THE INVARIANT SUBSPACES OF $S \oplus S^*$

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ABSTRACT. Using the tools of Sz.-Nagy–Foias theory of contractions, we describe in detail the invariant subspaces of the operator $S \oplus S^*$, where $S$ is the unilateral shift on a Hilbert space. This answers a question of Câmara and Ross.

1. Introduction

The recent series of papers [2, 3, 4] explore the class of so-called dual truncated Toeplitz operators, which act on a subspace of the usual Lebesgue space $L^2$ on the unit circle $T$. In the preprint [1] Câmara and Ross discuss the invariant subspaces of one of these operators, the dual of the compressed shift. In their investigation they encounter the problem of determining the invariant subspaces of the operator $S \oplus S^*$, where $S$ is the usual unilateral shift operator, acting as multiplication by the variable on the Hardy-Hilbert space $H^2$, and they state it as an open question.

It turns out that the answer can be given through the Sz.Nagy–Foias theory of characteristic functions of contractions on a Hilbert space [6]. That theory includes a general result about the relation between invariant subspaces of a contraction and regular factorizations of its characteristic function. In particular, we may use it in order to obtain an explicit description of all invariant subspaces of $S \oplus S^*$, giving thus a complete answer to the open question in [1].

The plan of the paper is the following. After some preliminaries, in Section 3 we provide a short presentation of the relevant part of the Sz.Nagy–Foias theory. Section 4 contains the main result, the description of the invariant subspaces. Section 5 provides an example related to [1], while Section 6 details the most interesting class of invariant subspaces.

2. Preliminaries

We denote shortly $L^2 = L^2(T, dm)$, where $m$ is Lebesgue measure on the unit circle $T$. Its subspace $H^2$ is the Hardy-Hilbert space of functions that can be analytically extended to the unit disk $D$; then $H^2 = L^2 \ominus H^2$. The orthogonal projections in $L^2$ onto $H^2$ and $H^2_-$ will be denoted by $P_+$ and $P_-$ respectively. The map $f \mapsto \overline{f}$, with $\overline{f}(z) = \overline{f(z)}$ is an involution on $H^2$.

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An inner function \( \theta \in H^2 \) is characterized by \( |\theta(e^{it})| = 1 \) for almost all \( t \). If \( \theta \) is inner function, then \( \bar{\theta} \) is also inner.

We will also use Lebesgue and Hardy spaces defined on the unit circle with values in a Hilbert space \( E \); they will be denoted with \( L^2(E) \) and \( H^2(E) \) respectively.

If \( S \) is the shift operator on \( H^2 \), defined by \( (Sf)(z) = zf(z) \), Beurling’s Theorem states that the invariant subspaces of \( S \) are \( \{0\} \) and the spaces \( \theta H^2 \) with \( \theta \) inner. We denote \( K_\theta = H^2 \ominus \theta H^2 \); so the invariant subspaces for \( S^* \) are \( H^2 \) and \( K_\theta \) for \( \theta \) inner. The map \( C_\theta \) defined by \( C_\theta f = \bar{\theta} \bar{z} \bar{f} \) is a conjugation on \( K_\theta \); in particular,

\[
C_\theta(K_\theta) = K_\theta.
\]

It will be convenient in the sequel to consider, rather than \( S^* \), the operator \( S^* \), acting on \( H^2_- \) as the compression of multiplication by \( z \) to \( H^2_- \). This is unitarily equivalent to \( S^* \), and the precise unitary operator that implements this equivalence is \( J : H^2 \to H^2_- \), \( (Jf)(z) = \bar{zf}(\bar{z}) \); we have \( JS^* = S^*J \).

The invariant subspaces of \( S_s \) are then \( H^2_\theta \) together with \( J(K_\theta) = J(H^2 \ominus J(\theta H^2)) = H^2 \ominus J(\theta H^2) \) for \( \theta \) inner. Since

\[
J(\theta H^2) = \{J(\theta f) : f \in H^2\} = \{\bar{zf}(\bar{z}) : f \in H^2\} = \{\bar{\theta}(\bar{z})\bar{zf}(\bar{z}) : f \in H^2\} = \bar{\theta} H^2_\theta,
\]

we have

\[
J(K_\theta) = \bar{\theta} H^2_- \ominus \bar{\theta} H^2 = \bar{\theta} K_\theta.
\]

Our purpose in this paper will be the determination of the invariant subspaces of \( S \oplus S_s^* \); we will see below that this is a model operator in the sense of Sz.Nagy and Foias. The invariant subspaces of \( S \oplus S^* \) are then immediately obtained by applying the operator \( J \).

Some of these invariant subspaces of \( S \oplus S_s \) may easily be described; namely, the subspaces \( X \oplus X' \), where \( X \subseteq H^2 \) is invariant to \( S \), while \( X' \subseteq H^2_- \) is invariant to \( S_s \). We will call them splitting invariant subspaces. The next lemma summarizes the above remarks.

**Lemma 2.1.** The splitting invariant subspaces of \( S \oplus S_s \) acting on \( H^2 \ominus H^2_- \) are of the form \( X \oplus X' \), where \( X \) is either \( \{0\} \) or \( \theta H^2 \) for some inner function \( \theta \), while \( X' \) is either \( H^2_- \) or \( \bar{\theta} K_\theta \) for some inner function \( \theta' \).

One may say that these are the obvious invariant subspaces of \( S \oplus S_s \). There is, however, a large variety of nonsplitting invariant subspaces, for whose determination we will have to bring into play the Sz.-Nagy–Foias theory of contractions [6].

We end the preliminaries with a lemma that will be helpful.

**Lemma 2.2.** If \( \theta \) is inner, then \( P_- (\bar{\theta} H^2) = \bar{\theta} K_\theta \).
Proof. The decomposition $H^2 = K_\theta \oplus \theta H^2$ yields $\bar{\theta} H^2 = \bar{\theta} K_\theta \oplus H^2$, and so $P_-(\bar{\theta} H^2) = \bar{\theta} K_\theta$. But $\bar{\theta} K_\theta = zK_\theta$ follows from equality (2.1).

3. Sz.Nagy–Foias theory of contractions and invariant subspaces

The general reference for this section is the monograph [4]. Suppose $\Theta : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{E}^*)$ is an analytic function in the unit disc $\mathbb{D}$ with values in the algebra of bounded operators from $\mathcal{E}$ to $\mathcal{E}^*$, with $\|\Theta(z)\| \leq 1$ for all $z \in \mathbb{D}$; we will call it a contractive analytic function. $\Theta$ has boundary values almost everywhere on $\mathbb{T}$, that will be denoted by $\Theta(e^{it})$. A contractive analytic function is called pure if $\|\Theta(0)x\| < \|x\|$ for any $x \in \mathcal{E}$. Any contractive analytic function admits a decomposition in a direct sum $\Theta = \Theta_p \oplus \Theta_u$, where $\Theta_p$ is pure and $\Theta_u$ is a constant unitary operator; then $\Theta_p$ is called the pure part of $\Theta$.

To a pure contractive analytic function corresponds a functional model, defined as follows. Denote $\Delta(e^{it}) = (I - \Theta(e^{it})^* \Theta(e^{it}))^{1/2}$. Then the model space is

$$H_\Theta = (H^2(\mathcal{E}_s) \oplus \overline{\Delta L^2(\mathcal{E})}) \ominus \{\Theta f \oplus \Delta f : f \in H^2(\mathcal{E})\},$$

on which acts the model operator $S_\Theta$, defined as the compression to $H_\Theta$ of multiplication with $e^{it}$ on both components of $H^2(\mathcal{E}_s) \oplus \overline{\Delta L^2(\mathcal{E})}$.

If $\mathcal{H}$ is a Hilbert space, a completely nonunitary contraction $T \in \mathcal{L}(\mathcal{H})$ is a linear operator that satisfies $\|T\| \leq 1$, and there is no reducing subspace of $T$ on which it is unitary. The defect of $T$ is the operator $D_T = (I - T^*T)^{1/2}$, and the defect space is $D_T = D_T \mathcal{H}$. It is shown in [4] that any completely nonunitary contraction $T$ is unitarily equivalent to $S_{\Theta_T}$, where $\Theta_T$ is the pure contractive analytic function with values in $\mathcal{L}(D_T, D_T^*)$ defined by

$$\Theta_T(z) = -T + D_T^*(I - zT^*)^{-1}D_T|D_T.$$  

Note that the domain of $\Theta_T(z)$ is $D_T$.

The invariant subspaces of $S_{\Theta}$ are in correspondence with the regular factorizations of $\Theta$, that we will define in the sequel. Suppose $\mathcal{F}$ is a third Hilbert space and $\Theta_1 : \mathbb{D} \to \mathcal{L}(\mathcal{E}, \mathcal{F})$, $\Theta_2 : \mathbb{D} \to \mathcal{L}(\mathcal{F}, \mathcal{E}_s)$ are two other contractive analytic functions such that $\Theta = \Theta_2 \Theta_1$. If $\Delta(e^{it}) = (I - \Theta(e^{it})^* \Theta(e^{it}))^{1/2}$ for $i = 1, 2$, then the map

$$\Delta f \mapsto \Delta_2 \Theta_1 f \oplus \Delta_1 f, \quad f \in \Delta L^2(\mathcal{E})$$

is isometric, and may thus be completed to an isometry

$$Z : \overline{\Delta L^2(\mathcal{E})} \to \overline{\Delta_2 L^2(\mathcal{F})} \oplus \Delta_1 L^2(\mathcal{E}).$$

The factorization $\Theta = \Theta_2 \Theta_1$ is called regular if $Z$ is unitary.
The relation between invariant subspaces and regular factorization is summed up in the next theorem, which follows from [6, Theorem VII.1.1], [6, Theorem VII.4.3], and the remark following it.

**Theorem 3.1.** To any regular factorization \( \Theta = \Theta_2 \Theta_1 \) corresponds an invariant subspace of \( S_{\Theta} \), defined by the formula

\[
Y = \{ \Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(\mathcal{F}), v \in \overline{\Delta L^2(\mathcal{E})} \} \\
\oplus \{ \Theta w \oplus \Delta w : w \in H^2(\mathcal{E}) \}.
\]  

(3.2)

The characteristic function of \( S_{\Theta} \) is the pure part of \( \Theta_1 \).

Conversely, any invariant subspace determines a regular factorization \( \Theta = \Theta_2 \Theta_1 \), such that \( Y \) is given by (3.2).

If \( \Theta = \Theta_2 \Theta_1' \), with \( \Theta_1 : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{E}, \mathcal{F}') \), \( \Theta_2 : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{F}', \mathcal{E}_w) \) produces through (3.2) the same subspace \( H_1 \), then there exists \( \Omega \in \mathcal{L}(\mathcal{F}, \mathcal{F}') \) unitary, such that \( \Theta_1' = \Omega \Theta_1, \Theta_2' = \Theta_2 \Omega^* \).

In general, the main difficulty in the application of Theorem 3.1 is the identification of the regular factorizations of a given contractive analytic function. Fortunately, this can be done explicitly in the case that interests us.

Let us denote by \( 0_{m \rightarrow n} \) the zero contractive analytic function considered as acting from \( \mathbb{C}^m \) to \( \mathbb{C}^n \). The functional model associated to \( \Theta = 0_{1 \rightarrow 1} \) is the space

\[
(H^2 \oplus L^2) \oplus \{ 0 \oplus f : f \in H^2 \} = H^2 \oplus H^2_1,
\]
on which the model operator is precisely \( S \oplus S_w \). Noting that \( \Delta(e^{it}) = 1 \) for all \( t \), we obtain

\[
Y = \{ \Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(\mathcal{F}), v \in \overline{\Delta_1 L^2} \} \oplus \{ 0 \oplus H^2 \} \\
= P_{H^2 \oplus H^2_1} \{ \Theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(\mathcal{F}), v \in \overline{\Delta_1 L^2} \} \\
= \{ \Theta_2 u \oplus P_-(Z^{-1}(\Delta_2 u \oplus v)) : u \in H^2(\mathcal{F}), v \in \overline{\Delta_1 L^2} \}.
\]

Theorem 3.1 yields then the next corollary.

**Corollary 3.2.** (i) To any regular factorization \( 0_{1 \rightarrow 1} = \Theta_2 \Theta_1 \), where \( \Theta_1 : \mathbb{D} \rightarrow \mathcal{L}(\mathbb{C}, \mathcal{F}), \Theta_2 : \mathbb{D} \rightarrow \mathcal{L}(\mathcal{F}, \mathbb{C}) \), corresponds an invariant subspace of \( S_{\Theta} \), defined by the formula

\[
Y = \{ \Theta_2 u \oplus P_-(Z^{-1}(\Delta_2 u \oplus v)) : u \in H^2(\mathcal{F}), v \in \overline{\Delta_1 L^2} \},
\]

(3.3)

where

\[
Z : L^2 \rightarrow \overline{\Delta_2 L^2(\mathcal{F})} \oplus \overline{\Delta_1 L^2}, \quad Zv = \Delta_2 \Theta_1 v \oplus \Delta_1 v.
\]

The characteristic function of \( T_Y := S \oplus S_w |Y \) is the pure part of \( \Theta_1 \).

(ii) Conversely, any invariant subspace determines a regular factorization \( 0_{1 \rightarrow 1} = \Theta_2 \Theta_1 \).
(iii) If another factorization $0 = \Theta'_2 \Theta'_1$, with $\Theta_1 : \mathbb{D} \to \mathcal{L}(\mathbb{C}, \mathcal{F'})$, $\Theta_2 : \mathbb{D} \to \mathcal{L}(\mathcal{F'}, \mathbb{C})$, produces by (3.3) the same invariant subspace $Y$, then there exists $\Omega \in \mathcal{L}(\mathcal{F}, \mathcal{F'})$ unitary, such that $\Theta'_1 = \Omega \Theta_1$, $\Theta'_2 = \Theta_2 \Omega^*$. 

In order to obtain a concrete description of the invariant subspaces in Corollary 3.2 we have to know the regular factorizations of the function $0_{1\to 1}$. This can be obtained from another result of Sz.-Nagy and Foias, namely [6, Proposition VII.3.5], which describes all regular factorizations of a scalar contractive analytic function. Applying it to the null function yields the next statement.

**Lemma 3.3.** The regular factorizations of the function $\Theta = 0_{1\to 1}$ are of the following three types, corresponding to $\dim \mathcal{F} = 0, 1$ or $2$:

1. $\dim \mathcal{F} = 0$. There is a unique possibility: $0_{1\to 1} = 0_{0\to 0_1 0_1\to 0}$.
2. $\dim \mathcal{F} = 1$. Then $0 = \Theta_2 \Theta_1$ with $\Theta_i$ scalar functions, and there are two cases:
   1. $\dim \mathcal{F} = 0_{1\to 1}$, $\Theta_2$ inner.
   2. $\dim \mathcal{F} = 0_{1\to 1}$, $\Theta_1$ inner.
3. $\dim \mathcal{F} = 2$. Then $\Theta_1 = \begin{pmatrix} \theta_{11} \\ \theta_{12} \end{pmatrix}$, $\Theta_2 = \begin{pmatrix} \theta_{21} & \theta_{22} \end{pmatrix}$, where

\[
|\theta_{11}|^2 + |\theta_{12}|^2 = 1, \quad |\theta_{21}|^2 + |\theta_{22}|^2 = 1, \quad \theta_{11} \theta_{21} + \theta_{12} \theta_{22} = 0.
\]

**Remark 3.4.** There is an alternate way to state conditions (3.5): they say precisely that the $2 \times 2$ matrix

\[
\Theta(e^{it}) := \begin{pmatrix} \theta_{11}(e^{it}) & \theta_{21}(e^{it}) \\ \theta_{12}(e^{it}) & \theta_{22}(e^{it}) \end{pmatrix}
\]

is unitary for almost all $t$. Moreover, if $\Theta'_1 = \begin{pmatrix} \theta'_{11} \\ \theta'_{12} \end{pmatrix}$, $\Theta'_2 = \begin{pmatrix} \theta'_{21} & \theta'_{22} \end{pmatrix}$ also satisfy (3.5), then $\Theta'_1 = \Omega \Theta_1$, $\Theta'_2 = \Theta_2 \Omega^*$ if and only if $\Theta' = \Omega \Theta$.

4. **The invariant subspaces**

We may now use the information provided by Corollary 3.2 and Lemma 3.3 in order to obtain the desired description of invariant subspaces. The next theorem is the main result of the paper.

**Theorem 4.1.** The invariant subspaces of $S \oplus S^*$ acting on $H^2 \oplus H^2_-$ are the following:

1. Splitting invariant subspaces; that is,

\[
Y = X \oplus X'
\]

with $X \subset H^2$ is invariant to $S$, $X' \subset H^2_-$ is invariant to $S^*$. 

(II) Nonsplitting invariant subspaces. These are of the form

\begin{equation}
Y = \{(θ_{21}u + θ_{22}v) + P_{\perp}(θ_{11}u + θ_{12}v) : u_1, u_2 ∈ H^2\},
\end{equation}

where θ_{ij} are functions in the unit ball of \(H^∞\), such that θ_{11} and θ_{12} are not proportional, and the matrix

\begin{equation}
Θ(e^{it}) := \begin{pmatrix}
θ_{11}(e^{it}) & θ_{21}(e^{it}) \\
θ_{12}(e^{it}) & θ_{22}(e^{it})
\end{pmatrix}
\end{equation}

is unitary almost everywhere (equivalently, θ_{ij} satisfy (3.5)).

Two matrices \(Θ, Θ'\) define the same invariant subspace if and only if there exists a unitary \(2 × 2\) matrix Ω with scalar entries, such that \(Θ = ΩΘ'\).

**Proof.** We take one by one the possibilities displayed in the statement of Lemma 3.3.

**Case (1).** We have \(dim F = 0\), so \(Δ_2 = 0\), and \(Δ_1 = I_{L^2}\), so \(Z : L^2 → \{0\} ⊕ L^2, Z(v) = 0 ⊕ v\).

\[Y = \{0 ⊕ v : v ∈ L^2\} ⊕ (\{0\} ⊕ H^2) = \{0\} ⊕ H^2.\]

The invariant subspace \(Y\) is the second component (the space on which acts \(S^*\)); it is obviously splitting.

**Case (2.1).** Here \(dim F = 1\), \(Δ_1 = I_{L^2}\), \(Δ_2 = 0\). \(Z\) is the same operator as in the previous case. We have

\[Y = \{Θ_2u ⊕ v : u ∈ H^2, v ∈ L^2\} ⊕ (\{0\} ⊕ H^2) = Θ_2H^2 ⊕ H^2.\]

**Case (2.2).** Again \(dim F = 1\), but \(Δ_1 = 0\), \(Δ_2 = I_{L^2}\). So \(Z : L^2 → L^2 ⊕ \{0\}\), \(Zv = Θ_1v ⊕ 0, Z^{-1}(w ⊕ 0) = Θ_1w\). Then

\[Y = \{0 ⊕ Θ_1v : v ∈ H^2\} ⊕ (\{0\} ⊕ H^2).\]

Since the projection of \(Θ_1H^2\) onto \(H^2\) is \(Θ_1K_{Θ_1}\), whence

\[Y = \{0\} ⊕ Θ_1K_{Θ_1}.\]

One sees that both cases (2.1) and (2.2) lead to splitting invariant subspaces.

**Case (3).** Here we have \(dim F = 2\). From (3.5) it follows that \(Θ_1^*Θ_1 = Θ_2Θ_2^* = I_{C^2}\) a.e., so \(Δ_1 = 0\), while \(Δ_2\) is a projection a.e.; that is, \(Δ_2 = Δ_2^2\).

Also, \(Θ\) unitary a.e. implies that \(ΘΘ^* = I_{C^2}\) a.e, which is equivalent to

\[Θ_1Θ_1^* + Θ_2Θ_2^* = I.\]

Therefore

\[Δ_2Θ_1 = Δ_2^2Θ_1 = (I - Θ_2Θ_2)Θ_1 = Θ_1Θ_1^*Θ_1 = Θ_1.\]

It follows that \(Z : L^2 → Δ_2L^2 ⊂ L^2(C^2)\) is defined by \(Zw = Θ_1w\), and

\[Z^{-1}(Δ_2u) = Z^{-1}(Δ_2^2u) = Z^{-1}(Θ_1Θ_1^*u) = Θ_1^*u.\]
Denoting \( u \in H^2(\mathbb{C}^2) \) by \( u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \), we have, according to (3.3),

\[
4.3 \quad Y = \{ (\theta_{21} u_1 + \theta_{22} u_2) \oplus P_{\perp} (\bar{\theta}_{11} u_1 + \bar{\theta}_{12} u_2) : u \in H^2(\mathbb{C}^2) \}.
\]

We have again to discuss two cases.

**Case (3.1).** Suppose \( \theta_{11}, \theta_{12} \) are proportional; then from (3.5) it follows that \( \theta_{11} = \alpha_1 \theta, \theta_{12} = \alpha_2 \theta \) for some inner function \( \theta \) and complex numbers \( \alpha_i \) with \(|\alpha_1|^2 + |\alpha_2|^2 = 1\). Then again by (3.5) we have \( \alpha_1 \theta_{21} + \alpha_2 \theta_{22} = 0 \).

Therefore \( \Theta \) defined by (4.2) is

\[
\Theta = \begin{pmatrix} \alpha_1 \theta & \bar{\theta}_{21} \\ \alpha_2 \theta & \bar{\theta}_{22} \end{pmatrix}.
\]

The \( 2 \times 2 \) matrix \( \Omega = \begin{pmatrix} \bar{\alpha}_1 & \bar{\alpha}_2 \\ \alpha_2 & -\alpha_1 \end{pmatrix} \) is unitary, and

\[
\Omega \Theta = \begin{pmatrix} \theta & 0 \\ 0 & \alpha_2 \bar{\theta}_{21} - \alpha_1 \bar{\theta}_{22} \end{pmatrix}.
\]

Since \( \Omega \Theta \) must be unitary almost everywhere, it follows that \( \theta' := \bar{\alpha}_2 \theta_{21} - \bar{\alpha}_1 \theta_{22} \) is inner. By Corollary 3.2 (iii) and Remark 3.4, the invariant subspace obtained in (4.3) can also be defined by \( \Theta' = \Omega \Theta \), and so

\[
4.4 \quad Y = \{ \theta' u_2 \oplus \bar{\theta} u_1 : u_1, u_2 \in H^2 \} = \theta' H^2 \oplus \bar{\theta} K_{\theta}.
\]

Note that this case completes the list of splitting invariant subspaces described in Lemma 2.1.

**Case (3.2).** The last case appears when \( \theta_{11} \) and \( \theta_{12} \) are not proportional, which leads us precisely to the invariant subspaces of type (II). To finish the proof, we have to show that these do not split. We already know the subspaces that split, so we must show that subspaces of type (II) do not coincide with any of them. By Corollary 3.2 (iii) any other factorization producing the same invariant subspace must be of type (3), with the associated invariant subspace given by (4.4). So we must have two inner functions \( \theta, \theta' \) and a \( 2 \times 2 \) unitary matrix \( \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \) such that

\[
\begin{pmatrix} \theta_{11} & \bar{\theta}_{21} \\ \theta_{12} & \bar{\theta}_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} \theta & 0 \\ 0 & \bar{\theta}' \end{pmatrix}.
\]

It follows that \( \theta_{11} = a_{11} \theta, \theta_{12} = a_{12} \theta \). So \( \theta_{11} \) and \( \theta_{12} \) are proportional, contrary to the assumption. The proof is thus finished. \( \square \)
Remark 4.2. (i) An alternate compact way to write the nonsplitting subspaces in (4.1) can be obtained if we consider $\Theta$ as a multiplication operator on $L^2 \oplus L^2$. Then

$$ Y = (P_+ \oplus P_-)\Theta^*(H^2 \oplus H^2). $$

(ii) The case (I) of Theorem 4.1, that is, the identification of all splitting subspaces, is stated in [1, Theorem 8.1].

Remark 4.3. As noted in Corollary 3.2, the characteristic function of $T_Y$ is the pure part of $\Theta_1^1$. If we examine the factorizations in Lemma 3.3, we see that $\Theta_1^1$ is pure in cases (1) and (2.1). In case (2.2), $\Theta_1^1$ is pure if it is nonconstant. In case (3), $\Theta_1^1$ is pure, except when $\theta_{11} = t_1$ and $\theta_{12} = t_2$ are constant scalars satisfying $|t_1|^2 + |t_2|^2 = 1$. Consequently, in all these cases the characteristic function of $T_Y$ is $\Theta_1^1$ and $\dim D_{T_Y} = 1$.

The remaining cases, which lead to $\dim D_{T_Y} = 0$, are then:

- (2.2), with $\Theta_1$ constant of modulus 1. The invariant subspace is $\{0\}$.
- (3), with $\theta_{11} = t_1$ and $\theta_{12} = t_2$ constant scalars satisfying $|t_1|^2 + |t_2|^2 = 1$. Arguing as in the proof of 4.1 Case 3.1, we obtain a $2 \times 2$ unitary matrix $\Omega$ and an inner function $\theta'$ such that

$$ \Omega \Theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta' \end{pmatrix}. $$

The invariant subspace is $Y = \theta' H^2 \oplus \{0\}$, and the characteristic function of $T_Y$ is the pure part of $\Omega \Theta_1 = \begin{pmatrix} 1 & \theta' \\ 0 & 1 \end{pmatrix}$, which is $0_{0 \to 1}$.

Using Remark 4.3 we may identify the reducing subspaces of $S \oplus S^*$. 

Theorem 4.4. The only reducing subspaces for $S \oplus S^*$ are $H^2 \oplus \{0\}$ and $\{0\} \oplus H^2$.

Proof. Suppose $H^2 \oplus H^2 = Y_1 + Y_2$, with $Y_1 \perp Y_2$ nontrivial and both invariant with respect to $S \oplus S^*$. Then $1 = \dim D_{S \oplus S^*} = \dim D_{T_{Y_1}} + \dim D_{T_{Y_2}}$. We may assume that $\dim D_{T_{Y_1}} = 0$, $\dim D_{T_{Y_2}} = 1$. It follows from Remark 4.3 that either $Y_1 = \{0\}$ or $Y_1 = \theta_1 H^2 \oplus \{0\}$ with $\theta_1$ inner. The nontrivial case is the latter; then $Y_2 = K_{\theta_1} \oplus H^2$ is invariant only if $\theta_1 \equiv 1$. \[\square\]

5. An example

The following example exhibits a whole class of nonsplitting invariant subspaces, for which we may obtain a simpler form than that given by Theorem 4.1.

Example 5.1. Fix $\alpha, a \in \mathbb{D}$, and $\alpha \neq 0$. Denote $b_a(z) = \frac{z-a}{z-a^2}$, and define the functions $\theta_{ij}$, $i, j = 1, 2$, by

$$ \theta_{11}(z) = \theta_{22}(z) = \alpha b_a(z), \quad \theta_{12}(z) = -(1 - |\alpha|^2)^{1/2}, \quad \theta_{21}(z) = (1 - |\alpha|^2)^{1/2}. $$
By Theorem 4.1 we obtain the following nonsplitting subspace:
\[ Y = \{(1 - |\alpha|^2)^{1/2}u_1 + \alpha b_au_2) \oplus P_-(\overline{\alpha b}_au_1 - (1 - |\alpha|^2)^{1/2}u_2) : u_1, u_2 \in H^2 \}. \]

There is a simpler way to write this subspace. First, \( P_-u_2 = 0 \). Secondly, \( K_{\overline{ba}} \) is a one dimensional space generated by the reproducing kernel \( k_{\alpha}(z) = \overline{(1-\alpha \bar{z})} \), and the orthogonal projection onto \( K_{\overline{ba}} \) has the formula \( P_{K_{\overline{ba}}}f = (1 - |\alpha|^2)f(a)k_{\alpha} \). We have the orthogonal decomposition \( H^2 = K_{\overline{ba}} \oplus b_aH^2, \) according to which \( u_1 = (1 - |\alpha|^2)^{1/2}u_1(a)k_a + b_a\bar{u}_a \). Therefore
\[
(1 - |\alpha|^2)^{1/2}u_1 + \alpha b_au_2 = (1 - |\alpha|^2)^{1/2}(1 - |\alpha|^2)u_1(a)k_a + b_a((1 - |\alpha|^2)\bar{u}_a + \alpha u_2)
\]
and
\[
P_-(\overline{\alpha b}_au_1 - (1 - |\alpha|^2)^{1/2}u_2) = P_-(\overline{\alpha b}_a(1 - |\alpha|^2)u_1(a)k_a + \bar{\alpha}u_2 - (1 - |\alpha|^2)^{1/2}u_2)
\]
\[= P_-(\overline{\alpha b}_a(1 - |\alpha|^2)u_1(a)k_a) = \overline{\alpha}(1 - |\alpha|^2)u_1(a)\bar{z}. \]

If we denote \( u = (1 - |\alpha|^2)^{1/2}u_1 + \alpha b_au_2, \) then \( u(a) = (1 - |\alpha|^2)^{1/2}u_1(a) \); moreover, if \( u_1, u_2 \) are arbitrary functions in \( H^2 \), then \( u \) is also an arbitrary function in \( H^2 \). We may therefore write
\[
Y = \{u \oplus \overline{\alpha}(1 - |\alpha|^2)u(\bar{z}) : u \in H^2\}.
\]

It is easy to see that when \( \alpha \in \mathbb{D} \setminus \{0\}, \overline{\alpha}(1 - |\alpha|^2) \) covers \( \mathbb{C} \setminus \{0\} \). Let us denote \( \beta = \frac{\overline{\alpha}(1 - |\alpha|^2)}{1 - |\alpha|^2} \); the invariant subspace is then
\[
(5.1) \quad Y = \{u \oplus \beta \frac{u(\bar{z})}{1 - a\bar{z}} : u \in H^2\}.
\]

We have thus obtained in (5.1) a class of nonsplitting invariant subspaces parameterized by the nonzero complex number \( \beta \). For an appropriate value of this parameter, \( Y \) corresponds to the subspace appearing in Example 7.3 of [1] (after taking into account the unitary equivalence implemented by \( J: H^2 \to H^2 \)).

6. Parametrization of nonsplitting subspaces

The nonsplitting subspaces are the most interesting ones, so it is worth to obtain a more detailed description of this class. Equations (3.5) define \( \Theta_1 \) and \( \Theta_2 \) in an implicit manner; we will determine in this last section a parametrization of these two functions.

We start with a pair of nonproportional functions \( \theta_1, \theta_2 \in H^\infty \) that satisfy \( |\theta_1|^2 + |\theta_2|^2 = 1 \). First, if we denote by \( g_1, g_2 \) the outer parts of \( \theta_1, \theta_2 \) respectively, they satisfy \( |g_1|^2 + |g_2|^2 = 1 \). In fact, this means an outer function \( g_1 \) bounded by 1 and subject to the condition \( \int (1 - |g_1|^2) > -\infty \), which is equivalent to \( g_1 \) not being an extreme point of the unit ball of \( H^\infty \) (see [3]). Then \( g_1 \) determines \( g_2 \) up to a scalar of modulus 1.
Since $\Theta$ is unitary, $|\theta_{11}(e^{it})| = |\theta_{22}(e^{it})|$ and $|\theta_{12}(e^{it})| = |\theta_{21}(e^{it})|$ almost everywhere. Therefore $g_1, g_2$ are also the outer parts of $\theta_{22}, \theta_{21}$ respectively. We may then write

$$
\theta_{11} = \alpha_{11}g_1, \quad \theta_{12} = \alpha_{12}g_2, \quad \theta_{21} = \alpha_{21}g_2, \quad \theta_{22} = \alpha_{22}g_1,
$$

with $\alpha_{ij}$ inner; from the last formula in (3.5) it follows that $\alpha_{11} \alpha_{21} + \alpha_{12} \alpha_{22} = 0$.

By factoring common inner divisors, let us then write $\alpha_{ij} = \alpha_i \beta_{ij}$, with $(\beta_{11}, \beta_{12}) = 1$. It follows then that $\beta_{11} \beta_{21} = -\beta_{12} \beta_{22}$.

Divisibility implies then that $\beta_{22} = \lambda \beta_{11}$ and $\beta_{21} = -\lambda \beta_{12}$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. If we denote, for simplicity, $\beta_1 = \beta_{11}$ and $\beta_2 = \beta_{12}$, we may write

$$
Y = \{(\lambda \alpha_2 [-\beta_2 g_2 u_1 + \beta_1 g_1 u_2]) \oplus P_-(\bar{\alpha}_1 [\bar{\beta}_1 g_1 u_1 + \bar{\beta}_2 g_2 u_2])\}.
$$

So the nonsplitting invariant subspace $Y$ is determined by the following “free” objects:

(i) An outer function $g_1$ bounded by 1 that is not an extreme point of the unit ball of $H^\infty$.

(ii) Two arbitrary inner functions $\alpha_1, \alpha_2$.

(iii) Two arbitrary, but coprime inner functions $\beta_1, \beta_2$.

(iv) A complex number $\lambda$ of modulus 1.

To obtain from these parameters $\theta_{ij}$, note first that $g_1$ determines up to a constant of modulus 1 an outer function $g_2$, such that $|g_1|^2 + |g_2|^2 = 1$. Then we have

$$(6.1) \quad \theta_{11} = \alpha_1 \beta_1 g_1, \quad \theta_{12} = \alpha_1 \beta_2 g_2, \quad \theta_{21} = -\lambda \alpha_2 \beta_2 g_2, \quad \theta_{22} = \lambda \alpha_2 \beta_1 g_1.$$

The condition for two parametrizations to produce the same invariant subspace follows from the last statement of Theorem 4.1. One sees that there is a remarkable richness of nonsplitting subspaces.

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