The gauge invariance of the non-Abelian Chern-Simons action for D-branes revisited

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ABSTRACT

We present an elegant method to prove the invariance of the Chern-Simons part of the non-Abelian action for $N$ coinciding D-branes under the R-R and NS-NS gauge transformations, by carefully defining what is meant by a background gauge transformation in the non-Abelian world volume action. We study as well the invariance under massive gauge transformations of the massive Type IIA supergravity and show that no massive dielectric couplings are necessary to achieve this invariance. We show that this result is consistent with (massive) T-duality from the non-Abelian action for $N$ D9-branes.
1 Introduction

It is well known by now that the physics of a set of $N$ coincident $Dp$-branes can be very different from the physics of $N$ parallel but separated $Dp$-branes. The latter is described by a $(p + 1)$-dimensional Abelian world volume action, with bosonic field content $N U(1)$ vector fields $V_a^I$, with $a$ the world volume index, and $N$ times $(9 - p)$ scalars $X^{Ii}$, that can be arranged in $(9 - p)$ $N \times N$ diagonal matrices. The $U(1)$ vectors are of course the Born-Infeld vectors living on each brane, while a scalar $X^{Ii}$ represents the position of the $I$-th brane in the transversal direction $x^i$.

As the separation between the different D-branes decreases, the open strings stretched between two distinct branes grow shorter and lighter, so that in the limit where all D-branes coincide new massless states are generated, and new physics appears.

Witten showed \[1\] that these new massless states are arranged such that the $U(1)^N$ gauge symmetry of the system of D-branes is enhanced to a full non-Abelian $U(N)$ gauge symmetry. The $N$ Born Infeld vectors form a single $U(N)$ Yang-Mills vector $V_a$ and the transverse scalars, arranged in $N \times N$ matrices $X^i$, become non-Abelian matrices transforming in the adjoint representation of $U(N)$. The $I$-th eigenvalue of the matrix $X^i$ has still the interpretation of the position of the $I$-th D-brane in the direction $x^i$, but since in the $U(N)$ case not all matrices are simultaneously diagonalisable, the branes are no longer fully localisable in all transverse directions. Therefore, the geometry of the transverse space described in terms of the matrix-valued coordinates $X^i$ becomes that of a “fuzzy surface”.

The new physics associated to these extra massless string states has to be encoded in the world volume effective action describing the system of coincident branes. This action should now be written in terms of the matrix valued fields $V_a$ and $X^i$. Determining the exact form of the Born-Infeld action is a highly non-trivial problem, to which the solution is still not clear (see for instance \[2\]). A lot of progress has been made however over the last few years in the understanding of the structure of the non-Abelian Chern-Simons (or Wess-Zumino) action.

The first generalisation of the Chern-Simons term to the $U(N)$ case was proposed in \[3\]:

$$S_{Dp} = T_p \int P[C] \text{ Tr} \{e^F\} = T_p \int \sum_n P[C_{p-2n+1}] \text{ Tr} \{F^n\}. \quad (1.1)$$

Here the trace is taken over the Yang-Mills indices of the $N$-dimensional representation of $U(N)$ and $P[\Omega]$ denotes the pullback of the background field $\Omega$ to the world volume of the D-brane. The world volume field $F$ is given by $F = F + P[B]$, where $F_{ab} = 2\partial_{[\mu} V_{b\nu]} + i[V_a, V_b]$ is the non-Abelian field strength of the Born-Infeld vector and $B$ the NS-NS two-form.

The invariance of this action under the gauge transformations of the background NS-NS and R-R fields was further investigated in \[4\], where it was shown as well that in order to be invariant under the massive gauge transformations of massive Type IIA supergravity \[5, 6\], extra $m$-dependent terms were needed in the action. These extra terms were also obtained from the (massive) T-duality relations \[6, 7\] between the different D-brane actions, generalising to the non-Abelian case the Abelian calculation of \[7\].

Nowadays we know, however, that the Chern-Simons action for coincident D-branes presented in \[4\] is not the complete story. On the one hand, in the non-Abelian case the background fields in \[11\] must be functionals of the matrix-valued coordinates $X^i$ \[8\]. Explicit calculations of string scattering amplitudes \[9\] suggest that this dependence is given by a non-Abelian Taylor expansion

$$C_{\mu\nu}(x^a, X^i) = \sum_n \frac{1}{n!} \partial_{k_1} \ldots \partial_{k_n} C_{\mu\nu}(x^a, x^i)|_{x^i=0} X^{k_1} \ldots X^{k_n}. \quad (1.2)$$

On the other hand, in order to have invariance under $U(N)$ gauge transformations the pullbacks of the background fields into the world volume have to be defined in terms of $U(N)$ covariant
derivatives $D_a X^\mu = \partial_a X^\mu + i[V_a, X^\mu]$, rather than partial derivatives. For instance,
\[ P[C_2] = C_{\mu\nu} D_\mu X^\mu D_\nu X^\nu. \] (1.3)
This, together with the symmetrised trace prescription of Tseytlin, that we will denote by curly brackets $\{..\}$, assures the invariance of the action under $U(N)$ gauge transformations
\[ \delta V_a = D_a \chi, \quad \delta X^i = i[\chi, X^i]. \] (1.4)
One should note however that the presence of $U(N)$ covariant pullbacks has consequences on the invariance under NS-NS and R-R gauge transformations. Let us look for example at the variation $\delta C_{\mu\nu} = 2\partial_{[\mu} \Lambda_{\nu]}$ of the term given in (1.3). Naively filling in the variation in the pullback yields:
\[ \delta \{ P[C_2] \} = 2 \{ P[\partial \Lambda_1] \} = 2 \{ \partial_\mu \Lambda_\nu D_\mu X^\nu D_\nu X^\nu \}. \] (1.5)
In the Abelian limit this gauge variation is a total derivative, such that the $\Lambda_1$ gauge invariance is assured in the D1-brane Chern-Simons action (for world volumes without boundaries). In the non-Abelian case however the variation is not a total derivative
\[ 2 \{ \partial_\mu \Lambda_\nu D_\mu X^\nu D_\nu X^\nu \} \neq 2 \partial_{[\mu} \{ \Lambda_\nu D_\nu X^\nu + ... \} \]
and not even D1-branes with topologically trivial world volumes are described by a gauge invariant action. We need to think more carefully how a background field gauge transformation should be defined in the non-Abelian action.

The most important modification to the action, with interesting physical implications, was found by Myers and Taylor and Van Raamsdonk, when new dielectric couplings to higher order background field potentials were shown to arise as a consequence of T-duality in non-Abelian actions. Myers found that the full T-duality invariant form of the Chern-Simons action is given by:
\[ S_{Dp} = T_p \int \left\{ P[\epsilon^{iX}_{\mu_1..\mu_n} (C_{\mu_0..\mu_n})] \epsilon F \right\}, \] (1.6)
where $(iX)_{\mu_1..\mu_n}$ denotes the interior product $X^{\mu_0} C_{\mu_1..\mu_n}$. These terms induce dipole and higher moment couplings to R-R fields with rank higher than $p+1$, which give rise, in particular, to the many non-Abelian solutions of D$p$-branes expanding into fuzzy surfaces that have been constructed in the literature (see for instance ).

Although these new terms are indeed necessary for the consistency of the T-duality transformations, they make the issue of the gauge invariance of the non-Abelian action even more unclear, because a pullback of a variational parameter of the form $\{ P[(iX)_{\mu_1..\mu_n} \partial \Lambda_{n+1}] \}$ is by no means a total derivative.

From all this it is clear that the gauge invariance of the non-Abelian action for D-branes requires further study. Nevertheless, there has been remarkably few concern about this issue in the literature (see however ). The first serious but quite involved attempt to show that the Chern-Simons action can be written in a form which is invariant under $\delta C_p = p \partial \Lambda_{p-1}$ was made in . By expanding the R-R background fields in their non-Abelian Taylor expansion, and integrating each term by parts, it was proven that the Chern-Simons action could be written as an infinite series of terms involving only the R-R field strengths. It is remarkable however that the resulting series cannot be interpreted as a Taylor series, or as a pullback of
In spite of these non-trivial results, \[24\] can still not be the complete story, because it hardly looks at the NS-NS gauge transformations

\[\delta B_{\mu\nu} = 2\partial[\mu \Sigma_{\nu}], \quad (1.7)\]

neither does it address the massive gauge transformations of Romans’ theory (which originally concerned the authors of \[7\, 4\]).

The aim of this letter is twofold. First, by carefully defining what is meant by a background gauge transformation in the non-Abelian world volume theory, we will present an elegant method to prove the invariance of the world volume effective action under the R-R and NS-NS gauge transformations. Secondly, we will consider the most general case of a massive Type IIA supergravity background, which will require dealing as well with the massive gauge transformations of Romans’ theory \[7\, 4\] for the dielectric case. In this respect it is a priori not clear whether and how the action \[1.6\] will have to be modified to the case of coincident D-branes in backgrounds of massive Type IIA supergravity. The original derivation \[20\] of the action \[1.6\] started with \(N\) coincident D9-branes and generated the action for coinciding lower-dimensional D-branes through a chain of (massless) T-duality transformations, with more and more non-Abelian couplings being generated as the number of transverse dimensions grew bigger. Since the D8-branes live in massive IIA supergravity the appropriate setting would be to use the massive T-duality rules between Romans’ theory and Type IIB supergravity \[6\, 7\], at least in the T-duality between the D9- and the D8-branes and between the D8- and the D7-branes.

We will generalise the action \[1.6\] to include D-branes living in massive Type IIA in two ways. First of all we will show that certain (mass dependent) world volume couplings have to be added to the action to achieve invariance under massive gauge transformations. Remarkably, we will find that no massive dielectric couplings are necessary and that the only massive world volume couplings are the ones already given in \[4\]. Secondly we will start from the action for \(N\) D9-branes and rederive the equivalent to \[1.6\] in a massive background using massive T-duality. We will show that the extra massive terms previously added by demanding gauge invariance are precisely those generated by the massive T-duality transformations, and that no massive dielectric terms appear by this procedure either.

The organisation of this letter is as follows. In section 2 we discuss the gauge invariance of the “pre-dielectric” action \[1.6\], taking into account the contribution of the \(U(N)\) covariant pullbacks and the fact that the background fields are functionals of the non-Abelian scalars \(X^\mu\). This will clarify how to define background gauge transformations in the non-Abelian world volume theory. We show as well how this action has to be modified to achieve invariance under massive gauge transformations. In section 3 we generalise these results to the full non-Abelian action including dielectric couplings and in section 4 we show that these results agree with the ones obtained applying massive T-duality. Finally we summarise our conclusions in section 5. Our conventions for the gauge transformations of the background fields and the Abelian actions can be found in appendix A.

## 2 Gauge invariance of the “pre-dielectric” action

As mentioned in the introduction, in the non-Abelian case the gauge variation of the pullback of a background field can no longer be defined as the pullback of the gauge variation, simply because due to the covariant derivatives used in the pullback, a naive pullback of the variational parameter can not be written as a total derivative, and hence does not lead to a gauge invariant action. Moreover, adding more terms to the action via a Noether procedure in order to obtain a total derivative gauge variation would violate \(U(N)\) covariance.

In order to have an action invariant under the background gauge transformations, we need to fulfil two conditions. First, it must be possible to write the variation as a total derivative, and second, the variation has to be a scalar under \(U(N)\) gauge transformations. We will propose a definition for a background gauge transformation in the non-Abelian world volume theory that satisfies these conditions, and relates to the usual gauge transformations in the Abelian case.
We define the variation of the pullback of a R-R field $C_p$ under the background gauge transformation $\delta C_p = p\partial\Lambda_{p-1}$ as:

$$\delta[P[C_p]]_\Omega \equiv p\, DP[\Lambda_{p-1}]_\Omega = p\, D[a_1(\Lambda_{\mu_2...\mu_p} D_{a_2} X^{\mu_2}...D_{a_p} X^{\mu_p})]\Omega,$$

where $\Omega$ is any combination of world volume or pullbacked background fields and where it is understood that all $U(N)$-valued objects appear symmetrisés (though not in a trace). In particular for the simplest case with $\Omega = 1$ we find that

$$\delta[P[C_p]] = p\, D[a_1(\Lambda_{\mu_2...\mu_p} D_{a_2} X^{\mu_2}...D_{a_p} X^{\mu_p})]$$

$$= p\, \partial_{\nu} \Lambda_{\mu_1...\mu_{p-1}} D[b] X^{\nu} D[a_1] X^{\mu_1}...D_{a_{p-1}} X^{\mu_{p-1}}$$

$$+ p(p-1)\Lambda_{\mu_1...\mu_{p-1}} D[b] D[a_1] X^{\mu_1}...D_{a_{p-1}} X^{\mu_{p-1}}$$

$$= p\, P[\partial\Lambda_{p-1}] + \frac{1}{2} p(p-1)\Lambda_{\mu_1...\mu_{p-1}} [F|[a_1] X^{\mu_1}] D[a_2] X^{\mu_2}...D_{a_{p-1}} X^{\mu_{p-1}}.$$

With this definition we see that the variation is not just the pullback of the gauge parameter, but contains as well a non-Abelian correction term proportional to $[F, X]$, since the covariant derivative $D[a_1]$ not only acts on the background gauge parameter $\Lambda_{p-1}$, but also on the covariant derivatives in the pullback. For the Abelian case, the correction term disappears and we recover the well-known gauge transformation for Abelian D-brane actions. Furthermore once we consider terms in the action and trace over all $U(N)$ indices in the symmetrisated trace prescription the variation is in fact a total derivative:

$$\delta\{P[C_p]\} = p\, \{DP[\Lambda_{p-1}]\} = p\partial\{P[\Lambda_{p-1}]\}. \quad (2.2)$$

Analogously, it is straightforward to derive from (2.1) the variation of terms of the following forms:

$$\delta\{P[C_p]\} = \left\{ \sum_{n=0}^{p-1} \frac{p!}{2^n n!(p-2n-1)!} DP[\Lambda_{p-2n-1}] P[B^n]\right\}. \quad (2.3)$$

$$\delta\{P[C_p] F^k\} = \left\{ \sum_{n=0}^{p-1} \frac{p!}{2^n n!(p-2n-1)!} DP[\Lambda_{p-2n-1}] P[B^n] F^k\right\}. \quad (2.4)$$

$$\delta\{P[C_p] B^k\} = \left\{ \sum_{n=0}^{p-1} \frac{p!}{2^n n!(p-2n-1)!} DP[\Lambda_{p-2n-1}] P[B^{n+k}]\right\}. \quad (2.5)$$

Similarly the non-Abelian generalisation of the NS-NS gauge transformation (1.7) is defined as:

$$\delta P[B] = 2DP[\Sigma]. \quad (2.6)$$

The Born-Infeld field should then transform as well as $\delta V = -P[\Sigma]$, which is the non-Abelian generalisation of the Abelian transformation $\delta V_a = -\Sigma_a \partial_a X^\mu$. It is clear that in this way the non-Abelian field strength $F = F + P[B]$ is indeed invariant.

With these definitions, the computation of the gauge transformations of the action

$$\mathcal{L}_{Dp} = (-)^{[(p+1)/2]} \left\{ P \left[ \frac{1}{(p+1)!} C_{p+1} - \frac{1}{2(p-1)!} C_{p-1} F^2 + \frac{1}{8(p-3)!} C_{p-3} F^4 - \frac{1}{48(p-5)!} C_{p-5} F^6 + \frac{1}{384(p-7)!} C_{p-7} F^8 - \frac{1}{3840(p-9)!} C_{p-9} F^{10} \right] + (-)^{(p+2)/2} [p+1]!! \lambda_{p+1} \right\}, \quad (2.7)$$

is straightforward, since it formally reduces to the Abelian case. Note that an extra Chern-Simons-like term

$$\omega_{2n+1} = \sum_{k=0}^{n} \frac{1}{2^n n!(n+k+1)!} \frac{(n+1)!}{k!(n-k)!} V(\partial V)^{n-k} [V, V]^k, \quad (2.8)$$

is.
has to be added to the action of the even D-branes \[4\], in order to assure the invariance under the massive gauge transformations \([A.1]\) of massive Type IIA supergravity, which in the action act on the pullback of the fields as
\[
\delta\{P|C_{2p+1}\} = -(2p+1)!! \ m \ \{P[\Sigma \ B^p]\}. \tag{2.9}
\]
These Chern-Simons terms are constructed in such a way that they transform under the Yang-Mills gauge transformations as a total derivative, and under the \(\Sigma\) transformations as
\[
\delta\omega_{2n+1} = -\frac{n+1}{2^n} \Sigma F^n, \tag{2.10}
\]
and thus cancel the massive gauge transformation of the R-R background fields. They are the non-Abelian generalisation of the \(V(\partial V)^n\) terms in \([4]\).

So far we have rederived the results of \([4]\) on the gauge invariance of non-Abelian Chern-Simons actions, taking into account explicitly the \(U(N)\) covariant pullbacks and the fact that the background fields are functionals of the non-Abelian coordinates \(X^a\). As we have seen this forces a precise definition for what we mean by gauge variation of a non-Abelian pullback. A consistency check of our definitions \([2.2] - [2.6], [2.9]\) is that the variation of the pullback of a R-R 3-form should be T-dual to the variation of the pullback of a R-R \((p-1)\)-form field. In the next section we will check this and see that in this manner we can find a natural way to also prove the gauge invariance of the dielectric terms.

3 Gauge invariance of the dielectric couplings

In this section we show that the gauge transformation of the pullback of a R-R field \(C_p\), as defined in \([2.1]\), is consistently mapped under T-duality into the gauge transformation of the pullback of the T-dual of \(C_p\). To show this let us define a R-R field \(\tilde{C}_p\), being related to \(C_p\) via a gauge transformation\(^7\)
\[
\tilde{C}_{\mu_1...\mu_p} = C_{\mu_1...\mu_p} + p \ \partial_{[\mu_1} \Lambda_{\mu_2...\mu_p]}.
\]
We then have on the one hand by definition \([2.4]\) that
\[
\{\tilde{C}_{a_1...a_{p-1}\sigma}\} = \{C_{a_1...a_{p-1}\sigma}\} + p \ \partial_{[a_1} \{\Lambda_{a_2...a_{p-1}\sigma]}\}, \tag{3.1}
\]
while on the other hand we know \([20]\) that applying T-duality on \(\tilde{C}_p\) we get (for simplicity we truncate for now to the “diagonal approximation” \(g_{\mu\nu} = B_{\mu\nu} = 0\))
\[
\{\tilde{C}_{\mu_1...\mu_p} D_{[a_1} X^{\mu_1} \ldots D_{a_{p-1}} X^{\mu_{p-1}} D_{\sigma]} X^{\mu_p}\} \rightarrow \{C_{\mu_1...\mu_p} D_{[a_1} X^{\mu_1} \ldots D_{a_{p-1}} X^{\mu_{p-1}} X^{x, \mu_p}\} \tag{3.2}
\]
\[
\rightarrow \{C_{\mu_1...\mu_{p-1}} D_{[a_1} X^{\mu_1} \ldots D_{a_{p-1}} X^{\mu_{p-1}} + i \tilde{C}_{\mu_1...\mu_p} D_{[a_1} X^{\mu_1} \ldots D_{a_{p-1}} X^{\mu_{p-1}}|X^{x, \mu_p}\} = \{C_{\mu_1...\mu_p} D_{[a_1} X^{\mu_1} \ldots D_{a_{p-1}} X^{\mu_{p-1}} + i C_{\mu_1...\mu_p} D_{[a_1} X^{\mu_1} \ldots D_{a_{p-1}} X^{\mu_{p-1}}|X^{x, \mu_p}\} \tag{3.3.}
\]
\[
+ \partial_{[a_1} \{(p-1) \Lambda_{\mu_2...\mu_p} D_{a_2} X^{\mu_2} \ldots D_{a_{p-1}} X^{\mu_{p-1}} + (p+1) \ i \ \Lambda_{\mu_2...\mu_p} D_{a_2} X^{\mu_2} \ldots D_{a_{p-1}} X^{\mu_{p-1}}|X^{x, \mu_p}\}.
\]
where we used that \(\tilde{C}_{p-1}\) and \(\tilde{C}_{p+1}\) are related to, respectively, \(C_{p-1}\) and \(C_{p+1}\) by the same type of background gauge transformation that relates \(\tilde{C}_p\) to \(C_p\). We then find that the pullback of the gauge parameter transforms under T-duality as
\[
p \ \{D_{[a_1} \Lambda_{a_2...a_{p-1}\sigma]}\} \rightarrow \{(p-1) \ D_{[a_1} \Lambda_{a_2...a_{p-1}\sigma]} + (p+1) \ i \ D_{[a_1} \Lambda_{a_2...a_{p-1}\sigma]} \mu_p x|X^{x, \mu_p}\}. \tag{3.3}
\]
In other words, the variation of the pullback of a R-R \(p\)-form potential goes under T-duality to the variation of the pullback of a R-R \((p-1)\)-form potential plus the variation of the pullback of the first dielectric coupling term:
\[
\delta\{P|C_p]\} \rightarrow \delta\{P|C_{p-1}\} + i \ \delta\{P[|iX|X]C_{p+1}\}, \tag{3.4}
\]
\(^7\)In this section a hatted index indicates that it runs from 0 to 9, including the T-duality direction, which is denoted by \(x\). Unhatted indices exclude the T-duality direction. Similarly in the world volume indices we denote the T-duality direction by \(\sigma\) and the other directions by \(a\).
if we define:
\[
\delta \{ (i X_i X) C_{p+1} \} = \left( p + 1 \right) \partial \{ (i X_i X) \Lambda_p \}
\]
\[
= \left( p + 1 \right) \{ (i X_i X) \partial \Lambda \} + \left( p + 2 \right) \Lambda_{\rho \Sigma} [F_{a_1 a_2}, X^\rho] [X^\Sigma, X^\mu] + 2 \Lambda_{\rho \Sigma} [D_{a_1} X^\rho, X^\mu] \}.
\]

The derivation with the full T-duality rules (beyond the diagonal approximation) is straightforward and not very enlightening, so we rather concentrate on the generalisation of the variation for dielectric couplings, which can be derived in a similar way. Under general R-R gauge transformations, the dielectric terms vary as
\[
\delta \{ (i X_i X) C_p \} = \sum_{n=0}^{(p-1)/2} \left\{ \frac{(p-2n)!}{2^n n!(p-2n-2)!} DP [i X_i X \Lambda_{p-2n-1}] P [B^n] + \frac{(p-2)!}{2^{n-1} n!(p-2n-1)!} DP [i X_L \Lambda_{p-2n-1}] P [(i X) B^{n-1}] \right\}.
\]

Here the coefficients in front of each term are the different weights that arise when the inclusion factor \((i X_i X)\) acts on the various background fields.

Similarly, under massive gauge transformations, the dielectric terms transform as
\[
\delta \Sigma \{ (i X_i X) C_p \} = -p! m \{ (i X_i X) (\Sigma B^{(p-1)/2}) \},
\]
while the variation under the NS-NS gauge transformation is given by
\[
\delta \Sigma \{ (i X_i X) B \} = \{ 2 DP [i X \Sigma] \}, \quad \delta \Sigma \{ (i X_i X) B \} = \{ (i X_i X) \partial \Sigma \}.
\]

As an example let us now look at the gauge transformations of the non-Abelian action for D6-branes, being this the simplest non-trivial case in which both dielectric couplings and massive gauge transformations are present. We will leave the invariance of the general Dp-brane action to the reader.

The dielectric part in the non-Abelian Chern-Simons action describing a system of coincident D6-branes in a massive Type IIA background can be written as (see Appendix A)
\[
\mathcal{L}_{D6} \sim \left\{ \sum_{n=0}^{3} \frac{(-1)^n}{2^{n+1} n (2n-2)!} P \left[ (i X_i X) A_{0-2n} \right] F^n \right\},
\]
where the \(p\)-forms \(A_p\) are defined in \ref{eq:3.3}. This form is very convenient to show the R-R gauge invariance, because it is obvious from the Abelian case that each \(A_p\) is invariant under the R-R and massive gauge transformations. Therefore, in this form the invariance of the action \ref{eq:3.9} under the transformations \ref{eq:3.6} and \ref{eq:3.7} is straightforward. It is also clear that besides the massive terms \ref{eq:3.6}, introduced in \ref{eq:H}, no other dielectric mass terms are needed to assure gauge invariance.

The invariance under the NS-NS transformations \ref{eq:3.8} is however more subtle, due to the fact that \((i X_i X)\) acts on \(B\) but it does not act on \(F\), so that they do not combine in an obvious way into the interior product of the gauge invariant field strength \(\mathcal{F}\). In order to show the invariance under these transformations let us look at those terms with a given \(C_p\), for example the dielectric terms that couple to \(C_7\)
\[
\mathcal{L}_{D6} \sim \left\{ P [-21 (i X_i X) C_7 F - 36 (i X_i X) (C_7 B)] \right\}.
\]
Taking into account that the interior products \((i_X i_X)\) in the first term only act on \(C_7\), while in the second term they act both on \(C_7\) and on \(B\), we have, explicitly:

\[
\mathcal{L}_{D6} \sim \left\{ P[-21(i_X i_X C_7)F - 21(i_X i_X C_7)B - 14(i_X C_7)(i_X B) - C_7(i_X i_X B)] \right\},
\]  

(3.11)

so we see that the first two terms can be combined into the NS-NS invariant field strength, whereas the terms where \(B\) is contracted with one or two \(i_X\) cannot. The NS-NS gauge variation of the latter is given by

\[
\delta \Sigma \mathcal{L}_{D6} \sim \left\{ -28P[(i_X C_7)]D P[(i_X \Sigma)] - 2P[C_7](i_X i_X \partial \Sigma)] \right\},
\]  

(3.12)

and can not be canceled by any other term in the action. The only field that also transforms under NS-NS transformations is the Born-Infeld vector \(V_\mu\), which is however a world volume field and cannot be contracted with \(i_X\). Actually, these contractions of the gauge parameter with the transverse scalars are zero, due to reasons inherent to the construction of the action. Recall that the action for non-Abelian Dp-branes is derived from the action for coincident D9-branes using T-duality [20], so that the directions in which the T-dualities are performed have to be isometric. In the T-dualised action these isometry directions correspond to the transverse directions \(X^i\), which through the T-duality mapping \(V_i \rightarrow X^i\) inherit the gauge transformation of the \(i\)-th component of the Born-Infeld field: \(\delta \Sigma X^i = -\Sigma^i\). Since these directions are isometric, the contractions of \(\Sigma\) with the transverse scalars must vanish, and in this way the gauge invariance is guaranteed. Strictly speaking this kind of terms are already zero in the Abelian case. However in that case if we demand the action to be invariant under diffeomorphisms of the background we recover a fully invariant action. Yet, in the non-Abelian case there is no clear notion of general coordinate transformations (see for example [12, 14]), and it is not clear how the resulting isometries can be removed. The fact that the terms in (3.12) vanish is therefore a manifestation of the lack of diffeomorphism invariance of the action [A.4]. We expect that in a fully diffeomorphism invariant formulation extra terms will be generated whose variation will cancel the terms in (3.12).

4 The non-Abelian CS action and massive T-duality

In this section we derive the Chern-Simons action for coincident Type IIA Dp-branes in a massive background, applying the same method of [20]. We start with a system of \(N\) D9-branes and use the massive T-duality rules between the massive Type IIA and Type IIB theories to generate the action for lower-dimensional branes. We will show that the form of the action is consistent with the one obtained in the previous sections from gauge invariance. In particular we will show that, besides the mass terms (2.5) presented in [4] no extra dielectric mass terms are required.

The massive T-duality rules for general R-R potentials are given by [6, 25]

\[
C_{2k}|_x = \begin{cases} 
-C_{2k-1} + (2k-1) \frac{g_{xx}}{g_{ss}} C_{2k-1}|_x - \frac{m_{x}}{2^{(2k-1)!}} \frac{g_{xx}}{g_{ss}} B^{2k-1} & \\
C_{2k} = C_{2k+1}|_x - 2k C_{2k-1}|_x B|_x - 2k(2k-1) C_{2k-1}|_x B|_x \frac{g_{xx}}{g_{ss}} \\
-C\frac{2k!}{2^{(2k)!}} m_{x} B^k - \frac{g_{xx}}{g_{ss}} B^{2k-1} B|_x \\
C_{2k+1}|_x = C_{2k} + 2k \frac{g_{xx}}{g_{ss}} C_{2k}|_x + \frac{2k!}{2^{(2k)!}} m_{x} B^k + \frac{2k!}{2^{(2k-1)!}} m_{x} \frac{g_{xx}}{g_{ss}} B^{2k-1} B|_x & \\
-C_{2k+2}|_x + (2k+1) C_{2k} B|_x + (2k+1) 2k \frac{g_{xx}}{g_{ss}} C_{2k}|_x B|_x & 
\end{cases}
\]  

(4.1)

where \(x\) denotes the T-duality direction and \(|_x\) means that the last space-time index is \(x\).

The action for \(N\) D9-branes can be expressed in terms of the p-forms \(A_p\) of (A.3) as

\[
\mathcal{L}_{D9} = - \left\{ \sum_{n=0}^{5} \frac{(-1)^n}{2^n n!(10-2n)!} A_{10-2n} F^n \right\}.
\]  

(4.2)
5 Conclusions

We have presented an elegant way to demonstrate the invariance of the non-Abelian Chern-Simons action for D-branes under the gauge transformations of the background fields. By carefully defining what precisely is meant with a background gauge variation in the world volume theory, we have found that the action is indeed invariant under R-R transformations (including massive transformations) if a Chern-Simons-like mass term involving the non-Abelian Born-Infeld vector is added. This extra mass term appears only in the non-dielectric part of the action and its presence was already found in [4]. No extra dielectric mass terms are present. Our results are confirmed by the construction of the non-Abelian Chern-Simons action for D-branes in massive backgrounds via massive T-duality.  

The invariance under the NS-NS transformations is more subtle, but it is guaranteed because by (the T-duality) construction the transverse space directions must be isometric. As a consequence of this the variation of the terms where the NS-NS B-field is contracted with one or two inclusions iX vanishes. The fact that these terms are zero is intimately related to the fact that the non-Abelian Chern-Simons action is not invariant under general coordinate transformations. We would expect a fully diffeomorphism invariant action to contribute with extra terms whose variation under NS-NS gauge transformations would cancel the variations that in our case are set to zero by construction. The structure of the latter could give valuable hints in the construction of a diffeomorphism invariant Chern-Simons action for D-branes.

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A Conventions

In our conventions the gauge transformations of the R-R background fields are given by:

$$\delta C_p = \sum_{n=0}^{[p/2]} \left( \frac{(-1)^n}{(2n)!} \right) \partial \Lambda_{p-2n-1} B^n - \frac{m}{\Sigma} B^{(p-1)/2},$$  \hspace{1cm} (A.1)

where the square brackets in the summation indicate integer part for $p$ even, and the massive term is only non-vanishing for the odd R-R potentials of the massive Type IIA theory.

The Abelian Chern-Simons action for D-branes, involving massive terms for the case of Romans’ theory, is given by \cite{7, 4}:

$$L_{Dq} = \left( - \frac{1}{2} \right) \sum_{n=0}^{q+1} \left( \frac{(-1)^n}{2^n n!} \right) P[C_{q-2n+1}] F^n - \left( \frac{q}{2} \right) \sum_{n=0}^{q+1} \left( \frac{(-1)^n}{2^n n!} \right) \frac{m V}{\Sigma} (\partial V)^{q/2}/2.$$  \hspace{1cm} (A.2)

Here $P[...]$ denotes the (Abelian) pullback and $F$ is defined as $F = P[B]$, with $F = \partial V$.

A useful form for the non-Abelian Chern-Simons action can be obtained introducing the following $p$-forms

$$A_p = \sum_{k=0}^{[p/2]} \frac{(-1)^k}{2^{p-k} (p-2k)!} C_{p-2k} B_k,$$  \hspace{1cm} (A.3)

which contain basically (up to an overall factor) the background field dependence of the pullback of the Abelian D$(p-1)$-brane action \cite{2}. The non-Abelian action for, for example, a D6-brane is then given by

$$L_{D6} = - \left\{ \sum_{n=0}^{3} \left( \frac{(-1)^n}{2^n n!} \right) P[C_{7-2n}] F^n + \frac{1}{24} m \omega_7 - \sum_{n=0}^{3} \left( \frac{(-1)^n}{2^n n!} \right) P[i (i X_i X_j A_{9-2n}] F^n \right\}$$  \hspace{1cm} (A.4)

where now $F$ is the non-Abelian field strength $F = \partial V + i[V, V]$ and the pullback is taken with covariant derivatives. The first two terms constitute the “pre-dielectric” action presented in \cite{3, 4}, while the third term contains the dielectric couplings of \cite{20}. The fact that the dielectric couplings appear precisely through contractions of the $p$-forms $A_p$ is crucial for the gauge invariance of the action. Lower dimensional D-branes, with higher rank dielectric couplings, will contain analogous structures.

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