REMARKS ON SPECTRAL MULTIPLIER THEOREMS ON HARDY SPACES ASSOCIATED WITH SEMIGROUPS OF OPERATORS

JACEK DZIUBAŃSKI AND MARCIN PREISNER

Abstract. Let \( L \) be a non-negative, self-adjoint operator on \( L^2(\Omega) \), where \((\Omega, d, \mu)\) is a space of homogeneous type. Assume that the semigroup \( \{T_t\}_{t>0} \) generated by \(-L\) satisfies Gaussian bounds, or more generally Davies-Gaffney estimates. We say that \( f \) belongs to the Hardy space \( H^1_L \) if the square function

\[
S_h f(x) = \left( \int_0^\infty \left| \int_{\Gamma(x)} t^2 e^{-t^2 L} f(y) \frac{d\mu(y)}{\mu(B_d(x, t))} \right|^2 dt \right)^{1/2}
\]

belongs to \( L^1(\Omega, d\mu) \), where \( \Gamma(x) = \{(y, t) \in \Omega \times (0, \infty) : d(x, y) < t\} \). We prove spectral multiplier theorems for \( L \) on \( H^1_L \).

1. Introduction.

A classical Hörmander multiplier theorem [37] asserts that if \( m \) is a bounded function on \( \mathbb{R}^d \) such that for some \( \beta > d/2 \) and any radial function \( \eta \in C_0^\infty \), \( \sup \eta \subset \{\xi \in \mathbb{R}^d : 2^{-1} \leq |\xi| \leq 2\} \), one has

\[
\sup_{t>0} \| \eta(\cdot) m(t \cdot) \|_{W^{2,\beta}((\mathbb{R}^d^d)}} \leq C_{\eta},
\]

where \( \| \cdot \|_{W^{2,\beta}(\mathbb{R}^d)} \) is the standard Sobolev norm on \( \mathbb{R}^d \), then the multiplier operator \( f \mapsto \mathcal{F}^{-1}(m\mathcal{F}f) \), initially defined on \( L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \), is bounded on \( L^p(\mathbb{R}^d) \) for \( 1 < p < \infty \), and is of weak-type \((1,1)\). Here \( \mathcal{F} \) denotes the Fourier transform.

Let \((\Omega, d(x, y))\) be a metric space equipped with a positive measure \( \mu \). We assume that \((\Omega, d, \mu)\) is a space of homogeneous type in the sense of Coifman-Weiss [9], that is, there exists a constant \( C > 0 \) such that

\[
(1.1) \quad \mu(B_d(x, 2t)) \leq C \mu(B_d(x, t)) \quad \text{for every } x \in \Omega, \ t > 0,
\]

where \( B_d(x, t) = \{y \in \Omega : d(x, y) < t\} \). The condition \((1.1)\) implies that there exist constants \( C > 0 \) and \( q > 0 \) such that

\[
(1.2) \quad \mu(B_d(x, st)) \leq C_0 s^q \mu(B_d(x, t)) \quad \text{for every } x \in \Omega, \ t > 0, \ s > 1.
\]

Of course we wish to get \( q \) as small as possible even at the expense of large \( C_0 \).

Let \( \{T_t\}_{t>0} \) be a semigroup of linear operators on \( L^2(\Omega, d\mu) \) generated by \(-L\), where \( L \) is a non-negative, self-adjoint operator which is injective on its domain.

2000 Mathematics Subject Classification. 42B30, (primary), 42B35, 42B15 (secondary).

Supported by the European Commission Marie Curie Host Fellowship for the Transfer of Knowledge "Harmonic Analysis, Nonlinear Analysis and Probability" MTKD-CT-2004-013389 and by Polish Government funds for science - grant N N201 397137, MNiSW.
Assume the operators $T_t$ have the following form

$$T_t f(x) = \int_\Omega T_t(x,y)f(y)d\mu(y),$$

where the kernels $T_t(x,y)$ satisfy Gaussian bounds, that is, there exist constants $C_0$, $c_0 > 0$ such that for every $x, y \in \Omega, t > 0$, we have

$$|T_t(x,y)| \leq \frac{C_0}{V(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{c_0 t}\right),$$

where here and subsequently $V(x,t) = \mu(B_d(x,t))$. The estimate (1.4) implies that for every $k \in \mathbb{N}$ there exist constants $C_k, c_k > 0$ such that

$$\left|\frac{\partial^k}{\partial t^k} T_t(x,y)\right| \leq \frac{C_k}{t^kV(x,\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{c_k t}\right) \quad \text{for } x, y \in \Omega, t > 0.$$  

The constants $C_k, c_k$ in (1.5) depend only on $k$ and the constants $C, C_0, q, c$ in (1.2) and (1.4).

For a suitable function $f$ (e.g., from $L^2(\Omega)$) we consider the square function $S_h f$ associated with $L$ defined by

$$S_h f(x) = \left(\int_\Omega \int_{\Gamma(x)} |r^2LT_2 f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t}\right)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \Omega \times (0,\infty) : d(x,y) \leq t\}$.

Following [2], [3], [36] (see also [4], [23]) we define the Hardy space $H^1_L = H^1_{L,S_h}(\Omega)$ as the completion of $\{f \in L^2(\Omega) : \|S_h f\|_{L^1(\Omega)} < \infty\}$ in the norm $\|f\|_{H^1_L} = \|S_h f\|_{L^1(\Omega)}$.

It was proved in Hofmann, Lu, Mitrea, Mitrea, Yan [36] that the space $H^1_L$, where $-L$ generates a semigroup having Gaussian bounds, admits the following atomic decomposition.

Let $M \geq 1$, $M \in \mathbb{N}$. A function $a$ is a $(1, 2, M)$-atom for $H^1_L$ if there exist a ball $B = B_d(y_0, r) = \{y \in \Omega : d(y,y_0) < r\}$ and a function $b \in D(L^M_f)$ such that

$$a = L^M b,$$

$$\text{supp } L^k b \subset B, \quad k = 0, 1, \ldots, M;$$

$$\|r^2L^k b\|_{L^2(\Omega)} \leq r^{2M} \mu(B)^{-1/2}, \quad k = 0, 1, \ldots, M.$$ 

The atomic norm $\|f\|_{H^1_L-\text{atom}}$ is defined by

$$\|f\|_{H^1_L-\text{atom}} = \inf \sum_{j} |\lambda_j|,$$

where the infimum is taken over all representation $f = \sum_j \lambda_j a_j$, where $a_j$ are $(1, 2, M)$-atoms for $H^1_L$, $\lambda_j \in \mathbb{C}$. Theorem 7.1 of [36] asserts that there exists a constant $C > 0$ such that

$$C^{-1}\|f\|_{H^1_L} \leq \|f\|_{H^1_L-\text{atom}} \leq C\|f\|_{H^1_L}.$$

Let

$$Lf = \int_0^\infty \lambda dE_L(\lambda)f,$$
be the spectral resolution of $L$.

Our first goal in this paper is to present a simple proof of the following spectral multiplier theorem.

**Theorem 1.12.** Let $m$ be a bounded function defined on $(0, \infty)$ such that for some real number $\alpha > q/2$ and any nonzero function $\eta \in C_c(2^{-1}, 2)$ there exists a constant $C_\eta$ such that

$$\sup_{t>0} \|\eta(\cdot)m(t \cdot)\|_{W^{\infty, \alpha}(\mathbb{R})} \leq C_\eta,$$

where $\|F\|_{W^{p, \alpha}(\mathbb{R})} = \|(I - d^2/dx^2)^{\alpha/2}F\|_{L^p(\mathbb{R})}$. Then the spectral multiplier operator

$$m(L) = \int_0^\infty m(\lambda)dE_L(\lambda),$$

maps $(1, 2, 1)$-atoms for $H^1_L$ into $H^1_L$. Moreover, there exists a constant $C > 0$ such that

$$\|m(L)a\|_{H^1_L} \leq C \quad \text{for every} \ (1, 2, 1)\text{-atom}.$$

**Remark 1.16.** If we additionally assume that for every $y \in \Omega$ there exist constants $\kappa > 0$ and $c > 0$ such that $\mu(B_d(y, s)) \geq cs^\kappa$ for $s > 1$, then the operator $m(L)$ extends uniquely to a bounded operator on $H^1_L$ (see Section 5 for details).

**Remark 1.17.** It turns out that if we replace (1.13) by the stronger condition

$$\sup_{t>0} \|\eta(\cdot)m(t \cdot)\|_{W^{2, \alpha}(\mathbb{R})} \leq C_\eta,$$

with some $\alpha > (q+1)/2$, then the multiplier theorem holds for Hardy spaces associated with more general semigroups, that is, semigroups satisfying Davies-Gaffney estimates. This will be discussed in Section 4. We present two seemingly similar theorems with two different proofs. The first proof, thanks to [34], could be also adapted to cover the case of a broader class of semigroups with integral kernels of a very mild decay. The other one does not even require the existence of integral kernels of the semigroups under consideration, however depends very much on the finite speed propagation of the wave equation associated with generators which is in fact equivalent to Davies-Gaffney estimates, see, e.g., [45], [10].

Spectral multiplier theorems on various spaces attracted attention of many authors (see, for example, [1], [6], [7], [11], [12], [15], [18], [19], [30], [32], [33], [35], [40], [43], [44], [46], and references there). E. M. Stein [46] proved that if $-A$ is the infinitesimal generator of a symmetric diffusion semigroup and $m$ is of Laplace transform type, then $m(A)$ is bounded on $L^p$, $1 < p < \infty$. E. Stein and A. Hulanicki (see [30]) noticed that if $-A$ is a sublaplacian on a stratified Lie group $G$, then the convolution kernel of the operator $m(A)$ satisfies Calderón-Zygmund type estimates. This fact together with atomic decompositions of the Hardy spaces $H^p(G)$ leads to a spectral multiplier theorem on these spaces (see [30, Theorem 6.25]). The finite speed propagation of the wave equation was used by Sikora [44] and [45] for proving $L^p$ bounds for certain operators. Actually, the technique of the proof of Lemma 4.8 is taken from [44].

The development of the theory of real Hardy spaces in $\mathbb{R}^d$ had its origin in works Stein-Weiss [47] and Fefferman-Stein [29]. An important contribution to the theory were atomic decompositions proved by Coifman [8] for $d = 1$ and Latter [38] for $d > 1$. 

The extension of $H^p$ on the spaces of homogenous type is due to Coifman-Weiss [9] (see also [43]). Hardy spaces associated with various semigroups of linear operators were considered by many authors. For their properties and equivalent characterizations we refer the reader to [2]-[5], [13]-[28], [31], [36], [41], [42].

2. Functional calculi

For $\beta \geq 0$ let $\omega_\beta(x,y) = (1 + d(x,y))^{\beta}$. The function is submultiplicative, that is, $\omega_\beta(x,y) \leq \omega_\beta(x,z)\omega_\beta(z,y)$.

For an integral kernel $k(x,y)$ and $\beta > 0$ we define

$$\|k(x,y)\|_{\omega(\beta)} = \sup_{x \in \Omega} \int |k(x,y)| (1 + d(x,y))^{\beta} d\mu(y) + \sup_{y \in \Omega} \int |k(x,y)| (1 + d(x,y))^{\beta} d\mu(x).$$

The following theorem is a consequence of (1.4) and results of W. Hebisch [34, Theorem 2.10].

**Theorem 2.1.** Let $(\Omega, d, \mu)$ and $\{T_t\}_{t>0}$ satisfy (1.2) and (1.4) respectively. For $\alpha, \beta > 0$ with $\alpha > \beta + q/2$ there exists a constant $C' > 0$ such that for every function $\eta \in W^{\infty, \alpha}(\mathbb{R})$ with $\text{supp} \eta \subset (1/4, 4)$ the multiplier operator

$$\eta(L)f = \int_0^\infty \eta(\lambda) dE_L(\lambda)f$$

is of the form

$$\eta(L)f(x) = \int_\Omega \eta(L)(x,y)f(y) d\mu(y)$$

with

$$\|\eta(L)(x,y)\|_{\omega(\beta)} \leq C'\|\eta\|_{W^{\infty, \alpha}(\mathbb{R})}.$$  

The constant $C'$ in (2.2) depends only on $\alpha, \beta$ and the constants $C, q$ from (1.2) and constants $C_0, c_0$ from (1.4).

In this paper we shall use the following scaling argument. For $\tau > 0$, let $d^{(\tau)}(x,y) = \tau^{-1/2}d(x,y)$. Then the space $(\Omega, d^{(\tau)}(x,y), \mu)$ is the space of homogeneous type such that

$$V_\tau(x, st) = \mu(B_{d^{(\tau)}}(x, st)) \leq Cs^\tau \mu(B_{d^{(\tau)}}(x, t)), \quad s > 1,$$

with the same constants $C, q$ as in (1.2). Similarly, let $L^{(\tau)} = \tau L$ and $\{T^{(\tau)}_t\}_{t>0}$ be the semigroup generated by $-L^{(\tau)}$. Clearly, $T^{(\tau)}_t(x,y) = T_{\tau t}(x,y)$ are the integral kernels of $T^{(\tau)}_t$. Hence, for $k = 0, 1, 2, ...$, we have

$$\left| \frac{\partial^k}{\partial t^k} T^{(\tau)}_t(x,y) \right| \leq \frac{C_k}{k! V_\tau(x, \sqrt{t})} \exp \left( -\frac{d^{(\tau)}(x,y)^2}{c_k t} \right) \quad \text{for } x, y \in \Omega, \ t > 0,$$

with the same constants $C_k, c_k$ as in (1.4) and (1.5) independent of $\tau$. 

Therefore, from Theorem 2.1 we conclude that
\begin{equation}
\int_{\Omega} |\eta(\tau L)(x,y)| \left( 1 + \frac{d(x,y)}{\sqrt{\tau}} \right)^\beta \, d\mu(y) + \int_{\Omega} |\eta(\tau L)(x,y)| \left( 1 + \frac{d(x,y)}{\sqrt{\tau}} \right)^\beta \, d\mu(x) \leq C \|\eta\|_{W^{\infty,\alpha}(\mathbb{R})},
\end{equation}
provided \( \text{supp}\, \eta \subset (4^{-1}, 4) \), \( \alpha > \beta + q/2 \).

**Proposition 2.6.** Assume that \( m \) satisfies the assumptions of Theorem 1.12. For \( N = 1, 2 \), we set
\[ \Phi_t^{(N)}(\lambda) = (t^2 \lambda)^N e^{-t^2 \lambda} m(\lambda). \]
Then there exist \( \beta > 0 \) and \( C'' > 0 \) such that
\begin{equation}
\int_{\Omega} |\Phi_t^{(N)}(L)(x,y)| \left( 1 + \frac{d(x,y)}{t} \right)^\beta \, d\mu(y) \leq C'',
\end{equation}
\begin{equation}
\int_{\Omega} |\Phi_t^{(N)}(L)(x,y)| \left( 1 + \frac{d(x,y)}{t} \right)^\beta \, d\mu(x) \leq C''.
\end{equation}

**Proof.** It suffices to prove (2.7) for \( t = 1 \) and then use the scaling argument. Fix a \( C_0^\infty(\frac{1}{2}, 2) \) function \( \psi \) such that
\begin{equation}
\sum_{j \in \mathbb{Z}} \psi(2^{-j} \lambda) = 1 \quad \text{for } \lambda > 0.
\end{equation}

Denote \( n_j(\lambda) = \psi(2^{-j} \lambda) \lambda^N e^{-2^j \lambda} m(\lambda), \tilde{n}_j(\lambda) = n_j(2^j \lambda) = \psi(\lambda)(2^j \lambda)^N e^{-2^j \lambda} m(2^j \lambda) \). Clearly, \( \text{supp}\, \tilde{n}_j \subset (2^{-1}, 2) \) and
\begin{equation}
\|\tilde{n}_j\|_{W^{\infty,\alpha}(\mathbb{R})} \leq \begin{cases} C 2^{-j} & \text{for } j \geq 0; \\ C 2^{j N} & \text{for } j < 0. \end{cases}
\end{equation}

Let \( 0 < \beta \leq 1/2 \) be such that \( \alpha > \beta + q/2 \). Applying (2.5) combined with (2.10) we obtain
\begin{equation}
\int_{\Omega} |n_j(L)(x,y)| \left( 1 + 2^{j/2} d(x,y) \right)^\beta \, d\mu(x) \leq \begin{cases} C 2^{-j} & \text{for } j \geq 0; \\ C 2^{j N} & \text{for } j < 0, \end{cases}
\end{equation}
with the same bounds when integrating with respect to \( d\mu(y) \). Obviously,
\begin{equation}
\int_{\Omega} |\Phi_t^{(N)}(L)(x,y)| \, d\mu(x) \leq \sum_{j \in \mathbb{Z}} \int_{\Omega} |n_j(L)(x,y)| \, d\mu(x) \leq C'.
\end{equation}

Moreover, from (2.11) we also deduce that
\begin{equation}
\int_{\Omega} |\Phi_t^{(N)}(L)(x,y)| d(x,y)^\beta \, d\mu(x) \leq \sum_{j \in \mathbb{Z}} \int_{\Omega} |n_j(L)(x,y)| d(x,y)^\beta \, d\mu(x)
\end{equation}
\begin{equation}
\leq \sum_{j \geq 0} C 2^{-j - \beta/2} + \sum_{j < 0} C' 2^{j N - j \beta/2} \leq C',
\end{equation}
which implies (2.7) for \( t = 1 \). To prove (2.8) we proceed in the same way. \( \square \)
Lemma 2.14. For $N = 1$ or $N = 2$, let

\[(2.15)\]

$$\Theta_j^{(N)}(x, y) = \sup_{2^j \leq t < 2^{j+1}} \sup_{d(x, x') < t} |\Phi_t^{(N)}(x', y)|.$$ 

Then there exist constants $C' > 0$ and $\beta > 0$ such that

\[(2.16)\]

$$\int_{\Omega} \Theta_j^{(N)}(x, y) \left(1 + \frac{d(x, y)}{2^j}\right) \beta \ d\mu(x) \leq C'.$$

Proof. Fix $2^j \leq t < 2^{j+1}$ and let $d(x, x') < t$. Since

$$\Phi_t^{(N)}(\lambda) = (2^{1-j}t)^{2N} \min\{2^j, \beta \} \Phi_{2^{j-1}}^{(N)}(\lambda)$$

and $V(x', (t^2 - 2^{2(j-1)})^{1/2}) \sim V(x, 2j)$ for $d(x, x') < t$, we have

\[(2.17)\]

$$|\Phi_t^{(N)}(x', y)| = (2^{1-j}t)^{2N} \left|\int_{\Omega} T_{t-2^{j-1}}(x', z) \Phi_{2^{j-1}}^{(N)}(z, y) \ d\mu(z)\right|$$

$$\leq C' \int_{\Omega} \exp(-d(x, z)^2/c_0(t^2 - 2^{2(j-1)})^{1/2}) |\Phi_{2^{j-1}}^{(N)}(z, y)| \ d\mu(z)$$

$$\leq C \int_{\Omega} \frac{\exp(-d(x, z)^2/c_0t^{2j})}{V(x, 2j)} |\Phi_{2^{j-1}}^{(N)}(z, y)| \ d\mu(z).$$

Using (2.17) and Proposition 2.6 we obtain

$$\int_{\Omega} \Theta_j^{(N)}(x, y) \left(1 + \frac{d(x, y)}{2^j}\right) \beta \ d\mu(x)$$

$$\leq C \int_{\Omega} \int_{\Omega} \frac{\exp(-d(x, z)^2/c_0t^{2j})}{V(x, 2j)} |\Phi_{2^{j-1}}^{(N)}(z, y)| \left(1 + \frac{d(x, z)}{2^j}\right) \beta \left(1 + \frac{d(z, y)}{2^j}\right) \beta \ d\mu(z) \ d\mu(x)$$

$$\leq C'.$$

\[\square\]

3. PROOF OF THEOREM 1.12

It suffices to establish that there exists a constant $C$ such that for every $(1, 2, 1)$-atom $a$ for $H^1_L$ we have

\[(3.1)\]

$$\|S_h(m(L)a)\|_{L^1(\Omega)} \leq C.$$

Our proof of (3.1) borrows ideas from [17]. Let $a$ be a $(1, 2, 1)$-atom for $H^1_L$ and let $b$ and $B = B_d(y_0, r)$ be as in (1.7)–(1.9). Since $S_h$ is bounded on $L^2(\Omega)$, we have

\[(3.2)\]

$$\|S_h m(L)a\|_{L^1(B_d(y_0, 2r), \mu)} \leq C' \|m(L)a\|_{L^2(\Omega)} \mu(B)^{1/2} \leq C \|a\|_{L^2(\Omega)} \mu(B)^{1/2} \leq C.$$
It suffices to estimate $S_h m(L)a$ on $(2B)^c$, where $2B = B_d(y_0, 2r)$. Clearly, $\Phi_t^{(1)}(L)a = t^{-2}\Phi_t^{(2)}(L)b$. Set $j_0 = \log_2 r$. Then

\[
(S_h m(L)a(x))^2 = \int_{\Gamma(x)} |t^2(LT_{t^2}m(L)a)(x')|^2 \frac{d\mu(x')}{V(x', t)} \frac{dt}{t}
\]

\[
= \sum_{j \leq j_0} \int_{2^j}^{2^{j+1}} \frac{d\mu(x')}{V(x', t)} \frac{dt}{t}
\]

\[
= \sum_{j \leq j_0} \int_{2^j}^{2^{j+1}} \frac{d\mu(x')}{V(x', t)} \frac{dt}{t}
\]

\[
+ \sum_{j > j_0} \int_{2^j}^{2^{j+1}} \frac{d\mu(x')}{V(x', t)} \frac{dt}{t}
\]

Using (2.15) we have

\[
(3.3)
\]

\[
(S_h m(L)a(x))^2 \leq C \sum_{j \leq j_0} \int_{2^j}^{2^{j+1}} \frac{d\mu(x')}{V(x', t)} \frac{dt}{t}
\]

\[
+ C \sum_{j > j_0} \int_{2^j}^{2^{j+1}} \frac{d\mu(x')}{V(x', t)} \frac{dt}{t}
\]

\[
\leq C \sum_{j \leq j_0} \left( \int_{\Omega} \Theta_j^{(1)}(x, y) |a(y)| d\mu(y) \right)^2
\]

\[
+ C \sum_{j > j_0} \left( \int_{\Omega} 2^{-2j} \Theta_j^{(2)}(x, y) |b(y)| d\mu(y) \right)^2,
\]

because

\[
\int_{2^j}^{2^{j+1}} \frac{d\mu(x')}{V(x', t)} \frac{dt}{t} \leq C.
\]

From (3.3) we trivially get

\[
(S_h m(L)a)(x)
\]

\[
\leq C \left( \sum_{j \leq j_0} \int_{\Omega} \Theta_j^{(1)}(x, y) |a(y)| d\mu(y) + \sum_{j > j_0} \int_{\Omega} 2^{-2j} \Theta_j^{(2)}(x, y) |b(y)| d\mu(y) \right).
\]

Applying Lemma 2.14 we obtain
Theorem 4.2. Let $M > q/4$, $M \in \mathbb{N}$. Assume $m$ be a bounded function defined on $(0, \infty)$ such that for some real number $\alpha > (q + 1)/2$ and any nonzero function \( \eta \in C_c(\mathbb{R}) \) the condition (1.18) holds. Then there exists a constant $C > 0$ such that

\[
(4.3) \quad \|m(\mathcal{L})a\|_{H^1_\mathcal{L}} \leq C \quad \text{for every } (1, 2, 2M)\text{-atom } a \text{ for the space } H^1_\mathcal{L}.
\]
Fix $\varepsilon > 0$ and $M > q/4, M \in \mathbb{N}$. We say that a function $\tilde{a}$ is a $(1, 2, M, \varepsilon)$-molecule associated to $\mathcal{L}$ if there exist a function $\tilde{b} \in \mathcal{D}(\mathcal{L}^M)$ and a ball $B = B_d(y_0, r)$ such that

\[(4.4) \quad \tilde{a} = \mathcal{L}^M \tilde{b},\]

\[(4.5) \quad \| (r^2 \mathcal{L})^k \tilde{b} \|_{L^2(U_j, B)} \leq r^{2M} 2^{-j\varepsilon} V(y_0, 2^j r)^{-1/2}\]

for $k = 0, 1, \ldots, M$, $j = 0, 1, 2, \ldots$, where $U_0 = B$, $U_j(B) = B_d(y_0, 2^j r) \setminus B_d(y_0, 2^{j-1} r)$ for $j \geq 1$.

It was proved in [36, Corollary 5.2] that every $(1, 2, M, \varepsilon)$-molecule $\tilde{a}$ belongs to $H^1_L$ and

\[(4.6) \quad \| \tilde{a} \|_{H^1_L} \leq C_{\varepsilon, M}.\]

Of course the condition (1.18) is invariant under the change of variable $\lambda \mapsto \lambda^s$ in multipliers. Hence (4.3) will be established if we have proved the following proposition for $\sqrt{\mathcal{L}}$.

**Proposition 4.7.** Assume that $m$ satisfies (1.18). Fix $M > q/4, M \in \mathbb{N}$. Then there exists $\varepsilon > 0$ such that for every $(1, 2, 2M)$-atom $a$ for $H^1_L$ the function

\[\tilde{a}(x) = m(\sqrt{\mathcal{L}}) a(x)\]

is a multiple of $(1, 2, M, \varepsilon)$-molecule. The multiple constant is independent of $a$.

**Proof.** Let $a$ be a $(1, 2, 2M)$-atom for $H^1_L$ and let $b$ and $B = B_d(y_0, r)$ be as in (1.7)–(1.9). Set $\tilde{b} = m(\sqrt{\mathcal{L}}) \mathcal{L}^M b$. Clearly, $\tilde{a} = \mathcal{L}^M \tilde{b}$. In order to complete the proof of the proposition is suffices to verify (4.5). To do this we need the following lemma.

**Lemma 4.8.** Let $\gamma > 1/2, \beta > 0$. Then there exists a constant $C > 0$ such that for every even function $F \in W^{2, \gamma + \beta/2}(\mathbb{R})$ and every $g \in L^2(\Omega), \text{supp} \ g \subset B_d(y_0, r)$, we have

\[
\int_{d(x, y_0) > 2r} |F(2^{-j} \sqrt{\mathcal{L}}) g(x)|^2 \left( \frac{d(x, y_0)}{r} \right)^\beta d\mu(x) \leq C (r 2^j)^{-\beta} \| F \|_{W^{2, \gamma + \beta/2}}^2 \| g \|_{L^2(\Omega)}^2
\]

for $j \in \mathbb{Z}$.

**Proof of the lemma.** The lemma seems to be well-known. For the convenience of the reader we provide a proof. To this end we borrow methods from [44]. Since $F$ is even,

\[F(2^{-j} \sqrt{\mathcal{L}}) g = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(\xi) \cos(2^{-j} \xi \sqrt{\mathcal{L}}) g d\xi,\]

where $\hat{F} = \mathcal{F} F$ is the Fourier transform of $F$. The Davies-Gaffney estimates (4.1) imply the finite speed propagation of the wave equation $\mathcal{L} u + u_{tt} = 0$ (see, e.g., [45], [10]), which means that there exists a constant $C' > 0$ that

\[\text{supp} \cos(2^{-j} \xi \sqrt{\mathcal{L}}) g \subset B_d(y_0, r + C' 2^{-j} |\xi|).\]
Hence,

\[
\left( \int_{d(x,y_0) > 2r} |F(2^{-j} \sqrt{L})g(x)|^2 \left( \frac{d(x,y_0)}{r} \right)^\beta \, d\mu(x) \right)^{1/2}
\]

\[
= \left( \int_{d(x,y_0) > 2r} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(\xi) \cos(2^{-j} \xi \sqrt{L})g(x) \, d\xi \right)^{1/2} \left( \frac{d(x,y_0)}{r} \right)^\beta \, d\mu(x) \right)^{1/2}
\]

\[
\leq C \int C^{2^{-j} |\xi| > 2r} \left( \int_{2r < d(x,y_0) < C2^{-j} |\xi|} |\hat{F}(\xi)|^2 |\cos(2^{-j} \xi \sqrt{L})g(x)|^2 \frac{d(x,y_0)}{r^\beta} \, d\mu(x) \right)^{1/2} \, d\xi
\]

\[
\leq C \int \left| \hat{F}(\xi) \right| \left( \frac{2^{-j} |\xi|}{r} \right)^{\beta/2} \|g\|_{L^2(\Omega)} \, d\xi
\]

\[
\leq C' (r2^j)^{-\beta/2} \|F\|_{W^{2,\gamma+\beta/2}(\Omega)} \|g\|_{L^2(\Omega)}.
\]

\[
\square
\]

We are now in a position to complete the proof of Proposition 4.7. Fix \( \varepsilon > 0 \) and \( \gamma > 1/2 \) such that \( \gamma + \varepsilon + q/2 = \alpha \). Set \( \beta = q + 2\varepsilon \). Then \( \gamma + \beta/2 = \alpha \). Let \( j_0 = -\log_2 r \). For an integer number \( k \), \( 0 \leq k \leq M \), write

\[
(4.9) \quad (r^2 \mathcal{L})^k \tilde{b} = r^{2k} \sum_{j \geq j_0} \psi(2^{-j} \sqrt{L})m(\sqrt{L})\mathcal{L}^{k+M}b + r^{2k} \sum_{j < j_0} \psi(2^{-j} \sqrt{L})\mathcal{L}^Mm(\sqrt{L})\mathcal{L}^k b
\]

\[
= r^{2k} \sum_{j \geq j_0} \psi(2^{-j} \sqrt{L})m(\sqrt{L})g_1 + r^{2k} \sum_{j < j_0} \psi(2^{-j} \sqrt{L})\mathcal{L}^Mm(\sqrt{L})g_2,
\]

where \( g_1 = \mathcal{L}^{k+M}b \), \( g_2 = \mathcal{L}^k b \). Since \( a \) is a \((1,2,2M)\)-atom for \( \mathcal{L} \) associated with \( B = B_d(y_0,r) \) and \( b \) (see (1.7)-(1.9)), we have

\[
(4.10) \quad \|g_1\|_{L^2(\Omega)} \leq r^{2M-2k} \mu(B)^{-1/2}, \quad \|g_2\|_{L^2(\Omega)} \leq r^{4M-2k} \mu(B)^{-1/2}.
\]

Put

\[
(4.11) \quad F_j(\lambda) = \begin{cases} m(2^j \lambda)\psi(\lambda) & \text{for } j \geq j_0; \\ 2^{2Mj}m(2^j \lambda)\lambda^{2M}\psi(\lambda) & \text{for } j < j_0; \end{cases}
\]

and extend each \( F_j \) to the even function. Clearly,

\[
(4.12) \quad \|F_j\|_{W^{2,\alpha}(\mathbb{R})} \leq \begin{cases} C & \text{for } j \geq j_0; \\ C2^{2Mj} & \text{for } j < j_0. \end{cases}
\]
Using Lemma 4.8 combined with (4.9) – (4.12), we get

\[
\left( \int_{d(x,y_0) > 2r} |(r^2 \mathcal{L})^k b(x)|^2 \frac{d(x,y_0)^\beta}{r^\beta} d\mu(x) \right)^{1/2} \\
\leq C r^{2k} \sum_{j \geq j_0} (r2^j)^{-\beta/2} \|F_j\|_{W^{2,\infty}(\mathbb{R})}\|g_1\|_{L^2(\Omega)} \\
+ C r^{2k} \sum_{j < j_0} (r2^j)^{-\beta/2} \|F_j\|_{W^{2,\infty}(\mathbb{R})}\|g_2\|_{L^2(\Omega)} \\
\leq C r^{2M} \mu(B)^{-1/2}.
\]

Moreover,

\[
\|(r^2 \mathcal{L})^k b\|_{L^2(\Omega)} = \|r^{2k} m(\sqrt{\mathcal{L}}) \mathcal{L}^{k+M} b\|_{L^2(\Omega)} \\
\leq C r^{2k} \|m\|_{L^{\infty}(\mathbb{R})}\|g_1\|_{L^2(\Omega)} \leq C r^{2M} \mu(B)^{-1/2}.
\]

Let \( j \in \mathbb{Z}, j \geq 0 \). Applying (4.13) and (4.14) we obtain

\[
\|(r^2 \mathcal{L})^k b\|_{L^2(U_j(B))} \leq C \int_{U_j(B)} |(r^2 \mathcal{L})^k b(x)|^2 \left( 1 + \frac{d(x, y_0)}{r} \right)^\beta 2^{-j\beta} d\mu(x) \\
\leq C r^{4M} 2^{-j\beta} \mu(B)^{-1} \\
\leq C' r^{4M} 2^{-j\beta} V(y_0, 2^j r)^{-1},
\]

where in the last inequality we have used (1.2).

\[\square\]

5. Remarks

1. Assume that \(-L\) generates a semigroup with Gaussian bounds. If we additionally assume that the space \((\Omega, d, \mu)\) is such that for every \( y \in \Omega \) there exists \( \kappa = \kappa(y) \) and \( c = c(y) > 0 \) such that

\[
\mu(B_d(y, s)) \geq cs^\alpha \quad \text{for } s > 1,
\]

then the multiplier operator \( m(L) \) (see (1.14)) extends uniquely to a bounded operator on \( H^1 \). To see this we define the space \( \mathfrak{T} \) of test functions in the following way: a function \( g \) belongs to \( \mathfrak{T} \) if there exist \( t > 0 \), a ball \( B_d(y, r) \), and a function \( \zeta \in L^\infty(\Omega) \) such that \( \text{supp} \zeta \subset B_d(y, r) \) and \( g = T_t \zeta \). We say that \( g_n \) converge to \( g_0 \) in \( \mathfrak{T} \) if there exist \( t > 0 \), a ball \( B = B_d(y, r) \), and functions \( \zeta_n \) such that \( \text{supp} \zeta_n \subset B \), \( \sup \|\zeta_n\|_\infty < \infty \), \( g_n = T_t \zeta_n \), and \( \zeta_n(x) \to \zeta_0(x) \) a.e. Clearly, \( \mathfrak{T} \subset L^p(\Omega) \) for every \( 1 \leq p \leq \infty \). One can easily prove that if \( f \in L^1(\Omega) \) is such that \( \int_{\Omega} fg d\mu = 0 \) for every \( g \in \mathfrak{T} \), then \( f = 0 \).

Lemma 5.2. Assume that \( m \) satisfies (1.13). Then \( \check{m}(L) \) maps continuously \( \mathfrak{T} \) into \( L^\infty(\Omega) \).

Proof. Recall that \( \alpha > q/2 \). Observe that there exists a constant \( C > 0 \) such that for every function \( n(\lambda) \) such that \( n \in W^{\infty,\alpha}(\mathbb{R}) \), \( \text{supp} n \subset (2^{-j-1}, 2^j) \), one has

\[
|n(L)(x, y)| \leq C \mu(B_d(y, 2^{-j/2}))^{-1} n(2^j .) \|n(2^j \cdot)\|_{W^{\infty,\alpha}(\mathbb{R})}.
\]
It suffices to prove (5.3) for \( j = 0 \) and then use the scaling argument. Set \( \xi(\lambda) = e^\lambda n(\lambda) \). Then \( \|\xi\|_{W^{\infty,0}(\mathbb{R})} \sim \|n\|_{W^{\infty,0}(\mathbb{R})} \). Hence, by Theorem 2.1,

\[
|n(L)(x,y)| \leq \int |\xi(L)(x,z)|T_1(z,y)\,d\mu(z) \leq C\mu(B_d(y,1))^{-1}\|n\|_{W^{\infty,0}(\mathbb{R})}.
\]

Assume that \( g \in \mathcal{F} \). Then there are \( t > 0, B = B_d(y_0,r) \), and a bounded function \( \zeta \) such that \( g = T_t\zeta, \operatorname{supp}\zeta \subset B \). Of course we can assume that \( r > 1 \). Let \( \psi \) be as in (2.9). Let \( n_j(\lambda) = m(\lambda)\psi(2^{-j}\lambda)e^{-t\lambda} \). Then

\[
\bar{m}(L)g(x) = \sum_j \bar{n}_j(L)\zeta(x) = \sum_j \int \bar{n}_j(L)(x,y)\zeta(y)\,d\mu(y).
\]

Set \( j_0 = -2\log_2 r \). Obviously \( \|\bar{n}_j(2^j\cdot)\|_{W^{\infty,0}(\mathbb{R})} \leq Ce^{-ct2^j} \). Thus

\[
(5.4) \quad \sum_j |\bar{n}_j(L)(x,y)| \leq C \sum_{j \leq j_0} \mu(B_d(y,2^{-j/2}))^{-1} + C \sum_{j > j_0} e^{-ct2^j}\mu(B_d(y,2^{-j/2}))^{-1}.
\]

By (5.1) there exist \( c(y_0) \) and \( \kappa = \kappa(y_0) > 0 \) such that for \( y \in B_d(y_0,r) \) and \( j \leq j_0 \) we have

\[
(5.5) \quad \mu(B_d(y,2^{-j/2})) \sim \mu(B_d(y_0,2^{-j/2})) \geq c(y_0)2^{-j\kappa/2}.
\]

On the other hand, by (1.2), for \( y \in B_d(y_0,r) \) and \( j > j_0 \), we have

\[
(5.6) \quad \mu(B_d(y_0,r)) \sim \mu(B_d(y,r)) \leq C(2^{j/2}r)^q\mu(B_d(y,2^{-j/2})).
\]

From (5.4)-(5.6) we conclude that there exists a constant \( C(y_0,r) \) such that

\[
\sum_j |\bar{n}_j(L)(x,y)| \leq C(y_0,r) \quad \text{for } x \in \Omega \text{ and } y \in B_d(y_0,r).
\]

We are now in a position to define the action of \( m(L) \) on the space \( L^1(\Omega) \) in the weak (distributional) sense by putting

\[
\langle m(L)f,g \rangle = \int_\Omega f(x)\bar{m}(L)g(x)\,d\mu(x).
\]

Let us observe that \( m(L) \) is uniquely defined on \( H^1_L \). Indeed, if \( f = \sum_j \lambda_j a_j \), where \( a_j \) are \( (1,2,1) \)-atoms, \( \lambda_j \in \mathbb{C}, \sum |\lambda_j| \sim \|f\|_{H^1_L} < \infty \) then, by Theorem 1.12 and Lemma 5.2, for every \( g \in \mathcal{F} \) we have

\[
\langle m(L)f,g \rangle = \int_\Omega \left( \sum_j \lambda_j a_j(x) \right)\bar{m}(L)g(x)\,d\mu(x) = \sum_j \lambda_j \int_\Omega a_j(x)\bar{m}(L)g(x)\,d\mu(x) = \sum_j \lambda_j \int_\Omega m(L)a_j(x)\bar{g}(x)\,d\mu(x) = \int_\Omega \left( \sum_j \lambda_j m(L)a_j(x) \right)\bar{g}(x)\,d\mu(x).
\]
Since $\sum_j \lambda_j m(L)a_j$ belongs to $L^1(\Omega)$, we obtain that $m(L)f = \sum_j \lambda_j m(L)a_j$, which gives the required uniqueness. Obviously, $\|m(L)f\|_{H^1_L} \leq C\|f\|_{H^1_L}$.

2. One of distinguished examples of semigroups of linear operators with Gaussian bounds is that generated by a Schrödinger operator $-A = \Delta - V$ on $\mathbb{R}^d$, where $V$ is a nonnegative potential such that $V \in L^1_{\text{loc}}(\mathbb{R}^d)$. By the Feynman-Kac formula the integral kernels $p_t(x,y)$ of the semigroup $e^{-tA}$ satisfy

$$0 \leq p_t(x,y) \leq (4\pi t)^{-d/2} \exp(-|x-y|^2/4t).$$

Clearly, considering $(\mathbb{R}^d, d(x,y) = |x-y|, dx)$ as a space of homogeneous type, we have that (1.2) and (5.1) hold with $q = d$. Thus, as a corollary of Theorem 1.12, we obtain that any bounded function $m : (0, \infty) \to \mathbb{C}$ which satisfies (1.13) with $\alpha > d/2$ is an $H^1_A$ spectral multiplier for $A$.

We would like to remark that the space $H^1_A$ admits also characterization by means of maximal function from the semigroup $e^{-tA}$ (see [36]). Using arguments similar to those of [19] one can prove the spectral multiplier theorem on Hardy spaces associated with the Schrödinger operators by applying both atomic and maximal function characterizations.

Another molecule decomposition of Hardy space $H^1$ associated with semigroups generated by Schrödinger operators was communicated to us by Jacek Zienkiewicz [48]. These decompositions also lead to multiplier theorems.

Acknowledgment. The authors would like to thank Pascal Auscher, Frédéric Bernicot, and Paweł Głowacki for their valuable comments.

References

[1] G. Alexopoulos, Spectral multipliers on Lie groups of polynomial growth, Proc. Amer. Math. Soc. 120 (1994), no. 3, 973–979.

[2] P. Auscher, X.T. Duong, A. McIntosh, Boundedness of Banach space valued singular integral operators and Hardy spaces, preprint.

[3] P. Auscher, A. McIntosh, E. Russ, Hardy Spaces of Differential Forms on Riemannian Manifolds, J. Geom. Anal. 18 (2008), 192–248.

[4] F. Bernicot, J. Zhao, New abstract Hardy spaces, J. Funct. Anal. 255 (2008), 1761–1798.

[5] J. Betancor, J. Dziubański, and J. L. Torrea, On Hardy spaces associated with Bessel operators, J. Anal. Math. 107 (2009), 195–219.

[6] S. Blunck, A Hörmander-type spectral multiplier theorem for operators without heat kernel, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2 (2003) no. 3, 449-459.

[7] M. Christ, $L^p$ bounds for spectral multipliers on nilpotent groups, Trans Amer. Math. Soc. 328 (1991), 73–81.

[8] R. Coifman, A real variable characterisation of $H^p$, Studia Math., 51 (1974), 269–274.

[9] R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. 83 (1977), 569–645.
[10] T. Coulhon and A. Sikora, *Gaussian heat kernel upper bounds via Phragmén-Lindelöf theorem*, Proc. Lond. Math. Soc. (3) 96 (2008), no. 2, 507–544.

[11] M. Cowling, *Harmonic analysis on semigroups*, Ann. of Math. 117 (1983), 267–283.

[12] L. De Michele and G. Mauceri, *$H^p$ multipliers on stratified groups*, Ann. Mat. Pura Appl. 148 (1987), 353–366.

[13] D. Deng, X. Duong, A. Sikora, L. Yan, *Comparison of the classical BMO with the BMO spaces associated with operators and applications*, Rev. Mat. Iberoam. 24 (2008), no. 1, 267–296.

[14] X. Duong, L. Yan, *Duality of Hardy and BMO spaces associated with operators with heat kernel bounds*, J. Amer. Math. Soc. 18, Number 4 (2005), 943–973.

[15] X. T. Duong, E. M. Ouhabaz, and A. Sikora, *Plancherel-type estimates and sharp spectral multipliers*, J. Funct. Anal., 196(2) (2002), 443-485.

[16] J. Dziubański, *Atomic decomposition of $H^p$ spaces associated with some Schrödinger operators*, Indiana Univ. Math. J. 47 (1998), 75–98.

[17] J. Dziubański, *Spectral multipliers for $H^1$ spaces associated with some Schrödinger operators*, Proc. Amer. Math. Soc. 127 (1999), 3605–3613.

[18] J. Dziubański, *Spectral multipliers for Hardy spaces associated with Schrödinger operators with polynomial potentials*, Bull. London Math. Soc. 32 (2000), no. 5, 571–581.

[19] J. Dziubański, *A spectral multiplier theorem for $H^1$ spaces associated with Schrödinger operators with potentials satisfying a reverse Hölder inequality*, Illinois J. Math. 45 (2001), 1301-1313.

[20] J. Dziubański, *Note on $H^1$ spaces related to degenerate Schrödinger operators*, Ill. J. Math. 49.4, (2005), 1271–1297.

[21] J. Dziubański, *Hardy spaces for Laguerre expansions*, Constructive Approximation 27 (2008), 269-287.

[22] J. Dziubański, M. Preisner, *Spectral multiplier theorem on Hardy spaces associated with semigroups generated by Schrödinger operators with compactly supported potentials*, unpublished manuscript.

[23] J. Dziubański, J. Zienkiewicz, *Hardy spaces associated with some Schrödinger operators*, Studia Math. 126 (1997), 149–160.

[24] J. Dziubański and J. Zienkiewicz, *Hardy space $H^1$ associated to Schrödinger operator with potential satisfying reverse Hölder inequality*, Rev. Mat. Iberoamericana, 15 (1999), no. 2, 279–296.

[25] J. Dziubański, J. Zienkiewicz, *$H^p$ spaces for Schrödinger operators*, Fourier analysis and related topics (Bedlewo, 2000), 45–53, Banach Center Publ., 56, Polish Acad. Sci., Warsaw, 2002.

[26] J. Dziubański and J. Zienkiewicz, *$H^p$ spaces associated with Schrödinger operators with potentials from reverse Hölder classes*, Colloq. Math. 98 (2003), no. 1, 5–38.

[27] J. Dziubański, J. Zienkiewicz, *Hardy spaces $H^1$ for Schrödinger operators with certain potentials*, Studia Math. 164 (2004), no. 1, 39–53.
[28] J. Dziubański, G. Garrigós, T. Martínez, J.L. Torrea, J. Zienkiewicz, \textit{BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality}, Math. Z. 249 (2005), no. 2, 329–356.

[29] C. Fefferman and E.M. Stein, \textit{H}^p \textit{spaces of several variables}, Acta Math., 129 (1972), 137–195.

[30] G. Folland and E. Stein, \textit{Hardy Spaces on Homogeneous Groups}, Princeton University Press, Princeton, NJ, 1982.

[31] J. García-Cuerva, \textit{Weighted H}^p \textit{spaces}, Dissertationes Math. (Rozprawy Mat.) 162 (1979).

[32] J. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea, \textit{Spectral multipliers for the Ornstein-Uhlenbeck semigroup}, J. Anal. Math. 78 (1999), 281–305.

[33] W. Hebisch, \textit{A multiplier theorem for Schrödinger operators}, Colloq. Math. 60/61 (1990), 659–664.

[34] W. Hebisch, \textit{Functional calculus for slowly decaying kernels}, preprint, University of Wroclaw; also available at http://www.math.uni.wroc.pl/~hebisch.

[35] W. Hebisch, J. Zienkiewicz, \textit{Multiplier theorems on generalized Heisenberg groups II}, Colloq. Math. 69 (1995), 29-36.

[36] S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea, and L.X. Yan \textit{Hardy spaces associated with non-negative self-adjoint operators satisfying Davies-Gafney estimates}, preprint 2008.

[37] L. Hörmander, \textit{Estimates for translation invariant operators in L^p spaces}, Acta. Math., 104 (1960), 93-140.

[38] R.H. Latter, \textit{A decomposition of H}^p(R^n) \textit{in terms of atoms}, Studia Math., 62 (1977), 92–102.

[39] R. Macías and C. Segovia, \textit{A decomposition into atoms of distributions on spaces of homogeneous type}, Adv. in Math., 33 (1979), 271-309.

[40] G. Mauceri and S. Meda, \textit{Vector-valued multipliers on stratified groups}, Rev. Mat. Iberoamericana 6 (1990), 141–154.

[41] G. Mauceri, S. Meda, \textit{BMO and H}^1 \textit{for the Ornstein-Uhlenbeck operator}, J. Funct. Anal 252 (2007), 278–313.

[42] G. Mauceri, M. Picardello, F. Ricci, \textit{A Hardy space associated with twisted convolution}, Adv. in Math. 39, (1981), 270–288.

[43] D. Müller and E. M. Stein, \textit{On spectral multipliers on Heisenberg and related groups}, J. Math. Pures Appl. 73 (1994), 413–440.

[44] A. Sikora, \textit{Multiplicateurs associés aux souslaplaciens sur les groupes homogènes}, C.R. Acad. Sci. Paris 315 (1992), 417–419.

[45] A. Sikora, \textit{Riesz transform, Gaussian bounds and the method of wave equation}, Math. Z., 247 (2004), no. 3, 643–662.

[46] E. Stein, \textit{Topics in harmonic analysis related to the Littlewood-Paley theory}, Princeton University Press, Princeton, NJ, 1970.
[47] E. Stein, G. Weiss, *On the theory of harmonic functions of several variables. I. The theory of $H^p$-spaces*, Acta Math., 103 (1960), 25-62.

[48] J. Zienkiewicz, Personal communication.

**Instytut Matematyczny, Uniwersytet Wrocławski, 50-384 Wrocław, Pl. Grunwaldzki 2/4, Poland**

*E-mail address: jdziuban@math.uni.wroc.pl, preisner@math.uni.wroc.pl*