The Character of the Exceptional Series of Representations of

$SU(1, 1)$

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Abstract

The character of the exceptional series of representations of $SU(1, 1)$ is determined by using Bargmann’s realization of the representation in the Hilbert space $H_\sigma$ of functions defined on the unit circle. The construction of the integral kernel of the group ring turns out to be especially involved because of the non-local metric appearing in the scalar product with respect to which the representations are unitary. Since the non-local metric disappears in the ‘momentum space’ i.e. in the space of the Fourier coefficients the integral kernel is constructed in the momentum space, which is transformed back to yield the integral kernel of the group ring in $H_\sigma$. The rest of the procedure is parallel to that for the principal series treated in a previous paper. The main advantage of this method is that the entire analysis can be carried out within the canonical framework of Bargmann.
1. INTRODUCTION

It is well known that the traditional definition of character breaks down for infinite dimensional representations of locally compact groups. For such representations character was defined by Gel’fand\(^1\) and coworkers as

\[
Tr(T_x) = \int K(z, z)d\lambda(z)
\]

where \(K(z, z_1)\) is the integral kernel of the group ring

\[
T_x f(z) = \int K(z, z_1)f(z_1)d\lambda(z_1).
\]

The operator \(T_x\) of the group ring is defined by

\[
T_x = \int d\mu(g)x(g)T_g
\]

where \(x(g)\) is a test function on the group which vanishes outside a bounded set, \(d\mu(g)\) is the left and right invariant measure (assumed coincident) on the group and \(g \rightarrow T_g\) is a unitary representation of the group realized in the Hilbert space \(H\) of functions \(f(z)\) with the scalar product

\[
(f, g) = \int \overline{f(z)}g(z)d\lambda(z).
\]

It was shown by Gel’fand\(^1\) and coworkers that \(Tr(T_x)\) can be written in the form

\[
Tr(T_x) = \int x(g)\pi(g)d\mu(g).
\]

(1.1)

The function \(\pi(g)\) is the character of the representation \(g \rightarrow T_g\).

In a previous paper\(^2\) (I) the character problems of \(SU(2)\) and \(SU(1, 1)\) were reexamined from the standpoint of a physicist by employing the powerful Hilbert space method of Bargmann\(^3\) and Segal\(^4\) which was shown to yield a completely unified treatment for \(SU(2)\) and the discrete series of representations of \(SU(1, 1)\). The main advantage of this method is that the entire analysis can be carried out within the canonical framework of Bargmann\(^5\). The representations of the positive discrete series were realized in I in the Hilbert space of
functions analytic within the open unit disc. For the principal series the carrier space was chosen to be the Hilbert space of functions defined on the unit circle.

It is the object of this paper to extend the method of I to the exceptional or the supplementary series of representations of $SU(1, 1)$. This representation is realized, as for the principal series, in the space of functions defined on the unit circle. The construction of the integral kernel of the group ring is, however, more involved for the exceptional representations because the scalar product contains a nonlocal metric

$$ (f, g) = \frac{c}{4\pi^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \overline{f(\theta_1)} e^{i(\frac{1}{2} - \sigma)(\theta_2 - \theta_1)} [1 - e^{i(\theta_2 - \theta_1)}]^{(2\sigma - 1)} g(\theta_2) \quad (1.2) $$

where

$$ c = \frac{\pi 2(1-2\sigma) e^{i\pi(\sigma - \frac{1}{4})}}{B(\frac{1}{2}, \sigma)}. $$

This difficulty is resolved by expanding the functions in Fourier series

$$ f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} a_m e^{im\theta}. $$

Since the nonlocal metric disappears in the ‘momentum space’ i.e. in the space of the Fourier coefficients $a_m$ the integral kernel is constructed in the momentum space, which is transformed back to yield the integral kernel of the group ring

$$ T_x f(\theta) = \frac{c}{4\pi^2} \int \int Q(\theta, \theta_1) e^{i(\theta_2 - \theta_1)(\frac{1}{2} - \sigma)} [1 - e^{i(\theta_2 - \theta_1)}]^{(2\sigma - 1)} f(\theta_2) d\theta_1 d\theta_2 \quad (1.3) $$

in the space of the functions $f(\theta)$. The rest of the procedure is parallel to that for the principal series of representations as outlined in I. An important common feature of the principal and supplementary series is that the elliptic elements of $SU(1, 1)$ do not contribute to their character.

2. THE EXCEPTIONAL REPRESENTATIONS OF THE GROUP $SU(1,1)$

To make this paper self-contained we start with the basic properties of the group $SU(1, 1)$ which consists of pseudounitary unimodular matrices
\[
 u = \begin{pmatrix}
  \alpha & \beta \\
  \bar{\beta} & \bar{\alpha}
 \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1 \tag{2.1}
\]

and is isomorphic to the group SL(2, R) of real unimodular matrices,

\[
 g = \begin{pmatrix}
  a & b \\
  c & d
 \end{pmatrix}, \quad ad - bc = 1. \tag{2.2}
\]

A particular choice of the isomorphism kernel is

\[
 \eta = \frac{1}{\sqrt{2}} \begin{pmatrix}
  1 & i \\
  i & 1
 \end{pmatrix} \tag{2.3}
\]

so that

\[
 u = \eta g \eta^{-1}
\]

\[
 \alpha = \frac{1}{2} [(a + d) - i(b - c)], \quad \beta = \frac{1}{2} [(b + c) - i(a - d)]. \tag{2.4}
\]

The elements of the group SU(1,1) may be divided into three subsets: (a) elliptic, (b) hyperbolic and (c) parabolic. We define them as follows. Let \( \alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2 \) so that

\[
 \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 = 1.
\]

The elliptic elements are those for which

\[
 \alpha_1^2 - \beta_1^2 - \beta_2^2 > 0.
\]

Hence if we set

\[
 \alpha_2' = \sqrt{\alpha_1^2 - \beta_1^2 - \beta_2^2}
\]

we have

\[
 \alpha_1^2 + \alpha_2'^2 = 1
\]
so that $-1 < \alpha_1 < 1$.

On the other hand the hyperbolic elements of SU(1,1) are those for which

$$\alpha_2^2 - \beta_1^2 - \beta_2^2 < 0.$$ 

Hence if we write

$$\alpha'_2 = \sqrt{\beta_1^2 + \beta_2^2 - \alpha_2^2}$$

we have

$$\alpha_1^2 - \alpha'_2^2 = 1 \quad (2.5)$$

so that $|\alpha_1| > 1$.

We exclude the parabolic class corresponding to

$$\alpha_2 = \sqrt{\beta_1^2 + \beta_2^2}$$

as this is a submanifold of lower dimensions.

If we diagonalize the SU(1,1) matrix (2.1), the eigenvalues are given by

$$\lambda = \alpha_1 \pm \sqrt{\alpha_1^2 - 1}$$

For the elliptic elements $\alpha_1 = \cos(\frac{\theta_0}{2})$, $0 < \theta_0 < 2\pi$ so that $\lambda = \exp(\pm i\frac{\theta_0}{2})$. For the hyperbolic elements $|\alpha_1| > 1$ so that setting $\alpha_1 = \epsilon \cosh \frac{t}{2}$, $\epsilon = \text{sgn}\lambda$ we obtain the eigenvalues as $\epsilon \exp(\pm \epsilon \frac{t}{2})$. Since the diagonal matrix

$$\epsilon(t) = \begin{pmatrix} sgn\lambda \ e^{sgn\lambda(\frac{t}{2})} & 0 \\ 0 & sgn\lambda \ e^{-sgn\lambda(\frac{t}{2})} \end{pmatrix} \quad (2.6)$$

belongs to SL(2,R), it can be regarded as the diagonal form of the matrix $g$ given by equations (2.2) and (2.4) with $|\alpha_1| = \frac{|a+d|}{2}$.

Following Bargmann we realize the representations of the exceptional series in the Hilbert space $H_\sigma$ of functions defined on the unit circle. The finite element of the group in this realization is given by
\[ T_u f(z) = |\beta z + \bar{\alpha}|^{-2\sigma-1} f \left( \frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}} \right) \]  

(2.7)

where

\[-\frac{1}{2} < \sigma < \frac{1}{2}; \quad z = e^{i\theta}, \quad 0 < \theta < 2\pi. \]  

(2.8)

This representation is unitary with respect to the scalar product

\[ (f_1, f_2) = \frac{c}{4\pi^2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \overline{f_1(z_1)} \frac{2(1-2\sigma)}{B\left(\frac{1}{2}, \sigma\right)} \left(1 - \frac{z_2}{z_1}\right)^{2\sigma-1} f_2(z_2) \]  

(2.9)

where,

\[ z_k = e^{i\theta_k}, \quad k = 1, 2; \quad c = \frac{\pi 2^{1-2\sigma} e^{i\pi(\sigma-\frac{1}{2})}}{B\left(\frac{1}{2}, \sigma\right)}. \]  

(2.10)

The integral converges in the usual sense for \( 0 < \sigma < \frac{1}{2} \). For \(-\frac{1}{2} < \sigma < 0\) the integral is to be understood in the sense of its regularization.\(^7\)

Setting \( z = -ie^{i\theta} \), the finite group element takes the form,

\[ T_u f(-ie^{i\theta}) = |-i\beta e^{i\theta} + \bar{\alpha}|^{-2\sigma-1} f \left( \frac{-i\alpha e^{i\theta} + \bar{\beta}}{-i\beta e^{i\theta} + \bar{\alpha}} \right). \]  

(2.11)

We now introduce as usual the operator of the group ring

\[ T_x = \int d\mu(u) x(u) T_u \]  

(2.12)

where \( x(u) \) is an arbitrary test function on the group which vanishes outside a bounded set and \( d\mu(u) \) is the left and right invariant measure on the group. The action of the operator \( T_x \) is given by

\[ T_x f(-ie^{i\theta}) = \int d\mu(u) x(u) \left| -i\beta e^{i\frac{\theta}{2}} + \bar{\alpha} e^{-i\frac{\theta}{2}} \right|^{-2\sigma-1} f \left( \frac{-i\alpha e^{i\frac{\theta}{2}} + \bar{\beta} e^{-i\frac{\theta}{2}}}{-i\beta e^{i\frac{\theta}{2}} + \bar{\alpha} e^{-i\frac{\theta}{2}}} \right). \]  

(2.13)

We now make a left translation

\[ u \rightarrow \theta^{-1} u \]

where
We, therefore, obtain

\[
T_x f(-ie^{i\theta}) = \int d\mu(u)x(\theta^{-1}u) | -i\beta + \bar{\alpha} |^{-2\sigma-1} f \left( \frac{-i\alpha + \bar{\beta}}{-i\beta + \bar{\alpha}} \right). \tag{2.14}
\]

We now map the \(SU(1, 1)\) matrix \(u\) onto the \(SL(2, R)\) matrix \(g\) by using the isomorphism kernel \(\eta\) given by (2.3) and (2.4) and perform the Iwasawa decomposition

\[
g = k \theta_2 \tag{2.15}
\]

where

\[
k = \begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{pmatrix}, \quad k_{11}k_{22} = 1 \tag{2.16}
\]

belongs to the subgroup \(K\) of real triangular matrices of determinant unity and \(\theta_2 \in \Theta\) where \(\Theta\) is the subgroup of pure rotation matrices

\[
\theta_2 = \begin{pmatrix} \cos(\theta_2/2) - \sin(\theta_2/2) \\ \sin(\theta_2/2) & \cos(\theta_2/2) \end{pmatrix}. \tag{2.17}
\]

As in I we now introduce the following convention. The letters without a bar below it will denote the \(SL(2, R)\) matrices or its subgroups and those with a bar below it will denote their \(SU(1, 1)\) image. For instance

\[
k = \eta k \eta^{-1} = \frac{1}{2} \begin{pmatrix} k_{11} + k_{22} - ik_{12} & k_{12} - i(k_{11} - k_{22}) \\ k_{12} + i(k_{11} - k_{22}) & k_{11} + k_{22} + ik_{12} \end{pmatrix}
\]

\[
\theta_2 = \begin{pmatrix} e^{i\theta_2/2} & 0 \\ 0 & e^{-i\theta_2/2} \end{pmatrix}.
\]

Thus the decomposition (2.13) can also be written as

\[
u = k \theta_2 \tag{2.18}
\]
which yields

\[-i \alpha + \beta = -i k_{22} e^{i\theta_2/2}, \quad -i \beta + \bar{\alpha} = k_{22} e^{-i\theta_2/2}.\]

Hence setting \(f(-ie^{i\theta}) = g(\theta)\) we obtain,

\[T_x \, g(\theta) = \int x(\bar{\theta}^{-1} k \, \theta_2) \mid k_{22} \mid^{-2\sigma-1} g(\theta_2) \, d\mu(u). \tag{2.19}\]

It can be shown that under the decomposition (2.15) or equivalently (2.18) the invariant measure decomposes as

\[d\mu(u) = \frac{1}{2} \, d\mu_l(g) = \frac{1}{2} \, d\mu_r(g) = \frac{1}{4} \, d\mu_l(k) \, d\theta. \tag{2.20}\]

Substituting the decomposition (2.20) in eqn. (2.19) we have

\[T_x \, g(\theta) = \int K(\theta, \theta_2) \, g(\theta_2) \, d\theta_2 \tag{2.21}\]

where

\[K(\theta, \theta_2) = \frac{1}{4} \int x(\bar{\theta}^{-1} k \, \theta_2) \mid k_{22} \mid^{-2\sigma-1} \, d\mu_l(k). \tag{2.22}\]

It must be pointed out that eqn. (2.21) does not yield the integral kernel of the operator \(T_x\) of the group ring because \(T_x\) now is an operator in the space \(H_\sigma\) in which the scalar product is given by eqn.(2.9). It is, therefore, not clear a priori that

\[\int K(\theta, \theta) d\theta \tag{2.23}\]

is the trace of the operator \(T_x\). Nevertheless, we shall show that (2.23) is the trace of the operator \(T_x\).

To write the action of \(T_x\) on \(g(\theta)\) in the form consistent with the scalar product (2.9) we pass over to the ‘momentum space’ (space of the Fourier coefficients)

\[g(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} a_m e^{im\theta} \tag{2.24}\]

where
\[ a_m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g(\theta)e^{-im\theta} d\theta. \]  

(2.25)

We then obtain

\[ (g_1, g_2) = \sum_{m=-\infty}^{\infty} \overline{a}_m b_m \rho_m \]  

(2.26)

where \( b_m \) is the Fourier coefficient of \( g_2(\theta) \),

\[ b_m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} g_2(\theta)e^{-im\theta} d\theta \]  

(2.27)

and

\[ \rho_m = \frac{1}{2\pi} \frac{\Gamma(\frac{1}{2} + \sigma)\Gamma(\frac{1}{2} - \sigma + m)}{\Gamma(\frac{1}{2} - \sigma)\Gamma(\frac{1}{2} + \sigma + m)}. \]  

(2.28)

Hence

\[ ||g||^2 = \sum_{m=-\infty}^{\infty} |a_m|^2 \rho_m \]  

(2.29)

We can, therefore, define the scalar product in the momentum space as

\[ (a, b) = \sum \overline{a}_m b_m \rho_m \]  

(2.30)

where \( \rho_m \) as given by eqn. (2.28) is a positive metric.

The operator of the group ring in the momentum space is given by

\[ T_x a_m = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} a_n \int_0^{2\pi} \int_0^{2\pi} K(\theta, \theta_2) e^{-i(m\theta - n\theta_2)} d\theta d\theta_2. \]  

(2.31)

If we now expand \( K(\theta, \theta_2) \) in a Fourier series,

\[ K(\theta, \theta_2) = \frac{1}{2\pi} \sum L_{mn} e^{i(m\theta - n\theta_2)} \]  

(2.32)

we have

\[ T_x a_m = \sum_{n=-\infty}^{\infty} L_{mn} a_n \]  

(2.33)

where
\[ L_{mn} = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} K(\theta, \theta_2) e^{-i(m\theta - n\theta_2)} d\theta d\theta_2. \]  

(2.34)

We now define

\[ L_{mn} = \Gamma_{mn} \rho_n. \]  

(2.35)

Thus

\[ T_x a_m = \sum_n \Gamma_{mn} \rho_n a_n. \]  

(2.36)

We shall now show that

\[ \rho_m = \frac{c}{4\pi^2} \int_{0}^{2\pi} e^{i(\frac{1}{2} - \sigma + m)\alpha} (1 - e^{i\alpha})^{2\sigma - 1} d\alpha. \]  

(2.37)

Setting \( z = e^{i\alpha} \) the integral on the r.h.s. can be recast as an integral over the unit circle \( S \). Since the only singularities of the subsequent integrand are the branch points at \( z = 0 \) and \( z = 1 \) the unit circle \( S \) can be deformed to a contour \( \sum \) that starts from \( z = 1 \) along the positive real axis, encircles the point \( z = 0 \) once counterclockwise and returns to the point \( z = 1 \) along the positive real axis. The integral is, therefore, the contour integral representation of the beta function regularized at the origin,

\[ \int_{1}^{0} t^{(\frac{1}{2} - \sigma + m - 1)} (1 - t)^{2\sigma - 1} dt = \left[ e^{2\pi i(\frac{1}{2} - \sigma)} - 1 \right] B \left( \frac{1}{2} - \sigma + m, 2\sigma \right). \]  

(2.38)

The eqn. (2.38) in conjunction with eqn. (2.10) immediately yields eqn. (2.37).

We now pass over from the momentum space to the space of functions \( g(\theta) \):

\[ T_x g(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{im\theta} \Gamma_{mn} \rho_n a_n. \]  

(2.39)

We now substitute eqn.(2.37) and (2.25) in eqn.(2.39). Thus

\[ T_x g(\theta) = \frac{c}{8\pi^3} \int_{0}^{2\pi} \int_{0}^{2\pi} d\alpha d\theta_2 g(\theta_2) e^{i(\frac{1}{2} - \sigma)\alpha} (1 - e^{i\alpha})^{2\sigma - 1} \sum_{n} \frac{L_{mn}}{\rho_n} e^{i[n\theta - n(\theta_2 - \alpha)]}. \]  

(2.40)

Since \( L_{mn} \) is the Fourier coefficient of \( K(\theta, \theta_2) \) and \( \rho_n \) is given by eqn.(2.28) the function

\[ Q(\theta, \theta_2 - \alpha) = \frac{1}{2\pi} \sum_{n} \frac{L_{mn}}{\rho_n} e^{i[n\theta - n(\theta_2 - \alpha)]} \]  

(2.41)
is known and well defined. Hence we have

$$T x g(\theta) = \frac{c}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\alpha d\theta_2 g(\theta_2) Q(\theta, \theta_2 - \alpha) e^{i(\frac{1}{2} - \sigma)\alpha} (1 - e^{i\alpha})^{2\sigma - 1}. \quad (2.42)$$

Finally setting $\alpha = \theta_2 - \theta_1$ we have

$$T x g(\theta) = \frac{c}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta_1 d\theta_2 g(\theta_2) Q(\theta, \theta_1) e^{i(\frac{1}{2} - \sigma)(\theta_2 - \theta_1)} (1 - e^{i(\theta_2 - \theta_1)})^{2\sigma - 1} \quad (2.43)$$

which is in the form consistent with the scalar product \( (2.39) \). \( Q(\theta, \theta_1) \) is, therefore, the

integral kernel of the group ring. Comparing eqn.(2.43) with eqn.(2.21) we have

$$K(\theta, \theta_2) = \frac{c}{4\pi^2} \int_0^{2\pi} Q(\theta, \theta_1) e^{i(\frac{1}{2} - \sigma)(\theta_2 - \theta_1)} (1 - e^{i(\theta_2 - \theta_1)})^{2\sigma - 1} d\theta_1. \quad (2.44)$$

From eqn.(2.33) \( Tr(T_x) \) is given by

$$Tr(T_x) = \sum_{n=\infty}^{\infty} L_{nn} = \sum_{n=-\infty}^{\infty} \Gamma_{nn} \rho_n. \quad (2.45)$$

Substituting the integral representation (2.37) in eqn.(2.45) we obtain

$$Tr(T_x) = \frac{c}{4\pi^2} \int d\alpha e^{i(\frac{1}{2} - \sigma)\alpha} (1 - e^{i\alpha})^{2\sigma - 1} \sum_n \Gamma_{nn} e^{in\alpha}. \quad (2.46)$$

Now from eqns. (2.33) and (2.41)

$$\Gamma_{nn} = \frac{1}{2\pi} \int Q(\theta, \theta_1) e^{-in(\theta - \theta_1)} d\theta d\theta_1. \quad (2.47)$$

The above equation in conjunction with eqn.(2.46) yields,

$$Tr(T_x) = \frac{c}{4\pi^2} \int \int d\theta d\theta_1 Q(\theta, \theta_1) \int_0^{2\pi} d\alpha e^{i(\frac{1}{2} - \sigma)\alpha} (1 - e^{i\alpha})^{2\sigma - 1} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\alpha - \theta + \theta_1)}. \quad (2.48)$$

The summation over \( n \) appearing on the r.h.s. of eqn.(2.48) can be carried out using the formula

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\beta} = \delta(\beta). \quad (2.49)$$

Eqn.(2.49) immediately leads to

$$Tr(T_x) = \frac{c}{4\pi^2} \int Q(\theta, \theta_1) e^{i(\frac{1}{2} - \sigma)(\theta - \theta_1)} [(1 - e^{i(\theta - \theta_1)})^{2\sigma - 1} d\theta d\theta_1. \quad (2.50)$$
Finally using eqn. (2.44) we have

\[ Tr (T_x) = \int K(\theta, \theta) d\theta \]

\[ = \frac{1}{4} \int x(\theta^{-1} k \theta) |k_{22}|^{-2\sigma-1} d\mu(k) d\theta. \]  

\( (2.51) \)

The rest of the calculation is parallel to that of the principal series outlined in I. However, to make the paper self-contained we reproduce the steps here.

We assert that

\[ u = \theta^{-1} k \theta \]

represents a hyperbolic element of \( SU(1,1) \) because the equation

\[ 2\alpha_1 = k_{11} + k_{22} = \frac{1}{k_{22}} + k_{22} \]  

\( (2.52) \)

has no real solution for \( \alpha_1 = \cos \frac{\theta_0}{2} \). Thus the elliptic elements of \( SU(1,1) \) do not contribute to the character of the exceptional series of representations.

We shall now show that every hyperbolic element of \( SU(1,1) \) (i.e those with \( |\alpha_1| = \frac{|(\alpha+\delta)|}{2} > 1 \)) can be represented as

\[ u = \theta^{-1} k \theta \]  

\( (2.53) \)

or equivalently as

\[ g = \theta^{-1} k \theta. \]  

\( (2.54) \)

Here \( k_{11} = \lambda^{-1}, k_{22} = \lambda \) are the eigenvalues of the matrix \( g \) taken in any order.

We recall that every \( g \in SL(2,R) \) for the hyperbolic case can be diagonalized as

\[ v' g v'^{-1} = \delta \]

where

\[ \delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}, \quad \delta_1\delta_2 = 1, \quad \delta_1, \delta_2 \text{ real}, \]
belongs to the subgroup $D$ of real diagonal matrices of determinant unity and $v' \in SL(2, \mathbb{R})$. If we write the Iwasawa decomposition for $v'$,

$$v' = k' \theta$$

then

$$g = \theta^{-1} k'^{-1} \delta k' \theta.$$ 

Now $k'^{-1} \delta \in K$ so that writing $k = k'^{-1} \delta k'$ we have the decomposition (2.54) in which

$$k_{11} = \delta_1 = \lambda^{-1}, \quad k_{22} = \delta_2 = \lambda.$$ 

If these eigenvalues are distinct then for a given ordering of them the matrices $k, \theta$ are determined uniquely by the matrix $g$. It follows that for a given choice of $\lambda$ the parameters $\theta$ and $k_{12}$ are uniquely determined. We note that there are exactly two representations of the matrix $g$ by means of formula (2.54) corresponding to two distinct possibilities

$$k_{11} = \text{sgn} \lambda \left| \lambda \right|^{-1} = \text{sgn} \lambda \ e^{t/2}, \quad k_{22} = \text{sgn} \lambda \ | \lambda | = \text{sgn} \lambda \ e^{-t/2}$$

$$k_{11} = \text{sgn} \lambda \ | \lambda | = \text{sgn} \lambda \ e^{-t/2}, \quad k_{22} = \text{sgn} \lambda \ | \lambda |^{-1} = \text{sgn} \lambda \ e^{t/2}.$$ 

Let us now remove from $K$ the elements with $k_{11} = k_{22} = 1$. This operation cuts the group $K$ into two connected disjoint components. Neither of these components contains two matrices which differ only by permutation of the two diagonal elements. In correspondence with this partition the integral over $K$ is represented in the form of a sum of two integrals,

$$\text{Tr} \ T_x = \frac{1}{4} \int_{\Theta} d\theta \int_{K_1} d\mu_i(k) \ | k_{22} |^{-2\sigma + 1} x(\theta^{-1} k \ \theta) + \frac{1}{4} \int_{\Theta} d\theta \int_{K_2} d\mu_i(k) \ | k_{22} |^{-2\sigma - 1} x(\theta^{-1} k \ \theta).$$

(2.55)

As $\theta$ runs over the subgroup $\Theta$ and $k$ runs over the components $K_1$ or $K_2$ the matrix $u = \theta^{-1} k \ \theta$ runs over the hyperbolic elements of the group $SU(1, 1)$. We shall now prove that in $K_1$ or $K_2$
\[ d\mu_1(k) \, d\theta = \frac{4 \left| k_{22} \right|}{\left| k_{22} - k_{11} \right|} \, d\mu(u). \] (2.56)

To prove this we start from the left invariant differential

\[ dw = g^{-1} \, dg \]

where \( g \in SL(2, R) \) and \( dg \) is the matrix of the differentials \( dg_{pq} \), i.e.

\[ dg = \begin{pmatrix} da & db \\ dc & dd \end{pmatrix}. \]

The elements \( dw \) are invariant under the left translation \( g \rightarrow g_0 g \). Hence choosing a basis in the set of all \( dg \) we immediately obtain a left invariant measure. For instance choosing \( dw_{12}, \, dw_{21} \) and \( dw_{22} \) as independent invariant differentials we arrive at the left invariant measure on \( SL(2, R) \),

\[ d\mu_1(g) = dw_{12} dw_{21} dw_{22}. \] (2.57)

We now write the decomposition (2.54) as

\[ \theta \, g = k \, \theta \]

which yields

\[ dw = g^{-1} \, dg = \theta^{-1} \, d\mu \, \theta \] (2.58)

where

\[ d\mu = k^{-1} \, dk + d\theta \, \theta^{-1} - k^{-1} \, d\theta \, \theta^{-1} \, k. \] (2.59)

In accordance with the choice of the independent elements of \( dw \) as mentioned above we choose the independent elements of \( d\mu \) as \( d\mu_{12}, \, d\mu_{21} \) and \( d\mu_{22} \). Eqn.(2.58) then leads to

\[ dw_{11} + dw_{22} = d\mu_{11} + d\mu_{22} \] (2.60)

\[ dw_{11} dw_{22} - dw_{12} dw_{21} = d\mu_{11} d\mu_{22} - d\mu_{12} d\mu_{21}. \] (2.61)
Further since $Tr(d\mu) = Tr(dw) = 0$ we obtain from eqns. (2.60) and (2.61)
\[ dw_{22}^2 + dw_{12}dw_{21} = d\mu_{22}^2 + d\mu_{12}d\mu_{21} \]
which can be written in the form
\[ d\eta_1^2 + d\eta_2^2 - d\eta_3^2 = d\eta'_1^2 + d\eta'_2^2 - d\eta'_3^2 \quad (2.62) \]
where
\[
\begin{align*}
    d\eta_1 &= (dw_{12} + dw_{21})/2 \\
    d\eta_2 &= dw_{22} \\
    d\eta_3 &= (dw_{12} - dw_{21})/2 \\
    d\eta'_1 &= (d\mu_{12} + d\mu_{21})/2 \\
    d\eta'_2 &= d\mu_{22} \\
    d\eta'_3 &= (d\mu_{12} - d\mu_{21})/2.
\end{align*}
\]
Eqn. (2.62) implies that the set $d\eta$ and the set $d\eta'$ are connected by a Lorentz transformation. Since the volume element $d\eta_1d\eta_2d\eta_3$ is invariant under such a transformation we have
\[ d\eta_1d\eta_2d\eta_3 = d\eta'_1d\eta'_2d\eta'_3. \quad (2.63) \]
But the l.h.s. of eqn. (2.63) is $dw_{12}dw_{21}dw_{22}/2$ and the r.h.s. is $d\mu_{12}d\mu_{21}d\mu_{22}/2$. Hence using eqn. (2.57) we easily obtain
\[ d\mu_i(g) = d\mu_{12}d\mu_{21}d\mu_{22}. \]
We now write eqn. (2.59) in the form
\[ d\mu = du + dv \quad (2.64) \]
where $du = k^{-1}dk$ is the left invariant differential element on $K$ and
\[ dv = d\theta \theta^{-1} - k^{-1}d\theta \theta^{-1}k. \quad (2.65) \]
In eqn. (2.64) $du$ is a triangular matrix whose independent nonvanishing elements are chosen to be $du_{12}$ and $du_{22}$ so that
\[ d\mu_i(k) = du_{12}du_{22}. \]
On the other hand $dv$ is a $2 \times 2$ matrix having one independent element which is chosen to be $dv_{21}$. Since the Jacobian connecting $d\mu_1 d\mu_2 d\mu_2$ and $du_{12} du_{22} dv_{21}$ is a triangular determinant having 1 along the main diagonal we obtain

$$d\mu_l(g) = d\mu_l(k) \, dv_{21}. \quad (2.66)$$

It can now be easily verified that each element $k \in K$ with distinct diagonal elements (which is indeed the case for $K_1$ or $K_2$) can be represented uniquely in the form

$$k = \zeta^{-1} \delta \, \zeta \quad (2.67)$$

where $\delta$ belongs to the subgroup of real diagonal matrices with unit determinant and $\zeta \in Z$ where $Z$ is a subgroup of $K$ consisting of matrices of the form

$$\zeta = \begin{pmatrix} 1 & \zeta_{12} \\ 0 & 1 \end{pmatrix}.$$ 

Writing eqn. $(2.67)$ in the form $k = \delta \, \zeta$ we obtain

$$k_{pp} = \delta_p, \quad \zeta_{12} = \frac{k_{12}}{(\delta_1 - \delta_2)}. \quad (2.68)$$

Using the decomposition $(2.67)$ we can now write eqn. $(2.63)$ in the form

$$dv = \zeta^{-1} \, dp \, \zeta$$

where

$$dp = d\lambda - \delta^{-1} d\lambda \, \delta$$

$$d\lambda = \zeta \, d\sigma \, \zeta^{-1}, \quad d\sigma = d\theta \, \theta^{-1}.$$ 

From the above equations it now easily follows

$$dv_{21} = \frac{\left| \delta_2 - \delta_1 \right|}{\left| \delta_2 \right|} \, d\theta. \quad (2.69)$$
Substituting eqn.(2.69) in eqn.(2.66) and using eqns.(2.20) and (2.68) we immediately obtain eqn.(2.56).

Now recalling that in $K_1$, $|k_{22}| = e^{-t/2}$ and in $K_2$, $|k_{22}| = e^{t/2}$ eqn.(2.55) in conjunction with (2.56) yields

$$Tr(T_x) = \int x(u) \pi(u) d\mu(u)$$

where the character $\pi(u)$ is given by

$$\pi(u) = \frac{e^{\sigma t} + e^{-\sigma t}}{e^{t/2} - e^{-t/2}} = \frac{\cosh \sigma t}{|\sinh \frac{t}{2}|}$$
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