Harmonic mappings and distance function

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Abstract. We prove the following theorem: every quasiconformal harmonic mapping between two plane domains with $C^{1,\alpha}$ ($\alpha < 1$) and, respectively, $C^{1,1}$ compact boundary is bi-Lipschitz. This theorem extends a similar result of the author [10] for Jordan domains, where stronger boundary conditions for the image domain were needed. The proof uses distance function from the boundary of the image domain.

Mathematics Subject Classification (2010): 58E20 (primary); 30C62 (secondary).

1. Introduction and statement of the main result

We say that a function $u : D \to \mathbb{R}$ is ACL (absolutely continuous on lines) in the region $D \subset \mathbb{R}^2$, if for every closed rectangle $R \subset D$ with sides parallel to the $x$ and $y$-axes, $u$ is absolutely continuous on a.e. horizontal and a.e. vertical line in $R$. Such a function has, of course, partial derivatives $u_x$ and $u_y$ a.e. in $D$. A homeomorphism $f : D \to G$, where $D$ and $G$ are subdomains of the complex plane $\mathbb{C}$, is said to be $K$-quasiconformal ($K$-q.c), for $K \geq 1$, if $f$ is ACL and

$$|\nabla f(z)| \leq Kl(\nabla f(z)) \quad \text{a.e. on } D,$$

(1.1)

where

$$|\nabla f(x)| := \max_{|h|=1} |\nabla f(x)h| = |f_z| + |f_{\bar{z}}|$$

and

$$l(\nabla f(z)) := \min_{|h|=1} |\nabla f(z)h| = |f_z| - |f_{\bar{z}}|$$

(cf. [1, pages 23–24] and [22]). Note that, condition (1.1) can be written as

$$|f_{\bar{z}}| \leq k|f_z| \quad \text{a.e. on } D,$$

where $k = \frac{K-1}{K+1}$ i.e. $K = \frac{1+k}{1-k}$.
or in its equivalent form

$$|\nabla f(z)|^2 \leq K J_f(z), \ z \in \mathbb{U},$$  \quad (1.2)$$

where $J_f$ is the Jacobian of $f$.

A function $w$ is called harmonic in a region $D$ if it has form $w = u + iv$ where $u$ and $v$ are real-valued harmonic functions on $D$. If $D$ is simply connected, then there are two analytic functions $g$ and $h$ defined on $D$ such that $w$ has the representation

$$w = g + \overline{h}.$$  

If $w$ is a harmonic univalent function then, by Lewy’s theorem (see [23]), $w$ has a non-vanishing Jacobian and consequently, according to the inverse mapping theorem, $w$ is a diffeomorphism.

Let

$$P(r, x) = \frac{1 - r^2}{2\pi(1 - 2r \cos x + r^2)}$$

denote the Poisson kernel. Then every bounded harmonic function $w$ defined on the unit disc $\mathbb{U} := \{z : |z| < 1\}$ has the representation

$$w(z) = P[F](z) = \int_0^{2\pi} P(r, x - \varphi) F(e^{i\varphi})dx,$$  \quad (1.3)$$

where $z = re^{i\varphi}$ and $F$ is a bounded integrable function defined on the unit circle $S^1$.

In this paper we continue to study quasiconformal harmonic mappings. See [25] for the pioneering work on this topic, and [8] for related earlier results. In some recent papers, a lot of work have been done on this class of mappings ([3, 10–17, 19–21, 24, 26, 28–29]). In these papers for the Lipschitz and the co-Lipschitz character is established quasiconformal harmonic mappings between plane domains with certain boundary conditions. In [32] the same problem is considered for hyperbolic harmonic quasiconformal selfmappings of the unit disk. Notice that, in general, quasi-symmetric self-mappings of the unit circle do not have a quasiconformal harmonic extension to the unit disk. In [25] an example is given of $C^1$ diffeomorphism of the unit circle onto itself whose Euclidean harmonic extension is not Lipschitz. Alessandrini and Nesi proved in [2] the following:

**Proposition 1.1.** Let $F : S^1 \to \gamma \subset \mathbb{C}$ be an orientation-preserving diffeomorphism of class $C^1$ of $S^1$ onto a simple closed curve $\gamma$. Let $D$ be the bounded domain such that $\partial D = \gamma$. Let $w = P[F] \in C^1(\overline{\mathbb{U}}; \mathbb{C})$. The mapping $w$ is a diffeomorphism of $\mathbb{U}$ onto $D$ if and only if

$$J_w > 0 \text{ everywhere on } S^1.$$  \quad (1.4)$$

From the inequalities (1.2) and (1.4), we easily deduce the following:

**Corollary 1.2.** Under the assumption of Proposition 1.1 the harmonic mapping $w$ is a diffeomorphism if and only if it is $K$-quasiconformal for some $K \geq 1$. 

In contrast to the case of the Euclidean metric, in the case of the hyperbolic metric, if \( f : S^1 \mapsto S^1 \) is \( C^1 \) diffeomorphism, or more generally if \( f : S^{n-1} \mapsto S^{m-1} \) is a mapping with non-vanishing energy, then its hyperbolic harmonic extension is \( C^1 \) up to the boundary ([4,5]).

To continue we need the definition of \( C^{k,\alpha} \) Jordan curves \( (k \in \mathbb{N}, 0 < \alpha \leq 1) \). Let \( \gamma \) be a rectifiable curve in the complex plane. Let \( l \) be the length of \( \gamma \). Let \( g : [0, l] \mapsto \gamma \) be an arc-length parametrization of \( \gamma \). Then \( |g'(s)| = 1 \) for all \( s \in [0, l] \). We will say that \( \gamma \in C^{k,\alpha} \) \( k \in \mathbb{N}, 0 < \alpha \leq 1 \) if \( g \in C^k \), and \( M(k, \alpha) := \sup_{t \neq s} \frac{|g^{(k)}(t) - g^{(k)}(s)|}{|t - s|^\alpha} < \infty \). Notice this important fact: if \( \gamma \in C^{1,1} \) then \( \gamma \) has a curvature \( \kappa_z \) for a.e. \( z \in \gamma \) and \( \text{ess sup}(|\kappa_z| : z \in \gamma) \leq M(1, 1) < \infty \).

This definition can be easily extended to an arbitrary \( C^{k,\alpha} \) compact 1-dimensional manifold (not necessarily connected).

The starting point of this paper is the following proposition.

**Proposition 1.3.** Let \( w = f(z) \) be a \( K \)-quasiconformal harmonic mapping between a Jordan domain \( \Omega_1 \) with \( C^{1,\alpha} \) boundary and a Jordan domain \( \Omega \) with \( C^{1,\alpha} \) (respectively \( C^{2,\alpha} \)) boundary. Consider in addition \( b \in \Omega_1 \) and set \( a = f(b) \). Then \( w \) is Lipschitz (respectively co-Lipschitz). Moreover there exists a positive constant \( c = c(K, \Omega, \Omega_1, a, b) \geq 1 \) such that

\[
|f(z_1) - f(z_2)| \leq c|z_1 - z_2|, \quad z_1, z_2 \in \Omega_1
\]

and

\[
\frac{1}{c}|z_1 - z_2| \leq |f(z_1) - f(z_2)|, \quad z_1, z_2 \in \Omega_1,
\]

respectively.

See [13] for the first part of Proposition 1.3 and [10] for its second part. In [10], it was conjectured that the second part of Proposition 1.3 remains true if we assume that \( \Omega \) has \( C^{1,\alpha} \) boundary only. Notice that the proof of Proposition 1.3 relies on the Kellogg-Warschawski theorem ([6, 33, 34]) from the theory of conformal mappings, which asserts that if \( w \) is a conformal mapping of the unit disk onto a domain \( \Omega \in C^{k,\alpha} \), then \( w^{(k)} \) has a continuous extension to the boundary \( (k \in \mathbb{N}) \). It also depended on Mori’s theorem from the theory of quasiconformal mappings, which deals with the Hölder character of quasiconformal mappings between plane domains (see [1,31]). In addition, Lemma 3.2 below is needed.

Using a different approach, we will extend here as stated in Theorem 1.4 the second part of Proposition 1.3 to the case of image domains with \( C^{1,1} \) boundary. The proof of Theorem 1.4, given in the last section, is different form the proof of second part of Proposition 1.3, and the use of the Kellogg-Warschawski theorem for the second derivative ([34]) is avoided. The distance function is used and hence a “weaker” smoothness of the boundary of image domain is needed.

**Theorem 1.4 (The main theorem).** Let \( w = f(z) \) be a \( K \)-quasiconformal harmonic mapping from the unit disk \( \mathbb{D} \) to a Jordan domain \( \Omega \) with \( C^{1,1} \) boundary. Set
\( a = f(0) \). Then \( w \) is co-Lipschitz. More precisely, there exists a positive constant \( c = c(K, \Omega, a) \geq 1 \) such that
\[
\frac{1}{c} |z_1 - z_2| \leq |f(z_1) - f(z_2)|, \quad z_1, z_2 \in \Omega.
\]
(1.7)

Since the composition of a quasiconformal harmonic and a conformal mapping is itself quasiconformal harmonic, using Theorem 1.4 and Kellogg’s theorem for the first derivative we obtain:

**Corollary 1.5.** Let \( w = f(z) \) be a \( K \)-quasiconformal harmonic mapping between a plane domain \( \Omega_1 \) with \( C^{1,\alpha} \) compact boundary and a plane domain \( \Omega \) with \( C^{1,1} \) compact boundary. Consider \( a_0 \in \Omega_1 \) and set \( b_0 = f(a_0) \). Then \( w \) is bi-Lipschitz. Moreover there exists a positive constant \( c = c(K, \Omega, \Omega_1, a_0, b_0) \geq 1 \) such that
\[
\frac{1}{c} |z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c |z_1 - z_2|, \quad z_1, z_2 \in \Omega_1.
\]
(1.8)

**Proof of Corollary 1.5.** Let \( b = f(a) \in \partial \Omega \). Since \( \partial \Omega \in C^{1,1} \), it follows that there exists a \( C^{1,1} \) Jordan curve \( \gamma_b \subset \Omega \), whose interior \( D_b \) lies in \( \Omega \), and \( \partial \Omega \cap \gamma_b \) is a neighborhood of \( b \). See [13, Theorem 2.1] for an explicit construction of such a Jordan curve. Let \( D_a = f^{-1}(D_b) \), and take a conformal mapping \( g_a \) of the unit disk onto \( D_a \). Then \( f_a = f \circ g_a \) is a quasiconformal harmonic mapping from the unit disk onto the \( C^{1,1} \) domain \( D_b \). From Theorem 1.4 it follows that \( f_a \) is bi-Lipschitz, and from Kellogg’s theorem it follows that \( f = f_a \circ g_a^{-1} \) and its inverse \( f^{-1} \) are Lipschitz in some small neighborhood of \( a \) and of \( b = f(a) \) respectively. This means that \( \nabla f \) is bounded in some neighborhood of \( a \). Since \( \partial \Omega_1 \) is a compact, we deduce that \( \nabla f \) is bounded in \( \partial \Omega_1 \). The same holds for \( \nabla f^{-1} \) with respect to \( \partial \Omega \). This implies that \( f \) is bi-Lipschitz. \( \square \)

**Acknowledgements.** I thank the referee for providing constructive comments and help in improving the contents of this paper.

### 2. Auxiliary results

Let \( \Omega \) be a domain in \( \mathbb{R}^2 \) having non-empty boundary \( \partial \Omega \). The distance function from the boundary is defined by
\[
d(x) = \text{dist} (x, \partial \Omega).
\]
(2.1)

Let \( \Omega \) be bounded and assume \( \partial \Omega \in C^{1,1} \). These conditions on \( \Omega \) imply that \( \partial \Omega \) satisfies the following: at a.e. point \( z \in \partial \Omega \) there exists a disk \( D = D(w_z, r_z) \) depending on \( z \) such that \( D \cap (\mathbb{C} \setminus \Omega) = \{z\} \). Moreover \( \mu := \text{ess inf} \{r_z, z \in \partial \Omega \} \).
\[ \partial \Omega \] > 0. It is easy to show that \( \mu^{-1} \) bounds the curvature of \( \partial \Omega \), which means that \( \frac{1}{\mu} \geq \kappa_z \), for \( z \in \partial \Omega \). Here \( \kappa_z \) denotes the curvature of \( \partial \Omega \) at \( z \in \partial \Omega \). Under the above conditions, we have \( d \in C^{1,1}(\Gamma_\mu) \), where \( \Gamma_\mu = \{ z \in \overline{\Omega} : d(z) < \mu \} \) and for \( z \in \Gamma_\mu \) there exists \( \omega(z) \in \partial \Omega \) such that

\[
\nabla d(z) = v_{\omega(z)}, \tag{2.2}
\]

where \( v_{\omega(z)} \) denotes the inner normal vector to the boundary \( \partial \Omega \) at the point \( \omega(z) \). See [7, Section 14.6] for the details.

**Lemma 2.1.** Let \( w : \Omega_1 \mapsto \Omega \) be a \( K \)-quasiconformal mapping and set \( \chi = -d(w(z)) \). Then

\[
|\nabla \chi| \leq |\nabla w| \leq K|\nabla \chi| \quad \tag{2.3}
\]

in \( w^{-1}(\Gamma_\mu) \) for \( \mu > 0 \) such that \( 1/\mu > \kappa_0 = \text{ess sup}\{|\kappa_z| : z \in \partial \Omega\} \).

**Proof.** Observe first that \( \nabla d \) is a unit vector. From the identity \( \nabla \chi = -\nabla d \cdot \nabla w \) it follows that

\[
|\nabla \chi| \leq |\nabla d||\nabla w| = |\nabla w|.
\]

For a non-singular matrix \( A \) we have

\[
\inf_{|x|=1} |Ax|^2 = \inf_{|x|=1} \langle Ax, Ax \rangle = \inf_{|x|=1} \left( A^T Ax, x \right) = \inf\{\lambda : \exists x \neq 0, A^T Ax = \lambda x\} = \inf\{\lambda : \exists x \neq 0, AA^T Ax = \lambda Ax\} = \inf\{\lambda : \exists y \neq 0, AA^T y = \lambda y\} = \inf_{|x|=1} |A^T x|^2.
\]

We next denote that \( (\nabla \chi)^T = -(\nabla w)^T \cdot (\nabla d)^T \), therefore for \( x \in w^{-1}(\Gamma_\mu) \) we obtain

\[
|\nabla \chi| \geq \inf_{|e|=1} |(\nabla w)^T e| = \inf_{|e|=1} |\nabla w e| = l(w) \geq K^{-1}|\nabla w|.
\]

The proof of (2.3) is complete. \[ \square \]

**Lemma 2.2.** Let \( \{e_1, e_2\} \) be the canonical basis of the space \( \mathbb{R}^2 \). Let \( w : \Omega_1 \mapsto \Omega \) be a twice differentiable mapping and let \( \chi = -d(w(z)) \). Then

\[
\Delta \chi(z_0) = \frac{\kappa_{w_0}}{1 - \kappa_{w_0}d(w(z_0))}[(O_{z_0} \nabla w(z_0))^T e_1]^2 - |(\nabla d)(w(z_0))^T e_1|^2, \quad \tag{2.5}
\]

where \( z_0 \in w^{-1}(\Gamma_\mu), \omega_0 \in \partial \Omega \) with \( |w(z_0) - \omega_0| = \text{dist}(w(z_0), \partial \Omega), \mu > 0 \) such that \( 1/\mu > \kappa_0 = \text{ess sup}\{|\kappa_z| : z \in \partial \Omega\} \) and \( O_{z_0} \) is an orthogonal transformation.
Proof. Let $\nu_{\omega_0}$ be the inner unit normal vector to $\gamma$ at the point $\omega_0 \in \gamma$. Let $O_{z_0}$ be an orthogonal transformation that takes the vector $e_2$ to $\nu_{\omega_0}$. In complex notations one has:

$$O_{z_0} w = -i \nu_{\omega_0} w.$$ 

Take $\tilde{\Omega} := O_{z_0} \Omega$. Let $\tilde{d}$ be the distance function for $\tilde{\Omega}$. Then

$$d(w) = \tilde{d}(O_{z_0} w) = \text{dist}(O_{z_0} w, \partial \tilde{\Omega}).$$

Therefore $\chi(z) = -\tilde{d}(O_{z_0}(w(z)))$. Furthermore

$$\Delta \chi(z) = -\sum_{i=1}^{2} (D^2 \tilde{d})(O_{z_0}(w(z)))(O_{z_0} \nabla w(z)) e_i, O_{z_0} \nabla w(z)) e_i)$$

$$- \langle \nabla d(w(z)), \Delta w(z) \rangle.$$ (2.6)

To continue, we make use of the following proposition.

**Proposition 2.3 ([7, Lemma 14.17]).** Let $\Omega$ be bounded and assume $\partial \Omega \in C^{1,1}$. Then, with notation as in Lemma 2.2, we have

$$(D^2 \tilde{d})(O_{z_0}(w(z_0))) = \text{diag} \left( \begin{array}{cc} \frac{-\kappa_{\omega_0}}{1 - \kappa_{\omega_0} \tilde{d}} & 0 \\ 0 & 0 \end{array} \right),$$ (2.7)

where $\kappa_{\omega_0}$ denotes the curvature of $\partial \Omega$ at $\omega_0 \in \partial \Omega$.

Applying (2.7) we have

$$\sum_{i=1}^{2} (D^2 \tilde{d})(O_{z_0}(w(z_0)))(O_{z_0} \nabla w(z_0)) e_i, O_{z_0} \nabla w(z_0)) e_i)$$

$$= \sum_{i=1}^{2} \sum_{j,k=1}^{2} D_{j,k} \tilde{d}(O_{z_0}(w(z_0))) D_i(O_{z_0} w j(z_0)) \cdot D_i(O_{z_0} w k(z_0))$$

$$= \sum_{j,k=1}^{2} D_{j,k} \tilde{d}(O_{z_0}(w(z_0))) \left( (O_{z_0} \nabla w(z_0))^T e_j, (O_{z_0} \nabla w(z_0))^T e_k \right)$$

$$= \frac{-\kappa_{\omega_0}}{1 - \kappa_{\omega_0} \tilde{d}} |(O_{z_0} \nabla w(z_0))^T e_1|^2.$$ (2.8)

Finally we obtain

$$\Delta \chi(z_0) = \frac{\kappa_{\omega_0}}{1 - \kappa_{\omega_0} \tilde{d}} |(O_{z_0} \nabla w(z_0))^T e_1|^2 - \langle \nabla d(w(z_0)), \Delta w \rangle. \quad \Box$$
3. Proof of the main theorem

The main step to establish the main theorem is the following lemma.

**Lemma 3.1.** Let \( w = f(z) \) be a \( K \)-quasiconformal mapping of the unit disk onto a \( C^{1,1} \) Jordan domain \( \Omega \) satisfying the differential inequality

\[
|\Delta w| \leq B|\nabla w|^2, \quad B \geq 0
\]

for some \( B \geq 0 \). Assume in addition that \( w(0) = a_0 \in \Omega \). Then there exists a constant \( C(K, \Omega, B, a) > 0 \) such that

\[
\left| \frac{\partial w}{\partial r}(t) \right| \geq C(K, \Omega, B, a_0) \text{ for almost every } t \in S^1.
\]

**Proof.** Let us find \( A > 0 \) such that the function \( \varphi_w(z) = -\frac{1}{A} + \frac{1}{A}e^{-Ad(w(z))} \) is subharmonic on \( \{z : d(w(z)) < \frac{1}{2\kappa_0}\} \), where

\[
\kappa_0 = \text{ess sup}\{|\kappa_w| : w \in \gamma\}.
\]

Let \( \chi = -d(w(z)) \). Combining (2.3), (2.5) and (3.1) we get

\[
|\Delta \chi| \leq 2\kappa_0|\nabla w|^2 + B|\nabla w|^2 \leq (2\kappa_0 + B)K^2|\nabla \chi|^2.
\]

Take

\[
g(t) = -\frac{1}{A} + \frac{1}{A}e^{At}.
\]

Then \( \varphi_w(z) = g(\chi(z)) \). Thus

\[
\Delta \varphi_w = g''(\chi)|\nabla \chi|^2 + g'(\chi)\Delta \chi.
\]

Since

\[
g'(\chi) = e^{-Ad(w(z))}
\]

and

\[
g''(\chi) = Ae^{-Ad(w(z))},
\]

it follows that

\[
\Delta \varphi_w \geq (A - (2\kappa_0 + B)K^2)|\nabla \chi|^2 e^{-Ad(w(z))}.
\]

In order to have \( \Delta \varphi_w \geq 0 \), it is enough to take

\[
A = (2\kappa_0 + B)K^2.
\]

Choosing

\[
\varrho = \max\left\{|z| : \text{dist}(w(z), \gamma) = \frac{1}{2\kappa_0}\right\},
\]

we have that \( \varphi_w \) satisfies the conditions of the following generalization of the Hopf lemma ([9]):
Lemma 3.2 ([10]). Let \( \varphi \) satisfy \( \Delta \varphi \geq 0 \) in \( R_{\varrho} = \{ z : \varrho \leq |z| < 1 \} \), \( 0 < \varrho < 1 \), \( \varphi \) be continuous on \( \overline{R_{\varrho}} \), \( \varphi(t) = 0 \) for \( t \in S^1 \). Assume that the radial derivative \( \frac{\partial \varphi}{\partial r} \) exists almost everywhere on \( S^1 \). Set \( M(\varphi, \varrho) = \max_{|z|=\varrho} \varphi(z) \). Then the following inequality holds

\[
\frac{\partial \varphi(t)}{\partial r} > \frac{2M(\varphi, \varrho)}{\varrho^2(1 - e^{1/\varrho^2 - 1})} \quad \text{for a.e. } t \in S^1.
\]

We will make use of (3.9), but under some improvement for the class of quasiconformal harmonic mappings. The idea is to make the right-hand side of (3.9) independent of the mapping \( w \) for \( \varphi = \varphi_w \).

We will say that a quasiconformal mapping \( f : \Omega \mapsto \varrho \) is normalized if
\( f(1) = w_0, f(e^{2\pi i/3}) = w_1 \) and \( f(e^{-2\pi i/3}) = w_2 \), where \( w_0 w_1, w_1 w_2 \) and \( w_2 w_0 \) are arcs of \( \varrho = \partial \Omega \) having the same length \( |\varrho|/3 \).

In what follows we will prove that, for the class \( \mathcal{H}(\Omega, K, B) \) of normalized \( K \)-quasiconformal mappings, satisfying (3.1) for some \( B \geq 0 \), and mapping the unit disk onto the domain \( \Omega \), the inequality (3.9) holds uniformly (see (3.10)).

Let
\[
\varrho := \sup \left\{ |z| : \text{dist}(w(z), \varrho) = \frac{1}{2\kappa_0}, w \in \mathcal{H}(\Omega, K, B) \right\}.
\]

Then there exists a sequence \( \{w_n\}, w_n \in \mathcal{H}(\Omega, K, B) \) such that
\[
\varrho_n = \max \left\{ |z| : \text{dist}(w_n(z), \varrho) = \frac{1}{2\kappa_0} \right\},
\]
and
\[
\varrho = \lim_{n \to \infty} \varrho_n.
\]

Now notice that if \( w_n \) is a sequence of normalized \( K \)-quasiconformal mappings of the unit disk onto \( \Omega \) then, up to taking a subsequence, \( w_n \) is a locally uniformly convergent sequence converging to some quasiconformal mapping \( w \in \mathcal{H}(\Omega, K, B) \). Under the condition on the boundary of \( \Omega \), by [27, Theorem 4.4] this sequence is uniformly convergent on \( \Omega \). Then there exists a sequence \( z_n \) such that \( \text{dist}(w_n(z_n), \varrho) = \frac{1}{2\kappa_0} \), \( \lim_{n \to \infty} z_n = z_0 \) and \( \varrho = |z_0| \). Since \( w_n \) converges uniformly to \( w \), it follows that \( \lim_{n \to \infty} w_n(z_n) = w(z_0) \), and \( \text{dist}(w(z_0), \varrho) = \frac{1}{2\kappa_0} \). This implies that \( \varrho < 1 \). Let now
\[
M(\varrho) := \sup\{M(\varphi_w, \varrho), w \in \mathcal{H}(\Omega, K, B)\}.
\]

Using a similar argument we obtain that there exists a uniformly convergent sequence \( w_n \), converging to a mapping \( w_0 \), such that
\[
M(\varrho) = \lim_{n \to \infty} M(\varphi_{w_n}, \varrho) = M(\varphi_{w_0}, \varrho).
\]
Thus \[ M(\varphi) < 0. \]

Placing \( M(\varphi) \) instead of \( M(\varphi, \varphi) \) and \( \varphi_\omega \) instead of \( \varphi \) in (3.9), we obtain

\[
\frac{\partial \varphi_\omega(t)}{\partial r} > \frac{2M(\varphi)}{\varphi^2(1 - e^{1/\varphi^2 - 1})} := C(K, \Omega, B) \quad \text{for a.e. } t \in S^1. \tag{3.10}
\]

To continue observe that

\[
\frac{\partial \varphi_\omega(t)}{\partial r} = e^{A_d(\omega(z))} |\nabla d| \left| \frac{\partial w}{\partial r}(t) \right| = e^{A_d(\omega(z))} \left| \frac{\partial w}{\partial r}(t) \right|. 
\]

Combining (3.8) and (3.10) we obtain for a.e. \( t \in S^1 \)

\[
\left| \frac{\partial w}{\partial r}(t) \right| = e^{-A_d(\omega(z))} \frac{\partial \varphi_\omega(t)}{\partial r} \geq e^{-K^2} \frac{2M(\varphi)}{\varphi^2(1 - e^{1/\varphi^2 - 1})}. 
\]

Lemma 3.1 is now proved for a normalized mapping \( w \). If \( w \) is not normalized then we take the composition of \( w \) and an appropriate Möbius transformation in order to obtain the desired inequality. The proof of Lemma 3.1 is complete.

**Conclusion of the proof of Theorem 1.4.** In this setting \( w \) is harmonic, therefore \( B = 0 \). Assume first that \( w \in C^1(\overline{U}) \). Let \( l(\nabla w)(t) = ||w_z(t)|| - |w_\bar{z}(t)|| \). Since \( w \) is \( K \)-quasiconformal, according to (3.2) we have

\[
l(\nabla w)(t) \geq \frac{|\nabla w(t)|}{K} \geq \frac{\left| \frac{\partial w}{\partial r}(t) \right|}{K} \geq \frac{C(K, \Omega, 0, a_0)}{K} 
\]

for \( t \in S^1 \). Therefore, having in mind Lewy’s theorem ([23]), which states that \( |w_z| > |w_\bar{z}| \) for \( z \in \mathbb{U} \), we obtain for \( t \in S^1 \) that \( |w_z(t)| \neq 0 \) and hence

\[
\frac{1}{|w_z|} \frac{C(K, \Omega, 0, a_0)}{K} + \frac{|w_\bar{z}|}{|w_z|} \leq 1, \quad t \in S^1.
\]

Since \( w \in C^1(\overline{U}) \), it follows that the functions

\[
a(z) := \frac{w_\bar{z}}{w_z}, \quad b(z) := \frac{1}{w_z} \frac{C(K, \Omega, 0, a_0)}{K}
\]

are well-defined holomorphic functions in the unit disk having a continuous extension to the boundary. As \( |a| + |b| \) is bounded on the unit circle by 1, it follows that it is bounded on the whole unit disk by 1 because

\[
|a(z)| + |b(z)| \leq P[|a|_\delta 1](z) + P[|b|_\delta 1](z) = P[|a|_\delta 1 + |b|_\delta 1](z), \quad z \in \mathbb{U}.
\]
This in turn implies that for every $z \in \mathbb{U}$
\[ l(\nabla w)(z) \geq \frac{C(K, \Omega, 0, a_0)}{K} =: C(\Omega, K, a_0). \]  \hfill (3.12)
This yields that
\[ C(K, \Omega, a_0) \leq \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|}, \quad z_1, z_2 \in \mathbb{U}. \]
Assume now that $w \notin C^1(\mathbb{U})$. We begin with a definition.

**Definition 3.3.** Let $G$ be a domain in $\mathbb{C}$ and let $a \in \partial G$. We will say that $G_a \subset G$ is a $\partial$-neighborhood of $a$ if there exists a disk $D(a, r) := \{z : |z - a| < r\}$ such that $D(a, r) \cap G \subset G_a$.

Let $t = e^{i\beta} \in S^1$, so that $w(t) \in \partial\Omega$. Let $\gamma$ be an arc-length parametrization of $\partial\Omega$ with $\gamma(s) = w(t)$. Since $\partial\Omega \in C^{1,1}$, there exists a $\partial$-neighborhood $\Omega_t$ of $w(t)$ with $C^{1,1}$ Jordan boundary such that
\[ \Omega_t^\tau := \Omega_t + i\gamma'(s) \cdot \tau \subset \Omega, \quad \text{and} \quad \partial\Omega_t^\tau \subset \Omega \quad \text{for} \quad 0 < \tau \leq \tau_t \quad (\tau_t > 0). \]  \hfill (3.13)
An example of a family $\Omega_t^\tau$ such that $\partial\Omega_t^\tau \subset C^{1,1}$ and with the property (3.13) has been given in [13].

Let $a_t \in \Omega_t$ be arbitrary. Then $a_t + i\gamma'(s) \cdot \tau \in \Omega_t^\tau$. Take $U_t = f^{-1}(\Omega_t^\tau)$. Let $\eta_t^\tau$ be a conformal mapping of the unit disk onto $U_t$ such that $\eta_t^\tau(0) = f^{-1}(a_t + i\gamma'(s) \cdot \tau)$, and $\arg(\frac{d\eta_t^\tau}{dz}(0)) = 0$. Then the mapping
\[ f_t^\tau(z) := f(\eta_t^\tau(z)) - i\gamma'(s) \cdot \tau \]
is a harmonic $K$-quasiconformal mapping of the unit disk onto $\Omega_t$ satisfying the condition $f_t^\tau(0) = a_t$. Moreover
\[ f_t^\tau \in C^1(\mathbb{U}). \]
Using the case $w \in C^1(\mathbb{U})$, it follows that
\[ |\nabla f_t^\tau(z)| \geq C(K, \Omega_t, a_t). \]
On the other hand
\[ \lim_{\tau \to 0^+} \nabla f_t^\tau(z) = \nabla(f \circ \eta_t)(z) \]
on the compact sets of $\mathbb{U}$ as well as
\[ \lim_{\tau \to 0^+} \frac{d\eta_t^\tau}{dz}(z) = \frac{d\eta_t}{dz}(z), \]
where $\eta_t$ is a conformal mapping of the unit disk onto $U_0 = f^{-1}(\Omega_t)$ with $\eta_t(0) = f^{-1}(a_t)$. It follows that

$$|\nabla f_t(z)| \geq C(K, \Omega_t, a_t).$$

Applying the Schwarz reflection principle to the mapping $\eta_t$ and using the formula

$$\nabla (f \circ \eta_t)(z) = \nabla f \cdot \frac{d\eta_t}{dz}(z)$$

it follows that in some $\partial$-neighborhood $\tilde{U}_t$ of $t \in S^1$ with smooth boundary where $(D(t, r_t) \cap \Omega \subset \tilde{U}_t$ for some $r_t > 0$), the function $f$ satisfies the inequality

$$|\nabla f(z)| \geq \frac{C(K, \Omega_t, a_t)}{\max\{|\eta_t'(\xi)| : \xi \in \tilde{U}_t\}} =: \tilde{C}(K, \Omega_t, a_t) > 0. \quad (3.14)$$

Since $S^1$ is a compact set, it can be covered by a finite family $\partial \tilde{U}_{t_j} \cap S^1 \cap D(t, r_t/2)$, $j = 1, \ldots, m$. It follows that the inequality

$$|\nabla f(z)| \geq \min\{\tilde{C}(K, \Omega_{t_j}, a_{t_j}) : j = 1, \ldots, m\} =: \tilde{C}(K, \Omega, a_0) > 0 \quad (3.15)$$

holds in the annulus

$$\tilde{R} = \left\{z : 1 - \frac{\sqrt{3}}{2} \min_{1 \leq j \leq m} r_{t_j} < |z| < 1 \right\} \subset \bigcup_{j=1}^{m} \tilde{U}_{t_j}.$$

This implies that the subharmonic function $S = |a(z)| + |b(z)|$ is bounded in $\Omega$. According to the maximum principle, it is bounded by 1 in the whole unit disk. This in turn implies again (3.12) and consequently

$$\frac{C(K, \Omega, a_0)}{K} |z_1 - z_2| \leq |w(z_1) - w(z_2)|, \quad z_1, z_2 \in \Omega. \quad \square$$

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