Second-order velocity slip with axisymmetric stagnation point flow and heat transfer due to a stretching vertical plate in a Copper-water nanofluid

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Abstract. The steady axisymmetric stagnation point flow with second-order velocity slip due to a stretching vertical plate with the existence of copper-water nanofluid was investigated. Similarity transformation has been applied to reduce the governing partial differential equations to ordinary differential equations. Then the self-similar equations are solved numerically using solver bvp4c available in Matlab with Prandtl number, \( Pr = 6.2 \). It is found that the dual solutions exist for the certain range of mixed convection parameter. The effects of the governing parameters on the velocity and temperature profile, skin friction coefficient and the local Nusselt number are observed. The results show that the inclusion of nanoparticle copper, will increase the shear stress on the stretching sheet and decrease the heat transfer rate for the slip parameters.

1. Introduction
Significant findings from the study of axisymmetric stagnation point flow can greatly increase the efficiency of the transportation industries such as submarines and aircraft. Many studies have been done to meet the needs of industrial growth. The problem of steady two-dimensional stagnation point flow was first studied by Hiemenz [1] to obtain the exact solution. Then, the study was continued by Homann [2] to the axisymmetric case. No-slip condition on the solid surface is applied for these kinds of problems. However, the slip effect on a solid surface cannot be neglected since it is important for certain situation such as flow over coated or lubricated surfaces (Teflon), rough or striated surfaces [3] and superhydrophobic nano-surfaces [4]. Therefore, no-slip condition should be replaced by the partial slip condition to overcome the situation.

Combination of the problem in two-dimensional stagnation point flow and the stretching sheet was done by Chiam [5]. Since then, the studies keep evolving among researchers. Several issues that were emphasized regarding the problem on axisymmetric stagnation point flow and heat transfer in a nanofluid [6-7], towards a moving plate [8-9], heat flux effect [10] and unsteady cases [11-12]. In the last several years, different flows with the slip effect have eagerly studied by researchers. One of the researcher, Wang [13] interested to extend the work previously done by other researchers on stagnation point flows with first-order velocity slip. From his observations, the high value of slip parameter will give effect to the flow behaviour. He also investigated the stagnation slip flow on a moving plate [14] by obtaining the exact similarity solutions. He found that slip dramatically change the surface resistances and velocity profiles. A new second-order velocity slip model was introduced by Wu [15], has been extended by Wang and Ng [16] by considering the first-order velocity slip of...
stagnation flow on a heated vertical plate. Next, the second-order velocity slip with mixed convection stagnation point flow past a vertical flat plate were examined by Rosca and Pop [17].

Recently, Soid et al. [18] interested to study the second-order velocity slip of axisymmetric stagnation point flow on vertical plate. Motivation regarding the problem studied by Soid et al. [18] leads to this study to observe the effect of copper-water nanofluid to such a problem using Tiwari and Das nanofluid model [19]. Our results are in good agreement with reported data by Wang and Ng [16] and Soid et al. [18]. The numerical results obtained of the nanoparticle volume fraction, mixed convection parameter and the slip parameters on the fluid velocity component, temperature distribution, skin friction coefficient and local Nusselt number were discussed in detail through figures.

2. Problem formulation

Consider axisymmetric stagnation point flow in three-dimensional phase due to a stretching surface in a copper-water nanofluid with the vertical plate, $x$ and $y$ plane. As seen in figure 1, the stagnation point flow is symmetrical about the $z$-axis and the position of the $x$-axis is in the opposite direction to the gravity. The stretching velocities of the surface in the $x$ and $y$ directions are referred as $\lambda u_w(x)$ and $\lambda v_w(y)$ with the stretching parameter $\lambda$ in the range $\lambda > 0$. The velocity distribution far from the plate are $u_e(x) = ax, v_e(y) = ay$ and $w_e(z) = -2az$, where $a > 0$ is the flow strength as $z \to \infty$. The surface temperature was assumed to be $T_w(x) = T_\infty + T_0 x$, where the constant ambient temperatures is $T_w$ and the wall temperature measure is $T_0$. The Navier-Stokes and energy equations are [16-17]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}
\]

\[
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + g\beta(T - T_\infty), \tag{2}
\]

\[
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \tag{3}
\]

\[
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \tag{4}
\]

\[
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right), \tag{5}
\]

subject to the boundary conditions

\[
u = \lambda u_w(x) + \nu_{slip}(x), \quad \lambda v_w(y) + \nu_{slip}(y), \quad T = T_w(x) + T_{slip}(z) \quad \text{on } z = 0, \tag{6}
\]

\[
u_e \to ax, \quad \nu_e \to ay, \quad w \to -2az, \quad T \to T_\infty \quad \text{as } z \to \infty.
\]
where \( T_{slip} = C \frac{\partial T}{\partial z} \) and \( C \) is the temperature jump coefficient.

The velocity components along the \( x \), \( y \) and \( z \)-axes are \( u, v \) and \( w \). The temperature of the nanofluid denoted by \( T \), \( g \) is the gravitational acceleration, \( p \) is the pressure, \( \beta \) is the thermal expansion coefficient, \( \nu_{nf} \) is the kinematic viscosity of the nanofluid, \( \alpha_{nf} \) is the thermal diffusivity of the nanofluid and \( \rho \) is the constant density of the fluid which are given by Oztop and Abu-Nada [20]

\[
\alpha_{nf} = \frac{k_{nf}}{(\rho C_p)_{nf}}, \quad \rho_{nf} = (1 - \varphi)\rho_f + \varphi \rho_s, \quad \mu_{nf} = \frac{\mu_f}{(1 - \varphi)^{2.5}}, \quad \frac{(\rho C_p)_{nf}}{k_f} = \frac{(k_s + 2k_f) - 2\varphi(k_f - k_s)}{(k_s + 2k_f) + \varphi(k_f - k_s)}.
\]

From equation (7) above, \( k_{nf} \) is the thermal conductivity of the nanofluid, \( (\rho C_p)_{nf} \) is the heat capacitance of the nanofluid, \( \rho_{nf} \) is the density of the nanofluid, \( \rho_f \) and \( \rho_s \) are the densities of the fluid and solid fractions, respectively, \( \varphi \) is the nanoparticle volume fraction, \( C_p \) is the specific heat at constant pressure, \( k_f \) and \( k_s \) are the thermal conductivities of the fluid and solid fractions, respectively.

Further, \( u_{slip} \) and \( v_{slip} \) are the slip velocities at the stretching surface and are defined as [15]

\[
u_{slip}(u) = \frac{2}{3} \left( \frac{3 - \varphi^2}{\varepsilon} - \frac{3(1 - \varphi^2)}{2K_n} \right) \frac{\partial u}{\partial z} - \frac{1}{4} \left( \varepsilon^4 + \frac{2(1 - \varphi^2)}{K_n^2} \right) \frac{\partial^2 \varphi}{\partial z^2} - \frac{\partial^2 \varphi \partial^2 u}{\partial z^2} = A \frac{\partial u}{\partial z} + B \frac{\partial^2 u}{\partial z^2},
\]
\[
u_{slip}(v) = \frac{2}{3} \left( \frac{3 - \varphi^2}{\varepsilon} - \frac{3(1 - \varphi^2)}{2K_n} \right) \frac{\partial v}{\partial z} - \frac{1}{4} \left( \varepsilon^4 + \frac{2(1 - \varphi^2)}{K_n^2} \right) \frac{\partial^2 \varphi}{\partial z^2} - \frac{\partial^2 \varphi \partial^2 v}{\partial z^2} = A \frac{\partial v}{\partial z} + B \frac{\partial^2 v}{\partial z^2},
\]

where \( A \) and \( B \) are constants \((B < 0)\), \( K_n \) is the Knudsen number, \( \ell = \min\{K_n^{-1}, 1\} \), \( 0 \leq \ell \leq 1 \) and \( \delta \) \((\delta > 0)\) is the molecular mean free path.

The governing equations, from equation (1) to equation (4) subject to the boundary conditions, equation (6) can be expressed in a simpler form by introducing the following transformation:

\[
\eta = z\sqrt{\frac{\alpha}{\nu}}, \quad T - T_\infty = \Delta T \theta(\eta),
\]

where \( \eta \) is the similarity variable, \( \Delta T = T_0 - T_\infty \) where \( \Delta T \) is the characteristic temperature and \( \theta(\eta) \) is the dimensionless parameter. The velocity components \( u = \alpha f'(\eta), \quad v = \alpha f(\eta) \) and \( w = -2\alpha f' \) were identically satisfies equation (1). The pressure \( p \) can be determined by integrating equation (4) to get \( p - p_\infty = \mu \frac{\partial w}{\partial z} - \rho \frac{\partial^2 w}{\partial z^2}/2 \).

Employing transformation equations (2), (3) and (5) will reduced to the following ordinary differential equations:

\[
\frac{1}{(1 - \varphi)^{2.5}} \left[ (1 - \varphi) + \varphi \rho_s/\rho_f \right] f'' + 2ff' + 1 - f'^2 + \sigma \theta = 0,
\]
\[
\frac{k_{nf}/k_f}{(1 - \varphi) + \varphi (\rho C_p)_{nf}/(\rho C_p)_{f}} \theta'' + Pr(2ff' - f' \theta) = 0,
\]

subject to the boundary conditions, from equation (6)

\[
f(0) = 0, \quad f'(0) = \lambda + \Delta f''(0) + A f''(0), \quad \theta(0) = 1 + K \theta'(0), \quad f'(\eta) \rightarrow 1, \quad \theta(\eta) \rightarrow 0, \quad \text{as} \ \eta \rightarrow \infty
\]

where \( Pr = \nu/\alpha \) is the Prandtl number, \( \Delta = A \sqrt{\alpha/\nu} \) where \( \Delta > 0 \) is the first-order velocity slip parameter, \( \Lambda = B a/\nu < 0 \) where \( \Lambda < 0 \) is the second-order velocity slip parameter, \( K = C\sqrt{a/\nu} \)
where $K > 0$ is the temperature parameter and $\sigma$ is the mixed convection parameter, $\sigma = g\beta T_0/a^2$. Noticed that for the assisting flow case $\sigma > 0$, opposing flow case $\sigma < 0$ and situation when no buoyancy $\sigma = 0$.

The present problem, highlighted two important physical quantities which is the skin friction coefficients $C_f$ and the local Nusselt number $N_u$, defined as

$$C_f = \frac{\tau_w}{\rho u_x^2}, \quad N_u = \frac{xq_w}{k_f(T_w - T_x)},$$  \hspace{1cm} (13)

where $\tau_w$ is the skin friction or the shear stresses on the stretching sheet, and $q_w$ is the heat flux from the surface of the plate, which are given by

$$\tau_w = \mu_0 \left( \frac{\partial u}{\partial z} \right)_{z=0}, \quad q_w = -k_0 \left( \frac{\partial T}{\partial z} \right)_{z=0}. \hspace{1cm} (14)$$

Substitute equation (9) into equations (13) and (14), we obtain

$$C_f \ Re_x^{1/2} = \frac{1}{(1-\varphi)^{2/3}} f''(0), \quad N_u Re_x^{-1/2} = -\frac{k_0}{k_f} \theta'(0),$$  \hspace{1cm} (15)

where $Re_x = u_x x/\nu$ is the local Reynolds number.

3. Results and Discussion

In this present study, the problem of the nonlinear ordinary differential equations, from equation (1) to equation (5) were solved numerically using Matlab bvp4c function. The results are presented graphically in the form of diagrams as well as in tabular form for some values of the nanoparticle volume fraction $\varphi$, the mixed convection parameter $\sigma$, first-order and second-order velocity slip parameters, $\Delta$ and $\Lambda$, respectively. The Prandtl number taken as $Pr = 6.2$ (for water) and the range of the nanoparticle volume fraction $\varphi$ chosen is $0 \leq \varphi \leq 0.2$ where $\varphi = 0$ referred to regular Newtonian fluid. The thermophysical properties of water and nanoparticle copper was taken from Oztop and Abu-Nada [20]. By comparing some of the results obtained with the previously published studies [16, 18] for regular fluid with some values of $\Delta$, we validate the results and it is give a favourable agreement as shown in table 1.

Figure 2 and figure 3 show the variation of $f''(0)$ and $-\theta'(0)$ for the different values of the first-order velocity slip parameter $\Delta$ with the mixed convection parameter $\sigma$. These figures show that the range of the mixed convection parameter $\sigma$ for which the solution exists decreases with the increasing of the first-order velocity slip parameter $\Delta$.

**Table 1.** Comparison of the values $f''(0)$ and $-\theta'(0)$ for $K = 1.0$, $Pr = 0.7$, $\lambda = 0$, $\varphi = 0$, $\Lambda = 0$ and $\sigma = 0$ in the axisymmetric case.

| $\Delta$ | Wang and Ng [16] | Soid et al. [18] | Present results |
|---------|-----------------|-----------------|-----------------|
| $f''(0)$ | $f''(0)$ | $-\theta'(0)$ | $f''(0)$ | $-\theta'(0)$ |
| 0 | 1.31194 | 1.311938 | 0.45110655 | 1.31193771 | 0.45110655 |
| 0.1 | 1.21009 | 1.210087 | 0.46867805 | 1.21008662 | 0.46867805 |
| 0.5 | 0.86688 | 0.866879 | 0.50548912 | 0.86687943 | 0.50548912 |
| 1 | 0.61730 | 0.617300 | 0.52304734 | 0.61729955 | 0.52304734 |
| 5 | 0.17928 | 0.179287 | 0.54568595 | 0.17928361 | 0.54568595 |
| 10 | 0.09460 | 0.094597 | 0.54929689 | 0.09459716 | 0.54929689 |
Figure 4 and figure 6 illustrate that the value of the skin friction coefficient enhances almost linearly with the increasing of the nanoparticle volume fraction $\phi$. Figure 4 demonstrates that the influence of the first-order velocity slip $\Delta$ on skin friction coefficient is stronger compared to local Nusselt number, same goes to the second-order velocity slip $\Lambda$, as seen in figure 6.

Figure 2. Variation of $f'(0)$ with $\sigma$ for $\Delta = 0.1, 1.5$ when $\Lambda = -0.3$, $K = 0.2$, $\lambda = 1$, $\phi = 0.1$.

Figure 3. Variation of $-\theta'(0)$ with $\sigma$ for $\Delta = 0.1, 1.5$ when $\Lambda = -0.3$, $K = 0.2$, $\lambda = 1$, $\phi = 0.1$.

Figure 4. Variation of skin friction coefficient for $\Delta = 0.1, 1.5$ with $\sigma = -0.5$ and $\Lambda = -0.3$.

Figure 5. Variation of Nusselt number for $\Delta = 0.1, 1.5$ with $\sigma = -0.5$ and $\Lambda = -0.3$.

Figure 6. Variation of skin friction coefficient for $\Lambda = -0.1, -1, -5$ with $\sigma = -0.5$ and $\Delta = 0.3$.

Figure 7. Variation of Nusselt number for $\Lambda = -0.1, -1, -5$ with $\sigma = -0.5$ and $\Delta = 0.3$.

As the first-order velocity slip parameter $\Delta$ increase, the value of skin friction coefficient will decrease as well as for the local Nusselt number as seen in figure 4 and figure 5. Increasing the nanoparticle volume fraction $\phi$ is to decrease the local Nusselt number for the first-order and second-order velocity slip. The skin friction coefficient and the local Nusselt number rise with the decreased of the second-order velocity slip parameter $\Lambda$, see figure 6 and figure 7.
Figure 8. Velocity profile $f'(\eta)$ for $\sigma$ when $K = 0.2, \varphi = 0.1, \Delta = 5, \Lambda = -0.3$ and $\lambda = 1$.

Figure 9. Temperature profile $\theta(\eta)$ for $\sigma$ when $K = 0.2, \varphi = 0.1, \Delta = 5, \Lambda = -0.3$ and $\lambda = 1$.

From figure 8 and figure 9, we obtained that the boundary layer thickness of the second solution is larger than the first solution for the magnitude of the mixed convection parameter $\sigma$ in opposing flow. Figure 8 shows that decrease the mixed convection parameter $\sigma$ will decrease the first solution and increase the second solution for the velocity profile. Figure 9 gives the different analysis, where we can see that decrease the mixed convection parameter $\sigma$ is to increase the first solution and decrease the second solution for the temperature profile.

4. Conclusion
The problem that has been studied in this paper using a second-order slip flow model proposed by Wu [15]. The variations of skin friction coefficient, Nusselt number, velocity profile and temperature profile have been discussed in this paper for some values of nanoparticle volume fraction, mixed convection parameter and slip parameters. The following conclusion are drawn from this study:

- dual solutions exist for the certain range of mixed convection parameter,
- the critical value increases as the mixed convection parameter is increasing on the variation of $-\theta'(0)$ for the first-order slip parameter,
- increasing the nanoparticle volume fraction, will increase the shear stress on the stretching sheet for the first-order and second-order slip parameter,
- the rate of the heat transfer will decrease as the nanoparticle volume fraction increase for the slip parameters.

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