On differential relations of 2-orthogonal polynomials

T. A. Mesquita∗

Escola Superior de Tecnologia e Gestão, Instituto Politécnico de Viana do Castelo, Rua Escola Industrial e Comercial de Nam’Alvares, 4900-347, Viana do Castelo, Portugal, & Centro de Matemática da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

Abstract

A generic differential operator on the vectorial space of polynomial functions was presented in [17] and applied in the study of differential relations fulfilled by polynomial sequences either orthogonal or 2-orthogonal.

Using the techniques therein developed, we prove an identity fulfilled by different differential operators and apply it in a systematic approach to the problem of finding polynomial eigenfunctions, assuming that those polynomials constitute a 2-orthogonal polynomial sequence.

In particular, we analyse a third order differential operator that does not increase the degree of polynomials.

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1 Notation and basic concepts

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual space. We denote by $\langle u, p \rangle$ the action of the form or linear functional $u \in \mathcal{P}'$ on $p \in \mathcal{P}$. In particular, $\langle u, x^n \rangle := (u)_n, n \geq 0$ represent the moments of $u$. In the following, we will call polynomial sequence (PS) to any sequence $\{P_n\}_{n \geq 0}$ such that $\deg P_n = n$, $n \geq 0$, that is, for all non-negative integer. We will also call monic polynomial sequence (MPS) to a PS so that all polynomials have leading coefficient equal to one.

If $\{P_n\}_{n \geq 0}$ is a MPS, there exists a unique sequence $\{u_n\}_{n \geq 0}, u_n \in \mathcal{P}'$, called the dual sequence of $\{P_n\}_{n \geq 0}$, such that,

$$\langle u_n, P_m \rangle = \delta_{n,m}, \ n, m \geq 0. \quad (1.1)$$

∗Corresponding author (tauugusta.mesquita@gmail.com)
On the other hand, given a MPS \( \{ P_n \}_{n \geq 0} \), the expansion of \( xP_{n+1}(x) \), defines sequences in \( \mathbb{C} \), \( \{ \beta_n \}_{n \geq 0} \) and \( \{ \chi_{n,\nu} \}_{0 \leq \nu \leq n, n \geq 0} \), such that

\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0, \tag{1.2}
\]

\[
xP_{n+1}(x) = P_{n+2}(x) + \beta_{n+1}P_{n+1}(x) + \sum_{\nu=0}^{n} \chi_{n,\nu}P_{\nu}(x). \tag{1.3}
\]

This relation is usually called the structure relation of \( \{ P_n \}_{n \geq 0} \), and \( \{ \beta_n \}_{n \geq 0} \) and \( \{ \chi_{n,\nu} \}_{0 \leq \nu \leq n, n \geq 0} \) are called the structure coefficients (SCs) \[11\]. Another useful presentation is the following.

\[
P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) + \sum_{\nu=0}^{n} \chi_{n,\nu}P_{\nu}(x),
\]

\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0.
\]

When the structure coefficients fulfill \( \chi_{n,\nu} = 0 \), \( 0 \leq \nu \leq n - 1 \), \( \chi_{n,n} \neq 0 \), identities (1.2), (1.3) refer to the well known three-term recurrence associated to an orthogonal MPS. More generally, identity (1.3) may furnish a recurrence relation for a higher order corresponding to the following notion of orthogonality with respect to \( d \) given functionals.

**Definition 1.1.** \[8, 13, 20\] Given \( \Gamma_1, \Gamma_2, \ldots, \Gamma_d \in \mathcal{P}' \), \( d \geq 1 \), the polynomial sequence \( \{ P_n \}_{n \geq 0} \) is called \( d \)-orthogonal polynomial sequence (d-OPS) with respect to \( \Gamma = (\Gamma_1, \ldots, \Gamma_d) \) if it fulfills

\[
\langle \Gamma^\alpha, P_mP_n \rangle = 0, \quad n \geq md + \alpha, \quad m \geq 0, \tag{1.4}
\]

\[
\langle \Gamma^\alpha, P_mP_{md+\alpha-1} \rangle \neq 0, \quad m \geq 0, \tag{1.5}
\]

for each integer \( \alpha = 1, \ldots, d \).

**Lemma 1.2.** \[12\] For each \( u \in \mathcal{P}' \) and each \( m \geq 1 \), the two following propositions are equivalent.

a) \( \langle u, P_{m-1} \rangle \neq 0, \quad \langle u, P_m \rangle = 0, \quad n \geq m. \)

b) \( \exists \lambda_\nu \in \mathbb{C}, \ 0 \leq \nu \leq m - 1, \ \lambda_{m-1} \neq 0 \) such that \( u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu. \)

The conditions (1.4) are called the \( d \)-orthogonality conditions and the conditions (1.5) are called the regularity conditions. In this case, the functional \( \Gamma \), of dimension \( d \), is said regular.

The \( d \)-dimensional functional \( \Gamma \) is not unique. Nevertheless, from Lemma \[12\] we have:

\[
\Gamma^\alpha = \sum_{\nu=0}^{\alpha-1} \lambda_\nu^\alpha u_\nu, \quad \lambda_{\alpha-1}^\alpha \neq 0, \quad 1 \leq \alpha \leq d.
\]
Therefore, since $U = (u_0, \ldots, u_{d-1})$ is unique, we use to consider the canonical functional of dimension $d$, $U = (u_0, \ldots, u_{d-1})$, saying that $\{P_n\}_{n \geq 0}$ is $d$-orthogonal (for any positive integer $d$) with respect to $U = (u_0, \ldots, u_{d-1})$ if
\[
\langle u_\nu, P_mP_n \rangle = 0, \quad n \geq md + \nu + 1, \quad m \geq 0,
\]
\[
\langle u_\nu, P_mP_{md+\nu} \rangle \neq 0, \quad m \geq 0,
\]
for each integer $\nu = 0, 1, \ldots, d-1$.

**Theorem 1.3.** [13] Let $\{P_n\}_{n \geq 0}$ be a MPS. The following assertions are equivalent:

a) $\{P_n\}_{n \geq 0}$ is $d$-orthogonal with respect to $U = (u_0, \ldots, u_{d-1})$.

b) $\{P_n\}_{n \geq 0}$ satisfies a $(d + 1)$-order recurrence relation $(d \geq 1)$:
\[
P_{m+d+1}(x) = (x - \beta_{m+d})P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \quad m \geq 0,
\]
with initial conditions
\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0 \quad \text{and if } d \geq 2:
\]
\[
P_n(x) = (x - \beta_{n-1})P_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} P_{n-2-\nu}(x), \quad 2 \leq n \leq d,
\]
and regularity conditions: $\gamma_{m+1}^0 = 0$, $m \geq 0$.

In this paper, we will focus on 2-orthogonal MPSs, thus fulfilling the recurrence relation
\[
P_{n+3}(x) = (x - \beta_{n+2})P_{n+2}(x) - \gamma_{n+2}^1 P_{n+1}(x) - \gamma_{n+1}^0 P_n(x),
\]
\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0, \quad P_2(x) = (x - \beta_1)P_1(x) - \gamma_1^1, \quad n \geq 0.
\]
While working solely with 2-orthogonality it is usual to rename the gammas as follows (cf. [4])
\[
P_{n+3}(x) = (x - \beta_{n+2})P_{n+2}(x) - \alpha_{n+2} P_{n+1}(x) - \gamma_{n+1}^0 P_n(x), \quad (1.6)
\]
\[
P_0(x) = 1, \quad P_1(x) = x - \beta_0, \quad P_2(x) = (x - \beta_1)P_1(x) - \alpha_1, \quad n \geq 0. \quad (1.7)
\]

## 2 Differential operators on $\mathcal{P}$ and technical identities

In this section, we list the main results indicated in [17] that will be applied along the text. Later on, we also prove new identities that are the fundamental utensils for the strategy pursued.

Given a sequence of polynomials $\{a_\nu(x)\}_{\nu \geq 0}$, let us consider the following linear mapping $J : \mathcal{P} \to \mathcal{P}$ (cf. [15], [19]).
\[
J = \sum_{\nu \geq 0} \frac{a_\nu(x)}{\nu!} D^\nu, \quad \deg a_\nu \leq \nu, \quad \nu \geq 0. \quad (2.1)
\]
Expanding \(a_\nu(x)\) as follows:

\[
a_\nu(x) = \sum_{i=0}^\nu a_{i}^{[\nu]} x^i,
\]

and recalling that \(D^\nu (\xi^n) (x) = \frac{n!}{(n-\nu)!} x^{n-\nu}\), we get the next identities about \(J\):

\[
J (\xi^n) (x) = \sum_{\nu=0}^n a_\nu(x) \binom{n}{\nu} x^{n-\nu}, \quad (2.2)
\]

\[
J (\xi^n) (x) = \sum_{\tau=0}^n \left( \sum_{\nu=0}^\tau \binom{\tau}{\nu} a_{[n-\nu]}^{[\tau-\nu]} \right) x^\tau, \quad n \geq 0. \quad (2.3)
\]

Most in particular, a linear mapping \(J\) is an isomorphism if and only if

\[
\text{deg} \left( J (\xi^n) (x) \right) = n, \quad n \geq 0, \quad \text{and} \quad J (1) (x) \neq 0. \quad (2.4)
\]

The next result establishes that any operator that does not increase the degree admits an expansion as (2.1) for certain polynomial coefficients.

**Lemma 2.1.** [17] For any linear mapping \(J\), not increasing the degree, there exists a unique sequence of polynomials \(\{a_n\}_{n \geq 0}\), with \(\text{deg} a_n \leq n\), so that \(J\) is read as in (2.1). Further, the linear mapping \(J\) is an isomorphism of \(P\) if and only if

\[
\sum_{\mu=0}^n \binom{n}{\mu} a_{[\mu]}^{[\mu]} \neq 0, \quad n \geq 0. \quad (2.5)
\]

The technique that we will implement in the next section require the knowledge about the \(J\)-image of the product of two polynomials \(fg\). The polynomial \(J (fg)\) is then given by a Leibniz-type development [17] as mentioned in the next Lemma.

**Lemma 2.2.** [17] For any \(f, g \in P\), we have:

\[
J (f(x)g(x)) (x) = \sum_{n \geq 0} J^{(n)} (f) (x) g^{(n)} (x) \frac{x^n}{n!} = \sum_{n \geq 0} J^{(n)} (g) (x) f^{(n)} (x) \frac{x^n}{n!}, \quad (2.6)
\]

where the operator \(J^{(m)}\), \(m \geq 0\), on \(P\) is defined by

\[
J^{(m)} = \sum_{n \geq 0} a_{n+m} x^n n! D^m. \quad (2.7)
\]

Let us suppose that \(J\) is an operator expressed as in (2.1), and acting as the derivative of order \(k\), for some non-negative integer \(k\), that is, it fulfils the following conditions.

\[
J (\xi^k) (x) = a_0^{[k]} \neq 0 \quad \text{and} \quad \text{deg} \left( J \left( \xi^{n+k} \right) (x) \right) = n, \quad n \geq 0; \quad (2.8)
\]

\[
J (\xi^i) (x) = 0, \quad 0 \leq i \leq k - 1, \quad \text{if} \ k \geq 1. \quad (2.9)
\]
Lemma 2.3. \[17\] An operator $J$ fulfills \((2.8)-(2.9)\) if and only if the next set of conditions hold.

a) $a_0(x) = \cdots = a_{k-1}(x) = 0$, if $k \geq 1$;

b) $\deg (a_\nu(x)) \leq \nu - k$, $\nu \geq k$;

c) 
\[
\lambda_{n+k}^{[k]} := \sum_{\nu=0}^{n} \binom{n+k}{n + k - \nu} a_{n+\nu}^{[n+k-\nu]} \neq 0, \quad n \geq 0. \tag{2.10}
\]

Remark 2.4. Note that in \((2.10)\) we find $\lambda_{k}^{[k]} = a_{k}^{[k]}$.

If $k = 0$, then it is assumed that $\lambda_{0}^{[0]} \neq 0, n \geq 0$, matching \((2.5)\), so that $J$ is an isomorphism.

If $k = 1$, then $J$ imitates the usual derivative and is commonly called a lowering operator (e.g. \([10, 16]\)).

Applying Lemma 2.2 to different pairs of polynomials, we obtain immediately the next identities.

\[
J(xp(x)) = xJ(p(x)) + J^{(1)}(p(x)) \tag{2.11}
\]

\[
J(x^2p(x)) = x^2J(p(x)) + 2xJ^{(1)}(p(x)) + J^{(2)}(p(x)) \tag{2.12}
\]

\[
J(x^3p(x)) = x^3J(p(x)) + 3x^2J^{(1)}(p(x)) + 3xJ^{(2)}(p(x)) + J^{(3)}(p(x)) \tag{2.13}
\]

Proposition 2.5. Given an operator $J$ defined by \((2.1)\), and taking into account the definition of the operator $J^{(m)}$, $m \geq 0$:

\[
J^{(m)} = \sum_{n \geq 0} \frac{a_{n+m}(x)}{n!} D^n,
\]

the following identities hold.

\[
J^{(i)}(xp(x)) = J^{(i+1)}(p(x)) + xJ^{(i)}(p(x)) , \quad i = 0, 1, 2, \ldots. \tag{2.14}
\]

Proof. Reading $i = 0$ in \((2.14)\) we find the identity stated in \((2.11)\). Let us now consider \((2.11)\) with $p(x)$ filled by the product $xp(x)$:

\[
J(x^2p(x)) = x^2J(p(x)) + xJ(xp(x)). \tag{2.15}
\]

The last term $xJ(xp(x))$ can be rephrased taking into account \((2.11)\), yielding

\[
J(x^2p(x)) = x^2J(p(x)) + xJ^{(1)}(p(x)) + J^{(1)}(xp(x)). \tag{2.15}
\]

Confronting \((2.12)\) with \((2.15)\), we conclude \((2.14)\) with $i = 1$:

\[
J^{(1)}(xp(x)) = J^{(2)}(p(x)) + xJ^{(1)}(p(x)).
\]

Let us assume as induction hypotheses over $k \geq 2$ that

\[
J^{(i)}(xp(x)) = J^{(i+1)}(p(x)) + xJ^{(i)}(p(x)) , \quad i = 0, \ldots, k - 1.
\]
In view of Lemma 2.2, we learn that for any polynomial $p = p(x)$

$$J(x^{k+1}p) = \sum_{n \geq 0} J^{(n)}(p) \frac{(x^{k+1})^{(n)}}{n!};$$

and thus we may write:

$$J(x^{k+1}p) = \sum_{\mu = 0}^{k+1} J^{(\mu)}(p) \binom{k+1}{\mu} x^{k+1-\mu};$$

$$J(x^k p) = \sum_{\nu = 0}^{k} J^{(\nu)}(p) \binom{k}{\nu} x^{k-\nu}. \tag{2.17}$$

Let us now consider (2.17) with $p$ filled by the product $xp$ as follows:

$$J(x^{k+1}p) = \sum_{\nu = 0}^{k} J^{(\nu)}(xp) \binom{k}{\nu} x^{k-\nu}. \tag{2.18}$$

By means of the induction hypotheses, identity (2.18) asserts the following.

$$J(x^{k+1}p) = \sum_{\nu = 0}^{k-1} \left( J^{(\nu+1)}(p) + x J^{(\nu)}(p) \right) \binom{k}{\nu} x^{k-\nu} + J^{(k)}(xp)$$

$$= \sum_{\nu = 0}^{k-1} J^{(\nu+1)}(p) \binom{k}{\nu} x^{k-\nu} + \sum_{\nu = 1}^{k-1} J^{(\nu)}(p) \binom{k}{\nu} x^{k+1-\nu} + J^{(k)}(xp) + J(p)x^{k+1}$$

$$= \sum_{\nu = 0}^{k-2} J^{(\nu+1)}(p) \left( \binom{k}{\nu} + \binom{k}{\nu+1} \right) x^{k-\nu} + J^{(k)}(p) \binom{k}{k-1} x + J^{(k)}(xp) + J(p)x^{k+1}$$

$$= \sum_{\nu = 0}^{k-2} J^{(\nu+1)}(p) \binom{k+1}{\nu+1} x^{k-\nu} + J^{(k)}(p)kx + J^{(k)}(xp) + J(p)x^{k+1}$$

$$= \sum_{\nu = 0}^{k-1} J^{(\nu)}(p) \binom{k+1}{\nu} x^{k+1-\nu} + J^{(k)}(p)kx + J^{(k)}(xp)$$

In brief

$$J(x^{k+1}p) = \sum_{\nu = 0}^{k-1} J^{(\nu)}(p) \binom{k+1}{\nu} x^{k+1-\nu} + J^{(k)}(p)kx + J^{(k)}(xp). \tag{2.19}$$

Comparing (2.19) with (2.16), we get

$$J^{(k)}(p) \binom{k+1}{k} x^{k+1-k} + J^{(k+1)}(p) \binom{k+1}{k+1} = kx J^{(k)}(p) + J^{(k)}(xp)$$

hence $x J^{(k)}(p) + J^{(k+1)}(p) = J^{(k)}(xp),$

which ends the proof.
3 An isomorphism applied to a 2-orthogonal sequence

In the sequel, we consider that \( J \) is an isomorphism and \( a_\nu(x) = 0 \), \( \nu \geq 4 \), thus

\[
J = a_0(x)I + a_1(x)D + \frac{a_2(x)}{2}D^2 + \frac{a_3(x)}{3!}D^3,
\]

where

\[
a_0(x) = a_0^0, \quad a_1(x) = a_0^1 + a_1^1 x, \quad a_2(x) = a_0^2 + a_1^2 x + a_2^2 x^2,
\]

\[
a_3(x) = a_0^3 + a_1^3 x + a_2^3 x^2 + a_3^3 x^3,
\]

and we suppose that the MPS \( \{P_n\}_{n \geq 0} \) is 2-orthogonal and fulfills

\[
J(P_n(x)) = \lambda_n^0 P_n(x), \quad \text{with } \lambda_n^0 \neq 0, \quad n \geq 0.
\]  

(3.2)

where

\[
\lambda_n^0 = a_0^0 + \binom{n}{1} a_1^1 + \binom{n}{2} a_2^2 + \binom{n}{3} a_3^3, \quad n \geq 0.
\]

In view of \( a_\nu(x) = 0 \), \( \nu \geq 4 \), the operators \( J^{(1)} \), \( J^{(2)} \) and \( J^{(3)} \) have the following definitions as indicated in (2.7).

\[
J^{(1)}(p) = \left( a_1(x)I + a_2(x)D + \frac{a_3(x)}{2}D^2 \right)(p)
\]  

(3.3)

\[
J^{(2)}(p) = (a_2(x)I + a_3(x)D)(p)
\]  

(3.4)

\[
J^{(3)}(p) = a_3(x)p
\]  

(3.5)

\[
J^{(m)}(p) = 0, \quad m \geq 4.
\]

Broadly speaking, in this section we will intertwine the action of operators \( J^{(k)} \), for initial values of \( k \), with the simple multiplication by the monomial \( x \), herein called \( T_x \):

\[
T_x : p \mapsto xp,
\]

in order to obtain the expansions of polynomials \( J^{(1)}(P_n(x)) \), \( J^{(2)}(P_n(x)) \) and \( J^{(3)}(P_n(x)) \) in the basis formed by the 2-orthogonal MPS \( \{P_n(x)\}_{n \geq 0} \).

Most importantly, we review (1.6)-(1.7) by establishing the following definition, considering henceforth \( P_{-i}(x) = 0 \), \( i = 1, 2, \ldots \).

\[
T_x(P_n(x)) = P_{n+1}(x) + \beta_n P_n(x) + \alpha_n P_{n-1}(x) + \gamma_{n-1} P_{n-2}(x), \quad n \geq 0.
\]  

(3.6)

Additionally, we can use the knowledge provided by Proposition 2.5, valid for all operators not decreasing the degree (2.7), that asserts

\[
J^{(i)}(T_x(p)) = J^{(i+1)}(p) + T_x\left(J^{(i)}(p)\right), \quad i = 0, 1, 2, \ldots.
\]  

(3.7)
First step: applying $J$ to the four-term recurrence

Let us apply the operator $J$ to the recurrence relation (1.6), using both (3.7), with $i = 0$, and (3.2):

$$
\lambda_{n+2}^{[0]} T_x (P_{n+2}(x)) + J^{(1)} (P_{n+2}(x)) = \lambda_{n+3}^{[0]} P_{n+3}(x)
$$

$$
+ \beta_{n+2} \lambda_{n+1}^{[0]} P_{n+2}(x) + \alpha_{n+2} \lambda_{n+1}^{[0]} P_{n+1}(x) + \gamma_{n+1} \lambda_{n}^{[0]} P_{n}(x).
$$

Next, by (3.6) we get $J^{(1)} (P_{n+2}(x))$ in the basis $\{ P_n(x) \}_{n \geq 0}$:

$$
J^{(1)} (P_{n+2}(x)) = \left( \lambda_{n+3}^{[0]} - \lambda_{n+2}^{[0]} \right) P_{n+3}(x)
$$

$$
+ \alpha_{n+2} \left( \lambda_{n+1}^{[0]} - \lambda_{n+2}^{[0]} \right) P_{n+1}(x) + \gamma_{n+1} \left( \lambda_{n}^{[0]} - \lambda_{n+1}^{[0]} \right) P_{n}(x), \ n \geq 0.
$$

Taking into account the information retained in identities $J (P_0(x)) = \lambda_0^{[0]} P_0(x)$, $J (P_1(x)) = \lambda_1^{[0]} P_1(x)$, it is easy to verify that

$$
a_{1}(x) = J^{(1)} (P_{0}(x)) = \left( \lambda_{1}^{[0]} - \lambda_{0}^{[0]} \right) P_{1}(x),
$$

$$
a_{1}(x) P_{1}(x) + a_{2}(x) = J^{(1)} (P_{1}(x)) = \left( \lambda_{2}^{[0]} - \lambda_{1}^{[0]} \right) P_{2}(x) + \alpha_{1} \left( \lambda_{1}^{[0]} - \lambda_{1}^{[0]} \right) P_{0}(x),
$$

and thus we may define the image of every $P_n(x)$ through the operator $J^{(1)}$ as follows:

$$
J^{(1)} (P_{n}(x)) = \left( \lambda_{n+1}^{[0]} - \lambda_{n}^{[0]} \right) P_{n+1}(x)
$$

$$
+ \alpha_{n} \left( \lambda_{n}^{[0]} - \lambda_{n}^{[0]} \right) P_{n-1}(x) + \gamma_{n} \left( \lambda_{n}^{[0]} - \lambda_{n}^{[0]} \right) P_{n-2}(x), \ n \geq 0.
$$

Second step: applying $J^{(1)}$ to the four-term recurrence

Let us now apply operator $J^{(1)}$ to the recurrence relation (1.6) fulfilled by $\{ P_n(x) \}_{n \geq 0}$:

$$
J^{(1)} (T_x (P_{n+2}(x))) = J^{(1)} (P_{n+3}(x))
$$

$$
+ \beta_{n+2} J^{(1)} (P_{n+2}(x)) + \alpha_{n+2} J^{(1)} (P_{n+1}(x)) + \gamma_{n+1} J^{(1)} (P_{n}(x)).
$$

We may then perform the following transformations:

$$
G_1(n) : \ J^{(1)} (T_x (P_{n+2}(x))) \rightarrow J^{(2)} (P_{n+2}(x)) + T_x \left( J^{(1)} (P_{n+2}(x)) \right),
$$

$$
I_1(n) : \ J^{(1)} (P_{n}(x)) \rightarrow \left( \lambda_{n+1}^{[0]} - \lambda_{n}^{[0]} \right) P_{n+1}(x)
$$

$$
+ \alpha_{n} \left( \lambda_{n}^{[0]} - \lambda_{n}^{[0]} \right) P_{n-1}(x) + \gamma_{n} \left( \lambda_{n}^{[0]} - \lambda_{n}^{[0]} \right) P_{n-2}(x),
$$

$$
M(n) : \ T_x (P_{n}(x)) \rightarrow P_{n+1}(x) + \beta_{n} P_{n}(x) + \alpha_{n} P_{n-1}(x) + \gamma_{n-1} P_{n-2}(x).
$$

These transformations are defined in a suitable computer software, allowing a symbolic implementation that executes the adequate positive increments on the
variable $n$. In this manner, it is possible for us to obtain the expansion of the image of $P_{n+2}(x)$ by operator $J^{(2)}$, in the basis $\{P_n(x)\}_{n\geq 0}$.

As a result of these computations, (3.11) corresponds to the next identity.

$$J^{(2)} (P_{n+2}(x)) = A_{n+4} P_{n+4}(x) + B_{n+3} P_{n+3}(x) + C_{n+2} P_{n+2}(x) + D_{n+1} P_{n+1}(x) + F_{n} P_{n}(x) + G_{n-1} P_{n-1}(x) + H_{n-2} P_{n-2}(x),$$

(3.12)

where

$$A_n = \lambda_n^{[0]} - 2\lambda_{n-1}^{[0]} + \lambda_{n-2}^{[0]};$$

$$B_n = (\beta_{n-1} - \beta_n) \left( \lambda_n^{[0]} - \lambda_{n-1}^{[0]} \right);$$

$$C_n = 2\alpha_{n+1} \left( \lambda_n^{[0]} - \lambda_{n+1}^{[0]} \right) + 2\alpha_n \left( \lambda_n^{[0]} - \lambda_{n-1}^{[0]} \right);$$

$$D_n = \alpha_{n+1} (\beta_{n+1} - \beta_n) \left( \lambda_n^{[0]} - \lambda_{n+1}^{[0]} \right) + \gamma_{n+1} \left( \lambda_n^{[0]} - 2\lambda_{n+2}^{[0]} + \lambda_{n+1}^{[0]} \right) + \gamma_n \left( \lambda_n^{[0]} - 2\lambda_{n-1}^{[0]} + \lambda_{n+1}^{[0]} \right);$$

$$F_n = \alpha_{n+2} \alpha_{n+1} \left( \lambda_n^{[0]} - 2\lambda_{n+1}^{[0]} + \lambda_{n+2}^{[0]} \right) + \gamma_{n+1} (\beta_{n+2} - \beta_n) \left( \lambda_n^{[0]} - \lambda_{n+2}^{[0]} \right);$$

$$G_n = \alpha_{n+3} \gamma_{n+1} \left( \lambda_n^{[0]} - 2\lambda_{n+2}^{[0]} + \lambda_{n+3}^{[0]} \right) + \alpha_{n+1} \gamma_{n+2} \left( \lambda_n^{[0]} - 2\lambda_{n+1}^{[0]} + \lambda_{n+3}^{[0]} \right);$$

$$H_n = \gamma_{n+3} \gamma_{n+1} \left( \lambda_n^{[0]} - 2\lambda_{n+2}^{[0]} + \lambda_{n+4}^{[0]} \right).$$

Once more, taking into account that $J (P_i(x)) = \lambda_i^{[0]} P_i(x), i = 0, 1,$ and also (3.10) for $n = 0, 1, 2,$ we are able to confirm that the following initial identities hold:

$$J^{(2)} (P_0(x)) = A_2 P_2(x) + B_1 P_1(x) + C_0 P_0(x),$$

$$J^{(2)} (P_1(x)) = A_3 P_3(x) + B_2 P_2(x) + C_1 P_1(x) + D_0 P_0(x),$$

and, hence:

$$J^{(2)} (P_n(x)) = A_{n+2} P_{n+2}(x) + B_{n+1} P_{n+1}(x) + C_n P_n(x) + D_{n-1} P_{n-1}(x) + F_{n-2} P_{n-2}(x) + G_{n-3} P_{n-3}(x) + H_{n-4} P_{n-4}(x), n \geq 0.$$  

(3.14)

**Third step:** applying $J^{(2)}$ to the four-term recurrence

Let us now apply operator $J^{(2)}$ to the recurrence relation (1.6) fulfilled by $\{P_n(x)\}_{n\geq 0}$:

$$J^{(2)} (T_x (P_{n+2}(x))) = J^{(2)} (P_{n+3}(x)),$$

$$\beta_{n+2} J^{(2)} (P_{n+2}(x)) + \alpha_{n+2} J^{(2)} (P_{n+1}(x)) + \gamma_{n+1} J^{(2)} (P_n(x)).$$

We may perform the following transformations:

$$G_2(n) : J^{(2)} (T_x (P_{n+2}(x))) \to J^{(3)} (P_{n+2}(x)) + T_x \left( J^{(2)} (P_{n+2}(x)) \right),$$

$$I_2(n) : J^{(2)} (P_n(x)) = A_{n+2} P_{n+2}(x) + B_{n+1} P_{n+1}(x) + C_n P_n(x) + D_{n-1} P_{n-1}(x) + F_{n-2} P_{n-2}(x) + G_{n-3} P_{n-3}(x) + H_{n-4} P_{n-4}(x),$$

$$M(n) : T_x (P_n(x)) \to P_{n+1}(x) + \beta_n P_n(x) + \alpha_{n-1} P_{n-1}(x) + \gamma_{n-1} P_{n-2}(x).$$
As before, these transformations and consequent simplifications, permit to express $J^{(3)} (P_{n+2}(x))$ as follows.

\[
J^{(3)} (P_{n+2}(x)) = a_3^{[3]} P_{n+5}(x) + (A_{n+4} \beta_{n+2} - A_{n+4} \beta_{n+4} - B_{n+3} + B_{n+4}) P_{n+4}(x) \\
+ (A_{n+3} \alpha_{n+2} - A_{n+4} \alpha_{n+4} + B_{n+3} \beta_{n+2} - B_{n+3} \beta_{n+4} - C_{n+2} + C_{n+3}) P_{n+3}(x) \\
+ (A_{n+2} \gamma_{n+1} - A_{n+4} \gamma_{n+3} + B_{n+2} \alpha_{n+2} - B_{n+3} \alpha_{n+3} - D_{n+1} + D_{n+2}) P_{n+2}(x) \\
+ (B_{n+1} \gamma_{n+1} - B_{n+3} \gamma_{n+2} + C_{n+1} \alpha_{n+2} - C_{n+2} \alpha_{n+2} \\
- D_{n+1} \beta_{n+1} + D_{n+1} \beta_{n+2} - F_n + F_{n+1}) P_{n+1}(x) \\
+ (C_{n} \gamma_{n+1} - C_{n+2} \gamma_{n+1} - D_{n+1} \alpha_{n+1} + D_{n} \alpha_{n+2} \\
- F_n \beta_n + F_n \beta_{n+2} - G_{n-1} + G_n) P_n(x) \\
+ (-D_{n+1} \gamma_n + D_{n-1} \gamma_{n+1} - F_n \alpha_n + F_{n-1} \alpha_{n+2} \\
- G_{n-1} \beta_{n-1} + G_{n-1} \beta_{n+2} - H_{n-2} + H_{n-1}) P_{n-1}(x) \\
+ (-F_n \gamma_{n-1} + F_{n-2} \gamma_{n+1} \\
- G_{n-1} \alpha_{n-1} + G_{n-2} \alpha_{n+2} - H_{n-2} \beta_{n-2} + H_{n-2} \beta_{n+2}) P_{n-2}(x) \\
+ (-G_{n-1} \gamma_{n-2} + G_{n-3} \gamma_{n+1} - H_{n-2} \alpha_{n-2} + H_{n-3} \alpha_{n+2}) P_{n-3}(x) \\
+ (H_{n-4} \gamma_{n+1} - H_{n-2} \gamma_{n-3}) P_{n-4}(x) , n \geq 0 ,
\]

with initial conditions:

\[
J^{(3)} (P_0(x)) = a_3^{[3]} P_3(x) + \left( (\beta_0 + \beta_1 + \beta_2) a_1^{[3]} + a_2^{[3]} \right) P_2(x) \\
+ \left( a_3^{[3]} (a_1 + a_2 + \beta_1 + \beta_0 + \beta_2) + (\beta_0 + \beta_1) a_2^{[3]} + a_1^{[3]} \right) P_1(x) \\
+ \left( a_3^{[3]} (a_1 (2\beta_0 + \beta_1) + \beta_0 + \gamma_1) + a_1 a_2^{[3]} + \beta_0 (\beta_0 a_2^{[3]} + a_1^{[3]}) + a_1^{[3]} \right) ;
\]

\[
J^{(3)} (P_1(x)) = a_3^{[3]} P_4(x) + \left( (\beta_1 + \beta_2 + \beta_3) a_1^{[3]} + a_2^{[3]} \right) P_3(x) \\
+ \left( a_3^{[3]} (a_1 + a_2 + a_3 + \beta_2 + \beta_1 + \beta_2) + (\beta_1 + \beta_2) a_2^{[3]} + a_1^{[3]} \right) P_2(x) \\
+ \left( a_3^{[3]} (2 (a_1 + a_2) \beta_1 + a_2 \beta_2 + \beta_1 + \gamma_1 + \gamma_2) + a_1 \beta_0 a_3^{[3]} + (a_1 + a_2) a_2^{[3]} \\
+ \beta_1 (\beta_1 a_2^{[3]} + a_1^{[3]}) + a_0^{[3]} \right) P_1(x) \\
+ \left( a_1 a_2^{[3]} (a_1 + a_2 + \beta_1 + \beta_2) + (\beta_0 + \beta_1) a_2^{[3]} + a_1^{[3]} \right) + a_1 a_2^{[3]} \\
+ \gamma_1 (\beta_0 + \beta_1 + \beta_2) a_1^{[3]} + a_2^{[3]} \right) .
\]

Recalling that $J^{(3)} (p) = a_3(x)p = \left( a_3^{[3]} x^3 + a_2^{[3]} x^2 + a_1^{[3]} x + a_0^{[3]} \right) p$, identity (3.15) enables the computation of the recurrence coefficients $(\beta_n)_{n \geq 0}$, $(\alpha_n)_{n \geq 1}$ and $(\gamma_n)_{n \geq 1}$ of a 2-orthogonal $\{P_n\}_{n \geq 0}$ that is the solution of $J (P_n) = \lambda_0^{[n]} P_n(x)$, $n \geq 0$, for a third-order $J$. We will pursue with such computations in the next section for particular cases.
4 Finding the 2-orthogonal solution of some third-order differential equations

Let us now assume that the 2-orthogonal MPS \( \{ P_n \}_{n \geq 0} \) fulfills \( J(P_n) = \lambda_n^{[0]} P_n(x) \), \( n \geq 0 \), where \( J \) is defined by (2.11) with \( a_\nu(x) = 0, \nu \geq 4 \).

Initially, we consider that \( \deg(a_3(x)) = 0 \), though \( a_3(x) \neq 0 \), \( \deg(a_2(x)) \leq 1 \) and \( \deg(a_1(x)) = 1 \). In other words:

\[
\begin{align*}
(a_0(x)I + a_1(x)D + \frac{a_2(x)}{2}D^2 + \frac{a_3(x)}{3!}D^3)(P_n(x)) &= \lambda_n^{[0]} P_n(x), \\
(a_0(x) = a_0^{[0]}, a_1(x) = a_1^{[1]} + a_1^{[2]}x, a_1^{[1]} \neq 0, \\
a_2(x) = a_2^{[2]} + a_2^{[3]}x, \\
a_3(x) = a_3^{[3]} \neq 0.
\end{align*}
\]

Consequently, \( \lambda_n^{[0]} = na_1^{[1]} + a_0^{[0]} \), which we are assuming as nonzero for all non-negative integer \( n \). Taking into account this set of hypotheses, identity (3.15) provides several difference equations due to the linear independence of \( \{ P_n \}_{n \geq 0} \).

In particular, the coefficient of \( P_{n+4}(x) \) on the right hand of (3.15) is expressed by

\[ -a_1^{[1]} (\beta_{n+2} - 2\beta_{n+3} + \beta_{n+4}) \]

and thus we get the equation

\[ \beta_{n+4} - 2\beta_{n+3} + \beta_{n+2} = 0, \quad n \geq 0. \]  \( (4.2) \)

Also, the coefficients of \( P_{n+3}(x) \) and \( P_{n+2}(x) \) on the right hand of (3.15) provide the following two identities

\[ a_1^{[1]} (-2\alpha_{n+2} + 4\alpha_{n+3} - 2\alpha_{n+4} + (\beta_{n+2} - \beta_{n+3})^2) = 0, \]  \( (4.3) \)

\[ -3a_1^{[1]} (\gamma_{n+1} - 2\gamma_{n+2} + \gamma_{n+3}) = a_0^{[3]} . \]  \( (4.4) \)

Taking into account the results presented in the previous sections 2 and 3, we have proved the following Proposition.

**Proposition 4.1.** Let us consider a 2-orthogonal polynomial sequence \( \{ P_n \}_{n \geq 0} \) fulfilling

\[ J(P_n(x)) = \lambda_n^{[0]} P_n(x) \]

where \( J \) is defined by (2.11) with \( a_\nu(x) = 0, \nu \geq 4, \) and such that \( a_0(x) = a_0^{[0]}, a_1(x) = a_1^{[1]} + a_1^{[2]}x, a_1^{[1]} \neq 0, \ a_2(x) = a_0^{[2]} + a_2^{[3]}x, \ a_3(x) = a_0^{[3]} \neq 0. \)
Then the coefficient $a_1^{[2]}$ of polynomial $a_2(x)$ is zero and the recurrence coefficients of the sequence $\{P_n\}_{n \geq 0}$ are the following.

$$\beta_n = -\frac{a_0^{[1]}}{a_1^{[1]}}, \ n \geq 0, \quad (4.5)$$

$$\alpha_{n+1} = -\frac{a_0^{[2]}}{2a_1^{[2]}}(n+1), \ n \geq 0, \quad (4.6)$$

$$\gamma_{n+1} = -\frac{a_0^{[3]}}{a_1^{[3]}}\left(\frac{1}{3} + \frac{1}{2}n + \frac{1}{6}n^2\right) = -\frac{a_0^{[3]}}{6a_1^{[3]}}(n+1)(n+2), \ n \geq 0. \quad (4.7)$$

Conversely, the 2-orthogonal polynomial sequence $\{P_n\}_{n \geq 0}$ defined by the recurrence coefficients $(4.5)$-$(4.7)$ fulfills the third order differential equation

$$J(P_n(x)) = \lambda_n^{[0]}P_n(x), \ n \geq 0,$$

where $a_0(x) = a_0^{[0]}$, $a_1(x) = a_0^{[1]} + a_1^{[1]}x$, $a_1^{[1]} \neq 0$, $a_2(x) = a_0^{[2]}$, $a_3(x) = a_0^{[3]} \neq 0$, and $\alpha_{\nu}(x) = 0$, $\nu \geq 4$.

Concerning the assumptions of this last Proposition, it is worth mention that, later on, it is clarified in Proposition 4.3 that if $\deg(a_3(x)) = 0$, though $a_3(x) \neq 0$, and $\deg(a_2(x)) = 0$, then $a_1^{[1]} \neq 0$.

It is also important to remark that the 2-orthogonal sequence described in Proposition 4.1 corresponds to a case, called $E$, of page 82 of [8]. We then conclude that the single 2-orthogonal polynomial sequence fulfilling the differential identity described in Proposition 4.1 is classical in Hahn’s sense, which means that the sequence of the derivatives $Q_n(x) := \frac{1}{n+1}DP_{n+1}(x)$, $n \geq 0$, is also a 2-orthogonal polynomial sequence. Furthermore, we read in [8] (p. 104) that this sequence is an Appell sequence, in other words, $Q_n(x) = P_n(x)$, $n \geq 0$. We review this detail while working with the intermediate relations $(3.9)$ and $(3.12)$ along the proof of Proposition 4.1 and based on those two identities we may indicate as corollary the following two differential identities.

**Corollary 4.2.** Let us consider the 2-orthogonal polynomial sequence $\{P_n\}_{n \geq 0}$ described in Proposition 4.1 that fulfills

$$J(P_n(x)) = \lambda_n^{[0]}P_n(x)$$

where $J$ is defined by $(2.1)$ with $\alpha_{\nu}(x) = 0$, $\nu \geq 4$, and such that $a_0(x) = a_0^{[0]}$, $a_1(x) = a_0^{[1]} + a_1^{[1]}x$, $a_1^{[1]} \neq 0$, $a_2(x) = a_0^{[2]}$, $a_3(x) = a_0^{[3]} \neq 0$. The sequence $\{P_n\}_{n \geq 0}$ also fulfills the following two identities

$$\left(a_1(x)I + a_0^{[2]}D + \frac{1}{2}a_0^{[3]}D^2\right)(P_n(x)) = a_1^{[1]}P_{n+1}(x)$$

$$+ \frac{1}{2}n a_0^{[2]}P_{n-1}(x) + \frac{1}{3}(n-1)n a_0^{[3]}P_{n-2}(x),$$

$$DP_n(x) = nP_{n-1}(x), \ n \geq 0, \ P_{-1}(x) = 0.$$
In the next proposition, we sum up a list of further conclusions pointed out by the application of the approach detailed in section 3.

**Proposition 4.3.** Let us consider a 2-orthogonal polynomial sequence \( \{P_n\}_{n \geq 0} \) fulfilling

\[
J(P_n(x)) = \lambda_n^0 P_n(x),
\]

where \( J \) is defined by (2.1) with \( a_\nu(x) = 0, \nu \geq 4 \).

a) If \( a_2(x) = a_0^2 \) (constant) and \( \deg(a_3(x)) \leq 2 \), though \( a_3(x) \neq 0 \), then \( a_1^1 \neq 0 \).

b) If \( a_2(x) = 0 \) and \( \deg(a_3(x)) = 1 \), then there isn’t a 2-orthogonal polynomial sequence \( \{P_n\}_{n \geq 0} \) such that \( J(P_n(x)) = \lambda_n^0 P_n(x), n \geq 0 \).

c) If \( a_3(x) = 0 \), then the only solution of \( J(P_n(x)) = \lambda_n^0 P_n(x) \) corresponds to \( J = a_0^1 D + a_0^0 I \).

Applying the identities shown in sections 2 and 3, we are able to prove the forthcoming results. In particular we bring to light the 2-orthogonal sequence defined in Corollary 4.5, as well as other differential identities fulfilled by a 2-orthogonal solution besides the prefixed one \( J(P_n(x)) = \lambda_n^0 P_n(x), n \geq 0 \).

In Theorem 4.4, we find the description of the 2-orthogonal sequence that is the solution of the problem posed with respect to the third order operator \( J \) defined by the conditions \( a_2(x) = 0 \) and \( \deg(a_3(x)) \leq 2 \). Taking into consideration Proposition 4.3 we have assured that \( \deg(a_1(x)) = 1 \), or \( a_1^1 \neq 0 \).

**Theorem 4.4.** Let us consider a 2-orthogonal polynomial sequence \( \{P_n\}_{n \geq 0} \) fulfilling

\[
J(P_n(x)) = \lambda_n^0 P_n(x), \quad n \geq 0,
\]

where \( J \) is defined by (2.1) with \( a_\nu(x) = 0, \nu \geq 4 \), and such that \( a_0(x) = a_0^0, \quad a_1(x) = a_0^1 + a_1^1 x, \quad a_1^1 \neq 0, \quad a_2(x) = 0, \quad a_3(x) = a_0^3 + a_1^3 x + a_2^3 x^2 \).

Then the recurrence coefficients of the sequence \( \{P_n\}_{n \geq 0} \) are the following and the coefficients of the polynomial \( a_3(x) = a_2^3 x^2 + a_1^3 x + a_0^3 \) fulfill

\[
(a_1^3)^2 - 4a_2^3 a_0^3 = 0.
\]
\[
\beta_n = -\frac{a_2^{[3]}}{2a_1^{[1]}}(n-1)n - \frac{a_0^{[1]}}{a_1^{[1]}}, \; n \geq 0, \quad (4.8)
\]

\[
\alpha_n = -\frac{a_2^{[3]}}{2a_1^{[1]}} + \frac{a_0^{[1]}a_2^{[3]}}{(a_1^{[1]})^2} + (n-2) \left( -\frac{3a_2^{[3]}}{4a_1^{[1]}} + \frac{a_2^{[3]}(9a_0^{[1]} + a_2^{[3]})}{6(a_1^{[1]})^2} \right) + (n-2)^2 \left( b_0 + b_1(n-2) + b_2(n-2)^2 \right), \; n \geq 1, \quad (4.9)
\]

\[
\gamma_n = -\frac{1}{3a_1^{[1]}} \left( a_0^{[4]} + \frac{a_1^{[1]}(-a_1^{[1]}a_1^{[3]} + a_0^{[1]}a_2^{[3]})}{(a_1^{[1]})^2} \right)
- (n-1) \left( \frac{(a_1^{[1]})^2a_0^{[3]} - a_0^{[1]}a_1^{[1]}a_1^{[3]} + (a_1^{[1]})^2a_2^{[3]}}{2(a_1^{[1]})^3} \right)
+ (n-1)^2 \left( f_0 + f_1(n-1) + f_2(n-1)^2 + f_3(n-1)^3 + f_4(n-1)^4 \right), \; n \geq 1;
\]

where

\[
f_0 = \frac{-18a_0^{[3]}(a_1^{[1]})^2 + 6a_1^{[3]}a_1^{[4]} + 3a_0^{[1]} + a_2^{[3]} + a_2^{[3]}(-18(a_1^{[1]})^2 - 12a_1^{[3]}a_0^{[1]} + (a_2^{[3]})^2)}{108(a_1^{[1]})^3},
\]

\[
f_1 = \frac{a_2^{[3]}(6a_1^{[3]}a_1^{[1]} + a_2^{[3]}(a_2^{[3]} - 12a_0^{[1]}))}{72(a_1^{[1]})^3},
\]

\[
f_2 = -\frac{a_2^{[3]}(a_2^{[3]}(12a_0^{[1]} + a_2^{[3]}) - 6a_1^{[1]}a_1^{[3]})}{216(a_1^{[1]})^3},
\]

\[
f_3 = -\frac{(a_2^{[3]})^3}{72(a_1^{[1]})^3},
\]

\[
f_4 = -\frac{(a_2^{[3]})^3}{216(a_1^{[1]})^3}.
\]

Conversely, the 2-orthogonal polynomial sequence \(\{P_n\}_{n \geq 0}\) defined by the recurrence coefficients (4.8) - (4.10), under the assumption \(\gamma_n \neq 0, \; n \geq 1\), fulfills the differential equation \(J(P_n(x)) = \lambda_n^{[0]}P_n(x), \; n \geq 0\), where \(a_0(x) = a_0^{[0]}, \; a_1(x) = a_0^{[1]} + a_1^{[1]}x, \; a_1^{[1]} \neq 0, \; a_0(x) = a_1(x) = a_3(x) = a_3^{[3]}x^2 + a_1^{[3]}x + a_0^{[3]}\) with \((a_1^{[3]})^2 - 4a_2^{[3]}a_0^{[3]} = 0\), and \(a_0(x) = 0, \; \nu \geq 4\).
The content of Theorem 4.4 provides an entire solution written in terms of the polynomial coefficients of the operator $J$. In the next Corollary we read a specific case endowed with Hahn’s property, as we may prove analytically using the functionals of the dual sequence.

**Corollary 4.5.** Let us consider the 2-orthogonal polynomial sequence $\{P_n\}_{n \geq 0}$ fulfilling

$$J(P_n(x)) = \lambda_n^{[0]} P_n(x), \quad n \geq 0,$$

where $J$ is defined by (2.1) with $a_\nu(x) = 0$, $\nu \geq 4$, and such that $a_0(x) = a_0^{[0]}$, $a_1(x) = \frac{1}{24} x$, $a_2(x) = 0$, $a_3(x) = (x-1)^2$.

Then the recurrence coefficients of the sequence $\{P_n\}_{n \geq 0}$ are the following.

$$\beta_n = -12(n-1)n, \quad n \geq 0, \quad (4.11)$$

$$\alpha_n = 12(n-1)n(2n-3)^2, \quad n \geq 1, \quad (4.12)$$

$$\gamma_n = -4n(n+1)(2n-3)^2(2n-1)^2, \quad n \geq 1. \quad (4.13)$$

Conversely, the 2-orthogonal polynomial sequence $\{P_n\}_{n \geq 0}$ defined by the recurrence coefficients $\{4.11\}$-$\{4.13\}$ fulfils the differential equation

$$\left(\frac{1}{6}(x-1)^2D^3 + \frac{1}{24}xD + a_0^{[0]}I\right)(P_n(x)) = \lambda_n^{[0]} P_n(x), \quad n \geq 0,$$

where $\lambda_n^{[0]} = \frac{1}{24}n + a_0^{[0]}, \quad n \geq 0$.

Furthermore, we remark that the polynomial sequence $\{P_n\}_{n \geq 0}$ defined by $\{4.11\}$-$\{4.13\}$, fulfils the following two differential relations obtained by (3.9) and (3.12).

\[
\begin{align*}
\left(\frac{1}{24}xI + \frac{1}{2}(x-1)^2D^2\right)(P_n(x)) &= \frac{1}{24}P_{n+1}(x) \\
-\frac{1}{2}(3-2n)^2(n-1)nP_{n-1}(x) + \frac{1}{3}(n-1)n(15-16n+4n^2)^2P_{n-2}(x) \quad (4.14)
\end{align*}
\]

\[
\begin{align*}
(x-1)^2D(P_n(x)) &= nP_{n+1}(x) - 2n(5+4n(2n-3))P_n(x) \\
+ (3-2n)^2(n+24(n-2)n+25)P_{n-1}(x) - 8(5-2n)^2(n-1)n(2n-3)^3P_{n-2}(x) \\
+ 4(3-2n)^2(5-2n)^2(7-2n)(n-2)(n-1)nP_{n-3}(x), \quad n \geq 0, \quad P_{-1}(x) = 0.
\end{align*}
\]

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