THE STABLE RANK OF $\mathbb{Z}[x]$ IS 3

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Abstract. Let $\mathbb{Z}[x]$ be the ring of univariate polynomials over $\mathbb{Z}$ and denote by $sr(\mathbb{Z}[x])$ its stable rank in the sense of Bass. Grunewald, Mennicke and Vaserstein proved that

$$sr(\mathbb{Z}[x]) = 3.$$  

As the inequality $sr(\mathbb{Z}[x]) \leq 3$ follows immediately from Bass’s stable range theorem, the above identity is equivalent to the existence of a non-stable unimodular row of size 3. This note addresses minor errors found in the existing proof of the latter fact in the literature. Using the same methods, we show that the unimodular row $(3, x + 1, x^2 + 16)$ is not stable.

Rings are assumed to be commutative and unital. Let $R$ be a ring. A row $(r_1, \ldots, r_n) \in R^n$ is unimodular if $\sum_i Rr_i = R$, or equivalently, if the ideal generated by $r_1, \ldots, r_n$ is $R$. We denote by $Um_n(R)$ the set of unimodular rows of size $n$. A row $(r_1, \ldots, r_{n+1}) \in Um_{n+1}(R)$ $(n > 0)$ is stable if there is $(s_1, \ldots, s_n) \in R^n$ such that

$$(r_1 + s_1 r_{n+1}, \ldots, r_n + s_n r_{n+1})$$

belongs to $Um_n(R)$. An integer $n > 0$ lies in the stable range of $R$ if every row in $Um_{n+1}(R)$ is stable. If $n$ lies in the stable range of $R$, then so does $k$ for every $k > n$ [MR87, Lemma 11.3.3]. The Bass stable rank $sr(R)$ of $R$ is the least integer in the stable range of $R$. This rank is key in several direct sum cancellation results [Mag02, Theorems 4.26 and 4.28]; stably free $R$-modules of stably free rank at least $sr(R)$ are free [Mag02, Corollary 4.23]. Bass’s stable rank is also used to simplify the computation of the algebraic $K$-groups $K_n(R)$ through surjectivity and injectivity theorems. For instance, the value of $sr(R)$ is an upper bound on the order of the matrices than can be used to represent elements of $K_1(R)$ [Mag02, Theorem 10.3].

Grunewald, Mennicke and Vaserstein proved the following:

Theorem A. ([GMV94, Proposition 1.9]) $sr(\mathbb{Z}[x]) = 3$. 
The inequality $sr(Z[x]) \leq 3$ is provided by Bass’s upper bound on the stable rank of finite-dimensional rings. Indeed, we have $sr(Z[x]) \leq \dim_{Krull}(Z[x]) + 1$ by Bass’s stable range theorem [MR87, Corollary 6.7.4] and $\dim_{Krull}(Z[x]) = 2$ [MR87, Proposition 6.5.4].

The authors of [GMV94, Proposition 1.9] claimed that the row $(21+4x, 12, x^2 + 20)$ is unimodular but not stable, which would therefore establish Theorem A. However, it is unknown if the previous row is stable: because of a typographical error, one should read $(21 + 2x, 12, x^2 + 20)$ instead of $(21 + 4x, 12, x^2 + 20)$. Besides, the proof of [GMV94, Proposition 1.9] uses, the false statement “$SK_1(O, fO) \neq 1$” (see Definition 1.11) where $O = Z + Z\sqrt{-5}$ and $f = 2$; Remark 3.4 below explains this mistake in detail. Thus, none of the previous unimodular rows has been proven to be unstable.

In Section 2, we present a proof of Theorem A which addresses these shortcomings. Our proof follows closely the lines of the original. It consists in exhibiting a quotient $R$ of $Z[x]$ such that the special Whitehead group $SK_1(R)$ (see Definition 1.6) is not trivial. This suffices to show that $sr(Z[x]) > 2$. Indeed, if $sr(Z[x]) \leq 2$, then the natural map $SK_1(Z[x]) \to SK_1(R)$ would be surjective (Corollary 1.7 below), which is impossible as $SK_1(Z[x]) = 1$ [BHS64, Theorem 1].

In Section 3, we show in addition:

**Proposition B.** The unimodular row $(3, x + 1, x^2 + 16)$ of $Z[x]$ is not stable.

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1. **Bass’s stable rank, and surjective homomorphisms**

In this section, we prove Corollary 1.7, which is a simple, but important argument in the proof of Theorem A. We also introduce the relative special Whitehead group $SK_1(R, I)$ with Definition 1.11, the group $K_2(R)$ and an exact sequence, called the relative sequence, which binds these groups. This sequence is used to show the existence of an isomorphism $SK_1(R) \cong SK_1(R, I)$ for suitable $R$ and $I$, in the proof of Theorem A, see Lemma 2.3 below.
Bass’s stable rank. Recall that rings are supposed to be commutative and unital. The Bass stable rank can be characterized in terms of a lifting property for unimodular rows of quotient rings.

**Proposition 1.1.** Let $R$ be a ring. Then the following are equivalent.

(i) $\text{sr}(R) \leq n$.

(ii) For every ideal $I \subseteq R$, the map $\text{Um}_n(R) \to \text{Um}_n(R/I)$ sending $(r_1, \ldots, r_n)$ to $(r_1 + I, \ldots, r_n + I)$ is surjective.

**Proof.** $(i) \Rightarrow (ii)$. Let $I$ be an ideal of $R$ and let $(r_1, \ldots, r_n) \in R^n$ be such that the ideal generated by $r_1 + I, \ldots, r_n + I$ is $R/I$. By hypothesis, there are $(s_1, \ldots, s_n) \in R^n$ and $a \in I$ such that $s_1r_1 + \cdots + s_nr_n + a = 1$. Thus $(r_1, \ldots, r_n, a) \in \text{Um}_{n+1}(R)$. Since $\text{sr}(R) \leq n$, we can find $(\lambda_1, \ldots, \lambda_n) \in R^n$ such that $(r_1 + \lambda_1a, \ldots, r_n + \lambda_n a) \in \text{Um}_n(R)$, which yields the result.

$(ii) \Rightarrow (i)$. Let $r = (r_1, \ldots, r_{n+1}) \in \text{Um}_{n+1}(R)$. We certainly have $(r_1 + I, \ldots, r_n + I) \in \text{Um}_n(R/I)$ for $I = Rr_{n+1}$. Assumption $(ii)$ entails that there is $(\lambda_1, \ldots, \lambda_n) \in R^n$ such that $(r_1 + \lambda_1 r_{n+1}, \ldots, r_n + \lambda_n r_{n+1}) \in \text{Um}_n(R)$, which shows that $r$ is stable. $\square$

Specializing $n$ to 1, we obtain:

**Proposition 1.2.** [EO67, Lemma 6.1] Let $R$ be a ring. Then the following are equivalent.

(i) $\text{sr}(R) \leq 1$.

(ii) For every ideal $I \subseteq R$, the natural map $R^* \to (R/I)^*$ is surjective.

The condition $\text{sr}(R) \leq 1$ received special attention in [EO67] and [Vas84]. We shall see with Proposition 1.4 below that the condition $\text{sr}(R) \leq 2$ can also be interpreted in terms of surjective group homomorphisms. We denote by $\text{Jac}(R)$, the Jacobson radical of $R$, that is, the intersection of the maximal ideals of $R$. We record the following proposition for later use.

**Proposition 1.3.** ([MR87, Lemma 11.4.6] and [Mag02, Exercise 4D.8]) Let $I$ be an ideal of $R$. Then we have $\text{sr}(R/I) \leq \text{sr}(R)$ and equality holds if $I \subseteq \text{Jac}(R)$.

Rings of stable rank at most 2. Rings of stable rank at most 2 enjoy the following characterization.

**Proposition 1.4.** Let $R$ be a ring. Then the following are equivalent:

(i) $\text{sr}(R) \leq 2$.

(ii) The natural map $\text{SL}_2(R) \to \text{SL}_2(\overline{R})$ is surjective for every quotient $\overline{R}$ of $R$.

(iii) The natural map $\text{SL}_n(R) \to \text{SL}_n(\overline{R})$ is surjective for every quotient $\overline{R}$ of $R$ and every $n \geq 2$. 
Our proof of Proposition 1.4 relies on

**Lemma 1.5.** ([GMV94, Lemma 6.2], [EO67, Corollary 8.3]) Let \( R \) be a ring. Let \( (a, b, c) \in \text{Um}_3(R) \) and let \( r \mapsto \overline{r} \) denote the natural map from \( R \) onto \( R/Rc \). Then the following are equivalent:

(i) The row \( (a, b, c) \) is stable.

(ii) Every matrix \( \left( \begin{array}{ccc} a & b & c \\ \overline{a} & \overline{b} & \overline{c} \end{array} \right) \in \text{SL}_2(R/\overline{R}) \) has a lift in \( \text{SL}_2(R) \).

Proof. (i) \( \Rightarrow \) (ii). Let \( A = \left( \begin{array}{ccc} \overline{a} & \overline{b} & \overline{c} \\ a & b & c \end{array} \right) \in \text{SL}_2(R/\overline{R}) \). Since \( (a, b, c) \) is stable, we can find \( (r, s) \in \text{Um}_2(R) \) such that \( \overline{r} = \overline{a} \) and \( \overline{s} = \overline{b} \). Let \( u, v \in R \) be such that \( \left( \begin{array}{cc} r & s \\ u & v \end{array} \right) \in \text{SL}_2(R) \). Then \( \left( \begin{array}{cc} \overline{a} & \overline{b} \\ \overline{w} & \overline{v} \end{array} \right) A^{-1} = \left( \begin{array}{cc} 1 & 0 \\ w & 1 \end{array} \right) \) for some \( w \in R \). Therefore \( \left( \begin{array}{cc} 1 & 0 \\ w & 1 \end{array} \right)^{-1} \left( \begin{array}{cc} r & s \\ u & v \end{array} \right) \) is a lift of \( A \).

(ii) \( \Rightarrow \) (i). By assumption, we can find \( (r, s) \in \text{Um}_2(R) \) such that \( \overline{r} = \overline{a} \) and \( \overline{s} = \overline{b} \), i.e., we have \( r = a + \lambda c \) and \( s = b + \mu c \) for some \( \lambda, \mu \in R \). It follows immediately that \( (a, b, c) \) is stable. \( \square \)

Let \( R \) be a ring. Let \( n \geq 2 \) and denote by \( I_n \) the \( n \times n \) identity matrix. For \( 1 \leq i, j \leq n \), let \( e_{ij} \) be the \( n \times n \) matrix whose \((i, j)\) entry is 1 and whose other entries are zero. For \( a \in R \) and \( i \neq j \), we set \( e_{ij}(a) = I_n + ae_{ij} \) and call any such matrix an *elementary matrix*. Let us denote by \( E_n(R) \) the subgroup of \( \text{SL}_n(R) \) generated by the elementary matrices.

**Proof of Proposition 1.4.** Clearly, we have \((iii) \Rightarrow (ii)\). The implication \((ii) \Rightarrow (i)\) is given by Lemma 1.5. Hence it only remains to show that \((i) \Rightarrow (iii)\).

Assume that \((i)\) holds and let \( n \geq 2 \), \( A \in \text{SL}_n(\overline{R}) \) where \( \overline{R} = R/I \) for some ideal \( I \) of \( R \). If \( n = 2 \), then it follows from Lemma 1.5 that \( A \) has a lift in \( \text{SL}_2(R) \).

Indeed, write \( A = \left( \begin{array}{cc} a + I & b + I \\ d + I & e + I \end{array} \right) \in \text{SL}_2(\overline{R}) \). Putting \( c := 1 - (ae - bd) \in I \), then \( (a, b, c) \in \text{Um}_3(R) \) and \( \overline{A} = \left( \begin{array}{cc} a + Rc & b + Rc \\ d + Rc & e + Rc \end{array} \right) \in \text{SL}_2(R/Rc) \). By hypothesis, we have \( \text{sr}(\overline{R}) \leq 2 \), so that Lemma 1.5 applies and provides us with a lift of \( \overline{A} \) in \( \text{SL}_2(R) \) which is also a lift of \( A \).

We can now assume that \( n > 2 \) and proceed by induction on \( n \). Since \( \text{sr}(\overline{R}) \leq 2 \) by Proposition 1.3, and because the first row of \( A \) is unimodular, we can find \( E, E' \in E_n(\overline{R}) \) such that \( A = E \left( \begin{array}{cc} 1 & 0 \\ 0 & A' \end{array} \right) E' \) with \( A' \in \text{SL}_{n-1}(\overline{R}) \).
By induction hypothesis, the matrix $A'$ has a lift in $\text{SL}_{n-1}(R)$. Clearly, the matrices $E$ and $E'$ lift to $\text{SL}_n(R)$. Thus $A$ has a lift in $\text{SL}_n(R)$. □

The next result refers to the special Whitehead group $\text{SK}_1(R)$ of a ring $R$. We define this group as follows. Let $E(R) := \bigcup_n E_n(R)$ and $\text{SL}(R) := \bigcup_n \text{SL}_n(R)$ be the ascending unions for which the embeddings $E_n(R) \to E_{n+1}(R)$ and $\text{SL}_n(R) \to \text{SL}_{n+1}(R)$ are defined through $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Then $E(R)$ is a normal subgroup of $\text{SL}(R)$ [Mag02, Whitehead Lemma 9.7].

**Definition 1.6.** $\text{SK}_1(R) := \text{SL}(R)/E(R)$.

The following corollary is an immediate consequence of Proposition 1.4.

**Corollary 1.7.** Let $R$ be a ring satisfying $\text{sr}(R) \leq 2$. Let $\overline{R}$ be a quotient of $R$. Then the natural map $\text{SK}_1(R) \to \text{SK}_1(\overline{R})$ is surjective.

Following [Coh66], we call a ring $R$ a $GE_2$-ring if $\text{SL}_2(R) = E_2(R)$.

The next corollaries are of independent interest.

**Corollary 1.8.** Let $R$ be a $GE_2$-ring satisfying $\text{sr}(R) \leq 2$. Then every quotient of $R$ is a $GE_2$-ring.

**Proof.** Let $\overline{R}$ be a quotient of $R$ and let $A \in \text{SL}_2(\overline{R})$. By Proposition 1.4, the matrix $A$ is the image of some matrix in $\text{SL}_2(R)$. Since $\text{SL}_2(R) = E_2(R)$ by assumption, we infer that $A \in E_2(\overline{R})$. □

**Remark 1.9.** Corollary 1.8 provides a straightforward proof of [Guy18b, Theorem A].

**Corollary 1.10.** [McG08, Theorem 3.6] Let $R$ be a ring such that any proper quotient of $R/\text{Jac}(R)$ has stable rank 1. Then $\text{sr}(R) \leq 2$.

**Proof.** It is easy to check that rings of stable rank 1 are $GE_2$-rings. The result follows by combining Propositions 1.4 and 1.3. □

**The groups $\text{SK}_1(R, I)$, $K_2(R)$ and the relative sequence.** In this section, we define the relative special Whitehead group $\text{SK}_1(R, I)$ for $I$ an ideal of a ring $R$ and the group $K_2(R)$ from algebraic K-theory. We describe then the relative sequence, an exact sequence relating them to one another. This sequence will come in handy for the K-theoretical computations of Section 2.

Recall that $E_n(R)$ is the subgroup of $\text{SL}_n(R)$ generated by the elementary matrices $e_{ij}(r)$ with $r \in R$. Let $I$ be an ideal of $R$. We denote by $E_n(R, I)$ the normal subgroup of $E_n(R)$ which is normally generated by the matrices $e_{ij}(a)$ with $a \in I$, $1 \leq i \neq j \leq n$. We denote by $\text{SL}_n(R, I)$ the kernel of the natural map $\text{SL}_n(R) \to \text{SL}_n(R/I)$. Let $E(R, I) := \bigcup_n E_n(R, I)$ and $\text{SL}(R, I) := \bigcup_n \text{SL}_n(R, I)$
be the ascending unions for which the embeddings $E_n(R, I) \to E_{n+1}(R, I)$ and
$\text{SL}_n(R, I) \to \text{SL}_{n+1}(R, I)$ are defined through $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Then $E(R, I)$ is a
normal subgroup of $\text{SL}(R, I)$ [Mag02, Relative Whitehead Lemma 11.1].

Definition 1.11. $\text{SK}_1(R, I) := \text{SL}(R, I)/E(R, I)$.

We now turn to the definition of $K_2(R)$. The Steinberg group $\text{St}_n(R)$ is the
group with generators $x_{ij}(r)$, with $i \neq j$, $1 \leq i, j \leq n$ and $r \in R$ subject to the
defining relations:

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r + s),$$

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 
1, & \text{if } i \neq l, j \neq k, \\
x_{il}(rs), & \text{if } i \neq l, j = k.
\end{cases}$$

Removing the restrictions $i \leq n$ and $j \leq n$ on the generators $x_{ij}(r)$, the
same presentation defines the Steinberg group $\text{St}(R)$. Because the elementary
matrices $e_{ij}(r)$ obey the above standard relations and generate $E(R)$, the map
$x_{ij}(r) \mapsto e_{ij}(r)$ induces a surjective group homomorphism $\text{St}(R) \to E(R)$. The
kernel of this homomorphism is $K_2(R)$.

Finally, we introduce the so-called relative sequence:

$$K_2(R/I) \xrightarrow{\partial_1} \text{SK}_1(R, I) \to \text{SK}_1(R) \to \text{SK}_1(R/I),$$

where the second and third arrows are induced respectively by the inclusion
$\text{SL}(R, I) \subseteq \text{SL}(R)$ and the natural map $\text{SL}(R) \to \text{SL}(R/I)$. For the connecting
homomorphism $\partial_1$, we refer the reader to [Mag02, Theorem 13.20 and Example
13.22] where this sequence is shown to be exact.

2. Proof of Theorem A

We shall establish

Proposition 2.1. $\text{SK}_1(Z + 4iZ) \neq 1$ where $i = \sqrt{-1}$.

Observe that $Z + 4iZ$ naturally identifies with $Z[x]/(x^2 + 16)$. As outlined
in the introduction, Theorem A immediately follows from the combination of
Proposition 2.1 with Corollary 1.7 and the following theorem:

Theorem 2.2. [BHS64, Theorem 1] $\text{SK}_1(Z[x]) = 1$.

Our proof of Proposition 2.1 relies on

Lemma 2.3. Let $I$ be an ideal of a ring $R$ such that $R/I \cong Z/nZ$ for some
$n \geq 0$. Then the natural map

$$\text{SK}_1(R, I) \to \text{SK}_1(R)$$

induced by the inclusion $\text{SL}(R, I) \subseteq \text{SL}(R)$ is an isomorphism.
Proof. In the exact sequence (2) introduced in Section 1

\[ K_2(R/I) \xrightarrow{\partial_1} SK_1(R, I) \to SK_1(R) \to SK_1(R/I) \]

the last term, namely \( SK_1(R/I) \), is trivial since \( R/I \) is Artinian and hence of stable rank 1 [Bas64, Corollary 10.5]. In addition, the image of \( K_2(R/I) \) in \( SK_1(R, I) \) is also trivial. Indeed \( K_2(R/I) \) is generated by the Steinberg symbol \( \{-1 + I, -1 + I\}_{R/I} \) because \( R/I \cong \mathbb{Z}/n\mathbb{Z} \) [Mag02, Definition 12.23, Exercises 12B.6 and 13A.9]. As \( \{-1, -1\}_R \) is a lift of the previous symbol in \( K_2(R) \) [Mag02, Exercise 12B.7] our claim follows from the definition of \( \partial_1 \) [Mag02, Theorem 13.20], which completes the proof. \( \square \)

Proof of Proposition 2.1. Let \( S = \mathbb{Z}[i], I = 4S \) and \( R = \mathbb{Z} + I \). By Lemma 2.3 we have \( SK_1(R) \cong SK_1(R, I) \). The inclusion \( R \subset S \) induces the identity map on \( SL(R, I) = SL(S, I) \) and hence a surjective group homomorphism

\[ (4) \quad SK_1(R, I) \twoheadrightarrow SK_1(S, I). \]

As \( SK_1(S, I) \cong \mathbb{Z}/2\mathbb{Z} \) by the Bass-Milnor-Serre Theorem [Mag02, Theorem 11.33] (Theorem 3.2 below), we conclude that \( SK_1(R) \neq 1 \). \( \square \)

Remark 2.4. As an alternative to Proposition 2.1, one can show that

\[ SK_1(\mathbb{Z} + 3\zeta_3\mathbb{Z}) \neq 1, \text{ where } \zeta_3 = e^{2\pi i/3}. \]

This is a direct consequence of [Swa71, Lemma 3.2 and subsequent remark] and Lemma 2.3.

3. Proof of Proposition B

We shall prove that the unimodular row

\[ (3, x + 1, x^2 + 16) \in Um_3(\mathbb{Z}[x]) \]

is not stable. We apply the strategy devised in [GMV94, Proof of Proposition 1.9]: we look for an explicit matrix in \( SL_2(R) \) that defines a non-trivial element of \( SK_1(R) \) for a suitable quotient \( R \) of \( \mathbb{Z}[x] \). To do so, we resort to the Bass-Milnor-Serre Theorem [BMS67, Theorems 3.6 and 4.1] and its description of \( SK_1(S) \) for \( S \) the ring of integers of a totally imaginary number field, in terms of power residue symbols. The following definitions are required to state the latter theorem.

Let \( m > 0 \) be a rational integer. We denote by \( \mu_m \) the group of \( m \)-th roots of unity in the field of complex numbers. Let \( S \) be the ring of integers of a number field and suppose that \( S \) contains \( \mu_m \) for some \( m > 0 \). For \( b \in S \) and \( a \) an ideal of \( S \) such that \( a + Sbm = S \), define the \( m \)-th power residue symbol

\[ \left( \frac{b}{a} \right)_m := \prod_{p | a} \left( \frac{b}{p} \right)^{\text{ord}_p(a)}_m \]
where \( p \) ranges in the set of prime ideals of \( S \) dividing \( a \) and where \( \left( \frac{b}{p} \right)_m \) is the unique element of \( \mu_m \) satisfying
\[
b^{\frac{a-1}{m}} \equiv \left( \frac{b}{p} \right)_m \mod p
\]
where \( q \) is the number of elements in the residue field of \( p \) [BMS67, Appendix on number theory, pages 86 and 89] [Mag02, Theorem 11.33 and Proposition 15.40]. Note that \( m \) divides \( q - 1 \) by virtue of Lagrange’s theorem. If \( a = Sa \) with \( a \in S \), we simply write \( \left( \frac{b}{a} \right)_m \). The latter symbol depends on \( b \) only modulo \( a \) and is evidently multiplicative in \( b \), i.e., we have
\[
\left( \frac{bc}{a} \right)_m = \left( \frac{b}{a} \right)_m \left( \frac{c}{a} \right)_m
\]
for every \( b, c \in S \) such that \( Sa + Sbcm = S \).

The following computation will come soon in handy.

**Lemma 3.1.** Let \( p \) be the principal ideal of \( \mathbb{Z}[i] \) generated by \( 1 + 4i \). Then \( p \) is prime, \( 12 \notin p \) and we have \( \left( \frac{12}{1+4i} \right)_2 = -1 \).

**Proof.** As the norm \( N_{\mathbb{Q}(i)/\mathbb{Q}}(1 + 4i) = 17 \), the ideal \( p \) is prime and \( \mathbb{Z}[i]/p \) is the field with 17 elements. Clearly, we have \( 12 \notin p \) so that \( \left( \frac{12}{1+4i} \right)_2 \) is well-defined.

It follows immediately from the multiplicative law (5) that \( \left( \frac{12}{1+4i} \right)_2 = \left( \frac{3}{1+4i} \right)_2 \).

Observing that \( 3 \frac{1+4i}{2} = 3^8 \equiv -1 \mod 17 \), we infer that \( \left( \frac{3}{1+4i} \right)_2 = -1 \), which completes the proof. \( \Box \)

**Theorem 3.2.** [BMS67, Theorems 3.6 and 4.1] Let \( S \) be the ring integers of a totally imaginary number field. Let \( m = m(S) \) denote the number of roots of unity in \( S \). If \( I \) is a non-zero ideal of \( S \), define the divisor \( r = r(I) \) of \( m \) by
\[
\text{ord}_p(r) = j_p(I), \text{ for every prime divisor } p \text{ of } m,
\]
where \( j_p(I) \) is the nearest integer in the interval \([0, \text{ord}_p(m)]\) to
\[
\min_{p \mid m} \left[ \frac{\text{ord}_p(I)}{\text{ord}_p(Sp)} - 1 \right]
\]
(where \( |x| \) denotes the greatest integer \( \leq x \) and \( p \) ranges over the prime ideals of \( S \) containing \( p \)).

Then the map \( \left( \begin{array}{cc} a & b \\ \ast & \ast \end{array} \right) \in \text{SL}_2(S, I) \mapsto \left( \frac{b}{a} \right)_r \) induces an isomorphism from \( \text{SK}_1(S, I) \) onto \( \mu_r \).

The first item of the next proposition was implicit in the above theorem.

**Proposition 3.3.** [Mag02, Lemma 11.24 and Proposition 11.25] Let \( R \) be a ring and let \( I \) be an ideal of \( R \).
(i) If two matrices of $SL_2(R, I)$ have the same first row, then they represent the same element of $SK_1(R, I)$. For $A = \begin{pmatrix} a & b \\ * & * \end{pmatrix} \in SL_2(R, I)$, define $[a, b]_I = A \cdot E(R, I) \in SK_1(R, I)$.

(ii) Let $(a, b) \in Um_2(R)$ with $a \in 1 + I$ and $b \in I$. Then $(a, b)$ is the first row of some matrix in $SL_2(R, I)$.

(iii) If $(a, b')$ is the first row of a matrix in $SL_2(R, I)$, then we have $[a, bb']_I = [a, b]_I [a, b']_I$.

Remark 3.4. Set $S = \mathbb{Z}[\sqrt{-5}]$ and $I = 2S$. The statement \cite{GMV94}, "$f = 2$ has property (⋆)" on page 191 implies that $SK_1(S, I) \simeq \mu_2$, which is false. Indeed, we have $m(S) = 2$ and it follows from Theorem 3.2 that $r(I) = 1$ and hence $SK_1(S, I) = 1$.

As a stepping stone to Proposition B, we shall establish:

**Proposition 3.5.** The unimodular row $(12, x + 1, x^2 + 16)$ of $\mathbb{Z}[x]$ is not stable.

For the remainder of this section, we set $S = \mathbb{Z}[i], I = 4S$, and $R = \mathbb{Z} + I$, with a view to apply Theorem 3.2 and Proposition 3.3. The reason why we first consider the row $(12, x + 1, x^2 + 16)$, and not $(3, x + 1, x^2 + 16)$, is that $[1 + 4i, 12]_I$ is well-defined since $12 \in I$ whereas $3 \notin I$.

**Proof of Proposition 3.5.** By Theorem 3.2, we have $r(I) = 2$. By Theorem 3.2 and Proposition 3.3.ii, the power residue symbol $\left(\frac{12}{1+4i}\right)_2$ is the image of $A \cdot E(S, I) \in SK_1(S, I)$ for some matrix $A = \begin{pmatrix} 1 + 4i & 12 \\ * & * \end{pmatrix} \in SL_2(S, I)$. By Lemma 3.1, we have $A \cdot E(S, I) \neq 1$. The matrix $A$ can be lifted from $SK_1(S, I)$ to $SK_1(R, I)$ via (4), and certainly maps to a non-trivial element of $SK_1(R)$ via the isomorphism (3).

Considering the surjective ring homomorphism from $\mathbb{Z}[x]$ onto $R$ induced by $x \mapsto 4i$, we infer from Theorem 2.2 and Lemma 1.5 that the row $(1 + x, 12, x^2 + 16)$ cannot be stable. This trivially implies the result. \qed

**Proof of Proposition B.** Let $\gamma = [1 + 4i, 12]_I [1 + 4i, 4]^{-1}_I \in SK_1(R, I)$. Thanks to Proposition 3.3.iii, we observe that $\gamma$ is the image of $[1 + 4i, 3]_R \in SK_1(R, R) = SK_1(R)$ by the inverse of the isomorphism (3). Mapping $\gamma$ to $SK_1(S, I)$ via (4), we obtain again, by means of Theorem 3.2, a non-trivial element of $SK_1(S, I)$ since $\left(\frac{4}{1+4i}\right)_2 = 1$. Reasoning as in the proof of Proposition 3.5, we conclude that $(3, x + 1, x^2 + 16)$ is not stable. \qed
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