Some Connections Between (Sub)Critical Branching Mechanisms and Bernstein Functions

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Abstract

We describe some connections, via composition, between two functional spaces: the space of (sub)critical branching mechanisms and the space of Bernstein functions. The functions $e_\alpha : x \mapsto x^\alpha$ where $x \geq 0$ and $0 < \alpha \leq 1/2$, and in particular the critical parameter $\alpha = 1/2$, play a distinguished role.

1 Introduction

This note is a prolongation of [8] where the following remarkable property of the function $e_\alpha : x \mapsto x^\alpha$ was pointed at for $\alpha = 1/2$: if $\Psi$ is a (sub)critical branching mechanism, then $\Psi \circ e_{1/2}$ is a Bernstein function (see the next section for the definition of these notions). In the present work, we first show that this property extends to every $\alpha \in ]0, 1/2]$. Then we characterize the class of so-called internal functions, i.e. that of Bernstein functions $\Phi$ such that the compound function $\Psi \circ \Phi$ is again a Bernstein function for every (sub)critical branching mechanism $\Psi$. In the final section, we gather classical results on transformations of completely monotone functions, Bernstein functions and (sub)critical branching mechanisms which are used in our analysis.

2 Some functional spaces

2.1 Completely monotone functions

For every Radon measure $\mu$ on $[0, \infty[$, we associate the function $L_\mu : [0, \infty[ \to [0, \infty]$ defined by

$$L_\mu(q) := \int_{[0, \infty[} e^{-qx} \mu(dx),$$

i.e. $L_\mu$ is the Laplace transform of $\mu$. We denote by

$$CM := \{L_\mu : L_\mu(q) < \infty \text{ for all } q > 0\},$$

where $CM$ is the space of completely monotone functions.

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which is an algebraic convex cone (i.e. a convex cone which is further stable under inner product). The celebrated theorem of Bernstein (see for instance Theorem 3.8.13 in [6]) identifies \(\text{CM}\) with the space of completely monotone functions, i.e. functions \(f : [0, \infty[ \rightarrow [0, \infty[\) of class \(C^\infty\) such that for every integer \(n \geq 1\), the \(n\)-th derivative \(f^{(n)}\) of \(f\) has the same sign as \((-1)^n\).

Recall from monotone convergence that \(\mathcal{L}_\mu\) has a (possibly infinite) limit at \(0^+\) which coincides with the total mass of \(\mu\).

We shall focus on two natural sub-cones of \(\text{CM}\):

\[
B_1 := \left\{ \mathcal{L}_\mu : \int_{[0,\infty[} (1 \wedge x^{-1}) \mu(dx) < \infty \right\} \quad (3)
\]

We further denote by \(B_1^+\) the sub-space of functions in \(B_1\) which are the Laplace transforms of absolutely continuous measures with a decreasing density:

\[
B_1^+ := \left\{ \mathcal{L}_\mu : \mu(dx) = g(x)dx, g \text{ decreasing and } \int_0^\infty (1 \wedge x^{-1}) g(x)dx < \infty \right\}. \quad (4)
\]

Note that the density \(g\) then has limit 0 at infinity.

### 2.2 Bernstein functions

For every triple \((a, b, \Lambda)\) with \(a, b \geq 0\) and \(\Lambda\) a positive measure on \([0, \infty[\) such that

\[
\int_{[0,\infty[} (x \wedge 1) \Lambda(dx) < \infty, \quad (5)
\]

we associate the function \(\Phi_{a,b,\Lambda} : [0, \infty[ \rightarrow [0, \infty[\) defined by

\[
\Phi_{a,b,\Lambda}(q) := a + bq + \int_{[0,\infty[} (1 - e^{-qx}) \Lambda(dx), \quad (6)
\]

and call \(\Phi_{a,b,\Lambda}\) the Bernstein function with characteristics \((a, b, \Lambda)\). We denote the convex cone of Bernstein functions by

\[
B_2 := \{ \Phi_{a,b,\Lambda} : a, b \geq 0 \text{ and } \Lambda \text{ positive measure fulfilling (5)} \}. \quad (7)
\]

It is well-known that \(B_2\) can be identified with the space of real-valued \(C^\infty\) functions \(f : [0, \infty[ \rightarrow [0, \infty[\) such that for every integer \(n \geq 1\), the \(n\)-th derivative \(f^{(n)}\) of \(f\) has the same sign as \((-1)^{n-1}\). See Definition 3.9.1 and Theorem 3.9.4 in [6].

Bernstein functions appear as Laplace exponents of subordinators, see e.g. Chapter 1 in [3], Chapter 6 in [9], or Section 3.9 in [6]. This means that \(\Phi \in B_2\) if and only if there exists an increasing process \(\sigma = (\sigma_t, t \geq 0)\) with values in \([0, \infty[\) (\(\infty\) serves as absorbing state) with independent and stationary increments as long as \(\sigma_t < \infty\), such that for every \(t \geq 0\)

\[
\mathbb{E}(\exp(-q\sigma_t)) = \exp(-t\Phi(q)), \quad q > 0.
\]

In this setting, \(a\) is known as the killing rate, \(b\) as the drift coefficient, and \(\Lambda\) as the Lévy measure.

We shall further denote by \(B_2^+\) the subspace of Bernstein functions for which the Lévy measure is absolutely continuous with a monotone decreasing density, viz.

\[
B_2^+ := \{ \Phi_{a,b,\Lambda} : a, b \geq 0 \text{ and } \Lambda(dx) = g(x)dx, g \geq 0 \text{ decreasing and } \int_0^\infty (x \wedge 1) g(x)dx < \infty \}. \]

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2.3 (Sub)critical branching mechanisms

For every triple \((a, b, \Pi)\) with \(a, b \geq 0\) and \(\Pi\) positive measure on \([0, \infty[\) such that
\[
\int_{[0,\infty]} (x \wedge x^2)\Pi(dx) < \infty
\]
we associate the function \(\Psi_{a,b,\Pi} : [0, \infty[ \to [0, \infty[\) defined by
\[
\Psi_{a,b,\Pi}(q) := aq + bq^2 + \int_{[0,\infty[} (e^{-qx} - 1 + qx)\Pi(dx),
\]
and denote the convex cone of such functions by
\[
B_3 := \{\Psi_{a,b,\Pi} : a, b \geq 0 \text{ and } \Pi \text{ a positive measure such that (8) holds}\}
\]
Functions in \(B_3\) are convex increasing functions of class \(C^\infty\) that vanish at 0; they coincide with the class of branching mechanisms for (sub)critical continuous state branching processes, where (sub)critical means critical or sub-critical. See Le Gall [7] on page 132.

Alternatively, functions in the space \(B_3\) can also be viewed as Laplace exponents of Lévy processes with no positive jumps that do not drift to \(-\infty\) (or, equivalently, with nonnegative mean). In this setting, \(a\) is the drift coefficient, \(2b\) the Gaussian coefficient, and \(\Pi\) the image of the Lévy measure by the map \(x \to -x\). See e.g. Chapter VII in [2].

3 Composition with \(e_\alpha\)

Stable subordinators correspond to a remarkable one-parameter family of Bernstein functions denoted here by \((e_\alpha, 0 < \alpha < 1)\), where
\[
e_\alpha(q) := q^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-qx})x^{-1-\alpha}dx, \quad q > 0.
\]

Theorem 1 The following assertions are equivalent:
(i) \(\alpha \in ]0, 1/2]\).
(ii) For every \(\Psi \in B_3\), \(\Psi \circ e_\alpha \in B_2\).

The implication (ii) \(\Rightarrow\) (i) is immediate. Indeed, \(\Psi_{0,1,0} : q \to q^2\) belongs to \(B_3\), but \(e_{2\alpha} = \Psi_{0,1,0} \circ e_\alpha\) is in \(B_2\) if and only if \(2\alpha \leq 1\). However, the converse (i) \(\Rightarrow\) (ii) is not straightforward and relies on the following technical lemma, which appears as Lemma VI.1.2 in [8]. Here, for the sake of completeness, we provide a proof.

Lemma 2 For \(\alpha \in ]0, 1/2]\), let \(\sigma^{(\alpha)} = (\sigma_x^{(\alpha)}, x \geq 0)\) be a stable subordinator with index \(\alpha\) with Laplace transform
\[
\mathbb{E} \left( \exp \left( -q \sigma_x^{(\alpha)} \right) \right) = \exp(-xq^\alpha), \quad x, q > 0.
\]
Denote by \(p^{(\alpha)}(x, t)\) the density of the law of \(\sigma_x^{(\alpha)}\). Then for every \(x, t > 0\), we have
\[
p^{(\alpha)}(x, t) \leq \frac{\alpha}{\Gamma(1-\alpha)} xt^{-(1+\alpha)}.
\]
Remark : The bound in Lemma 2 is sharp, as it is well-known that for any $0 < \alpha < 1$ and each fixed $t > 0$

$$p^{(\alpha)}(x, t) \sim \alpha \frac{x}{\Gamma(1-\alpha) t^{1+\alpha}}, \quad x \to \infty.$$ 

More precisely, there is a series representation of $p^{(\alpha)}(x, t)$, see Formula (2.4.7) on page 90 in Zolotarev [10]:

$$p^{(\alpha)}(x, 1) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \sin(\pi n \alpha) x^{-n\alpha-1}.$$ 

Using the identity

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha \pi)},$$

this agrees of course with the above estimate. It is interesting to note that the second leading term in the expansion,

$$-\frac{\Gamma(2\alpha+1)}{2\pi} \sin(2\pi \alpha) x^{-2\alpha-1},$$

is negative for $\alpha < 1/2$, but positive for $\alpha > 1/2$. So the bound in Lemma 2 would fail for $\alpha > 1/2$.

Proof: In the case $\alpha = 1/2$, there is an explicit expression for the density

$$p^{(1/2)}(x, t) = \frac{x}{2\sqrt{\pi} t^{1/2}} \exp \left(-\frac{x^2}{4t}\right),$$

from which the claim is obvious (recall that $\Gamma(1/2) = \sqrt{\pi}$).

In the case $\alpha < 1/2$, we start from the identity

$$\exp(-x q^\alpha) = \int_0^\infty e^{-qt} p^{(\alpha)}(x, t) dt,$$

and take the derivative in the variable $q$ to get

$$\alpha q^{\alpha-1} \exp(-x q^\alpha) = \int_0^\infty e^{-qt} \left( -\frac{x}{t} \right) p^{(\alpha)}(x, t) dt,$$

and then

$$\alpha q^{\alpha-1} (1 - \exp(-x q^\alpha)) = \int_0^\infty e^{-qt} \left( \frac{\alpha}{\Gamma(1-\alpha)} t^{-\alpha} - \frac{t}{x} \right) p^{(\alpha)}(x, t) dt.$$ 

Denote the left hand-side by $g(x, q)$, and take the derivative in the variable $x$. We obtain

$$\frac{\partial g(x, q)}{\partial x} = \alpha q^{2\alpha-1} e^{-x q^\alpha} = \alpha q^{2\alpha-1} \int_0^\infty e^{-qt} p^{(\alpha)}(x, t) dt.$$ 

On the other hand, since $1 - 2\alpha > 0$,

$$q^{2\alpha-1} = \frac{1}{\Gamma(1-2\alpha)} \int_0^\infty e^{-qs} s^{-2\alpha} ds,$$

and hence

$$\frac{\partial g(x, q)}{\partial x} = \frac{\alpha}{\Gamma(1-2\alpha)} \int_0^\infty ds \int_0^\infty dte^{-q(s+t)} p^{(\alpha)}(x, t).$$
The change of variables \( u = t + s \) yields
\[
\frac{\partial g(x, q)}{\partial x} = \frac{\alpha}{\Gamma(1 - 2\alpha)} \int_0^\infty du \int_0^u ds \frac{e^{-qu}}{s^{2\alpha}} p^{(\alpha)}(x, u - s); \]
and since \( g(0, t) = 0 \), we finally obtain the identity
\[
\int_0^\infty e^{-qt} \left( \frac{\alpha}{\Gamma(1 - \alpha)} t^{-\alpha} - \frac{t}{x} p^{(\alpha)}(x, t) \right) dt = \frac{\alpha}{\Gamma(1 - 2\alpha)} \int_0^x dy \int_0^u ds \frac{e^{-qu}}{s^{2\alpha}} p^{(\alpha)}(x, u - s).
\]
Inverting the Laplace transform, we conclude that
\[
\frac{\alpha}{\Gamma(1 - \alpha)} t^{-\alpha} - \frac{t}{x} p^{(\alpha)}(x, t) = \frac{\alpha}{\Gamma(1 - 2\alpha)} \int_0^x dy \int_0^t ds \frac{e^{-qu}}{s^{2\alpha}} p^{(\alpha)}(x, t - s),
\]
which entails our claim.

We are now able to prove Theorem 1.

**Proof:** Let \( \Psi_{a, \beta, \Pi} \in B_3 \). Since both \( \alpha e_\alpha \) and \( \beta e_{2\alpha} \) are Bernstein functions, there is no loss of generality in assuming that \( a = b = 0 \). Set for \( t > 0 \)
\[
\nu_\alpha(t) := \frac{\alpha}{\Gamma(1 - \alpha)t^{1+\alpha}} \int_0^\infty \Pi(dx)x \left( 1 - \frac{\Gamma(1 - \alpha)t^{1+\alpha}}{\alpha x} p^{(\alpha)}(x, t) \right).
\]
It follows from Lemma 2 that \( \nu_\alpha(t) \geq 0 \). We have for every \( q > 0 \)
\[
\int_0^\infty (1 - e^{-qt})\nu_\alpha(t) dt = \int_0^\infty \Pi(dx)x \int_0^\infty dt \left( \frac{\alpha(1 - e^{-qt})}{\Gamma(1 - \alpha)t^{1+\alpha}} - \frac{p^{(\alpha)}(x, t)}{x} + e^{-qt} \frac{p^{(\alpha)}(x, t)}{x} \right)
\]
\[
= \int_0^\infty \Pi(dx)x \left( q^\alpha - \frac{1}{x} + \frac{e^{-qx}}{x} \right)
\]
\[
= \Psi_{0,0,\Pi}(\alpha_\alpha(q)).
\]
As this quantity is finite for every \( q > 0 \), this shows that \( \Psi_{0,0,\Pi} \circ e_\alpha \in B_2 \).

**Remark:** The proof gives a stronger result than that stated in Theorem 1. Indeed, we specified the Lévy measure \( \nu_\alpha \) of \( \Psi_{0,0,\Pi} \circ e_\alpha \). Furthermore, in the case \( \alpha = 1/2 \), this expression shows that \( \Psi_{0,0,\Pi} \circ e_{1/2} \in B_2^\downarrow \). It is interesting to combine this observation with the forthcoming Proposition 7: for every \( \Psi \in B_3 \), \( \Psi \circ e_{1/2} \in B_2^\downarrow \), thus \( \text{Id} \times (\Psi \circ e_{1/2}) : q \rightarrow q\Phi(\sqrt{q}) \) is again in \( B_3 \), and in turn \( e_{1/2} \times (\Psi \circ e_{1/4}) \in B_2^\downarrow \). More generally, we have by iteration that for every integer \( n \)
\[
e_{2^{-2n}} \times (\Psi \circ e_{2^{-n}}) \in B_3,
\]
and
\[
e_{1-2^{-n}} \times (\Psi \circ e_{2^{-n-1}}) \in B_2^\downarrow.
\]
4 Internal functions

It is well-known that the cone \(\text{CM}\) of completely monotone functions and the cone \(B_2\) of Bernstein functions are both stable by right composition with a Bernstein function; see Proposition 8 below. Theorem 1 incites us to consider also compositions of (sub)critical branching mechanisms and Bernstein functions; we make the following definition:

**Definition 3** A Bernstein function \(\Phi \in B_2\) is said internal if \(\Psi \circ \Phi \in B_2\) for every \(\Psi \in B_3\).

Theorem 1 shows that the functions \(e_{\alpha}\) are internal if and only if \(\alpha \in \left[0, \frac{1}{2}\right]\). The critical parameter \(\alpha = \frac{1}{2}\) plays a distinguished role. Indeed, we could also prove Theorem 1 using the following alternative route. First, we check that \(e_{1/2}\) is internal (see [8]), and then we deduce by subordination that for every \(\alpha < \frac{1}{2}\) that \(\Psi \circ e_{\alpha} = \Psi \circ e_{1/2} \circ e_{2\alpha}\) is again a Bernstein function for every \(\Psi \in B_3\). Developing this argument, we easily arrive at the following characterization of internal functions:

**Theorem 4** Let \(\Phi = \Phi_{a,b,\Lambda} \in B_2\) be a Bernstein function. The following assertions are then equivalent:

(i) \(\Phi\) is internal,

(ii) \(\Phi^2 \in B_2\),

(iii) \(b = 0\) and there exists a subordinator \(\sigma = (\sigma_t, t \geq 0)\) such that

\[
\Lambda(dx) = c \int_0^\infty t^{-3/2} \mathbb{P}(\sigma_t \in dx) dt.
\]

**Proof:** (i) \(\Rightarrow\) (ii) is obvious as \(\Psi_{0,1,0} \circ \Phi = \Phi^2\).

(ii) \(\Rightarrow\) (i). We know from Theorem 1 or [8] that for every \(\Psi \in B_3\), \(\Psi \circ e_{1/2} \in B_2\). It follows by subordination that for every Bernstein function \(\kappa \in B_2\), \(\Psi \circ e_{1/2} \circ \kappa \in B_2\). Take \(\kappa = \Phi^2\), so \(e_{1/2} \circ \kappa = \Phi\), and hence \(\Phi\) is internal.

(iii) \(\Rightarrow\) (ii) Let \(\kappa\) denote the Bernstein function of \(\sigma\). We have

\[
\Phi(q) = a + \int_{[0,\infty]} (1 - e^{-qx}) \Lambda(dx)
\]

\[
= a + c \int_{[0,\infty]} \int_0^\infty dt (1 - e^{-qt}) t^{-3/2} \mathbb{P}(\sigma_t \in dx)
\]

\[
= a + c \int_0^\infty dt (1 - e^{-t\kappa(q)}) t^{-3/2}.
\]

The change of variables \(tk(q) = u\) yields

\[
\Phi(q) = a + c' \sqrt{\kappa(q)}
\]

and hence

\[
\Phi^2(q) = a^2 + 2ac' \sqrt{\kappa(q)} + c'^2 \kappa(q).
\]
Since $\kappa^{1/2} = e_{1/2} \circ \kappa$ is again a Bernstein function, we thus see that $\Phi^2 \in B_2$.

(ii) $\Rightarrow$ (iii) Recall that the drift coefficient $b$ of $\Phi_{a,b,\Lambda}$ is given by
\[
\lim_{q \to \infty} \Phi_{a,b,\Lambda}(q)/q = b;
\]
see e.g. page 7 in [3]. It follows immediately that $b = 0$ whenever $\kappa := \Phi_{a,b,\Lambda}^2 \in B_2$. Recall from Sato [9] on page 197-8 that if $\tau^{(1)}$ and $\tau^{(2)}$ are two independent subordinators with respective Bernstein functions $\Phi^{(1)}$ and $\Phi^{(2)}$, then the compound process $\tau^{(1)} \circ \tau^{(2)} := \tau^{(3)}$ is again a subordinator with Bernstein function $\Phi^{(3)} := \Phi^{(2)} \circ \Phi^{(1)}$; moreover its Lévy measure $\Lambda^{(3)}$ is given by
\[
\Lambda^{(3)}(dx) = \int_0^\infty \mathbb{P}(\tau^{(1)}_t \in dx)\Lambda^{(2)}(dt),
\]
where $\Lambda^{(2)}$ denotes the Lévy measure of $\tau^{(2)}$. As $\Phi_{a,b,\Lambda} = e_{1/2} \circ \kappa$, and the Lévy measure of $e_{1/2}$ is $ct^{-3/2}dt$ with $c = 1/(2\sqrt{\pi})$, we deduce that
\[
\Lambda(dx) = c \int_0^\infty \mathbb{P}(\sigma_t \in dx)t^{-3/2}dt.
\]

The proof of Theorem 4 is now complete.

It is noteworthy that if $\Phi_{a,b,\Lambda}$ is internal and $\Lambda \neq 0$, then
\[
\int_{0,\infty} x\Lambda(dx) = \infty.
\]
Indeed,
\[
\int_{0,\infty} x\Lambda(dx) = c \int_0^\infty \int_{0,\infty} x\mathbb{P}(\sigma_t \in dx)t^{-3/2}dt = c \int_0^\infty \mathbb{E}(\sigma_1)t^{-1/2}dt = \infty.
\]
For instance, the Bernstein function $q \to \log(1 + q)$ of the gamma subordinator is not internal.

**Corollary 5** For every $\Psi \in B_3$, we rewrite $\Phi$ for the inverse function of $\Psi$ and then $\Phi'$ for its derivative. Then $1/\Phi'$ is internal.

**Proof:** It is known (see Corollary 10 below) that $1/\Phi'$ is a Bernstein function; let us check that its square is also a Bernstein function.

We know that $\Psi'' \in B_1$ (Proposition 6 below) and $\Phi \in B_2$ (Proposition 9 below); we deduce from Proposition 8 that $\Psi'' \circ \Phi \in B_1$. If we write $I(f) : x \to \int_0^x f(y)dy$ for every locally integrable function $f$, then again by Proposition 6, we get that $I(\Psi'' \circ \Phi)$ is a Bernstein function.

Now
\[
\Psi'' = -\frac{\Phi'' \circ \Psi}{(\Phi' \circ \Psi)^{3}},
\]
so
\[
\Psi'' \circ \Phi = -\frac{\Phi''}{(\Phi')^3},
\]
and we conclude that
\[
\frac{1}{2(\Phi')^2} = I(\Psi'' \circ \Phi) \in B_2.
\]
5 Some classical results and their consequences

For convenience, this section gathers some classical transformations involving $B_j, j \in \{1, 2, 3\}$ and related subspaces, which have been used in the preceding section. We start by considering derivatives and indefinite integrals. The following statement is immediate.

**Proposition 6** Let $j = 2, 3$ and $f : ]0, \infty[ \to ]0, \infty[$ be a $C^\infty$-function with derivative $f'$. For $j = 3$, we further suppose that $\lim_{q \to 0} f(q) = 0$. There is the equivalence

$$f \in B_j \iff f' \in B_{j-1}.$$

The next statement is easily checked using integration by parts.

**Proposition 7** Let $j = 2, 3$ and consider two functions $f, g : ]0, \infty[ \to ]0, \infty[$ which are related by the identity $f(q) = qg(q)$. Then there is the equivalence

$$f \in B_j \land \lim_{q \to 0} f(q) = 0 \iff g \in B_{j-1}^\perp.$$

Proposition 7 has well-known probabilistic interpretations. First, let $\sigma$ be a subordinator with Bernstein function $f \in B_2$ with unit mean, viz. $E(\sigma_1) = 1$, which is equivalent to $f'(0+) = 1$. Then the completely monotone function $g(q) := f(q)/q$ is the Laplace transform of a probability measure on $\mathbb{R}_+$. The latter appears in the renewal theorem for subordinators (see e.g. [4]); in particular it describes the weak limit of the so-called age process $A(t) = t - g_t$ as $t \to \infty$, where $g_t := \sup \{\sigma_s : \sigma_s < t\}$. Second, let $X$ be a Lévy process with no positive jumps and Laplace exponent $f \in B_3$. The Lévy process reflected at its infimum, $X_t - \inf_{0 \leq s \leq t} X_s$, is Markovian; and if $\tau$ denotes its inverse local time at 0, then $\sigma = -X \circ \tau$ is a subordinator called the descending ladder-height process. The Bernstein function of the latter is then given by $g(q) = f(q)/q$; see e.g. Theorem VII.4(ii) in [2].

We next turn our attention to composition of functions; here are some classical properties

**Proposition 8** Consider two functions $f, g : ]0, \infty[ \to ]0, \infty[$. Then we have the implications

$$f, g \in B_2 \implies f \circ g \in B_2,$$

$$f \in \text{CM} \land g \in B_2 \implies f \circ g \in \text{CM},$$

$$f \in B_1 \land g \in B_2 \implies f \circ g \in B_1.$$

The first statement in Proposition 8 is related to the celebrated subordination of Bochner (see, e.g. Section 3.9 in [6] or Chapter 6 in [9]); more precisely if $\sigma$ and $\tau$ are two independent subordinators with respective Bernstein functions $f_\sigma$ and $f_\tau$, then $\sigma \circ \tau$ is again a subordinator whose Bernstein function is $f_\tau \circ f_\sigma$. The second statement is a classical result which can be found as Criterion 2 on page 441 in Feller [5]; it is also related to Bochner’s subordination.

Finally we turn our attention to inverses.
Proposition 9  Consider a function \( f : [0, \infty[ \rightarrow [0, \infty[ \). Then
\[
f \in B_2 \cup B_3 \implies 1/f \in CM.
\]

Further, if \( f^{-1} \) denotes the inverse of \( f \) when the latter is a bijection, then
\[
f \in B_3, f \not\equiv 0 \implies f^{-1} \in B_2.
\]

We mention that if \( f \in B_3 \), the completely monotone function \( 1/f \) is the Laplace transform of the so-called scale function of the Lévy process \( X \) with no positive jumps which has Laplace exponent \( f \). See Theorem VII.8 in [2]. On the other hand, \( f^{-1} \) is the Bernstein function of the subordinator of first-passage times \( T_t := \inf \{s \geq 0 : X_s > t\} \); see e.g. Theorem VII.1 in [2]. Finally, in the case when \( f \in B_2 \) is a Bernstein function, the completely monotone function \( 1/f \) is the Laplace transform of the renewal measure \( U(dx) = \int_0^\infty P(\sigma_t \in dx)dt \), where \( \sigma \) is a subordinator with Bernstein function \( f \).

Corollary 10  Let \( \Psi \not\equiv 0 \) be a function in \( B_3 \), and denote by \( \Phi = \Psi^{-1} \in B_2 \) its inverse bijection. Then \( q \mapsto 1/\Phi'(q) \) and \( \Id/\Phi : q \mapsto q/\Phi(q) \) are Bernstein functions. Furthermore \( 1/(\Phi(\Phi'(q))) \) is completely monotone.

Proof:  We know from Propositions 6 and 9 that both \( \Phi \) and \( \Psi' \) are Bernstein functions. We conclude from Proposition 8 that \( 1/\Phi' = \Psi' \circ \Phi \) is again in \( B_2 \).

Similarly, we know from Proposition 7 that \( q \mapsto \Psi(q)/q \) is a Bernstein function, and composition on the right by the Bernstein function \( \Phi \) yields \( \Id/\Phi \) that is again in \( B_2 \).

Finally, we can write \( 1/(\Phi \Phi') = f \circ \Phi \) where \( f(q) = \Psi'(q)/q \). We know from Proposition 6 that \( \Psi' \in B_2 \), so \( f \in CM \) by Proposition 7. Since \( \Phi \in B_2 \), we conclude from Proposition 8 that \( f \circ \Phi \in CM \).

If \( \Phi = \Psi^{-1} \) is the Bernstein function given by the inverse of a function \( \Psi \in B_3 \), the Bernstein function \( 1/\Phi' \) is the exponent of the subordinator \( L^{-1} \) defined as the inverse of the local time at 0 of the Lévy process with no positive jumps and Laplace exponent \( \Psi \). See e.g. Exercise VII.2 in [2]. On the other hand, \( \Id/\Phi \) is then the Bernstein function of the decreasing ladder times, see Theorem VII.4(ii) in [2]. The interested reader is also referred to [1] for further factorizations for Bernstein functions which arise naturally for Lévy processes with no positive jumps, and their probabilistic interpretations.

Next, recall that a function \( f : [0, \infty[ \rightarrow \mathbb{R}_+ \) is called a Stieltjes transform if it can be expressed in the form
\[
f(q) = b + \int_{[0,\infty[} \frac{\nu(dt)}{t + q}, \quad q > 0,
\]
where \( b \geq 0 \) and \( \nu \) is a Radon measure on \( \mathbb{R}_+ \) such that \( \int_{[0,\infty[}(1 \wedge t^{-1})\nu(dt) < \infty \). Equivalently, a Stieltjes transform is the Laplace transform of a Radon measure \( \mu \) on \( \mathbb{R}_+ \) of the type \( \mu(dx) = b\delta_0(dx) + h(x)dx \), where \( b \geq 0 \) and \( h \) is a completely monotone function which belongs to \( L^1(e^{-qx}dx) \) for every \( q > 0 \); see e.g. Section 3.8 in [6].
Corollary 11 Let $f \in B_2$ be a Bernstein function such that its derivative $f'$ is a Stieltjes transform. Then for every Bernstein function $g \in B_2$, the function $f \circ \frac{1}{g}$ is completely monotone.

Proof: We can write

$$f(q) = a + bq + \int_0^q dr \int_0^\infty dx e^{-rx} h(x), \quad q > 0,$$

where $a, b \geq 0$ and $h \in B_1$. Thus

$$f(q) = a + bq + \int_0^\infty dx (1 - e^{-q x}) \frac{h(x)}{x}, \quad q > 0,$$

and then

$$f \circ \frac{1}{g} (q) = a + \frac{b}{g(q)} + \int_0^\infty dx (1 - e^{-x/g(q)}) \frac{h(x)}{x}.$$

We already know from Proposition 9 that $a + b/g \in \text{CM}$. The change of variable $y = x/g(q)$ yields

$$\int_0^\infty dx (1 - e^{-x/g(q)}) \frac{h(x)}{x} = \int_0^\infty (1 - e^{-y}) h(yg(q)) \frac{dy}{y}.$$

For each fixed $y > 0$, $yg$ is a Bernstein function, so by Proposition 8, the function $q \to h(yg(q))$ is completely monotone.

We conclude that for every integer $n \geq 0$,

$$(-1)^n \frac{\partial^n}{\partial q^n} (f \circ \frac{1}{g})(q) = \int_0^\infty (-1)^n \frac{\partial^n}{\partial q^n} (h(yg(\cdot)))(q) (1 - e^{-y}) \frac{dy}{y} \geq 0,$$

which establishes our claim.

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