Almost Everywhere Regularity for the Free Boundary of the Normalized p-harmonic Obstacle problem $p > 2$.

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March 10, 2022

Abstract

Let $u$ be a solution to the normalized p-harmonic obstacle problem with $p > 2$. That is, $u \in W^{1,p}(B_1(0))$, $2 < p < \infty$, $u \geq 0$ and

$$\text{div} (|\nabla u|^{p-2} \nabla u) = \chi_{\{u>0\}} \text{ in } B_1(0)$$

where $u(x) \geq 0$ and $\chi_A$ is the characteristic function of the set $A$. Our main result is that for almost every free boundary point, with respect to the $(n-1)-$Hausdorff measure, there is a neighborhood where the free boundary is a $C^{1,\beta}$-graph. That is, for $H^{n-1}$-a.e. point $x^0 \in \partial \{u > 0\} \cap B_1(0)$ there is an $r > 0$ such that $B_r(x^0) \cap \partial \{u > 0\} \in C^{1,\beta}$.

1 Introduction.

In this article we will consider the following $p-$harmonic obstacle problem

$$\text{div} (|\nabla u|^{p-2} \nabla u) = \chi_{\{u>0\}} \text{ in } D$$

$$u(x) = g(x) \geq 0 \text{ on } \partial D \tag{1}$$

where $g(x) \geq 0$ is the restriction of a $W^{1,p}(D)$ function to $\partial D$. We will denote $\Omega_u = \{u > 0\}$ and the free boundary $\Gamma = \partial \Omega \cap D$.

A solution to (1) can be found by minimizing

$$\int_D \frac{1}{p} |\nabla u|^p + \max(u,0) \, dx$$

in $K = \{u \in W^{1,p}(D); \ u-g \in W^{1,p}_0(D)\}$. It is well known that solutions to (1) are $C^{1,\alpha}_{\text{loc}}(D)$ for some $\alpha > 0$, see for instance [5].

Our main interest will be the free boundary $\Gamma = \partial \Omega \cap D$. In the special, linear, case $p = 2$ it is known that the free boundary is a $C^{1,\beta}$ (even analytic) graph around almost every point, with respect to $H^{n-1}$, of the free boundary [3]. However, the techniques to prove regularity of the free boundary in the linear case breaks down when $p \neq 2$. In particular, it seems difficult to use the strong comparison and boundary comparison methods of [3] for $p-$harmonic problems. Moreover, due to the nonlinearity of the $p-$laplace operator, the solution, $u$, solves a different PDE compared to the derivative of the solution $\frac{\partial u}{\partial x_i}$. 
New techniques are needed to investigate the free boundary for the $p$–harmonic obstacle problem for $p \neq 2$. In this article we develop these techniques for $p > 2$.

We will use a flatness improvement argument and assume that the solution is almost one dimensional:

$$\|\nabla' u\|_{L^2(\mathbb{B}_1)} \leq \epsilon_0$$

where $L^2_A$ is the normal $L^2$ space with a suitable weight and $\nabla' = (\partial_1, \partial_2, ..., \partial_{n-1}, 0)$. We prove that when (2) is satisfied for an $\epsilon_0$ small enough then, for some $\tau < 1$,

$$\|\nabla' \frac{u(sx)}{s^{p/(p-1)}}\|_{L^2(\mathbb{B}_1)} \leq \tau \|\nabla' u\|_{L^2(\mathbb{B}_1)}.$$  (3)

A standard iteration of (3) implies that the free boundary is $C^{1, \beta}$ in $B_{1/2}$.

The ideas we use to prove the $C^{1, \beta}$ regularity of the free boundary are standard flatness improvement arguments from geometric measure theory and non-linear systems in the calculus of variations. However, the techniques will look somewhat different since we are dealing with regularity of the free boundary and not with the regularity of a function.

We will assume that we have a sequence of solutions $u^j$ to (1) in $B_1(0)$ satisfying (2) with constants $\epsilon_j \to 0$. We then show, more or less, that the limit function $u^0 = \lim_{j \to \infty} u^j$ satisfies an appropriate estimate of the kind (3). It is then a matter of showing that the limit $u^j \to u^0$ is strong enough to draw the conclusion that (3) holds for all $u^j$ with small enough $\epsilon_j$. For details and precise statements we refer the reader to the main text.

The second main result of this paper is that (2) is actually satisfied at almost every free boundary point, $x^0 \in \Gamma$, in a small enough ball $B_{r(x^0)}(x^0)$. This result is more subtle than it appears to be. We know, [12] ($p > 2$, see also [14] for these results in a much more general situation), that $\Gamma$ has finite $(n-1)$–Hausdorff measure so a famous result of de Giorgi implies that the free boundary has a measure theoretic normal at a.e. free boundary point. This means that $u^0 = \lim_{r \to 0+} \frac{u(x+\eta)}{s(x+\eta)}$ will converge to a solution to (1) in $\mathbb{R}^n$ with $\{u^0 > 0\} = \{x; x \cdot \eta \geq 0\}$, that is $\Omega_{u^0}$ is a half-space and the free boundary a hyperplane. However, since the equation is not uniformly elliptic we cannot use the Cauchy-Kovalevskaya Theorem and conclude that $u^0$ is one dimensional. There should be such a simple argument to show that $u^0$ is one dimensional - but we could not find any. Instead we will construct a Carleson measure related to $u^0$ and use that to show that there is a blow-up of $u^0$ that is one dimensional. The argument is interesting in its own right and based on an idea from [7]. A Lemma from geometric measure theory (see for instance [3]), used in the free boundary context in [11], then implies that a blow-up of $u^0$ is in fact also a blow-up of $u$ at $x^0$ - at least for almost every $x^0$ in the free boundary.

Our main Theorem is

**Theorem 1.** Let $u$ be a solution to the $p$–harmonic obstacle problem, $p \geq 2$, in $B_1(0)$ then there exists an open set $\Gamma_0 \subset \Gamma$ such that $\mathcal{H}^{n-1}(\Gamma \setminus \Gamma_0) = 0$ and for every $x^0 \in \Gamma_0$ there exists an $r = r(x^0)$ such that $\Gamma_u \cap B_r(x^0)$ is a $C^{1, \alpha}$ graph.

The plan of the paper is as follows. In the next section we gather some known regularity results for $p$–harmonic obstacle problem. In section [6] we
recall some results related to blow-ups of solutions. In section 4, we construct the Carleson measure that is vital to show that (2) is satisfied in a small ball centered at a.e. free boundary point. In the following section we show that any global solution with the free boundary being a hyperplane has a blow-up that is one dimensional. In section 6, we show that we can linearize the problem and that the convergence of the linearizing sequence is strong enough. We are then ready to prove Theorem 1.

1.1 Notation.
We will use the following notation:

1. We will use $B_r(x^0) = \{ x \in \mathbb{R}^n; |x - x^0| < r \}, B_r^+(x^0) = \{ x \in \mathbb{R}^n; |x - x^0| < r, x_n \geq 0 \}$.

2. An $' \$ will indicate that the $x_n -$coordinate is excluded: $x' = (x_1, ...x_{n-1}, 0)$, $B_r'(z') = \{ x'; |x' - z'| < r \}, \nabla' = (\partial_1, ..., \partial_{n-1}, 0)$ et c.

3. On occasion we are going to use, for a unit vector $\eta \in \mathbb{R}^n$, $\nabla' \eta = \nabla - \eta(\eta \cdot \nabla)$ (the gradient operator restricted to the subspace orthogonal to $\eta$).

4. $W^{1,p}(D)$ will be the usual Sobolev space and $W^{1,p}_A(D)$ will denote the Sobolev space with weight $A(x)$; That is, the space of all weakly differentiable functions with norm $\|u\|_{W^{1,p}_A(D)}^p = \int_D A(x)|u|^pdx + \int_D A(x)|\nabla u|^pdx$.

5. By $\Omega$ and $\Gamma$ we denote the non-coincidence set $\Omega = \{ x; u(x) > 0 \}$ and the free boundary $\Gamma = \partial \Omega \cap D$. At times we will write $\Gamma_u$ and $\Omega_u$ to indicate the function whose free boundary we are considering.

6. A blow-up of $u$ at $x^0 \in D$ will be the limit of any convergent sequence $\frac{u(r_j(x^0 + x^0))}{r_j^{p/(p-1)}}$ as $r_j \to 0$.

7. Characteristic function of the set $S$ will be denoted by $\chi_S$.

2 Regularity and known results.
In this section we gather some known regularity results for $p$-harmonic obstacle problems.

The following regularity results are well known for solutions to the $p$-harmonic obstacle problem.

Lemma 1. Let $u$ be a solution to the $p$–harmonic obstacle problem in $B_2(0)$ then there exists an $\alpha > 0$ and a constant $C$ such that

$$\|u\|_{C^{1,\alpha}(B_{3/4}(0))} \leq C\|u\|_{L^\infty(B_2(0))}.$$ (4)

Furthermore there exists constants $C, c > 0$ such that if $x^0 \in \Gamma \cap B_1(0)$ then

$$c r^{p/(p-1)} \leq \sup_{B_r(x^0)} u \leq C r^{p/(p-1)}.$$ (5)
The $C^{1,\alpha}$–regularity can be found in [6]. The growth estimate [5] was proved in [11] or [4].

We will also need a weak regularity result which can be proved by a standard difference quotient argument. Later we will use the same ideas to prove Lemma 3.

Lemma 2. Assume that $u$ is a solution to the $p$-harmonic obstacle problem in $D$ and $D \subset \subset D$. Let $\delta = \text{dist}(D, D^c)$ and $\psi$ be a standard cut off function: $\psi \in C^\infty_c(D)$, $\psi = 1$ in $D$ and $|\nabla \psi| \leq \frac{C}{\delta}$.

Then there exists a constant $C_1 = C_1(n, p)$ depending only on the dimension and $p$ such that

$$\int_D \psi^2 |\nabla u|^{p-2} \left| \frac{\partial u}{\partial x_i} \right|^2 \, dx \leq \frac{C_1 C_0^2}{\delta^2} \int_{D \setminus D} \psi^2 |\nabla u|^{p-2} \left| \frac{\partial u}{\partial x_i} \right|^2 \, dx.$$

2.1 Eigenfunction Expansions for the Linearized problem.

Consider the Rayleigh functional

$$J(u) = \frac{\int_{\partial B_1^+} |x_n|^{-p-2} |\nabla \phi u|^2}{\int_{\partial B_1^+} |x_n|^{-p-2} u^2}$$

(6)

defined on all functions in $W^{1,2}(\partial B_1^+, |x_n|^{(p-2)/(p-1)})$ that vanish on $\{x_n = 0\}$. Here $\nabla \phi$ is the gradient restricted to the sphere.

Notice that there is not any problem to define the boundary values of $u \in W^{1,2}(\partial B_1^+, |x_n|^{(p-2)/(p-1)})$ in the sense of traces since $W^{1,1}(\partial B_1^+(0))$ has a trace operator and for every $u \in W^{1,2}(\partial B_1^+, |x_n|^{(p-2)/(p-1)})$

$$\int_{\partial B_1^+} |\nabla \phi u| dx \leq \left( \int_{\partial B_1^+} |x_n|^{-p-2} \right)^\frac{1}{p} \left( \int_{\partial B_1^+} |x_n|^{-p-2} |\nabla u|^2 \right)^\frac{1}{2} < \infty.$$

A minimizer to (6) may be extended homogeneously to a global solution in $\mathbb{R}^n_+$ to

$$\text{div} \left( |x_n|^{-p-2} \nabla u \right) = 0 \quad \text{in } \mathbb{R}^n_+$$

$$u(x) = 0 \quad \text{on } x_n = 0.$$

Using a standard argument from functional analysis, see appendix D in [9], gives the following lemma.

Lemma 3. Let $v$ be a solution to

$$\text{div} \left( |x_n|^{-p-2} \nabla u \right) = 0 \quad \text{in } B_1^+(0)$$

$$v(x) = 0 \quad \text{on } \{x_n = 0\}$$

(7)

then there is a sequence of solutions $q_j$ to (6) such that

1. $q_j$ is $\lambda_j$–homogeneous,
2. $\frac{1}{p-1} = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \ldots$,
3. $\int_{B_1^+} |x_n|^{-p-2} \nabla q_i \cdot \nabla q_j = 0$ for $i \neq j$,
4. $v(x) = \sum_{j=1}^\infty a_j q_j(x)$ for some constants $a_j$. 

3 Blow-ups

In this section we analyze the blow-ups of the solutions. One of the main goals of this paper is to show that blow-ups are unique, that is that $u^0$ is independent of the sequence $r_j \to 0$. In this section we will gather some known results. But we will begin by define what we mean by a blow-up.

**Definition 1.** If $u$ is a solution to the $p$-harmonic obstacle problem in $B_1(0)$ and $x^0 \in \Gamma_u$. Then we say that $u^0(x)$ is a blow-up of $u$ at $x^0$ if there is a sequence $r_j \to 0$ such that

$$
\lim_{j \to \infty} \frac{u(r_j x + x^0)}{r_j^{p-1}} = u^0(x),
$$

where the limit is considered weakly in $W_{loc}^{1,p}(\mathbb{R}^n)$.

We denote the set of blow-ups of $u$ at $x^0$ by $\text{Blo}(u,x^0)$. That is

$$
\text{Blo}(u,x^0) = \left\{ u^0(x); \exists r_j \to 0, u^0(x) = \lim_{j \to \infty} \frac{u(r_j x + x^0)}{r_j^{p-1}} \right\}.
$$

The following properties (in Lemma 4 and 5) are well known for blow-ups.

**Lemma 4.** Suppose that $u$ solves the $p$-harmonic obstacle problem in $B_1(0)$ and $x^0 \in \Gamma_u$. Then for any sequence $r_j \to 0$ there exists a subsequence $r_{j_k} \to 0$ such that

$$
\lim_{k \to \infty} \frac{u(r_{j_k} x + x^0)}{r_{j_k}^{p-1}} = u^0(x),
$$

exists. That is $\text{Blo}(u,x^0)$ is not empty. If we call the limit $v(x)$ then $v \neq 0$ and $v$ is a solution to the $p$-harmonic obstacle problem.

Again see [11] or [4] for a proof.

**Lemma 5.** Let $u$ be a solution to (1). Then

1. The free boundary $\Gamma_u$ has locally finite $(n-1)$-Hausdorff measure $\mathcal{H}^{n-1}$.
2. The set $\{u > 0\}$ has locally finite perimeter.
3. For $\mathcal{H}^{n-1}$-a.e. point $x^0 \in \Gamma_u$ there exists a measure theoretic normal $\eta_{x^0}$ such that

$$
\Omega_{x^0} = \left\{ \frac{u(rx + x^0)}{r^{p-1}} > 0 \right\} \to \{x, x \cdot \eta \geq 0\}
$$

in the sense of Hausdorff convergence.

The first two points can be found in [12] or in [4] for the last point see [11]. The following Lemma was proved in [1] for a slightly different problem. But the proof is line for line the same for $p$-harmonic obstacle problems, there are some slight typos in the proof in [1] and the reader might benefit from considering the forthcoming paper [2] where the some more details are given. The proof is based on ideas from geometric measure theory, see for instance [8].
Lemma 6. Suppose that \( u \) is a solution to the \( p \)-harmonic obstacle problem. Then for \( \mathcal{H}^{n-1} \)-almost every free boundary point \( x^0 \) it holds that if \( u^{x^0} \in \text{Blo}(u, x^0) \) and \( x^1 \in \Gamma_{u, x^0} \) then

\[
\lim_{r \to 0} \frac{u^{x^0}(rx + x^1)}{r^{\frac{p}{p-1}}} \in \text{Blo}(x^0, u).
\]

Remark: Informally the Lemma states that “Blow-ups of blow-ups at a point \( x^0 \) are also blow-ups at \( x^0 \).”

4 A Carleson measure.

In this section we prove a simple, but technical, lemma. Later it will be used to prove that any global solution, with free boundary being a hyperplane, has a blow-up that is one dimensional. In particular, if the functions \( g^j \) in Lemma 6 are derivatives \( \frac{\partial^m u}{\partial x_1^m} \), \( j = 1, 2, ..., n - 1 \), then the lemma states that there is a blow-up sequence \( u^{(r_x x + y^k)} \rightarrow u^0 \) such that \( u^0 \) is independent of the \( x' \) variables (see Proposition 4).

The proof is very similar to a proof found in [7] relating to the Mumford-Shah problem.

Lemma 7. If \( g^j(x) \in L^2(Q^+_t) \), for \( j = 1, 2, ..., N \), are \( N \) functions satisfying, for every \( t \in (0, 1) \), \( \|g^j(t, t)\|_{L^2(Q^+_t \times \{x_1 = 1\})} \leq C_0 \sqrt{t} \). Then there is a sequence of cubes \( Q_{r_k}(y^k) \), \( y^k \in \{x_n = 0\} \) such that as \( k \rightarrow \infty \), \( r_k \rightarrow 0 \), and

\[
\mathcal{H}^1(r_k, y^k) = \frac{1}{|Q^+_t(y^k)|} \int_{Q^+_t} |g^j(x)|^2 \, dx \rightarrow 0,
\]

for all \( j = 1, 2, ..., N \).

Proof: We make the following calculation, where we use the definition of \( \mathcal{H}^1 \) in the first step,

\[
\int_{y' \in Q'_t \times \{x_n = 0\}} \int_{t = 0}^1 \frac{1}{t} \mathcal{H}^1(t, y) \, dt \, dy' = 
\]

\[
= \int_{y' \in Q'_t \times \{x_n = 0\}} \left( \int_{t = 0}^1 \frac{1}{t} \int_{Q^+_t(y')} |g^j(x)|^2 \, dx \right) \, dy' = \tag{8}
\]

\[
= \int_{y' \in Q'_t \times \{x_n = 0\}} \left( \int_{t = 0}^1 \frac{1}{t} \int_{x_n = 0}^{x'} \int_{x' \in Q^+_t(y')} |g^j(x', x_n)|^2 \, dx \right) \, dy' =
\]

\[
= \int_{x_n = 0}^1 \int_{t = x_n}^1 \frac{1}{t} \int_{y' \in Q'_t \times \{x_n = 0\}} \int_{x' \in Q^+_t(y')} |g^j(x', x_n)|^2 \, dx \, dy' \, dt \, dx_n,
\]

where we used Fubini’s Theorem on the two middle integrals in the last equality.

We may continue to estimate the right side of (8) by noticing that integrating over \( x' \in Q^+_t(y') \) and then over \( y' \in Q'_t(0) \) may be estimated by \( t^{n-1} \) times an integration of \( y' \) over \( Q^+_1 \subset Q^+_t \) for small \( t \). This gives that (8) can be estimated from above by

\[
\int_{x_n = 0}^1 \left( \int_{t = x_n}^1 \frac{1}{t} \int_{x' \in Q^+_t(0)} |g^j(x', x_n)|^2 \, dx' \right) \, dx_n =
\]

\[
= \int_{x_n = 0}^1 \int_{t = x_n}^1 \frac{1}{t} \int_{x' \in Q^+_t(0)} |g^j(x', x_n)|^2 \, dx' \, dx_n.
\]
\[ = \int_{x_n=0}^{1} \left( \int_{t=x_n}^{1} \frac{1}{t^2} \left( \int_{x' \in Q_1^2(0)} \left| g^i(x', x_n) \right|^2 dx' \right) \right) dx_n = \tag{9} \]

\[ = \int_{x_n=0}^{1} \left( \frac{1}{x_n} - 1 \right) \left( \int_{x' \in Q_1^2(0)} \left| g^i(x', x_n) \right|^2 dx' \right) dx_n \leq C_0^2, \]

where we used the assumption that \( \| g^i(\cdot, t) \|_{L^2} \leq C_0^2 t \) in the last inequality.

Putting the estimates (8) and (9) together we have shown that

\[ \int_{y' \in Q_1^2 \times \{x_n = 0\}} \left( \frac{1}{t} \int_{t=0}^{1} H_j(t, y) \, dt \right) \, dy \leq C_0^2. \tag{10} \]

Observe that this implies that, for every \( \epsilon > 0 \), if we define the the set

\[ K_\epsilon = \left\{ (t, y); \ y \in Q_1 \times \{x_n = 0\}, \ t \in (0, 1), \sup_j \left( H_j(t, y) \right) \geq \epsilon \right\}. \]

then

\[ \epsilon \chi_{K_\epsilon}(t, y) \leq H_j(t, y). \tag{11} \]

From (11) and (10) it follows that

\[ \int_{y' \in Q_1^2 \times \{x_n = 0\}} \int_{t=0}^{1} \frac{1}{t} \chi_{K_\epsilon} \, dt \, dy \leq \frac{C_0^2}{\epsilon}. \tag{12} \]

This implies that, for every \( \delta > 0 \),

\[ \{(t, y); \ y \in Q_1, \ 0 < t < \delta\} \setminus K_\epsilon \neq \emptyset. \tag{13} \]

Since if (13) was not true then \( \chi_{K_\epsilon}(t, y) = 1 \) for \( 0 < t < \delta \) which would imply that

\[ \int_{y' \in Q_1^2 \times \{x_n = 0\}} \int_{t=0}^{1} \frac{1}{t} \chi_{K_\epsilon} \, dt \, dy \geq \int_{y' \in Q_1^2 \times \{x_n = 0\}} \int_{t=0}^{\delta} \frac{1}{t} \, dt \, dy, \]

the right integral diverges which would contradict (12); thus (13) has to hold.

We may therefore for every \( \epsilon_j > 0 \), say \( \epsilon_j = \frac{1}{j} \), and \( \delta_k > 0 \), say \( \delta_k = \frac{1}{k} \) find a \((t_{j,k}, y_{j,k}) \notin K_{\epsilon_j}\) and \( 0 < t_{j,k} < \delta_k \). The diagonal sequence \((r_k, y^k) = (t_{k,k}, y_{k,k})\) has the desired property of the Lemma.

\[ \square \]

5 Blow-ups of Global Solutions.

In this section we prove that if \( u \) is a global solution in \( \mathbb{R}^n \) and \( \Omega_\|u\| \) is a half space, \( \{x_n \geq 0\} \), then \( u \) has a blow-up that is one dimensional. The proof is based on Lemma 7, in particular we will show that the tangential derivatives \( \frac{\partial u}{\partial x_i}, \ i = 1, 2, \ldots, n - 1 \) satisfies the assumptions in Lemma 7. This implies that there is a blow-up whose \( x_j \) derivatives, \( j = 1, 2, \ldots, n - 1 \), all vanish.

The argument is quite delicate and will partly be done in the space \( H^{-\frac{1}{2}}(\partial D) \) for a Lipschitz domain \( D \). The space \( H^{-\frac{1}{2}}(\partial D) \) is the dual space of \( H^{\frac{1}{2}}(\partial D) \) that is the space of traces of \( W^{1,2}(D) \) functions. In the next lemma we define a trace operator \( \gamma \) that gives a \( H^{-\frac{1}{2}}(\partial D) \) functional for every vector field \( v \) with divergence in \( L^2 \). The proof is standard.
Lemma 8. Let $D$ be a bounded $C^{0,1}$ domain, $v \in L^2(D)$ be a vector field with $L^2$ divergence $\text{div}(v) \in L^2(D)$. Then there is a unique $\gamma(v) \in H^{-1/2}(\partial D)$ such that $\gamma(v) = v|_{\partial D}$ for each $v \in C^\infty(D)$.

Furthermore:

$$\|\gamma(v)\|_{H^{-1/2}(\partial D)} \leq \|v\|_{L^2} + \|\text{div}(v)\|_{L^2}.$$  

Proof: We define the pairing $\gamma(v, \phi)$ for $\phi \in H^{1/2}(\partial D)$ according to

$$\gamma(v, \phi) = \int_D \left( v \cdot \nabla \tilde{\phi} + \text{div}(v) \tilde{\phi} \right),$$

where $\tilde{\phi}$ is the unique $W^{1,2}(D)$ harmonic function satisfying $\tilde{\phi} = \phi$ on $\partial D$.

Then

$$|\gamma(v, \phi)| \leq (\|v\|_{L^2(D)} + \|\text{div}(v)\|_{L^2(D)}) \|\phi\|_{H^{1/2}(\partial D)}.$$  

So $\gamma(v)$ is a bounded functional on $H^{1/2}(\partial D)$ and thus, by Riesz representation Theorem, representable by a function in $H^{-1/2}(\partial D)$.

That $\gamma$ equals the restriction on $C^\infty$ vector fields follows from a simple integration by parts in equation (13).

Next we make sure that the tangential derivatives of $u$ is controlled in a way that makes Lemma 7 applicable.

Lemma 9. Let $u \in W^{1,p}_{loc}(\mathbb{R}^n)$ be a solution to (1) in $\mathbb{R}^n$ and $\Omega = \mathbb{R}^n_+$. Assume that $u$ satisfies the following growth condition

$$|\nabla u| \leq C_0|x_n|^{\frac{1}{p-1}}$$

(15)

for some $C_0$.

Then, for each $i = 1, 2, ..., n$ and $j = 1, 2, ..., n-1$,

$$\int_{B_1(0) \cap \{0 < x_n < t\}} |\nabla u|^2(p-2) \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 \leq C_1 t,$$

Where $C_1$ depend only on $n$, $p$ and $C_0$.

Proof: Fix a direction $j = 1, 2, ..., n-1$ and let $\partial_h$ be the difference quotient in the $j$-direction: $\partial_h f(x) = \frac{f(x + h e_j) - f(x)}{h}$. Then for any function $\phi \in C^\infty_c(B_1(0))$ we may use $\partial_h(\phi^2 \partial_h u)$ as a test function in the weak formulation of (1) and derive,

$$\int_{\mathbb{R}^n} \partial_h(\phi^2 \partial_h u) = -\int_{\mathbb{R}^n} \nabla \left( \partial_h(\phi^2 \partial_h u) \right) \cdot (|\nabla u|^{p-2} \nabla u) =$$

$$= \int_{\mathbb{R}^n} \nabla \left( (\phi^2 \partial_h u) \right) \cdot \partial_h (|\nabla u|^{p-2} \nabla u).$$

The right side is 0 since it involves integrating a function minus a translate of the function. So a standard rearrangement of terms implies that, where we use $I$ for the identity matrix,

$$\int_{\mathbb{R}^n} \phi^2 \frac{\partial u}{\partial x_j} \cdot ((p-2)|\nabla u|^{p-4} \nabla u \otimes \nabla u + |\nabla u|^{p-2} I) \frac{\partial u}{\partial x_j} \leq$$
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\[ \leq 2 \int_{B^2} \phi^2 \left( (p - 2) |\nabla u|^p \nabla u + |\nabla u|^p \nabla u \right) \partial u \partial x_j, \]

which implies that

\[ \int_{B^2} \phi^2 \left( (p - 2) |\nabla u|^p \nabla u + |\nabla u|^p \nabla u \right) \partial u \partial x_j \leq C \int_{B^2} |\nabla \phi|^2 |\nabla u|^p. \]

Making a standard choice of \( \phi \) and noticing that \( (p - 2) |\nabla u|^p \nabla u + |\nabla u|^p \nabla u \) is comparable to \( |\nabla u|^p \nabla I \) implies that

\[ \int_{B^1(0) \cap \{0 < x_n < t\}} |\nabla u|^{p-2} \left| \nabla \partial u \partial x_j \right|^2 \leq C \int_{B^2(0) \cap \{0 < x_n < 2t\}} |\nabla u|^p \leq C t^{\frac{p-1}{p}}, \]

where we used (15). Using (15) again gives the result. \( \square \)

**Proposition 1.** Assume that \( u \) is a solution to (1) in \( \mathbb{R}^n \). Assume furthermore that \( \Omega = \mathbb{R}^n_+ \) and that

\[ \sup_{B^2(0)} |u| \leq CR^\frac{p}{p-1}. \]

Then there exists an \( x^0 \in \Gamma = \{ x_n = 0 \} \) and a sequence \( r_j \to 0 \) such that

\[ \lim_{j \to \infty} \frac{u(r_j x + x^0)}{r_j} = \frac{p - 1}{p} (x^+_{n}) \frac{p}{p-1} \]

strongly in \( W^{1,p}_0(\mathbb{R}^n) \).

**Proof:** Taking a derivative, which is well defined in the weak sense, of

\[ \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = \chi_{\{u > 0\}} \]

with respect to \( x_i, i = 1, 2, \ldots, n \), shows that

\[ \text{div} \left( (p - 2) |\nabla u|^p \nabla u \nabla u + |\nabla u|^p \nabla I \right) = 0 \quad \text{in} \ \mathbb{R}^n_+ \]

(16)

where \( \nabla u \nabla u \nabla u \) is the matrix with entities \( u_i u_j \).

With the notation

\[ \omega^i = \left( (p - 2) |\nabla u|^p \nabla u \nabla u + |\nabla u|^p \nabla I \right) \nabla v^i \]

it follows, from Lemma[9] that \( \omega^i \in L^2_{\text{loc}}(\mathbb{R}^n_+) \) and, from (16), that \( \omega^i \) is divergence-free in \( \mathbb{R}^n_+ \).

We can thus conclude from Lemma[8] that \( \omega^i \) has a trace

\[ \gamma_t(\omega^i) \in H^-_{\text{loc}}(\partial (\mathbb{R}^n_+ \cap \{ x_n > t \})). \]

First we claim that \( \gamma_0(\omega^i) = 0 \) for \( i = 1, 2, \ldots, n - 1 \). To see this we notice that for any test-function \( \phi \in C^\infty_c(B_1(x^0)) \) we have

\[ \int_{B_1(x^0)} \nabla \phi \cdot \omega^i + \phi \text{div}(\omega^i) dx = \int_{B_1(x^0)} \nabla \phi \cdot \left( \frac{\partial}{\partial x_i} |\nabla u|^{p-2} \nabla u \right) dx = \]


for any $\phi$ holds for $\phi$ fact that the embedding $C$ which implies that $\gamma$ since solutions to the obstacle problem are $C^{1,\alpha}$, in the last equality. Similarly, for $i = n$, we get

$$
\int_{B_1(x^0)} \nabla \phi \cdot w^i + \phi \text{div}(w^i) dx = \int_{B_1(x^0)} \nabla \phi \cdot \left( \frac{\partial}{\partial x_n} |\nabla u|^{p-2} \nabla u \right) dx =
$$

$$
= \int_{B_1(x^0) \cap \{x_n > 0\}} \frac{\partial \phi}{\partial x_n} dx + \int_{\{x_n = 0\} \cap B_1} \phi |\nabla u|^{p-2} \frac{\partial u}{\partial x_n} dx = 0,
$$

where we use an integration by parts in several of the steps and the definition of $w^i$, and that $\text{div}(w^i) = 0$, in the first equality and that $|\nabla u|^{p-2} \frac{\partial u}{\partial x_n} = 0$ on $\{x_n = 0\}$, since solutions to the obstacle problem are $C^{1,\alpha}$, in the last equality.

Using that, for $i = 1, 2, ..., n - 1$,

$$
\int_{B_1(x^0)} \nabla \phi \cdot w dx = \int_{B_1(x^0)} \nabla \phi \cdot w^i dx =
$$

$$
= \int_{\partial B_1^+} \phi(x)\gamma_0(w^i) dx - \int_{B_1^+ (0)} \phi(x)\text{div}(w^i(x)) dx = 0
$$

for any $\phi \in W_0^{1,2}(B_1(x^0))$ we can conclude that

$$
\left| \int_{B_1(x^0) \cap \{x_n > t\}} \nabla \phi \cdot w^i \right| = \int_{B_1(x^0) \cap \{0 < x_n < t\}} \nabla \phi \cdot w^i \leq \left(17\right)
$$

$$
\leq \|w^i\|_{L^2(B_1(x^0) \cap \{0 < x_n < t\})} \|\nabla \phi\|_{L^2(B_1(x^0) \cap \{0 < x_n < t\})}.
$$

From Lemma 9 and (17) we can conclude that

$$
\langle \gamma_1(w^i), \phi \rangle \leq Ct^\frac{1}{2} \|\nabla \phi\|_{L^2(B_1(x^0) \cap \{0 < x_n < t\})}.
$$

(18)

Similarly, for $i = n$ we can conclude that

$$
\langle \gamma_1(w^n) - 1, \phi \rangle \leq Ct^\frac{1}{2} \|\nabla \phi\|_{L^2(B_1(x^n) \cap \{0 < x_n < t\})}.
$$

(19)

Let us remind ourselves, before we continue with the proof, that we may characterize functions in $H^{-\frac{1}{2}}(\mathbb{R}^{n-1})$ by $L^2$-functions. In particular, for any $\kappa \in H^{-\frac{1}{2}}$ there is a $g \in L^2$ such that if $\phi \in H^\frac{1}{2}$ then

$$
\langle \kappa, \phi \rangle = \int g(x) \Lambda^\frac{1}{2} \phi dx
$$
where $\Lambda^{\frac{1}{2}}$ is defined

$$\Lambda^{\frac{1}{2}} \phi = \mathcal{F}^{-1} \left( (1 + | \xi |^2)^{\frac{1}{2}} \mathcal{F}(\phi) \right)$$

where $\mathcal{F}$ denotes the Fourier transform.

We may thus identify $\gamma_t(w^j)$ and $\gamma_t(w^n) = 1$ with $L^2$-functions $g^j(x',t)$ satisfying, in view of the inequalities (18) and (19),

$$\int_{Q'_1} |g^j(x',t)|^2 dx \leq C_0 t$$

for some constant $C_0$. Using Lemma 7 we can find a sub-sequence $y_k \in Q_1(0) \cap \{x_n = 0\}$ and $r_k \to 0$ such that $\frac{1}{r_k} ||g^j||_{L^2(Q_{r_k}^+(y^j))} \to 0$.

We claim that by, possibly considering another sequence, $r_k \to 0$ we may assume that $\frac{1}{r_k} ||g^j||_{L^2(Q_{r_k}^+(y^j))} \to 0$ (20) for any $R > 0$. This is easy to see: if $\frac{1}{r_k} ||g^j||_{L^2(Q_{r_k}^+(y^j))} = \epsilon_k \to 0$ then, with $s_k = \epsilon_k^{-\frac{1}{2}} r_k$, a simple calculation gives

$$\frac{1}{r_k} ||g^j||_{L^2(Q_{r_k}^+(y^j))} < \epsilon_k^{1/2} \to 0,$$

for every $R < \frac{1}{\epsilon_k^{1/2} r_k} \to \infty$. We may assume that $r_k$ already satisfies (20).

Since $\sup_{B_R} u^k \leq CR^{p/(p-1)}$ by Lemma [1] we may conclude that, for some subsequence,

$$u^k(x) = \frac{u(r_k x + y^j)}{r_k} \to u^0(x).$$

We will show that the the limit $u^0$ satisfies

$$\Delta_p u^0 = \chi_{\{u^0 > 0\}} \text{ in } \mathbb{R}^n \quad (21)$$

$$\sup_{B_R(0)} |u^0| \leq C \left( 1 + R^{\frac{p}{p-1}} \right) \text{ for every } R > 0 \quad (22)$$

$$\gamma_t(w^j_{u^0}) = 0 \text{ for } j = 1, 2, \ldots, n-1 \text{ and all } t > 0, \quad (23)$$

$$\gamma_t(w^n_{u^0}) = 1 \text{ for all } t > 0, \quad (24)$$

where $w^j_{u^0} = [(p-2)|\nabla u^0|^{p-2} \nabla u^0 \otimes \nabla u^0 + |\nabla u^0|^{p-2} I] \nabla \frac{\partial u^0}{\partial x_j}$. That (21) holds is trivial, (22) follows from Lemma [1] and (23) and (24) follows from the fact that $g^j \to 0$ by (20).

But since $w^j_{u^0} \in L^2$ we can conclude, from (23), that $w^j_{u^0} = 0$ almost everywhere, for $j = 1, 2, \ldots, n-1$. Similarly, by (24), $w^n_{u^0} = e_n$ almost everywhere.
Since $w^j_{u^0} = \frac{\partial}{\partial x_j}|\nabla u^0|^{p-2}\nabla u^0$, for $j = 1, 2, \ldots, n$, and $w^j_{u^0} = 0$ in $x_n < 0$ we can conclude that $|\nabla u^0|^{p-2}\frac{\partial u^0}{\partial x_n} = x_n$.

It directly follows, from the definition of $w^j_{u^0}$, that

$$\nabla' \cdot (|\nabla u^0|^{p-2}\nabla u^0) = 0$$

for each $x_n > 0$ and that $|\nabla u^0| \geq x_n^{-\frac{1}{p-1}}$ for $x_n > 0$.

Since $u^0(x', 0) = 0$ it follows from Lemma 1 that $u^0(x', x_n)$ is uniformly bounded in $x'$ for each fixed $x_n$. From (25) it also follows that $\frac{\partial u^0(x', x_n)}{\partial x_j}$ is constant in $x'$ for each fixed $x_n$; this together with the boundedness of $u^0(x', x_n)$ implies that $u^0(x', x_n)$ is constant in $x'$. That is $u^0(x) = u^0(x_n)$, and

$$\Delta_p u^0(x) = \frac{\partial}{\partial x_n} \left( \frac{|\partial u^0(x_n)|^{p-2}}{\partial x_n} \right) = 0$$

Solving (26) gives that $u^0(x_n) = \frac{p-1}{p}x_n^{\frac{1}{p-1}}$ in the set $\{x_n > 0\}$.

\section{Linearization.}

In this section we show that we may linearize solutions to the p-harmonic obstacle problem around points where the solution is close to the ground-state solution $\frac{p-1}{p}(x_n)^{\frac{1}{p-1}}$. Before we do that we fix some notation that we will use in this section. Notice that if $u$ is a solution to the p-harmonic obstacle problem then $v_i = \frac{\partial u}{\partial x_i}$ will be a weak solution to the following linear equation

$$\text{div}(A(x)\nabla v_i) = 0 \quad \text{in } \Omega = \{u > 0\},$$

where $A(x) = (p-2)|\nabla u|^{p-4}\nabla u \otimes \nabla u + |\nabla u|^{p-2}I$.

We will use $L^2_A(\Omega)$ for the Hilbert space with the norm

$$\|v\|_{L^2_A(\Omega)} = \left( \int_\Omega |A(x)||v|^2dx \right)^{1/2}.$$  

At times we will be somewhat informal and say that $\nabla v \in L^2_A(\Omega)$ if

$$\left( \int_\Omega \langle \nabla v, A(x)\nabla v \rangle dx \right)^{1/2} < \infty,$$

which is justified since $|\langle \nabla v, A(x)\nabla v \rangle|$ is comparable to $|A||\nabla v|^2$ up to a multiplicative constant that only depends on $(p-2)$. Thus we can form the Hilbert space $W^{1,2}_A$ with norm $\|v\|_{L^2_A} + \|\nabla v\|_{L^2_A}$.

In this section we will assume that we have a sequence of solutions $u^j$ in $B_1(0)$ in a normalized coordinate system defined as follows.

\begin{definition}
Let $u \in W^{1,p}(B_1(0))$ be a solution to (1). Then we say that the coordinate system is normalized (with respect to $u$) if

$$\left\|\nabla' \left( u - \frac{p-1}{p}x_n^{\frac{1}{p-1}} \right) \right\|_{W^{1,2}_A(\mathbb{R}^n \backslash B_1)} \leq$$

\end{definition}
6 LINEARIZATION.

\[ \begin{align*}
\leq \left\| \nabla' \left( u - \frac{p - 1}{p} (\nu \cdot x + \gamma) \right) \right\|_{W_A^{1,2}(\Omega \cap B_1)} \\
\text{and } \gamma \text{ is chosen so that} \\
\left\| \left( u - \frac{p - 1}{p} x_n \right) \right\|_{L_A^2(\Omega \cap B_1)} \leq \\
\leq \left\| \left( u - \frac{p - 1}{p} (x_n + \gamma) \right) \right\|_{L_A^2(\Omega \cap B_1)} \\
\end{align*} \]

for any unit vector \( \nu \) and constant \( \gamma \), where

\[ A = (p - 2) |\nabla u|^{-4} \nabla u \otimes \nabla u + |\nabla u|^{p - 2} I, \]

and \( \nabla' = \nabla - \nu (u \cdot \nabla) \) is the gradient on the subspace orthogonal to \( \nu \).

Notice that if \( u \) is a solution to (1) in some set we can easily choose a normalized coordinate system by making a translation and rotation of the coordinate system. Also, the term \( \frac{p - 1}{p} x_n \) is redundant in the first set of inequalities since it disappears under the operation of \( \nabla' \), we have included the term as an indication of the heuristic idea that the normalized coordinate system is the coordinate system where \( u \) is closest to a solution only depending on the \( x_n \) coordinate.

For the rest of this section we will assume that \( u \) is a sequence of solutions to (1) and \( v^j_i \) for \( i = 1, 2, ..., n - 1 \) are well defined.

Proposition 2. Assume that \( p > 2 \). The sequence \( v^j_i \) defined by (27), for \( i = 1, 2, ..., n - 1 \), converges to \( v^0_0 \) where \( v^0_0 \) satisfies

\[ \text{div} \left( \frac{|x_n|^{-\frac{p-2}{2-p}}}{|x_n|} \nabla u^0 \right) = 0 \quad \text{in } \{ x_n > 0 \} \cap B_1(0) \]
and $v_0 = 0$ on $\{x_n = 0\} \cap B_1(0)$. The convergence $v_j^1 \to v_0^0$ is in the sense that for any $\epsilon > 0$

$$v_j^1 \to v_0^0 \quad \text{in } C^{1,\alpha}(\{x_n > \epsilon\})$$

and $\limsup_{j \to \infty} \|v_j^1\|_{L^2(B_{1/\epsilon}(0))} = \|v_0^0\|_{L^2(B_{1/\epsilon}(0))}$.

**Proof:** That $\nabla w^j \to \nabla u^0$ in $C^\beta$ and weakly in $L^p$, for a sub-sequence, follows from the $C^{1,\alpha}$ regularity of $w^j$ together with the Arzela-Ascoli theorem and weak compactness in $L^p$ spaces. Moreover, by Lemma 2 and and $C^\beta$ convergence of $A_j$, it follows that $A_j \cdot \nabla v_j^0 \to A_0 \cdot \nabla v_0^0$ weakly and that

$$0 = \int_{B_1} \langle \nabla \phi \cdot A_j, \nabla v_j^0 \rangle = \int_{B_1} \langle \nabla \phi \cdot A_0, \nabla v_0^0 \rangle.$$

The only thing that remains to prove is the strong convergence of $v_j^1$ specified at the end of the Lemma. We will prove (25) by means of a hole filling argument.

In order to prove the strong convergence in $L^2$ we fix any $\epsilon > 0$. Then, since $w^1 \to \frac{x_n}{p}(x_n)_{+}^{-1}$ in $C^{1,\alpha}$ and $w^j$ is non-degenerate by Lemma 1 (in particular 25), if $j$ is large enough it follows that $\partial \{w^j > 0\} \cap B_1 \subset \{|x_n| < \epsilon\}$. We can thus, for any $\delta > 0$, assume, if $j$ is large enough, that the entire free boundary is contained in the strip $|x_n| < \delta$.

Let us define the following cubes

$$Q_k = \{x \in B_1(0); x' \in B_{1/2+2^{k+1}-1} \delta(0) \text{ and } |x_n| < (2^{k+1}-1)\delta\}$$

then $Q_k \subset B_{3/4}$ for $k \leq \frac{\ln(\delta)}{\ln(2)}$. Furthermore we have

1. There exists cut-off functions $\Psi_k \in C^\infty$ such that $\psi^k = 1$ in $Q_k$, $\psi^k = 0$ in $Q_{k+1}$ and $|\nabla \psi_k| \leq C \frac{1}{\delta^\alpha}$.

2. $|w^j| \leq C_2 2^{\frac{\alpha}{\alpha-\gamma}} \delta^{\frac{\alpha-\gamma}{\alpha}}$ and $|\nabla w^j| \leq C_2 2^{\frac{\alpha}{\alpha-\gamma}} \delta^{\frac{\alpha-\gamma}{\alpha}}$ for some constant $C$ (depending only on $n$ and $p$) in $Q_k$. This follows from Lemma 11 and that $\Gamma_u \subset \{|x_n| < \delta\}$.

**Lemma 10.** There exists a constant $\tau < 1$, depending only on $n$ and $p \geq 2$, such that for every $k \geq 0$

$$\int_{Q_k} |v_j^0|^2 dx \leq \tau \int_{Q_{k+1}} |v_j^0|^2 dx.$$

**Proof of Lemma 10.** We use $\phi = \psi_k^2 (w^j)^2$ as a test function in the variational integral for $w^j$, this is well defined since $v_j^1 \in C^\alpha \cap W^{1,2}_{A_j}$ by Lemma 1 and Lemma 2. That is

$$0 = \int_{B_1} \left( \nabla \left( \psi_k^2 (v_j^1)^2 \right) \cdot \nabla w^j |\nabla u^j|^{p-2} + \left( \psi_k^2 (v_j^1)^2 \right) \right) dx =$$

$$= \int_{B_1} \left( 2 \psi_k (v_j^1)^2 \nabla \psi_k \cdot \nabla w^j |\nabla u^j|^{p-2} \right) +$$

$$+ \left( 2 \psi_k v_j^1 \nabla v_j^1 \cdot \nabla w^j |\nabla u^j|^{p-2} \right) + \left( \psi_k^2 (v_j^1)^2 \right) dx.$$
Rearranging terms we can deduce that
\[
\int_{B_1} \psi_k^2 (v_i^j)^2 \, dx \leq \left| \int_{B_1} \left( 2 \psi_k (v_i^j)^2 \nabla \psi_k \cdot \nabla u^j \right) \, dx \right| + \\
\left| \int_{B_1} (2 \psi_k^2 v_i^j \nabla u^j \cdot \nabla u^j |^{p-2}) \, dx \right| = I_1 + I_2.
\]

(29)

We will estimate \(I_1\) and \(I_2\) independently.

In order to estimate \(I_1\) we use that \(\nabla \psi_k\) is supported in \(Q_{k+1} \setminus Q_k\) where \(|\nabla u^j|^{p-2} \leq C2^k \delta \leq \frac{C}{p-1} \). Therefore
\[
I_1 = \left| \int_{Q_{k+1} \setminus Q_k} (2 \psi_k (v_i^j)^2 \nabla \psi_k \cdot \nabla u^j |^{p-2}) \, dx \right| \leq C \int_{Q_{k+1} \setminus Q_k} \psi_k (v_i^j)^2 \, dx.
\]

(30)

In order to estimate \(I_2\) we use a small trick and notice that \(\nabla u^j |^{p-1} = \frac{1}{p-1} A_j \cdot \nabla u^j\). We may therefore estimate \(I_2\)
\[
I_2 = \frac{1}{p-1} \left| \int_{B_1} (2 \psi_k^2 v_i^j \langle \nabla v_i^j \cdot A_j, \nabla u^j \rangle) \, dx \right| = \\
\left| \int_{Q_{k+1} \setminus Q_k} (2 \psi_k^2 u^j \langle \nabla v_i^j \cdot A_j, \nabla u^j \rangle) \, dx \right| + \\
\left| \int_{Q_{k+1} \setminus Q_k} (4 \psi_k v_i^j u^j \langle \nabla v_i^j \cdot A_j, \nabla \psi_k \rangle) \, dx \right|
\]

(31)

where we used an integration by parts and \(\text{div}(A_j \nabla v_i^j) = 0\).

Continuing (31) by using the triangle inequality and then Hölder’s inequality it follows that
\[
I_2 \leq \frac{1}{p-1} \left| \int_{B_1} (2 \psi_k^2 u^j \langle \nabla v_i^j \cdot A_j, \nabla u^j \rangle) \, dx \right| + \\
\left| \int_{Q_{k+1} \setminus Q_k} (2 \psi_k^2 u^j \langle \nabla v_i^j \cdot A_j, \nabla \psi_k \rangle) \, dx \right| \leq \\
\leq C \frac{1}{p-1} \left( 2 \int_{Q_{k+1} \setminus Q_k} \psi_k \langle \nabla v_i^j \cdot A_j, \nabla u^j \rangle \, dx \right) + \\
\left( 2 \int_{Q_{k+1} \setminus Q_k} \psi_k \langle \nabla v_i^j \cdot A_j, \nabla u^j \rangle \, dx \right)^{\frac{1}{2}} \left( \int_{Q_{k+1} \setminus Q_k} (v_i^j)^2 \, dx \right)^{\frac{1}{2}}
\]

where we also used that \(|u^j| \leq C2^k \delta \), \(|\nabla \psi_k| \leq \frac{C}{2k \delta}\) and \(|\nabla u| \leq C2^k \delta \), see point 1 and 2 in the statement just before this Lemma, in \(Q_{k+1}\) in the last step.

From Lemma 2 it follows that, since \(p \geq 2\),
\[
\int_{Q_{k+1}} \psi_k^2 \langle \nabla v_i^j \cdot A_j, \nabla u^j \rangle \, dx \leq \frac{C \sup_{Q_{k+1}} |\nabla u|^{p-2}}{2^{2k} \delta^2} \int_{Q_{k+1} \setminus Q_k} (v_i^j)^2 \, dx \leq \frac{C \sup_{Q_{k+1}} |\nabla u|^{p-2}}{2^{2k} \delta^2} \int_{Q_{k+1} \setminus Q_k} (v_i^j)^2 \, dx \leq \frac{C \sup_{Q_{k+1}} |\nabla u|^{p-2}}{2^{2k} \delta^2} \int_{Q_{k+1} \setminus Q_k} (v_i^j)^2 \, dx \leq
\]

\[\leq \frac{C \sup_{Q_{k+1}} |\nabla u|^{p-2}}{2^{2k} \delta^2} \int_{Q_{k+1} \setminus Q_k} (v_i^j)^2 \, dx \leq \]
We may therefore estimate $I_2$ is according to

$$I_2 \leq C \int_{Q_{k+1} \setminus Q_k} (v^j_i)^2 \, dx.$$  \hfill (32)

From \([29]\), \([30]\) and \([32]\) we can conclude that

$$\int_{Q_k} (v^j_i)^2 \, dx \leq C \int_{Q_{k+1} \setminus Q_k} (v^j_i)^2 \, dx,$$

the result follows by adding $C \int_{Q_k} (v^j_i)^2 \, dx$ to both sides and dividing by $C + 1$.

**The end of the proof of Proposition 2.** It is enough to show that for every $\epsilon > 0$ there exists a $J_{\epsilon}$ such that $j > J_{\epsilon}$ implies that

$$\|v^j_i\|_{L^2(B_{1/2})} - \epsilon \leq \|v^0_i\|_{L^2(B_{1/2}(0))} \leq \|v^j_i\|_{L^2(B_{1/2})} + \epsilon.$$

Since $v^j_i \to v^0_i$ in $C^\infty(B_{1/2}(0) \setminus Q_0)$ for $\delta > 0$ it is enough to show that there exists an $J_{\epsilon}$ such that $\|v^j_i\|^2_{L^2(B_{1/2}(0) \cap Q_0)} < \epsilon$ for every $j > J_{\epsilon}$. But by Lemma 10 we know that

$$\|v^j_i\|^2_{L^2(B_{1/2}(0) \cap Q_0)} \leq \tau^k \|v^j_i\|^2_{L^2(Q_k)} \leq \tau^k$$  \hfill (33)

for every $k \leq \frac{\ln(\delta)}{\ln(2)}$.

We may thus, for every $\epsilon > 0$ choose $k_0$ large enough so that $\tau^{k_0} < \epsilon$ and then choose $\delta > 0$ so that $k_0 \leq \frac{\ln(\delta)}{\ln(2)}$. Then, if $j$, is large enough we may apply \([33]\) and conclude that

$$\|v^j_i\|^2_{L^2(B_{1/2}(0) \cap Q_0)} \leq \tau^{k_0} \|v^j_i\|^2_{L^2(Q_k)} \leq \tau^{k_0} < \epsilon.$$

This finishes the proof.

We also need to control the $x_n-$derivative. To that end we prove the following simple lemma.

**Lemma 11.** Given $M$ there exists a modulus of continuity $\sigma$ depending only on $2 < p < \infty$ and the dimension $n$ such that

$$\left\| \frac{\partial(u - (p - 1)/p(x_n)^{n/(p-1)})}{\partial x_n} \right\|_{L^2(\Omega_n \cap B_1)} \leq \sigma \left( \|\nabla u\|_{L^2(B_1)} \right)$$  \hfill (34)

for any solution $u$ to \([11]\) for which $0 \in \Gamma_u$.

**Proof:** We argue by contradiction and assume that $u^j$ is a sequence of solutions satisfying the assumptions in the Lemma and

$$\|\nabla u^j\|_{L^2(B_1)} \to 0$$

and

$$\left\| \frac{\partial(u^j - ((p - 1)/p)(x_n)^{n/(p-1)})}{\partial x_n} \right\|_{L^2(\Omega_n \cap B_1)} \geq \delta$$  \hfill (35)
for some $\delta > 0$.

Since $0 \in \Gamma_{u_j}$ it follows from Lemma 1 that $\|u^j\|_{L^\infty(B_{5/4}(0))}$ is bounded and therefore, also by Lemma 1 that $\|u^j\|_{C^{1,\alpha}(B_{6/5}(0))}$ is bounded.

We may therefore choose a subsequence, still denoted by $u_j$, such that $u_j \to u_0$ in $C^{1,\beta}$. We may therefore choose a subsequence, still denoted by $u_j$, such that $u_j \to u_0$ in $C^{1,\beta}$ where $u_0$ is a solution satisfying $\|\nabla' u_0\|_{L^2(B_1)} = 0$. In particular, $u_0$ is a one dimensional solution, with $0 \in \Gamma_{u_0}$, and thus $u_0 = p^{-1}x_n^p$ in its support. This together with $C^{1,\alpha}$ convergence contradicts (35).

## 7 Almost Everywhere Uniqueness of Blow-ups.

We are now ready to prove geometric decay for the tangential derivatives which will lead to $C^{1,\alpha}$-regularity of the free boundary.

**Proposition 3.** For every $M > 0$ there exists constants $\epsilon_0 > 0$, $\tau < 1$ and $s > 0$, depending only on $2 \leq p < \infty$ and $n$, such that if $u$ is a solution to (1) in a normalized coordinate system such that $\Gamma_u \cap B_{1/4}(0) \neq \emptyset$ and $\|\nabla' u\|_{L^2(B_1)} \leq \epsilon < \epsilon_0$ then there exists a coordinate system and an $0 < s < 1$ such that

$$\|\nabla' u(sx)\|_{L^2(B_1(0))} \leq \tau \epsilon.$$  \hspace{1cm} (36)

**Proof:** We will argue indirectly and assume that $u^j$ is a sequence of solutions as in the Proposition with

$$\|\nabla' u^j\|_{L^2(B_1(0))} \leq \frac{1}{j}.$$  

Then, by Lemma 11,

$$\left\|\frac{\partial}{\partial x_n} \left( u - \frac{p-1}{p} x_n^{p/(p-1)} \right) \right\|_{L^2(\Omega_u \cap B_1)} \leq \sigma \left( \frac{1}{j} \right) \to 0.$$  

From Proposition 2 we may therefore conclude that, for $i = 1, 2, ..., n-1$,

$$v_i^j = \frac{1}{\delta_{i,j}} \frac{\partial u^j}{\partial x_i} \to v_i^0,$$  \hspace{1cm} (37)

in $L^2(B_r(0))$ for every $r < 1$, where

$$\delta_{i,j} = \left\| \frac{\partial u^j}{\partial x_i} \right\|_{L^2(B_1)}$$

and $v_i^0$ solves

$$\text{div} \left( |x_n|^{\frac{p-1}{2}} \nabla v_i^0 \right) = 0 \quad \text{in} \quad \{x_n > 0\} \cap B_1(0).$$

By Lemma 3 we may write

$$v_i^0(x) = \sum_{k=1}^{\infty} a_k^i q_j(x),$$  \hspace{1cm} (38)
where \( q_i \) are \( \lambda_i \)-homogeneous eigenfunctions on the sphere. Also \( q_1(x) = x \).

We will show that \( a_1^i = 0 \) for \( i = 1, 2, ..., n - 1 \). As we will show at the end of the proof, this implies that \( v_i^j \) decays faster than \( s^{p/(p-1)} \) which will imply \( 30 \) for \( u^j \) when \( j \) is large. In order to show that \( a_1^i = 0 \) for \( i = 1, 2, ..., n - 1 \) we define

\[
G(\eta) = \int_{B_1(0)} A_j(x) \left| \nabla' \eta \left( u(x) - \frac{p-1}{p} (\eta \cdot x)^{p/(p-1)} \right) \right|^2 \, dx,
\]
on the set of unit vectors \( \eta \in \mathbb{R}^n \). Since we assume that the coordinate system is normalized with respect to \( u \) it follows that

\[
G(e_n) \leq G(\eta), \tag{39}
\]
for all unit vectors \( \eta \). If we let \( \eta' = (\eta_1, ..., \eta_{n-1}, 0) \) be a unit vector then \( \eta(\theta) = \cos(\theta)e_n + \sin(\theta)\eta' \) will also be a unit vector and \( 30 \) states that \( G(\eta(0)) \leq G(\eta(\theta)) \). Taking a derivative with respect to \( \theta \) at \( \theta = 0 \) gives

\[
0 = -2 \int_{B_1(0)} A_j(x) \frac{\partial u^j}{\partial x_n} \nabla' u^j \cdot \eta' \, dx =
-2 \int_{B_1(0)} A_j(x) \frac{\partial u^j}{\partial x_n} \left( \sum_{i=1}^{n-1} \delta_{i,j} v_i^j(x) \eta_i \right) \, dx, \tag{40}
\]
for any unit vector \( \eta' \).

Since \( \frac{\partial u^i}{\partial x_n} \rightarrow (x_n)^{1/(p-1)} = q_1(x) \) strongly and \( v_i^j \rightarrow v_i^0 \), defined as in \( 38 \) we may conclude, by choosing \( \eta' = e_i \) dividing by \( \delta_{i,j} \) and use that \( (x_n)^{p/(p-1)} = q_1(x) \), that

\[
0 = \int_{B_1(0)} |x_n|^{\frac{p-2}{p-1}} \sum_{k=1}^{\infty} a_k^i q_k(x) q_1(x) \, dx = a_1^i \int_{B_1(0)} |x_n|^{\frac{p-2}{p-1}} |q_1(x)|^2 \, dx,
\]
where we used that \( q_k \) and \( q_1 \) are orthogonal, by Lemma \( 3 \) in the last equality. It follows that \( a_1^i = 0 \) for \( i = 1, 2, ..., n - 1 \).

We have therefore shown that, for any \( i = 1, 2, ..., n - 1 \),

\[
v_i^j \rightarrow v_i^0 = \sum_{k=2}^{\infty} a_k^i q_k(x)
\]
strongly (by Proposition \( 2 \)) in \( L^2(B_{1/2}) \) where

\[
\left\| \sum_{k=2}^{\infty} a_k^i q_k(x) \right\|_{L^2(B_{1/2})} \leq 1.
\]

Since \( q_k \) is homogeneous of order \( \lambda_k > \frac{1}{p-1} \) for \( k \geq 2 \) by Lemma \( 3 \) it follows that

\[
\left\| v_i^0(sx) \right\|_{L^2(B_1)} \leq s^{\lambda_k - \frac{1}{p-1}} \left\| v_i^0 \right\|_{L^2(B_1)} \leq s^{\lambda_k - \frac{1}{p-1}}. \tag{41}
\]
Finally we notice that
\[ \epsilon = \left\| \nabla' u^j(x) \right\|_{L^2(B_1(0))}^2 = \sum_{i=1}^{n-1} \delta_{i,j}^2 \left\| v^j_i(x) \right\|_{L^2(B_1)}^2 = \sum_{i=1}^{n-1} \delta_{i,j}^2 \]
so if \( \epsilon \) is small enough then
\[ \left\| \nabla' \frac{u^j(sx)}{s^{p/(p-1)}} \right\|_{L^2(B_1(0))}^2 = \sum_{i=1}^{n-1} \delta_{i,j}^2 \left\| \frac{v^j_i(sx)}{s^{1/(p-1)}} \right\|_{L^2(B_1)}^2 = \sum_{i=1}^{n-1} \delta_{i,j}^2 \]
\[ \leq \sum_{i=1}^{n-1} s^{\lambda - \frac{1}{p-1}} \delta_{i,j}^2 \left\| v^j_i(x) \right\|_{L^2(B_1)}^2 + o(\epsilon) = \]
\[ \leq s^{\lambda - \frac{1}{p-1}} \left\| \nabla' u^j(x) \right\|_{L^2(B_1(0))}^2 + o(\left\| \nabla' u^j(x) \right\|_{L^2(B_1(0))}^2) \]
where we used (37) and (41). This implies the Proposition with
\[ \tau = s^{\lambda - \frac{1}{p-1}} + 1 < 1. \]

**Lemma 12.** Let \( u \) be a solution to the \( p \)-harmonic obstacle problem in a normalized coordinate system. Then there exists an \( \epsilon_0 > 0 \) such that if \( 0 \in \Gamma_u \) and \( \left\| \nabla u \right\|_{L^2(B_1(0))} \leq \epsilon < \epsilon_0 \) then there exists a vector \( \eta \) such that
\[ \lim_{r \to 0} \frac{u(rx)}{r^{p-1}} = \frac{p - 1}{p} \left( \eta \cdot x \right)_+^{p-1}. \]
Furthermore, there exists a constant, \( C \), such that
\[ |e_n - \eta| \leq C\epsilon. \]  \( \ldots \tag{42} \)

**Proof:** By Proposition 3 it follows that
\[ \left\| \nabla' \frac{u(sx)}{s^{p/(p-1)}} \right\|_{L^2(B_1(0))} \leq \tau \epsilon. \]
We may thus re-normalize the coordinate system (rotate the coordinate axes to a new set of basis vectors \( \{e_1^1, e_2^1, \ldots, e_n^1\} \)) so that
\[ u_s = \frac{u(sx)}{s^{p/(p-1)}} \]
satisfies
\[ \left\| \nabla' u_s \right\|_{L^2(B_1(0))} \leq \tau \epsilon \]
in the new coordinate system.

Next we claim that \(|\cos^{-1}(e_n \cdot e_n^k)| \leq C \tau \epsilon\). The argument is elementary so we will only give a rough sketch. From Lemma 14 we know that \(\partial_n u_\alpha \approx (x_n)_+^{1/(p-1)} + o(1)\). We also know that \(\|\nabla' u_\alpha\|_{L^p(B_1)} \leq C_\epsilon\) since \(\|\nabla' u\|_{L^p(B_1(0))} \leq \epsilon\). Therefore

\[
C \tau \epsilon \geq \|\nabla e_n^k u_\alpha\|_{L^2(B_1(0))} \geq (1 - o(1)) \left\| \nabla e_n^k \frac{p-1}{p} (x_n)_+^{p/(p-1)} \right\|_{L^2(B_1(0))} - C_\epsilon \epsilon.
\]

A direct calculation, and using a Taylor expansion, shows that

\[
\|\nabla e_n^k u_\alpha\|_{L^2(B_1(0))} \geq c_\epsilon (1 - e_n \cdot e_n^k)
\]

for \(e_n^k \approx e_n\). Using (14) in (43) and some simple calculus calculations implies that \(|\cos^{-1}(e_n \cdot e_n^k)| \leq C \tau \epsilon\), where \(C\) depend on \(\tau\) and on \(s\).

We may thus repeat the argument and conclude that for each \(k \geq 0\) there is a coordinate system \(\{e_1^k, e_2^k, \ldots, e_n^k\}\) such that

\[
\|\nabla' u_\alpha\|_{L^2(B_1(0))} \leq \tau^k \epsilon \rightarrow 0 \text{ as } k \rightarrow \infty
\]

and the rotation of of \(\{e_1^{k-1}, \ldots, e_n^{k-1}\}\) with respect to \(\{e_1^k, \ldots, e_n^k\}\) is of order \(C \tau^k \epsilon\), that is \(|\cos^{-1}(e_n^{k-1} \cdot e_n^k)| \leq C \tau^k \epsilon\).

In particular,

\[
|\cos^{-1}(e_n \cdot e_n^k)| \leq \sum_{l=1}^k |\cos^{-1}(e_n^{l-1} \cdot e_n^l)| \leq C \epsilon \sum_{l=1}^k \tau^l \leq C \epsilon.
\]

That means that the coordinate system converges: \(e_n^k \rightarrow \eta\) for some vector \(\eta\). And (43) implies that \(\gamma \cdot \nabla u_\alpha \rightarrow 0\) for any \(\gamma\) orthogonal to \(\eta\), \(\gamma \cdot \eta = 0\).

That means that \(u_\alpha \rightarrow u_0\) where \(u_0\) depend only on the \(\eta\) direction. That is \(u_0(x) = \frac{p-1}{p}(\eta \cdot x)^{\frac{p}{p-1}}\).

The estimate (42) follows directly from passing to the limit \(k \rightarrow \infty\) in (46).

**Corollary 1.** Let \(u\) be as in Lemma 7.2 maybe with a smaller \(\epsilon_0\), then \(\Gamma_u\) is a \(C^{1,\alpha}\) graph in \(B_1(0)\) for some \(\tau_0 > 0\). Furthermore the \(C^{1,\alpha}\) norm of the graph is bounded by \(C \epsilon\).

**Proof:** Notice that if \(u\) is as in the Corollary and \(x^1 \in \Gamma_u \cap B_{1/2}(0)\) then \(u_{x^1,1/2} = \frac{u(x_0 + x^1 + \epsilon / 2)}{u(x_0 + \epsilon / 2)}\) will satisfy the hypotheses of Lemma 12 in \(B_1(0)\) with \(C_n \epsilon\) in place of \(\epsilon\). This implies, if \(\epsilon\) is small enough, that the blow-up of \(u_{x^1,1/2}\) is unique:

\[
\lim_{r \rightarrow 0} \frac{u_{x^1,1/2}(rx)}{r^{p/(p-1)}} = \frac{p-1}{p}(\eta_{x^1} \cdot x)^{\frac{p}{p-1}}.
\]

This defines the measure theoretic normal of \(\Gamma_u\) at the point \(x^1 \in \Gamma_u\). Furthermore, by (42), \(|\eta_{x^1} - \eta| \leq C \epsilon\). Therefore the normal of \(\Gamma_u\) is well defined at every point in \(B_{1/2}(0)\).
It only remains to show that $\eta_x$ is Hölder continuous in $x$. We will show that
\[ |\eta_{x^1} - \eta_0| \leq C|x^1|^{\alpha} \epsilon \]
with $\alpha = \frac{\ln(\tau)}{\ln(s)} > 0$. A similar estimate for of $|\eta_x - \eta_y|$ for two arbitrary points $x, y \in \Gamma_u$ follows by a simple translation translating $y \in B_{r_0}(0)$ to the origin.

To that end assume that $s^{k+1} < |x^1| \leq s^k$. Then $\frac{x^1}{s^k} \in \Gamma_{u, s^k}$ and $u_{s^k}$ satisfies the hypothesis of Lemma 12 with $\tau^k \epsilon$ in place of $\epsilon$. This implies, again by (42), that
\[ |\eta_{x^1} - \eta_0| \leq C\tau^k \epsilon \leq C|x^1|^{\frac{\ln(\tau)}{\ln(s)}} \epsilon. \]

**Theorem 1.** Let $u$ be a solution to the $p$-harmonic obstacle problem in $B_1(0)$ then there exists an open set $\Gamma_0 \subset \Gamma$ such that $\mathcal{H}^{n-1}(\Gamma \setminus \Gamma_0) = 0$ and for every $x^0 \in \Gamma_0$ there exists an $r = r(x^0)$ such that $\Gamma_u \cap B_r(x^0)$ is a $C^{1,\alpha}$ graph.

**Proof:** Since $\Omega_u$ has finite perimeter, by Lemma 5 it follows that for $\mathcal{H}^{n-1}$-a.e. free boundary point $x^0 \in \Gamma_u$ blow-up
\[ \lim_{r \to 0} \frac{u(rx + x^0)}{r^{p/(p-1)}} = u^0 \]
has support in a half space which we may assume (after possibly rotation the coordinate system) to be $\mathbb{R}^n_+^\alpha$.

By Proposition 4 we know that for a.e. $y^0$ in $\Gamma_{u^0}$
\[ \lim_{r \to 0} \frac{u(rx + y^0)}{r^{p/(p-1)}} \to \frac{p-1}{p} (\eta \cdot x)^{p-1} \]
for some vector $\eta$. That is
\[ \frac{p-1}{p} (\eta \cdot x)^{p-1} \in \text{Blo}(u, y^0). \]

From Lemma 6 we can conclude that for $\mathcal{H}^{n-1}$-a.e. $x^0 \in \Gamma_u$
\[ \frac{p-1}{p} (\eta \cdot x)^{p-1} \in \text{Blo}(u, x^0). \]

If we normalize the coordinate system this means that $\frac{u(rx + x^0)}{r^{p/(p-1)}}$ satisfies the conditions of Corollary 1 if $r$ is small enough. The Theorem follows.

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