The determination of a doubly resolving set with the minimum size for $C_n\square P_k$
and some minimal resolving parameters for Double Graph of $C_n\square P_k$

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Abstract

Applications of resolving sets in graph theory and chemistry have a long history, and if we consider a graph as a chemical compound then the determination of a doubly resolving set with the minimum size is very useful to analysis of chemical compound. In this work, we will consider the computational study of a doubly resolving set of the cartesian product $C_n\square P_k$ and its double graph of the cartesian product $C_n\square P_k$, will be denoted by $D[C_n\square P_k]$. Indeed, we will show that if $n$ is an even or odd integer, then the minimum size of a doubly resolving set in $C_n\square P_k$ is 3, and more we compute some minimal resolving parameters for the double graph of the cartesian product $C_n\square P_k$. In particular, we will show that if $n$ is an even or odd integer, then the minimum size of a doubly resolving set in $D[C_n\square P_k]$ is 4.

Keywords: cartesian product, double graph, resolving set, doubly resolving set.

1. Introduction

All graphs considered in this work are assumed to be finite and connected. A graphical representation of a vertex $v$ of a connected graph $G$ relative to an arranged subset $W = \{w_1, ..., w_k\}$ of vertices of $G$ is defined as the $k$-tuple $(d(v, w_1), ..., d(v, w_k))$, and this $k$-tuple is denoted by $r(v|W)$, where $d(v, w_i)$ is considered as the minimum distance of a shortest path from $v$ to $w_i$. If any vertices $u$ and $v$ that belong to $V(G) - W$ have various representations with respect to the set $W$, then $W$ is called a resolving set for $G$ [6]. Slater [24] considered the concept and notation of the metric dimension problem under the term locating set. Also, Harary and Melter [13] considered these problems under the term metric dimension as follows: A resolving set of the minimum size or cardinality is called the metric dimension of $G$ and this minimum size denoted by $\beta(G)$. Resolving parameters in graphs have been studied in [1, 4, 5, 16, 17, 18, 19, 25].

Cáceres [7] considered the concept and notation of a doubly resolving set of graph $G$, and we can see that a subset $W = \{w_1, w_2, ..., w_k\}$ of vertices of a graph $G$ is a doubly resolving set of $G$ if for any various vertices $x, y \in V(G)$ we have $r(x|W) \neq r(y|W)$, where $I$ is an integer, and $I$ indicates the unit $I$-vector $(1, ..., 1)$, see [2]. Doubly resolving sets have played a special role in the study of resolving sets. In particular, a doubly resolving set in graph $G$ with the minimum size, is denoted by $\psi(G)$. The applications of above concepts and related parameters are very useful to analysis of a chemical compound and note that these problems are NP hard, see [3, 8, 9, 10, 15].

The cartesian product of two graphs $G$ and $H$, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and with edge set $E(G \times H)$ so that $(g_1, h_1)(g_2, h_2) \in E(G \square H)$, whenever $h_1 = h_2$ and $g_1g_2 \in E(G)$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$. The double graph of graph $G$ is denoted by $D(G)$, as a graph obtained by taking two copies of $G$ and joining each vertex in one copy with the neighbours of corresponding vertex in another copy, see [11, 12, 20, 21]. It is easy to verify that, a graph $G$ is bipartite, and connected if and only if $D(G)$ is bipartite, and connected, respectively.

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Now, we use $C_n$ and $P_k$ to denote the cycle on $n \geq 3$ and the path on $k \geq 3$ vertices, respectively. In this article, we will consider the computational study of a doubly resolving set of the cartesian product $C_n \square P_k$ and its double graph of the cartesian product $C_n \square P_k$. Indeed, in section 3.1, we define a graph isomorphic to the cartesian product $C_n \square P_k$, and we will consider the determination of a doubly resolving set with the minimum size of the cartesian product $C_n \square P_k$. In particular, in section 3.2, we define the double graph of the cartesian product $C_n \square P_k$, will be denoted by $D[C_n \square P_k]$, and more we compute some resolving parameters for the double graph of the cartesian product $C_n \square P_k$. More details about this graph are given in section 3.2.

2. Definitions and Preliminaries

**Definition 2.1.** Consider two graphs $G$ and $H$. If there is a bijection, $\theta : V(G) \rightarrow V(H)$ so that $u$ is adjacent to $v$ in $G$ if and only if $\theta(u)$ is adjacent to $\theta(v)$ in $H$, then we say that $G$ and $H$ are isomorphic.

**Definition 2.2.** [23] Let $G$ be a graph. A vertex $w$ of $G$ strongly resolves two vertices $u$ and $v$ of $G$ if $u$ belongs to a shortest $v - w$ path or $v$ belongs to a shortest $u - w$ path. A set $W = \{w_1, w_2, \ldots, w_n\}$ of vertices of $G$ is a strong resolving set of $G$ if every two distinct vertices of $G$ are strongly resolved by some vertex of $W$. A strong resolving set of the minimum size is called the strong metric dimension of $G$ and this minimum size denoted by $sdm(G)$.

**Remark 2.1.** Suppose that $n$ is an even natural number greater than or equal to 6 and $G$ is the cycle graph $C_n$. Then $\beta(G) = 2$, $\psi(G) = 3$ and $sdm(G) = \lceil \frac{n}{2} \rceil$.

**Remark 2.2.** Suppose that $n$ is an odd natural number greater than or equal to 3 and $G$ is the cycle graph $C_n$. Then $\beta(G) = 2$, $\psi(G) = 2$ and $sdm(G) = \lceil \frac{n}{2} \rceil$.

**Theorem 2.1.** Suppose that $n$ is an odd integer greater than or equal to 3. Then the minimum size of a resolving set in the cartesian product $C_n \square P_k$ is 2.

**Theorem 2.2.** Suppose that $n$ is an even integer greater than or equal to 4. Then the minimum size of a resolving set in the cartesian product $C_n \square P_k$ is 3.

**Theorem 2.3.** If $n$ is an even or odd integer is greater than or equal to 3, then the minimum size of a strong resolving set in the cartesian product $C_n \square P_k$ is $n$.

3. Main Results

3.1. The determination of a doubly resolving set with the minimum size for $C_n \square P_k$

Although, some resolving parameters such as the minimum size of resolving sets and the minimum size of strong resolving sets calculated for the cartesian product $C_n \square P_k$, see [7, 22], but in this section we will determine the minimum size of a doubly resolving set in $C_n \square P_k$. Thus for this purpose, we first label the vertices of the $C_n \square P_k$ in a way that helps us and we introduce some notation which is used throughout this section. Suppose $n$ and $k$ are natural numbers greater than or equal to 3, and $[n] = \{1, \ldots, n\}$. Now, suppose that $G$ is a graph with vertex set $\{x_1, \ldots, x_{nk}\}$ on layers $V_1, V_2, \ldots, V_k$, where $V_p = \{x_{(p-1)n+1}, x_{(p-1)n+2}, \ldots, x_{(p-1)n+n}\}$ for $1 \leq p \leq k$, and the edge set of graph $G$ is $E(G) = \{x_i x_j \mid x_i, x_j \in V_p, 1 \leq i < j \leq nk, j - i = 10rj - i = n - 1\} \cup \{x_i x_j \mid x_i \in V_p, x_j \in V_{p+1}, 1 \leq i < j \leq nk, 1 \leq q \leq k - 1, j - i = n\}$. We can see that this graph is isomorphic to the cartesian product $C_n \square P_k$. So, we can assume throughout this article $V(C_n \square P_k) = \{x_1, \ldots, x_{nk}\}$. Now, in this section, we give a more elaborate description of the cartesian product $C_n \square P_k$, that are required to prove of Theorems. We use $V_p$, $1 \leq p \leq k$, to indicate a layer of the cartesian product $C_n \square P_k$, where $V_p$ is defined already. Also, for every two vertices $x_i$ and $x_j$ in $C_n \square P_k$, we say that $x_i$ and $x_j$ are compatible in $C_n \square P_k$, if $n|j - i$. We can see that the degree of a vertex in the layers $V_1$ and $V_4$ is 3, also the degree of a vertex in the layer $V_5$ is $1 < p < k$ is 4, and hence $C_n \square P_k$ is not regular. We say that two layers of $C_n \square P_k$ are congruous, if the degree of compatible vertices in two layers are identical. Note that, if $n$ is an even natural number, then $C_n \square P_k$ contains no cycles of odd length, and hence in this case $C_n \square P_k$ is bipartite. For more result of families of graphs with constant metric, see [3, 14]. The cartesian product $C_5 \square P_2$ is depicted in Figure 1.
Theorem 3.1. Consider the cartesian product $C_n \Box P_k$. If $n$ is an odd integer greater than or equal to 3, then the minimum size of a doubly resolving set in the cartesian product $C_n \Box P_k$ is 3.

Proof. In the following cases we show that the minimum size of a doubly resolving set in the cartesian product $C_n \Box P_k$ is 3.

Case 1. First, we show that the minimum size of a doubly resolving set in $C_n \Box P_k$ must be greater than 2. Consider the cartesian product $C_n \Box P_k$ with the vertex set \{x_1, ..., x_{nk}\} on the layers $V_1, V_2, ..., V_k$, which is defined already. Based on Theorem 2.1, we know that $\beta(C_n \Box P_k) = 2$. We can show that if $n$ is an odd integer then all the elements of every minimum resolving set of $C_n \Box P_k$ must lie in exactly one of the congruous layers $V_1$ or $V_k$. Without lack of theory if we consider the layer $V_1$ of the cartesian product $C_n \Box P_k$ then we can show that all the minimum resolving sets in the layer $V_1$ of $C_n \Box P_k$ are the sets as to form $M_i = \{x_i, x_{\lceil \frac{n}{2}\rceil + i - 1}\}, 1 \leq i \leq \lceil \frac{n}{2}\rceil$ and $N_j = \{x_j, x_{\lceil \frac{n}{2}\rceil + j}\}, 1 \leq j \leq \lfloor \frac{n}{2}\rfloor$. On the other hand, we can see that the arranged subsets $M_i$ cannot be doubly resolving sets for $C_n \Box P_k$ because for $1 \leq i \leq \lceil \frac{n}{2}\rceil$ and two compatible vertices $x_{i+n}$ and $x_{i+2n}$ with respect to $x_i$, we have $r(x_{i+n}|M_i) - r(x_{i+2n}|M_i) = -I$, where $I$ indicates the unit 2-vector (1, 1). By applying the same argument we can show that the arranged subsets $N_j$ cannot be doubly resolving sets for $C_n \Box P_k$. Hence, the minimum size of a doubly resolving set in $C_n \Box P_k$ must be greater than 2.

Case 2. Now, we show that the minimum size of a doubly resolving set in the cartesian product $C_n \Box P_k$ is 3. For $1 \leq i \leq \lceil \frac{n}{2}\rceil$, let $x_i$ be a vertex in the layer $V_1$ of $C_n \Box P_k$ and $x_c$ be a compatible vertex with respect to $x_i$, where $x_c$ lie in the layer $V_k$ of $C_n \Box P_k$, then we can show that the arranged subsets $M_i = M_i \cup x_c = \{x_i, x_{\lceil \frac{n}{2}\rceil + i - 1}, x_c\}$ of vertices in the cartesian product $C_n \Box P_k$ are the minimum doubly resolving sets for the cartesian product $C_n \Box P_k$. It will be
enough to show that for any compatible vertices \( x_0 \) and \( x_d \) in \( C_n \square P_k \), \( r(x_0|A_l) - r(x_d|A_l) \neq \lambda I \). Suppose \( x_0 \in V_p \) and \( x_d \in V_q \) are compatible vertices in the cartesian product \( C_n \square P_k \), \( 1 \leq p < q \leq k \). Hence, \( r(x_0|M_l) - r(x_d|M_l) \neq -\lambda I \), where \( \lambda \) is a positive integer, and \( I \) indicates the unit 2-vector (1, 1). Also, for the compatible vertex \( x_i \) with respect to \( x_0 \), \( r(x_i|x_0) - r(x_d|x_0) \neq \lambda I \). So, \( r(x_0|A_l) - r(x_d|A_l) \neq \lambda I \), where \( I \) indicates the unit 3-vector (1, 1, 1). Especially, for \( 1 \leq j \leq \lceil \frac{q}{2} \rceil \) if we consider the arranged subsets \( B_j = N_j \cup x = \{x_j, x_{j+p}, x_d\} \) of vertices in the cartesian product \( C_n \square P_k \), where \( x_j \) lie in the layer \( V_j \) of the cartesian product \( C_n \square P_k \) and \( x_d \) is a compatible vertex with respect to \( x_j \), then by applying the same argument we can show that the arranged subsets \( B_j = N_j \cup x = \{x_j, x_{j+p}, x_d\} \) of vertices in the cartesian product \( C_n \square P_k \) are the minimum doubly resolving sets for the cartesian product \( C_n \square P_k \).

### Theorem 3.2
Suppose that \( n \) is an even integer greater than or equal to 4. Then the minimum size of a doubly resolving set in the cartesian product \( C_n \square P_k \) is 3.

**Proof.** Consider the cartesian product \( C_n \square P_k \) with the vertex set \( \{x_1, \ldots, x_{nk}\} \) on the layers \( V_1, V_2, \ldots, V_k \), which is defined already. Based on Theorem 2.2, if \( n \) is even, then \( \beta(C_n \square P_k) = 3 \) and it is well known that \( \beta(C_n \square P_k) \leq \psi(C_n \square P_k) \). Especially, we show that \( \psi(C_n \square P_k) = 3 \). Suppose \( S_1 = \{x_1, x_2\} \) is a set of vertices in the layer \( V_1 \) of the cartesian product \( C_n \square P_k \) and \( x_1 \) is a compatible vertex with respect to \( x_0 \), where \( x_0 \) lies in the layer \( V_0 \) of the cartesian product \( C_n \square P_k \). We can show that the arranged subset \( S_2 = S_1 \cup x = \{x_1, x_2, x_0\} \) of vertices in the cartesian product \( C_n \square P_k \) is one of the minimum resolving sets for the cartesian product \( C_n \square P_k \). In particular, we show that the arranged subset \( S_2 = S_1 \cup x = \{x_1, x_2, x_0\} \) of vertices in the cartesian product \( C_n \square P_k \) is one of the minimum doubly resolving sets for the cartesian product \( C_n \square P_k \). It will be enough to show that for any compatible vertices \( x_j \) and \( x_d \) in \( C_n \square P_k \), \( r(x_j|S_2) - r(x_d|S_2) \neq \lambda I \). Suppose \( x_j \in V_p \) and \( x_d \in V_q \) are compatible vertices in the cartesian product \( C_n \square P_k \), \( 1 \leq p < q \leq k \). Hence, \( r(x_j|S_2) - r(x_d|S_2) \neq -\lambda I \), where \( \lambda \) is a positive integer, and \( I \) indicates the unit 2-vector (1, 1). Also, for \( x_j \in S_2 \), \( r(x_j|x_0) - r(x_d|x_0) = \lambda I \). So, \( r(x_j|S_2) - r(x_d|S_2) \neq \lambda I \), where \( I \) indicates the unit 3-vector (1, 1, 1).

### Remark 3.1
It is noteworthy that, if \( n \) is an odd integer greater then 3, then by the similar manner which is done in the previous Theorem we can show that the arranged subset \( S_2 = S_1 \cup x = \{x_1, x_2, x_0\} \) of vertices in the cartesian product \( C_n \square P_k \) is also one of the minimum doubly resolving sets for the cartesian product \( C_n \square P_k \), where the set \( S_2 \) is defined in the previous Theorem.

### Lemma 3.1
If \( n \) is an even or odd integer is greater than or equal to 3, then the minimum size of a strong resolving set in the cartesian product \( C_n \square P_k \) is \( n \).

**Proof.** Although, the minimum size of strong resolving sets in the cartesian product \( C_n \square P_k \) calculated, but by another way we show that the minimum size of a strong resolving set in the cartesian product \( C_n \square P_k \) is \( n \). Suppose \( T_1 = V_2 \cup \ldots \cup V_{k-1} \) is an arranged subset of vertices in \( C_n \square P_k \), where \( V_p \cup 2 \leq p \leq k-1 \) which is defined already. If \( k = 3 \) then \( T_1 = V_2 \) cannot be a resolving set for \( C_n \square P_k \). If \( k \geq 4 \) then we can prove that the set \( T_1 \) is a resolving set for \( C_n \square P_k \). Now, by considering various vertices \( x_{1} \in V_1 \) and \( x_{m} \in V_1, n(k-1) + 1 \leq m \leq nk \), there is not a \( w \) \in \( T_1 \) so that \( x_1 \) belongs to a shortest \( x_m - w \) path or \( x_m \) belongs to a shortest \( x_1 - w \) path. Thus \( T_1 = V_2 \cup \ldots \cup V_{k-1} \) cannot be a strong resolving set for \( C_n \square P_k \). Now, suppose that \( T_2 = \{x_1, \ldots, x_{nk}\} \) of vertices in \( C_n \square P_k \) cannot be a strong resolving set for \( C_n \square P_k \). Hence, if \( T \) is a strong resolving set in \( C_n \square P_k \), then the minimum size of \( T \) must be greater than or equal to \( n \). So, suppose that \( T = \{x_1, \ldots, x_{nk}\} \) of vertices in the layer \( V_1 \) of the cartesian product \( C_n \square P_k \), we prove that this subset is a strong resolving set in \( C_n \square P_k \). For \( 1 < p < q \leq k \), if both vertices \( x_0 \in V_p \) and \( x_d \in V_q \) are compatible in \( C_n \square P_k \) relative to \( x_0 \), then \( x_0 \) belongs to a shortest \( x_0 - x_d \) path. For \( 1 < p < q \leq k \), if both vertices \( x_0 \in V_p \) and \( x_d \in V_q \) are not compatible in \( C_n \square P_k \) and lie in various layers in \( C_n \square P_k \), then there is exactly one compatible vertex \( x_0 \) relative to \( x_0 \). So, suppose that \( x_0 \in V_1 \) to the layer \( V_1 \) say \( x_1 \) so that \( x_1 \) belongs to a shortest \( x_1 - x_0 \) path. Thus the
set $T = \{x_1, ..., x_d\}$ is one of the minimum strong resolving sets for $C_n \square P_k$, and hence the minimum size of a strong resolving set in the cartesian product $C_n \square P_k$ is $n$.

3.2. The determination of some minimal resolving parameters for Double Graph of $C_n \square P_k$

Consider the cartesian product $C_n \square P_k$ with the vertex set $\{x_1, ..., x_{dk}\}$ on the layers $V_1, V_2, ..., V_k$, where $V_p, 1 \leq p \leq k$, which is defined in section 3.1. If we consider one copy of the cartesian product $C_n \square P_k$ with the vertex set $\{y_1, ..., y_{dk}\}$ on layers $U_1, U_2, ..., U_k$, where it can be defined $U_p$ as similar $V_p$ on the vertex set $\{y_1, ..., y_{dk}\}$. Now we define the double graph of the cartesian product $C_n \square P_k$ as follows: The double graph of the cartesian product $C_n \square P_k$, is denoted by $D(C_n \square P_k)$ as the vertex set $V(D(C_n \square P_k)) = \{x_1, ..., x_{dk}\} \cup \{y_1, ..., y_{dk}\}$. By definition of double graph for $1 \leq i \leq nk$, the vertex $x_i$ is adjacent to $y_j$ in $D(C_n \square P_k)$. So, we can assume that $D(C_n \square P_k)$ contains $k$ layers $Z_1, ..., Z_k$, where $Z_p = V_p \cup U_p$, $1 \leq p \leq k$; also $V_p$ and $U_p$, denote internal and external layers of $D(C_n \square P_k)$, on the sets $\{x_1, ..., x_{dk}\}$ and $\{y_1, ..., y_{dk}\}$, respectively. In particular, we can see that the degree of a vertex in the layer $Z_p$ is 5, and hence $D(C_n \square P_k)$ is not regular. In this section, we compute some resolving parameters for $D(C_n \square P_k)$.

**Theorem 3.3.** If $n$ is an odd integer greater than or equal to 3, then the minimum size of a resolving set in $D(C_n \square P_k)$ is 3.

**Proof.** Suppose $V(D(C_n \square P_k)) = \{x_1, ..., x_{dk}\} \cup \{y_1, ..., y_{dk}\}$. Based on Theorem 2.1, we know that if $n$ is an odd integer greater than or equal to 3, then the minimum size of a resolving set in $C_n \square P_k$ is 2. Also, by definition of double graph of the cartesian product $C_n \square P_k$ we can verify that for $1 \leq t \leq nk$, every vertex $y_t$ is adjacent to $x_t$, and hence none of minimal resolving sets of $C_n \square P_k$ can be a resolving set for $D(C_n \square P_k)$. Therefore, the minimum size of a resolving set in $D(C_n \square P_k)$ must be greater than 2.

Now, we show that the minimum size of a resolving set in $D(C_n \square P_k)$ is 3. For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, let $x_i$ be a vertex in the internal layer $V_i$ of $D(C_n \square P_k)$, and $x_i$ be a compatible vertex with respect to $x_i$, where $x_i$ lie in the internal layer $V_i$ of $D(C_n \square P_k)$.

Based on Theorem 3.1, we know that the arranged subsets $A_i = \{x_i, x_t^{i+1}, x_i\}$ of vertices in internal layers of $D(C_n \square P_k)$ are resolving sets for internal layers of $D(C_n \square P_k)$, and hence the arranged subsets $A_i = \{x_i, x_t^{i+1}, x_i\}$ are the minimum resolving sets for $D(C_n \square P_k)$ because for every vertex $x_i$ in external layer of $D(C_n \square P_k)$, we have $r(y_j|A_i) = (d(x_i, x_j) + 1, d(x_t^{i+1}, x_j) + 1, d(x_i, x_j) + 1)$, so all the vertices in the external layers $U_p$ have various representations with respect to the sets $A_i$. In the same way for $1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil$, if we consider the arranged subsets $B_j = N_j \cup x_c = \{x_j, x_t^{j+1}, x_c\}$ of vertices in internal layers of $D(C_n \square P_k)$, where $x_c$ lie in the internal layer $V_c$ of $D(C_n \square P_k)$ and $x_i$ be a compatible vertex with respect to $x_i$, then by applying the same argument we can show that the arranged subsets $B_j = N_j \cup x_c = \{x_j, x_t^{j+1}, x_c\}$ of vertices in internal layers of $D(C_n \square P_k)$ are the minimum resolving sets for $D(C_n \square P_k)$.

**Lemma 3.2.** If $n$ is an odd integer greater than or equal to 3, then the minimum size of a doubly resolving set in $D(C_n \square P_k)$ is greater than 3.

**Proof.** Suppose $V(D(C_n \square P_k)) = \{x_1, ..., x_{dk}\} \cup \{y_1, ..., y_{dk}\}$. For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, let $x_i$ be a vertex in the internal layer $V_i$ of $D(C_n \square P_k)$ and $x_i$ be a compatible vertex with respect to $x_i$, where $x_c$ lie in the internal layer $V_c$ of $D(C_n \square P_k)$. Based on proof of Theorem 3.3, we know that the arranged subsets $A_i = M_i \cup x_c = \{x_i, x_t^{i+1}, x_i\}$ of vertices in internal layers of $D(C_n \square P_k)$ cannot be doubly resolving sets for $D(C_n \square P_k)$ because $r(y_j|A_i) = (d(x_i, x_j) + 1, d(x, x_t^{i+1}) + 1, d(x, x_j) + 1)$. In the same way for $1 \leq j \leq \left\lceil \frac{n}{2} \right\rceil$, if we consider the arranged subsets $B_j = N_j \cup x_c = \{x_j, x_t^{j+1}, x_c\}$ of vertices in internal layers of $D(C_n \square P_k)$, where $x_c$ lie in the internal layer $V_c$ of $D(C_n \square P_k)$ and $x_c$ is a compatible vertex with respect to $x_c$, then we can show that the arranged subsets $B_j$ cannot be doubly resolving sets for $D(C_n \square P_k)$. Hence the minimum size of a doubly resolving set in $D(C_n \square P_k)$ is greater than 3.

**Lemma 3.3.** If $n$ is an even integer greater than or equal to 4, then the minimum size of a resolving set in $D(C_n \square P_k)$ is greater than 3.

**Proof.** Suppose $V(D(C_n \square P_k)) = \{x_1, ..., x_{dk}\} \cup \{y_1, ..., y_{dk}\}$. Based on Theorem 2.2, we know that if $n$ is an even integer greater than or equal to 4, then the minimum size of a resolving set in $C_n \square P_k$ is 3. By the same manner which is done in Theorem 3.3, we can show that the minimum size of a resolving set in $D(C_n \square P_k)$ must be greater than 3.
Theorem 3.4. If \( n \) is an even or odd integer greater than or equal to 3, then the minimum size of a doubly resolving set in \( D[C_n \square P_3] \) is 4.

Proof. Based on Theorem 3.3, we know that if \( n \) is an even integer greater than or equal to 4, then \( \beta(D[C_n \square P_3]) \geq 3 \). Also based on Theorem 3.3, we know that if \( n \) is an odd integer greater than or equal to 3, then \( \beta(D[C_n \square P_3]) = 3 \). Thus the minimum size of a doubly resolving set in \( D[C_n \square P_3] \) is greater than 3. In particular, it is well known that \( \beta(D[C_n \square P_3]) \leq \psi(D[C_n \square P_3]) \). Now, we show that if \( n \) is an even or odd integer greater than or equal to 3, then the minimum size of a doubly resolving set in \( D[C_n \square P_3] \) is 4. Let \( S_2 = \{x_1, x_2, x_3\} \) be an arranged subset of vertices in internal layers of \( D[C_n \square P_3] \), where \( x_c \in V_k \) is a compatible vertex with respect to \( x_1 \) and suppose that \( S_3 = S_2 \cup y_c = \{x_1, x_2, x_3, y_1\} \) is an arranged subset of vertices in \( D[C_n \square P_3] \) such that the vertex \( y_c \) lies in the external layer \( U_k \) and \( y_1 \) is adjacent to \( x_3 \). It will be enough to show that for \( 1 \leq t \leq nk \), every two adjacent vertices \( x_t \) and \( y_t \) so that \( x_t \) and \( y_t \) lie in internal and external layers of \( D[C_n \square P_3] \), respectively, \( r(x_t|S_3) - r(y_t|S_3) \neq -I \), where \( I \) indicates the unit 4-vector \((1, \ldots, 1)\). We can verify that, \( r(x_t|S_3) - r(y_t|S_3) = -I \), where \( I \) indicates the unit 3-vector, and \( r(x_t|y_1) - r(y_t|y_1) = 1 \). Therefore, the arranged subset \( S_3 \) is one of the minimum doubly resolving sets for \( D[C_n \square P_3] \). Thus the minimum size of a doubly resolving set in \( D[C_n \square P_3] \) is 4. \( \square \)

Theorem 3.5. If \( n \) is an even or odd integer greater than or equal to 3, then the minimum size of a strong resolving set in \( D[C_n \square P_3] \) is \( 2n \).

Proof. Suppose \( V(D[C_n \square P_3]) = \{x_1, \ldots, x_{nk}\} \cup \{y_1, \ldots, y_{nk}\} \) and suppose that \( O_1 = Z_2 \cup \ldots \cup Z_{k-1} \) is an arranged subset of vertices in \( D[C_n \square P_3] \), where \( Z_p \), \( 2 \leq p \leq k - 1 \) which is defined already. It is easy to verify that, the subset \( O_1 = Z_2 \cup \ldots \cup Z_{k-1} = Z_k \) is a strong resolving set for \( D[C_n \square P_3] \). By the same manner which is done in proof of the Lemma 3.1, it is also easy to verify that, every subset of vertices in the layer \( Z_t \) of \( D[C_n \square P_3] \), of cardinality \( 2n - 1 \) cannot be a strong resolving set for \( D[C_n \square P_3] \). Thus the minimum size of a strong resolving set in \( D[C_n \square P_3] \) must be greater than or equal to \( 2n \). So, suppose that \( O_2 = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \) is an arranged subset of vertices in the layer \( Z_t \) of \( D[C_n \square P_3] \), by the same manner which is done in proof of the Lemma 3.1, we again can show that the subset \( O_2 \) is a strong resolving set in \( D[C_n \square P_3] \), because for \( 1 \leq t \leq nk \), the vertex \( x_t \) is adjacent to \( y_t \), and hence the subset \( O_2 \) is one of the minimum strong resolving sets in \( D[C_n \square P_3] \). \( \square \)

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