Collective Lamb shift of superradiant cascade emissions in an atomic ensemble

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We calculate the collective Lamb shift of the cascade spontaneous emissions from an atomic ensemble driven by two-color classical fields. The correlated pair of photons (signal and idler) is generated by adiabatically driving the system with large-detuned light fields in four-wave-mixing condition. The signal photon from the upper transition of the diamond-type atomic levels is followed by the idler one which can be superradiant due to light-induced dipole-dipole interactions. We demonstrate that the collective Lamb shift of the idler photon is a cumulative effect of interaction energy, and investigate its dependence on a cylindrical geometry. Manipulating the collective frequency of cascade emissions enables frequency qubits that provide alternative robust elements in quantum network.

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I. INTRODUCTION

Superradiance [1] is a coherent and collective radiation from a multi-atom system. The radiation intensity is proportional to the square of particle number along with a shorter time variation, which conserves radiation energy. This collective radiation originates from a common light-matter interaction, through which spontaneously emitted photons can be reabsorbed by the atoms [2, 3] if they are close to each other, and interatomic phase correlations build up [4]. This spontaneous emission decays in an enhanced rate proportional to the number of particles, which is very different from spontaneous emissions of independent and uncorrelated single atoms [5].

Many theoretical works from different perspectives have investigated the superradiant emission. The microscopic mechanism of superradiance is shown to be related to induced dipole-dipole interactions [6, 7] between two atoms, in which spontaneous decay rate and frequency shift [8, 9] depend on their spatial separation. A master equation is formulated [10, 11] to study its statistical property [11], and a diagonalization of coupling matrix [12, 13] reveals the essential information of superradiance decay constants. One of the characteristics in superradiance is its directionality of emission [12, 14], and there is also a semiclassical treatment [15, 16] incorporating propagation effect, which indicates a threshold condition of cooperative emission in an extended medium. The QED approach for Lorentz-Lorenz shift [17, 18] of superradiance by a local field correction also applies to the extended medium of a slab and an ellipsoid [19]. An alternative approach of quantum trajectory method [20, 21] for superradiance tries to unravel the physics of successive sub- and superradiant photons, and positive-P phase-space method simulations show an enhanced decay in second order correlation function by including quantum fluctuations in diamond-type atomic system [22].

The preparation of single-photon absorption in the atomic ensemble followed by collective [23] and directional emission [24] raises the interests of single-photon superradiance. This singly-inverted system reduces Dicke’s full eigenstates of spin-1/2 system into only N state bases, and shows dynamical evolutions [25, 26] from a specified initial state. They demonstrate the initially symmetrical excited state for two-level atoms radiates superradiantly, while other less symmetrical excited states radiate in a slower rate with smaller probabilities [25]. Some recent experiments demonstrate superradiance or collective Lamb shift [27] in a variety of atomic systems, including atoms in a planar cavity [28], atomic vapor layers [34], cold atoms [35], and cascade atomic systems [36, 37].

The cascade atomic system provides a source for telecommunication bandwidth in its upper transition [38]. This correlated photon pair can realize long-distance quantum communication [39, 40] enabling a low-loss quantum repeater [40] in the (Duan-Lukin-Cirac-Zoller) DLCZ protocols [41]. Here we investigate the superradiance of correlated photon pair from a diamond-type atomic ensemble driven by two-color classical fields, and specifically we calculate the collective Lamb shift in an optically-thick cylindrical atomic system. Single atomic excitation adiabatically follows two excitation fields, and subsequently decays through cascade transitions in four-wave mixing (FWM) condition. Non-rotating wave approximation (RWA) in the Hamiltonian is required to correctly deduce the frequency shift which is a cumulative dipole-dipole interaction energy. We then investigate the geometry and density dependence of the frequency shift.

In this paper, we solve for Schrödinger equations of N identical atoms interacting with two-color laser fields and two quantum fields. We derive two-photon state function in bare state and symmetrical state bases in Sec. II. In Sec. III we calculate and study the collective Lamb shift of two-photon state in a cylindrical geometry. We conclude and summarize in Sec. IV, and the inclusion of virtual photons exchange and adiabatic approximation of excitation process are detailed in Appendixes A and B respectively.

II. TWO-PHOTON STATE FUNCTION

A. Hamiltonian and Schrödinger equation of motion

We consider an ensemble of N four-level atoms excited by two classical fields, and subsequently the signal and idler pho-
tons are spontaneously emitted as shown in Fig. 1. These identical atoms distribute randomly in a cylindrical geometry with a uniform density. We use dipole approximation of light-matter interaction and rotating wave approximation (RWA), and the Hamiltonian in interaction picture reads

\[
V_I = -\hbar \Delta_1 \sum_{\mu=1}^{N} |1 \rangle_\mu \langle 1| - \hbar \Delta_2 \sum_{\mu=1}^{N} |2 \rangle_\mu \langle 2|
\]

\[
- \frac{\hbar}{2} \left\{ \Omega_a \hat{P}_k^a + \Omega_b \hat{P}_k^b + \text{h.c.} \right\} - i \hbar \sum_{\mu \nu} g_{k_\mu} \epsilon_{k_\mu, \lambda_\mu, \lambda_\nu} \hat{S}_{k_\mu}^+ e^{-i(\omega_{k_\mu} - \omega_{k_\nu} - \Delta_2) t} + \sum_{k_\mu, \lambda_\mu} g_{k_\mu} \epsilon_{k_\mu, \lambda_\mu, \lambda_\nu} \hat{S}_{k_\mu} e^{-i(\omega_{k_\mu} - \omega_{k_\nu}) t} - \text{h.c.},
\]

where h.c. denotes Hermitian conjugate. Note that the RWA Hamiltonian is valid for deriving state functions to calculate transition probabilities while non-RWA or counter-RWA terms are required to account for a complete derivation of collective Lamb shift (see in Appendix A). The collective dipole operators are defined as

\[
\hat{P}_k^\dagger = \sum_\mu |1 \rangle_\mu \langle 0| e^{ik_\mu \cdot r_\mu}, \quad \hat{P}_k^2 = \sum_\mu |2 \rangle_\mu \langle 1| e^{ik_\mu \cdot r_\mu},
\]

\[
\hat{S}_{k_\mu}^\dagger = \sum_\mu |2 \rangle_\mu \langle 3| e^{ik_\mu \cdot r_\mu}, \quad \hat{S}_{k_\mu} = \sum_\mu |3 \rangle_\mu \langle 2| e^{ik_\mu \cdot r_\mu},
\]

where single photon detuning \( \Delta_1 = \omega_{a_1} - \omega_1 \), two-photon detuning \( \Delta_2 = \omega_{a_2} + \omega_b - \omega_2 \), and transition frequency \( \omega_{23} = \omega_2 - \omega_3 \). Driving Rabi frequencies are \( \Omega_a \equiv (1|\hat{d}|0) \mathcal{E}(k_a)/\hbar \), \( \Omega_b \equiv (2|\hat{d}|1) \mathcal{E}(k_b)/\hbar \), and coupling coefficients are \( g_{k_\mu} \equiv (3|\hat{d}|2) \mathcal{E}(k_\mu)/\hbar \), \( g_{k_\mu} \equiv (0|\hat{d}|3) \mathcal{E}(k_\mu)/\hbar \). The double matrix element of the dipole moment \( \hat{d} \) is independent of the hyperfine structure, and \( \mathcal{E}(k) = \sqrt{\hbar \nu k/(2e_0 V)} \) where \( V \) is the quantization volume. Polarizations of signal and idler fields are \( \epsilon_{k_\mu, \lambda_\mu, \lambda_\nu} \), and the unit directions of dipole operators are \( d_{s,i} \), respectively.

In the limit of large detuned and weak driving fields, \( \Delta_1 \gg \sqrt{N|\Omega_a|} \), we consider only single excitation, and neglect spontaneous decay during excitations (\( \Delta_1 \gg \Gamma_1, \Delta_2 \gg \Gamma_2 \)). We may express the state function as

\[
|\psi(t)\rangle = \mathcal{E}(t)|0, \text{vac}\rangle + \sum_{\mu=1}^{N} A_{\mu}(t)|1_\mu, \text{vac}\rangle
\]

\[
+ \sum_{\mu=1}^{N} B_{\mu}(t)|2_\mu, \text{vac}\rangle + \sum_{\mu=1}^{N} C_{s,\mu}(t)|3_\mu, \lambda_\mu, \lambda_\mu\rangle
\]

\[
+ \sum_{s,i} D_{s,i}(t)|0, 1_{k_s, \lambda_\mu}, 1_{k_s, \lambda_\mu}\rangle,
\]

where the indices \( s = (k_s, \lambda_s), i = (k_i, \lambda_i) \) for light fields, \( |m_\mu\rangle \equiv |m\rangle_{\mu=0} \otimes |N-1\rangle \) for atomic levels \( m = 1, 2, 3 \), and \( |\text{vac}\rangle \) is the vacuum photon state. These bare states make a complete basis of collective excitations for identical particles, which describe a complete cycle of a single atom following the excitation and spontaneously emitted fields. We apply Schrödinger equation \( i \hbar \frac{\partial}{\partial t}|\psi(t)\rangle = V_I(t)|\psi(t)\rangle \), and the coupled equations of motion are

\[
i \dot{\mathcal{E}} = -\frac{\Omega_a^2}{2} \sum_\mu e^{-ik_\mu \cdot r_{\mu}} A_{\mu},
\]

\[
i \dot{A}_{\mu} = -\frac{\Omega_b^2}{2} e^{-ik_\mu \cdot r_{\mu}} \mathcal{E} - \frac{\Omega_b^*}{2} e^{-ik_\mu \cdot r_{\mu}} B_{\mu} - \Delta_1 A_{\mu},
\]

\[
i \dot{B}_{\mu} = -\frac{\Omega_b}{2} \epsilon_{k_\mu, \lambda_\mu, \lambda_\nu} e^{-i(\omega_{k_\mu} - \omega_{k_\nu} - \Delta_2) t} C_{s,\mu}^*,
\]

\[
i \dot{C}_{s,\mu} = i g_{k_\mu} \epsilon_{k_s, \lambda_s, \lambda_\mu} \hat{d}_{s} e^{-i(\omega_{k_s} - \omega_{k_\mu} - \Delta_2) t} D_{s,i},
\]

\[
i \dot{D}_{s,i} = i g_{k_s} \epsilon_{k_s, \lambda_s, \lambda_\mu} \hat{d}_{s} \sum_{s, i} e^{-i(\omega_{k_s} - \omega_{k_\mu} - \omega_{k_\nu}) t} C_{s,\mu}^*,
\]

In the next subsection, we proceed to solve for two-photon state function from the above equation of motion in the bare state basis.

### B. Two-photon state in bare state basis

Firstly we solve for signal photon state function by substituting Eq. (8) into Eq. (7), and the time derivative of \( C_{s,\mu}(t) \) becomes

\[
\dot{C}_{s,\mu}^* = g_{k_s}^* \epsilon_{k_\mu, \lambda_s} \hat{d}_{s} e^{-i(\omega_{k_s} - \omega_{k_\mu} - \omega_{k_\nu} - \Delta_2) t} B_{\mu} + \sum_{\nu \neq \mu} |g_{s}|^2 |\epsilon_{k_\nu, \lambda_\nu} \hat{d}_{s}^\dagger|^2 e^{i(\omega_{k_\mu} - \omega_{k_\nu} - \omega_{k_\nu}) t} \times \int_0^t dt' e^{i(\omega_{k_\nu} - \omega_{k_\mu}) (t' - t)} C_{s,\mu}(t'),
\]
We use Weisskopf-Wigner approach \cite{42}, and the summation of field modes in the above leads to the frequency shift and decay rate which are

\[
\sum_{\nu} \sum_{k,l} \left| g_{\nu} \right|^2 \langle \epsilon_{k_{\nu},\lambda_{\nu}} \cdot \hat{d}^{\dagger}_{k_{\nu},(r_{\nu},-r_{\nu})} \rangle \times \int_0^\infty dt' e^{i(\omega_{k_{\nu},\lambda_{\nu}}(t'-t))C_{\nu}^{\mu}(t')},
\]

\[
= \sum_{\nu} \sum_{k,l} \frac{|d_{\nu}|^2 \omega_{3}^3}{2 \pi} 3 \pi \epsilon_0 \varepsilon_0^3 C_{\nu}^{\mu}(t') \times \left[ \frac{\delta(\omega_l - \omega_3)}{i P.V. (\omega_l - \omega_3)^{-1}} \right],
\]

\[
= \sum_{\nu} C_{\nu}^{\mu}(t) \left[ \frac{\Gamma_3}{2} F_{\mu\nu}(\xi) - i \Omega_{\mu\nu}^{\frac{-1}{2}}(\xi) \right],
\]

where $\xi = k_{\beta} r_{\mu\nu}$, and $r_{\mu\nu} = |r_{\mu} - r_{\nu}|$. $\Omega_{\mu\nu}^{\frac{-1}{2}}(\xi)$ involves the Lamb shift ($\nu = \mu$) and collective radiation shift from dipole-dipole interaction ($\mu \neq \nu$) which we describe in details in Appendix B. The other frequency shift of $\Omega_{\mu\nu}^{\frac{-1}{2}}(\xi)$ can be derived by including non-RWA terms as shown in Appendix A.

Therefore the differential equation of probability amplitude of $C_{\nu}^{\mu}(t)$ becomes

\[
\frac{d}{dt} C_{\nu}^{\mu}(t) = g_{\nu}^* \langle \epsilon_{k_{\nu},\lambda_{\nu}} \cdot \hat{d} \rangle e^{-ik_{\nu} \cdot r_{\nu} e^{i(\omega_{k_{\nu},\lambda_{\nu}} - \omega_3 - \Delta_2)}} B_{\mu}(t)
\]

\[
- \frac{\Gamma_3}{2} \sum_{\nu'} C_{\nu'}^{\mu}(t) F_{\mu\nu'}(\xi) + i \sum_{\nu' \neq \mu} 2 C_{\nu'}^{\mu}(t) G_{\mu\nu'}(\xi)
\]

\[
+ i C_{\nu}^{\mu}(t) \Omega),
\]

where $\tilde{\Omega} \equiv \Omega / (\Gamma_3 / 2)$, and $\Omega$ is Lamb shift. $F_{\mu\nu}$ and $G_{\mu\nu}$ represent the collective contributions to spontaneous decay rate and frequency shift, which are induced from the couplings between N states as shown schematically in Fig. 12.

The differential equation of Eq. (11) can be expressed in a matrix form as an eigenvalue problem, and we can solve for $C_{\nu}^{\mu}(t)$ by a similarity transformations $U = (u_1, u_2, \ldots, u_N)$ comprised of eigenvectors $u_i$,

\[
C_{\nu}^{\mu}(t) = g_s^* \langle \epsilon_{k_{\nu},\lambda_{\nu}} \cdot \hat{d} \rangle \int d't' e^{i(\omega_{k_{\nu},\lambda_{\nu}} - \omega_3 - \Delta_2)} t' \times \sum_{l,m} U_{\mu l} e^{i\lambda_l(t-t')} \tilde{U}_{lm} B_{\mu}(t') e^{-i\hat{k}_{\nu} \cdot \tau_m},
\]

where $\lambda_l$ and $u_l$ are eigenvalues and eigenvectors respectively of $N \times N$ matrix $M$,

\[
M \equiv \frac{-\Gamma_3}{2} \times \begin{pmatrix}
1 & F_{12} + i2G_{12} & \cdots & F_{1N} + i2G_{1N} \\
F_{12} + i2G_{12} & 1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
F_{1N} + i2G_{1N} & \cdots & \cdots & 1
\end{pmatrix},
\]

Note that we have absorbed Lamb shift into the optical frequency $\omega_3$, and the properties $F_{\alpha\beta} = F_{\beta\alpha}$, $G_{\alpha\beta} = G_{\beta\alpha}$ have been used. The diagonal element demonstrates a single atomic spontaneous decay, and off-diagonal elements represent the couplings between atoms. The cross couplings are in similar order of magnitude indicating highly dynamical interactions in the system.

We then substitute Eq. (12) into Eq. (9), and derive the two-photon state function in the bare state basis

\[
D_{s,i}(t) = g_s^* g_s^* \langle \epsilon_{k_{1},\lambda_{1}} \cdot \hat{d}_1 \rangle \langle \epsilon_{k_{2},\lambda_{2}} \cdot \hat{d}_2 \rangle \int d't' \int dt'' dt'' e^{i(\omega_{k_{1},\lambda_{1}} - \omega_3)} e^{i(\omega_{k_{2},\lambda_{2}} - \omega_3) - \Delta_2)} t'' \sum_{\mu l, m} e^{-i\hat{k}_{1} \cdot r_{\mu l}}
\]

\[
\times \hat{U}_{\mu l} e^{\lambda_l(t-t')} \tilde{U}_{lm} e^{i\hat{k}_{1} \cdot \tau_m} B_{\mu}(t') e^{-i\hat{k}_{2} \cdot \tau_m}.
\]

In the adiabatic approximation of the excitation process described in Appendix B, we may derive $B_{\mu}(t')$, and the single and two-photon probability amplitudes become

\[
C_{\nu}^{\mu}(t) = g_s^* \langle \epsilon_{k_{\nu},\lambda_{\nu}} \cdot \hat{d} \rangle \int d't' e^{i(\omega_{k_{\nu},\lambda_{\nu}} - \omega_3 - \Delta_2)} t' \times \sum_{l,m} \hat{U}_{\mu l} e^{i\lambda_l(t-t')} \tilde{U}_{lm} e^{i\hat{k}_{\nu} \cdot \tau_m},
\]

\[
D_{s,i}(t) = g_s^* g_s^* \langle \epsilon_{k_{1},\lambda_{1}} \cdot \hat{d}_1 \rangle \langle \epsilon_{k_{2},\lambda_{2}} \cdot \hat{d}_2 \rangle \int t' \int dt'' dt'' e^{i(\omega_{k_{1},\lambda_{1}} - \omega_3)} e^{i(\omega_{k_{2},\lambda_{2}} - \omega_3) - \Delta_2)} t'' \sum_{\mu l, m} e^{-i\hat{k}_{1} \cdot r_{\mu l}}
\]

\[
\times \hat{U}_{\mu l} e^{\lambda_l(t-t')} \tilde{U}_{lm} e^{i\hat{k}_{1} \cdot \tau_m} B_{\mu}(t') e^{-i\hat{k}_{2} \cdot \tau_m}.
\]

The cascade two-photon state is expressed of discrete sum of N eigenvalues $\lambda_l$ shown in the time evolution and sum of eigenvectors sandwiched by phase factors induced by four fields interacting with the system. In the above treatment where we use the bare state basis, the signal spontaneous emission comes from the adiabatic transfer of detuned two-color driving fields, and the idler spontaneous emission comes from N intermediate excited states. These solutions of probability amplitudes demonstrate a dynamical coupling between N atoms, and two-photon state is generated from all possible excitations of one of the atomic ensemble. In the next section, we rotate the coupling matrix $M$ to the new basis, and we find the superradiant idler photon with its decay constant coming from one of the eigenvalues.

The superradiance ($\lambda_l > \Gamma_3$) and subradiance ($\lambda_l < \Gamma_3$) are embedded in these eigenvalues $\lambda_l$, which depend on the density and geometry of atomic ensemble. For a low density where $r_{\alpha\beta} \gg 1/k_3$, $F_{\alpha\beta} \approx 0$, $G_{\alpha\beta} \approx 0$, and the coupling matrix $M$ becomes an identity matrix times an overall constant $-\Gamma_3 / 2$. The transformation matrix $\hat{U}$ is then also an identity matrix. Therefore the two-photon state form N diamond-type atoms behaves as no difference from a single atom. On the contrary in Dicke’s limit where the dimensions of ensemble are smaller than radiation wavelength, $F_{\alpha\beta} \approx 1$, and $G_{\alpha\beta}$ becomes divergent when $r_{\alpha\beta} \to 0$. Without considering this divergent collective Lamb shift, one of the eigenvalues becomes
where $\Re(\lambda_1) = -\Delta N/2$ proportional to the number of particles, which is reminiscent of Dicke’s state $|l = N/2, m = 1-N/2 \rangle$ \cite{dicke1954coherence, Messiah} where $m$ is the half difference of atomic populations in the excited and ground states.

In the next section, we construct the basis in terms of the symmetrical and $N-1$ unsymmetrical states. In this basis we are able to investigate the superradiance and collective Lamb shift of the two-photon state.

### C. Two-photon state in phased symmetrical state

Here we investigate the collective radiation decay and energy shift of two-photon state in the basis constructed from the phased symmetrical state \cite{Glauber, kinoshita} which provides a preference for the system in the limit of large number of atoms. Consider a unitary transformation $\hat{S}$ where a phase factor in an extended ensemble is introduced into basis \cite{Reed-Res}. \[S_{\mu \lambda} = e^{i k N} |3\rangle \otimes |0\rangle_{\lambda \neq \mu}^\otimes N-1,\]

where $k = k_a + k_b - k_s$, and the extra phase factor introduced above is from the observation in Eq. (16). The phased symmetrical state in the new basis is \[\phi_N = \frac{1}{\sqrt{N}} \sum_{\mu} e^{i k N} |3\rangle \otimes |0\rangle_{\lambda \neq \mu}^\otimes N-1,\]

and the other $N-1$ unsymmetrical states are

\[\phi_l = -\frac{\lambda_1}{\sqrt{N}} |3\rangle |0\rangle_0^\otimes N-1 + \sum_{j=1}^{N-1} \left( \frac{1+1/\sqrt{N}}{N-1} - \delta_{lj} \right) e^{i k N} |3\rangle |0\rangle_0^\otimes N-1,\]

where $l = 1, 2, ..., N-1$, and they are orthogonal to each other and normalized to one.

In this new rotational basis, the signal state function has the identity

\[\sum_{l=1}^{N} \sum_{k_s, \lambda_s} C_{l}(t) |3\rangle |0\rangle_{1, k_s, \lambda_s} = \sum_{l=1}^{N} \sum_{k_s, \lambda_s} C_{l}(t) |\phi_l, 1, k_s, \lambda_s \rangle,

where these coefficients are related by the unitary transformation matrix $C_{l}^t = \sum_{l} S_{l \mu} C_{l \mu}^t$ and $C_{s}^l = \sum_{l} S_{l \mu}^t C_{l \mu}^s$. A new set of equation of motion in this basis can be derived, and we have for the equation of $C_{s}$,

\[\dot{C}_s^t = \sqrt{N} g_{k_s} (\epsilon_{k_s, \lambda_s} \cdot \hat{d}_s) b(t) e^{i (\omega_{k_s} - \omega_{\Delta}) t} \delta_{lN} - \sum_{l} \sum_{\mu'} S_{l \mu}^t M_{\mu \mu'} S_{l \mu'} C_{l \mu'}^s,

where we have used the property of $\sum_{\mu} f_{k \mu} = \sqrt{N} \delta_{k N}$, and $M$ is the coupling matrix in bare state basis. The coupling matrix in this new rotational basis becomes

\[M = (SAS)^T \]

where the properties $\sum_{\mu} f_{l \mu} = \sqrt{N} \delta_{lN}$, $\sum_{\mu} f_{l \mu} f_{l' \mu} = \delta_{l l'}$, $M = M^T$ are used, and $T$ means transpose. The diagonal element $A_{ii}$, $i \leq N-1$ has the order of 1, which indicates a spontaneously decayed single atom, and $A_{NN}$ involves the decay rate and frequency shift of the symmetrical state.

Similarly we may diagonalize the above matrix with a similarity transformation matrix $\hat{U}$, and the solutions for probability amplitudes of single and two-photon states become

\[C_{l}^t = \sqrt{N} g_{k_s} (\epsilon_{k_s, \lambda_s} \cdot \hat{d}_s) \int dt' e^{i (\omega_{k_s} - \omega_{\Delta}) t'} b(t') \times \sum_{j} \hat{U}_{lj} e^{\lambda_j (t-t')} \hat{U}_{lj}^{-1},\]
results from adiabatic driving process (See Appendix B). The dependence on mixing (FWM) mismatch. The state functions show a specific presence of the new rotational basis from bare state one.

The above is one of the central results in this paper. The real part retrieves Dicke’s result when atoms are confined within the radiating wavelength and \( \tilde{\mu} \to 0 \). The frequency shift of this symmetrical state can be

\[\Omega_3 = \text{the solid angle of idle photon with wavevector } k_3, \text{ and } \Gamma(k_p, k'_p) \text{ is the ensemble average of } e^{i(k_p-k'_p) \cdot (r-r')} \text{ with randomly distributed atomic positions } r \text{ and } r'. \]
III. COLLECTIVE LAMB SHIFT OF THE PHASED SYMMETRICAL STATE

In this section, we calculate the collective Lamb shift of phased symmetrical state, which is

\[
\text{Im}(\lambda_N) = \frac{\Gamma_3}{2N} \sum_{\mu \neq \nu} G_{\mu \nu}(k_3 r_{\mu \nu}) e^{-ik(r_{\mu} - r_{\nu})},
\]

\[
= \frac{\Gamma_3}{N k_3^2} \text{P.V.} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^3}{k^3 - k_3} \sum_{\mu \neq \nu} F_{\mu \nu}(k r_{\mu \nu}) e^{-ik(r_{\mu} - r_{\nu})},
\]

\[
= \frac{\Gamma_3}{k_3^2} \text{P.V.} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^3}{k^3 - k_3} N \bar{\mu}(k),
\]

(29)

where P.V. denotes principal value, and we note that geometrical constant \( \bar{\mu}(k) \) has a momentum dependence in the integral. Interestingly the collective Lamb shift \( \text{Im}(\lambda_N) \) relates to the integral of spontaneous decay rate proportional to \( N \bar{\mu}(k) \), which is a Hilbert transform [43] if we put back the Lamb shift into our expression. The Lamb shift is an intrinsic level shift induced by vacuum fluctuation while the collective Lamb shift depends on the size and geometry of the atomic ensemble indicating a very different physical mechanism.

To calculate the integral, we may start from the analytic form of \( \mu(k) \), and re-express Eq. (29) by renormalizing \( k \to k_3 k \),

\[
\text{Im}(\lambda_N) = \frac{\Gamma_3}{2N} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{k^3}{k^3 - k_3} \left( k + 1 \right) \frac{6(N - 1) k^3 A H^2}{1 + x^2} \left[ \frac{1}{1 - x^2} \right] \frac{1}{2kH(1 - x)} dx.
\]

(30)

The function of integral over \( x \) changes slowly compared to the change of \( k \), and we may move it outside of the integral similar to the nonperturbative treatment in the radiation level shift [43]. Taking \( k = 1 \) for the integral of \( x \) where the integral of \( k \) is most appreciable, we have

\[
\text{Im}(\lambda_N) = N \bar{\mu}(k_3) \Gamma_3 \times \text{P.V.} \int_{k_m}^{k_M} \frac{dk}{k} \left( \frac{1}{k - 1} + \frac{1}{k + 1} \right),
\]

(31)

where we introduce the infrared \( (k_m) \) and ultraviolet \( (k_M) \) energy cutoffs [43, 44] to the integral, which represent the lowest and largest energy scales in the system. Then we have the collective Lamb shift in the leading order of \( k_m/k_3 \) and \( k_3/k_M \),

\[
\text{Im}(\lambda_N) = 2N \bar{\mu}(k_3) \Gamma_3 \left( \frac{k_m}{k_3} - \frac{k_3}{k_M} \right),
\]

(32)

which indicates a redshift [32, 34, 35] for the level \( |3\rangle \) if we let \( k_M \to \infty \). In Fig. 3 collective Lamb shift shows a geometrical dependence on height and radius in a cylindrical atomic ensemble. We use \( k_m = 2\pi/\sqrt{\pi} R^2 L \) as a long-wavelength momentum cutoff for the system, and use atomic radii [45] as a measure for maximal momentum cutoff which is reasonable for we can not probe atom’s internal structure in our quantum optical treatment. For a needle-like cylinder where \( H \gg A \), we have a redshift of several kHz that can be observable in conventional experimental setup. A blue shift is also shown in Fig. 3 for a disk-like geometry \( (A \ll H) \) which is due to the counteracting term from system energy cutoff in Eq. (32).

In Fig. 3 we show the collective Lamb shift as a dependence of radius, and demonstrate the crossing points of red-blue shifts for other alkali metals in the subplot (a). Due to different D1 transition wavelengths of alkali metals, sodium atoms reach the zero frequency shift at a smaller radius compared to rubidium and cesium ones. By scanning through the geometrical dependence and comparing with different atomic species, we may determine the system energy cutoff. The introduction of energy cutoff means lacking knowledge of the system momentum distribution as in the calculation of Eq. (31). If the experiment shows a redshift as radius varies, it means an unnecessary introduction of the short-wavelength energy cutoff, which suggests that collective Lamb shift is only
Note that when $H$ or $A$ approaches the condition when $N \to 1$, collective Lamb shift is vanishing as can be seen in the general form of Eq. (20) where $N \mu(k) \to 0$. In Fig. 2(b), we specifically show a short radius dependence where there is a sharp increase of collective Lamb shift. The maximum appears at about $a \approx 20 \, \mu m$, which can be significant if we have an even higher atomic density. An estimated of frequency shift of MHz is in the reach in the D1 transition of Rb atoms for atomic density $\rho \sim 10^{12} \, \text{cm}^{-3}$. Similar red to blue shifts in cooperative Lamb shift of an ellipsoid geometry is demonstrated by QED approach [19].

IV. CONCLUSION

We investigate the collective Lamb shift of correlated two-photon emissions in a diamond-type atomic ensemble. The photon pair is generated from the cascade transitions through four-wave mixing of two classical excitation fields. The super-radiant idler photon is well described by one of the eigenvalues of coupling matrix constructed by the phased symmetrical state. This symmetrical state is significantly relevant when the number of atoms is large. We specifically analyze the geometrical dependence of frequency shift on a cylindrical atomic ensemble. We find a redshift in a needle-like geometry, and suggest it could be blue-shifted in a disk-like one. We further demonstrate a comparison of the frequency shift for different alkali metals, and propose to determine the system energy cutoff in experiments.

We can take advantage of manipulating system’s geometry and density to control the collective Lamb shift, which enables robust frequency qubits in quantum network. In the same spirits of using dual-species matter and light-frequency qubits from a $^{85}\text{Rb},^{87}\text{Rb}$ isotopic mixture [46], we may prepare atomic ensembles with two different geometries or densities that generate entangled photon-pair states $(\hat{a}_{s,1}^\dagger \hat{a}_{1,1}^\dagger + \hat{a}_{s,2}^\dagger \hat{a}_{1,2}^\dagger)|\text{vac}\rangle$ assuming the same excitation processes, where we denote (1.2) for two different frequencies and (s,i) for signal and idler bosonic fields $\hat{a}_{s,i}$ respectively. We may construct multimode frequency qubits without acquiring multiple atomic species, and control the qubits via optical means of atom trapping which can be dynamical and efficient. The multimode frequency qubits may offer a speedup of entanglement generation as in temporal modes of rare-earth-metal ions [47, 48], spatial quantum registers [49, 50], and multiplexing quantum repeaters [51].

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Appendix A: Interaction energy

In this section, we derive the self-interaction energy shift for single and many two-level atoms interacting with a single photon. We demonstrate that a correct formulation of the interaction energy, including Lamb shift and collective Lamb shift, is derived when non-RWA terms are included in the Hamiltonian. These non-RWA terms indicate the energy nonconserving process. It involves virtual excitations which are unphysical but crucial in determining the frequency shift.

In interaction picture for single-atom light-matter interactions, we express the Hamiltonian as

$$V = \hbar \sum_k g_k \{ |1\rangle \langle 1| \hat{a}_k e^{-i(\omega - \omega_1)t} + |1\rangle \langle 1| \hat{a}_k \dagger e^{i(\omega + \omega_1)t} \}
+ |0\rangle \langle 0| \{ \hat{a}_k e^{-i(\omega + \omega_1)t} + |0\rangle \langle 0| \hat{a}_k \dagger e^{i(\omega - \omega_1)t} \}, \quad (A1)$$

where $|0\rangle$ and $|1\rangle$ are ground and excited states respectively, and high frequency parts $[\text{Exp}(\pm i(\omega + \omega_1)t)]$ are non-RWA terms. If the atom is in the excited state, it may decay to the ground state with an emitted single photon, whereas the atom in the ground state may be excited also by emitting a photon from non-RWA terms. The state function can be expressed as

$$|\psi\rangle = C_0(t)|0\rangle + C_1(t)|1\rangle + \sum_k C_{1,k}(t)|1,1_k\rangle$$

$$+ \sum_k C_{0,k}(t)|0,1_k\rangle. \quad (A2)$$

The coupled equations from Schrödinger equation are

$$\dot{C}_0 = -i \sum_k g_k e^{-i(\omega + \omega_1)t} C_{1,k},$$
$$\dot{C}_1 = -i \sum_k g_k e^{-i(\omega - \omega_1)t} C_{0,k},$$
$$\dot{C}_{1,k} = -i g_k e^{i(\omega + \omega_1)t} C_0,$$
$$\dot{C}_{0,k} = -i g_k e^{i(\omega - \omega_1)t} C_1. \quad (A3)$$

Solving for the probability amplitudes of atomic energy levels, we have

$$\dot{C}_0 = -\sum_k |g_k|^2 e^{-i(\omega + \omega_1)t} \int t e^{i(\omega + \omega_1)t'} C_0(t') dt',$$
$$\dot{C}_1 = -\sum_k |g_k|^2 e^{-i(\omega - \omega_1)t} \int t e^{i(\omega - \omega_1)t'} C_1(t') dt'. \quad (A4)$$

The summation of spontaneous emission modes in the above for the ground (sign $+$) and the excited (sign $-$) states becomes

$$\int d\omega \frac{\Gamma}{2\pi} \left[ \pi \delta(\omega \pm \omega_1) - i \text{P.V.}(\omega \pm \omega_1)^{-1} \right],$$

where the part of delta function simply represents the spontaneous decay rate, $\Gamma \equiv |d|^2 \omega^2 / (3\pi \hbar \epsilon_0 c^3)$, with the dipole moment $d$ for the transition. P.V. denotes principal value. Moreover the Lamb shift can be identified as the difference of two
level shifts,
\[
\int d\omega \frac{\Gamma}{2\pi} \left[ P.V.(\omega - \omega_1)^{-1} - P.V.(\omega + \omega_1)^{-1} \right]
= \Omega^-_{\alpha\alpha} - \Omega^+_{\alpha\alpha} = \Omega.
\]

Now for many atoms interacting with a single photon including non-RWA terms, the Hamiltonian becomes
\[
V = \hbar \sum_{\mu, k} g_k \left\{ |1\rangle_\mu \langle 0| \hat{a}_k e^{i k \cdot r_\mu} e^{-i(\omega + \omega_1)t} + |1\rangle_\mu \langle 0| \hat{a}_k e^{i k \cdot r_\mu} e^{-i(\omega - \omega_1)t} + |0\rangle_\mu \langle 1| \hat{a}_k e^{i k \cdot r_\mu} e^{-i(\omega + \omega_1)t} + |0\rangle_\mu \langle 1| \hat{a}_k e^{i k \cdot r_\mu} e^{-i(\omega - \omega_1)t} \right\},
\]
where \( \mu \) denotes the atomic indices. The state function can be written as
\[
|\psi\rangle = C_0(t)|0\rangle + \sum_{\mu, k} C^\mu_{0,k}(t)|1\rangle_\mu + \sum_{\mu, k} C^\mu_{0,k}(t)|1\rangle_\mu + \sum_{\mu, k} C^{\mu\nu}_{0,k}(t)|1\rangle_\mu, 1_\nu, 1_k,
\]
where the last unphysical state (two atomic excitations with one photon present) is introduced to couple back to the singly-excited state \( |\bar{g}\rangle \). The coupled equations of motion for the probability amplitudes are
\[
\dot{C}^\mu_0 = -i \sum_k g_k e^{-i(\omega + \omega_1)t} e^{i k \cdot r_\mu} C^\mu_{1,k},
\]
\[
\dot{C}^{\mu\nu}_{1,k} = -i \sum_k g_k e^{-i(\omega + \omega_1)t} C^{\mu\nu}_{0,k} - i \sum_{\nu \neq \mu} \sum_k g_k e^{-i(\omega + \omega_1)t} \left[ C^{\mu\nu}_{2,k} + (\mu \leftrightarrow \nu) \right],
\]
\[
\dot{C}^{\mu}_{0,k} = -i g_k e^{i(\omega + \omega_1)t} e^{-i k \cdot r_\mu} C^{\mu}_{0,k},
\]
\[
\dot{C}^{\mu\nu}_{2,k} = -i g_k e^{i(\omega + \omega_1)t} e^{-i k \cdot r_\mu} C^{\mu\nu}_{1,k},
\]
where \( \mu \), \( \nu \), \( \omega \), and \( \varphi \) are the ground state energy level has a Lamb shift contribution similar to single atomic one except that an extra factor of \( \Gamma \), the number of atoms, appears when substituting Eq. (A9) into Eq. (A7), which shows its intrinsic, not collective, property by virtual fluctuations.

Carefully calculating the contribution from nonconserving energy states in the excited state by substituting Eq. (A11) into Eq. (A9), we have
\[
N \Gamma \sum_{\nu \neq \mu} \int dt' e^{-i(\omega + \omega_1)t'} \left[ C^{\mu\nu}_{2,k} + (\mu \leftrightarrow \nu) \right],
\]

where we can combine the first term in the above with the Lamb shift contribution from the ground state (a factor of \( N \)), and we deduce the positive frequency part of Lamb shift \( \Omega^+_{\alpha\alpha} \). By substituting Eq. (A10) into Eq. (A8) for the energy conserving term and picking out the atomic index \( \mu \), we deduce the negative part of Lamb shift \( \Omega^-_{\alpha\alpha} \). Again the Lamb shift is derived as \( \Omega \equiv \Omega^-_{\alpha\alpha} - \Omega^+_{\alpha\alpha} \), which is the same as single atom case.

The second term in Eq. (A12) is the dipole-dipole interaction energy from energy nonconserving terms. Along with the energy conserving terms by substituting Eq. (A10) into Eq. (A8) and picking out the atomic indices other than \( \mu \), we deduce the collective Lamb shift and spontaneous decay rate as
\[
- \sum_k |g_k|^2 \left\{ e^{-i(\omega - \omega_1)t} \int dt' e^{i(\omega - \omega_1)t'} \times \sum_{\nu \neq \mu} C^{\mu\nu}_{1}(t') e^{i k \cdot (r_\mu - r_\nu)} + e^{-i(\omega + \omega_1)t} \int dt' e^{-i(\omega + \omega_1)t'} \times \sum_{\nu \neq \mu} C^{\mu\nu}_{1}(t') e^{i k \cdot (r_\mu - r_\nu)} \right\}
= - \sum_{\nu \neq \mu} C^{\mu\nu}_{1}(t) \left[ \frac{\Gamma}{2} F_{\mu\nu}(\xi) - i(\Omega^-_{\mu\nu}(\xi) + \Omega^+_{\mu\nu}(\xi)) \right],
\]
where the property of frequency shift, \( \Omega_{\alpha\beta} = \Omega_{\beta\alpha} \), is used, and \( \xi = |k| r_{\mu\nu}, r_{\mu\nu} = |r_\mu - r_\nu| \) with the transition wave vector \( |k| \). Here we show that the treatment of Schrödinger’s equation is equivalent to Heisenberg’s picture that \( F_{\alpha\beta} \) and \( G_{\alpha\beta} \) are defined as
\[
F_{\alpha\beta}(\xi) \equiv \frac{3}{2} \left\{ 1 - (\hat{d} \cdot \hat{r}_{\alpha\beta})^2 \right\} \frac{\sin \xi}{\xi} + \left[ 1 - 3(\hat{d} \cdot \hat{r}_{\alpha\beta})^2 \right] \times \left( \frac{\cos \xi}{\xi^2} - \frac{\sin \xi}{\xi^3} \right),
\]
\[
G_{\alpha\beta}(\xi) \equiv \frac{\Omega_{\alpha\beta}}{\Gamma^3} \equiv - (\Omega^-_{\alpha\beta} + \Omega^+_{\alpha\beta})/\Gamma^3,
\]
\[
\equiv \frac{3}{4} \left\{ 1 - (\hat{d} \cdot \hat{r}_{\alpha\beta})^2 \right\} \cos \xi + \left[ 1 - 3(\hat{d} \cdot \hat{r}_{\alpha\beta})^2 \right] \times \left( \frac{\sin \xi}{\xi^2} + \frac{\cos \xi}{\xi^3} \right), \text{ for } \alpha \neq \beta,
\]
where \( \hat{d} \) is the unit direction of electric dipole. The dependence of \( |\xi|^3 \) in the above arises the name of dipole-dipole interaction which is induced by the common light-matter interaction and radiation reaction.

Appendix B: Adiabatic approximation of excitation process

In this section, we derive the probability amplitudes of the excitation process in the adiabatic approximation. Firstly, we substitute the signal photon state of Eq. (12) into Eq. (5), and
the summation of signal field modes in Eq. (6) becomes

\[ \sum \left| g_{k, \lambda} \right|^2 \epsilon_{k, \lambda} \left( \hat{d}_s \right)^2 \int_0^t dt' e^{i(\omega_{k_s} - \omega_{23} - \Delta_2)(t' - t)} \times \sum e^{i \lambda_1 (r_m - r_m)} \hat{U}_{l_m} e^{\lambda_1 (t' - t)} \hat{U}_{l_m}^{-1} B_m(t'), \]

\[ \approx \sum \left( \frac{\Gamma_2}{2} F_{\mu m}(\xi') - i \Omega_{\mu m}(\xi') \right) \hat{U}_{l_m}^{-1} B_m(t), \]

\[ = \frac{\Gamma_2}{2} B_{\mu}(t) + \text{Lamb shift term}, \quad (B1) \]

where \( \xi' = (\omega_{23} + \Delta_2) |r_m - r_m|/c \). The summation of the signal field modes is determined by the fast oscillating exponential factor of optical frequency, which is valid for the eigenvalues \( \lambda_1 \ll \omega_{23} + \Delta_2 \). In the last step of derivation, we may absorb Lamb shift into the signal transition frequency, and the above result indicates that the upper excited state radiates in a single atomic decay rate.

In the limit of large detunings,

\[ |\Delta_1|, |\Delta_2| \gg \Omega_\alpha, \Omega_\beta, \Gamma_2, \]

we can solve the coupled equations of motion by adiabatically eliminating the intermediate and upper excited states in the excitation process. In Eqs. (5) and (6), we use integration by parts to express probability amplitudes in the first order of \( 1/\Delta_1 \). Note that we allow time-varying excitation fields, and let \( \hat{A}_\mu(t) \equiv e^{-i k_s r_m} A_\mu(t), \quad \hat{B}_\mu(t) \equiv e^{-i(k_{s'} + k_s) r_m} B_\mu(t), \) we have

\[ \hat{A}_\mu(t) = e^{i \Delta_1 t} \left[ \frac{i}{2} \int_{-\infty}^t e^{-i \Delta_1 t'} \Omega_\alpha(t') \mathcal{E}(t') dt' + \frac{i}{2} \int_{-\infty}^t e^{-i \Delta_1 t'} \Omega_\beta(t') \hat{B}_\mu(t') dt' \right], \]

\[ = \frac{\Omega_\alpha(t) \mathcal{E}(t)}{2 \Delta_1} - \frac{\Omega_\beta(t) \hat{B}_\mu(t)}{2 \Delta_1} + \mathcal{O}\left( \frac{1}{\Delta_1^2} \right), \quad (B2) \]

\[ \hat{B}_\mu(t) = e^{i (\Delta_2 + i \Gamma_2/2) t} \left[ \frac{i}{2} \int_{-\infty}^t e^{i (\Delta_2 + i \Gamma_2/2) t'} \right. \times \left. \Omega_\beta(t') \hat{A}_\mu(t') dt' \right], \]

\[ = -\frac{\Omega_\beta(t) \hat{A}_\mu(t)}{2 (\Delta_2 + i \Gamma_2/2)} + \mathcal{O}\left( \frac{1}{\Delta_2^2} \right), \quad (B3) \]

where the initial conditions \( B(-\infty) = A(-\infty) = 0 \) are applied. The adiabatic approximation requires the driving pulses to be smoothly turned on. Therefore in the first order of adiabatic approximation, we derive

\[ \hat{A}_\mu(t) \approx -\frac{\Omega_\alpha(t) \mathcal{E}(t)}{2 \Delta_1}, \quad (B4) \]

\[ \mathcal{E}(t) \approx 1, \quad (B5) \]

\[ \hat{B}_\mu(t) \approx \frac{\Omega_\alpha(t) \Omega_\beta(t)}{4 \Delta_1 \Delta_2} = b(t). \quad (B6) \]

The above results show that the probability amplitudes develop by following the driving fields, and the ground state is approximately unity in the limit of large detunings. The AC Stark shift in the ground state can be ignored if \( \int_{-\infty}^t |\Omega_\alpha(t')|^2 dt' \ll \Delta_1 \) which is also required for the validity of assumption of single excitation in our scheme.

[1] R. H. Dicke, “Coherence in spontaneous radiation processes,” Phys. Rev 93, 99-110 (1954).
[2] C. R. Stroud, Jr., J. H. Eberly, W. L. Lama, and L. Mandel, “Superradiant effects in systems of two-level atoms,” Phys. Rev. A 5, 1094-1104 (1972).
[3] M. Gross and S. Haroche, “Superradiance: An essay on the theory of collective spontaneous emission,” Phys. Rep. 93, 301-396 (1982).
[4] L. I. Men'shkov, “Superradiance and related phenomena,” Phys. Usp. 42, 107-147 (1999).
[5] L. Mandel and E. Wolf, Optical Coherence and Quantum Optics (Cambridge University Press, 1995).
[6] M. J. Stephen, “First-order dispersion forces,” J. Chem. Phys. 40, 669-673 (1964).
[7] R. H. Lehmlberg, “Radiation from an N-atom system. I. General formalism,” Phys. Rev A 2, 883-888 (1970).
[8] F. T. Arecchi and D. M. Kim, “Line shifts in cooperative spontaneous emission,” Opt. Commun. 2, 324-328 (1970).
[9] H. Morawitz, “Superradiant level shift and its possible detection in a transient optical experiment,” Phys. Rev A 7, 1148-1159 (1973).
[10] G. S. Agarwal, “Master-equation approach to spontaneous emission,” Phys. Rev A 2, 2038-2046 (1970).
[11] R. Bonifacio, P. Schwendimann, and F. Haake, “Quantum statistical theory of superradiance. I,” Phys. Rev A 4, 302-313 (1971).
[12] V. Ernst and P. Stehle, “Emission of radiation from a system of many excited atoms,” Phys. Rev 176, 1456-1479 (1968).
[13] E. Ressayre and A. Tallet, “Quantum theory for superradiance,” Phys. Usp. 15, 2410-2423 (1977).
[14] N. E. Rehler and J. H. Eberly, “Superradiance,” Phys. Rev A 3, 1735-1751 (1971).
[15] R. Bonifacio and L.A. Lugiato, “Cooperative radiation processes in two-level systems: Superfluorescence,” Phys. Rev A 11, 1507-1521 (1975).
[16] J. C. MacGillivray and M. S. Feld, “Theory of superradiance in an extended, optically thick medium,” Phys. Rev A 14, 1169-1189 (1976).
[17] R. Friedberg, S. R. Hartmann, J. T. Manassah, “Frequency shifts in emission and absorption by resonant systems of two-level atoms,” Phys. Rep. 7, 101-179 (1973).
[18] J. T. Manassah, “Statistical quantum electrodynamics of reso-
nant atoms,” Phys. Rep. 101 359-427 (1983).

[19] R. Friedberg and J. T. Manassah, “Cooperative Lamb shift in an ellipsoid,” Phys. Rev. A 81, 063822 (2010).

[20] H. J. Carmichael and K. Kim, “A quantum trajectory unraveling of the superradiance master equation,” Opt. Commun. 179, 417-427 (2000).

[21] J. P. Clemens, L. Horvath, B. C. Sanders and H. J. Carmichael, “Collective spontaneous emission from a line of atoms,” Phys. Rev A 68, 023809 (2003).

[22] H. H. Jen, “Positive-P phase-space-method simulation of superradiant emission from a cascade atomic ensemble,” Phys. Rev. A 85, 013835 (2012).

[23] J. H. Eberly, “Emission of one photon in an electric dipole transition of one among N atoms,” J. Phys. B: At. Mol. Opt. Phys. 39, S599-S604 (2006).

[24] M. O. Scully, E. S. Fry, C. H. Raymond Ooi, and K. Wódkiewicz, “Directed spontaneous emission from an extended ensemble of N atoms: Timing is everything,” Phys. Rev. Lett. 96, 010501 (2006).

[25] I. E. Mazets and G. Kurizki, “Multiatom cooperative emission theory” [Phys. Lett. A 372 (2008) 2514]”, Phys. Lett. A 372, 5732-5733 (2008).

[26] A. Svidzinsky and J.-T. Chang, “Cooperativity and eigenvalues of a scalar photon collective emission theory,” [Phys. Lett. A 372 (2008) 5732]”, Phys. Lett. A 372, 5734-5740 (2008).

[27] R. Friedberg and J. T. Manassah, “Time-dependent directionality of cooperative emission after short pulse excitation,” Opt. Commun. 281, 4391-4397 (2008).

[28] M. O. Scully, “Collective Lamb shift in single photon Dicke superradiance,” Phys. Rev. Lett. 102, 143601 (2009).

[29] R. Röhrsberger, K. Schlage, B. Sahoo, S. Couet, R. Rüffer, “Collective Lamb shift in single-photon superradiance,” Science 328, 1248-1251 (2010).

[30] J. Keaveney, A. Sargsyan, U. Krohn, I. G. Hughes, D. Sarkisyans, and C. S. Adams, “Cooperative Lamb shift in an atomic vapor layer of nanometer thickness,” Phys. Rev. Lett. 108, 173601 (2012).

[31] J. Pellegrino, R. Bourgain, S. Jennewein, Y. R. P. Sortais, A. Browaeys, S. D. Jenkins, and J. Ruostekoski, “Observation of suppression of light scattering induced by dipole-dipole interactions in a cold-atom ensemble,” Phys. Rev. Lett. 113, 133602 (2014).

[32] T. Chaneilère, D. N. Matsukevich, S. D. Jenkins, T. A. B. Kennedy, M. S. Chapman, and A. Kuzmich, “Quantum telecommunication based on atomic cascade transitions,” Phys. Rev. Lett. 96, 093604 (2006).

[33] B. Srivathsan, G. K. Gulati, B. Chng, G. Maslennikov, D. Matsukevich, and C. Kurtsiefer, “Narrow band source of transform-limited photon pairs via four-wave mixing in a cold atomic ensemble,” Phys. Rev. Lett. 111, 123602 (2013).

[34] A. G. Radnaev, Y. O. Dudin, R. Zhao, H. H. Jen, S. D. Jenkins, A. Kuzmich, and T. A. B. Kennedy, “A quantum memory with telecom-wavelength conversion,” Nat. Phys. 6, 894-899 (2010).

[35] R. Röhrsberger, K. Schlage, B. Sahoo, S. Couet, R. Rüffer, “Collective spontaneous emission of a single photon from a uniformly excited cloud of N atoms,” Phys. Rev. Lett. 100, 160504 (2008).

[36] A. Svidzinsky and J.-T. Chang, “Cooperative spontaneous emission as a many-body eigenvalue problem,” Phys. Rev A 77, 043833 (2008).

[37] R. Friedberg and J. T. Manassah, “Effects of including the counterrotating term and virtual photons on the eigenfunctions and eigenvalues of a scalar photon collective emission theory,” Phys. Lett. A 372, 2514-2521 (2008).

[38] A. Svidzinsky and J.-T. Chang, “Comment on: "Effects of including the counterrotating term and virtual photons on the eigenfunctions and eigenvalues of a scalar photon collective emission theory" [Phys. Lett. A 372 (2008) 2514]”, Phys. Lett. A 372, 5732-5733 (2008).

[39] A. Svidzinsky and J.-T. Chang, “Reply to: "Comment on: "Effects of including the counterrotating term and virtual photons on the eigenfunctions and eigenvalues of a scalar photon collective emission theory" [Phys. Lett. A 372 (2008) 2514]" [Phys. Lett. A 372 (2008) 5732],” Phys. Rev. Lett. A 372, 5734-5740 (2008).

[40] R. Friedberg and J. T. Manassah, “Time-dependent directionality of cooperative emission after short pulse excitation,” Opt. Commun. 281, 4391-4397 (2008).

[41] M. O. Scully, “Collective Lamb shift in single photon Dicke superradiance,” Phys. Rev. Lett. 102, 143601 (2009).

[42] R. Röhrsberger, K. Schlage, B. Sahoo, S. Couet, R. Rüffer, “Collective Lamb shift in single-photon superradiance,” Science 328, 1248-1251 (2010).

[43] J. Keaveney, A. Sargsyan, U. Krohn, I. G. Hughes, D. Sarkisyana, and C. S. Adams, “Cooperative Lamb shift in an atomic vapor layer of nanometer thickness,” Phys. Rev. Lett. 108, 173601 (2012).