EXISTENCE OF MINIMIZERS FOR THE SDRI MODEL IN 2D:
WETTING AND DEWETTING REGIME WITH MISMATCH STRAIN

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Abstract. The model introduced in [45] in the framework of the theory on Stress-Driven Rearrangement Instabilities (SDRI) [3, 43] for the morphology of crystalline materials under stress is considered. As in [45] and in agreement with the models in [50, 55], a mismatch strain, rather than a Dirichlet condition as in [16], is included into the analysis to represent the lattice mismatch between the crystal and possible adjacent (supporting) materials. The existence of solutions is established in dimension two in the absence of graph-like assumptions and of the restriction to a finite number of connected components for the free boundary of the region occupied by the crystalline material, thus extending previous results for epitaxially strained thin films and material cavities [6, 34, 35, 45]. Due to the lack of compactness and lower semicontinuity for the sequences of minimizers, i.e., minimizers among configurations with at most \( m \) connected boundary components, a minimizing candidate is directly constructed, and then shown to be a minimizer by means of uniform density estimates and the convergence of minimizers' energies to the energy infimum as \( m \to \infty \). Finally, regularity properties for the morphology satisfied by every minimizer are established.

1. Introduction

In this paper we establish existence and regularity properties for the solutions of the variational model for Stress-Driven Rearrangement Instabilities (SDRI) [3, 23, 43] that was introduced in [45]. Under the name of SDRI are included all those material morphologies, such as boundary irregularities, cracks, filaments, and surface patterns, which a crystalline material may exhibit in the presence of external forces, such as in particular the chemical bonding forces with adjacent materials. In order to release the induced stresses, atoms rearrange from the material optimal crystalline order and instabilities may develop.

The main advancement provided by the results in this manuscript with respect to [45] is the absence of the unphysical restriction on the number of connected components for the boundary of the region occupied by the crystalline material, by also avoiding graph-like assumptions for such boundaries assumed for the specific settings of epitaxially strained thin films in [6, 16, 34] and material voids in [35]. In particular, with respect to [16] we include into the analysis the dewetting regime, i.e., the presence of other fixed materials with possibly different boundary surface tensions, even if by only treating the two dimensional case, and we establish regularity results for the crystalline morphologies and instabilities satisfied by every minimizer. Furthermore, our strategy stems from the approach used in [22] for the Mumford-Shah functional, and hence differs from the method introduced in [16], which instead is based on allowing displacements to attain a limit value \( \infty \) on
sets with positive measure (and on technically assigning a zero cost to the elastic-energy contribution related to those sets).

The SDRI model of [45] is a variational model introduced in the framework of the SDRI theory initiated in the seminal papers of [3] and [43], and on the basis of the subsequent analytical descriptions provided in the context of epitaxially strained thin films [6, 24, 25, 34], crystal cavities [8, 35], capillarity droplets [26, 30], fractures [7, 11, 17, 19, 36], and boundary debonding and delamination [4, 49]. All such settings are included and can be treated simultaneously in the SDRI model [45] (see Section 2.5). In agreement with [3, 43] since SDRI morphologies relate to the boundary of crystalline materials and depend on the bulk rearrangements, the energy \( F \) characterizing the SDRI model displays both an elastic bulk energy and a surface energy denoted by \( W \) and \( S \), respectively. More precisely, the energy \( F \) is defined as

\[
F(A, u) := S(A, u) + W(A, u) \tag{1.1}
\]

for any admissible configurational pair \((A, u)\) consisting of a set \( A \) that represents the region occupied by the crystalline material in a fixed container \( \Omega \subset \mathbb{R}^d \) for \( d \in \mathbb{N} \), i.e.,

\[
A \in \mathcal{A} := \{ A \subset \Omega : A \text{ is } L^2\text{-measurable and } \partial A \text{ is } H^1\text{-rectifiable}\},
\]

and of a displacement function \( u \) of the bulk materials (with respect to the optimal crystal arrangement) given by

\[
u \in GSBD^2(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^d) \cap H^1_{\text{loc}}(\text{Int}(A) \cup S; \mathbb{R}^d),
\]

where \( S \subset \mathbb{R}^d \setminus \Omega \) is the region occupied by a fixed material, which we denote substrate in analogy with the thin-film setting and we consider possibly different from the material in the container, and

\[
\Sigma := \partial S \cap \partial \Omega
\]

represents the contact surface between the container \( \Omega \) and the substrate \( S \). In the following we refer to \( C \) as the configurational space and to each configuration \((A, u) \in C\) as a free crystal with \( A \) and \( u \) as the free-crystal region and the free-crystal displacement, respectively (see Figure 1).

The bulk elastic energy \( W \) in (1.1) is defined in [45] by

\[
W(A, u) = \int_{A \cup S} W(z, e(u) - M_0) \, dz,
\]

where the elastic density \( W \) is given by

\[
W(z, M) := C(z) M : M \tag{1.2}
\]

for any \( z \in \Omega \cup S \) and any \((d \times d)\)-symmetric matrix \( M \in \mathbb{M}^{d \times d}_{\text{sym}} \), and for a positive-definite elasticity tensor \( C \), and attains its minimum value zero for every \( z \) at a fixed strain \( M_0 \in M \in \mathbb{M}^{d \times d}_{\text{sym}} \) in the following referred to as mismatch strain. The inclusion in (1.2) of a mismatch strain \( M_0 \) defined by

\[
M_0 := \begin{cases} 
   e(u_0) & \text{in } \Omega, \\
   0 & \text{in } S,
\end{cases} \tag{1.3}
\]

for a fixed \( u_0 \in H^1(\mathbb{R}^d; \mathbb{R}^d) \), together with the fact that both \( M_0 \) and \( C \) are let free of jumping across \( \Sigma \), allows to model the presence of two different materials in the substrate and in the free crystals, and in particular to take into account the lattice mismatch between their optimal crystalline lattices that is crucial, e.g., in the setting of heteroepitaxy [24, 25].

The surface energy \( S \) in (1.1) is defined as

\[
S(A, u) = \int_{\partial A} \psi(z, u, \nu) \, d\mathcal{H}^{d-1},
\]
Figure 1. An admissible free-crystal region $A$ is displayed in light blue in the container $\Omega$, while the substrate $S$ is represented in dark blue. The boundary of $A$ (with the cracks) is depicted in black, the container boundary in green, the contact surface $\Sigma$ in red (thicker line) while the free-crystal delamination region $J_u$ with a white dashed line.

where the surface tension $\psi$ is given by

$$\psi(z, u, \nu) := \begin{cases} 
\varphi(z, \nu_A(z)) & z \in \Omega \cap \partial^* A, \\
2\varphi(z, \nu_A(z)) & z \in \Omega \cap (A^{(1)} \cup A^{(0)}) \cap \partial A, \\
\varphi(z, \nu_S(z)) + \beta(z) & z \in \Sigma \cap A^{(0)} \cap \partial A, \\
\beta(z) & z \in \Sigma \cap \partial^* A \setminus J_u, \\
\varphi(z, \nu_S(z)) & z \in J_u,
\end{cases}$$

(1.4)

with $\varphi \in C(\overline{\Omega} \times \mathbb{R}^d; [0, +\infty))$ being a Finsler norm such that $c_1|\xi| \leq \varphi(x, \xi) \leq c_2|\xi|$ for some $c_1, c_2 > 0$ and representing the anisotropy of the free-crystal material, $\beta$ denoting the relative adhesion coefficient on $\Sigma$ such that, as for capillarity problems [26, 30],

$$|\beta(z)| \leq \varphi(z, \nu_S(z))$$

for every $z \in \Sigma$, $\nu$ coinciding with the exterior normal on the reduced boundary $\partial^* A$, and $A^{(\delta)}$ denoting the set of points of $A$ with density $\delta \in [0, 1]$.

The anisotropic form of $\psi$ in (1.4) distinguishes various portions of the free-crystal topological boundary $\partial A$: the free boundary $\partial^* A \cap \Omega$, the family of internal cracks $A^{(1)} \cap \Omega \cap \partial A$, the family of external filaments $A^{(0)} \cap \Omega \cap \partial A$, the delaminated region $J_u$, i.e., the portion on the contact surface $\Sigma$ where there is no bonding between the free crystal and the substrate (even if they are adjacent), the adhesion area where the free-crystal displacement is continuous through $\Sigma$, i.e., $\Sigma \cap \partial^* A \setminus J_u$, and the wetting layer represented by the filaments on $\Sigma$, i.e., $\Sigma \cap A^{(0)}$. In particular, $\psi$ weights the different portions of $\partial A$ in relation to the active chemical bondings present at each portion, i.e., $\varphi$ when there is no extra chemical bonding, such as at the free profile and at the delaminated region, and $\beta$ at the adhesion contact area with the substrate, while both the cracks and at external filaments are counted $2\varphi$ and the wetting layer sees the contribution of both $\psi$ and $\beta$. 
We consider the case $d = 2$ as in [45], with the fixed sets $\Omega$ and $S$ being bounded Lipschitz open connected sets such that $\Sigma$ is a Lipschitz 1-manifold. For $d \geq 3$ results are available for the isotropic Griffith model with $L^p$-fidelity term (of the type (2.19)) in [11] and with Dirichlet conditions for the displacements at the boundary in [12]. Moreover, a similar energy as the SDRI energy introduced in [45] was subsequently found in [16] as a relaxation formula separately for thin films and material voids, for the different setting with a Dirichlet condition imposed at $\partial \Omega$, and in the wetting regime, i.e., the case where free crystals are expected to cover the substrate. Unfortunately the strategy employed in [16] is not implementable in our setting, where rather than prescribing a Dirichlet condition as in [16], the mismatch strain (1.3) (which depends on the substrate region $S$) is considered in the elastic energy in analogy with the models in [55] and [50, Section 4.2.2] (see also the mathematical treatments [24, 25, 34, 47]).

In fact, the existence results in [16] are achieved by working (in the proofs) with displacements in a larger space than the classical framework of small displacements of linearized elasticity, namely the space $\text{GSBD}^p_\infty$ for $p > 1$ that includes displacements attaining a limit value $\infty$ in a set of finite perimeter (on which their strain $\varepsilon(u)$ is defined to be zero [16, Page 1055]). Such a method works well with a Dirichlet condition that keeps the displacements anchored, while in our setting it would be always convenient for the displacements in $\text{GSBD}^p_\infty$ of the minimizing sequences to escape to infinity, as this would result with the definition of the energy in [16] in the minimum (zero) value of the elastic energy for the limiting free-crystal region. A treatment for $d \geq 3$ of the model under consideration in this paper with mismatch strain (and without Dirichlet conditions) is under preparation [46] by implementing the ideas in this manuscript together with the ones in [45], but without the need of Golab’s Theorem (and without employing the space $\text{GSBD}^p_\infty$ for the displacements).

Therefore, we must proceed differently here and we rely on the results of [45] for $d = 2$. We begin by observing that, as shown in [45], the specific weights of (1.4) are crucial to obtain the lower semicontinuity of the energy $\mathcal{F}$ under the constraint on a fixed number $m \in \mathbb{N}$ of boundary connected components for the free-crystal regions, which represented an extension of the more restrictive graph condition assumed in [34] for the particular setting of epitaxially strained thin films and the starshapedness condition in [35] for material cavities. More precisely, by considering the subfamily $\mathcal{C}_m$ of configurations with free crystals presenting at most $m \in \mathbb{N}$ boundary connected components, namely

$$\mathcal{C}_m := \left\{ (A, u) \in \mathcal{C} : \partial A \text{ has at most } m \text{ connected components} \right\},$$

in [45, Theorem 2.8] it is shown that

$$\liminf_{k \to \infty} \mathcal{F}(A_k, u_k) \geq \mathcal{F}(A, u)$$

for every sequence $\{(A_k, u_k)\} \subset \mathcal{C}_m$ converging in a properly chosen topology $\tau_\mathcal{C}$ to a configuration $(A, u) \in \mathcal{C}_m$. In particular, the convergence with respect to $\tau_\mathcal{C}$ prescribes that $H^1(\partial A_k)$ are equibounded, $\text{sdist}(\cdot, \partial A_k) \to \text{sdist}(\cdot, \partial A)$ locally uniformly in $\mathbb{R}^2$ with $\text{sdist}$ representing the signed distance function (recall definition at (2.2)), and $u_m \to u$ a.e. in $\text{Int}(A) \cup S$. We notice that the restriction to the subfamily $\mathcal{C}_m$ was needed in [45] to establish not only the lower semicontinuity, but also the compactness with respect to $\tau_\mathcal{C}$, which indeed fails in $\mathcal{C}$ (see Remark 2.3), so that by means of the direct method of the calculus of variations, the existence of minimizers $(A_m, u_m) \in \mathcal{C}_m$ of $\mathcal{F}$ among all configurations in $\mathcal{C}_m$ followed in [45, Theorem 2.6].

The aim of the investigation contained in this paper is to recover the full generality avoiding any extra hypothesis on the admissible free-crystal regions. This is achieved by retrieving compactness with respect to the free-crystal regions at least for any sequence $m$-minimizers $(A_m, u_m) \in \mathcal{C}_m$, and by combining the strategies of [22] and [45]. More
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precisely, the use in [45] of the Golab-type Theorem [40] is avoided for the compactness of the free-crystal regions by adapting to our setting the classical density-estimate arguments first introduced for surface energies and the Mumford-Shah functional (see, e.g., [2, 28, 52]), and then extended to the Griffith functional [12, 19], which in turns allow us also to establish some regularity results. Moreover, in our setting there is the extra difficulty with respect to [22] that the compactness and lower semicontinuity along sequences of $m$-minimizers (with respect to the topology used to find such $m$-minimizers through the direct method) are both missing. We overcome this issue, by directly constructing a minimizing candidate, proving that it belongs to the class

$$\tilde{\mathcal{A}} := \left\{ A \subset \Omega : A \text{ is } L^2\text{-measurable and } H^1(\partial A) < +\infty \right\},$$

and establishing a “lower-semicontinuity inequality” (see (1.7) below) along the selected sequence of $m$-minimizers $(A_m, u_m)$ (see Subsection 1.1 for more details).

Since $A \subset \tilde{\mathcal{A}}$, for proving such lower-semicontinuity property we introduce an auxiliary energy $\tilde{\mathcal{F}}$ defined in the larger family $\tilde{\mathcal{C}}$ of configurations $(A, u)$ for which $A \in \tilde{\mathcal{A}}$, i.e.,

$$\tilde{\mathcal{F}}(A, u) := \tilde{\mathcal{S}}(A, u) + \mathcal{W}(A, u),$$

with auxiliary surface energy $\tilde{\mathcal{S}}$ defined as

$$\tilde{\mathcal{S}}(A, u) = \int_{\partial A} \tilde{\psi}(z, u, \nu)dH^{d-1},$$

where the surface tension $\tilde{\psi}$ is given by

$$\tilde{\psi}(z, u, \nu) := \begin{cases} 
\varphi(z, \nu_A(z)) & z \in \Omega \cap \partial^* A, \\
2\varphi(z, \nu_A(z)) & z \in S^A_u, \\
\beta(z) & z \in \Sigma \cap \partial^* A \setminus J_u, \\
\varphi(z, \nu_S(z)) & z \in J_u
\end{cases}$$

for $S^A_u$ denoting the jump set of $u$ along the $H^1$-rectifiable portion of the cracks (see (2.6) for the precise definition).

The results of this paper are twofold: The existence results contained in Theorem 2.6 and the regularity properties of Theorem 2.7. More precisely, in Theorem 2.6 we prove the existence of a minimum configuration of $F$ and $\tilde{\mathcal{F}}$ among all configurations in $C$ and $\tilde{\mathcal{C}}$, respectively, with free-crystal region satisfying a volume constraint, i.e., we solve the minimum problems

$$\inf_{(A, u) \in \mathcal{C}, |A| = v} F(A, u) \quad (1.5)$$

and

$$\inf_{(A, u) \in \tilde{\mathcal{C}}, |A| = v} \tilde{\mathcal{F}}(A, u) \quad (1.6)$$

for a fixed volume parameter $v \in (0, |\Omega|)$ or, if $S = \emptyset$, $v = |\Omega|$. Furthermore, the minimum problems (1.5) and (1.6) are proven to be equivalent to the unconstraint minimum problems consisting in minimizing volume-penalized versions $F^\lambda$ and $\tilde{\mathcal{F}}^\lambda$ of the functionals $F$ and $\tilde{\mathcal{F}}$, for a penalization constant $\lambda > 0$ provided that $\lambda \geq \lambda_1$ for some uniform constant $\lambda_1 > 0$.

In Theorem 2.7 regularity properties shared by all solutions of (1.5) and (1.6) are found. Notice that we cannot directly apply the arguments of [34, 35] based on the external sphere condition considered in [15] because of the absence of graph and star-shapedness assumptions on the admissible free-crystal regions. As a byproduct of Theorem 2.6 and Proposition 5.1 given a configuration $(A, u)$ minimizing (1.5) resp. (1.6), we can construct a configuration $(A', u) \in \mathcal{C}$ which minimizes both minimum problems (1.5) and (1.6) such that $A'$ is an open set with cracks coinciding in $\Omega$ with the jump set of the corresponding minimizing free-crystal displacement $u$, and boundary $\partial A'$ satisfying uniform upper and
lower density estimates. Furthermore, we also observe that, any connected component $E$ of $A'$ that does not intersect $\Sigma \setminus J_u$ (up to $\mathcal{H}^1$-negligible sets), must have a sufficiently large area, i.e.,

$$|E| \geq (c_1 \sqrt{4\pi/\lambda_1})^2,$$

and must satisfy $u = u_0$ in $E$ up to adding a rigid displacement.

1.1. Paper organization and detail of the proofs. The paper is organized in 5 sections. In Section 2 we introduce the mathematical setting, recall the SDRI model from [45], and carefully state the main results of the paper.

In Section 3 we prove the upper and lower density estimates for the local decay of the energy $\mathcal{F}$ on any sequence of $m$-minimizers $(A_m, u_m) \in \mathcal{C}_m$ (see Theorem 3.1) by considering a local version of $\mathcal{F}^\lambda$ (see (2.9)), adapting arguments of [2, 12, 19] to our setting with displacements paired with free-crystal regions, and paying extra care to the fact that $\mathcal{C}$ is possibly not constant (but in $L^\infty(\Omega \cup S) \cap C^0(\Omega)$).

In Section 4 we prove compactness and lower-semicontinuity properties for a sequence of $m$-minimizers. We begin by establishing in Proposition 4.1 the compactness for a sequence of $m$-minimizers $\{(A_m, u_m)\}$ with free-crystal regions $A_m$ not containing isolated points of such free-crystal regions to a limiting set of finite perimeter $A \subset \Omega$ by means of both the Blaschke-type selection principle [45, Proposition 3.1] and the density estimates established in Section 3. Then, in Proposition 4.3, we further extend the (already generalized) Golab-type Theorem [40, Theorem 4.2] to a priori not-connected $\mathcal{H}^1$-measurable (not necessarily $\mathcal{H}^1$-rectifiable) sets satisfying uniform density estimates (see [22] for the isotropic case). The compactness of the displacements in $\{(A_m, u_m)\}$ is then proved in Propositions 4.4 by carefully constructing the limiting displacement $u$ in view of the property that for every connected component $E_i$ of $A$ the set in which displacements $u_m$ diverge is either the whole component $E_i$ or $\emptyset$, which follows from [45, Theorem 3.7]. Finally, in Proposition 4.6 we establish the lower-semicontinuity property

$$\liminf_{h \to \infty} \mathcal{F}(A_{m_h}^h, u_{m_h}) \geq \mathcal{F}(A, u),$$

(1.7)

by treating separately the elastic and the surface energy. For the latter we employ a blow-up method differently performed for each portion of the $\partial A$ where $\bar{\psi}$ is supported. In particular extra care is needed for the jump set $J_u$ and jump set along cracks $S_u^A$ (since there is no bound on the number of connected components), where we need to extend some ideas from [45, Proposition 4.1].

In Section 5 we prove the main results of the manuscript, i.e., the existence and regularity results that are contained in Theorems 2.6 and 2.7, respectively. In order to prove Theorem 2.6 we first establish in Proposition 5.1 the equalities

$$\inf_{(B, v) \in \mathcal{C}} \bar{\mathcal{F}}(B, v) = \inf_{(B, v) \in \mathcal{C}, |B| = v} \mathcal{F}(B, v) = \lim_{m \to \infty} \inf_{(B, v) \in \mathcal{C}_m, |B| = v} \mathcal{F}(B, v).$$

(1.8)

(recall that the second equality follows from [45, Theorem 2.6]) by using similar arguments previously used in [45, Theorem 2.6]. In particular, (1.7) and (1.8) imply that the configuration $(A, u) \in \mathcal{C}$ is a minimizer of $\bar{\mathcal{F}}$ in $\hat{\mathcal{C}}$. In Theorem 5.3 we establish the uniform density estimates for the jump set $S_u^A$ of $u$ along cracks for a minimizer $(A, u)$ of $\bar{\mathcal{F}}$. In particular, $S_u^A$ is then essentially closed, and using this fact in Proposition 5.4 we construct a configuration $(A', u) \in \mathcal{C}$, which minimizes both $\bar{\mathcal{F}}$ and $\mathcal{F}$, starting from a minimizer $(A, u)$ of $\bar{\mathcal{F}}$ in $\hat{\mathcal{C}}$. Moreover, $(A', u)$ solves both (1.5) and (1.6) and satisfies the properties stated in Theorem 2.7. Theorem 2.7 is then a direct consequence of Proposition 5.4, comparison arguments, the isoperimetric inequality in $\mathbb{R}^2$, and the equivalence of the constrained minimum problems and the unconstrained penalized minimum problem related to the energies $\mathcal{F}^\lambda$ and $\bar{\mathcal{F}}^\lambda$. 
We conclude the manuscript with Appendix A that contains some subsidiary results recalled for Reader’s convenience since very relevant in the arguments used throughout the paper.

2. Mathematical setting

In this section we recall the SDRI model from [45], collect all the definitions and hypotheses and state the main results of the paper. Since our model is two-dimensional, unless otherwise stated, all sets we consider are subsets of \( \mathbb{R}^2 \). We choose the standard basis \( \{ e_1 = (1, 0), e_2 = (0, 1) \} \) in \( \mathbb{R}^2 \) and denote the coordinates of \( x \in \mathbb{R}^2 \) with respect to this basis by \( (x_1, x_2) \). We denote by Int(\( A \)) the interior of \( A \subset \mathbb{R}^2 \). Given a Lebesgue measurable set \( E \), we denote by \( \chi_E \) its characteristic function and by \( |E| \) its Lebesgue measure. The set
\[
E(\alpha) := \left\{ x \in \mathbb{R}^2 : \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \alpha \right\}, \quad \alpha \in [0, 1],
\]
where \( B_r(x) \) denotes the ball in \( \mathbb{R}^2 \) centered at \( x \) of radius \( r > 0 \), is called the set of points of density \( \alpha \) of \( E \). Clearly, \( E(\alpha) \subset \partial E \) for any \( \alpha \in (0, 1) \), where
\[
\partial E := \{ x \in \mathbb{R}^2 : B_r(x) \cap E \neq \emptyset \text{ and } B_r(x) \setminus E \neq \emptyset \text{ for any } r > 0 \}
\]
is the topological boundary. The set \( E^{(1)} \) is the Lebesgue set of \( E \) and \( |E^{(1)} \Delta E| = 0 \). We denote by \( \partial^* E \) the reduced boundary of a set \( E \) of finite perimeter [2, 41], i.e.,
\[
\partial^* E := \left\{ x \in \mathbb{R}^2 : \exists \nu_E(x) := -\lim_{r \to 0^+} \frac{D\chi_E(B_r(x))}{|D\chi_E(B_r(x))|}, \quad |\nu_E(x)| = 1 \right\}.
\]
The vector \( \nu_E(x) \) is called the generalized outer normal to \( E \).

Remark 2.1. If \( E \) is a set of finite perimeter, then
\begin{itemize}
  \item \( \overline{\partial E} = \partial E^{(1)} \) (see e.g., [52, Eq. 15.3]);
  \item \( \partial^* E \subseteq E^{(1/2)} \) and \( \mathcal{H}^1(E^{(1/2)} \setminus \partial^* E) = 0 \) (see e.g., [52, Theorem 16.2]);
  \item \( P(E, B) = \mathcal{H}^1(B \cap \partial^* E) = \mathcal{H}^1(B \cap E^{(1/2)}) \) for any Borel set \( E \);
\end{itemize}
where \( P(E, B) \) and \( \mathcal{H}^1 \) denote the perimeter of \( E \) in \( B \) and the 1-dimensional Hausdorff measure, respectively.

An \( \mathcal{H}^1 \)-measurable set \( K \) is called \( \mathcal{H}^1 \)-rectifiable if \( \mathcal{H}^1(K) < \infty \) and there exist countably many Lipschitz functions \( f_i : \mathbb{R} \to \mathbb{R}^2 \) such that
\[
\mathcal{H}^1 \left( K \setminus \bigcup_{i \geq 1} f_i(\mathbb{R}) \right) = 0 \quad (2.1)
\]
(see e.g., [2, Definition 2.57]). Notice that one can assume in (2.1) that the functions \( f_i \) are \( C^1 \), since Lipschitz functions are a.e. differentiable. By the Besicovitch-Marstrand-Mattila Theorem ([2, Theorem 2.63] a Borel set \( K \subset \mathbb{R}^2 \) with \( \mathcal{H}^1(K) < \infty \) is \( \mathcal{H}^1 \)-rectifiable if and only if \( \theta^*(K, x) = \theta_*(K, x) = 1 \) for \( \mathcal{H}^1 \)-a.e. \( x \in K \), where
\[
\theta^*(K, x) := \limsup_{r \to 0^+} \frac{\mathcal{H}^1(B_r(x) \cap K)}{2r} \quad \text{and} \quad \theta_*(K, x) := \liminf_{r \to 0^+} \frac{\mathcal{H}^1(B_r(x) \cap K)}{2r}.
\]
In particular, any \( \mathcal{H}^1 \)-rectifiable set \( K \) admits a approximate tangent line at \( \mathcal{H}^1 \)-a.e. \( x \in K \), see e.g., [52, Remark 10.3]. When \( \theta_*(K, x) = 1 \), we write for simplicity \( \theta(K, x) = 1 \). A Borel set \( K \subset \mathbb{R}^2 \) with \( \mathcal{H}^1(K) < \infty \) is said purely unrectifiable if \( \mathcal{H}^1(K \cap \Gamma) = 0 \) for every 1-dimensional Lipschitz graph \( \Gamma \subset \mathbb{R}^2 \) (see e.g., [2, Definition 2.64]).

Moreover, by [29, Theorem 5.7], if \( K \subset \mathbb{R}^2 \) is an arbitrary Borel set with \( \mathcal{H}^1(E) < \infty \), then there exist Borel subsets \( K^r \) and \( K^o \) of \( K \) such that \( K = K^r \cup K^o \), \( K^r \) is \( \mathcal{H}^1 \)-rectifiable
and $K^u$ is purely unrectifiable, and such a decomposition is unique up to a $\mathcal{H}^1$-negligible set. More precisely, if $K = L' \cup L''$ with $\mathcal{H}^1$-rectifiable $L'$ and purely unrectifiable $L''$, then $\mathcal{H}^1(K' \Delta L') = \mathcal{H}^1(K'' \Delta L'') = 0$. In what follows we call $K'$ and $K''$ the rectifiable and purely unrectifiable parts of $K$, respectively. When $A \subset \mathbb{R}^2$ with $\mathcal{H}^1(\partial A) < +\infty$, we denote by $\partial^r A$ and $\partial^a A$ the $\mathcal{H}^1$-rectifiable and purely unrectifiable parts of $\partial A$, respectively.

The notation $\text{dist}(\cdot, E)$ stands for the distance function from the set $E \subset \mathbb{R}^2$ with the convention that $\text{dist}(\cdot, \emptyset) \equiv +\infty$. Given a set $A \subset \mathbb{R}^2$, we consider also signed distance function from $\partial A$, negative inside, defined as

$$
\text{sdist}(x, \partial A) := \begin{cases} 
\text{dist}(x, A) & \text{if } x \in \mathbb{R}^2 \setminus A, \\
-\text{dist}(x, \mathbb{R}^2 \setminus A) & \text{if } x \in A.
\end{cases} 
$$

(2.2)

**Remark 2.2.** The following assertions are equivalent:

(a) $\text{sdist}(x, \partial E_k) \rightarrow \text{sdist}(x, \partial E)$ locally uniformly in $\mathbb{R}^2$;

(b) $E_k \xrightarrow{K} E$ and $\mathbb{R}^2 \setminus E_k \xrightarrow{K} \mathbb{R}^2 \setminus (\text{Int}(E))$, where $K$ denotes the Kuratowski convergence of sets [20].

Moreover, either assumption implies $\partial E_k \xrightarrow{K} \partial E$.

Given $r > 0$, $\nu \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ we denote by $Q_{r, \nu}(x)$ the square of sidelength $2r$ centered at $x$ whose sides are either parallel or perpendicular to $\nu$. When $\nu = e_1$, or $\nu = e_2$, we drop the dependence on $\nu$ and write $Q_r(x)$. If in addition $x = 0$, we write just $Q_r$. We also set

$$
I_r := [−r, r] \times \{0\}, \quad Q_r^+ = \{x \in Q_r : x \cdot e_2 > 0\}, \quad \text{and} \quad Q_r^- = \{x \in Q_r : x \cdot e_2 < 0\}. 
$$

(2.3)

Given $x \in \mathbb{R}^2$ and $r > 0$, the blow-up map $\sigma_{x, r}$ is defined as

$$
\sigma_{x, r}(y) = \frac{y - x}{r}. 
$$

(2.4)

The blow-up of $K \subset \mathbb{R}^2$ is defined as $\sigma_{x, r}(K)$.

Given an open set $U \subset \mathbb{R}^2$ and a metric space $X$ we denote by $\text{Lip}(U; X)$ the family of all Lipschitz functions $\psi : U \rightarrow X$. We denote by $\text{Lip}(\psi)$ the Lipschitz constant of $\psi \in \text{Lip}(U; X)$. Furthermore, $\text{GSBD}(U; \mathbb{R}^2)$ denotes the collection of all *generalized special functions of bounded deformation* (see [14, 21] for their definition and properties). Given $u \in \text{GSBD}(U; \mathbb{R}^2)$ we denote with $e(u) \in \mathbb{M}_{\text{sym}}^{2 \times 2}$ the approximate symmetric gradient of $u$, for which

$$
\lim_{y \rightarrow x} \text{ap lim} \frac{|u(y) - u(x) - e(u)(x)(y - x)| \cdot (y - x)}{|y - x|^2} = 0
$$

holds for a.e. $x \in U$ by [21, Theorem 9.1], and with $J_u$ the jump set of $u$, which is $\mathcal{H}^1$-rectifiable by [21, Theorem 6.2]. Let us also define

$$
\text{GSBD}^2(U; \mathbb{R}^2) := \{u \in \text{GSBD}(U; \mathbb{R}^2) : e(u) \in L^2(U; \mathbb{M}_{\text{sym}}^{2 \times 2})\}.
$$

Given a $\mathcal{H}^1$-rectifiable set $M \subset \overline{U}$, we consider a normal vector $\nu_M$ to its approximate tangent line and we denote by $u_M^+$ and $u_M^-$ the approximate limits of $u \in \text{GSBD}^2(U; \mathbb{R}^2)$ with respect to $\nu_M$, i.e.,

$$
u_M^+(x) := \text{ap lim} \frac{u(y)}{(y - x) \cdot \nu_M^+ > 0}, \quad \text{and} \quad \nu_M^-(x) := \text{ap lim} \frac{u(y)}{(y - x) \cdot \nu_M^- < 0}. 
$$

(2.5)

for every $x \in M$ whenever they exist (see [21, Definition 2.4]). We refer to $u_M^+$ and $u_M^-$ as the *two-sided traces* of $u$ at $M$ and we notice that they are uniquely determined up to a permutation when changing the sign of $\nu_M$. If $U = \text{Int}(A)$ for some measurable set $A$ with $\mathcal{H}^1(\partial A) < +\infty$ and $M := \partial^r A$, we use the simplified notations $u_{\partial A}$ on $A^{(1)} \cap \partial^r A$,.
and $\text{tr}_A u := u^{+}_{\partial A}$ on $\partial^* A$, where on $\partial^* A$ we always choose $\nu_A$ in (2.5) as the generalized outer unit normal to $A$. Moreover, we define 

$$S^4_u := \{ x \in A^{(1)} \cap \partial^* A : u^{+}_{\partial A}(x) \neq u^{-}_{\partial A}(x) \}.$$  \hfill (2.6) 

Note that $S^4_u$ is $H^1$-rectifiable. We refer to $S^4_u$ the jump set of $u$ along the cracks of $A$.

A linear function $a : \mathbb{R}^2 \to \mathbb{R}^2$, defined as $Ax = Mx + b$, where $M$ is $2 \times 2$-matrix and $b \in \mathbb{R}^2$, is an (infinitesimal) rigid displacement if $M = -M^T$.

2.1. The SDRI model. Given two nonempty bounded Lipschitz connected open sets $\Omega \subset \mathbb{R}^2$ and $S \subset \mathbb{R}^2 \setminus \Omega$ such that $\overline{\Omega} \cap \overline{S} \neq \emptyset$ and the set $\Sigma := \partial S \cap \partial \Omega$ is a Lipschitz 1-manifold, we define the family of admissible regions for the free crystal and the space of admissible configurations by

$$A := \{ A \subset \overline{\Omega} : A \text{ is } C^2\text{-measurable and } \partial A \text{ is } H^1\text{-rectifiable} \}$$

and

$$C := \{ (A, u) : A \in A, \quad u \in GSBD^2(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^2) \cap H^1_{\text{loc}}(\text{Int}(A) \cup S; \mathbb{R}^2) \},$$

respectively. By Proposition A.1 any $A \in A$ has finite perimeter. Furthermore, $J_u \subset \Sigma \cap \partial^* A$ since $u \in H^1_{\text{loc}}(\text{Int}(A) \cup S; \mathbb{R}^2)$.

The energy of admissible configurations is given by 

$$\mathcal{F} : C \to [-\infty, +\infty],$$

$$\mathcal{F} := S + W,$$  \hfill (2.7) 

where $S$ and $W$ are the surface and elastic energies of the configuration, respectively. The surface energy of $(A, u) \in C$ is defined as

$$S(A, u) := \int_{\overline{\Omega} \cap \partial^* A} \varphi(x, \nu_A(x)) dH^1(x)$$

$$+ \int_{\overline{\Omega}^{(1)} \cup A^{(0)} \setminus \partial A} \left( \varphi(x, \nu_A(x)) + \varphi(x, -\nu_A(x)) \right) dH^1(x)$$

$$+ \int_{\Sigma \cap A^{(0)} \setminus \partial A} \left( \varphi(x, \nu_\Sigma(x)) + \beta(x) \right) dH^1(x)$$

$$+ \int_{\Sigma \cap \partial^* A \setminus J_u} \beta(x) dH^1(x) + \int_{J_u} \varphi(x, -\nu_\Sigma(x)) dH^1(x),$$  \hfill (2.8) 

where $\varphi : \overline{\Omega} \times S^1 \to [0, +\infty)$ and $\beta : \Sigma \to \mathbb{R}$ are Borel functions denoting the anisotropy of crystal and the relative adhesion coefficient of the substrate, respectively, and $\nu_\Sigma := \nu_S$. In the following we refer to the first term in (2.8) as the free-boundary energy, to the second as the energy of internal cracks and external filaments, to the third as the wetting-layer energy, to the fourth as the contact energy, and to the last as the delamination energy.

In applications instead of $\varphi(x, \cdot)$ it is more convenient to use its positively one-homogeneous extension $|\xi| \varphi(x, \xi / |\xi|)$. With a slight abuse of notation we denote this extension also by $\varphi$.

The elastic energy of $(A, u) \in C$ is defined as

$$W(A, u) := \int_{\overline{\Omega} \cup S} W(x, e(u(x)) - M_0(x)) dx,$$

where the elastic density $W$ is determined as the quadratic form

$$W(x, M) := C(x) M : M,$$

by the so-called stress-tensor, a measurable function $x \in \Omega \cup S \to C(x)$, where $C(x)$ is a nonnegative fourth-order tensor in the Hilbert space $\mathbb{M}^{2 \times 2}_{\text{sym}}$ of all $2 \times 2$-symmetric matrices
with the natural inner product
\[ M : N = \sum_{i,j=1}^{2} M_{ij} N_{ij} \]
for \( M = (M_{ij})_{i,j=1}^{2}, N = (N_{ij})_{i,j=1}^{2} \in \mathbb{M}_{2 \times 2}^{\text{sym}}. \)

The mismatch strain \( x \in \Omega \cup S \mapsto M_0(x) \in \mathbb{M}_{2 \times 2}^{\text{sym}} \) is given by
\[
M_0 := \begin{cases} 
  e(u_0) & \text{in } \Omega, \\
  0 & \text{in } S,
\end{cases}
\]
for a fixed \( u_0 \in H^1(\mathbb{R}^2; \mathbb{R}^2) \).

Given \( m \in \mathbb{N} \), let \( A_m \) be a collection of all \( A \in \mathcal{A} \) such that \( \partial A \) has at most \( m \) connected components and let
\[
\mathcal{C}_m := \left\{ (A, u) \in \mathcal{C} : A \in A_m \right\}
\]
to be the set of constrained admissible configurations. For simplicity, we assume that \( \mathcal{C}_\infty = \mathcal{C} \).

**Remark 2.3.** The reason to introduce \( \mathcal{C}_m \) is that \( \mathcal{C}_m \) is both closed under \( \tau_C \)-convergence (see [45, Definition 2.5]) and \( \mathcal{F} \) is lower semicontinuous with respect to \( \tau_C \) in \( \mathcal{C}_m \) (see [45, Theorems 2.7 and 2.8]). Such two properties do not apply instead to \( \mathcal{C} \) as the following examples show.

We begin by recalling that a sequence \( \{ (A_k, u_k) \} \subset \mathcal{C} \) is said to \( \tau_C \)-converge to \( (A, u) \subset \mathcal{C} \) and we denote by \( (A_k, u_k) \overset{\tau_C}{\longrightarrow} (A, u) \), if
- \( \sup_{k \geq 1} \mathcal{H}^1(\partial A_k) < \infty \),
- \( \text{sdist}(\cdot, \partial A_k) \rightarrow \text{sdist}(\cdot, \partial A) \) locally uniformly in \( \mathbb{R}^2 \) as \( k \to \infty \),
- \( u_k \rightarrow u \) a.e. in \( \text{Int}(A) \cup S \).

Let \( X := \{ x_n \} \) be a countable dense set in \( \Omega \) and \( A \in \mathcal{A} \) such that \( |A| = v \in (0, |\Omega|] \). Then the sets \( A_k := A \setminus \{ x_1, \ldots, x_k \} \in \mathcal{A} \), \( k \in \mathbb{N} \), are such that \( |A_k| = v \in (0, |\Omega|] \), \( \mathcal{H}^1(\partial A_k) = \mathcal{H}^1(\partial A) \), and \( (A_k, 0) \overset{\tau_C}{\longrightarrow} (A \setminus X, 0) \) as \( k \to \infty \), but \( A \setminus X \notin \mathcal{A} \) since \( \partial(A \setminus X) = \overline{A} \setminus A \). Therefore, compactness with respect to \( \tau_C \) fails in \( \mathcal{C} \).

Furthermore, let \( \Gamma \subset A \) be a segment such that \( \mathcal{H}^1(\Gamma) > 0 \), \( B := A \setminus \Gamma \), \( B_k := A \setminus (\Gamma \cap \{ x_1, \ldots, x_k \}) \) for every \( k \in \mathbb{N} \), and assume that \( X \) is dense in \( \Gamma \). We notice that \( \{ (B_k, 0) \} \subset \mathcal{C} \), \( (B, 0) \in \mathcal{C} \), \( |B_k| = |B| = |A| \), \( (B_k, 0) \overset{\tau_C}{\longrightarrow} (B, 0) \) as \( k \to \infty \). However,
\[
\mathcal{F}(B_k, 0) = \mathcal{F}(A, 0) < \mathcal{F}(A \setminus \Gamma, 0) = \mathcal{F}(B, 0).
\]
Therefore, lower semicontinuity of \( \mathcal{F} \) with respect to \( \tau_C \) fails in \( \mathcal{C} \).

### 2.2. Localized energies

In this section we introduce the notion of quasi minimizers of \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) in \( \Omega \) and the localized version \( \mathcal{F}(\cdot; O) : \mathcal{C}_m \to \mathbb{R} \) of \( \mathcal{F} \) for open sets \( O \subset \Omega \) and for \( m \in \mathbb{N} \cup \{ \infty \} \) with the convention \( \mathcal{C}_\infty := \mathcal{C} \). We define
\[
\mathcal{F}(A, u; O) := S(A; O) + W(A, u; O),
\]
where
\[
S(A; O) := \int_{O \cap \partial A} \varphi(y, \nu_A) d\mathcal{H}^1 + 2 \int_{O \cap \partial \alpha \cap (A^1 \cup A^0) \cap \partial \alpha} \varphi(y, \nu_A) d\mathcal{H}^1
\]
and
\[
W(A, u; O) = \int_{O \cap A} C(y) e(u) : e(u) dy,
\]
are the localized versions of the surface and elastic energies, respectively. Since we define the localized energy \( \mathcal{F}(\cdot; O) \) only for open subsets \( O \) of \( \Omega \), the localized surface energy...
Given $\Lambda \geq 0$ and $m \in \mathbb{N} \cup \{\infty\}$, the configuration $(A, u) \in \mathcal{C}_m$ is a local $(A, m)$-minimizer of $F : \mathcal{C}_m \to \mathbb{R}$ in $O$ if

$$F(A, u; O) \leq F(B, v; O) + \Lambda|A\Delta B|$$

whenever $(B, v) \in \mathcal{C}_m$ with $A\Delta B \subset O$ and supp $(u - v) \subset O$. Furthermore, we define

$$\Phi(A, u; O) := \inf \left\{ F(B, v; O) : (B, v) \in \mathcal{C}_m, B\Delta A \subset O, \text{ supp } (u - v) \subset O \right\}$$

and

$$\Psi(A, u; O) := F(A, u; O) - \Phi(A, u; O)$$

for every $(A, u) \in \mathcal{C}_m$ and every open set $O \subset \Omega$.

**Remark 2.5.** By [45, Theorem 2.6] (see also (3.1) below) for any minimizer $(A, u)$ of $F$ in $\mathcal{C}_m$, the configuration $(A, u - u_0)$ is a $(\lambda_0, m)$-minimizer of $F(\cdot, \cdot, \Omega)$. Indeed, since $(A, u)$ is a minimizer of $F^\lambda_0$ in $\mathcal{C}_m$, the function $\tilde{u} := u - u_0$ minimizes $\mathcal{C}_m \ni (B, v) \mapsto \tilde{F}^\lambda_0(B, v) := F^\lambda_0(B, v + u_0)$. Hence, for any open set $O \subset \Omega$ and $(B, v) \in \mathcal{C}_m$ with $A\Delta B \subset O$ and supp $(u - u_0 - v) \subset O$ we have $\tilde{F}^\lambda_0(A, u - \tilde{u}_0) \leq \tilde{F}^\lambda_0(B, v)$ so that

$$F(A, u - u_0; O) \leq F(B, v; O) + \lambda_0||A| - |B|| \leq F(B, v; O) + \lambda_0|A\Delta B|.$$

Similarly, if $(A, u)$ is a minimizer of $\tilde{F}$ in $\tilde{\mathcal{C}}$, the configuration $(A, u - u_0)$ is a $\lambda_0$-minimizer of $\tilde{F}(\cdot, O)$.

### 2.3. Auxiliary model

We also introduce a *weak* formulation of the SRDI model defined in Section 2.1 for which the more general family $\tilde{\mathcal{C}}$ of admissible configurations, given by

$$\tilde{\mathcal{C}} := \{(A, u) : A \in \tilde{A}, u \in GSBD^2(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^2) \cap H^1_{\text{loc}}(\text{Int}(A) \cup S; \mathbb{R}^2)\},$$

is considered, where

$$\tilde{A} := \left\{ A \subset \overline{\Omega} : A \text{ is } \mathcal{L}^2\text{-measurable and } \mathcal{H}^1(\partial A) < +\infty \right\}.$$  

The auxiliary energy $\tilde{F} : \tilde{\mathcal{C}} \to \mathbb{R}$ is defined as

$$\tilde{F} := \tilde{S} + \mathcal{W},$$

where

$$\tilde{S}(A, u) := \int_{\Omega \cap \partial^* A} \varphi(x, \nu_A(x))d\mathcal{H}^1(x)$$

$$+ \int_{\mathcal{S}^A_u} (\varphi(x, \nu_A(x)) + \varphi(x, -\nu_A(x)))d\mathcal{H}^1(x)$$

$$+ \int_{\mathcal{S}^\Lambda \cap \partial^* A \setminus J_u} (\beta(x)d\mathcal{H}^1(x) + \int_{J_u} \varphi(x, -\nu_\Sigma(x))d\mathcal{H}^1(x),$$

for $S^A_u \subset \Omega$ by definition (2.6).
2.4. Main results. We begin by stating the hypotheses which will be assumed throughout the paper:

\( \text{(H1) } \varphi \in C(\overline{\Omega} \times \mathbb{R}^2) \) and is a Finsler norm, i.e., there exist \( c_2 \geq c_1 > 0 \) such that for every \( x \in \overline{\Omega} \), \( \varphi(x, \cdot) \) is a norm in \( \mathbb{R}^2 \) satisfying

\[
\text{(2.13)} \quad c_1 |\xi| \leq \varphi(x, \xi) \leq c_2 |\xi|
\]

for any \( x \in \overline{\Omega} \) and \( \xi \in \mathbb{R}^2 \);

\( \text{(H2) } \beta \in L^\infty(\Sigma) \) and satisfies

\[
\text{(2.14)} \quad - \varphi(x, \nu_\Sigma(x)) \leq \beta(x) \leq \varphi(x, \nu_\Sigma(x))
\]

for \( H^1 \)-a.e. \( x \in \Sigma \);

\( \text{(H3) } C \in L^\infty(\Omega \cup S) \cap C^0(\overline{\Omega}) \) and there exists \( c_4 \geq c_3 > 0 \) such that

\[
\text{(2.15)} \quad 2c_3 M : M \leq C(x)M : M \leq 2c_4 M : M
\]

for any \( x \in \Omega \cup S \) and \( M \in \mathbb{M}^2_{\text{sym}} \);

\( \text{(H4) } \) Either \( v \in (0, |\Omega|) \) or \( S = \emptyset \).

Given \( G \in \{ F, \tilde{F} \} \), we use the notation:

\[
\mathcal{X}_G := \begin{cases} 
\mathcal{C} & \text{if } G = F, \\
\tilde{\mathcal{C}} & \text{if } G = \tilde{F}.
\end{cases}
\]

The first result is the existence of solutions without constraint on the number of free-crystal boundary components.

**Theorem 2.6 (Existence).** Assume \((\text{H1)-(H4)}\). Let \( G \in \{ F, \tilde{F} \} \). Then the minimum problem

\[
\text{(2.16)} \quad \inf_{(B,v) \in \mathcal{X}_G, |B|=\nu} G(B,v)
\]

admits a solution. Moreover, there exists \( \lambda_1 > 0 \) such that \((A,u) \in \mathcal{X}_G\) is a solution of (2.16) if and only if it solves

\[
\inf_{(B,v) \in \mathcal{X}_G} G^\lambda(B,v)
\]

for every \( \lambda \geq \lambda_1 \), where

\[
G^\lambda(B,v) := G(B,v) + \lambda|B| - \nu|;
\]

For simplicity we call the solutions of (2.16) global minimizers.

The second result is a partial regularity of the free-crystal boundaries. We recall that the definition of \( S_u^\lambda \) is provided in (2.6).

**Theorem 2.7 (Properties of global minimizers).** Assume \((\text{H1)-(H4)}\). Let \( G \in \{ F, \tilde{F} \} \) and \((A,u) \in \mathcal{X}_G\) be a solution of (2.16). Define

\[
A' := \text{Int}(A^{(1)}) \setminus \overline{\Gamma},
\]

where \( \overline{\Gamma} \) is the closure of \( \{ x \in S_u^\lambda : \theta_\lambda(S_u^\lambda, x) > 0 \} \), and, with a slight abuse of notation, consider \( u \) as defined in \( A' \cup S \) (and so, also on the \( L^2 \)-negligible set \( A' \setminus \text{Int}(A) \)). Then:

1. \( A' \) is open, \( \theta_\lambda(S_u^{A'}, x) > 0 \) for all \( x \in S_u^{A'}, |A'\Delta A| = 0, \mathcal{H}^1(\partial A\Delta \partial A') = 0, \mathcal{H}^1(S_u \Delta S_u^{A'}) = 0, (A', u) \in \mathcal{C}, \) and

\[
G(A,u) = F(A',u) = \inf_{(B,v) \in \mathcal{C}, |B|=\nu} F(B,v) = \inf_{(B,v) \in \mathcal{C}, |B|=\nu} \tilde{F}(B,v);
\]
(2) for any $x \in \Omega$ and $r \in (0, \min\{1, \text{dist}(x, \partial \Omega)\})$,
\[
\mathcal{H}^1(Q_r(x) \cap \partial A') \geq \frac{16c_2 + 4\lambda_1}{c_1}.
\]
(3) there exist $s_0 = s_0(c_3, c_4) \in (0, 1)$ and $R_0 = R_0(c_1, c_2, c_3, c_4, \lambda_1) > 0$, where $\lambda_1 > 0$

is given in Theorem 2.6, with the following property: if $x \in \Omega \cap \partial A'$, then
\[
\mathcal{H}^1(Q_r(x) \cap \partial A') \geq s_0
\]
for any square $Q_r(x) \subset \Omega$ with $r \in (0, R_0)$.

(4) $A^{(1)} \cap \partial A' = S_u^A$ and
\[
\mathcal{H}^1(S_u^A \setminus S_u^{A'}) = 0,
\]
hence cracks essentially coincide with the jump set for the displacement $u$;

(5) If $E \subset A'$ is any connected component of $A'$ with $\mathcal{H}^1(\partial E \cap \Sigma \setminus J_u) = 0$, then

$|E| \geq (c_1\sqrt{4\pi}/\lambda_1)^2$ and $u = u_0 + a$ in $E$, where $a$ is a rigid displacement.

In what follows we refer to the estimates in (2) and (3) as the (uniform) upper and lower density estimate, respectively. Note that by assertion (1), the assertions (3) and (5) directly hold also for solutions $(A, u)$ of (2.16).

2.5. Examples. We recall from [45] that the SDRI energy (2.7) coincides with the functionals of the following free-boundary problems considered in the Literature when restricted to the corresponding subfamilies of admissible configurations in $C$:

(a) Epitaxially strained thin films, e.g., [6, 24, 25, 34, 39, 47]: $\Omega := (a, b) \times (0, +\infty)$, $S := (a, b) \times (-\infty, 0)$ for some $a < b$, free crystals in the subfamily

\[\mathcal{A}_{\text{subgraph}} := \{A \subset \Omega : \exists h \in BV(\Sigma; [0, \infty)) \text{ and l.s.c. such that } A = A_h \} \subset A_1,\]

where $A_h := \{(x^1, x^2) : 0 < x^2 < h(x^1)\}$, and admissible configurations in the subspace

\[\mathcal{C}_{\text{subgraph}} := \{(A, u) : A \in \mathcal{A}_{\text{subgraph}}, u \in H^1_{\text{loc}}(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^2)\} \subset C_1\]

(see also [5, 42]);

(b) Crystal cavities, e.g., [35, 38, 54, 56]: $\Omega \subset \mathbb{R}^2$ smooth set containing the origin, $S := \mathbb{R}^2 \setminus \Omega$, free crystals in the subfamily

\[\mathcal{A}_{\text{starshaped}} := \{A \subset \Omega : \text{open and } \Omega \setminus A \text{ starshaped w.r.t. } (0, 0)\} \subset A_1,\]

and the space of admissible configurations

\[\mathcal{C}_{\text{starshaped}} := \{(A, u) : A \in \mathcal{A}_{\text{starshaped}}, u \in H^1_{\text{loc}}(\text{Int}(A \cup S \cup \Sigma); \mathbb{R}^2)\} \subset C_1;\]

(b) Capillarity droplets, e.g., [9, 26, 30]: $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz open set (or a cylinder), admissible configurations in the collection

\[\mathcal{C}_{\text{capillarity}} := \{(A, u_0) : A \in A\} \subset C \quad \text{or} \quad \tilde{\mathcal{C}}_{\text{capillarity}} := \{(A, u_0) : A \in \tilde{A}\} \subset \tilde{C};\]

(d) Griffith fracture model, e.g., [7, 11, 12, 17, 19, 36, 37]: $S = \Sigma = \emptyset, E_0 = 0$, and the space of configurations

\[\mathcal{C}_{\text{Griffith}} := \{(\Omega \setminus K, u) : K \text{ closed, } \mathcal{H}^1\text{-rectifiable, } u \in H^1_{\text{loc}}(\Omega \setminus K; \mathbb{R}^2)\} \subset C;\]

(e) Mumford-Shah model, e.g., [2, 22, 51]: $S = \Sigma = \emptyset, E_0 = 0$, $C$ is such that the elastic energy $\mathcal{W}$ reduces to the Dirichlet energy, and the space of configurations

\[\mathcal{C}_{\text{Mumford-Shah}} := \{(\Omega \setminus K, u) \in \mathcal{C}_{\text{Griffith}} : u = (u_1, 0)\} \subset C;\]
Boundary delaminations, e.g., [4, 31, 44, 48, 49, 57]: the SDRI model includes also the setting of debonding and edge delamination in composites [57]. The focus is here on the 2-dimensional film and substrate vertical section, while in [4, 48, 49] a reduced model for the horizontal interface between the film and the substrate is derived.

For the cases (a) and (b), the existence results for the SDRI model in $C$ subgraph and $C$ starshaped can be found for example in [45, Theorem 2.9 and Remark 2.10]. For (c), the same statements of Theorems 2.6 and 2.7 hold with $X := C$ capillarity if $G = F$ or $X := \tilde{C}$ capillarity if $G = \tilde{F}$ (note that $S_u$ and $\Gamma$ are empty in this case). For (d)-(f), we postpone the analysis to future investigations since some modifications in the proofs is needed to include boundary Dirichlet conditions or fidelity terms of type

$$\kappa \int_{\Omega \setminus K} |u - g|^p dx$$

(2.19)

for $p \in (1, \infty)$, $\kappa > 0$, and $g \in L^\infty(\Omega)$, which are generally considered (and needed) in these mechanical applications.

3. DECAY ESTIMATES FOR $m$-MINIMIZERS

In this section we always assume (H4). We recall that by [45, Theorem 2.6] under the hypotheses (H1)-(H3) both the volume-constrained minimum problem

$$\inf_{(A,u) \in C_m, |A| = v} F(A, u),$$

and the unconstrained minimum problem

$$\inf_{(A,u) \in C_m} F^\lambda(A, u)$$

admit a solution for any $m \in \mathbb{N}$. Moreover, by [45, Theorem 2.6] there exists $\lambda_0 > 0$ such that

$$\inf_{(A,u) \in C, |A| = v} F(A, u) = \inf_{(A,u) \in C} F^\lambda(A, u) = \lim_{m \to \infty} \inf_{(A,u) \in C_m, |A| = v} F(A, u)$$

(3.1)

for every $\lambda \geq \lambda_0$.

The main results of this section are the following density estimates for the quasi-minimizers of $F$ in $C_m$ with $m \in \mathbb{N} \cup \{\infty\}$.

**Theorem 3.1 (Density estimates for $(\Lambda, m)$-minimizers).** There exist $\varsigma_\ast = \varsigma_\ast(c_3, c_4) \in (0, 1)$ and $R_\ast = R_\ast(c_1, c_2, c_3, c_4, \lambda_0) > 0$, where $c_i$ are given by (2.13) and (2.15), with the following property. Let $(A,u) \in C_m$ be a $(\Lambda, m)$-minimizer of $F(\cdot, \cdot; \Omega)$ in $C_m$ for some $m \in \mathbb{N} \cup \{\infty\}$. Then for any $x \in \Omega$ and $r \in (0, \text{dist}(x, \partial \Omega))$,

$$\frac{\mathcal{H}^1(Q_r(x) \cap \partial A)}{r} \leq \frac{16c_2 + 4A}{c_1}.$$  

(3.2)

Moreover, if $x \in \Omega$ belongs to the closure of the set $\{y \in \Omega \cap \partial A : \theta_\ast(\partial A, y) > 0\}$, then

$$\frac{\mathcal{H}^1(Q_r(x) \cap \partial A)}{r} \geq \varsigma_\ast$$

(3.3)

for any square $Q_r(x) \subset \subset \Omega$ with $r \in (0, R_\ast)$.

To prove Theorem 3.1 we start with the following adaptation of [11, Theorem 3] to our setting (of set-function pairs).
Lemma 3.2. There exist $\eta \in (0, 1/32)$ and $c_0 > 0$ with the following property: For any $m \in \mathbb{N} \cup \{\infty\}$, any admissible $(A,u) \in E_m$, and any square $Q_R(x_0) \subset \Omega$ of sidelength $2R > 0$ with

$$\delta := \left( \frac{H^1(Q_R(x_0) \cap \partial^* A)}{R} \right)^{1/2} < \eta$$

(3.4)

there exist $v \in GSBD^2(\text{Int}(\Omega \cup S \cup \Sigma); \mathbb{R}^2)$, $B \in A$ with $(B,v|_B) \in E_m$, $R' \in (R(1 - \sqrt{\delta}), R)$ and a Lebesgue measurable set $\omega \subset \subset Q_R(x_0)$ such that

1. $v \in C^\infty(Q_R(1-\sqrt{\delta})(x_0))$, $A\Delta B \subset \subset Q_R(x_0) \setminus Q_R(1-\sqrt{\delta})(x_0)$ and supp$(\tilde{u} - v) \subset \subset Q_R(x_0)$, where

$$\tilde{u} := u\chi_{Q_R(x_0) \cap A} + \xi \chi_{Q_R(x_0) \setminus A},$$

(3.5)

where $\xi \in Q_R$ is chosen such that $Q_R \cap \partial^* A \subset J_5$;

2. $H^1(\partial B \setminus \partial A) \leq c_0 R^2 \sqrt{\delta}$ $H^1((Q_R(x_0) \setminus Q_R(1-\sqrt{\delta})(x_0)) \cap \partial A)$;

3. $|\omega| \leq c_0 \delta H^1(Q_R(x_0) \cap \partial A)$ and

$$\int_{Q_R(x_0) \cap \omega} |v - \tilde{u}|^2 dx \leq c_0 \delta^2 R^2 \int_{Q_R(x_0)} |\tilde{u}|^2 dx;
$$

(4)

for any $\psi \in \text{Lip}(Q_R \cap [0,1])$ and elasticity tensor $C \in L^\infty(Q_R)$ with

$$d_1 M : M \leq C(x) M : M \leq d_2 M : M, \quad (x,M) \in Q_R \times M_{2 \times 2}^{\text{sym}},$$

(3.6)

there exist $d_3 := d_3(c_0,d_1,d_2) > 0$ and $s := s(c_0,d_1,d_2) \in (0,1/2)$ such that

$$\int_{Q_R(x_0)} \psi \mathbb{C}(x) e(v) : e(v) dx \leq \int_{Q_R(x_0) \cap A} \psi \mathbb{C}(x) e(u) : e(u) dx + d_3 \delta^s (1 + R \text{Lip}(\psi)) \int_{Q_R(x_0) \cap A} |e(u)|^2 dx.$$

The proof of Lemma 3.2 is an adaptation of the arguments of [11, Theorem 3] to our situation of functional depending on set-function pairs with extra care paid for the constraint on the number of boundary connected components. The idea is to treat the boundary of each admissible region as a jump of a properly defined displacement. In particular, we choose such displacement of the type (3.5), where $\xi$ is selected as in the construction used in the proof of [45, Lemma 3.10]. We also notice that the constants $\eta$ and $c := c_0/(1 + \sqrt{2}/24) > 0$ are given by [11, Theorem 3].

Proof of Lemma 3.2. By translating and rescaling if necessary, we assume that $x_0 = 0$ and $R = 1$. Notice that since $H^1(Q_1 \cap \partial A) < +\infty$, by Proposition A.2 there exists $\xi \in (0,1)^2$ such that the set

$$\{ x \in Q_1 \cap \partial^* A : \text{tr}_A(u) \text{ exists and is equal to } \xi \}$$

is $H^1$-negligible. By [41, Theorem 4.4] up to a $H^1$-negligible set we can cover $Q_1 \cap \partial^* A$ with $C^1$-maps so that by [21, Theorem 5.2] $\text{tr}_A(u)$ exists $H^1$-a.e. on $Q_1 \cap \partial^* A$.

Let

$$\tilde{u} := u\chi_{Q_1 \cap A} + \xi \chi_{Q_1 \setminus A}.$$

Note that $\tilde{u} \in GSBD^2(Q_1; \mathbb{R}^2)$ and by the choice of $\xi$ and by [21, Definition 2.4] $Q_1 \cap \partial^* A \subset J_5$. In addition, by possibly adding to $\tilde{u}$ a function in $SB\Omega^2(Q_1; \mathbb{R}^2) \cap W^{1,\infty}(Q_1 \setminus \partial A; \mathbb{R}^2)$ with small $W^{1,\infty}(Q_1 \setminus \partial A; \mathbb{R}^2)$ norm, jump on the set $Q_1 \cap \partial A$, and supported near $Q_1 \cap \partial A$, we can assume without loss of generality that $Q_1 \cap J_5 \supset Q_1 \cap \partial^* A$ up to a $H^1$-negligible set*. Notice that

$$\delta := H^1(Q_1 \cap \partial^* A)^{1/2} = H^1(Q_1 \cap J_5)^{1/2},$$

*A similar argument was used in [13, p. 1359, above Eq. 4.19]
and set $N := [1/\delta]$ so that $(-N\delta, N\delta)^2 \subset Q_1$. For $i := 0, 1, \ldots, N - 1$ let $Q^i := (-N - i)\delta, (N - i)\delta)^2$ and $Q^i := Q^i \setminus Q^{i+1}$ (assuming $C^{N-1} := Q^{N-1}$). Up to a slight translation of $Q^i$ we assume that $H^1(\partial A \cap \partial Q^i) = 0$ for all $i$. By [11, Lemma 3.3] we find $i_0 \geq 1$ such that

$$
\begin{aligned}
\int_{C^{i_0} \cup C^{i_0+1}} |e(\tilde{u})|^2 \, dx \leq 8\sqrt{\delta} \int_{Q_1 \setminus \sqrt{\delta}} |e(\tilde{u})|^2 \, dx,
\int H^1(\partial A \cap (C^{i_0} \cup C^{i_0+1})) \leq 8\delta^2 H^1(\partial A \cap (Q_1 \setminus Q_1 - \delta)).
\end{aligned}
$$

We partition $Q^{i_0+1}$ into pairwise disjoint squares with sidelength $\delta$ and divide the slice $C^{i_0}$ into dyadic slices

$$
G_j := (-N - i_0 - 2^{-j}) \delta, (N - i_0 - 2^{-j}) \delta)^2 \setminus (-N - i_0 - 2^{-j+1}) \delta, (N - i_0 - 2^{-j+1}) \delta)^2,
$$

then we partition each slice $G_j$ into pairwise disjoint squares $Q_{j,l}$ of sidelength $2^{-j}\delta$ whose sides are parallel to the coordinate axis. Let $V_0$ be the collection of all squares of sidelength $\delta$ that cover the central square $Q^{i_0+1}$ and let $V$ be the union of $V_0$ and of the collection of all $Q_{j,l}$. Following [11] we differentiate between “good” and “bad” squares in $V$. A square $Q \in V$ is “good” if

$$
H^1(Q^m \cap \partial A) \leq \eta \delta Q,
$$

where $Q^m$ is the square with the same center as $Q$ and dilated by $7/6$, and $\delta_Q := \delta$ if $Q \in V_0$ and $\delta_Q := 2^{-j}\delta$ if $Q \subset G_j$. A square $Q$ is “bad” if it does not satisfy (3.7). By (3.4) $\delta^2 = H^1(Q_1 \cap \partial A) < \eta \delta$, hence, by definition, all squares in $V_0$ are good and by [11, Eq. 12] the sum of the perimeters of all bad squares satisfies

$$
\sum_{Q \in V_{\text{bad}}} H^1(\partial^* Q) \leq \tilde{c}_0 \sqrt{\delta} H^1((Q_1 \setminus Q_1 - \sqrt{\delta}) \cap \partial A)
$$

for some $\tilde{c}_0 > 0$. Since $\delta < \eta$, by [11, Theorem 3] there exist $\tilde{v} \in GSBD^2(Q_1; \mathbb{R}^2)$, $r \in (1 - \sqrt{\delta}, 1)$ and a Lebesgue measurable set $\tilde{\omega} \subset Q_r$ such that

(a1) $\tilde{v} \in C^{\infty}(Q_1 \setminus \sqrt{\delta})$, $\tilde{v} = \tilde{v}$ in $Q_1 \setminus Q_r$ and $H^1(J_{\tilde{v}} \cap \partial Q_r) = H^1(J_{\tilde{v}} \cap \partial Q_r) = 0$;

(a2) $H^1(J_{\tilde{v}} \cap J_\delta) \leq \tilde{c}_0 \sqrt{\delta} H^1((Q_1 \setminus Q_1 - \sqrt{\delta}) \cap J_{\tilde{v}})$;

(a3) $|\tilde{\omega}| \leq \tilde{c}_0 \delta H^1(Q_r \cap \partial A)$ and

$$
\int_{Q_1 \setminus \tilde{\omega}} |\tilde{v} - \tilde{u}|^2 \, dx \leq \tilde{c}_0 \delta^2 \int_{Q_1} |e(\tilde{u})|^2 \, dx;
$$

(a4) for any $\psi \in \text{Lip}(Q_1; [0, 1])$ and elasticity tensor $C \in L^\infty(Q_1)$ satisfying (3.6) there exists $d_3 := d_3(\tilde{c}_0, d_1, d_2) > 0$ such that

$$
\int_{Q_1} \psi C(x)e(\tilde{v}) : e(\tilde{v}) \, dx \leq \int_{Q_1} \psi C(x)e(\tilde{v}) : e(\tilde{u}) \, dx + d_3 \delta^s (1 + \text{Lip}(\psi)) \int_{Q_1} |e(\tilde{u})|^2 \, dx
$$

with $s \in (0, 1)$ depending only on $\tilde{c}_0$, $d_1$ and $d_2$;

(a5) $J_{\tilde{v}} \subset \partial^* D \cup (J_{\tilde{u}} \setminus Q^{i_0+1})$ and $J_{\tilde{u}} \setminus J_{\tilde{v}} \subset \partial^* D$, where $D$ is the union of all bad squares.

Note that for proving (a4) in [11] a mollifying argument is used (together with the fact that $C$ is assumed to be constant in [11]). As in our setting $C$ is in general not constant, we revised such argument (see [11, Eq. 23]), by using the fact that the energy

$$
w \in GSBD^2(O) \Rightarrow \int O \mathbb{C}(w) : e(w) \, dx
$$
is quadratic with respect to the \( e(w) \) and hence, we have convexity and we can employ Cauchy-Schwartz inequality for positive semidefinite bilinear forms to obtain

\[
\int_{O} C(x)e(\tilde{v}) : e(\tilde{v})dx \leq \int_{O} C(x)e(\tilde{u}) : e(\tilde{u})dx + 2 \int_{O} C(x)e(\tilde{v}) : [e(\tilde{v}) - e(\tilde{u})]dx
\]

\[
\leq \int_{O} C(x)e(\tilde{u}) : e(\tilde{u})dx + 2 \int_{O} C(x)e(\tilde{v}) : [e(\tilde{v}) - e(\tilde{u})]dx \quad \times \quad \left[ \int_{O} C(x)[e(\tilde{v}) - e(\tilde{u})] : [e(\tilde{v}) - e(\tilde{u})]dx \right]^{1/2}
\]

for any open set \( O \subset Q_1 \). Since the inequality \( a^2 \leq b^2 + 2ac \), where \( a, b, c \geq 0 \), implies \( a \leq b + 2c \), we get

\[
\left[ \int_{O} C(x)e(\tilde{v}) : e(\tilde{v})dx \right]^{1/2} \leq \left[ \int_{O} C(x)e(\tilde{u}) : e(\tilde{u})dx \right]^{1/2} + 2 \left[ \int_{O} C(x)[e(\tilde{v}) - e(\tilde{u})] : [e(\tilde{v}) - e(\tilde{u})]dx \right]^{1/2}
\]

\[
\leq (1 + c\delta^x) \left[ \int_{O} C(x)e(\tilde{u}) : e(\tilde{u})dx \right]^{1/2}
\]

so that Eq.23 of [11] holds also in our setting.

Let \( V_i \) be the family of all bad squares \( Q \) intersecting \( \text{Int}(A) \) and \( D_i := \bigcup_{Q \in V_i} Q \). For every \( Q \in V_i \) we define \( I_Q \) as the segment of smallest length connecting \( (Q'''' \cap \partial A) \setminus \overline{Q} \) to \( \partial Q \) with the convention that \( I_Q = \emptyset \) if \( (Q'''' \cap \partial A) \setminus \overline{Q} = \emptyset \) or \( Q \cap \text{Int}(\Omega \setminus A) \neq \emptyset \). By the definition of \( Q'''' \) and \( Q_1 \), \( H^1(I_Q) \leq \sqrt{\frac{2}{\pi}} H^1(\partial Q) \).

Let

\[
B := \left( A \setminus D_i \right) \cup \partial D_i \setminus \bigcup_{Q \in V_i} I_Q
\]

and

\[
v := \tilde{v} \chi_{Q_1} + \tilde{u} \chi_{(\Omega \setminus \Omega_1)}
\]

We claim that \( B, v \) and \( \tilde{w} \) satisfy the assertions of the lemma.

Indeed, from (a4) applied with \( \psi \equiv 1 \) and \( C = I \) it follows that \( v \in GSB\overline{D}^2(\text{Int}(B); \mathbb{R}^2) \). Moreover, by (a5) \( v \in H^1(\text{Int}(B); \mathbb{R}^2) \), thus, \( (B, v) \in C \). Let us show that if \( A \in A_m \) for some \( m \in \mathbb{N} \), then \( B \in A_m \). Indeed, by the construction of \( B \), for each bad square \( Q \), the dilated square \( Q'''' \) contains inside “large” portions of the boundary \( \partial A \). Now if \( \partial A \) intersects \( \overline{Q} \), then \( I_Q = \emptyset \) and the modification \( [A \setminus \overline{Q}] \cup \partial Q \setminus I_Q \) does not increase the number of boundary components. Otherwise, if \( \partial A \) does not increase \( \overline{Q} \), so that it intersects only \( Q'''' \setminus \overline{Q} \), then adding a small segment \( I_Q \) to connect \( \partial A \cap [Q'''' \setminus \overline{Q}] \) to \( \overline{Q} \) again does not increase the number of boundary components of \( [A \setminus \overline{Q}] \cup \partial Q \setminus I_Q \). Now, from the disjointness of the cubes \( Q \in V_i \) it follows that \( B \in C_m \). Therefore, if \( (A, u) \in C_m \) for some \( m \in \mathbb{N} \), then \( (B, v|_{\partial B}) \in C_m \).

By (a1) it follows that \( v \in C^\infty(Q_{1-\sqrt{3}}) \). Moreover, by the definition of \( B \), \( A\Delta B \subset Q_{r_h} \cap Q_{1-\sqrt{3}} \) for some \( r_h \in (1-\sqrt{3}, 1) \) such that \( D_i \subset Q_{r_h} \). Also, by (a1) \( \text{supp} (\tilde{u} - \tilde{v}) \subset Q_1 \) so that \( \text{supp} (\tilde{u} - v) \subset Q_1 \), and (1) follows.

---

*Note that \( a \leq b + 2c \) follows from \( a^2 \leq b^2 + 2ac \) as it yields \( (a - 2c)^2 \leq a(a - 2c) \leq b^2 \).
Moreover, by the definition of \( B, \ I_Q \) and (3.8)
\[
\mathcal{H}^1(\partial B \setminus \partial A) \leq \sum_{Q \in V_i} P(Q) + \sum_{Q \in V_i} \mathcal{H}^1(I_Q)
\]
\[
\leq \left(1 + \frac{\sqrt{2}}{24}\right) \sum_{Q \in V_i} P(Q) \leq c_0 \sqrt{\delta} \mathcal{H}^1((Q_1 \setminus Q_1 \setminus \partial A) \cap \partial A),
\]
where \( c_0 := \tilde{c}_0(1 + \sqrt{2}/24) \), and (2) follows.

Next, by (a3) \( |\omega| \leq c_0 \delta \mathcal{H}^1(Q_1 \cap \partial A) \), and
\[
\int_{Q_1 \setminus \omega} |v(x) - \tilde{u}(x)|^2 dx = \int_{Q_1 \setminus \omega} |\tilde{\omega}(y) - \tilde{u}(y)|^2 dy \leq \tilde{c}_0 \delta^2 \int_{Q_1} |e(\tilde{u})|^2 dy
\]
\[
\leq c_0 \delta^2 \int_{Q_1} |e(\tilde{u})|^2 dy.
\]

Finally, by (a4) and the definition of \( v \) (i.e., \( v = \tilde{v} \) in \( Q_1 \)) for any \( \psi \in \text{Lip}(Q_1) \) and \( C \in L^\infty(Q_1) \) satisfying (3.6) we have
\[
\int_{Q_1} \psi(x) C(x) e(v) : e(v) dx = \int_{Q_1} \psi(x) C(x) e(\tilde{u}) : e(\tilde{u}) dx
\]
\[
\leq \int_{Q_1} \psi(x) C(x) e(\tilde{u}) : e(\tilde{u}) dx + d_3 (1 + \text{Lip}(\psi)) \int_{Q_1} |e(\tilde{u})|^2 dx
\]
\[
= \int_{Q_1 \cap A} \psi(x) C(x) e(u) : e(u) dx + d_3 \delta^s (1 + \text{Lip}(\psi)) \int_{Q_1 \cap A} |e(u)|^2 dx,
\]
since \( \tilde{u} \) is constant in \( Q_1 \setminus A \). Hence, (4) follows. \( \square \)

The following proposition is a generalization to our setting of [11, Theorem 4] established for the Griffith model.

**Proposition 3.3.** Let \( Q_R(x_0) \subset \Omega \) be a square of side length \( 2R > 0 \). Consider sequences \( \{m_h\} \subset \mathbb{N} \cup \{\infty\} \), Finsler norms \( \{\varphi_h\} \) and ellipticity tensors \( \{C_h\} \) such that \( \{C_h\} \) is equicontinuous in \( Q_R(x_0) \) and there exist \( d_3, d_4, d_5 > 0 \) with
\[
d_3 M : M \leq C_h(x) M : M \leq d_4 M : M \quad \text{for all} \ (x, M) \in \overline{Q_R(x_0)} \times M_2^{\text{sym}},
\]
and
\[
d_5 \sup_{(x, \nu) \in Q_R \times S^1} \varphi_h(x, \nu) \leq \inf_{(x, \nu) \in Q_R \times S^1} \varphi_h(x, \nu),
\]
and define \( F_h \) and \( \Psi_h \) in \( C_{m_h} \) as in (2.9) and (2.11), respectively, with \( \varphi_h, C_h \) and \( m_h \) in places of \( \varphi, C \) and \( m \). Let \( \{ (A_h, u_h) \} \subset C_{m_h} \) be such that
\[
\lim_{h \to \infty} \Psi_h(A_h; u_h; Q_R(x_0)) = 0,
\]
\[
\lim_{h \to \infty} \mathcal{H}^1(Q_R(x_0) \cap \partial A_h) = 0,
\]
\[
\sup_{h \geq 1} F_h(A_h; u_h; Q_R(x_0)) =: M < \infty.
\]
Then there exist \( u \in H^1(Q_R(x_0)) \), an elasticity tensor \( C \in C^0(\overline{Q_R(x_0)}; M_2^{\text{sym}}) \), sequences \( \{\xi_j\} \subset (0, 1]^2 \) of vectors and \( \{a_j\} \) of rigid displacements and subsequences \( \{(A_{h_j}, u_{h_j})\} \), \( \{\varphi_{h_j}\} \) and \( \{C_{h_j}\} \) such that
(a) \( C_{h_j} \to C \) uniformly in \( \overline{Q_R(x_0)} \) and \( w_j := u_{h_j} + \xi_j Q_R(x_0) \cap A_{h_j} \to u \)
pointwise a.e. in \( Q_R(x_0) \), and \( e(w_j) \to e(u) \) in \( L^2(Q_R(x_0)) \) as \( j \to \infty \);
(b) for all \( v \in u + H^1_0(Q_R(x_0)) \)

\[
\int_{Q_R(x_0)} \mathbb{C}(y) e(u) : e(v) dy \leq \int_{Q_R(x_0)} \mathbb{C}(y) e(v) : e(v) dy;
\]

(c) for any \( r \in (0, R] \)

\[
\lim_{j \to \infty} \mathcal{F}_h(A_{h_j}, u_{h_j}; Q_r(x_0)) = \int_{Q_r(x_0)} \mathbb{C}(x) e(u) : e(u) dx.
\]

**Proof.** Without loss of generality, we suppose \( R = 1 \) and \( x_0 = 0 \). Let

\[
c_{1,h} := \inf_{(x,v) \in Q_1 \times \mathbb{S}^1} \varphi_h(x,v), \quad c_{2,h} := \sup_{(x,v) \in Q_1 \times \mathbb{S}^1} \varphi_h(x,v);
\]

by (3.10) we have \( d_5 c_{2,h} \leq c_{1,h} \). Since \( \sup_h \mathcal{H}^1(Q_1 \cap \partial A_h) < \infty \), by Proposition A.2 for every \( h \geq 1 \) there exists \( \xi_h \in (0, 1)^2 \) such that

\[
\mathcal{H}^1(\{ y \in Q_1 \cap \partial A_h : \text{tr}_{A_h}(u_h) \text{ exists and equals to } \xi_h \text{ at } y \}) = 0.
\]

Therefore

\[
\tilde{u}_h := \begin{cases} u_h & \text{in } Q_1 \cap A_h, \\ \xi_h & \text{in } Q_1 \setminus A_h \end{cases}
\]

belongs to \( GSBD^2(Q_1; \mathbb{R}^2) \) with \( J_{\tilde{u}_h} \subset Q_1 \cap \partial A_h \) and

\[
\lim_{h \to \infty} \mathcal{H}^1(J_{\tilde{u}_h}) = 0
\]

in view of (3.12). Further we suppose \( \mathcal{H}^1(J_{\tilde{u}_h}) < 1/4 \) for any \( h \geq 1 \).

By [10, Proposition 2] and (3.9), there exist a constant \( c \) (depending only on \( d_5 \)) and sequences \( \{ \tilde{\omega}_h \} \) of Lebesgue measurable subsets of \( Q_1 \) with \( |\tilde{\omega}_h| \leq c \mathcal{H}^1(Q_1 \cap \partial A_h) \) and \( \{ a_h \} \) of rigid motions such that

\[
\int_{Q_1 \setminus \tilde{\omega}_h} |\tilde{u}_h - a_h|^2 dx \leq c \int_{Q_1} \mathbb{C}_h(x) e(\tilde{u}_h) : e(\tilde{u}_h) dx.
\]

By (3.9) and (13.13), there exists \( u \in L^2(Q_1) \) such that up to a subsequence \( (\tilde{u}_h - a_h)\chi_{Q_1 \setminus \tilde{\omega}_h} \to u \) weakly in \( L^2(Q_1) \). Furthermore from (3.9) and (13.13) we obtain

\[
\sup_{h \geq 1} \int_{Q_1} |e(\tilde{u}_h - a_h)|^2 dx + \mathcal{H}^1(J_{\tilde{u}_h}) < \infty,
\]

and hence, by [14, Theorem 1.1] there exist a subsequence still denoted by \( \{ \tilde{u}_h - a_h \} \) for which the set

\[
E := \{ y \in Q_1 : \lim_{h \to \infty} |\tilde{u}_h(y) - a_h(y)| \to 0 \}
\]

has finite perimeter and \( \tilde{u} \in GSBD^2(Q_1 \setminus E; \mathbb{R}^2) \) with \( \tilde{u} = 0 \) in \( E \) such that

\[
\tilde{u}_h - a_h \to \tilde{u} \quad \text{a.e. in } Q_1 \setminus E,
\]

\[
e(\tilde{u}_h - a_h) \to e(\tilde{u}) \quad \text{in } L^2(Q_1 \setminus E; \mathbb{M}^{2 \times 2}_{\text{sym}}),
\]

\[
\mathcal{H}^1((Q_1 \setminus \partial^* E) \cap J_{\tilde{u}}) + \mathcal{H}^1(Q_1 \cap \partial^* E) = \mathcal{H}^1(J_{\tilde{u}} \cup \partial^* E) \leq \liminf_{h \to \infty} \mathcal{H}^1(J_{\tilde{u}_h}) = 0.
\]

In particular, \( P(E, Q_1) = 0 \) so that by the relative isoperimetric inequality either \( |E| = |Q_1| \) or \( |E| = 0 \). By the definition of \( E \), (3.12), the uniform \( L^2(Q_1) \)-boundedness of \( \{ (\tilde{u}_h - a_h)\chi_{Q_1 \setminus \tilde{\omega}_h} \} \) which is a consequence of (3.19) and (13.13), and Fatou’s Lemma it follows that \( |E| = 0 \). Hence, from (3.20) we get \( \tilde{u}_h - a_h \to \tilde{u} \) a.e. in \( Q_1 \) and \( e(\tilde{u}_h - a_h) \to e(\tilde{u}) \) in \( L^2(Q_1; \mathbb{M}^{2 \times 2}_{\text{sym}}) \), and all relations in (3.20) hold in \( Q_1 \) and \( \tilde{u} = u \) a.e. in \( Q_1 \). In particular, since \( \mathcal{H}^1(J_u) = 0 \), by Proposition A.3 we have that \( u \in H^1(Q_1; \mathbb{R}^2) \). In view of the fact that our elastic energy is invariant under rigid deformations, we suppose \( a_h = 0 \) for any \( h \geq 1 \).
Next we prove (3.14). Let $v \in H^1(Q_1; \mathbb{R}^2)$ be such that $\text{supp}(u - v) \subset Q_r$ for some $r \in (0, 1)$. Let $\psi \in C_c^2(Q_1; [0, 1])$ be a cut-off function with $\{0 < \psi < 1\} \subset \{u = v\} \cap Q_r$ and $\text{supp}(u - v) \subset \{\psi \equiv 1\} \subset Q_{r''}$ for some $r'' < r' < r$. By (3.18) and Lemma 3.2 applied with $(A_h, u_h)$ and $Q_r$, there exist $\bar{v}_h \in GSBD^2(\text{Int}(\Omega \cup S \cup \Sigma); \mathbb{R}^2)$, $B_h \in \mathcal{A}_{m_h}$ with $(B_h, \bar{v}_h|_{B_h}) \in C_m$, $r_h \in (r(1 - \sqrt{\delta_h}), r)$ and a Lebesgue measurable set $\omega_h \subset Q_r$ such that

(a1) $\bar{v}_h \in C^\infty(Q_{r(1 - \sqrt{\delta_h})})$, $A_h \Delta B_h \subset Q_{r_0} \setminus Q_{r(1 - \sqrt{\delta_h})}$ and $\text{supp}(\bar{u}_h - \bar{v}_h) \subset Q_r$;

(a2) $H^1(\partial B_h \setminus \partial A_h) \leq c_0 \sqrt{\delta_h} H^1([Q_r \setminus Q_{r(1 - \sqrt{\delta_h})}] \cap \partial A_h)$;

(a3) $|\omega_h| \leq c_0 \delta_h H^1(Q_r \cap \partial A_h)$ and

$$\int_{Q_r \setminus \omega_h} \|\bar{v}_h - \bar{u}_h\|^2 \, dx \leq c_0 \delta_h^2 r^2 \int_{Q_r \cap A_h} |e(u_h)| \, dx;$$

(a4) for any $\eta \in \text{Lip}(Q_r; [0, 1])$

$$\int_{Q_r} \eta C_h e(\bar{v}_h) : e(\bar{v}_h) \, dx \leq \int_{Q_r \cap A_h} \eta C_h e(u_h) : e(u_h) \, dx + d_3 \delta_h^2 (1 + r \text{Lip}(\eta)) \int_{Q_r \cap A_h} |e(u_h)| \, dx,$$  \hspace{1cm} (3.21)

where $\delta_h := r^{-1/2}H^1(Q_r \cap \partial A_h)^{1/2} \to 0$, and $d_3$ and $s$ are constants. We assume that $h$ is large enough so that $r_h > r'$. Set

$$v_h := (1 - \psi)\bar{v}_h + \psi v.$$

We observe that $\text{supp}(u_h - v_h|_{B_h}) \subset Q_r$; by (a1) and the definition of $\psi$, there exists $r_0 \in (r_h, r)$ such that $A_h \setminus Q_{r_0} = B_h \setminus Q_{r_0}$ and $\bar{u}_h = \bar{v}_h = v_h$ in $Q_r \setminus Q_{r_0}$ and hence, $u_h|_{Q_r \cap A_h \setminus Q_{r_0}} = \bar{u}_h|_{Q_r \cap A_h \setminus Q_{r_0}} = v_h|_{Q_r \cap B_h \setminus Q_{r_0}}$. Thus, $(B_h, v_h)$ is an admissible configuration in (2.10) and from (3.11) and the definition of deviation it follows that

$$F_h(A_h, u_h; Q_1) \leq F_h(B_h, v_h; Q_1) + o(1),$$  \hspace{1cm} (3.22)

where $o(1) \to 0$ as $h \to \infty$. We observe that

$$S_h(B_h; Q_1) - S_h(A_h; Q_1) \leq S_h(B_h; Q_{r(1 - \sqrt{\delta_h})}) - S_h(A_h; Q_{r(1 - \sqrt{\delta_h})}) \leq \int (\partial^* B_h \setminus \partial^* A_h) \cap Q_{r(1 - \sqrt{\delta_h})} \varphi(x, \nu_{B_h}) \, dH^1$$

$$\leq 2c_{2, h} H^1(\partial B_h \setminus \partial A_h) \leq 2c_{2, h} \sqrt{\delta_h} H^1([Q_r \setminus Q_{r(1 - \sqrt{\delta_h})}] \cap \partial A_h)$$

$$\leq \frac{2c_0 \sqrt{\delta_h}}{d_5} S_h(A_h; Q_1) = o(1)$$

as $h \to +\infty$, where we used in the first inequality (a1), in the second the definition and nonnegativity of $S_h$, in the third (3.16), in the fourth (a2) in the last again (3.16) and the definition of $S_h$, and finally in the equality we used (3.13). Thus, (3.22) is rewritten as

$$W_h(A_h, u_h; Q_1) \leq W_h(B_h, v_h; Q_1) + o(1).$$  \hspace{1cm} (3.23)

Note that by (a1), (a3), (3.18), (3.20) and Fatou’s Lemma, $\bar{v}_h|_{Q_r \setminus \omega_h} \to u$ a.e. in $Q_r$ and by (a3) $\chi_{Q_r \setminus \omega_h} \to 1$ a.e. in $Q_r$. Therefore, for a.e. $x \in Q_r$ there exists $h_x \geq 1$ such that $\chi_{Q_r \setminus \omega_h}(x) = 1$ for every $h > h_x$ and $\bar{v}_h(x) = \bar{v}_h(x)\chi_{Q_r \setminus \omega_h}(x) \to u(x)$. So

$$\bar{v}_h \to u \text{ a.e. in } Q_r.$$  \hspace{1cm} (3.24)
We claim that $\tilde{v}_h \to u$ strongly in $L^2_{\text{loc}}(Q_r)$. To see this we fix $\rho \in (0, r)$, and, since $\delta_h \to 0$ by (a1), there exists $h_\rho \geq 1$ such that $\tilde{v}_h \in H^1(Q_{\rho})$ for every $h > h_\rho$. From (3.9), (3.13) and (3.21) as well as the Korn-Poincaré inequality
\[
\sup_{h > h_\rho} \|\tilde{v}_h - b_h\|_{H^1(Q_{\rho})} < \infty
\]
for some sequence $\{b_h\}$ of rigid displacements. On the one hand, by Rellich-Kondrachov Theorem there exist $z \in H^1(Q_{\rho}; \mathbb{R}^2)$ and not relabelled subsequence such that $\tilde{v}_h - b_h \to z$ in $L^2(Q_{\rho}; \mathbb{R}^2)$ and a.e. in $Q_{\rho}$. On the other hand, by (3.24) $b_h = \tilde{v}_h - (\tilde{v}_h - b_h)$ converges to $b := u - z$ a.e. in $Q_{\rho}$. Since $b_h$ is a rigid displacement, so is $b$ and hence $b_h \to b$ uniformly in $Q_{\rho}$. Therefore,
\[
\limsup_{h \to \infty} \|\tilde{v}_h - u\|_{L^2(Q_{\rho})} \leq \limsup_{h \to \infty} \|\tilde{v}_h - b_h - z\|_{L^2(Q_{\rho})} + \limsup_{h \to \infty} \|b_h - b\|_{L^2(Q_{\rho})} = 0,
\]
and the claim follows.

Since $u = v$ out of $\{\psi = 1\}$, the claim implies $\tilde{v}_h \to v$ strongly in $L^2(\{0 < \psi < 1\})$, and hence,
\[
\lim_{h \to \infty} \int_{Q_r} |\nabla \psi \circ (v - \tilde{v}_h)|_{A_h}^2 \leq \liminf_{h \to \infty} \int_{\{0 < \psi < 1\}} |\nabla \psi \circ (v - \tilde{v}_h)|^2 = 0,
\]
where $X \circ Y = (X \otimes Y + Y \otimes X)/2$. Thus, by the definition of $v_h$ and the equality
\[
e(v_h) = (1 - \psi)e(\tilde{v}_h) + \psi e(v) + \nabla \psi \circ (v - \tilde{v}_h),
\]
we estimate
\[
\int_{Q_r} C_h e(v_h) : e(v_h)dx = \int_{Q_r} (1 - \psi)^2 C_h e(\tilde{v}_h) : e(\tilde{v}_h)dx + \int_{Q_r} \psi^2 C_h e(v) : e(v)dx + \int_{Q_r} C_h (\nabla \psi \circ (v - \tilde{v}_h)) : (\nabla \psi \circ (v - \tilde{v}_h))dx + 2 \int_{Q_r} (1 - \psi) C_h e(\tilde{v}_h) : (\nabla \psi \circ (v - \tilde{v}_h))dx
\]
\[
+ 2 \int_{Q_r} \psi C_h e(v) : (\nabla \psi \circ (v - \tilde{v}_h))dx \leq \int_{Q_r \cap A_h} (1 - \psi)^2 C_h e(u_h) : e(u_h)dx + \int_{Q_r} \psi^2 C_h e(v) : e(v)dx + o(1),
\]
where in the second equality we use (3.13), (3.21) with $\eta \equiv 1$, (3.25), (3.9) and the Hölder inequality, while in the last inequality we use (3.21) with $\eta = (1 - \psi)^2$ and (3.17). Now (3.23), (3.26) and (3.17) imply
\[
\int_{Q_r} (2\psi - \psi^2) C_h e(\tilde{u}_h) : e(\tilde{u}_h)dx \leq \int_{Q_r} \psi^2 C_h e(v) : e(v)dx + o(1).
\]
Since $\{C_h\}$ is equibounded (see (3.9)) and equicontinuous, by the Arzelà-Ascoli Theorem, there exist a (not relabelled) subsequence and an elasticity tensor $C \in C^0(Q_1; \mathbb{R}^{2 \times 2}_{\text{sym}})$ such that $C_h \to C$ uniformly in $Q_1$. Hence, letting $h \to \infty$ in (3.27) and using the convexity of the elastic energy and (3.20), we obtain
\[
\int_{Q_r} (2\psi - \psi^2) C(y) e(u) : e(u)dy \leq \int_{Q_r} \psi^2 C(y) e(v) : e(v)dy.
\]
By the choice of $\psi$, (3.28) implies
\[ \int_{Q_{r''}} C(y)e(u) : e(u)dy \leq \int_{Q_r} C(y)e(v) : e(v)dy. \] (3.29)

Since $r''$ is arbitrary, letting $r'' \nearrow r$ we deduce that (3.29) holds also with $r'' = r$. Since $\text{supp } (u - v) \subset \subset Q_r$, this implies (3.14).

It remains to prove (3.15). If we take $v = u$ in (3.27) and use $0 \leq \psi \leq 1$ and $\psi = 1$ in $Q_{r''}$ we get
\[
\int_{Q_{r''}} C e(u) : e(u)dx \leq \liminf_{h \to \infty} \int_{Q_{r''}} C_h e(\tilde{u}_h) : e(\tilde{u}_h)dx
\]
\[\leq \limsup_{h \to \infty} \int_{Q_{r''}} C_h e(\tilde{u}_h) : e(\tilde{u}_h)dx \leq \int_{Q_r} C e(u) : e(u)dx.
\]
Since $r''$ is arbitrary, letting $r'' \nearrow r$ we deduce
\[
\lim_{h \to \infty} \int_{Q_r} C_h e(\tilde{u}_h) : e(\tilde{u}_h)dx = \int_{Q_r} C e(u) : e(u)dx.\] (3.30)

Now we prove that
\[
\lim_{h \to \infty} S_h(A_h; Q_r) = 0
\] (3.31) for any $r \in (0, 1)$. By (3.12), we can find $h_r > 0$ such that
\[
\mathcal{H}^1(Q_1 \cap \partial A_h) < (1 - r)/5
\] (3.32) for any $h > h_r$, and hence there is no connected component of $\partial A_h$ intersecting both $\partial Q_r$ and $\partial Q_1$. Also by the relative isoperimetric inequality, passing to further subsequence we suppose that either
\[
\lim_{h \to \infty} |Q_1 \cap A_h| = 0
\] (3.33)
or
\[
\lim_{h \to \infty} |Q_1 \setminus A_h| = 0.
\] (3.34)

First assume that (3.33) holds. Let $E_h \subset A_h$ be the set consisting of all connected components of $\overline{A_h}$ not intersecting $\partial Q_1$. Then, $(A_h \setminus E_h, u_h|_{A_h \setminus E_h})$ is an admissible configuration in (2.10), thus,
\[
\mathcal{F}_h(A_h, u_h; Q_1) \leq \Phi_h(A_h, u_h; Q_1) + o(1) \leq \mathcal{F}_h(A_h \setminus E_h, u_h; Q_1) + o(1),
\] (3.35)
where in the first inequality we use (3.11) and in the second we use the definition of $\Phi_h$.
Hence,
\[
S(A_h; Q_r) \leq S(E_h; Q_1) = S_h(A_h; Q_1) - S_h(A_h \setminus E_h; Q_1)
\]
\[\leq \mathcal{F}_h(A_h; Q_1) - \mathcal{F}_h(A_h \setminus E_h; Q_1) \leq o(1),
\]
where we used in the first inequality the definition of $E_h$, which entitles that $U_r \cap \partial A_h \subset \partial E_h$, in the equality the disjointness of $\overline{A_h} \setminus \overline{E_h}$ and $\overline{E_h}$ which follows by (3.32), and in the second inequality the nonnegativity of the elastic energy and in the third (3.35). Hence, (3.31) follows.

Now assume that (3.34) holds and let $\delta_h := r^{-1/2} \sqrt{\mathcal{H}^1(Q_r \cap \partial A_h)} \to 0$. Fix any $\rho \in (0, r)$. By (3.12), we can find $h_{r, \rho} > 0$ such that $\delta_h < \min\{1 - r, r - \rho\}/5$ for any $h > h_{r, \rho}$. Since $A_h \in \mathcal{A}_{m_h}$, no connected component of $\partial A_h$ intersects both $\partial Q_r$ and $\partial Q_{r'}$. Let $F_h \subset Q_1 \setminus A_h$ be the union of all connected components of $Q_1 \setminus A_h$ lying strictly inside $Q_1$ (so $F_h$ is a union of “holes” and $\overline{F_h} \subset \partial A_h$). Let $\psi \in C^1_0([0, 1])$ be a cut-off function with $\{0 < \psi < 1\} \subset Q_{r'}$ and $\{\psi \equiv 1\} \subset Q_{r''}$ for some $r'' < r' < r$.
Set $A_h' := A_h \cup \overline{F_h}$. Applying Lemma 3.2 with $(A_h', \tilde{u}_h|_{A_h'})$, $Q_r$ and $m = m_h$ we find
\(\tilde{v}'_h \in GSBD^2(\text{Int}(\Omega \cup S \cup \Sigma); \mathbb{R}^2)\), \(B'_h \in \mathcal{A}_{m_h}\) with \(B'_h, \tilde{v}'_h |_{B'_h} \in \mathcal{C}_{m_h} \), \(r_h \in (r - \sqrt{h}, r)\) and a Lebesgue measurable set \(\omega'_h \subset \subset Q_r\), such that

(b1) \(\tilde{v}'_h \in C^\infty(Q_{r(1 - \sqrt{\delta_h})})\), \(A'_h \Delta B'_h \subset \subset Q_{r(1 - \sqrt{\delta_h})}\) and \(\text{supp}(\tilde{u}_h - \tilde{v}'_h) \subset \subset Q_r\); (b2) \(H^1(\partial B'_h \setminus \partial A'_h) \leq c_0 \sqrt{\delta_h} H^1([Q_r \setminus Q_{r(1 - \sqrt{\delta_h})}] \cap \partial A'_h)\);

(b3) \(|\omega'_h| \leq c_0 \delta_h H^1(Q_r \cap \partial A'_h)\)

\[
\int_{Q_r \setminus \omega'_h} |\tilde{v}'_h - \tilde{u}_h|^2 \, dx \leq c_0 \delta_h^2 \int_{Q_r \cap A'_h} |\epsilon(u_h)|^2 \, dx;
\]

(b4) for any \(\eta \in \text{Lip}(Q_r; [0, 1])\)

\[
\int_{Q_r} \eta \mathcal{C}_h(\tilde{v}'_h) : \epsilon(\tilde{v}'_h) \, dx \leq \int_{Q_r \cap A'_h} \eta \mathcal{C}_h(\tilde{u}_h) : \epsilon(\tilde{u}_h) \, dx + d_3 \delta_h (1 + r \text{Lip}(\eta)) \int_{Q_r \cap A'_h} |\epsilon(u_h)|^2 \, dx,
\]

where \(d_3\) and \(s\) are constants. Set

\(v'_h := (1 - \psi)\tilde{v}'_h + \psi u\).

By the definition of \(A'_h\) and (b1) \((B'_h, v'_h |_{B'_h})\) is an admissible configuration for \(\Phi_h(A_h, u_h; Q_1)\) in (2.10). Thus from (3.11) and (3.34)

\[
\mathcal{F}_h(A_h, u_h; Q_1) \leq \mathcal{F}_h(B'_h, v'_h |_{B'_h}; Q_1) + o(1). \tag{3.36}
\]

Now as in the proof of (3.27)

\[
\mathcal{W}_h(B'_h, v'_h |_{B'_h}; Q_1) - \mathcal{W}_h(A_h, u_h; Q_1)
\leq \int_{Q_r} \psi^2 \mathcal{C}_h(e(u)) : \epsilon(u) \, dx - \int_{Q_r} (2\psi - \psi^2) \mathcal{C}_h(e(\tilde{u}_h)) : \epsilon(\tilde{u}_h) \, dx + o(1)
\leq \int_{Q_r} \mathcal{C}_h(e(u)) : \epsilon(u) \, dx - \int_{Q_r} \mathcal{C}_h(e(\tilde{u}_h)) : \epsilon(\tilde{u}_h) \, dx + o(1). \tag{3.37}
\]

Moreover,

\[
\mathcal{S}_h(B'_h; Q_1) - \mathcal{S}_h(A_h; Q_1) = \left( \mathcal{S}_h(B'_h; Q_1) - \mathcal{S}_h(A'_h; Q_1) \right) + \left( \mathcal{S}_h(A'_h; Q_1) - \mathcal{S}_h(A_h; Q_1) \right)
\leq 2c_2 h \mathcal{H}^1(\partial B'_h \setminus \partial A'_h) - \mathcal{S}_h(A_h; Q_\rho) \leq 2c_2 h \mathcal{H}^1(\partial B'_h \setminus \partial A'_h) - \mathcal{S}_h(A_h; Q_\rho)
\leq 2c_2 h \mathcal{H}^1([Q_r \setminus Q_{r(1 - \sqrt{\delta_h})}] \cap \partial A_h) - \mathcal{S}_h(A_h; Q_\rho)
\leq 2c_0 \sqrt{\delta_h} \mathcal{S}_h(A_h; Q_1) - \mathcal{S}_h(A_h; Q_\rho) = o(1) - \mathcal{S}_h(A_h; Q_\rho), \tag{3.38}
\]

where we used in the first inequality (b1) and the definition of \(A'_h\), in the second and in the last inequalities the definition of \(\mathcal{S}_h\), (3.16) and (3.10), in the third inequality (b2), and in the last equality (3.13) and that \(\delta_h \to 0\) by (3.12). Hence, (3.36), (3.37) and (3.38) imply

\[
\mathcal{S}_h(A_h; Q_\rho) + \int_{Q_{r''}} \mathcal{C}_h(e(\tilde{u}_h)) : \epsilon(\tilde{u}_h) \, dx \leq \int_{Q_r} \mathcal{C}_h(e(u)) : \epsilon(u) \, dx + o(1).
\]

Thus, letting \(h \to \infty\) and using (3.30) we get

\[
\limsup_{h \to \infty} \mathcal{S}_h(A_h; Q_\rho) + \int_{Q_{r''}} \mathcal{C}(e(u)) \, dx \leq \int_{Q_r} \mathcal{C}(e(u)) \, dx.
\]

Now letting \(r'' \to r\) we get

\[
\limsup_{h \to \infty} \mathcal{S}_h(A_h; Q_\rho) = 0. \tag{3.39}
\]
Observe that the function \( B \mapsto S_h(A_h; B) \) defined for Borel sets \( B \subset Q_1 \) extends to a bounded nonnegative Radon measure \( \mu_h \) in \( Q_1 \). Since (3.39) holds for any \( \rho \in (0, r) \), \( \mu_h \) converges to 0 in the weak* sense, and thus (3.31) follows. \qed

Recall that by [18, Proposition 3.4] if the elasticity tensor \( C \) is constant, then for any \( \gamma \in (0, 2) \) there exists \( \kappa_\gamma := c_\gamma(c_3, c_4) > 0 \) such that for every local minimizer \((\Omega, u) \in \mathcal{C}\) of \( \mathcal{F}(\cdot; \Omega) \), \( u \) is analytic in \( \Omega \) and for any square \( Q_R(x) \subset \subset \Omega \) and \( r \in (0, R) \),
\[
\int_{Q_r(x)} \mathcal{C}(u) : e(u) \, dx \leq \kappa_\gamma \left( \frac{r}{R} \right)^{2-\gamma} \int_{Q_R(x)} \mathcal{C}(u) : e(u) \, dx. \tag{3.40}
\]

Given \( \gamma \in (0, 1) \) let
\[
\tau_0 = \tau_0(\gamma, c_3, c_4) := \min \{ 1, \frac{1}{2} c_\gamma^{-\frac{1}{1-\gamma}} \},
\]
where \( c_\gamma \) is the constant appearing in (3.40). Using Proposition 3.3 and repeating similar arguments of [12, 19] we get the following decay property of the functional \( \mathcal{F} \).

**Proposition 3.4.** For any \( \tau \in (0, \tau_0) \) there exist \( \varsigma = \varsigma(\tau) \in (0, 1) \) and \( \vartheta := \vartheta(\tau) \in (0, 1) \) with the following property: If there exist \( m \in \mathbb{N} \cup \{ \infty \} \), \( (A, u) \in \mathcal{C}_m \) and a square \( Q_\rho(x) \subset \subset \Omega \) such that
\[
\mathcal{H}^1(Q_\rho(x) \cap \partial A) \leq 2\varsigma \rho \quad \text{and} \quad \mathcal{F}(A, u; Q_\rho(x)) \leq (1 + \vartheta) \Phi(A, u; Q_\rho(x)),
\]
then
\[
\mathcal{F}(A, u; Q_{\tau \rho}(x)) \leq \tau^{2-\gamma} \mathcal{F}(A, u; Q_\rho(x)).
\]

**Proof.** We argue by contradiction. Assume that there exists \( \tau \in (0, \tau_0) \) such that for all \( \varsigma, \vartheta \in (0, 1) \) we can find \( m := m(\varsigma, \vartheta) \in \mathbb{N} \cup \{ \infty \} \), \( (A, u) := (A(\varsigma, \vartheta), u(\varsigma, \vartheta)) \in \mathcal{C}_m \) and \( Q_\rho(x) \subset \subset \Omega \) with \( \rho := \rho(\varsigma, \vartheta) \) and \( x := x(\varsigma, \vartheta) \) satisfying
\[
\mathcal{H}^1(Q_\rho(x) \cap \partial A) \leq 2\varsigma \rho \quad \text{and} \quad \mathcal{F}(A, u; Q_\rho(x)) \leq (1 + \vartheta) \Phi(A, u; Q_\rho(x)), \tag{3.41}
\]
but
\[
\mathcal{F}(A, u; Q_{\tau \rho}(x)) > \tau^{2-\gamma} \mathcal{F}(A, u; Q_\rho(x)). \tag{3.42}
\]
Let us choose any positive real numbers \( \varsigma_h, \vartheta_h \to 0 \), and denote for simplicity \( m_h := m(\varsigma_h, \vartheta_h) \), \( (A_h, u_h) := (A(\varsigma_h, \vartheta_h), u(\varsigma_h, \vartheta_h)) \), \( \rho_h := \rho(\varsigma_h, \vartheta_h) \), \( x_h = x(\varsigma_h, \vartheta_h) \). By (3.41) and (3.42),
\[
\mathcal{H}^1(Q_{\rho_h}(x_h) \cap \partial A_h) \leq 2\varsigma_h \rho_h, \tag{3.43}
\]
\[
\mathcal{F}(A_h, u_h; Q_{\rho_h}(x_h)) \leq (1 + \vartheta_h) \Phi(A_h, u_h; Q_{\rho_h}(x_h)), \tag{3.44}
\]
but
\[
\mathcal{F}(A_h, u_h; Q_{\tau \rho_h}(x_h)) > \tau^{2-\gamma} \mathcal{F}(A_h, u_h; Q_{\rho_h}(x_h)) \tag{3.45}
\]
for any \( h \). Note that \( \mathcal{F}(A_h, u_h; Q_{\rho_h}(x_h)) > 0 \). Let us define the rescaled energy \( \mathcal{F}_h(\cdot; Q_1) : \mathcal{C}_{m_h} \to \mathbb{R} \) as in (2.9) with
\[
\varphi_h(y, \nu) := \frac{\rho_h \varphi(x_h + \rho_h y, \nu)}{\mathcal{F}(A_h, u_h; Q_{\rho_h}(x_h))}
\]
in place of \( \varphi(y, \nu) \) and
\[
\mathcal{C}_h(y) := \mathcal{C}(x_h + \rho_h y)
\]
in place of \( \mathcal{C}(y) \), for \( y \in Q_1 \). We notice that
\[
\mathcal{F}_h(E_h, v_h; Q_1) = 1 \tag{3.46}
\]
for
\[
E_h := \sigma_{x_h, \rho_h}(A_h)
\]
By (3.43) we obtain
\[ \mathcal{H}^1(Q_1 \cap \partial E_h) < 2\delta_h \]
while (3.44) and (3.46) entails
\[ \Psi_h(E_h, v_h; Q_1) \leq \partial_h \Phi_h(E_h, v_h; Q_1) \leq \partial_h \mathcal{F}(E_h, v_h; Q_1) = \partial_h, \]
where \( \Phi_h \) and \( \Psi_h \) are defined as in (2.10) and (2.11) (again with \( \varphi_h \) and \( C_h \) in places of \( \varphi \) and \( C \), respectively). By (2.15) \( \{ C_h \} \) is equibounded. Since \( \Omega \) is bounded, there exists \( x_0 \in \Omega \) such that, up to extracting a subsequence, \( x_h \to x_0 \) as \( h \to +\infty \). As \( \rho_h \to 0 \), one has \( x_h + \rho_h y \to x_0 \) for every \( y \in Q_1 \). Thus \( \{ C_h \} \) is also equicontinuous and \( C_h \to C_0 := C(x_0) \) uniformly in \( Q_1 \). In view of (3.43), (3.44) and (3.46), we can apply Proposition 3.3 to find \( v \in H^1(Q_1; \mathbb{R}^2) \), vectors \( \xi_h \in (0, 1)^2 \), and infinitesimal rigid displacements \( a_h \) such that, up to a subsequence,
\[ w_h := v_h \chi_{Q_1 \cap E_h} + \xi_h \chi_{Q_1 \setminus E_h} - a_h \to v \]
pointwise a.e. in \( Q_1 \), \( e(w_h) \to e(v) \) in \( L^2(Q_1) \) as \( h \to +\infty \), and
\[ \lim_{h \to +\infty} \mathcal{F}(E_h, w_h; Q_r) = \lim_{h \to +\infty} \mathcal{F}(E_h, v_h; Q_r) = \int_{Q_r} C_0(x) e(v) : e(v) dx \quad (3.47) \]
for any \( r \in (0, 1] \). In particular, from (3.47) and (3.45) it follows that
\[ \int_{Q_r} C_0(x) e(v) : e(v) dx = \lim_{h \to +\infty} \mathcal{F}(E_h, v_h; Q_r) \]
\[ \geq \lim_{h \to +\infty} \tau^{2-\gamma} \mathcal{F}(E_h, v_h; Q_1) = \tau^{2-\gamma} \int_{Q_1} C_0(x) e(v) : e(v) dx. \]
Since \( C_0 \) is constant, applying (3.40) with \( r := \tau \) and \( R := 1 \) we get
\[ c_\tau \tau^{2-\gamma} \int_{Q_1} C_0(x) e(v) : e(v) dx \geq \int_{Q_1} C_0(x) e(v) : e(v) dx \]
\[ \geq \tau^{\gamma-2} \int_{Q_1} C_0(x) e(v) : e(v) dx. \]
Now recalling that \( \mathcal{F}(E_h, v_h; Q_1) = 1 \), by (3.47) we get \( \int_{Q_1} C_0(x) e(v) : e(v) dx = 1 \), thus,
\[ \tau^{2-\gamma} \geq c_\gamma^{-1/2} > \tau^{2-\gamma_0} \]
a contradiction.

By employing the arguments of [53, Section 4.3] and using Proposition 3.4 we establish the following lower bound for \( \mathcal{F} \).

**Proposition 3.5.** Given \( \tau \in (0, \tau_0) \), let \( \varsigma := \varsigma(\tau) \in (0, 1) \) and \( \vartheta := \vartheta(\tau) \in (0, 1) \) be as in Proposition 3.4. Let \( (A, u) \in C_{\lambda} \) be a \((\Lambda, m)\)-minimizer of \( \mathcal{F} \) in \( Q_{r_0}(x_0) \) for some \( m \in \mathbb{N} \cup \{ \infty \} \) and \( r_0 > 0 \), and let
\[ J^*_A := \{ y \in Q_{r_0}(x_0) \cap \partial A : \theta_A(\partial A, y) > 0 \}. \]
Then,
\[ \mathcal{F}(A, u; Q_{\rho}(x)) \geq 2c_1 \varsigma \rho \quad (3.48) \]
for every \( x \in J^*_A \) and for every square \( Q_{\rho}(x) \subset Q_{r_0}(x_0) \) with \( \rho \in (0, R_0) \), where
\[ R_0 := R_0(r_0, \Lambda, c_1, \tau) := \min \{ r_0, \sqrt{\pi} c_1 \vartheta / \Lambda(2 + \vartheta) \}. \]
Proof. Fix $m \in \mathbb{N} \cup \{\infty\}$. Note that for any $(C, w), (D, v) \in \mathcal{C}_m$ and $O \subset \Omega$ with $C\Delta D \subset O$
\[
\sqrt{4\pi |C\Delta D|^{1/2}} \leq \mathcal{H}^1(\partial^*(C\Delta D)) \leq \mathcal{H}^1(O \cap \partial^*C) + \mathcal{H}^1(O \cap \partial^*D)
\leq \frac{S(C, O) + S(D, O)}{c_1},
\]
where in the first inequality we used the isoperimetric inequality, in the second $\partial^*(C\Delta D) \subset O \cap (\partial^*C \cup \partial^*D)$, in the third (2.13) and the definition of $S(\cdot; O)$ and in the last the nonnegativity of $\mathcal{W}(\cdot; O)$. Thus, from the $(\Lambda, m)$-minimality of $(A, u)$ in $Q_{r_0}(x_0)$ we deduce that
\[
\mathcal{F}(A, w; Q_r(x)) \leq \mathcal{F}(B, v; Q_r(x)) + \Lambda |A\Delta B|^\frac{1}{2} |A\Delta B|^\frac{1}{2}
\leq \mathcal{F}(B, v; Q_r(x)) + \frac{\Lambda r}{\sqrt{\pi c_1}} \left( \mathcal{F}(A, u; Q_r(x)) + \mathcal{F}(B, v; Q_r(x)) \right)
\]
for any $Q_r(x) \subset Q_{r_0}(x_0)$ and $(B, v) \in \mathcal{C}_m$ with $A\Delta B \subset Q_r(x)$ and $\text{supp} (u-v) \subset Q_r(x)$, where in the last inequality we used (3.49) and the inequality $|A\Delta B| \leq |Q_r| = 4r^2$. Let $r > 0$ be small enough so that $\frac{\Lambda r}{\sqrt{\pi c_1}} \leq \frac{\vartheta}{2 + \vartheta}$, where $\vartheta := \vartheta(\tau) \in (0, 1)$ is given by Proposition 3.4. From (3.50) we obtain
\[
\mathcal{F}(A, w; Q_r(x)) \leq (1 + \vartheta)\mathcal{F}(B, v; Q_r(x)),
\]
which by the arbitrariness of $(B, v)$ is equivalent to
\[
\mathcal{F}(A, w; Q_r(x)) \leq (1 + \vartheta)\Phi(A, u; Q_r(x)).
\]  
Now we prove (3.48). Let $x \in J_A$. For simplicity we suppose that $x = 0$. Assume by contradiction that for such $m \in \mathbb{N} \cup \{\infty\}$, $(A, u) \in \mathcal{C}_m$ and for some $Q_{\rho} \subset Q_{r_0}(x_0)$ with $\rho \in (0, R_0)$ we have
\[
\mathcal{F}(A, w; Q_{\rho}) < 2c_1 \varsigma \rho.
\]
Then by the nonnegativity of the elastic energy and (2.13),
\[
2c_1 \varsigma \rho > \mathcal{F}(A, w; Q_{\rho}) \geq \int_{Q_{\rho} \cap \partial A} \varphi(x, \nu_A) d\mathcal{H}^1 \geq c_1 \mathcal{H}^1(Q_{\rho} \cap \partial A)
\]
so that
\[
\mathcal{H}^1(Q_{\rho} \cap \partial A) < 2\varsigma \rho.
\]  
By (3.52) and (3.51) we can apply Proposition 3.4 and obtain that
\[
\mathcal{F}(A, w; Q_{\tau\rho}) \leq \tau^{2-\gamma} \mathcal{F}(A, w; Q_{\rho}) \leq 2c_1 \tau^{2-\gamma} \rho
\]
Hence,
\[
\mathcal{H}^1(Q_{\tau\rho} \cap \partial A) \leq 2\varsigma \tau^{2-\gamma} \rho < 2\varsigma \tau \rho,
\]
where we used $\gamma, \tau \in (0, 1)$, and by induction
\[
\mathcal{H}^1(Q_{\tau^n \rho} \cap \partial A) \leq 2\varsigma \tau^{(2-\gamma)n} \rho < 2\varsigma \tau^n \rho, \quad n \in \mathbb{N}.
\]
However, by the choice of $x$
\[
0 < \theta_*(\partial A, x) = \liminf_{n \to +\infty} \frac{\mathcal{H}^1(Q_{\tau^n \rho} \cap \partial A)}{2\tau^n \rho} \leq \lim_{n \to +\infty} \frac{2c_1 \varsigma \tau^{(1-\gamma)n}}{2c_1} = 0,
\]
a contradiction. This contradiction implies (3.48) for $x \in J_A^*$.

Now consider any $x \in Q_{r_0}(x_0) \cap \overline{J_A}$ and $\rho \in (0, R_0)$ with $Q_{\rho}(x) \subset Q_{r_0}(x_0)$, and let us choose a sequence $\{Q_{\rho_k}(x_k)\}$ of squares with $x_k \in J_A^*$ and $\rho_1 \leq \rho_2 \leq \ldots \leq \rho$ such that
\[
Q_{\rho_1}(x_1) \subset Q_{\rho_2}(x_2) \subset \ldots Q_{\rho}(x) \quad \text{and} \quad Q_{\rho}(x) = \bigcup_k Q_{\rho_k}(x_k).
\]
Notice that \( x_k \to x \) and \( \rho_k \to \rho \). By De Giorgi-Letta Theorem [2, Theorem 1.53], both maps

\[
O \mapsto \int_{O \cap \partial^* A} \varphi(x, \nu_A) d\mathcal{H}^1 + 2 \int_{O \cap (A(0) \cup A(1)) \cap \partial A} \varphi(x, \nu_A) d\mathcal{H}^1
\]

and

\[
O \mapsto \int_{O \cap A} C(y) e(u) : e(u) dy,
\]

defined at open sets \( O \subset \subset \Omega \), uniquely extend to positive Borel measures \( \mu_1 \) and \( \mu_2 \) in \( \Omega \). Therefore, from the continuity of \( \mu_1 \) and \( \mu_2 \) (see e.g. [2, Remark 1.3]) and the validity of (3.48) with \( x_k \) and \( \rho_k \) it follows that

\[
F(A, u; Q_\rho(x)) = \mu_1(Q_\rho(x)) + \mu_2(Q_\rho(x)) = \lim_{k \to +\infty} [\mu_1(Q_\rho(x_k)) + \mu_2(Q_\rho_k(x_k))]
\]

\[
= \lim_{k \to +\infty} F(A, u; Q_\rho_k(x_k)) \geq \lim_{k \to +\infty} (2c_2 \varsigma \rho_k) = 2c_2 \varsigma \rho.
\]

\[ \square \]

Now we are ready to prove (3.2) and (3.3).

**Proof of Theorem 3.1.** Let \( m \in \mathbb{N} \cup \{\infty\} \) and \( (A, u) \) be a \((\Lambda, m)\)-minimizer of \( F(\cdot, \cdot; \Omega) \).

We begin by establishing (3.2). Let \( x \in \Omega \), \( r \in (0, \min\{1, \text{dist}(x, \partial \Omega)\}) \), and \( Q_r := Q_r(x) \).

Since (3.2) is trivial if \( Q_r \cap \partial A = \emptyset \), then we assume that \( Q_r \cap \partial A \neq \emptyset \) and so \( E := (A \setminus Q_r) \cup \partial Q_r \in \mathcal{A}_m \). By the \((\Lambda, m)\)-minimality of \((A, u)\)

\[
F(A, u; Q_r) \leq F(E, u; Q_r) + \Lambda \vert Q_r \vert.
\]

Hence, by the nonnegativity \( W(A \cap Q_r, u; Q_r) \)

\[
\int_{Q_r \cap \partial A} \varphi(x, \nu_A) d\mathcal{H}^1 \leq 2 \int_{Q_r} \varphi(x, \nu_{Q_r}) d\mathcal{H}^1 + 4\Lambda r^2
\]

and hence (2.13) entails (3.2). In particular, since \( E \Delta A \subset \subset Q_\rho \) for every \( \rho \in (r, \text{dist}(x, \partial \Omega)) \), we also have

\[
F(A, u; Q_\rho) \leq F(E, u; Q_\rho) + \Lambda \vert Q_r \vert = F(E, u; Q_\rho \setminus Q_r) + S(E, u; Q_r) + 4\Lambda r^2
\]

\[
\leq F(E, u; Q_\rho \setminus Q_r) + 2 \int_{Q_r} \varphi(x, \nu_{Q_r}) d\mathcal{H}^1 + 4\Lambda r^2
\]

\[
\leq F(E, u; Q_\rho \setminus Q_r) + 16c_2 r + 4\Lambda r^2
\]

and hence, letting \( \rho \downarrow r \) and using \( r \leq 1 \) we get

\[
F(A, u; Q_r) \leq (16c_2 + 4\Lambda) r.
\]

(3.53)

Now assuming that \( x \) belongs to the closure of the set \( \{y \in \Omega \cap \partial A : \theta_\ast(\partial A, y) > 0\} \), we prove (3.3). For \( \tau_\ast := \tau_0/2 \), let \( \varsigma_\ast = \varsigma(\tau_\ast) \in (0, 1) \) and \( R_\ast = R_0(1, \Lambda, c_1, \tau_\ast) > 0 \) be as in Proposition 3.5. Then by (3.48),

\[
F(A, u; Q_{\tau_\ast r}) \geq 2c_1 \varsigma_\ast k r
\]

(3.54)

for \( k \in (0, 1] \) and for any square \( Q_r \subset \Omega \) with \( r \in (0, R_\ast) \). We consider \( \varsigma_\ast := \varsigma(\tau_\ast) \), \( \vartheta_\ast := \vartheta(\tau_\ast) \), and \( R_\ast := \min\{R(1, \Lambda, c_1, \tau_\ast), R_0\} \) as given by Proposition 3.4 for \( \tau_\ast := \min\{\frac{R_0}{2}, \left(\frac{1+c_1}{16c_2 + 4\Lambda}\right)^{\frac{1}{\gamma}}\} \). By contradiction, if \( \mathcal{H}^1(Q_r \cap \partial A) < \varsigma_\ast r \), then by applying (3.51) with \( k = \tau_\ast \) we obtain

\[
F(A, u; Q_r) \leq (1 + \vartheta_\ast) F(A, u; Q_r).
\]

Then by Proposition 3.4,

\[
F(A, u; Q_{\tau_\ast r}) \leq \tau_\ast^{2-\gamma} F(A, u; Q_r).
\]
so that by (3.54) and (3.53)
\[ t^{1-\gamma} \geq \frac{2c_1\varsigma_o}{16c_2 + 4\Lambda}, \]
which is a contradiction. \(\square\)

4. Compactness and Lower-semicontinuity properties

For the convenience of the reader, we divide the prove into several propositions. We start by showing the compactness of free crystal regions of the sequence of constrained minimizers \(\{(A_m, u_m)\}\).

**Proposition 4.1.** Assume that either \(v \in (0, |\Omega|)\) or \(S = \emptyset\). There exist \(m_h \nearrow +\infty\), \((A_m, u_m) \in C_m\) and \(A \in \tilde{\mathcal{A}}\) such that

(a) for any \(n \in \mathbb{N}\), \((A_{mh}, u_{mh})\) is a minimizer of \(\mathcal{F}\) in \(C_m\) with \(|A| = v\) such that \(\partial A_{mh}\) does not contain isolated points;
(b) \(\text{sdist}(\cdot, \partial A_{mh}) \to \text{sdist}(\cdot, \partial A)\) locally uniformly in \(\mathbb{R}^2\) as \(h \to \infty\);
(c) for any \(x \in \Omega \cap \partial A\) and \(r \in (0, \min\{R_x, \text{dist}(x, \partial \Omega)\})\)
\[ \frac{c_1\varsigma}{8\pi c_2} \leq \frac{\mathcal{H}^1(Q_r(x) \cap \partial A)}{2r} \leq \frac{2\pi c_2}{c_1\varsigma}, \]

where \(\varsigma := \varsigma(c_3, c_4) \in (0, 1)\) and \(R_x := R_x(c_1, c_2, c_3, c_4) > 0\) are given in Theorem 3.1.

**Proof.** By [45, Theorem 2.6] there exists a minimizer \((A_m, u_m) \in C_m\) for every \(m \in \mathbb{N}\). Without loss of generality we assume that \(\partial A_m\) does not contain isolated points. In fact, if \(\partial A_m\) has an isolated point \(x \in A_m^{(0)}\), then \(A_m \setminus \{x\} \in A_m\) and \(\mathcal{F}(A_m, u_m) = \mathcal{F}(A_m \setminus \{x\}, u_m)\). Analogously, if \(\partial A_m\) has an isolated point in \(A_m^{(1)}\), then there exists \(r > 0\) such that \(B_r(x) \cap \partial A_m = \{x\}\) (and \(B_r(x) \subset A_m \cup \{x\} \in C_m\)). In view of Proposition A.3 the function \(u_m\), arbitrarily extended to \(x\) belongs to \(H^1_{\text{loc}}(B_r(x))\), hence, the configuration \((A_m \cup \{x\}, u_m) \in C_m\) and satisfies \(\mathcal{F}(A_m, u_m) = \mathcal{F}(A_m \cup \{x\}, u_m)\).

In view of Remark 2.5 \((A_m, u_m - u_0)\) is a \((\lambda_0, m)\)-minimizer of \(\mathcal{F}(\cdot, \cdot, \Omega)\). Moreover, since \(\partial A_m\) does not contain isolated points \(\theta_0(\partial A_m, x) > 0\) for any \(x \in \partial A_m\), hence by Theorem 3.1 the density estimates (3.2) and (3.3) hold for all \(x \in \Omega \cap \partial A_m\).

By [45, Proposition 3.1], there exist \(A \subset \Omega\) and a subsequence \(\{(A_{mh}, u_{mh})\}\) such that \(\text{sdist}(\cdot, \partial A_{mh}) \to \text{sdist}(\cdot, \partial A)\) as \(h \to \infty\). Consider the sequence \(\mu_h := \mathcal{H}^1\mathcal{L}\partial A_{mh}\) of positive Radon measures. By Theorem 3.1
\[ \frac{\varsigma}{2} \leq \frac{\mu_h(Q_r(x))}{2r} \leq \frac{2\pi c_2}{c_1}, \]
for every \(x \in \Omega \cap \partial A_{mh}\) and \(Q_r(x) \subset \subset \Omega\) with \(r \in (0, R_x)\). By (2.13), (2.14) and (3.1),
\[ \mu_h(\mathbb{R}^2) = \mathcal{H}^1(\partial A_{mh}) \leq \mathcal{H}^1(\partial \Omega) + \frac{\mathcal{F}(A_{mh}, u_{mh}) + 2c_2\mathcal{H}^1(\Sigma)}{c_1} \leq \mathcal{H}^1(\partial \Omega) + \frac{\mathcal{F}(A_1, u_1) + 2c_2\mathcal{H}^1(\Sigma)}{c_1}, \]
hence, by compactness, there exist a not relabelled subsequence and a positive Radon measure \(\mu\) in \(\mathbb{R}^2\) such that \(\mu_h \rightharpoonup^\ast \mu\) as \(h \to \infty\). We claim that \(\overline{\Omega \cap \partial A} \subseteq \text{supp } \mu \subseteq \partial A\).
Indeed, let \( x \in \Omega \cap \partial A \) and \( r \in (0, \min\{\text{dist}(x, \partial \Omega), R_*\}) \). By the \( \text{sdist} \)-convergence, there exists \( x_h \in Q_r(x) \cap \partial A_{A_{m_h}} \) with \( x_h \to x \) such that \( Q_{r/2}(x_h) \subset Q_r(x) \) and hence, by the weak* convergence and (4.2),

\[
\mu(Q_r(x)) \geq \limsup_{h \to \infty} \mu_h(Q_r(x)) \geq \limsup_{h \to \infty} \mu_h(Q_{r/2}(x_h)) \geq \varsigma_* r.
\]

This implies \( x \in \text{supp} \mu \). Conversely, if, by contradiction, there exists \( x \in \text{supp} \mu \setminus \partial A \), then we can find \( r > 0 \) such that \( Q_r(x) \cap \partial A = \emptyset \). From the \( \text{sdist} \)-convergence it follows that \( Q_{r/2}(x) \cap \partial A_{m_h} = \emptyset \) for \( h \) large enough, and hence,

\[
0 < \mu(Q_{r/2}(x)) \leq \liminf_{h \to \infty} \mu_h(Q_{r/2}(x)) = 0,
\]

which is a contradiction.

From (4.2) it follows that

\[
\frac{\varsigma_*}{2} \leq \frac{\mu(Q_r(x))}{2r} \leq \frac{2\pi c_2}{c_1} \tag{4.3}
\]

for any \( x \in \Omega \cap \text{supp} \mu \) any \( r \in (0, R_*) \) with \( Q_r(x) \subset \subset \Omega \). Indeed, let \( x \in \Omega \cap \text{supp} \mu \) and let \( R(x) := \min\{R_*, \text{dist}(x, \partial \Omega)\} \). Then by the weak* convergence \( \mu_h(Q_r(x)) \to \mu(Q_r(x)) = 0 \) for a.e. \( r \in (0, R(x)) \). In particular, (4.3) holds for a.e. \( r \in (0, R(x)) \). Since \( \mu \) is a Radon measure, (4.3) extends to all \( r \in (0, R(x)) \) by the left-continuity of the map \( r \mapsto \mu(Q_r(x)) \).

From (4.3) and [2, Theorem 2.56] it follows that

\[
\varsigma_* \mathcal{H}^1(\Omega \cap \text{supp} \mu) \leq \mu \mathcal{L} \Omega \leq \frac{4\pi c_2}{c_1} \mathcal{H}^1(\Omega \cap \text{supp} \mu). \tag{4.4}
\]

Thus, \( \mu \mathcal{L} \Omega \) is absolutely continuous with respect to \( \mathcal{H}^1(\Omega \cap \text{supp} \mu) \) and \( \mathcal{H}^1(\text{supp} \mu) < \infty \). By (4.4),

\[
\mathcal{H}^1(\partial A) \leq \mathcal{H}^1(\Omega \cap \partial A) + \mathcal{H}^1(\partial \Omega \cap \partial A) \leq \frac{1}{\varsigma_*} \mu(\Omega) + \mathcal{H}^1(\partial \Omega) \leq \infty.
\]

Finally let us prove (4.1). Fix any \( x \in \Omega \cap \partial A \) and let \( R(x) := \min\{R_*, \text{dist}(x, \partial \Omega)\} \). Then by (4.4)

\[
\frac{\varsigma_* \mathcal{H}^1(Q_r(x) \cap \partial A)}{2r} \leq \frac{\mu(Q_r(x))}{2r} \leq \frac{4\pi c_2}{c_1} \frac{\mathcal{H}^1(Q_r(x) \cap \partial A)}{2r}.
\]

This and (4.3) imply

\[
\frac{\varsigma_* \mathcal{H}^1(Q_r(x) \cap \partial A)}{2r} \leq \frac{2\pi c_2}{c_1} \quad \text{and} \quad \frac{\varsigma_*}{2} \leq \frac{4\pi c_2}{c_1} \frac{\mathcal{H}^1(Q_r(x) \cap \partial A)}{2r},
\]

and hence, (4.1) follows. \( \square \)

We notice that by Proposition A.1 the limit set \( A \) in Proposition 4.1 is of finite perimeter. However, a priori, by the arguments of Proposition 4.1, its topological boundary \( \partial A \) does not need to be \( \mathcal{H}^1 \)-rectifiable, and so in \( A \). This issue is overcome by introducing the extended class \( \tilde{A} \) and the auxiliary model \( \tilde{F} \) in Section 2.3.

**Corollary 4.2.** Let \( \{A_{m_h}\} \) and \( A \) be as in Proposition 4.1. Then \( A_{m_h} \to A \) in \( L^1(\mathbb{R}^2) \) as \( h \to \infty \).

**Proof.** Since \( \mathcal{H}^1(\partial A) < \infty \) and \( A_{m_h} \overset{K_*}{\rightharpoonup} A \) as \( h \to \infty \), one has \( \chi_{A_{m_h}}(x) \to \chi_A(x) \) as \( h \to \infty \) for a.e. \( x \in \mathbb{R}^2 \). Now Corollary 4.2 follows from the Dominated Convergence Theorem. \( \square \)

The following result generalizes [40, Theorem 4.2] since it applies to set \( \Gamma \) a priori not connected and even not necessarily \( \mathcal{H}^1 \)-rectifiable), but satisfying uniform density estimates. Recall that we denote by \( \Gamma^r \) and \( \Gamma^u \) the \( \mathcal{H}^1 \)-rectifiable and purely unrectifiable parts of a Borel 1-set \( \Gamma \).
Proposition 4.3. Let $\Gamma \subset \mathbb{R}^2$ be a Borel set such that $H^1(\Gamma) < +\infty$ and for some $r_0, c, C > 0$ and for all $x \in \Gamma$
\[ c \leq \frac{H^1(Q_r(x))}{2r} \leq C, \quad r \in (0, r_0). \quad \tag{4.5} \]
Then for any $R > 0$ and a.e. $x \in \Gamma^r$ one has
\[ \frac{Q_{R, \nu_T(x)}(x) \cap \sigma_{x, \rho}(\Gamma)}{Q_{R, \nu_T(x)}(x) \cap T_x} \xrightarrow{K} \frac{Q_{R, \nu_T(x)}(x) \cap \Gamma}{Q_{R, \nu_T(x)}(x) \cap T_x} \quad \tag{4.6} \]
and
\[ H^1(\sigma_{x, \rho}(\Gamma)) \xrightarrow{\rho \to 0} H^1(\Gamma) \quad \tag{4.7} \]
as $\rho \to 0$, where $\sigma_{x, \rho}$ is the blow-up map defined in (2.4) and $T_x$ is the generalized tangent line to $\Gamma$ at $x$. Moreover, for any $H^1$-measurable $\Gamma' \subset \Gamma$ and $H^1$-a.e. $x \in [\Gamma']^r$ the relations (4.6) and (4.7) hold with $\Gamma'$ in place of $\Gamma$.

Proof. By [33, Theorem 3.3], $\Gamma^r$ (and hence $[\Gamma']^r$) has a approximate tangent line at $\mathcal{H}^1$-a.e. $x$, therefore, (4.7) follows from [2, Remark 2.80]. To prove (4.6) with $\Gamma$ choose $x \in \Gamma$ such that $\sigma_T(x) = 1$ and $T_x$ exists. Without loss of generality we assume that $x = 0$ and $\nu_T(x) = e_2$ is the unit normal to $T_x$. First we prove
\[ \sigma_{0, r}(\Gamma) \xrightarrow{K} T_0 \quad \tag{4.8} \]
as $r \searrow 0$. Indeed, let $\mu_r := H^1(\sigma_{0, r}(\Gamma))$ and $\mu_0 := H^1(\Gamma)$. Given $r > 0$, since $\mu_r(Q_r(x)) = H^1(Q_r(x))$, by (4.5) for all $x \in \sigma_{0, r}(\Gamma)$ and $\rho \in (0, r_0/r)$ one has
\[ c \leq \frac{\mu_r(Q_r(x))}{2\rho} \leq C. \quad \tag{4.9} \]
Let $r_k \searrow 0$ be any sequence. By compactness of sets in the Kuratowski convergence, passing to a further not relabelled subsequence if necessary we suppose that
\[ \sigma_{0, r_k}(\Gamma) \xrightarrow{K} L \quad \tag{4.10} \]
for some closed set $L \subset \mathbb{R}^2$ as $k \to \infty$. We claim that $L = T_0$. If there exists $x \in T_0 \setminus L$, then for some $\rho > 0$, $Q_r(x) \cap L = \emptyset$. By (4.10), $Q_r(x) \cap \sigma_{0, r_k}(\Gamma) = \emptyset$ for all large $k$ so that $\mu_{r_k}(Q_r(x)) = 0$. Then by (4.7)
\[ 0 = \lim_{k \to \infty} \mu_{r_k}(Q_r(x)) \geq \mu_0(Q_{r/2}(x)) \geq \rho, \]
a contradiction. If there exists $x \in L \setminus T_0$, then for some $Q_r(x) \cap T_0 = \emptyset$ for some $\rho > 0$ and there exists a sequence $x_k \in \sigma_{0, r_k}(\Gamma)$ such that $x_k \to x$. Then $Q_{r/2}(x_k) \subset Q_r(x)$ for all large $k$ so that by (4.7) and (4.9),
\[ 0 = \mu_0(Q_{r/2}(x)) \geq \lim_{k \to \infty} \sup_k \mu_{r_k}(Q_{r/2}(x_k)) \geq \lim_{k \to \infty} \sup_k \mu_{r_k}(Q_{r/2}(x_k)) \geq c\rho, \]
a contradiction. Thus, $L = T_0$. Since the sequence $r_k \searrow 0$ is arbitrary, (4.8) follows. Now (4.6) is obvious.

To prove the assertion for $\Gamma'$, fix any $x \in \Gamma'$ such that $\theta(\Gamma', x) = \theta(\Gamma, x) = 1$ and both generalized tangents $T_x^\Gamma$ and $T_x^{\Gamma'}$ of $\Gamma$ and $\Gamma'$ exist. Note that $T_x^\Gamma = T_x^{\Gamma'} =: T_x$. For shortness, assume that $x = 0$ and $\nu_T(x) = e_2$. Since in general $\Gamma'$ does not satisfy the uniform density estimates of type (4.5), we cannot argue as above.

Let $r_k \searrow 0$ be arbitrary sequence such that $\sigma_{0, r_k}(\Gamma') \to L$ for some closed set $L \subset \mathbb{R}^2$. Then for every $x \in L$ there exists a sequence $x_k \in \sigma_{0, r_k}(\Gamma')$ such that $x_k \to x$. Since $\Gamma' \subset \Gamma$ and by (4.8) $\sigma_{0, r_k}(\Gamma) \xrightarrow{K} T_0$, we have $x_k \in \sigma_{0, r_k}(\Gamma)$ and $x_k \to x \in T_0$. Thus, $L \subset T_0$. To prove the converse inclusion, assume that there exists $x \in T_0 \setminus L$. Since $L$ is closed there
exists $r > 0$ such that $B_{2r}(x) \cap L = \emptyset$. As we mentioned in the beginning of the proof, for $
exists_k := \mathcal{H}^1(\sigma(0, r_k) \Gamma'))$ we have $\mu_k \to \mathcal{H}^1(T_0)$. In particular, for every $\rho \in (0, r)$

$$\lim_{k \to +\infty} \mu_k(B_\rho(x)) = \mathcal{H}^1(B_\rho(x) \cap T_0) = 2\rho.$$ 

Hence, $B_\rho(x) \cap \sigma(0, r_k) \Gamma')) \neq \emptyset$ for each such $\rho$ and thus, taking a sequence $\rho_n \to 0$ and using a diagonal argument we obtain a sequence $x_n \in \sigma(0, r_k) \Gamma'))$ converging to $x$. So $x \in L$, a contradiction.

Since $r_k \to 0$ is arbitrary, one has $\sigma(0, r(\Gamma')) \to T_0$ as $r \to 0$. $\square$

Next we turn to the compactness of the sequence of constrained minimizers $\{(A_m, u_m)\}$. 

**Proposition 4.4.** Let $A_{m_{i}}$ and $A$ be as in Proposition 4.1. Let $\{E_i\}_{i \in \mathbb{N}}$ be the family of all connected components of $\text{Int}(A)$. There exist a further (not relabelled) subsequence of $\{(A_{m_{i}}, u_{m_{i}})\}$, a sequence $\{a_i\}$ of rigid displacements, a subset $N \subseteq \mathbb{N}$, a function $v_0 \in H^1(S)$ and a family $\{v_i \in GSBD^2(\text{Int}(E_i)) \cap H^1_\text{loc}(\text{Int}(E_i) \cup S)\}_{i \in N}$ such that

$$|u_{m_{i}} + a_i| \to +\infty$$

a.e. in $\bigcup_{i \in N \setminus N} E_i$,

$$u_{m_{i}} + a_i \to v_0 \chi_S + \sum_{i \in N} v_i \chi_{E_i}$$

weakly in $H^1_\text{loc}(\bigcup_{i \in N} E_i) \cup S)$ (and hence a.e. in $(\bigcup_{i \in N} E_i) \cup S)$,

$$e(u_{m_{i}}) \to e(v_0) \chi_S + \sum_{i \in N} e(v_i) \chi_{E_i}$$

weakly in $L^2_\text{loc}(\bigcup_{i \in N} E_i) \cup S)$.

The main difference of our compactness result from [14, Theorem 1.1] is not only that in our setting we have the set-function coupling, but also we need to select those components of limiting free crystal region where the displacements diverge and those in which they don’t. This first requires to actually prove that the behavior is consistent inside each component of the limiting free-crystal region, which is achieved using [45, Proposition 3.7].

**Proof.** Since $S$ is connected and Lipschitz, by the Korn-Poincaré inequality and the Rellich-Kondrachov Theorem there exists a further not relabelled subsequence $\{u_{m_{i}}\}$, a sequence $\{a_i\}$ of infinitesimal rigid displacements and $v_0 \in H^1(S; \mathbb{R}^2)$ such that $u_{m_{i}} + a_i \to v_0$ weakly in $H^1(S; \mathbb{R}^2)$ and a.e. in $S$.

We define the set $N \subseteq \mathbb{N}$ as follows: For each $i \in \mathbb{N}$ fix some ball $B_i \subseteq E_i$. Since $A_{m_{i}} \to A$, there exists $h_0^i > 0$ such that $B_i \subseteq \subset A_{m_{i}}$ for all $h > h_0^i$. By (2.15) and (3.1)

$$\sup_{h > h_0^i} \int_{B_i} |e(u_{m_{i}} + a_i)|^2 dx \leq \frac{1}{2c3} \sup_{h > h_0^i} \int_{A_{m_{i}} \cup S} C(x)e(u_{m_{i}}) : e(u_{m_{i}}) dx < +\infty,$$

and thus, by [45, Proposition 3.7] either $|u_{m_{i}} + a_i| \to +\infty$ a.e. in $B_i$ or up to a subsequence, $u_{m_{i}} + a_i$ converges a.e. in $B_i$. By a diagonal argument, we choose a further not relabelled subsequence $\{u_{m_{i}}\}$ and the subset $N$ of indices $i \in \mathbb{N}$ such that for every $i \in N$ the sequence $w_h := u_{m_{i}} + a_i \to v_i$ converges a.e. in $B_i$ as $h \to +\infty$.

We claim that for every $i \in N$ there exists $v_i \in H^1_\text{loc}(E_i; \mathbb{R}^2) \cap GSBD^2(E_i; \mathbb{R}^2)$ such that $w_h \to v_i$ weakly in $H^1_\text{loc}(E_i; \mathbb{R}^2)$ and a.e. in $E_i$ as $h \to +\infty$. To prove the claim we fix $i \in N$ and let $D \subseteq E_i$ be an arbitrary connected open set containing $B_i$. Since dist($\cdot, \partial A_{m_{i}}$) →
where we recall that \(E\) is the displacement defining the mismatch strain and a.e. in \(D\), Now letting \(h \to \infty\) and using a diagonal argument we choose a (not relabeled) subsequence \(\{w_h\}\) and \(v_i \in H^1_{loc}(E_i) \cap GSBD^2_{loc}(E_i)\) such that \(w_h \to v_i\) weakly in \(H^1_{loc}(E_i)\) and a.e. in \(E_i\). In particular, \(e(w_h) \to e(v_i)\) weakly in \(L^2_{loc}(E_i)\) and hence, by convexity and (4.11)

\[
\int_{D_j} |e(v_i)|^2 dx \leq \liminf_{h \to +\infty} \int_{D_j} |e(w_h)|^2 dx \leq C.
\]

Hence, letting \(j \to \infty\) we get \(v_i \in GSBD^2(E_i)\).

Let us now show that by the choice of \(N\), for every \(j \in \mathbb{N} \setminus \{N\} \) one has \(|u_{m_h} + a_h| \to +\infty\) a.e. in \(E_j\) as \(h \to +\infty\). Indeed, by definition, if \(i \notin N\), then \(|u_{m_h} + a_h| \to +\infty\) a.e. in \(B_i \subset \subset E_i\). Let \(D \subset \subset E_i\) be any connected open set containing \(B_i\). As in (4.11) we can show \(\|e(u_{m_h} + a_h)\|_{L^2(D)}^2\) is uniformly bounded for all sufficiently large \(h\), and therefore, by [45, Proposition 4.7] \(|u_{m_h} + a_h| \to +\infty\) a.e. in \(D\).

Finally, since \(u_{m_h} + a_h \to u\) weakly in \(H^1_{loc}((\cup_{i \in N} E_i) \cup S)\), it follows that \(e(u_{m_h}) = e(u_{m_h} + a_h) \to e(u)\) weakly in \(L^2_{loc}((\cup_{i \in N} E_i) \cup S)\). \(\square\)

Proposition 4.4 allows us to define a “limit” displacement.

**Proposition 4.5.** Let \(\{(A_m, u_{m_h})\}, \{a_h\}, A, N\) and \(\{v_i\}_{i \in \mathbb{N} \cup \{0\}}\) satisfy the assertion of Proposition 4.4 and let

\[
u := v_0 \chi_S + \sum_{i \in N} v_i \chi_{E_i} + \sum_{j \in \mathbb{N} \setminus N} u_0 \chi_{E_j},
\]

where \(u_0\) is the displacement defining the mismatch strain \(M_0\). Then

\[
\liminf_{h \to \infty} W(A_{m_h}, u_{m_h}) \geq W(A, u).
\]

**Proof.** Fix arbitrary open set \(D \subset \subset \text{Int}(A) \cup S\). By Proposition 4.4 \(u_{m_h} + a_h \to u\) weakly in \(L^2(D \cap [(\cup_{i \in N} E_i) \cup S])\), hence, by the convexity of the elastic energy

\[
\liminf_{h \to \infty} W(A_{m_h}, u_{m_h}) = \liminf_{h \to \infty} \int_{A_{m_h} \cup S} W(x, e(u_{m_h}) - M_0) dx \\
\geq \liminf_{h \to \infty} \left( \int_{D \cap S} W(x, e(u_{m_h}) - M_0) dx + \sum_{j \in N} \int_{D \cap E_j} W(x, e(u_{m_h}) - M_0) dx \right) \\
\geq \int_{D \cap S} W(x, e(u) - M_0) dx + \sum_{i \in \mathbb{N} \setminus N} \int_{D \cap E_i} W(x, e(u) - M_0) dx,
\]

where we recall that \(M_0 = e(u_0)\). Since \(e(u) - M_0 = 0\) a.e. in \(\cup_{j \in \mathbb{N} \setminus N} E_j\), this inequality can also be rewritten as

\[
\liminf_{h \to \infty} W(A_{m_h}, u_{m_h}) \geq \int_{D \cap (A \cup S)} W(x, e(u) - M_0) dx.
\]

Now letting \(D \not\supset \text{Int}(A) \cup S\) and using \(|A \setminus \text{Int}(A)| \leq |\partial A| = 0\) we get (4.12). \(\square\)

Now we establish the following “lower semicontinuity” of \(\mathcal{F}(A_m, u_m)\).
Proposition 4.6. Let $\{(A_{m_k}, u_{m_k})\}$.  \( \text{A and u be as in Proposition 4.5. Then} \) \( (\text{Int}(A), u) \in \hat{\mathcal{C}} \) and

\[
\liminf_{h \to \infty} S(A_{m_k}, u_{m_k}) \geq \bar{S}(\text{Int}(A), u),
\]

where \( \bar{S} \) is defined in (2.12).

We postpone the proof of this proposition after the following auxiliary lemma, needed to treat the delamination and jumps along the cracks.

Lemma 4.7. Recall the definition of the sets \( I_r \) and \( Q^\pm_r \) from (2.3). Let \( \phi \) be any norm in \( \mathbb{R}^2 \). \( \text{Let} \) \( \{D_k\} \) and \( \{m_k\} \) be sequences of subsets of \( Q_4 \) and of natural numbers, respectively, satisfying

(a) the number of connected components \( \partial D_k \) lying strictly inside \( Q_4 \) does not exceed \( m_k \);
(b) \( \text{sdist}(\cdot, \partial D_k) \to \text{dist}(\cdot, I_4) \) uniformly in \( Q_4 \) and

\[
\sup_k \mathcal{H}^1(Q_4 \cap \partial D_k) < +\infty;
\]

(c) there exists a sequence \( \{w_k\} \subset GSBD^2(Q_4) \) such that \( J_{w_k} \subset Q_1 \cap \partial D_k \) and

\[
\sup_k \int_{Q_1} |e(w_k)|^2 dx < +\infty;
\]

(d) there exist \( \xi^\pm \in \mathbb{R}^2 \) such that \( w_k \to w_0 := \xi^+\chi_{Q_1}^- + \xi^-\chi_{Q_1}^+ \cup U_1^\infty \) a.e. in \( Q_1 \setminus U_1^\infty \)

and

\[
|w_k| \to +\infty \quad \text{a.e. in } U_1^\infty,
\]

where \( U_1^\infty \) is either \( \emptyset \) or \( Q_1^+ \).

Then there exists a subsequence \( \{k_h\} \subset \mathbb{N} \) such that for any \( \delta \in (0, 1) \) we can find \( h_\delta > 0 \) for which

\[
\int_{Q_1 \cap \partial D_{k_h}} \phi(\nu_{D_{k_h}}) d\mathcal{H}^1 + 2 \int_{Q_1 \cap D_{k_h}} \phi(\nu_{D_{k_h}}) d\mathcal{H}^1 \geq 2 \int_{I_1} |e_2| d\mathcal{H}^1 - \delta
\]

for all \( h > h_\delta \).

Before the proof of Lemma 4.7 we recall some notations and results from [14]. Given \( \xi \in \mathbb{R}^2 \setminus \{0\} \), let \( \Pi_\xi := \{y \in \mathbb{R}^2 : y \cdot \xi = 0\} \). For every set \( B \subset \mathbb{R}^2 \) and for every \( y \in \Pi_\xi \) we define

\[
B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\}.
\]

Moreover, for every \( u : B \to \mathbb{R}^2 \) we define \( \hat{u}^\xi_y : B^\xi_y \to \mathbb{R} \) by

\[
\hat{u}^\xi_y(t) := u(y + t\xi) \cdot \xi.
\]

When \( u \in GSBD^2(Q_1) \), then \( \hat{u}^\xi_y \in SBV^2_{\text{loc}}(Q_1) \) for \( \mathcal{H}^1 \)-a.e. \( \pi_\xi(Q_1) \) and for all \( \xi \in \mathbb{R}^2 \setminus \{0\} \). In this case we define

\[
I^\xi_y(u) := \int_{|Q_1|_y^|e_2|} dt,
\]

where \( (\hat{u})^\xi_y \) is the density of the absolutely continuity part of \( D\hat{u}^\xi_y \) and also

\[
I_{\hat{u}^\xi_y}^\xi(u) := |D(\tau(u \cdot \xi)^\xi_y)|(|Q_1|_y^\xi),
\]
where \( \tau(t) := \arctan(t) \). Recall that
\[
\int_{\Pi_\xi} I^\pi_y(u)\mathcal{H}^1(y) + \int_{\Pi_\xi} |e(u)|dx + \int_{Q_1} |e(u)|^2dx + \mathcal{H}^1(J_u)
\]
(see e.g. [14, Eq. 3.8 and 3.9]).

**Proof.** The proof is similar to [45, Lemma 4.7]. Since \( \phi \) is even,
\[
\phi(y) = \sup_{\eta \in \mathbb{R}^2} |\xi \cdot \eta|, \quad \xi \in \mathbb{R}^2,
\]
where \( \phi^o \) is the dual norm of \( \phi \). By the compactness of \( B^{\phi^o} := \{ \eta \in \mathbb{R}^2 : \phi^o(\eta) = 1 \} \), for any countable set \( \{ \eta_i \} \) dense in \( B^{\phi^o} \) and for any \( \mathcal{H}^1 \)-rectifiable set \( K \subset \mathbb{R}^2 \)
\[
\phi(\nu_K(x)) = \sup_{i \geq 1} |\nu_K(x) \cdot \eta_i| \quad \text{for } \mathcal{H}^1 \text{-a.e. } x \in K.
\]
Hence, by [27, Lemma 6] for any open set \( U \subset \mathbb{R}^2 \)
\[
\int_{U \cap K} \phi(\nu_K) d\mathcal{H}^1 = \sup_k \left\{ \sum_{i=1}^k \int_{A_i \cap K} |\nu_K \cdot \eta_i| d\mathcal{H}^1 : A_i \subset U \text{ open and pairwise disjoint} \right\}.
\]
Moreover, by the area formula for any Borel set \( B \)
\[
\int_{B \cap K} |\nu_K \cdot \xi| d\mathcal{H}^1 = |\xi| \int_{\pi_\xi(B)} \mathcal{H}^0(K \cap B^\xi_y) d\mathcal{H}^1(y),
\]
where \( \pi_\xi(z) = z - (z \cdot \xi) \frac{\xi}{|\xi|} \) and given \( y \in \pi_\xi(B) \), \( B^\xi_y = \pi_\xi^{-1}(y) \cap B \).

**Step 1:** There exists an at most countable set \( Y \subset B^{\phi^o} \) such that
\[
\lim_{k \to +\infty} \mathcal{H}^1(\pi_\xi(I) \cap \pi_\xi(J_{w_k})) = 0 \quad (4.15)
\]
for any \( \xi \in B^{\phi^o} \setminus Y \). Indeed, let \( Y \) be the set of all \( \xi \in B^{\phi^o} \) for which there exists \( y \in \pi_\xi(I_1) \) such that
\[
\mathcal{H}^1(\pi_\xi^{-1}(y) \cap \partial D_k) > 0.
\]
By assumption (b) and Proposition A.2 the set \( Y \) is at most countable. Let \( \{ w_k \} \) be arbitrary not relabelled subsequence of \( \{ w_k \} \). In view of [14, Eq. 3.23] (applied with \( A = U_1^\infty \)) for any \( \xi \in B^{\phi^o} \setminus Y, \epsilon > 0 \) and for \( \mathcal{H}^1 \text{-a.e. } y \in \pi_\xi(Q_1) \) there exists a further subsequence \( w_{k_l} \) (possibly depending on \( \xi, \epsilon \) and \( y \))
\[
\mathcal{H}^0(J_{[w_0]_y}^\xi \cap [Q_1 \setminus U_1^\infty]_y) + \mathcal{H}^0(\partial [U_1^\infty]_y^\xi) \leq \liminf_{h \to +\infty} \left[ \mathcal{H}^1(J_{[w_{k_l}]_y}^\xi) + \epsilon(I^\pi_y(w_{k_l}) + I^\pi_y w_{k_l}) \right].
\]
(4.16)
By the definition of \( w_0 \) and \( U_1^\infty \), the left-side of (4.16) is equal to 1 for \( \mathcal{H}^1 \text{-a.e. } y \in \pi_\xi(I_1) \). Therefore, for such \( y \) and for sufficiently small \( \epsilon > 0 \) we have \( \liminf_{h \to +\infty} \mathcal{H}^1(J_{[w_{k_l}]_y}^\xi) \geq 1 \). Hence, for \( \mathcal{H}^1 \text{-a.e. } y \in \pi_\xi(I_1) \) the line \( \pi_\xi^{-1}(y) \) intersects \( J_{w_{k_l}} \) for all \( h \) and (4.15) follows.

Note that by [45, Proposition 4.6]
\[
\liminf_{k \to +\infty} \int_{Q_1 \cap J_{w_k}} \phi(\nu_{J_{w_k}}) d\mathcal{H}^1 \geq \int_{I_1} \phi(e_2) d\mathcal{H}^1. \quad (4.17)
\]

**Step 2:** Now we improve (4.17) by including coefficient 2 on the right-hand side of the inequality in the presence of a small error term.

We proceed in three substeps. We redefine the displacement \( w_k \) in the convex envelope \( V_k^1 \) of each connected component \( K_k^1 \) of \( \partial D_k \) in such a way that \( \partial V_k^1 \) become jump sets with the left-hand side of (4.14) lowered up to a small error.

**Substep 2.1:** First we identify \( \{ V_k^1 \} \).
Fix any $\delta \in (0,1)$. By (b) there exists $k_1 > 0$ such that $([-2,2] \times [-2,-\delta]) \cup ([2,2] \times [\delta,2]) \subseteq \text{Int}(D_k)$ for any $k \geq k_1$. Let $F_k := Q_1 \cap D_k$. Note that $\partial F_k \subset (Q_1 \cap \partial D_k) \cup \{(\pm 1) \times [-\delta,\delta]\}$ and since $D_k \in A_{m_k}$, the number of connected components $\{L'_k\}_{j \geq 1}$ of $\partial F_k$ does not exceed $m_k$. Note that $F_k \subset [-1,1] \times [-\delta,\delta]$ and
\begin{align*}
\alpha_k & := \int_{Q_1 \cap \partial^{*} E_k} \phi(\nu_{E_k})d\mathcal{H}^1 + 2 \int_{Q_1 \cap E^{(1)}_k \cap \partial E_k} \phi(\nu_{E_k})d\mathcal{H}^1 \\
& \geq \int_{Q_2 \cap \partial^{*} F_k} \phi(\nu_{F_k})d\mathcal{H}^1 + 2 \int_{Q_2 \cap E^{(1)}_k \cap \partial F_k} \phi(\nu_{F_k})d\mathcal{H}^1 - 4\delta \\
& = \sum_{j \geq 1} \left[ \int_{Q_2 \cap \partial^{*} F_k \cap L'_k} \phi(\nu_{F_k})d\mathcal{H}^1 + 2 \int_{Q_2 \cap E^{(1)}_k \cap \partial F_k \cap L'_k} \phi(\nu_{F_k})d\mathcal{H}^1 \right] - 4\delta := \alpha'_k. \quad (4.18)
\end{align*}
Next repeating the same arguments of Step 1 in the proof of [45, Lemma 4.7] we can find a family $\{V_k\}_i$ of at most countably many pairwise disjoint closed convex sets with non-empty interior such that for each $L'_k$ there exists a unique $V_i$ with $L'_k \subseteq V'_i$ and
\begin{equation}
\alpha'_k \geq \sum_{i \geq 1} \int_{\partial V'_i} \phi(\nu_{V'_i})d\mathcal{H}^1 - 6\delta \quad (4.19)
\end{equation}
see e.g. [45, Eq. 4.34].

Substep 2.2: Now we replace $w_k$ with another function $v_k$ associated to $V'_k$. Fix $\xi_0 \in \mathbb{R}^2$ such that the jump set of the function
\[ v_k := w_k \chi_{Q_1 \cup \cup_i V'_i} + \xi_0 \chi_{\cup_i V'_i} \]
coincide with $\cup_i \partial V'_i$ (up to a $\mathcal{H}^1$-negligible set).

By assumption (b) $\cup_i \partial V'_i \rightarrow I_k$ as $k \rightarrow +\infty$. Moreover, as in Step 1 we can find a countable set $\Upsilon' \subset B_{d_D}$ such that by assumption (b) and (4.15)
\[
\limsup_{k \rightarrow +\infty} \mathcal{H}^1(\pi_\xi(I_k) \setminus \pi_\xi(\cup_i \partial V'_i)) \leq \limsup_{k \rightarrow +\infty} \mathcal{H}^1(\pi_\xi(I_k) \setminus \pi_\xi(\partial D_k)) \\
\leq \lim_{k \rightarrow +\infty} \mathcal{H}^1(\pi_\xi(I_k) \setminus \pi_\xi(J_{w_k})) = 0
\]
for all $\xi \in B_{d_D} \setminus (\Upsilon \cup \Upsilon')$. Moreover, by assumption (d) $v_k \rightarrow w_0$ a.e. in $Q_1 \setminus U_{1}^{\infty}$ and $|v_k| \rightarrow +\infty$ a.e. in $U_{1}^{\infty}$.

Substep 2.3: By convexity of each $V'_i$ we observe that
\[
\liminf_{k \rightarrow +\infty} \mathcal{H}^0(\pi_{\xi}^{-1}(y) \cap J_{v_k}) \geq 2 = 2 \mathcal{H}^0(J_{\pi_{\xi}} \cap [Q_1 \setminus U_{1}^{\infty}]_{y}).
\]
for all $\xi \in B_{d_D} \setminus (\Upsilon \cup \Upsilon')$ and $\mathcal{H}^1$-a.e. $y \in \pi_\xi$. Thus, by repeating the arguments of Step 1 in the proof of [45, Proposition 4.6] we get
\[
\liminf_{k \rightarrow +\infty} \int_{\cup_i \partial V'_i} \phi(\nu_{\cup_i V'_i})d\mathcal{H}^1 = \liminf_{k \rightarrow +\infty} \int_{J_{v_k}} \phi(\nu_{J_{v_k}})d\mathcal{H}^1 \geq 2 \int_{I_k} \phi(e_2) d\mathcal{H}^1,
\]
which together with (4.18) and (4.19) implies the assertion of the lemma. \qed

Now we are ready to prove (4.13).

Proof of Proposition 4.6. For shortness, let $G := \text{Int}(A)$.

We define
\[ \tilde{u}_h := (u_{m_h} + a_h) \chi_{A_{m_h}} + \eta \chi_{\Omega \setminus A_{m_h}} \]

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which holds for every $h$ and

$$\eta \quad \text{for } \{ \} \quad \text{part of } \partial G.$$ 

Let $\mu$ and $\lambda$ be the sequence of positive Radon measures defined at Borel sets $B \subset \mathbb{R}^2$ as

$$\mu_h(B) := \int_{B \cap \partial^* A_{m_h}} \varphi(x, v_{A_{m_h}}) d\mathcal{H}^1 + 2 \int_{B \cap \partial(\{A_{m^1} \cup A_{m^0}\}) \cap \partial A_{m_h}} \varphi(x, v_{A_{m_h}}) d\mathcal{H}^1$$

$$+ \int_{B \cap \partial^* A_{m^0_h} \cap \partial A_{m_h}} \varphi(x, v_{\Sigma}) \varphi(x, v_{G}) + \varphi(x, v_{\Sigma}) + \varphi(x, v_{\Sigma}) \varphi(x, v_{\Sigma}) d\mathcal{H}^1$$

$$+ \int_{B \cap \partial^* A_{m^0_h} \cap J_{m_h}} \varphi(x, v_{\Sigma}) \varphi(x, v_{\Sigma}) + \varphi(x, v_{\Sigma}) d\mathcal{H}^1$$

$$+ \int_{B \cap \partial^* A_{m^0_h} \cap J_{m_h}} \varphi(x, v_{\Sigma}) d\mathcal{H}^1$$

and let $\mu$ be the positive measure defined at Borel sets $B \subset \mathbb{R}^2$ as

$$\mu(B) := \int_{B \cap G \cap \partial^* G} \varphi(x, v_{\Sigma}) d\mathcal{H}^1 + 2 \int_{B \cap G \cap \partial^* G \cap \partial G \cap J_{\tilde{u}}} \varphi(x, v_{\Sigma}) d\mathcal{H}^1$$

$$+ \int_{B \cap G \cap \partial G \cap \partial G \cap J_{\tilde{u}}} g(x, 0) d\mathcal{H}^1$$

Since $S^A_{\tilde{u}} := G \cap \partial G \cap J_{\tilde{u}}$ and $\Sigma \cap J_{\tilde{u}} = \Sigma \cap J_u$, we have

$$\mu_h(\mathbb{R}^2) = S(A_{m_h}, u_{m_h}) + \int_{\Sigma} \varphi(x, v_{\Sigma}) d\mathcal{H}^1$$

and

$$\mu(\mathbb{R}^2) = \tilde{S}(G, u) + \int_{\Sigma} \varphi(x, v_{\Sigma}) d\mathcal{H}^1.$$
Hence, to establish (4.13) it suffices to prove
\[
\liminf_{h \to \infty} \mu_h(\mathbb{R}^2) \geq \mu(\mathbb{R}^2). \tag{4.24}
\]
Since \(\sup_h \mu_h(\mathbb{R}^2) < +\infty\), by compactness, there exists a positive Radon measure \(\mu_0\) in \(\mathbb{R}^2\) such that (up to a subsequence) \(\mu_h \rightharpoonup \mu_0\) as \(h \to \infty\). We show
\[
\mu_0 \geq \mu \tag{4.25}
\]
and we observe that (4.24) immediately follows from (4.25). To establish (4.25) it suffices to prove
\[
\frac{d\mu_0}{d\mathcal{H}^1(\Omega \cap \partial^* G)}(x) \geq \varphi(x, \nu_G(x)) \quad \text{for a.e. } x \in \Omega \cap \partial^* G, \tag{4.26a}
\]
\[
\frac{d\mu_0}{d\mathcal{H}^1(\Sigma \cap \partial^* G)}(x) \geq g(x, 1) \quad \text{for a.e. } x \in \Sigma \cap \partial^* G, \tag{4.26b}
\]
\[
\frac{d\mu_0}{d\mathcal{H}^1(\Sigma \setminus \partial G)}(x) \geq \varphi(x, \nu_\Sigma(x)) \quad \text{for a.e. } x \in \Sigma \setminus \partial G, \tag{4.26c}
\]
\[
\frac{d\mu_0}{d\mathcal{H}^1(S^1_{\mathcal{U}})}(x) \geq 2\varphi(x, \nu_G(x)) \quad \text{for a.e. } x \in S^1_{\mathcal{U}}, \tag{4.26d}
\]
\[
\frac{d\mu_0}{d\mathcal{H}^1(\Sigma \setminus J^r_{\mathcal{U}})}(x) \geq 2\varphi(x, \nu_\Sigma(x)) \quad \text{for a.e. } x \in \Sigma \setminus J^r_{\mathcal{U}} \tag{4.26e}
\]
since \(g(x, 0) = \varphi(x, \nu_\Sigma)\).

The proof of the estimates (4.26a)-(4.26e) follows from similar arguments used in [45, Proposition 4.1] with special care needed for (4.26d). In fact for (4.26d) we cannot employ the strategy used for [45, Eq. 4.40c] that was hinged on the uniform bound on the number of boundary components, which here we do not have. We instead adapt the arguments employed in [45, Eq. 4.40g] by using Lemma 4.7.

Next we detail the proofs of (4.26a)-(4.26e).

**Proof of (4.26a).** Note that \(A = G\) up to a negligible set. By Corollary 4.2 \(A_{m_h} \to A\) in \(L^1(\mathbb{R}^2)\), thus, the proof of (4.26a) can be done following the arguments of [45, Eq. 4.40a] using Reshetnyak lower semicontinuity Theorem [2, Theorem 2.38].

**Proof of (4.26b).** Since \(A_{m_h} \to G\) in \(L^1(\mathbb{R}^2)\), we have \(D\chi_{A_{m_h}} \rightharpoonup^* D\chi_G\). Thus, the proof of (4.26b) directly follows from [1, Lemma 3.8] (see also the proof of [45, Eq. 4.40d]).

**Proof of (4.26c).** Let \(x_0 \in \Sigma \setminus \partial G\) and let \(r_0 := \text{dist}(x_0, \partial G) > 0\). Since \(\mathbb{R}^2 \setminus \text{Int}(A_{m_h}) \to \mathbb{R}^2 \setminus \text{Int}(A) = \mathbb{R}^2 \setminus G\), there exists \(r_1 \in (0, r_0)\) such that \(B_r(x_0) \cap \text{Int}(A_{m_h})\) for any \(r \in (0, r_1)\). Therefore, for any \(r \in (0, r_1)\)
\[
\mu_h(B_r(x_0)) = \int_{B_r(x_0) \cap \Sigma \cup A_{m_h} \cup \partial A_{m_h}} [\varphi(x, \nu_\Sigma) + g(x, 1)] d\mathcal{H}^1(x) + \int_{B_r(x_0) \cap \Sigma \cup \partial A_{m_h}} g(x, 0) d\mathcal{H}^1
\]
\[
\geq \int_{B_r(x_0) \cap \Sigma \cup A_{m_h} \cup \partial A_{m_h}} g(x, 0) d\mathcal{H}^1(x) + \int_{B_r(x_0) \cap \Sigma \cup \partial A_{m_h}} g(x, 0) d\mathcal{H}^1
\]
\[
= \int_{B_r(x_0) \cap \Sigma} g(x, 0) d\mathcal{H}^1,
\]
where in the inequality we used (4.23). Thus, taking \(\limsup\) as \(h \to +\infty\) and using \(\mu_h \rightharpoonup^* \mu_0\) we get
\[
\mu_0(B_r(x_0)) \geq \int_{B_r(x_0) \cap \Sigma} g(x, 0) d\mathcal{H}^1.
\]
Now (4.26c) follows from the Besicovitch Differentiation Theorem.
Proofs of (4.26d) and (4.26e). We establish
\[
\frac{d\mu_0}{d\mathcal{H}^1|_K} \geq 2\phi(x, \nu_K) \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in K,
\] (4.27)
where
\[
K = S_\alpha^1 \cup (\Sigma \cap J_\alpha).
\]

Let \(x \in K\) be such that \(\theta(K, x) = 1\). Then either \(x \in S_\alpha \subset G^{(1)} \cap \partial^r G\) or \(x \in \Sigma \cap J_\alpha\). By setting \(E_0 := S\) and recalling that \(\text{Int}(A) = \bigcup_{i \in \mathbb{N}} E_i\), in view of Proposition 4.4 we have one of the following:

(b1) there exists \(i_0 \in N\) such that \(x \in E_{i_0}^{(1)} \cap \partial^r E_{i_0}, \theta(E_{i_0}^{(1)} \cap \partial^r E_{i_0}, x) = 1\) and \(m_{i_0} \to u\) a.e. in \(E_{i_0}\);

(b2) there exist \(i_0 \in N \setminus \{0\}\) and \(j_0 \in \mathbb{N} \setminus N\) such that \(x \in \partial^s E_{i_0} \cap \partial^s E_{j_0}\) and \(m_{i_0} + a_{m_{i_0}} \to u\) a.e. in \(E_{i_0}\) and \(|m_{i_0} + a_{m_{i_0}}| \to \infty\) a.e. in \(E_{j_0}\);

(b3) there exist \(i_1, i_2 \in N \setminus \{0\}\) with \(i_1 \neq i_2\) such that \(x \in \partial^s E_{i_1} \cap \partial^s E_{i_2}\) and \(m_{i_0} + a_{m_{i_0}} \to u\) a.e. in \(E_{i_1} \cup E_{i_2}\).

Let \(L\) denote the set among \(E_{i_0}^{(1)} \cap \partial^r E_{i_0}, \partial^s E_{i_0} \cap \partial^s E_{j_0}\) and \(\partial^s E_{i_1} \cap \partial^s E_{i_2}\) containing \(x\). Without loss of generality we assume that \(x \in Y \subset L\), where \(Y\) is defined as the set of points \(y \in L \subset \partial A\) satisfying:

(c1) \(\theta(\partial G, y) = \theta(\partial A, y) = \theta(L, y) = 1\) and \(\nu_G(y) = \nu_A(y) = \nu_L(y)\) exists. If \(y \in \Sigma\), then additionally, \(\theta(S, y) = 1\) and \(\nu_S\) also exists;

(c2) as \(\rho \to 0\) the sets \(Q_{\rho,2L}(y) \cap \sigma_{p,y}(\partial A), Q_{\rho,2L}(x) \cap \sigma_{p,x}(\partial G)\) and \(\overline{Q}_{\rho,2L}(y) \cap \sigma_{p,y}(\partial G)\) converge \(Q_{\rho,2L}(y) \cap T_y\) in the Kuratowski sense, where \(R > 0\) and \(T_y\) is the generalized tangent line to \(\partial A\) at \(y\);

(c3) one-sided traces \(\tilde{u}^+(y)\) and \(\tilde{u}^-(y)\) of \(u\) w.r.t. \(L\) exist and are not equal;

(c4) \(\frac{d\mu_0}{d\mathcal{H}^1|_K}(y)\) exists and is finite.

In fact, \(\mathcal{H}^1(L \setminus Y) = 0\) since for (c1) we notice that \(Y \subset L \subset \partial^s A\) and \(\partial^s A\) is \(\mathcal{H}^1\)-rectifiable, for (c2) we use Proposition 4.3 by observing that the points of \(\Sigma\) and \(\Omega \cap \partial A\) satisfy uniform density estimates in view of the Lipschitzianity of \(\Sigma\) and Proposition 4.1, respectively, for (c3) we use [21, Definition 2.4] and the existence of traces of \(GBD\)-functions along \(C^1\)-manifolds [21, Theorem 6.2] and the fact that being a jump set of \(\tilde{u}\), the set \(K\) (and also \(L\)) can be covered by at most countably many one-dimensional \(C^1\)-graphs (up to a \(\mathcal{H}^1\)-negligible set), and finally for (c4) we use Besicovitch Differentiation Theorem.

Without loss of generality, we assume \(x = 0\), \(\nu_K(x) = e_2\), \(T_x\) is the \(x_1\)-axis and \(e_2\) is the outer normal of \(E_0\).

Let \(4r_0 := \text{dist}(0, \partial \Omega)\) if \(0 \in \Omega\) and \(4r_0 := \text{dist}(0, \partial \Sigma)\) if \(0 \in \Sigma\); since \(\Sigma\) is Lipschitz, it consists of at most countably many open connected components in \(\partial \Omega\), and hence, \(r_0 > 0\). By weak convergence,
\[
\lim_{h \to \infty} \mu_h(Q_r) = \mu_0(Q_r)
\] (4.28)
for a.e. \(r \in (0, r_0)\). By assumption (b3), [21, Definition 2.4] and [21, Remark 2.2] separately applied to \(Q_1^+ := Q_1 \cap \{x_2 > 0\}\) and \(Q_1 \setminus Q_1^+\) we have
\[
\lim_{r \to 0} \int_{Q_1} |\tau(\tilde{u}(rx)) - \tau(u_0(x))|dx = 0,
\] (4.29)
where
\[
u_0 := \tilde{u}^+(0)\chi_{Q_1^+} + \tilde{u}^-(0)\chi_{Q_1 \setminus Q_1^+}
\]
and
\[
\tau(z) = (\arctan z_1, \arctan z_2), \quad z = (z_1, z_2) \in \mathbb{R}^2.
\]
For every \( r \in (0, r_0) \) let
\[
U^\infty_r := \{ x \in Q_1 : \lim_{h \to \infty} |\tilde{u}_h(rx)| = +\infty \}.
\]

Unlike the proof of [45, Eq. 4.40g], (4.22) implies that \( U^\infty_r \) can have positive measure. By (4.21) and the Dominated Convergence Theorem
\[
\lim_{h \to \infty} \int_{Q_1 \setminus U^\infty_r} |\tau(\tilde{u}_h(rx)) - \tau(\tilde{u}(rx))|dx = 0. 
\]

By (c2) applied with \( R = 8, \) Proposition 4.3 and (c1)-(c3)
\[
Q_8 \cap \sigma_r(\partial A) \xrightarrow{K} I_8 \quad \text{and} \quad \mathcal{H}^1(Q_8 \cap \sigma_r(\partial A)) \xrightarrow{\ast} \mathcal{H}^1 L I_8,
\]
and
\[
Q_8 \cap \sigma_r(L) \xrightarrow{K} I_8 \quad \text{and} \quad \mathcal{H}^1(L(Q_8 \cap \sigma_r(L))) \xrightarrow{\ast} \mathcal{H}^1 L I_8
\]
as \( r \to 0. \) Hence, by [45, Proposition A.5]
\[
\text{sdist}(. \cap \sigma_r(\partial A) \to -\text{dist}(. \cap T_0), (4.31a)
\]
\[
\text{sdist}(. \cap \sigma_r(\partial E_{i0}) \to -\text{dist}(. \cap T_0), (4.31b)
\]
\[
\text{sdist}(. \cap \sigma_r(\partial E_{j0} \cup E_{i1}) \to -\text{dist}(. \cap T_0), (4.31c)
\]
\[
\text{sdist}(. \cap \sigma_r(\partial E_{i1} \cup E_{i2}) \to -\text{dist}(. \cap T_0) (4.31d)
\]
locally uniformly in \( Q^1_4 \) as \( r \to 0. \)

By the definitions of \( E_{i0}, E_{j0}, E_{i1} \) and \( E_{i2} \) and (4.31b)-(4.31d)
\[
\lim_{r \to 0} |U^\infty_r \Delta U^\infty_0| = 0. \quad (4.32)
\]

**Step 1:** We choose sequences \( h_k \uparrow \infty \) and \( r_k \searrow 0 \) as follows. By (4.28), (4.29), (4.31a) and (4.32) for any \( k \in \mathbb{N} \) there exists \( r_k \in (0, 1/k) \) such that (4.28) holds with \( r = r_k \) and
\[
|\text{sdist}(. \cap \sigma_r(\partial A)) + \text{dist}(. \cap T_0)|_{L^\infty(Q_4)} \leq \frac{1}{k^2}, \quad (4.33a)
\]
\[
\int_{Q_1} |\tau(\tilde{u}(r_kx)) - \tau(u_0(x))|dx < \frac{1}{k^2}, \quad (4.33b)
\]
\[
|U^\infty_{r_k} \Delta U^\infty_0| < \frac{1}{k^2}. \quad (4.33c)
\]

Given \( k \geq 1 \) and \( r_k \), since \( A_{m_h} \) \( \text{sdist} \)-converges to \( A \) and the function \( \tau \) is bounded, by (4.30), (4.33c) and (4.28) we can choose \( h_k \) such that
\[
\frac{1}{h_k r_k} < \frac{1}{k}, \quad (4.34a)
\]
\[
|\text{sdist}(. \cap \sigma_{r_k}(\partial A_{m_h})) - \text{sdist}(. \cap \sigma_{r_k}(\partial A))|_{L^\infty(Q_4)} \leq \frac{1}{k}, \quad (4.34b)
\]
\[
\int_{Q_1 \setminus U^\infty_0} |\tau(\tilde{u}_{h_k}(r_kx)) - \tau(\tilde{u}(r_kx))|dx < \frac{1}{k}, \quad (4.34c)
\]
\[
\mu_{h_k}(Q_{r_k}) \leq \mu_0(Q_{r_k}) + r_k^2 \quad (4.34d)
\]

Notice that by (4.34a), \( h_k \to \infty \) as \( k \to \infty \).

Let
\[
D_k := \sigma_{r_k}(A_{m_h} \cup S)
\]
and
\[
w_k(x) := \tilde{u}_{h_k}(r_kx), \quad x \in Q_1.
\]
Then the number of connected components of $\partial D_k$ lying strictly inside $Q_4$ does not exceed $m_{h_k}$ and $w_k \in GSBD^2(Q_1)$ with $J_{w_k} \subset Q_1 \cap \partial D_k$. By (4.34b) and (4.33a),

$$\text{dist}(\cdot, \partial D_k) \to -\text{dist}(\cdot, T_0)$$ uniformly in $Q_4$ as $k \to \infty$.

Moreover, by (4.33b) and (4.34c) $w_k \to u_0$ a.e. in $Q_1 \setminus U_1^\infty$ and $|w_k| \to +\infty$ a.e. in $U_1^\infty$. By the finiteness of

$$\frac{d\mu_0}{\mathcal{H}^1 L}(0) = \lim_{k \to \infty} \frac{\mu_0(Q_{r_k})}{2r_k}$$

and (4.34d)

$$\frac{\mathcal{H}^1(Q_1 \cap \partial D_k)}{r_k} \leq \frac{\mu_{h_k}(Q_{r_k})}{c_1 r_k} \leq C := \frac{2}{c_1} \frac{d\mu_0}{\mathcal{H}^1 L}(0) + 1$$ (4.35)

for all large $k$. Moreover, by changing variables as $r = x r_k y$ and using (4.20) we get

$$\int_{Q_1} |e(w_k)|^2 dx = \int_{Q_{r_k}} |e(\tilde{w}_k)|^2 dy \leq M$$

for all $k$; note that the first equality holds only in dimension two.

Fix $\delta \in (0, 1)$. Since $\varphi$ is uniformly continuous, there exists $k^0_0 > 0$ such that

$$|\varphi(x, \nu) - \varphi(0, \nu)| < \delta, \quad x \in Q_{r_k}, \ \nu \in S^1.$$

Therefore, by the definitions of $D_k$ and $\mu_h$, the nonnegativity of $g$ as well as (4.35)

$$\frac{\mu_{h_k}(Q_{r_k})}{r_k} \geq \int_{Q_1 \cap \partial^* D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 + 2 \int_{Q_1 \cap D_k^{(1)} \cap \partial D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 - 2Cc_2 \delta,$$ (4.36)

where

$$\phi(\nu) = \varphi(0, \nu).$$

By Lemma 4.7 applied with sequences $\{D_k\}$ and $\{m_{h_k}\}$ we find $k^2_0 > k^1_0$ such that

$$\int_{Q_1 \cap \partial^* D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 + 2 \int_{Q_1 \cap D_k^{(1)} \cap \partial D_k} \phi(\nu_{D_k}) d\mathcal{H}^1 \geq 2 \int_{I_1} \phi(e_2) d\mathcal{H}^1 - \delta.$$

Thus, by (4.36) and (4.34d) we get

$$\frac{\mu_0(Q_{r_k})}{2r_k} + \frac{r_k}{2} \geq \int_{I_1} \phi(e_2) d\mathcal{H}^1 - \frac{2Cc_1 + 1}{2} \delta$$

for all $k > k^3_0$. Now letting first $k \to +\infty$ and then $\delta \to 0$ we get (4.27). \hfill \Box

5. Proof of the main results

The aim of this section is to prove theorems of Section 2.4. We start by showing that the volume-constraint infima of $\mathcal{F}$ in $\mathcal{C}$ and of $\tilde{\mathcal{F}}$ in $\tilde{\mathcal{C}}$ in fact coincide.

**Proposition 5.1.** Assume hypotheses (H1)-(H3) and let $v \in (0, |\Omega|)$ or $\Theta = \emptyset$. Then

$$\inf_{(A, u) \in \mathcal{C}, |A|=v} \mathcal{F}(A, u) = \inf_{(A, u) \in \mathcal{C}, |A|=v} \tilde{\mathcal{F}}(A, u) = \inf_{(A, u) \in \tilde{\mathcal{C}}} \tilde{\mathcal{F}}^\lambda(A, u)$$ (5.1)

for any $\lambda \geq \lambda_0$, where $\lambda_0$ is given by [45, Theorem 2.6] and $\tilde{\mathcal{F}}^\lambda$ is given by (5.17).

**Proof.** We repeat similar arguments to [45, Section 5]. Note that for any $\lambda > 0$

$$\inf_{(A, u) \in \mathcal{C}, |A|=v} \mathcal{F}(A, u) \geq \inf_{(A, u) \in \mathcal{C}, |A|=v} \tilde{\mathcal{F}}(A, u) \geq \inf_{(A, u) \in \tilde{\mathcal{C}}} \tilde{\mathcal{F}}^\lambda(A, u).$$ (5.2)

Further we fix any $\lambda \geq \lambda_0$. Recall that from [45] for such $\lambda$

$$\inf_{(A, u) \in \mathcal{C}, |A|=v} \mathcal{F}(A, u) = \lim_{m \to +\infty} \min_{(A, u) \in \mathcal{C}_m, |A|=v} \mathcal{F}(A, u) = \lim_{m \to +\infty} \min_{(A, u) \in \mathcal{C}_m} \mathcal{F}^\lambda(A, u),$$
where $\mathcal{F}^\lambda$ is given by (2.17). Thus, in view of (5.2) to prove (5.1) it is enough to establish that for $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ and $(A_\epsilon, u_\epsilon) \in \mathcal{C}_{n_\epsilon}$ such that
\[
\inf_{(A, u) \in \mathcal{C}} \bar{\mathcal{F}}^\lambda(A, u) + \epsilon > \mathcal{F}^\lambda(A_\epsilon, u_\epsilon).
\] (5.3)

To prove the existence of $n_\epsilon$ and $(A_\epsilon, u_\epsilon) \in \mathcal{C}_{n_\epsilon}$, we repeat essentially the same arguments of the proof of [45, Eq. 5.4]. For the convenience of the reader we give the detailed proof. Given $\epsilon > 0$ let $(B_1, v_1) \in \mathcal{C}$ be such that
\[
\inf_{(A, u) \in \mathcal{C}} \mathcal{F}^\lambda(A, u) > \mathcal{F}^\lambda(B_1, v_1) - \epsilon.
\] (5.4)

Since $|B_1| = |\text{Int}(B_1)|$ and $\mathcal{F}^\lambda(B_1, v_1) \geq \mathcal{F}^\lambda(\text{Int}(B_1), v_1)$, we may assume that $B_1 = \text{Int}(B_1)$, i.e., $B_1$ is open.

**Step 1:** First we remove the jump set $J_{v_1}$ of $v_1$ on $\Sigma$ making a hole in $\Omega$. Recall that by our choice, $\nu_{\Sigma}$ is always directed towards $\Omega$. Since $\Sigma$ is Lipschitz, by the regularity of $\mathcal{H}^1(\Sigma)$, there exists a relatively open set $\Sigma' \subset \Sigma$ such that $\mathcal{H}^1(J_{v_1} \setminus \Sigma') = 0$ and $\mathcal{H}^1(\Sigma' \setminus J_{v_1}) < \frac{\epsilon}{2}$. Let $r \in (0, \frac{\epsilon}{2H^1(\Sigma')})$ be such that whenever $|x - y| < 4r$. Since $\Sigma$ is Lipschitz, by Vitaly Covering Lemma we can find an at most countable family $\{Q_{r, \nu_{\Sigma}(x_j)}(x_j)\}_{j \geq 1}$ of disjoint open squares such that $x_j \in \Sigma$, $r_j \in (0, r)$, $\Sigma \cap Q_{r, \nu_{\Sigma}(x_j)}(x_j)$ is a graph in $\nu_{\Sigma}(x_j)$-direction, $\Sigma$ crosses two opposite sides of each $Q_{r, \nu_{\Sigma}(x_j)}(x_j)$ parallel to $\nu_{\Sigma}(x_j)$ and
\[
\mathcal{H}^1(\Sigma' \setminus \bigcup_j Q_{r, \nu_{\Sigma}(x_j)}(x_j)) = 0.
\] (5.6)

Note that $\sum_j r_j < \mathcal{H}^1(\Sigma)$. For each $j$ define
\[
\Sigma_j := (\Sigma \cap Q_{r, \nu_{\Sigma}(x_j)}(x_j)) + \rho_j \nu_{\Sigma}(x_j),
\]
where $\rho_j \in (0, r_j)$ is such that $\Sigma_j$ still connects two vertical sides of $Q_{r, \nu_{\Sigma}(x_j)}(x_j)$ and $\sum_j \rho_j < \frac{\epsilon}{2H^1(\Sigma')}$. Let $U_j$ be the open set whose boundaries are $\Sigma_j$, $\Sigma \cap Q_{r, \nu_{\Sigma}(x_j)}(x_j)$ and two vertical sides of $Q_{r, \nu_{\Sigma}(x_j)}(x_j)$. Note that $\{U_j\}_j$ is a countable family of pairwise disjoint open sets.

Let $B_2 := B_1 \setminus \bigcup_j U_j$ and $v_2 := v_1\big|_{B_2 \setminus S}$. Then using the localized version of $\mathcal{S}$ we get
\[
\mathcal{S}(B_2, v_2) \leq \mathcal{S}(B_1, v_1 - u_0; \Omega \setminus \bigcup U_j) + \sum_j \left( \int_{\Sigma_j} \varphi(x, \nu_{\Sigma}(x)) d\mathcal{H}^1 + 2c_2 \rho_j \right).
\] (5.7)

By (5.5) and the definition of $\Sigma_j$
\[
\int_{\Sigma_j} \varphi(x, \nu_{\Sigma}(x)) d\mathcal{H}^1 \leq \int_{\Sigma \setminus Q_{r, \nu_{\Sigma}(x_j)}(x_j)} \varphi(y, \nu_{\Sigma}(y)) d\mathcal{H}^1 + \frac{\epsilon \mathcal{H}^1(\Sigma \setminus Q_{r, \nu_{\Sigma}(x_j)}(x_j))}{\mathcal{H}^1(\Sigma)}.\]

Thus summing this inequality in $j$ and using pairwise disjointness of $Q_{r, \nu_{\Sigma}(x_j)}(x_j)$ and (5.6) we get
\[
\sum_j \int_{\Sigma_j} \varphi(x, \nu_{\Sigma}(x)) d\mathcal{H}^1 \leq \int_{\Sigma'} \varphi(y, \nu_{\Sigma}(y)) d\mathcal{H}^1 + \frac{\epsilon \mathcal{H}^1(\Sigma')}{\mathcal{H}^1(\Sigma)}.
\]
Using the definition of $\Sigma'$ we obtain
\[ \sum_j \int_{\Sigma_j} \varphi(x, \nu_{\Sigma}(x))dH^1 \leq \int_{\partial E_1} \varphi(y, \nu_{\Sigma}(y))dH^1 + 2\epsilon. \]
Inserting this in (5.7) and using the inequality $\sum_j \rho_j < \frac{\epsilon}{2\epsilon^2}$ we get
\[ S(B_2, v_2) \leq S(B_1, v_1 - u_0; \Omega \setminus \bigcup_j U_j) + \int_{\partial E_1} \varphi(y, \nu_{\Sigma}(y))dH^1 + 3\epsilon \leq S(B_1, v_1) + 3\epsilon. \]
Then by the nonnegativity of the elastic energy, for $(B_2, v_2)$ we get
\[ \tilde{F}(B_2, v_2) \leq \tilde{F}(B_1, v_1) + 3\epsilon. \]
Notice that by our construction $\Sigma \cap J_{v_2}$ is $H^1$-negligible, hence by Proposition A.3 $v_2 \in H^1_{\text{loc}}(\text{Int}(B_2 \cup S \cup \Sigma))$.
Finally we estimate the volume contribution of $B_2$. Since $U_j \subset Q_{\rho, \nu}(x_j)$ and $r_j \leq r < \frac{\epsilon}{\lambda H^1(\Sigma)}$, using $\sum_j r_j < H^1(\Sigma)$ we get
\[ |B_1 \setminus B_2| \leq \sum_j |U_j| \leq \sum_j r_j^2 \leq r \sum_j r_j < \frac{\epsilon}{\lambda}. \]
Therefore,
\[ \tilde{F}^\lambda(B_1, v_1) \geq \tilde{F}^\lambda(B_2, v_2) - 4\epsilon. \quad (5.8) \]

Step 2: Let $\{E_i\}_{i \geq 1}$ be all open connected components of $B_2$ (recall that $B_2$ is open).
We remove all sufficiently small connected components of $B_1$. Using the localized versions of $S$ and $\mathcal{W}$ we have
\[ \mathcal{W}(B_2, v_2 - u_0; \Omega) = \sum_{i \geq 1} \mathcal{W}(E_i, v_2 - u_0; \Omega). \]
Since $\partial E_i \cap \partial E_j \subset B_{2}^{(1)} \cap \partial B_2$ and $\varphi(x, \cdot)$ is even,
\[ S(B_2, v_2; \Omega) = \sum_{i \geq 1} S(E_i, v_2; \Omega). \]
Hence, there exists $N_1 \in \mathbb{N}$ such that the set $B_3 := \bigcup_{i = 1}^{N_1} E_i$ satisfies
\[ S(B_2, v_2; \Omega) + \mathcal{W}(B_2, v_2 - u_0; \Omega) + \epsilon > S(B_3, v_2; \Omega) + \mathcal{W}(B_3, v_2 - u_0; \Omega), \]
\[ 0 \leq |B_2| - |B_3| < \frac{\epsilon}{\lambda}. \]
Thus,
\[ F^\lambda(B_2, v_2) > F^\lambda(B_3, v_3) - 2\epsilon, \quad (5.9) \]
where $v_3 := v_2 \big|_{B_3}$. 

Step 3: Let $\{F_i\}_{i \geq 1}$ be all connected components of $\Omega \setminus \overline{B_3}$ such that $\partial F_i \subset \partial B_3$ (hence, $F_i$ are holes in $B_3$). We fill in all sufficiently small holes. Since $S(B_3, v) < +\infty$, there exists $N_2 \geq 1$ such that
\[ \sum_{i > N_2} S(F_i, v_3; \Omega) + \sum_{i > N_2} \mathcal{W}(F_i, v_3 - u_0; \Omega) < \epsilon, \sum_{i > N_2} |F_i| < \frac{\epsilon}{\lambda}. \]
Then the set $B_4 := B_3 \cup (\bigcup_{i > N_2} F_i)$ and the function $v_4 := v_3 \chi_{B_2 \cup S} + u_0 \chi_{\cup_{i > N_2} F_i}$ satisfies
\[ F^\lambda(B_3, v_3) > F^\lambda(B_4, v_4) - 2\epsilon. \quad (5.10) \]
By construction, $\partial^* \overline{B_4}$ has at most $N_1 + N_2$ connected components.

Step 5: Finally we construct $(A_e, u_e) \in C_n$ satisfying (5.3) for some $n_e \in \mathbb{N}$. Let $B_5 := \text{Int}(\overline{B_3})$. Since $B_4$ can have finitely many “substantial” holes $B_5 \cap \partial B_4 = \emptyset$. In
particular, if we extend \( v_4 \) arbitrarily to the set \( B_4^{(1)} \cap \partial B_4 \) and denote the extension by \( v_5 \), then \( v_5 \in GSBD^2(\text{Int}(B_5 \cup S \cup \Sigma)) \) and \( J_{v_5} = S_{v_5}^{B_4} \) up to a \( H^1 \)-negligible set, where \( S_u^A \) is defined in \((2.6)\). Since \( \nu_5 = v_4 \) a.e. in \( B_5 \), by \((5.4)-(5.10)\)
\[
\int_{B_5 \cup S} C(x)e(v_5) : e(v_5) = W(B_4, v_4) \leq \bar{F}(B_4, v_4) + c_2 \mathcal{H}^1(\Sigma) \leq C + 9\epsilon,
\]
where \( C := \max\{1, \inf \frac{\epsilon}{\bar{F}}\} \) is independent of \( \epsilon \).

By [13, Theorem 1.1] there exists \( u_\epsilon \in SBV^2(\text{Int}(B_5 \cup S \cup \Sigma)) \cap L^\infty(\text{Int}(B_5 \cup S \cup \Sigma)) \) such that \( J_{u_\epsilon} \) is contained in a union \( \Gamma \) of finitely many closed connected pieces of \( C^1 \)-curves in \( \text{Int}(B_5 \cup S \cup \Sigma) \), \( u_\epsilon \in W^{1,\infty}(\text{Int}(B_5 \cup S \cup \Sigma)) \) and
\[
\int_{B_5 \cup S} |e(u_\epsilon) - e(v_5)|^2 dx \leq \frac{\epsilon}{4(C + 11\epsilon)(||C||_\infty + 1)} \quad (5.11)
\]
and
\[
\mathcal{H}^1(J_{u_\epsilon} \Delta J_{v_5}) < \frac{\epsilon}{2\epsilon_2}. \quad (5.12)
\]
Since \( J_{v_5} \subset B_5 \), we can assume that the squares \( \{Q_j\}_{j \geq 1} \) of Vitali cover in [13, Eq. 4.3a] satisfies \( Q_j \subset B_5 \). Therefore, we may assume that \( \Gamma \subset \overline{B_5} \). Since \( \mathcal{H}^1 \Gamma \) is regular, we may extract finitely many intervals of \( \Gamma \) whose union \( \Gamma' \) still covers \( J_{u_\epsilon} \) and satisfies \( \mathcal{H}^1(\Gamma' \setminus J_{u_\epsilon}) \leq \frac{\epsilon}{2\epsilon_2} \). Now we define
\[
A_\epsilon := B_5 \setminus \overline{\Gamma'}.
\]
Recall that both \( \Sigma \cap J_{v_5} \) and \( \Sigma \cap J_{v_5} - \epsilon \) are \( H^1 \)-negligible. By the definition of \( B_5 \) and \( \Gamma' \), there exists \( n_\epsilon \in \mathbb{N} \) such that \((A_\epsilon, u_\epsilon) \in \mathcal{C}_{n_\epsilon} \). By the definition of \( \overline{S} \), \( B_5 \) and \( v_5 \) as well as by \((5.12)\) we have
\[
\tilde{S}(B_4, v_4) = \int_{\Omega \setminus \partial^* B_5} \varphi(x, \nu_{B_5})dH^1 + 2 \int_{B_5 \cap J_{v_5}} \varphi(x, \nu_{J_{v_5}})dH^1 + \int_{\Sigma \setminus \partial^* B_5} \beta dH^1 \geq \int_{\Omega \setminus \partial^* B_5} \varphi(x, \nu_{B_5})dH^1 + 2 \int_{B_5 \cap J_{u_\epsilon}} \varphi(x, \nu_{J_{u_\epsilon}})dH^1 + \int_{\Sigma \setminus \partial^* B_5} \beta dH^1 - \epsilon.
\]
Thus, by the definition of \( A_\epsilon \) and \( \Gamma' \)
\[
\tilde{S}(B_4, v_4) \geq \int_{\Omega \setminus \partial^* A_\epsilon} \varphi(x, \nu_{A_\epsilon})dH^1 + 2 \int_{A_\epsilon^{(1)} \cap \Gamma'} \varphi(x, \nu_{\Gamma'})dH^1 + \int_{\Sigma \setminus \partial^* A_\epsilon} \beta dH^1 - 2\epsilon.
\]
Moreover, using the relations \(|A_\epsilon \Delta B_4| = 0\) and \( v_4 = v_5 \) a.e. in \( B_5 \) and Cauchy-Schwartz inequality for nonnegative symmetric forms we obtain
\[
W(A_\epsilon, u_\epsilon) \leq W(B_4, v_4) + 2 \int_{B_5 \cup S} C(x)e(u_\epsilon) : (e(u_\epsilon) - e(v_5)) \leq W(B_4, v_4) + 2 \left( \int_{B_5 \cup S} C(x)e(u_\epsilon) : e(u_\epsilon) dx \right)^{1/2} \times \left( \int_{B_5 \cup S} C(x)(e(u_\epsilon) - e(v_5)) : (e(u_\epsilon) - e(v_5)) dx \right)^{1/2}.
\]
Similarly,
\[
\int_{B_5 \cup S} C(x)e(u_\epsilon) : e(u_\epsilon) dx \leq W(B_4, v_4) + 2 \left( W(B_4, v_4) \right)^{1/2} \left( \int_{B_5 \cup S} C(x)(e(u_\epsilon) - e(v_5)) : (e(u_\epsilon) - e(v_5)) dx \right)^{1/2} \leq (C + 9\epsilon) + 2\sqrt{(C + 9\epsilon) ||C||_\infty} ||e(u_\epsilon) - e(v_5)||_{L^2} \leq C + 10\epsilon.
\]
where in the last inequality we used (5.11). Therefore, by (5.14) and again by (5.11)
\[ W(A, u) \leq W(B_4, v_4) + 2\sqrt{(C + 10\varepsilon)}\|\varepsilon(u) - \varepsilon(v)\|_{L^2} \leq W(B_4, v_4) + \varepsilon. \]  
(5.15)

Now combining (5.13) and (5.15) as well as using \(|B_5| = |A|\) we get
\[ \tilde{F}^\lambda(B_4, v_4) \geq \mathcal{F}^\lambda(A, u) - 3\varepsilon. \]  
(5.16)

Since \((A, u) \in C_{n_\varepsilon}\), by (5.4), (5.8), (5.9), (5.10) and (5.16) we get
\[ \inf_{(A, u) \in \tilde{C}} \tilde{F}(A, u) + 12\varepsilon \geq \mathcal{F}(A, u), \]
and (5.3) follows. \(\square\)

Proposition 5.1 implies that the configuration \((A, u)\) given by Proposition 4.6 is a volume-constraint minimizer of \(\tilde{F}\) in \(\tilde{C}\).

**Proposition 5.2.** Let \((A, u) \in \tilde{C}\) be given by Proposition 4.6. Then \((\text{Int}(A), u)\) is a minimizer of \(\tilde{F}\) in \(\tilde{C}\) under the volume constraint \(|A|\) = \(v\). Moreover, let \(\lambda_0\) be as in Proposition 5.1 and let \((\tilde{A}, \tilde{u}) \in \tilde{C}\) be any volume-constraint minimizer of \(\tilde{F}\). Then \((\tilde{A}, \tilde{u})\) is a minimizer of \(\tilde{F}^\lambda\) for all \(\lambda \geq \lambda_0\), where
\[ \tilde{F}^\lambda(B, v) := \tilde{F}(B, v) + \lambda|B| - v, \quad (B, v) \in \tilde{C}, \quad \lambda > 0. \]  
(5.17)

**Proof.** Note that since \(|\text{Int}(A)\Delta A| = 0\) and \((\text{Int}(A), u) \in \tilde{C}\), by Propositions 4.5, 4.6 and 5.1
\[ \tilde{F}((\text{Int}(A), u) = \inf_{(B, v) \in \tilde{C}} \tilde{F}(B, v) = \inf_{(B, v) \in \tilde{C}} \tilde{F}(B, v) = \inf_{(B, v) \in \tilde{C}} \tilde{F}^\lambda(B, v) \]
for all \(\lambda \geq \lambda_0\). Thus, \((\text{Int}(A), u)\) is a minimizer of both \(\tilde{F}\) and \(\tilde{F}^\lambda_{00}\). The same is true for every minimizer \((B, v)\) of \(\tilde{F}\). \(\square\)

**Theorem 5.3 (Density estimates for minimizers of \(\tilde{F}^\lambda\)).** Given \(\lambda > 0\), let \((A, u) \in \tilde{C}\) be any minimizer of \(\tilde{F}^\lambda(\cdot, \cdot)\) in \(\tilde{C}\) and let \(\xi \in \mathbb{R}^2\) be such that for the function
\[ \tilde{u} := u\chi_{A \cup S} + \xi\chi_{\Omega \setminus A}, \]
one has \(\Omega \cap \partial^* A \subset J_{\tilde{u}}\). Then for any \(x \in \Omega\) and \(r \in (0, \text{dist}(x, \partial \Omega))\),
\[ \mathcal{H}^1(Q_r(x) \cap J_{\tilde{u}}) \leq \frac{16\varepsilon_2 + 4\lambda}{c_1}. \]  
(5.18)

Moreover, there exist \(\varsigma^* = \varsigma^* \in (0, 1)\) and \(R^* > 0\) not depending on \((A, u)\) with the following property. If \(x \in \Omega\) belongs to the closure \(J_{\tilde{u}}^c\) of the set \(\{y \in \Omega \cap J_{\tilde{u}}^c : \theta_{\varsigma}(J_{\tilde{u}}^c, y) > 0\}\), then
\[ \frac{\mathcal{H}^1(Q_r(x) \cap J_{\tilde{u}}^c)}{r} \geq \varsigma^* \]  
(5.19)

for any square \(Q_r(x) \subset \subset \Omega\) with \(r \in (0, \min\{R^*, \text{dist}(x, \partial \Omega)\})\), and if \(x \in \Omega\) belongs to the closure \(S_{\tilde{u}}^c\) of \(\{x \in S_u^A : \theta_{\varsigma}(S_u^A, x) > 0\}\), then
\[ \frac{\mathcal{H}^1(Q_r(x) \cap S_{\tilde{u}}^A)}{r} \geq \varsigma^* \]  
(5.20)

for any \(r \in (0, \min\{R^*, \text{dist}(x, \partial^* A)\})\). In particular,
\[ \mathcal{H}^1(\Omega \cap (J_{\tilde{u}}^c \setminus J_{\tilde{u}})) = \mathcal{H}^1(\text{Int}(A^{(1)}) \cap (S_{\tilde{u}}^c \setminus S_{\tilde{u}}^A)) = 0. \]  
(5.21)
Proof of Theorem 5.3. As in Remark 2.5 \( (A, u) \) is a minimizer of \( \tilde{F}^\lambda \) if and only if \( (A, u + u_0) \) minimizes the \( \tilde{F}^\lambda(u) := \tilde{F}^\lambda(u - u_0) \). Thus, we can introduce the following localized version of \( \tilde{F} \) in open subsets \( O \) of \( \Omega \) which does not see the substrate:

\[
\tilde{F}(B, v; O) := \tilde{S}(B; O) + W(B, v; O)
\]

where

\[
\tilde{S}(B, v; O) := \int_{O \cap \partial^* B} \varphi(y, v_B) d\mathcal{H}^1 + 2 \int_{O \cap B \cap \partial B \cap \mathcal{S}_v} \varphi(y, v_B) d\mathcal{H}^1,
\]

the \( W(\cdot; O) \) is given as in (2.9) and \( S^A_v \) is defined as in (2.6). Then the minimality of \( (A, u) \) implies that \( (A, u + u_0) \) is a quasi-minimizer of \( \tilde{F}(\cdot; O) \) in \( O \), namely,

\[
\tilde{F}(A, u + u_0; O) \leq \tilde{F}(B, v; O) + \lambda_0 |A \Delta B|
\]

whenever \( (B, v) \in \tilde{C} \) with \( A \Delta B \subset\subset O \) and \( \text{supp} \ (u + u_0 - v) \subset\subset O \). Now the proof of the existence of \( \zeta^* \) and \( R^* \) satisfying (5.18) and (5.19) runs along the same lines of the proof of Theorem 3.1 for \( m = \infty \), therefore, we do not repeat it here. Note that \( \zeta^* \) and \( R^* \) depend only on \( c_1 \) and \( \lambda \).

Let \( A_0 := \text{Int}(A^{(1)}) \). We claim that

\[
\partial A_0 = \partial^* A.
\]

Indeed, note that \( A^{(1)} \setminus A_0 \subset \partial A^{(1)} = \partial^* A \), where in the equality we used \( \partial^* A = \partial^* A^{(1)} = \partial A^{(1)} \) see e.g., [52, Eq. 15.3]. Thus, \( A_0 \) is also equivalent to \( A \), and hence, \( \partial^* A_0 = \partial^* A = \partial^* A^{(1)} \). In particular, \( \partial A^{(1)} = \partial^* A_0 \subset \partial A_0 \). On the other hand, assume that there exists \( x \in \partial A_0 \setminus \partial A^{(1)} \). Since \( \partial A^{(1)} \) is closed, there exists \( r > 0 \) such that \( B_r(x) \cap \partial A^{(1)} = \emptyset \). Hence, either \( B_r(x) \subset \text{Int}(A^{(1)}) = A_0 \) or \( B_r(x) \cap A^{(1)} = \emptyset \). Since \( A_0 \) is open and \( x \in \partial A_0 \), the inclusion \( B_r(x) \subset A_0 \) is not possible. On the other hand, since \( A_0 \subset A^{(1)} \) and \( x \in \partial A_0 \), the relation \( B_r(x) \cap A^{(1)} = \emptyset \) is also not possible. Thus, \( \partial A_0 \subset \partial A^{(1)} \).

To prove (5.20) we fix \( \Omega' \subset\subset \Omega \). We claim that \( \tilde{u}|_{A_0} \) is a minimizer of Griffith functional \( G : \text{GSBD}^2(\text{Int}(A_0 \cup S \cup \Sigma)) \rightarrow \mathbb{R} \),

\[
G(v) := \int_{A_0 \cap J_v} \varphi(x, \nu_{J_v}) d\mathcal{H}^1 + \int_{A_0} \mathcal{C}(x)e(v) : e(v) dx
\]

with Dirichlet boundary condition \( v = \tilde{u} = u \in A_0 \setminus \Omega' \). Indeed, for every \( v \in \text{GSBD}^2(A_0) \) with \( \tilde{u} = v \) in \( A_0 \setminus \Omega' \) we define \( B := A_0 \setminus \overline{\Omega'} \). Then \( (B, v) \in \tilde{C} \) and by the minimality of \( (A, u) \)

\[
G(u) - G(v) = \tilde{F}(A, u) - \tilde{F}(B, v) \leq 0.
\]

Since \( S^B_v = J_{\tilde{u}}|_{A_0} \) up to a \( \mathcal{H}^1 \)-negligible set, (5.20) directly follows from the density estimates for the jump set of Griffith minimizers (see e.g. [12]) with possibly smaller \( \zeta^* \in (0, 1) \) and \( R^* > 0 \).

Finally, we prove (5.21) only for \( S^A_u \), the other being similar. Let \( \Gamma := \{ x \in S^A_u : \theta \ast (S^A_u, x) > 0 \} \). Note that \( S^\ast_u = \Gamma \).

We claim that

\[
\mathcal{H}^1(A_0 \cap (\Gamma \setminus \Gamma)) = 0.
\]

Indeed, let \( \mu := \mathcal{H}^1 \setminus \Gamma \). Then \( \mu(\Gamma \setminus \Gamma) = 0 \). By the regularity of \( \mu \), for every \( \epsilon > 0 \) there exists an open set \( U \subset \mathbb{R}^2 \) such that \( L := A_0 \cap (\Gamma \setminus \Gamma) \subset U \) and \( \mu(U) = \mathcal{H}^1(U \cap \Gamma) < \epsilon \). Note that \( \Gamma \subset \{ y \in \Omega \cap J_B : \theta \ast (J_B, x) > 0 \} \), where \( \tilde{u} \) is given by Theorem 5.3. Hence, for (5.19) holds for all points of \( \Gamma \). By the definition of the closure, and Vitali Covering Lemma we can find at most countable pairwise disjoint family \( \{ B_{r_i}(x_i) \} \) of closed balls \( B_{r_i}(x_i) \) with \( x_i \in A_0 \cap \Gamma \), \( r_i \leq \min\{ R^*, \epsilon, \text{dist}(x, \partial A) \} \) such that \( A_0 \cap (\Gamma \setminus \Gamma) \subset \cup_i B_{r_i}(x_i) \). Without
loss of generality we may assume that \( B_r(x_i) \subset U \). Since \( Q_{r_j}(x_i) \subset B_r(x_i) \subset Q_{r_j}(x_i) \), from the definition of Hausdorff premeasure, \( (5.19) \) and disjointness of \( \{B_r(x_i)\} \) as well as the choice of \( U \) we obtain

\[
\mathcal{H}_{10^*}(A_0 \cap (\overline{\Gamma} \setminus \Gamma)) \leq \sum_{i \geq 1} 2\pi(5r_i) \leq \frac{10\pi \sqrt{3}}{\xi^*} \sum_{i \leq 1} \mathcal{H}^1(Q_{r_i/\sqrt{2}}(x_i) \cap \Gamma) = \frac{10\pi \sqrt{2}}{\xi^*} \mathcal{H}^1(U \cap \Gamma) < \frac{10\pi \sqrt{2} \epsilon}{\xi^*}.
\]

Now letting \( \epsilon \to 0 \) we get \( (5.22) \).

In the following proposition we construct a “regular” minimizer of \( F \) starting from a minimizer of \( \widetilde{F} \) in \( \widetilde{C} \).

**Proposition 5.4.** Given \( \lambda > 0 \), let \((A, u) \in \widetilde{C} \) be any minimizer of \( \widetilde{F}^\lambda \). Define

\[
A' := \text{Int}(A(1)) \setminus \overline{\Gamma},
\]

where \( \Gamma := \{x \in S_u^A : \theta_x(S_u^A, x) > 0\} \), and, with a slight abuse of notation, consider \( u \) as defined in \( A' \cup S \) (and so, also on the \( L^2 \)-negligible set \( A' \setminus \text{Int}(A) \)). Then \((A', u) \in C \) is such that \( \widetilde{F}(A, u) = F(A', u) \) and satisfy the following assertions:

1. \( A' \) is open, \( \theta_x(S_u^A, x) > 0 \) for all \( x \in S_u^A \), \( |A\Delta A'| = 0 \) and \( u\chi_{A' \cup S} = u\chi_{A' \cup S} \) a.e. in \( \Omega \cup S \).
2. The closure of \( A'(1) \cap \partial A' \) coincide with \( S_u^{A'} \) and \( \mathcal{H}^1(S_u^{A'} \setminus S_u^{A'}) = 0 \).
3. Let \( \varsigma^* \) and \( R^* \) be given by Theorem 5.3. Then

\[
\frac{\mathcal{H}^1(Q_{r}(x) \cap \partial A')}{r} \leq \frac{16c_2 + 4\lambda_0}{c_1} \quad \text{for every square } Q_{r}(x) \subset \Omega \quad \text{and} \quad \frac{\mathcal{H}^1(Q_{r}(x) \cap \partial A')}{r} \geq \varsigma^* \quad \text{for every } Q_{r}(x) \subset \Omega \text{ with for any } x \in \partial A' \text{ and } r \in (0, R^*).
\]

**Proof.** Note that by definition \( A' \) is open and \( |A\Delta A'| = 0 \). Moreover, \( S_u^{A'} \subset \Gamma \), and by \( (5.20) \) all points of \( \Omega \cap \overline{\Gamma} \) satisfy uniform lower density estimates, hence, \( \theta_x(S_u^A, x) > 0 \) for any \( x \in S_u^A \).

We claim that \( A' \in A \). Indeed, let \( \widetilde{u} \) be given as in Theorem 5.3. By definition

\[
\Omega \cap J_u^\varsigma = \Omega \cap A' \quad \text{and} \quad \partial A' \subset J_u^\varsigma \cup \Sigma,
\]

where \( J_u^\varsigma \) is the closure of the set \( \{x \in J_u^\varsigma : \theta_x(J_u^\varsigma, x) > 0\} \). Since \( J_u^\varsigma \) is \( \mathcal{H}^1 \)-rectifiable, so is \( J_u^{A'} \) in view of \( (5.21) \). Therefore, \( \partial A' \) is \( \mathcal{H}^1 \)-rectifiable, i.e., \( A' \in A \). Note that by construction \( \mathcal{H}^1(A' \cap J_u^\varsigma) = 0 \) hence, by Proposition A.3 \( \widetilde{u} \in H_{loc}^1(A') \) and, since \( u = \widetilde{u} \) a.e. in \( A' \) it follows that \( u \in H_{loc}^1(A') \).

Since \( |A\Delta A'| = 0 \) and \( u = u \) a.e. in \( A' \), it follows that

\[
\mathcal{W}(A, u) = \mathcal{W}(A', u).
\]

Moreover, by the definition of \( \Gamma \) and \( S_u^A \),

\[
|S(A', u) - S(A, u)| = \int_{\text{Int}(A(1)) \cap (\overline{\Gamma} \setminus S_u^A)} \varphi(x, \nu_T)d\mathcal{H}^1 \leq c_2 \mathcal{H}^1(\text{Int}(A(1)) \cap (\overline{\Gamma} \setminus \Gamma)) = 0,
\]

where in the last equality we used \( (5.21) \). Finally, \( (5.23) \) and \( (5.24) \) follows from \( (5.25) \) and density estimates of Theorem 5.3. \( \square \)

Now we are ready to prove the existence of global minimizers of \( F \).
Proof of Theorem 2.6. First we prove the assertion for $\mathcal{G} = \mathcal{F}$.

Let $(A_m, u_m) \in C_m$ be a minimizer of $F$ satisfying the volume constraint $|A_m| = v$ and let $(A_{m_h}, u_{m_h})$, and $A$ and $u$ be as in Proposition 4.6. By (3.1), (4.13) and (4.12) we have

$$\inf_{(B,v) \in C, |B| = v} F(B,v) = \lim_{h \to +\infty} F(A_{m_h}, u_{m_h}) \geq \tilde{F}(\text{Int}(A), u).$$

Since $|\text{Int}(A)| = v$, by Propositions 5.1 and 5.2

$$\inf_{(B,v) \in C, |B| = v} \tilde{F}(B,v) = \inf_{(B,v) \in C, |B| = v} \tilde{F}^{\lambda_0}(B,v) = \tilde{F}^{\lambda_0}(\text{Int}(A), u) = \tilde{F}(\text{Int}(A), u), \quad (5.26)$$

hence, $(\text{Int}(A), u)$ is a minimizer of $\tilde{F}^{\lambda_0}$ in $\tilde{C}$. Then by Proposition 5.4 there exists $(A', u) \in C$ such that

$$\tilde{F}(\text{Int}(A), u) = F(A', u),$$

and hence, in view of (5.26), $(A', u)$ is a solution to (2.16).

The proof of the second assertion (i.e., the existence of $\lambda_1$ for which the set of minimizers in $C$ of both $F$ and $F^{\lambda}$ coincide for all $\lambda \geq \lambda_1$) can be done using the first one and also following the arguments of [32, Theorem 1.1] and [45, Proposition A.6]. Without loss of generality we assume that $\lambda_1 \geq \lambda_0$, where $\lambda_0$ is given by Proposition 5.1.

Now we prove Theorem 2.6 for $\mathcal{G} = \tilde{F}$. We have already shown above that the configuration $(\text{Int}(A), u)$ given by Proposition 4.6 solves the minimum problem (2.16) with $\mathcal{G} = \tilde{F}$. In view of (5.1) every volume-constraint minimizer of $\tilde{F}$ also minimizer of $\tilde{F}^{\lambda}$ for all $\lambda \geq \lambda_1$. To prove the converse assertion, we fix any minimizer $(A, u) \in \tilde{C}$ of $\tilde{F}^{\lambda}$ for $\lambda \geq \lambda_1$. By Proposition 5.4 there exists $(A', u) \in C$ such that $|A'| = |A|$ and $F(A', u) = \tilde{F}(A, u)$. By the first part of the proof and (5.1) we know that

$$\inf_{(B,v) \in C} \tilde{F}^{\lambda}(B,v) = \inf_{(B,v) \in C, |B| = v} F(B,v) = \inf_{(B,v) \in C} \tilde{F}^{\lambda}(B,v) = \tilde{F}^{\lambda}(A', u).$$

Hence, $(A', u)$ is the minimizer of $F^{\lambda}$. Since $\lambda \geq \lambda_1$ according to the first part of the proof, $|A'| = v$. Hence, $|A| = v$ and $(A, u)$ minimizer of (2.16).

We are ready now to study the properties of the minimizers of $F$ in $C$ provided by Theorem 2.6.

Proof of Theorem 2.7. First we properties (1)-(4) the assertion for $\mathcal{G} = \tilde{F}$.

Consider any solution $(A, u) \in \tilde{C}$ of (2.16). By Proposition 5.4 there exists a $(A', u) \in C$ with $A'$ defined as in (2.18), such that the properties (1)-(4) hold except the conditions $\mathcal{H}^1(\partial A\Delta\partial A') = 0$ and $\mathcal{H}^1(S_u^A\Delta S_u^{A'}) = 0$ of (1). To prove these two equations it is enough to observe that

$$0 = |F(A, u) - F(A', u)| = 2 \int_{A(1) \cap (\partial A\Delta\partial A')} \varphi(x, \nu_A) d\mathcal{H}^1$$

and

$$0 = |\tilde{F}(A, u) - \tilde{F}(A', u)| = 2 \int_{A(1) \cap (S_u^A\Delta S_u^{A'})} \varphi(x, \nu_A) d\mathcal{H}^1.$$

Now we assume that $\mathcal{G} = F$ and let $(A, u) \in C$ be a solution to (2.16). Since $(A, u) \in \tilde{C}$, by Proposition 5.1

$$\inf_{(B,v) \in C, |B| = v} \tilde{F}(B,v) = F(A, u) \geq \tilde{F}(A, u).$$

Therefore, $(A, u)$ is also a volume-constraint minimizer of $\tilde{F}$. Thus, applying first part of the proof we establish that $(A', u) \in C$ satisfies (1)-(4).
Finally, notice that if $E \subset A'$ is a connected component of $A'$ with $\mathcal{H}^1(\partial E \cap \Sigma \setminus J_u) = 0$, then for $(A', v)$ with $v = u\chi_{(A \cup S) \setminus E} + (u_0 + a)\chi_E$, where $a$ is any rigid displacement, we have

$$S(A', u) \geq S(A', v)$$

and

$$\mathcal{W}(A', u) \geq \mathcal{W}(A', v),$$

where in (5.27) equality holds if and only if $u = u_0 + a$ in $E$. Therefore, by the minimality of $(A', u)$ it follows that $u = u_0 + a$ in $E$. It remains to prove

$$|E| \geq 4\pi \left(\frac{c_1}{\lambda_1}\right)^2.$$  

(5.28)

Consider the competitor $(A' \setminus E, v) \in C$. By minimality and Theorem 2.6, $\mathcal{F}^{\lambda_1}(A', u) \leq \mathcal{F}^{\lambda_1}(A' \setminus E, u)$, so that by (5.27) and the additivity of the surface energy, $S(E, u) \leq \lambda_1|E|$. Then by (2.13) and the isoperimetric inequality in $\mathbb{R}^2$

$$\lambda_1|E| \geq c_1\mathcal{H}^1(\partial E) \geq c_1\sqrt{4\pi}|E|^{1/2}.$$

Hence, (5.28) follows. 

\appendix

\section*{Appendix A}

We include in this section auxiliary results used in the paper for the convenience of the Reader. We begin by a property satisfied by the free-crystal regions in $\mathcal{A}$ and $\Lambda$.

\begin{proposition}
Let $A \subset \mathbb{R}^2$ be a bounded $\mathcal{L}^2$-measurable set with $\mathcal{H}^1(\partial A) < +\infty$. Then $A$ is a set of finite perimeter in $\mathbb{R}^2$.
\end{proposition}

\begin{proof}
Since $A \Delta \text{Int}(A) \subset \overline{A} \setminus \text{Int}(A) = \partial A$, we have $|A \Delta \text{Int}(A)| \leq |\partial A| = 0$, and hence, it suffices to prove that the open set $E := \text{Int}(A)$ has finite perimeter in $\mathbb{R}^2$. Note that by construction, $\partial E \subset \partial A$ and $\mathcal{H}^1(\partial E) \leq \mathcal{H}^1(\partial A) < +\infty$.

We divide the proof of $E \in BV(\mathbb{R}^2; \{0, 1\})$ into three steps.

\textbf{Step 1.} We claim that if $E$ is simply connected, then $E \in BV(\mathbb{R}^2; \{0, 1\})$. Indeed, in this case $\partial E$ is a connected compact set with $\mathcal{H}^1(\partial E) \leq \mathcal{H}^1(\partial A) < +\infty$ and by [33, Lemma 3.12] it contains a closed curve $\Gamma$ enclosing $E$. Since $\mathcal{H}^1(\Gamma) < +\infty$, it is rectifiable in the sense of [33, Section 3.2]: its length $\mathcal{H}^1(\Gamma)$ is well-approximated by the length of closed polygonal curves $\pi_k$ whose vertices lie on $\Gamma$, i.e., $\mathcal{H}^1(\pi_k) \to \mathcal{H}^1(\Gamma)$. Let $E_k$ be the set enclosed by $\pi_k$ and observe that $\pi_k \to E$ in $\mathcal{H}^1$ and $E_k \subset E$ for large $k$.

\textbf{Step 2.} We claim that if $E$ is connected, then $E \in BV(\mathbb{R}^2; \{0, 1\})$. Indeed, let $E'$ be the smallest simply connected open set containing $E$ (basically, $E'$ is contructed by filling in all “holes” in $E$) and let

$$F := E' \setminus \overline{E}$$

be the union of all holes. Since $\partial E' \subset \partial E$ and $\mathcal{H}^1(\partial E) \leq \mathcal{H}^1(\partial A) < +\infty$, by Step 1 $E' \in BV(\mathbb{R}^2; \{0, 1\})$. Observing $E = E' \setminus F$, to conclude this step it is enough to prove that $F$ has finite perimeter. Since every open set in $\mathbb{R}^2$ is a union of at most countably
many connected components, we have \( F = \bigcup_j F_j \), where \( \{ F_j \} \) are open, connected and \( F_i \cap F_j = \emptyset \) for \( i \neq j \). Since \( E \) is connected, each \( F_j \) is simply connected, and hence, by Step 1 \( F_j \in BV(\mathbb{R}^2; \{0, 1\}) \). Moreover, the set \( \partial F_i \cap \partial F_j \), \( i \neq j \), can have at most one point. Indeed, otherwise, by the definition of \( F \) and the connectedness of \( E \) we could find a curve \( \gamma \subset \partial F_i \cap \partial F_j \cap \partial E \) with \( H^1(\gamma) > 0 \), which contradicts the equality \( E = \text{Int}(E) \). Therefore, observing \( \partial F = \bigcup \partial F_j \subset \partial E \), we obtain

\[
\sum_j P(F_j) \leq \sum_j H^1(\partial F_j) = H^1\left( \bigcup_j \partial F_j \right) = H^1(\partial F) \leq H^1(\partial E) < +\infty.
\]

Thus, \( F = \bigcup_j F_j \) has finite perimeter in \( \mathbb{R}^2 \).

**Step 3.** Now we prove that \( E \in BV(\mathbb{R}^2; \{0, 1\}) \) (without assuming any extra connectedness assumption). Let \( \{ E_j \} \) be the family of connected components of \( E \). Since \( H^1(\partial E_j) \leq H^1(\partial E) < +\infty \), by Step 2 \( E_j \in BV(\mathbb{R}^2; \{0, 1\}) \). Therefore, since \( \partial E = \bigcup \partial E_j \) we obtain that

\[
\sum_j P(E_j) \leq \sum_j H^1(\partial E_j) \leq H^1\left( \bigcup_j \partial E_j \right) + \sum_i H^1(\partial E_i \cap \partial E_j) \leq 2H^1\left( \bigcup_j \partial E_j \right) = 2H^1(\partial E),
\]

and hence, by the finiteness of \( H^1(\partial E) \), the set \( E = \bigcup_j E_j \) has finite perimeter in \( \mathbb{R}^2 \).

The following proposition, which is based on [52, Proposition 2.16], is used throughout the paper.

**Proposition A.2.** Let \( K \subset \mathbb{R}^2 \) be such that \( H^1(K) < +\infty \) and let \( \{ E_i \}_{i \in I} \) be a family of sets parametrized by \( t \in T \) such that

\[
H^1(K \cap E_i \cap E_j) = 0 \quad \text{(A.1)}
\]

and \( H^1(K \cap E_i) > 0 \). Then \( I \) is at most countable.

**Proof.** The proof runs along the lines of the proof of [52, Proposition 2.16]. For \( j \in \mathbb{N} \) let \( T_j \subset T \) be the set of all \( t \in T \) such that \( H^1(K \cap E_j) > \frac{1}{j} \). Then by (A.1) \( T_j \) cannot contain more than \( jH^1(K) \) elements. Since \( T = \bigcup T_j \), the set \( T \) is at most countable.

We finally state a regularity property of GSBD functions with \( H^{d-1} \)-negligible jump.

**Proposition A.3.** Let \( U \subset \mathbb{R}^d \) be a connected bounded open set and \( u \in GSBD^2(U) \) be such that \( H^{d-1}(J_u) = 0 \). Then \( u \in H^{1}_{\text{loc}}(U) \).

**Proof.** Indeed, for \( r > 0 \) let \( Q := x_0 + (-r, r)^d \subset U \) be any cube centered at \( x \in U \) and let \( 0 < \theta^d < \theta^d < 1 \). For shortness, write \( Q' := x_0 + (\theta^d, \theta^d)^d \) and \( Q'' := x_0 + (-\theta^d, \theta^d)^d \). By [11, Proposition 3.1 (1)] (see also [10, Theorem 1.1]) there exists a \( C^2 \)-measurable set \( \omega \subset Q' \) and a rigid displacement \( a : \mathbb{R}^d \to \mathbb{R}^d \) such that \( |\omega| \leq c_{\varepsilon} r H^{d-1}(J_u) = 0 \) and

\[
\int_{Q'} |u - a|^{2\frac{d}{d-1}} dx = \int_{Q' \setminus \omega} |u - a|^{2\frac{d}{d-1}} dx \leq c_{\varepsilon} r^2 \left( \int_Q |e(u)|^2 \right)^{\frac{d}{d-1}},
\]

where \( c_{\varepsilon} \) depends only on \( d \). Hence, \( u \in L_{\text{loc}}^{\frac{2d}{d-1}}(Q) \). Next, fix any mollifier \( \rho_1 \in C^\infty(B_r(0)) \) with \( \rho_\varepsilon \in C^\infty_c(B_{\theta^d r}(x))) \), where \( \rho_\varepsilon(x) := \rho_1(x/\varepsilon) \), \( \varepsilon \in (0, r) \). By [11, Proposition 3.1] there exists \( \tilde{r} > 0 \) depending on \( n \) and \( \varepsilon \) such that

\[
\int_{Q''} |e(u * \rho_\varepsilon) - e(u) * \rho_\varepsilon|^2 dx \leq c \left( \frac{H^{d-1}(J_u)}{r^{d-1}} \right)^{\tilde{r}} \int_Q |e(u)|^2 dx = 0,
\]

\(^{1}\)This property easily follows by fact that we can always choose in each connected component a different point with rational coordinates.
where $c$ depends on $n$, $\rho_1$ and $\epsilon$. Hence,

$$e(u * \rho_\epsilon) = e(u) * \rho_\epsilon \quad \text{a.e. in } Q''.$$  \hfill (A.2)

Recall that $u * \rho_\epsilon \in C^\infty(Q'')$. Since $e(u) \in L^2(Q)$, $e(u) * \rho_\epsilon \in C^\infty(Q'') \cap L^2(Q'')$ in particular, $e(u) * \rho_\epsilon \in C^\infty(Q'') \cap L^2(Q'')$. By Poincaré-Korn inequality $u * \rho_\epsilon \in H^1(Q'')$. Since $e(u) * \rho_\epsilon \to e(u)$ in $L^2(Q'')$ as $\epsilon \to 0$, in view of (A.2) there exists $\epsilon_0 > 0$ such that

$$\|e(u * \rho_\epsilon)\|_{L^2(Q'')} \leq \|e(u)\|_{L^2(Q'')} + 1 \quad \text{for all } \epsilon \in (0, \epsilon_0).$$

Moreover, by Poincaré-Korn inequality for any $\epsilon \in (0, \epsilon_0)$ there exists a rigid displacement $a_\epsilon$ such that

$$\|e(u * \rho_\epsilon - a_\epsilon)\|_{H^1(Q'')} \leq C\|e(u * \rho_\epsilon)\|_{L^2(Q'')} \leq C(\|e(u)\|_{L^2(Q'')} + 1),$$

where $C$ is the Poincaré-Korn constant for a cube. Thus, the family $\{u * \rho_\epsilon\}$ is uniformly bounded in $H^1(Q'')$. Since $u * \rho_\epsilon \to u$ in $L^2(Q'')$, there exists a rigid displacement $a$ such that $a_\epsilon \to a$ in $L^2(Q'')$. Then $u * \rho_\epsilon - a_\epsilon$ weakly converges to $u - a$ in $H^1(Q'')$, i.e., $u - a \in H^1(Q'')$. Since $a$ is linear and $\theta''$ is arbitrary, $u \in H^1_{\text{loc}}(Q)$. Now covering $U$ with finitely many cubes of edgelength $2r$ we get $u \in H^1_{\text{loc}}(U)$. \hfill \Box

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