Mal’tsev algebras and triality

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Abstract. A concept of the Mal’tsev pair is presented. This concept is based on the generalized Maurer-Cartan equations of a local analytic Moufang loop. The triality can be seen as a fundamental property of such pairs. Based on triality, the Yamagutian is constructed. Properties of the Yamagutian are studied.

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1. Introduction
In [2], the generalized Maurer-Cartan equations for a local analytic Moufang loop were found. In this paper, we propose a concept of the Mal’tsev pair which is based on the generalized Maurer-Cartan equations of a local analytic Moufang loop and the structural properties of the Mal’tsev algebras [1]. Such a concept may also be inferred from the theory of alternative algebras [3, 4]. Triality can be seen as a fundamental property of such pairs. Based on triality, the Yamagutian is constructed. Properties of the Yamagutian are studied.

2. Mal’tsev pairs
Let $\mathcal{M}$ be an anti-commutative algebra and let $\mathcal{L}$ be a Lie algebra. Throughout this paper we assume that both algebras have the same base field $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$; as a matter of fact, only $\text{char} \mathbb{F} \neq 2, 3$ is essential). Denote by $(S, T)$ a pair of the linear maps $S, T : \mathcal{M} \to \mathcal{L}$.

Definition 2.1 (Mal’tsev pair). We call $(S, T)$ a Mal’tsev pair if

\[ [S_x, S_y] = S_{[y,x]} - 2[S_x, T_y] \]  \hspace{1cm} (2.1a)

\[ [T_x, T_y] = T_{[x,y]} - 2[T_x, S_y] \]  \hspace{1cm} (2.1b)

for all $x, y$ in $\mathcal{M}$.

We call (2.1a,b) the Mal’tsev relations. Note that the same brackets $[\cdot, \cdot]$ are used to denote multiplications in $\mathcal{M}$ and $\mathcal{L}$.

In an anti-commutative algebra $\mathcal{M}$ define the Jacobi map $J : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ by

\[ J(x, y, z) := [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \]
An anti-commutative algebra \( \mathcal{M} \) is said to be the Mal’tsev algebra [1] if the Mal’tsev identity holds in \( \mathcal{M} \):

\[
[J(x, y, z), x] = J(x, y, [x, z]), \quad \forall x, y, z \in M
\]

**Proposition 2.2.** Let \( (S, T) \) a Mal’tsev pair. Then

\[
[S_x, T_y] = [T_x, S_y]
\]

for all \( x, y \) in \( \mathcal{M} \).

**Proof.** Use anti-commutativity in \( L \) and \( \mathcal{M} \).

**Corollary 2.3** (minimality conditions). The Mal’tsev conditions read

\[
2[S_x, T_y] = S_{[y,x]} - [S_x, S_y] = T_{[x,y]} - [T_x, T_y] = 2[T_x, S_y]
\]

**Remark 2.4.** One can see that deviations of \( S \) from algebra map and \( T \) from anti-algebra may be considered as ”minimal” for the Mal’tsev pair.

3. Triality

**Proposition 3.1.** Let \( (S, T) \) be a Mal’tsev pair. Then \( (-T, -S) \) is a Mal’tsev pair as well.

**Proof.** By using anti-commutativity in \( M \), rewrite the Mal’tsev conditions of \( (S, T) \) as follows:

\[
[-T_x, -T_y] = -T_{[y,x]} - 2[-T_x, -S_y]
\]

\[
[-S_x, -S_y] = -S_{[x,y]} - 2[-S_x, -T_y]
\]

Consider a triple \( (S, T, P) \) of the linear maps \( S, T, P : M \to L \), such that

\[
S + T + P = 0
\]

With a given triple \( (S, T, P) \) we can associate the pairs

\[
(S, T), \quad (-T, -S), \quad (T, P), \quad (-P, -T), \quad (P, S), \quad (-S, -P) \quad (3.1)
\]

**Lemma 3.2** (triality). Let some pair from (3.1) be a Mal’tsev pair. Then all other pairs from (3.1) are the Mal’tsev pairs as well.

**Proof.** Assume that \( (S, T) \) is a Mal’tsev pair. Then we know from above Proposition 3.1 that \( (-T, -S) \) is a Mal’tsev pair. The required minimality conditions for \( (T, P) \) and \( (P, S) \) read, respectively,

\[
2[T_x, P_y] = T_{[y,x]} - [T_x, T_y] = P_{[x,y]} - [P_x, P_y] = 2[P_x, T_y]
\]

\[
2[P_x, S_y] = P_{[y,x]} - [P_x, P_y] = S_{[x,y]} - [S_x, S_y] = 2[S_x, P_y]
\]

As an example, calculate

\[
-2[S_x, P_y] = 2[P_x + T_x, P_y]
\]

\[
= 2[P_x, P_y] + 2[P_x, T_y]
\]
\[ [P_x, P_y] + [P_x, P_y] + 2[P_x, T_y] = [P_x, P_y] + [S_x + T_x, S_y + T_y] - 2[S_x + T_x, T_y] = [P_x, P_y] + [S_x, S_y] + [S_x, T_y] + [T_x, S_y] + [T_x, T_y] - 2[S_x, T_y] - 2[T_x, T_y] = [P_x, P_y] + [S_x, S_y] - [T_x, T_y] = [P_x, P_y] + S_{[y,x]} - 2[S_x, T_y] - T_{[x,y]} + 2[T_x, S_y] = [P_x, P_y] - P_{[x,y]} \]

All other required equalities in (3.1) can be verified in the same way. The result tells us that \((T, P)\) and \((P, S)\) are the Mal’tsev pairs, which in turn implies that \((-P, -T)\) and \((-S, -P)\) must be the Mal’tsev pairs as well.

It follows from Lemma 3.2 that a given Mal’tsev pair \((S, T)\) is invariant under the substitutions

\[
\sigma \doteq (S \to T \to P \to S) \\
\tau \doteq (S \to -T \to S)(P \to -P) \\
\sigma^2 = (S \to P \to T \to S) \\
\sigma \circ \tau = (S \to -P \to S)(T \to -T) \\
\sigma^2 \circ \tau = (T \to -P \to T)(S \to -S)
\]

which we call the *triality substitutions*. So it is natural to proclaim:

**Theorem 3.3** (principle of triality). *All algebraic consequences of the Mal’tsev conditions are triality invariant.*

Such a symmetry we call *triality*. It suggests that we should try to handle the Mal’tsev pairs in triality symmetric manner.

In particular, by using the *triality conjugation*,

\[
P^+ \doteq S - T = P + 2S = -P - 2T \\
S^+ \doteq T - P = S + 2T = -S - 2P \\
T^+ \doteq P - S = T + 2P = -T - 2S
\]

with the evident property

\[ S^+ + T^+ + P^+ = 0 \]

one can rewrite the Mal’tsev conditions as follows:

\[
[S_x, S_y^+] = [S_x^+, S_y] = S_{[x,y]} \\
[T_x, T_y^+] = [T_x^+, T_y] = T_{[x,y]} \\
[P_x, P_y^+] = [P_x^+, P_y] = P_{[x,y]}
\]

Note that

\[
3P = T^+ - S^+ = P^+ + 2T^+ = -P^+ - 2S^+ \\
3S = P^+ - T^+ = S^+ + 2P^+ = -S^+ - 2T^+ \\
3T = S^+ - P^+ = T^+ + 2S^+ = -T^+ - 2P^+
\]

which means that the triality conjugation is invertible.
4. Yamagutian

We introduced the triple \((S, T, P)\) via the triality symmetric identity (3.1). Following triality, it is natural to search for other but nontrivial triality invariant combinations of the maps from the triple \((S, T, P)\).

**Definition 4.1** (Yamagutian [5]). The *Yamagutian* of \((S, T)\) is the skew-symmetric bilinear map \(Y: M \otimes M \to L\) defined (cf ([5])) by

\[
6Y(x; y) = [S_x, S_y] + [T_x, T_y] + [P_x, P_y] = -Y(y; x)
\]

We can see the evident but important

**Proposition 4.2.** *The Yamagutian* \(Y\) *is triality invariant.*

By triality symmetry, the Yamagutian \(Y\) can be redefined in several useful ways. In particular,

\[
6Y(x; y) = 3[S_x, S_y] - S^{x,y} = 3[T_x, T_y] - T^{x,y} = 3[P_x, P_y] - P^{x,y}
\]

and one can also verify that

\[
6Y(x; y) = 2P^{x,y} - 6[S_x, T_y]
\]

Later we shall need the

**Proposition 4.3.** Let \((S, T)\) be a Mal’tsev pair. Then

\[
6Y(x; y) = [S^{x,y} + S^{x,y} + S^{x,y}] + [T^{x,y} + T^{x,y} + T^{x,y}] + [P^{x,y} + P^{x,y} + P^{x,y}]
\]

for all \(x, y\) in \(M\).

**Proof.** Due to triality, check only the last formula (4.1c):

\[
[P_x, P_y] = [S_x - T_x, S_y - T_y]
= [S_x, S_y] - 2[S_x, T_y] + [T_x, T_y]
= \frac{1}{3} Y(x; y) + \frac{1}{3} S^{x,y} + \frac{1}{3} Y(x; y) - \frac{2}{3} P^{x,y} + \frac{1}{3} Y(x; y) + \frac{1}{3} T^{x,y}
= Y(x; y) - P^{x,y}
\]
Remark 4.4. Formulae (4.1a–c) tell us that the Yamagutian $Y$ measures the deviation of $S^+$, $T^+$, and $P^+$ from the anti-algebra maps.

Corollary 4.5. We have

$$18Y(x, y) = [S^+_x, S^+_y] + [T^+_x, T^+_y] + [P^+_x, P^+_y]$$

Theorem 4.6. Let $(S, T)$ be a Mal’tsev pair. Then

$$[S_x, S_y] = 2Y(x; y) - \frac{1}{3}S_{[x,y]} - \frac{2}{3}T_{[x,y]}$$
$$[S_x, T_y] = -Y(x; y) - \frac{1}{3}S_{[x,y]} + \frac{1}{3}T_{[x,y]}$$
$$[T_x, T_y] = 2Y(x; y) + \frac{2}{3}S_{[x,y]} + \frac{1}{3}T_{[x,y]}$$

Proof. Evident.

Corollary 4.7. By triality, we have

$$[T_x, T_y] = 2Y(x; y) - \frac{1}{3}T_{[x,y]} - \frac{2}{3}P_{[x,y]}$$
$$[T_x, P_y] = -Y(x; y) - \frac{1}{3}T_{[x,y]} + \frac{1}{3}P_{[x,y]}$$
$$[P_x, P_y] = 2Y(x; y) + \frac{2}{3}T_{[x,y]} + \frac{1}{3}P_{[x,y]}$$

and

$$[P_x, P_y] = 2Y(x; y) - \frac{1}{3}P_{[x,y]} - \frac{2}{3}S_{[x,y]}$$
$$[P_x, S_y] = -Y(x; y) - \frac{1}{3}P_{[x,y]} + \frac{1}{3}S_{[x,y]}$$
$$[S_x, S_y] = 2Y(x; y) + \frac{2}{3}P_{[x,y]} + \frac{1}{3}S_{[x,y]}$$

Proposition 4.8. Let $(S, T)$ be a Mal’tsev pair. Then

$$6[Y(x; y), S_z] = 3[[S_x, S_y], S_z] - S_{[x,y],z}$$
$$6[Y(x; y), T_z] = 3[[T_x, T_y], T_z] - T_{[x,y],z}$$
$$6[Y(x; y), P_z] = 3[[P_x, P_y], P_z] - P_{[x,y],z}$$

for all $x, y, z$ in $M$.

Proof. Due to triality, only the first identity must be checked:

$$6[Y(x; y), S_z] = 3[[S_x, S_y], S_z] - S_{[x,y],z}$$
$$= 3[[S_x, S_y], S_z] - [S_{[x,y]}^+, S_z]$$
$$= 3[[S_x, S_y], S_z] - S_{[x,y],z}$$

□
Corollary 4.9. Adding formulae (4.2a–c) we obtain (cf (3.1)) the triality symmetric identity

\[ [[S_x, S_y], S_z] + [[T_x, T_y], T_z] + [[P_x, P_y], P_z] = 0 \]

Corollary 4.10. In (4.2a–c) make twice cyclic permutation of \( x, y, z \) and add the resulting equalities with the original ones. Then we obtain

\[
\begin{align*}
6[Y(x; y), S_z] &+ 6[Y(y; z), S_x] + 6[Y(z; x), S_y] = S_J(x,y,z) \\
6[Y(x; y), T_z] &+ 6[Y(y; z), T_x] + 6[Y(z; x), T_y] = T_J(x,y,z) \\
6[Y(x; y), P_z] &+ 6[Y(y; z), P_x] + 6[Y(z; x), P_y] = P_J(x,y,z)
\end{align*}
\]

Proposition 4.11. Let \((S, T)\) be a Mal’tsev pair. Then

\[
\begin{align*}
[S^+_x, S^+_y, S^+_z] &+ [S^+_y, S^+_z, S^+_x] + [S^+_z, S^+_x, S^+_y] = P^+_J(x,y,z) \\
[T^+_x, T^+_y, T^+_z] &+ [T^+_y, T^+_z, T^+_x] + [T^+_z, T^+_x, T^+_y] = T^+_J(x,y,z) \\
[P^+_x, P^+_y, P^+_z] &+ [P^+_y, P^+_z, P^+_x] + [P^+_z, P^+_x, P^+_y] = P^+_J(x,y,z)
\end{align*}
\]

for all \( x, y, z \) in \( M \).

Proof. Subtracting (4.2b) from (4.2a), we obtain

\[ 6[Y(x; y), P^+_z] = 3[[S_x, S_y], S_z] - 3[[T_x, T_y], T_z] - P^+_{[x,y],z} \]

On the other hand, using (4.2c), we have

\[ 6[Y(x; y), P^+_z] = [[[P^+_x, P^+_y], P^+_z] + [P^+_x, P^+_y], P^+_z] \]

and so we obtain

\[ [P^+_x, P^+_y] = -P^+_{[x,y],z} + 3[[S_x, S_y], S_z] - 3[[T_x, T_y], T_z] - [[P^+_x, P^+_y], P^+_z] \]

Now make twice cyclic permutation of \( x, y, z \) and add the resulting equalities with the original one. Then, using Jacobi conditions (in \( L \)) and the definition of \( J \) on the right hand-side of the resulting equality we obtain (4.3c). The remaining identities (4.3a,b) are evident from triality.

Lemma 4.12. Let \((S, T)\) be a Mal’tsev pair. Then

\[ Y(\{x, y\}; z) + Y(\{y, z\}; x) + Y(\{z, x\}; y) = 0, \quad \forall x, y, z \in M \]

Proof. Use (4.1c) to obtain

\[ 6Y(\{x, y\}; z) = [P^+_x, P^+_y, P^+_z] + P^+_{[x,y],z} \]

Make here twice the cyclic permutation of \( x, y, z \) and add the resulting equalities with the original one. Then use (4.3c) and the definition of \( J \) to obtain the desired identity (4.4).
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References
[1] Mal’tsev A 1955 Matem. Sb. 36 569 (in Russian)
[2] Paal 2008 Introduction to Moufang symmetry I. Generalized Lie and Maurer-Cartan equations Preprint 0802.3471
[3] Schafer R D 1952 Trans. Amer. Math. Soc. 72 1
[4] Schafer R D 1966 An Introduction to Nonassociative Algebras (New York - London: Academic Press)
[5] Yamaguti K 1963 Kumamoto J. Sci. A6 9