ and $C_3$ appropriate constants (note that $C_3$ can be put to zero for sufficiently high $n$, as in [2], since $\text{Re}(\omega_n) \to \infty$ as $n \to \infty$; we prefer to keep it to account for intermediate asymptotics [49]). It is suggestive that this makes direct contact with the possible universality of perturbed BH QNMs. Indeed, as shown in Ref. [129], $w$-modes of BH QNMs stars present exactly this logarithmic pattern [131]. Even more, this makes (an unexpected) contact with Pöschl-Teller, where the spectral instability discussed in section VTD is explained [31] in terms of so-called broad “Regge resonances” (not to confuse with “Regge poles”), precisely described by such a logarithmic dependence [132].

Beyond specific models, this kind of universal behaviour, independent of the high-frequency perturbation detailed nature and for a large class of potentials, invites for systematic semi-classical analyses of highly-damped scattering resonances, in terms of the wave operator principal part [133], including boundary behaviors. In the spirit adopted in this work, we expect asymptotic tools in the semiclassical analysis of the pseudospectrum to provide a systematic approach to assess the universality of perturbed BH QNMs branches [134].
2. Overall perspective on Schwarzschild QNM instability

The main result of this article is summarized in Fig. 17. Specifically, it combines Figs. 12, 13, and 14 to demonstrate QNM spectral (in)stability through their respective three distinct calculations: i) the calculation of the eigenfunctions of the exact spectral problem to calculate condition numbers $\kappa_n$’s, ii) the evaluation of operator matrix norms to generate the pseudospectrum, and iii) the calculation of eigenvalues of the perturbed spectral problem. Calculations i) and ii) work at the level of the unperturbed problem, whereas iii) deals with the perturbed problem. The three calculations fit consistently through the Bauer-Fike theorem that constrains through Eq. (39) the relation between the pseudospectrum and the tubular regions around the spectrum. They lead to these main results:

i) QNM overtones:

i.1) QNM overtones are ultraviolet unstable, including the lowest overtones. The pseudospectrum provides a systematic explanatory and predictive framework for QNM spectral instability, confirming the result by Nollert and Price [1][2]. Such instability is indeed realised by physical high-frequency perturbations in the effective potential $V$, reaching the first overtone for sufficiently high frequencies and/or amplitudes in the perturbation.

i.2) QNM overtones are stable under low frequency perturbations. No instability appears for low/intermediate frequency perturbations of $V$, consistently with studies [40][42][44][84] on astrophysical BH environments.

ii) Slowest decaying (fundamental) QNM:

ii.1) The slowest decaying QNM is ultraviolet stable. This feature critically relies on keeping a faithful description of the asymptotic structure at infinity through the compactified hyperboloidal approach. This result is in contrast with conclusions in [1][2], but no contradiction appears since the latter implement a step-potential approx-
imination fundamentally modifying $V$ at large distances, resulting rather in an “infrared probe” into QNMs.

ii.2) The slowest decaying QNM is stable under low and intermediate frequency perturbations in the potential. This property is shared by the whole QNM spectrum.

ii.3) The slowest decaying QNM is “infrared unstable”. The instability of the fundamental QNM observed in [1] 2] is physical inasmuch as fundamental modifications of the large-distance structure of the potential are allowed.

iii) Structural stability and QNM isospectrality.

iii.1) 'Nollert-Price BH QNM branches' track pseudospectrum contour lines. The QNM BH spectrum is ultraviolet structurally unstable, migrating to perturbed branches tracking $\epsilon$-contour lines of pseudospectra. Such migration pattern is largely independent of the detailed nature of high-frequency perturbations and potential. Once on such ‘Nollert-Price branches’, QNMs are spectrally stable. These structural stability properties result in the universality of perturbed QNM branches.

iii.1) QNM isospectrality ultraviolet loss. High-frequency perturbations spoil the integrability of Regge-Wheeler and Zerilli potentials, resulting in a slightly enhanced damping of axial modes with respect to polar ones.

VII. CONCLUSIONS AND PERSPECTIVES

A. Conclusions

We have demonstrated: i) the fundamental BH QNM is stable under high-frequency (ultraviolet) perturbations, while unstable under (infrared) modifications of the asymptotics, the latter consistent with [1]; ii) (all) BH QNM overtones are unstable under high-frequency (ultraviolet) perturbations, quantifiable in terms of the energy content (norm) of the perturbation, extending results in [1] 2] to show isospectrality loss; and iii) pseudospectrum contour lines provide the rationale underlying the structurally stable pattern of perturbed ‘Nollert-Price QNM BH branches’. Pseudospectra, together with tools from the analysis of non-selfadjoint operators, have revealed the analytic structure underlying such (in)stability properties of BH QNMs, offering an integrating and systematic approach to encompass a priori disparate phenomena. The soundness of the results relies on the use of a compactified hyperboloidal approach to QNMs, with the key identification of the relevant scalar product in the problem as associated with the physical energy, combined with accurate spectral numerical methods.

1. Caveats in the current approach to QNM (in)stability

Beyond the soundness of the results, key questions remain:

i) How much does the instability depend on the hyperboloidal approach? In other words, is the instability a property of the equation or rather of the employed scheme to cast it? This is a legitimate and crucial question, requiring specific investigation. In spite of this, we are confident in the soundness of our conclusions: as discussed in detail, the same qualitative behaviour is found systematically by other studies not relying on the hyperboloidal approach, in particular Nollert and Price’s pioneer work. Details may change from scheme to scheme, but the (in)stability properties seem robust.

ii) A numerical demonstration is not a proof. Moreover, numerical discretizations introduce their own difficulties and limitations. In particular, spectral issues in the passage from matrix approximations to the actual differential operator is a most delicate question. Again, we are confident in our results, as a consequence of mutual consistency of existing results and non-trivial tests like the ones described in the text. Definitely, proofs will require the use of other methods and techniques.

iii) Could the observed QNM spectral instability be an effect of regularity loss, namely a $C^p$ effect? It may be, but it is difficult to conclude at this stage. $C^p$ regularity provides indeed a sufficient condition for logarithmic branches (76) that can be traced to works by Regge [132], Berry [135, 136] or Zworski [92] and manifests in our setting in Nollert & Price’s analysis of BH QNM instability [2] (complemented in [39]), broad “Regge resonances” in Pöschl-Teller QNM instability [51, 52, 91], or in neutron star $w$-modes [129] (cf. also [137] in related Regge poles). But we also attest the same instability phenomenon for regular sinusoidal perturbations of sufficiently high-frequency. Moreover, the pseudospectrum already informs of the instability (cf. contour lines) at the unperturbed “regular” stage. If high-frequency is actually the basic mechanism, then $C^p$ would provide a sufficient, but not necessary condition for QNM instability. This point must be addressed.

B. Perspectives

While the pseudospectrum framework is already employed in physics (cf. e.g. [4, 8, 10, 11, 13]), there seems to be (up to our knowledge) no systematic application in the gravitational context. The introduction of pseudospectra in gravitational physics opens an avenue to interbreed the study of (in)stability and transients with other domains in physics (and beyond), by using pseudospectrum analysis as a common methodological frame. In the following we mention some possible lines of exploration in different gravitational settings, from astrophysics and fundamental gravity physics to mathematical relativity, closing the discussion with a perspective beyond gravity.
1. **Astrophysics and cosmology**

The astrophysical status of the ultraviolet QNM overtone instability, that reaches the lowest overtones for generic perturbations of sufficiently high frequency and energy, requires to assess whether actual astrophysical (and/or fundamental spacetime) perturbations are capable of triggering it. Some problems in which this question is relevant are the following:

a) **BH spectroscopy.** If such instability is actually present, this should be taken into account in current approaches to BH spectroscopy. The stability of the slowest decaying QNM guarantees that the dominating ringdown frequency is unaltered. But regarding QNM overtones, note that in we have not referred at all to late time ringdown frequencies, but to QNM frequencies: since such two sets of frequencies can actually decouple [1][2][38][39][43][48] and, as already noticed by Nollert [1], the propagating (scattered) field itself is not much affected by high-frequency perturbations, finding the signature of perturbed QNMs in the gravitational wave signal may pose a very challenging problem [39]. Awareness of this potential effect in the GW signal may however lead to specifically tailored data analysis tools.

b) **BH environment.** The arrangement of perturbed QNM branches along (a priori known) ϵ-contour lines of pseudospectra opens the possibility of probing, in an ‘inverse scattering’ spirit, environmental BH perturbations. One can envisage to read the “size” of the physical perturbations by comparing observational QNM data with the “a priori” calibrated pseudospectrum. This may help to assess “dry” versus “wet” BH mergers, a point of cosmological relevance in LISA science.

c) **Universality of compact object QNMs.** The combination of the “universality” of the perturbed “Nollert-Price QNM BH branches” with Nollert’s remark on their similarity to neutron star “ω-modes”, together with the demonstrated loss of BH QNM axial/polar isospectrality, poses a natural question: do QNM spectra of all generic compact objects share a same pattern?

Schemes such as [25] may provide a systematic frame for the analysis of the astrophysical implications.

d) **BH QNM (in)stability in generic BHs.** A natural and necessary extension of the present work is the study of QNM (in)stability in the full BH Kerr-Newman family, in particular understanding how it intertwines with superradiance instability and the approach to extremality.

2. **Fundamental gravitational physics**

We note some possible prospects at the fundamental level:

a) **(Sub)Planckian-scale physics.** Planck scale spacetime fluctuations seem a robust prediction of different models of quantum gravity. They represent “irreducible” ultraviolet perturbations potentially providing a probe into Planck scale physics that, given the universality of BH QNM overtone instability, may be ‘agnostic’ to an underlying theory of quantum gravity. Such a search of quantum gravity signatures in BH gravitational wave physics is akin to [138]. Actually, it would suffice that a Planck scale “cut-off” induces an effective CP regularity in the otherwise smooth low-energy description, to trigger the instability phenomenon. BH QNM instability might then provide a particular probe into ‘discreteness’ of spacetime (e.g. [139] are references therein).

b) **QNMs and (strong) cosmic censorship.** In the setting of cosmological BHs, the assessment of the extendibility through the Cauchy horizon in Reissner-Nordström de Sitter is controlled by the parameter $\beta = \alpha/\kappa_\infty$, where $\alpha$ is the spectral gap (the imaginary part of the fundamental QNM in our setting) and $\kappa_\infty$ is the surface gravity of the Cauchy horizon [31][140]. Therefore, a good understanding in this setting of the (in)stability properties of the slowest decaying QNM, and more generally of the QNM spectrum, may be enlightening in the assessment of the thresholds for Cauchy horizon stability.

c) **Random perturbations and spacetime semiclassical limit.** The “regularization effect” of random perturbations [11][98][107] in the scattering Green’s function is an intriguing phenomenon that may play a role in the transition to a semiclassical smooth effective description of fundamental gravitational degrees of freedom described in a more basic (quantum) theory, possibly including an irreducible randomness ingredient. Again, the universality of the phenomenon may play a key role.

3. **Mathematical relativity**

The presented numerical evidences need to be transformed into actual proofs. Some mathematical issues to address are:

a) **Regularity conditions and QNM characterization.** The mathematical study of QNMs entails subtle functional analysis issues. In the present hyperboloidal approach this involves, in particular, the choice of appropriate regularity conditions and the associated functional space. This connects our pseudospectrum study with the identification in [67] of the full upper-complex plane as the actual QNM spectrum, if general $C^\infty$ eigenfunctions are allowed. More regularity must therefore be enforced. An analysis along the lines in [70][72], where Gevrey classes are identified as the proper functional spaces to define QNMs, is therefore required. Likewise, a systematic comparison with QNM stability in the framework of [31][69] is needed (cf. also [51][52]).

b) **Semiclassical analysis and QNM (in)stability.** The interest of asymptotic tools, in the study of QNM stability, is twofold. On the one hand, an “asymptotic reasoning” [141] built on the semiclassical analysis of QNMs...
(a subject taken to full maturity in Sjöstrand’s works \cite{142,146}, with a small parameter defined in terms of highly-damped QNM frequencies, can help to assess universality patterns of perturbed Nollert-Price BH QNM branches. On the other hand, asymptotic analysis provides powerful tools to prove rigorously spectral instability and non-trivial pseudospectra (cf. e.g. \cite{147}). In particular, the recent work \cite{148} provides an explicit example of scattering resonance (or QNM) instability, sharing much of the spirit of the discussion in this work.

4. Beyond gravitation: “gravity as a crossroad in physics”

The disclosure of BH QNM instability \cite{1} resulted from the fluent interchange between gravitational and optical physics \cite{40,45,152}, again a key ‘flow channel’ in our work, e.g. to understand the ‘infrared’ instability of the fundamental QNM \cite{97}. In this spirit, the present work can offer some hints for further boosting such kind of transversal research in physics.

The hyperboloidal approach, with its explicit formulation of the dynamics in terms of a non-selfadjoint operator, provides a scheme of interest whenever dealing with an open physical system with losses at a radiation zone, a recurrent situation throughout physics (e.g. in optics, acoustics, physical oceanography, to cite some settings). A specific lesson of the present work, to be exported to other physical contexts, is the identification of the relevant scalar product in terms of the system’s energy, thus casting an a priori technical issue into neat physical terms. Moreover, when studying QNMs, the normalizability of the QNM eigenfunctions in the hyperboloidal approach may open an alternative avenue to the characterization of the so-called ‘mode volume’ $V_n$ of a QNM. This is relevant e.g. in the setting of photonic/plasmonic resonances \cite{52}, together with the notion of ‘quality factor’ $Q_n$, given in terms of the ratio between the real and imaginary parts of a QNM (see e.g. \cite{153} for its connection with BH gravity physics), it characterizes the Purcell factor $F_p \sim Q_n/V_n$ controlling the enhancement of spontaneous emission of a quantum system, a key notion in ‘cavity quantum electrodynamics’ \cite{154}.

Regarding the pseudospectrum, this notion is relevant whenever a non-Hermitian (or more generally non-selfadjoint operator) enters into scene, as it is typically the case in open systems \cite{12}. In the context of non-Hermitian quantum mechanics, it has been proposed \cite{10} to endow the pseudospectrum with a guiding central role in the theory, in a setting in which spectral instability makes insufficient the standard notion of spectrum to fully characterize the relevant operators. Apart from spectral instability, the pseudospectrum underlies purely dynamical phenomena \cite{4,8}, in particular accounting for so-called nonmodal instability \cite{155} in the setting of hydrodynamic stability theory and turbulence. Beyond hydrodynamics, the latter feature turns the pseudospectrum into a powerful tool for studying both spectral and dynamical stability issues in (open) physical systems that “trace” over a part of the total degrees of freedom and, as a result, are governed by non-selfadjoint operators. Such systems occur all over physics (e.g. condensed matter, optics, plasmonics, acoustics, nanophysics... \cite{13}), offering a natural arena for extending the already large range of applications of pseudospectra \cite{12}.

Gravitational physics is remarkable in its capacity to “provide a framework that calls for the interchange of ideas, concepts and methodologies from very different communities” \cite{156} in physics. The hyperboloidal approach and the pseudospectrum here discussed realize an instance of this understanding of “gravity as a crossroad in physics” \cite{156}.

Acknowledgments. We thank M. Ansorg, P. Bizoń, O. Reula and J. Sjöstrand for key insights. We also thank J. Olmedo, C. Barceló, L. Garay (and the rest of Carramplas-2019 participants), L. Andersson, A. Ashtekar, E. Berti, N. Besset, I. Booth, Y. Boucher, V. Cardoso, G. Colas des Francs, M. Colbrook, A. Coutant, G. Cox, T. Daudé, K. Destounis, G. Dito, J. Frauendiener, H. Friedrich, D. Gajic, S. Guérin, D. Häfner, M. Hitrik, A. Ianthenko, H.R. Jauslin, J. Jeziorski, B. Krishnan, J. Lampart, J. Lewandowski, M. Maliborski, M. Mokdad, J.-P. Nicolas, I. Racz, B. Raffaelli, A. Rostworowski, B. Sah, O. Sarbach, B.S. Sathyaprakash, J. Slipantschuk, A. Soumaila, J.A. Valiente-Kroon and A. Zenginoglu. This work was supported by the French “Investissements d’Avenir” program through project ISITE-BFC (ANR-15-IDEX-03), the ANR “Quantum Fields interacting with Geometry” (QFG) project (ANR-20-CE40-0018-02), the EIPHI Graduate School (ANR-17-EURE-0002), the Spanish FIS2017-86497-C2-1 project (with FEDER contribution), the European Research Council Grant ERC-2014-StG 639022-NewNGR “New frontiers in numerical general relativity” and the European Commission Marie Skłodowska-Curie grant No 843152 (Horizon 2020 programme). The project used Queen Mary’s Apocrita HPC facility, supported by QMUL Research-IT, and CCoB computational resources (université de Bourgogne).

Appendix A: Energy scalar product and adjoint operator $L^\dagger$

1. Energy scalar product

We start by considering the energy contained in the hyperboloidal slice $\Sigma_\tau$, defined by $\tau = \text{const}$ in Eq. (6), and associated with a mode $\phi_{\ell m}$ satisfying the effective Eq. (5), namely propagation in Minkowski with a potential $V_t$ (see also \cite{74}). In this stationary situation this energy is given \cite{59} by Eq. (18)

$$E = \int_{\Sigma_\tau} T_{ab} u^a n^b d\Sigma_\tau . \quad (A1)$$

The stress-energy tensor $T_{ab} = T_{ab}(\phi_{\ell m}, \nabla \phi_{\ell m})$ of a (generally complex) scalar field is given by Eq. (17), with $\eta_{ab}$ the Minkowski metric in arbitrary coordinates (dropping ($\ell,m$))

$$T_{ab} = \frac{1}{2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} \eta_{ab} \left( \nabla^c \phi \nabla_c \phi + V V_\phi + c.c. \right) \right) + c.c. \quad (A2)$$

with “c.c.” denoting “complex – conjugate”. Coming back to ($A1$), and using coordinates ($\tau, x$) adapted to $\Sigma_\tau$ and defined in Eq. (4), the timelike Killing is $t^a = \partial_t = \frac{1}{\lambda} \partial_{\tau}$, and we

\[ \partial_i = \nabla_i, \quad \partial_\tau = \frac{1}{\lambda} \partial_\tau \]
have

\[ n^a = \frac{1}{\sqrt{g^{\gamma^2 - h^2}}} \left( g^{\gamma^2 - h^2} \frac{\partial x}{|g'|} \partial_\nu - h^2 \frac{\partial x}{|g'|} \partial_\nu \right) \]  

(A3)

\[ = \frac{1}{\sqrt{g^{\gamma^2 - h^2}}} (w(x) \partial x - \gamma(x) \partial x), \]  

(A4)

for the timelike normal \( n^a \), with \( w(x) \) and \( \gamma(x) \) defined in Eq. (12). Finally, the radial part of the metric integration measure \( d\Sigma_\tau \) induced in the hyperboloidal slice \( \Sigma_\tau \) (see details in \[25\]) for the handling of the angular terms is given by

\[ d\Sigma_\tau = \lambda \sqrt{g^{\gamma^2 - h^2}} \, dx. \]  

(A5)

Inserting these elements in (A1), a straightforward calculation leads to Eq. (19), that we can rewrite as

\[ E = \frac{1}{2} \int_a^b \left( \frac{g^{\gamma^2 - h^2}}{|g'|} \partial_\nu \partial_\nu \phi + \frac{1}{|g'|} \partial_\nu \partial_\nu \phi + |g'| \hat{V} \phi \right) dx \]

\[ = \frac{1}{2} \int_a^b \left( w(x) \partial_\nu \partial_\nu \phi + p(x) \partial_\nu \partial_\nu \phi + q(x) \phi \right) dx. \]  

(A6)

Identifying \( \psi = \partial_\nu \phi \), and taking \( E \) for the square of the norm of the vector \( u = (\phi, \psi) \), i.e. prescribing \( ||u||^2 := E \), we recover expression (20). Considering only the \( \phi \)-part, this “energy norm” is a \( H^2 \)-like norm, so that it takes into account the frequency of the mode \( \phi \), a most important ingredient in our setting, giving the role of high-frequency perturbations in the ultraviolet instability of QNM overtones. Finally, considering the whole \( u = (\phi, \psi) \) vector, this norm is an \( L^2 \)-norm coming from the energy scalar product \( \langle.,.\rangle_\epsilon \) (for \( q(x) > 0 \))

\[ \langle \phi_1, \phi_2 \rangle_\epsilon \]

\[ = \frac{1}{2} \int_a^b \left( w(x) \bar{\psi}_1 \psi_2 + p(x) \partial_\nu \bar{\psi}_1 \partial_\nu \psi_2 + q(x) \phi_1 \phi_2 \right) dx, \]  

(A7)

that coincides with (22) upon identification \( q(x) = \hat{V} \). Note that \( \gamma(x) \) plays no role in the energy scalar product \( \langle.,.\rangle_\epsilon \).

2. Adjoint operator \( L^\dagger \)

A very important object in our discussion of QNM spectral instability and the pseudospectrum construction is the adjoint \( L^\dagger \) of the operator \( L \). The definition of \( L^\dagger \) depends on the choice of scalar product and we shall adopt here the energy scalar product (A7). The full construction of the adjoint \( L^\dagger \) requires a discussion of its domain of dependence. This is a delicate question intimately linked with the boundary and regularity conditions determining the functional space on which \( L \) and \( L^\dagger \) are defined. This functional analysis issue will be addressed elsewhere, and here we focus on the construction of the so-called “formal adjoint”, formally satisfying the relation

\[ \langle L^\dagger \phi_1, \phi_2 \rangle_\epsilon = \langle \phi_1, L \phi_2 \rangle_\epsilon, \]  

(A8)

for all \( u_1 = (\phi_1, \psi_1) \) and \( u_2 = (\phi_2, \psi_2) \). Taking into account the definition in Eq. (10) of the operator \( L \), this writes

\[ \langle L^\dagger \phi_1, \phi_2 \rangle_\epsilon = \langle \phi_1, \psi_1 \rangle_\epsilon \cdot \frac{1}{L_1} \left( \frac{0}{L_1 + L_2} \right) \phi_2 + \frac{1}{L_1} \phi_2 \phi_2 \psi_2 \]

\[ = \frac{1}{L_1} \phi_1 \phi_1 \psi_1 \psi_2 + \frac{1}{L_1 + L_2} \phi_2 \phi_2 \psi_2 \]  

(A9)

where we have used the expressions for \( L_1 \) and \( L_2 \) in Eq. (11). Using the energy scalar product (A7) and integrating by parts

\[ \langle L^\dagger \phi_1, \phi_2 \rangle_\epsilon = \frac{1}{L_1} \int_0^1 \left( \frac{0}{L_1} \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 + \frac{1}{L_1 + L_2} \phi_2 \phi_2 \phi_2 \right) \]  

(A10)

\[ + \frac{1}{L_1} \int_0^1 \left( \frac{0}{L_1} \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \right) \]  

\[ + \int_a^b \left( \frac{1}{L_1 + L_2} \phi_2 \phi_2 \right) \]  

\[ + \int_a^b \left( \frac{1}{L_1} \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \phi_1 \right) \psi_2 dx. \]
where we have used \( p(a) = p(b) = 0 \), the real character of \( u(x), p(x), q(x) \) and \( \gamma(x) \) and the Dirac-delta \( \delta(x) \) distribution to formally evaluate the boundary terms. This allows us to rewrite

\[
\left( L^\dagger \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} \right)_E = \left\langle \frac{1}{i} \begin{pmatrix} 0 \\ L_1 \end{pmatrix} L_2 + \frac{1}{w(x)} \left( \delta(x-a) - \delta(x-b) \right) \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} \right\rangle_E ,
\]

(A11)

so that, introducing the operator \( L^\phi_2 \) as in Eq. (25)

\[
L^\phi_2 = \frac{2}{w(x)} \left( \delta(x-a) - \delta(x-b) \right) ,
\]

(A12)

we can write the formal adjoint in Eqs. (23) and (24)

\[
\frac{\partial}{\partial \omega} \left( \langle \omega, A x, x \rangle_{A\sigma} \right) = \left\langle \omega, A^\dagger x, x \right\rangle_{A\sigma} ,
\]

(A13)

In general \( \gamma(x) \) does not vanish at the boundaries, so \( L \) is not even symmetric and therefore cannot be selfadjoint. Eq. (A13) identifies neatly the loss of selfadjointness with such non-vanishing \( \gamma(x) \), specifically linking spectral instability with a boundary phenomenon, formally cast through the presence of the Dirac-delta terms. This form also explains the (formal) selfadjoint case of the adjoint \( L^\dagger \) (we notice that in the problem studied in this work, the \( \epsilon \) condition is satisfied).

More generally, evaluation of adjoints play a key role in all aspects of our discussion of spectral instability: i) calculation of conditions numbers \( \kappa \)’s, involving the spectral problem of the adjoint \( L^\dagger \), cf. Eq. (26); ii) evaluation of the pseudospectrum, involving the calculation of (generalized) singular values of \( R_L(\omega) \) and therefore the spectral problem of \( R_L(\omega)R_L(\omega)^\dagger \), cf. Eqs. (43) and (B2); and iii) the prescription of the norm \( \| \delta V \|_\epsilon \) by \( \epsilon \) in the exploration of perturbed spectral QNM problems, again involving the spectral problem of the operator \( \delta V^\dagger \delta V \). Details of the calculation of adjoints in our discretised approach are given in appendices B and C.

Appendix B: Pseudospectrum in the energy norm

We derive here the relevant expressions for the construction of pseudospectra in the discretised version of the energy norm.

1. Scalar product and adjoint

Let us consider a general hermitian-scalar product in \( \mathbb{C}^n \) as

\[
\langle u, v \rangle_A = (u^*)^T G_{ij} v^j = u^* \cdot G \cdot v ,
\]

(B1)

with \( G \) a positive-definite Hermitian matrix

\[
G^* = G , \quad x^* \cdot G \cdot x > 0 \text{ if } x \neq 0 ,
\]

(B2)

where \( * \) denotes conjugate-transpose, i.e. \( u^* = \bar{u}^T \) and \( G^* = G^T \) (we notice that in the problem studied in this work, the Hermitian positive-definite matrix \( G \) is actually a real symmetric positive-definite matrix \( G^T = G \), but we keep the discussion in full generality). Using (B1) and (B2) in the relation

\[
\langle A^\dagger u, v \rangle_A = \langle u, Av \rangle ,
\]

(B3)

characterising the adjoint \( A^\dagger \) of \( A \) with respect to the scalar product (B1), we immediately get

\[
A^\dagger = G^{-1} A^* G .
\]

(B4)

2. Induced matrix norm from a scalar product norm

The (vector) norm \( \| \cdot \|_A \) in \( \mathbb{C}^n \) associated with the scalar product \( \langle \cdot, \cdot \rangle_A \) in (B1), namely

\[
\| v \|_A = \left( \langle v, v \rangle_A \right)^{\frac{1}{2}} ,
\]

(B5)

induces a matrix norm \( \| \cdot \|_A \) in \( M_n(\mathbb{C}) \) defined as

\[
\| A \|_A = \max_{\| x \|_A = 1, x \in \mathbb{C}^n} \left\{ \| Ax \|_A \right\} , \quad A \in M_n(\mathbb{C}) .
\]

(B6)

A more useful characterisation of this \( L^2 \) induced matrix norm is given in terms of the spectral radius \( \rho(A^\dagger A) \) of \( A^\dagger A \), where

\[
\rho(M) = \max_{\lambda \in \sigma(M)} \{ |\lambda| \} .
\]

(B7)

Indeed, we can write

\[
\| A \|_A^2 = \left( \max_{\| x \|_A = 1, x \in \mathbb{C}^n} \left\{ \langle Ax, A^* x \rangle_A \right\} \right)^2 = \max_{\| x \|_A = 1, x \in \mathbb{C}^n} \left\{ \langle Ax, A^* x \rangle_A \right\} = \max_{\| x \|_A = 1, x \in \mathbb{C}^n} \left\{ \langle A^\dagger Ax, x \rangle_A \right\} .
\]

(B8)

The rest of the argument essentially follows from Rayleigh-Ritz formula for self-adjoint operators. Explicitly, the (self-adjoint) matrix \( A^\dagger A \) is unitarily diagonalisable and non-negative definite (that is, \( \langle x, A^\dagger A x \rangle_A \geq 0, \forall x \in \mathbb{C}^n \)), so that we can find an orthonormal basis of eigenvectors \( \{ e_i \} \)

\[
A^\dagger A e_i = \lambda_i e_i , \quad \langle e_i, e_j \rangle_A = \delta_{ij} ,
\]

(B9)

with real non-negative eigenvalues \( \lambda_i \) that we order as

\[
0 \leq \lambda_1 \leq \lambda_2 \ldots \leq \lambda_n .
\]

(B10)

Expanding \( x = \sum_i x^i e_i \) for an arbitrary \( x \in \mathbb{C}^n \), we write

\[
\langle A^\dagger A x, x \rangle_A = \sum_i \lambda_i |x^i|^2 \leq \lambda_n \sum_i |x^i|^2 = \lambda_n \| x \|_A^2
\]

(B11)

that we can recast as

\[
\langle A^\dagger A \frac{x}{\| x \|_A}, \frac{x}{\| x \|_A} \rangle_A \leq \lambda_n = \rho(A^\dagger A) .
\]

(B12)
Inserting this in Eq. (B8), we conclude
\[ \|A\|_{1,\mathcal{G}}^2 \leq \rho(A^\dagger A). \] (B13)
To prove that the inequality is actually saturated, it suffices to show that there exists a vector \( x \), \( \|x\|_{\mathcal{G}} = 1 \), that realizes the equality, i.e. \( \|Ax\|_{\mathcal{G}}^2 = \rho(A^\dagger A) \). If we consider \( x = e_n \)
\[ \|Ax\|^2_{\mathcal{G}} = \langle Ax, Ax \rangle_{\mathcal{G}} = \langle A^\dagger A e_n, e_n \rangle_{\mathcal{G}} = \lambda_n = \rho(A^\dagger A), \] (B14)
and we can finally conclude
\[ \|A\|_{\mathcal{G}} = \left( \rho(A^\dagger A) \right)^{\frac{1}{2}}. \] (B15)

3. Characterization of the pseudospectrum

Given an invertible matrix \( A \in \mathbb{M}_n(\mathbb{C}) \) and a non-vanishing eigenvalue \( \lambda \), then \( 1/\lambda \) is an eigenvalue of \( A^{-1} \) and
\[ \max_{\lambda \in \sigma(A^{-1})} \{ |\lambda| \} = \left( \min_{\lambda \in \sigma(A)} \{ |\lambda| \} \right)^{-1}. \] (B16)
Then, for an invertible \( M \in \mathbb{M}_n(\mathbb{C}) \), we can write for the squared norm \( \| \cdot \|_{\mathcal{G}} \) of its inverse \( M^{-1} \)
\[ \|M^{-1}\|^2_{\mathcal{G}} = \rho\left( (M^{-1})^\dagger M^{-1} \right) = \rho\left( M M^\dagger \right)^{-1} \] (B17)
\[ = \left( \min_{\lambda \in \sigma(M M^\dagger)} \{ \lambda \} \right)^{-1} = \left( \min_{\lambda \in \sigma(M^\dagger M)} \{ \lambda \} \right)^{-1}, \]
where in the passage from the first line to the second we have used (B16) and the definition (B7) of the spectral radius, whereas in the last equality we have used that a matrix \( AB \) has the same eigenvalues as the matrix \( BA \).

We consider now the \( \epsilon \)-pseudospectrum characterisation in Definition 2, namely Eq. (32), applied to the discretised energy norm \( \| \cdot \|_{\mathcal{G}} \)
\[ \sigma_{\mathcal{G}}^\epsilon(A) = \{ \lambda \in \mathbb{C} : \|\lambda \mathbb{I} - A\|_{\mathcal{G}} > 1/\epsilon \}. \] (B18)
Using (B17), with \( M = \lambda \mathbb{I} - A \), we can write
\[ \|\lambda \mathbb{I} - A\|_{\mathcal{G}} > 1/\epsilon \iff \epsilon > \left( \min_{\lambda \in \sigma(M M^\dagger)} \{ \lambda \} \right)^{\frac{1}{2}} \] (B19)
Finally, \( \sigma_{\mathcal{G}}^\epsilon(A) \) can be written as
\[ \sigma_{\mathcal{G}}^\epsilon(A) = \{ \lambda \in \mathbb{C} : \sigma_{\mathcal{G}}^\min(\lambda \mathbb{I} - A) < \epsilon \}, \] (B20)
where \( \sigma_{\mathcal{G}}^\min(M) \) is the minimum of a set of “generalized singular values” of \( M \), related to the \( \langle \cdot, \cdot \rangle_{\mathcal{G}} \) scalar product
\[ \sigma_{\mathcal{G}}^\min(M) := \min \{ \sqrt{\lambda} : \lambda \in \sigma(M^\dagger M) \}. \] (B21)
When choosing the energy scalar product in section [IV], that is with \( G = G^E \) (see explicit expression in appendix [C]), we recover expression (45) for \( \sigma_{\mathcal{G}}^\epsilon(A) \). When using the canonical \( L^2 \) product we recover the standard \( \sigma_2^\epsilon(A) \) in (40), where
\[ \sigma_2^\min(M) = \min \{ \sqrt{\lambda} : \lambda \in \sigma(M^* M) \} =: \sigma_2^\min, \] (B22)
is the smallest of the singular values \( \sigma(M) = \sqrt{\lambda_i} : \lambda_i \in \sigma(M^* M) \), in the standard singular value decomposition of \( M \).

Appendix C: Elements in the Chebyshev discretization

1. Chebyshev spectral decomposition

The Chebyshev’s polynomial of order \( k \) is given by
\[ T_k(x) = \cos(k \arccos x), \ x \in [-1, 1]. \] (C1)

Chebyshev’s polynomials provide an orthogonal basis for functions \( f \in L^2([-1, 1], w(x)dx) \), with \( w(x) = 1/\sqrt{1-x^2} \), so that we can write the spectral expansion
\[ f(x) = c_0/2 + \sum_{k=1}^{\infty} c_k T_k(x). \] (C2)

For sufficiently regular functions \( f(x) \), coefficients \( c_k \) decay exponentially in \( k \). A \( f_N(x) \) approximate of \( f(x) \) is obtained by truncating the series to order \( N \)
\[ f_N(x) = c_0/2 + \sum_{k=1}^{N} c_k T_k(x). \] (C3)

The function \( f \) is therefore approximated by the vector \((c_0, c_1, \ldots, c_N)\) in \( \mathbb{C}^N \), with \( n = N + 1 \). In particular, we can evaluate the integral of \( f \) in the interval \([-1, 1]\) as
\[ \int_{-1}^{1} f_N(x) dx = c_0 - \sum_{k=1}^{\infty} c_k \frac{2k}{4k^2 - 1}. \] (C4)

2. Collocation methods: Chebyshev-Lobatto grid

When dealing with the product of functions, as it is the case in our setting, the description in terms of spectral coefficients \( c_i \)'s is not convenient. Instead, one constructs a Chebyshev’s interpolant \( f_N(x) \) from the evaluation of \( f(x) \) on points \( x_i \)
\[ f_N(x_i) = f(x_i), \ i \in \{0, 1, \ldots, N\}, \] (C5)
while \( x_i \in [-1, 1] \) define an appropriately chosen \( n \)-point quadrature grid. For concreteness, in the following we focus on the Chebyshev-Lobatto collocation grid including the interval boundaries \( x = \pm 1 \), in the spirit of including horizon and null infinity points in our compactified picture. The Chebyshev-Lobatto \((N + 1)\)-grid is given by the extrema of \( T_{N}(x) \) (i.e. the \( N \) = 1 zeros of \( T_{N}(x) \) together with both extreme points \( x_0 = 1 \) and \( x_N = -1 \), resulting in the values
\[ x_i = \cos \left( \frac{\pi i}{N} \right), \ i \in \{0, 1, \ldots, N\}. \] (C6)

We can enforce (C5) on this grid by constructing a \( f_N(x) \) interpolant in the functional form (C3), with coefficients \( c_i \)
\[ c_i = \frac{2 - \delta_{iN}}{2N} \left[ f(x_0) + (-1)^i f(x_N) + 2 \sum_{j=1}^{N-1} f(x_j)T_i(x_j) \right]. \] (C7)
with \( i \in \{0, 1, \ldots, N\} \). In the construction of our differential operator \( L \), the interpolant of the product of two functions \( f \) and \( g \) is obtained then by multiplication on grid points, that is

\[
(fg)_N(x_i) = f_N(x_i)g_N(x_i) .
\]

(C8)

In addition to that, we need an expression for the interpolant of the derivative \( f'_N(x) \), which is determined by

\[
f'_N(x_i) = \sum_{j=0}^{N} D_{ij}N f_N(x_j) ,
\]

(C9)

with

\[
D_{ij} = \begin{cases} 
- \frac{2N^2 + 1}{6}, & i = j = N \\
\frac{2N^2 + 1}{6}, & i = j = 0 \\
- \frac{2(1-x_j)^2}{\alpha_i (1-i^2)}, & 0 < i = j < N \\
\frac{\alpha_j}{x_i - x_j}, & i \neq j 
\end{cases}
\]

(C10)

where

\[
\alpha_i = \begin{cases} 
2, & i \in \{0, N\} \\
1, & i \in \{1, \ldots, N - 1\}
\end{cases}
\]

(C11)

3. Energy scalar product: Gram matrix \( G^E \)

Let us first consider the integral

\[
I_\mu(f,g) = \int_{-1}^{1} f(x)g(x) d\mu(x) ,
\]

(C12)

with \( d\mu(x) = \mu(x) dx \). We can get a quadrature approximation \( I^N_\mu(f,g) \) to \( I_\mu(f,g) \) by using expression \( \text{(C4)} \) for \( N \)-interpolants \( f_N(x) \) and \( g_N(x) \), combined with the particular expression \( \text{(C7)} \) for coefficients in the Chebyshev-Lobatto grid and the grid multiplication \( \text{(C8)} \). We obtain then

\[
I^N_\mu(f,g) = f_N^T \cdot C^N_\mu \cdot g_N ,
\]

(C13)

for the respective functions \( f \) and \( g \), and \( C^N_\mu \) the diagonal matrix given by

\[
(C^N_\mu)_{ij} = (C^N_\mu)_{ii} \delta_{ij}
\]

(C14)

\[
(C^N_\mu)_{ii} = \frac{2\mu(x_i)}{\alpha_i N} \left( 1 - \sum_{k=1}^{\frac{N}{2}} T_{2k}(x_i) \frac{2 - \delta_{2k,N}}{4k^2 - 1} \right)
\]

where we have used \( T_0(x) = 1 \), \( T_1(x) = 1 \) and \( T_{-1} = (-1)^k \). Then, dropping the indices \( N \), we can write the discrete version of the scalar product \( \langle \cdot, \cdot \rangle_E \) in \( \text{(C23)} \) as

\[
\langle u_1, u_2 \rangle_E = \left\langle \phi_1, \psi_1 \right\rangle_E \left\langle \phi_2, \psi_2 \right\rangle_E
\]

(C15)

\[
= \frac{1}{2} \left( \psi^*_1 \cdot C_\mu \cdot \psi_2 + (D\phi_1)^* \cdot C_\mu \cdot D\phi_2 + \phi^*_1 \cdot C_\mu \cdot \phi_2 \right)
\]

that can be rewritten in matrix form as

\[
\langle u_1, u_2 \rangle_E = u_1^T \cdot G^E \cdot u_2
\]

(C16)

\[
= (\phi_1, \psi_1) \begin{pmatrix} G^F & 0 \\ 0 & G^F \end{pmatrix} (\phi_2 \psi_2)
\]

with (here, the matrices \( C_{\mu} \), \( C_\nu \), and \( C_\omega \) are given by \( \text{(C14)} \) for the respective functions \( \mu(x) = \mu(x) p(x), w(x) \))

\[
G^F = \frac{1}{2} \left( C_{\nu} + D \cdot C_\mu \cdot D \right)
\]

(G17)

\[
G^E = \frac{1}{2} C_\mu w
\]

These expressions define the Gram matrix \( G^E \) for the discretised version of the energy scalar product (C2) in the basis determined from the Chebyshev-Lobatto spectral grid.

a. Grid interpolation

An important aspect to observe when performing the numerical integration is that Eq. \( \text{(C4)} \) is exact whenever the original function \( f(x) \) is a polynomial of order \( \leq N \). With this in mind, and assuming that \( f(x) \) and \( g(x) \) are polynomials, Eq. \( \text{(C13)} \) is exact only for the case where the product \( f(x)g(x) \) yields polynomials of order \( \leq N \). In practical terms, the procedure described above hampers the accuracy of the scalar product’s numerical integration whenever the order gets \( > N \).

As an illustrative example, take \( f(x) = P_{\ell}(x) \) and \( g(x) = P_{\ell'}(x) \), with \( P_{\ell}(x) \) the Legendre polynomials. Then the integral \( \text{(C12)} \) — with \( \mu(x) = 1 \) omitted of the expression — yields \( \int_{-1}^{1} f(x)g(x) dx = 2 \delta_{\ell,\ell'} / (2\ell + 1) \). If we now consider the discrete version \( I^N_\mu(f,g) \) given by Eq. \( \text{(C13)} \), one observes that the exact result is obtained only for the cases \( \ell = \ell' \leq N \), even though each individual function \( f(x) \) and \( g(x) \) is exactly represented for \( \ell \leq N \) and \( \ell' \leq N \), respectively.

To mitigate this issue, we modify the integration matrix \( C^N_\mu \) — or equivalently the Gram matrix \( G^E \) — by incorporating the following interpolation strategy. Given an interpolant vector \( f_N(x_i) \) associated with a Chebyshev-Lobatto grid \( \{ x_i \}_{i=0}^{N} \), one can obtain a second interpolant vector \( f_N(\bar{x}_i) \) associated with another Chebyshev-Lobatto grid \( \{ \bar{x}_i \}_{i=0}^{N} \) with a resolution \( N \neq \bar{N} \) via

\[
f_N(\bar{x}_i) = \sum_{i=0}^{N} \bar{\Pi}_{ii} f_N(x_i) .
\]

(C18)

Components \( \bar{\Pi}_{ii} \) of the interpolation matrix \( \bar{\Pi} \) are obtained by evaluating Eq. \( \text{(C30)} \) at the grid \( \{ \bar{x}_i \}_{i=0}^{N} \), with the coefficients \( \{ c_i \}_{i=0}^{N} \) expressed in terms of \( f_N(x_i) \) via Eq. \( \text{(C7)} \). Then

\[
\bar{\Pi}_{ii} = \frac{1}{\alpha_i \bar{N}} \left( 1 + \sum_{j=1}^{\bar{N}} (2 - \delta_{i,j})T_j(\bar{x}_i)T_j(x_i) \right)
\]

(C19)

Note that the interpolation matrix \( \bar{\Pi} \) has size \( \bar{N} \times \bar{N} \), which reduces to a square matrix only if \( N = \bar{N} \). In this case, Eq. \( \text{(C19)} \) is actually the identity matrix as expected.
Then, for a fixed $N$, we consider the discrete integration \((C_{ij}^{N})\) in terms of a higher resolution $\bar{N} = 2N$ and interpolate the expression back to the original resolution $N$. In other words, defining $T^N_{\mu}(f,g) := T^\bar{N}_{\mu}(f,g)$, we can consider the grid-interpolated new discrete integration
\[
T^N_{\mu}(f,g) = f_N \cdot C^N_{\mu} \cdot g_N ,
\]
where $C^N_{\mu} = I^\ell \cdot C^\bar{N}_{\mu} \cdot I$ or, in terms of its components
\[
(C^N_{\mu})_{ij} = \sum_{i=0}^{N} \sum_{j=0}^{N} (I^\ell)_{ii} (C^\bar{N}_{\mu})_{ij} \varpi_{jj} .
\]
Going back to the illustrative example where $f(x) = P_{\ell}(x)$ and $g(x) = P_{\ell'}(x)$, we now obtain $T^N_{\mu}(f,g) = 2\delta_{\ell,\ell'}/(2\ell + 1)$ exactly whenever $\ell, \ell' \leq N$.

In the same way, we grid-interpolate the Gram matrices
\[
G^E_1 = I^\ell \cdot G^E_1 \cdot I^\ell , \quad G^E_2 = I^\ell \cdot G^E_2 \cdot I^\ell ,
\]
that allows to perform the scalar product \((u_1, u_2)_{\mu} = u^*_1 \cdot G^E_{\mu} \cdot u_2\) via
\[
\langle u_1, u_2 \rangle_{\mu} = u^*_1 \cdot G^E_{\mu} \cdot u_2 = (\bar{\phi}_1, \bar{\psi}_1) \left( \begin{array}{c} G^E_1 \\ 0 \\ G^E_2 \end{array} \right) \left( \begin{array}{c} \bar{\phi}_2 \\ 0 \\ \bar{\psi}_2 \end{array} \right) .
\]

**Appendix D: Pöschl-Teller QNMs and regularity**

We give here the derivation of Pöschl-Teller QNM frequencies (and QNM eigenfunctions in our setting). This is done for completeness and, more importantly, to illustrate with an explicit example the role of regularity in the enforcement of outgoing boundary conditions in the hyperboloidal scheme.

We start from the Fourier transform in time of the Pöschl-Teller wave equation in Bizoń-Mach coordinates, i.e. Eq. (56)
\[
\left( (1 - x^2) \frac{d^2}{dx^2} - 2(i\omega + 1)x \frac{d}{dx} - i\omega(i\omega + 1) - 1 \right) \phi = 0
\]
This equation can be solved in terms of hypergeometric functions. Making the change $x = 1 - 2z$, it is rewritten as
\[
\left( z(1 - z) \frac{d^2}{dz^2} + \left( (1 + i\omega) - 2(1 + i\omega) \right) \frac{d}{dz} - (i\omega(i\omega + 1) + 1) \right) \phi = 0 ,
\]

namely Euler’s hypergeometric differential equation
\[
\left( (1 - z) \frac{d^2}{dz^2} + (c - (a + b + 1)z) \frac{d}{dz} - ab \right) \phi = 0
\]
for the values
\[
c = 1 + i\omega, \quad a = \frac{(2i\omega + 1) \pm i\sqrt{3}}{2}
\]
(4D)

For each choice of $\omega$, this equation admits two linearly independent solutions that can be built from the Gauss hypergeometric function $2F_1(a,b;c;z)$. It is only when we enforce some regularity in the solution, that the spectral parameter $\omega$ is discretised and we recover the QNM frequencies. In this particular case, it is when we truncate the hypergeometric series $2F_1(a,b;c;z)$ to a polynomial, that we recover Pöschl-Teller QNM frequencies. Such truncation occurs when either $a$ or $b$ is a non-positive integer. From (4D) we can write
\[
\omega = \frac{\sqrt{3}}{3} + i\left(-a + \frac{1}{2}\right) = \pm \frac{\sqrt{3}}{3} + i\left(-a + \frac{1}{2}\right)
\]
(5D)

Therefore, imposing either $a = -n$ or $b = -n$, with $n \in \mathbb{N} \cup \{0\}$, we finally get
\[
\omega_n^\pm = \pm \frac{\sqrt{3}}{3} + i\left(n + \frac{1}{2}\right)
\]
(6D)

Choosing the $a = -n$ version, the corresponding eigenvectors can be written as Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined as
\[
P_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_n}{n!} 2^{\alpha + \beta + \frac{1}{2}} \frac{1 - x}{\Gamma(n + 1)}
\]
(7D)

with the Pochhammer symbol (i.e. $(y)_n = \prod_{k=0}^{n-1} (y - k)$).

Inserting, for a given $n \in \mathbb{N} \cup \{0\}$, the values $a, b$, and $c$ into $2F_1(a,b;c;z)$ we get, upon comparison with (4D) and (6D),
\[
\alpha = \beta = i\omega_n
\]
(8D)

so that Pöschl-Teller QNM eigenfunctions write, in Bizoń-Mach coordinates, as
\[
\phi_n^\pm(x) = I_n^{(\omega_n^\pm)}(x), \quad x \in [-1,1] .
\]
(9D)

---

[1] Nollert, H.P.: About the significance of quasinormal modes of black holes. Phys. Rev. D53, 4397–4402 (1996). doi: 10.1103/PhysRevD.53.4397
[2] Nollert, H.P., Price, R.H.: Quantifying excitations of quasinormal mode systems. J. Math. Phys. 40, 980–1010 (1999). doi: 10.1063/1.532698
[3] Ashida, Y., Gong, Z., Ueda, M.: Non-Hermitian Physics. arXiv:2006.01837 (2020).
[4] Trefethen, L.N., Trefethen, A.E., Reddy, S.C., Driscoll, T.A.: Hydrodynamic stability without eigenvalues. Science 261(5121), 578–584 (1993). doi: 10.1126/science.261.5121.578
[5] Trefethen, L.N.: Pseudospectra of linear operators. SIAM Rev. 39(3), 383–406 (1997).
[6] Davies, E.B.: Pseudospectra of differential operators. J. Oper. Th 43, 243–262 (2000).
[7] Sjöstrand, J.: Pseudospectra for differential operators. Sémin. Équ. Dériv. Partielles, Éc. Polytech., Cent. Math. Lau-
dimensional wave equation. Trans. Am. Math. Soc. 373(6), 4051–4083 (2020). doi:10.1090/tras/8075
[90] Boosnern, P., Visser, M.: Quasi-normal frequencies: Key analytic results. JHEP 03, 073 (2011). doi:10.1007/JHEP03(2011)073
[91] Bindel, D., Zworski, M.: Theory and computation of resonances in 1D scattering. http://www.cs.cornell.edu/~ebindel/cima/resonant1d/
[92] Zworski, M.: Distribution of poles for scattering on the real line. Journal of Functional Analysis 73(2), 277 – 296 (1987). doi:https://doi.org/10.1016/0022-1236(87)90069-3
[93] In fact, as far as we are aware of the historical development, the path towards the interest in QNM instability followed the opposite way: concerns about BH QNM spectra stability were raised only after modifications/approximations of the potential gave rise to unexpected results [1][2] (Nollert’s study being the “cut potential” in section V D 4, since method “iii)” in [1] “regularizes” Poeschl-Teller transformation in black hole perturbation theory. Phys. Rev. D 2, 2141–2160 (1970). doi:10.1103/PhysRevD.2.2141
[94] More properly and generally [8], one should distinguish the “normal” (indeed selfadjoint in the present discussion in the present work) and the “non-normal” operator cases.
[95] Such an operator is relevant by itself, since it corresponds actually to the asymptual mode $m = 0$ of a wave propagating on a sphere with a constant unit potential, indeed a conservative system. The eigenfunctions are nothing more than the Legendre polynomials $P_n(x)$, with real eigenvalues $\omega_n = \pm \sqrt{1+\ell(\ell+1)}$. This provides a robust test case.
[96] Compare this decrease of the error as numerical resolution increases (the “expected” behaviour) with the anomalous growth in Fig. 2. This reflects that the “perturbed operator” has indeed improved spectral stability properties, as compared with the spectrally unstable “unperturbed” Poeschl-Teller operator.
[97] Al Sheikh, L., Jaramillo, J.L.: A geometric approach to QNMs in optics: application to pseudospectrum and structural stability. In preparation.
[98] Hager, M.: Instabilite spectrale semiclassique d’operateurs non-autoadjoints. Theses, Ecole Polytechnique X (2005).
[99] Hager, M.: Instabilité spectrale semiclassique pour des opérateurs non-autoadjoints. I: un modèle. Ann. Fac. Sci. Toulouse, Math. (6) 15(2), 243–280 (2006).
[100] Hager, M.: Instabilité spectrale semiclassique d’opérateurs non-autoadjoints. II. Ann. Henri Poincaré 7(6), 1035–1064 (2006).
[101] Hager, M., Sjöstrand, J.: Eigenvalue asymptotics for randomly perturbed non-selfadjoint operators. arXiv Mathematics e-prints math/0601381 (2006).
[102] Bordeaux Montieux, W.: Loi de Weyl presque sûre et résolvante pour des opérateurs non-autoadjoints. Theses, Ecole Polytechnique X (2008).
[103] Montieux, W.B., Sjöstrand, J.: Almost sure Weyl asymptotics for non-self-adjoint elliptic operators on compact manifolds. Ann. Fac. Sci. Toulouse, Math. (6) 19(3-4), 567–587 (2010).
[104] Bordeaux Montieux, W.: Almost sure weyl law for a differential system in dimension 1. Annales Henri Poincaré 11 (2011).
[105] Montieux, W.B.: Estimation de résolvante et construction de quasimode près du bord du pseudospectre arXiv:1301.3102 (2019).
[106] Vogel, M.: Spectral statistics of non-selfadjoint operators subject to small random perturbations. Séminaire Laurent Schwartz — EDP and applications (2016-2017). doi:10.5802/siespd.113
[107] Nonnenmacher, S., Vogel, M.: Local eigenvalue statistics of one-dimensional random non-self-adjoint pseudo-differential operators. arXiv:1711.05850 (2018).
[108] Sjöstrand, J.: Weyl law for semi-classical resonances with randomly perturbed potentials. No. 136 in Mémoires de la Société Mathématique de France. Société mathématique de France (2014). doi:10.24033/msmf.446
[109] Panioso Macedo, R.: Comment on “Some exact quasinormal frequencies of a massless scalar field in Schwarzschild spacetime”. Phys. Rev. D99(8), 088,501 (2019). doi:10.1103/PhysRevD.99.088501
[110] Regge, T., Wheeler, J.A.: Stability of a Schwarzschild Singularity. Physical Review 108(4), 1063–1069 (1957). doi:10.1103/PhysRev.108.1063
[111] Zenlli, F.J.: Gravitational field of a particle falling in a schwarzschild geometry analyzed in tensor harmonics. Phys. Rev. D 2, 2141–2160 (1970). doi:10.1103/PhysRevD.2.2141
[112] Cook, G.B., Zalubski, M.: Purely imaginary quasinormal modes of the Kerr geometry. Class. Quant. Grav. 33(24), 245,008 (2016). doi:10.1088/0264-9381/33/24/245008
[113] Berti, E.: (Personal Website)
[114] Cardoso, V.: (Personal Website)
[115] Black Hole Perturbation Toolkit. (bhowttok.org)
[116] Stein, L.C.: qnm: A Python package for calculating Kerr quasinormal modes, separation constants, and spherical-spheroidal mixing coefficients. J. Open Source Softw. 4(42), 1083 (2019). doi:21105/joss.01683
[117] Lin, K., Qian, W.L.: A Matrix Method for Quasinormal Modes: Schwarzschild Black Holes in Asymptotically Flat and (Anti-) Sitter Spacetimes. Class. Quant. Grav. 34(9), 095,004 (2017). doi:10.1088/1361-6382/aa6043
[118] Jansen, A.: Overdamped modes in Schwarzschild- de Sitter, and a Mathematica package for the numerical computation of quasinormal modes. Eur. Phys. J. Plus 132(12), 546 (2017). doi:10.1140/epjp/i2017-11825-9
[119] Fortuna, S., Vega, I.: Bernstein spectral method for quasinormal modes and other eigenvalue problems. arXiv:2003.06232 (2020).
[120] Leaver, E.: An analytic representation for the quasi-normal modes of Kerr black holes. Proc. R. Soc. London, Ser. A 402, 285–298 (1985).
[121] Warburton, N., et. al.: The Black Hole Perturbation Toolkit. In preparation.
[122] Bizon, P., Maliborski, M.: Dynamics at the threshold for blowup for supercritical wave equations outside a ball. arXiv:1909.01626 (2019).
[123] Chandrasekhar, S., Detweiler, S.: The quasi-normal modes of Kerr black holes. Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 344(1639), 441–452 (1975).
[124] Anderson, A., Price, R.H.: Intertwining of the equations of black hole perturbations. Phys. Rev. D43, 3147–3154 (1991). doi:10.1103/PhysRevD.43.3147
[125] Glampedakis, K., Johnson, A.D., Kennefick, D.: Darboux transformation in black hole perturbation theory. Phys. Rev. D96(2), 024,036 (2017). doi:10.1103/PhysRevD.96.024036
[126] Dunajski, M.: Solitons, instantons, and twistors. Oxford, UK: Univ. Pr. (2010).
[127] Such suppression must be stronger than exponential, since Poeschl-Teller shows stability of the fundamental QNM.
[128] The Nollert case $\lambda = 1$ in his method “iii)” seems special. It corresponds precisely to the “cut potential” in section Y1D3 and may require a separate discussion. It connects also with section VD4 since method “iii)” in [1] “regularizes” Schwarzschild with a Poeschl-Teller factor, cf. Eq. (7) in [1].
[129] Zhang, Y.J., Wu, J., Leung, P.T.: High-frequency behavior of...
Beyond $\omega$-modes of compact objects, such perturbed BH ‘universal’ branches share also features with QNMs of convex obstacles, where the asymptotic form of QNM branches (under a ‘pinched curvature assumption’) can be established as $\text{Im}(\omega_n) \sim K[\text{Re}(\omega_n)]^{1/2} + C$, for $n \gg 1$. Focusing on the spherical obstacle case (see also [31],[52]), if considering all angular $\ell$ modes and taking $\omega$ as the spectral parameter (while keeping $n$ fixed), the similar qualitative pattern between the corresponding branches and the perturbed BH QNM branches raises an intriguing question about a possible duality between QNM and Regge poles (c.f. e.g. [160],[163] in a complex angular momentum setting). In particular, the asymptotic logarithmic pattern of perturbed-BH [2] and compact object [129] QNMs is exactly recovered for Regge poles of compact objects in [137] (cf. [164] for related asymptotics).

We thank B. Raffaelli for signaling this point. Such an approach is very much in the spirit of the “asymptotic reasoning” advocated in [141], where asymptotic analysis is understood as an efficient and systematic tool to unveil structurally stable patterns underlying universality behaviour.

Berry, M.V.: Semiclassically weak reflections above analytic barriers. Journal of Physics A: Mathematical and General 15(12), 3693–3704 (1982). doi: 10.1088/0305-4470/15/12/021

Berry, M.V., Mount, K.E.: Semiclassical approximations in wave mechanics. Reports on Progress in Physics 35(1), 315–397 (1972).

Ould El Hadj, M., Stratton, T., Dolan, S.R.: Scattering from compact objects: Regge poles and the complex angular momentum method. Phys. Rev. D 101(10), 104,035 (2020). doi: 10.1103/PhysRevD.101.104035

Gualandri, L., Cardoso, V., del Rio, A., Maggiore, M., Pullin, J.: Gravitational-wave signatures of quantum gravity. arXiv:2007.13761 (2020).

Perez, A., Sudarsky, D.: Dark energy from quantum gravity discreteness. Phys. Rev. Lett. 122(22), 221,302 (2019). doi: 10.1103/PhysRevLett.122.221302

Cardoso, V., Costa, J.A., Destounis, K., Hintz, P., Jansen, A.: Quasinormal modes and strong cosmic censorship. Phys. Rev. Lett. 120, 031,103 (2018).

Jansen, A.: Quasinormal modes and strong cosmic censorship. Phys. Rev. Lett. 120, 031,110 (2018).

Batterman, R.W.: The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction, and Emergence. Oxford University Press (2001).

Helffer, B., Sjöstrand, J.: Résonances en limite semi-classique. No. 24-25 in Mémoires de la Société Mathématique de France. Société mathématique de France (1986).

Sjöstrand, J.: Lectures on resonances. version préliminaire, printemps (2002). http://sjostrand.perso.math.cnrs.fr/Coursqgb.pdf

Dimassi, M., Sjöstrand, J.: Spectral Asymptotics in the Semi-Classical Limit. London Mathematical Society Lecture Note Series. Cambridge University Press (1999).

Dencker, N., Sjöstrand, J., Zworski, M.: Pseudospectra of semiclassical (pseudo-) differential operators. Commun. Pure Appl. Math. 57(3), 384–415 (2004).

Davies, E.: Semi-classical analysis and pseudo-spectra. Journal of Differential Equations 216(1), 153–187 (2005).

Davies, E.B.: Pseudo-spectra, the harmonic oscillator and complex resonances. Proc. R. Soc. Lond., Ser. A. Math. Phys. Eng. Sci. 455(1982), 585–599 (1999).

Bony, J.F., Fajije, S., Ramond, T., Zerzeri, M.: An example of resonance instability. arXiv:2005.10035 (2020).

Leung, P.T., Liu, S.Y., Tong, S.S., Young, K.: Time-independent perturbation theory for quasinormal modes in leaky optical cavities. Phys. Rev. A 49, 3068–3073 (1994). doi: 10.1103/PhysRevA.49.3068

Leung, P.T., Liu, S.Y., Young, K.: Completeness and orthogonality of quasinormal modes in leaky optical cavities. Phys. Rev. A 49, 3057–3067 (1994). doi: 10.1103/PhysRevA.49.3057

Ching, E.S.C., Leung, P.T., Suen, W.M., Young, K.: Quasinormal mode expansion for linearized waves in gravitational systems. Phys. Rev. Lett. 74, 4588–4591 (1995). doi: 10.1103/PhysRevLett.74.4588

Ching, E.S.C., Leung, P.T., Maassen van den Brink, A., Suen, W.M., Tong, S.S., Young, K.: Quasinormal-mode expansion for waves in open systems. Rev. Mod. Phys. 70, 1545–1554 (1998). doi: 10.1103/RevModPhys.70.1545

Pook-Kolb, D., Birnholtz, O., Jaramillo, J.L., Krishnan, B., Schnetter, E.: Horizons in a binary black hole merger II: Fluxes, multipole moments and stability (2020).

Walther, H., Varcoe, B.T.H., Englert, B.G., Becker, T.: Cavity quantum electrodynamics. Reports on Progress in Physics 69(5), 1325–1382 (2006). doi: 10.1088/0034-4885/69/5/315

Smid, P.J.: Nonmodal stability theory. Annual Review of Fluid Mechanics 39(1), 129–162 (2007). doi: 10.1146/annurev.fluid.39.050306.092139

Aldaya, V., Barceló, C., Jaramillo, J.: Spanish relativistic meet- (ERE 2010): Gravity as a crossroad in physics. Journal of Physics: Conference Series 314 (2011). doi: 10.1088/1742-6596/314/1/011101

Note that the resulting associated interpolant $f(x)$ does not exactly coincide with the $N$-degree polynomial truncation from $[\mathbb{C}]$, since $c_0$’s in $[\mathbb{C}]$ are obtained from the orthogonal projection of the exact $f$ on the full Chebyshev complete basis. Both sets of $c_0$’s converge as $N \rightarrow \infty$.

Sjöstrand, J., Zworski, M.: Asymptotic distribution of resonances for convex obstacles. Acta Mathematica 183(2), 191–253 (1999). doi: 10.1007/BF02392828

Stefanov, P.: Sharp upper bounds on the number of the scattering poles. Journal of Functional Analysis 231(1), 111 – 142 (2006). doi: https://doi.org/10.1016/j.jfa.2005.07.007

Decanini, Y., Folacci, A., Raffaelli, B.: Unstable circular null geodesics of static spherically symmetric black holes, Regge poles and quasinormal frequencies. Phys. Rev. D 81, 104,039 (2010). doi: 10.1103/PhysRevD.81.104039

Decanini, Y., Folacci, A., Raffaelli, B.: Fine structure of high-energy absorption cross sections for black holes. Class. Quant. Grav. 28, 175,021 (2011). doi: 10.1088/0264-9381/28/17/175021

Raffaelli, B.: Strong gravitational lensing and black hole quasinormal modes: Towards a semiclassical unified description. Gen. Rel. Grav. 48(2), 16 (2016). doi: 10.1007/s10714-016-2572-x

Dolan, S.R., Ottewill, A.C.: On an Expansion Method for Black Hole Quasinormal Modes and Regge Poles. Class. Quant. Grav. 26, 225,003 (2009). doi: 10.1088/0264-9381/26/22/225003

Daudé, T., Nicoleau, F.: Local inverse scattering at a fixed
energy for radial schrödinger operators and localization of the regge poles. Annales Henri Poincaré 17(10), 2849–2904 (2015). doi:10.1007/s00023-015-0453-6