Operations of graphs and unimodality of independence polynomials \(^*\)

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Abstract

Given two graphs \(G\) and \(H\), assume that \(\mathcal{C} = \{C_1, C_2, \ldots, C_q\}\) is a clique cover of \(G\) and \(U\) is a subset of \(V(H)\). We introduce a new graph operation called the clique cover product, denoted by \(G^{C} \ast H^U\), as follows: for each clique \(C_i \in \mathcal{C}\), add a copy of the graph \(H\) and join every vertex of \(C_i\) to every vertex of \(U\). We prove that the independence polynomial of \(G^{C} \ast H^U\)

\[
I(G^{C} \ast H^U; x) = I^q(H; x)I(G; \frac{xI(H - U; x)}{I(H; x)}),
\]

which generalizes some known results on independence polynomials of corona and rooted products of graphs obtained by Gutman and Rosenfeld, respectively. Based on this formula, we show that the clique cover product of some special graphs preserves symmetry, unimodality, log-concavity or reality of zeros of independence polynomials. As applications we derive several known facts in a unified manner and solve some unimodality conjectures and problems.

MSC: 05C69; 05A20

Keywords: Independence polynomials; Unimodality; Log-concavity; Real zeros; Symmetry

\(^*\)Supported partially by the National Science Foundation of China (Nos.11071030,11201191), Natural Science Foundation of Jiangsu Higher Education Institutions (No.12KJB110005) and PAPD of Jiangsu Higher Education Institutions.

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1 Introduction

For the graph theoretical terms used but not defined, we follow Bondy and Murty [4].

Let $G = (V(G), E(G))$ be a finite and simple graph. By $G - U$ we mean the induced subgraph $G[V - U]$, if $U \subseteq V(G)$. We also denote by $G - e$ the subgraph of $G$, obtained by deleting an edge $e$ of $E(G)$. For $v \in V(G)$, denote $N(v) = \{w : w \in V(G) \text{ and } vw \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. The join of two disjoint graphs $G_1$ and $G_2$, denoted by $G_1 + G_2$, with $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$ as the edge set and $V(G_1) \cup V(G_2)$ as the vertex set. An independent set in a graph $G$ is a set of pairwise non-adjacent vertices. A maximum independent set in $G$ is a largest independent set and its size is denoted $\alpha(G)$. Let $i_k(G)$ denote the number of independent sets of cardinality $k$ in $G$. Then its generating function

$$I(G; x) = \sum_{k=0}^{\alpha(G)} i_k(G)x^k; \quad i_0(G) = 1$$

is called the independence polynomial of $G$ (Gutman and Harary [17]). In general, it is an NP-complete problem to determine the independence polynomial, since evaluating $\alpha(G)$ is an NP-complete problem ( [15]). Thus, a classical question is to compute the independence polynomial of a graph. It is a good way to obtain the independence polynomial of a graph in term of its subgraphs. One can easily deduce (e.g., Gutman and Harary [17]) that

$$I(G_1 \cup G_2; x) = I(G_1; x)I(G_2; x), \quad I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1.$$  

As we know, to study properties of graphs, there are some useful and important operations of graphs in the graph theory. Motivated by the above mentioned examples, one may further ask which operation of graphs is good to compute the independence polynomial. Recall that a clique cover of a graph $G$ is a spanning graph of $G$, each component of which is a clique. We here define an operation of graphs called the clique cover product. Given two graphs $G$ and $H$, assume that $\mathcal{C} = \{C_1, C_2, \ldots, C_q\}$ is a clique cover of $G$ and $U$ is a subset of $V(H)$. Construct a new graph from $G$, as follows: for each clique $C_i \in \mathcal{C}$, add a copy of the graph $H$ and join every vertex of $C_i$ to every vertex of $U$. Let $G^{\mathcal{C}} \star H^U$ denote the new graph. In fact, the clique cover product of graphs is a common generalization of some known operations of graphs. For instance: If each clique $C_i$ of the clique cover $\mathcal{C}$ is a vertex, then $G^{V(G)} \star H^{V(H)}$ is the corona of $G$ and $H$ defined by Frucht and Harary [14], denoted by $G \circ H$. If each clique $C_i$ of the clique cover $\mathcal{C}$ is a vertex and a vertex $v$ is the root of $H$, then $G^{V(G)} \star (H - v)^{N(v)}$ is the rooted
product of $G$ and $H$ introduced by Godsil and MacKay \cite{16}, denoted by $G \circ H$. If we take $H = 2K_1$ and $U = V(2K_1)$, then $G \ast H^U$ is the graph $\mathcal{C}(G)$ obtained by Stevanović \cite{43} using the clique cover construction. Note that $G \ast H^V(H)$ also contains the compound graph $G \Delta(H)$ as the special case \cite{41}. As the basic result of this paper, we formulate the independence polynomial for the clique cover product as follows.

**Theorem 1.1.** Given two graphs $G$ and $H$, assume that $C$ is a clique cover of $G$. Let $U$ be a subset of $V(H)$. Assume that $C \subseteq V(G)$. Then we have

$$I(G^C \ast H^U; x) = I(H; x)^q I(G; x) I(H - U; x).$$

In view of Theorem 1.1, the following is immediate.

**Corollary 1.1.** Given two graphs $G$ and $H$, assume that $C$ is a clique cover of $G$ and $U$ is a subset of $V(H)$. Let $|C| = q$. Then $I^q(G; x) I(H; x) I(H - U; x)$ divides $I(G^C \ast H^U; x)$.

In fact, Theorem 1.1 has generalized some known results, e.g., the following corollaries obtained by Gutman \cite{18} and Rosenfeld \cite{39}, respectively. In addition, our method is different from theirs.

**Corollary 1.2.** \cite{18} Gutman] For any graph $G$ of order $n$, $I(G \circ H; x) = I(G; x) I(H; x)$.

**Corollary 1.3.** \cite{18, 39} If $G$ is a graph of order $n$ and $H$ is a graph with root $v$, then

$$I(G \circ H; x) = I^n(H - v; x) I(G; x) I(H - v; x).$$

**Corollary 1.4.** \cite{39} Let $H$ be a graph with root $v$, where $v$ is a pedant vertex and $N(v) = u$. If $G$ is a graph of order $n$, then

$$I(G \circ H; x) = I^n(H - v; x) I(G; x) I(H - v; x).$$

Let $a_0, a_1, \ldots, a_n$ be a sequence of nonnegative numbers. It is **unimodal** if there is some $m$, called a mode of the sequence, such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n.$$

It is **log-concave** if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq n - 1$. It is **symmetric** if $a_k = a_{n-k}$ for $0 \leq k \leq n$. Clearly, a log-concave sequence of positive numbers is unimodal (see, e.g., Brenti \cite{7}). We say that a polynomial $\sum_{k=0}^n a_k x^k$ is **unimodal** (log-concave, symmetric, respectively) if the sequence of its coefficients $a_0, a_1, \ldots, a_n$ is unimodal (log-concave, symmetric, respectively). A mode of the sequence $a_0, a_1, \ldots, a_n$ is also called
a mode of the polynomial $\sum_{k=0}^{n} a_k x^k$. Unimodality problems arise naturally in many branches of mathematics and have been extensively investigated. See Stanley’s survey [42] and Brenti’s supplement [6] for known results and open problems on log-concave and unimodal sequences arising in algebra, combinatorics and geometry. It is well known that if the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros for the nonnegative sequence $\{a_i\}$, then by Newton’s inequalities the sequence $\{a_n\}$ is log-concave and unimodal (see Hardy, Littlewood and Pólya [20, p. 104]).

Recently, the open problems on the unimodality (Read [38, p. 68]) and log-concavity (Welsh [48, p. 266]) of the chromatical polynomial of a graph has been solved by Huh [21]. On the other hand, unimodality problems and zeros of independence polynomials have been investigated, e.g., see [1–3,9,10,12,13,21,22,26,31,33,34,37,47,49] for an extensive literature in recent years. It is well known that the matching polynomial of a graph has only real zeros [23,10]. In fact, the independence polynomial can be regarded as a generalization of the matching polynomial because the matching polynomial of a graph and the independence polynomial of its line graph are identical. Wilf asked whether the independence polynomials are also unimodal. However Alavi, Malde, Schwenk, Erdős [1] gave a negative example. They further conjectured the following.

**Conjecture 1.1.** The independence polynomial of any tree or forest is unimodal.

Similarly, Levit and Mandrescu [32] also make the next conjecture.

**Conjecture 1.2.** The independence polynomial of any very well-covered graph is log-concave.

Whereas the independence polynomials for certain special classes of graphs are unimodal and even have only real zeros. For instance, the independence polynomial of a line graph has only real zeros. More generally, the independence polynomial of a claw-free graph has only real zeros (Chudnovsky and Seymour [13]). Although the independence polynomial of almost every graph of order $n$ has a nonreal zero, the average independence polynomials always have all real and simple zeros (Brown and Nowakowski [10]). Hence an interesting problem naturally arises.

**Problem 1.1.** [11] When does the independence polynomial of a graph have only real zeros?

The symmetry of the matching polynomial and the characteristic polynomial of a graph were observed (see [16,25] for instance). Thus, we naturally study the symmetric independence polynomial. Stevanović [43] showed the next result.
Theorem 1.2. If there is an independent set $S$ in $G$ such that $|N(A) \cap S| = 2|A|$ holds for every independent set $A \subseteq V(G) - S$, then $I(G; x)$ is symmetric.

By virtue of this result, Stevanović deduced the following corollary.

Corollary 1.5. (i) If a graph $G$ has $i_{\alpha(G)}(G) = 1$, $i_{\alpha(G)-1}(G) = |V(G)|$ and the unique maximum independence set $S$ satisfies $|N(u) \cap S| = 2$ for every vertex $u \in V - S$, then $I(G; x)$ is symmetric.

(ii) If $G$ is a claw-free graph with $i_{\alpha(G)} = 1$, $i_{\alpha(G)-1} = |V(G)|$, then $I(G; x)$ is symmetric.

According to this corollary, Stevanović further obtained a few ways to construct graphs having symmetric independence polynomials, e.g., $G \circ 2K_1$ and $\mathcal{C}(G)$. However, the following general problem is still open.

Problem 1.2. When is the independence polynomial of a graph symmetric?

The organization of this paper is as follows. In Section 2, we give the proof of Theorem 1.1. In Section 3, based on the formula in Theorem 1.1, we present various results that the clique cover product of some special graphs preserves symmetry, unimodality, log-concavity or reality of zeros of independence polynomials. As applications we derive several known facts and solve some unimodality conjectures and problems in a unified manner. Finally, in the concluding remarks, we also give the similar result for another new graphs operation called the cycle cover product.

2 The proof of Theorem 1.1

Proof. We here give a combinatorial proof. Let $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$, $I(H; x) = \sum_{k=0}^{\alpha(H)} a_k x^k$, $I(H - U; x) = \sum_{k=0}^{\alpha(H-U)} b_k x^k$ and $I(G^G \ast H^U; x) = \sum_{k \geq 0} t_k x^k$. We can give an explicit expression of $t_k$ in the following method. For each $k$, select $k$ independent elements from $V(G^G \ast H^U)$ in a two-stage process. First, let us choose $m$ independent elements from the $V(G)$. And then select the remaining $(k - m)$ independent elements from $V(m(H - U) \cup (q - m)H)$. The number of ways in which we make two choices is $s_m$ and

$$\sum_{i_1 + i_2 + \ldots + i_m + j_1 + j_2 + \ldots + j_{q-m} = k-m} b_{i_1} b_{i_2} \cdots b_{i_m} a_{j_1} a_{j_2} \cdots a_{j_{q-m}}$$
respectively. In consequence, we obtain that

\[ t_k = \sum_{m=0}^{k} s_m \sum_{i_1+i_2+\ldots+i_m+j_1+j_2+\ldots+j_{q-m}=k-m} b_{i_1} b_{i_2} \ldots b_{i_m} a_{j_1} a_{j_2} \ldots a_{j_{q-m}}. \]

Thus we have

\[
I(G^\mathcal{C} \ast H^U; x) = \sum_{k \geq 0} t_k x^k = \sum_{k \geq 0} \left( \sum_{m=0}^{k} s_m \sum_{i_1+i_2+\ldots+i_m+j_1+j_2+\ldots+j_{q-m}=k-m} b_{i_1} b_{i_2} \ldots b_{i_m} a_{j_1} a_{j_2} \ldots a_{j_{q-m}} \right) x^k
\]

\[
= \alpha(G) \sum_{m=0}^{\alpha(G)} s_m I^m(H - U; x) I^{q-m}(H; x) x^m
\]

\[
= I^q(H; x) I(G; \frac{xI(H - U; x)}{I(H; x)}),
\]

which is desired.

\[ \square \]

3 Various results for symmetry, unimodality, log-concavity or reality of zeros of independence polynomials.

In this section, based on the formula in Theorem 1.1, we present various results that the clique cover product of some special graphs preserves symmetry, unimodality, log-concavity or reality of zeros of independence polynomials. We not only derive several known facts in a unified manner, but also settle some unimodality conjectures and problems in the literature.

Note that if we can give the factorization for the independence polynomial, then the next result will be very useful in solving unimodality problems for independence polynomials. We refer the reader to Stanley’s survey article [42] for further information.

Lemma 3.1. Let \( f(x) \) and \( g(x) \) be polynomials with positive coefficients.

(i) If both \( f(x) \) and \( g(x) \) have only real zeros, then so does their product \( f(x)g(x) \).

(ii) If both \( f(x) \) and \( g(x) \) are log-concave, then so is their product \( f(x)g(x) \).
If \( f(x) \) is log-concave and \( g(x) \) is unimodal, then their product \( f(x)g(x) \) is unimodal.

If both \( f(x) \) and \( g(x) \) are symmetric and unimodal, then so is their product \( f(x)g(x) \).

In addition, the mode of \( f(x)g(x) \) is the sum of modes of \( f(x) \) and \( g(x) \).

Let \( P(x) \) be a real polynomial of degree \( n \). Define its reciprocal polynomial by

\[
P^*(x) := x^n P \left( \frac{1}{x} \right).
\]

The following facts are elementary but very useful in the sequel.

(i) If \( P(0) \neq 0 \), then \( \deg P^*(x) = \deg P(x) \) and \( (P^*)^*(x) = P(x) \).

(ii) \( P(x) \) has only real zeros if and only if \( P^*(x) \) has only real zeros.

(iii) \( P(x) \) is log-concave if and only if \( P^*(x) \) is log-concave.

(iv) \( P(x) \) is symmetric if and only if \( P^*(x) = P(x) \).

3.1 Symmetry and unimodality of independence polynomials

The next result gives a characterization of the graphs having the symmetric or unimodal independence polynomials.

**Proposition 3.1.** Given two graphs \( G \) and \( H \), let \( \mathcal{C} \) be a clique cover of \( G \) and \( U \) be a subset of \( V(H) \). Let \( |\mathcal{C}| = q \) and \( \alpha(H) = \alpha(H - U) + 2 \). Then we have the following.

(i) If both \( I(H; x) \) and \( I(H - U; x) \) are symmetric, then so is \( I(G^\mathcal{C} * H^U; x) \).

(ii) If both \( I(H; x) \) and \( I(H - U; x) \) are symmetric and unimodal, then so is \( I(G^\mathcal{C} * H^U; x) \).

**Proof.** To show that \( I(G^\mathcal{C} * H^U; x) \) is symmetric, we only need to prove that \( I^*(G^\mathcal{C} * H^U; x) = I(G^\mathcal{C} * H^U; x) \). In view of Theorem 1.1, we have

\[
I(G^\mathcal{C} * H^U; x) = I^q(H; x) I \left( G; \frac{xI(H - U; x)}{I(H; x)} \right).
\]
Thus it is clear that

\[
I^*(G^\ell \star H^U; x) = x^{q(H)} I \left( G^\ell \star H^U; \frac{1}{x} \right)
\]

\[
= [I^*(H; x)]^q I \left( G; \frac{1}{x} I(H - U; \frac{1}{x}) \right)
\]

\[
= [I^*(H; x)]^q I \left( G; \frac{x I^*(H - U; x)}{I^*(H; x)} \right)
\]

\[
= I^q(H; x) I \left( G; \frac{x I(H - U; x)}{I(H; x)} \right)
\]

\[
= I \left( G^\ell \star H^U; x \right)
\]

since \( I^*(H; x) = I(H; x) \) and \( I^*(H - U; x) = I(H - U; x) \). Thus, we have shown that (i) holds.

Now we will prove that (ii) holds. Assume that

\[
I(G; x) = \sum_{i=0}^{\alpha(G)} s_i x^i.
\]

In view of Theorem 1.1, we have

\[
I(G^\ell \star H^U; x) = I^{q-\alpha(G)}(H; x) \sum_{i=0}^{\alpha(G)} s_i [x I(H - U; x)]^i [I(H; x)]^{\alpha(G)-i}.
\]

(3.1)

By Lemma 3.1 (iv), we can deduce that \([I(H - U; x)]^i [I(H; x)]^{\alpha(G)-i} \) is symmetric and unimodal and the mode of \([x I(H - U; x)]^i [I(H; x)]^{\alpha(G)-i} \) is equal to \( \frac{\alpha(G)q(H)}{2} \) for any \( i \in [0, \alpha(G)] \). Thus, we obtain that

\[
\sum_{i=0}^{\alpha(G)} s_i [x I(H - U; x)]^i [I(H; x)]^{\alpha(G)-i}
\]

is symmetric and unimodal. Hence, \( I(G^\ell \star H^U; x) \) is symmetric and unimodal by virtue of equality (3.1) and Lemma 3.1 (iv).

Let \( k \geq 1 \) and \( d \geq 0 \). Following Bahls and Salazar [3], the \( K_t \)-path of length \( k \), denoted by \( P(t, k) \), is the graph \((V, E)\) in which \( V = \{v_1, v_2, \ldots, v_{t+k-1}\} \) and

\[
E = \{\{v_i, v_{i+j}\} | 1 \leq i \leq t + k - 2, 1 \leq j \leq \min\{t - 1, t + k - i - 1\}\}.
\]

Such a graph consists of \( k \) copies of \( K_t \), each glued to the previous one by identifying certain prescribed subgraphs isomorphic to \( K_{t-1} \). The \( d \)-augmented \( K_t \) path, denoted
\[ P(t, k, d), \] is obtained from \[ P(t, k) \] by adding new vertices \( \{u_{i,1}, u_{i,2}, \ldots, u_{i,d}\} \) for \( t, k, d \) and edges
\[ \{v_i u_{i,j} \} \cup \{v_{i+1} u_{i,j} \} | j = 1, \ldots, d \} \]

By the complicated computation, Bahls and Salazar [3] showed the following corollary, which clearly follows from Proposition 3.1 (ii). In fact, we only need to assume that \( I(G - v; x) \) and \( I(G - N[v]; x) \) are symmetric and unimodal, and that \( \alpha(G) = \alpha(G - N[v]) + 2 \).

**Corollary 3.1.** [3, Theorem 2.1] Given a graph \( G \) and the vertex \( v \in V(G) \), assume that \( I(G - v; x) \), \( I(G; x) \) and \( I(G - N[v]; x) \) are symmetric and unimodal, and that \( \alpha(G) = \alpha(G - N[v]) + 2 \). Then \( I(P(t, k, d)^V(P(t,k,d)) * (G - v)^N(v); x) \) is symmetric and unimodal for any \( t \geq 2, k \geq 1 \) and \( d \geq 0 \).

Actually, Proposition 3.1 (ii) has given a generalized answer to the following question of Bahls and Salazar [3].

**Problem 3.1.** [3, Question 5.2] Can we obtain an explicit description of the most general family of graphs to which Corollary 3.1 may be applied in order to show symmetry and unimodality?

Note that if a graph \( G \) satisfies the conditions of Theorem 1.2, then we have
\[ I(G; x) = \sum_{k=0}^{\lfloor |S|/2 \rfloor} i_k(G[V - S])x^k(x + 1)^{|S| - 2k} \]
using the similar method of Theorem 1.1. From the proof of Proposition 3.1 (ii), it can be seen that the following results hold, which strengthens Stevanović’s results [3].

**Theorem 3.1.** If there is an independent set \( S \) in \( G \) such that \( |N(A) \cap S| = 2|A| \) holds for every independent set \( A \subseteq V(G) - S \), then \( I(G; x) \) is symmetric and unimodal.

**Corollary 3.2.** (i) If a graph \( G \) has \( i_{\alpha(G)}(G) = 1, i_{\alpha(G) - 1}(G) = |V(G)| \) and the unique maximum independence set \( S \) satisfies \( |N(u) \cap S| = 2 \) for every vertex \( u \in V - S \), then \( I(G; x) \) is symmetric and unimodal.

(ii) If \( G \) is a claw-free graph with \( i_{\alpha(G)} = 1, i_{\alpha(G) - 1} = |V(G)| \), then \( I(G; x) \) is symmetric and unimodal.

### 3.2 Log-concavity and reality of zeros of independence polynomials

The following criterion for log-concavity is very useful.
Lemma 3.2. [6, Brenti] If $P(x)$ is a log-concave polynomial with nonnegative coefficients and with no internal zeros, then $P(x+r)$ is log-concave for any positive integer $r$.

Similar to Proposition 3.1, we can also demonstrate the following result.

Proposition 3.2. Given two graphs $G$ and $H$, let $\mathcal{C}$ be a clique cover of $G$ and $U \subseteq V(H)$. Let $I(H; x) = I(H - U; x)(ax^2 + bx + 1)$, where $a, b$ are nonnegative integer.

(i) If both $I(G; x)$ and $I(H; x)$ have only real zeros, then so does $I(G^{\mathcal{C}} \star H^U; x)$.

(ii) Assume that $I(G; x)$ has only real zeros. If both $I(H - U; x)$ and $ax^2 + bx + 1$ are log-concave, then so is $I(G^{\mathcal{C}} \star H^U; x)$.

(iii) Assume that $I(G; x)$ is log-concave and $a = 0$. If $I(H - U; x)$ is log-concave, then so is $I(G^{\mathcal{C}} \star H^U; x)$.

Proof. We first show that (i) and (ii) hold. Let $|\mathcal{C}| = q$ and $I(G; x) = \prod_{i=1}^{\alpha(G)} (a_i x + 1)$ since $I(G; x)$ has only real zeros. By Theorem 1.1 we have

$$I(G^{\mathcal{C}} \star H^U; x) = I^q(H; x) \prod_{i=1}^{\alpha(G)} \left( 1 + \frac{a_i x}{ax^2 + bx + 1} \right)$$

$$= I^{q-\alpha(G)}(H; x) I^{\alpha(G)}(H - U; x) \prod_{i=1}^{\alpha(G)} \left[ ax^2 + (b + a_i) x + 1 \right].$$

Thus, we obtain that (i) and (ii) follow from Lemma 3.3 (i) and (ii), respectively.

In what follows, we will prove that (iii) holds. To show log-concavity of $I(G^{\mathcal{C}} \star H^U; x)$, it suffices to show that $I^*(G^{\mathcal{C}} \star H^U; x)$ is log-concave. Since $ax^2 + bx + 1 = 1 + bx$ for $a = 0$, we have

$$I(G^{\mathcal{C}} \star H^U; x) = I^q(H; x) I \left( G; \frac{x}{1 + bx} \right)$$

by virtue of Theorem 1.1. Thus we obtain that

$$I^*(G^{\mathcal{C}} \star H^U; x) = x^{q\alpha(H)} I^q(G^{\mathcal{C}} \star H^U; \frac{1}{x})$$

$$= [I^*(H; x)]^q I \left( G; \frac{1}{x + b} \right)$$

$$= [I^*(H - U; x)(x + b)]^q I \left( G; \frac{1}{x + b} \right)$$

$$= [I^*(H - U; x)]^q (x + b)^{q-\alpha(G)} I^*(G; x + b). \quad (3.2)$$

Applying Lemma 3.2 and Lemma 3.1 (ii) to the equality (3.2), we obtain that log-concavity of $I^*(G^{\mathcal{C}} \star H^U; x)$ follows from log-concavity of $I^*(H - U; x)$ and $I^*(G; x)$. Therefore the proof is complete. \qed
Now we give a simple application of Proposition 3.2. A well-covered spider graph $S_n$ is one of $\{K_1, K_2, K_{1,m} \circ K_1 : m \geq 1\}$. Levit and Mandrescu [28] proved that the independence polynomial of the well-covered spider $S_n$ is log-concave. By the complicated calculation, Chen and Wang [12] demonstrated that $I(K_{2,n} \circ K_1; x)$ is unimodal and log-concave. Further, they proposed the following general conjecture, which can be confirmed by our results.

**Conjecture 3.1** ([12]). $I(K_{t,n} \circ K_1; x)$ is log-concave for every $t$ and is therefore unimodal.

**Proof.** To show that $I(K_{t,n} \circ K_1; x)$ is log-concave, we only need to prove that $I(K_{t,n}; x)$ is log-concave by virtue of Proposition 3.2 (ii) for $H = K_1$. It is easy to obtain that

$$I(K_{t,n}; x) = (x + 1)^t + (1 + x)^n - 1.$$ 

Without loss of generality, we can assume $t \leq n$. Thus,

$$(x + 1)^t + (1 + x)^n = (x + 1)^t [1 + (1 + x)^{n-t}].$$

It follows from Lemma 3.1 (ii) that $(x + 1)^t + (1 + x)^n$ is log-concave. Consequently, it is clear that $I(K_{t,n}; x)$ is log-concave. This completes the proof. 

**Remark 3.1.** The graph $H$ in Propositions 3.1 and 3.2 can be a disconnected graph. In addition, we can also easily find more examples for $H$ applied to Propositions 3.1 and 3.2.

### 3.3 Claw-free graphs

Recently, Chudnovsky and Seymour [13] showed that the independence polynomial of a claw-free graph has only real zeros. Recall that two real polynomials $f(x)$ and $g(x)$ are compatible if $af(x) + bg(x)$ has only real zeros for all $a, b \geq 0$. Chudnovsky and Seymour actually proved the next result.

**Lemma 3.3** ([13]). Let $G$ be a claw-free graph. Then

(i) $I(G - v; x)$ and $xI(G - N[v]; x)$ are compatible for any $v \in V(G);$ 

(ii) $I(G; x)$ has only real zeros.

In [2], Bahls proved the following result. In fat, it clearly follows from Proposition 3.2 (i) and Lemma 3.3 since $P(t, k)$ is a claw-free graph.
Corollary 3.3. Given a graph $G$ and $U \subset V(G)$, assume that $I(G; x)$ has only real zeros and $I(G; x) = I(G-U; x)(ax^2+bx+1)$, where $0 < a, b \in \mathbb{N}$. Then $I(P(t, n)^V(P(t, n)) \star G^U; x)$ has only real zeros and is therefore log-concave for any $t \geq 2$ and $n \geq 1$.

Noting that the graph $H$ is claw-free for $\alpha(H) \leq 2$, we immediately obtain the following corollary by virtue of Proposition 3.1 and Proposition 3.2.

Corollary 3.4. Let $H$ be a graph with $\alpha(H) \leq 2$ and $\mathcal{C}$ be a clique cover of another graph $G$. Let $P_3$ denote the path with three vertices and $p \geq 1$.

(i) If $I(G; x)$ has only real zeros, then so does $I(G^\mathcal{C} \star H^V(H); x)$. In particular, if $I(G; x)$ has only real zeros, then so do $I(G \circ H; x)$ and $I(G \sigma K_p; x)$.

(ii) If $I(G; x)$ is log-concave, then so is $I(G^\mathcal{C} \star K_p^V(K_p); x)$. In particular, if $I(G; x)$ is log-concave, then $I(G \circ K_p; x)$ is log-concave.

(iii) If $H = K_p - e$ and $U = V(K_p - e)$, then $I(G^\mathcal{C} \star H^U; x)$ is symmetric and unimodal for $p \geq 2$. In particular, $I(G \circ 2K_1; x)$ and $I(G \circ P_3; x)$ are symmetric and unimodal.

Remark 3.2. If $H = K_2 - e = 2K_1$ and $U = V(2K_1)$, then $I(G^\mathcal{C} \star H^U; x)$ is symmetric, which was also obtained by Stevanović [43]. Mandrescu [37] demonstrated a particular example of Corollary 3.4(i): If $I(G; x)$ has only real zeros, then so does $I(G \circ (K_p \cup K_q); x)$. Rosenfeld [39] also showed that if $I(G; x)$ has only real zeros, then so does $I(G \sigma K_p; x)$.

Example 3.1. Centipede Graphs and Caterpillar Graphs

Let $W_n$ (see Figure 1) and $H_n$ (see Figure 2) be the centipede graph and the caterpillar graph, respectively. Levit and Mandrescu [26] showed that $I(W_n; x)$ is unimodal and further conjectured that $I(W_n; x)$ has only real zeros. Zhu [49] settled the conjecture and demonstrated that $I(H_n; x)$ is symmetric and unimodal. Now we can use Corollary 3.4 to give a unified proof of these results. Since $P_n$, i.e., the path with $n$ vertices, is a claw-free graph. We obtain that $I(W_n; x) = I(P_n \circ K_1; x)$ has only real zeros, and $I(H_n; x) = I(P_n \circ 2K_1; x)$ has only real zeros and is symmetric by Corollary 3.4. Thus $I(H_n; x)$ is log-concave and unimodal. In fact, we can further obtain that the independence polynomials of more graphs have only real zeros using Corollary 3.4 repeatedly.

![Figure 1: W_n](image1.png)  
![Figure 2: H_n](image2.png)
Example 3.2. N-sunlet Graphs

The \textit{n-sunlet graph} is the graph with \(2n\) vertices obtained by attaching pendant edges to a cycle graph, \(i.e., C_n \circ K_1\), where \(C_n\) is the cycle with \(n\) vertices. Applying Corollary 3.4, we have \(I(C_n \circ K_1; x)\) has only real zeros since \(C_n\) is a claw-free graph. Therefore \(I(C_n \circ K_1; x)\) is log-concave and unimodal. In addition, we also can verify that \(I(C_n \circ K_p; x)\) and \(I(C_n \circ 2K_p; x)\) have only real zeros for \(p \geq 1\).

Example 3.3. A Conjecture of Levit and Mandrescu

In [33], Levit and Mandrescu constructed a family of graphs \(H_n\) from the path \(P_n\) by the “clique cover construction”, as shown in Figure 3. By \(H_0\) we mean the null graph.

Levit and Mandrescu [33] proved that the independence polynomials of \(H_n\) is symmetric and unimodal. They further made the next conjecture.

\textbf{Conjecture 3.2 ([33]).} \textit{The independence polynomial of } \(H_n\textit{ is log-concave and has only real zeros for } n \geq 1\textit{.}

Recently, this conjecture has been confirmed by Wang and Zhu [47]. Now we also can use Corollary 3.4 to give a simple proof. In view of Corollary 3.4, we easily see that \(I(H_n; x)\) has only real zeros since \(P_n\) is claw-free and \(\alpha(H) = \alpha(2K_1) = 2\). Consequently, \(I(H_n; x)\) is log-concave and unimodal.

Finally, we also show the following general result, which preservers the reality of zeros of the independence polynomial. In particular, it extends the result of Chudnovsky and Seymour [13].

\textbf{Proposition 3.3.} \textit{Let } \(\mathcal{C}\textit{ be a clique cover of a graph } G\textit{ and } |\mathcal{C}| = q\textit{. Assume that } H\textit{ is a claw-free graph with the root } v\textit{ and } U = N(v)\textit{. If } I(G; x)\textit{ has only real zeros, then so does } I(G^{\mathcal{C}} \ast (H - v)^U; x)\textit{. In particular, if } I(G; x)\textit{ has only real zeros, then so does } I(G \circ H; x)\textit{.}
Proof. Let \( I(G; x) = \prod_{i=1}^{\alpha(G)} (r_i x + 1) \) since \( I(G; x) \) has only real zeros. Thus by Theorem 1.1, we obtain that

\[
I(G^{\ell} \ast (H - v)^U; x) = I^{q - \alpha(G)}(H - v; x) \prod_{i=1}^{\alpha(G)} [I(H - v; x) + r_i x I(H - N[v]; x)].
\]

According to the hypothesis, the graph \( H \) is claw-free, so are the induced graphs \( H - v \) and \( H - N[v] \). By virtue of Lemma 3.3, \( I(H - v; x) \) and \( x I(H - N[v]; x) \) have only real zeros and are compatible. Hence \( I(H - v; x) + r_i x I(H - N[v]; x) \) has only real zeros for every \( i \). Consequently, \( I(G^{\ell} \ast (H - v)^U; x) \) has only real zeros, which is desired.

In particular, as a corollary of Proposition 3.3, we obtain the next result.

**Corollary 3.5.** [47, Proposition 3.3] Let \( G \) be a rooted claw-free graph and \( P_n \) be the path with \( n \) vertices. Then \( I(P_n \circ G; x) \) has only real zeros.

**Remark 3.3.** Proposition 3.3 can be repeatedly used to construct infinite graphs with claw, whose independence polynomials have only real zeros.

## 4 Concluding Remarks

In this paper we define the clique cover product of graphs. Based on the formula of the independence polynomial of the graph so-formed, we show some operations of graphs, which preserve symmetry, unimodality, log-concavity or reality of zeros of the independence polynomials.

In fact, we can find that \( I(T; x) \) has only real zeros or is log-concave if we calculate the independence polynomial of the tree \( T \) with fewer vertices. Therefore we can confirm the unimodality conjectures in the literature for more graphs using the construction \( T \circ K_1 \) and \( T \circ 2K_1 \) repeatedly. In particular, \( I(T \circ 2K_1; x) \) is symmetric and unimodal for any tree \( T \). Generally speaking, if we can give the factorization for \( I(G^{\ell} \ast H^U; x) \) in Theorem 1.1 and can show that its every factor is symmetry or log-concave, or has only real zeros, then we can obtain that \( I(G^{\ell} \ast H^U; x) \) has the same property in view of Lemma 3.1. In particular, if both \( G \) and \( H \) are some particular graphs, then such result may easily be proved.

From this paper, we can see that it is a good way to construct graphs with independence polynomial being symmetric, unimodal or log-concave, or having only real zeros by using an operation of graphs. Thus, it is possible to define some other operations of
graphs. Recall that a cycle cover of a graph $G$ is a spanning graph of $G$, each connected component of which is a vertex (called a vertex-cycle), an edge (called an edge-cycle), or a proper cycle. Here we give a new operation of graphs called the cycle cover product: Given two graphs $G$ and $H$, let $\Gamma$ be a cycle cover of $G$ and $U \subseteq V(H)$. Construct a new graph, denoted by $G^\Gamma \otimes H^U$, as follows: If a cycle $C \in \Gamma$ is

(i) a vertex-cycle, say $v$, then add two copies of $H$ and join each vertex in two $U$ to $v$;

(ii) an edge-cycle, say $uv$, then add two copies of $H$ and join each vertex in two $U$ to both $u$ and $v$;

(iii) a proper cycle, with $V(C) = \{v_i : 1 \leq i \leq s\}$, $E(C) = \{v_i v_{i+1} : 1 \leq i \leq s - 1\} \cup \{v_1 v_s\}$, then add $s$ copies of $H$, say $\{H_i : 1 \leq i \leq s\}$ and each vertex in $i$th copy of $U$ is joined to two consecutive vertices $v_i, v_{i+1}$ on $C$ (each vertex in $s$th copy of $U$ is joined to two consecutive vertices $v_s, v_1$ on $C$).

Actually, using the similar technique of the clique cover product of graphs, we can also prove the following result. We here omit its proof for brevity.

Theorem 4.1. Given two graphs $G$ and $H$, assume that $\Gamma$ is a cycle cover of $G$ containing $k$ vertex-cycles and $U$ is a subset of $V(H)$. Let $|V(G)| = n$.

(i) The independence polynomial $I(G^\Gamma \otimes H^U; x) = I^{n+k}(H; x) I(G; \frac{xP^2(H-U;x)}{P(H;x)})$.

(ii) Let $\alpha(H) = \alpha(H-U) + 1$. If both $I(H; x)$ and $I(H-U; x)$ are symmetric, then so is $I(G^\Gamma \otimes H^U; x)$.

(iii) Let $\alpha(H) = \alpha(H-U) + 1$. If both $I(H; x)$ and $I(H-U; x)$ are symmetric and unimodal, then so is $I(G^\Gamma \otimes H^U; x)$.

(iv) If $I(G; x)$ has only real zeros and $I(H; x) = I(H-U; x)(1+ax)$, then $I(G^\Gamma \otimes H^U; x)$ has only real zeros.

(v) If $I(G; x)$ has only real zeros and $H = K_p$, then $I(G^\Gamma \otimes H^{V(K_p)}; x)$ has only real zeros.

Remark 4.1. From (ii) of this theorem, it is clear that $I(G^\Gamma \otimes K_1^{V(K_1)}; x)$ is symmetric, which also obtained by Stevanović [43] using the “cycle cover construction”. In fact, $I(G^\Gamma \otimes K_1^{V(K_1)}; x)$ is symmetric and unimodal by (iii) of this theorem.
Remark 4.2. Given two graphs $G$ and $H$, assume that $\Gamma$ is a cycle cover of $G$ containing no proper cycle, $\mathcal{C}$ is a clique cover of $G$ only containing $K_1, K_2$ and $U$ is a subset of $V(H)$. Then it is obvious that

$$I(G^\mathcal{C} \ast (2H)^{2U}; x) = I(G^\Gamma \otimes H^U; x).$$

At the end of this paper, we refer the reader to [5–7, 35, 36, 42, 44–46] for more results about unimodality problems of sequences and polynomials.

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