GEOMETRIC SYMMETRIC POWERS IN THE UNSTABLE
MOTIVIC HOMOTOPY CATEGORY

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Abstract. Symmetric powers of quasi-projective schemes can be extended, in
terms of left Kan extensions, to geometric symmetric powers of motivic spaces. In
this paper, we prove that the left derived geometric symmetric powers provide a
$\lambda$-structure on the motivic homotopy category over a field.

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1. Introduction

Symmetric powers play an important role in algebraic topology and algebraic geometry. The Dold-Thom theorem says that the group completion of the infinite symmetric power of a pointed connected $CW$-complex is weak equivalent to a product of the Eilenberg-MacLane spaces associated to its homology groups. Voevodsky has developed a motivic version of this theorem in terms of geometric symmetric powers of motivic spaces and the motivic Eilenberg-MacLane spaces. Geometric symmetric powers are Kan extensions of the symmetric powers of schemes considered in an admissible category of schemes over a field $\mathbb{F}$. An admissible category $\mathbb{F}$ is closed under taking quotients of schemes by finite groups and contains the affine line as an interval. Categoric symmetric powers are the quotients of Cartesian powers of motivic spaces by the action of symmetric groups. Categoric symmetric powers preserve $\mathbb{A}^1$-weak equivalences, and their left derived functors provide a $\lambda$-structure on the pointed and unpointed (unstable) motivic homotopy categories of an admissible category. The goal of the present paper is to prove that geometric symmetric powers also provide a $\lambda$-structure on the pointed unstable motivic homotopy category of an admissible category.

For that purpose we first consider a certain cofibrant resolution on the category of simplicial Nisnevich sheaves on an admissible category, induced by the generating
cofibrations of the category of simplicial sets. It allows us to deduce that every motivic space is $\mathbb{A}^1$-weak equivalent to a simplicial sheaf, given termwise by coproducts of representable sheaves, as it was showed by Voevodsky in the context of radditive functors, see [9, 10]. It turns out that every cofibre sequence is isomorphic, in the pointed motivic homotopy category over the admissible category, to a split cofibre sequence of simplicial sheaves, which termwise are coproducts of representable sheaves. Geometric symmetric powers in split cofibre sequences induce the corresponding Künneth towers and thus provide a $\lambda$-structure on the motivic homotopy category. This gives the following result (Theorem 34 in the text):

The left derived geometric symmetric powers provide a $\lambda$-structure on the pointed unstable motivic homotopy category of an admissible category of quasi-projective schemes over a field.

The paper is organized as follows. In Section 2, we give a survey of admissible categories and basic properties of simplicial Nisnevich sheaves. Section 3 is devoted to the study of geometric symmetric powers and Künneth towers associated to split cofibre sequences. Section 4 is about the $\mathbb{A}^1$-localization of geometric symmetric powers. In section 5 we recall the notion of a $\lambda$-structure and prove the main result (Theorem 34).

ACKNOWLEDGEMENTS. I am grateful to Vladimir Guletskiǐ for his inspiring explanations and helpful suggestions, and to Anwar Alameddin for useful comments on the paper. The paper is written in the framework of the EPSRC grant EP/I034017/1 “Lambda-structures in stable categories”. I am grateful to EPSRC for funding my PhD project.

2. ADMISSIBLE CATEGORIES AND NISnevich SHEAVES

Throughout the paper $k$ will denote a field of arbitrary characteristic and we write $\mathbb{A}^1$ for the affine line over Spec($k$). Let $\mathcal{Sch}/k$ be the category of schemes over $k$. For two $k$-schemes $X$ and $Y$, we write $X \times Y$ to mean the Cartesian product $X \times_{\text{Spec}(k)} Y$. We also denote by $X \amalg Y$ the disjoint union of the $X$ and $Y$, as schemes. We recall that the point Spec($k$) is the terminal object of $\mathcal{Sch}/k$, whereas the empty scheme $\emptyset$ is its initial object.

We say that a full subcategory $\mathcal{C}$ of $\mathcal{Sch}/k$ is called admissible, if it satisfies the following five axioms: (i) the point Spec($k$) and the affine line $\mathbb{A}^1$ are objects of $\mathcal{C}$, (ii) for any two objects $X$ and $Y$ of $\mathcal{C}$, the product $X \times Y$ is in $\mathcal{C}$, (iii) for any two objects $X$ and $Y$ of $\mathcal{C}$, the disjoint union $X \amalg Y$ is in $\mathcal{C}$, (iv) if $U \to X$ is an étale morphism of $k$-schemes such that $X$ is in $\mathcal{C}$, then $U$ is in $\mathcal{C}$, (v) If $G$ is finite group acting on an object $X$ of $\mathcal{C}$, then the quotient $X/G$ is in $\mathcal{C}$ whenever it exists. In [9], this definition appears as $f$-admissible category.

A typical example of an admissible category is the category of quasi-projective schemes over $k$.

Remark 1. It is not true that all admissible categories of schemes over a field contain the projective line $\mathbb{P}^1$ over a field. For example, the subcategory of normal quasi-affine schemes over a perfect field is admissible, but the projective line $\mathbb{P}^1$ is not quasi-affine.
Unless otherwise mentioned, \( \mathcal{C} \) will be a small, admissible category contained in the category of quasi-projective schemes over \( k \).

An elementary distinguished square in \( \mathcal{C} \) is a Cartesian square of the form

\[
\begin{array}{ccc}
Y & \rightarrow & V \\
\downarrow & & \downarrow \scriptstyle{p} \\
U & \rightarrow & X \\
\downarrow \scriptstyle{j} & & \downarrow \\
& & 
\end{array}
\]

where \( j \) is a open embedding and \( p \) is an étale morphism such that the induced morphism \( p^{-1}(X - U)_{\text{red}} \rightarrow (X - U)_{\text{red}} \) of reduced schemes is an isomorphism.

We recall that a family of étale morphisms \( \{ f_i : U_i \rightarrow X \}_{i \in I} \) of \( \mathcal{C} \) is a Nisnevich covering if for every point \( x \in X \), there exists an index \( i \in I \) and a point \( y \in U_i \) such that \( f_i(y) = x \) and the corresponding morphism of residual fields \( k(x) \rightarrow k(y) \) is an isomorphism. The Nisnevich topology on \( \mathcal{C} \) can be described as the smallest Grothendieck topology generated by families of the form \( \{ j : U \rightarrow X, p : V \rightarrow X \} \) associated to an elementary distinguished square (2) (see [11, page 1400]). We denote by \( \mathcal{C}_{\text{Nis}} \) the site consisting of \( \mathcal{C} \) and the Nisnevich topology on it.

We denote by \( \text{Pre}(\mathcal{C}) \) the category of presheaves on \( \mathcal{C} \), i.e., the category of functors from \( \mathcal{C}^{\text{op}} \) to the category of sets. A presheaf \( F \) on \( \mathcal{C} \) is a sheaf in the Nisnevich topology if and only if for each elementary distinguished square as (2), the square

\[
\begin{array}{ccc}
F(X) & \rightarrow & F(V) \\
\downarrow \scriptstyle{F(p)} & & \downarrow \\
F(U) & \rightarrow & F(U \times_X V) \\
\end{array}
\]

is Cartesian. Unless otherwise specified, \( \mathcal{S} \) will denote the category of sheaves on \( \mathcal{C}_{\text{Nis}} \).

We denote by \( h : \mathcal{C} \rightarrow \text{Pre}(\mathcal{C}) \) the Yoneda embedding. Since representable sheaves are Nisnevich sheaves, sometimes we use the same letter \( h \) to denote the full embedding of \( \mathcal{C} \) into \( \mathcal{S} \). We recall that the forgetful functor from \( \mathcal{S} \) to \( \text{Pre}(\mathcal{C}) \) has a left adjoint which we denote by \( \text{a}_{\text{Nis}} \). The category \( \mathcal{S} \) is complete and cocomplete, its terminal object is \( h_{\text{Spec}(k)} \), and filtered colimits of Nisnevich sheaves in the category of presheaves are Nisnevich sheaves.

Let \( \{ F_i \}_{i \in I} \) be a family of objects in \( \mathcal{S} \). The coproduct of this family in \( \mathcal{S} \) is the sheafification \( \text{a}_{\text{Nis}}(\coprod_{i \in I} F_i) \) of the coproduct \( \coprod_{i \in I} F_i \) in \( \text{Pre}(\mathcal{C}) \). We abusively denote it by \( \coprod_{i \in I} F_i \), if no confusion arises.

In this paper, we shall consider the injective model structure on the category of simplicial sheaves \( \Delta^{\text{op}} \mathcal{S} \), where the class of cofibrations is the class of monomorphisms, a weak equivalence is a stalk-wise weak equivalence and fibrations are morphisms having the right lifting property with respect to trivial cofibrations.

The category \( \Delta^{\text{op}} \mathcal{S} \) is a simplicial category. For a simplicial sheaf \( \mathcal{X} \) and a simplicial set \( K \), we define the product \( \mathcal{X} \times K \) to be the simplicial sheaf, such that for every \( n \in \mathbb{N} \), its term \( (\mathcal{X} \times K)_n \) is defined to be the coproduct \( \coprod_{K_n} \mathcal{X}_n \) in \( \mathcal{S} \).
For a couple of sheaves \((\mathcal{X}, \mathcal{Y})\), the function complex \(\text{Map}(\mathcal{X}, \mathcal{Y})\) is defined to be simplicial set which assigns an object \([n]\) of \(\Delta\) to the set
\[
\text{Hom}_{\Delta^{\text{op}}, \mathcal{S}}(\mathcal{X} \times \Delta[n], \mathcal{Y}).
\]
Then, for every pair of simplicial sheaves \((\mathcal{X}, \mathcal{Y})\) and every simplicial set \(K\), one has a natural bijection,
\[
(1) \quad \text{Hom}_{\Delta^{\text{op}}, \mathcal{S}}(\mathcal{X} \times K, \mathcal{Y}) \simeq \text{Hom}_{\Delta^{\text{op}}, \mathcal{S}_{\text{et}}}(K, \text{Map}(\mathcal{X}, \mathcal{Y})),
\]
which is functorial in \(\mathcal{X}, \mathcal{Y}\) and \(K\).

For each object \(U\) of \(\mathcal{C}\), we denote by \(\Delta_U[0]\) the constant functor from \(\Delta^{\text{op}}\) to \(\mathcal{S}\) with value \(h_U\). For each \(n \in \mathbb{N}\) and each object \(U\) of \(\mathcal{C}\), we denote by \(\Delta_U[n]\) the simplicial sheaf \(\Delta_U[0] \times \Delta[n]\). Similarly, we denote by \(\partial\Delta_U[n]\) the simplicial sheaf \(\Delta_U[0] \times \partial\Delta[n]\).

Notice that Yoneda lemma implies an isomorphism \(\text{Map}(\Delta_U[0], \mathcal{Y}) \simeq \mathcal{Y}(U)\), for every object \(U\) of \(\mathcal{C}\) and every simplicial sheaf \(\mathcal{Y}\). Hence, replacing \(\mathcal{X}\) by \(\Delta_U[0]\) in (1), we obtain an isomorphism
\[
(2) \quad \text{Hom}_{\Delta^{\text{op}}, \mathcal{S}}(\Delta_U[0] \times K, \mathcal{Y}) \simeq \text{Hom}_{\Delta^{\text{op}}, \mathcal{S}_{\text{et}}}(K, \mathcal{Y}(U)).
\]

We write \(\mathcal{C}_+\) to denote the full subcategory of the pointed category \(\mathcal{C}_+\) generated by objects of the form \(X_+ := X \amalg \text{Spec}(k)\). We denote by \(\mathcal{S}_+\) the pointed category of \(\mathcal{S}\). The symbols \(\lor\) and \(\land\) denote, respectively, the coproduct and the smash product in \(\mathcal{S}_+\).

In the next paragraph, we recall the notion of \(\Delta\)-closed classes introduced by Voevodsky (see [10]) which allow us to describe weak equivalences as a class generated by morphisms having a simpler form.

Let \(\mathcal{D}\) be a category with finite coproducts. A class of morphisms \(E\) of \(\Delta^{\text{op}}\mathcal{D}\) is called \(\Delta\)-closed, if it satisfies the following four axioms: (i) \(E\) contains all \(\Delta[1]\)-homotopy equivalences in \(\Delta^{\text{op}}\mathcal{D}\), (ii) \(E\) has the 2-out-of-3 property, (iii) if \(f\) is a morphism of bi-simplicial objects on \(\mathcal{D}\) such that for every \(n \in \mathbb{N}\), either \(f([n], -)\) or \(f(-, [n])\) belongs to \(E\), then the diagonal morphism of \(f\) belongs to \(E\), and (iv) \(E\) is closed under filtered colimits. For any class of morphisms \(E\) in \(\Delta^{\text{op}}\mathcal{D}\), we denote by \(cl_\Delta(E)\) the smallest \(\Delta\)-closed class containing the class \(E\).

A morphism from \(A\) to \(X\) in \(\mathcal{D}\) is called coprojection, if there exists an object \(Y\) of \(\mathcal{D}\) such that this morphism is isomorphic to the canonical morphism from \(A\) to \(A \amalg Y\). A morphism \(f\) in \(\Delta^{\text{op}}\mathcal{D}\) is called a termwise coprojection, if for each natural \(n\), its term \(f_n\) is a coprojection.

**Example 2.** The class of \(\mathbb{A}^1\)-weak equivalences in \(\Delta^{\text{op}}\mathcal{S}\) coincides with the \(\Delta\)-class \(cl_\Delta(\mathcal{W}_{\text{Nis}} \cup \mathcal{P}_{\mathbb{A}^1})\), where \(\mathcal{W}_{\text{Nis}}\) is the class of local equivalences with respect to the Nisnevich topology and \(\mathcal{P}_{\mathbb{A}^1}\) is the class of projections from \(\Delta_X[0] \times \Delta_{\mathbb{A}^1}[0]\) to \(\Delta_X[0]\), for \(X \in \mathcal{C}\) (see [3], Th. 4, page 378). Similarly, the class of \(\mathbb{A}^1\)-weak equivalences in \(\Delta^{\text{op}}\mathcal{S}_+\) coincides with the class \(cl_\Delta(\mathcal{W}_{\text{Nis},+} \cup \mathcal{P}_{\mathbb{A}^1,+})\), where \(\mathcal{W}_{\text{Nis},+}\) is the image of \(\mathcal{W}_{\text{Nis}}\) and \(\mathcal{P}_{\mathbb{A}^1,+}\) is the image of \(\mathcal{P}_{\mathbb{A}^1}\) through the functor which sends a simplicial sheaf \(\mathcal{X}\) to the pointed simplicial sheaf \(\mathcal{X}_+\).

**Example 3.** Let \(\mathcal{C}\) be as in the beginning and let \(\mathcal{S}\) be a simplicial sheaf on \(\mathcal{C}_{\text{Nis}}\). If \(K \subseteq L\) is an inclusion of simplicial sets, then the canonical morphism from \(\mathcal{S} \times K\) to \(\mathcal{S} \times L\) is a termwise coprojection. Indeed, for each natural \(n\), the \(n\)-simplex \((\mathcal{S} \times K)_n\) is equal to the coproduct of sheaves \(\coprod_{K_n} \mathcal{S}_n\), similarly,
$$(\mathcal{X} \times L)_n$$ is equal to $$\coprod_{L_n} \mathcal{X}_n$$. In view of the inclusion $$K_n \subset L_n$$, we have a canonical isomorphism

$$\coprod_{L_n} \mathcal{X}_n \simeq \left( \coprod_{K_n} \mathcal{X}_n \right) \amalg \left( \coprod_{L_n \setminus K_n} \mathcal{X}_n \right),$$

which allows us to deduce that $$(\mathcal{X} \times K)_n \to (\mathcal{X} \times L)_n$$ is a coprojection for all $$n \in \mathbb{N}$$.

We recall that $$\omega$$ usually denotes the countable ordinal. This notation will be used in Lemma 4, Corollary 5, and Corollary 8.

**Lemma 4.** Let $$\mathcal{D}$$ be a cocomplete category. Then, the class of termwise coprojections in $$\Delta^{\text{op}} \mathcal{D}$$ is stable under pushouts, small coproducts, and countable transfinite compositions.

**Proof.** Since colimits in $$\Delta^{\text{op}} \mathcal{D}$$ are termwise, it is enough to prove that the class of coprojections in $$\mathcal{D}$$ is closed under (a) pushouts, (b) arbitrary coproducts, and (c) countable transfinite compositions of coprojections. Indeed, (a) follows from the fact that the pushout square of a coprojection $$i_A : A \to A \amalg Y$$ and a morphism $$f : A \to B$$ is a cocartesian square of the form

$$\begin{array}{ccc}
A & \xrightarrow{i_A} & A \amalg Y \\
\downarrow f & & \downarrow f \amalg \text{id}_Y \\
B & \xrightarrow{i_B} & B \amalg Y
\end{array}$$

where the bottom horizontal and the right vertical arrows are the canonical morphisms. To prove (b), we give a family of canonical coprojections $$\{A_i \to A_i \amalg X_i\}_{i \in I}$$, where $$I$$ is a set of indices. We can assume that $$I$$ is an ordered set. Then, the coproduct $$\coprod_{i \in I}(A_i \amalg X_i)$$ is isomorphic to the coproduct $$\left( \coprod_{i \in I} A_i \right) \amalg \left( \coprod_{i \in I} X_i \right)$$, which allows us to deduce (b). Finally, the class of coprojections is closed under countable coprojection because one can deduce that any countable sequence $$X_0 \to X_1 \to X_2 \to \cdots (n < \omega)$$ has terms of the form $$\coprod_{i=0}^n X_i$$ for all $$n < \omega$$. Therefore, its transfinite composition coincide with the canonical morphism from $$X_0$$ to $$\coprod_{i<\omega} X_i$$. This shows (c). □

We denote by $$\mathcal{C}$$ the full subcategory of $$\mathcal{I}$$ generated by objects isomorphic to coproducts of representable functors in $$\mathcal{I}$$. The full embedding of $$\mathcal{C}$$ into $$\mathcal{I}$$ induces a full embedding of $$\Delta^{\text{op}} \mathcal{C}$$ into $$\Delta^{\text{op}} \mathcal{I}$$.

**Corollary 5.** Let $$I$$ be a set of morphisms in $$\Delta^{\text{op}} \mathcal{C}$$ consisting of termwise coprojections. Then any countable transfinite composition of pushouts of coproducts of elements of $$I$$, is a termwise coprojection with terms of the form $$\mathcal{Y}_n \to \mathcal{Y}_n \amalg \mathcal{Z}_n$$, where $$\mathcal{Z}_n$$ is in $$\mathcal{C}$$, for $$n \in \mathbb{N}$$.

**Proof.** Let $$\mathcal{X} : \omega \to \Delta^{\text{op}} \mathcal{I}$$ be a $$\omega$$-sequence such that each $$n < \omega$$, the morphism $$\mathcal{X}_n \to \mathcal{X}_{n+1}$$ is a pushout of coproducts of elements of $$I$$. Since $$I$$ consists of termwise coprojections in $$\Delta^{\text{op}} \mathcal{C}$$, by Lemma 4, the coproduct of elements of $$I$$ is a termwise coprojection in $$\Delta^{\text{op}} \mathcal{C}$$. Since termwise coprojection are closed under pushouts, we
deduce that terms of each morphism \( X_n \to X_{n+1} \) are canonical morphisms of the form \((X)_i \to (X)_i \amalg Y_{n,i}\), where \(Y_{n,i}\) is an object in \(\mathcal{C}\), for \(i \in \mathbb{N}\). Finally, by the same Lemma, we conclude the each term of the transfinite composition of \(X\) is a canonical morphism of the form \((X_0)_i \to (X_0)_i \amalg Y_i\), where \(Y_i\) is an object of \(\mathcal{C}\).

We define the following sets of morphisms of simplicial sheaves

\[ I_{proj} := \{ \partial \Delta_k [n] \to \Delta_k [n] | U \in \mathcal{C}, n \in \mathbb{N} \} \]

By Example 3, the morphisms \(\partial \Delta_k [n] \to \Delta_k [n]\) are termwise coprojections in \(\Delta^\text{op}\mathcal{C}\) for all \(U \in \mathcal{C}\) and \(n \in \mathbb{N}\).

**Lemma 6.** For any object \(U \in \mathcal{C}\) and every finite simplicial set \(K\), the object \(\Delta_U[0] \times K\) is finite relative to \(\Delta^\text{op}\mathcal{I}\), in the sense of Definition 2.1.4 of [6].

**Proof.** Let us fix an object \(U \in \mathcal{C}\) and a finite simplicial set \(K\). Since \(K\) is finite, there is a finite cardinal \(\kappa\) such that \(K\) is \(\kappa\)-small relative to all morphisms of \(\Delta^\text{op}\mathcal{I}\). We claim that \(\Delta_U[0] \times K\) is \(\kappa\)-small relative to all morphisms in \(\Delta^\text{op}\mathcal{I}\). Indeed, let \(\lambda\) be a \(\kappa\)-filtered ordinal and let

\[ X_0 \to X_1 \to \cdots \to X_\beta \to \cdots \] \((\beta < \lambda)\)

be a \(\lambda\)-sequence of simplicial sheaves on \(\mathcal{C}\). Since filtered colimits of Nisnevich sheaves (computed in the category of presheaves) are sheaves, we obtain a \(\lambda\)-sequence of simplicial sets,

\[ X_0(U) \to X_1(U) \to \cdots \to X_\beta(U) \to \cdots \] \((\beta < \lambda)\)

Then, we have a commutative diagram

\[
\begin{array}{ccc}
\text{colim}_{\beta < \lambda} \text{Hom}_{\Delta^\text{op}\mathcal{I}}(\Delta_U[0] \times K, X_\beta) & \to & \text{Hom}_{\Delta^\text{op}\mathcal{I}}(\Delta_U[0] \times K, \text{colim}_{\beta < \lambda} X_\beta) \\
\downarrow & & \downarrow \\
\text{colim}_{\beta < \lambda} \text{Hom}_{\Delta^\text{op}\mathcal{I}}(K, X_\beta(U)) & \to & \text{Hom}_{\Delta^\text{op}\mathcal{I}}(K, \text{colim}_{\beta < \lambda}(X_\beta(U)))
\end{array}
\]

where the vertical arrows are bijections. Since \(K\) is \(\kappa\)-small relative to all morphisms of \(\Delta^\text{op}\mathcal{I}\), the below arrow of the preceding diagram is bijective, hence the top arrow is so. This completes the proof. \(\square\)

**Lemma 7.** Every morphism in \((I_{proj})\)-inj is a section wise trivial fibration.

**Proof.** Let \(f : X \to Y\) be a morphism in \((I_{proj})\)-inj and let us fix an object \(U\) of \(\mathcal{C}\). By the naturality of the isomorphism [2], a commutative diagram

\[
\begin{array}{ccc}
\partial \Delta[n] & \to & X(U) \\
\downarrow & & \downarrow \\
\Delta[n] & \to & Y(U)
\end{array}
\]

(3)
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in $\Delta^{\text{op}} \mathcal{I}et$, corresponds biunivocally to a diagram

\[
\begin{array}{ccc}
\partial \Delta_U[n] & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\Delta_U[n] & \longrightarrow & \mathcal{Y}
\end{array}
\]

in $\Delta^{\text{op}} \mathcal{S}$. As the left vertical arrow is an element of $I_{\text{proj}}$, the above diagram has a lifting. Therefore, the bijection (2) induces a lifting of (3). □

The following corollary is a consequence of the small object argument. It will be useful to show the cofibrant resolution takes its values in the category $\Delta^{\text{op}} \mathcal{G}$.

Corollary 8. There exist a functorial factorization $(\alpha, \beta)$ on $\Delta^{\text{op}} \mathcal{S}$ such that for every morphism $f$ is factored as $f = \beta(f) \circ \alpha(f)$, where $\beta(f)$ is section wise trivial fibration and $\alpha(f)$ is a termwise coprojection with terms form $\mathcal{X}_n \to \mathcal{X}_n \amalg \mathcal{Y}_n$, where $\mathcal{Y}_n$ is an object of $\mathcal{G}$.

Proof. By Lemma 6, the objects $\partial \Delta_U[n]$ and $\Delta_U[n]$ are finite relative to $\Delta^{\text{op}} \mathcal{S}$. Since the countable ordinal $\omega$ is $\kappa$-filtered, the small object argument provides the factorization such that $\beta(f)$ in $(I_{\text{proj}})^{\text{inj}}$ and $\alpha(f)$ is a countable transfinite composition of pushouts of coproducts of elements of $I_{\text{proj}}$. By Example 5, every morphism $\partial \Delta_U[n] \to \Delta_U[n]$ of $I_{\text{proj}}$ is a termwise coprojection in $\Delta^{\text{op}} \mathcal{G}$. Therefore, Corollary 5 provides the desired factorization. □

We denote by $Q^{\text{proj}}$ the endofunctor of $\Delta^{\text{op}} \mathcal{S}$ which sends a simplicial sheaf $\mathcal{X}$ to the codomain of the morphism $\alpha(\emptyset \to \mathcal{X})$, where $\emptyset$ is the initial object of $\Delta^{\text{op}} \mathcal{S}$. The endofunctor $Q^{\text{proj}}$ will be called cofibrant resolution. In particular, for every object $\mathcal{X}$ of $\Delta^{\text{op}} \mathcal{S}$, the canonical morphism from $Q^{\text{proj}}(\mathcal{X})$ to $\mathcal{X}$ is a section-wise trivial fibration.

Corollary 9. The functor $Q^{\text{proj}}$ takes values in $\Delta^{\text{op}} \mathcal{G}$.

Proof. It follows from Corollary 8 applied to each morphism of simplicial sheaves $\emptyset \to \mathcal{X}$, where $\emptyset$ is the initial object of $\Delta^{\text{op}} \mathcal{S}$. □

Lemma 10. The class of $A^1$-weak equivalences in $\Delta^{\text{op}} \mathcal{S}_*$ is closed under finite coproducts and smash products.

Proof. By Example 2, the class of $A^1$-weak equivalences in $\Delta^{\text{op}} \mathcal{S}_*$ is $\Delta$-closed. Then, it is closed under finite coproducts. Let us consider the case of smash products. By the cube lemma (see [3] Lemma 5.2.6), one reduces the problem to the unpointed case, i.e., for products in $\Delta^{\text{op}} \mathcal{S}$. Using standard simplicial methods, the problem is reduced to show that: for every $A^1$-weak equivalence and every simplicial sheaf $\mathcal{X}$ of the $\Delta_U[0]$ for $U$ in $\mathcal{G}$, the product $f \times \text{id}_\mathcal{X}$ is an $A^1$-weak equivalence. But it follows from Example 2 and Lemma 2.20 of [10] applied to the functor $(-) \times \text{id}_\mathcal{X}$. □

3. GEOMETRIC SYMMETRIC POWERS IN COFIBRE SEQUENCES

In this section, we study the geometric symmetric powers on the category of simplicial Nisnevich sheaves. Here, we prove the Künneth’s rule for geometric symmetric powers.
Let \( \mathcal{C} \subset \mathcal{S} \) be an admissible category, and fix an object \( X \) of \( \mathcal{C} \) and an integer \( n \geq 1 \). By definition of an admissible category, \( \mathcal{C} \) is closed under finite products and quotients under finite groups. Then \( n \)-th fold product \( X^{\times n} / \Sigma_n \) is an object of \( \mathcal{C} \), hence, the quotient \( X^{\times n} / \Sigma_n \) is also in \( \mathcal{C} \). Denote this quotient by \( \text{Sym}^n(X) \). Then, we have a functor \( \text{Sym}^n : \mathcal{C} \to \mathcal{C} \). It is immediate to observe that \( \text{Sym}^n(\text{Spec}(k)) \) is isomorphic to the point \( \text{Spec}(k) \) for \( n \geq 1 \). By convention, \( \text{Sym}^0 \) will be the constant endofunctor of \( \mathcal{C} \) which sends an object \( X \) of \( \mathcal{C} \) to the point \( \text{Spec}(k) \).

Let us fix \( n \in \mathbb{N} \). Since \( \mathcal{C} \) is a small category and \( \Delta^{op} \mathcal{S} \) is cocomplete, Theorem 3.7.2 of [1] asserts the existence of the left Kan extension of the composite \( \mathcal{C} \xrightarrow{\text{Sym}^n} \mathcal{C} \xrightarrow{h} \mathcal{S} \) along the Yoneda embedding \( h \). We denote it by \( \text{Sym}^n \) and called it the \( n \)-th fold geometric symmetric power of Nisnevich sheaves. Explicitly, \( \text{Sym}^n \) is described as follows. For an sheaf \( \mathcal{X} \) in \( \mathcal{S} \), we denote by \( (h \downarrow \mathcal{X}) \) the comma category whose objects are arrows of the form \( h_U \to \mathcal{X} \) for \( U \in \text{ob}(\mathcal{C}) \), and let \( F_{\mathcal{X}} : (h \downarrow \mathcal{X}) \to \mathcal{S} \) be the functor which sends a morphism \( h_U \to \mathcal{X} \) to the representable sheaf \( h_{\text{Sym}^n U} \). Then, \( \text{Sym}^n(\mathcal{X}) \) is nothing but the colimit of the functor \( F_{\mathcal{X}} \).

The endofunctor \( \text{Sym}^n \) of \( \mathcal{S} \) induces an endofunctor of \( \Delta^{op} \mathcal{S} \) defined termwise. By abuse of notation, we denote this functor by the same symbol \( \text{Sym}^n \), if no confusion arises.

Example 11. Let \( n \) be a natural number. For each \( k \)-scheme \( X \) in \( \mathcal{C} \), the \( n \)-th fold geometric symmetric power \( \text{Sym}^n(\text{Spec}(k)) \) coincides with the representable functor \( \text{Spec}(k) \). Then, we deduce that the section \( \text{Sym}^n(\text{Spec}(k)) \) is nothing but the set of effective zero cycles of degree \( n \) on \( X \).

Next, we provide Lemmas 12, 13 and Proposition 14 in order to prove Corollary 15.

Lemma 12. Let \( \mathcal{C} \) be a symmetric monoidal category with finite coproducts. Let \( G \) be a finite group and let \( H \) be a subgroup of \( G \). If \( X \) is an \( H \)-object of \( \mathcal{C} \), then

\[
\text{cor}^G_H(X)/G \simeq X/H .
\]

Proof. Suppose \( X \) is an \( H \)-object of \( \mathcal{C} \). We recall that \( \text{cor}^G_0(X) \) coincides with the coproduct of \( |G| \)-copies of \( X \), it is usually denoted by \( G \times X \). Then, we observe that the group \( G \times H \) acts canonically on \( G \times X \), and by definition, \( \text{cor}^G_H(X) \) is equal to \( \text{colim}_G(G \times X) \). One can also notice that \( \text{colim}_H(G \times X) = X \). Then, we have,

\[
\text{cor}^G_H(X)/G = \text{colim}_G \text{cor}^G_H(X) = \text{colim} \text{cor}^G_H(X) = \text{colim} \text{colim}(G \times X) = \text{colim} X .
\]

By definition, \( X/H \) is equal to \( \text{colim}_H X \), thus we deduce that \( \text{cor}^G_H(X)/G \) is isomorphic to \( X/H \). 

Lemma 13. Let \( n, i, j \) be a three natural numbers such that \( i, j \leq n \) and \( i + j = n \), and let \( X_0, X_1 \) be two objects of a symmetric monoidal category with finite coproducts. Then, the symmetric group \( \Sigma_n \) acts on the coproduct \( \bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n} \) by permuting the indices of the factors, and one has an isomorphism

\[
\left( \bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n} \right) / \Sigma_n \simeq \text{Sym}^i X_0 \wedge \text{Sym}^j X_1
\]

Proof. After reordering of factors in a suitable way, we can notice that the coproduct \( \bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n} \) is isomorphic to the coproduct of \( \binom{n}{i} \)-copies of the term \( X_0^\wedge i \wedge X_1^\wedge j \), in other words, it is isomorphic to \( \text{cor}^{\sum_i} (X_0^\wedge i \wedge X_1^\wedge j) \) which is an \( \Sigma_n \)-object. By Lemma 12, we have an isomorphism

\[
\left( \text{cor}^{\sum_i} (X_0^\wedge i \wedge X_1^\wedge j) \right) / \Sigma_n \simeq (X_0^\wedge i \wedge X_1^\wedge j) / (\Sigma_i \times \Sigma_j)
\]

and the right-hand-side is isomorphic to \( \text{Sym}^i X_0 \wedge \text{Sym}^j X_1 \), which proves the expected isomorphism. \( \square \)

Proposition 14. Let \( X_0, X_1 \) be two objects of a symmetric monoidal category with finite coproducts. For any integer \( n \geq 1 \), there is an isomorphism

\[
(4) \quad \text{Sym}^n (X_0 \vee X_1) \simeq \bigvee_{i+j=n} (\text{Sym}^i X_0 \wedge \text{Sym}^j X_1).
\]

Proof. Let us fix an integer \( n \geq 1 \). We have the following isomorphism,

\[
(X_0 \vee X_1)^\wedge n \simeq \bigvee_{0 \leq j \leq n} \left( \bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n} \right),
\]

and for each index \( 0 \leq j \leq n \), the object \( \bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n} \) is invariant under the action of \( \Sigma_n \). Hence, we deduce that \( (X_0 \vee X_1)^\wedge n / \Sigma_n \) is isomorphic to the coproduct of the quotient \( \bigvee_{k_1+\cdots+k_n=j} X_{k_1} \wedge \cdots \wedge X_{k_n} / \Sigma_n \) for \( j = 0, \ldots, n \). Finally, by Lemma 13, we obtain that \( \text{Sym}^n (X_0 \vee X_1) \) is isomorphic to the coproduct \( \bigvee_{0 \leq j \leq n} (\text{Sym}^{n-j} X_0 \wedge \text{Sym}^j X_1) \), thus we have the isomorphism \( \square \).

Corollary 15. Let \( \mathcal{C} \subset \text{Sch}/k \) be an admissible category. Then for every integer \( n \geq 1 \) and for any two objects \( X, Y \) of \( \mathcal{C} \), we have an isomorphism

\[
\text{Sym}^n (X \amalg Y) \simeq \coprod_{i+j=n} (\text{Sym}^i X \times \text{Sym}^j Y).
\]

Remark 16. Let \( f \) be a morphism of the form \( X \to X \amalg Y \) in \( \mathcal{C} \). Then, for every integer \( n \geq 1 \), we have a commutative diagram

\[
\begin{array}{ccc}
\text{Sym}^n X & \xrightarrow{\text{Sym}^n (f)} & \text{Sym}^n (X \amalg Y) \\
\downarrow & & \downarrow \\
\text{Sym}^n X & \xrightarrow{\coprod_{i+j=n} (\text{Sym}^i X \times \text{Sym}^j Y)} & \coprod_{i+j=n} (\text{Sym}^i X \times \text{Sym}^j Y)
\end{array}
\]

where the right vertical arrow is the isomorphism given in Corollary 13 and the bottom arrow is the canonical morphism.
Lemma 17. Let $F$, $G$ be two objects in $\mathcal{S}$. For any integer $n \geq 1$, there is an isomorphism
\[ \text{Sym}^n_g(F \amalg G) \simeq \bigsqcup_{i+j=n} (\text{Sym}^i_g F \times \text{Sym}^j_g G) . \]

Proof. Let us fix an integer $n \geq 1$. By Corollary 15, for any two objects $X$ and $Y$ of $\mathcal{C}$, we have an isomorphism
\[ \text{Sym}^n(X \amalg Y) \simeq \bigsqcup_{i+j=n} (\text{Sym}^i X \times \text{Sym}^j Y) . \]
Since the Yoneda embedding $h : \mathcal{C} \to \mathcal{S}$ preserves finite product and coproduct, we get an isomorphism
\[ h_{\text{Sym}^n(X \amalg Y)} \simeq \bigsqcup_{i+j=n} (h_{\text{Sym}^i X} \times h_{\text{Sym}^j Y}) . \]
By definition, we have $\text{Sym}^n_g(h_X) = h_{\text{Sym}^n(X)}$, $\text{Sym}^n_g(h_Y) = h_{\text{Sym}^n(Y)}$ and $\text{Sym}^n_g(h_{X \amalg Y})$ is equal to $h_{\text{Sym}^n(X \amalg Y)}$. Hence, we deduce an isomorphism
\[ (5) \quad \text{Sym}^n_g(h_X \amalg h_Y) \simeq \bigsqcup_{i+j=n} \left( \text{Sym}^i_g(h_X) \times \text{Sym}^j_g(h_Y) \right) . \]
Let us consider the functor $\Phi_1 : \mathcal{C} \times \mathcal{C} \to \mathcal{S}$ which sends a pair $(X,Y)$ to $\text{Sym}^n_g(h_X \amalg h_Y)$ and the functor $\Phi_2 : \mathcal{C} \times \mathcal{C} \to \mathcal{S}$ which sends a pair $(X,Y)$ to $\bigsqcup_{i+j=n} (\text{Sym}^i_g(h_X) \times \text{Sym}^j_g(h_Y))$. Let
\[ \text{Lan} \Phi_1, \text{Lan} \Phi_2 : \mathcal{S} \times \mathcal{S} \to \mathcal{S} \]
be the left Kan extension of $\Phi_1$ and $\Phi_2$, respectively, along the embedding $h \times h$ from $\mathcal{C} \times \mathcal{C}$ into $\mathcal{S} \times \mathcal{S}$. By [2, Prop. 3.4.17] $\mathcal{S}$ is a Cartesian closed, hence, one deduces that the functor $\text{Lan} \Phi_1$ is nothing but the functor which sends a pair $(F,G)$ to $\text{Sym}^n_g(F \amalg G)$. Similarly, $\text{Lan} \Phi_2$ sends a pair $(F,G)$ to $\bigsqcup_{i+j=n} (\text{Sym}^i_g F \times \text{Sym}^j_g G)$ Finally, from the isomorphism (5), we have $\Phi_1 \simeq \Phi_2$, which implies that $\text{Lan} \Phi_1$ is isomorphic to $\text{Lan} \Phi_2$. This proves the lemma. 

Corollary 18. Let $\mathcal{X}$, $\mathcal{Y}$ be two objects in $\Delta^{\text{op}} \mathcal{S}$. For any integer $n \geq 1$, there is an isomorphism
\[ \text{Sym}^n_g(\mathcal{X} \amalg \mathcal{Y}) \simeq \bigsqcup_{i+j=n} (\text{Sym}^i_g \mathcal{X} \times \text{Sym}^j_g \mathcal{Y}) . \]

Remark 19. Let $f : \mathcal{X} \to \mathcal{X} \amalg \mathcal{Y}$ be a coprojection in $\Delta^{\text{op}} \mathcal{S}$. Using left Kan extensions, we deduce from Remark 16 that for every integer $n \geq 1$, we have a commutative diagram
\[ \begin{array}{ccc}
\text{Sym}^n \mathcal{X} & \xrightarrow{\text{Sym}^n(f)} & \text{Sym}^n(\mathcal{X} \vee \mathcal{Y}) \\
\downarrow & & \downarrow \\
\text{Sym}^n \mathcal{X} & \xrightarrow{\vee_{i+j=n} (\text{Sym}^i \mathcal{X} \wedge \text{Sym}^j \mathcal{Y})} & \end{array} \]
where the right vertical arrow is the isomorphism given in Corollary 18 and the bottom arrow is the canonical morphism.
We recall that $\Delta_{\text{Spec}(k)[0]}$ is the terminal object of $\Delta^{\text{op}}$. From the definition, we observe that the functor $\text{Sym}^n$ preserves terminal object $\Delta_{\text{Spec}(k)[0]}$, for $n \in \mathbb{N}$. Hence the endofunctor $\text{Sym}^n$ of $\Delta^{\text{op}}$ extends to an endofunctor of $\Delta^{\text{op}}$, denoted by the same symbol $\text{Sym}^n$, if no confusion arises.

**Corollary 20.** Let $\mathcal{X}$, $\mathcal{Y}$ be two objects in $\Delta^{\text{op}}$. For any integer $n \geq 1$, there is an isomorphism

$$\text{Sym}^n(\mathcal{X} \vee \mathcal{Y}) \simeq \bigvee_{i+j=n} (\text{Sym}^i \mathcal{X} \land \text{Sym}^j \mathcal{Y}).$$

**Proof.** It follows from Corollary 18. □

Let $\mathcal{X}$ and $\mathcal{Z}$ be two objects in an admissible category $\mathcal{C}$ and let $f : \mathcal{X} \to \mathcal{X} \vee \mathcal{Z}$ be a coprojection in $\Delta^{\text{op}}$. For every pair $(n, j) \in \mathbb{N}^2$ such that $j \leq n$, we denote by $L^n_j(f)$ the coproduct $\bigvee_{(n-j) \leq i \leq n} (\text{Sym}^i \mathcal{X} \land \text{Sym}^{n-i} \mathcal{Z})$. Then, for any coprojection $f$ as the previous definition, we have a sequence of morphisms

$$L^n_0(f) \to L^n_1(f) \to \cdots \to L^n_n(f).$$

called geometric Künneth tower of $f$. By the universal property of coproduct, the composite of the sequence (6) coincides with the canonical morphism from $L^n_0(f)$ to $L^n_n(f)$. Then, by Remark 19, we get a commutative diagram

$$
\begin{array}{ccc}
\text{Sym}^n \mathcal{X} & \xrightarrow{\text{Sym}^n(f)} & \text{Sym}^n(\mathcal{X} \vee \mathcal{Z}) \\
L^n_0(f) & \to & L^n_1(f) \\
& \to & \cdots \\
& \to & L^n_n(f)
\end{array}
$$

where the left vertical arrows is the isomorphism of Corollary 20.

A **coprojection sequence** in $\Delta^{\text{op}}$ is a diagram of the form

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Y} / \mathcal{X},$$

where $\mathcal{X} \to \mathcal{Y}$ is a termwise coprojection.

**Lemma 21.** Let $\mathcal{X} \xrightarrow{f} \mathcal{Y} \to \mathcal{Z}$ and $\mathcal{X}' \xrightarrow{f'} \mathcal{Y}' \to \mathcal{Z}'$ be two termwise coprojection sequences in $\Delta^{\text{op}}$ and let

$$
\begin{align*}
\text{Sym}^n(\mathcal{X}) &= L^n_0(f) \to L^n_1(f) \to \cdots \to L^n_n(f) = \text{Sym}^n(\mathcal{Y}), \\
\text{Sym}^n(\mathcal{X}') &= L^n_0(f') \to L^n_1(f') \to \cdots \to L^n_n(f') = \text{Sym}^n(\mathcal{Y}')
\end{align*}
$$

be their corresponding geometric Künneth towers, suppose we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\
\end{array}
$$
Then, one has a commutative diagram
\[
\begin{array}{cccccc}
\text{Sym}^n_g(X) &=& L^n_0(f) &
\longrightarrow & L^n_1(f) &
\longrightarrow & \cdots &
\longrightarrow & L^n_n(f) = \text{Sym}^n_g(Y) \\
\text{Sym}^n_g(Y') &=& L^n_0(f') &
\longrightarrow & L^n_1(f') &
\longrightarrow & \cdots &
\longrightarrow & L^n_n(f') = \text{Sym}^n_g(Y')
\end{array}
\]
in $\Delta^{op}_n$.

**Proof.** Since $f$ and $f'$ are termwise coprojections, we can assume that $f$ is of form $X \to X \vee X$ and $f'$ of the form $X \to X \vee X$. Let us fix an index $i \in \mathbb{N}$ such that $i \leq n$. Then, the morphisms $X \to X'$ and $X \to X'$ of diagram (9), induce two morphisms $\text{Sym}^j_g(X) \to \text{Sym}^j_g(X')$ and $\text{Sym}^{n-j}_g(X) \to \text{Sym}^{n-j}_g(X')$ for all indices $i \leq j \leq n$. Hence, we obtain a morphism
\[
\bigvee_{(n-i) \leq j \leq n} (\text{Sym}^j_g(X) \wedge \text{Sym}^{n-j}_g(X')) \to 
\bigvee_{(n-i) \leq j \leq n} (\text{Sym}^j_g(X') \wedge \text{Sym}^{n-j}_g(X'))
\]
and by definition this is a morphism $L^n_i(f) \to L^n_i(f')$. In view of the decompositions
\[
L^n_{i+1}(f) = L^n_i(f) \vee (\text{Sym}^{i+1}_g(X) \wedge \text{Sym}^{n-i-1}_g(X)) ,
\]
\[
L^n_{i+1}(f') = L^n_i(f') \vee (\text{Sym}^{i+1}_g(X') \wedge \text{Sym}^{n-i-1}_g(X')) ,
\]
the commutativity of (10) is clear. □

**Lemma 22.** Let $f : X \to Y$ be a coprojection in $\Delta^{op}_n$ and a let $i : X \to Y$ be a morphism in $\Delta^{op}_n$, such that $f \vee i : X \vee X \to Y$ is an isomorphism, and fix an integer $n \geq 1$. Then, for every index $1 \leq j \leq n$, we have an isomorphism
\[
L^n_j(f) / L^n_{j-1}(f) \simeq \text{Sym}^{n-j}_g(X) \wedge \text{Sym}^j_g(X') .
\]

**Proof.** For every index $1 \leq j \leq n$, $L^n_j(f)$ is, by definition, equal to the coproduct $\bigvee_{(n-j) \leq i \leq n} (\text{Sym}^i_g(X) \wedge \text{Sym}^{n-i}_g(X'))$. Hence,
\[
L^n_j(f) = L^n_{j-1}(f) \vee (\text{Sym}^{n-j}_g(X) \wedge \text{Sym}^j_g(X')) .
\]
Then, we have a cocartesian square
\[
\begin{array}{ccc}
L^n_{j-1}(f) & \longrightarrow & L^n_j(f) \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & \text{Sym}^{n-j}_g(X) \wedge \text{Sym}^j_g(X')
\end{array}
\]
thus, we have an isomorphism $L^n_j(f) / L^n_{j-1}(f) \simeq \text{Sym}^{n-j}_g(X) \wedge \text{Sym}^j_g(X')$. □
4. Geometric symmetric powers under $\mathbb{A}^1$-localization

In this section, we follow the ideas of Voevodsky in order to prove that geometric symmetric powers preserve $\mathbb{A}^1$-weak equivalences between simplicial sheaves which termwise are coproducts of representable sheaves. We also prove the existence of the left derived functors associated to geometric symmetric powers.

Let $\mathcal{C}$ be an admissible category of schemes over a field $k$. For an integer $n \geq 1$, the category $\mathcal{C}^{|\Sigma_n|}$ denotes the category of functors $\Sigma_n \to \mathcal{C}$. We recall that $\mathcal{C}^{|\Sigma_n|}$ can be viewed as the category of $\Sigma_n$-objects of $\mathcal{C}$. Let $X$ be an $\Sigma_n$-object on $\mathcal{C}$ and let $x \in X$. The stabilizer of $x$ is by definition the subgroup $\text{stab}(x)$ consisting of elements $\sigma \in \Sigma_n$ such that $\sigma.x = x$. Let $X$ be an object $\mathcal{C}^{|\Sigma_n|}$. A family of morphisms $\{f_i : U_i \to X\}_{i \in I}$ in $\mathcal{C}^{|\Sigma_n|}$ is called $\Sigma_n$-equivariant Nisnevich covering, if each morphism $f_i$ viewed as a morphism of $\mathcal{C}$ is $\acute{e}tale$ and we have the following property: for each point $x \in X$, viewed as an object of $\mathcal{C}$, there exist an index $i \in I$ and a point $y \in U_i$ such that: $f_i(y) = x$, the canonical homomorphism of residual fields $k(x) \to k(y)$ is an isomorphism, and the induced homomorphisms of groups $\text{stab}(y) \hookrightarrow \text{stab}(x)$ is an isomorphism. We shall denote by $(\mathcal{C}^{|\Sigma_n|})_{\text{Nis}}$ the site consisting of $\mathcal{C}^{|\Sigma_n|}$ and the Grothendieck topology formed by the $\Sigma_n$-equivariant Nisnevich coverings. We will the symbol $\mathcal{P}^{|\Sigma_n|}$ to denote the category of sheaves on $(\mathcal{C}^{|\Sigma_n|})_{\text{Nis}}$.

**Remark 23.** For $n = 1$, a $\Sigma_n$-equivariant Nisnevich covering is a usual Nisnevich covering in $\mathcal{C}$.

A Cartesian square in $\mathcal{C}^{|\Sigma_n|}$ of the form (2) is an *elementary distinguished square* if $p$ is an $\acute{e}tale$ morphism and $j$ is an open embedding when we forget the action of $\Sigma_n$, such that the morphism of reduced schemes $p^{-1}(X - U)_{\text{red}} \to (X - U)_{\text{red}}$ is an isomorphism. Notice that when $n = 1$, this definition coincide with the usual one. The above square induces a diagram

$$
\Delta_Y[0]_+ \lor \Delta_Y[0]_+ \quad \Delta_Y[0]_+ \land \Delta[1]_+
$$

$$
\Downarrow
$$

$$
\Delta_U[0]_+ \lor \Delta_U[0]_+
$$

We denote by $K_\mathcal{Q}$ the pushout in $\mathcal{D}^{|\text{op}|,\mathcal{P}^{|\Sigma_n|}}$ of the above diagram and denote by $\mathcal{G}_{\Sigma_n,\text{Nis}}$ the set of morphisms in $\mathcal{C}^{|\Sigma_n|}$ of canonical morphisms from $K_\mathcal{Q}$ to $\Delta_X[0]_+$. The set $\mathcal{G}_{\Sigma_n,\text{Nis}}$ is called *set of generating Nisnevich equivalences*. On the other hand, we denote by $\mathcal{P}_{\Sigma_n,\mathbb{A}^1}$ the set of morphisms in $\mathcal{C}^{|\Sigma_n|}$ which is isomorphic to the projection from $\Delta_X[0]_+ \land \Delta_{\mathbb{A}^1}[0]_+$ to $\Delta_X[0]_+$, for $X$ in $\mathcal{C}^{|\Sigma_n|}$. By Lemma 13 [3] page 392, the class of weak equivalences of $\mathcal{P}^{|\Sigma_n|}$ coincides with the class

$$(11)\quad \text{cl}_\Delta(\mathcal{G}_{\Sigma_n,\text{Nis}} \cup \mathcal{P}_{\Sigma_n,\mathbb{A}^1}).$$

We denote by $\text{Const} : \mathcal{C} \to \mathcal{C}^{|\Sigma_n|}$ the functor which sends $X$ to the $\Sigma_n$-object $X$ with the trivial action. Let $\text{colim}_{\Sigma_n} : \mathcal{C}^{|\Sigma_n|} \to \mathcal{C}$ be the functor which sends $X$ to $\text{colim}_{\Sigma_n}X = X / \Sigma_n$. By definition of colimit, the functor $\text{colim}_{\Sigma_n}$ is left adjoint to
Const. It turns out that the functor Const preserves finite limits and it sends Nisnevich coverings to \(\Sigma_n\)-equivariant Nisnevich coverings. In consequence, the functor Const is continuous and the functor \(\text{colim}_{\Sigma_n}\) is cocontinuous.

Let \(\Lambda_n : \mathcal{C} \to \mathcal{C}^{\Sigma_n}\) be the functor which sends \(X\) to the \(n\)th fold product \(X^{\times n}\). Then, the endofunctor \(\text{Sym}^n\) of \(\mathcal{C}\) is nothing but the composition of \(\text{colim}_{\Sigma_n}\) with \(\Lambda_n\).

**Proposition 24.** The cocontinuous functor \(\text{colim}_{\Sigma_n} : (\mathcal{C}^{\Sigma_n})_{\text{Nis}} \to \mathcal{C}_{\text{Nis}}\) is continuous. In consequence, it is a morphism of sites.

**Proof.** See [3, Prop. 43] \(\square\)

By previous proposition implies that the functor \(\text{colim}_{\Sigma_n}\) is a morphism of sites, then it induces an adjunction between the inverse and direct image functors,

\[
\text{(colim)}_{\Sigma_n}^* \dashv \text{colim}_{\Sigma_n} : \mathcal{C}_{\text{Nis}} \rightleftarrows (\mathcal{C}^{\Sigma_n})_{\text{Nis}}
\]

moreover, one has a commutative diagram up to isomorphisms

\[\begin{array}{ccc}
(\mathcal{C}^{\Sigma_n})_{\text{Nis}} & \xrightarrow{\text{colim}_{\Sigma_n}} & \mathcal{C}_{\text{Nis}} \\
\downarrow h & & \downarrow h \\
\mathcal{C}^{\Sigma_n} & \xrightarrow{(\text{colim}_{\Sigma_n})^*} & \mathcal{I}
\end{array}\]

where \(h\) is the Yoneda’s embedding. We denote by

\[
\gamma_n : \Delta^{\text{op}} \mathcal{I}^{\Sigma_n} \longrightarrow \Delta^{\text{op}} \mathcal{I}
\]

the functor induced by \((\text{colim}_{\Sigma_n})^*\) defined termwise. From the diagram \([12]\), we deduce that \(\gamma_n\) preserve terminal object, then it induces a functor

\[
\gamma_{n,+} : \Delta^{\text{op}} \mathcal{I}^{\Sigma_n} \longrightarrow \Delta^{\text{op}} \mathcal{I}^*.
\]

We write \(\tilde{\Lambda}_n\) for the left Kan extension of the composite \(\mathcal{C} \xrightarrow{\Lambda_n} \mathcal{C}^{\Sigma_n} \xrightarrow{h} \mathcal{I}^{\Sigma_n}\) along the Yoneda embedding \(h : \mathcal{C} \to \mathcal{I}\). Denote by

\[
\lambda_n : \Delta^{\text{op}} \mathcal{I} \longrightarrow \Delta^{\text{op}} \mathcal{I}^{\Sigma_n}
\]

the functor induced by \(\tilde{\Lambda}_n\) defined termwise. Since \(\tilde{\Lambda}_n\) preserves terminal objects, the functor \(\lambda_n\) does so, hence it induces a functor

\[
\lambda_{n,+} : \Delta^{\text{op}} \mathcal{I}^* \longrightarrow \Delta^{\text{op}} \mathcal{I}^{\Sigma_n}.
\]

The following lemma will be used in Lemma 26.

**Lemma 25.** Left adjoint functors preserves left Kan extensions, in the following sense. Let \(L : \mathcal{E} \to \mathcal{E}'\) be a left adjoint functor. If \(\text{Lan}_G F\) is the left Kan extension of a functor \(F : \mathcal{C} \to \mathcal{E}\) along a functor \(G : \mathcal{C} \to \mathcal{I}\), then the composite \(L \circ \text{Lan}_G F\) is the left Kan extension of the composite \(L \circ F\) along \(G\).

**Proof.** See [8, Lemma 1.3.3]. \(\square\)

**Lemma 26.** For every natural \(n\), the endofunctor \(\text{Sym}^n\) of \(\Delta^{\text{op}} \mathcal{I}\) is isomorphic to the composition \(\gamma_n \circ \lambda_n\). In consequence, \(\text{Sym}^n\) as an endofunctor of \(\Delta^{\text{op}} \mathcal{I^*}\) is isomorphic to the composition \(\gamma_{n,+} \circ \lambda_{n,+}\).
Proof. Since the functors $\text{Sym}^n_g$, $\gamma_n$ and $\lambda_n$ are defined termwise, it is enough to show that $\text{Sym}^n_g$ as an endofunctor of $\mathcal{S}$, is isomorphic to the composition of $\hat{\Lambda}_n$ with $(\text{colim}_{\Sigma_n})^*$. Since the functor $(\text{colim}_{\Sigma_n})^*$ is left adjoint, Lemma 25 implies that the composite

$$
\mathcal{S} \xrightarrow{\hat{\Lambda}_n} \mathcal{S}_{\Sigma_n} \xrightarrow{(\text{colim}_{\Sigma_n})^*} \mathcal{S}
$$

is the left Kan extension of the composite

$$
\mathcal{C} \xrightarrow{\Lambda_n} \mathcal{C}_{\Sigma_n} \xrightarrow{h} \mathcal{C}_{\Sigma_n} \xrightarrow{(\text{colim}_{\Sigma_n})^*} \mathcal{C}
$$

along the embedding $h : \mathcal{C} \to \mathcal{S}$. Now, in view of the commutativity of diagram (12), the preceding composite is isomorphic to the composite

$$
\mathcal{C} \xrightarrow{\Lambda_n} \mathcal{C}_{\Sigma_n} \xrightarrow{\text{colim}_{\Sigma_n}} \mathcal{C} \xrightarrow{h} \mathcal{S}
$$

but it is isomorphic to the composite $\mathcal{C} \xrightarrow{\text{Sym}^n_g} \mathcal{C} \xrightarrow{h} \mathcal{S}$, which implies that the composite (13) is isomorphic to $\text{Sym}^n_g$, as required. □

We denote by $\mathcal{C}_+^+$ the full subcategory of coproducts of pointed objects of the form $(h_X)_+$ in $\mathcal{S}$ for objects $X$ in $\mathcal{C}$. For every object $X$ in $\mathcal{C}$, the pointed sheaf $(h_X)_+$ is isomorphic to $h_{(X, +)}$. Indeed, $(h_X)_+$ is by definition equal to the coproduct $h_X \amalg h_{\text{Spec}(k)}$ and this coproduct is isomorphic to the representable functor $h_{(X, +)\text{Spec}(k)}$ which is equal to $h_{(X, +)}$.

Similarly, we denote by $\mathcal{C}_+^{\Sigma_n}$ the full subcategory of coproducts of pointed objects $(h_X)_+$ in $\mathcal{S}_{\Sigma_n}$ for objects $X$ in $\mathcal{C}_{\Sigma_n}$.

Theorem 27 (Voevodsky). Let $f : \mathcal{S} \to \mathcal{Y}$ be a morphism in $\Delta^{\text{op}} \mathcal{C}_+$. If $f$ is an $\mathbb{A}^1$-weak equivalence in $\Delta^{\text{op}} \mathcal{S}$, then $\text{Sym}^n_g(f)$ is an $\mathbb{A}^1$-weak equivalence.

Proof. We give an outline of the proof. By Lemma 26, $\text{Sym}^n_g$ is the composition $\gamma_{n, +} \circ \lambda_{n, +}$. The idea is to prove that $\gamma_{n, +}$ and $\lambda_{n, +}$ preserve $\mathbb{A}^1$-weak equivalences between objects which termwise are coproducts of representable sheaves. The functor $\lambda_{n, +}$ sends morphisms of $\mathcal{W}_{\text{Nis}, +} \cup \mathcal{P}_{\text{Nis}, +}$ between objects in $\Delta^{\text{op}} \mathcal{C}_+$ to $\mathbb{A}^1$-weak equivalences between objects in $\Delta^{\text{op}} \mathcal{C}_+^{\Sigma_n}$. Since $\Lambda_n$ preserves colimits, Lemma 2.20 of [10] implies that $\gamma_{n, +}$ preserves $\mathbb{A}^1$-weak equivalence as claimed. Similarly, in view of the class given in [11], we use the same lemma to prove that $\gamma_{n, +}$ sends an $\mathbb{A}^1$-weak equivalence between objects in $\Delta^{\text{op}} \mathcal{C}_+^{\Sigma_n}$ to an $\mathbb{A}^1$-weak equivalence. This completes the proof. □

We define the functor $\Phi : \Delta^{\text{op}} \mathcal{C}_+ \to \mathcal{H}_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1)$ as the composite

$$
\Delta^{\text{op}} \mathcal{C}_+ \hookrightarrow \Delta^{\text{op}} \mathcal{S} \to \mathcal{H}_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1).
$$

where the first arrow is the inclusion functor and the second arrow is the localization functor with respect to the $\mathbb{A}^1$-weak equivalences.

Lemma 28. Let $\mathcal{C}$ be an admissible category. The functor

$$
\Phi : \Delta^{\text{op}} \mathcal{C}_+ \to \mathcal{H}_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1)
$$

is a strict localization, that is, for every morphism $f$ in $\mathcal{H}_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1)$, there is a morphism $g$ of $\Delta^{\text{op}} \mathcal{C}_+$ such that the image $\Phi(g)$ is isomorphic to $f$. 


Proof. By Theorem 2.5 of [7, page 71], the category $\mathcal{H}_e(\mathcal{C}_{Nis}, A^1)$ is the localization of the category $\mathcal{H}_e(\mathcal{C}_{Nis})$ with respect to the image of $A^1$-weak equivalences through the canonical functor. Then, it is enough to prove that the canonical functor $\Delta^{op}\tilde{\mathcal{C}}_+ \to \mathcal{H}_e(\mathcal{C}_{Nis})$ is a strict localization. Indeed, let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of pointed simplicial sheaves on the site $\mathcal{C}_{Nis}$ representing a morphism in $\mathcal{H}_e(\mathcal{C}_{Nis})$. The functorial resolution $Q^{proj}$ gives a commutative square

$$
\begin{array}{ccc}
Q^{proj}(\mathcal{X}) & \xrightarrow{Q^{proj}(f)} & Q^{proj}(\mathcal{Y}) \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
$$

where the vertical arrows are object-wise weak equivalences. Since the object-wise weak equivalences are local weak equivalences, the vertical arrows of the above diagram are weak equivalences. This implies that $f$ is isomorphic to $Q^{proj}(f)$ in $\mathcal{H}_e(\mathcal{C}_{Nis})$. Moreover, by Corollary 9, the morphism $Q^{proj}(f)$ is in $\Delta^{op}\tilde{\mathcal{C}}_+$. □

Corollary 29. For each integer $n \geq 1$, there exists the left derived functor $L\text{Sym}_g^n$ from $\mathcal{H}_e(\mathcal{C}_{Nis}, A^1)$ to itself such that we have a commutative diagram up to isomorphism

$$
\begin{array}{ccc}
\Delta^{op}\tilde{\mathcal{C}}_+ & \xrightarrow{\text{Sym}_g^n} & \Delta^{op}\mathcal{I}_* \\
\downarrow & & \downarrow \\
\mathcal{H}_e(\mathcal{C}_{Nis}, A^1) & \xrightarrow{L\text{Sym}_g^n} & \mathcal{H}_e(\mathcal{C}_{Nis}, A^1)
\end{array}
$$

where the right arrow is the localization functor.

Proof. By theorem 27 the functor $\text{Sym}_g^n$ preserves $A^1$-weak equivalences between objects in $\Delta^{op}\tilde{\mathcal{C}}_+$. Hence, the composite

$$
\Delta^{op}\tilde{\mathcal{C}}_+ \xrightarrow{\text{Sym}_g^n} \Delta^{op}\mathcal{I}_* \xrightarrow{\phi} \mathcal{H}_e(\mathcal{C}_{Nis}, A^1)
$$

sends $A^1$-weak equivalences to isomorphisms. Then, by Lemma 28 there exists a functor $L\text{Sym}_g^n$ such the diagram (14) commutes and for every simplicial sheaf $\mathcal{X}$, the object $L\text{Sym}_g^n(\mathcal{X})$ is isomorphic to $\text{Sym}_g^n(Q^{proj}(\mathcal{X}))$ in $\mathcal{H}_e(\mathcal{C}_{Nis}, A^1)$. □

Lemma 30. The endofunctor $L\text{Sym}_g^0$ of $\mathcal{H}_e(\mathcal{C}_{Nis}, A^1)$ is the constant functor with value $1$, where $1$ is the object $\Delta_{\text{Spec}(k)[0]}$ in $\mathcal{H}_e(\mathcal{C}_{Nis}, A^1)$, and the endofunctor $L\text{Sym}_g^1$ is the identity functor on $\mathcal{H}_e(\mathcal{C}_{Nis}, A^1)$.

Proof. Since $\text{Sym}^0 X = \text{Spec}(k)$ for every scheme $X$ in $\mathcal{C}$, the endofunctor $\text{Sym}^0$ of $\mathcal{C}$ is constant with value $\text{Spec}(k)$. By the left Kan extension, we deduce that $\text{Sym}^0$ extends to an endofunctor $\text{Sym}_g^0$ of $\Delta^{op}\mathcal{I}$ given by $\mathcal{X} \mapsto \Delta_{\text{Spec}(k)[0]}$. Hence, we deduce that $L\text{Sym}_g^0$ is the endofunctor of $\mathcal{H}_e(\mathcal{C}_{Nis}, A^1)$ given by $\mathcal{X} \mapsto 1$. On the other hand, for every scheme $X$ in $\mathcal{C}$, we have $\text{Sym}^1 X = X$. By the left Kan
extension, we deduce that the endofunctor \( \text{Sym}^1_g \) of \( \Delta^{op} \) is the identity functor, then \( L\text{Sym}^0_g \) is the identity functor on \( \mathcal{H}_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1) \).

\[ \square \]

5. CATEGORIC LAMBDA STRUCTURES AND THE MAIN RESULT

Our goal in this section is to prove the main Theorem 34 which asserts that the left derived geometric symmetric powers \( L\text{Sym}^n_g \), for \( n \in \mathbb{N} \) (see Corollary 29), induce a \( \lambda \)-structure on the pointed motivic homotopy category \( \mathcal{H}_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1) \). We start by giving the definition of \( \lambda \)-structure on the homotopy category of a symmetric monoidal model category appearing in [4].

Let \( \mathcal{C} \) be a closed symmetric monoidal model category with unit \( 1 \). A \( \lambda \)-structure on \( \text{Ho}(\mathcal{C}) \) is a sequence \( \Lambda^* = (\Lambda^0, \Lambda^1, \Lambda^2, \ldots) \) consisting of endofunctors \( \Lambda^n \) of \( \text{Ho}(\mathcal{C}) \) for \( n \in \mathbb{N} \), satisfying the following axioms:

(i) \( \Lambda^0 = 1 \), \( \Lambda^1 = \text{id} \)

(ii) (Künneth tower axiom). For any cofibre sequence \( X \to Y \to Z \) in \( \text{Ho}(\mathcal{C}) \), and any \( n \in \mathbb{N} \), there is a unique sequence

\[ \Lambda^n(X) = L_0^n \to L_1^n \to \cdots \to L_i^n \to \cdots \to L_n^n = \Lambda^n(Y) \]

called Künneth tower, such that for any index \( 0 \leq i \leq n \), the quotient \( L_i^n/L_{i-1}^n \) in \( \mathcal{C} \) is weak equivalent to the product \( \Lambda^{n-i}(X) \wedge \Lambda^i(Z) \).

(iii) (Functoriality axiom). For any morphism of cofibre sequences

\[ X \to Y \to Z \]

\[ X' \to Y' \to Z' \]

in \( \text{Ho}(\mathcal{C}) \), there is a commutative diagram

\[ \Lambda^n(X) = L_0^n \to L_1^n \to L_2^n \to \cdots \to L_{i-1}^n \to L_i^n \to \cdots \to L_{n-1}^n \to L_n^n = \Lambda^n(Y) \]

\[ \Lambda^n(X') = L_0^m \to L_1^m \to L_2^m \to \cdots \to L_{i-1}^m \to L_i^m \to \cdots \to L_{n-1}^m \to L_n^m = \Lambda^n(Y') \]

in \( \text{Ho}(\mathcal{C}) \), were the horizontal sequences are their respective Künneth towers.

Let \( \text{Sym}^n \) be the abstract \( n \)th fold symmetric power defined on \( \Delta^{op} \), for \( n \in \mathbb{N} \), which is defined for every pointed simplicial sheaf \( \mathcal{F} \) to be the quotient \( \text{Sym}^n(\mathcal{F}) := (\mathcal{F}^{*n})/\Sigma_n \). Then, the left derived functors \( L\text{Sym}^n \), for \( n \in \mathbb{N} \), provide a \( \lambda \)-structure on \( \mathcal{H}_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1) \) (this result appears in [4] Theorem 57) in the context Nisnevich sheaves on the category of smooth schemes). Indeed, the morphism \( \Delta_{\mathbb{A}^1}[0] \to \Delta_{\text{Spec}(k)}[0] \) is a diagonalizable interval, meaning that \( \Delta_{\mathbb{A}^1}[0] \) has a structure of symmetric co-algebra in the category \( \Delta^{op} \). Notice that all objects of \( \Delta^{op} \) are cofibrant. We claim that the class of cofibrations and the class of trivial cofibrations in \( \Delta^{op} \) are symmetrizable. Since cofibrations in \( \Delta^{op} \) are section-wise cofibrations of simplicial sets, it follows from Proposition 55 of [4] that cofibrations are
symmetrizable. Let $f$ be a trivial cofibration in $\Delta^{op}\mathcal{S}$. By the previous case $f$ is a symmetrizable cofibration. For every point $P$ of the site $\mathcal{C}_{\text{Nis}}$, the induced morphism $f_P$ is a weak equivalence of simplicial sets. By [1] Lemma 54, the $n$th fold symmetric power $\text{Sym}^n(f_P)$ is also a weak equivalence. Since the morphism $\text{Sym}^n(f_P)$ coincide with $\text{Sym}^n(f_P)$, we deduce that the $n$th fold symmetric power $\text{Sym}^n(f)$ is a weak equivalence too. Hence, by [1] Corollary 54, $f$ is a symmetrizable trivial cofibration. Finally, Theorem 38 and Theorem 22 of [1] imply the existence of left derived functors $L\text{Sym}^n$, for $n \in \mathbb{N}$, and they provide a $\lambda$-structure on $H_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1)$.

**Lemma 31.** Let $f : X \to Y$ be a termwise coprojection in $\Delta^{op}\mathcal{C}_{+}$. Then, the quotient $Y/X$ is in $\Delta^{op}\mathcal{C}_{+}$ and the sequence $X \to Y \to Y/X$ splits, that is, $Y$ is isomorphic to $X \vee (Y/X)$.

**Proof.** Let $f : X \to Y$ be a termwise coprojection in $\Delta^{op}\mathcal{C}_{+}$. Then, for each natural number $n$, there exist two sets of indices $I_n$ and $J_n$ such that $I_n \subset J_n$, and a family of objects $\{X_{n,i}\}_{i \in J_n}$ of objects of $\mathcal{C}$ such that $X_n = \bigvee_{i \in I_n} (hX_{n,i})_+$ and $Y_n = \bigvee_{i \in J_n} (hX_{n,i})_+$. If we put $Z_n := \bigvee_{i \in J_n \setminus I_n} (hX_{n,i})_+$, then we have $Y_n = X_n \vee Z_n$. Hence, we get a cocartesian square

$$
\begin{array}{ccc}
X_n & \to & Y_n \\
\downarrow & & \downarrow \\
Z_n & \to & \ast
\end{array}
$$

By the functoriality of pushouts, the get an simplicial object $Z$ in $\Delta^{op}\mathcal{C}_{+}$ such that $X \simeq Y/X$ and $Y \simeq X \vee Z$. \hfill \Box

**Proposition 32.** Let $\mathcal{C}$ be an admissible category. We have the following assertions:

(a) Every cofibre sequence $X \to Y \to Z$ in $H_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1)$ is isomorphic to a split coprojection sequence of the form

$$
Q^{\text{proj}}(X) \to Q^{\text{proj}}(X) \vee Q^{\text{proj}}(Y) \to Q^{\text{proj}}(Z).
$$

(b) Every morphism of cofibre sequences

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow{f'} & Y'
\end{array}
$$

in $\mathcal{H}_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1)$, is isomorphic to a morphism of split cofibre sequences of objects in $\Delta^{op}\mathcal{C}_{+}$.

**Proof.** (a). Let $X \to Y \to Z$ be a cofibre sequence in $\mathcal{H}_*(\mathcal{C}_{\text{Nis}}, \mathbb{A}^1)$, where $f$ is a cofibration from $X$ to $Y$ in $\Delta^{op}\mathcal{S}$, such that $Z = Y/X$. Let $Q^{\text{proj}}$ be the cofibrant resolution given in section [2] and let us consider the induced morphism $Q^{\text{proj}}(X) \to X$. By Corollary [3] the composition of $Q^{\text{proj}}(X) \to X$ with $f$ induces
a commutative diagram

\[
\begin{array}{c}
\begin{tikzcd}
\mathcal{X} \arrow[r, \alpha(f)] \arrow[d, f] & \mathcal{A} \arrow[d, \beta(f)] \\
\mathcal{X} \arrow[r, \psi] & \mathcal{Y}
\end{tikzcd}
\end{array}
\]

where \( \beta(f) \) is a section wise trivial fibration and \( \alpha(f) \) is a coprojection such that each term \( \alpha(f)_n \) has the form \( Q_{\text{proj}}(\mathcal{X})_n \to Q_{\text{proj}}(\mathcal{X})_n \vee F_n \), where \( F_n \) is a coproduct of representable functors. By Corollary \( \Box \) \( Q_{\text{proj}}(\mathcal{X}) \) is an object of \( \Delta^{\text{op}} \mathcal{E}^+ \). Hence, \( \mathcal{A} \) is an object of \( \Delta^{\text{op}} \mathcal{E}^+ \). Thus, \( \alpha(f) \) is a termwise coprojection in \( \Delta^{\text{op}} \mathcal{E}^+ \), and by Lemma \( \Box \) it induces a split sequence. We put \( \mathcal{B} := \mathcal{A}/Q_{\text{proj}}(\mathcal{X}) \). Then \( \mathcal{A} \) is isomorphic to \( Q_{\text{proj}}(\mathcal{X}) \vee \mathcal{B} \). By Lemma \( \Box \) the canonical morphism

\[
Q_{\text{proj}}(\mathcal{X}) \vee Q_{\text{proj}}(\mathcal{B}) \to Q_{\text{proj}}(\mathcal{X}) \vee \mathcal{B},
\]

induced by \( Q_{\text{proj}}(\mathcal{B}) \to \mathcal{B} \), is a weak equivalence (hence an \( \mathbb{A}^1 \)-weak equivalence). Combining the above morphism with above square, we deduce a commutative square

\[
\begin{array}{c}
\begin{tikzcd}
\mathcal{X} \arrow[r, \varphi] \arrow[d, f] & Q_{\text{proj}}(\mathcal{X}) \vee Q_{\text{proj}}(\mathcal{B}) \arrow[d] \\
\mathcal{X} \arrow[r, \psi] & \mathcal{Y}
\end{tikzcd}
\end{array}
\]

where the vertical arrows are \( \mathbb{A}^1 \)-weak equivalences and \( \varphi \) is the canonical morphism. This square induces a commutative cube of cofibrant objects

\[
\begin{array}{c}
\begin{tikzcd}
\mathcal{X} \arrow[r, \varphi] \arrow[d, f] & Q_{\text{proj}}(\mathcal{X}) \vee Q_{\text{proj}}(\mathcal{B}) \arrow[d] \\
\mathcal{X} \arrow[r, \psi] & \mathcal{Y}
\end{tikzcd}
\end{array}
\]

in \( \Delta^{\text{op}} \mathcal{E}^+ \), where the squares on the front and on the back are cocartesian. Now, notice that \( f \) and \( \varphi \) are cofibrations, and both morphisms \( Q_{\text{proj}}(\mathcal{X}) \to \mathcal{X} \) and \( Q_{\text{proj}}(\mathcal{X}) \vee Q_{\text{proj}}(\mathcal{B}) \to \mathcal{Y} \) are \( \mathbb{A}^1 \)-weak equivalences. Then the cube lemma (see
Lemma 5.2.6) guarantees that the morphism $Q^{proj}(\mathcal{X}) \to \mathcal{Z}$ is also an $A^1$-weak equivalence in $\Delta^{op}_*$. Finally, by Lemma 6.2.5, the cofibre sequence

$$Q^{proj}(\mathcal{X}) \to Q^{proj}(\mathcal{X}) \cup Q^{proj}(\mathcal{B}) \to Q^{proj}(\mathcal{B})$$

is in $\Delta^{op}_{\mathcal{C}^+}$ and is isomorphic to the cofibre sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ in $\mathcal{H}_*(\mathcal{C}_{Nis}, A^1)$.

(b). By (a), a commutative diagram (15) induces a commutative diagram

$$\begin{array}{ccc}
Q^{proj}(\mathcal{X}) & \xrightarrow{\varphi} & Q^{proj}(\mathcal{X}) \cup Q^{proj}(\mathcal{B}) \\
\downarrow & & \downarrow \\
Q^{proj}(\mathcal{X}') & \xrightarrow{\varphi'} & Q^{proj}(\mathcal{X}') \cup Q^{proj}(\mathcal{B}')
\end{array}$$

where square on the front is a commutative square in $\Delta^{op}_{\mathcal{C}^+}$ such that $\varphi$ and $\varphi'$ are the canonical morphisms. Therefore, diagram (15) is isomorphic to the induced commutative diagram

$$\begin{array}{ccc}
Q^{proj}(\mathcal{X}) & \xrightarrow{\varphi} & Q^{proj}(\mathcal{X}) \cup Q^{proj}(\mathcal{B}) \to Q^{proj}(\mathcal{B}) \\
\downarrow & & \downarrow \\
Q^{proj}(\mathcal{X}') & \xrightarrow{\varphi'} & Q^{proj}(\mathcal{X}') \cup Q^{proj}(\mathcal{B}') \to Q^{proj}(\mathcal{B}')
\end{array}$$

in $\mathcal{H}_*(\mathcal{C}_{Nis}, A^1)$.

Proposition 33. For every termwise coprojection $\mathcal{X} \to \mathcal{Y}$ in $\Delta^{op}_{\mathcal{C}^+}$ and every index $1 \leq i \leq n$, we have a canonical isomorphism

$$L^n_i(f)/L^n_{i-1}(f) \simeq \text{Sym}^{n-i}(\mathcal{X}) \wedge \text{Sym}^i_g(\mathcal{Y}/\mathcal{X}).$$

Proof. By Lemma 31, the coprojection sequence $\mathcal{X} \to \mathcal{Y} \to \mathcal{Y}/\mathcal{X}$ splits, that is, we have an isomorphism $\mathcal{Y} \simeq \mathcal{X} \cup (\mathcal{Y}/\mathcal{X})$. Hence, by Lemma 22 we deduce an isomorphism

$$L^n_i(f)/L^n_{i-1}(f) \simeq \text{Sym}^{n-i}(\mathcal{X}) \wedge \text{Sym}^i_g(\mathcal{Y}/\mathcal{X}).$$
Let $C$ be an admissible category. By Lemma [10] the class of $\mathbb{A}^1$-weak equivalences in $\Delta^{\text{op}}C$ is closed under coproducts and smash products, then, by [5] Prop. 8.4.8, the functors
\[(− \vee −), (− \wedge −) : \Delta^{\text{op}}C \times \Delta^{\text{op}}C \to \Delta^{\text{op}}C\]
extend, respectively, to the left derived functors
\[(− \wedge L −), (− \vee L −) : H_*(C_{\text{Nis}}, \mathbb{A}^1) \times H_*(C_{\text{Nis}}, \mathbb{A}^1) \to H_*(C_{\text{Nis}}, \mathbb{A}^1) .\]

Let $f : X \to Y$ be a coprojection in $\Delta^{\text{op}}C$ of the form $X \to X \vee Z$. Then, for every index $1 \leq i \leq n$, we have
\[L^n_i(f) = \bigvee_{(n-i) \leq j \leq n} (\text{Sym}^j X \wedge \text{Sym}^{n-j} Z) .\]

Let us consider the functors $L\text{Sym}^n_i$, for $i \in \mathbb{N}$, of Corollary [29]. We define the object $L^n_i(f)$ in $H_*(C_{\text{Nis}}, \mathbb{A}^1)$ to be
\[\bigvee_{(n-i) \leq j \leq n} (L\text{Sym}^j X \wedge L\text{Sym}^{n-j} Z) ,\]
where the coproduct is the coproduct $\vee^L$ is defined above.

Now, we are ready to state and prove our main theorem.

**Theorem 34.** Let $C$ be an admissible category. The endofunctors $L\text{Sym}^n_i$, for $n \in \mathbb{N}$, provides a $\lambda$-structure on $H_*(C_{\text{Nis}}, \mathbb{A}^1)$.

**Proof.** By Lemma [30] $L\text{Sym}^0_i$ is the functor $X \mapsto 1$ and the $L\text{Sym}^1_i$ is the identity functor. Let $X \to Y \to Z$ be a cofibration sequence in $H_*(C_{\text{Nis}}, \mathbb{A}^1)$, induced by a cofibration $f : X \to Y$ in the injective model structure of $\Delta^{\text{op}}C$. By Proposition [32](a), we can assume that $f$ is in $\Delta^{\text{op}}C$, $Y$ has the form $X \vee Z$ and $Z$ is isomorphic to the quotient $Y / X$. Then, by Lemma [21](a), there is a sequence,
\[\text{Sym}^n_i(X) = L^n_0(f) \to L^n_1(f) \to \cdots \to L^n_n(f) = \text{Sym}^n_i(Y) ,\]
which induces a sequence,
\[L\text{Sym}^n_i(Z) = L^n_0(f) \to L^n_1(f) \to \cdots \to L^n_n(f) = L\text{Sym}^n_i(Y) ,\]
of morphisms in $H_*(C_{\text{Nis}}, \mathbb{A}^1)$. By Proposition [33] for each index $1 \leq i \leq n$, we get a canonical isomorphism $L^n_i(f) / L^n_{i-1}(f) \simeq \text{Sym}^{n-i}(X) \wedge \text{Sym}^i(X)$, as pointed simplicial sheaves. It follows that each object $L^n_i(f) / L^n_{i-1}(f)$, viewed as an object of $H_*(C_{\text{Nis}}, \mathbb{A}^1)$, is isomorphic to $L\text{Sym}^{n-i}(X) \wedge^L L\text{Sym}^i(Z)$. This proves the Künneth tower axiom. Finally, the functoriality axiom follows from Proposition [32](b) and Lemma [21].

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