Monogamy inequalities for the EPR paradox and quantum steering

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Monogamy inequalities for the way bipartite EPR steering can be distributed among $N$ systems are derived. One set of inequalities is based on witnesses with two measurement settings, and may be used to demonstrate correlation of outcomes between two parties, that cannot be shared with more parties. It is shown that the monogamy for steering is directional. Two parties cannot independently demonstrate steering of a third system, using the same two-setting steering witness, but it is possible for one party to steer two independent systems. This result explains the monogamy of two-setting Bell inequality violations, and the sensitivity of the continuous variable (CV) EPR criterion to losses on the steering party. We generalise to $m$ settings. A second type of monogamy relation gives the quantitative amount of sharing possible, when the number of parties is less than or equal to $m$, and takes a form similar to the Coffman-Kundu-Wootters (CKW) relation for entanglement. The results enable characterisation of the tripartite steering for CV Gaussian systems and qubit GHZ and W states.

I. INTRODUCTION

Entanglement is a major resource for quantum communication and information processing [1]. An important advantage of quantum communication is the potential for an unprecedented security [2, 3]. In quantum information, the security is based on properties, like the no-cloning theorem [4], that are fundamental to quantum mechanics, but not classical mechanics.

A property closely connected to the no-cloning theorem is the lack of shareability of entanglement between a number of parties. A quantitative formulation for the way entanglement can be shared among three qubit systems was presented by Coffman, Kundu and Wootters (CKW) [7] and represented an advance in understanding multipartite entanglement. Their formulation was expressed in terms of inequalities. The monogamy relation is

$$C_{AB}^2 + C_{AC}^2 \leq C_{A(BC)}^2$$

where $C_{A(BC)}$ is the concurrence of the bi-partition $A$ with the group $\{BC\}$. This relation illustrates that maximum entanglement can be shared between two parties only.

Despite the importance of monogamy relations for quantum information, the knowledge of quantitative relations for other forms of entanglement is so far rather limited. It is known that the CKW relation can be extended to $N$ qubits [8], and that a violation of the two-setting Bell inequalities [10] is completely monogamous [11–13], a property that underpins the extra “device independent” security provided by quantum cryptography using Bell states [3]. Two-party monogamy does not however apply to Bell inequalities involving three measurement settings per site [14]. There is also a relative lack of quantitative knowledge about the shareability of non-locality in the more complex continuous variable (CV) systems, although there have been new investigations for Svetlichny’s nonlocality [15] and much progress has been made for the entanglement of CV Gaussian states [16], for which quantitative monogamy relations have been worked out [17]. Recent work [18, 19] analyses the reasons for the difficulty in developing monogamy relations using more general measures of entanglement, such as entanglement of formation. In Gaussian CV cryptography [1, 20], because Bell inequalities are not directly violated [21], the monogamy of other forms of nonlocality is likely to be especially useful.

The objective of this paper is to understand more about the monogamy associated with the Einstein-Podolsky-Rosen (EPR) paradox [22, 23]. This is the subclass of entanglement called “quantum steering” [24] that was first formalised as a distinct type of nonlocality by Wiseman, Jones and Doherty [25, 26]. Comparatively little is known about the shareability of this nonlocality, which we refer to as “EPR steering” [21].

Here, we will derive monogamy relations that quantify the amount of bipartite EPR steering that can be shared by a number of parties. As might be expected, we find the lack of shareability is greater than for entanglement. An important feature of EPR steering monogamy is its directionality. While entanglement is defined symmetrically with respect to both parties, this is not true of steering or the EPR paradox [27]. “One-way” steering or the EPR paradox [28, 29] has been realised [30, 31]: that party $A$ may “steer” another system $B$, does not imply the converse. This property has implications for the way EPR steering can be used to achieve secure quantum communication [32]. In this paper, we identify the directionality associated with steering monogamy.

Like the CKW result, the relations derived here are expressed in terms of inequalities. The monogamy relations are specific to particular EPR steering witnesses. We introduce two- and three-setting “steering parameters” $S_{B|A}^{(2)}$ and $S_{B|A}^{(3)}$, that involve the variances of Pauli spin matrices, and prove monogamy relations that apply to three qubits $A$, $B$ and $C$:

$$S_{B|A}^{(2)} + S_{B|C}^{(2)} \geq 2 \max\{1, S_{B|AC}^2\}$$

where $S_{A(BC)}$ is the concurrence of the bi-partition $A$ with the group $\{BC\}$. This relation illustrates that maximum entanglement can be shared between two parties only.
and $S_{B|A}^{(3)} + S_{B|C}^{(3)} + S_{B|D}^{(3)} \geq 3 \max\{1, S_{B|\{AC\}}^{2}\}$, and

$$S_{B|A}^{(3)} + S_{B|C}^{(3)} \geq 2 S_{B|\{AC\}}^{(3)}$$

Here $S_{B|A}^{(2)}, S_{B|A}^{(3)}, S_{B|A}^{(4)} < 1$ are criteria sufficient to demonstrate an EPR steering of system $B$, by measurements made on system $A$. $S_{B|A}^{(2)}, S_{B|A}^{(3)} \to 0$ implies maximum steering. Similar results are derived for $m$-setting steering inequalities. These relations imply a tight monogamy: steering of a system $B$ can only be confirmed by $m - 1$ other parties, using the same $m$-setting inequality. The monogamy of Bell inequality violations is explained, because Bell inequalities are also steering inequalities. Using a graphical representation based on that of Plesch and Buzek [35], we apply these results to depict the distribution of bipartite steering associated with the tripartite GHZ and W states.

More fundamental are monogamy relations based on criteria for EPR steering that are necessary and sufficient for detecting steering. Restricting to CV Gaussian systems, we find that such monogamy relations are possible. EPR steering of $B$ by $A$ exists if one can show that a parameter $E_{B|A}$ involving conditional variances for Bob's system (given measurements by Alice) satisfies $E_{B|A} < 1$ [23, 26, 36]. This is the EPR paradox criterion in which “elements of reality” deduced for Bob's system show incompatibility between local realism and the completeness of quantum mechanics [23, 36]. For any three parties $A, B$ and $C$, we will see that

$$E_{B|A} E_{B|C} \geq \max\{1, E_{B|\{AC\}}^{2}\}$$

and $E_{B|A} + E_{B|C} \geq 2 \max\{1, E_{B|\{AC\}}^{2}\}$. If steering is shared between more than two Gaussian sites, then it becomes directional. Two systems $A$ and $C$ cannot both (Gaussian) steer a third system $B$, but we will show by example that the converse is not true.

A lack of robustness of the EPR criterion to losses on the steering party, but not on the party being steered, has been noted in experiments [37, 38]. This effect is now explained in terms of the monogamy relation, and is seen to be a fundamental one, independent of the mechanism of generation of the EPR fields, or the way in which losses are implemented. This very tight form of monogamy comes about because the witness $E_{B|A}$ is based only on two observables — position and momentum.

We conclude the paper with a brief discussion. The steering monogamy inequalities are likely to be useful in establishing threshold efficiency bounds [39] and one-sided device-independent quantum communication security [4].

II. MONOGAMY OF TWO-SETTING AND CV GAUSSIAN STEERING

A. CV EPR steering

Consider the situation of three distinct and separated systems, labelled $A, B$ and $C$. For each system, quadratures $X, P$ are defined: $X_A, P_A$ for system $A$, and similarly for $B$ and $C$. We now examine the monogamy result for the two-observable EPR criterion used to verify the CV EPR paradox [21, 23].

We begin by defining a “steering parameter”, that enables confirmation of the EPR paradox between two systems $A$ and $B$ [36]. The steering parameter is

$$E_{B|A} = \Delta_{in} X_{B|A} \Delta_{in} P_{B|A}$$

where $(\Delta_{in} X_{B|A})^2$ is the variance of the conditional distribution for the measurement $X_B$, given a measurement at $A$. We normally assume that the measurement at $A$ has been optimised, to minimise the conditional variance value. Here, $X$ and $P$ are scaled position and momentum quadratures, so that the Heisenberg relation for system $B$ is given by $\Delta X_B \Delta P_B \geq 1$. A confirmation of the EPR paradox [36], and quantum steering [23, 26], is given when

$$E_{B|A} < 1$$

This type of inequality is called an “EPR steering” or “steering” inequality. EPR steering inequalities based on entropic uncertainty relations have also been derived [40].

Following and summarising the work of Refs. [23, 26, 31], we will use the terminology that measurements of the system $A$ (of Alice) “steer” another system $B$ (of Bob), if some conditions are satisfied, that imply the ensemble for $B$ has been affected by those measurements. This relates closely to EPR’s notion of “spooky action-at-a-distance”, and steering is illustrated by an EPR paradox, when Alice’s inferences about Bob’s system cannot be reconciled for consistency between local realism premises and the completeness of quantum mechanics. Steering manifests as a failure of a local hidden variable (LHV) theory that additionally constrains Bob’s local hidden variable system to be describable as a quantum state. If the conditional variances for $B$ are reduced, so that $E_{B|A} < 1$, then this implies a directional EPR paradox, whereby the measurements of $A$ steer the system $B$. Throughout this paper, we abbreviate this last phrase, to say that “$A$ steers $B$”, or “$A$ EPR steers $B$”.

Now we come to the monogamy result for the CV EPR steering.

Result (1): If $A$ steers $B$, in such a way that (6) holds, then it is certain that $\Delta_{in} X_{B|C} \Delta_{in} P_{B|C} > 1$ i.e. $C$ cannot be shown to steer $B$ by the EPR criterion. This result is expressed as the monogamy relation

$$E_{B|A} E_{B|C} \geq 1$$

The relation has been stated and the proved in a previous paper [38]. We present the full details here again, because the nature of the proof is central to the results that follow.

Proof: The observer (Alice) at $A$ can make a local measurement $O_A$ to infer a result for an outcome of $X_B$ at $B$. Denoting the outcomes of Alice’s measurement by $A_i$, we can evaluate the variance of each conditional
and (c) also hold for steering detected by two-setting inequalities. The relations are ensured because a conditional quantum state has a positive Gaussian Wigner function and the measurements are restricted to be Gaussian [16]. For such systems, the Result (1) is particularly useful, since in this case the optimised EPR steering inequality is necessary and sufficient to detect bipartite EPR steering of $B$ by $A$ [22, 24, 25]. (The optimised inequality is that which optimises the measurement at $A$, to minimise the conditional variances.) Thus, in the Gaussian case we can make the stronger statement, that a system can be steered by only one other system i.e. two distinct systems cannot independently steer the same system (Figure 1a). We will see in Section II.C that this sort of monogamy is one-way only. The properties of tripartite Gaussian steering are therefore directional. Also, we note that the monogamy inequality can be saturated: $E_{B|A} = 1$ was measured by Bowen et al [37] and Buono et al [38] for a CV Gaussian state, with 50% loss on the mode $A$, which implies a second mode $C$ satisfying $E_{B|A} = E_{B|C} = 1$.

**B. Monogamy of two-observable EPR steering**

The crucial aspect to the proof of the monogamy relation Result (1) is that the steering inequality involves only two observables (measurement “settings”) at each site, e.g., position and momentum ($X$ and $P$). Similar monogamy relations can therefore be established for other two-observable steering inequalities.

Let us consider three systems $A$, $B$ and $C$ of a fixed dimension corresponding to a spin $J$. We define the “steering parameter”

$$S_{B|A}^{(2)} = \left( (\Delta_{inf} J^{X}_{B|A})^2 + (\Delta_{inf} J^{Y}_{B|A})^2 \right)/C_{J}$$

using the notation explained for (8). Here $J^{X}$, $J^{Y}$ and $J^{Z}$ are the spin components, and the constant $C_{J}$ is defined by the uncertainty relation $\Delta J^{X}^2 + \Delta J^{Y}^2 \geq C_{J}$ [42]. EPR steering of $B$ by $A$ is confirmed when $S_{B|A}^{(2)} < 1$ [44]. This inequality detects what we will refer to as “two-observable EPR steering”, since the inequality involves only two measurement settings, for $J^{X}$ and $J^{Y}$, at each site.

**Result (2):** The monogamy relation

$$S_{B|A}^{(2)} + S_{B|C}^{(2)} \geq 2$$

holds. The proof follows as a straightforward extension of the proofs given for Result (1) and Result (3), below.

The relation has the same consequences for monogamy as Result (1). If EPR steering of $B$ by $A$ is confirmed by
Within the monogamy Result (1), we immediately tell us that the configurations of 1(b) are impossible, for Gaussian systems, and where steering is detected via two-observable inequalities. The configurations (d) are likely to be achieved by adding thermal noise to the single sites.

\[ S_{B|A}^{(2)} < 1, \]  

then it follows that \( S_{B|C}^{(2)} > 1 \) i.e. the system \( C \) cannot be shown to steer \( A \) by using the same steering inequality.

The case \( J = 1/2 \) is especially important, since it relates to the original Bell states on which many experiments and quantum information protocols are based. In terms of Pauli spin matrices \( \sigma^X \) and \( \sigma^Y \), we find that \( S_{B|A}^{(2)} = (\Delta_{in} \sigma^X_{B|A})^2 + (\Delta_{in} \sigma^Y_{B|A})^2 \). If bipartite EPR steering of \( B \) by \( A \) is observed as \( \sigma_{B|A}^{(2)} = (\Delta_{in} \sigma^X_{B|A})^2 + (\Delta_{in} \sigma^Y_{B|A})^2 < 1 \), then we know that for any third site \( C \), there is no such steering: that is \( \sigma_{B|C}^{(2)} = (\Delta_{in} \sigma^X_{B|C})^2 + (\Delta_{in} \sigma^Y_{B|C})^2 \geq 1 \). The last inequality (11) gives us information about the minimum noise levels for Bob’s qubit values as inferred by any third “eavesdropper” observer at \( C \), given that we know the noise levels for Bob’s qubit values as inferred by Alice at \( A \).

C. Categories of tripartite Gaussian and two-observable EPR steering

The Figures 1 and 2 show diagrammatically the possible distributions of bipartite steering for a tripartite CV Gaussian system. The restrictions and possibilities apply also to steering detected by a two-observable steering inequality. These depictions are useful, because the steering or lack of steering, for a specific inequality can give important information about the way correlations or noise inference levels are shared among three parties.

Before discussing possible bipartite distributions, we recall several properties of steering. First, steering requires entanglement. We say the EPR steering is “maximum”, if the EPR conditional variances go to zero, i.e. \( E_{B|A}, S_{B|A}^{(2)} \rightarrow 0 \). For some pure bipartite systems, the EPR steering can achieve the maximum value and this corresponds also to the “strongest” entanglement, as measured either by concurrence, or logarithmic negativity. This is true for the two-mode squeezed state \( (E_{B|A} \rightarrow 0) \) and for the qubit Bell-Bohm EPR state \( (S_{B|A}^{(2)} \rightarrow 0) \). As not all entanglement will show EPR steering, two systems can be entangled even if there is no EPR steering between them.

The possibilities for steering shared between three systems are therefore limited by the possibilities for entanglement. Two distinct types of pure tripartite entangled qubit states exist. These are the Greenberger-Horne-Zeilinger (GHZ) and W states. Similar states have been defined for the CV case. Here, we discuss the bipartite distribution for specific CV Gaussian states only, leaving the qubit case until Section VI, since for qubits it is important to also consider steering detected by three-observable inequalities.

The tripartite GHZ state allows no pairwise bipartite entanglement between any of the three systems, \( A, B \) and \( C \). The same will be true for the EPR steering of a GHZ state (i.e. \( E_{B|A} = E_{B|C} \geq 1 \)), since steering is a special sort of entanglement. The GHZ state however has bipartite entanglement between \( A \) and the composite system \( B – C \). A tripartite CV GHZ state is a simultaneous eigenstate of \( X_i – X_j (i,j = A,B,C \) and \( i \neq j \) and \( P_A + P_B + P_C \) with eigenvalues 0. Party \( A \) can choose to predict either of two noncommuting observables (a single position and or the sum of momenta) of the combined system \( B – C \). A tripartite CV GHZ state is a simultaneous eigenstate of \( X_i – X_j (i,j = A,B,C \) and \( i \neq j \) and \( P_A + P_B + P_C \) with eigenvalues 0. Thus, there is a (maximum) “two-way” steering i.e. the system \( A \) can steer the composite system \( B – C \) (e.g. \( E_{A|(BC)} = 0 \)) and vice versa (e.g. \( E_{(BC)|A} = 0 \)). This situation is depicted in Figures 1b and 3.

Bipartite steering between two individual sites is possible for other sorts of tripartite CV Gaussian states. However, we deduce that this bipartite steering, in order to be consistent with the monogamy relation Result (1), must be “one-way” only. We find that the outward “dual” steering, where \( A \) steers both \( B \) and \( C \), is possible (Figure 1c).

This type of tripartite steering can be created between modes \( A, B, C \) as in Figure 4. We argue as follows. The final bipartite steering between the pair \( A, B \) and \( (B, C) \) is equivalent to that between a mode \( A \) with no loss and a second mode \( B \) that has been subject to 50% loss. That the EPR paradox (and hence steering) of the lossy system \( B \) by \( A \) remains possible was summarised in Refs. 23, 32. The systems \( B \) and \( C \) are symmetric, and hence both systems \( B \) and \( C \) can be steered by \( A \).

The monogamy rule Result (1) negates the possibility of the steering “the other way”, that is the “lossy” Gaussian
system $B$ (of Figure 4) steers the “lossless” Gaussian system $A$. The monogamy rule tells us that steering of $A$ by both $B$ and $C$ is ruled out. With $50\%$ loss on the original $B'$ channel, there will be symmetry of the correlation between $A$ and $C$, and $A$ and $B$, in which case if $B$ can steer $A$, then so can $C$. This would lead to a contradiction of Result (1). That the EPR paradox cannot be demonstrated with $50\%$ loss on the steering channel was noted experimentally \cite{37, 38}.

There are some open questions. The monogamy Result (1) tells us that if $A$ can steer $B$, and $B$ can steer $C$, then $A$ cannot steer $C$, so that two-observable steering cannot be “passed on” (Figure 1d and 2a). It is left unaddressed however whether the scenario of Figure 2a and c is possible, although for qubits, the state discussed by Plesch and Buzek \cite{35} will give this possibility. The arrangement of Figure 2c can be shared. We might expect that “once split” the degree of steering would be reduced, in accordance with a monogamy rule like that of CKW.

III. MULTI-OBSERVABLE QUBIT AND QUDIT STEERING MONOGAMY RELATIONS

More monogamy relations may be derived for EPR steering inequalities that involve $m$ observables i.e. $m$ measurement settings, at each site. We show that no more than $m-1$ independent parties can demonstrate “steering” of a system $B$, using the same $m$-observable steering inequality.

A. Bohm’s EPR paradox monogamy

We consider a bipartite EPR steering inequality that involves 3 observables: $J^X, J^Y, J^Z$. We define the steering parameter

$$S_{B|A}^{(3)} = \left(\Delta_{inf}J_B^X\right)^2 + \left(\Delta_{inf}J_B^Y\right)^2 + \left(\Delta_{inf}J_B^Z\right)^2 \right)/J \geq 1$$

(12)

EPR steering of system $A$ by $B$ is obtained when $S_{B|A}^{(3)} < 1$, which confirms Bohm’s EPR paradox for spins when $J = 1/2$ \cite{31, 12}. This steering inequality was derived from the uncertainty relation $\Delta J^X \Delta J^Y + \Delta J^Z \geq J$ that applies to all quantum states of fixed spin $J$ i.e. to qubit systems of dimension $d = 2J + 1$ \cite{12}. For two qubit systems, $S_{B|A}^{(3)} = \left(\Delta_{inf}\sigma_B^X\right)^2 + \left(\Delta_{inf}\sigma_B^Y\right)^2 + \left(\Delta_{inf}\sigma_B^Z\right)^2 \right)/2$.

**Result (3):** We can apply the method of proof of Result (1), to derive the monogamy steering relation.

$$S_{B|A}^{(3)} + S_{B|C}^{(3)} + S_{B|D}^{(3)} \geq 3 \geq J$$

(13)

**Proof:** The observer at $A$ (Alice) can make the measurement that gives her the value of Bob’s observable $J_B^X$ with uncertainty $\Delta_{inf}J_B^X$. The observer at $C$ (Charlie) can make the measurement that gives the result for Bob’s $J_B^Y$ with uncertainty $\Delta_{inf}J_B^Y$, and the observer at $D$ can make the measurement that gives the result for Bob’s $J_B^Z$ with uncertainty $\Delta_{inf}J_B^Z$. Since the three observers can measure simultaneously, using similar arguments as for the proof of result 1 we see that the quantum uncertainty relation for spins (as given above) constrains the variances to satisfy

$$\left(\Delta_{inf}J_B^X\right)^2 + \left(\Delta_{inf}J_B^Y\right)^2 + \left(\Delta_{inf}J_B^Z\right)^2 \geq J$$

(14)
Similarly,

\[(\Delta_{inf} J_{B|A}^X)^2 + (\Delta_{inf} J_{B|C}^Z)^2 + (\Delta_{inf} J_{B|D}^X)^2 \geq J\] (15)

and also

\[(\Delta_{inf} J_{B|A}^Z)^2 + (\Delta_{inf} J_{B|C}^X)^2 + (\Delta_{inf} J_{B|D}^Y)^2 \geq J\] (16)

We then see that the monogamy relation \(E\) follows, upon adding the three inequalities. □

The monogamy Result (3) does not exclude 2 observers from being able to steer \(B\). However, the relation certainly prevents all 3 observers from being able to demonstrate steering of the same system \(B\) via the violation of the 3-observable steering inequalities \((14-16)\) (i.e. we can not attain \(S_{B|A}^{(3)} < 1\), \(S_{B|C}^{(3)} < 1\) and \(S_{B|D}^{(3)} < 1\). We can extend the proof of Result (3), to derive similar results involving \(m\)-observable steering inequalities.

B. Steering inequalities with \(m\) observables

Steering inequalities for two qubit systems have been derived in Refs. \cite{saunders2021local, bennet2011no, evans2021steering, toplin2021steering}. The multi-observable steering inequalities derived by Saunders et al \cite{saunders2021local} and Bennett, Evans et al \cite{bennet2011no} have been used in experiments that confirm steering without fair sampling assumptions \cite{saunders2021local, evans2021steering}. Expressed in terms of correlation rather than as a noise reduction, these steering inequalities, similar to Bell inequalities, take the general form \(S_{B|A}^{(m)} \leq 1\), where

\[S_{B|A}^{(m)} = \frac{1}{C_m} \sum_{j=1}^{m} c_j \langle \sigma^j_A \sigma^{p_j}_B \rangle\] (17)

Here, \(\sigma^j_A, \sigma^{p_j}_B\) is the Pauli spin component at angle \(\theta_j\), \(\theta_j\) for system \(A/B\) respectively (where \(p_j\) is a function of \(j\), \(|c_j| = 1\), \(C_m\) is a constant, and \(m\) is the number of measurement settings at each site. Steering is obtained when \(S_{B|A}^{(m)} > 1\).

Result (4): The 2-observable monogamy relation is \(S_{B|A}^{(2)} + S_{B|C}^{(2)} \leq 2\), which generalises to

\[\sum_{k=1}^{m} S_{B|A_k}^{(m)} \leq m\] (18)

where the different parties (distinct from \(B\)) are labelled \(A_k\). The result also applies to the two-observable Bell-Clauser-Horne-Shimony-Holt (CHSH) inequality

\[S_{B|A}^{Bell} = \langle \sigma^X_B \sigma^A_{A'} \rangle - \langle \sigma^Y_B \sigma^A_{A''} \rangle + \langle \sigma^X_B \sigma^A_{A''} \rangle + \langle \sigma^Y_B \sigma^A_{A'} \rangle \leq 2\]

which is also an EPR steering inequality \cite{bennet2011no}. EPR steering is observed when \(S_{B|A}^{Bell} > 2\), and the monogamy relation is \(S_{B|A}^{Bell} + S_{B|C}^{Bell} \leq 4\).

Proof: To prove \(E\), we recall that steering is a failure of a special type of separable model, called a Local Hidden State (LHS) model \cite{bennet2011no, toplin2021steering}. For any LHS model

\[\langle \sigma^X_A \sigma^Y_B \rangle = \int \rho(\lambda) \langle \sigma^X_A \rangle \lambda \langle \sigma^Y_B \rangle d\lambda\] (19)

where \(\langle \sigma^X_A \rangle \lambda, \langle \sigma^Y_B \rangle\) is the predicted average of the measurement \(\sigma^X_A, \sigma^Y_B\) for the local state \(\lambda\), and the local state for the system \(B\) is to be consistent with a \textit{local quantum state} (LQS). If the LHS model is valid, the steering parameter can be written

\[S_{B|A}^{(m)} = \int \rho(\lambda) \langle \sigma^j_A \rangle \lambda \langle \sigma^{p_j}_B \rangle d\lambda\] (20)

where \(S_{B|A}^{(m)}(\lambda) = \frac{1}{C_m} \sum_{j=1}^{m} c_j \langle \sigma^j_A \rangle \lambda \langle \sigma^{p_j}_B \rangle\). The steering inequality \(S_{B|A}^{(m)} \leq 1\) follows from this assumption. A similar result holds for the Bell-CHSH inequality.

Consider an experiment where the \(m\) parties \(A_1, ..., A_k\) measure simultaneously \(\sigma^1_{A_1}, ..., \sigma^m_{A_k}\) respectively, and the party at \(B\) measures \(\sigma^j_B\). We denote the outcomes of the measurements by the symbols \(\sigma^j_A\) but note they are in fact numbers, and will be identified as a “hidden” variable set \(\{\lambda_1, ..., \lambda_m\} = \{\sigma^1_{A_1}, ..., \sigma^m_{A_k}\}\). The state at \(B\) conditioned on these outcomes is definable by a quantum density matrix \(\rho_B(\lambda)\) and has moments (an expectation value for \(\sigma^j_B\)) which we once again denote by \(\langle \sigma^j_B | \{\sigma^p_{A_k}\} \rangle\) (we drop the parentheses for convenience of notation). The linear combination \(\sum_{k=1}^{m} S_{B|A_k}^{(m)}\) can be written in the form of an LHS model, where the probability \(\rho(\lambda)\) is established as the probability \(P\) of obtaining the outcomes \(\{\sigma^p_{A_k}\}\) of the simultaneous measurements. Explicitly, we can write

\[\sum_{j=1}^{m} c_j \langle \sigma^j_B | \{\sigma^p_{A_k}\} \rangle \sigma^{p_j}_B = \sum_{j=1}^{m} \sum_{k=1}^{m} P(\{\sigma^p_{A_k}\}) \sigma^{p_j}_B\] (21)

which becomes

\[\sum_{j=1}^{m} c_j \langle \sigma^j_B | \{\sigma^p_{A_k}\} \rangle = \sum_{j=1}^{m} \int \rho(\lambda) \langle \sigma^j_B \rangle \lambda \langle \sigma^{p_j}_B \rangle \lambda d\lambda\] (22)

where we see that the moments \(\langle \sigma^j_B \rangle \lambda\) are those of the quantum state \(\rho_B(\lambda)\), and that \(\langle \sigma^{p_j}_B \rangle / \lambda = \sigma^{j}_B / \lambda_j\). The last line satisfies the LHS model \cite{bennet2011no} and hence must be less than or equal to \(C_m\). This is true regardless of the choice of \(P\). The \(\sum_{k=1}^{m} S_{B|A_k}^{(m)}\) contains \(m\) groups of \(m\) terms like \(22\), but where different choices of simultaneous measurements are used for a given \(j\). In this way, the result follows.

To prove the Bell-CHSH result, we consider an experiment where the parties at \(A\) and \(C\) measure simultaneously \(\sigma^X_A\) and \(\sigma^Y_A\), and the party at \(B\) measures
\( \sigma^X \) or \( \sigma^Y \). The state at \( B \) conditioned on these outcomes is definable by a quantum density matrix \( \rho_{B\lambda} \), and has moments which we denote by \( \langle \sigma^X_B | \sigma^X_A, \sigma^Y_C \rangle \) and \( \langle \sigma^Y_B | \sigma^X_A, \sigma^Y_C \rangle \). Now we see that the linear combination

\[
\tilde{S}_{B|A} + \tilde{S}_{B|C},
\]

namely

\[
\langle \sigma^X_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle
\]

\[
+ \langle \sigma^X_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle - \langle \sigma^Y_B | \sigma^X_A \rangle - \langle \sigma^Y_B | \sigma^X_A \rangle,
\]

can be written as consistent with a LHS model, since the probability \( \rho(\lambda) \) can be established as the probability of obtaining the outcomes \( \sigma^X_A \) and \( \sigma^Y_A \) of the simultaneous measurements. Explicitly, we can write

\[
\langle \sigma^X_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle = \sum_{\sigma^X_A, \sigma^Y_A} P(\sigma^X_A, \sigma^Y_A)
\]

\[
\times \{ \langle \sigma^X_B | \sigma^X_A \rangle \sigma^Y_A \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \sigma^Y_A \}
\]

which takes the form

\[
\langle \sigma^X_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle = \int \rho(\lambda) \{ \langle \sigma^X_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

and similarly

\[
\langle \sigma^X_B | \sigma^X_A \rangle - \langle \sigma^Y_B | \sigma^X_A \rangle = \int \rho(\lambda) \{ \langle \sigma^X_B | \sigma^X_A \rangle \}
\]

\[
- \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

where we see that the moments \( \langle \sigma^X_B | \sigma^X_A \rangle \), \( \langle \sigma^Y_B | \sigma^X_A \rangle \) are those of the quantum state \( \rho_{B\lambda} \), and \( \langle \sigma^X_A \rangle = \sigma^X_A = \lambda_1 \) and \( \langle \sigma^Y_A \rangle = \sigma^Y_A = \lambda_2 \). In this way, we can write

\[
\langle \sigma^X_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle - \langle \sigma^Y_B | \sigma^X_A \rangle
\]

\[
= \int \rho(\lambda) \{ \langle \sigma^X_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
= \int \rho(\lambda, \lambda_2) \{ \langle \sigma^X_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
+ \langle \sigma^Y_B | \sigma^X_A \rangle \}
\]

\[
\] and \( \langle \sigma^X_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle \leq \sqrt{2} \) \[27, 31, 46\]. If either of these inequalities is violated, steering is confirmed. Result (4) allows us to immediately write down monogamy relations associated with these steering inequalities:

\[
\langle \sigma^X_B | \sigma^X_A \rangle - \langle \sigma^Y_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle \leq 2 \sqrt{2}
\]

\[
\] and \( \langle \sigma^Y_B | \sigma^X_A \rangle + \langle \sigma^X_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle \leq 2 \sqrt{2} \).

\section{IV. CHSH-BELL NONLOCALITY MONOGAMY}

Since the Bell inequalities are also steering inequalities, the monogamy of steering implies the monogamy of the two-setting CHSH Bell inequalities. The CHSH Bell inequalities are

\[
\tilde{S}^B_{B|A} = \langle \sigma^X_B | \sigma^X_A \rangle - \langle \sigma^Y_B | \sigma^X_A \rangle + \langle \sigma^X_B | \sigma^X_A \rangle + \langle \sigma^Y_B | \sigma^X_A \rangle \leq 2
\]

\[
\]

Any Bell inequality is also an EPR steering inequality \[29\]. Using Result (4), we can therefore deduce the monogamy relation for the CHSH Bell inequality:

\[
\tilde{S}^B_{B|A} + \tilde{S}^B_{B|C} \leq 4
\]

\[
\]

The symmetry of the Bell-CHSH inequalities implies that in an experiment (where there is a fixed choice of measurement settings at each location) \( \tilde{S}^B_{B|A} > 2 \) is equivalent to \( \tilde{S}^B_{A|B} > 2 \). That is, as indeed must be true generally given the definition of Local hidden Variable theories \[25\], the violation of a Bell inequality implies "two-way" steering. The monogamy relations \( \tilde{S}^B_{A|B} + \tilde{S}^B_{A|C} \leq 4 \) and \( \tilde{S}^B_{B|A} + \tilde{S}^B_{B|C} \leq 4 \) also hold. If two parties \( A-B \) can violate the Bell-CHSH inequality, then the pairs \( A-C \) and \( C-B \) cannot.

The result for the monogamy of Bell-CHSH violations result is not new \[11, 13\]. What we have discovered from our analysis is that the monogamy follows as a result of steering monogamy. All two-observable (setting) steering inequalities possess one-way monogamy. Since the Bell-CHSH violations imply two-way steering, this is enough to explain Bell-CHSH monogamy.

Our results explain the shareability, with respect to three sites, of the three-observable Bell inequality violation of Collins and Gisin \[14\]. Being three-observable steering inequalities, we can expect however, using Result (4), that these violations cannot be shared among four sites.

\section{V. SHARING OF BIPARTITE STEERING}

We have seen that a very tight monogamy arises for the correlations of a witness when the number of parties equals or exceeds \( m + 1 \) where \( m \) is the number of observables that need to be measured at each site. Now, we
examine the constraints on the distribution of bipartite steering, when the number of systems is less than \( m + 1 \).

In this Section, we therefore derive relations for steering monogamy that are similar to the CKW inequalities, for particular witnesses. We quantify how the "total amount of steering" is shared among the subgroups. Similar to the result for sharing of entanglement with qubits, we find that the strongest steering exists only when all the steering is shared between two parties. Once steering is distributed over a series of systems, the pairwise steering will diminish. In this paper, we prove such a rule for steering in one direction only.

### A. CV bipartite sharing

We begin with the CV EPR steering relation \( S_{B|A} \). Given the definition of the steering parameter \( E_{B|A} \), it must be true that

\[
E_{B|\{AC\}} \leq E_{B|A}
\]

This simple result follows, because \( E_{B|A} \) is the lowest variance product possible, that arises from the best inference of Bob’s \( X_B \) or \( P_B \) by the group \( A \) of Alice. Alice can use any local observable, defined as a measurement performed on the system \( A \). The inference of Bob’s measurement by the group \( AC \), which includes both \( A \) and \( C \), must be at least as good as that of \( A \) alone, since the observables of system \( A \) are a subset of those of the combined system \( AC \). The steering of \( B \) by a combined group cannot be less effective than that of a subset. It is also true that \( E_{B|\{AC\}} \leq E_{B|C} \). On multiplying the two inequalities together, we can easily derive several new monogamy relations.

**Result (5):** For the three systems \( A \), \( B \) and \( C \), it follows that

\[
E_{B|A}E_{B|C} \geq E_{B|\{AC\}}^2
\]

We can express the product relation in terms of a sum relation, similar to CKW, by using the simple identity \( x^2 + y^2 \geq 2xy \).

**Result (6):** It is also true that

\[
E_{B|A} + E_{B|C} \geq 2E_{B|\{AC\}}
\]

This follows, since we can let \( x = \sqrt{E_{B|A}} \) and \( y = \sqrt{E_{B|C}} \), and use that \( E_{B|\{AC\}} \leq \sqrt{E_{B|A}E_{B|C}} \). Since the maximal steering is defined when \( E_{B|A} = 0 \), and \( E_{B|A} = 1 \) is the threshold for steering, the direction of the inequality is reversed as compared to that for the CKW relation. We note also that we could choose \( x = E_{B|A} \) and \( y = E_{B|C} \) from which we derive the monogamy result:

\[
E_{B|A}^2 + E_{B|C}^2 \geq 2E_{B|\{AC\}}^2
\]

The relations \( 30, 32 \) express a type of conservation law for steering. If there is steering of \( B \) by a group \( AC \) that has components \( A \) and \( C \), so that \( E_{B|\{AC\}} \leq 1 \), then the steering is shared among the components. The individual steering of \( B \) by \( A \), or \( B \) by \( C \), is reduced and bounded by the monogamy relations.

If the property \( \{A \} \) is specified as a condition for a witness for EPR steering, then the relation holds for all such witnesses. The monogamy relations \( 30, 31 \) would then become fundamental results for steering monogamy, that are non-specific to a particular steering witness or uncertainty relation.

The monogamy relation of Result (1) is stronger than Result (5) when steering is present, since steering requires \( E_{B|A} < 1 \). We thus write the monogamy relation for the CV EPR witness \( 3 \) as

\[
E_{B|A}E_{B|C} \geq \max\{1, E_{B|\{AC\}}^2\}
\]

One could test this relation experimentally, by adding noise to mode \( B \) so that \( E_{B|\{AC\}} > 1 \). We have not demonstrated saturation of the inequality, except where \( E_{B|\{AC\}} = 1 \), which was discussed in Section II.

### B. Qubits and qudits

The qubit case is more interesting. Following the same approach, we can deduce that \( S_{B|A}^{(2)} \geq S_{B|\{AC\}}^{(2)} \) and \( S_{B|C}^{(2)} \geq S_{B|\{AC\}}^{(2)} \) which implies

\[
S_{B|A}^{(2)} + S_{B|C}^{(2)} \geq \max\{2, 2S_{B|\{AC\}}^{(2)}\}
\]

and similarly

\[
S_{B|A}^{(3)} + S_{B|C}^{(3)} + S_{B|D}^{(3)} \geq \max\{3, 3S_{B|\{ACD\}}^{(3)}\}
\]

Also,

\[
S_{B|A}^{(3)} + S_{B|C}^{(3)} \geq 2S_{B|\{AC\}}^{(3)}
\]

The relation \( 36 \) for sharing of steering is significant for qubit systems, since it will apply to limit the steering detected using three-observable steering inequalities, for tripartite systems (here, the number of sites is less than \( m + 1 \)). This relation resembles the CKW relation for entanglement. We use the relation \( 36 \) in the next Section, to derive the steering properties of the tripartite qubit W state.

### VI. STEERING MONOGAMY OF TRIPARTITE GHZ AND W STATES

We are now in a position to analyse the distribution of bipartite steering for the tripartite qubit GHZ and W states. Consider the GHZ state, for three qubit (spin 1/2) systems:

\[
|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\uparrow\rangle_B|\downarrow\rangle_C - |\downarrow\rangle_A|\downarrow\rangle_B|\uparrow\rangle_C)
\]
The spins can be measured for each system, by measuring component cannot both be specified simultaneously in a Bohm’s EPR paradox, and hence steering, since two spin components of the collective system \( AC \), three-observable steering of any one party (e.g.: \( A \)) by the group \( AC \) allows perfect inference of the orthogonal spin components (we use the shortened notation \( | ↑↓⟩ = | ↑⟩_A | ↑⟩_B \)). Conditional variances for Alice inferring Bob’s results of measurement of spin are calculated in the Appendix. If Alice measures \( \sigma^2_Y \) then the average conditional variance is \((\Delta \sigma^2_{B|A})^2 = 2/3\). If she measures either \( \sigma^X_A \) or \( \sigma^X_B \), then respectively \((\Delta \sigma^X_{B|A})^2 = 5/9\), and \((\Delta \sigma^Y_{B|A})^2 = 5/9\). Though no steering can be deduced from the two-observable inequalities of Section II.B, the values are enough to confirm three-observable bipartite steering since (using the expression from Section III.A) \( S^{(3)}_{B|A} \leq 8/9 < 1 \). From the symmetry of the \( W \) state, we can deduce that this steering must be two-way (Figure 5). We note that the values are consistent with the monogamy relation \( |S^{(3)}_{B|A} + S^{(3)}_{B|C}| \geq S^{(3)}_{B|AC} \), that applies in this case.

The two-observable steering behaviour is different. Here, the stricter monogamy inequality \( |S^{(2)}_{B|A} + S^{(2)}_{B|C}| \geq 2 \). For the \( W \) state, we deduce that no steering is detectable via two-observable inequalities. The \( W \) state has complete symmetry with respect to the three sites. Hence if there is steering of \( B \) by \( A \), then there must be steering of \( B \) by \( C \), which we have seen is impossible for two-observable inequalities (Results 2 and 4).

VII. DISCUSSION AND CONCLUSION

The monogamy inequalities for EPR steering are likely to be useful. For example, in order to observe EPR steering with two-setting inequalities, we understand why it is necessary for the steering party to have greater than 50% efficiency for detection of data \( \eta > 1/m \). Otherwise, an eavesdropper could detect the steering also, which is forbidden by the two-setting monogamy relation. The argument extends to the \( m \)-setting inequalities, where the bound for efficiency \( \eta \) is \( \eta > 1/m \).

Monogamy relations give a simple way to understand security in quantum communication. If it can be shown that \( A \) steers \( B \) via a two-observable inequality, so that \( E_{B|A} \) or \( S^{(2)}_{B|A} < 1 \), then it is guaranteed that for a third (eavesdropper) observer \( C \), \( E_{B|A} \) or \( S^{(2)}_{B|A} \geq 1 \). Where the steering witness is directly related to the variance of the conditional inference for Bob’s values of qubits or amplitudes, given Alice’s measurements, the monogamy relations quantify the minimum noise levels for an eavesdropper to infer Bob’s values. This aids our understanding of QKD schemes based on a shared quadrature amplitude value, or a shared qubit value.
The new feature associated with quantum steering is the potential to implement one-sided device independent cryptographic security [4, 22]. The noise levels for the eavesdropper are quantified based on the uncertainty relation only, and do not depend on the details of a particular protocol. The device-independent security is one-way, since it is Alice’s inference of Bob’s amplitudes or spin values that are secured by the steering monogamy relations.

The monogamy with respect to steering witnesses has explained the monogamy of violations of Bell inequalities. Bell monogamy arises because Bell inequalities are also steering inequalities. As such, the degree of monogamy will depend on the number of observables (settings) of the Bell inequality.

Finally, the results presented here have enabled a characterisation of the bipartite sharing for the tripartite CV Gaussian states, and for qubit GHZ and W states, and several experimental tests and realisations have been proposed. Open questions remain. For example, the monogamy results given in this paper give a quantification of how the steering of a single system by a group is shared, but the nature of the reverse monogamy has not been examined.

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APPENDIX

From (39), if Alice measures \( \sigma_A^X \) then the average conditional variance is

\[
(\Delta \sigma^Z_{B|A})^2 = \sum_i P(\sigma_A^X = i)(\Delta(\sigma^Z_B|\sigma_A^X))^2
\]

\[
= \frac{1}{3} \times 0 + \frac{2}{3} \times 1 = \frac{2}{3}
\]

The joint probabilities for measurement are: 1/3 for both Alice and Bob with spins down; 1/3 for Alice’s spin down and Bob’s up; and 1/3 for Alice’s spin up and Bob’s down. If Alice measures spin +1, then Bob’s state is | ↓ ⟩ and the conditional variance is 0. If Alice measures −1 then Bob’s spin is up and down with probability 1/2, and the conditional variance is 1. We can rewrite in the basis of spin \( X \)

\[
\rho_{AB} = \frac{1}{3} \{ |ψ⟩⟨ψ| + |ψ_m⟩⟨ψ_m| \}
\]

Here |ψ⟩ = (| ↑↑ ⟩−| ↓↓ ⟩)/\(\sqrt{2}\) and |ψ_m⟩ = \(\frac{1}{3}(| ↑↑ ⟩+| ↓↓ ⟩−| ↑↓ ⟩−| ↓↑ ⟩)\). If Alice measures \( \sigma_A^X \) spin +1 (with probability 1/2) then the probability is 5/6 for Bob’s up and 1/6 down, for which the mean is 2/3 and the conditional variance is 1 − 4/9 = 5/9. The same variance is obtained for outcome −1. Thus, \( (\Delta \sigma^Y_{B|A})^2 = 5/9 \). Rewriting in the basis of \( Y \), we obtain the same conditional variance, \( (\Delta \sigma^Y_{B|A})^2 = 5/9 \), as for spin \( X \).

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