Deletion theorem and combinatorics of hyperplane arrangements

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Abstract

We show that the deletion theorem of a free arrangement is combinatorial, i.e., whether we can delete a hyperplane from a free arrangement keeping freeness depends only on the intersection lattice. In fact, we give an explicit sufficient and necessary condition for the deletion theorem in terms of characteristic polynomials. This gives a lot of corollaries including the existence of free filtrations. The proof is based on the result about the form of minimal generators of a logarithmic derivation module of a multiarrangement which satisfies the \( b_2 \)-equality.

1 Introduction

Let \( \mathcal{A} \) be a central arrangement of hyperplanes in \( V = \mathbb{K}^\ell \) for an arbitrary field \( \mathbb{K} \). In this section, we use the notation in §2 to explain the background of this article, and to state the main results. In the study of hyperplane arrangements, the most important problem is to determine whether some property of \( \mathcal{A} \) depends only on its combinatorial data (i.e., the intersection lattice \( L(\mathcal{A}) \)) or not. For example, when \( \mathbb{K} = \mathbb{C} \), the cohomology ring of the complement of \( \mathcal{A} \) is known to be combinatorial by Orlik-Solomon in [11], but the fundamental group of it is known to be not combinatorial by Rybnikov in [13]. On the other hand, the freeness of arrangements, the most important algebraic property of arrangements, is not yet known to be combinatorial or not when \( \ell \geq 3 \). This problem is called Terao’s conjecture. In general, whether some property is combinatorial or not is not known in most cases,

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and they are open problems. The aim of this article is to determine the deletion theorem of free arrangements is combinatorial, by giving the explicit condition for it. To state it, let us recall Terao’s addition-deletion theorem.

**Theorem 1.1** ([15], Terao’s addition-deletion theorem)

Let \( H \in \mathcal{A}, \mathcal{A}' := \mathcal{A} \setminus \{H\} \) and let \( \mathcal{A}'' := \mathcal{A}^H := \{L \cap H \mid L \in \mathcal{A}'\} \). Then two of the following imply the third:

1. \( \mathcal{A} \) is free with \( \exp(\mathcal{A}) = (d_1, \ldots, d_{\ell - 1}, d_\ell) \).
2. \( \mathcal{A}' \) is free with \( \exp(\mathcal{A}') = (d_1, \ldots, d_{\ell - 1}, d_\ell - 1) \).
3. \( \mathcal{A}'' \) is free with \( \exp(\mathcal{A}'') = (d_1, \ldots, d_{\ell - 1}) \).

Moreover, all the three hold true if \( \mathcal{A} \) and \( \mathcal{A}' \) are both free.

A lot of freeness have been checked and showed by using Theorem 1.1. In Theorem 1.1, if \( \mathcal{A} \) is free, then to show the freeness of the deletion \( \mathcal{A}' \), it suffices to check the algebraic structure of \( D(\mathcal{A}'') \) and the inclusion between two exponents. In fact, we can show that no algebra, but just a combinatorics is necessary for the deletion theorem. We summarize it as the main theorem in this article in the following.

**Theorem 1.2**

Let \( \mathcal{A} \) be a free arrangement and \( H \in \mathcal{A} \). Then \( \mathcal{A}' := \mathcal{A} \setminus \{H\} \) is free if and only if \( \chi(\mathcal{A}''; t) \) divides \( \chi(\mathcal{A}; t) \) for all \( X \in L_i(\mathcal{A}'') \) with \( 2 \leq i \leq \ell - 1 \).

Theorem 1.2 follows immediately when \( \ell \leq 3 \) by [1], so it can be regarded as a higher dimensional version of it. Though the statement in Theorem 1.2 is purely combinatorial, the proof heavily depends on algebra and algebraic geometry. An important corollary of Theorem 1.2 is the following.

**Corollary 1.3**

Assume that \( \mathcal{A} \) is free and take \( H \in \mathcal{A} \). Then the freeness of \( \mathcal{A} \setminus \{H\} \) depends only on \( L(\mathcal{A}) \).

Also, we have the following.

**Corollary 1.4**

Let \( \ell \leq 4 \), \( H \in \mathcal{A} \) and \( \chi(\mathcal{A}^H; t) \) divides \( \chi(\mathcal{A}; t) \). Then \( \mathcal{A} \) is free if and only if \( \mathcal{A}^H \) is free.

In [1], a free filtration of an arrangement was introduced. Namely, \( \mathcal{A} \) has a free filtration

\[
\emptyset = \mathcal{A}_0 \subset \cdots \subset \mathcal{A}_n = \mathcal{A}
\]
if \( \{ A_i \} \) satisfies that \( |A_i| = i \), \( |A| = n \) and \( A_i \) is free for all \( i \). Naively, a free arrangement with a free filtration is the arrangement which can be constructed from the empty arrangement by only using Terao’s addition theorem. Some free arrangements are known not to have any free filtration, the most famous one is the cone of the affine line arrangement consisting of all edges and diagonals of a regular pentagon, see [12] for example. We can make it clear that this property is combinatorial.

**Corollary 1.5**
For a free arrangement \( A \), whether it has a free filtration or not depends only on \( \mathcal{L}(A) \).

To show the results above, we need the \( b_2 \)-inequality with respect to \( H \in A \) which was shown in [2] and [3]:

\[
b_2(A) \geq b_2(A^H) + (|A| - |A|)|A^H|.
\]

It is easy to check that the \( b_2 \)-inequality becomes a equality if \( \chi(A^H; t) | \chi(A; t) \). This (in)equality played a key role in the proof of the division theorem in [2]. When the \( b_2 \)-equality holds, the following structure theorem on the logarithmic derivation modules holds, which is essential for the proof of our main results.

**Theorem 1.6**
Let \((A, m)\) be multiarrangements such that \( A \neq \emptyset \) and that

\[
b_2(A, m) = b_2(A) + (|m| - |A|)(|A| - 1).
\]

Then

1. there is a minimal generator \( \theta_E, \theta_1, \ldots, \theta_s \) for \( D(A) \) such that
   \[
   \frac{Q(A, m)}{Q(A)} \theta_E, \theta_1, \ldots, \theta_s
   \]
   form a generator for \( D(A, m) \), and they form a minimal generator for \( D(A, m) \) if and only if \( \langle \theta_1, \ldots, \theta_s \rangle_S \neq D(A, m) \),

2. if \((A, m)\) is free and \( A \) is not free, then there exists a minimal generator \( \theta_E, \theta_1, \ldots, \theta_\ell \in D(A) \) for \( D(A) \) such that \( \theta_1, \ldots, \theta_\ell \) form a basis for \( D(A, m) \), and

3. if \((A, m)\) and \( A \) are both free, then there is a free basis \( \theta_E, \theta_1, \ldots, \theta_{\ell-1} \) for \( D(A) \) such that \( \theta_1, \ldots, \theta_{\ell-1}, (Q(A, m)/Q(A))\theta_E \) form a basis for \( D(A, m) \).
The most important application of Theorem 1.6 is the following.

**Corollary 1.7**

Let $\mathcal{A}$ be an $\ell$-arrangement and $(\mathcal{A}^H, m^H)$ the Ziegler restriction of $\mathcal{A}$ onto $H \in \mathcal{A}$. Assume that the $b_2$-equation

$$b_2(\mathcal{A}) = b_2(\mathcal{A}^H) + |\mathcal{A}^H|(|\mathcal{A}| - |\mathcal{A}^H|)$$

holds true. Let $\theta_E := (Q(\mathcal{A}^H, m^H)/Q(\mathcal{A}^H))\theta_E \in D(\mathcal{A}^H, m^H)$, and let $\pi : D_H(\mathcal{A}) \to D(\mathcal{A}^H, m^H)$ be the Ziegler restriction.

1. Then there are minimal generators $\theta_E, \theta_1, \ldots, \theta_s$ for $D(\mathcal{A}^H)$ such that $\theta_E, \theta_1, \ldots, \theta_s$ form a generator for $D(\mathcal{A}^H, m^H)$.

2. Assume that $\mathcal{A}$ is free, and $\mathcal{A}^H$ is not free. Then there is a generator $\theta_E, \theta_2, \ldots, \theta_\ell$ for $D(\mathcal{A}^H)$ such that $\theta_2, \ldots, \theta_\ell$ form a basis for $D(\mathcal{A}^H, m^H)$, the preimage of $\theta_2, \ldots, \theta_\ell$ in $D_H(\mathcal{A})$ by $\pi$ form a free basis for $D_H(\mathcal{A})$, and the relation among $\theta_E, \theta_2, \ldots, \theta_\ell$ in $D(\mathcal{A}^H)$ is in the degree $d := |\mathcal{A}| - |\mathcal{A}^H|$ of the form $\theta_E^H = \sum_{i=2}^\ell f_i\theta_i$ for $f_i \in S/\alpha_H S$, and no other relation exists. In other words, we have a free resolution

$$0 \to S[-d] \to \bigoplus_{i=1}^\ell S[-d_i] \to D(\mathcal{A}^H) \to 0,$$

where $\deg \theta_i := d_i$.

We investigate several properties of $D(\mathcal{A})$ by using Theorem 1.6 including the modified Orlik’s conjecture (Problem 4.3).

The organization of this article is as follows: In §2 we introduce a notation and several results used for the proof. In §3 we prove Theorem 1.6 and show several related results. In §4 we prove main results.

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## 2 Preliminaries

In this section let us summarize several definitions and results used in this article. We refer [12] for a general reference in this section. Let $K$ be an arbitrary field and $\mathcal{A}$ a central arrangement of hyperplanes in $V = K^\ell$, i.e., a finite set of linear hyperplanes in $V$. Assume that every hyperplane $H \in \mathcal{A}$ is defined by a linear form $\alpha_H = 0$. Let $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H$. Without any specification, we assume that $\mathcal{A} \neq \emptyset$. Let $S = K[x_1, \ldots, x_\ell]$ be a coordinate
ring of $V$ and $\text{Der} S := \oplus_{i=1}^\ell S\partial_{x_i}$. Then the logarithmic vector field $D(A)$ of $A$ is defined as

$$D(A) := \{\theta \in \text{Der} S \mid \theta(\alpha_H) \in S\alpha_H \ (\forall H \in A)\}.$$ 

$D(A)$ is a reflexive module, and not free in general. We say that $A$ is free with $\exp(A) = (d_1, \ldots, d_\ell)$ if $D(A)$ is a free $S$-module of rank $\ell$ with homogenous basis $\theta_1, \ldots, \theta_\ell$ such that $\deg \theta_i = d_i$. Since $A$ is not empty, the lowest degree basis element $\theta_1$ can be chosen as the Euler derivation $\theta_E = \sum_{i=1}^\ell x_i \partial_{x_i}$ which is always contained in $D(A)$. Also, for $H \in A$, define $D_H(A) := \{\theta \in D(A) \mid \theta(\alpha_H) = 0\}$.

Next let us introduce combinatorics and topology of arrangements. Let $L(A) := \{\bigcap_{H \in B} H \mid B \subset A\}$ be the intersection lattice of $A$ with a partial order induced from the reverse inclusion. Define $L_i(A) := \{X \in L(A) \mid \dim_X X = i\}$. The Möbius function $\mu : L(A) \to \mathbb{Z}$ is defined by $\mu(V) = 1$, and by $\mu(X) := -\sum_{X \subseteq Y \subsetneq V} \mu(Y)$ for $L(A) \ni X \subsetneq V$. Define the characteristic polynomial $\chi(A; t)$ by

$$\chi(A; t) := \sum_{X \in L(A)} \mu(X)t^{\dim X},$$

and the Poincaré polynomial $\pi(A; t)$ by

$$\pi(A; t) := \sum_{X \in L(A)} \mu(X)(-t)^{\dim X}.$$ 

For $X \in L(A)$, the localization $A_X$ of $A$ at $X$ is defined by

$$A_X := \{H \in A \mid X \subset H\},$$

and the restriction $A^X$ of $A$ onto $X$ is defined by

$$A^X := \{H \cap X \mid H \in A \setminus A_X\}.$$ 

It is easy to check that $A_X$ is free if $A$ is free for any $X \in L(A)$. Also, we say that $A$ is locally free if $A_X$ is free for any $X \in L(A)$ with $X \neq \emptyset$. $A$ is locally free if and only if the sheaf $\widetilde{D_H(A)}$ is a vector bundle on $\mathbb{P}^{\ell-1} = \mathbb{P}(V^*)$ for any $H \in A$. Define the Euler restriction map $\rho : D(A) \to D(A^H)$ by taking modulo $\alpha_H$. Then it is known (e.g., see [12]) that there is an exact sequence

$$0 \to D(A \setminus \{H\}) \xrightarrow{\alpha_H} D(A) \xrightarrow{\rho} D(A^H).$$
The most useful inductive method to compute $\chi(A; t)$ is so called the deletion-restriction formula as follows:

$$\chi(A; t) = \chi(A'; t) - \chi(A^H; t).$$

We may apply this to compute $\chi(A; t)$ efficiently.

Let $\chi(A; t) = \sum_{i=0}^{\ell}(-1)^i b_i(A)t^{\ell-i}$. When $A \neq \emptyset$, it is known that $\chi(A; t)$ is divisible by $t-1$. Define $\chi_0(A; t) := \chi(A; t)/(t-1) = \sum_{i=0}^{\ell-1}(-1)^i b_i^0(A)t^{\ell-i-1} = \sum_{i=0}^{\ell-1}(-1)^i b_i(dA)t^{\ell-i-1}$, where $dA$ is the deconing of $A$ by any line $H \in A$.

It is known that $b_i(A)$ is the $i$-th Betti number of $V \setminus \bigcup_{H \in A} H$ when $\mathbb{K} = \mathbb{C}$. Then we may relate the exponents of free arrangements and the combinatorics and topology as follows:

**Theorem 2.1 (Terao's factorization, [16])**

Assume that $A$ is free with $\exp(A) = (d_1, \ldots, d_\ell)$. Then $\chi(A; t) = \prod_{i=1}^{\ell} (t - d_i)$.

A pair $(A, m)$ is a multiarrangement if $m : A \to \mathbb{Z}_{\geq 1}$. Let $|m| := \sum_{H \in A} m(H)$ and $Q(A, m) := \prod_{H \in A} \alpha_H^{m(H)}$. For two multiplicities $m$ and $k$ on $A$, $k \leq m$ means that $k(H) \leq m(H)$ for all $H \in A$. We may define its logarithmic derivation module $D(A, m)$ as

$$D(A, m) := \{\theta \in \text{Der} S \mid \theta(\alpha_H) \in S\alpha_H^{m(H)} \ (\forall H \in A)\}.$$  

We may define the freeness and exponents of $(A, m)$ in the same way as for $m \equiv 1$. Also, we can define the characteristic polynomial $\chi(A, m; t) = \sum_{i=0}^{\ell}(-1)^i b_i(A, m)t^{\ell-i}$ of $(A, m)$ in algebraic way, see [5] for details. Now let us introduce the fundamental method to determine the freeness of $(A, m)$.

**Theorem 2.2 (Saito’s criterion, [14], [18])**

Let $\theta_1, \ldots, \theta_\ell$ be a homogeneous element in $D(A, m)$. Then $D(A, m)$ has a free basis $\theta_1, \ldots, \theta_\ell$ if and only if they are $S$-independent, and $|m| = \sum_{i=1}^{\ell} \deg \theta_i$.

We can construct the multiarrangement canonically from an arrangement $A$ in the following manner:

**Definition 2.3 ([18])**

For an arrangement $A$ in $\mathbb{K}^\ell$ and $H \in A$, define the Ziegler multiplicity $m^H : A^H \to \mathbb{Z}_{\geq 0}$ by $m^H(X) := |\{L \in A \setminus \{H\} \mid L \cap H = X\}|$ for $X \in A^H$. The pair $(A^H, m^H)$ is called the Ziegler restriction of $A$ onto $H$. Also, there is a canonical Ziegler restriction map

$$\pi : D_H(A) \to D(A^H, m^H)$$

by taking modulo $\alpha_H$. 


A remarkable property of Ziegler restriction maps is the following.

**Theorem 2.4 ([18])**

Assume that \( A \) is free with \( \exp(A) = (1, d_2, \ldots, d_\ell) \). Then for any \( H \in A \), the Ziegler restriction \((A^H, m^H)\) is also free with \( \exp(A^H, m^H) = (d_2, \ldots, d_\ell) \).

In particular, \( \pi \) is surjective when \( A \) is free.

Moreover, a converse of Theorem 2.4 holds true with additional conditions.

**Theorem 2.5 (Yoshinaga’s criterion, [17], [7])**

In the notation of Definition 2.3, it holds that

\[
b_0^2(A) \geq b_2(A^H, m^H).
\]

Moreover, \( A \) is free if and only if the above inequality is the equality, and \((A^H, m^H)\) is free.

An immediate consequence of Theorem 2.5 with the addition-deletion theorem for multiarrangements in [6] is the following inequality, which also induces the improvement of Terao’s addition theorem, and the inequality is the key of this article.

**Theorem 2.6 (**\( b_2 \)-inequality and the division theorem, [2])**

It holds that

\[
b_0^2(A) \geq b_2(A^H) + (|A^H| - 1)(|A| - |A^H|) - 1),
\]

which is equivalent to

\[
b_2(A) \geq b_2(A^H) + |A^H|(|A| - |A^H|).
\]

The equality holds if and only if \( A_X \) is free for all \( X \in L(A^H) \) with \( \text{codim}_V X = 3 \). This equality holds true if \( \chi(A^H; t) \mid \chi(A; t) \). Moreover, \( A \) is free if the \( b_2 \)-inequality is an equality, and \( A^H \) is free for some \( H \in A \).

**Definition 2.7**

For \( H \in A \), define the derivation \( \theta_E^H \in D(A^H, m^H) \) by

\[
\theta_E^H := \frac{Q(A^H, m^H)}{Q(A^H)} \rho(\theta_E).
\]

Not only the Ziegler restriction in Definition 2.3, but also we have the other restriction, called the **Euler restriction**. See [6], Definition 0.2 for details. Let \((A^H, m^*)\) be the Euler restriction of \((A, m)\) onto \( H \). Then we have the following.
Proposition 2.8 ([6], Definition 3.3, Lemma 3.4)
Let \((A^H, m^*)\) be the Euler restriction of \((A, m)\) onto \(H \in A\). Then there is a polynomial \(B\) of degree \(|m| - |m^*| - 1\) such that
\[
\theta(\alpha_H) \in (\alpha_H^{m(H)^{-1}}, B).
\]

3 Proof of Theorem 1.6 and related results

First let us show Theorem 1.6 which will play the key roles in the rest of this article.

Proof of Theorem 1.6 (1) Let \(\theta_E, \theta_1, \ldots, \theta_s\) be a minimal generator for \(D(A)\). Let \(1 \leq m' < m\). We show that \(D(A, m')\) has a generator
\[
\varphi_{m'} := \frac{Q(A, m')}{Q(A)}\theta_E, \theta_1', \ldots, \theta_s',
\]
where \(\theta_i' = \theta_i + f_i\theta_E\) for some \(f_i \in S\). We show by the induction on \(|m'|\).

When \(m' \equiv 1\), then there is nothing to show. Assume that the statement holds true for \(m'\), and we show the same is true for \(k := m' + \delta_L\) with \(L \in A\) such that \(k \leq m\), where \(\delta_L(K) = 1\) when \(K = L\), and 0 otherwise.

By Proposition 2.8, there is a homogeneous polynomial \(B\) of degree \(|m'| - |m^*|\) such that, for any \(\theta \in D(A, m')\),
\[
\theta(\alpha_L) \in (\alpha_L^{m'(L)^{-1}}, B),
\]
where \((A^L, m^*)\) is the Euler restriction of \((A, m')\) onto \(L\). Note that \(b_2(A, k) = b_2(A) + (|k| - |A|)(|A| - 1)\) since we have the \(b_2\)-equality \(b_2(A, m) = b_2(A) + (|A| - 1)(|m| - |A|)\), and \(m' < k \leq m\). Hence the proof of Theorem 1.7 in [2], or Proposition 2.5 in [3] implies that \((A^L, m^*) = (A^L, m^L)\), where the latter is the Ziegler restriction of \(A\) onto \(L\). Hence \(|m^*| = |m^L| = |A| - 1\). In particular,
\[
|m' - |m^*| = |m'| - |A| + 1 = \deg B = \deg \varphi_{m'}.
\]

Note that \(\varphi_{m'}(\alpha_L) \not\in (\alpha_L^{m'(L)^{-1}})\) by definition. Hence we may assume that
\[
\varphi_{m'}(\alpha_L) \equiv B \pmod{\alpha_L^{m'(L)^{-1}}},
\]

Hence when \(\theta_i'(\alpha_L) = a_i\alpha_L^{m'(L)^{-1}} + b_i B\) for some \(a_i, b_i \in S\), replacing \(\theta_i'\) by \(\theta_i'' := \theta_i' - b_i\varphi_{m'}\), it holds that \(\langle \varphi_{m'}, \theta_1'', \ldots, \theta_s'' \rangle_S = D(A, m')\) and that
\[
\alpha_L\varphi_{m'}, \theta_1''', \ldots, \theta_s''' \in D(A, k).
\]
Conversely, let $\theta \in D(\mathcal{A}, k)$. Since $D(\mathcal{A}, k) \subset D(\mathcal{A}, m')$, it holds that

$$\theta = g_0 \varphi_{m'} + \sum_{i=1}^{s} g_i \theta_i''$$

here $g, g_i \in S$ and we used the fact that $\varphi_{m'}, \theta_1'', \ldots, \theta_s''$ form a generator for $D(\mathcal{A}, m')$ too. Hence

$$\theta(\alpha_L) = g_0 \varphi_{m'}(\alpha_L) + \sum_{i=1}^{s} g_i \theta_i''(\alpha_L)$$

for some $g, g_i \in S$. Note that $\alpha_L^{m'(L)+1} \mid \theta(\alpha_L)$ and $\alpha_L^{m'(L)+1} \mid \theta_i(\alpha_L)$ for $i = 1, \ldots, s$. Since $\alpha_L^{m'(L)+1} \mid \varphi_{m'}(\alpha_L)$ by the definition, it holds that $\alpha_L \mid g$, which shows that $\alpha_L \varphi_{m'}, \theta_1'', \ldots, \theta_s''$ form a generator for $D(\mathcal{A}, k)$. The minimality is clear by the construction, which completes the proof.

(2) Assume that $\mathcal{A}$ is not free, $(\mathcal{A}, m)$ free and $s > \ell$. By the above, there is a minimal generator $\theta_E, \theta_1, \ldots, \theta_s$ for $D(\mathcal{A})$ such that $Q'\theta_E, \theta_1, \ldots, \theta_s$ form a generator for $D(\mathcal{A}, m)$, where $Q' := \frac{Q(\mathcal{A}, m)}{Q(\mathcal{A})}$. Since $D(\mathcal{A}, m)$ is free with rank $\ell$, $Q'\theta_E, \theta_1, \ldots, \theta_s$ is not a minimal generator for $D(\mathcal{A}, m)$. Assume that $\theta_s$ is removable. Then $\theta_E, \theta_1, \ldots, \theta_{s-1}$ form a generator for $D(\mathcal{A})$, which contradicts the minimality of the generator. Hence $Q'\theta_E$ is removable, and no $\theta_i$ is. Hence $\theta_1, \ldots, \theta_s$ has to be a minimal generator (thus a free basis) for $D(\mathcal{A}, m)$, which completes the proof. Hence $s = \ell$, which completes the proof.

(3) Assume that $\mathcal{A}$ and $(\mathcal{A}, m)$ are both free. Then by (1), there is a generator (in fact, a basis) $\theta_E, \theta_1, \ldots, \theta_{\ell-1}$ for $D(\mathcal{A})$ such that $Q'\theta_E, \theta_1, \ldots, \theta_{\ell-1}$ form a generator for $D(\mathcal{A}, m)$, which completes the proof. □

**Proof of Corollary 1.7** (1) Immediate from Theorem 1.6 since the $b_2$-equality holds, and

$$b_2^0(\mathcal{A}) \geq b_2(\mathcal{A}^H, m^H) \geq b_2(\mathcal{A}^H) + (|\mathcal{A}^H| - 1)(|\mathcal{A}| - |\mathcal{A}^H| - 1)$$

by Theorems 2.3 and 2.6.

(2) Let $\theta_E, \theta_2, \ldots, \theta_{\ell}$ be a minimal generator for $D(\mathcal{A}^H)$ as in Theorem 1.6

(3). Then $\theta_2, \ldots, \theta_{\ell} \in D(\mathcal{A}^H, m^H)$ form a basis for $D(\mathcal{A}^H, m^H)$ by Theorem 1.6 (3), and $\pi$ is surjective by Theorem 2.4. Let $\varphi_i \in D_H(\mathcal{A})$ be a preimage of $\theta_i$ by $\pi$. Assume that $\varphi_2, \ldots, \varphi_{\ell}$ are $S$-dependent, i.e.,

$$\sum_{i=2}^{\ell} g_i \varphi_i = 0$$

for some $g_i \in S$. We may assume that not all $g_i$’s are divisible by $\alpha_H$. Then this relation contradicts the $S/\alpha_H$-independency of $\theta_2, \ldots, \theta_{\ell}$. Hence they
are \( S \)-independent, and form a basis for \( D_H(A) \) by the reason of degrees and Saito’s criterion. Since \( \theta_H^E \in D(A^H, m^H) \), and \( \theta_2, \ldots, \theta_\ell \) form a basis for \( D(A^H, m^H) \), there are \( f_i \in S/\alpha_H \) such that

\[
\theta_H^E = \sum_{i=2}^{\ell} f_i \theta_i
\]

in degree \( |A| - |A^H| \). Assume that there is \( \theta \in D(A^H) \) such that \( \theta \mid \theta_H^E \), and

\[
\theta = \sum_{i=2}^{\ell} f'_i \theta_i
\]

for some \( f'_i \in S/\alpha_H \). Since the right hand side is in \( D(A^H, m^H) \), so is the left hand side. Thus \( \theta = \theta_H^E \). Also, there are no relation among \( \theta_2, \ldots, \theta_\ell \) since they are basis for \( D(A^H, m^H) \), which completes the proof. \( \square \)

The following is immediate.

**Corollary 3.1**

Assume that \( A \) is free and the \( b_2 \)-equality holds for \( H \in A \). Then \( D(A^H) \) is generated by at most \( \ell \)-elements.

**Example 3.2**

Recall the Edelman-Reiner’s arrangement \( A \) in \( \mathbb{R}^5 \) in [10] defined by

\[
(\prod_{i=1}^{5} x_i)(x_1 \pm x_2 \pm x_3 \pm x_4 \pm x_5) = 0.
\]

This consists of 21-hyperplanes. Let \( H \) be \( x_1 - x_2 - x_3 - x_4 - x_5 = 0 \). Then Edelman and Reiner showed in [10] that \( A \) is free with \( \exp(A) = (1, 5, 5, 5, 5) \) but \( \chi_0(A^H; t) = (t-4)(t-10t+26) \), hence not free. However, since \( b_2(A) = 150 \) and \( b_2(A^H) = 80 \), it is clear that the \( b_2 \)-equality holds for \( H \in A \). Hence we may apply Corollary 3.1 to show that \( A^H \) is generated by the Euler derivations and 4-derivations of degree 5, and this is a minimal generator. In particular, since the relation between them are in degree 6 = \( |A| - |A^H| \) by Theorem 1.6, we can see that \( A^H \) is a nearly free surface in the sense of [8] and [9]. Also, since the known counter examples to Orlik’s conjecture are very few, we may pose the following problem.

**Problem 3.3**

Let \( A \) be a free arrangement. Then is \( A^H \) either free or nearly free for \( H \in A \)?
Remark 3.4
To Problem 3.3, we learned a counter example from Michael DiPasquale as follows. Let $A$ be defined by
$$(x_1^2 - x_0^2)(x_2^2 - x_0^2)(x_3^2 - x_0^2)(x_4^2 - x_0^2)(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 + x_1)x_0 = 0$$
in $\mathbb{R}^5$. Let $H := \{x_0 = 0\} \in A$. Then $A$ is free with $\exp(A) = (1, 3, 3, 3, 3)$, but $A^H$ is not free with $\text{pd}_A D(A^H) = 2$, which implies that $A^H$ is not nearly free in the sense of [9].

4 Proof of the main results

Now let us go to the proof of Theorem 1.2. The following is the starting point of the proof.

Theorem 4.1
Let $A$ be a free arrangement, $H \in A$ and $\chi(A^H; t)$ divides $\chi(A; t)$. If $A^H$ is locally free, then $A^H$ is free. In particular, by Terao’s addition-deletion theorem, $A \setminus \{H\}$ is free too.

Proof. Assume that $A^H$ is not free. Let $\exp(A) = (d_1, \ldots, d_\ell)$ and $|A| - |A^H| = d_\ell$. By Corollary 1.7 (2), $D(A^H)$ is generated by $\rho(\theta_E), \pi(\theta_2), \ldots, \pi(\theta_\ell)$, where $\theta_2, \ldots, \theta_\ell$ form a basis for $D_H(A)$ of degree $\theta_i = d_i$. $\pi: D_H(A) \to D(A^H)$ the Ziegler restriction and $\rho: D(A) \to D(A^H)$ the Euler restriction. Since $A$ is free, $D(A^H, m^H)$ is free with basis $\pi(\theta_2), \ldots, \pi(\theta_\ell)$, and by Corollary 1.7 (2), $\theta_E := Q(A^H, m^H) / Q(A^H) \rho(\theta_E)$ can be expressed as a linear combination of $\pi(\theta_2), \ldots, \pi(\theta_\ell)$ as follows:

$$\theta_E^H = \sum_{i=2}^{\ell} f_i \pi(\theta_i)$$

for $f_i \in S/\alpha_H$. Note that

$$\deg \theta_E^H = |A| - 1 - |A^H| + 1 = d_\ell.$$

Assume that some $f_i \neq 0$ for some $i$ with $d_i = d_\ell$. Then it is clear that

$$\pi(\theta_2), \ldots, \pi(\theta_{i-1}), \theta_E^H, \pi(\theta_{i+1}), \ldots, \pi(\theta_\ell)$$

are $S/\alpha_H$-independent, and the sum of degrees coincides with $|m^H|$. Hence Saito’s criterion shows that they form a basis for $D(A^H, m^H)$. Moreover, the $S/\alpha_H$-linear derivations

$$\pi(\theta_2), \ldots, \pi(\theta_{i-1}), \rho(\theta_E), \pi(\theta_{i+1}), \ldots, \pi(\theta_\ell)$$
form a basis for $D(\mathcal{A}^H)$ since the sum of degrees coincide with $|m^H| - (\deg Q(\mathcal{A}^H, m^H) - \deg Q(\mathcal{A}^H)) = |\mathcal{A}^H|$. Hence $\mathcal{A}^H$ is free, a contradiction.

Assume that $f_i = 0$ if $d_i = d_\ell$. Let $\theta_1 := \theta_E$. Then by Corollary 1.7 (2), there is a resolution

$$0 \to \mathcal{O}_H(-d_\ell) \xrightarrow{f} \bigoplus_{i=1}^{\ell} \mathcal{O}_H(-d_i) \to D(\mathcal{A}^H) \to 0.$$  

Here $f = (f_1, \ldots, f_\ell) \in \bigoplus_{i=1}^{\ell} \mathcal{O}_H(d_i - d_\ell).$ By the above argument, $f_i = 0$ if $d_i \geq d_\ell$. Let $I := \{i \mid 2 \leq i \leq \ell, \ d_i < d_\ell\}$. By definition, $|I| \leq \ell - 2$, thus

$$\mathbb{P}^{\ell-2} = \mathbb{P}(H) \supset Z_f := \cap_{i \in I} \{f_i = 0\} \neq \emptyset.$$

Recall the form of the relation:

$$\frac{Q(\mathcal{A}^H, m^H)}{Q(\mathcal{A}^H)} \rho(\theta_E) = f_1 \rho(\theta_E) = \sum_{i=2}^{\ell} f_i \theta_i.$$

Since $\rho(\theta_E) \neq 0$ for any points $x \in \mathbb{P}(H) \simeq \mathbb{P}^{\ell-2}$, $f_2 = \cdots = f_\ell = 0$ at some $x \in \mathbb{P}(H)$ implies that $f_1 = 0$ at $x$. Hence $Z_f \cap \{f_1 = 0\} = Z_f \neq \emptyset$. In other words, there is a point $x \in \mathbb{P}(H)$ such that $f_i = 0$ at $x$ for all $i$. Now we show that this contradicts the assumption that $\mathcal{A}^H$ is locally free. Take $x \in Z_f$ and consider the following exact sequence:

$$\mathcal{O}_H(-d_\ell)x \otimes k_x \xrightarrow{f} \bigoplus_{i=1}^{\ell} \mathcal{O}_H(-d_i)x \otimes k_x \to D(\mathcal{A}^H)_x \otimes k_x \to 0,$$

where $k_x$ is the residue field of the stalk $\mathcal{O}_{x,H}$. The above is

$$k_x \xrightarrow{f} k^\ell_x \to k^{\ell-1}_x \to 0.$$

By the choice of $x$, $f$ is a zero map at $x$, a contradiction. \qed

**Proof of Theorem 1.2** First let us show the “if” part by induction on $\ell$. By Theorem 1.1, this is true when $\ell \leq 4$. Assume that $\ell \geq 5$. Note that $\mathcal{A}_X$ is free since $A$ is free. Then the assumption that $\chi(\mathcal{A}_X^H; t)$ divides $\chi(\mathcal{A}_X; t)$ for all $X \in \mathcal{L}(\mathcal{A}^H)$ with $2 \leq i \leq \ell - 1$ implies that $\mathcal{A}_X \setminus \{H\}$ is free by the freeness of $\mathcal{A}_X$ and the induction hypothesis. Hence $\mathcal{A}^H$ is locally free by Theorem 1.1, thus Theorem 1.1 completes the proof.

Next let us show the “only if” part. Since $A \setminus \{H\}$ is free, both $A_X$ and $(A \setminus \{H\})_X = A_X \setminus \{H\}$ are free too. Hence Theorem 1.1 says that $\mathcal{A}_X^H$ is also free, and $\chi(\mathcal{A}_X^H; t)$ divides $\chi(\mathcal{A}_X; t)$ by Theorem 2.1. \qed

Now Corollaries 1.3 and 1.4 follow immediately.

**Proof of Corollary 1.5** Induction on $|A|$. It is trivial if $|A| \leq 1$. Assume that $|A| > 1$. If $A$ has a free filtration, then at least one hyperplane can be
removable from \( A \) keeping freeness, which is combinatorial by Theorem 1.2. Then apply the induction hypothesis to the deleted free arrangements. \( \square \)

Also from the proof, the following is immediate too.

**Corollary 4.2**

Let \( A \) be free and \( H \in A \). Then \( A \setminus \{ H \} \) is free if and only if the \( b_2 \)-equality holds for \( H \in A_X \), and \(|A_X| - |A_X^H|\) is a root of \( \chi(A_X; t) \) for all \( X \in L(A^H) \).

Now let us relate the results above to the modified Orlik’s conjecture. Orlik’s conjecture asserts that the restriction \( A^H \) of a free arrangement \( A \) onto \( H \in A \) is free, which was settled negatively by Edelman and Reiner in [10]. Based on the division theorem in [2], the author posed the following modified Orlik’s conjecture:

**Problem 4.3** ([3])

Let \( A \) be a free arrangement, \( H \in A \) and \( \chi(A^H; t) \mid \chi(A; t) \). Then \( A^H \) is free.

If we replace the freeness of \( A \) by that of \( A^H \) in Problem 4.3, then the statement is true by Theorem 2.6. We cannot show Problem 4.3 without any assumption. What we can say from the main results is as follows.

**Theorem 4.4**

Problem 4.3 is combinatorial, i.e., in the assumption in Problem 4.3 whether \( A^H \) is free or not depends only on \( L(A) \).

**Corollary 4.5**

The modified Orlik’s conjecture holds true for \( \ell \leq 4 \).

The following gives a sufficient condition for the freeness of the modified Orlik’s conjecture.

**Corollary 4.6**

Assume that \( A \) is free with \( \exp(A) = (d_1, \ldots, d_\ell) \), \( H \in A \) and \( \chi(A^H; t) \) divides \( \chi(A; t) \). Then \( A^H \) is free if there is a free subarrangement \( B \subset A \) such that \( B \ni H \) and \( B^H = A^H \).

**Proof.** Assume that \( A^H \) is not free. Then by Corollary 1.7 (2), there is a basis \( \theta_2, \ldots, \theta_\ell \in D_H(A) \) such that \( \theta_2^H \) can be expressed as a linear combination of \( \pi(\theta_2), \ldots, \pi(\theta_\ell) \) in degree \( \deg \theta_2^H = |A| - |A^H| \). Since the \( b_2 \)-equality holds true between \( B \) and \( H \in B \), the same holds true in degree \( |B| - |A^H| < |A| - |A^H| \), a contradiction. \( \square \)
Remark 4.7

The counter example to the original Orlik’s conjecture in [10] says that even when the $b_2$-equation holds, and $A$ free, $A^H$ could be non-free (see Example 3.2). However, in that example, $\chi(A^H; t)$ does not divide $\chi(A; t)$, though the $b_2$-equality holds. Hence Problem 4.3 is still open.

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