Exact Counts of $C_4$s in Blow-Up Graphs

S. Y. Chan* K. Morgan* J.Ugon*

Abstract

Cycles have many interesting properties and are widely studied in many disciplines. In some areas, maximising the counts of $k$-cycles are of particular interest. A natural candidate for the construction method used to maximise the number of subgraphs $H$ in a graph $G$, is the blow-up method. Take a graph $G$ on $n$ vertices and a pattern graph $H$ on $k$ vertices, such that $n \geq k$, the blow-up method involves an iterative process of replacing vertices in $G$ with a copy of the $k$-vertex graph $H$. In this paper, we apply the blow-up method on the family of cycles. We then present the exact counts of cycles of length 4 for using this blow-up method on cycles and generalised theta graphs.

1 Introduction

In graph theory, the family of cycles are considered to have very rich structures. Cycles have many interesting properties and are of interest in many disciplines. Some of the many applications of cycles include periodic scheduling, the identification of weak interdependence in ecosystems and to determine chemical pathways in chemical networks [1, 9].

In network analysis, cycles are used in modelling and studying communication systems, improve consensus network performance, to investigate faults and reliability and also study the topological features of such networks [16]. In some cases, counting cycles is used as part of network analysis or message-passing algorithms [8, 11, 15]. Some other interesting problems arise in relation to counting $k$-cycles in graphs. For example, counting the number of 4-cycles in a tournament [10] and enumerating cycles of a given length [2].

Further, some problems looks into maximising the number of cycles in graphs. There exists a kidney exchange program (KEP), which involves maximising the number of (directed) cycles to maximise the expected number of transplants [3, 4, 13].

In the areas of extremal graph theory, a construction method used to maximise counts of graphs is the blow-up method. This method has been applied in graphs to study graph spectra [12] and even the maximum induced density of graphs [6]. This method was also applied as an approach to the Caccetta-Häggkvist conjecture [5] and Johansson conjecture [7].

Suppose we have graphs $G$ and $H$ on $n$ and $k$ vertices respectively, such that $n \geq k$. The blow-up method is an iterative process of replacing each vertex in $G$ with copies of $H$. If all vertices in $G$ have been replaced with a copy of $H$, this is also known as a balanced blow-up of $G$.

In this paper, we are interested in determining the exact number of induced $C_4$s in the nested blow-up graph. We will find the exact counts of $C_4$s in two different blow-up graphs, one in the nested blow-up of $C_4$s and the other in the theta graph $\Theta_{2,2,2}$. We give the associated formulas with respect to the level of blow-up $N$, which is defined in the later section.

2 Notations and Definitions

In this section, we present some basic definitions and notations that are used in this paper. All graphs in this paper are simple.

A graph $G = (V, E)$ consists of the (finite) vertex set $V$ and edge set $E$, which is a subset of all unordered pair of vertices. The order of a graph is the number of vertices, whereas the size of a graph is the number of edges. Let $u, v \in V(G)$, we say that $u$ is adjacent to $v$ if there exists an edge $\{u,v\} \in E(G)$. We say that the edge $\{u,v\}$ is incident to vertices $u$ and $v$.

*Deakin University, Geelong, Australia, School of Information Technology, Faculty of Science Engineering & Built Environment, Australia

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In this section, we show the exact counts of $n$ and $G$. We show the following lemma:

**Lemma 1.** The number of non-edges $m_N^*$ in the nested blow-up graph $G_N$ is

$$m_N^* = \binom{4^{N+1}}{2} - 4^{N+1} \sum_{i=0}^N 4^i = \frac{4^{N+1}(4^{N+1} - 1)}{6}, N \in \mathbb{N} \cup \{0\}.$$
Proof. We prove the equation from Lemma 1 by induction.

**Base case:** For \( N = 0 \), the number of non-edges is 
\[
m_0^c = \binom{4^0}{2} - 4 \sum_{i=0}^{6} 4^0 = 6 - 4 = \frac{4^1(4^1 - 1)}{6} = 2,
\]
which is precisely the number of non-edges in \( C_4 \).

Assume the induction hypothesis that for a particular \( N \), the case \( n = N \) holds, that is:
\[
m_N^c = \binom{4^{N+1}}{2} - 4^{N+1} \sum_{i=0}^{N} 4^i = \frac{4^{N+1}(4^{N+1} - 1)}{6}.
\]

The number of non-edges in \( G_{N+1} \) is:
\[
m_{N+1}^c = \binom{4^{N+2}}{2} - 4 \times |E(G_N)| - 4 \times \text{edges between pairs of blobs}
\]

We obtain \( |E(G_N)| \) using \( m_N^c \) such that \( |E(G_N)| = \binom{4^{N+1}}{2} - m_N^c \). Thus,
\[
m_{N+1}^c = \binom{4^{N+2}}{2} - 4 \cdot 4^{N+1} \sum_{i=0}^{N} 4^i - 4 \cdot (4^{N+1})^2
\]
\[
= \binom{4^{N+2}}{2} - 4^{N+2} \sum_{i=0}^{N} 4^i - 4^{N+2} 4^{N+1}
\]
\[
= \binom{4^{N+2}}{2} - 4^{N+2} \sum_{i=0}^{N+1} 4^i.
\]

Since both the base case and inductive step have been proven as true, thus by mathematical induction \( m_N^c \) holds for all \( N \).

Thus, we state the following theorem.

**Theorem 1.** The nested blow-up graph \( G_N, N \geq 0 \), of a \( C_4 \) has precisely
\[
T_N = \frac{8 \times 4^N}{6670} \times (1280 \times 4^{3N} + 672 \times 4^{2N} + 105 \times 4^N - 713)
\]
induced subgraphs isomorphic to \( C_4 \).

**Proof.** First, we will show that,
\[
T_N = \begin{cases} 
4(T_{N-1}) + (4^N)^4 + 4 \times m_{N-1}^c \times (4^N)^2 + 4 \times (m_{N-1}^c)^2, & n > 0 \ 
1 & n = 0.
\end{cases}
\]

Since \( G_0 = C_4 \), we have \( T_0 = 1 \). We show that we can obtain \( T_N \) from \( T_{N-1} \) and prove each term from \( T_N \) respectively.

![Figure 3](image.png)

Figure 3: Illustration of \( G_N \) with four blobs labelled \( B_1, B_2, B_3 \) and \( B_4 \).
Figure 4: [Left to right] Illustration of ways to obtain $C_4$ from either 2, 3, or all 4 blobs respectively. The selected vertices are shown in red.

We know that $G_N$ has four blobs (see Figure 3) isomorphic to $G_{N-1}$. Each of these blobs has $T_{N-1}$ copies of $C_4$, which contributes to $4T_{N-1}$ induced copies of $C_4$. Thus giving the first term $4T_{N-1}$.

For simplicity of the remainder of the proof, we use Figure 4 to show the multiple ways of obtaining an induced $C_4$ from either 2, 3 or all 4 blobs. Example of how vertices can be chosen are shown in red.

We select a vertex from each of the 4 blobs that are isomorphic to $G_{N-1}$ in $G_N$ which results in an induced $C_4$. There are $4^N$ vertices in each $G_{N-1}$ which gives $(4^N)^4$ choices. This results in the second term $(4^N)^4$.

For the third term $4 \times m_{N-1} \times (4^N)^2$, we choose 1 blob that is isomorphic to $G_{N-1}$. We choose a non-edge from this blob. Then, we select two adjacent blobs and from each blob we select a vertex. There are $(4^N)$ vertices in a blob. As we choose a different vertex from each of the 2 blobs, there are $(4^N)^2$ choices of vertices, as well as $4 \times m_{N-1}^2$ non-edges, that can be chosen from the third blob.

Lastly, we choose 2 adjacent blobs and a non-edge from each. There are 4 choices for adjacent blobs in the blow-up of $C_4$. Thus, resulting in $4 \times (m_{N-1}^2)^2$ choices.

We now expand and simplify $T_N$ to find the recurrence relation.
Expanding the first few terms, we obtain:

\[ T_0 = 1 \]
\[ T_1 = 4(T_0) + 4^4 + 4 \times m_0^6 \times 4^2 + 4 \times (m_0^2)^2 = 4 + 4^4 + 4^3 \times m_0^6 + 4 \times (m_0^2)^2 \]
\[ = 4^1 \times \sum_{i=0}^{1} 4^{3i} + \sum_{i=1}^{1} (4^{i+1} \times m_{i-1}^6) + \sum_{i=1}^{1} (4^i \times (m_{i-1}^2)^2) \]
\[ T_2 = 4(T_1) + 4^8 + 4 \times m_1^6 \times 4^4 + 4 \times (m_1^2)^2 \]
\[ = 4 \left( 4^1 \times \sum_{i=0}^{1} 4^{3i} + \sum_{i=1}^{1} (4^{i+1} \times m_{i-1}^6) + \sum_{i=1}^{1} (4^i \times (m_{i-1}^2)^2) \right) + 4^8 + 4 \times m_1^6 \times 4^4 + 4 \times (m_1^2)^2 \]
\[ = 4^2 \times \sum_{i=0}^{2} 4^{3i} + 2 \sum_{i=1}^{2} (4^{i+1} \times m_{i-1}^6) + \sum_{i=1}^{2} (4^i \times (m_{i-1}^2)^2) \]
\[ T_3 = 4(T_2) + 4^{12} + 4 \times m_2^6 \times 4^6 + 4 \times (m_2^2)^2 \]
\[ = 4 \left( 4^2 \times \sum_{i=0}^{2} 4^{3i} + 2 \sum_{i=1}^{2} (4^{i+1} \times m_{i-1}^6) + \sum_{i=1}^{2} (4^i \times (m_{i-1}^2)^2) \right) + 4^{12} + 4 \times m_2^6 \times 4^6 + 4 \times (m_2^2)^2 \]
\[ = 4^3 \times \sum_{i=0}^{3} 4^{3i} + 3 \sum_{i=1}^{3} (4^{i+1} \times m_{i-1}^6) + \sum_{i=1}^{3} (4^i \times (m_{i-1}^2)^2) \]

\[ \vdots \]
\[ T_N = 4^N \times \sum_{i=0}^{N} 4^{3i} + \sum_{i=1}^{N} (4^{N+i+1} \times m_{i-1}^6) + \sum_{i=1}^{N} (4^i \times (m_{N-i}^2)^2) \]

We simplify for each \( Q_N, R_N \) and \( S_N \).

Simplifying using geometric sum,
\[ Q_N = 4^N \times \sum_{i=0}^{N} 4^{3i} = 4^N \times \left( \frac{4^{3(N+1)} - 1}{4^3 - 1} \right) = \frac{4^N \times (64 \times 4^{3N} - 1)}{63}. \] (1)

Using Lemma 1,
\[ R_N = \sum_{i=1}^{N} (4^{N+i+1} \times m_{i-1}^6) \]
\[ = \sum_{i=1}^{N} \left( \frac{4^{N+i+1}}{6} \times \frac{4^i(4^i - 1)}{6} \right) = 4^{N+1} \times \sum_{i=1}^{N} \frac{4^{3i} - 4^{2i}}{6} = \frac{4^{N+1}}{6} \times \left( \sum_{i=1}^{N} 4^{3i} - \sum_{i=1}^{N} 4^{2i} \right). \]

Again, simplify using geometric sum,
\[ R_N = \frac{4^{N+1}}{6} \left( \frac{4^3(4^{3N} - 1) - 4^2(4^{2N} - 1)}{4^3 - 1} \right) \]
\[ = \frac{4^{N+1}}{1890} \left( 320 \times (4^{3N} - 1) - 336 \times (4^{2N} - 1) \right) \]
\[ = \frac{4^{N+1}}{1890} \left( 320 \times 4^{3N} - 336 \times 4^{2N} + 16 \right). \] (2)
Lastly,
\[ S_N = \sum_{i=1}^{N} (4^i \times (m_{N-1}^i)^2) \]

By Lemma 1,
\[
S_N \sum_{i=1}^{N} \left( 4^i \times \left( \frac{4^{N-i+1} (4^{N-i+1} - 1)}{6} \right)^2 \right)
\]
\[
= \sum_{i=1}^{N} \left( 4^i \times \frac{4^{2N-2i+2} (4^{N-i+1} - 1)^2}{36} \right)
\]
\[
= \frac{4^{N+1}}{9} \sum_{i=1}^{N} 4^{N-i} \left( 4^{2N-2i+2} - 2 \times 4^{N-i+1} + 1 \right)
\]
\[
= \frac{4^{N+1}}{9} \left( \sum_{i=1}^{N} 4^{3N-3i+2} - 2 \times \sum_{i=1}^{N} 4^{2N-2i+1} + \sum_{i=1}^{N} 4^{N-i} \right)
\]
\[
= \frac{4^{N+1}}{9} \left( 16 \times \sum_{i=1}^{N} 4^{3(N-i)} - 8 \times \sum_{i=1}^{N} 4^{2(N-i)} + \sum_{i=1}^{N} 4^{N-i} \right). \tag{3}
\]

Simplify using geometric sum,
\[
S_N = \frac{4^{N+1}}{9} \left( 16 \times \frac{4^{3N} - 1}{63} - 8 \times \frac{4^{2N} - 1}{15} + \frac{4^N - 1}{3} \right)
\]
\[
= \frac{4^{N+1}}{2835} \left( 80 \times 4^{3N} - 168 \times 4^{2N} + 105 \times 4^N - 1 \right)
\]
\[
= \frac{4^{N+1}}{2835} \left( 80 \times 4^{3N} - 168 \times 4^{2N} + 105 \times 4^N - 17 \right). \tag{3}
\]

Thus,
\[
T_N = \frac{4^N}{63} (64 \times 4^{3N} - 1) + \frac{4^{N+1}}{1890} (320 \times 4^{3N} - 336 \times 4^{2N} + 16)
\]
\[
+ \frac{4^{N+1}}{2835} \left( 80 \times 4^{3N} - 168 \times 4^{2N} + 105 \times 4^N - 17 \right)
\]
\[
= \frac{4^N}{5670} (10240 \times 4^{3N} - 5376 \times 4^{2N} + 840 \times 4^N - 34). \tag{3}
\]

### 3.1 Counting $C_4$ in the theta graph $\Theta_{2,2,2}$

In this section, we construct the nested blow-up graph of the theta graph $\Theta_{2,2,2}$ by replacing each vertex $v_i$ in $\Theta_{2,2,2}$ with a $\Theta_{2,2,2}$.

Recall that the $N$ level nested blow-up of a graph $G$ is denoted $G_N$. We compute the number of induced $C_4$ in $\Theta_{2,2,2}$. We denote $T_N$ as the number of induced $C_4$s and $n_N$ as the number of distinct vertices in $G_N$ respectively. Figure 5 gives the theta graph $\Theta_{2,2,2}$.

![Figure 5: Illustration of theta graph $\Theta_{2,2,2}$](image)

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Lemma 2. Let $G = \Theta_{2,2,2}$. The number of non-edges $m^c_N$ in each level of the blow-up graph $G_N$ is given by

$$m^c_N = 4 \cdot 5^N \sum_{i=0}^{N} 5^i = 5^N (5^{N+1} - 1).$$

Proof. We prove the equation by induction.

**Base case:** When $N = 0$, there are $m^c_0 = 5^0(5^1 - 1) = 4$ non-edges which is precisely the number of non-edges in a $\Theta_{2,2,2}$.

Assume the induction hypothesis that for a particular $N$, the single case $n = N$ holds, that is,

$$m^c_N = 4 \cdot 5^N \sum_{i=0}^{N} 5^i = 5^N (5^{N+1} - 1).$$

The number of non-edges in $G_{N+1}$ is

$$m^c_{N+1} = \binom{5^{N+2}}{2} - |E(G_{N+1})|$$

The number of edges in $G_N$ is calculated as follows: There are 5 vertices and 6 edges in $\Theta_{2,2,2}$. At the level $N$ blow-up, there are $6 \cdot 5^N$ edges in each blob, and also $6 \cdot 5^{N+1}$ edges between blobs, thus giving the term $|E(G_N)| = 6 \cdot 5^N \sum_{i=0}^{N} 5^i$. Thus,

$$m^c_{N+1} = \binom{5^{N+2}}{2} - 6 \cdot 5^{N+1} \sum_{i=0}^{N+1} 5^i$$

$$= \frac{5^{N+2}(5^{N+2} - 1)}{2} - \frac{(6 \cdot 5^{N+1})(5^{N+2} - 1)}{4}$$

$$= 5^{N+1}(5^{N+2} - 1).$$

Since both the base case and inductive step has been proven as true, thus by mathematical induction $m^c_N$ holds for all $N$. 

We state the following theorem.

Theorem 2. The nested blow-up graph of a $\Theta_{2,2,2}$ has precisely $T_N = \frac{5^N}{1240}(6300 \times 5^{3N} - 2945 \times 5^{2N} + 372 \times 5^N - 3877)$ induced subgraphs isomorphic to $C_4$.

Proof. First, we will show that,

$$T_N = \begin{cases} 5 \times (T_{N-1}) + 3 \times (5^N)^4 + 6 \times (m^c_{N-1})^2 + 9 \times m^c_{N-1} \times (n_{N-1})^2, & n > 0 \\ \frac{3}{3}, & n = 0. \end{cases}$$

Since $G_0 = \Theta_{2,2,2}$, we have $T_0 = 3$. We show that we can obtain $T_N$ from $T_{N-1}$ and prove each term from $T_N$ respectively.

Note that at level $N$ blow-up, we replace each of the $5^{N+1}$ vertices in $G_{N-1}$ with $G_0$, this contributes to $5 \times T_{N-1}$ induced copies of $C_4$ at $G_N$. Thus giving the first term $5 \times T_{N-1}$.

![Figure 6: The $\Theta_{2,2,2}$ graph containing 5 blobs, with each blob labelled respectively.](image)
We know that $G_N$ has five blobs isomorphic to $G_{N-1}$. For simplicity of the rest of the proof, we will refer to the blobs as labelled $B_1, B_2, B_3, B_4$ and $B_5$, see Figure 6.

We select 4 of these 5 blobs and one vertex from each blob. There are five ways to select 4 blobs, but only 3 of these induce copies of $C_4$. Each of these three combinations of blobs contribute to 3 $C_4$ in $G_N$. There are $5^N$ vertices in each $G_{N-1}$ which gives $3 \times (5^N)^4$ choices. This results in the second term $3 \times (5^N)^4$.

For the third term $6 \times (m_{N-1}^e)^2$, we choose 1 blob. We choose a non-edge from this blob and another non-edge from any adjacent blob. There are six pairs of blobs that are adjacent, thus resulting in the term $6 \times (m_{N-1}^e)^2$.

Finally, we pick a non-edge in a blob $B$ and two vertices, one from each of two different blobs each adjacent to $B$. There are nine ways that we can select these (refer to Figure 6), namely, $\{B_1, B_2, B_3\}, \{B_1, B_4, B_5\}, \{B_2, B_3, B_4\}, \{B_1, B_2, B_4\}, \{B_1, B_3, B_5\}, \{B_2, B_3, B_5\}, \{B_2, B_4, B_5\}$, and $\{B_3, B_4, B_5\}$.

Each combination has $m_{N-1}^e$ choices of a non-edge from one blob and then a choice of a vertex from the $n_{N-1}$ vertices in each of the other two blobs. This results in the term $9 \times m_{N-1}^e \times (n_{N-1})^2$.

We now expand and simplify $T_N$ to find the recurrence relation for $T_N$. Expanding the first few terms, we obtain:

\[
T_0 = 3 \\
T_1 = 5T_0 + 3 \times 5^4 + 6 \times (m_0^e)^2 + 9 \times m_0^e \times 5^2 = 3 \times (5 + 5^4) + 6 \times (m_0^e)^2 + 9 \times 5^2 \times m_0^e \\
= 3 \times 5^4 \times \sum_{i=0}^{1} 5^i + 6 \times \sum_{i=1}^{1} (5^{i=1} \times (m_{1-i}^e)^2) + 9 \times \sum_{i=1}^{1} (5^{1+i} \times m_{1-i}^e) \\
T_2 = 5T_1 + 3 \times 5^8 + 6 \times (m_1^e)^2 + 9 \times m_1^e \times 5^4 \\
= 5 \left( 3 \times 5^4 \times \sum_{i=0}^{1} 5^i + 6 \times \sum_{i=1}^{1} (5^{i=1} \times (m_{1-i}^e)^2) + 9 \times \sum_{i=1}^{1} (5^{1+i} \times m_{1-i}^e) \right) + 3 \times 5^8 + 6 \times (m_1^e)^2 + 9 \times m_1^e \times 5^4 \\
= 3 \times 5^8 \times \sum_{i=0}^{2} 5^i + 6 \times \sum_{i=1}^{2} (5^{i=1} \times (m_{2-i}^e)^2) + 9 \times \sum_{i=1}^{2} (5^{2+i} \times m_{2-i}^e) \\
T_3 = 5T_2 + 3 \times 5^{12} + 6 \times (m_2^e)^2 + 9 \times m_2^e \times 5^6 \\
= 5 \left( 3 \times 5^8 \times \sum_{i=0}^{2} 5^i + 6 \times \sum_{i=1}^{2} (5^{i=1} \times (m_{2-i}^e)^2) + 9 \times \sum_{i=1}^{2} (5^{2+i} \times m_{2-i}^e) \right) + 3 \times 5^{12} + 6 \times (m_2^e)^2 + 9 \times m_2^e \times 5^6 \\
= 3 \times 5^{12} \times \sum_{i=0}^{3} 5^i + 6 \times \sum_{i=1}^{3} (5^{i=1} \times (m_{3-i}^e)^2) + 9 \times \sum_{i=1}^{3} (5^{3+i} \times m_{3-i}^e) \\
\vdots \\
T_N = 3 \times 5^N \times \sum_{i=0}^{N} 5^i + 6 \times \sum_{i=1}^{N} (5^{i=1} \times (m_{N-i}^e)^2) + 9 \times \sum_{i=1}^{N} (5^{N+i} \times m_{N-i}^e).
\]
We simplify for each $Q_N$, $R_N$ and $S_N$. Simplifying using geometric sum,

$$Q_N = 3 \cdot 5^N \times \sum_{i=0}^{N} 5^{3i} = 3 \cdot 5^N \times \frac{5^3(5^{3N} - 1)}{5^3 - 1} = \frac{3 \cdot 5^N}{124} (125 \times 5^{3N} - 1). \quad (4)$$

Using Lemma 2,

$$R_N = 6 \times \sum_{i=1}^{N} (5^{i-1} \times (m_{N-i}^r)^2)$$

$$= 6 \times \sum_{i=1}^{N} (5^{i-1} \times (5^{N-i}(5^{N-i+1} - 1))^2)$$

$$= 6 \cdot 5^N \left( 5 \cdot \sum_{i=1}^{N} 5^{3(N-i)} - 2 \cdot \sum_{i=1}^{N} 5^{2(N-i)} + \frac{1}{5} \cdot \sum_{i=1}^{N} 5^{N-i} \right). \quad (5)$$

Again, simplify using geometric sum,

$$R_N = 6 \cdot 5^N \left( 5 \times \frac{5^{3N} - 1}{5^3 - 1} - 2 \times \frac{5^{2N} - 1}{5^2 - 1} + \frac{1}{5} \times \frac{5^N - 1}{5 - 1} \right)$$

$$= \frac{5^N}{620} (150 \times 5^{3N} - 310 \times 5^{2N} + 186 \times 5^N - 26). \quad (5)$$

Lastly,

$$S_N = 9 \times \sum_{i=1}^{N} (5^{N+i} \times m_{i-1}^r).$$

By Lemma 2,

$$S_N = 9 \times \sum_{i=1}^{N} (5^{N+i} \times (5^{i-1} - 1))) = 9 \cdot 5^N \times \sum_{i=1}^{N} (5^{3i-1} - 5^{2i-1}) = \frac{9 \cdot 5^N}{5} \times \left( \sum_{i=1}^{N} 5^{3i} - \sum_{i=1}^{N} 5^{2i} \right).$$

Simplify using geometric sum,

$$S_N = \frac{9 \cdot 5^N}{5} \times \left( \frac{5^3(5^{3N} - 1)}{5^3 - 1} - \frac{5^2(5^{2N} - 1)}{5^2 - 1} \right) = \frac{3 \cdot 5^N}{1240} (750 \times 5^{3N} - 775 \times 5^{2N} + 25). \quad (6)$$

Thus,

$$T_N = \frac{3 \cdot 5^N}{124} (125 \times 5^{3N} - 1) + \frac{5^N}{620} (150 \times 5^{3N} - 310 \times 5^{2N} + 186 \times 5^N - 26)$$

$$+ \frac{3 \cdot 5^N}{1240} (750 \times 5^{3N} - 775 \times 5^{2N} + 25)$$

$$= \frac{5^N}{1240} (6300 \times 5^{3N} - 2945 \times 5^{2N} + 372 \times 5^N - 7). \quad (7)$$

4 Conclusion

In this paper, we gave exact counts of $C_4$s in two different graph structures: (i) the nested blow-up graphs of $C_4$s and (ii) the theta graph $\theta_{2,2,2}$. Previously, only bounds were found for the blow-ups of graphs [14]. We improved the bounds to give the exact counts of such graphs. In a general case, we can adapt a similar formula to construct the equations for blow-up graphs of higher order $k$s, to find the exact counts of cycles of higher order $k$. Future direction of this work could include finding a generalised formula for any types of blow-up graphs, with some cyclic property.
References

[1] F. Adriaens, C. Aslay, T. De Bie, A. Gionis, and J. Lijffijt. Discovering interesting cycles in directed graphs. In Proceedings of the 28th ACM International Conference on Information and Knowledge Management, page 1191–1200. Association for Computing Machinery, 2019.

[2] N. Alon, R. Yuster, and U. Zwick. Finding and counting given length cycles. Algorithmica, 17:209–223, 1997.

[3] F. Alvelos, X. Klimentova, A. Rais, and A. Viana. Maximizing expected number of transplants in kidney exchange programs. Electron. Notes Discrete Math., 52:269–276, 2016.

[4] P. Biró, D.F. Manlove, and R. Rizzi. Maximum weight cycle packing in directed graphs, with application to kidney exchange programs. Discrete Math. Algorithms Appl., 1(4):499–517, 2009.

[5] J.A. Bondy. Counting subgraphs a new approach to the caccetta-häggkvist conjecture. Discrete Math., 165-166:71–80, 1997.

[6] H. Hatami, J. Hirst, and S. Norine. The inducibility of blow-up graphs. J. Combin. Theory Ser. B, 109:196–212, 2014.

[7] R. Johansson. Triangle-factors in a balanced blown-up triangle. Discrete Math., 211(1-3):249–254, 2000.

[8] M. Karimi and A.H. Banihashemi. Message-passing algorithms for counting short cycles in a graph. IEEE Trans. Commun., 61(2):485–495, 2013.

[9] T. Kavitha, C. Liebchen, K. Mehlhorn, D. Michail, R. Rizzi, T. Ueckerdt, and K.A. Zweig. Cycle bases in graphs characterization, algorithms, complexity, and applications. Comput. Sci. Rev., 3(4):199–243, 2009.

[10] N. Linial and A. Morgenstern. On the number of 4-cycles in a tournament. J. Graph Theory, 83(3):266–276, 2016.

[11] H. Liu and J. Wang. A new way to enumerate cycles in graph. In Advanced Int’l Conference on Telecommunications and Int’l Conference on Internet and Web Applications and Services (AICT-ICIW’06), pages 57–57, 2006.

[12] C.S. Oliveira, L. Silva de Lima, and V. Nikiforov. Spectra of blow-up graphs. arXiv, 2014.

[13] J.P. Pedroso. Maximizing expectation on vertex-disjoint cycle packing. In Computational Science and Its Applications (ICCSA 2014), page 32–46, 2014.

[14] N. Pippenger and M.C. Golumbic. The inducibility of graphs. J. Combin. Theory Ser. B, 19 (3):189–283, 1975.

[15] M.H. Safar, I.Y. Sorkhoh, H.M. Farah, and K.A. Mahdi. On maximizing the entropy of complex networks. Procedia Computer Science, 5:480–488, 2011.

[16] D. Zelazo, S. Schuler, and F. Allgöwer. Performance and design of cycles in consensus networks. Systems & Control Letters, 62(1):85–96, 2013.