Strategies for particle resampling in PIC simulations

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Abstract

In particle-in-cell simulations, excessive or even unfeasible computational demands can be caused by the growth of particle number in the course of prolific ionization or cascaded pair production due to the effects of quantum electrodynamics. Here we discuss how one can arrange a dynamic shrinking of the macro-particle ensemble to maintain an acceptable sampling of arbitrary particle distribution. The approaches of merging and thinning as well as their variants are discussed and the aspects of their use are considered.

Keywords: particle-in-cell, resampling, merging, thinning, QED cascades

1. Introduction

In particle-in-cell (PIC) simulations and some other statistical computations the use of so-called macro-particles, that sample the distribution of some quantity, can be supplemented by the process of adding new macro-particles. For example, in PIC plasma simulations this can be done to account for the continuous ionization of matter \cite{1,2} or for the electron-positron pair production due to the effects of quantum electrodynamics (QED)\cite{3,4,5,6,7,8}. When such particle sources become prolific the number of macro-particles can grow significantly. This can slow down the simulation or/exhaust the memory available for the allocation of macro-particles data. The natural solution is to resample the modeled distribution by a smaller number of macro-particles with
increased weight of their contribution. This procedure, referred to as down-sampling, can be repeatedly applied to combat the rise of the macro-particles or, alternatively, to reduce the computational expenses for highly populated regions of phase space, where the representation became excessive with time. In the latter case, the released resources can be used to reduce the computational noise in underpopulated regions by introducing more macro-particles for better sampling, i.e. performing so-called up-sampling.

The implementation of down-sampling has been considered by several authors and a number of methods have been proposed. One can distinguish three main approaches. According to the first approach, referred to as merging or coalescing, one (or two) macro-particle is introduced to replace a subset of macro-particles that are close in the phase space. According to the second approach, referred to as thinning, we do not introduce new macro-particles, but remove one or several macro-particles and redistribute their weight among the others, either globally or locally within the given subset. Finally, one can totally replace the given subset of macro-particles with a new subset of appropriately introduced new macro-particles. This is the third approach, which is referred to as complete resampling in this article. In all cases, for the selected subset of closely located macro-particles the procedure can potentially lead to the change of intrinsically conserved quantities, such as the total mass (weight), charge, energy or momentum. In addition, the procedure can potentially lead to the sudden change of grid values for the charge and current density, leading to artificial noise, heating or other systematic effects. That is why the possibility of preserving such quantities has been considered with special care by many authors.

Within the approach of merging Lapenta and Brackbill proposed a method for coalescing two particles into one so that the charge assignment to the grid nodes is preserved [9]. Merging several macro-particles of dense clusters selected with the use of the Voronoi diagram has been suggested by Luu et al. [10]. To conserve both energy and momentum Vranic et al. proposed to perform merging into a pair of particles with the appropriately chosen momenta [11]. This can
be arranged using the selection of highly populated volumes in the phase space. A similar method with some modification has been used in Ref. [8].

The approach of thinning provides various options, including rather straightforward ones. For example, Timokhin developed a procedure that repeatedly selects a random particle, deletes it and uniformly redistributes its weight among the others of the same kind [12]. Nerush et al. used a similar global thinning but with the redistribution of mass, charge and energy [3]. Although this procedure preserves the mentioned quantities globally, it implies their stochastic local sudden variations. One way of preventing such variations is to restrict the redistribution to a dense cluster or a highly populated volume of the phase space. A way to perform thinning with conservation of several arbitrary particle and grid quantities is proposed in Ref. [13].

Within the approach of complete resampling, Lapenta and Brackbill proposed a method of replacing the macro-particles in a given cell with a new subset, preserving the contributions to the grid quantities and also maximizing the uniformity of the distribution of these quantities over the new subset [14]. A way of doing such resampling with the conservation of grid values for charge and current density, as well as of the total energy of the resampled macro-particles, has been proposed by Assous et al. [15] and further developed in Ref. [16]. Pfeiffer et al. proposed a statistical method that conserves momentum and energy [17]. Faghihi et al. reported on the development of an algorithm for both down- and up-sampling that preserves any number of particle and grid quantities [18].

Recently, the methods of down-sampling became highly demanded for the numerical studies of QED cascades that will be inherent for the upcoming experiments at the next generation high-intensity laser facilities [19] [20] [21] [22]. The numerical studies indicate that, apart from the drastic increase of particle number by several orders of magnitude, the physics in strong laser fields includes a variety of new phenomena [23] [24] [25] [26] [27] [28] [29]. The absence of prior knowledge about the minimal scales of new phenomena raises a new difficulty for the implementation of down-sampling: merging macro-particles within dense clusters or volumes of predetermined scale may erase or affect smaller
scale peculiarities that are essential for the modelled phenomena. Although the coordinates have the cell size as a natural limiting scale, the momentum does not have any natural resolution limit according to the PIC method. At the same time, shrinking the momentum gate for merging in a given cell, increases the needed density for the selection of several particles to be possible.

In this article we consider how one can use the thinning approach or modify the merging procedure to combat the outlined difficulty. In addition, we consider the aspect of reducing the difference between the weights of particles as a way to increase the efficiency of sampling. For our study we develop, compare and analyse several methods that we release as open-source tools available within the hi-χ framework [30]. The article is arranged as follows. In section 2 we introduce the principle of agnostic down-sampling for developing the thinning procedures that are applicable without prior knowledge about minimal scales attributed to the modelled process. In section 3 we describe several thinning methods that comply this principle and also discuss how to modify the merging methods to improve their applicability in this context. In section 4 we perform a basic comparison of the methods. In section 5, we show realistic examples. We conclude in section 6.

2. The principle of agnostic down-sampling

When several macro-particles are merged we assume that all these particles sample a uniform part of the particles distribution without any complex trends or peculiarities represented by some of them. For example, the macro-particles shown in Fig. 1 are assumed to sample the distribution shown with the solid curve, not the ones shown with the dashed or dotted curves. Although one can probably mitigate the risks of such misinterpretation by restricting the choice of particles in a cluster to sufficiently small phase space volumes, let us consider an alternative down-sampling methodology that is applicable without any prior knowledge and thus called agnostic here.

Firstly, note that simple merging into the average position of the selected
macro-particles will likely reduce the variance of the peaks that are more narrow than the selection scale. To mitigate this one can introduce a probabilistic mismatch with the variance determined over the merged macro-particles. However, such a procedure will likely cause filling the gap in the distribution shown with the dashed curve in Fig. [1]. Looking at the distribution shown with the dotted curve we can see that the only way to not introduce any particles in any potentially empty phase space region is to use only the existing particles. In other words, we should try to restrict our action to the removal of one or several macro-particles in combination with the change of the weights of the other macro-particles.

Secondly, we should not affect any distribution functions. In order to do this, we can arrange a probabilistic procedure so that the chance of removing any particle in the given subset is compensated by the chance of increasing its weight. If the expectation value for the weights of each particle exactly equal to its initial weight, all the possible distribution functions remains unchanged on average (see rigorous consideration below). This means that if some peculiarity is erased in one subset, it has a chance to be increased in a different subset.

Figure 1: An illustrative clarification of the difficulty in arranging non-destructive merging without prior knowledge about the peculiarities in the particle distribution and their minimal scales. The merging of four macro-particles (grey rectangular peaks) can be non-destructive if they sample the distribution show with black solid curve. However it can affect the modelled processes if the sampled distribution has a more complex shape. The examples of such potential shapes are shown with dashed blue and dotted green curves.
In this way, the procedure only increases the statistical variation but does not remove peculiarities at any scale.

We can now formulate the principle of down-sampling that is based on this idea. A down-sampling is called agnostic if it is restricted to the probabilistic change of weights so that (1) at least one macro-particle receives zero weight and can be removed and (2) the expectation value for the weight of each macro-particle is exactly equal to its initial weight.

If we have initially \( n \) macro-particles and the weight of \( i \)-th is \( w_i \), the principle of agnostic down-sampling implies determining a number of outcomes so that the probability \( p_k \) of choosing the \( k \)-th outcome provides

\[
\langle \hat{w}_i \rangle = \sum_{k \in \text{outcomes}} w_i^k p_k = w_i \quad \text{for} \quad i = 1, \ldots, n, \tag{1}
\]

where \( \langle \hat{w}_i \rangle \) is the resulting weight averaged over the outcomes, and \( w_i^k \) is the weight assigned to the \( i \)-th particle in the \( k \)-th outcome.

Let us demonstrate that any agnostic resampling has the property of preserving all the distribution functions on average, i.e. any distribution averaged over the outcomes of probabilistic resampling procedure coincides exactly with that we had before the resampling. In order to show this, we consider an arbitrary distribution \( F = \partial N / \partial f \), which can be numerically represented as a sequence of values \( F_j \):

\[
F_j (\mathcal{A}) = \frac{1}{V(d_j)} \sum_{f(x_i, p_i, \sigma_i) \in d_j} w_i, \tag{2}
\]

where \( f(x_i, p_i, \sigma_i) \) is the quantity (can be multidimensional) over which the distribution is computed, \( \sigma_i \) is a generalized vector of the particle’s parameters other than coordinate and momentum (e.g. spin, polarization, etc.), \( \mathcal{A} \) is the state of ensemble, \( d_j \) denotes the sub-regions used to discretize the space of \( f \) values and \( V(d_j) \) is the volume of \( j \)-th sub-region. In other words, \( F_j (\mathcal{A}) \) is the number of real particles, for which the quantity \( f \) falls into \( d_j \), divided by the volume of the element \( d_j \). As one can see, in such a way we can define coordinate-, momentum-, energy-, angular- and other distributions on uniform
as well as non-uniform grids defined by $d_j$. We can now formally compute the value $F_j$ averaged over outcomes of the resampling procedure that turns $A$ into $R(A)$:

$$\langle F_j (R(A)) \rangle_R = \frac{1}{V(d_j)} \sum_{f(x_i, p_i, \sigma_i) \in d_j} \langle \hat{w}_i \rangle_R = F_j (A).$$

(3)

Here the first equality is provided by the fact that an agnostic resampling neither changes internal state $(x_i, p_i, \sigma_i)$ of nor adds new macro-particles, whereas the second equality follows from the fact that it preserves the weight on average, i.e. $\forall i : \langle \hat{w}_i \rangle_R = w_i$. As one can see, we proved the statement without requiring any knowledge about (1) the quantity $f$, (2) the numerical intervals $d_j$ and (3) the distribution of macro-particles (either over the position in phase space or over the values of their weights). In this context, the term agnostic indicates that the procedure preserves all the distribution functions independently of the outlined entities.

3. Strategies of agnostic down-sampling

In this section we propose a number of methods that comply with the principle of agnostic down-sampling. We start with the simplest methods and then describe more advanced ones and their potential benefits. We quantify the rate of resampling by a parameter $k$ being the target ratio of number of macroparticles in the ensemble before resampling to their number after resampling. Each method was given a shortened name in parenthesis for designation on graphics.

1. Simple thinning (simple).

According to this method each macro-particle is either removed, with equal probability $p = 1 - k^{-1}$, or has its weight increased by a factor of $k$. This method is agnostic because $\langle \hat{w}_i \rangle = 0 \cdot (1 - k^{-1}) + kw_i \cdot k^{-1} = w_i$. The method does not strictly conserve any quantities such as total weight, energy or momentum, but conserves all quantities on average as any agnostic method. The method can be applied to subsets of any size, which makes it possible to apply it to very small volumes in phase space. This method is the easiest to implement and analyze.
theoretically. If the initial total number of macroparticles is \( n \), the number of macroparticles remaining after resampling is approximately \( n/k \).

2. *Leveling thinning* (leveling).

In this method, we first calculate the average weight \( \bar{w} \) of particles in a given cell. Then, for all particles with weight \( w_i < k \bar{w} \) the weight \( k \bar{w} \) is assigned with probability \( w_i / (k \bar{w}) \) and the particle is removed otherwise. This method is agnostic because \( \langle \tilde{w}_i \rangle = k \bar{w} \cdot w_i / (k \bar{w}) + 0 \cdot (1 - w_i / (k \bar{w})) = w_i \). It is clear that this procedure gets rid of macro-particles with weight below \( kW \). This may help to balance and optimize the distribution of computational resources. The method does not strictly conserve any quantities. The actual number of remaining macroparticles can be less or greater than \( n/k \) depending on the initial weight distribution among macroparticles. If all macroparticles initially have the same weight, the number of macroparticles remaining after resampling is approximately \( n/k \).

3. *Global leveling thinning* (globalLev) is a modification of the previously described leveling thinning method. The method works similarly, except it uses all particles in the entire calculation area to compute the average weight \( W \).

4. *Number-conservative thinning* (numberT).

In this method we select random macro-particles with probability proportional to their weight, i.e. \( w_i/W \), where \( W = \sum w_i \). We repeat this selection \( m \) times and count the number of times \( c_i \) we have chosen the \( i \)-th macro-particle. After that the particles that have not been selected even once \( (c_i = 0) \) are removed and the others are assigned with a new weight equal to \( \tilde{w}_i = c_iW/m \). It is clear that the \( i \)-th macro-particle will be selected \( \langle c_i \rangle = mw_i/W \) times on average and thus the mathematical expectation of the change in the macro-particle’s weight is zero: \( \langle \tilde{w}_i \rangle = \langle c_i \rangle W/m = w_i \), i.e. the method is agnostic. This procedure strictly conserves the total weight of macro-particles in a cell: \( \tilde{W} = \sum \tilde{w}_i = \sum c_iW/m = W \). In addition, it favors the removal of macro-particles with small weight. This may also contribute to the efficiency of sampling. The number of macro-particles \( \tilde{n} \) after this procedure is probabilistic but obviously cannot exceed \( m \). The average number of remaining macro-particles
is given by the expression

\[ \hat{n} = \sum_{i=1}^{n} \left( 1 - \left( 1 - \frac{w_i}{W} \right)^m \right). \] (4)

If we assume that in our distribution macro-particles have similar weights we can estimate \( \hat{n} \approx n \left( 1 - (1 - n^{-1})^m \right) \). Assuming also that \( n \) is large, we can estimate that \( \hat{n} = n/k \) is achieved for

\[ m \approx -n \ln \left( 1 - k^{-1} \right). \] (5)

This means, for example, that for large \( n \) if we need to remove roughly half of the macro-particles we need \( m \approx \ln(2) n \). This method is useful when the total charge/number of particles needs to be strictly conserved.

5. **Energy-conservative thinning** (energyT) is a modification of the previously described number-conservative thinning. According to this method we also select a random macro-particle \( m \) times, but do this with probability proportional to energy, i.e. the \( i \)-th macroparticle is selected with probability \( e_i w_i / E \), where \( e_i \) is the energy of the particle represented by the \( i \)-th macroparticle and \( E = \sum e_i w_i \). The macroparticles that have been selected \( c_i \neq 0 \) times are assigned with the weight \( c_i E / (e_i m) \) and the others are removed. One can check that this procedure complies with the principle of agnostic down-sampling and also strictly conserves the total energy \( E \) in each cell. However, this method does not strictly conserve the total weight \( W \). This method is useful when the total kinetic energy of particles needs to be strictly conserved.

6. **Conservative thinning** (conserv). This method, described in [13], can be configured to preserve several invariants simultaneously. Each invariant \( (A) \) can be represented by a linear equation: \( A = \sum a_i w_i \), where \( a_i \) and \( w_i \) are the contribution and the weight of the \( i \)-th particle, respectively. The conservation of several invariants sets a system of linear equations, where the number of variables (weights \( w_i \)) can be controlled by the number of particles involved in the thinning procedure. If the number of particles is greater than the number of invariants, the system is undetermined. It turns out that it is possible to find two solutions with one of the weights being equal to zero and others being
positive, so that the probabilistic choice of one of these solutions results in an agnostic resampling that reduces the number of macroparticles by one. The procedure can be repeated several times for a given set of macroparticles to reduce the number of particles down to \( n/k \) (assuming that it is still larger than the number of invariants). This method is useful when a number of physical properties of particles need to be strictly conserved.

For comparison we also consider methods that do not comply with the principle of agnostic down-sampling. Most of these methods revolve around merging of dense clusters.

7. **Merging to averaged location (mergeAv).** According to this method, for each cell we determine \( n_{\text{cell}}/k \) (where \( n_{\text{cell}} \) is the number of particles in a cell), but at least 3, clusters using the k-means method with respect to location of particles in the momentum space. Next we replace all particles in each cluster with a new macro-particle that has the mean coordinate and momentum of particles in the cluster and weight equal to the total weight \( W \). The number of selected clusters determines the number of remaining particles. Since the complexity of the k-means method is \( (n_{\text{cell}}^3/k) \), the algorithm may require significant computational resources. This method is useful when the phase space can be adequately represented by a number of dense clusters. If \( n_{\text{cell}}/k >> 1 \), the overall resulting number of macroparticles is approximately \( n/k \). Due to computational time restrictions, the recommended value of \( k \) for merge-based methods is such that no more than 30 clusters are formed.

8. **Merging to random particle (merge).** As we mentioned earlier, the merging procedure can naturally result in the systematic relocation of particles towards denser regions. An indicative example is the case of a bulk of particles (or a particle beam) with a narrow distribution in coordinate space (as compared to the coordinate scale of clusters used for merging). In this case merging naturally favors bringing macro-particles to the peak of that distribution, removing the macro-particles in its tails. We can avoid this by introducing the following modification to the previous method. The weight of particles in each determined cluster is brought to a random particle in the cluster. Note, however, that this
does not prevent the reduction of particles’ spread in the momentum space.

As a general note we would like to highlight the following. Any downsampling results in the loss of information since the amount of unique macro-particles decreases. The inevitable consequence of this is the increase of noise in the distributions of particles. The more macro-particles are used the less noise we can expect and vice versa. In practice this leads to the trade-off between the accuracy of results and the computational demands. The goal of arranging appropriate resampling is to avoid systematic deviations and minimize computational demands for reaching the accuracy necessary for the problem of interest.

To not spend computational resources for resampling when it is not needed, we use the trigger for starting of the resampling procedure: the number of macroparticles in a shared-memory computational domain must reach a certain threshold value. All methods except global leveling are applied to each cell independently.

4. Comparison of methods on model problems

In this section we present the comparison of the resampling methods described above using model problems where QED effects are negligible: a steady-state plasma and the development of a Weibel instability in two counter-streaming plasma flows.

4.1. Steady-state plasma

In our numerical experiments we observe an additional decrease of plasma temperature caused by the ensemble resampling. To clarify the reason for this effect, we start from a brief phenomenological analysis.

For simplicity we consider the resampling of electrons, whereas the ions (or positrons) are modelled by a uniform positively charged background. We can outline two basic reasons for the change of temperature to happen due to resampling. Firstly, if the applied resampling does not preserve the total kinetic energy of particles and the method is not agnostic, then we can potentially
have an asymmetric net acquisition of energy mismatches, either positive or negative. Secondly, even if the used resampling is agnostic, the change of weights effectively means that we draw the plasma out of the equilibrium state. There could be two cases.

If the Poisson's equation is solved at each iteration (as it happens in some spectral codes), this abrupt local relocation of charges builds up an additional local variation of electric field, the energy of which can eventually add up to the temperature increase. Note that if the resampling does not preserve charge within each cell, the local variation of charge density can build up a non-zero global electric field (especially in the 1D case), which can have a significant energy. This explains why preserving charge may be beneficial.

In case the Poisson's equation is not solved and the field is only affected by the charge currents, the local removal of macro-particles would effectively mean adding compensating charges that are fixed in space (the added positive charge is compensated by the increase of weight of the remaining macro-particles). In this case the plasma will tend to a new equilibrium state relative to the positively charged background with corresponding local variations of charge density. Again, if the charge is not preserved locally, a global effective potential can be formed and the placement of compensating charges can cause a significant energy change. Now, let us imagine a situation when the plasma is leaving some part of computational region. The added effective positive charges in this part will show up as unchanged noise in electric field. Since this field would contain a strictly positive energy, we can conclude that this energy is effectively deducted from the thermal motion of charges due to the application of resampling. Hence, resampling can cause an effective cooling of plasma. This is observed in our numerical experiments.

In order to compare how strongly different resampling strategies can affect simulations we choose to compare the change of temperature of a steady state plasma after the application of resampling using different methods.

Particularly, we consider a 3D region represented by $32 \times 32 \times 32$ cells with periodic boundary conditions filled with homogeneous quasineutral electron-
Figure 2: Change of equilibrium temperature for a plasma with initially $N_0 = 100$ ppc after a single instance of resampling depending on the target resampling coefficient $k$, agnostic methods. Dashed lines: methods’ results, Solid: linear fit.

positron plasma with initial temperature $T_0 = 0.001mc^2$ (here $m$ is the electron charge, $c$ is the speed of light), cell size equal to 2 Debye radius, and physical density derived from these values. The considered values of the initial number of macro-particles are $N_0 = 100$ and $N_0 = 1000$ particles per cell (ppc). The time step is set to $1/128$ of the period of cold plasma oscillations. After $t = 1$ oscillation period we perform resampling, at $t = 10$ oscillation periods we calculate the temperature of the plasma (as the average kinetic energy of particles) in comparison to the initial temperature $T_0$. This procedure was performed for values of $N_0$ mentioned above for every method of resampling and for a set of resampling coefficients $k$, which indicate the target decrease ratio in the amount of macroparticles, equal to $(1.1; 3; 10; 30; 100; 300; 1000)$ where possible due to method limitations. To identify the temperature decrease induced by resampling the temperature decrease in the case without resampling is subtracted from the value obtained using various methods.

For all agnostic methods the results are similar and lie within a narrow range of each other. These results are shown also in Fig. 2 and 3 for the cases with $N_0 = 100$ and $N_0 = 1000$, respectively.
Interestingly, the results show a linear trend in $\Delta T(k)$. In addition, close trends with respect to $N_0/k$ values (see Fig. 2 and 3) indicate that the temperature decrease depends solely on the number of macroparticles per cell remaining after resampling rather than on the initial ppc number and the coefficient $k$ separately. The results of agnostic methods can be roughly fitted by $\Delta T = -\left(0.12/\text{ppcf}\right) \times T_0$, where $\text{ppcf}$ is the final number of macroparticles per cell after resampling. Actual values of $\Delta T$ for each method vary by about 10% from this rough estimate depending on the particular method in question and the value of $k$.

Merge-based methods, on the other hand, show considerably poorer performance according to our chosen metric (see Fig. 4 and 5). Even in the best-case scenario where merge methods perform the closest to agnostic methods, $\Delta T$ shown by merge methods is approximately 15-20 times greater than $\Delta T$ shown by agnostic methods. Particularly, while agnostic methods show a linear trend towards a 10% temperature drop at $k = N_0$ (which means the target number of particles after resampling is $N_0/k = 1$ ppc), merge methods yield a whole 65% temperature decrease already at ppc=3. The curve for merge methods is
Figure 4: Change of equilibrium temperature for a plasma with initially $N_0 = 100$ ppc after a single instance of resampling depending on the target resampling coefficient $k$, all methods. Dashed lines: methods’ results, Solid: linear fit.

Figure 5: Change of equilibrium temperature for a plasma with initially $N_0 = 1000$ ppc after a single instance of resampling depending on the target resampling coefficient $k$, all methods. Dashed lines: methods’ results, Solid: linear fit.

slightly concave up, so for lower $k$ the result is even worse in relative comparison to the agnostic methods.

We conclude that from the point of view of numerical cooling, merge-based methods are at least an order of magnitude worse (i.e. these methods yield
numerical cooling an order of magnitude stronger) than agnostic methods.

4.2. Weibel instability in counter-streaming plasma flows

The second test problem is the development of Weibel instability [31] in counter-streaming plasma flows [32]. This instability results in an exponential growth of perturbations in plasma density, current and magnetic field along the direction transverse to the plasma stream. To make our experiment robust we introduce a periodic modulation of density in the transverse direction to act as a systematic seed for the instability. For each method we carry out an individual simulation and perform a single resampling procedure near the beginning of the growth. In such a way we intend (1) to see whether the introduced random perturbations in density can affect the process, and (2) compare different resampling methods in terms of the introduced perturbations. To quantify the strength of these perturbations we measure the variance of plasma density computed for individual cells of the computational grid.

Let us first note that the reduction of the number of macro-particles should naturally result in the increase of variance for the number of real particles in each cell. The extent of this increase, however, depends on the method. For example, number conservative thinning does not change the number of real particles (although the variance can grow after the migration of particles between the cells). In this cases the impact of resampling on macro-particles is coordinated within each cell. To estimate the worst case scenario, let us consider the case of simple thinning, for which the impact is totally uncoordinated. With probability $p_1 = 1/k$ the particles weight $w$ is increased by a factor of $k$ to $w_1 = kw$, otherwise (with probability $p_2 = 1 - k^{-1}$) the macro-particle is deleted. The expected value of the number of physical particles $N_{phys}$ among different realizations of this random process must remain unchanged: $E[N_{phys}] = \frac{1}{k} w_1 = w$ (hereafter by $E[\cdot]$ we denote the value averaged over the realization of resampling). For the contribution of individual macro-particles, we can compute the variance $D[N_{phys}] = E[N_{phys}^2] - E[N_{phys}]^2$, and $E[N_{phys}^2] = \sum p_i w_i^2$, where $p_i$ is the probability of the $i$-th outcome and
\[ N_{\text{phys}} = w_i \] is the number of physical particles in that outcome. For the simple thinning we can write \( E[N_{\text{phys}}^2] = k^{-1}w_1^2 + (1 - k^{-1}) \cdot 0 = kw^2 \).

Finally, the variance is \( D[N_{\text{phys}}] = kw^2 - w^2 = (k - 1)w^2 \). Since the variance is additive, considering a cell with \( N \) macroparticles with factor \( w \), we obtain:

\[
D[N_{\text{phys}}] = N(k - 1)w^2.
\]

We are interested in macroparameters, such as particle density \( n = N_{\text{phys}} \Delta V^{-1} = Nw\Delta V^{-1} \), where \( \Delta V \) is the cell volume. We can calculate

\[
D[n] = D[N_{\text{phys}}]/(\Delta V)^2 = n(k - 1)/\Delta V = n^2(k - 1)/N \quad (6)
\]

Although this expression is the variance of \( n \) over different realizations of random events, the independence of such events in different cells allows us to use it to calculate the dispersion of \( n \) over coordinate space.

For our study, we performed a series of 2D simulations of the Weibel instability development in counter-streaming flows of quasineutral electron-ion plasma. We considered the following parameters: initial density \( n_0 = 10^{22} \text{ cm}^{-3} \), plasma flow velocity \( V_{\pm} = \pm 0.99995c \), which corresponds to a Lorentz-factor of \( \gamma_0 = 100 \), where \( c \) is the light velocity and the "+" and "-" signs denote the streams directed along and opposite to the \( x \) axis in our simulations. The initial density of ions in both streams was set to be uniform. The initial density of electrons and their local momentum were modulated harmonically across the transverse direction (\( y \) axis):

\[ n_{\pm} = \pm \delta n_0 \cos(k_y y), \quad p_{y,\pm} = \pm \delta m_e \omega_p V_0^2 (k\Gamma)^{-1} c^{-2} \sin(k_y y), \]

where \( \delta = 0.02 \) is the modulation amplitude, \( k_y = \frac{2\pi}{L} \), \( L = 2.5 \cdot 10^{-5} \text{ cm} \) is the spatial scale of the modulation, \( \omega_p = \sqrt{8\pi e^2 n_0/m_e} \) is the plasma frequency (of total density of both streams), \( m_e \) and \( e \) are the electron mass and charge. In the considered case of small-scale spatial modulation, the growth rate of the Weibel instability is \( \Gamma \approx \omega_p \gamma_0^{-1/2} V_0/c \). The modulation of electron density leads to the electromagnetic field variation of the following form:

\[
\vec{B} = 8\pi \delta \frac{en_0 V_0}{c} \cos(k_y y) \hat{z}_0, \\
\vec{E} = -8\pi \delta \frac{en_0 V_0}{c} \cos(k_y y) \hat{y}_0.
\]

The size of the simulation region was \( 2 \mu m \times 4 \mu m \) and the region was represented by \( 96 \times 384 \) cells. The boundary conditions were periodic. The time step was set to \( 1/(128\Gamma) \). The computation time was set to match the duration
Figure 6: $\Delta D(t)$ for different methods of resampling and $k = 50$. Group 1: simple (dark blue), leveling (green), global leveling (blue); Group 2: number thinning (red), energy thinning (cyan); Group 3: merge (purple), mergeAv (yellow).

of the linear regime during which the plasma density perturbation is negligible compared to the plasma density itself, which was the case until $t \approx 3.75/\Gamma$. The computation was performed for each method of resampling except conserv and each value of the resampling coefficient $k$ from the set (1.1;2;5;10;20;50), as well as for the case without resampling. In every computation the resampling procedure was applied once at $t = 1.25/\Gamma$. In order to identify the dispersion induced by the resampling procedure, for each method and each value of $k$ the difference $\Delta D(t) = D[n](t) - D_0[n](t)$ was calculated, where $D[n](t)$ is the time dependence of the variance of particle density $n$ in that calculation and $D_0[n](t) \sim e^{2\Gamma t}$ is the density variance in the simulation, which was entirely performed without resampling.

Let us first compare the results of different methods using equal values of $k = 50$, see Fig. 6. According to these results, the methods can be divided into 3 groups. Within each group the results are very similar. Specifically, for

\footnote{The methods merge and mergeAv could not complete for coefficients $k = 1.1$ and $k = 2$ due to computational time restrictions.}
methods in Group 1 (simple, leveling, Globally) $\Delta D(t)$ takes the form close to a step function. In other words, resampling causes a single leap in $\Delta D(t)$ at the time of resampling, after that the difference in dispersion compared to the case without resampling stays constant up until the end of the linear regime, despite the fact that the variance itself grows exponentially. This result confirms that in this case the influence of resampling on the variance can be considered independently of other (physical) processes affecting it.

For methods in Group 2 (numberT, energyT) the results are of the same order of magnitude and follow a similar trend, but these results show a notable periodic oscillation in $\Delta D(t)$, which is not negligible, but still of low amplitude compared to the value itself.

Group 3 methods (merge, mergeAv) show a considerably lower $\Delta D(t)$ (especially the merge method), but oscillations are of the same order of magnitude as the value itself.

Now let us compare the results of each method depending on the value of $k$. In order to assess that, we present values of $\Delta D(t)$ immediately after resampling. Since before resampling all computations in our series are identical, this value is exactly the leap in dispersion caused by resampling. We present this value depending on the value of the resampling coefficient $k$ for all resampling methods except conserv, as well as the estimate (6) for the simple thinning method, see Fig. 7.

As evident, Group 1 results follow the estimate (6): a linear dependence proportional to $k^{-1}$. The results in group 2 have a slightly lower slope: compared to Group 1 the variance increase is overall lower, with except of some range of low values of $k$. The merge method performed close to linear in the range of coefficients where we were able to complete it, and yields notably lower increase than all other methods: the slope is $\sim 1/4$ of the slope for Group 1 methods. The mergeAverage method stands out as highly nonlinear, yielding higher increase of variance at low coefficients, but for $k = 50$ the increase is lower than that of Group 1.

The lowest increase of variance is systematically provided by the method
merge. This can be related to the fact that this method provides the most advanced coordinated replacement of macroparticles giving, in some way, most efficient representation of the ensemble. Nevertheless, we should highlight that this is possible because we know the coordinate and momentum scales of the simulated processed and ensure that they are larger than that of the merge procedure. Apart from this, we must note that even for the highest coefficient $k = 50$ the computational time of $merge/mergeAv$ resampling exceeds that of other methods by at least a factor of 7. For $k = 10$ a single application of resampling doubles the total execution time of the whole simulation, and for $k = 5$ resampling takes up 80% of the computational time. Such characteristics are highly dependant on the particular problem at hand, of course, but it can be firmly said that merging methods may be highly demanding as compared to the other methods under similar accuracy requirements.

5. Comparison of methods on pertinent physical problems

A large class of tasks that require resampling methods is the study of cascaded production of electron-positron pairs and high-energy photons in laser
fields of high intensities. The description of these processes is based on the rates of transitions between quantum states computed within quantum electrodynamics (QED) and thus the cascades are commonly referred to as QED cascades. One of the most widely used numerical approaches for the simulation of QED cascades is based on extended PIC codes, also known as QED-PIC codes. With the development of an electrodynamic cascade, the number of particles can increase exponentially in time and significantly increase the computational demands. This means that some problems are impossible to compute without the use of resampling, so there is no benchmark to compare results to. In this case we have to rely entirely on results acquired using one or more methods of resampling.

A cascade can have two stages: linear and nonlinear. At the linear stage, the density of the generated electron-positron plasma is not sufficient to significantly affect the structure of the electromagnetic field and its intensity. At the nonlinear stage, on the contrary, the generated plasma has a higher density and significantly affects the electromagnetic field. Therefore, at the linear stage resampling can affect only the particle distribution function, affecting, potentially, the local rate of cascade development, or the particle distribution function. At the nonlinear stage resampling may also affect the structure and magnitude of the fields, changing the plasma-field dynamics. Below, using several examples, we consider various stages of the QED cascade and show how different types of resampling can affect the simulated processes. For simulations, we used the PICADOR code with the Adaptive Event Generator module described in [7], which, when necessary, subdivides the time step in order to resolve the QED cascade.

5.1. Linear cascade in a standing linearly-polarized plane wave

To investigate the operation of resampling methods at the linear stage of the cascade, we chose a well-studied problem of QED-cascade development in the field of a standing linearly polarized plane wave (see, for example, [33, 34, 35]). In order to have a benchmark for comparison - results of a simulation without
the use of resampling (‘w/o’) - we chose a relatively small wave amplitude $E_0 = 1000 \text{m} \omega_0 c$, where $\omega_0 = 2.35 \times 10^{15} \text{s}^{-1}$ is the laser frequency for the wavelength $\lambda_0 = 0.8 \mu\text{m}$. The wave has only $E_z$ and $B_y$ non-zero field components, which causes particles to move along the $x$ and $z$ axes. At this wave amplitude, electrons and positrons tend to the vicinity of the nodes of the electric field, but due to the stochasticity of photon emission and a photon decay into pairs, electrons and positrons can reach the vicinity of the electric field antinode [36].

Initially, electrons and positrons with the density of approximately $10^{17} \text{cm}^{-3}$ are uniformly distributed in a $\lambda_0 \times \lambda_0 \times \lambda_0$ simulation box represented by $128 \times 2 \times 2$ cells. The initial number of particles of each type in a cell was approximately 976. Initially, there are no photons, but they can be emitted by electrons and positrons. The boundary conditions for the fields and particles are periodic. The development of the cascade was considered during the time period of $7T$, where $T = 2.66 \text{ fs}$ is the wave period. The time step was $dt = 1.33 \times 10^{-17} \text{s}$.

The same parameters were used for all resampling methods: every second iteration, starting from zero, if the amount of macroparticles of any type exceeds the resampling threshold of $5 \times 10^5$ particles, the particles of that type undergo resampling with $k = 2$, and the amount of these macroparticles decreases by approximately half.

To analyze the effect of resampling on the accuracy of each simulation, we consider the temporal evolution of the total number of electrons and positrons $N_e$ (Fig.8a)). We used the following parameters for our analysis. The first one is the cascade growth rate $\Gamma = \frac{\ln N_e(t=7T)}{4T}$, the second one is the relative mean-square deviation of $N_e(t)$ acquired with resampling from $N_e(t) \text{w/o} - \text{acquired without resampling} - \eta = \sqrt{\frac{\int_0^{7T} (N_e,\text{res} - N_{e,\text{w/o}})^2 dt}{\int_0^{7T} N_{e,\text{w/o}}^2 dt}}$, where $\text{res}$ denotes a certain type of resampling. In order to calculate $\Gamma$, the moment $t = 3T$ was used as the initial one, since at $t = 0$ there are no photons and it takes about $2.5T$ for steady exponential growth to establish.

Note, that the principle of agnostic resampling ensures that the growth rate is not affected on average, but for this we need to know that the process is
Figure 8: Comparison of different resampling types via simulations of the QED cascade development in a standing linearly-polarized plane wave. (a) Time dependence of the total number of electrons and positrons $N_e$ in the simulation box. (b) The cascade growth rate $\Gamma$. (c) The relative mean square deviation $\eta$ of $N_e(t)$ acquired with use of resampling from $N_e(t)$ acquired without resampling. The colors in figures (b) and (c) correspond to the color of the lines in figure (a).

It should be noted that at a given wave amplitude, electrons and positrons emit plenty of photons, but only a small fraction of all photons decays into electron-positron pairs. Therefore, during the whole simulation resampling was initiated about 100 times for photons, but only once for electrons and positrons (Table 5.1). Thus, hefty macroparticles are added to the ensemble of electrons and positrons as the result of photon decay. The simulation without resampling shows that $\Gamma T = 0.445$ and the number of electrons and positrons increased by about a factor of 12 during the whole simulation (Fig 8(a)).

Based on the comparison of Fig. FCsGr (a), (b) and (c), all considered resampling methods can be divided into 4 groups. The first group includes the most accurate thinning method leveling with $\eta = 0.006$ and an error in the value of $\Gamma$ of about 0.3%. The second group also includes fairly accurate
thinning methods \textit{globalLev}, \textit{conserv} and \textit{energyT}, for which \( \eta \approx 0.018 \), and \( \Gamma \) can be determined with an accuracy of 0.3\%, 2.2\% and 3\%, respectively. The third group includes merge methods \textit{merge}, \textit{mergeAv} and thinning method \textit{numberT} with \( \eta \approx 0.08; 0.08; 0.1 \) and accuracy of \( \Gamma \) equal to 7.2\%, 7.9\% and 4\%, respectively. The fourth group corresponds to the most inaccurate thinning method 'simple'. This method within the performed simulations yields \( \eta = 0.5 \) and an error in estimation of \( \Gamma \) of 72\%, thus the accuracy of this method is unacceptably low.

Apart from the accuracy of simulations of physical processes, a computational speedup provided by different resampling methods must also be considered (Table 5.1). Without the use of resampling the simulation run time was approximately 22000 sec. The run time using thinning methods was approximately 1000 sec., most of which is owing to the resampling of photons. The only exception is the 'simple' method which contains only a single loop over the particle ensemble. In general, thinning methods speed up the simulation by about a factor of 20. At the same time, the run time using the \textit{merge} and \textit{mergeAv} methods is significantly higher - up to about 11000 seconds. This is primarily due to the use of the 'k-means' method for merging of particles, and also due to a more frequent triggering of resampling. The merge methods yield a speedup of about 2 times in comparison with the 'w/o' simulation, and they have the same or even worse accuracy than thinning methods (the method 'simple' being an exception). Thus, in this setup, thinning methods, providing at least equal accuracy (or much greater accuracy in some cases), noticeably benefit in performance in comparison with the merge methods. In addition, thinning methods trigger the resampling procedure less frequently. It is especially worth noting that the most accurate method for the considered problem is the \textit{leveling} method. It provides a fractions of a percent accuracy and a more than twentyfold simulation speedup.
Table 1: The influence of different types of resampling on run time and frequency of resampling.

| Type of resampling | Run Time, s | Resampling of Photons | Resampling of $e^-, e^+$ |
|--------------------|-------------|-----------------------|-------------------------|
| conserv            | 1101.7      | 215                   | 1                       |
| globalLev          | 1060.5      | 207                   | 1                       |
| energyT            | 915.4       | 99                    | 1                       |
| leveling           | 965.7       | 134                   | 1                       |
| merge.Av           | 11274.5     | 264                   | 1                       |
| merge              | 11367.1     | 265                   | 1                       |
| numberT            | 848.3       | 96                    | 1                       |
| simple             | 978.9       | 214                   | 1                       |
| w/o                | 21929.6     | 0                     | 0                       |

5.2. Nonlinear cascade in a linearly-polarized standing wave

In this problem we perform a full 3D3P simulation of irradiation of a seed plasma target by two colliding half-infinite linearly-polarized ultraintense laser pulses. As shown previously [37], in such a configuration the electromagnetic cascade [3, 4, 5, 6, 7, 8] can result in a rapid generation of plasma on the cascade’s linear stage followed by the formation of ultra-thin current sheets on the nonlinear stage. We perform the simulation for this problem using different methods of resampling.

It is important to note that since in this setup plasma density can increase significantly over a single half-period of the electromagnetic pulse, the setup is highly sensitive to the phase of the standing electromagnetic wave at the moment in time when the plasma density becomes sufficient to affect the field structure of the standing wave. Consequently, the probabilistic nature of electromagnetic cascading may lead to a discrepancy in results (in physical and numerical experiments alike) over different realizations of probabilistic physical events. We stress that a slight discrepancy in observed parameters in calcula-
tions with the same initial conditions may represent different realizations of a probabilistic physical process and it alone does not indicate that the resampling process leads to the distortion of the result.

The chosen initial parameters for this problem are: laser pulse wavelength $\lambda = 800$ nm, field amplitude $a_0 = 3500$ in relativistic units, width $D = 5\lambda$, initial seed plasma density at $10^{-3}$ of the critical plasma density $N_{cr}$.

Below we present the total electromagnetic field energy in the $7\lambda \times 8\lambda \times 8\lambda$ simulation box for different methods of resampling as a function of time since the start of the nonlinear stage, during which the current sheets begin to form. Agnostic methods were compared using the same default value $k = 2$. This value of $k$ is well outside the recommended range for merge-based methods dictated by computational time restrictions, so here merge-based methods were not considered.

As shown in Fig. 9, agnostic methods excluding simple yield qualitatively similar results varying within 5%, thus cross-confirming each other’s results. For the method simple we performed five attempts with different seeds of the random function. Nevertheless, all these attempts have been terminated at an early stage with a clearly unphysical surge in field energy.

Figure 9: Total Field Energy as a function of time for different methods of resampling
Deeper investigation of the performance of the method simple shows that although average density is conserved well (up to a point), maximal density is not (also see Sec. 5.3). Since each particle’s weight is probabilistically increased or zeroed independently, simple resampling with coefficient $k$ may result in the total weight in a certain cell increasing by a factor of $k$. This phenomena is a lot more likely to occur if much of that cell’s weight is carried by a single particle or several particles. If $k$ is large enough, or if the procedure is applied many times (our case), this can result in a significant increase of the total particle weight in some cells (effectively at the cost of other cells). Since on average properties are conserved, intuition may suggest this should not lead to physically incorrect results, apart from some increased numerical noise. However, it can be shown that this is not the case. The reason behind this lies in the discrete nature of the particle-in-cell code. To operate correctly, all physically relevant time scales have to be resolved by the PIC time step. The artificial increase of total weight (and thus physical density) in some cells leads to the increase of the local plasma oscillation frequency. If this frequency exceeds the frequency resolved by the PIC code’s time step, an unphysical instability develops. The current generated by a very hefty particle or cluster of particles creates an electric field that inverts the momentum of these particles in the next iteration. The process is additionally fed by new particles created in an electromagnetic cascade, which results in an unphysical exponential growth in particle density, energy and field values, resulting in the termination of the computation, often in as few as several iterations. Other methods either strictly conserve certain values or exclude particles with a large weight from the procedure, helping to avoid this problem.

To summarize, we see that the simple thinning may not be applicable if a local stochastic variation of density can cause numerical instability. All other agnostic methods show adequate results similar to each other in this setup.
5.3. Pinching of electron-positron plasma in a multi-10PW dipole wave

To study the effect of various resampling techniques on the dynamics of the particle ensemble in another pertinent problem, we have investigated the interaction of a multi-10PW-level laser radiation with plasma targets. Such a problem is of great interest due to the fact that this kind of laser systems will soon be ready to be used in experiments \[38, 39, 40\] and even more powerful systems are being developed \[41, 42\]. These experimental setups will allow the studies of QED cascade development in converging fields of petawatt power level, which is discussed in the previous section. Moreover, these laser systems will be capable of driving various self-consistent nonlinear regimes during the interaction of a QED-produced electron-positron plasma with ultraintense fields.

As shown earlier \[28\] and \[29\], in this configuration there are two main nonlinear regimes of interaction. In the first regime, which is realized when the laser power is less than 20 PW, thin electron-positron plasma sheets are formed as a result of the development of a current instability \[28\]. If the threshold power exceeds 20 PW, pinching of electron-positron plasma is possible as a result of current contraction \[29\]. In this paper in order to study the influence of resampling method we consider the interaction of an ideal dipole wave with total power \(P = 27\,PW\) with a low-density plasma target, which acts as a seed for the development of a QED cascade. The choice of this value of power can be justified by two factors. First, the development of a QED cascade in such a configuration was discussed in detail in \[29\], and, second, such a formulation allows studying the dynamics of the system at both (linear and nonlinear) stages of evolution.

In the numerical simulation, performed with the PICADOR Particle-in-Cell code, the interaction of an ideal 27 PW dipole wave with a plasma target with a 3 \(\mu\)m diameter and a density of \(10^{13}\,cm^{-3}\) was simulated. The size of the simulation area was 4 x 4 x 4 \(\mu\)m\(^3\) with a grid size of 512 x 512 x 512, which for a radiation wavelength of \(\lambda = 0.9\,\mu\)m corresponds to a resolution of 115 points per wavelength. The time step was 0.015 fs, which corresponds to 200 steps per period of laser radiation. This resolution is sufficient to resolve the
dynamics of the electron-positron plasma. The processes of photon production and their decay into an electron-positron plasma were modeled with steps split accordingly to resolve the corresponding time scales. In this section we employed the previously discussed methods to resample the particle ensemble using either thinning or merging techniques. All methods except mergeAv were compared among each other. The resampling coefficient \(k = 2\) was used in all schemes.

In papers [28] and [29], the globalLev method was used. For completeness of presentation, this method is also compared with other methods discussed in this paper.

During the interaction the system evolves through several stages thoroughly discussed in related papers. First, the target is compressed towards the center of symmetry, which can be seen as the peak in maximum density at 7–8 T in Fig. 10(a). The next stage, if initial target density is low enough, is a linear QED cascade in a given field. During this stage the maximum pair density and total number of particles increase exponentially. This stage is marked in blue color in Fig. 10. When pair plasma density becomes comparable with the critical density, the transition to the nonlinear regime occurs, which manifests itself in the form of saturation of dependence of the total particle number and plasma density on time in Fig. 10. The origin of such behaviour is high absorption in overdense electron-positron pair plasma which leads to a significant drop of the field amplitude. This region is marked in red in Fig. 10.

As was noted previously, such a formulation of the problem allows us to study both linear and nonlinear stages of the QED cascade. It should be expected that the choice of the resampling algorithm should have the most significant impact on the dynamics of the system in the linear regime, since upon transition to the nonlinear stage the growth rate significantly decreases and resampling of the ensemble of particles occurs less frequently.

According to the results of this section, the resampling methods can be divided into several groups, results are summarized in Table 2.

The first group constitutes of the simple method. As discussed previously, disadvantages of this method include the formation of macro-particles with large
Figure 10: Temporal evolution of QED cascade in the field of a converging 27 PW dipole wave. (a) The maximum electron-positron plasma density during the development of the QED cascade. (b) The total number of electrons (positrons) in the cylinder with a diameter and height equal to $\lambda$. Methods with similar results are grouped accordingly.

Figure 11: Spatial distribution of electron-positron plasma density at $t = 14T$ for different resampling methods: (a, e) simple; (b, f) leveling, globalLeveling, conservative; (c, g) merge, energyThinning; (d, h) numberThinning. The density of electron-positron plasma is plotted to a logarithmic scale.

weights. Figure 10(a) shows the maximum density of the electron-positron plasma during the development of the QED cascade. It is clear that for this method the maximum value of pair plasma density exceeds the value obtained for all other methods and is exceptionally noisy. At the same time, the total number of particles is almost identical to other methods (see Fig. 10(b)) which indicates that the problem is the formation of super-particles. It can also be seen that the presence of huge macroparticles leads to an earlier transition to the
Table 2: Comparison of resampling methods for the problem of pinching of electron-positron plasma in a 27 PW dipole wave. Run time, amount of resampling instances, and the average number of particles for the time interval of 100 iterations at linear and nonlinear stages in domain are given for the central process. Simulation of the nonlinear regime for method simple failed due to a numerical instability.

| Type          | Linear regime | Nonlinear regime |
|---------------|---------------|------------------|
|               | Time, sec     | # of resampling  | # of particles ×10^6 | Time, sec | # of resampling | # of particles ×10^6 |
| simple        | 3.07          | 2373             | 31                  | 5.1       | -              | -                  |
| globalLev     | 3.38          | 1771             | 28                  | 3.11      | 734            | 6                  | 3.07                |
| leveling      | 3.38          | 1956             | 23                  | 3.6       | 869            | 7                  | 2.96                |
| conserve      | 3.38          | 3744             | 65                  | 4.45      | 1965           | 23                 | 3.22                |
| merge         | 3.19          | 3219             | 36                  | 4.73      | 23983          | 23                 | 4.3                 |
| energyT       | 3.21          | 1874             | 25                  | 5.1       | 830            | 9                  | 3.17                |
| numberT       | 3.27          | 2181             | 6                   | 4.2       | 1100           | 7                  | 2.6                 |

nonlinear regime. Presence of super-particles and the consequent development of a numerical instability results in the termination of the numerical simulation when the simple method is used at the nonlinear stage. Also, the cascade growth rate is underestimated by about 10%. The difference is not as high as for the slow linear cascade in Sec 5.1, but it may still lead to incorrect results. Spatial distributions of pair plasma at the intermediate stage in comparison with other methods are shown in Fig. [11] (a, e), the formation of a large number of dense super-particles is apparent.

The second group of methods includes the globalLev, conserve, and leveling. These methods behave almost identically to each other, both at the linear and nonlinear stages of development of the QED cascade. The dependencies of the total number of particles are shown in Fig. [10] (b). The growth rate yielded by these methods differs by about 10^{-3}. The plasma distribution in this case is
symmetric with respect to the azimuthal angle, as can be seen from the Fig. 11 (b, f). In this problem due to a high growth rate, unlike the problem in Sec 5.1, it is impossible to complete the simulation without resampling. However, taking into account the conclusions made in that section regarding the direct comparison of the growth rate with and without resampling, we can assume that this group of methods gives the best possible accuracy.

The third group includes the methods merge and energyT. These methods underestimate the growth rate of the cascade at the linear stage by about 3%, and the number of particles over time within this group is also identical. At the same time, surprisingly, these methods lead to the appearance of spatial modulation of plasma density, which is absent in the methods considered earlier (see Fig. 11). At the nonlinear stage, these modulations are amplified and start to significantly affect the dynamics of the system. The nature of these modulations requires a separate investigation. It should also be noted that the characteristic simulation time of the merge method in this task is almost an order of magnitude higher than the simulation time of all other methods, see Table 2. This, combined with other factors, makes it less convenient for practical use.

The last group of methods consists of the numberT method, which can be considered somewhat intermediate. The plasma modulation is present, but it is not as pronounced, and the growth rate is underestimated less than for the third group (see Fig. 11 (c, g)).

6. Conclusion

The principle of agnostic down-sampling that is applicable without any prior knowledge about the problem was formulated and several resampling methods complying with the agnostic principle were presented. Results acquired with use of these methods were compared among each other, and also to theoretical results, results acquired without resampling (where possible) and to the results acquired by some non-agnostic methods. The comparison was performed
first using simple model problems, and then using pertinent problems involving generation of plasma via QED cascade and thus often requiring extensive resampling.

It was shown that the relative accuracy of various methods highly depends on the problem at hand and the criteria for determining this accuracy. Therefore we conclude that there is no universal method of resampling which would show the best performance in all cases. However, it can be noted that several methods provide stable performance on all problems that we have considered. These methods are leveling, globalLev, conserv, and to a lesser extent energyT and numberT. The methods simple and merge/mergeAv have at least one example where the method in question significantly alters the physical outcome even though in certain other conditions these methods might be the most advantageous. Most of the considered methods are released as open-source tools within the hi-χ framework [30].

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