ON A QUESTION OF SILVER ABOUT GAP-TWO CARDINAL TRANSFER PRINCIPLES

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Abstract. Assuming the existence of a Mahlo cardinal, we produce a generic extension of Gödel’s constructible universe $L$, in which the $GCH$ holds and the transfer principles $(\aleph_2, \aleph_0) \rightarrow (\aleph_3, \aleph_1)$ and $(\aleph_3, \aleph_1) \rightarrow (\aleph_2, \aleph_0)$ fail simultaneously. The result answers a question of Silver from 1971. We also extend our result to higher gaps.

1. Introduction

In this paper we study cardinal transfer principles introduced by Vaught [6], [7], and prove some consistency results related to them.

Assume $L$ is a first order language which contains a unary predicate $U$. By a $(\kappa, \lambda)$-model for $L$, we mean a model $M = (M, U^M, \ldots)$, where $|M| = \kappa$ and $|U^M| = \lambda$, where $U^M$ is the interpretation of $U$ in $M$. Following Devlin [2], we use the notation

$$(\kappa, \lambda) \rightarrow (\kappa', \lambda')$$

to mean the following transfer principle:

For every countable first order language $L$ as above, and every first order theory $T$ of $L$, if $T$ has a $(\kappa, \lambda)$-model, then it has a $(\kappa', \lambda')$-model.

For any natural number $n \geq 1$, by the gap-$n$-cardinal transfer principle we mean the statement

$$\forall \kappa \forall \lambda (\kappa + n, \kappa) \rightarrow (\lambda + n, \lambda).$$

In [5], Silver proved the independence of gap-2-cardinal transfer principle. Starting from an inaccessible cardinal, he was able to produce a model in which the cardinal transfer $(\aleph_3, \aleph_1) \rightarrow (\aleph_2, \aleph_0)$ fails. His proof is simply as follows: By a result of Vaught [7], there

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exists a sentence $\phi_{KH}$ in a suitable first order language, such that for any infinite cardinal $\beta$,

$$\phi_{KH} \text{ has a } (\beta++, \beta)\text{-model } \iff \text{there exists a } \beta^+\text{-Kurepa tree.}$$

Now, starting from an inaccessible cardinal $\kappa$, Silver shows that in the generic extension by the Levy collapse $Col(\aleph_1, < \kappa)$, there are no $\aleph_1$-Kurepa trees. If we start with $V = L$, then in the resulting extension, there are $\aleph_2$-Kurepa trees, and so the transfer principle $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$ fails in it. Similarly if we force with $Col(\aleph_2, < \kappa)$, then in the extension there are no $\aleph_2$-Kurepa trees, and we can use it to prove the independence of $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$. The following question was asked by Silver [5].

**Question 1.1.** Is it consistent with $GCH$ that both transfer principles $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$ and $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$ fail simultaneously?

**Remark 1.2.** If we drop the $GCH$ assumption from the question, then one can easily answer the above question. Assume $\kappa$ is an inaccessible cardinals and let $G * H$ be $Col(\aleph_1, < \kappa) * Add(\aleph_0, \kappa)$-generic over $L$. In the generic extension $L[G * H]$ there are no $\aleph_1$-Kurepa trees (see Devlin [3]) but there exists an $\aleph_2$-Kurepa tree, and hence by the remarks above, the transfer principle $(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0)$ fails in $L[G * H]$.

On the other hand $L[G * H]$ satisfies “$2^{\aleph_0} = 2^{\aleph_1} = \kappa = \aleph_2$”. Let $\mathcal{L} = (U, F)$, where $U$ is a unary predicate symbol and $F$ is a binary predicate symbol. let $T$ be an $\mathcal{L}$-theory which says the following:

1. $\forall x, y F(x, y) \to U(y)$. In particular, for each $x$, $F$ determines a subset $F_x$ of $U$, namely, $F_x = \{y : F(x, y)\}$.
2. For all $x \neq x'$, $F_x \neq F_{x'}$.

Then $T$ has an $(\aleph_2, \aleph_0)$ model but it does not have an $(\aleph_3, \aleph_1)$-model (as otherwise we should have $2^{\aleph_1} \geq \aleph_3$). Thus the transfer principle $(\aleph_2, \aleph_0) \to (\aleph_3, \aleph_1)$ fails in $L[G * H]$.

We give an affirmative answer to this question by proving the following theorem:

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1On page 388 of 5, Silver writes “One can also get a $GCH$ model in which $(\aleph_7, \aleph_5) \to (\aleph_3, \aleph_1)$ fails and a $GCH$ model which $(\aleph_3, \aleph_1) \to (\aleph_7, \aleph_5)$ fails (though I don’t see how to get the $\to$ both ways to fail simultaneously)”. 

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Theorem 1.3. Assume $\kappa$ is a Mahlo cardinal. Then there is a generic extension of $L$, the Gödel’s constructible universe, in which the GCH holds and the cardinal transfer principles $(\aleph_2, \aleph_0) \rightarrow (\aleph_3, \aleph_1)$ and $(\aleph_3, \aleph_1) \rightarrow (\aleph_2, \aleph_0)$ fail.

Then we prove a general model theoretic fact, and use it to extend the above result to higher gaps:

Theorem 1.4. Assume $\kappa$ is a Mahlo cardinal. Then there is a generic extension of $L$ in which the GCH holds and for all $n \geq 2$, the cardinal transfer principles $(\aleph_n, \aleph_0) \rightarrow (\aleph_{n+1}, \aleph_1)$ and $(\aleph_{n+1}, \aleph_1) \rightarrow (\aleph_n, \aleph_0)$ fail.

Remark 1.5. Our proofs can be easily extended to get the following consistency result: assume $\alpha < \beta$ are regular cardinals and assume there exists a Mahlo cardinal above them. Then in a generic extension of $L$, the GCH holds and both transfer principles $(\alpha^{+n}, \alpha) \rightarrow (\beta^{+n}, \beta)$ and $(\beta^{+n}, \beta) \rightarrow (\alpha^{+n}, \alpha)$ fail.

In Section 2 we prove Theorem 1.3 and in Section 3 we prove Theorem 1.4. In the last section, we discuss the same problem for the case of gap-1.

2. Proof of Theorem 1.3

In this section we prove Theorem 1.3.

2.1. On a result of Jensen. In this subsection we state a result of Jensen [4] and mention some of its basic properties which are needed. Let $\mathcal{L} = \{\in, A, C\}$, where $A$ is a unary predicate and $C$ is a function symbol. Let $T_J$ be the following theory in $\mathcal{L}$:

"$\text{ZFC}^- + \text{GCH} + A^\uparrow \text{ is the largest cardinal} + C \text{ is a } \square_A^\uparrow \text{-sequence"}.

By a $(\kappa, \lambda)$-model of $T_J$ we mean a model $\mathcal{M} = (M, \in^\mathcal{M}, A^\mathcal{M}, C^\mathcal{M})$ of $T_J$, where $|M| = \kappa$ and $|A^\mathcal{M}| = \lambda$.

Theorem 2.1. (Jensen [4]) Assume $\text{GCH} + \exists^\beta \dagger$ holds, where $\beta$ is a regular cardinal, and suppose $\kappa > \beta$ is a Mahlo cardinal. Then there is a forcing notion $\mathbb{P}_{\beta, \kappa}$ such that if $K$ is $\mathbb{P}_{\beta, \kappa}$-generic over $V$, then the following hold in $V[K]$:

(a) $V[K] \models \text{“GCH”}$. 
(b) The principle \(\diamondsuit_{\beta}^+\) holds.

(c) The theory \(T_J\) does not have any \((\beta^{++}, \beta)\)-model.

**Proof.** As requested by the referees, we sketch the proof of the theorem, by providing the forcing construction \(P_{\beta, \kappa}\), and refer to [4] for details. Let \(G\) be \(\text{Col}(\beta^+, < \kappa)\)-generic over \(V\), where

\[
\text{Col}(\beta^+, < \kappa) = \{p : \beta^+ \times \kappa \to \kappa : |p| \leq \beta \text{ and for all } (\alpha, \lambda) \in \text{dom}(p), p(\alpha, \lambda) < \lambda\}
\]

is the Levy collapse. The next claim is standard.

**Claim 2.2.**

(a) The forcing \(\text{Col}(\beta^+, < \kappa)\) is \(\beta^+\)-closed and \(\kappa\)-c.c.

(b) \(V[G] \models \text{“GCH} + \diamondsuit_{\beta^+} \text{”}\).

(c) \(V[G] \models \text{“}\kappa = \beta^{++} \text{ and } \Box_{\beta^{++}} \text{ fails”}\).

In [4], the following strengthening of Claim 2.2(c) is proved.

**Claim 2.3.** In \(V[G]\), the theory \(T_J\) has no \((\beta^{++}, \beta)\)-model.

From now on we work in \(V[G]\). Let \(S = \langle S_\alpha : \alpha < \beta^+ \rangle\) witness \(\diamondsuit_{\beta^+}\). For each \(\alpha < \beta^+\) let \(d_\alpha : \beta \to \alpha\) be an onto function and set \(d = \langle d_\alpha : \alpha < \beta^+ \rangle\). For \(\alpha < \beta^+\) set

\[
M_\alpha = L_{\gamma_\alpha}[S \upharpoonright \alpha + 1, d \upharpoonright \alpha + 1],
\]

where \(\gamma_\alpha\) is the least ordinal \(\gamma > \alpha\) such that \(\gamma > \sup_{\nu < \alpha} \gamma_\nu\) and

\[
L_{\gamma}[S \upharpoonright \alpha + 1, d \upharpoonright \alpha + 1] \models \text{“ZF C”}.
\]

Define

\[
S^* = \langle S^*_\alpha : \alpha < \beta^+ \rangle,
\]

where \(S^*_\alpha = P(\alpha) \cap M_\alpha\). We find a generic extension of \(V[G]\) in which \(S^*\) is a \(\diamondsuit_{\beta^+}\)-sequence.

Let \(A \subseteq \kappa\) be such that \(L_\kappa[A] = H(\kappa)\) and define the sequence \(\langle \rho_\nu : \nu < \kappa \rangle\) by recursion on \(\nu\) as follows: \(\rho_\nu\) is the least ordinal \(\rho > \beta^+\) such that

- \(\rho > \sup_{\xi < \nu} \rho_\xi\).
- \(\langle M_\alpha : \alpha < \beta^+ \rangle \in L_\nu[A]\).
- \(cf(\rho) = \beta^+\).
• \( L_{\rho}[A] = "\text{ZF}^- + \forall x, |x| \leq \beta^+. \)

Set \( \hat{\rho}_\nu = \beta^+ \cup \sup_{\xi < \nu} \rho_\xi, \)

\[
\mathcal{U}_\nu = \langle L_{\rho_\nu}[A], \in, A \cap \rho_\nu, \langle M_\alpha : \alpha < \beta^+ \rangle \rangle,
\]

and for \( \nu > 0 \) set

\[
\hat{\mathcal{U}}_\nu = \bigcup_{\xi < \nu} \hat{\mathcal{U}}_\xi = \langle L_{\hat{\rho}_\nu}[A], \in, A \cap \hat{\rho}_\nu, \langle M_\alpha : \alpha < \beta^+ \rangle \rangle.
\]

Then set

\[
f_\nu = \text{the } \mathcal{U}_\nu\text{-least bijection } f : \beta^+ \leftrightarrow \hat{\rho}_\nu.
\]

\[
a_\xi = \text{the } \xi\text{-th } a \subseteq \beta^+ \text{ in } L_\kappa[A].
\]

\[
\tilde{a}_\nu = \{(\xi, \mu) : \xi \in a_{f_\nu(\mu)}\}.
\]

We are now ready to define the desired forcing notion, that we denote by \( \text{Add}(\diamond_{\beta^+}^+) \). First we define the forcing notions \( \text{Add}(\diamond_{\beta^+}^+)_{\nu}, \nu < \kappa, \) which are the building blocks of the main forcing construction.

A condition in \( \text{Add}(\diamond_{\beta^+}^+)_{\nu} \) is a subset \( p \) of \( \beta^+ \) such that

1. \( p \subseteq \beta^+ \) is closed and bounded.
2. \( \alpha \in p \implies \tilde{a}_\nu \cap \alpha \in M_\alpha. \)

\( \text{Add}(\diamond_{\beta^+}^+)_{\nu} \) is ordered by end extension:

\[
p \leq q \iff q = p \cap (\max(p) + 1).
\]

Let us now define \( \text{Add}(\diamond_{\beta^+}^+) \). A condition in \( \text{Add}(\diamond_{\beta^+}^+) \) is a function \( p \) such that

1. \( \text{dom}(p) \subseteq \kappa \) and \( |\text{dom}(p)| \leq \beta. \)
2. \( \forall \nu \in \text{dom}(p), p(\nu) \in \text{Add}(\diamond_{\beta^+}^+)_{\nu}. \)
3. If \( \nu \in \text{dom}(p), \) then
   1. \( f''_{\nu}[\max(p(\nu))] \subseteq \text{dom}(p). \)
   2. For each \( \xi \in f''_{\nu}[\max(p(\nu))], \max(p(\xi)) \geq \max(p(\nu)). \)
   3. \( \alpha \in p(\nu) \implies \tilde{C}_{p,\nu} \cap \alpha \in M_\alpha, \) where

\[
\tilde{C}_{p,\nu} = \{(\mu, \xi) \in \max(p(\nu)) \times \max(p(\nu)) : \mu \in p(f_\nu(\xi))\}.
\]
The forcing \( \text{Add}(\mathcal{O}^{+}_{\beta}) \) is ordered as follows: \( p \leq q \) if and only if
\[
\text{dom}(p) \supseteq \text{dom}(q) \quad \text{and for all } \nu \in \text{dom}(q), \quad p(\nu) \leq \text{Add}(\mathcal{O}^{+}_{\beta})\nu \ q(\nu).
\]

Let \( H \) be \( \text{Add}(\mathcal{O}^{+}_{\beta}) \)-generic over \( V[G] \). The next claim is proved in [4].

Claim 2.4. 
(a) \( \text{Add}(\mathcal{O}^{+}_{\beta}) \) is \( \beta^{+} \)-distributive and \( \kappa = \beta^{++} \)-c.c.”.
(b) \( V[G \ast H] \models “GCH”. “ \).
(c) \( S^* \) witnesses that \( \mathcal{O}^{+}_{\beta} \) holds in \( V[G \ast H] \).
(d) The theory \( T_J \) does not have a \( (\beta^{++}, \beta) \)-model in \( V[G \ast H] \).

Then \( P_{\beta, \kappa} = \text{Col}(\beta^+, < \kappa) \ast \text{Add}(\mathcal{O}^{+}_{\beta}) \) is as required. \( \square \)

Suppose \( K = G \ast H \) is \( P_{\beta, \kappa} \)-generic over \( L[G] \). As \( \mathcal{O}^{+}_{\beta} \) implies the existence of a \( \beta^{+} \)-Kurepa tree [2], in \( V[K] \), we have \( \beta^{+} \)-Kurepa trees.

2.2. Completing the proof of Theorem 1.3 In this subsection we complete the proof of Theorem 1.3. Thus assume \( V = L \) and let \( \kappa \) be a Mahlo cardinal. Let \( \lambda \) be the least inaccessible cardinal. So \( \lambda < \kappa \). Let \( G \) be \( \text{Col}(\aleph_1, < \lambda) \)-generic over \( L \). Then:

Lemma 2.5. 
(a) \( L[G] \models “There are no \( \aleph_1 \)-Kurepa trees”. “ \).
(b) \( L[G] \models “GCH holds”. “ \).
(c) \( L[G] \models “\kappa \text{ is a Mahlo cardinal}. “ \).

Proof. (a) and (b) hold by [5], and (c) is clear, as the forcing \( \text{Col}(\aleph_1, < \lambda) \) has size \( < \kappa \). \( \square \)

Let \( K \) be \( P_{\aleph_1, \kappa}^{L[G]} \)-generic over \( L[G] \). We show that \( L[G \ast K] \) is the required model. First note that by Theorem 2.1

\[
L[G \ast K] \models “\text{there exists an } \aleph_2 \text{-Kurepa tree}”. \nonumber
\]

But by Lemma 2.5 \( L[G] \models “\text{There are no } \aleph_1 \text{-Kurepa trees}. “ \). On the other hand, \( L[G] \models “P_{\aleph_1, \kappa} \text{is } \lambda = \aleph_2 \text{-distributive}”. “ \), in particular

\[
L[G \ast K] \models “\text{There are no } \aleph_1 \text{-Kurepa trees}. “ \)

It follows that

\[
L[G \ast K] \models “(\aleph_3, \aleph_1) \to (\aleph_2, \aleph_0) \text{ fails ”}. \nonumber
\]
On the other hand, by Theorem 2.1(b), \( L[G] \models “T_J \) does not have an \( (\aleph_3, \aleph_1) \)-model”.

We show that \( T_J \) has an \( (\aleph_2, \aleph_0) \)-model in \( L[G] \). First note that \( \aleph_2^{L[G]} = \lambda \), which is inaccessible but not Mahlo in \( L \), so it follows from results of Jensen and Solovay (see [2]) that \( \Box_{\aleph_1} \) holds in both \( L[G] \) and \( L[G*K] \). Let \( C = (C_\alpha : \alpha < \lambda, \lim(\alpha)) \in L[G] \) witness this. Consider the model

\[ M = (H(\lambda)^{L[G]}, \in, \aleph_0, C) \]

where \( \aleph_0 \) is considered as the interpretation of \( A \). Then \( M \) is an \( (\aleph_2, \aleph_0) \)-model of \( T \). So

\[ L[G*K] \models “(\aleph_2, \aleph_0) \rightarrow (\aleph_3, \aleph_1) \ fails” \]

The theorem follows.

3. A general model theoretic fact and the proof of Theorem 1.4

In this section we prove a general model theoretic fact, and use it to prove Theorem 1.4.

3.1. A general model theoretic fact. In this subsection we prove the following lemma and consider some of its consequences.

**Lemma 3.1.** Assume \( n \geq 1 \), \( \mathcal{L} \) is a first order language which contains a unary predicate \( U \), and \( T \) is a theory in \( \mathcal{L} \). Then there are \( \mathcal{L}^+ \supseteq \mathcal{L} \) and a theory \( T^+ \) in \( \mathcal{L}^+ \), such that for all infinite cardinals \( \beta \):

\[ T \text{ has a } (\beta^+, \beta)\text{-model } \iff T^+ \text{ has a } (\beta^{n+1}, \beta)\text{-model}. \]

**Proof.** Let \( \mathcal{L}^+ = \mathcal{L} \cup \{<, W_0, \ldots, W_n, F_{-1}, F_0, \ldots, F_n \} \) where \( < \) is a binary predicate symbol, \( W_i \)'s are unary predicate symbols, \( F_{-1} \) is a binary predicate symbol and \( F_i \)'s, \( 0 \leq i \leq n \), are ternary predicate symbols. Let \( T^+ \) consists of the following axioms:

1. \( \phi^{W_n}, \) for each \( \phi \in T \), where \( \phi^{W_n} \) is the relativization of \( \phi \) to \( W_n \).
2. \( < \) is a linear ordering of the universe.
3. Under \( <, \) each \( W_i \) is an initial segment of \( W_{i+1}, i < n \), and \( W_n \) is an initial segment of the universe (in particular \( W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n \)).
4. \( U \subseteq W_n \) (i.e., \( \forall x(U(x) \rightarrow W_n(x)) \)).
5. \( F_{-1} \subseteq U \times W_0 \) defines a bijection from \( U \) onto \( W_0 \).
(6) For each $0 \leq i < n$, $F_i \subseteq (W_{i+1} \setminus W_i) \times W_i \times W_{i+1}$ is such that if $x \in W_{i+1} \setminus W_i$, then $\{(y,z) : F_i(x,y,z)\}$ is a bijection from $W_i$ onto $\{z \in W_{i+1} : z < x\}$.

(7) $F_n$ is such that if $x \not\in W_n$, then $\{(y,z) : F_n(x,y,z)\}$ is a bijection from $W_n$ onto $\{z \in W_{n+1} : z < x\}$.

Now suppose that $T$ has a $(\beta^n, \beta)$-model $M = (\beta^n, U^M, \ldots)$. Consider the model $M = (\beta^{n+1}, \beta, \ldots, \beta^n, f_{n-1}, \ldots, f_0)$, where $f_{-1} : U^M \leftrightarrow \beta$, each $f_i, 0 \leq i \leq n$ is such that for each $\beta^i \leq \gamma < \beta^{i+1}, \{(\zeta, \eta) : (\gamma, \zeta, \eta) \in f_i\}$ defines a bijection $\beta^i \leftrightarrow \gamma$. It is easily seen that $M$ is a $(\beta^{n+1}, \beta)$-model for $T$.

Conversely assume that $M$ is a $(\beta^{n+1}, \beta)$-model for $T$. Consider the model $M$ which is obtained from $M \upharpoonright L_\gamma$ by replacing its universe with $W^M_n$. It follows from (1) that $M$ is a model of $T$. We show that it is a $(\beta^n, \beta)$-model. We have $U^M = U^M_\beta$, which has size $\beta$. On the other hand, axioms (4)-(6) can be used to show that $|W^M_0| = \beta$, $|W^M_{i+1}| \leq |W^M_i| + \beta^n$, and $|W^M_m| \geq \beta^n$, so by induction on $i \leq n$, we have $|W^M_i| = \beta^i$. In particular $|W^M_n| = \beta^n$, and the result follows. □

**Corollary 3.2.** For each $n \geq 1$, the gap-$(n + 1)$-cardinal transfer principle implies the gap-$n$-cardinal transfer principle.

**Remark 3.3.** In personal communication, Ali Enayat informed us that Corollary 3.2 is an immediate consequence of the downward Löwenheim-Skolem theorem, i.e., the fact that if $M = (M, \ldots)$ is an infinite structure in a countable language and $X$ is any subset of $M$, then there is an elementary substructure $M_0 = (M_0, \ldots)$ of $M$ that includes $X$ and whose cardinality is max$\{\aleph_0, |X|\}$. Using this theorem, it is easy to see that every model $M$ that exhibits a gap-$m$ model, say $(\kappa^m, \kappa)$, for some $m > 0$ has an elementary sub-model $M_0$ that exhibits a gap-$n$ model $(\kappa^n, \kappa)$ for all $n < m$.

**3.2. Proof of Theorem 1.4.** In this subsection we complete the proof of Theorem 1.4. Let $L[G * H]$ be the model obtained in Subsection 2.2. So in $L[G * H]$ both transfer principles $(\aleph_3, \aleph_1) \rightarrow (\aleph_2, \aleph_0)$ and $(\aleph_2, \aleph_0) \rightarrow (\aleph_3, \aleph_1)$ fail. So, by induction, and using Lemma 3.1 for
each $n \geq 2$, the transfer principles

$$(\kappa_n, \kappa_0) \rightarrow (\kappa_{n+1}, \kappa_1)$$

and

$$(\kappa_{n+1}, \kappa_1) \rightarrow (\kappa_n, \kappa_0)$$

fail in $L[G \ast H]$.

4. **The case of gap-1 and some problems**

In general, we can not hope to prove a result as above for gap-1-cardinal transfer principles. This is because of Vaught’s theorem [7] that the transfer principle $(\beta^+, \beta) \rightarrow (\kappa_1, \kappa_0)$ is a theorem of $ZFC$. However we do not know the answer to the following question:

**Question 4.1.** *Is it consistent that both transfer principles $(\kappa_2, \kappa_1) \rightarrow (\kappa_3, \kappa_2)$ and $(\kappa_3, \kappa_2) \rightarrow (\kappa_2, \kappa_1)$ fail simultaneously?*

As we showed in Corollary 3.2, the gap-$(n + 1)$-cardinal transfer principle implies the gap-$n$-cardinal transfer principle.

On the other hand if $L[G]$ is a generic extension of $L$ by the Levy collapse of an inaccessible cardinal $\kappa$ to $\kappa_2$, then it follows from results of Vaught [7], Chang [1] and Jensen [2] that the gap-1-cardinal transfer principle holds in $L[G]$, while by Silver’s result stated in the introduction, the gap-2-cardinal transfer principle fails in $L[G]$. We do not know the answer for higher gaps.

**Question 4.2.** *Assume $n > 1$. Is it consistent that the gap-$n$-cardinal transfer principle holds while the gap-$(n + 1)$-cardinal transfer principle fails?*

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