Let $L(s, \chi)$ be the Dirichlet $L$-function associated to a non-principal primitive Dirichlet character $\chi$ defined mod $q$, where $q \geq 3$. We prove, under the assumption of the Generalised Riemann Hypothesis, the validity of estimates given by Lamzouri, Li, and Soundararajan on $|L(1, \chi)|$. As a corollary, we have that similar estimates hold for the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$, $q \geq 5$.

1. Introduction

Let $q \geq 3$ be an integer, $\chi$ be a Dirichlet character mod $q$ and $L(s, \chi)$ be the associated Dirichlet $L$-function. Assuming the Riemann Hypothesis for $L(s, \chi_d)$ holds, where $\chi_d$ is a quadratic Dirichlet character, in 1928 Littlewood [7] proved, for $d \neq m^2$, that

$$\left(\frac{12e^y}{\pi^2}(1 + o(1)) \log \log d\right)^{-1} < L(1, \chi_d) < 2e^y(1 + o(1)) \log \log d$$

as $d$ tends to infinity, where $\gamma$ is the Euler–Mascheroni constant. In 2015 Lamzouri, Li, and Soundararajan [5, Theorem 1.5] proved an effective form of Littlewood’s inequalities: assuming the Generalised Riemann Hypothesis (GRH) holds, for every integer $q \geq 10^{10}$ and for every non-principal primitive character $\chi$ mod $q$, they obtained that

$$|L(1, \chi)| \leq 2e^y \left(\log \log q - \log 2 + \frac{1}{2} + \frac{1}{\log \log q}\right), \quad (2)$$

and

$$\frac{1}{|L(1, \chi)|} \leq \frac{12e^y}{\pi^2} \left(\log \log q - \log 2 + \frac{1}{2} + \frac{1}{\log \log q} + \frac{14 \log \log q}{\log q}\right). \quad (3)$$

Such inequalities were numerically verified, for every prime $3 \leq q \leq 10^7$, by Languasco [6]. We remark that the recent papers [1] and [3] used (2) and (3) in the region $q \geq 10^{10}$.

The main goal of this paper is to enlarge the $q$-range of validity of (2)-(3) in the following

**Theorem 1.** Assume the Generalised Riemann Hypothesis. Then, for $q \geq 3$ both (2) and (3) hold for every non-principal primitive Dirichlet character mod $q$.

The first step in the proof of Theorem 1 will be to extend the range of validity of the proof of Theorem 1.5 of [5] from $q \geq 10^{10}$ to $q \geq 404$. Then using a computer program we will verify that (2)-(3) both hold for every non-principal Dirichlet character mod $q$, where $q$ is composite, $3 \leq q \leq 1000$. Combining this result with the computations already performed for every prime $3 \leq q \leq 10^7$, by Languasco [6], we obtained that (2)-(3) hold for every integer $q \geq 3$.

Moreover, letting

$$f(q) := \log \log q - \log 2 + 1/2 + 1/\log \log q, \quad g(q) := f(q) + 14(\log \log q)/\log q, \quad (4)$$
we numerically obtained that
\[
\frac{\pi^2}{12e^\gamma} \frac{3.25}{g(q)} < |L(1, \chi)| < 0.47 \cdot 2e^\gamma f(q)
\] (5)
hold for \(3 \leq q \leq 1000\), and, concerning Littlewood’s estimates in (1), that
\[
0.425 \cdot \frac{\pi^2}{12e^\gamma} \log \log q < |L(1, \chi)| < 0.71 \cdot 2e^\gamma \log \log q,
\] (6)
holds for \(4 \leq q \leq 1000\). We note that Chowla [2] proved that there are infinitely many quadratic characters \(\chi \mod q\) such that \(L(1, \chi) \geq (1 + o(1))e^\gamma \log \log q\). The coefficient of \(\log \log q\) on the left side of (6) is 0.1962..., which is a long way from the Chowla’s \(e^\gamma = 1.781\). We refer the reader to the excellent article by Granville and Soundararajan [4] for more on this topic.

Let now \(q \geq 5\), \(-q\) be a discriminant and \(\chi_{-q}(n) = (-q \mid n)\) be the Kronecker symbol, which is a primitive character \(\mod q\). Moreover, let \(h(-q)\) denote the class number of the imaginary quadratic field \(\mathbb{Q}(\sqrt{-q})\). From Theorem [1] and the famous Dirichlet class number formula, \(i.e.,\)
\[
h(-q) = \frac{\sqrt{q}}{\pi} L(1, \chi_{-q}),
\]
we obtain the following

**Corollary 1.** Assume the Generalised Riemann Hypothesis. Let \(-q\) be a discriminant and let \(h(-q)\) denote the class number of \(\mathbb{Q}(\sqrt{-q})\). For every \(q \geq 5\) we have that
\[
h(-q) \leq \frac{2e^\gamma}{\pi} \sqrt{q} \left(\log \log q - \log 2 + \frac{1}{2} + \frac{1}{\log \log q}\right)
\]
and
\[
h(-q) \geq \frac{\pi}{12e^\gamma} \sqrt{q} \left(\log \log q - \log 2 + \frac{1}{2} + \frac{1}{\log \log q} + \frac{14 \log \log q}{\log q}\right)^{-1}.
\]

Corollary [1] extends the range of validity of Corollary 1.3 of [5] from \(q \geq 10^{10}\) to \(q \geq 5\).

It is natural to ask whether similar results hold for bounds on \(|\zeta(1 + it)|\). This was noted in [5, p. 2394], where bounds analogous to (2) and (3) were stated to hold for \(t \geq 10^{10}\), under the assumption of the Riemann hypothesis. There are some additional computational difficulties in extending the work of this article to cover \(\zeta(1 + it)\). We leave this to future work.

The paper is organised as follows: in Section [2] we state the needed lemmas from [5] and in Section [3] we prove that (2)-(3) hold for every \(q \geq 404\). Then in Section [3,3] we use a computer program to verify the validity of (2)-(3) for every \(q\) with \(3 \leq q \leq 1000\). In Section [4] after the references, we include Tables [1,2] about the values of \(|L(1, \chi)|\), \(\chi \mod q\), \(3 \leq q \leq 1000\). These are augmented by Figures [1,6] which contain several scatter plots.

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2. Definitions and Lemmas

We require the following definitions and lemmas from Lamzouri, Li, and Soundararajan [5]. Throughout the paper we assume the truth of the Generalised Riemann Hypothesis (GRH).

We let \(\vartheta\) stand for a complex number of magnitude at most one. In each occurrence \(\vartheta\) may stand for a different value, so that we may write \(\vartheta - \vartheta = 2\vartheta, \vartheta \cdot \vartheta = \vartheta\) and so on. We recall that
\[
\xi(s) = s(s - 1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),
\]
where \( \zeta(s) \) is the Riemann zeta-function and \( \Gamma(s) \) is Euler’s function. The function \( \xi(s) \) is an entire function of order 1, satisfying the functional equation \( \xi(s) = \xi(1-s) \), and for which the Hadamard factorisation formula gives

\[
\xi(s) = e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right)e^{s/\rho},
\]

where \( \rho \) runs over the non-trivial zeros of \( \xi(s) \), and \( B \) is a real number given by

\[
B = -\sum_{\rho} \Re \frac{1}{\rho} = \frac{1}{2} \log(4\pi) - 1 - \frac{\gamma}{2} = -0.0230957089 \ldots.
\]

Let \( L(s, \chi) \) be the Dirichlet \( L \)-function associated to a non-principal primitive Dirichlet character \( \chi \) defined mod \( q \), where \( q \geq 3 \) is an integer. Let \( a = 0 \) if \( \chi \) is even and \( a = 1 \) if \( \chi \) is odd. Let \( \xi(s, \chi) \) be the completed \( L \)-function

\[
\xi(s, \chi) = \left( \frac{q}{\pi} \right)^{s/2} \frac{\Gamma\left( \frac{s + a}{2} \right)}{\Gamma\left( \frac{s}{2} \right)} L(s, \chi)
\]

which satisfies the functional equation

\[
\xi(s, \chi) = e_\chi \xi(1-s, \chi^*),
\]

where \( e_\chi \) is a complex number of size 1. The zeros of \( \xi(s, \chi) \) are the non-trivial zeros of \( L(s, \chi) \), and letting \( \rho_\chi = 1/2 + i\gamma_\chi \) denote such a zero, we have Hadamard’s factorisation formula:

\[
\xi(s, \chi) = \exp(A(\chi) + sB(\chi)) \prod_{\rho_\chi} \left(1 - \frac{s}{\rho_\chi}\right)e^{s/\rho_\chi},
\]

where \( A(\chi) \) and \( B(\chi) \) are constants. In particular

\[
\Re B(\chi) = \Re B(\chi^*) = \Re \frac{\xi'(0, \chi)}{\xi(0, \chi)} = -\sum_{\rho_\chi} \Re \frac{1}{\rho_\chi}.
\]

Recall now the digamma function \( \psi(z) = \Gamma'/\Gamma(z) \); we will need the following special values

\[
\psi(1) = \gamma, \quad \psi\left(\frac{1}{2}\right) = -2 \log 2 - \gamma.
\]

Using such definitions, we now state the lemmas we need from [5].

**Lemma 1 (Lemma 2.3 of [5]).** Assume GRH. Let \( q \geq 3 \) and let \( \chi \) mod \( q \) be a primitive Dirichlet character. For any \( x > 1 \), we have, for some \( |\theta| \leq 1 \), that

\[
\frac{\xi'(0, \chi)}{\xi(0, \chi)} - \frac{1}{x} \frac{\xi'(0, \chi)}{\xi(0, \chi)} + \frac{2\theta}{\sqrt{x}} |\Re B(\chi)| = \frac{1}{2} \left(1 - \frac{1}{x}\right) \log \frac{q}{\pi} - \sum_{n \leq x} \frac{\Lambda(n)\chi(n)}{n} \left(1 - \frac{n}{x}\right) + E_0(x),
\]

where

\[
E_0(x) := - \log 2 - \frac{\gamma}{2} \left(1 - \frac{1}{x}\right) + \log \frac{x + 1}{x} - \sum_{k=1}^{\infty} \frac{x^{-2k-1}}{2k(2k + 1)}
\]

and

\[
E_1(x) := - \sum_{k=0}^{\infty} \frac{x^{-2k-2}}{(2k + 1)(2k + 2)} - \frac{\gamma}{2} \left(1 - \frac{1}{x}\right) + \log \frac{2}{x}.
\]

In particular, \( |\Re B(\chi)| \) equals

\[
\left(1 + \frac{2\theta}{\sqrt{x}} + \frac{1}{x}\right)^{-1} \left(\frac{1}{2} \left(1 - \frac{1}{x}\right) \log \frac{q}{\pi} - \Re \sum_{n \leq x} \frac{\Lambda(n)\chi(n)}{n} \left(1 - \frac{n}{x}\right) + E_0(x)\right).
\]
Lemma 2 (Lemma 2.4 of [5]). Assume RH. For \( x > 1 \) we have, for some \(|\theta| \leq 1\), that
\[
\Sigma_2 := \sum_{n \leq x} \frac{\Lambda(n)}{n} \left( 1 - \frac{n}{x} \right) = \log x - (1 + \gamma) + \frac{\log(2\pi)}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + 2\theta \frac{B}{\sqrt{x}}.
\]

Lemma 3 (Lemma 2.5 of [5]). Let \( q \geq 3 \) and let \( \chi \mod q \) be a primitive Dirichlet character. Suppose that GRH holds for \( L(s, \chi) \). For any \( x \geq 2 \), there exists a real number \(|\theta| \leq 1\) such that
\[
\log |L(1, \chi)| = \text{Re} \sum_{n \leq x} \frac{\chi(n)\Lambda(n)}{n \log n} \frac{\log(x/n)}{\log x} + \frac{1}{2 \log x} \left( \log \frac{q}{\pi} + \psi \left( \frac{1 + \alpha}{2} \right) \right) - \left( \frac{1}{\log x} + \frac{2\theta}{\sqrt{x} \log(x)^2} \right) |\text{Re} B(\chi)| + \frac{2\theta}{x \log(x)^2}.
\]

Lemma 4 (Lemma 2.6 of [5]). Assume RH. For all \( x \geq e \), there exists a real number \(|\theta| \leq 1\) such that
\[
\Sigma_1 := \sum_{n \leq x} \frac{\Lambda(n)}{n \log n} \frac{\log(x/n)}{\log x} = \log x + \gamma - 1 + \frac{\gamma}{\log x} + \frac{2B\theta}{\sqrt{x} \log(x)^2} + \frac{\theta}{3x^3 \log(x)^2}.
\]

Lemma 5 (Lemma 5.1 of [5]). Let \( q \geq 3 \) and let \( \chi \mod q \) be a primitive Dirichlet character. For \( x \geq 3 \) we have
\[
\text{Re} \sum_{n \leq x} \Lambda(n)\chi(n) \left( \frac{1}{n \log n} - \frac{1}{x \log x} \right) \geq \sum_{p^k \leq x} \Lambda(p^k)(-1)^k \left( \frac{1}{p^k \log p^k} - \frac{1}{x \log x} \right).
\]

We remark that Lemma 5 was stated just for \( x \geq 100 \) in [5], but it is easy to verify that its proof holds for \( x \geq 3 \).

3. Proof of Theorem 1

From now on we assume \( 3 \leq q < 10^{10} \) since the result was already proved for \( q \geq 10^{10} \) in [5].

3.1. Upper bound. Let \( x \geq 2 \). Using Lemma 5 and recalling that \(|\theta| \leq 1\) we have
\[
\log |L(1, \chi)| \leq \text{Re} \sum_{n \leq x} \frac{\chi(n)\Lambda(n)}{n \log n} \frac{\log(x/n)}{\log x} + \frac{1}{2 \log x} \left( \log \frac{q}{\pi} + \psi \left( \frac{1 + \alpha}{2} \right) \right) - \left( \frac{1}{\log x} + \frac{2\theta}{\sqrt{x} \log(x)^2} \right) |\text{Re} B(\chi)| + \frac{2\theta}{x \log(x)^2}.
\]

For \( x \geq 2 \), we can use Lemma 1 so that
\[
|\text{Re} B(\chi)| \geq \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} \left( \frac{1}{2} \left( 1 - \frac{1}{x} \right) \log \frac{q}{\pi} - \text{Re} \sum_{n \leq x} \frac{\Lambda(n)\chi(n)}{n} \left( 1 - \frac{n}{x} \right) + E_a(x) \right).
\]

We wish to show, for \( a \in \{0, 1\} \) and \( x \geq 9 \), that
\[
C^+_a(x) := -E_a(x) \left( 1 + \frac{1}{\sqrt{x}} \right)^{-2} \left( \frac{1}{2 \log x} - \frac{2}{\sqrt{x} \log(x)^2} \right) + \frac{1}{2 \log x} \psi \left( \frac{1 + \alpha}{2} \right) + \frac{2}{x \log(x)^2} \leq 0.
\]

This is equivalent to proving that \( D^+_a(x) := C^+_a(x) \log x \leq 0 \) for \( a \in \{0, 1\} \) and \( x \geq 9 \).
We first study the case \( a = 0 \) of (12). We begin by proving that \( E_0(x) < 0 \) for \( x \geq 2 \). To do this, let us consider upper and lower bounds on the sum over \( k \) in \( E_0(x) \) in (8). We have, for any \( K \geq 1 \), that
\[
\sum_{k=1}^{\infty} \frac{x^{-2k-1}}{2k(2k+1)} = \sum_{k=1}^{K} \frac{x^{-2k-1}}{2k(2k+1)} + \sum_{k \geq K+1} \frac{x^{-2k-1}}{2k(2k+1)} \\
\leq \sum_{k=1}^{K} \frac{x^{-2k-1}}{2k(2k+1)} + \frac{1}{x^3+2K} \sum_{k \geq K+1} \frac{1}{2k(2k+1)} \\
\leq \sum_{k=1}^{K} \frac{x^{-2k-1}}{2k(2k+1)} + \frac{1}{4k^3+2K} \sum_{k \geq K+1} \frac{1}{k^2} \\
\leq \sum_{k=1}^{K} \frac{x^{-2k-1}}{2k(2k+1)} + \frac{1}{4Kx^3+2K}.
\] (13)

Clearly, for any \( K \geq 1 \), we also have
\[
\sum_{k=1}^{\infty} \frac{x^{-2k-1}}{2k(2k+1)} \leq \sum_{k=1}^{K} \frac{x^{-2k-1}}{2k(2k+1)}.
\]

Hence
\[
-E_0(x) \geq \log 2 + \frac{\gamma}{2}(1 - \frac{1}{x}) - \frac{\log x + 1}{x} + \sum_{k=1}^{K} \frac{x^{-2k-1}}{2k(2k+1)}.
\] (14)

Taking \( K = 2 \) we see that right-hand side in (14) is positive for all \( x \geq 2 \). Hence, remarking that \((1 + 1/\sqrt{x})^{-2} \leq 1 \) and that \( 1 - 2/(\sqrt{x}\log x) > 0 \) for every \( x \geq 4 \), we have that
\[
-E_0(x) \leq F_0^+(x) := \log 2 + \frac{\gamma}{2}(1 - \frac{1}{x}) - \frac{\log x + 1}{x} + \sum_{k=1}^{K} \frac{x^{-2k-1}}{2k(2k+1)} + \frac{1}{4Kx^3+2K}.
\] (15)

and that
\[
D_0^+(x) = -\frac{2}{x\log x} F_0^+(x),
\]
\[
= G_0^+(x) - \frac{2}{\sqrt{x}\log x} F_0^+(x),
\] (16)

say. Choosing \( K = 2 \) in (15) and (16), we immediately have that \( G_0^+(x) < 0 \) and \( F_0^+(x) > 0 \) for every \( x \geq 3 \). This proves that \( D_0^+(x) < 0 \) for every \( x \geq 4 \). Hence (12) holds for \( a = 0 \) and \( x \geq 4 \).

We now study the case \( a = 1 \) of (12). This is done in the same way, noting that \( \psi(1) = -\gamma \) in (12) accounts for the main term in \( E_1(x) \) in (9). We estimate the sum over \( k \) in the definition of \( E_1(x) \) in (9) in the same way as in (13). This gives
\[
\sum_{k=0}^{\infty} \frac{x^{-2k-2}}{(2k+1)(2k+2)} \leq \sum_{k=0}^{K} \frac{x^{-2k-2}}{(2k+1)(2k+2)} + \frac{1}{4Kx^4+2K}.
\] (17)

Using this we see that \( E_1(x) \leq 0 \) for all \( x \geq 4 \). We therefore use \((1 + 1/\sqrt{x})^{-2} \leq 1 \) and have
\[
-E_1(x) \leq F_1^+(x) := \frac{\gamma}{2}(1 - \frac{1}{x}) - \frac{\log 2}{x} + \sum_{k=0}^{K} \frac{x^{-2k-2}}{(2k+1)(2k+2)} + \frac{1}{4Kx^4+2K}.
\] (18)
We proceed to obtain a bound similar to (16), namely
\[
D_1^+(x) \leq -\frac{\gamma}{2x} - \frac{\log 2}{x} + \frac{2}{x \log x} + \frac{\Lambda(n) \chi(n)}{n} \left(1 - \frac{n}{x}\right) + E_0(x) + E_1(x)
\]
where we use Lemma 4 to bound \(\Sigma_1\) and Lemma 2 to bound \(\Sigma_2\), taking just one term in the sum over \(n\). Putting all of this together, we set \(x = 1/4(\log q)^2\), as in [5], and exponentiate. We aim at showing that the resultant expression does not exceed the bound in (2), subject to the restriction that \(x \geq 9\). We have that (20) implies (2) whenever \(q \geq 15\), however, the bound \(x \geq 9\) means that we only win for \(q \geq 404\).

### 3.2. Lower bound

Let \(x \geq 2\). We begin with a lower bound on \(\log |L(1, \chi)|\) similar to that obtained in (11). Again for \(x \geq 2\), we can use Lemma 1 so that
\[
|\text{Re } B(\chi)| \leq \left(1 - \frac{1}{\sqrt{x}}\right)^2 \left(1 - \frac{1}{x}\right) \log \frac{q}{\pi} - \text{Re} \sum_{n \leq x} \Lambda(n) \chi(n) \left(1 - \frac{n}{x}\right) + E_0(x).
\]
We wish to show, for \(a \in \{0, 1\}\) and \(x \geq 5\), that
\[
C_a^-(x) := -E_0(x) \left(1 - \frac{1}{\sqrt{x}}\right)^2 \left(1 - \frac{1}{\log x + \frac{2}{\sqrt{x}(\log x)^2}}\right) + \frac{1}{2} \log x \psi \left(1 + \frac{a}{2}\right) - \frac{2}{x (\log x)^2} \geq 0.
\]
This is equivalent to proving that \(D_0^-(x) := C_0^-(x) \log x \geq 0\) for \(a \in \{0, 1\}\) and \(x \geq 9\). We recall that in \(\S 3.1\) we proved that \(E_0(x) < 0\) for \(x \geq 2\) and \(E_1(x) < 0\) for \(x \geq 4\).

We first study the case \(a = 0\) of (21). We use \((1 - 1/\sqrt{x})^{-2} \geq 1 + 2/\sqrt{x}\) and take \(K = 1\) to see that \(C_0^-(x) \geq 0\) for \(x \geq 6\).

We now study the case \(a = 1\) of (21). Again, we need to use the inequality \((1 - 1/\sqrt{x})^{-2} \geq 1 + 2/\sqrt{x}\) rather than just the trivial bound of \(\geq 1\). Taking \(K = 1\) shows that \(C_1^-(x) \geq 0\) for \(x \geq 9\).

Now we proceed to prove the lower bound. Keeping all terms together we invoke Lemmas 2 and 4 and only require \(x \geq 9\) since we just proved that then \(C_a^-(x) \geq 0\) for \(a \in \{0, 1\}\).

Note that Lemma 5 holds for any \(x \geq 3\). Note also that we only require lower bounds on \(\sum_{n \geq 1} x^{-2n-1}/(2n(2n + 1))\), whence we just take the term \(n = 1\). Arguing as on page 2408 of [5] and putting everything together we arrive at the following, for \(x \geq 9\),
\[
\log |L(1, \chi)| \geq - \left(\log \log x + \gamma - 1 - \frac{\gamma}{\log x} + \frac{2|B|}{\sqrt{x} \log^2 x}\right) - \frac{1}{\log x} \left(\log x - (1 + \gamma) + \frac{\log 2 \pi}{x} - \frac{1}{6x^3} + \frac{2|B|}{\sqrt{x}}\right) + \log \zeta(2) - \frac{3}{2\sqrt{x}}.
\]
The analysis on these data to verify the inequalities in (5)-(6) were performed using
we will obtain
≤
Tables 1 and 2 we present such data for 3
https://www.math.unipd.it/~languasc/LLS_ineq.html
downloaded from the following
seconds of computation time for each table. The machine we used was a Dell OptiPlex-3050,
equipped with an Intel i5-7500 processor, 3.40GHz, 16 GB of RAM and running Ubuntu
3.3.
\sum_{n \leq x} \frac{\Lambda(n)}{n} (1 - \frac{n}{x}) = \log x - (1 + \gamma) + \frac{\log 2\pi}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + \frac{2\theta B}{\sqrt{x}}
\leq \log x - (1 + \gamma) + \frac{\log 2\pi}{x} - \frac{1}{6x^3} - \frac{2B}{\sqrt{x}}
\Rightarrow \log x - (1 + \gamma) + a(x).
(23)
Note then, that \( a(x) \) is positive and tends to zero as \( x \rightarrow \infty \). We now use the definition of \( a(x) \) from (22) in (22), and exponentiate, taking \( x = 1/4(\log q)^2 \). We find that the bound obtained by (22) implies (3) whenever \( q \geq 120 \). However, as before, we must satisfy the condition that \( x \geq 9 \). With our choice of \( x = 1/4(\log q)^2 \) we may only deduce the result in the theorem when \( q \geq 404 \).

3.3. Computation of \(|L(1,\chi)|\) for \( 3 \leq q \leq 1000 \). Letting
\[ M_q := \max_{x \neq x_0} |L(1,\chi)| \quad \text{and} \quad m_q := \min_{x \neq x_0} |L(1,\chi)|, \]
we will obtain \( M_q, m_q \) using PARI/GP, v. 2.13.0, since it has the ability to generate Dirichlet
\( L \)-functions (and many other \( L \)-functions). This can be done with few instructions of the
\( \text{gp} \) scripting language. Such a computation has a linear cost in the number of calls of the
\texttt{lfun} function of PARI/GP. We were able to get the values of \( M_q, m_q \) for every integer \( q, 3 \leq q \leq 1000 \), with a precision of 30 decimal digits within 19 minutes and 57 seconds of
computation time (or 13 minutes and 4 seconds for just the composite integers we needed). In
Tables 1 and 2 we present such data for \( 3 \leq q \leq 200 \); to get them we needed less than
24 seconds of computation time for each table. The machine we used was a Dell OptiPlex-3050,
equipped with an Intel i5-7500 processor, 3.40GHz, 16 GB of RAM and running Ubuntu
18.04.4. The analysis on these data to verify the inequalities in (5)-(6) were performed using
a python-pandas program. All the programs used here can be downloaded from the following
address: https://www.math.unipd.it/~languasc/LLS_ineq.html

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UNIFORM EFFECTIVE ESTIMATES FOR $|L(1, \chi)|$

### 4. Tables and figures

| $M_r$ | $M_s$ |
|-------|-------|
| 0.6021374611542069183984091702 | 1.16199535095728554886466125749 |
| 0.75839139436489743809156685381 | 1.12573441141745605427398794372 |
| 0.888576876137249430167196801 | 0.7205111179275849631423587892 |
| 0.6045997080726768664066257524 | 2.021671441520481993389417082 |
| 0.1784014117237229487625795259 | 0.144200953255670685782972430737 |
| 1.1107207351614523739550509 | 0.34659841230939254927926513 |
| 0.885876876137249430167196801 | 0.129214852856159369991478664 |
| 0.42640418224018352099831537338 | 0.72068969171758748564391828799 |
| 0.78539139436489743809156685381 | 0.49865862066262104571951253 |
| 0.7613478807703972296194009 | 0.41570760737649961821239136169 |
| 0.1784014117237229487625795259 | 0.3512296750783991026647857 |
| 0.42640418224018352099831537338 | 0.41570760737649961821239136169 |
| 0.78539139436489743809156685381 | 0.41570760737649961821239136169 |
| 0.42640418224018352099831537338 | 0.41570760737649961821239136169 |
| 0.7613478807703972296194009 | 0.41570760737649961821239136169 |

Table 1. Values of $M_r$ for every $3 \leq r \leq 200$ with 30-digit precision (the last printed digit is rounded by PARI/GP); computed with PARI/GP, v. 2.13.0. Total computation time: 23 sec., 471 milliseconds.
Table 2. Values of $m_q$ for every $3 \leq q \leq 200$ with 30-digit precision (the last printed digit is rounded by PARI/GP); computed with PARI/GP, v. 2.13.0. Total computation time: 23 sec., 696 milliseconds.
The values of $M_q, 3 \leq q \leq 1000$. The minimal value for $M_q$ is $0.604599 \ldots$ attained at $q = 3$ and the maximal one is $3.652103 \ldots$ attained at $q = 959$. The blue line represents $0.47 \cdot L_2 f(q)$, $L_2 = 2e^7$, where $f(q)$ is defined in \[4\]. The green line represents $0.71 \cdot L_2 \log \log q$.

The values of $M'_q := M_q / f(q), 3 \leq q \leq 1000$, where $f(q)$ is defined in \[4\]. The minimal value for $M'_q$ is $0.057396 \ldots$ attained at $q = 3$ and the maximal one is $1.648945 \ldots$ attained at $q = 479$. The blue line represents $0.47 \cdot L_2, L_2 = 2e^7$. 
The values of $M''_q := M_q / \log \log q$, $3 \leq q \leq 1000$. The minimal value for $M''_q$ is $0.799902 \ldots$ attained at $q = 144$ and the maximal one is $6.428641 \ldots$ attained at $q = 3$. The green line represents $0.71 \cdot L_2$, where $L_2 = 2e^γ$.

The values of $m_q$, $3 \leq q \leq 1000$. The minimal value for $m_q$ is $0.246068 \ldots$ attained at $q = 163$ and the maximal one is $0.785398 \ldots$ attained at $q = 4$. The red line represents $3.25L_1/g(q)$, where $g(q)$ is defined in (4) and $L_1 = \frac{\pi^2}{12}\gamma$. The orange line represents $0.425L_1/\log \log q$. 

\[ m(q)g(q) \]

**Figure 5.** The values of \( m_q' := m_q g(q), 3 \leq q \leq 1000, \) where \( g(q) \) is defined in (4). The minimal value for \( m_q' \) is 1.520104... attained at \( q = 978 \) and the maximal one is 7.093329... attained at \( q = 3 \). The red line represents \( 3.25L_1 \), where \( L_1 = \frac{\pi^2}{12\pi^2} \).

\[ m_q \log \log q \]

**Figure 6.** The values of \( m_q'' := m_q \log \log q, 3 \leq q \leq 1000 \). The minimal value for \( m_q'' \) is 0.056861... attained at \( q = 3 \) and the maximal one is 1.148889... attained at \( q = 864 \). The orange line represents \( 0.425L_1 \), where \( L_1 = \frac{\pi^2}{12\pi^2} \).