Homogenization of Symmetric Lévy Processes on $\mathbb{R}^d$

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Abstract

In this short note we study homogenization of symmetric $d$-dimensional Lévy processes. Homogenization of one-dimensional pure jump Markov processes has been investigated by Tanaka et al. in [4]; their motivation was the work by Benssousan et al. [1] on the homogenization of diffusion processes in $\mathbb{R}^d$, see also [2] and [10]. We investigate a similar problem for a class of symmetric pure-jump Lévy processes on $\mathbb{R}^d$ and we identify – using Mosco convergence – the limit process.

A symmetric Lévy process $(X_t)_{t \geq 0}$ is a stochastic process in $\mathbb{R}^d$ with stationary and independent increments, càdlàg paths and symmetric laws $X_t \sim -X_t$. We can characterize the (finite-dimensional distributions of the) process by its characteristic function $E e^{i \langle \xi, X_t \rangle}$, $\xi \in \mathbb{R}^d$, $t > 0$, which is of the form $\exp(-t \psi(\xi))$; due to the symmetry of $X_t$, the characteristic exponent $\psi$ is real-valued. It is given by the Lévy–Khintchine formula

$$\psi(\xi) = \frac{1}{2} \langle \xi, \Sigma \xi \rangle + \int_{h \neq 0} \left(1 - \cos \langle \xi, h \rangle\right) \nu(h) \, dh, \quad \xi \in \mathbb{R}^d. \quad (1)$$

$\Sigma \in \mathbb{R}^{d \times d}$ is the positive semidefinite diffusion matrix and $\nu(h)$ is the Lévy measure, that is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{h \neq 0} (1 \land |h|^2) \nu(h) \, dh$ is finite. It is clear from (1) that we have $\nu(h) = \nu(-h)$. Throughout this paper we assume $\Sigma \equiv 0$ and that $\nu(h)$ has a (necessarily symmetric) density w.r.t. Lebesgue measure; in abuse of notation we write $\nu(h) = \nu(h) \, dh$.

Let $Q = (0,1)^d$ be the open unit cube in $\mathbb{R}^d$ and $a : \mathbb{R}^d \to \mathbb{R}$ a function in $L^p_{\text{loc}}(\mathbb{R}^d)$ for some $1 < p \leq \infty$. We assume that $a$ is $Q$-periodic in the sense that

$$a(h + ke_i) = a(h) > 0 \quad \text{for all } k \in \mathbb{Z}^d, \; i = 1, 2, \ldots, d \; \text{and a.a. } h \in Q; \quad (2)$$
as usual, $e_i$ denotes the $i$th unit vector of $\mathbb{R}^d$. Moreover, we assume that

$$\int_{h \neq 0} (1 \land |h|^2) a(h) \nu(h) \, dh < \infty \quad \text{and} \quad \nu(h) = \nu(-h) > 0, \; h \neq 0. \quad (3)$$

By $\bar{a}$ we denote the mean value of $a$,

$$\bar{a} := \int_Q a(h) \, dh; \quad (4)$$
moreover, we assume that $a_\delta(h) := a(\delta^{-1}h)$ satisfies

\[
\int_{h \neq 0} (1 \wedge |h|^2) a_\delta(h) \nu(h) \, dh < \infty \quad \text{for all } \delta > 0. \tag{5}
\]

For each $\delta > 0$ we consider the following quadratic form in $L^2(\mathbb{R}^d)$ which is defined for Lipschitz continuous functions with compact support $u, v \in C_0^\text{lip}(\mathbb{R}^d)$

\[
E^\delta(u, v) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))(u(x) - u(y)) a_\delta(y - x) \nu(y - x) \, dy \, dx. \tag{6}
\]

From the assumptions (2) and (5), we easily see that $(E^\delta, C_0^\text{lip}(\mathbb{R}^d))$ is a closable symmetric form in $L^2(\mathbb{R}^d)$ which is translation invariant, see [3]. Its closure $(E^\delta, F^\delta)$ is a translation invariant regular symmetric Dirichlet form in $L^2(\mathbb{R}^d)$, and the associated stochastic process is a symmetric Lévy process. If we use (1) and some elementary Fourier analysis, we obtain the following characterization of the Dirichlet form $(E^\delta, F^\delta)$ based on the characteristic exponent $\psi_\delta$, cf. [5, Example 4.7.28] and [3, Example 1.4.1],

\[
\begin{align*}
E^\delta(u, v) &= \int_{\mathbb{R}^d} \hat{u}(\xi) \overline{\hat{v}(\xi)} \psi_\delta(\xi) \, d\xi \\
F^\delta &= \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \psi_\delta(\xi) \, d\xi < \infty \right\},
\end{align*}
\]

\[
\hat{u}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i(\xi, x)} u(x) \, dx \quad \text{denotes the Fourier transform and}
\]

\[
\psi_\delta(\xi) := \int_{h \neq 0} (1 - \cos(\xi, h)) a_\delta(h) \nu(h) \, dh, \quad \xi \in \mathbb{R}^d. \tag{7}
\]

Condition (5) ensures that $a_\delta(h) \nu(h)$ is the density of a Lévy measure. If $\nu(h)$ is the density of a Lévy measure and if $a$ is a bounded, nonnegative (and 1-periodic) function, then (5) clearly holds. The following example illustrates that for unbounded functions $a$ the situation is different.

1 Example. a) Let $0 < \beta < 2$ and pick some $\delta$ such that $0 < \delta < 1 \wedge (2 - \beta)$. Define functions $\alpha_0$ on $[0, 1/2]$ and $\alpha_1$ on $[0, 1]$ by

\[
\alpha_0(x) := \begin{cases} 0, & x = 0, \\
x^{-\delta}, & 0 < x \leq \frac{1}{4}, \quad \text{and} \quad \alpha_1(x) := \begin{cases} \alpha_0(x), & 0 \leq x \leq \frac{1}{2}, \\
\alpha_0(1 - x), & \frac{1}{2} \leq x \leq 1, \end{cases}
\end{cases}
\]

Denote by $a : \mathbb{R} \to \mathbb{R}$ the 1-periodic extension of $\alpha_1$ to the real line. It is obvious that $a \in L_p^\text{loc}(\mathbb{R})$ for all $1 < p < 1/\delta$. Define a further function $b = b(x)$ on $\mathbb{R}$ by $b(x) := a(x - 1/2)$ for $x \in \mathbb{R}$ and set

\[
\nu(h) = \frac{b(h)}{|h|^{1 + \beta}}, \quad h \neq 0.
\]

Clearly, $\nu(h) = \nu(-h)$; let us show that $\nu(h)$ is the density of a Lévy measure, i.e. $\int_{h \neq 0} (1 \wedge h^2) a(h) \nu(h) \, dh < \infty$. 

Since \( a \) and \( \nu \) are even functions, we see
\[
\int_{h \neq 0} (1 \wedge h^2) a(h) \nu(h) \, dh = 2 \int_0^1 h^2 a(h) \nu(h) \, dh + 2 \sum_{\ell = 1}^{\infty} \int_{\ell}^{\ell + 1} a(h) \nu(h) \, dh.
\]
For the first term we get
\[
\int_0^1 h^2 a(h) \nu(h) \, dh = \int_0^1 h^2 a(h) b(h) h^{-1-\beta} \, dh
= 4^\delta \int_0^{1/4} h^{1-\delta - \beta} \, dh + 4^\delta \int_{1/4}^{1/2} h^{1-\beta} (1/2 - h)^{-\delta} \, dh
+ 4^\delta \int_{1/2}^{3/4} h^{1-\beta} (h - 1/2)^{-\delta} \, dh + 4^\delta \int_{3/4}^1 h^{1-\beta} (1 - h)^{-\delta} \, dh
= c(\delta) < \infty.
\]
The integrals under the sum appearing in the second term can be estimated using
the periodicity of \( a \) and \( b \); for all \( \ell \geq 1 \) we have
\[
\int_{\ell}^{\ell + 1} a(h) \nu(h) \, dh = \int_0^1 (h + \ell) b(h + \ell) (h + \ell)^{-1-\beta} \, dh
= \int_0^1 a(h) b(h) (h + \ell)^{-1-\beta} \, dh
\leq \ell^{-1-\beta} \int_0^1 a(h) b(h) \, dh.
\]
As in the previous calculus, noting \( 0 < \delta < 1 \) we see that
\[
\int_0^1 a(h) b(h) \, dh = 4^\delta \int_0^{1/4} h^{-\delta} \, dh + 4^\delta \int_{1/4}^{1/2} (1/2 - h)^{-\delta} \, dh
+ 4^\delta \int_{1/2}^{3/4} h^{-\delta} \, dh + 4^\delta \int_{3/4}^1 (1 - h)^{-\delta} \, dh < \infty.
\]
So \( c := \int_0^1 a(h) b(h) \, dh < \infty \). Thus,
\[
\int_{h \neq 0} (1 \wedge h^2) a(h) \nu(h) \, dh \leq 2c(\delta) + c \sum_{\ell = 1}^{\infty} \ell^{-1-\beta} < \infty.
\]
On the other hand, we also find that
\[
\int_{h \neq 0} (1 \wedge h^2) a_{1/2}(h) \nu(h) \, dh = \int_{h \neq 0} (1 \wedge h^2) a(2h) b(h) |h|^{-1-\beta} \, dh
\geq \int_{3/8}^{1/2} h^2 a(2h) b(h) h^{-1-\beta} \, dh
= \int_{3/8}^{1/2} h^{1-\beta} (1 - 2h)^{-\delta} (1/2 - h)^{-\delta} \, dh
= 2^\delta \int_{3/8}^{1/2} h^{1-\beta} (1 - 2h)^{-2\delta} \, dh.
\]
and this integral blows up if $0 < \beta < 3/2$ and $1/2 \leq \delta < 1 \wedge (2 - \beta)$. In a similar way we can show that
\[
\int_{h \neq 0} (1 + h^2) a_\delta(h) \nu(h) \, dh = \infty
\]
for infinitely many $\delta > 0$.

b) Let $a = a(x)$ on $\mathbb{R}$ be as in part (a)]. Set $\nu(h) = |h|^{-1-\beta}$ for $h \neq 0$. Then we can show that this pair $(a, \nu)$ satisfies the conditions (2)–(5).

We will now discuss the limit of $(C^\delta, F^\delta)$ as $\delta \downarrow 0$. To this end, we take a sequence of positive numbers $\{\delta_n\}_{n \in \mathbb{N}}$ such that $\delta_n \downarrow 0$ as $n \to \infty$.

2 Lemma. Suppose that (2) and (5) hold for the function $a$. The measures $\{a_{\delta_n}(h) \, dh\}_{n \in \mathbb{N}}$ converge to the measure $\bar{a} \, dh$ in the vague topology, i.e. for all compactly supported continuous functions $g \in C_0(\mathbb{R}^d)$ one has
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} g(h)a_{\delta_n}(h) \, dh = \bar{a} \int_{\mathbb{R}^d} g(h) \, dh, \quad g \in C_0(\mathbb{R}^d). \tag{8}
\]

Proof. We will show this lemma only in dimension $d = 1$, the case $d > 1$ adds only complications in notation. Fix $n \in \mathbb{N}$ and take any $g \in C_0(\mathbb{R})$. We have
\[
\int_{\mathbb{R}} g(h)a_{\delta_n}(h) \, dh = \int_{\mathbb{R}} g(h)a(\delta_n^{-1}h) \, dh = \delta_n \int_{\mathbb{R}} g(\delta_n h)a(h) \, dh = \delta_n \sum_{k=-\infty}^{\infty} \int_{k}^{k+1} g(\delta_n h)a(h) \, dh.
\]
Because of the periodicity of $a$, it follows that
\[
\int_{\mathbb{R}} g(h)a_{\delta_n}(h) \, dh = \delta_n \sum_{k=-\infty}^{\infty} \int_{0}^{1} g(\delta_n (h + k))a(h + k) \, dh = \delta_n \sum_{k=-\infty}^{\infty} \int_{0}^{1} g(\delta_n (h + k))a(h) \, dh.
\]
Since $g$ has compact support,
\[
g_{n,h}(\xi) := \delta_n \sum_{k=-\infty}^{\infty} g(\delta_n (h + k))1_{(\delta_n k, \delta_n (k+1))}(\xi)
\]
is, for fixed $h \in [0, 1]$, a family of step functions (each with finitely many values) indexed by $\delta_n$. It converges uniformly to $\int_{\mathbb{R}} g(\xi) \, d\xi$ as $n \to \infty$. Therefore,
\[
\int_{\mathbb{R}} g(h)a_{\delta_n}(h) \, dh = \int_{0}^{1} \left( \delta_n \sum_{k=-\infty}^{\infty} g(\delta_n (h + k)) \right) a(h) \, dh = \int_{0}^{1} \int_{\mathbb{R}} g_{n,h}(\xi) \, d\xi \, a(h) \, dh \xrightarrow{n \to \infty} \int_{0}^{1} \left( \int_{\mathbb{R}} g(\xi) \, d\xi \right) a(h) \, dh.
\]
This proves (8) for $d = 1$. \qed
A similar argument leads to the following variant of Lemma 2.

3 Corollary. Suppose that \( \| \) and \( \| \) hold. The family \( \{a_{\delta_n}\}_{n \in \mathbb{N}} \) converges to the constant \( \bar{a} := \int_Q a(h) \, dh \) weakly in \( L^p_{\text{loc}}(\mathbb{R}^d) \), 1 < \( p < \infty \), i.e. for any compact set \( K \) of \( \mathbb{R}^d \),
\[
\lim_{n \to \infty} \int_K g(x) a_{\delta_n}(x) \, dx = \bar{a} \int_K g(x) \, dx, \quad g \in L^q(K),
\]
where \( p \) and \( q \) are conjugate.

Proof. Let \( K \subset \mathbb{R}^d \) be a compact set. We may regard \( g1_K \) as an element of \( L^q(\mathbb{R}^d) \). Since \( C_0^\infty(\mathbb{R}^d) \) is dense in \( L^q(\mathbb{R}^d) \), there is for each \( \epsilon > 0 \), some \( \phi = \phi_\epsilon \in C_0^\infty(\mathbb{R}^d) \) such that \( \| \phi - g1_K \|_{L^q(\mathbb{R}^d)} < \epsilon \) and \( \supp \phi \subset K_1 := \{ x + y : x \in K, |y| \leq 1 \} =: K_1 \). So,
\[
\left| \int_K g(x) a_{\delta_n}(x) \, dx - \bar{a} \int_K g(x) \, dx \right| = \left| \int_{\mathbb{R}^d} g(x)1_K(x) a_{\delta_n}(x) \, dx - \bar{a} \int_{\mathbb{R}^d} g(x)1_K(x) \, dx \right|
\leq \int_{K_1} \left| g(x)1_K(x) - \phi(x) \right| a_{\delta_n}(x) \, dx + \int_{\mathbb{R}^d} \phi(x) \left( a_{\delta_n}(x) - \bar{a} \right) \, dx
\leq \left\| g1_K - \phi \right\|_{L^q(\mathbb{R}^d)} \left( \int_{K_1} a_{\delta_n}(x)^p \, dx \right)^{1/p} + \int_{\mathbb{R}^d} \phi(x) \left( a_{\delta_n}(x) - \bar{a} \right) \, dx
\leq \int_{\mathbb{R}^d} \left( a_{\delta_n}(x)^p \, dx \right)^{1/p} + \int_{\mathbb{R}^d} \phi(x) \left( a_{\delta_n}(x) - \bar{a} \right) \, dx.
\]

By the previous lemma, the second term on the right hand side tends to 0 as \( n \to \infty \). The expression \( \left\| g1_K - \phi \right\|_{L^q(\mathbb{R}^d)} \) can be made arbitrarily small if we choose \( \phi \) accordingly. This means that it is enough to show that \( \int_{K_1} a_{\delta}(x)^p \, dx \) is bounded for \( 0 < \delta < 1 \). Again, we consider the one-dimensional case, the arguments for \( d > 1 \) just have heavier notation.

Without loss of generality we may assume that \( K_1 = [-N, N] \) for some \( N \in \mathbb{N} \). Take \( k := \lfloor N/\delta \rfloor + 1 \in \mathbb{N} \), the smallest integer which is bigger or equal \( N/\delta \). We have
\[
I = \int_{-N}^N a_{\delta}(x)^p \, dx = \delta \int_{-N/\delta}^{N/\delta} a(x)^p \, dx \leq \delta \int_{-k}^{k-1} a(x)^p \, dx = \delta \sum_{\ell = -k}^{k-1} \int_{\ell}^{\ell+1} a(x)^p \, dx,
\]
and, because of the periodicity of \( a \),
\[
I \leq \delta \sum_{\ell = -k}^{k-1} \int_0^1 a(x + \ell)^p \, dx = 2k\delta \int_0^1 a(x)^p \, dx \leq 2(N + 1) \int_0^1 a(x)^p \, dx.
\]

4 Corollary. Assume that \( \square \)–\( \square \) hold and let \( \{\delta_n\}_{n \in \mathbb{N}} \) be a monotonically decreasing sequence of positive numbers such that \( \delta_n \to 0 \) as \( n \to \infty \). For any compact set \( K \subset \mathbb{R}^d \times \mathbb{R}^d \), let \( g_n \in L^q(K) \) be a sequence of functions which converges in \( L^q \) to some \( g \in L^q(K) \). Then the following limit exists
\[
\lim_{n \to \infty} \int_K g_n(x, y) a_{\delta_n}(x - y) \, dx \, dy = \bar{a} \int_K g(x, y) \, dx \, dy.
\]

5
Proof. Note that
\[
\left| \int_K g_n(x, y) a_\delta_n(x - y) \, dx \, dy - \bar{a} \int_K g(x, y) \, dx \, dy \right| \\
\leq \left| \int_K (g_n(x, y) - g(x, y)) a_\delta_n(x - y) \, dx \, dy \right| + \left| \int_K g(x, y) (a_\delta_n(x - y) - \bar{a}) \, dx \, dy \right| \\
\leq \left[ \int_K |g_n(x, y) - g(x, y)|^q \, dx \, dy \right]^{\frac{1}{q}} \left[ \int_K a_\delta_n(x - y)^p \, dx \, dy \right]^{\frac{1}{p}} + \left| \int_{\mathbb{R}^d} H(z) (a_\delta_n(z) - \bar{a}) \, dz \right| 
\]
where we use
\[
H(z) := \int_{\mathbb{R}^d} 1_K(y + z, y)g(y + z, y) \, dy, \quad z \in \mathbb{R}^d.
\]
Since \( K \) is a compact set of \( \mathbb{R}^d \times \mathbb{R}^d \), we have (i) \( \sup_{n \in \mathbb{N}} \int_K a_\delta_n(x - y)^p \, dx \, dy < \infty \) as in the proof of Corollary 3 and (ii) the function \( H \) has compact support, hence \( H \in L^q(\mathbb{R}^d) \). Therefore, the first term on the right hand side converges to 0 as \( n \to \infty \), while the second term tends to 0 because of Corollary 3.

Recall that a sequence of quadratic forms \( \{(\mathcal{E}^n, \mathcal{F}^n)\}_{n \in \mathbb{N}} \) defined on \( L^2(\mathbb{R}^d) \) is called Mosco-convergent, if the following two conditions are satisfied

(M1) For all \( u \in L^2(\mathbb{R}^d) \) and all sequences \( (u_n)_{n \in \mathbb{N}} \subset L^2(\mathbb{R}^d) \) such that \( u_n \rightharpoonup u \) (weak convergence in \( L^2 \)) we have \( \liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u, u) \).

(M2) For every \( u \in \mathcal{F} \) there exist elements \( u_n \in \mathcal{F}^n, n \in \mathbb{N} \), such that \( u_n \to u \) (strong convergence in \( L^2 \)) and \( \limsup_{n \to \infty} \mathcal{E}^n(u_n, u_n) \leq \mathcal{E}(u, u) \).

Note that (M1) entails that we have \( \limsup_{n \to \infty} \mathcal{E}^n(u_n, u_n) = \mathcal{E}(u, u) \) in (M2).

We can now state the main result of our paper.

5 Theorem. Assume that (2)–(5) hold for the functions \( a \) and \( \nu \), and let \( \nu \) be locally bounded as a function defined on \( \mathbb{R}^d \setminus \{0\} \). Let \( \{\delta_n\}_{n \in \mathbb{N}} \) be a monotonically decreasing sequence of positive numbers such that \( \delta_n \to 0 \) as \( n \to \infty \). For each \( n \in \mathbb{N} \) we consider the Dirichlet forms \( (\mathcal{E}^n, \mathcal{F}^n) := (\mathcal{E}^{\delta_n}, \mathcal{F}^{\delta_n}) \) defined in (3). The Dirichlet forms \( (\mathcal{E}^n, \mathcal{F}^n) \) converge to \( (\mathcal{E}, \mathcal{F}) \) in the sense of Mosco. The limit \( (\mathcal{E}, \mathcal{F}) \) is the closure of \( (\mathcal{E}, C_0^{lip}(\mathbb{R}^d)) \) which is given by
\[
\mathcal{E}(u, v) := \bar{a} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)) \nu(y - x) \, dy \, dx, \quad u, v \in C_0^{lip}(\mathbb{R}^d).
\]

Proof. We will check the conditions (M1) and (M2) of Mosco convergence. For (M1) we take any \( u \in L^2(\mathbb{R}^d) \) and any sequence \( \{u_n\} \subset L^2(\mathbb{R}^d) \) such that \( u_n \rightharpoonup u \) as \( n \to \infty \). Without loss, we may assume that \( \liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) < \infty \).

We will use the Friedrichs mollifier. This is a family of convolution operators
\[
J_\epsilon[u](x) = \int_{\mathbb{R}^d} u(x - y) \rho_\epsilon(y) \, dy, \quad x \in \mathbb{R}^d, \; \epsilon > 0,
\]
given by the kernels \( \{ \rho \}_{\epsilon > 0} \) for a \( C^\infty \)-kernel \( \rho : \mathbb{R}^d \rightarrow [0, \infty) \) satisfying
\[
0 \leq \rho(x) = \rho(-x), \quad \int_{\mathbb{R}^d} \rho(x) \, dx = 1, \quad \text{supp} \{ \rho \} = \{ x \in \mathbb{R}^d : |x| \leq 1 \}
\]
and \( \rho_\epsilon(x) := \rho(x/\epsilon) \), for \( \epsilon > 0 \) and \( x \in \mathbb{R}^d \).

We then have
\[
\mathcal{E}^n(u_n, u_n)
= \iint_{x \neq y} (u_n(x) - u_n(y))^2 \alpha_{\delta_n}(y - x) \nu(y - x) \, dy \, dx
= \int_{\mathbb{R}^d} \left( \iint_{x \neq y} (u_n(x) - u_n(y))^2 \alpha_{\delta_n}(y - x) \nu(y - x) \, dy \, dx \right) \rho_\epsilon(z) \, dz
= \int_{\mathbb{R}^d} \left( \iint_{x \neq y} (u_n(x) - u_n(y))^2 \alpha_{\delta_n}(y - x) \nu(y - x) \, dy \, dx \right) \rho_\epsilon(z) \, dz,
\]
and using the Fubini theorem and Jensen’s inequality yields, for any compact set \( K \) so that \( K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \),
\[
\mathcal{E}^n(u_n, u_n)
= \iint_{x \neq y} \left( \int_{\mathbb{R}^d} (u_n(x) - u_n(y))^2 \rho_\epsilon(z) \, dz \right) \alpha_{\delta_n}(y - x) \nu(y - x) \, dy \, dx
\geq \iint_{x \neq y} \left( \int_{\mathbb{R}^d} (u_n(x) - u_n(y)) \rho_\epsilon(z) \, dz \right)^2 \alpha_{\delta_n}(y - x) \nu(y - x) \, dy \, dx
\geq \iint_{K} (J_\epsilon[u_n](x) - J_\epsilon[u_n](y))^2 \alpha_{\delta_n}(y - x) \nu(y - x) \, dy \, dx.
\]
Note that \( \sup_{x \in \mathbb{R}^d} \| u_n \|_{L^2} < \infty \) because of the weak convergence \( u_n \rightharpoonup u \). Again by weak convergence, \( u_n \rightharpoonup u \), and we conclude that \( u_{n,\epsilon} = J_\epsilon[u_n] \) converges pointwise to \( u_\epsilon := J_\epsilon[u] \). Using the local boundedness of \( \nu \) on \( \mathbb{R}^d \setminus \{0\} \) and the fact that \( K \) is a compact set satisfying \( K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \), we see that \( (u_{n,\epsilon}(x) - u_{n,\epsilon}(y))^2 \nu(y - x) \) converges in \( L^2(K) \) to
\[
(u_\epsilon(x) - u_\epsilon(y))^2 \nu(y - x) \quad \text{as} \quad n \to \infty.
\]
From (10) we get
\[
\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) \geq \liminf_{n \to \infty} \mathcal{E}^n(u_{n,\epsilon}, u_{n,\epsilon})
\geq \liminf_{n \to \infty} \iint_{K} (u_{n,\epsilon}(x) - u_{n,\epsilon}(y))^2 \alpha_{\delta_n}(y - x) \nu(y - x) \, dy \, dx
= \bar{a} \iint_{K} (u_\epsilon(x) - u_\epsilon(y))^2 \nu(y - x) \, dy \, dx.
\]
Since \( K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \) is an arbitrary compact set, we can approximate \( \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \) by such sets. Using monotone convergence and the fact that the left hand side is independent of \( K, L \), we arrive at
\[
\sup_{0 < \epsilon < 1} \mathcal{E}(u_\epsilon, u_\epsilon) = \sup_{0 < \epsilon < 1} \sup_{K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\}} \bar{a} \iint_{K} (u_\epsilon(x) - u_\epsilon(y))^2 \nu(y - x) \, dy \, dx
\leq \liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) < \infty.
\]
Theorem 2.4 in [8] now shows that $u_\epsilon \in \mathcal{F} \cap C^\infty(\mathbb{R}^d)$ for each $\epsilon \in (0, 1)$. Since $J_\epsilon$ is an $L^2$-contraction operator for each $\epsilon > 0$, we see that the family $\{u_\epsilon\}_{\epsilon > 0}$, $u_\epsilon = J_\epsilon[u]$, is bounded w.r.t. $\mathcal{E}_1(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + (\cdot, \cdot)_{L^2}$ by (11). The Banach–Alaoglu theorem guarantees that there is an $\mathcal{E}_0$-weakly convergent subsequence $u_{\epsilon(n)}$, $\epsilon(n) \downarrow 0$, and a function $v$ so that $u_{\epsilon(n)}$ converges $\mathcal{E}_1$-weakly to $v \in \mathcal{F}$. Using the Banach–Saks theorem shows that the Cesàro means $\frac{1}{n} \sum_{k=1}^n u_{\epsilon(n_k)}$ of a further subsequence converge $\mathcal{E}_1$-strongly, hence in $L^2(\mathbb{R}^d)$, to $v$. As $u_\epsilon$ converges to $u$ in $L^2(\mathbb{R}^d)$, we can identify the limit as $u = v$. In particular, $u \in \mathcal{F}$ and

$$\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u, u).$$

In order to see (M2), we use the regularity of the Dirichlet form $(\mathcal{E}, \mathcal{F})$; therefore, it is enough to consider $u \in C_0^{lip}(\mathbb{R}^d)$. Set $u_n = u \in C_0^{lip}(\mathbb{R}^d)$ for each $n$. We have

$$\mathcal{E}^n(u_n, u_n) = \mathcal{E}^n(u, u) = \int\int_{x \neq y} (u(x) - u(y))^2 a_{\delta_n}(y - x) \nu(y - x) \, dy \, dx,$$

and we conclude with (8) that $\lim_{n \to \infty} \mathcal{E}^n(u_n, u_n) = \mathcal{E}(u, u)$. □

**6 Remark.** Suppose that the function $a$ on $\mathbb{R}$ satisfies (2–5), and $\nu$ is given by $\nu(x) = |x|^{-1 - \alpha}$, $x \in \mathbb{R} \setminus \{0\}$, for some $0 < \alpha < 2$. Then the following quadratic form defines a translation invariant regular symmetric Dirichlet form on $L^2(\mathbb{R})$:

$$\tilde{\mathcal{E}}(u, v) := \int\int_{x \neq y} (u(x) - u(y))(u(x) - u(y)) \frac{a(x - y)}{|x - y|^{1+\alpha}} \, dx \, dy, \quad u, v \in C_0^{lip}(\mathbb{R}).$$

Let $\tilde{X} = (\tilde{X}(t))_{t \geq 0}$ be the symmetric Lévy process on $\mathbb{R}$ associated with the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\mathbb{R})$. For any $n \in \mathbb{N}$, set

$$X^{(n)}(t) := \epsilon_n \tilde{X}(\epsilon_n^{-\alpha} t), \quad t > 0.$$ 

Then $X^{(n)} = (X^{(n)}(t))_{t \geq 0}$ is also a symmetric Lévy process and we denote for each $n \in \mathbb{N}$ by $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ the corresponding Dirichlet form. The semigroup $\{T_t^{(n)}\}_{t \geq 0}$ generated by $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ is given by

$$T_t^{(n)} f(x) = \mathbb{E} \left[ f(X^{(n)}(t)) \mid X^{(n)}(0) = x \right]$$

$$= \mathbb{E}_{x/\epsilon_n} \left[ f(\epsilon_n \tilde{X}(\epsilon_n^{-\alpha} t)) \right] = \left( \tilde{T}_{\epsilon_n^{-\alpha} t} f(\epsilon_n \cdot) \right)(\epsilon_n^{-1} x), \quad x \in \mathbb{R}.$$ 

Since the Dirichlet form $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ can be obtained by

$$\mathcal{E}^{(n)}(u, v) = \lim_{t \downarrow 0} \frac{1}{t} (u - T_t^{(n)} u, v)_{L^2},$$

it follows for $t > 0$ that

$$\frac{1}{t} (u - T_t^{(n)} u, v)_{L^2} = \frac{1}{t} \int_{\mathbb{R}} [u(x) - T_t^{(n)} u(x)] v(x) \, dx$$

$$= \frac{1}{t} \int_{\mathbb{R}} [u(\epsilon_n \cdot, \epsilon_n^{-1} x) - (\tilde{T}_{\epsilon_n^{-\alpha} t} u(\epsilon_n \cdot)) (\epsilon_n^{-1} x)] v(x) \, dx$$

$$= \frac{1}{\epsilon_n^{\alpha}} \cdot \frac{1}{s} \int_{\mathbb{R}} [u(\epsilon_n \xi) - (\tilde{T}_s u(\epsilon_n \cdot)) (\xi)] v(\epsilon_n \xi) \epsilon_n \, d\xi$$

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where we use the notation $\xi = \epsilon_n^{-1}x$ and $s = \epsilon_n^{-\alpha}t$. Letting $s \to 0$, hence $t \to 0$, yields

$$\lim_{t \to 0} \frac{1}{t} \langle u - T_t^{(n)} u, v \rangle_{L^2} = \epsilon_n^{1-\alpha} \cdot \tilde{\mathcal{E}}(u(\epsilon_n \cdot), v(\epsilon_n \cdot))$$

$$= \epsilon_n^{1-\alpha} \int_{\mathbb{R}} (u(\epsilon_n x) - u(\epsilon_n y)) (v(\epsilon_n x) - v(\epsilon_n y)) \frac{a(x - y)}{|x - y|^{1+\alpha}} \, dx \, dy$$

$$= \epsilon_n^{1-\alpha} \int_{\mathbb{R}} (u(x) - u(y)) (v(x) - v(y)) \frac{a(\epsilon_n^{-1}(x - y))}{|x - y|^{1+\alpha}} \, dx \, dy$$

$$= \int_{\mathbb{R}} (u(x) - u(y)) (v(x) - v(y)) \frac{a(\epsilon_n^{-1}(x - y))}{|x - y|^{1+\alpha}} \, dx \, dy$$

$$= \mathcal{E}^{(n)}(u, v).$$

Since Mosco convergence entails the convergence of the semigroups, hence the finite-dimensional distributions (fdd) of the processes, we may combine the above calculation with Theorem 5 to get the following result: The processes $X^{(n)}$ associated with $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ – these are obtained by scaling $t \mapsto \epsilon_n^{-\alpha}t$ and $x \mapsto \epsilon_n x$ from the process $\tilde{X}$ given by $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ – converge, in the sense of fdd, to the process $X$ associated with $(\mathcal{E}, \mathcal{F})$. This is the Dirichlet form approach to the problem discussed in [4] (see also [10, 7, 6] for related works and [9] for Mosco convergence of Dirichlet forms).

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