A NONLINEAR FOURTH-ORDER PDE FOR MULTI-FRAME IMAGE SUPER-RESOLUTION ENHANCEMENT

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Abstract. The multiframe super-resolution (SR) techniques are considered as one of the active research fields. More precisely, the construction of the desired high resolution (HR) image with less artifacts in the SR models, which are always ill-posed problems, requires a great care. In this paper, we propose a new fourth-order equation based on a diffusive tensor that takes the benefit from the diffusion model of Perona-Malik in the flat regions and the Weickert model near boundaries with a high diffusion order. As a result, the proposed SR approach can efficiently preserve image features such as corner and texture much better with less blur near edges. The existence and uniqueness of the proposed partial differential equation (PDE) are also demonstrated in an appropriate functional space. Finally, the given experimental results show the effectiveness of the proposed PDE compared to some competitive methods in both visually and quantitatively.

1. Introduction. Multi-frame super-resolution image reconstruction is frequently used in various image-processing tools, where the aim is to reconstruct high-resolution images from their corresponding low-resolution (LR) sequences. The resolution means the number of pixels per unit area. Then an HR image has a larger pixel number compared with a LR one. The need of HR quality in many applications increases day after day, since that such images contain more information and facilitate the comprehension and interpretation. For instance, to reach high recognition rates in machine learning and computer vision tasks, detailed images are inevitable to extract critical and tiny features, such as edges, corners and contours. Moreover the SR process enhances the reconstructed image features such as reducing noise and blur, which improve the quality of the HR image. Other relevant applications use the SR techniques to address the hardware limitations and the

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very expensive price that hardware components require, namely in areas like medical diagnostics [51, 20, 18, 21], satellite imaging [27, 57] and military surveillance, etc.

The multi-frame super-resolution process required three main steps as denoted in [44]. Firstly, the motion between each two given LR is calculated using image registration techniques; then, an alignment of these images onto an HR grid is achieved using the calculated motion vectors; and, finally, a reconstitution step is required to reduce the noise and blur. After the first frequency domain approach proposed in [58], several similar techniques have been introduced and studied to improve the multi-frame SR problems [69, 25, 24, 16, 5, 8, 29]. However, these approaches are very limited by the considered image observation model, while real problems are much more sophisticated. Then, the interpolation methods were proposed with less contribution since it does not take into consideration blur and noise [1]. In order to overcome the difficulties encountered by the frequency domain and interpolation methods, the SR image reconstruction algorithms based on the spatial domain are considered. Currently, the regularization functions are widely considered in the SR context and are showing their robustness and adaptive capacity in preserving image edges. In fact, SR reconstruction is always ill-conditioned and considered as an ill-posed inverse problem. Since the existence of the solutions of the inverse problem is, in some cases, not assured or the solution is not unique or is unstable, some prior functions are then employed to obtain a good approximation of the original image. This technique imposes some constraints on the solution space, which can help the final algorithm to remove different artifacts in the restored image and also improve the convergence rate. The SR algorithms are formulated as an optimization problem with a likelihood function which measures the difference between the LR frames and the obtained HR one, and a regularization term that impose some prior knowledge on the desired HR image. One of the widely-used prior functions was the Total Variation (TV) regularization [52, 42] which gives promising results. A simplest choice of this function in the SR process was the Bilateral Total Variation (BTV) [17], which is a generalized discrete form of TV regularization. This prior function is constructed by replacing every pixel with a weighted average of its neighborhood. Even if the BTV term preserves edges and smooth areas, it produces artificial edges in the flat surfaces. A more robust form of the BTV term was proposed in [68], which is called by the adaptive Bilateral Edge-Preserving (BEP) norm. This regularization stops the diffusion process of the BTV term in smooth areas of the image, however, it doesn’t preserve texture well. Just before, other derived terms of the BTV regularization are also presented such as the locally adaptive BTV (LABTV) which brings some consistency of the gradient between the constructed HR image and the LR image [33]. Based on the advantages of the TV and BTV norms, a combined term was also introduced in [31] which increase the quality of the restoration step of the SR process. Before that, a convolutional model with a constrained TV regularization was proposed in [39]. In this work, the authors introduced the Bregman algorithm as an iterative optimization approach to resolve the SR optimization problem. To preserve texture and fine details, a fractional order TV regularization was introduced with fine results [50], which reduce some weaknesses of the classical TV model. A more robust SR model was proposed using a spatially weighted TV model [65], where the authors introduce a spatial information indicator that identifies the spatial properties of the image region based on the difference curvature coefficient. This can provide the necessary information to preserve sharp edges and
avoids smoothing flat areas. After that, another adaptive TV-driven SR approach was formulated which gives promising results compared with those obtained by the classical TV model [67]. Other prior functions are also studied, such as the Gaussian Markov Random Field (MRF) SR [13, 28, 49], the Leclerc model [53], and the adaptive space-time regularization method [54] for multiple video sequence super-resolution reconstruction. Recently, two regularization terms were introduced for SR image reconstruction, namely, hybrid regularizations [56, 70, 37] and time regularization [14] with pleasant results. Other relevant SR approaches were proposed in [48], where the authors used the Bregman iteration for rapid SR image reconstruction with TV regularization and morphological regularization respectively. In the same principle, two new types of regularization terms were introduced in the SR context [19], based on a local weighted anisotropy regularization and successive regularization toward iteration process using the Bregman iteration, which reduces efficiently noise but not tackle the blur effect. A more recent SR approach based on Huber-Norm using Bregman distances was proposed in [32] with more consistency against contrast loss while strong edges and contours are well preserved with less blur.

On another side, many regularizations based on the non-local feature similarity were proposed to preserve the texture of the reconstructed HR image. One of the most successful models was the nonlocal means proposed in [47], where each pixel is estimated through all the similar patches in a given region, which avoids the locality constraint used in the local models. The Non-Local Total Variation (NLTV) [11] is also proposed in the SR context [46] but suffers from the creation of artificial edges. Recently, motivated by the advantages of the BTV model and NLTV model, an adaptive non-locale edge-preserving image prior model was proposed in [73] which preserve edges and also avoid blur creation. A more recent nonlocal Laplace regularization was proposed and have shown very promising results [30].

Lately, the Euler-Lagrange equation associated to an adaptive diffusion-based regularizer was treated to preserve edges [38]. Even if this approach preserves image features, it suffers from the blurring effect. In order to avoid this effect, El Mourabit et al. introduced a new partial differential equation [15] that performs the coherence-enhancing property and avoids blur. However, when the blur and noise levels are high, the restored image still contains some artefacts.

The main contribution of this paper consists of reducing the apparition of the noise and blur in the obtained HR image with respect to the methods discussed above. To increase the smoothness near edges, we use a nonlinear fourth-order PDE derived from the proposed one in [15] using Weickert filter [60]. In fact, the weakness of the second-order partial differential equations resulting in the appearance to a class of high-order diffusion models [40, 63, 62, 26, 22] that in general outperform the second order ones. Indeed, to restore corners, curvatures and for matching edges across large distances, more regularity is needed in the used PDE. Moreover, the supplementary information on the level lines directions, which is a characterization of the boundary conditions, restores and improves the obtained HR image. While, in the flat regions, the high-order of the operator reduces the noise faster than any second-order one. In addition, the use of the nonlinearity with the Weickert process takes into account the coherence-enhancing property near corners without destroying sharp edges. This approach takes into consideration flat regions and sharp edges avoiding the staircasing effect and blur.
We note that this paper is an improvement of the previous work [15]. There are many differences stated in the paper concerning the order of the proposed equation and also the choices of the function \(f^+\) and \(f^-\). In summary, the main contributions of the paper can be stated as follows:

- The use of the high order PDE instead of the second order equation presented in [15] to avoid much better the staircasing.
- The existence of a weak solution using a different compactness theorem.
- The elaboration of the estimation of the solution with precise constants.
- The computation on the eigenvalues without using the exponential form.

This paper is organized as follows. In section 2, we present the general multi-frame super-resolution task. Then, we recall some used regularization terms in the SR context. After, we introduce the proposed fourth-order PDE and we check the existence of a unique solution using Schauder fixed point theorem. Finally, In section 3, we present some real and synthetic results, while we give a comparison with some available methods.

2. The main multi-frame SR problem.

2.1. The SR problem formulation. The process by which the observed LR images have been obtained in the SR context is always the same. In general, the image acquisition process takes into consideration an inevitable set of degrading factors, such as blur, under-sampling, motion, and noise. We consider in the SR process that the degradation factors during image acquisition involves warping, blurring, down-sampling, and additive noise. We assume that all frames are taken under the same environmental conditions using the same sensor. The relationship between the HR image \(X\) (where \(X \in L^2(\Omega)\)) and the corresponding LR ones \(Y_k\) (\(Y_k \in L^2(\Omega)\), for \(k = 1, \ldots, n\)), is described by the following model

\[
Y_k = W F_k H X + V_k \quad \forall k = 1, 2, \ldots, n,
\]

where

- \(n\): the number of LR frames.
- \(H\): the blurring operator which supposed to be a Gaussian filer.
- \(W\): represents the decimation operator.
- \(F_k\): is geometric regular warp operators, representing non-parametric transformations that differs in all frames.
- \(V_k\): represents the additive noise for each image.

Under the above assumptions, we can deduce that the operator \(WF_k H \in L^\infty(\Omega)\).

The aim of the multiframe super-resolution process is to reconstruct an ideal HR image from a set of warped, blurred, noisy, and under-sampled frames: \(Y_k\), \(k = 1, \ldots, m\). Since the super-resolution model presented in (1) is ill-conditioned, SR is considered as an ill-posed inverse problem (since the solution is not unique). Based on maximum a posteriori (MAP) theory [1], the SR problem can be formulated to a minimization problem with some prior knowledge about the image \(X\) in a Bayesian framework. The estimation of HR image is given through the following minimization problem

\[
\hat{X} = \arg \min_X \left\{ \frac{1}{2m} \sum_{k=1}^{m} \|WF_k H X - Y_k\|_2^2 + \delta R(X) \right\},
\]
where $R(X)$ is a regularization term representing a prior knowledge on $X$. As discussed above, there is a great number of proposed regularizations in the multi-frame super-resolution context with different properties. The resolution of the problem (2) can be obtained through optimization methods such as in [32, 39], or using the Euler-Lagrange equation associated to the minimization problem (2). The resolution of this equation is related to the gradient descent optimization method. Recently, Maiseli et al. propose in [38] the resolution of the evolution equation associated to the Euler-Lagrange form defined as

$$\frac{\partial X}{\partial t} = \frac{1}{m} \sum_{k=1}^{m} (WF_k H)^T (WF_k H X - Y_k) + \text{div} \left( \frac{2 + |\nabla X|}{1 + \left( |\nabla X| \beta \right)^2} \nabla X \right). \quad (3)$$

This equation is related to the regularization term defined by

$$R(X) = \int_{\Omega} \beta |\nabla X| + \beta^2 \ln \left( 1 + \left( \frac{|\nabla X|}{\beta} \right)^2 \right) - \beta^2 \arctan \left( \frac{|\nabla X|}{\beta} \right) \, d\Omega, \quad (4)$$

where $\beta$ is a shape-defining tuning constant. The purpose of this choice is to use the edge preserving property of the total variation [52] and also the characterization of the Perona-Malik diffusion along the contours [45], taking into account the effect smoothing of backward diffusion anisotropic proposed in [64]. The obtained evolution PDE can preserve edges and details but it suffers from the blurring effect, especially in homogeneous regions and around corners. To address this problem, El Mourabit et al. [15], propose a new evolution PDE that performs the coherence-enhancing property and avoids blur. This approach uses the best of Perona-Malik diffusion process in flat regions [45], and the smoothness effect of the Weickert filter near boundaries and corners [60]. This PDE is proposed with Neumann boundary conditions and given as follows

$$\begin{cases} \frac{\partial X}{\partial t}(t, x) - \nabla (D(J_\rho(\nabla X_\sigma)) \nabla X) - \frac{1}{m} \sum_{k=1}^{m} (WF_k H)^T (WF_k H X - Y_k) = 0 \\
\text{on } [0, T] \times \partial \Omega, \\
\langle D(J_\rho(\nabla X_\sigma)) \nabla X, n \rangle = 0 \quad \text{on } [0, T] \times \partial \Omega, \\
X(t, x) = 0, \quad \text{on } (0, T) \times \partial \Omega, \\
X(0, x) = X_0, \quad \text{on } \Omega, \end{cases} \quad (5)$$

where $X_0$ is obtained using a bicubic interpolation of the LR image $Y_1$. $D$ is an anisotropic diffusion tensor and $J_\rho$ is the structure tensor defined by

$$J_\rho(\nabla X_\sigma) = K_\rho * (\nabla X_\sigma \otimes \nabla X_\sigma) = K_\rho * (\nabla K_\sigma \nabla X_\sigma) = K_\rho * (\nabla K_\sigma \otimes X \nabla K_\sigma), \quad (6)$$

where $X_\sigma$ is constructed using a convolution of $X$ with a Gaussian kernel. While $K_\rho$ and $K_\sigma$ represent two Gaussian convolution kernels such as $K_\sigma(x) = \frac{1}{2\pi \sigma^2} \exp(-\frac{x^2}{2\sigma^2})$. The function $D$ is calculated using the tensor $J_\rho$, eigenvalues and the eigenvectors as follows

$$D := f_+(\lambda_+, \lambda_-) \theta_+ \theta_+^T + f_- (\lambda_+, \lambda_-) \theta_- \theta_-^T, \quad (7)$$

where $\lambda_{+-}$ and $\theta_{+-}$ are respectively the eigenvalues and the eigenvectors of the tensor structure $J_\rho$, the eigenvalues $\lambda_{+-}$ are calculated as

$$\lambda_{+-} = \frac{1}{2} \left( \text{trace}(J_\rho) \pm \sqrt{\text{trace}^2(J_\rho) - 4 \text{det}(J_\rho)} \right). \quad (8)$$
While the functions $f_+$ and $f_-$ indicate the isotropic or anisotropic behavior of the smoothing on the image regions. For that, these functions are chosen carefully in order to satisfy the different smoothing constraints. It is worth noting that there are several choices of these functions in the literature. For example the classical model of Perona-Malik consists in taking the following choices of $f_+$ and $f_-$ such that

$$f_+(\lambda_+, \lambda_-) = f_-(\lambda_+, \lambda_-) = \exp \left( -\frac{(\lambda_++\lambda_-)^2}{K} \right),$$

where $K$ is a strictly positive constant. Taking these functions, the diffusion matrix $D$ has no specific direction and stops the diffusion in the vicinity of the edges and corners before smoothing them, which characterize the Perona-Malik equation. In order to improve the behavior of this model, Weickert proposes a new model based on the following functions

$$f_+(\lambda_+, \lambda_-) = \begin{cases} \alpha + (1-\alpha) \exp(-\frac{K}{\lambda_+ - \lambda_-}), & \text{if } \lambda_+ \neq \lambda_- \\ \alpha, & \text{else} \end{cases}$$

$$f_-(\lambda_+, \lambda_-) = \alpha.$$  

Based on this choice, a diffusion has been associated whose main direction is the vector $\theta_+$. This makes it possible to reinforce the coherence of the edges and features of the image. It is also noted that the Weickert model is characterized by diffusion coefficients $f_+$ and also $f_-$. This leads to a disturbance of the corners and singularities of the image. To overcome this disadvantage, it is necessary to look for new diffusion coefficients to smooth an image with its details (contours and corners) without destroying them. For this, the authors in [15] consider the behavior of the model of Weickert according to the direction $\theta_+$, then they change the diffusion coefficient $f_-$ in order to take into consideration the direction $\theta_-$. The proposed coefficients take into account the diffusion in the vicinity of the contours and corners where the eigenvectors $\lambda_+$ and $\lambda_-$ are very high. This choice is proposed as follows:

$$f_+(\lambda_+, \lambda_-) = \exp(-\frac{\lambda_+}{k_1}),$$

$$f_-(\lambda_+, \lambda_-) = \exp(-\frac{\lambda_-}{k_2})(1 - \exp(-\frac{\lambda_+}{k_1})).$$

where $k_1$ and $k_2$ are two threshold defining the diffusion with respect to the directions $\theta_+$ and $\theta_-$ respectively. Even if this choice permits a considerable reduction of noise, especially in uniform zones and smooth edges, the proposed PDE in (5) needs more regularity to better enhance image details with a different choice of $f_+$ and $f_-$ to keep safe the contours. For this reason, we propose a fourth-order evolution PDE with more smoothness effect

$$\frac{\partial X}{\partial t}(t, x) - \Delta (D(J_\rho(\nabla X_\sigma))\Delta X) - \frac{1}{m} \sum_{k=1}^{m} (WF_k H)^T (WF_k H X - Y_k) = 0$$

on $[0, T] \times \Omega$, 

$$\langle D(J_\rho(\nabla X_\sigma))\nabla X, n \rangle = 0 \text{ on } [0, T] \times \partial \Omega,$$

$X(t, x) = 0, \text{ on } (0, T) \times \partial \Omega,$

$X(0, x) = X_0, \text{ on } \Omega,$

where the function $f_+$ and $f_-$ are chosen with less regularity to preserve sharp edges and singularities. For that, we construct these functions thoroughly as follows
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\[
\begin{align*}
  f_+ (\lambda_+, \lambda_-) &= \sqrt{\frac{1}{1 + \left(\frac{\lambda_+}{k_1}\right)^2}}, \\
  f_- (\lambda_+, \lambda_-) &= \sqrt{\frac{1}{1 + \left(\frac{\lambda_-}{k_2}\right)^2}}.
\end{align*}
\]

(13)

Using this choice, we take the benefit of the Weickert diffusion oriented in the direction of \(\theta_+\) while we take into account a slightly diffusion in direction \(\theta_-\) with respect to the \(\lambda_+\) value. As a result, the diffusion near a contour or a corner is privileged where the values of \(\lambda_+\) or \(\lambda_-\) are large. We can see now that by this choice, if \(\lambda_+\) is too high, \(f_+\) will tend to zero which promotes the diffusion in the direction of \(\theta_-\). However, if the value of \(\lambda_-\) is high the diffusion is oriented in the direction of \(\theta_+\). When the two values \(\lambda_+\) and \(\lambda_-\) are high, which is the case of corner region, the two threshold \(k_1\) and \(k_2\) will be carefully selected to identify the diffusion along the respective directions \(\theta_+\) and \(\theta_-\).

Before resolving the equation (12), we have to check the existence and uniqueness of a solution in an appropriate framework. Indeed, since the SR process is an ill-posed problem, the existence of a unique solution must be assured. There are numerous works which have treated the existence and uniqueness of a (state and evolution) fourth-order equation, see for examples \([12, 71, 23, 34, 7, 6, 3]\). Recently, a theoretical study was investigated for an adaptive fourth-order partial differential equation in the denoising case with a rigorous space choice \([72]\). In our case the situation is different since the nonlinearity is related to a tensor diffusion term with a second member depending on the solution \(X\). For that we have to impose some regularity assumptions on the diffusion operator \(D\).

2.2. Assumptions on data. In order to establish the existence of the solution \(X\) associated to the problem (12), we introduce the following set of assumptions.

**A1:** - The tensors \(D\) is included in \(C^\infty (\mathbb{R})\), coercive and positive-definite matrix.

**A2:** - The initial condition \(X_0\) in \(L^\infty (\Omega)\). For more details and definition for such space and also the other used spaces in the rest of this paper, we suggest to see \([9, 4]\).

2.3. Variational formulation and a priori estimates. In this subsection, we give the variational formulation of the problem (12) and a priori estimates of the solution \(X\). The variational formulation of the problem (12) is stated as follows

\[
\begin{align*}
  \text{Find } X \in L^2 (0, T; H_0^2 (\Omega)) \text{ and } \frac{\partial X}{\partial t} \in L^2 (0, T; H^{-2} (\Omega)) \text{ such that } \\
  \langle \frac{\partial X}{\partial t}, \phi \rangle_{H^{-2} (\Omega), H_0^2 (\Omega)} + \int_\Omega D(J_\phi (\nabla X)) \Delta X \Delta \phi \, dx \\
  - \int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^t (WF_i H) X \phi \, dx \\
  - \int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^t Y_i \phi \, dx, \quad \forall \phi \in L^2 (0, T; H_0^2 (\Omega)).
\end{align*}
\]

(14)

In order to show the existence of a solution to the problem (14), we use Schauder fixed point theorem \([10]\). For this reason, we need to show a priori estimate of the solution to the problem (14) in \(L^2 (0, T; H_0^2 (\Omega))\).

**Lemma 2.1.** Assume that assumptions A1–A2 are satisfied, then there exists three constants \(C_i > 0\) for \(i = 1, 2, 3\) depending on \(X_0\) and \(\nu\), such that the weak solution of the problem (14) satisfies the following a priori estimations

\[
\|X\|_{L^\infty (0, T; L^2 (\Omega))} \leq C_1
\]

(15)
by using the Young and Grönwall inequalities, we obtain
\[ \|X\|_{L^2(0,T;H^2_0(\Omega))} \leq C_2. \]  
(16)
\[ \|\frac{\partial X}{\partial t}\|_{L^2(0,T;H^{-2}(\Omega))} \leq C_3. \]  
(17)

**Proof.** In the variational formulation (14), by taking \( \phi = X \) and integrating on \( t \in (0, \tau) \) with \( \tau \in (0, T] \), we get
\[
\int_0^\tau \frac{1}{2} \|X(t)\|^2_{L^2(\Omega)} dt + \int_0^\tau \int_\Omega D(J_\sigma(\nabla X)) \Delta X \Delta X dx dt \\
\leq \int_0^\tau \int_\Omega \nu |X|^2 dx dt + \frac{1}{2} \|X(0)\|^2_{L^2(\Omega)} + \frac{1}{m} \sum_{i=1}^m (WF_iH)^i (WF_iH) XX dx dt.
\]  
(18)
Using the Hölder inequality and the fact that
\[
\frac{1}{m} \sum_{i=1}^m (WF_iH)^i (WF_iH) XX \leq \nu |X|^2,
\]  
(19)
then, we have
\[
\frac{1}{2} \|X(\tau)\|^2_{L^2(\Omega)} + \frac{1}{m} \sum_{i=1}^m (WF_iH)^i (WF_iH) XX dx dt \\
\leq \int_0^\tau \int_\Omega |\Delta X|^2 dx dt + \frac{1}{2} \|X(0)\|^2_{L^2(\Omega)} + \frac{1}{m} \sum_{i=1}^m (WF_iH)^i (WF_iH) XX dx dt.
\]  
(20)
In the other hand, we have that \( D(J_\sigma) \) is coercive and the constant of coercivity is \( \alpha \), then
\[
\frac{1}{2} \|X(\tau)\|^2_{L^2(\Omega)} + \alpha \int_0^\tau \int_\Omega |\Delta X|^2 dx dt \leq \int_0^\tau \int_\Omega \nu |X|^2 dx dt + \frac{1}{2} \|X(0)\|^2_{L^2(\Omega)} + \frac{1}{m} \sum_{i=1}^m (WF_iH)^i (WF_iH) XX dx dt,
\]  
(21)
by using the Young and Grönwall inequalities, we obtain
\[ \|X\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1, \]  
(22)
with
\[
C_1 = \sqrt{\left( \frac{1}{m} \sum_{i=1}^m (WF_iH)^i (WF_iH) XX dx dt \right)^2 + \|X(0)\|^2_{L^2(\Omega)}^2 \exp(2\nu + 1)},
\]
and we have also
\[ \|X\|_{L^2(0,T;H^2_0(\Omega))} \leq C_2, \]  
(23)
with
\[
C_2 = \sqrt{C_1^2 T + \frac{1}{2} \|X(0)\|^2_{L^2(\Omega)} + \frac{1}{m} \sum_{i=1}^m (WF_iH)^i (WF_iH) XX dx dt \exp(2\nu + 1).}
\]
Let’s prove now the estimation of \( \frac{\partial X}{\partial t} \). From the variational formulation (14), we have
Integrating on $t \in (0, \tau]$, $\forall \tau \in (0, T]$, using the Hölder inequality and since we have $D(J_\tau(\nabla X))$ is bounded in $L^\infty(O, T; C^\infty(\Omega))$ (i.e $\|D(J_\tau(\nabla X))\|_{L^\infty(0, \tau; C^\infty(\Omega))} \leq C_4$), for every $X$ bounded in $L^\infty(O, T, L^2(\Omega))$, we find

$$\|\frac{\partial X}{\partial t}\|_{L^2(0, T, H^{-2}(\Omega))} \leq \left( \int_0^\tau \|D(J_\tau(\nabla X))\|_{C^\infty(\Omega)} \|\Delta X\|_{L^2(\Omega)} \right. + C \frac{1}{m} \sum_{i=1}^m \| (WF_i H)^t (WF_i H) \|_{L^\infty(0, \tau; L^\infty(\Omega))} \|X\|_{L^2(\Omega)} \left. + C \frac{1}{m} \sum_{i=1}^m \| (WF_i H)^t \|_{L^\infty(0, \tau; L^\infty(\Omega))} \|Y_i\|_{L^2(0, T, L^2(\Omega))} \right)^{\frac{1}{2}},$$

then, we have

$$\|\frac{\partial X}{\partial t}\|_{L^2(0, T, H^{-2}(\Omega))} \leq CC_4 \|\Delta X\|_{L^2(0, T, L^2(\Omega))}$$

$$+ C \frac{1}{m} \sum_{i=1}^m \| (WF_i H)^t (WF_i H) \|_{L^\infty(0, \tau; L^\infty(\Omega))} \|X\|_{L^2(0, T, L^2(\Omega))}$$

finally, we obtain

$$\|\frac{\partial X}{\partial t}\|_{L^2(0, T, H^{-2}(\Omega))} \leq CC_4 C_2$$

$$+ C \frac{1}{m} \sum_{i=1}^m \| (WF_i H)^t (WF_i H) \|_{L^\infty(0, \tau; L^\infty(\Omega))} T C_1$$

$$+ C \frac{1}{m} \sum_{i=1}^m \| (WF_i H)^t \|_{L^\infty(0, \tau; L^\infty(\Omega))} \|Y_i\|_{L^2(0, T, L^2(\Omega))},$$

This achieves the proof. \hfill \Box

2.4. Existence and uniqueness of the proposed PDE. In this subsection, based on the above estimates, we will show the existence and uniqueness of the solution for the problem (14). The main difficulties encountered in this study comes from the strong nonlinearity of the conductivity part of the equation (12). To overcome these difficulties, we opt for Schauder fixed point, which is more powerful and general than the classical fixed point theorems [43].

**Theorem 2.2.** The problem (14) admits a unique solution in $L^2(0, T; H^1_0(\Omega))$.

**Proof.** Before starting to show the existence of a solution to the problem (14), we note first that for all fixed $X \in L^2(0, T; H^1_0(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, the variational problem

$$\begin{aligned}
&\text{with } X \in L^2(0, T; H^1_0(\Omega)) \text{ and } \frac{\partial X}{\partial t} \in L^2(0, T; H^{-2}(\Omega)) \text{ such that } \\
&\frac{\partial X}{\partial t} = J_\tau(\nabla X)^t \Delta X \phi dx + \int_\Omega D(J_\tau(\nabla X))^t \Delta X \phi dx + \int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^t (WF_i H) X \phi dx \\
&\left. \begin{array}{l}
\int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^t Y_i \phi dx, \quad \forall \phi \in L^2(0, T; H^1_0(\Omega)) \\
\end{array}\right\}
\end{aligned}$$

and $\frac{\partial X}{\partial t} \in L^2(0, T; H^{-2}(\Omega))$, admits a unique solution (thanks to [36]) $X \in L^2(0, T; H^1_0(\Omega)) \rightarrow L^2(0, T; H^1_0(\Omega))$. Let's now define the following operator $G$ by

$$\begin{aligned}
G : V &\rightarrow V \\
X &\rightarrow X,
\end{aligned}$$
where $X$ is the unique solution of (28) and

$$V = \{ U \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \mid \| U \|_{L^\infty(0, T; L^2(\Omega))} \leq C_1, \\
\| U \|_{L^2(0, T; H_0^2(\Omega))} \leq C_2 \text{ and } \| \frac{\partial U}{\partial t} \|_{L^2(0, T; H^{-2}(\Omega))} \leq C_3 \}.$$  

We have then to prove that $G$ admits a unique fixed point $X$, which is the solution of (14). To prove the existence of a fixed point $G$, we have to show that $G$ is compact and continuous.

The proof is done in the following steps:

- First, using the same technics as in the proof of Lemma 2.1, it’s easy to prove that for all $\mathbf{x} \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, we have $X \in L^2(0, T, H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$. Thus the operator $G$ in now well defined in $L^2(0, T, H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$.

- In order to prove the continuity of the map $G$, let us consider the sequence $\mathbf{x}_n$ in $L^2(0, T, H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, such that

$$\mathbf{x}_n = G(\mathbf{x}_n) \underset{n \to \infty}{\rightarrow} \mathbf{x} \quad \text{in} \quad L^2(0, T, H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)),$$

we have to prove that

$$X_n = G(\mathbf{x}_n) \underset{n \to \infty}{\rightarrow} X = G(\mathbf{x}) \quad \text{in} \quad L^2(0, T, H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).$$

Let $X_n$ (respectively $X$) be the unique solution associated to $\mathbf{x}_n$ (respectively $\mathbf{x}$) for the formulation (28). Which means

$$\begin{cases}
\left( \frac{\partial X_n}{\partial t}, \phi \right)_{H^{-2}(\Omega), H_0^2(\Omega)} + \int_\Omega D(J_\sigma(\nabla X_n)) \Delta X_n \Delta \phi \, dx \\
+ \int_\Omega -\frac{1}{m} \sum_{i=1}^m (WF_i H)^t (WF_i H) X \phi \, dx \\
= \int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^t Y_i \phi \, dx, \quad \forall \phi \in L^2(0, T; H_0^2(\Omega)).
\end{cases}$$

Using equations (33) and (34), and by taking $\phi = X_n - X$, we find

$$\begin{cases}
\left( \frac{\partial X_n - X}{\partial t}, X_n - X \right)_{H^{-2}(\Omega), H_0^2(\Omega)} + \int_\Omega D(J_\sigma(\nabla X_n)) \Delta (X_n - X) \Delta (X_n - X) \, dx \\
= \int_\Omega (D(J_\sigma(\nabla X)) - D(J_\sigma(\nabla X_n))) \Delta X_n \Delta (X_n - X) \, dx \\
+ \int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^t (WF_i H) (X_n - X)^2 \, dx.
\end{cases}$$

Integrating on $t \in (0, \tau], \forall \tau \in (0, T]$, using the fact that $D$ is coercive, the Hölder inequality and also the following inequality

$$\| D(J_\sigma(\nabla X_n)) - D(J_\sigma(\nabla X)) \|_{L^\infty(0, \tau; L^2(\Omega))} \leq C \| X_n - X \|_{L^\infty(0, \tau; L^2(\Omega))},$$

we find

$$\begin{aligned}
\frac{1}{2} \| X_n(t) - X(t) \|_{L^2(\Omega)}^2 + \alpha \| \Delta (X_n - X) \|_{L^2(0, \tau; L^2(\Omega))}^2 &\leq C \| X_n - X \|_{L^2(0, \tau; L^2(\Omega))}^2 \\
+ C \| X_n - X \|_{L^\infty(0, \tau; L^2(\Omega))} \| \Delta X_n \|_{L^2(0, \tau; L^2(\Omega))} \| \Delta (X_n - X) \|_{L^2(0, \tau; L^2(\Omega))}.
\end{aligned}$$

Using the Young and Grönwall inequalities and since we have

$$\mathbf{x}_n \longrightarrow \mathbf{x} \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)),$$
we find
\[ X_n \to X \text{ in } L^2(0, T, H^0_0(\Omega)) \cap L^\infty(0, T, L^2(\Omega)). \] (39)
By the equivalence between the norms \( \| \cdot \|_{H^0_0(\Omega)} \) and \( \| \cdot \|_{H^2(\Omega)} \), we find that the operator \( G \) is continuous.

- Now, let us show that \( G \) is compact. For this, let \( (X_n)_n \) be a bounded sequence in \( L^2(0, T; H^0_0(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \), and let \( X_n = G(X_n) \) be the unique solution of (28) associated to \( X_n \). Indeed by taking \( \phi = X_n \) and integrating in \( t \) and using the same proof as in Lemma 2.1, we have
\[
\frac{1}{2} \| X_n(t) \|^2_{L^2(\Omega)} + \alpha \| \Delta X_n \|^2_{L^2(0, \tau; L^2(\Omega))} \leq \frac{1}{2} C_1 + \alpha C_2, \forall \tau \in [0, T],
\] (40)
and we have also
\[
\| \frac{\partial X_n}{\partial t} \|^2_{L^2(0, \tau; H^{-2}(\Omega))} \leq C_3^2.
\] (41)

Now, thanks to the compact embedding of \( H^0_0(\Omega) \) in \( H^1_0(\Omega) \) and the continuous embedding of \( H^1_0(\Omega) \) in \( H^{-2}(\Omega) \), from Aubin-Lions lemma \([35]\), we can extract a subsequence denoted again \( (X_n)_n \) which converges in \( L^2(0, T; H^1_0(\Omega)) \). Which implies that the operator \( G \) is compact.

The continuity and the compactness of \( G \) in \( V \), implies the existence of the fixed point.

To prove the uniqueness of the solution, we assume that \( X_1 \) and \( X_2 \) are two distinct solutions of the problem (14). By subtracting the weak formulations associated to the solutions \( X_1 \) and \( X_2 \), we obtain
\[
\begin{align*}
&\langle \frac{\partial X_1}{\partial t} - \frac{\partial X_2}{\partial t}, \phi \rangle_{H^{-2}(\Omega), H^0_0(\Omega)} \\
&\quad + \int_\Omega (D(J_\sigma(\nabla X_1)) \Delta X_1 - D(J_\sigma(\nabla X_2)) \Delta X_2) \Delta \phi \, dx \\
&\quad + \int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^i (WF_i H)(X_2 - X_1) \phi \, dx = 0,
\end{align*}
\] (42)

taking \( \phi = X_1 - X_2 \), we get
\[
\begin{align*}
&\langle \frac{\partial X_1}{\partial t} - \frac{\partial X_2}{\partial t}, X_1 - X_2 \rangle_{H^{-2}(\Omega), H^0_0(\Omega)} \\
&\quad + \int_\Omega (D(J_\sigma(\nabla X_1)) \Delta X_1 - D(J_\sigma(\nabla X_2)) \Delta X_2) \Delta (X_1 - X_2) \, dx \\
&\quad + \int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^i (WF_i H)(X_2 - X_1)(X_1 - X_2) \, dx = 0,
\end{align*}
\] (43)

Then, we have
\[
\begin{align*}
&\langle \frac{\partial X_1}{\partial t} - \frac{\partial X_2}{\partial t}, X_1 - X_2 \rangle_{H^{-2}(\Omega), H^0_0(\Omega)} \\
&\quad + \int_\Omega D(J_\sigma(\nabla X_1)) \Delta (X_1 - X_2) \Delta (X_1 - X_2) \, dx \\
&\quad = \int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^i (WF_i H)(X_2 - X_1)^2 \, dx \\
&\quad + \int_\Omega (D(J_\sigma(\nabla X_1)) - D(J_\sigma(\nabla X_2))) \Delta X_2 \Delta (X_2 - X_1),
\end{align*}
\] (44)
by using the fact that $D$ is bounded and Hölder inequality, we find
\[
\frac{d}{dt} \|X_1(t) - X_2(t)\|^2_{L^2(\Omega)} + \alpha \int_\Omega \Delta (X_1 - X_2)^2 dx \\
\leq \|D(J_\sigma(\nabla X_1(t))) - D(J_\sigma(\nabla X_2(t)))\|_{L^\infty(\Omega)} \|\Delta X_2(t)\|_{L^2(\Omega)} \\
+ \int_\Omega \frac{1}{m} \sum_{i=1}^m (WF_i H)^t (WF_i H)(X_2 - X_1)^2 dx.
\]
(45)

Since the operator $D(J_\sigma)$ is smooth enough, we have
\[
\|D(J_\sigma(\nabla X_1(t))) - D(J_\sigma(\nabla X_2(t)))\|_{L^\infty(\Omega)} \\
\leq C \|X_1(t) - X_2(t)\|_{L^2(\Omega)}, \forall t \in [0, T],
\]
(46)

and
\[
\frac{1}{m} \sum_{i=1}^m (WF_i H)^t (WF_i H)(X_2 - X_1)^2 dx \leq \nu(X_2 - X_1)^2.
\]
(47)

Then
\[
\frac{d}{dt} \|X_1(t) - X_2(t)\|^2_{L^2(\Omega)} + \alpha \int_\Omega \Delta (X_1 - X_2)^2 dx \\
\leq C \|X_1(t) - X_2(t)\|_{L^2(\Omega)} \|\Delta X_2(t)\|_{L^2(\Omega)} \|\Delta (X_1(t) - X_2(t))\|_{L^2(\Omega)} \\
+ \int_\Omega \nu(X_2 - X_1)^2 dx.
\]
(48)

Using the Young inequality, we find
\[
\frac{d}{dt} \|X_1(t) - X_2(t)\|^2_{L^2(\Omega)} + \left(\alpha - \frac{C\epsilon}{2}\right) \int_\Omega \Delta (X_1 - X_2)^2 dx \\
\leq C \|X_1(t) - X_2(t)\|^2_{L^2(\Omega)} \|\Delta X_2(t)\|^2_{L^2(\Omega)} + \nu \|X_2(t) - X_1(t)\|^2_{L^2(\Omega)}.
\]
(49)

by taking a very small value of $\epsilon$ and by using the Grönwall inequality, we conclude that
\[
\|X_1 - X_2\|^2_{L^\infty(0,T;L^2(\Omega))} \leq 0.
\]
(50)

Finally, we have the uniqueness of the solution.

Since we have assured the existence and uniqueness of the solution to the proposed PDE (12), the convergence to the desired solution using a discretization scheme is then verified. Indeed, we solve the PDE using a fixed point technique and an explicit finite difference scheme [60]. We will not detail this part, since it is similar to previous works [72, 22, 63, 2], where an appropriate scheme is used to discretize the proposed PDE. In other hand, to compute the warping matrix $F_k$ related to the registration step of the SR technique, we used a sub-pixel accurate optical flow method [61].

3. **Numerical results.** In this section, we present both simulated and real results to perform the proposed SR approach. we compare our method with some available and competitive multi-frame SR methods, such as the TV SR with Bregman iteration [39], the BTV regularization [17], SR with the spatial adaptive prior model proposed in [67] and also SR with the nonlinear PDE [15]. We have tested the proposed multi-frame SR method on a large number of images, we show in the following experiments only seven tested images presented in the Fig.1. In the sim-
ulated experiment, we construct the LR frames through the original HR images presented in the Fig. 1. To obtain these sequences, the original images are firstly blurred by a Gaussian low-pass filter with a $3 \times 5$ and standard deviation of 2 which is presented in the image SR model (1) by the operator $H$. Then a translation operation is applied to each frame which is designed by the operators $F_k$. After that, the blurred and transformed frames are down-sampled vertically and horizontally by a factor of $r = 4$ which is represented by the operator $W$. Finally, a Gaussian noise was added to each down-sampled frames with high intensity such as $\sigma_{\text{noise}} = 30$ (the first six images of the obtained sequences for each image are presented in Fig. 2). Afterwards, we try to recover the HR image by applying various multi-frame SR method discussed above and also the proposed one. The obtained images are represented in Figs. 1 to 5. In addition, to better perform the proposed SR with respect to the others, a quantitative evaluation is used. In
Table 1. The PSNR table

| Image | BTV reg. [17] | TV reg. [39] | Adaptive reg. [67] | GDP SR [73] | The PDE in [15] | Our Method |
|-------|---------------|--------------|--------------------|-------------|-----------------|-------------|
| Build | 27.00         | 28.10        | 28.93              | 28.50       | 29.66           | 31.08       |
| Lion  | 26.90         | 26.95        | 27.42              | 27.88       | 28.33           | 29.51       |
| Penguin | 31.77       | 31.50        | 31.85              | 32.09       | 33.00           | 34.02       |
| Man   | 29.55         | 29.03        | 29.80              | 30.40       | 31.10           | 32.55       |
| Surf  | 30.01         | 30.05        | 30.88              | 30.91       | 31.44           | 32.12       |
| Zebra | 29.96         | 29.92        | 30.61              | 30.84       | 31.22           | 32.18       |

Table 2. The SSIM table

| Image | BTV reg. [17] | TV reg. [39] | Adaptive reg. [67] | GDP SR [73] | The PDE in [15] | Our Method |
|-------|---------------|--------------|--------------------|-------------|-----------------|-------------|
| Build | 0.786         | 0.783        | 0.796              | 0.802       | 0.812           | 0.830       |
| Lion  | 0.805         | 0.807        | 0.837              | 0.845       | 0.866           | 0.884       |
| Penguin | 0.848       | 0.833        | 0.846              | 0.860       | 0.888           | 0.900       |
| Man   | 0.776         | 0.772        | 0.800              | 0.819       | 0.836           | 0.874       |
| Surf  | 0.802         | 0.789        | 0.816              | 0.840       | 0.855           | 0.872       |
| Zebra | 0.804         | 0.802        | 0.811              | 0.823       | 0.859           | 0.890       |

Fact, we used three metrics: peak-signal-to-noise ratio (PSNR) [66], the information fidelity criterion (IFC) index [55] and mean structure similarity (SSIM) [59]. The PSNR measures signal strength relative to noise in the image while the IFC index gives a highest correlation with perceptual scores for super resolution evaluation, as demonstrated in [41]. Also the SSIM metric gives an indication on the quality of the image based on the known characteristics of the human visual system.

The obtained results in Figs. (3)-(9) show the robustness of the proposed PDE against noise and blur visually, which confirms its theoretical aspect and behaviour. To assure that, the qualitative study, represented by the three tables 1, 2 and 3 confirms that the proposed model is always with the higher PSNR, SSIM and IFC values. Note that we take the optimal parameters according to the best IFC value in all the experiments for the compared methods. Concerning the execution time of the proposed method, it is relatively high compared to the other methods since computing the function $f_+$ and $f_-$ values needs much more time (see table 4 for more details). We recall that the main implementation of the algorithm is done on an i7 3.4 GHz Quad-Core computer using Matlab R2014.

In the real experiment, the three sequences used are: “Book”, “Disk” and “Paint” videos, which are challenging examples, since it contains a high level of blur and noise. We select the first sixteen low-resolution degraded images from each video and we use the approach in [30] to estimate the motion for our method and the comparable ones. The reconstructed image by a factor of $r = 4$ by different methods are shown in the Figs. 10, 11 and 12. We can observe clearly that the restored image by our SR method is visually better than the others. We can see that near boundaries and in the texture, where the proposed approach produces a sharper image with less blur and noise (see for example 11 where the difference is obvious).

4. Conclusion. In this paper, we have presented a fourth-order PDE in the multi-frame super-resolution task which restores an ideal HR image from a much degraded LR sequence. This PDE takes the advantages of the Perona-Malik equation with much regularity and the efficiency of a nonlinear filter to restore tiny and sharp
The LR sequence of the *Penguin* sequence.

The *Surf* sequence.

The *Lion* sequence.

The *Old* sequence.

The *Build* sequence.

The *Zebra* sequence.

The *Barbara* sequence.

**Figure 2.** The First six LR images used for the super-resolution process.

edges. The existence and uniqueness of the proposed PDE is proved using the Schauder fixed point theorem. The experimental part demonstrates, visually and quantitatively the performance and the contribution of this new PDE over the other super-resolution approaches.
Figure 3. Comparisons of different SR methods (Build image).

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Compliance with ethical standards.

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Figure 5. Comparisons of different SR methods (Penguin image).

- Neither human participants nor animals are involved in this research.
Figure 6. Comparisons of different SR methods (Man image).

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Figure 7. Comparisons of different SR methods (Surf image).

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Figure 8. Comparisons of different SR methods (Zebra image).

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Figure 9. Comparisons of different SR methods (*Barbara* image).

Table 3. The IFC table

| Image | BTV reg. [17] | TV reg. [39] | Adaptive reg. [67] | GDP SR [73] | The PDE in [15] | Our Method |
|-------|---------------|--------------|-------------------|-------------|----------------|-------------|
| Build | 1.712         | 1.700        | 1.820             | 1.844       | 1.901          | **1.980**   |
| Lion  | 1.777         | 1.770        | 1.778             | 1.830       | 1.860          | **1.889**   |
| Penguin | 1.825   | 1.802        | 1.933             | 1.930       | 1.964          | **2.111**   |
| Man   | 1.790         | 1.768        | 1.900             | 1.893       | 1.992          | **2.155**   |
| Surf  | 1.788         | 1.785        | 1.801             | 1.825       | 1.885          | **2.090**   |
| Zebra | 1.800         | 1.802        | 1.863             | 1.860       | 1.933          | **2.200**   |

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Table 4. CPU times (in seconds) of different super-resolution methods and the proposed method when the magnification factor is 4.

| Image  | Size     | BTV reg. [17] | TV reg. [39] | Adaptive reg. [67] | GDP SR [73] | The PDE in [15] | Our Method |
|--------|----------|---------------|--------------|---------------------|-------------|----------------|------------|
| Build  | 472 × 312| 9.84          | 8.26         | 12.02               | 9.88        | 24.96          | 26.44      |
| Lion   | 472 × 312| 6.82          | 13.64        | 6.62                | 7.34        | 6.61           | 52.60      |
| Penguin| 472 × 312| 8.31          | 14.48        | 7.16                | 7.62        | 8.08           | 48.33      |
| Man    | 312 × 472| 10.66         | 9.75         | 13.92               | 9.84        | 25.53          | 26.10      |
| Surf   | 504 × 504| 11.05         | 11.18        | 14.22               | 10.86       | 25.80          | 25.93      |
| Zebra  | 472 × 312| 9.86          | 9.14         | 12.22               | 10.32       | 20.24          | 22.02      |
| Barbara| 512 × 512| 10.87         | 9.96         | 13.88               | 10.66       | 26.60          | 27.10      |
| Average|          | 13.77         | 20.06        | 12.33               | 14.10       | 14.66          | 58.68      |

Figure 10. Comparisons of different SR methods (Book image).
Figure 11. Comparisons of different SR methods (Disk image).

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Figure 12. Comparisons of different SR methods (Paint image).

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