The nonlinear Schrödinger equation with nonzero backgrounds: 
Bilinearisation and reduction approach

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Abstract

In this paper we develop a bilinearisation-reduction approach to derive solutions to the classical and nonlocal nonlinear Schrödinger (NLS) equations with nonzero backgrounds. We start from the second order Ablowitz-Kaup-Newell-Segur coupled equations as an unreduced system. With a pair of solutions $(q_0, r_0)$ we bilinearize the unreduced system and obtain solutions in terms of quasi double Wronskians. Then we implement reductions by introducing constraints on the column vectors of the Wronskians and finally obtain solutions to the reduced equations, including the classical NLS equation and the nonlocal NLS equations with reverse-space, reverse-time and reverse-time-space, respectively. With a special set of solution $(q_0, r_0)$ as a background solution, we present explicit formulae for these column vectors. As examples, we analyze and illustrate solutions to the focusing NLS equation and the reverse-space nonlocal NLS equation. In particular, we present formulae for the rouge waves of arbitrary order for the focusing NLS equation.

Keywords: nonlinear Schrödinger equation, nonlocal, bilinear, reduction, quasi double Wronskian, rogue wave, nonzero background.

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1 Introduction

For most of integrable (1+1)-dimensional evolution equations of the form $u_t = K(u)$, if $u$ is a conserved density, then $u$ asymptotically goes to zero as $|x| \to \infty$. Thus their solutions may behavior like the waves with an asymptotically zero background. Some soliton equations admit solutions with either zero or nonzero backgrounds. For example, the Korteweg-de Vries (KdV) equation, its nonzero background can be introduced via the Galilean transformation. As one of the most popular nonlinear integrable models, the focusing nonlinear Schrödinger (NLS) equation,

$$i\frac{q}{t} = q_{xx} + 2|q|^2q,$$

(1.1)

where $i$ is the imaginary unit, $|q|^2 = qq^*$ and $q^*$ stands for the complex conjugate of $q$, its nonzero background (in terms of $|q|$) can be introduced by replacing $q$ with $qe^{-2i\alpha t}$ where $\alpha$ is a positive number. This leads equation (1.1) to

$$i\frac{q}{t} = q_{xx} + 2(|q|^2 - \alpha)q.$$

(1.2)

The defocusing NLS equation,

$$i\frac{q}{t} + q_{xx} - 2|q|^2q = 0,$$

(1.3)

provides dark solitons with a nonzero background. Zakharov and Shabat are pioneers who studied the two NLS equations using integrable approaches [60, 61]. For more references about the NLS equations one can refer to [8] and the references therein.

For the NLS equation (1.1) with nonzero backgrounds, early investigations were due to Kuznetsov [36], Kawata and Inoue [34, 35] and Ma [41]. They solved the NLS equation (1.1) with $q_0 = e^{-2i\alpha t}$ as a background solution by means of the inverse scattering transform. Faddeev and Takhtajan have also done important work in this area (see for instance the monograph Ref. [25] and references therein). Besides, the NLS equation with different asymmetric nonzero backgrounds has been studied in [14, 20, 21, 33, 53]. There are also some research of the NLS equation on the background of elliptic functions, e.g. [17, 26].

Comparing the two equations (1.1) and (1.2), the background solution $q_0 = e^{-2i\alpha t}$ provides the envelope (or carrier wave) $|q|^2$ a positive lift $\alpha$. This does bring more interesting behavior of $|q|$. The simplest solution (corresponding to one-soliton) of the equation (1.2) is a breather [35, 36, 41], not the usual soliton. In a special limit the breather yields a localized rational solution [46], which is nowadays used to describe a rogue wave. The rational solution was also
derived by Matveev and Salle via Darboux transformation (see §4.2 in [42], where the rogue wave is called “exulton” solution). The second order rational solution of equation (1.2) was derived in 1985 in [10], using a similar way as in [46]. The rational solution of arbitrary order of the NLS equation was first constructed in 1986 in [24], where explicit formula of the solution was presented in an elegant way and nonsingular property of the solution was proved as well (see also [23] for an alternative proof). “Rogue waves” is the name given by oceanographers to isolated large amplitude waves, which occur more frequently than expected for normal, Gaussian distributed, statistical events (cf.[45]). After rogue wave was observed in optic experiment in 2007 [51], it started to draw new attention and the research on rogue waves has become a hot topic. One can refer to the review [45] for more references. Mathematically, higher order rational solutions of the NLS equation (1.2) can be obtained using Darboux transformation via a special limit procedure [28, 30], from a bilinear approach using reduction of Kadomtsev-Petviashvili $\tau$ functions [44], and from inverse scattering transform [13]. In this paper, we will derive them using bilinear method but in a completely different way from [44].

Our idea is to solve the second order Ablowitz-Kaup-Newell-Segur (AKNS) coupled equations

\begin{align}
    iq_t &= q_{xx} - 2q^2 r, & (1.4a) \\
    ir_t &= -r_{xx} + 2r^2 q & (1.4b)
\end{align}

as an unreduced system, which, for instance, yields the NLS equation (1.1) via reduction $r = -q^*$. We can bilinearize this unreduced system and present solutions of the bilinear equations in terms of double Wronskian. Then, we impose constraints on the column vectors of the double Wronskians so that the desired reduction hold and thus we get solutions to the reduced equation. Such an idea was first proposed in [18, 19] for obtaining solutions for the nonlocal integrable equations. Nonlocal integrable systems were first systematically proposed by Ablowitz and Musslimani in 2013 [5] and have drawn intensive attention (e.g.[1, 4, 7, 15, 29, 40, 47–49, 57, 65]). The bilinearisation-reduction approach has proved effective in deriving solutions not only to the nonlocal systems but also to the classical equations (e.g. [22, 37–39, 50, 54, 55]). In this paper, we introduce transformation

\begin{equation}
    q = q_0 + \frac{g}{f}, \quad r = r_0 + \frac{h}{f} \tag{1.5}
\end{equation}

for the unreduced system (1.4). Here $(q_0, r_0)$ are a set of solution of (1.4). It will be seen that in some cases $|q_0|^2$ does act as a background of the envelope $|q|^2$, see equation (5.1) for example). In this context we call $(q_0, r_0)$ a set of background solutions of the system (1.4). We will employ (1.5) to bilinearize the unreduced system (1.4) and present (quasi) double Wronskian solutions to the bilinear equations. Then we will implement reduction technique to obtain solutions to the reduced equations listed in (2.3)-(2.6).

The paper is organized as follows. In Sec.2 we recall the classical and nonlocal reductions of the unreduced AKNS system (1.4). In Sec.3 we derive the bilinear form of (1.4) with backgrounds solution $(q_0, r_0)$ and derive (quasi) double Wronskian solutions to the bilinear equations. In Sec.4 the reduction technique is implemented and explicit form of solutions are obtained for the reduced equations. Then in Sec.5 we investigate dynamics of some obtained solutions for the the classical NLS equation and the reverse-space nonlocal NLS equation with nonzero backgrounds. Finally, Sec.6 serves for presenting conclusions.
2 The second order AKNS system and its reductions

The second order coupled AKNS equations (1.4), where \( q = q(x,t) \) and \( r = r(x,t) \) are functions of \((x,t) \in \mathbb{R}^2\), has been studied as a classical couple system in past decades. Recently it was found that this system is related to the cubic nonlinear Klein-Gordon equation, see [7]. Its Lax pairs consist of the well known AKNS (or Zakharov-Shabat (ZS)-AKNS) spectral problem [2, 3, 60],

\[
\begin{pmatrix}
\Phi \\
\Psi
\end{pmatrix}_x = M \begin{pmatrix}
\Phi \\
\Psi
\end{pmatrix}, \quad M = \begin{pmatrix}
\lambda & q \\
r & -\lambda
\end{pmatrix},
\]

and the corresponding time evolution part

\[
i \left( \begin{pmatrix}
\Phi \\
\Psi
\end{pmatrix} \right)_t = N \begin{pmatrix}
\Phi \\
\Psi
\end{pmatrix}, \quad N = \begin{pmatrix}
2\lambda^2 - qr & 2\lambda q + q_x \\
2\lambda r - r_x & -2\lambda^2 + qr
\end{pmatrix},
\]

in which \( \lambda \) is spectral parameter, \( \lambda_t = 0 \), \( \Phi \) and \( \Psi \) are wave functions.

In the following we list possible one-component equations reduced from equation (1.4). These equations will be considered in this paper. Equation (1.4) admits the following reductions (see [6] and reference therein)

\[
iq_t = q_{xx} - 2\delta q^2 q^*, \quad r = \delta q^*, \quad (2.3)
\]

\[
iq_t = q_{xx} - 2\delta q^2 q(-x), \quad r = \delta q^*(-x), \quad (2.4)
\]

\[
iq_t = q_{xx} - 2\delta q^2 q(-t), \quad r = \delta q(-t), \quad (2.5)
\]

\[
iq_t = q_{xx} - 2\delta q^2 q(-x,-t), \quad r = \delta q(-x,-t), \quad (2.6)
\]

where \( \delta = \pm 1 \), \( q(-x) = q(-x,t) \), \( q(-t) = q(x,-t) \) and \( q(-x,-t) \) indicate the reverse-space, reverse-time and reverse-space-time, respectively.

3 Bilinearisation and solutions of the AKNS system (1.4)

In this section, we develop the double Wronskian technique to construct exact solutions of the second order AKNS system (1.4) with nonzero background solution \((q_0, r_0)\).

3.1 Bilinearisation

Suppose that \((q_0, r_0)\) are a set of solution to the second order AKNS system (1.4). To introduce nonzero backgrounds, we consider the dependent variable transformation (i.e. (1.5))

\[
q = q_0 + \frac{q}{f}, \quad r = r_0 + \frac{h}{f},
\]

with which the system (1.4) can be decoupled into the following bilinear form of \( f, g \) and \( h \),

\[
D_x^2 f \cdot f = -2gh - 2q_0 hf - 2r_0 gf,
\]

\[
(D_x^2 - iD_t - 2q_0 r_0) g \cdot f + q_0 D_x^2 f \cdot f = 0,
\]

\[
(D_x^2 + iD_t - 2q_0 r_0) h \cdot f + r_0 D_x^2 f \cdot f = 0,
\]

where \( D_x \) and \( D_t \) are the well known Hirota bilinear operators defined as [32]

\[
D_x^n D_t^n f \cdot g = (\partial_x - \partial_{x'})^n (\partial_t - \partial_{t'})^n f(x,t) g(x',t') |_{x'=x, t'=t}.
\]
Note that when $q_0 = r_0 = 0$ the above bilinear form (3.2) degenerates to the case of zero background (cf. equations (1.5.1)-(1.5.3) in [16]).

To have solutions of (3.2), we expanding $f, g$ and $h$ as the series

$$
 f(x, t) = 1 + f^{(2)} \varepsilon^2 + f^{(4)} \varepsilon^4 + \cdots + f^{(2j)} \varepsilon^{2j} + \cdots, \tag{3.3a}
$$

$$
 g(x, t) = g^{(1)} \varepsilon + g^{(3)} \varepsilon^3 + \cdots + g^{(2j+1)} \varepsilon^{2j+1} + \cdots, \tag{3.3b}
$$

$$
 h(x, t) = h^{(1)} \varepsilon + h^{(3)} \varepsilon^3 + \cdots + h^{(2j+1)} \varepsilon^{2j+1} + \cdots, \tag{3.3c}
$$

where $\varepsilon$ is an arbitrary number, $\{f^{(j)}, g^{(k)}, h^{(l)}\}$ are functions to be determined. Consider a special case,

$$
 q_0 = A_0 e^{2iA_0^2 t}, \quad r_0 = A_0 e^{-2iA_0^2 t}, \tag{3.4}
$$

where $A_0$ is an arbitrary constant. By calculation we can find out 1-, 2- and 3-soliton solutions, which agree with the following general expressions,

$$
 g = A_0 e^{2iA_0^2 t} \sum_{\mu = 0, 1} \exp \left[ \sum_{j=1}^{N} \mu_j (\xi_j + a_j) + \sum_{1 \leq i < j} \mu_i \mu_j B_{ij} \right], \tag{3.5a}
$$

$$
 h = A_0 e^{-2iA_0^2 t} \sum_{\mu = 0, 1} \exp \left[ \sum_{j=1}^{N} \mu_j (\xi_j + b_j) - \sum_{1 \leq i < j} \mu_i \mu_j B_{ij} \right], \tag{3.5b}
$$

$$
 f = \sum_{\mu = 0, 1} \exp \left[ \sum_{j=1}^{N} \mu_j \xi_j + \sum_{1 \leq i < j} \mu_i \mu_j A_{ij} \right], \tag{3.5c}
$$

where

$$
 e^{\alpha_{ij}} = \frac{-2k_j^4 - 2ik_j^2 w_j}{k_j^4 + w_j^2}, \quad e^{\beta_{ij}} = \frac{-2k_j^4 + 2ik_j^2 w_j}{k_j^4 + w_j^2}, \tag{3.5d}
$$

$$
 \xi_j = k_j x + \omega_j t + \xi_j^{(0)}, \quad \omega_j = \pm \sqrt{4A_0^2 k_j^2 - k_4^4}, \tag{3.5e}
$$

$$
 e^{A_{ij}} = \frac{-k_i^2 k_j^2 + 2A_0^2 \left( k_i^2 + k_j^2 \right) - \sqrt{4A_0^2 k_i^2 - k_4^4} \sqrt{4A_0^2 k_j^2 - k_4^4}}{2A_0^2 \left( k_i + k_j \right)^2}, \tag{3.5f}
$$

$$
 e^{B_{ij}} = \frac{i(k_i - k_j) \left( -k_i^2 \sqrt{4A_0^2 k_i^2 - k_4^4} + k_i^2 \sqrt{4A_0^2 k_j^2 - k_4^4} \right)}{2k_i^2 k_j^2 \left( k_i + k_j \right)}, \tag{3.5g}
$$

the summation of $\mu$ means to take all possible $\mu_j = \{0, 1\}$ for $j = 1, 2, \cdots, N$.

### 3.2 Quasi double Wronskian solutions of the AKNS system (1.4)

We now derive double Wronskian solutions of the second order AKNS system (1.4).

We extend the Lax pair (1.4) to the following matrix system

$$
 \begin{pmatrix}
 \phi \\
 \psi
 \end{pmatrix}_x = \mathcal{M} \begin{pmatrix}
 \phi \\
 \psi
 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix}
 A & q_0 I_{2m+2} \\
 r_0 I_{2m+2} & -A
 \end{pmatrix}, \tag{3.6}
$$

$$
 i \begin{pmatrix}
 \phi \\
 \psi
 \end{pmatrix}_t = \mathcal{N} \begin{pmatrix}
 \phi \\
 \psi
 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix}
 2A^2 - q_0 r_0 I_{2m+2} & 2A q_0 + q_{0,x} I_{2m+2} \\
 2A r_0 - r_{0,x} I_{2m+2} & -2A^2 + q_0 r_0 I_{2m+2}
 \end{pmatrix}, \tag{3.7}
$$
where \( A \in \mathbb{C}_{(2m+2)\times (2m+2)} \), \( I_{2m+2} \) is the \((2m+2)\)-th order identity matrix, \( \phi \) and \( \psi \) are two \((2m+2)\)-th column vectors. Introduce vectors \( \phi_k \) and \( \psi_k \) by
\[
\phi_k = A^k \phi, \quad \psi_k = (-A)^k \psi,
\]
and define determinants\(^1\)
\[
f = |\hat{\phi}_m; \hat{\psi}_m|, \quad g = 2|\hat{\phi}_{m+1}; \hat{\psi}_{m-1}|, \quad h = -2|\hat{\phi}_{m-1}; \hat{\psi}_{m+1}|,
\]
where \( \hat{\phi}_m \) stands for the consecutive columns \((\phi_0, \phi_1, \ldots, \phi_m)\). Then, solutions of the bilinear system \((3.2)\) are described as the following.

**Theorem 1.** The bilinear system \((3.2)\) has solutions \((3.9)\), where \( \phi \) and \( \psi \) in \((3.8)\) satisfy \((3.6)\) and \((3.7)\), and \((q_0, r_0)\) are given solutions of the system \((1.4)\). Furthermore, matrix \( A \) and any matrix that is similar to it lead to the same solution of the AKNS system \((1.4)\) through the transformation \((3.1)\).

The proof will be sketched in Appendix A.

Strictly speaking, the above \( f, g, h \) in \((3.9)\) are not double Wronskians that are defined by arranging columns by increasing the order of derivatives of \( \phi \) and \( \psi \). We may call them quasi double Wronskians. We note that when the background solution \((q_0, r_0)\) is independent of \( x \), we may covert \( f, g, h \) given in \((3.9)\) to double Wronskians.

**Theorem 2.** Suppose that the \((2m+2) \times (2m+2)\) double Wronskians\(^2\)
\[
f = (-1)^m |\hat{n}; \hat{m}|, \quad g = 2|m+1; m-1|, \quad h = 2|m-1; m+1|,
\]
where \( |\hat{n}; \hat{m}| \) denotes a \((m+n+2)\) double Wronskian defined as (see [43])
\[
|m+j; m-j| = |0, 1, \ldots, m+j; 0, 1, \ldots, m-j| = |\phi, \partial_x \phi, \ldots, \partial_x^{m+j} \phi; \psi, \partial_x \psi, \ldots, \partial_x^{m-j} \psi|,
\]
\( \phi \) and \( \psi \) are \((2m+2)\)-th order column vectors. When \( \phi \) and \( \psi \) meet the condition
\[
\phi_x = A\phi + q_0 \psi, \quad i\phi_t = 2\phi_{xx} - q_0 r_0 \phi - 2q_0 \psi x,
\]
\[
\psi_x = r_0 \phi - A\psi, \quad i\psi_t = -2\psi_{xx} + q_0 r_0 \psi + 2r_0 \phi_x,
\]
where \( A \) is a \((2m+2) \times (2m+2)\) complex matrix, \((q_0, r_0)\) satisfy \((1.4)\) but are independent of \( x \), then \( f, g, h \) defined in \((3.10)\) are solutions to the bilinear equations \((3.2)\). Furthermore, matrix \( A \) and any matrix that is similar to it lead to the same solution to the AKNS system \((1.4)\) through the transformation \((3.1)\).

The proof will be given in Appendix B.

4\ Reduction and solutions

For convenience we call \((1.4)\) the unreduced system and \((2.3)-(2.6)\) the reduced equations. In the previous section we have already obtained solutions in terms of quasi double Wronskians \((3.9)\) (see Theorem 1 for the unreduced system \((1.4)\)). In this section we implement reductions by imposing constraints on \( A \) and \( \psi \) so that \((3.9)\) can provides solutions to the reduced equations \((2.3)-(2.6)\). Such a reduction technique was first introduced in [18, 19].

\(^1\)When \( m = 0 \) we have \( g = 2|\phi_0, \phi_1|, \quad h = -2|\psi_0, \psi_1| \).

\(^2\)When \( m = 0, \) \( g \) and \( h \) take the form \( g = -2|\phi, \partial_x \phi|, \quad h = 2|\psi, \partial_x \psi| \).
4.1 Reduction of the Wronskian solution

Let us directly present results and then prove them.

**Theorem 3.** Let $A$ and $T$ be matrices in $\mathbb{C}^{(2m+2) \times (2m+2)}$. Solutions of the reduced equations (2.3)-(2.6) are given in the following, respectively.

1. The classical NLS equation (2.3) has solution

\[ q = q_0 + \frac{g}{f}, \]
\[ f = |\hat{\phi}_m; T\hat{\phi}_m^*|, \quad g = 2|\hat{\phi}_{m+1}; T\hat{\phi}_{m-1}^*|, \]

where $q_0$ is a solution of equation (2.3) such that $r_0^* = \delta q_0$, vector $\phi$ is a solution of matrix equations

\[ \phi_x = A\phi + q_0 T\phi^*, \]
\[ i\phi_t = (2A^2 - \delta q_0 q_0^* I_{2m+2})\phi + (2Aq_0 + q_{0,x} I_{2m+2}) T\phi^*, \]

and $A$ and $T$ obey the relation

\[ AT + TA^* = 0, \quad TT^* = \delta I_{2m+2}, \quad \delta = \pm 1. \] (4.3)

2. For the reverse-space nonlocal NLS equation (2.4), its solution is given by

\[ q = q_0 + \frac{g}{f}, \]
\[ f = (-1)^{\frac{m(m+1)}{2}} |\hat{\phi}_m; T\hat{\phi}_m^* (-x)|, \quad g = 2(-1)^{\frac{m(m-1)}{2}} |\hat{\phi}_{m+1}; T\hat{\phi}_{m-1}^* (-x)|, \]

where $q_0$ is a solution of equation (2.4) such that $r_0^*(x) = \delta q_0(-x)$, vector $\phi$ is a solution of matrix equations

\[ \phi_x(x) = A\phi(x) + q_0(x) T\phi^*(-x), \]
\[ i\phi_t(x) = (2A^2 - \delta q_0(x) q_0^*(-x) I_{2m+2})\phi(x) + (2Aq_0 + q_{0,x}(x) I_{2m+2}) T\phi^*(-x), \]

and $A$ and $T$ obey the relation

\[ AT - TA^* = 0, \quad TT^* = -\delta I_{2m+2}, \quad \delta = \pm 1. \] (4.6)

3. For the reverse-time nonlocal NLS equation (2.5), its solution is given by

\[ q = q_0 + \frac{g}{f}, \]
\[ f = |\hat{\phi}_m; T\hat{\phi}_m(-t)|, \quad g = 2|\hat{\phi}_{m+1}; T\hat{\phi}_{m-1}(-t)|, \]

where $q_0$ is a solution of equation (2.5) such that $r_0^*(t) = \delta q_0(-t)$, $\phi$ is a solution of matrix equations

\[ \phi_x(t) = A\phi(t) + q_0(t) T\phi(-t), \]
\[ i\phi_t(t) = (2A^2 - \delta q_0(t) q_0(-t) I_{2m+2})\phi(t) + (2Aq_0(t) + q_{0,x}(t) I_{2m+2}) T\phi(-t), \]

and $A$ and $T$ obey the relation

\[ AT + TA = 0, \quad T^2 = \delta I_{2m+2}, \quad \delta = \pm 1. \] (4.9)
(4) For the reverse-space-time nonlocal NLS equation (2.6), its solution is given by

\[ q = q_0 + \frac{g}{f}, \quad (4.10a) \]

\[ f = (-1) \frac{m(m+1)}{2} \vert \hat{\phi}_m; T \hat{\phi}_m(-x, -t) \vert, \quad g = 2(-1) \frac{m(m+1)}{2} \vert \hat{\phi}_{m+1}; T \hat{\phi}_{m-1}(-x, -t) \vert, \quad (4.10b) \]

where \( q_0 \) is a solution of equation (2.6) such that \( r_0^*(x, t) = \delta q_0(-x, -t) \), vector \( \phi \) is a solution of matrix equations

\[ \phi_x(x, t) = A\phi(x, t) + q_0(x, t)T\phi(-x, -t), \quad (4.11a) \]

\[ i\phi_t(x, t) = (2A^2 - \delta q_0(x, t)q_0(-x, -t))\phi(x, t) + (2Aq_0(x, t) + q_0(x, t))T\phi(-x, -t), \quad (4.11b) \]

and \( A \) and \( T \) obey the relation

\[ AT - TA = 0, \quad T^2 = -\delta I_{2m+2}, \quad \delta = \pm 1. \quad (4.12) \]

**Proof.** We employ the classical NLS equation (2.3) as an illustrating example. Introduce constraint on \( \psi \):

\[ \psi = T\phi^*, \quad (4.13) \]

where \( T \) is a certain matrix in \( \mathbb{C}^{(2m+2) \times (2m+2)} \). First, it can be verified that when \( r_0 = \delta q_0^* \) and \( A \) and \( T \) satisfy (4.3), the constraint (4.13) reduces (3.6) and (3.7) to (4.5). In fact, taking (3.6) as an example, under (4.13) and \( r_0 = \delta q_0^* \), we rewrite (3.6) as

\[ \phi_x = A\phi + q_0 T \phi^*, \quad (4.14a) \]

\[ T \phi_x^* = \delta q_0^* \phi - AT \phi^*, \quad (4.14b) \]

where (4.14a) is nothing but (4.5a). Making use of (4.3), equation (4.14b) multiplied by \( \delta T^* \) from the left gives rise to the complex conjugate of (4.14a). This indicates (3.6) reduces to (4.5a). In a similar way one can find (3.7) reduces to (4.5b) in this case.

Next, with the constraint (4.13), we can rewrite the quasi double Wronskians (3.9) as

\[ f = \vert \hat{\phi}_m; \hat{\psi}_m \vert = \vert \hat{\phi}_m; T \hat{\phi}_m^* \vert, \quad (4.15a) \]

\[ g = 2\vert \hat{\phi}_{m+1}; \hat{\psi}_{m-1} \vert = 2\vert \hat{\phi}_{m+1}; T \hat{\phi}_{m-1}^* \vert, \quad (4.15b) \]

\[ h = -2\vert \hat{\phi}_{m-1}; \hat{\psi}_{m+1} \vert = -2\vert \hat{\phi}_{m-1}; T \hat{\phi}_{m+1}^* \vert. \quad (4.15c) \]

Making use of (4.3) we find that

\[ f = \vert \hat{\phi}_m; T \hat{\phi}_m^* \vert = \vert T \vert \delta T^* \hat{\phi}_m; \hat{\phi}_m^* \vert. \]

Then, switching the first \((m+1)\) columns and the last \((m+1)\) columns and picking the parameter \( \delta \) out yield

\[ f = (-\delta)^{m+1} \vert T \vert \hat{\phi}_m^*; T^* \hat{\phi}_m \vert = (-\delta)^{m+1} \vert T \vert f^*. \]

In a similar way we can prove

\[ h = -(-\delta)^m \vert T \vert g^*. \]

Thus we have

\[ \frac{r}{q^*} = \frac{r_0 + h/f}{q_0^* + g^*/f^*} = \frac{\delta q_0^* + \delta g^*/f^*}{q_0^* + g^*/f^*} = \delta, \]

8
i.e. $r = \delta q^*$, which is the reduction by which we get (2.3) from (1.4).

The proof of nonlocal cases is similar to the classical one. For the reverse-space nonlocal NLS equation (2.4), the reduction is implemented by taking

$$\psi = T\phi^*(-x)$$

(4.16)

together with (4.6). Here and below we note that we do not write out independent variables unless the inverse of them are involved. Relations between Wronskians are

$$f = \delta^{m+1}|T|f^*(-x), \quad h = \delta^{m}|T|g^*(-x),$$

which yield

$$\frac{r}{q^*(-x)} = \frac{r_0 + h/f}{q_0^*(-x) + g^*(-x)/f^*(-x)} = \frac{\delta q_0^*(-x) + \delta g^*(-x)/f^*(-x)}{q_0^*(-x) + g^*(-x)/f^*(-x)} = \delta,$$

i.e. $r = \delta q^*(-x)$, which reduces the unreduced system (1.4) to equation (2.4).

For the reverse-time nonlocal NLS equation (2.5), the reduction is implemented by taking

$$\psi = T\phi(-t)$$

(4.17)

together with (4.9). Relations between Wronskians are

$$f = (-\delta)^{m+1}|T|f(-t), \quad h = (-\delta)^{m}|T|g(-t),$$

which yield

$$\frac{r}{q(-t)} = \frac{r_0 + h/f}{q_0(-t) + g(-t)/f(-t)} = \frac{\delta q_0(-t) + \delta g(-t)/f(-t)}{q_0(-t) + g(-t)/f(-t)} = \delta,$$

i.e. $r = \delta q(-t)$, which reduces (1.4) to (2.5).

For the reverse-time-space nonlocal NLS equation (2.6), we start from

$$\psi = T\phi(-x,-t)$$

(4.18)

and (4.12). Relations between Wronskians are

$$f = \delta^{m+1}|T|f(-x,-t), \quad h = \delta^{m}|T|g(-x,-t),$$

which yield

$$\frac{r}{q(-x,-t)} = \frac{r_0 + h/f}{q_0(-x,-t) + g(-x,-t)/f(-x,-t)} = \frac{\delta q_0(-x,-t) + \delta g(-x,-t)/f(-x,-t)}{q_0(-x,-t) + g(-x,-t)/f(-x,-t)} = \delta,$$

i.e. $r = \delta q(-x,-t)$, which reduces (1.4) to (2.6).

\[\square\]

### 4.2 Matrices $A$ and $T$

We look for explicit forms of $A$ and $T$ in Theorem 3. Equations (4.3) and (4.6) can be unified to be

$$AT + \sigma TA^* = 0, \quad TT^* = \sigma\delta I, \quad \sigma,\delta = \pm 1;$$

(4.19)

and equations (4.9) and (4.12) can be unified to be

$$AT + \sigma TA = 0, \quad T^2 = \sigma\delta I, \quad \sigma,\delta = \pm 1.$$

(4.20)
can be listed out as in Table 1 and 2, cf. [18].

Consider special solutions to these matrix equations, i.e. $A$ and $T$ are block matrices

$$A = \begin{pmatrix} K_1 & 0 \\ 0 & K_4 \end{pmatrix}, \quad T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix},$$  \hspace{1cm} (4.21)$$

where $T_i$ and $K_i$ are $(m + 1) \times (m + 1)$ matrices. Then solutions to equations (4.19) and (4.20) can be listed out as in the Table 1 and 2, cf. [18].

In addition, equation (4.6) admits real solution in the form (4.21) for the case $\delta = -1$, where

$$K_1 = K_{m+1}, \quad K_4 = H_{m+1}, \quad K_{m+1}, H_{m+1} \in \mathbb{R}^{(m+1) \times (m+1)},$$  \hspace{1cm} (4.22a)$$

$$T_1 = -T_4 = I_{m+1}, \quad T_2 = T_3 = 0_{m+1},$$  \hspace{1cm} (4.22b)$$

or

$$T_1 = T_4 = I_{m+1}, \quad T_2 = T_3 = 0_{m+1}.$$  \hspace{1cm} (4.22c)$$

Besides, equation (4.3) with $\delta = 1$ can have pure imaginary solution (4.21) and (4.22) where in

(4.22a) $K_{m+1}, H_{m+1} \in i\mathbb{R}^{(m+1) \times (m+1)}$.

Due to the fact that $A$ and any matrix that is similar to it generate same solutions to the system (1.4) (see Theorem 1), we only need to consider the canonical forms\(^\text{3}\) of $A$. That is, $K_{m+1}$ can either be

$$K_{m+1} = \text{Diag}(k_1, k_2, \cdots, k_{m+1}), \quad k_i \in \mathbb{C},$$  \hspace{1cm} (4.23)$$

or $K_{m+1} = J_{m+1}(k), \quad k \in \mathbb{C}$, where

$$J_{m+1}(k) = \begin{pmatrix} k & 0 & 0 & \cdots & 0 & 0 \\ 1 & k & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & k \end{pmatrix}_{(m+1) \times (m+1)}.$$  \hspace{1cm} (4.24)$$

\(^3\)A general case for $K_{m+1}$ is the block diagonal form

$$K_{m+1} = \text{Diag}(J_{h_1}(k_1), J_{h_2}(k_2), \cdots, J_{h_s}(k_s), \text{Diag}(k_{s+1}, \cdots, k_{s+n})),$$

where each $J_{h_j}(k_j)$ is an $h_j \times h_j$ Jordon block matrix defined as (4.24), $\text{Diag}(k_{s+1}, \cdots, k_{s+n})$ is an $n \times n$ diagonal matrix and $n + \sum_{j=1}^s h_j = m + 1$. In this case, $\phi$ is just composed accordingly since (3.6) and (3.7) are linear system of $\phi$ and $\psi$. 

---

Table 1: $T$ and $A$ for equation (4.19)

| $(\sigma, \delta)$ | $T$ | $A$ |
|-------------------|---------|---------|
| (4.3)             | $(1, 1)$ | $T_1 = T_4 = 0_{m+1}, T_2 = T_3 = I_{m+1}$ | $K_1 = K_{m+1}, K_4 = -K^*_{m+1}$ |
|                   | $(1, -1)$ | $T_1 = T_4 = 0_{m+1}, T_2 = -T_3 = I_{m+1}$ | $K_1 = K_{m+1}, K_4 = -K^*_{m+1}$ |
| (4.6)             | $(-1, 1)$ | $T_1 = T_4 = 0_{m+1}, T_2 = -T_3 = I_{m+1}$ | $K_1 = K_{m+1}, K_4 = K^*_{m+1}$ |
|                   | $(-1, -1)$ | $T_1 = T_4 = 0_{m+1}, T_2 = T_3 = I_{m+1}$ | $K_1 = K_{m+1}, K_4 = K^*_{m+1}$ |

Table 2: $T$ and $A$ for equation (4.20)

| $(\sigma, \delta)$ | $T$ | $A$ |
|-------------------|---------|---------|
| (4.9)             | $(1, 1)$ | $T_1 = T_4 = 0_{m+1}, T_2 = T_3 = I_{m+1}$ | $K_1 = K_{m+1}, K_4 = -K_{m+1}$ |
|                   | $(1, -1)$ | $T_1 = T_4 = 0_{m+1}, T_2 = -T_3 = I_{m+1}$ | $K_1 = K_{m+1}, K_4 = -K_{m+1}$ |
| (4.12)            | $(-1, 1)$ | $T_1 = -T_4 = iI_{m+1}, T_2 = T_3 = 0_{m+1}$ | $K_1 = K_{m+1}, K_4 = H_{m+1}$ |
|                   | $(-1, -1)$ | $T_1 = -T_4 = I_{m+1}, T_2 = T_3 = 0_{m+1}$ | $K_1 = K_{m+1}, K_4 = H_{m+1}$ |
4.3 Background solution \( q_0 \)

Considering the expression (3.1) (and also (5.1)) we can call \( q_0 \) and \( r_0 \) to be background solutions of \( q \) and \( r \), respectively. The unreduced system (1.4) admits a set of plane-wave solutions

\[
q_0 = A_0 e^{i[a x + (a^2 + 2A_0B_0)t]}, \quad r_0 = B_0 e^{-i[a x + (a^2 + 2A_0B_0)t]},
\]

where \( A_0, B_0 \) and \( a \) are arbitrary complex constants. It is easy to find that the reduced classical and nonlocal NLS equations (2.3)-(2.6) admit the following solutions, respectively,

\[
q_0 = A_0 e^{i[a x + (a^2 + 2|A_0|^2)t]}, \quad (A_0 \in \mathbb{C}, \ a \in \mathbb{R}), \tag{4.26a}
\]
\[
q_0 = A_0 e^{-ax+i(-a^2+2|A_0|^2)t}, \quad (A_0 \in \mathbb{C}, \ a \in \mathbb{R}), \tag{4.26b}
\]
\[
q_0 = A_0 e^{2i\delta A_0^2 t}, \quad (A_0 \in \mathbb{C}), \tag{4.26c}
\]
\[
q_0 = A_0 e^{i(ax+(a^2+2|A_0|^2)t)}, \quad (A_0, a \in \mathbb{C}). \tag{4.26d}
\]

Our purpose is to write out explicit Wronskian vectors \( \phi \) that respectively satisfy the conditions (4.2), (4.5), (4.8) and (4.11), for given background solutions \( q_0 \). We are going to consider the simple case where \( q_0 \) are given in (4.26). If making use of some symmetries, we may start from a simpler background solution

\[
q_0 = e^{2i\delta t}, \tag{4.27}
\]

instead of (4.26).

**Proposition 1.** The classical and nonlocal NLS equations (2.3)-(2.6) admit the following symmetries.

(1) Classical NLS equation (2.3):

- **Galilean symmetry:** if \( q(x, t) \) solves the NLS equation (2.3) with the background solution \( q_0 \) given in (4.26a), then \( Q(X,Y) = q(x,t) e^{-iax - ia^2 t} \), where \( X = x + 2at \) and \( Y = t \), solves the NLS equation (2.3) with \( Q(X,Y) \), i.e.

\[
iQ_Y = Q_{XX} - 2\delta Q^2 Q^*,
\]

of which \( Q_0(X,Y) = q_0 e^{-iax + ia^2 t} = A_0 e^{2i\delta|A_0|^2 Y} \) is a solution.

- **Scaling symmetry:** if \( q(x,t) \) solves the NLS equation (2.3) with a background solution \( q_0(x,t) = A_0 e^{2i\delta|A_0|^2 t} \), then \( Q(X,Y) = \frac{1}{A_0^2} q(x,t) \), where \( X = |A_0|x \) and \( Y = |A_0|^2 t \), also solves the NLS equation (4.28), of which \( Q_0(X,Y) = \frac{1}{A_0^2} q_0 = e^{2i\delta Y} \) is a solution.

(2) Nonlocal reverse-space NLS equation (2.4):

- **Galilean symmetry:** if \( q(x,t) \) solves the reverse-space NLS equation (2.4) with the background solution \( q_0(x,t) \) given in (4.26b), then \( Q(X,Y) = q(x,t) e^{ax + ia^2 t} \), where \( X = x + 2iat \) and \( Y = t \), also solves the reverse-space NLS equation (2.4) with \( Q(X,Y) \), i.e.

\[
iQ_Y = Q_{XX} - 2\delta Q^2 Q^*(-X),
\]

of which \( Q_0(X,Y) = q_0 e^{ax + ia^2 t} = A_0 e^{2i\delta|A_0|^2 Y} \) is a solution.

- **Scaling symmetry:** if \( q(x,t) \) solves the reverse-space NLS equation (2.4) with a background solution \( q_0(x,t) = A_0 e^{2i\delta|A_0|^2 t} \), then \( Q(X,Y) = \frac{1}{A_0} q(x,t) \), where \( X = |A_0|x, \ Y = |A_0|^2 t \), also solves the reverse-space NLS equation (4.29) of which \( Q_0(X,Y) = \frac{1}{A_0} q_0 = e^{2i\delta Y} \) is a solution.
(3) Nonlocal reverse-time NLS equation (2.5):

- Scaling symmetry: if \( q(x, t) \) solves the reverse-time NLS equation (2.5) with the background solution \( q_0(x, t) \) given in (4.26c), then \( Q(X, Y) = \frac{1}{A_0} q(x, t) \), where \( X = A_0 x \) and \( Y = A_0^2 t \), also solves the reverse-time NLS equation (2.5) with \( Q(X, Y) \), i.e.

\[
iQ_Y = Q_{XX} - 2\delta Q^2 Q(-Y),
\]

of which \( Q_0(X, Y) = \frac{1}{A_0} q_0 = e^{2i\delta Y} \) is a solution.

(4) Nonlocal reverse space-time NLS equation (2.6):

- Galilean symmetry: if \( q(x, t) \) solves the reverse space-time NLS equation (2.6) with the background solution \( q_0(x, t) \) given in (4.26d), then \( Q(X, Y) = q(x, t)e^{-iax - ia^2 t} \), where \( X = x + 2at \) and \( Y = t \), also solves the reverse space-time NLS equation (2.6) with \( Q(X, Y) \), i.e.

\[
iQ_Y = Q_{XX} - 2\delta Q^2 Q(-X, -Y),
\]

of which \( Q_0(X, Y) = q_0e^{-iax - ia^2 t} = A_0 e^{2i\delta Y} \) is a solution.

Based on these symmetries of equations (2.3)-(2.6), we only need to consider the unified background solution \( q_0 \) given in (4.27).

4.4 Wronskian column vectors \( \phi \) and \( \psi \)

4.4.1 Vectors \( \phi \) and \( \psi \) for the unreduced system (1.4)

We start with a pair of background solutions

\[
q_0 = e^{2i\delta t}, \quad r_0 = \delta e^{-2i\delta t}, \quad \delta = \pm 1
\]

(4.32)
of the unreduced system (1.4). Note that the background solution \((q_0, r_0)\) agrees with the reductions used in the equations (2.3)-(2.6). Substituting \((q, r) = (q_0, r_0)\) into the matrix equations (2.1) and (2.2), we find the following solutions of wave functions,

\[
\Phi(\lambda, c, d) = \delta \left[ c \left( \lambda - \sqrt{\lambda^2 + \delta} \right) e^{-\sqrt{\lambda^2 + \delta}(x-2i\lambda t)} + d \left( \lambda + \sqrt{\lambda^2 + \delta} \right) e^{\sqrt{\lambda^2 + \delta}(x-2i\lambda t)} \right] e^{i\delta t}, \tag{4.33a}
\]

\[
\Psi(\lambda, c, d) = ce^{-\sqrt{\lambda^2 + \delta}(x-2i\lambda t)} + de^{\sqrt{\lambda^2 + \delta}(x-2i\lambda t)} e^{-i\delta t}, \tag{4.33b}
\]
in which \( c \) and \( d \) are constants (or functions of \( \lambda \)). Define

\[
\phi = \left( \Phi(k_1, 1, d_1), \Phi(k_2, c_2, d_2), \ldots, \Phi(k_{2m+2}, c_{2m+2}, d_{2m+2}) \right)^T, \tag{4.34a}
\]

\[
\psi = \left( \Psi(k_1, 1, d_1), \Psi(k_2, c_2, d_2), \ldots, \Psi(k_{2m+2}, c_{2m+2}, d_{2m+2}) \right)^T. \tag{4.34b}
\]

Then, the quasi double Wronskians (3.9) composed by the above \( \phi \) and \( \psi \) provide solutions to the unreduced system (1.4) via the transformation (3.1) where the background solutions take
with regards to (4.34), the matrix \( A = \text{Diag}(k_1, k_2, \ldots, k_{2m+2}) \) in (3.6) and (3.7). One can also take

\[
\phi = \left( \Phi(k_1, c_1, d_1), \frac{\partial k_1}{1!} \Phi(k_1, c_1, d_1), \ldots, \frac{\partial^{2m+1} k_1}{(2m+1)!} \Phi(k_1, c_1, d_1) \right)^T, \tag{4.35a}
\]

\[
\psi = \left( \Psi(k_1, c_1, d_1), \frac{\partial k_1}{1!} \Psi(k_1, c_1, d_1), \ldots, \frac{\partial^{2m+1} k_1}{(2m+1)!} \Psi(k_1, c_1, d_1) \right)^T, \tag{4.35b}
\]

to get multiple pole solutions corresponding to \( A = J_{2m+2}(k_1) \), defined as in (4.24).

### 4.4.2 Vector \( \phi \) for the reduced equations

To present vector \( \phi \) for the reduced equations (2.3)-(2.6), let us define

\[
\phi = (\phi^+, \phi^-)^T, \quad \phi^\pm = (\phi^\pm(1), \phi^\pm(2), \ldots, \phi^\pm(m+1))^T, \tag{4.36a}
\]

\[
\psi = (\psi^+, \psi^-)^T, \quad \psi^\pm = (\psi^\pm(1), \psi^\pm(2), \ldots, \psi^\pm(m+1))^T, \tag{4.36b}
\]

where \( \phi^\pm(j) \) and \( \psi^\pm(j) \) are scalar functions. Note that a general form for the constraints (4.13), (4.16), (4.17) and (4.18) is

\[
\phi = T C^\epsilon \psi(\alpha x, \beta t), \quad \alpha, \beta = \pm 1, \quad \epsilon = 0, 1, \tag{4.37}
\]

where \( C \) stands for an operator for complex conjugation: \( C^\epsilon \psi = \psi^* \) when \( \epsilon = 1 \) and \( C^\epsilon \psi = \psi \) when \( \epsilon = 0 \).

There are only two types of \( T \) in Sec. 4.2, block skew-diagonal or block diagonal. We can present vector \( \phi \) according to the type of \( T \).

**Case 1: \( T \) being block skew-diagonal:** In this case,

\[
T = \begin{pmatrix} 0 & I_{m+1} \\ \gamma I_{m+1} & 0 \end{pmatrix}, \quad \gamma = \pm 1, \tag{4.38}
\]

which is for equation (2.3), (2.4) and (2.5), see Table 1 and Table 2. Vector \( \phi \) takes the form

\[
\phi = (\phi^+, C^\epsilon \psi^+ (\alpha x, \beta t))^T, \tag{4.39}
\]

where \( (\epsilon, \alpha, \beta) \) takes \((1, 1, 1)\) for (2.3), \((1, -1, 1)\) for (2.4) and \((0, 1, -1)\) for (2.5). When \( K_{m+1} \) is diagonal as given in (4.23), we have

\[
\phi^+_j = \Phi(k_j, c_j, d_j), \quad \psi^+_j = \Psi(k_j, c_j, d_j), \quad j = 1, 2, \ldots, m + 1, \tag{4.40}
\]

where \( \Phi \) and \( \Psi \) are defined in (4.33). When \( K_{m+1} = J_{m+1}(k_1) \) is the Jordan matrix as given in (4.24), we take

\[
\phi^+_j = \frac{\partial^{j-1}}{(j-1)!} \Phi(k_1, c, d), \quad \psi^+_j = \frac{\partial^{j-1}}{(j-1)!} \Psi(k_1, c, d), \quad j = 1, 2, \ldots, m + 1. \tag{4.41}
\]

**Case 2: \( T \) being block diagonal:** In this case,

\[
T = \begin{pmatrix} I_{m+1} & 0 \\ 0 & \gamma I_{m+1} \end{pmatrix}, \quad \gamma = \pm 1 \text{ or } \gamma = -i, \tag{4.42}
\]

which is associated with equation (2.6), equation (2.4) with \( \delta = -1 \) and equation (2.3) with \( \delta = 1 \), see Table 2 and (4.22).
Consider diagonal case where
\[
K_{m+1} = \text{Diag}(k_1, k_2, \cdots, k_{m+1}), \quad H_{m+1} = \text{Diag}(h_1, h_2, \cdots, h_{m+1}).
\] (4.43)

For the reverse-space-time NLS equation (2.6), we have
\[
\delta = 1: \quad \phi^+_{(j)} = \Phi(k_j, c_j^+, ic_j^+), \quad \phi^-_{(j)} = \Phi(h_j, c_j^-, -ic_j^-),
\] (4.44)
\[
\delta = -1: \quad \phi^+_{(j)} = \Phi(k_j, c_j^+, c_j^+), \quad \phi^-_{(j)} = \Phi(h_j, c_j^-, c_j^-),
\] (4.45)
where \(\Phi\) is defined in (4.33a), \(k_j, h_j \in \mathbb{C}, c_j^+\) and \(c_j^-\) can be arbitrary complex functions of \(k_j\) and \(h_j\), respectively. For the classical NLS equation (2.3) with \(\delta = 1\) and the reverse-space NLS equation (2.4) with \(\delta = -1\), we have
\[
\phi(j)^+ = \Phi(k_j, c_j^+, c_j^+), \quad \phi(j)^- = \Phi(h_j, c_j^-, c_j^-), \quad \text{when } T \text{ is (4.22b)},
\] (4.46)
\[
\phi(j)^+ = \Phi(k_j, c_j^+, ic_j^+), \quad \phi(j)^- = \Phi(h_j, c_j^-, -ic_j^-), \quad \text{when } T \text{ is (4.22c)},
\] (4.47)
where \(k_j, h_j \in \mathbb{iR}\) for equation (2.3) with \(\delta = 1\) and \(k_j, h_j \in \mathbb{R}\) for equation (2.4) with \(\delta = -1\).

When
\[
K_{m+1} = J_{m+1}(k_1), \quad H_{m+1} = J_{m+1}(h_1),
\]
we have
\[
\phi^+_{(s)} = \frac{\partial^{s-1}}{(s-1)!} \phi^+_{(1)}; \quad \phi^-_{(s)} = \frac{\partial^{s-1}}{(s-1)!} \phi^-_{(1)}, \quad s = 1, 2, \cdots, m + 1,
\] (4.48)
where \(\phi^+_{(1)}\) are defined as in (4.44), (4.45), (4.46) or (4.47), varying with the equation considered.

5 Dynamics of solutions

Nonzero background can bring new features for the classical and nonlocal NLS equations. In this section we analyze some solutions and illustrate their dynamics. The classical NLS equation (1.1) and nonlocal reverse-space NLS equation (2.4) with \(\delta = -1\) will serve as models.

5.1 The classical focusing NLS equation

It follows from the transformation (3.1) and the bilinear form (3.2) that the envelope \(|q|\) of the solution to the focusing NLS equation (1.1) with a background solution \(q_0\) can be expressed as (also see [12, 52])
\[
|q|^2 = |q_0|^2 + \partial_x^2 \ln f,
\] (5.1)
where \(f = |\hat{\phi}_m; T\hat{\phi}_m^*|\) is the quasi double Wronskian. For the focusing NLS equation (1.1), \(A = \text{Diag}(K_{m+1}, -K_{m+1}^*)\). \(T\) is given by (4.38) with \(\gamma = -1\), \(\phi\) is given by (4.39) with \((\epsilon, \alpha, \beta) = (1, 1, 1)\). In principle, solutions to equation (1.1) can be determined by the eigenvalue structure of \(K_{m+1}\). One can investigate these solutions according to the canonical form of \(K_{m+1}\).

5.1.1 Breathers

Case 1: \(K_{m+1}\) being a complex diagonal matrix

When \(K_{m+1}\) is a diagonal matrix (4.23), following (4.39) we have \(\phi = (\phi^+, \phi^-)^T\) where the entries in \(\phi^\pm\) are
\[
\phi^+_\(j\) = \Phi(k_j, c_j, d_j), \quad \phi^-_\(j\) = \psi^{+*}_\(j\), \quad \psi^+_\(j\) = \Psi(k_j, c_j, d_j),
\]
Φ and Ψ are given as (4.33). When the background solution takes \( q_0 = e^{-2it} \), we have

\[
\phi^+_j(\omega) = \begin{pmatrix} \phi_1^+ \\ \psi_1^+ \end{pmatrix} = \begin{pmatrix} (-k_j + \sqrt{k_j^2 - 1})e^{-\sqrt{k_j^2 - 1}(x - 2iku_j t) - \xi_j^{(0)}} - (k_j + \sqrt{k_j^2 - 1})e^{\sqrt{k_j^2 - 1}(x - 2iku_j t) + \xi_j^{(0)}} \\ e^{-\sqrt{k_j^2 - 1}(x - 2iku_j t) - \xi_j^{(0)}} + e^{\sqrt{k_j^2 - 1}(x - 2iku_j t) + \xi_j^{(0)}} \end{pmatrix} e^{-it},
\]

(5.2a)

\[
\psi^+_j(\omega) = \begin{pmatrix} \phi_1^+ \\ \psi_1^+ \end{pmatrix} = \begin{pmatrix} (-k_j + \sqrt{k_j^2 - 1})e^{-\sqrt{k_j^2 - 1}(x - 2iku_j t) - \xi_j^{(0)}} - (k_j + \sqrt{k_j^2 - 1})e^{\sqrt{k_j^2 - 1}(x - 2iku_j t) + \xi_j^{(0)}} \\ e^{-\sqrt{k_j^2 - 1}(x - 2iku_j t) - \xi_j^{(0)}} + e^{\sqrt{k_j^2 - 1}(x - 2iku_j t) + \xi_j^{(0)}} \end{pmatrix} e^{it},
\]

(5.2b)

where we have taken \( c_j = e^{-\xi_j^{(0)}}, d_j = e^{\xi_j^{(0)}} \) with \( \xi_j^{(0)} \) being an arbitrary functions of \( k_j \).

When \( m = 0 \) we have from (4.1b) that

\[
f = |\phi, T\phi^*|,
\]

(5.3)

where

\[
\phi = \begin{pmatrix} \phi_1^{(1)} \\ \psi_1^{(1)} \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

(5.4)

Note that in this case we have

\[-f = |\phi_1^{(1)}|^2 + |\psi_1^{(1)}|^2,
\]

which is positive definite when \( \phi_1^{(1)} \) and \( \psi_1^{(1)} \) are defined as in (5.2).

By calculation we find

\[
f = -(A_1 \cosh 2X_1(x, t) + A_2 \sinh 2X_1(x, t) + A_3 \cos 2X_2(x, t) - A_4 \sin 2X_2(x, t)),
\]

(5.5)

where

\[
A_1 = 2(1 + a_1^2 + b_1^2 + u_{11}^2 + u_{12}^2), \quad A_2 = 4(a_1 u_{11} + b_1 u_{12}),
\]

\[
A_3 = 2(1 + a_1^2 + b_1^2 + u_{11}^2 + u_{12}^2), \quad A_4 = 4(a_1 u_{12} - b_1 u_{11}),
\]

\[
X_1(x, t) = u_{11} x + 2B_1 t + \xi^{(0)}_{1R}, \quad B_1 = a_1 u_{12} + b_1 u_{11},
\]

\[
X_2(x, t) = u_{12} x + 2B_2 t + \xi^{(0)}_{1I}, \quad B_2 = b_1 u_{12} - a_1 u_{11},
\]

\[
k_1 = a_1 + ib_1, \quad \sqrt{k_1^2 - 1} = u_{11} + i u_{12}, \quad \xi^{(0)}_1 = \xi^{(0)}_{1R} + i \xi^{(0)}_{1I},
\]

and \( a_1, b_1, u_{11}, u_{12}, \xi^{(0)}_{1R}, \xi^{(0)}_{1I} \in \mathbb{R} \). Since \( \sqrt{k_1^2 - 1} \) is a double-valued function of \( k \), here we consider the branch

\[
u_{11} = \sqrt{(a_1^2 - b_1^2 - 1)^2 + (2a_1 b_1)^2} \cos \left( \frac{1}{2} (\arg(a_1 + 1 + ib_1) + \arg(a_1 - 1 + ib_1)) \right),
\]

\[
u_{12} = \sqrt{(a_1^2 - b_1^2 - 1)^2 + (2a_1 b_1)^2} \sin \left( \frac{1}{2} (\arg(a_1 + 1 + ib_1) + \arg(a_1 - 1 + ib_1)) \right)
\]

without loss of generality. Further we introduce

\[
\tan \theta = \frac{A_4}{A_3},
\]

such that (5.5) is rewrite as

\[
f = - \left( A_1 \cosh 2X_1(x, t) + A_2 \sinh 2X_1(x, t) + \sqrt{A_3^2 + A_4^2} \cos(2X_2(x, t) + \theta) \right).
\]

(5.6)

Noticing that \( A_1 > |A_2| > 0 \) for all \( k_1 \neq 0 \), from the above expression and (5.1), \(|q|^2\) behaves like a wave traveling along the line \( X_1 = 0 \) and oscillating periodically with a period determined
by \(2X_2 + \theta = 2j\pi, j \in \mathbb{Z}\). Note that the case \(a_1 = 0\) or \(a_1 = \pm 1, b_1 = 0\) yields \(|q|^2 = 1\), which is trivial and we do not consider.

To see more details, we rewrite (5.6) in terms of the following coordinates,
\[
(x, z = t + \frac{u_{11}}{2B_1} x + \frac{\xi^{(0)}_{1R}}{2B_1}), \tag{5.7}
\]
which gives rise to
\[
f = - \left\{ A_1 \cosh(4B_1 z + 2\xi^{(0)}_{1R}) + A_2 \sinh(4B_1 z + 2\xi^{(0)}_{1R}) + \sqrt{A_3^2 + A_4^2} \cos \left( 4B_2 \left( z + \frac{a_1(u_{11}^2 + u_{12}^2)}{2B_1 B_2} x + \frac{\xi^{(0)}_{1I}}{2B_2} - \frac{\xi^{(0)}_{1R}}{B_1} + \theta \right) \right) \right\}. \tag{5.8}
\]
In terms of (5.7) we can see that (5.1) with (5.8) provides a breather traveling along the straight line \(z = \) constant and oscillating with a period with respect to \(x\):
\[
P = \left| \frac{2\pi B_1}{a_1(u_{11}^2 + u_{12}^2)} \right|. \tag{5.9}
\]
An illustration is given in Fig.1(a), which describes a moving breather. Such a breather is also known as the Tajiri-Watanabe breather (see Fig.4 in [52]). In 1998 Tajiri and Watanabe derived and studied breathers of the focusing NLS equation using Hirota’s bilinear method [52].

Back to the expression (5.6). Stationary breathers appear when \(b_1 = 0\). More precisely, when \(|a_1| > 1\) and \(b_1 = 0\), which leads to \(u_{11} = \sqrt{a_1^2 - 1}\) and \(u_{12} = 0\), we have \(B_1 = 0\) and then \(X_1(x, t) = u_{11} x + \xi^{(0)}_{1R}\). In this case we can have a breather stationary with respect to \(x\), where
\[
f = - \left( 2a_1^2 \cosh 2(\sqrt{a_1^2 - 1} x + \xi^{(0)}_{1R}) + 4a_1 \sqrt{a_1^2 - 1} \sinh 2(\sqrt{a_1^2 - 1} x + \xi^{(0)}_{1R}) + 4 \cos 2(-2a_1 \sqrt{a_1^2 - 1} + \xi^{(0)}_{1I}) \right). \tag{5.10}
\]
In this case, we have a stationary breather oscillating in time with period \(P_t = \frac{\pi}{|2a_1 \sqrt{a_1^2 - 1}|}\), which is known as the Kuznetsov-Ma breather [36, 41]. It is described in Fig.1(b). In another case where \(|a_1| < 1\) and \(b_1 = 0\), which leads to \(u_{11} = 0\) and \(u_{12} = \sqrt{1 - a_1^2}\), from (5.6) we have
\[
f = - \left( 2 \cosh 2 \left( 2a_1 \sqrt{1 - a_1^2} t + \xi^{(0)}_{1R} \right) + 4a_1^2 \cos \left( 2(\sqrt{1 - a_1^2} x + \xi^{(0)}_{1I}) + \theta \right) \right), \tag{5.11}
\]
with \(\tan \theta = \frac{1 - a_1^2}{2a_1}\). This will gives rise to a breather traveling along the line \(t = -\frac{\xi^{(0)}_{1I}}{2a_1 \sqrt{1 - a_1^2}}\) and being periodic with respect to \(x\) with the period \(P_x = \frac{\pi}{\sqrt{1 - a_1^2}}\). Such a breather is known as the Akhmediev breather [11], which was first studied by Akhmediev in [11] and then bear his name. The Akhmediev breather is perpendicular to the Kuznetsov-Ma breather, as depicted in Fig.1 (b) and (c).

The envelop of two-breather solution can be obtained via (5.1) by taking \(m = 1\) in quasi double Wronskians (4.1b), i.e.
\[
f = |\phi, \phi^*; \psi, \psi^*|, \tag{5.12a}
\]
with
\[
\phi = \left( \phi^{(1)}_1, \phi^{(2)}_1; \psi^{(1)}_1, \psi^{(2)}_1 \right)^T, \quad \psi = \left( \psi^{(1)}_1, \psi^{(2)}_1; -\phi^{(1)*}_1, -\phi^{(2)*}_1 \right)^T. \tag{5.12b}
\]
Fig. 1. Shape and motion of one breather solution of the focusing NLS equation (1.1) with a background solution \( q_0 = e^{-2it} \). The envelope is given by (5.1) where \( f \) is (5.5). (a) 3D-plot for a moving breather associated with (5.8) for \( a_1 = 0.3, b_1 = -0.3, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = 0 \). (b) 3D-plot for the Kuznetsov-Ma breather associated with (5.10) for \( a_1 = 1.2, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = 0 \). (c) 3D-plot for the Akhmediev breather associated with (5.11) for \( a_1 = 0.5, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = 0 \).

in which \( \phi_{(j)}^+ \) and \( \psi_{(j)}^+ \) are defined as in (5.2). There are various types of two-breather interaction. As examples Fig. 2 illustrates interactions between two Tajiri-Watanabe breathers, interaction of the Akhmediev breather and Kuznetsov-Ma breather and interaction of two Akhmediev breathers, in Fig. 2 (a), (b) and (c), respectively.

Fig. 2. Shape and motion of two breather interactions of the focusing NLS equation (1.1) with a background solution \( q_0 = e^{-2it} \). The envelope is given by (5.1) where \( f \) is (5.12). (a) 3D-plot for \( a_1 = 0.3, b_1 = 0.5, a_2 = 0.3, b_2 = -0.5, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = \xi_{2R}^{(0)} = \xi_{2I}^{(0)} = 0 \). (b) 3D-plot for \( a_1 = 1.2, b_1 = 0, a_2 = 0.8, b_2 = 0, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = 0, \xi_{2R}^{(0)} = 1, \xi_{2I}^{(0)} = 0 \). (c) 3D-plot for \( a_1 = 0.3, b_1 = 0, a_2 = 0.5, b_2 = 0, \xi_{1R}^{(0)} = 2, \xi_{1I}^{(0)} = 0, \xi_{2R}^{(0)} = -2, \xi_{2I}^{(0)} = 0 \).

Case 2: \( K_{m+1} \) being a Jordan matrix

Let \( \phi_{(1)}^+ \) and \( \psi_{(1)}^+ \) be defined as in (5.2), and we define

\[
\phi_{(j)}^+ = \frac{\partial^{j-1}_{k_1}}{(j-1)!} \phi_{(1)}^+, \quad \psi_{(j)}^+ = \frac{\partial^{j-1}_{k_1}}{(j-1)!} \psi_{(1)}^+,
\]

\[
\phi_{(j)}^- = \psi_{(j)}^{+*}, \quad \psi_{(j)}^- = -\phi_{(j)}^{+*}, \quad j = 1, 2, \cdots, m + 1.
\]

The corresponding \( f \) composed by the above elements yields breathers when \( K_{m+1} \) is the Jordan matrix \( J_{m+1}(k_1) \) as given in (4.24). For the simplest Jordan block solution of the focusing NLS
equation (1.1) with the background solution \( q_0 = e^{-2it} \), we have \( m = 1 \) and \( f \) composed by

\[
\phi = \left( \phi^+_1, \partial_{k_1} \phi^+_1; \psi^+_1, \left( \partial_{k_1} \psi^+_1 \right)^* \right)^T, \quad \psi = \left( \psi^+_1, \partial_{k_1} \psi^+_1; -\phi^+_1, \left( -\partial_{k_1} \phi^+_1 \right)^* \right)^T. \tag{5.13}
\]

Such a breather is described in Fig. 3.

**Fig. 3.** Shape and motion of Jordan block solution of the focusing NLS equation (1.1) with a background solution \( q_0 = e^{-2it} \). The envelope is given by (5.1) where \( f \) is composed by (5.13). (a) 3D-plot for \( a_1 = 0.8, b_1 = -0.15, s_{1R} = \xi_1^{(0)} = 0, \delta_1^{(0)} = 0 \). (b) 3D-plot for \( a_1 = 1.2, b_1 = 0, s_{1R} = \xi_1^{(0)} = \xi_1^{(0)} = 0 \).

### 5.1.2 Rational solutions and rogue waves

Rational solutions can be obtained as a limit case of breathers when taking \( k_j \to 1 \). This can be seen from the expression (5.2). Since the Akhmediev breathers and Kuznetsov-Ma breathers are generated when \( b_j = 0 \), rational solutions can also be understood as a limit of these two types of breathers. In principle, for getting rational solutions, in \( A \) we should take \( K_{m+1} = J_{m+1}(1) \), but the limit procedure needs to be elaborated.

Let us consider (4.33) and rewrite them in the form

\[
\Phi(\kappa, c, d) = \left( c(\kappa)(-\sqrt{\kappa^2 + 1} + \kappa)e^{-\kappa(x-2i\sqrt{\kappa^2+1}t)} - d(\kappa)(\sqrt{\kappa^2 + 1} + \kappa)e^{\kappa(x-2i\sqrt{\kappa^2+1}t)} \right)e^{-it},
\]

\[
\Psi(\kappa, c, d) = \left( c(\kappa)e^{-\kappa(x-2i\sqrt{\kappa^2+1}t)} - \xi_j^{(0)} + d(\kappa)e^{\kappa(x-2i\sqrt{\kappa^2+1}t)} + \xi_j^{(0)} \right)e^{it},
\]

where we have taken \( \delta = -1 \) and introduce \( \kappa = \sqrt{\kappa^2 - 1} \), and we take \( c(\kappa) \) and \( d(\kappa) \) to be functions of \( \kappa \). Impose constraint \( c(\kappa) = -d(-\kappa) \) and take formal expressions

\[
c(\kappa) = \sum_{j=0}^{\infty} s_j \kappa^j, \quad d(\kappa) = \sum_{j=0}^{\infty} (-1)^{j+1} s_j \kappa^j,
\]

where \( s_j \) are arbitrary complex parameters. We denote the above \( \Phi(\kappa, c, d) \) and \( \Psi(\kappa, c, d) \) with (5.15) by \( \Phi_{\text{odd}} \) and \( \Psi_{\text{odd}} \) respectively. Expand them in terms of \( \kappa \) as

\[
\Phi_{\text{odd}} = \sum_{j=0}^{\infty} R_{2j+1} \kappa^{2j+1}, \quad \Psi_{\text{odd}} = \sum_{j=0}^{\infty} S_{2j+1} \kappa^{2j+1},
\]

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in which
\[ R_{2j+1} = \frac{1}{(2j+1)!} \frac{\partial^{2j+1}}{\partial t^{2j+1}} \Phi_{\text{odd}}|_{\kappa=0}, \quad S_{2j+1} = \frac{1}{(2j+1)!} \frac{\partial^{2j+1}}{\partial t^{2j+1}} \Psi_{\text{odd}}|_{\kappa=0}, \quad (j = 0, 1, 2, \cdots). \]

Define
\[ \phi^{\text{odd}} = (R_1, R_3, \cdots, R_{2m+1}, S_1^\ast, S_3^\ast, \cdots, S_{2m+1}^\ast)^T, \]
\[ \psi^{\text{odd}} = (S_1, S_3, \cdots, S_{2m+1}, -R_1^\ast, -R_3^\ast, \cdots, -R_{2m+1}^\ast)^T = T(\phi^{\text{odd}})^*, \]
where \( T \) takes the form \((4.38)\) with \( \gamma = -1 \). It can be verified that \( \phi^{\text{odd}} \) satisfies equation \((4.2)\) where \( q_0 = e^{-2it} \), \( \delta = -1 \), \( T \) is given by \((4.38)\) with \( \gamma = -1 \), and \( A = \text{Diag}(K_{m+1}, -K_{m+1})^\ast \)
\[ K_{m+1} = \begin{pmatrix}
\alpha_0 & 0 & 0 & \cdots & 0 \\
\alpha_2 & \alpha_0 & 0 & \cdots & 0 \\
\alpha_4 & \alpha_2 & \alpha_0 & \cdots & 0 \\
\vdots & \vdots & \iddots & \iddots & \vdots \\
\alpha_{2m} & \alpha_{2m-2} & \cdots & \alpha_2 & \alpha_0
\end{pmatrix}, \]
in which \( \alpha_{2j} = \frac{1}{(2j)!} \frac{\partial^j}{\partial t^j} \sqrt{\kappa^2 + 1}|_{\kappa=0}, \quad (j = 0, 1, 2, \cdots) \). Note also that \( A \) and \( T \) satisfy \((4.3)\) with \( \delta = -1 \). Thus, the quasi double Wronskians
\[ f = |\hat{\phi}_m^{\text{odd}}, \hat{\psi}_m^{\text{odd}}|, \quad g = 2|\hat{\phi}_{m+1}^{\text{odd}}, \hat{\psi}_{m-1}^{\text{odd}}|, \]
provide rational solutions to the focusing NLS equation \((1.1)\) via \((3.1)\) and the envelop via \((5.1)\).

The first order rational solution (for \( m = 0 \)) is
\[ q = -\left( 1 + \frac{4it - 1}{\tilde{x}^2 + 4t^2 + \frac{1}{4}} \right) e^{-2it}, \]
where \( \tilde{x} = x + \frac{s_0 - 2s_1}{2s_0} \) with \( s_0, s_1 \) being coefficients of \( c(k) \). Here we take \( s_0, s_1 \in \mathbb{R} \) for simplicity. We refer to it as the Peregrine soliton since it was first derived by Peregrine in \([46]\). Its envelope \( |q| \) is localized in both space and time. It is also known as a rogue wave of the focusing NLS equation. The maximum value of \( |q| \) is 3, occurring at \((x, t) = \left( -\frac{s_0 - 2s_1}{2s_0}, 0 \right) \), three times hight of the background \(|q_0| = 1\). The envelope is depicted in Fig.4(a).

The general second order rational solution can be obtained from
\[ q = q_0 + \frac{g}{f}, \quad g = 2|\phi^{\text{odd}}, \phi^{\text{odd}}_1, \phi^{\text{odd}}_2, \psi^{\text{odd}}|, \quad f = |\phi^{\text{odd}}, \phi^{\text{odd}}_1, \psi^{\text{odd}}, \psi^{\text{odd}}_1|, \]
where
\[ \phi^{\text{odd}} = (R_1, R_2; S_1^\ast, S_2^\ast)^T, \quad \psi^{\text{odd}} = (S_1, S_2; -R_1^\ast, R_2^\ast)^T. \]
We skip explicit expression of \( q \). The envelope of a typical second order rational solution is shown in Fig.4(b) with a symmetric shape and having a single maximum 5. In general, the maximum amplitude of a \( n \)-th-order rogue wave with one central main peak is \( 2n + 1 \) times of the height of the amplitude of the background plane wave \([9, 56]\), (also see \([56]\) where rogue wave with such pattern is called a “fundamental rogue wave”). The envelope of another typical second order rational solution has three peaks, as shown in Fig.4(c).

The third order rational solutions is obtained by taking \( m = 2 \) in \((5.20)\). Without presenting formulae, we depict some different patterns of the envelope of these solutions in Fig.5. Fig.5 (a)
Fig. 4. Shape and motion of rational solutions of the focusing NLS equation (1.1). (a) Envelope of the first order rational solution given by (5.21) with \( s_0 = 1, s_1 = 0.5 \). (b) Envelope of the second order rational solution given by (5.22) with \( s_0 = 2, s_1 = 0.5 \). (c) Envelope of the second order rational solution given by (5.22) with \( s_0 = 1, s_1 = 0, s_2 = 10, s_3 = -20 \).

shows the pattern where there is only one central main peak, Fig.5 (d) and Fig.5 (e) show the pattern consisting basically of 6 well-separated fundamental part on a unit background, which are located on a triangle and a pentagon, respectively. Another two interesting patterns are shown in Fig.5 (b) and Fig.5 (c). Thus, it indicates that higher-order rogue waves contain richer structures. Note that recently it was found the pattern of rogue waves is related to the roots of Yablonskii-Vorob’ev polynomials [58, 59].

Apart from (5.18), one can also introduce Wronskian entries by imposing \( c(\kappa) = d(-\kappa) \), i.e.,

\[
c(\kappa) = \sum_{j=0}^{\infty} s_j \kappa^j, \quad d(\kappa) = \sum_{j=0}^{\infty} (-1)^j s_j \kappa^j,
\]

such that \( \Phi(\kappa, c, d) \) and \( \Psi(\kappa, c, d) \) given in (5.14) (denoted by \( \Phi_{\text{even}} \) and \( \Psi_{\text{even}} \), respectively) can be expanded as

\[
\Phi_{\text{even}} = \sum_{j=0}^{\infty} R_{2j} \kappa^{2j}, \quad \Psi_{\text{even}} = \sum_{j=0}^{\infty} S_{2j} \kappa^{2j},
\]

where

\[
R_{2j} = \frac{1}{(2j)!} \frac{\partial^{2j} \Phi_{\text{even}}}{\partial \kappa^{2j}}|_{\kappa=0}, \quad S_{2j} = \frac{1}{(2j)!} \frac{\partial^{2j} \Psi_{\text{even}}}{\partial \kappa^{2j}}|_{\kappa=0}, \quad (j = 0, 1, 2, \cdots).
\]

Then the vectors for the quasi double Wronskian are taken as

\[
\phi_{\text{even}} = (R_0, R_2, \cdots, R_{2m}, S_0^*, S_2^*, \cdots, S_{2m}^*)^T, \quad \psi_{\text{even}} = (S_0, S_2, \cdots, S_{2m}, -R_0^*, -R_2^*, \cdots, -R_{2m}^*)^T = T(\phi_{\text{even}})^*, \quad (5.26a)
\]

where \( T \) is given by (4.38) with \( \gamma = -1 \). In this case \( m = 0 \) does not lead to a nontrivial solution but the solutions obtained by taking \( m = 1 \) and \( m = 2 \) correspond to (5.21) and (5.22a), which are the first order and second order rational solutions derived using \( \phi_{\text{odd}} \) and \( \psi_{\text{odd}} \).

One may conjecture that the \( m \)-th order rational solution derived using \( \phi_{\text{odd}} \) and \( \psi_{\text{odd}} \) corresponds to the \( (m+1) \)-th order rational solution derived using \( \phi_{\text{even}} \) and \( \psi_{\text{even}} \). Similar connection is proved in the rational solutions of the discrete KdV-type equations, see [62]. We also note that the parameters \( \{s_j\} \) (or \( c(\kappa) \)) play the same roles as the lower triangular Toeplitz
Fig. 5. Shape and motion of the envelope of the third order rational solution of the focusing NLS equation (1.1). (a) 3D plot for $s_0 = 1, s_1 = 0, s_2 = 0, s_3 = 0, s_4 = 0, s_5 = 0$. (b) 3D plot for $s_0 = 1, s_1 = 1, s_2 = 0, s_3 = 0, s_4 = 0, s_5 = 0$. (c) 3D plot for $s_0 = 1, s_1 = 0, s_2 = 0, s_3 = 1, s_4 = 0, s_5 = 0$. (d) 3D plot for $s_0 = -1, s_1 = 0, s_2 = 0, s_3 = 100, s_4 = -200, s_5 = 0$. (e) 3D plot for $s_0 = 1, s_1 = 0, s_2 = 100, s_3 = 1, s_4 = 0, s_5 = 200$.

matrices, cf.\cite{63, 64}. An $(m+1)$-th order lower triangular Toeplitz matrix $P_{m+1}$ is defined as

$$P_{m+1} = T_{m+1}[t_j]_{j=0}^{m} = \begin{pmatrix} t_0 & 0 & 0 & \cdots & 0 \\ t_1 & t_0 & 0 & \cdots & 0 \\ t_2 & t_1 & t_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ t_m & t_{m-1} & \cdots & t_1 & t_0 \end{pmatrix}, \quad t_j \in \mathbb{C}, \quad (5.27)$$

which commutes with $K_{m+1}$ defined in (5.19). For the block diagonal matrix $Q = \text{Diag}(P_{m+1}, P_{m+1}^*)$, when $T$ is given by (4.38) with $\gamma = -1$, and $A = \text{Diag}(K_{m+1}, -K_{m+1}^*)$ with (5.19), we have

$$AQ = QA, \quad QT = TQ^*.$$  

This indicates that for any $\phi$ that satisfies (4.2) with the above $A$ and $T$, $\tilde{\phi} = Q\phi$ is also a solution of (4.2). Moreover, if $\phi^{\text{odd}}$ in (5.18a) is derived with $c(\kappa) = 1$, then in $\tilde{\phi} = Q\phi$, the parameters $\{t_j\}$ play the exactly same roles as $\{s_j\}$. In \cite{63}, for the KdV equation, the relation between $P_{m+1}$ and $c(\kappa)$ is described, see Sec.2 of \cite{63}.

5.2 The defocusing reverse-space nonlocal NLS equation

In this section we investigate solutions of the defocusing reverse-space nonlocal NLS equation

$$iq_t = q_{xx} - 2q^2q^*(-x) \quad (5.28)$$

with the background solution $q_0 = e^{2it}$. Note that this is the equation (2.4) with $\delta = 1$. 

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5.2.1 Solitons and doubly periodic solutions

Solution to equation (5.28) with a background solution \( q_0 \) is written as
\[
q = q_0 + \frac{g}{f}, \quad f = |\hat{\phi}_m; \hat{\psi}_m|, \quad g = 2|\hat{\phi}_{m+1}; \hat{\psi}_{m-1}|,
\]  
(5.29)
where we take \( q_0 = e^{2it} \). Consider the simplest case, \( m = 0 \). From the results in Table 1 and in Sec.4.4.2, we have
\[
f = \begin{vmatrix}
\phi_{(1)}^+(x) & \psi_{(1)}^+(x) \\
\phi_{(1)}^+(x) & -\psi_{(1)}^+(x)
\end{vmatrix}, \quad g = 2 \begin{vmatrix}
\phi_{(1)}^+(x) & k_1 \phi_{(1)}^+(x) \\
\psi_{(1)}^+(x) & k_1^* \psi_{(1)}^+(x)
\end{vmatrix},
\]
(5.30a)

where
\[
\phi_{(1)}^+(x) = \left( \alpha_1 e^{\sqrt{k_1^2 + 1} (x - 2ik_1 t)} + \beta_1 e^{-\sqrt{k_1^2 + 1} (x - 2ik_1 t)} \right) e^{it},
\]
(5.30b)
\[
\psi_{(1)}^+(x) = \left( \alpha_1 e^{\sqrt{k_1^2 + 1} (x - 2ik_1 t)} - \beta_1 (\sqrt{k_1^2 + 1} + k_1) e^{-\sqrt{k_1^2 + 1} (x - 2ik_1 t)} \right) e^{-it},
\]
(5.30c)

and in \( \Phi(k, c, d) \) and \( \Psi(k, c, d) \) defined in (4.33) we have taken \( \delta = 1 \),
\[
c(k) = -\beta_1 (k) (\sqrt{k_1^2 + 1} + k), \quad d(k) = \alpha_1 (k) (\sqrt{k_1^2 + 1} - k)
\]
with \( \alpha_1 \) and \( \beta_1 \) as arbitrary functions of \( k \).

The envelope \(|q|\) of some solutions resulting from (5.30) is depicted in Fig.6, which exhibits features of two-soliton interactions, although the solution is from the simplest case, \( m = 0 \). In the following we implement asymptotic analysis so as to understand such features. To avoid singular and trivial solutions, we consider the special case where \( k_1 = ib, \ b \in \mathbb{R} \). It turns out that the solution can be classified according to the sign of \( 1 - b^2 \).

**Case 1: \(|b| < 1\)**

We write solution \( q \) in terms of the following coordinates,
\[
(X_1 = x + 2bt, \ t).
\]

This gives rise to
\[
q = \frac{G}{F},
\]
where
\[
G = e^{2it} \left\{ 1 + 2ib \left[ (\sqrt{1 - b^2} + ib)e^{4\sqrt{1 - b^2}t} - \beta^* (\sqrt{1 - b^2} - ib)e^{-4\sqrt{1 - b^2}t} \\
+ \beta (\sqrt{1 - b^2} + ib)e^{-2\sqrt{1 - b^2}X_1 + 4ib\sqrt{1 - b^2}t} - \beta^* (\sqrt{1 - b^2} - ib)e^{2\sqrt{1 - b^2}X_1 - 4ib\sqrt{1 - b^2}t} \right] \right\},
\]
\[
F = e^{4\sqrt{1 - b^2}t} + \beta^* e^{-4\sqrt{1 - b^2}t} + \beta b (b - i\sqrt{1 - b^2}) e^{-2\sqrt{1 - b^2}X_1 + 4ib\sqrt{1 - b^2}t} \]
\[+ \beta^* b (b + i\sqrt{1 - b^2}) e^{2\sqrt{1 - b^2}X_1 - 4ib\sqrt{1 - b^2}t},
\]
and we have taken \( \beta = \frac{\beta_1}{\alpha_1} \). When keeping \( X_1 \) to be constant, we find
\[
|q|^2 \to 1 - \frac{2\sqrt{1 - b^2} \beta_I}{\text{sgn}(b) \left( |\beta| \cosh(2\sqrt{1 - b^2} X_1 - \ln(|\beta| b^{\pm 1})) + \text{sgn}(b) (\sqrt{1 - b^2} \beta_I + b \beta_R) \right)}, \quad bt \to \pm \infty.
\]

Similarly, in terms of the coordinate
\[
(X_2 = x - 2bt, \ t),
\]

**Case 2:**
we get

\[ |q|^2 \to 1 + \frac{2((2b^2 - 1)\beta_I - 2b\sqrt{1 - b^2}\beta_R)\sqrt{1 - b^2}}{\text{sgn}[b]\left(|\beta| \cosh(2\sqrt{1 - b^2}X_2 + \ln(|\beta b^2|)\right) + \text{sgn}[b](\sqrt{1 - b^2}\beta_I + b\beta_R)} \}, \quad bt \to \pm \infty. \]

Here we have taken \( \beta = \beta_R + i\beta_I, \beta_R, \beta_I \in \mathbb{R} \). Note that here we do not have formula (5.1), which indicates the background \( |q_0| \) for the envelope \( |q| \) in the classical case, however, since the background solution is \( q_0 = e^{2it} \) which yields \( |q_0| = 1 \), the above asymptotic results indicate that \( |q| \) has a background plane \( |q| = 1 \), which is equal to \( |q_0| \).

For convenience let us call the above two solitons \( X_1 \)-soliton and \( X_2 \)-soliton, respectively. We further impose restriction \( \text{sgn}[b]\left(\sqrt{1 - b^2}\beta_I + b\beta_R\right) > 0 \) so that the solution has no singularity. Then we have the following results on the asymptotic behaviors of \( X_j \)-solitons.

**Theorem 4.** Assume that \( \text{sgn}[b]\left(\sqrt{1 - b^2}\beta_I + b\beta_R\right) > 0 \). In case \( |b| < 1 \) and when \( bt \to \pm \infty \), the envelope of \( X_1 \)-soliton asymptotically travels on a background \( |q| = 1 \) and with characteristics

trajectory: \( x(t) = \frac{1}{2\sqrt{1 - b^2}} \ln |\beta b^{\pm 1}| - 2bt \),

velocity: \( x'(t) = -2b \),

amplitude: \( 1 - \frac{2\sqrt{1 - b^2}\beta_I}{\text{sgn}[b]\left(|\beta| + \text{sgn}[b](\sqrt{1 - b^2}\beta_I + b\beta_R)\right)} \);

when \( bt \to \pm \infty \), the \( |q|^2 \) of \( X_2 \)-soliton asymptotically travels on a background \( |q|^2 = 1 \) and with characteristics

top trace: \( x(t) = -\frac{1}{2\sqrt{1 - b^2}} \ln |\beta b^{\pm 1}| + 2bt \),

velocity: \( x'(t) = 2b \),

amplitude: \( 1 + \frac{2((2b^2 - 1)\beta_I - 2b\sqrt{1 - b^2}\beta_R)\sqrt{1 - b^2}}{\text{sgn}[b]\left(|\beta| + \text{sgn}[b](\sqrt{1 - b^2}\beta_I + b\beta_R)\right)} \).

Each soliton gets a phase shift \(-2\ln |b|\) due to interaction.

The value of amplitude of each soliton can be either larger or less than the background \( |q_0| = 1 \). This indicates various types of interactions. Fig.6 exhibits three types of interactions. It is also notable that the value of amplitude of each soliton can be even equal to the background \( |q_0| = 1 \), which means the soliton can vanish on the background. This happens when \( \beta_I = 0 \) for the \( X_1 \)-soliton and when \( (2b^2 - 1)\beta_I - 2b\sqrt{1 - b^2}\beta_R = 0 \) for the \( X_2 \)-soliton. Illustrations are given in Fig.7. Note that such a behavior usually appear in some coupled system and known as “ghost soliton”, cf. [31].

**Case 2: |b| > 1**

In this case the one soliton solution of equation (5.28) resulting from (5.30) can be written as

\[ q = \frac{G}{F}, \quad (5.31a) \]
Fig. 6. Interactions of the $X_j$-soliton for the defocusing reverse-space NLS equation (5.28): shape and motion of the envelope $|q|^2$ resulting from (5.30). (a) 3D plot for $b = -0.8, \beta = 1.4 - i$. (b) 3D plot for $b = 0.3, \beta = -0.5 + 0.1i$. (c) 3D plot for $b = 0.3, \beta = -1.6 - 1.4i$.

Fig. 7. Interactions of the $X_j$-soliton for the defocusing reverse-space NLS equation (5.28): shape and motion of the envelope $|q|^2$ resulting from (5.30). (a) 3D plot for $b = 0.5, \beta = 0.5 - \sqrt{2}i$. (b) 3D plot for $b = 0.5, \beta = -0.5 + \sqrt{2}i$. (c) 3D plot for $b = 0.5, \beta = -0.5$. (d) 3D plot for $b = 0.5, \beta = 0.5$. 

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where
\[
G = e^{2it} \left\{ 1 + 2b \left[ (\sqrt{b^2 - 1} - b) - \kappa \kappa^* (\sqrt{b^2 - 1} + b) \cos(2\sqrt{b^2 - 1}x) \right] + i(\sqrt{b^2 - 1} - b + \kappa \kappa^* (\sqrt{b^2 - 1} + b) \sin(2\sqrt{b^2 - 1}x) \right] \right\},
\]
(5.31b)
\[
F = -b(\sqrt{b^2 - 1} - b) - \kappa \kappa^* (\sqrt{b^2 - 1} + b) \cos(2\sqrt{b^2 - 1}x - i(\kappa - \kappa^*) \sin(4\sqrt{b^2 - 1}t).
\]
(5.31c)
Solution (5.31) is doubly periodic with respect to both \(x\) and \(t\) and the periods are
\[
P_x = \frac{\pi}{\sqrt{b^2 - 1}}, \quad P_t = \frac{\pi}{2b\sqrt{b^2 - 1}}.
\]
(5.32)
The solution is plotted in Fig.8. Although there are some results on doubly periodic solutions, which are constructed by Jacobi elliptic functions in general, to our knowledge, the doubly periodic solution (5.31) to the defocusing reverse-space NLS equation (5.28) is not reported before.

**Fig. 8.** Envelope of doubly periodic solution (5.31) for the defocusing reverse-space NLS equation (5.28), for \(b = -1.25, \kappa = 1\).

### 5.2.2 Rational solutions

According to Table 1, for equation (5.28), we have \(A = \text{Diag}(K_{m+1}, K_{m+1}^*)\) and \(T\) given by (4.38) with \(\gamma = -1\). In the following we consider solutions resulting from \(K_{m+1} = J_{m+1}(i)\).

Consider \(\Phi(k, c, d)\) and \(\Psi(k, c, d)\) defined in (4.33) where we take \(\lambda = k\) and \(\delta = 1\). Expanding them in terms of \((k - i)\) yields
\[
\Phi(k, c, d) = \sum_{j=0}^{\infty} R_{j+1}(k - i)^j, \quad \Psi(k, c, d) = \sum_{j=0}^{\infty} S_{j+1}(k - i)^j,
\]
(5.33)
where
\[
R_{j+1} = \frac{1}{j!} \frac{\partial^j}{\partial k^j} \Phi(k, c, d) \bigg|_{k=i}, \quad S_{j+1} = \frac{1}{j!} \frac{\partial^j}{\partial k^j} \Psi(k, c, d) \bigg|_{k=i}, \quad (j = 0, 1, 2, \cdots).
\]
(5.34)
Define
\[
\phi = (R_1, R_2, \cdots, R_{m+1}, S_1(-x), S_2(-x), \cdots, S_{m+1}(-x))^T,
\]
(5.35a)
\[
\psi = (S_1, S_2, \cdots, S_{m+1}, -R_1(-x), -R_2(-x), \cdots, -R_{m+1}(-x))^T = T\phi^*(-x).
\]
(5.35b)
It can be verified that $\phi$ satisfies (4.5) with the above mentioned $A$, $T$, $\delta = 1$ and $q_0 = e^{2it}$. This means, with such $\phi$ and $\psi$ as basic column vectors, by the formula

$$q = q_0 + \frac{g}{f} = e^{2it} + \frac{g}{f}, \quad (5.36)$$

the quasi double Wronskians

$$f = |\hat{\phi}_m; \hat{\psi}_m|, \quad g = 2|\hat{\phi}_{m+1}; \hat{\psi}_{m-1}|$$

provide rational solutions to defocusing reverse-space NLS equation (5.28).

The first order rational solution (for $m = 0$) is provided by

$$f = \left| \frac{R_1}{S_1^*(x)} - \frac{S_1}{R_1^*(x)} \right|, \quad g = 2 \left| \frac{R_1}{S_1^*(x)} - i \frac{S_1}{R_1^*(x)} \right| \quad (5.37a)$$

with

$$R_1 = 1 + (i + d)(x + 2t)e^{it}, \quad S_1 = (d + (1 - id)(x + 2t))e^{-it}, \quad (5.37b)$$

where we have taken $c = 1$ and $d$ being a constant. Explicit form of the first order rational solution is given by

$$q = \frac{G}{F}, \quad (5.38a)$$

where

$$G = e^{2it} \left\{ 1 + 4i[d^* + (i + d)d^*(x + 2t)] + i(-i + d^*)(-x + 2t) \right\},$$

$$F = 1 + |d|^2 + (i + d)(1 - id^*)(x + 2t) + (-i + d^*)(1 + id)(-x + 2t) + 2i + d|^2(x + 2t)(-x + 2t). \quad (5.38b)$$

Some illustrations are given in Fig.9 which exhibit soliton interactions. To understand the dynamics we investigate asymptotic behaviors of the above rational solution. We introduce a new coordinate

$$(X_1 = x + 2t, \quad t),$$

then rewrite (5.38) in this coordinate, keep $X_1$ to be constant and let $t \to \pm \infty$. It follows that

$$|q(X_1, t)|^2 \to 1 + \frac{8(1 + d_I)}{4|d|^2(X_1 + \frac{d_R}{|d|^2})^2 + \frac{|d|^2-1}{|d|^2}}, \quad t \to \pm \infty, \quad (5.39)$$

where $d = d_R + id_I$. It indicates that, asymptotically, this is a wave traveling on the background plane $|q|^2 = 1$, along the line $x = -2t - \frac{d_R}{|d|^2},$ with amplitude $1 + \frac{8|d|^2(1 + d_I)}{(|d|^2-1)^2}$ and without phase shift due to interaction. The wave can be above the background plane when $1 + d_I > 0$, or below the background plane when $1 + d_I < 0$, or vanishes in the background plane when $1 + d_I = 0$. We further introduce a second coordinate frame

$$(X_2 = x - 2t, \quad t),$$

in terms of which we rewrite (5.38). Then keeping $X_2$ to be constant and letting $t \to \pm \infty$, we find

$$|q(X_2, t)|^2 \to 1 + \frac{8(d_I + |d|^2)}{4|d|^2(X_2 - \frac{d_R}{|d|^2})^2 + \frac{|d|^2-1}{|d|^2}}, \quad t \to \pm \infty. \quad (5.40)$$
This implies that, when $t \to \pm \infty$, there is a wave traveling on the background plane $|q|^2 = 1$, along the line $x = 2t + \frac{dp}{|q+d|^2}$ with amplitude $1 + \frac{8i + d^2(d_I + |d|^2)}{(|d|^2 - 1)^2}$ and without phase shift due to interaction. The wave can be either above or below or vanishes in the background plane, depending on the sign of $d_I + |d|^2$.

Various types of interactions are illustrated in Fig. 9, which coincide with the above results of asymptotic analysis. Note that, considering the signs of $1 + d_I$ and $d_I + |d|^2$, it is impossible to have both waves below the background plane, neither one wave below the background plane and another vanishing.

![Fig. 9.](image)

**Fig. 9.** Shape and motion of the envelope of the solution (5.38) to the defocusing reverse-space NLS equation (5.28). (a) 3D plot for $d = 0.5$.  (b) 3D plot for $d = -2i$.  (c) 3D plot for $d = 0.2 - 0.5i$.  (d) 3D plot for $d = 0.5 - i$.  (e) 3D plot for $d = 0.5 - 0.5i$.

### 6 Conclusions

In this paper, by means of the bilinearisation-reduction approach, solutions for classical and nonlocal NLS equations with nonzero backgrounds were constructed in a systematical way. Solutions are presented in terms of quasi double Wronskians. Asymptotic analysis and illustrations were provided to understand dynamics of solutions, in particular breathers and rogue waves of the classical focusing NLS equation (2.3) and solitons and rational solutions of the reverse-space nonlocal NLS equation (2.4). One can see that the nonzero backgrounds bring more interesting behaviors in the dynamics of solutions. In addition, although the solutions are given in terms of quasi double Wronskians (not standard double Wronskians), the reduction technique is still effective. In light of Theorem 2 one can also use the double Wronskians given in Theorem 2 if $q_0$ is independent of $x$. This bilinearisation-reduction technique can also be extended to the other integrable equations with nonzero backgrounds, which will be investigated elsewhere.
Acknowledgments

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A Proof of Theorem 1

To prove Theorem 1, we first recall the following lemmas.

Lemma A.1. [27]
Suppose that $D$ is an arbitrary $s \times (s - 2)$ matrix, and $a$, $b$, $c$ and $d$ are column vectors of order $s$, then

$$|D, a, b| |D, c, d| - |D, a, c| |D, b, d| + |D, a, d| |D, b, c| = 0. \quad (A.1)$$

Lemma A.2. [63, 64]
Suppose that $\Xi = (a_{js})_{M \times M}$ is an $M \times M$ matrix with column vector set $\{\alpha_j\}$ and row vector set $\{\beta_j\}$. $P = (P_{js})_{M \times M}$ is an $M \times M$ operator matrix where each $P_{js}$ is an operator. Then we have

$$\sum_{s=1}^{M} |\alpha_1, \cdots, \alpha_{s-1}, C_s \circ \alpha_s, \alpha_{s+1}, \cdots, \alpha_M| = \sum_{j=1}^{M} \begin{vmatrix} \beta_1 \\ \vdots \\ \beta_j-1 \\ R_j \circ \beta_j \\ \beta_{j+1} \\ \vdots \\ \beta_M \end{vmatrix}, \quad (A.2)$$

where $\{C_s\}$ and $\{R_j\}$ are respectively the column and row vector sets of $P$, and "$\circ$" denotes $C_s \circ \alpha_s = (P_{1s}a_{1s}, P_{2s}a_{2s}, \cdots, P_{Ms}a_{Ms})^T$ and $R_j \circ \beta_j = (P_{j1}a_{j1}, P_{j2}a_{j2}, \cdots, P_{jM}a_{jM})$.

Proof of Theorem 1
Direct calculation yields

\[
\begin{align*}
    f_x &= [\hat{\phi}_{m-1}, \hat{\phi}_{m+1}; \hat{\psi}_m] + [\hat{\phi}_m; \hat{\psi}_m, \psi_{m+1}], \\
    f_{xx} &= [\hat{\phi}_{m-2}, \phi_m, \phi_{m+1}; \hat{\psi}_m] + [\hat{\phi}_{m-1}, \phi_{m+2}; \hat{\psi}_m] + 2[\hat{\phi}_{m-1}, \phi_{m+1}; \hat{\psi}_m, \psi_{m+1}] \\
        &\quad + [\hat{\phi}_m; \hat{\psi}_m, \psi_{m+1}] + [\hat{\phi}_m; \hat{\psi}_{m-1}, \psi_{m+2}] \\
        &\quad + 2q_0[\hat{\phi}_{m-1}; \hat{\psi}_{m+1}] - 2r_0[\hat{\phi}_{m+1}; \hat{\psi}_{m-1}], \\
    i f_x &= -2[\hat{\phi}_{m-2}, \phi_m, \phi_{m+1}; \hat{\psi}_m] + 2[\hat{\phi}_{m-1}, \phi_{m+2}; \hat{\psi}_m] \\
        &\quad + 2[\hat{\phi}_m; \hat{\psi}_{m-2}, \psi_{m+1}] - 2[\hat{\phi}_m; \hat{\psi}_{m-1}, \psi_{m+2}] \\
        &\quad + 2q_0[\hat{\phi}_{m-1}; \hat{\psi}_{m+1}] + 2r_0[\hat{\phi}_{m+1}; \hat{\psi}_{m-1}], \\
    i g_x/2 &= -2[\hat{\phi}_{m-1}, \phi_m, \phi_{m+2}; \hat{\psi}_m] + 2[\hat{\phi}_m, \phi_{m+3}; \hat{\psi}_m] \\
        &\quad + 2[\hat{\phi}_{m+1}, \hat{\psi}_{m-3}, \psi_{m-1}] - 2[\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \psi_{m+1}] \\
        &\quad + 2q_0([\hat{\phi}_m; \hat{\psi}_{m+1}, \psi_{m+2}] + [\hat{\phi}_{m-1}, \phi_{m+1}; \hat{\psi}_{m-1}, \psi_{m+1}] + [\hat{\phi}_{m-2}, \phi_m, \phi_{m+1}; \hat{\psi}_m]) \\
        &\quad - q_0 r_0 g - q_0 f_x, \\
    g_x/2 &= [\hat{\phi}_m, \phi_{m+1}; \psi_m] + [\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \psi_{m+1}] - q_0 f_x, \\
    g_{xx}/2 &= [\hat{\phi}_{m-1}, \phi_m, \phi_{m+1}; \hat{\psi}_m] + [\hat{\phi}_m, \phi_{m+2}; \hat{\psi}_m, \psi_{m+1}] \\
        &\quad + [\hat{\phi}_{m+1}, \hat{\psi}_{m-3}, \psi_{m-1}] + [\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \psi_{m+1}] \\
        &\quad + q_0([\hat{\phi}_m; \hat{\psi}_{m-1}, \psi_{m+2}] - [\hat{\phi}_{m-1}, \phi_{m+2}; \hat{\psi}_m] \\
        &\quad - [\hat{\phi}_m; \hat{\psi}_{m-2}, \psi_m] - [\hat{\phi}_{m+2}, \phi_m, \phi_{m+1}; \hat{\psi}_m]) - q_0 f_{xx} - q_0 f_x.
\end{align*}
\]

Next, using Lemma A.2 we derive some relations of quasi double Wronskians. Taking \( \Xi = [\hat{\phi}_m; \hat{\psi}_m] \) and (for \( 1 \leq j \leq 2m+2 \))

\[
P_j = \begin{cases} 
    \partial_x - (-1)^m q_0 \phi_m \circ \phi_m^{-1} \circ, & 0 \leq s \leq m, \\
    -\partial_x + (-1)^m r_0 \phi_m \circ \phi_m^{-1} \circ, & m + 1 \leq s \leq 2m + 2,
\end{cases}
\]

where the “\( \circ \)” is defined as in Lemma A.2. One can find from the relation (A.2) that

\[
(TrA) f = [\hat{\phi}_{m-1}, \phi_{m+1}; \hat{\psi}_m] + [\hat{\phi}_m; \hat{\psi}_{m-1}, \psi_{m+1}],
\]

where TrA stands for the trace of matrix A. Similarly, we can get

\[
(TrA)^2 f = [\hat{\phi}_{m-2}, \phi_m, \phi_{m+1}; \hat{\psi}_m] + [\hat{\phi}_{m-1}, \phi_{m+2}; \hat{\psi}_m] - 2[\hat{\phi}_{m-1}, \phi_{m+1}; \hat{\psi}_{m-1}, \psi_{m+1}] \\
+ [\hat{\phi}_m; \hat{\psi}_{m-2}, \psi_m, \psi_{m+1}] + [\hat{\phi}_m; \hat{\psi}_{m-1}, \psi_{m+2}].
\]

Substituting them into the left hand side of (3.2a) one obtains

\[
ff_{xx} - f_{xx} f_x = 4[\hat{\phi}_{m-1}, \phi_{m+1}; \hat{\psi}_m, \psi_{m+1}] f - 4[\hat{\phi}_{m-1}, \phi_{m+1}; \hat{\psi}_m] [\hat{\phi}_m; \hat{\psi}_{m-1}, \psi_{m+1}] \\
+ 2q_0[\hat{\phi}_{m-1}; \hat{\psi}_{m+1}] f - 2r_0[\hat{\phi}_{m+1}; \hat{\psi}_{m-1}] f \\
= 4[\hat{\phi}_{m-1}; \psi_{m+1}] [\hat{\phi}_{m+1}; \hat{\psi}_m] - 2q_0[\hat{\phi}_{m-1}; \hat{\psi}_{m+1}] f - 2r_0[\hat{\phi}_{m+1}; \hat{\psi}_{m-1}] f \\
= -g h - q_0 h f - r_0 g f,
\]

where we have made use of

\[
f([TrA] f) = (TrA) f^2
\]

and the identity

\[
[\hat{\phi}_{m-1}, \phi_{m+1}; \hat{\psi}_m, \psi_{m+1}] f - 4[\hat{\phi}_{m-1}, \phi_{m+1}; \hat{\psi}_m] [\hat{\phi}_m; \hat{\psi}_{m-1}, \psi_{m+1}] \\
+ [\hat{\phi}_{m-1}; \psi_{m+1}] [\hat{\phi}_{m+1}; \hat{\psi}_m] = 0,
\]

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which is derived from Lemma A.1. Thus, (3.2a) is proved.

For (3.2b), we first derive the following relations using Lemma A.2,

\[(\text{Tr} A)[\hat{\phi}_{m+1}; \hat{\psi}_{m-1}] = |\hat{\phi}_m, \hat{\phi}_{m+2}; \hat{\psi}_{m-1} - |\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \hat{\psi}_m|,
\]
\[(\text{Tr} A)^2[\hat{\phi}_{m+1}; \hat{\psi}_{m-1}] = |\hat{\phi}_{m-1}, \hat{\phi}_{m+1}; \hat{\psi}_{m-1} + |\hat{\phi}_m, \hat{\phi}_{m+3}; \hat{\psi}_{m-1} - 2|\hat{\phi}_m, \hat{\phi}_{m+2}; \hat{\psi}_{m-2}, \hat{\psi}_m|
\]
\[+ |\hat{\phi}_{m+1}; \hat{\psi}_{m-3}, \hat{\psi}_{m-1}, \hat{\psi}_m| + |\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \hat{\psi}_{m+1}|.\]

Substituting the derivatives of \(f\) and \(g\) into equation (3.2b) we have

\[(f_{xx}g + fg_{xx} - 2f_xg_x - i(gf + ifg)/2 = (|\hat{\phi}_{m-2}, \hat{\phi}_m, \hat{\phi}_{m+1}; \hat{\psi}_m| + 3|\hat{\phi}_{m-1}, \hat{\phi}_{m+2}; \hat{\psi}_m| + 2|\hat{\phi}_{m-1}, \hat{\phi}_{m+1}; \hat{\psi}_{m-1}, \hat{\psi}_m|)
\]
\[+ 3|\hat{\phi}_m; \hat{\psi}_{m-2}, \hat{\psi}_m, \hat{\psi}_{m+1}| - |\hat{\phi}_m; \hat{\psi}_{m-1}, \hat{\psi}_{m+2}||\hat{\phi}_{m+1}; \hat{\psi}_{m-1}| + 4q_0|\hat{\phi}_{m-1}; \hat{\psi}_{m+1}||\hat{\phi}_{m+1}; \hat{\psi}_{m-1}|
\]
\[+ (3|\hat{\phi}_{m-1}, \hat{\phi}_{m+1}, \hat{\phi}_{m+2}; \hat{\psi}_{m-1} - |\hat{\phi}_m, \hat{\phi}_{m+3}; \hat{\psi}_{m-1}| + 2|\hat{\phi}_m, \hat{\phi}_{m+2}; \hat{\psi}_{m-2}, \hat{\psi}_m|
\]
\[- |\hat{\phi}_{m+1}; \hat{\psi}_{m-3}, \hat{\psi}_{m-1}, \hat{\psi}_m| + 3|\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \hat{\psi}_{m+1}|) f
\]
\[- q_0(|\hat{\phi}_{m-2}, \hat{\phi}_m, \hat{\phi}_{m+1}; \hat{\psi}_m| + |\hat{\phi}_{m-1}, \hat{\phi}_{m+2}; \hat{\psi}_m| + 2|\hat{\phi}_{m-1}, \hat{\phi}_{m+1}; \hat{\psi}_{m-1}, \hat{\psi}_m|)
\]
\[+ 3|\hat{\phi}_m; \hat{\psi}_{m-2}, \hat{\psi}_m, \hat{\psi}_{m+1}| + |\hat{\phi}_m; \hat{\psi}_{m-1}, \hat{\psi}_{m+2}||f - q_0f_{xx}f + 2q_0\rho_0|\hat{\phi}_{m+1}; \hat{\psi}_{m-1}| f
\]
\[- 2(|\hat{\phi}_{m-1}, \hat{\phi}_{m+1}; \hat{\psi}_m| + |\hat{\phi}_m; \hat{\psi}_{m-1}, \psi_{m+1}|||\hat{\phi}_m, \hat{\phi}_{m+2}; \hat{\psi}_{m-1} + |\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \hat{\psi}_m| - q_0f_x)
\]
\[= - 4(|\hat{\phi}_{m-1}, \hat{\phi}_{m+1}; \hat{\psi}_m| + |\hat{\phi}_m; \hat{\psi}_{m-1}, \psi_{m+1}|||\hat{\phi}_m, \hat{\phi}_{m+2}; \hat{\psi}_{m-1} + |\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \hat{\psi}_m|)
\]
\[+ 4(|\hat{\phi}_{m-1}, \hat{\phi}_{m+1}, \hat{\phi}_{m+2}; \hat{\psi}_{m-1} + |\hat{\phi}_m; \hat{\psi}_{m-2}, \hat{\psi}_{m+1}||f + 4q_0|\hat{\phi}_{m-1}; \hat{\psi}_{m+1}||\hat{\phi}_{m+1}; \hat{\psi}_{m-1}|
\]
\[= - 2q_0(f_{xx}f - f_x^2) + q_0h = 0, \]
\[= - q_0(f_{xx}f - f_x^2) + q_0h f - q_0gh = 0,
\]
in which the identity

\[f(\text{Tr} A)^2g = g(\text{Tr} A)^2f = [(\text{Tr} A)f][\text{Tr} A]g\]

and relations

\[|\hat{\phi}_{m-1}, \hat{\phi}_{m+2}; \hat{\psi}_m|/2 - |\hat{\phi}_{m-1}, \hat{\phi}_{m+1}; \hat{\psi}_{m+1}||\hat{\phi}_m, \hat{\phi}_{m+2}; \hat{\psi}_{m-1} + |\hat{\phi}_{m-1}, \hat{\phi}_{m+1}, \hat{\phi}_{m+2}; \hat{\psi}_{m-1}| f = 0,
\]
\[|\hat{\phi}_m; \hat{\psi}_{m-2}, \hat{\psi}_m, \hat{\psi}_{m+1}|/2 - |\hat{\phi}_m; \hat{\psi}_{m-1}, \hat{\psi}_{m+2}||\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \hat{\psi}_m + |\hat{\phi}_{m+1}; \hat{\psi}_{m-2}, \hat{\psi}_{m+1}| f = 0
\]
have also been used. Thus, (3.2b) has been proved. The third equation can be proved in a similar way.

Suppose that \(A = P^{-1}JP\), i.e. \(A\) is similar to \(J\). We introduce \(\phi' = P\phi\), \(\psi' = P\psi\), which satisfy (3.6) and (3.7) with \(J\) in place of \(A\). Then, for the quasi double Wronskians, we have \(f(\phi', \psi') = |P|f(\phi, \psi), g(\phi', \psi') = |P|g(\phi, \psi)\) and \(h(\phi', \psi') = |P|h(\phi, \psi)\), which means \(A\) and any matrix that is similar to it can generate same \(q\) and \(r\).

**B Proof of Theorem 2**

For simplicity we denote

\[F = [\hat{m}; \hat{m}], \quad s = [\hat{m}+1; \hat{m}-1], \quad H = [\hat{m}-1; \hat{m}+1].\]
Direct calculation yields

\[
F_x = |\bar{m} - 1, m + 1; \hat{m}| + |\hat{m}; \bar{m} - 1, m + 1|,
\]

\[
F_{xx} = |\bar{m} - 2, m, m + 1; \hat{m}| + |\hat{m} - 1, m + 2; \hat{m}| + 2|\bar{m} - 1, m + 1; \bar{m} - 1, m + 1| + |\bar{m}; \bar{m} - 2, m, m + 1| + |\hat{m}; \bar{m} - 1, m + 1|,
\]

\[
s_x = |\hat{m}, m + 2; m - 1| + |m + 1; \bar{m} - 2, m|,
\]

\[
s_{xx} = |\bar{m} - 1, m + 1, m + 2; m - 1| + |\hat{m}, m + 3; \bar{m} - 1| + 2|\hat{m}, m + 2; \bar{m} - 2, m| + |\bar{m} + 1; m - 3, m - 1, m| + |m + 1; \bar{m} - 2, m + 1|,
\]

\[
iF_t = -2|\bar{m} - 2, m, m + 1; \hat{m}| + 2|m - 1, m + 2; \hat{m}| + 2|\bar{m}; \bar{m} - 2, m, m + 1| - 2|m; \bar{m} - 1, m + 2| + 2q_0(-1)^m|m - 1; m + 1| + 2r_0(-1)^m|m + 1; \bar{m} - 1|,
\]

\[
is_t = -2q_0r_0|m + 1; (m - 1)| - 2|m - 1, m + 1, m + 2; \bar{m} - 1| + 2|\hat{m}, m + 3; \bar{m} - 1| + 2|\bar{m} + 1; m - 3, m - 1, m| - 2|m + 1; \bar{m} - 2, m + 1|
+ 2q_0(-1)^m(-|\bar{m} - 2, m, m + 1; \hat{m}| + |\bar{m} - 1, m + 1; \bar{m} - 1, m + 1| - |\bar{m} + 1; m - 2, m + 1|).
\]

Taking \( \Xi = |\hat{m}; \hat{m}| \) and (for \( 1 \leq j \leq 2m + 2 \))

\[
P_{js} = \begin{cases} 
\partial_x - q_0(\partial^2_x \psi) \circ (\partial^2_x \phi)^{-1}, & 0 \leq s \leq m, \\
-\partial_x + r_0(\partial^2_x \psi) \circ (\partial^2_x \phi)^{-1}, & m + 1 \leq s \leq 2m + 2,
\end{cases}
\]

using (A.2) we can have

\[
(\text{Tr}A)F = |\bar{m} - 1, m + 1; \hat{m}| - |\hat{m}; \bar{m} - 1, m + 1|,
\]

and

\[
(\text{Tr}A)s = |\hat{m}, m + 2; \bar{m} - 1| - |m + 1; \bar{m} - 2, m|
+ q_0(-1)^m(|\bar{m} - 1, m + 1; \hat{m}| - |\hat{m}; \bar{m} - 1, m + 1|),
\]

\[
(\text{Tr}A)^2F = |\bar{m} - 2, m, m + 1; \hat{m}| + |\bar{m} - 1, m + 2; \hat{m}| + 2|\hat{m}, m + 3; \bar{m} - 1| + 2|\hat{m} - 1, m + 1; \bar{m} - 1, m + 1| + 2q_0(-1)^m|m - 1; m + 1| + 2r_0(-1)^m|m + 1; \bar{m} - 1|,
\]

\[
(\text{Tr}A)^2s = |\bar{m} - 1, m + 1, m + 2; \bar{m} - 1| + |\hat{m}, m + 3; \bar{m} - 1| - 2|m, m + 2; \bar{m} - 1| + 2|\hat{m} - 1, m + 1; \bar{m} - 1, m + 1| - 2q_0(q_0|m - 1, m + 1; \bar{m} - 1, m + 1|)
+ 2q_0(q_0|m - 1, m + 1; \bar{m} - 1, m + 1| - r_0|m + 1; \bar{m} - 1|).
\]

Substituting the derivatives of \( f \) into the left-hand side of (3.2a) one obtains

\[
ff_{xx} - f_x^2 = F^2 - F_{xx}^2
\]

\[
= 4(|\hat{m}; \hat{m}| - |\bar{m} - 1, m + 1; \hat{m}| + |\hat{m}; \bar{m} - 1, m + 1|) - 2q_0(-1)^m|\hat{m}; \hat{m}| + 2r_0(-1)^m|\hat{m}; \hat{m}| - 2q_0(-1)^m|m - 1; m + 1| + 2r_0(-1)^m|m + 1; m - 1|
\]

\[
= 4Hs - 2q_0Hf + 2r_0sf,
\]
Then we have
\[ F[(\text{Tr}A)F] = [(\text{Tr}A)F]^2 \]
and the relation
\[
|\hat{m} - 1; m + 1| |\hat{m} - 1| - |\hat{\hat{m}}; \hat{\hat{m}}||m - 1, m + 1| + |m - 1, m + 1| \]
\[ + |\hat{m} - 1, m + 1; \hat{m}; \hat{m}||m - 1, m + 1| = 0. \]

Meanwhile, a direct calculation of the right-hand side of (3.2a) gives rise to \(4(2Hs - q_0 Hf + r_0 sf)\). Thus, equation (3.2a) is proved.

For equation (3.2b), let us first consider \((D_x^2 - iD_t)s \cdot F\). We have
\[
s_{xx}F - 2sxF_x + sF_{xx} - i(s_t F - sF_t) = (F(s_{xx} - is_t) + s(F_{xx} + IF_t) - 2s_x F_x) \]
\[ = (3|m - 1, m + 1, m + 2; m - 1| - |\hat{m}, m + 3; m - 1| + 2|\hat{m}, m + 2; m - 2, m| - |m + 1; m - 3, m - 1, m| + 3|m + 1; m - 2, m + 1|)F + 2q_0 r_0 sf \]
\[ + 2q_0 (-1)^m(|\hat{m}, m - 1, m + 1| + |\hat{m}, m - 1, m + 1| + 2|m, m - 1, m + 1|)F + 2q_0 r_0 sf \]
\[ = |m - 1, m + 1; \hat{m}||\hat{m}; m - 1, m + 1||m, m - 1, m + 1| + |m, m - 1, m + 1|. \]

Utilizing identity
\[ F[(\text{Tr}A)^2]s = s[(\text{Tr}A)F] = [(\text{Tr}A)F][(\text{Tr}A)s], \]

(3.2b) gives rise to
\[
-4(|\hat{m} - 1, m + 1; \hat{\hat{m}}||\hat{\hat{m}}, m + 2; m - 1| + |\hat{\hat{m}}, m - 1, m + 1||m + 1, m - 2, m|) \]
\[ + 4F(|\hat{m} - 1, m + 1, m + 2; m - 1| + |m + 1, m - 2, m + 1|) \]
\[ + 4s(|\hat{m} - 1, m + 2; \hat{\hat{m}}| + |\hat{\hat{m}}, m - 2, m, m + 1|) \]
\[ - 2q_0^2 HF + 4q_0 r_0 sf + 4q_0 (-1)^m Hs \]
\[ = -2q_0^2 HF + 4q_0 r_0 sf + 4q_0 (-1)^m Hs. \]

Then we have
\[
(D_x^2 - iD_t)s \cdot f \]
\[ = (-1)^m(-2q_0^2 HF + 4q_0 r_0 sf + 4q_0 (-1)^m Hs) \]
\[ = -q_0^2 Hf - 2q_0 r_0 gf - q_0 gh, \]

which proves (3.2b). Equation (3.2c) can be verified similarly.

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