Normal Modes, Quasi-normal Modes and Super-radiant Modes for Scalar Fields in Kerr anti-de Sitter Spacetime

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Abstract

Normal modes, quasi-normal modes and super-radiant modes are studied to clarify the total dynamics for complex scalar fields in Kerr anti-de Sitter black hole spacetime. Orthonormal relations and quasi-orthonormal relations are obtained for normal modes and quasi-normal modes. Mode expansions are done and the conserved quantities are studied. Any modes are shown to be separated into two groups, physical modes and unphysical modes, by the zero mode line. Zero modes themselves do not exist as normalizable modes with the correct boundary condition. The allowed physical modes exclude the super-radiant instability modes in rotating black hole spacetime. The result is consistent with the co-rotating frame consideration.

1 Introduction

Black holes have many interesting features in theoretical as well as in observational investigations. As the development of observation techniques and devices, many candidates of black holes have been observed including super-massive black holes in the center of galaxies [1][2][3]. These black holes are expected to be described well by the exact axisymmetric solutions of Einstein’s equation [4]. Schwarzschild, Kerr and Reissner-Nördstrom solutions are known as massive, rotating and charged black holes in (3+1)-dimensional spacetime. Incorporating the cosmological constant to them, Kerr de Sitter and Kerr anti-de Sitter (Kerr-AdS) solutions are also known, which are interesting in views of the recent observation of cosmological term in WMAP [5] and AdS/CFT correspondence [6]. Higher dimensional multi-rotating black hole solutions [7][8][9] are also interesting in views of recent development in string theory [10][11], brane world [12] and M-theory [13].

The interaction of matter fields with black holes are important in observational and theoretical understanding of black holes. Perturbation of matter fields in the black hole spacetime is investigated substantially [14][15][16]. The massless field equations for

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the field of scalar, Dirac, Maxwell, Rarita-Schwinger and tensor are summarized as Teukolsky equations \[17\] and studied extensively \[18, 19\]. One of important investigations is the stability problem of black holes. Perturbative stability has been studied in (3+1) and higher dimensional spacetime \[21\]. The present status of the stability issue for (3+1)-dimensional black holes is that the black holes of Schwarzschild, Kerr, Reisner-Nördstrom are stable \[2\]. As to Kerr-AdS black hole spacetime, large rotating black holes are studied to be stable \[25\] but small black holes are unstable under the condition giving rise to super-radiant instability in the scattering of scalar fields with black holes \[26, 27, 28\]. Asymptotic AdS spacetime plays the role of a natural reflecting mirror which amplifies the scattered wave repeatedly to lead the instability, which is called as black hole bomb \[29\]. For the incident wave with the frequency $\omega$ and the azimuthal angular momentum $m$ upon a target rotating black hole, the scattered wave is amplified if the super-radiant instability condition is satisfied:

$$\text{Re} \omega - \Omega_H m < 0 \quad \text{with} \quad 0 < \text{Re} \omega ,$$

where $\Omega_H$ denotes the angular velocity of black holes at horizon (see Eq.(3.23)). In higher dimensions, the situation is similar to (3+1)-dimensional spacetime so that rapidly rotating Kerr-AdS black holes are reported to be unstable \[30\].

The black hole thermodynamics \[31, 32, 33, 34\] is also an important issue in the interaction of matter fields with black holes. The microscopic understanding of the black hole thermodynamics has been studied extensively in string theory \[35\], conformal field theory and brick wall model \[36\]. The brick wall model is the model to built the brick wall at the horizon in order to confine the scalar fields around the black hole and sum up all the eigenstates to calculate the entropy according to the standard statistical mechanics. In the brick wall model, the problem is that the Boltzmann weight cannot be well-defined if the super-radiant instability occurs in rotating black hole spacetime in (3+1)-dimensions \[37, 38\] and in (2+1)-dimensional BTZ \[39\] black holes \[40, 41, 42, 43\].

In view of these severe super-radiant instability problem, the purpose of this paper is to make clear the dynamics of scalar fields in rotating black hole spacetime as exact as possible in analytical method. We have already studied eigenvalue problem of normal modes for scalar fields analytically and numerically in (2+1)-dimensional BTZ black hole spacetime \[44, 45\] to show that the super-radiant instability does not occur, which is consistent with the negative imaginary part of the quasi-normal frequency by Birmingham \[46\]. We extend our previous method to (3+1)-dimensional rotating black hole spacetime, especially to Kerr-AdS black holes. Our strategy is to study the total dynamics of scalar fields with the Dirichlet or the Neumann boundary condition at horizon for normal modes (see Eq.(3.1)) and with the ingoing (into black hole) boundary condition at horizon for quasi-normal modes (see Eq.(4.1)), in order to clarify the independence and the completeness of each modes. Any modes are

\[1\] Teukolsky equation is categorized as Heun’s equation in Mathematics \[20\].

\[2\] The instability of scalar fields in rotating spacetime without cosmological constant is reported \[22, 23, 24\].
separated into two groups, physical modes and unphysical modes, by the zero mode line defined as

\begin{equation}
0 = \text{Re} \omega - \Omega_{\text{H}m} .
\end{equation}

Zero modes themselves do not exist as normalizable modes with the correct boundary condition. By non-existence of zero modes and assumed analyticity of rotation parameter, the allowed physical mode region is derived:

\begin{equation}
0 < \text{Re} \omega - \Omega_{\text{H}m} ,
\end{equation}

where the rotation parameter is defined for the real value of the horizon $0 < r_- < r_+$ (see Eq. (2.10)). The allowed physical modes exclude the super-radiant instability modes in rotating black hole spacetime.

The organization of this paper is the following. In section 2, Kerr-AdS spacetime and equations of scalar fields will be reviewed and summarized for the following convenience. In section 3, orthonormal relations and completeness relations of normal modes will be studied. Normal mode expansion, quantization and the conserved quantities will also be studied. In section 4, quasi-orthonormal relations for quasi-normal modes will be studied in the similar method for normal modes. The fluxes of energy and angular momentum are also studied. In section 5, zero modes and super-radiant modes will be studied in detail. This section is one of the most important parts of this paper. Final section is to summarize the results. Rotating black holes in co-rotating frame will be studied in appendix.

2 Kerr-AdS spacetime and equations of scalar fields

In this section, Kerr-AdS spacetime and equations of scalar fields are reviewed and summarized for the following convenience. Throughout this paper, the natural unit is used: $c = \hbar = G = 1$.

2.1 Kerr-AdS spacetime

The Einstein-Hilbert action with negative cosmological constant $\Lambda$ is

\begin{equation}
I_G = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - 2\Lambda) .
\end{equation}

The vacuum Einstein’s equations for this action are

\begin{equation}
R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R + g_{\mu\nu}\Lambda = 0 .
\end{equation}
The (3+1)-dimensional Kerr-AdS metric is given by Carter,\(^3\)

\[
ds^2 = -\frac{\Delta r}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\Delta_\theta}{\rho^2} \sin^2 \theta \left( adt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2 + \frac{\Delta_\theta}{\rho^2} \sin^2 \theta \rho^2 \left( a \sin^2 \theta - \frac{r^2 + a^2}{\Xi} d\phi \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 ,
\]

(2.3)

where

\[
\Delta_r = (r^2 + a^2)(1 + r^2 \ell^{-2}) - 2Mr , \quad \Delta_\theta = 1 - a^2 \ell^{-2} \cos^2 \theta , \\
\rho^2 = r^2 + a^2 \cos^2 \theta , \quad \Xi = 1 - a^2 \ell^{-2} ,
\]

(2.4)

with \( \ell = \sqrt{-3/\Lambda} \) denotes the cosmological parameter and \( a = J/M \) denotes the rotation parameter per unit black hole mass \( M \). The metrics in the standard form, \( ds^2 = g_{tt} dt^2 + g_{\phi\phi} d\phi^2 + 2g_{t\phi} dt d\phi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 \), is given by

\[
g_{tt} = \frac{1}{\rho^2} (-\Delta_r + a^2 \sin^2 \theta \Delta_\theta) , \quad g_{t\phi} = \frac{a \sin^2 \theta}{\rho^2 \Xi} (\Delta_r - (r^2 + a^2) \Delta_\theta) , \\
g_{\phi\phi} = \frac{\sin^2 \theta}{\rho^2 \Xi^2} (-a^2 \sin^2 \theta \Delta_r + (r^2 + a^2)^2 \Delta_\theta) , \quad g_{rr} = \frac{\rho^2}{\Delta_r} , \quad g_{\theta\theta} = \frac{\rho^2}{\Delta_\theta} ,
\]

(2.5)

and the square of determinant of the metrics is \( \sqrt{-g} = \rho^2 \sin \theta / \Xi \).

### 2.2 Equations of scalar fields in Kerr-AdS spacetime

The action and the Lagrangian density of complex scalar field \( \Phi \) with mass \( \mu \) as a matter field in the Kerr-AdS spacetime is

\[
I_M = \int d^4 x \sqrt{-g} L_M ,
\]

(2.6)

\[
L_M = -g^{\mu\nu} \partial_\mu \Phi^*(x) \partial_\nu \Phi(x) - \mu^2 \Phi^*(x) \Phi(x) - \xi R \Phi^* \Phi ,
\]

(2.7)

where the non-minimal coupling constant is denoted by \( \xi \) and the scalar curvature takes the value \( R = -12/\ell^2 \). Field equations of scalar fields is

\[
\left( \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu - \bar{\mu}^2 \right) \Phi = 0 ,
\]

(2.8)

where \( \bar{\mu}^2 = \mu^2 - 12\xi/\ell^2 \) denotes the effective mass of scalar field, which is assumed to take a non-negative value. The contravariant metrics of the Kerr-AdS spacetime are

\[
g^{tt} = \frac{1}{\rho^2} \left( a^2 \sin^2 \theta \Delta_\theta - \frac{(r^2 + a^2)^2}{\Delta_r} \right) , \quad g^{t\phi} = \frac{a \Xi}{\rho^2} \left( \frac{1}{\Delta_\theta} - \frac{r^2 + a^2}{\Delta_r} \right) , \\
g^{\phi\phi} = \frac{\Xi^2}{\rho^2} \left( \frac{1}{\Delta_\theta} - \frac{a^2}{\Delta_r} \right) , \quad g^{rr} = \frac{1}{g_{rr}} , \quad g^{\theta\theta} = \frac{1}{g_{\theta\theta}} .
\]

(2.9)

\(^3\)The Carter metric is related to the Boyer-Lindquist metric \( 48 \): \( t_{BL} = \Xi t_{Carter} \) and \( \omega_{BL} = \omega_{Carter} / \Xi \).
One of zeros of $1/g^{tt}$ and $g^{rr}$ denote the horizons of black hole, namely,

$$
\Delta_r = (r - r_-)(r - r_+)(r - r_\ell)(r - r_\ell^*)/\ell^2 ,
$$

(2.10)

where $r_-, r_+$ take real numbers, $r_\ell, r_\ell^*$ take complex numbers, which are assigned such that $r_\pm \to M \pm \sqrt{M^2 - a^2}, r_\ell \to i\ell$ and $r_\ell^* \to -i\ell$ in the limit $\ell \to \infty (\Lambda \to 0)$. The event horizon corresponds to $r_+$. As the background Kerr-AdS metrics do not depend on time and azimuthal angle variables, the scalar field solution is put in the form of separation of variables:

$$
\Phi = \frac{1}{\sqrt{2\pi}} e^{-i\omega t} e^{im\phi} S(\theta) R(r) ,
$$

(2.11)

where the frequency and azimuthal angular momentum are denoted by $\omega$ (complex value in general) and $m$ (integer value). Field equations for angular and radial parts become in the form:

$$
\left( \frac{\partial \sin \theta \Delta_r \partial \theta}{\sin \theta} - \frac{(a\omega \sin \theta - \Xi m/\sin \theta)^2 - \bar{\mu}^2 a^2 \cos^2 \theta + \lambda}{\Delta_r} \right) S(\theta) = 0 \quad (2.12)
$$

$$
\left( \partial_r \Delta_r \partial_r + \frac{(r^2 + a^2)\omega - \Xi a m)^2}{\Delta_r} - \bar{\mu}^2 r^2 - \lambda \right) R(r) = 0, \quad (2.13)
$$

where $\lambda$ denotes the separation parameter $^4$.

### 3 Normal modes of scalar fields

In this section, we consider the eigenstate problem of normal modes for scalar fields in Kerr-AdS spacetime, which provides to analyze the super-radiant instability problem in the following sections. The boundary conditions on field equations are imposed and orthonormal relations among eigenfunctions for normal modes will be derived. The final part of this section, scalar fields are quantized and conserved quantities are obtained.

#### 3.1 Boundary conditions

For the radial wave function, the Dirichlet boundary condition is imposed at infinity, because the spacetime is asymptotically AdS spacetime, and the Dirichlet or the Neumann boundary condition is imposed at the horizon to obtain normalized states:

$$
R(r) \to 0 \quad \text{for} \quad r \to \infty ,
$$

$$
R(r) = 0 \quad \text{or} \quad \frac{d}{dr} R(r) = 0 \quad \text{for} \quad r = r_+ .
$$

(3.1)

$^4$ It should be noted that field equations for angular and radial parts of Eqs.(7)-(8) in the paper by Cardoso et al. [27] (similarly in [28]) are different from ours of Eqs. (2.12)-(2.13) though the same metric notation is used: Eqs.(2)-(3) in [27] and Eqs.(2.3)-(2.4) in our paper. Their field equations cannot derive the general form of the Klein-Gordon inner product relations of Eq.(3.11) nor the current conservation of particle number of Eq.(4.13).
For the azimuthal angular function, the periodic boundary condition is imposed at 
\( \phi = 0 \) and \( 2\pi \), which is satisfied for integer values of \( m \). For the polar angular function, 
the Dirichlet or the Neumann boundary condition is imposed at \( \theta = 0 \) and \( \pi \).

### 3.2 Orthonormal relations

From field equations of Eqs. (2.13) and (2.13), two identity equations in bi-linear forms 
of fields are obtained:

\[
\int_0^\pi \! d\theta \sin \theta \left( (\omega^* - \omega')(\omega^* + \omega') a^2 \sin^2 \theta - 2ma\Xi \right) \frac{\Delta \theta}{\Delta r} \times S_{\omega,m,\lambda}(\theta) S_{\omega',m,\lambda'}(\theta) = 0, \tag{3.2}
\]

for angular part and

\[
\int_{r_+}^\infty \! dr \left( (\omega^* - \omega')(\omega^* + \omega') (r^2 + a^2)^2 - 2ma\Xi (r^2 + a^2) \right) \frac{\Delta r}{\Delta \theta} \times R_{\omega,m,\lambda}(r) R_{\omega',m,\lambda'}(r) = 0, \tag{3.3}
\]

for radial part. From these identity equations, relations for the product of bi-linear 
forms are obtained:

\[
(\omega^* - \omega') X_{(\omega,\omega',\lambda,\lambda')} = (\lambda^* - \lambda') X_{(\omega,\omega',\lambda,\lambda')} = 0, \tag{3.4}
\]

where \( X_{(\omega,\omega',\lambda,\lambda')} \) is defined as

\[
\begin{align*}
X_{(\omega,\omega',\lambda,\lambda')} & := \int_0^\pi \! d\theta \int_{r_+}^\infty \! dr \sqrt{-g} \\
& \quad \times \left( (\omega^* + \omega') \left( -a^2 \sin^2 \theta \frac{\Delta \theta}{\Delta r} + \frac{(r^2 + a^2)^2}{\Delta \theta} \right) + 2ma\Xi \left( \frac{1}{\Delta \theta} - \frac{r^2 + a^2}{\Delta r} \right) \right) \\
& \quad \times S_{\omega,m,\lambda}(\theta) S_{\omega',m,\lambda'}(\theta) R_{\omega,m,\lambda}(r) R_{\omega',m,\lambda'}(r). \tag{3.5}
\end{align*}
\]

According to the usual eigenvalue problems, the following cases are considered:

(i) If \( \omega = \omega' \) and \( \lambda = \lambda' \), real values of frequencies and separation parameters are 
obtained: \( \omega^* = \omega \) and \( \lambda^* = \lambda \), because \( X_{(\omega,\omega',\lambda,\lambda')} \neq 0 \).

(ii) If \( \omega \neq \omega' \) and \( \lambda \neq \lambda' \), orthogonal relations among different frequencies and 
separation parameters are obtained: \( X_{(\omega,\omega',\lambda,\lambda')} = 0 \).

Combining (i) and (ii), orthonormal relations among angular and radial eigenfunctions 
for same azimuthal angular momenta \( m \) are obtained: \( X_{(\omega,\omega',\lambda,\lambda')} = \delta_{\omega,\omega'} \delta_{\lambda,\lambda'} \), or in an 
explicit form\(^5\)

\[
\int_0^\pi \! d\theta \int_{r_+}^\infty \! dr \sqrt{-g} \left( -g^{tt}(\omega^* + \omega') + 2g^{t\phi} m \right)
\]

\(^5\)In deriving Eq.(3.6), the following relation is used:

\[
- g^{tt}(\omega^* + \omega') + 2g^{t\phi} m = \frac{(\omega^* + \omega')}{\rho^2} \left( -a^2 \sin^2 \theta \frac{\Delta \theta}{\Delta r} + \frac{(r^2 + a^2)^2}{\Delta \theta} \right) + 2ma\Xi \left( \frac{1}{\Delta \theta} - \frac{r^2 + a^2}{\Delta r} \right). \]
\[ S_{\omega,m,\lambda}(\theta)S_{\omega',m,\lambda'}(\theta)R^*_{\omega,m,\lambda}(r)R_{\omega',m,\lambda'}(r) = \delta_{\omega,\omega'}\delta_{\lambda,\lambda'} \cdot (3.6) \]

Similarly orthogonal relations for positive and negative values of azimuthal angular momenta, \( m, -m \) are obtained:
\[
\int_{r_+}^\infty dr \int_0^\pi d\theta \sqrt{-g} \left( -g^{tt}(\omega - \omega') + 2g^{t\phi}m \right) \times S_{\omega,m,\lambda}(\theta)S_{\omega',-m,\lambda'}(\theta)R^*_{\omega,m,\lambda}(r)R_{\omega',-m,\lambda'}(r) = 0 \cdot (3.7) \]

The full eigenfunctions are defined:
\[
f_{\omega,m,\lambda} := \frac{1}{\sqrt{2\pi}} e^{-i\omega t} e^{im\phi} S_{\omega,m,\lambda}(\theta)R_{\omega,m,\lambda}(r) \cdot (3.8)\]

and the full orthonormal relations are obtained from Eqs. (3.6)-(3.7) as
\[
\int_\Sigma d^3x \sqrt{-g} \left( -ig^{t\nu} \left( A^*(t,x)\partial_\nu B(t,x) - \partial_\nu A^*(t,x)B(t,x) \right) \right) = 0 \cdot (3.9)\]

where the integration region \( \Sigma \) is \( 0 \leq \phi < 2\pi, 0 \leq \theta \leq \pi \) and \( r_+ + \epsilon \leq r < \infty \). The cutoff parameter \( \epsilon \) is introduced to regularize the divergent integration region due to the factor \( g^{tt} \), which is considered in the brick wall model [36]. In the following, the parameter \( \epsilon \) is omitted to write explicitly but it is understood to be recovered if necessary.

The general Klein-Gordon inner product is introduced as a compact notation:
\[
< A , B > := \int_\Sigma d^3x \sqrt{-g} \left( -ig^{t\nu} \left( A^*(t,x)\partial_\nu B(t,x) - \partial_\nu A^*(t,x)B(t,x) \right) \right) = (3.10)\]

where space coordinates are denoted as \( x = (r, \theta, \phi) \) and time coordinate as \( t \). The general form of orthonormal relations are written as
\[
< f_\alpha , f_\alpha' > = - < f_\alpha^* , f_\alpha' > = \delta^{(3)}_{\alpha,\alpha'} \cdot < f_\alpha^* , f_\alpha' > = < f_\alpha , f_\alpha' > = 0 \cdot (3.11)\]

where the eigenvalues for normal modes are denoted: \( \alpha := (\omega, m, \lambda) \), \( \alpha' := (\omega', m', \lambda') \) and \( \delta^{(3)}_{\alpha,\alpha'} := \delta_{\omega,\omega'}\delta_{m,m'}\delta_{\lambda,\lambda'} \). It is worthwhile to note that orthonormal relations hold in product forms of angular and radial parts (3.6)-(3.7) but not in separate forms. It is also note that general orthonormal relations (3.11) are the same as those of the general Klein-Gordon inner product relations [14].

### 3.3 Normal mode expansion and quantization

The canonical momentum conjugate to scalar field is given by
\[
\Pi := \frac{\partial L_M}{\partial \dot{\Phi}} = -g^{t\nu}\partial_\nu\Phi^\dagger = -(g^{tt}\partial_t\Phi^\dagger + g^{t\phi}\partial_\phi\Phi^\dagger) \cdot (3.12)\]
where $\dagger$ denotes the Hermitian conjugate operation. Scalar fields and conjugate momenta are expressed in the normal mode expansion as
\[
\Phi(t, x) = \sum_\alpha \left( a_\alpha f_\alpha(t, x) + b_\alpha^\dagger f^*_\alpha(t, x) \right),
\]
\[
\Pi(t, x) = -i \sum_\alpha \left( g^{tt} \omega - g^{t\phi} m \right) (a_\alpha^\dagger f^*_\alpha(t, x) - b_\alpha f_\alpha(t, x)).
\]
(3.13)
The equal time commutation relations among fields and their momenta are imposed:
\[
[\Phi(t, x), \Pi(t, x')] = \Pi^\dagger(t, x), \Pi^\dagger(t, x') = i \sqrt{-g} \delta^{(3)}(x - x'),
\]
(3.14)
and others are zeros. From these commutation relations, commutation relations among annihilation and creation operators are derived:
\[
[a_\alpha, a^\dagger_{\alpha'}] = [b_\alpha, b^\dagger_{\alpha'}] = \delta^{(3)}_{\alpha, \alpha'},
\]
(3.15)
and others are zeros, where the completeness relations are used:
\[
\sum_\alpha (g^{tt} \omega + g^{t\phi} m) (f_\alpha(t, x)f^*_{\alpha'}(t, x') + f^*_{\alpha'}(t, x)f_\alpha(t, x')) = \frac{1}{\sqrt{-g}} \delta^{(3)}(x - x'),
\]
\[
\sum_\alpha (f_\alpha(t, x)f^*_{\alpha'}(t, x') - f^*_{\alpha'}(t, x)f_\alpha(t, x')) = 0.
\]
(3.16)
It is noted that the commutation relations in the Kerr-AdS spacetime are the same form as those in the Minkowski spacetime except for the metric determinant factor $1/\sqrt{-g}$.

### 3.4 Energy and angular momentum

Because the metrics are independent of time and azimuthal angle, two Killing vectors exist: $\xi^\mu_{(t)} = \partial/\partial t$, $\xi^\mu_{(\phi)} = \partial/\partial \phi$. Defining the energy-momentum tensor
\[
T_{\mu\nu} := -\frac{2}{\sqrt{-g}} \delta I_M \delta g^{\mu\nu} - \partial_\mu \Phi \partial_\nu \Phi + \partial_\nu \Phi^\dagger \partial_\mu \Phi - g_{\mu\nu} (g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi + \bar{\mu}^2 \Phi^\dagger \Phi) ,
\]
(3.17)
local conservation laws hold for each Killing vector
\[
\partial_\nu (\sqrt{-g} \xi^\mu_{(i)} T^\nu_\mu) = 0 , \text{ for } i = t, \phi.
\]
(3.18)
Corresponding conservative quantities are energy and angular momentum:
\[
E = -\int_{\Sigma} d^3 x \sqrt{-g} \xi^\mu_{(t)} T^t_\mu = \int_{\Sigma} d^3 x \sqrt{-g} \left( -g^{tt} \partial_t \Phi \partial_t \Phi + g^{t\phi} \partial_t \Phi \partial_\phi \Phi \right)
\]
\[
+ g^{rr} \partial_r \Phi \partial_r \Phi + g^{\theta\phi} \partial_\phi \Phi \partial_\theta \Phi + 2 g^{t\phi} \partial_\phi \Phi \partial_t \Phi \right) ,
\]
(3.19)
\[
L = \int_{\Sigma} d^3 x \sqrt{-g} \xi^\mu_{(\phi)} T^t_\mu = \int_{\Sigma} d^3 x \sqrt{-g} \left( g^{t\phi} \partial_t \Phi \partial_\phi \Phi + 2 g^{t\phi} \partial_\phi \Phi \partial_t \Phi \right) .
\]
(3.20)
They are expressed by creation and annihilation operators using the normal node expansion as

\[ E = \sum_\alpha \omega (a_\alpha^\dagger a_\alpha + b_\alpha b_\alpha^\dagger) , \quad L = \sum_\alpha m (a_\alpha^\dagger a_\alpha + b_\alpha b_\alpha^\dagger) . \quad (3.21) \]

In combining these conserved quantities, the effective energy is defined, which is the energy taking into the rotation effect on the horizon, as

\[ E - \Omega_H L = \sum_\alpha (\omega - \Omega_H m) (a_\alpha^\dagger a_\alpha + b_\alpha b_\alpha^\dagger) , \quad (3.22) \]

where the angular velocity on the horizon is defined:

\[ \Omega_H := \lim_{r \to r_+} \frac{g^{rr}}{g^{\phi\phi}} = \frac{a\Xi}{r_+^2 + a^2} . \quad (3.23) \]

From the expression in Eq.(3.22), the effective energy \( E - \Omega_H L \) is positive definite if \( 0 < \omega - \Omega_H m \). This condition is important on the super-radiant instability and the definition of the statistical mechanics for scalar fields in Kerr-AdS spacetime, which will be studied in section 5 and in appendix.

4 Quasi-normal modes of scalar fields

In this section, we consider quasi-normal modes of scalar fields in Kerr-AdS spacetime to extend normal modes results in the previous section. Quasi-normal modes are also important in analyzing the stability problem. Our treatment of boundary conditions and initial conditions for quasi-normal modes is natural extension of the case for normal modes.

4.1 Boundary conditions and initial condition

We impose the Dirichlet boundary condition at infinity and the ingoing (into black holes) boundary condition at horizon on the radial wave function for quasi-normal modes:

\[ R(r) \to \begin{cases} 0 & \text{for } r \to \infty \\ \exp(-i(\omega - \Omega_H m)r_*) & \text{for } r = r_+ \end{cases} \quad (4.1) \]

where \( r_* \) denotes the Regge-Wheeler tortoise coordinate:

\[ r_* := \int^r \frac{dr}{\Delta r} \simeq \begin{cases} \frac{\ell^2(r_+^2 + a^2) \ln(r - r_+)}{(r_+ - r_ - r_\ell)(r_+ - r_+^*)} & \text{for } r \approx r_+ \\ -\frac{\ell^2}{r} & \text{for } r \to \infty \end{cases} . \quad (4.2) \]

As the initial condition for orthonormal relations of quasi-normal modes, we impose them to be coincident with those for normal modes in Eq.(3.9) or Eq.(3.11) in case of real values of \( \omega \) and \( \lambda \).
4.2 Quasi-orthonormal relations

In order to obtain quasi-orthonormal relations for quasi-normal modes, we follow calculations of orthonormal relations for normal modes in keeping the boundary term in Eq.(4.3). The bi-linear identity is obtained with the boundary contribution as

\[ (\omega^* - \omega') \int_{r_+}^{\infty} dr \int_{0}^{\pi} d\theta \sqrt{-g} \left( -g^{tt}(\omega^* + \omega') + 2g^{t\phi}m \right) S_{t\alpha}^*(\theta) R_{\alpha}(r) S_{t\alpha'}(\theta) R_{\alpha'}(r) \]

\[ = \int_{0}^{\pi} d\theta \sin \theta \left( S_{t\alpha}^*(\theta) S_{t\alpha'}(\theta) \Delta_r \left( R_{t\alpha}^*(r) \frac{d}{dr} R_{t\alpha'}(r) - \frac{d}{dr} R_{t\alpha}(r) R_{t\alpha'}(r) \right) \right) \bigg|_{r_+}^{\infty}, \quad (4.3) \]

where \( \bar{\alpha} = (\omega, m, \lambda) \) and \( \bar{\alpha}' = (\omega', m, \lambda') \) (same \( m \)) with complex values of \( \omega \) and \( \lambda \). Using the general form of eigenfunctions \( f_\alpha \) in Eq.(4.8), the bi-linear identity in Eq.(4.3) is rewritten in the form

\[ \frac{d}{dr} \int_{\Sigma} dr d\theta d\phi \sqrt{-g} g^{\nu\tau} (f_\alpha^* \partial_\nu f_{\alpha'} - \partial_\nu f_\alpha^* f_{\alpha'}) \]

\[ = \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\phi \sqrt{-g} g^{\nu\tau} (f_\alpha^* \partial_\nu f_{\alpha'} - \partial_\nu f_\alpha^* f_{\alpha'}) \bigg|_{r_+}^{\infty} + C_{\alpha,\alpha'}, \quad (4.4) \]

with \( \alpha = (\omega, m, \lambda) \) and \( \alpha' = (\omega', m, \lambda') \). Integrating this equation, the general form of the identity is obtained:

\[ -i \int_{\Sigma} dr d\theta d\phi \sqrt{-g} g^{\nu\tau} (f_\alpha^* \partial_\nu f_{\alpha'} - \partial_\nu f_\alpha^* f_{\alpha'}) \]

\[ = i \int_{0}^{t} dt \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\phi \sqrt{-g} g^{\nu\tau} (f_\alpha^* \partial_\nu f_{\alpha'} - \partial_\nu f_\alpha^* f_{\alpha'}) \bigg|_{r_+}^{\infty} + C_{\alpha,\alpha'}, \quad (4.5) \]

where the integration constant is determined by the initial value of bilinear forms:

\[ C_{\alpha,\alpha'} := -i \int d\Sigma d\theta d\phi \sqrt{-g} g^{\nu\tau} (f_\alpha^* \partial_\nu f_{\alpha'} - \partial_\nu f_\alpha^* f_{\alpha'}) \bigg|_{t=0}, \quad (4.6) \]

with \( C_{\alpha,\alpha} = 1 \) as the normalization condition for quasi-normal modes. Defining the quasi-inner product by

\[ <\!\! A, B \!\!> := \int d^3x \sqrt{-g} (-ig^{\nu\tau})(A^*(t,x) \partial_\nu B(t,x) - \partial_\nu A^*(t,x) B(t,x)) \]

\[ + \int_{0}^{t} dt \int_{0}^{\pi} d\theta \int_{0}^{2\pi} d\phi \sqrt{-g} (-ig^{\nu\tau})(A^*(t,x) \partial_\nu B(t,x) - \partial_\nu A^*(t,x) B(t,x)) \bigg|_{r_+}^{\infty}, \quad (4.7) \]

the quasi-orthonormal relations are expressed:

\[ <\!\! f_\alpha, f_{\alpha'} \!\!> = C_{\alpha,\alpha'} . \quad (4.8) \]

Similar relations are also obtained:

\[ <\!\! f_\alpha^*, f_{\alpha'}^* \!\!> = C_{\alpha,\alpha'}^* , \quad <\!\! f_\alpha^*, f_{\alpha'} \!\!> = D_{\alpha,\alpha'}, \quad <\!\! f_{\alpha}, f_{\alpha'} \!\!> = D_{\alpha,\alpha'}^* . \quad (4.9) \]
where the other integration constants are defined:

\[
D_{\alpha,\alpha'} := -i \left. \int_{\Sigma} \rho d\phi \sqrt{-g} g^{\nu\mu} (f_\alpha \partial_\nu f_{\alpha'} - \partial_\nu f_\alpha f_{\alpha'}) \right|_{t=\alpha}.
\] (4.10)

Here we consider a single quasi-normal mode of \(0 < \text{Re} \omega - \Omega H m\), which is normalized to one at the initial time

\[
\int d^3x \sqrt{-g} (-2)(g^{tt} \text{Re} \omega - g^{\phi \phi} m)|f_\alpha|^2 = 1 \quad \text{at} \quad t = 0,
\] (4.11)

according to the initial condition in Eq.(4.6). This mode decreases with time to tend to zero at \(t = \infty\). On the other hand, the boundary term increases from zero to one as the flux flows into black holes, which gives the sum rule for the imaginary part of the frequency:

\[
\text{Im} \omega = -\frac{1}{2} \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{-g} (-i g^{rr})(f^*_\alpha \partial_r f_\alpha - \partial_r f^*_\alpha f_\alpha) \bigg|_{r_+}^\infty
\]
\[
= -\int_0^\pi d\theta \frac{\sin \theta}{2} (r_+^2 + a^2)(\text{Re} \omega - \Omega H m)|S_\alpha(\theta) R_\alpha(r_+)|^2.
\] (4.12)

This sum rule means that the total flux flowing into black holes determines the imaginary part of the quasi-normal mode. We see that \(\text{Im} \omega\) is negative for \(0 < \text{Re} \omega - \Omega H m\). We will discuss this point again in the connection with the non-existence of zero mode in section 5. It is worthwhile to note that the conditions \(0 < \text{Re} \omega - \Omega H m\) and \(0 < \text{Im} \omega\) are supported by the co-rotating frame consideration: Eqs.(A.9)-(A.10) in appendix.

These quasi-inner product relations for quasi-normal modes are the extension of the inner products for normal modes to include boundary terms. It is noted that the quasi-orthonormal relations of Eqs.(4.8)-(4.9) are consistent with the current conservation of particle number:

\[
\partial_{\mu} (-i g^{\mu\nu}(\Phi^* \partial_\nu \Phi - \partial_\nu \Phi^* \Phi)) = 0.
\] (4.13)

Related to quasi-orthonormal relations, it is also noted that the completeness of quasi-normal modes in the sense of normal modes was studied by Kokkotas and Schmidt [16] and by Price and Husain [49].

### 4.3 Flux of energy and angular momentum

The energy and angular momentum of scalar fields are conserved in time for normal modes but they vary with time for quasi-normal modes. The time derivative of energy and momentum are expressed by the flux of them:

\[
\frac{dE}{dt} = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{-g} \epsilon^\mu_s(t) T_\mu \bigg|_{r_+}^\infty
\]
\[
= \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{-g} g^{rr} \left( \partial_\phi \Phi^\dagger \partial_r \Phi + \partial_r \Phi^\dagger \partial_\phi \Phi \right) \bigg|_{r_+}^\infty,
\] (4.14)
\[
\frac{dL}{dt} = - \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{-g} \left[ \left. \frac{\partial_r \Phi^* \partial_r \Phi + \partial_r \Phi \partial_r \Phi}{T_r^\mu} \right|_{\theta=0} \right]_r^\infty.
\]

(4.15)

From these expressions, the time derivative of the effective energy is shown to be negative definite as

\[
\frac{d(E - \Omega H L)}{dt} = -2 \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{\sin \theta (r_+^2 + a^2)}{2} |(\partial_t + \Omega H \partial_\phi)\Phi|^2.
\]

(4.16)

For a single quasi-normal mode with \( \alpha = (\omega, m, \lambda) \), the time derivative of the effective energy satisfies the relation

\[
\frac{d(E_\alpha - \Omega H L_\alpha)}{dt} = 2\text{Im} \omega (E_\alpha - \Omega H L_\alpha),
\]

(4.17)

where

\[
E_\alpha - \Omega H L_\alpha = (\text{Re} \omega - \Omega H m) \int d^3x \sqrt{-g} (-2)(g^{tt}\text{Re} \omega - g^{t\phi}m)|f_\alpha|^2,
\]

(4.18)

which is positive definite for \( 0 < \text{Re} \omega - \Omega H m \). Together with the negative definiteness of \( d(E - \Omega H L)/dt \) of Eq. (4.16) and the positive definiteness of the effective energy of Eq. (4.18) for \( 0 < \text{Re} \omega - \Omega H m \), \( \text{Im} \omega \) is negative definite consistent with the sum rule of Eq. (4.12).

5 Zero modes and super-radiant modes

In this section, we study zero modes defined as \( \text{Re} \omega - \Omega H m = 0 \) \( (-\infty < m < \infty) \) and super-radiant unstable modes \( \text{Re} \omega - \Omega H m \leq 0 \) \( (0 < \text{Re} \omega \text{ and } -\infty < m < \infty) \) relating to normal modes and quasi-normal modes studied in previous sections. Each mode is specified by a set of values; \( \omega, m \) and \( \lambda \), where \( \omega \) and \( \lambda \) can be complex values and \( m \) takes a integer number. The separation parameter \( \lambda \) is considered to be fixed value throughout in this section. Some statements are provided in the following.

**Statement 1** Boundary terms for zero modes vanish, and the frequency and the separation parameter of zero modes take real values.

[Proof] The identity for the boundary term

\[
-ig^{rr} (f^*_\alpha(x)\partial_r f^\alpha(x) - \partial_r f^*_\alpha(x)f^\alpha(x))
\]

\[
= -2(\text{Re} \omega - \Omega H m) \frac{r^2 + a^2}{r_+^2 + a^2 \cos^2 \theta} f^*_\alpha(x)f^\alpha(x) \text{ at } r = r_+,
\]

(5.1)

shows that boundary contributions in Eqs. (4.13) and (4.31) vanish for zero modes regardless of boundary conditions. The zero modes are special modes in the sense that
they are not ingoing or outgoing modes (or oscillating modes) but the linear function with respect to the tortoise coordinate:

$$R_{\text{zero}} \simeq d_1 + d_2 r_\ast \text{ near } r \simeq r_+,$$

(5.2)

where $d_1, d_2$ are integration constants. The frequency and the separation parameter for zero modes take real values according to the similar argument for normal modes in section 3.

**Statement 2** A mode of $(\omega, m)$ has its reflection symmetric partner mode $(-\omega, -m)$. The zero mode line $0 = \Re \omega - \Omega_H m$ in $\omega - m$ plane is invariant under the reflection transformation.

**Proof** Field equations of Eqs. (2.12) - (2.13) are invariant under the reflection transformation: $\omega \rightarrow -\omega, m \rightarrow -m$ and any solutions form pair modes: $(\omega, m)$ and $(-\omega, -m)$. The zero mode line $0 = \Re \omega - \Omega_H m$ in $\omega - m$ plane is transformed into itself by the reflection transformation and is reflection symmetric.

**Statement 3** If a mode $(\omega, m)$ is a physical mode, its reflection partner $(-\omega, -m)$ is an unphysical mode. Zero modes do not exist as physical modes or unphysical modes.

**Proof** Generally if a mode $(\omega, m)$ is a physical mode described by a particle annihilation operator $a_\alpha$, its reflection partner $(-\omega, -m)$ is an unphysical mode described by an antiparticle creation operator $b^+_\alpha$ (see Eq. (3.13)) according to the usual relativistic theory [50]. The zero mode line $0 = \Re \omega - \Omega_H m$ in $\omega - m$ plane cannot be divided into two parts, physical and unphysical modes, because the zero mode line is invariant under the reflection transformation.

As another evidence for the non-existence of zero modes, the constant zero mode of the first term in Eq. (5.2) (a candidate of zero modes) cannot be normalized because the suppression factor $r_2^2 - r_\ast^2$ appears in the integrand of the normalization equation (3.6):

$$\int_{r_\ast}^{\infty} dr \int_0^\pi d\theta \sqrt{-g} \frac{2g^{tt} \Xi m (r_2^2 - r_\ast^2)}{(r_2^2 + a^2) (r_\ast^2 + a^2)} |S_{\text{zero}}(\theta) R_{\text{zero}}(r)|^2 \simeq 1,$$

(5.3)

where the near horizon approximation is applied because of the enhanced factor $g^{tt}$ in the integrand. The second term in Eq. (5.2) (another candidate of zero modes) do not satisfy the boundary conditions. Therefore any zero mode solutions cannot satisfy the boundary conditions or the normalization condition.

**Statement 4** The allowed Physical mode region is $0 < \Re \omega - \Omega_H m$ (Re $\omega < 0$ or $0 < \Re \omega$ and $-\infty < m < \infty$).

**Proof** Consider first the non-rotating case with zero black hole rotation parameter: $J = aM = 0$. Physical modes are $0 < \Re \omega$ ($-\infty < m < \infty$) and unphysical modes are reflection symmetric modes: $\Re \omega < 0$ ($-\infty < m < \infty$). We assume that physical modes are analytic with respect to the rotation parameter $J$. Consider next the rotating case. Physical modes shift from $0 < \Re \omega$ to $0 < \Re \omega - \Omega_H m$ ($-\infty < m < \infty$) because any physical modes cannot cross the zero mode line during the change of the
rotation parameter from $0 = J$ to $J \neq 0$. It is noted that the value of $J$ is defined for the real value of horizon: $0 < r_- < r_+$. We know that one of important roles of the zero mode line is to separate physical modes from unphysical modes. This statement is consistent with the co-rotating frame consideration of Eq.(A.9) in appendix.

**Statement 5** The imaginary part of the frequency of quasi-normal modes become negative and scalar fields in Kerr-AdS spacetime is stable.

[Proof] The sum rule for the imaginary part in Eq.(4.12) and the effective energy flow in Eqs.(4.17)-(4.18) combining the negative definite expression of the effective energy in Eq.(4.16) show that the imaginary part of the frequency for zero modes is negative. This statement is also consistent with the co-rotating frame consideration of Eq.(A.10) in the appendix.

**Statement 6** The super-radiant instability for $\text{Re}\omega - \Omega Hm < 0$ with $0 < \text{Re}\omega$ does not occur though the stable super-radiant modes for $0 < \text{Re}\omega - \Omega Hm$ with $\text{Re}\omega < 0$ exist as physical modes.

[Proof] According to Statement 4, unstable super-radiant modes for $\text{Re}\omega - \Omega Hm < 0$ with $0 < \text{Re}\omega$ are unphysical modes and their reflection partners $0 < \text{Re}\omega - \Omega Hm$ with $\text{Re}\omega < 0$ are physical modes, which are considered as stable super-radiant modes.

**Statement 7** The statistical mechanics for scalar fields around Kerr-AdS spacetime is well-defined, namely, the partition function by Hartle and Hawking \[51\]

\[
Z = \text{Tr} \exp \left( -\beta_H (E - \Omega H L) \right), \tag{5.4}
\]

becomes well-defined.

[Proof] The effective energy in Eq.(3.22) becomes positive definite $0 < E - \Omega H L$ for the allowed physical mode region $0 < \text{Re}\omega - \Omega Hm$. This establishes the brick wall model for rotating black holes \[52\] \[53\].

6 **Summary**

We have studied normal modes, quasi-normal modes, zero modes and super-radiant modes for scalar fields in (3+1)-dimensional Kerr-AdS in order to make clear the scalar perturbation around rotating black hole spacetime.

For normal modes, radial eigenfunctions and polar angle eigenfunctions are shown to satisfy the orthonormal relations in the product forms Eqs.(3.6)-(3.7) but not in the separate forms. Each normal mode is specified by a set of real values $(\omega, m, \lambda)$. Quantum scalar fields are expressed in the normal mode expansion with the creation and annihilation operators. The effective energy $E - \Omega H L$ is shown to be positive definite for $0 < \omega - \Omega Hm$.

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6 Related to the non-normalizability of zero modes (Statement 3) and the allowed physical mode region (Statement 4), orthonormal relation of Eq.(3.9) or Eq.(3.11) and quasi-orthonormal relations of Eqs. (1.8)-(1.9) are recognized to be valid for physical modes: $0 < \text{Re}\omega - \Omega Hm$. 

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For quasi-normal modes, the quasi-orthonormal relations are obtained taking account of the boundary effect and are consistent with the current conservation of particle number. The sum rule for the imaginary part of frequency is obtained in Eq. (4.12), which shows the negative value of $\text{Im}\,\omega$. The flux of the effective energy is shown to be negative definite in Eq. (4.16). The imaginary part of the frequency is again to be negative using the relation of Eq. (4.17) and the positive value of $E - \Omega_H L$ in Eq. (4.18) for $0 < \text{Re}\,\omega - \Omega_H m$. The negative value of $\text{Im}\,\omega$ indicates that the scalar fields in the Kerr-AdS spacetime is stable.

For zero modes: $0 = \omega - \Omega_H m$, they are shown not to exist because the zero mode line in $0 = \omega - m$ plane is invariant under the reflection transformation and cannot be identified as physical modes or unphysical modes. Other evidence for the non-existence of zero mode is that they cannot satisfy the boundary condition or the normalization condition. Physical modes are shown to be $0 < \omega - \Omega_H m$, because of the non-existence of zero modes and the analyticity of rotation parameter $J$. (The value of $J$ is allowed for well-defined values of the horizon $0 < r_- < r_+$. ) This fact implies that the important role of the zero mode line is to separate physical modes from unphysical modes.

For super-radiant modes, unstable super-radiant modes $\text{Re}\,\omega - \Omega_H m < 0$ with $0 < \text{Re}\,\omega$ do not occur but stable super-radiant modes $0 < \text{Re}\,\omega - \Omega_H m$ with $\text{Re}\,\omega < 0$ can occur.

The results are consistent with orthonormal relations (Eq. (3.9)) or Eq. (3.11), quasi-orthonormal relations Eqs. (4.8)-(4.9) and the co-rotating frame consideration: $0 < \text{Re}\,\omega - \Omega_H m$ (Eq. (A.9)) and $\text{Im}\,\omega < 0$ (Eq. (A.10)) in appendix. The results for (3+1)-dimensional Kerr-AdS spacetime are also consistent with those for (2+1)-dimensional BTZ black hole spacetime \cite{44, 52}. As our method does not essentially depend on spacetime dimensionality, the application to higher dimensional rotating black holes will be possible and interesting \cite{54}. Explicit construction and numerical analysis of eigenvalue solutions for scalar fields in Kerr-AdS spacetime will be published in a separate paper \cite{55}.

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\section*{A Co-rotating frame}

In this appendix, we consider the co-rotating coordinate, which diagonalize the metric in $t - \phi$ spacetime as

$$g_{tt}dt^2 + g_{t,\phi}dt d\phi + g_{\phi,\phi} d\phi^2 = g_{\tilde{t},\tilde{t}}d\tilde{t}^2 + g_{\tilde{\phi},\tilde{\phi}}d\tilde{\phi}^2,$$

where the transformation is defined:

$$(d\tilde{t}, d\tilde{\phi}) = (dt, d\phi) S,$$
with the transformation matrix
\[ S := \frac{\sqrt{r^2 + a^2}}{\rho} \begin{pmatrix} 1 & -a\Xi/(r^2 + a^2) \\ -a\sin^2\theta/\Xi & 1 \end{pmatrix}, \]  
(A.3)
and the diagonal metrics are obtained:
\[ \begin{pmatrix} g_{\tilde{t}\tilde{t}} & 0 \\ 0 & g_{\tilde{\phi}\tilde{\phi}} \end{pmatrix} = \begin{pmatrix} -\Delta_r/(r^2 + a^2) & 0 \\ 0 & \Delta_\theta (r^2 + a^2)\sin^2\theta/\Xi^2 \end{pmatrix}. \]  
(A.4)
Under the transformation of Eq.(A.3), the Klein-Gordon operator for \( t - \phi \) components is also diagonalized:
\[ g^{tt}\partial_t^2 + 2t\phi\partial_t\partial_\phi + g^{\phi\phi}\partial_\phi^2 = \tilde{g}^{tt}\partial_{\tilde{t}}^2 + \tilde{g}^{\tilde{\phi}\tilde{\phi}}\partial_{\tilde{\phi}}^2, \]  
(A.5)
with the diagonal inverse metrics: \( g_{\tilde{t}\tilde{t}} = 1/g^{\tilde{t}\tilde{t}}, \ g_{\tilde{\phi}\tilde{\phi}} = 1/g^{\tilde{\phi}\tilde{\phi}}. \) The diagonal form of the Klein-Gordon operator leads to the field equations in separation of variables of Eqs.(2.12)-(2.13).

A invariant is formed by the combination of coordinates and differential operators:
\[ dt \partial_t + d\phi \partial_\phi = d\tilde{t} \partial_{\tilde{t}} + d\tilde{\phi} \partial_{\tilde{\phi}}. \]  
(A.6)
If differential operators are operated to scalar fields \( \Phi \) in a form of Eq.(2.11), the invariant relation becomes for fixed \( \theta \) and \( r = r_+ \) (on the horizon) as
\[ -\omega t + m\phi = -\tilde{\omega} \tilde{t} + \tilde{m} \tilde{\phi}, \]  
(A.7)
where the co-rotating coordinates, frequency and azimuthal angular momentum are
\[ \tilde{t} = N_\theta \left( t - \frac{a\sin^2\theta}{\Xi} \phi \right), \ \tilde{\phi} = N_\theta \left( \phi - \frac{a\Xi}{r_+^2 + a^2} t \right), \]  
\[ \tilde{\omega} = N_\theta (\omega - \Omega_H m), \ \tilde{m} = N_\theta \left( m - \frac{a\sin^2\theta}{\Xi} \omega \right), \]  
(A.8)
with the normalization factor \( N_\theta = \sqrt{(r_+^2 + a^2)/(r_+^2 + a^2\cos^2\theta)} \). The relations between the co-rotating frame observer at the black hole horizon \( r_+ \) and the infinite frame observer in Eq.(A.8) leads the positive value of \( \text{Re} \omega - \Omega_H m \):
\[ 0 < \text{Re} \tilde{\omega} \Rightarrow 0 < \text{Re} \omega - \Omega_H m, \]  
(A.9)
and the negative value of \( \text{Im} \omega \):
\[ \text{Im} \tilde{\omega} < 0 \Rightarrow \text{Im} \omega < 0. \]  
(A.10)
This result suggests that the unstable super-radiant modes do not occur. It is worthwhile to note that the stable type of super-radiant modes can occur for the negative values of frequency \( \text{Re} \omega < 0 \) under the condition of Eq.(A.9).

\[ \]  
\footnote{This observer is called as zero angular momentum observer (ZAMO).}
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