BRAID GROUP ACTIONS AND TENSOR PRODUCTS

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Abstract

We define an action of the braid group of a simple Lie algebra on the space of imaginary roots in the corresponding quantum affine algebra. We then use this action to determine an explicit condition for a tensor product of arbitrary irreducible finite-dimensional representations is cyclic. This allows us to determine the set of points at which the corresponding $R$-matrix has a zero.

0. Introduction

In this paper we give a sufficient condition for the tensor product of irreducible finite-dimensional representations of quantum affine algebras to be cyclic. This condition is obtained by defining a braid group action on the imaginary root vectors. We make the condition explicit in Section 5 and see that it is a natural generalization of the condition in $\mathbb{CP}^1$ given in the $sl_2$ case. This allows us for instance, to determine the finite set of points at which a tensor product of fundamental representations can fail to be cyclic. Our result proves a generalization of a recent result of Kashiwara [K], [HKOTY]. Further, it also establishes a conjecture stated in [K], [VV], [N1].

Let $\pi(u) \in C(q)[u]$ be a polynomial that splits in $C(q)$. Any such polynomial $\pi$ can be written uniquely as a product

$$\pi(u) = \sum_{r=1}^{k} (1 - a_r q^{m_r} u) \cdots (1 - a_r q^{m_r} u),$$

where $a_r \in C(q)$ and $m_r \in \mathbb{Z}$ satisfy

$$a_r \neq q^{\pm (m_r + m_l - 2m)}, \quad 0 \leq m < \min(m_r, m_l),$$

if $r < l$. Let $S(\pi)$ be the collection of the pairs $(a_r, m_r)$. $1 \leq r \leq k$ defined above. Say that a polynomial $\pi'(u)$ is in general position with respect to $\pi(u)$, if

$$a'_r \neq q^{-(m'_r + m_l - 2p)}, \quad 0 \leq p < m_r,$$

for all pairs $(a'_r, m'_r) \in S(\pi')$ and $(a_r, m_r) \in S(\pi)$.

Our main result Theorem 3 is the following, we restrict ourselves to the simply laced (only in the introduction). Let $s_1, s_2, \ldots, s_n$ be the set of simple reflections in $W$ and let $s_{i_1} \cdots s_{i_N}$ be a reduced expression for the longest element $w_0 \in W$.

The tensor product $V(\pi') \otimes V(\pi)$ is cyclic on $v_{\pi'} \otimes v_{\pi}$ if for all $1 \leq j \leq N$ the polynomial $(T_{i_{j+1}} \cdots T_{i_N} \pi')_{i_j}$ is in general position with respect to $\pi_{i_j}$. More generally, let $V_{i_1} \cdots V_{i_r}$
be irreducible finite-dimensional representations such that $V_j \otimes V_i$ is cyclic if $j \neq i$. Then $V_j \otimes \cdots \otimes V_r$ is cyclic.

Of course, we first have to prove that $(T_{j+1} \cdots T_N \pi)^j_j$ is a polynomial, we do this in Section 2. In Section 5, we write down the polynomials $(T_{j+1} \cdots T_N \pi)^j_j$, for all $\mathfrak{g}$ and a specific reduced expression of $w_0$ thus making the condition for a tensor product to be cyclic explicit and we see that it is the appropriate generalization of the result in the case when $g = sl_2$.

To make the connection with Kashiwara’s conjectures, we consider the case

$$\pi_j(u) = 1 \quad (j \neq i), \quad \pi_i(u) = \prod_{a=1}^{m} (1 - q^{m+1-2a}au) \quad (a \in C(q)).$$

Denoting this $n$-tuple of polynomials as $\pi_{i,a}$ and the corresponding representation by $V(\pi^i_{i,a})$, we prove the following: Let $l \geq 1$ and let $i_j \in I$, $m_j \in \mathbb{Z}_+$, $a_j \in C(q)$ for $1 \leq j \leq l$. The tensor product $V(\pi^{i_1}_{i_1,a_1}) \otimes V(\pi^{i_2}_{i_2,a_2}) \otimes \cdots \otimes V(\pi^{i_l}_{i_l,a_l})$ is cyclic on the tensor product of highest weight vectors if for all $r \leq s$,

$$\frac{a_r}{a_s} \neq q^{m_r-m_s-p}, \quad \forall \quad p \geq 0.$$

In Corollary 5.1, we write down the precise values of $a_r/a_s$ at which the tensor product is not cyclic. In the case of two when $\ell = 2$, this is the set of all possible zeros of the $R$-matrix, $R(a): V(\pi^i_{i_1,a_1}) \otimes V(\pi^i_{i_2,a_2}) \rightarrow V(\pi^i_{i_1,a_1}) \otimes V(\pi^i_{i_2,a_2})$.

The case when $m_j = 1$ was originally conjectured and partially proved in [AK] and completely proved in [K] and [V] for the simply-laced case) [N1, N2]. The result in the case when the $m_i$ are arbitrary but $a_i = 1$ for all $i$ was conjectured in [K], [HKOTY]. In the case of $A_n$ and $C_n$, and when $m_j = 1$ the values of $a_r/a_s$ where the tensor product is not cyclic was written in [AK].

In the exceptional case the explicit calculation of the possible points of reducibility can be used to write down the precise $\mathfrak{g}$-module structures of the exceptional algebras for all nodes of the Dynkin diagram. Details of this will appear elsewhere.

Finally, recall that a tensor product of two irreducible finite-dimensional representations is irreducible if both $V \otimes V'$ and its dual are cyclic on the tensor product of highest weight vectors. Thus our theorem gives us a sufficient condition for the tensor product $V \otimes V'$ to be irreducible. When $g = sl_2$, this condition is the same as the one given in [CP1].

1. Preliminaries

In this section we recall the definition of quantum affine algebras and several results on the classification of their irreducible finite-dimensional representations.

Let $q$ be an indeterminate, let $C(q)$ be the field of rational functions in $q$ with complex coefficients. For $r, m \in \mathbb{N}$, $m \geq r$, define

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_q! = [m]_q[m - 1]_q \cdots [2]_q[1]_q, \quad \frac{[m]_q!}{[r]_q!} = \frac{[m]_q!}{[r]_q!}.\frac{[m]_q}{[r]_q!}.$$

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra of rank $n$, set $I = \{1, 2, \cdots, n\}$, let $\{\alpha_i : i \in I\}$ (resp. $\{\omega_i : i \in I\}$) be the set of simple roots (resp. fundamental weights) of $\mathfrak{g}$ with respect to $\mathfrak{h}$. As usual, $Q^+$ (resp. $P^+$) denotes the non-negative root (resp. weight) lattice of $\mathfrak{g}$. Let $\Lambda = (\alpha_{ij})_{i,j \in I}$ be the $n \times n$ Cartan matrix of $\mathfrak{g}$ and let $\hat{\Lambda} = (\hat{a}_{ij})$ be the $(n+1) \times (n+1)$ extended Cartan matrix associated to $\mathfrak{g}$. Let $\hat{I} = I \cup \{0\}$. Fix non-negative integers $d_i$ (i $\in \hat{I}$) such that the matrix $(d_i a_{ij})$ is symmetric. Set $q_i = q^{d_i}$ and $[m]_q = [m]_{q_i}$. 
Proposition 1.1. There is a Hopf algebra $\hat{U}_q$ over $\mathbb{C}(q)$ which is generated as an algebra by elements $E_{\alpha_i}, F_{\alpha_i}, K_i^{-1}$ ($i \in I$), with the following defining relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$

$$K_i E_{\alpha_i} K_i^{-1} = q_i^{a_{ij}} E_{\alpha_i},$$

$$K_i F_{\alpha_i} K_i^{-1} = q_i^{-a_{ij}} F_{\alpha_i},$$

$$[E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} K_i - K_i^{-1}.$$

$$\left(\sum_{r=0}^{1-a_{ij}} (-1)^r \left[1 - \frac{a_{ij}}{r}\right] \right) \left(E_{\alpha_i}\right)^r F_{\alpha_j} \left(E_{\alpha_i}\right)^{1-a_{ij}-r} = 0 \quad \text{if } i \neq j,$$

$$\left(\sum_{r=0}^{1-a_{ij}} (-1)^r \left[1 - \frac{a_{ij}}{r}\right] \right) E_{\alpha_i} \left(F_{\alpha_j}\right)^{1-a_{ij}-r} = 0 \quad \text{if } i \neq j.$$

The comultiplication of $\hat{U}_q$ is given on generators by

$$\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes 1 + K_i \otimes E_{\alpha_i}, \quad \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes K_i^{-1} + 1 \otimes F_{\alpha_i}, \quad \Delta(K_i) = K_i \otimes K_i,$$

for $i \in \hat{I}$. \hfill \Box

Set $K_0 = \prod_{i=1}^{m} K_i^{r_i/\theta_i}$, where $\theta = \sum r_i \alpha_i$ is the highest root in $R^+$. Let $U_q$ be the quotient of $\hat{U}_q$ by the ideal generated by the central element $K_0 K_q^{-1}$; we call this the quantum loop algebra of $\mathfrak{g}$.

It follows from [Dr], [B], [J] that $U_q$ is isomorphic to the algebra with generators $x_{i,r}^\pm$ ($i \in I, r \in \mathbb{Z}$), $K_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I, r \in \mathbb{Z}\{0\}$) and the following defining relations:

$$K_i K_i^{-1} = K_j^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$

$$K_i h_{j,r} K_i^{-1} = h_{j,r} K_i,$$

$$K_i x_{j,r} K_i^{-1} = q_i^{a_{ij}} x_{j,r},$$

$$[h_{i,r}, h_{j,s}] = 0, \quad [h_{i,r}, x_{j,s}] = \pm \frac{1}{r} [r a_{ij}] x_{j,r+s},$$

$$x_{i,r}^\pm x_{j,r+s}^\pm = q_i^{a_{ij}} x_{j,s}^\pm x_{i,r+s}^\pm = q_i^{a_{ij}} x_{i,s}^\pm x_{j,r+s}^\pm,$$

$$x_{i,r}^+, x_{j,s}^- = \delta_{ij} q_i^{1/2} - \delta_{ij} q_i^{-1/2}.$$

$$\sum_{i \in \Sigma_m} \sum_{k=0}^{m} (-1)^k \left\{ \frac{m}{k} \right\} \hat{x}_{i,r}, \ldots, \hat{x}_{i,r^k} = 0, \quad i \neq j,$$

for all sequences of integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$, $\Sigma_m$ is the symmetric group on $m$ letters, and the $\psi_{i,r}^\pm$ are determined by equating powers of $u$ in the formal power series

$$\sum_{r=0}^{\infty} \psi_{i,r}^\pm u^r = K_i^{\pm 1} \exp \left( \pm (q_i - q_i^{-1}) \sum_{s=1}^{\infty} h_{i,s} u^s \right).$$

For $i \in I$, the preceding isomorphism maps $E_{\alpha_i}$ to $x_{i,0}^+$ and $F_{\alpha_i}$ to $x_{i,0}^-$. The subalgebra generated by $E_{\alpha_i}, F_{\alpha_i}, K_i^{\pm 1}$ ($i \in I$) is the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ associated to $\mathfrak{g}$. Let $U_q(\mathfrak{g})$ be the subalgebra generated by the elements $x_{i,k}^\pm$ ($i \in I, k \in \mathbb{Z}$). For $i \in I$, let $U_i$ be the subalgebra of $U_q$ generated by the elements $\{x_{i,k}^\pm : k \in \mathbb{Z}\}$; the subalgebra $U_q(\mathfrak{sl}_2)$ is defined in the same way. Notice that $U_i$ is isomorphic to the quantum affine algebra $U_q(\mathfrak{sl}_2)$. Let $\Delta_i$ be the comultiplication of $U_q(\mathfrak{sl}_2)$. 


An explicit formula for the comultiplication on the Drinfeld generators is not known. Also, the subalgebra \( U_i \) is not a Hopf subalgebra of \( U_q \). However, partial information which is sufficient for our needs is given in the next proposition. Let

\[
X^\pm = \sum_{i, k \in \mathbb{Z}} C(q)x_{i,k}^\pm, \quad X^\pm(i) = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}} C(q)x_{j,k}^\pm.
\]

Proposition 1.2. The restriction of \( \Delta \) to \( U_i \) satisfies,

\[
\Delta(x) = \Delta(x) \mod (U_q \otimes (U_q \setminus U_i)).
\]

More precisely:

(i) \( \text{Modulo } U_q X^- \otimes U_q(X^+)^2 + U_q X^- \otimes U_q X^+(i) \), we have

\[
\Delta(x_{i,k}^+) = x_{i,k}^+ \otimes 1 + K_i \otimes x_{i,k}^+ + \sum_{j=1}^{k} \psi_{i,j}^+ \otimes x_{i,k-j}^+ \quad (k \geq 0),
\]

\[
\Delta(x_{i,-k}^-) = K_i^{-1} \otimes x_{i,-k}^+ + x_{i,-k}^- \otimes 1 + \sum_{j=1}^{k} \psi_{i,j}^- \otimes x_{i,k+j}^- \quad (k > 0).
\]

(ii) \( \text{Modulo } U_q(X^-)^2 \otimes U_q X^+ + U_q X^- \otimes U_q X^+(i) \), we have

\[
\Delta(x_{i,k}^-) = x_{i,k}^- \otimes K_i + 1 \otimes x_{i,k}^- + \sum_{j=1}^{k-1} x_{i,k-j}^- \otimes \psi_{i,j}^+ \quad (k > 0),
\]

\[
\Delta(x_{i,-k}^+) = x_{i,-k}^+ \otimes K_i^{-1} + 1 \otimes x_{i,-k}^- + \sum_{j=1}^{k} x_{i,-k+j}^- \otimes \psi_{i,j}^- \quad (k \geq 0).
\]

(iii) \( \text{Modulo } U_q X^- \otimes U_q X^+ \), we have

\[
\Delta(h_{i,k}) = h_{i,k} \otimes 1 + 1 \otimes h_{i,k} \quad (k \in \mathbb{Z}).
\]

Proof. Part (iii) was proved in [Da]. The rest of the proposition was proved in [CP3]. \( \square \)

We conclude this section with some results on the classification of irreducible finite-dimensional representations of quantum affine algebras. Let

\[
\mathcal{A} = \{ f \in \mathbb{C}(\mathbb{Q})[[u]] : f(0) = 0 \}.
\]

For any \( U_q \)-module \( V \) and any \( \mu = \sum_{\lambda} \mu_\lambda \omega_i \in P \), set

\[
V_\mu = \{ v \in V : K_i.v = q^{\mu_i}v, \quad \forall \quad i \in I \}.
\]

We say that \( V \) is a module of type 1 if

\[
V = \bigoplus_{\mu \in P} V_\mu.
\]

From now on, we shall only be working with \( U_q \)-modules of type 1. For \( i \in I \), set

\[
h_i^+(u) = \sum_{k=1}^{\infty} q^{\mu_i k} h_{i,k} u^k.
\]

Definition 1.1. We say that a \( U_q \)-module \( V \) is (pseudo) highest weight, with highest weight \( (\lambda, \mathbf{h}^\pm) \), where \( \lambda = \sum_{i \in I} \lambda_i \omega_i \), \( \mathbf{h}^\pm = (h_i^+(u), \cdots, h_i^+(u)) \in \mathcal{A}_i \), if there exists \( 0 \neq v \in V_\lambda \) such that \( V = U_q.v \) and

\[
x_{i,k}.v = 0, \quad K_i.v = q^{\lambda_i}v, \quad h_i^+(u).v = h_i^+(u)v,
\]

for all \( i \in I, k \in \mathbb{Z} \). \( \square \)
If \( V \) is any highest weight module, then in fact \( V = U_q(\langle \cdot \rangle_\nu) \) and so
\[
V_\mu \neq 0 \implies \mu = \lambda - \eta \quad (\eta \in \mathbb{Q}^+).
\]
Clearly any highest weight module has a unique irreducible quotient \( V(\lambda, h^\pm) \).

The following was proved in [CP2].

**Theorem 1.** Assume that the pair \( (\lambda, h^\pm) \in P \times A^n \) satisfies, the following: \( \lambda = \sum_{i \in I} \lambda_i \omega_i \in P^+ \), and there exist elements \( a_{i,r} \in C(q) \) \( (1 \leq r \leq \lambda_i, i \in I) \) such that
\[
h_i^\pm(u) = -\sum_{r=1}^{k_i} \ln(1 - a_{i,r}^2 u).
\]

Then, \( V(\lambda, h^\pm) \) is the unique (up to isomorphism) irreducible finite-dimensional \( U_q \)-module with highest weight \( (\lambda, h^\pm) \).

**Remark.** This statement is actually a reformulation of the statement in [CP2]. Setting \( \pi_i(u) = \prod_{r=1}^{k_i} (1 - a_{i,r} u) \) and calculating the eigenvalues of the \( \psi_{i,k} \) gives the result stated in [CP2], see also [CP4].

From now on, we shall only be concerned with the modules \( V(\lambda, h^\pm) \) satisfying the conditions of Theorem 1. In view of the preceding remark, it is clear that the isomorphism classes of such modules are indexed by an \( n \)-tuple of polynomials \( \pi = (\pi_1, \ldots, \pi_n) \), which have constant term 1, and which are split over \( C(q) \). We denote the corresponding module by \( V(\pi) \) and the highest weight vector by \( v_\pi \), where \( \lambda = \sum_{i=1}^n \deg \pi_i \).

For all \( i, k \in \mathbb{Z} \), we have
\[
(1.1) \quad x_i^{+k} v_\pi = 0, \quad K_i v_\pi = q_i^{\deg \pi_i} v_\pi,
\]
and
\[
(1.2) \quad \frac{q_i^{\pm k}}{[k]} h_i^{\pm k} v_\pi = h_i^{\pm k} v_\pi, \quad (x_i^{-k})^{\deg \pi_i+1} v_\pi = 0,
\]
where the \( h_i^{\pm k} \) are determined from the functional equation
\[
\exp \left( -\sum_{k>0} h_i^{\pm k} u^k \right) = \pi_i^\pm(u),
\]
with \( \pi_i^+(u) = \pi_i(u) \) and \( \pi_i^-(u) = u^{\deg \pi_i} \pi_i(u) u^{-1} / (u^{\deg \pi_i} \pi_i(u^{-1})) \). For all \( i \in I \), \( k \in \mathbb{Z} \), depending only on \( \pi \),
\[
(1.3) \quad V(\pi)^* \cong V(\pi), \quad \pi^* = (\pi_1(q' u), \ldots, \pi_n(q' u)),
\]

Analogous statements hold for right duals [CP3]. Recall also, that if a module and its dual are highest weight then they must be irreducible.

Finally, let \( \omega : U_q \to U_q \) be the algebra automorphism and coalgebra anti-automorphism obtained by extending the assignment \( \omega(x_i^{\pm1}) = -x_{i,-k}^\pm \). If \( V \) is any \( U_q \)-module, let \( V^\omega \) be the pull–back of \( V \) through \( \omega \). Then, \( (V \otimes V^\omega)^* \cong (V^\omega)^* \otimes V^\omega \) and
\[
(1.4) \quad V(\pi)^\omega = V(\pi^\omega)
\]
where
\[
\pi^\omega = (\pi_1(q_1^2 k u), \ldots, \pi_n(q_n^2 k u)),
\]
for a fixed \( \kappa \) depending only on \( \mathfrak{g} \).

Throughout this paper, we shall only work with polynomials in \( C(q)[u] \) which are split and have constant term 1. For any \( 0 \neq a \in C(q) \), \( m \in \mathbb{Z}^+ \), set
\[
(1.5) \quad \pi_{m,a}(u) = \prod_{r=1}^m (1 - a q^{m-2r+1} u),
\]
It is a simple combinatorial fact [CP1] that any such polynomial can be written uniquely as a product

$$\pi(u) = \prod_{j=1}^{s} \pi_{m_j,a_j},$$

where $m_j \in \mathbb{Z}_+$, $a_j \in \mathbb{C}(q)$ and

$$j < \ell \implies \frac{a_j}{a_\ell} \neq q^{\pm(m_j + m_\ell - 2p)}, \quad 0 \leq p < \min(m_j, m_\ell).$$

If $\pi$ and $\pi'$ are two such polynomials, then we say that $\pi$ is in general position with respect to $\pi'$ if for all $1 \leq j \leq s$, $1 \leq k \leq s'$, we have

$$\frac{a_j}{a_k} \neq q^{- (m_j + m_\ell' - 2p)},$$

for any $0 \leq p < m_j$. This is equivalent to saying that for all roots $a$ of $\pi$ we have,

$$\frac{a}{a_k} \neq q^{-1-m_\ell'}, \quad \forall \ 1 \leq k' \leq s'.$$

We conclude this section with some results in the case when $g = sl_2$.

**Theorem 2.**

(i) The irreducible module $V(\pi_{m,a})$ with highest weight $\pi_{m,a}$ is of dimension $m + 1$ and is irreducible as a $U_q^{\text{fin}}$-module.

(ii) Assume that $a_1, \ldots, a_\ell \in \mathbb{C}(q)$ are such that if $r < s$ then $a_r/a_s \neq q^{-2}$ (i.e. the polynomial $(1 - a_r u)$ is in general position with respect to $(1 - a_s u)$ for all $1 \leq r \leq s \leq \ell$). The module $W(\pi) = V(\pi_{1,a_1}) \otimes \cdots \otimes V(\pi_{1,a_\ell})$ is the universal finite–dimensional highest weight module with highest weight

$$\pi(u) = \prod_{r=1}^{\ell} (1 - a_r u),$$

i.e. any other finite–dimensional highest weight module with highest weight $\pi$ is a quotient of $W(\pi)$.

(iii) For $1 \leq r \leq \ell$, let $a_r \in \mathbb{C}(q)$ and $m_r \in \mathbb{Z}_+$ be such that $\pi_{m_r,a_r}$ is in general position with respect to $\pi_{m_r,a_r}$ for all $1 \leq r \leq s \leq \ell$. The module $V(\pi_{m_1,a_1} \otimes \cdots \otimes V(\pi_{m_\ell,a_\ell})$ is a highest weight module with highest weight $\pi_{m_1,a_1} \cdots \pi_{m_\ell,a_\ell}$ and highest weight vector $v_{m_1,a_1} \otimes \cdots \otimes v_{m_\ell,a_\ell}$. In particular, the module $W(\pi_{m_1,a_1}) \otimes V(\pi_{m_2,a_2}) \otimes \cdots \otimes V(\pi_{m_\ell,a_\ell})$ is highest weight.

(iv) The module $V(\pi) \otimes V(\pi')$ is irreducible iff

$$\frac{a_r}{a_s} \neq q^{\pm(m_r + m_s - 2p)} \quad \forall 0 \leq p < \min(m_r, m_s).$$

**Proof.** Parts (i) and (iv) were proved in [CP1]. Part (ii) was proved in [CP1] (see also [VV, Theorem 2] in the case when $\pi(u) \in \mathbb{C}(q, q^{-1}, u)$. In the general case, choose $v \in \mathbb{C}(q, q^{-1})$ so that $v(u) = \pi(u)$ has all its roots in $\mathbb{C}(q, q^{-1})$. Let $\tau_v : U_q \to U_q$ be the algebra and coalgebra automorphism defined by sending $x^+_i \to v^i x^+_i$. The pull back of $V(\pi)$ through $\tau_v$ is $V(\pi)$ and hence $W(\pi(u)) \cong W(\pi(u v))$. This proves (ii). Part (iii) is proved as in Lemma 4.9 in [CP1]. In fact, the proof given there establishes the stronger result stated here. □
2. Braid group action

Let $W$ be the Weyl group of $\mathfrak{g}$ and let $B$ be the corresponding braid group. Thus, $B$ is the group generated by elements $T_i$ $(i \in I)$ with defining relations:

\[
T_i T_j = T_j T_i \quad \text{if} \quad a_{ij} = 0, \\
T_i T_j T_i = T_j T_i T_j \quad \text{if} \quad a_{ij} a_{ji} = 1, \\
(T_i T_j)^2 = (T_j T_i)^2 \quad \text{if} \quad a_{ij} a_{ji} = 2, \\
(T_i T_j)^3 = (T_j T_i)^3 \quad \text{if} \quad a_{ij} a_{ji} = 3, \\
\]

where $i, j \in \{1, 2, \ldots, n\}$ and $A = (a_{ij})$ is the Cartan matrix of $\mathfrak{g}$.

A straightforward calculation gives the following proposition.

**Proposition 2.1.** For all $r \geq 1$, the formulas

\[
T_i e_j = e_j - q_i^r (a_{ij}) j e_i, \\
\]

define a representation $\eta_r : B \to \text{end}(V_r)$, where $V_r \cong \mathbb{C}(q)^n$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $V_r$. Further, identifying

\[
\mathcal{A}^n \cong \prod_{r=1}^{\infty} V_r, \\
\]

we get a representation of $B$ on $\mathcal{A}^n$ given by

\[
(T_i h)_{ij} = h_{ij}(u), \quad \text{if} \quad a_{ji} = 0, \\
(T_i h)_{ji} = h_{ij}(u) + h_i(q u), \quad \text{if} \quad a_{ji} = -1, \\
(T_i h)_{ij} = h_{ij}(u) + h_i(q^3 u) + h_i(q u), \quad \text{if} \quad a_{ji} = -2, \\
(T_i h)_{ij} = h_{ij}(u) + h_i(q^5 u) + h_i(q^3 u) + h_i(q u), \quad \text{if} \quad a_{ji} = -3, \\
(T_i h)_{ij} = -h_i(q^2 u), \\
\]

for all $i, j \in I$, $h \in \mathcal{A}^n$. \qed

Let $s_i, i \in I$ be a set of simple reflections in $W$. For any $w \in W$, let $\ell(w)$ be the length of a reduced expression for $w$. If $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced expression for $w$, set $I_w = \{i_1, i_2, \ldots, i_k\} \subset I$ and let $T_w = T_{i_1} \cdots T_{i_k}$. It is well–known that $T_w$ and $I_w$ are independent of the choice of the reduced expression. Given $h \in \mathcal{A}^n$ and $w \in W$, we have

\[
T_w h = T_{i_1} T_{i_2} \cdots T_{i_k} h = ((T_w h)_{i_1}, \ldots, (T_w h)_{i_n}). \\
\]

We can now prove:

**Proposition 2.2.** Suppose that $w \in W$ and $i \in I$ is such that $\ell(s_i w) = \ell(w) + 1$. There exists an integer $M \equiv M(i, w, h) \geq 0$ and non–negative integers $p_{r,j}$ $(j \in I, 1 \leq r \leq M)$ such that

\[
(T_w h)_i = \sum_{j \in I_w \cup \{i\}} \sum_{r=1}^{M} p_{r,j} h_j (q^r u). \\
\]

Further, if $i \notin I_w$, then

\[
(T_w h)_i = h_i(u) + \sum_{j \in I_w} \sum_{r=1}^{M} p_{r,j} h_j(q^r u). \\
\]

**Proof.** Proceed by induction on $\ell(w)$; the induction clearly starts at $\ell(w) = 0$. Assume that the result is true for $\ell(w) < k$. If $\ell(w) = k$, write $w = s_j w'$ with $\ell(w') = k - 1$. 

To see this, observe that
\[ h_i T (T_{w'} h_i) = \sum_{s=0}^{[a_{i,j}]-1} (T_{w'} h_j) (u) + \sum_{s=0}^{[a_{i,j}]-1} (T_{w'} h_j) (q^{2[a_{i,j}]-2s-1} u). \]

If \( \ell (s_i w') = \ell (w') + 1 \), in particular this happens if \( i /u \), the proposition follow by induction. If \( \ell (s_i w') = \ell (w') - 1 \), we have \( w = s_j s_i w'' \). Suppose that \( a_{i,j} a_{j,i} = -1 \). Then, \( \ell (s_j w'') = \ell (w'') + 1 \) and we get
\[ (T_j T_{s_i} h_i) = (T_j T_{s_i} h_i) + (T_j T_{s_i} h_j) (qu) = (T_{w''} h_j) (q_i u). \]

The result again follows by induction. The cases when \( a_{i,j} a_{j,i} = 2, 3 \) are proved similarly. \( \square \)

3. The main theorem

Our goal in this section is to obtain a sufficient condition for a tensor product of two highest weight representations to be highest weight.

Let \( V \) be any highest weight finite–dimensional \( U_q \mathfrak{g} \) module for \( \pi (\lambda, h^\mathfrak{g}) \) as in Theorem 3.1. For all \( w \in W \), we have
\[ \dim \ V_{\lambda} = 1. \]

If \( s_{i_1} \cdots s_{i_k} \) is a reduced expression for \( w \), and \( \lambda = \sum_i \lambda_i \omega_i \), set \( m_k = \lambda_k \) and define non–negative integers \( m_j \) (depending on \( w \)) for \( 1 \leq j \leq k \), by
\[ s_{i_{j+1}} s_{i_{j+2}} \cdots s_{i_k} \lambda = m_j \omega_j + \sum_{i \neq j} n_i \omega_i. \]

Let \( v_\lambda \) be the highest weight vector in \( V \). For \( w \in W \), set
\[ v_{w\lambda} = (x_{i_1,0}^{m_1}) \cdots (x_{i_k,0}^{m_k}) v_\lambda. \]

If \( i \in I \) is such that \( \ell (s_i w) = \ell (w) + 1 \), then
\[ x_{i,k} v_{w\lambda} = 0, \quad \forall k \in \mathbb{Z}. \]

To see this, observe that \( w\lambda + \alpha_i \) is not a weight of \( V \), since \( w^{-1} \alpha_i \in R^+ \) if \( \ell (s_i w) = \ell (w) + 1 \).

It is now easy to see that \( v_{w\lambda} \neq 0 \), \( V_{\lambda} = C(q)v_{w\lambda} \) and
\[ V = U_q v_{w\lambda}. \]

Since \( h_{i,k} V_{w\lambda} \subseteq V_{w\lambda} \) for all \( i \in I \), \( k \in \mathbb{Z} \) it follows that
\[ \frac{h_{i,k}}{[k]} v_{w\lambda} = h_{i,k} v_{w\lambda}, \quad \forall i \in I, \quad 0 \neq k \in \mathbb{Z}, \]

where \( h_{i,k} \in C(q) \). Set
\[ h^w(u) = \sum_{k=1}^\infty h_{i,k}^w u^k, \quad h^w = (h^1(u), \cdots, h^w(u)). \]

Recall that \( h^1 = -(\ln \pi_1(u), \cdots, \ln \pi_n(u)) \).

Proposition 3.1. If \( w \in W \), then
\[ h^w = T_w h^1. \]

Proof. We proceed by induction on \( \ell (w) \). If \( \ell (w) = 0 \) then \( w = id \) and the result follows by definition. Suppose that \( \ell (w) = 1 \), say \( w = s_j \). Writing \( \lambda = \sum_j \lambda_j \omega_j \), we have \( v_{s_j \lambda} = (x_{j,0}^-)^{\lambda_j} v_{\lambda} \). We first show that
\[ h_{j,k} v_{s_j \lambda} = -h_{j,q_j^2 u} v_{s_j \lambda} = (T_j h) (u) v_{s_j \lambda}. \]

The subspace spanned by the elements \( \{ x_{j,0}^- v_{\lambda} : 0 \leq r \leq \lambda_j \} \) is a highest weight module for \( U_j \), hence it is enough to prove (3.2) for highest weight representations of
quantum affine $sl_2$. In fact it is enough to prove it for the module $W(\pi)$ of Theorem 2. Using Proposition 1.2, we see that the eigenvalue of $h_{i,k}$ on the tensor product of the lowest (and the highest) weight vectors is just the sum of the eigenvalues values in each representation. This reduces us to the case of the two-dimensional representation, which is trivial.

Next consider the case $\ell(w) = s_i$, with $i \neq j$. Recall that

$$[h_{i,r}, x_{j,0}^-] = -\frac{[ra_i]}{r} x_{j,r}, \quad [h_{j,r}, x_{j,0}^-] = -\frac{[2r]}{r} x_{j,r}.$$  

Hence,

$$h_{i,r}(x_{j,0}^-)^\lambda h_{i,r} + [h_{i,r}, (x_{j,0}^-)^\lambda] = (x_{j,0}^-)^\lambda h_{i,r} + \frac{[ra_i]}{[2r]}[h_{j,r}, (x_{j,0}^-)^\lambda].$$  

This gives

$$\frac{h_{i,r}}{[r]}(x_{j,0}^-)^\lambda v_\lambda = h_{i,r}(x_{j,0}^-)^\lambda v_\lambda + \frac{[ra_i]}{[2r]} \left( \frac{h_{j,r}}{[r]} (x_{j,0}^-)^\lambda \right) v_\lambda,$$

$$= h_{i,r}(x_{j,0}^-)^\lambda v_\lambda + \frac{[ra_i]}{[2r]} \left( \frac{h_{j,r}}{[r]} (x_{j,0}^-)^\lambda \right) v_\lambda - \frac{[ra_i]}{[2r]} \left( \frac{h_{j,r}}{[r]} (x_{j,0}^-)^\lambda \right) v_\lambda,$$

$$= (h_{i,r} - q_{ij} h_{j,r}) (x_{j,0}^-)^\lambda v_\lambda,$$

where the third equality follows from (3.2).

This proves the result when $\ell(w) = 1$. Proceeding by induction on $\ell(w)$, write $w = s_j w'$ with $\ell(w') = \ell(w) - 1$. Since $v_{w,\lambda} = (x_{j,0}^-)^{m_j} v_{w',\lambda}$ for some $m_j \geq 0$, the inductive step is proved exactly as in the case $\ell(w) = 1$, with $v_\lambda$ being replaced by $v_{w',\lambda}$. This completes the proof of the proposition.

Let $w_0 \in W$ be the longest element of the Weyl group of $g$.

**Lemma 3.1.** Let $V$, $V'$ be finite-dimensional highest weight representations with highest weights $\pi$ and $\pi'$ and highest weight vectors $v_\lambda$ and $v_{\lambda'}$, respectively. Assume that $v_{w,\lambda} \otimes v_{\lambda'} \in U_q(v_\lambda \otimes v_{\lambda'})$. Then, $V \otimes V'$ is highest weight with highest weight vector $v_\lambda \otimes v_{\lambda'}$ and highest weight $\pi \pi' = (\pi_1 \pi'_1, \ldots, \pi_n \pi'_n)$.

**Proof.** It is clear from Proposition 1.2 that the element $v_\lambda \otimes v_{\lambda'}$ is a highest weight vector with highest weight $\pi \pi'$. It suffices to prove that

$$V \otimes V' = U_q(v_\lambda \otimes v_{\lambda'}).$$

Since $x_{i,k} v_{w,\lambda} = 0$ for all $i \in I$ and $k \in \mathbb{Z}$, it follows from Proposition 1.2 that

$$\Delta(x_{i,k}^-) (v_{w,\lambda} \otimes v_{\lambda'}) = v_{w,\lambda} \otimes x_{i,k}^- v_{\lambda'},$$

Repeating this argument we see that $v_{w,\lambda} \otimes V' \subset U_q(v_\lambda \otimes v_{\lambda'})$. Now applying the generators $E_{\alpha_i}$, $F_{\alpha_i}$ ($i \in I$) repeatedly, we see that $V \otimes V' \subset U_q(v_\lambda \otimes v_{\lambda'})$. This proves the lemma.

**Lemma 3.2.** Let $w \in W$ and assume that $i \in I$ is such that $\ell(s_i w) = \ell(w) + 1$. Then, $v_{w,\lambda} \otimes v_{\lambda'}$ generates a $U_q$-highest weight module with highest weight $(T_w h_i) h_i'$. 

**Proof.** This is immediate from Proposition 1.2 and Proposition 3.1.
We can now prove our main result. Given $\pi = (\pi_1, \ldots, \pi_n)$, and $w \in W$, set
\[ T_w \pi = (\exp - (T_w \ln \pi_1(u)))_1, \ldots, \exp - (T_w \ln \pi_n(u))_n. \]

**Theorem 3.** Let $s_{i_1} \cdots s_{i_N}$ be a reduced expression for $w_0$. The module $V(\pi_1) \otimes \cdots \otimes V(\pi_r)$ is highest weight if for all $1 \leq j \leq N$, $1 \leq m \leq \ell \leq r$, the polynomial $(T_{i_{j+1}} T_{i_{j+2}} \cdots T_{i_N} \pi_m)^{i_j}$ is in general position with respect to $(\pi_r)^{i_j}$.

**Proof.** First observe that by Proposition 2.2, $(\pi_1)_{i_j}$ is in general position with respect to $(\pi_r)^{i_j}$. By Lemma 3.2, it suffices to prove that for all $1 \leq j \leq N$, $1 \leq m \leq \ell \leq r$, the polynomial $(T_{i_{j+1}} T_{i_{j+2}} \cdots T_{i_N} \pi_m)^{i_j}$ is in general position with respect to $(\pi_r)^{i_j}$.

If $g = s_{i_j}$, the theorem was proved in Theorem 2 (ii). For arbitrary $g$, we proceed by induction on $r$. If $r = 1$ there is nothing to prove. Let $r > 1$ and let $V' = V(\pi_2) \otimes \cdots \otimes V(\pi_r)$. Then $V'$ is highest weight module with highest weight vector $v' = v_{\pi_2} \otimes \cdots \otimes v_{\pi_r}$ and highest weight $\pi' = \pi_2 \cdots \pi_r$. Setting $\lambda = \deg \pi_1 = ((\deg \pi_1), \ldots, (\deg \pi_1)_n)$, it is enough by Lemma 3.2 to prove that
\[ v_{w_0 \lambda} \otimes v' \in U_g(v_{\pi_1} \otimes v'). \]

By Lemma 3.2, it suffices to prove that for all $1 \leq j \leq N$,
\[ v_{s_{i_j} s_{i_{j+1}} \cdots s_{i_N} \lambda} \otimes v' \in U_{i_j}(v_{s_{i_j} s_{i_{j+1}} \cdots s_{i_N} \lambda} \otimes v'). \]

In fact it suffices to prove that
\[ U_{i_j}(v_{s_{i_j} s_{i_{j+1}} \cdots s_{i_N} \lambda}) \otimes v' = U_{i_j}, v_{s_{i_j} s_{i_{j+1}} \cdots s_{i_N} \lambda} \otimes U_{i_j} v', \]
as $U_{i_j}$–modules.

By Theorem 2 (iii), we know that $U_{i_j}, v_{s_{i_j} s_{i_{j+1}} \cdots s_{i_N} \lambda}$ is a quotient of $W((T_{i_{j+1}} \cdots T_n \pi_1)^{i_j})$. Further we claim that
\[ U_{i_j}, v' = U_{i_j}, v_{\pi_2} \otimes \cdots \otimes U_{i_j}, v_{\pi_r}. \]

To see this, notice that the left hand side is clearly contained in the right hand side. Since $V'$ is highest weight it follows that
\[ U_{i_j}, v_{\pi_2} \otimes \cdots \otimes U_{i_j}, v_{\pi_r} \subset U_g(<) v. \]

Since any element in $U_{i_j}, v_{\pi_2} \otimes \cdots \otimes U_{i_j}, v_{\pi_r}$ has weight $\sum p \deg (\pi_m) - p a_{i_j}$, for some $p \geq 0$, it now follows that
\[ U_{i_j}, v_{\pi_2} \otimes \cdots \otimes U_{i_j}, v_{\pi_r} \subset U_{i_j}(<) v', \]
thus establishing our claim. Since, $U_{i_j}, v_{\pi_m})^{i_j} \simeq V((\pi_m)^{i_j})$ as $U_g(s_{i_j})$–modules, we have therefore proved that $U_{i_j}, v_{s_{i_j} s_{i_{j+1}} \cdots s_{i_N} \lambda} \otimes U_{i_j} v'$ is a quotient of the tensor product of $U_{i_j}$–modules $W((\pi_2)^{i_j}) \otimes V((\pi_3)^{i_j}) \otimes \cdots \otimes V((\pi_r)^{i_j})$. Since we have proved that the polynomial $(\pi_m)^{i_j}$ is in general position with respect to $(\pi_r)^{i_j}$ if $m < \ell$, the result now follows from Theorem 2 (iii).
4. Relationship with Kashiwara’s results and conjectures

Let us consider the special case when $h$ has the following form,

$$h^j_k(u) = 0, \quad j \neq i,$$

and denote the corresponding $n$–tuple of power series by $h^i_{m,a}$ and the $n$–tuple of polynomials by $\pi^i_{m,a}$. We shall prove the following result.

**Theorem 4.** Let $k_1, k_2, \ldots, k_l \in I$, $a_1, \ldots, a_l \in \mathbb{C}(q)$, $m_1, \ldots, m_l \in \mathbb{Z}_+$, and assume that

$$r < s \implies \frac{a_r}{a_s} \neq q^{d_{k_r}m_r-d_{k_s}m_s-d_{k_r}p} \quad \forall \quad p \geq 0.$$

Then, the tensor product $V(\pi^k_{m_1,a_1}) \otimes \cdots \otimes V(\pi^k_{m_l,a_l})$ is a highest weight module.

Assume the theorem for the moment.

**Remark.** In the special case when $m_j = 1$ for all $j$, it was conjectured in [AK] that such a tensor product is cyclic if $a_j/a_i$ does not have a pole at $q = 0$ if $j < \ell$, and this was proved when $g$ is of type $A_n$ or $C_n$; subsequently, a geometric proof of this conjecture was given in [VV] when $g$ is simply–laced; a complete proof was given using crystal basis methods in [K].

The following corollary to Theorem 4 was conjectured in [K], [HKOTY].

**Corollary 4.1.** The tensor product $V = V(\pi^k_{m_1,a_1}) \otimes \cdots \otimes V(\pi^k_{m_\ell,a_\ell})$ is an irreducible $U_q$–module.

**Proof.** First observe that if $d_{k_1}m_1 \leq d_{k_2}m_2 \leq \cdots \leq d_{k_\ell}m_\ell$ then the tensor product is cyclic by Theorem 4. We claim that it suffices to prove the corollary in the case when $\ell = 2$. For then, by rearranging the factors in the tensor product we can show that both $V$ and its dual are highest weight and hence irreducible. To see that $V = V(\pi^k_{m_1,a_1}) \otimes V(\pi^k_{m_2,a_2})$ is cyclic if $d_{k_1}m_1 > d_{k_2}m_2$, we have to consider the case when $d_{k_1}m_1 - d_{k_2}m_2 - d_{k_1} - d_{k_2} \geq 0$.

It suffices to show that $V^\omega$ is cyclic, since $\omega$ is an algebra automorphism. Now, $V^\omega = V(\pi^k_{m_2,a_2}q^{2\kappa}) \otimes V(\pi^k_{m_1,a_1}q^{2\kappa})$ for some fixed $\kappa$ depending only on $g$. By Theorem 4, this is cyclic, since $2d_{k_2} - 2d_{k_1} \neq d_{k_2}m_2 - d_{k_1}m_1 - d_{k_1} - d_{k_2} - p$ for any $p \geq 0$. This proves the result.

\[\square\]

It remains to prove the theorem, for which we must show that, if $w_0 = s_{i_1} \cdots s_{i_N}$ and $w = s_{i_{j+1}} \cdots s_{i_N}$, then the polynomial $(T_w^r \pi^k_{m_r,a_r})_{i_j}$ is in general position with respect to $(\pi^k_{m_\ell,a_\ell})_{i_j}$ for all $r < \ell$.

Using Proposition 2.3, we see that

$$(T_w^r \pi^k_{m_r,a_r})_{i_j} = \prod_{s \geq 0} \pi_{m_r,a_r}(q^s u),$$

where $s$ varies over a finite subset of $\mathbb{Z}_+$ with multiplicity. This means that any root of $(T_w^r \pi^k_{m_r,a_r})_{i_j}$ has the form $q^{d_{k_r}(m_r-2p+1)+s}a_r$ where $s \geq 0$ and $1 \leq p \leq m_r$. If $\ell \neq i_j$, there is nothing to prove since $(\pi^k_{m_\ell,a_\ell})_{i_j} = 1$. If $\ell = i_j$, then the assumption on $a_r/a_\ell$ implies that

$$\frac{q^{d_{k_r}(-m_r+2p+1)+s}a_r}{a_\ell} \neq q^{-d_{k_\ell}(1+m_\ell)}.$$

This proves the theorem.
5. The cyclicity condition made explicit

In this section we work with a specific reduced expression for the longest element $w_0$ of the Weyl group, and give the condition explicitly for the tensor product $V(\pi) \otimes V(\tilde{\pi})$ to be cyclic. In what follows we assume that the nodes of the Dynkin diagram of $g$ are numbered so that 1 is the short root (resp. long root) when $g = B_n$ (resp. $g = C_n$) and that 1 and 2 are the spin nodes when $g = D_n$.

5.1. The classical algebras. The main result is:

**Theorem 5.** (i) Assume that $g$ is of type $A_n$. Then $V(\pi) \otimes V(\pi')$ is cyclic if for all $1 \leq j \leq i \leq n$, we have that the polynomial

$$\prod_{r=1}^{i} \pi_r(q^{2i-j-r}u)\pi_{i-j+2}(q^{i-2}u)\cdots \pi_1(q^{1-j}u)$$

is in general position with respect to $\pi_j'$.

(ii) Assume that $g$ is of type $B_n$. Then $V(\pi) \otimes V(\tilde{\pi})$ is cyclic if for all $i \geq 1$ we have that the polynomial

$$\pi_1(q^{4i-4}u)\prod_{r=2}^{i} \pi_r(q^{4i-1-2r}u)\pi_i(q^{4i-3-2r}u)$$

is in general position with respect to $\pi_i'$ and for all $i \geq j \geq 2$, the polynomials

$$\pi_1(q^{4i-2j-3}u)\prod_{r=2}^{i} \pi_r(q^{4i-2j-2r}u)\prod_{r=2}^{j-1} \pi_r(q^{4i-2j-6+2r}u),$$

$$\prod_{r=j}^{i} \pi_r(q^{4i-6+2j-2r}u)$$

are in general position with respect to $\pi_j'$.

(iii) Assume that $g$ is of type $C_n$. Then $V(\pi) \otimes V(\pi)$ is cyclic if for all $i \geq 1$ we have that the polynomial

$$\prod_{r=1}^{i} \pi_r(q^{2i-1-r}u)$$

is in general position with respect to $\pi_i'$ and for all $i \geq j \geq 2$ the polynomials

$$\pi_1(q^{2i-j-1}u)\pi_1(q^{2i-j+1}u)\prod_{r=2}^{i} \pi_r(q^{2i-j-r}u)\prod_{r=2}^{j-1} \pi_r(q^{2i-j+r}u)\prod_{r=j}^{i} \pi_r(q^{2i+r}u)$$

are in general position with respect to $\pi_j'$.

(iv) Assume that $g$ is of type $D_n$. Then $V(\pi) \otimes V(\tilde{\pi})$ is cyclic if for all $i \geq 3$, and $i$ even (resp. $i$ odd), we have

$$\pi_1(q^{2i-4}u)\prod_{r=3}^{i} \pi_r(q^{2i-2-r}u),$$

$$\text{(resp. } \pi_2(q^{2i-4}u)\prod_{r=3}^{i} \pi_r(q^{2i-2-r}u))$$

is in general position with respect to $\pi_i'$.

$$\pi_2(q^{2i-4}u)\prod_{r=3}^{i} \pi_r(q^{2i-2-r}u), \text{ (resp. } \pi_1(q^{2i-4}u)\prod_{r=3}^{i} \pi_r(q^{2i-2-r}u)),$$
is in general position with respect to \( \pi' \) and and for all \( i \geq j \geq 3 \), the polynomials

\[
\pi_1(q^{2i-j-2}u)\pi_2(q^{2i-j-2}u) \prod_{r=3}^{i} \pi_r(q^{2i-j-r}u) \prod_{r=3}^{j-1} \pi_r(q^{2i-j+r-4}u),
\]

are in general position with respect to \( \pi'_j \).

We note the following corollary of this theorem which gives us a condition for a tensor product \( V(\pi_{m_1,a_1}^{k_1}) \otimes \cdots \otimes V(\pi_{m_\ell,a_\ell}^{k_\ell}) \) to be cyclic. It suffices in view of Corollary 5.1 to give the condition for a pair of such representations to be cyclic. Recall also that the representation \( V(\pi_{m_1,a_1}^{k_1}) \otimes V(\pi_{m_2,a_2}^{k_2}) \) is cyclic if and only if the representation \( V(\pi_{m_2,a_2}^{k_2} \otimes \pi_{m_2,a_2}^{k_2}) \) is cyclic.

**Corollary 5.1.** The tensor product \( V(\pi_{m_1,a_1}^{k_1}) \otimes V(\pi_{m_2,a_2}^{k_2}) \) is cyclic if \( a_1^{-1}a_2 \notin S(\pi_{m_1,a_1}^{k_1}, \pi_{m_2,a_2}^{k_2}) \), where

\[
S(\pi_{m_1,a_1}^{k_1}, \pi_{m_2,a_2}^{k_2}) = \bigcup_{p=1}^{\min(m_1,m_2)} q_{i_2}^{d_1 m_1 + d_2 m_2 + d_1 + d_2 - 2p} S(i_1, i_2)
\]

and \( S(i_1, i_2) \) is defined as follows:

(i) \( \mathfrak{g} = A_n \),

\[
S(i_1, i_2) = \{ q^{2k-i-j} : 1 \leq i_1, i_2 \leq k, i_1 + i_2 \geq k + 1 \}.
\]

(ii) \( \mathfrak{g} = B_n \),

\[
S(1, 1) = \{ q^{4k-4} : 1 \leq k \leq n \},
\]

\[
S(1, i_2) = \{ q^{4k-2i_2-3} : 2 \leq i_2 \leq n \},
\]

\[
S(i_1, 1) = \{ q^{4k-2i_1-3}, q^{4k-2i_2-1} : 2 \leq i_1 \leq n \},
\]

\[
S(i_1, i_2) = \{ q^{4k-2i_1-i_2}, q^{4k-2i_2-i_1} : 2 \leq i_1, i_2 \leq k \leq n \}.
\]

(iii) \( \mathfrak{g} = C_n \),

\[
S(i_1, 1) = \{ q^{2k-1-i_1} : 1 \leq i_1 \leq k \},
\]

\[
S(1, 1) = \{ q^{2k-1-i} \cdot q^{2i-1} : 1 \leq i \leq k \},
\]

\[
S(i_1, i_2) = \{ q^{2k-1-i_2}, q^{2k-1-i_1} : 2 \leq i_1, i_2 \leq k \leq n \}.
\]

(iv) \( \mathfrak{g} = D_n \),

\[
S(1, 1) = \{ q^{2k-4} : 1 \leq k \leq n, k \equiv 0 \mod 2 \} = S(2, 2),
\]

\[
S(1, 2) = \{ q^{2k-4} : 1 \leq k \leq n, k \equiv 1 \mod 2 \}
\]

\[
S(i_1, 1) = \{ q^{2k-1-i_1} : 3 \leq i_1 \leq k \leq n \} = S(1),
\]

\[
S(i_1, i_2) = \{ q^{2k-1-i_2}, q^{2k-1-i_1} : 3 \leq i_1, i_2 \leq k \leq n \}.
\]

To prove Theorem 5.1, we fix a reduced expression for the longest element \( w_0 \). Thus, we take \( w_0 = \gamma_n \gamma_{n-1} \cdots \gamma_1 \), where

\[
\gamma_i = \begin{cases} s_1 s_2 \cdots s_i, & \mathfrak{g} = A_n, \\ s_i s_{i-1} \cdots s_1 s_2 s_3 \cdots s_i, & \mathfrak{g} = B_n, C_n, \\ s_i s_{i-1} \cdots s_2 s_1 s_3 \cdots s_i, & \mathfrak{g} = D_n. \end{cases}
\]

It is convenient also to set \( w_i = \gamma_{n-i} \gamma_{n-1} \cdots \gamma_1 \) for \( 0 \leq i \leq n - 1 \).

Theorem 5.1 is an immediate consequence of the following proposition, which is easily established by an induction on \( i \). We state the proposition only for algebras of type \( A_n \) and \( B_n \), the results for the other algebras are entirely similar.
Proposition 5.1. Let $h \in A^n$.

(i) $g = A_n$. For all $i \geq j$, we have

$$(T_{j+1} \cdots T_{T_{w_i-1}} h) j = \sum_{r=1}^{i-j} h_r(q^{2i-j-r} u),$$

$T_{w_i}(h) = (-h_1(q^{i+1} u), \ldots, -h_1(q^{i+1} u), \sum_{r=1}^{i+1} h_r(q^{i+1-r} u), h_{i+2}(u), \ldots, h_n(u)).$

(ii) $g = B_n$. For all $i \geq j$,

$$(T_{j+1} T_{j+2} \cdots T_{T_{w_i-1}} h) j = h_1(q^{4i-2j-3} u) + \sum_{r=2}^{i} h_r(q^{4i-2j-2r} u) + \sum_{r=2}^{j-1} h_r(q^{4i-2j-6+2r} u),$$

$$(T_2 T_3 \cdots T_{T_{w_i-1}} h) i = h_1(q^{4i-4} u) + \sum_{r=2}^{i} h_r(q^{4i-1-2r} u) + h_r(q^{4i-3-2r} u),$$

$$(T_{j-1} \cdots T_2 T_1 \cdots T_{T_{w_i-1}} h) j = \sum_{r=2}^{i} h_r(q^{4i-2j-6-2r} u),$$

$$(T_{w_i} h) j = \begin{cases} -h_1(q^{4i-2} u) & \text{if } j \leq i, \\ -h_1(q^{2i-1} u) + \sum_{r=2}^{i+1} h_r(q^{2i-2r+2} u) + \sum_{r=2}^{j-1} h_r(q^{3i-2r-4} u) & \text{if } j = i + 1, \\ -h_j(u) & \text{if } j > i + 2. \end{cases}$$

5.2. The Exceptional algebras. We content ourselves with writing down the set $S(i_1, i_2)$, $i_1 \leq i_2$ which gives the values of $a_1^{i_1} a_2$ for which the tensor product $V(i_1, a_1) \otimes V(i_2, a_2)$ is not cyclic, as usual if $i_1 > i_2$ the condition is that $q_1^2 q_2^{-2} a_1^{i_1} a_2 \notin S(i_2, i_1)$ . The values were obtained by using mathematica, the program can be used to write down the conditions for an arbitrary tensor product of representations to be irreducible.

We first consider the algebras $E_n$, $n = 6, 7, 8$, the nodes are numbered as in [39] so that 2 is the special node. The reduced expression for the longest element is chosen as in [3] namely:

$$E_6 \ 81838428584381868548283848586u_1,$$

$$E_7 \ 87868584283486878183485824838186854828384858687u_2,$$

$$E_8 \ 8887868584283486878183485824838186854828384858687u_3,$$

where $u_1$ (resp. $u_2$, $u_3$) are the longest elements of $D_5$ obtained by dropping the node 6 (resp. 7, 8) (resp. $E_6$, $E_7$) chosen previously.
Proposition 5.2. (i) Assume that $g = E_6$. Then,

$$S_6(1, 1) = \{q^2, q^4\} = q^{-4}S_6(1, 6),$$
$$S_6(1, 2) = \{q^5, q^9\},$$
$$S_6(1, 3) = \{q^3, q^7, q^{13}\},$$
$$S_6(1, 4) = \{q^4, q^6, q^8, q^{10}\},$$
$$S_6(1, 5) = \{q^7, q^7, q^{11}\},$$
$$S_6(2, 2) = \{q^2, q^6, q^8, q^{12}\},$$
$$S_6(2, 3) = \{q^4, q^6, q^8, q^{10}\} = S_6(2, 5),$$
$$S_6(2, 4) = \{q^3, q^7, q^7, q^{11}\},$$
$$S_6(2, 6) = \{q^5, q^9\},$$
$$S_6(3, 3) = \{q^2, q^4, q^6, q^8, q^{10}\},$$
$$S_6(3, 4) = \{q^3, q^7, q^7, q^9, q^{11}\},$$
$$S_6(3, 5) = \{q^4, q^6, q^8, q^{10}, q^{12}\},$$
$$S_6(3, 6) = \{q^5, q^7, q^{11}\}$$
$$S_6(4, 4) = \{q^2, q^4, q^6, q^8, q^{10}, q^{12}\},$$
$$S_6(4, 5) = \{q^3, q^5, q^7, q^9, q^{11}\},$$
$$S_6(4, 6) = \{q^4, q^6, q^8, q^{10}\},$$
$$S_6(5, 5) = \{q^2, q^4, q^6, q^8, q^{10}\},$$
$$S_6(5, 6) = \{q^3, q^7, q^9\},$$
$$S_6(6, 6) = \{q^2, q^8\}.$$

(ii) If $g = E_7$, then $S(i, j)$ is the union of the set $S_6(i, j)$ with the sets $S_7(i, j)$ defined below.

$$S_7(1, 1) = \{q^{12}, q^{18}\},$$
$$S_7(1, 2) = \{q^{11}, q^{13}\},$$
$$S_7(1, 3) = \{q^{11}, q^{13}, q^{15}\},$$
$$S_7(1, 4) = \{q^{12}, q^{14}, q^{16}\},$$
$$S_7(1, 5) = \{q^9, q^{13}, q^{15}\},$$
$$S_7(1, 6) = \{q^8, q^{14}\}$$
$$S_7(1, 7) = \{q^7\}.$$
\[ S_7(2, 2) = \{q^{10}, q^{14}, q^{18}\}, \]
\[ S_7(2, 3) = \{q^{12}, q^{16}\}, \]
\[ S_7(2, 4) = \{q^9, q^{13}, q^{15}, q^{17}\}, \]
\[ S_7(2, 5) = \{q^8, q^{12}, q^{14}, q^{16}\} \]
\[ S_7(2, 6) = \{q^7, q^{11}, q^{13}, q^{15}\}, \]
\[ S_7(2, 7) = \{q^6, q^{10}, q^{14}\} \]
\[ S_7(3, 3) = \{q^{12}, q^{14}, q^{16}, q^{18}\} \]
\[ S_7(3, 4) = \{q^{13}, q^{15}, q^{17}\}, \]
\[ S_7(3, 5) = \{q^{14}, q^{16}\}, \]
\[ S_7(3, 6) = \{q^9, q^{13}, q^{15}\}, \]
\[ S_7(3, 7) = \{q^6, q^8, q^{12}, q^{14}\} \]
\[ S_7(4, 4) = \{q^{14}, q^{16}, q^{18}\} \]
\[ S_7(4, 5) = \{q^{13}, q^{15}, q^{17}\} \]
\[ S_7(4, 6) = \{q^{12}, q^{14}, q^{16}\}, \]
\[ S_7(4, 7) = \{q^5, q^7, q^9, q^{11}, q^{13}, q^{15}\} \]
\[ S_7(5, 5) = \{q^{12}, q^{14}, q^{16}, q^{18}\} \]
\[ S_7(5, 6) = \{q^3, q^{11}, q^{13}, q^{15}, q^{17}\}, \]
\[ S_7(5, 7) = \{q^4, q^8, q^{10}, q^{12}, q^{16}\} \]
\[ S_7(6, 6) = \{q^4, q^{10}, q^{12}, q^{16}, q^{18}\}, \]
\[ S_7(6, 7) = \{q^3, q^9, q^{11}, q^{17}\} \]
\[ S_7(7, 7) = \{q^2, q^{10}, q^{18}\}. \]
(iii) If $g = E_8$, then $S(i,j)$ is the union of the set $S_6(i,j) \cup S_{7}(i,j)$ with the sets $S_8(i,j)$ defined below.

\[
S_8(1, 1) = \{q^{14}, q^{20}, q^{24}, q^{30}\}, \\
S_8(1, 2) = \{q^{17}, q^{19}, q^{21}, q^{23}, q^{27}\}, \\
S_8(1, 3) = \{q^{17}, q^{19}, q^{21}, q^{23}, q^{25}, q^{29}\}, \\
S_8(1, 4) = \{q^{18}, q^{20}, q^{22}, q^{24}, q^{26}, q^{28}\}, \\
S_{7}(1, 5) = \{q^{11}, q^{17}, q^{19}, q^{21}, q^{23}, q^{25}, q^{27}\}, \\
S_8(1, 6) = \{q^{10}, q^{16}, q^{18}, q^{20}, q^{24}, q^{28}\}, \\
S_{8}(1, 7) = \{q^{9}, q^{17}, q^{19}, q^{25}\}, \\
S_{8}(1, 8) = \{q^{9}, q^{14}, q^{18}\} \\
S_{8}(2, 2) = \{q^{12}, q^{16}, q^{20}, q^{22}, q^{24}, q^{26}, q^{30}\}, \\
S_{8}(2, 3) = \{q^{14}, q^{18}, q^{20}, q^{22}, q^{24}, q^{26}, q^{28}\}, \\
S_{8}(2, 4) = \{q^{19}, q^{21}, q^{23}, q^{25}, q^{27}, q^{29}\}, \\
S_{8}(2, 5) = \{q^{18}, q^{20}, q^{22}, q^{24}, q^{26}, q^{28}\}, \\
S_{8}(2, 6) = \{q^{17}, q^{19}, q^{21}, q^{23}, q^{25}, q^{27}\}, \\
S_{8}(2, 7) = \{q^{12}, q^{16}, q^{18}, q^{20}, q^{22}, q^{24}, q^{26}, q^{28}\} \\
S_{8}(2, 8) = \{q^{7}, q^{11}, q^{17}, q^{21}, q^{25}\}
\]
If \( g = F_4 \), then we work with the following reduced expression for \( w_0 \),
\[
\begin{align*}
& s_4 s_3 s_2 s_4 s_1 s_2 s_1 s_4 s_3 s_4 s_1 s_2 s_3 s_1 s_2 s_3 s_2 s_3 s_4 s_3 s_2 s_1 s_2 s_3 s_2 s_3 s_2 s_3.
\end{align*}
\]
We assume here that the nodes 1 and 2 correspond to the short simple roots. Assume that \( d_i \geq d_j \), then \( V(i,a) \otimes V(j,b) \) is cyclic if \( ab^{-1} \in S(i,j) \), where
\[
S(1,1) = \{ q^2, q^6, q^{12}, q^{18} \},
S(1,2) = \{ q^7, q^9, q^{13}, q^{17} \},
S(1,3) = \{ q^6, q^{10}, q^{16} \},
S(1,4) = \{ q^8, q^{14} \}.
\]
\[
S(2,2) = \{ q^2, q^6, q^8, q^{10}, q^{12}, q^{14}, q^{16}, q^{18} \},
S(2,3) = \{ q^5, q^7, q^9, q^{11}, q^{13}, q^{15}, q^{17} \},
S(2,4) = \{ q^7, q^{11}, q^{13}, q^{17} \}.
\]
\[
S(3,3) = \{ q^6, q^8, q^{10}, q^{12}, q^{14}, q^{16}, q^{18} \},
S(3,4) = \{ q^6, q^8, q^{10}, q^{14}, q^{16} \},
\]
\[
S(4,4) = \{ q^4, q^{10}, q^{12}, q^{18} \}
\]
If \( g = G_2 \), then we choose the following reduced expression for \( w_0 \), where 1 is corresponds to the short simple root.
\[
\begin{align*}
& s_2 s_1 s_2 s_1.
\end{align*}
\]
We have,
\[
\begin{align*}
S(1,1) &= \{ q^2, q^6, q^8, q^{12} \},
S(1,2) &= \{ q^7, q^{11} \}
S(2,2) &= \{ q^6, q^8, q^{10}, q^{12} \}.
\end{align*}
\]

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