Supersymmetry, holonomy and Kundt spacetimes

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Abstract
Supersymmetric solutions of supergravity theories, and consequently metrics with special holonomy, have played an important role in the development of string theory. We describe how a Lorentzian manifold is either completely reducible, and thus essentially known, or not completely reducible so that there exists a degenerate holonomy invariant lightlike subspace and consequently admits a covariantly constant or a recurrent null vector and belongs to the higher-dimensional Kundt class of spacetimes. These Kundt spacetimes (which contain the vanishing and constant curvature invariant spacetimes as special cases) are genuinely Lorentzian and have a number of interesting and unusual properties, which may lead to novel and fundamental physics.

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1. Introduction

We discuss the geometrical properties of spacetimes in the context of string theory. In the study of supersymmetry, there are two classes of solutions. If the spacetime admits a covariantly constant time-like vector, the spacetime is static and \((1+10)\)-decomposable, where the ten-dimensional transverse space is Riemannian. The second class of solutions consists of spacetimes which admit a covariantly constant light-like vector, and belong to the higher-dimensional Kundt class of spacetimes. These spacetimes are genuinely Lorentzian and have many mathematical properties quite different from their Riemannian counterparts, which can lead to interesting and novel physics. It is within string theory that the full richness of Lorentzian geometry is consequently realized, where the Kundt spacetimes play a fundamental role. Indeed, in gravitational physics the richness of Lorentzian spacetimes and general relativity is often not fully realized. For example, in many applications in cosmology Newtonian gravity, or small deviations thereof, suffices. In many applications of quantum field theory on a curved background or quantum gravity there exists a unique time, and space...
and time are essentially treated independently (and the structure of the Lorentzian manifold is not fully utilized).

In particular, we discuss the work of [1–3], and show that it follows that a Lorentzian manifold is either completely reducible, and thus essentially known, or not completely reducible, which is equivalent to the existence of a degenerate invariant subspace and entails the existence of a holonomy invariant light-like subspace. In this latter case the Lorentzian manifold decomposes into irreducible or flat Riemannian manifolds and a Lorentzian manifold with indecomposable, but non-irreducible holonomy representation; i.e., with (a one-dimensional) invariant light-like subspace. Such Lorentzian spacetimes admit a covariantly constant or a recurrent null vector (CCNV/RNV), and contain the vanishing and constant curvature invariant spacetimes (VSI/CSI) as special cases.

1.1. Supersymmetry

Supersymmetric solutions of supergravity theories have played an important role in the development of string theory. For example, supersymmetric compactifications may lead to realistic models of particle physics and a microscopic interpretation of black hole entropy in string theory is best understood for supersymmetric black holes. The existence of parallel (Killing) spinor fields, which play a central role in supersymmetry, accounts for much of the interest in metrics with special holonomy in mathematical physics. The existence of a parallel spinor on a Lorentzian manifold defines a parallel vector field which can be null. Hence the manifold has an indecomposable, non-irreducible factor. In the physically important dimensions below 12 the maximal indecomposable Lorentzian holonomy groups admitting parallel spinors are known [4, 5].

A systematic classification of supersymmetric M-theory vacua with zero flux (that is, 11-dimensional Lorentzian manifolds with vanishing Ricci curvature and admitting covariantly constant spinors) was provided in [5]. There are two classes of solutions. If the spacetime admits a covariantly constant time-like vector, the spacetimes are static and their classification reduces to the classification of 10-dimensional Riemannian manifolds with holonomy contained in \( SU(5) \) (i.e., to the classification of Calabi–Yau 5-folds). The second class of solutions consists of spacetimes which are not static but admit a covariantly constant light-like vector. Supersymmetric solutions of 11-dimensional supergravity can be classified according to the holonomy of the supercovariant derivative arising in the Killing spinor condition. This class can be extended to M-theory [6, 7].

The following results are useful in 11D supergravity. Let \((\mathcal{M}, g_{ab})\) be a Lorentzian manifold. Then there is a parallel (i.e., covariantly constant) vector field on \(\mathcal{M}\) if and only if the holonomy has a trivial subrepresentation [8]. Let \((\mathcal{M}, g_{ab})\) be a simply connected, indecomposable, reducible Lorentzian manifold with Abelian holonomy algebra \( \mathbb{R}^{m-2} \). Then the manifold admits parallel spinors whose corresponding vectors are light-like [3]. The classification can be further refined by the holonomy group of the spacetime. In the Lorentzian case there are subgroups of the Lorentz group which act reducibly yet indecomposably on the spacetime that plays an important role. We shall discuss these results in more detail later.

Summarizing the situation in 11-dimensional supergravity [5]:

1.1.1. Static vacua. If the spacetime admits a covariantly constant time-like vector, then the holonomy group must be contained in the subgroup \( \text{Spin}(10) \subset \text{Spin}(10, 1) \), leaving a time-like vector invariant. Spacetimes with holonomy group \( H \subset SU(5) \) always admit a Ricci-flat metric and hence, equipped with this metric, they satisfy the supergravity equations of motion. Such spacetimes contain a time-like Killing vector and are static and consequently...
locally isometric to a product $\mathbb{R} \times X$ with metric
\[
\text{d} s^2 = -\text{d} t^2 + \text{d} s^2(X),
\]
where $X$ is any Riemannian ten-manifold with holonomy contained in $SU(5)$, that is, a Calabi–Yau 5-fold.

1.1.2. Non-static vacua. If the spacetime admits a covariantly constant null vector, the isotropy subgroup of a null spinor is contained in the isotropy subgroup of the null vector, which in arbitrary dimensions is isomorphic to the spin cover of
\[
ISO(n - 2) = SO(n - 2) \ltimes \mathbb{R}^{n-2} \subset SO(n - 1, 1).
\]
For $n \leq 5$ this means the holonomy group is $\mathbb{R}^{n-2}$, which implies that the metric is Ricci-null. Bryant [4] has recently written the most general local metric with this holonomy (particularly for a 11-dimensional spacetime).

1.2. Recurrent null vector
Another group of interest in Lorentzian signature in $n$ dimensions is the maximal proper subgroup of the Lorentz group, $\text{Sim}(n - 2)$. The Einstein metric of $\text{Sim}(n - 2)$ holonomy, with and without a cosmological constant, was studied in [9].

In general, time-dependent solutions cannot be supersymmetric. However, a time-dependent solution can arise from the dimensional reduction of a time-independent solution in one higher dimension that admits a Killing spinor. Because the Killing spinor is not boost invariant, it does not descend to the lower-dimensional spacetime which is not supersymmetric.

Multi-centre metrics can arise by dimensional reduction from a metric with a reduced holonomy group. An example is when the Killing vector field is (covariantly constant and) null. The holonomy is then reduced to the Abelian subgroup $\mathbb{R}^{n-2}$ of the $n$-dimensional Lorentz group $SO(n - 1, 1)$. These Brinkmann waves [10] (or CCNV spacetimes) necessarily have vanishing Ricci scalar curvature. They can be used to obtain time-independent and time-dependent extremal Kaluza–Klein multi black holes and multi Kaluza–Klein monopole metrics.

Dimensional reduction on a null Killing vector that is not necessarily covariantly constant has also been studied. Indeed, solutions with non-vanishing spatial curvature from Einstein metrics in one higher dimension with reduced holonomy and with non-vanishing cosmological constant must necessarily be obtained for a holonomy group which is larger than that of $\mathbb{R}^{n-2}$.

The $\text{Sim}(n - 2)$ holonomy metrics are of interest in $M$-theory and string theory. In any of the metrics of $\text{Sim}(n - 2)$ holonomy there exists a preferred spinor which generalizes the covariantly-constant spinor that gives rise to supersymmetric CCNV metrics; however, this preferred spinor is not associated with any supersymmetry.

2. Holonomy theorem
In the Riemannian case, the de Rham decomposition theorem [11] states that if the holonomy group (see the appendix) of a simply-connected Riemannian manifold $M$ at $m \in M$, $\Phi$, preserves a proper subspace of the tangent space (i.e., is reducible), then the tangent space is decomposable (into holonomy invariant subspaces) and $M$ is (locally) isometric to a product manifold. That is, a Riemannian manifold is locally a product of Riemannian manifolds with irreducible holonomy algebras. This allows one to restrict oneself to groups acting irreducibly, and leads to the classification of Riemannian holonomy groups of Berger [12].
The possible restricted holonomy groups of irreducible Riemannian manifolds are known [4, 12]. In the familiar positive-definite case, Berger’s classification [12] implies that a simply connected, irreducible, non-symmetric manifold can be Calabi–Yau, with SU(n) holonomy in 2n dimensions, hyper-Kähler, with Sp(n) holonomy in 4n dimensions, have G2 holonomy in seven dimensions and spin(7) holonomy in eight dimensions. In each case, there exists a metric with the given holonomy and vanishing Ricci tensor. An exceptional case with Sp(n)Sp(1) holonomy and Einstein metric also exists.

2.1. Lorentz manifolds

The Ambrose–Singer theorem [13] says that the Lie algebra of the holonomy group of the manifold is determined by the curvature of the corresponding connection; thus the existence of a parallel tensor constrains the holonomy algebra and determines the product structure (a precise statement, and a number of appropriate definitions, are given in the appendix).

The classification of holonomy groups in Lorentzian spacetimes is quite different since the Riemannian decomposition theorem does not apply without modification [1]. It is important to extend the de Rham decomposition theorem to the Lorentzian case [2, 11, 13].

The classification of the holonomy-irreducible case proceeds much as in the positive definite case. In the Lorentzian case, there are spacetimes such that no proper subgroup of the Lorentz group acts irreducibly. A general survey of the pseudo-Riemannian case when the holonomy acts irreducibly, particularly regarding the existence of parallel spinor fields, was given in [14]. It is this difference that makes classifying the possible pseudo-Riemannian metrics having parallel spinors more difficult. Thus, the Lorentzian analogue of the decomposition theorem, due to Wu [2], requires the weaker notion of weak irreducibility.

Wu’s theorem [2] asserts that every simply-connected, complete semi-Riemannian manifold is isometric to a product of simply-connected, complete semi-Riemannian manifolds, of which one can be flat and all others are indecomposable or ‘weakly-irreducible’ (i.e., with no non-degenerate invariant subspace under holonomy representation). For a Riemannian manifold this theorem asserts that the holonomy representation is completely reducible; i.e., decomposes into factors which are trivial or irreducible, and are again Riemannian holonomy representations. For pseudo-Riemannian manifolds indecomposability is not the same as irreducibility. We can have degenerate invariant subspaces under holonomy representation.

This exotic class of holonomy groups, acting reducibly but indecomposably, is of particular interest in supersymmetry. The classification of these spacetimes is still not complete [5] (see also [1, 15, 16]). All simply connected irreducible non-locally symmetric Lorentzian manifolds admitting parallel spinors were studied by [15]. All irreducible factors are known by the Berger classification of possible irreducible semi-Riemannian holonomy groups [12]. Bryant determined the local generality of pseudo-Riemannian metrics with parallel spinors, with and without imposing the Ricci-flat condition [4].

2.2. Classification

Thus in order to classify holonomy groups of simply-connected Lorentzian manifolds it is necessary to find the possible holonomy groups of indecomposable (i.e., weakly-irreducible), but non-irreducible Lorentzian manifolds. The holonomy algebra of such a manifold of dimension n is contained in the Sim(n−2) algebra. The projections of this holonomy algebra, classified into four types based on the possible projections on R and R(n−2), were studied in [1]. A decomposition property for the so(n−2)-projection was also found in [1] (i.e., there
is a decomposition of the representation space into irreducible components and of the Lie algebra into ideals which act irreducibly on the components).

Thus, it suffices to study irreducibly acting groups or algebras, a fact which is necessary for obtaining a classification. An algebraic criterion on the so(n − 2)-component of an indecomposable, reducible, simply-connected Lorentzian manifold, in analogy with the well-known Berger criterion for holonomy algebras, was derived by Leistner [3]. Furthermore, it was shown that every irreducible weak-Berger algebra which is contained in u((n − 2)/2) is a Berger algebra, and from the decomposition property is, in particular, a Riemannian holonomy algebra. In addition, the result was then repeated for a simple weak-Berger algebra which is not contained in u((n − 2)/2) [3]. Finally, it was shown that there are no semisimple, nonsimple, irreducibly acting Lie algebras not contained in u((n − 2)/2) which are weak-Berger but not Berger [16] (the converse results were also discussed).

It therefore follows that every SO(n − 2)-projection of an indecomposable, non-irreducible Lorentzian holonomy group is a Riemannian holonomy group [3].

The holonomy group at a point p in an indecomposable non-irreducible Lorentzian manifold, acting on TpM, then has a null, one-dimensional invariant subspace, which is equivalent to the existence of a recurrent null vector field. The existence of the recurrent vector leads to the holonomy algebra \((R \oplus \mathfrak{so}(n − 2)) \times \mathbb{R}^{n−2}\), which is the Sim(n − 2) algebra. The corresponding group is the maximal proper subgroup of the Lorentz group \(SO(n − 1, 1)\), a subgroup of dimension \((n^2 − 3n + 4)/2\). Sim(n − 2) holonomy implies the existence of a recurrent null vector [9].

With respect to the four projections of the holonomy algebra on the \(R\)- and on the \(\mathbb{R}^{(n−2)}\)-components, we have the following [1]: For the types I and II the holonomy is equal to \((R \oplus g) \times \mathbb{R}^{(n−2)}\) and \(g \times \mathbb{R}^{(n−2)}\), respectively. In the case of types II and IV the projection on \(R\) is zero, which implies the existence not only of a recurrent null vector field but also of a parallel one. In the case of types III and IV the \(R\)- and the \(\mathbb{R}^{(n−2)}\)-components are coupled to the \(\mathfrak{so}(n − 2)\)-component.

All candidates to the weakly-irreducible not irreducible holonomy algebras of Lorentzian manifolds are known. To complete the classification of holonomy algebras it was proved that all Berger algebras can be realized as holonomy algebras of Lorentzian manifolds [16].

2.2.1. Summary. As a result, we have the following [1–3]:

For a Lorentzian manifold, \(M\), the de Rham/Wu-decomposition yields the following two cases. (1) Completely reducible: here \(M\) decomposes into irreducible or flat Riemannian manifolds and a manifold which is an irreducible or flat Lorentzian manifold or \((\mathbb{R}, −dt)\). The irreducible Riemannian holonomies are known, as well as the irreducible Lorentzian one, which has to be the whole of \(SO(1, n − 1)\). (2) Not completely reducible: this is equivalent to the existence of a degenerate invariant subspace and entails the existence of a holonomy invariant light-like subspace. The Lorentzian manifold decomposes into irreducible or flat Riemannian manifolds and a Lorentzian manifold with indecomposable, but non-irreducible holonomy representation, i.e., with (a one-dimensional) invariant light-like subspace.

2.3. The 4D case

In four dimensions (4D) there are 15 holonomy subgroups, labelled \(R_1−R_{15}\) [17]. \(R_1\) is the trivial group and \(R_5\) cannot exist as a spacetime holonomy. \(R_{15}\) is the full Lorentz group and corresponds to the irreducible case. The non-degenerately reducible subgroups are \(R_2, R_3, R_4, R_6, R_7, R_{10}\) and \(R_{13}\) (the corresponding spacetimes include the 1 + 3 and
2 + 2 decomposable spacetimes; e.g., $R_{13}$). Finally, the reducible but degenerate subgroups are $R_8$, $R_9$, $R_{11}$, $R_{12}$ and $R_{14}$; spacetimes with $R_8$ and $R_{11}$ holonomy admit a CCNV, while spacetimes with $R_9$, $R_{12}$ and $R_{14}$ holonomy admit a RNV.

The holonomy group in a 4D Lorentzian spacetime was reviewed by Hall [17]; the holonomy algebras and associated RNV and CCNV in 4D were given explicitly in the table therein. The holonomy group can be used to find all metrics on $M$ [18]. The well-known classification of pseudo-Riemannian metrics with parallel spinors in 4D has been reviewed by Bryant [4]; the Lorentzian case was studied in [19].

3. Kundt spacetimes

3.1. Higher-dimensional Kundt spacetimes

The $n$-dimensional Kundt metric can be written [20] as

$$ds^2 = 2du [dv + H(v, u, x^k) du + W_i(v, u, x^k) dx^i] + g_{ij}(u, x^k) dx^i dx^j. \tag{1}$$

The metric functions $H$, $W_i$ and $g_{ij}$ satisfy the Einstein equations ($i, j = 2, \ldots, N - 2$). The metric (1) possesses a null vector field $\ell = \frac{\partial}{\partial v}$ which is geodesic, non-expanding, shear-free and non-twisting (i.e., the Ricci rotation coefficients $L_{ij} \equiv \ell_i \ell_j = 0$). Kundt spacetimes are of Riemann type II (or simpler) [20].

3.1.1. Covariantly constant null vector (CCNV). In general, the generalized Kundt metric has the non-vanishing Ricci rotation coefficients $L$ and $L_i$. From $L = 0$ we obtain $H_v = 0$ and $L_v = 0$ implies $W_{i,v} = 0$ [21]. The remaining transformations can be used to further simplify the remaining non-trivial metric functions. Thus, the aligned, repeated, null vector $\ell$ of (1) is a null Killing vector (KV) if and only if $H_v = 0$ and $W_{i,v} = 0$ (whence the metric no longer has any $v$ dependence). Furthermore, it follows that if $\ell$ is a null KV then it is also covariantly constant. Without any further restrictions, the higher-dimensional metrics admitting a null KV are of Ricci and Weyl type II. Therefore, the most general metric that admits a covariantly constant null vector (CCNV) is (1) with $H = H(u, x^k)$ and $W_i = W_i(u, x^k)$ [21]; we shall refer to this as a Kundt–CCNV metric.

3.1.2. Recurrent null vector (RNV). As shown in [9], the null vector field of a metric with Sim$(n - 2)$ holonomy is recurrent, and it follows that the null congruence is geodesic, twist-free, expansion-free and shear-free; i.e., the metrics belong to the class of Kundt metrics [20]. Metrics with Sim$(n - 2)$ holonomy may be cast in the form of (1) with $W_i = W_i(u, x^k)$ (i.e., independent of $v$), called a Kundt–RNV metric or a metric of Walker form. For the Sim$(n - 2)$ metrics there are no further restrictions on the metric functions.

3.1.3. Constant curvature invariants (CSI). If we require that all of the curvature invariants be constants, it follows that the metric function $H(v, u, x^k)$ in the Kundt metric (1) only contains terms polynomial and second order in $v$; i.e.,

$$H(v, u, x^k) = v^2 \hat{\sigma} + v H^{(1)}(u, x^k) + H^{(0)}(u, x^k),$$

where $\hat{\sigma} = (4\sigma + W^{(1)} W^{(1)})/8$ and $\sigma$ is a constant. In addition, for a Kundt–CSI spacetime there exists a ($u$-dependent) diffeomorphism such that the transverse metric can be made $u$-independent. Furthermore, the transverse metric $d\tilde{s}^2$ is locally homogeneous. Hence, there is no loss of generality in assuming that $d\tilde{s}^2$ is a locally homogeneous space, $M_{\text{Hom}}$. 
3.1.4. Vanishing curvature invariants (VSI). In this case all curvature invariants to all orders vanish, which implies that we can set 
\[ \sigma = 0, \quad g_{ij} dx^i dx^j = \delta_{ij} dx^i dx^j. \]
(This is the \( W^{(1)} = 0 \) case of the general VSI metrics.) The Riemann tensor is of type III (or simpler).

3.2. Supergravity theories

The VSI and CSI spacetimes are of particular interest since they are solutions of supergravity or superstring theory, when supported by appropriate bosonic fields. The supersymmetry properties of these spacetimes have also been discussed.

In [22] it was shown that the higher-dimensional VSI spacetimes with fluxes and dilaton are solutions of type IIB supergravity. Exact VSI solutions of IIB supergravity with NS–NS and RR fluxes and dilaton have been constructed. The solutions are classified according to their Ricci type (N or III). The Ricci type N solutions are generalizations of pp-wave type IIB supergravity solutions. The Ricci type III solutions are characterized by a non-constant dilaton field. In particular, it was shown that all Ricci type N VSI spacetimes are solutions of supergravity (and that Ricci type III VSI spacetimes are also supergravity solutions if supported by appropriate sources), and similar results are expected in all supergravity theories. It was also argued that, in general, the VSI spacetimes are exact string solutions to all orders in the string tension. It is known that the higher-dimensional pp-wave spacetimes are exact solutions in string theory, in type IIB superstrings with an R–R five-form, and also with NS–NS form fields [23, 24].

A number of CSI spacetimes are also known to be solutions of supergravity theory when supported by appropriate bosonic fields [25]. It is known that \( AdS_d \times S^{(N-d)} \) (in short \( AdS \times S \)) is an exact solution of supergravity (and preserves the maximal number of supersymmetries) for certain values of \( (N, d) \) and for particular ratios of the radii of curvature of the two space forms. Such spacetimes (with \( d = 5, N = 10 \)) are supersymmetric solutions of IIB supergravity (and there are analogous solutions in \( D = 11 \) supergravity). \( AdS \times S \) is an example of a CSI spacetime [26]. There are a number of other CSI spacetimes known to be solutions of supergravity and admit supersymmetries; namely, generalizations of \( AdS \times S \) and (generalizations of) the chiral null models [27]. The Weyl type III \( AdS \) gyratons [28] (which is a CSI spacetime with the same curvature invariants as pure \( AdS \)) is a solution of gauged supergravity (the \( AdS \) gyraton can be cast in the Kundt form [25]).

Some explicit examples of CSI supergravity spacetimes were constructed in [25] by generalizing a homogeneous Einstein spacetime, \((M_{\text{Hom}}, g)\), of Kundt form to an inhomogeneous spacetime, \((M, g)\), by including arbitrary functions \( W_i^{(0)}(u, x^k) \), \( H^{(1)}(u, x^k) \) and \( H^{(0)}(u, x^k) \). In the examples \((M_{\text{Hom}}, g)\) was chosen to be a regular Lorentzian Einstein solvmanifold or the transverse space was chosen to be the Heisenberg group, \( SL(d, \mathbb{R}) \), or the 3-sphere, \( S^3 \).

The supersymmetry properties of VSI spacetimes have also been studied. It is known that in general if a spacetime admits a Killing spinor, it necessarily admits a null or time-like Killing vector. Therefore, a necessary (but not sufficient) condition for a particular supergravity solution to preserve some supersymmetry is that the spacetime possess such a Killing vector. Therefore the supersymmetry properties of VSI type IIB supergravity solutions with a CCNV were studied [22]. Supersymmetry was also studied in the CSI-CCNV subclass of supergravity spacetimes [25].
4. Discussion

A Lorentzian manifold admitting an indecomposable but non-irreducible holonomy representation (i.e., with a one-dimensional invariant light-like subspace) is a CCNV or RNV (Kundt) spacetime, which contains the VSI and CSI subclasses as special cases. Therefore, the Kundt spacetimes that are of particular physical interest are degenerately reducible, which leads to complicated holonomy structure and various degenerate mathematical properties. Such spacetimes have a number of other interesting and unusual properties, which may lead to novel and fundamental physics. Indeed, a complete understanding of string theory is not possible without a comprehensive knowledge of the properties of the Kundt spacetimes.

For example, in general a Lorentzian spacetime is completely classified by its set of scalar polynomial curvature invariants. However, this is not true for Kundt spacetimes [29] (i.e., they have important geometrical information that is not contained in the scalar invariants). All VSI spacetimes and CSI spacetimes that are not locally homogeneous (including the important CCNV and RNV subcases) belong to the Kundt class [20]. In these spacetimes all of the scalar invariants are constant or zero. This leads to interesting problems with any physical property that depends essentially on scalar invariants, and may lead to ambiguities and pathologies in models of quantum gravity or string theory.

As an illustration, in many theories of fundamental physics there are geometric classical corrections to general relativity. Different polynomial curvature invariants (constructed from the Riemann tensor and its covariant derivatives) are required to compute different loop-orders of renormalization of the Einstein–Hilbert action [30]. In specific quantum models such as supergravity there are particular allowed local counterterms (such as, e.g., the square of the quadratic Bel–Robertson tensor at three-loop order in $D=4$, $N=1$ SUGRA; two-loop divergences involving terms quartic in curvature in $D=11$ SUGRA, etc [31]).

In particular, a classical solution is called universal if the quantum correction is a multiple of the metric. In [32] metrics of holonomy $\text{Sim}(n-2)$ were investigated, and it was found that all four-dimensional $\text{Sim}(2)$ metrics are universal and consequently can be interpreted as metrics with vanishing quantum corrections and are automatically solutions to the quantum theory. The RNV and CCNV (Kundt) spacetimes therefore play an important role in the quantum theory, regardless of what the exact form of this theory might be.

Finally, in the domain of Planck scale curvatures, the character of gravity may change radically due to its underlying quantum nature. The expectation is that singularities will be ‘resolved’ in the correct theory of quantum gravity. Indeed, spacetimes which are singular in general relativity can be completely nonsingular in string theory [33]. However, it is not true that all singularities are removed in string theory. The VSI and CSI Kundt spacetimes with arbitrary $u$-dependence are exact solutions to string theory [23]. However, if any of the metric functions diverge as $u \rightarrow u_0$, then the Kundt spacetime is singular. By studying the string propagation in this background, the string does not always have well-behaved propagation through this singularity since the divergent tidal forces cause the string to become infinitely excited [23]. Indeed, it has been argued that on physical grounds, any reasonable theory will not ‘resolve’ certain classes of time-like singularities, since the elimination of these singularities would lead to a theory without a stable ground state [34].

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Appendix

A.1. Definitions

The holonomy group of \( M \) is denoted by \( \Phi \). If one restricts to members which are continuously or smoothly homotopic to zero one arrives at the restricted holonomy group of \( M \) denoted abstractly by \( \Phi^0 \). In fact \( \Phi^0 \) is the identity component Lie subgroup of \( \Phi \). If \( M \) is simply connected \( \Phi^0 = \Phi \). The associated holonomy algebra is denoted by \( \phi \).

The infinitesimal holonomy algebra of \( M \) at \( p, \phi'_{p} \) is a subalgebra of \( \phi \) for each \( p \in M \). By definition, \( \phi'_{p} \) is the Lie algebra spanned by \( R(X, Y), \nabla R(X, Y, Z), \nabla \nabla R(X, Y, Z, W), \ldots \). If the infinitesimal holonomy algebra is determined by the Riemann tensor alone, the manifold is called perfect. The corresponding unique connected Lie subgroup arising from \( \phi'_{p} \) is referred to as the infinitesimal holonomy group of \( M \) at \( p \) and is denoted by \( \Phi'_{p} \). Under weak assumptions on the manifold, \( \Phi^0 = \Phi \) [35].

The holonomy group \( \Phi \) of \( M \) is called reducible if for some (and hence any) \( p \in M \) and for some non-trivial proper subspace \( V \subseteq \mathbb{T}_pM, V \) is invariant under each member of \( \Phi_p \). Otherwise \( \Phi \) is called irreducible. Such a subspace \( V \) is called holonomy invariant. Further, \( \Phi \) is called non-degenerately reducible if a (non-trivial proper) non-null holonomy invariant subspace of \( \mathbb{T}_pM \) exists at some (and hence every) \( p \in M \). In such a situation \( \mathbb{T}_pM \) is said to be decomposable, weakly reducible or non-degenerately reducible. If one instead considers the restricted holonomy group \( \Phi^0 \), one speaks of strictly irreducible and strictly holonomy-indecomposable manifolds.

If \( \Phi \) is reducible, but not non-degenerately reducible, it is called degenerately reducible. Therefore, in the Lorentzian case, there is a distinction to be made between a metric being holonomy-irreducible (no parallel subbundles of the tangent bundle), being holonomy-indecomposable (no parallel splitting of the tangent bundle), and being indecomposable (no local product decomposition of the metric) [2].

A.2. The Ambrose–Singer theorem

Theorem 1 (Ambrose–Singer). The Lie algebra \( \phi_p \) of \( \Phi_p \) (and of \( \Phi \)) is generated by

\[
R_{abcd} X^c_q Y^d_q
\]

where \( X^q \) and \( Y^q \) range over all possible tangent vectors at \( q \) and the point \( q \) ranges over all points that can be joined to \( p \) by a parallel-transported curve.

References

[1] Bérard-Bergery L and Ikemakhen A 1993 Proc. Symp. Pure Math. (AMS) 54 27
[2] Wu H 1967 Proc. J. Math. 20 351  
Wu H 1964 Illinois J. Math. 8 291
[3] Leistner T 2003 Preprint math.dg/0305139  
Leistner T 2003 Preprint math.dg/0309274
[4] Bryant R L 1987 Ann. Math. 126 525  
Bryant R L 2000 Preprint math.dg/0004073
[5] Figueroa-O’Farrill J 2000 Class. Quantum Grav. 17 2925
[6] Duff M J and Liu J T 2003 Nucl. Phys. B 674 217
[7] Hull C 2003 Preprint hep-th/0305039
[8] Besse A L 1987 Einstein Manifolds (Berlin: Springer)
[9] Gibbons G W and Pope C N 2007 Preprint arXiv:0709.2440
[10] Brinkmann H W 1925 Math. Ann. 94 119
[11] De Rham G 1952 Commun. Math. Helv. 26 328
[12] Berger M M 1955 Bull. Soc. Math. France 83 279
Berger M M 1975 Ann. Sci. Ecole Norm. Sup. 74 85
[13] Kobayashi S and Nomizu K 1963 Foundations of Differential Geometry and its Applications vol 1 (New York: Interscience)
[14] Baum H and Kath I 1999 Ann. Global Anal. Geom. 17 1
[15] Ikemakhen A 1996 Ann. Sci. Math. Québec 20 53
[16] Galaev A S 2003 Preprint math.dg/0304407
Galaev A S 2005 Preprint math.dg/0502575
[17] Hall G S 2004 Symmetries and Curvature Structure in General Relativity (Singapore: World Scientific)
[18] Hall G S 1988 Gen. Rel. Grav. 20 399
[19] Tod K P 1983 Phys. Lett. B 121 241
Tod K P 1995 Class. Quantum Grav. 12 1801
[20] Coley A 2008 Class. Quantum Grav. 25 033001
[21] Ortin T 2000 Gravity and Strings (Cambridge: Cambridge University Press)
Coley A, Fuster A, Hervik S and Pelavas N 2006 Class. Quantum Grav. 23 7431
Coley A, Pelavas N and McNutt D 2008 Preprint arXiv:0809.0707
[22] Coley A, Fuster A, Hervik S and Pelavas N 2007 J. High Energy Phys. JHEP05(2007)032
[23] Amati D and Klimčík C 1989 Phys. Lett. B 219 443
Horowitz G T and Steif A R 1990 Phys. Rev. Lett. 64 260
Coley A A 2002 Phys. Rev. Lett. 89 281601
[24] Metsaev R R and Tseytlin A A 2002 Phys. Rev. D 65 126004
Blau M et al 2002 J. High Energy Phys. JHEP01(2002)047
Meessen P 2002 Phys. Rev. D 65 087501
Russo J G and Tseytlin A A 2002 J. High Energy Phys. JHEP09(2002)035
Maldacena J and Mazoz L 2002 J. High Energy Phys. JHEP12(2002)046
[25] Coley A, Fuster A and Hervik S 2007 Preprint arXiv:0707.0957
[26] Coley A, Hervik S and Pelavas N 2006 Class. Quantum Grav. 23 3053
[27] Horowitz G T and Tseytlin A A 1995 Phys. Rev. D 51 2896
[28] Frolov V P and Zelnikov A 2005 Phys. Rev. D 72 104005
Frolov V P and Fursaev D V 2005 Phys. Rev. D 71 104034
[29] Coley A, Hervik S and Pelavas N 2008 Class. Quantum Grav. 25 025008
[30] Dixon L, Harvey J, Vafa C and Witten E 1985 J. Nucl. Phys. B 261 678
Rocek M and Verlinde E 1992 J. Nucl. Phys. B 373 630
[31] Deser S, Kay J and Stelle K 1977 Phys. Rev. Lett. 38 527
Barnet et al 1998 Nucl. Phys. B 530 401
Deser S and Seminara D 2000 Phys. Rev. D 62 084010
[32] Coley A, Gibbons G W, Hervik S and Pope C N 2008 Preprint arXiv:0803.2438
[33] Horowitz G T 2005 New J. Phys. 7 201
[34] Horowitz G T and Myers R 1995 Gen. Rel. Grav. 27 915
[35] Ozeki H 1956 Nagoya Math. J. 10 105