GENERALIZED $(\tilde{\kappa} \neq -1, \tilde{\mu})$-PARACONTACT METRIC MANIFOLDS WITH $\xi(\tilde{\mu}) = 0$

I. KÜPELI ERKEN

Abstract. We give a local classification of generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifold $(M, \tilde{\phi}, \xi, \eta, \tilde{\gamma})$ which satisfies the condition $\xi(\tilde{\mu}) = 0$. An example of such manifolds is presented.

1. Introduction

The study of paracontact geometry was introduced by Kaneyuki and Williams in [11]. A systematic study of paracontact metric manifolds started with the paper [19], were the Levi-Civita connection, the curvature and a canonical connection (analogue to the Tanaka Webster connection of the contact metric case) of a paracontact metric manifold have been described. However such structures were studied before [17], [4], [5]. Note also [3]. These authors called such structures almost para-coHermitian. The curvature identities for different classes of almost paracontact metric manifolds were obtained e.g. in [10], [18], [19]. The importance of paracontact geometry, and in particular of para-Sasakian geometry, has been pointed out especially in the last years by several papers highlighting the interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics (cf. e.g. [1], [2], [7], [8], [9]). Paracontact metric manifolds have been studied under several different points of view. The case when the Reeb vector field satisfies a nullity condition was studied in [7].

A remarkable class of paracontact metric manifolds $(M^{2n+1}, \tilde{\phi}, \xi, \eta, \tilde{\gamma})$ is that of paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-spaces, which satisfy the nullity condition

$$\tilde{R}(X,Y)\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y),$$

for all $X,Y$ vector fields on $M$, where $\tilde{\kappa}$ and $\tilde{\mu}$ are constants and $\tilde{h} = \frac{1}{2}\mathcal{L}_{\xi}\tilde{\varphi}$.

This new class of pseudo-Riemannian manifolds was introduced in [6]. In [7], the authors showed that while the values of $\tilde{\kappa}$ and $\tilde{\mu}$ change the form of (1.1) remains unchanged under $\mathcal{D}$-homothetic deformations. There are differences between a contact metric $(\kappa, \mu)$-space $(M^{2n+1}, \varphi, \xi, \eta, g)$ and a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-space $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{\gamma})$. Namely, unlike in the contact Riemannian case, a paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold such that $\tilde{\kappa} = -1$ in general is not para-Sasakian. In fact, there are paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifolds such that $\tilde{h}^2 = 0$ (which is equivalent to take $\tilde{\kappa} = -1$) but with $\tilde{\mu} \neq 0$. For 5-dimensional, Cappelletti Montano and Di Terlizzi gave the first example of paracontact metric $(-1, 2)$-space $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{\gamma})$ with $\tilde{h}^2 = 0$ but $\tilde{h} \neq 0$ in [6] and then Cappelletti Montano et al. gave the first paracontact metric structures defined on the tangent sphere bundle and constructed an example with arbitrary $n$ in [7]. Later, for 3-dimensional, the first numerical

2010 Mathematics Subject Classification. Primary 53B30, 53C15, 53C25; Secondary 53D10.

Key words and phrases. Paracontact metric manifold, $(\tilde{\kappa}, \tilde{\mu})$-paracontact metric manifold, nullity distributions.
example was given in [15]. Another important difference with the contact Riemannian case, due to the non-positive definiteness of the metric, is that while for contact metric \((\kappa, \mu)\)-spaces the constant \(\kappa\) cannot be greater than 1, paracontact metric \((\tilde{\kappa}, \tilde{\mu})\)-space has no restriction for the constants \(\tilde{\kappa}\) and \(\tilde{\mu}\).

Koufogiorgos and Tischlias [14] gave a local classification of a non-Sasakian generalized \((\kappa, \mu)\)-contact metric manifold with \(\xi(\mu) = 0\). This has been our motivation for studying generalized \((\tilde{\kappa}, \tilde{\mu})\)-paracontact metric manifolds with \(\xi(\tilde{\mu}) = 0\). We would like to emphasize that, as will be shown in this paper, the class of generalized \((\tilde{\kappa}, \tilde{\mu})\)-paracontact metric manifolds with \(\xi(\tilde{\mu}) = 0\) is much more different than the class of generalized \((\kappa, \mu)\)-contact metric manifolds with \(\xi(\mu) = 0\).

By a generalized \((\tilde{\kappa}, \tilde{\mu})\)-paracontact metric manifold we mean a 3-dimensional paracontact metric manifold satisfying (1.1) where \(\tilde{\kappa}\) and \(\tilde{\mu}\) are non constant smooth functions.

In the special case, where \(\kappa\) and \(\mu\) are constant, then \((M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})\) is called a \((\tilde{\kappa}, \tilde{\mu})\)-paracontact metric manifold.

In [15], Kupeli Erken and Murathan proved the existence of a new class of paracontact metric manifolds: the so called \((\tilde{\kappa}, \tilde{\mu}, \tilde{v})\)-paracontact metric manifolds. Such a manifold \(M\) is defined through the condition

\[
\tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) + \tilde{v}(\eta(Y))\tilde{\varphi}\tilde{h}X - \eta(X)\tilde{\varphi}\tilde{h}Y,
\]

where \(\kappa, \mu\) and \(v\) are smooth functions on \(M\). Furthermore, it is proved that these manifolds exist only in the dimension 3, whereas such a manifold in dimension greater than 3 is a \((\kappa, \mu)\)-paracontact metric manifold.

The paper is organized in the following way. In Section 2, we will report some basic information about paracontact metric manifolds. Some results about generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifolds will be given in Section 3. In Section 4, we shall locally classify generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifold with \(\xi(\tilde{\mu}) = 0\) (i.e. the function \(\mu\) is constant along the integral curves of the characteristic vector field \(\xi\)). We will prove that we can construct in \(R^3\) two families of such manifolds. All manifolds are assumed to be connected.

2. Preliminaries

The aim of this section is to report some basic facts about paracontact metric manifolds. All manifolds are assumed to be connected and smooth. We may refer to [11], [19] and references therein for more information about paracontact metric geometry.

An \((2n + 1)\)-dimensional smooth manifold \(M\) is said to have an almost paracontact structure if it admits a \((1, 1)\)-tensor field \(\tilde{\varphi}\), a vector field \(\xi\) and a 1-form \(\eta\) satisfying the following conditions:

(i) \(\eta(\xi) = 1, \quad \tilde{\varphi}^2 = I - \eta \otimes \xi,\)

(ii) the tensor field \(\tilde{\varphi}\) induces an almost paracomplex structure on each fibre of \(D = \ker(\eta)\), i.e. the \(\pm 1\)-eigendistributions, \(D^\pm := D_{\tilde{\varphi}}(\pm 1)\) of \(\tilde{\varphi}\) have equal dimension \(n\).

From the definition it follows that \(\tilde{\varphi}\xi = 0, \eta \circ \tilde{\varphi} = 0\) and the endomorphism \(\tilde{\varphi}\) has rank \(2n\). When the tensor field \(N_{\tilde{\varphi}} := [\tilde{\varphi}, \tilde{\varphi}] - 2d\eta \otimes \xi\) vanishes identically the almost paracontact manifold is said to be normal. If an almost paracontact manifold admits a pseudo-Riemannian metric \(\tilde{g}\) such that

\[
\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y),
\]

we say that \(\tilde{g}\) is normal.
for all $X, Y \in \Gamma(TM)$, then we say that $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is an almost paracontact metric manifold. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n + 1, n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis \(\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}\) such that \(\tilde{g}(X_i, X_j) = \delta_{ij}, \tilde{g}(Y_i, Y_j) = -\delta_{ij}\) and \(Y_i = \tilde{\varphi}X_i\), for any \(i, j \in \{1, \ldots, n\}\). Such basis is called a \(\tilde{\varphi}\)-basis.

If in addition \(d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)\) for all vector fields \(X, Y\) on \(M\), \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) is said to be a paracontact metric manifold. In a paracontact metric manifold one defines a symmetric, trace-free operator \(\tilde{\xi}\) proved that \(\tilde{h}\) anti-commutes with \(\tilde{\varphi}\) and satisfies \(\tilde{h}\xi = 0\), \(\text{tr} \tilde{h} = \text{tr} h \tilde{\varphi} = 0\) and

\[
\tilde{\nabla} \xi = -\tilde{\varphi} + \tilde{\varphi} h,
\]

where \(\tilde{\nabla}\) is the Levi-Civita connection of the pseudo-Riemannian manifold \((M, \tilde{g})\).

Moreover \(\tilde{h} = 0\) if and only if \(\xi\) is a Killing vector field and in this case \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) is said to be a \(K\)-paracontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold. Moreover, \(K\)-paracontact metric manifolds are studied by Kupeli Erken and Murathan \cite{15}. In the same study, they characterized the 3-dimensional manifolds in terms of \((\tilde{\varphi}, \xi, \eta, \tilde{g})\)-paracontact metric manifolds. In particular, they proved the following theorem.

\textbf{Theorem 1.} \cite{15} Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. If the characteristic vector field \(\xi\) is harmonic map then the paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-paracontact manifold always exists on every open and dense subset of \(M\). Conversely, if \(M\) is a
paracontact metric \((\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})\)-manifold then the characteristic vector field \(\xi\) is harmonic map.

It is shown that condition (2.4) is meaningless for \(\kappa \neq -1\) in dimension higher than three, because the functions \(\tilde{\kappa}, \tilde{\mu}\) are constants and \(\tilde{\nu}\) is the zero function.

Given a paracontact metric structure \((\tilde{\varphi}, \xi, \eta, \tilde{g})\) and \(\alpha > 0\), the change of structure tensors
\[
\tilde{\eta} = \alpha \eta, \quad \tilde{\xi} = \frac{1}{\alpha} \xi, \quad \tilde{\varphi} = \tilde{\varphi}, \quad \tilde{g} = \alpha \tilde{g} + \alpha(\alpha - 1)\eta \otimes \eta
\]
is called a \(D_\alpha\)-homothetic deformation. One can easily check that the new structure \((\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) is still a paracontact metric structure [19]. We now show that while \(D_\alpha\)-homothetic deformations destroy conditions like \(\tilde{R}_{XY}\xi = 0\), they preserve the class of paracontact \((\tilde{\kappa}, \tilde{\mu})\)-spaces.

Kupeli Erken and Murathan analyzed the different possibilities for the tensor field \(\tilde{h}\) in [15]. If \(\tilde{h}\) has
\[
\begin{pmatrix}
\tilde{\lambda} & 0 & 0 \\
0 & -\tilde{\lambda} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
the form (2.6) respect to local orthonormal \(\tilde{\varphi}\)-basis \(\{X, \tilde{\varphi}X, \xi\}\), the authors called the operator \(\tilde{h}\) is of \(h_1\) type.

If the tensor \(\tilde{h}\) has the form
\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
relative a pseudo orthonormal basis \(\{e_1, e_2, e_3\}\). In this case, the authors called \(\tilde{h}\) is of \(h_2\) type.

If the matrix form of \(\tilde{h}\) is given by
\[
\tilde{h} = \begin{pmatrix}
0 & -\tilde{\lambda} & 0 \\
\tilde{\lambda} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
with respect to local orthonormal basis \(\{X, \tilde{\varphi}X, \xi\}\). In this case, the authors said that \(\tilde{h}\) is of \(h_3\) type.

3. Generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifolds

In this section, we will give some basic facts about generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifolds.

**Lemma 1.** Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifold. The following identities hold:
\[
\begin{align*}
(3.1) \quad & h^2 = (1 + \tilde{\kappa})\tilde{\varphi}^2, \\
(3.2) \quad & \xi(\tilde{\kappa}) = 0, \\
(3.3) \quad & \tilde{Q}\xi = 2\tilde{\kappa}\xi, \\
(3.4) \quad & \tilde{Q} = \left(\frac{\tau}{2} - \tilde{\kappa}\right)I + \left(-\frac{\tau}{2} + 3\tilde{\kappa}\right)\eta \otimes \xi + \tilde{\mu}\tilde{h}, \quad \tilde{\kappa} \neq -1,
\end{align*}
\]
where \(\tilde{Q}\) is the Ricci operator of \(M\), \(\tau\) denotes scalar curvature of \(M\) and \(\tilde{l} = \tilde{R}(,\xi)\xi\).
Proof. For the proof of (3.1)-(3.3) are similar to that of [[15], Lemma 3.2]. The relation (3.4) is an immediate consequence of [[15], Lemma 4.4 and Lemma 4.14]. □

Lemma 2. Let \((M, \bar{\varphi}, \xi, \eta, \bar{g})\) be a generalized \((\bar{\kappa} \neq -1, \bar{\mu})\)-paracontact metric manifold. Then, for any point \(P \in M\), with \(\bar{\kappa}(P) > -1\) there exist a neighborhood \(U\) of \(P\) and an \(\bar{h}\)-frame on \(U\), i.e. orthonormal vector fields \(\xi, X, \bar{\varphi}X, \eta\), defined on \(U\), such that
\[
(3.5) \quad \bar{h}X = \bar{\lambda}X, \quad \bar{h}\bar{\varphi}X = -\bar{\lambda}\bar{\varphi}X, \quad \bar{h}\xi = 0, \quad \bar{\lambda} = \sqrt{1 + \bar{\kappa}}
\]
at any point \(q \in U\). Moreover, setting \(A = X\bar{\lambda}\) and \(B = \bar{\varphi}X\bar{\lambda}\) on \(U\) the following formulas are true :
\[
(3.6) \quad \bar{\nabla}_X\xi = (\bar{\lambda} - 1)\bar{\varphi}X, \quad \bar{\nabla}_{\bar{\varphi}X}\xi = -(\bar{\lambda} + 1)X,
\]
\[
(3.7) \quad \bar{\nabla}_\xi X = -\frac{\bar{\mu}}{2}\bar{\varphi}X, \quad \bar{\nabla}_{\bar{\varphi}X}\bar{\varphi}X = -\frac{\bar{\mu}}{2}X,
\]
\[
(3.8) \quad \bar{\nabla}_X\bar{\varphi}X = -\frac{B}{2\bar{\lambda}}\bar{\varphi}X, \quad \bar{\nabla}_{\bar{\varphi}X}\bar{\varphi}X = -\frac{A}{2\lambda}X,
\]
\[
(3.9) \quad \bar{\nabla}_{\bar{\varphi}X}X = -\frac{A}{2\bar{\lambda}}\bar{\varphi}X - (\bar{\lambda} + 1)\xi, \quad \bar{\nabla}_X\bar{\varphi}X = -\frac{B}{2\lambda}X + (1 - \bar{\lambda})\xi,
\]
\[
(3.10) \quad [\xi, X] = (1 - \bar{\lambda} - \frac{\bar{\mu}}{2})\bar{\varphi}X, \quad [\xi, \bar{\varphi}X] = (\bar{\lambda} + 1 - \frac{\bar{\mu}}{2})X,
\]
\[
(3.11) \quad [X, \bar{\varphi}X] = -\frac{B}{2\lambda}X + \frac{A}{2\lambda}\bar{\varphi}X + 2\xi,
\]
\[
(3.12) \quad \bar{h} \text{ grad} \bar{\mu} = \text{ grad} \bar{\kappa},
\]
\[
(3.13) \quad X\bar{\mu} = 2A,
\]
\[
(3.14) \quad \bar{\varphi}X\bar{\mu} = -2B,
\]
\[
(3.15) \quad \xi(A) = (1 - \bar{\lambda} - \frac{\bar{\mu}}{2})B,
\]
\[
(3.16) \quad \xi(B) = (\bar{\lambda} + 1 - \frac{\bar{\mu}}{2})A,
\]
\[
(3.17) \quad [\xi, \bar{\varphi}\text{ grad} \bar{\lambda}] = 0.
\]
Proof. The proofs of (3.6)-(3.11) are given in [[15]]. For the proof of (3.12), we will use well known formula
\[
\frac{1}{2} \text{ grad } \tau = \sum_{i=1}^{3} \varepsilon_i(\bar{\nabla}_X, \bar{\varphi})X_i,
\]
I. KÜPELİ ERKEN

where \( \{X_1 = X, X_2 = \tilde{\varphi}X, X_3 = \xi \} \). Using the equations (2.2) and (1), since \( tr h = tr \tilde{h} \tilde{\varphi} = 0 \), we obtain

\[
\sum_{i=1}^{3} \varepsilon_i (\nabla_{X_i} \tilde{Q})X_i = \sum_{i=1}^{3} \varepsilon_i X_i \left( \frac{r}{2} - \tilde{k} \right) X_i + \sum_{i=1}^{3} \varepsilon_i (X_i (\tilde{\mu}) \tilde{h} X_i)
+ \tilde{\mu} \sum_{i=1}^{3} \varepsilon_i (\nabla_{X_i} \tilde{h}) X_i
\]

(3.18)

\[
= \frac{1}{2} \text{grad} \tau - \text{grad} \tilde{k} + \tilde{h} \text{grad} \tilde{\mu}
+ \tilde{\mu} \sum_{i=1}^{3} \varepsilon_i (\nabla_{X_i} \tilde{h}) X_i - \frac{1}{2} \varepsilon (\tau) \xi
\]

The relations (3.5), (3.8) and (3.9) yield \( \sum_{i=1}^{3} (\nabla_{X_i} \tilde{h}) X_i = 0 \). Using the last relation in (3.18), one has

(3.19) \( \frac{1}{2} \text{grad} \tau = \frac{1}{2} \text{grad} \tau - \text{grad} \tilde{k} + \tilde{h} \text{grad} \tilde{\mu} - \frac{1}{2} \varepsilon (\tau) \xi \)

that is

(3.20) \( - \text{grad} \tilde{k} + \tilde{h} \text{grad} \tilde{\mu} - \frac{1}{2} \varepsilon (\tau) \xi = 0 \).

Since the vector field \( - \text{grad} \tilde{k} + \tilde{h} \text{grad} \tilde{\mu} \) is orthogonal to \( \xi \). So, we get (3.12). The equations (3.13) and (3.14) are immediate consequences of (3.12).

By virtue of (3.2) and (3.10), we have

\[
\xi (A) = \xi X \lambda = [\xi, X] \lambda + X \xi \lambda = (1 - \tilde{\lambda} - \frac{\tilde{\mu}}{2}) \tilde{\varphi} X \lambda
= (1 - \tilde{\lambda} - \frac{\tilde{\mu}}{2}) B
\]

The relation (3.16) is proved similarly. Using (3.2), we have

(3.21) \( \text{grad} \tilde{\lambda} = -AX + B \tilde{\varphi} X, \quad \tilde{\varphi} \text{grad} \tilde{\lambda} = -A \tilde{\varphi} X + BX \).

From the relations (3.21), (3.10), (3.15) and (3.16) we obtain

\[
\begin{bmatrix} \xi, \tilde{\varphi} \text{grad} \tilde{\lambda} \end{bmatrix} = [\xi, -A \tilde{\varphi} X + BX]
= - (\xi A) \tilde{\varphi} X - A [\xi, \tilde{\varphi} X] + (\xi B) X + B [\xi, X] = 0.
\]

Lemma 3. Let \( (M, \tilde{\varphi}, \xi, \eta, \tilde{\gamma}) \) be a generalized \( (\tilde{\kappa} \neq -1, \tilde{\mu}) \)-paracontact metric manifold. Then, for any point \( P \in M \), with \( \tilde{\kappa}(P) < -1 \) there exist a neighborhood \( U \) of \( P \) and an \( \tilde{h} \)-frame on \( U \), i.e. orthonormal vector fields \( \xi, X, \tilde{\varphi} X, \) defined on \( U \), such that

(3.22) \( \tilde{h}X = \lambda \tilde{\varphi} X, \quad \tilde{h}\tilde{\varphi} X = -\lambda X, \quad h \xi = 0, \quad \tilde{\lambda} = \sqrt{1 - \tilde{\kappa}} \)

at any point \( q \in U \). Moreover, setting \( A = X \lambda \) and \( B = \tilde{\varphi} X \lambda \) on \( U \) the following formulas are true :

(3.23) \( \tilde{\nabla}_X \xi = -\tilde{\varphi} X + \lambda \xi, \quad \tilde{\nabla}_{\tilde{\varphi} X} \xi = -X - \tilde{\lambda} \tilde{\varphi} X, \)

(3.24) \( \tilde{\nabla}_\xi X = -\frac{\tilde{\mu}}{2} \tilde{\varphi} X, \quad \tilde{\nabla}_\xi \tilde{\varphi} X = -\frac{\tilde{\mu}}{2} X, \)
\[(3.25)\quad \hat{\nabla}_X X = -\frac{B}{2\lambda} \hat{\phi} X + \hat{\lambda} \xi, \quad \hat{\nabla}_{\hat{\phi} X} \hat{\phi} X = -\frac{A}{2\lambda} X + \hat{\lambda} \xi,\]

\[(3.26)\quad \hat{\nabla}_{\hat{\phi} X} X = -\frac{A}{2\lambda} \hat{\phi} X - \xi, \quad \hat{\nabla}_X \hat{\phi} X = -\frac{B}{2\lambda} X + \xi,\]

\[(3.27)\quad [\xi, X] = -\hat{\lambda} X + (1 - \frac{\hat{\mu}}{2}) \hat{\phi} X, \quad [\xi, \hat{\phi} X] = (1 - \frac{\hat{\mu}}{2}) X + \hat{\lambda} \hat{\phi} X,\]

\[(3.28)\quad [X, \hat{\phi} X] = -\frac{B}{2\lambda} X + \frac{A}{2\lambda} \hat{\phi} X + 2\xi,\]

\[(3.29)\quad \hat{h} \text{ grad } \hat{\mu} = \text{ grad } \hat{\kappa},\]

\[(3.30)\quad X \hat{\mu} = 2B,\]

\[(3.31)\quad \hat{\phi} X \hat{\mu} = -2A,\]

\[(3.32)\quad \xi(A) = -\hat{\lambda} A + (1 - \frac{\hat{\mu}}{2}) B,\]

\[(3.33)\quad \xi(B) = (1 - \frac{\hat{\mu}}{2}) A + \hat{\lambda} B,\]

\[(3.34)\quad \left[ \xi, \hat{\phi} \text{ grad } \hat{\lambda} \right] = 0.\]

**Proof.** The proofs of (3.23)-(3.28) are given in [15]. The proof of (3.29) is similar to proof of Lemma 2, equation (3.12). The equations (3.30) and (3.31) are immediate consequences of (3.29).

By virtue of (3.2) and (3.27), we have

\[\xi(A) = \xi X \hat{\lambda} = [\xi, X] \hat{\lambda} + X \xi \hat{\lambda} = -\hat{\lambda} X \hat{\lambda} + (1 - \frac{\hat{\mu}}{2}) \hat{\phi} X \hat{\lambda} = -\hat{\lambda} A + (1 - \frac{\hat{\mu}}{2}) B\]

The relation (3.33) is proved similarly. Using (3.2), we have

\[(3.35)\quad \text{ grad } \hat{\lambda} = -AX + B\hat{\phi} X, \quad \hat{\phi} \text{ grad } \hat{\lambda} = -A\hat{\phi} X + BX .\]

From the relations (3.35), (3.27), (3.32) and (3.33) we obtain

\[\left[ \xi, \hat{\phi} \text{ grad } \hat{\lambda} \right] = [\xi, -A\hat{\phi} X + BX] = -[\xi, \hat{\phi} X] - A [\xi, \hat{\phi} X] + (\xi B) X + B [\xi, X] = 0.\]

\[\square\]
4. Generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifolds with \(\xi(\tilde{\mu}) = 0\)

We shall give a local classification of generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifolds with \(\tilde{\kappa} > -1\) which satisfy the condition \(\xi(\tilde{\mu}) = 0\).

**Theorem 2** (Main Theorem). Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifold with \(\tilde{\kappa} > -1\) and \(\xi(\tilde{\mu}) = 0\). Then

1) At any point of \(M\), precisely one of the following relations is valid: \(\tilde{\mu} = 2(1 + \sqrt{1 + \tilde{\kappa}})\), or \(\tilde{\mu} = 2(1 - \sqrt{1 + \tilde{\kappa}})\).

2) At any point \(P \in M\) there exists a chart \((U, (x, y, z))\) with \(P \in U \subseteq M\), such that

   i) the functions \(\tilde{\kappa}, \tilde{\mu}\) depend only on the variable \(z\).

   ii) if \(\tilde{\mu} = 2(1 + \sqrt{1 - \kappa})\), (resp. \(\tilde{\mu} = 2(1 - \sqrt{1 - \kappa})\)), the tensor fields \(\eta, \xi, \tilde{\varphi}, \tilde{g}, \tilde{h}\) are given by the relations,

\[
\begin{align*}
\xi &= \frac{\partial}{\partial x}, \quad \eta = dx - adz \\
\tilde{g} &= \begin{pmatrix}
1 & 0 & -a \\
0 & 1 & -b \\
-a & b & -1 + a^2 + b^2
\end{pmatrix} \\
\tilde{\varphi} &= \begin{pmatrix}
0 & a & -ab \\
0 & b & 1 - b^2 \\
0 & 1 & -b
\end{pmatrix} \\
\tilde{h} &= \begin{pmatrix}
0 & 0 & a\tilde{\lambda} \\
0 & -\tilde{\lambda} & 2\lambda b \\
0 & 0 & \tilde{\lambda}
\end{pmatrix}
\end{align*}
\]

with respect to the basis \(\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\), where \(a = -2y + f(z)\) (resp. \(a = 2y + f(z)\)), \(b = -\frac{y}{2} \frac{r'(z)}{r(z)} - 2x r(z) + s(z), \tilde{\lambda} = \tilde{\lambda}(z) = r(z)\) and \(f(z), r(z), s(z)\) are arbitrary smooth functions of \(z\).

Proof of the Main Theorem: Let \(\{\xi, X, \tilde{\varphi}X\}\) be an \(\tilde{h}\)-frame, such that

\[\tilde{h}X = \tilde{\lambda}X, \quad \tilde{h}\tilde{\varphi}X = -\tilde{\varphi}\tilde{\varphi}X, \quad \tilde{\lambda} = \sqrt{1 + \tilde{\kappa}}\]

in an appropriate neighbourhood of an arbitrary point of \(M\). Using the hypothesis \(\xi(\tilde{\mu}) = 0\) and equations \((3.13)-(3.17)\) and \((3.21)\) we have the following relations,

\[
\begin{align*}
(\tilde{\varphi}\text{grad}\lambda)\tilde{\mu} &= 4AB, \\
[\xi, \tilde{\varphi}\text{grad}\lambda] \tilde{\mu} &= 0, \\
\xi(AB) &= 0, \\
A\xi B + B\xi A &= 0, \\
A^2(\lambda + 1 - \frac{\tilde{\mu}}{2}) + B^2(1 - \lambda - \frac{\tilde{\mu}}{2}) &= 0.
\end{align*}
\]

Differentiating the relation \((4.5)\) with respect to \(\xi\) and using the equations \((3.2)\), \(\xi(\tilde{\mu}) = 0\), \((3.15)\), \((3.16)\) and \((4.5)\), we obtain

\[
1 + \lambda - \frac{\tilde{\mu}}{2}(-\lambda + 1 - \frac{\tilde{\mu}}{2})AB = 0.
\]
We put \( F = (1 + \tilde{\lambda} - \frac{\tilde{k}}{2})(-\tilde{\lambda} + 1 - \frac{\tilde{k}}{2}) \) and consider the set \( N = \{ p \in M \mid (\text{grad } \tilde{\lambda})(p) \neq 0 \} \).

We will prove that \( F = 0 \) at any point of \( N \). Let \( p \in N \) be such that \( F(p) \neq 0 \). From \( (4.10) \) we get \( (AB)(p) = 0 \). We consider cases \( \{ A(p) = B(p) = 0 \}, \{ A(p) \neq 0, B(p) = 0 \} \) and \( \{ A(p) = 0, B(p) \neq 0 \} \). Now we will examine the first case. In this case, by \( (3.2) \), we get \( (\xi(\tilde{\lambda})))(p) = 0 \). As a result we obtain \( (\text{grad } \tilde{\lambda})(p) = 0 \) which is a contradiction with \( (\text{grad } \tilde{\lambda})(p) \neq 0 \). So, the first case is impossible. We assume that \( \{ A(p) \neq 0, B(p) = 0 \} \). Since the function \( F \) is continuous, we find that a neighbourhood \( V \subseteq N \) exists, with \( p \in V \) such that \( F \neq 0 \) at any point of \( V \). Similarly, due to the fact that the function \( A \) is continuous on its domain, a neighbourhood \( W \subseteq p \in W \subseteq V \), such that \( A \neq 0 \) at any point of \( W \), and thus \( B = 0 \) on \( W \). From \( (4.10) \), we have \( (1 - \lambda - \frac{\kappa}{2}) = 0 \) at any point of \( W \) and thus \( F = 0 \) on \( W \), which is a contradiction. Since the last case is similar to the second case we omit it. Therefore, \( F = 0 \) at any point of \( N \). In what follows, we will work on the complement \( N^C \) of set \( N \), in order to prove that \( F = 0 \) on \( M \). If \( N^C = \emptyset \), then \( F = 0 \) on \( M \). Let us suppose that \( N^C \neq \emptyset \). Then we have \( \text{grad } \tilde{\lambda} = 0 \) on \( N^C \) and thus the function of \( \tilde{\lambda} \) is constant at any connected component of the interior \( (N^C)^o \). From the constancy of \( \tilde{\lambda} \) and the relations \( (5.10) \) and \( (5.11) \), the function \( \tilde{\mu} \) is also constant. As a result we find that \( F \) is constant on any connected component of \( (N^C)^o \). Because \( M \) is connected and \( F = 0 \) on \( N \) and \( F = 0 \) on any connected component of \( (N^C)^o \) we conclude that \( F = 0 \), or equivalently \( (1 + \tilde{\lambda} - \frac{\tilde{k}}{2})(-\tilde{\lambda} + 1 - \frac{\tilde{k}}{2}) = 0 \) at any point of \( M \).

Now we consider the open disjoint sets \( U_0 = \{ p \in M \mid (\tilde{\lambda} + 1 - \frac{\tilde{k}}{2})(p) \neq 0 \} \) and \( U_1 = \{ p \in M \mid (1 - \tilde{\lambda} - \frac{\tilde{k}}{2})(p) \neq 0 \} \). We have \( U_0 \cup U_1 = M \). Due to the fact that \( M \) is connected, we conclude that \( \{ M = U_0 \) and \( U_1 = \emptyset \} \) or \( \{ U_0 = \emptyset \) and \( U_1 = M \} \). Regarding the set \( U_0 \) we have \( \tilde{\mu} = 2(1 + \tilde{\lambda}) \), or equivalently \( \tilde{\mu} = 2(1 + \sqrt{1 + \kappa}) \) at any point \( M \). Similarly, regarding the set \( U_1 \) we obtain \( \tilde{\mu} = 2(1 - \tilde{\lambda}) = 2(1 - \sqrt{1 + \kappa}) \). Therefore, \( (1) \) is proved. Now, we will examine the cases \( \tilde{\mu} = 2(1 + \sqrt{1 + \kappa}) \) and \( \tilde{\mu} = 2(1 - \sqrt{1 + \kappa}) \).

Case 1. \( \tilde{\mu} = 2(1 + \sqrt{1 + \kappa}) \).

Let \( p \in M \) and \( \{ \xi, X, \tilde{\varphi}X \} \) be an \( \tilde{h} \)-frame on a neighborhood \( U \) of \( p \). Using the assumption \( \tilde{\mu} = 2(1 + \sqrt{1 + \kappa}) \) and \( (4.5) \) we obtain \( B = 0 \) and thus the relations \( (3.10) \) and \( (3.11) \) are reduced to

\[
\begin{align*}
(4.7) \quad [\xi, X] &= \frac{A}{2\lambda} \tilde{\varphi}X, \\
(4.8) \quad [\xi, \tilde{\varphi}X] &= 0, \\
(4.9) \quad [X, \tilde{\varphi}X] &= \frac{A}{2\lambda} \tilde{\varphi}X + 2\xi.
\end{align*}
\]

Since \( [\xi, \tilde{\varphi}X] = 0 \), the distribution which is spanned by \( \xi \) and \( \tilde{\varphi}X \) is integrable and so for any \( q \in V \) there exist a chart \( (V, (x, y, z)) \) at \( p \in V \subseteq U \), such that

\[
(4.10) \quad \xi = \frac{\partial}{\partial x}, \quad X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}, \quad \tilde{\varphi}X = \frac{\partial}{\partial y},
\]

where \( a, b, \) and \( c \) are smooth functions on \( V \). Since \( \xi, X \) and \( \tilde{\varphi}X \) are linearly independent we have \( c \neq 0 \) at any point of \( V \). By using \( (4.10), (3.2) \) and \( B = 0 \) we obtain

\[
(4.11) \quad \frac{\partial \tilde{\lambda}}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \tilde{\lambda}}{\partial y} = 0.
\]
From (4.11) we find
\[ \tilde{\lambda} = r(z), \]
where \( r(z) \) is smooth function of \( z \) defined on \( V \). By using (4.7), (4.9) and (4.10) we have following partial differential equations:
\[ \frac{\partial a}{\partial x} = 0, \quad \frac{\partial b}{\partial x} = -2\tilde{\lambda}, \quad \frac{\partial c}{\partial x} = 0, \]
\[ \frac{\partial a}{\partial y} = -2, \quad \frac{\partial b}{\partial y} = -\frac{A}{2\tilde{\lambda}}, \quad \frac{\partial c}{\partial y} = 0. \]
From \( \frac{\partial c}{\partial x} = \frac{\partial c}{\partial y} = 0 \) it follows that \( c = c(z) \) and because of the fact that \( c \neq 0 \), we can assume that \( c = 1 \) through a reparametrization of the variable \( z \). For the sake of simplicity we will continue to use the same coordinates \( (x, y, z) \), taking into account that \( c = 1 \) in the relations that we have occurred. From \( \frac{\partial a}{\partial x} = 0 \) we obtain
\[ a = a(x, y, z) = -2y + f(z), \]
where \( f(z) \) is smooth function of \( z \) defined on \( V \). Differentiating \( \tilde{\lambda} \) with respect to \( X \) and using (4.11) and (4.12) we have
\[ A = r'(z), \]
where \( r'(z) = \frac{dr}{dz} \). By using the relations \( \frac{\partial b}{\partial x} = -2\tilde{\lambda}, \frac{\partial b}{\partial y} = -\frac{A}{2\tilde{\lambda}} \) and (4.12) we get
\[ b = -\frac{y}{2} r'(z) - 2xy(z) + s(z), \]
where \( s(z) \) is arbitrary smooth function of \( z \) defined on \( V \). We will calculate the tensor fields \( \eta, \tilde{\varphi}, \tilde{g} \) and \( \tilde{h} \) with respect to the basis \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \). For the components \( \tilde{g}_{ij} \) of the Riemannian metric, using (4.10) we have
\[ \tilde{g}_{11} = \tilde{g}(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 1, \quad \tilde{g}_{12} = \tilde{g}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}X) = 1, \]
\[ \tilde{g}_{13} = \tilde{g}_{31} = \tilde{g}(\frac{\partial}{\partial x}, X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}) = \tilde{g}(\tilde{\varphi}X, X) - a\tilde{g}_{11} = -a, \]
\[ \tilde{g}_{23} = \tilde{g}_{32} = \tilde{g}(\frac{\partial}{\partial y}, X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}) = \tilde{g}(\tilde{\varphi}X, X) - a\tilde{g}_{12} - b\tilde{g}_{22} = -b, \]
\[ -1 = \tilde{g}(X, X) \Rightarrow a^2\tilde{g}_{11} + 2ab\tilde{g}_{13} + b^2\tilde{g}_{22} + 2ab\tilde{g}_{12} + 2b\tilde{g}_{23} + \tilde{g}_{33} = -1 \]
\[ = a^2 - 2a^2 + b^2 - 2b^2 + \tilde{g}_{33} = \tilde{g}_{33} - a^2 - b^2, \]
from which we obtain \( \tilde{g}_{33} = -1 + a^2 + b^2 \).

The matrix form of \( \tilde{g} \) is given by
\[ \tilde{g} = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & -1 + a^2 + b^2 \end{pmatrix}. \]
The components of the tensor field $\tilde{\varphi}$ are immediate consequences of

$$\tilde{\varphi}(\xi) = \tilde{\varphi}(\partial_x) = 0, \quad \tilde{\varphi}(\partial_y) = X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

$$\tilde{\varphi}(\partial_z) = \tilde{\varphi}(X - a \frac{\partial}{\partial x} - b \frac{\partial}{\partial y}) = \tilde{\varphi}X = a \tilde{\varphi}(\frac{\partial}{\partial x}) - b \tilde{\varphi}(\frac{\partial}{\partial y})$$

$$= \tilde{\varphi}X - b(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + \frac{\partial}{\partial z})$$

$$= \frac{\partial}{\partial y} - ab \frac{\partial}{\partial x} - b^2 \frac{\partial}{\partial y} - b \frac{\partial}{\partial z}$$

$$= -ab \frac{\partial}{\partial x} + (1 - b^2) \frac{\partial}{\partial y} - b \frac{\partial}{\partial z}.$$  

The matrix form of $\tilde{\varphi}$ is given by

$$\tilde{\varphi} = \begin{pmatrix} 0 & a & -ab \\ 0 & b & 1 - b^2 \\ 0 & c & -b \end{pmatrix}.$$  

The expression of the 1-form $\eta$, immediately follows from $\eta(\xi) = 1, \eta(X) = \eta(\tilde{\varphi}X) = 0$

$$\eta = dx - adz.$$  

Now we calculate the components of the tensor field $\tilde{h}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$:

$$\tilde{h}(\xi) = \tilde{h}(\frac{\partial}{\partial x}) = 0, \quad \tilde{h}(\frac{\partial}{\partial y}) = -\lambda \frac{\partial}{\partial y}, \quad \tilde{h}(\frac{\partial}{\partial z}) = 0.$$  

$$\tilde{h}(\frac{\partial}{\partial z}) = \tilde{h}(X) = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}$$

$$= \tilde{\lambda} a \frac{\partial}{\partial x} + 2b \tilde{\lambda} \frac{\partial}{\partial y} + \tilde{\lambda} \frac{\partial}{\partial z}.$$  

The matrix form of $\tilde{h}$ is given by

$$\tilde{h} = \begin{pmatrix} 0 & 0 & a \tilde{\lambda} \\ 0 & -\tilde{\lambda} & 2b \tilde{\lambda} \\ 0 & 0 & \tilde{\lambda} \end{pmatrix}.$$  

Thus the proof of the Case 1 is completed.

Case 2. $\tilde{\mu} = 2(1 - \sqrt{1 + \kappa}).$

As in the Case 1, we consider an $\tilde{h}$-frame $\{\xi, X, \tilde{\varphi}X\}$. Using the assumption $\tilde{\mu} = 2(1 - \sqrt{1 + \kappa})$ and (4.5) we obtain $A = 0$ and thus the relation (3.10) and (3.11) is written as

$$[\xi, X] = 0,$$  

$$A = 0.$$  

Thus the proof of the Case 1 is completed.
12 I. Küpeli Erken

(4.17) \[ [\xi, \tilde{\varphi}X] = 2\tilde{\lambda}X, \]

(4.18) \[ [X, \tilde{\varphi}X] = -\frac{B}{2\lambda}X + 2\xi. \]

Because of (4.16) we find that there is a chart \((V', (x, y, z))\) such that

\[ \xi = \frac{\partial}{\partial x}, \quad X = \frac{\partial}{\partial y} \]

on \(V'\). We put

\[ \tilde{\varphi}X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}, \]

where \(a, b, c\) are smooth functions defined on \(V'\). As in the Case 1, we can directly calculate the tensor fields \(\eta, \tilde{\varphi}, \tilde{g}\) and \(\tilde{h}\) with respect to the basis \(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\). This completes the proof of the main theorem.

Now, we give an example of a generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifold with \(\xi(\tilde{\mu}) = 0\) which satisfy the conditions of Main Theorem (Case 1).

**Example 1.** We consider the 3-dimensional manifold

\[ M = \{ (x, y, z) \in \mathbb{R}^3, z \neq 0 \} \]

and the vector fields

\[ \xi = \frac{\partial}{\partial x}, \quad \tilde{\varphi}X = \frac{\partial}{\partial y}, \quad X = (-2y + 1)\frac{\partial}{\partial x} + (-\frac{y}{2z} - 2xz + 2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \]

The 1-form \(\eta = dx - (-2y + 1)dz\) defines a contact structure on \(M\) with characteristic vector field \(\xi = \frac{\partial}{\partial x}\). Let \(\tilde{g}, \tilde{\varphi}\) be the pseudo-Riemannian metric and the \((1,1)\)-tensor field given by

\[
\tilde{g} = \begin{pmatrix}
1 & 0 & 2y - 1 \\
0 & 1 & \frac{y}{2z} + 2xz - 2 \\
2y - 1 & \frac{y}{2z} + 2xz - 2 & -1 + (-2y + 1)^2 + (-\frac{y}{2z} - 2xz + 2)^2
\end{pmatrix},
\]

\[
\tilde{\varphi} = \begin{pmatrix}
0 & -2y + 1 & -(-2y + 1)(-\frac{y}{2z} - 2xz + 2) \\
0 & -\frac{y}{2z} - 2xz + 2 & 1 - (-\frac{y}{2z} - 2xz + 2)^2 \\
0 & 1 & \frac{y}{2z} + 2xz - 2
\end{pmatrix},
\]

\[
\tilde{h} = \begin{pmatrix}
0 & 0 & -2y + 1 \\
0 & -z & 2z(-\frac{y}{2z} - 2xz + 2) \\
0 & 0 & z
\end{pmatrix}, \quad \tilde{\lambda} = z,
\]

with respect to the basis \(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\).

Let \(\{\xi, X, \tilde{\varphi}X\}\) be an \(\tilde{h}\)-frame, such that

\[ \tilde{h}X = \lambda \tilde{\varphi}X, \quad \tilde{h}\tilde{\varphi}X = -\tilde{\lambda}X, \quad \tilde{\lambda} = \sqrt{-1 - \tilde{\kappa}} \]

in an appropriate neighbourhood of an arbitrary point of \(M\). Using the hypothesis \(\xi(\tilde{\mu}) = 0\) and equations (3.30)-(3.34) and (3.35) we have the following relations,

(4.19) \[ (\tilde{\varphi}\text{grad}\lambda)\tilde{\mu} = 2(A^2 + B^2), \]

(4.20) \[ [\xi, \tilde{\varphi}\text{grad}\lambda] \tilde{\mu} = 0, \]

(4.21) \[ \xi(A^2 + B^2) = 0, \]
(4.22) \[ A\xi A + B\xi B = 0, \]

(4.23) \[ -\tilde{\lambda}A^2 + 2AB(1 - \frac{\tilde{\mu}}{2}) + \tilde{\lambda}B^2 = 0. \]

Differentiating the relation (4.23) with respect to \( \xi \) and using the equations (3.2), \( \xi(\tilde{\mu}) = 0 \), (3.32), (3.33) and (4.23), we obtain

(4.24) \[ (A(1 - \frac{\tilde{\mu}}{2}) + \tilde{\lambda}B)^2 + (B(1 - \frac{\tilde{\mu}}{2}) - \tilde{\lambda}A)^2 = 0. \]

From (4.24), precisely following cases occurs.

- (4.25) \[ A = 0 \] and \[ B = 0, \]
- (4.26) \[ A \neq 0 \] and \[ \tilde{\lambda}^2 + (1 - \frac{\tilde{\mu}}{2})^2 = 0, \]
- (4.27) \[ B \neq 0 \] and \[ \tilde{\lambda}^2 + (1 - \frac{\tilde{\mu}}{2})^2 = 0, \]
- (4.28) \[ A = 0 \] and \[ \tilde{\lambda}^2 + (1 - \frac{\tilde{\mu}}{2})^2 \neq 0, \]
- (4.29) \[ B = 0 \] and \[ \tilde{\lambda}^2 + (1 - \frac{\tilde{\mu}}{2})^2 \neq 0. \]

We now check, case by case, whether (4.24) give rise to a local classification of generalized \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-paracontact metric manifolds with \( \tilde{\kappa} < -1 \). From (4.25) we get \( \tilde{\kappa} \) and \( \tilde{\mu} \) constants. So the manifold returns to a \((\tilde{\kappa}, \tilde{\mu})\)-paracontact metric manifold. But we want to give a local classification for generalized \((\tilde{\kappa}, \tilde{\mu})\)-paracontact metric manifolds. So we omit this case. (4.26) and (4.27) hold if and only if \( \tilde{\kappa} = -1 \) and \( \tilde{\mu} = 2 \). But this is a contradiction with \( \tilde{\kappa} < -1 \). If we use (4.28) in (4.24) we obtain \[ B^2((1 - \frac{\tilde{\mu}}{2})^2 + \tilde{\lambda}^2) = 0. \]

But the solution of this equation contradicts with the type of manifold and choosing of \( \tilde{\kappa} \). So we can give following corollary.

**Corollary 1.** There is not exist any generalized \((\tilde{\kappa} < -1, \tilde{\mu})\)-paracontact metric manifolds which satisfy the condition \( \xi(\tilde{\mu}) = 0 \).

In the following theorem, we will locally construct generalized \((\tilde{\kappa}, \tilde{\mu})\)-paracontact metric manifolds with \( \tilde{\kappa} > -1 \) and \( \xi(\tilde{\mu}) = 0 \).

**Theorem 3.** Let \( \tilde{\kappa} : I \subset \mathbb{R} \to \mathbb{R} \) be a smooth function defined on an open interval \( I \), such that \( \tilde{\kappa}(z) > -1 \) for any \( z \in I \). Then, we can construct two families of generalized \((\tilde{\kappa}, \tilde{\mu})\)-paracontact metric manifolds \((M, \tilde{\varphi}_i, \xi_i, \eta_i, \tilde{g}_i), i = 1, 2, \) in the set \( M = \mathbb{R}^2 \times I \subset \mathbb{R}^3 \), so that, for any \( P(x, y, z) \in M \), the following are valid:

\[ \tilde{\kappa}_1(P) = \tilde{\kappa}_2(P) = \tilde{\kappa}(z), \]

\[ \tilde{\mu}_1(P) = 2(1 + \sqrt{1 + \tilde{\kappa}(z)}) \quad \text{and} \quad \tilde{\mu}_2(P) = 2(1 - \sqrt{1 + \tilde{\kappa}(z)}) \]

Each family is determined by two arbitrary smooth functions of one variable.

**Proof.** We put \( \tilde{\lambda}(z) = \sqrt{1 + \tilde{\kappa}(z)} > 0 \) and consider on \( M \) the linearly independent vector fields

(4.30) \[ \xi_1 = \frac{\partial}{\partial x}, \quad X_1 = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \text{and} \quad Y_1 = \frac{\partial}{\partial y}, \]
where \( a(x, y, z) = -2y + f(z), b(x, y, z) = -\frac{2\tilde{\lambda}(z)}{\lambda(z)} - 2x\tilde{\lambda}(z) + s(z), \) \( f(z), s(z) \) are arbitrary smooth functions of \( z \) and \( \tilde{g}_1(X_1, X_1) = \tilde{g}_1(Y_1, Y_1) = \tilde{g}_1(\xi_1, \xi_1) = 1. \) The structure tensor fields \( \eta_1, \tilde{g}_1, \tilde{\varphi}_1 \) are defined by \( \eta_1 = dx - (-2y + f(z))dz, \tilde{g}_1 = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1 + a^2 + b^2 \end{pmatrix} \) and \( \tilde{\varphi}_1 = \begin{pmatrix} 0 & a & -ab \\ 0 & b & 1 - b^2 \\ 0 & 1 & -b \end{pmatrix} \), respectively. From (4.30), we can easily obtain

\[
\begin{align*}
\{ \xi_1, X_1 \} &= -2\tilde{\lambda}(z)Y_1, \quad \{ \xi_1, Y_1 \} = 0, \\
\{ X_1, Y_1 \} &= \frac{\tilde{\lambda}'(z)}{2\tilde{\lambda}(z)}Y_1 + 2\xi_1.
\end{align*}
\]

Since \( \eta_1 \wedge d\eta_1 = 2dx \wedge dy \wedge dz \neq 0 \) everywhere on \( M \), we conclude that \( \eta_1 \) is a contact form. By using just \( \tilde{g}_1 \) and \( \tilde{\varphi}_1 \), we find \( \eta_1 = \tilde{g}_1(., \xi_1), \tilde{\varphi}_1X_1 = Y_1, \tilde{\varphi}_1Y_1 = X_1, \tilde{\varphi}_1\xi_1 = 0 \) and \( d\eta_1(Z, W) = \tilde{g}_1(Z, \tilde{\varphi}_1W), \tilde{g}_1(\tilde{\varphi}_1Z, \tilde{\varphi}_1W) = -\tilde{g}_1(Z, W) + \eta_1(Z)\eta_1(W) \) for any \( Z, W \in \Gamma(M) \). Hence \( M(\eta_1, \xi_1, \tilde{\varphi}_1, \tilde{g}_1) \) is a paracontact metric manifold. From the well-known Koszul’s formula \( 2\tilde{g}_1(\nabla_Z W, T) = Z\tilde{g}_1(W, T) + W\tilde{g}_1(T, Z) - T\tilde{g}_1(Z, W) - \tilde{g}_1(Z, [W, T]) + \tilde{g}_1(W, [T, Z]) + \tilde{g}_1(T, [Z, W]) \) and (2.22), we have the following equations

\[
\nabla_{\xi_1}X_1 = (\tilde{\lambda}(z) - 1)Y_1, \quad \tilde{\nabla}_{\xi_1}Y_1 = -((1 + \tilde{\lambda}(z))X_1,
\]

(4.33) \( \tilde{\nabla}_{\xi_1}X_1 = 0, \quad \tilde{\nabla}_{\xi_1}Y_1 = -(1 + \tilde{\lambda}(z))X_1, \)

(4.34) \( \tilde{\nabla}_{\xi_1}X_1 = 0 \), \( \tilde{\nabla}_{\xi_1}Y_1 = \frac{\tilde{\lambda}'(z)}{2\tilde{\lambda}(z)}X_1, \)

(4.35) \( \tilde{\nabla}_{X_1}X_1 = 0, \quad \tilde{\nabla}_{Y_1}Y_1 = \frac{\tilde{\lambda}'(z)}{2\tilde{\lambda}(z)}Y_1, \)

(4.36) \( \tilde{\nabla}_{Y_1}X_1 = -\frac{\tilde{\lambda}'(z)}{2\tilde{\lambda}(z)}Y_1 - (1 + \tilde{\lambda}(z))\xi_1, \)

(4.37) \( \tilde{\nabla}_{X_1}Y_1 = -\tilde{\lambda}(z) + 1)\xi_1, \)

\( \tilde{h}_1\tilde{\varphi}_1X_1 = -\tilde{\lambda}(z)\tilde{\varphi}_1X_1 \) and \( \tilde{h}_1X_1 = \tilde{\lambda}(z)X_1 \), where \( \tilde{\nabla} \) is Levi-Civita connection of \( \tilde{g}_1 \). By using the relations (4.33)+(4.37) we obtain

\[
\begin{align*}
\tilde{R}(\xi_1, \xi_1)\xi_1 &= 0, \\
\tilde{R}(X_1, \xi_1)\xi_1 &= \tilde{\kappa}_1X_1 + \tilde{\mu}_1\tilde{h}_1X_1, \\
\tilde{R}(Y_1, \xi_1)\xi_1 &= \tilde{\kappa}_1Y_1 + \tilde{\mu}_1\tilde{h}_1Y_1, \\
\tilde{R}(X_1, X_1)\xi_1 &= 0, \quad \tilde{R}(Y_1, Y_1)\xi_1 = 0, \\
\tilde{R}(X_1, Y_1)\xi_1 &= 0.
\end{align*}
\]

From the above relations and by virtue of the linearity of the curvature tensor \( \tilde{R} \), we conclude that

\[
\tilde{R}(Z, W)\xi_1 = (\tilde{\kappa}_1 I + \tilde{\mu}_1\tilde{h}_1)(\eta_1(Z)W - \eta_1(W)Z)
\]

for any \( Z, W \in \Gamma(M) \), i.e. \( (M, \tilde{\varphi}_1, \xi_1, \eta_1, \tilde{g}_1) \) is a generalized \((\tilde{\kappa}_1, \tilde{\mu}_1)\)-paracontact metric manifold with \( \xi(\tilde{\mu}_1) = 0 \) and thus the construction of the first family is completed. For the second construction, we consider the vector fields

\[
\begin{align*}
\xi_2 &= \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y},
\end{align*}
\]
and define the tensor fields $\eta_2, \tilde{g}_2, \tilde{\varphi}_2, \tilde{h}_2$ as follows:

$$
\eta_2 = dx - (2y + f(z))dz
$$

$$
\tilde{g}_2 = \begin{pmatrix}
1 & 0 & -a \\
0 & -1 & -b \\
-a & -b & 1 + a^2 + b^2
\end{pmatrix},
\tilde{\varphi}_2 = \begin{pmatrix}
0 & a & -ab \\
0 & b & 1 - b^2 \\
0 & 1 & -b
\end{pmatrix},
\tilde{h}_2 = \begin{pmatrix}
0 & 0 & -a\lambda_2 \\
0 & \lambda_2 & -2\lambda_2b \\
0 & 0 & -\lambda_2
\end{pmatrix}
$$

with respect to the basis $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, where $a = 2y + f(z)$, $b = (-\frac{y\lambda'(z)}{\lambda(z)} - 2x\lambda(z) + s(z))$ and $-\tilde{g}_2(X_2, X_2) = \tilde{g}_2(Y_2, Y_2) = \tilde{g}_2(\xi_2, \xi_2) = 1$. As in first construction, we say that $(M, \tilde{\varphi}_2, \xi_2, \eta_2, \tilde{g}_2)$ is a generalized $(\tilde{\kappa}, \tilde{\mu})$-paracontact metric manifold with $\xi(\tilde{\mu}) = 0$, where $\tilde{\kappa}_2(z) = \lambda(z)^2 - 1$ and $\tilde{\mu}_2(x, y, z) = 2(1 - \sqrt{1 + \kappa_2(z)})$. This completes the proof of the theorem. \hfill \Box

In the following theorem, we give an analytic expression of the scalar curvature $\tau$ of generalized $(\tilde{\kappa} \neq -1, \tilde{\mu})$-paracontact metric manifolds. It is interesting that the same formula holds both for the case $\tilde{\kappa} < -1$ and $\tilde{\kappa} > -1$.

**Theorem 4.** Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a generalized $(\tilde{\kappa}, \tilde{\mu})$-paracontact metric manifold. Then,

$$
\triangle \tilde{\lambda} = -X(A) + \tilde{\varphi}_X(B) + \frac{1}{2\lambda}(A^2 - B^2)
$$

and

$$
\tau = \frac{1}{\lambda}(\triangle \tilde{\lambda}) - \frac{1}{\lambda^2} \| \text{grad} \tilde{\lambda} \|^2 + 2(\tilde{\kappa} + \tilde{\mu}),
$$

where $\triangle \tilde{\lambda}$ is Laplacian of $\tilde{\lambda}$.

**Proof.** We will give the proof for $\tilde{\kappa} > -1$. The proof for $\tilde{\kappa} < -1$ is similar to $\tilde{\kappa} > -1$. Using the definition of the Laplacian and equations (3.2) and (3.8) we obtain

$$
\triangle \tilde{\lambda} = -XX(\tilde{\lambda}) + \tilde{\varphi}_X \tilde{\varphi}_X(\tilde{\lambda}) + \xi(\tilde{\lambda})
+ (\nabla X)\tilde{\lambda} - (\nabla X)\tilde{\lambda} - (\nabla \xi)\tilde{\lambda}
= -X(A) + \tilde{\varphi}_X(B) + \frac{1}{2\lambda}(A^2 - B^2).
$$
In order to compute scalar curvature \( \tau \) of \( M \), we will use (3.6)-(3.9). Defining the curvature tensor \( \tilde{R} \), after some calculations we get

\[
\tilde{R}(X, \phi X)\phi X = \tilde{\nabla}_X \tilde{\nabla}_{\phi X} \phi X - \tilde{\nabla}_{\phi X} \tilde{\nabla}_X \phi X - \tilde{\nabla}_{[X, \phi X]} \phi X
\]

\[
= \tilde{\nabla}_X \left(-\frac{A}{2\lambda} X\right) - \tilde{\nabla}_{\phi X} \left(-\frac{B}{2\lambda} X + (1 - \tilde{\lambda})\xi\right) - \tilde{\nabla}_{\phi X} - \frac{1}{2\lambda} X + \frac{1}{2\lambda} (\phi X + 2\xi) \phi X
\]

\[
= -X \left(\frac{A}{2\lambda} X - \frac{A}{2\lambda} \tilde{\nabla}_X X + \phi X \left(\frac{B}{2\lambda}\right) X + \frac{B}{2\lambda} \tilde{\nabla}_{\phi X} X\right)
\]

\[
+ \phi X (\tilde{\lambda})\xi - (1 - \tilde{\lambda})\tilde{\nabla}_{\phi X} \xi - \frac{B}{2\lambda} \tilde{\nabla}_X \phi X - \frac{A}{2\lambda} \tilde{\nabla}_{\phi X} \phi X - 2\tilde{\nabla}_\xi \phi X
\]

\[
= -X \left(\frac{A}{2\lambda} X + \frac{A}{2\lambda} \frac{B}{2\lambda} \phi X + \phi X \left(\frac{B}{2\lambda}\right) X\right)
\]

\[
+ \frac{B}{2\lambda} \left(-\frac{B}{2\lambda} X + (1 - \tilde{\lambda})\xi\right) + \frac{A}{2\lambda} \left(\frac{A}{2\lambda} X\right) + 2 \left(\frac{\mu}{2}\right) X
\]

\[
= \left[-X \left(\frac{A}{2\lambda} + \phi X \left(\frac{B}{2\lambda}\right) X\right) - \frac{B^2}{4\lambda^2} + \frac{A^2}{4\lambda^2} (-\tilde{\lambda}^2 + 1) + \tilde{\mu}\right] X
\]

\[
= \frac{1}{2} \left(\frac{X(A)\lambda - A^2}{\lambda^2} + \frac{\phi X (B)\lambda - B^2}{\lambda^2}\right) + \frac{1}{4\lambda^2} (A^2 - B^2) + (-\tilde{\lambda}^2 + 1) + \tilde{\mu}\right] X
\]

\[
= \frac{1}{2\lambda} \triangle \tilde{\lambda} - \frac{1}{2\lambda^2} \parallel grad\tilde{\lambda} \parallel^2 - \tilde{\kappa} + \tilde{\mu}\right] X
\]

and namely

\[
\tilde{g}(\tilde{R}, \phi X) \phi X, X) = -\frac{1}{2\lambda} \triangle \tilde{\lambda} + \frac{1}{2\lambda^2} \parallel grad\tilde{\lambda} \parallel^2 + \tilde{\kappa} - \tilde{\mu}.
\]

By using definition of scalar curvature, i.e. \( \tau = TrQ = -\tilde{g}(Q X, X) + \tilde{g}(Q \phi X, \phi X) + \tilde{g}(Q \xi, \xi) \), and using (3.9), we have

\[
\tau = -2\tilde{g}(\tilde{R}, \phi X) \phi X, X) + 2\tilde{g}(Q \xi, \xi)
\]

\[
= \frac{1}{\lambda} \triangle \tilde{\lambda} - \frac{1}{\lambda^2} \parallel grad\tilde{\lambda} \parallel^2 - 2(\tilde{\kappa} - \tilde{\mu}) + 4\tilde{\kappa}
\]

\[
= \frac{1}{\lambda} \triangle \tilde{\lambda} - \frac{1}{\lambda^2} \parallel grad\tilde{\lambda} \parallel^2 + 2(\tilde{\kappa} + \tilde{\mu}).
\]

The last equation gives (4.41). \( \square \)

References

[1] D.V. Alekseevski, V. Cortés, A.S. Galaev, T. Leistner, Cones over pseudo-Riemannian manifolds and their holonomy, J. Reine Angew. Math. 635 (2009), 23-69.
GENERALIZED \((\tilde{\kappa} \neq -1, \tilde{\mu})\)-PARACONTACT METRIC MANIFOLDS WITH \(\xi(\tilde{\mu}) = 0\)

[2] D.V. Alekseevski, C. Medori, A. Tomassini, *Maximally homogeneous para-CR manifolds*, Ann. Glob. Anal. Geom. **30** (2006), 1–27.

[3] C.L. Bejan, *Almost parahermitian structures on the tangent bundle of an almost para-coHermitian manifold*, In: The Proceedings of the Fifth National Seminar of Finsler and Lagrange Spaces (Brașov, 1988), 105–109, Soc. Stiințe Mat. R. S. Romania, Bucharest, 1989.

[4] K. Buchner, R. Rosca, *Variétés para-coKählerian à champ concirculaire horizontale*, C. R. Acad. Sci. Paris **285** (1977), Ser. A, 723–726.

[5] K. Buchner, R. Rosca, *Co-isotropic submanifolds of a para-coKählerian manifold with concicular vector field*, J. Geometry **25** (1985), 164–177.

[6] B. Cappelletti Montano, L. Di Terlizzi, *Geometric structures associated to a contact metric \((\kappa, \mu)\)-space*, Pacific J. Math. **246** no:2 (2010) 257–292.

[7] B. Cappelletti-Montano, I. Kupeli Erken, C. Murathan, *Nullity conditions in paracontact geometry*, Diff. Geom. Appl. **30** (2012), 665–693.

[8] V. Cortés, C. Mayer, T. Mohaupt, F. Saueressing, *Special geometry of Euclidean supersymmetry. 1. Vector multiplets*, J. High Energy Phys. (2004) 03:028: 73.

[9] V. Cortés, M.A. Lawn, L. Schäfer, *Affine hyperspheres associated to special para-Kähler manifolds*, Int. J. Geom. Methods Mod. Phys. **3** (2006), 955–1009.

[10] P. Dacko, *On almost para-cosymplectic manifolds*, Tsukuba J. Math. **28** (2004), 193–213.

[11] S. Kaneyuki, F. L. Williams, *Almost paracontact and parahodge structures on manifolds*, Nagoya Math. J. **99** (1985), 173–187.

[12] T. Koufogiorgos and C. Tsichlias, *On the existence of a new class of contact metric manifolds*, Canad. Math. Bull. Vol **43** (2000), 449–447.

[13] T. Koufogiorgos and C. Tsichlias, *Generalized \((\kappa, \mu)\)-contact metric manifolds with \(\|\text{grad} \kappa\| = \text{constant}\)*, J. Geom. **78** (2003), 83–91.

[14] T. Koufogiorgos and C. Tsichlias, *Generalized \((\kappa, \mu)\)-contact metric manifolds with \(\xi(\mu) = 0\)*, Tokyo J. Math. Vol **31** (2008), 39–57.

[15] I. Kupeli Erken, C. Murathan, *A Complete Study of Three-Dimensional Paracontact \((\kappa, \mu, \nu)\)-spaces*, Submitted. Available in [arXiv:1305.1511 [math. DG]].

[16] B. O’Neill, *Semi-Riemann Geometry*, Academic Press. New York, 1983.

[17] R. Rosca, L. Vanhecke, *Sur une variété presque parakählerienne munie d’une connexion self-orthonale involutive*, Ann. Sti. Univ. “Al. I. Cuza” Iași **22** (1976), 49–58.

[18] J. Węczeko, *On basic curvature identities for almost (para)contact metric manifolds*. Available in Arxiv: 1209.4731v1 [math. DG].

[19] S. Zamkovoy, *Canonical connections on paracontact manifolds*, Ann. Glob. Anal. Geom. **36** (2009), 37–60.

**Art and Science Faculty, Department of Mathematics, Uludag University, 16059 Bursa, Turkey**

E-mail address: iremkupeli@uludag.edu.tr