BIFURCATION ANALYSIS OF A GENERAL ACTIVATOR-INHIBITOR MODEL WITH NONLOCAL DISPERAL

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ABSTRACT. In this paper, we are mainly concerned with the effect of nonlocal diffusion and dispersal spread on bifurcations of a general activator-inhibitor system in which the activator has a nonlocal dispersal. We find that spatially inhomogeneous patterns always exist if the dispersal rate of the activator is sufficiently small, while a larger dispersal spread and an increase of the activator diffusion inhibit the formation of spatial patterns. Compared with the “spatial averaging” nonlocal dispersal model, our model admits a larger parameter region supporting pattern formations, which is also true if compared with the local reaction-diffusion one when the dispersal spread is small. We also study the existence of nonconstant positive steady states through bifurcation theory and find that there could exist finite or infinite steady state bifurcation points of the inhibitor diffusion constant. As an example of our results, we study a water-biomass model with nonlocal dispersal of plants and show that the water and plant distributions could be inphase and antiphase.

1. Introduction. Since the pioneering work of Turing [34], reaction-diffusion equations as activator-inhibitor systems have attracted much attention of many researchers to study the formation of spatially inhomogeneous patterns. The diffusion constants are known to be the important factors in generation of the complex spatial-temporal structure. To be more precise, a spatially homogenous steady state is stable with respect to homogenous perturbation, but is unstable with respect to inhomogeneous perturbation if the activator-inhibitor system has a slow diffusion rate for activator and a fast diffusion rate for inhibitor [15, 25], which is called Turing instability or diffusion-driven instability.
A classical activator-inhibitor system is usually described by coupled reaction-diffusion equations:

\[
\begin{aligned}
    u_t &= d_u u_{xx} + f(u, v), \quad x \in \Omega, \ t > 0, \\
v_t &= d_v v_{xx} + g(u, v), \quad x \in \Omega, \ t > 0,
\end{aligned}
\]  
(1.1)

where \( u(x, t) \) and \( v(x, t) \) denote the local concentrations of the activator and inhibitor species in a given spatial region \( \Omega \), which is assumed to be a bounded, connected and smooth domain in \( \mathbb{R}^n (n \geq 1) \) with smooth boundary. The terms \( d_u u_{xx} \) and \( d_v v_{xx} \) represent the random diffusion of the activator and inhibitor species, respectively, where \( d_u \) and \( d_v \) are the dispersal rate of each one and are both positive-defined. The functions \( f(u, v), g(u, v) \in C^2(\mathbb{R}^n \times \mathbb{R}^n) \) stand for the local dynamics of the system. We assume that \((u_0, v_0)\) is a constant equilibrium solution which is stable for the corresponding kinetic system of (1.1), then the Jacobian

\[
J = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}
\]

at \((u_0, v_0)\) satisfies \( f_u + g_v < 0 \) and \( f_u g_v - f_v g_u > 0 \). We also assume \( f_u > 0 \) and \( g_v < 0 \) as the activator auto-catalytically promotes its own production and the inhibitor inhibits its own formation \[27\]. Thus, there are two possibilities which can be shown with the sign patterns of the Jacobian matrix:

\[
(a) \begin{pmatrix} + & + \\ + & - \end{pmatrix}, \quad (b) \begin{pmatrix} + & - \\ + & + \end{pmatrix}.
\]  
(1.2)

For model (1.1), Turing instability occurs if \( d_v > k_0 d_u \), where

\[
k_0 = \frac{f_u g_v - 2 f_v g_u + 2 \sqrt{-f_v g_u (f_u g_v - f_v g_u)}}{f_u^2}.
\]

Furthermore, under certain conditions, there are finite bifurcation points of the diffusion rate for the inhibitor where spatially nonhomogeneous steady states are bifurcated. We note that reaction-diffusion systems with a non-linear dispersion can also admit Turing instability \[5\].

In model (1.1) the Laplacian operator essentially describes the random diffusion of each individual in a small range area. However, the motion of individuals is often free and usually occurs in a long range area. Then it is reasonable to introduce the nonlocal diffusion to describe the movement of individuals. Assume \( \phi(x, t) \) is the density of a single population at location \( x \in \Omega \) and time \( t \). The nonlocal dispersal can be described by the convolution operator

\[
(J * \phi - \phi)(x) = \int_{\Omega} J(x - y) \phi(y) dy - \phi(x),
\]  
(1.3)

where \( x, y \in \Omega \). Here, the function \( J(x) \) satisfies \( J(x) \in C^1(\mathbb{R}^n) \), \( J(x) > 0 \) in \( \Omega \), \( J(x) = 0 \) in \( \mathbb{R}^n \setminus \Omega \), \( J(-x) = J(x) \), and \( \int_{\Omega} J(x) dx = 1 \). As pointed out in \[13\], the integral operator (1.3) can be written as \( \int_{\Omega} J(x - y) (\phi(y) - \phi(x)) dy \) which describes diffusion processes. In this term, \( J(x - y) \) is interpreted as the probability of “jumping” from location \( y \) to location \( x \), the convolution \( (J * \phi)(x) = \int_{\Omega} J(x - y) \phi(y) dy \) is the rate at which individuals arrive to position \( x \) from all other positions, while \( -\int_{\Omega} J(x - y) \phi(y) dy = -\int_{\Omega} J(y - x) \phi(x) dy = -\phi(x) \) is the rate at which they leave position \( x \) to reach any other position. For example, in water-biomass models the kernel function \( J(x - y) \) means the probability per unit length of seeds originating at the point \( y \) being dispersed to point \( x \) \[1, 12, 30, 31\]; in epidemic models the kernel
function $J(x-y)$ weights the contributions of the susceptible/infective/recovered individuals at location $y$ to the susceptible/infective/recovered individuals at location $x$ \cite{4, 26, 37, 38}.

In recent years, the nonlocal dispersal has attracted the attention of a great number of investigators \cite{2, 3, 7, 8, 9, 10, 14, 20, 21, 36} and the references therein for more details. A general activator-inhibitor model in which the activator has a nonlocal dispersal is

\[
\begin{align*}
    u_t &= d_u(J * u - u) + f(u, v), \quad x \in \Omega, \ t > 0, \\
    v_t &= d_v v_{xx} + g(u, v), \quad x \in \Omega, \ t > 0.
\end{align*}
\]

(1.4)

In \cite{6}, the authors assumed the nonlocal dispersal operator to be “spatial averaging” and have studied its effect on the spatial pattern formation of model (1.4) with Neumann boundary condition in one dimensional domain. For this simplified operator, they found that, from a spectrum analysis of the corresponding linearized operator, the constant steady state is unstable if the activator dispersal is slow and any non-constant steady states caused by slow activator dispersal or symmetry-breaking bifurcations are unstable.

We point out that the “spatial averaging” dispersal operator only approximately describes the individual motion for the sufficiently large spreading scale. To appropriately reflect some important individual dispersal traits, we use a more general convolution term which depends on the spread parameter to explore the effect of nonlocal dispersal traits on spatial-temporal dynamics of the population. Assume $\Omega = (-l, l) \subset \mathbb{R}$. The dependence of the kernel function upon the spread parameter is given by

\[
J(x) = \begin{cases} 
\frac{1}{2\sqrt{d_w \sinh(l/\sqrt{d_w})}} \cosh\left(\frac{l-|x|}{\sqrt{d_w}}\right), & -l < x < l, \\
0, & \text{otherwise}.
\end{cases}
\]

(1.5)

There are two main parameters that control the shape of the kernel: $l$, which is the half length of the habitat, and $d_w$, which is the spread of dispersal that characterises the dispersal distance \cite{20}. A typical example of the kernel (1.5) is shown in Figure 1 in which the horizontal and vertical axes correspond to the position $x \in [-l, l]$ and the value of $J(x)$. Obviously, the spread parameter $d_w$ can be used to describe the level of being contracted or stretched. For fixed $l$, the maximum of $J(x)$ becomes smaller and its width is larger with the increase of $d_w$ which implies the individuals can move in a longer range area.

![Figure 1](image_url)
The model (1.4) with periodic boundary condition reads as follows

\[
\begin{align*}
& u_t = d_u(J * u - u) + f(u, v), \quad x \in [-l, l], \; t > 0, \\
& v_t = d_v v_{xx} + g(u, v), \quad x \in (-l, l), \; t > 0, \\
& v(-l, t) = v(l, t), \; v_x(-l, t) = v_x(l, t), \; t > 0, \\
& u(x, 0) = u_0(x) \geq 0, \quad v(x, 0) = v_0(x) \geq 0, \quad x \in [-l, l],
\end{align*}
\]

where \( J(x) \) is given in (1.5). We mainly investigate the effects of the nonlocal spatial dispersal and the spread of dispersal on Turing bifurcation and steady state bifurcation of model (1.6). We want to know how the nonlocal dispersal and the spread of dispersal affect the spatial pattern formation. Recall that \( d_u \) is the rate of dispersal, which measures the rate of flow of the individuals at any point, and \( d_v \) is the spread of dispersal, which is a measure of the spread of the individuals [20]. Our study shows that spatially inhomogeneous patterns always exist if the dispersal rate of the activator is sufficiently small, while a larger dispersal spread and an increase of activator diffusion inhibit the formation of spatially inhomogeneous patterns. Compared with the nonlocal dispersal model with “spatial averaging” dispersal operator [6], the nonlocal model (1.6) admits more complex patterns and always predicts a larger parameter region supporting pattern formation, which is also true if it is compared with the local dispersal reaction-diffusion model (1.1) when the dispersal spread is small.

The paper is organized as follows. The Turing bifurcation analysis of model (1.6) is given in Section 2. In Section 3, we give the steady state bifurcation analysis of model (1.6). In Section 4, as an example of our results, we obtain the Turing instability region, the effect of the nonlocal dispersal, and the existence of non-constant steady states through bifurcation analysis of a nonlocal modified Klausmeier-Gray-Scott model. We end with some more discussions in Section 5.

Throughout this paper, we use the following notations.

\( N_0 = \mathbb{N} \cup \{0\} \).

\( \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} \).

\( X_C = X \oplus iX = \{x_1 + ix_2 : x_1, x_2 \in X\} : \) the complexification of \( X \).

\( \sigma(L), \sigma_p(L) : \) the spectrum and point spectrum of the linear operator \( L \), respectively.

\( \mathcal{D}(L), \mathcal{R}(L), \mathcal{N}(L) : \) the domain, range and kernel of the linear operator \( L \), respectively.

\( C([-l, l]) : \) the Banach space of all continuous on the interval \([-l, l]\).

\( C^2_P[-l, l] : \) the Banach space of all twice continuously differential functions on the interval \([-l, l]\) satisfying periodic boundary conditions.

\( C^2_{PE}[-l, l] : all \ even \ functions \ in \ C^2_P[-l, l] \).

2. Turing instability. In this section, we aim to study the linear stability and show the effect of the nonlocal dispersal and the dispersal spread on the Turing instability of system (1.6).

For system (1.6), define a mapping \( G \) by

\[
G \left( \begin{array}{c}
u \\
u
\end{array} \right) = \left( \begin{array}{c}
d_u(J * u - u) + f(u, v) \\
n(v_{xx} + g(u, v))
\end{array} \right), \quad (2.1)
\]
Proof. Define an operator integral operator \( K \) zero simultaneously. Then from (2.6), we obtain
\[
\mathcal{L} \equiv \partial_t(u,v)G(u_0, v_0)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} d_u(J*\phi - \phi) + f_u\phi + f_v\psi \\ d_v\psi \end{pmatrix}.
\]
Then \( \mathcal{L} \) is a closed linear operator in \( Y \) with domain \( D(\mathcal{L}) = X \). Define
\[
K\phi := J*\phi - \phi, \tag{2.2}
\]
where \( J(x) \) has the form of (1.5). Firstly, we give the results about the spectral set of the convolution operator \( K \).

**Lemma 2.1.** Let \( \mathcal{K} \) be defined as in (2.2). Then the spectral set \( \sigma(\mathcal{K}) = \sigma_p(\mathcal{K}) = \{-1\} \cup \left\{-\frac{d\mu}{d\mu + 1}, k \in \mathbb{N}_0 \right\}. \]

**Proof.** Define an operator \( \mathcal{K}_0 : \phi \to J*\phi \) for \( \phi \in C[-l, l] \). Then the Fredholm integral operator \( \mathcal{K}_0 \) is compact on \( C[-l, l] \), and Riesz-Schauder theory implies that
\[
0 \in \sigma(\mathcal{K}_0), \quad \sigma(\mathcal{K}_0) \setminus \{0\} \subset \sigma_p(\mathcal{K}_0).
\]

Obviously,
\[
\mathcal{N}(\mathcal{K}_0) = \left\{ \phi \in C[-l, l] : \int_{-l}^{l} \frac{1}{2\sqrt{a_w} \sinh \sqrt{a_w}} \cosh \left( \frac{l-x-y}{\sqrt{a_w}} \right) \phi(y)dy = 0 \right\} \neq \emptyset.
\]
Then \( 0 \in \sigma_p(\mathcal{K}_0) \). Therefore, \( \sigma(\mathcal{K}_0) = \sigma_p(\mathcal{K}_0) \). To find the nonzero point spectrum of the linear operator \( \mathcal{K}_0 \), we assume there is a \( \eta \in C \cap \sigma_p(\mathcal{K}_0) \setminus \{0\} \) and a function \( \phi(x) \in C[-l, l] \), such that
\[
\begin{cases}
J*\phi = \eta\phi, & x \in (-l, l), \\
\phi(-l) = \phi(l), \phi_x(-l) = \phi_x(l).
\end{cases} \tag{2.3}
\]

Considering the Green kernel of the problem
\[
\begin{align*}
-d_u w_{xx} + w &= \phi, & x \in (-l, l), \\
w(-l) &= w(l), w_x(-l) = w_x(l).
\end{align*} \tag{2.4}
\]
If \( J(x) \) has the form of (1.5), then we have \( w(x) = (J*\phi)(x) \) [28]. Combining (2.4) with (2.3), we have
\[
\begin{align*}
w &= \eta\phi, \\
d_u w_{xx} - w + \phi &= 0, & x \in (-l, l), \\
w(-l) &= w(l), w_x(-l) = w_x(l), \tag{2.5}
\end{align*}
\]
which is equivalent to
\[
\begin{align*}
d_w w_{xx} - w + \frac{1}{\eta} w &= 0, & x \in (-l, l), \\
w(-l) &= w(l), w_x(-l) = w_x(l). \tag{2.6}
\end{align*}
\]

It is well known that the eigenvalue problem
\[
-\varphi'' = \mu \varphi, \quad x \in (-l, l), \quad \varphi(-l) = \varphi(l), \quad \varphi'(-l) = \varphi'(l) \tag{2.7}
\]
has eigenvalues \( \mu_k = \left( \frac{2\pi}{l} \right)^2 (k \in \mathbb{N}_0) \) with corresponding eigenfunctions \( \varphi_k(x) = c_{1k} \cos(\pi k x/l) + c_{2k} \sin(\pi k x/l) \), where \( c_{1k}, c_{2k} \) are constants which are not equal to zero simultaneously. Then from (2.6), we obtain
\[
\frac{1}{d_w} \left( \frac{1}{\eta} - 1 \right) = \mu_k, \quad k \in \mathbb{N}_0.
\]
Therefore, we obtain the sequence of eigenvalues of problem (2.3)
\[
\eta_k = \frac{1}{d_w \mu_k + 1}, \quad k \in \mathbb{N}_0,
\]
with corresponding eigenfunction \( \varphi_k(x) \). Then \( \sigma(K_0) = \sigma_p(K_0) = \{0\} \cup \left\{ \frac{1}{d_w \mu_k + 1}, k \in \mathbb{N}_0 \right\} \). Therefore, \( \sigma(K) = \sigma_p(K) = \{-1\} \cup \left\{ -\frac{d_w \mu_k}{d_w \mu_k + 1}, k \in \mathbb{N}_0 \right\} \). This completes the proof. 

Now we are in the position to study the Turing instability of system (1.6). Actually, we have the following results.

**Theorem 2.2.** Assume \((u_0, v_0)\) is a constant steady state of system (1.6), the Jacobian matrix at \((u_0, v_0)\) has the sign patterns of (a) and (b) as in (1.2), and satisfies \(f_u + g_v < 0\) and \(f_v g_v - f_u g_u > 0\). Then

(i): If \(d_u < f_u\), then \((u_0, v_0)\) is an unstable steady state with respect to system (1.6).

(ii): If \(d_u = f_u\), then \((u_0, v_0)\) is a stable steady state when \(d_v \leq d_v^*\), but an unstable steady state when \(d_v > d_v^*\) with respect to system (1.6). Here,
\[
d_v^* = -\frac{d_w f_v g_u}{f_u}.
\]

(iii): If \(d_u > f_u\), then \((u_0, v_0)\) is a stable steady state when \(d_v < d_v^{**}\), but an unstable steady state when \(d_v > d_v^{**}\) with respect to system (1.6).
\[
d_v^{**} := \frac{d_v (d_u - f_u) (f_u g_v - f_v g_u) - f_v g_v d_u + 2 \sqrt{-d_u f_v g_u (d_u - f_u) (f_u g_v - f_v g_u)}}{f_u^2}.
\]

**Proof.** For \(\lambda \in \mathbb{C}\), the associated eigenvalue problem of \(\mathcal{L}\) is
\[
\partial_{(u,v)} G(u_0, v_0) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix},
\]
which is equivalent to
\[
\begin{align*}
\lambda \phi &= d_u (J \ast \phi - \phi) + f_u \phi + f_v \psi, \\
\lambda \psi &= d_v \psi'' + g_u \phi + g_v \psi.
\end{align*}
\]

Let
\[
\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} \phi_k \\ \psi_k \end{pmatrix} \varphi_k(x)
\]
be an eigenfunction for \(\mathcal{L}\) with eigenvalue \(\lambda\), i.e. \(\mathcal{L} (\phi, \psi)^T = \lambda (\phi, \psi)^T\). Then there exists \(k \in \mathbb{N}_0\) such that \(\mathcal{L}_k (\phi_k, \psi_k)^T = \lambda (\phi_k, \psi_k)^T\), where \(\mathcal{L}_k\) is defined by
\[
\mathcal{L}_k := \begin{pmatrix} -\frac{d_u d_v \mu_k}{d_w \mu_k + 1} + f_u f_v \\
g_u - d_v \mu_k + g_v
\end{pmatrix}
\]
for all other parameters. The corresponding characteristic equation of \(\mathcal{L}_k\) is
\[
\text{Det}(\lambda I - \mathcal{L}_k) = \lambda^2 - \text{Tr}_k \lambda + \text{Det}_k = 0, \quad k \in \mathbb{N}_0,
\]
Since \( d_u, d_v, d_w \) are all positive-defined, then \( \text{Tr}_k < 0 \) is always true. Hence, \((u_0, v_0)\) is a stable steady state with respect to (1.6) if \( \text{Det}_k > 0 \) for all \( k \in \mathbb{N}, \) but an unstable steady state with respect to (1.6) if \( \text{Det}_k < 0 \) for some \( k \in \mathbb{N}. \)

Define

\[
D(d_v, p) := d_v d_w (d_u - f_u) p^2 - [(g_v d_w (d_u - f_u) + f_v g_u d_w + d_v f_u) p + (f_u g_v - f_v g_u)],
\]

and

\[
\Delta := (g_v d_w (d_u - f_u) + f_v g_u d_w + d_v f_u)^2 - 4d_v d_w (d_u - f_u) (f_u g_v - f_v g_u).
\]

The discussion is divided into three cases.

**Case 1.** If \( d_u < f_u \), then \( D(d_v, p) < 0 \) for \( p > p_0 \) with

\[
p_0 = \frac{g_v d_w (d_u - f_u) + f_v g_u d_w + d_v f_u - \sqrt{\Delta}}{2d_v d_w (d_u - f_u)},
\]

which implies \( \text{Det}_k < 0 \) for all \( k > k_0 \) with \( \mu_{k_0} \geq p_0. \)

**Case 2.** If \( d_u = f_u \), then \( D(d_v, p) > 0 \) for all \( p > 0 \) when \( d_v \leq d_v^* \), and \( D(d_v, p) < 0 \) for \( p > p^* \) with

\[
p^* = \frac{f_u g_v - f_v g_u}{f_u d_v + d_w f_v g_u},
\]

when \( d_v > d_v^* \), which implies \( \text{Det}_k > 0 \) for all \( k \in \mathbb{N} \) when \( d_v \leq d_v^* \), and \( \text{Det}_k < 0 \) for all \( k > k^* \) with \( \mu_{k^*} \geq p^* \) when \( d_v > d_v^*. \)

**Case 3.** If \( d_u > f_u \), then \( D(d_v, p) > 0 \) for all \( p > 0 \) when \( d_v < d_v^{**} \), and \( D(d_v, p) < 0 \) for \( p_- < p < p_+ \) when \( d_v > d_v^{**} \), where

\[
p_- = \frac{g_v d_w (d_u - f_u) + f_v g_u d_w + d_v f_u - \sqrt{\Delta}}{2d_v d_w (d_u - f_u)},
\]

\[
p_+ = \frac{g_v d_w (d_u - f_u) + f_v g_u d_w + d_v f_u + \sqrt{\Delta}}{2d_v d_w (d_u - f_u)},
\]

which implies \( \text{Det}_k > 0 \) for all \( k \in \mathbb{N} \) when \( d_v < d_v^{**} \), and \( \text{Det}_k < 0 \) for some \( k \) satisfying \( p_- < \mu_k < p_+ \) if the spatial scale \( l \) is appropriately chosen when \( d_v > d_v^{**} \).

**Remark 2.3.** (1) Turing instability occurs if \( d_v > k_0 d_u \) for the local model (1.1), but for the nonlocal model (1.6) it can also occur when \( d_v < k_0 d_u \) if \( d_u < f_u \) because of the effect of the nonlocal dispersal.

(2) For the nonlocal model (1.6), the region for Turing instability does not change when \( d_u < f_u \) whatever \( d_w \) is.

(3) For the nonlocal model (1.6), the region for Turing instability becomes smaller with the increase of \( d_w \) when \( d_u \geq f_u \).
In a conclusion, the spatial pattern formation for model (1.6) under the schemes (a) and (b) is possible when the dispersal parameter pair \((d_u, d_v)\) is in the instability region \(R_1 = R_1^1 \cup R_2^2 \cup R_3^3\), where

\[
\begin{align*}
R_1^1 &= \{ (d_u, d_v) : 0 < d_u < f_u, d_v > 0 \}, \\
R_2^2 &= \{ (d_u, d_v) : d_u = f_u, d_v > d_v^* \}, \\
R_3^3 &= \{ (d_u, d_v) : d_u > f_u, d_v > d_v^* \}.
\end{align*}
\]

Compared with random diffusion models, spatially inhomogeneous patterns always exist if \(d_u\) (the disperse rate of the activator) is sufficiently small, and the parameter region for pattern formation becomes smaller as the dispersal spread \(d_w\) increases if \(d_u\) is appropriately large. This implies that a larger \(d_w\) and a larger \(d_u\) inhibit the formation of patterns. On the other hand, if compared with the “spatial averaging” dispersal operator, the nonlocal dispersal with the kernel function (1.5) could cause more complicated results and predicts a larger parameter region supporting pattern formations.

We are also interested in the cases of \(d_w \to 0\) and \(d_w \to +\infty\), and the following results can be easily obtained.

\textbf{Corollary 2.4.} Assume \((u_0, v_0)\) is a constant steady state of system (1.6), the Jacobian matrix at \((u_0, v_0)\) has the sign patterns of (a) and (b) as in (1.2), and satisfies \(f_u + g_v < 0\) and \(f_ug_v - f_vg_u > 0\). Then \((u_0, v_0)\) is an unstable steady state with respect to system (1.6) as \(d_w \to 0\).

\textbf{Corollary 2.5.} Assume \((u_0, v_0)\) is a constant steady state of system (1.6), the Jacobian matrix at \((u_0, v_0)\) has the sign patterns of (a) and (b) as in (1.2), and satisfies \(f_u + g_v < 0\) and \(f_ug_v - f_vg_u > 0\). If \(d_w \to +\infty\), then

(i): if \(d_u < f_u\), then \((u_0, v_0)\) is an unstable steady state with respect to system (1.6);

(ii): if \(d_u \geq f_u\), then \((u_0, v_0)\) is a stable steady state with respect to system (1.6).

\textbf{Remark 2.6.} If \(J(x) = \frac{1}{2l}\), then \((J * u)(x) = \frac{1}{2l} \int_{-l}^{l} u(y)dy\). Define the operator \(\hat{\mathcal{L}}\phi = \frac{1}{2l} \int_{-l}^{l} \phi(y)dy - \phi\). Then \(\sigma(\hat{\mathcal{L}}) = \sigma_P(\hat{\mathcal{L}}) = \{0, -1\}\). We can obtain that the results about the linear stability of the constant steady state \((u_0, v_0)\), which has been studied in [6], are the same as the case \(d_w \to +\infty\). This result can be easily explained because

\[
\frac{1}{2\sqrt{d_w} \sinh \frac{1}{\sqrt{d_w}}} \cosh \left( \frac{l - |x|}{\sqrt{d_w}} \right) \to \frac{1}{2l}, \quad \text{as} \quad d_w \to +\infty.
\]

3. Steady state bifurcation. In this section, we study the steady state bifurcation of (1.6). The corresponding stationary problem for system (1.6) is equivalent to the following problem:

\[
\begin{align*}
d_u(J * u - u) + f(u, v) &= 0, \quad x \in [-l, l], \\
d_vv_{xx} + g(u, v) &= 0, \quad x \in (-l, l), \\
v(-l) &= v(l), v_x(-l) = v_x(l).
\end{align*}
\]

Define \(C_P^2[-l, l] := \{ \phi \in C^2[-l, l] : \phi(-l) = \phi(l), \phi'(-l) = \phi'(l), \phi(-x) = \phi(x) \}\), and a mapping \(G_1\) by

\[
G_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} d_u(J * u - u) + f(u, v) \\ d_vv_{xx} + g(u, v) \end{pmatrix},
\]

\[
\begin{align*}
\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} 0 \\ l \end{pmatrix}, \\
\begin{pmatrix} u \end{pmatrix} &= \begin{pmatrix} 0 \end{pmatrix},
\end{align*}
\]

\[
(1.6)\quad \begin{align*}
d_u(J * u - u) + f(u, v) &= 0, \\
d_vv_{xx} + g(u, v) &= 0, \\
v(-l) &= v(l), v_x(-l) = v_x(l).
\end{align*}
\]
where \((u, v) \in C[-l, l] \times C^2_0 E[-l, l] \equiv X_1\). Then \(G_1 : X_1 \rightarrow Y\) is Fréchet differentiable, and at a constant steady state \((u_0, v_0)\),
\[
\mathcal{L}_1 \equiv \partial_{(u, v)} G_1(u_0, v_0) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} d_u (J \ast \phi - \phi) + f_v \phi + f_v \psi \\ d_v \psi'' + g_u \phi + g_v \psi \end{pmatrix}.
\]

We will give the bifurcation analysis of \(\mathcal{L}_1\) from simple eigenvalue by applying the classical Crandall-Rabinowitz bifurcation theorem \([11, 32]\). These results have been strengthened by Pejsachowicz and Rabier in \([29]\) and Shi and Wang in \([33]\).

Note that for system (3.1) the determinant of the Jacobian matrix at a constant steady state \((u_0, v_0)\) can be rewritten \(\text{Det}_k\) as
\[
\text{Det}_k = \frac{D(d_v, p)}{d vp + 1}
\]
where \(D(d_v, p)\) is defined in (2.14) and \(p := \mu_k = (\pi k/l)^2\). Then \(\text{Det}_k = 0\) if and only if \(D(d_v, p) = 0\). Firstly, we give the possible steady state bifurcation points if the dispersal parameter pair \((u, v)\) is in the instability region \(R_1\).

If \((u, v) \in R_1^1\), solving \(d_v\) from the equation \(D(d_v, p) = 0\), we have
\[
d_v = d_v(p) := \frac{[d_u g_v (f_u - d_u) - d_u f_v g_u]p + f_u g_v - f_v g_u}{d_u (f_u - d_u)p^2 + f_u p}.
\]

Obviously, the function \(d_v(p)\) is a strictly monotonically decreasing function in the interval \((0, +\infty)\). Furthermore, \(d_v(p) \rightarrow 0\) as \(p \rightarrow +\infty\) and \(d_v(p) \rightarrow +\infty\) as \(p \rightarrow 0\).

Define
\[
d_{v, k} := d_v(\mu_k) = \frac{[d_u g_v (f_u - d_u) - d_u f_v g_u]\mu_k + f_u g_v - f_v g_u}{d_u (f_u - d_u)\mu_k^2 + f_u \mu_k}, \quad k \in \mathbb{N}.
\]

Then the function \(d_v(p)\) is a strictly monotonically decreasing function in the interval \((0, +\infty)\). Furthermore, \(d_v(p) \rightarrow -d_u f_v g_u / f_u\) as \(p \rightarrow +\infty\) and \(d_v(p) \rightarrow +\infty\) as \(p \rightarrow 0\).

Define
\[
d_{v, k} := d_v(\mu_k) = -\frac{d_u f_v g_u}{f_u} + \frac{f_u g_v - f_v g_u}{f_u \mu_k}, \quad k \in \mathbb{N}.
\]

If \((u, v) \in R_1^2\), solving \(d_v\) from the equation \(D(d_v, p) = 0\), we have
\[
d_v = d_v(p) := \frac{[d_u g_v (u - f_u) + d_u f_v g_u]p - (f_u g_v - f_v g_u)}{d_u (u - f_u)p^2 - f_u p}.
\]

Obviously, \(d_v(p) \rightarrow +\infty\) as \(p \rightarrow 0\) and \(d_v(p) \rightarrow +\infty\) as \(p \rightarrow \frac{f_u}{d_u (u - f_u)}\). Furthermore, there exists a \(p_c \in (0, \frac{f_u}{d_u (u - f_u)})\), such that \(d_v(p)\) is decreasing in \((0, p_c)\) and increasing in \((p_c, \frac{f_u}{d_u (u - f_u)})\), where \(p_c = \frac{(d_u - f_u)(f_u g_v - f_v g_u)\sqrt{f_u g_v - f_v g_u}}{(d_u - f_u)(f_u g_v + f_u g_v) - (f_u - f_v)(f_u g_v - f_v g_u)}\).

Define
\[
d_{v, k} := d_v(\mu_k) = \frac{[d_u g_v (u - f_u) + d_u f_v g_u]\mu_k - (f_u g_v - f_v g_u)}{d_u (u - f_u)\mu_k^2 - f_u \mu_k}, \quad 0 < \mu_k < \frac{f_u}{d_u (u - f_u)}.
\]

Now we will show that \(d_v = d_{v, k}\), which are defined as in (3.3), (3.4) and (3.5), are steady state bifurcation points, from which spatially nonhomogeneous steady states will emerge.
Theorem 3.1. Assume that the parameters \( d_u, d_v, d_w \) are all positive, \((u_0, v_0)\) is a constant steady state of (3.1). Let \( \mu_k = (\pi k/l)^2 \) be an eigenvalue of (2.7) with the corresponding even eigenfunction \( \varphi_k(x) = \cos(k\pi x/l) \) such that

(i) \( \mu_k \) is a simple eigenvalue;
(ii) \( 0 < \mu_k < +\infty \) if \((d_u, d_v) \in R_1^1 \cup R_2^1 \) and \( 0 < \mu_k < \frac{f_w}{d_u(d_u - f_u)} \) if \((d_u, d_v) \in R_3^1 \).

Let \( d_v = d_{v,k} \) be defined as in (3.3), (3.4) and (3.5), respectively. Then there is a smooth curve \( \Gamma \) of the steady state solutions of (3.1) bifurcating from \((u_0, v_0)\), \((d_{v,k} + d_v, s)\). In a neighborhood of the bifurcation point, the bifurcating branch \( \Gamma \) can be parameterized as \( \Gamma = \{(u_0(s), v_0(s)), d_{v,k} + d_v(s) : s \in (-\epsilon, \epsilon)\} \), where

\[
\begin{align*}
u_0(s) &= u_0 + \frac{s f_v(d_w \mu_k + 1)}{d_w (d_u - f_u) \mu_k - f_u} \cos \left( \frac{k \pi}{l} x \right) + s \phi(s), \\
v_0(s) &= v_0 + s \cos \left( \frac{k \pi}{l} x \right) + s \psi(s),
\end{align*}
\]

and \( d_v(s) : (-\epsilon, \epsilon) \to \mathbb{R} \), \( \phi(s), \psi(s) : (-\epsilon, \epsilon) \to \mathbb{Z} \) are \( C^1 \) functions, such that \( d_v(0) = 0, \phi(0) = \psi(0) = 0 \). Here, \( \mathbb{Z} \) is any closed complement of one-dimensional space spanned by \( y_0 \).

Proof. From the properties of \( d_v(p) \) we know that \( 0 \) is a simple eigenvalue of the linearized operator \( L_1 \) when \( d_v = d_{v,k} \). Obviously, the following problem has a nontrivial solution because \( 0 \) is an eigenvalue of the linearized operator \( L_1 \):

\[
\begin{align*}
d_u(J \ast \phi - \phi) + f_u \phi + f_v \psi &= 0, & x \in [-l, l], \\
d_v \psi'' + g_v \phi + g_v \psi &= 0, & x \in (-l, l), \\
\psi(-l) &= \psi(l), & \psi'(l) = \psi'(l).
\end{align*}
\]

Note that \( \varphi_k(x) = \cos(k\pi x/l) \) satisfying \( \varphi_k(-x) = \varphi_k(x) \) is the eigenfunction of \( -d^2/dx^2 \) corresponding the eigenvalue \( \mu_k = (k\pi/l)^2 \) with periodic boundary condition in \((-l, l)\). Note that \( d_w(d_u - f_u) \mu_k - f_u < 0 \) holds. Then direct calculation shows that

\[
y_0 := (\phi_0, \psi_0) = \left( \frac{f_v(d_w \mu_k + 1)}{d_w (d_u - f_u) \mu_k - f_u}, 1 \right) \cos \left( \frac{k \pi}{l} x \right)
\]

is the corresponding eigenvector which belongs to the eigenvalue \( 0 \). Therefore,

\[
\mathcal{N}(L_1) = \text{span} \left\{ \left( \frac{f_v(d_w \mu_k + 1)}{d_w (d_u - f_u) \mu_k - f_u}, 1 \right) \cos \left( \frac{k \pi}{l} x \right) \right\}.
\]

This means \( \dim\mathcal{N}(L_1) = 1 \) when \( d_v = d_{v,k} \).

In order to find the range of \( L_1 \), denoted by \( \mathcal{R}(L_1) \), we consider the following non-homogeneous problem:

\[
\begin{align*}
d_u(J \ast \phi - \phi) + f_u \phi + f_v \psi &= \sigma, & x \in [-l, l], \\
d_v \psi'' + g_v \phi + g_v \psi &= \tau, & x \in (-l, l), \\
\psi(-l) &= \psi(l), & \psi'(l) = \psi'(l).
\end{align*}
\]

If \( \mu_k \) satisfies (i) and (ii), then \( -f_u/d_u \notin \sigma(\mathcal{K}) \). From (3.9a), we have

\[
\phi = (d_u \mathcal{K} + f_u I)^{-1}(\sigma - f_v \psi).
\]

Substituting (3.10) into (3.9b), we have

\[
d_v(f_u - d_u) \psi'' + d_u d_v J \ast \psi'' + d_u g_v J \ast \psi + (f_u g_v - f_v g_u - d_w g_v) \psi = d_u (J \ast \tau - \tau) + f_u \tau - g_v \sigma.
\]
Since \( d_v(f_u - d_w)\psi_0'' + d_v d_u J * \psi_0'' + d_u g_v J * \psi_0 + (f_u g_v - f_v g_u - d_u g_v)\psi_0 = 0 \), the equation (3.11) has a solution if and only if
\[
\int_{-l}^{l} (d_u (J * \tau - \tau) + f_u \tau - g_u \sigma)\psi_0 dx = 0. \tag{3.12}
\]

Let us expand \( \sigma = \sum_{n=0}^{\infty} \sigma_n \cos(n\pi x/l) \) and \( \tau = \sum_{n=0}^{\infty} \tau_n \cos(n\pi x/l) \). Since \( \psi_0 = \cos(k\pi x/l) \), (3.12) is equivalent to the following condition
\[
\left( f_u - \frac{d_u d_w \mu_k}{d_u \mu_k + 1} \right) \tau_k - g_u \sigma_k = 0. \tag{3.13}
\]

Hence, \( \text{codim} \mathcal{R}(L_1) = 1 \).

Calculating \( \partial_{d_u} \partial_{(u,w)} G_1((u_0, v_0), d_v, k) \), we have
\[
\partial_{d_u} \partial_{(u,w)} G_1((u_0, v_0), d_v, k) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}.
\]

Then \( \partial_{d_u} \partial_{(u,w)} G_1((u_0, v_0), d_v, k) y_0 = (0, \psi_0'') \), where \( y_0 \) is defined as in (3.7). Note that
\[
\psi_0'' = -\mu_k \cos \left( \frac{k\pi}{l} x \right).
\]

Since \( \frac{d_u d_w \mu_k}{d_u \mu_k + 1} - f_u < 0 \) from the definition of \( d_v^k \), whence
\[
\int_{-l}^{l} (d_u (J * \psi_0'' - \psi_0'') + f_u \psi_0'')\psi_0 dx = \left( \frac{d_u d_w \mu_k}{d_u \mu_k + 1} - f_u \right) \mu_k \int_{-l}^{l} \cos^2 \left( \frac{k\pi}{l} x \right) dx < 0.
\]

This means \( \partial_{d_u} \partial_{(u,w)} G_1((u_0, v_0), d_v, k) y_0 \not\in \mathcal{R}(\partial_{d_u} \partial_{(u,w)} G_1((u_0, v_0), d_v, k)) \).

Thus we complete the proof and obtain the results about the existence of a one-parameter family of non-constant solutions bifurcated from \((u_0, v_0), d_v, k)\) from Theorem 1.7 of [11].

From Theorem 2.2 the spatial pattern formation for model (1.6) under the schemes (a) and (b) is possible for the dispersal parameter pair \((d_u, d_w)\) in the instability region \(R_1\). From Theorem 3.1, if \((d_u, d_w) \in R_1\), then \(d_v = d_v, k\), which are defined as in (3.3), (3.4) and (3.5), are all bifurcation points, where nonconstant positive steady states of (3.1) bifurcate from \((u_0, v_0)\). Furthermore, the unstable constant steady state \((u_0, v_0)\) is in the mode \(-k\) (corresponding to \(\cos(k\pi x/l)\)) when the spatial scale \(l\) is chosen to satisfy
\[
l \in \begin{cases} 
0, \frac{k\pi}{\sqrt{p_0}}, & \text{if } d_u < f_u, \\
0, \frac{k\pi}{\sqrt{p_+}}, & \text{if } d_u = f_u, \\
\frac{k\pi}{\sqrt{p_-}}, \frac{k\pi}{\sqrt{p_+}}, & \text{if } d_u > f_u,
\end{cases}
\]

where \(p_0, p_+, p_-, p_+\) are defined in (2.15), (2.16) and (2.17).
Application and numerical simulation. In this section, we will apply the obtained theoretical results to a nonlocal Klausmeier-Gray-Scott model which is based on Klausmeier model in [24] and Gray-Scott model (see [17, 18, 19, 22, 35]). The Klausmeier-Gray-Scott model is as follows:

\[
\begin{aligned}
  u_t &= d_u u_{xx} + u^2 v - Bu, \\
  v_t &= d_v v_{xx} + A - v - u^2 v, \\
\end{aligned}
\]  
(4.1)

where \( u(x,t) \) and \( v(x,t) \) are water and plant biomass, respectively; \( A \) controls water input; \( B \) measures plant losses; \( d_u \) and \( d_v \) are diffusion rates of plant and water respectively. To account for the possibility of long-range dispersal of seeds, Sherratt imposed a nonlocal dispersal on plant in model (4.1) [12]. With periodic boundary condition and the kernel function which is defined in (1.5), the model in [12] is modified as follows:

\[
\begin{aligned}
  u_t &= d_u(J * u - u) + u^2 v - Bu, & x \in [-l,l], t > 0, \\
  v_t &= d_v v_{xx} + A - v - u^2 v, & x \in (-l,l), t > 0, \\
  v(-l,t) &= v(l,t), v_x(-l,t) = v_x(l,t), & t > 0, \\
  u(x,0) &= u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in [-l,l].
\end{aligned}
\]  
(4.2)

In this model, the parameter \( d_w \) in the kernel function \( J(x) \) describes the disperse spread of seeds away from the “mother” plant position. Clearly, model (4.2) has a trivial steady state \((u,v) = (0, A)\), which is always stable for all parameters. If \( A \geq 2B \), model (4.2) admits two positive constant steady states \((u_\pm, v_\pm)\) with

\[
  u_\pm = \frac{A \pm \sqrt{A^2 - 4B^2}}{2B}, \quad v_\pm = \frac{A + \sqrt{A^2 - 4B^2}}{2}.
\]  
(4.3)

The Jacobian matrix of the corresponding kinetic system at a positive equilibrium \((u,v)\) is

\[
J = \begin{pmatrix}
  B & u^2 \\
  -2B & -1 - u^2
\end{pmatrix}.
\]  
(4.4)

Then the corresponding characteristic equation is \( \lambda^2 - T_0 \lambda + D_0 = 0 \), with \( T_0 := \text{Trace}(J) = B - (1 + u^2) \), \( D_0 := \text{Det}(J) = B(u^2 - 1) \). It is easy to obtain that \((u_-, v_-)\) is always unstable whenever it exists and \((u_+, v_+)\) is stable with respect to the corresponding kinetic model if \( 0 < B < 2 \).

Assume \( A \geq 2B \) and \( 0 < B < 2 \). From Theorem 2.2 we give the following results about the linear stability of \((u_+, v_+)\) and the Turing instability of model (4.2).

**Theorem 4.1.** Assume \( d_u, d_v, d_w, A, B \) are all positive, and \( A \geq 2B \), \( 0 < B < 2 \). Then we have the following statements.

(i): If \( d_u < f_u \), then \((u_+, v_+)\) is an unstable steady state with respect to system (4.2).

(ii): If \( d_u = f_u \), then \((u_+, v_+)\) is a stable steady state when \( d_v \leq d_v^* \), but an unstable steady state when \( d_v > d_v^* \) with respect to system (4.2). Here,

\[
d_v^* = 2d_u u_+^2.
\]

(iii): If \( d_u > f_u \) then \((u_+, v_+)\) is a stable steady state when \( d_v < d_v^{**} \), but an unstable steady state when \( d_v > d_v^{**} \) with respect to system (4.2). Here,

\[
d_v^{**} = \frac{d_w[(d_u - B)(u_+^2 - 1) + 2d_u u_+^2 + 2d_u^2 (d_u - B)(u_+^2 - 1)]}{B}.
\]
Setting $A = 1, B = 0.45, d_w = 1$, we give a diagram to show the results of Theorem 4.1, see Figure 2 (a). In this figure, the spatial pattern formation for model (4.2) is possible if $(d_u, d_v) \in R_1$. Here, $R_1$ is between the polygonal line (composed of the vertical cyan line $d_u = 0.45$ and the cyan curve $d_v = 14.7472d_u - 1.5454 + 8.8149\sqrt{d_u(d_u - 0.45)}$), the $d_v$-axis and the $d_u$-axis. We also show the effect of the nonlocal dispersal and the dispersal spread on the parameter region of Turing instability in Figure 2 (b). In this figure, the dotted blue line is $d_v = 23.5621d_u$ which serves as the separatrix of Turing instability of model (4.1). On the right of the vertical line $d_u = 0.45$, the cyan polygonal line is $d_v = 14.7472d_u - 1.5454 + 8.8149\sqrt{d_u(d_u - 0.45)}$ and the green polygonal line is $d_v = 3(14.7472d_u - 1.5454 + 8.8149\sqrt{d_u(d_u - 0.45)})$, which are the separatrices of Turing instability region corresponding to $d_w = 1$ and $d_w = 3$, respectively. Compared with existing results for the original reaction-diffusion model (4.1), the following changes can be found for the Turing instability region of model (4.2) because of the effect of nonlocal dispersal:

1. Turing instability can also occur when $d_v < 23.5621d_u$ if $d_u < 0.45$;
2. The region for Turing instability does not change when $d_u < 0.45$ whatever $d_w$ is;
3. Whether the region for Turing instability becomes smaller or larger is subject to $d_w$ when $d_u \geq 0.45$;
4. The region for Turing instability becomes smaller with the increase of $d_w$ when $d_u \geq 0.45$.

Biologically, these differences show that in the nonlocal model (4.1) pattern formations always exist if the dispersal spread of seeds is sufficiently small ($d_u$ is small), while a wider dispersal spread of seeds ($d_u$ is large) and an increase in dispersal rate ($d_u$ is large) are helpful for the formation of the vegetation state. Furthermore, the long-range dispersal of seeds predicts a stronger tendency and a larger parameter region for the formation of patterns if the dispersal spread of seeds is small ($d_w$ is small).

If $(d_u, d_v) \in R_1$, we can study the steady state bifurcation of model (4.2). The steady states of (4.2) satisfy...
where (d), (e) and (f). We also remind the reader (b), (c) and (d)). As the plant dispersal rate increases, two bumps of plant with same maximum height emerge (see Figure 4 (e) and (f)). We also remind the reader (1) and (2) of (4.5) as follows.

\[
\begin{align*}
&d_u(J * u - u) + u^2v - Bu = 0, \quad x \in [-l, l], \\
&d_v v_{xx} + A - v - u^2v = 0, \quad x \in (-l, l), \\
&v(-l) = v(l), v_x(-l) = v_x(l).
\end{align*}
\]

Choosing \( A = 1, B = 0.45, d_w = 1, l = 20 \), we get steady state bifurcation points of model (4.5) as follows.

(1) For the case \( d_u < f_u \), we set \( d_u = 0.2 \). There are infinite steady state bifurcation points satisfying \( d_w,j > d_w,k \) with \( j < k \) (see Figure 3 (a)):

\[
d_{w,1} = 64.9286 > d_{w,2} = 17.8192 > d_{w,3} = 8.9806 > d_{w,4} = 5.7745 > d_{w,5} = 4.1934 > d_{w,6} = 3.2571 > \cdots.
\]

(2) For the case \( d_u = f_u \), we set \( d_u = 0.45 \). There are also infinite steady state bifurcation points satisfying \( d_w,j > d_w,k \) with \( j < k \) (see Figure 3 (b)):

\[
d_{w,1} = 67.7874 > d_{w,2} = 20.7650 > d_{w,3} = 12.0571 > d_{w,4} = 9.0093 > d_{w,5} = 7.5987 > d_{w,6} = 6.8324 > \cdots.
\]

(3) For the case \( d_u > f_u \), we set \( d_u = 1 \). There are finite steady state bifurcation points (see Figure 3 (c)):

\[
d_{w,1} = 74.3609 > d_{w,5} = 48.3396 > d_{w,2} = 28.5371 > d_{w,4} = 25.7592 > d_{w,3} = 22.4877.
\]

With the increase of \( d_w \), the steady state bifurcation occurs at much larger values of \( d_u \) if \( d_u < f_u \), and there exist less number of bifurcation points if \( d_u > f_u \).

According to Theorem 3.1, under certain conditions there exist steady state solutions of (1.6) bifurcating from steady state bifurcation points \((u_+, v_+, d_{w,k})\), where \( d_{w,k} \) are the values as above. The initial functions are fixed as follows

\[
u_0(x) = u_+ + 0.1 \sin x, \quad v_0(x) = v_+ + 0.1 \cos x, \quad x \in [-20, 20].
\]

By the finite difference method, we show the pattern formations for three cases: (1) \( d_u < f_u \) (see Figure 4 (a) and (b)); (2) \( d_u = f_u \) (see Figure 4 (c) and (d)); (3) \( d_u > f_u \) (see Figure 4 (e) and (f)). Here, we choose \( A = 1, B = 0.45, d_w = 30, d_u = 1 \) and \( d_u = 0.2 \) in Figure 4 (a) and (b), \( d_u = 0.45 \) in Figure 4 (c) and (d), \( d_u = 1 \) in Figure 4 (e) and (f), respectively. Simulations show that there exist spiky solutions when the plant dispersal is comparatively small, and multiple spikes will not be symmetric and in general have different amplitudes (see Figure 4 (a), (b), (c) and (d)). As the plant dispersal rate increases, two bumps of plant with same maximum height emerge (see Figure 4 (e) and (f)). We also remind the reader
that in water-biomass models an “antiphase” pattern means that the water content in the patch with denser biomass decreases, and a “inphase” pattern means that the water content in the patch with denser biomass increases, see for example [23]. Then the plant and the water distributions are antiphase in Figure 4 (b) and (d), while the plant and the water distributions are inphase in Figure 4 (f). In a classical activator-inhibitor system, the pattern such as in Figure 4(f) can also be obtained, which is called a plateau-like pattern [16].

5. **Conclusions.** In this paper, we have studied the effect of the nonlocal dispersal and the dispersal spread on the Turing bifurcation and the steady state bifurcation of a general activator-inhibitor system with a nonlocal activator dispersal. In particular, our results have shown the existence of Turing patterns and inhomogeneous periodic oscillatory patterns. We have found that a larger dispersal spread and an increase of activator diffusion inhibit the formation of spatially inhomogeneous patterns. Compared with the “spatial averaging” nonlocal dispersal model, our model
admits a larger parameter region supporting pattern formations, which is also true when compared with the original reaction-diffusion model if the dispersal spread is small. By numerical simulation, we have shown that the activator and inhibitor distributions could be inphase or antiphase. Therefore, the activator-inhibitor system with a more general nonlocal activator dispersal gives rise to a much more complex dynamics than the original activator-inhibitor system and the “spatial averaging” nonlocal dispersal one. These results have also exhibited the critical role of the nonlocal dispersal and the dispersal spread in leading to the formation of spatiotemporal patterns.

In our model, we suppose that the activator has a nonlocal dispersal. If the inhibitor has a nonlocal dispersal, then the sign patterns of the Jacobian matrix are shown with the following two possibilities:

\[
(c) \begin{pmatrix} - & - \\ + & + \end{pmatrix}, \quad (d) \begin{pmatrix} - & + \\ - & + \end{pmatrix}.
\]

In these cases, similar results can be obtained for the Turing bifurcation and the steady state bifurcation. For the model in which the dispersal of the inhibitor is nonlocal the spatial pattern formation under the schemes (c) and (d) is possible for the dispersal parameter pair \((d_u, d_v)\) is in the instability region (see Figure 5 (c))

\[
R_2 = \{ (d_u, d_v) : 0 < d_v < \tilde{d}_v, d_u > 0 \},
\]

where

\[
\tilde{d}_v := \frac{d_w[(d_u - f_u)(f_v g_v - f_v g_u) - f_v g_u d_u - 2\sqrt{-d_u f_v g_u (d_u - f_u)(f_u g_v - f_v g_u)}}{f_u^2}.
\]

Moreover, nonconstant positive steady states of (1.6) bifurcate from the constant steady state \((u_0, v_0, \tilde{d}_v,k)\), where

\[
\tilde{d}_{v,k} := \frac{[d_w g_v (d_u - f_u) + d_w f_v g_u] \mu_k - (f_u g_v - f_v g_u)}{d_v (d_u - f_u) \mu_k^2 - f_u \mu_k}.
\]

Biologically, if the dispersal of the activator is diffusive and the one for the inhibitor is nonlocal, then a larger dispersal spread and an increase of inhibitor diffusion induce the formation of spatially inhomogeneous patterns.
For the local reaction-diffusion model, the Turing instability region where the spatial scale could induce unstable modes is between the line 
\[ d_v = \tilde{k}_0 d_u \] 
and the \( d_u \)-axis where 
\[ \tilde{k}_0 = \frac{f_u g_v - 2 f_v g_u - 2 \sqrt{-f_v g_u (f_u g_v - f_u g_u)}}{f_u^2}. \] 
(see Figure 5 (a)). Compared with the original reaction-diffusion model, the model in which the dispersal of the inhibitor is nonlocal and the activator has a diffusive dispersal always predicts a larger parameter region supporting pattern formation (see Figure (5) (a) and (c)). We also exhibit the Turing instability region of the “spatial averaging” nonlocal dispersal model when the inhibitor has a nonlocal dispersal in Figure 5 (b), where a small nonlocal inhibitor dispersal rate inhibits the formation of patterns [6]. Compared with the “spatial averaging” nonlocal dispersal model when the inhibitor has a nonlocal dispersal, a small nonlocal inhibitor dispersal rate can also induce the formation of patterns in the model where the dispersal of the inhibitor is nonlocal and the activator has a diffusive dispersal if the activator dispersal rate is properly small.

In our model, we impose the periodic boundary on the activator-inhibitor system and consider a special kernel function \( J(x) \) in form of (1.5). Because this kernel function is just the Green kernel of problem (2.4), we can obtain the spectral set of the convolution operator 
\[ J \ast \phi \] 
This may give an insight on the study of the following general activator-inhibitor system in which both the activator and the inhibitor have a nonlocal dispersal:

\[
\begin{align*}
  u_t &= d_u (J_1 \ast u - u) + f(u, v), & x \in [-l, l], \ t > 0, \\
  v_t &= d_v (J_2 \ast v - v) + g(u, v), & x \in [-l, l], \ t > 0, \\
  u(x, 0) &= u_0(x) \geq 0, & v(x, 0) = v_0(x) \geq 0, & x \in [-l, l],
\end{align*}
\]  
(5.2)

where \( J_1, J_2 \) have the form (1.5) with the corresponding dispersal spread \( d_w(1) \) and \( d_w(2) \). Because the introduction of nonlocal diffusion term leads to the lack of smoothing, asymptotic compactness of the semiflow generated by model (5.2) does not hold. Thus it is difficult to analyze the global dynamics of these nonlocal models. Thanks to [26, 37, 38], by Lyapunov functional and LaSalle’s invariance principle, some results about the existence, uniqueness and stability of steady states of nonlocal SIS epidemic models have been obtained if the dispersal spread is held fixed, i.e. \( d_w(1) = d_w(2) \). But it is unclear whether or not the spatially inhomogeneous patterns (via a Turing instability) will have a different result. We conjecture that the same results for Turing instability as for the reaction-diffusion model holds. The results may be much more complicated if the rate of dispersal is fixed while the dispersal spreads are different or if both the dispersal rates and the dispersal spreads are varied. We will consider these questions in the future investigations.

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