EXPONENTIAL STABILITY FOR THE COMPRESSIBLE 
NEMATIC LIQUID CRYSTAL FLOW 
WITH LARGE INITIAL DATA 

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Abstract. In this paper, we consider the asymptotic behavior of the global 
spherically or cylindrically symmetric solution for the compressible nematic 
liquid crystal flow in multi-dimension with large initial data. Using the un-
iform point-wise positive lower and upper bounds of the density, we obtain the 
exponential stability of the global strong solution.

1. Introduction. In this paper, we consider the following compressible nematic 
liquid crystal flow

\begin{align}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla (P(\rho)) &= \mu \Delta u + (\lambda + \mu) \nabla \text{div} u \\
&\quad - \nu \text{div}(\nabla \n \odot \n - \frac{|\nabla \n|^2}{2} \mathbb{I}_N), \\
\n_t + (u \cdot \nabla) \n &= \theta (\Delta \n + |\nabla \n|^2 \n),
\end{align}

in $\Omega \times (0, +\infty)$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$. Here $\rho : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ 
denotes the density, $u : \Omega \times [0, \infty) \rightarrow \mathbb{R}^N$ denotes the velocity field and $\n : \Omega \times 
 [0, \infty) \rightarrow \mathbb{S}^2$ denotes the optical axis vector of the liquid crystal which is a unit 
vector $|\n| = 1$. $P(\rho) = A \rho^\gamma$ is the pressure, where $\gamma > 1$ and $A$ is a positive 
constant. $\lambda$ and $\mu$ are the bulk and shear viscosity coefficients, respectively, which 
satisfy the physical conditions $\mu > 0$ and $2\mu + N\lambda \geq 0$. The constants $\nu > 0, \theta > 0$ 
present the competition between the kinetic energy and the potential energy, and the 
microscopic elastic relaxation time for the molecular orientation field, respectively. 
$\mathbb{I}_N$ is the identity matrix of order $N$. $\otimes$ and $\odot$ represent the tensor product satisfy 

$$u \otimes u = (u^i u^j)_{N \times N}, \quad \nabla \n \odot \n = (n_{xi} \cdot n_{xj})_{N \times N}. $$

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Liquid crystals are substances that exhibit a phase of matter that has properties between those of a conventional liquid, and those of a solid crystal [2]. The continuum theory of liquid crystals was established by Ericksen [5] and Leslie [15] during the period of 1958 through 1968. Since then, the mathematical theory is still progressing and the study of the full Ericksen-Leslie model presents relevant mathematical difficulties. When the fluid containing nematic liquid crystal materials is at rest, we have the well-known Ossen-Frank theory for static nematic liquid crystals, see the pioneering work by Hardt-Lin-Kinderlehrer [8] on the analysis of energy minimal configurations of nematic liquid crystals. In general, the motion of fluid always takes place. The so-called Ericksen-Leslie system is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow velocity field $u$ and the macroscopic description of the microscopic orientation configurations $n$ of rod-like liquid crystals.

When the density function $\rho$ is a positive constant, then (1)-(3) becomes the hydrodynamic flow equation of incompressible liquid crystals. The pioneering work comes from Lin and his partners. Lin [16] first derived a simplified Ericksen-Leslie equations modeling the liquid crystal flows in 1989. Later, Lin and Liu [17, 18] made some important analytic studies, such as the existence of weak/strong solutions and the partial regularity of suitable solutions of the simplified Ericksen-Leslie system, under the assumption that the liquid crystal director field is of either varying length by Leslie’s terminology or variable degree of orientation by Ericksen’s terminology.

For the compressible case, the Ericksen-Leslie system becomes more complicate and there are only a few literatures on the global solutions and asymptotic behavior. The authors in [4, 3] first consider the solutions to the initial-boundary value problem of (1)-(3) with nonnegative initial density. They obtained the global existence and uniqueness for classical, weak or strong solutions in 1-dimensional. Qin and Huang also study the 1-dimensional model, in [21] they showed the existence of global solutions in $H^i(i = 1, 2, 4)$ in Lagrangian coordinates and then established regularity results. For the multi-dimensional case, in [9], the global existence and uniqueness of strong solution to the Cauchy problem are obtained in critical Besov spaces. Inspired by [1], for the radially symmetric initial data and non-negative initial density, Huang-Ding [10] proved the global existence and uniqueness of global strong solution to the problem of (1)-(3). Recently, in [22], by establishing a uniform point-wise positive lower and upper bounds of the density, the first author and his collaborators derived the existence, uniqueness and regularity of global strong solution with large initial data in the cylindrically symmetric case. And the following long time behavior of the global strong solution has also been established,

$$
\|(\rho - \bar{\rho}, u, n - \bar{n})(t)\|_{L^\infty} + \|(\rho_r, u_r, n_r)(t)\|_{L^2} \to 0, \text{ as } t \to \infty,
$$

where $\bar{\rho}$ is a certain positive constant and $\bar{n}$ is a certain constant vector with $|\bar{n}| = 1$.

For more results on the compressible nematic liquid crystal flows, readers can refer to [19]-[20] and references therein.

Our aim in this paper is to study the asymptotic behavior of the compressible nematic liquid crystal flow in multi-dimension with spherically symmetric and cylindrically symmetric. More precisely, if the initial data is spherically symmetric, the space dimension can be taken not less than two, and if the initial data is cylindrically symmetric, the space dimension is three. Base on the uniform point-wise positive lower and upper bounds of the density, we obtain the exponential stability of the global strong solution with large initial data, which improves the long time
behavior of the global strong solution in [22]. The difficulty encountered here is to deal with the high-order derivative of \( n \).

From now on, we consider \( \Omega = \{ x \mid 0 < a < |x| < b < \infty \} \) which is a symmetric domain. The reduced system is now of the form (see [10] and [22]):

\[
\begin{align*}
\rho_t + (\rho u)_r + \frac{m}{r} \rho u &= 0, \\
\rho (u_t + uu_r - \frac{u^2}{r}) + P_r - \kappa (u_r + \frac{m}{r} u) + \frac{\nu}{2} (|n_r|^2)_r + \nu \frac{m}{r} |n_r|^2 &= 0, \\
\rho (v_t + uv_r + \frac{uv}{r}) - \mu (v_r + \frac{m v}{r}) &= 0, \\
\rho (w_t + uw_r) - \mu (w_{rr} + \frac{mw_r}{r}) &= 0, \\
(n_t + un_r - \theta n_r - \theta |n_r|^2 n - \theta \frac{m}{r} n_r) &= 0,
\end{align*}
\]

where \( \rho = \rho(r, t) \), \( u = u(r, t) \), \( \kappa = 2\mu + \lambda \), the positive constants \( \nu \) and \( \theta \) are competitive between kinetic and potential energy, and microscopic elastic relaxation time, respectively. \( u \) is the component of the velocity vector \( u \) along the radial variable \( r \in (0, b) \), \( v \) is the angular component of \( u \), \( w \) is the axial component of \( u \). In the spherically symmetric case, \( m = N - 1, r = |x|, u(x, t) = u(r, t) \frac{x}{r} \), \( v = w \equiv 0 \). In the cylindrically symmetric case, \( m = 1, r = \sqrt{x_1^2 + x_2^2}, u(x, t) = u(r, t) \frac{x_1 x_2}{r^2} + v(r, t) \frac{-x_2, x_1}{r^2} + w(r, t)(0, 0, 1) \). The initial boundary value problem (4)-(8) subjected to the following initial and boundary conditions

\[
\begin{align*}
(\rho, u, v, w, n)|_{t=0} &= (\rho_0, u_0, v_0, w_0, n_0)(r), \quad r \in \Omega, \\
(u, v, w, n_r)|_{r=a,b} &= 0, \quad t \geq 0,
\end{align*}
\]

where \( n_0 : \Omega \rightarrow S^2 \).

In what follows, we use \( C \) (and \( C_i, i = 0, 1, 2, 3, 4, 5 \)) to denote a generic positive constant depending only on \( a, b \), the parameters of the system and the bounds of the initial data, but independent of \( t \). Without loss of generality, let \( A = \theta = \nu = 1 \). For simplicity, we will use the abbreviation \( \| \cdot \| = \| \cdot \|_{L^2} \) and denote \( v_0 = (u_0, v_0, w_0) \).

Since the global existence and uniqueness of strong solution has been derived in [10] and [22], we state the main result concerning the exponential stability of the solution in this paper.

**Theorem 1.1.** Assume that the initial data satisfies \( (\rho_0, v_0, n_0) \in H^2 \times (H^1 \cap H^2)^3 \times H^3 \) with \( 0 < c_0 \leq \rho_0 \) and \( |n_0| = 1 \) in \( \Omega \) for some constant \( c_0 \) and are compatible with boundary conditions. Then for any \( t \geq 0 \), the strong solution of problem (4)-(10) has the following estimate,

\[
\| (\rho - \bar{\rho}, u, v, w)(t) \|_{L^\infty} + \| (\rho_t, u_r, v_r, w_r, n_r)(t) \| \leq C e^{-Ct},
\]

where \( \bar{\rho} = \frac{1}{b-a} \int_a^b \rho dr \) is a certain positive constant.

2. The proof of Theorem 1.1. In this section, we shall give the proof for Theorem 1.1. To this end, we first need some important lemmas. The following lemma is very useful when we deal with the high-order derivative of \( n \).

**Lemma 2.1** (see [7]). There exists a positive constant \( C_0 \), such that the following inequality holds for all \( f \in H^1_0(R_0, \infty) \) and \( r^\alpha f, r^\alpha f_r \in L^2(R_0, \infty) \) with \( R_0 > 0 \):

\[
\| r^\beta f \|_{L^p(R_0, \infty)} \leq C_0 \| r^\alpha f \|_{L^2(R_0, \infty)} \| r^\alpha f_r \|_{L^2(R_0, \infty)}^{1-b} \| r^\alpha f \|_{L^2(R_0, \infty)}^{b} \quad (11)
\]
if and only if the following two relations hold:

\begin{align*}
(1) \quad & \frac{1}{p} + \beta = \frac{1}{2} + \alpha - b, \\
(2) \quad & \alpha - \sigma \begin{cases} \geq 0, & \text{if } b > 0, \\ \leq 1, & \text{if } b > 0 \text{ and } \alpha - \frac{1}{2} = \frac{1}{p} + \beta, \end{cases}
\end{align*}

where \( p > 0, 0 \leq b \leq 1, \alpha > -\frac{1}{2}, \beta > -\frac{1}{p} \) and \( \beta = b\sigma + (1 - b)\alpha. \)

In order to show the long time behavior of the solution, we need to recall some uniform estimates which have been established in [22]. Hence, we only state it here in the following lemma.

**Lemma 2.2.** Assume that the assumptions of Theorem 1.1 are satisfied. Then for any \( t \geq 0 \), there exist a constant \( C > 0 \), which is independent of \( t \), such that

\begin{equation}
C^{-1} \leq \rho(r,t) \leq C, \quad \int_a^b \left( \rho^2 r \left( u_r^2 + v_r^2 + w_r^2 + |n_r|^2 + |n_{rr}|^2 \right) \right) (t) dr \leq C. \tag{12}
\end{equation}

**Remark 1.** In [22], we only obtained the above results in the cylindrically symmetric case. In a similar argument, one can get the same results in the spherically symmetric case with modification.

At the same time, we have the basic energy equality.

**Lemma 2.3** (see [10] and [22]). For any \( t \geq 0 \), it holds that

\begin{equation}
\frac{d}{dt} \int_a^b \left[ \rho \left( u_r^2 + v_r^2 + w_r^2 + \frac{n_r^2}{r^2} \right) + \rho^\gamma - \frac{1}{\gamma - 1} + \frac{\rho}{2} \right] dr \\
+ \int_a^b \left[ \kappa (u_r^2 + \frac{m_u^2}{r^2}) + \mu (v_r^2 + \frac{m_v^2}{r^2}) + \mu w_r^2 \right] dr \\
+ \int_a^b \left[ (n_{rr} + |n_r|^2 n_r)^2 + \frac{|n_r|^2}{r^2} \right] dr = 0. \tag{14}
\end{equation}

Now, we are ready to prove the exponential stability of the global solution. Let us define

\begin{align*}
\mathcal{U}(t) &= \int_a^b \rho \left( u_r^2 + v_r^2 + w_r^2 + \frac{n_r^2}{2} \right) dr, \quad \mathcal{K}(t) = \frac{\kappa}{2} \int_a^b \left. \rho \left( \frac{1}{\rho} \right) r \right|_r^2 dr, \\
\mathcal{V}(t) &= \int_a^b (u_r^2 + v_r^2 + w_r^2 + |n_r|^2) dr, \quad \mathcal{M}(t) = \int_a^b \rho u \left( \frac{1}{\rho} \right) r. 
\end{align*}

The forms of the four-time-dependent functions are technical. \( \mathcal{U} \) and \( \mathcal{V} \) come from the basic energy equality (14). The definitions of \( \mathcal{K} \) and \( \mathcal{M} \) are motivated by Lemma 3.6 in [22] on the uniform estimate of derivative for \( \rho \). We observe from (14) that

\begin{equation}
\frac{d}{dt} \mathcal{U} + C_0 \mathcal{V} \leq 0. \tag{15}
\end{equation}

**Lemma 2.4.** For any \( t \geq 0 \), it holds that

\begin{equation}
\frac{d}{dt} (\mathcal{U} + \mathcal{K} - \mathcal{M}) + C_1 \mathcal{K} \leq C_2 \mathcal{V}, \tag{16}
\end{equation}

\begin{equation}
\frac{d}{dt} \rho u \left( \frac{1}{\rho} \right) r \leq \frac{1}{2} \rho u \left( \frac{1}{\rho} \right) r, \tag{17}
\end{equation}

\begin{equation}
\mathcal{M}(t) \leq \int_a^b \rho u \left( \frac{1}{\rho} \right) r dr. \tag{18}
\end{equation}
Proof. It follows from (4) that

\[
\frac{d}{dt} \int_a^b \rho \left( \frac{1}{\rho} \right)_r^2 dr
\]

\[
= \int_a^b \rho u \left( \frac{1}{\rho} \right)_r^2 dr + 2 \int_a^b \rho \left( \frac{1}{\rho} \right)_r \left( \frac{1}{\rho} \right)_r t dr
\]

\[
= - \int_a^b (\rho u)_r \left( \frac{1}{\rho} \right)_r^2 dr - \int_a^b \frac{m \rho u}{r} \left( \frac{1}{\rho} \right)_r^2 dr
\]

\[+ 2 \int_a^b \rho \left( \frac{1}{\rho} \right)_r \left( \frac{1}{\rho} \right)_r u - \left( \frac{1}{\rho} \right)_r u_r dr = - \int_a^b (\rho u)_r \left( \frac{1}{\rho} \right)_r^2 dr + 2 \int_a^b \left( \frac{1}{\rho} \right)_r (u_r + \frac{m \rho u}{r}) dr
\]

\[= 2 \int_a^b \left( \frac{1}{\rho} \right)_r (u_r + \frac{m \rho u}{r}) dr + \int_a^b m \rho u \left( \frac{1}{\rho} \right)_r^2 dr.
\]

Multiplying (5) by \( \left( \frac{1}{\rho} \right)_r \) and using (4), we have

\[
\frac{d}{dt} \int_a^b \rho \left( \frac{1}{\rho} \right)_r dr
\]

\[
= \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr + \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr + \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr
\]

\[
= \kappa \int_a^b \left( \frac{1}{\rho} \right)_r (u_r + \frac{m \rho u}{r}) dr - \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr + \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr
\]

\[
+ \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr + \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr + \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr.
\]

By virtue of (4) and (17), we get

\[
\frac{\kappa}{2} \frac{d}{dt} \int_a^b \rho \left( \frac{1}{\rho} \right)_r^2 dr + \gamma \int_a^b \rho^\gamma - 3 \rho_r^2 dr - \frac{d}{dt} \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr
\]

\[
= \frac{\kappa}{2} \int_a^b \frac{m \rho u}{r} \left( \frac{1}{\rho} \right)_r^2 dr + 2 \int_a^b \rho u \left( \frac{1}{\rho} \right)_r - \int_a^b \rho u \left( \frac{1}{\rho} \right)_r dr - \int_a^b \frac{\rho u}{\rho_2} (\rho u)_r dr
\]

\[+ \int_a^b (\rho u^2 + \frac{m \rho u^2}{r} - \rho v^2) \left( \frac{1}{\rho} \right)_r dr + \int_a^b (\frac{|\mathbf{n}_r|^2}{2} + m|\mathbf{n}_r|^2) \left( \frac{1}{\rho} \right)_r dr
\]

\[= \frac{\kappa}{2} \int_a^b \frac{m \rho u}{r} \left( \frac{1}{\rho} \right)_r^2 dr + \int_a^b \left( u_r^2 + \frac{m \rho u}{r} \right) dr
\]

\[+ \int_a^b \left( \mathbf{n}_r \mathbf{n}_{rr} + \frac{m|\mathbf{n}_r|^2}{r} - \rho v^2 \right) \left( \frac{1}{\rho} \right)_r dr.
\]
Thus, by (12) and Sobolev inequalities, we have
\[
\frac{d}{dt}(K - M) + \frac{\gamma}{\rho} \int_a^b \rho^{-3} \rho^2 dr \\
\leq \frac{\kappa}{2} \int_a^b \frac{m\rho u}{r} \left(\frac{1}{\rho} \right) r^2 dr + C \int_a^b (u^2 + v^2 + |n_r|^2) dr + C \int_a^b r^m|n_{rr}|^2 dr.
\]
(18)

From Lemma 2.2, we get
\[
\frac{\kappa}{2} \int_a^b \frac{m\rho u}{r} \left(\frac{1}{\rho} \right) r^2 dr \leq \epsilon \int_a^b \rho^{-3} \rho^2 dr + C(\epsilon) \int_a^b u^2 \rho^3 dr \\
\leq \epsilon \int_a^b \rho^{-3} \rho^2 dr + C(\epsilon) \int_a^b (u^2 + u_r^2) dr.
\]
(19)

On the other hand, if we choose \( \alpha = \beta = \sigma = \frac{m}{2}, b = \frac{1}{4} \) and \( p = 4 \), it follows from Lemma 2.1 that
\[
\|r^{m/2}n_r\|^4_{L^4} \leq C\|r^{m/2}n_r\|^3\|r^{m/2}n_{rr}\|.
\]
(20)

With the help of \( |n| = 1 \) and (20), we get
\[
\int_a^b r^m|n_{rr}|^2 dr = \int_a^b r^m(n_{rr} + |n_r|^2 n) dr + \int_a^b r^m|n_r|^4 dr \\
\leq \int_a^b r^m(n_{rr} + |n_r|^2 n)^2 dr + \frac{1}{a^m} \int_a^b r^{2m}|n_r|^4 dr \\
\leq \int_a^b r^m(n_{rr} + |n_r|^2 n)^2 dr \\
+ \frac{1}{2} \int_a^b r^m|n_{rr}|^2 dr + C \left( \int_a^b r^m|n_r|^2 dr \right)^3,
\]
which implies that
\[
\int_a^b r^m|n_{rr}|^2 dr \leq -\frac{d}{dt}U + CV.
\]
(21)

Introducing (19) and (21) in (18), we arrive at (16). The proof is now complete.

Lemma 2.5. For any \( t \geq 0 \), it holds that
\[
\frac{d}{dt}(U + V) \leq C(\dot{K} + V).
\]
(22)

Proof. Utilizing (5), (12), (13) and the boundary condition, we have
\[
\frac{d}{dt} \int_a^b u_r^2 dr \\
= -2 \int_a^b u_r u_{rr} dr \\
= -2 \int_a^b u_{rr} \left(-uu_r + \frac{u^2}{r} - \gamma\rho^{-2}\rho_r + \frac{\kappa}{\rho}(u_r + \frac{mu}{r})_r + \frac{1}{2\rho}(|n_r|^2)_r + \frac{m}{\rho r}|n_r|^2 \right) dr \\
\leq -\int_a^b \frac{\kappa}{\rho} u_r^2 dr + C \int_a^b r^m|n_{rr}|^2 dr + C \int_a^b (\rho_r^2 + v^4 + u^4 + u_r^2 + |n_r|^2) dr.
\]
By virtue of (21), we get
\[
\frac{d}{dt} \int_a^b u^2 \, dr + \frac{d}{dt} U \leq C(K + \mathcal{V}).
\] (23)

Recalling (6), (7), (8) and (12), a simple computation gives
\[
\frac{d}{dt} \int_a^b (v^2 + w^2 + |\mathbf{n}_r|^2) \, dr \leq CV.
\] (24)

Adding (23) and (24) yields (22), which completes the proof.

Proof of Theorem 1.1. Let \( \bar{\rho} = \frac{1}{b-a} \int_a^b \rho \, dr \). By virtue of conservation of mass and the mean value theorem of integrals, it is clear that there exists a \( \xi \in [a, b] \) such that
\[
\bar{\rho} = \frac{1}{b-a} \int_a^b \frac{1}{r^m} (r^m \rho) \, dr = \frac{1}{(b-a)^{m-1}} \int_a^b r^m \rho \, dr,
\]
which yields \( \bar{\rho} > 0 \) is a constant. By Gagliardo-Nirenberg-Sobolev inequality and (12), we have
\[
\|\rho - \bar{\rho}\|_{L^\infty}^2 \leq C\|\rho - \bar{\rho}\|\|\rho_r\|,
\]
which implies
\[
\mathcal{K} \geq C_0 > 0,
\]
and then we can get
\[
U \leq C_4(K + \mathcal{V}).
\] (25)

From the definition of \( \mathcal{M} \), we have
\[
|\mathcal{M}| \leq \frac{1}{2} K + C \int_a^b u^2 \, dr \leq \frac{1}{2} K + C_5 \mathcal{V}.
\] (26)

Multiplying (16) by a small constant \( \epsilon_1 \) with \( 0 < \epsilon_1 < \frac{C_0}{2C_2} \) and adding the result to (15), we deduce
\[
\frac{d}{dt} [(1 + \epsilon_1) U + \epsilon_1 K - \epsilon_1 \mathcal{M}] + \epsilon_1 C_1 K + (C_0 - \epsilon_1 C_2) \mathcal{V} \leq 0.
\] (27)

Then, multiplying (22) by a small constant \( \epsilon_2 > 0 \) and adding the result to (27), we have
\[
\frac{d}{dt} [(1 + \epsilon_1 + \epsilon_2) U + \epsilon_2 \mathcal{V} + \epsilon_1 K - \epsilon_1 \mathcal{M}] + \epsilon_1 C_1 K + (C_0 - \epsilon_1 C_2) \mathcal{V} \leq \epsilon_2 C_3(K + \mathcal{V}).
\] (28)

Using (25) and (26), for choosing sufficiently small positive constants \( \epsilon_2, \epsilon_3, \epsilon_4 \), we have
\[
\epsilon_2 C_3(K + \mathcal{V}) \leq \frac{1}{4} \epsilon_1 C_1 K + (C_0 - \epsilon_1 C_2) \mathcal{V} |, \]
\[
- \epsilon_3 \epsilon_1 \mathcal{M} \leq \epsilon_3 \left( \frac{C_1}{2} K + \epsilon_1 C_5 \mathcal{V} \right) \leq \frac{1}{4} \epsilon_1 C_1 K + (C_0 - \epsilon_1 C_2) \mathcal{V} \)
\] (29)

and
\[
\epsilon_4 (1 + \epsilon_1 + \epsilon_2) U \leq \epsilon_4 (1 + \epsilon_1 + \epsilon_2) C_4(K + \mathcal{V}) \leq \frac{1}{4} \epsilon_1 C_1 K + (C_0 - \epsilon_1 C_2) \mathcal{V} \]
\] (31)
Combining (28)-(31), we obtain

\[
\frac{d}{dt} \left[ (1 + \epsilon_1 + \epsilon_2) U + \epsilon_2 V + \epsilon_1 K - \epsilon_1 M \right] \\
+ C \left[ (1 + \epsilon_1 + \epsilon_2) U + \epsilon_2 V + \epsilon_1 K - \epsilon_1 M \right] \leq 0, \tag{32}
\]

which implies that

\[
(1 + \epsilon_1 + \epsilon_2) U + \epsilon_2 V + \epsilon_1 K - \epsilon_1 M \leq C e^{-Ct}. \tag{33}
\]

Thus, for sufficiently small \( \epsilon_1 \), we conclude the proof by (26), Gagliardo-Nirenberg-Sobolev inequality and the above inequality.

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