On partition function and Weyl anomaly of conformal higher spin fields

A.A. Tseytlin

Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.

Abstract

We study 4-dimensional higher-derivative conformal higher spin (CHS) fields generalising Weyl graviton and conformal gravitino. They appear, in particular, as “induced” theories in the AdS/CFT context. We consider their partition function on curved Einstein-space backgrounds like (A)dS or sphere and Ricci-flat spaces. Remarkably, the bosonic (integer spin $s$) CHS partition function appears to be given by a product of partition functions of the standard 2nd-derivative “partially massless” spin $s$ fields, generalising the previously known expression for the 1-loop Weyl graviton ($s = 2$) partition function. We compute the corresponding spin $s$ Weyl anomaly coefficients $a_s$ and $c_s$. Our result for $a_s$ reproduces the expression found recently in arXiv:1306.5242 by an indirect method implied by AdS/CFT (which relates the partition function of a CHS field on $S^4$ to a ratio of known partition functions of massless higher spin field in AdS$_5$ with alternate boundary conditions). We also obtain similar results for the fermionic CHS fields. In the half-integer $s$ case the CHS partition function on (A)dS background is given by the product of squares of “partially massless” spin $s$ partition functions and one extra factor corresponding to a special massive conformally invariant spin $s$ field. It was noticed in arXiv:1306.5242 that the sum of the bosonic $a_s$ coefficients over all $s$ is zero when computed using the $\zeta$-function regularization, and we observe that the same property is true also in the fermionic case.

1Also at Lebedev Institute, Moscow. e-mail: tseytlin@imperial.ac.uk
1 Introduction

Free Lagrangians of massless spin $\frac{1}{2}$ fermion and vector spin 1 fields are conformally invariant in $D = 4$. This property is not shared by the standard spin $\frac{3}{2}$, 2 and higher spin massless fields. In view of potential importance of the conformal invariance condition one may wonder if there are alternative higher-spin models sharing the conformal invariance property with the familiar $s = \frac{1}{2}, 1$ theories. Giving up manifest unitarity, one may indeed view higher-derivative Weyl graviton and conformal gravitino [1, 2] as such $s = 2$ and $s = \frac{3}{2}$ examples. Conformal higher spin (CHS) models [3] are their $s > 2$ generalisations. Regardless possible more fundamental role of such theories [3, 4] they naturally appear in the context of AdS/CFT correspondence [5, 6, 7, 8, 9, 10, 11].

CHS theory describes pure spin $s$ field by a local action with maximal gauge invariance. Locality requires higher derivatives (unless one gives up minimal field content requirement and introduces auxiliary fields): free kinetic operator is $D_s = \partial^{2s} P_s$ where $P_s$ is pure spin $s$ (transverse traceless) field projector. When coupled to a metric, i.e. considered on a curved background, the corresponding action should be given by order $2s$ Weyl and reparametrization covariant differential operator $D_s(g) = \nabla^{2s} P_s + \ldots$ that generalizes the $s = 2$ 4-derivative operator appearing in the linearization of the Weyl gravity action or the 3-derivative $s = tri$ conformal gravitino operator.

The corresponding quantum 1-loop effective action (given by $\log \det D_s(g)$) will not, in general, be Weyl-invariant – there will be a non-trivial Weyl anomaly. The well-known low spin $s = \frac{1}{2}, 1, \frac{3}{2}, 2$ conformal field examples (massless fermion, Maxwell vector, Weyl graviton and conformal gravitino) have non-zero Weyl anomaly coefficients found for $s = \frac{1}{2}, 1$ in [20] (see also [21, 22]) and for $s = \frac{3}{2}, 2$ in [23, 24, 3].

The key consistency requirement is the preservation of all gauge symmetries, including the conformal symmetry, at the quantum level and thus the cancellation of the Weyl anomaly [3]. This may be possible to achieve by combining several higher spin fields together. The computation of Weyl anomaly in conformal supergravities [24, 25] led to the conclusion that the only known Weyl-invariant theory with spins $\leq 2$ is $\mathcal{N} = 4$ conformal supergravity [2] coupled to $\mathcal{N} = 4$ super Yang-Mills theory with any gauge group of rank 4. An interesting question is if there are other special CHS models involving spins $\geq 2$ that are also conformal at the quantum level.\(^2\)

To find the Weyl anomaly of a CHS field in $D = 4$ one would need to start with the corresponding Weyl-covariant $2s$-derivative kinetic operator in a background metric and compute the corresponding heat kernel coefficient $b_4$ (often called also $a_2$). One immediate problem is

\(^1\)In particular, CHS fields are “sources” for higher conserved conformal currents in the free limit of the boundary conformal theory and there are “kinematical” relations between ordinary massless higher spins in the bulk of AdS$_{D+1}$ space and conformal higher spins on the $D$-dimensional boundary discussed, e.g., in [12, 13, 14] (for some other discussions of conformal higher spins see also [15, 16]). There are also connections of conformal (super)gravity with twistor string theory [17] and with scattering in dS space [18, 19].

\(^2\)One option may be to sum over infinite number of higher spin modes (like in ordinary massless spin theories of Fradkin-Vasiliev type in AdS space), or one may try to explore possible existence of anomaly-free irreducible models with finite number of higher spins like hypothetical $N \geq 5$ conformal supergravities [3].
that such kinetic operator in curved metric is not known explicitly so far for \( s > 2 \).\(^3\) Moreover, even if such kinetic operators were known, computing their Weyl-anomaly coefficients would be extremely complicated given the absence of general higher-derivative algorithms for \( b_4 \) for \( s > 2 \) (cf. \([23, 24]\)).

This suggests to take a short-cut route originally used in \([26, 27, 3]\) as an efficient way to reproduce the Weyl anomaly of conformal graviton and conformal gravitino.\(^4\) First, to find the two \( D = 4 \) Weyl anomaly coefficients (\( a \) and \( c \)) it is sufficient to consider just two particular curved Einstein-space backgrounds (e.g., conformally-flat \((\mathrm{A})\text{dS}_4\) or \( S^4 \) one and a Ricci-flat one). Second, in the case of Einstein-space background one may expect the covariant higher-derivative operator to factorize into the product of standard 2-nd order differential operators whose anomaly can be computed using the standard algorithm. This factorization was indeed observed for the conformal graviton and conformal gravitino on the Ricci-flat and on conformally-flat background \([26, 27, 29, 30]\).\(^5\)

One may expect that such factorization should happen in general for CHS fields on Einstein-space backgrounds. In flat space one may replace (e.g., in bosonic case) a higher-derivative CHS Lagrangian \( \phi_i \partial^2 \phi_s \) by “ordinary-derivative” Lagrangian \( \sum_{n,m} (\epsilon_{nm} \phi_n \partial^2 \phi_m + \mu_{mn} \phi_m \phi_n) \) for a set of fields \( \phi_n \) (some of which are of course ghost-like) such that integrating all but one of them leads back to the original higher-derivative action. Such gauge-covariant “ordinary-derivative” formulations of both bosonic and fermionic CHS models in flat space were explicitly constructed in \([33, 34]\). Applying conformal transformation (by redefining the fields by powers of the conformal factor according to their conformal weights) one should then get the corresponding action on a conformally-flat background. We shall assume that such actions should exist also on a Ricci-flat background.

Then for \((\mathrm{A})\text{dS}\) or Ricci-flat backgrounds the higher-derivative CHS kinetic \((2s\) derivative bosonic or squared fermionic) operator should factorize into product of standard 2nd-derivative operators, so that the CHS Weyl anomaly can be found as a sum of Weyl anomalies of an effective set of “ordinary-derivative” fields.

In more detail, in 4 dimensions the non-trivial part of the Weyl anomaly is determined by \([32]\) (we omit the \( \nabla^2 R \) term)

\[
b_4 = \beta_1 R^* R^* + \beta_2 W = -a R^* R^* + c C^2 ,
\]

\(^3\)Existence of conformal supergravity implies that \( s = 2 \) and \( s = \frac{3}{2} \) kinetic operators are consistent (have background gauge invariance) provided the background metric satisfies the Weyl gravity equations of motion (e.g., is an Einstein-space metric). Existence of interacting conformal higher-spin theory was explored in \([7, 10]\). Consistent cubic coupling of two higher spins with a conformal spin 2 field or the metric (e.g., via a product of two generalized Weyl tensors and standard linearized Weyl tensors) implies that higher spin gauge invariance should hold provided the metric satisfies the linearized Weyl gravity equations of motion. In general, one may expect that CHS kinetic operator in background metric will have consistent background gauge invariance provided the metric satisfies the non-linear Weyl gravity equations. We are grateful to K. Mkrtchyan and M. Vasiliev for useful remarks on this issue.

\(^4\)See \([28]\) for a recent use of similar idea in \( D = 6 \) context.

\(^5\)This factorization is not surprising given that the Weyl gravity Lagrangian \( R^2 - \frac{1}{5} R^2 \) can be written in terms of the “ordinary-derivative” fields as \( 2 R_{mn} \phi^m \phi^n + R \phi - \phi^2 + 3 \phi^2 \) where \( \phi_{mn} \) and \( \phi \) are the auxiliary traceless tensor and scalar fields. Similarly, the conformal gravitino action can be written in terms of three auxiliary spin \( \frac{3}{2} \) fields, symbolically, \( \varphi D \varphi + \psi D \psi + \chi (\varphi - D \psi) \) where \( \varphi \) will be the gravitino field strength on-shell \([1]\) (see also \([31]\]).
\[ a = -\beta_1 + \frac{1}{2} \beta_2 , \quad c = \frac{1}{2} \beta_2 , \quad \beta_1 = c - a , \quad \beta_2 = 2c . \tag{1.2} \]

Here \( R^* R^* \) is \( 32\pi^2 \) times the Euler number integrand and \( C^2 \) is the square of the Weyl tensor,

\[ C^2 = R^* R^* + 2W , \quad W = R^2_{mn} - \frac{1}{3} R^2 . \tag{1.3} \]

The integral of \( b_4 \) is related to the coefficient of the UV logarithmic divergence of the corresponding curved-space partition function \(^6\)

\[ \ln Z = -\frac{1}{2} \ln \det D = \frac{1}{2} \left( \frac{1}{4} L^4 B_0 + L^2 B_2 + B_4 \ln L^2 \right) + \text{finite} , \quad B_p = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} b_p . \tag{1.4} \]

Here \( b_0 = \nu \) is the number of effective degrees of freedom and \( b_2 = 0 \) for conformally covariant operators. The known values of the anomaly coefficients for low conformal spin cases \( s = 1, 2 \) and \( s = \frac{1}{2}, \frac{3}{2} \) are: \(^7\)

\[ \begin{align*}
  s &= 1 : \quad \beta_1 = -\frac{13}{180} , \quad \beta_2 = \frac{1}{5} , \quad a = \frac{31}{180} , \quad c = \frac{1}{10} , \\
  s &= 2 : \quad \beta_1 = -\frac{137}{60} , \quad \beta_2 = \frac{199}{15} , \quad a = \frac{87}{20} , \quad c = \frac{199}{30} , \\
  s &= \frac{1}{2} : \quad \beta_1 = \frac{7}{720} , \quad \beta_2 = \frac{1}{20} , \quad a = \frac{11}{720} , \quad c = \frac{1}{40} , \\
  s &= \frac{3}{2} : \quad \beta_1 = -\frac{173}{180} , \quad \beta_2 = -\frac{149}{30} , \quad a = -\frac{137}{90} , \quad c = -\frac{149}{60} .
\end{align*} \tag{1.5-1.8} \]

To find the two anomaly coefficients it should thus be sufficient to compute the logarithmically divergent part of the partition function on two particular curved backgrounds that have different values of \( R^* R^* \) and \( C^2 \). Obvious examples are provided by the Einstein spaces

\[ R_{mn} = \Lambda g_{mn} , \tag{1.9} \]

which solve the Weyl gravity equations of motion and, as was already mentioned above, should thus allow for a consistent higher-spin coupling. \(^8\) It is sufficient to consider the two special cases: a conformally-flat Einstein space (de Sitter \( \epsilon > 0 \) or Anti de Sitter \( \epsilon < 0 \)) \(^9\)

\[ C_{mnkl} = 0 , \quad R = 4\Lambda = 12\epsilon , \quad b_4 = -a R^* R^* = -24a \epsilon^2 , \tag{1.10} \]

\(^6\)Here \( \det D \) is assumed to include ghost contributions and \( L \to \infty \) is UV cutoff. Overall sign is changed in the case of fermions.

\(^7\)For standard real conformally-coupled scalar \( \beta_1 = \frac{1}{180} , \quad \beta_2 = \frac{1}{60} \), while for 4-derivative real conformal scalar \([24] \beta_1 = \frac{1}{30} , \quad \beta_2 = -\frac{1}{20} \). Let us recall also that for standard \( N = 4 \) conformal supergravity \( \beta_1 = 0 , \quad \beta_2 = -2 \) so that coupling it to exactly four \( N = 4 \) SYM multiplets (that have \( \beta_1 = 0 , \quad \beta_2 = \frac{1}{2} \)) leads to anomaly-free theory \([3] \). The cancellation is possible due to the fact that \( a \) and \( c \) of conformal gravitino have negative sign.

\(^8\)To have gauge independence of partition function of Weyl gravity in curved background one is to expand near solutions of the Weyl theory equations of motion. All solutions of (1.9) are solutions of these equations. Same applies to conformal gravitino: the existence of conformal supergravity \([1] \) implies consistent coupling of conformal gravitino to Weyl gravity and thus its partition function is gauge-independent provided the background metric satisfies the Weyl gravity equations of motion. It is natural to assume that all CHS fields can be consistently coupled to Weyl gravity, and then their kinetic operator should have background gauge invariance provided the metric satisfies Weyl equations, and, in particular, (1.9).

\(^9\)We shall mostly consider \( D = 4 \) case with Minkowski signature (for some notation and useful relations see Appendix) but all results will have straightforward continuation to the Euclidean signature case: in particular, (A)dS_4 can be replaced by S^4 background.
and a Ricci-flat space

\[ R_{mn} = 0 , \quad b_4 = \beta_1 R^* R^* = (c - a) C^2 . \quad (1.11) \]

Computing \( b_4 \) for (A)dS\(_4\) background will determine \( a \) and then computing \( b_4 \) for a Ricci-flat background will allow also to find \( c \).

As discussed above, the second key simplification is that for such special backgrounds the CHS kinetic operators should factorize into products of “ordinary-derivative” operators whose Weyl anomaly can be computed using the standard algorithm [35]. In (A)dS\(_4\) background (1.10) this factorization was observed for the Weyl graviton and conformal gravitino in [26, 27] and nearly simultaneously in [29, 30]. For example, the Weyl graviton operator factorizes into the product of the usual “massless” transverse traceless graviton operator and a massive transverse traceless spin 2 operator. The latter was noticed in [29] to have special conformal invariance property and was called “partially massless” spin 2 field. “Partially massless” (PM) higher spin fields in (A)dS\(_D\) exist also for \( s > 2 \) [36, 37, 38, 39, 40, 41, 42, 43] (see also [44, 45, 49]) and, as we shall suggest, are directly related to the factorization of the CHS operators on (A)dS\(_4\) background for all \( s \).

Starting with the “ordinary-derivative” formulation [33, 34], applying a conformal transformation and solving for all of the auxiliary fields one should end up with the factorized form of the CHS kinetic operator in the (A)dS\(_4\) background. As we shall suggest below, the \( s \) of the 2nd-derivative operator factors that form the \( 2s \)-derivative bosonic spin \( s \) CHS kinetic operator can be identified precisely with the \( s \) species of the PM spin \( s \) fields with the mass parameters found in [37, 38, 39].\(^{10}\) As we shall see, a similar relation appears to exists also between the fermionic CHS operator and the fermionic PM fields [38, 43] in (A)dS\(_4\) plus an additional massive spin \( s \) field.

As was noted above, kinetic operators for the Weyl graviton and the conformal gravitino factorize also in a Ricci-flat background [27, 3]: here the factorization is even simpler – into the relevant powers (second and third) of the standard covariant massless graviton and massless gravitino operators. Given that CHS kinetic operator is defined on traceless and transverse symmetric tensors it is natural to expect that the same factorization pattern should apply also for all CHS cases with \( s > 2 \).

Assuming these factorization relations and including the relevant ghost determinant factors one is then able to express the CHS partition function on a conformally-flat or Ricci-flat background in terms of products of powers of determinants of ordinary 2nd-derivative operators and thus to compute the corresponding values of the Weyl anomaly coefficients using the standard algorithm for \( b_4 \) for \( \Delta_2 = -\nabla^2 + X \) type operators. This is the strategy that we will implement below.

Our results that generalize the low-spin expressions in (1.5)-(1.8) can be summarized as follows. In the bosonic integer spin \( s = 1, 2, 3, \ldots \) case the Weyl anomaly coefficients in (1.1) for a CHS field read

\[ a_s^{(b)} = \frac{1}{720} \nu_s (3 \nu_s + 14 \nu_s^2) , \quad \nu_s = \nu_s^{(b)} = s(s + 1) , \quad (1.12) \]

\[ c_s^{(b)} = \frac{1}{720} \nu_s (4 - 42 \nu_s + 29 \nu_s^2) , \quad (1.13) \]

\(^{10}\)A relation between a general \( s > 2 \) conformal higher spin field and partially massless fields in (A)dS was anticipated by E. Skvortsov and M. Vasiliev (unpublished) and also conjectured in [16].
while in the fermionic case of $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$ we found
\[ a_s^{(f)} = \frac{1}{2880} \nu_s (12 + 45\nu_s + 14\nu_s^2), \quad \nu_s = \nu_s^{(f)} = -2\left(s + \frac{1}{2}\right)^2, \tag{1.14} \]
\[ c_s^{(f)} = \frac{1}{2880} \nu_s (118 + 135\nu_s + 29\nu_s^2). \tag{1.15} \]

Here $\nu_s$ is the number of dynamical degrees of freedom of the corresponding CHS field (with fermionic modes counted with negative sign, i.e. $\nu_1 = 2$, $\nu_{\frac{1}{2}} = -2$, etc.).

Remarkably, the bosonic $a_s$ coefficient in (1.12) that we will find below directly in $D = 4$ matches the expression found recently in [11] by a completely different indirect method based on AdS$_5$/CFT$_4$ duality.\footnote{Ref. [11] started with the expected relation between the spin $s$ CHS partition function on $S^D$ and the ratio of the standard massless higher spin $s$ partition functions on AdS$_{D+1}$ [50, 51] with alternate boundary conditions which is implied by the consideration of the RG flow induced by the “double-trace” deformation [52, 53, 54, 55] of the boundary CFT by the square of the corresponding spin $s$ conformal current. The conformal anomaly $a$-coefficient was then extracted from the singular $\ln L$ term (cf. (1.4)) in the predicted CHS partition function on $S^4$.\footnote{The same idea was previously used in [56] to reproduce the $a$-coefficient in Weyl anomaly of the conformal 4-derivative scalar operator [24, 3, 57] and of its higher-dimensional generalizations (see Appendix B).\footnote{Note that our normalization of $a_s$ is different from the one used in [11] by factor of 4.}} Ref.[11] started with the expected relation between the spin $s$ CHS partition function on $S^D$ and the ratio of the standard massless higher spin $s$ partition functions on AdS$_{D+1}$ [50, 51] with alternate boundary conditions which is implied by the consideration of the RG flow induced by the “double-trace” deformation [52, 53, 54, 55] of the boundary CFT by the square of the corresponding spin $s$ conformal current. The conformal anomaly $a$-coefficient was then extracted from the singular $\ln L$ term (cf. (1.4)) in the predicted CHS partition function on $S^4$.\footnote{The idea of computation of $\beta_{1,s}$-coefficient using the assumption of factorization of $s > 2$ CHS operators on Ricci-flat background came up in our discussions with S. Giombi who independently found (1.13).\footnote{Instead of $\zeta$-function regularization one may, in fact, use any consistent analytic regularization (see discussion in section 5 below). Use of such regularization in computing infinite sums over spins appears to be natural and necessary in the AdS/CFT context, see [11, 58].\footnote{Such theory would be originating as an induced one starting with a large $N$ free complex scalar CFT with all $s > 0$ currents being gauged. One may then expect the induced CHS theory to be consistent [11], i.e. at least do not have logarithmic UV divergence in partition function on a sphere or IR divergence in the corresponding ratio of AdS partition functions with alternate boundary conditions, implying the vanishing of the sum of the $a_s$ coefficients.\footnote{Note that our normalization of $a_s$ is different from the one used in [11] by factor of 4.}}}}

It was observed in [11] that if one sums $a_s$ in (1.12) over all spins $s = 1, 2, ...$ and computes the (formally power-divergent) result using the $\zeta$-function regularization the final expression vanishes, suggesting that a theory containing each bosonic CHS field just once may be quantum-consistent.\footnote{It was observed in [11] that if one sums $a_s$ in (1.12) over all spins $s = 1, 2, ...$ and computes the (formally power-divergent) result using the $\zeta$-function regularization the final expression vanishes, suggesting that a theory containing each bosonic CHS field just once may be quantum-consistent.\footnote{Interestingly, the fermionic CHS coefficient $a_s$ in (1.14) also has the same property, i.e.}\footnote{Note that our normalization of $a_s$ is different from the one used in [11] by factor of 4.}} Interestingly, the fermionic CHS coefficient $a_s$ in (1.14) also has the same property, i.e.$\sum_{s=1}^{\infty} a_s^{(f)} \big|_{\text{reg}} = 0,$ $\sum_{s=\frac{1}{2}}^{\infty} a_s^{(f)} \big|_{\text{reg}} = 0$.\footnote{Instead of $\zeta$-function regularization one may, in fact, use any consistent analytic regularization (see discussion in section 5 below). Use of such regularization in computing infinite sums over spins appears to be natural and necessary in the AdS/CFT context, see [11, 58].\footnote{The idea of computation of $\beta_{1,s}$-coefficient using the assumption of factorization of $s > 2$ CHS operators on Ricci-flat background came up in our discussions with S. Giombi who independently found (1.13).\footnote{Such theory would be originating as an induced one starting with a large $N$ free complex scalar CFT with all $s > 0$ currents being gauged. One may then expect the induced CHS theory to be consistent [11], i.e. at least do not have logarithmic UV divergence in partition function on a sphere or IR divergence in the corresponding ratio of AdS partition functions with alternate boundary conditions, implying the vanishing of the sum of the $a_s$ coefficients.\footnote{Note that our normalization of $a_s$ is different from the one used in [11] by factor of 4.}}}}

At the same time, similar sums of the $c_s$ coefficients in (1.13),(1.15) do not appear to vanish (and thus same applies also the sums of the corresponding $\beta_{1,s} = c_s - a_s$ coefficients in (1.1)). That may be suggesting that either some subtlety was overlooked in our computation of $\beta_{1,s}$ or that to achieve the vanishing of the full Weyl anomaly one is to consider a particular combination of the (bosonic and fermionic) CHS fields.

This paper is organized as follows. We shall start in section 2 with a review of CHS fields in flat space extending the discussion in [3] and clarifying the structure of the corresponding free partition functions.
In section 3 we shall discuss the bosonic CHS fields in curved space. We shall first consider (in section 3.1) the case of a Ricci-flat background and suggest a natural factorized representation for the quantum partition function for a spin $s$ CHS field that generalizes both the known conformal graviton expression [24, 26, 27] and the flat space expression. The corresponding $\beta_1$ Weyl anomaly coefficient is then computed as in [35, 22].

In section 3.2 we shall turn to the case of a CHS field on conformally-flat $(A)dS_4$ background and will argue that the corresponding spin $s$ partition function can be represented as a product of $s$ factors which are quantum partition functions of the standard 2nd-derivative massless and $s - 1$ “partially massless” spin $s$ fields in $(A)dS$. These PM partition functions correspond to quantization of Lagrangians in [39, 43] were not given previously in the literature. This leads to the $a_s$ expression in (1.12), and, combined with $\beta_{1,s}$ found in section 3.1, to the $c_s$ in (1.13).

The analysis of the fermionic CHS case in section 4 is similar: we start with Ricci-flat case and determine $\beta_{1,s}$ (section 4.1), and then turn to the conformally-flat case. The partition function on $(A)dS$ background (section 4.2) is again expressed in terms of the massless spin $s$ partition function and the product of “partially massless” ones but here there is also one extra factor which corresponds to a massive spin $s$ mode without any residual gauge invariance. The resulting expression is direct generalization of the conformal gravitino partition function in [26, 27, 3] and also of the flat-space fermionic CHS partition function. The corresponding $a_s$ coefficient is given by (1.14) and together with $\beta_{1,s}$ from section 4.1 it leads to $c_s$ in (1.15).

In section 5 we shall make some concluding remarks discussing in particular the finiteness property of the sums over spins in (1.16) and the one-loop relation [11] between ratio of massless higher-spin determinants in AdS$_{D+1}$ and CHS partition function on $S^D$.

Appendix contains some definitions and useful relations. In Appendix B we review factorization of conformal higher derivative scalar operators on Einstein space background into product of 2nd-derivative scalar Laplacians which is analogous to the one discussed in section 3 for CHS operators.

## 2 Flat space background

Conformal higher spin theories have free kinetic Lagrangians with maximal degree of gauge invariance and irreducibility consistent with locality, i.e. describe pure spin states even off shell [3]  

$$L_s = \Phi_s D_s \Phi_s , \quad D_s = P_s \partial^{2s} . \tag{2.1}$$

Here we assume that space-time dimension is $D = 4$ and for integer $s = 1, 2, ...$ we have bosonic field $\Phi_s = \phi_s = (\phi_{m_1...m_s})$ which is real totally symmetric tensor of rank $s$, and $\partial^{2s} = (\partial^2)^s$. For half-integer spin $s = s + \frac{1}{2}$ ($s = 0, 1, 2, ...$) the corresponding field is fermionic $\Phi_s = \psi_s = (\psi_{m_1...m_s})$ which is Majorana totally symmetric spinor-tensor of rank $s = s - \frac{1}{2}$, and

\footnote{Note that here the scalar $s = 0$ case is that of a trivial non-propagating field with no dynamical degrees of freedom. There are of course 2- and 4-derivative conformally-covariant scalar operators but they are not natural members of the CHS family we are going to discuss.}
\[ \partial^{2s} = (\partial^2)^s \gamma^m \partial_m. \quad P_s = (P_{m_1 \ldots m_s}^{n_1 \ldots n_s}) \] is totally symmetric, traceless (\( \gamma \)-traceless in half-integer spin case) and transverse projector in each of the two sets of indices.\(^{17}\)

To find a formal generalization of (2.1) to any dimension \( D \) (see, e.g., [7]) one is to replace the kinetic operator \( P_s \partial^{2s} \) by

\[ D_s = P_s \partial^{2s+D-4}, \tag{2.2} \]
i.e. to shift the power of the Laplacian \( s \rightarrow s + \frac{1}{2}(D - 4) \). The corresponding field \( \Phi_s \) has dimension \( 2 - s \) for all \( D \). For even \( D > 4 \) the \( s = 0 \) scalar field will have a higher-derivative conformal kinetic operator \((\partial^2)^{D-4}\) (see Appendix B).

The structure of (2.1) implies the presence of both differential (analogs of Maxwell or reparametrization) gauge invariances with parameters \( \xi_{s-1} \) and algebraic (analogs of Weyl or conformal supersymmetry) gauge invariances with parameters \( \tilde{\eta}_{s-2} \); symbolically, \( \delta \phi_s = \partial \xi_{s-1} + g_2 \tilde{\eta}_{s-2} \) (\( g_2 \) stands for the metric factor). The presence of higher-derivative \( \partial^{2s} \) factor in the kinetic term ensures the locality of these actions.

In what follows we shall always consider only symmetric traceless tensors \( \phi_s \) in the case of bosons and symmetric \( \gamma \)-traceless tensors \( \psi_s \) in the case of fermions. Then the algebraic part of the gauge group will be automatically taken into account and the remaining differential gauge freedom will be parametrized by symmetric traceless tensors \( \xi_{s-1} \). The number of components of a totally symmetric traceless rank-\( s \) tensor in \( D \) dimensions is

\[ N_s \equiv N(\phi_s) = \left( \frac{s + D - 1}{s} \right) - \left( \frac{s + D - 3}{s - 2} \right), \quad N_s \big|_{D=4} = (s + 1)^2. \tag{2.3} \]

2.1 Bosons

Let us start with bosonic CHS fields having integer \( s = 1, 2, 3, \ldots \). In what follows we shall mostly be interested in the \( D = 4 \) case but will quote some relations for general \( D \). The number of off-shell degrees of freedom (i.e. the number of components minus dimension of gauge group) of a bosonic CHS field of spin \( s \) is given by

\[ N_{s \perp} = N_s - N_{s-1} = 2s + 1, \tag{2.4} \]

where \( N_{s \perp} \equiv N(\phi_{s \perp}) \) is the number of components of transverse \((\partial \cdot \phi_{s \perp} = 0)\) traceless rank \( s \) tensor field. The number of dynamical (or on-shell) degrees of freedom \( \nu_s \) can be defined in terms of the corresponding free partition function

\[ Z_s = (\det D'_s)^{-1/2} = (Z_0)^{\nu_s}, \quad Z_0 = (\det \partial^2)^{-1/2}, \tag{2.5} \]

where prime indicates proper gauge fixing and \( Z_0 \) is a standard real scalar field partition function. Since the action (2.1) depends only on the transverse part of \( \phi_s \), changing the variables

\(^{17}\)For comparison, to get a Lagrangian of an ordinary higher spin field one is to start with an operator \( D_s = P_s \partial^2 \) for bosons and \( D_s = P_s \gamma^m \partial_m \) for fermions and choose \( P_s = P_s + a_1 P_{s-1} + \ldots \) as particular combinations of lower-spin projectors such that the resulting Lagrangian is local. For example, in the Maxwell, Einstein and the standard gravitino cases one has \( \bar{P}_1 = P_1, \quad \bar{P}_2 = P_2 - 2P_0, \quad \bar{P}_3 = P_3 - 2P_2 \).
\( \phi_s \rightarrow (\phi_{s,\perp}, \xi_{s-1}) \) as \( \phi_s = \phi_{s,\perp} + \partial \xi_{s-1} \) with Jacobian \((\det \Delta_{s-1})^{1/2}\), where \( \Delta_k = -\partial^2 \) is defined on totally symmetric traceless rank \( k \) tensors, and dividing over the volume of gauge group (i.e. omitting spurious integral over \( \xi_{s-1} \)) we find for the CHS partition function

\[
Z_s = \left[ \frac{\det \Delta_{s-1}}{(\det \Delta_{s,\perp})^s} \right]^{1/2} , \tag{2.6}
\]

Here \( \Delta_{k,\perp} = -\partial^2 \) is defined on symmetric traceless \textit{transverse} rank \( k \) tensors. Using that

\[
\det \Delta_k = \det \Delta_{k,\perp} \det \Delta_{k-1} , \tag{2.7}
\]

we can rewrite (2.6) in two alternative forms: in terms of unprojected operators \( \Delta_k \)

\[
Z_s = \left[ \frac{(\det \Delta_{s-1})^{s+1}}{(\det \Delta_s)^s} \right]^{1/2} , \tag{2.8}
\]

or in terms of transverse-projected operators

\[
Z_s = \prod_{k=0}^{s-1} Z_{s,k} , \quad Z_{s,k} = \left[ \frac{\det \Delta_{k,\perp}}{(\det \Delta_{s,\perp})^s} \right]^{1/2} . \tag{2.9}
\]

Here the indices of \( Z_{s,k} \) indicate ranks of tensors on which the two 2nd-order operators (in the denominator and numerator) are defined. In particular,

\[
Z_{s,s-1} = \left[ \frac{\det \Delta_{(s-1),\perp}}{(\det \Delta_{s,\perp})^s} \right]^{1/2} = \left[ \frac{(\det \Delta_{s-1})^2}{(\det \Delta_{s} \det \Delta_{s-2})} \right]^{1/2} \tag{2.10}
\]

is the partition function of a standard massless higher spin field.

In general, \( Z_{s,k} \) in (2.9) can be formally interpreted as a partition function of a spin \( s \) field with gauge invariance \( \delta \phi_s = \partial^{s-k} \xi_k \) involving higher derivatives but lower-rank parameter tensor (becoming the standard \( \delta \phi_s = \partial \xi_{s-1} \) for \( k = s-1 \) corresponding to (2.10)). The number of the associated dynamical degrees of freedom is then

\[
\nu_{s,k} = N_{s,\perp} - N_{k,\perp} = 2(s-k) , \tag{2.11}
\]

where \( N_{s,\perp} \) was given in (2.4). Thus \( \nu_{s,k} \) ranges from \( \nu_{s,0} = 2s \) to \( \nu_{s,s-1} = 2 \) in the massless case. As follows from (2.8) or (2.9), the number of dynamical degrees of freedom of a bosonic CHS field is then \([3]\)

\[
\nu_s = sN_s - (s+1)N_{s-1} = \sum_{k=0}^{s-1} \nu_{s,k} = s(s+1) . \tag{2.12}
\]

For example, \( \nu_1 = 2 \) for a vector, \( \nu_2 = 6 \) for a Weyl graviton \([23, 24]\), etc.

As we shall see below, the representation (2.9) of the flat-space partition function has a natural generalization to the case when the CHS field is propagating in an Einstein-space background, in particular, (A)dS\(_4\) one, when \( Z_{s,k} \) become partition functions of "partially massless" fields. The representation (2.8) will also have a direct generalization to a Ricci-flat background.
The expressions (2.6), (2.8), (2.9) have the following generalization to even \( D > 4 \):

\[
Z_s \equiv \left[ \frac{\det \Delta_{s-1}}{\left( \det \Delta_{s-1} \right)^{s+\frac{3}{2}(D-4)}} \right]^{1/2} = \left[ \frac{\det \Delta_{s-1}}{\left( \det \Delta_{s-1} \right)^{s+\frac{3}{2}(D-4)}} \right]^{1/2} = \prod_{k=0}^{s-1} \left[ \frac{\det \Delta_{s-1}}{\left( \det \Delta_{s-1} \right)^{s}} \right]^{1/2} \left( \frac{1}{\left( \det \Delta_{s-1} \right)^{1/2}} \right)^{\frac{3}{2}(D-4)}. \tag{2.13}
\]

### 2.2 Fermions

Next, let us consider the case of fermionic CHS fields with half-integer \( s = s + \frac{1}{2} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \). The number of components of Majorana \( \gamma \)-traceless (and thus also traceless) spinor-tensor \( \psi_s = (\psi_{m_1 \ldots m_s}) \) in \( D = 4 \) is

\[
N_s \equiv N(\psi_s) = 4 \left[ \left( \frac{s + 3}{s} \right) - \left( \frac{s + 2}{s - 1} \right) \right] = 2(s + 1)(s + 2) = 2(s + 1)(s + \frac{3}{2}), \tag{2.14}
\]

and thus the number of off-shell degrees of freedom is (cf. (2.4)) \([3]\)

\[
N_{s,\perp} \equiv N(\psi_{s,\perp}) = N_s - N_{s-1} = 4(s + 1) = 2(2s + 1). \tag{2.15}
\]

Squaring the fermionic CHS kinetic operator \( \partial^2 \) \( P_s \) and writing the corresponding partition function in terms of 2nd-order Laplacians we find the following analog of the bosonic expression (2.16)

\[
Z_s = \left[ \frac{(\det \Delta_{s-1})^{2s}}{(\det \Delta_{s-1})^{2s}} \right]^{-1/4} = \left[ \frac{\det \Delta_{s-1}}{(\det \Delta_{s-1})^{s}} \right]^{-2/4} \left[ \frac{1}{\det \Delta_{s-1}} \right]^{-1/4}, \tag{2.16}
\]

where \( \Delta_s = -\partial^2 \) is defined on totally symmetric \( \gamma \)-traceless spinor-tensors \( \psi_s \) and we used that as in the bosonic case the Jacobian of transformation from \( \psi_s \) to \( \psi_{s,\perp} \) (i.e. \( \psi_s = \psi_{s,\perp} + \partial \xi_{s-1} \)) produces the factor \( (\det \Delta_{s-1})^{-1/2} \) (with extra -1 power). Depending on interpretation, the structure of the expression (2.16) is formally different from that of (2.6) in either having \( \det \Delta_{s-1} \) in the numerator being squared or in having an extra factor of \( \det \Delta_{s,\perp} \).

The equivalent forms of (2.16) are (cf. (2.8) and (2.9))

\[
Z_s = \left[ \frac{(\det \Delta_{s-1})^{s+1}}{(\det \Delta_{s})^{s}} \right]^{-1/4}, \tag{2.17}
\]

or

\[
Z_s = \prod_{k=1/2}^{s-1} (Z_{s,k})^2 Z_{s,0}, \quad Z_{s,k} = \left[ \frac{\det \Delta_{s-1}}{\det \Delta_{s,\perp}} \right]^{-1/4}, \quad Z_{s,0} = \left[ \frac{1}{\det \Delta_{s,\perp}} \right]^{-1/4}. \tag{2.18}
\]

Like in the bosonic case in (2.10), here \( Z_{s, s-1} \) is the partition function of the standard (1-st derivative) massless fermionic spin \( s \) field, while \( Z_{s,k} \) may be interpreted as corresponding to a spin \( s \) field with smaller gauge group but with \( s - k \) derivatives in the gauge transformation. The corresponding number of fermionic d.o.f. is then

\[
\nu_{s,k} = -\frac{1}{2} (N_{s,\perp} - N_{s,\perp}) = -2(s - k), \quad \nu_{s,0} = -\frac{1}{2} N_{s,\perp} = -(2s + 1), \tag{2.19}
\]
where \( k = k + \frac{1}{2} \), \( k = 0, 1, \ldots \). A peculiarity of the fermionic CHS case is the second power of \( Z_{s,k} \) in (2.18) and the presence of an extra “purely-massive” mode (with no remaining gauge invariance) represented by the \( Z_{s,\emptyset} \) factor. As in the bosonic case, the representations (2.18) will have natural generalizations to curved Einstein-space backgrounds.

The fermionic CHS field number of dynamical degrees of freedom following from either (2.17) or (2.19) is thus (cf. (2.12))

\[
\nu_s = -s N_s + (s + 1) N_{s-1} = 2 \sum_{k=1/2}^{s-1} \nu_{s,k} + \nu_{s,\emptyset} = -2(s + \frac{1}{2})^2 = -2(s + 1)^2 .
\]

(2.20)

For example, this gives the standard values for spin 1/2 field and spin 3/2 (conformal gravitino) fields: \( \nu_{\frac{1}{2}} = -2 \) and \( \nu_{\frac{3}{2}} = -8 \) [3]. Explicitly, according to (2.18) the partition function of conformal gravitino \( Z_{\frac{3}{2}} \) is a product of 2 partition functions of ordinary massless gravitino and one “purely massive” gravitino, implying \( -(2 \times 2 + 4) = -8 \) for the d.o.f. count. For \( s = \frac{5}{2} \) CHS field we get 2 massless spin \( \frac{5}{2} \) factors \( Z_{\frac{5}{2}, \frac{5}{2}} \), 2 “partially massless” spin \( \frac{5}{2} \) factors \( Z_{\frac{5}{2}, \frac{1}{2}} \) and one “purely massive” factor \( Z_{\frac{5}{2}, \emptyset} \), etc.

For an \( \mathcal{N} = 1 \) CHS supermultiplet containing spins \( \{ s, s + \frac{1}{2}, s + 1 \} \) (i.e. \( \{ \frac{1}{2}, 1 \}, \{ 1, \frac{3}{2}, 2 \}, \) etc.) one then finds from (2.4),(2.12),(2.20) the expected result: \( \sum \{ s \} n_s = 0, \sum \{ s \} \nu_s = 0 \) [3].

3 Bosonic conformal higher spins in curved background

Here we shall consider the expressions for the integer \( s \) CHS partition functions in Einstein-space (Ricci-flat and conformally-flat backgrounds) starting first with the known cases of low spins \( s = 1 \) (Maxwell vector) and \( s = 2 \) (Weyl graviton) and then suggesting natural generalizations to \( s > 2 \).

3.1 Ricci-flat background

The Maxwell vector partition function in a curved background has familiar form

\[
Z_1 = \left[ \frac{\det \Delta_0}{\det \Delta_{1,\perp}} \right]^{1/2} = \left[ \frac{(\det \Delta_0)^2}{\det \Delta_1} \right]^{1/2} , \quad \Delta_0 = -\nabla^2 , \quad (\Delta_1)_{mn} = -(\nabla^2)_{mn} + R_{mn} . \tag{3.1}
\]

Considering a conformal spin \( s > 1 \) field in an external metric (i.e. coupled to \( s = 2 \) conformal field) one, in general, should get a complicated 2s-derivative reparametrization and Weyl covariant differential operator with coefficients depending on the background curvature. However, this operator may simplify for specific backgrounds, reducing to a product of lower-dimensional operators.

This indeed happens for the 4-derivative Weyl graviton operator in Einstein-space backgrounds. One finds [26, 27, 3] that in the Ricci-flat background the Weyl graviton 4-th order operator factorizes, becoming the square of the traceless Einstein graviton operator. Then the Weyl gravity 1-loop partition function takes formally the same form as its flat-space counterpart in (2.8). It can be expressed also as a product of the familiar one-loop partition functions.
of the two Einstein gravitons [59] and one Maxwell vector in the $R_{mn} = 0$ background:

$$Z_2 = \left[ \frac{\det \Delta_0}{(\det \Delta_2)^2} \right]^{1/2} = \left[ \frac{(\det \Delta_1)^2}{(\det \Delta_2)^2} \right]^{1/2} = (Z_{2,1})^2 Z_1 , \quad (3.2)$$

$$Z_{2,1} = \left[ \frac{\det \Delta_{1,1}}{\det \Delta_2} \right]^{1/2} = \left[ \frac{(\det \Delta_1)^2}{\det \Delta_2 \det \Delta_0} \right]^{1/2} . \quad (3.3)$$

Here $\Delta_0$ and $(\Delta_1)_{mn}$ are as in (3.1), the operator $(\Delta_2)_{mn,kl} = -\nabla^2_{mn,kl} - 2R_{mnkl}$ is assumed to be defined on traceless symmetric 2-tensors and $\Delta_{k\perp}$ are defined on transverse symmetric traceless tensors.

Note that $Z_2$ in (3.2) and $Z_1$ in (3.1),(3.3) have the same structure as the flat-space partition function (2.8) but here with covariant differential operators $\Delta_s$. It is then natural to expect that for any conformal higher spin field in a Ricci-flat background the kinetic operator should be factorizing into $s$ factors of the “massless” spin $s$ 2nd-order operators. Then the partition function should be given by

$$Z_s = \left[ \frac{(\det \Delta_{s-1})^{s+1}}{(\det \Delta_s)^s} \right]^{1/2} , \quad (3.4)$$

where $\Delta_k$ are covariant 2nd-order differential operators defined on traceless rank $k$ tensors corresponding to standard massless spin $k$ fields. This appears to be the simplest “minimal”-coupling assumption extended to the CHS fields.

In general, one may define (following, e.g., [21, 22]) an operator acting on a field in an irreducible $SO(1,3)$ representation $(A, B)$ of dimension $N_{(A,B)} = (2A+1)(2B+1)$ as $(A, B$ are positive half-integers)

$$\Delta_{(A,B)} = -\nabla^2(V) + X , \quad X = -R^{ab}_{\phantom{ab}mn} \Sigma_{ab} \Sigma^{mn} . \quad (3.5)$$

Here $V_m = \omega_m^{ab} \Sigma_{ab}$, $\omega_m^{ab}$ is the standard spin connection and $\Sigma_{ab}$ are the corresponding generators of $SO(1,3)$. In the present case of symmetric traceless rank $s$ tensor fields we have

$$(A, B) = (\frac{s}{2}, \frac{s}{2}), \quad N_{(A,B)} \equiv (2A+1)(2B+1) = N_s = (s+1)^2 , \quad \Delta_{(A,B)} = \Delta_s , \quad (3.6)$$

where $\Delta_s$ is defined on symmetric traceless rank $s$ tensors and corresponds to standard “harmonic-gauge” massless higher spin operators on a curved background, generalizing the $s = 2$ Lichnerowicz operator. Explicitly [60, 61], for the symmetric traceless tensor representation the Lorentz generators are $(\Sigma_{mn})^{a_1 \ldots a_s}_{b_1 \ldots b_s} = s \delta^{(a_1 \ldots a_s)}_{(b_1 \ldots b_s)} \delta^{a_2 \ldots a_s}_{b_2 \ldots b_s}$ so that

$$(X \phi_s)^{a_1 \ldots a_s} = -s(s-1) R^{(a_1}_{m \phantom{a_1} a_2 \ldots a_s} \phi^{a_2 \ldots a_s} \phi^{mn} + s R^{(a_1}_{m \phantom{a_1} \delta^{a_2 \ldots a_s}_{b_2 \ldots b_s}} \phi^{b_2 \ldots b_s} \phi^{mn} . \quad (3.7)$$

As follows from (3.4), one can then express the value of the $\beta_1$ coefficient in (1.1) for a CHS field as a combination of $\beta_1$ coefficients for the operators $\Delta_s$ by a relation similar to the one for the number of degrees of freedom (2.12)

$$\beta_{1,s} = s \beta_1 [\Delta_s] - (s+1) \beta_1 [\Delta_{s-1}] , \quad (3.8)$$
where $\beta_1(\Delta_k)$ is the $\beta_1$ coefficient in the expression (1.1) for $b_4(\Delta_k)$. The latter can be computed using the standard algorithm [35] for the 2-nd order operators $\Delta = -\nabla^2(V) + X$ defined in curved space on fields $\Phi^i$ with $\nabla_m$ containing the matrix connection $(V^i_j)_m$.\(^{18}\)

$$b_4[\Delta] = \frac{1}{180} \text{Tr} \left[ 15 F^2_{mn}(V) + 90 X^2 - 30 R X - 30 \nabla^2 X + 1(R^* R^* + 3 R^2_{mn} + \frac{3}{2} R^2 + 6 \nabla^2 R) \right] \quad (3.9)$$

Following [21, 22] one finds in the Ricci flat case ($\text{Tr} 1 = N_{(A,B)}$)

$$\beta_1[\Delta(A,B)] = \frac{1}{180} N_{(A,B)} \left( 1 + A(A + 1)[6A(A + 1) - 7] + B(B + 1)[6B(B + 1) - 7] \right) , \quad (3.10)$$

so that in the present case of (3.6)

$$\beta_1[\Delta_s] = \frac{1}{120} N_s (21 - 20 N_s + 3 N_s^2) . \quad (3.11)$$

Finally, for the bosonic CHS field the $\beta_1$ coefficient (3.8) is thus given by

$$\beta_{1,s}^{(b)} = \frac{1}{720} \nu_s (4 - 45 \nu_s + 15 \nu_s^2) , \quad \nu_s = s(s + 1) . \quad (3.12)$$

This expression agrees with the known values for $s = 1$ and $s = 2$ in (1.5),(1.6) (and it vanishes as it should for non-dynamical $s = 0$ case).

It is interesting to note that while on general grounds (cf. (3.9)) the Weyl anomaly coefficients in (1.1) for a CHS field of spin $s$ should be 6-th order polynomials in $s$, $\beta_1$ is actually a cubic polynomial in the number of dynamical degrees of freedom $\nu_s = s(s + 1)$. The same will apply also to the expression for the second Weyl anomaly coefficient $\beta_2$ discussed below.

### 3.2 Conformally-flat background

Next, let us determine $a$ in (1.1) (and thus $\beta_2$) by considering a constant curvature (A)dS$_4$ background (1.10). The Maxwell vector partition function (3.1) in this case may be written as

$$Z_1 = \left[ \frac{\det \hat{\Delta}_0(0)}{\det \hat{\Delta}_1(3)} \right]^{1/2} = \left[ \frac{(\det \hat{\Delta}_0(0))^2}{\det \hat{\Delta}_1(3)} \right]^{1/2} . \quad (3.13)$$

Here and in what follows the operator

$$\hat{\Delta}_s(M^2) \equiv -\nabla^2_s + M^2 , \quad M^2 \equiv M^2 \epsilon \quad (3.14)$$

will be defined on symmetric traceless tensors and $\hat{\Delta}_{s\perp}(M^2)$ will stand for $\hat{\Delta}_s(M^2)$ defined on transverse symmetric traceless tensors. The parameter $\epsilon = \pm \frac{1}{2}$ is equal to 1 for unit-radius dS space and -1 for unit-radius AdS space.\(^{19}\)

As in the Ricci-flat case, one finds that the Weyl graviton kinetic operator again factorizes [26, 27, 29, 30]:\(^{20}\)

$$C^2 = \frac{1}{2} \phi_2 \hat{\Delta}_{2\perp}(2) \hat{\Delta}_{2\perp}(4) \phi_2 + O(\phi_2^3) . \quad (3.15)$$

---

\(^{18}\)In the present cases $V_m$ will be expressed in terms of the spin connection so that $\text{Tr} F^2_{mn}(V)$ will give contraction of two curvatures.

\(^{19}\)Note also that the notation $M^2$ does not mean that this dimensionless parameter is always positive.

\(^{20}\)Note, in particular, that $\Delta_2 \phi_{mn} = -\nabla^2 \phi_{mn} - 2 R_{mknl} \phi_{kl} \rightarrow \hat{\Delta}_2(2) \phi_{mn} = (-\nabla^2_s + 2 \epsilon) \phi_{mn}$. 

13
As a result, the 1-loop partition function of the Weyl theory can be written as \[ Z_2 = Z_{2,1} Z_{2,0} = \left[ \frac{\det \hat{\Delta}_{1\perp}(-3)}{\det \Delta_{2\perp}(2)} \right]^{1/2} \left[ \frac{\det \hat{\Delta}_0(-4)}{\det \Delta_{2\perp}(4)} \right]^{1/2}. \] (3.16)

For \( \Lambda = 3\epsilon \to 0 \) the mass terms in (3.14) disappear and this reduces to the flat-space expression in (2.9).

In contrast to the flat and Ricci-flat case in (3.2) here the two standard graviton operator factors are not the same: the degeneracy is lifted by the curvature. The first factor is the usual (A)dS \( \text{“massless”} \) graviton contribution equal to the 1-loop partition function of the Einstein gravity with cosmological term \[ \text{(A)dS}_4 \]

For \( \Lambda = 3\epsilon \to 0 \) the mass terms in (3.14) disappear and this reduces to the flat-space expression in (2.9).

In contrast to the flat and Ricci-flat case in (3.2) here the two standard graviton operator factors are not the same: the degeneracy is lifted by the curvature. The first factor is the usual (A)dS \( \text{“massless”} \) graviton contribution equal to the 1-loop partition function of the Einstein gravity with cosmological term \[ \text{(A)dS}_4 \]

For \( \Lambda = 3\epsilon \to 0 \) the mass terms in (3.14) disappear and this reduces to the flat-space expression in (2.9).

The second factor

\[ Z_{2,0} = \left[ \frac{\det \hat{\Delta}_0(-4)}{\det \Delta_{2\perp}(4)} \right]^{1/2} = \left[ \frac{\det \hat{\Delta}_1(-1) \det \hat{\Delta}_0(-4)}{\det \Delta_2(4)} \right]^{1/2}. \] (3.18)

This corresponds to the “partially massless” spin 2 field found in [29, 30, 41] to have a special conformal covariance property (allowing to transform its equation of motion to the massless flat space \( \partial^2 \) form and thus ensuring its “null-cone” propagation). This field has on-shell 2nd-derivative gauge invariance with a scalar parameter, \( \delta \phi_{mn} = (\nabla_m \nabla_n + 4\epsilon_{mn})\xi \) [37, 38]. This field can be described by a local Lagrangian [39] involving the standard 2-tensor and vector fields with gauge invariance \( \delta \phi_{mn} = \nabla(m\xi_n) + \mu g_{mn}\eta, \delta A_m = \nabla_m\eta + \mu \xi_m, \mu^2 = -4\epsilon \) (so that it effectively describes same number of dynamical d.o.f. \( 4 = 2 + 2 \) as a massless spin 2 plus a massless spin 1 system). Quantization of this system leads to the partition function (3.18).\(^{22}\)

Partially massless (PM) fields in (A)dS\(_4\) that admit local gauge-invariant description after introduction of some extra lower-spin modes exist for all \( s > 2 \). For given value of \( s \) there is one massless and \( s - 1 \) PM fields which in the general case of (A)dS\(_D\) with curvature given in (A.1) are described by the following kinetic operators [38, 39]

\[
\hat{\Delta}(M_{s,k}^2) = -\nabla_s^2 + M_{s,k}^2, \quad k = 0, 1, ..., s - 1, \quad (3.19)
\]

\[
M_{s,k}^2 = s - (k - 1)(k + D - 2), \quad M_{s,k}^2\big|_{D=4} = 2 + s - k - k^2. \quad (3.20)
\]

Here \( k = s - 1 \) corresponds to the massless field in (A)dS\(_D\) [46, 47] with the mass parameter\(^{23}\)

\[
m_{s,0}^2 \equiv M_{s,s-1}^2 = s - (s - 2)(s + D - 3), \quad M_{s,s-1}^2\big|_{D=4} = 2 + 2s - s^2. \quad (3.21)
\]

\(^{21}\)For some general relations between \( \hat{\Delta}_s \) and \( \hat{\Delta}_{s\perp} \) operators see Appendix.

\(^{22}\)This can be made more apparent by rewriting the partition function (3.18) as

\[
Z_{2,0} = \left[ \frac{\det \hat{\Delta}_{1\perp}(-1)}{\det \Delta_{2\perp}(4)} \right]^{1/2} \left[ \frac{\det \hat{\Delta}_0(-4)}{\det \Delta_{1\perp}(-1)} \right]^{1/2}.
\]

\(^{23}\)Separating the massless spin \( s \) contribution to \( M^2 \) one may write the mass parameters of the PM fields as \( M_{s,k}^2 = m_{s,0}^2 + \mu_{s,k}^2, \mu_{s,k}^2 = (s - k - 1)(s + k + D - 4) \). Discussion of massless fields in AdS\(_D\) in the frame-like, metric-like, and BRST approaches may be found in the respective references [48].
Interestingly, the “transpose” of (3.20)

\[ M_{k,s}^2 = k - (s - 1)(s + D - 2) , \quad M_{k,s}^2 \big|_{D=4} = 2 + k - s - s^2 \] (3.22)
gives

\[ M_{s-1,s}^2 = -(s - 1)(s + D - 3) , \quad M_{s-1,s}^2 \big|_{D=4} = 1 - s^2 , \] (3.23)

which is exactly the mass parameter of the “ghost” factor in the partition function of the massless spin \( s \) field in (A)dS \([50, 51]\)

\[ Z_{s,s-1} = \left[ \frac{\det \Delta_{s-1 \perp} (M_{s-1,s}^2)}{\det \Delta_{s \perp} (M_{s,s-1}^2)} \right]^{1/2} . \] (3.24)

This of course agrees with the \( s = 1, 2 \) expressions (3.13) and (3.17) in \( D = 4 \).\(^{24}\)

Our key observation is that CHS kinetic operator in conformally-flat background should factorize into precisely \( s \) factors of “partially massless” kinetic operators (3.19), i.e.

\[ \phi_s D_s \phi_s = \phi_s \left[ \prod_{k=0}^{s-1} \Delta_{s \perp} (M_{s,k}^2) \right] \phi_s , \] (3.25)

thus generalizing the familiar Maxwell and Weyl theory (3.15) cases. One possible derivation of this relation may start from the “ordinary-derivative” formulation of the CHS theory in flat space \([33]\), then explicitly transforming to conformally-flat metric and finally solving for all auxiliary fields.

To obtain the quantum CHS partition function in a conformally-flat background it remains then to find the corresponding “ghost” factor. As in the low-spin and massless spin examples \([62, 63, 44, 65, 26, 24, 66, 67, 50, 51]\) it is found using the Jacobian of transformation from the traceless field \( \phi_s \) to its transverse component \( \phi_{s \perp} \) and other lower-spin transverse fields (cf. Appendix for some examples). One is also to divide over the volume of the gauge transformation group with unconstrained traceless parameters. The final expression for the CHS partition function in (A)dS\(_4\) background then takes the following remarkably simple form which is a generalization of the flat-space expression in (2.9)

\[ Z_s = \prod_{k=0}^{s-1} Z_{s,k} , \quad Z_{s,k} = \left[ \frac{\det \Delta_{k \perp} (M_{k,s}^2)}{\det \Delta_{s \perp} (M_{s,k}^2)} \right]^{1/2} . \] (3.26)

Here the \( k = s - 1 \) factor is precisely the massless spin \( s \) partition function (3.24) and other factors correspond to the PM fields.

As follows from the structure of the flat-space partition function in (2.13), to find a generalization of (3.26) to general even dimension \( D \) one needs to multiply (3.26) (now with \( D \)-dependent parameters).

\(^{24}\)Note also that the “maximal-depth” PM field with \( k = 0 \) (in \( D = 4 \)) plays a somewhat special role being conformally-invariant \([41]\) and having the highest-derivative (order \( s \)) gauge invariance with a scalar gauge parameter. A discussion of this field and some hints of its connection to CHS fields appear in \([49]\).
masses (3.20),(3.22)) by extra $\frac{1}{2}(D-4)$ “purely-massive” (no residual gauge invariance) factors. These have the following general form:

$$Z_{\text{extra}} = \prod_{i=1}^{D-4} \left[ \frac{1}{\det \Delta_{s,i}(m_{s,i}^2)} \right]^{-1/2}, \quad \hat{\Delta}_{s,i}(m_{s,i}^2) = -\nabla_s^2 + m_{s,i}^2 \hat{\epsilon}, \quad (3.27)$$

where the mass coefficients $m_{s,i}^2$ remain to be determined. They are actually known in the special case of $s = 0$ when the CHS operator becomes the conformal scalar operator $\Delta_{(2r)}$ with $r = \frac{1}{2}(D-4)$ (see Appendix B): from (B.7) we have

$$m_{0,i}^2 = -(i - \frac{1}{2}D)(i + \frac{1}{2}D - 1), \quad i = 1, \ldots, \frac{1}{2}(D-4). \quad (3.28)$$

Comparing this with “partially-massless” mass formula (3.20) with $s = 0$ we observe that it coincides with (3.28) if we set $k = i - \frac{1}{2}D + 1$, i.e. $i = 1, \ldots, \frac{1}{2}(D-4)$ correspond to $k = -\frac{1}{2}(D-4), \ldots, -1$. Then a natural conjecture is that in general one should have

$$m_{s,i}^2 = s - (i - \frac{1}{2}D)(i + \frac{1}{2}D - 1), \quad i = 1, \ldots, \frac{1}{2}(D-4), \quad (3.29)$$

i.e. the massive factors in (3.27) may be interpreted as the “partially-massless” contributions with masses $M_{s,k}$ in (3.20) extended to negative values of $k = -\frac{1}{2}(D-4), \ldots, -1$ (and without “ghost” factors $\det \Delta_{k,i}(M_{s,k}^2)$ in (3.26)).

The partition function (3.26) can be written also in terms of unconstrained operators as in (3.13),(3.16) using the following relation (valid for any $k = 1, 2, \ldots$)

$$\det \Delta_{k,i}(M^2) = \frac{\det \hat{\Delta}_k(M^2)}{\det \Delta_{k-1}(M^2 - \delta_k)}, \quad \delta_k = 2k + D - 3, \quad \delta_k|_{D=4} = 2k + 1. \quad (3.30)$$

For example, for the massless factor (3.24) we then get

$$Z_{s,s-1} = \left[ \frac{(\det \Delta_{s-1}(M_{s-1,s}^2))^2}{\det \Delta_s(M_{s,s-1}^2) \det \Delta_{s-2}(M_{s+2,s+1}^2)} \right]^{1/2}, \quad (3.31)$$

where we used that

$$M_{s,s-1}^2 - \delta_s = M_{s-1,s}^2, \quad M_{s-1,s}^2 - \delta_{s-1} = M_{s+2,s+1}^2 = 2 - s^2 - s(D-2). \quad (3.32)$$

In particular, for $s = 1$ and $s = 2$ and $D = 4$ the expression (3.26) agrees with (3.13) and (3.16). Also, for $s = 3$ in $D = 4$ we find from (3.20) that $M_{3,k}^2 = 5 - k - k^2, \quad M_{3,3}^2 = k = 10$ where $k = 2, 1, 0$ and thus

$$Z_3 = Z_{3,2}Z_{3,1}Z_{3,0} = \left[ \frac{\det \hat{\Delta}_{2,1}(-8)}{\det \Delta_{3,1}(-1)} \right]^{1/2} \left[ \frac{\det \hat{\Delta}_{1,1}(-9)}{\det \Delta_{3,1}(3)} \right]^{1/2} \left[ \frac{\det \hat{\Delta}_0(-10)}{\det \Delta_{3,1}(5)} \right]^{1/2} \quad (3.33)$$

$$= \left[ \frac{(\det \Delta_2(-8))^2}{\det \Delta_3(-1) \det \Delta_1(-13)} \right]^{1/2} \left[ \frac{\det \hat{\Delta}_2(-4) \det \hat{\Delta}_1(-9)}{\det \Delta_3(3) \det \Delta_0(-12)} \right]^{1/2} \left[ \frac{\det \hat{\Delta}_2(-2) \det \hat{\Delta}_0(-10)}{\det \Delta_3(5)} \right]^{1/2}. \quad$$

---

25 The presence of such extra massive degrees of freedom in the $D \neq 4$ case is implied also by the structure of the “ordinary-derivative” formulation of [33] (we are grateful to R. Metsaev for pointing this out).

26 For $s = 0$ the product in (3.26) is to be set to 1 as there $k \geq 0$ and $Z_{0,0} = 1$.

27 There appears to be a group-theoretic argument leading to this relation (E. Skvortsov and M. Vasiliev, private communication).

28 Note that here and in (3.16) we write the factors in the opposite order to (3.26) so that the massless spin factor appears first.
In the second line we used (3.30). Here the first factor is the massless spin 3 partition function (cf. (3.31)). Note that in the limit \( \epsilon \to 0 \) all spin 0 and spin 1 factors cancel and we recover the flat-space expression in (2.8).

It is now straightforward to apply the \( b_4 \) algorithm (3.9) to each of the 2nd order operator in the CHS partition function (3.26) in conformally-flat \( D = 4 \) space to compute the corresponding \( a \)-coefficient according to (1.10). Let us first consider a generic unconstrained operator (3.14) defined on symmetric traceless tensors with an arbitrary dimensionless "mass" constant \( M^2 \) and with \( \nabla_m = \nabla_m(V) \) with connection \( V_m \) corresponding to an \( SO(1,3) \) representation \((A,B)\) (in particular, to the one in (3.6)). In the conformally flat Einstein-space background case we get from (3.9) (here \( R = 12 \epsilon \))

\[
\frac{1}{150} \text{Tr} \left[ 15F_{mn}^2(V) + 1(90M^4 - 360M^2 + 348)\epsilon^2 \right],
\]

where \( \text{Tr} F_{mn}^2(V) = -4N_{(A,B)}[A(A + 1) + B(B + 1)]\epsilon^2 \). (3.34)

Using (3.6) we find \( \text{Tr} F_{mn}^2(V) = -2Ns(s + 2)\epsilon^2, \quad N_s = (s + 1)^2 \). Then we may determine the contribution to the \( a \)-coefficient corresponding to the operator (3.14) according to (1.10)

\[
a[\hat{\Delta}_s(M^2)] = \frac{1}{1140}N_s \left( N_s - 3M^4 + 12M^2 - \frac{64}{9} \right).
\]

(3.36)

Using (3.30) we may then find also the \( a \)-coefficient corresponding to the transverse operator as (cf. (2.4))\(^{29}\) overall factor here

\[
a[\hat{\Delta}_{s\perp}(M^2)] = a[\hat{\Delta}_s(M^2)] - a[\hat{\Delta}_{s-1}(M^2 - 2s - 1)]
\]

\[
= \frac{1}{720}(2s + 1) \left[ 30s^3 + 85s^2 + 10s - 58 - 30(s^2 - 2)M^2 - 15M^4 \right].
\]

(3.37)

It is now straightforward to compute the resulting total value of the \( a \)-coefficient corresponding to the CHS partition function (3.26) with the mass parameters given in (3.21).

Computing the finite sum over \( k \) as implied by the representation (3.26) we end up with a simple expression for the bosonic CHS anomaly coefficient \( a_s \)

\[
a_s^{(b)} = \sum_{k=0}^{s-1} \left( a[\hat{\Delta}_{s\perp}(2 + s - k - k^2)] - a[\hat{\Delta}_{k\perp}(2 + k - s - s^2)] \right)
\]

\[
= \frac{1}{720} \nu_s^2(3 + 14\nu_s), \quad \nu_s = s(s + 1).
\]

(3.38)

Like the \( \beta_1 \) coefficient found earlier in (3.12), it depends on \( s \) only through the corresponding number of dynamical degrees of freedom \( \nu_s \) in (2.12). As already mentioned in the Introduction, (3.38) matches the expression for \( a_s^{(b)} \) found in [11] by an indirect method.

\(^{29}\)The use of the \( b_4 \) for the unprojected operator means that we are effectively computing the anomaly of the partition function expressed in terms of unprojected operators like in the second line of (3.33), thus avoiding a subtlety with zero-mode contributions if one computes the anomaly using \( \zeta(0) \) for the projected operators (cf. [65, 26, 27]).
For comparison, the contribution of just massless spin $s$ part of (3.24) (with 2 degrees of freedom) is\textsuperscript{30}
\[
a^{(b)}_{s,s-1} = \frac{1}{360}(2 - 15s^2 + 75s^4) .
\] (3.39)
It of course agrees with (3.38) for $s = 1$ when $a_1 = a_{1,0} = \frac{31}{180}$.

Combining the results for $\beta^{(b)}_{1,s}$ (3.12) and $a^{(b)}_s$ (3.38) we conclude from (1.2) that
\[
e^{(b)}_s = \frac{1}{2}\beta^{(b)}_{2,s} = \beta^{(b)}_{1,s} + a^{(b)}_s = \frac{1}{1720}\nu_s(4 - 42\nu_s + 29\nu_s^2) .
\] (3.40)

4 Fermionic conformal higher spins in curved background

Let us now consider the fermionic CHS fields with half-integer spin $s = s + \frac{1}{2}$ ($s = 0, 1, 2, \ldots$) described by symmetric $\gamma$-traceless spinor-tensors $\psi_{m_1 \ldots m_s}$. We shall follow the same strategy as in the previous section, first discussing the Ricci-flat and then the conformally-flat backgrounds.

4.1 Ricci-flat background

The simplest examples of the fermionic CHS fields are the Majorana spinor\textsuperscript{31}
\[
Z_1 = \left[\frac{1}{\det\Delta_1}\right]^{-1/4} , \quad \Delta_1 = -\nabla_1^2 + \frac{1}{4}R \to -\nabla_1^2 ,
\] (4.1)
and the conformal gravitino [26, 27, 3]
\[
Z_3 = \left[\frac{\det\Delta_3}{\det\Delta_2}\right]^{5/4} , \quad \Delta_3 = (Z_{\frac{3}{2}})^3 Z_{\frac{3}{2}} = (Z_{\frac{3}{2}})^2 Z_{\frac{3}{2},0} ,
\] (4.2)
\[
Z_{\frac{3}{2},1} = \left[\frac{\det\Delta_{\frac{3}{2}}}{\det\Delta_2}\right]^{-1/4} , \quad (\Delta_{\frac{3}{2}})_{mn} = -(\nabla_{\frac{3}{2}})_{mn} - \frac{1}{2}\gamma^{kl}R_{kmn} ,
\] (4.3)
where $Z_{\frac{3}{2},1}$ is the partition function of the standard massless gravitino. Like for the Weyl graviton, the kinetic operator of conformal gravitino factorizes on the Ricci-flat background – here into the product of the three standard gravitino operators (with transverse projection) and the resulting partition function is equivalent to the one for 3 massless gravitino and one massless spin 1/2 field (with total of 8 fermionic degrees of freedom, cf. (2.20)).

As in the bosonic case (3.4), one may conjecture that in general the fermionic CHS operator will factorize into the product of $s$ massless (transverse-projected) operators $\gamma^m \nabla_m$ so that the partition function will have again the same form as the flat-space one (2.16), now with covariant 2nd-order (squared 1st order) operators $\Delta_s$ containing only “minimal” curvature couplings
\[
Z_s = \left[\frac{(\det\Delta_{s-1})^{s+1}}{(\det\Delta_s)^s}\right]^{-1/4} , \quad s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots .
\] (4.4)
\textsuperscript{30}The same expression was recently used in [58], leading to the conclusion that the logarithmic divergence in the 1-loop partition function of (type-A/B) Vasiliev’s higher-spin theories in AdS\textsubscript{4} vanishes assuming one uses the $\zeta$-function to define the sum over all integer spins $s$.
\textsuperscript{31}We represent the partition functions in terms of squared fermionic operators.
As in the bosonic case, these $\Delta_s$ operators will be assumed to have the form (3.5) acting on totally-symmetric $\gamma$-traceless real spinor-tensors $\psi_{m_1...m_s}$ corresponding to the $(A, B) \oplus (B, A)$ representation of the Lorentz group with $(A, B)$ contained in $(\frac{5}{2}, \frac{5}{2}) \otimes (\frac{5}{2}, 0)$, i.e.\(^{32}\)

$$(A, B) = \left( \frac{s+1}{2}, \frac{s}{2} \right), \quad N_{(A,B)} = (s+1)(s+2), \quad s = s + \frac{1}{2}.$$  \hspace{1cm} (4.5)

Then the corresponding $\beta_1$ coefficient is given by the same expression as in (3.8) up to an overall minus sign. Applying (3.10) we get

$$\beta_1[\Delta_s] = \beta_1[\Delta_{(A,B)}] + \beta_1[\Delta_{(B,A)}] = \frac{1}{2880} N_s (50 - 28N_s + 3N_s^2), \quad N_s = 2N_{(A,B)} = 2(s+1)(s+2),$$

so that the $\beta_1$ Weyl anomaly coefficient for the fermionic CHS field is given by

$$\beta_{1,s}^{(f)} = -s \beta_1[\Delta_s] + (s+1)\beta_1[\Delta_{s-1}] = \frac{1}{2880} \nu_s (106 + 90\nu_s + 15\nu_s^2), \quad \nu_s = -2(s + \frac{1}{2})^2.$$ \hspace{1cm} (4.7)

As in the bosonic case (3.12), the anomaly coefficient $\beta_{1,s}$ is again a cubic polynomial in the number of dynamical degrees of freedom $\nu_s$ (see (2.20)). For $s = \frac{1}{2}$ ($\nu_s = -2$) and $s = \frac{3}{2}$ ($\nu_s = -8$) $\beta_{1,s}$ in (4.7) reproduces the previously known values in (1.7),(1.8).

It is interesting to note that the expression for $\beta_1$ simplifies for a combination of CHS fields with spins $(s, s + \frac{1}{2}, s + 1)$ (with integer $s$) forming an $\mathcal{N} = 1$ supermultiplet: from (3.12) and (4.7) one then finds

$$\beta_{1,\mathcal{N}=1} = \beta_{1,s}^{(b)} + \beta_{1,s+\frac{1}{2}}^{(f)} + \beta_{1,s+1}^{(b)} = \frac{1}{16} (s+1)^2 (4s^2 + 2s - 1).$$ \hspace{1cm} (4.8)

The choice of $s = 0$ corresponds to $\mathcal{N} = 1$ vector multiplet $(\frac{1}{2}, 1)$ where $\beta_1 = -\frac{1}{16}$ and $s = 1$ to $\mathcal{N} = 1$ conformal supergravity multiplet $(1, \frac{3}{2}, 2)$ where $\beta_1 = \frac{5}{2}$, in agreement with previously known values.

### 4.2 Conformally-flat background

Let us first recall the known low-spin cases. In the case of the conformally-flat background (1.10) one finds for $s = \frac{1}{2}, \frac{3}{2}$ partition functions (cf. (3.13)–(3.16))[26, 27, 3]

$$Z_{\frac{1}{2}} = \left[ \frac{1}{\det \tilde{\Delta}_{\frac{1}{2}}(3)} \right]^{-1/4}, \quad \tilde{\Delta}_{s}(M^2) = -\nabla_s^2 + M^2 \epsilon,$$ \hspace{1cm} (4.9)

\(^{32}\)Here we shall follow [21, 68] but not [22]: the fermionic higher-spin operator assumed in [22] in the $A > B$ case contained extra $1/A$ factor in $X$ in (3.5) that may seem somewhat unnatural in the context of factorizing higher-derivative CHS kinetic operators (it may, however, in principle appear upon squaring of 1st-order fermionic operators, cf. [69]). The operator in [22] required strong consistency conditions that rule out non-trivial backgrounds when applied to real fermions and thus do not allow to compute the coefficient $\beta_1$ in the conformal anomaly for $s > 2$. Here we will not worry about consistency conditions of the factor-operators like (3.5) as such conditions on the total operator should be weaker in the conformal higher spin case (and Einstein-space background should be a consistent one). Needless to say, the structure of “minimal” fermionic factor-operators in Ricci-flat background for higher spins $s \geq \frac{5}{2}$ deserves further clarification.
\[ Z_{\frac{3}{2}} = (Z_{\frac{1}{2},\frac{1}{2}})^2 Z_{\frac{3}{2},\theta} = \left[ \frac{\det \hat{\Delta}_3 (1)}{\det \hat{\Delta}_{3/2} (3)} \right]^{-2/4} \left[ \frac{1}{\det \hat{\Delta}_{3/2} (4)} \right]^{-1/4} \]  
(4.10)  
\[ = \left[ \frac{(\det \hat{\Delta}_3 (1))^2}{\det \hat{\Delta}_{3/2} (3)} \right]^{-2/4} \left[ \frac{\det \hat{\Delta}_3 (0)}{\det \hat{\Delta}_{3/2} (4)} \right]^{-1/4} . \]  
(4.11)  

Here \( \hat{\Delta}_s (M^2) \equiv -\nabla_2^2 + M^2 \epsilon \) is defined on \( \gamma \)-traceless spinor-tensors, while \( \hat{\Delta}_{s \perp} (M^2) \) is, in addition, restricted to transverse spinor-tensors. The relation between (4.10) and (4.11) is based on (cf. (3.30),(A.5))

\[ \det \hat{\Delta}_{3/2} (M^2) = \det \hat{\Delta}_{3\perp} (M^2) \det \hat{\Delta}_3 (M^2 - 4) . \]  
(4.12)  

One notes [26] that the conformal gravitino partition function (4.11) contains two factors of the standard “massless” Einstein gravitino partition function (with “cosmological” mass parameter \( m^2 = -\frac{\Lambda}{3} = -\epsilon \)) [70, 65, 66]

\[ Z_{\frac{3}{2} \perp} = \left[ \frac{\det \hat{\Delta}_{3/2} (1)}{\det \hat{\Delta}_{3\perp} (3)} \right]^{-1/4} = \left[ \frac{(\det \hat{\Delta}_3 (1))^2}{\det \hat{\Delta}_{3/2} (3)} \right]^{-1/4} . \]  
(4.13)  

In (4.10) the \( \det \hat{\Delta}_{3/2} (1) \) factor comes from the Jacobian of transformation from \( \psi_m \) to its transverse part plus pure-gauge gradient part (A.10) while \( \hat{\Delta}_{3\perp} \) operators appear from factorization of the 3rd-derivative conformal gravitino kinetic operator in conformally flat background [26, 27, 29, 30]:

\[ \psi_{\frac{3}{2}} \hat{D} \psi_{\frac{3}{2}} = \psi_{\frac{3}{2} \perp} \hat{\Delta}_{\frac{3}{2}} (3) \hat{\nabla}_{\frac{3}{2} \perp} \psi_{\frac{3}{2} \perp} , \quad \hat{\nabla}_{\frac{3}{2}} \equiv (\gamma^k \nabla_k)_{\frac{3}{2}} . \]  
(4.14)  

In general, “squaring” the “massive” 1st-order gravitino operator \( \hat{\nabla}_{\frac{3}{2}} + m \) gives

\[ (\hat{\Delta}_{\frac{3}{2}})_{mn} = - (\nabla_{\frac{3}{2}}^2)_{mn} + \frac{1}{4} R g_{mn} - \frac{1}{2} \gamma^k \gamma^l R_{klnm} + m^2 g_{mn} \]
\[ = - (\nabla_{\frac{3}{2}}^2)_{mn} + (4 \epsilon + m^2) g_{mn} , \]  
(4.15)  

where in the second line we assumed the conformally-flat background (A.1).\(^{33}\) Thus the second power of \( \hat{\Delta}_{3\perp} (3) \) (with \( m^2 = -\epsilon \)) in (4.10) has to do with its appearance already in the original fermionic action, while the extra factor \( \hat{\Delta}_{3\perp} (4) \) comes from squaring of the \( \hat{\nabla}_{\frac{3}{2}} \) operator in (4.14), i.e. (4.15) with \( m = 0 \). The fact that this “extra” operator is just the square of the standard “mass-zero” transverse \( \gamma \)-traceless gravitino operator \( \hat{\nabla}_{\frac{3}{2}} \) explains also its special conformal-invariance property (implying “null-cone” propagation) [29, 30]. Thus this operator is a direct counterpart of \( \hat{\Delta}_{2\perp} (4) \) in the conformal graviton \( s = 2 \) case (3.15),(3.16) but lacking extra effective gauge invariance (reflected in the trivial numerator in the second factor of (4.10))

\(^{33}\)Note that in this paper we use different notation compared to [26, 27, 3]: there this operator was denoted as \( \hat{\Delta}_{\frac{3}{2}} (m^2) \) while here it is called \( \hat{\Delta}_{\frac{3}{2}} (4 + \epsilon^{-1} m^2) \). Similarly, \( \hat{\Delta}_{\frac{3}{2}} (m^2) = - \nabla_{\frac{3}{2}}^2 + \frac{1}{4} R + m^2 = - \nabla_{\frac{3}{2}}^2 + 3 \epsilon + m^2 \) here is called \( \hat{\Delta}_{\frac{3}{2}} (3 + \epsilon^{-1} m^2) \).
the corresponding field was not called “partially massless” in [37, 38] and it describes a massive state (with 2s + 1 = 4 dynamical d.o.f.) in (A)dS_4.\(^{34}\)

The above discussion suggests the following natural generalization of the flat-space fermionic CHS partition function (2.18) to the conformally-flat Einstein-space case which is a direct counterpart of the bosonic expression (3.26). We shall assume that for all s ≥ \( \frac{3}{2} \) the 2s = 2s+1-derivative fermionic CHS field kinetic operator factorizes, like in the bosonic case (3.25), into the product of “squares” of all s “partially massless” (PM) 1st-order fermionic spin s operators \( \hat{\nabla}_s + m_{s,k} \) with special mass parameters \( m_{s,k} \) (k = \( \frac{1}{2}, \ldots, s - 1 \)) and also one extra “mass-zero” operator \( \hat{\nabla}_s \) (which, in fact, represents a massive state in (A)dS)

\[
\psi_s D_s \psi_s = \psi_{s,\perp} \left[ \prod_{k=1/2}^{s-1} (\hat{\nabla}_s + m_{s,k}) (\hat{\nabla}_s + m_{s,k}) \right] \hat{\nabla}_s \psi_{s,\perp}, \quad \hat{\nabla}_s \equiv (\gamma^k \nabla_k)_s . \tag{4.16}
\]

As in the bosonic case (3.25), this factorization is suggested, in particular, by the existence of an “ordinary-derivative” formulation of the fermionic CHS fields [34].

The values of the fermionic PM mass parameters (first conjectured in \( D = 4 \) in [38] and confirmed and extended to any \( D \) in [43]) are

\[
m_{s,k}^2 = -(k + \frac{1}{2} + \frac{D-4}{2})^2 \epsilon , \quad m_{s,k}^2 \big|_{D=4} = -(k + \frac{1}{2})^2 \epsilon , \quad k = \frac{1}{2}, \ldots, s - 1 , \tag{4.17}
\]

where \( k = s - 1 \) corresponds to the massless field in (A)dS_\( D \). The PM fields admit a local gauge-covariant description upon introducing extra lower-spin fields [43]; eliminating the latter gives residual gauge transformations with higher \( \nabla^{s-k} \) derivatives acting on lower-rank spinor-tensor parameters \( \xi_k \).

Starting with generic operator \( \hat{\nabla}_s + m \) describing massive on-shell spinor-tensors (s = \( s + \frac{1}{2} \))

\[
(\gamma^k \nabla_k + m) \psi_{m_1 \ldots m_s} = 0 , \quad \gamma^{m_1} \psi_{m_1 \ldots m_s} = 0 , \quad \nabla^{m_1} \psi_{m_1 \ldots m_s} = 0 , \tag{4.18}
\]

and “squaring” it gives in conformally-flat case the following operator (generalizing the s = 1 one in (4.15))\(^{35}\)

\[
\hat{\Delta}_s(M^2) = (\hat{\nabla}_s + m)(\hat{\nabla}_s + m) = -\nabla_s^2 + M^2 , \quad M^2 = (s + 3)\epsilon + m^2 \equiv M^2 \epsilon . \tag{4.19}
\]

Thus the family of PM fermionic operators in \( D = 4 \) is represented by the following set of 2nd-order operators (cf. (3.19))

\[
\hat{\Delta}_s(M_{s,k}^2) = -\nabla_s^2 + M_{s,k}^2 \epsilon , \quad M_{s,k}^2 = s + 3 - (k + 1)^2 , \quad s = s - \frac{1}{2} = 0, 1, \ldots , \quad k = k - \frac{1}{2} = k = 0, \ldots, s - 1 . \tag{4.20}
\]

\(^{34}\)Let us recall that the definition of “mass” is ambiguous in (A)dS and truly massless field (with 2 degrees of freedom) corresponds to maximal amount of gauge invariance (and thus smallest number of propagating modes). PM fields have less gauge invariance (and thus more degrees of freedom), with generic massive fields having no residual gauge invariance.

\(^{35}\)A derivation of the \( (s + 3)\epsilon \) contribution to the mass term can be given, e.g., by considering \( X = -\frac{1}{2} \gamma^m \gamma^n [\nabla_m, \nabla_n] = -\frac{1}{2} \gamma^m \gamma^n \hat{K}_{mn} \hat{\Sigma}_{ab} \) where \( \hat{\Sigma}_{ab} \) corresponds to the representation describing spinor-tensor \( \psi_{m_1 \ldots m_s} \) (generalizing s = 1 case in (4.15)).
\[ k = s - 1 \] corresponds to the standard massless case (here \( D = 4 \), cf. (3.21))

\[ m^2_{s0} \equiv M^2_{s,s-1} = s + 3 - s^2 = 2s - s^2 + \frac{9}{4} , \]  
(4.21)

which is the only choice for \( s = \frac{3}{2} \) (\( s = 1 \)) case. The first non-trivial PM field appears for \( s = \frac{5}{2} \) where we get for \( k = 1 \) and \( k = 0 \):

\[ M^2_{\frac{5}{2},\frac{3}{2}} = 1, \quad M^2_{\frac{5}{2},\frac{1}{2}} = 4. \]

One extra “genuinely-massive” operator that we should add corresponds to \( m^2 = 0 \) in (4.19): it can be viewed as a natural member of an “extended” PM family (4.20) were we allow also the \( k = -1 \) (\( k = -\frac{1}{2} \)) value:

\[ \hat{\Delta}_s(M^2_{s,\emptyset}) = -\nabla^2_s + M^2_{s,\emptyset} \epsilon , \quad M^2_{s,\emptyset} \equiv M^2_{s,-\frac{1}{2}} = s + 3 . \]  
(4.22)

As in (2.18) here the index \( \emptyset \) indicates that there is no associated gauge invariance, i.e. this field describes \( 2s + 1 \) degrees of freedom (2.19). The set of such \((\text{A)dS})\)-massive but conformally-invariant fields includes the standard fermion (4.9) (\( s = 0 \)) and the \( \hat{\Delta}_{\frac{3}{2}}(4) \) gravitino in (4.10) (\( s = 1 \)).

As a result, the fermionic CHS partition function in conformally-flat background should have the following representation (that directly reduces to (2.18) in the flat-space \( \epsilon = 0 \) limit)

\[ Z_s = \prod_{k=\frac{3}{2}}^{s-1} (Z_{s,k})^2 Z_{s,\emptyset} , \]  
(4.23)

\[ Z_{s,k} = \left[ \frac{\det \hat{\Delta}_{k\bot}(M^2_{k,s})}{\det \hat{\Delta}_{s\bot}(M^2_{s,k})} \right]^{-1/4} , \quad Z_{s,\emptyset} = \left[ \frac{1}{\det \hat{\Delta}_{s\bot}(M^2_{s,\emptyset})} \right]^{-1/4} . \]  
(4.24)

Here we used that as in the corresponding bosonic expression in (3.26) the part of the Jacobian of transformation from \( \psi_s \) to \( \psi_{s\bot} \) and other low-rank reducible components that remains after the division over the volume of gauge group is given by

\[ \prod_{k=\frac{3}{2}}^{s-1} \left[ \det \hat{\Delta}_{k\bot}(M^2_{k,s}) \right]^{-1/2} , \quad M^2_{k,s} = k + 3 - (s + 1)^2 , \]  
(4.25)

where \( M^2_{k,s} \) is again the “transpose” of the PM mass matrix in (4.21). Let us note also that for half-integer \( k \) one has the following counterpart of (3.30)

\[ \det \hat{\Delta}_{k\bot}(M^2) = \frac{\det \hat{\Delta}_k(M^2)}{\det \hat{\Delta}_{k-1}(M^2 - \delta_k)} , \quad \delta_k \bigg|_{D=4} = 2(k + 1) , \]  
(4.26)

which generalizes (4.12).

As already mentioned, the \( k = s - 1 \) factor in (4.23) is the square of the standard massless spin \( s \) partition function in \((\text{A)dS}_4)\) (cf. (3.24),(3.31) and (4.13))

\[ Z_{s,s-1} = \left[ \frac{\det \hat{\Delta}_{(s-1)\bot}(M^2_{s,s-1})}{\det \hat{\Delta}_{s\bot}(M^2_{s,s-1})} \right]^{-1/4} = \left[ \frac{\det \hat{\Delta}_{(s-1)\bot}(1 - s - s^2)}{\det \hat{\Delta}_{s\bot}(3 + s - s^2)} \right]^{-1/4} . \]  
(4.27)
The special cases of (4.23) for $s = \frac{1}{2}$ and $s = \frac{3}{2}$ of course agree with (4.9) and (4.10) while, e.g., for $s = \frac{5}{2}$ we get

$$Z_{\frac{5}{2}} = (Z_{\frac{3}{2}+\frac{1}{2}})^2(Z_{\frac{1}{2}+\frac{3}{2}})^2Z_{\frac{3}{2},0} = \left[ \frac{\det \hat{\Delta}_s(-5)}{\det \Delta_{\frac{1}{2}}(1)} \right]^{-2/4} \left[ \frac{\det \hat{\Delta}_{\frac{1}{2}}(-6)}{\det \Delta_{\frac{3}{2}}(4)} \right]^{-2/4} \left[ \frac{1}{\det \Delta_{\frac{5}{2}}(5)} \right]^{-1/4} .$$

(4.28)

To compute the Weyl-anomaly coefficient $a_s$ corresponding to (4.23) we start again with the general relations (3.34),(3.35) applied now to the case of the representation (4.5). Then the counterparts of (3.36),(3.37) in the half-inter spin $s$ case are (see (4.26))

$$a[\hat{\Delta}_s(M^2)] = -\frac{1}{144}N_s(N_s - 3M^4 + 12M^2 - \frac{121}{10}) , \quad N_s = (s + 1)(s + 2) ,$$

(4.29)

$$a[\hat{\Delta}_{s-1}(M^2)] = a[\hat{\Delta}_s(M^2)] - a[\hat{\Delta}_s(M^2 - 2s - 2)]$$

$$= -\frac{1}{720}(s + 1)[ -101 + 20s(3s^2 + 13s + 11) - 60(s^2 + s - 2)M^2 - 30M^4 ] .$$

(4.30)

Here we already accounted for an extra $\frac{1}{2}$ factor and fermionic minus sign, i.e., for example, $a[\hat{\Delta}_{\frac{1}{2}}(3)] = \frac{11}{720}$ gives the contribution of a single $s = \frac{1}{2}$ fermion in (4.9). Applying this to operators in (4.24) with mass parameters given in (4.20),(4.22),(4.25) and performing the sum over $k = 0, ..., s - 1$ as required by (4.23) we end up with the following expression for the fermionic CHS $a_s$-coefficient which is a counterpart of the bosonic expression in (3.38):

$$a_s^{(f)} = 2 \sum_{k=0}^{s-1} \left( a[\hat{\Delta}_{s-1}(2 + s - 2k - k^2)] - a[\hat{\Delta}_{s-1}(2 + k - 2s - s^2)] \right) + a[\hat{\Delta}_{s-1}(3 + s)]$$

$$= \frac{1}{2880}\nu_s(12 + 45\nu_s + 14\nu_s^2) , \quad \nu_s = -2(s + \frac{1}{2})^2 .$$

(4.31)

In particular, $a_{\frac{1}{2}}^{(f)} = \frac{11}{720}$, $a_{\frac{3}{2}}^{(f)} = -\frac{137}{90}$ in agreement with (1.7),(1.8), also $a_{\frac{5}{2}}^{(f)} = -\frac{1869}{80}$, etc.

Combining the results for $\beta_{1,s}^{(f)}$ in (4.7) and $a_s^{(f)}$ in (4.31) we conclude that

$$c_s^{(f)} = \frac{1}{2}\beta_{2,s}^{(f)} = \beta_{1,s}^{(f)} + a_s^{(f)} = \frac{1}{2880}\nu_s(118 + 135\nu_s + 29\nu_s^2) .$$

(4.32)

Like the bosonic expression in (3.38), the cubic polynomial (4.31) in $\nu_s$ turns out to be special: when summed over all spins $s = \frac{1}{2}, ..., \infty$ and analytically ($\zeta$-function) regularized it gives zero. This will not be true, however, for the sum of $c_s^{(f)}$ in (4.32). We shall discuss this in more detail in the next section.

5 Concluding remarks

Our final results for the Weyl anomaly coefficients (1.1) of $D = 4$ conformal higher spin fields were already summarized in (1.12)–(1.15). As was mentioned in the Introduction, ref.[11] made a remarkable observation that the sum over all spins of the bosonic $a_s$ coefficient in (1.12) gives zero if computed using $\zeta$-function prescription, suggesting the existence of an anomaly-free
theory. The same happens to be true also for the corresponding sum in the fermionic case (see (1.16)).

Let us now discuss this vanishing of the regularized sum of $a_s$ anomaly coefficients in more detail. As is well known, to define a power-divergent sum like $P_n = \sum_{s=0}^\infty p_n(s)$ where $p_n$ is an order $n$ polynomial in $s$ one should not, in general, use a sharp cutoff (like $0 \leq s \leq M$, $M \to \infty$) but should consider a smooth analytic regularization with a cut-off function $f$, i.e. define (see, e.g., [71] or [72] for a recent discussion)

\[ P_n = P_n(\epsilon \to 0) \bigg|_{\text{fin}} , \quad P_n(\epsilon) \equiv \sum_{s=1}^\infty f(\epsilon s) \, p_n(s) , \quad f(0) = 1 , \quad f(\infty) = 0 , \quad (5.1) \]

one should compute the regularized sum, take the limit $\epsilon \to 0$ and drop all singular $\frac{1}{\epsilon^n}$ terms. For example, one may use an exponential cutoff $f(\epsilon s) = e^{-\epsilon s}$. Then $(P_n)_{\text{reg}}$ will be the same as found by computing each term $\sum_{s=1}^\infty s^k$ using the $\zeta$-function regularization.

One may wonder what is the physical meaning of this regularization prescription in the present “sum over spin” context. A possible answer is that it is required to preserve some hidden symmetries of the higher-spin system (cf. [58]). In fact, one can draw an analogy with string theory which describes an infinite set of fields of growing spins and masses which are effectively summed over in the world-sheet description. Indeed, a standard example is that the use of an analytic or $\zeta$-function regularization of oscillator sums in computing, e.g., 2d central charge and vacuum energy [71] gives, for example, the right (zero) value for the mass of the first excited level state in bosonic open string (photon) and is thus required for a consistent realisation of target space symmetries of string theory.

Computing the sum of $a_s^{(b)}$ in (1.12) using the exponential cutoff we get

\[
\sum_{s=1}^\infty e^{-\epsilon s} \, a_s^{(b)} = \frac{e^{2(566\epsilon^4 + 1326\epsilon^2 + 566\epsilon^3 + 31\epsilon^4 + 31)}}{180(\epsilon^4 - 1)^7} = \frac{4}{\epsilon^7} + \frac{7}{\epsilon^5} + \frac{3}{2\epsilon^3} + \frac{1}{6\epsilon} + \frac{1}{120\epsilon^3} + \frac{\epsilon}{7560} + O(\epsilon^2) . \quad (5.2)
\]

Thus the finite part of this sum vanishes as claimed in (1.16). The same result is, of course, found [11] using $\zeta$-function regularization.

Similarly, in the case of the fermionic $a_s$ coefficient in (1.14) we get

\[
\sum_{s=1}^\infty e^{-\epsilon s} \, a_s^{(f)} = \sum_{s=0}^\infty e^{-\epsilon(s+\frac{1}{2})} \, a_s^{(f)}
= \frac{\epsilon^{\frac{5}{2}} \left( -1173e^\epsilon - 8918e^{2\epsilon} - 8918e^{3\epsilon} - 1173e^{4\epsilon} + 11e^{5\epsilon} + 11 \right)}{720(\epsilon^4 - 1)^7}
= -\frac{28}{\epsilon^7} - \frac{14}{\epsilon^5} - \frac{2}{\epsilon^3} + \frac{1}{6\epsilon} + \frac{47}{480\epsilon^3} + \frac{1}{64\epsilon} + \frac{7}{5760\epsilon} + \frac{3607e}{7741440} + O(\epsilon^2) \quad (5.3)
\]

---

36Here we consider just on the conformal spin 2 gauge symmetry preservation: the reparametrization invariance should be manifest, while the Weyl invariance anomaly should cancel out. Anomalies of higher spin analogs of these $s=2$ symmetries (in particular, higher spin trace anomalies, cf.[10, 76]) should also be absent for the full consistency of the theory.

37Computing $P_n(\epsilon) = \sum_{s=0}^\infty e^{-\epsilon s} \, s^n$, and dropping all singular terms in $\epsilon \to 0$ one gets the finite part $P_n|_{\text{reg}} = \zeta(-n)$. Explicitly, for odd $n$: $P_1 = \frac{1}{\epsilon} - \frac{1}{12} + O(\epsilon)$, $P_3 = \frac{6}{\epsilon} + \frac{1}{120} + O(\epsilon^2)$, $P_5 = \frac{120}{\epsilon} - \frac{1}{252} + O(\epsilon^2)$, etc., while for even $n$: $P_0 = \frac{1}{\epsilon} - \frac{1}{12} + O(\epsilon)$, $P_2 = \frac{2}{\epsilon} - \frac{1}{20} + O(\epsilon^2)$, $P_4 = \frac{24}{\epsilon} + \frac{1}{576} + O(\epsilon^2)$, etc.
so that again there is no left-over finite part. Note that here the original sum goes over half-integer spins, so that to apply the equivalent $\zeta$-function regularization prescription one needs to use $\zeta(z,q) = \sum_{n=0}^{\infty} (n+q)^{-z}$ with $q = \frac{1}{2}$, i.e. $\zeta(z,\frac{1}{2}) = (2^z-1)\zeta(z)$.\footnote{This is again similar to the prescription one uses in string theory when computing, e.g., the vacuum energy in NS sector where 2d fermions are anti-periodic. Explicitly, the cancellation of the finite part in (5.3) is due to the following relations: $\zeta(0,\frac{1}{2}) = \zeta(-2,\frac{1}{2}) = \zeta(-4,\frac{1}{2}) = 0$ and $\frac{1}{60}\zeta(-1,\frac{1}{2}) + \frac{1}{360}\zeta(-3,\frac{1}{2}) - \frac{7}{60}\zeta(-5,\frac{1}{2}) = 0$.}

Let us stress that this vanishing of the regularized sums of the $a_s$-coefficients is non-trivial. Together with the known lower spin results for $a_s$ in (1.5)–(1.8) this property uniquely fixes the expression for $a_s$ in both the bosonic and the fermionic cases. First, one may argue on general grounds (from the structure of $b_4$ in (3.9) and the form of the partition functions (3.26),(4.23)) that the conformal anomaly coefficients $(a_s, c_s)$ should be given by cubic homogeneous polynomials of $\nu_s$, i.e. of the physical number of d.o.f. of a spin $s$ field. In the bosonic case demanding agreement with the known $s = 1, 2$ values in (1.5),(1.6) then leads to the following predictions

$$a_s^{(b)} = \frac{1}{23040} \nu_s \left[ \nu_s(3 + 14\nu_s) + q^{(b)}(\nu_s - 2)(\nu_s - 6) \right],$$

$$c_s^{(b)} = \frac{1}{1080} \nu_s \left[ \nu_s(-59 + 43\nu_s) + r^{(b)}(\nu_s - 2)(\nu_s - 6) \right], \quad \nu_s = s(s+1).$$

Similarly, in the fermionic case the expressions that match the known $s = \frac{1}{2}, \frac{3}{2}$ values in (1.7),(1.8) are

$$a_s^{(f)} = \frac{1}{23040} \nu_s \left[ \nu_s(300 + 106\nu_s) + q^{(f)}(\nu_s + 2)(\nu_s + 8) \right],$$

$$c_s^{(f)} = \frac{1}{23040} \nu_s \left[ \nu_s(490 + 173\nu_s) + r^{(f)}(\nu_s + 2)(\nu_s + 8) \right], \quad \nu_s = -2(s + \frac{1}{2})^2.$$  \hspace{0.5cm} (5.7)

Here $q^{(b)}, r^{(b)}$ and $q^{(f)}, r^{(f)}$ are so far arbitrary coefficients. Now imposing the additional condition of the vanishing of finite parts the corresponding sums of $a_s$ over all spins (integer in the bosonic case and half-integer in the fermionic case) fixes (after computing the sums as in (5.2), (5.3)) the coefficients $q^{(b)}$ and $q^{(f)}$ uniquely

$$q^{(b)} = 0, \quad q^{(f)} = 6.$$ \hspace{0.5cm} (5.8)

Then (5.4) and (5.6) become precisely to the expressions for $a_s^{(b)}$ (3.38) and and $a_s^{(f)}$ (4.31) that were independently found above from the detailed structure of the CHS partition functions (3.26) and (4.23) in (A)dS$_4$ background.

Applying the same requirement of zero finite part to the regularized sums of the $c_s$-coefficients in (5.5),(5.7), i.e. $\sum_{s=1}^{\infty} e^{-\epsilon s} c_s^{(b)}$ and $\sum_{s=\frac{1}{2}}^{\infty} e^{-\epsilon s} c_s^{(f)}$, gives

$$r^{(b)}_0 = -1, \quad r^{(f)}_0 = \frac{3597}{367}.$$ \hspace{0.5cm} (5.9)

The results for $c_s^{(b)}$ in (3.40) and $c_s^{(f)}$ in (4.32) that we have found above correspond, however, to different values \footnote{It may be interesting to note that the difference between our value of $c_s^{(b)}$ with $r^{(b)} = \frac{1}{2}$ and the “zero-sum” value with $r^{(b)} = -1$ is an integer (a binomial coefficient)

$$c_s^{(b)} \bigg|_{r^{(b)}=\frac{1}{2}} - c_s^{(b)} \bigg|_{r^{(b)}=-1} = \frac{1}{720} \nu_s(\nu_s - 2)(\nu_s - 6) = \frac{1}{61} s(s^2 - 1)(s^2 - 4)(s + 3) = \left( \frac{s + 3}{6} \right).$$}

$$r^{(b)} = \frac{1}{2}, \quad r^{(f)} = 59.$$ \hspace{0.5cm} (5.10)
This non-vanishing of sums of $c_s$ we have found suggests that the expressions in (3.40) and (4.32) may deserve further checks.

Let us note also that the vanishing (1.16) of the regularized sums of the $a_s$ coefficients (5.2), (5.3) means also the UV finiteness of the products of the (A)dS partition functions: $\prod_{s=1}^{\infty} Z_s^{(b)}$ corresponding to (3.26) and $\prod_{s=1}^{\infty} \frac{1}{2} Z_s^{(f)}$ corresponding to (4.23).

Here we discussed only the $b_4$ heat kernel coefficient of the logarithmically divergent part of CHS free energies $\ln Z_s$ but the corresponding partition functions on (A)dS$_D$ or $S^D$ may be computed explicitly as, e.g., in [64, 65, 67, 11]. This should allow one to prove directly the relation between, e.g., the bosonic conformal higher spin $s$ partition function (3.26) on $S^4$ and the ratio of partition functions of massless spin $s$ field in AdS$_5$ with alternate boundary conditions as implied by the AdS/CFT in the context of “double-trace” deformation construction [11].

Let us briefly review some underlying ideas. Coupling, e.g., the $D = 4$ conformal $\mathcal{N} = 4$ SYM theory to a background conformal supergravity multiplet and integrating out the SYM fields one finds an induced action for the conformal supergravity fields [25, 5, 73]: $S_{\text{eff}} \sim \int C_{mnkl} \ln (L^{-2} \nabla^2) C_{mnkl} + ... \sim (C_{mnkl}^2 + ...) +$ non-local terms. The quadratic and cubic terms in this action expanded in powers of the fields summarize information about the protected 2- and 3-point SYM correlators like $\langle T_{mn} T_{kl} \rangle$ and $\langle T_{mn} T_{kl} T_{sr} \rangle$. The “protected” part of this induced action (which is the same at strong and weak coupling) appears also upon solving the Dirichlet problem in the 5-d $\mathcal{N} = 8$ gauged supergravity on the AdS$_5$ background. This relation can be generalized [6] by starting with the free $\mathcal{N} = 4$ gauge theory and coupling it to a higher spin generalization of the conformal supergravity multiplet. Let us consider, e.g., the bosonic conserved traceless bilinear currents [74] $J_{m_1...m_s} \sim X_r \partial_{m_1}...\partial_{m_s} P_{m_1...m_s} X_r$ (cf.(2.1); $X_r$ stand for the CFT fields) of dimension $\Delta = 2 + s$.\footnote{Note that in general the mass dimensions of different fields involved are (we assume that $D$ is even): boundary scalars with action $\int d^D x \, X_r \partial^2 X_r$: $\Delta = \frac{1}{2} (D - 2)$; conformal current $J_s \sim X_r \partial^s X_r$: $\Delta = s + D - 2$; the corresponding “source” field ($\int d^D x \, J_s \phi_s$) – conformal field with action $\int d^D x \, \phi_s P_s \partial^{2s+D-4} \phi_s$: $\Delta = 2 - s$.}

Coupling them to a background higher spin conformal field $\phi_s$, integrating out the free SYM fields and expanding the resulting induced effective action for $\phi_s$ to quadratic order one then gets the logarithmically divergent term proportional to the CHS Lagrangian $\int d^D x \, \phi_s P_s \partial^2 \phi_s$. It can be matched with the term originating from the classical free action of the corresponding “dual” higher spin massless field $\varphi_s$ in AdS$_5$ evaluated on the solution of a Dirichlet problem with $\phi_s$ as the boundary data. As in the $s = 2$ case of conformal (super)gravity multiplet, this agreement between the free bulk massless higher spin action and the induced boundary conformal higher spin action is essentially kinematical, i.e. is guaranteed by symmetries (see also [75]) and applies, of course, not only in $D = 4$ but also in other dimensions (see [9, 10, 13, 11]).

In addition to this “tree-level” relation between free action of massless higher spins on AdS$_{D+1}$ and free action of conformal higher spins on $S^D$ there is also a “one-loop” relation [11] motivated by the AdS/CFT correspondence in the presence of the “double-trace” deformation [52, 53, 54, 55]. Namely, the 1-loop determinant of the CHS kinetic operator on $S^D$ should be equal to the ratio of the massless higher spin 1-loop determinants in euclidean AdS$_{D+1}$ with alternate boundary conditions. This “one-loop” relation is more subtle than the “tree-level” one mentioned above but it should be possible to prove it directly by comparing the correspond-
ing heat kernel representations (cf. [54, 55]), now that the expression for the CHS partition function in terms of the standard 2nd-derivative operator determinants on $S^D$ is known (3.26).

To motivate this relation one starts with a large $N$ CFT free energy $F = -\ln Z$ on $S^D$ and considers its change upon RG flow from UV to IR induced by “double-trace” $\gamma(J_s)^2$ deformation. This change corresponds to alternate $\Delta_{\pm}$ boundary conditions for a massless higher spin in $\text{AdS}_{D+1}$. Considering, e.g., as a boundary CFT $N$ free scalars on $S^D$ and introducing an auxiliary field $\phi_s$ one may replace $\gamma(J_s)^2 \to J_s \phi_s - \frac{1}{3!}(\phi_s)^2$. The resulting ratio of large $N$ partition functions with and without the “double-trace” deformation is given by the path integral over $\phi_s$ with the action being the induced effective action for $\phi_s$ found by integrating out the original CFT scalars, $N \int \phi_s P_s \partial^{2s+D-4} \phi_s$ plus the $(\phi_s)^2$ term. Then for large $N$ the latter can be ignored and the leading-order result should is given just by the 1-loop partition function of the conformal higher spin field on $S^D$.

Given a generic massive spin $s$ field equations in $\text{AdS}_{D+1}$ with the transverse traceless kinetic operator (3.14) with $M^2 = m_{s0}^2 + m^2$ (with $m_{s0}^2$ given by (3.21) with $D \to D+1$) one finds the solutions behaving near the AdS boundary $(z \to 0)$ as $\varphi_s \sim z^\delta$, $\delta = \Delta - s$, where $[\Delta - (2-s)](\Delta - (s + D - 2)) = m^2$, i.e. $\Delta_{\pm} = \frac{1}{2} D \pm \left(\frac{1}{2} D + s - 2\right)^2 + m^2)^{1/2}$ [47]. These two values of dimensions correspond to the dimensions of $J_s$ in the two boundary CFT’s which are the end-points of the RG flow induced by the “double-trace” deformation [11] ($\Delta_+$ corresponds to the original free IR CFT and $\Delta_-$ to the UV CFT). In the $m = 0$ case the two values $\Delta_{\pm}$ are equal to the dimensions of the conserved current $J_s$ ($\Delta_+ = s + D - 2$) and $\phi_s$ ($\Delta_- = 2 - s$).

From the $\text{AdS}_{D+1}$ theory side the corresponding order $N^0$ term in the partition function should be given by the 1-loop partition function of the AdS massless spin $s$ field with the appropriate boundary conditions. One is then led to the following relation between the ratio of the 1-loop massless higher spin $\text{AdS}_{D+1}$ partition functions evaluated with alternate $\Delta_{\pm}$ boundary conditions and the conformal higher spin partition function in $S^D$.

\[
\frac{Z_{s0}^{(-)}}{Z_{s0}^{(+)}}|_{\text{AdS}_{D+1}} = Z_s|_{S^D}, \quad (5.11)
\]

\[
Z_{s0}|_{\text{AdS}_{D+1}} = \left[ \frac{\det[-\nabla^2 + (s-1)(s+D-2)]_{s-1\perp}}{\det[-\nabla^2 - s + (s-2)(s+D-2)]_{s\perp}} \right]^{1/2}, \quad (5.12)
\]

\[
Z_s|_{S^D} = \prod_{k=0}^{s-1} \left[ \frac{\det[-\nabla^2 + k - (s-1)(s+D-2)]_{k\perp}}{\det[-\nabla^2 + s - (k-1)(k+D-2)]_{s\perp}} \right]^{1/2} \times \prod_{k'=-\frac{1}{2}(D-4)}^{s-1} \left[ \frac{1}{\det[-\nabla^2 + s - (k'-1)(k'+D-2)]_{s\perp}} \right]^{1/2}. \quad (5.13)
\]

$Z_s$ is the CHS partition function on $S^D$ given in (3.26),(3.19),(3.20) (we set the radii to 1, i.e. $\epsilon_{\text{AdS}} = -1$, $\epsilon_S = 1$). In $Z_s$ we included the extra “massive” factor (3.27),(3.29) present for $D > 4$. $Z_{s0}$ is the massless spin $s$ partition function in $\text{AdS}_{D+1}$ (given by (3.24) with $D \to D + 1$).

\footnote{As explained in [11], the choice of the boundary conditions for “ghost” determinant in $Z_{s0}|_{\text{AdS}_{D+1}} (\xi_s \sim z^{\delta_{\pm}}, \delta_+ = D, \delta_- = 2 - 2s)$ is correlated with the $\Delta_\pm$ choice for the physical operator.}
Acknowledgments

We are grateful to S. Giombi and R. Metsaev for important discussions, suggestions and initial collaboration. We also thank V. Didenko, M. Grigoriev, I. Klebanov, G. Korchemsky, K. Mkrtchyan, R. Roiban, E. Skvortsov and M. Vasiliev for useful discussions, and S. Giombi, I. Klebanov and R. Metsaev for helpful comments on the draft. This work was supported by the ERC Advanced grant No.290456 and also by the STFC grant ST/J000353/1.
Appendix A: Notation and some useful relations

In this paper we always use symmetric traceless tensors (or \(\gamma\)-traceless spinor-tensors) and do not explicitly indicate the tracelessness condition.

The curvature tensor of conformally-flat Einstein backgrounds that we assume is

\[
R_{mnkl} = \epsilon (g_{mk}g_{nl} - g_{mk}g_{nl}) , \quad \epsilon = \pm r^{-2} = \frac{2\Lambda}{(D - 1)(D - 2)} ,
\]

where \(\epsilon > 0\) for \(dS_D\) (or \(S^D\) in the case of euclidean signature) and \(\epsilon < 0\) for AdS\(_D\) spaces. In \(D = 4\) one has \(\epsilon = \frac{\Lambda}{3}\), \(R = 4\Lambda = 12\epsilon\). One may assume that the curvature radius is \(r = 1\) so that \(\epsilon = \pm 1\).

Generic covariant second-order operator defined on rank \(s\) tensors in such constant curvature space can be put into the form

\[
\hat{\Delta}^s = \hat{\Delta}^s(M^2) \equiv -\nabla^2 + M^2 \epsilon .
\]

\(\hat{\Delta}_{\perp}^s(M^2)\) will stand for \(-\nabla^2 + M^2 \epsilon\) defined on transverse traceless tensors of rank \(s\). In general (modulo zero-mode contributions)

\[
\det \hat{\Delta}^s(M^2) = \det \hat{\Delta}_{\perp}^s(M^2) \det \hat{\Delta}_{s-1}(M^2 - (2s + D - 3)) .
\]

In particular, in \(D = 4\)

\[
\begin{align*}
\det \hat{\Delta}_2(M^2) &= \det \hat{\Delta}_{2\perp}(M^2) \det \hat{\Delta}_1(M^2 - 5) , \\
\det \hat{\Delta}_1(M^2) &= \det \hat{\Delta}_{1\perp}(M^2) \det \hat{\Delta}_0(M^2 - 3) .
\end{align*}
\]

in agreement with (A.5).

One may decompose \(\phi_{m_1...m_s} = \phi_{m_1...m_s\perp} + [\nabla_{(m_1} \xi_{m_2...m_s)} - \text{traces}]\) and compute the Jacobian of the corresponding transformation. For example,

\[
\begin{align*}
\phi_m &= \phi_{m\perp} + \nabla_m \xi , \\
\phi_{mn} &= \phi_{mn\perp} + \nabla_{(m} \xi_{n)} + (\nabla_m \nabla_n - D^{-1} \nabla^2) \xi , \\
J_1 &= [\det \hat{\Delta}_0(0)]^{1/2} , \\
J_2 &= [\det \hat{\Delta}_{1\perp}(-D + 1) \ det \hat{\Delta}_0(-D)]^{1/2} [\det \hat{\Delta}_0(0)]^{1/2} .
\end{align*}
\]

Here \(\hat{\Delta}_1(-D + 1)\) is the familiar ghost operator \(\hat{\Delta}_{1mn} = -\nabla^2_{mn} - R_{mn}\) in the case of (A.1). The first factor in (A.8) is the ghost determinant, while the second factor is cancelled against similar factor in the volume of the gauge group (the gauge group vector parameters are not transverse).

Similar decomposition for the rank 3 tensor \(\phi_{mnk} = (\phi_{mnk\perp}, \xi_{mn\perp}, \xi_{m\perp}, \xi)\) gives

\[
J_3 = [\det \hat{\Delta}_{2\perp}(-2D) \ det \hat{\Delta}_{1\perp}(-2D - 1) \ det \hat{\Delta}_0(-2D - 2)]^{1/2} \\
\times [\det \hat{\Delta}_{1\perp}(-D + 1) \ det \hat{\Delta}_0(-D) \ det \hat{\Delta}_0(0)]^{1/2} .
\]

Again, the first factor is the ghost determinant (for details in the \(D = 3\) case see [50]).

Similar relations are found for the fermions, e.g., for the \(D = 4\) gravitino

\[
\psi_m = \psi_{m\perp} + (\nabla_m - \frac{1}{4} \gamma_m \gamma^k \nabla_k) \xi , \quad J_{\frac{3}{2}} = \det \hat{\Delta}_{\frac{3}{2}}(-1) .
\]
Appendix B: Comments on higher-order conformal scalar operators

In this paper we considered a family of conformal higher-spin operators defined on symmetric traceless tensors $\phi_s$, i.e.

$$S = \int d^D x \sqrt{g} \phi^{m_1...m_s} (\nabla^{2s+D-4} + ...) \phi_{m_1...m_s},$$  \hspace{1cm} (B.1)

where $\phi_s$ has dimension $2 - s$. We suggested that on Einstein-space background this kinetic operator takes factorized form like (3.25) in $D = 4$.

Similar factorization is known for higher-order conformal operators $\Delta_{(2r)}$ defined on scalars

$$S_{2r} = \int d^D x \sqrt{g} \phi \Delta_{(2r)} \phi, \quad \Delta_{(2r)} = (-\nabla^2)^r + ... \hspace{1cm} (B.2)$$

In fact, a special case of such scalar action with $r = \frac{1}{2}(D - 4)$ appears as the special case of (B.1) with $s = 0$.

In general, in addition to the familiar conformal scalar operator

$$\Delta_{(2)} = -\nabla^2 + \frac{D-2}{4(D-1)} R,$$  \hspace{1cm} (B.3)

one may define also $\Delta_{(4)} = \nabla^4 + ...$, etc. In $D = 4$ there are two choices: $\Delta_{(2)} = -\nabla^2 + \frac{1}{6} R$ and $\Delta_{(4)}$ introduced in [24] and independently (for $D \geq 4$) in [57]:

$$S_4 = \int d^4 x \sqrt{g} \phi \Delta_{(4)} \phi = \int d^4 x \sqrt{g} \left[ \nabla^2 \phi \nabla^2 \phi - 2(R^{mn} - \frac{1}{3} g^{mn} R) \nabla_m \phi \nabla_n \phi \right].$$  \hspace{1cm} (B.4)

As is clear from (B.4), on Ricci-flat background $\Delta_{(4)} = (\Delta_{(2)})^2 = (\nabla^2)^2$ while on the constant curvature background (A.1) one gets $\Delta_{(2)} = -\nabla^2 + \frac{1}{6} R = -\nabla^2 + 2 \epsilon$ and

$$\Delta_{(4)} = (-\nabla^2) (-\nabla^2 + 2 \epsilon) = \hat{\Delta}_0(0) \hat{\Delta}_0(2),$$ \hspace{1cm} (B.5)

where we used the notation in (A.2). This leads to a simple derivation of the corresponding Weyl anomaly in (1.1) [24, 25, 3]:\footnote{A discussion of this anomaly in mathematics literature appeared also in [80].} it is just a sum of anomalies of minimal scalar Laplacian $-\nabla^2$ and the conformally-coupled one, i.e. $\beta_1 = 2 \beta_1[-\nabla^2] = \frac{1}{90}$, $a = a[\hat{\Delta}_0(0)] + a[\hat{\Delta}_0(2)] = -\frac{7}{90}$ (and thus $c = -\frac{1}{18}$).

In any even dimension $D$ one may consider a family of higher-order conformal scalar Laplacians $\Delta_{(2)}, \Delta_{(4)}, \Delta_{(6)}, ..., \Delta_{(D)}$ [81] usually referred to as GJMS operators. The operator $\Delta_{(2)}$ appearing in the $s = 0$ case of (B.1) is thus a special member of this family.

\footnote{Let us note that in addition to $(2s + D - 4)$-derivative CHS operators discussed in this paper one may formally consider also operators defined on symmetric traceless tensors with smaller number of derivatives that have Weyl symmetry (see, e.g., [30, 77, 78] for some 2-derivative cases and [79, 16] for general discussions). Such 2-derivative operators effectively appear in the factorisation of CHS operators on Einstein background (as $k = 0$ term in (3.25) in $D = 4$ [30, 41]). In general $D$ the operator discussed in [78] restricted to transverse fields and considered on conformally-flat backgrounds corresponds to the “maximal negative depth” operator in (3.19),(3.20) with $k = -\frac{1}{2}(D - 4)$ or $i = 1$ in (3.27),(3.29), i.e. is given by $-\nabla^2 + [s + \frac{1}{4} D(D - 2)] \epsilon$.}
In the case of a Ricci-flat background the GJMS operator $\Delta_{(2r)}$ is simply the $r$-th power of the standard Laplacian $-\nabla^2$ while for a conformally-flat background (a sphere $S^D$) the operators $\Delta_{(2r)}$ were explicitly constructed in [82] by arguing that they should take the following factorized form\footnote{Here we set $\epsilon = 1$ in (A.1) so that $\Delta_{(2)} = -\nabla^2 + \frac{D-2}{4(D-1)} R = -\nabla^2 + \frac{D}{2} D(D-2)$.}

$$\Delta_{(2r)} = \prod_{i=1}^{r} [\Delta_{(2)} - i(i - 1)] = \prod_{i=1}^{r} \hat{\Delta}_0 (m_i^2) , \quad \hat{\Delta}_0 (m_i^2) = -\nabla^2 + m_i^2 \epsilon , \quad (B.6)$$

$$m_i^2 = \frac{1}{4} D(D - 2) - i(i - 1) = \left( \frac{1}{2} D - i \right) \left( \frac{1}{2} D + i - 1 \right) . \quad (B.7)$$

The operator (B.6) is positive for $r \leq \frac{1}{2} D$. Eq.(B.6) is thus a generalization of the relation (B.5) in the $r = 2$ case.

The factorized structure of (B.6) is obviously similar to that in the case of higher-spin conformal operators in (3.25),(3.20). The direct derivation of (B.6) in [83] that uses stereographic projection from flat space should certainly have an analog in the CHS case of (3.25).

The representation (B.6) has a straightforward generalization [84] to any Einstein-space background $R_{mn} = \frac{1}{n} R g_{mn}$:

$$\Delta_{(2r)} = \prod_{i=1}^{r} \left[ -\nabla^2 + q_i R \right] , \quad q_i = \frac{\left( \frac{1}{2} D - i \right) \left( \frac{1}{2} D + i - 1 \right)}{D(D - 1)} , \quad (B.8)$$

so that, in particular, $\Delta_{(D)} = [-\nabla^2 + \frac{D-2}{4(D-1)} R] \ldots [-\nabla^2]$. For some other general properties of GJMS operators see also [85].\footnote{See also [86] for a computation of the corresponding conformal anomaly and determinant on $S^D$.}

References

[1] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, “Properties of conformal supergravity,” Phys. Rev. D 17, 3179 (1978).
[2] E. Bergshoeff, M. de Roo and B. de Wit, “Extended conformal supergravity,” Nucl. Phys. B 182, 173 (1981).
[3] E. S. Fradkin and A. A. Tseytlin, “Conformal Supergravity,” Phys. Rept. 119, 233 (1985).
[4] E. S. Fradkin and V. Y. Linetsky, “Superconformal higher spin theory in the cubic approximation,” Nucl. Phys. B 350, 274 (1991).
[5] H. Liu and A. A. Tseytlin, “D = 4 superYang-Mills, D = 5 gauged supergravity, and D = 4 conformal supergravity,” Nucl. Phys. B 533, 88 (1998) [hep-th/9804083].
[6] A. A. Tseytlin, “On limits of superstring in AdS(5) x S5,” Theor. Math. Phys. 133, 1376 (2002) [Teor. Mat. Fiz. 133, 69 (2002)] [hep-th/0201112].
[7] A. Y. Segal, “Conformal higher spin theory,” Nucl. Phys. B 664, 59 (2003) [hep-th/0207212].
[8] V. Balasubramanian, E. G. Gimon, D. Minic and J. Rahmfeld, “Four-dimensional conformal supergravity from AdS space,” Phys. Rev. D 63, 104009 (2001) [hep-th/0007211]. R. G. Leigh and A. C. Petkou, “SL(2,Z) action on three-dimensional CFTs and holography,” JHEP 0312, 020 (2003) [hep-th/0309177]. G. Compere and D. Marolf, “Setting the boundary free in AdS/CFT,” Class. Quant. Grav. 25, 195014 (2008) [arXiv:0805.1902].

[9] R. R. Metsaev, “Gauge invariant two-point vertices of shadow fields, AdS/CFT, and conformal fields,” Phys. Rev. D 81, 106002 (2010) [arXiv:0907.4678].

[10] X. Bekaert, E. Joung and J. Mourad, “Effective action in a higher-spin background,” JHEP 1102, 048 (2011) [arXiv:1012.2103].

[11] S. Giombi, I. R. Klebanov, S. S. Pufu, B. R. Safdi and G. Tarnopolsky, “AdS description of induced higher-spin gauge theory,” arXiv:1306.5242.

[12] S.E. Konstein, M.A. Vasiliev and V.N. Zaikin, “Conformal higher spin currents in any dimension and AdS/CFT correspondence,” JHEP 0012, 018 (2000) [arXiv:hep-th/0010239]. X. Bekaert and M. Grigoriev, “Manifestly conformal descriptions and higher symmetries of bosonic singletons,” SIGMA 6, 038 (2010) [arXiv:0907.3195]. “Notes on the ambient approach to boundary values of AdS gauge fields,” J. Phys. A 46, 214008 (2013) [arXiv:1207.3439]. V. E. Didenko and E. D. Skvortsov, “Towards higher-spin holography in ambient space of any dimension,” J. Phys. A 46, 214010 (2013) [arXiv:1207.6786].

[13] E. Joung and J. Mourad, “Boundary action of free AdS higher-spin gauge fields and the holographic correspondence,” JHEP 1206, 161 (2012) [arXiv:1112.5620]. X. Bekaert, E. Joung and J. Mourad, “Comments on higher-spin holography,” Fortsch. Phys. 60, 882 (2012) [arXiv:1202.0543].

[14] M. A. Vasiliev, “Holography, Unfolding and Higher-Spin Theory,” J. Phys. A 46, 214013 (2013) [arXiv:1203.5554].

[15] M. A. Vasiliev, “Bosonic conformal higher-spin fields of any symmetry,” Nucl. Phys. B 829, 176 (2010) [arXiv:0909.5226]. R. Marnelius, “Lagrangian conformal higher spin theory,” arXiv:0805.4686 .

[16] E. Joung and K. Mkrtchyan, “A note on higher-derivative actions for free higher-spin fields,” JHEP 1211, 153 (2012) [arXiv:1209.4864].

[17] N. Berkovits and E. Witten, “Conformal supergravity in twistor-string theory,” JHEP 0408, 009 (2004) [hep-th/0406051].

[18] J. Maldacena, “Einstein Gravity from Conformal Gravity,” arXiv:1105.5632 .

[19] T. Adamo and L. Mason, “Conformal and Einstein gravity from twistor actions,” arXiv:1307.5043 .

[20] D. M. Capper, M. J. Duff and L. Halpern, “Photon corrections to the graviton propagator,” Phys. Rev. D 10, 461 (1974). L. S. Brown and J. P. Cassidy, “Stress tensor trace anomaly in a gravitational metric: general theory, Maxwell field,” Phys. Rev. D 15, 2810 (1977).

[21] S. M. Christensen and M. J. Duff, “Axial and conformal anomalies for arbitrary spin in gravity and supergravity,” Phys. Lett. B 76, 571 (1978).

[22] S. M. Christensen and M. J. Duff, “New gravitational index theorems and supertheorems,” Nucl. Phys. B 154, 301 (1979).
E. S. Fradkin and A. A. Tseytlin, “Renormalizable asymptotically free quantum theory of gravity,” Nucl. Phys. B 201, 469 (1982).

E. S. Fradkin and A. A. Tseytlin, “One Loop Beta Function In Conformal Supergravities,” Nucl. Phys. B 203, 157 (1982). “Asymptotic Freedom In Extended Conformal Supergravities,” Phys. Lett. B 110, 117 (1982).

E. S. Fradkin and A. A. Tseytlin, “Conformal Anomaly in Weyl Theory and Anomaly Free Superconformal Theories,” Phys. Lett. B 134, 187 (1984).

A. A. Tseytlin, “Effective Action In De Sitter Space And Conformal Supergravity,” Sov.J.Nucl.Phys. 39(6), 1984, 1018-1023, Yad. Fiz. 39, 1606 (1984).

E. S. Fradkin and A. A. Tseytlin, “Instanton Zero Modes And Beta Functions In Supergravities. 2. Conformal Supergravity,” Phys. Lett. B 134, 307 (1984).

Y. Pang, “One-Loop Divergences in 6D Conformal Gravity,” Phys. Rev. D 86, 084039 (2012) [arXiv:1208.0877].

S. Deser and R. I. Nepomechie, “Gauge Invariance Versus Masslessness In De Sitter Space,” Annals Phys. 154, 396 (1984).

S. Deser and R. I. Nepomechie, “Anomalous Propagation Of Gauge Fields In Conformally Flat Spaces,” Phys. Lett. B 132, 321 (1983).

R. R. Metsaev, “Ordinary-derivative formulation of conformal low spin fields,” JHEP 1201, 064 (2012) [arXiv:0707.4437].

S. Deser, M. J. Duff and C. J. Isham, “Nonlocal conformal anomalies,” Nucl. Phys. B 111, 45 (1976). S. Deser and A. Schwimmer, “Geometric classification of conformal anomalies in arbitrary dimensions,” Phys. Lett. B 309, 279 (1993) [hep-th/9302047]. M. J. Duff, “Twenty years of the Weyl anomaly,” Class. Quant. Grav. 11, 1387 (1994) [hep-th/9308075]. S. Deser, “Conformal anomalies: Recent progress,” Helv. Phys. Acta 69, 570 (1996) [hep-th/9609138].

R. R. Metsaev, “Ordinary-derivative formulation of conformal totally symmetric arbitrary spin bosonic fields,” JHEP 1206, 062 (2012) [arXiv:0709.4392].

R. R. Metsaev, “Conformal totally symmetric arbitrary spin fermionic fields,” arXiv:1211.4498.

B. S. DeWitt, “Dynamical theory of groups and fields,” Conf. Proc. C 630701, 585 (1964) [Les Houches Lect. Notes 13, 585 (1964)]. G. ‘t Hooft and M. J. G. Veltman, “One loop divergencies in the theory of gravitation,” Annales Poincare Phys. Theor. A 20, 69 (1974). P. B. Gilkey, “The Spectral geometry of a Riemannian manifold,” J. Diff. Geom. 10, 601 (1975).

A. Higuchi, “Forbidden mass range for spin-2 field theory in De Sitter space-time,” Nucl. Phys. B 282, 397 (1987). “Symmetric tensor spherical harmonics on the N sphere and their application to the De Sitter group So(n,1),” J. Math. Phys. 28, 1553 (1987) [Erratum-ibid. 43, 6385 (2002)].

S. Deser and A. Waldron, “Partial masslessness of higher spins in (A)dS,” Nucl. Phys. B 607, 577 (2001) [hep-th/0103198].
[38] S. Deser and A. Waldron, “Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations,” Phys. Lett. B 513, 137 (2001) [hep-th/0105181].

[39] Y. M. Zinoviev, “On massive high spin particles in AdS,” hep-th/0108192.

[40] L. Dolan, C. R. Nappi and E. Witten, “Conformal operators for partially massless states,” JHEP 0110, 016 (2001) [hep-th/0109096].

[41] S. Deser and A. Waldron, “Conformal invariance of partially massless higher spins,” Phys. Lett. B 603, 30 (2004) [hep-th/0408155].

[42] E. D. Skvortsov and M. A. Vasiliev, “Geometric formulation for partially massless fields,” Nucl. Phys. B 756, 117 (2006) [hep-th/0601095].

[43] R. R. Metsaev, “Gauge invariant formulation of massive totally symmetric fermionic fields in (A)dS space,” Phys. Lett. B 643, 205 (2006) [hep-th/0609029].

[44] K. Alkalaev and M. Grigoriev, “Unified BRST approach to (partially) massless and massive AdS fields of arbitrary symmetry type,” Nucl. Phys. B 853, 663 (2011) [arXiv:1105.6111].

[45] S. Deser, E. Joung and A. Waldron, “Partial Masslessness and Conformal Gravity,” J. Phys. A 46, 214019 (2013) [arXiv:1208.1307]. “Gravitational- and Self- Coupling of Partially Massless Spin 2,” Phys. Rev. D 86, 104004 (2012) [arXiv:1301.4181].

[46] C. Fronsdal, “Massless Fields with Integer Spin,” Phys. Rev. D 18, 3624 (1978). “Massless Fields with Half Integral Spin,” Phys. Rev. D 18, 3630 (1978).

[47] R. R. Metsaev, “Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space,” Lect. Notes Phys. 524, 331 (1997) [hep-th/9810231]. R.R. Metsaev, “Massive totally symmetric fields in AdS(d),” Phys. Lett. B 590, 95 (2004) [hep-th/0312297]. “Anomalous conformal currents, shadow fields and massive AdS fields,” Phys. Rev. D 85, 126011 (2012) [arXiv:1110.3749].

[48] V. E. Lopatin and M. A. Vasiliev, “Free massless bosonic fields of arbitrary spin in d-dimensional de Sitter space”, Mod. Phys. Lett. A 3, 257 (1988). R.R. Metsaev, “Light cone form of field dynamics in anti-de Sitter spacetime and AdS/CFT correspondence, Nucl. Phys. B 563, 295 (1999) [hep-th/9906217]. I. L. Buchbinder, A. Pashnev and M. Tsulaia, “Lagrangian formulation of the massless higher integer spin fields in the AdS background,” Phys. Lett. B 523, 338 (2001) [hep-th/0109067].

[49] X. Bekaert and M. Grigoriev, “Higher order singletons, partially massless fields and their boundary values in the ambient approach,” arXiv:1305.0162.

[50] M. R. Gaberdiel, R. Gopakumar and A. Saha, “Quantum W-symmetry in AdS$_3$,” JHEP 1102, 004 (2011) [arXiv:1009.6087].

[51] R. K. Gupta and S. Lal, “Partition Functions for Higher-Spin theories in AdS,” JHEP 1207, 071 (2012) [arXiv:1205.1130].

[52] E. Witten, “Multitrace operators, boundary conditions, and AdS / CFT correspondence,” hep-th/0112258. S. S. Gubser and I. Mitra, “Double trace operators and one loop vacuum energy in AdS / CFT,” Phys. Rev. D 67, 064018 (2003) [hep-th/0210093].

[53] S. S. Gubser and I. R. Klebanov, “A Universal result on central charges in the presence of double trace deformations,” Nucl. Phys. B 656, 23 (2003) [hep-th/0212138].
[54] T. Hartman and L. Rastelli, “Double-trace deformations, mixed boundary conditions and functional determinants in AdS/CFT,” JHEP 0801, 019 (2008) [hep-th/0602106].
[55] D. E. Diaz and H. Dorn, “Partition functions and double-trace deformations in AdS/CFT,” JHEP 0705, 046 (2007) [hep-th/0702163].
[56] D. E. Diaz, “Polyakov formulas for GJMS operators from AdS/CFT,” JHEP 0807, 103 (2008) [arXiv:0803.0571].
[57] S. Paneitz, “A quartic conformally covariant differential operator for arbitrary pseudo-riemannian manifolds”, preprint (1983). Summary appeared in SIGMA 4 (2008) 036, arXiv:0803.4331.
[58] S. Giombi and I. R. Klebanov, “One Loop Tests of Higher Spin AdS/CFT,” arXiv:1308.2337.
[59] G. W. Gibbons, S. W. Hawking and M. J. Perry, “Path Integrals and the Indefiniteness of the Gravitational Action,” Nucl. Phys. B 138, 141 (1978).
[60] A. Cucchieri, M. Porrati and S. Deser, “Tree level unitarity constraints on the gravitational couplings of higher spin massive fields,” Phys. Rev. D 51, 4543 (1995) [hep-th/9408073]. I. Giannakis, J. T. Liu and M. Porrati, “Massive higher spin states in string theory and the principle of equivalence,” Phys. Rev. D 59, 104013 (1999) [hep-th/9809142].
[61] I. Cortese, R. Rahman and M. Sivakumar, “Consistent Non-Minimal Couplings of Massive Higher-Spin Particles,” arXiv:1307.7710 [hep-th].
[62] G. W. Gibbons and M. J. Perry, “Quantizing Gravitational Instantons,” Nucl. Phys. B 146, 90 (1978).
[63] S. M. Christensen and M. J. Duff, “Quantizing Gravity with a Cosmological Constant,” Nucl. Phys. B 170, 480 (1980).
[64] B. Allen, “Phase Transitions in de Sitter Space,” Nucl. Phys. B 226, 228 (1983).
[65] E. S. Fradkin and A. A. Tseytlin, “One Loop Effective Potential In Gauged O(4) Supergravity,” Nucl. Phys. B 234, 472 (1984).
[66] E. S. Fradkin and A. A. Tseytlin, “Instanton Zero Modes And Beta Functions In Supergravities. 1. Gauged Supergravity,” Phys. Lett. B 134, 301 (1984).
[67] R. Camporesi and A. Higuchi, “Arbitrary spin effective potentials in anti-de Sitter spacetime,” Phys. Rev. D 47, 3339 (1993). R. Gopakumar, R. K. Gupta and S. Lal, “The Heat Kernel on AdS,” JHEP 1111, 010 (2011) [arXiv:1103.3627].
[68] M. J. Duff, “Ultraviolet divergences in extended supergravity,” in: Supergravity ’81: proceedings. Ed. by S. Ferrara and J.G. Taylor. Cambridge Univ. Press, 1982. 489p. arXiv:1201.0386.
[69] J. S. Dowker, “Arbitrary Spin Theory In The Einstein Universe,” Phys. Rev. D 28, 3013 (1983). J. S. Dowker and Y.P. Dowker, “Particles of arbitrary spin in curved spaces”, Proc. Phys. Soc. London 87, 65 (1966)
[70] S. M. Christensen, M. J. Duff, G. W. Gibbons and M. Rocek, “Vanishing one-loop beta function in gauged N greater than 4 supergravity” Phys. Rev. Lett. 45, 161 (1980).
[71] L. Brink and H. B. Nielsen, “A Simple Physical Interpretation of the Critical Dimension of Space-Time in Dual Models,” Phys. Lett. B 45, 332 (1973). L. Brink and D. B. Fairlie, “Pomeron Singularities In The Fermion Meson Dual Model,” Nucl. Phys. B 74, 321 (1974). W. Nahm, “Functional Integrals for the Partition Functions of Dual Strings,” Nucl. Phys. B 124, 121 (1977).

[72] A. Bilal and F. Ferrari, “Multi-Loop Zeta Function Regularization and Spectral Cutoff in Curved Spacetime,” arXiv:1307.1689.

[73] I. L. Buchbinder, N. G. Pletnev and A. A. Tseytlin, “Induced’ N=4 conformal supergravity,” Phys. Lett. B 717, 274 (2012) [arXiv:1209.0416].

[74] F. A. Berends, G. J. Burgers and H. van Dam, “Explicit Construction of Conserved Currents for Massless Fields of Arbitrary Spin,” Nucl. Phys. B 271, 429 (1986).

[75] M.A. Vasiliev, “Conformal higher spin symmetries of 4D massless supermultiplets and osp(L, 2M) invariant equations in generalized (super)space,” arXiv:hep-th/0106149. “Cubic interactions of bosonic higher spin gauge fields in AdS(5),” Nucl. Phys. B 616, 106 (2001) [arXiv:hep-th/0106200]. E. Sezgin and P. Sundell, “Towards massless higher spin extension of D = 5, N = 8 gauged supergravity,” JHEP 0109, 025 (2001) [arXiv:hep-th/0107186]. “Doubletons and 5D higher spin gauge theory,” JHEP 0109, 036 (2001) [arXiv:hep-th/0105001]. A. Mikhailov, “Notes on higher spin symmetries,” arXiv:hep-th/0201019.

[76] R. Manvelyan and W. Ruhl, “The Structure of the trace anomaly of higher spin conformal currents in the bulk of AdS(4),” Nucl. Phys. B 751, 285 (2006) [hep-th/0602067]. “Generalized curvature and Ricci tensors for a higher spin potential and the trace anomaly in external higher spin fields in AdS(4) space,” Nucl. Phys. B 796, 457 (2008) [arXiv:0710.0952].

[77] V. P. Gusynin and V. V. Romankov, “Conformally Covariant Operators and Effective Action in External Gravitational Field,” Sov. J. Nucl. Phys. 46, 1097 (1987) [Yad. Fiz. 46, 1832 (1987)]. A. Iorio, L. O’Raifeartaigh, I. Sachs and C. Wiesendanger, “Weyl gauging and conformal invariance,” Nucl. Phys. B 495, 433 (1997) [hep-th/9607110].

[78] J. Erdmenger and H. Osborn, “Conformally covariant differential operators: Symmetric tensor fields,” Class. Quant. Grav. 15, 273 (1998) [gr-qc/9708040].

[79] O. V. Shaynkman, I. Y. Tipunin and M. A. Vasiliev, “Unfolded form of conformal equations in M dimensions and o(M + 2) modules,” Rev. Math. Phys. 18, 823 (2006) [hep-th/0401086].

[80] T. Branson, “An anomaly associated with 4-dimensional quantum gravity,” Commun. Math. Phys. 178, 301 (1996).

[81] G. R. Graham, R. Jenne, L. Mason, G. Sparling, “Conformally invariant powers of the Laplacian. I. Existence”, J. London Math. Soc. 46 (1992), 3, 557

[82] T. Branson, “Sharp inequalities, the functional determinant, and the complementary series”, Trans. Amer. Math. Soc. 347 (1995), 3671

[83] C.R. Graham, “Conformal Powers of the Laplacian via Stereographic Projection”, SIGMA 3 (2007) 121 [arXiv:0711.4798]
[84] A.R. Gover, “Laplacian operators and Q-curvature on conformally Einstein manifolds”, arXiv:math/0506037

[85] A. Juhl, “Explicit formulas for GJMS-operators and Q-curvatures”, arXiv:1108.0273; “On conformally covariant powers of the Laplacian”, arXiv:0905.3992

[86] J. S. Dowker, “Determinants and conformal anomalies of GJMS operators on spheres,” J. Phys. A 44, 115402 (2011) [arXiv:1010.0566].