Free path groupoid grading on Leavitt path algebras

Daniel Gonçalves* and Gabriela Yoneda†

Department de Matemática, UFSC
Florianópolis, SC - 88040-900, Brazil
*daemig@gmail.com
†yonedagabriela@gmail.com

Received 5 May 2016
Accepted 24 June 2016
Published 5 August 2016

Communicated by B. Steinberg

In this work, we realize Leavitt path algebras as partial skew groupoid rings. This yields a free path groupoid grading on Leavitt path algebras. Using this grading, we characterize free path groupoid graded isomorphisms of Leavitt path algebras that preserves generators.

Keywords: Leavitt path algebras; partial skew groupoid rings; free path groupoid; isomorphism of Leavitt path algebras.

Mathematics Subject Classification 2010: 16W50, 16W55, 16G99

1. Introduction

Leavitt path algebras, introduced in [1, 2] as generalizations of Cuntz–Krieger algebras, have been the focus of intense research in recent years. Part of the interest in these algebras come from the fact that the Leavitt path algebra associated to a graph encodes much of the combinatorics of the graph and, therefore, their algebraic properties are often linked to combinatorial properties of the underlying graph. To mention a few results in the field, in [7, 24, 26] the ideal structure of Leavitt path algebras is studied, in [8, 17–19, 21, 23] the theory of representations is approached, in [20] the graded structure of Leavitt path algebras is explored and in [9, 10] Leavitt path algebras are realized as Steinberg algebras. Of particular interest to our work, in [16], Leavitt path algebras are realized as partial skew group ring (a notion introduced in [11, 12]) and in [15] this realization is used to give new proofs of the simplicity criteria of these algebras.

One direction of research in the field regards the study of isomorphisms of Leavitt path algebras and its connections with isomorphisms of their counterpart in C*-algebras, namely graph C*-algebras. This is a topic included in the graph algebra open problem webpage kept by Mark Tomforde and was initially proposed...
Recent developments on the subject can be found in [25]. Also in a very recent development, graph C*-algebras are shown to be closely related to orbit equivalence of graphs, see [6]. Given the above, the study of isomorphisms of Leavitt path algebras gains extra importance. It is our goal in this paper to give the reader a new insight into Leavitt path algebras and show how this insight can be applied to characterize a class of isomorphisms between Leavitt path algebras.

A key asset in our work is the theory of partial skew groupoid rings. These rings were defined in [4, 5] as generalizations of partial skew group rings, which in turn were defined in [11, 12]. As we mentioned before, in [16] Leavitt path algebras were realized as partial skew group rings. Building from the ideas in [16] we realize Leavitt path algebras as partial skew groupoid rings. While in the group case the group acting is the free group in the edges, in our case the free path groupoid will act. Among the differences between the groupoid approach and the one in [16] we mention that, in the partial groupoid skew ring case, each local unit of a Leavitt path algebra is represented on its own fiber and the definition of the appropriate partial action happens in a more natural ideal. We should also mention that the realization of Leavitt path algebras as Steinberg algebras, see [9] for the row finite case and [10] for general graphs, involves a groupoid, namely the graph groupoid, which although related is not the same as the free path groupoid.

With our description of Leavitt path algebras as partial skew groupoid rings, we can obtain a characterization of (free path groupoid) graded isomorphisms of Leavitt path algebras that preserves generators. More precisely, given two graphs $E_1$ and $E_2$ we show that if there exists a homomorphism between the associated free path groupoids, say $G_1$ and $G_2$, that preserves the set of finite words and is injective in this set, then there exists a groupoid graded isomorphism between the associated partial skew groupoid rings (which preserves the generators). Furthermore, in the presence of condition (L) for $E_1$, we show the converse of our statement, that is, we show that if there exists a groupoid graded isomorphism, that preserves generators, from the partial skew groupoid ring associated to $E_1$ to the partial skew groupoid ring associated to $E_2$, then there exists an homomorphism between the associated free path groupoids that preserves the set of finite words and is injective in this set.

We organize our work as follows: In Sec. 2, we include background material, in order to make the paper as self-contained as possible. The characterization of Leavitt path algebras as partial skew groupoid rings is done in Sec. 3 and in Sec. 4 we show the applications mentioned in the previous paragraph.

2. Background

2.1. Partial skew groupoid rings

The notion of partial skew groupoid rings was introduced in [4, 5], derived from the work by Dokuchaev and Exel in [11]. Below we recall the notions leading to partial skew groupoid rings.
Definition 2.1. A partial action of a group \( G \) on a set \( \Omega \) is a pair \( \alpha = (\{D_t\}_{t \in G}, \{\alpha_t\}_{t \in G}) \), where for each \( t \in G \), \( D_t \) is a subset of \( \Omega \) and \( \alpha_t : D_{t^{-1}} \rightarrow D_t \) is a bijection such that \( D_e = \Omega \), \( \alpha_e \) is the identity in \( \Omega \), \( \alpha_t(D_{t^{-1}} \cap D_e) = D_t \cap D_{t^{-1}} \) and \( \alpha_t(\alpha_s(x)) = \alpha_{ts}(x) \), for all \( x \in D_{s^{-1}} \cap D_{s^{-1}} \). In case \( \Omega \) is an algebra or a ring, then the subsets \( D_t \) should also be ideals and the maps \( \alpha_t \) should be isomorphisms.

Associated to a partial action of a group \( G \) in a ring \( A \), the partial skew group ring \( A \rtimes G \) is defined as the set of all finite formal sums \( \sum_{t \in G} a_t \delta_t \), where, for all \( t \in G \), \( a_t \in D_t \) and \( \delta_t \) are symbols. Addition is defined in the usual way and multiplication is determined by \( (a_t \delta_t)(b_t \delta_t) = \alpha_t(\alpha_{-t}(a_t)b_t)\delta_{ts} \).

Definition 2.2 (as in [4]). A groupoid is a set \( G \) equipped with a binary partially defined operation such that:

(i) for all \( g, h, l \in G \), \( g(hl) \) exists if, and only if, \( (gh)l \) exists. In this case, \( g(hl) = (gh)l \);
(ii) for every \( g, h, l \in G \), \( g(hl) \) exists if, and only if, \( gh \) and \( hl \) exist;
(iii) for all \( g \in G \), there exist (unique) elements \( d(g), t(g) \in G \) such that \( gd(g) \) and \( t(g)g \) exist and \( gd(g) = g = t(g)g \);
(iv) for each \( g \in G \) there exists \( g^{-1} \in G \) such that \( d(g) = g^{-1}g \) and \( t(g) = gg^{-1} \).

Remark 2.3. Let \( G \) be a groupoid. The set of admissible pairs is defined as \( G^2 = \{(g, h) \in G \times G : d(g) = t(h)\} \). An element \( e \in G \) is an identity in \( G \) if \( e = d(g) = t(g^{-1}) \), for some \( g \in G \). We denote by \( G^{(0)} \) the set of all identities in \( G \). Notice that \( G^{(0)} = \{g^{-1}g : g \in G\} \).

Next, following [4, 5] we define partial groupoid actions and their associated partial skew groupoid rings.

Definition 2.4. A partial action \( \alpha \) of a groupoid \( G \) on a set \( X \) is a pair \( \alpha = ((X_g)_{g \in G}, (\alpha_g)_{g \in G}) \) such that:

(I) for all \( g \in G \), \( X_{t(g)} \) is a subset of \( X \) and \( X_g \) is a subset of \( X_{t(g)} \);

(II) \( \alpha_g : X_{t^{-1}} \rightarrow X_g \) are bijections that satisfies:

(i) \( \alpha_e \) is the identity \( \text{Id}_{X_e} \) of \( X_e \), for all \( e \in G^{(0)} \);
(ii) \( \alpha_{h^{-1}}(X_{g^{-1}} \cap X_h) \subseteq X_{(gh)^{-1}} \), whenever \( (g, h) \in G^2 \);
(iii) \( \alpha_g(\alpha_h(x)) = \alpha_{gh}(x) \), for all \( x \in \alpha_{h^{-1}}(X_{g^{-1}} \cap X_h) \) and \( (g, h) \in G^2 \).

For partial actions on a ring \( R \), we further ask that each \( D_{t(g)} \) be an ideal of \( R \), each \( D_g \) be an ideal of \( D_{t(g)} \) and the maps \( \alpha_g : D_{t^{-1}} \rightarrow D_g \) be isomorphisms.

Definition 2.5. Let \( \alpha = ((D_g)_{g \in G}, (\alpha_g)_{g \in G}) \) be a partial action of the groupoid \( G \) on a ring \( R \). The partial skew groupoid ring \( R \rtimes G \) is the set of all formal sums of the form \( \sum_{g \in G} a_g \delta_g \), where \( a_g \in D_g \), with addition defined in the usual way and multiplication given by

\[
a_g \delta_g \cdot b_h \delta_h = \begin{cases} 
\alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}, & \text{if } (g, h) \in G^2; \\
0, & \text{otherwise}.
\end{cases}
\]
2.2. Leavitt path algebras as partial skew group rings

For readers convenience in this section, we recall the realization of Leavitt path algebras as partial skew group rings, as done in [16].

A directed graph $E = (E^0, E^1, r, s)$ consists of a set $E^0$ of vertices, a set $E^1$ of edges, a range map $r : E^1 \to E^0$ and a source map $s : E^1 \to E^0$ which may be used to read off the direction of an edge. Given a field $K$ and a directed graph $E$, the so called Leavitt path algebra associated with $E$ (see e.g. [1, 2]) is denoted by $L_K(E)$. To be more precise, $L_K(E)$ is the universal $K$-algebra generated by a set $\{v, e, e^* : v \in E^0, e \in E^1\}$ of elements satisfying the following five assertions:

1. for all $v, w \in E^0$, $v^2 = v$, and $vw = 0$ if $v \neq w$;
2. $s(e)e = er(e) = e$ for all $e \in E^1$;
3. $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$;
4. for all $e, f \in E^1$, $e^*e = r(e)$, and $e^*f = 0$ if $e \neq f$;
5. $v = \sum_{e \in E^1 : s(e) = v} e e^*$ for each vertex $v \in E^0$ which satisfies $0 < \#\{e \in E^1 : s(e) = v\}$.

In [16], it was showed that each Leavitt path algebra can be realized as a partial skew group ring. We shall review this construction by first defining a partial action at the level of sets.

Let $E = (E^0, E^1, r, s)$ be a directed graph. A path of length $n$ in $E$ is a sequence $\xi_1 \xi_2 \cdots \xi_n$ of edges in $E$ such that $r(\xi_i) = s(\xi_{i+1})$ for $i \in \{1, 2, \ldots, n-1\}$. If $\xi$ is a path of length $n$, then we write $|\xi| = n$. The set of all finite paths in $E$ is denoted by $W$. An infinite path in $E$ is an infinite sequence $\xi_1 \xi_2 \cdots$ of edges in $E$ such that $r(\xi_i) = s(\xi_{i+1})$ for $i \in \mathbb{N}$. The set of all infinite paths in $E$ is denoted by $W^\infty$. Note that $W$ (resp. $W^\infty$) is a subset of the set of all finite (respectively infinite) words in the alphabet $E^1$. As usual, the range and source maps can be extended from $E^1$ to $W \cup W^\infty \cup E^0$ by defining $s(\xi_1) := s(\xi_1)$ for $\xi = \xi_1 \xi_2 \cdots \in W^\infty$ or $\xi = \xi_1 \cdots \xi_n \in W$, $r(\xi) := r(\xi_n)$ for $\xi = \xi_1 \cdots \xi_n \in W^\infty$ or $\xi = \xi_1 \cdots \xi_n \in W$, and $r(v) = s(v) = v$ for $v \in E^0$. A finite path $\eta$ is said to be an initial subpath of a (possibly infinite) path $\xi$, if there is a path $\xi'$ such that $r(\eta) = s(\xi')$ and $\xi = \eta \xi'$ hold.

The partial action that we are about to define, takes place on the set

$$X = \{\xi \in W : r(\xi) \text{ is a sink} \} \cup \{v \in E^0 : v \text{ is a sink} \} \cup W^\infty$$

which is acted upon by $F$, the free group generated by the set $E^1$ (notice that, since $F$ is generated by $E^1$, some elements of $F$ can be thought of as coming directly from $W$).

In order to have a partial action of $F$ on $X$, for each $e \in F$, we need to define a set $X_e$ and a map $\theta_e : X_{e^{-1}} \to X_e$, such that they comply with the definition of a partial action. This is done as follows:

- $X_0 := X$, where $0$ is the neutral element of $F$.
- $X_{b^{-1}} := \{\xi \in X : s(\xi) = r(b)\}$, for all $b \in W$.
- $X_a := \{\xi \in X : \xi_1 \xi_2 \cdots \xi_n = a\}$, for all $a \in W$. 
• $X_{a^{-1}} := \{ \xi \in X : \xi_1 \xi_2 \cdots \xi_{|a|} = a \} = X_a$, for $ab^{-1} \in \mathbb{F}$ with $a, b \in W$, $r(a) = r(b)$ and $ab^{-1}$ in its reduced form.

• $X_c := \emptyset$, for all other $c \in \mathbb{F}$.

3. Leavitt Path Algebras as Partial Skew Groupoid Rings

Associated to a graph, one can construct the free path groupoid. In this section, we use this groupoid to realize the Leavitt path algebra associated to a graph as a partial skew ring groupoid ring. It is interesting to note that, under this realization, there exists a standard set of local units of the Leavitt path algebra such that each local unit lives on its own fiber, while if the Leavitt path algebra is realized as a partial skew group ring, all local units live in the fiber associated to the neutral element of the free group.
The definition of the free path groupoid can be found in [22]. Since this is the main groupoid in our work, we recall its construction below.

Let \( E = (E^0, E^1, r, s) \) be a graph and let \( W \) denotes the set of finite paths in \( E \), where we now see the vertices of the graph as paths of length zero. Let \((E^1)^* := \{e^* : e \in E^1\}\) and extend the maps \( r, s \) to \((E^1)^*\) by \( s(e^*) = r(e) \) and \( r(e^*) = s(e) \), for all \( e \in E^1\). For \( \alpha = \alpha_1 \ldots \alpha_n \in W \setminus E^0 \) define \( \alpha^* := \alpha_n^* \ldots \alpha_1^* \). Let

\[
P = \{ \alpha_1 \ldots \alpha_n : \alpha_i \in E^1 \cup (E^1)^* \cup E^0 \text{ and } r(\alpha_i) = s(\alpha_{i+1}) \}.
\]

**Remark 3.1.** In the previous section, we did not include the paths of length zero in the set of words because we were recalling the notions introduced in [16], but for the reminder of this work, we will always consider \( W \) to contain the paths of length zero as described above.

**Example 3.2.** In the graph below, the element \( e_1 e_2 e_3^* e_4 \) belongs to \( P \).

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
e_1 & e_2 & e_3 & e_4 & \\
\end{array}
\]

In \( P \), we define a concatenation operation, denoted by \( \cdot \), so that

\[
e_1 \cdot e_2 = e_1 e_2 \quad \text{(concatenation)};
\]

\[
e \cdot r(e) = s(e) \cdot e = e;
\]

\[
e \cdot e^* = s(e);
\]

\[
e^* \cdot e = r(e);
\]

and we extend this operation to finite paths in the expected way.

**Definition 3.3.** Given \( a_1, \ldots, a_n, b \in E^1 \cup (E^1)^* \cup E^0 \) and sequences of the form

\[
a_1 \cdot a_k b b^* a_{k+1} a_n
\]

or

\[
a_1 \cdot a_k r(a_k) a_{k+1} \cdots a_n = a_1 \cdots a_k s(a_{k+1}) a_{k+1} \cdots a_n
\]

in \( P \), we say that \( a_1 \cdots a_k a_{k+1} a_n \) is a reduction of these sequences. A sequence in \( P \) is said irreducible if it cannot be reduced.

**Definition 3.4.** Let \( G \) be the set of all irreducible sequences in \( P \), \( G^2 = \{ (\alpha, \beta) \in G \times G : r(\alpha) = s(\beta) \} \) and define the partial groupoid operation \( G^2 \to G \) by

\[
\alpha \cdot \beta = \text{irr}(\alpha \beta),
\]

where \( \text{irr}(\alpha \beta) \) denotes the reduction of \( \alpha \beta \). Then \( G \) is the free path groupoid associated to the graph \( E \).

**Remark 3.5.** To emphasize the notations, we are using notice that, for all \( \alpha \in G \), we have \( s(\alpha) = t(\alpha), r(\alpha) = d(\alpha), \alpha^* = s(\alpha) \) and \( \alpha^* \alpha = r(\alpha) \). This means that the domain of a free path groupoid element is in fact its graphical range and its range is its graphical source. This unfortunate notation could be avoided.
if we defined paths as sequences $\xi_1\xi_2\cdots\xi_n$ of edges such that $r(\xi_{i+1}) = s(\xi_i)$ for $i \in \{1, 2, \ldots, n-1\}$ (and make the necessary adjustments in the Leavitt path algebra definition), which is becoming the more popular choice in C*-algebras papers. But in the algebra world, the original definition of paths is still the most used and hence we decided to adopt it in this paper.

We now proceed to realize a Leavitt path algebra as a partial skew groupoid ring. The main ideas follow what was done for groups. We start by defining a partial groupoid action in the level of sets.

Let $E = (E^0, E^1, r, s)$ be a graph and $G$ the associated free path groupoid, as in Definition 3.4. Let $X = \{\xi \in W : r(\xi) \text{ is a sink}\} \cup W^\infty$. For $a \in W \setminus E^0$, define $X_a := \{\xi \in X : \xi_1 \cdots \xi_{|a|} = a\}$ and $X_{a^{-1}} := \{\xi \in X : r(\xi) = r(a)\}$. For $ab^{-1} \in G$, with $a, b \in W$ and $a, b \notin E^0$, define $X_{ab^{-1}} := X_a$. For $v \in E^0$, define $X_v := \{\xi \in X : s(\xi) = v\}$. Finally, for any other $g \in G$, let $X_g = \emptyset$.

Remark 3.6. Notice that the sets $X_v$ were not part of the definition of the partial action of the free group. So, to obtain the Leavitt path algebra, it was necessary to artificially add the span of the characteristic functions of the sets $X_v$ to the algebra where the free group acts (see the definition of $D_0$ in the previous section). This procedure will be avoided with our groupoid approach.

Next, we define the groupoid partial action.

For $v \in E^0$, define

$$
\theta_v : X_v \rightarrow X_v
$$

$$
\xi \mapsto \xi.
$$

For $b \in W \setminus E^0$, define

$$
\theta_b : X_{b^{-1}} \rightarrow X_b
$$

$$
\xi \mapsto b\xi,
$$

and $\theta_{b^{-1}} : X_b \rightarrow X_{b^{-1}}$, by

$$
\theta_{b^{-1}}(\xi) = \begin{cases} 
\xi_{|b|+1}\xi_{|b|+2} \cdots, & \text{if } r(b) \text{ is not a sink} \\
r(b), & \text{if } r(b) \text{ is a sink.}
\end{cases}
$$

For $ab^{-1} \in G$ with $a, b \in W \setminus E^0$, define

$$
\theta_{ba^{-1}} : X_{ab^{-1}} \rightarrow X_{ba^{-1}}
$$

$$
\xi \mapsto b\xi_{|a|+1}\xi_{|a|+2} \cdots
$$

$$
\theta_{ab^{-1}} : X_{ba^{-1}} \rightarrow X_{ab^{-1}}
$$

$$
\xi \mapsto a\xi_{|b|+1}\xi_{|b|+2} \cdots.
$$

Proposition 3.7. $\theta = \{X_g\}_{g \in G}, \{\theta_g\}_{g \in G}$ is a partial action of the groupoid $G$ in the set $X$.
Proof. It is straightforward to check that $X_{1(g)}$ is a subset of $X$, that $X_g$ is a subset of $X_{1(g)}$ and that $\theta_g : X_{g^{-1}} \to X_g$ is a bijection. We need to prove the remaining conditions a partial groupoid action must satisfy.

(i) For all $v \in E^0$, we have that $\theta_v$ is the identity in $X_v$.

Given $\xi \in X_v$, then $s(\xi) = v = r(\xi)$ and hence

$$\theta_v(\xi) = v_\xi = s(\xi)_\xi = \xi.$$ 

(ii) $\theta_h^{-1}(X_{g^{-1}} \cap X_h) \subseteq X_{(gh)^{-1}}$.

Let $g, h \in G$. We will verify the above statement when $g = ab^{-1}, h = cd^{-1}$ with $a, b, c$ and $d \in W \setminus E^0$.

Since $X_{g^{-1}} \cap X_h = X_{ba^{-1}} \cap X_{cd^{-1}}$, we need to consider three cases:

Case 1: If $b$ is not an initial subpath of $c$ and $c$ is not an initial subpath of $b$.

In this case, $X_{ba^{-1}} \cap X_{cd^{-1}} = \emptyset$.

Case 2: If $b$ is an initial subpath of $c$ (we write $c = bc'$).

Note that in this case, we have that

$$X_{ba^{-1}} \cap X_{cd^{-1}} = \begin{cases} 
X_{a^{-1}} \cap X_{d^{-1}}, & \text{if } b = r(a), c = r(d) \in E^0 \\
X_{a^{-1}} \cap X_c, & \text{if } b \in E^0, c \notin E^0 \\
X_b \cap X_c, & \text{if } b, c \notin E^0
\end{cases}$$

Then,

$$\theta_h^{-1}(X_{g^{-1}} \cap X_h) = \begin{cases} 
\theta_{d^{-1}}^{-1}(X_{a^{-1}} \cap X_{d^{-1}}) & (1) \\
\theta_{cd^{-1}}^{-1}(X_{a^{-1}} \cap X_c) & (2) \\
\theta_{c^{-1}}^{-1}(X_c) & (3)
\end{cases}$$

In (1), we have that $(gh)^{-1} = h^{-1}g^{-1} = dc^{-1}ba^{-1} = da^{-1}$ and hence $X_{(gh)^{-1}} = X_{da^{-1}} = X_d$. In (2), we have that $(gh)^{-1} = h^{-1}g^{-1} = dc^{-1}ba^{-1} = dc^{-1}a^{-1}$ and hence $X_{(gh)^{-1}} = X_{dc^{-1}a^{-1}} = X_d$. Finally, in (3) we have that $(gh)^{-1} = dc^{-1}ba^{-1} = dc^{-1}b^{-1}ba^{-1} = dc^{-1}r(b)a^{-1} = dc^{-1}a^{-1}$. In any of the three cases we obtain that

$$\theta_{dc^{-1}} \subseteq X_{dc^{-1}} = X_{dc^{-1}b^{-1}} = X_{(gh)^{-1}},$$

as desired.

Case 3: $c$ is an initial subpath of $b$ (we write $b = cb'$).

This case is analogous to Case 2.

(iii) For all $\xi \in \theta_g^{-1}(X_g \cap X_{h^{-1}})$ we have that $\theta_h \circ \theta_g(\xi) = \theta_{bh}(\xi)$.

This can be proved in a similar way to (ii).
We can now define the partial action in the ring level:

Let $K$ be a field and let $\mathcal{F}(X)$ denote the ring of all functions from $X$ to $K$, with ring structure given by point-wise sum and multiplication. For each $X_g \neq 0$, let $\mathcal{F}(X_g) = \{ f \in \mathcal{F}(X) : f \text{ vanishes outside of } X_g \} \subseteq \mathcal{F}(X)$ and for $X_g = \emptyset$, let $\mathcal{F}(X_g) = \{ \text{null function} \}$. Now define

$$\alpha_g : \mathcal{F}(X_g^{-1}) \rightarrow \mathcal{F}(X_g)$$

$$f \mapsto f \circ \theta_{g^{-1}}$$

and

$$\alpha_v : \mathcal{F}(X_v) \rightarrow \mathcal{F}(X_v)$$

$$f \mapsto f \circ \theta_v$$

**Proposition 3.8.** $\alpha = (\{ \mathcal{F}(X_g) \}_{g \in G}, \{ \alpha_g \}_{g \in G})$ is a partial action of the groupoid $G$ on the ring $\mathcal{F}(X)$.

**Proof.** The proof of this proposition follows the usual techniques used in the partial skew group ring case, see for example [7, 13] or [14].

As it happened in the group case, the partial action above is too “large”. To get the Leavitt path algebra, we need to do the following. Let

$$D(X) = \text{span}\{1_g : g \in G\},$$

where $1_g$ denotes the characteristic function of the set $X_g$, let

$$D_p = 1_p D(X) = \text{span}\{1_p 1_g : g \in G\},$$

for all $p \in G$, and consider the restriction of $\alpha_p$ to the ideal $D_p$ (that each $D_p$ is an ideal follows from the intersection properties satisfied by the sets $X_g$):

$$\alpha_p : D_p^{-1} \rightarrow D_p$$

$$1_p^{-1}1_q \mapsto \alpha_p(1_p^{-1}1_q) = 1_p 1_{pq}$$

One can check that $\tilde{\alpha} = (\{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ is a partial action of the groupoid $G$ on the ring $D(X)$.

**Definition 3.9.** We will denote the partial skew groupoid ring associated to $\tilde{\alpha}$ by $D(X) \rtimes G$.

**Remark 3.10.** Note that for any vertex $v$ the function $1_v \delta_v \in D(X) \rtimes G$.

**Theorem 3.11.** Let $E$ be a graph. Then $L_K(E)$ is isomorphic to $D(X) \rtimes G$ via an isomorphism $\phi$ such that, for all $e \in E^1$, $\phi(e) = 1_e \delta_e$, $\phi(e^*) = 1_{e^{-1}} \delta_{e^{-1}}$ and, for all $v \in E^0$, $\phi(v) = 1_v \delta_v$.

**Proof.** Consider the family $\{1_e \delta_e, 1_{e^{-1}} \delta_{e^{-1}}, 1_v \delta_v : e \in E^1, v \in E^0\}$ in $D(X) \rtimes G$. This family satisfies the relations defining the Leavitt path algebra $L_K(E)$. To give
the reader an idea of the techniques involved, we show how to prove the Cuntz–
Krieger relation.

Let \( v \in E^0 \) be such that \( 0 < \# \{ e : s(e) = v \} < \infty \). Since \( X_v = \bigcup_{e \in E_1 : s(e) = v} X_e \), we have that

\[
\sum_{e \in E_1 : s(e) = v} 1_e \delta_e 1_{e^{-1}} \delta_{e^{-1}} = \sum_{e \in E_1 : s(e) = v} 1_e \delta_{s(e)} = \left( \sum_{e \in E_1 : s(e) = v} 1_e \right) \delta_v = 1_v \delta_v.
\]

Hence, by the universal property of \( L_K(E) \) and the fact that \( L_K(E) \cong D_0 \times F \) via an isomorphism such that, for all \( e \in E_1, \phi(e) = 1_e \delta_e, \phi(e^*) = 1_{e^{-1}} \delta_{e^{-1}} \) and, for all \( v \in E^0, \phi(v) = 1_v \delta_v \) (see Sec. 2), we obtain an unique homomorphism

\[
\Gamma : (D_0 \times F) \to D(X) \times G
\]

such that \( \Gamma(1_e \delta_e) = 1_e \delta_e, \Gamma(1_{e^{-1}} \delta_{e^{-1}}) = 1_{e^{-1}} \delta_{e^{-1}}, \) for all \( e \in E_1 \), and \( \Gamma(1_v \delta_v) = 1_v \delta_v \), for all \( v \in E^0 \).

It is clear that \( \Gamma \) is surjective, since \( \{1_e \delta_e, 1_{e^{-1}} \delta_{e^{-1}} : e \in E_1\} \) and \( \{1_v \delta_v : v \in E^0\} \) generates \( D(X) \times G \). We will show that \( \Gamma \) is injective.

Let \( x \in \text{Ker} \Gamma \). Then we can write \( x = a_0 \delta_0 + \sum a_g \delta_g \), where \( a_g \in D_g \) and \( a_0 = (\sum \lambda_{ab}^{-1} 1_{ab}^{-1} + \sum \beta_w 1_w) \in D_0 \).

For \( 0 \neq g \), since \( \Gamma \) is a homomorphism, it is straightforward to check that \( \Gamma(a_g \delta_g) = a_g \delta_g \). We need to compute \( \Gamma(a_0 \delta_0) \).

Note that given \( 1_a \delta_0 \in D_0 \delta_0, \) with \( a \in W \), we have that

\[
1_a \delta_a \cdot 1_{a^{-1}} \delta_{a^{-1}} = a_a (a_{a^{-1}} (1_a) 1_{a^{-1}}) \delta_0
= a_a (1_a^{-1} 1_a^{-1}) \delta_0
= 1_a \delta_0
\]

and hence

\[
\Gamma(1_a \delta_0) = \Gamma(1_a \delta_a) \Gamma(1_{a^{-1}} \delta_{a^{-1}})
= 1_a \delta_a \cdot 1_{a^{-1}} \delta_{a^{-1}}
= a_a (a_{a^{-1}} (1_a) 1_{a^{-1}}) \delta_{a a^{-1}}
= a_a (1_{a^{-1}}) \delta_{a (a)}
= 1_a \delta_{(a)}.
\]

Therefore, since \( 1_c d^{-1} = 1_c \) and \( 1_d^{-1} = 1_r(d) \) for \( c, d \in W \), we have that

\[
\Gamma(\alpha_0 \delta_0) = \Gamma \left( \left( \sum \lambda_{ab}^{-1} 1_{ab}^{-1} + \sum \beta_w 1_w \right) \delta_0 \right)
= \sum \lambda_{ab}^{-1} 1_{ab}^{-1} \delta_{\alpha (a)} + \sum \beta_w 1_w \delta_w
= \sum v \in V' v \delta_v,
\]

where \( V' = \{ v \in V^0 : v = s(ab^{-1}) \) for \( \lambda_{ab}^{-1} \) or \( \beta_w \neq 0 \}.\)
We conclude that
\[ \Gamma(x) = 0 \iff \sum_{v \in V'} 1_v a_0 \delta_v + \sum_{g \in G} a_g \delta_g = 0 \]
\[ \iff a_0 = 0 \text{ and } a_g = 0, \text{ for all } g \]
and hence \( \Gamma \) is injective as desired. \( \square \)

4. Applications to Free Path Groupoid Graded Isomorphisms of Leavitt Path Algebras

Given two graphs, say \( E_1 \) and \( E_2 \), by the results of the previous section, we can see the associated Leavitt path algebras as groupoid graded rings. In this section, we show that if there exists a homomorphism between the associated free path groupoids, say \( G_1 \) and \( G_2 \), that preserves the set of finite words and is injective in this set, then there exists a (free path) groupoid graded isomorphism between the Leavitt path algebras (which preserves the generators). Furthermore, in the presence of condition (L) for \( E_1 \), we show the converse of our statement, that is, we show that if there exists a free path groupoid graded isomorphism, that preserves generators, from \( L_K(E_1) \) to \( L_K(E_2) \), then there exists an homomorphism between the associated free path groupoids that preserves the set of finite words and is injective in this set.

We start the work by giving a few necessary definitions and showing a series of auxiliary results.

**Definition 4.1.** Let \( G \) and \( H \) be groupoids. A map \( h : G \to H \) is a groupoid homomorphism if \((g_1, g_2) \in G^2\) implies that \((h(g_1), h(g_2)) \in H^2\) and \( h(g_1g_2) = h(g_1)h(g_2) \).

**Definition 4.2.** Given a graph \( E \), let \( G \) be the associated free path groupoid and \( \theta = (\{X_g\}_{g \in G}, \{\theta_g\}_{g \in G}) \) the partial groupoid action of \( G \) in the set \( X \), as defined in Section 3. Define \( S \subseteq G \) by
\[ S = \{ g \in G : X_g \neq \emptyset \}. \]

Throughout this section, for \( i = 1, 2 \), \( E_i = (E_i^0, E_i^1, r, s) \) is a graph, \( G_i \) is the free path groupoid associated to \( E_i \) (as defined in Definition 3.4), \( W_i \subseteq G_i \) is the set of all finite paths in \( E_i \) (see Remark 3.1), \( \theta_i = (\{X_g\}_{g \in G_i}, \{\theta_g\}_{g \in G_i}) \) is the partial groupoid action of \( G_i \) in the set \( X_i \), as defined in Sec. 3. \( S_i \) is the set associated with \( G_i \), as defined in Definition 4.2 and \( h : G_1 \to G_2 \) is a groupoid homomorphism. Furthermore, we will see make use of the identifications given in Remark 3.5 without warning, in order to not overload the notation.

**Lemma 4.3.** Let \( G_i \) and \( W_i \) be as above and \( h : G_1 \to G_2 \) be a groupoid homomorphism. Then for all \( v \in E^0_i \), we have that \( h(v) \in E^0_2 \). Furthermore, for all \( \alpha \in W_i \), we have that \( h(s(\alpha)) = s(h(\alpha)) \) and \( h(r(\alpha)) = r(h(\alpha)) \).
Proof. Let $\beta \in G_2$ be such that $h(v) = \beta$, where $v \in E_1^0$. Then

$$\beta = h(v) = hv = h(v)h(v) = \beta \beta$$

and therefore $\beta \in E_2^0$.

Now let $\alpha \in W_1$. Since $h(\alpha) = h(s(\alpha)) = h(s(\alpha))h(\alpha)$, we have that $h(s(\alpha)) = h(s(\alpha))$. The equality involving $r(\alpha)$ follows analogously. \hfill \square

Corollary 4.4. Given $\alpha \in W_1$, it follows that $h(\alpha^{-1}) = h(\alpha)^{-1}$ and hence, for every $(\alpha, \beta^{-1}) \in G_2^1$, it holds that $h(\alpha \beta^{-1}) = h(\alpha) h(\beta)^{-1}$.

Assuming that a homomorphism $h : G_1 \rightarrow G_2$ is injective in $W_1$, we can also prove the following:

Lemma 4.5. Suppose that $h : G_1 \rightarrow G_2$ is a groupoid homomorphism that is injective in $W_1$. Then $(\alpha, \beta) \in G_2^1$ if, and only if, $(h(\alpha), h(\beta)) \in G_2^2$. Furthermore, for every $\alpha, \beta \in W_1$, we have that $r(\alpha) = r(\beta)$ if, and only if, $r(h(\alpha)) = r(h(\beta))$.

Proof. Suppose that $(\alpha, \beta) \notin G_2^1$. Then $r(\alpha) \neq s(\beta)$ and, since $h$ is injective in $W_1$, it follows that

$$r(h(\alpha)) = h(r(\alpha)) \neq h(s(\beta)) = h(h(\beta))$$

and hence $(h(\alpha), h(\beta)) \notin G_2^2$.

Now suppose that $r(\alpha) = r(\beta)$. Then $(\alpha, \beta^{-1}) \in G_2^1$ and, since $h$ is a homomorphism, we have that $(h(\alpha), h(\beta^{-1})) \in G_2^2$. Therefore, $r(h(\alpha)) = s(h(\beta^{-1})) = r(h(\beta^{-1}))^{-1} = r(h(\beta))$.

Finally, suppose that $r(\alpha) \neq r(\beta)$. Then, since $h$ is injective in $W_1$, we have that $h(r(\alpha)) \neq h(r(\beta))$ and hence

$$r(h(\alpha)) = h(r(\alpha)) \neq h(r(\beta)) = h(h(\beta)).$$

If we further assume that a homomorphism $h$ (as in the previous lemma) preserves the set of finite paths in the graph $E_1$, then we can prove the following:

Lemma 4.6. Let $h : G_1 \rightarrow G_2$ be a groupoid homomorphism such that $h$ is injective in $W_1$ and $h(W_1) = W_2$. Then, for every $\alpha \in W_1$, we have that $|\alpha| = 1$ if and only if $|h(\alpha)| = 1$.

Proof. Let $e \in E_1^1$. Suppose that $h(e) = \alpha_1 \alpha_2 \in W_2$, with $\alpha_1, \alpha_2 \notin E_2^0$. Then there exist $f_1, f_2 \in W_1$ such that $h(f_1) = \alpha_1$ and $h(f_2) = \alpha_2$. Note that by Lemma 4.3 $f_1, f_2 \notin E_1^0$. Therefore, since

$$h(f_1 f_2) = h(f_1)h(f_2) = \alpha_1 \alpha_2 = h(e),$$

we have that $f_1 f_2 = e$ and hence $f_1 \in E_1^0$ or $f_2 \in E_1^0$, what contradicts our assumption. Now, if $h(e) \in E_2^0$ then $h(e e) = h(e) h(e) = h(e)$ and hence $e \in E_1^0$, which is a contradiction. The only possibility left is that $|h(e)| = 1$. 


Conversely, suppose that $\alpha \in W_1$ is such that $|h(\alpha)| = 1$. Notice that if $\alpha \in E^0_1$ then $h(\alpha) \in E^0_2$ and $|h(\alpha)| = 0$. So suppose that $|\alpha| = n, n \geq 2$. Then $\alpha = \alpha_1 \cdots \alpha_n$, for $\alpha_i \in E^1_1$, and hence $h(\alpha) = h(\alpha_1) \cdots h(\alpha_n) = h(\alpha_1) \cdots h(\alpha_n)$. So $1 = |h(\alpha)| = |h(\alpha_1) \cdots h(\alpha_n)| = n$, a contradiction. Therefore, $|\alpha| = 1$.

Corollary 4.7. Let $h : G_1 \to G_2$ be a groupoid homomorphism as in the lemma above. Then $|\alpha| = |h(\alpha)|$ for all $\alpha \in W_1$.

With the above conditions on the homomorphism $h : G_1 \to G_2$, we can actually show that $h$ is a bijection. We do this below.

Proposition 4.8. Suppose that $h : G_1 \to G_2$ is a groupoid homomorphism such that $h|_{W_1}$ is injective and $h(W_1) = W_2$. Then $h$ is bijective.

Proof. First, we prove that $h$ is surjective. Let $\gamma \in G_2$. Then we can write $\gamma = \gamma_1 \cdots \gamma_n$, where $\gamma_i \in E^0_2 \cup E^1_2 \cup (E^2_2)^\ast$. Since $h(W_1) = W_2$, $h(W^n_1) = W^n_2$ and $h(E^0_1) = E^0_2$, we have that for each $i$, there exists an unique $e_i \in W_1 \cup W^*_1$ such that $\gamma_i = h(e_i)$. By Lemma 4.6, we have that $e_i \in E^0_1 \cup E^1_2 \cup (E^2_1)^\ast$ and, by Lemma 4.5, the element $e_1 \cdots e_n \in G_1$. Therefore

$$h(e_1 \cdots e_n) = h(e_1) \cdots h(e_n) = \gamma_1 \cdots \gamma_n = \gamma.$$  

Now suppose that there exists another $f_1 \cdots f_k \in G_1$ such that $\gamma = h(f_1) \cdots h(f_k)$. Clearly $k \geq n$, since $\gamma$ is in reduced form. If $k > n$, then $h(f_1) \cdots h(f_k)$ is not in reduced form and therefore there exists $j$ such that $h(f_j)h(f_{j+1})$ is equal to either $s(h(f_j))$ or $r(h(f_{j+1}))$. Suppose, without loss of generality, that $h(f_j)h(f_{j+1}) = s(h(f_j))$. Then $h(f_jf_{j+1}) = h(s(f_j))$ and, since $h$ is injective in $W_1$, we have that $f_jf_{j+1} = s(f_j)$. Hence $f_1 \cdots f_k$ is not in reduced form, what is a contradiction? Therefore $k = n$ and it follows by Lemma 4.6 that $h(f_i) = \gamma_i = h(e_i)$ for all $i$ and hence $f_i = e_i$ for all $i$ (from injectivity of $h$ in $W_1$).

Our main results in this section concern groupoid graded homomorphisms. We give this definition below and then proceed to prove the two theorems of the section.

Definition 4.9. Let $A = \bigoplus_{g \in G_1} A_g$ and $B = \bigoplus_{h \in G_2} B_h$ be graded algebras by the groupoids $G_1$ and $G_2$, respectively. A graded groupoid homomorphism between $A$ and $B$ is a pair $(\varphi, h)$, where $\varphi : A \to B$ is an algebra homomorphism, $h : G_1 \to G_2$ is a groupoid homomorphism and $\varphi(A_g) \subseteq B_{h(g)}$, for all $g \in G_1$.

Theorem 4.10. Let $E_1$ and $E_2$ be graphs and $G_1, G_2$ the associated free path groupoids, respectively. If there exists a groupoid homomorphism $h : G_1 \to G_2$ such that $h|_{W_1}$ is injective and $h(W_1) = W_2$ (where $W_i$ denotes the set of finite paths in $E_i$, $i = 1, 2$), then the associated partial skew groupoid rings $D(X_1) \rtimes G_1$ and $D(X_2) \rtimes G_2$ are groupoid graded isomorphic, via an isomorphism $\varphi : D(X_1) \rtimes G_1 \to D(X_2) \rtimes G_2$ such that $\varphi(1_e \delta_e) = 1_{h(e)} \delta_{h(e)}$ and $\varphi(1_e \delta_e) = 1_{h(e)} \delta_{h(e)}$. 

Consider in $D(X_2) \times G_2$ the family

$$\mathcal{F} = \{1_{h(e)}\delta_{h(e)}, 1_{h(e^{-1})}\delta_{h(e^{-1})}, 1_{h(v)}\delta_{h(v)} : v \in E_1^0, e \in E_1^1\}.$$ 

We will show that $\mathcal{F}$ satisfy the relations defining the Leavitt path algebra $L_K(E_1)$ (which is isomorphic to $D(X_1) \times G_1$).

(i) Since for every vertex $v$, the action $\alpha_v$ is the identity we have that

$$1_{h(s(e))}\delta_{h(s(e))} \cdot 1_{h(e)}\delta_{h(e)} = 1_{s(h(e))}\delta_{s(h(e))} \cdot 1_{h(e)}\delta_{h(e)}
= \alpha_{s(h(e))}(\alpha^{-1}_{s(h(e))}(1_{s(h(e))})1_{h(e)})\delta_{s(h(e))}h(e)
= 1_{h(e)}\delta_{h(e)}.$$ 

Also,

$$1_{h(e)}\delta_{h(e)} \cdot 1_{r(h(e))}\delta_{r(h(e))} = 1_{h(e)}\delta_{h(e)} \cdot 1_{r(h(e))}\delta_{r(h(e))}
= \alpha_{h(e)}(\alpha^{-1}_{h(e)}(1_{h(e)})1_{r(h(e))})\delta_{r(h(e))}h(e)
= \alpha_{h(e)}(1_{h(e)}1_{r(h(e))})\delta_{r(h(e))}h(e)
= \alpha_{h(e)}(1_{h(e)}\delta_{h(e)}h(e)
= 1_{h(e)}\delta_{h(e)}.$$ 

(ii) For the second relation, note that

$$1_{h(r(e))}\delta_{h(r(e))} \cdot 1_{h(e^{-1})}\delta_{h(e^{-1})} = 1_{r(h(e))}\delta_{r(h(e))} \cdot 1_{h(e^{-1})}\delta_{h(e^{-1})}
= \alpha_{r(h(e))}(\alpha^{-1}_{r(h(e))}(1_{r(h(e))})1_{h(e^{-1})})\delta_{r(h(e))}h(e^{-1})
= 1_{h(e^{-1})}\delta_{h(e^{-1})}$$

and

$$1_{h(e^{-1})}\delta_{h(e^{-1})} \cdot 1_{s(h(e))}\delta_{s(h(e))} = 1_{h(e^{-1})}\delta_{h(e^{-1})} \cdot 1_{s(h(e))}\delta_{s(h(e))}
= \alpha_{h(e^{-1})}(\alpha^{-1}_{h(e^{-1})}(1_{h(e^{-1})})1_{s(h(e))})\delta_{h(e^{-1})}s(h(e^{-1}))
= \alpha_{h(e^{-1})}(1_{h(e^{-1})}1_{s(h(e))})\delta_{s(h(e^{-1})}
= \alpha_{h(e^{-1})}(1_{h(e^{-1})}\delta_{h(e^{-1})}
= 1_{h(e^{-1})}\delta_{h(e^{-1})}.$$ 

(iii) Next we check the third relation:

$$1_{h(e^{-1})}\delta_{h(e^{-1})}1_{h(g)}\delta_{h(g)} = \alpha_{h(e^{-1})}(\alpha^{-1}_{h(e^{-1})}(1_{h(e^{-1})})1_{h(g)})\delta_{h(e^{-1})}h(g)
= \alpha_{h(e^{-1})}(\alpha_{h(e)}(1_{h(e^{-1})})1_{h(g)})\delta_{h(e^{-1})}h(g).$$
Theorem 4.11. Let $h$ be an exit, we can prove the converse of the above result. We do this in two steps.

Namely, consider in $G$ such that $h \in G$ (note that $h$ then $h$).

(iv) Finally, we check the Cuntz–Krieger relation. Let $e \in E^0_1$ be such that $0 < \# \{ e : s(e) = v \} < \infty$. Note that

$$\{ e : s(e) = v \} = \{ e : h(s(e)) = h(v) \} = \{ e : s(h(e)) = h(v) \}.$$ 

Furthermore,

$$\sum_{s(e) = v} 1_h \delta_{h(e)} 1_h \delta_{h(e)} = \sum_{s(e) = v} 1_h \delta_{s(h(e))} = \sum_{s(e) = v} 1_h \delta_{h(v)}.$$ 

Therefore, it is enough to show that $\sum_{e : s(e) = v} 1_h = 1_h(v)$. But this follows since

$$\{ e \in E^1_2 : s(e) = h(v) \} = \{ h(f) : f \in E^1_2 \text{ and } s(f) = v \}$$

and $1_h(v) = \sum_{f : s(f) = h(v)} 1_f$.

We conclude that there exists a homomorphism $\varphi : D(X_1) \rtimes G_1 \to D(X_2) \rtimes G_2$ such that $\varphi(1_e \delta_e) = 1_h \delta_{h(e)}$, $\varphi(1_e \delta_{e^{-1}}) = 1_h \delta_{h(e^{-1})}$, and $\varphi(1_e \delta_{e}) = 1_h \delta_{h(e)}$. To show that this $\varphi$ is bijective, we construct its inverse in an analogous way to what was done above. Namely, consider in $D(X_1) \rtimes G_1$ the family

$$\mathcal{H} = \{ 1_{h^{-1}(e)} \delta_{h^{-1}(e)} 1_{h^{-1}(e)} \delta_{h^{-1}(e)} : v \in E^0_2, e \in E^1_2 \}.$$ 

Since $h : G_1 \to G_2$ is a bijection such that $h^{-1}(W_2) = W_1$ and $h^{-1} |_{W_1} = W_1$, we obtain, proceeding as above, the inverse homomorphism of $\varphi$.

Under condition (L) for the graph $E_1$, that is, when every closed cycle in $E_1$ has an exit, we can prove the converse of the above result. We do this in two steps.

Theorem 4.11. Let $E_1$ and $E_2$ be two graphs and suppose that $E_1$ satisfies condition (L). If $(\varphi, h)$ is a groupoid graded homomorphism between the associated partial skew groupoid rings, where $\varphi : D(X_1) \rtimes G_1 \to D(X_2) \rtimes G_2$ is an isomorphism and $h : G_1 \to G_2$ is a homomorphism such that $\varphi(1_e \delta_e : v \in E^0_1) = \{ 1_w \delta_w : w \in E^0_2 \}$, then $h \mid_{W_1}$ is injective and $h(S_1) = S_2$ (where $S_i$ are as in Definition 4.2).

Proof. We show first that $h(S_1) = S_2$.

Given $g \in S_1$ we have that $1_g \delta_g \neq 0$ in $D(X_1) \rtimes G_1$. Therefore, since $\varphi$ is injective, $0 \neq 1_g \delta_g \in D(h(g) \delta_{h(g)}$ and hence $h(g) \in S_2$.

For the other inclusion, let $c \in S_2$. Then $1_e \delta_e \neq 0$ and, since $\varphi$ is an isomorphism, there exists $x = \sum \alpha_g \delta_g$ such that $1_e \delta_e = \varphi(x) = \sum \varphi(\alpha_g \delta_g)$. Since $\varphi(\alpha_g \delta_g) \in D(h(g) \delta_{h(g)}$ for all $g$, we have that there exists $g$ such that $h(g) = c$ and $\varphi(\alpha_g \delta_g) \neq 0$ (note that $g \in S_1$, since $\alpha_g \delta_g \neq 0$).
Next, we prove that $h|_{W_1}$ is injective. We divide this in three steps. Recall that, by Lemma 4.3, $h(E_1^0) \subseteq E_2^0$.

**Step 1:** Show that $h$ is injective in the vertices, that is, $h|_{E_1^0}$ is injective. Let $v, w \in E_1^0$. Notice that

$$\varphi(1_v \delta_v 1_w \delta_w) = \varphi(1_v \delta_v) \varphi(1_w \delta_w) = 1_h(v) \delta_h(v) \cdot 1_h(w) \delta_h(w) = \delta_h(v) h(v) \delta_h(w)$$

On the other hand, $\varphi(1_v \delta_v 1_w \delta_w) = \varphi(\delta_v w 1_v \delta_v) = \delta_v w 1_h(v) \delta_h(v)$. Therefore, $h(v) = h(w)$ if, and only if, $v = w$.

**Step 2:** Let $\alpha \in S_1$. Show that if $h(\alpha) \in E_2^0$ then $\alpha \in E_1^0$.

Suppose that $\alpha \in S_1$ is such that $h(\alpha) \in E_2^0$. Then we have that

$$h(s(\alpha)) = s(h(\alpha)) = r(h(\alpha)) = h(r(\alpha)).$$

Therefore, since $h$ is injective on the vertices (see Step 1), we have that $s(\alpha) = r(\alpha)$ and $\alpha$ must be a closed cycle.

Next note that, since the partial action on the ideals associated to vertices is the identity, we have that

$$\varphi(1_v \delta_v 1_{\alpha^{-1}} \delta_{\alpha^{-1}}) = \varphi(1_v \delta_\alpha) \varphi(1_{\alpha^{-1}} \delta_{\alpha^{-1}}) = a b \delta_h(\alpha)$$

where $a, b \in D_h(\alpha) = D_h(\alpha^{-1})$ are such that $\varphi(1_v \delta_\alpha) = a \delta_h(\alpha)$ and $\varphi(1_{\alpha^{-1}} \delta_{\alpha^{-1}}) = b \delta_h(\alpha^{-1}) = b \delta_h(\alpha)$. Analogously, $\varphi(1_{\alpha^{-1}} \delta_{\alpha^{-1}} 1_\alpha \delta_{\alpha^{-1}}) = b a \delta_h(\alpha) = a b \delta_h(\alpha)$. Since $\varphi$ is an isomorphism, we have that $1_\alpha \delta_\alpha 1_{\alpha^{-1}} \delta_{\alpha^{-1}} = 1_{\alpha^{-1}} \delta_{\alpha^{-1}} 1_\alpha \delta_{\alpha^{-1}}$, that is,

$$1_\alpha \delta_s(\alpha) = 1_{\alpha^{-1}} \delta_r(\alpha).$$

Hence, since $s(\alpha) = r(\alpha)$, the above equality implies that $1_\alpha = 1_{\alpha^{-1}} = 1_r(\alpha)$, what can only happen if $\alpha \in E_1^0$ (as we show below).

Suppose that $\alpha \in W_1$ or $\alpha \in W_1^*$. Without loss of generality, we can assume that $\alpha \in W_1$. Since $E_1$ satisfies condition $(L)$, the cycle $a$ has an exit $e \in E_1^1$. So we can choose a subpath $\alpha'$ of $\alpha$ such that $\alpha' = \alpha'_{\alpha' e} \alpha'_{e}$. Then $1_\alpha(\alpha' e) = 0$ and $1_r(\alpha)(\alpha' e) = 1$ and hence $\alpha \notin W_1 \cup W_1^*.$

Finally, suppose that $\alpha = \beta \gamma^{-1}$, where $\beta, \gamma \in W_1 \setminus E_1^0$ and $r(\beta) = r(\gamma)$, in reduced form. Then $1_\beta = 1_{\gamma} = 1_{\alpha^{-1}} = 1_\gamma$, what is impossible.

We conclude that $\alpha \in E_1^0$ as desired.

**Step 3:** Show that $h$ is injective in $W_1$.

Let $\alpha$ and $\beta \in W_1$ be such that $h(\alpha) = h(\beta)$. Then

$$h(r(\alpha)) = r(h(\alpha)) = r(h(\beta)) = h(r(\beta)),$$

and by Step 1 we have that $r(\alpha) = r(\beta)$. Therefore, $\alpha \beta^{-1}$ is composable and

$$h(\alpha \beta^{-1}) = h(\alpha) h(\beta)^{-1} = h(\alpha) h(\alpha)^{-1} = s(h(\alpha)) \in E_2^0.$$

By Step 2 we have that $\alpha \beta^{-1} \in E_1^0$ and hence $\alpha = \beta$ as desired.

**Corollary 4.12.** If in Theorem 4.11 we further assume that $\varphi(\{1_c \delta_c : e \in E_1^1\}) = \{1_f \delta_f : f \in E_2^1\}$, then $h(W_1) = W_2$. 

Free path groupoid grading on Leavitt path algebras

Proof. Notice that with the additional hypothesis, since \((\varphi, h)\) is a graded homomorphism, then \(\varphi(1_e \delta_e) = 1_{h(e)} \delta_{h(e)}\), for all \(e \in E_1^1\).

Let \(e_1 \cdots e_n \in W_1\). Then \(h(e_1 \cdots e_n) = h(e_1) \cdots h(e_n)\) and since \(h(e_i) \in E_2^1\), for all \(i \in \{1, \ldots, n\}\), we have that \(h(e_1 \cdots e_n) \in W_2\) and hence \(h(W_1) \subseteq W_2\).

For the other inclusion, let \(f \in E_2^1\). Since \(\varphi(\{1_e \delta_e : e \in E_1^1\}) = \{1_f \delta_f : f \in E_1^2\}\), there exists \(e \in E_1^1\) such that \(1_f \delta_f = \varphi(1_e \delta_e) = 1_{h(e)} \delta_{h(e)}\) and hence \(h(e) = f\). So, given \(f_1 \cdots f_n \in W_2\) there exist \(e_1, \ldots, e_n \in W_1\) such that \(h(e_i) = f_i\) and hence, by Lemma 4.5

\[
f_1 \cdots f_n = h(e_1) \cdots h(e_n) = h(e_1 \cdots e_n).
\]

Therefore, \(W_2 \subseteq h(W_1)\) as desired. \(\square\)

We end our work with an example that shows that the additional hypothesis included in Corollary 4.12 is essential.

Example 4.13. Let \(E_1 = \{E_1^0, E_1^1, r_1, s_1\}\), where \(E_1^0 = \{v_1, v_2, v_3\}\), \(E_1^1 = \{e_1, e_2\}\), \(r_1(e_1) = v_2 = s_1(e_2)\), \(s_1(e_1) = v_1\) and \(r_1(e_2) = v_3\). We picture this graph below.

\[
\begin{array}{c}
v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3.
\end{array}
\]

Let \(E_2 = \{E_2^0, E_2^1, r_2, s_2\}\), where \(E_2^0 = \{w_1, w_2, w_3\}\), \(E_2^1 = \{f_1, f_2\}\), \(r_2(f_1) = w_3 = r_2(f_2)\), \(s_2(f_1) = w_1\) and \(s_2(f_2) = w_2\). We picture this graph below.

\[
\begin{array}{c}
w_1 \xrightarrow{f_1} w_3 \xrightarrow{f_2} w_2.
\end{array}
\]

Note that \(G_1 = \{v_1, v_2, v_3, e_1, e_2, e_1^{-1}, e_2^{-1}, e_1 e_2, e_2^{-1} e_1^{-1}\}\).

Define \(h : G_1 \to G_2\) so that \(h(e_1) = f_1^{-1}\) and \(h(e_2) = f_1 f_2^{-1}\). Note that

\[
\begin{align*}
h(v_1) &= h(e_1 e_1^{-1}) = h(e_1) h(e_1)^{-1} = f_1^{-1} f_1 = r(f_1) = w_3, \\
h(v_2) &= h(e_2^{-1} e_1) = h(e_1)^{-1} h(e_1) = f_1 f_2^{-1} = s(f_1) = w_1, \\
h(v_3) &= h(e_2^{-1} e_2) = h(e_2)^{-1} h(e_2) = (f_1 f_2^{-1})^{-1} (f_1 f_2^{-1}) \\
&= f_2 (f_1 f_2^{-1}) = s(f_2) = w_2.
\end{align*}
\]

Using the universal property of Leavitt path algebras, one can construct a groupoid graded isomorphism \((\varphi, h)\), where \(\varphi : L_K(E_1) \to L_K(E_2)\) is such that \(\varphi(1_e \delta_e) = 1_f \delta_f^{-1}\), \(\varphi(1_e \delta_{e_2}) = 1_{f_1} f_2^{-1} \delta_{f_1} f_2^{-1}\), \(\varphi(1_{v_1} \delta_{v_3}) = 1 w_3 \delta w_3\), \(\varphi(1_{v_2} \delta_{v_2}) = 1 w_2 \delta w_2\).

Notice that \(\varphi(\{1_{v_1} \delta_{v_1}, 1_{v_2} \delta_{v_2}, 1_{v_3} \delta_{v_3}\}) = \{1 w_1 \delta w_1, 1 w_2 \delta w_2, 1 w_3 \delta w_3\}\), but \(\varphi(\{1_{v_3} \delta_{e_1}, 1_{v_2} \delta_{e_2}\}) \neq \{1_{f_1} \delta_{f_1}, 1_{f_2} \delta_{f_2}\}\), that is, the additional hypothesis of Corollary 4.12 is not satisfied. Furthermore, it is clear that \(h(W_1) \neq W_2\).
Acknowledgments

The first author was partially supported by CNPq-Brazil.

References

[1] G. Abrams and G. Aranda-Pino, The Leavitt path algebra of a graph, *J. Algebra* **293** (2005) 319–334.
[2] G. Abrams and G. Aranda-Pino, The Leavitt path algebras of arbitrary graphs, *Houston J. Math.* **34**(2) (2008) 423–442.
[3] G. Abrams and M. Tomforde, Isomorphism and Morita equivalence of graph algebras, *Trans. Amer. Math. Soc.* **363** (2011) 3733–3767.
[4] D. Bagio, D. Flores and A. Paques, Partial actions of ordered groupoids on rings, *J. Alg. and its Appl.* **9**(3) (2010) 501–517.
[5] D. Bagio and A. Paques, Partial Groupoid Actions: Globalization, Morita theory, and Galois theory, *Commun. Algebra* **40** (2012) 3658–3678.
[6] N. Brownlowe, T. M. Carlsen, M. Whittaker and T. M. Carlsen, Graph algebras and orbit equivalence, *Erg. Th. and Dyn. Sys.* doi:10.1017/etds.2015.52, to appear.
[7] V. Beuter and D. Gonçalves, Partial crossed products as equivalence relation algebras, *Rocky Mountain J. Math.* **46** (2016) 85–104.
[8] X. Chen, Irreducible representations of Leavitt path algebras, *Forum Mathematicum* **27**(1) (2012) 549–574.
[9] L. O. Clark, C. Farthing, A. Sims and M. Tomforde, A groupoid generalization of Leavitt path algebras, *Semigroup Forum* **89** (2014) 501–517.
[10] L. O. Clark and A. Sims, Equivalent groupoids have Morita equivalent Steinberg algebras, *J. Pure Appl. Algebra* **219** (2015) 2062–2075.
[11] M. Dokuchaev and R. Exel, Associativity of crossed products by partial actions, enveloping actions and partial representations, *Tans. Amer. Math. Soc.* **357** (2005) 1931–1952.
[12] R. Exel, Circle actions on C*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences, *J. Funct. Anal.* **122**(3) (1994) 361–401.
[13] R. Exel, M. Laca and J. Quigg, Partial dynamical systems and C*-algebras generated by partial isometries, *J. Operator Theory* **47** (2002) 169–186.
[14] D. Gonçalves, Simplicity of partial skew group rings of abelian groups, *Canad. Math. Bull.* **57**(3) (2014) 511–519.
[15] D. Gonçalves, J. Oinert and D. Royer Simplicity of partial skew group rings with applications to Leavitt path algebras and topological dynamics, *J. Algebra* **420** (2014) 201–216.
[16] D. Gonçalves and D. Royer, Leavitt path algebras as partial skew group rings, *Comm. Algebra* **42** (2014) 3578–3592.
[17] D. Gonçalves and D. Royer On the representations of Leavitt path algebras, *J. Algebra* **333** (2011) 258–272.
[18] D. Gonçalves and D. Royer, Unitary equivalence of representations of graph algebras and branching systems, *Functional Anal. Appl.* **45** (2011) 117–127.
[19] D. Gonçalves and D. Royer, Branching systems and representations of Cohn-Leavitt path algebras of separated graphs, *J. Algebra* **422** (2015) 413–426.
[20] R. Hazrat, The graded structure of Leavitt path algebras, *Israel J. Math.* **195** (2013) 833–895.
[21] R. Hazrat and M. Rangaswamy, On graded irreducible representations of Leavitt path algebras, *J. Algebra* **450** (2016) 458–486.
[22] P. J. Higgins, Categories and Groupoids, *Reprints in Theory and Appl. of Categories*, 7 (1971) 1–195.
[23] K. Rangaswamy, Leavitt path algebras with finitely presented irreducible representations, *J. Algebra* 447 (2016) 624–648.
[24] M. Tomforde and E. Ruiz, Ideals in graph algebras, *Algebr. Represent. Theory* 17 (2014) 849–861.
[25] M. Tomforde and E. Ruiz, Ideal-related K-theory for Leavitt path algebras and graph C*-algebras, *Indiana Univ. Math. J.* 62(5) (2013) 1587–1620.
[26] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, *J. Algebra* 318 (2007) 270–299.