COHOMOLOGICAL INVARIANTS OF GROUPS OF ORDER $p^3$

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Abstract. We show that the nonabelian groups of order $p^3$ for odd prime $p$ have no nontrivial cohomological invariants of degree 3.

1. Introduction

For $p$ an odd prime, there are precisely two nonabelian groups of order $p^3$, both arising as central extensions of $\mathbb{Z}/p \times \mathbb{Z}/p$ by $\mathbb{Z}/p$. Throughout, let $G_i$ be the nonabelian group of order $p^3$ and exponent $p^i$, for $i = 1, 2$. (These groups go by various notations and names; my notation here is by no means standard, but hopefully the subscripts provide a bit of a mnemonic.) These groups have the presentations

$$G_i = \langle a, b, c : c^p = b^p = [a, c] = [b, c] = 1, [a, b] = c, a^p = c^{i-1} \rangle.$$ 

In other words, $c$ generates the center, and $a$ and $b$ generate the two factors under the projection map to $\mathbb{Z}/p \times \mathbb{Z}/p$.

Both the integral and mod $p$ cohomology of these groups have been computed (see for example [5] and [6]), as have the Chow groups [8]. The complete computation of their motivic cohomology, and in particular a complete description of their rings of cohomological invariants, has not been given. Here we contribute to that endeavor by tying together several recent results to compute cohomological invariants of small degree. In particular, we show that these groups have no nontrivial invariants of degree 3.

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2. A bit of background

Write $\zeta = \exp(2\pi i/p^2)$ and $\omega = \exp(2\pi i/p)$. There are $p$-dimensional complex representations $\rho_i$ of the groups $G_i$ given explicitly as follows:

For $G_1$, the representation $\rho_1$ is given by sending

$$a \mapsto \text{diag}(1, \omega, \omega^2, \ldots, \omega^{p-1}),$$

$$b \mapsto (\delta_{i,j-1}),$$

$$c \mapsto \text{diag}(\omega, \ldots, \omega),$$

where $\delta_{i,j} = 1$ if $i \equiv j \pmod{p}$ and 0 else.
For $G_2$, the representation $\rho_2$ is given by sending
\[
\begin{align*}
a &\mapsto \text{diag}(\zeta, \zeta^{1+p}, \zeta^{1+2p}, \ldots, \zeta^{1+p(p-1)}), \\
b &\mapsto (\delta_{i,j}), \\
c &\mapsto \text{diag}(\omega, \ldots, \omega).
\end{align*}
\]

In both cases, the action is free outside of a finite number of lines. Thinking of $G_i$ as a complex algebraic group, the representation $\rho_i$ gives a free action of $G_i$ on the variety $U = \mathbb{A}^p_c - S$, where $S$ is the set on which the action has nontrivial stabilizers. As Guillot describes in [4], the generic fiber of the map $U \to U/G_i$ is a versal $G_i$-torsor; therefore the ring $\text{Inv}^*(G_i)$ of cohomological invariants can be thought of as sitting inside $H^*_{\text{et}}(\mathcal{C}(U/G_i), \mathbb{Z}/p)$, since a cohomological invariant is determined by its value on a versal torsor. Note that since $U/G_i$ is $p$-dimensional, this implies that $\text{Inv}^n(G_i)$ is trivial for $n > p$.

The image of a versal torsor in $H^*_{\text{et}}(\mathbb{C}(U/G_i), \mathbb{Z}/p)$ will be unramified along every valuation arising from a codimension 1 irreducible subscheme. Since $\text{codim}(S) > 1$, Totaro showed that every such unramified class gives rise to an invariant (letter to Serre reprinted in [3]). Therefore, letting $X$ be an appropriate approximation to the classifying space $BG_i$, the invariants $\text{Inv}^s(G_i)$ appear as the $E_2^{0,s}$ term of the Bloch-Ogus spectral sequence [1]:
\[
E_2^{r,s} = H^r(X, \mathcal{H}^s) \Rightarrow H^{r+s}_{\text{et}}(X, \mathbb{Z}/p),
\]
where the sheaf $\mathcal{H}^s$ is the Zariski sheaf induced by the presheaf $U \mapsto H^s_{\text{et}}(U, \mathbb{Z}/p)$ for $U \subset X$ Zariski open. (A note about the choice of $X$: We may take $X = U/G_i$ as defined above, but we may want the codimension of $S$ to be larger. Totaro showed that there is a sequence of quasi-projective varieties $(V_n - S_n)/G_i$ where $V_n$ is an affine representation and $G_i$ acts freely outside $S_n$ such that $\text{codim} S_n \to \infty$ with $n$ ([7] Remark 1.4). We will take $X$ to be one of these varieties with codim $S_n$ as high as required.)

The diagonal terms $E_2^{r,s}$ are isomorphic to $CH^r(X) \otimes \mathbb{Z}/p$. The following proposition (corollary 5.1.5 in [4]) is obvious from examination of the low-degree terms of the spectral sequence:

**Proposition 2.1.** For $X = U/G_i$ as described above, we have
\[
\text{Inv}^1(G_i) \cong H^1_{\text{et}}(X, \mathbb{Z}/p)
\]
and
\[
\text{Inv}^2(G_i) \cong \frac{H^2_{\text{et}}(X, \mathbb{Z}/p)}{CH^1(X) \otimes \mathbb{Z}/p}.
\]
Here we identify $CH^1(X) \otimes \mathbb{Z}/p$ with its image inside $H^2_{\text{et}}(X, \mathbb{Z}/p)$ under the map induced by the spectral sequence, which coincides with the standard mod $p$ cycle class map. Indeed, as we are over the complex numbers, $CH^1(X) \cong H^2(X, \mathbb{Z})$ and by the Bockstein we get that $\text{Inv}^2(G_i) \cong H^3(X, \mathbb{Z})[p]$.

The other concept we will need is that of stable cohomology, and stable cohomological dimension, as defined by Bogomolov and Böhm in [2]. They...
define the stable cohomology of a group $G$ over $\mathbb{C}$ with coefficients in $\mathbb{Z}/p$ (denoted $H^*_s(G,\mathbb{Z}/p)$) as the quotient of $H^*(BG,\mathbb{Z}/p)$ by the kernel of the map

$$H^*(BG,\mathbb{Z}/p) \to H^*_{\text{ét}}(\mathbb{C}(V/G),\mathbb{Z}/p),$$

where $V$ is any generically free complex linear representation of $G$. (They prove that this is independent of choice of representation $V$.) This map coincides with the composition

$$H^*(BG,\mathbb{Z}/p) \to \text{Inv}^*(G) \hookrightarrow H^*_{\text{ét}}(\mathbb{C}(V/G),\mathbb{Z}/p),$$

where the first map is induced from the Bloch-Ogus spectral sequence, and the second is inclusion of the invariants into $H^*_{\text{ét}}(\mathbb{C}(V/G),\mathbb{Z}/p)$ as the classes that are unramified along all valuations defined by codimension one subvarieties of $(V - S)/G$ for $S$ as described above. Since this second map is injective, the kernel of the composition is the kernel of the first map. Therefore we have for each $i$ that

$$H^i_s(G,\mathbb{Z}/p) \cong \text{im}(H^i(BG,\mathbb{Z}/p) \to \text{Inv}^i(G)).$$

The stable cohomological dimension of $G$, $\text{scd}(G)$, is defined as the lowest integer $d$ such that $H^i_s(G,F) = 0$ for all $i > d$ and all $G$-modules $F$.

3. Degree 3 Invariants

We now have the ingredients to prove our main result.

**Theorem 3.1.** For either group $G_i$ as above, we have $\text{Inv}^3(G_i) = 0$.

**Proof.** From the Bloch-Ogus spectral sequence for appropriate $X$, we get a differential

$$d : \text{Inv}^3(G_i) \to CH^2(BG_i) \otimes \mathbb{Z}/p.$$ 

Here we’ve taken $X$ such that $CH^2(X) \cong CH^2(BG_i)$. The theorem will follow from showing first that $\text{im} d = 0$ and second that $\ker d = 0$.

The first step follows immediately from a result of Yagita’s and the Bockstein long exact sequence. The following lemma is Theorem 1.1 in [8]:

**Lemma 3.2.** The cycle class map is an isomorphism $CH^*(BG_i) \cong H^{2r}(BG_i,\mathbb{Z})$.

Therefore in this case, the diagonal entries $E_2^{r,r}$ of the Bloch-Ogus spectral sequence, for appropriate $X$ (that is, with the codimension of $S$ sufficiently large) are isomorphic to $H^{2r}(BG_i,\mathbb{Z}) \otimes \mathbb{Z}/p$, and the spectral sequence provides a map

$$e : H^{2r}(BG_i,\mathbb{Z}) \otimes \mathbb{Z}/p \to H^{2r}(BG_i,\mathbb{Z}/p).$$

But this map coincides with the normal mod $p$ cycle class map (see [4]), which is given by composing the cycle map with the change of coefficients from $\mathbb{Z}$ to $\mathbb{Z}/p$. Since the change of coefficients map arises from the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0,$$

the only classes in $H^{2r}(BG_i,\mathbb{Z})$ that are sent to zero are those divisible by $p$; therefore the map is injective on $H^{2r}(BG_i,\mathbb{Z}) \otimes \mathbb{Z}/p$. Since the kernel of this
map is precisely im(d), this shows that im(d) = 0. (Indeed, it shows that the image of any differential into a diagonal entry of the spectral sequence must be zero.)

The second claim, that ker(d : Inv^3(G_i) → CH^2(BG_i) ⊗ Z/p) = 0, comes as a consequence of Bogomolov and Böhning’s computations of the stable cohomological dimension of the groups G_i.

**Lemma 3.3.** For each of the groups G_i, the stable cohomological dimension is at most 2.

**Proof.** The G_1 case: The group G_1 is the Heisenberg group H_p discussed by Bogomolov and Böhning. By corollary 5.8 in [2], scd(G_1) = 2.

The G_2 case: Consider the normal series

\[ \{1\} < \langle a \rangle < G_2. \]

This is indeed a normal series, since \([a, b] = a^p\), meaning that bab^{-1} = a^{1-p} is a power of a. Further, each quotient is clearly cyclic. Therefore theorem 3.1 of [2] applies, which says that the length of the nontrivial part of the series provides an upper bound on the stable cohomological dimension. In this case, that bound is 2.

Therefore we conclude that the image of the map H^3(BG_i, Z/p) → Inv^3(G_i) arising from the Bloch-Ogus spectral sequence is trivial. Since this map should be onto the kernel of d, that kernel is trivial as well. As we have shown that d : Inv^3(G_i) → CH^2(BG_i) ⊗ Z/p has both trivial image and trivial kernel, we conclude that Inv^3(G_i) = 0 for i = 1, 2.

\[ \square \]

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