SCALING LAWS IN
HIERARCHICAL CLUSTERING MODELS
WITH POISSON SUPERPOSITION

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ABSTRACT  Properties of cumulant- and combiant ratios are studied for multihadron final states composed of Poisson distributed clusters. The application of these quantities to “detect” clusters is discussed. For the scaling laws which hold in hierarchical clustering models (void scaling, combinant scaling) a generalization is provided. It is shown that testing hierarchical models is meaningful only for phase-space volumes not larger than the characteristic correlation length introduced by Poisson superposition. Violation of the scaling laws due to QCD effects is predicted.

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1. Introduction

In galaxy clustering studies there is accumulating observational evidence in favour
of the so-called hierarchical models (see e.g. refs. [1-5] and ref. [6] for a historical
account). Initiated by the work of Carruthers and Sarcevic [7] and the Tucson
group [8,9] considerable interest has been devoted to this subject over the last few
years also in studying the nature of correlations of multihadron final states [10-13].
In the field of multiparticle dynamics one speaks about Linked Pair Approximation
(LPA) because of the special hierarchy of the higher-order correlation functions.
Throughout this paper we shall use both naming conventions.

In the framework of the LPA the two-particle cumulant correlations, defined
in terms of the inclusive single- and two-particle density correlations \( \rho_1 \) and \( \rho_2 \) as

\[
\mathcal{K}_2(1,2) = \rho_2(1,2) - \rho_1(1)\rho_1(2),
\]

(1.1)

provide the building blocks of the higher-order cumulants [7,8]. In eq. (1.1) the
arguments denote coordinates e.g. on the rapidity axis. The cumulants(densities) are
known also as irreducible(reducible) correlations since the lower-order background
correlation terms involved by \( \rho_q \) are subtracted in \( \mathcal{K}_q \). The cumulants measure
genuine \( q \)-particle correlations, the degree of independence of their arguments. In
the case of full statistical independence (each argument is independent of all the
others) the \( q \)-particle density \( \rho_q \) factorizes into the product of \( q \) single-particle
densities and the \( q \)-particle cumulant \( \mathcal{K}_q \) vanishes. In the LPA the normalized
cumulants

\[
\kappa_q(1, ..., q) = \mathcal{K}_q(1, ..., q)/\rho_1(1)\ldots\rho_1(q)
\]

(1.2)

are built up as sums of products of linked two-particle normalized cumulants. For
example, \( \kappa_3 \) is composed according to

\[
\kappa_3(1, 2, 3) = \frac{A_3}{3} [\kappa_2(1, 2)\kappa_2(2, 3) + \kappa_2(2, 3)\kappa_2(3, 1)
+ \kappa_2(3, 1)\kappa_2(1, 2)]
\]

(1.3)

where the constant \( A_3 \) is a free parameter to be determined (by definition, \( A_1 =
A_2 = 1 \)). In the astrophysical literature the constants \( A_q \) are known as hierarchical
amplitudes. The Linked Pair Approximation is in fact a special case of the hier-
archical models [11]. Some types of topologically distinct graphs allowed in the
construction of higher-order galaxy correlation functions are absent in the LPA.
These are the so-called non-snake graphs. In lowest order the two schemes are
equivalent, leading to eq. (1.3), and they coincide at the level of the integrated
cumulants.
Two years before the discovery of the successfulness of eq. (1.3) for galaxy correlation data [2] Mandelbrot arrived at the same functional form by constructing a simple model for galaxy clustering [14]. In the model the positions of galaxies are stepping points of a Rayleigh-Lévy random walk which gives rise to an unbounded fractal galaxy distribution. Later it was realized that the distribution of galaxies cannot be a pure, unbounded fractal; it contradicts to the fine structure of the observed galaxy distribution [6]. The model was modified by introducing many Rayleigh-Lévy random walk fractals distributed according to a Poisson process [1]. The Poisson superposition of bounded fractal clumps well reproduces the visual appearance of the galaxy distribution in the sky and it is in agreement with the hierarchical structure of higher-order correlation functions [1,6].

According to recent work of Giovannini, Lupia and Ugoccioni the hierarchical structure of cumulant correlations and the Poisson superposition principle play an important role in multiparticle dynamics too. The Torino group studies the nature of inside-jet correlations in $e^+ e^-$ annihilations using JETSET 7.2 Monte Carlo events in a wide c.m. energy range up to $\sqrt{s} = 1000$ GeV [13, 15-17]. One of the main results is the identification of jets originated by a quark(antiquark) and a gluon. The correlations are in agreement with the hierarchical ansatz in quark- and gluon-jets. Moreover, it is found that both types of jets consist of Poisson distributed clumps of particles called clans. Within gluon-jets the correlations are stronger (the clans have larger particle content) which is attributed to QCD effects. In gluon initiated jets the dominant mechanism is the self-interaction of gluons whereas quark-jets are controlled by gluon bremsstrahlung emission [13].

The Poisson superposition introduces a characteristic correlation length in hierarchical models. The main goal of the present paper is the study of the consequences of this newly appearing length scale. Although correlations of multihadron final states will be investigated some of the results may have applications in other fields, particularly in galaxy clustering studies.

2. Basic definitions

Currently available data for hadrons and galaxies do not allow precise determination of the higher-order correlation functions. But testing the validity of hierarchical models is possible through the integrated cumulants that provide the factorial cumulant moments of the underlying count distributions. Moments can be determined up to 8th order for galaxy catalogs [4,5] and up to 5th order for multihadron final states [8]. In this section we collect some basic definitions and formulae concerning count distributions and various moments. For more details, see e.g. refs. [10] and [18].
Let us start with the generating function for the multiplicity- or count distributions \( P_n \), conveniently defined by

\[
G(z) = \sum_{n=0}^{\infty} P_n z^n. \tag{2.1}
\]

The connection with the density correlation functions \( \rho_q \) is given by the power expansion of \( G(z) \):

\[
G(z) = 1 + \sum_{q=1}^{\infty} \frac{(z - 1)^q}{q!} \xi_q \tag{2.2}
\]

where the \( \xi_q \) are the integrated densities providing the factorial moments of \( P_n \). The integrated cumulants \( K_q \), being the factorial cumulant moments \( f_q \) of \( P_n \), are defined by the power expansion of the logarithm of \( G(z) \):

\[
\ln G(z) = \sum_{q=1}^{\infty} \frac{(z - 1)^q}{q!} f_q. \tag{2.3}
\]

In the analysis of various count distributions occurring in nature the so-called infinitely divisible distributions play a distinguished role. In multiparticle dynamics their importance was emphasized by Giovannini and Van Hove [19,20]. A discrete distribution is said to be infinitely divisible if its generating function has the property that for all \( k > 0 \) integer \( \sqrt[k]{G(z)} \) is again the generating function of a certain distribution [21]. \( G(z) \) satisfies this property if and only if \( G(1) = 1 \) and

\[
\ln G(z) = \ln G(0) + \sum_{q=1}^{\infty} C_q z^q. \tag{2.4}
\]

In eq. (2.4) the \( C_q \) are the combinants of Gyulassy and Kauffmann [22,23]. The combinants should obey \( C_q \geq 0 \) and

\[
\sum_{q=1}^{\infty} C_q = -\ln G(0) < \infty \tag{2.5}
\]

for infinitely divisible distributions. From eq. (2.5) one sees that the probability of detecting no particles at all in a certain phase-space volume, the so-called void probability, is \( P_0 = G(0) > 0 \) for discrete distributions satisfying the conditions of infinite divisibility. The generating function takes the form

\[
G(z) = \exp \left( \sum_{q=1}^{\infty} C_q (z^q - 1) \right) \tag{2.6}
\]
in terms of combinants. The $C_q$ can be expressed as combinations of the count probability ratios $P_q = P_q/P_0$,

$$C_q = P_q - \frac{1}{q} \sum_{r=1}^{q-1} r C_r P_{q-r}, \quad (2.7)$$

hence their name. Eq. (2.7) shows that the knowledge of $C_q$ requires only a finite number of count probabilities. Furthermore, we need not know the probabilities themselves: the combinants follow directly from the unnormalized topological cross sections since they involve only ratios of probabilities. It is also seen from eq. (2.7) that in general the $C_q$ can take negative values as well for $q \geq 2$. In this case a necessary condition of the infinite divisibility of $P_n$ is not satisfied. The combinants have some advantageous features, e.g. they share common properties with the cumulant moments. For further details see refs. [23,24].

### 3. Cumulant- and combinant ratios in Poisson cluster models

In order to study hierarchical models with Poisson superposition we shall utilize the so-called Poisson cluster models. These are closely related to the infinitely divisible distributions discussed in the previous section. An advantageous feature of these models is the absence of free parameters. The clan production picture of hadronization developed by Giovannini and Van Hove [19,20] is a well known example of Poisson cluster models. We start with the basic equations.

Assume that the observed events (the phase-space distribution of multihadron final states produced in a certain collision process or the galaxy distribution in the sky as a single “event”) can be decomposed into Poisson distributed clusters. In the Poisson cluster models the generating function of the total event multiplicity distribution $P_n$ takes the form

$$G(z) = \exp \left( \tilde{C} (H(z) - 1) \right) = \mathcal{P}(H(z)) \quad (3.1)$$

where $\mathcal{P}(z)$ and $H(z)$ stand for the generating functions of the Poissonian cluster distribution and the arbitrary distribution of particles inside the clusters. $G(z)$ is obtained as the convolution of $\mathcal{P}(z)$ and $H(z)$. Discrete distributions having a generating function of the above form are known as compound Poisson distributions [21]. Eq. (3.1) is another way of writing $G(z)$ for infinitely divisible distributions, eq. (2.6), with $\tilde{C} = -\ln G(0)$ and

$$H(z) = 1 - \frac{\ln G(z)}{\ln G(0)} = \sum_{q=1}^{\infty} p_q z^q. \quad (3.2)$$
In eq. (3.2) the $q$-particle count probability within a single cluster, $p_q$, is found to be [24, 25]

$$p_q = \frac{C_q}{\sum_q C_q},$$

(3.3)

reminiscent of the definition of count probabilities $P_n$ in terms of the topological cross sections $\sigma_n$, $P_n = \sigma_n / \sum_n \sigma_n$.

Since $\mathcal{H}(0) = 0$ each cluster must contain at least one particle, i.e. $p_0 = C_0 = 0$. Accordingly, $\mathcal{G}(0) = \mathcal{P}(0)$ and the probability of detecting no particles, $P_0$, provides also the probability of detecting no clusters in a certain phase-space volume. Hence the two basic parameters of the model, the average cluster multiplicity, $\bar{C}$, and the average multiplicity within the clusters, $\bar{q}$, are found to be [12, 26]

$$\bar{C} = -\ln P_0 \quad \text{and} \quad \bar{q} = \bar{n} / \bar{C}$$

(3.4)

with $\bar{n} = f_1 = \xi_1$ being the total event average multiplicity. It is worth noticing that eq. (2.6) can be written also in the form

$$\mathcal{G}(z) = \prod_{q=1}^{\infty} \exp \left( C_q (z^q - 1) \right),$$

(3.5)

i.e. as the product of the generating function of a Poisson distribution of particle singlets having mean $C_1$, a Poisson distribution of particle pairs having mean $C_2$, and so on [21]. Thus a formally equivalent derivation of the Poisson cluster models is possible in which the total events are composed of Poisson distributed particle $q$-tuples with average multiplicities $C_q$ instead of identical clusters or clans. This view is closely reminiscent of the Mayer cluster expansion of statistical mechanics. Recall that the $C_q$ can take negative values as well for $q \geq 2$. In this case the interpretation of combinants as unnormalized count probabilities within identical Poisson clusters or as the average multiplicities of Poisson distributed particle $q$-tuples loses its meaning.

On the basis of eq. (3.1) let us consider the relationship between quantities corresponding to the observed events and quantities at the level of the individual clusters. For the total event factorial cumulants we get [27]

$$f_q = \frac{d^q}{dz^q} \ln \mathcal{G}(z) \bigg|_{z=1} = C_q \frac{d^q}{dz^q} \mathcal{H}(z) \bigg|_{z=1} = \bar{C} \zeta_q$$

(3.6)

with $\zeta_q$ denoting the factorial moments within a single cluster. As obtained in eq. (3.3), the combinants of the total events are related to the count probabilities
inside the clusters:

\[ C_q = \frac{1}{q!} \frac{d^q}{dz^q} \ln G(z) \bigg|_{z=0} = \bar{C}_q \frac{1}{q!} \frac{d^q}{dz^q} H(z) \bigg|_{z=0} = \bar{C}_p. \]  

(3.7)

We see that ratios of quantities derived from \( \ln G(z) \) are equivalent to ratios of quantities derived from \( H(z) \) and characterize the individual clusters. This feature is well suited to “detect” clusters. When both \( p \)- and \( q \)-particle correlations exist any of the ratios \( f_q/f_p \), \( C_q/C_p \) or \( C_q/f_p \) should become independent of changes in the size of the phase-space volume if a certain characteristic correlation length is exceeded. Cuts in phase-space larger than the typical cluster size correspond to cuts in the number of clusters and do not influence single cluster statistics. Therefore a flattening is expected in the large-scale behaviour of the above quantities, provided that translation invariance holds for extended volumes. Flattening of the ratios \( f_q/f_1 \) for wide rapidity intervals has already been studied by Dias de Deus using the assumption that multihadron final states consist of a fixed number of uncorrelated clusters [28]. In this case the constancy follows from the additivity property of the cumulant moments. The large binsize \( (\delta y \geq 2) \) plateau in the behaviour of the \( f_q/f_1 \) was indeed observed for NA22 data at \( \sqrt{s} = 22 \) GeV and for UA5 data at \( \sqrt{s} = 546 \) GeV [28,29].

4. Hierarchical models, Poisson superposition and scaling

In this section we consider properties of the cumulant- and combinant ratios again but the Poisson superposition principle is combined with the simultaneous validity of the Linked Pair Approximation. Assuming translation invariance and the linked-pair hierarchy of the higher-order cumulant correlation functions the strip-integration [8] of \( \kappa_q \) produces normalized factorial cumulant moments obeying the recurrence relation

\[ K_q \equiv \frac{f_q}{f_1} = A_q K_2^{q-1}. \]  

(4.1)

The hierarchical amplitudes \( A_q \) should be independent of collision energy and phase-space volume. Eq. (4.1) is valid also for hierarchical models that allow not only snake-type graphs in the construction of higher-order cumulant correlation functions. The integrated amplitudes in eq. (4.1) may deviate from the local ones appearing in the expressions for \( \kappa_q \). In addition to the direct test of eq. (4.1) it is possible to check the validity of the LPA through various scaling laws. For the ratio of two factorial cumulants eq. (4.1) yields

\[ \frac{f_q}{f_p} = \frac{A_q}{A_p} (f_2/f_1)^{-p}. \]  

(4.2)
Expressing the combinants in terms of factorial cumulant moments \([24,25,30]\),

\[
C_q = \sum_{r=q}^{\infty} \binom{r}{q} \frac{(-1)^{r-q}}{r!} f_r,
\]  

(4.3)

one sees that the cumulant- and combinant ratios considered in the previous section obey a universal behaviour in hierarchical models. Any of these quantities should depend on collision energy and phase-space volume not arbitrarily but only through the combination \(f_2/f_1\). Beside eq. (4.2) we have e.g.

\[
\frac{C_q}{f_p} = \sum_{r=q}^{\infty} \binom{r}{q} \frac{(-1)^{r-q}}{r!} \frac{A_r}{A_p} (f_2/f_1)^{r-p}
\]

\[\equiv \chi_{qp}(f_2/f_1),\]

(4.4)

the combinant-to-cumulant ratios chosen at fixed \(q, p\) from different energies and volumes scale to a universal curve when plotted against \(f_2/f_1\). In eq. (4.4) the contribution of an infinite number of factorial cumulant moments is confined into the scaling variable \(f_2/f_1\). The \(\chi_{qp}\) provide the generalization of the scaling functions introduced in ref. [25] which are the \(p = 1\) special cases of eq. (4.4).

The generalization of the famous void scaling function \([31,32]\) can be obtained in a similar manner. The probability of detecting no particles in a certain phase-space volume obeys the scaling law

\[\frac{-\ln P_0}{f_p} = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r!} \frac{A_r}{A_p} (f_2/f_1)^{r-p}\]

\[\equiv \chi_{0p}(f_2/f_1),\]

(4.5)

as is seen from eq. (2.3) by substituting \(z = 0\). The zero subscript of \(\chi\) in eq. (4.5) refers to the property that these scaling functions involve only the void probability. Through eq. (2.5) we get \(\chi_{0p} = \sum_{q=1}^{\infty} \chi_{qp}\). Similarly to the void scaling functions, the \(\chi_{qp}\) constructed from the combinants characterize the distribution of empty regions in phase-space. The \(C_q\) can be interpreted as minus the logarithm of the probability that a certain phase-space volume is empty of particle \(q\)-tuples, see eq. (3.5). The advantage of \(\chi_{qp}\) over \(\chi_{0p}\) lies in the fact that the contribution of low-order factorial cumulants can be excluded in a systematic manner by increasing \(q\) (observe that \(f_1\) and \(f_2\) involved by \(P_0\) carry no information on the validity of the LPA).

Since the relationship between count probabilities and factorial moments is the same as the relationship between combinants and factorial cumulants \([23,30]\)
a scaling law analogous to eq. (4.4) holds for the count probabilities $P_n$ if the recurrence relation

$$F_q \equiv \frac{\xi_q}{\xi_1} = A_q F^{q-1}_2$$

(4.6)

with constant coefficients $A_q$ is satisfied for the normalized factorial moments:

$$\frac{P_n}{\xi_p} = \sum_{r=n}^{\infty} \binom{r}{n} \frac{(-1)^{r-n}}{r!} \frac{A_r}{A_p} (\xi_2/\xi_1)^{r-p}$$

(4.7)

with $n, p \geq 1$ since $\xi_0 = 1$. That is to say, if one plots the ratios $P_n/\xi_p$ chosen at fixed $n, p$ against the combination $\xi_2/\xi_1$ the validity of eq. (4.6) results in a universal curve, $\eta_{np}(\xi_2/\xi_1)$, instead of many different behaviours corresponding to different collision energies and volume sizes. Count probability ratios not involving $P_0$ constitute another set of scaling functions. The analogue of void scaling for eq. (4.6) is provided by $\eta_{0p} \equiv (1 - P_0)/\xi_p = \sum_{n=1}^{\infty} \eta_{np}$.

Eq. (4.6) characterizes monofractal density fluctuations. They are expected to occur e.g. in second-order phase transitions. Hence the scaling law of the multiplicity distributions expressed by eq. (4.7) can in principle signal the formation of quark-gluon plasma if QCD undergoes a second-order transition [30]. It should be emphasized that eq. (4.6) holds for any field theory with dimensionless coupling [33]. For example, in QCD with fixed coupling constant $\alpha_s$ the density fluctuations are concentrated into randomly distributed monofractal patches of phase-space [34]. With running coupling $\alpha_s$ multifractal fluctuations appear and the exponent on the rhs. of eq. (4.6) gets multiplied by a $q$-dependent factor. QCD effects violate the above scaling rule of the count probabilities $P_n$.

We turn our attention to the validity of the LPA and the resulting scaling laws if the observed events are composed of Poisson distributed clusters. First of all, from eq. (3.6) we get $F_q = \tilde{C}^{q-1} K_q$ and

$$A_q \equiv \frac{K_q}{K_2^{q-1}} = \frac{F_q}{F_2^{q-1}}$$

(4.8)

with $F_q$ denoting the normalized factorial moments within a single cluster, $\xi_q/\xi_1^q$. One sees that the hierarchical amplitudes $A_q$ are equivalent to single cluster statistics. Consequently, they should be independent of changes in the size of the phase-space volume if the typical cluster size is exceeded, regardless of the validity of the LPA. For the scaling laws we have similar equivalence relations. According to eqs. (3.6) and (3.7) the scaling functions of eq. (4.4) at the level of the observed events are equivalent to the scaling functions of eq. (4.7) at the level of the
individual clusters:

\[ \chi_{qp}(f_2/f_1) \big|_{\text{total event}} = \eta_{qp}(\xi_2/\xi_1) \big|_{\text{single cluster}}. \quad (4.9) \]

Moreover, the void scaling functions of eq. (4.5) are equivalent to \(1/\zeta_q\), the reciprocal of the unnormalized factorial moments inside the clusters. Thus for multihadron final states composed of Poisson distributed clusters testing eq. (4.1) and the scaling laws expressed by eqs. (4.4) and (4.5) is meaningful only for phase-space volumes not exceeding the characteristic correlation length introduced by Poisson superposition. Cuts in phase-space larger than the typical cluster size are equivalent to cuts in the number of clusters and leave single cluster statistics unchanged. We mention that the cluster size may vary with energy, e.g. in \(hh\) collisions the threshold binsize in rapidity at which the constancy of the ratios \(f_q/f_1\) appears is increasing from \(\delta y \approx 2\) to \(\delta y \approx 3\) between \(\sqrt{s} = 22\) and 546 GeV [28,29]. In the \(\chi\)-scaling analyses already performed [11-13] the chosen cuts in rapidity do not exceed the above values.

Closing this section let us reconsider the promising results of the Torino group mentioned in the Introduction. The finding that quark- and gluon-jets consist of Poisson distributed groups of particles, the clans, and the confirmation of hierarchical inside-jet correlations provide a direct connection to our results. The hierarchical pattern of cumulants in quark- and gluon-jets was deduced from the validity of the void scaling law eq. (4.5) for \(p = 1\) [13]. Due to the Poisson superposition this scaling rule should follow from the validity of eq. (4.6) at the level of the individual clans. Since clan properties inside gluon originated jets are controlled by the self-interaction of gluons, deviations are expected from the void- and combinant scaling functions in gluon-jets. For running QCD coupling constant \(\alpha_s\) eq. (4.6) is no longer valid [33,34] and this violates the inside-clan scaling rules of count probabilities and unnormalized factorial moments being equivalent to the inside-jet combinant- and void scaling laws. In the \(\sqrt{s} = 1000\) GeV c.m. energy range investigated by the Torino group the self-interaction of gluons may cause observable scaling violation for gluon initiated jets. Deviations from the void scaling function \(\chi_{01}\) are indeed observed [13] and they are attributed to the absence of translation invariance. The separation of the two sources of scaling violation would be of particular importance. According to preliminary results of De Wolf [35] the breakdown of scaling can be seen also in the behaviour of the higher-order \(\chi_{q1}\) combinant-to-cumulant ratios for JETSET Monte Carlo events.

5. Summary

One of the common problems in the study of the very large and very small length-scale phenomena is the formation of structures — the texture of multihadron final
states in phase-space and the distribution of galaxies in the sky. The assumption of randomly superimposed objects (clusters, clans) with cumulant correlation functions obeying hierarchical structure at the level of the total matter distribution plays a distinguished role in multiparticle physics and particularly in galaxy clustering studies. We have investigated the properties of cumulant- and combinator ratios for hierarchical models with Poisson superposition. According to our results the behaviour of the ratios \( f_q/f_p, \ C_q/C_p \) and \( C_q/f_p \) among others considerably simplifies with the two assumptions:

1) From eqs. (3.6) and (3.7) one obtains that for Poisson distributed clusters (clans) the above quantities are equivalent to single cluster statistics. Consequently, they should be independent of changes in the size of the phase-space volume if the characteristic correlation length introduced by Poisson superposition is exceeded. Cuts in phase-space larger than the typical cluster size correspond to cuts in the number of clusters and do not influence single cluster statistics. Therefore large-scale constancy of the cumulant- and combinator ratios is expected at a fixed energy.

2) The consequence of the LPA-relation, eq. (4.1), is that the cumulant- and combinator ratios should depend on collision energy and phase-space volume not arbitrarily but only through the momentum combination \( f_2/f_1 \). One arrives e.g. at the nontrivial scaling laws expressed by eqs. (4.4) and (4.5). These are generalizations of the scaling laws introduced in refs. [31] and [25]. According to 1) for a set of Poisson distributed clusters all of the hierarchical statistics \( (A_q, \chi_{qp}, \chi_{0p}) \) are equivalent to statistics characterizing the individual clusters. Therefore testing the validity of eq. (4.1) and the resulting scaling laws is meaningful only for volume sizes not larger than the typical size of the clusters.

By eq. (4.8) we are led to the conclusion that in hierarchical models with Poisson superposition the linked-pair hierarchy of higher-order correlation functions manifests itself primarily at the level of the individual clusters. Within a single cluster the density correlation functions and factorial moments (the reducible statistics) obey the hierarchical structure. The cumulant correlation functions and factorial cumulant moments (the irreducible statistics) exhibit the hierarchical pattern at the level of the total events as the consequence of Poisson superposition. Eq. (4.1) and the resulting combinator- and void scaling laws hold for any field theory with dimensionless coupling if randomly superimposed clusters are present, e.g. in QCD with fixed coupling constant \( \alpha_s \). Restoration of the running of \( \alpha_s \) causes violation of the scaling laws. This can be most significant in \( e^+e^- \) annihilations for gluon initiated jets controlled by gluon self-interaction.
We close with a list of some measurements that may shed more light on the validity of hierarchical models and Poisson superposition:

- The Poisson cluster models discussed in the present paper can be tested e.g. by measuring the $C_q$. Observation of negative combinants signals that the underlying count distributions are not infinitely divisible. In this case the Poisson cluster models are inappropriate to represent the data.

- The presence of clusters can be revealed by correlation length measurements. They consist of determining the threshold volume size at which the constancy of the cumulant- and combinant ratios appears. The threshold provides an estimation for the typical size (e.g. rapidity extent) of the Poisson distributed clusters.

- Measurement of the higher-order combinant scaling functions, e.g. the $\chi_{q1}$ for $q \geq 3$, is important to confirm eq. (4.1). These have the advantage over the void scaling function $\chi_{01}$ that the contribution of $f_1$ and $f_2$ is excluded. The first two factorial cumulants have considerable influence on the shape of $\chi_{01}$ but they carry no information on the validity of eq. (4.1).

- It would be interesting to see how clear is the violation of $\eta$-scaling if $\chi$-scaling holds valid. This provides information on the selective power of scaling laws that have different origin. The simplest possibility is the comparison of $-\ln P_0/f_1$ and $(1 - P_0)/\xi_1$ as functions of $f_2/f_1$ and $\xi_2/\xi_1$ respectively.

Lack of translation invariance may considerably influence the proposed measurements. Estimation of its significance is important to gain reliable information on higher-order correlations obeying hierarchical structure with the possible presence of clusters distributed according to a Poisson process. On the basis of the experimental data that will be available at LEP200 and TEVATRON, at twice as large c.m. energies as present, we have the chance of answering the open questions in the near future.

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