Isometric immersions via compensated compactness
for slowly decaying negative Gauss curvature and rough data

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Abstract. In this paper, the method of compensated compactness is applied to the problem of isometric immersion of a two-dimensional Riemannian manifold with negative Gauss curvature into three-dimensional Euclidean space. Previous applications of the method to this problem have required decay of order $t^{-4}$ in the Gauss curvature. Here, we show that the decay of Hong (Commun Anal Geom 1:487–514, 1993) $t^{-2-\delta/2}$ where $\delta \in (0, 4)$ suffices.

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1. Introduction

In two recent papers, Chen et al. [2] and Cao et al. [1] have used the method of compensated compactness to establish global isometric immersions into $\mathbb{R}^3$ for two-dimensional Riemannian manifolds for rough data. In the examples considered, the Gauss curvature was negative and decayed at least as $t^{-4}$ where initial data were given at $t = 0$. Needless to say that leaves open the question as whether the compensated compactness method will work for a slower rate of decay. Certainly based on the paper of Hong [11] which has also been exposited in the book of Han and Hong [10], we expect the result to be true for decay of order $t^{-2-\delta/2}$ where $\delta$ is between 0 and 4. The proof of Hong is a careful study of the hyperbolic system of two balance laws (the two Codazzi equations) and one closure relation (the Gauss equation) and requires two separate steps. The first step is to establish existence of smooth solutions to the balance laws for small, smooth data prescribed at a large enough time $t = T_1$. The reason for this part is that it is only after large time that the decay of the Gauss curvature may be exploited to obtain the relevant $C^1$ a priori estimates. The second part of the proof is rather standard and simply asks for the initial data at $t = 0$ to be sufficiently small and smooth to enable us to get a solution up to $t = T_1$. Here, no reference is given in Han and Hong, but a standard existence, uniqueness theorem for quasi-linear hyperbolic systems will suffice. Such a theorem may be found in Janenko and Rozdestvenskii [12, Chapter 1, Sect. 8] where the growth in $C^1$ of solutions is governed by a coupled pair of ordinary differential equations, one which is of Riccati type. Hence, just as in the classical theory of ordinary differential equations, small data allow for a longer time of existence. In this paper as a compliment to Hong’s result, we reconsider the first part of Hong’s program and show that in fact that rough $L^\infty$ data suffice at the initial time $t = T_1$ and that the method of compensated compactness will yield existence of weak solutions to the Gauss–Codazzi system for $t > T_1$. Of course, this provides a new global result only if the second part of Hong’s proof is established in the non-classical sense. Indeed, we verify this part by carefully applying the existence result by Dafermos and Hsiao [7] in the setting of weak solutions of bounded variation in order to reach time.
$t = T_1$ from $t = 0$. Therefore, a new global immersion different from the smooth immersion in Hong [11] is constructed, which is actually $C^{1,1}$. It should be emphasized that the application of the compensated compactness method is the core of this article.

Before continuing a short historical view is in order. First, we note the relevant Gauss–Codazzi system can be written as a linearly degenerate system or what is termed “weakly nonlinear quasi-linear” system in the monograph of Janenko and Rozdestvenskii [12] and is discussed in Chap. 4, Section 4 of that book. They note that such systems possess the property that uniform boundedness of solutions on $0 \leq t \leq T$ and strict hyperbolicity imply uniform boundedness of first derivatives on $0 \leq t \leq T$ if these derivatives are initially bounded. Thus, it appears that the crucial estimates will be a uniform bound on the dependent variables and in addition a proof that the strict hyperbolicity is not lost. It was this path that was followed by Hong [11] and yields the following result:

**Theorem 1.1.** (Han and Hong [10, Theorem 10.2.2]) For a complete simply connected two-dimensional Riemannian manifold $(M, g)$ with negative Gauss curvature $K$ and metric $g = dt^2 + B^2(x, t)dx^2$. Assume for some constant $\delta > 0$

1. $t^{2+\delta}|K|$ is decreasing in $|t|$, $|t| > T$;
2. $\partial_x^i \ln|K|$, for $i = 1, 2$ and $t\partial_t \partial_x \ln|K|$ are bounded;
3. $K$ is periodic in $x$ with period $2\pi$.

Then, $(M, g)$ admits a smooth isometric immersion in $\mathbb{R}^3$.

But as noted above, a search for “corrugated immersions” would ask that the data be “rough” and not in $C^1$ and that is the issue pursued here. In particular, we study data in $L^\infty$. It should be mentioned that for discontinuous data of bounded variation, isometric immersions have been established using a different method in [4] but again with decay rate at least as $t^{-4}$. An exposition of the current state of the theory of systems of balance laws can be found in the book [6].

The paper contains six sections after this introduction. Section 2 provides a review of the isometric embedding problem and exposit the theorem of S. Mardare on non-smooth embeddings. In Sect. 3, we give a viscous approximation scheme for resolving the relevant balance laws: the equations of Gauss and Codazzi. In Sect. 4, we derive a priori $L^\infty$ estimates for the viscous approximations, and in Sect. 5, we show the viscous system possesses a crucial $H^{-1}_\text{loc}$ estimate which is needed to apply the method of compensated compactness. In Sect. 6, we recall the compensated compactness framework [2,3] and show that passage to the inviscid limit may be accomplished, hence yielding the desired non-smooth global immersion. Combining with the local existence result in [7] using the random choice method, we construct a global isometric immersion that is not necessarily smooth. Finally, Sect. 7 pursues the issue as to whether the decay rate of the Gauss curvature can be reduced, say as given by the choice

$$K = -\frac{1}{(3+t)^2(\ln(3+t))^p}, \quad t > 0, \quad p \text{ sufficiently large.}$$

Here, we show that for this choice, one of the key a priori estimates—the preservation of strict hyperbolicity—is retained. Hence, any lack of non-smooth embedding must be due to lack of $L^\infty$ bounds on two of the three components on the second fundamental form (the third component is a priori bounded).

### 2. Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be an open set. Consider a map $y : \Omega \to \mathbb{R}^3$ having the tangent plane of the surface $y(\Omega) \subset \mathbb{R}^3$ at $y(x_1, x_2)$ spanned by the vectors $\{\partial_1 y, \partial_2 y\}$. Then, the unit normal vector $n$ to the surface $y(\Omega)$ is given by

$$n = \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|}, \quad (2.1)$$
and the corresponding metric is
\[ ds^2 = dy \cdot dy, \]
or equivalently,
\[ ds^2 = (\partial_1 y \cdot \partial_1 y)(dx_1)^2 + 2(\partial_1 y \cdot \partial_2 y)dx_1 \, dx_2 + (\partial_2 y \cdot \partial_1 y)(dx_2)^2. \] 
(2.2)

The isometric immersion problem is an inverse problem: Given \((g_{ij})\) \(i, j = 1, 2\) functions in \(\Omega\), with \(g_{12} = g_{21}\), find a map \(y : \Omega \rightarrow \mathbb{R}^3\) so that
\[ dy \cdot dy = g_{11}(dx_1)^2 + 2g_{12}dx_1dx_2 + g_{22}(dx_2)^2, \] 
(2.3)
or equivalently,
\[ \partial_1 y \cdot \partial_1 y = g_{11}, \quad \partial_1 y \cdot \partial_2 y = g_{12}, \quad \partial_2 y \cdot \partial_2 y = g_{22} \] 
(2.4)
with a linearly independent set \(\{\partial_1 y, \partial_2 y\}\) in \(\mathbb{R}^3\). Hence, the isometric immersion problem is fully nonlinear in the three unknowns being the three components of the map \(y\).

We recall that a two-dimensional manifold \((M, g)\) parameterized by \(\Omega\) with associated metric \(g = (g_{ij})\) admits two fundamental forms: The first fundamental form \(I\) for \(M\) on \(\Omega\) is
\[ I = g_{11}(dx_1)^2 + 2g_{12}dx_1dx_2 + g_{22}(dx_2)^2, \] 
(2.5)
and the second fundamental form \(II\) is
\[ II = -dn \cdot dy = h_{11}(dx_1)^2 + 2h_{12}dx_1 \, dx_2 + h_{22}(dx_2)^2 \] 
(2.6)
with \(n\) being the unit normal vector to \(M\). The coefficients \((h_{ij})\) represent the orthogonality of \(n\) to the tangent plane and are associated with the second derivatives of \(y\), and since \(n \cdot dy = 0\), it follows
\[ II = (n \cdot \partial_1^2 y)(dx_1)^2 + (n \cdot \partial_1 \partial_2 y)dx_1 \, dx_2 + (n \cdot \partial_2^2 y)(dx_2)^2. \]

By equating the cross-partial derivatives of \(y\), the isometric immersion problem as stated above reduces to the Gauss–Codazzi system
\[ \partial_1 M - \partial_2 L = \Gamma_{22}^{(2)} L - 2\Gamma_{12}^{(2)} M + \Gamma_{11}^{(2)} N, \]
\[ \partial_1 N - \partial_2 M = -\Gamma_{22}^{(1)} L + 2\Gamma_{12}^{(1)} M - \Gamma_{11}^{(1)} N \] 
(2.7)
with the condition
\[ LN - M^2 = K, \] 
(2.8)
where
\[ L = \frac{h_{11}}{\sqrt{|g|}}, \quad M = \frac{h_{12}}{\sqrt{|g|}}, \quad N = \frac{h_{22}}{\sqrt{|g|}} \] 
(2.9)
and \(|g| = det(g_{ij}) = g_{11}g_{22} - g_{12}^2\). The Gauss curvature \(K = K(x_1, x_2)\) is given by
\[ K(x_1, x_2) = \frac{R_{1212}}{|g|}, \] 
(2.10)
where \(R_{ijkl}\) is the curvature tensor
\[ R_{ijkl} = g_{lm} \left( \partial_k \Gamma_{ij}^{(m)} - \partial_j \Gamma_{ik}^{(m)} + \Gamma_{ij}^{(n)} \Gamma_{nk}^{(m)} - \Gamma_{ij}^{(n)} \Gamma_{nk}^{(m)} \right), \] 
(2.11)
and \(\Gamma_{ij}^{(k)}\) is the Christoffel symbol
\[ \Gamma_{ij}^{(k)} = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}) . \]
(2.12)
Here, the indices \(i, j, k, l = 1, 2\), \((\partial_1, \partial_2) = (\partial_{x_1}, \partial_{x_2})\), and the summation convention is used. Also, \((g^{kl})\) is the inverse of \((g_{ij})\).

The fundamental theorem of surface theory states that given forms \(I\) and \(II\) with \((g_{ij})\) being positive definite and smooth coefficients, \((g_{ij})\) and \((h_{ij})\) that satisfy the Gauss–Codazzi system \((2.7)–(2.9)\); then, there exists a surface embedded into \(\mathbb{R}^3\) with first and second fundamental forms \(I\) and \(II\). This result
has been extended by Mardare [13] when $(h_{ij}) \in L^\infty_{\text{loc}}(\Omega)$ for given $(g_{ij}) \in W^{1,\infty}_{\text{loc}}(\Omega)$, and then, the surface immersed is $C^{1,1}(\Omega)$ locally. Thus, the isometric immersion problem reduces to solving the Gauss–Codazzi system (2.7)–(2.9) for $(h_{ij}) \in L^\infty_{\text{loc}}(\Omega)$ with a given positive definite metric $(g_{ij}) \in W^{1,\infty}_{\text{loc}}(\Omega)$, and then, immediately, we recover the immersion surface $y(\Omega)$, which is $C^{1,1}$ locally. We refer the reader to books [8,10] for an exposition of the surface theory and to Mardare [13,14] for the extension of the aforementioned result to $(h_{ij}) \in L^\infty_{\text{loc}}$.

However, for completeness of our presentation, we state Mardare’s result in full:

**Theorem 2.1.** (Mardare [13]) Assume that $\Omega$ is a connected and simply connected open subset of $\mathbb{R}^2$, and that the matrix fields $(g_{ij}) \in W^{1,\infty}_{\text{loc}}(\Omega)$ being symmetric positive definite and $(h_{ij}) \in L^\infty_{\text{loc}}(\Omega)$ symmetric satisfy the Gauss and Codazzi-Mainardi equations in $D'(\Omega)$. Then, there exists a mapping $y \in W^{2,\infty}_{\text{loc}}(\Omega,\mathbb{R}^3)$ such that

\[
g_{ij} = \partial_i y \cdot \partial_j y, \\
h_{ij} = \partial_{ij} y \cdot \frac{\partial_1 y \times \partial_2 y}{|\partial_1 y \times \partial_2 y|}
\]
a.e. in $\Omega$. Moreover, the mapping $y$ is unique in $W^{2,\infty}_{\text{loc}}(\Omega,\mathbb{R}^3)$ up to proper isometries in $\mathbb{R}^3$.

By virtue of the embedding of $W^{2,\infty}$ into $C^{1,1}$, the immersion $y$ is locally in $C^{1,1}$, cf. Evans [9, Chapter 5]. Now, let us recall the following definition.

**Definition 2.2.** $(\mathcal{M}, g)$ is a geodesically complete Riemannian manifold if and only if every geodesic can be extended indefinitely.

It is perhaps useful to note that the concept of geodesically complete Riemannian manifold is equivalent to the Riemannian manifold defining a complete metric space. This is a consequence of Hopf–Rinow theorem, cf. [8, Chapter 7]. In fact, do Carmo notes “intuitively, this means that the manifold does not have any holes or boundaries.”

Under the assumption that our two-dimensional manifold is geodesically complete and simply connected, we can simplify the structure of our metric. The exact result is as follows and is essentially due to Hadamard, but we use the presentation given in Han and Hong [10].

**Lemma 2.3.** (Han and Hong [10, Lemma 10.2.1]) Let $(\mathcal{M}, g)$ be a geodesically complete simply connected smooth two-dimensional Riemannian manifold with non-positive Gauss curvature. Then, there exists a global geodesic coordinate system $(x, t)$ in $\mathcal{M}$ with metric

\[
g = dt^2 + B^2(x, t)dx^2 \tag{2.13}
\]

where $B$ is a smooth function satisfying $B(x, 0) = 1$ and $\partial_t B(x, 0) = 0$ for $x \in \mathbb{R}$.

A direct substitution of (2.13) in (2.7) then yields that $L, M, N$ satisfy the Gauss–Codazzi system in the form

\[
\partial_t L - \partial_x M = L\partial_t \ln B - M\partial_x \ln B + NB\partial_t B, \\
\partial_t M - \partial_x N = -M\partial_t \ln B, \\
LN - M^2 = KB^2, \tag{2.14}
\]

with

\[
\partial_t B = -KB \text{ defines the Gauss curvature } K \text{ in terms of the metric.}
\]
3. The viscous approximation

As we will be dealing with non-smooth data, a natural approach is to embed our initial value problem into viscous approximating system with viscosity $\mu > 0$ and attempt to recover our solution as limit for $\mu \to 0^+$. More precisely, in this section, we first study the viscous approximation of the scaled variables of $(L, M, N)$ to system (2.14) and then establish properties for a family of metrics of the form (2.13) that correspond to the class of negative Gauss curvature with decay rate of the order of $t^{-2-\frac{\delta}{2}}$ with $\delta \in (0, 4)$.

First, it is easy to check that the scaled variables
\[ l = \frac{L}{B^2 \sqrt{|K|}}, \quad m = \frac{M}{B \sqrt{|K|}}, \quad n = \frac{N}{\sqrt{|K|}} \] (3.1)
satisfy the system
\[
\begin{align*}
\partial_t l - \frac{1}{B} \partial_x m + (l - n) \partial_t \ln B + \frac{1}{2} \partial_t \ln |K| - \frac{m}{2B} \partial_x \ln |K| = 0, \\
\partial_t m - \frac{1}{B} \partial_x n + 2m \partial_t \ln B + \frac{m}{2} \partial_t \ln |K| - \frac{n}{2} \partial_x \ln |K| = 0,
\end{align*}
\]
with
\[ \ln m^2 = -1. \] (3.3)
The eigenvalues associated with system (3.2) are
\[ \lambda_1 = \frac{m - 1}{lB}, \quad \lambda_2 = \frac{m + 1}{lB}, \] (3.4)
and we see that each characteristic field is linear degenerate. System (3.2) is \textit{strictly hyperbolic} if $\lambda_1 < \lambda_2$, or equivalently if $l$ is finite.

Consider the viscous approximations $(l^\mu, m^\mu, n^\mu)$ that satisfy system
\[
\begin{align*}
\partial_t l^\mu - \frac{1}{B} \partial_x m^\mu + (l^\mu - n^\mu) \partial_t \ln B + \frac{\mu}{2} \partial_t \ln |K| - \frac{m^\mu}{2B} \partial_x \ln |K| &= \mu \partial_{xx} l^\mu, \\
\partial_t m^\mu - \frac{1}{B} \partial_x n^\mu + 2m^\mu \partial_t \ln B + \frac{m^\mu}{2} \partial_t \ln |K| - \frac{n^\mu}{2} \partial_x \ln |K| &= \mu \partial_{xx} m^\mu,
\end{align*}
\] (3.5)
with
\[ l^\mu m^\mu - (m^\mu)^2 = -1. \] (3.6)
Here, $\mu > 0$ is a constant “viscosity.” For convenience, we drop the index $\mu$ from the viscous approximate solutions $(l^\mu, m^\mu, n^\mu)$ from here and on, and we use it again in Sect. 6 when studying the limit $\mu \to 0^+$.

Set the Riemann invariants
\[
\begin{align*}
 u &= \frac{m}{l} + \frac{1}{l}, \quad v = \frac{m}{l} - \frac{1}{l}, \\
 l &= \frac{2}{u - v}, \quad m = -\left( \frac{u + v}{u - v} \right), \quad n = \frac{2uv}{u - v}.
\end{align*}
\] (3.7)
(3.8)

Multiply system (3.5) from the left by $(\partial_t u, \partial_m u)^T$ and $(\partial_t v, \partial_m v)^T$ to obtain the viscous equations of $(u, v)$:
\[
\begin{align*}
\partial_t u + \frac{v}{B} \partial_x u + u (1 + u^2) \partial_t \ln B - \frac{(u - v)}{4} \left( \partial_t \ln |K| + \frac{u}{B} \partial_x \ln |K| \right) &= \frac{\mu}{(v - u)} \left\{ \left( \partial_x u \right)^2 - \left( \partial_x v \right)^2 \right\} - \frac{2u}{v - u} \left( \partial_x u - \partial_x v \right)^2 - u \partial_{xx} u, \\
\partial_t v + \frac{u}{B} \partial_x v + u (1 + v^2) \partial_t \ln B - \frac{(v - u)}{4} \left( \partial_t \ln |K| + \frac{v}{B} \partial_x \ln |K| \right) &= \frac{\mu}{(v - u)} \left\{ \left( \partial_x u \right)^2 - \left( \partial_x v \right)^2 \right\} - \frac{2v}{v - u} \left( \partial_x u - \partial_x v \right)^2 + v \partial_{xx} v.
\end{align*}
\] (3.9) (3.10)
It is easy to check that strict hyperbolicity in the \((u, v)\) variables is equivalent to \(u \neq v\). Now, the system is uniformly strictly hyperbolic if \(v - u\) is uniformly bounded away from zero.

In the following sections, we prove uniform \(L^\infty\) bounds independent of \(\mu\) to \((u, v)\) and therefore to \((l^\mu, m^\mu, n^\mu)\) via (3.8).

### 3.1. The metric for a special case

Let \((\mathcal{M}, g)\) be a geodesically complete simply connected smooth two-dimensional Riemannian manifold with non-positive Gauss curvature \(K\) and a metric of the form (2.13).

To keep our ideas clear and the presentation relatively simple, we here consider only the following special case: Let \(h = B > 0\) and \(k^* = |K|\), and we assume that \(k^*, h\) are taken to be independent of \(x\). Then, \(h\) and \(k^*\) satisfy

\[
\partial_t h = k^* h, \quad h(0) = 1, \quad \partial_t h(0) = 0. \tag{3.11}
\]

Also, we choose \(k^*\) to be

\[
k^* = \frac{C}{(1 + |t|)^{2 + \frac{\delta}{2}}}, \quad C > 0 \tag{3.12}
\]

as taken in Hong [11]. We note that our method could be generalized beyond this case to cover the more general hypotheses on \(g\) and \(K\) as stated in Theorem 1.1 at the cost of greater complications and technicalities. For simplicity, we restrict ourselves in the aforementioned special case.

Now, let \(\phi = \phi(t)\) be the solution of

\[
\begin{align*}
\partial_t \phi &= \phi(1 + \phi^2) \partial_t \ln h + \frac{\phi}{2} \partial_t \ln k^*, \quad t > T \\
\phi(T) &= \psi_0
\end{align*} \tag{3.13}
\]

with \(\psi_0\) a constant. As noted in [10, 10.2.36], the explicit solution \(\phi\) of the above problem is given by the expression

\[
\phi(t) = \frac{bh\sqrt{k^*}}{(1 - 2b^2 \int_0^t h \partial_t h k^* ds)^{1/2}} \tag{3.14}
\]

with

\[
b = \frac{\psi_0}{h(T)\sqrt{k^*(T)}}. \tag{3.15}
\]

Notice that for \(\phi\) to be defined for all \(t > T\), we must have \(h \partial_t h k^* \in L^1(T, \infty)\). By the choice (3.12) and \(\psi_0\) small enough, then we have the formula

\[
(1 + \phi^2) \partial_t \ln h + \frac{1}{2} \partial_t \ln k^* = -\left(\frac{\delta - 4\phi^2}{4t}\right) + O(\frac{1}{t^{1+\delta/2}}) \leq 0 \tag{3.16}
\]

for \(t > T\). Thus, for \(\psi_0\) small enough, equation (3.13) implies that \(\partial_t \phi < 0, \phi > 0\) for \(t > T\). In fact, formula (3.16) is the key to the rest of our analysis.

In what follows, we establish estimates on \(h\) and \(\partial_t h\) for the chosen decay rate of the curvature (3.12) that are used in the following sections.

**Lemma 3.1.** If \(k^*(s)\) and \(sk^*(s)\) are in \(L^1(0, \infty)\), then

\[
\int_0^t k^*(s) ds \leq \partial_t h \leq C_1 \tag{3.17}
\]
and

\[ 1 + \int_0^t \int_0^s k'(\tau) d\tau ds \leq h(t) \leq 1 + C_1 t \quad (3.18) \]

**Proof.** The proof is given in Han and Hong [10, Lemma 10.2.3]. \qed

Next, an important estimate for \( \partial_t \ln h \) is given in the following lemma.

**Lemma 3.2.** Let \( |k^* t^{2+\delta}| \) be decreasing in \( |t| \) for \( |t| > T \). Then,

\[ \partial_t \ln h = \frac{1}{t} + O \left( \frac{1}{|t|^{1+\delta/2}} \right) \quad (3.19) \]

for sufficiently large \( |t| \) and \( \partial_t \ln h \) is bounded.

**Proof.** Again, the proof is given in Han and Hong [10, Lemma 10.2.3]. \qed

4. Invariant regions: \( L^\infty \) bounds

In this section, we establish \( L^\infty \) bounds on the solutions to the viscous system (3.5)–(3.6). By (3.16) and the choice of curvature (3.12), we can get information on the sign of \( \partial_t \ln h + \frac{1}{2} \partial_t \ln k^* \). In fact, this follows from Lemma 3.2. Simply write

\[ \partial_t \ln h + \frac{1}{2} \partial_t \ln k^* = -\frac{\delta}{4t} + O \left( \frac{1}{|t|^{1+\delta/2}} \right) < 0 \quad (4.1) \]

for large \( t \). Here, however, we shall use a related equality which we call the sign–switch property. Specifically compute

\[ \partial_t \ln h + \frac{1}{4} \partial_t \ln k^* = \frac{1}{t} + O \left( \frac{1}{|t|^{1+\delta/2}} \right) - \frac{1}{4} \left( 2 + \frac{\delta}{2} \right) \frac{1}{t} \]

\[ = \frac{1}{2t} \left( 1 - \frac{\delta}{4} \right) + O \left( \frac{1}{|t|^{1+\delta/2}} \right) \quad (4.2) \]

Hence, when \( 0 < \delta < 4 \), we have the sign–switch

\[ \partial_t \ln h + \frac{1}{4} \partial_t \ln k^* > 0 \quad \text{for } t \text{ large} \quad (4.3) \]

We now establish the following lemma under the choice \( k^* \) given by (3.12) and \( 0 < \delta < 4 \).

**Lemma 4.1.** Assume that \( h \) and \( k^* \) are independent of \( x \) satisfying (3.11) and \( k^* \) is given by (3.12). Then, if initially at time \( t = T_1 \), with \( T_1 \) sufficiently large, the solutions \( (u, v) \) of system (3.9)–(3.10) satisfy the bounds

\[ -\psi_0 \leq u \leq -e^{-t} \psi_0, \quad e^{-t} \psi_0 \leq v \leq \psi_0 \quad (4.4) \]

then, these bounds on \( (u, v) \) persist for all \( t > T_1 \).

**Proof.** Without loss of generality, let us first establish the bound on \( u \) from above. If at some first passage time \( t^* > T_1 \), we have for some \( x^* \) that \( u(x^*, t^*) = -e^{-t^*} \psi_0 \); then, at \( (x^*, t^*) \), we have \( \partial_x u = 0, \partial_{xx} u \leq 0, \partial_t (u + e^{-t} \psi_0) > 0 \) and \( e^{-t} \psi_0 \leq v \leq \psi_0 \). From (3.9), we have at \( (x^*, t^*) \)

\[ \partial_t u \leq -v \left[ (1 + e^{-2t^*} \psi_0^2) \partial_t \ln h + \frac{1}{4} \partial_t \ln k^* \right] - \frac{1}{4} e^{-t^*} \psi_0 \partial_t \ln k^* \quad (4.5) \]

From (4.4) and (4.3), we have

\[ \partial_t u \leq -e^{-t} \psi_0 \left( e^{-2t} \psi_0^2 \partial_t \ln h + \frac{1}{4} \partial_t \ln k^* \right) \quad (4.6) \]
for \( T_1 \) sufficiently large. This is equivalent to
\[
\partial_t (u + e^{-t} \psi_0) \leq -e^{-t} \psi_0 \left( 1 + e^{-2t} \psi_0^2 \partial_t \ln h - \left( \frac{1}{2} + \delta \right) \frac{1}{1 + t^*} \right)
\]  
(4.7)
at \((x^*, t^*)\). Hence, for \( T_1 \) sufficiently large \( \partial_t (u + e^{-t} \psi_0) < 0 \), which is a contradiction of our assumption \( \partial_t (u + e^{-t} \psi_0) > 0 \). A similar argument yields the bound on \( v \) from below.

Next, we establish the bound on \( u \) from below. Again, let \( t^* \) denote the first passage time. Hence, we have some point \((x^*, t^*)\) for which \( \partial_x u = 0, \partial_{xx} u \geq 0, \partial_t u > 0 \) with \( e^{-t} \psi_0 \leq v \leq \psi_0 \). From (3.9), we have at \((x^*, t^*)\)
\[
\partial_t u = -v(1 + \psi_0^2) \partial_t \ln h + \frac{1}{4}(-\psi_0 - v) \partial_t \ln k^*
\]
\[+ \frac{\mu}{v + \psi_0} \left\{ (\partial_x v)^2 (1 + \frac{2\psi_0}{v + \psi_0}) + \psi_0 \partial_{xx} u \right\}
\]  
(4.8)
Adding to this expression, the identity (3.13) yields
\[
\partial_t (u + \phi) > \phi(1 + \phi^2) \partial_t \ln h + \frac{\phi}{2} \partial_t \ln k^*
\]
\[+ v(1 + \psi_0^2) \partial_t \ln h + \frac{1}{4}(-\psi_0 - v) \partial_t \ln k^*
\]  
(4.9)
at \((x^*, t^*)\). Now, choose the data for \( \phi \) to be \( \phi(t^*) = \psi_0 \). Hence, at \((x^*, t^*)\)
\[
\partial_t (u + \phi) \geq (\psi_0 - v) \left[ (1 + \psi_0^2) \partial_t \ln h + \frac{1}{4} \partial_t \ln k^* \right]
\]  
(4.10)
By (4.4) and the fact that \( v \leq \psi_0 \), we have that \( \partial_t (u + \phi) \geq 0 \) at \((x^*, t^*)\). Since \( u(x^*, t^*) + \phi(t^*) = -\psi_0 + \psi_0 = 0 \), we have \( u(x^*, t) \geq -\phi(t) \) for \( t > t^* \), \( |t - t^*| \) small. Recalling the analysis in (3.16) for the choice (3.12), \( \phi(t) \) is decreasing. Thus, we have \( u(x^*, t) \geq -\phi(t^*) = -\psi_0 \) which is a contradiction. A similar argument holds for \( v \). This completes the proof of the lemma. \( \square \)

**Remark 4.2.** Of course, the proof is motivated by the one given in Han and Hong [10]. The advantage of the one given above is its relative simplicity and the precise estimates for \((u, v)\) from above, below (respectively).

5. \(H^{-1}\) compactness

In this section, we prove the \(H^{-1}\) compactness of the sequence
\[
\partial_t l - \partial_x \left( \frac{m}{h} \right), \quad \partial_t m - \partial_x \left( \frac{n}{h} \right)
\]
(5.1)
constructed from the viscous approximation (3.5)–(3.6) with data satisfying (4.4). First from the formulas (3.8) and the bounds established in Lemma 4.1, we immediately have

**Lemma 5.1.** Under the assumptions of Lemma 4.1, for \( u, v \) satisfying (4.4) at \( t = T_1 \), \( T_1 \) sufficiently large, we have \((l, m, n) \in L^{\infty}(\mathbb{R} \times [T_1, T_2])\) where \( T_2 \) is any value \( T_2 > T_1 \).

For convenience, let us set \( \Omega = \mathbb{R} \times [T_1, T_2] \) so that \((l, m, n) \in L^{\infty}(\Omega)\). Inspection of (3.5) shows that \( \partial_t l - \partial_x \left( \frac{m}{h} \right), \partial_t m - \partial_x \left( \frac{n}{h} \right) \) will lie in a compact subset of \( H^{-1}_{loc}(\Omega) \) if the viscous terms \( \mu \partial_{xx} l, \mu \partial_{xx} m \) will lie in a compact subset of \( H^{-1}_{loc}(\Omega) \). To show this, we follow a standard argument say as given in the paper of Cao et al. [1].

Define the entropy, entropy flux pair
\[
\eta = -\frac{m^2 + 1}{l}, \quad q = \frac{m^3 - m}{lh^2}.
\]
(5.2)
Notice that the Hessian of $\eta$ is given by
\[
\begin{bmatrix}
\eta_{t} & \eta_{lm} \\
\eta_{lm} & \eta_{mm}
\end{bmatrix} = -\frac{2}{T} \begin{bmatrix}
m^2 + 1 & -m \\
-m & 1
\end{bmatrix}.
\] (5.3)

Since $v - u > 0$, we have $l < 0$, and therefore, the Hessian will be positive definite. Hence, $\eta$ is a convex entropy. Next, multiply system (3.5) by $(\eta_t, \eta_m)$ to get
\[
\eta_t + q_x = C(x, t) + \mu(\eta_t \partial_x l + \eta_m \partial_x m)
\] (5.4)
where $C(x, t)$ lies in a bounded set of $L^\infty(\Omega)$. Write $\eta_x = \eta_t \partial_x l + \eta_m \partial_x m$ and
\[
\eta_{xx} = \eta_t (\partial_x l)^2 + 2\eta_m \partial_x l \partial_x m + \eta_{mm} (\partial_x m)^2 + \eta_t \partial_x l + \eta_m \partial_x m
\]
so that
\[
\eta_t + q_x = C(x, t) - \mu(\eta_t (\partial_x l)^2 + 2\eta_m \partial_x l \partial_x m + \eta_{mm} (\partial_x m)^2) + \mu \eta_{xx}
\] (5.5)
Let $V$ be a compact subset of $\Omega$ and $\chi$ be a $C^\infty$ function with compact support in $\Omega$ and $\chi|_V = 1$. Multiply (5.5) by $\chi$ and integrate over $\Omega$ to see
\[
\int_\Omega -\chi t \eta - \chi x q \, dt \, dx = \int_\Omega \chi C \, dt \, dx + \mu \int_\Omega \chi_{xx} \eta \, dt \, dx
\]
\[- \mu \int_\Omega \chi (\eta_t (\partial_x l)^2 + 2\eta_m \partial_x l \partial_x m + \eta_{mm} (\partial_x m)^2) \, dt \, dx.
\] (5.6)

Thus, from the convexity of $\eta$, we have that
\[
\sqrt{\mu} \partial_x l, \quad \sqrt{\mu} \partial_x m
\] (5.7)
belong to a bounded subset of $L^2_{loc}(\Omega)$. Next, compute
\[
\mu \left| \int_\Omega \chi \partial_x l \, dx \, dt \right| = \mu \left| \int_\Omega \chi_x \partial_x l \, dx \, dt \right|
\leq \sqrt{\mu} \left( \int_{\text{supp}(\chi)} \mu (\partial_x l)^2 \, dx \, dt \right)^{1/2} \left( \int_\Omega (\chi_x)^2 \, dx \, dt \right)^{1/2}
\to 0 \quad \text{as} \quad \mu \to 0^+. \tag{5.8}
\]

Thus, $\mu \partial_{xx} l \to 0$ weakly in $L^2_{loc}(\Omega)$ and hence strongly in $H^{-1}_{loc}(\Omega)$. A similar statement holds for $\mu \partial_{xx} m$.

Hence, we have proven

**Lemma 5.2.** Under the assumptions of Lemma 4.1, for $u$, $v$ satisfying (4.4) at $t = T_1$, $T_1$ sufficiently large, we have that the sequences (5.1) parameterized by $\mu$ lie in compact subsets of $H^{-1}_{loc}(\Omega)$.

### 6. Compensated compactness

In this section, we use the a priori estimates of Sect. 5 and the compensated compactness framework to pass to the limit as $\mu \to 0+$ for our viscous system (3.5). Notice the results of Sect. 5 show that for initial data at $t = T_1$, $T_1$ sufficiently large, which satisfy (4.4), we have
\[
|l(\mu, m^\mu, n^\mu)| \leq A \quad \text{for} \quad (x, t) \in \Omega \tag{6.1}
\]
where $A$ is a constant independent of $\mu$ and
\[
\partial_t l^\mu - \partial_x \left( \frac{m^\mu}{h} \right), \quad \partial_t m^\mu - \partial_x \left( \frac{n^\mu}{h} \right)
\] (6.2)
confined in a compact subset of $H^{-1}_{loc}(\Omega)$. Since (6.1)–(6.2) are satisfied, Theorem 4.1 of Chen et al. [2] is applicable. We quote that result here.

**Theorem 6.1.** (Compensated Compactness Framework [2]) Let a sequence $(l^\mu, m^\mu, n^\mu)$ satisfy (3.5) and properties (6.1)–(6.2). There exists a subsequence, still labeled $(l^\mu, m^\mu, n^\mu)$ that converges weak* in $L^\infty(\Omega)$ to $(\hat{l}, \hat{m}, \hat{n})$ as $\mu \to 0$ such that

(i) $|(\hat{l}, \hat{m}, \hat{n})(x,t)| \leq A$ a.e. in $\Omega$

(ii) the Gauss equation (3.6) is weakly continuous with respect to the sequence $(l^\mu, m^\mu, n^\mu)$ that converges weak* in $L^\infty(\Omega)$ to $(\hat{l}, \hat{m}, \hat{n})$

(iii) the Codazzi equations hold for $(\hat{l}, \hat{m}, \hat{n})$.

Specifically, the limit $(\hat{l}, \hat{m}, \hat{n})$ is a bounded weak solution of the Gauss–Codazzi system in the domain $\Omega$.

The main result of this article is:

**Theorem 6.2.** Let $(\mathcal{M}, g)$ be a geodesically complete simply connected smooth two-dimensional Riemannian manifold with non-positive Gauss curvature $K$ and a metric of the form (2.13) defined on $\Omega = \mathbb{R} \times [T_1, T_2]$. Assume that $h = B$ and $k^* = |K|$ are independent of $x$ satisfying (3.11) and $k^*$ is given by (3.12) for $t > T_1$. Then, for $T_1$ large enough, there exists $y \in W_{loc}^{2,\infty}(\Omega)$ satisfying the embedding equations of Theorem 2.1 for any $T_2 > T_1$.

**Proof.** At time $t = T_1$, $T_1$ large enough, we choose data for $(l, m, n)$ to satisfy (4.4) via (3.7). By previous analysis, the sequence $(l^\mu, m^\mu, n^\mu)$, $\mu > 0$, satisfies the statement of Theorem 6.1. Thus, we can apply Mardare’s Theorem (see Theorem 2.1) to the limit $(\hat{l}, \hat{m}, \hat{n})$, and the proof is complete. □

The aforementioned theorem provides an isometric immersion on $\Omega = \mathbb{R} \times [T_1, T_2]$ for some $T_1 > 0$ large enough. To achieve a global result different from Hong’s smooth case, one has to choose non-smooth data at $t = 0$ and get a solution up to $t = T_1$ in the non-classical sense. This is accomplished in the following proposition, and the two pieces are glued together to provide a global immersion on the plane.

**Theorem 6.3.** Let $(\mathcal{M}, g)$ be a geodesically complete simply connected smooth two-dimensional Riemannian manifold with negative Gauss curvature $K$ and induced metric of the form (2.13). For some $T_1 > 0$ large enough as chosen in Theorem 6.2, assume that $h = B$ and $k^* = |K|$ are independent of $x$ satisfying (3.11), and $k^*$ is given by (3.12) for $t > T_1$. Then, $(\mathcal{M}, g)$ admits a global isometric immersion in $\mathbb{R}^3$, which is locally $C^{1,1}(\mathbb{R}^2)$.

**Proof.** Under the assumptions of the theorem, the goal is to solve Gauss–Codazzi system (3.2)–(3.3) first for $t > 0$ and obtain the solution $(l^+, m^+, n^+)$ in $[0, \infty) \times \mathbb{R}$ and then for $t < 0$ and obtain the solution $(l^-, m^-, n^-)$ in $\mathbb{R} \times (-\infty, 0]$. We solve both Cauchy problems with the same initial data $(l_0, m_0, n_0)$ satisfying relation (3.3) at $t = 0$ and choose the data $U_0 \equiv (l_0, m_0, n_0) \in C^1(\mathbb{R})$ with some properties to be determined in the sequel. By local existence and uniqueness results, we can glue together the two solutions and get a solution $(l, m, n)$ in $\mathbb{R}^2$. Indeed, just consider

$$
(l, m, n)(x,t) = (l^+, m^+, n^+) \cdot 1_{t \geq 0} + (l^-, m^-, n^-) \cdot 1_{t < 0}.
$$

(6.3)

The reader should note that smoothness of $U_0$ is not necessary, and it is enough to assume that $U_0$ is $C^1$ to glue the two pieces together. Thus, it remains to establish the existence of a weak solution $(l^+, m^+, n^+)(\cdot, t)$, for $t > 0$ to (3.2)–(3.3) with the aforementioned data. Then, similarly we construct $(l^-, m^-, n^-)$ for $t < 0$. Also, recall that Hong’s smooth immersion admits special constant data $u_0 = -\varepsilon$, $v_0 = +\varepsilon$ for a fixed small parameter $\varepsilon > 0$, where $u_0$, $v_0$ are the Riemann invariants at $t = 0$. We choose more general data to obtain a different solution, not necessarily smooth.

Fix $T_1 > 0$ as chosen in Theorem 6.2. Consider the smooth solution $(l^*, m^*, n^*)$ constructed by Hong as stated in Theorem 1.1 on the time interval $[0, T_1)$. Define the perturbed solution

$$
\hat{l} = l - l^*, \quad \hat{m} = m - m^*, \quad \hat{n} = n - n^*
$$

(6.4)
with \((l, m, n)\) satisfying system \((3.5)\). Then, \((\hat{l}, \hat{m}, \hat{n})\) satisfies the system
\[
\begin{align*}
\partial_t \hat{l} - \frac{1}{B} \partial_x \hat{m} + (\hat{l} - \hat{n}) \partial_t \ln B + \frac{\hat{l}}{2} \partial_t \ln |K| &= 0, \\
\partial_t \hat{m} - \frac{1}{B} \partial_x \hat{n} + 2\hat{m} \partial_t \ln B + \frac{\hat{m}}{2} \partial_t \ln |K| &= 0,
\end{align*}
\]
(6.5)
with
\[
\hat{n} = \frac{\hat{m}^2 + 2m^* \hat{m} - \hat{l}n^*}{\hat{l} + l^*}.
\]
(6.6)
We view system \((6.5)\) as a system of balance laws with state vector \(U = (\hat{l}, \hat{m})^T\), and flux \(F\) and source \(G\) given by
\[
F(U, t) = \left(-\frac{1}{B} \hat{m}, -\frac{1}{B} \hat{n} \right)^T, \quad G(U, t) = \left( (\hat{l} - \hat{n}) \partial_t \ln B + \frac{\hat{l}}{2} \partial_t \ln |K|, 2\hat{m} \partial_t \ln B + \frac{\hat{m}}{2} \partial_t \ln |K| \right)^T,
\]
(6.7)
respectively. Note here the dependence of the \(F\) and \(G\) on time due to the inhomogeneity of system \((6.5)\).

Also, this system admits an equilibrium solution at \((\hat{l}, \hat{m}, \hat{n}) = (0, 0)\). By Theorem 1 in Dafermos and Hsiang \([7]\), if we assign initial data \((\hat{l}_0, \hat{m}_0)\) of small total variation, we can obtain an entropy weak solution \((\hat{l}, \hat{m}, \hat{n})(. , t)\) of bounded variation to \((6.5)\) on the time interval \([0, T_1]\). In particular, choose data \((\hat{l}_0, \hat{m}_0)\) in a ball \(B_\rho\) centered at \((0, 0)\) and of radius \(\rho > 0\) with \(TV\{\hat{l}_0, \hat{m}_0\} < \delta_0\). By \(TV\{V\}\), we denote the total variation in the function \(V\) in \(x \in \mathbb{R}\). It is easy to check that the system is strictly hyperbolic for \(t \in [0, T_1]\) since \(\rho\) is small, and the function \(B = h(t)\) is bounded away from zero (see \((3.18)\)). By \([7, \text{Theorem 1}]\), there exists \(\delta_0 > 0\) small enough, depending on \(\rho\) and \(T_1\), and we have a solution \((\hat{l}, \hat{m}, \hat{n})(x,t)\) to \((6.5)\) on the strip \(\mathbb{R} \times (0, T_1)\). The solution \((\hat{l}, \hat{m}, \hat{n})(x,t)\) has bounded total variation depending on \(T_1\). It is worth mentioning that under stronger assumptions on the inhomogeneity of a system of balance laws, one can obtain a global in time weak solution; for example, see \([5]\). However, these hypotheses are not fulfilled by \((6.7)\) uniformly in time.

Using now \((6.4)\), we immediately obtain an entropy weak solution, denoted by \((l, m, n) = (l^+, m^+, n^+)\) \((\cdot, t)\), to \((3.2)-(3.3)\) for all times \(t \in [0, T_1]\). The goal is to prolong \((l^+, m^+, n^+)\) from \(t = T_1\) up to \(t = T_2\) for any \(T_2 > T_1\) by applying Theorem 6.2 from \(t = T_1\). To accomplish this, we only need the Riemann invariants \((u^+, v^+)\) associated with \((l^+, m^+, n^+)\) at time \(t = T_1\) to satisfy bounds \((4.4)\). Condition \((4.4)\) at \(t = T_1\) is fulfilled by choosing \(\rho\) small enough or \(T_1\) even larger. Thus, we deduce the existence of a solution \((l, m, n) = (l^+, m^+, n^+)\) \((\cdot, t)\) for any \(t > 0\). Also note that the data \((l_0, m_0, n_0)\) are determined via \((6.4)\) using the data \((\hat{l}_0, \hat{m}_0, \hat{n}_0)\) for system \((6.5)\); thus, the data differ from those chosen in the smooth case of Theorem 1.1. Similarly, we obtain \((l^-, m^-, n^-)\) for \(t < 0\).

In view of the previous analysis, we deduce the existence of a solution \((L, M, N) \in L^\infty_{loc}(\mathbb{R}^2)\) to \((2.14)-(2.15)\), which is a \(BV\) function in \(x\)-variable for every \(t\) fixed, \(t \in (-T_1, T_1)\) and belongs in \(L^\infty\) for any \(|t| > T_1\). Then, Mardare’s result stated in Sect. 2 yields immediately the existence of an isometric immersion in \(\mathbb{R}^3\), which is locally \(C^{1,1}\); cf. Evans \([9, \text{Chapter 5, Theorem 4}]\). The proof of the theorem is complete.

7. Weaker decay and preservation of strict hyperbolicity

Immediate inspection of the proof of Lemma 4.1 shows that the bounds
\[
u \geq -e^{-t}\psi_0, \quad \psi \geq e^{-t}\psi_0
\]
(7.1)
for \(t\) sufficiently large only follow from inequality \((4.3)\). Hence, a natural question is whether we can produce \(k^*, h\) with weaker decay than given by \((3.12)\) and still satisfy \((4.3)\) as well as \(k^*, sk^* \in L^1[0, \infty)\).
In fact, the answer is yes as provided in the following example. Take
\[
k^* = \frac{1}{(3 + t)^2(\ln(3 + t))^p}, \quad t > 0, p > 1.
\]
(7.2)

Note that \(k^*, tk^*(t)\) are in \(L^1[0, \infty)\).

A direct computation shows
\[
\frac{k^*(t)}{k^*(t)} = \frac{2}{3 + t} - \frac{p}{(3 + t)\ln(3 + t)}.
\]

Also recall from (3.17)–(3.18) that
\[
\frac{h'(t)}{h(t)} \geq \frac{\int_0^t k^*(s)ds}{1 + C_1t}
\]
where \(C_1\) is as given in Han and Hong [10, Lemma 10.2.3] by the expression
\[
C_1 = \int_0^\infty k^*(s)ds \exp \left\{ \int_0^\infty sk^*(s)ds \right\}.
\]

Hence, (4.3) will be satisfied if
\[
\frac{\int_0^t k^*(s)ds}{1 + C_1t} + \frac{1}{4} \left( -\frac{2}{3 + t} - \frac{p}{(3 + t)\ln(3 + t)} \right) > 0,
\]
(7.3)
or alternatively
\[
\int_0^t k^*(s)ds - \frac{(1 + C_1t)}{4} \left( \frac{2}{3 + t} + \frac{p}{(3 + t)\ln(3 + t)} \right) > 0.
\]
(7.4)

Thus, for large \(t\), it suffices that
\[
\int_0^\infty k^*(s)ds > \frac{C_1}{2}.
\]
(7.5)

From the definition of \(C_1\), we see (7.5) will be satisfied when
\[
2 > \exp \left\{ \int_0^\infty k^*(s)ds \right\}.
\]
(7.6)

An easy estimate shows
\[
\int_0^\infty sk^*(s)ds \leq \int_0^\infty (s + 3)k^*(s)ds
\]
\[
\leq \int_0^\infty ds \frac{\ln(3 + s)}{(s + 3)^p}
\]
\[
= \frac{1}{p - 1}(\ln 3)^{1-p}.
\]
(7.7)

Hence, (7.5) will be satisfied if
\[
2 > \exp \left\{ \frac{1}{(p - 1)(\ln 3)^{p-1}} \right\}.
\]
As \( \ln 3 = 1.0986 \ldots \), we want

\[
2 > \exp\left\{\frac{1}{(p - 1)(1.09)^{p-1}}\right\}
\]

and as \( p \to \infty \) the right-hand side of (7.8) approaches 1, and the inequality (7.8) is satisfied for \( p \) large enough.

We summarize our observations in the following theorem.

**Theorem 7.1.** Assume that \( h \) and \( k^* \) are independent of \( x \) satisfying (3.11) and \( k^* \) is as given by (7.2). Then, if (7.1) is satisfied at \( t = T_1 \), \( T_1 \) sufficiently large, then (7.1) is satisfied for all \( t \geq T_1 \). Hence, strict hyperbolicity of (3.2)–(3.3) would not be lost and furthermore \( \frac{-2e^t}{\psi_0} < l < 0 \).

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