On semiclassical approximation for correlators of closed string vertex operators in AdS/CFT

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Abstract

We consider the 2-point function of string vertex operators representing string state with large spin in $AdS_5$. We compute this correlator in the semiclassical approximation and show that it has the expected (on the basis of state-operator correspondence) form of the strong-coupling limit of the 2-point function of single trace minimal twist operators in gauge theory. The semiclassical solution representing the stationary point of the path integral with two vertex operator insertions is found to be related to the large spin limit of the folded spinning string solution by a euclidean continuation, transformation to Poincare coordinates and conformal map from cylinder to complex plane. The role of the source terms coming from the vertex operator insertions is to specify the parameters of the solution in terms of quantum numbers (dimension and spin) of the corresponding string state. Understanding further how similar semiclassical methods may work for 3-point functions may shed light on strong-coupling limit of the corresponding correlators in gauge theory as was recently suggested by Janik et al in arXiv:1002.4613.

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1 Introduction

The AdS/CFT duality [1] between the $\mathcal{N} = 4$ SYM theory and the superstring theory in $AdS_5 \times S^5$ predicts, in particular, a relation between planar correlation functions of single-trace conformal primary operators in gauge theory and correlation functions of the corresponding closed-string vertex operators on a 2-sphere. By analogy with the relation between correlators of BPS operators at strong coupling and the supergravity correlators [2,3] one may conjecture the following equality of the associated generating functionals

$$\langle e^{\Phi_2} \rangle_{4d} = \langle e^{\Phi_2} \rangle_{2d}.$$  \hspace{1cm} (1.1)

The l.h.s. in this relation is computed in 4d gauge theory and the r.h.s – in the 2d superstring sigma model. Here $\Phi(x)$ is a source 4d field, $O$ is a primary gauge theory operator of dimension $\Delta$ and $V$ is the corresponding marginal string vertex operator\(^1\) [4,5] with

$$\Phi \cdot O = \int d^4 x' \phi(x') O(x'),$$  \hspace{1cm} (1.2)

$$\Phi \cdot V = \int d^4 x' \phi(x') V(x') = \int d^2 \xi \phi(x(\xi), z(\xi)) U[x(\xi), z(\xi), ...] ,$$  \hspace{1cm} (1.3)

$$\hat{\phi}(x, z) = \int d^4 x' K(x - x'; z) \phi(x'),$$  \hspace{1cm} (1.4)

Here $\hat{x} = (z, x_m)$ are the Poincare patch coordinates, $ds^2 = z^{-2}(dz^2 + dx_m dx_m)$, and

$$K(x - x'; z) = c[z + z^{-1}(x - x')^2]^{-\Delta}, \quad K(x - x'; z)_{z \to 0} = \delta^{(4)}(x - x'),$$  \hspace{1cm} (1.5)

$U$ stands for a nontrivial part of the vertex operator that encodes information about other quantum numbers. In general, $U$ depends also on the $S^5$ coordinates and the fermions present in the superstring action [6].\(^2\)

The structure of the 2-point and 3-point correlators in gauge theory is essentially constrained by the 4d conformal invariance,

$$\langle O_{\Delta_1}(x) O_{\Delta_2}(x') \rangle_{4d} = \frac{C_2 \delta_{\Delta_1, \Delta_2}}{|x - x'|^{2\Delta_1}} ,$$  \hspace{1cm} (1.6)

$$\langle O_{\Delta_1}(x) O_{\Delta_2}(x') O_{\Delta_3}(x'') \rangle_{4d} = \frac{C_3}{|x - x'|^{\Delta_1 + \Delta_2 - \Delta_3} |x - x'|^{\Delta_1 + \Delta_3 - \Delta_2} |x' - x''|^{\Delta_2 + \Delta_3 - \Delta_1}} ,$$  \hspace{1cm} (1.7)

Here $C_2$ depends on normalization of $O_{\Delta}(x)$. The normalization-independent combination $C_3^{\prime} \equiv C_3[C_2(\Delta_1)C_2(\Delta_2)C_2(\Delta_3)]^{-1/2}$ is, in general, a non-trivial function of ’t Hooft coupling $\lambda$.

\(^1\)Precise relation between gauge theory conformal primary operators and marginal vertex operators remains an open problem, though its solution should be aided by recent progress in understanding the spectrum of states based on integrability.

\(^2\)Here we indicated just schematic form of the vertex operator. In general, there may be “mixing” between the $K$-part and $U$-part (see [5]); this will not be important in the leading semiclassical approximation $\Delta \sim \sqrt{\lambda} \gg 1$ discussed below.
dimensions $\Delta_1, \Delta_2, \Delta_3$ and other quantum numbers like spins. So far $C_3$ is explicitly known for the BPS operators only [7, 8].

On the string side, the world-sheet conformal invariance and the target space $SO(2, 4)$ global symmetry of the string sigma model imply that for 2d marginal operators that represent highest-weight states of $SO(2, 4)$ one should get similar relations (here $J$ stands for some “compact” spin quantum number that label the vertex operator)

$$
\langle V_{\Delta_1, J_1}(\xi_1; x)V_{\Delta_2, J_2}(\xi_2; x') \rangle_{2d} = \frac{C_2}{|x - x'|^{2\Delta}} \frac{\delta(\Delta_1 - \Delta_2)\delta_{J_1 + J_2, 0}}{|\xi_1 - \xi_2|^{2\delta}},
$$

(1.8)

$$
\langle V_{\Delta, J}(x)V_{\Delta, -J}(x') \rangle_{2d} = \frac{C_2}{|x - x'|^{2\Delta}}, \quad C_2 = C_2\delta(0) \int \frac{d^2\xi_1d^2\xi_2}{|\xi_1 - \xi_2|^{2\delta}}.
$$

(1.9)

Here

$$
\delta(\Delta, J) = 2
$$

(1.10)

is the marginality condition that should be satisfied for a physical vertex operator $V_{\Delta, J}$.

As usual, the definition of the string vertex operator correlators on a sphere (complex plane) should include division over the infinite volume of the $SL(2, C)$ Mobius group (which is part of the residual reparametrization gauge freedom left out in the conformal gauge). In flat space this implies the vanishing of the 2-point correlators (which has an interpretation of the vanishing of the tree-level inverse propagator when evaluated on the mass shell). This should no longer be so in the $AdS$ case where the 2-point correlator should have a non-trivial gauge-theory interpretation (1.6). The resolution of this puzzle (suggested in the NS-NS $AdS_3/CFT_2$ case [9, 10] with the $AdS_3$ part described by the $SL(2, R)$ WZW model in [11, 12]) should be due to a cancellation between the divergent factor $\delta(0) \int \frac{d^2\xi_1d^2\xi_2}{|\xi_1 - \xi_2|^{2\delta}}$ and the Mobius group volume factor contained in $C_2$. Here the factor $\delta(0) = \delta(\Delta_1 - \Delta_2)\delta_{\Delta_1 = \Delta_2}$ has its origin in the non-compactness of the symmetry group $SO(1, 5)$ of Euclidean $AdS_5$ space which is a global symmetry of the string sigma model (this factor should effectively come out of the integral over the zero mode of the “Liouville” field $\varphi \equiv \ln z$ where $z$ is the Poincare patch coordinate).

For a 3-point function of marginal vertex operators one should get a similar expression as in

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3Note that while $\Delta_i$ take, in general, continuous values, in the correlator (1.6) computed in perturbation gauge theory $\delta_{\Delta_1, \Delta_2}$ stands effectively for the Kronecker delta-symbol of integer-valued canonical dimensions of the two operators.

4Equivalently, the volume of the target space conformal symmetries with 2 marked points should cancel against that of the world-sheet conformal transformations with 2 marked points. It would be useful to make this cancellation argument more explicit by regularizing the divergent factors. Let us note also that the prescription of [2, 3] for correlators of the BPS operators at strong coupling used the Einstein (or the full supergravity) action to define the correlators on the AdS side. The 10d supergravity action follows from the full string theory (expanded near a consistent vacuum) after one subtracts Mobius divergences and massless exchanges and takes the “low-energy” $\alpha' \to 0$ limit.
\( \langle V_{\Delta_1,J_1}(x)V_{\Delta_2,J_2}(x')V_{\Delta_3,J_3}(x'') \rangle_{2d} \)

\[
= \frac{C_3}{|x-x'|^{\Delta_1+\Delta_2-\Delta_3}|x-x''|^{\Delta_1+\Delta_3-\Delta_2}|x'-x''|^{\Delta_2+\Delta_3-\Delta_1}}, \quad (1.11)
\]

\[
C_3 = C_3 \int \frac{d^2\xi d^2\xi' d^2\xi''}{|\xi - \xi'|^2|\xi - \xi''|^2|\xi' - \xi''|^2}. \quad (1.12)
\]

Here \( C_3 \) should again contain the inverse of the Mobius group factor so that \( C_3 \) should be a finite function of \( \Delta_i, J_i \) and string tension and should match \( C_3 \) in (1.7) (for \( AdS_3/CFT_2 \) examples see [12, 13]).

One may hope to gain some important information about string-theory correlators in semiclassical approximation by considering states with large quantum numbers of order of string tension \( T = \sqrt{\lambda} \gg 1 \), i.e. \( \Delta_i \sim \sqrt{\lambda} \), \( J \sim \sqrt{\lambda} \). That may then allow one to predict the strong-coupling behaviour of the corresponding gauge theory correlators for non-BPS states. Indeed, as in the analogous limit in flat space [14], if each of the vertex operators scales effectively as an exponent of the string tension, then the leading \( \sqrt{\lambda} \gg 1 \) contribution to the corresponding string path integral should be determined by a classical stationary point. In the case of the 2-point functions of vertex operators (and related Wilson loop correlators) such semiclassical approximation was discussed in [4, 15, 16, 17, 5, 18, 19, 20, 21].

Recently, a similar idea was suggested [22] for the 3-point correlators. While the semiclassical approximation should capture the \( x \)-coordinate dependent factors in (1.11) which scale as \( e^{\sqrt{\lambda}(\ldots)} \) for \( \Delta_i \sim \sqrt{\lambda} \) (as was, indeed, demonstrated in an example in [22]) it is not a priori clear if it is sufficient to shed light on the strong-coupling asymptotics of the non-trivial coefficient \( C_3 \), i.e. whether it also scales as \( e^{\sqrt{\lambda}(\ldots)} \) at \( \sqrt{\lambda} \gg 1 \) (such scaling does not happen in the case of correlators of three BPS operators [7]).

The discussion in [22] was qualitative in that it did not specify precisely which are the string states under consideration, beyond the fact that they should correspond to known classical spinning string solutions on a 2-cylinder. To be able to address the question about strong-coupling asymptotics of correlators systematically one needs to define the states using vertex operators. One should then gain a better understanding of how a particular structure of the vertex operators representing string states with large quantum numbers translates into the boundary conditions for the corresponding semiclassical world surface, apart from a heuristic requirement that it should end at boundary points in the Poincare patch [3, 22] that are \( x \)-space locations of the vertex operators.

This will be our aim below. We shall mainly concentrate on the example of 2-point function of vertex operators representing spinning string state in \( AdS_5 \) with \( S = \sqrt{\lambda}S, \ S \gg 1 \). As was suggested in [5], computing this 2-point function semiclassically and demanding the 2d conformal invariance, i.e. the consistency of the result with (1.9), (1.10), should lead to the

\[ \text{Let us mention also a related discussion of semiclassical approximation with vertex operators in } AdS_3 \text{ context in [10]. In that case of string sigma model being } SL(2,R) \text{ WZW theory the situation is simpler as the WZW equations of motion can be solved explicitly.} \]

\[ ^5 \]
same relation between the energy (dimension $\Delta$) and the spin as found for the corresponding folded string classical solution [15] in the large spin limit. In [5] this was shown to be true in the flat space limit; a detailed argument for a large spin limit of the spinning string in $AdS_3$ (and similar orbiting string states in $S^5$) was recently presented in [21]. Ref. [21] used global coordinates while here we shall show how a similar argument can be given in the Poincare coordinates, including also the dependence of vertex operators on the boundary points ignored in [21]. This will allow us to clarify a relation of this vertex operator approach to the approach of [22]. We shall see that the corresponding semiclassical world surface is closely related to the euclidean counterpart of the Poincare patch image [23] of the long folded spinning string – the null cusp surface [24].

We shall review the basic ideas [5, 21] of the semiclassical approach to the 2-point function of the spinning string vertex operators in section 2. In section 3 we shall consider the case when vertex operators represent a string which is point-like in $AdS_5$ but may be extended in $S^5$. In section 4 we shall discuss our main example – the semiclassical computation of the 2-point function of vertex operators representing spinning string in $AdS_5$. Some concluding remarks will be made in Section 5. Appendix A contains details of the argument in section 4 that the stationary point solution for the 2-point correlator is the same as conformally mapped euclidean continuation of the large spin limit of the folded string in $AdS_5$.

2 Semiclassical string states and vertex operators

Let us start with a review of the relation between semiclassical string states and 2-point functions of the corresponding vertex operators. Given a quantum closed string state on a Minkowski 2d cylinder $(\tau, \sigma)$ one may first switch to euclidean time $\tau_e = i\tau$ and map the cylinder into complex plane with two punctures at 0 ($\tau_e = -\infty$) and $\infty$ ($\tau_e = +\infty$) by $z = e^{\tau_e + i\sigma}$. According to the standard state-operator correspondence such state may be thought of as created from the vacuum by a vertex operator inserted at the origin of the complex plane.6 A semiclassical string state with large quantum numbers of order of string tension can be described by a classical solution on original Minkowski 2d cylinder. The Virasoro condition for the classical solution (that relates its energy to quantum numbers like spins) is then equivalent (for large quantum numbers) to the marginality condition for the vertex operator.7

6There is a similar relation between 4d CFT state on $R \times S^3$ and the corresponding primary operator at the origin of $R^4$. The correspondence between a closed string state on $R \times S^3$ in global $AdS_5$ and a gauge theory state on $R \times S^3$ thus translates into a correspondence between a single trace primary operator at the origin of $R^4$ and a marginal vertex operator at the origin of $R^2$.

7Below we shall use the string action in the conformal gauge. In general, to recover the two conditions following from the Virasoro constraints one should keep the 2d metric arbitrary and extremise with respect to it before fixing the conformal gauge.
2.1 General ideas

When rotating to euclidean world-sheet time it is natural to do similar rotation in the target space, i.e. rotate the global AdS time \( t = e^{it} \).

\[
\tau_{e} = i\tau, \quad t_{e} = it. \tag{2.1}
\]

Considering two vertex operators it is useful to map the euclidean 2d cylinder into the complex plane so that the punctures appear at arbitrary finite positions \( \xi_1 \) and \( \xi_2 \)

\[
e^{\tau_{e} + i\sigma} = \frac{\xi - \xi_2}{\xi - \xi_1}, \tag{2.2}
\]

or, explicitly,

\[
\tau_{e} = \frac{1}{2} \ln \frac{(\xi - \xi_2)(\xi - \bar{\xi}_2)}{(\xi - \xi_1)(\xi - \bar{\xi}_1)}, \quad \sigma = \frac{1}{2i} \ln \frac{(\xi - \xi_2)(\bar{\xi} - \xi_1)}{(\xi - \xi_2)(\bar{\xi} - \xi_1)}. \tag{2.3}
\]

One may expect that the analytically continued version of a classical solution on 2d cylinder mapped onto complex plane should then be the same as the world-surface which is the stationary trajectory of the path integral representing the 2-point correlation function of the vertex operators. This stationary point solution should solve the rotated (euclidean-signature) string equations of motion on the complex plane in conformal gauge with “delta-function” sources at \( \xi_1 \) and \( \xi_2 \). The role of matching onto these source terms is to relate the parameters of the semiclassical solution to the quantum numbers (energy, spins, ...) that label the vertex operators. The requirement that the resulting correlator evaluated on the stationary point solution takes the conformally invariant form (1.8),(1.10) proportional to \( |\xi_1 - \xi_2|^{-4} \) is equivalent to the condition of marginality of the vertex operators. It will provide a relation between the parameters which is the same (for large quantum numbers) as the one that follows from the Virasoro condition for the original solution on the cylinder.

This equivalence between the analytically continued and conformally mapped classical world surface solution and the stationary point in the corresponding 2-point correlator of the vertex operators was explicitly demonstrated in [5] in flat space on the example of the vertex operator that describes bosonic string state on the leading Regge trajectory with spin \( S \) and energy \( E \)

\[
V(\xi) = e^{-iEt} (\partial X \bar{\partial} X)^{S/2}, \quad X = x_1 + ix_2 \tag{2.4}
\]

with the marginality condition being \( S + \frac{1}{2} \alpha' E^2 = 2 \), i.e. \( E = \sqrt{2(S - 2)/\alpha'} \). The relevant classical solution on Minkowski 2d cylinder is

\[
t = \kappa \tau, \quad X = x_1 + ix_2 = r(\sigma) e^{i\phi(\tau)} = r_0 \sin \sigma e^{i\tau}, \tag{2.5}
\]

where

\[
\kappa = r_0, \quad \text{i.e.} \quad E = \sqrt{2S/\alpha'}. \tag{2.6}
\]

\[^8\text{That preserves, in particular, the reality of the space-time energy for a simple class of solutions. In AdS}_5 \text{ case this allows the classical trajectory (e.g. massive geodesic) to reach the boundary [18].}\]
follows from the conformal gauge constraint. Assuming $S$ is large and transforming (2.5) using (2.1),(2.2) one gets a solution that indeed solves the string equations with source terms coming from the two vertex operator insertions, with the $E(S)$ relation (2.6) now following from the 2d conformal invariance of the resulting semiclassical value of the correlator.\footnote{The shift by 2 ($S + \frac{1}{2} \alpha' E^2 = 0 \rightarrow S + \frac{1}{2} \alpha' E^2 = 2$) in the marginality condition is of course not visible in the leading semiclassical large $S$ approximation.}

It was suggested in [5] that the same relation between the classical solution and the stationary point world surface in the vertex operator correlator should generalise to the case of the spinning folded string solution in $AdS_5$ [15]. The latter should correspond to a vertex operator like (cf. (1.4))\footnote{We mixing with other structures [5] and fermionic contributions that should be irrelevant at leading order of large spin or semiclassical expansion.}

$$V(\xi) = (Y_5 + iY_0)^{-E} (\partial Y \bar{\partial} Y)^{S/2}, \quad Y \equiv Y_1 + iY_2. \quad (2.7)$$

Here $Y_m$ are embedding coordinates of $AdS_5$ (see below). By discrete symmetry of the euclidean $AdS_5$ $(iY_0 = Y_0 \leftrightarrow Y_4)$ this operator is related to [5]

$$V(\xi) = (Y_5 + Y_4)^{-\Delta} (\partial Y \bar{\partial} Y)^{S/2}, \quad (2.8)$$

with $\Delta$ being the analog of $E$ in the 54 boost plane.

The original folded-string solution solves Minkowski $AdS_5$ string equations defined on a Minkowski 2d cylinder. It describes propagation of a particular semiclassical closed string mode in real time. The vertex-operator 2-point correlator defined on a Euclidean 2-plane may be mapped onto Euclidean 2-cylinder with the vertex operators specifying a particular string state propagating on the cylinder. As in flat space case the relevant semiclassical trajectory should then be a (complex and conformally transformed) analog of the folded string solution.\footnote{As was already apparent in the BMN state case with $\Delta = J$ case [4, 5, 18], one should not attribute a special meaning to the complex nature of the resulting semiclassical trajectory saturating the path integral. Its imaginary nature of is related to external sources one puts in to specify the required boundary/initial conditions. Like in the case of a euclidean gaussian path integral with imaginary sources the result is an analytic function of the quantum numbers, i.e. one is allowed to make analytic continuations.}

Despite the general expectation that the role of the two vertex operators should simply be to implement a proper choice of boundary conditions in mapping the 2-plane back onto the cylinder, i.e. their detailed pre-exponential form should not be that important, the non-linear nature of the string equations and the non-trivial elliptic function form of the solution of [25, 15] precluded the direct verification of this relation in [5].

One simplification that one can make is to consider the large spin limit of the folded string solution in $AdS_5$ in which it is stretched all the way to the boundary and takes a simple ("homogeneous") form [26]. As was recently shown in [21], in this case one can verify that the euclideanized and conformally transformed (2.2) large spin solution represents indeed the semiclassical trajectory saturating the 2-point correlation function of the vertex operators (2.7), with the marginality condition being equivalent to $E = S + \frac{\sqrt{\lambda}}{\pi} \ln S$, $\frac{S}{\sqrt{\lambda}} \gg 1$.

In [21] the vertex operator was chosen in the simplest global coordinate form (2.7) with $Y_5 + iY_0 = \cosh \rho \ e^{it}$ in which there was no dependence on a boundary point $x_m$. This is
analogous to ignoring extra spatial momentum dependence in the exponent of the flat-space vertex operator in (2.4). As a result, the $x$-dependence of the 2-point correlator in (1.9) was not determined explicitly.

Motivated by a possibility of generalization to 3-point correlators and in order to establish the relation to the approach of [22] (were similar semiclassical computations of 2-point functions were done without using explicit vertex operators) here we shall reconsider the case of the 2-point correlator of large spin operators using the Poincare coordinates and explicitly including the dependence on the boundary points. We shall show that the large spin limit of the folded string transformed into the euclidean Poincare coordinates and mapped into complex plane using (2.2) represents the semiclassical surface saturating the large tension limit of the 2-point correlation function (1.9).

2.2 $AdS_5$ coordinates and vertex operators

Let us first recall the basic definitions of global and Poincare coordinates in $AdS_5$, their euclidean continuation and the form of the corresponding vertex operators [5]. For the standard Minkowski signature $AdS_5$ we have $(m = 0, 1, 2, 3; \ i = 1, 2, 3)$

\[
Y_5 + iY_0 = \cosh \rho \ e^{it}, \quad Y_1 + iY_2 = \sinh \rho \ \cos \theta \ e^{i\phi_1}, \quad Y_3 + iY_4 = \sinh \rho \ \sin \theta \ e^{i\phi_2},
\]

\[
Y^M Y_M = -Y_5^2 + Y^m Y_m + Y_4^2 = -1, \quad Y_m Y^m = -Y_0^2 + Y_i Y_i,
\]

(2.9)

where $(t, \rho, \theta, \phi_1, \phi_2)$ are the global angular coordinates related to the Poincare coordinates by

\[
Y_m = \frac{x_m}{z}, \quad Y_4 = \frac{1}{2z}(-1 + z^2 + x^m x_m), \quad Y_5 = \frac{1}{2z}(1 + z^2 + x^m x_m),
\]

(2.10)

with $x^m x_m = -x_0^2 + x_i x_i$. Assuming a highest-weight state of $SO(2,4)$ is labelled by the three Cartan generators $(E, S_1, S_2)$ in the 50,12,34 planes, a wave function or a vertex operator representing a state with AdS energy $E$ should contain a factor

\[
(Y_5 + iY_0)^{-E} = (\cosh \rho)^{-E} \ e^{-iEt}.
\]

(2.11)

This does not have a particularly simple form when written in the Poincare coordinates. If instead we label representations by an $SO(1,1)$ generator in the 54 plane then the corresponding factor will be

\[
(Y_5 + Y_4)^{-\Delta} = (z + z^{-1} x^m x_m)^{-\Delta},
\]

(2.12)

where $\Delta$ is the dilatation generator $(z \rightarrow k z, \ x_m \rightarrow k x_m)$.

The euclidean continuation corresponds to changing the time-like coordinates to

\[
t_e = it, \quad Y_{0e} = iY_0, \quad x_{0e} = ix_0,
\]

(2.13)

so that eq. (2.9) takes the form

\[
Y^M Y_M = -Y_5^2 + Y_{0e}^2 + Y_i Y_i + Y_4^2 = -1.
\]

(2.14)
Then the \(SO(2, 4)\) symmetry becomes \(SO(1, 5)\) which contains the discrete transformation
\[
Y_{0e} \leftrightarrow Y_4, \quad E \leftrightarrow \Delta
\] (2.15)
that relates the factors in (2.11) and (2.12).

To construct a vertex operator parametrized by 4 coordinates of a point at the boundary of the euclidean Poincare patch we should shift \(x_m = (x_{0e}, x_i)\) by a constant vector \(x'_m\), getting the same expression as in (1.4),(1.5)\(^{12}\)
\[
V(x') = \int d^2 \xi \left[ z(\xi) + z^{-1}(\xi) \right] |x_m(\xi) - x'_m|^2 \Delta U[x(\xi), z(\xi), ...].
\] (2.16)

It should be noted that the choice of the Poincare coordinate form of the vertex operators is natural for comparison to the standard form of correlators of primary operators on the gauge theory side (cf. (1.6),(1.7)) but is not a priori the only one possible on the string theory side: one may choose any coordinates as long as one uses the general form of the vertex operators that includes dependence on the boundary data.\(^{13}\)

3 Semiclassical approximation: case of point-like string in \(AdS_5\)

Let us start the discussion of semiclassical computation of 2-point function of operators (2.16) with the case when \(V\) represents a state which is point-like in \(AdS_5\) (e.g., it may be a string spinning only in \(S^5\)). Then \(U\) will not depend on \(z, x_m\) coordinates. The simplest example is the BMN state when \(U = (X_1 + iX_2)^{-J}\) where \(X_I\) are embedding coordinates of \(S^5\).\(^{14}\) The \(AdS_5\) part of the corresponding classical solution \(t = \kappa \tau, \rho, \theta, \phi_1, \phi_2 = 0\) (with \(\kappa\) related to the spins of the \(S^5\) part via the conformal gauge constraints) will be simply a massive geodesic running through the center of \(AdS_5\) but after the euclidean continuation it will be able to reach the boundary (see [18, 22]). Indeed, continued to the Euclid and written in the Poincare coordinates (2.10) this background takes the form (see, e.g., [27])
\[
z = \frac{1}{\cosh(\kappa \tau_e)}, \quad x_{0e} = \tanh(\kappa \tau_e), \quad x_i = 0,\] (3.1)
so that for \(\tau_e \to \pm \infty\) we have \(z \to 0\), i.e. the euclidean trajectory reaches the boundary at the two points: \(x_{0e} = -1, x_i = 0\) and \(x_{0e} = 1, x_i = 0\). By a dilatation and translation, we can put the position of the end-points at \(x_{0e} = 0\) and \(x_{0e} = a\); the corresponding solution is then
\[
z = \frac{a}{2 \cosh(\kappa \tau_e)}, \quad x_{0e} = \frac{a}{2} \left[ \tanh(\kappa \tau_e) + 1 \right], \quad x_i = 0.\] (3.2)

Let us now show that mapping this solution onto the complex plane by (2.2) gives a singular configuration that coincides with the semiclassical trajectory for the correlator of two vertex operators (2.16) inserted at the points \(x_m = (0, 0, 0, 0)\) and \(x'_m = (a, 0, 0, 0)\). Assuming

\(^{12}\)Since translations of \(x_m\) are part of the conformal group this corresponds to a particular \(SO(1, 5)\) transformation applied to (2.12).

\(^{13}\)For example, to construct the analog of (2.16) in global coordinates one is to apply the inverse of the transformation (2.10) to \(z(\xi)\) and \(x_m(\xi)\) there.

\(^{14}\)The euclidean stationary point solution for coordinates of \(S^5\) will be in general complex, see [4, 16, 18, 5, 21].
$\Delta \sim \sqrt{\lambda} \gg 1$ so that the correlator can be approximated by a semiclassical trajectory, the corresponding euclidean string action in conformal gauge including the contributions of the source terms coming from the two vertex operator insertions is $(\partial = \frac{1}{\pi}(\partial_1 + i \partial_2))$

\[
A_e = \frac{\sqrt{\lambda}}{\pi} \int d^2 \xi \frac{1}{z^2} (\partial z \bar{\partial} z + \partial \bar{\partial} x_m) - \Delta \int d^2 \xi \delta^2(\xi - \xi_1) \ln \frac{z}{z^2 + (x_{0e} - a)^2} - \Delta \int d^2 \xi \delta^2(\xi - \xi_2) \ln \frac{z}{z^2 + x_{0e}^2} + A_e(S^5) . \tag{3.3}
\]

Here $A_e(S^5)$ is the $S^5$ part of the action which will be relevant only for the $|\xi_1 - \xi_2|$-dependent part of the correlator, i.e. for the check of the marginality condition relating $\Delta$ with $S^5$ quantum numbers as in the examples discussed in [5, 21]. The resulting equations are solved by $x_i = 0$ while the equation for $x_{0e}$ reads

\[
\frac{\partial}{\partial z} \frac{\bar{\partial} x_{0e}}{z^2} + \frac{\bar{\partial}}{\partial z} \frac{x_{0e} - a}{z^2 + (x_{0e} - a)^2} \delta^2(\xi - \xi_1) + \frac{x_{0e}}{z^2 + x_{0e}^2} \delta^2(\xi - \xi_2) = 0 . \tag{3.4}
\]

Using (2.3) and (3.2) we have $\frac{\partial x_{0e}}{z^2} = \frac{2a}{\lambda} \partial \tau_e$ and then the l.h.s. of eq. (3.4) is just

\[
\frac{2\pi \lambda}{a} \left[ \delta^2(\xi - \xi_2) - \delta^2(\xi - \xi_1) \right] . \tag{3.5}
\]

On the solution (3.2) we also have $\frac{x_{0e}}{z^2 + x_{0e}^2} = -\frac{x_{0e} - a}{z^2 + (x_{0e} - a)^2} = \frac{1}{a}$ and then eq. (3.4) is satisfied by (3.2) provided

\[
k = \frac{\Delta}{\sqrt{\lambda}} . \tag{3.6}
\]

It is straightforward to check that the equation for $z$ following from (3.3)

\[
\frac{\partial}{\partial z} \frac{\bar{\partial} z}{z^2} + \frac{\bar{\partial}}{\partial z} \frac{2z^2 (\partial z \bar{\partial} z + \partial \bar{\partial} x_m \bar{x}_m)}{z^3} = \frac{\pi \Delta}{\sqrt{\lambda}} \frac{1}{z} \left[ \frac{z^2 - (x_{0e} - a)^2}{z^2 + (x_{0e} - a)^2} \delta^2(\xi - \xi_1) + \frac{z^2 - x_{0e}^2}{z^2 + x_{0e}^2} \delta^2(\xi - \xi_2) \right] \tag{3.7}
\]

is again satisfied by (3.2) with (3.6). Evaluating the action (3.3) on the classical solution (3.2) mapped according to (2.3) we get (using (3.6) and that on the solution $\frac{z}{z^2 + (x_{0e} - a)^2} = \frac{1}{a} e^{\kappa \tau_e}, \quad \frac{z}{z^2 + x_{0e}^2} = \frac{1}{a} e^{-\kappa \tau_e}$)

\[
A_e = 2\Delta \ln a - \frac{\Delta^2}{\sqrt{\lambda}} \ln |\xi_1 - \xi_2| + A_e(S^5) . \tag{3.8}
\]

The final expression for the semiclassical approximation for the 2-point correlator is then consistent with (1.9)

\[
\langle V_{\Delta,J}(a) V_{\Delta,-J}(0) \rangle \sim \frac{1}{a^{2\Delta}} \int d^2 \xi_1 d^2 \xi_2 |\xi_1 - \xi_2|^{\frac{\Delta^2}{\sqrt{\lambda}} - \gamma(J)} , \tag{3.9}
\]
where $\gamma(J) = \frac{J^2}{\sqrt{\lambda}} + \ldots$ stands for the $S^5$ contribution. The condition of the world-sheet conformal invariance, i.e. the marginality condition (1.10) then relates $\Delta$ and $S^5$ spins $(J, \ldots)$. This will be the same relation as between the energy $E$ and the spins that follows from the Virasoro condition for the original Minkowski-signature real solution representing the corresponding semiclassical string state (see also [21] for some explicit examples).

The scaling $\sim a^{-2\Delta}$ of the 2-point correlator with the boundary point coordinate is of course a simple consequence of the conformal invariance of the classical string action and of the fact that the vertex operators have definite scaling dimensions: it follows from doing the rescaling $z \rightarrow az, \quad x_m \rightarrow ax_m$ in the path integral. The same will be true in the spinning string case discussed in the next section.

### 4 Semiclassical approximation: case of folded spinning string in $AdS_5$

Let us now consider the semiclassical approximation for the correlator of two vertex operators (2.8) that represent semiclassical string states with spin $S \sim \sqrt{\lambda} \gg 1$. Written in the Poincare coordinates (using (2.10),(2.12),(2.16)) the euclidean vertex operator labelled by the boundary point $x'_m = (a, 0, 0, 0)$ reads

$$V_{\Delta,S}(a) = c \int d^2 \xi \left[ z + z^{-1}(x_{0e} - a)^2 \right]^{-\Delta} \left[ \partial \left( \frac{r^{i\phi}}{z} \right) \overline{\partial} \left( \frac{r^{i\phi}}{z} \right) \right]^{S/2}, \quad (4.1)$$

where we set (cf. (2.5))

$$Y_1 + iY_2 = \frac{x_1 + ix_2}{z} = \frac{r}{z} e^{i\phi}. \quad (4.2)$$

We shall consider the correlator

$$\langle V_{\Delta,S}(a) V_{\Delta,-S}(0) \rangle, \quad V_{\Delta,-S} = (V_{\Delta,S})^* \quad (4.3)$$

in the limit of $\Delta \sim S \sim \sqrt{\lambda} \gg 1$, with $S = \frac{S}{\sqrt{\lambda}}$ being large. We are going to demonstrate that the semiclassical trajectory that saturates the 2-point correlator is equivalent to the conformally transformed (2.2) euclidean continuation (2.1) of the asymptotic large spin limit [26] of the spinning folded string solution in $AdS_3$ [25, 15], i.e.

$$t = \kappa \tau, \quad \phi = \phi_1 = \kappa \tau, \quad \rho = \mu \sigma, \quad \kappa = \mu. \quad (4.4)$$

Here $\theta = \phi_2 = 0$ and $(\tau, \sigma)$ are coordinates of 2-cylinder. The background (4.4) approximates the elliptic function solution [15] in the limit $\kappa = \mu \gg 1$ on the interval $\sigma \in [0, \frac{\pi}{2}]$; to get the formal periodic solution on $0 < \sigma \leq 2\pi$ one needs to combine together 4 stretches $\rho = \mu \sigma$ of the folded string as in Figure 1 (see also [28, 29]). The condition $\kappa = \mu$ follows from the Virasoro constraint and implies the well-known relation between the corresponding energy and spin [15]

$$E = S \frac{\sqrt{\lambda}}{S} \ln \frac{S}{\sqrt{\lambda}} + \ldots, \quad \frac{S}{\sqrt{\lambda}} \gg 1. \quad (4.5)$$
Below we shall formally treat $\kappa$ and $\mu$ as independent, recovering their equality from the requirement of 2d conformal invariance of the semiclassical value of the 2-point correlator.

The formal euclidean continuation (2.1) of the solution (4.4) is complex\(^{15}\)

\[
t_e = \kappa \tau_e, \quad \phi = i\kappa \tau_e, \quad \rho = \mu \sigma.
\] (4.6)

Written in the embedding coordinates (2.14) it becomes [23]

\[
\begin{align*}
Y_5 &= \cosh(\kappa \tau_e) \cosh(\mu \sigma), \quad Y_{0e} = \sinh(\kappa \tau_e) \cosh(\mu \sigma), \quad Y_4 = 0, \\
Y_1 &= \cosh(\kappa \tau_e) \sinh(\mu \sigma), \quad Y_2 = i \sinh(\kappa \tau_e) \sinh(\mu \sigma), \quad Y_3 = 0.
\end{align*}
\] (4.7)

In the euclidean Poincare coordinates (2.10) we then find\(^{16}\)

\[
\begin{align*}
z &= \frac{1}{\cosh(\kappa \tau_e) \cosh(\mu \sigma)}, \\
x_{0e} &= \tanh(\kappa \tau_e), \quad x_\pm \equiv x_1 \pm ix_2 = re^{\pm i\phi} = \frac{\tanh(\mu \sigma)}{\cosh(\kappa \tau_e)} e^{\mp \kappa \tau_e}, \\
z^2 + x_+ x_- + x_{0e}^2 &= 1.
\end{align*}
\] (4.8)

While in the Minkowski Poincare coordinates the string moves towards the center of AdS, rotating and stretching, after the euclidean continuation the world surface described by (4.8) approaches the boundary ($z \to 0$) at $\tau_e \to \pm\infty$ with $x_{0e}(\pm\infty) = \pm 1$. Performing a simple dilatation and translation to ensure that $x_{0e}(-\infty) = 0$ and $x_{0e}(\infty) = a$ as above in (3.2) we get

\[
\begin{align*}
z &= \frac{2}{a} \frac{1}{\cosh(\kappa \tau_e) \cosh(\mu \sigma)}, \\
x_{0e} &= \frac{a}{2} [\tanh(\kappa \tau_e) + 1], \quad x_\pm \equiv x_1 \pm ix_2 = re^{\pm i\phi} = \frac{a}{2} \frac{\tanh(\mu \sigma)}{\cosh(\kappa \tau_e)} e^{\mp \kappa \tau_e}.
\end{align*}
\] (4.10)

---

\(^{15}\)Herein an addition to euclidean continuation we (purely for notational convenience) also change the sign of $\phi$ which corresponds to $S \to -S$ and simply interchanges the roles of the two operators in the correlator.

\(^{16}\)For generic background ($t, \rho, \phi$) in Minkowski $AdS_3$ coordinates one gets [27]: $z = (\cos t \cosh \rho)^{-1}, \quad x_0 = \tan t, \quad x_1 + ix_2 = re^{i\phi}, \quad r = (\cos t)^{-1} \tanh \rho.$
Here the values of the argument $\rho = \mu \sigma$ are as in Figure 1. Explicitly, the end-points of the euclidean world cylinder are mapped to

$$
\begin{align*}
\tau_e \to +\infty & : \quad z \to 0, \quad x_0 \to a, \quad r \to 0, \quad x_+ \to 0, \quad x_- \to a \tanh(\mu \sigma), \\
\tau_e \to -\infty & : \quad z \to 0, \quad x_0 \to 0, \quad r \to 0, \quad x_+ \to a \tanh(\mu \sigma), \quad x_- \to 0.
\end{align*}
$$

Note that if we further analytically continue $x_2 \to ix_2$ to make the solution real in the Poincare patch (which will have again the Minkowski signature) then the resulting surface will be ending not on points but on null lines. It will be, in fact, equivalent to the null cusp [24] surface which was already shown [23] to be related by similar transformations$^{17}$ to the asymptotic form of the folded string solution.

The final step is to transform the solution (4.10) to the complex $\xi$-plane by (2.2), i.e. to express $\tau_e$ and $\sigma$ in (4.10) in terms of $\xi$ using (2.3), with $\tau_e \to \pm \infty$ corresponding to $\xi \to \xi_{1,2}$. As demonstrated explicitly in Appendix the resulting solution is the semiclassical trajectory for the correlator (4.3) provided the parameters $\kappa$ and $\mu$ are related to the quantum numbers $\Delta, S$ carried by the vertex operators as in the original folded string solution

$$
\kappa = \frac{\Delta - S}{\sqrt{\lambda}}, \quad \mu = \frac{1}{\pi} \ln \frac{S}{\sqrt{\lambda}}.
$$

The value of the effective semiclassical action evaluated on this solution leads to (see Appendix)

$$
\langle V_{\Delta,S}(a) \, V_{\Delta,-S}(0) \rangle \sim \frac{1}{a^{2\Delta}} \int d^2 \xi_1 d^2 \xi_2 |\xi_1 - \xi_2|^{\sqrt{\lambda}(\kappa^2 - \mu^2)}.
$$

The marginality condition (cf. (1.9),(1.10)) implies, for large $\kappa, \mu$,

$$
\kappa \approx \mu, \quad \text{i.e.} \quad \Delta \approx S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}}.
$$

We thus find that the 2-point function (4.14) scales with $a$ as expected for a 2-point function of the minimal twist gauge theory operators at strong coupling.

5 Concluding remarks

We have seen how to relate the solution representing a semiclassical string state with large AdS spin with the stationary point surface in the 2-point correlator of the corresponding vertex operators. Similar relation should apply to other semiclassical states and associated vertex operators. That may help determine the structure of the latter.

Since classical closed string solutions have integrability-based description as finite gap solutions, it should be possible to give an equivalent description of the stationary point surfaces “ending” at 2 points on the boundary of the Poincare patch. It would be interesting to make

$^{17}$Note that here $Y_5$ and $Y_0$ are interchanged compared to [23], i.e. to relate the two backgrounds one needs to do conformal transformations combined with analytic continuations.
this description explicit, keeping in mind though that generic stationary point surfaces will be complex.

An obvious next question [22] concerns surfaces associated to 3-point correlators and if integrability may help in constructing them. They should be related to classical solutions describing semiclassical decay of a spinning string into two strings where one needs to consider the world sheet as a cylinder with a singular point where interaction takes place. The resulting surface will be equivalent to a sphere with 3 punctures. Such solutions may be possible to construct following [30, 31]. A related interesting problem is how to construct Wilson loop surfaces ending on 3 finite contours.

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Appendix: Spinning string solution as semiclassical trajectory for 2-point correlator (4.3)

The Euclidean action including the contribution of the vertex operators in (4.3) has the form

$$A_e = \frac{\sqrt{\lambda}}{\pi} \int d^2 \xi \left( \frac{1}{z^2} (\partial z \partial \bar{z} + \partial x_0 \partial \bar{x}_0 + \partial r \partial \bar{r} + i^2 \partial \phi \partial \bar{\phi}) \right)$$

$$- \Delta \int d^2 \xi \left[ \delta^2(\xi - \xi_1) \ln \frac{z}{z^2 + r^2 + (x_0 - a)^2} + \delta^2(\xi - \xi_2) \ln \frac{z}{z^2 + r^2 + x_0^2} \right]$$

$$- \frac{S}{2} \int d^2 \xi \left[ \delta^2(\xi - \xi_1) \ln \partial \frac{e^{i\phi}}{z} + \delta^2(\xi - \xi_1) \ln \partial \frac{e^{i\phi}}{z} \right]$$

$$+ \delta^2(\xi - \xi_2) \ln \partial \frac{e^{-i\phi}}{z} + \delta^2(\xi - \xi_2) \ln \partial \frac{e^{-i\phi}}{z} \right].$$ (A.1)

We have set $x_3 = 0$ as it does not contribute to the semiclassical solution. Our aim is to show that the solution to the equations of motion following from (A.1) is given by (4.10) transformed to the $\xi$-plane using (2.3).

Let us start with the equation for $\phi$. The "source" part of it will contain terms like

$$\frac{e^{i\phi}}{z} \partial \left( \frac{\delta^2(\xi - \xi_1)}{\partial e^{i\phi}/z} \right) = -\delta^2(\xi - \xi_1) + \partial \left( \frac{\delta^2(\xi - \xi_1)}{\partial \ln \frac{e^{i\phi}}{z}} \right).$$ (A.2)

In the limit of large $\mu$, $\frac{e^{i\phi}}{z} \sim e^{\mu(\xi - \xi_2)}$. Then using (2.3) we find

$$\partial \ln \frac{e^{i\phi}}{z} \sim \frac{1}{(\xi - \xi_1)(\xi - \xi_2)},$$ (A.3)

and hence

$$\frac{\delta^2(\xi - \xi_1)}{\partial \ln \frac{e^{i\phi}}{z}} = 0.$$ (A.4)
All delta-function terms can be simplified in the same way. Then the equation for $\phi$ becomes

$$\partial \left( \frac{r^2}{z^2} \partial \phi \right) + \bar{\partial} \left( \frac{r^2}{z^2} \partial \phi \right) = -\frac{i\pi S}{\sqrt{\lambda}} \left[ \delta^2(\xi - \xi_1) - \delta^2(\xi - \xi_2) \right],$$  \hspace{1cm} (A.5)

or, using (4.10),

$$\partial \left[ \sinh^2(\mu \sigma) \partial \phi \right] + \bar{\partial} \left[ \sinh^2(\mu \sigma) \partial \phi \right] = -\frac{i\pi S}{\sqrt{\lambda}} \left[ \delta^2(\xi - \xi_1) - \delta^2(\xi - \xi_2) \right].$$  \hspace{1cm} (A.6)

Exactly the same equation was already discussed in a slightly different context in [21]. For completeness, we will repeat its analysis here. For large $\mu$ we can replace $\sinh^2(\mu \sigma)$ with $e^{2\mu \sigma}$.

Next, we know that away from singularities this equation is satisfied and, hence, our aim is understand how to match the singular contributions. For this we have to recall that the actual solution must be a periodic function of $\sigma$, that is $e^{2\mu} = e^{2\mu \sigma}$ should be understood in the sense of Figure 1. This periodic function may be expanded in Fourier series in $\sigma \in (0, 2\pi)$, i.e.

$$f(\sigma) = \sum_n c_n e^{in\sigma}$$

and then $\sigma$ may be replaced by $\xi$ using (2.3). It is straightforward to see that if $\phi$ has a logarithmic behavior, the singular contribution to the l.h.s. of (A.5) may only come from the constant Fourier mode of

$$\left( e^{2\mu \sigma} \right)_0 = \int_0^{2\pi} \frac{d\sigma}{2\pi} e^{2\mu \sigma} = 4 \int_0^{\pi/2} \frac{d\sigma}{2\pi} e^{2\mu \sigma} = \frac{1}{\pi \mu} (e^{\pi \mu} - 1).$$ \hspace{1cm} (A.7)

If we now substitute (A.7) into (A.6) we find the expected solution for $\phi$

$$\phi = i\kappa \tau = \frac{i}{2} \kappa \left( \ln |\xi - \xi_2|^2 - \ln |\xi - \xi_1|^2 \right)$$ \hspace{1cm} (A.8)

provided $\mu$ is related to $S$ as

$$\mu = \frac{1}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \ldots.$$ \hspace{1cm} (A.9)

Now let us consider the equation for $x_{0e}$

$$\partial \frac{\partial x_{0e}}{z^2} + \bar{\partial} \frac{\partial x_{0e}}{z^2} = \frac{2\pi \Delta}{\sqrt{\lambda}} \left[ \frac{x_{0e} - a}{z^2 + r^2 + (x_{0e} - a)^2} \delta^2(\xi - \xi_1) + \frac{x_{0e}}{z^2 + r^2 + x_{0e}^2} \delta^2(\xi - \xi_2) \right]$$ \hspace{1cm} (A.10)

Using the form of the solution in $(\tau, \sigma)$ coordinates (4.10) we can write this equation as

$$\kappa \partial \left[ \cosh^2(\mu \sigma) \partial \tau \right] + \kappa \bar{\partial} \left[ \cosh^2(\mu \sigma) \partial \tau \right] = \frac{\pi \Delta}{\sqrt{\lambda}} \left[ \delta^2(\xi - \xi_2) - \delta^2(\xi - \xi_1) \right].$$ \hspace{1cm} (A.11)

Comparing this with (A.6) with $\phi$ given in (A.8) we conclude that eq. (A.11) is satisfied if

$$\kappa = \frac{\Delta - S}{\sqrt{\lambda}}.$$ \hspace{1cm} (A.12)
Next, let us turn to the equation for \( r \). We can simplify the delta-function terms in it by the same transformation as in eqs. (A.2)-(A.4) to obtain

\[
\begin{align*}
& r \frac{\partial}{\partial z^2} \frac{\partial}{\partial z^2} - 2 \frac{r^2}{z^2} \partial \phi \partial \phi = - \frac{\pi S}{\sqrt{\lambda}} \left[ \delta^2(\xi - \xi_1) + \delta^2(\xi - \xi_2) \right] \\
& + \frac{2 \pi \Delta}{\sqrt{\lambda}} \left[ \frac{r^2}{z^2 + r^2} \delta^2(\xi - \xi_1) + \frac{r^2}{z^2 + r^2 + (x_{0e} - a)^2} \delta^2(\xi - \xi_2) \right].
\end{align*}
\]  

(A.13)

Recalling that \( \xi = \xi_1 \) corresponds to \( \tau_e \to \infty \) and \( \xi = \xi_2 \) to \( \tau_e \to -\infty \) we can simplify the last two delta-functions as follows

\[
\begin{align*}
& \frac{r^2}{z^2 + r^2 + (x_{0e} - a)^2} \delta^2(\xi - \xi_2) = \tanh^2(\mu \sigma) \delta^2(\xi - \xi_2) \approx \delta^2(\xi - \xi_2), \\
& \frac{r^2}{z^2 + r^2 + (x_{0e} - a)^2} \delta^2(\xi - \xi_1) = \tanh^2(\mu \sigma) \delta^2(\xi - \xi_1) \approx \delta^2(\xi - \xi_1),
\end{align*}
\]

(A.14) (A.15)

where we have used that \( \tanh^2(\mu \sigma) \approx 1 \) in the limit of large \( \mu \). In solving eq. (A.13) we need to pay attention only to the singular terms. Then we obtain (again using that \( \mu \) is large)

\[
\kappa \tanh(\kappa \tau_e) \left[ \partial (e^{2\mu \sigma} \partial \tau_e) + \overline{\partial} (e^{2\mu \sigma} \partial \tau_e) \right] = -\frac{\pi (2\Delta - S)}{\sqrt{\lambda}} \left[ \delta^2(\xi - \xi_1) + \delta^2(\xi - \xi_2) \right].
\]  

(A.16)

Repeating the same analysis as for the equation for \( \phi \), using that

\[
\tanh(\kappa \tau_e)|_{\xi \to \xi_1} = 1, \quad \tanh(\kappa \tau_e)|_{\xi \to \xi_2} = -1,
\]

(A.17)

and taking into account that to leading order \( \Delta \sim S \) we find that eq. (A.16) is satisfied provided one imposes again the relation (A.9).

The discussion of the equation for \( z \) is simplified if we consider the sum of this equation and the equation for \( r \) (A.13). Using similar manipulations as above it can be written in the form

\[
\begin{align*}
& \frac{1}{2} \partial \left( \frac{z^2 + r^2}{z^2} \right) + \frac{1}{2} \partial \left( \frac{z^2 + r^2}{z^2} \right) + \frac{2}{z^2} \partial x_{0e} \partial x_{0e} = \frac{\pi \Delta}{\sqrt{\lambda}} \left[ \delta^2(\xi - \xi_1) + \delta^2(\xi - \xi_2) \right].
\end{align*}
\]

(A.18)

Using the solution (4.10) we obtain

\[
\kappa \tanh(\kappa \tau_e) \left[ \partial (\cosh^2(\mu \sigma) \partial \tau_e) + \overline{\partial} (\cosh^2(\mu \sigma) \partial \tau_e) \right] = -\frac{\pi \Delta}{\sqrt{\lambda}} \left[ \delta^2(\xi - \xi_1) + \delta^2(\xi - \xi_2) \right],
\]

(A.19)

or, in view of (A.17),

\[
\kappa \left[ \partial (\cosh^2(\mu \sigma) \partial \tau_e) + \overline{\partial} (\cosh^2(\mu \sigma) \partial \tau_e) \right] = \frac{\pi \Delta}{\sqrt{\lambda}} \left[ \delta^2(\xi - \xi_2) - \delta^2(\xi - \xi_1) \right].
\]

(A.20)

This is the same as equation (A.11) for \( x_{0e} \) which was proven to be consistent provided \( \kappa \) is related to \( \Delta \) and \( S \) as in (A.12).

This finishes our proof that the solution in (4.10) transformed by the conformal map (2.2) is indeed the semiclassical trajectory with singularities prescribed by the operators in (4.1)-(4.3).
Let us now evaluate the action (A.1) on this solution. The string action (the first line in (A.1)) is easier to compute if we go back to the \((\tau_e, \sigma)\) cylinder coordinates:

\[
A_{str} = \frac{\sqrt{\lambda}}{4\pi} (\kappa^2 + \mu^2) \int d\tau_e d\sigma = \frac{\sqrt{\lambda}}{2} (\kappa^2 + \mu^2) (\tau_{e,\infty} - \tau_{e,-\infty}),
\]

(A.21)

where

\[
\tau_{e,\pm\infty} = \frac{1}{2} \left( \ln |\xi - \xi_2|^2 - \ln |\xi - \xi_1|^2 \right) |_{\xi \to \xi_{1,2}}
\]

(A.22)

Ignoring the obvious one-point function divergence \((\sim \ln(0))\) we obtain

\[
A_{str} = \sqrt{\lambda} (\kappa^2 + \mu^2) \ln |\xi_1 - \xi_2|.
\]

(A.23)

It is straightforward to evaluate the remaining delta-function terms in (A.1) using the relation between \(\kappa, \Delta\) and \(S\) (A.12); one finds

\[
2\Delta \ln a - 2\sqrt{\lambda} \kappa^2 \ln |\xi_1 - \xi_2|,
\]

(A.24)

where we again neglected the \((\sim \ln(0))\) terms. Combining (A.23) and (A.24) we end up with

\[
A_e \approx 2\Delta \ln a - \sqrt{\lambda} (\kappa^2 - \mu^2) \ln |\xi_1 - \xi_2|,
\]

(A.25)

so that \(e^{-A_e}\) leads to the expression in (4.14).

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9808150].

[4] A. M. Polyakov, “Gauge fields and space-time,” Int. J. Mod. Phys. A 17S1, 119 (2002) [hep-th/0110196]. talk at Strings 2002, Cambridge, July 15-20, 2002, www.damtp.cam.ac.uk/strings02/avt/polyakov/

[5] A. A. Tseytlin, “On semiclassical approximation and spinning string vertex operators in \(AdS_5 \times S^5\),” Nucl. Phys. B 664, 247 (2003) [arXiv:hep-th/0304139].

[6] R. R. Metsaev and A. A. Tseytlin, “Type IIB superstring action in \(AdS_5 \times S^5\) background,” Nucl. Phys. B 533, 109 (1998) [hep-th/9805028].

[7] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, “Three-point functions of chiral operators in \(D = 4, N = 4\) SYM at large N,” Adv. Theor. Math. Phys. 2, 697 (1998) [arXiv:hep-th/9806074].
[8] E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS/CFT correspondence,” arXiv:hep-th/0201253.

[9] A. Giveon, D. Kutasov and N. Seiberg, “Comments on string theory on AdS(3),” Adv. Theor. Math. Phys. 2, 733 (1998) [arXiv:hep-th/9806194].

[10] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, “String theory on AdS(3),” JHEP 9812, 026 (1998) [arXiv:hep-th/9812046].

[11] D. Kutasov and N. Seiberg, “More comments on string theory on AdS(3),” JHEP 9904, 008 (1999) [arXiv:hep-th/9903219].

[12] J. M. Maldacena and H. Ooguri, “Strings in AdS(3) and the SL(2,R) WZW model. III: Correlation functions,” Phys. Rev. D 65, 106006 (2002) [arXiv:hep-th/0111180].

[13] M. R. Gaberdiel and I. Kirsch, “Worldsheet correlators in AdS(3)/CFT(2),” JHEP 0704, 050 (2007) [arXiv:hep-th/0703001]. A. Dabholkar and A. Pakman, “Exact chiral ring of AdS(3)/CFT(2),” Adv. Theor. Math. Phys. 13, 409 (2009) [arXiv:hep-th/0703022].

[14] D. J. Gross and P. F. Mende, “String Theory Beyond the Planck Scale,” Nucl. Phys. B 303, 407 (1988). D. J. Gross and J. L. Manes, “The high energy behaviour of open string scattering,” Nucl. Phys. B 326, 73 (1989).

[15] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B 636, 99 (2002) [hep-th/0204051].

[16] K. Zarembo, “Open string fluctuations in AdS5 × S5 and operators with large R charge,” Phys. Rev. D 66, 105021 (2002) [arXiv:hep-th/0209095].

[17] V. Pestun and K. Zarembo, “Comparing strings in AdS5 × S5 to planar diagrams: An example,” Phys. Rev. D 67, 086007 (2003) [arXiv:hep-th/0212296].

[18] S. Dobashi, H. Shimada and T. Yoneya, “Holographic reformulation of string theory on AdS(5) × S5 background in the PP-wave limit,” Nucl. Phys. B 665, 94 (2003) [arXiv:hep-th/0209251]. “Resolving the holography in the plane-wave limit of AdS/CFT correspondence,” Nucl. Phys. B 711, 3 (2005) [arXiv:hep-th/0406225]. A. Miwa and T. Yoneya, “Holography of Wilson-loop expectation values with local operator insertions,” JHEP 0612, 060 (2006) [arXiv:hep-th/0609007].

[19] A. Tsuji, “Holography of Wilson loop correlator and spinning strings,” Prog. Theor. Phys. 117, 557 (2007) [arXiv:hep-th/0606303].

[20] M. Sakaguchi and K. Yoshida, “A Semiclassical String Description of Wilson Loop with Local Operators,” Nucl. Phys. B 798, 72 (2008) [arXiv:0709.4187 [hep-th]].

[21] E. I. Buchbinder, “Energy-Spin Trajectories in AdS5 × S5 from Semiclassical Vertex Operators,” JHEP 1004, 107 (2010) [arXiv:1002.1716].

[22] R. A. Janik, P. Surowka and A. Wereszczynski, “On correlation functions of operators dual to classical spinning string states,” arXiv:1002.4613.

[23] M. Kruczenski, R. Roiban, A. Tirziu and A. A. Tseytlin, “Strong-coupling expansion of cusp anomaly and gluon amplitudes from quantum open strings in AdS5 × S5,” Nucl. Phys. B 791, 93 (2008) [arXiv:0707.4254].
[24] M. Kruczenski, “A note on twist two operators in N = 4 SYM and Wilson loops in Minkowski signature,” JHEP 0212, 024 (2002) [arXiv:hep-th/0210115].

[25] H. J. de Vega and I. L. Egusquiza, “Planetoid String Solutions in 3 + 1 Axisymmetric Spacetimes,” Phys. Rev. D 54, 7513 (1996) [arXiv:hep-th/9607056].

[26] S. Frolov, A. Tirziu and A. A. Tseytlin, “Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT,” Nucl. Phys. B 766, 232 (2007) [arXiv:hep-th/0611269].

[27] A. A. Tseytlin, “Semiclassical quantization of superstrings: AdS$_5$ × S$^5$ and beyond,” Int. J. Mod. Phys. A 18, 981 (2003) [arXiv:hep-th/0209116].

[28] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in AdS$_5$ × S$^5$, ” JHEP 0206, 007 (2002) [arXiv:hep-th/0204226].

[29] M. Beccaria, V. Forini, A. Tirziu and A. A. Tseytlin, “Structure of large spin expansion of anomalous dimensions at strong coupling,” Nucl. Phys. B 812, 144 (2009) [arXiv:0809.5234] M. Beccaria, G. V. Dunne, V. Forini, M. Pawellek and A. A. Tseytlin, “Exact computation of one-loop correction to energy of spinning folded string in AdS$_5$ × S$^5$ ” J. Phys. A 43, 165402 (2010) [arXiv:1001.4018].

[30] R. Iengo and J. G. Russo, “Semiclassical decay of strings with maximum angular momentum,” JHEP 0303, 030 (2003) [arXiv:hep-th/0301109].

[31] K. Peeters, J. Plefka and M. Zamaklar, “Splitting spinning strings in AdS/CFT,” JHEP 0411, 054 (2004) [arXiv:hep-th/0410275].