MINIMAL PERIOD SOLUTIONS IN ASYMPTOTICALLY LINEAR HAMILTONIAN SYSTEM WITH SYMMETRIES

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Abstract. In this paper, applying the Maslov-type index theory for periodic orbits and brake orbits, we study the minimal period problems in asymptotically linear Hamiltonian systems with different symmetries. For the asymptotically linear semipositive even Hamiltonian systems, we prove that for any given $T > 0$, there exists a central symmetric periodic solution with minimal period $T$. Moreover, if the Hamiltonian systems are also reversible, we prove the existence of a central symmetric brake orbit with minimal period being either $T$ or $T/3$. Also we give some other lower bound estimations for brake orbits case.

1. Introduction. Let $H \in \mathcal{C}^{2}(\mathbb{R}^{2n}, \mathbb{R})$ and let us consider the following problem

$$\dot{z} = JH'(z), \quad z \in \mathbb{R}^{2n},$$

where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

and $I$ is the $n \times n$ identity matrix. Denote by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ the standard norm and inner product in $\mathbb{R}^{2n}$, respectively. Set

$$\mathcal{L}_a(\mathbb{R}^{2n}) = \{X \in \text{GL}(2n, \mathbb{R}) : X^T = X\}.$$ 

Let $H$ satisfy the following asymptotically linear behavior:

(AH1): $H(z) = o(|z|^2)$ at $z = 0$ and $H'(z) = Bz + o(|z|)$ as $|z| \to +\infty$, where $B \in \mathcal{L}_a(\mathbb{R}^{2n})$ is semi-positive, i.e., $\langle Bz, z \rangle \geq 0$ for all $z \in \mathbb{R}^{2n}$.

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Let $z$ be a solution of (1.1). $(z, T)$ is called a periodic solution of (1.1) if $z(0) = z(T)$. In 1980, H. Amann and E. Zehnder proved the existence of nontrivial periodic solution of asymptotic linear system with constant $B$ nondegenerate at zero and infinity (cf. [1], [2]). Their results was extended to the situation of nonconstant $B$ (cf. [5], [17] and references therein). In 1990, Y. Long removed the assumption of non-degeneracy at zero (cf. [24], [26]). The nondegeneracy at infinity was removed by Chang-Liu-Liu (cf. [7]) and Fei-Qiu (cf. [14]), independently.

For system (1.1), P. Rabinowitz raised the question that whether there is a non-constant solution with minimal period $T$ for any given $T > 0$ in 1978. In [8], F. Clarke and I. Ekeland proved the existence of periodic solutions with minimal period $T$ for any given $T > 0$ under $\sqrt{2}$-pinching condition for convex system. In [11], I. Ekeland and H. Hofer proved the existence of solutions with minimal period $T$ for any given $T > 0$ under the strictly convex assumption. We refer to [3], [23] for various results of convex Hamiltonian system. In [25], Y. Long considered the system without convex assumption (cf. [26]). In [13], G. Fei and Q. Qiu showed that the lower bound of the minimal period is $\frac{T}{2\pi}$ for semi-positive definite system. In [31], the second author of this paper proved that the minimal period is $T$ if $H$ is even for semi-positive and super-quadratic Hamiltonian system.

For the minimal period problem of autonomous Hamiltonian system under the assumption of asymptotically linear, D. Dong and Y. Long proved the existence of solution $z$ with minimal period $T$ under the assumption that the Hamiltonian system is positive definite and nondegenerate with $i(z) \leq n + 1$ (cf. [9]).

Let

$$N = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$  

We define the following conditions:

- $(AH2)'$: $H(z) = H(Nz)$, for $z \in \mathbb{R}^{2n}$. 
- $(B1)$: $NBN = B$.

Suppose $H$ satisfies $(AH2)'$. A periodic orbit $(z, T)$ is called a brake orbit if $(z, T)$ is the solution of the following

$$\begin{cases} \dot{z} = JH'(z), \\ z(-t) = Nz(t), \quad t \in \mathbb{R}. \end{cases}$$

Given an asymptotically linear Lagrangian system, $(z, T)$ is called a brake orbit if $(z, T)$ is the solution of (1.2) and $B$ satisfies $(B1)$. C. Liu gave the multiplicity results of the solutions of certain asymptotically linear Hamiltonian systems with a Lagrangian boundary condition in [18]. The second author of this paper dealt with the case of symmetric periodic brake orbit solutions in [30].

For the minimal period problem of (1.2), C. Liu proved the existence of a non-constant solution of (1.2) with minimal period no less than $T/2$ for positive definite Hamiltonian system under super quadratic condition in [20]. In [32], the second author of this paper extended the result to the semi-positive case.

We define the following conditions, where $(AH2)$ is the symmetric requirement, $(AH3)$ and $(AH4)$ are the semi-positive restrictions for the periodic Hamiltonian systems and the brake Hamiltonian systems, respectively. $(B2)$, $(B2)'$ and $(B2)''$ can be considered as more precise requirement for the asymptotically linear Hamiltonian systems under symmetries. The definition of Maslov-type indices $i_\omega$ and $i_{\omega_0}$ are given in Appendix A.

- $(AH2)$: $H(z) = H(-z)$, for $z \in \mathbb{R}^{2n}$. 
- $(AH3)$: $H(z) = H(Nz)$, for $z \in \mathbb{R}^{2n}$. 
- $(AH4)$: $H(z) = H(Nz)$, for $z \in \mathbb{R}^{2n}$. 

The second author of this paper proved that the minimal period is $T$ if $H$ is even for semi-positive and super-quadratic Hamiltonian system.
(AH3): $H''(z) \geq 0$, for $z \in \mathbb{R}^{2n}$.
(B2): $i_{-1}(B, \frac{T}{2}) \geq 1$.
(AH4): $H''_{22}$ is positive definite, where $H''_{22}(z)$ denotes the Hessian matrix w.r.t $q$ with $z = (p, q)$, $p, q \in \mathbb{R}^n$.
(B2)': $i_{L_0}(B, \frac{T}{2}) \geq 1$.
(B2)'': $i_{L_0}(B, \frac{T}{4}) \geq 1$.

1.1. Main results.

Theorem 1.1. Suppose $H$ satisfies (AH1) – (AH3) and (B2). Then given any $T > 0$, equation (1.1) has a non-constant solution with minimal period $T$.

Theorem 1.2. Suppose $H$ satisfies (AH1), (AH2)', (AH3), (B1) and (B2)'.

Then given any $T > 0$,

1. Equation (1.2) has a non-constant brake solution with minimal period no less than $\frac{T}{2^n+n+2}$;
2. In addition, if also (AH4) holds, equation (1.2) has a non-constant brake orbit with minimal period no less than $\frac{T}{2}$;
3. Moreover, if also (AH2) and (B2)'' hold, equation (1.2) has a non-constant symmetric brake solution with minimal period no less than $\frac{T}{3}$.

1.2. Sketch of the proof. For asymptotically linear Hamiltonian system, in order to apply Saddle-point reduction Theorem (Theorem B.3) to varies boundary problems, we first truncate the Hamiltonian function suitably to satisfy (CH). Combining Theorem B.3, Theorem B.4 and Theorem B.11, we obtain the existence of a non-constant symmetric periodic orbit $(z, T)$ (resp. brake orbit, symmetric brake orbit) for the truncated Hamiltonian function, which is the solution of (1.1) (resp. (1.2)) if we carefully choose truncated function under the assumption (AH1).

Moreover, with the aid of Theorem B.11 and Theorem B.6 (resp. Theorem B.7, Theorem B.8), the Maslov-type indices of $(z, T)$ satisfy a certain condition, which is given by (3.8) (resp. (3.15), (3.23)).

Next, we give an estimation of the minimal period of $(z, T)$. Suppose that $z = z^p_r$, where $z^r_r$ is also a solution of (1.1) (resp. (1.2)) and $z^r_r$ has minimal period $\tau = T/p$ for some positive integer $p$. Combining Bott-type iteration formula (i.e., Theorem A.11, Theorem A.7) with Lemma A.10 (resp. Lemma A.8), we obtain the relation of the Maslov-type indices between $z$ and $z^r_r$, which gives an upper bound of $p$ for semi-positive Hamiltonian system. In addition, the upper bound of $p$ is related to the minimal period $\tau = T/p$ of $z^r_r$ under the assumption that $z = z^p_r$. Thus $z$ is a non-constant solution of (1.1) (resp. (1.2)) with minimal period $T/p$.

Furthermore, by applying Lemma A.13 to (3.13) (resp. (3.20), (3.21), (3.24)), we remove some exceptions of the estimation of $p$ and thus we obtain more precise upper bound of $p$. Finally, we obtain the expected minimal period under varies boundary conditions and we complete the proof.

1.3. Main difficulty. There are two difficulties we need to overcome: the first one is the relation between Maslov-type index and the Morse index of symmetric periodic solution of (1.1) (resp. brake solution, symmetric brake solution of (1.2)) obtained by the variational methods. We overcome it by truncating the Hamiltonian function to apply the saddle point reduction and homological link theorem, and thus we obtain the existence of symmetric $T$-periodic solution of (1.1) (resp. brake orbit
solution, symmetric brake orbit solution of (1.2)) with expected Maslov-type index information.

The second one is the precise estimation of the iteration number $p$ in the last step of the proof of Theorem 1.1 (resp. Theorem 1.2). More precisely, we need to exclude the case $k = 1$ (resp. $r = 1$ for Theorem 1.2). Indeed, combining Lemma A.10 (resp. Lemma A.8) and Theorem A.11 (resp. Theorem A.7), we obtain a rough estimation of $p$ under the assumption of (AH2)-(AH3) (resp. (AH2)$'$(AH3)). Based on the pioneer works of Long on $i_\omega$ index for symplectic paths (cf. [26, P. 204]) and Lemma A.13, we obtain that the inequality (3.13) (resp. (3.20), (3.21) and (3.24)) is strict. We finally obtain the upper bound of the iteration number and hence we obtain the lower bound of the minimal period.

**Organization:** In Section 2, we establish the basic setting for the application of Theorem B.3 for Hamiltonian systems under various boundary conditions. In Section 3 we prove the main results of this paper based on previous preparations. In Appendix A, we give a brief introduction of Maslov-type index theory and associated iteration theory for symplectic paths starting at identity matrix under various boundary conditions. Appendix B.1 contains 4 parts: the first part is a simple introduction of Saddle Point Reduction Theorem, the second part is the application of Saddle Point Reduction Theorem to Hamiltonian systems, the third part establishes the relations between Maslov-type indices and the Morse indices of a truncated Morse function under saddle point reduction approach and the last part aims to introduce the minimax theorem in critical point theorem.

Throughout this paper, let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{U}$ denote the set of natural integers, integers, rational numbers, real numbers, complex numbers and the unit circle in $\mathbb{C}$, respectively.

2. **The background setting.** In order to apply Theorem B.3 for symmetric orbits, brake orbits and symmetric brake orbits, we give an introductions of following notations and the basic properties. The details can be found in [6, Chapter 4], [26, Chapter 4, Chapter 6] and references therein.

Let $S_T = \mathbb{R}/(T\mathbb{Z})$, $L = L^2(S_T, \mathbb{R}^{2n})$ and $W = W^{1,2}(S_T, \mathbb{R}^{2n})$. Define

$$\hat{L} = \left\{ x \in L^2(S_T, \mathbb{R}^{2n}) : x(t + \frac{T}{2}) = -x(t) \right\},$$
$$\bar{L} = \left\{ x \in L^2(S_T, \mathbb{R}^{2n}) : x(-t) = Nx(t) \right\},$$
$$\hat{W} = \left\{ x \in W^{1,2}(S_T, \mathbb{R}^{2n}) : x(t + \frac{T}{2}) = -x(t) \right\},$$
$$\bar{W} = \left\{ x \in W^{1,2}(S_T, \mathbb{R}^{2n}) : x(-t) = Nx(t) \right\},$$

where $\hat{L}$ and $\bar{L}$ (resp. $\hat{W}$ and $\bar{W}$) are equipped with $L^2$-norm (resp. $W^{1,2}$-norm). For Hilbert spaces $L$ (resp. $\hat{L}$, $\bar{L}$) and $W$ (resp. $\hat{W}$, $\bar{W}$), let

$$\|x\|^2_L = \int_0^T |x(t)|^2 dt,$$
$$\|x\|^2_{\hat{L}} = \int_0^T |x(t)|^2 dt + \int_0^T |\dot{x}(t)|^2 dt,$$

and let $\langle \cdot , \cdot \rangle_L$ and $\langle \cdot , \cdot \rangle_{\hat{L}}$ be the inner product in $L$ and $\hat{L}$, respectively. In addition, $W$, $\hat{W}$ and $\bar{W}$ are dense subspaces of $L$, $\hat{L}$ and $\bar{L}$, respectively. Furthermore, by
the standard bootstrap argument (cf. [26, Chapter 4]), a weak solution of (1.1) is actually a classical solution.

Let $A = J \frac{d}{dt} : L \to L$ with $\text{dom}(A) = W$ and let $\hat{A} = A|_{\hat{L}}$, $\overline{A} = A|_{\overline{L}}$, $\hat{\overline{A}} = A|_{\hat{L} \cap \overline{L}}$ with domain respectively given by $\text{dom}(\hat{A}) = \hat{W}$, $\text{dom}(\overline{A}) = \overline{W}$, $\text{dom}(\hat{\overline{A}}) = \hat{W} \cap \overline{W}$. By [26, P.93-P.94], $A$ is self-adjoint on $W$ with closed image and the spectrum of $A$ is point spectrum, which each nonzero element of $\sigma(A_0)$ has the multiplicity $2n$. Thus $\hat{A}$ (resp. $\overline{A}$ and $\hat{\overline{A}}$) is self-adjoint on $\hat{W}$ (resp. $\overline{W}$ and $\hat{W} \cap \overline{W}$) with each nonzero spectrum of multiplicity $2n$ (resp. $n$ and $n$). Let

$$X_k = \left\{ \cos \left( \frac{2k\pi t}{T} \right) a - \sin \left( \frac{2k\pi t}{T} \right) \right\} \text{ for } a \in \mathbb{R}^{2n}$$

(2.1)

and

$$\overline{X}_k = \left\{ \cos \left( \frac{2k\pi t}{T} \right) a - \sin \left( \frac{2k\pi t}{T} \right) \right\} \text{ for } a \in \{0\} \times \mathbb{R}^n.$$  

(2.2)

Then $X_k$ (resp. $X_{2k+1}$, $\overline{X}_k$, $\overline{X}_{2k+1}$) is the eigenspace of $A$ (resp. $\hat{A}$, $\overline{A}$, $\hat{\overline{A}}$) corresponding to the eigenvalue $\frac{2k\pi}{T}$ (resp. $(\frac{(2k+1)\pi}{T})$, $(\frac{2k\pi}{T})$, $(\frac{(2k+1)\pi}{T})$).

Let $P_0 : L \to X_0 = \ker A = \mathbb{R}^{2n}$ be the projection map and let $\tilde{P}_0 = P_0|_{\hat{L}}$ (resp. $\overline{P}_0 = P_0|_{\overline{L}}$, $\tilde{\overline{P}}_0 = P_0|_{\hat{L} \cap \overline{L}}$) with $\text{Im}(\tilde{P}_0) = \{0\}$ (resp. $\text{Im}(\overline{P}_0) = \overline{X}_0 = \{0\}$ $\times \mathbb{R}^n$, $\text{Im}(\tilde{\overline{P}}_0) = \{0\}$). By taking $A_0 = A + P_0$, $\hat{A}_0 = A_0|_{\hat{W}}$, $\overline{A}_0 = A_0|_{\overline{W}}$, $\hat{\overline{A}}_0 = \hat{A}_0|_{\overline{W}}$, we obtain that $\hat{A}_0$ (resp. $\overline{A}_0$, $\hat{\overline{A}}_0$) is the self-adjoint operator on $\hat{W}$ (resp. $\overline{W}$, $\hat{W}$) and $\hat{A}_0$ (resp. $\overline{A}_0$, $\hat{\overline{A}}_0$) has closed image and compact resolvent. In addition, the spectrum of $\hat{A}_0$ (resp. $\overline{A}_0$, $\hat{\overline{A}}_0$) is given by $\sigma(A_0) = \{1\} \cup \frac{2\pi}{T} \mathbb{Z} \setminus \{0\}$ (resp. $\sigma(\overline{A}_0) = \{1\} \cup \frac{2\pi}{T} \mathbb{Z} + \frac{\pi}{2}$, $\sigma(\hat{\overline{A}}_0) = \{1\} \cup \frac{2\pi}{T} \mathbb{Z} + \frac{\pi}{2}$).

3. Proof of the main result.

3.1. Proof of Theorem 1.1. The proof is divided into 4 steps.

Step 1: In order to apply the saddle point reduction frame work, we truncate $H$ into $H_k$ such that it satisfies (CH) (cf. Section B.2) under the assumption (AH1). By (AH1), there exists a constant $\Lambda_0 > 0$ such that $0 \leq B \leq \Lambda_0 I$ and

$$H(z) = \frac{1}{2}Bz \cdot z + o(|z|), \quad \text{as } |z| \to \infty.$$  

Choose a constant $k_0 \geq 5$ such that

$$0 \leq H(z) \leq 2\Lambda_0 |z|^2, \quad \text{for } |z| \geq k_0.$$

For $k_0 \leq k \in \mathbb{Z}$, let $\chi_k \in C^\infty(\mathbb{R}, \mathbb{R})$ be a cut-off function such that

$$\chi_k(r) = \begin{cases} 1, & 1 \leq r \leq k, \\ 0, & r \geq k + b_k, \end{cases}$$

where $b_k \geq 1$ is chosen to satisfy $0 \leq -\chi'(r) \leq \frac{2}{T}$ for $k < r < k + b_k$. Set

$$H_k(z) = \chi_k(|z|)H(z) + \frac{1}{2}(1 - \chi_k(|z|))Bz \cdot z, \quad \text{for } z \in \mathbb{R}^{2n}.$$  

Then $H_k \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ and the followings hold:
1: There exists a constant $k_0 \geq k + b_k$ such that if $|z| \geq k_0$,
\[ 0 \leq H_k(z) \leq 2\Lambda_0|z|^2. \] (3.1)

2: Let $M_0 = \max_{|z| \leq k_0} H(z)$. Then
\[ -M_0 \leq H_k(z) \leq 2\Lambda_0|z|^2 + M_0, \quad \text{for } z \in \mathbb{R}^{2n}. \] (3.2)

3: $H_k(t, z)$ satisfies (AH1) and
\[ |H'_k(z) - Bz| \leq |H'(z) - Bz| + \frac{1}{|z|} |H(z) - \frac{1}{2} Bz \cdot z|, \quad \text{for } z \neq 0. \] (3.3)

4: There is a constant $c(H, k)$ such that
\[ \|H'_k(x)\|_{C(\mathbb{R}^{2n})} \leq c(H, k). \] (3.4)

We let
\[ f_k(z) = \frac{1}{2} (Az, z) - \int_0^T H_k(z) dt, \]
and we let $(\hat{a}_k, \hat{u}_k)$ be defined as in Theorem B.3 associated for $(A, f, L, Z, W) = (\hat{A}, \hat{f}, \hat{L}, \hat{Z}, \hat{W})$ with $c(H) = c(H, k)$. By Theorem B.3, for $H = H_k$, the critical point $z_k$ of $f_k$ satisfies $\hat{z} = JH'_k(z_k)$.

Step 2: We show that $\hat{a}_k$ satisfies (PS)-condition (cf. Definition B.9).

We first prove that for any critical point $z_k$ of $\hat{f}_k$, there is a constant $k_1 \geq k_0$ such that
\[ \|z_k\|_{W^{1,2}} \leq k_1, \] (3.5)
which means the critical points of $\hat{f}_k$ is uniformly bounded with $\hat{a}_k = \hat{f}_k \circ \hat{u}_k$. In addition, $u_k$ and $u_k^{-1}$ is bounded with $u_k^{-1} : \text{Im}(u_k) \to Z_k$. Combining (3.5), Theorem B.3 with the fact that $\text{dom}(u_k)$ is finite dimensional, we obtain that $\hat{a}_k$ satisfies (PS)-condition. Indeed, we first show the $C^0$-norm of the critical point $z_k$ of $\hat{f}_k$ is bounded from above, then we apply (3.3) to obtain that the $W^{1,2}$-norm is bounded from above. Let $z_k$ be a critical point of $\hat{f}_k$. If $||(\hat{A} - B)z_k||_L = 0$, then by Hölder inequality, we get
\[ |z_k(t)|^2 \leq T \int_0^t |\dot{z}_k(s)|^2 ds = T \int_0^T |J\dot{z}_k(s)|^2 ds \leq T \int_0^T |Bz_k|^2, \] (3.6)
which implies $\|z_k\|_{C^0} \leq M_2$ for some constant $M_2 > 0$. Furthermore, by (3.3), for $z_k \neq 0$,
\[ |\dot{z}_k(t)| \leq \left|H'_k(z_k) - Bz_k\right| + |Bz_k| \leq |H'(z_k) - Bz_k| + \frac{1}{|z_k|} |H(z_k) - \frac{1}{2} Bz_k \cdot z_k| + |Bz_k|, \] (3.7)
which yields that $\|z_k\|_{W^{1,2}} \leq M_3$ for some constant $M_3$. Otherwise, by the properties listed in Section 2 of $A$, there is a constant $\alpha = \alpha(z_k) > 0$ such that
\[ ||(\hat{A} - B)z_k||_L \geq 2\alpha \|z_k\|_L. \]
By (3.3), there is $M_1 \geq k_0$ such that for $||z||_L \geq M_1$, $||Bz - H'_k(z)||_L \leq \alpha ||z||_L$. We get that
\[ \|\hat{f}_k(z_k)\|_L \geq ||(\hat{A} - B)z_k||_L - ||Bz_k - H'_k(z_k)||_L \geq \alpha \|z_k\|_L, \]
which means that for any critical point $z_k$ of $\hat{f}_k$, $\|z_k\|_L \leq M_4$, for some constant $M_4$ (independent of $k$). By (3.6), we obtain
\[ |z_k(t)|^2 \leq T \int_0^t |\dot{z}_k(s)|^2 ds = T \int_0^t |J\dot{z}_k(s)|^2 ds \leq T\|\hat{\Lambda} - B\|_L + T\|Bz_k - H_k'(z_k)\|_L, \]
which implies that $\|z_k\|_{C^0} \leq M_5$ for some constant $M_5$. Combining (3.7) and (AH1), there is a constant $k_1 \geq k_0$ such that $\|z_k\|_{W^{1,2}} \leq k_1$.

**Step 3:** We aim to prove that there exists a non-constant solution $z_k$ of (1.1) such that
\[ i_{-1}(z_k) \leq 1 \leq i_{-1}(z_k) + \nu_{-1}(z_k). \] (3.8)
Indeed, if $H$ satisfies (AH1)-(AH3), then (3.4) holds. By applying Theorem B.3 and Theorem B.4 to the case of $(A,f,L,Z,W) = (\hat{A},\hat{f}_k,\hat{L},\hat{Z},\hat{W})$, we obtain
\[ \hat{a}_k(z) \leq \frac{1}{2} \langle \hat{\Lambda} (\hat{P}^{-\hat{w}_k}(z) + z), \hat{P}^{-\hat{w}_k}(z) + z \rangle_L \]
\[ - \hat{g}_k \left( \hat{P}^{-\hat{w}_k}(z) + z \right), \text{ for } z \in \hat{Z} \]
and
\[ \hat{a}_k(z) \geq \frac{1}{2} \langle \hat{\Lambda} (\hat{P}^{+\hat{w}_k}(z) + z), \hat{P}^{+\hat{w}_k}(z) + z \rangle_L \]
\[ - o(||z||^2), \text{ as } z \to 0 \text{ in } \hat{Z}, \]
where $\hat{w}_k(z) = \hat{u}_k(z) - z$. Then there exists a constant $\rho > 0$ small enough such that
\[ \hat{a}_k(z) \geq \frac{1}{2} \langle \hat{\Lambda} (\hat{P}^{+\hat{w}_k}(z) + z), \hat{P}^{+\hat{w}_k}(z) + z \rangle_L - o(||z||^2) \]
\[ \geq \frac{1}{4} ||z||^2 = \frac{1}{4} \rho^2, \text{ for } z \in \partial B_{\rho}(0,\hat{Z}^+), \]
where $Z^\pm$ are the eigenspaces of $A$ corresponding to the eigenvalues belonging to $\mathbb{R}^\pm$ and $\hat{Z}^\pm = \hat{Z} \cap Z^\pm$, respectively. In addition, there exists $y \in \hat{Z}^+$ with $||y|| = 1$ such that for some constant $\lambda_1 > 0$,
\[ (\hat{\Lambda} - B)y = -\lambda_1 y \text{ in } \hat{L}. \] (3.9)
Such $y$ does exist since we require $i_{-1}(B, \frac{T}{2}) \geq 1$ by (B2). More precisely, according to Theorem B.6,
\[ m^- (\hat{a}_k) = \frac{1}{2} \dim(\hat{Z}) + i_{-1} \left( B, \frac{T}{2} \right), \]
where $k_0 = [c(H,k)]$ and $\hat{a}_k = a_{-1}$ in Theorem B.6. Since for $y' \in \hat{Z}^-$, we have
\[ \langle \hat{a}_k y', y' \rangle = \langle (\hat{\Lambda} - B)\hat{u}_k(y'), \hat{u}_k'(y') \rangle < 0. \]
Combining Theorem B.3, Theorem B.6 with
\[ \dim(\hat{Z}^-) = \frac{1}{2} \dim \hat{Z}, \ (\hat{\Lambda} - B)_{2-} \leq 0, \]
the existence of such $y$ is obvious. Let
\[ Q_y = \left\{ ry + z_\cdot \in \hat{Z}: z_\cdot \in \hat{Z}^-, ||z_\cdot|| \leq R_0, \ 0 \leq r \leq R_0 \right\}, \]
where $R_0 > 0$ will be determined later. Thus $\hat{a}_k(z) \leq 0$ for $z = z_\tau \in \hat{Z}^+$. Furthermore, for $z = ry + z_\tau \in \partial Q_y$ with sufficiently large $r$, combining (AH1), (3.9) and Theorem B.4 implies

$$
\hat{a}_k(z) \leq \frac{1}{2} \langle \hat{A} \left( \hat{P}^- \hat{w}_k(z) + ry + z_\tau \right), \hat{P}^- \hat{w}_k(z) + ry + z_\tau \rangle_L - \frac{1}{2} \int_0^T B \left( \hat{P}^- \hat{w}_k(z) + ry + z_\tau \right) \cdot \left( \hat{P}^- \hat{w}_k(z) + ry + z_\tau \right) dt + o(\|\hat{P}^- \hat{w}_k(z) + z\|^2)
$$

$$
\leq -\frac{1}{2} \|\hat{P}^- \hat{w}_k(z) + z_\tau\|^2 L - \frac{1}{2} \lambda_1 r^2 \|y\|^2_L - r \int_0^T By \cdot (\hat{P}^- \hat{w}_k(z) + z_\tau) dt + o(\|\hat{P}^- \hat{w}_k(z) + z\|^2).
$$

Since $\hat{P}^- \hat{w}_k(z) + z_\tau \in \hat{L}^- \oplus \hat{Z}^+$ is orthogonal to $\hat{A}y \in \hat{Z}^+$, we have

$$
By \cdot (\hat{P}^- \hat{w}_k(z) + z_\tau) = - (\hat{A} - B)y \cdot (\hat{P}^- \hat{w}_k(z) + z_\tau) + \hat{A}y \cdot (\hat{P}^- \hat{w}_k(z) + z_\tau) = 0.
$$

Thus if $z = ry + z_\tau \in \partial Q_y$ with $\|z_\tau\| = R_0$ or $r = R_0$ for sufficiently large $R_0$, we obtain

$$
\hat{a}_k(z) \leq 0, \text{ for } z = ry + z_\tau \in \partial Q_y. \quad (3.10)
$$

Therefore, combining (3.9), (3.10) with Theorem B.11, there exists a critical point $z_k \in \hat{Z}$ with its Morse index $m^-(\hat{a}_k)$ at $z_k$ and the nullity $n^0(\hat{a}_k)$ at $z$ satisfying

$$
m^-(\hat{a}_k) \leq \dim Q_y + 1 \leq m^-(\hat{a}_k) + n^0(\hat{a}_k).
$$

Then by Theorem B.6, $i_{-1}(z_k) \leq 1 \leq i_{-1}(z_k) + \nu_{-1}(z_k)$. Furthermore, since $\hat{a}_k(z_k) = \hat{f}(\hat{u}(z_k)) > 0$, we have that $z_k$ is a non-constant solution. In addition, by choosing $k \geq k_1$ sufficiently large in step 1, such $z_k$ is $T$-periodic symmetric solution of (1.1).

**Step 4:** Suppose $(z, T)$ is a symmetric solution of (1.1) obtained from step 3 with minimal period $\tau$. Then $T/\tau = p \in \mathbb{Z}_+$. There exists a symmetric solution $z_\tau$ of (1.1) with minimal period $\tau$ such that $z(t) = z_\tau^p(t)$. Next we prove that $p = 1$, which implies $T = T$.

Denote by $\gamma_{z_\tau}$ and $\gamma_z$ the associated symplectic paths of $z_\tau$ and $z$, respectively. Then $\gamma_z = \gamma_{z_\tau}$ and its Maslov-type indices satisfy

$$
i_{-1}(\gamma_z) \leq 1 \leq i_{-1}(\gamma_z) + \nu_{-1}(\gamma_z). \quad (3.11)
$$

By (AH2), $\nu_{-1}(z_\tau) \geq 1$. In addition, since (AH3) holds, we have

$$
i_{-1}(\gamma_{z_\tau}) \geq 0 \quad \text{and} \quad i(\gamma_{z_\tau}) + \nu(\gamma_{z_\tau}) \geq n.
$$

Thus

$$
i(\gamma_{z_\tau}^2) + \nu(\gamma_{z_\tau}^2) = i(\gamma_{z_\tau}) + \nu(\gamma_{z_\tau}) + i_{-1}(\gamma_{z_\tau}) + \nu_{-1}(\gamma_{z_\tau}) \geq n + 1, \quad (3.12)
$$

where the equality holds by Theorem A.11. Moreover, according to Theorem A.11,

$$
i_{-1}(\gamma_{z_\tau}^{2k}) = i_{-1}(\gamma_{z_\tau}^{2k}) = \sum_{\omega^k = -1} i_{\omega}(\gamma_{z_\tau}^{2k}), \quad \text{for} \quad k \geq 1.
$$
Then
\[ i_{-1}(\gamma_{z_r}^{2k}) \geq \sum_{n=1}^{\infty} (i(\gamma_{z_r}^2) + \nu(\gamma_{z_r}^2) - n) \]
\[ = k(i(\gamma_{z_r}^2) + \nu(\gamma_{z_r}^2) - n) \geq k, \quad (3.13) \]
where the first inequality holds by Lemma A.13 and the second inequality holds by (3.12). Thus if \( p = 2k + 1 \geq 1 \), combining Lemma A.10 and (3.11),
\[ 1 \geq i_{-1}(\gamma_z) = i_{-1}(\gamma_{z_r}^{2k+1}) = i_{-1}(\gamma_{z_r}^k) \geq i_{-1}(\gamma_{z_r}^{2k}) \geq k, \]
which implies \( k \leq 1 \). It is sufficient to show \( k = 0 \) to complete the proof. Indeed, if \( k = 1 \), by (3.13), \( i_{-1}(\gamma_{z_r}^2) = 1 \). According to Lemma A.13, \( \gamma_{z_r}^2 \) can be connected to
\[ I_2^p \circ N(1, -1)^{\sigma_0} \circ -I_2^{\sigma} \circ N(1, -1)^{\sigma} \circ H \]
with \( \sigma(H) \subset U \setminus \{ \pm 1 \} \) and the eigenvalues belonging to \( \sigma(H) \cap U^+ \) (resp. \( \sigma(H) \cap U^- \)) are all Krein-negative (resp. Krein-positive) definite. By Theorem A.11 and Lemma A.13,
\[ i_{-1}(\gamma_{z_r}^2) = i(\gamma_{z_r}^2) + \nu(\gamma_{z_r}^2) - n = 1 \quad (3.14) \]
contradicts the fact
\[ i_{-1}(\gamma_{z_r}^2) = i\sqrt{-1}(\gamma_{z_r}) + i_{-1}\sqrt{-1}(\gamma_{z_r}) = 2i\sqrt{-1}(\gamma_{z_r}) \in 2\mathbb{Z}. \]
Thus we complete the proof of Theorem 1.1.
\[ \square \]

3.2. Proof of Theorem 1.2. We give the proof of conclusions 1 and 2 of Theorem 1.2 in 3 steps:

Step 1: Under the assumption of (AH1), (AH2)' and (B1), we truncate \( H \) into \( H_k \) in a similar way as in Section 3.1 such that \( H_k \) satisfy (3.1)-(3.4). Furthermore, since \( B \) satisfies (B1), we obtain that \( H_k \) satisfies (CH) and (AH2)'. Let \( f_k \) be given in Section 3.1 and let \( \overline{f}_k = f_k|_{\mathbb{P}} \). Suppose \( (\overline{\pi}_k, \overline{\pi}_k) \) is given in Theorem B.3 associated for the case of \( (A, f, L, Z, W) = (\overline{A}, \overline{f}, \overline{Z}, \overline{L}, \overline{W}) \) for \( c(H) = c(H, k) \).

Similarly, the following two facts hold:
- For any critical point \( z_k \) of \( \overline{f}_k \), there is a constant \( k_1 \geq k_0 \) such that \( \|z_k\|_{W^1,2} \leq k_1 \).
- \( \overline{\pi}_k \) satisfies (PS)-condition on \( \overline{Z} \).

Indeed, the above two facts can be obtained by a similar argument in Section 3.1. We omit the proof for simplicity.

Step 2: We aim to show that there exists a non-constant solution \( z_k \) of the solution of (1.2) such that its Maslov-type indices satisfy
\[ i_{L_0}(z_k) \leq 1 \leq i_{L_0}(z_k) + \nu_{L_0}(z_k). \quad (3.15) \]
Indeed, under the assumption (AH1), applying Theorem B.3 and Theorem B.4 to the case of \( (A, f, L, Z, W) = (\overline{A}, \overline{f}, \overline{Z}, \overline{L}, \overline{W}) \), we have
\[ \overline{u}_k(z) \leq \frac{1}{2} (A \overline{P}^{-1} \overline{w}_k(z) + z), P^{-1} \overline{w}_k(z) + z \rangle_L \]
\[ - \overline{g} (P^{-1} \overline{w}_k(z) + z), \quad \text{for } z \in \overline{Z} \]
and
\[ \overline{u}_k(z) \geq \frac{1}{2} (A \overline{P}^{+1} \overline{w}_k(z) + z), \overline{P}^{+1} \overline{w}_k(z) + z \rangle_L \]
\[ - o(\|z\|^2), \quad \text{as } z \to 0 \text{ in } \overline{Z}, \]
where \( \overline{w}_k(z) = z - \overline{n}_k(z) \) is given in Theorem B.3. Then there exists a constant \( \rho > 0 \) small enough such that

\[
\overline{\sigma}_k(z) \geq \frac{1}{2} \langle \mathcal{A} \mathcal{P}^+ w_k(z) + z, \mathcal{P}^+ \overline{w}_k(z) + z \rangle_L - o(\|z\|^2)
\geq \frac{1}{4} \|z\|^2 \geq \frac{1}{4} \rho^2, \quad \text{for } z \in \partial B_\rho(0, Z^+) , \tag{3.16}
\]

where \( Z^\pm = Z \cap Z^\pm \). By Theorem B.7,

\[
m^-(\overline{\sigma}_k) = \frac{\dim(Z) - n}{2} + i L_0 \left( B, \frac{T}{2} \right),
\]

where \( k_0 = [c(H,k)] \) and \( \overline{n}_k = \overline{n} \) is given in Theorem B.6. Since for \( y' \in Z^- \), we have

\[
\langle \overline{n}_k y', y \rangle = \langle (A - B) \overline{n}_k(y), \overline{n}_k(y) \rangle < 0.
\]

Combining Theorem B.3, (B2)' with

\[
\dim(Z^-) = \frac{\dim(Z) - n}{2}, \quad (A - B)|Z^- < 0,
\]

there exists \( y \in Z^+ \) with \( \|y\| = 1 \) and \( \lambda_1 > 0 \) such that \( (A - B)y = -\lambda_1 y \) in \( Z \). Let

\[
Q_y = \{ ry + z_0 + z_- \in Z: z_0 + z_- \in Z^0 \oplus Z^- , \|z_0 + z_-\| \leq R_0, \ 0 \leq r \leq R_0 \},
\]

where \( R_0 > 0 \) will be determined later. Thus

\[
\overline{\sigma}_k(z) \leq 0, \quad \text{for } z = z_0 + z_- \in X_0 \oplus Z^- = (X_0 \cap Z) \oplus (Z^- \cap Z).
\]

Furthermore, for \( z = ry + z_0 + z_- \in \partial Q_y \), since \( \mathcal{P}^- w(z) + z_0 + z_- \) is orthogonal to \( \overline{A} y \in Z^+ \), taking sufficiently large \( r \), we obtain

\[
\overline{\sigma}_k(z) \leq \frac{1}{2} \langle \mathcal{A} \left( \mathcal{P}^- \overline{w}_k(z) + ry + z_- \right), \mathcal{P}^- \overline{w}_k(z) + ry + z_- \rangle_L \\
- \frac{1}{2} \int_0^T B \left( \mathcal{P}^- \overline{w}_k(z) + ry + z_- \right) \cdot \left( \mathcal{P}^- \overline{w}_k(z) + ry + z_- \right) dt \\
+ o \left( \mathcal{P}^- \overline{w}_k(z) + ry + z_- \right) \\
\leq - \frac{1}{2} \| \mathcal{P}^- \overline{w}_k(z) + z_- \|^2 - \frac{1}{2} \lambda_1 r^2 \|y\|^2 \\
- r \int_0^T By \cdot \left( \mathcal{P}^- \overline{w}_k(z) + ry + z_- \right) dt + o \left( \mathcal{P}^- \overline{w}_k(z) + ry + z_- \right).
\]

Since \( \mathcal{P}^- w(z) + z_0 + z_- \in \mathcal{L}^- \oplus \overline{X}_0 \oplus Z^- \) is orthogonal to \( \overline{A} y \in Z^+ \),

\[
By \cdot (\mathcal{P}^- w(z) + z_0 + z_-) = - (A - B)y \cdot (\mathcal{P}^- w(z) + z_0 + z_-) \\
+ \overline{A} y \cdot (\mathcal{P}^- w(z) + z_0 + z_-) = 0.
\]

Thus if \( z = ry + z_0 + z_- \in \partial Q_y \) with \( \|z_0 + z_-\| = R_0 \) or \( r = R_0 \), we have

\[
\overline{\sigma}_k(z) \leq 0 \quad \text{for } z = ry + z_0 + z_- \in \partial Q_y. \tag{3.17}
\]

Therefore, applying (3.16) and (3.17) to Theorem B.11, there exists a critical point \( z_k \in Z \) of \( \overline{\sigma}_k \) with its Morse index \( m^-(\overline{\sigma}_k, z) \) and the nullity \( m^0(\overline{\sigma}_k, z) \) satisfying

\[
m^-(\overline{\sigma}_k, z) \leq \dim Q_y \leq m^-(\overline{\sigma}_k, z) + m^0(\overline{\sigma}_k, z).
\]

Thus by Theorem B.7, \(i_{L_0}(z) \leq 1 \leq i_{L_0}(z) + \nu_{L_0}(z)\). Furthermore, since \(\overline{f}_k(x) = \overline{f}_k(\alpha_k(z)) > 0\), \(\alpha_k\) is non-constant solution. By taking sufficiently large \(k_1\), \(\alpha_k\) is actually a brake solution of (1.2).

**Step 3:** Suppose \((z, T)\) is brake solution of (1.2) obtained from step 2 and \(z(t) = z^2(t)\) (up to a time shift by Remark A.3) for some \(z_\tau\) with minimal period \(\tau = T/p\). Next we give an estimation of \(p\). More precisely, we show that \(p \leq 2n + 2\) under the assumption of 1 of Theorem 1.2. If, in addition, (AH4) holds, we prove that \(p \leq 2\). Thus we complete the proof of 1 and 2 of Theorem 1.2.

Denote by \(\gamma_{z_\tau}\) and \(\gamma_z\) the symplectic paths associated with \(z_\tau\) and \(z\), respectively. Then \(\gamma_z = \gamma_{z_\tau}\). Besides, by step 2, \(i_{L_0}(\gamma_z) \leq 1 \leq i_{L_0}(\gamma_{z_\tau}) + \nu_{L_0}(\gamma_{z_\tau})\). Based on (AH1) and (AH3), we have \(i_{L_0}(\gamma_{z_\tau}) + \nu_{L_0}(\gamma_{z_\tau}) \geq 0\). Furthermore, according to (AH2), we claim \(\nu_{L_1}(\gamma_{z_\tau}) + i_{L_1}(\gamma_{z_\tau}) \geq 1\). Indeed, let \(L_1 = \mathbb{R}^n \times \{0\} = \hat{X} \oplus \hat{Y}\), where

\[
\hat{X} = \{z \in L_1 = \mathbb{R}^n \times \{0\} : Bz = 0, \quad H''(z_\tau)z = 0, \quad \text{for } t \in \mathbb{R}\}.
\]

Let \(a_\epsilon\) be the associated action function of Theorem B.3 with \(g(u) = \langle \epsilon Bu, u \rangle_L\). Then for \(\epsilon > 0\) sufficiently small and \(\xi \in \hat{Y}\),

\[
a_\epsilon''(z)\xi = (\overline{X}_\xi, \xi) - \epsilon dg'(u(z))(\xi + dw(z)\xi) = - \epsilon dg'(u(z))(\xi + dw(z)\xi) < 0.
\]

Thus for sufficiently small \(\epsilon > 0\), combining Theorem B.8 and

\[
\hat{Z} = \bigoplus_{|k| \leq \frac{1}{2}} \left\{ \cos \left(\frac{2k\pi t}{T}\right) a + \sin \left(\frac{2k\pi t}{T}\right) Ja : a \in \mathbb{R}^n \times \{0\} \right\},
\]

we have \(\hat{d} = n = \dim \hat{Z}\) and

\[
\hat{d} + i_{L_1} \left(\epsilon B, \frac{T}{2}\right) \geq \dim(\hat{Y}) = n - \dim \hat{X},
\]

which yields

\[
i_{L_1} \left(\epsilon B, \frac{T}{2}\right) \geq - \dim \hat{X}.
\]

Combining \(B \geq \epsilon B\) with Lemma A.10,

\[
i_{L_1} \left(B, \frac{T}{2}\right) \geq - \dim \hat{X}.
\]

Furthermore, since we are considering autonomous Hamiltonian system,

\[
\nu_{L_1} \left(B, \frac{T}{2}\right) \geq \dim \hat{X} + 1. \quad (3.18)
\]

Thus

\[
i_{L_1} \left(B, \frac{T}{2}\right) + \nu_{L_1} \left(B, \frac{T}{2}\right) \geq 1.
\]

Replacing \(B\) by \(H''\) in (3.18), we have \(i_{L_1}(\gamma_{z_\tau}) + \nu_{L_1}(\gamma_{z_\tau}) \geq 1\). Thus according to Theorem A.11 (cf. [26, Theorem 9.2.1]),

\[
i(\gamma_{z_\tau}^2) + \nu(\gamma_{z_\tau}^2) = i_{L_0}(\gamma_{z_\tau}) + \nu_{L_0}(\gamma_{z_\tau}) + i_{L_1}(\gamma_{z_\tau}) + \nu_{L_1}(\gamma_{z_\tau}) + n \geq n + 1. \quad (3.19)
\]
Furthermore, for any $k \geq 1$ and $\omega_k = e^{\frac{2\sqrt{-1}}{k} \pi}$,
\[
i_{L_0}(\gamma_{zr}^{2k}) = i_{L_0}(\gamma_{zr}) + i \sqrt{-1}(\gamma_{zr}) + \sum_{j=0}^{k-1} i_{\omega_k^j}(\gamma_{zr}^2)
\geq i_{L_0}(\gamma_{zr}) + \sum_{j=0}^{k-1} (i(\gamma_{zr}^2) + \nu(\gamma_{zr}^2) - n)
= (k-1)(i(\gamma_{zr}^2) + \nu(\gamma_{zr}^2) - n) - n
\geq k - 1 - n,
\] (3.20)
where the first equality holds by Theorem A.7, the first inequality holds by Lemma A.13 and the second inequality holds by (3.19). By Theorem A.7, Lemma A.13 and (3.19) again,
\[
i_{L_0}(\gamma_{zr}^{2k+1}) = i_{L_0}(\gamma_{zr}) + \sum_{j=0}^{k-1} i_{\omega_k^j}(\gamma_{zr}^2)
\geq i_{L_0}(\gamma_{zr}) + \sum_{j=0}^{k-1} (i(\gamma_{zr}^2) + \nu(\gamma_{zr}^2) - n)
= k(i(\gamma_{zr}^2) + \nu(\gamma_{zr}^2) - n) - n
\geq k - n.
\]

Thus applying $i_{L_0}(\gamma_{zr}^p) = i_{L_0}(\gamma_{zr}) \leq 1$ to (3.20) and (3.21) for $p = 2k$ and $p = 2k+1$, respectively, we obtain
\[
\begin{cases}
p \leq 2(n + 2), & p \text{ even}, \\
p \leq 2(n + 1) + 1 = 2n + 3, & p \text{ odd},
\end{cases}
\]
where the equalities hold if and only if the followings hold (cf. Lemma A.13).
\[
i_{L_0}(\gamma_{zr}) = -n \quad \text{and} \quad i_{\omega_k^{2j}}(\gamma_{zr}^2) = i(\gamma_{zr}^2) + \nu(\gamma_{zr}^2) - n = 1.
\] (3.21)

Hence, $i_{\omega_k^{2j}}(\gamma_{zr}^2) = 1$ contradicts the fact that
\[
i_{\omega_k^{2j}}(\gamma_{zr}^2) = i_{\omega_k^{2j}}(\gamma_{zr}) + i_{\omega_k^{2j} + \sqrt{-1}\pi}(\gamma_{zr}) \in 2\mathbb{Z}.
\]

Altogether,
\[
\begin{cases}
p \leq 2(n + 1), & p \text{ even}, \\
p \leq 2n + 1, & p \text{ odd}.
\end{cases}
\] (3.22)

Thus equation (1.2) has a non-constant brake solution with minimal period no less than $T/(2n + 2)$. Thus we complete the proof of conclusion 1 of Theorem 1.2. \qed

• If (AH4) holds, we have $i_{L_0}(\gamma_{zr}) \geq 0$. Therefore, the inequalities of (3.22) can be improved to
\[
\begin{cases}
p \leq 1, & p \text{ odd}, \\
p \leq 2, & p \text{ even}.
\end{cases}
\]

Thus equation (1.2) has a non-constant brake solution with minimal period no less than $T/2$, the proof of conclusion 2 of Theorem 1.2 is complete. \qed

In the following, we prove conclusion 3 of Theorem 1.2. The proof is divided into 2 steps.

**Step 1:** Under the assumption of Theorem 1.2, we truncate $H$ in a similar way as in Section 3.1 to make $H_k$ satisfy (3.1)-(3.4). Furthermore, since $B \in \mathcal{L}_s(\mathbb{R}^{2n})$
satisfies (B1), we obtain that \( H_k \) satisfies (CH), (AH2) and (AH2)'. Suppose that \( f_k \) is the same as in Section 3.1. Let 
\[
(\mathcal{F}_k)_{-1}(z) = f_k|_{\mathcal{Z}^{-1}},
\]
and let \((\pi_k)_{-1}, (\pi_k)_{-1}\) be given in Theorem B.3 for 
\[
(A, f, L, Z, W) = (\hat{A}, \hat{\mathcal{F}}_k, \hat{Z}, \hat{L} \cap \mathcal{T}, \hat{W} \cap \mathcal{W}).
\]
In the following, we apply the saddle-point reduction argument to equation (1.2) with \( c(H) = c(H, k) \). With the aid of arguments of Section 3.1 and the proof of 1, 2 of Theorem 1.2, we have

- For any critical point \( z_k \) of \((\pi_k)_{-1}\), there is a constant \( k_1 \geq k_0 \) such that 
\[
\|z_k\|_{\mathcal{W}^{1,2}} \leq k_1.
\]

- \((\pi_k)_{-1}\) satisfies (PS)-condition on \( \mathcal{Z} \).

- There exists a non-constant symmetric brake orbit \( z_k \) of (1.2) such that its Maslov-type indices satisfy 
\[
i^{\sqrt{-1}}_{L_0}(z_k) \leq 1 \leq i^{\sqrt{-1}}_{L_0}(z_k) + \nu^{\sqrt{-1}}_{L_0}(z_k). \tag{3.23}
\]

**Step 2:** Suppose \((z, T)\) is the symmetric brake solution of (1.2) obtained by (3.23) with minimal period \( \tau \). Then \( T/\tau \equiv p \in \mathbb{Z}_+ \) and there is a symmetric brake solution of (1.2) with minimal period \( \tau \) such that \( z(t) = z^p_\tau(t) \). Under the assumption of Theorem 1.2, we show that \( p \leq 3 \), and hence the minimal period of \((z, T) \geq T/3\). Indeed, let \( \gamma_z \) and \( \gamma_z \) be the symplectic paths associated with \( z_\tau \) and \( z \), respectively. Then \( \gamma_z = \gamma_z^p \). Besides, by (3.23),
\[
i^{\sqrt{-1}}_{L_0}(\gamma_z) \leq 1 \leq i^{\sqrt{-1}}_{L_0}(\gamma_z) + \nu^{\sqrt{-1}}_{L_0}(\gamma_z).
\]
By (AH3), the followings hold
\[
i_{L_1}(\gamma_{z_r}) + \nu_{L_1}(\gamma_{z_r}) \geq 0, 
\]
\[
i_{L_0}(\gamma_{z_r}) + \nu_{L_0}(\gamma_{z_r}) \geq 0,
\]
\[
i^{\sqrt{-1}}_{L_0}(\gamma_{z_r}) + \nu^{\sqrt{-1}}_{L_0}(\gamma_{z_r}) \geq 0.
\]
By (AH1)-(AH3),
\[
i^{\sqrt{-1}}_{L_1}(\gamma_{z_r}) + \nu^{\sqrt{-1}}_{L_1}(\gamma_{z_r}) \geq 1.
\]
Then combining (AH1)-(AH3) with Theorem A.11, we have
\[
i(\gamma_{z_r}^4) + \nu(\gamma_{z_r}^4)
= i_{L_0}(\gamma_{z_r}^2) + \nu_{L_0}(\gamma_{z_r}^2) + i_{L_1}(\gamma_{z_r}^2) + \nu_{L_1}(\gamma_{z_r}^2) + n
= i^{\sqrt{-1}}_{L_0}(\gamma_{z_r}) + \nu^{\sqrt{-1}}_{L_0}(\gamma_{z_r}) + i_{L_0}(\gamma_{z_r}) + \nu_{L_0}(\gamma_{z_r})
+ i^{\sqrt{-1}}_{L_1}(\gamma_{z_r}) + \nu^{\sqrt{-1}}_{L_1}(\gamma_{z_r}) + i_{L_1}(\gamma_{z_r}) + \nu_{L_1}(\gamma_{z_r}) + n
\geq n + 1.
Thus for $\omega_k = \exp(\frac{2\pi i n}{k})$ with $k \in \mathbb{Z}_+$, combining Lemma A.7 and Lemma A.13, we have

$$i_{L_0}^{-1}(\gamma_{z_2}^4) = i_{L_0}^{-1}(\gamma_{z_2}^2)^{2r} = \sum_{j=0}^{r-1} i_{L_0}^{-1}(\gamma_{z_2}^4) \geq \sum_{j=0}^{r-1} (i(\gamma_{z_2}^j) + \nu(\gamma_{z_2}^j) - n) = r(i(\gamma_{z_2}^4) + \nu(\gamma_{z_2}^4) - n) \geq r.$$

If $p = 4r + s$, we have

$$r \leq i_{L_0}^{-1}(\gamma_{z_2}^4) \leq i_{L_0}^{-1}(\gamma_{z_2}^{4r+s}) = i_{L_0}^{-1}(\gamma_{z_2}^2) = i_{L_0}^{-1}(\gamma_{z_2}) \leq 1,$$

which implies $r \leq 1$. By Lemma A.13, $r = 1$ holds if

$$1 = i_{L_0}^{-1}(\gamma_{z_2}^4) = i_{L_0}^{-1}(\gamma_{z_2}^4) = i(\gamma_{z_2}) + \nu(\gamma_{z_2}) - n, \quad (3.24)$$

and $\gamma^4(r)$ can be connected to

$$I_2^{op} \circ N(1, -1)^{op} \circ K$$

with $\sigma(K) \in \{ \exp(\sqrt{-1}\theta) \in U : 0 < \theta \leq \pi/2 \}$ and Krein-negative definite (cf. Lemma A.13). Since

$$\sigma(\gamma^4(r)) \in \{1\} \cup \{ e^{\sqrt{-1}\theta} \in U : 0 < \theta \leq \pi/2 \},$$

we obtain that

$$i_{-1}(\gamma_{z_2}^4) = i_{L_0}^{-1}(\gamma_{z_2}^4) + S_{\gamma_{z_2}^4}(r) (\sqrt{-1}) + \sum_{\frac{2k}{\pi} < \theta < \pi} \left( S_{\gamma_{z_2}^4}(r) (e^{\sqrt{-1}\theta}) - S_{\gamma_{z_2}^4}(r) (e^{\sqrt{-1}\theta}) \right) - S_{\gamma_{z_2}^4}(r)(-1)$$

$$= i_{L_0}^{-1}(\gamma_{z_2}^4) = 1,$$

which contradicts the fact that

$$i_{-1}(\gamma_{z_2}^4) = i_{L_0}^{-1}(\gamma_{z_2}^4) + i_{-1}(\gamma_{z_2}^2) + 2i_{L_0}^{-1}(\gamma_{z_2}^2) \in 2\mathbb{Z}.$$

Thus $r = 0$. Altogether, equation (1.2) has a non-constant symmetric brake solution with minimal period no less than $\frac{\pi}{2}$. We complete the proof of Theorem 1.2. \qed

A. Appendix: Maslov-type indices. Let

$$\text{Sp}(2n) = \{ M \in GL(2n, \mathbb{R}) \mid M^T JM = J \},$$

$$\mathcal{P}(2n) = \{ \gamma \in C([0, T], \text{Sp}(2n)) \mid \gamma(0) = I_{2n} \},$$

where we often omit $T$ if $\mathbb{R}_{\geq 0}$ for $\mathcal{P}(2n)$. Let $J$ be given in Introduction, $J$ is an almost complex structure of $\mathbb{R}^{2n}$, i.e.,

$$J : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \text{ with } J^2 = -id.$$

Let $\mathbb{R}^{2n}$ be equipped with the standard symplectic form $\omega_0$ such that $\omega_0(x,y) = \langle Jx, y \rangle$, thus $J$ is compatible with $\omega_0$ in the sense that for $x, y \in \mathbb{R}^{2n}$,

$$\omega_0(Jx, Jy) = \omega_0(x, y) \quad \text{and} \quad \omega_0(x, Jx) > 0 \quad \text{for} \quad x \neq 0.$$

A subspace $\Lambda \subseteq \mathbb{R}^{2n}$ is called Lagrangian if

$$\omega_0(x, y) = 0, \quad \text{for} \quad x, y \in \Lambda.$$
Let $F = \mathbb{R}^{2n} \bigoplus \mathbb{R}^{2n}$ be equipped with symplectic form $(-\omega_0) \bigoplus \omega_0$. Then $J = (-J) \bigoplus J$ is an almost complex structure on $F$ and $J$ is compatible with $(-\omega_0) \bigoplus \omega_0$. Denote by $\text{Lag}(F)$ the set of Lagrangian subspaces of $F$. Then for any $M \in \text{Sp}(2n)$, its graph

$$\text{Gr}(M) = \left\{ \begin{pmatrix} x \\ My \end{pmatrix} \bigg| x \in \mathbb{R}^{2n} \right\}$$

(A.1)

is a Lagrangian subspace of $F$. Let $L_0 = \{0\} \times \mathbb{R}^n$ and $L_1 = \mathbb{R}^n \times \{0\}$ be the two fixed Lagrangian subspaces of $\mathbb{R}^{2n}$ and let

$$V_0 = L_0 \times L_0, \quad V_1 = L_1 \times L_1,$$

$$\text{Gr}(M)|_{V_j} = \left\{ \begin{pmatrix} x \\ My \end{pmatrix} \bigg| x, y \in L_j \right\}.$$

Then both $V_j$ and $\text{Gr}(M)|_{V_j}$ are Lagrangian subspaces of $F$ for $M \in \text{Sp}(2n)$ and $j = 0, 1$.

Denote by $\mu^{CLM}_F(V, W; [a, b])$ the Maslov-type index for (ordered) pair of paths of Lagrangian subspaces $(V, W)$ in $F$ on $[a, b]$, which is defined by Cappel, Lee and Miller in [4] (cf. [10], [16], [28]).

**Definition A.1.** For any continuous path $\gamma \in \mathcal{P}_T(2n)$, we define the following Maslov-type indices:

$$i^{\omega}_L(\gamma) = \left\{ \begin{array}{ll}
\mu^{CLM}_F(\text{Gr}(e^{\theta J})|_{V_j}, \text{Gr}(\gamma(t)), t \in [0, \tau]), & \omega = e^{\sqrt{-1} \theta} \in U \setminus \{1\}, \\
\mu^{CLM}_F(V_j, \text{Gr}(\gamma(t)), t \in [0, \tau]) - n, & \omega = 1,
\end{array} \right.$$

$$\nu^{\omega}_L(\gamma) = \dim_{\mathbb{C}}(\gamma(\tau) L_j \cap e^{\theta J} L_j), \hspace{1cm} \omega = e^{\sqrt{-1} \theta} \in U.$$

for $j = 0, 1$. We often omit $\omega$ from the notation of $(i^{\omega}_L(\gamma), \nu^{\omega}_L(\gamma))$ if $\omega = 1$.

For a solution $(z, T)$ of (1.1), $\gamma \in \mathcal{P}_T(2n)$ is called the symplectic path associated with $(z, T)$ if $\gamma \in \mathcal{P}_T(2n)$ is the solution of the following

$$\begin{cases}
\dot{\gamma} = JH''(z)\gamma, \\
\gamma(0) = I_{2n},
\end{cases}$$

(A.2)

Suppose $(z, T)$ is the solution of (1.2) with the associated symplectic path $\gamma_z \in \mathcal{P}_T(2n)$. Define

$$(i^{\omega}_L(z), \nu^{\omega}_L(z)) := (i^{\omega}_L(\gamma_z), \nu^{\omega}_L(\gamma_z(\frac{T}{2}))), \hspace{1cm} \omega \in U.$$

Suppose $\gamma_B \in \mathcal{P}_T(2n)$ is the solution of (A.2) with $H'' = B$ for $B \in \mathcal{C}(S_T, L_s(\mathbb{R}^{2n}))$. Such $\gamma_B$ is called the symplectic path associated with $B$. For $\omega \in U$, define

$$\left( i^{\omega}_L(B, T), \nu^{\omega}_L(B, T) \right) := \left( i^{\omega}_L(\gamma_B), \nu^{\omega}_L(\gamma_B(T)) \right).$$

For any $x_i, y_i \in \mathbb{R}^{k_i}$ with $i = 1, 2$, define $(x_1, y_1) \circ (x_2, y_2) = (x_1, x_2, y_1, y_2)$. For any

$$X = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \text{Sp}(2n_1) \quad \text{and} \quad Y = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \in \text{Sp}(2n_2),$$

"
the symplectic sum $\diamond$ is given by

$$X \diamond Y = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix} \in \text{Sp}(2n_1 + 2n_2).$$

**Proposition A.2** (cf. [21, Theorem 2.2]).

1. Suppose $\gamma = \gamma_1 \diamond \gamma_2 \in P_T(2n)$ with corresponding decomposition of Lagrangian subspaces $L_j = L_j' \oplus L_j''$. Then

$$i_{L_j}(\gamma) = i_{L_j'}(\gamma_1) + i_{L_j''}(\gamma_2) \quad \text{and} \quad \nu_{L_j}(\gamma) = \nu_{L_j'}(\gamma_1) + \nu_{L_j''}(\gamma_2).$$

2. Suppose that $\gamma$ is the fundamental solution of $\dot{x}(t) = JB(t)x(t)$ with symmetric matrix for every $t \in \mathbb{R}$

$$B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}. $$

Then we have

$$i_{L_0}(\gamma) = \sum_{0<s<1} \nu_{L_0}(\gamma(st)), \quad \text{(A.3)}$$

provided that $b_{22}(t)$ is strictly positive definite for all $t \in \mathbb{R}$ and

$$i_{L_1}(\gamma) = \sum_{0<s<1} \nu_{L_1}(\gamma(st)), \quad \text{(A.4)}$$

provided that $b_{11}(t)$ is strictly positive definite for all $t \in \mathbb{R}$.

3. For $\gamma \in P_T(2n)$, we have

$$i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2} \text{sgn} M_{\varepsilon}(\gamma(T)), \quad \text{for} \quad 0 < \varepsilon \ll 1;$$

$$i_{L_0}(\gamma) + \nu_{L_0}(\gamma) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma)) = \frac{1}{2} \text{sgn} M_{\varepsilon}(\gamma(T)), \quad \text{for} \quad 0 < -\varepsilon \ll 1,$$

where $M_{\varepsilon}(P)$ is given by

$$P_T \left( \begin{array}{cc} \sin(2\varepsilon)I_n & -\cos(2\varepsilon)I_n \\ -\cos(2\varepsilon)I_n & \sin(2\varepsilon)I_n \end{array} \right) + \left( \begin{array}{cc} \sin(2\varepsilon)I_n & \cos(2\varepsilon)I_n \\ \cos(2\varepsilon)I_n & -\sin(2\varepsilon)I_n \end{array} \right).$$

The statement 1 of Proposition A.2 can be found in [4, Property IV (Symplectic Additivity)]. The statement 2 can be found in [19, Lemma 5.1], which is obtained by applying the method developed in [10], [16] and [28] for positive-definite Hamiltonian system. The statement 3 concerns the difference between $i_{L_0}$ and $i_{L_1}$ (cf. [32, Theorem 2.3]), which mainly relies on the homotopy invariant property of Maslov-type index.

**Remark A.3.** For a brake solution $(z,T)$ of (1.2), let $\gamma_z$ be the symplectic path associated with $z$. Then

$$(y(\cdot),T) = (z\left(\frac{T}{2} + \cdot\right),T)$$

is a brake solution of (1.2) and the symplectic path associated with $y$ is given by

$$N_{\gamma_z} \left(\frac{T}{2} - \cdot\right) \gamma_z \left(\frac{T}{2}\right)^{-1} N.$$
Proof. Since \( z \left( \frac{T}{2} + t \right) = Nz \left( \frac{T}{2} - t \right) \) and \( JN = -NJ \), we have
\[
\dot{y}(t) = \dot{z} \left( \frac{T}{2} + t \right) = \frac{d}{dt} \left( Nz \left( \frac{T}{2} - t \right) \right) = N \left( -JH' \left( z \left( \frac{T}{2} - t \right) \right) \right) \\
= -NJH' \left( Nz \left( \frac{T}{2} + t \right) \right) = JH' \left( z \left( \frac{T}{2} + t \right) \right) = JH'(y(t)).
\]

Besides, we have
\[
\left( N\gamma_z \left( \frac{T}{2} - t \right) \gamma_z \left( \frac{T}{2} \right)^{-1} N \right) \bigg|_{t=0} = I
\]
and
\[
\frac{d}{dt} \left( N\gamma_z \left( \frac{T}{2} - t \right) \gamma_z \left( \frac{T}{2} \right)^{-1} N \right) \\
= N \left( \frac{d}{dt} \gamma_z \left( \frac{T}{2} - t \right) \right) \gamma_z \left( \frac{T}{2} \right)^{-1} N \\
= -N \left( JH'' \left( z \left( \frac{T}{2} - t \right) \right) \gamma_z \left( \frac{T}{2} - t \right) \right) \gamma \left( \frac{T}{2} \right)^{-1} N \\
= -N \left( JH'' \left( Nz \left( \frac{T}{2} + t \right) \right) \gamma_z \left( \frac{T}{2} - t \right) \right) \gamma \left( \frac{T}{2} \right)^{-1} N \quad \text{(by (1.2))} \\
= -NJNH'' \left( z \left( \frac{T}{2} + t \right) \right) N\gamma_z \left( \frac{T}{2} - t \right) \gamma \left( \frac{T}{2} \right)^{-1} N \\
= JH''(y(t)) \left( N\gamma_z \left( \frac{T}{2} - t \right) \gamma_z \left( \frac{T}{2} \right)^{-1} N \right) \quad \text{(since } NJN = -J). \]

Then \( N\gamma_z \left( \frac{T}{2} - t \right) \gamma_z \left( \frac{T}{2} \right)^{-1} N \) is symplectic path associated with \( y \). \( \square \)

The following lemma shows the relation of Maslov-type indices between \( z \) and \( y \) under the assumption of Remark A.3.

**Lemma A.4.** Let \( z, y \) be given in Remark A.3. Then for \( j = 0, 1 \),
\[
(i_{L_j}(\gamma_y), \nu_{L_j}(\gamma_y)) = (i_{L_j}(\gamma_z), \nu_{L_j}(\gamma_z)).
\]
Moreover, if \( z \left( t + \frac{T}{2} \right) = -z(t) \), then for \( \omega \in U \), we have \( \nu^*_{U_j}(\gamma_y) = \nu^*_{U_j}(\gamma_z) \).

**Proof.** We have proved that in [12], here we give a proof for completeness. First, we have
\[
\nu_{L_j}(\gamma_y) = \dim \ker(\gamma_y(T) \cdot L_j \cap L_j) \\
= \dim \ker(NM^{-1}N \cdot L_j \cap L_j) \\
= \dim \ker(M^{-1} \cdot L_j \cap L_j) \\
= \dim \ker(M \cdot L_j \cap L_j) = \nu_{L_j}(\gamma_z),
\]
Further more, by Proposition A.2, the followings hold
\[
\begin{align*}
& \begin{cases}
  i_{L_0}(\gamma_z) - i_{L_1}(\gamma_z) = \frac{1}{2} \text{sgn}M_z(\gamma_z(T)) = \frac{1}{2} \text{sgn}M_z(M), \\
  i_{L_0}(\gamma_z) + i_{L_1}(\gamma_z) = i_1(\gamma_z^2) - n = i_1(\gamma_y^2) - n = i_{L_0}(\gamma_y) + i_{L_1}(\gamma_y), \quad (A.5) \\
  i_{L_0}(\gamma_y) - i_{L_1}(\gamma_y) = \frac{1}{2} \text{sgn}M_z(\gamma_y(T)) = \frac{1}{2} \text{sgn}M_z(NM^{-1}N). 
\end{cases}
\end{align*}
\]
To prove the first statement, it is sufficient to show
\[ \frac{1}{2} \text{sgn} M_\varepsilon(M) = \frac{1}{2} \text{sgn} M_\varepsilon(NM^{-1}N). \]
Indeed, we have
\[ M_\varepsilon(M) = M^T \begin{pmatrix} \sin(2\varepsilon)I_n & -\cos(2\varepsilon)I_n \\ \cos(2\varepsilon)I_n & -\sin(2\varepsilon)I_n \end{pmatrix} M + \begin{pmatrix} \sin(2\varepsilon)I_n & \cos(2\varepsilon)I_n \\ -\cos(2\varepsilon)I_n & -\sin(2\varepsilon)I_n \end{pmatrix} \]
and
\[ N \begin{pmatrix} \sin(2\varepsilon)I_n & \cos(2\varepsilon)I_n \\ \cos(2\varepsilon)I_n & -\sin(2\varepsilon)I_n \end{pmatrix} N = \begin{pmatrix} \sin(2\varepsilon)I_n & -\cos(2\varepsilon)I_n \\ -\cos(2\varepsilon)I_n & -\sin(2\varepsilon)I_n \end{pmatrix}. \]
Then we get
\[ (M^{-1}N)^TM_\varepsilon(M)M^{-1}N = M_\varepsilon(NM^{-1}N). \]
Thus
\[ \frac{1}{2} \text{sgn} M_\varepsilon(M) = \frac{1}{2} \text{sgn} M_\varepsilon(NM^{-1}N). \] (A.6)
Applying (A.6) to (A.5), we have \( i_{L_j}(\gamma_y) = i_{L_j}(\gamma_z) \), for \( j = 0, 1 \). In addition, if \( z(t + T/2) = -z(t) \), then we have
\[ y(t) = z \left( t + \frac{T}{2} \right) = -z(t) \]
and
\[ \dot{y}(t) = -\dot{z}(t) = -JH'(z(t)) = JH'(-z(t)), \]
which imply
\[ \dot{\gamma}_y(t) = JH''(y(t))\gamma_y(t) = JH''(-z(t))\gamma_y(t) = JH''(z(t))\gamma_y(t), \]
\[ \dot{\gamma}_z(t) = JH''(z(t))\gamma_z(t). \]
Thus \( \dot{\gamma}_z(t) = \dot{\gamma}_y(t) \) and for \( \omega \in U \), we obtain \( i_{L_j}^\omega(\gamma_y) = i_{L_j}^\omega(\gamma_z) \) for \( j = 0, 1 \). We complete the proof. \( \square \)

We define \( k \)-th iteration in brake orbit boundary sense as follows:

**Definition A.5** (cf. [22, Definition 2.9]). For any \( \gamma \in \mathcal{P}_T(2n) \), the \( k \)-th iteration \( \gamma^k \) of \( \gamma \) in brake orbit boundary sense is defined by \( \dot{\gamma} \mid_{[0,kT]} \) with \( \dot{\gamma} : \mathbb{R} \to \mathcal{P}(2n) \):
\[
\dot{\gamma}(t) = \begin{cases} 
\gamma(t - 2jT)(NM^{-1}NM)^j, & t \in [2jT, (2j + 1)T], j \in \mathbb{N}, \\
N\gamma((2j + 1)T - t)NM^{-1}NM)^{j+1}, & t \in [(2j + 1)T, (2j + 2)T], j \in \mathbb{N}.
\end{cases}
\]

**Remark A.6.** The \( k \)-th iteration \( \gamma^k \) of \( \gamma \) with \( \gamma \in \mathcal{P}_T(2n) \) in periodic sense was defined in [26, Chapter 8]. Without loss of generality, if \( z \) is a solution of (1.1), the iteration of associated symplectic path \( \gamma_z \in \mathcal{P}_T(2n) \) means the iteration in the periodic boundary sense (cf. [26, P. 177]). If \( z \) is a solution of (1.2), the iteration of associated symplectic path \( \gamma_z \in \mathcal{P}_T(2n) \) means the iteration in the brake orbit boundary sense (cf. Definition A.5).

The following lemma is Bott-type iteration formula for \( (i_{L_0}(\gamma^k), \nu_{L_0}(\gamma^k)) \):
Lemma A.7 (cf. [21, Theorem 4.1]). Suppose \( \omega_k = \exp(\pi \sqrt{-1}/k) \) for \( k \in \mathbb{N} \). For \( \gamma \in \mathcal{P}_T(2n) \),

\[
i_{L_0}(\gamma^k) = \begin{cases} 
  i_{L_0}(\gamma) + \sum_{l=1}^{k-1} i_{\omega_k^l}(\gamma^2), & \text{if } k \text{ is odd}, \\
  i_{L_0}(\gamma) + i_{\gamma_{L_0}}(\gamma) + \sum_{l=1}^{k-2} i_{\omega_k^l}(\gamma^2), & \text{if } k \text{ is even}.
\end{cases}
\]

\[
\nu_{L_0}(\gamma^k) = \begin{cases} 
  \nu_{L_0}(\gamma) + \sum_{l=1}^{k-1} \nu_{\omega_k^l}(\gamma^2), & \text{if } k \text{ is odd}, \\
  \nu_{L_0}(\gamma) + \nu_{\gamma_{L_0}}(\gamma) + \sum_{l=1}^{k-2} \nu_{\omega_k^l}(\gamma^2), & \text{if } k \text{ is even}.
\end{cases}
\]

Similarly, for the pair \((i_{L_0}^{-1}, \nu_{L_0}^{-1})\), we have

\[
i_{L_0}^{-1}(\gamma^k) = \begin{cases} 
  i_{L_0}^{-1}(\gamma) + \sum_{l=1}^{k-1} i_{\omega_k^l}(\gamma^2), & \text{if } k \text{ is odd}, \\
  \frac{1}{2} \sum_{l=1}^{k-1} i_{\omega_k^l}(\gamma^2), & \text{if } k \text{ is even}.
\end{cases}
\]

\[
\nu_{L_0}^{-1}(\gamma^k) = \begin{cases} 
  \nu_{L_0}^{-1}(\gamma) + \sum_{l=1}^{k-1} \nu_{\omega_k^l}(\gamma^2), & \text{if } k \text{ is odd}, \\
  \frac{1}{2} \sum_{l=1}^{k-1} \nu_{\omega_k^l}(\gamma^2), & \text{if } k \text{ is even}.
\end{cases}
\]

We have followings monotonicity property for \( i_{L_0} \)-index:

Lemma A.8 (cf. [32, Theorem 3.2]). Suppose \( \gamma_B \) is the symplectic path associated with semi-positive definite \( B \in \mathcal{C}(S_T, \mathcal{L}_s(\mathbb{R}^{2n})) \) and

\[
B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}.
\]

Then for any two positive integer \( p > q \),

\[
i_{i_{L_0}}^p(\gamma_B^p) \geq i_{i_{L_0}}^q(\gamma_B^q), \quad \omega = \pm 1, \quad \sqrt{-1}.
\]

The following definition is often referred as Maslov index in [26]:

Definition A.9. For any \( \gamma \in \mathcal{P}_T(2n) \) and \( \omega \in \mathbb{U} \), \( \nu_{\omega}(\gamma) \in \{0,1,\cdots,2n\} \) is given by

\[
\nu_{\omega}(\gamma) = \dim_{\mathbb{C}} \ker(\gamma(t) - \omega I)
\]

and \( i_{\omega}(\gamma) \in \mathbb{Z} \) is given in [26, Definition 5.2.7, Definition 5.4.2, Section 6.2]. We often omit \( \omega \) from the notation of \((i_{\omega}(\gamma), \nu_{\omega}(\gamma))\) if \( \omega = 1 \).

Assume that \((z,T)\) is the solution of (1.2) with associated symplectic path \( \gamma_z \). Define

\[
(i_{\omega}(z), \nu_{\omega}(z)) := (i_{\omega}(\gamma_z), \nu_{\omega}(\gamma_z)), \quad \omega \in \mathbb{U}.
\]

Suppose \( \gamma_B \in \mathcal{P}_T(2n) \) is the symplectic path associated with \( B \in \mathcal{C}(S_T, \mathcal{L}_s(\mathbb{R}^{2n})) \). For \( \omega \in \mathbb{U} \), define

\[
(i_{\omega}(B,T), \nu_{\omega}(B,T)) := (i_{\omega}(\gamma_B), \nu_{\omega}(\gamma_B(T))).
\]

We have following monotonicity property for \( i_{\omega} \)-index:
Lemma A.10 ([26, Theorem 15.3.2]). Let \( \gamma_B \) be the symplectic path associated with \( B \) semi-positive definite. Then for any two positive integer \( p > q \),
\[
i_\omega(\gamma_B^p) \geq i_\omega(\gamma_B^q), \quad \omega \in U.
\]

The Bott-type iteration formula for \((i(\gamma^k), \nu(\gamma^k))\) is given as:

Lemma A.11 (cf. [26, Theorem 9.2.1]). Suppose \( \omega_k = \exp(2\pi \sqrt{-1}/k) \) for \( k \in \mathbb{N} \). Then for \( \gamma \in \mathcal{P}_T(2n) \),
\[
i(\gamma^k) = \sum_{i=1}^{k} i_\omega(\gamma) \quad \text{and} \quad \nu(\gamma^k) = \sum_{i=1}^{k} \nu_\omega(\gamma).
\]

For any \( \gamma \in \mathcal{P}_T(2n) \) and \( \omega \in U \), the splitting number is defined by
\[
S_M^+(\omega) = \lim_{\epsilon \to \pm 0} i_{\exp(\sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma), \quad (A.7)
\]
where \( M = \gamma(T) \). By the definition of splitting number, for \( \omega = e^{\sqrt{-1}\theta_0} \in U \) and \( M = \gamma(T) \), we have
\[
i_\omega(\gamma) = i_1(\gamma) + S_M^+(1) + \sum_{\theta < \theta_0} (-S_M^-(e^{\sqrt{-1}\theta}) + S_M^+(e^{\sqrt{-1}\theta})) - S_M^-(e^{\sqrt{-1}\theta_0}). \quad (A.8)
\]

The abstract iteration formula of \( i_{L_0}(\gamma^k) \) is given as follows:

Lemma A.12 (cf. [21, Theorem 1.3]). Suppose \( \gamma \in \mathcal{P}_T/2(2n) \) and \( M = \gamma^2(T) \), where \( \gamma^k \) is given in Definition A.5 for \( k \in \mathbb{Z}_+ \). Then
\[
i_{L_0}(\gamma^{2k+1}) = i_{L_0}(\gamma) + k(i(\gamma^2) + S_M^+(1) - C(M))
\]  
\[+ \sum_{\theta \in (0,2\pi)} E \left( \frac{(2k+1)\theta}{2\pi} \right) S_M^-(e^{\sqrt{-1}\theta}) - C(M),
\]
\[
i_{L_0}(\gamma^{2k+2}) = i_{L_0}(\gamma^2) + k(i(\gamma^2) + S_M^+(1) - C(M))
\]  
\[+ \sum_{\theta \in (0,2\pi)} E \left( \frac{(2k+2)\theta}{2\pi} \right) S_M^-(e^{\sqrt{-1}\theta})
\]  
\[+ C(M) - \sum_{\theta \in (\pi,2\pi)} S_M^-(e^{\sqrt{-1}\theta}),
\]
where
\[
C(M) = \sum_{\theta \in (0,2\pi)} S_M^-(e^{\sqrt{-1}\theta}),
\]
\[
E(\kappa) = \min \{ l \in \mathbb{Z} : l \geq \kappa \}, \quad \kappa \in \mathbb{R}.
\]

Furthermore,
\[
i_{L_0}(\gamma) = \frac{1}{2}(i(\gamma^2) + S_M^+(1) - C(M)) + \sum_{\theta \in (0,2\pi)} \frac{\theta}{2\pi} S_M^-(e^{\sqrt{-1}\theta}) = \frac{1}{2} i(\gamma^2).
\]

Denote by \( \sigma(P) \) the collection of the eigenvalues of a matrix \( P \) and \( U^\pm = \{ z \in \mathbb{C} : |z| = 1, \ z = x + \sqrt{-1}y \text{ with } \pm y \geq 0 \} \). We have the followings:

Lemma A.13 (cf. [26, Theorem 10.1.1, Theorem 10.1.3]).

1. For any \( \gamma \in \mathcal{P}_T(2n) \) and \( \omega \in U \setminus \{ 1 \} \),
\[
i(\gamma) + \nu(\gamma) - n \leq i_\omega(\gamma) \leq i(\gamma) + n - \nu_\omega(\gamma).
\]  
\[\quad \text{(A.9)}
\]
• The left equality of (A.9) holds for \( \omega \in U^+ \setminus \{1\} \) (resp. \( U^- \setminus \{1\} \)) if and only if \( \gamma(\tau) \) can be connected to

\[
I_{2}^{\bar{p}} \circ N(1, -1)^{\bar{q}} \circ K,
\]

where \( 0 \leq p + q \leq n \) and \( K \in \text{Sp}(2(n - p - q)) \) with \( \sigma(K) \subset U \setminus \{1\} \), and all eigenvalues of \( K \) located within the arc between 1 and \( \omega \) with total multiplicity \( n - p - q \). If \( \omega \neq -1 \), all eigenvalues of \( K \) are in \( U \) which those in \( U^+ \setminus \{\pm 1\} \) (resp. \( U^- \setminus \{\pm 1\} \)) are all Krein-positive (resp. Krein-negative) definite. If \( \omega = -1 \), the equality holds if and only if \( K \) can be connected to

\[
-I_{2}^{\ast} \circ N(-1, 1)^{\ast} \circ H,
\]

where \( 0 \leq s + t + p + q \leq n \) and \( H \in \text{Sp}(2n - 2(p + q + s + t)) \) with \( \sigma(H) \subset U \setminus \{\pm 1\} \) and the eigenvalues belonging to \( \sigma(H) \cap U^+ \) (resp. \( \sigma(H) \cap U^- \)) are all Krein-negative (resp. Krein-positive) definite.

• The right equality of (A.9) holds for all \( \omega \in U \setminus \{1\} \) if and only if \( \gamma(\tau) \) can be connected to

\[
I_{2}^{\bar{p}} \circ N(1, -1)^{\bar{q}} - p
\]

with \( \nu(\gamma(\tau)) = n + p \).

• The right equality of (A.9) holds for \( \omega \in U^+ \setminus \{1\} \) (resp. \( U^- \setminus \{1\} \)) if and only if \( \gamma(\tau) \) can be connected to

\[
I_{2}^{\bar{p}} \circ N(1, 1)^{\bar{q}} \circ K,
\]

where \( 0 \leq p + r \leq n \) and \( K \in \text{Sp}(2(n - p - r)) \) with \( \sigma(K) \subset U \setminus \{1\} \), and all eigenvalues of \( K \) located within the arc between 1 and \( \omega \) with total multiplicity \( n - p - r \). If \( \omega \neq -1 \), all eigenvalues of \( K \) are in \( U \) which those in \( U^+ \) (resp. \( U^- \)) are all Krein-positive (resp. Krein-negative) definite. If \( \omega = -1 \), the equality holds if and only if \( K \) can be connected to

\[
-I_{2}^{\ast} \circ N(-1, 1)^{\ast} \circ H,
\]

where \( 0 \leq s + t + p + r \leq n \) and \( H \in \text{Sp}(2n - 2(p + r + s + t)) \) with \( \sigma(H) \subset U \setminus \{\pm 1\} \) with the eigenvalues belonging to \( \sigma(H) \cap U^+ \) (resp. \( \sigma(H) \cap U^- \)) are all Krein-negative (resp. Krein-positive) definite.

• The right equality of (A.9) holds for all \( \omega \in U \setminus \{1\} \) if and only if \( \gamma(\tau) \) can be connected to

\[
I_{2}^{\bar{p}} \circ N(1, 1)^{\bar{q}} - p
\]

with \( \nu(\gamma(\tau)) - n + p \).

• Both equalities of (A.9) hold for \( \omega \in U \setminus \{1\} \) if and only if \( \gamma(\tau) = I_{2n} \).

2. For any \( \gamma \in \mathcal{P}_{\gamma}(2n) \) and \( m \in \mathbb{N} \),

\[
m(i(\gamma) + \nu(\gamma) - n) + n - \nu(\gamma) \\
\leq i(\gamma^{m}) - m(i(\gamma) + n) - n - (\nu(\gamma^{m}) - \nu(\gamma)). \tag{A.10}
\]

• The first equality of (A.10) holds for \( m \geq 3 \) if and only if \( \gamma(\tau) \) can be connected to

\[
I_{2}^{\bar{p}} \circ N(1, -1)^{\bar{q}} \circ K,
\]

where \( 0 \leq p + q \leq n \) and \( K \in \text{Sp}(2(n - p - q)) \) with \( \sigma(K) \subset U \setminus \{1\} \), \( K \) can be connected to

\[
R(\theta_{1}) \circ \cdots \circ R(\theta_{r}) \text{ with } 0 < \frac{\text{max}_{\theta} \theta}{2\pi} \leq 1, \text{ for } 1 \leq j \leq r.
\]
The eigenvalues on $\sigma(K) \cap U^+$ (resp. $\sigma(K) \cap U^-$) located on the arc between 1 and
\[ e^{2\pi i m} \left( \text{resp. } e^{-2\pi i m} \right) \]
are all Krein-negative (resp. Krein-positive) definite. If $m = 2$, the equality holds if and only if $\gamma(\tau)$ can be connected to
\[ I_{2p}^p \circ N(1, -1)^{\sigma q} \circ N(-1, -1)^{\sigma(n - p - q)}. \]
- The second equality of (A.10) holds for $m \geq 3$ if and only if $\gamma(\tau)$ can be connected to
\[ I_{2p}^p \circ N(1, -1)^{\sigma r} \circ K, \]
where $0 \leq p + q \leq n$ and $K \in \text{Sp}(2(n - p - r))$ with $\sigma(K) = \{-1\}$. If $m = 2$, the equality holds if and only if $\gamma(\tau)$ can be connected to
\[ I_{2p}^p \circ N(1, -1)^{\sigma r} \circ N(-1, -1)^{2s} \circ N(-1, 1)^{ot} \]
with $n - p - q = s + t$.
- The two equalities of (A.10) hold for some $m_1$ and $m_2 \geq 2$ respectively if and only if
\[ \gamma(\tau) = I_{2p}^p \circ N(-1, -1)^{\sigma(n - p)} \]
for some $0 \leq p \leq n$. $p < n$ happens if and only if $m_1 = m_2 = 2$.

B. Appendix: Saddle point reduction and Minimax theorem.

B.1. General theory. Most materials in this section can be found in [1], [6], [26]. We give an introduction in order to apply it for infinite dimensional functions of periodic Hamiltonian systems.

For a Hilbert space $L$, let $A$ be a self-adjoint operator with $\text{dom}(A) \subset L$ and $\Phi \in C^1(L, \mathbb{R})$ with $d\Phi = F, \Phi(0) = 0$. Assume the followings

(C): There exists $c_\lessgtr c_+$ such that $c_\lessgtr \notin \sigma(A)$ and $\sigma(A) \cap [c_-, c_+]$ contains at most finitely many eigenvalues with finite multiplicities.

(F): $F$ is Gateaux differential in $L$ with
\[ c_- \leq \|dF(u)\|_W \leq c_+, \text{ for any } u \in L, \]
where $\| \cdot \|_L$ is the norm in $L$.

By assumption 1, there exists a sufficiently small $\varepsilon > 0$ such that $0 \notin \sigma(A_\varepsilon)$ where $A_\varepsilon = A + \varepsilon Id$. Next we apply the Lyapunov-Schmidt reduction for $A_\varepsilon$. Let $\{E_\lambda\}$ be the spectral resolution of $A_\varepsilon$ and let
\[ P = \int_{c_-}^{c_+} dE_\lambda, \quad P^+ = \int_{c_+}^{\infty} dE_\lambda, \quad P^- = \int_{-\infty}^{c_-} dE_\lambda, \]
\[ Z = PL, \quad L^\pm = P^\pm L. \]

Then
\[ L = L^- \oplus Z \oplus L^+, \]
where $Z$ is finite dimensional since $A_\varepsilon$ is self-adjoint and $c_\perp$ is bounded. Define
\[ R = \int_{c_-}^{c_+} |\lambda|^{-\frac{1}{2}} dE_\lambda, \quad S^+ = \int_{c_+}^{\infty} \lambda^{-\frac{1}{2}} dE_\lambda, \quad S^- = \int_{-\infty}^{c_-} (-\lambda)^{-\frac{1}{2}} dE_\lambda, \]

Then the followings hold
1. $R, S^-, S^+$ are pairwise commuting.
2. $R$ and $S^\pm$ are injective restricted on $Z$ and $L^\pm$, respectively.
3. \(-(S^-)^2 + R^2 + (S^+)^2 = A_{\varepsilon}^{-1}\).
4. \(A_{\varepsilon}(S^\pm)^2 = P^\pm \) and \(A_{\varepsilon}R^2 = P\).

Define a subspace \(V \subset L\)

\[
V = V^- \oplus V_0 \oplus V^+,
\]

(B.1)

where

\[
V_0 = RZ, \quad V^\pm = S^\pm L^\pm.
\]

The norm \(\| \cdot \|_V\) in \(V\) is given by

\[
\|v\|_V^2 = \|(S^-)^{-1}v^\pm\|_L^2 + \|R^{-1}v^0\|_L^2 + \|(S^+)^{-1}v^+\|_L^2
\]

with \(v = v^- + v^0 + v^+ \in V^- \oplus V_0 \oplus V^+\). Then \(V^0\) and \(V^\pm\) are isomorphic to \(Z\) and \(L^\pm\), respectively. For \(x = x^- + z + x^+ \in L^- \oplus Z \oplus L^+\), we define two functionals \(f\) and \(\Phi_{\varepsilon}\) as follows:

\[
\Phi_{\varepsilon}(v) = \Phi(x) + \frac{\varepsilon}{2}\|x\|_L^2,
\]

\[
f(x) = \frac{1}{2}\left(\|x^+\|_L^2 - \|x^-\|_L^2 + \|Q^+z\|_L^2 - \|Q^-z\|_L^2\right) - \Phi_{\varepsilon}(v),
\]

where

\[
Q^+ = \int_0^\infty dE_{\lambda}, \quad Q^- = \int_{-\infty}^0 dE_{\lambda}, \quad \text{and} \quad v = S^-x^- + Rz + S^+x^+.
\]

Assume

(D): \(\Phi \in C^2(V, \mathbb{R})\).

Then by the assumption of \(\Phi\), \(f\) is \(C^1\) on \(L\). Let \(F_{\varepsilon} = F + \varepsilon Id\). The critical point \(x\) of \(f\) satisfies

\[
0 = x^+ - x^- + (PQ^+)x - (PQ^-)x
- P^+S^+F_{\varepsilon}(v) - P^-S^-F_{\varepsilon}(v) - RF_{\varepsilon}(v),
\]

which implies

\[
x^\pm = \pm S^\pm P^\pm F_{\varepsilon}(v),
\]

(B.2)

\[
PQ^\pm x = RQ^\pm F_{\varepsilon}(v).
\]

Thus \(x = x^- + z + x^+\) is a critical point of \(f\) if and only if \(v\) is the solution of

\[
A_{\varepsilon}x = F_{\varepsilon}(x), \quad x \in \text{dom}(A).
\]

In addition, (B.2) is equivalent

\[
v^\pm = A_{\varepsilon}^{-1}P^\pm F_{\varepsilon}(v),
\]

(B.3)

where \(v = v^- + v_0 + v^+ = S^-x^- + Rz + S^+x^+\). By (F), for all \(u, v \in L\),

\[
\|F_{\varepsilon}(u) - F_{\varepsilon}(v)\|_L = \|F(u) - F(v) + \varepsilon(u - v)\|_L
\]

\[
\leq (c_+ + \varepsilon)\|u - v\|_L.
\]

By (C), there is a constant \(c_\varepsilon > c_+ + \varepsilon\) such that

\[
\|A_{\varepsilon}^{-1}\|_{L^2} \leq \frac{1}{c_\varepsilon}.
\]
Then by

Since

Next we prove that \( F_\varepsilon = A_\varepsilon^{-1}(P^+ + P^-)F_\varepsilon \) is contractible with respect to variables in \( V^+ \oplus V^- \). Indeed, suppose \( u = u^- + z + u^+ \), \( v = v^- + z + v^+ \) for fixed \( z \in V_0 \). Then by \( A_\varepsilon(S^\pm)^2 = P^\pm \),

\[
\|F_\varepsilon(u) - F_\varepsilon(v)\|_V
= \|A_\varepsilon^{-\frac{1}{2}}(P^+ + P^-)(F_\varepsilon(u) - F_\varepsilon(v))\|_L
\leq \|A_\varepsilon^{-\frac{1}{2}}(P^+ + P^-)\| \cdot \|(F_\varepsilon(u) - F_\varepsilon(v))\|_L
\leq \frac{1}{\sqrt{c_\varepsilon}}(c_+ + \varepsilon)\|u - v\|_L.
\]

Since for \( v^\pm \in V^\pm \), we have

\[
\|v^\pm\|_L = \|S^\pm v^\pm\|_V \leq \frac{1}{\sqrt{c_\varepsilon}}\|v^\pm\|_V.
\]

Finally,

\[
\|F_\varepsilon(u) - F_\varepsilon(v)\|_V \leq \frac{1}{\sqrt{c_\varepsilon}}(c_+ + \varepsilon)\|u - v\|_L \leq \frac{c_+ + \varepsilon}{c_\varepsilon}\|u - v\|_V.
\]

Since \( c_\varepsilon > c_+ + \varepsilon \), \( F_\varepsilon \) is a contraction on \( V \) and \( F_\varepsilon \) is \( C^1 \) by (D). Applying the implicit function theorem, there exists a solution \( w^\pm(z) \) for each \( v_0 \in V \) and \( w^\pm(v_0) \in C^1(V_0, V^\pm) \). Since \( \dim V_0 \) is finite, we have that

\[
w^\pm(v_0) = (S^\pm)^{-1}w^\pm(v_0) \in C^1(Z, L)
\]

is the solution of (B.2). Let \( z = v_0 \) and let

\[
u(z) = z + w(z), \quad w(z) = w^-(z) + w^+(z).
\]

Since \( w^\pm \) is \( C^1 \), \( u \in C^1(Z, L) \). By (B.2), \( w(z) \in \text{dom}(A) \). Let \( u_0 = R^{-1}z \) and let

\[
\begin{align*}
a(z) &= f(u^+(z) + u^-(z) + u_0(z)) \\
&= f((S^+)^{-1}w^+(z) + (S^-)^{-1}w^-(z) + R^{-1}z) \\
&= \frac{1}{2} \left( \|S^+\|^{-1}w^+\|_L^2 - \|S^-\|^{-1}w^-\|_L^2 \right) \\
&\quad + \frac{1}{2} \left( \|Q^+R^{-1}z\|_L^2 - \|Q^-R^{-1}z\|_L^2 \right) - \Phi_\varepsilon(u(z)).
\end{align*}
\]

Applying (B.4) to \( a(z) \),

\[
a(z) = \langle A_\varepsilon u(z), u(z) \rangle - \Phi_\varepsilon(u(z)) = \langle Au(z), u(z) \rangle - \Phi(u(z)). \tag{B.5}
\]

Furthermore, by (B.3), we have

\[
w'(z) = A_\varepsilon^{-1}(P^+ + P^-)F'_\varepsilon(u(z))u'(z),
\]

which implies

\[
A_\varepsilon w'(z) = (Id - P)F'(u(z))u'(z) + \varepsilon w'(z),
\]

\[
Au'(z) = (Id - P)F'(u(z))u'(z).
\]

The following Saddle Point Reduction Theorem can be found in [1, Proposition 4.1, Proposition 4.5], [6, Theorem 4.2.1] and references therein.

**Theorem B.1.** Assume (C), (F) and (D) hold and \( a \) is given in (B.5). Then there exists a one-to-one correspondence

\[
z \mapsto u = u(z) = z + w(z)
\]
between the critical points of the \( a \in C^2(Z, \mathbb{R}) \) and the solutions of the equation
\[
Ax = F(x), \quad x \in \text{dom}(A).
\]

In addition, \( a' \) and \( a'' \) are given by
\[
a'(z) = Au(z) - F(u(z)) = Az - PF(u(z)) = Au(z) - F(u(z))
\]
and
\[
a''(z) = (A - P')(u'(z)) = (AP - PF')(u'(z)) = (A|_Z - PF'(u(z)))(u'(z)).
\]

We have following estimations (cf. [1, Lemma 7.2], [6, Lemma 4.2.2]).

**Lemma B.2.** Assume \((C), (F)\) and \((D)\) hold and \(F(0) = 0\).

1. If there exists a self-adjoint operator \( C^- \) on \( L \) commuting with \( P \) and \( P^- \) such that
   \[
   \min(\sigma(A) \cap [c_-, c_+]) \text{Id} \leq C^- \leq F'(0),
   \]
   then
   \[
a(z) \leq \frac{1}{2} \left( ((A - C^-)z, z) + o(\|z\|_L) \right), \quad \text{as } \|z\|_L \to 0;
   \]

2. If there exists a self-adjoint operator \( C^+ \) on \( L \) commuting with \( P \) and \( P^+ \) such that
   \[
   F'(0) \leq C^+ \leq \max(\sigma(A) \cap [c_-, c_+]) \text{Id},
   \]
   then
   \[
a(z) \geq \frac{1}{2} \left( ((A - C^+)z, z) + o(\|z\|_L) \right), \quad \text{as } \|z\|_L \to 0;
   \]

**B.2. Application to Hamiltonian system.** In this section, we apply the saddle point reduction Theorem to Hamiltonian system in suitable setting. Let \( S_T, L, W, A, P_0 \) and \( A_0 \) be given in Section 2 and let \( Q^\pm \), be given in Section B.1. Define \((CH)\) as follows:

\((CH)\): \( H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \) satisfies \( H(z) = o|z|^2 \) at \( z = 0 \) and
\[
c(H) \notin \sigma(A_0) \quad \text{and} \quad \|H^0\| \leq c(H)
\]
with constant \( c(H) > 1 \).

Since \( A_0 \) is the self-adjoint operator on \( W \) and has closed image and compact resolvent, the eigenvalues \( A_0 \) are nonzero with spectrum given by \( \sigma(A_0) = \{1\} \cup \frac{\mathbb{Z}}{T} \setminus \{0\} \). Let \( E_A \) be the spectral resolutions of the self-adjoint operators \( A_0 \) and let \( P, P^\pm, S^\pm, R \) be the projections in Section B.1 with respect to \((c_-, c_+) = (-c(H), c(H))\). We let \( g : L \to \mathbb{R} \) be defined as
\[
g(x) = \int_0^T H(x(t))dt, \quad x \in L.
\]

By \((CH)\), \( g \in C^1(L, \mathbb{R}) \) and the derivative \( g'(x) : T_xL \to \mathbb{R} \) is given by
\[
g'(x)[y] = \left< \nabla H(x), y \right>_L,
\]
where \( x, y \in L, \left< \cdot, \cdot \right>_L \) means the inner product and \( \nabla H(*) : L \to T_xL \) is given by
\[
\left( \nabla H(x), y \right)_L = \int_0^T H'(x(t)) \cdot y(t)dt,
\]
By \( H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \), \( g' \) is Gateaux differentiable and the Gadeaux derivative
\[
(dg'(x))(y)[z] : T_y(T_xL) \to \mathbb{R}, \quad (dg'(x))(y)[z] = \left< \nabla^2 H(x)y, z \right>_L,
\]
where $x, y, z \in L$ and $(\nabla^2 H(x(t)) : T_x L \to T_x(T_x L)$ is given by

$$(\nabla^2 H(x)y, z)_L = \int_0^T H''(x(t))y(t) \cdot z(t)dt.$$ 

Then by (CH), $f \in C^1(W, \mathbb{R})$ and $f'$ is Gateaux differentiable with $f'$ and $df'$ are given by

$$f'(z) = Az - g'(z) \quad \text{and} \quad df'(z)[y] = Ay - dg'(z)[y].$$

Let $A_0$ and $X_0$ be given as in Section 2. We apply Theorem B.1 to the case of symmetric periodic orbits, brake orbits and symmetric brake orbits (cf. [26, Theorem 4.5.1, Proposition 4.5.2]).

**Theorem B.3** (cf. [26, Theorem 4.4.1]). Suppose that $H$ satisfies (CH). Then there exists a function $a \in C^2(Z, \mathbb{R})$ and an injection map $u \in C^1(Z, L)$ so that $u : Z \to W = \text{dom}(A)$ and

1. The map $u$ can be written as $u(z) = w(z) + z$, where $P(w(z)) = 0$.
2. The functional $a$ satisfies

$$a(z) = f(u(z)) = \frac{1}{2}(Au(z), u(z))_L - g(u(z)),$$

$$a'(z) = Au(z) - g'(u(z)),$$

$$a''(z) = f(u(z)) = (A - dg'(u(z)))u'(z).$$

3. $z \in Z$ is a critical point of $a$ if and only if $u(z)$ is a solution of (1.1), if and only if $Au(z) - g'(u(z)) = 0$.

4. Suppose $g(u) = \frac{1}{2}(Bu, u)_L$, where $B \in \mathcal{L}_c(2n)$, then $a(z) = \frac{1}{2}((A - B)z, z)_L.$

5. Suppose $z \in Z$ is a critical point of a with associated symplectic path $\gamma \in \mathcal{P}(2n)$. Then

$$\dim \ker a''(z) = \dim(\gamma(T) : X_0 \cap X_0).$$

Since (CH) holds, we apply Lemma B.2 to the case of $F(0) = 0$ as follows:

**Theorem B.4** (cf. [26, Theorem 4.5.1, Proposition 4.5.2]). Suppose that $H$ satisfies (CH). Then for every $z \in Z$, the following inequalities hold

$$a(z) \leq \frac{1}{2}\langle A \left(P^- w(z) + z\right), P^- w(z) + z\rangle_L - g \left(P^- w(z) + z\right),$$

$$a(z) \geq \frac{1}{2} \langle A \left(P^+ w(z) + z\right), P^+ w(z) + z\rangle_L - o(\|z\|^2), \quad \text{as } z \to 0 \quad \text{in } Z.$$

**Remark B.5.** We denote the restrictions of $g$ to $\mathcal{L}$, $\mathcal{L}$ and $\mathcal{L} \cap \mathcal{L}$ as follows

$$\hat{g} = g|_\mathcal{L}, \quad \bar{g} = g|_\mathcal{L} \quad \text{and} \quad \tilde{g} = g|_{\mathcal{L} \cap \mathcal{L}}.$$

Then by (CH), $\hat{g}, \bar{g}$ and $\tilde{g}$ are $C^1$ and $\hat{g}', \bar{g}'$ and $\tilde{g}'$ is Gateaux differentiable w.r.t. $\mathcal{L}$, $\mathcal{L}$ and $\mathcal{L} \cap \mathcal{L}$. Let

$$\hat{Z} = \text{Im}(P|_\mathcal{L}), \quad \bar{Z} = \text{Im}(P|_\mathcal{L}), \quad \tilde{Z} = \text{Im}(P|_{\mathcal{L} \cap \mathcal{L}}),$$

$$\hat{\mathcal{L}}^\pm = \text{Im}(P^\pm|_\mathcal{L}), \quad \bar{\mathcal{L}}^\pm = \text{Im}(P^\pm|_\mathcal{L}), \quad \tilde{\mathcal{L}}^\pm = \text{Im}(P^\pm|_{\mathcal{L} \cap \mathcal{L}}).$$

Then for the case of symmetric periodic orbits, brake orbits and symmetric brake orbits, similar results can be obtained by applying Theorem B.3 and Theorem B.4 to
the case of \( (A, f, L, Z, W) = (\hat{A}, f|_{W}, \hat{L}, \hat{Z}, \hat{W}) \), \( (A, f, L, Z, W) = (\hat{A}, f|_{W'} \hat{L}, \hat{Z}, \hat{W}, \hat{W}) \) and \( (A, f, L, Z, W) = (\hat{A}, f|_{W|W'} \hat{L}, \hat{Z}, \hat{W} \cap \hat{W}) \), respectively.

B.3. The relation between Morse indices and Maslov-type indices. In this subsection, we give the relation between the Morse indices of \( a \) and the Maslov-type indices of \( B \) provided that \( f \) in Theorem B.3 is given by

\[
f(z) = \frac{1}{2} \langle (A - B)x, x \rangle_L = \int_0^T \left( (J \frac{d}{dt} - B)x \right) dt,
\]

where \( B \in L_s(\mathbb{R}^{2n}) \). More precisely, we obtain Theorem B.6, Theorem B.7 and Theorem B.8 for the case of symmetric periodic orbits, brake orbits and symmetric brake orbits, respectively. Let \( X_k \) be given in (2.1) and for \( k_0 \in \mathbb{N} \), let

\[
Z_s = \bigoplus_{|k| \leq k_0} X_{2k+1}.
\]

Thus we have \( \dim \mathbb{R} Z_s = 2|Z| \) for each \( k \in \mathbb{N} \) and we let \( 2d_s = \dim \mathbb{R} Z_s \). Applying Theorem B.3, if \( k_0 \in \mathbb{N} \) is sufficiently large, then there exists an injection map \( u_s \in C^\infty(Z_s, L^2(S_T, \mathbb{R}^{2n})) \) such that \( u_s \) satisfies the conclusion of Theorem B.3. Let

\[
a_s(z) = \hat{f}(u_s(z)), \quad z \in Z_s.
\]

For symmetric periodic orbits, we have

**Theorem B.6** (cf. [26, Theorem 6.1.1]). Suppose \( B \in L_s(\mathbb{R}^{2n}) \). Then

\[
\begin{align*}
m^- (a_s, z) &= d_s + \nu(B), \\
m^0 (a_s, z) &= \nu(B), \\
m^+ (a_s, z) &= d_s - \nu(B),
\end{align*}
\]

where \( m^*(a_s, z) \) is the Morse indices of \( a_s \) at \( z \in Z_s \) for Theorem B.3 with \( * = +, 0, - \), respectively.

For a sufficiently large \( k_0 \in \mathbb{N} \), let

\[
Z^{L_0} = \bigoplus_{|l| \leq k_0} \overline{X}_l \quad \text{and} \quad \dim \mathbb{R} Z^{L_0} = 2d + n,
\]

where \( \overline{X}_l \) is given in (2.2). Let \( \overline{a} = a|_{Z^{L_0}} \), where the functional \( a \) is defined in Theorem B.3 with \( f \) given by (B.6). Let \( m^*(\overline{a}, 0) \) be the Morse indices of \( \overline{a} \) at \( z \in Z^{L_0} \) of 3 in Theorem B.3 with \( * = +, - , 0 \), respectively. Then for brake orbit case, we have

**Theorem B.7** (cf. [27, Theorem 5.1]). Suppose \( B \) satisfies (B1). Then

\[
\begin{align*}
m^- (\overline{a}, z) &= \overline{d} + i_{L_0} \left(B, \frac{T}{2}\right), \\
m^0 (\overline{a}, z) &= \nu_{L_0} \left(B, \frac{T}{2}\right), \\
m^+ (\overline{a}, z) &= \overline{d} + n - i_{L_0} \left(B, \frac{T}{2}\right).
\end{align*}
\]
For a sufficiently large $k_0 \in \mathbb{N}$, let

$$Z_s = \bigoplus_{|l| \leq \left\lfloor \frac{k_0^2}{2} \right\rfloor} \mathbb{X}_{2l+1}$$

and $\dim Z_s = 2d_s + n$,

where $\mathbb{X}_l$ is given in (2.2). Let $a_s = a|_{Z_s}$, where the functional $a$ is defined in Theorem B.3. Set $m^*(a_s, z)$ to be the Morse indices of $a_s$ at $z \in Z_s$ of 3 in Theorem B.3 with $* = +, -, 0$, respectively. For symmetric brake orbit case, we have

**Lemma B.8.** Suppose $B \in L_s(\mathbb{R}^{2n})$ satisfies (B1). Then

$$m^-(a_s, z) = d_s + i_{L_0}^{-1}(B, \frac{T}{4}),$$

$$m^0(a_s, z) = \nu_{L_0}^{-1}(B, \frac{T}{4}),$$

$$m^+(a_s, z) = d_s - i_{L_0}^{-1}(B, \frac{T}{4}).$$

**Proof.** By Lemma A.7,

$$i_{L_0} \left( B, \frac{T}{2} \right) = i_{L_0}(B, \frac{T}{4}) + i_{L_0}^{-1}(B, \frac{T}{4}),$$

$$\nu_{L_0} \left( B, \frac{T}{2} \right) = \nu_{L_0}(B, \frac{T}{4}) + \nu_{L_0}^{-1}(B, \frac{T}{4}).$$

For sufficiently large $k_0 \in \mathbb{N}$, since

$$Z_s = \bigoplus_{|l| \leq \left\lfloor \frac{k_0^2}{2} \right\rfloor} \mathbb{X}_{2l+1},$$

$$Z_{L_0} = \bigoplus_{|l| \leq k_0} \mathbb{X}_l = \bigoplus_{|l| \leq \left\lfloor \frac{k_0^2}{2} \right\rfloor} \mathbb{X}_{2l+1} \oplus \bigoplus_{|l| \leq \left\lfloor \frac{k_0^2}{2} \right\rfloor} \mathbb{X}_{2l},$$

where $\mathbb{X}_l$ is given in (2.2). Combining Theorem A.7, Theorem B.7 with (B1), we obtain the conclusion.

\[\Box\]

### B.4. Minimax theorem

In this section, we give a brief introduction of some basic concepts and minimal theorems, which the details of proofs can be found in [6], [15], [29].

**Definition B.9.** Suppose $E$ is a Banach space and $f \in C^1(E, \mathbb{R})$. $f$ is said to satisfy (PS)$_c$-condition if for every sequence $\{x_k\}$ satisfying

$$f'(x_k) = 0, \quad f(x_k) \to c, \quad \text{as} \quad k \to \infty,$$

implies that there exists a subsequence of $\{x_k\}$ which is convergent in $E$. $f$ is said to satisfy (PS)-condition if $f$ satisfies (PS)$_c$-condition for every $c \in \mathbb{R}$.

**Definition B.10.** Let $E$ be a real Banach space and let $D$ be a closed subset of $E$. Denote by $\phi(\alpha)$ a family of subsets of $E$ with

$$\phi(\alpha) = \{ G \subseteq E : G \text{ compact, } D \subseteq G, \ \alpha \in \text{Im}(i_* : H_q(G, D) \to H_q(E, D)) \}.$$

$\phi(\alpha)$ is said to be a homological family of dimension $q$ with boundary $D$ if there exist some nontrivial classes $\alpha \in H_q(E, D)$.

The following theorem can be found in [15], [26, Theorem 13.1.7], [6, Theorem 2.1.3, Theorem 2.1.5] and references therein.
Theorem B.11. Let $W$ be a real Hilbert space with orthogonal decomposition $W = X \oplus Y$, where $\dim X < +\infty$. Suppose $f \in C^2(W; \mathbb{R})$ satisfies the following conditions

1. $f$ satisfies $(PS)$ condition.
2. There exist $\rho, \delta > 0$ such that $f(w) \geq \delta$, $w \in \partial B_\rho(0) \cap Y$.
3. There exist $c \in \partial B_1(0) \cap Y$ and $r_0 > \rho > 0$ such that $f(w) < \delta$, $\forall w \in \partial Q$,

where $Q = (B_{r_0}(0) \cap X) \oplus \{re : 0 \leq r \leq r_0\}$, $B_r(0) = \{w \in W : \|w\| \leq r\}$.

Then

1. $f$ possesses a critical value $c \geq \delta$ which is given by $c = \inf_{h \in \Gamma} \max_{w \in Q^1} f(h(w))$, where $\Gamma = \{h \in C(Q,W) : h|_{\partial Q^1} = id\}$.
2. For $h \in \Gamma$, $h(Q)$ is a family of homological dimension $\dim X + 1$ with boundary $\partial Q$.
3. For $w_0 \in K_c(f) = \{w \in W : f'(w) = 0, f(w) = c\}$, the Morse index $m^-(f,w_0)$ of $f$ at $w_0$ and the nullity $m^0(f,w_0)$ of $f$ at $w_0$ satisfy $m^-(f,w_0) \leq \dim X + 1 \leq m^-(f,w_0) + m^0(f,w_0)$.
4. Suppose that there is a $S^1$-action on $X$ and $f$ is $S^1$-invariant. Under the assumption of 2 for $w_0 \in K_c(f)$, the inequality can be improved to $m^-(f,w_0) \leq \dim X + 1 \leq m^-(f,w_0) + m^0(f,w_0) - 1$.

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