Abstract

The purpose of this tutorial is to give a brief introduction to linear quantum control systems. The mathematical model of linear quantum control systems is presented first, then some fundamental control-theoretic notions such as stability, controllability and observability are given, which are closely related to several important concepts in quantum information science such as decoherence-free subsystems, quantum non-demolition variables, and back-action evasion measurements. After that, quantum Gaussian states are introduced, in particular, an information-theoretic uncertainty relation is presented which often gives a better bound for mixed Gaussian states than the well-known Heisenberg uncertainty relation. The quantum Kalman filter is presented for quantum linear systems, which is the quantum analogy of the Kalman filter for classical (namely, non-quantum-mechanical) linear systems. The quantum Kalman canonical decomposition for quantum linear systems is recorded, and its application is illustrated by means of a recent experiment. As single- and multi-photon states are useful resources in quantum information technology, the response of quantum linear systems to these types of input is presented. Finally, coherent feedback control of quantum linear systems is briefly introduced, and a recent experiment is used to demonstrate the effectiveness of quantum linear systems and networks theory.

Keywords: quantum linear control systems, quantum Kalman filter, quantum Kalman canonical form, quantum coherent feedback networks

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Contents

1 Introduction 2
2 Quantum linear systems 4
3 Hurwitz stability, controllability and observability 10
4 Quantum Gaussian states 12
4.1 An introduction 12
4.2 Pure Gaussian state generation 15
4.3 Skew information and information-theoretic uncertainty relation 15

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1. Introduction

The dynamics of quantum systems are governed by quantum-mechanical laws. The temporal evolution of a quantum system can be described by its state evolution in the Schrödinger picture. Alternatively, it can also be described by the evolution of system variables for example position and momentum of a quantum harmonic oscillator, this is the so-called Heisenberg picture. System variables are operators in a Hilbert space, instead of ordinary functions. Therefore, the operations of these system variables may not commute. Specifically, let $A, B$ be two system variables (operators), $AB \neq BA$ may occur. Non-commutativity renders quantum systems fundamentally different from classical systems where system variables are functions of time and two system variables always commute.

A linear quantum system is a quantum system whose temporal evolution in the Heisenberg picture can be described by a set of linear differential equations of system variables. Many physical systems can be well modeled as linear quantum systems, for instance, quantum optical systems \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]\), circuit quantum electro-dynamical (circuit QED) systems \([12, 13, 14, 15]\), cavity QED systems \([16, 17, 18]\), quantum opto-mechanical systems \([19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]\), atomic ensembles \([32, 33, 24, 34, 28]\), and quantum memories \([35, 36, 37, 38, 39]\).

Quantum linear systems have been studied extensively, and many results have been recorded in the well-known books \([3, \text{ chapter 7}], [9, \text{ chapter 6}]\), and a recent survey paper \([40]\). The aim of this tutorial is to give a concise introduction to quantum linear systems with emphasis on recent development.

This tutorial is organized as follows. Quantum linear systems are introduced in Section 2. Some important structural properties, such as stability, controllability and observability, are summarized in Section 3. It is shown that these concepts, widely used in systems and control theory, are closely related to important properties of quantum linear systems such as decoherence-free subsystems, quantum non-demolition...
variables, quantum mechanics-free subsystems and quantum back-action evasion measurement. In Section 4, quantum Gaussian states are introduced. The Wigner function is given, and an example is used to demonstrate the Heisenberg uncertainty relation. Skew information and an information-theoretic uncertainty relation is presented. In Section 6, the quantum Kalman filter is introduced. A general introduction to quantum filters is first presented in Subsection 5.1, after that the quantum Kalman filter for quantum linear systems as well as a derivation procedure is given in Subsection 5.2. The purpose of providing a derivation procedure is to illustrate some commonly used techniques in the study of quantum linear systems such as Eqs. (2) and (107). An example is given in Subsection 5.3 which illustrates the quantum Kalman filter and also demonstrates measurement back-action effect. In Section 6, several interesting structural properties of quantum linear systems are summarized, then the quantum Kalman canonical form is presented. An example, taken from a recent experiment [31, Fig. 1(A)], is analyzed in Subsection 6.2. In Subsection 7, continuous-mode single-photon states are introduced, and the response of quantum linear systems to quantum filters is first presented in Subsection 5.1, after that the quantum Kalman filter for quantum linear networks is presented in Subsection 8.1, and a recent experiment is analyzed based on the proposed theory in Subsection 8.2. Some concluding remarks are given in Section 9.

Notation.

- \(i = \sqrt{-1}\) is the imaginary unit. \(I_k\) is the identity matrix and \(0_k\) the zero matrix in \(\mathbb{C}^{k \times k}\). \(\delta_{ij}\) denotes the Kronecker delta; i.e., \(I_k = [\delta_{ij}]\). \(\delta(t)\) is the Dirac delta function.
- \(x^*\) denotes the complex conjugate of a complex number \(x\) or the adjoint of an operator \(x\). Clearly, \((xy)^* = y^*x^*\). Given two operators \(x, y\), their commutator is defined to be \([x, y] \triangleq xy - yx\).
- For a matrix \(X = [x_{ij}]\) with number or operator entries, \(X^\top = [x_{ji}]\) is the matrix transpose. Denote \(X^\# = [x^*_{ij}]\), and \(X^\top = (X^\#)^\top\). For a vector \(x\), we define \(\tilde{x} \triangleq [x^\#]\).
- Given two column vectors of operators \(X\) and \(Y\) of the same length, their commutator is defined as
  \[ [X, Y]^\top \triangleq ([X_j, Y_k]) = XY^\top - (YX^\top)^\top. \] (1)
  If \(X\) is a row vector of operators of length \(m\) and \(Y\) is a column vector of operators of length \(n\), their commutator is defined as
  \[ [X, Y] \triangleq \left( \begin{array}{cccc}
  [x_1, y_1] & \cdots & [x_m, y_1] \\
  \vdots & \ddots & \vdots \\
  [x_1, y_n] & \cdots & [x_m, y_n]
  \end{array} \right)_{m \times m}^\top = (X^\top Y^\top)^\top - YX. \] (2)
- Let \(J_k \triangleq \text{diag}(I_k, -I_k)\). For a matrix \(X \in \mathbb{C}^{2k \times 2r}\), define its \(\flat\)-adjoint by \(X^\flat \triangleq J_r X^\top J_k\). The \(\flat\)-adjoint operation enjoys the following properties:
  \[(x_1A + x_2B)^\flat = x_1^*A^\flat + x_2^*B^\flat, \quad (AB)^\flat = B^\flat A^\flat, \quad (A^\flat)^\flat = A, \] (3)
  where \(x_1, x_2 \in \mathbb{C}\).
- Given two matrices \(U, V \in \mathbb{C}^{k \times r}\), define their doubled-up \(\Delta\) as \(\Delta(U, V) \triangleq \left[ \begin{array}{c}
  U \\
  Y^\# U^\# 
  \end{array} \right]\). The set of doubled-up matrices is closed under addition, multiplication and \(\flat\) adjoint operation.
A matrix $T \in \mathbb{C}^{2k \times 2k}$ is called Bogoliubov if it is doubled-up and satisfies $TT^\dagger = T^\dagger T = I_{2k}$. The set of Bogoliubov matrices forms a complex non-compact Lie group known as the Bogoliubov group.

- A matrix $S \in \mathbb{C}^{2k \times 2k}$ is called symplectic, if $SS^\# = S^\# S = I_{2k}$. Symplectic matrices forms a complex non-compact group known as the symplectic group. The subgroup of real symplectic matrices is one-to-one homomorphic to the Bogoliubov group.

2. Quantum linear systems

Mathematically, a linear quantum system, as shown in Fig. 1, describes the dynamics of a collection of $n$ quantum harmonic oscillators which are driven by $m$ bosonic fields, for example light fields. The $j$th quantum harmonic oscillator is represented by its annihilation operator $a_j$ and creation operator $a_j^\dagger$ (the Hilbert space adjoint of $a_j$). If the $j$th harmonic oscillator is in the number state $|k\rangle$ for $k \in \mathbb{Z}^+$, then $a_j |k\rangle = \sqrt{k} |k-1\rangle$ and $a_j^\dagger |k\rangle = \sqrt{k+1} |k+1\rangle$. In particular, $|0\rangle$ is the vacuum state and $a_j |0\rangle = 0$. From these it is easy to see that the commutator $[a_j, a_k^\dagger] = 1$. In general, the operators $a_j, a_k^\dagger$ satisfy the canonical commutation relations

$$[a_j(t), a_k^\dagger(t)] = [a_j^\dagger(t), a_k(t)] = 0, \quad [a_j(t), a_k^\dagger(t)] = \delta_{jk}, \quad \forall j, k = 1, \ldots, n, \forall t \in \mathbb{R}. \tag{5}$$

Let $\mathbf{a} = [a_1 \cdots a_n]^\top$ and $\mathbf{\dot{a}} = [a^\dagger (a^\#)^\dagger]^\top$ as introduced in the Notation part. Then Eq. (5) can be rewritten as

$$[\mathbf{\dot{a}}(t), \mathbf{\dot{a}}^\dagger(t)] = J_n. \tag{6}$$

The Hamiltonian of the quantum linear system is at most quadratic in terms of $\mathbf{a}$ and $\mathbf{a}^\#$, thus it is of the form

$$\mathbf{H} = \frac{1}{2} \mathbf{\dot{a}}^\dagger \Omega \mathbf{a} + \mathbf{\dot{a}}^\dagger K \mathbf{\ddot{a}} + \mathbf{\ddot{a}}^\dagger K^\dagger \mathbf{\dot{a}}, \tag{7}$$

where $\Omega = \Delta(\Omega_-, \Omega_+)$ is a Hermitian matrix with $\Omega_-, \Omega_+ \in \mathbb{C}^{n \times n}$, $K = [K_1 \ K_2] \in \mathbb{C}^{2n \times 2l}$, and $\mathbf{\ddot{a}} \in \mathbb{C}^{2l}$ is a vector of classical signals used to model the laser or other classical signals that drive the system.
Eq. (20), [2, Chapter 3], [9, Section 5.1.1], [30, Eq. (1)], [43, Eq. (C6)], [44, Eq. (9)]). For example, the quantum system can be a particle moving in a potential well whose potential can be controlled. In the literature of measurement-based quantum feedback control [5], the quantum system can be a particle moving in a potential well whose potential can be controlled. In the Itô quantum stochastic differential equation (QSDE), first developed by [45], the quantum system can be measured and the measurement data can be processed by a classical controller that generates the classical control signal $v$ which is sent back to the quantum system to modulate its dynamics. The system is coupled to the input fields via the operator $L = [C_- C_+]\hat{a}$, with $C_-, C_+ \in \mathbb{C}^{m \times m}$. The input boson field $k, k = 1, \ldots, m$, is described in terms of an annihilation operator $b_{in,k}(t)$ and a creation operator $b_{in,k}^*(t)$, which is the adjoint operator of $b_{in,k}(t)$. If there are no photons in an input channel, this input channel is in the vacuum state denoted $|\Phi_0\rangle$. Annihilation and creation operators of these free traveling fields satisfy the following commutation relations

$$
[b_{in,j}(t)|\Phi_0\rangle = 0, \quad [b_{in,j}(t), b_{in,k}(r)] = [b_{in,j}^*(t), b_{in,k}^*(r)] = 0,
$$

$$
[b_{in,j}(t), b_{in,k}^*(r)] = \delta_{jk}\delta(t - r), \quad \forall j, k = 1, \ldots, m, \quad t, r \in \mathbb{R}.
$$

(8)

Eq. (8) can be re-written in the vector form

$$
b_{in}(t)|\Phi_0\rangle = 0, \quad [\dot{b}_{in}(t), \dot{b}_{in}^\dagger(r)] = \delta(t - r)J_m, \quad t, r \in \mathbb{R}.
$$

(9)

Before interacting with the system, the input fields may pass through some static devices, for example beamsplitters and phase shifters. This is modeled by a unitary matrix $S \in \mathbb{C}^{m \times m}$. $S$ is often referred to as a scattering matrix in the quantum optics literature.

The integrated input annihilation, creation, and gauge processes (counting processes) are given by

$$
B_{in}(t) = \int_{t_0}^t b_{in}(r)dr, \quad B_{in}^\#(t) = \int_{t_0}^t b_{in}^\#(r)dr, \quad A_{in}(t) = \int_{t_0}^t b_{in}^\#(r)b_{in}^\dagger(r)dr.
$$

(10)

respectively, where $t_0$ is the initial time when the system and the input fields start interaction. In this article, the input fields are assumed to be canonical fields which satisfy the Itô table [1].

### Table 1: Quantum Itô table

| $\times$ | $dB_{in,j}(t)$ | $dB_{in,m}^\dagger(t)$ | $dA_{in,lm}(t)$ | $dt$ |
|----------|----------------|------------------------|----------------|------|
| $dB_{in,j}(t)$ | 0 | $\delta_{jm}dt$ | $\delta_{jl}dB_{in}(t)$ | 0 |
| $dB_{in,j}^\dagger(t)$ | 0 | 0 | 0 | 0 |
| $dA_{in,km}(t)$ | 0 | $\delta_{km}dB_{in,j}(t)$ | $\delta_{kl}dA_{in,jm}(t)$ | 0 |
| $dt$ | 0 | 0 | 0 | 0 |

In the Heisenberg picture, in terms of the triple of parameters $(S, L, H)$ introduced above, the dynamics of the open quantum system in [1] is governed by a unitary operator $U(t)$ that is the solution to the following Itô quantum stochastic differential equation (QSDE), first developed by [45],

$$
dU(t) = \{-L^\dagger L/2 + iH\}dt + dB_{in}^\dagger(t)L
- L^\dagger SdB_{in}(t) + Tr[(S - I)dA_{in}^\dagger(t)]\} U(t), \quad t \geq t_0
$$

(11)
with the initial condition $U(t_0) = I$ (the identity operator). Let $X$ be a system operator. In the Heisenberg picture, we have
\begin{equation}
X(t) = U^*(t)(X \otimes I_{\text{field}})U(t), \quad t \geq t_0.
\end{equation}
According to Eq. (11) and by quantum Itô calculus, we have
\begin{equation}
dX(t) = \mathcal{L}_{\mathbf{L},\mathbf{H}}(X(t))dt + dB^\dagger_{\text{in}}(t)S[X(t),\mathbf{L}(t)] + [L^\dagger(t),X(t)]SdB_{\text{in}}(t) + \text{Tr}[(S^\dagger X(t)S - X(t))d\Lambda^\top_{\text{in}}(t)],
\end{equation}
where the superoperator
\begin{equation}
\mathcal{L}_{\mathbf{L},\mathbf{H}}(X(t)) \triangleq -i[X(t),\mathbf{H}(t)] + \frac{1}{2}[L^\dagger(t),X(t),\mathbf{L}(t)] + \frac{1}{2}[L^\dagger(t),X(t)]\mathbf{L}(t).
\end{equation}
(Notice that even initially $X$ is an operator on the system’s state space, $X(t)$ is an operator on the state space of the joint system-field system).

On the other hand, after interaction, the integrated output annihilation operators and gauge processes
\begin{equation}
\begin{align*}
B_{\text{out}}(t) = U^*(t)(I_{\text{system}} \otimes B_{\text{in}}(t))U(t), \\
\Lambda_{\text{out}}(t) = U^*(t)(I_{\text{system}} \otimes \Lambda_{\text{in}}(t))U(t)
\end{align*}
\end{equation}
are generated, and their dynamical evolution is given by
\begin{equation}
\begin{align*}
dB_{\text{out}}(t) = \mathbf{L}(t)dt + SdB_{\text{in}}(t), \\
\d\Lambda_{\text{out}}(t) = \mathbf{L}^\#(t)\mathbf{L}^\top(t)dt + S^\#d\mathbf{B}_{\text{in}}^\#(t)L^\top(t) + \mathbf{L}(t)d\mathbf{B}_{\text{in}}^\#(t)S(t) + S^\#d\Lambda_{\text{in}}(t)S^\top.
\end{align*}
\end{equation}

**Remark 2.1.** According to Eq. (11), the quantum Itô table is satisfied by the Bosonic fields that are in the vacuum state $|\Phi_0\rangle$. In quantum optics, the input $B_{\text{in}}$ may be of the from $dB_{\text{in}}(t) = w(t)dt + dB_{\text{in}}(t)$, where $w(t)$ can be an operator of another quantum linear system, and $B_{\text{in}}(t)$ is in the vacuum state $|\Phi_0\rangle$. For example, $B_{\text{in}}$ may be the output of a quantum linear system $K$. In this case, in analogy to Eq. (10), $B_{\text{in}}(t)$ will be of the form $L_K(t)dt + S_KdB_{\text{in},K}(t)$ with $L_K,S_K,B_{\text{in},K}(t)$ being the coupling operator, scattering matrix and input filed for the quantum system $K$. Thus, $w(t) = L_K(t)$ and $dB_{\text{in}}(t) = S_KdB_{\text{in},K}(t)$. More detail can be found in Section 8. The input field $dB_{\text{in}}(t)$ of the above form also satisfies the quantum Itô table. Moreover, as will be shown in Subsection 7.1, fields in continuous-mode single-photon states also satisfy the quantum Itô table. However, there do exist fields that do not satisfy the quantum Itô table. For example, the coherent state $|\alpha\rangle$ with $\alpha \neq 0$ to be introduced in Section 4 does not satisfy the quantum Itô table. In such cases, another quantum system, often referred to as a modulating filter may be designed which is driven by a vacuum field and generates the desired input fields. Consequently, by cascading the modulating filter and the original quantum system, the quantum Itô QSDE (11) still holds. The problem of generating various nonclassical quantum input field states with modulating filters has been discussed in [47, 48], and [49].

By means of Eqs. (13) and (16), it can be shown that the evolution of the quantum linear system is governed by a system of QSDEs
\begin{equation}
\begin{align*}
\dot{\mathbf{a}}(t) &= \mathbf{Aa}(t) + \mathbf{E}(t) + \mathbf{Bb}(t), \\
\dot{\mathbf{b}}_{\text{out}}(t) &= \mathbf{Ca}(t) + \mathbf{Db}(t), \quad t \geq t_0,
\end{align*}
\end{equation}
where the constant system matrices are given by
\[
D = \Delta(S, 0), \quad C = \Delta(C_-, C_+), \quad B = -C^\dagger D, \\
A = -iJ_n \Omega - \frac{1}{2} C^\dagger C, \quad E = -i \left( J_n K + J_n K^\# \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \right).
\]  
(18)

The form of the system Eq. (17) is often referred to as the quantum Langevin equation. It should be understood in the Itô form
\[
d\hat{a}(t) = A\hat{a}(t)dt + E\hat{v}(t)dt + Bd\hat{B}_{in}(t), \\
d\hat{B}_{out}(t) = C\hat{a}(t)dt + Dd\hat{B}_{in}(t), \quad t \geq t_0,
\]  
(19)

Remark 2.2. Notice that \(v(t)\) in system (17) is a classical signal, for example photocurrent. On the other hand, in the literature of quantum coherent feedback control, two quantum systems may directly couple to each other via some interaction Hamiltonian, where the involved signals are all quantum operators; see, e.g., [50, Figs. 8(b) and 16], [1, Eqs. (3.1) and (4.1)], [51, Section II.B], and [52, Fig. 3.1].

The constant system matrices in Eq. (18) are parametrized by the physical parameters \(\Omega_-, \Omega_+, C_-, C_+, S\) and they satisfy the following physical realizability conditions
\[
A + A^\dagger + BB^\dagger = 0, \\
B = -C^\dagger D.
\]  
(20)

Remark 2.3. Broadly speaking, if its system parameters satisfy the physical realizability conditions (20), the system (17) can be physically realized by a genuine quantum-mechanical system. For example, the fundamental commutation relations are preserved during temporal evolution
\[
[\hat{a}(t), \hat{a}^\dagger(t)] = [\hat{a}(t_0), \hat{a}^\dagger(t_0)], \quad t \geq t_0, \\
[\hat{a}(t), \hat{B}_{out}^\dagger(r)] = 0, \quad t_0 \leq r < t.
\]  
(21)

The problem of physical realization of quantum linear systems was first addressed in [46] and [53]. A comprehensive study of physical realization of quantum linear systems is nicely summarized in [9, Chapter 3], see also [54], [55], [56] and [40, Sections 2.4 and 3]. Further development can be found in [57] and [58]. This physical realization theory for quantum systems can be regarded as an generalization of the network analysis and synthesis theory for classical systems [61].

As in the classical linear systems theory, the impulse response function from \(\hat{B}_{in}(t)\) to \(\hat{B}_{out}(t)\) is defined as
\[
g_G(t) \triangleq \begin{cases} 
\delta(t)D - Ce^{At}C^\dagger D, & t \geq 0, \\
0, & t < 0.
\end{cases}
\]  
(22)

Define matrix functions
\[
g_{G^-}(t) \triangleq \begin{cases} 
\delta(t)S - [C_- C_+]e^{At}\begin{bmatrix} C_+^\dagger \\ -C_+^\dagger \end{bmatrix}S, & t \geq 0, \\
0, & t < 0,
\end{cases}
\]  
(23)

\[
g_{G^+}(t) \triangleq \begin{cases} 
-[C_- C_+]e^{At}\begin{bmatrix} -C_+^T \\ C_+^T \end{bmatrix}, & t \geq 0, \\
0, & t < 0.
\end{cases}
\]
It is easy to show that the impulse response function \( g_G(t) \) defined in Eq. (22) has a nice structure of the form
\[
g_G(t) = \Delta(g_G^-(t), g_G^+(t)). \tag{24}
\]

Define the bilateral Laplace transform \([62, \text{Chapter 10}]\)
\[
a[s] \triangleq \int_{-\infty}^{\infty} e^{-st} a(t) dt \tag{25}
\]
Conjugating both sides of Eq. (25) yields
\[
a[s]^\# = \int_{-\infty}^{\infty} e^{-s^*t} a^\#(t) dt \tag{26}
\]
Denote
\[
a^\#[s] \equiv a[s^*]^\#, \quad \tilde{a}[s] \equiv \begin{bmatrix} a[s] \\ a^\#[s] \end{bmatrix}. \tag{27}
\]

Then
\[
\tilde{a}[s] = \int_{-\infty}^{\infty} e^{-st} \tilde{a}(t) dt. \tag{28}
\]

Using similar definitions and notations for other operators or functions, the transfer function from \( f \tilde{b}_{\text{in}}[s] \) to \( \tilde{b}_{\text{out}}[s] \) is
\[
\Xi_G[s] = C(sI - A)^{-1} \mathcal{B} + \mathcal{D}. \tag{29}
\]

Remark 2.4. The notation in Eq. (27) is consistent with that in \([41]\) and \([9, \text{Section 2.3.3}]\), but is slightly different from that in \([50]\). For example, \( b_2^*[s^*] \) in \([50]\, \text{Eq. (3)}\) is \( b_{\text{in},2}[s] \) in our notation. The same is true for the other operators. Also, in this article we use \( [s] \) to indicate the frequency domain and \( (t) \) to indicate the time domain, as have been adopted in \([63, 64]\) and \([65]\).

Due to the structure of the impulse response function (24), the transfer function \( \Xi_G[s] \) defined in Eq. (29) enjoys the following nice properties
\[
\Xi_G[-s^*]^\# \Xi_G[s] = I_{2m}, \quad \forall s \in \mathbb{C}, \tag{30}
\]
and
\[
\Xi_G[i\omega]^\# \Xi_G[i\omega] = \Xi_G[i\omega] \Xi_G[i\omega]^\# = I_{2m}, \quad \forall \omega \in \mathbb{R}. \tag{31}
\]
(A derivation of Eq. (31) can be seen in \([41, \text{Section VI.H}]\), and Eq. (31) follows Eq. (30) by setting \( s = \omega \).

If \( C_+ = 0, \quad \Omega_+ = 0, \quad K_2 = 0 \) and \( K_3 = 0 \), the resulting quantum linear system is said to be passive \([51, 56, 9, 65]\). Specifically, the Itô QSDEs for a passive linear quantum system are
\[
da(a(t)) = Aa(t) dt + Ev(t) dt + BdB_{\text{in}}(t),
\]
\[
dB_{\text{out}}(t) = Ca(t) dt + DdB_{\text{in}}(t), \quad t \geq t_0, \tag{32}
\]
where
\[
A = -i\Omega_--\frac{1}{2} C^\dagger C_-, \quad E = -i(K_1 + K_4^\#), \quad B = -C^\dagger S, \quad C = C_-, \quad D = S.
\]
In the passive case, the physical realizability conditions \([20]\) reduce to
\[
A + A^\dagger + BB^\dagger = 0, \quad B = -C^\dagger S. \tag{33}
\]
Moreover, in the passive case, $\Xi_{G^+}[s] \equiv 0$ and

$$\Xi_{G^-}[s] = S - C_-(sI + i\Omega_- + \frac{1}{2} C_-^\dagger C_-)^{-1} C_-^\dagger S. \quad (34)$$

In other words, the dynamics of a quantum linear passive system are completely characterized by its annihilation operators. Finally, it can be easily verified that for a quantum linear passive system, the following holds

$$\Xi_{G^-}[i\omega]^\dagger \Xi_{G^-}[i\omega] \equiv I_m, \quad \forall \omega \in \mathbb{R}. \quad (35)$$

As a result, a quantum linear passive system does not change the amplitude of the input signal, but modifies its phase.

Besides the annihilation–creation operator representation (17), a quantum linear system can also be described in the (real) quadrature operator representation. For a positive integer $k$, define the unitary matrix

$$V_k \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ -iI_k & iI_k \end{bmatrix}. \quad (36)$$

The following unitary transformations

$$\begin{bmatrix} q \\ p \end{bmatrix} \equiv \begin{bmatrix} x \equiv V_n \tilde{a} \\ u \equiv V_l \tilde{v} \end{bmatrix}, \quad \begin{bmatrix} q_{in} \\ p_{in} \end{bmatrix} \equiv \begin{bmatrix} u \equiv V_m \tilde{b} \\ q_{out} \equiv p_{out} \equiv y \equiv V_m \tilde{b}_{out} \end{bmatrix}, \quad (37)$$

generate real quadrature operators of the system, the classical signal and the fields. The counterparts of the commutation relations (6) and (8) are

$$[x, x^\top] = iJ_n, \quad (38)$$

and

$$[u(t), u^\top(r)] = i\delta(t - r)J_m, \quad t, r \in \mathbb{R} \quad (39)$$

respectively.

**Remark 2.5.** In some works, for example [46, 33], and [3], the transformations $q = a + a^\#$, and $p = -ia + ia^\#$ are adopted. In this case, Eq. (38) becomes

$$[x, x^\top] = 2iJ_n. \quad (40)$$

In terms of unitary transformations in Eq. (37), the coupling operator $L$ and the Hamiltonian $H$ are transformed to

$$L = \Lambda x,$$

$$H = \frac{1}{2} x^\top \Xi x + x^\top \Xi u + u^\top \Xi^\dagger x,$$

where

$$\Lambda = [C_- C_+] V_n^\dagger, \quad \Xi = V_n \Omega V_n^\dagger, \quad \Xi = V_n \Omega V_n^\dagger. \quad (41)$$
The QSDEs that describe the dynamics of the linear quantum system in the real quadrature operator representation are the following:

\[
\begin{align*}
\dot{x} &= Ax + Eu + Bu, \\
y &= Cx + Du,
\end{align*}
\] (43)

where

\[
\begin{align*}
D &= V_m D V_m^\dagger = \begin{bmatrix} \text{Re}(S) & -\text{Im}(S) \\ \text{Im}(S) & \text{Re}(S) \end{bmatrix}, \\
C &= V_m C V_n^\dagger = \begin{bmatrix} \text{Re}(C_- + C_+) & \text{Im}(-C_- + C_+) \\ \text{Im}(C_- + C_+) & \text{Re}(C_- - C_+) \end{bmatrix}, \\
B &= V_n B V_m^\dagger = -C^2 D, \\
A &= V_n A V_n^\dagger = J_n H - \frac{1}{2} C^2 C, \\
E &= J_n (K + K^\#).
\end{align*}
\] (44)

It is easy to verify that \( D D^\dagger = I_{2m} \). Define Itô increments

\[
\begin{align*}
dQ_{\text{in}}(t) &= \int_t^{t+dt} q_{\text{in}}(r)dr, \quad dP_{\text{in}}(t) = \int_t^{t+dt} p_{\text{in}}(r)dr, \\
dQ_{\text{out}}(t) &= \int_t^{t+dt} q_{\text{out}}(r)dr, \quad dP_{\text{out}}(t) = \int_t^{t+dt} p_{\text{out}}(r)dr.
\end{align*}
\] (45)

And denote

\[
\begin{align*}
\mathcal{U}(t) &= \begin{bmatrix} Q_{\text{in}}(t) \\ P_{\text{in}}(t) \end{bmatrix}, \\
\mathcal{Y}(t) &= \begin{bmatrix} Q_{\text{out}}(t) \\ P_{\text{out}}(t) \end{bmatrix}.
\end{align*}
\] (46)

Then system (43) can be re-written as

\[
\begin{align*}
d\mathbf{x}(t) &= A\mathbf{x}(t)dt + Eu(t)dt + Bd\mathcal{U}(t), \\
d\mathcal{Y}(t) &= C\mathbf{x}(t)dt + Dd\mathcal{U}(t)
\end{align*}
\] (47)

3. Hurwitz stability, controllability and observability

Hurwitz stability, controllability and observability are fundamental concepts of classical linear systems. Interestingly, these concepts can naturally be generalized to linear quantum systems. In the following discussions of this section, we assume the classical signal \( u = 0 \) in Eq. (43).

If we take expectation on both sides of Eq. (43) with respect to the initial joint system-field state we get a classical linear system

\[
\begin{align*}
\frac{d}{dt} \langle \mathbf{x}(t) \rangle &= A \langle \mathbf{x}(t) \rangle + \mathbb{E} \langle u(t) \rangle, \\
\langle \mathbf{y}(t) \rangle &= C \langle \mathbf{x}(t) \rangle + \mathbb{D} \langle u(t) \rangle.
\end{align*}
\] (48)

Thus we can define controllability, observability, and Hurwitz stability for the quantum linear system (43) using those for the classical linear system (48).

**Definition 3.1.** The quantum linear system (43) is said to be Hurwitz stable (resp. controllable, observable) if the corresponding classical linear system (48) is Hurwitz stable (resp. controllable, observable).
Decoherence-free subsystems for linear quantum systems have recently been studied in e.g., [70, 22, 24, 25, 56, 71, 72, 73] and [65] and references therein. It turns out that decoherence-free subsystems are uncontrollable/unobservable subspaces in the linear quantum systems setting.

**Definition 3.2 ([65, Definition 2.1]).** The linear span of the system variables related to the uncontrollable/unobservable subspace of a linear quantum system is called its decoherence-free subsystem (DFS).

In quantum information science, decoherence-free subspaces are widely used for protecting useful quantum information; see for example [74] and [75]. The relation between decoherence-free subsystems and decoherence-free subspaces are discussed in [76].

In principle, an observable can be measured. However, the measurement may perturb the future evolution of this observable; this is the so-called quantum measurement back-action. Interestingly, sometimes one can engineer a quantum system so that measurement will not affect the evolution of the desired observable. Observables having this property are referred to as quantum non-demolition (QND) variables; see e.g., [77, 78, 79, 80, 25, 31] and [65].

**Definition 3.3.** An observable $F$ is called a continuous-time QND variable if

$$[F(t_1), F(t_2)] = 0$$

for all time instants $t_1, t_2 \in \mathbb{R}^+$.  

A natural extension of the notion of a QND variable is the following concept [80].

**Definition 3.4 ([80]; [65, Definition 2.3]).** The span of a set of observables $F_i$, $i = 1, \ldots, r$, is called a quantum mechanics-free subsystem (QMFS) if

$$[F_i(t_1), F_j(t_2)] = 0$$

for all time instants $t_1, t_2 \in \mathbb{R}^+$, and $i, j = 1, \ldots, r$.

QND variables and QMFS subsystems are $p_k$ in the Kalman canonical form of a quantum linear system; see Theorem 6.1 below.

Examples of physical realization of QMFS subsystems can be found in [31] and [65, Example 5.2].

The transfer function of the quantum linear system (43) from $u$ to $y$ is

$$\Xi_{u \rightarrow y}[s] = D - C(sI - A)^{-1}B.$$  

The transfer function relates the overall input $u$ to the overall output $y$. However, in many applications, we are interested in a particular subvector $u'$ of the input vector $u$ and a particular subvector $y'$ of the output vector $y$. This motivates us to introduce the following concept.

**Definition 3.5 ([65, Definition 2.4]).** For the linear quantum system (13), let $\Xi_{u' \rightarrow y'}[s]$ be the transfer function from a subvector $u'$ of the input vector $u$ to a subvector $y'$ of the output vector $y$. We say that system (13) realizes the back-action evasion (BAE) measurement of the output $y'$ with respect to the input $u'$ if $\Xi_{u' \rightarrow y'}[s] = 0$ for all $s$.

More discussions on BAE measurements can be found in, e.g., [79, 13, 81, 25, 27, 82, 83], and the references therein.

We shall see that all of these notions can be nicely revealed by the Kalman decomposition of a linear quantum system, see Section 6.
4. Quantum Gaussian states

In this section, quantum Gaussian states are briefly introduced. More discussions can be found in, e.g., [2, 84, 42, 44, 85, 5, 86, 87, 64, 8, 88, 89, 90, 91] and references therein.

4.1. An introduction

Define the displacement operator

\[ D(\alpha) \triangleq \exp(\hat{a}^\dagger J_n \alpha), \quad \forall \alpha \in \mathbb{C}^n. \] (52)

Define the counterpart of \( \hat{a} \) in the real domain,

\[ \beta = V_n \hat{a} \in \mathbb{R}^{2n}, \] (53)

where \( V_n \) is the unitary matrix defined in Eq. (36). Then the displacement operator defined in Eq. (52) can be re-written as

\[ D(\alpha) = \exp(\mathbf{x}^\top J_n \beta). \] (54)

Given a density matrix \( \rho \) of a quantum linear system, define its quantum characteristic function to be

\[ \chi_\rho \triangleq \text{Tr}[\rho D(\alpha)]. \] (55)

For Gaussian states, we have, [85, Eq. (3.1)],

\[ \chi_\rho = \exp \left( -i\mu^\top J_n \beta - \frac{1}{2} \beta^\top \mathbb{V} \beta \right), \] (56)

where

\[ \mu = \text{Tr}[\rho \mathbf{x}] \in \mathbb{R}^{2n}, \]
\[ \mathbb{V} = \frac{1}{2} \text{Tr}\{\rho[(\mathbf{x}_t - \mu)(\mathbf{x}_t - \mu)^\top + ((\mathbf{x}_t - \mu)(\mathbf{x}_t - \mu))^\top]\} \in \mathbb{R}^{2n \times 2n}. \] (57)

are the mean and covariance, respectively. Define the complex domain counterpart of \( \mu \) and \( \mathbb{V} \) to be

\[ \hat{\gamma} \triangleq V_n^\dagger \mu, \]
\[ \Pi \triangleq V_n^\dagger \mathbb{V} V_n. \] (58)

Then the quantum characteristic function in Eq. (56) can be re-written as

\[ \chi_\rho = \exp \left( -\hat{\gamma}^\dagger J_n \hat{\alpha} - \frac{1}{2} \hat{\alpha}^\dagger \Pi \hat{\alpha} \right), \] (59)

Moreover, for \( \rho \) to be a Gaussian state of a quantum linear system, it is required that, [85, Theorem 3.1],

\[ \mathbb{V} \geq \pm \frac{i}{2} J_n; \] (60)

or equivalently,

\[ \Pi \geq \pm \frac{1}{2} J_n. \] (61)
In terms of the characteristic function $\chi_\rho$ defined in Eq. (55), we can define the Wigner function via the multi-dimensional Fourier transform

$$W_\rho(w) = \frac{1}{\sqrt{(2\pi)^{2n}}} \int_{\mathbb{R}^{2n}} \exp(-iw^\top J_\beta) \text{Tr}[\rho \exp(iw^\top J_n \beta)] d\beta, \ \forall w \in \mathbb{R}^{2n}. \quad (62)$$

In particular, if $\rho$ is a Gaussian state with the characteristic function in Eq. (56), the Wigner function is of the form

$$W_\rho(w) = \frac{1}{\sqrt{(2\pi)^{2n}}} \frac{1}{\det(V)} \exp \left( -\frac{1}{2}(w - \mu)^\top V^{-1}(w - \mu) \right), \quad (63)$$

with the mean $\mu$ and covariance matrix $V$ given in Eq. (57). In other words, a Gaussian state is uniquely determined by its first and second moments.

**Example 4.1.** When $n = 1$ and the system state is the quantum vacuum state $|0\rangle$, then $\rho = |0\rangle \langle 0|$, and

$$\mu = \text{Tr}(\rho x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{Tr}(\rho q^2) = \text{Tr}(\rho p^2) = \frac{1}{2}, \quad \text{Tr}(\rho pq) = -\frac{i}{2}. \quad (64)$$

The covariance matrix $V$ in Eq. (57) becomes

$$V = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (65)$$

Thus,

$$V \pm \frac{i}{2} J_n = \frac{1}{2} \begin{bmatrix} 1 & \pm i \\ -\pm i & 1 \end{bmatrix} \geq 0, \quad (66)$$

which verifies Eq. (60). Moreover, the Wigner function (63) is now

$$W_\rho(w) = \frac{1}{\pi} \exp \left( -\frac{1}{2} w^\top w \right). \quad (67)$$

Finally, from Eq. (64) we have

$$\sqrt{\text{Tr}(\rho q^2)} \sqrt{\text{Tr}(\rho p^2)} = \frac{1}{2}. \quad (68)$$

For any state $\rho$ and observables $X$ and $Y$, the Heisenberg uncertainty relation is

$$\sqrt{\text{Tr}(\rho X^2) - (\text{Tr}(\rho X))^2} \sqrt{\text{Tr}(\rho Y^2) - (\text{Tr}(\rho Y))^2} \geq \frac{1}{2} |\text{Tr}(\rho [X, Y])|. \quad (69)$$

According to Eq. (68), the vacuum state $|0\rangle$ saturates the Heisenberg uncertainty relation (69) when $X = q$ and $Y = p$. In the literature, states saturating the Heisenberg uncertainty relation (69) are often called **minimum uncertainty states**. For example, a special type of Gaussian states, coherent states, defined as

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} |k\rangle, \quad \alpha \in \mathbb{C}, \quad (70)$$

are minimum uncertainty states as Eq. (69) is saturated when $X = q$ and $Y = p$.

**Example 4.2.** In this example, we show that the vacuum state $|0\rangle$ is a Gaussian state. For simplicity, we look at the single-oscillator case ($n = 1$). In this case, the displacement operator $D(\alpha)$ defined in Eq. (52) becomes

$$D(\alpha) = \exp(\alpha a^* - \alpha^* a). \quad (71)$$
If two operators \( A \) and \( B \) satisfy \([A, [A, B]] = [B, [A, B]] = 0\), then the Baker-Campbell-Hausdorff formula is
\[
e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} = e^B e^A e^{\frac{1}{2}[A,B]}.
\] (72)

Therefore,
\[
D(\alpha) = e^{\alpha a^*} e^{-\alpha^* a} e^{-\frac{1}{2} |\alpha|^2}.
\] (73)

As
\[
e^{\alpha a^*} e^{-\alpha^* a} |0\rangle = e^{\alpha a^*} |0\rangle = \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} |k\rangle,
\] (74)
we have
\[
D(\alpha) |0\rangle = e^{-\frac{1}{2} |\alpha|^2} \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}} |k\rangle = |\alpha\rangle.
\] (75)

Consequently, the characteristic function for the vacuum state \(|0\rangle\) is
\[
\chi = \langle 0 | D(\alpha) | 0 \rangle = e^{-\frac{1}{2} |\alpha|^2},
\] (76)
which is of the form Eq. (59) with
\[
\gamma = 0, \quad \Pi = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\] (77)

Clearly, \( \Pi \) derived above satisfies the inequality (61), hence the vacuum state \(|0\rangle\) is a Gaussian state.

In [92], Section 2.7 and [64], another form of characteristic functions is defined for a Gaussian system state \( \rho \), which is
\[
\text{Tr} \left[ \rho \exp(i \hat{z}^\dagger \hat{a}) \right] = \exp \left( i \hat{z}^\dagger \hat{\gamma} - \frac{1}{2} \hat{z}^\dagger \Sigma \hat{z} \right), \quad \forall z \in \mathbb{C}^n,
\] (78)
where \( \hat{\gamma} = \text{Tr} [\rho \hat{a}] \), and \( \Sigma = \text{Tr} [\rho (\hat{a} - \hat{\gamma})(\hat{a} - \hat{\gamma})^\dagger] \) is a non-negative Hermitian matrix. In general, \( \Sigma \) has the form
\[
\Sigma = \begin{bmatrix} I_n + \frac{1}{2} N^T M \\ M^\dagger \end{bmatrix}.
\] (79)

In particular, the ground or vacuum state \(|0\rangle\) is specified by \( \gamma = 0 \) and \( \Sigma = \begin{bmatrix} I_n & 0 \\ 0 & 0_n \end{bmatrix} \). Clearly,
\[
\frac{1}{2} \Sigma \geq \frac{i}{2} J_n.
\] (80)

However, the following is not true:
\[
\frac{1}{2} \Sigma \geq -\frac{i}{2} J_n.
\] (81)

Consequently, to be consistent with \( V \) for the real quadrature operator representation, it is better to use the covariance matrix \( \Pi \) given in Eq. (58), instead of \( \Sigma \) in Eq. (79).
4.2. Pure Gaussian state generation

Gaussian states are very useful resources in quantum signal processing,\cite{93, 94, 95} and \cite{84, 96}. Thus, the problem of Gaussian state generation has been studied intensively in the quantum control literature. In this subsection, we present one result for pure Gaussian state generation by means of environment engineering.

A Gaussian state is a pure state if the determinant of its associated covariance $V$ satisfies $\det(V) = 1/2^n$, where $n$ is the number of the oscillators. The covariance matrix $V$ of a pure Gaussian state can be decomposed as (\cite{87, Eq. (16)}; \cite{86, Eq. (2.18)})

$$V = \frac{1}{2} SS^\top,$$  \hspace{1cm} (82)

where

$$S = \begin{bmatrix} Y^{-1/2} & 0 \\ XY^{-1/2} & Y^{1/2} \end{bmatrix}$$  \hspace{1cm} (83)

with $X = X^\top \in \mathbb{R}^n$ and $Y = Y^\top \in \mathbb{R}^n$ being positive definite. As $SJ_nS^\top = J_n$, $S$ is symplectic (See the Notation part). Let $Z = X + iY$.

**Theorem 4.1** (\cite{87}). The pure Gaussian state associated with the covariance matrix $V$ in Eq. (82) can be generated by the linear quantum system (43) if and only if

$$H = \begin{bmatrix} XRX + YRY - \Gamma Y^{-1}X - XY^{-1}\Gamma^\top & -XR + \Gamma Y^{-1} \\ -RX + \Gamma^\top Y^{-1} & R \end{bmatrix},$$  \hspace{1cm} (84)

and

$$\Lambda = P^\top [-Z \ I],$$  \hspace{1cm} (85)

where $R = R^\top$, $\Gamma = -\Gamma^\top$, and the matrix pair $(P, Q)$ is controllable with $Q = -iRY + Y^{-1}\Gamma$.

The problem of Gaussian state generation by means of environment engineering has been studied insensitively by the quantum control community. Interest reader may refer to \cite[Section 6.1]{9} and further development \cite{97, 98, 99, 89, 90, 91}.

4.3. Skew information and information-theoretic uncertainty relation

In addition to Heisenberg’s uncertainty relation \cite{69}, an information-theoretic uncertainty relation was proposed in \cite{100} based on skew information. In what follows, we use the notation in \cite{100}. Given a density operator $\rho$ and an observable $X$, the Wigner–Yanase skew information \cite{101} is defined as

$$I(\rho, X) \triangleq -\frac{1}{2} \text{Tr}([\sqrt{\rho}, X]^2).$$  \hspace{1cm} (86)

Skew information was originally defined for Hamiltonians of closed (namely, isolated) quantum systems \cite{101}, and was later generalized to arbitrary observables of open quantum systems. Roughly speaking, $I(\rho, X)$ measures the quantum uncertainty of the observable $X$ with respect to the density operator $\rho$.

The variance of $X$ with respect to the density operator $\rho$ is

$$V(\rho, X) = \text{Tr}(\rho X^2) - (\text{Tr}(\rho X))^2.$$  \hspace{1cm} (87)

When the state is pure, it is easy to see that $I(\rho, X) = V(\rho, X)$. However, $I(\rho, X) \leq V(\rho, X)$ when the state is a mixed one. To quantify quantum uncertainty, the following quantity has been defined in \cite{100}:

$$U(\rho, X) \triangleq \sqrt{V^2(\rho, X) - (V(\rho, X) - I(\rho, X))^2}.$$  \hspace{1cm} (88)
The information-theoretic uncertainty relation is [100, Eq. (2)]

\[ U(\rho, X)U(\rho, Y) \geq \frac{1}{4} |\text{Tr}(\rho[X, Y])|^2. \] (89)

Interestingly, it is proved in [102] that when \( n = 1 \) for the single-mode oscillator case, all Gaussian states, pure or mixed, are minimum uncertainty states, i.e., states that saturate (89). On the other hand, minimum uncertainty states are Gaussian states. However, in general mixed Gaussian states do not saturate the Heisenberg’s uncertainty relation (69). In this sense, the information-theoretic uncertainty relation (89) better characterizes Gaussian states. More studies on quantum skew-information and information-theoretic uncertainty relation can be found in, e.g., [103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 102] and references therein.

5. Quantum Kalman filter

Simply speaking, a quantum filter describes the temporal evolution of an open quantum system under repeated measurement. A general form of quantum filters is first presented in Subsection 5.1 after that the quantum filter for linear quantum systems is given in Subsection 5.2, which is of the form of the Kalman filter for classical linear systems. Finally, an example is given in Subsection 5.3 for demonstration. In this section, it is assumed that \( S = I_m \) and the initial time \( t_0 = 0 \).

5.1. Quantum filter

In this subsection, we present a quantum filter for open quantum systems.

By Eq. (37), we have

\[
[q_{\text{in}}(t_1), q_{\text{in}}^\dagger(t_2)] = \frac{1}{2} [\mathbf{b}_{\text{in}}(t_1) + \mathbf{b}_{\text{in}}^\dagger(t_1), \mathbf{b}_{\text{in}}^\dagger(t_2) + \mathbf{b}_{\text{in}}(t_2)] \\
= \frac{1}{2} [\mathbf{b}_{\text{in}}(t_1), \mathbf{b}_{\text{in}}^\dagger(t_2)] + \frac{1}{2} [\mathbf{b}_{\text{in}}^\dagger(t_1), \mathbf{b}_{\text{in}}(t_2)] \\
= \frac{1}{2} \delta(t_1 - t_2)I_m - \frac{1}{2} \delta(t_1 - t_2)I_m \\
= 0. \] (90)

According to Eq. (13),

\[ q_{\text{out}}(t) = U(t)^\ast (I_{\text{system}} \otimes q_{\text{in}}(t))U(t). \] (91)

Moreover, the unitary operator \( U(t) \) has the following property [114, Section 5.2]

\[ U(t_2)^\ast (I_{\text{system}} \otimes q_{\text{in}}(t_1))U(t_2) = U(t_1)^\ast (I_{\text{system}} \otimes q_{\text{in}}(t_1))U(t_1), \quad t_1 \leq t_2. \] (92)

Consequently, \( q_{\text{out}}(t) \) enjoys the self-non-demolition property:

\[ [q_{\text{out}}(t_1), q_{\text{out}}^\dagger(t_2)] = 0, \quad 0 \leq t_1 \leq t_2. \] (93)

Moreover, from Eqs. (21) and (92) the following non-demolition property can be derived:

\[ [X(t), q_{\text{out}}(r)^\dagger] = 0, \quad t_0 \leq r \leq t. \] (94)
It can be easily verified that the integrated quadrature operator $Q_{\text{out}}(t)$ defined in Eq. (45) also enjoys the self-non-demolition property (93) and non-demolition property (94). Due to the self-non-demolition property (93), \{ $Q_{\text{out}}(r) : 0 \leq r \leq t$ \} can be regarded as a classical stochastic process. (Strictly speaking, measuring $Q_{\text{out}}(t)$ gives rise to a classical stochastic process.) Moreover, due to the non-demolition property (94), $X(t)$ lives in the $\sigma$-field generated by this classical stochastic process. Hence, it is meaningful to define the expectation of $X(t)$ conditioned on this $\sigma$-field. We denote this conditional expectation by $E[X(t)|\{Q_{\text{out}}(r) : 0 \leq r \leq t]\}$. Then, one can define the conditional density operator $\rho_{\text{c}}(t)$ by means of

$$\text{Tr}(\rho_{\text{c}}(t)X) = E[X(t)|\{Q_{\text{out}}(r) : 0 \leq r \leq t\}]$$

Clearly, $\rho_{\text{c}}(0)$ is the initial joint system density matrix denoted $\rho_S(0)$. The dynamics of the conditioned density operator is given by the stochastic master equation (SME) (also called quantum trajectories $[115, 116]$)

$$d\rho_{\text{c}}(t) = L^*_L H(\rho_{\text{c}}(t))dt + \{L^\top \rho_{\text{c}}(t) + \rho_{\text{c}}(t)L^\top - \text{Tr}[\rho_{\text{c}}(t)(L^\top + L^\top)]\rho_{\text{c}}(t)\}d\nu(t),$$

where

$$d\nu(t) \triangleq dQ_{\text{out}}(t) - \pi_t(L + L^\#)dt$$

is an innovation process, and the superoperator

$$L^*_L H(\rho_{\text{c}}(t)) = -i[H, \rho_{\text{c}}(t)] + L^\top \rho_{\text{c}}(t)L^\# - \frac{1}{2}L^\top L\rho_{\text{c}}(t) - \frac{1}{2}\rho_{\text{c}}(t)L^\top L.$$

For a given system operator $X$, define the conditioned mean vector

$$\pi_t(X) \triangleq \text{Tr}(\rho_{\text{c}}(t)X).$$

It turns out that $\pi_t(X)$ is the solution to the Belavkin quantum filtering equation, which is a classical stochastic differential equation $[117, 118]$

$$d\pi_t(X) = \pi_t(L^*_L H(X))dt + [\pi_t(XL^\top + L^\top X) - \pi_t(L^\top + L^\top)\pi_t(X)]d\nu(t),$$

with the initial condition $\pi_0(X) = \text{Tr}(\rho_S(0)X)$, where $L^*_L H(X)$ is the superoperator defined in Eq. (44).

**Remark 5.1.** In classical control systems theory, measurement noise is always supposed to be decoupled from the system dynamics. This is no longer true in the quantum regime. As shown by Eq. (96), measurement affects the dynamics of the system which is being monitored. This is often called measurement back-action. Moreover, Eq. (43) show linear dynamics of the system. However, its conditioned dynamics (96) is nonlinear. This is essentially different from classical linear dynamics.

**Remark 5.2.** The measurement used in this section is homodyne measurement. There are other types of quantum measurements used in quantum filtering and feedback control, for instance, heterodyne measurement, photodetection and general positive operator valued measurements (POVMs). The experimental realization of a real-time POVM measurement-based feedback control of the 2012 Nobel prize winning photon-box is described in $[13]$ and $[17]$. A comprehensive study of quantum measurement and feedback control is presented in $[114]$ and $[5]$. 

17
Finally, denote
\[\rho(t) \equiv E[\rho_c(t)].\] (101)
Then the unconditioned system dynamics are given by the Lindblad master equation
\[d\rho(t) = \mathcal{L}_{L,H}^\dagger(\rho(t))dt.\] (102)

More discussions of quantum filters can be found in \[117, 16, 119, 120, 121, 114, 122, 123, 124, 125, 126, 127, 128, 129, 130,\] and \[5,\] among others.

5.2. Quantum Kalman filter

The above formulations hold for general open quantum systems, in this subsection we present their specific forms for linear quantum systems. In this case, the quantum filter consists of Eqs. (104) and (106) to be given below, which is in the same form of a classical Kalman filter.

Define the conditional covariance matrix
\[V_t \equiv \text{Tr} \left[ \rho_c(t) \frac{(x - \pi_t(x))(x - \pi_t(x))^{\top} + ((x - \pi_t(x))(x - \pi_t(x))^{\top})}{2} \right].\] (103)

**Theorem 5.1.** The quantum Kalman filter for the quantum linear system (43) is of the form
\[d\pi_t(x) = A\pi_t(x)dt + Eu\,d\nu(t),\] (104)
with the initial condition \[\pi_0(x) = \text{Tr} \left[ \rho_S(0)x \right],\] where the matrices
\[C_1 = [I_m \, 0_m]C, \quad M = \frac{1}{\sqrt{2}}B \left[ I_m \, 0_m \right],\] (105)
and the conditional covariance matrix \(V_t\) solves the following differential Riccati equation
\[\dot{V}_t = AV_t + V_tA^{\top} + \frac{1}{2}BB^{\top} - (V_tC_1^{\top} + M)(V_tC_1^{\top} + M)^{\top}.\] (106)

The quantum Kalman filter appears very close to the Kalman filter for classical linear systems [67, 131, Chapter 3].

In what follows, a proof of Theorem 5.1 is given,

**Proof of Theorem 5.1.** Look at Eq. (104) first.

**Step 0.** Let \(X, Y\) and \(Z\) be vectors of operators of dimension \(l, m,\) and \(n,\) respectively. Let \(M \in \mathbb{C}^{n \times n}\). If the commutators \([a, b] \in \mathbb{C}\) where \(a\) and \(b\) are arbitrary elements of the vectors \(X, Y\) and \(Z\). Then
\[\begin{align*}
[X, Y^{\top}MZ] &= [X, Y^{\top}]MZ + [X, Z^{\top}]M^{\top}Y. \\
(107)
\end{align*}\]

**Step 1.** Substituting the vector \(x(t)\) in Eq. (43) into Eq. (14) and using Eqs. (12) and (107) we get
\[
\begin{align*}
\pi_t(-i[x(t), H(t)] &+ \frac{1}{2}L_{t}^{\dagger}(t)[x(t), L(t)] + \frac{1}{2}[L_{t}^{\dagger}(t), x(t)]L(t)) \\
= J_n\pi_t(x) + J_n(K + K#)u + \frac{1}{2t}J_n(\Lambda^\dagger\Lambda - \Lambda^{\top}\Lambda^#)\pi_t(x) \\
= \left(J_n\pi_t - \frac{1}{2}C^{\dagger}C\right)\pi_t(x) + J_n(K + K#)u \\
= A\pi_t(x) + Eu.
\end{align*}
\] (108)
Step 2. By Eqs. (109) and (113), we have
\[ V_t = \pi_t(xx^T) - \frac{1}{2}J_n - \pi_t(x)\pi_t(x)^T. \] (109)

Step 3. Noticing that
\[ L^\dagger q = qx^T\Lambda^\dagger - \iota [0 \ I] \Lambda^\dagger, \]
\[ L^\dagger p = px^T\Lambda^\dagger + \iota [I \ 0] \Lambda^\dagger, \] (110)
we have
\[ L^\dagger x = xL^\dagger - iJ_n\Lambda^\dagger. \] (111)

Step 4. By Eqs. (109) and (111), we get
\[ \pi_t(xL^\dagger + L^\dagger x) - \pi_t(L^\dagger + L^\dagger)\pi_t(x) \]
\[ = \pi_t(xx^T)\Lambda^\dagger + \pi_t(xL^\dagger - iJ_n\Lambda^\dagger) - \pi_t(L^\dagger + L^\dagger)\pi_t(x) \]
\[ = \pi_t(xx^T)(\Lambda^\dagger + \Lambda^\dagger) - iJ_n\Lambda^\dagger - \pi_t(L^\dagger + L^\dagger)\pi_t(x) \]
\[ = (V_t + \frac{1}{2}J_n)(\Lambda^\dagger + \Lambda^\dagger) - iJ_n\Lambda^\dagger \]
\[ = V_t(\Lambda^\dagger + \Lambda^\dagger) + \frac{1}{2}J_n(\Lambda^\dagger - \Lambda^\dagger), \] (112)
where
\[ \pi_t(L^\dagger + L^\dagger)\pi_t(x) = \pi_t(x)\pi_t(L^\dagger + L^\dagger) = \pi_t(x)\pi_t(x)^T(\Lambda^\dagger + \Lambda^\dagger) \] (113)
has been used in the derivation.

Combining (108), (112) with (107), we have
\[ d\pi_t(x) = (\mathbb{A}\pi_t(x) + \mathbb{E}u)dt + \left(V_tC_1^T + \frac{1}{2}J_n(\Lambda^T - \Lambda^\dagger)\right)d\nu_t \]
\[ = (\mathbb{A}\pi_t(x) + \mathbb{E}u)dt + (V_tC_1^T + M)d\nu_t, \] (114)
which is Eq. (104).

Next, we derive Eq. (106).

By Eqs. (105) and (108), we have
\[ V_t = E \left[ \frac{(x_t - \pi_t(x))(x_t - \pi_t(x))^T + ((x_t - \pi_t(x))(x_t - \pi_t(x))^T)^T}{2} \right] \}
\[ \{Q_{out}(r) : 0 \leq r \leq t \}. \] (115)

As a result, by the property of conditional expectation,
\[ E[V(t)] \]
\[ = E \left[ \frac{(x_t - \pi_t(x))(x_t - \pi_t(x))^T + ((x_t - \pi_t(x))(x_t - \pi_t(x))^T)^T}{2} \right] \]
\[ = \text{Tr} \left[ \rho_{S \otimes F}(0) \frac{(x_t - \pi_t(x))(x_t - \pi_t(x))^T + ((x_t - \pi_t(x))(x_t - \pi_t(x))^T)^T}{2} \right], \] (116)
where \( \rho_{S \otimes F}(0) \) is the initial joint system-field density matrix. By the following property of Gaussian random variables: \( V(t) = E[V(t)] \) almost surely (a.s.), see e.g., [132, Chapter 10], we have
\[ V_t = \text{Tr} \left[ \rho_{S \otimes F}(0) \frac{(x_t - \pi_t(x))(x_t - \pi_t(x))^T + ((x_t - \pi_t(x))(x_t - \pi_t(x))^T)^T}{2} \right]. \] (117)
Moreover, it can be easily checked that

$$\langle d(x_t - \pi_t(x))(x_t - \pi_t(x))^\top \rangle.$$  

According to Eqs. (43) and (114), we have

$$d \langle (x_t - \pi_t(x))(x_t - \pi_t(x))^\top \rangle = \langle d(x_t - \pi_t(x))(x_t - \pi_t(x))^\top \rangle + \langle (x_t - \pi_t(x))d(x_t - \pi_t(x))^\top \rangle,$$

which is Eq. (106).

Differentiating $$\langle x - \pi(x) \rangle$$ with respect to $$t$$, yields

$$\frac{d}{dt} \langle (x_t - \pi_t(x))(x_t - \pi_t(x))^\top \rangle$$

and recall the form of conditional covariance matrix $$V_t$$ in Eq. (117), we have

$$d(x_t - \pi_t(x)) = A(x_t - \pi_t(x))dt + dBu - (V_tC_1^T + M)dv_t.$$  

Moreover, it can be easily checked that

$$\text{Tr}[\rho_{\otimes F}(0)dBudv_t^\top] = -iJ_nA^\dagger dt,$$

$$\text{Tr}[\rho_{\otimes F}(0)dv_tudv_t^\top] = iA^\dagger J_n^\dagger dt,$$

$$\text{Tr}[\rho_{\otimes F}(0)dBudv_t^\top B^\top] = J_nA^\dagger A^\dagger_n^\top dt.$$  

By Eqs. (120), (118) can be calculated as

$$\frac{d}{dt} \langle (x_t - \pi_t(x))(x_t - \pi_t(x))^\top \rangle = \Delta T - \langle (x_t - \pi_t(x))(x_t - \pi_t(x))^\top \rangle C_1^T(V_tC_1^T + M)^T$$

$$+ J_nA^\dagger J_n^\dagger + (V_tC_1^T + M)(V_tC_1^T + M)^T$$

$$+ iJ_nA^\dagger(V_tC_1^T + M)^T - i(V_tC_1^T + M)A^\dagger_n^\top.$$  

Step 1. Transposing both sides of (121), yields

$$\frac{d}{dt} \langle (x_t - \pi_t(x))(x_t - \pi_t(x))^\top \rangle^\top = \Delta^\top - \langle (x_t - \pi_t(x))(x_t - \pi_t(x))^\top \rangle^\top C_1^T(V_tC_1^T + M)^\top$$

$$+ J_nA^\dagger A^\dagger_n^\top + (V_tC_1^T + M)(V_tC_1^T + M)^\top$$

$$+ i(V_tC_1^T + M)A^\dagger_n^\top - iJ_nA^\dagger(V_tC_1^T + M)^\top.$$  

Step 2. Combine (121) with (122), and recall the form of conditional covariance matrix $$V_t$$ in Eq. (117), we have

$$\frac{d}{dt} V_t = AV_t + V_tA^\top + \frac{1}{2}(J_nA^\dagger A^\dagger_n^\top + J_nA^\top A^\dagger_n^\top)$$

$$+ (V_tC_1^T + M)(V_tC_1^T + M)^\top - (V_tC_1^T + M)C_1V_t - V_tC_1^T(V_tC_1^T + M)^\top$$

$$+ \frac{i}{2}J_n(\Lambda^\top - \Lambda^\top J_n^\dagger + J_n\Lambda^\dagger - \Lambda)J_n^\top$$

$$= AV_t + V_tA^\top + \frac{1}{2}BB^\top - (V_tC_1^T + M)(V_tC_1^T + M)^\top,$$

which is Eq. (106).
5.3. An example

In this subsection, we use a simple example to illustrate the quantum Kalman filter.

Consider a single-mode oscillator with system parameters $S = 1$, $L = \sqrt{\kappa} a$, and $H = \omega a^* a$. Then by Eq. (47)

$$
dx(t) = \Lambda x(t)dt + \mathbb{B} d\mathcal{U}(t),
\quad d\mathcal{Y}(t) = \mathbb{C} x(t)dt + \mathbb{D} d\mathcal{U}(t)
$$

where

$$
\Lambda = \begin{bmatrix}
-\frac{\kappa}{2} & \omega \\
-\omega & -\frac{\kappa}{2}
\end{bmatrix}, \quad \mathbb{B} = -\sqrt{\kappa} I_2, \quad \mathbb{C} = \sqrt{\kappa} I_2, \quad \mathbb{D} = I_2.
$$

Let the system be initialized in the vacuum state $|0\rangle$. Assume that $Q_{\text{out}}(t)$ is continuously measured. Thus, a sequence of measurement data $\{Q_{\text{out}}(r), 0 \leq r \leq t\}$ is available at the present time $t$. By the quantum Kalman filter presented in the previous subsection, we have

$$
d\pi_t(q) = -\sqrt{\kappa} \pi_t(q)dt + \omega \pi_t(p)dt - \sqrt{\kappa} \left(\frac{\sqrt{2}}{2} - V_1\right) d\nu(t)
$$

$$
= \left(\frac{\sqrt{2}\kappa - \sqrt{\kappa}}{2} - \kappa V_1\right) \pi_t(q)dt + \omega \pi_t(p)dt - \sqrt{\kappa} \left(\frac{\sqrt{2}}{2} - V_1\right) dQ_{\text{out}}(t),
$$

$$
d\pi_t(p) = -\omega \pi_t(q)dt - \sqrt{\kappa} \pi_t(p)dt + \sqrt{\kappa} V_2 d\nu(t)
$$

$$
= -\left(\omega + \sqrt{\kappa} V_2\right) \pi_t(q)dt - \kappa \pi_t(p)dt + \sqrt{\kappa} V_2 dQ_{\text{out}}(t),
$$

where the innovation process is

$$
d\nu(t) = dQ_{\text{out}}(t) - \sqrt{\kappa} \pi_t(q)dt,
$$

and the entries of the covariance matrix $V$ evolve according to

$$
\dot{V}_1 = (\sqrt{2} - 1)\kappa V_1 + 2\omega V_2 - \kappa V_1^2
$$

$$
\dot{V}_2 = -\kappa(1 - \frac{\sqrt{2}}{2}) V_2 - \omega (V_1 - V_3) - \kappa V_1 V_2
$$

$$
\dot{V}_3 = \frac{\kappa}{2} - 2\omega V_2 - \kappa V_3 - \kappa V_2^2
$$

with the initial condition $V_1(0) = V_3(0) = 1$ and $V_2(0) = 0$.

When the detuning $\omega = 0$, by Eq. (128a), $V_2(t) \equiv V_2(0) = 0$ provided that the solution is unique. Then by Eq. (126b), we have $\pi_t(p) \equiv \pi_0(p)$, which is not disturbed directly by the continuous measurement of $Q_{\text{out}}(t)$. On the other hand, when the detuning $\omega \neq 0$, $V_2 \neq 0$. From Eq. (126b) we can see that the measurement of $Q_{\text{out}}(t)$ affects $\pi_t(p)$, which in the sequel affects $\pi_t(q)$. This clearly demonstrates the quantum back-action effect.

6. Quantum Kalman canonical decomposition

In this section, we discuss the quantum Kalman canonical decomposition of quantum linear systems in Subsection 6.1. In Subsection 6.2, an example taken from a recent experiment [31] is used to illustrate the procedures and main results. Kalman canonical decomposition of classical linear systems can be found in, e.g., [68 Section 2.2] and [69 Section 3.3].
6.1. Quantum Kalman canonical form

The following result reveals the structure of quantum linear systems; see [56, 65] and [133] for more details.

**Proposition 6.1.** The quantum linear system (43) has the following properties.

(i) Its controllability and observability are equivalent to each other; see [56, Proposition 1], and see [134] for the passive case.

(ii) If it is Hurwitz stable, then it is both controllable and observable; see [133, Theorem 3.1], and see [134] for the passive case.

(iii) In the passive case Eq. (32), its Hurwitz stability, controllability and observability are all equivalent; see [56, Lemma 2].

(iv) All its poles corresponding to an uncontrollable and unobservable subsystem are on the imaginary axis; see [65, Theorem 3.2].

Interestingly, the equivalence between stabilizability and detectability of quantum linear systems is pointed out in the physics literature [44].

The following result presents the Kalman canonical form of quantum linear systems.

**Theorem 6.1 ([65, Theorem 4.4]).** Assume \( u = 0 \) in Eq. (43). Also suppose the scattering matrix \( S = I_m \). There exists a real orthogonal and blockwise symplectic matrix \( T \) that facilitates the following coordinate transformation

\[
\begin{bmatrix}
q_h \\
p_h \\
x_{co} \\
x_{\bar{c}o}
\end{bmatrix} = \tilde{x} = T^\top x,
\]

and transform the linear quantum system (43) into the form

\[
\begin{bmatrix}
\dot{q}_h(t) \\
\dot{p}_h(t) \\
\dot{x}_{co}(t) \\
\dot{x}_{\bar{c}o}(t)
\end{bmatrix} = \tilde{A} \begin{bmatrix}
q_h(t) \\
p_h(t) \\
x_{co}(t) \\
x_{\bar{c}o}(t)
\end{bmatrix} + \tilde{B} u(t),
\]

\[
y(t) = \tilde{C} \begin{bmatrix}
q_h(t) \\
p_h(t) \\
x_{co}(t) \\
x_{\bar{c}o}(t)
\end{bmatrix} + u(t),
\]

where the system matrices are

\[
\tilde{A} = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
0 & A_{22} & 0 \\
0 & A_{21} & A_{co} \\
0 & A_{31} & 0
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}
B_h \\
0 \\
B_{co}
\end{bmatrix}, \quad \tilde{C} = \begin{bmatrix}
0 & C_h & C_{co} & 0
\end{bmatrix}.
\]
After a re-arrangement, the system (130)-(131) becomes

\[
\begin{bmatrix}
\dot{q}_h(t) \\
\dot{x}_{co}(t) \\
\dot{x}_{\bar{c}o}(t) \\
\dot{p}_h(t)
\end{bmatrix} =
\begin{bmatrix}
A_{h}^{11} & A_{12} & A_{13} & A_{h}^{12} \\
0 & A_{co} & 0 & A_{21} \\
0 & 0 & A_{\bar{c}o} & A_{31} \\
0 & 0 & 0 & A_{h}^{22}
\end{bmatrix}
\begin{bmatrix}
q_h(t) \\
x_{co}(t) \\
x_{\bar{c}o}(t) \\
p_h(t)
\end{bmatrix} +
\begin{bmatrix}
B_h \\
B_{co} \\
0 \\
0
\end{bmatrix} u(t),
\] (133)

\[
y(t) =
\begin{bmatrix}
0 & C_{co} & 0 & C_h
\end{bmatrix}
\begin{bmatrix}
q_h(t) \\
x_{co}(t) \\
x_{\bar{c}o}(t) \\
p_h(t)
\end{bmatrix} + u(t).
\] (134)

A block diagram for the system (130)-(131) is given in Fig. 2.

A refinement of the quantum Kalman canonical form is given in \([133, \text{Theorem 3.3}]\),

![Diagram](image_url)

The following result characterize the system parameters corresponding to the Kalman canonical form (130).

**Theorem 6.2.** [133, Proposition 2.1] Denote

\[
\tilde{H} = T^\top H T, \quad \tilde{\Lambda} = \Lambda T,
\] (135)

where the matrices \(H\) and \(\Lambda\) are given in Eq. (42). The real symmetric matrix \(\tilde{H}\) corresponding to the Kalman canonical form (130) must be of the form

\[
\tilde{H} =
\begin{bmatrix}
0 & H_h^{12} & 0 & 0 \\
H_h^{12\top} & H_h^{22} & H_{12} & H_{13} \\
0 & H_{12}^\top & H_{co} & 0 \\
0 & H_{13}^\top & 0 & H_{\bar{c}o}
\end{bmatrix},
\] (136)

...
and the complex matrix $\tilde{\Lambda}$ corresponding to the Kalman canonical form (130) must satisfy

$$
\begin{bmatrix}
\Lambda & \Lambda_h \\
\Lambda^\# & \Lambda_{co}
\end{bmatrix}
= 
\begin{bmatrix}
0 & \Lambda_{co} \\
0 & 0
\end{bmatrix},
$$

where

$$
\Lambda_{co} = V_m^\dagger C_{co},
$$

and

$$
\Lambda_h = V_m^\dagger C_h.
$$

**Remark 6.1.** The matrices $\Lambda_{co}$ and $\Lambda_h$ in Eq. (138) are respectively of the form

$$
\Lambda_{co} = 
\begin{bmatrix}
\Lambda_{co,q} & \Lambda_{co,p} \\
\Lambda^\#_{co,q} & \Lambda^\#_{co,p}
\end{bmatrix},
$$

and

$$
\Lambda_h = 
\begin{bmatrix}
\Lambda_{h,p} \\
\Lambda^\#_{h,p}
\end{bmatrix}.
$$

The Kalman canonical form (130) presented above can be used to investigate BAE measurements of quantum linear systems.

**Theorem 6.3.** [133, Theorem 4.1]

(i) The quantum Kalman canonical form (130) realizes the BAE measurements of $q_{out}$ with respect to $p_{in}$; i.e., $\Xi_{p_{in}\rightarrow q_{out}}(s) = 0$ if and only if

$$
\begin{bmatrix}
\text{Re}(\Lambda_{co,q}) & \text{Re}(\Lambda_{co,p})
\end{bmatrix}
(sI - J_{n_1} H_{co})^{-1}
\begin{bmatrix}
\text{Re}(\Lambda^\dagger_{co,p}) \\
-\text{Re}(\Lambda^\#_{co,q})
\end{bmatrix} = 0;
$$

(ii) The quantum Kalman canonical form (130) realizes the BAE measurements of $p_{out}$ with respect to $q_{in}$; i.e., $\Xi_{q_{in}\rightarrow p_{out}}(s) = 0$ if and only if

$$
\begin{bmatrix}
\text{Im}(\Lambda_{co,q}) & \text{Im}(\Lambda_{co,p})
\end{bmatrix}
(sI - J_{n_1} H_{co})^{-1}
\begin{bmatrix}
\text{Im}(\Lambda^\dagger_{co,p}) \\
-\text{Im}(\Lambda^\#_{co,q})
\end{bmatrix} = 0.
$$

6.2. An example

An opto-mechanical system has recently been physically realized in [31]. In this subsection, we analyze this system by means of the quantum Kalman canonical form presented in the previous subsection. An excellent introduction to quantum opto-mechanical systems can be found in [26].

In [31, Fig. 1(A)], an effective positive-mass oscillator and effective negative-mass oscillator coupled to a cavity are constructed to generate a quantum mechanics-free subsystem (QMFS). To be specific, the linearized Hamiltonian is [31, Eq. (S3)]

$$
H = \omega(a_1^* a_1 - a_2^* a_2) + g_1[(\alpha_1 - a_1 + \alpha_1^* a_1^*)a_3^* + (\alpha_1^* - a_1^* + \alpha_1 a_1) a_3]
+ g_2[(\alpha_2 - a_2 + \alpha_2^* a_2^*)a_3^* + (\alpha_2^* - a_2^* + \alpha_2 a_2) a_3],
$$

where the reduced Planck constant is omitted and $\omega$ is the detuning frequency. $a_j, a_j^*$, $(j = 1, 2)$ denote the annihilation and creation operators for the mechanical oscillators, and their corresponding coupling strengths (to the cavity) are $G_{j\pm} = g_j \alpha_{j\pm}$. The cavity is characterized by the damping rate $\kappa$ and described...
by the annihilation and creation operators $a_3, a_3^\dagger$. Choose equal effective couplings $|G_{j\pm}| \equiv G$ and let $G_{j\pm} = Ge^{-\theta_{j\pm}}$, where $\theta_{j\pm} \in \mathbb{C}$, $j = 1, 2$. Then the interaction Hamiltonian (142) can be re-written as the following quadrature form [31, Eq. (S4)]

$$H_{\text{int}} = \frac{G}{2} a_3 [A_- q_- + A_+ q_+ + B_- p_- + B_+ p_+]$$

$$+ \frac{G}{2} a_3^\dagger [A_- q_- + A_+ q_+ + B_- p_- + B_+ p_+]$$

where

$$q_j = \frac{a_j^* + a_j}{\sqrt{2}}, \quad p_j = \frac{i(a_j^* - a_j)}{\sqrt{2}}, \quad j = 1, 2,$$

$$q_\pm = \frac{q_1 \pm q_2}{\sqrt{2}}, \quad p_\pm = \frac{p_1 \pm p_2}{\sqrt{2}},$$

and

$$A_- = e^{-i\theta_1} + e^{-i\theta_2} - e^{-i\theta_2} - e^{-i\theta_1},$$

$$A_+ = e^{-i\theta_1} + e^{-i\theta_2} + e^{-i\theta_2} + e^{-i\theta_1},$$

$$B_- = i[-e^{-i\theta_1} + e^{-i\theta_1} + e^{-i\theta_2} - e^{-i\theta_2}],$$

$$B_+ = i[-e^{-i\theta_1} + e^{-i\theta_1} - e^{-i\theta_2} + e^{-i\theta_2}],$$

which are [31, Eq. (S5)].

Let $\theta_1 = \theta_2 = 0$ and $\theta_1 \mp = \theta_2 \mp = \phi$. Eq. (143) becomes, [31, Eq. (S6)]

$$H_{\text{int}} = 2G(e^{-i\phi/2} a_3 + e^{i\phi/2} a_3^\dagger) \left( q_+ \cos \frac{\phi}{2} + p_\mp \sin \frac{\phi}{2} \right).$$

Moreover, by omitting the dissipation term and choosing $\phi = 0$, Eq. (146) becomes

$$H_{\text{int}} = 2\sqrt{2}Gq_c q_+,$$

where $q_c = \frac{a_3^* + a_3}{\sqrt{2}}$.

**Remark 6.2.** When $\theta_1 = \theta_2 = \theta_1 \mp = \theta_2 \mp = 0$, all coupling strengths $G_{j\pm} = G$ are equal. This setting is used below.

Assume that the coupling strengths are equal and by the framework presented in Section 2, the system Hamiltonian (142) can be written as

$$\Omega = \Delta(\Omega_-, \Omega_+),$$

where

$$\Omega_- = \begin{bmatrix} \omega & 0 & G \\ 0 & -\omega & G \\ G & G & 0 \end{bmatrix}, \quad \Omega_+ = \begin{bmatrix} 0 & 0 & G \\ 0 & 0 & G \\ G & G & 0 \end{bmatrix}.$$ 

As the optical coupling is $L = \sqrt{\kappa a_3}$ what describes energy dissipation from the cavity, we have

$$C = \Delta(C_-, C_+),$$

25
where $C_- = \begin{bmatrix} 0 & 0 & \sqrt{\kappa} \end{bmatrix}$, $C_+ = 0$. By Eq. (18), the system matrices can be calculated as

$$\begin{align*}
B &= -C^\circ = -\begin{bmatrix} 0 & 0 & \sqrt{\kappa} & 0 & 0 & 0 \end{bmatrix}^\top, \\
A &= -iJ_3\Omega - \frac{1}{2}C^\circ C = \\
&= \begin{bmatrix}
-i\omega & 0 & -iG & 0 & 0 & -iG \\
0 & i\omega & -iG & 0 & 0 & -iG \\
-iG & -iG & -\frac{\kappa}{2} & -iG & -iG & 0 \\
0 & 0 & iG & i\omega & 0 & iG \\
0 & 0 & iG & 0 & -i\omega & iG \\
iG & iG & 0 & iG & iG & -\frac{\kappa}{2}
\end{bmatrix},
\end{align*}$$

(151)

As a result, the opto-mechanical system composed of two oscillators and a cavity can be described by

$$\begin{align*}
\dot{a} &= A\dot{a} + Bb_{\text{in}}, \\
\dot{b}_{\text{out}} &= C\dot{a} + b_{\text{in}},
\end{align*}$$

(152)

where $a = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^\top$. The auxiliary matrix $O_s$ defined in the proof of [65, Theorem 3.1], originally in [56, Eq. 7], can be solved by

$$O_s = \begin{bmatrix}
C \\
C(J_3\Omega) \\
\vdots \\
C(J_3\Omega)^5
\end{bmatrix}.$$  

(153)

In what follows, $R_{co}$, $R_{c\bar{o}}$, $R_{c\bar{o}}$, and $R_{\bar{o}o}$ represent the controllable/observable (co), uncontrollable/unobservable ($\bar{o}$), controllable/unobservable (c$\bar{o}$), and uncontrollable/observable ($\bar{o}o$) subspaces of system (152), respectively. By [65, Lemma 4.2], these four subspaces can be calculated as

$$R_{co} = \text{span}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad R_{c\bar{o}} = \emptyset,$n(154)

$$R_{c\bar{o}} = \text{span}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad R_{\bar{o}o} = \text{span}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_{\bar{o}o} = \text{span}\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$  

Then by [65, Lemma 4.3] and [65, Lemma 4.7], the special orthonormal bases $T_{co}$, $T_{c\bar{o}}$, $T_{c\bar{o}}$, and $T_{\bar{o}o}$ can be
constructed. From \[\text{Eq. (47)}\], the blockwise Bogoliubov transformation matrix can be calculated as

\[
T = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

By \[\text{Eq. (47)}\], we have the transformed system matrices

\[
\bar{A} = T^\dagger A T, \quad \bar{B} = T^\dagger B, \quad \bar{C} = CT.
\]  

Recall the dimensions of the four subspaces introduced in \[\text{Eq. (47)}\], we have \(n_1 = 1\), \(n_2 = 0\), \(n_3 = 2\), and \(n_1 + n_2 + n_3 = n = 3\) in this example. By \[\text{Eq. (47)}\],

\[
\bar{V}_n = \text{diag} \{ \bar{V}_{n_3}, V_{n_1} \},
\]

where \(\bar{V}_{n_3} = \Pi V_{n_3}\), \(\Pi = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}\), and

\[
\bar{A} = \bar{V}_n \bar{A} \bar{V}_n^\dagger = \begin{bmatrix}
0 & -\omega & 0 & 0 & 0 & 0 \\
\omega & 0 & 0 & 0 & 2\sqrt{2}G & 0 \\
0 & 0 & 0 & -\omega & 0 & 0 \\
0 & 0 & \omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{\kappa}{2} & 0 \\
0 & 0 & 0 & -2\sqrt{2}G & 0 & -\frac{\kappa}{2}
\end{bmatrix},
\]

\[
\bar{B} = \bar{V}_n \bar{B} V_1^\dagger = -\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \sqrt{\kappa} \\
0 & 0 & 0 & 0 & \sqrt{\kappa} & 0
\end{bmatrix}^\top,
\]

\[
\bar{C} = V_1 \bar{C} \bar{V}_n^\dagger = -\bar{B}^\top.
\]

By \[\text{Eq. (47)}\], the real, orthogonal, and blockwise symplectic matrix \(T \triangleq V_n T \bar{V}_n^\dagger\). From \[\text{Eq. (47)}\], the transformed system operators

\[
T^\top V_n \hat{\mathfrak{a}} = \begin{bmatrix}
q_- \\
p_+ \\
-p_- \\
oslash p_+ \\
q_+ \\
q_c \\
p_c
\end{bmatrix} = \begin{bmatrix}
\hat{q}_h \\
\hat{p}_h \\
x_{co}
\end{bmatrix},
\]  

27
where \( p_c = \frac{i(a^*_c - a_c)}{\sqrt{2}} \). By [65, Theorem 4.4], the Kalman decomposition for the opto-mechanical system in the real quadrature operator representation can be expressed as

\[
\begin{align*}
\dot{q}_h &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} q_h + \begin{bmatrix} 0 & -2 \sqrt{2} G \\ 2 \sqrt{2} G & 0 \end{bmatrix} x_{co}, \\
\dot{p}_h &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} p_h, \\
\dot{x}_{co} &= -\frac{\kappa}{2} x_{co} - \begin{bmatrix} 0 & 0 \\ 0 & 2 \sqrt{2} G \end{bmatrix} p_h - \sqrt{\kappa} \begin{bmatrix} q_{in} \\ p_{in} \end{bmatrix}, \\
\begin{bmatrix} q_{out} \\ p_{out} \end{bmatrix} &= \sqrt{\kappa} x_{co} + \begin{bmatrix} q_{in} \\ p_{in} \end{bmatrix}.
\end{align*}
\]

(160)

Consequently, by omitting the dissipation term (\( \kappa = 0 \)), we have

\[
\begin{align*}
\dot{q}_- &= \omega p_+, \\
\dot{p}_+ &= -\omega q_- - 2 \sqrt{2} G q_c, \\
\dot{q}_+ &= -\omega q_+, \\
\dot{q}_c &= 0, \\
\dot{p}_c &= -2 \sqrt{2} G q_+,
\end{align*}
\]

(161)

where \( \{q_+, p_-\} \) forms an isolated QMFS, and Eq. (161) is consistent with [31, Eq. (S10)].

Moreover, by Eq. (160) it can be verified that the opto-mechanical system realizes a BAE measurement of \( q_{out} \) with respect to \( p_{in} \), and a BAE measurement of \( p_{out} \) with respect to \( q_{in} \).

7. Response to single-photon states

7.1. Continuous-mode single-photon states

In this subsection, we introduce continuous-mode single-photon states of a free traveling light field.

We look at the single-channel (\( m = 1 \)) case first. Denote \( |1_t⟩ = b^*_m(t) |\Phi_0⟩ \), i.e., a photon is generated at the time instant \( t \) by the creation operator \( b^*_m(t) \) from the vacuum field \( |\Phi_0⟩ \). By Eq. (8) we get \( \langle 1_t | 1_\tau⟩ = \delta(t - \tau) \); in other words, the entries in \( \{ |1_t⟩ : t \in \mathbb{R} \} \) are orthogonal. Actually, \( \{ |1_t⟩ : t \in \mathbb{R} \} \) form a complete basis of continuous-mode single-photon states of a free propagating light field, in the sense that a continuous-mode single-photon state \( |1_\xi⟩ \) of the temporal pulse shape \( \xi \in L^2(\mathbb{R}, \mathbb{C}) \) can be expressed as

\[
|1_\xi⟩ \equiv B^*_m(\xi) |\Phi_0⟩ \triangleq \int_{-\infty}^{\infty} \xi(t) |1_t⟩ dt.
\]

(162)

(Here, it is assumed that the \( L^2 \) norm \( \| \xi \| \triangleq \sqrt{\int_{-\infty}^{\infty} |\xi(t)|^2 dt} = 1 \). Then \( \langle 1_\xi | 1_\xi⟩ = 1 \).) The physical interpretation of the single-photon state \( |1_\xi⟩ \) is that the probability of detecting the photon in the time bin \([t, t + dt]\) is \( |\xi(t)|^2 dt \). In the frequency domain, we denote \( |1_\omega⟩ \triangleq b^*_m[i\omega] |0⟩ \). Hence, in the frequency domain (162) becomes

\[
|1_\xi⟩ = \int_{-\infty}^{\infty} \xi[i\omega] |1_\omega⟩ d\omega.
\]

(163)
Remark 7.1. Notice

\[ \langle 1_\xi | dB^*_\text{in}(t) dB_{\text{in}}(t) | 1_\xi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} drd\tau \xi(r) \xi(\tau) \int_{t}^{t+dt} dt_1 \int_{t}^{t+dt} dt_2 \langle \Phi_0 | b^*_\text{in}(r) b_{\text{in}}(t_1) b^*_\text{in}(t_2) b_{\text{in}}(\tau) \Phi_0 \rangle \]

\[ = \int_{t}^{t+dt} \int_{t}^{t+dt} dt_1 dt_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} drd\tau \xi(r) \xi(\tau) \delta(t_1 - r) \delta(t_2 - \tau) \]

\[ = \int_{t}^{t+dt} \xi^*(t_1) dt_1 \int_{t}^{t+dt} \xi(t_2) dt_2. \] (164)

Thus, for most functions \( \xi \in L_2(\mathbb{R}, \mathbb{C}) \),

\[ \langle 1_\xi | dB^*_\text{in}(t) dB_{\text{in}}(t) | 1_\xi \rangle = (O(dt))^2. \] (165)

In Itô stochastic calculus, \( \langle 1_\xi | dB^*_\text{in}(t) dB_{\text{in}}(t) | 1_\xi \rangle = 0 \). This shows that the field with a continuous-mode single-photon state \( |1_\xi \rangle \) is a canonical field.

In this tutorial, we investigate single-photon states from a control-theoretic perspective. For physical implementation of single photon generation, detection and storing, please refer to the physics literature [135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159] and references therein. A concise discussion can be found in [63, Section 3.1].

7.2. Response of quantum linear systems to continuous-mode single-photon states

Let the linear quantum system \( G \) be initialized in the coherent state \( |\alpha \rangle \) (defined in Section 4) and the input field be initialized in the vacuum state \( |\Phi_0 \rangle \). Then the initial joint system-field state is \( \rho_{0g} \triangleq |\alpha \rangle \langle \alpha | \otimes |\Phi_0 \rangle \langle \Phi_0 | \) in the form of a density matrix. Denote

\[ \rho_{\infty g} = \lim_{t \to \infty, t_0 \to -\infty} U(t, t_0) \rho_{0g} U(t, t_0)^*. \] (166)

Here, \( t_0 \to -\infty \) indicates that the interaction starts in the remote past and \( t \to \infty \) means that we are interested in the dynamics in the far future. In other words, we look at the steady-state dynamics. Define

\[ \rho_{\text{field}, g} \triangleq \langle \alpha | \rho_{\infty g} | \alpha \rangle. \] (167)

In other words, the system is traced off and we focus on the steady state of the output field.

Let the \( k \)th input channel be in a single photon state \( |1_{\mu_k} \rangle, k = 1, \ldots, m \). Thus, the state of the \( m \)-channel input is given by the tensor product

\[ |\Psi_\mu \rangle = |1_{\mu_1} \rangle \otimes \cdots \otimes |1_{\mu_m} \rangle. \] (168)

Denote \( \mu = [\mu_1 \ \cdots \ \mu_m]^\top \).

Theorem 7.1. Assume that the passive linear quantum system \( [32] \) is Hurwitz stable, initialized in the vacuum state \( |0 \rangle \) and driven by an \( m \)-photon input state \( |\Psi_\mu \rangle \) in Eq. (168). Then the steady-state output state is another \( m \)-photon \( |\Psi_\nu \rangle \) whose pulse \( \nu = [\nu_1 \ \cdots \ \nu_m]^\top \) is given by

\[ \nu[i\omega] = \Xi_G - [i\omega] \mu[i\omega]. \] (169)
If the linear system $G$ is not passive, or is not initialized in the vacuum state $|0\rangle$, the steady-state output field state $\rho_{\text{out}}$ in general is not a single- or multi-photon state. This new type of states has been named “photon-Gaussian” states in [64]. Moreover, it has been proved in [64] that the class of photon-Gaussian states is invariant under the steady-state action of a linear quantum system. In what follows we present this result.

**Definition 7.1.** [64, Definition 1] A state $\rho_{\xi,R}$ is said to be a photon-Gaussian state if it belongs to the set

$$
\mathcal{F} \triangleq \left\{ \rho_{\xi,R} = \prod_{k=1}^{m} \sum_{j=1}^{m} \left( B_{m,j}^{\dagger}(\xi_{jk}) - B_{m,j}(\xi_{jk}^{+}) \right) \rho_{R} \left( \prod_{k=1}^{m} \sum_{j=1}^{m} \left( B_{m,j}(\xi_{jk}) - B_{m,j}(\xi_{jk}^{+}) \right) \right)^{*} : \text{function } \xi = \Delta(\xi^{-},\xi^{+}) \text{ and density matrix } \rho_{R} \text{ satisfy } \text{Tr}[\rho_{\xi,R}] = 1 \right\}.
$$

**Theorem 7.2.** [64, Theorem 5] Let $\rho_{\text{in},R_{\text{in}}} \in \mathcal{F}$ be a photon-Gaussian input state. Also, assume that $G$ is Hurwitz stable and is initialized in a coherent state $|\alpha\rangle$. Then the linear quantum system $G$ produces in steady state a photon-Gaussian output state $\rho_{\text{out},R_{\text{out}}} \in \mathcal{F}$, where

$$
\xi_{\text{out}}[s] = \Xi_{s}[s]|\xi_{\text{in}}[s],
$$

$$
R_{\text{out}}[i\omega] = \Xi_{i\omega}[i\omega]R_{\text{in}}[i\omega]\Xi_{i\omega}[i\omega]^\dagger.
$$

### 7.3. Response of quantum linear systems to continuous-mode multi-photon states

Response of quantum linear systems to multi-photon states has been studied in [160, 161], as generalization to the single-photon case. In this subsection, we present one of the main results in these papers.

Let there be $\ell_{j}$ photons in the $j$th channel. The state for this channel is

$$
|\Psi_{j}\rangle = \frac{1}{\sqrt{N_{\ell_{j}}}} \int_{\ell_{j}} \Psi_{j}(t_{1},\ldots,t_{\ell_{j}}) b_{m,j}^{\dagger}(t_{1}) \cdots b_{m,j}^{\dagger}(t_{\ell_{j}}) dt_{1} \cdots dt_{\ell_{j}} |\Phi_{0}\rangle,
$$

where $\Psi_{j}$ is the pulse shape and $\frac{1}{\sqrt{N_{\ell_{j}}}}$ is the normalization coefficient. Here, for any integer $r > 1$, we write $\int_{r}$ for integration in the space $\mathbb{C}^{r}$. If $\ell_{j} = 0$, then Eq. (172) is understood as $|\Psi_{j}\rangle = |\Phi_{0}\rangle$. Then the state for the $m$-channel input field can be defined as

$$
|\Psi\rangle = \prod_{j=1}^{m} |\Psi_{j}\rangle.
$$

Next, we rewrite this $m$-channel multi-photon state into an alternative form; this will enable us to present the input and output states in a unified form. For $j = 1,\ldots,m$, $i = 1,\ldots,\ell_{j}$, and $k_{i} = 1,\ldots,m$, define functions

$$
\Psi_{j,k_{1},\ldots,k_{\ell_{j}}}(\tau_{1},\ldots,\tau_{\ell_{j}}) \triangleq \begin{cases} 
\Psi_{j}(\tau_{1},\ldots,\tau_{\ell_{j}}), & k_{1} = \cdots = k_{\ell_{j}} = j, \\
0, & \text{otherwise.}
\end{cases}
$$

Then we define a class of pure $m$-channel multi-photon states

$$
\mathcal{F}_{1} \triangleq \left\{ |\Psi\rangle = \prod_{j=1}^{m} \frac{1}{\sqrt{N_{\ell_{j}}}} \sum_{k_{1},\ldots,k_{\ell_{j}}=1}^{m} \int_{\ell_{j}} \Psi_{j,k_{1},\ldots,k_{\ell_{j}}}(\tau_{1},\ldots,\tau_{\ell_{j}}) \\
\times \prod_{i=1}^{\ell_{j}} b_{m,k_{i}}^{\dagger}(\tau_{i}) d\tau_{1} \cdots d\tau_{\ell_{j}} |\Phi_{0}\rangle : \langle \Psi|\Psi\rangle = 1 \right\}.
$$
Clearly, $|\Psi\rangle$ in Eq. (173) belongs to $\mathcal{F}_1$.

We need some operations between matrices and tensors. Let $g_G(t) = (g_G^{jk}(t)) \in \mathbb{C}^{m \times m}$ be the impulse response function of the quantum linear system $G$ in Fig. 1. For each $j = 1, \ldots, m$, let $\mathcal{V}_j(t_1, \ldots, t_{\ell_j})$ be an $\ell_j$-way $m$-dimensional tensor function that encodes the pulse information of the $j$th input field containing $\ell_j$ photons. Denote the entries of $\mathcal{V}_j(t_1, \ldots, t_{\ell_j})$ by $V_{j,k_1,\ldots,k_{\ell_j}}(t_1, \ldots, t_{\ell_j})$. Define an $\ell_j$-way $m$-dimensional tensor $\mathcal{W}_j$ with entries given by the following multiple convolution

$$
\mathcal{W}_j = \mathcal{V}_j \times_1 g_G \times_2 \cdots \times_{\ell_j} g_G, \quad \forall j = 1, \ldots, m,
$$

for all $1 \leq r_1, \ldots, r_{\ell_j} \leq m$. In compact form we write

$$
\mathcal{W}_j = \mathcal{V}_j \times_1 g_G \times_2 \cdots \times_{\ell_j} g_G, \quad \forall j = 1, \ldots, m,
$$

cf. [162], [163, Sec. 2.5], [164] and [165].

**Theorem 7.3.** ([164, Theorem 12]) Suppose that the quantum linear system $G$ is Hurwitz stable and passive. The steady-state output state of $G$ driven by a state $|\Psi_{\text{in}}\rangle \in \mathcal{F}_1$ is another state $|\Psi_{\text{out}}\rangle \in \mathcal{F}_1$ with wave packet transfer

$$
|\Psi_{\text{out},j}\rangle = |\Psi_{\text{in},j}\rangle \times_1 g_G^- \times_2 \cdots \times_{\ell_j} g_G^-, \quad \forall j = 1, \ldots, m,
$$

where the operation between the matrix and tensor is defined in Eq. (176).

More discussions on continuous-mode multi-photon states can be found in [160, 161]. An application to the amplification of optical Schrödinger cat states can be found in [166]. Simply speakin, the mathematial methods proposed in [160, 161] can be used to study photon-catalyzed quantum non-Gaussian states, which are useful resources in quantum information processing [167, 168, 169]. The problem of the response of quantum nonlinear systems to multi-photon states has been studied in [170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195]. It turns out that the linear systems theory plays a key role in some of these studies.

A continuous-mode single-photon field studied in Subsection 7.1 has statistical properties. Thus, it is natural to study the filtering problem of a quantum system driven by a continuous-mode single-photon states. Continuous-mode single-photon filters were derived in [196] and [197] first, and their multi-photon version was developed in [198, 199, 200] and [201]. A review of continuous-mode single or multi-photon states is given in [63].

8. Feedback control of quantum linear systems

Feedback control of quantum systems has been covered in several books, for example, [202] and [203]. In particular, the monograph [4] is devoted to the feedback control of quantum linear systems. Depending on whether the underlying quantum system (plant) is measured and the measurement data is used for the feedback control of the plant, feedback control methods of quantum systems can be roughly divided into two
categories: measurement feedback control and coherent feedback control. It is clear that the former makes use of measurement information, whereas in a coherent feedback network no measurement is involved and thus coherence of quantum signals is preserved. To our understanding, measurement feedback control has been studied intensively and well recorded in [5], [202], and [203]. In contrast, coherent feedback control is still a bit new to many researchers in the quantum control community, though it has advantages in many applications [204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230]. Thus, in this section, we describe briefly linear quantum coherent feedback networks and use a recent experiment as demonstration.

8.1. Quantum coherent feedback linear networks

For notational simplicity, in this and the next subsections, the subscript “in” for input fields is omitted. In Fig. 3 the closed-loop system contains a quantum plant \( P \) and a quantum controller \( K \) to be designed.

We look at the plant \( P \) first, which is a quantum linear system driven by three types of input channels. To be specific, \( b_{p1} \) describes the free input channels which may model quantum white noise such as the vacuum or thermal noise. For example, \( b_{p1} \) can be unmodeled quantum vacuum noise on the quantum plant due to imperfection of the physical system. As discussed in Remark 2.1, \( b_{p2} \) models the quantum vacuum fields and \( w_p \) represents quantum or classical signals. For example, \( w_p \) may be some undesired disturbance on the quantum plant \( P \); it may also represent classical signals from a classical controller such as \( u_2 \) in [9, Figure 5.1]. The third input, which is the output \( b_{out,kp} \) of the controller \( K \), is denoted by \( b_{p3} \). Correspondingly, we define three coupling operators \( L_{p1}, L_{p2}, L_{p3} \) and three scattering matrices \( S_{p1}, S_{p2}, S_{p3} \) for the input–output channels. By means of the theory introduced in Section 2, the physical output channels are \( b_{out,p1}, \)
$b_{\text{out,}p_2}$ and $b_{\text{out,}p_3}$, which in the integral form are given by

$$
\begin{align}
    dB_{\text{out,}p_1}(t) &= \mathbf{L}_{p_1}(t)dt + SP_d dB_{p_1}(t), \\
    dB_{\text{out,}p_2}(t) &= \mathbf{L}_{p_2}(t)dt + SP_d dB_{p_2}(t) + w_p dt, \\
    dB_{\text{out,}p_3}(t) &= \mathbf{L}_{p_3}(t)dt + SP_d dB_{p_3}(t),
\end{align}
$$

(177)

These physical channels can be put into three categories $b_{\text{out,}p_f}$, $b_{\text{out,}p_m}$, and $b_{\text{out,}p_k}$. Here, $b_{\text{out,}p_f}$ is a set of free output channels, $b_{\text{out,}p_m}$ represents a collection of output field channels which are to be measured, and $b_{\text{out,}p_k}$ is a set of output channels to be sent to the controller $K$. As a result, the dynamics of the plant $P$ can be described by the following QSEEs

$$
\begin{align}
    d\tilde{a}_p(t) &= A_p\tilde{a}_p(t)dt + E_p\tilde{v}_p(t)dt + B_{p_1} dB_{p_1}(t) \\
    &+ B_{p_2} (dB_{p_2}(t) + \tilde{w}_p(t)dt) + B_{p_3} dB_{p_3}(t), \\
    d\tilde{B}_{\text{out,}p_f}(t) &= C_{p_f}\tilde{a}_p(t)dt + D_{p_f_1} dB_{p_1}(t) \\
    &+ D_{p_f_2} (dB_{p_2}(t) + \tilde{w}_p(t)dt) + D_{p_f_3} dB_{p_3}(t), \\
    d\tilde{B}_{\text{out,}p_m}(t) &= C_{p_m}\tilde{a}_p(t)dt + D_{p_m_1} dB_{p_1}(t) \\
    &+ D_{p_m_2} (dB_{p_2}(t) + \tilde{w}_p(t)dt) + D_{p_m_3} dB_{p_3}(t), \\
    d\tilde{B}_{\text{out,}p_k}(t) &= C_{p_k}\tilde{a}_p(t)dt + D_{p_k_1} dB_{p_1}(t) \\
    &+ D_{p_k_2} (dB_{p_2}(t) + \tilde{w}_p(t)dt) + D_{p_k_3} dB_{p_3}(t).
\end{align}
$$

(178)

Similarly, the inputs $b_{k_1}$, $b_{k_2}$, $b_{k_3}$ and outputs $b_{\text{out,}k_p}$, $b_{\text{out,}k_f}$, $b_{\text{out,}k_m}$ of the controller $K$ are labeled in Fig. 3 respectively. The QSEEs for the controller $K$ is given by

$$
\begin{align}
    d\tilde{a}_k(t) &= A_k\tilde{a}_k(t)dt + E_k\tilde{v}_k(t)dt + B_{k_1} dB_{k_1}(t) \\
    &+ B_{k_2} (dB_{k_2}(t) + \tilde{w}_k(t)dt), \\
    d\tilde{B}_{\text{out,}k_p}(t) &= C_{k_p}\tilde{a}_k(t)dt + D_{k_p_1} dB_{k_1}(t) \\
    &+ D_{k_p_2} (dB_{k_2}(t) + \tilde{w}_k(t)dt) + D_{k_p_3} dB_{k_3}(t) + \tilde{w}_k(t)dt), \\
    d\tilde{B}_{\text{out,}k_f}(t) &= C_{k_f}\tilde{a}_k(t)dt + D_{k_f_1} dB_{k_1}(t) \\
    &+ D_{k_f_2} (dB_{k_2}(t) + \tilde{w}_k(t)dt) + D_{k_f_3} dB_{k_3}(t) + \tilde{w}_k(t)dt), \\
    d\tilde{B}_{\text{out,}k_m}(t) &= C_{k_m}\tilde{a}_k(t)dt + D_{k_m_1} dB_{k_1}(t) \\
    &+ D_{k_m_2} (dB_{k_2}(t) + D_{k_m_3} dB_{k_3}(t) + \tilde{w}_k(t)dt).
\end{align}
$$

(179)

Remark 8.1. As shown in Fig. 3 the input field $b_{k_1}$ of the controller $K$ is the output field $b_{\text{out,}p_k}$ of the plant $P$, and the output $b_{\text{out,}k_p}$ of the controller $K$ is the input $b_{p_3}$ of the plant $P$. As shown in Eq. (178) the output field $b_{\text{out,}p_k}$ may correspond to the input field $b_{p_3}$ which is the field $b_{\text{out,}p_k}$. To guarantee causality, the field $b_{\text{out,}k_p}$ must not contain the input field $b_{k_1} = b_{\text{out,}p_k}$. This is the reason why the evolution of $B_{\text{out,}k_p}(t)$ in Eq. (179) depends on the free traveling fields $b_{k_2}$ and $b_{k_3}$, but not on $b_{k_1}$. There are other possible configurations which can guarantee causality. One example is given in Subsection 8.2 to demonstrate one such possible configuration. In this example, $b_{\text{out,}p_k}$ corresponds to the input laser which is $b_{p_2} + w_p$. $b_{\text{out,}p_k}$ is sent to $K$ generating the corresponding output $b_{\text{out,}k_p}$. However, $b_{\text{out,}k_p}$ is the input $b_{p_3}$ of $P$ whose corresponding output is the output laser $b_{\text{out,}p_m}$ to be measured. This fundamental assumption is also used in quantum circuits [231, Section 1.3.4].

The plant $P$ and controller $K$ can also be directly coupled via an interaction Hamiltonian $H_{\text{int}}$ as labeled in Fig. 3 with the following form

$$
H_{\text{int}} = \frac{1}{2} \left( \tilde{a}_p^\dagger \Xi \tilde{a}_k + \tilde{a}_k^\dagger \Xi \tilde{a}_p \right),
$$

(180)
where $\Xi = \Delta(tK_-, tK_+)$ for complex matrices $K_-$ and $K_+$ with suitable dimensions. It is easy to see that the commutators $[\dot{a}_p, H_{\text{int}}]$ and $[\dot{a}_k, H_{\text{int}}]$ yield

$$B_{12} = -\Delta(K_-, K_+)^\dagger, \quad B_{21} = -B_{12} = \Delta(K_-, K_+).$$

(181)

On the other hand, indirect coupling refers to the coupling through field channels $\dot{b}_{\text{out}, p}$ and $\dot{b}_{\text{out}, k}$. More discussions on direct coupling and indirect coupling can be found in, e.g., [8] and [51, 6].

The controller matrices for direct and indirect couplings are to be found to optimize performance criteria defined in terms of the set of closed-loop performance variables

$$\dot{z}(t) = [C_p \quad C_k] \begin{bmatrix} \dot{a}_p(t) \\ \dot{a}_k(t) \end{bmatrix} + D_z \dot{w}(t).$$

(182)

An example of control performance variables is $z = a_p$ for a single-mode cavity. Then $z^* z = a_p^* a_p = \frac{z^2 + p^2 - 1}{2}$. Minimizing the mean value of $\int z^* z(t) dt$ means cooling the cavity oscillator; see [232]. The form of performance variables for $H^\infty$ control can be found in [46, 4]. More discussions can be found in books [5, Chapter 6] and [4].

By eliminating the in-loop fields $b_{\text{out}, p_k}$ and $b_{\text{out}, k_p}$, the overall plant-controller quantum system, including direct and indirect couplings, can be written as

$$\begin{bmatrix} \dot{d}_p(t) \\ \dot{d}_k(t) \end{bmatrix} = \begin{bmatrix} A_p & B_{p3}C_{kp} + B_{12} \\ B_{k1}C_{kp} + B_{21} & A_k + B_{k3}D_{kp3}C_{kp} \end{bmatrix} \begin{bmatrix} \dot{a}_p(t) \\ \dot{a}_k(t) \end{bmatrix} + \begin{bmatrix} B_{p2} \\ B_{k2}D_{kp2} + B_{k3}D_{kp3}D_{kp2} \end{bmatrix} \begin{bmatrix} \dot{u}_p(t) \\ \dot{u}_k(t) \end{bmatrix} dt + G_{cl} \begin{bmatrix} d\dot{B}_{p1}(t) \\ d\dot{B}_{p2}(t) \\ d\dot{B}_{k2}(t) \\ d\dot{B}_{k3}(t) \end{bmatrix},$$

(183)

where

$$G_{cl} = \begin{bmatrix} B_{p1} & B_{p2} & B_{p3}D_{kp2} & B_{p3}D_{kp3} \\ B_{k1}D_{kp1} & B_{k2}D_{kp2} + B_{k1}D_{kp3}D_{kp2} & B_{k3} + B_{k1}D_{kp3}D_{kp3} \end{bmatrix}.$$  

(184)

Because standard matrix algorithms commonly used in $H^\infty$ synthesis and LQG synthesis are for real-valued matrices, in what follows we resort to quadrature representation. Let $x_p, x_k, \tilde{w}_p, \tilde{w}_k, u_p, u_k, U_{p1}, U_{p2}, U_{k1}, U_{k2}$ be the quadrature counterparts of $\dot{a}_p, \dot{a}_k, \dot{w}_p, \dot{w}_k, \dot{v}_p, \dot{v}_k, \ddot{B}_{p1}, \ddot{B}_{p2}, \ddot{B}_{k2}, \ddot{B}_{k3}$, respectively. Then the closed-loop quantum system in the quadrature representation is given by

$$\begin{bmatrix} dx_p(t) \\ dx_k(t) \end{bmatrix} = A_{cl} \begin{bmatrix} x_p(t) \\ x_k(t) \end{bmatrix} dt + B_{cl} \begin{bmatrix} \tilde{w}_p(t) \\ \tilde{w}_k(t) \end{bmatrix} dt + E_{cl} \begin{bmatrix} u_p(t) \\ u_k(t) \end{bmatrix} dt + G_{cl} \begin{bmatrix} d\dot{U}_{p1}(t) \\ d\dot{U}_{p2}(t) \\ d\dot{U}_{k1}(t) \end{bmatrix},$$

(185)

$$\ddot{z}(t) = C_{cl} \begin{bmatrix} x_p(t) \\ x_k(t) \end{bmatrix} + D_{cl} \ddot{w}(t),$$

$$\ddot{x}_p(t) = C_{cl} \begin{bmatrix} x_p(t) \\ x_k(t) \end{bmatrix} + D_{cl} \ddot{w}(t),$$

$$\ddot{x}_k(t) = C_{cl} \begin{bmatrix} x_p(t) \\ x_k(t) \end{bmatrix} + D_{cl} \ddot{w}(t),$$
where

\[
\begin{align*}
A_{cl} &= \begin{bmatrix}
A_p & B_{p_3}C_{k_p} + B_{12} \\
B_{k_1}C_{p_k} + B_{21} & A_k + B_{k_1}D_{p_{k_3}}C_{k_p}
\end{bmatrix}, \\
B_{cl} &= \begin{bmatrix}
B_{p_2} & B_{p_3}D_{k_{p_3}} \\
B_{k_1}D_{p_{k_2}} & B_{k_3} + B_{k_1}D_{p_{k_3}}D_{k_{p_3}}
\end{bmatrix}, \\
E_{cl} &= \begin{bmatrix}
E_p & 0 \\
0 & E_k
\end{bmatrix}, \\
G_{cl} &= \begin{bmatrix}
B_{p_1} & B_{p_2} & B_{p_3}D_{k_{p_2}} & B_{p_3}D_{k_{p_3}} \\
B_{k_1}D_{p_{k_1}} & B_{k_1}D_{p_{k_2}} & B_{k_2} + B_{k_1}D_{p_{k_3}}D_{k_{p_2}} & B_{k_3} + B_{k_1}D_{p_{k_3}}D_{k_{p_3}}
\end{bmatrix}, \\
C_{cl} &= \begin{bmatrix}
C_p & C_k
\end{bmatrix}, \quad D_{cl} = D_z.
\end{align*}
\]

8.2. An example

In this subsection, we use one example to demonstrate the coherent feedback network in Fig. 3.

A hybrid atom-optomechanical system has recently been implemented [28, Fig. 1A], in which a laser beam is used to realize couplings between an atomic spin ensemble and a micromechanical membrane. Strong coupling between these two subsystems is successfully realized in a room-temperature environment. Interesting physical phenomena, such as normal-mode splitting, coherent energy exchange between the atomic ensemble and the micromechanical membrane, and two-mode thermal noise squeezing, are observed.

![Figure 4: The system experimentally realized in [28].](image)

This hybrid system is depicted in Fig. 4. Compared with the coherent feedback network in Fig. 3, the atomic ensemble corresponds to the plant \( P \), whereas the membrane corresponds to the controller \( K \). There is no direct coupling Hamiltonian \( H_{\text{int}} \) between the atomic ensemble and the membrane.

The atomic ensemble is modeled as a single-mode quantum mechanical oscillator which is parametrized
by

\[
S_p = I_3, \quad L_p = \begin{bmatrix}
L_{p1} \\
L_{p2} \\
L_{p3}
\end{bmatrix} = \begin{bmatrix}
\sqrt{\gamma_s}q_s \\
\sqrt{2\Gamma_s}q_s \\
\sqrt{2\Gamma_s}q_s
\end{bmatrix}, \quad H_p = \frac{\Omega_s}{2}(q_s^2 + p_s^2),
\]

(187)

where \(q_s\) and \(p_s\) are real quadrature operators of the atomic ensemble. The input laser beam is parametrized by \([8]\) and \([43, \text{Appendix C}]\)

\[
S_l = 1, \quad L_l = \sqrt{\kappa_{\text{ext}}}a_l + \alpha I, \quad H_l = 0,
\]

(188)

where \(a_l\) is the annihilation operator of the laser and \(\alpha \in \mathbb{C}\). Thus, \(w_p\) in Fig. 3 is \(L_l\) in Eq. (188). By the concatenation product and the series product, the cascaded \(P < \) laser system is

\[
S_c = I_3, \quad L_c = \begin{bmatrix}
L_{c1} \\
L_{c2} \\
L_{c3}
\end{bmatrix} = \begin{bmatrix}
\sqrt{\gamma_s}q_s \\
\sqrt{2\Gamma_s}q_s + \sqrt{\kappa_{\text{ext}}}a_l + \alpha I \\
\sqrt{2\Gamma_s}q_s
\end{bmatrix},
\]

\[
H_c = \frac{\Omega_s}{2}(q_s^2 + p_s^2) + \sqrt{\kappa_{\text{ext}}}\Gamma_s q_s p_l.
\]

(191)

It can be seen that the last term of \(H_c\), namely \(\sqrt{\kappa_{\text{ext}}}\Gamma_s q_s p_l\), describes the interaction Hamiltonian between the atomic ensemble and the light, which is consistent to the form given in \([28, \text{Eq. (S4)}]\).

Let \(x_c\) be the real quadrature operators of the cascaded system \(P < \) laser and \(u_c\) be the input quadrature operators. Specifically,

\[
x_c = \begin{bmatrix}
q_s & q_s & p_s & p_l
\end{bmatrix}^\top,
\]

\[
u_c = \begin{bmatrix}
q_s^{(\text{th})} & q_s^{(\text{th})} & p_s^{(\text{th})} & p_s^{(\text{th})}
\end{bmatrix}^\top,
\]

where \((q_s^{(\text{th})}, p_s^{(\text{th})})\) is input thermal noise, \((q_{s,\tilde{\zeta}_1}, p_{s,\tilde{\zeta}_1})\) (at position \(\zeta_1\)) and \((q_{s,\tilde{\zeta}_3}, p_{s,\tilde{\zeta}_3})\) (at position \(\zeta_3\)) denote the second and third inputs of the atomic spin ensemble, respectively. In the notation used in \([3]\) we have system parameters

\[
\begin{bmatrix}
q_s^{(\text{th})} \\
p_s^{(\text{th})}
\end{bmatrix} = V_1 \tilde{b}_{p1}, \quad \begin{bmatrix}
q_{s,\tilde{\zeta}_1} \\
p_{s,\tilde{\zeta}_1}
\end{bmatrix} = V_1 \tilde{b}_{p2}, \quad \begin{bmatrix}
q_{s,\tilde{\zeta}_3} \\
p_{s,\tilde{\zeta}_3}
\end{bmatrix} = V_1 \tilde{b}_{p3},
\]

(193)

\(^1\)Given two open quantum systems \(G_1 \triangleq (S_1, L_1, H_1)\) and \(G_2 \triangleq (S_2, L_2, H_2)\) where \(S_j, L_j, H_j\) are operators on the Hilbert space of the system \(G_j\) \((j = 1, 2)\), their concatenation product is defined to be

\[
G_1 \bowtie G_2 \triangleq \left( \begin{bmatrix}
S_1 \\
0
\end{bmatrix}, \begin{bmatrix}
L_1 \\
L_2
\end{bmatrix}, H_1 + H_2 \right),
\]

(189)

and their series product is defined to be

\[
G_2 \bowt G_1 \triangleq \left( S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \frac{1}{2\Gamma_l}(L_2^\dagger S_2 L_1 - L_1^\dagger S_2^\dagger L_2) \right).
\]

(190)

In this paper, the term \(\frac{1}{2\Gamma_l}(L_2^\dagger S_2 L_1 - L_1^\dagger S_2^\dagger L_2)\) is called the \textit{interaction Hamiltonian}. See \([210, 209, 6]\) and \([3]\) for more detailed discussions on coherent feedback connections.
where the unitary matrix $V_1$ is defined in Eq. (36). By the unitary transformations in (41), we have

$$
\Lambda_c = \begin{bmatrix}
\sqrt{\gamma_s} & 0 & 0 & 0 \\
\sqrt{2\Gamma_s} & \sqrt{2\kappa_2} & 0 & \sqrt{2\kappa_2} \\
\sqrt{2\Gamma_s} & 0 & 0 & 0 \\
\end{bmatrix},
$$

$$
H_c = \begin{bmatrix}
\Omega_s & 0 & 0 & \sqrt{\kappa_{ext}} \Gamma_s \\
0 & 0 & 0 & 0 \\
0 & 0 & \Omega_s & 0 \\
\sqrt{\kappa_{ext}} \Gamma_s & 0 & 0 & 0 \\
\end{bmatrix}, \quad K = 0.
$$

(194)

Consequently, by Eq. (44) the system matrices can be calculated as

$$
D_c = I_6, \quad C_c = \begin{bmatrix}
\sqrt{\gamma_s} & 0 & 0 & 0 \\
\sqrt{2\Gamma_s} & \sqrt{\kappa_{ext}} & 0 & 0 \\
\sqrt{2\Gamma_s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\kappa_{ext}} \\
0 & 0 & 0 & 0 \\
\end{bmatrix},
$$

$$
B_c = -\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\kappa_{ext}} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\gamma_s} & 2\sqrt{2\Gamma_s} & 2\sqrt{2\Gamma_s} & 0 \\
0 & 0 & 0 & \sqrt{\kappa_{ext}} & 0 & 0 \\
(1 - \sqrt{2})\sqrt{\kappa_{ext}} \Gamma_s & -\frac{\kappa_{ext}}{2} & 0 & 0 & 0 \\
-\Omega_s & 0 & 0 & -(1 + \sqrt{2})\sqrt{\kappa_{ext}} \Gamma_s & -\frac{\kappa_{ext}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
$$

$$
A_c = \begin{bmatrix}
0 & 0 & \Omega_s & 0 \\
0 & 0 & 0 & 0 \\
-\Omega_s & 0 & 0 & -(1 + \sqrt{2})\sqrt{\kappa_{ext}} \Gamma_s \\
0 & 0 & 0 & 0 \\
\end{bmatrix},
$$

(195)

which yield the linear QSDEs that describe the dynamics of the atomic spin ensemble in the real quadrature operator representation

$$
\dot{q}_s = \Omega_s p_s, \quad \dot{p}_s = -\Omega_s q_s - (1 + \sqrt{2})\sqrt{\kappa_{ext}} \Gamma_s p_t - 2\sqrt{2\Gamma_s} p_s, \zeta_1
$$

$$
- 2\sqrt{2\Gamma_s} p_s, \zeta_1 - 2\sqrt{\gamma_s} p_s^{(th)}.
$$

(196)

According to Eq. (43), the output of the atomic ensemble is given by

$$
C_c \mathbf{x}_c + \mathbf{u}_c = \begin{bmatrix}
\sqrt{2\gamma_s} q_s^{(th)} + q_s \\
2\sqrt{2\Gamma_s} q_s + \sqrt{\kappa_{ext}} q_t + q_s, \zeta_1 \\
2\sqrt{2\Gamma_s} q_s + q_s, \zeta_3 \\
\sqrt{\kappa_{ext}} p_t + p_s, \zeta_1 \\
\sqrt{\kappa_{ext}} p_t + p_s, \zeta_3 \\
\end{bmatrix}.
$$

(197)

As shown in Fig. 4, the second output of the atomic spin ensemble, which is

$$
\begin{bmatrix}
2\sqrt{2\Gamma_s} q_s + \sqrt{\kappa_{ext}} q_t + q_s, \zeta_1 \\
\sqrt{\kappa_{ext}} p_t + p_s, \zeta_1 \\
\end{bmatrix} = V_1 \mathbf{b}_{out, pk}
$$

(198)

37
with $b_{\text{out},k}$ being that in Fig. 3 is sent to the micromechanical membrane. As a result, we have

$$
\begin{bmatrix}
q_{m,\zeta_2}(t) \\
p_{m,\zeta_2}(t)
\end{bmatrix} = \begin{bmatrix}
2\sqrt{2}\Gamma_s q_s(t - \tau) + \sqrt{\kappa_{\text{ext}}} q_l(t - \tau) + q_{s,\zeta_1}(t - \tau) \\
\sqrt{\kappa_{\text{ext}}} p_l(t - \tau) + p_{s,\zeta_1}(t - \tau)
\end{bmatrix},
$$

(199)

where $\tau$ denotes the time delay from the atomic spin ensemble to the micromechanical membrane in the feedback loop. Here, $
\begin{bmatrix}
q_{m,\zeta_2}(t) \\
p_{m,\zeta_2}(t)
\end{bmatrix} = V_1 \bar{b}_k$, with $\bar{b}_k$ being that in Fig. 3 is the first input to the micromechanical membrane in Fig. 4.

On the other hand, the micromechanical membrane in Fig. 4, which is a single-mode quantum harmonic oscillator too, can be parametrized by

$$
S_m = I_2, \quad L_m = \begin{bmatrix} L_{k1} \\ L_{k2} \end{bmatrix}, \quad H_m = \frac{\Omega_m}{2} (q_m^2 + p_m^2),
$$

(200)

where $q_m$ and $p_m$ are real quadrature operators of the micromechanical membrane. Notice that

$$
\frac{L_{k1}^* L_{k2} - L_{k2}^* L_{k1}}{2t} = 2\sqrt{\Gamma_s \Gamma_m q_s q_m} + \sqrt{\Gamma_m \kappa_{\text{ext}}} q_m q_l + \sqrt{2\Gamma_m \kappa_{\text{ext}}} \frac{\alpha + \alpha^*}{2}.
$$

(201)

The coupling between the second output channel of the atomic ensemble and the micromechanical membrane generates several interaction Hamiltonian terms, among which the second term $\sqrt{\Gamma_m \kappa_{\text{ext}}} q_m q_l$ is consistent with the form given in [28, Eq. (S14)]. As the coupling happens at $\zeta_2$ in Fig. 4, $q_l$ can be written as $q_{l,\zeta_2}$ which corresponds to $X_L(\zeta_m)$ in [28, Eq. (S14)].

Let $x_m$ be the real quadrature operators of the micromechanical membrane $K$ and $u_m$ be the input, i.e.,

$$
x_m = \begin{bmatrix} q_m \\ p_m \end{bmatrix}^T, \quad u_m = \begin{bmatrix} q_{m,\zeta_2} \\ q_{m,\zeta_2}^{(th)} \\ p_{m,\zeta_2} \\ p_{m,\zeta_2}^{(th)} \end{bmatrix}^T,
$$

(202)

where $(q_{m}^{(th)}, p_{m}^{(th)})$ is input thermal noise, and $(q_{m,\zeta_2}, p_{m,\zeta_2})$ (at position $\zeta_2$) denotes the first input of the micromechanical membrane, as introduced in Eq. (199). In the notation used in Fig. 3 we have

$$
\begin{bmatrix}
q_{m,\zeta_2} \\
p_{m,\zeta_2}
\end{bmatrix} = V_1 \bar{b}_{k1}, \quad \begin{bmatrix}
q_{m,\zeta_2}^{(th)} \\
p_{m,\zeta_2}^{(th)}
\end{bmatrix} = V_1 \bar{b}_{k2},
$$

(203)

and there is no input channel associated with $b_{k3}$.

Similarly, the system matrices of the micromechanical membrane $K$ can be calculated as

$$
D_m = I_4, \quad C_m = \begin{bmatrix}
0 & 0 \\
\sqrt{2\gamma_m} & 0 \\
-2\sqrt{\Gamma_m} & 0 \\
0 & 0
\end{bmatrix},
$$

$$
B_m = \begin{bmatrix}
0 & 0 & 0 & 0 \\
2\sqrt{\Gamma_m} & 0 & 0 & \sqrt{2\gamma_m}
\end{bmatrix}, \quad A_m = \begin{bmatrix}
0 & \Omega_m \\
-\Omega_m & 0
\end{bmatrix},
$$

(204)

38
which yields the linear QSDEs that describe the dynamics of the micromechanical membrane in the real quadrature operator representation

$$\dot{q}_m = \Omega_m p_m,$$

$$\dot{p}_m = -\Omega_m q_m - 2\sqrt{\Gamma_m}q_{m,\zeta_2} - \sqrt{2\gamma_m}p_m^{(th)}.$$  \hspace{1cm} (205)

Substituting (199) into (205), we have

$$\dot{q}_m(t) = \Omega_m p_m(t),$$

$$\dot{p}_m(t) = -\Omega_m q_m(t) - 4\sqrt{2\Gamma_m}q_s(t-\tau) - 2\sqrt{\kappa_{ext}\Gamma_m}q_l(t-\tau)$$

$$- 2\sqrt{\Gamma_m}q_{s,\zeta_2}(t) - \sqrt{2\gamma_m}p_m^{(th)}(t).$$  \hspace{1cm} (206)

Moreover, the output of the micromechanical membrane is

$$C_m x_m + u_m = \begin{bmatrix} q_{m,\zeta_2} \\
2\sqrt{\gamma_m} q_m + q_m^{(th)} \\
-2\sqrt{\Gamma_m} q_m + p_{m,\zeta_2} \\
p_m^{(th)} \end{bmatrix}.$$  \hspace{1cm} (207)

The first output of the micromechanical membrane, which is

$$\begin{bmatrix} q_{m,\zeta_2} \\
-2\sqrt{\Gamma_m} q_m + p_{m,\zeta_2} \end{bmatrix} = V_1 \dot{b}_{out,k_p}$$  \hspace{1cm} (208)

with \(b_{out,k_p}\) being that in Fig. 4 is sent to the atomic spin ensemble. Noticing the phase shifter \(e^{i\phi}\) on the way, we have

$$b_{p_3}(t) = e^{i\phi}b_{out,k_p}(t-\tau).$$  \hspace{1cm} (209)

By Eqs. 207, 209 and noticing Eq. 193, we get

$$q_{s,\zeta_3}(t) = \cos \phi q_{m,\zeta_2}(t-\tau) - \sin \phi \left[-2\sqrt{\Gamma_m} q_m(t-\tau) + p_{m,\zeta_2}(t-\tau)\right],$$

$$p_{s,\zeta_3}(t) = \cos \phi \left[-2\sqrt{\Gamma_m} q_m(t-\tau) + p_{m,\zeta_2}(t-\tau)\right] + \sin \phi q_{m,\zeta_2}(t-\tau).$$  \hspace{1cm} (210)

Substituting Eq. 199 into Eq. 210, yields

$$p_{s,\zeta_3}(t) = -2\cos \phi \sqrt{\Gamma_m} q_{m}(t-\tau) + 2\sin \phi \sqrt{2\Gamma_s} q_s(t-2\tau)$$

$$+ \cos \phi \sqrt{\kappa_{ext}} p_l(t-2\tau) + \sin \phi \sqrt{\kappa_{ext}} q_l(t-2\tau)$$

$$+ \cos \phi p_{s,\zeta_4}(t-2\tau) + \sin \phi q_{s,\zeta_4}(t-2\tau).$$  \hspace{1cm} (211)

Thus, Eq. 196 can be rewritten as

$$\dot{q}_s(t) = \Omega_s p_s(t),$$

$$\dot{p}_s(t) = -\Omega_s q_s(t) - (1 + \sqrt{2}) \sqrt{\kappa_{ext}\Gamma_s} p_l(t-2\tau) - 2\cos \phi \sqrt{2\kappa_{ext}\Gamma_s} p_l(t-2\tau)$$

$$- 2\sin \phi \sqrt{2\kappa_{ext}\Gamma_s} q_l(t-2\tau) - 8\sin \phi \kappa_{s} q_s(t-2\tau)$$

$$+ 4\cos \phi \sqrt{2\Gamma_s \Gamma_m} q_m(t-\tau)$$

$$- 2\sqrt{2\Gamma_s} \left[p_{s,\zeta_4}(t) + \cos \phi p_{s,\zeta_3}(t) + \sin \phi q_{s,\zeta_3}(t)\right] - \sqrt{2\gamma_s p_s^{(th)}}(t).$$  \hspace{1cm} (212)

Despite of the terms containing laser loss \(\kappa_{ext}\), the dynamical equations of the micromechanical membrane 206 and the atomic spin ensemble 212 are consistent with the forms given in 28 Eqs. (S54–S58).
Remark 8.2. In the following, we only look into the input–output channel with the laser, i.e. ignoring the thermal noise inputs. Notice that $L_{p2} = \sqrt{2\Gamma_s}q_s$ is the coupling between the spin and the input light, and $L_{k1} = -i\sqrt{2\Gamma_m}q_m$ is the membrane–light coupling. The coherent feedback loop can be divided into two parts. The first part is from the spin to the membrane, according to Eq. (190) whose interaction Hamiltonian is given by

$$H^{(1)}_{\text{int}} = \frac{1}{2i}(L_{k1}^*L_{p2} - L_{p2}^*L_{k1}) = 2\sqrt{\Gamma_m\Gamma_s}q_m q_s,$$

and the cascaded coupling operator $L^{(1)} = L_{k1} + L_{p2}$. The second part is from the membrane to the spin with the phase shifter $e^{i\phi}$ on the way, the corresponding interaction Hamiltonian is

$$H^{(2)}_{\text{int}} = \frac{1}{2i}(L_{p3}^*e^{i\phi}L^{(1)} - L^{(1)*}e^{-i\phi}L_{p3}) = -2\sqrt{\Gamma_m\Gamma_s}\cos \phi q_s q_m + 2\sin \phi \Gamma_s q_s^2,$$

and the cascaded coupling operator

$$L^{(2)} = L_{p3} + e^{i\phi}L^{(1)} = -ie^{i\phi}\sqrt{2\Gamma_m}q_m + (1 + e^{i\phi})\sqrt{2\Gamma_s}q_s,$$

which is consistent with the collective jump operator $J$ used in [28, Eq. (1)]. Combining Eq. (213) with Eq. (214), the interaction Hamiltonian between the atomic spin ensemble and the micromechanical membrane is

$$H_{sm} = (1 - \cos \phi)2\sqrt{\Gamma_m\Gamma_s}q_s q_m + 2\sin \phi \Gamma_s q_s^2,$$

which is consistent with the effective interaction Hamiltonian $H_{\text{eff}}$ used in [28, Eq. (1)]. In summary, all the essential equations in [28] can be reproduced by means of the quantum linear systems and network theory introduced in this tutorial.

9. Conclusion

In this tutorial, we have given a concise introduction to linear quantum systems, for example, their mathematical models, relation between their control-theoretic properties and physical properties, Gaussian states, quantum Kalman filter, Kalman canonical form, and response to continuous-mode single-photon states. Several simple examples are designed to demonstrate some fundamental properties of linear quantum systems. Pointers to more detailed discussions are given in various places. It is hoped that this tutorial is helpful to researchers in the control community who are interested in quantum control of dynamical systems. Finally, an information-theoretic uncertainty relation has been recorded in this tutorial, which describes uncertainties of mixed quantum Gaussian states better than the Heisenberg uncertainty relation. It is an open question whether this uncertainty relation is useful for mixed quantum Gaussian state engineering.

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