Three-dimensional $\mathcal{N} = 2$ (AdS) supergravity and associated supercurrents

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Abstract

Long ago, Achúcarro and Townsend discovered that in three dimensions (3D) $\mathcal{N}$-extended anti-de Sitter (AdS) supergravity exists in several incarnations, which were called the $(p,q)$ AdS supergravity theories with non-negative integers $p \geq q$ such that $\mathcal{N} = p+q$. Using the superspace approach to 3D $\mathcal{N}$-extended supergravity developed in arXiv:1101.4013, we present three superfield formulations for $\mathcal{N} = 2$ supergravity that allow for well defined cosmological terms and supersymmetric AdS solutions. The conformal compensators corresponding to these theories are respectively: (i) a chiral scalar multiplet; (ii) a vector multiplet; and (iii) an improved complex linear multiplet. The theories corresponding to (i) and (iii) are shown to provide two dually equivalent realizations of the (1,1) AdS supergravity, while (ii) describes the (2,0) AdS supergravity. We associate with each supergravity formulation, with and without a cosmological term, a consistent supercurrent multiplet. The supercurrents in the (1,1) and (2,0) AdS backgrounds are derived for the first time. We elaborate on rigid supersymmetric theories in (1,1) and (2,0) AdS superspaces.

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1 Introduction

A great many $\mathcal{N} = 2$ supersymmetric theories in three dimensions (3D) can be obtained by dimensional reduction from 4D $\mathcal{N} = 1$ supersymmetric systems. Of particular interest, however, are those theories which do not allow for such a construction. They are characterized by purely 3D phenomena such as Chern-Simons couplings that

\[ \text{[1]} \]

The physical phenomena specific to three dimensions include the existence of real mass terms generated by a central charge. However, such mass terms can be obtained by dimensionally reducing a 4D $\mathcal{N} = 1$ system of chiral multiplets coupled to a certain background vector multiplet [1]. In the context of supersymmetric nonlinear $\sigma$-models, such mass terms were introduced for the first time in two dimensions [2] by using the Scherk-Schwarz mechanism for dimensional reduction [3].
are ubiquitous in three dimensions, both in pure gravity \cite{4, 5, 6, 7} and supergravity \cite{8, 9, 10, 11, 12, 13}. A non-trivial example of $\mathcal{N} = 2$ supersymmetric theories with Chern-Simons terms is the so-called (2, 0) anti-de Sitter (AdS) supergravity studied in \cite{10, 14}. More specifically, Achúcarro and Townsend \cite{10} discovered that in three dimensions $\mathcal{N}$-extended AdS supergravity exists in several incarnations. These were called the $(p, q)$ AdS supergravity theories where the non-negative integers $p \geq q$ are such that $\mathcal{N} = p + q$. It was shown in \cite{10} that these theories are naturally associated with the 3D AdS supergroups $\text{OSp}(p|2; \mathbb{R}) \times \text{OSp}(q|2; \mathbb{R})$. The $(0, 0)$ theory is simply 3D gravity with a negative cosmological term. The $(1, 0)$ theory coincides with the $\mathcal{N} = 1$ AdS supergravity first presented in \cite{15}. In the simplest extended case $\mathcal{N} = 2$, two different AdS supergravity theories emerge, $(1, 1)$ and $(2, 0)$, of which the former may be obtained by dimensional reduction from 4D $\mathcal{N} = 1$ AdS supergravity, while the latter is truly novel. It turns out that $(1, 1)$ and $(2, 0)$ AdS supergravity theories possess drastically different matter couplings. At the component level, certain matter couplings in $(2, 0)$ AdS supergravity were studied in \cite{14}. To the best of our knowledge, a superspace analysis of such problems has not yet appeared in the literature (a special off-shell version of 3D $\mathcal{N} = 2$ Poincaré supergravity was presented in \cite{16}). One of the goals of this paper is to fill this gap.

A robust approach to engineering Poincaré supergravity theories in diverse dimensions is to describe them as conformal supergravity coupled to certain compensating supermultiplet(s) \cite{17}. The same approach is clearly suitable to construct AdS supergravity models. In the case of 3D $\mathcal{N}$-extended conformal supergravity, conventional constraints on the superspace torsion were proposed in \cite{13}, and some of their implications were also analyzed. Starting from these constraints, in our recent work \cite{18} the superspace geometry of 3D $\mathcal{N}$-extended conformal supergravity was developed\footnote{The cases of $\mathcal{N} = 8$ and $\mathcal{N} = 16$ conformal supergravity theories have been worked out in \cite{19, 20} and \cite{21} respectively.} and then applied to construct general off-shell supergravity-matter couplings for the cases $\mathcal{N} \leq 4$. In the present paper we make use of the approach of \cite{18} in order to elaborate upon the case $\mathcal{N} = 2$. The main goals of this work are to study (i) the $(1, 1)$ and $(2, 0)$ AdS supergravity theories; and (ii) supersymmetric field theory in $(1, 1)$ and $(2, 0)$ AdS superspaces, including a thorough analysis of the consistent supercurrent multiplets corresponding to the two types of 3D $\mathcal{N} = 2$ AdS supersymmetry.

From the point of view of Poincaré supergravity, the 3D $\mathcal{N} = 2$ and 4D $\mathcal{N} = 1$ theories are very similar. A non-trivial difference between them proves to emerge only in the AdS
So let us first recall some general facts about the known off-shell versions of 4D $\mathcal{N} = 1$ supergravity (see [15, 22] for reviews), each of which can be realized as conformal supergravity coupled to a compensator [23, 15]. There exist three off-shell formulations of 4D $\mathcal{N} = 1$ Poincaré supergravity which are: (i) the old minimal ($n = -1/3$) [24] reviewed in [25]; the new minimal ($n = 0$) [26]; and (iii) the non-minimal ($n \neq -1/3, 0$) [27, 28].

In the conformal supergravity setting, they differ by the choice of compensator, which is respectively: (i) a chiral scalar multiplet; (ii) a massless tensor multiplet; or (iii) a non-minimal scalar multiplet described by a complex linear scalar and its conjugate. For a long time it was believed [15] that only the old minimal formulation is suitable to realize AdS supergravity by adding an appropriate cosmological term to the supergravity action (see [15, 22] for reviews). Recently it has been shown [29] that a certain version of non-minimal supergravity, $n = -1$, is equally suitable to describe AdS supergravity. However, this is achieved not by adding a cosmological term to the supergravity action, as in the $n = -1/3$ case, but instead by deforming the complex linear constraint obeyed by the compensator. The minimal and the non-minimal formulations of AdS supergravity are then dually equivalent [29]. As to the new minimal formulation of $\mathcal{N} = 1$ supergravity in four dimensions, $n = 0$, it cannot be used to describe AdS supergravity.

As regards 3D $\mathcal{N} = 2$ Poincaré supergravity, it also allows three different off-shell formulations [18] which are associated with the following choices of conformal compensator: (i) a chiral scalar multiplet; (ii) a massless vector multiplet; and (iii) a non-minimal scalar multiplet described by a complex linear scalar $\Sigma$ and its conjugate. They are 3D analogues of the old minimal, new minimal and non-minimal supergravity theories in four dimensions, respectively. The 3D supergravity versions (i) and (ii) will be called Type I minimal and Type II minimal, respectively, in what follows. As shown in [18], the 3D non-minimal theory is naturally parametrized by the super-Weyl weight of $\Sigma$, denoted $w$, which proves to be related to the 4D Siegel-Gates parameter $n$ as follows:

$$n = \frac{1 - w}{3w + 1}.$$ (1.1)

As in four dimensions, the Type I minimal and the $w = -1$ (or $n = -1$) non-minimal formulations can be used to describe AdS supergravity by modifying the supergravity action (in the Type I case) or deforming the complex linear constraint (in the non-minimal case). The two realizations turn out to be dually equivalent and lead to the same $(1,1)$ AdS supergravity. Unlike the situation in four dimensions, the Type II theory can also
be used to describe AdS supergravity, for now a cosmological term can be realized as the
supersymmetric Chern-Simons term associated with the compensating vector multiplet.
Adding such a cosmological term to the Type II supergravity action provides a superspace
description of (2,0) AdS supergravity!

For both (1,1) and (2,0) AdS supergravity theories, the equations of motion prove to
require the superspace geometry to have constant torsion and curvature. In the case of
Type I AdS supergravity, the on-shell geometry is described by covariant derivatives

$$\nabla_A = (\nabla_a, \nabla_{\alpha}, \nabla^{\alpha}) = E_A^M \partial_M + \frac{1}{2} \Omega_A^{cd} M_{cd}$$

(1.2)
observing the following algebra

$$\{\nabla_{\alpha}, \nabla_{\beta}\} = -4\mu M_{\alpha\beta}, \quad \{\nabla_{\alpha}, \nabla^{\beta}\} = 4\mu M_{\alpha\beta}, \quad \{\nabla_{\alpha}, \nabla^{\beta}\} = -2i \nabla_{\alpha\beta}, \quad (1.3a)$$

$$[\nabla_{\alpha\beta}, \nabla_{\gamma}] = -2i \mu \varepsilon_{\gamma(\alpha} \nabla_{\beta)} \quad \{\nabla_{\alpha}, \nabla^{\gamma}\} = 2i \mu \varepsilon_{\gamma(\alpha} \nabla_{\beta)}, \quad (1.3b)$$

$$[\nabla_a, \nabla_b] = -4\mu \mu M_{ab} \quad (1.3c)$$

with $\mu$ a constant complex parameter, and $M_{ab} = -M_{ba}$ and $M_{\alpha\beta} = M_{\beta\alpha}$ the Lorentz
generators with vector and spinor indices respectively (see section 2 for the explicit relation
between them). These (anti-)commutation relations define the geometry of (1,1) AdS
superspace. In the case of Type II AdS supergravity, the on-shell geometry is described by covariant derivatives

$$D_A = (D_a, D_{\alpha}, \bar{D}^{\alpha}) = E_A^M \partial_M + \frac{1}{2} \Omega_A^{cd} M_{cd} + i \Phi_A J$$

(1.4)
observing the following algebra:

$$\{D_{\alpha}, D_{\beta}\} = \{D_{\alpha}, \bar{D}_{\beta}\} = 0 \quad \{D_{\alpha}, D_{\beta}\} = -2i D_{\alpha\beta} - i \rho \varepsilon_{\alpha\beta} J + i \rho M_{\alpha\beta}, \quad (1.5a)$$

$$[D_{\alpha\beta}, D_{\gamma}] = -\frac{1}{2} \rho \varepsilon_{\gamma(\alpha} D_{\beta)} \quad [D_{\alpha\beta}, \bar{D}_{\gamma}] = -\frac{1}{2} \rho \varepsilon_{\gamma(\alpha} \bar{D}_{\beta)}, \quad (1.5b)$$

$$[D_a, D_b] = -\frac{1}{4} \rho^2 M_{ab} \quad (1.5c)$$

Here the constant real parameter $\rho$ determines the scale of the cosmological constant, and
$J$ is the generator of the $R$-symmetry group $U(1)_R$. The (anti-)commutation relations
(1.5) define the geometry of (2,0) AdS superspace.

Comparing the relations (1.3) and (1.5) shows that the two superspace geometries
are inequivalent, although the bosonic bodies of the two superspaces can be shown to
be identical and coincide with the ordinary AdS space. This indicates that properties
of supersymmetric field theory in the (1,1) AdS superspace may considerably differ from
those in the (2,0) case. In four dimensions, nontrivial information about supersymmetric theories defined on maximally symmetric superspaces is encoded in the structure of consistent supercurrent multiplets associated with these superspaces [30, 31]. Indeed, it has been shown that 4D $\mathcal{N} = 1$ rigid supersymmetric theories in AdS differ significantly from their counterparts defined in Minkowski space [32, 33, 31], and so do the corresponding supercurrent multiplets [30, 31]. This motivates us to study consistent supercurrents in the (1,1) and (2,0) AdS superspaces.

The supercurrent [34] is a supermultiplet containing the energy-momentum tensor and the supersymmetry current(s) as well as some other bosonic and fermionic operators. The supercurrent naturally originates as the source of supergravity [35, 36, 37], and this realization gives a powerful practical tool to compute this multiplet for a given supersymmetric field theory in Minkowski space (see [15, 22] for reviews). Specifically, if the theory under consideration can be coupled to an off-shell supergravity background, then its supercurrent and associated trace multiplet coincide with (covariantized) variational derivatives of the action with respect to the supergravity prepotentials evaluated at the background configuration corresponding to Minkowski superspace. Since there exist several off-shell formulations for 4D $\mathcal{N} = 1$ supergravity [24, 26, 27], there appear several consistent supercurrent multiplets, studied e.g. in [38, 39, 15], of which the Ferrara-Zumino multiplet [34] is usually considered to be universal. Another useful scheme to compute supercurrents is the superfield Noether procedure [40, 41] (which can in fact be derived from the off-shell supergravity techniques presented in [15, 22]).

Recently, there has been much interest in consistent $\mathcal{N} = 1$ supercurrents in four dimensions [42]–[50] inspired by two papers of Komargodski and Seiberg [42, 45]. These authors noticed the existence of certain rigid supersymmetric theories for which the Ferrara-Zumino (FZ) multiplet is not well defined. Such theories include (i) models with a Fayet-Iliopoulos term; and (ii) supersymmetric nonlinear $\sigma$-models with non-exact Kähler forms. In the case (i), the appropriate supercurrent was shown in [43, 44] to be the so-called $\mathcal{R}$-multiplet which is associated with the new minimal formulation of $\mathcal{N} = 1$ supergravity [26]. To furnish the case (ii) with a consistent supercurrent, Ref. [45] put forward the so-called $\mathcal{S}$-multiplet which incorporates both FZ and $\mathcal{R}$ multiplets as special limits. Although the $\mathcal{S}$-multiplet can be embedded in an even more general supercurrent [44, 49] of natural supergravity origin, it has recently been argued by Dumitrescu and Seiberg [50] that the $\mathcal{S}$-multiplet is the most general supercurrent modulo a well defined improvement.

\[ ^{4}\text{General } \mathcal{N} = 2 \text{ supercurrent multiplets in Minkowski and AdS space were constructed in [51] and [30] respectively.} \]
transformation. These authors have also derived a 3D $\mathcal{N} = 2$ super-Poincaré extension of the $\mathcal{S}$-multiplet. In spite of the fact that the $\mathcal{S}$-multiplet is fundamental in Poincaré supersymmetry, it does not have a natural extension to the AdS case in four dimensions [30, 31]. It is also to be expected that special care is required to construct consistent supercurrents for theories possessing the (1,1) and (2,0) AdS supersymmetry types in three dimensions. This problem is addressed in the present paper.

This paper is organized as follows. In section 2 we review and elaborate on the superspace geometry of $\mathcal{N} = 2$ conformal supergravity presented in [18]. In sections 3 to 5 we present three superfield formulations for $\mathcal{N} = 2$ supergravity that allow for well defined cosmological terms and supersymmetric AdS solutions. In section 6 we describe the realizations of (1,1) and (2,0) AdS superspaces as conformally flat supergeometries. Section 7 presents four off-shell formulations for linearized $\mathcal{N} = 2$ supergravity in Minkowski space. Using the explicit structure of the linearized supergravity actions, in section 8 we construct consistent supercurrent multiplets in Minkowski space and study their properties. Section 9 is devoted to rigid supersymmetric theories in (1,1) AdS superspace, and section 10 gives a similar analysis in the (2,0) case. Concluding comments are given in section 11. The main body of the paper is accompanied by an appendix in which we review the structure of 4D $\mathcal{N} = 1$ supercurrents in Minkowski space.

2 Geometry of $\mathcal{N} = 2$ conformal supergravity

In our recent work [18] the superspace geometry of three-dimensional $\mathcal{N}$-extended conformal supergravity was developed. In this section we review the formulation for $\mathcal{N} = 2$ conformal supergravity.

Consider a curved 3D $\mathcal{N} = 2$ superspace $\mathcal{M}^{3|8}$ parametrized by local bosonic ($x$) and fermionic ($\theta, \bar{\theta}$) coordinates $z^M = (x^m, \theta^\mu, \bar{\theta}_\mu)$, where $m = 0, 1, 2$, $\mu = 1, 2$. The Grassmann variables $\theta^\mu$ and $\bar{\theta}_\mu$ are related to each other by complex conjugation: $\bar{\theta}^\mu = \bar{\theta}_\mu$. The structure group is chosen to be $\text{SL}(2, \mathbb{R}) \times \text{U}(1)_R$ and the covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\alpha)$ have the form

$$\mathcal{D}_A = E_A + \Omega_A + i \Phi_A \mathcal{J}.$$ (2.1)

Here $E_A = E_A^M(z) \partial / \partial z^M$ is the supervielbein,

$$\Omega_A = \frac{1}{2} \Omega^a_{bc} \mathcal{M}_{bc} = \frac{1}{2} \Omega^\beta\gamma \mathcal{M}_{\beta\gamma},$$ (2.2)

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is the Lorentz connection, and $\Phi_A$ is the $U(1)_R$-connection. The Lorentz generators with two vector indices ($M_{ab} = -M_{ba}$), one vector index ($M_a$) and two spinor indices ($M_{\alpha\beta} = M_{\beta\alpha}$) are related to each other as follows:

$$M_a = \frac{1}{2} \varepsilon_{abc} M^{bc}, \quad M_{ab} = -\varepsilon_{abc} M^c, \quad M_{\alpha\beta} = (\gamma^a)_{\alpha\beta} M_a, \quad M_a = -\frac{1}{2} (\gamma_a)^{\alpha\beta} M_{\alpha\beta}.$$  

Here $\varepsilon_{abc}$ ($\varepsilon_{012} = -1$) is the Levi-Civita tensor and $(\gamma_a)_{\alpha\beta}$ are the symmetric and real gamma-matrices defined in subsection 7.1. The generators of $\text{SL}(2,\mathbb{R}) \times U(1)_R$ act on the covariant derivatives as follows:

$$[\mathcal{J}, D_a] = D_a, \quad [\mathcal{J}, \bar{D}^a] = -\bar{D}^a, \quad [\mathcal{J}, D_a] = 0,$$

$$[M_{\alpha\beta}, D_\gamma] = \varepsilon_{\gamma(\alpha} D_{\beta)}], \quad [M_{\alpha\beta}, \bar{D}_\gamma] = \varepsilon_{\gamma(\alpha} \bar{D}_{\beta)}], \quad [M_{ab}, D_c] = 2\eta_{[a} D_{b]}.$$.  

(2.3)

The supergravity gauge group is generated by local transformations of the form

$$\delta_K D_A = [K, D_A], \quad K = K^C(z) D_C + \frac{1}{2} K^{cd}(z) M_{cd} + i \tau(z) \mathcal{J},$$  

(2.4)

with the gauge parameters obeying natural reality conditions, but otherwise arbitrary. Given a tensor superfield $U(z)$, with its indices suppressed, it transforms as follows:

$$\delta_K U = K U.$$.  

(2.5)

The covariant derivatives obey (anti-)commutation relations of the form

$$[D_A, D_B] = T_{AB} C D_C + \frac{1}{2} R_{AB}^{cd} M_{cd} + i R_{AB} \mathcal{J},$$  

(2.6)

where $T_{AB}^C$ is the torsion, and $R_{AB}^{cd}$ and $R_{AB}$ constitute the curvature. According to the analysis given in [18], the conventional constraints [13] and the Bianchi identities lead to the spinor-spinor anti-commutation relations:

$$\{D_\alpha, D_\beta\} = -4\bar{R} M_{\alpha\beta}, \quad \{\bar{D}_\alpha, \bar{D}_\beta\} = 4R M_{\alpha\beta},$$

(2.7a)

$$\{D_\alpha, \bar{D}_\beta\} = -2i D_{\alpha\beta} - 2C_{\alpha\beta} \mathcal{J} - i\varepsilon_{\alpha\beta S} \mathcal{J} + i S M_{\alpha\beta} - 2\varepsilon_{\alpha\beta} C^{\gamma\delta} M_{\gamma\delta}.$$  

(2.7b)

The vector-spinor commutation relations are ($D_{\alpha\beta} = (\gamma^a)_{\alpha\beta} D_a$):

$$[D_{\alpha\beta}, D_\gamma] = -i\varepsilon_{\gamma(\alpha} C_{\beta)\delta} D^\delta + iC_{\gamma(\alpha} D_{\beta)} - \frac{1}{2} \varepsilon_{\gamma(\alpha} S D_{\beta)} - 2i\varepsilon_{\gamma(\alpha} \bar{R} D_{\beta)} +$$

$$+ 2\varepsilon_{\gamma(\alpha} C_{\beta)\rho} M^{\rho} - \frac{4}{3} \left( \frac{1}{2} D_{(\alpha} S + i D_{(\alpha} \bar{R} \right) M_{\beta)\gamma} + \frac{1}{3} \left( \frac{1}{2} D_{\alpha} S + i D_{\alpha} \bar{R} \right) M_{\beta\gamma}$$

$$+ \left( C_{\alpha\beta\gamma} + \frac{1}{3} \varepsilon_{\gamma(\alpha} (2 D_{\beta) S + i D_{\beta) \bar{R}}) \right) J.$$  

(2.8)

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5We refer the reader to [18] for more details on our conventions; see also subsection 7.1 of this paper. Note that the (anti)symmetrization of $n$ indices is defined to include a factor of $(n!)^{-1}$.

6For convenience, in the present paper the torsion superfield $S$ of [18] has been replaced by $S = 4S$.  

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Finally, the commutator of two vector covariant derivatives turns out to be\footnote{Note that the complete algebra of covariant derivatives is presented here for the first time. Eq. (2.9) was not given in [18].}

\[
[D_a, D_b] = \frac{1}{2} \varepsilon_{abc}(\gamma^c)^{\alpha\beta}\varepsilon^{\gamma\delta} \left( -i\bar{C}_{\alpha\beta\delta} + \frac{i}{3} \varepsilon_{\delta(\alpha}\bar{D}_{\beta)S} + \frac{2}{3} \varepsilon_{\delta(\alpha} D_{\beta)R} \right) D_{\gamma} \\
+ \frac{1}{2} \varepsilon_{abc}(\gamma^c)^{\alpha\beta}\varepsilon^{\gamma\delta} \left( -iC_{\alpha\beta\delta} + \frac{i}{3} \varepsilon_{\delta(\alpha} D_{\beta)S} - \frac{2}{3} \varepsilon_{\delta(\alpha} \bar{D}_{\beta)R} \right) \bar{D}_{\gamma} \\
- \varepsilon_{abc} \left[ \delta^d_c \left( \frac{1}{6} (D^2 R + \bar{D}^2 \bar{R}) + \frac{1}{3} D^a \bar{D}_a S - 4 \bar{R} R - \frac{1}{4} S^2 \right) \\
+ \frac{1}{4} (\gamma^d)^{\alpha\beta}(\gamma_c)^{\gamma\delta} (D_{(\alpha} \bar{C}_{\beta\gamma\delta)} + \bar{D}_{(\alpha} C_{\beta\gamma\delta)}) - 4 C^d C_c \right] M^c \\
+ \frac{i}{8} \varepsilon_{abc}(\gamma^c)^{\alpha\beta} \left( D^\gamma \bar{C}_{\gamma\alpha\beta} - \bar{D}^\gamma C_{\gamma\alpha\beta} + \frac{1}{3} [D_{\alpha}, D_{\beta}] S \right) J .
\] (2.9)

The algebra is parametrized by three dimension-1 torsion superfields: a real scalar $S$, a complex scalar $R$ and its conjugate $\bar{R}$, and a real vector $C_a (C_{\alpha\beta} := (\gamma^a)_{\alpha\beta} C_a)$. The superfields $S$ and $C_a$ are neutral under the group $U(1)_R$, while the $U(1)_R$ charge of $R$ is $-2$, $JR = -2R$ and $J\bar{R} = 2\bar{R}$. The torsion superfields obey differential constraints implied by the Bianchi identities. At dimension-3/2 these are

\[
\bar{D}_{\alpha} R = 0 , \quad D_{\alpha} C_{\beta\gamma} = iC_{\alpha\beta\gamma} + \frac{i}{3} \varepsilon_{\alpha(\beta} \left( iD_{\gamma)} \bar{R} - D_{\gamma) S \right) ,
\] (2.10)

together with their complex conjugates. These equations and their higher-dimension descendants are sufficient to solve the complete set of Bianchi identities. One dimension-2 descendant equation which is important for our subsequent analysis is

\[
(D^2 - 4\bar{R})S = (\bar{D}^2 - 4R)S = 0 .
\] (2.11)

This means that the torsion $S$ is a real covariantly linear superfield.

The rule for integration by parts in superspace is as follows: given a vector superfield $V = V^A E_A$, it holds that

\[
\int d^3x d^4\theta E (-1)^{\varepsilon_A} D_A V^A = 0 , \quad E^{-1} = \text{Ber}(E^M_A) .
\] (2.12)

Given a real scalar superfield $L$, the following chiral reduction rule also holds

\[
\int d^3x d^4\theta E L = \int d^3x d^4\theta \frac{E}{\bar{R}} \tilde{\Delta} L = \int d^3x d^2\theta \mathcal{E} \tilde{\Delta} L ,
\] (2.13)

where $\mathcal{E}$ denotes the chiral density, $\bar{D}_{\alpha} \mathcal{E} = 0$, and $\tilde{\Delta}$ the chiral projection operator

\[
\tilde{\Delta} := -\frac{1}{4} (\bar{D}^2 - 4R) .
\] (2.14)
We conclude by remarking that the algebra (2.7a)–(2.9) and the Bianchi identities (2.10) are invariant under super-Weyl transformations generated by a real unconstrained superfield $\sigma$. The invariance under super-Weyl transformations ensures that the geometry under consideration describes conformal supergravity. The super-Weyl transformation of the covariant derivatives is

\[ D'_\alpha = e^{\frac{i}{2}\sigma} \left( D_\alpha + (D^\gamma \sigma) M_{\gamma\alpha} - (D_\alpha \sigma) J \right), \quad (2.15a) \]
\[ \bar{D}'_\alpha = e^{\frac{i}{2}\sigma} \left( \bar{D}_\alpha + (\bar{D}^\gamma \sigma) M_{\gamma\alpha} + (D_\alpha \sigma) J \right), \quad (2.15b) \]
\[ D'_a = e^{\sigma} \left( D_a - \frac{i}{2} (\gamma_a)^{\gamma\delta} (D_\gamma \sigma) \bar{D}_\delta - \frac{i}{2} (\gamma_a)^{\gamma\delta} (D_\gamma \sigma) (D_\delta \sigma) + \varepsilon_{abc} (D^b \sigma) M^c \right. 
\left. + \frac{i}{2} (D_\gamma \sigma) (D^\gamma \sigma) M_a - \frac{i}{8} (\gamma_a)^{\gamma\delta} ([D_\gamma, \bar{D}_\delta] \sigma) J - \frac{3i}{4} (\gamma_a)^{\gamma\delta} (D_\gamma \sigma) (\bar{D}_\delta \sigma) J \right). \quad (2.15c) \]

To ensure that the algebra (2.7a)–(2.9) is invariant, the dimension-1 torsion components have to transform as

\[ S' = e^{\sigma} \left( S + i(D^\gamma \bar{D}_\gamma \sigma) \right), \quad (2.16a) \]
\[ C'_a = e^{\sigma} \left( C_a + \frac{1}{8} (\gamma_a)^{\gamma\delta} ([D_\gamma, \bar{D}_\delta] \sigma) + \frac{1}{4} (\gamma_a)^{\gamma\delta} (D_\gamma \sigma) (\bar{D}_\delta \sigma) \right), \quad (2.16b) \]
\[ R' = e^{\sigma} \left( R + \frac{1}{4} (D^2 \sigma) - \frac{1}{4} (D_\gamma \sigma) (D^\gamma \sigma) \right), \quad (2.16c) \]
\[ \bar{R}' = e^{\sigma} \left( \bar{R} + \frac{1}{4} (D^2 \sigma) - \frac{1}{4} (D_\gamma \sigma) (D^\gamma \sigma) \right). \quad (2.16d) \]

For later use, it is useful to rewrite the transformations of the dimension-1 torsion superfields in the following equivalent form

\[ S' = \left( e^{\sigma} S + i(D^\gamma \bar{D}_\gamma \sigma) e^{\sigma} - ie^{-\sigma} (D^\gamma \sigma) (D_\gamma \sigma) e^\sigma \right), \quad (2.17a) \]
\[ C'_a = \left( C_a + \frac{1}{8} (\gamma_a)^{\gamma\delta} (D_\gamma (\bar{D}_\delta \sigma) e^\sigma \right), \quad (2.17b) \]
\[ R' = -\frac{1}{4} e^{2\sigma} \left( D^2 - 4R \right) e^{-\sigma}, \quad \bar{R}' = -\frac{1}{4} e^{2\sigma} \left( \bar{D}^2 - 4\bar{R} \right) e^{-\sigma}. \quad (2.17c) \]

3 Type I minimal supergravity

This supergravity theory is a 3D analogue of the old minimal formulation for 4D $\mathcal{N} = 1$ supergravity [24] (see [15, 22, 23] for reviews). The corresponding conformal compensators

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8The infinitesimal super-Weyl transformations, that we will denote with $\delta_\sigma$, were given in our previous paper [18]. Here the full nonlinear result is presented for the first time.
are a covariantly chiral scalar $\Phi$ and its conjugate $\bar{\Phi}$, $\bar{\mathcal{D}}_{\alpha} \Phi = 0$. It is always possible to choose the chiral compensator $\Phi$ to have super-Weyl weight $1/2$, 

$$\Phi' = e^{\frac{i}{2}\sigma} \Phi .$$

(3.1)

This implies that its $U(1)_R$ charge must be $-1/2$, in accordance with the analysis in [18],

$$\mathcal{J} \Phi = -\frac{1}{2} \Phi .$$

(3.2)

The freedom to perform the super-Weyl and local $U(1)_R$ transformations can be used to impose the gauge

$$\Phi = 1 .$$

(3.3)

Such a gauge fixing is accompanied by the consistency conditions [18]

$$0 = \bar{\mathcal{D}}_{\alpha} \Phi = -i \frac{\Phi}{2} \Phi , \quad 0 = \{\mathcal{D}_{\alpha}, \bar{\mathcal{D}}_{\beta}\} \Phi = -\Phi_{\alpha\beta} + C_{\alpha\beta} - i \frac{\varepsilon_{\alpha\beta}\mathcal{S}}{2} ,$$

(3.4)

and therefore

$$\Phi_{\alpha} = \mathcal{S} = 0 , \quad \Phi_{\alpha\beta} = C_{\alpha\beta} .$$

(3.5)

The gauge conditions (3.3) and (3.5) are invariant under a combined set of super-Weyl and $U(1)_R$ transformations. The condition $\mathcal{S} = 0$ is preserved if the real superfield $\sigma$ satisfies

$$i \mathcal{D}^\gamma \bar{\mathcal{D}}_\gamma \sigma = 0 \quad \Leftrightarrow \quad \sigma = \lambda + \bar{\lambda} , \quad \bar{\mathcal{D}}_{\alpha} \lambda = 0 ,$$

(3.6)

with the parameter $\lambda$ being an arbitrary chiral superfield. The resulting residual super-Weyl and $U(1)_R$ transformations of the Type-I geometry turn out to be

$$\mathcal{D}'_{\alpha} = e^{\frac{i}{2}(3\lambda - \lambda)} \left( \mathcal{D}_{\alpha} + (\mathcal{D}^\gamma \lambda) \mathcal{M}_{\gamma\alpha} \right) ,$$

(3.7a)

$$\bar{\mathcal{D}}'_{\alpha} = e^{\frac{i}{2}(3\lambda - \lambda)} \left( \bar{\mathcal{D}}_{\alpha} + (\bar{\mathcal{D}}^\gamma \lambda) \mathcal{M}_{\gamma\alpha} \right) ,$$

(3.7b)

$$\mathcal{D}'_{\alpha} = e^{\lambda + \lambda} \left( \mathcal{D}_{\alpha} - i \frac{1}{2} (\gamma_a)^{\alpha\beta} (\mathcal{D}_{\alpha} \lambda) \bar{\mathcal{D}}_{\beta} - i \frac{1}{2} (\gamma_a)^{\alpha\beta} (\bar{\mathcal{D}}_{\alpha} \bar{\lambda}) \mathcal{D}_{\beta} 

+ \varepsilon_{abc} (\mathcal{D}^b (\lambda + \bar{\lambda})) \mathcal{M}_c + i \frac{1}{2} (\mathcal{D}^\gamma \lambda) \mathcal{M}_a \right) .$$

(3.7c)

The dimension-1 torsion superfields transform according to the following equations

$$\mathcal{C}'_{\alpha} = e^{\lambda + \bar{\lambda}} \left( \mathcal{C}_{\alpha} - i \frac{1}{2} (\mathcal{D}_{\alpha} (\lambda - \bar{\lambda})) (\mathcal{D}_{\alpha} \lambda) (\bar{\mathcal{D}}_{\beta} \bar{\lambda}) \right) ,$$

(3.8a)

$$R' = e^{3\lambda - \lambda} \left( R + \frac{1}{4} (\mathcal{D}^2 \lambda) - \frac{1}{4} (\mathcal{D}^\gamma \lambda) (\mathcal{D}^\gamma \bar{\lambda}) \right) = -\frac{1}{4} e^{3\lambda} (\mathcal{D}^2 - 4R) e^{-\lambda} ,$$

(3.8b)

$$\bar{R}' = e^{3\lambda - \lambda} \left( \bar{R} + \frac{1}{4} (\bar{\mathcal{D}}^2 \lambda) - \frac{1}{4} (\bar{\mathcal{D}}^\gamma \lambda) (\bar{\mathcal{D}}^\gamma \lambda) \right) = -\frac{1}{4} e^{3\lambda} (\bar{\mathcal{D}}^2 - 4\bar{R}) e^{-\lambda} .$$

(3.8c)
3.1 Supergravity without a cosmological term

The supergravity action is

\[ S = -4 \int d^3 x d^4 \theta \ E \Phi \Phi . \]  

(3.9)

The equation of motion for \( \Phi \) is

\[ (\bar{D}^2 - 4R)\Phi = 0 . \]  

(3.10)

In the gauge (3.3) it reduces to

\[ R = 0 . \]  

(3.11)

Modulo purely gauge degrees of freedom, the complete set of unconstrained prepotentials for the supergravity formulation under consideration comprises \( \Phi, \bar{\Phi} \) and a gravitational superfield \( H^{\alpha \beta} = H^{\beta \alpha} = \bar{H}^{\alpha \beta} \). In the gauge (3.3) the equation of motion for \( H^{\alpha \beta} \) can be shown to be

\[ C_{\alpha \beta} = 0 . \]  

(3.12)

The equations (3.5), (3.11) and (3.12) tell us that the on-shell superspace geometry is locally flat. Denoting the on-shell covariant derivatives by \( D_A = (D_a, D_\alpha, \bar{D}^\alpha) \), their algebra is

\[
\begin{align*}
\{D_\alpha, D_\beta\} &= 0 , & \{\bar{D}_\alpha, \bar{D}_\beta\} &= 0 , & \{D_\alpha, \bar{D}_\beta\} &= -2iD_{\alpha \beta} , \\
[D_a, D_\beta] &= 0 , & [D_a, \bar{D}_\beta] &= 0 , & [D_a, D_b] &= 0 .
\end{align*}
\]

(3.13)

3.2 Supergravity with a cosmological term

The supergravity action is

\[ S = -4 \int d^3 x d^4 \theta \ E \Phi \Phi + \mu \int d^3 x d^2 \theta \mathcal{E} \Phi^4 + \bar{\mu} \int d^3 x d^2 \bar{\theta} \bar{\mathcal{E}} \bar{\Phi}^4 . \]  

(3.14)

The equation of motion for \( \Phi \) is

\[ \frac{1}{4}(\bar{D}^2 - 4R)\Phi + \mu \Phi^3 = 0 . \]  

(3.15)

In the gauge (3.3) it reduces to

\[ R = \mu = \text{const} . \]  

(3.16)
The other supergravity equations (3.5) and (3.12) do not change. Therefore, this theory describes AdS supergravity. Any solution of the theory is locally an AdS superspace. In addition to a unique maximally symmetric solution (AdS superspace), there also exist supersymmetric versions \[14\] of the BTZ black hole \[52\]. The supersymmetry properties of the black holes in three dimensions were investigated in \[54\]. Here we will only be interested in the AdS superspace. Let \(\nabla_A = (\nabla_a, \nabla_\alpha, \bar{\nabla}^\alpha)\) be the resulting on-shell covariant derivatives, eq. (1.2), obeying the (anti)commutation relations (1.3a)–(1.3c). They describe, according to the classification given in \[10\], the (1,1) AdS superspace.

There is an alternative realization of the above supergravity formulation, in the spirit of \[55\] \[56\]. It is based on representing the chiral compensator \(\Phi\) as a composite field,

\[
\Phi^4 = -\frac{1}{4}(\bar{D}^2 - 4R) P, \quad \bar{P} = P, \tag{3.17}
\]

where \(P\) is a real unconstrained scalar with the super-Weyl transformation

\[
P \rightarrow e^{\sigma} P. \tag{3.18}
\]

The supergravity action turns into

\[
S = -4 \int d^3x d^4\theta E \bar{\Phi} \Phi + (\mu + \bar{\mu}) \int d^3x d^4\theta E P. \tag{3.19}
\]

The cosmological term looks like a Fayet-Iliopoulos term. However this interpretation is somewhat misleading since the action (3.19) is invariant under gauge symmetry

\[
\delta P = \mathbb{H}, \quad (\bar{D}^2 - 4R)\mathbb{H} = (D^2 - 4\bar{R})\mathbb{H} = 0 \tag{3.20}
\]

which do not describe a vector multiplet, but rather a scalar multiplet.

### 3.3 Matter-coupled supergravity

Matter couplings in Type I supergravity are analogous to those in the old minimal formulation for 4D \(\mathcal{N} = 1\) supergravity, see e.g. \[25\] for a review. As an example, we only consider a general locally supersymmetric nonlinear \(\sigma\)-model

\[
S = -4 \int d^3x d^4\theta E \bar{\Phi} \Phi e^{-K/4} + \int d^3xd^2\theta \mathcal{E} \Phi^4 W + \int d^3x d^2\bar{\theta} \bar{\mathcal{E}} \bar{\Phi}^4 \bar{W}. \tag{3.21}
\]

\[\text{The BTZ black hole is obtained as a discrete quotient of the 3D AdS space \[53\]. A similar realization exists for its supersymmetric extensions.}\]
Here the Kähler potential, \( K = K(\varphi^I, \bar{\varphi}^\bar{J}) \), is a real function of the covariantly chiral superfields \( \varphi^I \) and their conjugates \( \bar{\varphi}^\bar{J} \), obeying \( \bar{D}_\alpha \varphi^I = 0 \). The superpotential, \( W = W(\varphi^I) \), is a holomorphic function of \( \varphi^I \) alone. The matter superfields \( \varphi^I \) and \( \bar{\varphi}^\bar{J} \) are chosen to be inert under the super-Weyl and local U(1)\(_R\) transformations. This guarantees the super-Weyl invariance of the action.

The action (3.1) is invariant under Kähler transformations,

\[
K \to K + F + \bar{F}, \quad W \to e^{-F} W, \quad \Phi \to e^{F/4} \Phi ,
\]

with \( F(\varphi^I) \) an arbitrary holomorphic function.

### 4 Type II minimal supergravity

This supergravity theory is a 3D analogue of the new minimal formulation for 4D \( \mathcal{N} = 1 \) supergravity [26] (see [22, 15] for reviews). Its conformal compensator is a vector multiplet described by a real scalar superfield \( G \) which is defined modulo arbitrary gauge transformations of the form:

\[
\delta G = \lambda + \bar{\lambda} , \quad \mathcal{J} \lambda = 0 , \quad \bar{D}_\alpha \lambda = 0 .
\]

The gauge field is inert under the super-Weyl transformations,

\[
G' = G .
\]

Associated with \( G \) is the gauge-invariant field strength

\[
\mathbb{G} = i \bar{D}^\alpha D_\alpha G = \bar{\mathbb{G}}
\]

which is covariantly linear,

\[
(D^2 - 4\bar{R}) \mathbb{G} = (\bar{D}^2 - 4R) \mathbb{G} = 0 ,
\]

and is required to be nowhere vanishing, \( \mathbb{G} \neq 0 \). The expression (4.3) is the most general solution to the constraint (4.4).

In accordance with (4.2), the super-Weyl transformation of \( \mathbb{G} \) is

\[
\mathbb{G}' = e^\sigma \mathbb{G} .
\]

Since \( \mathbb{G} \) is nowhere vanishing, it is always possible to choose the super-Weyl gauge

\[
\mathbb{G} = 1 .
\]
This gauge condition will be often used in what follows.

As a consequence of (4.4), the gauge condition (4.6) implies that

\[ R = \bar{R} = 0 \]  \hspace{1cm} (4.7)

The curved superspace associated with the super-Weyl gauge choice (4.6)–(4.7) will be referred to as Type-II geometry.

Due to the equations (2.17c), it is clear that the Type-II geometry is invariant under residual super-Weyl transformations generated by a real superfield \( \sigma \) such that

\[ D^2 e^{-\sigma} = \bar{D}^2 e^{-\sigma} = 0 \]  \hspace{1cm} (4.8)

Not surprisingly, the residual super-Weyl transformations are generated by a vector multiplet.

### 4.1 Supergravity without a cosmological term

The pure supergravity action \[18\] is

\[ S = \int d^3 x d^4 \theta E L_{\text{Einst}} \]  \hspace{1cm} (4.9)

where

\[ L_{\text{Einst}} = 4 \left( G \ln G - G S \right) \]  \hspace{1cm} (4.10)

We recall that the torsion superfield \( S \) is covariantly real linear, eq. (2.11). Its super-Weyl transformation is given by (2.16a). Due to the relations (2.16a), (4.2) and (4.5), the action (4.9) is invariant under the super-Weyl transformations.

Consider the equation of motion for \( G \):

\[ i \bar{D}^\alpha D_\alpha \ln G - S = 0 \]  \hspace{1cm} (4.11)

Let us choose the super-Weyl gauge (4.6). Then, the equation of motion gives

\[ S = 0 \]  \hspace{1cm} (4.12)

It should be remembered that the relation (4.7) holds in the same gauge.
The compensator $G$ is one of the two supergravity prepotentials. The second prepotential is a gravitational superfield $H^\alpha{}^\beta = H^\beta{}^\alpha = \overline{H}^\alpha{}^\beta$. The corresponding equation of motion in the gauge (4.6) is

$$C_{\alpha\beta} = 0 \ .$$

(4.13)

The equations (4.12)–(4.13) tell us that the on-shell superspace geometry is locally flat. We conclude that this theory describes $\mathcal{N} = 2$ supergravity without a cosmological term.

The supergravity formulation introduced can equivalently be described by a Lagrangian that slightly differs in its form from (4.10). In order to derive such a Lagrangian, a few formal observations should be made. First of all, the constraint (2.7a) implies\footnote{A complete solution to the supergravity constraints in terms of unconstrained prepotentials will be given elsewhere.} that

$$D_\alpha = E_\alpha + \frac{1}{2} \Omega^\alpha{}_{cd} \mathcal{M}^{cd} - E_\alpha U \mathcal{J} \ ,$$

(4.14)

for some complex scalar prepotential $U$ defined modulo gauge transformations

$$U \to U + \bar{\lambda} \ , \quad D_\alpha \bar{\lambda} = 0 \ ,$$

(4.15)

with $\lambda$ an arbitrary chiral scalar of zero $\text{U}(1)_R$ charge. Our second observation is that the prepotential $U$ is characterized by the super-Weyl and local $\text{U}(1)_R$ transformation laws:

$$\delta_\sigma U = \sigma \ ,$$

(4.16a)

$$\delta_\tau U = i \tau \ .$$

(4.16b)

Finally, the third observation is that the constraint (2.7b) leads to the following relation

$$S = iD^\alpha \bar{D}_\alpha S \ , \quad S = \frac{1}{2} (U + \bar{U}) \ .$$

(4.17)

Now, integration by parts can be used to show that the Lagrangian (4.10) is equivalent to

$$\tilde{L}_{\text{Einst}} = 4G \left( \ln G - S \right) \ .$$

(4.18)

The above supergravity theory is dual to that described by the action (3.9). To prove this, it suffices to consider the following first-order model:

$$L_{\text{first-order}} = 4G \left( \ln G - 1 - S - \psi - \bar{\psi} \right) \ , \quad \mathcal{D}_\alpha \psi = 0 \ .$$

(4.19)
Here $G$ is a real unconstrained superfield, and $\psi$ a chiral scalar of zero $U(1)_R$ charge. Varying the action with respect to $\psi$ gives $G = G$, and then the model under consideration reduces to that described by the Lagrangian (4.18). On the other hand, the auxiliary superfield $G$ can be integrated out using its equation of motion, $\ln G = S + \psi + \bar{\psi}$. This lead to the supergravity theory (3.9) in which

$$\Phi := e^{\frac{1}{2}U} e^\psi.$$  \hspace{1cm} (4.20)

Using the super-Weyl and local $U(1)_R$ transformation laws of $U$, eqs. (4.16a) and (4.16b), one may see that $\Phi$ is a covariantly chiral superfield characterized by the properties (3.1) and (3.2).

### 4.2 Supergravity with a cosmological term

Consider a deformed supergravity action

$$S_{AdS} = \int d^3x d^4\theta E L_{AdS},$$  \hspace{1cm} (4.21)

where, up to a total derivative,

$$L_{AdS} = 4 \left( G \ln G - G S + \frac{1}{2} \rho G G \right) \simeq 4 G \left( \ln G - S + \frac{1}{2} \rho G \right),$$  \hspace{1cm} (4.22)

with $\rho$ a real coupling constant. This Lagrangian differs from (4.10), or its equivalent form (4.18), by the presence of a Chern-Simons term.

Now, the equation of motion for $G$ is

$$i \bar{D}^\alpha D_{\alpha} \ln G - S + \rho G = 0.$$  \hspace{1cm} (4.23)

Choosing the super-Weyl gauge (4.6) gives

$$S = \rho = \text{const}.$$  \hspace{1cm} (4.24)

The supergravity equations of motion (4.7) and (4.13) do not change. Therefore, the theory describes AdS supergravity. Any solution to the supergravity equations of motion is locally an AdS space-time.

The algebra of the on-shell covariant derivatives becomes (1.5a)–(1.5c). According to the classification given in [10], such an algebra describes the (2,0) AdS supergeometry.
4.3 Matter-coupled supergravity

The pure supergravity model (4.18) can be readily generalized to include supersymmetric chiral matter that is neutral under the local $U(1)_R$ group

$$L = 4G\left( \ln G - S + \frac{1}{4}K(\phi^I, \bar{\phi}^J) \right), \quad \bar{D}_\alpha \phi^J = 0 \quad (4.25)$$

with $K$ the Kähler potential of a Kähler manifold. The corresponding action is invariant under Kähler transformations,

$$K \rightarrow K + F + \bar{F}, \quad (4.26)$$

with $F(\phi^I)$ an arbitrary holomorphic function. The model (4.25) proves to be dual to (3.21) with $W(\phi) = 0$. This duality can be demonstrated by making use of a natural generalization of the first-order Lagrangian (4.19).

Similarly to the new minimal $\mathcal{N} = 1$ supergravity in four dimensions, Type II minimal supergravity can be coupled to $R$-invariant $\sigma$-models. Let us consider a system of self-interacting covariantly chiral superfields $\phi^I$, where $I = 1, \cdots, m$, with $U(1)_R$ charges

$$\mathcal{J} \phi^I = -r_I \phi^I \quad \text{(no sum)} \quad (4.27)$$

and hence their infinitesimal super-Weyl transformation laws are

$$\delta_\sigma \phi^I = r_I \sigma \phi^I. \quad (4.28)$$

In order to have an $R$-invariant system, the Kähler potential $K(\phi^I, \bar{\phi}^J)$ and the superpotential $W(\phi^I)$ should obey the equations

$$\sum_I r_I \phi^I K_I = \sum_I r_I \bar{\phi}^J K_I, \quad (4.29a)$$

$$\sum_I r_I \phi^I W_I = 2W. \quad (4.29b)$$

The complete supergravity-matter system is described by the action

$$S = 4 \int d^3x d^4\theta E \mathcal{G} \left( \ln \mathcal{G} - S + \frac{1}{4}K(\phi^I/G^I, \bar{\phi}^J/G^J) \right) + \left\{ \int d^3x d^2\theta E W(\phi^I) + \text{c.c.} \right\}. \quad (4.30)$$

The action can be seen to be super-Weyl invariant. In the case of a superconformal $\sigma$-model, such that $r_I = 1/2$ and $K(\phi^I, \bar{\phi}^J)$ obeys the homogeneity condition

$$\sum_I \phi^I K_I = K, \quad (4.31)$$
the matter sector in (4.30) decouples from the linear compensator $G$.

Given a system of Abelian vector multiplets described by gauge prepotentials $F^i$ and
gauge invariant field strengths $\mathbb{F}^i = iD^\alpha \bar{D}_\alpha F^i$, their coupling to supergravity can be described by an action of the form
\[
S = \int d^3 x d^4 \theta E \left( L(\mathbb{F}^i/G) + \frac{1}{2G} m_{ij} F^i \mathbb{F}^j + \xi_i F^i \right),
\]
where the parameters
\[
m_{ij} = m_{ji} = (m_{ij})^* = \text{const}
\]
describe Chern-Simons couplings, and $\xi_i$ correspond to Fayet-Iliopoulos terms. If the
Lagrangian $L(\mathbb{F}^i)$ corresponds to a superconformal system,
\[
\mathbb{F}^i \frac{\partial}{\partial \mathbb{F}^i} L(\mathbb{F}) = L(\mathbb{F}),
\]
and no Fayet-Iliopoulos term is present, $\xi_i = 0$, then the action (4.32) is independent of
the linear compensator $G$.

5 Non-minimal supergravity

In this section we present 3D analogues of the following 4D $\mathcal{N} = 1$ theories: (i) the
non-minimal supergravity without a cosmological term [27, 28]; and (ii) the non-minimal
AdS supergravity [29].

5.1 Supergravity without a cosmological term

This supergravity formulation involves the following conformal compensators: a com-
plex linear superfield $\Sigma$ and its conjugate $\bar{\Sigma}$. The superfield $\Sigma$ obeys the constraint
\[
(\bar{D}^2 - 4R)\Sigma = 0
\]
and no reality condition. If $\Sigma$ is chosen to transform homogeneously under the super-Weyl
transformations, then its $U(1)_R$ charge is determined by the super-Weyl weight [18]
\[
\delta_\sigma \Sigma = w \sigma \Sigma \quad \Rightarrow \quad \mathcal{J} \Sigma = (1 - w)\Sigma.
\]

We derive the non-minimal supergravity action by dualizing the Type I minimal action
(3.9). Let us consider the parent action
\[
S_{\text{parent}} = \int d^3 x d^4 \theta E \left\{ - 4\Phi \bar{\Phi} + \frac{2}{1 - w} \left( \Sigma \Phi^{2(1-w)} + \bar{\Sigma} \bar{\Phi}^{2(1-w)} \right) \right\},
\]
19
where $\Phi$ is complex unconstrained, and $\Sigma$ is complex linear. This action is super-Weyl invariant provided $\Phi$ transforms as in (3.1). It is also invariant under local $U(1)_R$ transformations if the $U(1)_R$ charge of $\Phi$ is chosen as in (3.2).

The theory (5.3) is equivalent to the Type I minimal supergravity, eq. (3.9). Indeed, varying (5.3) with respect to $\Sigma$ gives $\bar{D}_\alpha \Phi = 0$, and then (5.3) reduces to the action (3.9). On the other hand, we can start from (5.3) and integrate out the fields $\Phi$ and $\bar{\Phi}$. This yields

$$S_{\text{non-minimal}} = 4 \frac{w}{1 - w} \int d^3 x d^4 \theta E \left( \Sigma \bar{\Sigma} \right)^{\frac{1}{w}}. \quad (5.4)$$

This action is not defined if $w = 1$. This value proves to correspond to the Type II minimal supergravity. It may be seen that the case of Type I minimal supergravity corresponds to the limit $w \to \infty$. The last singular point of the action (5.4) is given by $w = 0$. In this case the complex linear superfield is super-Weyl invariant and cannot be used as a conformal compensator.

For later use, it is worth presenting a relationship between the 3D parameter $w$ and the 4D Siegel-Gates parameter $n$ [28]. By identifying the $U(1)_R$ charges of a complex linear superfield coupled to conformal supergravity respectively in 4D and 3D one gets the relation

$$\frac{4n}{3n + 1} = 1 - w \quad (5.5)$$

which is equivalent to (1.1).

### 5.2 Supergravity with a cosmological term

The non-minimal formulation developed in the previous subsection is not suitable to describe AdS supergravity, in complete analogy with the four-dimensional case [15]. In four dimensions, however, the way out has been found in [29]. The same idea can successfully be applied in three dimensions.

Our point of departure will be the following super-Weyl transformation law [18]

$$\delta_\sigma \left( (\bar{D}^2 - 4R)\Gamma \right) = (1 + w)\sigma(\bar{D}^2 - 4R)\Gamma, \quad (5.6)$$

which holds for any complex superfield $\Gamma$ with the transformation properties

$$\delta_\sigma \Gamma = w \sigma \Gamma, \quad J\Gamma = (1 - w)\Gamma. \quad (5.7)$$
The complex linear compensator \( \Sigma \) is an example of such a superfield. Eq. (5.6) tells us that \((\bar{D}^2 - 4R)\Gamma\) is super-Weyl invariant if and only if \(w = -1\). In that case, we may consistently deform the linear constraint, eq. (5.1). In what follows, we fix \(w = -1\).

We introduce a new conformal compensator \( \Gamma \) which has the transformation properties
\[
\delta_\sigma \Gamma = -\sigma \Gamma, \quad \mathcal{J} \Gamma = 2\Gamma \tag{5.8}
\]
and obeys the improved linear constraint\footnote{In global 4D \( \mathcal{N} = 1 \) supersymmetry, constraints of the form (5.9) were introduced for the first time by Deo and Gates [57]. In the context of supergravity, such constraints have recently been used in [58] to generate couplings of the Goldstino superfield to chiral matter.}
\[
-\frac{1}{4}(\bar{D}^2 - 4R)\Gamma = W(\varphi), \tag{5.9}
\]
with \(W(\varphi)\) the matter superpotential defined in subsection 3.3. Using \(\Gamma\) and its conjugate \(\bar{\Gamma}\), we can develop a dual formulation of the theory (3.21). In order to achieve that, we consider the first-order action
\[
S_{\text{first-order}} = \int d^3x d^4\theta E \left( -4\Phi \Phi e^{-K/4} + \Gamma \Phi^4 + \bar{\Gamma} \bar{\Phi}^4 \right), \tag{5.10}
\]
where \(\Phi\) is complex unconstrained, and \(\Gamma\) obeys the constraint (5.9). Varying \(S_{\text{first-order}}\) with respect to \(\Gamma\) yields \(\bar{D}_a \Phi = 0\), and then the action reduces to the supergravity matter action (3.21). On the other hand, we can integrate out the fields \(\Phi\) and \(\bar{\Phi}\) to end up with the dual model
\[
S = -2 \int d^3x d^4\theta E e^{-K/2}(\bar{\Gamma} \Gamma)^{-1/2}. \tag{5.11}
\]

To describe pure AdS supergravity, we have to set \(K = 0\) and \(W = \mu\). Now the compensator obeys the constraint
\[
-\frac{1}{4}(\bar{D}^2 - 4R)\Gamma = \mu = \text{const}, \tag{5.12}
\]
and the action (5.11) turns into AdS supergravity
\[
S_{\text{AdS}} = -2 \int d^3x d^4\theta E (\bar{\Gamma} \Gamma)^{-1/2}. \tag{5.13}
\]

By construction, this theory is dual to the Type I minimal AdS supergravity, eq. (3.14).
6 Conformal flatness of the AdS superspaces

It is well known that 4D $\mathcal{N} = 1$ AdS superspace is conformally flat, see e.g. [22] for a pedagogical review. The same property is characteristic of the 4D $\mathcal{N} = 2$ [63, 64] and 5D $\mathcal{N} = 1$ [65] AdS superspaces. At the same time, it was shown in [63] that the conventional superspace extensions of the coset manifolds $\text{AdS}_2 \times S^2$, $\text{AdS}_3 \times S^3$ and $\text{AdS}_5 \times S^5$, which arise as solutions of certain supergravity theories in four, six and ten dimensions, are not conformally flat. In this section we prove that the (1,1) and (2,0) AdS superspaces are conformally flat. Our proof is constructive and provides explicit realizations of the (1,1) and (2,0) AdS superspace geometries.

6.1 (1,1) AdS superspace

The super-Weyl and U(1)$_R$ transformations of the Type-I curved superspace geometry are given by eq. (3.7). Our goal is to show that the covariant derivatives $\nabla_A$ of the (1,1) AdS superspace can be brought to the form

$$\nabla_a = e^{\frac{i}{2}(\lambda - \overline{\lambda})} \left( \partial_a + (D^\gamma \lambda) M_{\gamma a} \right),$$

$$\nabla_\alpha = e^{\frac{i}{2}(\lambda - \overline{\lambda})} \left( \bar{D}_\alpha + (\bar{D}^\gamma \bar{\lambda}) M_{\gamma a} \right),$$

$$\nabla = e^{\lambda + \bar{\lambda}} \left( \partial_a - \frac{i}{2} (\gamma_a)^{\alpha \beta} (D_\alpha \lambda) \bar{D}_\beta - \frac{i}{2} (\gamma_a)^{\alpha \beta} (\bar{D}_\alpha \bar{\lambda}) D_\beta \right. + \varepsilon_{abc} \partial^b (\lambda + \bar{\lambda}) M^c + \left. \frac{i}{2} (D_\gamma \lambda) (\bar{D}^\gamma \bar{\lambda}) M_a \right),$$

for some chiral scalar $\lambda$. Here, $D_A = (\partial_a, D_\alpha, \bar{D}_\alpha)$ are the flat global covariant derivatives

$$\partial_a = \frac{\partial}{\partial x^a}, \quad D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i \theta^\beta (\gamma^a)^{\alpha \beta} \partial_a, \quad \bar{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}^\alpha} - i \bar{\theta}^\beta (\gamma^a)^{\alpha \beta} \partial_a.$$ (6.2)

They obey the (anti-)commutation relations (3.13).

Under the super-Weyl and U(1)$_R$ transformations, the dimension-1 torsion superfields transform according to eq. (3.8). Since Minkowski superspace has no dimension-1 torsion, the chiral parameter $\lambda$ and its conjugate $\bar{\lambda}$ in (6.1) must obey the following equations:

$$\mu = -\frac{1}{4} e^{3\lambda} \bar{D}^2 e^{-\bar{\lambda}}, \quad \bar{\mu} = -\frac{1}{4} e^{3\bar{\lambda}} D^2 e^{-\lambda},$$

$$0 = i \partial_{\alpha \beta} (\lambda - \bar{\lambda}) + (D_\alpha \lambda) \bar{D}_\beta \bar{\lambda}. $$ (6.3a, 6.3b)

Here the complex parameter $\mu$ is the constant curvature of the (1,1) AdS superspace (1.3a)–(1.3c).
To find a solution of the equations (6.3a) and (6.3b) we first observe that (6.3b) can equivalently be rewritten as

\[ D(\alpha, \bar{D}_\beta) e^{\lambda+\bar{\lambda}} = 0 \] (6.4)

We look for a Lorentz invariant solution of this equation of the form

\[ e^{\lambda+\bar{\lambda}} = 1 + a\mu \bar{\mu} x^2 + b\mu \bar{\theta}^2 + \frac{a}{2} \mu \bar{\mu} \theta^2 \bar{\theta}^2 , \] (6.5)

where

\[ x^2 := x^a x_a , \quad \theta^2 := \theta^\alpha \theta_\alpha , \quad \bar{\theta}^2 := \bar{\theta}^\alpha \bar{\theta}_\alpha = \bar{\theta}^2 . \] (6.6)

The right-hand side of (6.5) involves two parameters, \( a \) and \( b \), which are real and complex respectively. The superfield \( e^{\lambda+\bar{\lambda}} \) in (6.5) is reminiscent of that emerging in the 4D \( \mathcal{N} = 1 \) AdS superspace geometry [22]. The relation (6.5) implies that

\[ e^{\lambda} = (1 + a\mu \bar{\mu} x_L^2 + 2b\mu \bar{\theta}^2)^{\frac{1}{2}} , \quad e^{\bar{\lambda}} = (1 + a\mu \bar{\mu} x_R^2 + 2b\mu \bar{\theta}^2)^{\frac{1}{2}} , \] (6.7)

where we have introduced the (anti)chiral vector variables

\[ x^a_L = x^a + i(\gamma^a)_{\alpha\beta} \theta^\alpha \bar{\theta}^\beta , \quad \bar{D}_\alpha x^a_L = 0 , \] (6.8a)
\[ x^a_R = x^a - i(\gamma^a)_{\alpha\beta} \theta^\alpha \bar{\theta}^\beta , \quad D_\alpha x^a_R = 0 . \] (6.8b)

Plugging (6.7) into equations (6.3a), after some algebra we find that (6.7) is indeed a solution of (6.3a) provided \( a = -1 \) and \( b = -1 \). As a result, we have constructed an explicit conformally flat realization for the (1,1) AdS superspace. The covariant derivatives are given by the relations (6.1) with

\[ e^{\lambda} = (1 - \mu \bar{\mu} x_L^2 - 2\mu \bar{\theta}^2)^{\frac{1}{2}} , \quad e^{\bar{\lambda}} = (1 - \mu \bar{\mu} x_R^2 - 2\mu \bar{\theta}^2)^{\frac{1}{2}} . \] (6.9)

Using the expression for \( e^{\lambda+\bar{\lambda}} \) and the explicit form of the vector covariant derivative \( \nabla_a \), eq. (6.1c), we can read off the space-time metric

\[ ds^2 = dx^a dx_a \left( e^{-2(\lambda+\bar{\lambda})} \right) \bigg|_{\theta = 0} = \frac{dx^a dx_a}{(1 - \mu \bar{\mu} x^2)^2} . \] (6.10)

This coincides with a standard expression for the metric of AdS_3 computed using the stereographic projection for an AdS hyperboloid. As such, the conformally flat representation is defined only locally.

\[ \text{See, e.g., Appendix D of [64] for details about the stereographic projection for AdS_3.} \]
6.2 (2,0) AdS superspace

In three-dimensional $\mathcal{N} = 2$ supergravity, the super-Weyl transformation of the covariant derivatives is given by (2.15). Our goal in this subsection is to show that the covariant derivatives $D_A$ of the (2,0) AdS superspace can be brought to the conformally flat form:

$$D_\alpha = e^{\frac{i}{2}\sigma}(D_\alpha + (D^\gamma\sigma)\mathcal{M}_{\gamma\alpha} - (D_\alpha\sigma)\mathcal{J}),$$

$$\bar{D}_\alpha = e^{\frac{i}{2}\sigma}(\bar{D}_\alpha + (\bar{D}^\gamma\sigma)\mathcal{M}_{\gamma\alpha} + (\bar{D}_\alpha\sigma)\mathcal{J}),$$

$$D_a = e^{\sigma}\left(\partial_a - \frac{i}{2}(\gamma_a)^{\gamma\delta}(D_{(\gamma}\sigma)\bar{D}_{\delta)} - \frac{i}{2}(\gamma_a)^{\gamma\delta}(\bar{D}_{(\gamma}\sigma)D_{\delta)} + \varepsilon_{abc}(\partial^b\sigma)\mathcal{M}^c - \frac{i}{2}(\gamma_a)^{\gamma\delta}(\bar{D}_{(\gamma}\sigma)(\bar{D}_{\delta}\sigma)\mathcal{J}\right),$$

for some real scalar $\sigma$. Under the super-Weyl transformation, the dimension-1 components of the torsion transform according to (2.17). Since there is no dimension-1 torsion in Minkowski superspace, the super-Weyl parameter $\sigma$ must obey the following equations:

$$\rho = ie^{\sigma}D^\gamma\bar{D}_\gamma\sigma = i\left(D^\gamma\bar{D}_\gamma e^\sigma - e^{-\sigma}(D^\gamma e^\sigma)\bar{D}_\gamma e^\sigma\right),$$

$$0 = [D_\alpha, \bar{D}_\beta]e^\sigma, \quad (6.12b)$$

$$0 = \bar{D}^2 e^{-\sigma} = D^2 e^{-\sigma}. \quad (6.12c)$$

Here $\rho$ is the parameter which appears in the (anti-)commutation relations (1.5a)–(1.5c). We now turn to deriving a Lorentz invariant solution of the equations (6.12a)–(6.12c).

It should be remarked that the system (6.12a)–(6.12c) involves only two independent equations since eq. (6.12c) proves to be a consequence of (6.12a). Indeed, eq. (6.12c) states that the superfield $e^{-\sigma}$ is real linear, and this automatically holds if (6.12a) is satisfied. Therefore it suffices to focus on the equations (6.12a) and (6.12b) only.

Let us start by analyzing eq. (6.12b). Note that this equation has the same functional form as (6.4) with $\lambda + \bar{\lambda}$ replaced by $\sigma$. We recall that, in searching for a solution to the system of equations (6.3a) and (6.4), we started with a simple ansatz (6.5). That expression consists of three parts that separately satisfy (6.12b), which are: $x^2 + \frac{1}{2}\theta^2\bar{\theta}^2$, $\theta^2$ and $\bar{\theta}^2$. The term proportional to a linear combination of $\theta^2$ and $\bar{\theta}^2$ had to be included in (6.5), since $e^{\lambda+\bar{\lambda}}$ should be the product of a chiral and antichiral superfields. In the (2,0) case, however, this is not the case; in particular, the presence of such a term would be inconsistent with the real linear constraint on $e^{-\sigma}$. But a natural way to make an ansatz consistent with eq. (6.12b)–(6.12c) is to include a term proportional to $i\theta^2\bar{\theta}$. These
considerations lead to the ansatz
\[ e^\sigma = 1 + c \rho^2 x^2 + i d \rho \theta^\alpha \bar{\theta}_\alpha + \frac{c}{2} \rho^2 \theta^2 \bar{\theta}^2 , \]  
(6.13)
where \( c, d \) are two constant real parameters. Such a superfield trivially satisfies equation (6.12b). After some algebra, one can prove that the function (6.13) also satisfies equation (6.12a) provided the parameters \( b, c \) are fixed as follows: \( c = -1/16 \) and \( d = -1/2 \). We thus have constructed the Lorentz invariant solution to the equations (6.12a)–(6.12c):
\[ e^\sigma = 1 - \frac{1}{16} \rho^2 x^2 - \frac{i}{2} \rho \theta^\alpha \bar{\theta}_\alpha - \frac{1}{32} \rho^2 \theta^2 \bar{\theta}^2 . \]  
(6.14)
As pointed out earlier, the equation (6.12a) implies that
\[ e^{-\sigma} = \frac{1}{1 - \frac{1}{16} \rho^2 x^2} + \frac{i \rho \theta^\alpha \bar{\theta}_\alpha}{2 \left(1 - \frac{1}{16} \rho^2 x^2 \right)^2} - \frac{\rho^2 \theta^2 \bar{\theta}^2 \left(3 + \frac{1}{16} \rho^2 x^2 \right)}{32 \left(1 - \frac{1}{16} \rho^2 x^2 \right)^3} . \]  
(6.15)
This means that \( e^{-\sigma} \) can be interpreted as the field strength of a particular vector multiplet in flat superspace,
\[ G_{\text{flat}} \equiv e^{-\sigma} = i D^\alpha \bar{D}_\alpha G_0 , \quad G_0 = \frac{1}{\rho} \sigma , \]  
(6.16)
such that its prepotential, \( G_0 \), is proportional to
\[ \sigma = \log \left(1 - \frac{1}{16} \rho^2 x^2 \right) - \frac{i \rho \theta^\alpha \bar{\theta}_\alpha}{2 \left(1 - \frac{1}{16} \rho^2 x^2 \right)} + \frac{\rho^2 \theta^2 \bar{\theta}^2 \left(1 + \frac{1}{16} \rho^2 x^2 \right)}{32 \left(1 - \frac{1}{16} \rho^2 x^2 \right)^2} . \]  
(6.17)
Now, we should recall the super-Weyl transformation laws of (i) the field strength \( G \) of a vector multiplet, eq. (4.15); and (ii) the corresponding prepotential \( G \), eq. (4.2). Let us apply the super-Weyl transformation generated by (6.13), which takes us from Minkowski superspace to (2,0) AdS superspace, to the vector multiplet (6.16). We then end up with a vector multiplet in (2,0) AdS superspace which is characterized by the prepotential \( G_0 = \sigma / \rho \) and the field strength
\[ G_{\text{AdS}} = i D^\alpha \bar{D}_\alpha G_0 = 1 . \]  
(6.18)
The existence of such a frozen vector multiplet with constant field strength is of importance in the study of matter multiplets in (2,0) AdS superspace. It can be used to describe chiral scalar multiplets with a real mass generated by a central charge.
7 Linearized supergravity models in Minkowski space

In this section we derive linearized 3D $\mathcal{N} = 2$ supergravity actions by dimensional reduction and truncation of 4D $\mathcal{N} = 1$ supergravity models. Dimensional reduction of any off-shell 4D $\mathcal{N} = 1$ supergravity multiplet to three dimensions should result in an off-shell 3D $\mathcal{N} = 2$ supergravity theory coupled to a vector/scalar multiplet. At the linearized level, the reduced action should be equivalent to a sum of decoupled supergravity and vector/scalar multiplet actions.

According to the classification of linearized off-shell actions for 4D $\mathcal{N} = 1$ supergravity given in [59], there are three minimal models with $12 + 12$ degrees of freedom and one non-minimal model (parametrized by a real parameter $n \neq -1/3, 0$) with $20 + 20$ degrees of freedom. One can also consider reducible supergravity actions with $16 + 16$ degrees of freedom obtained as a linear combination of two minimal models.

7.1 Type II minimal supergravity

It appears that the procedure of dimensional reduction $4D \rightarrow 3D$ is simplest in the case of the linearized action of new minimal 4D $\mathcal{N} = 1$ supergravity. This action is (see [22, 59] for derivations)

$$S^{(II)}[H_{\dot{\alpha} \dot{\beta}}, F] = \int d^4x d^4\theta \left\{ -\frac{1}{16} H^{\alpha \dot{\beta}} D^\gamma D^2 D^\gamma H_{\alpha \dot{\beta}} - \frac{1}{4} (\partial_{\dot{\alpha} \dot{\beta}} H^{\alpha \dot{\beta}})^2 + \frac{1}{16} ([D_\alpha, D_{\dot{\beta}}] H^{\alpha \dot{\beta}})^2 + \frac{1}{2} F [D_\alpha, D_{\dot{\beta}}] H^{\alpha \dot{\beta}} + \frac{3}{2} F^2 \right\}.$$  \hfill (7.1)

It is described in terms of a gravitational superfield, $H_{\alpha \dot{\beta}} = \overline{H}_{\dot{\beta} \alpha}$, and a real linear compensator, $F = \overline{F}$, subject to the constraint $D^2 F = \overline{D}^2 F = 0$. The action is invariant under the gauge transformations

$$\delta H_{\alpha \dot{\beta}} = \bar{D}_{\dot{\beta}} L_{\alpha} - D_{\alpha} \bar{L}_{\dot{\beta}},$$  \hfill (7.2a)

$$\delta F = \frac{1}{4} (D^\alpha \bar{D}^2 L_{\alpha} + \bar{D}_{\dot{\alpha}} D^2 \bar{L}_{\dot{\alpha}}),$$  \hfill (7.2b)

with $L_{\alpha}$ an unconstrained spinor parameter.

Our goal is to dimensionally reduce the action (7.1) to three dimensions. We follow [18] to relate our 3D spinor formalism to the 4D sigma-matrices

$$(\sigma_m)_{\alpha \dot{\beta}} := (1, \sigma), \quad (\bar{\sigma}_m)^{\alpha \dot{\beta}} := \varepsilon^{\beta \gamma} \varepsilon^{\dot{\alpha} \dot{\delta}} (\sigma_m)_{\gamma \dot{\delta}} = (1, -\bar{\sigma}), \quad m = 0, 1, 2, 3,$$  \hfill (7.3)
where $\bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. By deleting the matrices with space index $m = 2$ we obtain the 3D gamma-matrices

$$
(\sigma_m)_{\alpha\beta} \rightarrow (\gamma_m)_{\alpha\beta} = (\gamma_m)_{\beta\alpha} = (1, \sigma_1, \sigma_3), \quad (7.4a)
$$

$$
(\sigma_m)_{\dot{\alpha}\dot{\beta}} \rightarrow (\gamma_m)_{\alpha\beta} = (\gamma_m)_{\beta\alpha} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(\gamma_m)_{\gamma\delta}, \quad (7.4b)
$$

where the spinor indices are raised and lowered using the $\text{SL}(2,\mathbb{R})$ invariant tensors

$$
\varepsilon_{\alpha\beta} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad \varepsilon^{\alpha\beta} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha_{\beta} \quad (7.5)
$$

as follows:

$$
\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta. \quad (7.6)
$$

By construction, the matrices $(\gamma_m)_{\alpha\beta}$ and $(\gamma_m)_{\alpha\beta}$ are real and symmetric.

Upon dimensional reduction, the gravitational superfield splits into two superfields

$$
H_{\alpha\beta} := (\sigma_m)_{\alpha\beta} H_m \rightarrow H_{3D,\alpha\beta} = H_{\alpha\beta} + i\varepsilon_{\alpha\beta} H, \quad H_{\alpha\beta} := (\gamma_m)_{\alpha\beta} H_m = H_{\beta\alpha}. \quad (7.7)
$$

Here $H_{\alpha\beta}$ is the three-dimensional gravitational superfield. The gauge transformations (7.2) turn into:

$$
\delta H_{\alpha\beta} = \bar{D}_{(\alpha} L_{\beta)} - D_{(\alpha} \bar{L}_{\beta)}, \quad (7.8a)
$$

$$
\delta H = -\frac{i}{2}(\bar{D}_{\alpha} L^\alpha - D^\alpha \bar{L}_{\alpha}), \quad (7.8b)
$$

$$
\delta F = \frac{1}{4}(D^\alpha \bar{D}^2 L_{\alpha} + \bar{D}_{\alpha} D^2 L^\alpha). \quad (7.8c)
$$

In three dimensions, the real linear scalar $F$ can be expressed in terms of a real unconstrained superfield $F = \bar{F}$,

$$
F = iD^\alpha \bar{D}_{\alpha} F, \quad (7.9)
$$

which is defined modulo arbitrary gauge transformations of the form

$$
\delta F = \lambda + \bar{\lambda}, \quad \bar{D}_{\alpha} \lambda = 0. \quad (7.10)
$$

This is the gauge transformation law of an Abelian vector multiplet, with $F$ being the gauge-invariant field strength. The supergravity gauge transformation (7.8c) implies that

$$
\delta F = \frac{i}{2}(\bar{D}_{\alpha} L^\alpha - D^\alpha \bar{L}_{\alpha}). \quad (7.11)
$$

---

13Given a vector multiplet, we always use blackboard bold style to denote its gauge invariant field strength (e.g. $F$) and italic style to denote the gauge prepotential (e.g. $F$). The same capital Latin letter is used for both the field strength and gauge prepotential.
Dimensionally reducing the action (7.1) gives

\[ S_{3D}^{(II)}[H_{\alpha\beta}, H, F] = \int d^3x d^4\theta \left\{ -\frac{1}{16} H^{\alpha\beta} D^\gamma \tilde{D}^2 D_\gamma H_{\alpha\beta} - \frac{1}{4} (\partial_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{16} ([D_\alpha, \tilde{D}_\beta] H^{\alpha\beta})^2 \\
+ \frac{1}{4} H [D_\alpha, \tilde{D}_\beta] H^{\alpha\beta} + F H + \frac{1}{2} F [D_\alpha, \tilde{D}_\beta] H^{\alpha\beta} + \frac{3}{2} F^2 \right\}. \]  

(7.12)

Here \( H \) denotes the real linear superfield

\[ H := i D^\alpha \bar{D}_\alpha H, \quad D^2 H = \bar{D}^2 H = 0. \]  

(7.13)

We see that \( H \) appears in the action only through its gauge invariant field strength \( \mathbb{H} \). Thus dimensional reduction provides us with a bonus gauge symmetry. The supergravity gauge transformation (7.8b) leads to

\[ \delta H = -\frac{1}{4} (D^\alpha D^2 L_\alpha + \bar{D}_\alpha D^2 L^\alpha). \]  

(7.14)

It is useful to introduce a new parametrization for the real linear superfields:

\[ \mathbb{G} := \mathbb{H} + 2F, \quad S := \mathbb{H} + F. \]  

(7.15)

As follows from (7.8c) and (7.14), the superfield \( S \) is invariant under the supergravity gauge transformations,

\[ \delta S = 0. \]  

(7.16)

In terms of the real linear superfields introduced, the action (7.12) becomes

\[ S_{3D}^{(II)}[H_{\alpha\beta}, H, F] = S^{(II)}[H_{\alpha\beta}, \mathbb{G}] - \frac{1}{2} \int d^3x d^4\theta S^2, \]  

(7.17)

where

\[ S^{(II)}[H_{\alpha\beta}, \mathbb{G}] = \int d^3x d^4\theta \left\{ -\frac{1}{16} H^{\alpha\beta} D^\gamma \tilde{D}^2 D_\gamma H_{\alpha\beta} - \frac{1}{4} (\partial_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{16} ([D_\alpha, \tilde{D}_\beta] H^{\alpha\beta})^2 \\
+ \frac{1}{4} \mathbb{G} [D_\alpha, \tilde{D}_\beta] H^{\alpha\beta} + \frac{1}{2} \mathbb{G}^2 \right\}. \]  

(7.18)

The second term in (7.17) describes a decoupled \( \mathcal{N} = 2 \) vector multiplet. Therefore, the action (7.18) describes linearized \( \mathcal{N} = 2 \) supergravity. It is invariant under the supergravity gauge transformations (7.8a) and

\[ \delta \mathbb{G} = \frac{1}{4} (D^\alpha \bar{D}^2 L_\alpha + \bar{D}_\alpha D^2 L^\alpha). \]  

(7.19)
The properties of the real linear compensator $G$ are identical to those of $F$. We can introduce a real gauge prepotential $G = \tilde{G}$ such that
\[ G = iD^\alpha \bar{D}_\alpha G . \tag{7.20} \]
The supergravity gauge transformation (7.19) is equivalent to
\[ \delta G = \frac{i}{2}(\bar{D}_\alpha L^\alpha - D^\alpha \bar{L}_\alpha) . \tag{7.21} \]

### 7.2 Type I minimal supergravity

The supergravity action (7.18) possesses a dual formulation. To construct it, we consider the first-order model
\[
S_{\text{II} \rightarrow \text{I}} = \int d^3x d^4\theta \left\{ -\frac{1}{16} H^{\alpha\beta} D^\gamma \bar{D}^2 D_\gamma H_{\alpha\beta} - \frac{1}{4}(\partial_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{16} ([D_\alpha, \bar{D}_\beta] H^{\alpha\beta})^2 
+ U \left( \frac{1}{4} [D_\alpha, \bar{D}_\beta] H^{\alpha\beta} - \frac{3}{2}(\sigma + \bar{\sigma}) \right) + \frac{1}{2} U^2 \right\}, \tag{7.22}
\]
where $U$ is unconstrained real, $U = \bar{U}$, and $\sigma$ is chiral,
\[ D_\alpha \sigma = 0 . \tag{7.23} \]
This action proves to be invariant under the supergravity gauge transformation (7.8a) accompanied by
\[
\delta U = \frac{1}{4}(D^\alpha \bar{D}^2 L_\alpha + \bar{D}_\alpha D^2 \bar{L}^\alpha) , \tag{7.24a}
\delta \sigma = -\frac{1}{12} \bar{D}^2 D^\alpha L_\alpha . \tag{7.24b}
\]
The superfield $\sigma$ act as a Lagrange multiplier for the real linear constraint. Varying $S_{\text{II} \rightarrow \text{I}}$ with respect to $\sigma$ gives $U = G$, and then the action reduces to (7.18).

On the other hand, if we integrate out $U$, we get the dual (Type I) action
\[
S^I[H_{\alpha\beta}, \sigma] = \int d^3x d^4\theta \left\{ -\frac{1}{16} H^{\alpha\beta} D^\gamma \bar{D}^2 D_\gamma H_{\alpha\beta} - \frac{1}{4}(\partial_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{32} ([D_\alpha, \bar{D}_\beta] H^{\alpha\beta})^2 
- \frac{3i}{4}(\sigma - \bar{\sigma}) \partial_{\alpha\beta} H^{\alpha\beta} - \frac{9}{4} \bar{\sigma}\sigma \right\}. \tag{7.25}
\]
The supergravity gauge freedom of this action is as follows:
\[
\delta H_{\alpha\beta} = \bar{D}_{(\alpha} L_{\beta)} - D_{(\alpha} \bar{L}_{\beta)} , \tag{7.26a}
\delta \sigma = -\frac{1}{12} \bar{D}^2 D^\alpha L_\alpha . \tag{7.26b}
\]
7.3 Type III minimal supergravity

In complete analogy to the four-dimensional case \[59\], there exist two inequivalent ways to dualize the chiral compensator of the Type I theory into a real linear superfield. One of these dualities leads to the Type II theory, while the other produces a new dual formulation which we are going to work out below.

Let us introduce a first-order model with action

\[
S^{1\to\text{III}} = \int d^3x d^4\theta \left\{ -\frac{1}{16} H^{\alpha\beta} D^\gamma \bar{D}^2 D\gamma H_{\alpha\beta} - \frac{1}{8} (\partial_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{32} ([D_\alpha, \bar{D}_\beta] H^{\alpha\beta})^2 
+ \frac{1}{4} P \left( \partial_{\alpha\beta} H^{\alpha\beta} + 3i(\sigma - \bar{\sigma}) \right) + \frac{1}{8} P^2 \right\},
\]

(7.27)

where \( P \) is unconstrained real. The action proves to be invariant under the supergravity gauge transformations \( (7.26) \) accompanied by

\[
\delta P = \frac{i}{4} (D^\alpha \bar{D}^2 L_\alpha - \bar{D}_\alpha D^2 \bar{L}_\alpha).
\]

(7.28)

The model \( (7.27) \) is equivalent to Type I supergravity. Indeed, if the field \( P \) is integrated out, using its equation of motion, then \( (7.27) \) reduces to the Type I action, eq. \( (7.25) \). On the other hand, the equation of motion for \( \sigma \) enforces \( P \) to be linear, \( P = V \), where \( V \) obeys the constraint \( D^2 V = \bar{D}^2 V = 0 \). As a result, the first-order action \( (7.27) \) turns into the Type III supergravity action

\[
S^{\text{III}}[H_{\alpha\beta}, V] = \int d^3x d^4\theta \left\{ -\frac{1}{16} H^{\alpha\beta} D^\gamma \bar{D}^2 D\gamma H_{\alpha\beta} - \frac{1}{8} (\partial_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{32} ([D_\alpha, \bar{D}_\beta] H^{\alpha\beta})^2 
+ \frac{1}{4} V \partial_{\alpha\beta} H^{\alpha\beta} + \frac{1}{8} V^2 \right\}.
\]

(7.29)

The corresponding gauge freedom is as follows:

\[
\delta H_{\alpha\beta} = \bar{D}_{(\alpha} L_{\beta)} - D_{(\alpha} \bar{L}_{\beta)},
\]

(7.30a)

\[
\delta V = \frac{i}{4} (D^\alpha \bar{D}^2 L_\alpha - \bar{D}_\alpha D^2 \bar{L}_\alpha).
\]

(7.30b)

Associated with the real linear scalar \( V \) is a real unconstrained prepotential \( V = \bar{V} \) which is introduced by the standard rule

\[
V = iD^\alpha \bar{D}_\alpha V.
\]

(7.31)

The supergravity gauge transformation \( (7.30b) \) is equivalent to

\[
\delta V = -\frac{1}{2} (\bar{D}_\alpha L^\alpha + D^\alpha \bar{L}_\alpha).
\]

(7.32)
7.4 Non-minimal supergravity

The chiral compensator of the linearized Type I minimal supergravity, eq. (7.25), can be dualized into a complex linear superfield. The resulting theory, which is derived below, describes linearized non-minimal supergravity in three dimensions.

To work out the action for linearized non-minimal supergravity, we introduce the following first-order action:

\[
S^{I\rightarrow \text{NM}} = \int d^3x d^4\theta \left\{ -\frac{1}{16} H^{\alpha\beta} D^\gamma D^2 D_\gamma H_{\alpha\beta} - \frac{1}{4} (\partial_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{32} ([D_\alpha, \bar{D}_\beta] H^{\alpha\beta})^2 \\
- \frac{3i}{2} (C - \bar{C})(\partial^{\alpha\beta} H_{\alpha\beta}) - \frac{3}{8} (C + \bar{C}) ([D^\alpha, \bar{D}^\beta] H_{\alpha\beta}) + \frac{9(1 - 2w)}{8}(C^2 + \bar{C}^2) \\
- \frac{9}{4} C \bar{C} + 3C \Sigma + 3\bar{C} \bar{\Sigma} \right\}.
\]

(7.33)

Here \( C \) is an unconstrained complex superfield, and \( \Sigma \) a complex linear superfield under the constraint

\[
D^2 \Sigma = 0.
\]

(7.34)

The action (7.33) proves to be invariant under the supergravity gauge transformation (7.8a) accompanied with the following variations of \( C \) and \( \Sigma \):

\[
\delta C = -\frac{1}{12} \bar{D}^2 D^\alpha L_\alpha,
\]

(7.35a)

\[
\delta \Sigma = -\frac{w + 1}{8} \bar{D}^2 D^\alpha L_\alpha - \frac{1}{4} \bar{D}_\alpha D^2 \bar{L}^\alpha.
\]

(7.35b)

The model (7.33) is equivalent to Type I supergravity. Indeed, the equation of motion for \( \Sigma \) enforces the field \( C \) to be chiral, \( \bar{D}_\alpha C = 0 \). After that, upon re-labelling \( C = \sigma \), the action (7.33) reduces to (7.25). On the other hand, when \( w \neq 0, 1 \), one can use the equations of motion for \( C \) and \( \bar{C} \) in order to algebraically express these fields in terms of the other dynamical variables in (7.33). This yields the dual non-minimal supergravity model

\[
S^{\text{NM}}[H_{\alpha\beta}, \Sigma] = \int d^3x d^4\theta \left\{ -\frac{1}{16} H^{\alpha\beta} D^\gamma D^2 D_\gamma H_{\alpha\beta} + \frac{w + 1}{32w} ([D_\alpha, \bar{D}_\beta] H^{\alpha\beta})^2 \\
- \frac{w + 1}{4(w - 1)} (\partial_{\alpha\beta} H^{\alpha\beta})^2 - \frac{i}{(w - 1)}(\Sigma - \bar{\Sigma}) \partial^{\alpha\beta} H_{\alpha\beta} - \frac{1}{4w} ([\Sigma + \bar{\Sigma}][D^\alpha, \bar{D}^\beta] H_{\alpha\beta}) \\
- \frac{1}{w(w - 1)} \Sigma \bar{\Sigma} + \frac{2w - 1}{2w(w - 1)} (\Sigma^2 + \bar{\Sigma}^2) \right\}.
\]

(7.36)
The corresponding gauge freedom is as follows:

$$
\delta H_{\alpha\beta} = \bar{D}_{(\alpha} L_{\beta)} - D_{(\alpha} \bar{L}_{\beta)} ,
$$

(7.37a)

$$
\delta \Sigma = -\frac{w + 1}{8} \bar{D}^2 D^\alpha L_\alpha - \frac{1}{4} \bar{D}_\alpha D^2 \bar{L}^\alpha .
$$

(7.37b)

By construction, the action (7.36) is not defined for $w = 0, 1$. The constraint (7.34) can be solved in terms of an unconstrained prepotential $\bar{\Psi}^\alpha$,

$$
\Sigma = \bar{D}_\alpha \bar{\Psi}^\alpha ,
$$

(7.38)

defined modulo arbitrary gauge transformations of the form

$$
\delta \bar{\Psi}^\alpha = \bar{D}^\beta \rho^{\alpha\beta} , \quad \bar{\rho}^{\beta\alpha} = \bar{\rho}^{\alpha\beta} .
$$

(7.39)

The parameter $w$ corresponds to the one introduced in section (5.1). The gauge transformation (7.37b) and its $w$ dependence can in fact be inferred in few simple steps. First, note that in four-dimensions the non-minimal supergravity gauge transformations for the complex linear compensator are given by [22]

$$
\delta \Sigma = -\frac{1}{4} \frac{n + 1}{3n + 1} \bar{D}^2 D^\alpha L_\alpha - \frac{1}{4} \bar{D}_{\dot{\alpha}} D^2 \bar{L}^{\dot{\alpha}} ,
$$

(7.40)

and are parametrized by the parameter $n$. Then, dimensionally reduce the previous transformations to 3D and use (5.5) to obtain (7.35b). Once the gauge transformations are determined, the first-order action (7.33) is uniquely determined from the requirement of its gauge invariance.

As in four dimensions (see the Appendix), there is a natural freedom to perform a field redefinition of $\Sigma$ of the form

$$
\Sigma \rightarrow \Sigma + \kappa \bar{D}_\alpha D_\beta H^{\alpha\beta} ,
$$

(7.41)

with $\kappa$ a constant parameter which we choose (for simplicity) to be real. Such a field redefinition will modify the transformation law (7.37b) to the form

$$
\delta \Sigma = -\frac{w + 1}{8} \bar{D}^2 D^\alpha L_\alpha - \frac{1}{4} (1 - 3\kappa) \bar{D}_\alpha D^2 \bar{L}^\alpha + \kappa \bar{D}_\alpha D_\beta D(\alpha L^\beta) .
$$

(7.42)

8 Variant supercurrents in Minkowski space

This section is devoted to the study of general 3D $\mathcal{N} = 2$ supercurrent multiplets in Minkowski space. The general 4D $\mathcal{N} = 1$ supercurrents in Minkowski space are reviewed in the Appendix.
8.1 Supercurrents associated with off-shell supergravity

Using the explicit structure of the three minimal actions for linearized supergravity constructed above, we can derive the most general 3D $\mathcal{N} = 2$ supercurrent multiplet in complete analogy with the four-dimensional analysis given in \cite{46, 49}. This general procedure leads to the following conservation equation:

$$\bar{D}^\beta J_{\alpha\beta} = D_\alpha X + \bar{D}_\alpha (\mathcal{Y} + i\mathcal{Z}) ,$$  \hspace{1cm} (8.1)

where $J_{\alpha\beta} = J_{\beta\alpha} = \overline{J_{\alpha\beta}}$ is the supercurrent, and the trace multiplets $X$, $\mathcal{Y}$ and $\mathcal{Z}$ are constrained as follows:

$$\bar{D}_\alpha X = 0 ,$$  \hspace{1cm} (8.2a)

$$\bar{D}^2 \mathcal{Y} = 0 , \quad \overline{\mathcal{Y}} = \mathcal{Y} ,$$  \hspace{1cm} (8.2b)

$$\bar{D}^2 \mathcal{Z} = 0 , \quad \overline{\mathcal{Z}} = \mathcal{Z} .$$  \hspace{1cm} (8.2c)

A 3D extension of the superfield Noether procedure \cite{41} can be argued to lead to the same 16+16 supercurrent. This multiplet is decomposable and can be viewed as a superposition of the supercurrent multiplets associated with the three minimal supergravity versions. The choice $\mathcal{Y} = \mathcal{Z} = 0$ corresponds to the Ferrara-Zumino (FZ) multiplet associated with the Type I supergravity. Choosing $X = \mathcal{Y} = 0$ gives the so-called $\mathcal{R}$-multiplet associated with the Type II supergravity. Finally, the option $X = \mathcal{Z} = 0$ corresponds to the supercurrent associated with the Type III supergravity model. Of course, there remains one more possibility, $X = \mathcal{Y} = \mathcal{Z} = 0$, which holds for all $\mathcal{N} = 2$ superconformal field theories.

Given a chiral spinor superfield $\eta_\alpha$, such that $\bar{D}_\beta \eta_\alpha = 0$, it can always be represented in the form $\eta_\alpha = \bar{D}_\alpha (\mathcal{Y} + i\mathcal{Z})$, for some real linear superfields $\mathcal{Y}$ and $\mathcal{Z}$ defined modulo constant shifts.

As an example, let us derive the $\mathcal{R}$-multiplet. For this we add source terms to the Type II action (7.18)

$$\mathcal{S}^{\text{II}}[H_{\alpha\beta}, \mathcal{G}] - \frac{1}{2} \int d^3x d^4\theta H^{\alpha\beta} \mathcal{R}_{\alpha\beta} + \int d^3x d^4\theta G \mathcal{Z} .$$  \hspace{1cm} (8.3)

This action should preserve the vector-multiplet gauge freedom

$$\delta G = \lambda + \bar{\lambda} , \quad \bar{D}_\alpha \lambda = 0$$  \hspace{1cm} (8.4)
which demands the source $Z$ to be real linear, eq. (8.2c). The action should also respect the linearized supergravity gauge freedom 

$$
\delta H_{\alpha\beta} = \bar{D}_{(\alpha L_{\beta})} - D_{(\alpha \bar{L}_{\beta})} , \quad \delta G = \frac{i}{2}(\bar{D}_{\alpha}L_{\alpha} - D^{\alpha}\bar{L}_{\alpha}) .
$$

(8.5)

This requires the sources to obey the conservation equation

$$
\bar{D}^\beta R_{\alpha\beta} = i \bar{D}^\alpha Z.
$$

(8.6)

The $16 + 16$ supercurrent multiplet, eq. (8.1), can be modified by an improvement transformation of the form

$$
\begin{align*}
J_{\alpha\beta} &\rightarrow J_{\alpha\beta} + D_{(\alpha \bar{\Upsilon}_{\beta})} - \bar{D}_{(\alpha \Upsilon_{\beta})} , \\
X &\rightarrow X + \frac{1}{2}D_{\alpha} \Upsilon^\alpha , \\
\Upsilon &\rightarrow \Upsilon - \frac{1}{2}(D^\alpha \bar{\Upsilon}_{\alpha} + \bar{D}_{\alpha} \Upsilon^\alpha) , \\
Z &\rightarrow Z + i(\bar{D}_{\alpha} \Upsilon^\alpha - D^{\alpha}\bar{\Upsilon}_{\alpha}) ,
\end{align*}
$$

(8.7a-d)

where the spinor superfield $\Upsilon_{\alpha}$ is constrained by

$$
D_{(\alpha \Upsilon_{\beta})} = 0 \rightarrow D^2 \Upsilon_{\alpha} = 0 .
$$

(8.8)

This constraint can locally be solved by

$$
\Upsilon_{\alpha} = D_{\alpha}(V + iU) , \quad \bar{V} = V , \quad \bar{U} = U ,
$$

(8.9)

where the scalars $V$ and $U$ are defined modulo a local shift

$$
V + iU \rightarrow V + iU + \bar{\lambda} , \quad D_{\alpha} \bar{\lambda} = 0 ,
$$

(8.10)

with an arbitrary chiral superfield $\lambda$. If $\Upsilon_{\alpha}$ is globally given by (8.9), for well defined operators $V$ and $U$, then the improvement transformation (8.7) takes the form

$$
\begin{align*}
J_{\alpha\beta} &\rightarrow J_{\alpha\beta} + [D_{(\alpha} \bar{D}_{\beta)}]V - 2\partial_{\alpha\beta}U , \\
X &\rightarrow X + \frac{1}{2}\bar{D}^2(V - iU) , \\
\Upsilon &\rightarrow \Upsilon + iD^\alpha \bar{D}_{\alpha}U , \\
Z &\rightarrow Z - 2iD^\alpha \bar{D}_{\alpha}V .
\end{align*}
$$

(8.11a-d)

This is a 3D analogue of the improvement transformation given in [49].
As mentioned before, the supercurrent (8.1) encodes information about the three minimal supergravity versions. We should also consider a supercurrent associated with the non-minimal model (7.36) for linearized supergravity. Direct calculations lead to the non-minimal supercurrent
\[ \bar{D}_\beta J^{\alpha \beta} = -\frac{w + 1}{4} D_\alpha \bar{D}_\beta \zeta^\alpha - \frac{1}{2} \bar{D}^2 \zeta_\alpha , \] (8.12)
where the trace multiplet \( \zeta_\alpha \) is constrained by
\[ D(\alpha \zeta_\beta) = 0 . \] (8.13)
This constraint is required by the gauge invariance (7.39). The conservation law (8.12) can be rewritten in the form (8.1) if we identify
\[ X = -\frac{w + 1}{4} \bar{D}_\alpha \zeta^\alpha , \quad Y = \frac{1}{2} (D^\alpha \bar{\zeta}_\alpha + \bar{D}_\alpha \zeta^\alpha) , \quad Z = \frac{1}{2} (D^\alpha \bar{\zeta}_\alpha - \bar{D}_\alpha \zeta^\alpha) . \] (8.14)
As follows from (8.7), it is always possible to improve at least one of \( Y \) and \( Z \) to zero.

It is also instructive to construct the non-minimal supercurrent associated with the modified transformation law (7.42). It is
\[ \bar{D}_\beta J^{\alpha \beta} = \left( \kappa - \frac{w + 1}{4} \right) D_\alpha \bar{D}_\beta \zeta^\alpha + \bar{D}_\alpha \left\{ \kappa D^\beta \bar{\zeta}_\beta + (1 - 3\kappa) \bar{D}_\beta \zeta^\beta \right\} , \] (8.15)
with \( \zeta_\alpha \) constrained as in eq. (8.13). This conservation law can be rewritten in the form (8.1) provided we identify
\[ X = \left( \kappa - \frac{w + 1}{4} \right) \bar{D}_\beta \zeta^\beta , \] (8.16a)
\[ Y = \frac{1}{2} (1 - 2\kappa) (D^\alpha \bar{\zeta}_\alpha + \bar{D}_\alpha \zeta^\alpha) , \] (8.16b)
\[ Z = \frac{1}{2} (1 - 4\kappa) (D^\alpha \bar{\zeta}_\alpha - \bar{D}_\alpha \zeta^\alpha) . \] (8.16c)
It follows from these expressions that one of the three trace multiplets \( X, Y \) and \( Z \) can be set to zero by appropriately choosing the deformation parameter \( \kappa \).

Our consideration shows that non-minimal supergravity does not lead to a more general supercurrent than the one defined by eq. (8.1). The reason for this is that the non-minimal action (7.36) can be represented as a linear combination of the three minimal actions (7.18), (7.25) and (7.29),
\[ S_{\text{NM}}^{\alpha \beta, \Sigma} = a_I S_I^{\alpha \beta, \sigma} + a_{II} S_{\text{II}}^{\alpha \beta, \mathcal{G}} + a_{III} S_{\text{III}}^{\alpha \beta, \mathcal{V}} , \] (8.17)
for some real parameters \( a_I \)'s such that \( a_I + a_{II} + a_{III} = 1 \), provided the complex linear compensator is represented as \( \Sigma = \alpha \sigma + \beta \mathcal{G} + i \gamma \mathcal{V} \), with constant real coefficients \( \alpha, \beta \) and \( \gamma \). The derivation of this result is completely similar to the four-dimensional analysis given in [59].

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8.2 The S-multiplet

The $S$-multiplet of Dumitrescu and Seiberg [50], $S_{\alpha\beta} = S_{\beta\alpha} = \bar{S}_{\alpha\beta}$, obeys the conservation equation

$$\bar{D}^\beta S_{\alpha\beta} = \chi_\alpha + \mathcal{Y}_\alpha ,$$  \hspace{1cm} (8.18)

where the trace multiplets $\chi_\alpha$ and $\mathcal{Y}_\alpha$ are constrained by

$$\bar{D}_\alpha \chi_\beta = \frac{1}{2} C \varepsilon_{\alpha\beta} , \hspace{0.5cm} D^\alpha \chi_\alpha = \bar{D}_\alpha \bar{\chi}_\alpha ,$$  \hspace{1cm} (8.19a)

$$D_{(\alpha} \mathcal{Y}_{\beta)} = 0 , \hspace{0.5cm} \bar{D}^\alpha \mathcal{Y}_\alpha = -C ,$$  \hspace{1cm} (8.19b)

with $C$ a complex constant. Our goal is to compare the $S$-multiplet to the most general supercurrent (8.1) derived from off-shell supergravity. It should be pointed out that the parameter $C$ is non-vanishing only in the presence of brane currents [50]. Since in this paper we are interested in those rigid supersymmetric theories that can be coupled to supergravity, we are forced (i) to set $C = 0$ and (ii) to restrict $\mathcal{Y}_\alpha$ to have the form

$$\mathcal{Y}_\alpha = D_\alpha X , \hspace{0.5cm} \bar{D}_\alpha X = 0 .$$  \hspace{1cm} (8.20)

As a result, the $S$-multiplet turns into (8.1) with

$$\mathcal{Y} = 0 , \hspace{0.5cm} i\bar{D}_\alpha Z = \chi_\alpha .$$  \hspace{1cm} (8.21)

When $\mathcal{Y} = 0$, the improvement transformation (8.7) is generated by a superfield $\Upsilon_\alpha$ constrained by [50]

$$D_{(\alpha} \Upsilon_{\beta)} = 0 , \hspace{0.5cm} D^\alpha \bar{\Upsilon}_\alpha + \bar{D}_\alpha \Upsilon^\alpha = 0 .$$  \hspace{1cm} (8.22)

A remarkable result of Dumitrescu and Seiberg [50] is that the trace multiplet $\mathcal{Y}$ can always be improved to zero. Although their proof is based on some nontrivial assumptions, the outcome proves to be correct for all known supersymmetric theories. This result has in fact a natural justification from the supergravity point of view, as first discussed in four dimensions [46]. The point is that the $\mathcal{Y}$ multiplet is associated with the Type III minimal supergravity which is known only at the linearized level and does not have a nonlinear extension. It is therefore to be expected that matter couplings to this supergravity formulation should be impossible.

\footnote{This result was actually derived in [50] for the four-dimensional case. However, their argument can be easily extended to three dimensions.}
8.3 Examples of supercurrents

We now give several examples of $\mathcal{N} = 2$ supercurrents in three dimensions. Modulo an improvement transformation, it holds that $\mathcal{Y} = 0$ for all models to be considered. Our first example is the most general supersymmetric nonlinear $\sigma$-model

$$S = \int d^3x d^4\theta \, K(\varphi^I, \bar{\varphi}^J) + \left\{ \int d^3x d^2\theta \, W(\varphi^I) + \text{c.c.} \right\}. \quad (8.23)$$

The corresponding supercurrent multiplet is

$$S_{\alpha\beta} = 2K_{IJ}D(\alpha \phi^I \bar{D}_\beta \bar{\phi}^J), \quad Z = -iD^a \bar{D}_a K, \quad X = 4W. \quad (8.24)$$

This is a 3D analogue of the 4D $\mathcal{N} = 1$ $\sigma$-multiplet given in [45]. All the superfields in (8.24) are invariant under arbitrary Kähler transformations.

An interesting subclass of nonlinear $\sigma$-models is the case in which the action is invariant under $U(1)_R$ symmetry. The $R$-symmetric Kähler potential $K(\phi^I, \bar{\phi}^J)$ and the superpotential $W(\phi^I)$ are respectively constrained by the equations (4.29a) and (4.29b) where the chiral superfields have $U(1)_R$ charges $J_{\phi^I} = -r_I \phi^I$, eq. (4.27). The equations of motion for $\phi^I$ are

$$\bar{D}^2 K_I = 4W_I. \quad (8.25)$$

This equation, together with (4.29a)–(4.29b), imply that on-shell the superpotential $W$ admit a real prepotential $V$ defined by

$$W = \frac{1}{16} \bar{D}^2 V, \quad V := 2 \sum_I r_I \phi^I K_I = 2 \sum_I r_I \bar{\phi}^I K_I. \quad (8.26)$$

For the $R$-symmetric $\sigma$-model the supertrace multiplet $X$ in the supercurrent (8.24) simplifies and takes the form $X = \frac{1}{4} \bar{D}^2 V$. It is clear that, by using the improvement transformations (8.7a)–(8.7d), we can set to zero either $Z$ or $X$. In fact, by applying the improvement transformation (8.7a)–(8.7d) with $V = -\frac{1}{2} K$ and $U = 0$ to the supercurrent (8.24) we obtain the FZ multiplet

$$J^{(\text{FZ})}_{\alpha\beta} = 2K_{IJ}D(\alpha \phi^I \bar{D}_\beta \bar{\phi}^J) - \frac{1}{2}[D(\alpha, \bar{D}_\beta)]K, \quad X = \frac{1}{4} \bar{D}^2 (V - K). \quad (8.27)$$

On the other hand, applying the improvement transformation (8.11) with $V = \frac{1}{2}(K - V)$ and $U = 0$ to the FZ multiplet leads to the $R$-multiplet

$$R_{\alpha\beta} = 2K_{IJ}D(\alpha \phi^I \bar{D}_\beta \bar{\phi}^J) - \frac{1}{2}[D(\alpha, \bar{D}_\beta)]V, \quad Z = iD^a \bar{D}_a (V - K). \quad (8.28)$$
The requirement that the $R$-symmetric $\sigma$-model be also superconformal is expressed as the condition $V = K$ \[66\]. In such a case, it follows from \eqref{8.27} and \eqref{8.28} that the FZ and $R$ multiplets coincide.

For the next example, consider a vector multiplet with Chern-Simons and Fayet-Iliopoulos terms

$$S = \int d^3x d^4\theta \left\{-\frac{1}{2e^2}G^2 + \frac{\kappa}{2}G\bar{G} + \xi G\right\}.$$ \hspace{1cm} (8.29)

Here the parameters $\kappa$ and $\xi$ correspond to the Chern-Simons and Fayet-Iliopoulos terms respectively. This model is characterized by the $R$-multiplet \[50\]

$$J_{\alpha\beta} = \frac{2}{e^2}D_{(\alpha}G\bar{D}_{\beta)}G, \hspace{1cm} (8.30a)$$

$$Z = -\frac{i}{2e^2}D^a\bar{D}_aG^2 - \frac{\xi}{2}G^2, \hspace{0.5cm} X = 0. \hspace{1cm} (8.30b)$$

This is the 3D analogue of the supercurrent for the free vector multiplet model with a Fayet-Iliopoulos term \[43, 44\]. The Chern-Simons coupling does not appear in \eqref{8.30}. This is due to the fact that the Chern-Simons term does not couple to the supergravity prepotentials.

It is also instructive to consider an improved vector multiplet with a Chern-Simons term (see e.g. \[18\])

$$S = \int d^3x d^4\theta \left\{-G\ln G + \frac{\kappa}{2}G\bar{G}\right\}.$$ \hspace{1cm} (8.31)

This model is $\mathcal{N} = 2$ superconformal. Its supercurrent proves to be

$$J_{\alpha\beta} = \frac{2}{G}D_{(\alpha}G\bar{D}_{\beta)}G - \frac{1}{2}[D_{(\alpha}, \bar{D}_{\beta)}]G. \hspace{1cm} (8.32)$$

It obeys the conservation equation

$$\bar{D}_\beta J_{\alpha\beta} = 0. \hspace{1cm} (8.33)$$

The previous two models \eqref{8.29} and \eqref{8.31} are special cases of a general system of self-interacting Abelian vector multiplets described by the gauge invariant action

$$S = \int d^3x d^4\theta \left\{L(F^i) + \frac{1}{2}m_{ij}F^i\bar{F}^j + \xi_iF^i\right\}. \hspace{1cm} (8.34)$$

Here $L$ is an arbitrary real function of the real linear field strengths $F^i$, with $i = 1, \ldots, n$, for which $F^i$ are the gauge prepotentials, $\bar{F}^i = iD^\gamma\bar{D}_\gamma F^i$. The real constants $m_{ij} = m_{ji} = \ldots$
(m_{ij})^* and \( \xi_i = (\xi_i)^* \) are respectively Chern-Simons and Fayet-Iliopoulos couplings. It can be shown that the \( \mathcal{R} \)-multiplet for this system is
\[
J_{\alpha\beta} = -2L_{ij}D_{(\alpha}F^{i}\bar{D}_{\beta)}F^j ,
\]
\[
Z = iD^\gamma\bar{D}_\gamma(F^iL_i - L) - \xi_iF^i .
\]

The action \((8.34)\) describes a superconformal theory if \( \xi_i = 0 \) and \( L(F) \) is a homogeneous function of first degree, \( F^iL_i = L \). In this case \( Z = 0 \).

Finally, we consider a scalar multiplet with a real central charge
\[
S = \int d^3x d^4\theta \bar{\Phi}e^{2mV_0}\Phi , \quad V_0 = i\theta^\alpha\bar{\theta}_\alpha , \quad m = \bar{m} = \text{const} . \tag{8.36}
\]
Here the dynamical variable \( \Phi \) is a chiral superfield, \( \bar{D}_\alpha\Phi = 0 \). The equations of motion imply that
\[
(\Box - m^2)\Phi = 0 . \tag{8.37}
\]

The supercurrent for this model is the \( \mathcal{R} \)-multiplet \[50\]
\[
J_{\alpha\beta} = [D_{(\alpha}D_{\beta)}](\bar{\Phi}e^{2mV_0}\Phi) - 4i \bar{\Phi}e^{2mV_0}\bar{\partial}_{\alpha\beta}\Phi , \quad Z = 8m\bar{\Phi}e^{2mV_0}\Phi . \tag{8.38}
\]

Although the trace multiplet \( Z \) is linear on-shell, \( \bar{D}^2Z = 0 \), it cannot be represented as \( iD^\alpha\bar{D}_\alpha Z \), for a well defined operator \( Z \).

\section{(1,1) AdS supersymmetry}

In this section we study rigid supersymmetric field theories in (1,1) AdS superspace. The geometry of this superspace is completely determined by the (anti-)commutation relations \((1.3a)-(1.3c)\). As shown in subsection \(4.2\) the (1,1) AdS superspace originates as a solution to the equations of motion of the Type I minimal and the non-minimal AdS supergravity theories. In order to derive consistent supercurrents corresponding to (1,1) AdS supersymmetry, we have to compute linearized supergravity actions around the (1,1) AdS background chosen.

\subsection{Linearized minimal supergravity}

Our first task is to derive a (1,1) AdS extension of the linearized Type I action in Minkowski superspace, eq. \((7.25)\). To achieve this we start with the following functional
in (1,1) AdS superspace

\[
\int d^3x d^4\theta E \left\{ -\frac{1}{16} H^{\alpha\beta} \nabla^2 H_{\alpha\beta} + \frac{1}{32} \left[ (\nabla_{\alpha}, \nabla_{\beta}) H^{\alpha\beta} \right]^2 \right.
\]
\[
- \frac{1}{4}(\nabla_{\alpha\beta} H^{\alpha\beta})^2 - \frac{3i}{4}(\sigma - \bar{\sigma})(\nabla_{\alpha\beta} H^{\alpha\beta}) - \frac{9}{4} \bar{\sigma} \sigma \} , \tag{9.1}
\]

where the compensator \(\sigma\) is covariantly chiral,

\[
\nabla_{\alpha} \sigma = 0 . \tag{9.2}
\]

The above functional is a minimal lift of the Type I action (7.25) to (1,1) AdS superspace. The desired linearized action for minimal AdS supergravity is expected to differ from (9.1) by some \(\mu\)-dependent terms required to guarantee invariance under the linearized supergravity gauge transformations in (1,1) AdS:

\[
\delta H_{\alpha\beta} = \bar{\nabla}(\alpha L_{\beta}) - \nabla(\bar{\alpha} L_{\beta}) , \tag{9.3a}
\]
\[
\delta \sigma = -\frac{1}{12} (\bar{\nabla}^2 - 4\bar{\mu}) \nabla_{\alpha} L_{\alpha} . \tag{9.3b}
\]

To derive the linearized action, we compute the variation of (9.1) under (9.3a) and (9.3b) and then iteratively add certain \(\mu\)-dependent terms in order to end up with an invariant action. In carrying out such calculations, one may find useful the following identities that derive from (1.3a)–(1.3c):

\[
\nabla_{\alpha} (\bar{\nabla}^2 - 4\bar{\mu}) = 4\bar{\mu} \nabla^\beta M_{\alpha\beta} , \tag{9.4a}
\]
\[
[\nabla^\alpha, \nabla^2] = 4i \nabla^\alpha \bar{\nabla} - 6\mu \nabla^\alpha , \quad [\nabla_{\alpha}, \nabla^2] = -i\bar{\mu}(\gamma_\alpha)_{\alpha\beta}[\nabla^\alpha, \bar{\nabla}^\beta] , \tag{9.4b}
\]
\[
[\nabla_{\alpha\beta}, \nabla_\gamma \bar{\nabla}_\gamma^2] = -2i\mu \varepsilon_{\gamma(\alpha} \nabla^\beta \bar{\nabla}) + 4\mu \nabla_{\alpha\beta} \nabla_\gamma - 4i\mu \bar{\mu} \varepsilon_{\gamma(\alpha} \nabla_\beta) - 8i\mu \bar{\mu} \varepsilon_{\gamma(\alpha} \bar{\nabla} \beta M_{\beta\gamma}) + 8i\mu \bar{\mu} \varepsilon_{\gamma(\alpha} \nabla_\beta \bar{\nabla}) , \tag{9.4c}
\]

Together with their complex conjugates. It is also important to keep in mind the rule for integration by parts, eq. (2.12).

The calculation procedure outlined leads to the following linearized action for Type I minimal AdS supergravity:

\[
S_{(1,1)}[H_{\alpha\beta}, \sigma] = \int d^3x d^4\theta E \left\{ -\frac{1}{16} H^{\alpha\beta} \nabla^\gamma (\nabla^2 - 6\mu) \nabla_\gamma H_{\alpha\beta} \right. \\
- \frac{1}{4}(\nabla_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{32} \left[ (\nabla_\alpha, \nabla_\beta) H^{\alpha\beta} \right]^2 - \frac{3i}{4}(\sigma - \bar{\sigma})(\nabla_{\alpha\beta} H^{\alpha\beta}) - \frac{9}{4} \bar{\sigma} \sigma \} . \tag{9.5}
\]

Due to the identity

\[
\nabla^\alpha (\nabla^2 - 6\mu) \nabla_\alpha = \nabla_\alpha (\nabla^2 - 6\bar{\mu}) \nabla^\alpha , \tag{9.6}
\]

the Lagrangian in (9.5) is manifestly real.
9.2 Linearized non-minimal supergravity

By analogy with the flat superspace case, now that we have derived the Type I action we can obtain a non-minimal one by dualization. The nonlinear analysis of section 5.2 tells us that this can be done only when $w = -1$. This can also be immediately understood by comparing the last term in the third line of (9.5) and the last term in the second line of (7.33). Because the $(C^2 + \bar{C}^2)$ terms in the flat first-order action (7.33) and its AdS analogue have to be the same, it follows that only for $w = -1$ it is possible to carry out a dualization procedure.

A (1,1) AdS extension of the first-order action (7.33) with $w = -1$ should involve a complex unconstrained superfield $C$ and a complex linear superfield $\Sigma$ constrained by

$$\nabla^2 \Sigma = 0. \quad (9.7)$$

This action has the form

$$S^{\text{I-\text{NM}}}_{(1,1)}[H_{\alpha\beta}, C, \Sigma] = \int d^3x d^4\theta E \left\{ -\frac{1}{16} H^{\alpha\beta} \nabla^\gamma (\nabla^2 - 6\mu) \nabla_\gamma H_{\alpha\beta} 
+ \frac{1}{4} \left( \nabla_\alpha H_{\beta}^{\alpha\beta} \right)^2 + \frac{3}{32} \left( [\nabla_\alpha, \nabla_\beta] H^{\alpha\beta} \right)^2 - \frac{3}{4} (C - \bar{C}) \nabla_\alpha \nabla_\beta H_{\alpha\beta} - \frac{9}{4} \bar{C} C + \frac{27}{8} (C^2 + \bar{C}^2)
+ 3C \Sigma + 3\bar{C}\bar{\Sigma} + \frac{3}{4} C \nabla^\alpha \nabla^\beta H_{\alpha\beta} - \frac{3}{4} \bar{C} \nabla^\alpha \nabla^\beta H_{\alpha\beta} \right\} \quad (9.8)$$

and is invariant under the supergravity gauge transformation (9.3a) in conjunction with

$$\delta C = -\frac{1}{12} (\nabla^2 - 4\mu) \nabla^\alpha L_\alpha, \quad (9.9a)$$
$$\delta \Sigma = -\frac{1}{4} \nabla_\alpha (\nabla^2 + 2\bar{\mu}) \bar{L}_\alpha. \quad (9.9b)$$

The equation of motion for $\Sigma$ enforces the field $C$ to be chiral $\nabla_\alpha C = 0$; with $C = \sigma$ the action (9.8) reduces to (9.5). On the other hand, integrating out $C$ and $\bar{C}$, we obtain the linearized action for non-minimal AdS supergravity:

$$S^{\text{NM}}_{(1,1)}[H_{\alpha\beta}, \Sigma] = \int d^3x d^4\theta E \left\{ -\frac{1}{16} H^{\alpha\beta} \nabla^\gamma (\nabla^2 - 6\mu) \nabla_\gamma H_{\alpha\beta} 
+ \frac{1}{2} (\Sigma - \bar{\Sigma}) \nabla^\alpha \nabla^\beta H_{\alpha\beta} + \frac{1}{4} (\Sigma + \bar{\Sigma})([\nabla_\alpha, \nabla_\beta] H_{\alpha\beta}) 
- \frac{1}{2} \Sigma \Sigma - \frac{3}{4} (\Sigma^2 + \bar{\Sigma}^2) \right\}. \quad (9.10)$$

By construction, this action is invariant under the gauge transformations

$$\delta H_{\alpha\beta} = \nabla_{(\alpha} L_{\beta)} - \nabla_{(\alpha} \bar{L}_{\beta)}, \quad (9.11a)$$
$$\delta \Sigma = -\frac{1}{4} \nabla_\alpha (\nabla^2 + 2\bar{\mu}) \bar{L}_\alpha. \quad (9.11b)$$
The reader may now ask the question: Is it possible to dualize the linearized Type I theory in (1,1) AdS to Type II and Type III theories? By looking at the first-order actions (7.22) and (7.27), which are used to carry out the duality transformations in the flat case, it is not surprising that the answer to this question is no. In fact, a necessary condition to perform the duality would be that the action (9.5) was a function only of the real (Type II) or imaginary (Type III) part of the chiral compensator $\sigma$. However, from the explicit form of the Type I action in (1,1) AdS, eq. (9.5), it is clear that this is not the case.

9.3 Matter couplings in (1,1) AdS superspace

To describe rigid supersymmetric field theories in (1,1) AdS superspace, we need to develop a superfield description of the corresponding isometry transformations. The isometries are generated by (1,1) AdS Killing vector fields, $\Lambda = \lambda^a \nabla_a + \lambda^\alpha \nabla_\alpha + \bar{\lambda}^{\dot{\alpha}} \bar{\nabla}_{\dot{\alpha}}$, which are defined to obey the master equation

$$\left[ \Lambda + \frac{1}{2} \omega^{ab} M_{ab}, \nabla_C \right] = 0.$$ (9.12)

This equation is equivalent to the relations

$$0 = \nabla_{(\alpha} \lambda_{\beta)} - \frac{1}{2} \omega_{\alpha\beta}, \quad 0 = \bar{\nabla}_{(\alpha} \lambda_{\beta)} + i \mu_{\alpha\beta}, \quad \nabla_\alpha \lambda^\alpha = \bar{\nabla}^\alpha \lambda_\alpha = 0,$$ (9.13a)

$$0 = \nabla_\beta \omega_{\alpha\beta} - 12 \bar{\mu}_\alpha \lambda^\alpha, \quad 0 = \bar{\nabla}^\beta \lambda_{\alpha\beta} + 6i \bar{\lambda}_\alpha, \quad \nabla_{(\alpha} \lambda_{\beta\gamma)} = \nabla_{(a} \omega_{b)} = 0$$ (9.13b)

and their complex conjugates. The (1,1) AdS Killing vector fields can be shown to generate the supergroup $\text{OSp}(1|2; \mathbb{R}) \times \text{OSp}(1|2; \mathbb{R})$.

Matter couplings in (1,1) AdS superspace are very similar to those in 4D $\mathcal{N} = 1$ AdS [32, 33, 31], and as such they are more restrictive than their counterparts in Minkowski space. As a nontrivial example, here we consider the most general supersymmetric nonlinear $\sigma$-model in (1,1) AdS superspace:

$$S = \int d^3x d^4 \theta E \mathcal{K}(\varphi^I, \bar{\varphi}^\dot{I}).$$ (9.14)

The dynamical variables $\varphi^I$ are covariantly chiral superfields, $\nabla_a \varphi^I = 0$, and at the same time local complex coordinates of a complex manifold $\mathcal{M}$. The action is invariant under (1,1) AdS isometry transformations

$$\delta \varphi^I = \Lambda \varphi^I.$$ (9.15)
Unlike in the Minkowski case, the action does not possess Kähler invariance since

$$\int d^3x \, d^4\theta \mathcal{E} F(\phi) = \int d^3x \, d^2\theta \mathcal{E} \mu F(\phi) \neq 0,$$

(9.16)

with \( \mathcal{E} \) the chiral density. Nevertheless, Kähler invariance naturally emerges if we represent the Lagrangian as

$$\mathcal{K}(\phi, \bar{\phi}) = \mathcal{K}(\phi, \bar{\phi}) + \frac{1}{\mu} W(\phi) + \frac{1}{\bar{\mu}} \bar{W}(\bar{\phi}).$$

(9.17)

Under a Kähler transformation, these transform as

$$K(\phi, \bar{\phi}) \rightarrow K(\phi, \bar{\phi}) + F(\phi) + F(\bar{\phi}), \quad W(\phi) \rightarrow W(\phi) - \mu F(\phi).$$

(9.18)

The Kähler metric defined by

$$g_{IJ} := \partial_I \partial_J K = \partial_I \partial_J \mathcal{K}$$

(9.19)

is obviously invariant under the Kähler transformations.

Because of (9.16), the Lagrangian \( \mathcal{K} \) in (9.14) should be a globally defined function on the Kähler target space \( \mathcal{M} \). This immediately implies that the Kähler two-form, \( \Omega = 2i g_{IJ} d\phi^I \wedge d\bar{\phi}^J \), associated with (9.19), is exact and hence \( \mathcal{M} \) is necessarily non-compact. We see that the \( \sigma \)-model couplings in (1,1) AdS are more restrictive than in the Minkowski case.

### 9.4 Supercurrents

The most general supercurrent multiplet in (1,1) AdS superspace is described by the conservation equation

$$\bar{\nabla}^\beta J_{\alpha \beta} = \nabla_\alpha X - \frac{1}{2} (\nabla^2 + 2\mu) \zeta_\alpha,$$

(9.20)

where \( J_{\alpha \beta} \) is the supercurrent, and \( X \) and \( \zeta_\alpha \) the trace multiplets constrained by

$$\nabla_\alpha X = 0,$$

(9.21a)

$$\nabla_{(\alpha} \zeta_{\beta)} = 0.$$

(9.21b)

The multiplet with \( \zeta_\alpha = 0 \) corresponds to the Ferrara-Zumino supercurrent which is associated with the Type I minimal AdS supergravity. The case \( X = 0 \) corresponds to
the non-minimal AdS supergravity. Similarly to four dimensions \[29\], the trace multiplets in \eqref{9.20} are defined modulo a gauge transformation of the form

\[ X \rightarrow X + \mu \Lambda , \quad \zeta_\alpha \rightarrow \zeta_\alpha + \frac{1}{4} \nabla_\alpha \Lambda , \quad \bar{\nabla}_\alpha \Lambda = 0 . \tag{9.22} \]

This can be used to set \( X = 0 \).

The general supercurrent \eqref{9.20} can be modified by an improvement transformation

\begin{align*}
J_{\alpha\beta} &\rightarrow J_{\alpha\beta} + \frac{1}{2} [\nabla(\alpha, \bar{\nabla}(\beta)] V - 2 \nabla_{\alpha\beta} U , \quad (9.23a) \\
X &\rightarrow X + \frac{1}{4} (\bar{\nabla}^2 - 4 \mu)(V - 2iU) , \quad (9.23b) \\
\zeta_\alpha &\rightarrow \zeta_\alpha - \nabla_\alpha (V + iU) , \quad (9.23c)
\end{align*}

with \( V \) and \( U \) well defined operators.

A specific feature of the \((1, 1)\) AdS geometry is that the constraint \eqref{9.21b} can always be solved as (compare with \[61\] \[62\])

\[ \zeta_\alpha = \nabla_\alpha (V + iU) , \tag{9.24} \]

for well defined operators \( V \) and \( U \). This property means that we can always set \( \zeta_\alpha = 0 \) by applying a certain improvement transformation \eqref{9.23}. Therefore, a Ferrara-Zumino multiplet exists for any supersymmetric field theory in the case of \((1, 1)\) AdS supersymmetry.

As an example, let us consider supercurrents for the supersymmetric \(\sigma\)-model \eqref{9.14}. The non-minimal supercurrent \((X = 0)\) can be shown to be

\begin{align*}
J^{(NM)}_{\alpha\beta} &= 2 \mathcal{K}_{IJ} \nabla(\alpha, \bar{\nabla}(\beta] \bar{\phi}^j , \quad (9.25a) \\
\zeta_\alpha &= -\nabla_\alpha \mathcal{K} . \tag{9.25b}
\end{align*}

The Ferrara-Zumino multiplet is

\begin{align*}
J^{(FZ)}_{\alpha\beta} &= 2 \mathcal{K}_{IJ} \nabla(\alpha, \bar{\nabla}(\beta] \bar{\phi}^j - \frac{1}{2} [\nabla(\alpha, \bar{\nabla}(\beta)] \mathcal{K} , \quad (9.26a) \\
X &= -\frac{1}{4} (\bar{\nabla}^2 - 4 \mu) \mathcal{K} . \tag{9.26b}
\end{align*}

The non-minimal supercurrent looks simpler than the Ferrara-Zumino one.
10 (2,0) AdS supersymmetry

In this section we study rigid supersymmetric field theories in (2,0) AdS superspace. Its geometry is determined by the (anti-)commutation relations (1.5a)-(1.5c). As shown in subsection 4.2, this superspace originates as a solution to the equations of motion of the Type II minimal supergravity with a cosmological term.

10.1 Linearized supergravity action

We start by deriving a (2,0) AdS extension of the Type II action in Minkowski space (7.18). To achieve this we follow the same strategy which was adopted in subsection 9.1. We start with the following functional in (2,0) AdS superspace

\[
\int d^3x d^4\theta E \left\{ -\frac{1}{16}H^{\alpha\beta}\nabla^2D_\gamma H_{\alpha\beta} - \frac{1}{4}(D_{\alpha\beta}H^{\alpha\beta})^2 + \frac{1}{16}([D_\alpha, \bar{D}_\beta]H^{\alpha\beta})^2 \\
+ \frac{1}{4}G([D_\alpha, \bar{D}_\beta]H^{\alpha\beta}) + \frac{1}{2}G^2 \right\},
\]

(10.1)

that reduces to the Type II action (7.18) in the flat superspace limit. The real linear compensator, \( G = G \), now satisfies the covariant constraint

\[
\bar{D}^2G = 0.
\]

(10.2)

As in Minkowski superspace, this constraint is solved in terms of a real unconstrained prepotential \( G \),

\[
G = iD^\alpha \bar{D}_\alpha G,
\]

(10.3)

which is defined modulo gauge shifts

\[
\delta G = \lambda + \bar{\lambda}, \quad \bar{D}_\alpha \lambda = 0.
\]

(10.4)

We further postulate linearized supergravity gauge transformations

\[
\delta H_{\alpha\beta} = D_{(\alpha}L_{\beta)} - D_{(\alpha}\bar{L}_{\beta)}, \quad D = iD^\alpha \bar{D}_\alpha G,
\]

(10.5a)

\[
\delta G = -\frac{1}{2}(\bar{D}^\alpha L_\alpha - \bar{D}_\alpha \bar{L}^\alpha),
\]

(10.5b)

\[
\delta G = \frac{1}{4}(D^\alpha D^2L_\alpha + D_\alpha D^2\bar{L}^\alpha).
\]

(10.5c)

Note that the gauge parameter \( L_\alpha \) is charged under the \( U(1)_R \):

\[
\mathcal{J}L_\alpha = 1, \quad \mathcal{J}\bar{L}_\alpha = -1.
\]

(10.6)
The functional (10.1) is not invariant under the gauge transformations (10.5a)–(10.5c), as can be seen using the following identities

\[ [D^\alpha, \bar{D}^2] = 4iD^{\alpha\beta}\bar{D}_\beta + i\rho D^\alpha J - 2i\rho\bar{D}_\alpha\mathcal{M}^{\alpha\beta}, \]  
\[ D_\alpha D_\beta D_\gamma = 0, \quad [D_\alpha, \bar{D}^2] = 0, \quad [D_{\alpha\beta}, D_\gamma \bar{D}^2] = -\frac{1}{2}\rho\varepsilon_{\gamma(\alpha}D_{\beta)}\bar{D}^2 \]  
and their complex conjugates. These identities can be easily derived by using the covariant derivative algebra (1.5a)–(1.5c). In order to get a gauge invariant action, we have to modify (10.1) by adding certain \( \rho \)-dependent terms. This procedure results in the linearized action for Type II AdS supergravity

\[ S_{II}^{(2,0)} = \int d^3 x d^4 \theta E \left\{ -\frac{1}{16} H^{\alpha\beta} D^{\gamma} \bar{D}^2 D_\gamma H_{\alpha\beta} - \frac{1}{4}(D_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{16}([D_\alpha, \bar{D}_\beta] H^{\alpha\beta})^2 \right. 
+ \frac{1}{4}G[D_\alpha, \bar{D}_\beta] H^{\alpha\beta} + \frac{1}{2}G^2 - \frac{i}{4}\rho H^{\alpha\beta} D^{\gamma} \bar{D}_\gamma H_{\alpha\beta} + \left. \frac{1}{2}\rho G^2 \right\}. \]  

(10.7a)

As compared with (10.1), the action involves two new structures. The Chern-Simons term coincides with that appearing in the nonlinear supergravity action (4.22). Because of its presence, the linearized Type II AdS action cannot be dualized to a Type I or non-minimal model.

10.2 Type III minimal action in (2,0) AdS superspace

As discussed in subsection 9.2, the Type III supergravity action (7.29) cannot be lifted to (1,1) AdS superspace in a gauge invariant way. It is quite remarkable that such an extension exists in (2,0) AdS superspace. It has the form:

\[ S_{III}^{(2,0)} = \int d^3 x d^4 \theta E \left\{ -\frac{1}{16} H^{\alpha\beta} D^{\gamma} \bar{D}^2 D_\gamma H_{\alpha\beta} - \frac{1}{8}(D_{\alpha\beta} H^{\alpha\beta})^2 + \frac{1}{32}([D_\alpha, \bar{D}_\beta] H^{\alpha\beta})^2 \right. 
+ \frac{1}{4}V D_{\alpha\beta} H^{\alpha\beta} + \frac{1}{8}V^2 - \frac{i}{8}\rho H^{\alpha\beta} D^{\gamma} \bar{D}_\gamma H_{\alpha\beta} - \left. \frac{1}{4}\rho V V \right\}. \]  

It can be shown that this action is invariant under the supergravity gauge transformations

\[ \delta H_{\alpha\beta} = D_{(\alpha}L_{\beta)} - D_{(\alpha} \bar{L}_{\beta)} , \quad \delta V = \frac{1}{2}(D^\alpha L_\alpha + D_\alpha \bar{L}^\alpha) , \quad \delta \bar{V} = \frac{i}{4}(D^\alpha \bar{D}^2 L_\alpha - D_\alpha D^2 \bar{L}^\alpha) , \quad \bar{V} = iD^\alpha \bar{D}_\alpha V . \]  

(10.10a)  

(10.10b)  

(10.10c)
10.3 Matter couplings in (2,0) AdS superspace

The isometries of (2,0) AdS superspace are generated by Killing vector fields, \( \tau = \tau^a D_a + \tau^\alpha D_\alpha + \bar{\tau}_\alpha \bar{D}_\alpha \), obeying the master equation

\[
\left[ \tau + i t \mathcal{J} + \frac{1}{2} t^{bc} \mathcal{M}_{bc}, \mathcal{D}_A \right] = 0 .
\]

(10.11)

This is equivalent to the following equations on the components:

\[
\rho \tau_\alpha = \bar{D}_\alpha t = \frac{i}{6} \rho \bar{D}^2 \tau_{\alpha\beta} = \frac{i}{3} \bar{D}^2 t_{\alpha\beta} ,
\]

(10.12a)

\[
D_\alpha \tau_\beta = D_{(\alpha} \tau_{\beta\gamma)} = D_{(\alpha \gamma} t_{\beta)} = 0 ,
\]

(10.12b)

\[
D_\gamma \tau_\gamma = - \bar{D}^\gamma \bar{\tau}_\gamma = 2i t ,
\]

(10.12c)

\[
D_{(\alpha} \tau_{\beta)} = - D_{(\alpha} \bar{\tau}_{\beta)} = \frac{1}{2} t_{\alpha\beta} + \frac{1}{4} \rho \tau_{\alpha\beta} .
\]

(10.12d)

The (2,0) AdS Killing vector fields prove to generate the supergroup OSp(2|2; \mathbb{R}) \times Sp(2, \mathbb{R}). Rigid supersymmetric field theories in (2,0) AdS superspace should be invariant under the isometry transformations.

Matter couplings in (2,0) AdS superspace significantly differ from those in the (1,1) case. In particular, only \( R \)-invariant nonlinear \( \sigma \)-models can be consistently defined in (2,0) AdS superspace. As an example, consider the \( \sigma \)-model action

\[
S = \int \! d^3 x d^4 \theta E K(\phi^I, \bar{\phi}^\dot{I}) + \left\{ \int \! d^3 x d^2 \theta E W(\phi^I) + \text{c.c.} \right\} .
\]

(10.13)

The dynamical variables \( \phi^I \) are covariantly chiral superfields, \( \mathcal{D}_\alpha \phi^I = 0 \), with definite \( U(1)_R \) charges \( r_I \)

\[
\mathcal{J} \phi^I = - r_I \phi^I ,
\]

(10.14)

(no sum).

In order for the action to be \( R \)-invariant, the Kähler potential \( K(\phi, \bar{\phi}) \) and the superpotential \( W(\phi) \) should obey the equations:

\[
\sum_I r_I \phi^I K_I = \sum_I r_I \bar{\phi}^{\dot{I}} K_{\dot{I}} \equiv \frac{1}{2} \mathcal{V}(\phi, \bar{\phi}) ,
\]

(10.15a)

\[
\sum_I r_I \phi^I W_I = 2 W .
\]

(10.15b)

The action is invariant under the isometry transformations

\[
\delta \phi^I = (\tau + i t \mathcal{J}) \phi^I .
\]

(10.16)
The equations of motion

\[ \bar{D}^2 K_I = 4W_I \quad (10.17) \]

imply that on-shell

\[ \sum_I r_I \phi^I \bar{D}^2 K_I = 8W. \quad (10.18) \]

An important class of \( \sigma \)-models in (2,0) AdS superspace is specified by the conditions \( r_i = 0 \) and \( W(\phi) = 0 \). In this case no restrictions on the Kähler target space occur. Unlike the \( \sigma \)-models in (1,1) AdS superspace, compact target spaces are allowed.

Another interesting theory is a system of self-interacting Abelian vector multiplets described by real linear field strengths \( F_i \), with \( i = 1, \ldots, n \). A general gauge invariant action is

\[ S = \int d^3x d^4\theta E \left\{ L(F^i) + \frac{1}{2} m_{ij} F^i F^j + \xi_i F^i \right\}, \quad (10.19) \]

with \( m_{ij} = m_{ji} = (m_{ij})^* \) and \( \xi \) being Chern-Simons and Fayet-Iliopoulos coupling constants respectively. Here \( F^i \) is the gauge prepotential for \( F^i \) and \( L \) is an arbitrary real function of \( F^i \). The isometry transformations of the scalar superfields \( F^i \) and \( \bar{F}^i \) are

\[ \delta F^i = \tau F^i, \quad \delta \bar{F}^i = \tau \bar{F}^i. \quad (10.20) \]

Fayet-Iliopoulos terms are not allowed in (1,1) AdS superspace.

We conclude by presenting a (2,0) AdS extension of the action \( (8.36) \) for a scalar multiplet with an Abelian central charge. Such an extension cannot be defined in the case of (1,1) AdS superspace in which a frozen vector multiplet with constant field strength simply does not exist (since the conditions \( (\bar{\nabla}^2 - 4\mu)G = 0 \) and \( \nabla_A G = 0 \) are inconsistent). On the other hand, in (2,0) superspace such a frozen vector multiplet has been explicitly constructed in subsection \( 6.2 \) It is described by a real gauge prepotential \( V_0 \) such that

\[ i \, D^a \bar{D}_a V_0 = -2. \quad (10.21) \]

In order to formulate our desired model, it is useful to introduce gauge covariant derivatives

\[ \Delta_A = (\Delta_a, \Delta_\alpha, \bar{\Delta}^\alpha) := D_A + i \Gamma_A Z, \quad (10.22) \]
where $\Gamma_A$ is the gauge connection and $Z$ the central charge operator, $[Z, \Delta_A] = 0$. The (anti-)commutation relations for the $\Delta$-derivatives look like those in (13a)–(13c) except for the following relation:

$$\{\Delta_\alpha, \bar{\Delta}_\beta\} = -2i\Delta_{\alpha\beta} - i\epsilon_{\alpha\beta}(\rho \mathcal{J} + 2Z) + i\rho \mathcal{M}_{\alpha\beta} .$$

(10.23)

The model is described in terms of a gauge-covariant chiral superfield $\phi$, $\bar{\Delta}_\alpha \phi = 0$, and its conjugate $\bar{\phi}$, which are eigenvectors of the central charge, $Z\phi = m\phi$ and $Z\bar{\phi} = -m\bar{\phi}$, with $m$ a real mass parameter. In practice, it is useful to work in the chiral representation defined by

$$\Delta_\alpha = e^{-2V_0^Z}D_\alpha e^{2V_0^Z} , \quad \bar{\Delta}_\alpha = \bar{D}_\alpha , \quad \phi = \Phi , \quad \bar{\phi} = e^{-2V_0^Z}\bar{\Phi} = \bar{\Phi}e^{2mV_0^Z} ,$$

(10.24)

with $\Phi$ an ordinary chiral superfield. In the chiral representation, the action has the form

$$S = \int d^3x d^4\theta E \bar{\phi}\phi = \int d^3x d^4\theta E \Phi e^{2mV_0}\Phi .$$

(10.25)

### 10.4 Supercurrents

The most general supercurrent multiplet in (2,0) AdS superspace is characterized by the conservation equation

$$\bar{D}^\beta J_{\alpha\beta} = D_\alpha X - i\rho \bar{\Lambda}_\alpha + D_\alpha (Y + iZ) ,$$

(10.26)

where the trace multiplets obey the constraints

$$\bar{D}_{(\alpha\bar{\Lambda}_\beta)} = 0 , \quad 4X = \bar{D}_\alpha \bar{\Lambda}^\alpha \quad \implies \quad \bar{D}_\alpha X = 0$$

(10.27a)

$$\bar{\Phi} - \bar{\Psi} = \bar{Z} - Z = 0 , \quad \bar{D}^2\bar{\Phi} = \bar{D}^2\bar{Z} = 0 .$$

(10.27b)

The trace multiplets are defined modulo gauge transformations

$$\delta \bar{\Lambda}_\alpha = -\frac{i}{\rho} \bar{D}_\alpha (S + iT) , \quad \delta \bar{\Psi} = S , \quad \delta \bar{Z} = T ,$$

(10.28)

with $S$ and $T$ real linear superfields,

$$\bar{S} - S = \bar{T} - T = 0 , \quad \bar{D}^2\bar{S} = \bar{D}^2\bar{T} = 0 .$$

(10.29)

This gauge freedom can be used to gauge away $\bar{\Psi}$ and $Z$. 

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The supercurrent can be modified by an improvement transformation of the form

\begin{align}
J_{\alpha\beta} &\rightarrow J_{\alpha\beta} + D_{(\alpha} \bar{\Upsilon}_{\beta)} - \tilde{D}_{(\alpha} \Upsilon_{\beta)} , \\
\bar{\Lambda}_{\alpha} &\rightarrow \bar{\Lambda}_{\alpha} + 2\bar{\Upsilon}_{\alpha} \quad \Rightarrow \quad X \rightarrow X + \frac{1}{2} \tilde{D}_{\alpha} \bar{\Upsilon}_{\alpha} , \\
\Upsilon &\rightarrow \Upsilon - \frac{1}{2} (D^\alpha \bar{\Upsilon}_\alpha + \tilde{D}_\alpha \Upsilon^\alpha) , \\
Z &\rightarrow Z + i (\tilde{D}_\alpha \Upsilon^\alpha - D^\alpha \bar{\Upsilon}_\alpha) ,
\end{align}

where the parameter \( \Upsilon_{\alpha} \) is constrained by

\[ D_{(\alpha} \Upsilon_{\beta)} = 0 . \] (10.31)

This freedom can be used to improve \( \bar{\Lambda}_{\alpha} \) to zero, thus resulting with the supercurrent multiplet

\[ \tilde{D}^\beta J_{\alpha\beta} = D_{\alpha} (\Upsilon + i Z) \] (10.32)

which is associated with the two linearized supergravity actions constructed in subsection \[10.31\]. We can still perform an improvement transformation generated by \( \bar{\Lambda}_{\alpha} = \tilde{D}_{\alpha} (S + i T) \), with \( S \) and \( T \) real linear superfields. This transformation results in a non-zero \( \bar{\Lambda}_{\alpha} \) which can be set to zero by applying a certain transformation \[10.28\]. We thus end up with the following improvement transformation

\begin{align}
J_{\alpha\beta} &\rightarrow J_{\alpha\beta} + [D_{(\alpha}, \tilde{D}_{\beta)}] S + 2 D_{\alpha\beta} T , \\
\Upsilon &\rightarrow \Upsilon - i D^\alpha \tilde{D}_{\alpha} T + 2 \rho T , \\
Z &\rightarrow Z - 2i D^\alpha \tilde{D}_{\alpha} S - 2 \rho S .
\end{align}

In accordance with the analysis given in \[50\], we expect that the trace multiplet \( \Upsilon \) can always be improved to zero. Thus any theory in (2,0) superspace should have a well defined \( \mathcal{R} \)-multiplet described by the conservation equation

\[ \tilde{D}^\beta J_{\alpha\beta} = i \tilde{D}_{\alpha} Z , \] (10.34)

where the trace multiplet is constrained by

\[ \tilde{Z} = Z , \quad \tilde{D}^2 Z = 0 . \] (10.35)

The supercurrent can be modified by an improvement transformation of the form \[10.33\] with \( \Upsilon = 0 \) and \( T = 0 \).
As an example, it can be shown that the nonlinear $\sigma$-model (10.13) is characterized by the supercurrent

$$J_{\alpha\beta} = 2K_{IJ}D_{(\alpha}\phi^{J}D_{\beta)}\bar{\phi}^{J} - \frac{1}{2}[D_{(\alpha}, D_{\beta)}]V ,$$  \hspace{1cm} (10.36a)

$$Z = -iD^{\alpha}\bar{D}_{\alpha}(K - V) ,$$  \hspace{1cm} (10.36b)

where $V$ is defined by (10.15a). An interesting special case of $\sigma$-models is $r_l = 0$ and $W(\phi) = 0$. Then the Kähler potential is arbitrary. The action is invariant under Kähler transformations

$$K \to K + F + \bar{F} ,$$  \hspace{1cm} (10.37)

with $F(\phi^I)$ an arbitrary holomorphic function. The supercurrent becomes

$$J_{\alpha\beta} = 2K_{IJ}D_{(\alpha}\phi^{J}D_{\beta)}\bar{\phi}^{J} ,$$  \hspace{1cm} (10.38a)

$$Z = -iD^{\alpha}\bar{D}_{\alpha}K .$$  \hspace{1cm} (10.38b)

The trace multiplet is clearly invariant under the Kähler transformations, and therefore it is a well defined operator.

As another example, we consider the system of self-interacting Abelian vector multiplets described by the action (10.19). The supercurrent for this model is

$$J_{\alpha\beta} = -2L_{ij}D_{(\alpha}F^{i}D_{\beta)}F^{j} ,$$  \hspace{1cm} (10.39a)

$$Z = iD^{\gamma}\bar{D}_{\gamma}(F^{i}L_{i} - L) - \xi_{i}F^{i} .$$  \hspace{1cm} (10.39b)

This theory is superconformal if $\xi_{i} = 0$ and $F^{i}L_{i} = L$, in which case $Z = 0$.

We conclude with the supercurrent for the scalar multiplet model (10.25). It is an instructive exercise to show that the supercurrent is given by

$$J_{\alpha\beta} = [\Delta_{(\alpha}, \bar{\Delta}_{\beta)}](\bar{\phi}\phi) - 4i\bar{\phi}\bar{\Delta}_{\alpha\beta}\phi = [D_{(\alpha}, D_{\beta)}](\bar{\phi}\phi) - 4i\bar{\phi}\bar{\Delta}_{\alpha\beta}\phi ,$$  \hspace{1cm} (10.40a)

$$Z = ((5 - 4r)\rho + 8m)\bar{\phi}\phi .$$  \hspace{1cm} (10.40b)

Here $(-r)$ denotes the $U(1)_{R}$ charge of $\phi$.

11 Conclusion

In this paper we have elaborated on different aspects of three-dimensional $\mathcal{N} = 2$ supergravity in superspace. One of the goals was to understand how the (1,1) and (2,0)
AdS supergravity theories \cite{10} can be described using the different off-shell versions of $\mathcal{N} = 2$ supergravity which were briefly introduced in \cite{18}. The other goal was to understand the general structure of 3D $\mathcal{N} = 2$ supercurrents from the supergravity point of view.

It was argued by Dumitrescu and Seiberg \cite{50} that their $\mathcal{S}$-multiplet, eq. (8.18), is the most general supercurrent in three dimensions. However, off-shell supergravity allows the existence of more general supercurrent described by eqs. (8.1) and (8.2). The same multiplet appears to emerge using a 3D analogue of the superfield Noether procedure \cite{41}. Making use of the observations given in \cite{50}, we expect that the trace multiplet $\mathcal{Y}$ in (8.1) can always be improved to zero in the case of Poincaré supersymmetry. This reduces then the supercurrent (8.1) to the $\mathcal{S}$-multiplet. However, we have shown that the $\mathcal{S}$-multiplet does not have a natural extension to the (1,1) and (2,0) AdS superspaces. In this sense the 3D picture is very similar to the 4D $\mathcal{N} = 1$ case studied in \cite{29, 30}.

Recently there has been much interest in 3D new massive gravity \cite{67} and its supersymmetric extension \cite{68, 69, 70}. The results reported here should offer insight into the structure of such theories.

Recently the problem of computing the partition function of gauge theories on non-trivial three- and four-dimensional constant-curvature backgrounds (mostly spheres) has arisen as a means to compute observables such as expectation values of Wilson loop and superconformal indices by using localization techniques (see \cite{71} and \cite{72} and references therein). The construction of supersymmetric theories on nontrivial backgrounds is itself an interesting problem and, as also pointed out in \cite{33}, off-shell supergravity is a perfect setting to address many related aspects. In a sense this is a natural top-down approach: once general supergravity-matter couplings in superspace are understood, applications to particular backgrounds arise just as an example. On the other hand, supercurrents may serve as a powerful censor to indicate which supersymmetric theories can be lifted from flat to certain curved backgrounds. We believe that the results of this paper can be extended to nontrivial 3D supersymmetric space-times distinct from AdS. Such applications in the case of $\mathcal{N} = 2$ supersymmetry, and extensions to the cases $\mathcal{N} = 3, 4$ using the supergravity techniques of \cite{18}, will be studied in the future.

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A 4D N = 1 supercurrents in Minkowski space

In this appendix we review the structure of 4D $\mathcal{N} = 1$ supercurrents in Minkowski space. The most general supercurrent multiplet is described by the conservation equation given in [46, 49]

$$\bar{D}^\dot{\alpha} J_{\dot{\alpha} \dot{\alpha}} = D_\alpha X + \chi_\alpha + i \eta_\alpha ,$$

(A.1)

$$\bar{D}_\dot{\alpha} \chi_\alpha = \bar{D}_\dot{\alpha} \eta_\alpha = \bar{D}_\dot{\alpha} X = 0 , \quad D^\alpha \chi_\alpha - \bar{D}_\dot{\alpha} \bar{\chi}^\dot{\alpha} = D^\alpha \eta_\alpha - \bar{D}_\dot{\alpha} \bar{\eta}^\dot{\alpha} = 0 .$$

Here $J_{\dot{\alpha} \dot{\alpha}} = J_{\dot{\alpha} \dot{\alpha}}$ denotes the supercurrent, and the chiral superfields $X, \chi_\alpha$ and $\eta_\alpha$ constitute the so-called multiplet of anomalies. The above multiplet coincides with that derived by Magro, Sachs and Wolf [41] using their superfield Noether procedure (see also [40]) provided $\chi_\alpha$ and $\eta_\alpha$ have the form:

$$\chi_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha F , \quad \bar{F} = F ,$$

(A.2a)

$$\eta_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha H , \quad \bar{H} = H .$$

(A.2b)

However, the prepotentials $F$ and $H$ are not always well defined operators, and in this sense the conservation law (A.1) is more general.\footnote{If the prepotentials $F$ and $H$ are well defined operators, then the supercurrent (A.2b) can be improved to a Ferrara-Zumino multiplet [49] (see below).}

Some of the superfields $X, \chi_\alpha$ and $\eta_\alpha$ are absent for concrete models, and all of them can be chosen to vanish in the case of superconformal theories. The three terms on the right of (A.1) emphasize the fact that there exist exactly three different linearized actions for minimal $(12 + 12)$ supergravity, according to the classification given in [59], which are related by superfield duality transformations. The case $\chi_\alpha = \eta_\alpha = 0$ describes the Ferrara-Zumino multiplet [34] which corresponds to the old minimal formulation for $\mathcal{N} = 1$ supergravity [24]. The choice $X = \eta_\alpha = 0$ corresponds to the new minimal supergravity [26]; this supercurrent is called the $\mathcal{R}$-multiplet [15, 45]. Finally, the third choice $X = \chi_\alpha = 0$ corresponds to the minimal supergravity formulation proposed in [60]; unlike the old minimal and the new minimal theories, this formulation is known at the linearized level only.

If only one of the superfields $\chi_\alpha, \eta_\alpha$ and $X$ in (A.1) is zero, the supercurrent multiplet describes $16+16$ components. Of the three such supercurrents studied in [46], the so-called $S$-multiplet, $S_{\dot{\alpha} \dot{\alpha}}$, introduced earlier by Komargodski and Seiberg [12] is of fundamental significance (see below). It is described by the conservation equation

$$\bar{D}^\dot{\alpha} S_{\dot{\alpha} \dot{\alpha}} = D_\alpha X + \chi_\alpha , \quad \bar{D}_\dot{\alpha} X = \bar{D}_\dot{\alpha} \chi_\alpha = 0 , \quad D^\alpha \chi_\alpha - \bar{D}_\dot{\alpha} \bar{\chi}^\dot{\alpha} = 0 .$$

(A.3)
The supercurrent multiplet (A.1) can be modified by improvement transformations of the form [49]:

\[ J_{\alpha\dot{\beta}} \rightarrow J_{\alpha\dot{\beta}} + [D_\alpha, \bar{D}_{\dot{\alpha}}]V - 2\partial_{\alpha\dot{\alpha}}U \]  
(A.4a)

\[ X \rightarrow X + \frac{1}{2}\bar{D}^2(V - iU) \]  
(A.4b)

\[ \chi_\alpha \rightarrow \chi_\alpha + \frac{3}{2}\bar{D}^2D_\alpha V \]  
(A.4c)

\[ \eta_\alpha \rightarrow \eta_\alpha + \frac{1}{2}\bar{D}^2D_\alpha U \]  
(A.4d)

In terms of the spinor superfield \( \Upsilon_\alpha = D_\alpha(V + iU) \), this improvement transformation can be rewritten as follows [50]:

\[ J_{\alpha\dot{\beta}} \rightarrow J_{\alpha\dot{\beta}} + D_\alpha \bar{\Upsilon}_{\dot{\alpha}} - \bar{D}_{\dot{\alpha}}\Upsilon_\alpha \]  
(A.5a)

\[ X \rightarrow X + \frac{1}{2}\bar{D}_{\dot{\alpha}}\bar{\Upsilon}_{\dot{\alpha}} \]  
(A.5b)

\[ \chi_\alpha \rightarrow \chi_\alpha + \frac{3}{4}\left(\bar{D}^2\Upsilon_\alpha - 2\bar{D}_{\dot{\alpha}}D_\alpha \bar{\Upsilon}_{\dot{\alpha}} - D_\alpha\bar{D}_{\dot{\alpha}}\bar{\Upsilon}_{\dot{\alpha}}\right) \]  
(A.5c)

\[ \eta_\alpha \rightarrow \eta_\alpha - \frac{1}{4}\left(\bar{D}^2\Upsilon_\alpha + 2\bar{D}_{\dot{\alpha}}D_\alpha \bar{\Upsilon}_{\dot{\alpha}} + D_\alpha\bar{D}_{\dot{\alpha}}\bar{\Upsilon}_{\dot{\alpha}}\right) \]  
(A.5d)

This improvement transformation is also defined for a general spinor operator \( \Upsilon_\alpha \) obeying only the constraint

\[ D_{(\alpha}\Upsilon_{\beta)} = 0 \]  
(A.6)

In the case of the \( S \)-multiplet, \( \eta_\alpha = 0 \) and the parameter \( \Upsilon_\alpha \) in (A.5) should be further constrained [50] by

\[ \bar{D}^2\Upsilon_\alpha + 2\bar{D}_{\dot{\alpha}}D_\alpha \bar{\Upsilon}_{\dot{\alpha}} + D_\alpha\bar{D}_{\dot{\alpha}}\bar{\Upsilon}_{\dot{\alpha}} = 0 \]  
(A.7)

It was argued in [45] that the \( S \)-multiplet exists in all rigid supersymmetric theories in Minkowski space. A remarkable result of Dumitrescu and Seiberg [50] is that the trace multiplet \( \eta_\alpha \) in (A.1) can always be improved to zero. Although their proof is based on some nontrivial assumptions, no counterexample is known. This result has in fact a natural justification from the supergravity point of view, as first discussed in [46]. The point is that the trace multiplet \( \eta_\alpha \) is associated with the minimal supergravity

\[ ^{16} \text{In some exotic supersymmetric theories, the chiral scalar } X \text{ is not a well defined operator [50]. In such case, the term } D_\alpha X \text{ in (A.3) should be replaced by a spinor operator } \Upsilon_\alpha \text{ constrained by } D_{(\alpha}\Upsilon_{\beta)} = 0 \text{ and } \bar{D}^2\Upsilon_\alpha = 0 [50]. \]
formulation proposed in [60], which is known only at the linearized level and does not have a nonlinear extension. It is therefore to be expected that matter couplings to this supergravity formulation should be impossible.

It is instructive to consider the supercurrent associated with the non-minimal formulation for $N = 1$ supergravity [27, 28]. It is described by the conservation equation (see, e.g., [46])

$$\bar{D}^\alpha J_{\alpha \dot{\alpha}} = -\frac{n + 1}{4n + 1} D_\alpha \bar{D}_\beta \tilde{\zeta}^\beta - \frac{1}{4} \bar{D}^2 \zeta_\alpha , \quad D_{(\alpha \zeta_\beta)} = 0 , \quad (A.8)$$

where $n$ is a real constant, $n \neq -1/3, 0$, parametrizing the different versions of non-minimal supergravity [28]. It should be pointed out that this conservation equation is based on the supergravity gauge transformation (7.40) of the complex linear compensator $\Sigma$. In non-minimal supergravity, there is a natural freedom to redefine $\Sigma$ as

$$\Sigma \rightarrow \Sigma + \kappa \bar{D}_\alpha D_\alpha H^{a\dot{\alpha}} , \quad (A.9)$$

with $H_{a\dot{\alpha}}$ the gravitational superfield, and $\kappa$ a constant parameter which can be chosen (for simplicity) real. The redefined compensator transforms as

$$\delta \Sigma = -\frac{n + 1}{4} \bar{D}^2 D^a L_\alpha + (\kappa - \frac{1}{4}) \bar{D}_\alpha D^a \bar{L}^\alpha - \kappa \bar{D}_\alpha D^a \bar{D}^\dot{\alpha} L_\alpha . \quad (A.10)$$

Adopting such a transformation law leads to the conservation equation

$$\bar{D}^\dot{\alpha} J_{a\dot{\alpha}} = -\frac{n + 1}{4n + 1} D_\alpha \bar{D}_\beta \tilde{\zeta}^\beta + (\kappa - \frac{1}{4}) \bar{D}^2 \zeta_\alpha - \kappa \bar{D}_\beta D_\alpha \tilde{\zeta}^\beta , \quad (A.11)$$

where the trace multiplet $\zeta_\alpha$ is again constrained by $D_{(\alpha \zeta_\beta)} = 0$. This conservation equation can be written in the general form (A.1) if we identify

$$X = \frac{1}{4} \left(2\kappa - \frac{n + 1}{3n + 1}\right) \bar{D}_\alpha \tilde{\zeta}^\alpha , \quad (A.12a)$$

$$\chi_\alpha = \frac{1}{4} \left(3\kappa - \frac{1}{2}\right) \left(\bar{D}^2 \zeta_\alpha - 2\bar{D}_\dot{\alpha} D_\alpha \tilde{\zeta}^\dot{\alpha} - D_\dot{\alpha} \bar{D}_\alpha \tilde{\zeta}^\dot{\alpha}\right) , \quad (A.12b)$$

$$\eta_\alpha = -\frac{i}{4} \left(\kappa - \frac{1}{2}\right) \left(\bar{D}^2 \zeta_\alpha + 2\bar{D}_\dot{\alpha} D_\alpha \tilde{\zeta}^\dot{\alpha} + D_\dot{\alpha} \bar{D}_\alpha \tilde{\zeta}^\dot{\alpha}\right) . \quad (A.12c)$$

There are two lessons we can learn from this example. First, the improvement transformation (A.5) can be used to get rid of either $\chi_\alpha$ or $\eta_\alpha$. Second, by an appropriate choice of the deformation parameter $\kappa$ we can set to zero one of the three trace multiplets $X$, $\chi_\alpha$ and $\eta_\alpha$. The choice $\kappa = 1/6$ was made in Superspace and recently used in [50]. Of course, if $\zeta_\alpha = D_\alpha \zeta$, for a well defined operator $\zeta$, then both $\chi_\alpha$ and $\eta_\alpha$ can be improved to zero.
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