THE LOCAL CATEGORICAL DT/PT CORRESPONDENCE

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Abstract. In this paper, we prove the categorical wall-crossing formula for certain quivers containing the three loop quiver, which we call DT/PT quivers. These quivers appear as Ext-quivers for the wall-crossing of DT/PT moduli spaces on Calabi-Yau 3-folds. The resulting formula is a semiorthogonal decomposition which involves quasi-BPS categories studied in our previous papers, and we regard it as a categorical analogue of the numerical DT/PT correspondence. As an application, we prove a categorical DT/PT correspondence for sheaves supported on reduced plane curves in the affine three dimensional space.

1. Introduction

The purpose of this paper is to give a categorical wall-crossing formula for certain quivers, called DT/PT quivers. Our main result is motivated by our pursuit of categorifying the DT/PT correspondence for curve counting theories on Calabi-Yau 3-folds, see Subsection 1.2 for its review. As we will explain in Subsection 1.3 the DT/PT quivers appear as Ext-quivers of wall-crossing of DT/PT moduli spaces on Calabi-Yau 3-folds. We regard the resulting categorical wall-crossing formula as a local categorical analogue of the DT/PT correspondence. We apply it to obtain a categorical (and K-theoretic) DT/PT correspondence in a geometric example, for sheaves supported on reduced plane curves in the affine three dimensional space. The main result of this paper will be applied in [PTc] to give a categorical DT/PT wall-crossing for reduced curve classes on local surfaces.

1.1. The categorical wall-crossing formula for DT/PT quivers. For $a \in \mathbb{N}$, we define the DT/PT quiver $Q^{af}$ to be the quiver with two vertices 0 and 1, three loops at 1, $a + 1$ edges from 0 to 1, and $a$ edges from 1 to 0, see Figure 1 for the picture of $Q^{2f}$.

We denote by $R_{af}^{a}(1, d)$ the affine space of representations of $Q^{af}$ of dimension $(1, d)$. Let $\chi_0: GL(V) \to \mathbb{C}^*$ be the determinant character $g \mapsto \det g$. Consider the GIT semistable stacks, which are smooth quasi-projective varieties:

$$I^a(d) := R_{af}^{a}(1, d)^{\chi_0\text{-}\text{ss}} / GL(d),$$

$$P^a(d) := R_{af}^{a}(1, d)^{\chi_0^{-1}\text{-}\text{ss}} / GL(d).$$
As we explain in Subsection 1.3, these moduli spaces together with some superpotentials give local models of the DT/PT moduli spaces on Calabi-Yau 3-folds. The following is the main result of this paper:

**Theorem 1.1.** Let \( \mu \in \mathbb{R} \) such that \( 2 \mu l \notin \mathbb{Z} \) for \( 1 \leq l \leq d \). There is a semiorthogonal decomposition

\[
D^b(I^a(d)) = \bigoplus_{k=1}^d M(d_i)_{w_i} \otimes D^b(P^a(d_i))
\]

The right hand side is after all \( d' \leq d \), partitions \( (d_i)_{k=1}^k \) of \( d - d' \), and integers \( (w_i)_{k=1}^k \) such that for \( v_i := w_i + d_i \left( d' + \sum_{j>i} d_j - \sum_{j<i} d_j \right) \), we have

\[
-1 - \mu - \frac{a}{2} < \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k} < -\mu - \frac{a}{2}.
\]

The order on the categories is discussed in Subsection 3.2.3, see also Subsection 2.5.

In the above, the categories \( M(d)_w \) are certain (twisted) noncommutative resolutions of singularities of \( gl(d) \oplus \mathbb{C}/GL(d) \) introduced by Špenko–Van den Bergh [ˇSVdB17]. We treated the case \( a = 0 \) in [PTa, Theorem 1.1], when \( P^0(d) \) is empty for \( d > 0 \) and is a point for \( d = 0 \). We note that while the proof of [PTa, Theorem 1.1] is based on the constructions of [P˘adc] (going back to [ˇSVdB21]) and on the use of decompositions of the weight space of \( T(d) \cong (\mathbb{C})^d \) in polytopes using the \((r,p)\)-invariant of a weight introduced by Špenko–Van den Bergh [ˇSVdB17, ˇSVdB21], a similar argument does not produce the semiorthogonal decomposition (1.1). The proof of Theorem 1.1 is still based on decompositions of polytopes in the weight space of \( T(d) \), but these decompositions are different from the ones considered in [ˇSVdB21, Corollary 8.5], [P˘adc, Theorem 1.1], [PTa, Theorem 1.1].

Using [PTa, Theorem 1.1], we also obtain the following decomposition in K-theory, where \( K \) denotes the Grothendieck group of a dg-category:

**Corollary 1.2.** Let \( d \in \mathbb{N} \). Then there is an isomorphism

\[
K(I^a(d)) \cong \bigoplus_{d' = 0}^d K(N\text{Hilb}(d - d')) \otimes K(P^a(d')).
\]

Compare the above isomorphism with [13]. We prove Theorem 1.1 and Corollary 1.2 in Section 3.

### 1.2. The DT/PT correspondence for Calabi-Yau 3-folds

We now review the DT/PT correspondence for Calabi-Yau 3-folds. For a smooth Calabi-Yau 3-fold \( X \), let \( \beta \in H_2(X, \mathbb{Z}) \) and \( n \in \mathbb{Z} \). The moduli space which defines the (rank one) Donaldson-Thomas (DT) invariant \( I_{n, \beta} \) is the classical Hilbert scheme

\[
I_X(\beta, n)
\]

which parametrizes closed subschemes \( C \subset X \) with \( \dim C \leq 1 \) and \( ([C], \chi(O_C)) = (\beta, n) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z} \). The rank one DT invariant is defined by

\[
\text{DT}_{\beta, n} = \int_{I_X(\beta, n)} \chi_B \, de,
\]

where \( \chi_B \) is the Behrend constructible function [Beh09]. The DT invariants are related to Gromov-Witten invariants by the Maulik-Nekrasov-Okounkov-Pandharipande (MNOP) conjecture [MNOP06].
In [PT09], Pandharipande–Thomas defined virtual counts of stable pairs on a smooth complex 3-fold, which we call PT invariants. In the Calabi-Yau case, PT invariants provide a geometric construction of the contribution of DT invariants in the statement of the MNOP conjecture. The moduli space of PT stable pairs is denoted by

$$P_X(\beta, n)$$

and it parametrizes pairs $$(F, s)$$, where $$F$$ is a pure one-dimensional coherent sheaf on $$X$$ with $$([F], \chi(F)) = (\beta, n) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$$ and $$s: \mathcal{O}_X \to F$$ is a section whose cokernel is at most zero-dimensional. The PT invariant is defined by

$$\text{PT}_{\beta,n} = \int_{P_X(\beta, n)} \chi_B \, de.$$ 

The DT/PT correspondence is the formula

$$\text{DT}_{\beta,n} = \sum_{k \geq 0} \text{DT}_{0,k} \cdot \text{PT}_{\beta,n-k}.$$ 

The above formula was conjectured in [PT09], its $$\chi_B$$-unweighted version was proved in [Tod10, ST11] using Joyce’s motivic Hall algebra machinery [Joy06, Joy07a, Joy07b, Joy08], and its version incorporating the Behrend function was proved in [Bri11] (see also [Tod20]) using the work of Joyce-Song [JS12] (see also [KS]).

There are several refinements of DT invariants in the Calabi-Yau case: motivic, cohomological, K-theoretic, or categorical, see [Sze16, Toda, PTa]. The motivic DT/PT correspondence was established in local cases by Davison–Ricolfi [DR21]. It is an interesting problem to study the cohomological DT/PT correspondence for quivers with potential (following [DM20]), or for global geometries.

In [Toda, Subsection 1.3, Conjecture 1.3.2], the second author proposed the existence of dg-categories $$\mathcal{D}T_X(\beta,n)$$ and $$\mathcal{P}T_X(\beta,n)$$ which categorify DT and PT invariants, and the existence of a fully-faithful functor

$$\mathcal{P}T_X(\beta,n) \hookrightarrow \mathcal{D}T_X(\beta,n).$$

We do not know how to construct such categories for a general Calabi-Yau 3-fold. In [Toda, Sections 3 and 4], the second author constructed such dg-categories in the case of local surfaces $$X = \text{Tot}_{S}(\omega_S)$$ (for $$S$$ a smooth surface), and proved the existence of a fully-faithful functor (1.4) when $$\beta$$ is a reduced curve class [Toda, Theorem 5.5.5]. It is natural to try to obtain a finer statement than (1.4) such as a semiorthogonal decomposition of $$\mathcal{D}T_X(\beta,n)$$ which categorifies the numerical DT/PT correspondence (1.3).

As we explain in the next subsection, the motivation of the main result in this paper is to establish such a categorical DT/PT wall-crossing for certain local models of the (still to be defined) dg-categories in (1.4) via Ext-quivers.

1.3. DT/PT wall-crossing and Ext-quivers. We now explain a construction of local models of the dg-categories (1.4) in terms of Ext-quivers associated with a wall-crossing diagram. For a collection of objects $$\{E_1, \ldots, E_k\}$$, the associated Ext-quiver has set of vertices $$\{1, \ldots, k\}$$ and the number of arrows from $$i$$ to $$j$$ is $$\dim \text{Ext}^1(E_i, E_j)$$.

The two moduli spaces $$I_X(\beta,n)$$ and $$P_X(\beta,n)$$ are related by wall-crossing in the derived category of $$X$$ by regarding them as moduli spaces of certain two-term
complexes. We have open immersions

$$I_X(\beta, n) \subset T_X(\beta, n) \supset P_X(\beta, n),$$

where $T_X(\beta, n)$ is the moduli stack of pairs $(F, s)$ such that $F$ is a (not necessary pure) one-dimensional sheaf and $s: \mathcal{O}_X \to F$ is a section with at most zero-dimensional cokernel. The stack $T_X(\beta, n)$ is the moduli stack of semistable objects on the DT/PT wall. Consider its good moduli space

$$(1.5) \quad \pi: T_X(\beta, n) \to T_X(\beta, n).$$

We have the following wall-crossing diagram

$$\begin{array}{ccc}
I_X(\beta, n) & \xrightarrow{\pi^+} & T_X(\beta, n), \\
\downarrow & & \downarrow \\
& \pi^- & P_X(\beta, n)
\end{array}$$

which is an example of a d-critical flip as defined in [Tod22].

A closed point $p \in T_X(\beta, n)$ corresponds to a direct sum

$$I_C \oplus \bigoplus_{j=1}^m V^{(j)} \otimes \mathcal{O}_{x^{(j)}}[-1]$$

where $C \subset X$ is a Cohen-Macaulay curve and $x^{(1)}, \ldots, x^{(m)}$ are distinct points in $X$. The formal fiber $\hat{T}_X(\beta, n)_p$ of the morphism $(1.5)$ at $p$ is described in terms of the Ext-quiver $Q_p$ associated with the collection

$$\{I_C, \mathcal{O}_{x^{(j)}}[-1], \ldots, \mathcal{O}_{x^{(m)}}[-1]\}.$$ 

The Ext-quiver $Q_p$ is given by gluing the quivers $Q^{d(j)}_{0}$ at the vertex 0, and adding $N$-loops at 0, where $d(j) = \text{ext}^1(I_C, \mathcal{O}_{x^{(j)}})$ and $N = \text{ext}^1(I_C, I_C)$, see [Toda, Proposition 5.5.2]. The moduli stack of the representations of the above Ext-quiver $Q_p$ with dimension vector $(1, d^{(1)}, \ldots, d^{(m)})$, where $d^{(j)} = \dim V^{(j)}$, is then given by

$$\mathcal{U}_p = \left( \mathbb{C}^N \times \prod_{j=1}^m \text{Rep}^{d(j)}_{(1, d^{(j)})} \right) / \prod_{j=1}^m \text{GL}(d^{(j)}).$$

Then there is a super-potential $w_p$ on some formal completion $\hat{\mathcal{U}}_p$ of $\mathcal{U}_p$ such that $\hat{T}_X(\beta, n)_p$ is isomorphic to $\text{Crit}(w_p)$, see [Tod18, Tod22, Proposition 9.11].

Let $\chi_0$ be the character of $\prod_{j=1}^k \text{GL}(d^{(j)})$ given by the product of each determinant character of $\text{GL}(d^{(j)})$. Then the formal fibers of $\pi^\pm$ are given by $\chi_0^\pm$-semistable loci on $\text{Crit}(w_p)$. Therefore the categories

$$\hat{T}_X(\beta, n)_p := \text{MF}(\hat{\mathcal{U}}_p, w_p), \quad \mathcal{P}\hat{T}_X(\beta, n)_p := \text{MF}(\hat{\mathcal{U}}_p^-, w_p)$$

are natural candidates of formal local models of the (yet to be defined) DT/PT categories in [1.3]. Theorem [1.1] (more precisely, its version allowing loops at 0 and a super-potential, see Corollary 3.3) implies that there is a semiorthogonal decomposition

$$(1.6) \quad \mathcal{D}\hat{T}_X(\beta, n)_p = \left\langle \bigotimes_{j=1}^m \mathcal{E}^{d(j)}_{(1, d^{(j)})}_{x^{(j)}} \bigotimes \mathcal{P}\hat{T}_X(\beta, n')_{p'} \right\rangle,$$
where $\sum d_i^{(j)} + n' = n$, the weights $w_{i}^{(j)}$ satisfy a condition analogous to (1.2), and the point $p' \in T_{w'}(X, \beta)$ satisfies $\sum d_i^{(j)}x_i^{(j)} + p' = p$. In the above, $\overline{S}(d)_w$ is the formal analogue of the quasi-BPS category $S(d)_w$ of $\mathbb{C}^3$ considered in [PTa, PTb].

We expect that (1.3) can be globalized to give a semiorthogonal decomposition of $DT_X(\beta, n)$. An issue here is that (1.3) depends on local data $a^{(j)}$ and a choice of $\mu$ in Theorem 1.1. In Theorem 1.1, if we take $\mu = -a/2 - \varepsilon$ for $0 < \varepsilon \ll 1$, then these data cancel and the condition (1.2) becomes

$$-1 < \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k} \leq 0.$$

From the above consideration, if the DT/PT categories in (1.4) exist, we expect a semiorthogonal decomposition of the form

$$DT_X(\beta, n) = \left< \bigoplus_{i=1}^k S_X(d_i)_{w_i} \boxtimes PT_X(\beta, n') \right>$$

where $d_1 + \cdots + d_k + n' = n$, the integers $w_i$ satisfy the condition (1.7) under the transformation $w_i \mapsto v_i$ in Theorem 1.1 and $S_X(d)_w$ is an analogue of the quasi-BPS category $S(d)_w$ for $X$.

In the local surface case, the DT/PT categories have been defined in [Toda] and the quasi-BPS categories in [PT]. We will give an example of $\mathbb{C}^3$ in Subsection 1.2. We are able to define the categories $DT$ and $PT$ in this case because of the global description as a critical locus of the relevant moduli spaces, see Theorem 1.3 and the discussion in Subsection 1.5.

Let $X = \text{Tot}_{\mathbb{P}^2}(\omega_{\mathbb{P}^2})$, and let $T_{C^3}(m, d)$ be the classical moduli stack of pairs $(F, s)$, where $F$ is a one-dimensional sheaf on $X$ with support a reduced plane curve of degree $m$ in $\mathbb{P}^2$, and $s: \mathcal{O}_X \rightarrow F$ is a section with at most zero-dimensional cokernel. The open immersion $\mathbb{C}^2 \subset \mathbb{P}^2$ determines an open immersion $\mathbb{C}^2 \subset X$. We consider the open substack

$$T_{\mathbb{C}^3}^{\text{red}}(m, d) \subset T_X^{\text{red}}(m, d)$$

consisting of $(F, s)$ such that $\text{Cok}(s)$ and the maximal zero-dimensional subsheaf of $F$ are supported on $\mathbb{C}^3$, and the one-dimensional support of $F$ is a plane curve from $(\mathbb{C}^{1\text{m}})^{\text{red}}$, see (1.6) and (1.19).

**Theorem 1.3.** (Theorem 1.5) Let $V$ be a vector space of dimension $d$. Then $T_{\mathbb{C}^3}^{\text{red}}(m, d)$ is the global critical locus of a regular function

$$T_{\mathbb{C}^3}^{\text{red}}(m, d) = \text{Crit} \left( \text{Tr } W_{m,d} : \left( V^{\mathbb{P}^2} \oplus V^\vee \oplus g^\otimes 3 \times (\mathbb{C}^{1\text{m}})^{\text{red}} \right) / \text{GL}(V) \rightarrow \mathbb{C} \right),$$

where $\text{Tr } W_{m,d}$ is explicitly given by, see (1.5) and (1.20):

$$\text{Tr } W_{m,d}(u_1, u_2, v, A, B, C, (\alpha_{ij})) = \sum_{1 \leq i+j \leq m} \alpha_{ij}a^iB^j(u_2) + \text{Tr } C(u_1 \circ v + [A, B]).$$
Using the above presentation of $\mathcal{T}^{\text{red}}_{C^3}(m, d)$ as a global critical locus, we define dg-categories

$$\mathcal{D}T^{\text{red}}_{C^3}(m, d), \text{ (resp. } \mathcal{P}T^{\text{red}}_{C^3}(m, d))$$

categorifying DT (resp. PT) invariants for the Hilbert schemes of 1-dimensional subschemes (resp. PT stable pair moduli spaces) with reduced supports. Using Theorem 1.1 for $a = 1$ and $\mu = -1/2 - \varepsilon$ for $0 < \varepsilon \ll 1$ and applying matrix factorizations for $\text{Tr} W_{d,m}$, see Corollary 3.14 we obtain the following:

**Theorem 1.4.** There is a semiorthogonal decomposition

$$\mathcal{D}T^{\text{red}}_{C^3}(m, d) = \left(\bigoplus_{i=1}^{k} S(d_{i})w_{i} \right) \boxtimes \mathcal{P}T^{\text{red}}_{C^3}(m, d') \right).$$

The right hand side is after all $d' \leq d$, partitions $(d_{i})_{i=1}^{k}$ of $d - d'$, and integers $(w_{i})_{i=1}^{k}$ such that for $v_{i} := w_{i} + d_{i} \left(d' + \sum_{j>i}^{k} d_{j} - \sum_{j<i}^{k} d_{j}\right)$, we have

$$-1 < \frac{v_{1}}{d_{1}} < \cdots < \frac{v_{k}}{d_{k}} \leq 0.$$

The quasi-BPS categories $S(d)_{w}$ are defined as categories of matrix factorizations for the potential $\text{Tr} X[Y, Z]$ with factors in $M(d)_{w}$, see [PTb] for results on these categories. Fix $0 \leq d' \leq d$. By [PTa] Theorem 1.1, the categories with which we tensor $\mathcal{P}T^{\text{red}}_{C^3}(m, d')$ in (1.9) form a semiorthogonal decomposition of $\mathcal{D}T(d - d')$, the DT category of $d - d'$ points in $C^{3}$. We thus obtain the following decomposition in K-theory, compare with (1.9):

**Corollary 1.5.** Let $d \in \mathbb{N}$. Then there is an isomorphism

$$K \left(\mathcal{D}T^{\text{red}}_{C^3}(m, d) \right) \cong \bigoplus_{d'=0}^{d} K \left(\mathcal{D}T(d - d') \boxtimes \mathcal{P}T^{\text{red}}_{C^3}(m, d') \right).$$

We do not know whether the K"unneth isomorphism holds above. In Corollary 4.5 we discuss the analogous statement in the localized equivariant situation (with respect to the two dimensional Calabi-Yau torus of $C^{3}$), when the K"unneth isomorphism holds.

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2. Preliminaries

2.1. **Notations.** The spaces considered in this paper are defined over the complex field $\mathbb{C}$ and are quotient stacks $\mathcal{X} = A/G$, where $A$ is a dg-scheme (or also called derived scheme), the derived zero locus of a section $s$ of a finite rank vector bundle $E$ on a finite type separated scheme $X$ over $\mathbb{C}$, and $G$ is a reductive group. For such a dg-scheme $A$, let $\dim A := \dim X - \text{rank}(E)$, let $A^{cl} := Z(s) \subset X$ be the (classical) zero locus, and let $\mathcal{X}^{cl} := A^{cl}/G$. We denote by $O_{\mathcal{X}}$ or $O_{A}$ the structure sheaf of $\mathcal{X}$. We denote by $D^{b}(\mathcal{X})$ the bounded derived category of coherent sheaves on $\mathcal{X}$.

For $G$ a reductive group and $A$ a dg-scheme as above, denote by $A/G$ the corresponding quotient stack and by $A//G$ the quotient dg-scheme with dg-ring of regular functions $O^{G}_{A}$.

In this paper, all the dg-categories are $\mathbb{C}$-linear pre-triangulated dg-categories, in particular their homotopy categories are triangulated categories. For $D$ a dg-category, we denote by $K(D)$ the Grothendieck group of the homotopy category.
of $D$. For $\mathcal{X}$ a stack as above, we denote by $G(\mathcal{X}) = G(\mathcal{X}^{cl})$ the Grothendieck group of $D^b\text{Coh}(\mathcal{X})$ and by $K(\mathcal{X})$ the Grothendieck group of the category of perfect complexes $\text{Perf}(\mathcal{X}) \subset D^b(\mathcal{X})$.

2.2. Weights and partitions.

2.2.1. Let $Q = (I, E)$ be the quiver with vertex set $I = \{1\}$ and edge set $E = \{x, y, z\}$. For $a \in \mathbb{N}$, let $Q^a = (J, E^a)$ be the quiver with vertex set $J = \{0, 1\}$ and edge set $E^a$ containing three loops $E = \{x, y, z\}$ at 1, $a$ edges from 0 to 1, and $a$ edges from 1 to 0. Then $Q^0$ is the disjoint union of $Q$ and the quiver with one vertex 0. Let $Q^{af} = (J, E^{af})$ be the quiver with edges $E^{af} = e \sqcup E^a$, where $e$ is an edge from 0 to 1, see Figure 1 for a picture of $Q^{3f}$.

For $d \in \mathbb{N}$, let $V$ be a $\mathbb{C}$-vector space of dimension $d$. We often write $GL(d)$ as $GL(V)$. Its Lie algebra is denoted by $\mathfrak{gl}(d) = \mathfrak{gl}(V) := \text{End}(V)$. When the dimension is clear from the context, we drop $d$ from its notation and write it as $\mathfrak{g}$. Consider the $GL(V) \cong GL(d)$ representations:

\[
R(d) := \mathfrak{gl}(V)^{\oplus 3},
\]

\[
R^a(1, d) := V^{\otimes a} \oplus (V^*)^{\otimes a} \oplus \mathfrak{gl}(V)^{\oplus 3},
\]

\[
R^{af}(1, d) := V^{\otimes (a+1)} \oplus (V^*)^{\otimes a} \oplus \mathfrak{gl}(V)^{\oplus 3}.
\]

Define the following stacks:

\[
\mathcal{X}(d) := R(d)/GL(d),
\]

\[
\mathcal{X}^a(1, d) := R^a(1, d)/GL(d),
\]

\[
\mathcal{X}^{af}(1, d) := R^{af}(1, d)/GL(d).
\]

2.2.2. Fix $T(d) \subset GL(d)$ the maximal torus consisting of diagonal matrices. Denote by $M(d)$ the weight space of $T(d)$ and let $M(d)_\mathbb{R} := M(d) \otimes \mathbb{R}$. Let $\beta_1, \ldots, \beta_d$ be the simple roots of $GL(d)$. A weight $\chi = \sum_{i=1}^d c_i \beta_i$ is dominant (resp. strictly dominant) if

\[
c_1 \leq \cdots \leq c_d, \quad (\text{resp. } c_1 < \cdots < c_d).
\]

We denote by $M^+ \subset M$ and $M^+_\mathbb{R} \subset M_\mathbb{R}$ the dominant chambers. When we want to emphasize the dimension vector, we write $M(d)$ etc. Denote by $N$ the coweight lattice of $T(d)$ and by $N_\mathbb{R} := N \otimes \mathbb{R}$. Let $\langle \ , \ \rangle$ be the natural pairing between $N_\mathbb{R}$ and $M_\mathbb{R}$.

Let $W = \mathfrak{S}_d$ be the Weyl group of $GL(d)$. For $\chi \in M(d)^+$, let $\Gamma_{GL(d)}(\chi)$ be the irreducible representation of $GL(d)$ of highest weight $\chi$. We drop $GL(d)$ from the notation if the dimension vector $d$ is clear from the context. Let $w*\chi := w(\chi+\rho) - \rho$ be the Weyl-shifted action of $w \in W$ on $\chi \in M(d)_\mathbb{R}$. We denote by $\ell(w)$ the length of $w \in W$.

2.2.3. Denote by $\mathcal{W}^a$ the multiset of $T(d)$-weights of $R^a(1, d)$ and by $\mathcal{W}^{af}$ the multiset of $T(d)$-weights of $R^{af}(1, d)$. Namely, we have

\[
\mathcal{W}^a = \{ (\beta_i - \beta_j)^x, (\pm \beta_i)^y \mid 1 \leq i, j \leq d \}, \quad \mathcal{W}^{af} = \mathcal{W}^a \cup \{ \beta_i \mid 1 \leq i \leq d \}.
\]

We denote by $\rho$ half the sum of positive roots of $GL(d)$. In our convention of the dominant chamber, it is given by

\[
\rho = \frac{1}{2} g^{\lambda < 0} = \frac{1}{2} \sum_{j < i} (\beta_i - \beta_j),
\]
where \( \lambda \) is the antiderominant cocharacter \( \lambda(t) = (t^d, t^{d-1}, \ldots, t) \). We denote by \( 1_d := z \cdot \text{Id} \) the diagonal cocharacter of \( T(d) \). We define the weights in \( M_k \):

\[
\sigma_d := \sum_{j=1}^{d} \beta_j, \quad \tau_d := \frac{1}{d} \sigma_d.
\]

### 2.2.4

Let \( G \) be a reductive group (in this paper, one may assume \( G = GL(d) \)), let \( X \) be a \( G \)-representation, and let \( \mathcal{X} = X/G \) be the corresponding quotient stack. Let \( \mathcal{V} \) be the multiset of \( T(d) \)-weights of \( X \). For a cocharacter \( \lambda \) of \( T(d) \), let \( X^\lambda \subset X \) be the subspace generated by weights \( \beta \in \mathcal{V} \) such that \( \langle \lambda, \beta \rangle = 0 \), let \( X^{\lambda \geq 0} \subset X \) be the subspace generated by weights \( \beta \in \mathcal{V} \) such that \( \langle \lambda, \beta \rangle \geq 0 \), and let \( G^\lambda \) and \( G^{\lambda \geq 0} \) be the Levi and parabolic groups associated to \( \lambda \). Consider the fixed and attracting stacks

\[
\mathcal{X}^\lambda := X^\lambda / G^\lambda, \quad \mathcal{X}^{\lambda \geq 0} := X^{\lambda \geq 0} / G^{\lambda \geq 0}
\]

with maps

\[
\mathcal{X}^\lambda \xrightarrow{\rho_\lambda} \mathcal{X}^{\lambda \geq 0} \xrightarrow{p_\lambda} \mathcal{X}.
\]

We abuse notation and denote by \( \mathcal{X}^{\lambda \geq 0} \) the class

\[
\left[ X^{\lambda \geq 0} \right] - \left[ g^{\lambda \geq 0} \right] \in K_0(T(d)) = M
\]

and by \( \langle \lambda, \mathcal{X}^{\lambda \geq 0} \rangle \) the corresponding integer. We use the similar notation for \( \mathcal{X}^{\lambda > 0}, \mathcal{X}^{\lambda < 0} \) etc.

### 2.2.5

Let \( d \in \mathbb{N} \). We call \( \underline{d} := (d_i)_{i=1}^{k} \) a partition of \( d \) if \( d_i \in \mathbb{N} \) are all non-zero and \( \sum_{i=1}^{k} d_i = d \). We similarly define partitions of \((d, w) \in \mathbb{N} \times \mathbb{Z} \). For a partition \((d_i)_{i=1}^{k} \) of \( d \), there is an antiderominant cocharacter \( \lambda \) of \( T(d) \) such that \( \mathcal{X}(d)^\lambda \cong \times_{i=1}^{k} \mathcal{X}(d_i) \).

For example we can take

\[
\lambda = \left( \bar{t}^{d_1}, \ldots, \bar{t}^{d_k}, \bar{t}^{k-1}, \ldots, \bar{t}, t, \ldots, t \right).
\]

We have the maps from (2.2)

\[
\times_{i=1}^{k} \mathcal{X}(d_i) \xrightarrow{q_\lambda} \mathcal{X}(d)^{\lambda \geq 0} \xrightarrow{p_\lambda} \mathcal{X}(d).
\]

We also use the notations \( p_\lambda = p_{\underline{d}}, q_\lambda = q_{\underline{d}} \). Conversely, given an antiderominant cocharacter \( \lambda \), there is an associated partition \((d_i)_{i=1}^{k} \) inducing the diagram above. Define the length \( \ell(\lambda) := k \).

The stack \( \mathcal{X}(d)^{\lambda \geq 0} \) is isomorphic to the moduli stack of filtrations of \( Q \)-representations

\[
0 = R_0 \subset R_1 \subset \cdots \subset R_k
\]

such that \( R_i / R_{i-1} \) has dimension \( d_i \). The morphism \( q_\lambda \) sends the above filtration to its associated graded, and \( p_\lambda \) sends it to \( R_k \). The categorical Hall product for \( Q \) is given by the functor \( p_\lambda q_\lambda^* = p_{\underline{d}} q_{\underline{d}}^* \) and denoted by

\[
*: D^b(\mathcal{X}(d_1)) \boxtimes \cdots \boxtimes D^b(\mathcal{X}(d_k)) \rightarrow D^b(\mathcal{X}(d)).
\]

We may drop the subscript \( \lambda \) or \( \underline{d} \) in the functors \( p_* \) and \( q^* \) when the cocharacter \( \lambda \) or the partition \( \underline{d} \) is clear.

We will also use the Hall products for the quivers \( Q^e \) and \( Q^{af} \). For \( e \leq d \), let \((d_i)_{i=1}^{k} \) be a partition of \( e \) and set \( d' = d - e \). Let \( \lambda \) be the antiderominant cocharacter
of $T(e)$ given by (2.3), and set $\chi' = (\lambda, 1, d)$ which is an antidominant cocharacter of $T(d)$. Then the diagrams of (2.2) for $X^a(1, d)$, $X^{a_f}(1, d)$ are given by

\begin{equation}
\chi_{i=1}^k X(d_i) \times X^a(1, d') \xrightarrow{\beta} X^a(1, d)^{\lambda \geq 0} \xrightarrow{\alpha} X^a(1, d),
\end{equation}

\begin{equation}
\chi_{i=1}^k X(d_i) \times X^{a_f}(1, d') \xrightarrow{\beta} X^{a_f}(1, d)^{\lambda \geq 0} \xrightarrow{\alpha} X^{a_f}(1, d)
\end{equation}

respectively. The functors $p_{\lambda, q}\chi_\tau$ give categorical Hall products

\begin{equation}
*: D^b(X(d_1)) \boxtimes \cdots \boxtimes D^b(X(d_k)) \boxtimes D^b(X^a(1, d')) \rightarrow D^b(X^a(1, d)),
\end{equation}

\begin{equation}
*: D^b(X(d_1)) \boxtimes \cdots \boxtimes D^b(X(d_k)) \boxtimes D^b(X^{a_f}(1, d')) \rightarrow D^b(X^{a_f}(1, d)).
\end{equation}

2.2.6. Let $(d_i)$ be a partition of $d$. There is an identification

$$\bigoplus_{i=1}^k M(d_i) \cong M(d),$$

where the simple roots $\beta_j$ in $M(d_1)$ correspond to the first $d_1$ simple roots $\beta_j$ of $M(d)$ etc.

2.3. Polytopes. We construct several polytopes in $M(d)\mathbb{R}$ which will be used to define categories in Subsection 2.7. The polytope $W(d)$ is defined as

\begin{equation}
W(d) := \frac{3}{2} \sum \{0, \beta_i - \beta_j\} + \mathbb{R} \tau_d \subset M(d)\mathbb{R},
\end{equation}

where the Minkowski sum is after all $1 \leq i, j \leq d$ and where $\tau_d$ is given by (2.1). Consider the hyperplane

\begin{equation}
W(d)_w := \frac{3}{2} \sum \{0, \beta_i - \beta_j\} + w \tau_d \subset W(d).
\end{equation}

The polytope $V(d)$ is defined as

\begin{equation}
V(d) := \frac{3}{2} \sum \{0, \beta_i - \beta_j\} + \sum \{-\beta_k, 0\} \subset M(d)\mathbb{R},
\end{equation}

where the Minkowski sum is after all $1 \leq i, j, k \leq d$. Note that the definition of the polytope $V(d)$ differs by the one used in [PTa] by a translation by $\sigma_d$. We let $V(d)_w \subset V(d)$ be the subspace of weights $\chi$ such that $(1, \chi) = w$.

Finally, define the (not necessarily closed) polytopes in $M(d)\mathbb{R}$:

$$W^a(1, d) := \frac{3}{2} \sum \{0, \beta_i - \beta_j\} + \frac{a}{2} \sum \{-\beta_k, \beta_k\},$$

$$V^a(1, d) := \frac{3}{2} \sum \{0, \beta_i - \beta_j\} + \frac{a}{2} \sum \{-\beta_k, \beta_k\} + \sum \{-\beta_k, 0\},$$

where the Minkowski sums are after all $1 \leq i, j, k \leq d$.

2.4. A corollary of the Borel-Weyl-Bott theorem. For future reference, we state a result from [HLS20, Section 3.2]. We continue with the notations from Subsection 2.2.4. For a weight $\chi \in M$, let $\chi^+$ be the dominant Weyl-shifted conjugate of $\chi$ if it exists, and let $\chi^+ = 0$ otherwise. Let $V$ be the multiset of weights in $X$. For a multiset $J \subset V$, let

$$\sigma_J := \sum_{\beta \in J} \beta.$$

For a weight $\chi \in M$, let $w$ be the element of the Weyl group such that $w * (\chi - \sigma_J)$ is dominant or zero. It has length $\ell(w) := \ell(J)$.
Proposition 2.1. Let $G$ be a reductive group, let $X$ be a $G$-representation, and let $\lambda$ be a cocharacter of the maximal torus $T \subset G$. Recall the fixed and attracting stacks and the corresponding maps

$$X^{\lambda}/G^{\lambda} \xrightarrow{q_{\lambda}} X^{\lambda \geq 0}/G^{\lambda \geq 0} \xrightarrow{p_{\lambda}} X/G.$$ 

Let $\chi$ be a weight of $T$. Then there is a quasi-isomorphism

$$\left( \bigoplus O_X \otimes \Gamma_G \left( (\chi - \sigma_j)^+ \right) \middle| |J| - \ell(J), d \right) \xrightarrow{\sim} p_{\lambda,*} q_{\lambda}^{-1} (O_X \otimes \Gamma_G(\chi)),$$

where the complex on the left hand side has terms (shifted) vector bundles for all multisets $J \subset \{ \beta \in V \mid \langle \lambda, \beta \rangle < 0 \}$.

2.5. Semiorthogonal decompositions. Let $R$ be a set. Consider a set $O \subset R \times R$ such that for any $i, j \in R$ we have $(i, j) \in O$, or $(j, i) \in O$, or both $(i, j) \in O$ and $(j, i) \in O$.

Let $T$ be a dg-category. We will construct semiorthogonal decompositions

$$T = \langle A_i \mid i \in R \rangle$$

with summands dg-subcategories $A_i$ indexed by $i \in R$ such that for any $i, j \in R$ with $(i, j) \in O$ and objects $A_i \in A_i, A_j \in A_j$, we have

$$\text{Hom}_T(A_i, A_j) = 0.$$

2.6. Matrix factorizations. Let $\mathcal{X}$ be a smooth and let $w: \mathcal{X} \rightarrow \mathbb{C}$ be a regular function. We denote by

$$\text{MF}(\mathcal{X}, w)$$

the category of matrix factorizations of $w$. It consists of objects $(\alpha: F \rightarrow G; \beta)$ with $F, G \in \text{Coh}(\mathcal{X})$ such that both of $\alpha \circ \beta$ and $\beta \circ \alpha$ are multiplications by $w$. We refer to [PTa, Subsection 2.6] for details about categories of matrix factorizations.

2.7. Categories of generators.

2.7.1. For $w \in \mathbb{Z}$, we denote by $D^b(\mathcal{X}(d))_w$ the subcategory of $D^b(\mathcal{X}(d))$ consisting of objects of weight $w$ with respect to the diagonal cocharacter $1_d$ of $T(d)$. We have the direct sum decomposition

$$D^b(\mathcal{X}(d)) = \bigoplus_{w \in \mathbb{Z}} D^b(\mathcal{X}(d))_w.$$ 

We define the dg-subcategories

$$\mathbb{M}(d) \subset D^b(\mathcal{X}(d)), \quad (\text{resp. } \mathbb{M}(d)_w \subset D^b(\mathcal{X}(d))_w)$$

to be generated by the vector bundles $O_{\mathcal{X}(d)} \otimes \Gamma_{GL(d)}(\chi)$, where $\chi$ is a dominant weight of $T(d)$ such that

$$\chi + \rho \in \mathcal{W}(d), \quad (\text{resp. } \chi + \rho \in \mathcal{W}(d)_w).$$ 

Note that $\mathbb{M}(d)$ decomposes into the direct sum of $\mathbb{M}(d)_w$ for $w \in \mathbb{Z}$. Moreover, taking the tensor product with the determinant character $\text{det}: GL(d) \rightarrow \mathbb{C}^\ast$ gives an equivalence

$$\otimes \text{det}: \mathbb{M}(d)_w \xrightarrow{\sim} \mathbb{M}(d)_{d+w}.$$

Let $\mu \in \mathbb{R}$ and let $\delta := \mu \sigma_d \in M(d)_{\mathbb{R}}$. We define the dg-subcategories

$$\mathbb{D}(d; \delta) \subset D^b(\mathcal{X}(d)), \quad (\text{resp. } \mathbb{D}(d; \delta)_w \subset D^b(\mathcal{X}(d))_w)$$

as suitable subcategories of $D^b(\mathcal{X}(d))$. The category of matrix factorizations. Let $\mathcal{X}$ be a smooth and let $w: \mathcal{X} \rightarrow \mathbb{C}$ be a regular function. We denote by
generated by the vector bundles $\mathcal{O}_{X(d)} \otimes \Gamma_{GL(d)}(\chi)$, where $\chi$ is a dominant weight of $T(d)$ such that
\begin{equation}
\chi + \rho + \delta \in \mathbf{V}(d), \text{ (resp. } \chi + \rho + \delta \in \mathbf{V}(d)_w) .
\end{equation}

Note that $\mathbb{D}(d; \delta)$ decomposes into the direct sum of $\mathbb{D}(d; \delta)_w$ for $w \in \mathbb{Z}$ defined as above. We will use the following proposition proved in [PTa]:

**Proposition 2.2.** ([PTa], Proposition 3.9) There is a semiorthogonal decomposition
\begin{equation}
\mathbb{D}(d; \delta) = \left\langle \bigoplus_{i=1}^{k} \mathbb{M}(d_i)_{w_i} \right\rangle .
\end{equation}

Here, the right hand side is after all partitions $d_1 + \cdots + d_k = d$ such that, by setting $v_i := w_i + d_i(\sum_{j>i} d_j - \sum_{j<i} d_j)$, we have
\begin{equation}
-1 - \mu \leq \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k} \leq -\mu .
\end{equation}

Moreover, each fully-faithful functor $\bigoplus_{i=1}^{k} \mathbb{M}(d_i)_{w_i} \to \mathbb{D}(d; \delta)$ is given by the restriction of the categorical Hall product (2.14).

2.7.2. We fix $\mu \in \mathbb{R}$ and set $\delta := \mu \sigma_d$. We define the dg-subcategories
\begin{equation}
\mathbb{M}^{a}(1, d; \delta) \subset D^b(X^a(1), d), \text{ (resp. } \mathbb{F}^{a}(1, d; \delta) \subset D^b(X^a(1), d))
\end{equation}
to be generated by the vector bundles $\mathcal{O}_{X^a(1,d)} \otimes \Gamma_{GL(d)}(\chi)$ (resp. $\mathcal{O}_{X^a(1,d)} \otimes \Gamma_{GL(d)}(\chi))$, where $\chi$ is a dominant weight of $T(d)$ such that
\begin{equation}
\chi + \rho + \delta \in \mathbf{W}^a(1, d) .
\end{equation}

We define the dg-subcategories
\begin{equation}
\mathbb{D}^{a}(1, d; \delta) \subset D^b(X^a(1), d), \text{ (resp. } \mathbb{E}^{a}(1, d; \delta) \subset D^b(X^a(1), d))
\end{equation}
to be generated by the vector bundles $\mathcal{O}_{X^a(1,d)} \otimes \Gamma_{GL(d)}(\chi)$ (resp. $\mathcal{O}_{X^a(1,d)} \otimes \Gamma_{GL(d)}(\chi))$, where $\chi$ is a dominant weight of $T(d)$ such that
\begin{equation}
\chi + \rho + \delta \in \mathbf{W}^a(1, d) .
\end{equation}

Note that for the projection $b: X^a(1, d) \to X^a(1, d)$, the pull-back $b^*$ restricts to the functors
\begin{equation}
b^*: \mathbb{M}^a(1, d; \delta) \to \mathbb{F}^{a}(1, d; \delta), \text{ } b^*: \mathbb{D}^{a}(1, d; \delta) \to \mathbb{E}^{a}(1, d; \delta)
\end{equation}
whose essential images generate $\mathbb{F}^{a}(1, d; \delta)$ and $\mathbb{E}^{a}(1, d; \delta)$, respectively.

2.7.3. Let $\text{Tr} W$ be the regular function
\begin{equation}
\text{Tr} W := \text{Tr} X[Y, Z]: \mathcal{X}(d) = \mathfrak{gl}(d)^{\otimes 3}/GL(d) \to \mathbb{C} .
\end{equation}

We define the subcategory
\begin{equation}
\mathbb{S}(d) := \text{MF}(\mathbb{M}(d), \text{Tr} W) \subset \text{MF}(\mathcal{X}(d), \text{Tr} W)
\end{equation}
to be the subcategory of matrix factorizations $(\alpha: F \Rightarrow G; \beta)$ with $F$ and $G$ in $\mathbb{M}(d)$. It decomposes into the direct sum of $\mathbb{S}(d)_w$ for $w \in \mathbb{Z}$, where $\mathbb{S}(d)_w$ is defined similarly to $\mathbb{S}(d)$ using $\mathbb{M}(d)_w$. There is an equivalence:
\begin{equation}
\otimes \det: \mathbb{S}(d)_w \sim \mathbb{S}(d)_{d+w} .
\end{equation}

The category $\mathbb{S}(d)_w$ is called the *quasi-BPS category* of $\mathbb{C}^d$ for $(d, w) \in \mathbb{N} \times \mathbb{Z}$ in [PTa], see [PTb] for its properties and computations of its K-theory.

2.8. More on weights.
2.8.1. Let $A$ be a partition $(d_i, w_i)_{i=1}^k$ of $(d, w)$ and consider its corresponding antidominant cocharacter $\lambda$. Define the weights

$$\chi_A := \sum_{i=1}^k w_i \tau_{d_i}, \quad \chi'_A := \chi_A + g^{\lambda>0}.$$ 

Consider weights $\chi'_i \in M(d_i)_{\mathbb{R}}$ such that

$$\chi'_A = \sum_{i=1}^k \chi'_i.$$ 

Let $v_i$ be the sum of coefficients of $\chi'_i$ for $1 \leq i \leq k$; alternatively, $v_i := \langle 1_{d_i}, \chi'_i \rangle$. We denote the above transformation by

$$A \mapsto A', \quad (d_i, w_i)_{i=1}^k \mapsto (d_i, v_i)_{i=1}^k.$$ 

This transformation explains how to change weights under the Koszul equivalence [Pada Proposition 3.1].

2.8.2. We recall a construction from [PTa], see [PTa, Subsections 3.3.2 and 3.3.3 and Propositions 3.4 and 3.7] which is also used in the proof of Proposition 2.2. Let $\mu \in \mathbb{R}$, and let $\chi$ be a dominant weight such that $\chi + \rho + \mu \sigma_d \in \mathcal{V}(d)$. Then there exists a unique partition $(d_i)_{i=1}^k$ of $d$, integers $(w_i)_{i=1}^k$ such that for $v_i$ as in (2.18), we have

$$-1 - \mu \leq \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k} \leq -\mu,$$

and weights $\chi_i \in M(d_i)$ such that

$$\chi = \sum_{i=1}^k \chi_i \text{ with } \chi_i + \rho_i + \mu \sigma_{d_i} \in \mathcal{W}(d_i)_{w_i} \text{ for } 1 \leq i \leq k.$$ 

Here, we denote by $\rho_i$ half the sum of positive roots of $GL(d_i)$ for $1 \leq i \leq k$.

The converse is also true, and it follows from [PTa Proposition 3.8]. Consider a partition $(d_i)_{i=1}^k$ of $d$, integers $(w_i)_{i=1}^k$ such that for $v_i$ as in (2.18), the inequality (2.19) holds, and weights $\chi_i$ with $\chi_i + \rho_i + \mu \sigma_{d_i} \in \mathcal{W}(d_i)_{w_i}$ for $1 \leq i \leq k$. Let $\chi := \sum_{i=1}^k \chi_i$. Then we have

$$\chi + \rho + \mu \sigma_d \in \mathcal{V}(d).$$

3. The Categorical Wall-Crossing Formula for DT/PT Quivers

Fix $a \geq 1$. Let $\chi_0 : GL(V) \to \mathbb{C}^*$ be the determinant character $g \mapsto \det g$. Recall the varieties:

$$I^n(d) := R^{a^n}(1, d)^{\chi_0-ss}/GL(d),$$

$$P^n(d) := R^{a^n}(1, d)^{\chi_0^{-1}-ss}/GL(d).$$
Remark 3.1. It is easy to describe the $\chi^+_{0}$-semistable loci on $R^{af}(1,d)$: a $\chi^0$-semistable representation (resp. $\chi^{-1}$-semistable representation) consists of tuples 

\[(u_1, \ldots, u_{a+1}, v_1, \ldots, v_a, A, B, C) \in V^\oplus(a+1) \oplus (V^\vee)^\oplus a \oplus gl(V)^\oplus 3
\]
such that (see \cite[Lemma 7.10]{Tod22})

\[C\langle A, B, C \rangle(u_1, \ldots, u_{a+1}) = V, \text{ (resp. } C\langle A, B, C \rangle(v_1, \ldots, v_a) = V^\vee)\].

The main result of this section is Theorem 1.1. The most important ingredient in its proof is the following semiorthogonal decomposition for subcategories of $D^b(X^a(1,d))$:

**Theorem 3.2.** Let $\mu \in \mathbb{R}$ and consider the weight $\delta := \mu \sigma_d \in M(d)_\mathbb{R}$. There is a semiorthogonal decomposition

\[(3.1) \quad D^a(1,d;\delta) = \left( \bigotimes_{i=1}^k M(d_i)_{w_i} \right) \otimes M^a \left( 1, d'; \delta' \right).
\]

The right hand side is after all $d' \leq d$, all decompositions $\sum_{i=1}^k d_i = d - d'$, and all integers $w_i \in \mathbb{Z}$ for $1 \leq i \leq k$ such that for

\[(3.2) \quad v_i := w_i + d_i \left( d' + \sum_{j>i} d_j - \sum_{j<i} d_j \right),
\]

we have

\[(3.3) \quad -1 - \mu - \frac{a}{2} \leq \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k} \leq -\mu - \frac{a}{2},
\]

and where $\delta' := (\mu - d + d') \sigma_{d'}$. The orthogonality of the categories is as in Subsections \ref{subsec:orthogonality}. The functors

\[\left( \bigotimes_{i=1}^k M(d_i)_{w_i} \right) \otimes M^a \left( 1, d'; \delta' \right) \to D^a(1,d;\delta)
\]

are given by the Hall product \eqref{eq:hall_product}. Given $\mu \in \mathbb{R}$ and a partition $(d_i)_{i=1}^k$ of $d' \leq d$, we refer to \eqref{eq:orthogonality_conditions} as a condition on the tuple of integers $(w_i)_{i=1}^k$.

### 3.1. Weight decompositions.

We discuss several preliminary results about decompositions of weights, or, alternatively, about decompositions of polytopes in $M(d)_\mathbb{R}$.

**Proposition 3.3.** Let $\chi$ be a strictly dominant weight such that $\chi \in V^a(1,d)$. There exists a decomposition

\[(3.4) \quad D : \chi = \sum_{1 \leq j < i \leq d} c_{ij}(\beta_i - \beta_j) + \sum_{i=1}^d c_i \beta_i
\]

with coefficients

\[(3.5) \quad 0 \leq c_{ij} \leq \frac{3}{2} \text{ and } -\frac{a+2}{2} \leq c_i \leq \frac{a}{2} \text{ for } 1 \leq j < i \leq d.
\]

**Proof.** The argument used in \cite[Proposition 3.5]{PTa} also applies here. \qed
Proposition 3.4. Let \( \chi \) be a strictly dominant weight such that \( \chi \in V^a(1,d) \). Then there exists a unique \( e \leq d \) such that

\[
(3.6) \quad \chi = \sum_{j<i\leq e} c_{ij}(\beta_i - \beta_j) + \sum_{e<j<i} c_{ij}(\beta_i - \beta_j) + \sum_{j\leq e<i} \frac{3}{2}(\beta_i - \beta_j) + \sum_{1\leq i\leq d} c_i\beta_i,
\]

where \( c_{ij} \) and \( c_i \) satisfy

\[
0 \leq c_{ij} \leq \frac{3}{2} \text{ for } j < i, \quad -\frac{a+2}{2} \leq c_i \leq -\frac{a}{2} \text{ if } i \leq e, \quad -\frac{a}{2} < c_i \leq \frac{a}{2} \text{ if } i > e.
\]

Let \( f := d - e \). Alternatively, we have that

\[
(3.7) \quad \chi \in \left( -\frac{a+3f}{2}\sigma_e + V(e) \right) + \left( \frac{3e}{2}\sigma_f + W^a(1,f) \right).
\]

**Step 1.** There exists such \( e \leq d \).

**Proof.** Consider a decomposition

\[
(3.8) \quad D: \chi = \sum_{1\leq j<i\leq d} c_{ij}(\beta_i - \beta_j) + \sum_{1\leq i\leq d} c_i\beta_i
\]

as in Proposition 3.3. A choice of the decomposition \( D \) corresponds to a point in the fiber of the following continuous map at \( \chi \in M(d) \):

\[
\gamma: \left[ 0, \frac{3}{2} \right]^{d(d-1)/2} \times \left[ -\frac{a+2}{2}, \frac{a}{2} \right]^d \to M(d)_R
\]

defined by sending \((c_{ij}, c_i)\) to the right hand side in (3.8). For each decomposition \( D \), we set

\[
(3.9) \quad I := \left\{ i \mid c_i > -\frac{a}{2} \right\}, \quad J := \left\{ i \mid c_i \leq -\frac{a}{2} \right\}.
\]

Define the function \( \sigma: \gamma^{-1}(\chi) \to \mathbb{R} \):

\[
\sigma(D) := \sum_{i \in I} \left( c_i + \frac{a}{2} \right).
\]

The function \( \sigma \) is continuous and the set \( \gamma^{-1}(\chi) \) is compact. Among all decompositions \( D \) for which \( \sigma \) is minimal, choose one for which \( |I| \) is as large as possible. Let \( m := \sigma(D) \).

We first show that

\[
(3.10) \quad c_{ij} = \frac{3}{2} \text{ if } j < i, (i,j) \in I \times J.
\]

Assume there exists \( j < i \) such that \((i,j) \in I \times J \) and \( c_{ij} < 3/2 \). We take \( 0 < \varepsilon \ll 1 \) and set \( c'_i = c_i - \varepsilon \), \( c'_j = c_j + \varepsilon \), \( c'_{ij} = c_{ij} + \varepsilon \). Consider the decomposition \( D' \) with the same coefficients as \( D \) except for \( c'_{ij} \), \( c'_i \), and \( c'_j \). Then \( D' \) is a decomposition of \( \chi \) because

\[
c_{ij}(\beta_i - \beta_j) + c_i\beta_i + c_j\beta_j = (c_{ij} + \varepsilon)(\beta_i - \beta_j) + (c_i - \varepsilon)\beta_i + (c_j + \varepsilon)\beta_j.
\]

The decomposition \( D' \) satisfies (3.5). If \( c'_j < -a/2 \), then we may assume that \( I' = I \) and then \( \sigma(D') < \sigma(D) \). If \( c'_j = -a/2 \), then \( I' = I \cup \{ j \} \) and \( \sigma(D') = \sigma(D) \), which contradicts the maximality of \( |I| \) among all decompositions with \( \sigma(D) = m \). Thus (3.10) holds.

Similarly, we show that

\[
(3.11) \quad c_{ij} = 0 \text{ if } j < i, (i,j) \in J \times I.
\]
Assume there exists \( i > j \) such that \((i, j) \in J \times I\) and \( c_{ij} > 0\). We take \( 0 < \varepsilon \ll 1\) and set \( \epsilon'_i = \epsilon_i + \varepsilon, \epsilon'_j = \epsilon_j - \varepsilon, \epsilon'_{ij} = \epsilon_{ij} - \varepsilon \). Consider the decomposition \( D' \) with the same coefficients as \( D \) except for \( \epsilon'_i, \epsilon'_j, \) and \( \epsilon'_{ij} \). Then \( D' \) is a decomposition of \( \chi \) because

\[
\epsilon'_{ij}(\beta_i - \beta_j) + \epsilon_i \beta_i + \epsilon_j \beta_j = (\epsilon_{ij} - \varepsilon)(\beta_i - \beta_j) + (\epsilon_i + \varepsilon)\beta_i + (\epsilon_j - \varepsilon)\beta_j
\]

and \( D' \) satisfies (3.10). If \( c_i < -a/2 \), then we can choose \( \varepsilon \) such that \( I' = I \) and then \( \sigma(D') < \sigma(D) \). If \( c_i = -a/2 \), then \( I' = I \cup \{i\} \) and \( \sigma(D') = \sigma(D) \), which contradicts the maximality of \( |I| \) among decompositions \( D \) with \( \sigma(D) = m \). Thus (3.11) holds.

We denote \( f = |I|, e = |J| = d - f \), and write

\[
\chi = \sum_{i=1}^d b_i \beta_i, \quad b_i = \sum_{j<i} c_{ij} - \sum_{j<i} c_{ji} + c_i.
\]

Let \( i' \in I \) and \( j' \in J \). Then from (3.10) and (3.11) and noting that \( c_{ij} \in [0, 3/2] \) for all \( j < i \), we have

\[
\begin{align*}
\epsilon'_{i'} > & -\frac{a}{2} + \frac{3}{2}|J \cap \{1, \ldots, i'-1\}| - \frac{3}{2}|I \cap \{i'+1, \ldots, d\}| \\
\epsilon_{j'} \leq & -\frac{a}{2} + \frac{3}{2}|J \cap \{1, \ldots, j'-1\}| - \frac{3}{2}|I \cap \{j'+1, \ldots, d\}|.
\end{align*}
\]

We claim that

\[
(3.13) \quad J = \{1, \ldots, e\} \text{ and } I = \{e + 1, \ldots, d\}.
\]

Assume the claim in (3.13) is false. Thus there exist \( i' \in I, j' \in J \) with \( i' + 1 = j' \). Then \( \epsilon_{i'} < \epsilon_{j'} \) because \( \chi \) is strictly dominant. Combining inequalities (3.12), we obtain:

\[
\begin{align*}
-\frac{a}{2} + \frac{3}{2}|J \cap \{1, \ldots, j'-1\}| - \frac{3}{2}|I \cap \{j'+1, \ldots, d\}| \\
> -\frac{a}{2} + \frac{3}{2}|J \cap \{1, \ldots, i'-1\}| - \frac{3}{2}|I \cap \{i'+1, \ldots, d\}|,
\end{align*}
\]

and so

\[
(3.14) \quad |J \cap \{i', \ldots, j'-1\}| + |I \cap \{i'+1, \ldots, j'\}| > 0.
\]

However, \( I \cap \{i' + 1, \ldots, j'\} = I \cap \{j'\} = \emptyset \) and \( J \cap \{i', \ldots, j'-1\} = J \cap \{i'\} = \emptyset \). Thus (3.14) is false and so the claim in (3.13) is true. Consider the weights in \( V(\varepsilon) \) and \( W(\sigma, 1, f) \), respectively:

\[
\begin{align*}
\chi_1 = & \sum_{j<i \leq e} c_{ij}(\beta_i - \beta_j) + \sum_{i \leq e} \left( \epsilon_i + \frac{a}{2} \right) \beta_i, \\
\chi_2 = & \sum_{e<j<i} c_{ij}(\beta_i - \beta_j) + \sum_{i \geq e+1} c_i \beta_i.
\end{align*}
\]

Then

\[
\chi = \left( -\frac{a + 3f}{2\sigma_e} \chi_1 + \chi_2 \right) + \left( \frac{3e}{2\sigma_f} \chi_2 \right).
\]

The conclusion thus follows.

\( \square \)

**Step 2.** The \( e \leq d \) with the desired property is unique.
**Proof.** Assume there exist $e' < e$ such that (3.7) is satisfied for $e$ and $e'$. Let $f = d - e$. Recall that $\chi = \sum_{i=1}^{d} b_i \beta_i$. The weight $\chi$ has a description (3.6) for $e'$, so

$$
(3.15) \quad \sum_{e' < i \leq e} b_i = \sum_{e' < i \leq e} \left( \sum_{e' < j < i} c_{ij} - \sum_{i < j} c_{ji} + \frac{3}{2} e' + c_i \right)
$$

$$
= - \sum_{e' < i < e} c_{ji} + \sum_{e' < i \leq e} \left( \frac{3}{2} e' + c_i \right)
$$

$$
> \left( -\frac{3}{2} f + \frac{3}{2} e' - \frac{1}{2} a \right) (e - e').
$$

Next, $\chi$ has a description (3.6) for $e$. We abuse notation and denote the coefficients of this description also by $c_{ij}$ and $c_i$. Similarly to (3.15), we have

$$
(3.16) \quad \sum_{e' < i \leq e} b_i = \sum_{e' < i \leq e} \left( - \sum_{i < j \leq e} c_{ji} + \sum_{i \leq j < e} c_{ij} - \frac{3}{2} f + c_i \right)
$$

$$
= - \sum_{j \leq e' < i \leq e} c_{ij} + \sum_{e' < i \leq e} \left( -\frac{3}{2} f + c_i \right)
$$

$$
\leq \left( \frac{3}{2} e' - \frac{3}{2} f - \frac{1}{2} a \right) (e - e').
$$

We obtain a contradiction by comparing (3.15) and (3.16). □

The decomposition of $V^a(1, 2)$ into chambers constructed in Proposition 3.4 is depicted in Figure 2.

3.2. **Invariants of weights.**

3.2.1. Let $e \leq d$ be two natural numbers and let $f = d - e$. Consider the antidominant cocharacter

$$
\tau_e := (t, \ldots, t, 1, \ldots, 1) : \mathbb{C}^* \to T(d).
$$

Fix $\mu \in \mathbb{R}$ and let $\delta := \mu \sigma_d$. Let $\chi$ be a dominant weight such that $\chi + \rho + \delta \in V^a(1, d)$. Let $e = e(\chi)$ be the integer $0 \leq e \leq d$ such that

$$
(3.17) \quad \chi + \rho + \delta = \sum_{j < i \leq e} c_{ij} (\beta_i - \beta_j) + \sum_{e < j < i} c_{ij} (\beta_i - \beta_j) + \sum_{j \leq e < i} \frac{3}{2} (\beta_i - \beta_j) + \sum_{i \leq j \leq d} c_i \beta_i,
$$

where $c_{ij}$ and $c_i$ are as in Proposition 3.4. We define $p(\chi)$ to be

$$
p(\chi) := \langle \tau_e, \chi + \rho + \delta \rangle + \frac{3}{2} ef + \frac{a}{2} e = \sum_{i=1}^{e} \left( c_i + \frac{a}{2} \right) \leq 0.
$$

The functions $e(\chi), p(\chi)$ depend on $\mu$. 
3.2.2. The following is a consequence of Proposition 3.4 and the discussion in Subsection 2.8.2.

**Proposition 3.5.** Let $\mu \in \mathbb{R}$, let $\delta = \mu \sigma_d$, and let $\chi$ be a dominant (integral) weight such that $\chi + \rho + \delta \in \mathcal{V}^a(1,d)$. Then there exists a unique $d' \leq d$, partition $(d_i)_{i=1}^k$ of $e := d - d'$, integers $(w_i)_{i=1}^k$ satisfying (3.3), and weights $\chi_i \in M(d_i)$, $\chi' \in M(d')$ such that $\chi = \chi_1 + \cdots + \chi_k + \chi'$ and

$$\chi_i + \rho_i \in \mathcal{W}(d_i)_{w_i}, \quad \chi' + \rho' + (\mu - e)\sigma_{d'} \in \mathcal{W}^a(1,d').$$

In the above, $\rho_i$ and $\rho'$ are half the sums of positive roots of $GL(d_i)$ and $GL(d')$, respectively, where $1 \leq i \leq k$.

Let $\mu \in \mathbb{R}$, let $\delta = \mu \sigma_d$, and let $\chi$ be a dominant weight such that $\chi + \rho + \delta \in \mathcal{V}^a(1,d)$. Let $S(\chi) = S$ be the ordered collection of pairs $S = (d_i, w_i)_{i=1}^k$ satisfying the condition in Proposition 3.5. We say $S$ is the type of $\chi$. Then $p(\chi)$ is alternatively written as

$$(3.18) \quad p(\chi) = \sum_{1 \leq i \leq k} w_i - (d - d') \left( -\mu - d' - \frac{a}{2} \right)$$

$$\quad = \sum_{1 \leq i \leq k} v_i - (d - d') \left( -\mu - \frac{a}{2} \right) \leq 0.$$

We also denote the above number by $p(S)$.

3.2.3. **Comparison of ordered pairs.** Fix $d \in \mathbb{N}$ and $\mu \in \mathbb{R}$. Let $R$ be the set

$$R = \{(d_i, w_i)_{i=1}^k \mid d_i \in \mathbb{N}, w_i \in \mathbb{Z}\}.$$
We define a subset $O \subset R \times R$ depending on $\mu$. A pair of types $(S', S)$ is in $O$ if either

- $p(S') > p(S)$, or
- $p(S') = p(S)$ and $\sum_{i=1}^{k'} d'_i < \sum_{i=1}^{k} d_i$, or
- $p(S') = p(S)$, and $\sum_{i=1}^{k'} d'_i = \sum_{i=1}^{k} d_i$, and $(S', S)$ is in the set $O$ from [PTa Subsection 3.4].

If $(S', S)$ is in $O$, we also write $S > S'$.

For $0 < \varepsilon \ll 1$, let $\mu = -a/2 - \varepsilon$. Recall the definition (3.12) of the integers $v_i$ from $u_i$. Then

$$p(S) = \sum_{i=1}^{k} v_i - \varepsilon \sum_{i=1}^{k} d_i.$$ 

In this case, the set $O$ contains pairs of types $(S', S)$ such that

- $\sum_{i=1}^{k'} v'_i > \sum_{i=1}^{k} v_i$, or
- $\sum_{i=1}^{k'} v'_i = \sum_{i=1}^{k} v_i$ and $\sum_{i=1}^{k'} d'_i < \sum_{i=1}^{k} d_i$, or
- $\sum_{i=1}^{k'} v'_i = \sum_{i=1}^{k} v_i$, and $\sum_{i=1}^{k'} d'_i = \sum_{i=1}^{k} d_i$, and $(S', S)$ is in the set $O$ from [PTa Subsection 3.4].

### 3.3. Weight comparison

We discuss some preliminary results on weights. We fix $\mu \in \mathbb{R}$ and let $\delta := \mu \sigma_d \in M(d)_{\mathbb{R}}$. For $\chi$ a weight and for $1 \leq l \leq d$, define

$$p_l(\chi) := (\tau_l, \chi + \rho + \delta) + \frac{3}{2} l(d-l) + \frac{a}{2} l.$$ 

**Lemma 3.6.** Let $\chi$ be a dominant weight such that $\chi + \rho + \delta \in V^\alpha(1, d)$ and let $e = e(\chi)$. Then $p_l(\chi) > p(\chi)$ and the inequality is strict if $l > e$.

**Proof.** Note that $p_e(\chi) = p(\chi)$. We write $\chi + \rho + \delta$ as (3.17), where $c_{ij}$ and $c_i$ are as in Proposition 3.4. We have

$$p_l(\chi) = - \sum_{i \leq l < j} c_{ij} + \sum_{i \leq l} c_i + \frac{3}{2} l(d-l) + \frac{a}{2} l$$

$$\geq \sum_{i \leq l} \left( c_i + \frac{a}{2} \right) \geq \sum_{i \leq e} \left( c_i + \frac{a}{2} \right) = p(\chi).$$

The inequality (*) holds from Lemma 3.7 and is strict if $l > e$. \qed

We have used the following lemma, whose proof is obvious:

**Lemma 3.7.** Let $x_1, \ldots, x_d \in \mathbb{R}$ and let $S := \{i \mid x_i \leq 0\} \subset \{1, \ldots, d\}$. Then for any subset $T \subset \{1, \ldots, d\}$, we have

$$\sum_{i \in T} x_i \geq \sum_{i \in S} x_i.$$ 

If equality holds, then $T \subset S$. If equality holds and $|T| = |S|$, then $T = S$.

For $e \leq d$, consider the multiset of weights $W^a_e := R^{a}(1, d)_{\mathbb{R}^{<0}}$, or explicitly

$$W^a_e = \{(\beta_i - \beta_j)^x \times \alpha | j \leq e < i\}.$$ 

**Proposition 3.8.** Let $\chi, \chi'$ be dominant weights such that $\chi + \rho + \delta \in V^\alpha(1, d)$ and $\chi' + \rho + \delta \in V^\alpha(1, d)$. Let $e = e(\chi)$, $e' = e(\chi')$, and let $I$ be a subset $W^a_{\rho}$. Let $e'' = e((\chi' - \sigma_I)^+)$. Suppose that either

(i) $p(\chi') > p(\chi)$ or

(ii) $p(\chi') > p(\chi)$. 

(ii) \( p(\chi') = p(\chi), e' < e \) or 
(iii) \( p(\chi') = p(\chi), e' = e, \) and \( I \neq \emptyset. \)

Then we have

\[
(3.19) \quad p_e((\chi' - \sigma_I)^+) \geq p((\chi' - \sigma_I)^+) \geq p(\chi).
\]

Moreover, if the second inequality is an equality, then \( p(\chi') = p(\chi) \) and \( e'' \leq e' \), and \( e'' < e' \) if \( I \neq \emptyset \). Hence \( e'' < e \), and thus the first inequality is strict by Lemma 3.7.

In particular, in each of the above three cases, we have

\[
\langle \tau_e, (\chi' - \sigma_I)^+ \rangle > \langle \tau_e, \chi \rangle.
\]

**Proof.** As the first inequality in (3.19) follows from Lemma 3.6 it is enough to prove the second inequality. Write

\[
\chi' + \rho + \delta = \sum_{1 \leq i,j \leq d} c'_{ij}(\beta_i - \beta_j) + \sum_{1 \leq i \leq d} c_i \beta_i,
\]

\[
\chi' - \sigma_I + \rho + \delta = \sum_{1 \leq i,j \leq d} \tilde{c}_{ij}(\beta_i - \beta_j) + \sum_{1 \leq i \leq d} \tilde{c}_i \beta_i,
\]

where \( c'_{ij}, c_i \) are as in Proposition 3.3. Then \( |\tilde{c}_{ij}| \leq 3/2 \) for all \( 1 \leq i, j \leq d, \) and \( \tilde{c}_i \geq c'_i \) for \( 1 \leq i \leq d \). Let \( w \in S_d \) be such that \( (\chi' - \sigma_I)^+ + \rho = w(\chi' - \sigma_I + \rho). \n\)

Then using Lemma 3.7 and the assumption (i) or (ii) or (iii), we have

\[
(3.20) \quad p((\chi' - \sigma_I)^+) = \sum_{w(i) \leq e''} \left( \tilde{c}_i + \frac{a}{2} \right) \geq \sum_{i=1}^{e''} \left( \tilde{c}_i + \frac{a}{2} \right) = \langle \tau_{e'}, (\chi' - \sigma_I)^+ \rangle = \langle \tau_{e'}, \chi' \rangle - \langle \tau_{e'}, \sigma_I \rangle.
\]

Therefore the second inequality in (3.19) also holds. Suppose that every inequality in (3.20) is an equality. Then \( p(\chi) = p(\chi') \) and \( e'' \leq e' \) from the equality of (\( \diamond \)) and Lemma 3.7. Assume that \( e'' = e' \) and \( I \neq \emptyset \). From the equality of (\( \star \)) and \( e'' = e' \), by Lemma 3.7 we have \( \{ w(1), \ldots, w(e') \} = \{ 1, \ldots, e' \} \). Then we have

\[
\langle \tau_{e'}, (\chi' - \sigma_I)^+ \rangle = \langle \tau_{e'}, \chi' \rangle - \langle \tau_{e'}, \sigma_I \rangle < 0.
\]

As \( I \in V_d^\mu \) is non-empty, we have \( \langle \tau_{e'}, \sigma_I \rangle < 0 \). Then \( p((\chi' - \sigma_I)^+) > p(\chi') \), which contradicts that (3.20) are equalities. \( \square \)

### 3.4. Generation

Fix \( \mu \in \mathbb{R} \) and let \( \delta := \mu \sigma_d \in M(d)_{\mathbb{R}}. \n\)

**Proposition 3.9.** The category \( \mathbb{D}^a(1, d; \delta) \) is generated by the images of the Hall products

\[
(3.21) \quad \left( \bigotimes_{i=1}^k \mathbb{M}(d_i)_{w_i} \right) \boxtimes \mathbb{M}^a (1, d'; \delta') \to \mathbb{D}^a(1, d; \delta)
\]

for all \( d' \leq d, \) all decompositions \( (d_i)_{i=1}^k \) of \( d - d' \), and tuples of integers \( (w_i)_{i=1}^k \) satisfying (3.20), and where \( \delta' := (\mu - d + d') \sigma_d. \n\)

We first explain that the images of the functors (3.21) are indeed in the category \( \mathbb{D}^a(1, d; \delta) \).

**Proposition 3.10.** Let \( (d_i, w_i)_{i=1}^k \) and \( d' \) be as above. Then the image of the Hall product

\[
\left( \bigotimes_{i=1}^k \mathbb{M}(d_i)_{w_i} \right) \boxtimes \mathbb{M}^a (1, d'; \delta') \to D^k(X^a(1, d))
\]
Proposition 3.10, we have $(\chi, \sigma)$, thus follows by the induction of $(\chi, \sigma)$. Consider weights $\chi_e \in M(e)$ and $\chi_{d'} \in M(d')$ such that $\chi_e + \rho_e + \delta_e \in V(e)$ and $\chi_{d'} + \rho_{d'} + \delta' \in W^a(1, d')$. It suffices to check that

$$\forall 0 < \rho \leq a, \quad \chi_e + \rho + \delta \in V(e) \Rightarrow \chi_{d'} + \rho_{d'} + \delta' \in W^a(1, d')$$

for all $0 < \rho \leq a$. Then we can write

$$\chi = \chi_e + \chi_{d'}$$

where $\chi_e$ is generated by the vector bundles $O$ generated by the images of these functors. The function $\chi_3$ satisfies

$$\chi_3 \in V(e) + W^a(1, d') - \left(\frac{3}{2}d' + \frac{a}{2}\right) \sigma_e + \frac{3}{2}c_e \sigma_{d'}.$$ 

Then we can write

$$\chi + \rho + \delta = \sum_{i,j \leq d} c_{ij} (\beta_i - \beta_j) + \sum_{i \leq d} c_i \beta_i$$

where $c_{ij}$ and $c_i$ satisfy $|c_{ij}| \leq 3/2$ with $c_{ij} = 3/2$ and $c_{ji} = 0$ for $j \leq e < i$, $-1 \leq c_i + a/2 \leq 0$ for $1 \leq i \leq e$, $|c_i| \leq a/2$ for $e < i \leq d$. We take $I \subset W^a_e$ and write $\sigma_I$ as

$$\sigma_I = \sum_{j \leq i \leq c} f_{ij} (\beta_i - \beta_j) - \sum_{i \leq c} f_i \beta_i$$

for $0 \leq f_{ij} \leq 3$ and $0 \leq f_i \leq a$. Then $\chi - \sigma + \rho + \delta$ is in $V^a(1, d)$. The polytope $V^a(1, d)$ is Weyl invariant, so $(\chi - \sigma_I)^+ + \rho + \delta$ is in $V^a(1, d)$. Thus we have (3.22) by Proposition 2.1.

**Proof of Proposition 3.10.** By Proposition 2.2, it suffices to check that the category $\mathbb{D}^a(d; \delta)$ is generated by the images of the Hall products

$$p_{e,d'} \circ q_{e,d'}^* : \mathbb{D}(e; \delta_e) \boxtimes M^a(1, d'; \delta') \to D^b(\chi^a(1, d))$$

for all $e \leq d$, $d' = d - e$, and $\delta_e := (\mu + d' + a/2) \sigma_e$. Let $\chi$ be a dominant weight such that $\chi + \rho + \delta \in W^a(1, d; \delta)$. We need to show that $O_{\chi(d)} \boxtimes \Gamma_{GL(d)}(\chi)$ is generated by the images of these functors. The function $p(\chi)$ takes finitely many values. We use induction on the pair $(-p(\chi), e(\chi))$ ordered lexicographically. Use Proposition 3.4 for $\chi + \rho + \delta$ and write $\chi = \chi_e + \chi_{d'}$ such that

$$\chi_e + \rho_e + \delta_e \in V(e), \chi_{d'} + \rho_{d'} + \delta' \in W^a(1, d').$$

By Proposition 2.1 the cone of the map

$$O_{\chi(d)} \boxtimes \Gamma_{GL(d)}(\chi) \to p_{e,d'} \circ q_{e,d'}^* (O_Y \boxtimes \Gamma L(\chi_e + \chi_{d'}))$$

is generated by the vector bundles $O_{\chi(d)} \boxtimes \Gamma_{GL(d)}((\chi - \sigma_I)^+)$ for non-empty multisets $I \subset W^a_e$.

If $(-p(\chi), e(\chi)) = (0, 0)$, then $\chi + \rho + \delta \in W^a(1, d; \delta)$. Assume $(-p(\chi), e(\chi))$ is different from $(0, 0)$ and let $I$ be a non-empty subset of $W^a_e$. Then as in the proof of Proposition 3.10 we have $(\chi - \sigma_I)^+ + \rho + \delta \in W^a(1, d)$. By Proposition 3.8 either $-p((\chi - \sigma_I)^+) < -p(\chi)$ or $p((\chi - \sigma_I)^+) = p(\chi)$ and $e((\chi - \sigma_I)^+) < e(\chi)$. The claim thus follows by the induction of $(-p(\chi), e(\chi))$. □
3.5. Orthogonality. In this subsection, we discuss orthogonality of the categories appearing on the right hand side of (3.1). For $S = (d_i, w_i)_{i=1}^k$, let $e = \sum_{i=1}^k d_i$ and $f := d - e$. We denote by $\lambda_S$ the antidual cocharacter

\begin{equation}
\lambda_S := (k^b, \ldots, k^b, 1, \ldots, 1, 1) : \mathbb{C}^* \to T(d).
\end{equation}

In order to simplify the notation, we set $X := X^a(1, d), Y := X^a(1, d)\lambda_S \geq 0$, and $Z := X^a(1, d)\lambda_S$. We have the maps from the attracting stacks for the antidual cocharacter $\lambda_S$, see (2.3)

\[ Z \xrightarrow{\varphi} Y \xrightarrow{p} X. \]

For another $S' = (d'_i, w'_i)$, we define $(e', f', \lambda_{S'}, X', Y', Z', p', q')$ in a similar way.

**Proposition 3.11.** Suppose that $(S', S) \in O$, see Subsection 3.2.3. Let $\mu \in \mathbb{R}$, and set $\delta_1 = (\mu - e)\sigma_f \in M(f)_\mathbb{R}$ and $\delta_2 = (\mu - e')\sigma_f \in M(f')_\mathbb{R}$. Let $A, \tilde{A}$ and $B$ be objects

\[ A, \tilde{A} \in \left( \bigotimes_{i=1}^k M(d_i, w_i) \right) \boxtimes M^a(1, f; \delta_f), \quad B \in \left( \bigotimes_{i=1}^{k'} M(d'_i, w'_i) \right) \boxtimes M^a(1, f'; \delta_{f'}). \]

We have

\[ \text{Hom}_X(p^*q^*B, p^*q^*A) = 0, \quad \text{Hom}_Z(\tilde{A}, A) \xrightarrow{\cong} \text{Hom}_X(p^*q^*\tilde{A}, p^*q^*A). \]

**Proof.** We discuss the first equality. Let $\chi$ and $\chi'$ be dominant weights such that

\[ \chi + \rho + \delta \in V^a(1, d), \quad \chi' + \rho + \delta \in V^a(1, d), \]

which are of types $S, S'$ respectively. Consider the Levi group $L := GL(d)^{\lambda_S}$, and define $L'$ analogously for $S'$. We may assume that $A = O_Z \otimes \Gamma_L(\chi)$ and $B = O_{Z'} \otimes \Gamma_{L'}(\chi')$. By adjunction, it suffices to check that

\[ \text{Hom}_Y(p^*p^*q^*B, q^*A) = 0. \]

Assume first that $e = e'$ and $\sum_{i=1}^k w_i = \sum_{i=1}^{k'} w'_i$. Then the conclusion follows as in [PTa, Proposition 3.9]. More explicitly, by [Pâdc, Proposition 4.2] and Proposition 2.1, it suffices to check that the $\lambda_S$-weights of the vector bundles in the resolution of $p^*q^*B$ are strictly greater than the $\lambda_S$-weight of $\chi$, and this is verified in [Pâdc, Proposition 4.3].

Assume next that either $p(S') > p(S)$ or $p(S) = p(S')$ and $e > e'$. By [Pâdc, Proposition 4.2] and Proposition 2.1, it suffices to check that

\begin{equation}
\langle \tau_{\epsilon_i}(\chi' - \sigma_f)^+ \rangle > \langle \tau_{\epsilon_i}, \chi \rangle
\end{equation}

for all $I \subset W^a_S$, where $W^a_S$ be the multiset of weights of $R^a(1, d)^{\lambda_S < 0}$. Since $W^a_S \subset W^a_{S'}$, the inequality (3.24) follows from Proposition 3.8. Note that [Pâdc, Proposition 4.2] can be applied ad litteram when comparing $\lambda_S$-weights, however the proof in loc. cit. applies to $\tau_{\epsilon}$-weights as well because $\tau_{\epsilon}$ fixes $R^a(d)^{\lambda_S}$ and acts with non-negative weights on $R^a(d)^{\lambda_S < 0}$.

We next discuss the second isomorphism. By [PTa, Proposition 3.9], it suffices to show that the functor

\[ p^*q^* : \mathbb{D}(\varepsilon; \delta_{\epsilon}) \boxtimes M^a(1, f; \delta_f) \to \mathbb{D}(1, d; \delta) \]

is fully faithful, where $\delta_{\epsilon} := (\mu + f + a/2)\sigma_{\epsilon}$. We may assume that $\tilde{A} = O_Z \otimes \Gamma_L(\tilde{\chi})$ for $\tilde{\chi}$ a dominant weight of type $S$. By adjunction, it suffices to show that

\[ \text{Hom}_Y(p^*p^*\tilde{A}, q^*A) \cong \text{Hom}_Y(q^*\tilde{A}, q^*A) \cong \text{Hom}_Z(\tilde{A}, A). \]
Let $C$ be the cone of the morphism $p^*p_*q^*\tilde{A} \to q^*\tilde{A}$. It suffices to check that $\text{Hom}_Y(C, q^*\tilde{A}) = 0$. By [Pac01 Proposition 4.2] and Proposition [2.1] it suffices to check that

$$\langle \tau_e, (\bar{\chi} - \sigma I)^{\dagger} \rangle > \langle \tau_e, \chi \rangle$$

for all non-empty $I \subset \mathcal{W}^\mu$. This follows from Proposition [3.8]. \hfill \square

3.6. **Proofs of the theorems.** In this subsection, we prove Theorems 3.2 and 1.1 and Corollary 1.2.

**Proof of Theorem 3.2.** The statement follows from Propositions 3.9 and 3.11. \hfill \square

Recall the subcategory $E^a(1, d; \delta)$ of $D^b(X^{af}(1, d))$ from Subsection 2.7.2. Consider the projection map $b: X^{af}(1, d) \to X^a(1, d)$.

**Proposition 3.12.** There is a semiorthogonal decomposition

$$E^a(1, d; \delta) = \left\langle \left( \bigoplus_{i=1}^k \mathbb{M}(d_i), \mathbb{P}^a(1, d'; \delta') \right) \right\rangle,$$

where the right hand side is as in Theorem 3.2.

**Proof.** By Theorem 3.2 the result follows as [PTa Corollary 3.11] using [PTa Proposition 3.10]. \hfill \square

**Proposition 3.13.** Let $\mu \in \mathbb{R}$ with $2\mu l \notin \mathbb{Z}$ for $1 \leq l \leq d$ and let $\delta := \mu \sigma d$. Consider the inclusions

$$\iota: I^a(d) \hookrightarrow X^{af}(1, d), \quad \iota': P^a(d) \hookrightarrow X^{af}(1, d).$$

Then the following functors are equivalences

$$E^a(1, d; \delta) \hookrightarrow D^b(X^{af}(1, d)) \overset{\iota^*}{\longrightarrow} D^b(I^a(d)),$$

$$\mathbb{P}^a(1, d; \delta) \hookrightarrow D^b(X^{af}(1, d)) \overset{\iota'^*}{\longrightarrow} D^b(P^a(d)).$$

**Proof.** The same proof as in [PTa Proposition 3.13] applies here. We refer to loc. cit. for full details. The main tools used in the proof are window categories [HL15] and the magic window theorem for symmetric representations [HLS20].

The Kempf-Ness loci for $I^a(d)$ and $P^a(d)$ are attracting loci for the cocharacters $\tau_e^{-1}$ and $\tau_e$, respectively, see [PTa Lemma 3.14]. We first explain fully faithfulness of the first functor. Let $I$ be the set of Kempf-Ness strata and consider the stratification

$$X^{af}(1, d) = I^a(d) \sqcup \bigcup_{i \in I} \mathcal{S}_i.$$ 

Let $\lambda_i = \tau_{e_i}^{-1}$ be the cocharacter corresponding to $\mathcal{S}_i$, let $Z_i := \mathcal{S}_i^{\lambda_i}$, and let

$$\eta_i := \langle \lambda_i, (X^{af}(1, d)^\vee)^{\lambda_i > 0} \rangle = -\langle \lambda_i, X^{af}(1, d)^{\lambda_i < 0} \rangle = (a + 1)e_i + 2e_i(d - e_i).$$

Let $\mathbb{G}_\eta \subset D^b(X^{af}(1, d))$ be the subcategory of complexes $\mathcal{F}$ such that

$$\text{wt}_{\lambda_i}(\mathcal{F}|_{Z_i}) + \left\langle \lambda_i, \delta + \frac{1}{2}d\tau_d \right\rangle \subset \left[ -\frac{1}{2} \eta_i, \frac{1}{2} \eta_i \right]$$

for all $i \in I$. By [HL15 Theorem 2.10], the restriction $\iota^*$ induces an equivalence

$$\iota^*: \mathbb{G}_\eta \xrightarrow{\sim} D^b(I^a(d)).$$
On the other hand, as \( \lambda_i = \tau_{e_i}^{-1} \), we have that \( \mathcal{X}^{af}(1, d)^{\lambda_i < 0} = \mathcal{X}^{a+1}(1, d)^{\lambda_i < 0} \), and thus
\[
\eta_i = \langle \lambda_i, \mathcal{X}^{a+1}(1, d)^{\lambda_i > 0} \rangle.
\]
(3.28)

By the assumption \( 2\mu l \notin \mathbb{Z} \) for \( 1 \leq l \leq d \) and \( \langle \lambda_i, \delta \rangle = -\mu c \), \( \langle \lambda_i, \delta \rangle = -e \), the condition (3.26) is equivalent to
\[
\text{wt}_{\lambda_i}(\mathcal{F}|_{Z_i}) + \left( \lambda_i, \delta + \frac{1}{2} d \tau_d \right) \subset \left[ \frac{1}{2} \eta, \frac{1}{2} \eta \right].
\]

Thus \( E(d; \delta) \subset \mathbb{G}_a \) by (3.28) and [HLS20, Lemma 2.9], noting that \( R^{a+1}(1, d) \) is a symmetric \( GL(d) \)-representation. Fully faithfulness of the second functor follows similarly.

We next discuss essential surjectivity. We have the projection maps
\[
\mathcal{X}^{a+1}(1, d) \xrightarrow{c} \mathcal{X}^{af}(1, d) \xrightarrow{b} \mathcal{X}^{a}(1, d)
\]
where \( c \) (resp. \( b \)) forgets one arrow from 1 to 0, (resp. 0 to 1). The maps \( c \), \( b \) are affine bundles of relative dimension \( d \). By Remark 3.1, the \( \chi_0 \)-stability (resp. \( \chi_0^{-1} \)-stability) on \( R^{af}(1, d) \) does not impose constraint on maps corresponding to edges from 1 to 0 (resp. 0 to 1). We have similar descriptions of \( \chi_0 \)-semistable loci for \( R^{a+1}(1, d) \) (resp. \( \chi_0^{-1} \)-semistable loci for \( R^a(1, d) \)), see [Tod22, Lemma 7.10]. Therefore the projection maps \( c \), \( b \) restrict to affine bundles of relative dimension \( d \)
\[
c: \mathcal{X}^{a+1}(1, d)^{\chi_0\text{-ss}} \to I^a(d), \quad b: P^a(d) \to \mathcal{X}^{a}(1, d)^{\chi_0^{-1}\text{-ss}}.
\]

Then the essential surjectivity follows from the magic window theorem [HLS20 Theorem 3.2] for the symmetric stacks \( \mathcal{X}^{a+1}(a, d) \), \( \mathcal{X}^{a}(1, d) \) as in [PTa, Proof of Proposition 3.13].

**Proof of Theorem 1.1.** The claim follows from Propositions 3.12 and 3.13 noting that the most left and right inequalities in (3.3) are never equalities by the condition \( 2\mu l \notin \mathbb{Z} \) for \( 1 \leq l \leq d \).

**Proof of Corollary 1.2.** Taking the Grothendieck group of the categories in (1.1), we obtain the isomorphism
\[
K(P^a(d)) \cong \bigoplus K \left( (\mathbb{Z}^k_{i=1} \mathcal{M}(d_i)_{w_i}) \boxtimes P^a(d') \right),
\]
(3.29)

where the sum of the right hand side is as in Theorem 1.1. Choose \( d' \leq d \). By [PTa, Theorem 1.1], there is a semiorthogonal decomposition
\[
D^b(\mathbb{NHilb}(d - d')) = \left( \bigoplus_{i=1}^k \mathcal{M}(d_i)_{w_i} \right)
\]
(3.30)

where the right hand side is after all partitions \( (d_i)_{i=1}^k \) of \( d - d' \) and integers \( (w_i)_{i=1}^k \) such that for \( u_i := w_i + d_i, f_i := d' + \sum_{j>i} d_j - \sum_{j<i} d_j \), the inequality (1.2) is satisfied. There is thus a semiorthogonal decomposition
\[
D^b \left( \mathbb{NHilb}(d - d') \times P^a(d') \right) = \left( \bigoplus_{i=1}^k \mathcal{M}(d_i)_{w_i} \right) \boxtimes D^b(P^a(d'))
\]
and thus a decomposition in K-theory:
\[
K \left( \mathbb{NHilb}(d - d') \times P^a(d') \right) \cong \bigoplus K \left( (\mathbb{Z}^k_{i=1} \mathcal{M}(d_i)_{w_i}) \boxtimes P^a(d') \right),
\]
where the right hand side is as for (3.30). By (3.29), we obtain

\[
\text{(3.31)} \quad K(P^n(d)) \cong \bigoplus_{d' = 0}^d K(\text{NHilb}(d - d') \times P^n(d')).
\]

We finally show that there is a Künneth isomorphism

\[
\text{(3.32)} \quad K(\text{NHilb}(d - d') \times P^n(d')) \cong K(\text{NHilb}(d - d')) \otimes K(P^n(d')),
\]

which implies the conclusion by (3.31). We use [HL15, Theorem 2.10] to choose window subcategories \( \mathbb{G} \subset D^b(\mathcal{X}^f(1, d - d')) \) and \( \mathbb{H} \subset D^b(\mathcal{X}^{af}(1, d')) \) such that the restriction maps to the stable locus give equivalences

\[
\mathbb{G} \sim D^b(\text{NHilb}(d - d')), \quad \mathbb{H} \sim D^b(P^n(d')).
\]

Then, by loc. cit., the category \( \mathbb{G} \boxtimes \mathbb{H} \) is part of a semiorthogonal decomposition of \( D^b(\mathcal{X}^f(1, d - d') \times \mathcal{X}^{af}(1, d')) \) such that the restriction map gives an equivalence

\[
\mathbb{G} \boxtimes \mathbb{H} \sim D^b(\text{NHilb}(d - d') \times P^n(d')).
\]

The following diagram commutes and the spaces on the left column are summands of the spaces of the right column:

\[
\begin{array}{ccc}
K(\mathbb{G}) \otimes K(\mathbb{H}) & \xrightarrow{a} & K(\mathcal{X}^f(1, d - d')) \otimes K(\mathcal{X}^{af}(1, d')) \\
\downarrow & & \downarrow \\
K(\mathbb{G} \boxtimes \mathbb{H}) & \xleftarrow{b} & K(\mathcal{X}^f(1, d - d') \times \mathcal{X}^{af}(1, d')).
\end{array}
\]

To show that \( a \) is an isomorphism, it suffices to show that \( b \) is an isomorphism, which is clear because \( K(\mathcal{X}^{af}(1, d')) \cong K_{GL(d')}(pt) \), \( K(\mathcal{X}^f(1, d - d')) \cong K_{GL(d - d')}(pt) \), and \( K(\mathcal{X}^f(1, d - d') \times \mathcal{X}^{af}(d')) \cong K_{GL(d - d') \times GL(d')(pt)} \). \qed

3.7. A variant of Theorem 1.1. We discuss a slight extension of Theorem 1.1 for quivers obtained from \( Q^{df} \) by adding loops at the vertex 0 and imposing a super-potential. We will use this extension in the proof of Theorem 1.4.

Let \( Q^{af,N} \) be the quiver adding \( N \)-loops at the vertex 0. Then the affine space of representations of \( Q^{af,N} \) of dimension \( (1, d) \) is

\[
R^{af,N}(1, d) = \mathbb{C}^N \times R^{af}(1, d).
\]

Let \( \widetilde{W} \) be a super-potential on \( Q^{af,N} \) satisfying

\[
\widetilde{W}|_Q = X[Y, Z],
\]

where \( Q \subset Q^{af,N} \) is the full subquiver consisting of the vertex \( \{1\} \), i.e. it is the triple loop quiver. We define

\[
\mathcal{D}T_{\widetilde{W}}^{a,N}(d) := \text{MF} \left( R^{af,N}(1, d)^{\text{\underline{\text{ss}}}} / GL(d), \text{Tr} \, \widetilde{W} \right),
\]

\[
\mathcal{P}T_{\widetilde{W}}^{a,N}(d) := \text{MF} \left( R^{af,N}(1, d)^{\text{\underline{\text{ss}}}} / GL(d), \text{Tr} \, \widetilde{W} \right).
\]

By applying \( D^b(\mathbb{C}^N) \boxtimes \) to the semiorthogonal decomposition in Theorem 1.1 and the super-potential \( \text{Tr} \, \widetilde{W} \), see [Päda, Proposition 2.1], we have the following corollary:

**Corollary 3.14.** Let \( \mu \in \mathbb{R} \) such that \( 2\mu \ell \notin \mathbb{Z} \) for \( 1 \leq \ell \leq d \). There is a semiorthogonal decomposition

\[
\mathcal{D}T_{\widetilde{W}}^{a,N}(d) = \left< \left( \bigoplus_{i=1}^b \mathbb{S}(d_i w_i) \right) \boxtimes \mathcal{P}T_{\widetilde{W}}^{a,N}(d') \right>.
\]
The right hand side is after all $d' \leq d$, partitions $(d_i)_{i=1}^k$ of $d - d'$, and integers $(w_i)_{i=1}^k$ such that for $v_i := w_i + d_i \left( d' + \sum_{j>i} d_j - \sum_{j<i} d_j \right)$, we have

$$-1 - \mu - \frac{a}{2} < \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k} < -\mu - \frac{a}{2}.$$ 

The order of the semiorthogonal summands is the same as in Theorem 1.4.

The analogous conclusion holds if we replace $\mathbb{C}^N$ by an open subset in all the constructions and statements above.

4. THE CATEGORICAL DT/PT CORRESPONDENCE FOR $\mathbb{C}^3$

In this section, we prove Theorems 1.3 and 1.4 and Corollary 1.5. We give explicit descriptions of DT/PT moduli spaces on $\mathbb{C}^3$ with reduced supports via extended ADHM quivers. The ADHM quiver is a quiver with a relation (depicted in Figure 3) which was used to construct framed moduli spaces of instantons on $\mathbb{P}^2$, see [Nak99].

**Figure 3. ADHM quiver**

$$[A, B] + u \circ v = 0.$$ 

4.1. Perverse coherent sheaves via the ADHM construction. Below we regard $\mathbb{C}^2$ as an open subset of $\mathbb{P}^2$ consisting of $[X : Y : Z]$ such that $Z \neq 0$, and set $l_{\infty} = (Z = 0) \subset \mathbb{P}^2$.

Let $\mathcal{T} \subset \text{Coh}(\mathbb{P}^2)$ be the subcategory of zero-dimensional sheaves and let $\mathcal{F} \subset \text{Coh}(\mathbb{P}^2)$ be the subcategory of sheaves $F$ such that $\text{Hom}(\mathcal{T}, F) = 0$. The pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\text{Coh}(\mathbb{P}^2)$, and we define $\mathcal{A}$ to be the associated tilting abelian category

$$\mathcal{A} = \langle \mathcal{F}, \mathcal{T}[-1] \rangle \subset D^b(\mathbb{P}^2).$$

An object in $\mathcal{A}$ is an example of a perverse coherent sheaf on $\mathbb{P}^2$ introduced in [Kas04, AB10]. We denote by $\mathcal{U}_{\mathbb{C}^2}(d)$ the derived moduli stack of objects $I \in \mathcal{A}$ with an isomorphism $I|_{l_{\infty}} \cong O_{l_{\infty}}$ such that $\text{ch}_2(I) = -d$. Note that $I$ fits into an exact sequence in $\mathcal{A}$:

$$0 \to I_Z \to I \to Q[-1] \to 0,$$

where $I_Z = \mathcal{H}^0(I)$ is an ideal sheaf of a zero-dimensional subscheme $Z \subset \mathbb{C}^2$ and $Q = \mathcal{H}^1(I)$ is zero-dimensional on $\mathbb{C}^2$ such that $|Q| + |Z| = d$. The stack $\mathcal{U}_{\mathbb{C}^2}(d)$ is
an Artin stack of finite type whose classical truncation $\mathcal{U}_{C^2}(d)$ admits a good moduli space

$$ \mathcal{U}_{C^2}(d) \to \text{Sym}^d(C^2), \ I \mapsto \text{Supp}(Q) + \text{Supp}(Z). $$

There is an explicit description of $\mathcal{U}_{C^2}(d)$ via the ADHM quiver. Let $V$ be a $d$-dimensional vector space and let $g = \text{Hom}(V, V)$. We consider the following $GL(V)$-equivariant morphism:

$$ \nu: V \oplus V^\vee \oplus g^\oplus 2 \to g, $$

$$(u, v, A, B) \mapsto [A, B] + u \circ v.$$ 

Let $\nu^{-1}(0)$ be the derived zero locus of $\nu$. The derived moduli stack of the ADHM quiver with relation is given by

$${\mathcal{N}}_{C^2}(d) := \nu^{-1}(0)/GL(V).$$

Then there is an equivalence of derived stacks, see [BFG06, Theorem 5.7]

$$(4.2) \quad \Upsilon: {\mathcal{N}}_{C^2}(d) \sim U_{C^2}(d).$$

For a point $(u, v, A, B) \in \nu^{-1}(0)$, the corresponding object in $U_{C^2}(d)$ is given by the complex on $\mathbb{P}^2$:

$$(4.3) \quad 0 \to V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\phi} (V^\oplus 2 \oplus \mathbb{C}) \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\psi} V \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0.$$ 

Here, the maps $\phi$ and $\psi$ are given by

$$(4.4) \quad \phi = \begin{pmatrix} ZA - X \text{id} \\ ZB - Y \text{id} \end{pmatrix}, \quad \psi = (-ZB + Y \text{id}, ZA - X \text{id}, Z) .$$

It is proved in [BFG06, Theorem 5.7] that the above correspondence gives an isomorphism (4.2) on classical truncations. The above construction and its inverse in [BFG06, Theorem 5.7] can be generalized to families over dg-rings, so giving the claimed equivalence (4.2).

**4.2. The extended ADHM stack.** For each $m \in \mathbb{N}$, let $I_m$ be the set

$$ I_m := \{(i, j) \in \mathbb{Z}^{\oplus 2} : 1 \leq i + j \leq m\}. $$

Note that we have $|I_m| = m^2/2 + 3m/2 = \dim(\mathcal{O}_{\mathbb{P}^2}(m))$. For each $\alpha = (a_{ij}) \in C^{I_m}$, consider the polynomial

$$(4.5) \quad f_\alpha = 1 + \sum_{(i,j) \in I_m} a_{ij}x^iy^j \in \mathbb{C}[x, y]$$

and let $C_\alpha \subset C^2$ be the plane curve defined by $f_\alpha = 0$. There is an open immersion

$$(4.6) \quad C^{I_m} \hookrightarrow |\mathcal{O}_{\mathbb{P}^2}(m)|, \ \alpha \mapsto \mathcal{T}_\alpha := Z^m f_\alpha(X/Z, Y/Z).$$

whose image consists of polynomials $\sum c_{ijk}X^iY^jZ^k$ such that $c_{000} \neq 0$.

Let $V$ be a $d$-dimensional vector space and $g = \text{Hom}(V, V)$. We consider the map

$$ \mu: V \oplus V^\vee \oplus g^\oplus 2 \oplus C^{I_m} \to g \oplus V^\vee $$(

$$(u, v, A, B, \alpha) \mapsto ([A, B] + u \circ v, v \circ f_\alpha(A, B)).$$

It is proved in [BFG06, Theorem 5.7] that the above correspondence gives an isomorphism (4.2) on classical truncations. The above construction and its inverse in [BFG06, Theorem 5.7] can be generalized to families over dg-rings, so giving the claimed equivalence (4.2).
The above map $\mu$ is $GL(V)$-equivariant, where $GL(V)$ acts on $g$ by conjugation and on $\mathbb{C}^{l_m}$ trivially. Let $\mu^{-1}(0)$ be the derived zero locus of $\mu$. We define the derived stack

$$\mathcal{M}_{\mathbb{C}^2}(m, d) := \mu^{-1}(0)/GL(V).$$

By forgetting $\alpha$, there is a morphism of derived stacks

$$(4.7) \quad g : \mathcal{M}_{\mathbb{C}^2}(m, d) \to \mathcal{M}_{\mathbb{C}^2}(d).$$

In what follows, we explain that $\mathcal{M}_{\mathbb{C}^2}(m, d)$ can be interpreted as a derived moduli stack of pairs of a one-dimensional sheaf on $\mathbb{C}^2$ with a section.

4.3. The derived moduli stack of pairs on $\mathbb{C}^2$. We consider pairs

$$(4.8) \quad (F, s), \quad F \in \text{Coh}(\mathbb{P}^2), \quad s : \mathcal{O}_{\mathbb{P}^2} \to F,$$

where $F$ is one-dimensional and the cokernel of $s$ is at most zero-dimensional. Let $l \subset \mathbb{P}^2$ be a line. We denote by

$$\mathcal{T}_{\mathbb{P}^2}(m, d) \leftarrow \mathcal{T}_{\mathbb{P}^2}^l(m, d)$$

the derived moduli stack of pairs $(4.8)$ and its classical truncation satisfying

$$[F] = m[l], \chi(F) = 3m/2 - m^2/2 + d.$$

The stack $\mathcal{T}_{\mathbb{P}^2}^l(\mathbb{P}^2, m)$ is an Artin stack of finite type which admits a good moduli space, see [Toda, Subsection 4.2.4].

Note that a pair $(\mathcal{O}_{\mathbb{P}^2} \to F)$ in $\mathcal{T}_{\mathbb{P}^2}(m, d)$ admits a filtration

$$(0 \to Q) \subset (\mathcal{O}_{\mathbb{P}^2} \to F') \subset (\mathcal{O}_{\mathbb{P}^2} \to F)$$

such that $Q$ and $Q' := F/F'$ are zero-dimensional sheaves with $|Q| + |Q'| = d$ and $F'/Q = \mathcal{O}_C$ for $C \in |\mathcal{O}_{\mathbb{P}^2}(m)|$. The good moduli space morphism is

$$\mathcal{T}_{\mathbb{P}^2}(m, d) \to T_{\mathbb{P}^2}(m, d) = |\mathcal{O}_{\mathbb{P}^2}(m)| \times \text{Sym}^d(\mathbb{P}^2),$$

$$(F, s) \mapsto (C, \text{Supp}(Q) + \text{Supp}(Q')).$$

We define the derived open substack $\mathcal{T}_{\mathbb{C}^2}(m, d) \subset \mathcal{T}_{\mathbb{P}^2}(m, d)$ whose classical truncation fits into the Cartesian square

$$\mathcal{T}_{\mathbb{C}^2}(m, d) \leftarrow \mathcal{T}_{\mathbb{P}^2}(m, d)$$

$$\mathbb{C}^{l_m} \times \text{Sym}^d(\mathbb{C}^2) \leftarrow |\mathcal{O}_{\mathbb{P}^2}(m)| \times \text{Sym}^d(\mathbb{P}^2).$$

Lemma 4.1. There is a natural morphism

$$(4.9) \quad h : \mathcal{T}_{\mathbb{C}^2}(m, d) \to \mathcal{U}_{\mathbb{C}^2}(d)$$

sending a pair $(F, s)$ to a two term complex $(\mathcal{O}_{\mathbb{P}^2}(m) \xrightarrow{\delta} F(m))$.

Proof. By the definition of $\mathcal{T}_{\mathbb{C}^2}(m, d)$, for a point $(F, s)$ in $\mathcal{T}_{\mathbb{C}^2}(m, d)$, the two term complex

$$I = (\mathcal{O}_{\mathbb{P}^2}(m) \xrightarrow{\delta} F(m))$$

is an object in $\mathcal{A}$ such that $H^0(I) = I_Z$ for a zero-dimensional subscheme $Z \subset \mathbb{C}^2$ and $H^1(I)$ is zero-dimensional with support contained in $\mathbb{C}^2$. In particular, there is an open neighborhood $l_{\infty} \subset U \subset \mathbb{P}^2$ such that $I|_U = (\mathcal{O}_U(m) \xrightarrow{\delta_U} \mathcal{O}_C(m)|_U)$ for $C \in |\mathcal{O}_{\mathbb{P}^2}(m)|$. Then $I|_U$ is naturally isomorphic to $\mathcal{O}_U$, giving a trivialization $I|_{l_{\infty}} \cong \mathcal{O}_{l_{\infty}}$. □
Lemma 4.2. For an object \( I \) in \( \mathcal{U}_{C^2}(d) \), we have
\[
\text{Hom}_{\mathcal{P}^2}(I, \mathcal{O}_{\mathcal{P}^2}(m)) = \ker \left( H^0(\mathcal{O}_{\mathcal{P}^2}(m)) \xrightarrow{\eta} \text{Ext}^2_\mathcal{P}^2(\mathcal{H}^1(I), \mathcal{O}_{\mathcal{P}^2}(m)) \right).
\]

In the above, \( \eta \) sends \( a \in H^0(\mathcal{O}_{\mathcal{P}^2}(m)) \) to \( \mathcal{H}^1(I) \to I_Z[2] \to \mathcal{O}_{\mathcal{P}^2}(m)[2] \) where the first arrow is the extension class of \( (4.7) \) and the second arrow is given by multiplication with \( a \).

Proof. The lemma is straightforward by applying \( \text{RHom}_{\mathcal{P}^2}(-, \mathcal{O}_{\mathcal{P}^2}(m)) \) to \( (4.11) \) and observing that \( \text{Hom}(I_Z, \mathcal{O}_{\mathcal{P}^2}(m)) = H^0(\mathcal{O}_{\mathcal{P}^2}(m)) \).

By Lemma 4.2, we have an injection
\[
(4.10) \quad \text{Hom}(I, \mathcal{O}_{\mathcal{P}^2}(m)) \hookrightarrow H^0(\mathcal{O}_{\mathcal{P}^2}(m))
\]
which sends \( t: I \to \mathcal{O}_{\mathcal{P}^2}(m) \) to \( \det(t): \det I = \mathcal{O}_{\mathcal{P}^2} \to \mathcal{O}_{\mathcal{P}^2}(m) \). We set
\[
\mathbb{P}(\text{Hom}(I, \mathcal{O}_{\mathcal{P}^2}(m)))^o := \mathbb{P}(\text{Hom}(I, \mathcal{O}_{\mathcal{P}^2}(m))) \cap \mathcal{C}^{lm}.
\]

Lemma 4.3. For an object \( I \) in \( \mathcal{U}_{C^2}(d) \), we have \( h^{-1}(I)^\text{cl} \cong \mathbb{P}(\text{Hom}(I, \mathcal{O}_{\mathcal{P}^2}(m)))^o \).

Proof. Given \( I \), let us take a non-zero morphism \( t: I \to \mathcal{O}_{\mathcal{P}^2}(m) \) corresponding to a point in \( \mathbb{P}(\text{Hom}(I, \mathcal{O}_{\mathcal{P}^2}(m)))^o \). Let \( C(t) \) be the cone of \( t \) and consider the distinguished triangle
\[
I \to \mathcal{O}_{\mathcal{P}^2}(m) \to C(t) \to I[1].
\]

By taking the associated long exact sequence of cohomologies, we see that \( C(t) \in \text{Coh}(\mathbb{P}^2) \) and it fits into the exact sequence
\[
0 \to \mathcal{O}_{\mathcal{P}^2}(m)/H^0(I) \to C(t) \to H^1(I) \to 0.
\]
It follows that \( C(t) \) is one-dimensional with support a curve represented by an element in \( \mathcal{C}^{lm} \). Moreover, \( I \) is isomorphic to the two-term complex
\[
I \cong (\mathcal{O}_{\mathcal{P}^2}(m) \xrightarrow{s} C(t))
\]
such that \( \text{Cok}(s) = H^1(I) \) is zero-dimensional. Suppose that for another morphism \( t': I \to \mathcal{O}_{\mathcal{P}^2}(m) \), there is a commutative diagram
\[
\begin{array}{ccc}
I & \xrightarrow{t} & \mathcal{O}_{\mathcal{P}^2}(m) \\
\downarrow & & \downarrow \\
I & \xrightarrow{t'} & \mathcal{O}_{\mathcal{P}^2}(m) \\
\end{array}
\]
\[
\cong C(t).
\]
Then there is an isomorphism \( \phi: I \xrightarrow{\cong} I \) which makes the above diagram commutative. Then \( \det t \det \phi = \det t' \). As \( 0 \neq \det \phi \in H^0(\mathcal{O}_{\mathcal{P}^2}) = \mathbb{C} \), and \( (4.11) \) is injective, we see that \( t \) and \( t' \) differ by a non-zero scalar multiplication. It follows that
\[
(4.11) \quad \mathbb{P}(\text{Hom}(I, \mathcal{O}_{\mathcal{P}^2}(m)))^o \subset h^{-1}(I)^\text{cl}.
\]

Conversely, let \( (\mathcal{O}_{\mathcal{P}^2} \xrightarrow{s} F) \) be a point in \( \mathcal{U}_{C^2}(m, d) \) such that the two-term complex \( (\mathcal{O}_{\mathcal{P}^2}(m) \xrightarrow{s} F(m)) \) is isomorphic to \( I \) in \( D^b(\mathbb{P}^2) \). Then there is a non-zero morphism \( I \to \mathcal{O}_{\mathcal{P}^2}(m) \) corresponding to a point in \( \mathbb{P}(\text{Hom}(I, \mathcal{O}_{\mathcal{P}^2}(m)))^o \). Therefore \( (4.11) \) is an isomorphism. □
4.4. Explicit description of the moduli stack of pairs on \(\mathbb{C}^2\). For \((i, j) \in I_m\) and \((A, B) \in g^{\otimes 2}\), we have

\[
A^i B^j - x^i y^j \text{id} = A^i (B^{j-1} + y B^{j-2} + \cdots + y^{i-1} \text{id})(B - y \text{id}) + y^j (A^{i-1} + x A^{i-2} + \cdots + x^{j-1} \text{id})(A - x \text{id}).
\]

Let \(\alpha \in C^{I_m}\). By the above equality, we can write

\[
f_\alpha(A, B) - f_\alpha \text{id} = g_\alpha(A - x \text{id}) + h_\alpha(B - y \text{id})
\]

for \(g_\alpha, h_\alpha \in \mathbb{C}[x, y] \otimes g\) with degree less than \(m\). Then for each element

\[(u, v, A, B, \alpha) \in V \oplus V^\vee \oplus g^{\otimes 2} \oplus C^{I_m}\]

such that \([A, B] + u \circ v = 0\) and \(v \circ f_\alpha(A, B) = 0\), we have the commutative diagram

\[
\begin{array}{ccc}
V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\phi} & (V^{\otimes 2} \oplus \mathbb{C}) \otimes \mathcal{O}_{\mathbb{P}^2} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\gamma} & \mathcal{O}_{\mathbb{P}^2}(m) & \xrightarrow{0}
\end{array}
\]

In the above diagram, \(\phi, \psi\) are given as (4.4), and \(\gamma\) is given by

\[
\gamma = (v \circ \mathcal{G}_\alpha, v \circ \mathcal{H}_\alpha, f_\alpha)
\]

Here \(\mathcal{G}_\alpha = Z^m g_\alpha(X/Z, Y/Z) \in \mathbb{C}[X, Y, Z] \otimes g\), etc. We note that the equation \(v \circ f_\alpha(A, B) = 0\) is used for the commutativity of the left square in (4.12).

The diagram (4.12) determines a morphism \(I \to \mathcal{O}_{\mathbb{P}^2}(m)\), where \(I\) is a perverse coherent sheaf represented by (4.3). From the proof of Lemma 4.3, the totalization of the diagram (4.12) is quasi-isomorphic to a one-dimensional sheaf \(F\) on \(\mathbb{P}^2\) with curve class \((\mathcal{F}_\alpha) = 0\) and with a morphism \(\mathcal{O}_{\mathbb{P}^2}(m) \to F\). Therefore, by applying \(\otimes \mathcal{O}_{\mathbb{P}^2}(-m)\), we obtain the morphism

\[
\Theta: \mathcal{M}_{\mathbb{C}^2}(m, d) \to \mathcal{S}_{\mathbb{C}^2}(m, d)
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{M}_{\mathbb{C}^2}(m, d) & \xrightarrow{\Theta} & \mathcal{S}_{\mathbb{C}^2}(m, d) \\
\downarrow g & & \downarrow h \\
\mathcal{N}_{\mathbb{C}^2}(d) & \xrightarrow{\sim} & \mathcal{U}_{\mathbb{C}^2}(d)
\end{array}
\]

**Proposition 4.4.** The morphism \(\Theta\) is an equivalence of derived stacks.

**Proof.** We first prove that \(\Theta\) is an isomorphism on classical truncations. Let \((u, v, A, B)\) be a point in \(\mathcal{U}_{\mathbb{C}^2}(d)\) and \(I\) be the associated perverse coherent sheaf via the construction (4.3). From the commutative diagram (4.13), it is enough to show that \(\Theta\) induces the isomorphism

\[
\Theta^{cl}: g^{-1}(u, v, A, B)^{cl} \xrightarrow{\sim} h^{-1}(I)^{cl}.
\]

Here, the right hand side is given by \(\mathbb{P}(\text{Hom}(I, \mathcal{O}_{\mathbb{P}^2}(m)))^{0}\) by Lemma 4.3 and the left hand side is given by the zero-locus of the map

\[
C^{I_m} \to V^\vee, \alpha \mapsto v \circ f_\alpha(A, B).
\]
Let $\Sigma \subset V$ be the intersection of all subspaces $S \subset V$ satisfying $A(S) \subset S$, $B(S) \subset S$ and $u \in S$. We set $N = V/\Sigma$. Then we have the commutative diagram

$$
\begin{array}{cccc}
\Sigma \otimes \mathcal{O}_{p^2}(-1) & \xrightarrow{\phi|_{\Sigma}} & (\Sigma^\oplus 2 \oplus \mathbb{C}) \otimes \mathcal{O}_{p^2} & \xrightarrow{\psi|_{\Sigma}} & \Sigma \otimes \mathcal{O}_{p^2}(1) \\
V \otimes \mathcal{O}_{p^2}(-1) & \xrightarrow{\phi} & (V^\oplus 2 \oplus \mathbb{C}) \otimes \mathcal{O}_{p^2} & \xrightarrow{\psi} & V \otimes \mathcal{O}_{p^2}(1) \\
N \otimes \mathcal{O}_{p^2}(-1) & \xrightarrow{\phi} & N^\oplus 2 \otimes \mathcal{O}_{p^2} & \xrightarrow{\psi} & N \otimes \mathcal{O}_{p^2}(1).
\end{array}
$$

(4.16)

Recall that the middle horizontal complex is isomorphic to $I$. By [JM11 Theorem 4.10], the top and bottom horizontal complexes are isomorphic to $I_Z$ and $Q[-1]$, respectively, and the vertical arrows between these complexes are identified with the exact sequence (4.1) in $A$. Moreover, we have $v|_{\Sigma} = 0$ by [Nak99 Proposition 2.8]. Take a non-zero element

$$c = \sum_{i+j+k=m} c_{ijk} X^i Y^j Z^k \in H^0(\mathcal{O}_{p^2}(m)).$$

Under the identification $\text{Hom}(I_Z, \mathcal{O}_{p^2}(m)) = H^0(\mathcal{O}_{p^2}(m))$, the morphism $I_Z \to \mathcal{O}_{p^2}(m)$ corresponding to $c$ is represented by the commutative diagram

$$
\begin{array}{cccc}
\Sigma \otimes \mathcal{O}_{p^2}(-1) & \xrightarrow{\phi|_{\Sigma}} & (\Sigma^\oplus 2 \oplus \mathbb{C}) \otimes \mathcal{O}_{p^2} & \xrightarrow{\psi|_{\Sigma}} & \Sigma \otimes \mathcal{O}_{p^2}(1) \\
0 & \xrightarrow{\delta} & (0,0) \xrightarrow{0,0,c} \mathcal{O}_{p^2}(m) & \xrightarrow{0} & 0.
\end{array}
$$

(4.17)

We compute $\eta(c)$ in Lemma 4.2 using the diagram 4.16. Since $\mathcal{H}^1(I)$ is supported on $\mathbb{C}^2$, via the isomorphism $Z_m: \mathcal{O}_{\mathbb{C}^2} \cong \mathcal{O}_{p^2}(m)|_{\mathbb{C}^2}$ we have

$$\text{Ext}^2_{\mathcal{O}_{\mathbb{C}^2}}(\mathcal{H}^1(I), \mathcal{O}_{p^2}(m)) \cong H^0(\mathcal{E}xt^2_{\mathcal{C}^2}(\mathcal{H}^1(I), \mathcal{O}_{\mathbb{C}^2})) \cong N^\vee \subset V^\vee.
$$

(4.18)

In order to compute $\eta(c)$ as an element of $V^\vee$, we extend the middle vertical arrow in (4.17) to $\varepsilon$ as follows

$$
\begin{array}{cccc}
(\Sigma^\oplus 2 \oplus \mathbb{C}) \otimes \mathcal{O}_{p^2} & \xrightarrow{\delta} & (V^\oplus 2 \oplus \mathbb{C}) \otimes \mathcal{O}_{p^2} \\
(0,0,c) & \xrightarrow{\varepsilon=(0,0,c)} & \mathcal{O}_{p^2}(m).
\end{array}
$$

We then compose it with the morphism $\phi$ in (4.16) to obtain the morphism

$$V \otimes \mathcal{O}_{p^2}(-1) \xrightarrow{\phi} (V^\oplus 2 \oplus \mathbb{C}) \otimes \mathcal{O}_{p^2} \xrightarrow{\varepsilon} \mathcal{O}_{p^2}(m).$$

The morphism above factors through $N \otimes \mathcal{O}_{p^2}(-1)$, giving a morphism

$$v \circ Zc: N \otimes \mathcal{O}_{p^2}(-1) \to \mathcal{O}_{p^2}(m).$$

The above morphism represents $\eta(c)$. Under the isomorphism (4.18), we have

$$\eta(c) = v \circ \sum_{i+j+k=m} c_{ijk} A^i B^j.$$
Note that we have $\alpha_{ij} = c_{ijk}/c_{000m}$ under the embedding (4.19), where $k = m - i - j$. Therefore comparing with (4.13), we see the isomorphism (4.14).

Finally, we show that $\Theta$ induces an equivalence of derived stacks. As $\Theta$ is an isomorphism on classical truncations and the bottom arrow in (4.13) is an equivalence, it is enough to show that $\Theta$ induces a quasi-isomorphism on relative tangent complexes $\mathcal{T}_g \to \Theta^* \mathcal{T}_h$. Let us take a point $x = (u, v, A, B, \alpha)$ in $\mathcal{M}_{\mathbb{C}^2}(m, d)$ such that $[A, B] + u \circ f_\alpha(A, B) = 0$. Then we have

$$\mathcal{T}_{g|x} = (\mathcal{C}^{I_m} \to V^\vee), \quad \iota: (e_{ij}) \mapsto \sum_{(i,j) \in I_m} e_{ij}v \circ A^iB^j.$$

Let $y = (\mathcal{O}_{\mathbb{P}^2} \to F)$ be the point in $\mathcal{X}_{\mathbb{C}^2}(m, d)$ corresponding to $\Theta(x)$. The tangent complex of $\mathcal{X}_{\mathbb{C}^2}(m, d)$ at $y$ is

$$\text{RHom}(\mathcal{O}_{\mathbb{P}^2} \to F, F) = \text{RHom}(I, F(m)),$$

where $I = (\mathcal{O}_{\mathbb{P}^2}(m) \to F(m))$. The tangent complex $\mathcal{T}_h|_y$ is

$$\text{Cone}(\text{RHom}(I, F(m)) \to \text{RHom}(I, I)_0[1]) \cong \text{Cone}(\mathbb{C} \to \text{RHom}(I, \mathcal{O}_{\mathbb{P}^2}(m))).$$

Here, $\text{RHom}(I, I)_0$ is the cone of the natural morphism $\mathbb{C} \to \text{RHom}(I, I)$, and the map $\mathbb{C} \to \text{RHom}(I, \mathcal{O}_{\mathbb{P}^2}(m))$ is the composition of the above natural morphism with $I \to \mathcal{O}_{\mathbb{P}^2}(m)$. From the calculation of $\eta$, we have

$$\mathcal{T}_h|_y = \left(\mathcal{C}^{I_m}/\mathbb{C}^\mathcal{F}_\alpha \xrightarrow{\iota'} V^\vee\right), \quad \iota': (c_{ijk}) \mapsto v \circ \sum_{i+j+k=m} c_{ijk}A^iB^j.$$

The required quasi-isomorphism is

$$\begin{array}{ccc}
\mathbb{C}^{I_m} & \xrightarrow{\iota} & V^\vee \\
\cong & & \\
\mathcal{C}^{I_m}/\mathbb{C}^\mathcal{F}_\alpha & \xrightarrow{\iota'} & V^\vee.
\end{array}$$

Here the left vertical arrow sends $(e_{ij})$ to $(c_{ijk})$ for $c_{ijk} = e_{ij}$ when $i + j > 0$ and $c_{00m} = 0$. □

### 4.5. The Categorical DT/PT Correspondence for $\mathbb{C}^3$

We define Zariski open subsets

(4.19) \[ |\mathcal{O}_{\mathbb{P}^2}(m)|^\text{red} \subset |\mathcal{O}_{\mathbb{P}^2}(m)|, \quad (\mathbb{C}^{I_m})^\text{red} := |\mathcal{O}_{\mathbb{P}^2}(m)|^\text{red} \cap \mathbb{C}^{I_m}, \]

where $|\mathcal{O}_{\mathbb{P}^2}(m)|^\text{red}$ is the locus of reduced plane curves in $\mathbb{P}^2$.

Let $X = \text{Tot}_{\mathbb{P}_2}(\omega_{\mathbb{P}^2})$. Let

$$\mathcal{T}_X^\text{red}(m, d)$$

be the classical moduli stack of pairs $(F, s)$, where $F$ is a one-dimensional sheaf on $X$ with support a reduced plane curve of degree $m$ in $\mathbb{P}^2$, and $s: O_X \to F$ is surjective in dimension one. The open immersion $\mathbb{C}^2 \subset \mathbb{P}^2$ determines the open immersion $\mathbb{C}^3 \subset X$. We have the open substack

$$\mathcal{T}_{\mathbb{C}^3}^\text{red}(m, d) \subset \mathcal{T}_X^\text{red}(m, d)$$

to be consisting of $(F, s)$ such that $\text{Cok}(s)$ and $F_{\text{tor}}$ are supported on $\mathbb{C}^3$, and the one-dimensional support of $F$ is an element from $(\mathbb{C}^{I_m})^\text{red}$. Here $F_{\text{tor}} \subset F$ is the maximal zero-dimensional subsheaf.
Let $\Tr W_{m,d}$ be the function
\[
\Tr W_{m,d} : \left( V^{\oplus 2} \oplus V^\vee \oplus g^{\oplus 3} \times (\mathcal{C}^m)^{\text{red}} \right) / GL(V) \to \mathbb{C}
\]
defined by the formula
\[
(4.20) \quad (u_1, u_2, v, A, B, C, \alpha) \mapsto v \circ f_\alpha(A, B)(u_2) + \Tr C(u_1 \circ v + [A, B]).
\]

The next theorem describes $\mathcal{T}_{C_3}^{\text{red}}(m, d)$ as a critical locus. Here it is essential to restrict to reduced curves, see Remark 4.6.

**Theorem 4.5.** We have
\[
(4.21) \quad \mathcal{T}_{C_3}^{\text{red}}(m, d) \cong \text{Crit}(\Tr W_{m,d}).
\]

**Proof.** We set
\[
\mathcal{T}_{C_3}^{\text{red}}(m, d) := \mathcal{T}_{C_3}(m, d) \times |O_{C_3}(m)|^{\text{red}},
\]
\[
\mathcal{T}_{C_2}^{\text{red}}(m, d) := \mathcal{T}_{C_2}(m, d) \times \mathcal{C}^m (\mathcal{C}^m)^{\text{red}}.
\]

By the proof of [Toda, Lemma 5.5.4], we have an isomorphism
\[
(4.22) \quad \mathcal{T}_X^{\text{red}}(m, d) \cong \Omega_{2}^{\text{red}}(m, d)[-1]^{\text{cl}}.
\]

The above isomorphism restricts to the isomorphism of open substacks
\[
\mathcal{T}_{C_3}^{\text{red}}(m, d) \cong \Omega_{C_3}^{\text{red}}(m, d)[-1]^{\text{cl}}.
\]

The lemma follows from the explicit description of the $(-1)$-shifted cotangent stack, see [Toda, Section 2.1.1].

**Remark 4.6.** We need to restrict to reduced curves in the above theorem as we use the isomorphism (4.22) from [Toda, Lemma 5.5.4]. Otherwise, a pair $(F, s)$ from $X$ with zero-dimensional cokernel may push-forward to a pair on $\mathbb{P}^2$ with one-dimensional cokernel, so the isomorphism (4.22) may not hold.

We have the open substacks, which are quasi-projective schemes:
\[
I_{C_3}^{\text{red}}(m, d) \subset \mathcal{T}_{C_3}^{\text{red}}(m, d) \supset P_{d}^{\text{red}}(C_3, m).
\]

Here $I_{d}^{\text{red}}(C_3, m)$ is the DT moduli space consisting of $(F, s)$ such that $s$ is surjective, and $P_{d}^{\text{red}}(C_3, m)$ is the PT moduli space consisting of $(F, s)$ such that $F$ is pure. Let $\chi_0 : GL(V) \to \mathbb{C}^*$ be the determinant character $g \mapsto \det g$. Under the isomorphism (4.21), the above open substacks correspond to GIT-semistable loci, see [Toda, Proposition 5.5.2]:
\[
I_{C_3}^{\text{red}}(m, d) = \mathcal{T}_{C_3}^{\text{red}}(m, d)^{\chi_0}\text{-ss}, \quad P_{d}^{\text{red}}(C_3, m) = \mathcal{T}_{C_3}^{\text{red}}(m, d)^{\chi_0^{-1}}\text{-ss}.
\]

We define
\[
(4.23) \quad \Tr W_{m,d}^\pm : \left( (V^{\oplus 2} \oplus V^\vee \oplus g^{\oplus 3})^{\chi_0}\text{-ss} \times (\mathcal{C}^m)^{\text{red}} \right) / GL(V) \to \mathbb{C}
\]
to be the restriction of $\Tr W_{m,d}$ to the GIT semistable loci. Then we have
\[
I_{C_3}^{\text{red}}(m, d) = \text{Crit}(\Tr W_{m,d}^+), \quad I_{C_3}^{\text{red}}(m, d) = \text{Crit}(\Tr W_{m,d}^-).
\]

We define the following dg-categories
\[
\mathcal{D}T_{C_3}^{\text{red}}(m, d) := \text{MF} \left( (V^{\oplus 2} \oplus V^\vee \oplus g^{\oplus 3})^{\chi_0}\text{-ss} \times (\mathcal{C}^m)^{\text{red}} / GL(V), \Tr W_{m,d}^+ \right),
\]
\[
\mathcal{P}T_{C_3}^{\text{red}}(m, d) := \text{MF} \left( (V^{\oplus 2} \oplus V^\vee \oplus g^{\oplus 3})^{\chi_0^{-1}}\text{-ss} \times (\mathcal{C}^m)^{\text{red}} / GL(V), \Tr W_{m,d}^- \right).
\]
The above dg-categories categorify DT invariants (resp. PT invariants) for Hilbert schemes of 1-dimensional subschemes (resp. PT stable pair moduli spaces) with reduced supports, see [Eli18, Theorem 1.1], [Toda, Subsection 3.3].

**Remark 4.7.** Following the definition of DT and PT categories in [Toda], one can also consider graded categories of matrix factorizations, with grading given by the weight two $\mathbb{C}^*$-action on $v$ and $C$. The graded version also recovers DT and PT invariants and they have the same Grothendieck group as the ungraded one, see [Toda, Proposition 3.3.6], [Tod13, Corollary 3.13].

**Proof of Theorem 1.4.** The statement follows from Corollary 4.8 for $a = 1$ and $\mu = -1/2 - \varepsilon$ for $0 < \varepsilon \ll 1$. □

**Proof of Corollary 1.5.** The claim follows as the isomorphism (3.31) from the proof of Corollary 1.2. □

In the context of Corollary 1.5 we do not know whether the following is an isomorphism

$$K \left( D_T(d - d') \otimes P_{C^3}(m, d') \right) \cong K \left( D_T(d - d') \right) \otimes K \left( P_{C^3}(m, d') \right).$$

To obtain a result analogous to the one of Corollary 1.2, we use equivariant K-theory, see [PTa, Section 4] for similar constructions and computations. Let $T \cong (\mathbb{C}^*)^2$ be the natural two dimensional torus acting on $\mathbb{C}^2$. We let $T$ act on $\mathbb{C}^3$ as follows:

$$(t_1, t_2) \cdot (x, y, z) = (t_1 x, t_2 y, t_1^{-1} t_2^{-1} z).$$

It induces the action on $V^{\otimes 2} \oplus V^\vee \oplus \mathbb{C}^{3 \oplus} \oplus \mathbb{C}^{m}$ by

$$(t_1, t_2) \cdot (u_1, u_2, v, A, B, C, \alpha_{ij}) = (u_1, u_2, v, t_1 A, t_2 B, t_1^{-1} t_2^{-1} C, t_1^{-1} t_2^{-1} \alpha_{ij}).$$

The function (4.20) is invariant under the above $T$-action. We consider the equivariant categories $D_T^{\text{red}}(C_t, T, m, d)$ and $P_{C^3}^{\text{red}}(m, d)$ and we denote their Grothendieck groups by

$$K_T(D_T^{\text{red}}(C_t, d, m, d)) \text{ and } K_T(P_{C^3}^{\text{red}}(m, d)).$$

Let $K := K_T(pt)$ and let $F := \text{Frac } K$. For $V$ a $K$-module, let $V_F := V \otimes_K F$.

**Corollary 4.8.** Let $d \in \mathbb{N}$. Then there is an isomorphism of $F$-vector spaces:

$$K_T \left( D_T^{\text{red}}(C_t, d, m, d) \right)_F \cong \bigoplus_{d' = 0}^d K_T \left( D_T(d - d') \right)_F \otimes_K K_T \left( P_{C^3}^{\text{red}}(m, d') \right)_F.$$

**Proof.** As in the proof of Corollary 1.5 there is an isomorphism of $F$-vector spaces:

$$K_T \left( D_T^{\text{red}}(C_t, d, m, d) \right)_F \cong \bigoplus_{d' = 0}^d K_T \left( D_T(d - d') \otimes P_{C^3}^{\text{red}}(m, d') \right)_F.$$

It suffices to show that there is a Künneth isomorphism:

$$K_T \left( D_T(d - d') \otimes P_{C^3}^{\text{red}}(m, d') \right)_F \cong K_T(D_T(d - d'))_F \otimes_K K_T \left( P_{C^3}^{\text{red}}(m, d') \right)_F.$$
the same Grothendieck group \cite{Todb}, Corollary 3.13. Further, \(S^P_T(d)_w\) appears as summands in \(D^b_T(C(d))_w\) \cite{Todb}, Corollary 3.3, where \(C(d)\) is the (derived) stack of pairs of commuting matrices of size \(d\). Consider the analogous graded category \(\mathcal{P}^T(m,d)\) with the same Grothendieck group as \(\mathcal{P}^T(m,d)\). By \cite[Theorem 2.10]{HL15} and the Koszul equivalence, see for example \cite[Theorem 2.3.3]{Toda}, the category \(\mathcal{P}^T(m,d)\) is admissible in \(D^b_T(C^3_T(m,d))\). By the argument in the proof of Corollary 1.2, it suffices to check the Künneth isomorphism

\[
G_T\left(\mathcal{D} \times C^3_T(m,d)\right)_F \cong G_T(\mathcal{D})_F \otimes F G_T\left(C^3_T(m,d)\right)_F,
\]

where \(\mathcal{D}\) is the product of finitely many commuting stacks \(C(e)\) for various \(e\). The statement follows from the localization theorem in K-theory (see \cite{Tak94}) with respect to \(T\) because \(\mathcal{D}^T = \text{pt}/L\), where \(L\) is a product of finitely many groups \(GL(e)\).

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