The Ostrogradsky prescription for BFV formalism

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Abstract

Gauge-invariant systems of a general form with higher order time derivatives of gauge parameters are investigated within the framework of the BFV formalism. Higher order terms of the BRST charge and BRST-invariant Hamiltonian are obtained. It is shown that the identification rules for Lagrangian and Hamiltonian BRST ghost variables depend on the choice of the extension of constraints from the primary constraint surface.

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1 Introduction

For a consistent quantization of a system it is desirable to have its Hamiltonian description. If the system under consideration is non-degenerate then this problem is in principle solved by the Legendre transformation that is, at least locally, a bijective mapping of the velocity phase space onto the (canonical) phase space. In this, the quantization procedure endows the phase spaces with the sense of the state space of the system.

But if the initial classical system is gauge-invariant then we have no more one-to-one correspondence between the points of the phase spaces of the Lagrangian and the Hamiltonian descriptions of the system: actually, gauge invariance, in general, gives rise to constraints in these two formalisms because the Hessian is a degenerate matrix for the case, and so, the Legendre transformation turns out to be a singular mapping [1]. This circumstance forces us to modify the notion of the state space. The most powerful method to work out this situation in a covariant way is the BRST formalism based on the concept of the BRST symmetry [2].

There are two approaches to constructing a covariant BRST-invariant effective theory. In the first one, called the Lagrangian BRST formalism [3], the initial object is a gauge-invariant Lagrangian $L(q, \dot{q})$ given on the corresponding velocity phase space. Here, the gauge symmetry transformations

$$\delta q^R = \epsilon^A \psi^R_A,$$

where $\psi^R_A$ are some functionals of the trajectories of the system, and $\epsilon^A$ are gauge parameters, are taken as a basis to construct the so-called BRST transformations:

$$\delta q^R \rightarrow \delta q^R = \lambda s(q^R).$$

Here $\lambda$ is an odd parameter of (global) BRST transformations, and $s$ is an odd vector field connected with the gauge transformations in such a way that

$$s(q^R) = c^A \psi^R_A,$$

and so that we have extended the initial configuration space of the system by adding globally defined ghost variables $c^A$ to initial generalized coordinates $q^R$. Finally, we have to introduce the so-called antighosts $\bar{c}_A$ that gives us the extended configuration space and the corresponding velocity phase space of the Lagrangian BRST formalism. In this, the ghost and antighost fields have the Grassmann parity being opposite to that the gauge parameters have:

$$|c^A| = |\bar{c}_A| = |\epsilon^A| + 1.$$

Besides, while requiring that the initial generalized coordinates $q^R$ have the ghost number to be zero, we get that ghosts and antighosts have ghost numbers which are opposite to each other:

$$gh(q^R) = 0, \quad gh(c^A) + gh(\bar{c}_A) = 0.$$

The effective BRST-invariant system is constructed on the extended velocity phase space.

From the other side, we have the Hamiltonian BRST formalism elaborated by Batalin, Fradkin and Vilkovisky [4, 5]. This one, called also BFV formalism, begins with a Hamiltonian description of a system. In this we have a Hamiltonian $h$ and a set of irreducible
constraints \( \varphi_a \) which are functions being in involution. Within the framework of the Hamiltonian BRST formalism we introduce a new set of canonical pairs – ghost variables \( \eta^a \) and \( \pi_a \) associated with the constraints \( \varphi_a \), and put them to have opposite ghost numbers:

\[
gh(\eta^a) + gh(\pi_a) = 0.
\]

The effective BRST-invariant theory is now defined on the extended phase space. The BRST transformations in the Hamiltonian approach are generated by the so-called BFV-BRST charge, that is a nilpotent odd operator globally defined on the extended phase space.

Thus, we arrive at a natural question: in which correspondence are the ghost and antighost variables of the Lagrangian BRST formalism and the ghost canonical pairs of the BFV formalism? This question seems to be more intricate if we look at the following circumstance. One can construct an equivalent Hamiltonian description of the system when introducing another set of Hamiltonian and constraints, i.e. new functions \( h' \) and \( \varphi'_a \) which are some linear combinations of \( h \) and \( \varphi_a \). This change of ingredients of the Hamiltonian description of the system is equivalent, within the framework of the BFV formalism, to a canonical transformation of the extended phase space \([5]\). And so, we get different sets of the ghost canonical pairs, while in the Lagrangian BRST formalism the ghost fields are fixed by the initial form of the gauge transformations.

Another question we consider in this paper, although being of a more technical character, is also connected with the above mentioned: this is the calculation of the higher order structure functions of the BFV formalism. In Refs.\([3, 7]\) the authors investigated systems whose local symmetry transformations contain higher order time derivatives of gauge parameters. These systems are interesting from the physical point of view. Moreover, as is shown in Ref.\([8]\), the gauge algebra corresponding to the model of the so-called rigid particle \([7]\) is equivalent to a particular case of \( W \)-algebras. Hence, it would be useful to have some general formulas providing BRST analysis of such systems ”in advance”.

The paper is organized as follows. In Sec. 2 the Hamiltonian BRST-BFV formalism for gauge-invariant systems with higher order time derivatives of gauge parameters is constructed. Higher order structure functions of the corresponding BRST charge and BRST-invariant Hamiltonian are obtained. The Ostrogradsky prescription \([9]\) is used to relate the BFV ghost canonical pairs with the ghost and antighost variables of the Lagrangian BRST formalism. In Sec. 3 it is shown that this relationship, called identification rules, depends on the choice of the extension of the constraints from the primary constraint surface.

Summation over repeated indices is assumed. All partial derivatives are understood as the left ones \([10]\). Everywhere in the text integers within parentheses over characters imply the corresponding order of time derivative, while superscripts and subscripts within square brackets just denote different functions.

## 2 The BFV formalism and higher order structure functions

Let a constrained system with a Hamiltonian \( h \) and irreducible constraints \( \varphi_a \) of the first class be given \([11]\):

\[
\{\varphi_a, \varphi_b\} = f^c_{ab} \varphi_c,
\]

(2.1)
\{h, \varphi_a\} = h_a^b \varphi_b. \quad (2.2)

We suppose the quantities entering Eqs. (2.1), (2.2) to be globally defined on the phase space.

To provide the Hamiltonian description of the system within the framework of the BFV formalism we have to construct the corresponding BFV-BRST charge and the BFV-BRST-invariant Hamiltonian \[4, 5\]. These functions are defined on the extended phase space and, in general, can be represented as a series over the ghost variables. In this, the BRST charge is an odd nilpotent operator having the ghost number 1, and the BRST-invariant Hamiltonian is defined as an even function with the ghost number 0.

For the constrained system of the first class introduced by Eqs. (2.1) and (2.2) we have the following general formulas. The BFV-BRST charge is given by the expression \[4, 5\]:

$$
\Omega_B = \sum_{n \geq 0} \Omega_{B}^{[n]} = \sum_{n \geq 0} \Omega_{a_1 \ldots a_{n+1}}^{b_1 \ldots b_n} \eta^{a_{n+1}} \cdots \eta^{a_1} \pi_{b_n} \cdots \pi_{b_1},
$$

(2.3)

where

$$
\Omega_{a_1}^{[n]} = \varphi_{a_1},
$$

(2.4)

and the quantities \(\Omega_{a_1 \ldots a_{n+1}}^{b_1 \ldots b_n}\) \((n > 0)\) are determined by the nilpotency condition \[4\]

\[\{\Omega_B, \Omega_B\} = 0.\]

(2.5)

The BFV-BRST invariant Hamiltonian for the case can be written in the form

$$
H_A = \sum_{n \geq 0} H_{[n]} = \sum_{n \geq 0} H_{a_1 \ldots a_n}^{b_1 \ldots b_n} \eta^{a_n} \cdots \eta^{a_1} \pi_{b_n} \cdots \pi_{b_1}.
$$

(2.6)

Assuming that

$$
H_{[0]} = h,
$$

(2.7)

we can find the quantities \(H_{a_1 \ldots a_n}^{b_1 \ldots b_n}\) from the BRST-invariance condition for \(H_A\) \[4\]

\[\{H_A, \Omega_B\} = 0.\]

(2.8)

The general theorem of the existence of the higher order structure functions \(\Omega_{a_1 \ldots a_{n+1}}^{b_1 \ldots b_n}\) and \(H_{a_1 \ldots a_n}^{b_1 \ldots b_n}\) of the BFV formalism is proved in Ref. \[5\]. In particular, we have for \(n = 1\)

$$
\Omega_{B}^{[1]} = -\frac{1}{2} f_{c b}^{c a} \eta^a \eta^b \pi_c, \quad H_{[1]} = h_a^b \eta^a \pi_b.
$$

(2.9)

For the second order structure functions \((n = 2)\) we have the expressions \[4, 5, 12\]:

$$
2 \Omega_{a_1 a_2 a_3}^{b_1 c} \varphi_c = \frac{1}{6} \left[ \{\varphi_{a_1}, f_{a_2 a_3}^{b_1}\} + \{\varphi_{a_2}, f_{a_3 a_1}^{b_1}\} + \{\varphi_{a_3}, f_{a_1 a_2}^{b_1}\} \right. \\
+ f_{a_1 a_2}^{c} f_{a_3 a_1}^{b_1} + f_{a_2 a_3}^{c} f_{a_1 a_2}^{b_1} + f_{a_3 a_1}^{c} f_{a_1 a_2}^{b_1}],
$$

(2.10)

$$
2 H_{a_1 a_2}^{b_1 c} \varphi_c = \frac{1}{2} \left[ \{h, f_{a_1 a_2}^{b_1}\} - \{h_{a_1}^{b_1}, \varphi_{a_2}\} + \{h_{a_2}^{b_1}, \varphi_{a_1}\} \right. \\
+ h_{a_1}^{c} f_{a_2 c}^{b_1} - h_{a_2}^{c} f_{a_1 c}^{b_1} + h_{a_1}^{b_1} f_{a_2 c}^{c} \left. + h_{a_2}^{b_1} f_{a_1 c}^{c} + h_{a_1}^{b_1} f_{a_2 c}^{c} \right].
$$

(2.11)
From the nilpotency of the BRST charge we get that Eq. (2.8) defines $H_A$ only up to a BRST exact term. Hence, the general form of the BRST invariant Hamiltonian is given by the expression

$$H_B = H_A - \{\Psi, \Omega_B\},$$

where $\Psi$ is an odd function, having the ghost number equal to $-1$. Thus, the gauge-fixing procedure within the framework of the BFV formalism consists in the choice of $\Psi$-function.

In Ref. [13] the Hamiltonian formalism was constructed for the gauge-invariant system given by the Lagrangian $L(q, \dot{q})$ with the gauge symmetry transformations of the form

$$\delta_\varepsilon q^r = \sum_{k=0}^{N} \varepsilon^{(k)} \psi^r_{\alpha}(q),$$

where $r = 1, \ldots, R$, $\alpha = 1, \ldots, A$ and $R > A$. In this we suppose the highest order $N$ of time derivatives of the gauge parameters to be more than 1, and that the symmetry transformations (2.13) form a closed gauge algebra. Let us briefly recall some necessary definitions and formulas from Ref. [13].

Introduce $2R$-dimensional phase space with generalized coordinates $q^r$ and generalized momenta $p_r$, $r = 1, \ldots, R$, having the Poisson bracket of the form:

$$\{p_r, q_s\} = -\delta_{sr},$$

and define the mapping of the velocity phase space to this phase space as usual:

$$p_r(q, \dot{q}) = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^r}.$$  

From the gauge invariance of the system it follows that this mapping is singular. Actually, we have from the Noether identities that the Hessian of the system has $A$ linearly independent null vectors $\psi^r_{\alpha}$. We suppose henceforth these vectors to be functions of the generalized coordinates only. One can show that under the action of the mapping (2.15) one or several $A$-dimensional surfaces having parametric representation of the form:

$$q^r(\tau) = q^r, \quad \dot{q}^r(\tau) = \dot{q}^r + \tau^\alpha \psi^r_{\alpha}(q),$$

are mapped into a point of the phase space. So, in our case, the image of the velocity phase space under the mapping (2.13) is a $(2R - A)$-dimensional surface in the phase space, which may be defined by the following relations

$$\Phi_{\alpha}(q, p) = 0, \quad \alpha = 1, \ldots, A,$$

where the functions $\Phi_{\alpha}$ are functionally independent. By Eq. (2.18) we have introduced the primary constraints of the system [11] and, respectively, the primary constraint surface [11]. Note that in the case under consideration, with the null vectors of the Hessian being
functions of the generalized coordinates only, the primary constraints turn out to be linear in the generalized momenta \([12]\).

Let \(F(q, p)\) be a function defined on the phase space. There exists a function \(f(q, \dot{q})\) on the velocity phase space, such that

\[
f(q, \dot{q}) = F(q, p(q, \dot{q})).
\]

In this, the function \(f\) is constant on the surfaces given by Eqs.\((2.16)\),\((2.17)\). This fact is expressed by the differential equations of the form

\[
\psi_{\alpha}^{[0]} \frac{\partial f}{\partial q^\alpha} = 0, \quad \alpha = 1, \ldots, A.
\]

We see however that not for any function \(f(q, \dot{q})\), given on the velocity phase space, we can find a function \(F(q, p)\) on the phase space, which is connected with \(f\) by the relation

\[
F(q, p(q, \dot{q})) = f(q, \dot{q}).
\]

In this, Eq.\((2.20)\) gives the necessary condition for the existence of such a function \(F(q, p)\). We assume these relations to be also sufficient for the validity of the corresponding Eq.\((2.21)\). It means that in our case any point of the primary constraint surface \((2.18)\) is the image of only one connected surface of the form \((2.16), (2.17)\) \([1]\).

Therefore, for any function \(f(q, \dot{q})\) satisfying the equalities \((2.20)\) one can find a function \(F(q, p)\), connected with \(f\) by Eq.\((2.21)\) . We call such a function \(f\) projectable to the primary constraint surface, or simply projectable, and write

\[
F \doteq f.
\]

We have by definition

\[
\Phi_{\alpha}^{[0]} = 0,
\]

an so, any function \(F\) of the form

\[
F = F_0 + F^\alpha \Phi_{\alpha}^{[0]},
\]

where \(F_0\) satisfies Eq.\((2.22)\) and \(F^\alpha\) are some arbitrary functions, satisfies the relation \((2.22)\) as well. Indeed, this equation determines the function \(F\) only on the primary constraint surface and its solution is defined up to a linear combination of the primary constraints. Hence, the expression \((2.24)\) gives the general solution of Eq.\((2.22)\). It means that the relation\((2.22)\) specifies the values of the function \(F\) at the points of the primary constraint surface only, and this function can be extended from this surface \((2.23)\) to the total phase space arbitrarily. Note that according to Eq.\((2.24)\) various extensions will differ from each other in linear combinations of the primary constraints \([1]\).

In this paper we use the notion of the standard extension introduced earlier in Ref.\([14]\) (on the definition and some useful properties of this method see also Ref.\([12]\)). Remember that a function \(F(q, p)\) is called standard if it satisfies the relation:

\[
\chi^\alpha \frac{\partial F}{\partial p^\alpha} = 0, \quad \alpha = 1, \ldots, A,
\]

5
where the vectors $\chi_\alpha^r(q)$ are dual to the null vectors of the Hessian, i.e., the matrix

$$v_\beta^\alpha(q) = \psi_\beta^r(q)\chi_\alpha^r(q)$$

is nonsingular.

For the Poisson bracket of two standard functions $F$ and $G$ we have the expression

$$\{F, G\} = \{F, G\}^0 + \frac{\partial F}{\partial p_r}(\partial \chi_\alpha^r - \partial \chi_\alpha^s \partial q_s - \partial \chi_\alpha^r \partial q_r)\frac{\partial G}{\partial p_s}.$$  (2.26)

For a notational convenience we suppose that the vectors $\chi_\alpha^r(q)$ can be chosen in such a way that

$$\frac{\partial \chi_\alpha^r}{\partial q_s} - \frac{\partial \chi_\alpha^s}{\partial q_r} = 0.$$  (2.27)

It is clear that this assumption is not very restrictive. Moreover, from consequences of the gauge algebra we see that for the systems with $N > 2$ the vectors $\chi_\alpha^r(q)$ can always be chosen in such a way that Eq.(2.28) is fulfilled.

Now the constraint algebra corresponding to the gauge-invariant system under consideration is given by the expressions:

$$\{[0] \Phi_\alpha, [0] \Phi_\beta\} = 0,$$  (2.29)

$$\{[1] \Phi_\alpha, [0] \Phi_\beta\} = \frac{[0] \mu_\alpha^\beta [N-l+1]_{\alpha\delta} [1] \Phi_\gamma^\delta}{[N-k+1]_{\alpha\gamma}},$$  (2.30)

$$\{[1] \Phi_\alpha, [1] \Phi_\beta\} = \left(\frac{[1] \mu_\alpha^\beta [N-l+1]_{\alpha\delta} - [0] \mu_\alpha^\beta [N-k+1]_{\alpha\delta}}{[N-k+1]_{\alpha\gamma}} + q_r \frac{\partial [N-l]_{\alpha\beta}}{\partial q_r} [2N-2k+l]_{\alpha\beta}\Phi_\gamma^\gamma\right)^0 [1] \Phi_\gamma^\gamma,$$  (2.31)

$$\{[1] \Phi_\alpha, [0] \Phi_\beta\} = \frac{[0] \mu_\alpha^\beta [1] \Phi_\beta^\gamma}{[1] \Phi_\gamma^\gamma},$$  (2.32)

$$\{[1] \Phi_\alpha, [1] \Phi_\beta\} = [k+1]_{\alpha\beta} \Phi_\alpha + \left(\frac{[1] \mu_\alpha^\beta [N-k+1]_{\alpha\delta}}{[N-k+1]_{\alpha\gamma}} + \mu_\alpha^\beta\frac{\partial [N-k+1]_{\alpha\gamma}}{\partial q_r} [2N-2k+l]_{\alpha\beta}\Phi_\gamma^\gamma\right)^0 [1] \Phi_\gamma^\gamma,$$  (2.33)

where $k, l = 1, \ldots, N > 1; i > N - k - l,$ and we use the notations

$$v_\alpha^\gamma v_\gamma^\beta = \delta_\beta^\alpha, \quad \mu_\alpha^\beta = \psi_r^\alpha \chi_r^\beta v_\alpha^\beta, \quad \psi_\beta^r \chi_r^\gamma v_\alpha^\gamma.$$  (2.34)

The symbol $f^0$ denotes the standard Hamiltonian counterpart of a function $f(q, \dot{q})$ satisfying the relation $\frac{\partial f}{\partial \dot{q}_r}$. In this, the secondary constraints $\Phi_\alpha$ and the Hamiltonian $H$ are the standard functions uniquely defined by the relations connecting them, respectively, with the Lagrangian constraints and the energy function of the system:

$$[k] \Phi_\alpha = \Lambda_\alpha, \quad H = E = \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L.$$  (2.35)
Thus, we have the first class constraint system and can apply to it the general BFV formalism \[4, 5\] in order to construct a BRST-invariant effective theory. To this end, let us enlarge the initial phase space by adding to the canonical pairs of even variables \(q^r, p_r\) the set of odd ghost coordinates \(\eta^\alpha, \eta^\beta\), associated with the constraints of the system, and ghost momenta \(\pi^\alpha, k = 0, \ldots, N, \alpha = 1, \ldots, A\), endowed with the ghost numbers, respectively, 1 and \(-1\). We suppose the nonzero Poisson brackets for the ghost variables to be of the form

\[
\{\eta^\alpha, \eta^\beta\} = -\delta^k l \delta^\alpha, \quad k, l = 0, \ldots, N.
\]

(2.36)

The BFV-BRST charge \(\Omega_B\) and the BFV-BRST-invariant Hamiltonian \(H_A\) are constructed on the extended phase space according to the general BFV prescriptions Eqs.(2.3)-(2.8). Using the constraint algebra in the standard extension we obtain for \(\Omega_B\):

\[
\begin{align*}
\text{[0]} \Omega_B &= \sum_{k=0}^{N} [k] \eta^\alpha \Phi_\alpha, \\
\text{[1]} \Omega_B &= \text{[0]} \delta \left( [1] A^\gamma_{\beta \delta} [N] \eta^\beta + [1] A^\gamma_{\delta \beta} [N-1] \eta^\beta \right) \text{[0]} [1] \eta^\alpha \pi_\gamma \\
&- \frac{1}{2} \sum_{k,l=1}^{N} \left( \text{[0]} u^\gamma_{\alpha \beta} [N-l+1] A^\gamma_{\delta \beta} - \text{[0]} u^\gamma_{\beta \delta} [N-k+1] A^\gamma_{\alpha \beta} + \frac{\partial}{\partial q^r} \text{[2N-k-l]} A^\gamma_{\alpha \beta} \right) \text{[0]} \eta^\beta \eta^\alpha \pi_\gamma \\
&- \frac{1}{2} \sum_{k,l=1}^{N} \sum_{i=0}^{2} \sum_{j=0}^{1} \left( 2N - k - l - i \right) \left( N - l - j \right) \left[ k, l - [N-k+1] \eta^\gamma \right] \text{[0]} \eta^\beta \eta^\alpha \pi_\gamma \pi_\gamma.
\end{align*}
\]

(2.37)

(2.38)

and for \(H_A\):

\[
\begin{align*}
\text{[0]} H &= H, \\
\text{[1]} H &= \text{[0]} \delta \left( [1] A^\gamma_{\beta \delta} [N] \eta^\beta + [1] A^\gamma_{\delta \beta} [N-1] \eta^\beta \right) \text{[0]} [1] \eta^\alpha \pi_\gamma \\
&- \frac{1}{2} \sum_{k,l=1}^{N} \left( \text{[0]} u^\gamma_{\alpha \beta} [N-l+1] A^\gamma_{\delta \beta} - \text{[0]} u^\gamma_{\beta \delta} [N-k+1] A^\gamma_{\alpha \beta} + \frac{\partial}{\partial q^r} \text{[2N-k-l]} A^\gamma_{\alpha \beta} \right) \text{[0]} \eta^\beta \eta^\alpha \pi_\gamma \\
&- \frac{1}{2} \sum_{k,l=1}^{N} \sum_{i=0}^{2} \sum_{j=0}^{1} \left( 2N - k - l - i \right) \left( N - l - j \right) \left[ k, l - [N-k+1] \eta^\gamma \right] \text{[0]} \eta^\beta \eta^\alpha \pi_\gamma \pi_\gamma.
\end{align*}
\]

(2.39)

(2.40)

From Eqs.(2.10) and (2.11) we see that to get the next order terms of \(\Omega_B\) and \(H_A\) we need the expressions for the Poisson brackets of the constraints and the Hamiltonian with the structure functions of the constraint algebra. The necessary formulas were obtained in Ref.[13]. Recall that the corresponding calculations are based on the notion of the pseudoinverse matrix [13]. In this we define the projector

\[
\Pi^r_s = \delta^r_s - \chi^r_s \text{[0]} [0] u^r \eta^r, \quad \Pi^t_s \Pi^r_s = \Pi^r_s,
\]

(2.41)

and then the pseudoinverse matrix \(W^{rs}\), corresponding to the Hessian of the system, is uniquely defined by the relations

\[
W^{rt} W_{ts} = \Pi^r_s, \quad W^{rs} \chi^s = 0.
\]

(2.42)
So, for an arbitrary standard function $F(q, p)$ connected with a function $f(q, \dot{q})$ by the relation (2.22) we have for our case \cite{13}:

$$\{\Phi_\alpha, F\} \doteq \{u_\alpha^\beta, \xi_\beta(f)\},$$

(2.43)

$$\{\Phi_\alpha, F\} \doteq \xi_\alpha(f) - \{u_\alpha^\beta, \xi_\beta(f)\} + \left(\frac{\partial f}{\partial q^r} W^{rs} \frac{\partial u_\alpha^r}{\partial q^s}\right) \Lambda_\beta,$$

(2.44)

$$\{H, F\} \doteq -T(f) + \mu^\alpha \xi_\alpha,$$

(2.45)

where the vector fields $\xi_\alpha$ are of the form

$$\xi_\alpha = \psi_\alpha^\tau \frac{\partial}{\partial q^\tau} + \left(\psi_\alpha^{\tau+1} + T(\psi_\alpha^\tau)\right) \frac{\partial}{\partial q^\tau}, \quad k = 0, 1, \ldots, N,$$

(2.46)

and we have used the notation

$$T = \dot{q}^i \frac{\partial}{\partial q^i} + R_s W^{st} \frac{\partial}{\partial q^t}.$$  

(2.47)

Note that the differential operator $T$ has the sense of the evolution operator of gauge-invariant systems \cite{14}:

$$T(f)|_{L_r=0} = \frac{d}{dt}(f),$$

(2.48)

and the algebra of the vector fields $\xi_\alpha$, $k = 0, 1, \ldots, N$, and $\psi_\alpha^\tau \frac{\partial}{\partial q^\tau}$ coincides with the gauge algebra \cite{13} on the trajectories of the system.

Making use of Eqs. (2.43) and (2.44) and consequences of the Jacobi identities \cite{17} for the gauge transformations \cite{13} we obtain from Eq.(2.10) that the only nonzero second order structure functions of the BRST charge are given by the expression:

$$\Omega^{\tau\rho}_{\mu\nu}(k, l, m) = \frac{1}{12} \left(\frac{\partial u_\mu^r}{\partial q^r} W^{rs} \frac{\partial (\psi_\rho^\tau \Pi^{\tau}_{\tau \rho})}{\partial q^s} \frac{\partial u_\rho^\sigma}{\partial q^\tau} + \frac{\partial u_\mu^r}{\partial q^r} W^{rs} \frac{\partial (\psi_\rho^\tau \Pi^{\tau}_{\tau \rho})}{\partial q^s} \frac{\partial u_\rho^\sigma}{\partial q^\tau}\right)$$

$$\times \frac{\partial u_\rho^\tau}{\partial q^\tau} W^{rs} \frac{\partial (\psi_\rho^\tau \Pi^{\tau}_{\tau \rho})}{\partial q^s} \frac{\partial u_\rho^\sigma}{\partial q^\tau} \right)_{0}, \quad k, l, m = 1, \ldots, N. $$

(2.49)

and so, we have that

$$\Omega_B = \frac{1}{4} \sum_{k,l,m=1}^N \left(\frac{\partial u_\mu^r}{\partial q^r} W^{rs} \frac{\partial (\psi_\rho^\tau \Pi^{\tau}_{\tau \rho})}{\partial q^s} \frac{\partial u_\rho^\sigma}{\partial q^\tau}\right)_{0}. \quad \eta^\rho \eta^\nu \eta^\mu \eta^\sigma \subset \Pi.$$  

(2.50)

Further, using additionally Eq.(2.45) we get from Eq.(2.11) that the only nonzero structure functions of the second order of the BRST-invariant Hamiltonian are

$$H^{\tau\rho}_{\mu\nu}(k, l) = -\frac{1}{4} \left(\frac{\partial u_\mu^\tau}{\partial q^\tau} W^{rs} \frac{\partial u_\nu^\rho}{\partial q^s} - \frac{\partial u_\mu^\rho}{\partial q^\rho} W^{rs} \frac{\partial u_\nu^\tau}{\partial q^s}\right)_{0}, \quad k, l = 1, \ldots, N. $$

(2.51)
From the last relation we get immediately

\[
\left[\hat{H}\right] = -\frac{1}{2} \sum_{k,l=1}^{N} \left( \frac{\partial L}{\partial \dot{q}^k \dot{q}^l} \right) W_{rs}^{\alpha \beta} \chi_{\alpha}^r \chi_{\beta}^s 0^{[\mu]} \eta^{[\nu]} \Pi_{\alpha}^{[1]} \Pi_{\beta}^{[1]},
\]

(2.52)

The expressions for the structure functions of the BFV formalism can be obtained also in the Lagrangian approach making use of the techniques elaborated in Ref.[12]. This way is more convenient when the initial object is a gauge-invariant Lagrangian. For the system under consideration one needs first to apply the Ostrogradsky prescription to the corresponding BRST-invariant Lagrangian \( L_B \) [17]

\[
L_B = L - \frac{1}{2} \chi^\alpha \gamma_{\alpha \beta} \chi^\beta - \bar{c}_\alpha s(q^r) \chi^\alpha_{,r} + \dot{\bar{c}}_\alpha s(q^r) \frac{\partial \chi^\alpha}{\partial q^r},
\]

(2.53)

where the notation \( \chi^\alpha_{,r} \) for the variation of the functions \( \chi^\alpha(q,q^r) = \dot{q}^r \chi^\alpha(q) + \nu^\alpha(q) \) over the trajectory \( q^r(t) \) is used. One of the results we get in this way is the identification rules for the BFV ghost canonical pairs and ghosts and antighosts introduced in the Lagrangian formalism. Namely, we introduce new odd variables putting

\[
\theta^\alpha = (c^k_\alpha)_0, \quad k = 1, \ldots, N.
\]

(2.54)

Now according to the Ostrogradsky formalism the extended configuration space is described by the set of even and odd variables \( q^r, \bar{c}_\alpha \) and \( \theta^\alpha \). Using the Ostrogradsky prescription we define the generalized momenta, corresponding to the system with the Lagrangian \( L_B \), by the formulas [17]:

\[
p_r = \frac{\partial L_B}{\partial \dot{q}^r} = \frac{\partial L}{\partial \dot{q}^r} - \chi^\alpha \gamma_{\alpha \beta} \chi^\beta - \bar{c}_\alpha \frac{\partial s(q^r)}{\partial q^r} \chi^\alpha_{,r} + \dot{\bar{c}}_\alpha \frac{\partial s(q^r)}{\partial q^r} \chi^\alpha_{,r},
\]

(2.55)

\[
p^\alpha = \frac{\partial L_B}{\partial \bar{c}_\alpha} = \sum_{k=0}^{N} \left( \chi^\alpha \gamma_{\beta} \chi^\beta \right)^{[k]} \psi^r_{\beta} \chi^\alpha_{,r},
\]

(2.56)

\[
\bar{p}_\alpha = \sum_{l=1}^{k} (-1)^{k-l} \frac{d^{k-l}}{dt^{k-l}} \frac{\partial L_B}{\partial \left( \psi^r_{\alpha} \chi^\beta \right)^{[k]}} = \sum_{l=1}^{k} (-1)^{k-l} \frac{q^{k-l}}{dt^{k-l}} \left[ \psi^r_{\alpha} \chi^\beta \right]_{,r},
\]

for any \( k = 1, \ldots, N \).

(2.57)

The Hamiltonian description of the system with the Lagrangian \( L_B \) is constructed on the phase space with canonically conjugate variables \( q^r, \bar{c}_\alpha, \theta^\alpha \) and \( p_r, p^\alpha, \bar{p}_\alpha \). When comparing the forms of the BRST charge obtained from the Lagrangian and the Hamiltonian BRST formalisms, we obtain the following correspondence between the odd variables:

\[
\eta^\alpha = p^\alpha, \quad \eta^\alpha = \theta^\alpha, \quad \pi^\alpha = \bar{c}_\alpha, \quad \pi^\alpha = \bar{p}_\alpha.
\]

(2.58)

However it is important to note that the BFV ghosts \( \eta^\alpha \) are associated with the standard constraints and the relations (2.58) should be modified for different extensions from the primary constraint surface. We deal with this question in the next section.
3 On the canonical transformations

In Ref. [5] it was shown that a transition from a set of constraints to another one can be realized in the Hamiltonian BRST formalism, at least locally, as a canonical transformation of the extended phase space. In order to demonstrate how this statement works in our case, let us consider a simplified situation with the vectors \( \psi^r_{\alpha} \), \( k = 0, 1, \ldots, N \), being functions of the generalized coordinates only.

One can easily find that the gauge algebra in this case is equivalent to the following Lie algebra of the vector fields \( \psi^r_{\alpha} = \psi^r_{\alpha} \frac{\partial}{\partial q^r} \):

\[
[\psi^r_{\alpha}, \psi^s_{\beta}] = \left( \frac{2N - k - l}{N - l} \right) A^\gamma_{\alpha \beta} \psi^\gamma_{\gamma},
\]

(3.1)

where \( k, l = 0, 1, \ldots, N \) and the only nonzero structure functions \( A^\gamma_{\alpha \beta} \) are constant.

From the corresponding constraint algebra we immediately get the expression of the BRST charge:

\[
\Omega_B = \sum_{k=1}^{N} \theta^\alpha \Phi^\alpha_{\alpha} + \sum_{k,l=1}^{N} \left( \frac{2N - k - l}{N - l} \right) A^\gamma_{\alpha \beta} \theta^\alpha \theta^\beta \bar{p}^\gamma.
\]

(3.2)

Remember that this form of the BRST charge (3.2) corresponds to the standardly extended constraints \( \Phi^\alpha_{\alpha} \), and we have the identification rules (2.58).

Let us now choose another set of the constraints. To this end, we take into account that for any non-degenerate matrix \( w^\beta_{\alpha} \) the equations

\[
w^\beta_{\alpha} \Phi^\beta_{\beta} = 0
\]

(3.3)

define one and the same surface in the phase space – the primary constraint surface given by Eq.(2.18), and consider the functions

\[
\bar{F}^\alpha_{\alpha} = w^\beta_{\alpha} \Phi^\beta_{\beta}
\]

(3.4)

as new primary constraints. Besides, let us choose the following non-standard extension for all other constraints:

\[
\bar{F}^\alpha_{\alpha} = \Phi^\alpha_{\alpha} + u^\beta_{\alpha} v^\gamma_{\gamma} \Phi^\gamma_{\beta}.
\]

(3.5)

In Eq.(3.3) we used that various extensions from the primary constraint surface to the total phase space differ from each other in a linear combination of the primary constraints.

The BRST charge corresponding to the new constraint algebra is given by the expression

\[
\Omega_B = \sum_{k=1}^{N} \theta^\alpha \bar{F}^\alpha_{\alpha} + \sum_{k,l=1}^{N} \left( \frac{2N - k - l}{N - l} \right) A^\gamma_{\alpha \beta} \theta^\alpha \theta^\beta \bar{p}^\gamma.
\]

(3.6)
While comparing the expressions (3.2) and (3.6) for the BRST charge, it is easy to show that the transition from the set of constraints \( \Phi_\alpha, \Phi_k \) to \( F_\alpha, F_k \) can be realized as the following canonical transformation of the extended phase space:

\[
\begin{align*}
\bar{c}_\alpha & \rightarrow v_\alpha^\beta \bar{c}_\beta, \\
p^\alpha & \rightarrow p^\alpha + \sum_{k=1}^N \theta^\beta_{|k|} \bar{u}^\alpha_{|k|}, \\
\bar{p}_\alpha & \rightarrow \bar{p}_\alpha - u^\gamma_{|k|} \bar{v}^\delta_{|k|} \bar{c}_\beta, \\
p_r & \rightarrow p_r - p^\alpha \frac{\partial u^\gamma_{|k|}}{\partial q^r} v_\gamma^\beta \bar{c}_\beta - \sum_{k=1}^N \theta^\alpha_{|k|} \frac{\partial u^\gamma_{|k|}}{\partial q^r} v_\gamma^\beta \bar{c}_\beta.
\end{align*}
\]

The canonical transformation (3.7)–(3.10) obviously changes the identification rules (2.58). To have the same correspondence between the new ghost variables, one needs to perform an equivalent transformation in the Lagrangian formalism. We see that the corresponding transformation consists of two steps: first we have to redefine the antighosts in \( L_B \) Eq.(2.53) as follows:

\[
\bar{c}_\alpha = [0]_{\alpha}^\delta c_\beta,
\]

and then to construct a new BRST-invariant Lagrangian

\[
L_B \rightarrow L_B'' = L_B' - \frac{d}{dt} \sigma_{can},
\]

where we have used the notation

\[
\sigma_{can} = \sum_{k=1}^N [k] \theta^\alpha_{|k|} u^\gamma_{|k|} [k] v_\gamma^\beta c_\beta.
\]

We see that redefinition of the primary constraints Eq.(3.3) is related to redefinition of the antighosts Eq.(3.11) and there is no generator of the corresponding global canonical transformation (3.7) and certain parts of Eqs.(3.8), (3.10). This is connected with the very property of the primary constraints, whose appearance is of a non-dynamical character. On the other hand, to the change of the standard extension, obtained by the relation (3.3), corresponds globally defined generator of the canonical transformation \( \sigma_{can} \).

One can also choose a non-standard extension for the Hamiltonian \( H \) as follows:

\[
H \rightarrow H + D^\alpha \Phi_\alpha,
\]

where \( D^\alpha \) are some functions on the phase space. We easily find that the transition (3.14) is equivalent in the BFV formalism not to a canonical transformation, but to the shift of the gauge-fixing fermion:

\[
\Psi \rightarrow \Psi + \bar{c}_\alpha D^\alpha.
\]
4 Conclusion

In this paper we have analyzed gauge-invariant systems with $N$-th order time derivatives of gauge parameters within the framework of the Hamiltonian BRST-BFV formalism. Higher order structure functions of the corresponding BFV-BRST charge and BFV-BRST-invariant Hamiltonian have been calculated on the basis of the results obtained earlier in Refs. [12, 13]. It has been shown that higher order terms for the systems with $N > 1$ are formally of the same form as for the systems with $N = 1$ [12]: one needs only to take into account an additional summation over the ghosts associated with the secondary constraints of $k$-th stage, $k = 1, \ldots, N$. The difference between these systems appears in the forms of the corresponding first order structure functions. Besides, for the systems with $N > 1$ we have an additional strong restriction on the structure functions of the (closed) gauge algebra [13]: for the case of $N = 2$ these structure functions depend on the generalized coordinates only, and for the systems with $N > 2$ they turn out to be constant, whereas for the case of $N = 1$ the structure functions of the gauge algebra may depend on all the velocity phase space coordinates. In this respect, the systems with higher order time derivatives of gauge parameters do not generalize the systems with $N = 1$, but present different classes of gauge-invariant systems.

Another principal property of the systems considered here is the problem of identifying Lagrangian and Hamiltonian ghost variables. This problem is solved with the use of the Ostrogradsky prescription. The corresponding identification rules turned out to be connected with the choice of the extension of the constraints from the primary constraint surface.

It would be interesting to generalize the results of our consideration to the systems, whose gauge symmetry transformations form an open gauge algebra.

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