Hagedorn transition for strings on pp-waves and tori with chemical potentials

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Abstract

It has been conjectured that string theory in a pp-wave background is dual to a sector of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We study the Hagedorn transition for free strings in this background. We find that the free energy at the transition point is finite suggesting a confinement/deconfinement transition in the gauge theory. In the limit of vanishing mass parameter the free energy matches that of free strings on an 8-torus with momentum/winding chemical potential. The entropy in the microcanonical ensemble with fixed energy and fixed momentum/winding is computed in each case.
1 Introduction

One of the original motivations for studying string thermodynamics was to model the confinement/deconfinement phase transition in QCD \cite{1}. With the advent of a direct correspondence between supersymmetric gauge theory and string theory in an anti-de Sitter background \cite{2}, this subject has been re-examined.

As argued in Refs. \cite{3,4}, a type of confinement/deconfinement transition in the strong coupling limit of large $N$ $SU(N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $R^1 \times S^3$ may be mapped to the Hawking-Page phase transition for black holes \cite{5}. True phase transitions only occur in systems with infinite numbers of degrees of freedom, so it is necessary to take $N \to \infty$ to see this transition on a compact $S^3$. This transition has been further studied in Refs. \cite{6,7,8,9,10}. Unfortunately it is difficult to quantize the string theory directly in the anti-de Sitter background due to the presence of Ramond-Ramond flux, so it has not been possible to extend this result to the weak coupling limit, where one expects to see a Hagedorn-like exponential increase of the density of states with energy below the deconfinement transition.

However recently BMN \cite{11} have described how to map a subsector of the large $N$ $SU(N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory to a simpler string theory background where exact quantization is possible, known as the parallel-plane-wave or pp-wave, with metric characterized by a mass parameter $\mu$,

\begin{equation}
\text{ds}^2 = -2dx^+dx^- - \mu^2 \sum_{i=1}^{8}(x^i)^2(dx^+)^2 + \sum_{i=1}^{8}(dx^i)^2 .
\end{equation}

One of the main results of the present work is to examine free string thermodynamics in this background, extending earlier results of Refs. \cite{12,13}.

In order to isolate the subsector appropriate in the BMN limit \cite{11}, it is necessary to introduce, in addition to the inverse temperature, $\beta$, a chemical potential, $\nu$, for the longitudinal momentum, corresponding to the large R-charge limit of the gauge theory. This string background exhibits Hagedorn-like behavior \cite{14}. We find that the free energy, as an analytic function of $\beta$, has a singularity

\begin{equation}
F(\beta, \nu; \mu) \sim c\sqrt{\beta - \beta_H(\nu; \mu)} + \text{regular part} ,
\end{equation}

where $c > 0$, and the Hagedorn temperature, $T_H \equiv \beta_H^{-1}$, is a function of both the chemical potential $\nu$ and the mass parameter. The free energy is finite at the transition point, suggesting a phase transition rather than a limiting temperature.

The mass parameter $\mu$ effectively confines oscillations of strings in the transverse directions $x^i$. In the limit that the curvature may be neglected (i.e. $\mu\sqrt{\alpha'} \ll 1$) we have found that all salient features are in fact already visible in the more familiar context of strings in a toroidal compactification, contrary to claims in Refs. \cite{12,13}. We therefore begin by first treating toroidal compactification with chemical potentials. This system has been used in the past to model QCD at finite baryon number chemical potential in Ref. \cite{13}. Here we obtain a number of results not previously noted in the literature.
As usual with free string dynamics in the canonical ensemble, fluctuations may diverge as the Hagedorn temperature is approached \([16]\), which can invalidate the thermodynamic limit. In this situation, the microcanonical ensemble is more physically appropriate. We compute the entropy in the microcanonical ensemble which clarifies many of our canonical ensemble results.

2 Canonical Ensemble and Toroidal Compactification

The canonical ensemble for free strings has been described in Refs. \([17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]\). We define the multi-string grand canonical ensemble for free strings in terms of a sum over single string partition functions for bosonic modes \(Z^B\) and (spacetime) fermionic modes \(Z^F\),

\[
\log Z(\beta, \mu, \nu) = \sum_{r=1}^{\infty} \frac{1}{r} \left( Z^B_1(r\beta, r\mu, r\nu) - (-1)^r Z^F_1(r\beta, r\mu, r\nu) \right). \tag{3}
\]

When we have non-compact spatial dimensions, the behavior near the Hagedorn temperature will be dominated by the single string \(r=1\) term, which we denote \(Z_1 = Z^B_1 + Z^F_1\).

We define the free energy as \(\beta F = -\log Z \approx -Z_1\). With all dimensions compactified, as we will see, it is necessary to be more careful in relating the single string thermodynamics potentials to the full thermodynamic potentials.

Consider a system of closed strings in \(D\) spatial dimensions, (e.g., \(D = 9\) for the Type II superstring). Of these, \(\tilde{d}\) dimensions are compactified on a \(\tilde{d}\)-torus, each with radius \(R_i\), with the remaining \(d\) directions uncompactified, i.e., \(D = d + \tilde{d}\). The single closed string free energy density is given by

\[
Z_1(\beta, \lambda, \nu) = \frac{\beta V_D}{(2\pi \sqrt{\alpha'})^{D+1}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1 e^{-\frac{\beta^2}{4\pi\alpha'\tau_2}} K(z, \bar{z}) W(\bar{z}, z, \{R\}, \{\lambda\}, \{\nu\}), \tag{4}
\]

where \(V_D\) is the spatial volume and \(K(z, \bar{z})\) is the standard modular invariant integrand for the uncompactified one-loop partition function, and we have defined \(z \equiv e^{2\pi i \tau}\) and \(\bar{z} \equiv e^{-2\pi i \bar{\tau}}\) with \(\tau = \tau_1 + i\tau_2\). \(W\) is the contribution due to momenta and windings in the compactified directions, with chemical potentials, \(\{\lambda_i, \nu_i\}\),

\[
W(\bar{z}, z, \{R\}, \{\lambda\}, \{\nu\}) = \prod_{i=1}^{\tilde{d}} W_0(\bar{z}, z, R_i, \lambda_i, \nu_i), \tag{5}
\]

\[
W_0(\bar{z}, z, R_i, \lambda_i, \nu_i) = \frac{\sqrt{\alpha'\tau_2}}{R_i} \sum_{p_i, w_i = -\infty}^\infty \bar{z}^{p_i} \left( \frac{w_i + \nu_i}{\lambda_i} \right)^2 z^{w_i} \left( \frac{\nu_i - p_i R_i}{\lambda_i} \right)^2 e^{-(\nu_i p_i + \lambda_i w_i)}. \tag{4'}
\]

In the following we assume the chemical potential terms are sufficiently small that the sums in these expressions converge.
2.1 Hagedorn Singularity without Chemical Potentials

We begin by first setting all chemical potentials to zero and identify the nature of Hagedorn singularity as the dimension of the uncompactified directions, \( d \), is varied. We shall demonstrate shortly that the only effect of the chemical potentials will be to shift the value of the Hagedorn temperature \( T_H = \beta_H^{-1} \). A Hagedorn singularity occurs as \( \beta \) is lowered due to a potential divergence of the integral in Eq. (4) at \( \tau_2 = 0 \). To extract the singularity at the transition temperature, we first perform the integral over \( \tau_1 \) to obtain the leading contribution in the limit \( \tau_2 \to 0 \). This is a standard stationary phase problem. The relevant factors coming from \( K(z, \bar{z}) \to (\sqrt{\tau_2}/|\tau|)^{-D+1} \exp\left(\frac{\beta^2 H \tau_2}{4 \pi \alpha'} |\tau| \right) \) and \( W \to V_d^{-1} (2\pi \sqrt{\alpha'})^d (\sqrt{\tau_2}/|\tau|)^{\bar{d}} \), lead to for the integrand of \( \tau_2 \) integral

\[
\tau_2^{\bar{d}/2} \int_{-1/2}^{1/2} d\tau_1 \left( \frac{|\tau|}{\sqrt{\tau_2}} \right)^{d-1} \exp\left( \frac{\beta^2 H \tau_2}{4 \pi \alpha' |\tau|} \right) 
\]

\[
\simeq \tau_2^{(d-3)/2} \int_{-\infty}^{\infty} dx \cdot (1 + x^2)^{(d-1)/2} \exp\left( \frac{\beta^2 H}{4 \pi \alpha' \tau_2} \frac{1}{1 + x^2} \right), \tag{6}
\]

where in the second expression we have changed the integration variable to \( x = \tau_1/\tau_2 \). The stationary point is \( x = 0 \) with Gaussian fluctuations \( x^2 = O(\tau_2) \) as \( \tau_2 \to 0 \). Doing the Gaussian integral and inserting the result of (6) into (4), we get

\[
Z_1 \sim \frac{\beta V_D}{(2\pi \sqrt{\alpha'})^d \beta_H V_d} \int_0^\Lambda d\tau_2 \tau_2^{d/2} \exp\left( \frac{\beta^2 H - \beta^2}{4 \pi \alpha' \tau_2} \right), \tag{7}
\]

where the integral is cut off at the upper limit. The dependence on \( d \), the number of uncompactified spatial dimension, is generic while the value for \( \beta_H \) depends on specific string theory in question. For Type II string theory, \( \beta_H = T_H^{-1} = \pi \sqrt{8\alpha'} \). As \( T \to T_H \) from below, one finds, for \( D \geq d \geq 0 \), \[27, 29\]

\[
Z_1 = \begin{cases} 
  c_d \left( \beta - \beta_H \right)^{d/2} + \text{regular} & : \text{d odd} \\
  c_d \left( \beta - \beta_H \right)^{d/2} \log(\beta - \beta_H) + \text{regular} & : \text{d even} 
\end{cases} \tag{8}
\]

with \( c_d(\beta_H) \sim (-1)^{(d+1)/2} \) and \( c_d(\beta_H) \sim (-1)^{d/2+1} \) for \( d \) odd and even respectively. The free energy always remains bounded for \( d \neq 0 \), e.g., for \( d = 1 \),

\[
F(\beta) \sim c \sqrt{\beta - \beta_H} + \text{regular part}, \tag{9}
\]

where \( c > 0 \). For the case where all spatial directions are compactified, \( d = 0 \), one can show that the coefficient is exactly \( c_0 = -1 \), leading to

\[
Z_1 = -\log(\beta - \beta_H) + \text{regular}, \tag{10}
\]

\[1\text{Some previous works in the literature do not do this integral over } \tau_1 \text{ correctly. For an alternative derivation where the } \tau_1 \text{ integral is done exactly, see Refs. [21, 27].}\]
and $Z \simeq \text{const}/(\beta - \beta_H)$. The interpretation for this case is a little subtle since now the partition function, no longer dominated by the single string excitations, receives essential contributions from multi-string states \[30, 27, 31].

All these potentials are consistent with positive specific heat, $C = \beta^2(\log Z)''$, as they must for $T < T_H$. As $T$ approaches $T_H$ from below, both internal energy, $E = -(\log Z)'$, and specific heat $C$ can diverge for low values of $d$. In particular, $C/E^2\beta^2$ blows up for $0 \leq d \leq 4$ as $T \to T_H$, indicating that energy fluctuations in the canonical ensemble diverge as the Hagedorn temperature is approached. For $d = 0$ the free energy $F$ diverges signaling a limiting temperature. For $d = 1$ and $2$, the average internal energy $E$ diverges. This alone does not signal a limiting temperature since the free energy $F$ remains finite in these cases, but is rather a product of the divergent energy fluctuations. For $d = 3$ and 4, the internal energy $E$ is finite, but the specific heat diverges at the transition point. For $d \neq 0$, the natural interpretation is in terms of a conventional, most likely second order, phase transition. Including interactions may modify this picture. The Hagedorn transition is associated with a state winding the imaginary time direction becoming massless \[21, 25, 32\]. As argued in Ref. \[26\] interactions can turn this into a first order transition with a temperature below the Hagedorn temperature.

### 2.2 Hagedorn Singularity with Chemical Potentials

We now proceed to generalize this result to the situation with nontrivial chemical potentials. For simplicity, we will focus on the case where chemical potentials are present for only one dimension. First consider the factor $W_0(\xi, z, R, \lambda, \nu)$. In the limit $\tau_2 \to 0$, we can perform the sum over $p$ and $w$ by approximating the sums by integrals. After rescaling variables as $x = \tau_1/\tau_2$, $y = \sqrt{\alpha'}\tau_2(p/R + wR/\alpha')$ and $z = \sqrt{\alpha'}\tau_2(p/R - wR/\alpha')$, one finds

$$Z_1 \sim \int_0^\Lambda d\tau_2/\tau_2 \int dxdydz \exp \left(-\frac{\beta^2}{4\pi\alpha'\tau_2} + \frac{1}{2\tau_2} \left(\frac{4\pi}{1 + x^2} - \pi(y^2 + z^2) - i\pi x(y^2 - z^2) - \gamma_+ y + \gamma_- z\right)\right)$$

where

$$\gamma_+ = (\nu R + \alpha' \lambda/R)/\sqrt{\alpha'}, \quad \gamma_- = (\nu R - \alpha' \lambda/R)/\sqrt{\alpha'}.$$  

The $x, y$ and $z$ integrals are performed first by the stationary phase approximation\[\[7\]\] leading to

$$Z_1 \sim \int_0^\Lambda d\tau_2/\tau_2 \exp \left(-\frac{\beta^2}{4\pi\alpha'\tau_2} + \frac{1}{16\pi\tau_2} \left(\sqrt{8\pi^2 + \gamma_+^2} + \sqrt{8\pi^2 + \gamma_-^2}\right)^2\right).$$

Note that this remains of the same form as (7), so the nature of Hagedorn singularity is unchanged. The inverse Hagedorn temperature is then given by

$$\beta_H = \left(\sqrt{\beta_0^2 + \alpha'\gamma_+^2} + \sqrt{\beta_0^2 + \alpha'\gamma_-^2}\right)/2,$$

\[\text{2}\] The dominate saddle point is at real values of $y = -(\gamma_+ / 2\pi)(1 + ix)^{-1}$ and $z = -(\gamma_- / 2\pi)(1 - ix)^{-1}$ where $x$ lies on the imaginary axis at $ix = (\sqrt{\beta_0^2 + \alpha'\gamma_+^2} - \sqrt{\beta_0^2 + \alpha'\gamma_-^2})/(2\beta_H)$. 

4
where $\beta_0 = \pi \sqrt{8\alpha'}$. As the chemical potential $\lambda$ or $\nu$ is increased, the Hagedorn temperature decreases. The general expression for several chemical potentials replaces $\gamma_+^2$ by $\gamma_+^2 = \sum_i (\nu_i R_i \pm \alpha' \lambda_i R_i)^2 / \alpha'$.

The free energy is still given by (8) and (10), but now with $\beta_H = \beta_H(\lambda, \nu)$ given by (14). The canonical ensemble remains stable, for $T < T_H = \beta_H^{-1}$. As in the case without chemical potential, for $0 \leq d \leq 4$, energy fluctuations become large as the Hagedorn temperature is approached.

Before proceeding to the more interesting situation of the pp-wave background, it is worth adding several comments. We first note that both the location and the nature of the Hagedorn singularity (14) can also be understood, as mentioned earlier, in terms of the occurrence of a thermal tachyon \cite{21, 25, 32}. We have verified that the analysis of Ref. \cite{21} remains valid when chemical potentials $(\lambda, \nu)$ are introduced.

Consider next the special situation where a nontrivial chemical potential is present only for the momenta, i.e., $\nu \neq 0$ and $\lambda = 0$. Equivalently, one can treat this direction as uncompactified so that the allowed momenta become continuous. In this case, due to Lorentz invariance, $Z_1$ can only depend on the combination $\beta^2 - \bar{\nu}^2$, with $\bar{\nu} \equiv \nu R$. It follows that the inverse Hagedorn temperature, $\beta_H(0, \nu)$, can be directly obtained from $\beta_H(0, 0) = \beta_0$ by a Lorentz boost, i.e.,

$$\beta_H^2(\nu) \equiv \beta_H^2(0, \nu) = \beta_0^2 + \nu^2 R^2. \quad (15)$$

Indeed, this is consistent with Eq. (14), since $\gamma_+ = \gamma_- = \nu R / \alpha'$ in this limit. Due to T-duality, it follows that $\beta_H(\lambda, 0)$ can also be directly obtained by such a symmetry consideration. In what follows, we shall treat only the case where no chemical potentials for winding modes are introduced, i.e., $\lambda_i = 0$.

We also note that conventional chemical potentials in statistical mechanics are dimensionful, scaled by $\beta$, i.e., $\nu \rightarrow \beta \bar{\nu}$. When this is done, one arrives at a Wigner’s semi-circle law for the Hagedorn transition, i.e., when $\lambda = 0$, \cite{13}

$$T_H^2 + \bar{\nu}^2 = T_0^2, \quad (16)$$

where $T_0 \equiv \beta_0^{-1} = 1 / \sqrt{8\pi^2\alpha'}$ is the Hagedorn temperature when chemical potential is absent, and a factor of $RT_0$ has also been absorbed by $\bar{\nu}$. In view of the previous comment, it is interesting to note that a semi-circle law can emerge as a consequence of Lorentz invariance involving extra dimensions.

### 3 Canonical Ensemble in the PP-Wave Background

The pp-wave background and the duality with a sector of super-Yang-Mills theory has been described in Ref. \cite{11} and references therein. The canonical ensemble with chemical potential for the longitudinal momentum was described in Ref. \cite{12}, and without chemical potential in Ref. \cite{13, 33},

$$Z_1(a, b; \mu) = \text{Tr}_H e^{-a p_+ - b p_+ - \mu}, \quad (17)$$
where \( p_+ = p^- = (p^0 - p^9)/\sqrt{2} \) and \( p_- = p^+ = (p^0 + p^9)/\sqrt{2} \). Hence we should identify
\[
\beta = \frac{b + a}{\sqrt{2}}, \quad R\nu = \frac{b - a}{\sqrt{2}},
\]
in order to compare with the results of the previous section. We can of course scale \( R \to \infty \) when the longitudinal direction is non-compact, keeping \( \tilde{\nu} = R\nu \) and \( p^9 = Q/R \) fixed. In what follows, we shall simply replace \( R\nu \) by \( \nu \) and \( p^9 \) by \( Q \), so that \( \nu \) and \( Q \) are now dimensionful.

The lightcone Hamiltonian is given by
\[
p_+ = \frac{1}{\alpha' p_-} \left( \omega_0(m) \left( N_0^B + N_0^F \right) + \sum_{n=1}^{\infty} \omega_n(m) \left( N_n^B + N_n^F + \tilde{N}_n^B + \tilde{N}_n^F \right) \right),
\]
where
\[
\omega_n(m) = \sqrt{n^2 + m^2},
\]
with mass \( m = 2\alpha' \mu p^+ \). It follows that
\[
Z_1(a, b; \mu) \propto \sum_{N_n, \cdots, \tilde{N}_n} \int_0^\infty dp_- e^{-bp_- - ap_+}
\]
subject to the level-matching constraint
\[
\sum_n n(N_n - \tilde{N}_n) = 0.
\]

So far we are treating the longitudinal direction as non-compact. However, we can allow for winding in this direction as well if we so choose. The relevant string compactifications have been described in Refs. [34, 35].

As shown in Ref. [12], we may then change variables to \( \tau_2 = a/2\pi\alpha'p_- \) and introduce an auxiliary integration variable \( \tau_1 \) to implement the level-matching constraint to give
\[
Z_1(a, b; \mu) = \frac{a V_L}{4\pi^2\alpha'} \int_{\tau_2}^\infty d\tau_2 \int_{-1/2}^{1/2} d\tau_1 e^{-\frac{ab}{2\pi^2\alpha'}} \left( \frac{\Theta_{1/2,0}(\tau, \tilde{\nu}, \mu a/2\pi\tau_2)}{\Theta_{0,0}(\tau, \tilde{\nu}, \mu a/2\pi\tau_2)} \right)^4
\]
where the generalized theta functions are defined in [36], \( \tau = \tau_1 + i\tau_2 \), and \( V_L \) is the longitudinal volume. The high temperature behavior may be read off by looking at the \( \tau_2 \to 0 \) limit. It is convenient to perform a modular transformation on the theta function as in Ref. [33] to analyze this limit. The integrand becomes
\[
\frac{1}{\tau_2^2} \exp \left( -\frac{ab}{2\pi\alpha'\tau_2} \right) \exp \left( \frac{16\mu a\tau_2}{4\pi^2} \left( f(\mu a|\tau|, 0) - f(\mu a|\tau|, 1/2) \right) \right)
\]
where
\[
f(x, \alpha) \equiv 2 \sum_{l=1}^{\infty} e^{2\pi i lx} K_1(2\pi lx)
\]
with \( K_1 \) a modified Bessel function of the second kind \([13, 12]\).

Performing the \( \tau_1 \) integral as in \([3]\), we are left with

\[
Z_1(a, b; \mu) \sim \int_0^\Lambda d\tau_2 \tau_2^{1/2} \exp \left( \frac{\beta_c^2 - 2ab}{4\pi \alpha' \tau_2} \right)
\]  

(26)

where \( \beta_c^2/\alpha' \) depends only on \( \mu a \),

\[
\beta_c^2(\mu a) = 16\alpha' \mu a \left( f\left(\frac{\mu a}{2\pi}, 0\right) - f\left(\frac{\mu a}{2\pi}, \frac{1}{2}\right) \right).
\]  

(27)

As the product \( ab \) is increased, contribution from \( \tau_2 \to 0 \) leads to a singularity for the free energy

\[
F \sim \left(2ab - \beta_c^2(\mu a)\right)^{1/2} + \text{regular},
\]  

(28)

the same basic form as on the torus with \( d = 1 \), \([3]\). Since the free energy is finite at the transition point, there may be a true phase transition at this temperature.

Our result \([28]\) differs from that of Refs. \([13, 12]\) due to our improved treatment of the \( \tau_1 \) integral which yields a different power of \( \tau_2 \) in the prefactor in \([20]\). The expression for the Hagedorn temperature is in complete agreement with that given in Refs. \([13, 12]\). Those papers also define the Hagedorn temperature as \( \beta_c^{-1} \), which corresponds to the proper temperature at which the partition function is singular. In the following we will stick with our definition of Hagedorn temperature as the critical value of \( \beta^{-1} \).

The inverse Hagedorn temperature \( \beta_H(\nu; \mu) \) is specified by the solution to

\[
\beta_H^2 - \nu^2 = \beta_c^2 \left( \frac{\mu(\beta_H - \nu)}{\sqrt{2}} \right).
\]  

(29)

To gain a qualitative understanding for the solution, let us consider two simple limits:\footnote{We are interested in the region where \( \nu < 0 \), corresponding to having positive \( Q > 0 \).}

\( |\nu| \ll \mu^{-1} \) and \( |\nu| \gg \mu^{-1} \). Here we simply address the behavior of \( \beta_H(\nu; \mu) \) while leaving the interpretation to the next section where we discuss the microcanonical ensemble.

For the first limit, we need \( \mu a \ll 1 \) when \([27]\) becomes

\[
\beta_c^2(\mu a) \approx \beta_c^2(0) = 8\pi^2 \alpha' = \beta_0^2.
\]  

(30)

Thus as we approach the flat space limit \footnote{In fact in this limit, there is an accumulation of sub-leading Hagedorn-like singularities, with spacing \( \sim \mu \alpha' \), causing a non-uniformity in the limits \( T \to T_H \) and \( \mu \to 0 \). In this paper, we consider only the “near flat space limit” where \( \mu \sqrt{\alpha} \ll 1 \) with \( \mu \neq 0 \). In the strict \( \mu = 0 \) case at fixed \( T < T_H \) the free energy will exhibit the singularity for zero compact dimensions, i.e., \( d = 9 \).} we reproduce the same Hagedorn temperature as obtained for torus compactification,

\[
\beta_H^2(\nu; \mu) \approx \beta_0^2 + \nu^2,
\]  

(31)

which led to the semi-circular law \([16]\). As in the torus case, the thermodynamic ensemble is stable in this limit.
In the second limit, one needs \( \mu a \gg 1 \), and (27) becomes
\[
\beta_c^2(\mu a) \simeq 32 \sqrt{2\pi \mu a e^{-\mu a/\pi}}.
\] (32)

For \( \nu < 0 \) and \( |\nu| \) large, this leads to
\[
\beta_H^2(\nu; \mu) \simeq \nu^2 + 4\beta_0^2 \sqrt{\sigma^3 \mu |\nu| e^{-\sigma |\nu|}},
\] (33)

where \( \sigma = \sqrt{2/\pi} \). In the limit \( \mu |\nu| \to \infty \), one has \( \beta_H(\nu; \mu) \simeq |\nu| + 0(e^{-\sigma |\nu|}) \). In particular, the Hagedorn temperature \( \beta_H^{-1} \) decreases as \( |\nu| \) increases. We have checked the eigenvalues of the matrix of second derivatives of (28) are all negative, as required for stability.

If instead of treating the longitudinal direction as noncompact, we had compactified and allowed for spatial windings, the prefactor \( 1/\tau^2 \) in (26) would change to \( 1/\tau_2 \), and (28) should be changed to
\[
\beta F \sim \log \left( 2ab - \beta_c^2(\mu a) \right).
\] (34)

The same conclusions hold regarding the stability of the ensemble. Now the free energy diverges at the transition point, so in this case the Hagedorn temperature could be interpreted as a limiting temperature for the ensemble. However once interactions are included we expect to see a first order phase transition as mentioned above, and this is naturally associated with a confinement/deconfinement phase transition.

### 3.1 Yang-Mills Dual

Finally let us discuss the implications of the results for the dual Yang-Mills theory. BMN [11] map string theory \( p_+ \) and \( p_- \) to Yang-Mills energy \( E_{YM} \) and R-charge \( Q_{YM} \) as
\[
\frac{\sqrt{2}p^-}{\mu} = E_{YM} - Q_{YM}, \quad \sqrt{2}\mu p^+ = \frac{E_{YM} + Q_{YM}}{R^2_{AdS}},
\] (35)

where one has in mind the large \( N \) limit with \( Q_{YM}^2/N, E_{YM} - Q_{YM} \), and the Yang-Mills coupling \( g_{YM} \) held fixed. Here the radius of curvature of the \( AdS^5 \) space is \( R^4_{AdS} = 4\pi g_{YM}^2 N \alpha'^2 \), which goes to infinity in this limit, so that \( p^+ \) and \( p^- \) remain fixed. Note that \( \mu \) may be scaled to 1 using the longitudinal boost symmetry \( p^- \to \alpha p^-, \mu \to \mu/\alpha, p^+ \to \alpha p^+ \) of the metric (1) and does not appear directly in the Yang-Mills theory.

In the Yang-Mills theory \( E_{YM} \) corresponds to the dimensionless energy on the space \( S^3 \times R \) in units of the radius, \( R_{S^3} \), and \( Q_{YM} \) corresponds to the R-charge. We define the Yang-Mills temperature to be conjugate to \( E_{YM} \) and the chemical potential to be conjugate to \( Q_{YM} \). With the identifications (35) we find
\[
\beta_{YM} = \frac{\mu a}{\sqrt{2}} + \frac{b}{\sqrt{2}R^2_{AdS} \mu}, \quad \nu_{YM} = \frac{b}{\sqrt{2}R^2_{AdS} \mu} - \frac{\mu a}{\sqrt{2}}.
\] (36)

We wish to interpret the Hagedorn transition as a kind of confinement/deconfinement transition for the large \( N \) gauge theory on a compact space. Since we have a conformal
field theory, the scale is set by the size of the $S^3$ rather than a dynamically generated scale as in real QCD. Note that both $\betaYM$ and $\nuYM$ are dimensionless, with $\betaYM + \nuYM = (\beta + \nu)/\muR^2_{\text{AdS}}$ and $\betaYM - \nuYM = \mu(\beta - \nu)$. Of course the invariant properties of the canonical ensemble are unaffected by this redefinition of parameters. For example we have

$$(\betaYM^2 - \nuYM^2) = (\beta^2 - \nu^2)/R^2_{\text{AdS}}.$$  \hfill (37)

In the near flat space limit, a semi-circle law still holds for the Yang-Mills parameters at the Hagedorn singularity,

$$T^2YM + \bar{\nu}^2YM = T^2_{0YM},$$  \hfill (38)

where the radius of the circle is set by $T_{0YM} = 1/\sqrt{8\pi^2\alphaYM'}$ with $\alphaYM' = R^2_{S^3}/\sqrt{g^2YMN}$, the inverse Yang-Mills string tension relative to the size of $S^3$, and $\bar{\nu}YM$, the chemical potential after an appropriate re-scaling, $\nuYM \rightarrow \betaYM\bar{\nu}YM/T_{0YM}$. Beyond the Hagedorn transition the perturbative string theory picture seems to break down at fixed small string coupling, and it is natural to conjecture the proper definition of the string theory is in terms of the large $N$ gauge theory in the deconfining phase. This is reminiscent of the type of transition envisaged in Ref. [26] but the details are rather different.

In a strict pp-wave limit, the inverse Yang-Mills string tension $\alphaYM'$ vanishes, so the Hagedorn behavior is better described in term of $a$ and $b$, which are fixed. In this limit, $\mu a/\sqrt{2} \simeq \betaYM \simeq -\nuYM$ and $\sqrt{2b/R^2_{\text{AdS}}\mu} = \betaYM + \nuYM \rightarrow 0$. It is useful to treat $a$ as a new inverse temperature, conjugate to $EYM - QYM$, and $b$ as a new chemical potential, conjugate to $EYM + QYM$. In this new language, as the temperature vanishes, $a \rightarrow \infty$, one approaches the “ground state” where $EYM - QYM = 0$.

In the limit $b$ large (large chemical potential), the Hagedorn transition now corresponds to the near flat space limit (30) with

$$a \approx \frac{\beta^2YM}{2b},$$  \hfill (39)

where the new transition temperature $a^{-1}$ increases without bound. In the small $b$ limit (small chemical potential), the transition is now the same as $\mu a >> 1$, (32), with

$$\mu a \sim -\pi \log(b/\mu\alpha'),$$  \hfill (40)

so that the transition temperature decreases as $b$ decreases.

It is also interesting to note the free energy given by Eq. (34) matches exactly with that calculated in a simplified model for the large $N$, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $S^3$ [37]. (See also Ref. [38] for closely related results.) The results of Ref. [39] show that when weak string interactions are included with non-compact spatial dimensions the density of states shifts to that of the fully compactified case. This supports the comparison of the result of Ref. [37] to Eq. (34) rather than Eq. (28). One might then wonder if the Hagedorn temperature should then be interpreted as an unattainable limiting temperature or whether a real phase transition happens. In Ref. [37] it was found for the model considered there that finite $N$ smooths out the singularity in the
free energy. A cross-over to a free energy proportional to $N^2$ occurs above the transition point, yielding an interpretation in terms of a confinement/deconfinement transition. As we have mentioned in the case at hand, string interactions also open up the possibility of a first order phase transition at a temperature just below the Hagedorn temperature which cuts off the limiting temperature behavior, and a very similar physical picture may emerge.

To describe the physics above the transition point we would need to generalize the Hawking-Page transition to the BMN limit. In fact, a pp-wave solution as a Penrose limit of the AdS Schwarzschild background has been constructed in Ref. [40]. However a pp-wave background cannot give rise to a background with a horizon [41]. Instead the background is singular, and difficult to quantize, and hence it is unclear how a gravitational entropy of order $N^2$ arises.

With the longitudinal direction of the string theory compactified, one should still have duality with the large $N$ gauge theory described in Ref. [34, 35], and the free energy up to the Hagedorn point should be described by the general form given by Eq. (34). As we have seen the nature of the singularity is dependent only on the number of non-compact directions, and only the Hagedorn temperature itself is sensitive to the details of the compactification. The free energy now diverges at the transition, and the same issues as already discussed above arise.

4 Microcanonical Ensemble

In the previous section, we studied the canonical ensemble and found that there are large energy fluctuations as the system approached the Hagedorn temperature, implying the system did not have a smooth thermodynamic limit for a fixed temperature. These difficulties may be circumvented by studying the system in the more fundamental microcanonical ensemble. The non-smoothness of the thermodynamic limit then shows up in the guise of long-string ensembles that dominate the entropy [29, 31, 39]. In particular, a better understanding can be achieved in this approach in terms of the behavior of the entropy as $T_H \to 0$.

Using the results for the canonical ensemble, we can apply an inverse Laplace transform to obtain the entropy in the microcanonical ensemble for fixed $E$ and fixed charge/angular momentum $Q$,

$$\Omega(E, Q) = \int_{-i\infty}^{i\infty} d\nu \int_{-i\infty}^{i\infty} \frac{d\beta}{2\pi i} e^{\beta E + \nu Q} Z(\beta, \nu),$$

(41)

where the contour in $\beta$ is taken to lie to the right of any singularity in $Z$ while the contour in $\nu$ is along the imaginary axis. In the following we will denote the integrand obtained after performing the $\beta$ integral but before performing the $\nu$ integral as $Z(E, \nu)e^{\nu Q}$, where $Z(E, \nu)$ may be regarded as the partition function of an ensemble with fixed total energy, but in contact with a charge reservoir with fixed chemical potential $\nu$. 
4.1 Torus Compactification

We will restrict our attention to the case of a single chemical potential $\nu$. For $d > 0$, at large $E$ the integral (41) receives its leading contribution from the nearest singularity, in $Z_1$ leading to

$$Z(E, \nu) \sim \frac{e^{\beta_H(\nu)E}}{E^{d/2+1}},$$

(42)

where $\beta_H(\nu)$ is given by (15). For $d = 0$, since the coefficient in front of $\log(\beta - \beta_H)$ in Eq. (10) is exactly $-1$, after exponentiation giving

$$Z(\beta, \nu) = \text{const}/(\beta - \beta_H(\nu)),$$

(43)

we obtain

$$Z(E, \nu) \sim e^{\beta_H(\nu)E}.$$  

(43)

Since we are discussing an ensemble in contact with a charge reservoir, it is necessary to check the ensemble is stable with respect to fluctuations. In the case at hand this reduces to the condition that $d^2 \beta_H(\nu)/d\nu^2 > 0$. It is straightforward to check this is satisfied for all values of $\nu$. This implies the integral (41) will be dominated by the stationary point

$$Q = -\frac{\partial \beta_H(\nu)}{\partial \nu} E, \quad \text{or} \quad \nu = -\beta_0 \frac{Q}{\sqrt{E^2 - Q^2}},$$

(44)

where, as noted earlier, we have absorbed the dependence on $R$ by replacing $Q/R$ by $Q$ and $\nu/R$ by $\nu$. Ignoring prefactors that are power-like in $E$ and $Q$ and $d$ dependent, we find

$$\Omega(E, Q) \sim e^{\beta_0 \sqrt{E^2 - Q^2}}.$$  

(45)

As $Q \to E$, the linear growth in the entropy $S = \log \Omega$ with energy disappears. The temperature in microcanonical ensemble, $T = (\partial S/\partial E)^{-1}$, which at this level of approximation is simply the Hagedorn temperature, becomes

$$T_H \simeq \beta_0^{-1} \sqrt{1 - (Q/E)^2},$$

(46)

and it goes to zero as $Q$ approaches its maximal value. As noted in Ref. [15] this is a useful model for the finite temperature and baryon number chemical potential phase diagram in QCD.

4.2 PP-Wave Background

Precisely the same arguments carry over to the microcanonical ensemble in the pp-wave background, with the replacement of $\beta_H(\nu)$ by $\beta_H(\nu; \mu)$ given by the solution of (29). Expressing this in terms of the thermodynamic potential $Z(E, \nu)$ we find

$$Z(E, \nu) \sim \begin{cases} 
\exp(\beta_H(\nu; \mu)E)/E^{3/2} & : \text{non-compact longitudinal direction} \\
\exp(\beta_H(\nu; \mu)E) & : \text{compact longitudinal direction}
\end{cases}$$

(47)

In the limit $\mu a \ll 1$, we recover the standard flat space results for the entropy, obtained by substituting $\beta_H = \sqrt{\beta_0^2 + \nu^2}$ into (44) and integrating over $\nu$ in (41). Therefore the entropy can be obtained simply from (45), subject to the condition

$$\mu \sqrt{\alpha'} \ll \sqrt{(E - Q)/(E + Q)}.$$  

(48)
This indicates the dual sector of the Yang-Mills theory should show an exponentially rising density of states with energy. The scale of the inverse string tension $\alpha'$ in string variables translates into inverse string tension $\alpha'_{YM} = (\sqrt{g_{YM}^2 N})^{-1} R_{S^3}^2$ in Yang-Mills variables \((35)\), where $R_{S^3}$ is the radius of the $S^3$, with Hagedorn temperature becoming

$$T_H \simeq (8 \sqrt{\pi \alpha'_{YM}})^{-1} \sqrt{1 - (Q_{YM}/E_{YM})^2}. \quad (49)$$

For finite $N$ we expect to see a cutoff on the range of energy exhibiting this behavior, as suggested by the toy model of Ref. [37], and already discussed above.

When $\mu a \gg 1$, $\beta_H(\nu; \mu)$ is given by \((33)\). We may then compute \((41)\) in a saddle point approximation where

$$\mu a \sim \pi \log \left( \frac{E + Q}{E - Q} \right) + \pi \log (\mu^2 \alpha'). \quad (50)$$

Ignoring prefactors that are powerlike in $E$ and $Q$, one arrives at

$$\Omega(E, Q) \sim \exp \left( \frac{\pi}{\mu} (E - Q)(\log \frac{E + Q}{E - Q} + \log (\mu^2 \alpha')) \right), \quad (51)$$

subject to the condition

$$\mu \sqrt{\alpha'} \gg \sqrt{(E - Q)/(E + Q)}. \quad (52)$$

This limit corresponds to $(E - Q)/(E + Q) \to 0$. As $Q \to E$, the entropy decreases as $(\pi/\mu)(E - Q) \log (E - Q)$, and the Hagedorn temperature decreases as

$$T_H \sim (\mu/\pi)(\log (E - Q))^{-1}. \quad (53)$$

It is interesting to note the pp-wave mass parameter $\mu$ now sets the scale of the Hagedorn-like growth in the number of states. When we translate this into Yang-Mills variables \((35)\) we obtain an inverse string tension proportional to $R_{S^3}^2$ only.

We note in passing that when the subleading corrections to the saddle point approximation are computed, we find the temperature can become negative in the microcanonical ensemble. Likewise the specific heat is not well-behaved. This is symptomatic of the long string states that dominate the microcanonical ensemble at high energies \([29, 31, 39]\), and is a sign of large fluctuations in the canonical ensemble, rather than some inconsistency in the microcanonical ensemble.

## 5 Conclusion

We have studied the thermodynamics of IIB free strings as one approaches the Hagedorn transition from the low temperature phase in both the pp-wave background and in flat space on compact tori with non-zero chemical potential. A comparison reveals that the pp-wave transition is well characterized by the flat space example on an 8-torus with a chemical potential in the longitudinal momentum playing the role of the R charge. Apparently the harmonic “potential” in the transverse co-ordinates, $\mu \sum x^i x^i$, of the pp-wave
background effectively acts like a compact space. This correspondence for the Hagedorn
temperature and the free energy become an identity if we take the near flat space limit
\( \mu \sqrt{\alpha'} \ll 1 \).

In both cases the free energy exhibits a singularity, \( F \sim c\sqrt{\beta - \beta_H} + \) regular, \( c > 0 \),
consistent with the onset of a phase transition rather than a limiting temperature. This
supports the interpretation of a Hagedorn transition as a close analogue of the conventional
picture for the confinement/deconfinement transition. Indeed the real interest in this
calculation for the pp-wave is the existence of a dual \( \mathcal{N} = 4 \) super Yang-Mills field theory
on \( R \times S^3 \). On the Yang-Mills side it is possible to go to the high temperature side of
the Hagedorn transition where the entropy is expected to rise sharply from \( O(1) \) to \( O(N^2) \).
At \( N = \infty \), the \( O(1) \) term which exhibits the Hagedorn singularity does not have to
be connected analytically to the \( N^2 \) term. However at finite \( N \) (when strings begin to
interact) the two regions should be part of a single thermodynamics description.

Understanding this transition on the string side is a long standing puzzle in interacting
string theory. A natural supposition as suggested by Witten [3, 4] is that the Hawking-
Page phase transition for black holes that occurs in the strong coupling description of
strings in \( AdS^5 \) causes interacting strings to deconfine. A recent paper by Zayas and
Sonnenschein [40] has taken the pp-wave limit of the black hole metric for the \( AdS^5 \)
space. A difficult but interesting problem is to explore this trans-Hagedorn phase as
strings propagate in this background. In view of the exact mapping for gravity/gauge
duality in the pp-wave background, this appears to be a good setting in which to further
explore these phenomena in string dynamics at extreme temperature. Ultimately one
goal is to shed some light on QCD at finite temperature and chemical potential [42,43], a
region of phase space out of reach of present non-perturbative (i.e. lattice) methods but
of growing experimental and phenomenological interest in heavy ion collisions and ultra
dense astronomical objects.

Acknowledgements
This research is supported in part by DOE grants DE-FE0291ER40688-Task A and DE-
FG02-91ER40676.

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