THE GEOMETRY OF
HIGHER-ORDER
HAMILTON SPACES
Applications to Hamiltonian Mechanics

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# Contents

1 Geometry of the $k$-Tangent Bundle $T^k M$
   1.1 The Category of $k$-Accelerations Bundles ........................................ 1
   1.2 Liouville Vector Fields. $k$-Semisprays ............................................. 4
   1.3 Nonlinear Connections ................................................................. 8
   1.4 The Dual Coefficients of a Nonlinear Connection .............................. 11
   1.5 The Determination of a Nonlinear Connection .................................. 15
   1.6 $d$-Tensor Fields. $N$-Linear Connections ...................................... 18
   1.7 Torsion and Curvature ................................................................. 22

2 Lagrange Spaces of Higher Order ......................................................... 27
   2.1 Lagrangians of Order $k$ ................................................................. 27
   2.2 Variational Problem ........................................................................ 29
   2.3 Higher Order Energies ..................................................................... 32
   2.4 Jacobi-Ostrogradski Momenta ......................................................... 33
   2.5 Higher Order Lagrange Spaces ....................................................... 34
   2.6 Canonical Metrical $N$-Connections ............................................... 37
   2.7 Generalized Lagrange Spaces of Order $k$ ....................................... 39

3 Finsler Spaces of Order $k$ ...................................................................... 43
   3.1 Spaces $F^{(k)} n$ ............................................................................ 43
   3.2 Cartan Nonlinear Connection in $F^{(k)} n$ ........................................ 48
   3.3 The Cartan Metrical $N$-Linear Connection .................................... 54

4 The Geometry of the Dual of $k$-Tangent Bundle .................................. 59
   4.1 The Dual Bundle $(T^\ast k M, \pi^k, M)$ .............................................. 59
   4.2 Vertical Distributions. Liouville Vector Fields ................................ 62
   4.3 The Structures $J$ and $J^\ast$ ........................................................... 65
   4.4 Canonical Poisson Structures on $T^\ast k M$ .................................... 69
   4.5 Homogeneity .................................................................................. 70

5 The Variational Problem for the Hamiltonians of Order $k$ .................... 77
   5.1 The Hamilton-Jacobi Equations ....................................................... 77
   5.2 Zermelo Conditions ....................................................................... 82
   5.3 Higher Order Energies. Conservation of Energy $\mathcal{E}^{k-1}(H)$ ...... 84
   5.4 The Jacobi-Ostrogradski Momenta ................................................. 88
## CONTENTS

5.5 Nöther Type Theorems ........................................ 87

### 6 Dual Semispray. Nonlinear Connections

6.1 Dual Semispray .................................................. 95
6.2 Nonlinear Connections ........................................... 99
6.3 The Dual Coefficients of the Nonlinear Connection $N$ .......... 103
6.4 The Determination of the Nonlinear Connection by a Dual $k$-Semispray ................................. 110
6.5 Lie Brackets. Exterior Differential .............................. 111
6.6 The Almost Product Structure $P$. The Almost Contact Structure $F$............... 116
6.7 The Riemannian Structure $G$ on $T^*kM$ .......................... 117
6.8 The Riemannian Almost Contact Structure $(G, F)$ ................. 119

### 7 Linear Connections on the Manifold $T^*kM$

7.1 The Algebra of Distinguished Tensor Fields ......................... 121
7.2 $N$-Linear Connections .......................................... 122
7.3 The Torsion and Curvature of an $N$-Linear Connection ......... 126
7.4 The Coefficients of a $N$-Linear Connection ...................... 128
7.5 The $h$, $v_\alpha$- and $w_k$-Covariant Derivatives in Local Adapted Basis 130
7.6 Ricci Identities. Local Expressions of $d$-Tensor of Curvature and Torsion. Bianchi Identities. .......................... 134
7.7 Parallelism of the Vector Fields on the Manifold $T^*kM$ .......... 138
7.8 Structure Equations of a $N$-Linear Connection .................. 143

### 8 Hamilton Spaces of Order $k \geq 1$

8.1 The Spaces $H^{(k)n}$ ........................................... 147
8.2 The $k$-Tangent Structure $J$ and the Adjoint $k$-Tangent Structure $J$* .................................................. 150
8.3 The Canonical Poisson Structure of the Hamilton Space $H^{(k)n}$ .... 153
8.4 Legendre Mapping Determined by a Lagrange Space $L^{(k)n} = (M, L)$ 155
8.5 Legendre Mapping Determined by a Hamilton Space of Order $k$ .... 159
8.6 The Canonical Nonlinear Connection of the Space $H^{(k)n}$ ........ 161
8.7 Canonical Metrical $N$-Linear Connection of the Space $H^{(k)n}$ ........ 163
8.8 The Hamilton Space $H^{(k)n}$ of Electrodynamics ................ 167
8.9 The Riemannian Almost Contact Structure Determined by the Hamilton Space $H^{(k)n}$ ......................... 170

### 9 Subspaces in Hamilton Spaces of Order $k$

9.1 Submanifolds $T^*kM$ in the Manifold $T^*kM$ .................. 173
9.2 Hamilton Subspaces $H^{(k)n}$ in $H^{(k)n}$. Darboux Frames ........ 177
9.3 Induced Nonlinear Connection .................................. 179
9.4 The Relative Covariant Derivative ................................ 182
9.5 The Gauss-Weingarten Formula .................................. 189
9.6 The Gauss-Codazzi Equations .................................. 191
CONTENTS

10 The Cartan Spaces of Order $k$ as Dual of Finsler Spaces of Order $k$ ........................................ 195
10.1 $C^{(k)n}$-Spaces ......................................................... 195
10.2 Geometrical Properties of the Cartan Spaces of Order $k$ ........................................ 197
10.3 Canonical Presymplectic Structures, Variational Problem of the
       Space $C^{(k)n}$ ......................................................... 199
10.4 The Cartan Spaces $C^{(k)n}$ as Dual of Finsler Spaces $F^{(k)n}$ .................. 201
10.5 Canonical Nonlinear Connection, $N$-Linear Connections .................. 205
10.6 Parallelism of Vector Fields in Cartan Space $C^{(k)n}$ .................. 209
10.7 Structure Equations of Metrical Canonical $N$-Connection ............. 212
10.8 Riemannian Almost Contact Structure of the Space $C^{(k)n}$ .......... 214

11 Generalized Hamilton and Cartan Spaces of Order $k$. Applications to Hamiltonian Relativistic Optics ............. 217
11.1 The Space $GH^{(k)n}$ .................................................... 217
11.2 Metrical $N$-Linear Connections ......................................... 219
11.3 Hamiltonian Relativistic Optics .......................................... 222
11.4 The Metrical Almost Contact Structure of the Space $GH^{(k)n}$ .......... 226
11.5 Generalized Cartan Space of Order $k$ ..................................... 228
Preface

As is known, the Lagrange and Hamilton geometries have appeared relatively recently [76, 86]. Since 1980 these geometries have been intensively studied by mathematicians and physicists from Romania, Canada, Germany, Japan, Russia, Hungary, U.S.A. etc.

Scientific prestigious manifestations devoted to Lagrange and Hamilton geometries and their applications have been organized in the above mentioned countries and have been published a number of books and monographs by specialists in the field: R. Miron [94, 95], R. Miron and M. Anastasiei [99, 100], R. Miron, D. Hrimiuc, H. Shimada and S. Sabău [115], P.L. Antonelli, R. Ingarden and M. Matsumoto [7]. Finsler spaces, which form a subclass of the class of Lagrange spaces, have been generated some excellent books, due to M. Matsumoto [76], M. Abate and G. Patrizio [1], D. Bao, S.S. Chern and Z. Shen [17] and A. Bejancu and H.R.Farran [20]. Also, we remark on the monographs of M. Crampin [34], O. Krupkova [72] and D. Opriş, I. Butulescu [125], D. Saunders [144] - which contain pertinent applications in Analytical Mechanics and in the Theory of Partial Differential Equations. But the direct applications in Mechanics, Cosmology, Theoretical Physics and Biology can be found in the well known books of P.L. Antonelli and T. Zawstaniak [11], G.S. Asanov [14], S. Ikeda [59], M. de Leone and P. Rodrigues [73].

The importance of Lagrange and Hamilton geometries consists of the fact that the variational problems for important Lagrangians or Hamiltonians have numerous applications in various fields, as: Mathematics, Theory of Dynamical Systems, Optimal Control, Biology, Economy etc.

In this respect, P.L. Antonelli’s remark is interesting:

‘There is now strong evidence that the symplectic geometry of Hamiltonian dynamical systems is deeply connected to Cartan geometry, the dual of Finsler geometry’, (see V.I. Arnold, I.M. Gelfand and V.S. Retach [13]).

But, all of the above mentioned applications have imposed also the introduction of the notions of higher order Lagrange spaces and, of course, of higher order Hamilton spaces. The base manifolds of these spaces are the bundles of accelerations of superior order. The methods used in the construction of these geometry are the natural extensions of the classical methods used in the edification of Lagrange and Hamilton geometries. These methods allows us to solve an old problem of differential geometry formulated by Bianchi and Bompani [94], more than 100 years ago. Namely, the problem of prolongation of a Riemann
structure \( g \) defined on the base manifold \( M \), to the tangent bundle \( T^k M, k > 1 \). By means of this solution of the previous problem, we can construct, for the first time, good examples of regular Lagrangians and Hamiltonians of higher order.

While the higher order Lagrange geometry has been sufficiently developed we cannot say the same thing about the higher order Hamilton geometry. However the beginning was done only for the case \( k = 2 \), in the book [115]. The reason comes from the fact that, in the year 2001 we hadn’t solved the variational problem for a Hamiltonian which depends on the higher order accelerations and momenta. This problem was solved this year, [98]. Another reason was due to the absence of a consistent theory of subspaces in the Hamilton space of order \( k, k \geq 1 \), such kind of theory being indispensable in applications.

In the present book, we give the general geometrical theory of the Hamilton spaces of order \( k \geq 1 \). This is not a simple generalization of the theory expounded for the case \( k = 2 \) in the monograph 'The Geometry of Hamilton and Lagrange Spaces' Kluwer Acad. Publ. FTPH, no. 118, written by the present author together with D. Hrimiuc, H. Shimada and V. S. Sabău, but it is a global picture of this new geometry, extremely useful in applications from Hamiltonian Mechanics, Quantum Physics, Optimal Control and Biology.

Consequently, this book must be considered a direct continuation of the monographs [94], [95], [99] and [115]. It contains new developments of the subjects: variational principles for higher order Hamiltonians; higher order energies; laws of conservations; Nöther theorems; the Hamilton subspaces of order \( k \) and their fundamental equations. Also, the Cartan spaces of order \( k \) are investigated in details as dual of Finsler spaces of the same order.

In this respect, a more explicit argumentation is as follows.

The geometry of Lagrange space of order \( k \geq 1 \) is based on the geometrical edifice of the \( k \)-accelerations bundle \( (T^k M, \pi^k, M) \).

In Analytical Mechanics the manifold \( M \) is the space of configurations of a physical system. A point \( x = (x^i), (i = 1, ..., n = \text{dim} M) \) in \( M \) is called a configuration. A mapping \( c : t \in I \rightarrow (x^i(t)) \in U \subset M \) is a law of moving (evolution), \( t \) is time, a pair \((t, x)\) is an event and the \( k \)-uple \( \left( \frac{dx^i}{dt}, \cdots, \frac{1}{h!} \frac{d^k x^i}{dt^k} \right) \) gives the velocity and generalized accelerations of order 1, ..., \( k - 1 \). The factors \( \frac{1}{h!} (h = 1, ..., k) \) are introduced here for the simplicity of calculus. In this book we omit the word 'generalized' and say shortly, the accelerations of order \( h \), for \( \frac{1}{h!} \frac{d^h x^i}{dt^h} \). A law of moving \( c : t \in I \rightarrow c(t) \in U \) will be called a curve parametrized by time \( t \).

A Lagrangian of order \( k \geq 1 \) is a real scalar function \( L(x, y^{(1)}, ..., y^{(k)}) \) on \( T^k M \), where \( y^{(h)i} = \frac{1}{h!} \frac{d^h x^i}{dt^h} \). This definition is for autonomous Lagrangians. A similar definition can be formulated for nonautonomous Lagrangians of order \( k \), by

\[
L : (t, x, y^{(1)}, ..., y^{(k)}) \in \mathbb{R} \times T^k M \rightarrow L(t, x, y^{(1)}, ..., y^{(k)}) \in \mathbb{R},
\]
Let $L$ be the scalar functions on the manifold $\mathbb{R} \times T^kM$.

The previous considerations can be done for the Hamiltonians of order $k$. Let $p_i = \frac{1}{2} \frac{\partial L}{\partial y^{(k)i}}$ be the ‘momenta’ determined by the Lagrangian $L$ of order $k$. Then a scalar function

$$H: (x, y^{(1)}, ..., y^{(k-1)}, p) \in T^kM \rightarrow H(x, y^{(1)}, ..., y^{(k-1)}, p) \in \mathbb{R},$$

is an autonomous Hamiltonian of order $k$. It is a function of the configurations $x$, accelerations $y^{(1)}$, ..., $y^{(k-1)}$ of order $1$, ..., $k-1$ and momenta $p$.

A similar definition can be formulated for nonautonomous Hamiltonian of order $k$.

For us it is preferable to study the autonomous Lagrangians and Hamiltonians, because the notions of Lagrange space of order $k$ or Hamilton space of order $k$ are geometrical concepts and one can construct these geometries over the differentiable manifolds $T^kM$ and $T^*M$, respectively. Of course, the geometries of nonautonomous Lagrangians $L(t, x, y^{(1)}, ..., y^{(k)})$ and nonautonomous Hamiltonians $H(t, x, y^{(1)}, ..., y^{(k-1)}, p)$ can be constructed by means of the same methods. One obtains the rheonomic Lagrange spaces of order $k$ and rheonomic Hamiltonian spaces of order $k$.

Now we know the usefulness of the geometry of Higher order Lagrange space (see the book [94]). But why do we need a geometry of Hamilton space of order $k$. This must be a natural extension of the classical Hamiltonian Mechanics, expounded by V.I. Arnold in the book [12] or R.M. Santilli in the book [139].

The problem is why did we use the manifold $T^kM$ as the background for the construction of the Hamilton geometry of order $k$. The answer is as follows. We need a ‘dual’ of the $k$ - acceleration bundle $(T^kM, \pi^k, M)$ denoted by $(T^*M, \pi^*, M)$ which must have the following properties:

1°. $T^1M = T^*M$, $((T^*M, \pi^*, M)$ is the cotangent bundle).

2°. $\dim T^kM = \dim T^*M = (k+1)n$.

3°. The manifold $T^*M$ carries a natural presymplectic structure.

4°. $T^*M$ carries a natural Poisson structure.

5°. $T^*M$ is local diffeomorphic to $T^kM$.

We solved this problem by considering the differentiable bundle $(T^*M, \pi^k, M)$ as the fibred bundle $(T^{k-1}M \times_M T^*M, \pi^{k-1} \times_M \pi^*, M)$. So we have

$$T^kM = T^{k-1}M \times_M T^*M, \quad \pi^k = \pi^{k-1} \times_M \pi^*.$$

A point $u \in T^*M$ is of the form $u = (x, y^{(1)}, ..., y^{(k-1)}, p)$. It is determined by a configuration $x = (x^i)$, the accelerations

$$y^{(1)i} = \frac{dx^i}{dt}, ..., y^{(k-1)i} = \frac{1}{(k-1)!} \frac{d^{k-1}x^i}{dt^{k-1}}$$

and the momenta $p = (p_i)$.

All previous conditions 1°-5° are satisfied. These considerations imply the fact that the geometries of higher order Lagrange space and higher order Hamilton space are dual.
The duality is obtained via a Legendre transformation.

For a good understanding of the important concept of duality we had to make a short introduction to the geometrical theory of Lagrange and Finsler spaces of order \( k \) and then continue with the main subject of the book, the geometry of Hamilton and Cartan spaces of order \( k \).

The Lagrange spaces of order \( k \) are defined as the pairs \( L^{(k)n} = (M, L) \) where \( L \) is a regular Lagrangian of order \( k \). By means of variational calculus the integral of action \( I(c) = \int_0^1 L \left( x(t), \frac{dx}{dt}(t), \ldots, \frac{d^k x}{dt^k}(t) \right) dt \) gives the Euler-Lagrange equations and the Craig-Synge equations. The last equations determine a canonical \( k \)-semispray \( S \). The geometry of the space \( L^{(k)n} \) can be developed by means of the fundamental function \( L \), of the fundamental tensor \( g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}} \) and of the canonical \( k \)-semispray \( S \). The lifting of the previous geometrical edifice to the total space \( T^kM \) will give us a metrical almost contact structure, canonically related to the Lagrange space of order \( k \), \( L^{(k)n} \). Of course, this structure involves the geometry of the space \( L^{(k)n} \).

An important problem was to find some remarkable examples of spaces \( L^{(k)n} \), for \( k > 1 \). By solving the problem of prolongations to \( T^kM \) of a Riemannian structures \( g \) given on the base manifold \( M \), we found interesting examples of Lagrange spaces of order \( k \).

For the applications, one studies the notions of energy of order \( 1, 2, \ldots, k \) and one proves the law of conservation for the energy of order \( k \) and a Nöther type theorem.

The spaces \( L^{(k)n} \) have two important particular cases.

The Finsler spaces of order \( k \), \( F^{(k)n} \), obtained when the fundamental function \( L \) is homogeneous with respect to accelerations \( y^{(1)}, \ldots, y^{(k)} \), and the Riemann spaces of order \( k \), \( R^{(k)n} \) are the spaces \( L^{(k)n} \) for which the fundamental tensor \( g_{ij} \) does not depend on the accelerations \( y^{(1)}, \ldots, y^{(k)} \).

Therefore we have the following sequence of inclusions, [94]:

\[
\left\{ R^{(k)n} \right\} \subset \left\{ F^{(k)n} \right\} \subset \left\{ L^{(k)n} \right\} \subset \left\{ GL^{(k)n} \right\}.
\]

In the case \( k = 1 \) this sequence admits a dual, which is obtained via Legendre transformation. In the book [115] we have introduced the 'dual' of the sequence (*) for the case \( k = 2 \).

Now, the main problem for us is to define and study a dual sequence of the inclusions (*) in the dual Hamilton space of order \( k \).

A Hamilton space of order \( k \) is a pair \( H^{(k)n} = (M, H) \), where \( H : (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^*kM \rightarrow H(x, y^{(1)}, \ldots, y^{(k-1)}, p) \in \mathbb{R} \) is a regular Hamiltonian. Here, the regularity means: the Hessian of \( H \), with respect to the momenta \( p_i \), is not singular. The elements of the Hessian matrix are \( g^{ij} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j} \). Thus \( H \) is called the fundamental function and \( g^{ij} \) the fundamental tensor of the space \( H^{(k)n} \). The geometry of the space \( H^{(k)n} \) can be based on
these two geometrical object fields: $H$ and $g^{ij}$. In the case when $g^{ij} = g^{ij}(x)$ we have a particular class of Hamilton spaces $\mathcal{R}^{(k)n}$ called Riemannian. If the fundamental function $H$ is $2k$-homogeneous on the fibres of the bundle $T^kM$, the spaces $H^{(k)n}$ are called Cartan spaces of order $k$ and denoted $C^{(k)n}$.

Finally, a pair $GH^{(k)n} = (M, g^{ij})$, where $g^{ij}(x, y^{(1)}, ..., y^{(k-1)}, p)$ is a symmetric, nonsingular, distinguished tensor field which is called a generalized Hamilton space of order $k$.

Consequently, we obtain the sequence of inclusions

\[
\{ \mathcal{R}^{(k)n} \} \subset \{ C^{(k)n} \} \subset \{ H^{(k)n} \} \subset \{ GH^{(k)n} \}.
\]

This is the dual sequence of the sequence (*) via the Legendre transformation.

The main goal of this book is to study the classes of spaces from the sequence (**). The chapters 4-11 of the book are devoted to this subject.

Therefore we begin with the geometry of the total space $T^kM$ of the dual bundle $(T^kM, \pi^k, M)$ of the $k$-tangent bundle $(T^kM, \pi^k, M)$ underline: vertical distributions; Liouville vector fields; Liouville 1-form $\omega = p_idx^i$; the closed 2-form $\theta = d\omega$ which defines a natural presymplectic structure on $T^kM$. In the chapter 5 a new theory of variational problem for the Hamiltonian $H$ of order $k$ is developed starting from the integral of action of $H$ defined by

\[
I(c) = \int_0^1 [p_i \frac{dx^i}{dt} - \frac{1}{2} H(x, \frac{dx}{dt}, ..., \frac{1}{(k-1)!} \frac{d^{k-1}x}{dt^{k-1}}, p)] dt.
\]

One proves that: the extremal curves are the solutions of the following Hamilton-Jacobi equations:

\[
\frac{dx^i}{dt} = \frac{1}{2} \frac{\partial H}{\partial p_i},
\]

\[
\frac{dp_i}{dt} = \frac{1}{2} \frac{\partial H}{\partial x^i} - \frac{d}{dt} \frac{\partial H}{\partial y^{(1)i}} + ... + (-1)^{k-1} \frac{1}{(k-1)!} \frac{d^{k-1}}{dt^{k-1}} \frac{\partial H}{\partial y^{(k-1)i}}.
\]

These equations are fundamental in the whole construction of the geometry of Hamiltonians of order $k$. They allow the introduction of the notion of energy of order $k - 1, ..., 1, \mathcal{E}^{k-1}(H), ..., \mathcal{E}^1(H)$ and prove a law of conservation for $\mathcal{E}^{k-1}(H)$ along extremal curves. Now we can introduce in a natural way the Jacobi-Ostrogradski momenta and the Hamilton-Jacobi-Ostrogradski equations.

A theory of symmetries of the Hamiltonians $H$ and the Nöther type theorems are investigated, too; a specific theory of tangent structure $J$ and its adjoint $J^*$; canonical Poisson structure; the notion of dual semispray, which can be defined only by $k \geq 2$; nonlinear connection $N$; the dual coefficients of $N$; the almost product structure $P$, almost contact structure $F$ and Riemannian structure $G$, are studied.

We pay special attention to the theory of $N$-linear connections; curvatures and torsions; parallelism and structures equations.
Chapter 8 is devoted to the main subject from the book: Hamilton spaces of order $k$, $H^{(k)n} = (M, H(x, y^{(1)}, \ldots, y^{(k-1)}, p))$. To begin with, we prove the existence of these spaces and the existence of a natural presymplectic structure, as well as of a natural Poisson structure. Using the Legendre mapping from a Lagrange space of order $k$, $L^{(k)n} = (M, L)$ to the Hamilton space of order $k$, $H^{(k)n} = (M, H)$ one proves that there is a local diffeomorphism between these spaces. A direct consequence of previous results one can determine some important geometric object fields on the Hamilton spaces $H^{(k)n}$, namely: the canonical nonlinear connection, the $N$-linear metrical connection given by generalized Christoffel symbols. Evidently, the structure equations, curvatures and torsions of above mentioned connections are pointed out. The Hamilton-Jacobi equations and an example from Electrodynamics end this chapter.

A theory of subspaces $H^{\forall(k)m} = (M, H)$ in the Hamilton spaces $H^{(k)n} = (M, H)$ appears for the first time in this book, ch. 9. Of course, it is absolutely necessary, especially for applications. But $M$ being a submanifold in the manifold $M$, the immersion $i : M \to M$ does not implies automatically an immersion of $T^*kM$ into the dual manifold $T^*kM$.

So, by means of an immersion of the cotangent bundle $T^*M$ into $T^*kM$ we construct $T^*kM$ as an immersed submanifold of the manifold $T^*kM$.

The Hamilton space $H^{(k)n} = (M, H)$ induces Hamilton subspaces $H^{\forall(k)m} = (M, H)$. So, we study the intrinsic geometrical object fields on $H^{\forall(k)m}$ and the induced geometrical object fields, as well as the relations between them. These problems are studied using the method of moving frame - suggested by the theory of subspaces in Lagrange spaces of order $k$. The Gauss-Weingarten formulae and the Gauss-Codazzi equations are important results.

In chapter 10 we investigate the notion of Cartan space of order $k \geq 1$ as dual of that of Finsler space of same order. We point out the canonical linear connection, N-metrical connection, structure equations, the fundamental equations of Hamilton Jacobi and the Riemannian almost contact model of these spaces.

The last chapter, Ch. 11, is devoted to the Generalized Hamilton spaces $GH^{(k)n}$, Generalized Cartan spaces $GC^{(k)n}$ and applications in the Hamiltonian Relativistic Optics.

Now, some remarks. The book can be divided in three parts: the Lagrange geometry of order $k$, presented in the first three chapters, the geometrical theory of the dual manifolds $T^*kM$ - chapters 4-7 and the geometry of Hamilton spaces of order $k$ and their subspaces, contained in the last four chapters. They are studied directly and as 'dual' geometry, via Legendre transformation. More details for Lagrange geometry of order $k$ can be found in the book [94]. Also, the particular case, $k = 2$, of the geometry of Hamilton spaces $H^{(k)n}$ can be found in the book [115].

For these reasons, the book is accessible for readers from graduate students to researchers in Mathematics, Mechanics, Physics, Biology, Informatics, etc.
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Chapter 1

Geometry of the $k$-Tangent Bundle $T^k M$

The notion of $k$-tangent bundle (or $k$-accelerations bundle or $k$-osculator bundle), $(T^k M, \pi^k, M)$ is sufficiently known. It was presented in the book [115].

The manifold $T^k M$ carries some geometrical object fields as the vertical distributions $V_1, ..., V_k$, the Liouville independent vector fields $\Gamma^1, ..., \Gamma^k$, with the properties $\Gamma^k$ belongs to $V_1$, $\Gamma^{k-1}$ belongs to $V_2$, ..., $\Gamma^1$ belongs to the distribution $V_k$. On $T^k M$ is defined a $k$-tangent structure $J$ which maps $\Gamma^1$ on $\Gamma^2$, $\Gamma^2$ on $\Gamma^3$, ..., $\Gamma^{k-1}$ on $\Gamma^k$, $\Gamma^k$ on $\Gamma$ and $J \Gamma^k = 0$.

Besides these fundamental notion on $T^k M$ we can introduce new concepts as the $k$-semisprays $S$, nonlinear connections $N$ and the $N$-linear connections $D$. But for $D$ we can get the curvatures, torsions, structure equations, geodesics, etc. The $k$-semispray $S$ is defined by the conditions $J S^k = \Gamma^k$. It is important to remark that $S$ is used for introducing those notions as nonlinear connection, or $N$-linear connection.

Concluding the geometry of $k$-accelerations bundle is basic for a geometrical theory of higher order Lagrange spaces or higher order Finsler spaces. In this book we need it for a theory of duality between higher order Lagrange spaces and higher order Hamilton spaces.

1.1 The Category of $k$-Accelerations Bundles

Let $M$ be a real $n$-dimensional manifolds of $C^\infty$ class and $(T^k M, \pi^k, M)$ its bundle of accelerations of order $k$. It can be identified with the $k$-osculator bundles [94] or with the tangent bundle of order $k$. In the case $k = 1$, $(T^1 M, \pi^1, M)$ is the tangent bundle of the manifold $M$.

A point $u \in T^k M$ will be written as $u = (x, y^{(1)}, ..., y^{(k)})$ and $\pi^k(u) = x$, $x \in M$. The canonical coordinates of $u$ are $(x^i, y^{(1)i}, ..., y^{(k)i})$, $i = 1, n$, $n = \dim M$. 

1
These coordinates have a geometrical meaning. If $c : I \to M$ is a differentiable curve, $c(0) = x_0 \in M$, and $Im \ c \subset U$, $U$ being a local chart of the base manifold $M$, and the mapping $c : I \to M$ is represented by $x^i = x^i(t)$, $t \in I$, then the osculating space of the curve $c$, in the point $x_0 = x^i(0)$ is characterized by the set of numbers:

\begin{align}
(1.1.1) \quad x^i &= x^i(0), \quad y_j^{(1)i} = \frac{dx^i}{dt}(0), ..., \quad y_j^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}(0) \\
&
\end{align}

Thus, the formulas (1.1.1) give us the canonical coordinates of a point $u_0 = (x_0, y_0^{(1)i}, ..., y_0^{(k)i})$ of the domain of the local chart $(\pi^k)^{-1}(U) \subset T^k M$.

Starting from (1.1.1) it is not difficult to see which are the changing rules of the local coordinates on $T^k M : (x^i, y_j^{(1)i}, ..., y_j^{(k)i}) \to (\bar{x}^i, \bar{y}_j^{(1)i}, ..., \bar{y}_j^{(k)i})$.

We deduce:

\begin{align}
(1.1.2) \quad \\
\bar{x}^i &= \bar{x}^i(x^1, ..., x^n), \quad \text{rank} \left\| \frac{\partial \bar{x}^i}{\partial x^j} \right\| = n \\
\bar{y}_j^{(1)i} &= \frac{\partial \bar{x}^i}{\partial x^j} y_j^{(1)i} \\
2 \bar{y}_j^{(2)i} &= \frac{\partial \bar{y}_j^{(1)i}}{\partial x^j} y_j^{(1)i} + 2 \frac{\partial \bar{y}_j^{(1)i}}{\partial y_j^{(1)j}} y_j^{(2)i} \\
&\quad \vdots \\
k \bar{y}_j^{(k)i} &= \frac{\partial \bar{y}_j^{(k-1)i}}{\partial x^j} y_j^{(1)i} + 2 \frac{\partial \bar{y}_j^{(k-1)i}}{\partial y_j^{(1)j}} y_j^{(2)i} + ... + k \frac{\partial \bar{y}_j^{(k-1)i}}{\partial y_j^{(k-1)j}} y_j^{(k)i}.
\end{align}

But we must remark the following identities:

\begin{align}
(1.1.3) \quad \frac{\partial \bar{y}_j^{(\alpha)i}}{\partial x^j} &= \frac{\partial \bar{y}_j^{(\alpha+1)i}}{\partial y_j^{(1)i}} = ... = \frac{\partial \bar{y}_j^{(k)i}}{\partial y_j^{(k-\alpha)j}}, \quad (\alpha = 0, ..., k-1; \ y_j^{(0)i} = x^i).
\end{align}

In the following $T^0 M$ is canonically identified to $M$. Sometimes we employ the notations $y_j^{(0)i} = x^i$. The projections:

\begin{align}
\pi^k_l : (x, y_j^{(1)i}, ..., y_j^{(k)i}) \in T^k M \to (x, y_j^{(1)i}, ..., y_j^{(l)i}) \in T^l M, \quad (0 \leq l < k)
\end{align}

are submersions. Clearly $\pi^k_0 = \pi^k$.

A section $S : M \to T^k M$ of the projection $\pi^k$ is a differentiable mapping with the property $\pi^k \circ S = 1_M$. It is a local section if $\pi^k \circ S |_U = 1_U$. Of course a section $S : M \to T^k M$ along a curve $c : I \to M$ has the property $\pi^k(S(c)) = c$.

If $c : I \to M$ is locally represent on $U \subset M$ by $x^i = x^i(t)$ then the mapping $\bar{c} : I \to T^k M$ given by:

\begin{align}
(1.1.4) \quad x^i &= x^i(t), \quad y_j^{(1)i} = \frac{1}{1!} \frac{dx^i}{dt}(t), ..., \quad y_j^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}(t), \quad t \in I
\end{align}

is the extension of order $k$ of $c$. Of course $\pi^k \circ \bar{c} = c$. So $\bar{c}$ is a section of $\pi^k$ along $c$. 

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**THE GEOMETRY OF HIGHER-ORDER HAMILTON SPACES**
Geometry of the $k$-Tangent Bundle $T^kM$

More general, if $V$ is a vector field on the domain of a chart $U$ and $c : I \rightarrow U$ is a curve, then the mapping 

$$S_V : c \rightarrow (\pi^k)^{-1}(U) \subset T^kM$$

defined by

$$(1.1.5) \quad S_V : x^i = x^i(t), \quad y^{(1)i} = V^i(x(t)), ..., \quad y^{(k)i} = \frac{1}{k!} \frac{d^{k-1}V^i(x(t))}{dt^{k-1}}, \quad t \in I$$

is a section of the projection $\pi^k$ along curve $c$.

Of course the notion of the section of $\pi^k_l$ along $T^lM$ can be defined, as in the previous case.

The following property hold:

**Theorem 1.1.1** If the differentiable manifold $M$ is paracompact, then $T^kM$ is a paracompact manifold.

We can see, that

$$T^k : Man \rightarrow Man,$$

where $Man$ is the category of differentiable manifolds, is a covariant functor.

Indeed we define:

$T^k : M \in ObMan \rightarrow T^kM \in ObMan$ and

$T^k : \{ f : M \rightarrow M' \} \rightarrow \{ T^k f : T^kM \rightarrow T^kM' \}$ as follows:

if $f(x)$ in the local coordinate of $M$ is given by $x^{i'} = x^i(x^1, ..., x^n)$, 
$i', j' = 1, ..., m = \dim M'$, then the morphism $T^k f : T^kM \rightarrow T^kM'$ is defined by:

$$(1.1.6) \quad \begin{cases} 
 x^{i'} = x^i(x^1, ..., x^n), \\
 y^{(1)i'} = \frac{\partial x^{i'}}{\partial x^j} y^{(1)j}, \\
 2y^{(2)i'} = \frac{\partial y^{(1)i'}}{\partial x^j} y^{(1)j} + 2\frac{\partial y^{(1)i'}}{\partial y^{(1)j}} y^{(2)j}, \\
 .........................
 ky^{(k)i'} = \frac{\partial y^{(k-1)i'}}{\partial x^j} y^{(1)j} + 2\frac{\partial y^{(k-1)i'}}{\partial y^{(1)j}} y^{(2)j} + ... + k \frac{\partial y^{(k-1)i'}}{\partial y^{(k-2)j}} y^{(k)j}.
\end{cases}$$

Remarking that

$$(1.1.7) \quad \frac{\partial y^{(0)i'}}{\partial x^j} = \frac{\partial y^{(\alpha+1)i'}}{\partial y^{(\alpha)j}} = ... = \frac{\partial y^{(k)i'}}{\partial y^{(k-\alpha)j}}, \quad (\alpha = 0, ..., k - 1; \quad y^{(0)} = x),$$

we can prove without difficulties that $T^k$ is a covariant functor.
### 1.2 Liouville Vector Fields. \( k \)-Semisprays

A local coordinate changing (1.1.2) transforms the natural basis of the tangent space \( T_u(T^kM) \) by the following rule:

\[
\begin{align*}
\frac{\partial}{\partial x^i} &= \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{y}^{(1)}_{ij}}{\partial x^i} \frac{\partial}{\partial \tilde{y}^{(1)}_{ij}} + \cdots + \frac{\partial \tilde{y}^{(k)}_{ij}}{\partial x^i} \frac{\partial}{\partial \tilde{y}^{(k)}_{ij}}, \\
\frac{\partial}{\partial y^{(1)}_{ij}} &= \frac{\partial \tilde{y}^{(1)}_{ij}}{\partial y^{(1)}_{ij}} \frac{\partial}{\partial \tilde{y}^{(1)}_{ij}} + \cdots + \frac{\partial \tilde{y}^{(k)}_{ij}}{\partial y^{(1)}_{ij}} \frac{\partial}{\partial \tilde{y}^{(k)}_{ij}}, \\
\frac{\partial}{\partial y^{(k)}_{ij}} &= \frac{\partial \tilde{y}^{(k)}_{ij}}{\partial y^{(1)}_{ij}} \frac{\partial}{\partial \tilde{y}^{(k)}_{ij}},
\end{align*}
\]

calculated at the point \( u \in T^kM \).

These formulas imply the transformation of the natural cobasis at the point \( u \in T^kM \) by the rule:

\[
\begin{align*}
d\tilde{x}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, \\
d\tilde{y}^{(1)}_{ij} &= \frac{\partial \tilde{y}^{(1)}_{ij}}{\partial x^j} dx^j + \frac{\partial \tilde{y}^{(1)}_{ij}}{\partial y^{(1)}_{ij}} dy^{(1)}_{ij}, \\
&\vdots \\
d\tilde{y}^{(k)}_{ij} &= \frac{\partial \tilde{y}^{(k)}_{ij}}{\partial x^j} dx^j + \frac{\partial \tilde{y}^{(k)}_{ij}}{\partial y^{(1)}_{ij}} dy^{(1)}_{ij} + \cdots + \frac{\partial \tilde{y}^{(k)}_{ij}}{\partial y^{(k)}_{ij}} dy^{(k)}_{ij}.
\end{align*}
\]

The matrix of coefficients of second member of (1.2.1) is the Jacobian matrix of the changing of coordinates (1.1.2). Since

\[
\frac{\partial \tilde{x}^i}{\partial x^j} = \frac{\partial \tilde{y}^{(1)}_{ij}}{\partial y^{(1)}_{ij}} = \cdots = \frac{\partial \tilde{y}^{(k)}_{ij}}{\partial y^{(k)}_{ij}},
\]

it follows.

**Theorem 1.2.1** If the number \( k \) is odd, then the manifold \( T^kM \) is orientable.

Also the formulae (1.2.1), (1.2.1’) allow to determine some important geometric object fields on the total space of accelerations bundle \( T^kM \).

The distribution \( V_1 : u \in T^kM \to V_{1,u} \subset T_u(T^kM) \) generated by the tangent vectors \( \left\{ \frac{\partial}{\partial y^{(1)}_{ij}}, \ldots, \frac{\partial}{\partial y^{(k)}_{ij}} \right\}_u \), \( \forall u \in T^kM \) is the vertical distribution on the bundle \( T^kM \). Its dimension is \( kn \).

\( V_2 : u \in T^kM \to V_{2,u} \subset T_u(T^kM) \) generated by \( \left\{ \frac{\partial}{\partial y^{(2)}_{ij}}, \ldots, \frac{\partial}{\partial y^{(k)}_{ij}} \right\}_u \), \( \forall u \in T^kM \) is a subdistribution of \( V_1 \) of local dimension \( (k-1)n, \ldots, \ldots \).
Geometry of the $k$-Tangent Bundle $T^kM$

$V_k : u \in T^kM \rightarrow V_k,u \subset T_u(T^kM)$ generated by $\{ \frac{\partial}{\partial y^{(k)i}} \}_u$, $\forall u \in T^kM$ is a subdistribution of dimension $n$ of the distribution $V_{k-1}$. All these subdistributions are integrable and the following sequence holds:

$$V_1 \supset V_2 \supset \ldots \supset V_k$$

Using again (1.2.1) we deduce:

**Theorem 1.2.2** The following operators in the algebra of functions $\mathcal{F}(T^kM)$:

$$\begin{align*}
\Gamma_1 &= y^{(1)i} \frac{\partial}{\partial y^{(k)i}}, \\
\Gamma_2 &= y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k)i}}, \\
&\hspace{1cm}\ldots \\
\Gamma_k &= y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(2)i}} + \ldots + ky^{(k)i} \frac{\partial}{\partial y^{(k)i}}
\end{align*}$$

are vector fields globally defined on $T^kM$ and independent on the manifold $\tilde{T}^kM = T^kM \setminus \{0\}$, $\Gamma_1$ belongs to distribution $V_k$, $\Gamma_2$ belongs to distribution $V_{k-1}$, ..., $\Gamma_k$ belongs to distribution $V_1$.

Taking into account (1.2.1') it is not hard to prove:

**Theorem 1.2.3** For any differentiable function $L(x, y^{(1)}, \ldots, y^{(k)})$ on the manifold $\tilde{T}^kM$, the following entries $d_0L, \ldots, d_kL$ are 1-form fields on $\tilde{T}^kM$:

$$\begin{align*}
d_0L &= \frac{\partial L}{\partial y^{(k)i}} dx^i, \\
d_1L &= \frac{\partial L}{\partial y^{(k-1)i}} dx^i + \frac{\partial L}{\partial y^{(k)i}} dy^{(1)i}, \\
&\hspace{1cm}\ldots \\
d_kL &= \frac{\partial L}{\partial x^i} dx^i + \frac{\partial L}{\partial y^{(1)i}} dy^{(1)i} + \ldots + \frac{\partial L}{\partial y^{(k)i}} dy^{(k)i}.
\end{align*}$$

Evidently, $d_kL = dL$ is the differential of the function $L$. But:

$d_0L$ vanish on the distribution $V_1$,

$d_1L$ vanish on the distribution $V_2$,

$\ldots$

$d_{k-1}L$ vanish on the distribution $V_k$.

In applications we shall use also the following nonlinear operator

$$\Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \ldots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}.$$
A $k$-tangent structure $J$ on $T^k M$ is defined as usual by the following $\mathcal{F}(T^k M)$-linear mapping $J : \mathcal{X}(T^k M) \to \mathcal{X}(T^k M)$:

\[ J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}; \quad J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}; \ldots; \]

\[ J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = \frac{\partial}{\partial y^{(k)i}}; \quad J\left(\frac{\partial}{\partial y^{(k)i}}\right) = 0. \]

$J$ is globally defined on $T^k M$. It is a tensor fields of type $(1,1)$ on $T^k M$, locally expressed by

\[ J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \ldots + \frac{\partial}{\partial y^{(k)i}} \otimes dy^{(k-1)i}. \]

The last form of $J$ implies that $J$ is an integrable structure. The $k$-tangent structure $J$ has the following properties:

1. $\text{Im} J = V_1$, $\text{Ker} J = V_k$.
2. $\text{rank} J = kn$.
3. $J^k = \Gamma^1 \ldots \Gamma^1$, $J^2 = \Gamma^2 \Gamma$, $J^1 = 0$.
4. $J \circ \ldots \circ J = 0$, ($k + 1$ factors).

A $k$-semispray on the manifold $T^k M$ is a vector field $S$ on $T^k M$ with the property

\[ JS = \Gamma^k. \]

The notion of local $k$-semispray is obvious.

Any $k$-semispray $S$ can be uniquely written in the form

\[ S = y^{(1)i} \frac{\partial}{\partial x^i} + \ldots + ky \left(\frac{k-1}{i}\right) - (k + 1)G^i (x, y^{(1)}, \ldots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}}, \]

or shortly

\[ S = \Gamma - (k + 1)G^i (x, y^{(1)}, \ldots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}}. \]

The set of functions $G^i$ is the set of coefficients of $S$. With respect to (1.1.2), $G^i$ are transformed as following:

\[ (k + 1) \tilde{G}^i = (k + 1)G^j \frac{\partial \tilde{x}^i}{\partial x^j} - \Gamma^i (k)^j. \]

A curve $c : I \to M$ is called a $k$-path on $M$ with respect to a $k$-semispray $S$ if its extension $\tilde{c}$ to $T^k M$ is an integral curve of $S$. If $S$ is given in the form (1.2.6), then its $k$-paths are characterized by the system of differential equations:

\[ \frac{d^{k+1}x^i}{dt^{k+1}} + (k + 1)!G^i (x^i, dx^i, \ldots, d^k x^i) = 0. \]
Indeed, the solutions curves of $S$ are given by the system of ordinary differential equations

$$\frac{dx^i}{dt} = y^{(1)i}, \quad \frac{dy^{(1)i}}{dt} = 2y^{(2)i}, \ldots, \quad \frac{dy^{(k-1)i}}{dt} = ky^{(k)i},$$

$$\frac{dy^{(k)i}}{dt} = -(k + 1)G^i(x, y^{(1)}, \ldots, y^{(k)}).$$

If we eliminate $y^{(1)i}, \ldots, y^{(k)i}$ we obtain (1.2.7).

The previous theory will be used in the geometry of higher order Lagrange spaces, which is based on the regular Lagrangians of higher order.

Now, let us considered the adjunct $k$-tangent structure $J^*$. It is the endomorphism of the module $\mathcal{X}^*(T^kM)$, defined by:

$$J^*(dy^{(k)i}) = dy^{(k-1)i} = dy^{(k)i} \circ J,$$

$$J^*(dy^{(k-1)i}) = dy^{(k-2)i} = dy^{(k-1)i} \circ J,$$

$$\vdots$$

$$J^*(dy^{(1)i}) = dx^i = dy^{(1)i} \circ J,$$

$$J^*(dx^i) = 0.$$

By using the formula (1.2.1') is not difficult to prove:

**Theorem 1.2.4** $J^*$ is globally defined on $T^kM$.

If $\omega \in X^*(T^kM)$ is given by

$$\omega = \omega^0 \ dx^i + \omega^1 \ dy^{(1)i} + \ldots + \omega^k \ dy^{(k)i},$$

then

$$J^*\omega = \omega^1 \ dx^i + \ldots + \omega^k \ dy^{(k-1)i}.$$

We put $J^*f = f$ for any function $f \in \mathcal{F}(T^kM)$ and observe that $J^*$ is a tensor field, of type $(1, 1)$ on $T^kM$. Namely, we have

$$J^* = dy^{(k-1)i} \otimes \frac{\partial}{\partial y^{(k)i}} + dy^{(k-2)i} \otimes \frac{\partial}{\partial y^{(k-1)i}} + \ldots + dx^i \otimes \frac{\partial}{\partial y^{(1)i}}.$$

The rank $\|J^*\| = kn$. $J^*$ can be extended to an endomorphism of the exterior algebra $\wedge(T^kM)$, by putting

$$J^*(\omega)(X_1, \ldots, X_p) = \omega(JX_1, \ldots, JX_p), \forall \omega \in \wedge^p(T^kM).$$

The existence of $J^*$ allow to introduce the so called vertical differential operators in the exterior algebra $\wedge(T^kM)$. 
Indeed, let us consider the operators of differentiation $d_k$:

$$d_k = d = \frac{\partial}{\partial x^i} dx^i + \frac{\partial}{\partial y^{(1)i}} dy^{(1)i} + \ldots + \frac{\partial}{\partial y^{(k)i}} dy^{(k)i}.$$ 

We get:

$$J^* d_k = d_{k-1} = \frac{\partial}{\partial y^{(1)i}} dx^i + \ldots + \frac{\partial}{\partial y^{(k-1)i}} dy^{(k-1)i},$$

$$J^* d_1 = d_0 = \frac{\partial}{\partial y^{(k)i}} dx^i.$$ 

It is not difficult to prove that the operators $d_k, d_{k-1}, \ldots, d_0$ do not depend on the changing of coordinates on the manifold $T^k M$. If $L(x, y^{(0)}, \ldots, y^{(k)})$ is a 1-form function on $T^k M$, then $d_k L, d_{k-1} L, \ldots, d_0 L$ are differentiable given by (1.2.2').

But $d_0, d_1, \ldots, d_k$ can be extended to the exterior algebra $\wedge (T M)$ giving them restrictions to $F(T^k M)$ and $\wedge^1 (T^k M)$. So, we will take $d_0 L, \ldots, d_k L$ expressed in (1.2.2') and (1.2.12)

$$d_\alpha (dy^{(\beta)i}) = 0, (\alpha, \beta = 0, 1, \ldots, k; y^{(0)} = x).$$

Consequently $d_0, \ldots, d_k$ are the antiderivations of degree 1 in the exterior algebra $\wedge (T^k M)$. For instance, if $\omega \in \wedge^1 (T^k M)$ and it is locally express by

$$\omega = \omega^{(0)}_i dx^i + \omega^{(1)}_i dy^{(1)i} + \ldots + \omega^{(k)}_i dy^{(k)i}$$

then

$$d_\alpha \omega = \sum_{\beta=0}^k d_\alpha (\omega^{(\beta)}_i \wedge dy^{(\beta)i}), \ (\alpha = 0, \ldots, k).$$

It is not so difficult to see that the following properties hold:

$$d_\alpha \circ d_\alpha = 0, \ (\alpha = 0, \ldots, k).$$

In the case $k = 1$, $\frac{1}{2} d_0 L = \frac{1}{2} \frac{\partial L}{\partial y^i} dx^i = p_i dx^i$ is the Liouville 1-form and $\frac{1}{2} (d_1 \circ d_2) L = dp_i \wedge dx^i$ is the symplectic structure on $TM$.

1.3 Nonlinear Connections

The notion of nonlinear connection is also known, [94]. A subbundle $HT^k M$ of the tangent bundle $(TT^k M, d\pi^k, T^k M)$ which is supplementary to the vertical subbundle $V_1 T^k M$:

$$TT^k M = HT^k M \oplus V_1 T^k M$$

is called a nonlinear connection.
Geometry of the $k$-Tangent Bundle $T^kM$

The fibres of $HT^kM$, determine a horizontal distribution $N : \forall u \in T^kM \rightarrow N_u = H_uT^kM \subset T_uT^kM$, supplementary to the vertical distribution $V_1$, i.e:

$$T_uT^kM = N_u \oplus V_{1,u}, \ \forall u \in T^kM.$$  \hspace{1cm} (1.3.1)

If the base manifold $M$ is paracompact then on $T^kM$ there exist nonlinear connections. The dimension of the horizontal distribution $N$ is $n = \dim M$.

Consider a nonlinear connection $N$ on $T^kM$ and denote by $h$ and $v$ the horizontal and vertical projectors with respect to the distributions $N$ and $V_1$:

$$h + v = I, h^2 = h, v^2 = v, hv = vh = 0.$$  \hspace{1cm} (1.3.2)

As usual we denote

$$X^H = hX, \ X^V = vX, \ \forall X \in \mathcal{X}(T^kM).$$

An horizontal lift, with respect to $N$ is a $\mathcal{F}(M)$--linear mapping $l_h : \mathcal{X}(M) \rightarrow \mathcal{X}(T^kM)$ which has the properties:

$$v \circ l_h = 0, \ d\pi^k \circ l_h = I_d$$

There exists an unique local basis adapted to the horizontal distribution $N$. It is given by

$$\frac{\delta}{\delta x^i} = l_h(\frac{\partial}{\partial x^i}), \ (i = 1, \ldots, n).$$  \hspace{1cm} (1.3.3)

The linearly independent vector fields $\frac{\delta}{\delta x^i}, \ (i = 1, \ldots, n)$ can be uniquely written in the form

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{j}_{(1)} \frac{\partial y(1)}{\partial x^j} - \ldots - N^{j}_{(k)} \frac{\partial y(k)}{\partial x^j}. \hspace{1cm} (1.3.4)$$

The system of functions $(N^{j}_{(1)}, \ldots, N^{j}_{(k)})$ gives the coefficients of the nonlinear connection $N$.

We remark that:

1) For $X = X^i(x)\frac{\partial}{\partial x^i} \in \mathcal{X}(M)$ the horizontal lifts is $l_h \ X = X^i \frac{\delta}{\delta x^i}.$

2) With respect to (1.1.2), we obtain

$$\frac{\delta}{\delta x^i} = \frac{\partial x^j}{\partial x^i} \frac{\delta}{\partial x^j}.$$  \hspace{1cm} (1.3.5)

3) With respect to (1.1.2) the coefficients of $N$ are transformed by the rule

$$\frac{\partial y^{(1)}_i}{\partial x^{(1)}_j} = N^{m}_{(1)} \frac{\partial x^{(1)}_i}{\partial x^m} - \frac{\partial y^{(1)}_m}{\partial x^{(1)}_i},$$

$$\frac{\partial y^{(k)}_i}{\partial x^{(k)}_j} = N^{m}_{(k)} \frac{\partial x^{(k)}_i}{\partial x^m} + \ldots + N^{m}_{(1)} \frac{\partial y^{(k-1)}_m}{\partial x^{(1)}_i} - \frac{\partial y^{(k)}_m}{\partial x^{(1)}_i}.$$
The $k$-tangent structure $J$, defined in (1.2.4) applies the horizontal distribution $N$ into a vertical distribution $N_1 \subset V_1$ of dimension $n$, supplementary to the distribution $V_2$. Then it applies the distribution $N_1$ in a distribution $N_2 \subset V_2$, supplementary to the distribution $V_3$ and so on. Of course, we have $\dim N_0 = \dim N_1 = \cdots = \dim N_{k-1} = n$.

Setting $N_0 = N$, we can write
\[(1.3.6)\quad N_1 = J(N_0), N_2 = J^2(N_0), \ldots, N_{k-1} = J^{k-1}(N_0)\]
and we obtain the following direct decomposition:
\[(1.3.7)\quad T_u T^k M = N_{0,u} \oplus N_{1,u} \oplus \cdots \oplus N_{k-1,u} \oplus V_{k,u}, \quad \forall u \in T^k M.\]

An adapted basis to the distributions $N_0, N_1, \ldots, N_{k-1}, V_k$ at a point $u \in T^k M$ is given by:
\[(1.3.8)\quad \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)} i}, \cdots, \frac{\delta}{\delta y^{(k)} i},\]
where
\[\frac{\delta}{\delta y^{(1)} i} = J \left( \frac{\delta}{\delta x^i} \right), \quad \frac{\delta}{\delta y^{(2)} i} = J^2 \left( \frac{\delta}{\delta x^i} \right), \quad \cdots, \quad \frac{\delta}{\delta y^{(k)} i} = J^k \left( \frac{\delta}{\delta x^i} \right).\]

Therefore, using (1.2.4) and (1.3.4) we get:
\[(1.3.9)\quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)}^j \frac{\partial}{\partial y^{(1)} j} - \cdots - N_{(k)}^j \frac{\partial}{\partial y^{(k)} j},\]
\[\frac{\delta}{\delta y^{(1)} i} = \frac{\partial}{\partial y^{(1)} i} - N_{(1)}^j \frac{\partial}{\partial y^{(2)} j} - \cdots - N_{(k-1)}^j \frac{\partial}{\partial y^{(k)} j},\]
\[\ddots\]
\[\frac{\delta}{\delta y^{(k-1)} i} = \frac{\partial}{\partial y^{(k-1)} i} - N_{(1)}^j \frac{\partial}{\partial y^{(k)} j},\]
\[\frac{\delta}{\delta y^{(k)} i} = \frac{\partial}{\partial y^{(k)} i}.\]

With respect to (1.1.2) we have:
\[(1.3.10)\quad \frac{\delta}{\delta y^{(\alpha)} i} = \frac{\partial x^j}{\partial x^i} \frac{\delta}{\partial y^{(\alpha)} j}, \quad (\alpha = 0, \ldots, k; y^{(0)} = x).\]

Taking into account the direct sum (1.3.1) and (1.3.7), it follows that the vertical distribution $V_1$ at a point $u$ gives rise to the direct decomposition:
\[(1.3.11)\quad V_{1,u} = N_{1,u} \oplus \cdots \oplus N_{k-1,u} \oplus V_{k,u}, \quad \forall u \in T^k M.\]
Let $h, v_1, \ldots, v_k$ be the projectors determined by (1.3.7):
\[ h + v_1 + \cdots + v_k = I, \quad h^2 = h, \quad v_\alpha v_\alpha = v_\alpha, \quad hv_\alpha = v_\alpha h = 0, \quad (\alpha = 1, \ldots, k), \]
\[ v_\alpha v_\beta = v_\beta v_\alpha = 0, \quad (\alpha \neq \beta; \alpha, \beta = 1, \ldots, k). \]

If we denote
\[
(1.3.12) \quad X^H = hX, \quad X^{V_\alpha} = v_\alpha X, \quad \forall X \in \mathcal{X}(T^k M)
\]
we have, uniquely,
\[
(1.3.13) \quad X = X^H + X^{V_1} + \ldots + X^{V_k}. \]

In the adapted basis (1.3.8) we can write:
\[
X^H = X^{(0)i} \frac{\delta}{\delta x^i}, \quad X^{V_\alpha} = X^{(\alpha)i} \frac{\delta}{\delta y^{(\alpha)i}}, \quad (\alpha = 1, \ldots, k).
\]

The following properties are important in applications.

1) The distribution $N = N_0$ is integrable if, and only if
\[
[X^H, Y^H]^{V_\alpha} = 0, \quad (\alpha = 1, \ldots, k).
\]

2) The distribution $N_\alpha$ is integrable, if and only if:
\[
[X^{V_\alpha}, Y^{V_\alpha}]^H = 0, \quad [X^{V_\alpha}, Y^{V_\alpha}]^{V_\beta} = 0, \quad (\alpha \neq \beta, \alpha, \beta = 1, \ldots, k).
\]

The notion of $h$- or $v_\alpha$- lift of a vector fields $X = \mathcal{X}(M)$, $X = X^i(x) \frac{\partial}{\partial x^i}$ is obvious. We have:
\[
(1.3.14) \quad l_h(X) = X^i(x) \frac{\delta}{\delta x^i}, \quad l_{v_\alpha}(X) = X^i(x) \frac{\delta}{\delta y^{(\alpha)i}}, \quad (\alpha = 1, \ldots, k).
\]

### 1.4 The Dual Coefficients of a Nonlinear Connection

Consider a nonlinear connection $N$, having the coefficients $(N^1_j, \ldots, N^k_j)$. The adapted basis $\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \ldots, \frac{\delta}{\delta y^{(k)i}} \right)$ to the direct decomposition (1.3.7) is expressed in the formulae (1.3.9). Its dual basis, denoted by
\[
(1.4.1) \quad \delta x^i, \delta y^{(1)i}, \ldots, \delta y^{(k)i},
\]
can be uniquely written in the form:
The system of functions \((M_j^i, \ldots, M_j^k)\) is called the system of dual coefficients of the nonlinear connection \(N\). They are determined entirely by means of the coefficients \((N_j^i, \ldots, N_j^k)\).

Conversely, if the dual coefficients are given, then the coefficients \((N_j^i, \ldots, N_j^k)\) are expressed by:

\[
M_j^i = N_j^i, \quad M_j^i = N_j^i + N_j^m M_j^m, \ldots, \quad (\text{1.4.3})
\]

The relations between the natural basis on the manifold \(T^k M\) and the
adapted basis are immediately:

\[ \frac{\partial}{\partial x^i} = \delta^i_{x^i} + M^j_{(1)} \delta^j_{(1)y^j} + \cdots + M^j_{(k)} \delta^j_{(k)y^j}, \]

\[ \frac{\partial}{\partial y^{(1)i}} = \delta^i_{y^{(1)i}} + M^j_{(1)} \delta^j_{(1)y^{(2)}j} + \cdots + M^j_{(k)} \delta^j_{(k)y^{(1)i}}, \]

\[ \frac{\partial}{\partial y^{(k)i}} = \delta^i_{y^{(k)i}}. \]

Similarly, we have:

\[ dx^i = \delta x^i, \]

\[ dy^{(1)i} = \delta y^{(1)i} - N_j^i \delta x^j, \]

\[ dy^{(k)i} = \delta y^{(k)i} - N_j^i \delta y^{(k-1)i} - \cdots - N_j^i \delta y^{(1)i} - N_j^i \delta x^j. \]

As an application we can prove:

**Theorem 1.4.1**

1) The Liouville vector fields \( \Gamma_1, \ldots, \Gamma_k \) can be expressed in the adapted basis \( (1.3.8) \) in the form

\[ 1\Gamma = z^{(1)i} \frac{\delta}{\delta y^{(k)i}}, \]

\[ 2\Gamma = z^{(1)i} \frac{\delta}{\delta y^{(k-1)i}} + 2z^{(2)i} \frac{\delta}{\delta y^{(k)i}}, \]

\[ k\Gamma = z^{(1)i} \frac{\delta}{\delta y^{(1)i}} + 2z^{(2)i} \frac{\delta}{\delta y^{(2)i}} + \cdots + kz^{(k)i} \frac{\delta}{\delta y^{(k)i}}, \]

where

\[ z^{(1)i} = y^{(1)i}, \quad 2z^{(2)i} = 2y^{(2)i} + M_j^i y^{(1)m}, \quad \ldots, \]

\[ kz^{(k)i} = ky^{(k)i} + (k-1)M_j^i y^{(k-1)m} + \cdots + M_j^i y^{(1)m} \]

2) With respect to \( (1.1.2) \) we have:

\[ \tilde{z}^{(\alpha)i} = \frac{\partial \bar{x}^i}{\partial x^j} z^{(\alpha)j}, \quad (\alpha = 1, \ldots, k). \]

We note that the formulas \( (1.4.7') \) express the geometrical meaning of each entry \( z^{(1)i}, \ldots, z^{(k)i} \). So, we call them the Liouville distinguished vector fields.
THE GEOMETRY OF HIGHER-ORDER HAMILTON SPACES

(shortly, \textit{d-vector fields}). These vectors are important in the geometry of higher order Lagrangians.

A field of 1-form $\omega \in \mathcal{X}^*(T^k M)$ can be uniquely written as

\begin{equation}
\omega = \omega^H + \omega^{V_1} + \cdots + \omega^{V_k}
\end{equation}

where

\begin{equation}
\omega^H = \omega \circ h, \quad \omega^{V_\alpha} = \omega \circ v_\alpha, \quad (\alpha = 1, \ldots, k).
\end{equation}

In the adapted cobasis (1.4.1) we get:

\begin{equation}
\omega^H = \omega^H_i \delta x^i, \quad \omega^{V_\alpha} = \omega^{V_\alpha}_{(\alpha)} \delta y^{(\alpha)i}, \quad (\alpha = 1, \ldots, k).
\end{equation}

For any function $f \in \mathcal{F}(T^k M)$, the 1-form $df$ has the components:

\begin{equation}
df = (df)^H + (df)^{V_1} + \cdots + (df)^{V_k}.
\end{equation}

Using (1.4.8”) we obtain:

\begin{equation}
(df)^H = \frac{\delta f}{\delta x^i} \delta x^i, \quad (df)^{V_\alpha} = \frac{\delta f}{\delta y^{(\alpha)i}} \delta y^{(\alpha)i}, \quad (\alpha = 1, \ldots, k).
\end{equation}

Let $\gamma : I \to T^k M$ be a parametrized curve, locally expressed by

\begin{equation}
x^i = x^i(t), \quad y^{(\alpha)i} = y^{(\alpha)i}(t), \quad t \in I, \quad (\alpha = 1, \ldots, k).
\end{equation}

The tangent vector field can be expressed as:

\[
\frac{d\gamma}{dt} = \left(\frac{d\gamma}{dt}\right)^H + \cdots + \left(\frac{d\gamma}{dt}\right)^{V_k} = \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{\delta y^{(1)i}}{\delta y^{(1)i}} \frac{\delta}{\delta y^{(1)i}} + \cdots + \frac{\delta y^{(k)i}}{\delta y^{(k)i}} \frac{\delta}{\delta y^{(k)i}},
\]

where, by means of (1.4.2) one can write:

\begin{equation}
\frac{\delta y^{(1)i}}{dt} = \frac{dy^{(1)i}}{dt} + M_j \frac{dx^j}{dt}, \ldots,
\end{equation}

\begin{equation}
\frac{\delta y^{(k)i}}{dt} = \frac{dy^{(k)i}}{dt} + M_j \frac{dy^{(k-1)i}}{dt} + \cdots + M_j \frac{dx^j}{dt}.
\end{equation}

A parametrized curve $\gamma$ is called \textit{horizontal} if $\left(\frac{d\gamma}{dt}\right)^{V_\alpha} = 0, \quad (\alpha = 1, \ldots, k)$. It is characterized by the system of differential equations:

\begin{equation}
\frac{\delta y^{(1)i}}{dt} = \cdots = \frac{\delta y^{(k)i}}{dt} = 0.
\end{equation}
A parametrized curve \(c : I \to M\) on the base manifold \(M\), analytically given by \(x^i = x^i(t), \ t \in I\), has its extension \(\tilde{c} : I \to T^k M\), given by:

\[
x^i = x^i(t), \ y^{(1)} i = \frac{dx^i}{dt}, ..., y^{(k)} i = \frac{1}{k!} \frac{d^k x^i}{dt^k}.
\]

A horizontal curve \(\tilde{c}\) is called an autoparallel curve of the nonlinear connection \(N\).

**Theorem 1.4.2** The autoparallel curves of the nonlinear connection \(N\) are characterized by the system of differential equations

\[
\begin{align*}
y^{(1)} i &= \frac{dx^i}{dt}, ..., y^{(k)} i = \frac{1}{k!} \frac{d^k x^i}{dt^k}, \\
\delta y^{(1)} i \frac{dt}{dt} &= 0, ..., \delta y^{(k)} i \frac{dt}{dt} = 0.
\end{align*}
\]

1.5 The Determination of a Nonlinear Connection

A nonlinear connection \(N\) on the manifold of accelerations of order \(k\), \(T^k M\) can be determined by a \(k\)-semispray \(S\) or a Riemannian structure \(g(x)\) defined on the base manifold, or by a Finslerian or Lagrangian structure over the manifold \(M\).

A first result is as follows:

**Theorem 1.5.1** (R. Miron and Gh. Atanasiu, [94]) If a \(k\)-semispray \(S\), with the coefficients \(G^i(x, y^{(1)}, ..., y^{(k)})\) is given on \(T^k M\), then the set of functions:

\[
\begin{align*}
M^i_j^{(1)} &= \frac{\partial G^i}{\partial y^{(k)} j}, \\
M^i_j^{(2)} &= \frac{1}{2} \left( SM^i_j^{(1)} + M^i_m^{(1)} M^m_j^{(1)} \right), ..., \\
M^i_j^{(k)} &= \frac{1}{k} \left( S M^i_j^{(k-1)} + M^i_m^{(1)} M^m_j^{(k-1)} \right)
\end{align*}
\]

(1.5.1)

is the set of dual coefficients of a nonlinear connection \(N\) determined only by the \(k\)-semispray \(S\).

The proof can be find in the book [94].

Other result obtained by I. Bucataru ([26,27]) is given in:

**Theorem 1.5.2** If \(S\) is a \(k\)-semispray with the coefficients \(G^i\), then the following set of functions

\[
\begin{align*}
M^i_j^{(1)} &= \frac{\partial G^i}{\partial y^{(k)} j}, \\
M^i_j^{(2)} &= \frac{\partial G^i}{\partial y^{(k-1)} j}, ..., \\
M^i_j^{(k)} &= \frac{\partial G^i}{\partial y^{(1)} j}
\end{align*}
\]

(1.5.2)

define the dual coefficients of a nonlinear connection \(N^*\) determined only by the \(k\)-semispray \(S\).
The problem is to determine a nonlinear connection \( N \) on \( T^k M \) from a Riemannian structure \( g_{ij}(x) \), given on the base manifold \( M \).

Let us consider \( \gamma_{ijk}^i(x) \) the Christoffel symbols of the tensor \( g_{ij}(x) \). Then, we obtain, ([94]), without difficulties:

**Theorem 1.5.3** The following set of functions

\[
M_j^i(x, y^{(1)}) = \gamma_{jm}^i(x)y^{(1)m},
\]

\[
M_j^i(x, y^{(1)}, y^{(2)}) = \frac{1}{2} \left( \Gamma^i_j (1) + M_j^i m M_m^m (1) \right),
\]

\[
\vdots
\]

\[
M_j^i(x, y^{(1)}, \ldots, y^{(k)}) = \frac{1}{k} \left( \Gamma^i_j (k) + M_j^i m M_m^m (1) \right),
\]

where \( \Gamma \) is the operator (1.2.3), has the properties:

1° It defines the dual coefficients of a nonlinear connection \( N \) on \( \widetilde{T}^k M \), determined only by the Riemannian structure \( g_{ij}(x) \).

2° \( M_j^i(x, y^{(1)}) \) depend linearly on \( y^{(1)i} \), \( M_j^i(x, y^{(1)}, \ldots, y^{(k)}) \) depend linearly on \( y^{(k)i} \).

In the same manner we can determine a nonlinear connection on \( \widetilde{T}^k M \) by means of a Finsler space. Namely, we can prove:

**Theorem 1.5.4** Let \( N_j^i(x, y^{(1)}) \) be the Cartan nonlinear connection of a Finsler space \( F^n = (M, F(x, y^{(1)})) \). Then, the following set of functions

\[
M_j^i(x, y^{(1)}) = N_j^i(x, y^{(1)}),
\]

\[
M_j^i(x, y^{(1)}, y^{(2)}) = \frac{1}{2} \left( \Gamma^i_j (1) + M_j^i m M_m^m (1) \right),
\]

\[
\vdots
\]

\[
M_j^i(x, y^{(1)}, \ldots, y^{(k)}) = \frac{1}{k} \left( \Gamma^i j (k) + M_j^i m M_m^m (1) \right),
\]

has the properties:

1° It defines the dual coefficients of a nonlinear connection \( N \) on \( \widetilde{T}^k M \), determined only by the fundamental function \( F(x, y^{(1)}) \) of the Finsler space \( F^n \).

2° \( M_j^i \) depend linearly on \( y^{(2)i} \), \( M_j^i \) depend linearly on \( y^{(k)i} \). \( \Gamma \) being the operator (1.2.3).
Remark 1.5.1 Theorems 1.5.3 and 1.5.4 prove the existence of the nonlinear connections on $\widetilde{T}^kM$ in the case when the base manifold is paracompact.

Let us consider the adapted basis (1.3.8) and adapted cobasis (1.4.1) corresponding to the nonlinear connection with the dual coefficients (1.5.3).

Let $X^i(x)$ be a vector field on the manifold $M$. We obtain

$$l_h X = X^i(x) \frac{\delta}{\delta x^i}, \quad l_{v_\alpha} X = X^i(x) \frac{\delta}{\delta y^{(\alpha)}i},$$

(1.5.5)

$$(\alpha = 1, \ldots, n), \quad \forall X = X^i(x) \frac{\partial}{\partial x^i}.$$ Of course $l_h X, l_{v_\alpha} X$ are $h$- and respectively $v_\alpha$-lifts of the vector field $X = X^i(x) \frac{\partial}{\partial x^i}$.

Theorem 1.5.5 If $g_{ij}(x)$ is a Riemannian structure on the base manifold $M$ and $N$ is a nonlinear connection with the dual coefficients (1.5.3) determined by $g_{ij}(x)$, then

$$G = g_{ij}(x) dx^i \otimes dx^j + g_{ij}(x) \delta y^{(1)}i \otimes \delta y^{(1)}j + \cdots + g_{ij}(x) \delta y^{(k)}i \otimes \delta y^{(k)}j$$

(1.5.6) is a Riemannian structure on $\widetilde{T}^kM$ induced only by $g_{ij}(x)$.

The proof is not difficult. The Riemannian structure (1.5.6) is the prolongation to $T^kM$ of the Riemannian structure $g_{ij}(x), [94]$.

Using the same nonlinear connection $N$ with the dual coefficients (1.5.3) we can consider the $\mathcal{F}(T^kM)$-linear mapping

$$F \left( \frac{\delta}{\delta x^i} \right) = - \frac{\delta}{\delta y^{(k)}i}, \quad F \left( \frac{\delta}{\delta y^{(\alpha)}i} \right) = 0, (\alpha = 1, \ldots, k), \quad F \left( \frac{\delta}{\delta y^{(k)}i} \right) = \frac{\delta}{\delta y^{(k)}i}$$

(1.5.7)

$$(i = 1, \ldots, n).$$
One proves:
1° $F$ is globally defined on the manifold $\tilde{T}^kM$.

2° $F$ is a tensor field of type $(1, 1)$:

\[(1.5.8)\]

\[F = -\frac{\delta}{\delta y^{(k)i}} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^{(k)i}.\]

3° $\text{Ker } F = N_1 \oplus \cdots \oplus N_{k-1}, \text{Im } F = N_0 \oplus V_k$.

4° $\text{rank } F = 2n$.

5° $F^3 + F = 0$.

6° $F$ is determined only by $g_{ij}(x)$.

Concluding we have:

**Theorem 1.5.6** A Riemannian structure $g_{ij}(x)$ given on the base manifold determine an almost $(k-1)n$-contact structure $F$ on $\tilde{T}^kM$ depending only on $g_{ij}(x)$.

Let \( \begin{pmatrix} \xi_a^{(1)} & \cdots & \xi_a^{(k-1)} \end{pmatrix}, \ (a = 1, ..., n) \) be a local basis adapted to the direct decomposition $N_1 \oplus \cdots \oplus N_{k-1}$ and $\begin{pmatrix} \eta^a_{(1)} & \cdots & \eta^a_{(k-1)} \end{pmatrix}$, $(a = 1, ..., n)$ its dual.

In the book [94] is proved the result:

**Theorem 1.5.7** The set $\begin{pmatrix} G, F, \xi_a^{(1)} & \cdots & \xi_a^{(k-1)}, \eta^a_{(1)} & \cdots & \eta^a_{(k-1)} \end{pmatrix}$ is a Riemannian almost $(k-1)n$-contact structure on the manifold $\tilde{T}^kM$ determined only by the Riemannian structure $g_{ij}(x)$ defined on the base manifold $M$.

A similar theory we can study for a Finsler space $F^n = (M, F(x, y^{(1)}))$ using the nonlinear connection (1.5.4) defined only by the fundamental function $F(x, y^{(1)})$.

Also, we can investigate the problem of determination of a nonlinear connection on $T^kM$ by means of a Riemannian structure $G$ given on the manifold $T^kM$.

One shows that: A Riemann structure $G$ on $T^kM$ determine a Riemannian almost $(k-1)n$-contact structure depending only on $G$ (K. Matsumoto and R. Miron, [111]).

### 1.6 $d$-Tensor Fields. $N$-Linear Connections

Let $N$ be a nonlinear connection on the manifold $T^kM$. Then, the direct decomposition (1.3.7) holds. With respect to (1.3.7) a vector field $X$ on $T^kM$ and a 1-form $\omega$ on $T^kM$ can be uniquely written in the form (1.3.13) and (1.4.8), respectively.
A distinguished tensor field (shortly \(d\)-tensor) \(T\) on \(T^k M\) of type \((r, s)\) is a tensor field \(T\) with the property

\[
T\left(\omega^{(1)}, ..., \omega^{(r)}, X^{(1)}, ..., X^{(s)}\right) = T\left(\omega^H, ..., \omega^V, X^{(1)}, ..., X^{(s)}\right)
\]

for any \(1\)-forms \(\omega^{(1)}, ..., \omega^{(r)}\) from \(\mathcal{X}^*(T^k M)\) and any vector fields \(X^{(1)}, ..., X^{(s)}\) from \(\mathcal{X}(T^k M)\).

In the adapted cobasis \((1.3.8)\) and its dual basis \((1.4.1)\) a \(d\)-tensor field \(T\) has the components:

\[
T^{i_1...i_r}_{j_1...j_s}(x, y^{(1)}, ..., y^{(k)}) = T\left(\delta x^{i_1}, ..., \delta y^{(k)}_{j_s}, \frac{\delta}{\delta x^{j_1}}, ..., \frac{\delta}{\delta y^{(k)}_{j_s}}\right).
\]

Using the formulas \((1.3.10)\) and \((1.4.3')\) we deduce:

A transformation of coordinates \((1.2)\) on \(T^k M\) implies a transformation of the components of the \(d\)-tensor field \(T\) by the classical rule

\[
(1.6.1) \quad \bar{T}^{i_1...i_r}_{j_1...j_s} = \frac{\partial x^{i_1}}{\partial z^{i_1}} \cdots \frac{\partial x^{i_r}}{\partial z^{i_r}} \frac{\partial x^{k_1}}{\partial z^{k_1}} \cdots \frac{\partial x^{k_s}}{\partial z^{k_s}} T^{h_1...h_s}_{k_1...k_s}.
\]

But, this fact is possible only for the components of a \(d\)-tensor in the adapted basis. The components of \(T\) in the natural basis \(\left(\frac{\partial}{\partial x^{i_1}}, \frac{\partial}{\partial y^{(1)}_{i_1}}, ..., \frac{\partial}{\partial y^{(k)}_{i_1}}\right)\) and natural cobasis \((dx^i, dy^{(1)}_{i_1}, ..., dy^{(k)}_{i_1})\) have very complicated rule of transformations with respect to the changing of local coordinates \((1.1.2)\) on the manifold \(T^k M\).

If \(X \in \mathcal{X}(T^k M)\), its projections \(X^H, X^V, ..., X^V_k\) are \(d\)-vector fields and their components \(X_{(0)i}, X_{(a)i}, (a = 1, ..., k)\) are called also \(d\)-vector fields.

As an example, we have that \(z^{(1)i}, ..., z^{(k)i}\) from \((1.4.6)\) are \(d\)-vector fields. They are called the Liouville \(d\)-vector fields.

For a \(1\)-form \(\omega, \omega^H, \omega^V\) from \((1.4.8)\) are \(d\)-1-forms and their components \(\omega_{(0)i}, ..., \omega_{(a)i}\) from \((1.4.8')\) are \(d\)-covector fields.

Of course, the set of \(d\)-tensor fields with respect to ordinary operation of addition and tensor product determines a tensor algebra on the ring of functions \(F(T^k M)\).

An \(N\)-linear connection on the total space of \(k\)-accelerations bundle \(T^k M\) is a linear connection \(D\) on \(T^k M\) with the properties:

(1) \(D\) preserves by parallelism the horizontal distribution \(N\);

(2) The \(k\)-tangent structure \(J\) is absolutely parallel with respect to \(D\).

As a consequence of this definition, the following theorems hold, [94]:

**Theorem 1.6.1** A linear connection \(D\) on the manifold \(T^k M\) is an \(N\)-linear connection if and only if the following properties hold:

\[
(1.6.2) \quad \left(D_X Y^H\right)^V_{\alpha} = 0, \quad \left(D_X Y^V_{\alpha}\right)^H = 0, \\
\left(D_X Y^V_{\alpha}\right)^V_{\beta} = 0, \quad (\alpha \neq \beta, \alpha, \beta = 1, ..., k),
\]
(1.6.2') \[ D_X(JY^H) = JD_XY^H, \quad D_X(JY^{\alpha}) = JD_XY^{\alpha}, \quad (\alpha = 1, \ldots, k), \]

for any \( X, Y \in \mathcal{X}(T^kM) \).

If \( X \in \mathcal{X}(T^kM) \) is written in the form (1.3.13), for any \( Y \in \mathcal{X}(T^kM) \) and \( D \) an \( N \)-linear connection, we have:

(1.6.3) \[ D_XY = D_XH^Y + D_XV_1^Y + \cdots + D_XV_k^Y. \]

Let us denote:

\[ D_X^H = D_XH, \quad D_X^V_1 = D_XV_1, \ldots, D_X^V_k = D_XV_k. \]

Consequently, we can write:

(1.6.3') \[ D_XY = D_X^H Y + D_X^V_1 Y + \cdots + D_X^V_k Y. \]

The operators \( D_X^H, \ldots, D_X^V_k \) are not covariant derivations in the \( d \)-tensor algebra, since \( D_X^H f = X^H f \neq Xf \).

But they have similar properties with the covariant derivative.

So, if \( T \) is a tensor field of type \((r, s)\) we have:

(1.6.4) \[
(D_X^H T) \left( \omega^{(1)}, \ldots, \omega^{(r)}, X^{(1)}, \ldots, X^{(s)} \right) = X^H T \left( \omega^{(1)}, \ldots, \omega^{(r)}, X^{(1)}, \ldots, X^{(s)} \right) - \\
- T \left( D_X^H \omega^{(r)} + \ldots + D_X^V_k \omega^{(r)} \right) X^H X^{(s)} - \\
- \left( \omega^{(r)}, X^{(s)} \right) D_X^V_1 X^{(s)} - \cdots - T \left( \omega^{(r)}, X^{(s)} \right) D_X^H X^{(s)}.
\]

(1.6.5) \[
(D_X^V_\alpha T) \left( \omega^{(1)}, \ldots, \omega^{(r)}, X^{(1)}, \ldots, X^{(s)} \right) = X^V_\omega T \left( \omega^{(1)}, \ldots, \omega^{(r)}, X^{(1)}, \ldots, X^{(s)} \right) - \\
- T \left( D_X^V_\alpha \omega^{(r)} + \ldots + D_X^V_k \omega^{(r)} \right) X^V_\omega X^{(s)} - \\
- \left( \omega^{(r)}, X^{(s)} \right) D_X^V_1 X^{(s)} - \cdots - T \left( \omega^{(r)}, X^{(s)} \right) D_X^V_\alpha X^{(s)}.
\]

For instance, the formula (1.6.4) for a 1-form \( \omega \in \mathcal{X}^*(T^kM) \) leads to the following expressions of \( D_X^H \omega \) and \( D_X^V_\alpha \omega \):

(1.6.5) \[
(D_X^H \omega)(Y) = X^H \omega(Y) - \omega(D_X^H Y), \\
(D_X^V_\alpha \omega)(Y) = X^V_\omega \omega(Y) - \omega(D_X^V_\alpha Y), \quad (\alpha = 1, \ldots, k).
\]
In the adapted basis (1.3.13) an $N$-linear connection $D$ can be uniquely represented in the form, [94]:

\[
\begin{aligned}
D \frac{\delta}{\delta x^i} & = L^{m}_{ij} \frac{\delta}{\delta x^m}, \\
D \frac{\delta}{\delta y^{(\alpha)m}} & = L^{m}_{ij} \frac{\delta y^{(\alpha)m}}{x^m}, \quad (\alpha = 1, k), \\
D \frac{\delta}{\delta y^{(\beta)l}} & = C^{m}_{ij} \frac{\delta}{\delta x^m}, \\
D \frac{\delta}{\delta y^{(\alpha)m}} & = C^{m}_{ij} \frac{\delta y^{(\alpha)m}}{x^m}, \quad (\alpha, \beta = 1, k).
\end{aligned}
\]

The system of functions $(L^{m}_{ij}, C^{m}_{ij})$ represents the coefficients of $D$.

With respect to (1.1.2), $L^{m}_{ij}$ are transformed by the same rule as the coefficients of a linear connection defined on the base manifold $M$. Others coefficients $C^{m}_{ij}(\alpha) = 1, \ldots, k$ are transformed like $d$-tensors of type $(1, 2)$.

If $T$ is a $d$-tensor field of type $(r, s)$, represented in adapted basis in the form

\[
T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\delta}{\delta x^{i_1}} \otimes \cdots \otimes \frac{\delta}{\delta y^{(k)j_s}} \otimes dx^{j_1} \otimes \cdots \otimes dy^{(k)j_s},
\]

and $X = X^H = X^i(x, y^{(1)}, \ldots, y^{(k)}) \frac{\delta}{\delta x^i}$, then the $h$-covariant derivative $D^H_X T$, by means of (1.6.6), is as follows

\[
D^H_X T = X^m T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\delta}{\delta x^m} \otimes \cdots \otimes \frac{\delta}{\delta y^{(k)j_s}} \otimes dx^{j_1} \otimes \cdots \otimes dy^{(k)j_s},
\]

where

\[
T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\delta T^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\delta x^m} + T^{i_1}_{hm} T^{h i_2 \ldots i_r}_{j_1 \ldots j_s} + \cdots - T^{i_r}_{j_s m} T^{i_1 \ldots i_{r-1}}_{j_1 \ldots j_{s-1}}.
\]

The operator "\( \frac{\delta}{\delta y^{(k)j_s}} \)" will be called the $h$-covariant derivative, with the same denomination as $D^H_X$.

Consider the $v_\alpha$-covariant derivatives $D^H_{X^\alpha}$, for $X = X^i \frac{\delta}{\delta y^{(\alpha)i}}$, $(\alpha = 1, \ldots, k)$. Then, using (1.6.6) and (1.6.7) we have:

\[
D^H_{X^\alpha} T^{(\alpha)} = X^m T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \mid_m \frac{\delta}{\delta x^m} \otimes \cdots \otimes \frac{\delta}{\delta y^{(k)j_s}} \otimes dx^{j_1} \otimes \cdots \otimes dy^{(k)j_s},
\]

where

\[
T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \mid_m = \frac{\delta T^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\delta y^{(\alpha)j_s}} + C^{i_1 h}_{jm} \frac{\delta T^{h i_2 \ldots i_r}_{j_1 \ldots j_s}}{\delta y^{(\alpha)j_s}} + \cdots - C^{i_r h}_{jm} T^{i_1 \ldots i_{r-1}}_{j_1 \ldots j_{s-1}}, \quad (\alpha = 1, \ldots, k).
\]

The operators "\( \mid_m \)" in number of $k$ are called $v_\alpha$-covariant derivatives, with the same denominations as $D^H_{X^\alpha}$. Each of them has similar properties to those of $h$-covariant derivative. For instance
\[(fT:\ldots)^{(\alpha)} | \substack{m = \delta f \\ \delta y^{(\alpha)} m} T:\ldots + fT:\ldots)^{(\alpha)} | m, \]

\[\left( T:\ldots \otimes T:\ldots \right)^{(\alpha)} | m = T:\ldots^{(\alpha)} | m \otimes T:\ldots + T:\ldots \otimes T:\ldots^{(\alpha)} | m. \]

The operators \(\|\) and \((\alpha)\) commute with the operation of contraction, etc.

These operators applied to the Liouville \(d\)-vector fields \(z^{(1)i}, \ldots, z^{(k)i}\) determine 'the deflection tensors' of \(D\):

\[\begin{align*}
(\alpha) & \quad D_j^i = z^{(\alpha)i}_{ij}, \\
(\beta\alpha) & \quad d_{ij}^k = z^{(\beta)i}_{jk}.
\end{align*}\]

### 1.7 Torsion and Curvature

The tensor of torsion \(\mathcal{T}\) of an \(N\)-linear connection \(D\):

\[\mathcal{T}(X, Y) = D_X Y - D_Y X - [X, Y], \quad \forall X, Y \in \mathcal{X}(T^k M), \]

for \(X = X^H + X^{V_i} + \cdots + X^{V_k}, Y = Y^H + Y^{V_i} + \cdots + Y^{V_k}\), uniquely determines the following vector fields:

\[\begin{align*}
(1.7.2) \quad & \mathcal{T}(X^H, Y^H), \mathcal{T}(X^H, Y^{V_\alpha}), \mathcal{T}(X^{V_\alpha}, Y^{V_\beta}), (\alpha, \beta = 1, \ldots, k).
\end{align*}\]

Therefore:

The tensor of torsion \(\mathcal{T}\) of the \(N\)-linear connection \(D\) is well determined by the following components, which are \(d\)-tensor fields of type \((1,2)\):

\[\mathcal{T}(X^H, Y^H) = h \mathcal{T}(X^H, Y^H) + \sum_{\alpha=1}^{k} v_\alpha \mathcal{T}(X^H, Y^{V_\alpha}), \]

\[\begin{align*}
(1.7.3) \quad & \mathcal{T}(X^H, Y^{V_\beta}) = h \mathcal{T}(X^H, Y^{V_\beta}) + \sum_{\alpha=1}^{k} v_\alpha \mathcal{T}(X^{V_\alpha}, Y^{V_\beta}), \\
& \mathcal{T}(X^{V_\alpha}, Y^{V_\beta}) = \sum_{\gamma=1}^{k} v_\gamma \mathcal{T}(X^{V_\alpha}, Y^{V_\beta}).
\end{align*}\]

It is not difficult to prove that \(h \mathcal{T}(X^{V_\alpha}, Y^{V_\beta}) = 0, \forall \alpha, \beta = 1, \ldots, k. \) The expressions of \(d\)-tensors of torsion are the following:
Geometry of the $k$-Tangent Bundle $T^kM$

\[ hT(X^H, Y^H) = D^H_X Y^H - D^H_Y X^H - [X^H, Y^H]^H, \]

\[ v_\alpha T(X^H, Y^H) = [X^H, Y^H]^{V\alpha}, \]

\[ hT(X^H, Y^{V\beta}) = -\nabla_{Y^{V\beta}} X^H - [X^H, Y^{V\beta}]^H, \]

\[ v_\alpha T(X^H, Y^{V\beta}) = (\nabla^H_X Y^{V\beta})^{V\alpha} - [X^H, Y^{V\beta}]^{V\alpha}, \]

\[ v_\gamma T(X^{V\alpha}, Y^{V\beta}) = v_\gamma \left( \nabla^{V\alpha}_X Y^{V\beta} - \nabla_{Y^{V\beta}} X^{V\gamma} - [X^{V\alpha}, Y^{V\beta}] \right). \]

In the adapted basis we set:

\[ hT \left( \frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^j} \right) = T^i_{j(h)} \frac{\delta}{\delta x^i}, \]

\[ v_\alpha T \left( \frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^j} \right) = T^i_{j(h)} \frac{\delta}{\delta x^i}, \]

\[ hT \left( \frac{\delta}{\delta y^{(\alpha)_h}}, \frac{\delta}{\delta y^{(\beta)_j}} \right) = T^i_{j(h)} \frac{\delta}{\delta y^{(\alpha)_i}} \]

Using (1.7.4) and (1.6.6) as well as the following expressions of the Lie brackets, we can calculate the components of $d$-tensors of torsion.

The Lie brackets of the vector fields of the adapted basis are given by:

\[ \left[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^h} \right] = R^i_{j(h)} \frac{\delta}{\delta y^{(1)i}} + \cdots + R^i_{j(k)} \frac{\delta}{\delta y^{(k)i}}, \]

\[ \left[ \frac{\delta}{\delta y^{(\alpha)_h}}, \frac{\delta}{\delta y^{(\beta)_j}} \right] = B^i_{j(h)} \frac{\delta}{\delta y^{(1)i}} + \cdots + B^i_{j(k)} \frac{\delta}{\delta y^{(k)i}}, \]

\[ \left[ \frac{\delta}{\delta y^{(\alpha)}}, \frac{\delta}{\delta y^{(\beta)h}} \right] = C^i_{j(h)} \frac{\delta}{\delta y^{(1)i}} + \cdots + C^i_{j(k)} \frac{\delta}{\delta y^{(k)i}}, \]

where the coefficients $R, B, C$ can be calculated by means of the coefficients of the nonlinear connection $N_i$, [94].

We remark the following $d$-tensor of torsion

\[ T^i_{jk} = L^i_{jk} - L^i_{k}, \]

\[ T^i_{jk} = C^i_{jk} - C^i_{kj}, \quad (\alpha = 1, \ldots, k). \]
For simplicity we denote

\[ T^i_{jk} = T^i_{jk}^{(0)}, \quad T^i_{jk} = S^i_{jk}^{(a)} \]

The notion of curvature can be investigated by the same method. The curvature tensor \( R \) of the \( N \)-linear connection \( D \) is given by

\[ R(X, Y)Z = (D_X D_Y - D_Y D_X)Z - D_{[X, Y]}Z, \quad \forall X, Y, Z \in \mathcal{X}(T^k M). \]

Using the formula (1.6.2) we obtain

\[ J(R(X, Y)Z) = R(X, Y)JZ, \quad J^k(R(X, Y)Z) = R(X, Y)J^kZ. \]

Setting \( Z^\alpha = J^\alpha Z^H \) we have

\[ R(X, Y)Z^\alpha = J^\alpha (R(X, Y)Z^H), \quad (\alpha = 1, ..., k). \]

The essential components of the curvature tensor field \( R \) are \( R(X, Y)Z^H \). It has the properties:

\[ v_\beta (R(X, Y)Z^H) = 0, \quad h (R(X, Y)Z^V) = 0, \]

\[ v_\beta (R(X, Y)Z^V) = 0, \quad (\alpha \neq \beta, \alpha, \beta = 1, ..., k). \]

Thus, the curvature tensor \( R \) of the \( N \)-linear connection \( D \) gives rise to the \( d \)-vector fields:

\[ R(X^H, Y^H)Z^H = [D^H_X, D^H_Y]Z^H - D^H_{[X^H, Y^H]}Z^H - \sum_{\gamma=1}^{k} D^V_{[X^H, Y^H]}Z^H, \]

\[ R(X^V, Y^H)Z^H = [D^V_X, D^H_Y]Z^H - D^H_{[X^V, Y^H]}Z^H - \sum_{\gamma=1}^{k} D^V_{[X^V, Y^H]}Z^H, \]

\[ R(X^V, Y^V)Z^H = [D^V_X, D^V_Y]Z^H - \sum_{\gamma=1}^{k} D^V_{[X^V, Y^V]}Z^H, \quad (\alpha, \beta = 1, ..., k; \beta \leq \alpha). \]

The \( d \)-tensor fields (1.7.10) are obtained applying the operators \( J, J^2, ..., J^k \) and taking into account \( J^\gamma Z^H = Z^V, \quad (\gamma = 1, ..., k). \)

In the applications it is suitable to consider the equalities (1.7.10) as Ricci identities, [94].

The local expressions of \( d \)-tensors of curvature in adapted basis are:

\[ R \left( \frac{\delta}{\delta x^m}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right) = R_{h}^{i} j m \frac{\delta}{\delta x^l}, \]

\[ R \left( \frac{\delta}{\delta y^{(a)m}}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right) = P_{(a)}^{i} h j m \frac{\delta}{\delta x^l}, \]

\[ R \left( \frac{\delta}{\delta y^{(b)m}}, \frac{\delta}{\delta y^{(a)m}}, \frac{\delta}{\delta x^k} \right) = S_{(b)}^{i} (a) h j m \frac{\delta}{\delta x^l}. \]
Note that the other $d$-tensors of curvature (1.7.10) have the same components. The $d$-tensors (1.7.11) are called also $d$-tensors of curvature. They have the expressions:

\[(1.7.12)\]
\[R^i_{\ jh} = \delta L^i_{\ hj} - \delta L^i_{\ hm} + L^p_{\ hj} L^i_{\ pm} - L^p_{\ hm} L^i_{\ pj} + \sum_{\gamma=1}^{k} C^i_{\ hm} R^p_{\ jm},\]

\[P^i_{\ (\alpha) \ jh} = \frac{\delta L^i_{\ hj}}{\delta y^{(\alpha)m}} - \delta C^i_{\ jm} + L^p_{\ hj} C^i_{\ pm} - C^p_{\ jm} L^i_{\ pj} + \sum_{\gamma=1}^{k} C^i_{\ hm} B^p_{\ jm},\]

\[S^i_{\ (\beta) \ jm} = \frac{\delta C^i_{\ hj}}{\delta y^{(\beta)m}} - \delta C^i_{\ hm} + L^p_{\ hj} C^i_{\ pm} - C^p_{\ hm} C^i_{\ pj} + \sum_{\gamma=1}^{k} C^i_{\ hp} C^p_{\ jm}.\]

The connection 1-forms of the $N$-linear connection $D$ given in ch. 7, §7 of the book [94] are:

\[(1.7.13)\]
\[\omega^i_{\ j} = L^i_{\ hj} dx^h + C^i_{\ hm} \delta y^{(1)h} + \cdots + C^i_{\ jh} \delta y^{(k)h}.\]

Therefore, the structure equations of $D$ are given by the following theorem, [94]:

**Theorem 1.7.1** The structure equations of an $N$-linear connection $D$ on the manifold $T^kM$ are given by:

\[(1.7.14)\]
\[d(dx^i) - dx^m \wedge \omega^i_{\ m} = -^0\Omega^i,\]
\[d(\delta y^{(\alpha)i}) - \delta y^{(\alpha)m} \wedge \omega^i_{\ m} = -^\alpha\Omega^i,\]
\[d\omega^i_{\ j} - \omega^m_{\ j} \wedge \omega^i_{\ m} = -^0\Omega^i_{\ j},\]
\[d\omega^i_{\ j} - \omega^m_{\ j} \wedge \omega^i_{\ m} = -^\alpha\Omega^i_{\ j},\]

where the 2-forms of torsion $\Omega^i$, $\Omega^i$ are
\( \Omega^i = \frac{1}{2} T^i_{jh} dx^j \wedge dx^h + C^i_{jh} dx^j \wedge \delta y^{(1)h} + \cdots + C^i_{jk} dx^j \wedge \delta x^{(k)h}, \)

\[ (1.7.15) \]

\( \Omega^i = \frac{1}{2} R^i_{jkh} dx^j \wedge dx^h + \sum_{\gamma=1}^{k} B^i_{jhn} dx^j \wedge \delta y^{(\gamma)h} + \cdots + \sum_{\gamma=1}^{k} C^i_{jhn} \delta y^{(\gamma)h} \wedge \delta y^{(\gamma)h} \]

\[ (1.7.16) \]

and where the 2-forms of curvature \( \Omega^i_{\cdot j} \) are:

\( \Omega^i_{\cdot j} = \frac{1}{2} R^i_{j pq} dx^p \wedge dx^q + \sum_{\gamma=1}^{k} P^i_{j pq} dx^p \wedge \delta y^{(\gamma)q} \)

\[ (1.7.16) \]

The Bianchi identities of \( D \) can be derived from (1.7.14) applying the operator of exterior differentiation and calculating \( d \Omega^i \), \( d \Omega^{i}_{\cdot j} \) from (1.7.15) and (1.7.16) modulo the system (1.7.14).
Chapter 2

Lagrange Spaces of Higher Order

We define the notion of higher order Lagrangian and the notion of integral of action. We investigate the variational problem for autonomous Lagrangians deriving the Euler-Lagrange equations. The notion of Lagrange space of order $k$ is introduced by means of regular nondegenerate Lagrangian defined on the total space of the $k$-accelerations bundle $T^kM$. In this case the Craig-Synge equations determine a $k$-semispray, which depend only on the considered Lagrangian. Therefore the geometry of the Lagrange space of order $k$ is based on the mentioned $k$-semispray, \cite{94}.

2.1 Lagrangians of Order $k$

Let us consider the $k$-accelerations bundle $(T^kM, \pi^k, M)$. In Analytical Mechanics $M$ is the space of configurations of a physical system. A point $x = (x^i) \in U \subset M$ is called a configuration, a mapping $c : t \in I \to (x^i(t)) \in U$ is a law of moving (evolution), $t$ is called time, a couple $(t, x)$ is an event and

$$
\left( \frac{dx^i}{dt}, \frac{1}{2!} \frac{d^2x^i}{dt^2}, \ldots, \frac{1}{k!} \frac{d^kx^i}{dt^k} \right)
$$

are the velocity and generalized accelerations of order 1, 2, $\ldots$, $k - 1$ (respectively). The factors $\frac{1}{h!}$ ($h = 1, \ldots, k$) are introduced for convenience.

Throughout this book we omit the word generalized and say shortly, accelerations of order $h$ for $\frac{1}{h!} \frac{d^h x^i}{dt^h}$, $h = 1, \ldots, k$. A law a moving $c : t \in I \to (x^i(t)) \in U \subset M$ will be called a curve parametrized by time $t$. As usual the curve $	ilde{c} : t \in I \to \tilde{c}(t) \in (\pi^k)^{-1}(U) \subset T^kM$,

$$
(2.1.1) \quad \tilde{c} : t \in I \to \left( x^i(t), \frac{dx^i}{dt}, \frac{1}{2!} \frac{d^2x^i}{dt^2}, \ldots, \frac{1}{k!} \frac{d^kx^i}{dt^k} \right)
$$

27
is the extension of the curve $c$ to the total space $T^kM$ of the $k$-acceleration bundle.

**Definition 2.1.1** A Lagrangian of order $k$, $(k \in \mathbb{N}^*)$ is a mapping $L : T^kM \rightarrow \mathbb{R}$.

This means that $L$ is a real function $L(x, y^{(1)}, \ldots, y^{(k)})$ on $T^kM$. In the other words, with respect to a change of local coordinates on $T^kM$ (1.1.2), we have

$$L(\tilde{x}, \tilde{y}^{(1)}, \ldots, \tilde{y}^{(k)}) = L(x, y^{(1)}, \ldots, y^{(k)}).$$

The previous definition is given for autonomous Lagrangians. A similar definition we have for nonautonomous Lagrangians. They are mappings $L : (t, x, y^{(1)}, \ldots, y^{(k)}) \in \mathbb{R} \times T^kM \rightarrow L(t, x, y^{(1)}, \ldots, y^{(k)}) \in \mathbb{R}$, which are real functions with respect to a change of coordinates on $\mathbb{R} \times T^kM$, $(t, x, y^{(1)}, \ldots, y^{(k)}) \rightarrow (\tilde{t}, \tilde{x}, \tilde{y}^{(1)}, \ldots, \tilde{y}^{(k)})$, where $t = \tilde{t}$.

For us it is preferable to study the autonomous Lagrangians, because the notion of Lagrange space of order $k$ is a geometrical one. But, one sees that the nonautonomous Lagrangians can be geometrized by means of the notion of rheonomic Lagrange space of order $k$. Such kind of geometry can be constructed by the same methods as in the autonomous case.

In the following we investigate the Lagrangians $L(x, y^{(1)}, \ldots, y^{(k)})$ which have the property (2.1.2).

A Lagrangian of order $k$, $L : T^kM \rightarrow \mathbb{R}$ is called differentiable if it is of $C^\infty$-class on $T^kM$ and continuous on the null section of the projection $\pi^k : T^kM \rightarrow M$.

The Hessian of a differentiable Lagrangian $L$, with respect to the variables $y^{(k)i}$ on $\widetilde{T^kM}$ is the matrix $||g_{ij}||$, where

$$(2.1.3) \quad g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}}.$$

One can prove, [94] that $g_{ij}$ is a $d$-tensor field, on the manifold $\widetilde{T^kM}$. This is covariant of order 2 and symmetric.

This property determines the geometrical covariance of the Hessian of $L$.

If

$$(2.1.4) \quad \text{rank } ||g_{ij}|| = n, \quad \text{on } \widetilde{T^kM}$$

we say that $L(x, y^{(1)}, \ldots, y^{(k)})$ is a regular (or nondegenerate) Lagrangian.

The existence of the regular Lagrangians of order $k$ is assured by the following example.

Let $\gamma_{ij}(x)$ a Riemannian tensor field on the base manifold $M$ and $z^{(k)i}$ the Liouville vector field on $\widetilde{T^kM}$ determined by the prolongation of order $k$ of the Riemannian spaces $\mathcal{R}^n = (M, \gamma_{ij}(x))$, an arbitrary covector field $b_i$ on $T^{k-1}M$ and a function $b$ on $T^{k-1}M$. Then the Lagrangian of order $k$

$$(2.1.5) \quad L(x, y^{(1)}, \ldots, y^{(k)}) = \gamma_{ij}(x)z^{(k)i}z^{(k)j} + b_i(x, y^{(1)}, \ldots, y^{(k-1)})z^{(k)i} + b(x, y^{(1)}, \ldots, y^{(k-1)})$$
is a regular Lagrangian on $T^k M$. We have $g_{ij}(x, y^{(1)}, ..., y^{(k)}) = \gamma_{ij}(x)$.

Let us consider the scalar fields

\begin{equation}
I^1(L) = \mathcal{L}_{\Gamma} L, ..., I^k(L) = \mathcal{L}_{\Gamma^k} L,
\end{equation}

where $^1 \Gamma, ..., ^k \Gamma$ are the Liouville vector fields on $T^k M$ and $\mathcal{L}$ is the Lie operator of derivation. $I^1(L), ..., I^k(L)$ will be called the main invariants of the Lagrangian $L$.

For a smooth parametrized curve $c : [0, 1] \to M$ represented in a domain of a local chart by $x^i = x^i(t), t \in [0, 1]$. The parameter $t$ is called time and its extension to $\tilde{T}^k M$ is $\tilde{c}$, given by (2.1.1).

The integral of action for the differentiable Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$ along curve $c$ is defined by

\begin{equation}
I(c) = \int_0^1 L \left( x(t), \frac{dx}{dt}, \frac{1}{2!} \frac{d^2 x}{dt^2}, \cdots, \frac{1}{k!} \frac{d^k x}{dt^k} \right) dt.
\end{equation}

Ones proves, [94] the following important results:

**Theorem 2.1.1** The necessary conditions for the integral of action be independent on the parametrization of the curve $c$ are

\begin{equation}
I^1(L) = \cdots = I^{k-1}(L) = 0, I^k(L) = L.
\end{equation}

The conditions (2.1.8) are called the Zermello conditions, [69, 94].

**Theorem 2.1.2** If the differentiable Lagrangian $L$ of order $k, k > 1$, satisfies the Zermello conditions (2.1.8), then it is degenerate (singular), i.e.

\begin{equation}
\text{rank } ||g_{ij}(x, y^{(1)}, ..., y^{(k)})|| < n, \text{ on } \tilde{T}^k M.
\end{equation}

**2.2 Variational Problem**

The variational problem concerning the functional $I(c)$ from (2.1.7) was studied in the book [94], ch. 8. So we shall present here only the corresponding results.

**Theorem 2.2.1** In order for the curve $c : t \in [0, 1] \to (x^i(t)) \in U \subset M$ to be an extremal curve for the integral of action $I(c)$ it is necessary that the following Euler-Lagrange equations hold:

\begin{equation}
\hat{E}_i(L) := \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^{(1)i}} + \cdots + (-1)^k \frac{1}{k!} \frac{d^k}{dt^k} \frac{\partial L}{\partial y^{(k)i}} = 0,
\end{equation}

\begin{equation}
y^{(1)i} = \frac{dx^i}{dt}, \cdots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}.
\end{equation}
An important remark regard the geometrical meaning of the system of functions $\tilde{E}_i (L)$. The following property holds:

Every $\tilde{E}_i (L)$ from (2.2.1) determines a $d$-covector field along curve $c$. This means that, with respect to a change of local coordinates on the manifold $T^k M$, $\tilde{E}_i (L)$ obeys the transformation

$$ (2.2.2) \quad \tilde{E}_i (L) \frac{\partial x^j}{\partial x^i} = \tilde{E}_j (L). $$

Let $c : t \in [0, 1] \to (x^i(t)) \in U \subset M$ be a smooth curve (or a law of moving) parametrized by the time $t$ and $V^i (x(t))$ a differentiable vector field on $c$.

The relations between the operators $d$ are scalar fields. For $\dot{V} = \frac{dx^i}{dt}$, then $\frac{dV}{dt} = \frac{dV^i}{dt} \frac{\partial}{\partial x^i}$ and for any Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$, $\frac{dV}{dt}$ is a scalar field.

The action of the $k$-tangent structure $J$ (cf. §2.4, ch. 1) on the operator $\frac{dV}{dt}$ leads to $k$ new operators:

$$ (2.2.3) \quad \begin{cases} \dot{x} = x^i(t), & t \in [0, 1], \\ \dot{y}^{(1)} = V^i (x(t)), & 2\dot{y}^{(2)} = \frac{1}{1!} \frac{dV^i}{dt}, ..., k\dot{y}^{(k)} = \frac{1}{(k-1)!} \frac{d^{k-1}V^i}{dt^{k-1}}, \end{cases} $$

is a section of the projection $\pi^k : T^k M \to M$.

It is not difficult to see that the operator

$$ (2.2.4) \quad \frac{dV}{dt} = V^i \frac{\partial}{\partial x^i} + \frac{dV^i}{dt} \frac{\partial}{\partial y^{(1)i}} + \cdots + \frac{1}{k!} \frac{d^kV^i}{dt^k} \frac{\partial}{\partial y^{(k)i}}, $$

is invariant with respect to the coordinate transformations (1.1.2).

If $V = \frac{dx^i}{dt}$, then $\frac{dV}{dt} = \frac{d}{dt}$ and for any Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$, $\frac{dV}{dt}$ is a scalar field.

The action of the $k$-tangent structure $J$ (cf. §2.4, ch. 1) on the operator $\frac{dV}{dt}$ leads to $k$ new operators:

$$ (2.2.5) \quad \begin{cases} I_V^k = J \left( \frac{dV}{dt} \right), & I_V^{k-1} = J \left( I_V^k \right), ..., I_V^1 = J \left( I_V^2 \right), \ 0 = J \left( I_V^1 \right), \end{cases} $$

where

$$ (2.2.5') \quad \begin{cases} \dot{I}_V^k = V^i \frac{\partial}{\partial y^{(1)i}} + \frac{dV^i}{dt} \frac{\partial}{\partial y^{(2)i}} + \cdots + \frac{1}{(k-1)!} \frac{d^{k-1}V^i}{dt^{k-1}} \frac{\partial}{\partial y^{(k)i}}, \end{cases} $$

But the previous operators are vector fields. Consequently, $I_V^1 (L), ..., I_V^k (L)$ are scalar fields. For $V = \frac{dx^i}{dt}$ they coincide with the main invariants $I^1 (L), ..., I^k (L)$.

The relations between the operators $\frac{dV}{dt}, I_V^1, ..., I_V^k$ are expressed cf. [94] by:

**Theorem 2.2.2** The following identities hold:

$$ (2.2.6) \quad \frac{dV}{dt} = V^i \tilde{E}_i (L) + \frac{d}{dt} I_V^k (L) - \frac{1}{2!} \frac{d^2}{dt^2} I_V^{k-1} (L) + \cdots + \big( -1 \big)^{k-1} \frac{1}{k!} \frac{d^k}{dt^k} I_V^1 (L). $$
Besides of the covector field \( E_i (L) \), Craig and Synge introduced other covectors, important in the variational problem of the integral of action. They are denoted by \( E_i, \ldots, E_i \) and are provided by:

**Lemma 2.2.1**

For any differentiable Lagrangian \( L(x, y^{(1)}, \ldots, y^{(k)}) \) and any differentiable function \( \Phi(t) \) we have:

\[
E_i (\Phi L) = \Phi E_i (L) + \frac{d\Phi}{dt} E_i (L) + \cdots + \frac{d^k\Phi}{dt^k} E_i (L),
\]

where \( E_i (L), \ldots, E_i (L) \) are \( d \)-covector fields. They are expressed by the actions on \( L \) of the following operators:

\[
E_i = \frac{\partial}{\partial x^i} - \frac{d}{dt} \frac{\partial}{\partial y^{(1)}}, \ldots + (-1)^{k} \frac{1}{k!} \frac{d^k}{dt^k} \frac{\partial}{\partial y^{(k)}},
\]

\[
E_i = \sum_{\alpha=1}^{k} (-1)^{\alpha} \frac{1}{\alpha!} \frac{d^{\alpha-1}}{dt^{\alpha-1}} \frac{\partial}{\partial y^{(\alpha)}},
\]

\[
E_i = \sum_{\alpha=2}^{k} (-1)^{\alpha} \frac{1}{\alpha!} \frac{d^{\alpha-2}}{dt^{\alpha-2}} \frac{\partial}{\partial y^{(\alpha)}},
\]

\[
E_i = (-1)^{k} \frac{1}{k!} \frac{\partial}{\partial y^{(k)}}.
\]

In the light of the above results we get:

**Theorem 2.2.3** For any differentiable Lagrangians \( L(x, y^{(1)}, \ldots, y^{(k)}) \) and any function \( F(x, y^{(1)}, \ldots, y^{(k-1)}) \) the following properties hold:

\[
0 = E_i (L + \frac{dF}{dt}) = E_i (L),
\]

\[
0 = E_i \left( \frac{dF}{dt} \right) = 0, \ E_i \left( \frac{dF}{dt} \right) = - E_i (F), \ldots, E_i \left( \frac{dF}{dt} \right) = - \frac{1}{k-1} E_i (F).
\]

A consequence of the property (2.2.9) is as follows.

**Theorem 2.2.4** The integral of action

\[
I(c) = \int_0^1 L dt, \quad I'(c) = \int_0^1 \left( L + \frac{dF}{dt} \right) dt
\]

determine the same Euler-Lagrange equations, \( F \) depending on \( (x, y^{(1)}, \ldots, y^{(k-1)}) \).
2.3 Higher Order Energies

In the book [94] we introduce the notion of higher order energies of a Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$.

**Definition 2.3.1** We call energies of order $k$, $k - 1$, ..., 1 of the differentiable Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$ the following invariants:

$$
E_k(L) = I^k(L) - \frac{1}{2!} \frac{dI^{k-1}(L)}{dt} + \cdots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-1}I^{1}(L)}{dt^{k-1}} - L,
$$

$$(2.3.1)
E^{k-1}(L) = -\frac{1}{2!} I^{k-1}(L) + \frac{1}{3!} \frac{dI^{k-2}(L)}{dt} + \cdots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-2}I^{1}(L)}{dt^{k-2}},
$$

$$(2.3.2)
E^1(L) = (-1)^{k-1} \frac{1}{k!} I^{1}(L).
$$

As we shall see, the energies $E_k(L)$, ..., $E^1(L)$ are involved in a Noether theory of symmetries of the higher order Lagrangians.

The next theorem is well known:

**Theorem 2.3.1** For any differentiable Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$ the following identity holds

$$
\frac{dE^k(L)}{dt} = -\frac{\partial}{\partial x^i} (L) \frac{dx^i}{dt}.
$$

An immediate consequence of the previous theorem is the following law of conservation:

**Theorem 2.3.2** For any differentiable Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$ the energy of order $k$, $E^k(L)$ is conserved along every extremal curve of the Euler-Lagrange equations $E_i (L) = 0$.

Theorem 2.2.4 says that the integrals of actions $I(c)$ and $I'(c)$ from (2.2.11) determine the same Euler-Lagrange equations, if $F$ is a Lagrangian with the property $\frac{\partial F}{\partial y^{(k)i}} = 0$.

**Definition 2.3.2** A symmetry of a differentiable Lagrangian $L(x, y^{(1)}, ..., y^{(k)})$ is a $C^\infty$-diffeomorphism $\varphi : M \times \mathbb{R} \rightarrow M \times \mathbb{R}$ which preserves the variational principle of the integral of action $I(c)$ from (2.1.7).

One can consider the notion of local symmetry of the Lagrangian $L$, taking $\varphi$ as local diffeomorphism. If $U \times (a, b)$ is a domain of a local chart on the manifold $M \times \mathbb{R}$, then we can express an infinitesimal diffeomorphism $\varphi$ in the form

$$
x^i = x^i + \varepsilon V^i(x, t), \quad (i = 1, ..., n)
$$

and

$$
t' = t + \varepsilon \tau(x, t),
$$

(2.3.3)
where \((V', \tau)\) is a vector field on \(U \times (a, b)\) defined along curve 
\(c : t \in [0, 1] \rightarrow (x'(t), t) \in U \times (a, b)\) and \(\varepsilon\) is a real number, sufficiently small in
absolute value, so that \(\text{Im} \ \varphi \subset U \times (a, b)\).

The infinitesimal transformation (1.3.3) is a symmetry for the differentiable
Lagrangian \(L\) if and only if for any differentiable function
\(F(x, y^{(1)}, \ldots, y^{(k-1)})\) the following equation holds:

\[
F \left( x, \frac{d x}{d t}, \ldots, \frac{1}{k!} \frac{d^k x}{d t^k} \right) dt = \left\{ L \left( x, \frac{d x}{d t}, \ldots, \frac{1}{k!} \frac{d^k x}{d t^k} \right) + \right. \\
+ \left. F \left( x, \frac{d x}{d t}, \ldots, \frac{1}{(k-1)!} \frac{d^{k-1} x}{d t^{k-1}} \right) \right\} dt
\]

One proves the following Nöther theorem, [94]:

**Theorem 2.3.3** For any infinitesimal symmetry (1.3.3) of a Lagrangian
\(L(x, y^{(1)}, \ldots, y^{(k)})\) and for any function \(\Phi(x, y^{(1)}, \ldots, y^{(k-1)})\), the function:

\[
\mathcal{E}^k(L, \Phi) = I^k_V(L) - \frac{d}{2!} I^k_{V'}(L) + \cdots + (-1)^{k-1} \frac{d^{k-1}}{k!} I^k_{V^{(k-1)}}(L) - \right.
\]

\[
\left. \tau \mathcal{E}^k \right) - \frac{d \tau}{dt} \mathcal{E}^{k-1}(L) + \cdots + (-1)^k \frac{d^{k-1} \tau}{dt^{k-1}} \mathcal{E}^1(L) - \Phi.
\]

is conserved along extremal curves of the Euler-Lagrange equations
\(\mathcal{E}_1(L) = 0\).

In particular, if the Zermello conditions (2.1.8) are satisfied, then the energies
\(\mathcal{E}^1(L), \ldots, \mathcal{E}^k(L)\) vanish and the previous theorem has a simpler form.

### 2.4 Jacobi-Ostrogradski Momenta

We introduce the Jacobi-Ostrogradski momenta and the Hamilton - Jacobi
Ostrogradski equations. The main results on these are given without proofs,
[94].

Consider the energy of order \(k\), \(\mathcal{E}^k(L)\) of the Lagrangian \(L\) from (2.3.1).

Noticing that \(\mathcal{E}^k(L)\) is a polynomial function of degree one in \(\frac{dx^i}{dt}, \ldots, \frac{d^k x^i}{d t^k}\) we
we can write:

\[
(2.4.1) \quad \mathcal{E}^k(L) = p_{(1)i} \frac{dx^i}{dt} + p_{(2)i} \frac{d^2 x^i}{dt^2} + \cdots + p_{(k)i} \frac{d^k x^i}{dt^k} - L,
\]

where

\[
p_{(1)i} = \frac{\partial L}{\partial y^{(1)i}} - \frac{1}{2!} \frac{d}{dt} \frac{\partial L}{\partial y^{(2)i}} + \cdots + (-1)^{k-1} \frac{1}{k!} \frac{d^{k-1}}{dt^{k-1}} \frac{\partial L}{\partial y^{(k)i}},
\]

\[
p_{(2)i} = \frac{1}{2!} \frac{\partial L}{\partial y^{(2)i}} - \frac{1}{3!} \frac{d}{dt} \frac{\partial L}{\partial y^{(3)i}} + \cdots + (-1)^{k-2} \frac{1}{k!} \frac{d^{k-2}}{dt^{k-2}} \frac{\partial L}{\partial y^{(k)i}},
\]

\[
p_{(k)i} = \frac{1}{k!} \frac{\partial L}{\partial y^{(k)i}}.
\]
p_{(1)i}, ..., p_{(k)i} \) are called the Jacobi-Ostrogradski momenta.

M. de Léon and others, \([73, 74]\) established the following important result.

**Theorem 2.4.1** Along extremal curves of Euler-Lagrange equations, \( \dot{E}_i (L) = 0 \), the following Hamilton-Jacobi-Ostrogradski equations hold:

\[
\begin{align*}
\frac{\partial \mathcal{E}^k(L)}{\partial p_{(\alpha i)}} &= \frac{d^\alpha x^i}{dt^\alpha}, \quad (\alpha = 1, ..., k), \\
\frac{\partial \mathcal{E}^k(L)}{\partial x^i} &= -\frac{dp_{(1)i}}{dt}, \\
\frac{\partial \mathcal{E}^k(L)}{\partial y^{(\alpha i)}} &= -\alpha! \frac{dp_{(\alpha+1)i}}{dt}, \quad (\alpha = 1, ..., k - 1).
\end{align*}
\]

The Jacobi-Ostrogradski momenta \( p_{(1)i}, ..., p_{(k)i} \) allow the introduction of the 1-forms:

\[
\begin{align*}
p_{(1)} &= p_{(1)i} dx^i + p_{(2)i} dy^{(1)i} + \cdots + p_{(k)i} dy^{(k-1)i}, \\
p_{(2)} &= p_{(2)i} dx^i + p_{(3)i} dy^{(1)i} + \cdots + p_{(k)i} dy^{(k-2)i}, \\
&\hspace{2cm} \vdots \\
p_{(k)} &= p_{(k)i} dx^i.
\end{align*}
\]

The following properties hold:

\[ J^* p_{(1)} = p_{(2)}, \quad \ldots, \quad J^* p_{(k-1)} = p_{(k)}. \]

### 2.5 Higher Order Lagrange Spaces

The notion of Lagrange space of order \( k \) is a natural extension of that of classical Lagrange space \( L^n = (M, L(x, y)) \). It was introduced by the author of this monograph, in the papers quoted in his book \([94]\). It was introduced by the author in the paper quoted in his book *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics*, (Kluwer, FTPH no. 82) entirely devoted to this subject.

Here we give, without proofs, the main results from the geometry of higher order Lagrange spaces, \([94]\).

We call a Lagrange space of order \( k \) a pair \( L^{(k)} = (M, L) \) formed by a real \( n \)-dimensional manifold \( M \) and a differentiable Lagrangian of order \( k \), \( L : (x, y^{(1)}, ..., y^{(k)}) \in T^k M \rightarrow L(x, y^{(1)}, ..., y^{(k)}) \in \mathbb{R} \) for which the Hessian with the entries

\[
g_{ij}(x, y^{(1)}, ..., y^{(k)}) = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(i)} \partial y^{(j)}}, \quad \text{on } \tilde{T^k M},
\]

has the property

\[ \text{rank } ||g_{ij}|| = n. \]
and the quadratic form $\Psi = g_{ij} \xi^i \xi^j$ on $\wedge^k M$ has constant signature.

$L$ is called the fundamental function and $g_{ij}$ the fundamental (or metric) tensor field of the space $L^{(k)n}$.

The geometry of the pair $(T^k M, L(x, y^{(1)}, \ldots, y^{(k)}))$ is called the geometry of the space $L^{(k)} = (M, L)$.

Notice that the geometry of Lagrange space of order $k$ is not coincident with the geometrization of the Lagrangians of order $k$, $L(x, y^{(1)}, \ldots, y^{(k)})$, which could be degenerate, i.e. $\text{rank } |g_{ij}| < n$.

We shall study this geometry using the methods suggested by the Higher Order Lagrangian Mechanics [35] and by the geometry of higher order Finsler spaces [95]. Consequently, we determine a canonical semispray $S$ and derive the main geometrical object field of the space $L^{(k)n}$ by means of $S$.

Consider the integral of action of the Lagrangian $L$, the fundamental function of the Lagrange space of order $k$, $L^{(k)n} = (M, L)$ and determine the covector fields $0 E_i(L), \ldots, k-1 E_i(L), k E_i(L)$.

Then $E_i(L) = 0$ are just the Euler-Lagrange equations. Thus, $\mathcal{E}^k(L), \ldots, \mathcal{E}^{1}(L)$ from (2.3.1) give us the energies of the space $L^{(k)n}$. Theorems 2.3.1 and 2.3.2 can be applied and Theorem 2.3.3 gives the infinitesimal symmetries of the considered space.

The following result is known, [94]:

**Theorem 2.5.1** The equations $g^{ij} E_i(L) = 0$ determine the $k$-semispray

\[ S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k + 1) G^i \frac{\partial}{\partial y^{(k)i}} \]

with the coefficients

\[ (k + 1) G^i = \frac{1}{2} g^{ij} \left\{ \Gamma \left( \frac{\partial L}{\partial y^{(k)j}} \right) - \frac{\partial L}{\partial y^{(k-1)j}} \right\}, \]

$\Gamma$ being the operator (1.2.3).

The $k$-semispray $S$ depends on the fundamental function $L$ of the space $L^{(k)n}$. It will be called canonical. If $L$ is globally defined on $T^k M$, then $S$ has the same property on $T^k M$.

Taking into account Theorem 1.5.1 we find, [94]:

**Theorem 2.5.2** The set of functions

\[ M^i_{(1)} = \frac{\partial G^i}{\partial y^{(k)i}}, \quad M^i_{(2)} = \frac{1}{2} \left( SM^i_{(1)} + M^i_{m} M_{m}^{m} \right), \quad \ldots, \]

\[ M^i_{(k)} = \frac{1}{k} \left( S M^i_{(1)} + M^i_{m} M_{m}^{m} \right) \]

are the dual coefficients of a nonlinear connection $N$ determined only by the canonical semispray $S$. 

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**Lagrange Spaces of Higher Order**

35
THE GEOMETRY OF HIGHER-ORDER HAMILTON SPACES

N is called canonical, too.

Theorem 1.5.2 give us new dual coefficients \( M_{ij}^{*}, \ldots, M_{ij}^{**} \), (I. Bucataru, [26]).

The coefficients \( N_{ij}^{(1)}, \ldots, N_{ij}^{(k)} \) of \( N \) are obtained from (1.4.3').

The existence of the spaces \( L^{(k)n} = (M, L) \), when \( M \) is a paracompact manifold is assured by the following examples.

**Example 5.1** Let \( g_{ij}(x) \) be a Riemannian tensor on \( M \) and the nonlinear connection \( N \) with the dual coefficients \( M_{ij}^{(1)}(x, y^{(1)}), \ldots, M_{ij}^{(k)}(x, y^{(1)}, \ldots y^{(k)}) \) given by (1.5.3). Consider the Liouville d-vector field, \( z^{(k)i} \) (Th. 21.4.1):

\[
(k)z = ky^{(k)i} + (k-1)M_{mj}^{(1)}y_{(k-1)m}^{(1)} + \ldots + M_{mj}^{(1)}y_{(1)m}^{(1)}
\]

and remark that \( z^{(k)i} \) is a d-vector field, linear in the vertical variables \( y^{(k)i} \). Consequence, the function

\[
L(x, y^{(1)}, \ldots y^{(k)}) = g_{ij}(x)z^{(k)i}z^{(k)j}
\]

is a fundamental function of a Lagrange space \( L^{(k)n} \). Its fundamental tensor field is exactly \( g_{ij}(x) \).

**Example 5.2** Let \( \tilde{L}(x, y^{(1)}) \) be the Lagrangian of electrodynamics

\[
\tilde{L}(x, y^{(1)}) = mc\gamma_{ij}(x)y^{(1)ij} + \frac{2e}{m}b_{i}(x)y^{(1)i}
\]

\( m, c, e \) being the well known physical constants, \( \gamma_{ij}(x) \) the gravitational potentials and \( b_{i}(x) \) the electromagnetic potentials.

Let us consider the nonlinear connection \( N \) determined like in previous example. Then

\[
L(x, y^{(1)}, \ldots y^{(k)}) = mc\gamma_{ij}(x)z^{(k)i}z^{(k)j} + \frac{2e}{m}b_{i}(x)z^{(k)i}
\]

is a fundamental function of a Lagrange space of order \( k \), \( L^{(k)n} = (M, L) \).

Evidently \( L \) from (2.5.7) is a particular case of the Lagrangian from (2.1.5). It will be called the prolongation of order \( k \) of the electrodynamic Lagrangian in (2.5.6).

In the end of this chapter we will study the geometry of the Lagrange space \( L^{(k)n} \), with fundamental function (2.5.7).

The adapted basis \( \left\{ \delta x^{i}, \delta y^{(1)i}, \ldots, \delta y^{(k)i} \right\} \) determined by the canonical nonlinear connection \( N \) is given by (2.3.9) and its dual basis \( \left\{ \delta x^{i}, \delta y^{(1)i}, \ldots, \delta y^{(k)i} \right\} \) is expressed in the formulae (2.4.2).

The horizontal curves of the space \( L^{(k)n} \) with respect to canonical nonlinear connection \( N \) are characterized by the system of differential equations

\[
\frac{\delta y^{(1)i}}{dt} = \ldots = \frac{\delta y^{(k)i}}{dt} = 0.
\]
In the light of these properties we determine the autoparallel curves of $N$ by adding the conditions to the previous differential equations

$$y^{(1)i} = \frac{dx^i}{dt},..., y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k}.$$  

Using the results from chapter 1 one can proves:

The canonical nonlinear connection $N$ is integrable if, and only if the following equations hold

$$R_{ijk}^{ij} = \cdots = R_{ij}^{jk} = 0.$$  

### 2.6 Canonical Metrical $N$-Connections

Consider the canonical nonlinear connection $N$ of the Lagrange space of order $k$, $L^{(k)n} = (M, L)$. A linear connection $D$ on $T^k M$ is called an $N$-linear connection if:

1) $D$ preserves by parallelism the horizontal distribution $N$,

2) $DJ = 0$.  

The coefficients of $D$ with respect to the adapted basis denoted by $D \Gamma(N) = (L^i_{jh}, C^i_{jh}, \ldots, C^i_{jh})$, can be uniquely determined if $D$ is compatible with the Riemannian (or pseudoriemannian) metric $G$ on $T^k M$, determined by the fundamental tensor field $g_{ij}$ of $L^{(k)n}$. Namely, $G$ is expressed in the adapted basis by

$$(2.6.1) \quad G = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + \cdots + g_{ij} \delta y^{(k)i} \otimes \delta y^{(k)j}.$$  

$D$ is compatible with $G$ if

$$D_X G = 0, \forall X \in \mathcal{X}(\widehat{T^k M}).$$  

**Theorem 2.6.1** The following properties hold:

1) There exists a unique $N$-linear connection $D$ on $\widehat{T^k M}$ verifying the axioms:

$$(2.6.2) \quad g_{ij|h^{(1)}} = 0, \quad g_{ij|h^{(k)}} = 0,$$

$$(2.6.2') \quad L^i_{jh} = L^i_{jh}, \quad C^i_{jh} = C^i_{hj}, \quad (\alpha = 1, \ldots, k).$$
2) The coefficients $CT(N) = (L^i_{jh}, C^i_{jh}(1), ..., C^i_{jh}(k))$ of $D$ are given by the generalized Christoffel symbols:

\[
\begin{aligned}
L^m_{ij} &= \frac{1}{2}g^{ms}(\frac{\delta g_{is}}{\delta x^j} + \frac{\delta g_{sj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^s}), \\
C^m_{ij}(\alpha) &= \frac{1}{2}g^{ms}(\frac{\delta g_{is}}{\delta y^{(\alpha)j}} + \frac{\delta g_{sj}}{\delta y^{(\alpha)i}} - \frac{\delta g_{ij}}{\delta y^{(\alpha)s}}), \quad (\alpha = 1, ..., k).
\end{aligned}
\]

3) $D$ depends only on the fundamental function $L$ of the space $L^{(k)n}$.

The connection $D$ from the previous theorem is called the canonical metrical $N$-connection of the Lagrange space of order $k$, $L^{(k)n}$.

The d-tensors of curvature of $D$ satisfy the following identities:

\[
\begin{aligned}
g_{sj}R^s_{i \, hm} + g_{is}R^s_{j \, hm} &= 0, \\
g_{sj}P^s_{(\alpha) \, i \, hm} + g_{is}P^s_{(\alpha) \, j \, hm} &= 0, \\
g_{sj}S^s_{(\alpha\beta) \, i \, hm} + g_{is}S^s_{(\alpha\beta) \, j \, hm} &= 0.
\end{aligned}
\]

The connection 1-forms of $D$ are

\[
\omega^i_j = L^i_{jh}dx^h + \sum_{\alpha=1}^{k} C^i_{jh}(\alpha)dy^{(\alpha)h}.
\]

Finally, the structure equations of the canonical metrical $N$-connection $D$ are given by Theorem 1.7.1 in the conditions (2.6.3).

The Bianchi identities of $D$ are obtained from the structure equations applying the operator of exterior differentiation and taking into account the same equations of structure.

Now let us consider the tensor field

\[
F = -\delta \delta y^{(k)i} \otimes dx^i + \delta \delta x^i \otimes dy^{(k)i}.
\]

We can see that $F$ is globally defined on $\tilde{T}^kM$ and $F^3 + F = 0$. The pair $(G, F)$ determines a Riemannian almost $(k-1)n$-contact structure on $\tilde{T}^kM$ depending only on then fundamental function $L$ of the space $L^{(k)n}$.

**Examples.**

1°. $R^{(k)n} = \text{Prol}^k R^n$.

Let $N$ be the nonlinear connection with the dual coefficients (2.5.3). It is determined only by $g_{ij}(x)$. If \{\delta x^i, \delta y^{(1)i}, ..., \delta y^{(k)i}\} is the adapted cobasis to $N$ and $V_1$, then the formula (2.6.1) gives us a Riemannian structure $G$ on \[38\]
\( \tilde{T}^k M \) which depend on \( g_{ij}(x) \), only. The space \( \mathcal{R}^{(k)n} = (\tilde{T}^k M, \mathcal{G}) \) is called the prolongation to \( \tilde{T}^k M \) of the Riemannian structure \((M, g_{ij}(x))\).

Consider the \( k \)-Liouville vector field \( z^{(k)i} \) from (2.5.4) constructed by means of (2.5.3).

In this case the pair \( L^{(k)n} = (M, L(x, y^{(1)}, \ldots, y^{(k)})) \) is a Lagrange space of order \( k \), where \( L = g_{ij}(x) z^{(k)i} z^{(k)j} \).

The fundamental tensor field of \( L^{(k)n} \) is exactly \( g_{ij}(x) \), because \( z^{(k)i} \) is of the form \( z^{(k)i} = y^{(k)i} + \lambda^i(x, y^{(1)}, \ldots, y^{(k-1)}) \). The canonical metrical \( N \)-connection has the coefficients

\[
L^{ij}_{jh} = \gamma^{ij}_{jh}(x), \quad C^{ij}_{jh} = 0, \quad (\alpha = 1, \ldots, k).
\]

\( 2^o, \quad F^{(k)n} = \text{Pro}^k F^n \). The prolongation of order \( k \) of a Finsler space \( F^n = (M, F(x, y^{(1)})) \) leads to a second example of Lagrange space of order \( k \).

Consider \( N \) the nonlinear connection with the dual coefficients (1.5.4). It is determined only by \( F^n \). The \( N \)-lift of the fundamental tensor \( g_{ij}(x, y^{(1)}) \) of the space \( F^n \) is given by (2.6.1). It is a Riemannian metric on \( \tilde{T}^k M \) determined by \( F^n \) only. Thus,

\[
L = g_{ij}(x, y^{(1)}) z^{(k)i} z^{(k)j},
\]

\( z^{(k)i} \) being the \( k \)-Liouville vector fields corresponding to \( N \), is the fundamental function of a space \( L^{(k)n} \).

The fundamental tensor of space is \( g_{ij}(x, y^{(1)}) \) and the canonical metrical \( N \)-connection has the coefficients

\[
L^{ij}_{jk} = F^{ij}_{jk}(x, y^{(1)}), \quad C^{ij}_{jk} = C^{ij}_{jk}, \quad C^{ij}_{jk} = 0, \quad (\alpha = 2, \ldots, k - 1).
\]

## 2.7 Generalized Lagrange Spaces of Order \( k \)

The notion of generalized Lagrange spaces of order \( k \) is a natural extension of that of space \( L^{(k)n} \). It was used, [94], in the geometrical theory of the higher order relativistic optics.

**Definition 2.7.1** A generalized Lagrange spaces of order \( k \) is a pair \( GL^{(k)n} = (M, g_{ij}(x, y^{(1)}, \ldots, y^{(k)})) \) formed by a real \( n \)-dimensional differentiable manifold \( M \) and a differentiable symmetric \( d \)-tensor field \( g_{ij} \) defined on \( \tilde{T}^k M \), having two properties:

\[ a. \quad g_{ij} \text{ has a constant signature on } \tilde{T}^k M; \]
\[ b. \quad \text{rank } ||g_{ij}|| = n \text{ on } \tilde{T}^k M. \]

Of course, \( GL^{(k)n} \) can be defined locally, in the case when \( g_{ij} \) is given on an open set \( (\pi^k)^{-1}(U), U \subset M \).

We say that \( g_{ij} \) is the fundamental tensor of the space \( GL^{(k)n} \).
Notice that any Lagrange space of order \( k \), \( L^{(k)n} = (M, L) \) is a generalized Lagrange space \( GL^{(k)n} = (M, g_{ij}) \) with

\[
g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}}.
\]

But not and conversely. Indeed, it is possible that the system of differential partial equations (2.7.1) does not admit any solution \( L(x, y^{(1)}, ..., y^{(k)}) \) when the \( d \)-tensor field \( g_{ij}(x, y^{(1)}, ..., y^{(k)}) \) is apriori given.

Let us consider the tensor field:

\[
C^{(k)}_{ijh} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^{(k)h}}.
\]

It is not hard to see that \( C^{(k)}_{ijh} \) is a covariant of order three \( d \)-tensor field.

Easily follows:

**Proposition 2.7.1**

A necessary condition so that the system of differential partial equation (2.7.1) admits a solution \( L(x, y^{(1)}, ..., y^{(k)}) \) is that the \( d \)-tensor field \( C^{(k)}_{ijh} \) be completely symmetric.

If the tensor \( C^{(k)}_{ijh} \) is not completely symmetric we say that the space \( GL^{(k)n} \) is not reducible to a Lagrange space of order \( k \).

**Example 2.7.1** Let \( \mathcal{R}^n = (M, \gamma_{ij}(x)) \) be a Riemannian space and \( \sigma(x, y^{(1)}, ..., y^{(k)}) \) a smooth function on \( T^kM \) with the property \( \frac{\partial \sigma}{\partial y^{(k)i}} \neq 0 \).

Consider the \( d \)-tensor field on \( T^kM \):

\[
g_{ij}(x, y^{(1)}, ..., y^{(k)}) = e^{2\sigma(x, y^{(1)}, ..., y^{(k)})}(\gamma_{ij} \circ \pi^k)(x, y^{(1)}, ..., y^{(k)}).
\]

We can prove that the pair \( GL^{(k)n} = (M, g_{ij}) \) with \( g_{ij} \) from (2.7.3) is a generalized Lagrange space of order \( k \), which is not reducible to a Lagrange space.

This example proves the existence of spaces \( GL^{(k)n} \) on the paracompact manifolds. For \( k = 1 \), this space was introduced by R. Miron and R. Tavakol [100], [114].

**Example 2.7.2** Consider again \( \mathcal{R}^n = (M, \gamma_{ij}(x)) \) and the Liouville \( d \)-vector field \( z^{(k)i} \) of the space \( Prol^k\mathcal{R}^n \). It is expressed by

\[
z^{(k)i} = k y^{(k)i} + (k - 1) M^i_m y^{(k-1)m} + ... + M^i_m y^{(1)m},
\]

where the dual coefficients \( M^i_j, ..., M^i_{(k-1)} \) are given by the formulae (2.5.3).

Evidently \( z^{(k)i} \) linearly depends on \( y^{(k)i} \). So that its covariant \( z^{(k)}_i = \gamma_{ij} z^{(k)j} \) linearly depends on \( y^{(k)i} \).
Assuming that there exists a function \( n(x, y^{(1)}, ..., y^{(k)}) \geq 1 \) on \( T^k M \), we can construct the following \( d \)-tensor field

\[
g_{ij}(x, y^{(1)}, ..., y^{(k)}) = \gamma_{ij}(x) + \left( 1 - \frac{1}{n^2(x, y^{(1)}, ..., y^{(k)})} \right) z^{(k)}_i z^{(k)}_j.
\]

One proves that the pair \( GL^{(k)n} = (M, g_{ij}) \), with \( g_{ij} \) from (2.7.5), is a generalized Lagrange space of order \( k \), which is not reducible to a Lagrange space of order \( k \).

In the case \( k = 1 \), this space was studied by the author [91, 100] and applied, together with T. Kawaguchi [104] in the relativistic geometrical optics.

In the previous two examples we have a natural canonical nonlinear connection, with the dual coefficients (2.5.3), determined by the space \( \text{Pro}^k R^n \).

Returning to the generalized Lagrange spaces of order \( k \), \( GL^{(k)n} \), we remark the difficulty to find a nonlinear connection derived only from the fundamental tensor \( g_{ij} \) of the space.

Therefore we assume that a nonlinear connection \( N \) on \( \tilde{T}^k M \) is apriori given.

Thus we shall study the pair \( (N, GL^{(k)n}) \) using the same methods like in the geometry of the space \( L^{(k)} \).

Indeed, considering the adapted basis \( \left( \delta x^i, \delta y^{(1)}_i, \ldots, \delta y^{(k)}_i \right) \) and its dual basis \( \left( \delta x_i, \delta y^{(1)}_i, \ldots, \delta y^{(k)}_i \right) \), determined by the nonlinear connection, we define the \( N \)-lift of the fundamental tensor \( g_{ij} \):

\[
G = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^{(1)}_i \otimes \delta y^{(1)}_j + \cdots + g_{ij} \delta y^{(k)}_i \otimes \delta y^{(k)}_j.
\]

A \( N \)-linear connection compatible to \( G \) is furnished by the following theorem:

**Theorem 2.7.1** 1° There exists an unique \( N \)-linear connection \( D \) for which

\[
g_{ij|h}^{(\alpha)} = 0, \quad g_{ij} \mid_h = 0, \quad (\alpha = 1, ..., k),
\]

\[
L_{jh}^i = L_{hj}^i, \quad C_{j\mid h}^i = C_{i\mid h}^j, \quad (\alpha = 1, ..., k).
\]

2° The coefficients \( \Gamma(N) \) of \( D \) are given by the generalized Christoffel symbols:

\[
L_{ij}^m = \frac{1}{2} g_{ms} \left( \frac{\delta g_{is}}{\delta x^j} + \frac{\delta g_{sj}}{\delta x^i} - \frac{\delta g_{js}}{\delta x^i} \right),
\]

\[
C_{ij}^{m(\alpha)} = \frac{1}{2} g_{ms} \left( \frac{\delta g_{is}}{\delta y^{(\alpha)}_j} + \frac{\delta g_{sj}}{\delta y^{(\alpha)}_i} - \frac{\delta g_{js}}{\delta y^{(\alpha)}_i} \right), \quad (\alpha = 1, ..., k).
\]
The previous $N$-linear connection of the space $GL^{(k)n}$ is called metrical canonical $N$-linear connection.

The structure equations can be written, exactly as in the chapter 1 (theorem 1.7.1).

The tensor $F$ on $\widehat{T^kM}$:

\begin{equation}
F = -\frac{\delta}{\delta y^{(k)i}} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^{(k)i},
\end{equation}

together with the metric tensor $G$ from (2.7.6) determine a Riemannian $(k - 1)n$-almost contact structure on $\widehat{T^kM}$. It is 'the geometrical model' of the generalized Lagrange space of order $k$, $GL^{(k)n} = (M, g_{ij})$. 

Chapter 3

Finsler Spaces of Order $k$

The geometry of Finsler spaces of order $k$, introduced by the author and presented in his book 'The geometry of Higher-Order Finsler Spaces' Hadronic Press, 1998 is a natural extension to $T^kM$ of the classical theory of Finsler Spaces. The impact of this geometry in Differential Geometry, Variational Calculus, Analytical Mechanics or Theoretical Physics is decisive. Finsler spaces play a role in applications to Biology, Engineering, Physics or Optimal Control. Also the introduction of the notion of Finsler space of order $k$ is demanded by the solution of problem of prolongation to $T^kM$ of the Riemannian or Finslerian structures defined on the base manifold $M$.

In the present chapter we will develop the geometrical theory of the Finsler spaces of order $k$, based on the geometry of Lagrange spaces of the same order. Such that the Finsler spaces $F^{(k)n}$ form a subclass of the class of spaces $L^{(k)n}$. Consequently we obtain an extension of the sequence $\{R^n\} \subset \{F^n\} \subset \{L^n\} \subset \{GL^n\}$ to the following sequence of the spaces of order $k$: $\{R^{(k)n}\} \subset \{F^{(k)n}\} \subset \{L^{(k)n}\} \subset \{GL^{(k)n}\}$.

In the next chapter we shall investigate the dual of this sequence, made by the Hamilton spaces of order $k$.

3.1 Spaces $F^{(k)n}$

In order to introduce the notion of Finsler space of order $k$ there are necessary some preliminaries. A functions $f : T^kM \to R$, of $C^\infty$ class on $\tilde{T}^kM$ and continuous on the null sections of the mapping $\pi^k : T^kM \to M$ is called homogeneous of degree $r \in Z$ on the fibres on $T^kM$ (briefly $r$-homogeneous) if for any positive constant $a$, we have:

$$f(x, ay^{(1)}, a^2 y^{(2)}, ..., a^k y^{(k)}) = a^r f(x, y^{(1)}, ..., y^{(k)}).$$

An Euler theorem holds:

A function $f \in F(T^kM)$, differentiable on $\tilde{T}^kM$ and continuous on the null
section of $\pi^k$ is $r-$homogeneous if and only if:

$$(3.1.2) \quad L_{\Gamma} f = rf,$$

$L_{\Gamma} f$ being the Lie operator of derivation with respect to the Lioville vector field $\Gamma$, i.e.:

$$(3.1.2') \quad L_{\Gamma} f = k \Gamma f = y(1)^i \frac{\partial f}{\partial y^{(1)i}} + ... + ky^{(k)i} \frac{\partial f}{\partial y^{(k)i}}.$$

Notice the following property:

If the function $f \in F(T^kM)$ is differentiable on $T^kM$ (inclusive in the points $(x,0,...,0)$) and $k-$homogeneous then $f$ is a polynomial of order $k$ in the variable $y^{(1)i}, y^{(2)i}, ..., y^{(k)i}$.

The notion of homogeneity can be extended to the vector fields and 1-form fields on $T^kM$. One proves:

A vector field $X \in X(T^kM)$ is $r-$homogeneous if and only if

$$(3.1.3) \quad L_{\Gamma} X = (r-1)X.$$  

Of course, $L_{\Gamma} X = [\Gamma, X]$.

For instance, the vector fields $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(1)i}}, ..., \frac{\partial}{\partial y^{(k)i}}$ are $1, 0, ..., 1-k$ homogeneous, respectively.

An homogeneous $k-$semispray $S$ is called a $k-$spray. The following sentences hold:

1°. A $k-$spray $S$ given by

$$(3.1.4) \quad S = y^{(1)i} \frac{\partial f}{\partial x^i} + ... + ky^{(k)i} \frac{\partial f}{\partial y^{(k)i}} - (k+1)G^i \frac{\partial \delta}{\partial y^{(k)i}}$$

is $2$-homogeneous if and only if its coefficients $G^i$ are $k+1-$homogeneous.

2°. The dual coefficients $M^{(1)}_j, M^{(2)}_j, ..., M^{(k)}_j$ of the nonlinear connection $N$, according to Theorem 2.5.2, ch.2) determined by a $2-$homogeneous spray $S$ are $1, 2, ..., k-$homogeneous, respectively.

3°. The same property have the coefficients $N^{(1)}_j, ..., N^{(k)}_j$ of the nonlinear connection $N$.

4°. The local adapted basis to $N$, $\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, ..., \frac{\delta}{\delta y^{(k)i}}$ has the following degree of homogeneity $1, 0, ..., 1-k$, respectively.

5°. If $X \in \mathcal{X}(T^kM)$ is $r-$homogeneous and $f \in \mathcal{F}(T^kM)$ is $s-$homogeneous, then $Xf$ is $s+r-1$-homogeneous.

Evidently the $1$-forms $dx^i, dy^{(1)i}, ..., dy^{(k)i}$ are homogeneous of degree $0, 1, ..., k$, respectively. The same degree of homogeneity have $\delta x^i, \delta y^{(1)i}, ..., \delta y^{(k)i}$ (the dual basis adapted to $N$).
A $q$-form $\omega \in \Lambda^q(\tilde{T}^kM)$ is $s$-homogeneous, if and only if

\begin{equation}
\mathcal{L}_\xi \omega = s\omega.
\end{equation}

Evidently, if $\omega$ is $s$-homogeneous and $\omega'$ is $s'$-homogeneous then $\omega \wedge \omega'$ is $s+s'$-homogeneous.

For any $s$-homogeneous $\omega \in \Lambda^q(\tilde{T}^kM)$ and any $r$-homogeneous $\tilde{T}^kM)$ the function $\omega(X, \ldots, X)$ is $r+(s-1)q$ homogeneous.

Now we can formulate:

**Definition 3.1.1** A Finsler space of order $k$, $k \geq 1$, is a pair $F^{(k)n} = (M, F)$ determined by a real differentiable manifold $M$ of dimension $n$ and a function $F : T^kM \to \mathbb{R}$ having the following properties:

1. $F$ is differentiable on $\tilde{T}^kM$ and continuous on the null section $0 : M \to T^kM$.

2. $F$ is positive.

3. $F$ is $k$-homogeneous on the fibres of the bundle $T^kM$.

4. The Hessian of $F^2$ with the entries

\begin{equation}
\label{eq:3.1.6}
g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(k)i} \partial y^{(k)j}}
\end{equation}

is positively defined on $\tilde{T}^kM$.

The $k$-homogeneous means positively $k$-homogeneous since $F(x, ay^{(1)}, \ldots, ay^{(k)}) = a^k F(x, y^{(1)}, \ldots, y^{(k)})$ holds for any $a > 0$.

It is not difficult to see that this definition has a geometrical meaning, $g_{ij}$ from (3.1.6) being a $d$-tensor field on $\tilde{T}^kM$. Such that the function $F$ is called the fundamental or metric function and the $d$-tensor field $g_{ij}$ is called fundamental or metric tensor of the Finsler space of order $k$, $F^{(k)n}$. Of course, the axiom 4° implies:

\begin{equation}
\label{eq:3.1.7}
\text{rank } \|g_{ij}\| = n
\end{equation}

Clearly, in the case $k = 1$, $F^{(1)n} = (M, F)$ is a Finsler space [115].

We can see, without difficulties that the following theorem holds:

**Theorem 3.1.1** The pair $L^{(k)n} = (M, F^2(x, y^{(1)}, \ldots, y^{(k)}))$ is a Lagrange space of order $k$. Conversely, if $L^{(k)n} = (M, L(x, y^{(1)}, \ldots, y^{(k)}))$ is a Lagrange space of order $k$, having the fundamental function $L$ positively, $2k$-homogeneous and the fundamental tensor $g_{ij}$ positively defined, then the pair $F^{(k)n} = (M, \sqrt{L})$ is a Finsler space of order $k$.

Consequently, the class of spaces $F^{(k)n}$ is a subclass of spaces $L^{(k)n}$.

Taking into account the $k$-homogeneity of the fundamental function $F$ and $2k$-homogeneity of $F^2$ we get:
1°. $p_i$ given by

\[ p_i = \frac{1}{2} \frac{\partial F^2}{\partial y^{(k)}_i} \]

is a $k$-homogeneous $d$-covector fields.

2°. $p = p_i dx^i$ is a $k$-homogeneous 1-form.

3°. The fundamental tensor $g_{ij}$ is $0$-homogeneous. Its contravariant $g^{ij}$ is $0$-homogeneous ($g_{ij}g^{jh} = \delta_i^h$), too.

These homogeneities imply:

\[ (3.1.8) \quad L_k \Gamma F^2 = 2kF^2, L_k \frac{\partial F^2}{\partial y^{(k)}} = k\frac{\partial F^2}{\partial y^k}, (or \quad L_k p_i = k p_i), \]

\[ (3.1.8') \quad L_k \Gamma g_{ij} = 0, \quad L_k \Gamma C_{ijh} = -kC_{ijh}, \]

where

\[ C_{ijh} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^{(k)}_h} = \frac{1}{4} \frac{\partial^3 F^2}{\partial y^{(k)}_i \partial y^{(k)}_j \partial y^{(k)}_h}. \]

Of course, the equation $L_k g_{ij} = 0$ can be written as follows:

\[ (3.1.8'') \quad L_k g_{rs} = (y^{(1)}_i \frac{\partial}{\partial y^{(1)}_i} + ... + y^{(k)}_i \frac{\partial}{\partial y^{(k)}_i}) g_{rs} = 0. \]

The integral of action of $F$ along a parametrized curve $c$:

\[ I(c) = \int_0^1 F(x, dx/dt, ..., 1/dk/dt) dt \]

can be considered for determining the length of $\tilde{c} : [0, 1] \rightarrow T^\infty M$ in the given parametrization.

Theorem 2.1.1 shows that the necessary conditions for the integral of action $I(c)$, (3.1.9), to be independent on the parametrization of a curve $c$ are given by the Zermelo conditions

\[ L_k F = ... = L_{k-1} F = 0, \quad L_k F = F. \]

But $L_k F = F$ and $L_k F = kF$, for $k > 1$ are contradictory.

Consequently, in a Finsler space of order $k$, $k > 1$, the integral of action (3.1.9) essentially depend on the parametrization of a curve $c$.

**Example 3.5.1** Let $F^n = (M, F(x, y^{(1)}))$ be a Finsler space having $g_{ij}(x, y^1)$ as the fundamental tensor and $M^i_j(x, y^1)$ as coefficients of the Cartan
nonlinear connection. Then, theorem 1.5.4 gives us the dual coefficients on $T^k M$:

$$M^i_j(x, y^{(1)}), M^i_j(x, y^{(2)}), \ldots, M^i_j(x, y^{(1)}, \ldots, y^{(k)})$$

of a nonlinear connection, which depends only on the fundamental function $F(x, y^{(1)})$ of the Finsler space $F^n$.

It is not hard to see that these coefficients are homogeneous of degree $1, 2, \ldots, k$ respectively. This property implies that the $d$-Liouville vector field $z^{(k)i}$:

$$kz^{(k)i} = ky^{(k)i} + (k - 1)M^i_j y^{(k-1)j} + \ldots + M^i_j y^{(1)j}$$

is linear in the variables $y^{(k)i}$ and it is $k$-homogeneous.

Consider the function

$$F(x, y^{(1)}, \ldots, y^{(k)}) = \left\{ g_{ij}(x, y^{(1)})z^{(k)i}z^{(k)j} \right\}^{1/2},$$

$g_{ij}(x, y^{(1)})$ being a d-tensor positively defined. It follows that $F$ from (3.1.11) is a positive differentiable function on $T^k M$ and continuous on the null section of $\pi^k$. It is $k$-homogeneous and has $g_{ij}(x, y^{(1)})$ as the fundamental tensor.

Consequently, the pair $F^{(k)n} = (M, F(x, y^{(1)}, \ldots, y^{(k)}))$ for $F$ from (3.1.11) is a Finsler space of order $k$.

Concluding, we have:

**Theorem 3.1.2** If the base manifold is paracompact then there exist a Finsler space of order $k$, $F^{(k)n}$.

The spaces $F^{(k)n}$ constructed in example (3.1.1) is called the Prolongation of order $k$ of the Finsler space $F^n$. It is denoted by $\text{Prol}^{(k)} F^n$.

In order to determine the geodesics of the space $F^{(k)n}$ we take the integral of action of the regular Lagrangian $F^2$. The variational problem leads to the Euler - Lagrange equations.

$$\bar{E}_i (F^2) = \frac{\partial F^2}{\partial x^i} - d \frac{\partial F^2}{\partial y^{(1)i}} + \ldots + (-1)^k \frac{1}{k!} \frac{d^k}{dt^k} \frac{\partial F^2}{\partial y^{(k)i}} = 0.$$

The integral curves of the previous equations are called the geodesics of the space $F^{(k)n}$. Applying the theory from the section 2, ch. 2 we can determine the infinitesimal symmetries of the spaces $F^{(k)n}$.

The energies of order $k, k - 1, \ldots, 1$ of the Finsler space of order $k$, $F^{(k)n} = (M, F(x, y^{(1)}, \ldots, y^{(k)}))$ are given by the formulae (2.3.1), for $L = F^2$.

In particular, Theorem 2.3.2 can be applied in order to obtain

**Theorem 3.1.3** The energy of order $k$, $\mathcal{E}^k (F^2)$ of the Finsler space $F^{(k)n}$ is conserved along every geodesic of this space.
Let us consider
\[ \tilde{p}_{(k)i} = \frac{\partial F^2}{\partial y^{(k)}_i}, \tilde{p}_{(k-1)i} = \frac{\partial F^2}{\partial y^{(k-1)}_i}, ..., \tilde{p}_{(1)i} = \frac{\partial F^2}{\partial y^{(1)}_i}, \tilde{p}_{(0)i} = \frac{\partial F^2}{\partial x^i}. \]

Then we have

**Theorem 3.1.4**

1°. The Cartan differential 1-forms are the followings

\[ d_0 F^2 = \tilde{p}_{(k)i} dx^i, \]

\[ d_1 F^2 = \tilde{p}_{(k-1)i} dx^i + \tilde{p}_{(1)i} dy^{(1)i}, \]

\[ d_k F^2 = \tilde{p}_{(0)i} dx^i + \tilde{p}_{(1)i} dy^{(1)i} + ... + \tilde{p}_{(k)i} dy^{(k)i}. \]

2°. And the Poincare 2-forms are given by

\[ dd_0 F^2 = d \tilde{p}_{(k)i} \wedge dx^i, \]

\[ dd_1 F^2 = d \tilde{p}_{(k-1)i} \wedge dx^i + d \tilde{p}_{(1)i} dy^{(1)i}, \]

\[ dd_{(k-1)} F^2 = d \tilde{p}_{(0)i} \wedge dx^i + ... + d \tilde{p}_{(1)i} dy^{(1)i}. \]

Here we have \( dd_k F^2 = d^2 F^2 = 0. \)

3°. The 1-forms \( d_0 F^2, d_1 F^2, ..., d_k F^2 \) are \( k, k+1, ..., 2k \) homogeneous, respectively.

### 3.2 Cartan Nonlinear Connection in \( F^{(k)n} \)

The considerations made in the previous chapter allow us to introduce in a Finsler space of order \( k \), \( F^{(k)n} = (M, F) \) the main geometrical object fields as: canonical \( k \)-spray, Cartan nonlinear connection, canonical \( N \)-linear connection etc. Canonical mean here that all these object fields depend only on the fundamental function \( F \).

Taking into account the operators \( ^0 E_i, ^1 E_i, ..., ^k E_i \) given by (2.2.8) we construct the system of \( d \)-covector fields

\[ ^0 E_i (F^2), ^1 E_i (F^2), ..., ^k E_i (F^2) \]

All equations

\[ ^0 E_i (F^2) = 0, ^1 E_i (F^2) = 0, ..., ^{k-1} E_i (F^2) = 0 \]

have geometrical meanings. The equation

\[ ^{k-1} E_i (F^2) = 0 \]
Finsler Spaces of Order \( k \) is important for us. It will be called the Craig-Synge equation. Using (2.2.8), this equation is expressed as follows

\[
\frac{\partial F^2}{\partial y^{(k-1)i}} - \frac{d}{dt} \frac{\partial F^2}{\partial y^{(k)i}} = 0.
\]

But

\[
\frac{d}{dt} \frac{\partial F^2}{\partial y^{(k)i}} = \Gamma \frac{\partial F^2}{\partial y^{(k)i}} + \frac{2}{k!} g_{ij} \frac{d^{k+1}x^i}{dt^{k+1}},
\]

where \( \Gamma \) is the operator (1.2.3).

Consequently, the Craig-Synge equations (3.2.2) is equivalent to the following equations

\[
g^{ij} E_j(F^2) = 0, \quad y^{(1)i} = \frac{dx^i}{dt}, \ldots, y^{(k)i} = \frac{1}{k!} \frac{d^k x^i}{dt^k},
\]

or

\[
\frac{d^{k+1}x^i}{dt^{k+1}} + (k + 1)G^i(x, \frac{dx}{dt}, \ldots, \frac{1}{k!} \frac{d^k x}{dt^k}) = 0,
\]

where

\[
(k + 1)G^i(x, y^{(1)}, \ldots, y^{(k)}) = \frac{1}{2} g^{ij} \{ \Gamma \left( \frac{\partial F^2}{\partial y^{(k)j}} \right) - \frac{\partial F^2}{\partial y^{(k-1)j}} \}.
\]

Applying the Theorem 2.5.1 one obtains:

**Theorem 3.2.1** The Craig-Synge equations (3.2.3) determines a canonical \( k \)- spray \( S \):

\[
S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \ldots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k + 1)G^i(x, y^{(1)}, \ldots, y^{(k)}) \frac{\partial}{\partial y^{(k)i}}
\]

with the coefficients \( G^i \) from (3.2.4).

Note that \( S \) depend only on the fundamental function \( F \) of the space \( F^{(k)n} \).

It is a \( k \)- spray, since it is a 2-homogeneous vector field. The paths of \( S \) are given by the differential equations (3.2.3).

By means of the Theorems 1.5.1 and 3.2.1 the dual coefficients of the nonlinear connection \( N \) determined by the canonical \( k \)- spray \( S \) are given by:

**Theorem 3.2.2** In a Finsler space of order \( k \), \( F^{(k)n} = (M, F) \) there exist nonlinear connections, depending only on the fundamental function \( F \). One of these, denoted by \( N \), has the dual coefficients:

\[
M^i_j = \frac{1}{2(k + 1)} \frac{\partial}{\partial y^{(k)j}} \{ g^{im} \left[ \Gamma \frac{\partial F^2}{\partial y^{(k)m}} - \frac{\partial F^2}{\partial y^{(k-1)m}} \right] \},
\]

\[
M^i_j = \frac{1}{2} \left( SM^i_j + M^i_m M^m_j \right),
\]

\[
M^i_j = \frac{1}{k} \left( SM^i_j + M^i_m M^m_j \right),
\]

............................
where $S$ is the canonical $k$-spray of the space $F^{(k)n}$.

$N$ is called the Cartan nonlinear connection of the space $F^{(k)n}$.

In the case $k = 1$, $N$ reduces to the classical Cartan nonlinear connection of the Finsler space $F^n = (M, F(x, y^{(1)}))$.

Some properties of $N$.

1°. The Cartan nonlinear connection $N$ is globally defined on $\widetilde{T^kM}$ (if $F(x, y^{(1)}, \ldots, y^{(k)})$ has this property).

2°. The dual coefficients (3.2.6) of $N$ are homogeneous of degree 1, 2, ..., $k$ i.e:

\[
\mathcal{L}_\Gamma N^i_j = \alpha M^i_j, \quad \alpha = 1, \ldots, k.
\]

The coefficients $N^i_j$, ..., $N^i_j$ of $N$, (ch.1) are expressed by

\[
N^i_j = M^i_j,
\]

\[
N^i_j = M^i_j - N^i_m M^m_j,
\]

\[
N^i_j = M^i_j - N^i_m M^m_j - \cdots - N^i_m M^m_j.
\]

These coefficients are homogeneous functions of degree 1, ..., $k$ respectively, i.e

\[
\mathcal{L}_\Gamma N^i_j = \alpha N^i_j, \quad (\alpha = 1, \ldots, k).
\]

The Cartan nonlinear connection $N$ gives rise to a distribution $N_u \subset T_u(\widetilde{T^kM})$ supplementary to the vertical distribution $V_u \subset T_u(\widetilde{T^kM})$, $\forall u \in \widetilde{T^kM}$ with the property:

\[
T_u(\widetilde{T^kM}) = N_u \oplus V_u, \quad \forall u \in \widetilde{T^kM}.
\]

If we consider the distributions

\[
N_0 = N, N_1 = J(N_0), \ldots, N_{k-1} = J(N_{k-2}), V_k = J(N_{k-1}),
\]

then according to the general theory we obtain the direct decomposition of the vector spaces:

\[
T_u(\widetilde{T^kM}) = N_{0,u} \oplus N_{1,u} \oplus \cdots \oplus N_{k-1,u} \oplus V_{k,u}, \forall u \in \widetilde{T^kM}.
\]

The local adapted basis to the direct decomposition (3.2.9) is

\[
\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)}}, \ldots, \frac{\delta}{\delta y^{(k)}}\right),
\]
where

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^{(1)}j} - \ldots - N^j_i \frac{\partial}{\partial y^{(k)}j},
\]

\[
(3.2.11)
\]

\[
\frac{\delta}{\delta y^{(1)}i} = \frac{\partial}{\partial y^{(1)}i} - N^j_i \frac{\partial}{\partial y^{(2)}j} - \ldots - N^j_i \frac{\partial}{\partial y^{(k)}j},
\]

\[
\frac{\delta}{\delta y^{(k)}i} = \frac{\partial}{\partial y^{(k)}i},
\]

\(N^j_i, \ldots, N^j_i\) being the coefficients (3.2.8) of the Cartan nonlinear connection.

Of course we have

\[
\frac{\delta}{\delta y^{(1)}i} = J(\frac{\delta}{\delta x^i}), \ldots, \frac{\delta}{\delta y^{(k)}i} = J(\frac{\delta}{\delta y^{(k-1)}i}), 0 = J(\frac{\delta}{\delta y^{(k)}i}).
\]

Taking into account section 1.4 ch.1, the dual (adapted) cobasis, of the basis (3.2.10) is:

\[
(3.2.12) \quad (\delta x^i, \delta y^{(1)}i, \ldots, \delta y^{(k)}i)
\]

where

\[
\delta x^i = dx^i,
\]

\[
\delta y^{(1)}i = dy^{(1)}i + M^j_i dx^j,
\]

\[
(3.2.12a)
\]

\[
\delta y^{(k)}i = dy^{(k)}i + M^j_i dy^{(k-1)}i + \ldots + M^j_i dx^j,
\]

\(M^j_i, \ldots, M^j_i\), being the dual coefficients (3.2.6) of the Cartan nonlinear connection \(N^j_i\).

It is not difficult to see that the following identities hold:

\[
(3.2.13) \quad \begin{cases} J^*(\delta y^{(k)}i) = \delta y^{(k-1)}i, & J^*(\delta y^{(k-1)}i) = \delta y^{(k-2)}i, \ldots, \\ J^*(\delta y^{(1)}i) = \delta x^i, & J^*(dx^i) = 0. \end{cases}
\]

\(J^*\) being the adjoint of the \(k\)-structure \(J\).

Now we can determine the differential operators \(d_k, d_{k-1}, \ldots, d_0\) defined in (1.2.11), using the expression of the operator of differentiation \(d_k = d\) in the adapted basis:

\[
(3.2.14) \quad d_k = \frac{\delta}{\delta x^i} \delta x^i + \frac{\delta}{\delta y^{(1)}i} \delta y^{(1)}i + \ldots + \frac{\delta}{\delta y^{(k)}i} \delta y^{(k)}i.
\]
Applying (3.2.13), we get $d_k$ in (3.2.14), and

$$d_{k-1} = J^*d_k, \ d_{k-2} = J^*d_{k-1}, \ldots, \ d_0 = J^*d_1.$$ 

One obtains:

$$d_{k-1} = \frac{\delta}{\delta y^{(1)}} \delta x^i + \frac{\delta}{\delta y^{(2)}} \delta y^{(1)}i + \ldots + \frac{\delta}{\delta y^{(k)}} \delta y^{(k-1)}i,$$

$$d_{k-2} = \frac{\delta}{\delta y^{(2)}} \delta x^i + \frac{\delta}{\delta y^{(3)}} \delta y^{(1)}i + \ldots + \frac{\delta}{\delta y^{(k+1)}} \delta y^{(k-2)}i,$$

(3.2.15)

$$d_1 = \frac{\delta}{\delta y^{(k)}} \delta x^i,$$

$$d_0 = \frac{\delta}{\delta y^{(k)}} \delta x^i.$$ 

Consequently, we get:

**Theorem 3.2.3** With respect to the direct decomposition (3.2.9), in adapted basis (3.2.11), (3.2.12), the Cartan 1-forms $d_0F^2, d_1F^2, \ldots, d_kF^2$ of a Finsler space of order $k$, $F^{(k)}_n = (M, F)$ can be expressed as follows:

$$d_0F^2 = (d_0F^2)^H,$$

(3.2.16)

$$d_1F^2 = (d_1F^2)^H + (d_1F^2)^V_1,$$

$$d_kF^2 = (d_kF^2)^H + (d_kF^2)^V_1 + \ldots + (d_kF^2)^V_k.$$

Equivalently,

$$d_0F^2 = \frac{\delta F^2}{\delta y^{(k)}} \delta x^i,$$

(3.2.17)

$$d_1F^2 = \frac{\delta F^2}{\delta y^{(k-1)}} \delta x^i + \frac{\delta F^2}{\delta y^{(k)}} \delta y^{(1)}i,$$

$$d_kF^2 = \frac{\delta F^2}{\delta x^i} \delta x^i + \frac{\delta F^2}{\delta y^{(1)}} \delta y^{(1)}i + \ldots + \frac{\delta F^2}{\delta y^{(k)}} \delta y^{(k)}i.$$ 

In the previous expressions every term is an 1-form field on $\tilde{T}^kM$. So we have the following main 1-form fields
\begin{align}
\theta_0 &= (d_0 F^2)_H = \frac{\delta F^2}{\delta y^{(k)} i} \delta x^i = p_{(k)i} \delta x^i, \\
\theta_1 &= (d_1 F^2)_{V^i} = \frac{\delta F^2}{\delta y^{(k)} i} \delta y^{(1)i} = p_{(k)i} \delta y^{(1)i}, \\
\theta_{k-1} &= (d_{k-1} F^2)_{V_{k-1}} = \frac{\delta F^2}{\delta y^{(k)} i} \delta y^{(k-1)i} = p_{(k)i} \delta y^{(k-1)i}, \\
\theta_k &= (d_k F^2)_{V_k} = \frac{\delta F^2}{\delta y^{(k)} i} \delta y^{(k)i} = p_{(k)i} \delta y^{(k)i}.
\end{align}

**Theorem 3.2.4** \(\Theta\). The 1-form fields \(\theta_0, \ldots, \theta_k\) depend only on the fundamental function \(F\) of the Finsler space \(F^{(k)n}\).

\(\Theta\). The exterior differentials of \(\theta_0, \ldots, \theta_k\),

\begin{align}
d\theta_0 &= d p_{(k)i} \wedge \delta x^i, \\
d\theta_1 &= d p_{(k)i} \wedge \delta y^{(1)i} + p_{(k)i} \wedge d\delta y^{(1)i}, \\
\vdots \\
d\theta_k &= d p_{(k)i} \wedge \delta y^{(k)i} + p_{(k)i} \wedge d\delta y^{(k)i}
\end{align}

have the same property of homogeneity.

The second terms of \(d\theta_0, \ldots, d\theta_k\), the exterior differentials of 1-forms \(\delta y^{(1)i}, \ldots, \delta y^{(k)i}\) are calculated by means of formulas:

\begin{align}
d\delta y^{(\alpha)i} &= \frac{1}{2} R^{i}_{(\alpha)\gamma} dx^m \wedge dx^j + \sum_{\gamma=1}^{k} B^{i}_{jm} dy^{(\gamma)m} \wedge dx^j + \\
&+ \sum_{\beta, \gamma=1}^{k} C_{jm}^{i(\beta)} dy^{(\gamma)m} \wedge dy^{(\alpha)j},
\end{align}

where \(C_{jm}^{i(\alpha)} = 0\), and the coefficients from the right hand side can be calculated.
using the following Lie brackets:

\[
\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)}_i} \right] = R^j_{\ h} \frac{\delta}{\delta y^{(1)}_h} + \ldots + R^j_{\ (0k)} \frac{\delta}{\delta y^{(k)}_i}, \\
\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(\alpha)}_h} \right] = B^j_{\ h} \frac{\delta}{\delta y^{(1)}_h} + \ldots + B^j_{\ (\alpha k)} \frac{\delta}{\delta y^{(k)}_i}, \\
\left[ \frac{\delta}{\delta y^{(\alpha)}_h}, \frac{\delta}{\delta y^{(\beta)}_k} \right] = C^j_{\ h} \frac{\delta}{\delta y^{(1)}_h} + \ldots + C^j_{\ (\alpha \beta k)} \frac{\delta}{\delta y^{(k)}_i},
\]

\[(\alpha, \beta = 1, \ldots, k).\]

The whole previous theory can be applied to the following Lagrangians associated to the Finsler space $F^{(k)}_n$ (3.2.22)

\[F^1_1 = g_{ij} z^{(1)i} z^{(1)j}, \ldots, F^k_k = g_{ij} z^{(k)i} z^{(k)j}\]

especially in the cases of the particular Finsler spaces of order $k$, $Prol^k \mathbb{R}^n$ or $Prol^k F^n$ (see ch.2). The Lagrangians $F^1_1, \ldots, F^k_k$ are positive functions and are $2, 4, \ldots, 2k$-homogeneous, respectively.

The autoparallel curves of the Cartan nonlinear connection $N$ are characterized by the system of differential equations

\[
\frac{\delta y^{(1)i}}{dt} = \ldots = \frac{\delta y^{(k)i}}{dt} = 0, \\
y^{(1)i} = \frac{dx^i}{dt}, \ldots, y^{(k)i} = \frac{1}{k!} \frac{dx^{(k)i}}{dt^k}.
\]

### 3.3 The Cartan Metrical $N$-Linear Connection

Let $N$ be the Cartan nonlinear connection of the Finsler space of order $k$, $F^{(k)}_n = (M, F)$ having the adapted basis (3.2.10) and its dual (3.2.12).

The lift of the fundamental tensor field $g_{ij}$ is given by (2.6.1),

\[(3.3.1) \quad \mathcal{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + \ldots + g_{ij} \delta y^{(k)i} \otimes \delta y^{(k)j}.\]

**Theorem 3.3.1** $\mathcal{G}$ from (3.3.1) is a Riemannian structure on $\tilde{T}^k M$ which depend only on the fundamental function $F$ of the space $F^{(k)}_n$. The terms of $\mathcal{G}$ are $0, 2, \ldots, 2k$-homogeneous, respectively.

Notice that $\mathcal{G}$ is not homogeneous. We can construct an homogeneous one using the Lagrangians (3.2.22).

Namely

\[(3.3.1a) \quad \mathcal{\hat{G}} = g_{ij} dx^i \otimes dx^j + \frac{1}{F^1_1} g_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + \ldots + \frac{1}{F^k_k} g_{ij} \delta y^{(k)i} \otimes \delta y^{(k)j}.\]
\[ Finsler \ Spaces \ of \ Order \ k \]

\[ \mathcal{G} \] is a Riemannian structure on the manifold \( \widetilde{T^kM} \) determined only by the fundamental function \( F \) and it is 0-homogeneous.

In the following we consider the Riemannian structure \( \mathcal{G} \) from (3.3.1).

An \( N \)-linear connection \( D \) is compatible with \( \mathcal{G} \) if

\[ D_X \mathcal{G} = 0, \forall X \in \mathfrak{X}(\widetilde{T^kM}). \]

Applying the Theorem 2.6.1, we have:

**Theorem 3.3.2** For a Finsler space of order \( k \), \( F^{(k)n} = (M, F) \), the following properties hold:

1. There exists an unique \( N \)-linear connection \( D \) on \( \widetilde{T^kM} \) verifying the axioms:
   - \( A_1 \) \( N \) is the Cartan nonlinear connection,
   - \( A_2 \) \( g_{ij \mid h} = 0, \) \( \alpha \)
   - \( A_3 \) \( g_{ij \mid h} = 0, \) \( \alpha \)
   - \( A_4 \) \( F_{jk}^i = F_{kj}^i, \) \( \alpha \)
   - \( A_5 \) \( C_{jk}^i = C_{kj}^i (\alpha = 1, ..., k). \) \( \alpha \)

2. The coefficients \( CT(N) = (F_{jk}^i, C_{jk}^i, ..., C_{jk}^i) \) of \( D \) are given by the generalized Christoffel symbols (2.6.3), \( (F_{jk}^i = L_{jk}^i). \)

3. \( D \) depends only on the fundamental function \( F \) of the space \( F^{(k)n} \).

The metrical \( N \) linear -connection \( D \) from the previous theorem will be called the Cartan metrical \( N \)-linear connection of the space \( F^{(k)n} \) and denoted by \( CT(N) \).

Of course, the torsion \( d \)-tensor fields and the curvature \( d \)-tensor fields of \( CT(N) \) can be written without difficulties. Such that we have

\[ (3.3.2) \quad T_{jk}^i = 0, \quad S_{jk}^i = 0, (\alpha = 1, ..., k). \]

Also we can calculate the deflection tensor of \( CT(N) \) :

\[ D_j^i = z_{(\alpha)}^{(\alpha)} |_{(\alpha)} \]

\[ D_j^i = z_{(\beta)}^{(\beta)} |_{(\alpha)} \]

The coefficients \( CT(N) = (F_{jk}^i, C_{jk}^i, ..., C_{jk}^i) \) are \( 0, -1, ..., -k \) -homogeneous.

The \( d \)-tensors of curvature of \( CT(N) \) satisfy the identities:

\[ g_{sj} R_{hm}^i + g_{is} R_{hm}^i = 0, \]

\[ g_{sj} P_{(\alpha)}^{i} |_{hm} + g_{is} P_{(\alpha)}^{i} |_{hm} = 0, \]

\[ g_{sj} S_{(\alpha \beta)}^{i} |_{hm} + g_{is} S_{(\alpha \beta)}^{i} |_{hm} = 0 \]
Notice that the equations $R^i_{\alpha jh} = 0, (\alpha = 1, ..., k)$ characterize the integrability of the Cartan nonlinear connection.

The connection 1-forms $\omega^i_j$ of the Cartan metrical $N$-linear connection $C_\Gamma(N)$ are given by:

\[
\omega^i_j = F^i_{jh}dx^k + C^i_{jh}\delta y^{(1)h} + ... + C^i_{jh}\delta y^{(k)h}.
\]

**Theorem 3.3.3** The structure equations of the Cartan metrical $N$-linear connection $C_\Gamma(N)$ of the Finsler space $F^{(k)n}$ are given by:

\[
\begin{align*}
\text{(3.3.5)} & \quad d(dx^i) - dx^m \wedge \omega^i_m = -\Omega^i, \\
\text{(3.3.5)} & \quad d(\delta y^{(\alpha)i}) - \delta y^{(\alpha)m} \wedge \omega^i_m = -\Omega^i, (\alpha = 1, ..., k), \\
\text{(3.3.5)} & \quad d\omega^i - \omega^m \wedge \omega^i_m = -\Omega^i,
\end{align*}
\]

where $\Omega^i, \Omega^i_i$ are the 2-forms of torsion:

\[
\begin{align*}
\text{(3.3.6)} & \quad \Omega^i = C^i_{jh}dx^j \wedge \delta y^{(1)h} + ... + C^i_{jh}dx^j \wedge \delta y^{(k)h} \\
\text{(3.3.6)} & \quad \Omega^i_i = \frac{1}{2} R^i_{jkh}dx^j \wedge dx^h + \sum_{\gamma=1}^{k} B^i_{jhm}dx^j \wedge \delta y^{(\gamma)h} + \sum_{\gamma=1}^{k} (C^i_{jgh}\delta y^{(\gamma)h}) \delta y^{(\alpha)h}
\end{align*}
\]

and $\Omega^i_j$ are the 2-forms of curvature:

\[
\text{(3.3.7)}
\Omega^i_j = \frac{1}{2} R^i_{jph}dx^p \wedge dx^h + \sum_{\gamma=1}^{k} P^i_{jgh}dx^h \wedge \delta y^{(\gamma)h} + \sum_{\beta,\gamma=1}^{k} S^i_{j\beta\gamma} dx^\beta \wedge \delta y^{(\gamma)p} \wedge \delta y^{(\gamma)q}.
\]

Now, we can obtain the Bianchi identities of the Cartan metrical $N$-linear connection $C_\Gamma(N)$ if we apply the exterior differential to the system of equations (3.3.5) and calculate $d\Omega^i, d\Omega^i_i$ and $d\Omega^i_j$ from (3.3.6) and (3.3.7), modulo the system (3.3.5).

Finally, consider the tensor field $F$ determined by the Cartan nonlinear connection $N$.

\[
\text{(3.3.8)}
F = -\frac{\delta}{\delta y^{(k)i}} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes \delta y^{(k)i}.
\]
It is not difficult to prove that $C\Gamma(N)$ has the property $D_X F = 0$. Indeed,

$$D_{\delta \frac{\delta}{\delta x^h}} \mathbb{F} = -(D_{\delta \frac{\delta}{\delta x^h}} \frac{\delta}{\delta y^{(k) i}}) \otimes dx^i - \frac{\delta}{\delta y^{(k) i}} \otimes D_{\delta \frac{\delta}{\delta x^h}} dx^i +$$

$$(D_{\delta \frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^i}) \otimes \delta y^{(k) i} + \frac{\delta}{\delta y^{(k) i}} \otimes D_{\delta \frac{\delta}{\delta y^{(k) i}}} \delta y^{(k) i} =$$

$$-F_{m h}^i \frac{\delta}{\delta y^{(k) m}} \otimes dx^i + F_{m b}^i \delta_{b y^{(k) i}} \otimes dx^m + F_{m b}^i \frac{\delta}{\delta x^m} \otimes \delta y^{(k) i} - F_{m b}^i \delta \frac{\delta}{\delta x^m} \otimes \delta y^{(k) i} = 0.$$
Chapter 4

The Geometry of the Dual of $k$-Tangent Bundle

In the book [115] one studies the geometry of the dual of the 2-tangent bundle, which has been used to investigate the Hamilton spaces of order 2. Now, we consider this problem for the general case, $k \geq 1$.

The dual bundle $T^*kM$ of the $k$-tangent bundle must have the same properties as the cotangent bundle $T^*M$ with respect to tangent bundle $TM$. Such that the manifold $T^*kM$ should have the same dimension $(k+1)n$ as the manifold $T^kM$. $T^*kM$ should carry a natural presymplectic structure and at least one Poisson structure. The manifolds $T^kM$ and $T^*kM$ should be locally diffeomorphic.

The dual bundle $T^*kM$ plays a main role in construction of the notion of Hamilton space of order $k$.

In the present chapter we introduce the bundle $T^*kM$ and point out the main geometrical natural object fields, that live on the differentiable manifold $T^kM$.

4.1 The Dual Bundle $(T^*kM, \pi^*k, M)$

**Definition 4.1.1** We call the dual bundle of $k$-tangent bundle $(T^kM, \pi^k, M)$ the differentiable bundle $(T^*kM, \pi^*k, M)$ whose total space is the fibered product:

$$T^*kM = T^{k-1}M \times_M T^*M$$

and for which the canonical projection $\pi^*k$ is

$$\pi^*k = \pi^{k-1} \times_M \pi^*.$$  

The previous fibered product has a differentiable structure given by that of the $(k - 1)$-tangent bundle $(T^{k-1}M, \pi^{k-1}, M)$ and the cotangent bundle $(T^*M, \pi^*, M)$. For $k = 1$, we have $T^*1M = T^*M$ and $\pi^*1 = \pi^*$. Sometimes we denote $(T^*kM, \pi^*k, M)$ by $T^*kM$. 

59
A point \( u \in T^*kM \) will be denoted by \( u = (x, y^{(1)}, \ldots, y^{(k-1)}, p), \pi^k(u) = x \).

The projections on the factors of the \( T^*kM \) from (4.1.1) are

\[
\pi^k_{k-1}: T^*kM \to T^{k-1}M, \quad \pi^k_{k-1}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = (x, y^{(1)}, \ldots, y^{(k-1)}),
\]

\[
\pi^*: T^*kM \to T^*M, \quad \pi^*(x, y^{(1)}, \ldots, y^{(k-1)}, p) = (x, p)
\]

and the canonical projection \( \pi^k: T^*kM \to M \).

Therefore, the following diagram is commutative:

\[
\begin{array}{ccc}
T^*kM & \xrightarrow{\pi^k} & T^*M \\
\downarrow{\pi^k_{k-1}} & & \downarrow{\pi^*} \\
T^{k-1}M & \xrightarrow{\pi^k_{k-1}} & M
\end{array}
\]

Let \((x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i), (i = 1, 2, \ldots, n = \dim M)\) be the coordinates of a point \( u = (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^*kM \) in a local chart \(((\pi^k)^{-1}(U), \Phi)\) on \( T^*kM \).

The change of coordinates on the manifold \( T^*kM \) is:

\[
\tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^n), \quad \det \left( \frac{\partial \tilde{x}^j}{\partial x^j} \right) \neq 0,
\]

\[
\tilde{y}^{(1)i} = \frac{\partial \tilde{x}^j}{\partial x^j} y^{(1)j},
\]

\[
\begin{array}{c}
\ldots \\
(k-1) \tilde{y}^{(k-1)i} = \frac{\partial \tilde{y}^{(k-2)i}}{\partial x^j} y^{(1)j} + \cdots + (k-1) \frac{\partial \tilde{y}^{(k-2)i}}{\partial y^{(k-1)j}} y^{(k-1)j},
\end{array}
\]

\[
\tilde{p}_i = \frac{\partial \tilde{x}^j}{\partial x^j} p_j.
\]

where the following equalities hold:

\[
\frac{\partial \tilde{y}^{(\alpha)i}}{\partial x^j} = \frac{\partial \tilde{y}^{(\alpha+1)i}}{\partial y^{(1)i}} = \cdots = \frac{\partial \tilde{y}^{(k-1)i}}{\partial y^{(k-1-\alpha)j}}, \quad (\alpha = 0, \ldots, k-2; y^{(0)} = x).
\]

Sometimes (cf. Ch. 1) \( y^{(1)i}, \ldots, y^{(k-1)i} \) will be called \( \text{accelerations of order} \ 1, 2, \ldots, k-1, \) respectively and \( p_i \) will be called \( \text{momenta} \). \( T^*kM \) is a real manifold of dimension \((k+1)n\). So it has the same dimension as that of the manifold \( T^kM \).

The natural basis of the vector space \( T_u(T^*kM) \) at the point \( u \in T^*kM \),
calculated at the point \( u \in T^k M \).

The Jacobian matrix of the transformation (4.1.2) at the point \( u \in T^k M \) is:

\[
\begin{align*}
\frac{\partial}{\partial x^i} & \begin{pmatrix} \frac{\partial \tilde{x}^j}{\partial x^i} & 0 & 0 & \cdots & 0 & 0 \\ \frac{\partial \tilde{y}^{(1)}_j}{\partial x^i} & \frac{\partial \tilde{y}^{(1)}_j}{\partial y^{(1)}_j} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \tilde{y}^{(k-1)}_j}{\partial x^i} & \frac{\partial \tilde{y}^{(k-1)}_j}{\partial y^{(k-1)}_j} & \frac{\partial \tilde{y}^{(k-1)}_j}{\partial y^{(k-1)}_j} & \cdots & \cdots & 0 \\ \frac{\partial \tilde{p}_j}{\partial x^i} & 0 & 0 & \cdots & 0 & \frac{\partial x^j}{\partial x^i} \end{pmatrix} 
\end{align*}
\]

(4.1.5) \( J_k \)

It follows

\[
\det J_k(u) = \left[ \det \left( \frac{\partial \tilde{x}^j}{\partial x^i} \right) \right]^{k-1}.
\]

(4.1.5a)

**Theorem 4.1.1**

1. If \( k \) is an odd number, then \( T^k M \) is an orientable manifold.

2. If \( k \) is an even number, the manifold \( T^k M \) is orientable if and only if the base manifold \( M \) is orientable.

The form (4.1.5) of the Jacobian matrix implies the following transformation, with respect to (4.1.2), of the natural cobasis \( \{ dx^i, dy^{(1)}_i, \ldots, dy^{(k-1)}_i, dp_i \} \), at
a point \( u \in T^*^k M \):

\[
dx^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j,
\]

\[
d\tilde{y}^{(1)} = \frac{\partial \tilde{y}^{(1)}_i}{\partial x^j} dx^j + \frac{\partial \tilde{y}^{(1)}_i}{\partial y^{(1)}_j} dy^{(1)}_j,
\]

\[
d\tilde{y}^{(k-1)} = \frac{\partial \tilde{y}^{(k-1)}_i}{\partial x^j} dx^j + \frac{\partial \tilde{y}^{(k-1)}_i}{\partial y^{(1)}_j} dy^{(1)}_j + \cdots + \frac{\partial \tilde{y}^{(k-1)}_i}{\partial y^{(k-1)}_j} dy^{(k-1)}_j,
\]

\[
d\tilde{p}_j = \frac{\partial \tilde{p}_j}{\partial x^i} dx^i + \frac{\partial \tilde{x}^i}{\partial x^j} dp_i.
\]

Also, we can prove without difficulties the following theorem:

**Theorem 4.1.2** If the differentiable manifold \( M \) is paracompact, then the differentiable manifold \( T^*^k M \) is paracompact.

Consider the category \( \text{Man} \) of differentiable manifolds.

There exists a covariant functor \( T^*^k : \text{Man} \to \text{Man} \) in which the differentiable mappings \( f : M \to N \), analytical expressed by \( x^i = x^i(x^1, ..., x^n), (i', j' = 1', 2', ..., n' = \dim N) \) give the mappings \( T^*^k f : T^*^k M \to T^*^k N \), in the form

\[
x'^i = x'^i(x^1, ..., x^n),
\]

\[
y'^{(1)} = \frac{\partial x'^i}{\partial x^j} y^{(1)}_j,
\]

\[
(k - 1)y'^{(k-1)} = \frac{\partial y'^{(k-2)}_i}{\partial x^j} y^{(1)}_j + \cdots + (k - 1) \frac{\partial y'^{(k-2)}_i}{\partial y^{(k-1)}_j} y^{(k-1)}_j,
\]

\[
\frac{\partial x'^i}{\partial x^j} p_i = p_j.
\]

This fact will be used in the theory of subspaces in Hamilton spaces of order \( k \).

### 4.2 Vertical Distributions. Liouville Vector Fields

The null section \( 0 : M \to T^*^k M \) of the projection \( \pi^k \) is defined by \( 0 : x \in M \to (x, 0, ..., 0) \in T^*^k M \). As usual we denote \( \pi^k M = T^*^k M \setminus \{0\} \).

The tangent bundle of the manifold \( T^*^k M \), \( (T^*^k M, d\pi^k, T M) \), allows to define the vertical subbundle \( V T^*^k M = \ker d\pi^k \). We get the vertical distribution \( V \), formed by the fibres of \( V T^*^k M \). \( V \) is locally generated by the set of vector fields \( \left\{ \frac{\partial}{\partial y^{(1)}_i}, \cdots, \frac{\partial}{\partial y^{(k-1)}_i}, \frac{\partial}{\partial p_i} \right\} \) at every point \( u \in T^*^k M \).
So, $V$ is an integrable distribution and its dimension is $kn$. It is convenient to adopt the notation:

\[(4.2.1) \quad \hat{\partial}^i = \frac{\partial}{\partial p_i}.\]

Taking into account the relations (4.1.4) we can consider the following sub-distributions of $V$:

$V_{k-1}$, locally generated by $\left\{ \frac{\partial}{\partial y^{(k-1)i}} \right\}$. Its dimension is $n$ and it is integrable.

$V_{k-2}$, locally generated by $\left\{ \frac{\partial}{\partial y^{(k-2)i}}, \frac{\partial}{\partial y^{(k-1)i}} \right\}$. This has dimension $2n$ and it is integrable, too and so on.

$V_1$, locally generating by $\left\{ \frac{\partial}{\partial y^{(1)i}}, \cdots, \frac{\partial}{\partial y^{(k-1)i}} \right\}$. Its dimension is $(k-1)n$ and it is also integrable. Of course, we have the sequence of inclusions:

$V_{k-1} \subset V_{k-2} \subset \cdots \subset V_1 \subset V$.

The transformation of vector field $\hat{\partial}^i$ from (4.1.4), $\hat{\partial}^i = \frac{\partial x^i}{\partial \tilde{x}^j} \partial^j$, shows that we have one more vertical distribution $W_k$, locally generated by the vector fields $\left\{ \hat{\partial}^i \right\}$ at the points $u \in (\pi^*_k)^{-1}(U)$. Its dimension is $n$ and it is an integrable distribution, too.

We conclude by

**Proposition 4.2.1** The following direct sum of vector spaces holds:

\[(4.2.2) \quad V_u = V_{1,u} \oplus W_{1,u}, \quad \forall u \in T^{*k}M.\]

Using again (4.1.4) we obtain without difficulties

**Theorem 4.2.1** 1º The following operators in the algebra of functions $\mathcal{F}(T^{*k}M)$:

\[(4.2.3) \quad \Gamma^1 = y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}}, \]

\[(4.2.3) \quad \Gamma^2 = y^{(1)i} \frac{\partial}{\partial y^{(k-2)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k-1)i}}, \]

\[\cdots\]

\[(4.2.3) \quad \Gamma^{k-1} = y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + (k-1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-1)i}}, \]

and

\[(4.2.4) \quad C^* = p_i \hat{\partial}^i.\]
are vector fields on $T^*kM$. They are independent vector fields on the manifold $\tilde{T}^*kM$.

$\varphi = p_iy^{(1)i}$ is a scalar function on the manifold $T^*kM$.

Evidently, $1^\Gamma$ belongs to the distribution $V_{k-1}$, $2^\Gamma$ belongs to $V_{k-2}$, ..., $k-1^\Gamma$ belongs to the distribution $W_k$.

$1^\Gamma, ..., k-1^\Gamma$ are called the Liouville vector fields and $C^*$ is the Hamilton vector field on $T^*kM$.

The Liouville vector fields $1^\Gamma, ..., k-1^\Gamma$ are linearly independent on $\tilde{T}^*kM$ and exactly as in ch. 1 we can prove:

Theorem 4.2.2 For any differentiable function $H : \tilde{T}^*kM \to \mathbb{R}$, $d_0H$, $d_1H$, ..., $d_{k-2}H$ defined by

\begin{equation}
    d_0H = \frac{\partial H}{\partial y^{(k-1)i}} dx^i,
\end{equation}
\begin{equation}
    d_1H = \frac{\partial H}{\partial y^{(k-2)i}} dx^i + \frac{\partial H}{\partial y^{(k-1)i}} dy^{(1)i},
\end{equation}
\begin{equation}
    d_{k-2}H = \frac{\partial H}{\partial y^{(1)i}} dx^i + \frac{\partial H}{\partial y^{(2)i}} dy^{(1)i} + \ldots + \frac{\partial H}{\partial y^{(k-1)i}} dy^{(k-2)i}
\end{equation}
are fields of 1-form on $\tilde{T}^*kM$.

Proposition 4.2.2 If $d_{k-1}H$ given by

\begin{equation}
    d_{k-1}H = \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial y^{(1)i}} dy^{(1)i} + \ldots + \frac{\partial H}{\partial y^{(k-1)i}} dy^{(k-1)i}
\end{equation}
is not a field of 1-form.

Under a transformation of coordinate on $T^*kM$, $d_{k-1}H$ transforms as follows

\begin{equation}
    d_{k-1}H = d_{k-1}\tilde{H} + \tilde{\partial} H \frac{\partial y_j}{\partial x^i} dx^i.
\end{equation}

If $\tilde{\partial} H = 0$, then $d_{k-1}H$ is a field of 1-form.

Proposition 4.2.3 The relation between the differential $dH$ and $d_{k-1}H$ are given by

\begin{equation}
    dH = d_{k-1}H + \tilde{\partial} H dp_i.
\end{equation}
Indeed,
\[ dH = \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial y^{(1)i}} dy^{(1)i} + \cdots + \frac{\partial H}{\partial y^{(k-1)i}} dy^{(k-1)i} + \partial^i H dp_i = \]
\[ = d_{k-1}H + \partial^i H dp_i. \]

**Remark** The differential \( dH \) being invariant under the coordinate transformations (4.1.2), i.e. \( dH = d\tilde{H} \), from (4.2.9) it follows the rule of transformation (4.2.8) of \( d_{k-1}H \).

If \( H = \varphi = p_i y^{(1)i} \), then \( d_0 H = \cdots = d_{k-3}H = 0 \) and
\[ (4.2.10) \quad \omega = d_{k-2}\varphi = p_i dx^i. \]

\( \omega \) is called the Liouville 1-form on the manifold \( \tilde{T}^*kM \).

The exterior differential \( d\omega \) of the Liouville 1-form \( \omega \) is expressed by
\[ (4.2.11) \quad \theta = d\omega = dp_i \wedge dx^i. \]

Using (4.1.6) we can prove the invariance of 2-form \( \theta \) with respect to (4.1.2).

Now, based on the previous results we obtain:

**Theorem 4.2.3**
1. The differential forms \( \omega \) and \( \theta \) are globally defined on the manifold \( \tilde{T}^*kM \).
2. \( \theta \) is a closed 2-form, i.e. \( d\theta = 0 \).
3. \( \theta \) is a 2-form of rank \( 2n \). It is a presymplectic structure on \( \tilde{T}^*kM \).

**Proof:**
1. \( \omega \) and \( \theta \) are invariant with respect to a change of coordinates (4.1.2).
2. \( \theta = d\omega \) implies \( d\theta = 0 \).
3. \( \theta = dp_i \wedge dx^i \) is a 2-form of rank \( 2n < (k+1)n = \dim T^*kM \), for \( k > 1 \).

Consequently \( \theta \) is a presymplectic structure on \( T^*kM \).

**Remarks**
1. For \( k = 1 \), \( \omega \) and \( \theta \) are the Poincaré-Cartan forms on the cotangent bundle \( T^*M \).
2. The previous theorem shows the existence of a natural presymplectic structure on \( T^*kM \).

### 4.3 The Structures \( J \) and \( J^* \)

There exists a tangent structure \( J \) on \( T^*kM \) defined as usual by the endomorphism \( J : \mathcal{X}(T^*kM) \to \mathcal{X}(T^*kM) \):

\[ J \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^{(1)i}}, J \left( \frac{\partial}{\partial y^{(1)i}} \right) = \frac{\partial}{\partial y^{(2)i}}, \cdots, \]
\[ (4.3.1) \quad J \left( \frac{\partial}{\partial y^{(k-2)n}} \right) = \frac{\partial}{\partial y^{(k-1)i}}, J \left( \frac{\partial}{\partial y^{(k-1)i}} \right) = 0, J(\partial^i) = 0 \]
at every point \( u \in \tilde{T^kM} \).

By means of (4.1.4) one proves without difficulties

**Theorem 4.3.1**

\( J^0 \) is globally defined on \( T^kM \).

\( J^0 \) is a tensor field of type \((1, 1)\), locally expressed by

\[
J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \cdots + \frac{\partial}{\partial y^{(k-1)i}} \otimes dy^{(k-2)i}.
\]

\( J \) is integrable.

\( J \circ J \circ \cdots \circ J = J^k = 0 \).

\( \ker J = V_{k-1} \oplus W_k, \text{Im } J = V_1 \).

\( \text{rank } ||J|| = (k-1)n. \)

According to the above theorem we may call \( J \) the \( k-1 \)-tangent structure.

The endomorphism \( J \) applied to the Liouville vector fields gives us:

\[
J(1\Gamma) = 0, \quad J(2\Gamma) = 1\Gamma, \quad \ldots, \quad J(k-1\Gamma) = k-2\Gamma,
\]

and

\[
J(C^*) = 0.
\]

Let us consider a vector field \( X \in \mathcal{X}(T^kM) \), locally expressed by:

\[
X = (0)i \frac{\partial}{\partial x^i} + (1)i \frac{\partial}{\partial y^{(1)i}} + \cdots + (k-1)i \frac{\partial}{\partial y^{(k-1)i}} + X_i \partial^i.
\]

**Proposition 4.3.1**

\( J^0 \) For any vector field \( X \in \mathcal{X}(T^kM) \), \( 1, \ldots, k-1 \)

\( X \) given by

\[
1X = JX, \quad 2X = J^2X, \quad \ldots, \quad k-1X = J^{k-1}X.
\]

are vector fields.

\( J^0 \) If \( X \) is given by (4.3.4), then we have:

\[
1X = (0)i \frac{\partial}{\partial y^{(1)i}} + \cdots + (k-2)i \frac{\partial}{\partial y^{(k-1)i}},
\]

\[
2X = (0)i \frac{\partial}{\partial y^{(2)i}} + \cdots + (k-3)i \frac{\partial}{\partial y^{(k-1)i}},
\]

\[
\vdots
\]

\[
k-1X = (0)i \frac{\partial}{\partial y^{(k-1)i}}.
\]

\( J^0 \) The vector field \( 1X \) belongs to the vertical distribution \( V_1 \), \( 2X \) belongs to the distribution \( V_2 \), \ldots, \( k-1X \) belongs to the distribution \( V_{k-1} \).
Now consider the adjoint of $k$-1-tangent structure $J$. It is the endomorphism $J^* : \mathcal{X}^*(T^*kM) \to \mathcal{X}^*(T^*kM)$ defined by

\begin{equation}
J^*(dy^{(k-1)i}) = dy^{(k-2)i}, \ldots, J^*(dy^{(1)i}) = dx^i,
\end{equation}

(4.3.5)

\begin{equation}
J^*(dx^i) = 0, \ J^*(dp_i) = 0.
\end{equation}

Using (4.3.5) and (4.1.6) we obtain:

**Theorem 4.3.2** $J^*$ is globally defined on $T^*kM$.

1. $J^*$ is a tensor field of type $(1,1)$ on $T^*kM$, i.e.

\begin{equation}
J^* = dx^i \otimes \frac{\partial}{\partial y^{(1)i}} + dy^{(1)i} \otimes \frac{\partial}{\partial y^{(2)i}} + \cdots + dy^{(k-2)i} \otimes \frac{\partial}{\partial y^{(k-1)i}}.
\end{equation}

(4.3.6)

2. $\text{rank } ||J^*|| = (k-1)n$.

3. $J^*$ is an integrable structure.

$J^*$ is called the $k-1$-adjoint tangent structure.

$J^*$ can be extended to an endomorphism of the exterior algebra $\Lambda(T^*kM)$ as follows:

\begin{equation}
J^* f = f, \ \forall f \in \mathcal{F}(T^*kM),
\end{equation}

(4.3.7)

\begin{equation}
(J^* \omega)(X_1, \ldots, X_q) = \omega(JX_1, \ldots, JX_q), \ \forall \omega \in \Lambda^q(T^*kM).
\end{equation}

Let be $\omega \in \Lambda^1(T^*kM)$ and consider

\begin{equation}
\omega^1 = J^* \omega, \ldots, \omega^{k-1} = J^{(k-1)} \omega.
\end{equation}

(4.3.7a)

Then $\omega^1, \ldots, \omega^{k-1}$ are 1-form fields.

In particular, we get

\begin{equation}
J^* dH = d_{k-2}H, \ldots, J^{(k-1)} dH = d_0 H.
\end{equation}

(4.3.8)

The $k-1$ adjoint structure $J^*$ allows to introduce the vertical differential operator in the exterior algebra $\Lambda(T^*kM)$, [74].

Taking into account the operator of differentiation:

\[ d = \frac{\partial}{\partial x^i} dx^i + \frac{\partial}{\partial y^{(1)i}} dy^{(1)i} + \cdots + \frac{\partial}{\partial y^{(k-1)i}} dy^{(k-1)i} + \partial^i_\partial dp_i \]
we define the following operators:

\[ d_{k-2} = J^* d = \frac{\partial}{\partial y^{(1)i}} dx^i + \cdots + \frac{\partial}{\partial y^{(k-1)i}} dy^{(k-2)i}, \]

\[ d_{k-3} = J^* d = \frac{\partial}{\partial y^{(2)i}} dx^i + \cdots + \frac{\partial}{\partial y^{(k-1)i}} dy^{(k-3)i}, \]

\[ d_1 = J^*(k-2) d = \frac{\partial}{\partial y^{(k-2)i}} dx^i + \frac{\partial}{\partial y^{(k-1)i}} dy^{(k-1)i}, \]

\[ d_0 = J^*(k-1) d = \frac{\partial}{\partial y^{(k-1)i}} dx^i. \]  

(4.3.9)

Clearly, these operators \( d_0, \ldots, d_{k-2} \) and \( d \) do not depend on the transformation of coordinates on the manifold \( T^k M \).

For any \( H \in \mathcal{F}(T^k M), d_0 H, \ldots, d_{k-2} H \) are given by Theorem 4.2.2. \( d_0, \ldots, d_{k-2} \) are the vertical operators of differentiation.

They can be extended to the exterior algebra \( \Lambda(T^k M) \) if we give their restrictions to \( \Lambda^0(T^k M) \) and \( \Lambda^1(T^k M) \).

As we already have seen \( d_0 H, \ldots, d_{k-2} H \) are expressed in (4.2.6) and \( dH \) is the differential of \( H \). The restrictions to \( \Lambda^1(T^k M) \) are defined by

\[
\begin{align*}
\alpha(d x^i) &= 0, \quad \alpha(dp_i) = 0, \quad (\alpha = 0, \ldots, k-2), \\
\alpha(dy^{(\beta)i}) &= 0, \quad (\alpha = 1, \ldots, k-2; \ \beta = 1, \ldots, k-1), \\
\alpha(dx^i) &= 0, \quad \alpha(dy^{(\beta)i}) = 0, \quad \alpha(dp_i) = 0.
\end{align*}
\]

(4.3.10)

In this case \( d_0, \ldots, d_{k-2} \) and \( d \) are the antiderivations of degree 1.

**Proposition 4.3.2** The vertical differential operators \( d_0, \ldots, d_{k-2} \) and \( d \) have the property

\[ d_\alpha \circ d_\beta = 0, \quad (\alpha = 0, \ldots, k-2), \quad d \circ d = 0. \]  

(4.3.11)

For instance, applying the exterior differential \( d \) to the 1-forms \( d_0 H, d_1 H, \ldots, d_{k-2} H \), written in the form

\[ d_0 H = p_i \, dx^i, \]

\[ d_1 H = p_i \, dx^i + p_i \, dy^{(1)i}, \]

(4.3.12)

\[ d_{k-2} H = p_i \, dx^i + p_i \, dy^{(1)i} + \cdots + p_i \, dy^{(k-2)i}, \]

with

\[ p_i = \frac{\partial H}{\partial y^{(k-1)i}}, \quad p_i = \frac{\partial H}{\partial y^{(k-2)i}}, \quad \ldots, \quad p_i = \frac{\partial H}{\partial y^{(1)i}}. \]  

(4.3.12a)
we obtain
\[ dd_0 H = d^{(0)} p_i \wedge dx^i, \]
\[ dd_1 H = d^{(1)} p_i \wedge dx^i + d^{(0)} p_i \wedge dy^{(1)i}, \]
\[ \cdots \]
\[ dd_{k-2} H = d^{(k-2)} p_i \wedge dx^i + d^{(k-3)} p_i \wedge dy^{(1)i} + \cdots + d^{(0)} p_i \wedge dy^{(k-2)i}. \]

The previous formulae (4.3.12) are similar to the Jacobi-Ostrogradski formulae (2.4.2) from the Lagrange spaces of order \( k, L^{(k)} M \). The equalities (4.3.13) are important in the geometrical theory of the Hamiltonians \( H(x, y^{(1)}, ..., y^{(k-1)}, p) \).

### 4.4 Canonical Poisson Structures on \( T^*k M \)

In this section we state that on the manifold \( T^*k M \) there exists at least a Poisson structure. Let us consider the brackets:

\[ \{ f, g \}_0 = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i}, f, g \in \mathcal{F}(T^*k M), \]

\[ \{ f, g \}_\alpha = -\frac{\partial f}{\partial y^{(\alpha)i}} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial y^{(\alpha)i}}, (\alpha = 1, ..., k-1). \]

**Theorem 4.4.1** The bracket \( \{ \cdot, \cdot \}_\alpha, (\alpha = k - 1) \) is a canonical Poisson structure on the manifold \( T^*k M \).

**Proof:** Indeed, remarking that with respect to a change of local coordinates on \( T^*k M, \frac{\partial f}{\partial y^{(\alpha)i}} \frac{\partial g}{\partial p_i} \) has a geometrical meaning, it follows

1. For any \( f, g \in \mathcal{F}(T^*k M), \{ f, g \}_{k-1} \) is a differentiable function on \( T^*k M \).
2. \( \{ f, g \}_{k-1} = -\{ g, f \}_{k-1}. \)
3. \( \{ f, g \}_{k-1} \) is \( \mathbb{R} \)-linear in every argument.
4. The Jacobi identities are verified:
   \[ \{ \{ f, g \}_{k-1}, h \}_{k-1} + \{ \{ g, h \}_{k-1}, f \}_{k-1} + \{ \{ h, f \}_{k-1}, g \}_{k-1} = 0. \]

Also we remark here

5. \( \{ \cdot, gh \}_{k-1} = \{ \cdot, g \}_{k-1} h + \{ \cdot, h \}_{k-1} g. \)

We can see, without difficulties that every bracket \( \{ f, g \}_0 \) and \( \{ f, g \}_\alpha, (\alpha = 1, ..., k-2) \) verifies 2\(^0\), 3\(^0\), 4\(^0\) and 5\(^0\) but they do not satisfy the property 1\(^0\).

The restrictions of these brackets to the special submanifolds immersed in \( T^*k M \) can induce Poisson structures. As we shall see in ch. 8, §3, the restriction of \( \{ f, g \}_0 \), \( f, g \in \mathcal{F}(\Sigma_0) \) to the submanifold \( \Sigma_0 \):

\[ \Sigma_0 = \{(x, y^{(1)}, ..., y^{(k-1)}, p) \in T^{(k)} M | y^{(1)i} = \cdots = y^{(k-1)i} = 0 \} \]
has this property.

Using the notion of nonlinear connection $N$, studied in chapter 6 of the present monograph, we can take an adapted basis $\delta x^i$, $\delta y^i(\alpha)$, $(\alpha = 1, \ldots, k-2)$, and remarking that $\frac{\delta f}{\delta x^i} \partial g/\partial p_i$ and $\frac{\delta f}{\delta y^i(\alpha)} \partial g/\partial p_i$, $(\alpha = 1, \ldots, k-2)$, have geometrical meaning we can construct the following brackets:

$$
\{f, g\}_0^N = \frac{\delta f}{\delta x^i} \partial g/\partial p_i - \frac{\delta g}{\delta x^i} \partial f/\partial p_i, f, g \in \mathcal{F}(T^*kM),
$$

$$
\{f, g\}_\alpha^N = \frac{\delta f}{\delta y^i(\alpha)} \partial g/\partial p_i - \frac{\partial f}{\partial p_i} \frac{\delta g}{\delta y^i(\alpha)}, (\alpha = 1, \ldots, k-2).
$$

We can check, without difficulties

**Proposition 4.4.1** Every bracket from (4.4.2) has the properties $t^0$, $\varphi^0$, $s^0$, $\phi^0$ from Theorem 4.4.1.

We shall see later that for some particular nonlinear connections, the brackets (4.4.2) have also the property $4^0$.

### 4.5 Homogeneity

The notion of homogeneity for the functions $H(x, y^{(1)}, \ldots, y^{(k-2)}, p)$ defined on the manifold $T^*kM$ can be considered with respect to the vertical variables $y^{(1)i}$, ..., $y^{(k-1)i}$, as well as with respect to the momenta $p_i$, respectively.

Indeed, any homothety $h_a : T^*kM \to T^*kM$

$$
h_a(x, y^{(1)}, y^{(2)}, \ldots, y^{(k-1)}, p) = (x, ay^{(1)}, a^2y^{(2)}, \ldots, a^{k-1}y^{(k-1)}, p)
$$

is preserved by the transformations of local coordinates (4.1.2) on $T^*kM$.

Let $H_y$ be a group of transformation on $T^*kM$:

$$
H_y = \{h_a | a \in \mathbb{R}^+ \}.
$$

The orbit of a point $u_0 = (x_0, y_0^{(1)}, \ldots, y_0^{(k-1)}, p^0)$ by $H_y$ is given by

$$
x^i = x_0^i, \quad y^{(1)i} = ay_0^{(1)i}, \quad \ldots, \quad y^{(k-1)i} = a^{k-1}y_0^{(k-1)i}, \quad p_i = p_0^i, \quad a \in \mathbb{R}^+.
$$

The tangent vector at the point $u_0 = h_1(u_0)$ is the Liouville vector field $\Gamma^{k-1}$ at a point $u_0$:

$$
\Gamma^{k-1}(u_0) = y_0^{(1)i} \frac{\partial}{\partial y^{(1)i}}|_{u_0} + 2y_0^{(2)i} \frac{\partial}{\partial y^{(2)i}}|_{u_0} + \cdots + (k-1)y_0^{(k-1)i} \frac{\partial}{\partial y^{(k-1)i}}|_{u_0}.
$$

**Definition 4.5.1** A function $H : T^*kM \to \mathbb{R}$ differentiable on $\overline{T^*kM}$ and continuous on the null section of the projection $\pi^*k$ is called homogeneous of degree $r \in \mathbb{Z}$ with respect to $(y^{(1)}, \ldots, y^{(k-1)})$ if

$$
H \circ h_a = a^rH, \quad \forall a \in \mathbb{R}^+.
$$
The Geometry of the Dual of \( k \)-Tangent Bundle

Exaxtly as in the section 3.1, ch. 3, it follows:

**Theorem 4.5.1** A function \( H \in \mathcal{F}(T^*kM) \), differentiable on \( \widetilde{T}kM \) and continuous on the null section is \( r \)-homogeneous with respect to \( (y^{(1)}, \ldots, y^{(k-1)}) \) if and only if

\[
\mathcal{L}_{k^{-1}} H = rH,
\]

where \( \mathcal{L}_{k^{-1}} \) is the Lie derivative with respect to the Liouville vector field \( k^{-1} \Gamma \).

Evidently, (4.5.2) can be written in the form

\[
y^{(1)i} \frac{\partial H}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial H}{\partial y^{(2)i}} + \cdots + (k-1)y^{(k-1)i} \frac{\partial H}{\partial y^{(k-1)i}} = rH.
\]

As is usually (see Ch.3, §3.1) the notion of homogeneity can be extended to the vector fields on \( \widetilde{T}kM \).

A vector field \( X \) on \( \widetilde{T}kM \) is \( r \)-homogeneous with respect to \( (y^{(1)}, \ldots, y^{(k-1)}) \) if

\[
X \circ h = a^{r-1} h^* \circ X, \forall a \in \mathbb{R}^+.
\]

One proves [115]:

**Theorem 4.5.2** A vector field \( X \in \mathcal{X}(\widetilde{T}kM) \) is \( r \)-homogeneous with respect to \( (y^{(1)}, \ldots, y^{(k-1)}) \) if and only if

\[
\mathcal{L}_{k^{-1}} X = (r-1)X.
\]

Of course, \( \mathcal{L}_{k^{-1}} X = [\Gamma, X] \).

Consequently, \( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(1)i}}, \ldots, \frac{\partial}{\partial y^{(k-1)i}}, \frac{\partial}{\partial p_i} \) are 0, 1, \( 2-k \), 0 homogeneous, respectively, with respect to \( (y^{(1)}, \ldots, y^{(k-1)}) \).

The following properties hold:

10 If \( H \) is \( s \)-homogeneous and \( X \in \mathcal{X}(\widetilde{T}kM) \) is \( r \)-homogeneous, then \( HX \) is \( s+r \)-homogeneous, with respect to \( (y^{(1)}, \ldots, y^{(k-1)}) \).

20 If \( H \) is \( s \)-homogeneous and \( X \in \mathcal{X}(\widetilde{T}kM) \) is \( r \)-homogeneous, then \( XH \) is \( s+r-1 \)-homogeneous, with respect to \( (y^{(1)}, \ldots, y^{(k-1)}) \).

A \( q \)-form \( \omega \in \Lambda^q(\widetilde{T}kM) \) is \( s \)-homogeneous, with respect to \( (y^{(1)}, \ldots, y^{(k-1)}) \) if

\[
\omega \circ h^* = a^s \omega, \forall a \in \mathbb{R}^+.
\]

As we know, [115] the following theorem holds:
Theorem 4.5.3 A q-form $\omega$ is $s$-homogeneous with respect to $(y^{(1)}, ..., y^{(k-1)})$ if and only if

\[(4.5.6) \quad \mathcal{L}_{k-1} \omega = s \omega.\]

The 1-forms $dx^i$, $dy^{(1)i}$, ..., $dy^{(k-1)i}$, $dp_i$ are homogeneous of degree 0, 1, ..., $k-1$, 0 respectively, with respect to $(y^{(1)}, ..., y^{(k-1)})$.

As we remarked above it is important to study the notion of homogeneity with respect to momenta.

Let $H_p$ be the group of homothety:

$$H_p = \left\{ h'_a : \widetilde{T}^*kM \to \widetilde{T}^*kM | a \in \mathbb{R}^+ \right\}$$

where $h'_a(x, y^{(1)}, ..., y^{(k-1)}, p) = (x, y^{(1)}, ..., y^{(k-1)}, ap)$. A function $H \in T^*kM$, differentiable on $\widetilde{T}^*kM$ and continuous on the null section of $\pi^*k$ is homogeneous of degree $r$, with respect to the momenta $p_i$, if

\[(4.5.7) \quad H \circ h'_a = a^r H, \quad \forall a \in \mathbb{R}^+.\]

In other words:

\[(4.5.7a) \quad H(x, y^{(1)}, ..., y^{(k-1)}, ap) = a^r H(x, y^{(1)}, ..., y^{(k-1)}, p), \quad \forall a \in \mathbb{R}^+.\]

We have, [115]:

Theorem 4.5.4 A function $H \in \mathcal{F}(T^*kM)$, differentiable on $\widetilde{T}^*kM$ and continuous on the null section is $r$-homogeneous with respect to $p_i$ if and only if

\[(4.5.8) \quad \mathcal{L}_{C^*} H = rH.\]

But $\mathcal{L}_{C^*} H = C^* H = p_i \frac{\partial H}{\partial p_i}$.

Remark It is not difficult to see that if $H$ is differentiable on $T^*kM$, then the $r$-homogeneous function $H$ is a polynomial of degree $r$ in the variables $p_i$.

A vector field $X \in \mathcal{X}(\widetilde{T}^*kM)$ is homogeneous of degree $r$ if

$$X \circ h'_a = a^{r-1} h'^r_a \circ X, \quad \forall a \in \mathbb{R}^+.\]

We have

Theorem 4.5.5 A vector field $X \in \mathcal{X}(\widetilde{T}^*kM)$ is $r$-homogeneous with respect to $p_i$ if and only if:

\[(4.5.9) \quad \mathcal{L}_{C^*} X = (r-1)X.\]
This result implies that the vector fields \( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(1)i}}, \ldots, \frac{\partial}{\partial y^{(k-1)i}}, \frac{\partial}{\partial p_i} \) are 1, 1, ..., 1, 0 homogeneous, respectively, with respect to \( p_i \).

If \( H \) is \( s \)-homogeneous and \( X \) is \( r \)-homogeneous, then \( fX \) is \( s+r \)-homogeneous and \( Xf \) is \( s+r-1 \)-homogeneous, with respect to \( p_i \).

For instance, if \( H(x, y^{(1)}, \ldots, y^{(k-1)}, p) \) is a function \( r \)-homogeneous with respect to \( p_i \), then

\[
1^0 \frac{\partial H}{\partial p_i} \text{ is } r-1 \text{-homogeneous;}
\]

\[
2^0 \frac{\partial^2 H}{\partial p_i \partial p_j} \text{ is } r-2 \text{-homogeneous.}
\]

A \( q \)-form \( \omega \in \Lambda^q(T^{*k}M) \) is \( r \)-homogeneous with respect to momenta \( p_i \) if

\[
\omega \circ h^r_a = a^r \omega, \ \forall a \in \mathbb{R}^+.
\]

It follows

**Theorem 4.5.6** A \( q \)-form \( \omega \) is \( r \)-homogeneous with respect to \( p_i \) if

\[
(4.5.10) \quad L_C^* \omega = r \omega.
\]

Consequently, \( dx^i, dy^{(1)i}, \ldots, dy^{(k-1)i}, dp_i \) are 0, 0, ..., 0, 1-homogeneous with respect to \( p_i \), respectively.

Finally, we determine the degree of homogeneity of the function \( \{f, g\}_{k=1} \).

**Proposition 4.5.1** If \( f \) and \( g \) are \( r \) and \( s \)-homogeneous functions with respect to \( p_i \), then the function \( \{f, g\}_{k-1} \) is homogeneous of degree \( s+r-1 \).

For applications to the geometry of Cartan spaces of order \( k \) is important to have a special notion of homogeneity on the fibres of the bundle \( T^{*k}M \).

More precisely, the homothety

\[
\bar{h}_a : (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^{*k}M \rightarrow (x, ay^{(1)}, \ldots, a^{k-1}y^{(k-1)}, ap) \in T^{*k}M
\]

is preserved by the transformation of local coordinates (4.1.2) on \( T^{*k}M \).

Let \( H_{y,p} \) be the group of transformation on \( T^{*k}M \):

\[
H_{y,p} = \left\{ \bar{h}_a : (x, y^{(1)}, \ldots, y^{(k-1)}, p) \rightarrow (x, ay^{(1)}, \ldots, a^{k-1}y^{(k-1)}, ap) | a \in \mathbb{R}^+ \right\}.
\]

The orbit of a point \( u_0 = (x_0, y_0^{(1)}, \ldots, y_0^{(k-1)}, p^0) \) by \( H_{y,p} \) is given by

\[
x^i = x_i, \ y^{(1)i} = ay_0^{(1)i}, \ldots, y^{(k-1)i} = a^{k-1}y_0^{(k-1)i}, \ p_i = a^kp_i^0, \ \forall a \in \mathbb{R}^+.
\]

The tangent vector at the point \( u_0 = \bar{h}_1(u_0) \) is the vector field \( \Gamma^{k-1} + kC^* \) at the point \( u_0 \):

\[
\Gamma^{k-1}(u_0) + kC^*(u_0) = y_0^{(1)i} \frac{\partial}{\partial y^{(1)i}} |_{u_0} + \cdots + (k-1)y_0^{(k-1)i} \frac{\partial}{\partial y^{(k-1)i}} |_{u_0} + kp_i^0 \frac{\partial}{\partial p_i} |_{u_0}.
\]
Definition 4.5.2 A function $H : T^{*k}M \to \mathbb{R}$ differentiable on $\tilde{T}^{*k}M$ and continuous on the null section of the projection $\pi^{*k}$ is called homogeneous of degree $r \in \mathbb{Z}$ on the fibres of the bundle $T^{*k}M$ if
\begin{equation}
H \circ \overline{h}_a = a^r H, \ \forall a \in \mathbb{R}^+.
\end{equation}

Applying the usual methods it follows:

Theorem 4.5.7 A function $H$ on $T^{*k}M$, differentiable on $\tilde{T}^{*k}M$ and continuous on the null section is $r$-homogeneous on the fibres of $T^{*k}M$ if and only if
\begin{equation}
\mathcal{L}_{k^{-1}} \Gamma_{+C^*} H = rH.
\end{equation}

If we expand (4.5.12), we can write it in the form
\begin{equation}
(y^{(1)i})_I \frac{\partial H}{\partial y^{(1)i}_I} + \cdots + (k - 1)y^{(k-1)i}_I \frac{\partial H}{\partial y^{(k-1)i}_I} + kp_i \partial^i H = rH.
\end{equation}

A vector field $X$ on $\tilde{T}^{*k}M$ is $r$-homogeneous on the fibres of $T^{*k}M$ if
\begin{equation}
X \circ \overline{h}_a = a^{r-1} \overline{h}_a \circ X, \ \forall a \in \mathbb{R}^+.
\end{equation}

It follows

Theorem 4.5.8 A vector field $X \in \mathcal{X}(\tilde{T}^{*k}M)$ is $r$-homogeneous on the fibres of $T^{*k}M$ if and only if
\begin{equation}
\mathcal{L}_{k^{-1}} \Gamma_{+C^*} X = (r - 1)X.
\end{equation}

Of course, (4.5.14) can be given in the form
\begin{equation}
\left[ k^{-1} \Gamma, X \right] + k [C^*, X] = (r - 1)X.
\end{equation}

Corollary 4.5.1 1° The vector fields $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial y^{(1)i}_I}$, ..., $\frac{\partial}{\partial y^{(k-1)i}_I}$ and $\dot{\partial} = \frac{\partial}{\partial p_i}$ are $1, 0, ..., 2 - k, 1 - k$ homogeneous on the fibres of $\tilde{T}^{*k}M$, respectively.

2° If $H \in \mathcal{F}(\tilde{T}^{*k}M)$ is s-homogeneous and $X \in \mathcal{X}(T^{*k}M)$ is r-homogeneous on the fibres of $T^{*k}M$ then
a. $HX$ is $r + s$-homogeneous;

b. $XH$ is $r + s - 1$-homogeneous.

A $q$-form $\omega \in \Lambda^q(\tilde{T}^{*k}M)$ is s-homogeneous on the fibres of $T^{*k}M$ if
\begin{equation}
\omega \circ \overline{h}_a = a^s \omega, \ \forall a \in \mathbb{R}^+.
\end{equation}

The following theorem holds:
Theorem 4.5.9 A q-form $\omega \in \Lambda^q(T^*kM)$ is $s$-homogeneous on the fibres of $T^*kM$ if and only if

\begin{equation}
(4.5.15) \quad \mathcal{L}_{\Gamma + kC^*} \omega = s\omega.
\end{equation}

Corollary 4.5.2 The 1-forms $dx_i$, $dy^{(1)i}$, ..., $dy^{(k-1)i}$, $dp_i$ are $0$, $1$, ..., $k-1$, $k$-homogeneous on the fibres of $T^*kM$.

We will apply these results in the study of the homogeneity on the fibres of $T^*kM$ of 1-forms $d_0H$, ..., $d_{k-2}H$ and, of course of the functions $\{f, g\}_{k=1}$. 
Chapter 5

The Variational Problem for the Hamiltonians of Order $k$

The theory of higher order Hamiltonian systems and its applications in Analytical Mechanics are consistent only if we study the variational problem for the differentiable Hamiltonians of order $k$, $H(x, y^{(1)}, \ldots, y^{(k-1)}, p)$, [96, 98].

In this case the integral of action of $H$ must be defined along curve $c$ on the cotangent manifold $T^*M$ by

$$I(c) = \int_0^1 \left[ \sum_i p_i \frac{dx^i}{dt} - \frac{1}{2} H(x, \frac{dx}{dt}, \ldots, \frac{1}{(k-1)!} \frac{d^{k-1}x}{dt^{k-1}}, p) \right] dt.$$

A local variation of $c$ is a curve $\tilde{c}(\varepsilon_1, \varepsilon_2)$ which depend on a vector field $V^i$ and a covector field $\eta_i$. The integral of action $I(\tilde{c}(\varepsilon_1, \varepsilon_2))$ depends on two parameters $\varepsilon_1, \varepsilon_2$. In order for the functional $I(c)$ to be an extremal value of the functionals $I(\tilde{c}(\varepsilon_1, \varepsilon_2))$ it is necessary that

$$\frac{\partial I(\tilde{c}(\varepsilon_1, \varepsilon_2))}{\partial \varepsilon_\alpha} \bigg|_{\varepsilon_1=\varepsilon_2=0} = 0, \quad (\alpha = 1, 2)$$

These conditions allow to determine the Hamilton-Jacoby equations (5.1.17).

Introducing the higher order energies of $H$, $E^{k-1}(H), \ldots, E^1(H)$, a law of conservation of the energy $E^{k-1}(H)$ is proved and a Nöther type theorem is formulated. This theory is valid in the case when the order $k$ is greater then 1.

5.1 The Hamilton-Jacobi Equations

A function $H : T^*kM \to \mathbb{R}$ differentiable on $\tilde{T^*kM}$ and continue on the nul section is called a differentiable Hamiltonian of order $k$. It depends on the
variables \((x^i, y^{(1)i}, \ldots, y^{(k-1)i}, p_i)\). So, it will be denoted by 
\[ H(x, y^{(1)}, \ldots, y^{(k-1)}, p). \]
Let us consider a curve
\[ c : t \in [0, 1] \rightarrow (x^i(t), p_i(t)) \in \tilde{T}^*M, \]
having the image in a local chart of the manifold \(\tilde{T}^*M\). The curve \(c\) can be analytically given by the equations:
\[
(5.1.1) \quad x^i = x^i(t), \quad p_i = p_i(t), \quad t \in [0, 1].
\]

The extension \(\tilde{c}\) to the dual bundle \(T^{*k}M\) is well determined. The extension \(\tilde{c}\) is given by the equations
\[
(5.1.2) \quad x^i = x^i(t), \quad y^{(\alpha)i}(t) = \frac{1}{\alpha!} \left( d^{\alpha}x^i \frac{dt^\alpha}{dt} + \varepsilon_1 d^{\alpha}V^i \frac{dt^\alpha}{dt} \right), \quad (\alpha = 1, \ldots, k - 1), \quad p_i = p_i(t), \quad t \in [0, 1].
\]

Also, we consider a vector field \(V^i(t)\) and a covector field \(\eta_i(t)\) along curve \(c\), having the properties:
\[
(5.1.3) \quad V^i(0) = V^i(1) = 0, \quad \eta_i(0) = \eta_i(1) = 0, \quad \frac{d^{\alpha}V^i}{dt^{\alpha}}(0) = \frac{d^{\alpha}V^i}{dt^{\alpha}}(1) = 0, \quad (\alpha = 1, \ldots, k - 2).
\]

The variation \(\tau(\varepsilon_1, \varepsilon_2)\) of a curve \(c\) determined by the pair \((V^i(t), \eta_i(t))\) is defined by
\[
(5.1.4) \quad \tau^i = x^i(t) + \varepsilon_1 V^i(t), \quad \tilde{\tau}_i = p_i(t) + \varepsilon_2 \eta_i(t), \quad t \in [0, 1],
\]
where \(\varepsilon_1\) and \(\varepsilon_2\) are constants, small in the absolute values, such that the image of the curve \(\tilde{c}(\varepsilon_1, \varepsilon_2)\) belongs to the same domain of chart on \(\tilde{T}^*M\) as the image of curve \(c\). The extension of \(\tau(\varepsilon_1, \varepsilon_2)\) is the curve \(\tilde{\tau}(\varepsilon_1, \varepsilon_2)\) given by the equations:
\[
(5.1.5) \quad \tau^i = x^i(t) + \varepsilon_1 V^i(t), \quad \tilde{\tau}_i = p_i(t) + \varepsilon_2 \eta_i(t), \quad t \in [0, 1].
\]
The integral of action for the Hamiltonian \( H(x, y^{(1)}, ..., y^{(k-1)}, p) \) along curve \( c \) is defined, like an extension of the classical form, by

\[
(5.1.6) \quad I(c) = \int_0^1 \left[ p_i(t) \frac{dx^i}{dt}(t) - \frac{1}{2} H(x(t), \frac{dx}{dt}(t), ..., \frac{1}{(k-1)!} \frac{d^{k-1}x}{dt^{k-1}}(t), p(t)) \right] dt.
\]

Evidently, \( p_i \frac{dx^i}{dt} - \frac{1}{2} H(x, \frac{dx}{dt}, ..., \frac{1}{(k-1)!} \frac{d^{k-1}x}{dt^{k-1}}, p) \) is a differentiable Hamiltonian on the curve \( c \).

The integral of action \( I(\overline{c}(\bar{\varepsilon}_1, \bar{\varepsilon}_2)) \) is:

\[
(5.1.7) \quad I(\overline{c}(\bar{\varepsilon}_1, \bar{\varepsilon}_2)) = \int_0^1 \left[ \left( p + \varepsilon_2 \eta \right) \left( \frac{dx}{dt} + \varepsilon_1 V \right) - \frac{1}{2} H(x + \varepsilon_1 V, \frac{dx}{dt} + \varepsilon_1 V, ..., 1 \right) \right] \eta dt.
\]

The necessary conditions in order that \( I(c) \) is an extremal value of \( I(\overline{c}(\bar{\varepsilon}_1, \bar{\varepsilon}_2)) \) are:

\[
(5.1.8) \quad \left. \frac{\partial I(\overline{c}(\bar{\varepsilon}_1, \bar{\varepsilon}_2))}{\partial \varepsilon_1} \right|_{\varepsilon_1=\varepsilon_2=0} = 0, \quad \left. \frac{\partial I(\overline{c}(\bar{\varepsilon}_1, \bar{\varepsilon}_2))}{\partial \varepsilon_2} \right|_{\varepsilon_1=\varepsilon_2=0} = 0.
\]

In our conditions of differentiability, using the equality (5.1.7), we get the equations:

\[
(5.1.9) \quad \int_0^1 \left[ p_i(t) \frac{dV^i}{dt}(t) - \frac{1}{2} H(x, \frac{dV}{dt}^{(1)} + \varepsilon_1 V, ..., \frac{1}{(k-1)!} \frac{d^{k-1}V^i}{dt^{k-1}}(t), p(t)) \right] dt = 0
\]

and

\[
(5.1.10) \quad \int_0^1 \left[ \frac{dx}{dt} - \frac{1}{2} \frac{\partial H}{\partial p_i}(x) \right] \eta_i dt = 0.
\]

So, we obtain:

**Theorem 5.1.1** The necessary conditions for \( I(c) \) to be an extremal value of the functional \( I(\overline{c}(\bar{\varepsilon}_1, \bar{\varepsilon}_2)) \) are given by the equations (5.1.9) and (5.1.10).

The previous equations imply the Hamilton-Jacobi equations of the Hamiltonian \( H \). To prove this we need to introduce some new notions.

Consider the following main invariants, \cite{96}:

\[
(5.1.11) \quad I^1(H) = \mathcal{L}_\Gamma H, \quad I^2(H) = \mathcal{L}_\Gamma^2 H, ..., I^{k-1}(H) = \mathcal{L}_\Gamma^{k-1}(H),
\]
in which $\mathcal{L}$ is the Lie operator of derivation and

$$
\Gamma = y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}},
$$

(5.1.12)

$$
\Gamma = y^{(1)i} \frac{\partial}{\partial y^{(k-2)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k-1)i}},
$$

$$
\Gamma = y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k-1)i}} + \ldots + (k-1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-1)i}},
$$

are the Liouville vector fields on the manifold $T^*M$. Also, we define the invariants:

(5.1.13)

$$
I^1_V(H) = V^i \frac{\partial H}{\partial y^{(k-1)i}},
$$

$$
I^2_V(H) = V^i \frac{\partial H}{\partial y^{(k-2)i}} + \frac{dV^i}{dt} \frac{\partial H}{\partial y^{(k-1)i}},
$$

$$
I^{k-1}_V(H) = V^i \frac{\partial H}{\partial y^{(1)i}} + \frac{dV^i}{dt} \frac{\partial H}{\partial y^{(2)i}} + \ldots + \frac{1}{(k-2)!} \frac{d^{k-2}V^i}{dt^{k-2}} \frac{\partial H}{\partial y^{(k-1)i}}.
$$

For $V^i = \frac{dx^i}{dt}$, the invariants (5.1.13) are the same with the invariants $I^1(H), \ldots, I^{(k-1)}(H)$ along curve $c$. An important notation is as follows:

(5.1.14)

$$
\overset{\circ}{E}_i (H) = \frac{dp_i}{dt} + \frac{1}{2} \frac{\partial H}{\partial x^i} - \frac{d}{dt} \frac{\partial H}{\partial y^{(1)i}} + \ldots + (-1)^{k-1} \frac{1}{(k-1)!} \frac{d^{k-1}V^i}{dt^{k-1}} \frac{\partial H}{\partial y^{(k-1)i}}
$$

Later we prove that $\overset{\circ}{E}_i (H)$ is a $d$-covector field along curve $c$.

By a straightforward calculus we can prove:

**Lemma 5.1.1** The following identity holds:

(5.1.15)

$$
p_i \frac{dV^i}{dt} - \frac{1}{2} \frac{\partial H}{\partial x^i} V^i + \frac{\partial H}{\partial y^{(1)i}} \frac{dV^i}{dt} + \ldots + \frac{1}{(k-1)!} \frac{\partial H}{\partial y^{(k-1)i}} \frac{d^{k-1}V^i}{dt^{k-1}} =
$$

$$
- \overset{\circ}{E}_i (H) V^i + \frac{d}{dt} (p_i V^i) - \frac{1}{2} \frac{d}{dt} I^{k-1}_V(H) - \frac{1}{2} \frac{d}{dt} I^{k-2}_V(H) + \ldots + (-1)^{k-2} \frac{1}{(k-1)!} \frac{d^{k-2}I^1}{dt^{k-2}}.
$$

Using the previous Lemma, we can prove:
The Variational Problem for the Hamiltonians of Order $k$

**Theorem 5.1.2** The equations (5.1.9) and (5.1.10) are equivalent to the equations

\[(5.1.16) \int_0^1 \overset{\circ}{E}_i (H) V^i dt = 0; \int_0^1 \left( \frac{dx^i}{dt} - \frac{1}{2} \frac{\partial H}{\partial p_i} \right) \eta_i dt = 0.\]

**Proof.** By means of the identity (5.1.15) the equations (5.1.9) can be written:

\[
\int_0^1 \{ - \overset{\circ}{E}_i (H)V^i + \frac{d}{dt} p_i V^i - \frac{1}{2} (I^k_{V} - 1) \frac{d}{dt} I^{k-2}_{V} (H) + ... + (-1)^{k-2} \frac{1}{(k-1)!} \frac{d^{k-2}}{dt^{k-2}} I^1_{V} \} dt = 0.
\]

Integrating and taking into account the conditions (5.1.3) and the expression of the invariants (5.1.13) we obtain the announced result. q.e.d.

Now, the equations (5.1.16) in which $V^i$ and $\eta_i$ are arbitrary lead to the following Hamilton-Jacobi equations:

**Theorem 5.1.3** In order for the integral of action $I(c)$, (5.1.6) to be an extreme for the functionals $I(\varphi(\varepsilon_1, \varepsilon_2))$, (5.1.7) it is necessary that the curve $c$ to satisfy the following Hamilton-Jacobi equations

\[(5.1.17) \frac{dx^i}{dt} = \frac{1}{2} \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{1}{2} \frac{\partial H}{\partial x^i} - \frac{d}{dt} \frac{\partial H}{\partial y^{(1)}_i} + ... + (-1)^{(k-1)} \frac{1}{(k-1)!} \frac{d^{k-1}}{dt^{k-1}} \frac{\partial H}{\partial y^{(k-1)}_i},\]

where

\[(5.1.17a) y^{(1)}_i = \frac{dx^i}{dt}, ..., y^{(k-1)}_i = \frac{1}{(k-1)!} \frac{d^{k-1}x^i}{dt^{k-1}}.\]

Evidently, the second equation (5.1.17) is equivalent to $\overset{\circ}{E}_i (H) = 0$.

Another important property is expressed in the following theorem:

**Theorem 5.1.4** $\overset{\circ}{E}_i (H)$ is a covector field.

This result can be proved by a direct calculation. Another way is as follows.

With respect to a change of local coordinates on $T^{*k}M$ we have

\[
\int_0^1 \overset{\circ}{E}_i (\overset{\circ}{H}) V^i dt - \int_0^1 \overset{\circ}{E}_i (H)V^i dt = \int_0^1 \left[ \overset{\circ}{E}_i (\overset{\circ}{H}) \frac{\partial x^i}{\partial x^j} - \overset{\circ}{E}_j (H) \right] V^j dt = 0.
\]

Since $V^i$ is an arbitrary vector field we obtain

\[
\overset{\circ}{E}_i (\overset{\circ}{H}) \frac{\partial x^i}{\partial x^j} = \overset{\circ}{E}_j (H).
\]
The previous property shows that the equation $E_i (H) = 0$ has a geometrical meaning.

### 5.2 Zermelo Conditions

The integral of action (5.1.6) is defined for the parametrized curves $c : t \in [0, 1] \to (x^i(t), p_i(t)) \in \tilde{T}^*\tilde{M}$. The problem is when it does not depend on the parametrization of curve $c$.

Consider a differentiable diffeomorphism $\tilde{t} = \tilde{t}(t), t \in [0, 1]$ which defines a new parametrization of the curve $c$. If $a = \tilde{t}(0), b = \tilde{t}(1)$, then $c$ will be represented by

$$c : \tilde{t} \in [a, b] \to (\tilde{x}^i(\tilde{t}), \tilde{p}_i(\tilde{t})) \in \tilde{T}^*\tilde{M}.$$  

In order that the integral of action $I(c)$ does not depend on the parametrization of curve $c$ is necessary that:

$$I^{k-1} (H) = H, I^{k-2} (H) = 0, ..., I^1 (H) = 0.$$

If we take again the derivative of (5.2.1) with respect to $\frac{d^2 \tilde{t}}{dt^2}$ and consider $\tilde{t} = t$ we have

(5.2.2a) 0 = $y^{(1)i} \frac{\partial H}{\partial y^{(1)i}} + ... + (k - 2)y^{(k-2)i} \frac{\partial H}{\partial y^{(k-2)i}}.$

And so on.

Therefore we have:

**Theorem 5.2.1** The necessary conditions that the integral of action $I(c)$, (5.1.6) does not depend on the parametrization of the curve $c$ are the following ones:

(5.2.3) $I^{k-1} (H) = H, I^{k-2} (H) = 0, ..., I^1 (H) = 0.$
Indeed, the equation (5.2.2) can be written as $H = L_\Gamma^* H = I^{k-1}(H)$. The equation (5.2.2') is expressed by $0 = L_\Gamma^* H = I^{(k-2)}(H)$, etc. Q.E.D.

The equations (5.2.3) will be called the Zermelo conditions. They were introduced for the higher order Lagrangians by Kazuo Kondo [70] and are fundamental for the definition of the Kawaguchi spaces [63].

These conditions are very restrictive for the Hamiltonians $H$. Indeed, let us consider the Hessian of $H$, with respect to momenta $p_i$.

\begin{equation}
(5.2.4) \quad g^{ij} = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j}.
\end{equation}

In next chapter we shall prove that $g^{ij}$ is a tensor field and its is called the fundamental tensor field of the Hamiltonian $H$.

Now is not difficult to prove the following result:

**Theorem 5.2.2** If the Hamiltonian $H$ satisfies the Zermelo conditions (5.2.3), then its fundamental tensor $g^{ij}$ has the properties:

\begin{equation}
(5.2.5) \quad \mathcal{L}_1^* g^{ij} = ... = \mathcal{L}_{k-2}^* g^{ij} = 0, \mathcal{L}_{k-1}^* g^{ij} = g^{ij}.
\end{equation}

**Proof.** Remarking that the invariants $I^1(H) = \mathcal{L}_1^* H, ..., I^{(k-1)} = \mathcal{L}_{k-1}^* H$ have the properties:

\[ \partial^i \partial^j I^{(k-1)}(H) = I^{(k-1)}(\partial^i \partial^j H), ..., \partial^i \partial^j I^1(H) = I^1(\partial^i \partial^j H) \]

and using the Zermelo conditions (5.2.3), it follows the equations (5.2.5).

The first conditions (5.2.3), $I^{k-1}(H) = H$ and theorem 4.5.1 allow to prove:

**Theorem 5.2.3** A necessary condition that the Zermelo conditions (5.2.3) be verified is that the Hamiltonian $H$ is 1-homogeneous with respect to the variables $y^{(1)i}, ..., y^{(k-1)i}$.

**Corollary 5.2.1** If the differentiable Hamiltonian $H$ is not 1-homogeneous with respect to $y^{(1)i}, ..., y^{(k-1)i}$ then the Zermelo conditions (5.2.3) are not verified.

**Corollary 5.2.2** If the differentiable Hamiltonian $H$ is 1-homogeneous with respect to $y^{(1)i}, ..., y^{(k-1)i}$ and 2-homogeneous with respect to $p_i$, then $H$ is $2k+1$-homogeneous on the fibres of $T^{*k}M$.

Indeed, $\mathcal{L}_{k-1}^{\Gamma^* + kC^*} H = (1 + 2k)H$.

**Corollary 5.2.3** If the differentiable Hamiltonian $H$ has the properties:

a. It satisfies the Zermelo conditions

b. $H$ is $mk + 1$-homogeneous on the fibres of $T^{*k}M$, $(m \in \mathbb{Z})$, then $H$ is $m$-homogeneous with respect to momenta $p_i$.

**Remark 5.2.1** The Hamiltonian $H = p_i y^{(1)i}$ satisfies the Zermelo conditions
5.3 Higher Order Energies. Conservation of Energy $\mathcal{E}^{k-1}(H)$

For a differentiable Hamiltonian $H$ the following invariants are important, [98]:

\begin{equation}
\mathcal{E}^{k-1}(H) = I^{k-1}(H) - \frac{1}{2!} \frac{d}{dt} I^{k-2}(H) + \ldots + (-1)^{k-2} \frac{1}{(k-1)!} \frac{d^{k-2}}{dt^{k-2}} I^1(H) - H,
\end{equation}

\begin{equation}
\mathcal{E}^{k-2}(H) = \frac{-1}{3!} \frac{d}{dt} I^{k-2}(H) + \frac{1}{4!} \frac{d}{dt} I^{k-3}(H) + \ldots + (-1)^{k-3} \frac{1}{(k-1)!} \frac{d^{k-3}}{dt^{k-3}} I^1(H),
\end{equation}

...........................................................................................................

\begin{equation}
\mathcal{E}^1(H) = (-1)^{k-2} \frac{1}{(k-1)!} I^1(H).
\end{equation}

The invariants $\mathcal{E}^{k-1}(H), \mathcal{E}^{k-2}(H), \ldots, \mathcal{E}^1(H)$ are called the energies of order $k-1, k-2, \ldots, 1$ of the Hamiltonian $H$, respectively.

Their expressions justify the invariant character for each. And these energies are essential for studying the Noether symmetries of $H$.

In order to prove the law of conservation for the energy $\mathcal{E}^{k-1}(H)$ we need some preliminary considerations.

**Lemma 5.3.1** If the differentiable Hamiltonian $H$ has the property

\begin{equation}
\frac{dx^i}{dt} = \frac{1}{2} \frac{\partial H}{\partial p_i},
\end{equation}

then we have

\begin{equation}
\frac{dH}{dt} = 2 \mathcal{E}_i (H) \frac{dx^i}{dt} + \frac{d}{dt} I^{k-1}(H) - \frac{1}{2!} \frac{d}{dt} I^{k-2}(H) + \ldots
\end{equation}

\begin{equation}
\ldots + (-1)^{k-2} \frac{1}{(k-1)!} \frac{d^{k-2}}{dt^{k-2}} I^1(H)].
\end{equation}

**Proof.** The condition (5.3.2) implies:

\begin{equation}
\frac{dH}{dt} = \frac{\partial H}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial H}{\partial p_i} \frac{dp_i}{dt} + \left( \frac{\partial H}{\partial y^{(1)}_i} \frac{dy^{(1)}_i}{dt} + \ldots + \frac{\partial H}{\partial y^{(k-1)}_i} \frac{dy^{(k-1)}_i}{dt} \right) =
\end{equation}

\begin{equation}
= \frac{2}{2} \frac{dp_i}{dt} \frac{dx^i}{dt} + \left\{ \frac{\partial H}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial H}{\partial y^{(1)}_i} \frac{dy^{(1)}_i}{dt} + \ldots + \frac{\partial H}{\partial y^{(k-1)}_i} \frac{dy^{(k-1)}_i}{dt} \right\} =
\end{equation}

\begin{equation}
= -2p_i \frac{d^2 x^i}{dt^2} + \left\{ \frac{\partial H}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial H}{\partial y^{(1)}_i} \frac{dy^{(1)}_i}{dt} + \ldots +
\end{equation}
The Variational Problem for the Hamiltonians of Order $k$

\[ + \frac{\partial H}{\partial y^{(k-1)i}} \frac{dy^{(k-1)i}}{dt} \} + 2 \frac{dt}{dt} \left( p_i \frac{dx^i}{dt} \right). \]

Using Lemma 4.1.1 for $V^i = \frac{dx^i}{dt}, \frac{dV^i}{dt} = \frac{d^2 x^i}{dt^2}, \ldots$ the previous equality leads to the formula (5.3.3). Q.E.D.

Now, we can prove without difficulties

**Theorem 5.3.1** If the differentiable Hamiltonian $H$ satisfies the first Hamilton-Jacobi equations (5.3.2) then the variation of the energy of order $k - 1$, $\mathcal{E}^{k-1}(H)$ is given by:

\[ (5.3.4) \quad \frac{d\mathcal{E}^{k-1}(H)}{dt} = - E_i(\mathcal{H}) \frac{dx^i}{dt}. \]

Clearly, this formula leads to an interesting result

**Theorem 5.3.2** Along every solution curve $c$ of the Hamilton-Jacobi equations (5.1.17) the energy of order $k - 1$, $\mathcal{E}^{k-1}(H)$, is conserved.

Finally, we notice the following theorems:

**Theorem 5.3.3** If the differentiable Hamiltonian $H$ verifies the Zermelo conditions (5.2.3), then the energies of $H$, $\mathcal{E}^{k-1}(H), \ldots, \mathcal{E}^1(H)$ vanish.

**Theorem 5.3.4** If the differentiable Hamiltonian $H$ verifies the Zermelo conditions (5.2.3), then along any curve $c$ which satisfies the first Hamilton-Jacobi equations (5.1.17) we have

\[ \mathcal{E}_i(\mathcal{H}) \frac{dx^i}{dt} = 0. \]

**5.4 The Jacobi-Ostrogradski Momenta**

The theory of Jacoby-Ostrogradski momenta of the Lagrangians of order $k$, briefly presented in the Chapter 2, can be extended to the Hamiltonians of order $k$.

Indeed, from (5.3.1) we see that the energy of order $k - 1$, $\mathcal{E}^{k-1}(H)$ is a polynomial function of degree one in the higher order accelerations $\frac{dx^i}{dt}, \ldots, \frac{d^{k-1}x^i}{dt^{k-1}}$. So, we have

\[ (5.4.1) \quad \mathcal{E}^{k-1}(H) = p_{(1)i} \frac{dx^i}{dt} + \ldots + p_{(k-1)i} \frac{d^{k-1}x^i}{dt^{k-1}} - H \]

A straightforward calculus shows that $p_{(1)i}, \ldots, p_{(k-1)i}$ are given by
THE GEOMETRY OF HIGHER-ORDER HAMILTON SPACES

(5.4.2)
\[ p^{(1)i} = \frac{\partial H}{\partial y^{(1)i}} - \frac{1}{2!} \frac{d}{dt} \frac{\partial H}{\partial y^{(2)i}} + \ldots + (-1)^{k-2} \frac{1}{(k-1)!} \frac{d^{k-2}}{dt^{k-2}} \frac{\partial H}{\partial y^{(k-1)i}}, \]
\[ p^{(2)i} = \frac{1}{2!} \frac{\partial H}{\partial y^{(2)i}} - \frac{1}{3!} \frac{d}{dt} \frac{\partial H}{\partial y^{(3)i}} + \ldots + (-1)^{k-3} \frac{1}{(k-1)!} \frac{d^{k-3}}{dt^{k-3}} \frac{\partial H}{\partial y^{(k-1)i}}, \]

\[ p^{(k-1)i} = \frac{1}{(k-1)!} \frac{\partial H}{\partial y^{(k-1)i}}. \]

The entries \( p^{(1)i}, \ldots, p^{(k-1)i} \) are called the Jacobi-Ostrogradski momenta of the Hamiltonian \( H \).

Using the rule of transformation (4.1.4) of the natural basis of the module \( \mathcal{X}(T^*kM) \) and the definition (5.4.2) of the Jacobi-Ostrogradski momenta \( p^{(1)i}, \ldots, p^{(k-1)i} \) we obtain

**Proposition 5.4.1** With respect to the transformations of local coordinates (4.1.2) on the manifold \( T^*kM \) the Jacobi-Ostrogradski momenta \( p^{(1)i}, \ldots, p^{(k-1)i} \) are transformed as follows:

\[ p^{(1)i} = \frac{\partial y^{(1)m}}{\partial y^{(1)i}} \tilde{p}^{(1)m} + \ldots + \frac{\partial y^{(k-1)m}}{\partial y^{(1)i}} \tilde{p}^{(k-1)m}, \]
\[ p^{(2)i} = \frac{\partial y^{(2)m}}{\partial y^{(2)i}} \tilde{p}^{(2)m} + \ldots + \frac{\partial y^{(k-1)m}}{\partial y^{(2)i}} \tilde{p}^{(k-1)m}, \]

\[ p^{(k-1)i} = \frac{\partial y^{(k-1)m}}{\partial y^{(k-1)i}} \tilde{p}^{(k-1)m}. \]

Let us consider the following 1-form fields on \( T^*kM \):

\[ p^{(1)} = p^{(1)i}dx^i + p^{(2)i}dy^{(1)i} + \ldots + p^{(k-1)i}dy^{(k-2)i}, \]
\[ p^{(2)} = p^{(2)i}dx^i + p^{(3)i}dy^{(1)i} + \ldots + p^{(k-1)i}dy^{(k-3)i}, \]

\[ p^{(k-1)} = p^{(k-1)i}dx^i. \]

**Proposition 5.4.2** With respect to the transformations of coordinates on the manifold \( T^*kM \) we have

\[ p^{(\alpha)} = \tilde{p}^{(\alpha)}, \quad (\alpha = 1, \ldots, k-1). \]

Indeed, (5.4.3), (5.4.4) and (4.1.6), have as consequence the equalities (5.4.5).

The relations between the momenta \( p_i \) and the Jacobi-Ostrogradski momenta \( p^{(1)i} \) are given by:
Lemma 5.4.1 The following identity holds:

\[
\frac{1}{2} \frac{dp_{(1)i}}{dt} = -\frac{\partial}{\partial p_{(1)i}} (H) + \left( \frac{dp_i}{dt} + \frac{1}{2} \frac{\partial H}{\partial x^i} \right).
\]

Indeed, (5.1.14) and (5.4.2) imply the last equality. Now, we can formulate

Theorem 5.4.1 Along the solution curves of the Hamilton-Jacobi equations (5.1.17), the following Hamilton-Jacobi-Ostrogradski equations hold:

\[
\frac{\partial C_{k-1}^{\alpha}(H)}{\partial p_{(\alpha)i}} = \frac{\partial^\alpha x^i}{d\alpha}, \quad (\alpha = 1, \ldots, k-1),
\]

\[
\frac{\partial C_{k-1}^{\alpha}(H)}{\partial x^i} = -\frac{\partial H}{\partial x^i} + \frac{1}{2} \frac{\partial p_{(1)i}}{dt}, \quad \alpha = (1, \ldots, k-1).
\]

Proof. The energy \( E_{k-1}(H) \), (5.4.1) can be written in the form

\[
E_{k-1}(H) = p_{(1)i} y_{(1)i} + 2! p_{(2)i} y_{(2)i} + \ldots + (k-1)! p_{(k-1)i} y_{(k-1)i} - H,
\]

where

\[
y_{(\alpha)i} = \frac{1}{\alpha!} \frac{d^\alpha x^i}{d\alpha^\alpha}, \quad (\alpha = 1, \ldots, k-1).
\]

Therefore the equations (5.4.7)_1 and (5.4.7)_3 hold. In order to prove (5.4.7)_2 we remark that (5.4.8) implies \( \frac{\partial C_{k-1}^{\alpha}(H)}{\partial x^i} = -\frac{\partial H}{\partial x^i} \) and using Lemma 5.4.1 we obtain the equations (5.4.7)_2. Q.E.D.

From Lemma 5.4.1 we deduce also

Theorem 5.4.2 Along the solution curves of the Hamilton-Jacobi equations (5.1.17) we have

\[
\frac{dx^i}{dt} = \frac{1}{2} \frac{\partial H}{\partial p_i} - \frac{1}{2} \frac{\partial H}{\partial x^i} + \frac{1}{2} \frac{\partial p_{(1)i}}{dt}.
\]

5.5 Nöther Type Theorems

In this section we will define the notion of symmetry of a differentiable Hamiltonian \( H \) and will prove two Nöther type theorems, using the model of symmetries from the Lagrangian theory, (Ch.2).

Consider a differentiable Hamiltonian \( H_0 \) with the properties:

\[
\frac{\partial H_0}{\partial y_{(k-1)i}} = 0, \quad \frac{\partial H_0}{\partial p_i} = 0
\]
Proposition 5.5.1 If $H_0$ is a differentiable Hamiltonian, having the properties (5.5.1), then the following identities hold:

\[
\begin{align*}
\frac{\partial}{\partial p_i} \frac{dH_0}{dt} &= \frac{d}{dt} \frac{\partial H_0}{\partial p_i} = 0, \\
\frac{\partial}{\partial x^i} \frac{dH_0}{dt} &= \frac{d}{dt} \frac{\partial H_0}{\partial x^i}, \\
\frac{\partial}{\partial y^{(1)i}} \frac{dH_0}{dt} &= \frac{d}{dt} \frac{\partial H_0}{\partial y^{(1)i}} + \frac{\partial H_0}{\partial x^i}, \\
\frac{\partial}{\partial y^{(2)i}} \frac{dH_0}{dt} &= \frac{d}{dt} \frac{\partial H_0}{\partial y^{(2)i}} + 2 \frac{\partial H_0}{\partial y^{(1)i}}, \\
&\vdots \\
\frac{\partial}{\partial y^{(k-2)i}} \frac{dH_0}{dt} &= \frac{d}{dt} \frac{\partial H_0}{\partial y^{(k-2)i}} + (k-2) \frac{\partial H_0}{\partial y^{(k-3)i}}, \\
\frac{\partial}{\partial y^{(k-1)i}} \frac{dH_0}{dt} &= (k-1) \frac{\partial H_0}{\partial y^{(k-2)i}}.
\end{align*}
\]

Indeed, along curve $c : t \in [0, 1] \rightarrow (x^i(t), p_i(t)) \in \widetilde{T^*M}$ we have

\[
\frac{dH_0}{dt} = \frac{\partial H_0}{\partial x^i} y^{(1)i} + 2 \frac{\partial H_0}{\partial y^{(1)i}} y^{(2)i} + \ldots + (k-1) \frac{\partial H_0}{\partial y^{(k-2)i}} y^{(k-1)i}.
\]

By a straightforward calculus we deduce the previous identities.

Proposition 5.5.2 If $H_0$ is a differentiable Hamiltonian which satisfies the equations (5.5.1), then the following identities hold:

\[
\begin{align*}
\frac{1}{2} \frac{\partial}{\partial p_i} \left( H + \frac{dH_0}{dt} \right) &= \frac{1}{2} \frac{\partial H_0}{\partial p_i}, \\
\overset{\circ}{E}_i \left( H + \frac{dH_0}{dt} \right) &= \overset{\circ}{E}_i \left( H \right).
\end{align*}
\]

Indeed, taking into account the proposition 5.5.1 and the expression (5.1.14) of the covector $\overset{\circ}{E}_i \left( H \right)$ we have (5.5.3)$_1$ and

\[
\overset{\circ}{E}_i \left( H + \frac{dH_0}{dt} \right) = \overset{\circ}{E}_i \left( H \right) + \frac{1}{2} \frac{\partial}{\partial x^i} \left[ \frac{\partial H_0}{\partial p_i} \right] + \frac{d^2}{dt^2} \frac{\partial}{\partial y^{(2)i}} \frac{dH_0}{dt} + \ldots + (-1)^{k-1} \frac{d^{k-1}}{(k-1)!} \frac{\partial}{\partial y^{(k-1)i}} \frac{dH_0}{dt} = \overset{\circ}{E}_i \left( H \right).
\]

Now, it is easy to prove
The Variational Problem for the Hamiltonians of Order $k$

**Theorem 5.5.1** $H_0$ being an arbitrary differentiable Hamiltonian with the properties (5.5.1), then the Hamiltonians $H$ and $H + \frac{dH_0}{dt}$ have the same Hamilton-Jacobi equations:

\[
(5.5.4) \quad \frac{dx^i}{dt} = \frac{1}{2} \frac{\partial H}{\partial p_i}, \quad \circ E^i (H) = 0
\]

Indeed, (5.5.4) are the consequence of the equations (5.5.3) Q.E.D.

**Corollary 5.5.1** If $H_0$ is an arbitrary differentiable Hamiltonian with the properties $\frac{\partial H_0}{\partial y^{(k-1)i}} = 0, \frac{\partial H_0}{\partial p_i} = 0$, then the integrals of action:

\[
(5.5.5) \quad I (c) = \int_0^1 [p_i \frac{dx^i}{dt} - H(x, \frac{dx}{dt}, ..., \frac{1}{(k-1)!} \frac{d^{k-1}x}{dt^{k-1}}, p)] dt
\]

and

\[
(5.5.6) \quad I' (c) = \int_0^1 [p_i \frac{dx^i}{dt} - \frac{1}{2} H(x, \frac{dx}{dt}, ..., \frac{1}{(k-1)!} \frac{d^{k-1}x}{dt^{k-1}}, p)] - \frac{1}{2} \frac{dH_0}{dt} (x, \frac{dx}{dt}, ..., \frac{1}{(k-2)!} \frac{d^{k-2}x}{dt^{k-2}}) dt
\]

determine the same Hamilton-Jacobi equations (5.5.4).

Be means of the previous result we can formulate

**Definition 5.5.1** A symmetry of the differentiable Hamiltonian $H(x, y^{(1)}, ..., y^{(k-1)}, p)$ is a $C^\infty$-diffeomorphism $\varphi : \tilde{T}^* M \times R \rightarrow \tilde{T}^* M \times R$ which preserves the variational principle of the integral of action, expressed in the Corollary 5.5.1.

We consider the local symmetries only. Therefore we study the infinitesimal symmetries defined on an open set $\pi^{*-1}(U) \times (a, b)$ in the infinitesimal form:

\[
x'^i (t') = x^i (t) + \varepsilon_1 V^i (t),
\]

\[
p'_i (t') = p_i (t) + \varepsilon_2 \eta_i (t),
\]

\[
t' = t + \varepsilon_1 \tau_1 (t) + \varepsilon_2 \tau_2 (t),
\]

where $\varepsilon_1, \varepsilon_2$ are real numbers, sufficiently small in absolute values so that the points $(x, p, t)$ and $(x', p', t')$ belong to the same set $\pi^{*-1}(U) \times (a, b)$, where the curve

\[
c : t \in [0, 1] \rightarrow (x^i (t), p_i (t), t) \in \pi^{*-1}(U) \times (a, b)
\]
is given. By \( V^i(t) \) we mean a vector field \( V^i(x(t)) \) and \( \eta(t) \) denotes a covector field \( \eta_i(x(t)) \), along \( c \). The pair \( (V^i(t), \eta_i(t)) \) satisfies (5.1.3) and the differentiable functions \( \tau_1(t), \tau_2(t) \) satisfy the conditions \( \tau_0(0) = \tau_0(1) = 0, (\alpha = 1, 2) \).

In the following considerations the terms of higher order in \( \varepsilon_1, \varepsilon_2 \), will be neglected.

The infinitesimal transformation (5.5.7) is a symmetry of the differentiable Hamiltonian \( H(x, y^{(1)}, ..., y^{(k-1)}, p) \) if for any differentiable Hamiltonian \( H_0(x, y^{(1)}, ..., y^{(k-2)}) \) the following equations hold:

\[
\frac{dH_0}{dt}(x, \frac{dx}{dt}, ..., \frac{d^{k-2}x}{dt^{k-2}}) = 0.
\]

From (5.5.7) we deduce

\[
\frac{d\tau_1}{dt} + \frac{d\tau_2}{dt} = 0,
\]

\[
\frac{dV_i}{dt} + \frac{d\varphi_i}{dt} = \frac{dV_i}{dt} + \frac{d\varphi_i}{dt} + \frac{d\psi_i}{dt},
\]

\[
\frac{d^{k-1}x^i}{dt^{k-1}} = \frac{d^{k-1}x^i}{dt^{k-1}} + \frac{d^{k-1}x^i}{dt^{k-1}} + \frac{d\varphi_{k-1}}{dt^{k-1}} + \frac{d\psi_{k-1}}{dt^{k-1}},
\]

where

\[
\varphi_1 = \frac{dx^i}{dt} \frac{d\tau_1}{dt},
\]

\[
\varphi_2 = \left( \frac{1}{2} \right) \frac{\frac{d^2x^i}{dt^2} \frac{d\tau_1}{dt}}{dt^2} + \left( \frac{3}{2} \right) \frac{dx^i}{dt} \frac{d^2\tau_1}{dt^2},
\]

\[
\varphi_{k-1} = \left( \frac{k-1}{k-1} \right) \frac{d^{k-1}x^i}{dt^{k-1}} \frac{d\tau_1}{dt} + \left( \frac{k-2}{k-2} \right) \frac{d^{k-2}x^i}{dt^{k-2}} \frac{d^2\tau_1}{dt^2} + \ldots
\]

\[
+ \left( \frac{k-1}{k-1} \right) \frac{dx^i}{dt} \frac{d^{k-1}\tau_1}{dt^{k-1}}
\]
The Variational Problem for the Hamiltonians of Order \(k\) and similarly, for \(\Psi_i^1, ..., \Psi_i^{k-1}\):

\[
\Psi_i^1 = \frac{dx_i}{dt} \frac{d\tau_2}{dt},
\]

\[
\Psi_i^2 = \left(\frac{1}{2}\right) \frac{d^2 x_i^1}{dt^2} \frac{d\tau_2}{dt} + \left(\frac{3}{2}\right) \frac{dx_i^1}{dt} \frac{d^2 \tau_2}{dt^2},
\]

\[
\Psi_i^{k-1} = \left(\frac{k-1}{2}\right) \frac{d^{k-1} x_i^1}{dt^{k-1}} \frac{d\tau_2}{dt} + \left(\frac{k-2}{2}\right) \frac{d^{k-2} x_i^1}{dt^{k-2}} \frac{d^2 \tau_2}{dt^2} + ...
\]

The previous formulas are proved starting from (5.5.7) and taking into account the following expressions:

\[
\begin{align*}
\frac{dx_i^1}{dt} &= \frac{dx_i}{dt} + \varepsilon_1 \frac{dV_i}{dt}, \\
\frac{d^2 x_i^1}{dt^2} &= \frac{dx_i}{dt} + \varepsilon_1 \frac{d^2 V_i}{dt^2}.
\end{align*}
\]

By means of the formulas (5.5.9) the equality (5.5.8) becomes:

\[
\begin{align*}
\{p_i + \varepsilon_2 \eta_i\} &\left[\frac{dx_i}{dt} + \varepsilon_1 \left(\frac{dV_i}{dt} - \phi_i\right) - \varepsilon_2 \Psi_i^1\right] - \frac{1}{2} \frac{\partial H}{\partial x_i} V_i - \varepsilon_2 \Psi_i^1 + 1 \frac{dH_0}{dt}(x, \frac{dx}{dt}, ..., 1 \frac{d^{k-2} x}{dt^{k-2}}) = 0.
\end{align*}
\]

Using the Taylor expansion with respect to \(\varepsilon_1, \varepsilon_2\) and neglecting the terms of higher order in \(\varepsilon_1, \varepsilon_2\) we obtain

\[
\begin{align*}
\frac{p_i}{dt} &- \frac{1}{2} \frac{\partial H}{\partial x_i} V_i + \frac{\partial H}{\partial y^{(1)i}} \frac{dV_i}{dt} + \ldots + \frac{1}{(k-1)!} \frac{\partial H}{\partial y^{(k-1)i}} \frac{d^{k-1} V_i}{dt^{k-1}} = 0, \\
\frac{1}{2} \frac{\partial H}{\partial y^{(1)i}} \phi_i^1 + \frac{1}{2!} \frac{\partial H}{\partial y^{(2)i}} \phi_i^2 + \ldots + \frac{1}{(k-1)!} \frac{\partial H}{\partial y^{(k-1)i}} \phi_i^{k-1} = 0, \\
\frac{1}{2} \frac{dH}{dt} &= \frac{1}{2} \frac{d\Phi}{dt}.
\end{align*}
\]
and

\[
\begin{aligned}
\left( \frac{dx^i}{dt} - \frac{1}{2} \frac{\partial H}{\partial p_i} \right) \eta_i + \frac{1}{2} \frac{\partial H}{\partial y^{(1)i}} \Psi^i_1 + \frac{1}{2!} \frac{\partial H}{\partial y^{(2)i}} \Psi^i_2 + \\
\ldots + \frac{1}{(k-1)!} \frac{\partial H}{\partial y^{(k-1)i}} \Psi_{k-1}^i - \frac{1}{2} H \frac{d\tau_2}{dt} = \frac{d\Phi_2}{dt} 
\end{aligned}
\]

(5.5.11a)

where

\[
(5.5.12) \quad \varepsilon_1 \Phi_1 + \varepsilon_2 \Phi_2 = H_0
\]

As a consequence we have that the functions \(\Phi_1\) and \(\Phi_2\) are depending only on the variables \((x^i, y^{(1)i}, \ldots, y^{(k-2)i})\) and, of course, these can be arbitrary chosen.

Taking into account of Lemma 5.1.1 and remarking the identity

\[
\frac{\partial H}{\partial y^{(1)i}} \varphi^i_1 + \frac{1}{2!} \frac{\partial H}{\partial y^{(2)i}} \varphi^i_2 + \ldots + \frac{1}{(k-1)!} \frac{\partial H}{\partial y^{(k-1)i}} \varphi^i_{k-1} =
\]

\[
= \frac{d\tau_1}{dt} I^{k-1}(H) + \frac{1}{2!} \frac{d^2 \tau_1}{dt^2} I^{k-2}(H) + \ldots + \frac{1}{(k-1)!} \frac{d^{k-1} \tau_1}{dt^{k-1}} I^1(H)
\]

and similar identities for \(\Psi^i_\alpha\) and \(\tau_2\), the equations (5.5.11) and (5.5.11') can be expressed more simple.

**Proposition 5.5.3** The equations (5.5.11) and (5.5.11') are equivalent to

\[
\begin{aligned}
- E_i (H) V^i + \frac{d}{dt} (p_i V^i) - \frac{1}{2} \frac{d}{dt} [I^{k-1}_V(H) - \frac{1}{2} I^{k-2}_V(H)] + \\
\ldots + \frac{1}{2!} \frac{d^{k-2} \tau_1}{dt^{k-2}} I^1(V)(H) + \\
\frac{1}{2} \frac{d \tau_1}{dt} I^{k-1}(H) + \frac{1}{2!} \frac{d^2 \tau_1}{dt^2} I^{k-2}(H) + \\
\ldots + \frac{1}{(k-1)!} \frac{d^{k-1} \tau_1}{dt^{k-1}} I^1(H) - \frac{1}{2} H \frac{d \tau_2}{dt} = \frac{1}{2} \frac{d \Phi_1}{dt}
\end{aligned}
\]

(5.5.13)

and

\[
\begin{aligned}
\left( \frac{dx^i}{dt} - \frac{1}{2} \frac{\partial H}{\partial p_i} \right) \eta_i + \frac{1}{2!} \frac{d \tau_2}{dt} I^{k-1}(H) + \frac{1}{2!} \frac{d^2 \tau_2}{dt^2} I^{k-2}(H) + \\
\ldots + \frac{1}{(k-1)!} \frac{d^{k-1} \tau_1}{dt^{k-1}} I^1(H) - \frac{1}{2} H \frac{d \tau_2}{dt} = \frac{1}{2} \frac{d \Phi_2}{dt}.
\end{aligned}
\]

(5.5.13a)

Now, we can prove by a straightforward calculus.
Lemma 5.5.1 The following identities hold
\[\frac{d\tau_\alpha}{dt} I^{k-1}(H) + \frac{1}{2!} \frac{d^2\tau_\alpha}{dt^2} I^{k-2}(H) + \ldots + \frac{1}{(k-1)!} \frac{d^{k-1}\tau_\alpha}{dt^{k-1}} I^{1}(H) - H \frac{d\tau_\alpha}{dt} = 0,\]

\[= -\tau_\alpha \frac{d}{dt} c^{k-1}(H) + \frac{d}{dt} [\tau_\alpha \mathcal{E}^{k-1}(H) - \frac{d\tau_\alpha}{dt} c^{k-2}(H) + \ldots + (-1)^{k-2} \frac{d^{k-2}\tau_\alpha}{dt^{k-2}} \mathcal{E}^1(H)], \quad (\alpha = 1, 2).\]

In the previous equality \(\mathcal{E}^{k-1}(H), \ldots \mathcal{E}^1(H)\) are the energies of order \((k-1, \ldots 1)\) respectively, expressed in (5.3.1).

Theorem 5.5.2 For any infinitesimal symmetry (5.5.7) the left hand sides of the following equalities do not depend on the variables \(y^{(k-1)i}\) and momenta \(p_i\).
\[\frac{d\tau_\alpha}{dt} I^{k-1}(H) + \frac{1}{2!} \frac{d^2\tau_\alpha}{dt^2} I^{k-2}(H) + \ldots + \frac{1}{(k-1)!} \frac{d^{k-1}\tau_\alpha}{dt^{k-1}} I^{1}(H) - H \frac{d\tau_\alpha}{dt} = 0,\]

\[= -\tau_\alpha \frac{d}{dt} c^{k-1}(H) + \frac{d}{dt} [\tau_\alpha \mathcal{E}^{k-1}(H) - \frac{d\tau_\alpha}{dt} c^{k-2}(H) + \ldots + (-1)^{k-2} \frac{d^{k-2}\tau_\alpha}{dt^{k-2}} \mathcal{E}^1(H)] = \frac{1}{2} \frac{d\Phi_1}{dt}.\]

\[\frac{d\tau_\alpha}{dt} I^{k-1}(H) + \ldots + (-1)^{k-2} \frac{d^{k-2}\tau_\alpha}{dt^{k-2}} \mathcal{E}^1(H) = \frac{1}{2} \frac{d\Phi_2}{dt}.\]

Proof. Indeed, the equalities (5.5.15) result from proposition 5.5.3 and Lemma 5.5.1. But the functions \(\Phi_1\) and \(\Phi_2\) satisfies the conditions
\[\frac{\partial \Phi_\alpha}{\partial y^{(k-1)i}} = 0, \quad \frac{\partial \Phi_\alpha}{\partial p_i} = 0, \quad (\alpha = 1, 2)\] Q.E.D.

Now, we can prove a Nöther type theorem:

Theorem 5.5.3 For any infinitesimal symmetry (5.5.7) of a differentiable Hamiltonian \(H(x, y^{(1)}, \ldots, y^{(k-1)}, p)\) and for any differentiable functions \(\Phi_1, \Phi_2\) with the properties
\[\frac{\partial \Phi_\alpha}{\partial y^{(k-1)i}} = \frac{\partial \Phi_\alpha}{\partial p_i} = 0, \quad (\alpha = 1, 2),\]
the following functions
\[\mathcal{F}_1^k(H, \Phi_1), \mathcal{F}_2^k(H, \Phi_2)\]
are conserved along the solutions curves of the Hamilton-Jacobi equations
\[\frac{dx^i}{dt} = \frac{1}{2} \frac{\partial H}{\partial p_i}, \quad E_1(H) = 0.\]
The functions $F_1^k(H, \Phi_1)$ and $F_2^k(H, \Phi_2)$ depend on the invariants $I_1^k(H)$, $..., I_{V-1}^k(H)$, the energies of order $1, ..., k - 1$, $\mathcal{E}^1(H), ..., \mathcal{E}^{k-1}(H)$ and the arbitrary functions $\Phi_\alpha(x, y^{(1)}, ..., y^{(k-2)})$, ($\alpha = 1, 2$).

In particular, if the Zermelo conditions (5.2.3) are verified then the energies $\mathcal{E}^1(H), ..., \mathcal{E}^{k-1}(H) = 0$. Assuming $\Phi_2 = 0$, the previous Nöther Theorem, becomes:

**Theorem 5.5.4** For any infinitesimal symmetry (5.5.7) of a differentiable Hamiltonian $H(x, y^{(1)}, ..., y^{(k-1)}, p)$, which satisfies the Zermelo conditions (5.2.3), and for any differentiable function $\Phi(x, y^{(1)}, ..., y^{(k-1)})$, along the solution curves of the Hamilton-Jacobi equations $\frac{dx^i}{dt} = \frac{1}{2} \frac{\partial H}{\partial p_i} \circ E_1(H) = 0$, the following function is constant:

$$\mathcal{F}^k(H, \Phi) = p_i V^i - \frac{1}{2} I_{V-1}^k(H) - \frac{1}{2!} I_{V-2}^k(H) + ... + (-1)^{k-2} \frac{1}{(k-1)!} \frac{dI_{V-1}^k(H)}{dt} + \Phi.$$  

(5.5.18)

The theory from this chapter will be applied to the geometrical study of the Hamilton space of order $k$. 

---

**Proof.** Taking into account the Theorem 5.5.2 and the expression of the function $F_1^k(H, \Phi_1)$ and $F_2^k(H, \Phi_2)$ it follows that the following equations hold:

$$\frac{d}{dt} \mathcal{F}_1^k(H, \Phi_1) = \frac{d}{dt} \frac{d\mathcal{F}_1^k(H, \Phi_1)}{dt} - \frac{1}{2} \frac{d\mathcal{E}^{k-1}(H)}{dt},$$

$$\frac{d}{dt} \mathcal{F}_2^k(H, \Phi_2) = \frac{d}{dt} \frac{d\mathcal{F}_2^k(H, \Phi_2)}{dt} - \frac{1}{2} \frac{d\mathcal{E}^{k-1}(H)}{dt}.$$  

(5.5.17)

According to the Hamilton-Jacobi equations (5.1.17) and the law of conservation of energy $\mathcal{E}^{k-1}(H)$, the last equations (5.5.17) implies

$$\frac{d\mathcal{F}_1^k(H, \Phi_1)}{dt} = 0, \frac{d\mathcal{F}_2^k(H, \Phi_2)}{dt} = 0.$$
Chapter 6

Dual Semispray. Nonlinear Connections

The notion of semispray from the geometry of higher order Lagrange spaces has a dual correspondent in the geometrical theory of the Hamilton spaces of order $k$. The same remark is true concerning the concept of nonlinear connection which is canonical related with that of semispray.

6.1 Dual Semispray

**Definition 6.1.1** A dual $k$-semispray on $\widetilde{T^kM}$ is a vector field $S \in \mathcal{X} (\widetilde{T^kM})$ with the property

$$JS = \Gamma^k_{-1},$$

where $\Gamma^k_{-1}$ is the Liouville vector field

$$\Gamma^k_{-1} = y^{(1)} i \frac{\partial}{\partial y^{(1)} i} + \cdots + (k-1)y^{(k-1)} i \frac{\partial}{\partial y^{(k-1)} i}.$$

We have:

**Proposition 6.1.1** A dual $k$-semispray $S$ can be locally represented by

$$S = y^{(1)} i \frac{\partial}{\partial x^i} + 2y^{(2)} i \frac{\partial}{\partial y^{(1)} i} + \cdots + (k-1)y^{(k-1)} i \frac{\partial}{\partial y^{(k-1)} i} + k \xi^{i}(x, y^{(1)}, ..., y^{(k-1)}, p) \frac{\partial}{\partial y^{(k-1)} i} + \eta^{i}(x, y^{(1)}, ..., y^{(k-1)}, p) \frac{\partial}{\partial p_i}.$$

Indeed, $S$ from (6.1.2) satisfies the equation (6.1.1) for an arbitrary system of functions $\{\xi^{i}\}$ and $\{\eta^{i}\}$ ($i = 1, ..., n$), $J$ being the $k-1$-tangent
endomorphism (§4.3). The systems of functions \( \{ \xi^i(x, y^{(1)}, \ldots, y^{(k-1)}, p) \} \) and
\( \{ \eta_i(x, y^{(1)}, \ldots, y^{(k-1)}, p) \} \) are called the coefficients of the dual k-semispray \( S \).

Since \( S \) is a vector field on \( T^*kM \) it follows that the functions \( \xi^i \) and \( \eta_i \),
deﬁned on every local chart of the manifold \( T^*kM \) are important geometrical
object ﬁelds on the manifold \( T^*kM \).

So, we have

**Theorem 6.1.1** With respect to the transformation (4.1.2) of the local coordinates on \( T^*kM \), the systems of functions \( \{ \xi^i \} \) and \( \{ \eta_i \} \) transform as follows:

\[
(6.1.3) \quad k\xi^i = k \frac{\partial \bar{y}^{(k-1)i}}{\partial y^{(k-1)i}} \xi^j + (k-1) \frac{\partial \bar{y}^{(k-1)i}}{\partial y^{(k-2)i}} y^{(k-1)j} + \cdots + 2 \frac{\partial \bar{y}^{(k-1)i}}{\partial y^{(1)i}} y^{(2)j} + \frac{\partial \bar{y}^{(k-1)i}}{\partial x^j} y^{(1)j},
\]

\[
(6.1.4) \quad \bar{\eta}_i = \frac{\partial x^i}{\partial x^j} \eta_j + \frac{\partial \bar{p}_i}{\partial x^j} y^{(1)i}.
\]

**Proof:** \( S \) being a vector ﬁeld, from (6.1.2), (6.1.4) we have:

\[
S = y^{(1)i} \left\{ \frac{\partial \bar{y}^m}{\partial x^i} \frac{\partial}{\partial x^m} + \frac{\partial y^{(1)m}}{\partial y^{(1)i}} \frac{\partial}{\partial y^{(1)m}} + \cdots + \frac{\partial y^{(k-1)m}}{\partial y^{(k-1)i}} \frac{\partial}{\partial y^{(k-1)m}} \right\} + \frac{\partial \bar{p}_m}{\partial x^i} \frac{\partial}{\partial \bar{p}_m}
\]

\[
+ 2y^{(2)i} \left\{ \frac{\partial \bar{y}^{(1)m}}{\partial y^{(1)i}} \frac{\partial}{\partial y^{(1)m}} + \frac{\partial y^{(2)m}}{\partial y^{(1)i}} \frac{\partial}{\partial y^{(2)m}} + \cdots + \frac{\partial y^{(k-1)m}}{\partial y^{(1)i}} \frac{\partial}{\partial y^{(k-1)m}} \right\} + \frac{\partial \bar{p}_m}{\partial x^i} \frac{\partial}{\partial \bar{p}_m}
\]

\[
+ (k-1)y^{(k-1)i} \left\{ \frac{\partial \bar{y}^{(k-2)m}}{\partial y^{(k-2)i}} \frac{\partial}{\partial y^{(k-2)m}} + \frac{\partial \bar{y}^{(k-1)m}}{\partial y^{(k-2)i}} \frac{\partial}{\partial y^{(k-1)m}} \right\} + \frac{\partial \bar{p}_m}{\partial x^i} \frac{\partial}{\partial \bar{p}_m}
\]

\[
+ k\xi^i \frac{\partial \bar{y}^{(k-1)m}}{\partial y^{(k-1)i}} \frac{\partial}{\partial y^{(k-1)m}} + \eta_i \frac{\partial x^i}{\partial x^m} \frac{\partial}{\partial \bar{p}_m}
\]

Taking into account (6.1.4) one obtains:

\[
S = \bar{y}^{(1)m} \frac{\partial}{\partial x^m} + 2y^{(2)m} \frac{\partial}{\partial y^{(1)m}} + \cdots + (k-1)\bar{y}^{(k-1)m} \frac{\partial}{\partial y^{(2)m}} + \frac{\partial \bar{p}_m}{\partial x^i} \frac{\partial}{\partial \bar{p}_m}
\]

\[
+ \left( y^{(1)i} \frac{\partial \bar{y}^{(k-1)m}}{\partial x^i} + 2y^{(2)i} \frac{\partial \bar{y}^{(k-1)m}}{\partial y^{(1)i}} + \cdots + (k-1)y^{(k-1)i} \frac{\partial \bar{y}^{(k-1)m}}{\partial y^{(1)i}} \right) \frac{\partial}{\partial y^{(1)m}} + \eta_i \frac{\partial x^i}{\partial x^m} \frac{\partial}{\partial \bar{p}_m}
\]

\[
+ k\xi^i \frac{\partial \bar{y}^{(k-1)m}}{\partial y^{(k-1)i}} \frac{\partial}{\partial y^{(k-1)m}} + \eta_i \frac{\partial x^i}{\partial x^m} \frac{\partial}{\partial \bar{p}_m}
\]

Now, writing \( S \) in the form

\[
S = \bar{y}^{(1)m} \frac{\partial}{\partial x^m} + 2y^{(2)m} \frac{\partial}{\partial y^{(1)m}} + \cdots + \frac{\partial \bar{p}_m}{\partial x^i} \frac{\partial}{\partial \bar{p}_m}
\]

\[
+ (k-1)\bar{y}^{(k-1)m} \frac{\partial}{\partial y^{(k-2)m}} + k\xi^m \frac{\partial}{\partial y^{(k-1)m}} + \bar{\eta}_m \frac{\partial}{\partial \bar{p}_m}
\]

and identifying with the previous expression of \( S \) we obtain the relations (6.1.3)
and (6.1.4). q.e.d.
Theorem 6.1.2 If on every domain of local chart of the manifold $T^{*k}M$ the systems of functions $\{\xi^i\}$ and $\{\eta_i\}$, $(i = 1, \ldots, n)$ are given, such that, with respect to (4.1.2) the formulae (6.1.3) and (6.1.4) hold, then $S$ from (6.1.2) is a dual $k$-semispray on $T^{*k}M$.

The proof follows the usual way, [115].

Proposition 6.1.2 The integral curves of the dual $k$-semispray $S$, from (6.1.2) are the solution curves of the system of differential equations:

\[
\begin{align*}
\frac{dx^i}{dt} &= y^{(1)}_i, \\
\frac{dy^{(1)}_i}{dt} &= 2y^{(2)}_i, \ldots, \\
\frac{dy^{(k-2)}_i}{dt} &= (k-1)y^{(k-1)}_i, \\
\frac{dy^{(k-1)}_i}{dt} &= k\xi^i(x, y^{(1)}, \ldots, y^{(k-1)}, p), \quad \frac{dp_i}{dt} = \eta_i(x, y^{(1)}, \ldots, y^{(k-1)}, p).
\end{align*}
\]

(6.1.5)

Notice that this system is equivalent to the following:

\[
\begin{align*}
\frac{dx^i}{dt} &= y^{(1)}_i, \\
\frac{d^2x^i}{dt^2} &= y^{(2)}_i, \ldots, \\
\frac{d^{k-1}x^i}{dt^{k-1}} &= y^{(k-1)}_i, \\
\frac{1}{k!} \frac{d^kx^i}{dt^k} &= \xi^i(x, \frac{dx}{dt}, \ldots, \frac{d^{k-1}x}{dt^{k-1}}, p), \\
\frac{dp_i}{dt} &= \eta_i(x, \frac{dx}{dt}, \ldots, \frac{d^{k-1}x}{dt^{k-1}}, p).
\end{align*}
\]

(6.1.5a)

The problem of integration of this system of differential equations is solved by usual methods. The solutions $\{(x^i(t), p_i(t)), \ t \in (a, b)\}$ are curves on the cotangent manifold $T^*M$.

Concerning the homogeneity of $S$, we see that every term which does not include $\xi^i$ or $\eta_i$ has the degree of homogeneity 2 on the fibres of $T^{*k}M$. Therefore we have:

Proposition 6.1.3 The dual $k$-semispray $S$, from (6.1.2) is 2-homogeneous on the fibres of $T^{*k}M$ if and only if the coefficients $\xi^i$ are $k$-homogeneous and $\eta_i$ are $k+1$-homogeneous on the fibres of $T^{*k}M$.

A dual $k$-semispray $S$, 2-homogeneous on the fibres of $T^{*k}M$ is called a $k$-spray.

Remarking that the rule (6.1.3) of transformation of the coefficients $\xi^i$ of a $k$-semispray $S$ is same rule with that of the coordinates $y^{(k)}_i$ on the total space of bundle of accelerations $T^kM$ we can give a remarkable geometrical meaning of the coefficients $\xi^i$.

Proposition 6.1.4 Every dual $k$-semispray $S$ on the manifold $T^{*k}M$ with the coefficients $\{(\xi^i, \eta_i)\}$ determine a bundle morphism

\[\xi : (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^{*k}M \rightarrow (x, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}) \in T^kM\]
defined by

\[(6.1.6)\]
\[x^i = x^i, \quad y^{(1)i} = y^{(1)i}, \quad \ldots, \quad y^{(k-1)i} = y^{(k-1)i}, \quad y^{(k)i} = \xi^i(x, y^{(1)}, \ldots, y^{(k-1)}, p).\]

Moreover, the bundle morphism \(\xi : T^*kM \to T^kM\) does not depend on the transformation of local coordinates. It is local invertible if and only if

\[
\text{rank} \left\| \partial^i \xi^j \right\| = n.
\]

Indeed, the mapping \(\xi : T^*kM \to T^kM\) does not depend on the transformation of local coordinates. It is local invertible if and only if

\[
\text{rank} \left\| \partial^i \xi^j \right\| = n. \quad \text{Q.E.D.}
\]

We shall see in Chapter 8, that the bundle morphism \(\xi\) defined in (6.1.6) is uniquely determined by the Legendre transformation between a Lagrange spaces of order \(k\), \(L^{(k)n} = (M, L)\) and a Hamilton spaces of order \(k\), \(H^{(k)n} = (M, H)\).

Consequently, if the bundle morphism \(\xi\) from (6.1.6) is apriori given we can consider the dual \(k\)-semispray

\[(6.1.7)\]
\[S_\xi = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + (k-1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-2)i}} + k\xi^i \frac{\partial}{\partial y^{(k-1)i}} + \eta_i \frac{\partial}{\partial \eta_i}.\]

In this case \(S_\xi\) is characterized only by the coefficients \(\eta_i(x, y^{(1)}, \ldots, y^{(k-1)}, p)\).

The values of 1-forms \(d_0H, \ldots, d_{k-2}H\) from (4.3.14), (4.3.5) on the vector fields \(S_\xi\) are as follows:

\[(6.1.8)\]
\[d_0H(S_\xi) = (0)\]
\[d_1H(S_\xi) = \xi^i + 2(0)\]
\[d_{k-2}H(S_\xi) = \frac{k}{p_i} y^{(1)i} + 2(0)\]

These scalar fields does not depend on the coefficients \(\xi^i\) and \(\eta_i\) of \(S_\xi\). Also the 1-form \(dH\) leads to the formula

\[(6.1.8a)\]
\[dH(S_\xi) = \frac{\partial H}{\partial x^i} y^{(1)i} + 2 \frac{\partial H}{\partial y^{(1)i}} y^{(2)i} + \cdots + (k-1) \frac{\partial H}{\partial y^{(k-2)i}} y^{(k-1)i} + \]
\[+ k \frac{\partial H}{\partial y^{(k-1)i}} \xi^i + \xi^i \eta_i = S_\xi(H).\]

The existence of a \(k\)-semispray on the manifold \(T^*kM\) is assured by the following theorem

**Theorem 6.1.3** If the base manifold \(M\) is paracompact, then on the manifold \(T^*kM\) there exists the dual \(k\)-semispray \(S_\xi\), with apriori given bundle morphism \(\xi\).
**Proof:** Assuming that the manifold $M$ is paracompact by the Theorem 4.1.2, the manifold $T^* M$ is paracompact, too. We shall see (Ch. 8) that a bundle morphism $\xi$, defined in (6.1.6) exists. Now, let $\gamma_{ij}(x), \ x \in M$, be a Riemann metric on $M$ and $\gamma^i_j(x)$ its Christoffel symbols.

Setting

$$\eta_j = \gamma^i_{jh}(x)p_i y^{(1)}h$$

we can prove that the rule of transformation of the system of functions $\{\eta_i\}$, with respect to (4.1.2) is just (6.1.4). Applying Theorem 6.1.2 we obtain a $k$-semispray $S_\xi$, with the coefficients $\eta_i$.

Another properties of $S$ are given in §4.3. We have

$$k-1 \Gamma = JS, \ k-2 \Gamma = J^2 S, \ ..., \ 1 \Gamma = J^{(k-1)} S.$$

### 6.2 Nonlinear Connections

The notion of nonlinear connection on the total space of the dual bundle $(T^* M, \pi^*, M)$ can be introduced by the classical method, [115].

**Definition 6.2.1** A nonlinear connection on the manifold $T^* M$ is a regular distribution $N$ on the $T^* M$ supplementary to the vertical distribution $V$, i.e.

$$T_u(T^* M) = N_u \oplus V_u, \ \forall u \in T^* M.\quad (6.2.1)$$

Taking into account the Proposition 4.2.1 it follows that the distribution $N$ has the property

$$T_u(T^* M) = N_u \oplus V_{1,u} \oplus W_{k,u}.\quad (6.2.1a)$$

Locally $V_1$ is generated by the system of vector fields

$$\left( \frac{\partial}{\partial y^{(1)}} \right), ..., \left( \frac{\partial}{\partial y^{(k-1)}} \right)$$

and $W_k$ is generated locally by $\left( \frac{\partial}{\partial p_i} \right)$.

As usual, we shall write these systems of tangent vectors

$$\left( \frac{\partial}{\partial y^{(1)}}, ..., \frac{\partial}{\partial y^{(k-1)}} \right)$$

as $(\partial_1, ..., \partial_{k-1})$ and $(\partial_{k})$ as $(\partial_k)$, respectively.

It follows that the local dimension of the distribution $N$ is $n$, local dimension of the distribution $V_1$ is $(k-1)n$ and that of distribution $W_k$ is $n$.

Consider a nonlinear connection $N$ on $T^* M$ and denote by $h$ and $v$ the projectors determined by direct decomposition (6.2.1). Then we have:

$$h + v = Id, \ h^2 = h, v^2 = v, hv = vh = 0.$$  

As usual we denote

$$X^H = hX, \ X^V = vX, \ \forall X \in \mathcal{X}(T^* M).\quad (6.2.2)$$
A horizontal lift, with respect to $N$ is a $\mathcal{F}(M)$-linear mapping $l_h : \mathcal{X}(M) \to \mathcal{X}(T^\ast k M)$ which has the properties
\[ v \circ l_h = 0, \quad d\pi^k \circ l_h = I_d. \]

There exists an unique local basis adapted to the horizontal distribution $N$. It is given by

\[ \frac{\delta}{\delta x^i} = l_h(\frac{\partial}{\partial x^i}), \quad (i = 1, \ldots, n). \]

The linearly independent vector fields $\frac{\delta}{\delta x^i}, \quad (i = 1, \ldots, n)$ can be uniquely written in the form:

\[ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(1) \frac{\partial}{\partial y^j(1)} - \ldots - N^j_i(k-1) \frac{\partial}{\partial y^j(k-1)} + N_{ij} \frac{\partial}{\partial p_j}. \]

The systems of functions $N^j_i, \ldots, N^j_i(k-1), N_{ij}$ are called the coefficients of the nonlinear connection $N$.

They determine an important object fields on the manifold $T^k M$. Indeed, a change of local coordinates (4.1.2) and (6.2.3) imply:

\[ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}, \]

since
\[ \frac{\delta}{\delta x^i}|_u = l_h(\frac{\delta}{\delta x^i}|_x) = l_h(\frac{\partial}{\partial x^i} \frac{\delta}{\partial \tilde{x}^j})_x = (\frac{\partial}{\partial x^i} \frac{\delta}{\partial \tilde{x}^j})_u, \quad x = \pi^k(u). \]

It is not difficult to prove that the formula (6.2.5) has the following consequence:

**Theorem 6.2.1** \( \text{iv.} \) With respect to changes of coordinate on $T^k M$, (4.1.2), the coefficients of a nonlinear connection $N$ are transformed by the rule

\[ \tilde{N}^m_i \frac{\partial \tilde{x}^m}{\partial x^j} = N^m_j \frac{\partial x^m}{\partial \tilde{x}^i} - \frac{\partial y^{(1)i}}{\partial x^j}, \]

\[ \tilde{N}^m_i \frac{\partial \tilde{x}^m}{\partial \tilde{x}^j} = N^m_j \frac{\partial \tilde{x}^m}{\partial x^j} + N^m_j \frac{\partial y^{(1)i}}{\partial x^m} - \frac{\partial y^{(2)i}}{\partial \tilde{x}^j}, \]

\[ \vdots \]

\[ \tilde{N}^m_i \frac{\partial \tilde{x}^m}{\partial \tilde{x}^j} = N^m_j \frac{\partial \tilde{x}^m}{\partial x^j} + N^m_j \frac{\partial y^{(1)i}}{\partial x^m} + \ldots + N^m_j \frac{\partial y^{(k-2)i}}{\partial x^m} - \frac{\partial y^{(k-1)i}}{\partial \tilde{x}^j}, \]

\[ \tilde{N}_{ij} = \frac{\partial x^r}{\partial x^i} \frac{\partial x^s}{\partial \tilde{x}^j} N_{rs} + p_r \frac{\partial^2 x^r}{\partial \tilde{x}^i \partial \tilde{x}^j}. \]
2°. Conversely, if the system of functions \((N^j_1, ..., N^j_i, N^j_{ij})\) are given on every domain of local chart of the manifold \(T^{*k}M\) such that the equations (6.2.6) are verified, then \((N^j_1, ..., N^j_i, N^j_{ij})\) are the coefficients of a nonlinear connection \(N\) on \(T^{*k}M\).

The \((k-1)\)-tangent structure \(J\), defined in the section 3, ch.4, maps the horizontal distribution \(N\) into a vertical distribution \(N_1 = J(N_0), N_0 = N\) which is supplementary to the distribution \(V_2\) in \(V_1\). If we continue this process, we obtain:

\[
N_0 = N, N_1 = J(N_0), N_2 = J^2(N_0), ..., N_{k-2} = J^{k-2}(N_0),
\]

\[
V_k = J^{k-1}(N_0).
\]

Therefore, we have the direct sum of vector spaces:

\[
V_{1,u} = N_{1,u} \oplus N_{2,u} \oplus ... \oplus N_{k-2,u} \oplus V_{k-1,u}, \forall u \in T^{*k}M
\]

and

\[
V_u = N_{1,u} \oplus N_{2,u} \oplus ... \oplus N_{k-2,u} \oplus V_{k-1,u} \oplus W_{k,u}, \forall u \in T^{*k}M.
\]

Taking into account the formula (6.1.1), we get:

**Theorem 6.2.2** 1°. A nonlinear connection \(N\) on the manifold \(T^{*k}M\) gives rise to the direct sum of linear spaces for the tangent space \(T_u(T^{*k}M)\) :

\[
T_u(T^{*k}M) = N_{0,u} \oplus N_{1,u} \oplus ... \oplus N_{k-2,u} \oplus V_{k-1,u} \oplus W_{k,u}, \forall u \in T^{*k}M
\]

2°. The adapted basis of every term of the previous direct sum, respectively is:

\[
\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^{(1)}j} - \cdots - N^j_{(k-1)} \frac{\partial}{\partial y^{(k-1)}j} + N^j_{ij} \frac{\partial}{\partial p^j},
\]

\[
\frac{\delta}{\delta y^{(1)}i} = \frac{\partial}{\partial y^{(1)}i} - N^j_i \frac{\partial}{\partial y^{(2)}j} - \cdots - N^j_{(k-1)} \frac{\partial}{\partial y^{(k-1)}j},
\]

\[
\vdots
\]

\[
\frac{\delta}{\delta y^{(k-1)}i} = \frac{\partial}{\partial y^{(k-1)}i},
\]

\[
\frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i}.
\]

**Proof:** 1°. The direct sum (6.2.9) is obtained from (6.2.1) and (6.2.8').

2°. \(J\left(\frac{\delta}{\delta x^i}\right) = \frac{\delta}{\delta y^{(1)}i}\) is obtained by means of the definition of the \(k-1\)-tangent structure \(J\). Also, \(\frac{\delta}{\delta y^{(2)}i} = J\left(\frac{\delta}{\delta y^{(1)}i}\right), \ldots \) leads to the formulae (6.2.10).

Remarking that (6.2.5) holds, it follows:
\textbf{Proposition 6.2.1} Under a change of local coordinates on $T^k M$, the vector fields of the adapted basis:

\begin{equation}
(6.2.11) \quad \left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \ldots, \frac{\delta}{\delta y^{(k-1)i}}, \frac{\delta}{\delta p_i} \right\}
\end{equation}

transform by the rule:

\begin{equation}
(6.2.11a) \quad \frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\partial \tilde{x}^j}, \quad \frac{\delta}{\delta y^{(1)i}} = \frac{\partial \tilde{y}^{(1)j}}{\partial y^{(1)i}} \frac{\delta}{\partial \tilde{y}^{(1)j}}, \ldots,
\end{equation}

According to (6.2.8') the vertical projector $v$ is expressed by means of the projectors $v_1, \ldots, v_{k-1}, w_k$ determined by the distributions $N_1, N_2, \ldots, N_{k-2}, V_k$ and $W_k$ as follows

$$v = v_1 + v_2 + \ldots + v_{k-1} + w_k.$$ 

If we consider also the horizontal projector $h$, determined by the distribution $N = N_0$ we have, for $\alpha = 1, \ldots, k - 1$

\begin{equation}
(6.2.12) \quad h + v_1 + \ldots + v_{k-1} + w_k = Id,
\end{equation}

$$h^2 = h, \quad v_\alpha^2 = v_\alpha, \quad w_k^2 = w_k,$$

$$h \circ v_\alpha = v_\alpha \circ h = 0, \quad h \circ w_\alpha = w_\alpha \circ h = 0, \quad v_\beta \circ v_\alpha = v_\alpha \circ v_\beta = 0, \quad \alpha \neq \beta.$$ 

As usual we put:

\begin{equation}
(6.2.13) \quad X^H = hX, \quad X^V_\alpha = v_\alpha X, \quad X^{W_k} = w_k X, \quad \forall X \in \mathcal{X}(T^k M).
\end{equation}

In adapted basis we get

\begin{equation}
(6.2.13a) \quad X^H = x^i \frac{\delta}{\delta x^i}, \quad X^V_\alpha = x^i \frac{\delta}{\delta y^{(1)i} x^i}, \quad X^{W_k} = x^i \frac{\delta}{\partial p_i}, (\alpha = 1, \ldots, k - 1).
\end{equation}

Therefore, the formulae (6.2.11') show that one has

\textbf{Proposition 6.2.2} With respect to (4.1.2), the coordinates of the vectors $X^H, X^V_\alpha, X^{W_k}$ are changed by the rule:

\begin{equation}
(6.2.13b) \quad x^i = \frac{\partial \tilde{x}^j}{\partial x^i} x^j, \quad x^i = \frac{\delta}{\partial y^{(1)i} x^i}, \quad x^i = \frac{\delta}{\delta p_i}, (\alpha = 1, \ldots, k - 1).
\end{equation}

The following result is important:

\textbf{Proposition 6.2.3} $\mathcal{P}$. The distribution $N_0$ is integrable if and only if for any $X, Y \in \mathcal{X}(T^k M)$:

$$[X^H, Y^H]^{V_\alpha} = 0, \quad [X^H, Y^H]^{W_k} = 0, \quad (\alpha = 1, \ldots, k - 1).$$

$\mathcal{P}$. The distributions $N_1, \ldots, N_{k-2}$ are integrable if and only if

$$[X^{V_\alpha}, Y^{V_\beta}]^H = [X^{V_\alpha}, Y^{V_\beta}]^{V_\delta} = [X^{V_\alpha}, Y^{V_\beta}]^{W_k} = 0,$$

for $\alpha = 1, \ldots, k - 2, \beta \neq \alpha, \beta = 1, \ldots, k - 1$, respectively.
Indeed, the previous equations appear if we express that $[X^H, Y^H]$ is a vector field which belongs to the distribution $N_0$, $[X^V_\alpha, Y^V_\alpha]$ is a vector field which belongs to the distribution $N_\alpha$, for $\alpha = 1, \ldots, k - 2$.

Notices that the distributions $V_{k-1}$ and $W_k$ are integrable. q.e.d

The notions of $h$-lift, $v_\alpha$-lift or $w_k$-lift of a vector field $X = X^i(x)\partial / \partial x^i \in \mathfrak{X}(M)$ are not difficult to define. If $l_h$ is the horizontal lift to $N_0$, $l_{v_\alpha}$ are the lifts to $N_\alpha$ and $l_{w_{k-1}}$ is the lift to $V_{k-1}$, then we have

\begin{equation}
X^H = l_h X = X^i(x) \frac{\delta}{\delta x^i},
X^V_\alpha = l_{v_\alpha} X = X^i(x) \frac{\delta}{\delta y_\alpha^i}
\end{equation}

at the point $u \in T^{*k}M$, $\pi^k(u) = x$.

In the case of an 1-form $\omega = \omega^i(x)dx^i$ from $\mathfrak{X}^*(M)$ the lift $l_{w_k}(\omega)$ to the distribution $W_k$ can be defined, at every point $u \in T^{*k}M$, with $\pi^k(u) = x$, by

\begin{equation}
\omega^{W_k} = l_{w_k} \omega = \omega^i \partial^i.
\end{equation}

### 6.3 The Dual Coefficients of the Nonlinear Connection $N$

Throughout this chapter, we consider the coefficients $(N^i_j, \ldots, N^i_j, N_{ij})$ of the nonlinear connection $N$. By means of these coefficients, the vector fields from the adapted basis (6.2.11), are expressed. The dual basis of (6.2.11) will be denoted by

\begin{equation}
(\delta x^i, \delta y^{(1)i}, \ldots, \delta y^{(k-1)i}, \delta p_i).
\end{equation}

Remark that the coefficients of the basis (6.3.1), called the dual coefficients of $N$, are expressed by the coefficients of the nonlinear connection but are not coincident with them.

First of all, the conditions of duality between the adapted basis (6.2.11) and its dual basis (6.3.1) impose the following form of the 1-form fields (6.3.1)

\begin{align}
\delta x^i &= dx^i, \\
\delta y^{(1)i} &= dy^{(1)i} + M^j_i dx^j, \\
\delta y^{(k-1)i} &= dy^{(k-1)i} + M^j_i dy^{(k-2)i} + \ldots + M^j_i dy^{(1)i} + M^j_i dx^j, \\
\delta p_i &= dp_i - N_{ji} dx^j.
\end{align}

We obtain, without difficulties, [115]
THE GEOMETRY OF HIGHER-ORDER HAMILTON SPACES

Proposition 6.3.1 The dual coefficients \((M^i_j, ..., M^i_j, N^i_j)\) are uniquely determined by the coefficients \((N^i_j, ..., N^i_j, N^i_j)\) through the following formulae:

\[
\begin{align*}
M^i_j &= N^i_j, \\
M^i_j &= N^i_j + N^i_m M^m_j, \\
(6.3.3) \\
M^i_j &= N^i_j + N^i_m M^m_j + ... + N^i_m M^m_j.
\end{align*}
\]

The same formulae determine the coefficients \((N^i_j, ..., N^i_j, N^i_j)\) as functions by the dual coefficients.

These two dual basis (6.2.11) and (6.3.1) and the Proposition 6.1.1 lead to:

Proposition 6.3.2 If we change the local coordinates on \(T^*kM\), then the 1-form of dual basis

\((\delta x^i, \delta y^{(1)i}, ..., \delta y^{(k-1)i}, \delta p_i)\)

are transformed as follows:

\[
\begin{align*}
\delta \tilde{x}^i &= \frac{\partial \tilde{x}^i}{\partial x^j} \delta x^j, \\
\delta \tilde{y}^{(1)i} &= \frac{\partial \tilde{x}^i}{\partial y^{(1)j}} \delta y^{(1)j}, ..., \\
\delta \tilde{y}^{(k-1)i} &= \frac{\partial \tilde{x}^i}{\partial y^{(k-1)j}} \delta y^{(k-1)j}, \\
\delta p_i &= \frac{\partial p_i}{\partial \tilde{x}^i}. \\
(6.3.4)
\end{align*}
\]

Evidently, the properties (6.3.4) imply some special rules of transformation of dual coefficients. In this respect we remark the formulae, which can be easily deduced

\[
\begin{align*}
\frac{\partial}{\partial x^i} &= \frac{\delta}{\delta x^i} + M^j_i \frac{\delta}{\delta y^{(1)j}} + ... + M^j_i \frac{\delta}{\delta y^{(k-1)j}} + N^i_j \frac{\delta}{\delta p_j}, \\
\frac{\partial}{\delta y^{(1)i}} &= \frac{\delta}{\delta y^{(1)i}} + M^j_i \frac{\delta}{\delta y^{(2)i}} + ... + M^j_i \frac{\delta}{\delta y^{(k-1)i}}, \\
\frac{\partial}{\delta y^{(k-1)i}} &= \frac{\delta}{\delta y^{(k-1)i}}, \\
\frac{\partial}{\delta p_i} &= \frac{\delta}{\delta p_i}.
\end{align*}
\]

(6.3.5)
and
\[ dx^i = \delta x^i, \]
\[ dy^{(1)i} = \delta y^{(1)i} - N^j_1 \delta x^j, \]
\[ dy^{(2)i} = \delta y^{(2)i} - N^j_2 \delta y^{(1)i} - N^j_1 \delta x^j, \]
\[ \ldots \]
\[ dy^{(k-1)i} = \delta y^{(k-1)i} - N^j_{k-2} \delta y^{(k-2)i} - \ldots - N^j_1 \delta y^{(1)i} - N^j_1 \delta x^j, \]
\[ dp_i = \delta p_i + N_{ji} \delta x^j. \]

Therefore, we obtain

**Theorem 6.3.1** 1°. A change of local coordinates on \( T^*kM \) implies for the dual coefficients of the nonlinear connection \( N \), the following rule of transformations (6.3.6)
\[
\frac{\partial \tilde{x}^i}{\partial x^m} = \tilde{M}^i_m \frac{\partial \tilde{x}^m}{\partial x^j} + \frac{\partial \tilde{y}^{(1)i}}{\partial x^j},
\]
\[
\frac{\partial \tilde{y}^{(2)i}}{\partial x^j} = \tilde{M}^i_m \frac{\partial \tilde{y}^{(1)m}}{\partial x^j} + \frac{\partial \tilde{y}^{(2)i}}{\partial x^j},
\]
\[
\ldots
\]
\[
\frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j} = \tilde{M}^i_m \frac{\partial \tilde{y}^{(k-2)m}}{\partial x^j} + \ldots + \tilde{M}^i_m \frac{\partial \tilde{y}^{(1)m}}{\partial x^j} + \frac{\partial \tilde{y}^{(k-1)i}}{\partial x^j},
\]
\[
\tilde{N}_{ij} = \frac{\partial \tilde{x}^r}{\partial x^r} \frac{\partial x^s}{\partial x^r} N_{rs} + p_r \frac{\partial^2 x^r}{\partial x^r \partial x^j}.
\]

2°. Conversely, if the system of function \( (M^i_j, N^i_j) \) are given on every domain a local chart of the manifold \( T^*kM \) such that the equations (6.3.6) are verified, then \( (M^i_j, N^i_j) \) are the dual coefficients of a nonlinear connection \( N \) on \( T^*kM \).

The proof can be done as in the usual cases, [115].

Some applications:

1) The vector fields \( ^1\Gamma, ^2\Gamma, \ldots, ^{k-1}\Gamma \) can be expressed in the adapted basis, in
the form:
\[ \Gamma = \frac{\delta}{\delta y^{(k-1)i}}z^{(1)i}, \]
\[ \Gamma = \frac{\delta}{\delta y^{(k-2)i}}z^{(1)i} + 2\frac{\delta}{\delta y^{(k-1)i}}z^{(2)i}, \]
\[ \vdots \]
\[ \Gamma = \frac{\delta}{\delta y^{(1)i}}z^{(1)i} + 2\frac{\delta}{\delta y^{(2)i}}z^{(2)i} + \cdots + (k-1)\frac{\delta}{\delta y^{(k-1)i}}z^{(k-1)i}, \]

where
\[ z^{(1)i} = y^{(1)i}, \]
\[ 2z^{(2)i} = 2y^{(2)i} + M^i y^{(1)m}_{(1)}, \]
\[ (k-1)z^{(k-1)i} = (k-1)y^{(k-1)i} + (k-2)M^i y^{(1)m}_{(1)} + \cdots + \]
\[ + M^i_{(k-2)} y^{(1)m}, \]

With respect to (4.1.2), we have
\[ \tilde{z}^{(\alpha)i} = \frac{\partial \tilde{x}^{(\alpha)i}}{\partial y^{(\alpha)i}} z^{(\alpha)i}, (\alpha = 1, \ldots, k-1) \]

From (6.3.7b) it follows that \( z^{(1)i}, \ldots, z^{(k-1)i} \) have a geometrical meaning. They will be called the \textit{Liouville d-vector fields}.

2) The operators \( d_0, \ldots, d_{k-2} \) and \( d \) are represented in the adapted basis by the formulae
\[ d_0 = \frac{\delta}{\delta y^{(k-1)i}} \delta x^i, \]
\[ d_1 = \frac{\delta}{\delta y^{(k-2)i}} \delta x^i + \frac{\delta}{\delta y^{(k-1)i}} \delta y^{(1)i}, \]
\[ \vdots \]
\[ d_{k-2} = \frac{\delta}{\delta y^{(1)i}} \delta x^i + \frac{\delta}{\delta y^{(2)i}} \delta y^{(1)i} + \cdots + \frac{\delta}{\delta y^{(k-1)i}} \delta y^{(k-2)i} \]

and
\[ d = \frac{\delta}{\delta x^i} \delta x^i + \frac{\delta}{\delta y^{(1)i}} \delta y^{(1)i} + \cdots + \frac{\delta}{\delta y^{(k-1)i}} \delta y^{(k-1)i} + \frac{\delta}{\delta p_i}. \]
If \( H \in \mathcal{F}(T^*M) \), then we have the 1-forms

\[
d_0 H = \delta H \frac{\partial}{\partial y^{(k-1)i}} \delta x^i,
\]

\[
d_1 H = \delta H \frac{\partial}{\partial y^{(k-2)i}} \delta x^i + \delta H \frac{\partial}{\partial y^{(k-1)i}} \frac{\partial y^{(1)i}}{\partial y^{(2)i}} \delta x^i
\]

\[
\ldots
\]

\[
d_{k-2} H = \delta H \frac{\partial}{\partial y^{(1)i}} \delta x^i + \delta H \frac{\partial}{\partial y^{(2)i}} \frac{\partial y^{(1)i}}{\partial y^{(3)i}} \delta x^i + \ldots + \delta H \frac{\partial}{\partial y^{(k-1)i}} \frac{\partial y^{(k-2)i}}{\partial y^{(k-3)i}} \delta x^i
\]

and

\[
d H = \delta H \frac{\partial}{\partial x^i} \delta x^i + \delta H \frac{\partial}{\partial y^{(1)i}} \frac{\partial y^{(1)i}}{\partial y^{(2)i}} \delta x^i + \ldots + \delta H \frac{\partial}{\partial y^{(k-1)i}} \frac{\partial y^{(k-2)i}}{\partial y^{(k-3)i}} \delta x^i + \delta H \frac{\partial}{\partial p_i} \delta p_i.
\]

In the previous formulas

\[
\delta H \frac{\partial}{\partial x^i}, \ldots, \delta H \frac{\partial}{\partial y^{(k-1)i}}
\]

are d-covector and \( \delta H \frac{\partial}{\partial p_i} \) is a d-vector.

3) In the adapted basis (6.2.11) a dual \( k \)-semispray \( S_\xi \) can be written as follows

\[
S_\xi = z^{(1)i} \frac{\partial}{\partial x^i} + 2 z^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \ldots + (k-1) z^{(k-1)i} \frac{\partial}{\partial y^{(k-2)i}} +
\]

\[
+ k \xi^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \eta_i \frac{\partial}{\partial p_i},
\]

where

\[
k \xi^{(1)i} = k \xi^i + (k-1) y^{(k-1)i} M^{(1)i} + \ldots + y^{(1)i} M^{(k-1)i},
\]

\[
\eta_i = \eta_i + y^{(1)i} N^{(1)i}.
\]

Evidently, \( \xi \) is a d-vector field, and \( \eta_i \) is a d-covector field. It is remarkable that the equation \( \eta_i = 0 \) has a geometrical meaning. Especially the sprays \( S_\xi \) with this property will be considered.

The interior products of the 1-forms \( d_0 H, \ldots, d_{k-2} H \) and \( d H \) with the vector field \( S_\xi \) are given by
\( d_0 H(S_\xi) = z^{(1)} \frac{\delta H}{\delta y^{(k-1)i}} \),

\( d_1 H(S_\xi) = z^{(1)} \frac{\delta H}{\delta y^{(k-2)i}} + 2 z^{(2)} \frac{\delta H}{\delta y^{(k-1)i}} \),

\( d_{k-2} H(S_\xi) = z^{(1)} \frac{\delta H}{\delta x^i} + 2 z^{(2)} \frac{\delta H}{\delta y^{(1)i}} + \ldots + (k-1) z^{(k-1)} \frac{\delta H}{\delta y^{(k-2)i}} \)

and

\( dH(S_\xi) = z^{(1)} \frac{\delta H}{\delta x^i} + 2 z^{(2)} \frac{\delta H}{\delta y^{(1)i}} + \ldots + (k-1) z^{(k-1)} \frac{\delta H}{\delta y^{(k-2)i}} + \)

\( + k \xi^i \frac{\delta}{\delta y^{(k-1)i}} + \eta_i \frac{\delta H}{\delta p_i} \).

Consequently, we have

\( dH(S_\xi) = S_\xi(H) \).

We shall use all these formulae in the theory of higher order Hamilton spaces. An 1-form fields \( \omega \in \mathcal{X}^*(T^*kM) \) can be uniquely written as

\( \omega = \omega^H + \omega^{V_1} + \ldots + \omega^{V_{k-1}} + \omega^{W_k} \),

where

\( \omega^H = \omega \circ h, \ \omega^{V_\alpha} = \omega \circ v_\alpha, \ (\alpha = 1, \ldots, k-1), \ \omega^{W_k} = \omega \circ w_k. \)

These components can be easily written in the adapted basis. For a function \( H \in F(T^*kM) \) we deduce

\( dH = (dH)^H + (dH)^{V_1} + \ldots + (dH)^{V_{k-1}} + (dH)^{W_k} \),

with

\( (dH)^H = \delta H/\delta x^i dx^i, (dH)^{V_1} = \delta H/\delta y^{(1)i}, \ldots, \delta H/\delta y^{(k-1)i}, \delta H/\delta p_i \).

In the case of 1-forms \( d_0 H, \ldots, d_{k-2} H \) we have:

\( (d_0 H)^{V_\alpha} = 0, (d_0 H)^{W_k} = 0, (\alpha = 1, \ldots, k-1), \)

\( (d_1 H)^{V_\alpha} = 0, (d_1 H)^{W_k} = 0, (\alpha = 2, \ldots, k-1), \)

\( \ldots \)

\( (d_{k-2} H)^{V_{k-1}} = 0, (d_{k-2} H)^{W_k} = 0. \)
4) The horizontal curves with respect to a nonlinear connection $N$, can be studied exactly as in Ch. 10, from the book [115].

Let $\gamma : I \to T^k M$ be a parametrized curve, locally expressed by

\begin{equation}
(6.3.20) \quad x^i = x^i(t), y^{(1)i} = y^{(1)i}(t), \ldots, y^{(k-1)i} = y^{(k-1)i}(t), p_i = p_i(t), t \in I.
\end{equation}

For the tangent vector field $\frac{d\gamma}{dt}, t \in I$ one can write:

\begin{equation}
(6.3.20a) \quad \frac{d\gamma}{dt} = \left(\frac{d\gamma}{dt}\right)^H + \left(\frac{d\gamma}{dt}\right)V_1 + \ldots + \left(\frac{d\gamma}{dt}\right)V_{k-1} + \left(\frac{d\gamma}{dt}\right)W_k.
\end{equation}

Using the basis (6.2.11), adapted to the direct decomposition (6.2.9), we obtain

\begin{equation}
(6.3.20b) \quad \frac{d\gamma}{dt} = \frac{\delta x^i}{dt} + \frac{\delta y^{(1)i}}{dt} + \ldots + \frac{\delta y^{(k-1)i}}{dt} + \frac{\delta p_i}{dt}.
\end{equation}

Taking into account the formulas (6.3.2) the coefficients of the tangent vector field $\frac{d\gamma}{dt}$ from (6.3.20b) have the following expressions

\begin{equation}
(6.3.21) \quad \frac{\delta x^i}{dt} = \frac{dx^i}{dt}, \quad \frac{\delta y^{(1)i}}{dt} = \frac{dy^{(1)i}}{dt} + M^n_{(1)} \frac{dx^m}{dt}, \quad \ldots, \quad \frac{\delta y^{(k-1)i}}{dt} = \frac{dy^{(k-1)i}}{dt} + M^n_{(k-2)} \frac{dy^{(1)i}}{dt} + M^n_{(k-1)} \frac{dx^m}{dt}, \quad \frac{\delta p_i}{dt} = \frac{dp_i}{dt} - N_{im} \frac{dx^m}{dt}.
\end{equation}

An horizontal curve $\gamma : I \to T^k M$, with respect to the nonlinear connection $N$ is defined by the conditions

\begin{equation}
(\frac{d\gamma}{dt})V_1 = \ldots = (\frac{d\gamma}{dt})V_{k-1} = 0, (\frac{d\gamma}{dt})W_k = 0.
\end{equation}

It follows:

**Theorem 6.3.2** A parametrized curve $\gamma : I \to T^k M$ is horizontal if and only if the following system of ordinary differential equations is verified:

\begin{equation}
(6.3.22) \quad \frac{\delta y^{(1)i}}{dt} = \ldots = \frac{\delta y^{(k-1)i}}{dt} = 0, \frac{\delta p_i}{dt} = 0.
\end{equation}

The horizontal curves with the property

\begin{equation}
(\ast) \quad y^{(1)i} = \frac{dx^i}{dt}, \ldots, y^{(k-1)i} = \frac{1}{(k-1)!} \frac{d^{k-1}x^i}{dt^{k-1}}
\end{equation}

are called the autoparallel curves of the nonlinear connection $N$. These curves are characterized by (6.3.22) in the conditions (\ast).
6.4 The Determination of the Nonlinear Connection by a Dual $k$-Semispray

The main problem concerning the notion of nonlinear connection $N$ is whether a dual $k$-semispray $S_\xi$ determines a nonlinear connection $N$ or not. The answer is yes. We have the following important result.

**Theorem 6.4.1** Any dual $k$-semispray $S_\xi$ with the coefficients $(\xi^i, \eta_i)$ determines:

1°. The dual coefficients $M^i_j(1), \ldots, M^i_j(1+k-2)$, of a nonlinear connection $N$, by the formulas:

\[
M^i_j(1) = -\frac{\partial \xi^i}{\partial y^{(k-1)}j}, \quad M^i_j(2) = -\frac{\partial \xi^i}{\partial y^{(k-2)}j}, \ldots, \quad M^i_j(1+k-2) = -\frac{\partial \xi^i}{\partial y^{(1)}j}.
\]

2°. The coefficients $N_{ij}$ by the formula

\[
N_{ij} = \frac{\delta \eta_i}{\delta y^{(1)}j},
\]

where the operators

\[
\frac{\delta}{\delta y^{(1)}j} = \frac{\partial}{\partial y^{(1)}j} - N^i_j(1)\frac{\partial}{\partial y^{(2)}j} - \ldots - N^i_j(k-2)\frac{\partial}{\partial y^{(k-1)}j} + N^i_j(\alpha),
\]

$(\alpha = 1, \ldots, k-2)$

are determined by $M^i_j(1), \ldots, M^i_j(1+k-2)$ from (6.4.1).

**Proof:** We prove that with respect to a change of local coordinates on the manifold $T^{*k}M$, the system of functions $M^i_j(1), \ldots, M^i_j(1+k-2)$ from (6.4.1) obey the transformation (6.3.6).

By means of (6.1.3), we have

\[
k\xi^i = k\frac{\partial \xi^i}{\partial x^j} \xi^j + (k-1)\frac{\partial y^{(1)}i}{\partial x^j} y^{(k-1)}j + \ldots + \frac{\partial y^{(k-1)}i}{\partial x^j} y^{(1)}j.
\]

Applying the formula \(\frac{\partial}{\partial y^{(k-1)m}} = \frac{\partial x^s}{\partial x^m} \frac{\partial}{\partial y^{(k-1)s}}\), we deduce the first formula (6.3.6). By the same method, we establish inductively the other formula (6.3.6), for the coefficients (6.4.1).
In order to prove (6.4.2), we remark that the vector fields \( \frac{\delta}{\delta y^{(1)i}} \), constructs by means of coefficients \( M^i_j \), ..., \( M^i_j \), from (6.4.1) has the law of transformation:

\[
\frac{\delta}{\delta y^{(1)i}} = \frac{\partial x^j}{\partial x^i} \frac{\delta}{\delta y^{(1)j}}.
\]

The rule of transformation of the coefficients \( \eta_i \) of \( S_\xi \) is

\[
\tilde{\eta}_i = \frac{\partial x^r}{\partial x^i} \eta_r + \frac{\partial \tilde{p}}{\partial x^i} y^{(1)r}.
\]

These formula have as a consequence:

\[
\frac{\delta \tilde{\eta}_i}{\delta y^{(1)i}} = \frac{\partial x^r}{\partial x^i} \frac{\partial x^m}{\partial x^j} \frac{\delta \eta_r}{\delta y^{(1)m}} + \frac{\partial^2 x^r}{\partial x^i \partial x^j}.
\]

This is the rule of transformation of the coefficients \( N_{ij} \) of a nonlinear connection \( N \) on the manifold \( T^* M \).

### 6.5 Lie Brackets. Exterior Differential

In the following it is important to determine the Lie brackets of the vector fields of the adapted basis (6.2.11) and the exterior differentials of the covector fields of adapted cobasis (6.3.1).

By a direct calculus we obtain

**Proposition 6.5.1** The following expressions of the Lie brackets hold true:

\[
\left[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^h} \right] = R^i_{jh} \frac{\delta}{\delta y^{(1)i}} + \ldots + R^i_{jh} \frac{\delta}{\delta y^{(k-1)i}} + R_{ijh} \frac{\delta}{\delta p_i},
\]

\[
\left[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta y^{(a)h}} \right] = B^i_{jh} \frac{\delta}{\delta y^{(1)i}} + \ldots + B^i_{jh} \frac{\delta}{\delta y^{(k-1)i}} + B_{ijh} \frac{\delta}{\delta p_i},
\]

\[
\left[ \frac{\delta}{\delta y^{(a)j}}, \frac{\delta}{\delta y^{(b)h}} \right] = C^i_{ab} \frac{\delta}{\delta y^{(1)i}} + \ldots + C^i_{ab} \frac{\delta}{\delta y^{(k-1)i}} + C^i_{ab} \frac{\delta}{\delta p_i},
\]

\[
\left[ \frac{\delta}{\delta x^j}, \frac{\delta}{\delta p_h} \right] = B^i_{ih} \frac{\delta}{\delta y^{(1)i}} + \ldots + B^i_{ih} \frac{\delta}{\delta y^{(k-1)i}} + B^i_{ih} \frac{\delta}{\delta p_i},
\]

\[
\left[ \frac{\delta}{\delta y^{(a)j}}, \frac{\delta}{\delta p_h} \right] = C^i_{aj} \frac{\delta}{\delta y^{(1)i}} + \ldots + C^i_{aj} \frac{\delta}{\delta y^{(k-1)i}} + C^i_{aj} \frac{\delta}{\delta p_i}.
\]
\(\alpha, \beta = 1, \ldots, k; \beta \leq \alpha\) in which:

\[\begin{align*}
(1) \quad C_{\beta \delta}^\alpha &= 0, \quad 1 \leq \alpha \leq \beta, \\
(2) \quad C_{\beta \delta}^\alpha &= 0, \quad 2 \leq \alpha \leq \beta, \\
& \vdots \\
(\gamma) \quad C_{\beta \delta}^\alpha &= 0, \quad \gamma \leq \alpha \leq \beta
\end{align*}\]

and the coefficients \(R, B, C\) can be obtained by a straightforward calculus, \(R_{\beta \delta} = 0_{\alpha}\) being expressed by:

\[\begin{align*}
R_{\beta \delta} &= r_{\beta \delta}, \\
R_{\beta \delta} &= r_{\beta \delta} + M_{\chi} r_{\delta \chi}, \\
& \vdots \\
R_{\beta \delta} &= r_{\beta \delta} + M_{\chi} r_{\delta \chi} + \ldots + M_{\chi} r_{\delta \chi} + \ldots
\end{align*}\]

and

\[\begin{align*}
R_{\beta \delta} &= \frac{\delta N_{\beta \delta}}{\delta x_{\alpha}} - \frac{\delta N_{\delta \alpha}}{\delta x_{\beta}}, \\
R_{\beta \delta} &= \frac{\delta N_{\beta \delta}}{\delta x_{\alpha}} - \frac{\delta N_{\delta \alpha}}{\delta x_{\beta}}, \quad (\alpha = 1, \ldots, k - 1)
\end{align*}\]

Taking into account the conditions of integrability of the nonlinear connection \(N\), (cf Prop. 6.2.3) we have:

**Theorem 6.5.1** The nonlinear connection \(N\) is integrable if and only if the following equations hold:

\[\begin{align*}
R_{\beta \delta} &= 0, \quad (\alpha = 1, \ldots, k - 1), R_{\beta \delta} = 0.
\end{align*}\]

Indeed, the distribution \(N\) is integrable if and only if the vector fields \(\delta x_{\alpha}, \delta x_{\beta}\) belong to \(N\). So, the equations (6.5.4) (which have a geometrical meaning) express the necessary and sufficient conditions for \(N\) to be integrable.
Similar characterizations hold for the case when each distribution \( N_1, ..., N_{k-2} \) is integrable. (cf. Prop.6.2.3, 2°)

In order to determine the exterior differentials of 1-forms \((\delta x^i, \delta y^{(1)i}, ..., \delta y^{(k-1)i}, \delta p_i)\), we start from the formulae (6.3.2) and express the mentioned differentials with respect to the base of \( \wedge^2(T^{*k}M) \) determined by \((\delta x^i, \delta y^{(1)i}, ..., \delta y^{(k-1)i}, \delta p_i)\).

From (6.3.2) one obtains:

\[
d\delta x^i = 0,
\]

\[
d\delta y^{(1)i} = dM^{i}_j \wedge dx^j,
\]

\[
d\delta y^{(2)i} = dM^{i}_j \wedge dy^{(1)j} + dM^{i}_j \wedge dx^j,
\]

\[
d\delta y^{(k-1)i} = dM^{i}_j \wedge dy^{(k-2)j} + dM^{i}_j \wedge dy^{(k-3)j} + ... + dM^{i}_j \wedge dx^j
\]

and

\[
d\delta p_i = -dN_{ji} \wedge dx^j.
\]

Substituting \((dx^i, dy^{(1)i}, ..., dy^{(k-1)i})\) from (6.3.5') we can write (6.5.5) in the following form

\[
d\delta y^{(\alpha)i} = P^{i}_{j(\alpha)} \wedge dx^j + P^{i}_{j(\alpha1)} \wedge dy^{(1)j} + ... + P^{i}_{j(\alpha,\alpha-1)} \wedge dy^{(\alpha-1)j},
\]

\[
(\alpha = 1, ..., k - 1),
\]

\[
d\delta p_i = P_{ij} \wedge dx^j,
\]

where \( P^{i}_{j(\alpha\beta)} \) are 1-forms, which should be calculated by means of formula

\[
d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]), \forall \omega \in \wedge^1(T^{*k}M)
\]

and using the Lie brackets (6.5.1).

**Theorem 6.5.2** The exterior differentials of the 1-forms
(\delta x^i, \delta y^{(1)i}, ..., \delta y^{(k-1)i}, \delta p_i) \text{ are given by}

\begin{align*}
(6.5.7) \\
d\delta x^i &= 0, \\
d\delta y^{(1)i} &= P^i_j \wedge dx^j, \\
(6.5.8) \\
d\delta y^{(2)i} &= P^i_j \wedge dx^j + P^i_j \wedge \delta y^{(1)j}, \\
&\vdots \\
d\delta y^{(k-1)i} &= P^i_j \wedge dx^j + P^i_j \wedge \delta y^{(1)j} + ... + P^i_j \wedge \delta y^{(k-2)j} \\
\text{and} \\
(6.5.7a) \\
d\delta p_i &= P^i_j \wedge dx^j,
\end{align*}

where the 1-forms \( P^i_j \) and \( P^i_j \) are given by

\begin{align*}
(6.5.8a) \\
P^i_j &= \frac{1}{2} R^i_{jm} dx^m + \sum_{\gamma=1}^{k-1} B^i_{jm} \delta y^{(\gamma)m} + B^i_{jm} \delta p_m, \\
(6.5.9) \\
P^i_j &= \frac{1}{2} R^i_{jm} dx^m + \sum_{\gamma=1}^{k-1} B^i_{jm} \delta y^{(\gamma)m} + B^i_{jm} \delta p_m.
\end{align*}
Remark 6.5.1 In order to give a proof of the equations (6.5.2) we remark that

\[ d\delta y^{(\gamma)i} \left( \frac{\delta}{\delta y^{(\alpha)j}}, \frac{\delta}{\delta y^{(\beta)h}} \right) = -C^{(\gamma)i}_{jhm} \]

\[ \text{From (6.5.6) it follows that the left hand side of the previous formula vanishes for } \gamma < \alpha \leq \beta. \]

Remark 6.5.2 By this method we can calculate the coefficients for the expression of the Lie brackets (6.5.1).

Indeed, \( d\delta y^{(\alpha)i} \) from (6.5.5) can be written in the adapted basis

\[ d\delta y^{(\alpha)i} = \{ dM^j_i - N^m_j dM^i_m - ... - N^i_j dM^j_m \} \wedge dx^j + \]

\[ \{ dM^j_i - N^m_j dM^i_m - ... - N^i_j dM^j_m \} \wedge dy^{(1)j} + \]

\[ + ... + \{ dM^j_i - N^m_j dM^i_m \} \wedge dy^{(\alpha-2)j} + dM^j_i \wedge dy^{(\alpha)j}. \]

(6.5.10)

Identifying to \( d\delta y^{(\alpha)i} \) for (6.5.6) we have

\[ P^i_{j} = dM^i_j - N^m_j dM^i_m - ... - N^i_j dM^j_m, \]

\[ P^i_{j} = dM^i_j - N^m_j dM^i_m - ... - N^i_j dM^j_m, \]

\[ \vdots \]

\[ P = dM^i_j. \]

(6.5.11)

Taking into account that, with respect to the adapted basis the 1-form \( dM^i_j \)

are given by

\[ dM^i_j = \frac{\delta M^i_j}{\delta x^h} \delta x^h + \sum_{\gamma=1}^{k-1} \frac{\delta M^i_j}{\delta y^{(\gamma)h}} \delta y^{(\gamma)h} + \frac{\delta M^i_j}{\delta \phi^h} \delta \phi^h. \]

(6.5.12)

Substituting (6.5.11) and identifying to (6.5.8) we completely determine the coefficients of the Lie brackets.

For instance, from \( P_{\alpha,\alpha-1} \), (6.5.8) \( \alpha \) and \( P \), (6.5.11) we obtain

\[ B_{m,j}^{i} = \frac{\delta M^i_j}{\delta x^m}, \quad C^{i}_{m,j} = -\frac{\delta M^i_j}{\delta y^{(\gamma)h}}, \quad C^{i}_{j} = \frac{\delta M^i_j}{\delta \phi^h}. \]
6.6 The Almost Product Structure $\mathbb{P}$. The Almost Contact Structure $F$

The $\mathcal{F}(T^{*k}M)$-linear mapping $\mathbb{P} : \mathcal{X}(T^{*p}M) \rightarrow \mathcal{X}(T^{*p}M)$ defined by

\begin{equation}
\mathbb{P}(X^H) = X^H, \quad \mathbb{P}(X^{V_\alpha}) = -X^{V_\alpha}, \quad \mathbb{P}(X^{W_k}) = -X^{W_k}, \quad (\alpha = 1, \ldots, k-1)
\end{equation}

determines an almost product structure on the manifold $T^{*k}M$. It is given by means of a nonlinear connection $N$.

We have

\begin{equation}
\begin{aligned}
\mathbb{P} \circ \mathbb{P} &= I, \\
\mathbb{P} &= I - 2(v_1 + \ldots + v_{k-1} + w_k), \\
\text{rank} \mathbb{P} &= (k+1)n.
\end{aligned}
\end{equation}

**Theorem 6.6.1** A nonlinear connection $N$ on $T^{*k}M$ is characterized by the existence of an almost product structure $\mathbb{P}$ on $T^{*k}M$ whose eigenspaces corresponding to the eigenvalue $-1$ coincides with the linear space of the vertical distribution $V$ on $T^{*k}M$.

The proof is same as in the case $k=2$ (see the book [115]).

**Theorem 6.6.2** The almost product structure $\mathbb{P}$, defined by (6.6.1) is integrable if and only if the horizontal distribution $N$ is integrable.

The proof is exactly as in Prop. 9.8.1 of the book [115].

Another important structure on $T^{*k}M$ is determined by the $\mathcal{F}(T^{*k}M)$-linear mapping

\begin{equation}
\mathbb{F}(\frac{\delta}{\delta x^i}) = -\frac{\partial}{\partial y^{(k-1)i}}, \mathbb{F}(\frac{\delta}{\partial y^{(\alpha)i}}) = 0, \quad (\alpha = 1, \ldots, k-2),
\end{equation}

where $\left(\frac{\delta}{\delta x^i}, \ldots, \frac{\delta}{\delta p_i}\right)$ is the adapted basis of a nonlinear connection $N$ and of the vertical distribution $V$. Now, is not difficult to prove

**Theorem 6.6.3** The mapping $\mathbb{F}$ has the following properties:

1° $\mathbb{F}$ is globally defined on $T^{*k}M$.

2° $\mathbb{F}$ is a tensor field of type $(1, 1)$:

\begin{equation}
F = -\frac{\partial}{\partial y^{(k-1)i}} \otimes dx^i + \frac{\delta}{\delta x^i} \otimes dy^{(k-1)i},
\end{equation}

3° $\text{Ker} \mathbb{F} = N_1 \oplus \ldots \oplus N_{k-2} \oplus W_k$, $\text{Im} \mathbb{F} = N_0 \oplus V_{k-1}$,

4° $\text{rank} \mathbb{F} = 2n$,

5° $\mathbb{F}^3 + \mathbb{F} = 0$. 

Looking at $5^\circ$, we can say that $F$ is an \textit{almost $(k-1)n$-contact structure} determined by the nonlinear connection $N$.

The Nijenhuis tensor of structure $F$ is expressed by:

$$N_F(X, Y) = [F^2[X, Y] + [FX, FY] - F[FX, Y] - F[X, FY],$$

and the condition of normality of the structure $F$ is as follows:

$$(6.6.5)$$

$$N_F(X, Y) + \sum_{i=1}^{n} \left[ \sum_{\alpha=1}^{k-1} d(\delta y^{(\alpha)}_i)(X, Y) + d(\delta p_i)(X, Y) \right] = 0,$$

$\forall X, Y \in \mathcal{A}(T^kM)$.

Using the formulas $(6.4.2)$ and $(6.4.5)$ we can obtain the explicit form of the last equation.

\section{6.7 The Riemannian Structure $G$ on $T^kM$}

Let $G$ be a Riemannian structure on the manifold $T^kM$. $G$ determines uniquely a nonlinear connection $N$ on $T^kM$. $N$ is the orthogonal distribution, with respect to $G$, to the vertical distribution $V$.

In the case when the base manifold $M$ is paracompact, Theorem 4.1.2 affirms that the manifolds $T^kM$ is paracompact, too. So on $T^kM$ there exists a Riemannian structure $G$. Consequently, we have:

\textbf{Theorem 6.7.1} \textit{If the base manifold $M$ is paracompact then on the manifold $T^kM$ there exist nonlinear connections $N$.}

Let $G$ be a Riemannian structure on $T^kM$ and $N$ the nonlinear connection, whose distribution is orthogonal to the vertical distribution $V$. The problem is to determine the local coefficients $N^i_1, ..., N^i_{k-1}, N_{ij}$ of $N$ by means of the local coefficients of $G$:

$$(6.7.1)$$

$$g_{ij}^{(00)} = G(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \quad g_{ij}^{(0k)} = G(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{(\alpha)}_j}),$$

$$g_{ij}^{(k-1)} = G(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}), \quad ..., \quad g_{ij}^{(k,k)} = G(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}).$$

It follows that $g_{ij}^{(k-1,k-1)}$ and $g_{ij}^{(k,k)}$ are d-tensor fields symmetric and positively defined. They are coefficients of the restrictions of $G$ to the distributions $V_{k-1}$ and $W_k$.

The coefficients of the nonlinear connection $N$ enter in the adapted basis $\{\delta_{\delta x^i}\}$ to the distribution $N = N_0$, $(6.2.4)$, in the adapted basis
\{ \delta \over \delta y^{(1)i} \}, ..., \{ \delta \over \delta y^{(k-1)i} \} \) to the distribution \( N_1, ..., V_{k-1} \) and in the adapted basis \( \{ \delta \over \delta p_i \} \) to the vertical distributions \( W_k \). They are uniquely determined by the conditions that each of the distributions \( \{ N_1, ..., N_{k-2} \} \) is orthogonal to \( V_{k-1} \) with respect to \( G \) and \( N_0 \) is orthogonal to \( V \).

Indeed, the conditions of orthogonality

\[ G( \delta \over \delta y^{(k-2)i} \), \delta \over \delta y^{(k-1)j} ) = 0 \]

give us

\[ (k-2,k-1) \ y_{ij} - N^m_i \ (k-1,k-1) \ y_{m,j} = 0. \]

But \( \text{rank}(k-1,k-1) \) \( y_{ij} \) = \( n \). So the previous equation uniquely gives the coefficients \( N^m_i^{(1)} \).

The following equations

\[ G( \delta \over \delta y^{(k-3)i} \), \delta \over \delta y^{(k-1)j} ) = 0 \]

lead to the equations

\[ (k-3,k-1) \ y_{ij} - N^m_i^{(k-2,k-1)} \ y_{m,j} - N^m_i^{(k-1,k-1)} \ y_{m,j} = 0. \]

These equations uniquely determine the coefficients \( N^m_i^{(2)} \), etc.

Now we prove the following result.

**Theorem 6.7.2** Any Riemannian structure \( G \) on the manifold \( T^* k M \) determines on this manifold a Riemannian almost contact structure \((G,F)\).

**Proof.** The structure \( G \) determine a nonlinear connection \( N \) with the local coefficients \( (N^m_i^{(1)}, ..., N^m_i^{(k-1)}, N_{ij}) \). With these coefficients we construct the adapted basis

\( \{ \delta \over \delta x^i \), \delta \over \delta y^{(1)i} \), ..., \delta \over \delta y^{(k-1)i} \), \delta \over \delta p_j \) \) to \( N \) and \( V_1 \). The restrictions of \( G \) to the distribution \( V_{k-1} \) and \( W_k \) give us the symmetric, positively defined d-tensor fields

\[ g_{ij} = G( \delta \over \delta y^{(k-1)i} \), \delta \over \delta y^{(k-1)j} ) \], \( h^{ij} = G( \delta \over \delta p_i \), \delta \over \delta p_j \)

Therefore \( G \) determines on the manifold \( T^* k M \) the Riemannian structure

\[ \overset{\circ}{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j} + ... + g_{ij} \delta y^{(k-1)i} \otimes \delta y^{(k-1)j} + h^{ij} \delta p_i \otimes \delta p_j. \]
Consider the almost \((k-1)n\)-contact structure \(F\) determined by \(N\). It is given by (6.6.3). Consequently, \(F\) is determined only by the Riemannian structure \(G\).

Therefore the pair \((\mathcal{G}, F)\) is a Riemannian almost contact structure determined only by \(G\).

Of course, the equation
\[
\mathcal{G}(FX, Y) = - \mathcal{G}(X, FY),
\]
is verified on the adapted basis to \(N\) and \(V\). q.e.d

**Remark 6.7.1** We shall use the Riemannian structure \(\mathcal{G}\), in the case of Hamilton space of order \(k\), for \(h_{ij} = g_{ij}\)

### 6.8 The Riemannian Almost Contact Structure \((\mathcal{G}, F)\)

Consider a \(d\)-tensor field \(g_{ij}\) on \(\widetilde{T^kM}\), symmetric and positively defined. It follows \(\det \|g_{ij}\| > 0\) on \(\widetilde{T^kM}\). Let \(N\) be an apriori given nonlinear connection on the manifold \(\widetilde{T^kM}\) and the adapted basis with respect to \(N\) and \(V\):

\[
\left( \delta_{\delta x^i}, \delta_{\delta y^{(1)}i}, \ldots, \delta_{\delta y^{(k-1)}i}, \delta_{\delta p^i} \right).
\]

As usual, the dual basis is \((dx^i, \delta y^{(1)}i, \ldots, \delta y^{(k-1)}i, \delta p^i)\). The \(N\)-lift of the \(d\)-tensor fields \(g_{ij}\) at every point \(u \in \widetilde{T^kM}\) is defined by

\[
\mathcal{G} = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^{(1)}i \otimes \delta y^{(1)}j + \ldots + g_{ij}\delta y^{(k-1)}i \otimes \delta y^{(k-1)}j + g^{ij}\delta p^i \otimes \delta p^j,
\]

where \(g^{ij}\) is the contravariant tensor of \(g_{ij}\). We have:

**Proposition 6.8.1** \(\mathcal{G}\) is a Riemannian structure on \(\widetilde{T^kM}\).

Indeed, \(\mathcal{G}\) is a tensor field on the manifold \(\widetilde{T^kM}\) covariant symmetric and positively defined.

The geometrical object fields \(N\) and \(g_{ij}\) allows to define the \(F(\widetilde{T^kM})\)-linear mapping

\[
\mathcal{F}: \mathcal{X}(\widetilde{T^kM}) \rightarrow \mathcal{X}(\widetilde{T^kM})
\]

by

\[
\mathcal{F} \left( \frac{\delta}{\delta x^j} \right) = -g_{ij} \frac{\delta}{\delta p^j}, \quad \mathcal{F} \left( \frac{\delta}{\delta y^{(\alpha)i}} \right) = 0, \alpha = 1, \ldots, k, \quad \mathcal{F} \left( \frac{\delta}{\delta p^i} \right) = g^{ij} \frac{\delta}{\delta x^j}
\]

Taking into account the rule of transformations of the d-tensor \(g_{ij}\) and of the vector fields \(\frac{\delta}{\delta x^j}, \ldots, \frac{\delta}{\delta p^j}\) it follows that \(\mathcal{F}\) has a geometrical meaning.

We have also:
Theorem 6.8.1 1° $\nabla F$ is a tensor field on $\tilde{T}^kM$ of type (6.1.1)
2° In adapted basis $\nabla F$ is expressed by

\[(6.8.3) \quad \nabla F = -g_{ij} \frac{\delta}{\delta p_j} \otimes dx^i + g^{ij} \frac{\delta}{\delta x^i} \otimes \delta p_j,\]

3° $\ker \nabla F = N_1 \oplus \ldots \oplus N_{(k-1)}$, $\text{Im} \nabla F = N_0 \oplus W_k$,
4° $\text{rank} \nabla F = 2n$,
5° $\nabla F^3 + \nabla F = 0$.

By means of (6.8.2) the proof is immediate.

Consequently, $\nabla F$ is an almost $(k-1)n$-contact structure on $T^kM$ determined by the nonlinear connection $N$ and d-tensor $g_{ij}$. Of course, it is useful in the case of Hamilton spaces of order $k$.

The conditions of normality of the structure $\nabla F$ can be written exactly as in (6.6.5).

Between the Riemannian structure $\nabla G$ and the structure $\nabla F$ there is a strongly relation.

Theorem 6.8.2 1° The pair $(\nabla F, \nabla G)$ is a Riemannian almost contact structure determined only by the pair $(N, g_{ij})$
2° The associated 2-form is

$\theta = \delta p_i \wedge dx^i$.

3° If the coefficients $N_{ij}$ of $N$ are symmetric, then $\theta$ reduces to the presymplectic structure

$\theta = dp_i \wedge dx^i$.

Proof:

1° The following formula $\nabla G(\nabla F X, Y) = -\nabla G(X, \nabla F Y)$ can be verified on the adapted basis, using (6.8.2).

2° $\theta(X, Y) = \nabla G(\nabla F X, Y)$ is satisfied, too.

3° From $N_{ij} = N_{ji}$ and $\delta p_i = dp_i - N_{ji} dx^j$ we deduce $\delta p_i \wedge dx^i = dp_i \wedge dx^i$, q.e.d.
Chapter 7

Linear Connections on the Manifold $T^*^k M$

In the chapter 10 of the book 'The Geometry of Hamilton and Lagrange Spaces', [115], we studied the notions of $d$-tensor algebra and $N$-linear connections on $T^*^2 M$. The corresponding theory will be extended in the present section to the case $k > 2$.

7.1 The Algebra of Distinguished Tensor Fields

Let $N$ be a nonlinear connection on the manifold $T^*^k M$. It determines, at every point $u \in T^*^k M$, the direct decomposition of the linear space $T_u(T^*^k M)$:

\begin{equation}
T_u(T^*^k M) = N_{0,u} \oplus N_{1,u} \oplus \ldots \oplus N_{k-2,u} \oplus V_{k-1,u} \oplus W_{k,u}.
\end{equation}

A vector field $X \in \mathcal{X}(T^*^k M)$ and an one form $\omega \in \mathcal{X}^*(T^*^k M)$ can be uniquely written in the form

\begin{equation}
\begin{cases}
X = X^H + X^{V_1} + \ldots + X^{V_{k-1}} + X^{W_k}, \\
\omega = \omega^H + \omega^{V_1} + \ldots + \omega^{V_{k-1}} + \omega^{W_k}.
\end{cases}
\end{equation}

Clearly if $h, v_1, v_2, \ldots, v_{k-1}, w_k$ are the projectors determined by the decomposition (7.1.1), we have

\begin{equation}
\begin{aligned}
X^H &= hX, & X^{V_\alpha} &= v_\alpha X, & (\alpha = 1, \ldots, k-1), & X^{W_k} &= w_k X, \\
\omega^H &= \omega \circ h, & \omega^{V_\alpha} &= \omega \circ v_\alpha, & (\alpha = 1, \ldots, k-1), & \omega^{W_k} &= \omega \circ w_k.
\end{aligned}
\end{equation}
Definition 7.1.1 A distinguished tensor field (briefly $d$-tensor field) on the manifold $T^*kM$ of type $(r,s)$ is a $d$-tensor field $T$ of type $(r,s)$ on $T^*kM$ with the property:

$$T(\omega^1, ..., \omega^r, X, ..., X_s) = T(\omega^1, ..., \omega^H, X^1, ..., X^W_k)$$

(7.1.4)

$$\forall \omega, ..., \omega \in X^*(T^*kM), \forall X, ..., X \in \mathcal{X}(T^*kM)$$

For instance, every components $X^H, X^V_i, X^{V_{k-1}}, X^{W_k}$ of a vector field $X$ is a $d$-vector field and every component of an 1-form $\omega^H, \omega^V_i, \omega^{V_{k-1}}, \omega^{W_k}$ is a $d$-1-form field.

In the adapted basis \((\delta x^1, \delta y^{(1)i}, ..., \delta y^{(k-1)i}, \delta p_i)\) to the decomposition (7.1.1) and in its dual basis \((dx^1, \delta y^{(1)i}, ..., \delta y^{(k-1)i}, \delta p_i)\), given by (4.1.10) and (4.2.2), a $d$-tensor field $T$ of type $(r,s)$ can be written in the form

$$T(u) = T^{i_1...i_r}_{j_1...j_s}(u) \frac{\delta}{\delta x^{i_1}} \otimes ... \otimes \frac{\delta}{\delta p_{j_s}} \otimes dx^{j_1} \otimes ... \otimes \delta p_i, \forall u \in T^*kM$$

(7.1.5)

It follows that the set \(\{1, \frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta y^{(1)i}}, ..., \frac{\delta}{\delta y^{(k-1)i}}, \frac{\delta}{\delta p_i}\}\) generates the algebra of the $d$-tensor fields over the ring of functions \(F(T^*kM)\).

With respect to a local transformation of the coordinates on $T^*kM$, the local coefficients $T^{i_1...i_r}_{j_1...j_s}$ of $T$ are transformed by the classical rule

$$T^{i_1...i_r}_{j_1...j_s} = \frac{\partial x^{i_1}}{\partial x^{h_1}} ... \frac{\partial x^{i_r}}{\partial x^{h_r}} \frac{\partial x^{h_1}}{\partial x^{j_1}} ... \frac{\partial x^{h_r}}{\partial x^{j_s}} T^{h_1...h_r}_{j_1...j_s}$$

(7.1.6)

For instance, if $f \in F(T^*kM)$, then each set of functions $\frac{\delta f}{\delta x^1}, ..., \frac{\delta f}{\delta y^{(1)i}}, ..., \frac{\delta f}{\delta p_i}, (i = 1, ..., n)$ is a $d$-covector field, and $\frac{\delta f}{\delta p_i}$ is a $d$-vector field.

7.2 $N$-Linear Connections

The $N$-linear connections, for $k = 2$, were studied in chapter 10, of the book [115]. Their definition for $k \geq 2$ is as follows:

Definition 7.2.1 A linear connection $D$ on the manifold $T^*kM$ is called an $N$-linear connection if:

(1) $D$ preserves by parallelism the distributions $N_1, N_1, ..., N_{k-2}, V_k, W_k$.

(2) The $k-1$ tangent structure $J$ is absolutely parallel with respect to $D$.

(3) The presymplectic structure $\theta$ is absolutely parallel with respect to $D$.

Directly from the definition we can establish, without difficulties the following characterization of an $N$-linear connection.
Theorem 7.2.1 A linear connection $D$ is an $N$-linear connection on the manifold $T^kM$ if and only if:

1. $D$ preserves by parallelism the distributions $N_0, N_1, ..., N_{k-2}, V_{k-1}$ and $W_k$.

2. 

$$D_X (JY^H) = J(D_X Y^H), \quad D_X (JY^V_\alpha) = J(D_X Y^V_\alpha),$$

$$(\alpha = 1, ..., k-1)$$

3. 

$$D_X (JY^{W_k}) = J(D_X Y^{W_k}), \quad \forall X, Y \in X(T^kM).$$

(7.2.1)

(7.2.2)

We remark that the equalities $D_X (JY^{V_{k-1}}) = J(D_X Y^{V_{k-1}})$ and $D_X (JY^{W_k}) = J(D_X Y^{W_k})$ are trivial, since $J(Y^{V_{k-1}}) = 0$, and $J(Y^{W_k}) = 0$.

We obtain also

Theorem 7.2.2 For any $N$-linear connection $D$ we have

$$D_X h = D_X v_\alpha = D_X w_k = 0, (\alpha = 1, ..., k-1),$$

(7.2.3)

(7.2.4)

Indeed, from $(D_X h)(Y) = D_X (hY) - hD_X Y$ if $Y = Y^H$ we obtain

$$(D_X h)(Y^H) = D_X Y^H - hD_X Y^H = 0$$

and for $Y = Y^V_\alpha$,

$$(D_X h)(Y^V_\alpha) = D_X (hY^V_\alpha) - hD_X Y^V_\alpha = 0.$$ 

Also $(D_X h)(Y^{W_k}) = 0$. That means $(D_X h)(Y) = 0, \forall Y \in \chi(T^kM)$. Similarly, we prove the other equalities (7.2.3).

Now, taking into account the formula (6.6.2), we deduce $D_X F = 0$. The last equality (7.2.4) can be proved by the formula $(D_X F)(Y) = D_X F(Y) - F(D_X Y)$, using the local expression of $D_X Y$ (see the section 5, from this chapter).

Let us consider the vector field $X$ written in the form (7.1.2). From the linearity of the operator $D_X Y$ with respect to $X$ we deduce

$$D_X Y = D_X nY + D_X v_1 Y + ... + D_X v_{k-1} Y + D_X w_k Y.$$ 

(7.2.5)

Here appears $(k + 1)$ new operators of derivation in the $d$-tensor algebra, defined by

$$D^H_X = D_X n, D^V_1, ..., D^V_{k-1}, D^V_{k-1} = D_X v_{k-1}, X^W_k = D_X w_k.$$ 

(7.2.6)

These operators are not covariant derivations in the $d$-tensor algebra, since

$$D^H_X f = X^H f \neq X f, D^V_\alpha f = X^V f \neq X f, (\alpha = 1, ..., k),$$

$$D^W_k f = X^W_k f \neq X f.$$
However, they have similar properties with the covariant derivatives.

From (7.2.5) and (7.2.6) we deduce

\[ D_X Y = D_X^{H} Y + D_X^{V_1} Y + \ldots + D_X^{V_{k-1}} Y + D_X^{W_k} Y. \]

By means of Theorem 7.2.2, we have:

**Theorem 7.2.3** The operators \( D_X^{H}, D_X^{V_1}, \ldots, D_X^{V_{k-1}} \) and \( D_X^{W_k} \) have the following properties:

1) Each operator \( D_X^{H}, D_X^{V_1}, \ldots, D_X^{V_{k-1}} \) and \( D_X^{W_k} \) maps a vector field that belongs to one of the distributions \( N_0, N_1, \ldots, N_{k-2}, V_{k-1} \) and \( W_k \) into a vector field that belongs to the same distribution,

2) \( D_X^{H} f = X^{H} f, \quad D_X^{V_1} f = X^{V_1} f, \quad (\alpha = 1, \ldots, k), \quad D_X^{W_k} f = X^{W_k} f, \)

3) \( D_X^{H} (fY) = X^{H} (fY) + f D_X^{H} Y, \quad D_X^{V_1} (fY) = X^{V_1} (fY) + f D_X^{V_1} Y, \)

\( (\alpha = 1, \ldots, k - 1), \quad D_X^{W_k} (fY) = X^{W_k} (fY) + f D_X^{W_k} Y, \)

4) \( D_X^{H} (Y + Z) = D_X^{H} Y + D_X^{H} Z, \quad D_X^{V_1} (Y + Z) = D_X^{V_1} Y + D_X^{V_1} Z, \)

\( (\alpha = 1, \ldots, k - 1), \quad D_X^{W_k} (Y + Z) = D_X^{W_k} Y + D_X^{W_k} Z, \)

5) \( D_{X+Y}^{H} = D_X^{H} + D_Y^{H}, \quad D_{X+Y}^{V_1} = D_X^{V_1} + D_Y^{V_1}, \quad (\alpha = 1, \ldots, k - 1), \quad D_{X+Y}^{W_k} = D_X^{W_k} + D_Y^{W_k}, \)

6) \( D_X^{H} f = f D_X^{H} f, \quad D_X^{V_1} f = f D_X^{V_1} f, \quad (\alpha = 1, \ldots, k - 1), \quad D_X^{W_k} f = f D_X^{W_k} f, \)

7) \( D_X^{H} (fY) = J D_X^{H} f Y, \quad D_X^{V_1} (fY) = J D_X^{V_1} f Y, \quad (\alpha = 1, \ldots, k - 1) \)

\( D_X^{W_k} (fY) = J D_X^{W_k} f Y, \)

8) \( D_X^{H} \theta = 0, \quad D_X^{V_1} \theta = 0, \quad (\alpha = 1, \ldots, k - 1), \quad D_X^{W_k} \theta = 0. \)

9) For any open set \( U \subset T^{*k} M \) the following properties hold:

\( (D_X^{H} Y)_{|U} = D_X^{H} Y_{|U}, \quad (D_X^{V_1} Y)_{|U} = D_X^{V_1} Y_{|U}, \quad (\alpha = 1, \ldots, k - 1), \quad (D_X^{W_k} Y)_{|U} = D_X^{W_k} Y_{|U}. \)

The proof of this theorem can be done by the classical methods, [115].

The operators \( D_X^{H}, D_X^{V_1}, D_X^{W_k} \) will be called the operators of \( h \)-, \( v_1 \)- and \( w_k \)-covariant derivation.
The action of these operators over the 1-form fields $\omega$ are given by

\[(7.2.8)\]
\[
(D^H_X \omega)(Y) = X^H \omega(Y) - \omega(D^H_X Y),
\]
\[
(D^V_\alpha X \omega)(Y) = X^V_\alpha \omega(Y) - \omega(D^V_\alpha X Y), (\alpha = 1, ...k - 1),
\]
\[
(D^W_k X \omega)(Y) = X^W_k \omega(Y) - \omega(D^W_k X Y).
\]

Of course, the action of the operators $D^H_X, D^V_\alpha X, D^W_k X$ can be extended to any tensor fields, particularly to any d-tensor field on $T^*k M$.

For instance, if the d-tensor $T$ verifies (7.1.4) we have

\[(7.2.9)\]
\[
(D^H_X T)(\omega^1, ..., \omega^s, X^H, ..., X^W_k) = X^H T(\omega^1, ..., \omega^s, X^H, ..., X^W_k)
\]
\[= -T(D^H_X \omega^1, ..., \omega^s, X^H, ..., X^W_k) - ... -
\]
\[= -T(\omega^1, ..., \omega^s, X^H, ..., D^H_X X^W_k).
\]

Now, let us consider a parametrized smooth curve $\gamma : t \in I \rightarrow \gamma(t) \in T^*k M$ having the image in a domain of local chart.

Its tangent vector field $\dot{\gamma} = \frac{d\gamma}{dt}$ is uniquely written in the form

\[(7.2.10)\]
\[
\dot{\gamma} = \gamma^H + \gamma^V_1 + \gamma^V_{k-1} + \gamma^W_k .
\]

If the curve $\gamma$ is analytically given by (6.3.2), then $\dot{\gamma}$, in the adapted basis is given by (6.3.20'), (6.3.21). The horizontal curves are defined by Theorem 6.3.2 and the autoparallel curves of the nonlinear connection $N$ are given by the equations (6.3.22) in the conditions

\[y^{(1)i} = \frac{dx^i}{dt}, ..., y^{(k-1)i} = \frac{1}{(k - 1)!} \frac{d^{k-1}x^i}{dt^{k-1}}.
\]

A vector field $Y(\gamma(t))$, along curve $\gamma$ has the covariant derivative

\[(7.2.11)\]
\[
D^\gamma Y = D^H_\gamma Y + D^V_1 Y + ... + D^V_{k-1} Y + D^W_k Y.
\]

The vector field $Y(\gamma(t))$ is parallel along curve $\gamma$ if

\[(7.2.12)\]
\[
D^\gamma Y = 0.
\]

The curve $\gamma$ is autoparallel with respect to the $N$-linear connection $D$ if $D^\gamma \gamma = 0$. This equation is equivalent to

\[(7.2.13)\]
\[
D^H_\gamma \dot{\gamma} + D^V_1 \dot{\gamma} + ... + D^V_{k-1} \dot{\gamma} + D^W_k \dot{\gamma} = 0
\]

In a next section we shall express this equation in an adapted basis.
7.3 The Torsion and Curvature of an N-Linear Connection

The torsion tensor field $T$ of an $N$-linear connection $D$ is expressed as usually by

\begin{equation}
T(X, Y) = D_X Y - D_Y X - [X, Y], \forall X, Y \in \mathcal{X}(T^k M).
\end{equation}

$T$ can be characterized by the vector fields

\begin{align*}
T(X^H, Y^H) &= hT(X^H, Y^H) + v_1 T(X^H, Y^H) + \ldots + v_{k-1} T(X^H, Y^H) + w_k T(X^H, Y^H), \\
T(X^H, Y^{V_\alpha}) &= hT(X^H, Y^{V_\alpha}) + v_1 T(X^H, Y^{V_\alpha}) + \ldots + v_{k-1} T(X^H, Y^{V_\alpha}) + w_k T(X^H, Y^{V_\alpha}), \\
T(X^{V_\alpha}, Y^{V_\beta}) &= hT(X^{V_\alpha}, Y^{V_\beta}) + v_1 T(X^{V_\alpha}, Y^{V_\beta}) + \ldots + v_{k-1} T(X^{V_\alpha}, Y^{V_\beta}) + w_k T(X^{V_\alpha}, Y^{V_\beta}), \\
T(X^{V_\alpha}, Y^{W_k}) &= hT(X^{V_\alpha}, Y^{W_k}) + v_1 T(X^{V_\alpha}, Y^{W_k}) + \ldots + v_{k-1} T(X^{V_\alpha}, Y^{W_k}) + w_k T(X^{V_\alpha}, Y^{W_k}), \\
T(X^{W_k}, Y^{W_k}) &= hT(X^{W_k}, Y^{W_k}) + v_1 T(X^{W_k}, Y^{W_k}) + \ldots + v_{k-1} T(X^{W_k}, Y^{W_k}) + w_k T(X^{W_k}, Y^{W_k}).
\end{align*}

Since $D$ preserves by parallelism the distributions $N_0, N_1, \ldots, N_{k-2}, V_{k-1}$ and $W_k$ we deduce
**Proposition 7.3.1** The following properties of the torsion $\mathbb{T}$ of the $N$-linear connection $D$ hold:

\[(7.3.3)\quad h\mathbb{T}(X^{V_{k-1}}, Y^{V_{k-1}}) = 0, h\mathbb{T}(X^{W_k}, X^{W_k}) = 0.\]

Indeed, the distributions $V_{k-1}$ and $W_k$ being integrable, the equations (7.3.3) are verified.

Now, using the formula (7.3.1), we can write the expression of $\mathbb{T}(X, Y)$ in the form

\[(7.3.4)\quad \mathbb{T}(X, Y) = h\mathbb{T}(X, Y) + v_1\mathbb{T}(X, Y) + \ldots + v_{k-1}\mathbb{T}(X, Y) + w_k\mathbb{T}(X, Y),\]

taking into account the components of $X$ and $Y$ from the decomposition (7.1.2).

The curvature of the $N$-linear connection $D$ is given by

\[(7.3.5)\quad \mathbb{R}(X, Y)Z = (D_X D_Y - D_Y D_X)Z - D_{[X,Y]}Z, \quad \forall X, Y, Z \in \mathcal{X}(T^*kM)\]

Taking into account the decomposition of vector fields $X, Y, Z$ in the form (7.1.2) we can write the curvature $\mathbb{R}$ in a similar manner as the torsion $\mathbb{T}$.

**Proposition 7.3.2** The following properties hold:

\[(7.3.5a)\quad J(\mathbb{R}(X, Y)Z) = \mathbb{R}(X, Y)JZ; \quad D_X\theta = 0.\]

Consequently, we have

**Theorem 7.3.1** The curvature $\mathbb{R}$ has the properties:

1° The essential components of $\mathbb{R}$ are

\[(7.3.6)\quad \mathbb{R}(X, Y)Z^H, \mathbb{R}(X, Y)Z^{V_\alpha}, (\alpha = 1, \ldots k-1), \mathbb{R}(X, Y)Z^{W_k}.\]

2° The vector field $\mathbb{R}(X, Y)Z^H$ belongs to the horizontal distribution $N = N_0$.

3° The vector field $\mathbb{R}(X, Y)Z^{V_\alpha}, (\alpha = 1, \ldots k-1)$ belongs to the distribution $N_\alpha$.

4° The vector field $\mathbb{R}(X, Y)Z^{W_k}$ belongs to the distribution $W_k$.

5° The following equations hold:

\[(7.3.7)\quad v_\alpha\{\mathbb{R}(X, Y)Z^H\} = 0, (\alpha = 1, \ldots k-1), w_k\{\mathbb{R}(X, Y)Z^H\} = 0,
\quad v_\alpha\{\mathbb{R}(X, Y)Z^{V_\beta}\} = 0, (\alpha \neq \beta; \alpha, \beta = 1, \ldots k-1),
\quad v_\alpha\{\mathbb{R}(X, Y)Z^{W_k}\} = 0,
\quad h\{\mathbb{R}(X, Y)Z^{V_\alpha}\} = 0, (\alpha = 1, \ldots k-1), h\{\mathbb{R}(X, Y)Z^{W_k}\} = 0.\]

Of course we can express the $d$-tensor of curvature by means of the operators $D_X^H, D_X^{V_\alpha}, D_X^{W_k}$. They will be written in the adapted basis in a next section.
Proposition 7.3.3 The Ricci identities of the \( N \)-linear connection \( D \) are:

\[
[D_X, D_Y]Z^H = \mathbb{R}(X, Y)Z^H + D_{[X,Y]}Z^H,
\]

(7.3.8) \[
[D_X, D_Y]Z^{V_\alpha} = \mathbb{R}(X, Y)Z^{V_\alpha} + D_{[X,Y]}Z^{V_\alpha}, \quad (\alpha = 1, ..., k - 1),
\]

\[
[D_X, D_Y]Z^{W_k} = \mathbb{R}(X, Y)Z^{W_k} + D_{[X,Y]}Z^{W_k}.
\]

Let us consider the Liouville vector fields \( \Gamma^1, \ldots, \Gamma^{k-1} \) and \( C^* \) from Theorem 4.2.1 and let us apply the previous Proposition.

Theorem 7.3.2 For any \( N \)-linear connection \( D \) the following identities hold:

\[
[D_X, D_Y] \Gamma^{\alpha} = \mathbb{R}(X, Y) \Gamma^{\alpha} + D_{[X,Y]} \Gamma^{\alpha}, \quad (\alpha = 1, ..., k - 1),
\]

(7.3.9) \[
[D_X, D_Y] C^* = \mathbb{R}(X, Y) C^* + D_{[X,Y]} C^*.
\]

Using the considerations from this chapter we can establish the Bianchi identities of an \( N \)-linear connection \( D \), by means of the operators \( D^H_X, D^{V_\alpha}_X, D^{W_k}_X \) taking into account the classical identities

\[
\sum_{(X,Y,Z)} \{(D_X T)(Y, Z) - \mathbb{R}(X, Y) Z + T(T(X, Y), Z)\} = 0,
\]

(7.3.10) \[
\sum_{(X,Y,Z)} \{(D_X \mathbb{R})(U, Y, Z) - \mathbb{R}(T(X, Y), Z) U\} = 0,
\]

where \( \sum_{(X,Y,Z)} \) means the cyclic sum.

7.4 The Coefficients of a \( N \)-Linear Connection

A \( N \)-linear connection \( D \) is characterized by its coefficients in the adapted basis \( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(\alpha)}} , \frac{\delta}{\delta p_i} \). As we shall see these coefficients obey particular rules of transformations with respect to the change of local coordinates on the manifold \( T^*kM \).

Taking into account Theorem 7.2.1 we can prove:

Theorem 7.4.1 We have:
**1°** An N-linear connection $D$ can be uniquely represented in the adapted basis in the following form:

$$\tag{7.4.1} \frac{\delta}{\delta x^i} \frac{\delta}{\delta x^j} = H^s_{ij} \frac{\delta}{\delta x^s}; \frac{\delta}{\delta y^{(\alpha s)}} = H^s_{ij} \frac{\delta}{\delta y^{(\alpha s)}}, (\alpha = 1, \ldots, k - 1);$$

$$\frac{\delta}{\delta x^i} = -H^i_{sj} \frac{\delta}{\delta p^s};$$

$$\frac{\delta}{\delta y^{(\alpha s)}} = C^s_{ij} \frac{\delta}{\delta x^s},$$

$$\frac{\delta}{\delta y^{(\beta s)}} = C^s_{ij} \frac{\delta}{\delta y^{(\beta s)}}, (\alpha, \beta = 1, \ldots, k - 1);$$

$$\frac{\delta}{\delta p^i} = C_{ij} \frac{\delta}{\delta p^j};$$

$$\frac{\delta}{\delta y^{(\alpha s)}} = -C^i_{sj} \frac{\delta}{\delta p^s}.$$

**2°** With respect to the transformation (4.1.2), the coefficients $H^i_{jk}$ obey the rule of transformation

$$\tag{7.4.2} \tilde{H}^i_{rs} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^s}{\partial x^h} = \frac{\partial \tilde{x}^i}{\partial x^r} H^r_{jh} - \frac{\partial^2 \tilde{x}^i}{\partial x^r \partial x^h}$$

**3°** The system of functions $C^i_{jk}, C^j_{ik}, (\alpha = 1, \ldots, k - 1)$ are d-tensor fields of type (1,2) and (2,1), respectively.

**Proof:** According to the definition 2.1, we can write

$$\frac{\delta}{\delta x^i} \frac{\delta}{\delta x^j} = H^s_{ij} \frac{\delta}{\delta x^s}; \frac{\delta}{\delta y^{(\alpha s)}} = H^s_{ij} \frac{\delta}{\delta y^{(\alpha s)}},$$

$$\frac{\delta}{\delta y^{(k-1)s}} = H^s_{ij} \frac{\delta}{\delta y^{(k-1)s}}; \frac{\delta}{\delta p^i} = H^i_{sj} \frac{\delta}{\delta p^s};$$

Applying the mapping $J$ and looking to (4.3.1) and to Theorem 7.2.3, we obtain

$$D \frac{\delta}{\delta x^i} (J \frac{\delta}{\delta x^j}) = J (H^s_{ij} \frac{\delta}{\delta x^s}), D \frac{\delta}{\delta y^{(k-2)s}} (J \frac{\delta}{\delta y^{(k-1)s}}).$$
Consequently,
\[ H^s_{ij} = H^s_{ij}, \quad H^s_{ij} = H^s_{ij}, \quad \ldots, \quad H^s_{ij} = H^s_{ij} \]
and, from \( D \frac{\partial}{\partial x^j} \theta = 0 \), we have \( H^s_{ij} = H^s_{ij} \). It follows that the formulae of the first line of (7.4.1) are valid. Similarly we prove the other equalities of (7.4.1).

By a direct calculus we prove that 2° and 3° from this theorem hold. q.e.d

The systems of functions
\[ \text{(7.4.3)} \quad D^\Gamma(N) = \{ H^i_jh, C^i_jh, C^j_ih \}, \quad (\alpha = 1, \ldots, k-1) \]
is the system of coefficients of the \( N \)-linear connection \( D \) in the adapted basis.

The converse of the previous theorem holds, too.

**Theorem 7.4.2** If the system of functions (7.4.3) are apriori given over every domain of a local chart on the manifold \( T^*kM \), having the rule of transformation mentioned in the previous theorem, then there exists an unique \( N \)-linear connection \( D \) whose local coefficients are just the system of given functions.

**Corollary 7.4.1** The following formulae hold:
\[ D \frac{\delta}{\delta x^j} dx^i = -H^i_jdx^s; \quad D \frac{\delta}{\delta y^{(\alpha)}j} dy^{(\alpha)i} = -H^i_jdy^{(\alpha)s}, \quad (\alpha = 1, \ldots, k-1); \]
\[ D \frac{\delta}{\delta p_i} = H^s_{ij} \delta p_s; \quad D \frac{\delta}{\delta y^{(\alpha)}j} \delta y^{(\alpha)i} = -H^i_j \delta y^{(\alpha)s}, \quad (\alpha = 1, \ldots, k-1); \]
\[ D \frac{\delta}{\delta p_i} = C^s_{ij} \delta p_s; \quad D \frac{\delta}{\delta y^{(\alpha)}j} \delta p_i = C^s_{ij} \delta y^{(\alpha)s}, \quad (\alpha = 1, \ldots, k-1); \]
\[ D \frac{\delta}{\delta p_i} = C^i_{js} \delta p_s; \quad D \frac{\delta}{\delta y^{(\alpha)}j} \delta p_i = C^i_{js} \delta y^{(\alpha)s}, \quad (\alpha = 1, \ldots, k-1); \]
\[ D \frac{\delta}{\delta p_i} = C^i_{js} \delta p_s. \]

Indeed, the formula (7.4.1) and the conditions of duality between the vector fields from the adapted basis and its dual 1-form basis lead to the formula (7.4.4).

### 7.5 The \( h\)-, \( v_{\alpha}\)- and \( w_k\)-Covariant Derivatives in Local Adapted Basis

In the adapted basis a tensor field \( T \) can be written in the form (7.1.5):
Applying the operator of covariant derivation $D_X$ for $X = X^H = X^i \frac{\delta}{\delta x^i}$ and taking into account the formulae (7.4.1), (7.4.4) and the properties of the operator $D_X^H T = X^i D^H \frac{\delta}{\delta x^i} T$, expressed in Theorem 7.2.3, we deduce

$$D_X^H T = X^m T^{i_1 \ldots i_r}_{j_1 \ldots j_s, m} \frac{\delta}{\delta x^{i_1}} \otimes \ldots \otimes \frac{\delta}{\delta x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dp_x,$$

where

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s, m} = \frac{\delta T^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\delta x^m} + T^{h i_2 \ldots i_r}_{j_1 \ldots j_s} H^{i_1}_{h m} + \ldots + T^{i_1 \ldots h}_{j_1 \ldots j_s} H^h_{j m} - T^{i_1 \ldots i_r}_{h j_s \ldots j_s} H^h_{j m} - \ldots - T^{i_1 \ldots i_r}_{j_1 \ldots h} H^h_{j m}.$$

The operator "\( i \)" is called the \( h \)-covariant derivative with respect to $D \Gamma(N)$.

Now, we put $X = X^{V \alpha} = X^i \frac{\delta}{\delta y^{(\alpha)i}}$, $(\alpha = 1, \ldots, k)$. From (7.5.1) we deduce

$$D_X^{V \alpha} T = X^m T^{i_1 \ldots i_r}_{j_1 \ldots j_s, m} \frac{\delta}{\delta x^{i_1}} \otimes \ldots \otimes \frac{\delta}{\delta x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dp_x,$$

where

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s, m} = \frac{\delta T^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\delta y^{(\alpha)i}} + T^{h i_2 \ldots i_r}_{j_1 \ldots j_s} C^{i_1}_{h 1 m} + \ldots + T^{i_1 \ldots h}_{j_1 \ldots j_s} C^h_{j m} - T^{i_1 \ldots i_r}_{h j_s \ldots j_s} C^h_{j m} - \ldots - T^{i_1 \ldots i_r}_{j_1 \ldots h} C^h_{j m}.$$

The operator "\( i \)" is called the \( v \)-covariant derivative with respect to $D \Gamma(N)$.

Finally, taking $X = X^{W \alpha} = X^i \frac{\delta}{\delta p_i}$, then $D_X^{W \alpha} T$ has the form:

$$D_X^{W \alpha} T = X^m T^{i_1 \ldots i_r}_{j_1 \ldots j_s, m} \frac{\delta}{\delta p_i} \otimes \ldots \otimes \frac{\delta}{\delta p_i} \otimes dx^{j_1} \otimes \ldots \otimes dp_x,$$

where

$$T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\delta T^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\delta p_i} + T^{h i_2 \ldots i_r}_{j_1 \ldots j_s} C^{i_1}_{h m} + \ldots + T^{i_1 \ldots h}_{j_1 \ldots j_s} C^h_{j m} - T^{i_1 \ldots i_r}_{h j_s \ldots j_s} C^h_{j m} - \ldots - T^{i_1 \ldots i_r}_{j_1 \ldots h} C^h_{j m}.$$
Proposition 7.5.1 The following properties hold:

\[ T_{j_1...j_s|m}, T^{i_1...i_r}(\alpha)_{j_1...j_s|m}, T^{i_1...i_r}|m, (\alpha = 1, ..., k - 1) \]

are \(d\)-tensor fields. The first two are of type \((r, s + 1)\) and the last one is of type \((r + 1, s)\).

As an application, the \(d\)-tensor field \(g_{ij}\) has the \(h-, v_\alpha- \) and \(w_k\)-covariant derivatives with respect to the \(N\)-linear connection with the coefficients \(D_\Gamma(N)\), given by:

\[
(7.5.5) \quad \begin{align*}
g_{ij}^{(\alpha)}|m &= \frac{\delta g_{ij}}{\delta x^m} - H^h_{im}g_{hj} - H^h_{jm}g_{ih}, \\
g_{ij}|m &= \frac{\delta g_{ij}}{\delta y^{(\alpha)m}} - C^h_{im}g_{hj} - C^h_{jm}g_{ih}, \quad (\alpha = 1, ..., k - 1), \\
g_{ij}|m &= \frac{\delta g_{ij}}{\delta p_m} - C^m_{im}g_{hj} - C^m_{jm}g_{ih}.
\end{align*}
\]

In a next chapter we shall determine the coefficients \(D_\Gamma(N)\) from the conditions that \(g_{ij}\) is covariant constant with respect \(D\).

Proposition 7.5.2 The operators \(\ |\), \(\ |\) and \(\ |\) have the properties:

1° \(f_{|m} = \frac{\delta f}{\delta x^m}, f\ | m = \frac{\delta f}{\delta y^{(\alpha)m}}, (\alpha = 1, ..., k - 1), f|m = \frac{\delta f}{\delta p_m}\)

2° These operators are distributive with respect to the addition of the \(d\)-tensor of the same type.

3° They commute with the operation of contraction

4° They verify the Leibniz rule to the tensor product.

As an application we study the \((z^i)\)-deflection tensor of \(D_\Gamma(N)\). They are defined by:

\[
(7.5.6) \quad \begin{align*}
D_{ij}^{(\alpha)} &= z^i_{|j}, \quad D_{ij}^{(\beta)} = z^i_{|j}, \quad (\alpha, \beta = 1, ..., k - 1), \\
D^{ij} &= z^i \ | j,
\end{align*}
\]

where \(z^i, (\alpha = 1, ..., k - 1)\) are the Liouville \(d\)-vector fields.

Evidently, they have the following expressions

\[
(7.5.6a) \quad \begin{align*}
D_{ij}^{(\alpha)} &= \frac{\delta z^i}{\delta x^j} + z^m H^{i}_{mj}^{(\alpha)}, \\
D_{ij}^{(\alpha)} &= \frac{\delta z^i}{\delta y^{(\beta)j}} + z^m C_{mj}^{i(\beta)}.
\end{align*}
\]
Similarly, we introduce the $(p)$-deflection tensors by

\[(7.5.7) \Delta_{ij} = p_{|ij|}, \quad \beta_{ij} = p_{i|j}, \quad \beta_{i}^j = p_{i|j}.\]

We deduce:

\[
\Delta_{ij} = -p_{h}H_{ij}^h, \quad \beta_{ij} = -p_{h}C_{ij}^h, \quad \beta_{i}^j = \delta_{i}^j - p_{h}C_{i|j}^h.
\]

The deflection tensors will be used in some important identities determined by the Ricci identities, applied to the Liouville $d$-tensor fields $z^i$, and to the $d$-covector $p_i$.

Some remarks:

1° In the adapted basis we can prove the equation

\[D_X F = 0, \forall X \in X(T^*kM)\]

2° A Berwald connection is an $N$-linear connection $D$ with the coefficients

\[(7.5.8) B^\Gamma (N) = (B_{jk}^i, 0, ..., 0, 0)\]

where $B_{jk}^i$ has the same rule of transformation as $H_{ij}^h$ from (7.4.2) and is determined by the nonlinear connection $N$.

We have:

**Theorem 7.5.1** Any nonlinear connection $N$, with the coefficients $(N_j^i, N_j^{i \downarrow}, N_{ij})$ determines the following Berwald connections:

\[(7.5.9) B^\Gamma (N) = (\partial^i N_{ij}, 0, ..., 0, 0)\]

\[(7.5.9a) B^\Gamma (N) = (\delta N_j^i, 0, ..., 0, 0)\]

**Proof:** The first one is given by the last formula (7.1.6) applying the derivation $\delta^i = \frac{\partial x^i}{\partial x^s} \partial^s$ and (7.5.9’) is obtained from the first formula (7.1.6), applying the derivation $\delta^i = \frac{\partial x^m}{\partial y^{i \downarrow}^h} \delta$, q.e.d.

If the base manifold $M$ is paracompact, then the manifold $T^*kM$ is paracompact, too. Consequently, on $T^*kM$ there exist nonlinear connections. Therefore, we have:

**Theorem 7.5.2** If the base manifold $M$ is paracompact, then on the manifold $T^*kM$ there exist $N$-linear connections $D$.

Indeed, on $T^*kM$ there exist nonlinear connections $N$. Applying the Theorem 7.5.1 the conclusion follows. Q.E.D.
7.6 Ricci Identities. Local Expressions of \( d \)-Tensor of Curvature and Torsion. Bianchi Identities.

Let \( D \) be an \( N \)-linear connection with the local coefficients

\[
D \Gamma(N) = (H^i_{jh}, C^i_{jh,1}, \ldots, C^i_{jh,(k-1)})
\]

The Ricci identities for a \( d \)-vector field \( X^i \) can be deduced from the formulae (7.3.8) written in the adapted basis. But we can obtain them by a straightforward calculus.

**Theorem 7.6.1** For any \( N \)-linear connection \( D \) and any \( d \)-vector field \( X^i \) the following Ricci formulae hold:

\[
X^i_{|j|h} - X^i_{|h|j} = X^m R^i_{mj,h} - X^i_{|m|T^i_{jh}} - \{ X^i_{|m} R^m_{jh} + X^i_{|m} H^m_{jh} \} -
\]

\[
+ X^j_{(k-1) mj} + X^i_{|m} R^m_{jh}, \quad (0)
\]

\[
X^i_{|j|h} - X^i_{|h|j} = X^m P^i_{mj,h} - X^i_{|m} C^m_{jh} - \{ X^i_{|m} R^m_{jh} + X^i_{|m} H^m_{jh} \} -
\]

\[
- \{ X^i_{|m} B^m_{jh} + \ldots + X^i_{(k-1)m} B^m_{jh} + X^i_{|m} B^m_{jh} \}, \quad (1)
\]

\[
X^i_{|j|h} - X^i_{|h|j} = X^m P^i_{mj,h} - X^i_{|m} C^m_{jh} - X^i_{|m} H^m_{jh} -
\]

\[
- \left\{ X^i_{|m} B^m_{jh} + \ldots + X^i_{(k-1)m} B^m_{jh} + X^i_{|m} B^m_{jh} \right\}, \quad (1)
\]

\[
X^i_{|j|} - X^i_{|h|j} = X^m S^i_{(\alpha) mj} - X^i_{|m} C^m_{jh} - X^i_{|m} B^m_{jh} -
\]

\[
- \left\{ X^i_{|m} C^m_{(\alpha) j} + \ldots + X^i_{(k-1)m} C^m_{(\alpha) j} + X^i_{|m} B^m_{(\alpha) j} \right\},
\]

\[
X^i_{|j|} - X^i_{|h|j} = X^m S^i_{(\beta) mj} - X^i_{|m} C^m_{jh} - X^i_{|m} B^m_{jh} -
\]

\[
- \left\{ X^i_{|m} C^m_{(\beta) j} + \ldots + X^i_{(k-1)m} C^m_{(\beta) j} + X^i_{|m} B^m_{(\beta) j} \right\},
\]

\[
X^i_{|j|} - X^i_{|h|j} = X^m S^i_{(\alpha\beta) mj} - X^i_{|m} C^m_{jh} - X^i_{|m} B^m_{jh} -
\]

\[
- \left\{ X^i_{|m} C^m_{(\alpha\beta) j} + \ldots + X^i_{(k-1)m} C^m_{(\alpha\beta) j} + X^i_{|m} B^m_{(\alpha\beta) j} \right\}.
\]
Linear Connection on the Manifold $T^*kM$

\[ (7.6.2_5) \]
\[ X^i |^h - X^i |^h = X^m S^i_m |^h - X^i |^m C^h_{mj} + X^i |^h C^m_{jh} - \left\{ X^i |^1_m C^m_{mjh} + \cdots + X^i |^k_{m(k-1)j} + X^i |^m C^h_{mj} \right\}, \]

\[ (7.6.2_6) \]
\[ X^i |^h - X^i |^h = X^m S^i_m |^h + X^i |^m S^j_h, \]

where all terms in $R^i_{mjh}, R_m^{i j h}, B^i_{mjh}, B_{ijh}, B_m^h$ are known from the Lie brackets (6.5.1).

The coefficients $D\Gamma(N)$ are given in (7.6.1). Supplementary we put:

\[ (7.6.3) \]
\[ T_{jh}^i = H_{jh}^i - H_{ijh}^s - C_{ijh}^h. \]

Here $R^i_{mjh}, \ldots$, are called \textit{d-tensor of curvature of} $D$ and they have the expressions:

\[ (7.6.4) \]
\[
\begin{align*}
R^i_{mjh} &= \frac{\delta H^i_{mj}}{\delta x^h} - \frac{\delta H^i_{mh}}{\delta x^j} + H^s_{mj} H^i_{sh} - H^s_{mh} H^i_{sj} + C^i_{m s (1) j h} + \\
&\quad + \cdots + C^i_{m (k-1) s j h} + C^i_{s m} R^s_{s jh},
\end{align*}
\[
\begin{align*}
P^i_{m jh} &= \frac{\delta H^i_{mj}}{\delta y^{(a)h}} - \frac{\delta C^i_{mh}}{\delta x^j} + H^s_{mj} C^i_{s h} - C^s_{mh} H^i_{sj} + C^i_{m s (1) j h} + \\
&\quad + \cdots + C^i_{m (k-1) s j h} + C^i_{s m} B^s_{s jh},
\end{align*}
\[
\begin{align*}
P^i_{m j} &= \frac{\delta H^i_{mj}}{\delta p_h} - \frac{\delta C^i_{mh}}{\delta x^j} - C^s_{m s} H^i_{sj} + C^i_{m s (1) j h} + \\
&\quad + \cdots + C^i_{m (k-1) s j} + C^i_{s m} B^s_{s jh}.
\end{align*}
\]
and

\[
\begin{align*}
S_{(\alpha\beta)^m_{ij}^h} &= \frac{\delta C_i^j}{\delta y^{(\beta)h}} - \frac{\delta C_i^j}{\delta y^{(\alpha)h}} + C^i_{mj} C^j_{sh} - C^i_{mh} C^j_{s\alpha} + C^i_{ms} C^j_{s\alpha h} + \\
&\quad + \cdots + C^i_{m(k-1)h} s_j + C^i_{m\alpha h}, \quad (\alpha \leq \beta, \alpha, \beta = 1, \ldots, k - 1) \\
S_{(m^i_j)^h} &= \frac{\delta C_i^j}{\delta p_h} - \frac{\delta C_i^j}{\delta p_{\alpha j}} + C^i_{m\alpha j} + C^i_{m\alpha h}, \\
S_{m^{ij}h} &= \frac{\delta C_{ij}}{\delta p_h} - \frac{\delta C_{ij}}{\delta p_{\alpha j}} + C^m_{ij} C^j_{sh} - C^m_{ij} C^sh.
\end{align*}
\]

As usually, we extend the Ricci identities (7.6.2) for any \( d \)-tensor field \( T^i_{j_1 \ldots j_r} \).

For instance, if \( g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p) \) is a \( d \)-tensor field, the Ricci identities of \( g^{ij} \), with respect to the \( N \)-linear connection \( D\Gamma(N) \), are:

\[
(7.6.6)
\]

\[
g_{ij}^{(1)} - g_{ij}^{(1)} (x, y^{(1)}, \ldots, y^{(k-1)}, p) = \delta R^{i}_{sj} - g^{is} R_{sj}^{j} - g^{is} T_{sh}^{s} - \left\{ g^{ij} \right\}_{s}^{(1)} R_{s}^{s} s_{h} + \\
\cdots + g^{ij} (k-1) s_{j} + g^{ij} s_{j} R_{sh}^{s} s_{h} + g^{ij} | s_{(0,k-1)}^{s} S_{s}^{s} h m.
\]

In particular if \( D\Gamma(N) \) satisfies the supplementary conditions:

\[
(7.6.7)
\]

\[
g_{ij}^{(1)} = 0, \quad g_{ij}^{(1)} | h = 0, \quad g^{ij} | h = 0, \quad (\alpha = 1, \ldots, k - 1),
\]

then the Ricci identities (7.6.6) give us:

\[
(7.6.8)
\]

\[
g^{is} R_{s}^{i} h m + g^{is} R_{s}^{j} h m = 0,
\]

\[
g^{is} P_{(s)}^{i} h m + g^{is} P_{(s)}^{j} h m = 0,
\]

\[
g^{is} S_{s}^{i} h m + g^{is} S_{s}^{j} h m = 0.
\]

Some important identities are obtained applying the Ricci identities to the \( d \)-covector \( p_i \) and to the Liouville vector fields \( z^{(\alpha)j} \), \( (\alpha = 1, \ldots, k - 1) \).
\section*{Linear Connection on the Manifold $T^*kM$}

\textbf{Theorem 7.6.2} Any $N$-linear connection $D\Gamma(N)$ satisfies the following identities:

(7.6.9) \[ \Delta_{ij|h} - \Delta_{ih|j} = -p_s R_i s^j h - \Delta_{is} T^s_{jh} - \left\{ \delta_{ij} R_i s^j h + \cdots + \delta_{ij} R_i s^j h + \delta_{ij} R_i s^j h \right\}; \]

\[ \Delta_{ij} | h - \beta_{ih} | j = -p_s P_i s^j h - \Delta_{is} C^i_{s} jh - \beta_{ij} H_s h - \left\{ \delta_{ij} B s^j h + \cdots + \delta_{ij} B s^j h + \delta_{ij} B s^j h \right\}; \]

\[ \Delta_{ij} | h - \beta_{i} h | j = -p_s P_i s^j h - \Delta_{is} C^i_{s} jh - \beta_{ij} H_s h - \left\{ \delta_{ij} B s^j h + \cdots + \delta_{ij} B s^j h + \delta_{ij} B s^j h \right\}; \]

\[ \beta_{ij} | h - \beta_{ih} | j = -p_s S_i s^j h - \Delta_{is} C^i_{s} jh - \beta_{ij} H_s h - \left\{ \delta_{ij} B s^j h + \cdots + \delta_{ij} B s^j h + \delta_{ij} B s^j h \right\}; \]

The similar identities are obtained applying the Ricci identities to the Liouville vector fields $z^{(a)i}$.

\textbf{Theorem 7.6.3} Any $N$-linear connection $D\Gamma(N)$ satisfies the following identities, obtained from (7.6.2) for $X^i = z^{(a)i}$:

(7.6.10) \[ \delta_{ij} R_i s^j h - \Delta_{is} T^s_{jh} - \left\{ \delta_{ij} R_i s^j h + \cdots + \delta_{ij} R_i s^j h + \delta_{ij} R_i s^j h \right\}; \]

\[ \delta_{ij} B s^j h + \cdots + \delta_{ij} B s^j h + \delta_{ij} B s^j h \right\}; \]

\[ \delta_{ij} B s^j h + \cdots + \delta_{ij} B s^j h + \delta_{ij} B s^j h \right\}; \]

The Bianchi identities for the $N$-linear connection $D\Gamma(N)$ can be obtained
using the methods described above. Namely, applying the Ricci identities to $d$-tensor field $X^i_{|j}$ we obtain:

\[
\begin{pmatrix}
X^i_{|j} |_{h|m} - X^i_{|j} |_{m|h} = X^r |_{r} R^i_{r |h m} - X^i |_{r} R^j_{r |h m} - X^i |_{r} T^r_{h m} -
\end{pmatrix}
\]

\[
(\sum_{\alpha=1}^{(k-1)} \delta x^\alpha).
\]

If we cyclically permute the indices $j, h, m$ and add the identities obtained we determine a first set of Bianchi identities:

\[
(7.6.11)
\]

\[
S_{(jhm)} \left\{ R^i_{j h m} - T^i_{j h|m} - T^i_{j r} T^r_{h m} + \sum_{\alpha=1}^{k-1} C^i_{\alpha} R^r_{\alpha h m} + C^i_{\alpha} R^r_{(0) h m} \right\} = 0,
\]

\[
S_{(jhm)} \left\{ R^i_{s j h|m} + R^i_{s j r} T^r_{h m} - \sum_{\alpha=1}^{k-1} P^i_{\alpha} R^r_{\alpha h m} - P^i_{\alpha} R^r_{(0) h m} \right\} = 0,
\]

\[
S_{(jhm)} \left\{ R^i_{s j h|m} - R^i_{s j r} T^r_{h m} + P^i_{\alpha} R^r_{\alpha h m} + B^i_{\alpha} R^r_{(0) h m} \right\} = 0,
\]

\[
S_{(jhm)} \left\{ R^i_{s j h|m} - R^i_{s j r} T^r_{h m} + \sum_{\alpha=1}^{k-1} B^i_{\alpha} R^r_{\alpha h m} + P^i_{\alpha} R^r_{(0) h m} \right\} = 0.
\]

\[
(7.6.12)
\]

\[
P^i_{(\alpha) j r} = H^i_{j r} + B^i_{(\alpha) j r}, \quad (\alpha = 1, ..., k - 1).
\]

In a similar way we have the second set of Bianchi identities:

\[
(7.6.13)
\]

\[
S_{(jhm)} \{ S^i_{s j h|m} - S^i_{s j r} S^r_{h m} \} = 0,
\]

\[
S_{(jhm)} \{ S^i_{s h m} - S^i_{s h|m} - S^i_{s r} S^r_{h m} \} = 0,
\]

where $S_{(jhm)}$ is the cyclic sum.

Similarly, we can get the other Bianchi identities.

### 7.7 Parallelism of the Vector Fields on the Manifold $T^*kM$

Consider a $N$-linear connection $D$ on the manifold $T^*kM$ with the coefficients $\Gamma(N)=(H^i_{j h}, C^i_{\alpha} j h, C^i_{j h})$, $(\alpha = 1, ..., k - 1)$ in the adapted basis

\[
\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(\alpha)i}}, \frac{\delta}{\delta p_i} \right).
\]

A smooth curve $\gamma : I \rightarrow T^*kM$ having the image in a domain of a local chart is given by

\[
x^i = x^i(t), \quad y^{(\alpha)i}(t), \quad p_i = p_i(t), \quad (\alpha = 1, ..., k - 1), \quad t \in I.
\]
The tangent vector field \( \dot{\gamma} = \frac{d\gamma}{dt} \) can be written by means of (7.2.10), in the form:

\[
\dot{\gamma} = \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{\delta y^{(1)i}}{dt} \frac{\delta}{\delta y^{(1)i}} + \cdots + \frac{\delta y^{(k-1)i}}{dt} \frac{\delta}{\delta y^{(k-1)i}} + \frac{\delta p_i}{dt} \frac{\partial}{\partial p_i},
\]

where, from (6.3.2),

\[
\begin{align*}
\frac{\delta}{\delta x^i} & = dx^i \\
\frac{\delta}{\delta y^{(1)i}} & = dy^{(1)i} \\
\frac{\delta}{\delta y^{(k-1)i}} & = dy^{(k-1)i} \\
\frac{\delta}{\delta p_i} & = dp_i - N_{ji} dx^j,
\end{align*}
\]

Let us denote

\[
D \cdot \gamma X = \frac{DX}{dt}, \quad DX = \frac{DX}{dt} dt, \quad \forall X \in \mathcal{X}(T^*kM).
\]

\( DX \) is called the covariant differential of the vector field \( X \) and \( \frac{DX}{dt} \) is the covariant differential of \( X \) along curve \( \gamma \).

If \( X \) is written in the form

\[
X = X^H + X^{V_i} + \cdots + X^{V_{k-1}} + X^{W_k} = \quad (7.7.4a)
\]

and \( \dot{\gamma} \) from (7.7.2) is \( \dot{\gamma} = \dot{\gamma}^H + \dot{\gamma}^{V_i} + \cdots + \dot{\gamma}^{V_{k-1}} + \dot{\gamma}^{W_k} \) we have

\[
D_{\gamma} = D_{\gamma}^H + D_{\gamma}^{V_i} + \cdots + D_{\gamma}^{V_{k-1}} + D_{\gamma}^{W_k}.
\]

Then \( DX \) has the final form:

\[
DX = \left( d X + X^{\omega_s} \right) \frac{\delta}{\delta x^i} + \left( d X + X^{\omega_s} \right) \frac{\delta}{\delta y^{(1)i}} + \cdots + \\
+ \left( d X + X^{\omega_s} \right) \frac{\delta}{\delta y^{(k-1)i}} + (dX_i - X_i^{\omega_s}) \frac{\partial}{\partial p_i},
\]

where

\[
\omega^i_j = H_{js}^i dx^s + C_{js}^{(1)i} dy^{(1)s} + \cdots + C_{js}^{(k-1)i} dy^{(k-1)s} + C_{js}^{k-1}s dp_s.
\]

The differential forms \( \omega^i_j \) are called the 1-forms connection of the connection \( D \).
Putting

\[ (7.7.7) \quad \frac{\omega_i}{dt} = H^i_{js} \frac{dx^s}{dt} + C^i_{(1)s} \frac{\delta y^{(1)s}}{dt} + \cdots + C^i_{(k-1)s} \frac{\delta y^{(k-1)s}}{dt} + C^i_j \frac{\delta p_s}{dt}. \]

the covariant differential \( \frac{DX}{dt} \) along curve \( \gamma \) is

\[ (7.7.8) \quad \frac{DX}{dt} = \left( \frac{d}{dt} + \frac{(0)i}{X} \frac{\omega_s^i}{dt} \right) \frac{\delta}{\delta x^i} + \left( \frac{d}{dt} + \frac{(1)i}{X} \frac{\omega_s^i}{dt} \right) \frac{\delta}{\delta y^{(1)i}} + \cdots + \left( \frac{d}{dt} + \frac{(k-1)i}{X} \frac{\omega_s^i}{dt} \right) \frac{\delta}{\delta y^{(k-1)i}} + \left( \frac{dX_i}{dt} - X_s \frac{\omega_s^i}{dt} \right) \frac{\delta}{\delta p_i}. \]

The theory of the parallelism of vector fields along curve \( \gamma \), presented in the chapter 10, section 7 of the book [115], in the case \( k = 2 \), can be extended without difficulties for \( k > 2 \).

Consequently, we define the notion of parallelism of a vector field \( X(\gamma(t)) \) along curve \( \gamma \), by the differential equation \( \frac{DX}{dt} = 0 \). We obtain:

**Theorem 7.7.1** The vector field \( X \), given by \((7.7.4')\) is parallel along the parametrized curve \( \gamma \), with respect to the \( N \)-linear connection \( D \), if and only if its components \( X, X_s, X_i (\alpha = 1, \ldots, k-1) \) are solutions of the differential equations

\[ (7.7.9) \quad \frac{d}{dt} + \frac{(0)i}{X} \frac{\omega_s^i}{dt} = 0, \quad \frac{d}{dt} + \frac{(\alpha)i}{X} \frac{\omega_s^i}{dt} = 0, \quad (\alpha = 1, \ldots, k-1), \]

\[ \frac{dX_i}{dt} - X_s \frac{\omega_s^i}{dt} = 0. \]

By means of the formula \((7.7.8)\), the proof of the previous theorem is immediate.

The vector field \( X \in X(T^kM) \) is called absolute parallel with respect to the \( N \)-linear connection \( D \) if the equation \( DX = 0 \) holds for any curve \( \gamma \). This equations \( DX = 0, \forall \gamma \) is equivalent to the integrability of the following system of Pfaff equations

\[ (7.7.10) \quad d X + X_s \omega_s^i = 0, \quad d X + X_s \omega_s^i = 0, \quad (\alpha = 1, \ldots, k-1), \quad dX_i - X_s \omega_s^i = 0. \]
But the previous system is equivalent to
\[(7.7.10a)\]
\[(0)^{ij}_{\alpha} X^i_j = X^i_j = 0, \ (\alpha = 1, ..., k-1), \ X^i_{ij} = 0, \]
\[(0)^{ij}_{\alpha\beta} X^i_j = X^i_j = 0, \ (\alpha = 1, ..., k-1), \ X^i_j = 0, \ (\beta = 1, ..., k-1), \]
\[(0)^{ij}_{\alpha} X^i_j = 0, \ (\alpha = 1, ..., k-1), \ X^i_j = 0, \]
which must be integrable.

Using the Ricci identities (7.6.2) the system (7.7.10') is integrable if and only if the coordinates \((0)^{ij}, (\alpha)^{ij}, X^i_j\) of the vector field \(X\) satisfy the following equations:
\[(7.7.11)\]
\[(\alpha)^s R^i_{sjh} = 0, \ (\alpha)^s P^i_{(\beta)^s}_{jh} = 0, \ (\alpha)^s P^i_{s h} = 0, \ (\alpha)^s S^i_{(\beta\gamma)^s}_{jh} = 0, \]
\[(\beta)^s S^i_{(\beta)^s h} = 0, \ (\alpha)^s S^i_{s jh} = 0, \ (\alpha = 0,1, ..., k-1; \beta, \gamma = 1, ..., k-1)\]
and
\[(7.7.11a)\]
\[X^* R^i_{sjh} = 0, \ X^* P^i_{s h} = 0, \ X^* S^i_{s jh} = 0, \]

The manifold \(T^{*k}M\) is called with absolute parallelism of vectors, with respect to the \(N\)-linear connection \(D\) if and only if all curvature \(d\)-tensors of \(D\) vanish, i.e.
\[R^i_{m jh} = 0, \ P^i_{m h} = 0, \ P^i_{m j} = 0, \ S^i_{(\alpha\beta) m jh} = 0, \ S^i_{(\alpha) m j} = 0, \ S^i_{m ij} = 0, \ (\alpha, \beta = 1, ..., k-1).\]

The previous theory can be applied to investigate the autoparallel curves with respect to a \(N\)-linear connection \(D\).

The parametrized curve \(\gamma: t \in I \rightarrow \gamma(t) \in T^{*k}M\), is an autoparallel curve with respect to \(D\) if \(D_{\gamma} \dot{\gamma} = 0\).
By means of (7.7.2), (7.7.8) we obtain

\[
D \gamma = \frac{\delta}{dt} \gamma = \left( \frac{d^2 x^i}{dt^2} + \frac{dx^i \omega^s}{dt} \frac{d}{dt} \frac{dx^s}{dt} \right) \frac{\delta}{\delta x^i} + \left( \frac{d}{dt} \frac{\delta y^{(1)i}}{dt} + \frac{\delta y^{(1)s} \omega^i}{dt} \frac{\delta}{\delta y^{(1)s}} \right) \frac{\delta}{\delta y^{(1)i}} + \cdots + \left( \frac{d}{dt} \frac{\delta y^{(k-1)i}}{dt} + \frac{\delta y^{(k-1)s} \omega^i}{dt} \frac{\delta}{\delta y^{(k-1)s}} \right) \frac{\delta}{\delta y^{(k-1)i}} + \left( \frac{d}{dt} \frac{\delta p_i}{dt} - \frac{\delta p_s \omega^i}{dt} \frac{\delta}{\delta p_s} \right) \frac{\delta}{\delta p_i}.
\]

**Theorem 7.7.3** A smooth parametrized curve \( \gamma, (7.7.1) \) is an autoparallel curve with respect to the \( N \)-linear connection \( D \) if and only if the functions \( x^i(t), y^{(\alpha)i}(t), p_i(t), (\alpha = 1, \ldots, k-1), t \in I \) verify the following system of differential equations:

\[
\begin{align*}
\frac{d^2 x^i}{dt^2} + \frac{dx^i \omega^s}{dt} \frac{d}{dt} \frac{dx^s}{dt} &= 0, \\
\frac{d}{dt} \frac{\delta y^{(\alpha)i}}{dt} + \frac{\delta y^{(\alpha)s} \omega^i}{dt} \frac{\delta}{\delta y^{(\alpha)s}} &= 0, (\alpha = 1, \ldots, k-1), \\
\frac{d}{dt} \frac{\delta p_i}{dt} - \frac{\delta p_s \omega^i}{dt} \frac{\delta}{\delta p_s} &= 0.
\end{align*}
\]

(7.7.13)

Of course, the theorem of existence and uniqueness for the autoparallel curves can be formulated taking into account the system of differential equations (7.7.13).

We recall that \( \gamma \) is an horizontal curve if \( \dot{\gamma} = \gamma^H \). The horizontal curves are characterized by

\[
x^i = x^i(t), \quad \frac{\delta y^{(\alpha)i}}{dt} = 0, (\alpha = 1, \ldots, k-1), \quad \frac{\delta p_i}{dt} = 0.
\]

(7.7.14)

**Definition 7.7.1** A horizontal path of an \( N \)-linear connection \( D \) is a horizontal autoparallel curve \( \gamma, \) with respect to \( D \).

So, a horizontal path \( \gamma \) is characterized by \( D_{\gamma^H} \gamma = 0 \). Taking into account (7.7.13) we get:

**Theorem 7.7.4** The horizontal paths of an \( N \)-linear connection \( D \) are characterized by the system of differential equations

\[
\frac{d^2 x^i}{dt^2} + H^h_{j\underline{n}} \frac{dx^j}{dt} \frac{dx^h}{dt} = 0, \quad \frac{\delta y^{(\alpha)i}}{dt} = 0, (\alpha = 1, \ldots, k-1), \quad \frac{\delta p_i}{dt} = 0.
\]

(7.7.15)

Indeed, (7.7.14) implies \( \frac{\omega^i}{dt} = H^h_{j\underline{n}} \frac{dx^h}{dt} \). But (7.7.13) gives us the mentioned equations (7.7.15).
A parametrized curve $\gamma : I \to T^k M$ is called $v_{\alpha}$-vertical at the point $x_0 \in M$ if $\gamma = \gamma_{V_{\alpha}}$. It is analytically given by

$$x^i = x^i_0, \quad y^{(\alpha)i}(t), \quad y^{(\beta)i}(t) = 0, \quad \beta \neq \alpha, \quad p_i = 0, \quad t \in I.$$ 

A $v_{\alpha}$-vertical path $\gamma$, with respect to $D$ is defined by $D_{\gamma_{V_{\alpha}}} \gamma = 0$.

In this case, the equations (7.7.13) are as follows

$$(7.7.16) \quad \frac{dx^i}{dt} = 0, \quad \frac{dy^{(\beta)i}}{dt} = 0, \quad \frac{dp_i}{dt} = 0 \quad \text{and} \quad \frac{d}{dt} \frac{dy^{(\alpha)i}}{dt} + C^i_s (x_0, 0, ..., 0, p) \frac{dp_j}{dt} \frac{dp_m}{dt} = 0.$$  

Similarly, a $w_k$-vertical curve $\gamma$ at the point $x_0 \in M$ is defined by the condition $\gamma = \gamma_{W_k}$. Analytically it is expressed by

$$x^i = x^i_0, \quad y^{(\alpha)i}(t) = 0, \quad (\alpha = 1, ..., k-1), \quad p_i = p_i(t), \quad t \in I.$$ 

A $w_k$-path $\gamma$, with respect to $D$ has the property $D_{\gamma_{W_k}} \gamma = 0$.

The $w_k$-paths, with respect to the $N$-linear connection $D$ are characterized by

$$(7.7.17) \quad \frac{dx^i}{dt} = \frac{dy^{(1)i}}{dt} = \cdots = \frac{dy^{(k-1)i}}{dt} = 0, \quad \frac{d^2p_i}{dt^2} - C^i_j (x_0, 0, ..., 0, p) \frac{dp_j}{dt} \frac{dp_m}{dt} = 0.$$ 

In the case when $D$ is a Berwald $N$-linear connection the previous theory is a simple one.

### 7.8 Structure Equations of a $N$-Linear Connection

Let us consider a $N$-linear connection $D$ with the coefficients $D\Gamma(N)$ in the adapted basis $\left(\frac{\delta}{\delta x^1}, \frac{\delta}{\delta y^{(1)i}}, ..., \frac{\delta}{\delta y^{(k-1)i}}, \frac{\delta}{\delta p_i}\right)$.

It is not difficult to prove:

**Lemma 7.8.1** 1° Each of the following object fields

$$d(dx^i) - dx^m \wedge \omega^i_m; \quad d(\delta y^{(\alpha)i}) - \delta y^{(\alpha)m} \wedge \omega^i_m, \quad (\alpha = 1, ..., k-1); \quad d(\delta p_i) + \delta p_m \wedge \omega^i_m$$

is a $d$-vector field, except the last one which is a $d$-covector field.

2° The geometrical object field

$$d\omega^i_j - \omega^j_m \wedge \omega^i_m$$

is a $d$-tensor field of type $(1,1)$.
Using this lemma we obtain a fundamental result in the geometry of the manifold $T^*kM$ and implicitly in the geometry of higher order Hamilton spaces.

**Theorem 7.8.1** For any $N$-linear connection $D$ with the coefficients

$$D\Gamma(N) = \left( H^i_{jh}, C^i_{(1)jh}, \ldots, C^i_{(k-1)jh}, C^i_{jh} \right)$$

the following structure equations hold:

\[
\begin{align*}
    d(\delta x^i) - dx^m \wedge \omega^i_m &= -\Omega^i, \\
    d(\delta y^{(\alpha)i}) - dy^{(\alpha)m} \wedge \omega^i_m &= -\Omega^{(\alpha)}_i, \quad (\alpha = 1, \ldots, k-1), \\
    d(\delta p_i) + \delta p_m \wedge \omega^i_m &= -\Omega^i
\end{align*}
\]

and

\[
\begin{align*}
    d\omega^j_i - \omega^m_j \wedge \omega^i_m &= -\Omega^i_j,
\end{align*}
\]

where $\Omega^i_i, \Omega^{(1)}_i, \ldots, \Omega^{(k-1)}_i$ and $\Omega_i$ are the 2-forms of torsion,

\[
\begin{align*}
    (0) &: \Omega^i_i = dx^j \left\{ \frac{1}{2} T^i_{jm} dx^m + \sum_{\alpha=1}^{k-1} C^i_{(\alpha)jm} dy^{(\alpha)m} + C^i_{jm} \delta p_m \right\}, \\
    (1) &: \Omega^{(1)}_i = dx^j \left\{ P^i_{(0)j} + \sum_{\gamma=1}^{k-1} \delta y^{(\gamma)j} \wedge P^i_{(\gamma)j} + \right. \\
    &\quad \left. + \delta y^{(\alpha j)} \wedge \left( H^i_{jm} dx^m + \sum_{\gamma=1}^{k-1} C^i_{jm} dy^{(\gamma)m} + C^i_{jm} \delta p_m \right) \right\}, \quad (\alpha = 1, \ldots, k-1), \\
    (k-1) &: \Omega^{(k-1)}_i = dx^j \left\{ \frac{1}{2} R^i_{ijm} dx^m + \sum_{\gamma=1}^{k-1} B^i_{ijm} dy^{(\gamma)m} + B^i_{(0)jm} \delta p_m \right\} - \\
    &\quad - \delta p_j \left\{ H^j_{im} dx^m + \sum_{\gamma=1}^{k-1} C^j_{im} dy^{(\gamma)m} + C^j_{im} \delta p_m \right\}, \\
    (\alpha, k-1) &: P^i_{(\alpha)j}, \ldots, P^i_{(\alpha, k-1)j}
\end{align*}
\]

being given by (5.4.8) and where $\Omega^i_j$ are the 2-forms of curvature:

\[
\begin{align*}
    \Omega^i_j &= \frac{1}{2} R^i_{hm} dx^h \wedge dx^m + \sum_{\gamma=1}^{k-1} P^i_{(\gamma)j} h_m dx^h \wedge dy^{(\gamma)m} + \\
    &\quad + P^i_{jh} dx^h \wedge \delta p_m + \sum_{\alpha \leq \beta}^{k-1} \sum_{(\alpha\beta)=1}^{k-1} S^i_{\alpha\beta} h_m dy^{(\alpha)h} \wedge dy^{(\beta)m} + \\
    &\quad + \sum_{\gamma=1}^{k-1} S^i_{(\gamma)h} dx^h \wedge \delta y^{(\gamma)h} \wedge \delta p_m + \frac{1}{2} S^i_{jh} h_m \delta p_h \wedge \delta p_m.
\end{align*}
\]

Indeed, by means of the exterior differential $d\delta y^{(\alpha)i}$ from (7.4.7), (7.4.8) and $\omega^i_j$ from (7.7.6) we get the formulas (7.8.3) and (7.8.4).
These formulas have a very simple form in the case of Berwald connection, where $C^i_{\alpha h} = 0$, $C^{\alpha h}_{jh} = 0$.

The structure equations will be used in a theory of submanifold of the Hamilton spaces, studied in chapter 9.
Chapter 8

Hamilton Spaces of Order $k \geq 1$

The Hamilton spaces of order 1 and 2 were investigated in the chapter 5 and 12 of the book [115]. In the present chapter we study the natural extension of this notion to order $k \geq 1$.

A Hamilton space of order $k$ is a pair $H^{(k)n} = (M, H^{(k)})$ in which $M$ is a real $n$-dimensional manifold and $H : T^{*k}M \to \mathbb{R}$ is a regular Hamiltonian function on the manifold $T^{*k}M = T^{(k-1)}M \times_M T^*M$.

The geometry of the spaces $H^{(k)n}$ can be developed step by step following the same ideas as in the cases $k = 1$ or $k = 2$ and using the geometrical theory of the manifold $T^{*k}M$ described in the last three chapters. Of course, $T^{*k}M$ being the dual of $T^kM$, the geometry of the Hamilton spaces of order $k$, $H^{(k)n} = (M, H)$, appears as dual of the geometry of Lagrange spaces of order $k$, $L^{(k)n} = (M, L)$, via a Legendre mapping.

Therefore, in this chapter we study the notion of Hamilton space $H^{(k)n} = (M, H)$, the canonical presymplectic structure and canonical Poisson structure, Legendre mappings, the nonlinear connection and canonical metrical connection. We end with the Riemannian almost contact model of this space.

8.1 The Spaces $H^{(k)n}$

Let us consider the dual bundle $(T^{*k}M, \pi^{*k}, M)$. The local coordinates of a point $u = (x, y^{(1)}, ..., y^{(k-1)}, p)$, $u \in T^{*k}M$, will be denoted as usually by $(x^i, y^{(1)i}, ..., y^{(k-1)i})$; $(x^i)$ being the coordinates of the particle $x$, $y^{(1)i}$, ..., $y^{(k-1)i}$ are seen as the coordinates of the accelerations of order 1, ..., $k - 1$, respectively and $p_i$ are the momenta. The coordinate transformations on $T^{*k}M$ are given by (4.1.2), (4.1.3).

On the manifold $T^{*k}M$ there are the vertical distributions $V_{k-1} \subset V_{k-2} \subset \cdots \subset V_1 \subset V$ and a vertical distribution $W_k$ such that $V_u = V_{1,u} \oplus W_{k,u}$, $\forall u \in T^{*k}M$. 

147
Also, on the manifold $T^*kM$ there exist the Liouville vector fields $\Gamma_1, \ldots, \Gamma_{k-1}$ and the Hamilton vector field $C^*$, linearly independent, expressed by

\[
\Gamma_1 = y^{(1)i} \frac{\partial}{\partial y^{(k-1)i}}, \\
\Gamma_2 = y^{(1)i} \frac{\partial}{\partial y^{(k-2)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(k-1)i}}, \\
................................. \\
\Gamma_{k-1} = y^{(1)i} \frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(2)i}} + \cdots + (k-1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-1)i}}
\]

and

\[
C^* = p_i \frac{\partial}{\partial p_i}.
\]

Theorem 4.2.1 stipulates that these vector fields are globally defined on the total space of the dual bundle.

The function

\[
\varphi = p_i y^{(1)i}
\]

is a scalar function on $T^*kM$.

A Hamiltonian is a scalar function $H : (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^*kM \to H(x, y^{(1)}, \ldots, y^{(k-1)}, p) \in \mathbb{R}$. 'Scalar' means that $H$ does not depend on the changing of coordinates on $T^*kM$.

As we know, the Hamiltonian $H$ is differentiable if it is differentiable on the manifold $\tilde{T}^*kM = T^*kM \setminus \{0\}$ (where 0 is the null section of the projection $\pi^*k$) and $H$ is continuous on the null section. Evidently,

\[
\tilde{T}^*kM = \left\{ (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^*kM | y^{(1)}, \ldots, y^{(k-1)}, p \text{ are not all null} \right\}.
\]

The null section $0 : M \to T^*kM$, having the property $\pi^*k \circ 0 = 1_M$ can be identified with the manifold $M$.

**Definition 8.1.1** A regular Hamiltonian $H : T^*kM \to \mathbb{R}$ is a differentiable Hamiltonian whose Hessian with respect to the momenta $p_i$, with the entries:

\[
g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = \frac{1}{2} \frac{\partial^2 H}{\partial p_i \partial p_j}
\]

is nondegenerate on the manifold $\tilde{T}^*kM$.

Of course, $g^{ij}$ from (8.1.3) is a $d$-tensor field, contravariant of order 2, symmetric.
The condition of regularity is expressed by

\[(8.1.3a) \quad \text{rank} \|g^{ij}\| = n, \quad \text{on} \widetilde{T^kM}.\]

If the base manifold \(M\) is paracompact, then the manifold \(T^kM\) is paracompact, too and on \(T^kM\) there exist regular Hamiltonians.

The \(d\)-tensor field \(g^{ij}\) being nonsingular on \(\widetilde{T^kM}\) there exists a \(d\)-tensor field \(g_{ij}\) covariant of order 2, symmetric, uniquely determined, at every point \(u \in T^kM\), by

\[(8.1.4) \quad g_{ij}g^{jk} = \delta^k_i.\]

**Definition 8.1.2** An Hamilton space of order \(k\) is a pair \(H^{(k)} = (M, H(x, y^{(1)}, ..., y^{(k-1)}, p))\), where \(M\) is a real \(n\)-dimensional manifold and \(H\) is a differentiable regular Hamiltonian having the property that the \(d\)-tensor field \(g^{ij}\) has a constant signature on \(\widetilde{T^kM}\).

As usually, \(H\) is called the fundamental function and \(g^{ij}\) the fundamental tensor of the space \(H^{(k)}\).

In the case when the fundamental tensor \(g^{ij}\) is positively defined, then the condition of regularity \((8.1.3')\) is verified.

**Theorem 8.1.1** Assuming that the base manifold \(M\) is paracompact, then there exists on \(\widetilde{T^kM}\) a regular Hamiltonian \(H\) such that the pair \((M, H)\) is a Hamilton space of order \(k\).

**Proof:** Let \(F^{(k-1)n} = (M, F(x, y^{(1)}, ..., y^{(k-1)}))\) be a Finsler space of order \(k-1\) on the manifold \(T^{k-1}M\), where \(T^{k-1}M = \pi_k^{-1}(T^kM)\), having \(\gamma_{ij}(x, y^{(1)}, ..., y^{(k-1)})\) as fundamental tensor. The manifold \(M\) being paracompact, the space \(F^{(k-1)n}\) exists.

Then, the function

\[
H(x, y^{(1)}, ..., y^{(k-1)}, p) = \alpha \gamma_{ij}(x, y^{(1)}, ..., y^{(k-1)})p_ip_j, \quad (\alpha \in \mathbb{R}, \alpha > 0),
\]

is well defined in every point \((x, y^{(1)}, ..., y^{(k-1)}, p) \in \widetilde{T^kM}\) and it is a fundamental function for a Hamilton space of order \(k\). Its fundamental tensor field is \(\alpha \gamma_{ij}\). Q.E.D.

One of the important \(d\)-tensor field derived from the fundamental function \(H\) of the space \(H^{(k)n}\) is:

\[(8.1.5) \quad C^{ijh} = -\frac{1}{2} \frac{\partial}{\partial i} g^{ih} = -\frac{1}{4} \frac{\partial}{\partial i} \frac{\partial}{\partial j} H, \quad \left(\frac{\partial}{\partial i} = \frac{\partial}{\partial p_i}\right).\]

**Proposition 8.1.1** We have:

1. \(\mathcal{U} C^{ijh}\) is a totally symmetric \(d\)-tensor field;
2. \(\mathcal{L} C^{ijh}\) vanishes if and only if the fundamental tensor \(g^{ij}\) does not depend on the momenta \(p_i\).
Other geometrical object fields which are entirely determined by the fundamental function $H$ are the coefficients $C^j_i{}^h$ of the $w_k$-covariant derivation, given by

\begin{equation}
C^j_i{}^h = -\frac{1}{2} g_{ls} \left( \partial^l g^{sh} + \partial^h g^{js} - \partial^s g^{jh} \right).
\end{equation}

**Proposition 8.1.2**

1. $C^j_i{}^h$ are the components of a $d$-tensor fields of type $(2,1)$.
2. They depend on the fundamental function $H$ only.
3. They are symmetric in the indices $j, h$.
4. The formula $C^j_i{}^h = g_{is} C^s{}^j{}^h$ holds.
5. The $w_k$-covariant derivative of the fundamental tensor $g^{ij}$, vanishes:

\begin{equation}
(g^{ij})^h = 0.
\end{equation}

The proof is not difficult.

### 8.2 The $k$-Tangent Structure $J$ and the Adjoint $k$-Tangent Structure $J^*$

For the Hamilton space of order $k$, $H^{(k)n} = (M,H)$, the structures $J$ and $J^*$ defined on the manifold $T^*kM$ in the section 3, Ch.4 have some special properties.

The $k$-tangent structure is the mapping:

\begin{equation}
J \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial y^{(1)i}}, \ldots, J \left( \frac{\partial}{\partial y^{(k-1)i}} \right) = \frac{\partial}{\partial y^{(k-1)i}}; \quad J \left( \frac{\partial}{\partial y^{(k-1)i}} \right) = 0, J \left( \frac{\partial}{\partial i} \right) = 0.
\end{equation}

Locally, it is expressed by (4.3.2):

\begin{equation}
J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \cdots + \frac{\partial}{\partial y^{(k-1)i}} \otimes dy^{(k-2)i}.
\end{equation}

The main properties of $J$ are explicitly given in Theorem 4.3.1.

Let $X$ be a vector field on $T^*kM$, locally expressed by

\begin{equation}
X = X^{(0)i} \frac{\partial}{\partial x^i} + X^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + X^{(k-1)i} \frac{\partial}{\partial y^{(k-1)i}} + X_i \frac{\partial}{\partial i}.
\end{equation}
Hamilton Spaces of Order $k \geq 1$

Here, $\partial_i = \frac{\partial}{\partial p_i}$.

Consider the following vector fields $0, 1, \ldots, k-1$

\begin{equation}
X = JX, \quad X = J^2X, \ldots, \quad X = J^{k-1}X.
\end{equation}

Taking into account (8.2.3), these vector fields have the form:

\begin{equation}
X = X \left( \frac{\partial}{\partial y^{(1)i}} \right) + \cdots + \frac{(k-2)i}{X} \frac{\partial}{\partial y^{(k-1)i}}.
\end{equation}

Now, the adjoint $J^*$ of $J$ is defined by

\begin{equation}
J^* (dx^i) = 0, \quad J^* (dy^{(1)i}) = dx^i, \ldots, \quad J^* (dy^{(k-1)i}) = dy^{(k-2)i}, \quad J^* (dp_i) = 0.
\end{equation}

$J^*$ is the following $d$-tensor field of type $(1, 1)$:

\begin{equation}
J^* = dx^i \otimes \frac{\partial}{\partial y^{(1)i}} + dy^{(1)i} \otimes \frac{\partial}{\partial y^{(2)i}} + \cdots + dy^{(k-2)i} \otimes \frac{\partial}{\partial y^{(k-1)i}}.
\end{equation}

$J^*$ is an integrable structure and $\text{rank } J^* = (k-1)n$.

If $\omega$ is an 1-form field on the manifold $T^*kM$ and

\begin{equation}
\omega = \omega_i dx^i + \cdots + \omega_i dy^{(k-1)i} + \omega_i dp_i,
\end{equation}

then by means of $J^*$ we obtain a number of $k-1$ 1-forms on $T^*kM$:

\begin{equation}
\omega = J^* \omega, \quad \ldots, \quad \omega = J^{(k-1)} \omega.
\end{equation}

The vertical differential operators $d_0, \ldots, d_{k-2}$ are introduced in §4.3 by

\begin{equation}
d_0 = J^{(k-1)}d, \quad \ldots, \quad d_{k-2} = J^*d,
\end{equation}

where $d$ is the operator of differentiation on the manifold $T^*kM$. We know from §4.3 that these operators are the antiderivations of degree 1 in the exterior algebra $\Lambda(T^*kM)$.

The following formula hold:

\begin{equation}
d \circ d = 0, \quad d_\alpha \circ d_\alpha = 0, \quad (\alpha = 0, 1, \ldots, k-2).
\end{equation}

We get:
Proposition 8.2.1 For any Hamilton space of order \( k \), \( H^{(k)} = (M, H) \) the 1-forms (8.2.8) of the form \( dH \) are given by

\[
d_0 H = (0)_{p_i} dx^i,
\]

(8.2.11)

\[
d_1 H = (1)_{p_i} dx^i + (0)_{\frac{\partial H}{\partial y^{(k-1)i}}} i,
\]

\[
d_{k-2} H = (k-2)_{p_i} dx^i + (k-3)_{\frac{\partial H}{\partial y^{(k-1)i}}} i + \cdots + (0)_{p_i} dy^{(k-2)i}.
\]

where

\[
(8.2.11a) \quad (0)_{p_i} = \frac{\partial H}{\partial y^{(k-1)i}}, \quad (1)_{p_i} = \frac{\partial H}{\partial y^{(k-2)i}}, \quad \ldots, \quad (k-2)_{p_i} = \frac{\partial H}{\partial y^{(1)i}}.
\]

Proposition 8.2.2 The following 2-forms depend only on the fundamental function \( H \) of the Hamilton space \( H^{(k)} \):

\[
\begin{align*}
\dd_0 H &= d (0)_{p_i} \wedge dx^i, \\
\dd_1 H &= d (1)_{p_i} \wedge dx^i + d (0)_{\frac{\partial H}{\partial y^{(k-1)i}}} i, \\
\dd_{k-2} H &= d (k-2)_{p_i} \wedge dx^i + \cdots + d (0)_{p_i} \wedge dy^{(k-2)i}.
\end{align*}
\]

We have, also:

\[
(8.2.13) \quad d_\alpha \circ d_\alpha H = 0, \quad \forall \alpha = 0, 1, \ldots, k-2.
\]

Clearly, \( \dd_\alpha H = 0 \), \( \alpha = 0, \ldots, k-2 \) are closed 2-forms.

As we know, the operator

\[
(8.2.14) \quad d_{k-1} = \frac{\partial}{\partial x^i} dx^i + \frac{\partial}{\partial y^{(1)i}} dy^{(1)i} + \cdots + \frac{\partial}{\partial y^{(k-1)i}} dy^{(k-1)i}
\]

is not a vertical differentiation.

Proposition 8.2.3 We have:

1. \( d_{k-1} H \) is not an 1-form;
2. Under a change of local coordinate on \( T^*kM \) we have

\[
d_{k-1} H = d_{k-1} \tilde{H} + \frac{\partial}{\partial x^i} \tilde{H} \frac{\partial \tilde{p}_j}{\partial x^i} dx^i;
\]

3. \( J^* \circ d_{k-1} = d_{k-2} \).
8.3 The Canonical Poisson Structure of the Hamilton Space $H^{(k)n}$

Consider a Hamilton space $H^{(k)n} = (M, H)$. As we know from the section 4 ch.4 on the manifold $T^*kM$ there exists a canonical Poisson structure $\{,\} _{k−1}$. Besides this natural Poisson structure on the submanifold $Σ_0$ of $T^*kM$ there exist a remarkable Poisson structure of the space $H^{(k)n} = (M, H)$.

Let $(T^*kM, π^*, T*M)$ be the bundle introduced in §1, ch.4. The projection $π^*$ is given by $π^*(x, y^{(1)}, ..., y^{(k−1)}, p) = (x, p)$. The canonical section $σ_0 : (x, p) \in T*M \rightarrow (x, 0, ..., 0, p)$ of the manifold $T^*kM$. The canonical presymplectic structure $θ = dp_i ∧ dx^i$ has its restriction $θ_0$ to $Σ_0$, given by $θ_0 = dp_i ∧ dx^i$ in every point $(x, p) \in Σ_0$. The equations of the submanifold $Σ_0$ being $y^{(α)} = 0$, ($α = 1, ..., k−1$), then $(x^i, p_i)$ are the coordinate of the points $(x, p) \in Σ_0$.

Theorem 8.3.1 The pair $(Σ_0, θ_0)$ is a symplectic manifold.

Proof: Indeed, $θ_0 = dp_i ∧ dx^i$ is a closed 2-form and rank $||θ_0|| = 2n = \dim Σ_0$. Q.E.D.

In a point $u = (x, p) \in Σ_0$ the tangent space $T_uΣ_0$ has the natural basis

\[
\left(\frac{∂}{∂x^i}, \frac{∂}{∂p_i}\right)_u, (i = 1, ..., n)\]

and natural cobasis $(dx^i, dp_i)_u$.

Let us consider $\mathcal{F}(Σ_0)$-module $\mathcal{X}(Σ_0)$ of vector fields and $\mathcal{F}(Σ_0)$-module $\mathcal{X}^*(Σ_0)$ of covector fields on the submanifold $Σ_0$.

The following $\mathcal{F}(Σ_0)$-linear mapping

$S_{θ_0} : \mathcal{X}(Σ_0) → \mathcal{X}^*(Σ_0)$ defined by

\[
S_{θ_0}(X) = i_Xθ_0, ∀X ∈ \mathcal{X}(Σ_0)
\]

gives us

\[
S_{θ_0}\left(\frac{∂}{∂x^i}\right) = −dp_i, \quad S_{θ_0}\left(\frac{∂}{∂p_i}\right) = dx^i.
\]

These equalities have as a consequence:

Proposition 8.3.1 The mapping $S_{θ_0}$ is an isomorphism.

The Hamilton space $H^{(k)n} = (M, H)$, allows to consider the restriction $H_0$ of the fundamental function $H$ to the submanifold $Σ_0$. $H_0(x, p) = H(x, 0, ..., 0, p)$.

Therefore the pair $(M, H_0(x, p))$ is a classical Hamilton space (cf. [115]) with fundamental tensor field

\[
g^{ij}(x, 0, ..., 0, p) = \frac{1}{2} \frac{∂}{∂x^i} \frac{∂}{∂x^j} H_0.
\]

By means of the last proposition it follows:
Proposition 8.3.2 \(1^0\) There exists a unique vector field \(X_{H_0} \in \mathcal{X}(\Sigma_0)\) such that
\[
S_{\theta_0}(X_{H_0}) = i_{X_{H_0}}\theta_0 = -dH_0.
\]
\(2^0\) \(X_{H_0}\) is given by
\[
(8.3.5) \quad X_{H_0} = \frac{\partial H_0}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial H_0}{\partial x^i} \frac{\partial}{\partial p_i}.
\]

Theorem 8.3.2 The integral curves of the vector field \(X_{H_0}\) are given by the \(\Sigma_0\)-canonical equations
\[
(8.3.6) \quad \frac{dx^i}{dt} = \frac{\partial H_0}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H_0}{\partial x^i}, \quad y^{(\alpha)i} = 0, \quad (\alpha = 1, \ldots, k).
\]

For two functions \(f, g \in \mathcal{F}(\Sigma_0)\), let \(X_f\) and \(X_g\) be the corresponding Hamilton vector fields given by
\[
i_{X_f}\theta_0 = -df, \quad i_{X_g}\theta_0 = -dg.
\]

Theorem 8.3.3 The following formula holds
\[
(8.3.7) \quad \{f, g\}_0 = \theta_0(X_f, X_g), \quad \forall f, g \in \mathcal{F}(\Sigma_0)
\]

Proof: We have
\[
\theta(X_f, X_g) = (i_{X_f}\theta_0)(X_g) = S_{\theta_0}(X_f)(X_g) = -df(X_g) = -X_g f =
\]
\[
= \left( \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} \right) = \{f, g\}_0. \quad \text{Q.E.D.}
\]

Remark \(1^0\) The previous theory can be extended to the other Poisson structures \(\{\cdot, \cdot\}_\alpha, \quad (\alpha = 1, \ldots, k-1)\) (cf. [4]).

\(2^0\) The triple \((T^*kM, H(x, y^{(1)}, \ldots, y^{(k-1)}, p), \theta)\) is an Hamiltonian system in which \(\theta\) is a presymplectic structure. Therefore we can apply Gotay’s method (cf. M. de Leon and Gotay, [115]) taking into account the considerations from the previous section, §8.2.

The equations (8.3.6) are particular. For a Hamilton space \(H(k)^n = (M, H)\), the integral of action, (see Ch.5):
\[
I(c) = \int_0^1 [p_i \frac{dx^i}{dt} - \frac{1}{2} H(x, \frac{dx}{dt}, \ldots, \frac{1}{(k-1)!} \frac{d^{k-1}}{dt^{k-1}} y^{(k-1)}, p)] dt
\]
leads, via the variational problem, to the fundamental equations of the space \(H(k)^n\), i.e. the Hamilton-Jacobi equations
\[
\frac{dx^i}{dt} = \frac{1}{2} \frac{\partial H}{\partial p_i},
\]
\[
\frac{dp_i}{dt} = -\frac{1}{2} \frac{\partial H}{\partial x^i} - \frac{d}{dt} \frac{\partial H}{\partial y^{(1)i}} + \cdots + (-1)^{k-1} \frac{d^{k-1}}{dt^{k-1}} \frac{\partial H}{\partial y^{(k-1)i}}.
\]
The energies of order $k-1$, $\mathcal{E}^{k-1}(H)$ of the considered space has the expression

$$\mathcal{E}^{k-1}(H) = I^{k-1}(H) - \frac{1}{2!} \frac{d}{dt} I^{k-2}(H) + \cdots + (-1)^{k-2} \frac{1}{(k-1)!} \frac{d^{k-2}}{dt^{k-2}} I^1(H) - H$$

and we have:

**Theorem 8.3.4** For a Hamilton space $H^{(k)}n = (M, H)$ the energy of order $k-1$, $\mathcal{E}^{k-1}(H)$ is constant along every solution curve of the Hamilton-Jacobi equations.

### 8.4 Legendre Mapping Determined by a Lagrange Space $L^{(k)}n = (M, L)$

Let $L^{(k)}n = (M, L(x, y^{(1)}, \ldots, y^{(k)}))$ be a Lagrange space of order $k$. It determines a local diffeomorphism $\varphi : T^kM \to \tilde{T}^kM$ which preserves the fibres. The mapping $\varphi$ transforms the canonical $k$-semispray $S$, (1.2.5), of $L^{(k)}n$ in the dual $k$-semispray $S_\xi$, where $\xi = \varphi^{-1}$ and determines a nonlinear connection $N^*$ on $T^*kM$. But $\varphi$ does not transform the regular Lagrangian $L$ in a regular Hamiltonian, in the case $k > 1$. We need some supplementary geometrical object fields for getting a regular Hamiltonian $H$ from the fundamental function $L$ of the space $L^{(k)}n$. In this section we investigate the above mentioned problems.

A point $(x, y^{(1)}, \ldots, y^{(k)})$ of the manifold $T^kM$ will be denoted by $(y^{(0)}, y^{(1)}, \ldots, y^{(k)})$ and its coordinates by $(y^{(0)i}, \ldots, y^{(k)i})$.

The fundamental tensor of the space $L^{(k)}n = (M, L(y^{(0)}, \ldots, y^{(k)}))$ will be given by

$$a_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)i} \partial y^{(k)j}}.$$  

### Proposition 8.4.1 The mapping $\varphi : u = (y^{(0)}, y^{(1)}, \ldots, y^{(k)}) \in T^kM \to u^* = (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^*kM$ given by

$$\begin{cases}
  x^i = y^{(0)i}, & y^{(1)i} = y^{(1)i}, & \ldots, & y^{(k-1)i} = y^{(k-1)i}, \\
  p_i = \frac{1}{2} \frac{\partial L}{\partial y^{(k)i}}.
\end{cases}$$

is a local diffeomorphism, which preserves the fibres.

**Proof:** The mapping $\varphi$ is differentiable on the manifold $T^kM$ and its Jacobian has the determinant equal to $|a_{ij}| \neq 0$. Of course, $\pi^k(y^{(0)}, \ldots, y^{(k)}) = \pi^k \circ \varphi(y^{(0)}, \ldots, y^{(k)}) = y^{(0)}$.

The diffeomorphism $\varphi$ is called the Legendre mapping (or Legendre transformation).
We denote
\[ p_i = \frac{1}{2} \frac{\partial L}{\partial y^{(k)i}} = \varphi_i(y^{(0)}, y^{(1)}, \ldots, y^{(k)}). \]

Clearly, \( \varphi_i \) is a \( d \)-covector field on \( \tilde{T}^kM \).

The local inverse diffeomorphism \( \xi = \varphi^{-1} : \tilde{T}^kM \to \tilde{T}^kM \) is expressed by
\[ y^{(0)i} = x^i, \ y^{(1)i} = y^{(1)i}, \ldots, \ y^{(k-1)i} = y^{(k-1)i}, \]
\[ y^{(k)i} = \xi^i(x, y^{(1)}, \ldots, y^{(k-1)}, p). \]

With respect to a change of local coordinates on the manifold \( T^kM \), \( \xi \) is transformed exactly as the variables \( y^{(k)i} \) from the formulas (1.1.2).

The mappings \( \varphi \) and \( \xi \) satisfy the conditions
\[ \xi \circ \varphi = 1_{\tilde{U}}, \varphi \circ \xi = 1_{\tilde{U}}, \ \tilde{U} = (\pi^k)^{-1}(U), \ U = (\pi^k)^{-1}(U), \ U \subset M. \]

Therefore we have the following identities
\[ a_{ij}(y^{(0)}, \ldots, y^{(k)}) = \frac{\partial \varphi_i}{\partial y^{(k)j}}; \ a_{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, \xi(x, y^{(1)}, \ldots, y^{(k-1)}, p)) = \frac{\partial \xi^i}{\partial p_j} \]
and
\[ \begin{aligned}
\frac{\partial \varphi_i}{\partial x^j} &= -a_{ij} \frac{\partial \xi^s}{\partial y^{(k)j}}; \ \frac{\partial \xi^s}{\partial y^{(k)j}} = -a_{is} \frac{\partial \varphi_i}{\partial y^{(k)j}}, \ (s = 1, \ldots, k-1); \ \frac{\partial \varphi_i}{\partial y^{(k)j}} = a_{ij}; \\
\frac{\partial \xi^i}{\partial x^j} &= -a_{is} \frac{\partial \varphi_s}{\partial y^{(k)j}}; \ \frac{\partial \varphi_s}{\partial y^{(k)j}} = -a_{is} \frac{\partial \xi^i}{\partial p_j}, \ (s = 1, \ldots, k-1); \ \frac{\partial \xi^i}{\partial p_j} = a_{ij}.
\end{aligned} \]

The differential \( d\varphi \) of the mapping \( \varphi, \varphi : T_u(T^kM) \to T_{\varphi(u)}(T^kM) \) in the natural basis is expressed by
\[ \varphi^*(\frac{\partial}{\partial y^{(0)i}}|_u) = \frac{\partial}{\partial x^i}|_{u^*} + \frac{\partial \varphi_m}{\partial x^i} \frac{\partial}{\partial p_m}|_{u^*}, \ u^* = \varphi(u) \]
(8.4.6)
\[ \varphi^*(\frac{\partial}{\partial y^{(\alpha)j}}|_u) = \frac{\partial}{\partial y^{(\alpha)j}}|_{u^*} + \frac{\partial \varphi_m}{\partial y^{(\alpha)j}} \frac{\partial}{\partial p_m}|_{u^*}, \ (\alpha = 1, \ldots, k-1), \]
\[ \varphi^*(\frac{\partial}{\partial y^{(k)i}}|_u) = a_{im} \frac{\partial}{\partial p_m}|_{u^*}. \]

**Theorem 8.4.1** The Legendre mapping \( \varphi : T^kM \to T^kM \) transforms the canonical \( k \)-semispray \( S \) of the Lagrange space \( L^{(k)n} \)
\[ S = y^{(1)i} \frac{\partial}{\partial y^{(0)i}} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i(y^{(0)}, \ldots, y^{(k)}) \frac{\partial}{\partial y^{(k+i)}} \]
(8.4.7)
in the dual $k$-semispray $S^*_\xi$ on $T^kM$:

\[
S^*_\xi = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + (k-1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-2)i}} +
\]

\[+ k\xi^i(x, y^{(1)}, \ldots, y^{(k-1)}, p) \frac{\partial}{\partial y^{(k-1)i}} + \eta_i(x, y^{(1)}, \ldots, y^{(k-1)}, p) \frac{\partial}{\partial p_i},
\]

with the coefficients $\xi^i$ from (8.4.3) and

\[
(8.4.9)
\]

\[
\eta_i = -\alpha_is \left( \frac{\partial \xi^s}{\partial x^r} y^{(1)r} + \cdots + (k-1) \frac{\partial \xi^s}{\partial y^{(k-1)r}} \xi^r + \right.
\]

\[
\left. + (k+1)G^s(x, y^{(1)}, \ldots, y^{(k-1)}, \xi(x, y^{(1)}, \ldots, y^{(k-1)}, p)) \right)
\]

**Proof:** If $X \in \mathcal{X}(\widetilde{T^kM})$ have the local expression

\[
X = X^{(0)i} \frac{\partial}{\partial y^{(0)i}} + X^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + X^{(k)i} \frac{\partial}{\partial y^{(k)i}} \text{ in every point } u \in \widetilde{T^kM},
\]

then

\[
X^* = (\varphi_* X)(u^*) = X^{(0)i}(u^*) \varphi_* \frac{\partial}{\partial x^i} + \cdots + X^{(k)i}(u^*) \varphi_* \frac{\partial}{\partial y^{(k)i}} =
\]

\[
= X^{(0)i}(u^*) \frac{\partial}{\partial x^i}|_{u^*} + \cdots + X^{(k-1)i}(u^*) \frac{\partial}{\partial y^{(k-1)i}} |_{u^*} +
\]

\[
+ \left( X^{(0)m}(u^*) \frac{\partial \varphi}{\partial x^m} |_{u^*} + \cdots + X^{(k-1)m}(u^*) \frac{\partial \varphi}{\partial y^{(k-1)m}} \right) \frac{\partial}{\partial p_i} |_{u^*} +
\]

\[
+ X^{(k)m}(u^*) \alpha_m(u^*) \frac{\partial}{\partial p_i} |_{u^*} \text{ in the points } u^* = \varphi(u).
\]

Consequently, $S^*_\xi = \varphi_* S$ holds.

Now, applying Theorem 5.3.1 we obtain without difficulties:

**Theorem 8.4.2** The Legendre mapping $\varphi : T^kM \rightarrow T^kM$ determined by a Lagrange space of order $k$, $L^{(k)n}$, transforms the canonical $k$-semispray $S$ in the dual $k$-semispray $S^*_\xi$ with the coefficients $\xi^i$, $\eta_i$. $S^*_\xi$ determines locally a nonlinear connection $N^*_\xi$ on $T^kM$ with dual coefficients:

\[
(8.4.10)
\]

\[
M^{* \, i}_{\,(1) \, j} = - \frac{\partial \xi^i}{\partial y^{(1)j}}, \ldots, M^{* \, i}_{\,(k-1) \, j} = - \frac{\partial \xi^i}{\partial y^{(k-1)j}}
\]

and

\[
(8.4.10a)
\]

\[
N^{*}_{ij} = \frac{\delta \eta_i}{\partial y^{(1)j}},
\]

where the operators $\frac{\delta}{\partial y^{(1)j}}$ are obtained by means of the coefficients, (8.4.10).

We may ask whether by means of the Legendre mapping $\varphi$ we can transfer the fundamental function $L$ of the Lagrange space $L^{(k)n}$ to a fundamental function $H$ of a Hamilton space of order $k$, $H^{(k)}$, like in the classical case of $k = 1$. 
THE GEOMETRY OF HIGHER-ORDER HAMILTON SPACES

Remark that the energy of order $k$, $\mathcal{E}_k^c(L)$ of the space $L^{(k)n}$ is not a regular Lagrangian on $T^kM$, the Legendre transformation $\varphi$ cannot apply $\mathcal{E}_k^c(L)$ in a regular Hamiltonian on $T^s^kM$. Therefore it is necessary to look for another way to solve this problem.

Let us consider a fixed nonlinear connection $\overset{\circ}{\nabla}$ on the submanifold $T^{k-1}M$ in $T^kM$. The dual coefficients of $\overset{\circ}{\nabla}$ are denoted by $M^{(1)}_{(k-1)}i(x,y^{(1)},...,y^{(k-1)})$, ...

\[ k\overset{\circ}{\nabla} = \partial_y^i(x,y^{(1)},...,y^{(k-1)}) \] is not difficult (cf. (1.4.7)) to prove that

\[ (8.4.11) \quad k\overset{\circ}{\nabla} = k\partial_y^i + (k - 1) \frac{\partial}{\partial y^{(k-1)m}} + \cdots + M^{(1)}_{(k-1)}\partial_{y^{(1)m}} \]

is a $d$-vector field at every point $u = (x,y^{(1)},...,y^{(k)})$.

The Legendre mapping

\[ \varphi : u \in T^kM \rightarrow u^* \in T^s^kM, \quad u^* = (x,y^{(1)},...,y^{(k-1)},p) \]

transform $z^{(k)i}$ in a $d$-vector field $\overset{\varphi}{\overset{\circ}{\nabla}}$ at $u^*$, given by

\[ (8.4.12) \quad k\overset{\varphi}{\overset{\circ}{\nabla}} = k\partial_y^i + (k - 1) \frac{\partial}{\partial y^{(k-1)m}} + \cdots + M^{(1)}_{(k-1)}\partial_{y^{(1)m}} \]

Let us consider the following Hamiltonian

\[ (8.4.13) \quad H(x,y^{(1)},...,y^{(k-1)},p) = 2p_i\overset{\varphi}{\overset{\circ}{\nabla}} - L(x,y^{(1)},...,y^{(k-1)},\xi(x,y^{(1)},...,y^{(k-1)},p)) \]

Clearly, $H$ is a differentiable Hamiltonian function on $T^s^kM$.

**Theorem 8.4.3** The Hamiltonian function $H$, (8.4.13), is the fundamental function of a Hamilton space of order $k$, $H^{(k)n}$ and its fundamental tensor field is

\[ (8.4.14) \quad g^{ij}(x,y^{(1)},...,y^{(k-1)},p) = a^{ij}(x,y^{(1)},...,y^{(k-1)},\xi(x,y^{(1)},...,y^{(k-1)},p)), \]

$a_{ij}$ being the fundamental tensor field of the Lagrange space of order $k$, $L^{(k)n} = (M,L)$.

**Proof:** From (8.4.13) we have

\[
\begin{align*}
\frac{1}{2} \frac{\partial}{\partial z} H &= z^{\overset{\circ}{\nabla}j} + p_m \frac{\partial}{\partial z} z^{\overset{\circ}{\nabla}m} - \frac{1}{2} \frac{\partial L}{\partial y^{(k)m}} \frac{\partial}{\partial z} \xi^{(k)m} = \\
&= z^{\overset{\circ}{\nabla}j} + p_m \frac{\partial}{\partial z} z^{\overset{\circ}{\nabla}m} - p_m \frac{\partial}{\partial z} z^{\overset{\circ}{\nabla}m} = z^{\overset{\circ}{\nabla}j}.
\end{align*}
\]

Consequently,

\[ g^{ij}(x,y^{(1)},...,y^{(k-1)},p) = \frac{1}{2} \frac{\partial}{\partial z} H = \frac{1}{2} \frac{\partial}{\partial z} H = \frac{1}{2} \frac{\partial}{\partial z} H = \]
Hamilton Spaces of Order \( k \geq 1 \)

\[
\frac{1}{2} \partial^i \xi^{(k)j} = a^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, \xi).
\]

It follows that \( H \) is a regular Hamiltonian and its fundamental tensor \( g^{ij} \) has a constant signature on \( T^k M \). Q.E.D.

The Hamilton space of order \( k \), \( H^{(k)n} = (M, H) \) is called the dual of the Lagrange space of order \( k \), \( L^{(k)n} = (M, L) \).

Remarks. All geometrical object fields of the space \( H^{(k)n} \), which are derived from the fundamental tensor field \( g^{ij} \) do not depend on the apriori fixed nonlinear connection \( \circ N \). Consequently, the geometry of the Hamilton space \( H^{(k)n} = (M, H) \) can be constructed by means of the dual \( k \)-semispray \( S^*_\xi \) and the fundamental tensor field \( g^{ij} \) from (8.4.14).

8.5 Legendre Mapping Determined by a Hamilton Space of Order \( k \)

Let us consider a converse for the previous problem: being given a Hamilton space of order \( k \), \( H^{(k)n} = (M, H(x, y^{(1)}, \ldots, y^{(k-1)}, p)) \) let us determine its dual like a Lagrange space of order \( k \), \( L^{(k)n} = (M, L(x, y^{(1)}, \ldots, y^{(k)})) \).

In this case, we start from the space \( H^{(k)n} \) and try to determine the local diffeomorphism in the form of \( \varphi^{-1} \) from (8.4.3). But \( y^{(k)i} \) is not a vector field on \( T^k M \) and we can not define it only by the \( d \)-vector field \( \partial^i H \). As in the previous section we assume that a nonlinear connection \( \circ N \) on the submanifold \( T^{k-1} M \) in \( T^k M \) is apriori fixed. The dual coefficients of \( \circ N \) being \( \circ M_{j}^{i}, \ldots, \circ M_{k-1}^{i} \) they are functions of the points \( (x, y^{(1)}, \ldots, y^{(k-1)}) \) from \( T^{k-1} M \).

Then the \( d \)-vector field \( z^{(k)i} \), given by (8.4.11) on \( T^k M \) can be considered.

The mapping

\[
\xi: u^* = (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^k M \rightarrow u = (y^{(0)}, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}) \in T^k M, \text{ defined as follows:}
\]

\[
y^{(0)i} = x^i, y^{(1)i} = y^{(1)i}, \ldots, y^{(k-1)i} = y^{(k-1)i},
\]

\[
y^{(k)i} = \xi^i (x, y^{(1)}, \ldots, y^{(k-1)}, p),
\]

where \( \xi^i \) is expressed from the formula:

\[
k \xi \xi_{j}^i (x, y^{(1)}, \ldots, y^{(k-1)}, p) + (k - 1) \circ M_{m}^{i} (x, y^{(1)}, \ldots, y^{(k-1)}) y^{(k-1)m} + \ldots + \circ M_{m}^{i} (x, y^{(1)}, \ldots, y^{(k-1)}) y^{(k-1)m} = \frac{k}{2} \partial^i H.
\]
Theorem 8.5.1  The mapping $^*\xi$ is a local diffeomorphism, which preserves the fibres of $T^*kM$ and $T^kM$.

Proof: The determinant of the Jacobian of $^*\xi$ is

$$\det |g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p)| \neq 0, \text{ since } \partial_j^*\xi = g^{ij} \text{ and } \pi^*k = \pi^k \circ ^*\xi. \text{ q.e.d.}$$

The formulae (8.5.1) and (8.5.2) imply:

$$z^{(k)}i(x, y^{(1)}, \ldots, y^{(k-1)}, ^*\xi) = \frac{1}{2} \partial_i^* H(x, y^{(1)}, \ldots, y^{(k-1)}, p),$$

(8.5.3)

$$g^{ij}$$ being the fundamental tensor field of space $H^{(k)n}$.

The inverse mapping $^*\varphi$: $u = (y^{(0)}, y^{(1)}, \ldots, y^{(k)}) \in T^kM \rightarrow u^* = (x, y^{(1)}, \ldots, y^{(k)}, p) \in T^*kM$ of the Legendre transformation $^*\xi$ can be written as follows:

$$x^i = y^{(0)i}, y^{(1)i} = y^{(1)i}, \ldots, y^{(k-1)i} = y^{(k-1)i},$$

(8.5.4)

$$p_i = ^*\varphi^i(y^{(0)}, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}).$$

¿From here we deduce

$$\frac{\partial \varphi^i}{\partial y^{(k)j}} = g_{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, \varphi(u)),$$

(8.5.6)

where $g_{ij}$ is the covariant tensor of the fundamental tensor $g^{ij}$ of $H^{(k)n}$.

The Legendre transformation $^*\xi$ maps the regular Hamiltonian $H$ into the differentiable Lagrangian $L = ^*\xi(H)$:

$$L(x, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}) = 2p_i z^{(k)i} - H(x, y^{(1)}, \ldots, y^{(k-1)}, p),$$

(8.5.7)

$$p_i = ^*\varphi^i(y^{(0)}, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}).$$

Theorem 8.5.2  The Lagrangian $L$ from (8.5.7) is a regular one. Its fundamental tensor field is $g_{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, \varphi^i(x, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)})).$

Indeed, $\frac{\partial z^{(k)i}}{\partial y^{(k)j}} = \delta^i_j$ implies

$$\frac{1}{2} \partial L \frac{\partial p_m}{\partial y^{(k)i}} = \frac{\partial z^{(k)m}}{\partial y^{(k)i}} + \frac{1}{2} \partial \frac{\partial p_m}{\partial y^{(k)i}} H.$$

Taking into account (8.5.4) it follows

$$p_i = \frac{1}{2} \partial L \frac{\partial z^{(k)i}}{\partial y^{(k)i}}, \quad p_i = ^*\varphi^i(x, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}).$$

(8.5.8)
Theorem 8.5.3 We have the following local properties:

Consider the Legendre transformation determined by the Lagrange space of order $k$, $L^{(k)n} = (M, L)$. Then, the dual Hamilton space of order $k$, $H^{(k)n} = (M, H)$, is defined by the local diffeomorphism $\varphi$, the inverse of the local diffeomorphism $\xi$ and $H = \varphi^* (L)$ is locally given by

$$H(x, y^{(1)}, ..., y^{(k-1)}, p) = 2p_i \xi^{(k)}(x, y^{(1)}, ..., y^{(k-1)}, y^{(k)}) - L(x, y^{(1)}, ..., y^{(k-1)}, y^{(k)}),$$

(8.5.10)

$$y^{(k)i} = \xi^i (x, y^{(1)}, ..., y^{(k-1)}, p).$$

Indeed, as the Legendre transformation $\varphi : T^k M \to T^* k M$ is defined by $\varphi : (x, y^{(1)}, ..., y^{(k-1)}, y^{(k)}) \in T^k M \to (x, y^{(1)}, ..., y^{(k-1)}, p) \in T^* k M$, with $p_i = \frac{1}{2} \frac{\partial L}{\partial y^{(k)i}} = \varphi^* (x, y^{(1)}, ..., y^{(k)}).$ It follows, locally, $\varphi = \varphi^*$, and $\varphi^{-1} = \xi$. But $H$ from (8.5.10) is $H = \varphi^* (L)$. So, locally, $H = \varphi^* (\xi H)$ and $L = \xi^* (\varphi L). Q.E.D.$

8.6 The Canonical Nonlinear Connection of the Space $H^{(k)n}$

There exists a nonlinear connection $N^*$ on the manifold $T^* k M$, which is determined only by a Hamilton space of order $k$, $H^{(k)n} = (M, H)$.

Such a property holds in the case of Lagrange spaces of order $k$, $L^{(k)n} = (M, L)$, (see ch. 2). Namely, $L^{(k)n}$ determines a canonical $k$-semispray $S$ and $S$ allows to construct a nonlinear connection $N$, which depends only on the fundamental function $L$ (Theorem 2.5.2.). Since $L^{(k)n}$ defines the Legendre mapping $\varphi$ from (8.4.2), (8.4.2'), then $\varphi^{-1} = \xi$ determines a bundle morphism and $S$ is transformed in a $k$-dual semispray $S^* = \varphi_*(S)$. Theorem 8.4.2. shows that $S^*_\xi$ determine a nonlinear connection $N^*$ on $T^* k M$, which depends on the fundamental function $L$ of the space $L^{(k)n}$.
Considering the Hamilton space $H^{(k)n} = (M, H)$, the dual of $L^{(k)n}$, its fundamental function $H = \varphi(L)$ is built locally by means of $L$ and by an apriori given nonlinear connection $\overset{\circ}{N}$ on the submanifold $T^{k-1}M$. The connection $N^*$ can be considered the $\overset{\circ}{N}$-canonical nonlinear connection of the space $H^{(k)n}$.

Assuming now that a Hamilton space of order $k$, $H^{(k)n} = (M, H)$ is considered, we construct a bundle morphism $\overset{\star}{\xi}$, (8.5.1), (8.5.2), by means of $H$ and $\overset{\circ}{N}$, and we determine locally the Lagrange space $L^{(k)n} = (M, L)$, where $L = \overset{\star}{\xi}(H)$ is from (8.5.7). It is the dual space of $H^{(k)n}$. The Legendre transformation $\overset{\star}{\varphi} = \overset{\star}{\xi}^{-1}$, with $\overset{\star}{\varphi}_i = \frac{1}{2} \frac{\partial L}{\partial y^{(k)n}}$, transforms the $k$-semispray $S$ of $L$ from (8.4.7) in the dual $k$-semispray $S^*_{\overset{\star}{\xi}}$ from (8.4.8):

\begin{equation}
S^*_{\overset{\star}{\xi}} = y^{(1)i} \frac{\partial}{\partial x^i} + \cdots + (k - 1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-2)i}} + k \xi \frac{\partial}{\partial y^{(k-1)i}} + \eta_i \frac{\partial}{\partial p_i},
\end{equation}

where $\xi^*$ is given by (8.5.2) and its coefficients are

\begin{equation}
\eta_i = -g^{rs} \left[ \frac{\partial \xi^*}{\partial x^r} y^{(1)r} + \cdots + (k - 1) \frac{\partial \xi^*}{\partial y^{(k-1)r}} \xi + (k + 1)G^s(x, ..., y^{(k-1)}, \overset{\star}{\xi}(u^*)) \right]
\end{equation}

So, we have obtained:

**Theorem 8.6.1** The dual coefficients of the $\overset{\circ}{N}$-canonical nonlinear connection $N^*$ of the Hamilton space of order $k$, $H^{(k)n}$, are expressed as follows:

\begin{equation}
M^{i}_{(1)} = -\frac{\partial \xi^*}{\partial y^{(1)i}}, ..., M^{i}_{(k-1)} = -\frac{\partial \xi^*}{\partial y^{(k-1)i}}
\end{equation}

and

\begin{equation}
N^*_{ij} = \frac{\delta \eta_i}{\delta y^{(1)j}},
\end{equation}

where the operator $\frac{\delta}{\delta y^{(1)j}}$ is constructed by means of the coefficients $N^i_{(1)j}, ..., N^i_{(k)j}$ given by the dual coefficients (8.6.3).

The previous theory was presented in the book [115], ch. 12, in the case $k = 2$.

It is useful to prove the existence of a nonlinear connection canonically determined by the fundamental function of the Hamilton space $H^{(k)n}$.
8.7 Canonical Metrical $N$-Linear Connection of the Space $H^{(k)n}$

For a Hamilton space of order $k$, $H^{(k)n} = (M, H)$, we consider the canonical nonlinear connection $N$ determined in the previous section.

We consider also the adapted basis $\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \cdots, \frac{\delta}{\delta y^{(k-1)i}}, \frac{\partial}{\partial p_i} \right)$ and its dual basis $\left( \delta x^i, \delta y^{(1)i}, \cdots, \delta y^{(k-1)i}, \delta p_i \right)$ determined by $N$ and by the distribution $W_k$ (cf. ch. 6).

We can define a $N$-linear connection $D$ by its coefficients $D_\Gamma(N) = \left( H^i_{j\ h}, C^{i}_{(\alpha) h} \right)$, $(\alpha = 1, 2, \ldots, k-1)$. Therefore the $h$, $v_{\alpha}$, $w_{k}$-operators of covariant derivations with respect to $D$, denoted by $'|$, $'|'$, $'|''|'$ can be applied to the $d$-tensor fields on the manifold $T^k M$.

Therefore if $g^{ij} = \frac{1}{2} \partial_{x^i} \partial_{x^j} \ H$ is the fundamental tensor of the Hamilton space $H^{(k)n}$, we have

\begin{equation}
\begin{aligned}
g^{ij} |_h &= \frac{\delta g^{ij}}{\delta x^h} + g^{mj} H^i_{mh} + g^{im} H^j_{mh}, \\
g^{ij} |_{(\alpha) h} &= \frac{\delta g^{ij}}{\delta y^{(\alpha) h}} + g^{mj} C^{i}_{(\alpha) mh} + g^{im} C^{j}_{(\alpha) mh}, \quad (\alpha = 1, \ldots, k-1), \\
g^{ij} |_{h} &= \frac{\partial g^{ij}}{\partial p_h} + g^{mj} C^{i}_{\ m h} + g^{im} C^{j}_{\ m h}.
\end{aligned}
\end{equation}

A $N$-linear connection $D_\Gamma(N)$ is called compatible with the fundamental tensor $g^{ij}$ of the Hamilton space of order $k$, $H^{(k)n} = (M, H)$, (or it is metrical) if $g^{ij}$ is covariant constant (or absolute parallel) with respect to $D_\Gamma(N)$, i.e.

\begin{equation}
\begin{aligned}
g^{ij} |_h &= 0, \quad g^{ij} |_{(\alpha) h} = 0, \quad g^{ij} |_{h} = 0.
\end{aligned}
\end{equation}

The previous equations have a geometrical meaning. Indeed, if the tensor field $g^{ij}$ is positively defined and $\omega_i$ is a $d$-covector field, then

$$||\omega||^2 = g^{ij} \omega_i \omega_j$$

is a scalar field. Then, locally, $\frac{d}{dt} ||\omega||^2 = 0$, along any curve $\gamma$ and for any parallel covector $\omega_i$ along $\gamma$, if and only if the equations (8.7.2) hold.

As usual, we can prove:

**Theorem 8.7.1** 1) In a Hamilton space of order $k$, $H^{(k)n} = (M, H)$, there exists a unique $N$-linear connection $D$, with the coefficients $D_\Gamma(N) = \left( H^i_{j\ h}^{(1)} C^{i}_{(\alpha) jh}, \ldots, C^{i}_{(k-1) jh}, C^{j}_{i h} \right)$ verifying the axioms:
$N$ is the canonical nonlinear connection of $H^{(k)n}$.

2. The fundamental tensor $g^{ij}$ is $h$-covariant constant:

$$g^{ij} |_{h} = 0.$$  \hspace{1cm} (8.7.3)

3. $g^{ij}$ is $v_{\alpha}$-covariant constant:

$$g^{ij} \big|_{h} = 0, \quad (\alpha = 1, \ldots, k - 1).$$  \hspace{1cm} (8.7.4)

4. $g^{ij}$ is $w_{k}$-covariant constant:

$$g^{ij} |_{h} = 0.$$  \hspace{1cm} (8.7.4a)

5. $D \Gamma(N)$ is $h$-torsion free:

$$T_{i}^{jh} = H_{i}^{jh} - H_{hj}^{i} = 0.$$  \hspace{1cm} (8.7.5)

6. $D \Gamma(N)$ is $v_{\alpha}$-torsion free:

$$S_{\alpha}^{(\alpha)}^{i}{}_{jh} = C_{\alpha}^{(\alpha)}^{i}{}_{jh} - C_{\alpha}^{(\alpha)}^{i}{}_{hj} = 0, \quad (\alpha = 1, \ldots, k - 1).$$  \hspace{1cm} (8.7.6)

7. $D \Gamma(N)$ is $w_{k}$-torsion free:

$$S_{jh}^{i} = C_{jh}^{i} - C_{hj}^{i} = 0.$$  \hspace{1cm} (8.7.7)

2) The connection $D \Gamma(N)$ has the coefficients given by the generalized Christoffel symbols:

$$H_{i}^{jh} = 1 \over 2 g^{is} \left( \frac{\delta g_{sh}}{\delta x^{j}} + \frac{\delta g_{js}}{\delta x^{h}} - \frac{\delta g_{jh}}{\delta x^{s}} \right),$$  \hspace{1cm} (8.7.8)

$$C_{\alpha}^{(\alpha)}^{i}{}_{jh} = 1 \over 2 g^{is} \left( \frac{\delta g_{sh}}{\delta y_{(\alpha)j}} + \frac{\delta g_{js}}{\delta y_{(\alpha)h}} - \frac{\delta g_{jh}}{\delta y_{(\alpha)s}} \right), \quad (\alpha = 1, \ldots, k - 1),$$

$$C_{s}^{jh} = -1 \over 2 g^{is} \left( \frac{\partial}{\partial y^{h}} g^{sh} + \frac{\partial}{\partial y^{s}} g^{jh} - \frac{\partial}{\partial y^{s}} g^{jh} \right).$$

3) This connection depends only by the fundamental function $H$ of the space $H^{(k)n}$ and by the canonical nonlinear connection $N$.

The connection $D \Gamma(N)$ with the coefficients (8.7.8) will be denoted by $CT(N)$ and called canonical for the space $H^{(k)n}$.

The canonical $N$-metrical connection $CT(N)$ has zero torsions $T_{i}^{jh}$, $S_{\alpha}^{(\alpha)}^{i}{}_{jh}$, $S_{i}^{jh}$. Of course, we can determine all $N$-linear metrical connections of the space $H^{(k)n}$. Therefore we consider the Obata’s operator, [115]:

$$\Omega_{hm}^{ij} = 1 \over 2 \left( \delta_{h}^{i} \delta_{m}^{j} - g_{hm} g^{ij} \right), \quad \Omega_{hm}^{*ij} = 1 \over 2 \left( \delta_{h}^{i} \delta_{m}^{j} + g_{hm} g^{ij} \right)$$  \hspace{1cm} (8.7.9)

and prove without difficulties:
Theorem 8.7.2 The set of all $N$-linear metrical connections $D\Gamma(N) = (\overline{H}^i_{jh}, \overline{C}^{i}_{(a)jh}, \overline{C}^{jh}_{i})$ of the space $H^{(k)n}$ are expressed by

$$\overline{H}^i_{jh} = H^i_{jh} + \Omega_{rjs}^{is}X^r_{sh},$$

(8.7.10)

$$\overline{C}^{i}_{(a)jh} = C^{i}_{(a)jh} + \Omega_{rjs}^{is}Y^r_{sh}, \quad (\alpha = 1, \ldots, k - 1),$$

$$\overline{C}^{jh}_{i} = C^{jh}_{i} + \Omega_{rjs}^{is}Z^r_{sh}.$$  

where $CT(N) = (H^i_{jh}, C^{i}_{(a)jh}, C^{jh}_{i})$ is the canonical $N$-metrical connection and $X^r_{sh}, Y^r_{sh}$ and $Z^r_{sh}$ are arbitrary $d$-tensors.

It is important to remark that a triple $\left(X^i_{jh}, Y^i_{(a)jh}, Z^j_{h} \right)$ determines by means of (8.7.10) a transformation of the $N$-metrical connections $D\Gamma(N) \to D\Gamma(N)$.

The set of transformations $\{D\Gamma(N) \to D\Gamma(N)\}$ and the composition of these mappings is an Abelian group.

The last theorems lead to the following result:

Theorem 8.7.3 There exists an unique $N$-linear connection $D\Gamma(N) = \left(\overline{H}^i_{jh}, \overline{C}^{i}_{(a)jh}, \overline{C}^{jh}_{i}\right)$, (\(\alpha = 1, \ldots, k - 1\)), metrical with respect to the fundamental tensor $g^{ij}$ of the space $H^{(k)n}$ having as torsion the $d$-tensor fields $T^i_{jh} (= -T^i_{hj}), S^i_{(a)jh} (= -S^i_{(a)hj}), S^{jh}_{i} (= -S^{hj})$, a priori given. The coefficients of $D\Gamma(N)$ have the following expressions:

$$\overline{H}^i_{jh} = \frac{1}{2}g^{is}\left(\frac{\delta g_{sh}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^h} - \frac{\delta g_{jh}}{\delta x^i}\right),$$

$$\overline{C}^{i}_{(a)jh} = \frac{1}{2}g^{is}\left(\frac{\delta g_{sh}}{\delta y^{(a)}j} + \frac{\delta g_{js}}{\delta y^{(a)}h} - \frac{\delta g_{jh}}{\delta y^{(a)}i}\right),$$

$$\overline{C}^{jh}_{i} = \frac{1}{2}g^{is}\left(\partial^j g^{sh} + \partial^h g^{js} - \partial^s g^{jh}\right) - \frac{1}{2}g^{is}\left(g^{sm}S^m_{(a)jh} - g^{jm}S^m_{sh} + g^{hm}S^m_{js}\right).$$

Now, consider the canonical $N$-metrical connection $CT(N)$ only. Then, we can write the Ricci identities, from Theorem 6.6.1, where $T^i_{jh} = 0$, $S^i_{(a)jh} = 0$, $(\alpha = 1, \ldots, k - 1)$, $S^{jh}_{i} = 0$. Since the metricity conditions (6.6.7) are verified it follows:
**Proposition 8.7.1** The curvature $d$-tensor fields of $\Gamma(N)$ satisfy the identities (6.6.8).

**Proposition 8.7.2** The tensors of deflection of the canonical $N$-metrical connection $\Gamma(N)$ satisfy the identities (6.6.9), with $T_{jk}^i = 0$, $S_{(\alpha)j}^i = 0$, $(\alpha = 1, \ldots, k - 1)$, $S_{i}^{j} = 0$.

Let $\omega_i^j$ be the 1-forms connection of $\Gamma(N)$

\[(8.7.12) \quad \omega_i^j = H_i^j ds + C_{(1)}^{i} j s \delta y^{(1)s} + \cdots + C_{(k-1)}^{i} j s \delta y^{(k-1)s} + C_{(k-1)}^{i} \delta p_s.\]

Theorem 6.7.1 is valid for $\Gamma(N)$.

In particular, Theorem 6.7.2 for $\Gamma(N)$ holds, too. Namely:

**Theorem 8.7.4** The manifold $T^k M$ is with absolute parallelism of vectors with respect to the canonical $N$-metrical connection $\Gamma(N)$ if and only if the curvature $d$-tensors of $\Gamma(N)$ vanishes.

Also, we have:

**Theorem 8.7.5** A smooth parametrized curve $\gamma$, given by $x^i = x^i(t)$, $y^{(\alpha)i} = y^{(\alpha)i}(t)$, $(\alpha = 1, \ldots, k - 1)$, $p_i = p_i(t)$, $t \in I$ is an autoparallel curve with respect to $\Gamma(N)$ if and only if the equations (6.7.13), are verified, $\omega_i^j$ being the 1-forms connection (8.7.12).

The horizontal curves of the space $H^{(k)n}$ are characterized by the equations (6.7.14).

The horizontal paths of $H^{(k)n}$ are expressed by the equations (6.7.15) for the connection $\Gamma(N)$.

The $v_\alpha$-vertical paths of $H^{(k)n}$ are characterized by (6.7.16) and $w_k$-vertical paths are given by (6.7.17).

The canonical $N$-metrical connection $\Gamma(N)$ of the Hamilton space of order $k$, $H^{(k)n} = (M, H)$ is $h$-, $v_\alpha$- and $w_k$-torsion free if the $d$-tensors of torsion vanish:

\[(8.7.13) \quad T_{jk}^i = H_{jk}^i - H_{kj}^i = 0, \quad S_{(\alpha)j}^i = C_{(\alpha)j}^i - C_{(\alpha)h}^i = 0, \quad (\alpha = 1, \ldots, k - 1), \quad S_{i}^{j} = C_{i}^{j} - C_{i}^{h}.

Therefore, we have:

**Theorem 8.7.6** The canonical $N$-metrical connection $\Gamma(N)$, with the coefficients (8.7.8), of the Hamilton space of order $k$, $H^{(k)n}$, has the structure equations given by the equations (7.8.1), (7.8.2), (7.8.3) and (7.8.4) from Theorem 7.8.1, in the conditions (8.7.13) and with 1-forms $\omega^i_j$ from (8.7.12).

The Bianchi identities of $\Gamma(N)$ can be obtained from (7.6.11), (7.6.12), (7.6.13) by means of the conditions (8.7.13).
8.8 The Hamilton Space $H^{(k)n}$ of Electrodynamics

The classical Lagrangian of electrodynamics has been considered in section 5 of chapter 2:

$$L(x, y^{(1)}) = mc\gamma_{ij}(x)y^{(1)i}y^{(1)j} + \frac{2e}{m}b_i(x)y^{(1)i}$$

in which $m, c, e$ are known physical constants, $\gamma_{ij}(x)$ are gravitational potentials and $b_i(x)$ are electromagnetic potentials. This can be extended to the manifold of accelerations of order $k$, $T^k M$ as follows:

$$(8.8.1) \quad L(x, y^{(1)}, ..., y^{(k)}) = mc\gamma_{ij}(x)z^{(k)i}z^{(k)j} + \frac{2e}{m}b_i(x)z^{(k)i},$$

$z^{(k)i}$ being the Liouville $d$-vector field

$$(8.8.2) \quad k z^{(k)i} = ky^{(k)i} + (k - 1) M^{(1)i} y^{(k-1)j} + \cdots + M^{(k-1)i} y^{(1)j}.$$  

The dual coefficients $M^{(1)i}, ..., M^{(k-1)i}$ are given by the prolongation to $T^k M$ of the Riemannian structure $\mathcal{R}^n = (M, \gamma_{ij}(x)).$

Consequently, $M^{(1)i}$ depends on the variables $(x, y^{(1)}); M^{(2)i}$ depends on the variables $(x, y^{(1)}, y^{(2)}); \ldots; M^{(k-1)i}$ depends on the variables $(x, y^{(1)}, ..., y^{(k-1)}).$

This is an important property, since $\left( M^{(1)i}, ..., M^{(k-1)i} \right)$ can be considered as the dual coefficients of a nonlinear connection $N$ on the manifold $T^{*k}M,$ determined only by the tensor field $\gamma_{ij}(x)$.

It follows that:

1) The Lagrangian $L$, (8.8.1), is well determined only by $\gamma_{ij}(x)$ and $b_i(x);$  
2) $L$ is differentiable on $T^k M;$  
3) $L$ is regular:

$$(8.8.3) \quad g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^{(k)j} \partial y^{(k)j}} = mc\gamma_{ij}(x);$$

4) The pair $L^{(k)n} = (M, L)$ is a Lagrange space of order $k$.

It is called the Lagrange space of order $k$ of electrodynamics. The geometry of the space $L^{(k)n}$ gives us a good geometrical model for the Analytical Mechanics based on the Lagrangian of electrodynamics. So, the integral of action $I(c) = \int_0^1 L(x, \frac{dx}{dt}, \ldots, \frac{d^{k}x}{dt^k})dt$ leads to the Euler-Lagrange equations $\frac{d}{dt} \dot{E}_i(L) = 0$ (2.2.1). The energy of higher order can be determined. The energy of order $k$, $\mathcal{E}_c^k(L),$ satisfies the law of conservation. A Nöther theorem holds.

In the same time the canonical $k$-semispray $S$ is given by

$$(8.8.4) \quad S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k + 1)G^i \frac{\partial}{\partial y^{(k)i}}.$$
whose coefficients $G^i$, calculated by means of (8.8.1) have the expressions

$$(8.8.4a) \quad (k + 1)G^i = \frac{1}{2mc} \gamma^{ij} \left\{ \Gamma \left( \frac{\partial}{\partial y^{(k)j}} \right) - \frac{\partial}{\partial y^{(k-1)j}} \right\}$$

with

$$(8.8.4b) \quad \Gamma = y^{(1)} \frac{\partial}{\partial x^i} + 2 y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + k y^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}$$

and

$$(8.8.4c) \quad \frac{\partial L}{\partial y^{(k)j}} = 2mc \gamma_{ih} (x) z^{(k)h} + \frac{2e}{m} b_j (x).$$

The Legendre transformation $\xi: T^k M \rightarrow T^* T^k M$ determined by the Lagrange space of electrodynamics $L^{(k)n}$, is defined by (8.4.2), (8.4.2'). This is:

$$(8.8.5) \quad p_i = \frac{1}{2} \frac{\partial L}{\partial y^{(k)j}} = mc \gamma_{ih} (x) z^{(k)h} + \frac{e}{m} b_i (x).$$

The mapping $\varphi$ is a local diffeomorphism. Its inverse $\varphi^{-1} = \xi$ is given by (8.4.3), where

$$(8.8.5a) \quad \xi^i (x, y^{(1)}, \ldots, y^{(k-1)}, p) = \frac{1}{mc} \gamma^{ih} (x) \left\{ p_h - \frac{e}{m} b_h (x) \right\} - \frac{1}{k} \left\{ (k - 1) M_{(1)i}^{(k)h} y^{(k-1)h} + \cdots + M_{(k-1)i}^{(1)h} y^{(k-1)h} \right\}.$$

The canonical $k$-semispray $S$, (8.8.4) is transformed by Legendre mapping $\varphi$ in the dual $k$-semispray $S^*_\xi$, from (8.4.8), with the coefficients $\eta_i$ from (8.4.9).

Theorem 8.4.2 allows to determine, locally, the canonical nonlinear connection $N^*$ on the manifold $T^k M$.

As we remarked above the connection $N$ of the prolongation of the Riemannian structure $(M, \gamma_{ij}(x))$, with the dual coefficients $M_{(1)i}^{(1)} (x, y^{(1)})$, ..., $M_{(k-1)i}^{(1)} (x, y^{(k-1)})$ defines a nonlinear connection on the submanifold $T^{k-1} M$ in $T^k M$.

Thus, the Liouville vector field $z^{(k)i}$ from (8.8.2) is transformed by the Legendre mapping $\varphi$ in the vector field $\zeta^{(k)i}$:

$$(8.8.6) \quad k \zeta^{(k)i} = k \xi^i + (k - 1) M_{(1)i}^{(1)m} y^{(k-1)m} + \cdots + M_{(k-1)i}^{(1)m} y^{(1)m}.$$

Let us consider the following Hamiltonian $H$, the dual of $L$:
Hamilton Spaces of Order $k \geq 1$

(8.8.7)
\[ H(x, y^{(1)}, ..., y^{(k-1)}, p) = 2p_i \frac{\partial}{\partial x_i} - L(x, y^{(1)}, ..., y^{(k-1)}, \xi(x, y^{(1)}, ..., y^{(k-1)}, p)) \]

We can state:

**Theorem 8.8.1** We have:

1. The dual of the Lagrange space of electrodynamics $L^{(k)} = (M, L)$, (8.8.1), is a Hamilton space $H^{(k)} = (M, H)$, (8.8.7).

2. The fundamental function $H$ of the space $H^{(k)}$ is given by

\[
H = \frac{1}{mc} \left\{ \gamma^{ij}(x)p_i p_j - \frac{2e}{m} \gamma^{ij} p_i b_j + \frac{e}{mc^2} \gamma^{ij} b_i b_j \right\}.
\]

3. $H$ depends only on the point $x$ and the momenta $p$. 

4. $H$ is the fundamental function of a Hamilton space of order 1, $H^1 = (M, H)$.

**Proof:** 1. Applying theorem 7.4.3 and the formulas (8.8.5), (8.8.6) we obtain:

\[
\frac{\partial}{\partial x_i} = \frac{1}{mc} \gamma^{ij}(x) p_j - \frac{e}{m} b_j(x)
\]

and

\[
L(x, y^{(1)}, ..., y^{(k-1)}, \xi) = mc \gamma^{ij} \left[ p_i - \frac{e}{m} b_i \right] \left[ p_j - \frac{e}{m} b_j \right] + \frac{2e}{mc^2} \gamma^{ij} b_j \left[ p_i - \frac{e}{m} b_i \right].
\]

Substituting in (8.8.7) we have the equality (8.8.8).

3. Looking at the function $H$ one remarks that $H$ does not depend on the variables $y^{(1)}, ..., y^{(k-1)}$.

4. $H$ is a Hamilton function on the cotangent bundle $T^* M$ and its Hessian, with respect to the momenta $p_i$, has the matrix $\frac{1}{mc} \gamma^{ij}(x)$. So, the pair $H^1 = (M, H)$ is a Hamilton space. Q.E.D.

**Remarks.** 1. The Hamilton space $H^1 = (M, \tilde{H}(x, p))$ is the dual of the Lagrange space of electrodynamics $L^1 = (M, \tilde{L}(x, y^{(1)}))$.

2. $H^{(k)}$ does not depend on the nonlinear connection $N$ of the prolongation of the Riemann space $R^{(n)} = (M, \gamma^{ij}(x))$.

For the Hamilton space $H^n = (M, H)$ with the fundamental function $H(x, p)$ from (8.8.8) we have a canonical nonlinear connection $N$, with the coefficients

(8.8.9)
\[
N_{ij} = \gamma^{h}_{ij}(x)p_h + \frac{e}{c} (b_{ij} + b_{ji}),
\]

where $b_{ij} = \frac{\partial b_i}{\partial x^j} - b_s \gamma^s_{ij}$, $\gamma^h_{ij}(x)$ being the Christoffel symbols of the metric $\frac{1}{mc} \gamma^{ij}(x)$.  

The canonical $N$-metrical connection $C^\Gamma(N) = (H^i_j, C^j_i)$ has the following coefficients

$$H^i_j = \gamma^i_j, \quad C^j_i = 0.$$  

The geometrical object fields $H, g_{ij} = \frac{1}{mc}\gamma_{ij}, N_{ij}$ and $H^i_j$ allow to develop the geometry of the space $H^n = (M, H)$.

### 8.9 The Riemannian Almost Contact Structure Determined by the Hamilton Space $H^{(k)n}$

Consider a Hamilton space of order $k$, $H^{(k)n} = (M, H)$ having $g^{ij}$ as fundamental tensor field. Let $N$ be a nonlinear connection canonical associated to $H^{(k)n}$, according to Theorem 8.6.1. As usually, we consider the adapted basis $(\delta x^i, \delta y^{(1)}_i, ..., \delta y^{(k-1)}_i, \delta p_i)$ and its dual basis $(\delta x^i, \delta y^{(1)}_i, ..., \delta y^{(k-1)}_i, \delta p_i)$.

Then, at every point $u^* = (x, y^{(1)}, ..., y^{(k-1)}, p) \in \text{T}^* M$ we can define the tensor

$$\nabla = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^{(1)}_i \otimes \delta y^{(1)}_j + \cdots + g_{ij} \delta y^{(k-1)}_i \otimes \delta y^{(k-1)}_j + g^{ij} \delta p_i \otimes \delta p_j$$

$\nabla$ is the $N$-lift of the fundamental tensor $g^{ij}$ of the Hamilton space of order $k$, $H^{(k)n} = (M, H)$, (cf. §6.7, ch. 6).

**Theorem 8.9.1**

1. $\nabla$ is a (pseudo)-Riemannian structure on the manifold $\text{T}^* M$, determined only by the space $H^{(k)n}$ and the nonlinear connection $N$.

2. The distributions $N_0, N_1, ..., N_{k-2}, V_{k-1}, W_k$ are mutual orthogonal with respect to $\nabla$.

Indeed: 1. Every term of $\nabla$ is defined on $\text{T}^* M$ because $g_{ij}$ is a $d$-tensor field and $\delta x^i, ..., \delta p_i$ have geometrical meaning, i.e. they are transformed as in (6.3.4), $\det |\nabla| \neq 0$. 2. This property is obvious, taking into account the form of $\nabla$. Q.E.D.

The tensor $\nabla$ is of the form:

$$\nabla = \nabla^H + \nabla^{V_1} + \cdots + \nabla^{V_{k-1}} + \nabla^{W_k},$$

$$\nabla^H = g_{ij} dx^i \otimes dx^j, \quad \nabla^{V_\alpha} = g_{ij} \delta y^{(\alpha)}_i \otimes \delta y^{(\alpha)}_j, (\alpha = 1, ..., k-1),$$

$$\nabla^{W_k} = g^{ij} \delta p_i \otimes \delta p_j.$$

Here $\nabla^H$ is the restriction of $\nabla$ to the distribution $N_0 = N^*$, $\nabla^{V_\alpha}$ is the restriction of $\nabla$ to the distribution $N_\alpha, \alpha = 1, ..., k-1, N_{k-1} = V_{k-1}$ and
Hamilton Spaces of Order \( k \geq 1 \)

\( GW_k \) is the restriction of \( G \) to the distribution \( W_k \). Moreover, \( G^H, G^{V_\alpha}, G^W_k \) are \( d \)-tensor fields.

**Theorem 8.9.2** The tensor fields \( G^H, G^{V_\alpha}, (\alpha = 1, \ldots, k-1) \), \( G^W_k \) are covariant constant with respect to any metrical \( N \)-connection \( D\Gamma(N) \).

Indeed, using the considerations of this section, it follows \( DX G^H = 0, DX G^{V_\alpha} = 0, DX G^W_k = 0 \) and, consequently \( DX \check{G}^H = 0 \). Q.E.D.

The geometry of the (pseudo)-Riemannian space \( (\check{T}^*kM, \check{G}) \) can be studied by means of a metrical \( N \)-linear connection, presented in the section 7 of this chapter.

As we know (§6, Ch.6) on \( \check{T}^*kM \), endowed with the nonlinear connection \( N \), there exists a natural almost \( (k-1)n \)-contact structure \( F \), defined by (6.6.3), having the properties from in Theorem 6.5.2. The condition of normality of \( F \) is (6.6.5). Exactly as in Theorem 6.7.2 we can deduce:

**Theorem 8.9.3** The pair \( (\check{G}, F) \) is a Riemannian \( (k-1)n \)-almost contact structure determined by the space \( H^{(k)n} \) and the nonlinear connection \( N \).

The considerations from section 6.8, ch. 8 show that the nonlinear connection \( N \) and the fundamental tensor \( g^{ij} \) of the Hamilton space of order \( k \), \( H^{(k)n} \), determine on the manifold \( T^*kM \), two important geometric object fields. One is the \( N \)-lift \( \check{G} \) from (8.9.1) and another one is \( F \) the almost \( (k-1)n \)-contact structure, given by (6.8.5).

Namely,

\[
\check{F} \left( \frac{\delta}{\partial x^i} \right) = -g_{ij} \frac{\partial}{\partial y^j}, \quad \check{F} \left( \frac{\delta}{\partial y^{(\alpha)n}} \right) = 0, \quad (\alpha = 1, \ldots, k-1), \quad \check{F} \left( \frac{\partial}{\partial \hat{\eta}} \right) = g^{ij} \frac{\delta}{\delta x^i}
\]

By means of Theorem 7.9.1 it follows that \( \check{F} \) is a tensor field of type \((1,1)\) on the manifold \( T^*kM \) given in (6.8.3), \( \text{rank} \check{F} = 2n \) and \( \check{F}^3 + \check{F} = 0 \).

The following version of the Theorem 6.8.2 holds:

**Theorem 8.9.4** For any Hamilton space of order \( k \), \( H^{(k)n} = (M, H) \), endowed with a canonical nonlinear connection \( N \) the following properties hold:

1. The pair \( (\check{G}, F) \) is a Riemannian almost \((k-1)n\)-contact structure.

2. The associated 2-form is

\[
\theta = \delta p_i \wedge dx^i.
\]

3. If the coefficients \( N_{ij} \) of \( N \) are symmetric then

\[
\theta = dp_i \wedge dx^i
\]
and $\theta$ is the canonical presymplectic structure on $\widetilde{T^*kM}$.

The condition of normality of structure $\widetilde{F}$ is

$$\mathcal{N}_{\nabla}(X, Y) + \sum_{i=1}^{n} \left[ \sum_{\alpha=1}^{k-1} d(\delta y^{(\alpha)i})(X, Y) + d\delta p_{i}(X, Y) \right] = 0, \quad \forall X, Y \in \mathcal{X}(\widetilde{T^*kM}),$$

where $\mathcal{N}_{\nabla}$ is the Nijenhuis tensor of $\widetilde{F}$.

Now, taking the local expression of the tensor field $\widetilde{F}$:

$$\widetilde{F} = -g_{ij} \partial_{j} \otimes dx^{i} + g^{ij} \delta \frac{\delta}{\delta x^{i}} \otimes \delta p_{j}$$

and applying the theory of $h$-, $v_{\alpha}$- and $w_{\lambda}$-covariant derivative from §7.5, ch. 7, it follows:

**Theorem 8.9.5** With respect to a metrical $N$-linear connection $D\Gamma(N)$, the Riemannian almost $(k-1)n$-contact structure $\left(\nabla G, \nabla F\right)$ is covariant constant, i.e.

$$D_{X} \nabla G = 0, \quad D_{X} \nabla F = 0.$$

It follows that the geometry of the Riemannian almost $(k-1)n$-contact space $\left(\widetilde{T^*kM}, \nabla G, \nabla F\right)$ can be studied by means of the metrical $N$-linear connection $D\Gamma(N)$. This space is called the Riemannian almost $(k-1)n$-contact model of the Hamilton space of order $k$, $H^{(k)n}$. 
Chapter 9

Subspaces in Hamilton Spaces of Order \( k \)

In this chapter we shall study the geometry of subspaces in a Hamilton space \( H^{(k)n} = (M, H) \). A submanifold \( \mathcal{V} \) of the manifold \( M \) determines a dual manifold \( T^{\mathcal{V}} M \). But the immersion \( \mathcal{V} M \to M \) does not automatically imply an immersion of \( T^{\mathcal{V}} M \), into the dual manifold \( T^{\mathcal{V}} M \), like in the case of manifold \( T^k M \) into the total space of \( k \) tangent bundle \( T^k M \), [94], since \( T^{\mathcal{V}} M = T^{k-1}_M \mathcal{V} \times_M T^* M \).

Therefore, by means of the immersion \( i \) and by an immersion of the cotangent manifold \( T^* M \mathcal{V} \) into \( T^* M \) we can define \( T^{\mathcal{V}} M \) as an immersed submanifold in \( T^{\mathcal{V}} M \). Thus, the main geometrical objects fields on \( T^{\mathcal{V}} M \) induce the corresponding geometrical object fields on submanifold \( T^{\mathcal{V}} M \). The Hamilton space \( H^{(k)m} = (M, H) \) induces a Hamilton subspaces \( \mathcal{V}^{(k)m} = (\mathcal{V}, \mathcal{V}) \). So we study the intrinsic geometrical object fields on \( \mathcal{V}^{(k)m} \) and the induced geometrical object fields, as well as the relations between them. These problems are approached by means of the method of moving frame, used in the case of subspaces \( \mathcal{L}^{(k)m} = (\mathcal{V}, \mathcal{V}) \) in the Lagrange spaces of order \( k \).

9.1 Submanifolds \( T^{\mathcal{V}} M \) in the Manifold \( T^{\mathcal{V}} M \)

Let \( M \) be a \( C^\infty \)-real, \( n \)-dimensional manifold and \( \mathcal{V} M \) be a \( C^\infty \)-real, \( m \)-dimensional manifold, \( 1 < m < n \), immersed in \( M \) through the immersion \( i : \mathcal{V} M \to M \). Locally \( i \) can be given in the form

\[
(9.1.1) \quad x^i = x^i(u^1, ..., u^m), \quad \text{rank} |\frac{\partial x^i}{\partial u^\alpha}| = m, \forall (u^\alpha) \in \mathcal{V} \subset \mathcal{V} M
\]
The indices $i, j, h, r, s, p, q$ run over the set \{1, 2, ..., $n$\}; the indices $\alpha, \beta, \gamma, ...$ run over the set \{1, 2, ..., $m$\} and $\alpha, \beta, \gamma, ...$ run over the set \{1, ..., $n - m$\}.

If $i : \overset{\vee}{M} \to M$ is an embedding, then we identify $\overset{\vee}{M}$ with $i(\overset{\vee}{M}) \subset M$ and say that $\overset{\vee}{M}$ is a submanifold of $M$. In this case (9.1.1) are called the parametric equations of the submanifold $\overset{\vee}{M}$ in the manifold $M$.

Let us consider the manifold $T^k \overset{\vee}{M}$ determined by $\overset{\vee}{M}$, i.e.

$$T^k \overset{\vee}{M} = T^{k-1} \overset{\vee}{M} \times_T T^* \overset{\vee}{M}. ~ A \text{ point } \overset{\vee}{u} \in T^k \overset{\vee}{M} \text{ will be denoted by } \overset{\vee}{u} = (u, v(1), ..., v(k-1), p) \text{ and its coordinates by } (u^\alpha, v(1)^\alpha, ..., v(k-1)^\alpha, p^\alpha).$$

A change of local coordinates on the manifold $T^k \overset{\vee}{M}$ is given by

\[
\begin{cases}
\tilde{u}^\alpha = \tilde{u}^\alpha(u^1, ..., u^m), \text{ rank } \left| \frac{\partial \tilde{u}^\alpha}{\partial u^\beta} \right| = m, \\
\tilde{v}^{(1)} \alpha = \frac{\partial \tilde{v}^{(1)} \alpha}{\partial u^\beta} v^{(1)} \beta, \\
\vdots \\
(k-1) \tilde{v}^{(k-1)} \alpha = \frac{\partial \tilde{v}^{(k-2)} \alpha}{\partial u^\beta} v^{(1)} \beta + ... + (k-1) \frac{\partial \tilde{v}^{(k-2)} \alpha}{\partial v^{(k-2)} \beta} v^{(k-1)} \beta, \\
\overset{\vee}{p}^\alpha = \frac{\partial v^\beta}{\partial u^\alpha} p^\beta,
\end{cases}
\]

with

$$\frac{\partial \tilde{v}^{(1)} \alpha}{\partial u^\beta} = \frac{\partial \tilde{v}^{(1)} \alpha}{\partial v^{(1)} \beta} = ... = \frac{\partial \tilde{v}^{(k-1)} \alpha}{\partial v^{(k-1)} \beta}.$$

Remarking that $T^k : Man \to Man$ is a covariant functor from the category of differentiable manifolds Man to itself, it follows that the immersion $i : \overset{\vee}{M} \to M$ uniquely determines the mapping $T^k i : T^k \overset{\vee}{M} \to T^k M$ (see Ch.4) analytically given by the equations

\[
\begin{cases}
x^i = x^i(u^1, ..., u^m), \text{ rank } \left| \frac{\partial x^i}{\partial u^\alpha} \right| = m, \\
y^{(1)} = \frac{\partial x^i}{\partial u^\alpha} u^{(1)} \alpha,
\end{cases}
\]

\[
\begin{cases}
(k-1) y^{(k-1)} = \frac{\partial y^{(k-2)} i}{\partial u^\alpha} v^{(1)} \alpha + 2 \frac{\partial y^{(k-2)} i}{\partial v^{(1)} \alpha} v^{(2)} \alpha + ... + (k-1) \frac{\partial y^{(k-2)} i}{\partial v^{(k-2)} \alpha} v^{(k-1)} \alpha,
\end{cases}
\]

and

\[
\frac{\partial x^i}{\partial u^\alpha} p_i = \overset{\vee}{p}_\alpha,
\]
where
\[
\frac{\partial x^i}{\partial u^\alpha} = \frac{\partial y^{(1)i}}{\partial v^{(1)\alpha}} = \cdots = \frac{\partial y^{(k-1)i}}{\partial v^{(k-1)\alpha}},
\]
(9.1.3b)
\[
\frac{\partial y^{(a)i}}{\partial u^\alpha} = \frac{\partial y^{(a+1)i}}{\partial u^{(1)\alpha}} = \cdots = \frac{\partial y^{(k-i)i}}{\partial u^{(k-1-a)\alpha}}, (a = 1, \ldots, k-2).
\]

We shall denote
\[
B_i^\alpha(u) = \frac{\partial x^i}{\partial u^\alpha}, B_i^{\alpha\beta}(u) = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \ldots
\]
(9.1.4)

In order to obtain an immersion \(i^*\) of \(T^* M = T^{k-1} M \times T^* M\) in \(T^k M = T^{k-1} M \times T^* M\), it is necessary that \(i^*\) to be of the form \(i^* = T^{k-1} i \times f\), with \(f : T^* M \to T^* M\). Analytically \(T^{k-1} i\) is defined by the equations (9.1.3). Consequently, taking into account (9.1.3'), it follows that the mapping \(f\) can be given in the form
\[
p_i = A_i^\alpha(u) v^\alpha, \text{rank} \|A_i^\alpha(u)\| = m, \forall u = (u^1, \ldots, u^m) \in U \subset M
\]
(9.1.5)

We obtain

**Theorem 9.1.1** The equations (9.1.3), (9.1.5) define a local immersion
\[
i^* : T^k M \to T^k M,
\]
i^* : \( (u, v^{(1)}, \ldots, v^{(k-1)}, p) \to u^* = (x, y^{(1)}, \ldots, y^{(k-1)}, p) \).

**Proof.** The Jacobian matrix of the mapping \(i^*\) given by the equations (9.1.3), (9.1.5) at a point \(\mathbf{u} \in \mathbf{U}\) is as follows:
\[
J(i^*)_\mathbf{u} = \begin{bmatrix}
\frac{\partial x^i}{\partial u^\alpha} & 0 & 0 & \cdots & 0 & 0 \\
\frac{\partial y^{(1)i}}{\partial u^\alpha} & \frac{\partial x^i}{\partial u^\alpha} & 0 & \cdots & 0 & 0 \\
\frac{\partial y^{(k-1)i}}{\partial u^\alpha} & \frac{\partial y^{(k-1)i}}{\partial u^\alpha} & \frac{\partial y^{(k-1)i}}{\partial u^\alpha} & \cdots & 0 & 0 \\
\frac{\partial A_i^\beta}{\partial u^\alpha} & \frac{\partial^2 x^i}{\partial u^\alpha \partial \beta} & 0 & \cdots & 0 & A_i^\alpha \end{bmatrix}
\]
(9.1.6)

Since the matrices \(\|\frac{\partial x^i}{\partial u^\alpha}\|\) and \(\|A_i^\alpha(u)\|\) have the rank \(m\) it follows that \(i^*\) defines a local immersion. Q.E.D.

The functions \(A_i^\alpha(u)\) from (9.1.5) are not arbitrary. They must satisfy some conditions.
Proposition 9.1.1 The following properties hold:

\[ B_i^\alpha(u)A_\beta^\alpha(u) = \delta_\alpha^\beta, \]

1° \( A_\alpha^\beta(u) \) is a \( d \)-covector field with respect to index \( \alpha \),
\( A_i^\alpha(u) \) is a \( d \)-vector field with respect to index \( i \).

Indeed, (9.1.7) follows from (9.1.3') and (9.1.5'), \( p_i \) being a \( d \)-covector it follows that \( A_\alpha^\beta(u) \) has the same quality with respect to index \( \alpha \). The same remark is valid for the index \( \alpha \) of \( A_i^\alpha(u) \).

Of course \( B_i^\alpha(u) \) is a \( d \)-vector with respect to index \( i \) and it is a covector with respect to index \( \alpha \).

The previous properties allow to call \( B_i^\alpha(u) \) and \( A_i^\alpha(u) \) the mixed \( d \)-tensor fields. Along \( M \) a mixed \( d \)-tensor will be given by the components \( T_{\beta_1...\beta_q\gamma_1...\gamma_s}^{\alpha_1...\alpha_p} \) having the property that it a \( d \)-tensor of type \( (p, q) \) with respect to indices \( \alpha_1...\alpha_p \), and \( \beta_1...\beta_q \) (to a change of coordinates on the submanifold \( T^k\hat{M} \) and it is a \( d \)-tensor of type \( (r, s) \) with respect to indices \( i_1...i_r \) and \( j_1...j_s \), to a change of coordinates on the enveloping manifold \( T^kM \).

Remark. The notion of mixed \( d \)-tensor will be extended in the section 3.

As usual we set
\[ \partial^i = \frac{\partial}{\partial p_i}, \partial^\alpha = \frac{\partial}{\partial p^\alpha}. \]

The differential \( di^* \) of the immersion \( i^* \) acts on the natural basis
\[ \left( \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^{(1)}\alpha}, ..., \frac{\partial}{\partial v^{(k-1)}\alpha}, \partial^\alpha \right) \]
by the rule:
\[ di^* \left[ \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^{(1)}\alpha}, ..., \frac{\partial}{\partial v^{(k-1)}\alpha}, \partial^\alpha \right] = \left[ \frac{\partial}{\partial x^i}, ..., \frac{\partial}{\partial y^{(k-1)}\alpha}, \partial^\alpha \right] J(i^*) \]
and for the natural cobasis:
\[ di^* \left[ dx^i, ..., dy^{(k-1)}\alpha, dp_i \right]^T = J(i^*) \left[ dx^\alpha, ..., dy^{(k-1)}\alpha, d\hat{p}_\alpha \right] \]

Therefore, it is not difficult to see that:

1°. We have
\[ \partial^\alpha = A_\alpha^\beta \partial^i; dp_i = dA_i^\alpha \hat{p}_\alpha + A_i^\alpha d\hat{p}_\alpha \]

2° Along submanifold \( T^k\hat{M} \) the vertical distributions \( v_1, ..., v_{k-1}, w_k \) are subdistributions of the vertical distributions \( V_1, ..., V_{k-1}, W_k \) from the manifold \( T^kM \).

3° We have for the Liouville vector fields \( \gamma_1, ..., \gamma^{k-1} \) from \( T^kM \) the relations
Subspaces in Hamilton Spaces of Order $k$

(9.1.10) \[ di^*(\gamma) = 1, ..., di^*(\gamma) = k^{-1} \]

These equations will be applied in the theory of Hamilton subspaces.

### 9.2 Hamilton Subspaces $H^{(k)m}$ in $H^{(k)n}$: Darboux Frames

Let $H^{(k)n} = (M, H)$ be a Hamilton spaces of order $k$, its fundamental function $H$ being defined on the manifold $T^{*k}M$ and an immersion $i^* : T^{*k}M \to T^{*k}M$ given locally by the equations (9.1.3), (9.1.5). For a point $u^* = (x, y^{(1)}, ..., y^{(k-1)}, p) \in T^{*k}M$ and a point $\tilde{u} = (u, v^{(1)}, ..., v^{(k-1)}, \tilde{p}) \in T^{*k}M$ with the property $u^* = i^*(\tilde{u})$ we obtain the restriction of the fundamental function $H$ expressed by

\[ H \circ i^*(\tilde{u}) = H(\tilde{u}), \forall \tilde{u} \in T^{*k}M. \]

It follows that $H(\tilde{u})$ is a differentiable Hamiltonian on the submanifold $T^{*k}M$.

Consider the $d$-tensor field

\[ (9.2.2) \quad \tilde{g}^{\alpha\beta} = \frac{1}{2} \partial^\alpha \partial^\beta H \text{ on } T^{*k}M. \]

From (9.2.1) we deduce that at the points $\tilde{u} \in T^{*k}M$ the fundamental tensor field $\tilde{g}^{\alpha\beta}$ is given by

\[ \tilde{g}^{\alpha\beta}(\tilde{u}) = A^\alpha_i(u)A^\beta_j(u)g^{ij}(i^* \tilde{u}). \]

It follows from previous equality that

\[ (9.2.3a) \quad \text{rang } \tilde{g}^{\alpha\beta} = m. \]

Therefore we have:

**Theorem 9.2.1** The pair $H^{(k)m} = (M, H)$ is a Hamilton space of order $k$.

Indeed, this property is a consequences of the equations (9.2.1), (9.2.2) and (9.2.3) and the fact that $A^\alpha_i(u^1, ..., u^m)$ is a mixed $d$-tensor, with rank $\|A^\alpha_i\| = m$.

The space $H^{(k)m}$ will be called the Hamilton subspace of the Hamilton space $H^{(k)n} = (M, H)$. 
In the point \( u \in U \subset T^k M \), the \( d \)-covector fields \( A^\alpha_i \) (\( \alpha = 1, \ldots, m \)) are defined by \( A^\alpha_i (\tilde{u}) = A^\alpha_i (u) \). So \( A^\alpha_i (\tilde{u}), \ldots, A^\alpha_n (\tilde{u}) \) are linearly independent. They are called tangent covectors to \( T^k M \). A \( d \)-covector \( \omega_i (u^*) \), \( u^* \in T^k M \) is called normal to \( T^k M \) if \( u^* = i^* (\tilde{u}) \) and
\[
g^{ij} (i^* (\tilde{u})) A^\alpha_i (\tilde{u}) \omega_j (i^* (\tilde{u})) = 0, \quad \forall \ u \in U \subset T^k M.
\]

Then we can determine a Darboux coframe \( R^* \), at every point \( \tilde{u} \in U \), of the form
\[
R^* = \{ \tilde{u}, A^1_i (\tilde{u}), \ldots, A^m_i (\tilde{u}); A^\alpha_1 (\tilde{u}), \ldots, A^\alpha_m (\tilde{u}) \}
\]
formed by \( m \)-tangent \( d \)-covectors \( A^\alpha_i (\tilde{u}) \) and by \( n - m \) normal unit \( d \)-covectors \( A^\alpha_i (\tilde{u}) \), \(( \alpha = 1, \ldots, n - m )\), which verify the conditions:
\[
g^{ij} (i^* (\tilde{u})) A^\alpha_i (\tilde{u}) A^\alpha_j (\tilde{u}) = 0, \quad (9.2.4a)
\]
\[
g^{ij} (i^* (\tilde{u})) A^\alpha_i (\tilde{u}) A^\alpha_j (\tilde{u}) = \delta^\alpha_j \quad \forall \ u \in U \quad (9.2.4b).
\]

Of course \( R^* \) exists and it has a geometrical meaning with respect to the change of local coordinates on the manifold \( T^k M \) and \( T^k M \) with respect to the orthogonal (transformation of the normal \( d \)-covectors \( A^\alpha_i \)) given by
\[
A^\alpha_i (\tilde{u}) = C^\alpha_i (\tilde{u}) \quad (9.2.5)
\]
the dual frame \( R \) of the coframe \( R^* \) is given by
\[
R = \{ \tilde{u}, A^\alpha_i (\tilde{u}), A^\alpha_\alpha (\tilde{u}) \}, \quad \forall \ u \in U \subset T^k M.
\]

\( R^* \) is called a moving coframe on the submanifold \( T^k M \). The dual frame \( R \) of the coframe \( R^* \) is given by
\[
R = \{ \tilde{u}, A^\alpha_i (\tilde{u}), A^\alpha_\alpha (\tilde{u}) \}, \quad \forall \ u \in U \subset T^k M.
\]

\( R^* \) is called a moving coframe on the submanifold \( T^k M \). The dual frame \( R \) of the coframe \( R^* \) is given by
\[
A^\alpha_i (\tilde{u}) = C^\alpha_i (\tilde{u}), \quad \forall \ u \in U \subset T^k M.
\]

\( R^* \) is called a moving coframe on the submanifold \( T^k M \). The dual frame \( R \) of the coframe \( R^* \) is given by
\[
R = \{ \tilde{u}, A^\alpha_i (\tilde{u}), A^\alpha_\alpha (\tilde{u}) \}, \quad \forall \ u \in U \subset T^k M.
\]

\( R^* \) is called a moving coframe on the submanifold \( T^k M \). The dual frame \( R \) of the coframe \( R^* \) is given by
\[
A^\alpha_i (\tilde{u}) = C^\alpha_i (\tilde{u}), \quad \forall \ u \in U \subset T^k M.
\]

\( R^* \) is called a moving coframe on the submanifold \( T^k M \). The dual frame \( R \) of the coframe \( R^* \) is given by
\[
R = \{ \tilde{u}, A^\alpha_i (\tilde{u}), A^\alpha_\alpha (\tilde{u}) \}, \quad \forall \ u \in U \subset T^k M.
\]

\( R^* \) is called a moving coframe on the submanifold \( T^k M \). The dual frame \( R \) of the coframe \( R^* \) is given by
\[
A^\alpha_i (\tilde{u}) = C^\alpha_i (\tilde{u}), \quad \forall \ u \in U \subset T^k M.
\]

\( R^* \) is called a moving coframe on the submanifold \( T^k M \). The dual frame \( R \) of the coframe \( R^* \) is given by
\[
R = \{ \tilde{u}, A^\alpha_i (\tilde{u}), A^\alpha_\alpha (\tilde{u}) \}, \quad \forall \ u \in U \subset T^k M.
\]
Indeed, \( g^{\alpha\beta} A^i_\beta = g^{rs} A^r_\alpha A^s_\beta = g^{rs} (\delta^r_i - A^r_\alpha A^i_\beta) = g^{ij} A^i_\beta \).

In the following investigation we use the Darboux frame \( R \) in order to represent in \( R \) the \( d \)-tensor from the Hamilton spaces \( H^{(k)n} \).

We get:

**Proposition 9.2.1** The fundamental tensor \( g^{ij} \) of the Hamilton space \( H^{(k)n} \) and its covariant \( g_{ij} \) are represented in the Darboux frame \( R \) by

\[
\begin{align*}
\left\{
\begin{array}{l}
g^{ij} = g^{\alpha\beta} A^i_\alpha A^j_\beta + \delta^{ij} A^r_\beta A^s_\alpha A^r_\alpha A^s_\beta \\
\end{array}
\right.
\tag{9.2.8}
\end{align*}
\]

Indeed, the formulas (9.2.8) are consequence of (9.2.7)

The moving frame \( R \) will be used in the next section in order to derive the Gauss-Weingarden formulae and Gauss-Codazzi equations of the Hamilton subspaces \( H^{(k)m} \) in \( H^{(k)n} \).

### 9.3 Induced Nonlinear Connection

Now, let us consider the canonical nonlinear connection \( N \) of the Hamilton space of order \( k, H^{(k)n} = (M, H). \) \( N \) has the dual coefficients \( \{ M^i_1, \ldots, M^{(k-1)}_i, N_{ij} \}. \) We shall prove that the restriction of \( N \) to the Hamilton subspaces \( \widetilde{H}^{(k)m} = (\tilde{M}, \tilde{H}) \) determines an induced nonlinear connection \( \tilde{N} \) on the manifold \( T^{*k} \tilde{M} \).

The nonlinear connection \( \tilde{N} \) is well determined by its dual coefficients or by means of its adapted cobasis \( (du^\alpha, \delta v^{(1)\alpha}, \ldots, \delta v^{(k-1)\alpha}, \delta \tilde{p}^\alpha). \)

**Definition 9.3.1** A nonlinear connection \( \tilde{N} \) is called **induced** by the canonical nonlinear connection \( \nabla \) if the following conditions hold:

\[
\left\{
\begin{align*}
\delta v^{(1)\alpha} = A^\alpha_i \delta y^{(1)i}, \ldots, \delta v^{(k-1)\alpha} = A^\alpha_i \delta y^{(k-1)i} \\
\delta \tilde{p}^\alpha = B^i_\alpha \delta \tilde{p}_i,
\end{align*}
\right.
\tag{9.3.1}
\]

The previous conditions have a geometrical meaning. In fact, to a change of coordinates on \( T^{*k} \tilde{M} \) the previous formulae are preserved. Tha same happens if we change the coordinates on the manifold \( T^{*k} \tilde{M} \).

**Theorem 9.3.1** The dual coefficients of the induced nonlinear connection \( \tilde{N} \) are given by the following formulae

\[
\begin{align*}
\nabla^\alpha_j = A^\alpha_i M^i_j, \quad \nabla^\alpha_j = A^\alpha_i M^i_j, \quad N_{ij} = N_{ai} B^i_j,
\end{align*}
\tag{9.3.2}
\]

Indeed, the formulas (9.3.2) are consequence of (9.3.1)
where

\[(9.3.3)\]

\[
\begin{align*}
\nabla M^i_{\beta} &= M^j_{(1)} \frac{\partial y^{(1)j}}{\partial u^{(1)\beta}} + \frac{\partial y^{(1)i}}{\partial u^\beta}, \\
\nabla M^i_{\beta} &= M^j_{(2)} \frac{\partial y^{(2)j}}{\partial u^{(2)\beta}} + M^j_{(1)} \frac{\partial y^{(2)i}}{\partial u^{(1)\beta}} + \frac{\partial y^{(2)i}}{\partial u^\beta}, \\
\cdots &
\end{align*}
\]

\[
\begin{align*}
\nabla M^i_{\beta} &= M^j_{(k-1)} \frac{\partial y^{(k-1)j}}{\partial u^{(k-1)\beta}} + M^j_{(k-2)} \frac{\partial y^{(k-2)j}}{\partial u^{(k-2)\beta}} + \cdots + M^j_{(1)} \frac{\partial y^{(1)j}}{\partial u^{(1)\beta}} + \frac{\partial y^{(k-1)i}}{\partial u^\alpha} \\
\end{align*}
\]

and

\[(9.3.4)\]

\[
\begin{align*}
\nabla N^i_{\alpha} &= N_{ji} B^j_{\alpha} - \frac{\partial p_i}{\partial u^\alpha}.
\end{align*}
\]

**Proof.** The first equations (9.3.1) lead to

\[
\begin{align*}
\nabla M^i_{\beta} &= M^j_{(1)} \frac{\partial y^{(1)j}}{\partial u^{(1)\beta}} + \frac{\partial y^{(1)i}}{\partial u^\beta}, \\
\nabla M^i_{\beta} &= M^j_{(2)} \frac{\partial y^{(2)j}}{\partial u^{(2)\beta}} + M^j_{(1)} \frac{\partial y^{(2)i}}{\partial u^{(1)\beta}} + \frac{\partial y^{(2)i}}{\partial u^\beta}, \\
\cdots &
\end{align*}
\]

Because of \(\frac{\partial y^{(1)i}}{\partial u^{(1)\alpha}} = B^i_{\alpha}\) and \(A^i_{\alpha} B^j_{\beta} = \delta^i_{\beta}\) the last equality leads to (9.3.2):

\[
\begin{align*}
\nabla M^i_{\beta} &= A^i_{\alpha} \frac{\partial y^{(1)i}}{\partial u^{(1)\beta}}, \\
\end{align*}
\]

Remark that

\[(9.3.5)\]

\[
\begin{align*}
\delta y_{(a)}i &= dy^{(a)i} + M^j_{(a-1)} \frac{\partial y^{(a-1)i}}{\partial u^{(1)\beta}} + \cdots + M^j_{(a)} \frac{\partial y^{(a)i}}{\partial u^{(1)\beta}}, (a = 1, \ldots, k-1)
\end{align*}
\]

and

\[(9.3.5a)\]

\[
\begin{align*}
dy^{(a)i} &= \frac{\partial y^{(a)i}}{\partial u^{(a)\alpha}} du^\alpha + \cdots + \frac{\partial y^{(a)i}}{\partial u^{(a-1)\alpha}} \frac{\partial y^{(a-1)i}}{\partial u^{(1)\beta}} + B^i_{\alpha} dv^{(a)\alpha}.
\end{align*}
\]

We can prove:

**Proposition 9.3.1** The following formulae hold:

\[
\begin{align*}
dx^i &= B^i_{\beta} du^\beta, \\
\delta y_{(1)i} &= B^i_{\beta} \delta v^{(1)\beta} + A^i_{\alpha} H^\beta_{\alpha} du^\beta, \\
\delta y_{(k-1)i} &= B^i_{\beta} \delta v^{(k-1)\beta} + \cdots + H^\beta_{\alpha} \frac{\partial y^{(k-1)i}}{\partial u^{(1)\beta}}, \\
\delta p_i &= A^i_{\beta} \delta p^\beta - A^\gamma_{\beta} H^\gamma_{\alpha} du^\beta,
\end{align*}
\]
where

\[ A_{i}^{\alpha}H_{\beta} = M_{i}^{\alpha} - B_{i}^{\alpha} M_{\beta}^{\alpha}; \ldots; A_{i}^{k}H_{\beta} = M_{i}^{k} - B_{i}^{k} M_{\beta}^{k}, \]

(9.3.7a)

\[ A_{i}^{k}H_{\beta} = N_{\beta i}. \]

**Proof.** The formulae (9.3.5), (9.3.5') and (9.3.3) have as consequence the formulae (9.3.6). Also, we have

\[ A_{i}^{k}H_{\beta} = -A_{i}^{k}p_{i} = N_{\beta i}. \quad q.e.d. \]

The previous expressions (9.3.6) is not convenient for us, because in the right hand sides we have the natural cobasis \( du^{\alpha}, dv^{(1)i}, \ldots, dv^{(k-2)i}. \) This means that the corresponding coefficients \( A_{i}^{k}H_{\beta}, (a = 1, \ldots, k-1) \) do not have a geometrical meaning.

So, we prove:

**Theorem 9.3.2** In every point \( u \in T^{*k}M \) the following formulas hold:

(9.3.8)

\[ dx^{i} = B_{i}^{\beta} du^{\beta}, \]

\[ \delta y^{(1)i} = B_{i}^{\beta} \delta y^{(1)\beta} + A_{i}^{\alpha}K_{\beta}^{\alpha} du^{\beta}, \]

\[ \delta y^{(2)i} = B_{i}^{\beta} \delta y^{(2)\beta} + A_{i}^{\alpha}(K_{\beta}^{\alpha} \delta y^{(1)\beta} + K_{\beta}^{\alpha} du^{\beta}), \]

\[ \delta y^{(k-1)i} = B_{i}^{\beta} \delta v^{(k-1)\beta} + A_{i}^{\alpha}(K_{\beta}^{\alpha} \delta v^{(k-2)\beta} + K_{\beta}^{\alpha} \delta v^{(k-3)\beta} + \ldots + K_{\beta}^{\alpha} du^{\beta}) \]

and

\[ \delta p_{i} = A_{i}^{\beta} \delta p_{\beta} - \bar{A}_{i}^{k}H_{\beta} du^{\beta}, \]

where \( \bar{A}_{i}^{k}H_{\beta} \) is given by (9.3.7') and

(9.3.9)

\[ K_{\beta}^{\alpha} = H_{\beta}^{\alpha}, \]

\[ K_{\beta}^{k} = H_{\beta}^{k} - H_{\gamma}^{k} N_{\gamma}^{\beta}, \]

\[ K_{\beta}^{k} = H_{\beta}^{k} - H_{\gamma}^{k} N_{\gamma}^{\beta} - \ldots - H_{\gamma}^{k} N_{\gamma}^{\beta} \]

Of course, \( \bar{N}_{\beta}^{\gamma}, \ldots, \bar{N}_{\beta}^{\gamma} \) are the coefficients of the induced connection \( \bar{N}. \)
Proof. Taking into account the formulas (6.3.5) we have
\[ dv^{(a)\alpha} = \delta v^{(a)\alpha} - N_\beta^\alpha \delta v^{(a-1)\beta} - \ldots - N_\beta^\alpha \delta u^\beta, \quad (a = 1, \ldots, k-1). \]

So, \( \delta y^{(a)i} \) from (9.3.6) is as follows
\[ \delta y^{(a)i} = B_i^\alpha \delta v^{(a-1)\gamma} - N_\beta^\alpha \delta v^{(a-2)\gamma} - \ldots - N_\beta^\gamma \delta u^\gamma \]
\[ + H_\beta^\gamma \delta v^{(a-2)\beta} - N_\beta^\gamma \delta v^{(a-3)\beta} - \ldots - N_\gamma^\beta \delta u^\beta + \ldots + \]
\[ + H_\gamma^\beta \delta v^{(a-3)\gamma} + \ldots + H_\gamma^\gamma \delta v^{(a-1)\gamma} \]
\[ + H_\gamma^\gamma \delta v^{(a)\gamma} \]

Identifying with \( \delta y^{(a)i} \) from (9.3.6):
\[ \delta y^{(a)i} = B_i^\alpha \delta v^{(a-1)\gamma} + A_i^\alpha \delta v^{(a-2)\beta} + \ldots + K_\beta^\gamma \delta v^{(a-1)\beta} \]
we obtain the formulae (9.3.9), q.e.d.

The previous theorem has an important consequence.

Corollary 9.3.1 With respect to the transformations of coordinates on the submanifold \( T^*k_\gamma M \) and to the transformation (9.2.5), \( K_\beta^\gamma, \ldots, K_\beta^\gamma \) and \( H_\beta^\alpha \) are the mixed \( d \)-tensor fields.

Generally a set of functions \( T^{i_1\ldots i_m}_{\ldots \beta \ldots \beta}(u) \) which are the components of a \( d \)-tensor in the indices \( i, j, \ldots \) and \( d \)-tensor in the indices \( \alpha, \beta, \ldots \) and tensor with respect to the transformation (9.2.5) in the indices \( \alpha, \beta, \ldots \) is called a mixed \( d \)-tensor field on the submanifolds \( T^*k_\gamma M \).

For instance \( B_i^\alpha, A_i^\alpha, A_i^\beta, \delta \gamma^\beta, g_{\alpha \beta}, g_{ij}, \delta \tau_{\beta \gamma}, K_\beta^\gamma, \ldots, K_\beta^\gamma \) and \( H_\beta^\alpha \) are mixed \( d \)-tensor fields. The previous definition of mixed \( d \)-tensor fields can be extended to any geometrical \( d \)-object fields.

9.4 The Relative Covariant Derivative

In this section we shall construct an operator \( \nabla \) of relative covariant derivation in the algebra of mixed \( d \)-tensor fields. It is clear that \( \nabla \) will be well determined if we know its action on the mixed \( d \)-vector field:

\[ (9.4.1) \quad X^{(a)}(\hat{u}), X^{(a)}(\hat{u}), X^{(a)}(\hat{u}), \forall \hat{u} = (u, v^{(1)}, \ldots, v^{(k-1)}, \hat{p}) \in T^*k_\gamma M. \]

Definition 9.4.1 We call a coupling of the canonical metrical \( N \)-connection \( D \) of the Hamilton spaces of order \( k \), \( H^{(k)\alpha} = (M, H) \) to the induced nonlinear
connection \( \nabla \), of the submanifold \( \nabla \), in the manifold \( M \) the operator \( \nabla \) with the property

\[
(9.4.2) \quad \nabla X^i = DX^i \text{ modulo } \{(9.3.8), (9.3.8')\},
\]

where

\[
(9.4.2a) \quad DX^i = dx^i + X^i \omega^i_j
\]

and where \( \omega^i_j \) are 1-forms connection:

\[
(9.4.2b) \quad \omega^i_j = H^i_{jh}dx^h + C^i_{jh\delta y}^{(1)h} + \ldots + C^i_{jh\delta y}^{(k-1)h} + C^i_{jh\delta y}^{(k)h}.
\]

Thus, in every point \( u \in T^k M \), \( \nabla X^i \) has the form

\[
(9.4.3) \quad \nabla X^i = dx^i + X^i \omega^i_j,
\]

where

\[
(9.4.4) \quad \omega^i_j = H^i_{jh\delta y}^{(1)h} + \ldots + C^i_{jh\delta y}^{(k-1)h} + C^i_{jh\delta y}^{(k)h}.
\]

**Theorem 9.4.1** The connection one forms \( \omega^i_j \) of \( \nabla \) are given by (9.4.4), where:

\[
H^i_{jh\delta y} = H^i_{jh\delta y}^{(1)h} + \ldots + C^i_{jh\delta y}^{(k-1)h} + C^i_{jh\delta y}^{(k)h}.
\]

\[
C^i_{jh\delta y} = C^i_{jh\delta y}^{(1)h} + \ldots + C^i_{jh\delta y}^{(k-1)h} + C^i_{jh\delta y}^{(k)h}.
\]

\[
(9.4.5)
\]

**Proof.** Using the formulae (9.3.8), (9.3.8'), \( \omega^i_j \) modulo (9.3.8), (9.3.8') from (9.4.2') leads to

\[
(9.4.4a) \quad \omega^i_j = H^i_{jh\delta y}^{(1)h} + \ldots + C^i_{jh\delta y}^{(1)h} + A^h_{\beta}K^\beta_{\delta y}^{(1)h}du^\beta + \ldots + \]

\[
C^i_{jh\delta y}^{(1)h} + A^h_{\beta}K^\beta_{\delta y}^{(1)h}du^\beta + \ldots + K^\beta_{\delta y}^{(1)h}du^\beta] +
\]

\[
C^i_{jh\delta y}^{(1)h} + A^h_{\beta}K^\beta_{\delta y}^{(1)h}du^\beta],
\]
Identifying with \( \omega^i_j \) from (9.4.4) we obtain the formulae (9.4.5). Evidently the coupling connection \( \hat{D} \) of the canonical metrical \( N \)-connection \( D \), depend only on the fundamental function \( H \) of the space \( H^{(k)n} \) and by the immersion \( i^*: T^k \hat{M} \rightarrow T^k M \).

Of course, we can write \( \hat{D} X^i \) in the form

\[
D^\alpha X^\alpha = A^\alpha_i \hat{D} X^i, \quad \text{for} \quad X^i = A^i_\alpha X^\alpha
\]

We can extend, without difficulties the action of the linear connection \( \hat{D} \) to the \( d \)-tensors \( T^i_{ji...j_a}(\hat{u}), \forall \hat{u} \in T^k \hat{M} \).

**Definition 9.4.2** We call the induced tangent connection on \( T^k \hat{M} \) by the canonical metrical \( N \)-connection \( D \) the operator \( D^T \) given by

\[
D^T X^\alpha = A^\alpha_i \hat{D} X^i, \quad \text{for} \quad X^i = A^i_\alpha X^\alpha
\]

We can see that:

\[
D^T X^\alpha = dX^\alpha + X^\delta \omega^\alpha_\delta,
\]

where \( \omega^\alpha_\delta \) are the connection one forms of \( D^T \):

\[
\omega^\alpha_\delta = H^\alpha_\delta \gamma du^\gamma + C^\alpha_\delta^a \delta v^{(a)}(1)^\gamma + \ldots + C^\alpha_\delta^{(k-1)} \gamma\delta v^{(k-1)}(1)^\gamma + C_{\delta \gamma}^\alpha \gamma \delta \hat{p}^\gamma
\]

**Theorem 9.4.2** 1° The coefficients \( (H^\alpha_\delta \gamma, C^\alpha_\delta^a, C^\alpha_\delta^{(1)} \gamma, \ldots, C^\alpha_\delta^{(k-1)} \gamma) \) of the one forms connection \( \omega^\alpha_\delta \) have the following expressions

\[
H^\alpha_\delta \gamma = A^\alpha_i \left( \frac{\delta A^i_\delta}{\delta u^\gamma} + A^i_\beta H^i_\beta \gamma \right),
\]

\[
C^\alpha_\delta^a \gamma = A^\alpha_i \left( \frac{\delta A^i_\delta}{\delta v^{(a)}(1)^\gamma} + A^h_\beta C^\delta_\gamma \gamma \right), \quad (a = 1, \ldots, k - 1),
\]

\[
C^\alpha_\delta^{(k-1)} \gamma = A^\alpha_i (\hat{\partial}^{\gamma} A^i_\beta + A^h_\beta C^i_\gamma h),
\]
Subspaces in Hamilton Spaces of Order $k$

$2^\circ$ With respect of the change of local coordinates on the manifold $T^k \overset{\nabla}{M}$, the coefficients

$$H_{\beta \gamma}^\alpha, C_{\beta \gamma}^\alpha, \ldots, C_{\beta \gamma}^{(k-1)}$$

have the rule of transformation as the coefficients of $\nabla$-linear connection on $T^k \overset{\nabla}{M}$.

**Proof.** $1^\circ$ The equalities (9.4.8) and (9.4.9) imply

$$dX^\alpha + X^\beta \omega^\alpha_\beta = A^\alpha_i (dX^i + X^j \omega^i_\beta) = A^\alpha_i [(dA^\beta_i + A^h_\beta \omega^i_h)X^\beta + A^\beta_i dx^\beta]$$

Consequently,

(9.4.12) $$\omega^\alpha_\beta = A^\alpha_i (dA^\beta_i + \omega^i_j A^\beta_j).$$

Thus, the equalities (9.4.4), (9.4.10) and (9.4.12) lead to the form (9.4.11) of the coefficients of induced tangent connection $D^T$.

$2^\circ$ Using the rule of transformations of the coefficients of $\omega^i_j$ it follows the sentence $2^\circ$ of theorem. q.e.d.

As in the case of $\nabla$ we can write:

(9.4.13) $$D^T X^\alpha = X^\alpha_\beta du^\beta + X^\alpha_\beta |^{(1)} \delta u^{(1)} \beta + \ldots + X^\alpha_\beta |^{(k-1)} \delta u^{(k-1)} \beta + X^\alpha_\beta |^{(k-1)} \delta \nabla_{\beta},$$

where

(9.4.14) $$X^\alpha_\beta |^{(a)} \gamma = \frac{\delta X^\alpha_\beta}{\delta u^{(a)} \gamma} + X^\beta C^\alpha_\beta_\gamma, (a = 1, \ldots, k-1),$$

$$X^\alpha_\beta |^{(a)} \gamma = \partial^\gamma X^\alpha_\beta + X^\beta C^\alpha_\beta_\gamma.$$
As before we set
\[(9.4.16)\]
\[D^\perp X^\alpha = dX^\alpha + X^\beta \omega^\alpha_{\beta j},\]
with the connection 1-forms of \(D^\perp\).
\[(9.4.17)\]
\[\omega^\alpha_{\beta j} = H^\alpha_{\beta j} du^j + C^\alpha_{\beta j} (1) \delta u^{(1)\gamma} + \ldots + C^\alpha_{\beta j} (k-1) \delta u^{(k-1)\gamma} + C^\alpha_{\beta j} \delta \bar{v}^\gamma .\]

Applying the same method as in the case of \(\bar{\nabla}\) and \(D_T\) we can calculate the coefficients of the 1-forms connection \(\omega^\alpha_{\beta j}\).

**Theorem 9.4.3** The coefficients \((H^\alpha_{\beta j}, C^\alpha_{\beta j}, \ldots, C^\alpha_{\beta j}, C^\alpha_{\beta j})\) of the induced normal connection \(D^\perp\) are given by
\[(9.4.18)\]
\[H^\alpha_{\beta j} = A^i_\alpha \left( \frac{\delta A^i_{\beta j}}{\delta u^j} + A^i_\beta \bar{\nabla}^j \right),\]
\[C^\alpha_{\beta j} (a) = A^i_\alpha \left( \frac{\delta A^i_{\beta j}}{\delta u^{(a)\gamma}} + A^i_{\beta} \bar{\nabla}^{(a)} \right), (a = 1, \ldots, k-1),\]
\[C^\alpha_{\beta j} = A^i_\alpha \left( \partial^\gamma A^i_{\beta j} + A^i_{\beta} \bar{\nabla}^{\gamma} \right).\]

**Proof.** Indeed, \((9.4.15)\) and \((9.4.16)\) have as consequence:
\[(9.4.19)\]
\[\bar{\omega}^\alpha_{\beta j} = A^i_\alpha \left( dA^i_{\beta j} + A^i_{\beta} \bar{\nabla}^j \right).\]

Therefore, \(\bar{\omega}^i_j\) from \((9.4.4)\) and \(\omega^\alpha_{\beta j}\) expressed in \((9.4.17)\) substituted in the equality \((9.4.19)\) leads to \((9.4.18)\), q.e.d.

As usual, we may set for the induced normal connection
\[(9.4.20)\]
\[D^\perp X^\alpha = X^\alpha_{\beta} du^\beta + X^\alpha_{\beta} (1) \delta u^{(1)\beta} + \ldots + X^\alpha_{\beta} (k-1) \delta u^{(k-1)\beta} + X^\alpha_{\beta} \delta \bar{v}^\beta .\]
\[
X^\alpha_{\beta} = \frac{\delta X^\alpha}{\delta u^\beta} + X^\gamma H^\alpha_{\beta j},
\]
\[(9.4.21)\]
\[X^\alpha_{\beta} (a) = \frac{\delta X^\alpha}{\delta u^{(a)\beta}}, (a = 1, \ldots, k-1),\]
\[X^\alpha_{\beta} = \partial^\beta X^\alpha + X^\gamma C^\alpha_{\beta j}.\]

These are the induced normal covariant derivations.

Now, we can define the relative (or mixed) derivation \(\nabla\) introduced at the beginning of this section.
Subspaces in Hamilton Spaces of Order $k$

**Definition 9.4.4** A relative (mixed) covariant derivation in the algebra of mixed $d$-tensor field is an operator $\nabla$ which has the following properties.

\[
\nabla f = df, \forall f \in \mathcal{F}(T^*kM)
\]

(9.4.22)

\[
\nabla X^i = \mathring{D} X^i, \nabla X^\alpha = D^TX^\alpha, \nabla X^\nabla = D^\perp X^\nabla
\]

for any mixed vector fields $X^i(\nabla), X^\alpha(\nabla)$ and $X^\nabla(\nabla)$.

The connection 1-forms $\nabla \omega^i_j, \omega_\alpha^\beta$ and $\omega^\nabla_\beta$ will be called the **connections 1-forms** of the relative covariant derivation $\nabla$. Of course, the operator $\nabla$ can be extended to any mixed $d$-tensor

\[
T^i...\alpha...\alpha^j...\beta...
\]

\[
(\nabla u), (\nabla u) \in T^*kM.
\]

We can write the Ricci identities for the relative covariant derivative $\nabla$, and its torsion and curvature tensor, taking into account every components $\mathring{D}, D^T, \text{and } D^\perp$ of $\nabla$.

Then, it is easy to prove:

**Theorem 9.4.4** The structure equations of the mixed covariant derivation $\nabla$ are as follows:

\[
d(\omega^\alpha_\beta) - \delta \omega^\alpha_\beta \wedge \omega^\beta_\gamma = -\Omega^\alpha_\beta,
\]

(9.4.23)

\[
d(\omega^{(a)\beta}) - \delta \omega^{(a)\beta} \wedge \omega^\beta_\gamma = -\Omega^{(a)}_\beta, \quad (\alpha = 1, ..., k-1),
\]

\[
d(\delta \mathring{\omega}_\alpha) - \delta \mathring{\omega}_\beta \wedge \omega^\beta_\gamma = -\Omega_\alpha,
\]

and

\[
d(\omega^i_j) - \omega^i_j \wedge \omega^j_h = -\Omega^i_j,
\]

(9.4.24)

\[
d(\omega^\nabla_\beta) - \omega^\nabla_\beta \wedge \omega^\nabla_\gamma = -\Omega^\nabla_\beta,
\]

\[
d(\omega^\nabla_\beta) - \omega^\nabla_\beta \wedge \omega^\nabla_\gamma = -\Omega^\nabla_\beta,
\]

in which $\Omega^\alpha_\beta, \Omega^\alpha_\beta$ and $\Omega_\alpha$ are the 2-forms of torsion:

(9.4.25)

\[
\Omega^{(0)}_\alpha = du^\beta \wedge \left\{ \frac{1}{2} T^\beta_\gamma du^\gamma + C^\beta_\gamma \delta u^{(1)\gamma} + \ldots + C^\beta_\gamma \delta u^{(k-1)\gamma} + C^\gamma_\beta \mathring{\delta} P^\gamma \right\},
\]

\[
\Omega^{(a)}_\alpha = du^\beta \wedge P^\beta_\alpha + \sum_{b=1}^{k-1} \delta u^{(b)\beta} \wedge P^\beta_\alpha + \delta u^{(a)\beta} \wedge \left\{ H^\beta_\gamma du^\gamma + \sum_{b=1}^{k-1} C^\alpha_\beta \delta u^{(b)\gamma} + C^\alpha_\beta \mathring{\delta} P^\gamma \right\}, \quad (a = 1, ..., k-1),
\]

(9.4.26)
with \( P^\alpha_\beta \) from (5.4.8) and where the 2-forms of curvature are given by

\[ \Omega^i_j = \frac{1}{2} R^i_j \alpha \beta \alpha \beta (a) \left( \sum_{a=1}^{k-1} P^i_j (a) \alpha \beta \right) \]

\[ + \sum_{a=1}^{k-1} S^i_j (a) \alpha \beta \alpha \beta (a) \left( \sum_{a=1}^{k-1} S^i_j (a) \alpha \beta \right) \]

\[ \Omega^\alpha \beta = \frac{1}{2} R^\alpha \beta \gamma \phi \alpha \beta \gamma \phi + \sum_{a=1}^{k-1} P^\alpha \beta \gamma \phi \alpha \beta \gamma \phi \]

\[ + \sum_{a=1}^{k-1} S^\alpha \beta \gamma \phi \alpha \beta \gamma \phi + \sum_{a=1}^{k-1} S^\alpha \beta \gamma \phi \alpha \beta \gamma \phi \]

\[ \Omega^\alpha \beta = \frac{1}{2} R^\alpha \beta \gamma \phi \alpha \beta \gamma \phi + \sum_{a=1}^{k-1} P^\alpha \beta \gamma \phi \alpha \beta \gamma \phi \]

\[ + \sum_{a=1}^{k-1} S^\alpha \beta \gamma \phi \alpha \beta \gamma \phi + \sum_{a=1}^{k-1} S^\alpha \beta \gamma \phi \alpha \beta \gamma \phi . \]

The previous equations can be particularized in the case when the induced tangent connection \( D_T \) has vanishing tensors of torsion \( T^\alpha \beta \gamma \), \( S^\alpha \beta \gamma \), and \( S^\alpha \gamma \beta \).

In the following we shall adopt the notations

\[ \Omega^i_j = \Omega^i_j, \quad \Omega^\alpha \beta = \Omega^\alpha \beta, \quad \Omega^\alpha \beta = \Omega^\alpha \beta . \]

These covariant 2-forms of curvature of the mixed covariant connection \( \nabla \) will be useful to write the fundamental equations of the immersion \( i^* : T^k M \rightarrow T^k M \).

As a direct consequence of the Theorems from the present section we get:

**Theorem 9.4.5** The mixed covariant derivation \( \nabla \) is a metrical one, i.e.

\[ \nabla g^{ij} = 0, \quad \nabla \delta^{ij} = 0, \quad \nabla \gamma^{\alpha \beta} = 0. \]

**Proof.** The first formula is immediate, because of \( \nabla g^{ij} = \delta^{ij} = 0 \) while

\[ \nabla \delta^{ij} = 0 \quad \nabla \gamma^{\alpha \beta} = 0 \]

can be proved by means of Gauss-Weingarten formulas, given in next section.
9.5 The Gauss-Weingarten Formula

We need to study the moving equations of the Darboux frame:

\[(9.5.1) \quad \mathbf{R} = \{ \overset{\cdot}{u}, A_i^\alpha(\overset{\cdot}{u}), A_i^\alpha(\overset{\cdot}{\omega}) \}, \forall \overset{\cdot}{u} \in T^*kM .\]

So, we obtain

**Theorem 9.5.1** *The following Gauss-Weingarten formulae hold good:*

\[(9.5.2) \quad \nabla A_i^\alpha = A_i^\alpha \Pi^\alpha_\beta, \quad \nabla A_i^\beta = -A_i^\alpha \Pi^\alpha_\beta,
\]

where

\[(9.5.3) \quad \Pi^\alpha_\beta = \frac{\partial}{\partial \gamma} \Pi^\beta_\gamma,
\]

and where

\[(9.5.4) \quad \Pi^\beta_\gamma = A_i^\beta(\frac{\delta A_i^\beta}{\delta u^\gamma} + A_j^\beta H_j^\gamma),
\]

\[(9.5.5) \quad \Pi^\alpha_\beta = A_i^\beta(\frac{\delta A_i^\beta}{\delta u^\alpha} + A_j^\beta C_j^\alpha).
\]

with $H_j^\gamma$, $C_j^\gamma$ and $C_j^h^\gamma$ are from (9.4.5).

**Proof.** Taking into account that $A_i^\beta$ is a mixed $d$-tensor we have for its relative covariant derivation:

\[
\nabla A_i^\beta = dA_i^\beta + A_i^\alpha \omega_j^\beta - A_i^\beta \omega_j^\alpha = dA_i^\beta + A_i^\alpha \omega_j^\beta \frac{\partial}{\partial u^\alpha} - A_i^\beta A_j^\gamma(dA_j^\gamma + \omega_j^h A_j^h) = dA_i^\beta + A_i^\alpha \omega_j^\beta - \frac{\delta A_i^\beta}{\delta u^\alpha} A_j^\gamma + A_j^\beta C_j^\gamma A_i^\alpha.
\]

So, we get

\[(9.5.5) \quad \Pi^\alpha_\beta = A_i^\beta(dA_i^\alpha + \omega_j^h A_j^h).
\]

Remark that

\[
dA_i^\beta = \frac{\delta A_i^\beta}{\delta u^\alpha} du^\alpha + \frac{\delta A_i^\beta}{\delta u^{(1)\alpha}} du^{(1)\beta} + \frac{\delta A_i^\beta}{\delta u^{(k-1)\beta}} du^{(k-1)\beta} + \frac{\delta A_i^\beta}{\delta v^\alpha} A_j^\alpha \delta p_j.
\]
and $\omega_j^i$ is from (9.4.4) we obtain

$$
\Pi^\alpha_\beta = A^\alpha_i \left( \frac{\delta A^h}{\delta u^h} + A^i_\alpha H^{h}_{j\beta} \right) du^\beta + \left( \frac{\delta A^h}{\delta v^{(1)} h} + A^i_\alpha C^{h}_{j\beta} \right) \delta v^{(1)} \beta + ...
$$

$$
... + \left( \frac{\delta A^h}{\delta v^{(k-1)} h} + A^i_\alpha C^{h}_{j\beta} \right) \delta v^{(k-1)} \beta + \left( \frac{\partial^\beta}{\partial A^h} A^i_\alpha + A^i_\alpha C^{h}_{j\beta} \right) \delta v^\beta.
$$

The last expression of the 1-forms $\Pi^\alpha_\beta$ can be written as in (9.5.3) with the coefficients (9.5.4).

In order to prove the second formula (9.5.2) we remark, using the same method, the formula

$$(9.5.6) \nabla A^i_\alpha = A^i_\alpha [A^\alpha_j (dA^j_\beta + A^k_\alpha \omega^j_k)] = -A^i_\alpha \Pi^\alpha_\beta.$$ 

So, we have

$$(9.5.6a) \Pi^\alpha_\beta = -A^i_\alpha (dA^j_\alpha + A^k_\alpha \omega^j_k).$$

The second relation between the 1-forms $\Pi^T_\alpha$ and $\Pi^\alpha_\beta$ is proved without difficulties by means of the equations $\nabla g_{ij} = 0$, $\nabla \omega_{ij} = 0$ and $\nabla \delta u^i = 0$, which will be proved in the next Lemma. q.e.d

**Lemma 9.5.1** The following properties of the relative covariant derivation $\nabla$ hold:

$$(9.5.7) \nabla g_{ij} = 0, \nabla g_{\alpha\beta} = 0, \nabla \delta u^i = 0.$$ 

Evidently, the equation $\nabla g_{ij} = 0$ is immediate. After the formula (9.2.8) we have

$$(9.5.8) \nabla g_{\alpha\beta} = g_{ij} A^i_\alpha A^j_\beta, \delta u^i \Pi^\alpha_\beta = g_{ij} A^i_\alpha A^j_\beta.$$ 

So, using (9.5.2) and $\nabla g_{ij} = 0$, we obtain $\nabla g_{\alpha\beta} = 0, \nabla \delta u^i = 0$. q.e.d

**Remark 9.5.1**

1° Theorem 8.4.5 is a consequence of the previous Lemma.

2° Because of $g_{ij} A^i_\alpha A^j_\beta = 0$, applying the operator $\nabla$ we get by means of (9.5.2) that $g_{ij}(\Pi^T_\alpha A^i_\alpha \Pi^\alpha_\beta - A^i_\alpha A^j_\beta \Pi^\alpha_\beta) = 0$. But, this is $\delta \Pi^\alpha_\beta = g_{\alpha\beta} \Pi^\alpha_\beta$ which imply the second formula (9.5.3).

The formulae (9.5.4) allows to prove that the coefficients

$$(9.5.9) H_{(0)}^{\alpha\gamma}, H_{(1)}^{\alpha\gamma}, \ldots, H_{(k-1)}^{\alpha\gamma}, H_{(k-1)}^{\gamma}$$

are mixed $d$-tensor. They will be called the second fundamental tensors of the Hamilton subspace $H^{(k)} = (\hat{M}, \hat{H})$ of the Hamilton space $H^{(k)} = (M, H)$.

As an application of the previous considerations we get:
Proposition 9.5.1 We have

\[ (9.5.10) \quad \Omega_{ij} = -\Omega_{ji}, \Omega_{\alpha\beta} = -\Omega_{\beta\alpha}, \Omega_{\alpha\beta\gamma\delta} = -\Omega_{\beta\gamma\delta\alpha} \]

The proof follows the same way as for the covariant curvature 2-forms \( \Omega_{ij} \) of a metrical connection.

Let us consider a parametrized smooth curve \( \mathring{c} \) on the Hamilton subspace \( H^{(k)m} = (\mathring{M}, \mathring{H}) \). Locally \( \mathring{c} \) can be given by

\[ u^\alpha = u^\alpha(t), v^{(1)} = v^{(1)}(t), ..., v^{(k-1)} = v^{(k-1)}(t), \mathring{p}_\alpha = \mathring{p}_\alpha(t) \]

A tangent vector \( X^i \) along \( \mathring{c} \) is given by \( X^i = A^i_\alpha X^\alpha \).

But, along curve \( \mathring{c} \), we have

\[ (9.5.11) \quad \frac{\nabla X^i}{dt} = A^i_\alpha \Pi^\alpha_\beta X^\beta + A^i_\alpha \frac{\nabla X^\alpha}{dt} \]

\( \frac{\nabla X^\alpha}{dt} = 0 \) along \( \mathring{c} \), implies \( \frac{\nabla X^i}{dt} = A^i_\alpha \Pi^\alpha_\beta X^\beta \). This means that \( \frac{\nabla X^i}{dt} \) is normal to the subspace \( H^{(k)m} \).

We say that the subspace \( H^{(k)n} \) is totally geodesic in the space \( H^{(k)n} \) if along any curve \( \mathring{c} \), \( \frac{\nabla X^\alpha}{dt} = 0 \) implies \( \frac{\nabla X^i}{dt} = 0 \). The geometrical meaning of the condition is evident.

Theorem 9.5.2 The Hamilton subspaces of order \( k \), \( H^{(k)n} = (\mathring{M}, \mathring{H}) \) is totally geodesic in the Hamilton space of order \( k \), \( H^{(k)n} = (H, M) \) if and only if, the second fundamental tensors \( H^\beta_\alpha \gamma_0 , H^\beta_\alpha \gamma_1 , ..., H^\beta_\alpha \gamma_{(k-1)} \) vanish.

Proof. If the second fundamental tensors vanish then \( \Pi^\alpha_\beta \) identically vanish and by (9.5.11) it follows that \( \frac{\nabla X^\alpha}{dt} = 0 \) imply \( \frac{\nabla X^i}{dt} = 0 \), for any \( X^i = A^i_\alpha X^\alpha \).

Conversely, the condition \( \frac{\nabla X^\alpha}{dt} = 0 \Rightarrow \frac{\nabla X^i}{dt} = 0 \) for any \( X^i = A^i_\alpha X^\alpha \) and the formula (9.5.11) leads to \( \Pi^\alpha_\beta = 0 \). Taking into account (9.5.3) and the fact that \( \mathring{c} \) is arbitrary, it follows that all second fundamental tensors vanish.

9.6 The Gauss-Codazzi Equations

The Gauss-Codazzi equations of a Hamilton subspaces of order \( k \), \( H^{(k)m} = (\mathring{M}, \mathring{H}) \) in the Hamilton spaces of order \( k \), \( H^{(k)n} = (H, M) \) endowed with a canonical metrical \( N \)-connection \( D \), are obtained from the integrability conditions of the system of equalities (9.5.2). We can deduce these equations using the structure equations (9.4.23) and (9.4.24).
Theorem 9.6.1 The Gauss-Codazzi equations of the Hamilton subspaces $H^{(k)m}$ in the Hamilton space $H^{(k)n}$ are as follows:

\begin{align}
A^i_\alpha A^j_\beta \Omega_{ij} - \Omega_{\alpha\beta} &= \Pi_{\alpha\gamma} \wedge \Pi^\gamma_{\alpha}, \\
A^i_\alpha A^j_\beta \Omega_{ij} - \Omega_{\alpha\beta} &= \Pi_{\beta\gamma} \wedge \Pi^\gamma_{\beta}, \\
- A^i_\alpha A^j_\beta \Omega_{ij} &= \delta^i_{\alpha}(d\Pi^\gamma_{\alpha} + \Pi^\gamma_{\beta} \wedge \omega^\beta_{\alpha} - \Pi^\gamma_{\alpha} \wedge \omega^\gamma_{\beta}),
\end{align}

where $\Pi_{\alpha\beta} = g_{\alpha\gamma} \Pi^\gamma_{\beta}$.

Proof. Consider the first equations (9.5.2), written in the form

\begin{align}
\frac{\partial A^i_\alpha}{\partial x^j} + A^j_\alpha \omega^i_j - A^j_\beta \omega^i_\alpha &= A^i_\alpha \Pi^\gamma_{\alpha}.
\end{align}

By exterior differentiation of the both side of this equation we get:

\begin{align}
\frac{\partial}{\partial x^j} A^i_\alpha \wedge \omega^j_\beta + A^j_\alpha \frac{\partial}{\partial x^j} \omega^i_\beta - \frac{\partial}{\partial x^j} A^j_\beta \wedge \omega^i_\alpha &= dA^i_\alpha \wedge \Pi^\gamma_{\alpha} + A^i_\alpha \Pi^\gamma_{\alpha}.
\end{align}

Looking to the equality (*) and at:

\begin{align}
\frac{\partial}{\partial x^j} A^i_\alpha \wedge \omega^j_\beta - \frac{\partial}{\partial x^j} A^j_\beta \wedge \omega^i_\alpha &= - A^i_\beta \Pi^\gamma_{\alpha},
\end{align}

the equality (**) becomes

\begin{align}
- A^i_\alpha A^j_\beta \Omega_{ij} &= \delta_{\alpha\beta}(d\Pi^\gamma_{\alpha} + \Pi^\gamma_{\beta} \wedge \omega^\beta_{\alpha} - \Pi^\gamma_{\alpha} \wedge \omega^\gamma_{\beta}).
\end{align}

Now, multiplying (9.6.2) by $g_{ih} A^h_\beta = g_{\beta\gamma} A^i_\gamma$ we obtain the first equations (9.6.1). The same operation, taking the factor $g_{ij} A^j_\beta = \delta_{\alpha\beta} A^i_\gamma$ leads to third equation (9.6.1). Of course, to deduce the second equations (9.6.1) we will take the exterior differential of both sides of equations (***). The proof is completed. q.e.d.

In order to obtain the system of all fundamental equations of the subspaces $H^{(k)m}_{(k)m}$ in $H^{(k)n}_{(k)n}$, we must find the relations between the torsion 2-forms $\Omega^i_{\alpha}, \Omega^i_{\beta}$, $(a = 1, ..., k - 1), \Omega_a$ of the canonical metrical N-connection $D$ of $H^{(k)m}_{(k)n}$ and the torsion 2-forms $\Omega^\alpha, \Omega^\beta$, $(a = 1, ..., k - 1), \Omega_a$ of the relative connection $\nabla$. Therefore, we obtain:

Theorem 9.6.2 The fundamental equation of the Hamilton subspaces of order $k, H^{(k)m}$ in a Hamilton space of order $k, H^{(k)n}$ endowed with the canonical metrical N-connection $D$ are: the Gauss-Codazzi equations (9.6.1), as well as the following equations:
Subspaces in Hamilton Spaces of Order $k$

\begin{align*}
  d(dx^i) - dx^j \wedge \omega_j^i &= (0) \Omega^i, \text{ modulo (9.3.8)}, \\
  d(\delta y^{(a)i}) - \delta y^{(a)j} \wedge \omega_j^i &= - \Omega^i \text{ modulo (9.3.8), (a = 1, ..., k - 1)}, \\
  d(\delta p_i) + \delta p_j \wedge \omega_j^i &= - \Omega_i, \text{ modulo (9.3.8')}.
\end{align*}

We end here the theory of Hamilton subspaces of order $k$ in a Hamilton space of the same order.

We underline the importance of this theory for applications in the Higher Order Hamiltonian Mechanics.

The particular case $m = n - 1$ of the hypersurfaces $\mathcal{M}$ in $M$ can be obtained from the previous study without difficulties. If $k = 1$ we have the theory of Hamilton subspaces $\mathcal{H}^n = (\mathcal{M}, \mathcal{H}(x, p))$ in a Hamilton space $H^n = (M, H(x, p))$. 
Chapter 10

The Cartan Spaces of Order $k$ as Dual of Finsler Spaces of Order $k$

The geometry of Finsler spaces of order $k$, $F^{(k)n} = (M, F(x, y^{(1)}, ..., y^{(k)}))$, studied in chapter 3, is a particular case of the geometry of Lagrange spaces of order $k$, $L^{(k)n} = (M, L(x, y^{(1)}, ..., y^{(k)}))$. The fundamental function $F$ being a $k$-homogeneous regular Lagrangian on the fibres of the bundle $T^kM$.

As we know, the 'dual' of the space $L^{(k)n}$, via Legendre transformation $\text{Leg}$, is a Hamilton space of order $k$, $H^{(k)n} = (M, H(x, y^{(1)}, ..., y^{(k-1)}, p))$. In this chapter we will prove that the restriction of the mapping $\text{Leg}$ to the Finsler spaces of order $k$ determines a new class of Hamilton spaces of order $k$, that are called the Cartan spaces of order $k$ and are denoted by $C^{(k)n} = (M, K(x, y^{(1)}, ..., y^{(k-1)}, p))$.

Also, the spaces $C^{(k)n}$ are the Hamilton spaces $H^{(k)n} = (M, H)$ in which the fundamental function $K(x, y^{(1)}, ..., y^{(k-1)}, p)$ is regular and $k$-homogeneous on the fibres of the dual bundle $T^*kM$.

For the spaces $C^{(k)n}$ it is important to determine the fundamental geometrical object fields which are important in their differential geometry.

10.1 $C^{(k)n}$-Spaces

Definition 10.1.1 A Cartan space of order $k \geq 1$ is a pair $C^{(k)n} = (M, K(x, y^{(1)}, ..., y^{(k-1)}, p))$ for which the following axioms hold:

1. $K$ is a real function on the manifold $T^*kM$, differentiable on $\widetilde{T^*kM}$ and continuous on the null section of the projection $\pi^*k : T^*kM \rightarrow M$.
2. $K > 0$ on $T^*kM$.
3. $K$ is positively $k$-homogeneous on the fibres of bundle $T^*kM$, i.e. $K(x, ay^{(1)}, ..., a^{k-1}y^{(k-1)}, ap) = a^kK(x, y^{(1)}, ..., y^{(k-1)}, p), \forall a \in \mathbb{R}^+$.
The Hessian of $K^2$, with respect to the momenta $p_i$, having the elements
\begin{equation}
(10.1.2) \quad g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = \frac{1}{2} \partial_i \partial_j K^2
\end{equation}

is positively defined.

It follows that the $k$-homogeneity of the function $K$ is not $k$-homogeneity of $K$ with respect to $p_i$. It is considered as in section 1, ch. 3. In the book [115], ch. 13 is considered only the case of homogeneity of $K$ with respect to $p_i$. We deduce from (10.1.2) that $g^{ij}$ is contravariant of order two, symmetric and nondegenerate, i.e.
\begin{equation}
(10.1.3) \quad \text{rank } g^{ij} = n, \text{ on } T^{*k}M.
\end{equation}

The function $K$ is called the fundamental (or metric) function of $\mathcal{C}^{(k)n}$ and $g^{ij}$ is the fundamental tensor of this space.

Theorem 10.1.1 If the base manifold $M$ is paracompact, then on $T^{*k}M$ there exist functions $K(x, y^{(1)}, \ldots, y^{(k-1)}, p)$ such that the pair $(M, K)$ is a Cartan space of order $k$.

Proof: The manifold $M$ being paracompact there exists a Finsler space of order $k-1$, $F^{(k-1)n} = (M, F(x, y^{(1)}, \ldots, y^{(k-1)}))$. Its fundamental tensor $a_{ij}:
\begin{align*}
a_{ij}(x, y^{(1)}, \ldots, y^{(k-1)}) &= \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(k-1)i} \partial y^{(k-1)j}}
\end{align*}
is positively defined on $T^{k-1}M$ and $0$-homogeneous on the fibres of bundle $(T^{k-1}M, \pi^{k-1}, M)$.

Now, we can construct on the dual bundle $T^{*k}M = T^{k-1}M \times_M T^*M$ the following Hamiltonian
\begin{equation}
(10.1.4) \quad K(x, y^{(1)}, \ldots, y^{(k-1)}, p) = \left\{a^{ij}(x, y^{(1)}, \ldots, y^{(k-1)})p_ip_j\right\}^{1/2},
\end{equation}
where $a^{ij}$ is the contravariant tensor of $a_{ij}$.

It follows, without difficulties, that $K$ is a scalar function (i.e. it does not depend of the transformations of coordinates on $T^{*k}M$) which satisfies the axioms of Definition 1.1.

The fact that $K$ is positively $k$-homogeneous on the fibres of the bundle $T^{*k}M$ follows directly. As $a^{ij}$ are $0$-homogeneous we have that
\begin{equation*}
K(x, ay^{(1)}, \ldots, a^{k-1}y^{(k-1)}, a^kp) = a^kK(x, y^{(1)}, \ldots, y^{(k-1)}, p).
\end{equation*}
The fundamental tensor $g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p)$ coincides with $a^{ij}(x, y^{(1)}, \ldots, y^{(k-1)})$. The conclusion is that the pair $(M, K)$, with $K$ from (10.1.4) is a Cartan space of order $k$. 

40
Remark 10.1.1 The Cartan space $C^{(k)n}$ with the fundamental function $K$ from (10.1.4) is a particular one, but it is an important and useful example of Cartan space of order $k$.

Remark 10.1.2 If $C^{(k)n} = (M, K)$ is a Cartan space of order $k$, then $H^{(k)n} = (M, K^2)$ is a Hamilton space of order $k$ having the same fundamental tensor $g^{ij}$ as the space $C^{(k)n}$. $H^{(k)n}$ will be called the Hamilton space associated to the Cartan space of order $k$, $C^{(k)n}$. All geometrical properties of $H^{(k)n} = (M, K^2)$ are geometrical properties of $C^{(k)n} = (M, K)$.

10.2 Geometrical Properties of the Cartan Spaces of Order $k$

First of all we shall study those properties of the spaces $C^{(k)n} = (M, K)$ which result from the $k$-homogeneity of the fundamental function $K$ expressed in the identity (10.1.1).

Consider the vector field on $T^*kM$:

\[ k^{-1} \Gamma + kC^* = g^{(1)i} \frac{\partial}{\partial y^{(1)i}} + \cdots + (k-1)g^{(k-1)i} \frac{\partial}{\partial y^{(k-1)i}} + kp_i \cdot \partial \]

and the Lie derivation with respect to this vector field $L_{k^{-1} \Gamma + kC^*}$.

The functions $K, K^2, \partial^j K^{2k}, g^{ij}$ ($i, j = 1, \ldots, n$) and $C^{ijh}$:

\[ C^{ijh} = \frac{1}{2} \partial^h g^{ij} = -\frac{1}{4} \partial^i \partial^j \partial^h K^2 \]

are $k, 2k, k, 0$ and $-k$ homogeneous, respectively on the fibres of the bundle $T^*kM$. Taking into account the Theorems 4.5.7 and 4.5.8 we have:

Proposition 10.2.1 The following identities hold:

\[ L_{k^{-1} \Gamma + kC^*} K = kK, \]
\[ L_{k^{-1} \Gamma + kC^*} K^2 = 2kK, \]
\[ L_{k^{-1} \Gamma + kC^*} \partial^h K^2 = k \partial^h K^2, \]
\[ L_{k^{-1} \Gamma + kC^*} g^{ij} = 0, \]
\[ L_{k^{-1} \Gamma + kC^*} C^{ijh} = -kC^{ijh}. \]
We remark, especially, 0-homogeneity of the functions $g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p)$. By virtue of (10.2.1) and (10.2.6) we have the identity:

$$y^{(1)i}rac{\partial g^{jh}}{\partial y^{(1)j}} + \cdots + (k-1)y^{(k-1)i}rac{\partial g^{jh}}{\partial y^{(k-1)j}} + kp_i \frac{\partial}{\partial y^{(1)i}} g^{jh} = 0.$$

Of course, the covariant tensor field $g_{ij}$ of the fundamental tensor $g_{ij}$ of the Cartan space $C^{(k)}n$ is 0-homogeneous on the fibres of $T^kM$. Indeed, the equality (10.2.6) implies $L_{\Gamma_{k-1}^{kC}} g_{ij} = 0$. Therefore we have the identity:

$$y^{(1)i}rac{\partial g^{jh}}{\partial y^{(1)j}} + \cdots + (k-1)y^{(k-1)i}rac{\partial g^{jh}}{\partial y^{(k-1)j}} + kp_i \frac{\partial}{\partial y^{(1)i}} g^{jh} = 0.$$

In any Cartan space of order $k$, $C^{(k)}n = (M, K)$, there exists two important tensors $C_{(k-1)i}^{jh}$, $C_i^{jh}$ which are the $v_{k-1}$- and $w_k$-coefficients of a canonical metrical connection.

We have:

**Theorem 10.2.1** For any Cartan space of order $k$, $C^{(k)}n = (M, K)$, $C_{(k-1)i}^{jh}$, $C_i^{jh}$ given by

$$C_{(k-1)i}^{jh} = \frac{1}{2} g^{js} \left( \frac{\partial g^{sh}}{\partial y^{(k-1)j}} + \frac{\partial g^{js}}{\partial y^{(k-1)h}} - \frac{\partial g^{jh}}{\partial y^{(k-1)s}} \right),$$

$$C_i^{jh} = -\frac{1}{2} g^{is} \left( \frac{\partial g^{sh}}{\partial p_j} + \frac{\partial g^{js}}{\partial p_h} - \frac{\partial g^{jh}}{\partial p_s} \right)$$

have the following properties:

1. They are $d$-tensor fields of type $(1,2)$ and $(2,1)$, respectively.
2. $C_{(k-1)i}^{jh}$ are $1-k$ homogeneous and $C_i^{jh}$ are $-k$ homogeneous on the fibres of $T^kM$.
3. They are the $v_{k-1}$- and $w_k$-coefficients of a canonical metrical connection $D$, i.e.:

$$g^{ij}(k-1)_{|n} = 0, \quad g^{ij}|^n = 0.$$

4. We have

$$C_i^{jh} = g_{is} C_s^{jh}.$$

5. $S_{(k-1)i}^{jh} = C_{(k-1)i}^{jh} - C_i^{jh} - C_{(k-1)hj} = 0, \quad S_i^{jh} = C_i^{jh} - C_i^{hj} = 0.$$

The proofs of these affirmations are not difficult.
10.3 Canonical Presymplectic Structures, Variational Problem of the Space $C^{(k)n}$

Consider a Cartan space of order $k$, $C^{(k)n} = (M, K(x, y^{(1)}, ..., y^{(k-1)}, p))$, and the canonical presymplectic structure

$$\theta = dp_i \wedge dx^i,$$

where

$$\theta = d\omega, \omega = p_i dx^i.$$

Of course the structure $\theta$ can be directly studied using the property of integrability, given by $d\theta = 0$. The relations between the structure $\theta$ and the Poisson structure $\{f, g\}_0$:

$$\{f, g\}_0 = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i}, \forall f, g, \in \mathcal{F}(\Sigma_0),$$

and

$$\Sigma_0 = \left\{(x, y^{(1)}, ..., y^{(k-1)}, p) | y^{(1)i} = \cdots = y^{(k-1)i} = 0\right\},$$

lead to some important geometrical results.

Recall that the submanifold $\Sigma_0$ has been introduced in the section 3 of chapter 8 as a section of the canonical projection of the differentiable bundle $(T^*kM, \pi^*, T^*M)$.

Let us consider the restriction $K_0$ of the fundamental function $K$ of space $C^{(k)n}$ to $\Sigma_0$:

$$K_0 = K|_{\Sigma_0}.$$

Thus we have

$$K_0(x, p) = K(x, 0, ..., 0, p).$$

It follows that $K_0$ is 1-homogeneous with respect to $p_i$.

Therefore the pair $C^{(1)n}_0 = (M, K_0)$ is a classical Hamilton space, [115], with the fundamental tensor field

$$g^{ij}_0(x, p) = g^{ij}(x, 0, ..., 0, p) = \frac{1}{2} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} K_0^2.$$

Theorem 7.3.1 is valid in the particular case of Cartan space $C^{(1)n}_0$:

**Theorem 10.3.1** The pair $(\Sigma_0, \theta_0)$, with $\theta_0 = \theta|_{\Sigma_0}$, is a symplectic manifold.
Since \( \theta_0 = dp_i \wedge dx^i \) in every point \((x, p) \in \Sigma_0\), it is a closed 2-form of rank \(2n = \dim \Sigma_0\).

The tangent space \(T_u \Sigma_0\) at a point \(u \in \Sigma_0\) has the natural basis \(\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_i}\right)\), \((i = 1, ..., n)\).

Consider the \(\mathcal{F}(\Sigma_0)\)-module \(\mathcal{X}(\Sigma_0)\) of vector fields and the \(\mathcal{F}(\Sigma_0)\)-module \(\mathcal{X}^*(\Sigma_0)\) of 1-form fields on the submanifold \(\Sigma_0\).

Taking into account the theory from ch. 8, section 3 we have:

1° The \(\mathcal{F}(\Sigma_0)\)-linear mapping \(S_{\theta_0} : \mathcal{X}(\Sigma_0) \to \mathcal{X}^*(\Sigma_0)\) given by
\[
S_{\theta_0}(X) = i_X \theta_0, \quad \forall X \in \mathcal{X}(\Sigma_0)
\]
is an isomorphism.

2° There exists an unique vector field \(X_{K^2_0} \in \mathcal{X}(\Sigma_0)\) such that
\[
S_{\theta_0}(X_{K^2_0}) = i_{X_{K^2_0}} \theta_0 = -dK^2_0.
\]

3° The Hamiltonian vector field \(X_{K^2_0}\) is given by
\[
X_{K^2_0} = \frac{\partial K^2_0}{\partial p_i} \frac{\partial}{\partial x^i} - \frac{\partial K^2_0}{\partial x^i} \frac{\partial}{\partial p_i}.
\]

Consequently, we can formulate:

**Theorem 10.3.2** The integral curves of the Hamiltonian vector field \(X_{K^2_0}\) are given by the \(\Sigma_0\)-canonical equations:

\[
\begin{align*}
\frac{dx^i}{dt} &= \frac{\partial K^2_0}{\partial p_i}, & \frac{dp_i}{dt} &= -\frac{\partial K^2_0}{\partial x^i}, & y^{(\alpha)i} = 0, & (\alpha = 1, ..., k - 1).
\end{align*}
\]

Now, let us consider two functions \(f, g \in \mathcal{F}(\Sigma_0)\) and the vector fields \(X_f, X_g\) given by
\[
i_X f \theta_0 = -df, \quad i_X g \theta_0 = -dg.
\]

**Proposition 10.3.1** The following relations between the structures \(\theta_0\) and \(\{\cdot, \cdot\}_0\) on the submanifold \(\Sigma_0\), hold:

\[
\{f, g\}_0 = \theta_0(X_f, X_g), \quad \forall f, g \in \mathcal{F}(\Sigma_0).
\]

**Remark 10.3.1** The triple \((T^{*k}M, K^{2}(x, y^{(1)}, ..., y^{(k-1)}, p), \theta)\) is a particular Hamiltonian system. It can be studied directly applying the method of Gotay, [115].

But, the equations (10.3.6) are extremely particular. For Cartan spaces \(C^{(k)n} = (M, K(x, y^{(1)}, ..., y^{(k-1)}, p))\) the integral of action (see Ch.5)

\[
I(c) = \int_0^1 [p_i \frac{dx^i}{dt} - K(x, \frac{dx}{dt}, ..., \frac{1}{(k-1)!} \frac{d^{k-1}y}{dtk^{k-1}}, p)]dt
\]
leads to the fundamental equations of the space:

\[
\begin{align*}
\frac{dx^i}{dt} &= \frac{1}{2} \frac{\partial K}{\partial p_i}, \\
\frac{dp_i}{dt} &= -\frac{1}{2} \frac{\partial K}{\partial x^i} - \frac{d}{dt} \frac{\partial K}{\partial y(1)i} + \cdots + (-1)^{k-1} \frac{d^{k-1}}{dt^{k-1}} \frac{\partial K}{\partial y(k-1)i}.
\end{align*}
\]

The integrand of the integral of action is \(k\)-homogeneous on the fibres of \(T^kM\). The Hamilton-Jacobi equations (10.3.9) are homogeneous on the fibres of \(T^kM\), too.

The energy of order \(k-1\), \(E^{k-1}\), of the Cartan space \(C^{(k)n} = (M, K)\), by means of formula (5.3.1), is given by:

\[
E^{k-1}(K) = I^{k-1}(K) - \frac{1}{2} \frac{d}{dt} I^{k-2}(K) + \cdots + (-1)^{k-2} \frac{1}{(k-1)!} \frac{d^{k-2}}{dt^{k-2}} I^{1}(K) - K.
\]

We have:

**Theorem 10.3.3** For a Cartan space \(C^{(k)n} = (M, K)\), the energy of order \(k-1\), \(E^{k-1}(K)\) is constant along every solution curve of the Hamilton-Jacobi equations.

### 10.4 The Cartan Spaces \(C^{(k)n}\) as Dual of Finsler Spaces \(F^{(k)n}\)

Let \(F^{(k)n} = (M, F(x, y^{(1)}, \ldots, y^{(k)}))\) be a Finsler space of order \(k\) having \(F(x, y^{(1)}, \ldots, y^{(k)})\) as fundamental function and

\[
a_{ij}(x, y^{(1)}, \ldots, y^{(k)}) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(k)i} \partial y^{(k)j}}
\]
as fundamental tensor.

Remembering the Definition 3.1.1, \(F\) is a function from \(T^kM\) to \(\mathbb{R}\), differentiable on the manifold \(\tilde{T^kM} = T^kM \setminus \{0\}\) and continuous on the null section. \(F\) is a positive function, \(k\)-homogeneous:

\[
F(x, ay^{(1)}, \ldots, a^k y^{(k)}) = a^k F(x, y^{(1)}, \ldots, y^{(k)}), \ \forall a \in \mathbb{R}^+
\]

and the tensor field \(a_{ij}\) is positively defined on \(\tilde{T^kM}\).

The Legendre mapping \(\varphi : \tilde{T^kM} \rightarrow \tilde{T^*kM}\) defined in section 4, ch.8, by

\[
\begin{align*}
x^i &= y^{(0)i}, y^{(1)i} = y^{(1)i}, \ldots, y^{(k-1)i} = y^{(k-1)i}, \\
p_i &= \frac{1}{2} \frac{\partial F^2}{\partial y^{(k)i}}
\end{align*}
\]
is a local diffeomorphism.
We denote
\[ p_i = \varphi_i(y^{(0)}, y^{(1)}, \ldots, y^{(k)}). \]
Evidently \( y^{(0)i} = x^i \) is the point of the base manifold \( M, x = \pi^k(x, y^{(1)}, \ldots, y^{(k)}). \)

The local inverse \( \xi = \varphi^{-1} : T^k M \to T^k M \) is expressed by (8.4.3):
\[
\begin{cases}
  y^{(0)i} = x^i, y^{(1)i} = y^{(1)i}, \ldots, y^{(k-1)i} = y^{(k-1)i}, \\
  y^{(k)i} = \xi_i(x, y^{(1)}, \ldots, y^{(k-1)}, p)
\end{cases}
\]

Remarking that the Legendre transformation \( \varphi \), from (10.4.2) is \( k \)-homogeneous, it follows that its local inverse \( \xi^{-1} \) is \( k \)-homogeneous on the fibres of bundle \( T^k M \). So, we have:
\[
\xi_i(x, ay^{(1)}, \ldots, a^{k-1}y^{(k-1)}, a^k p) = a^k \xi_i(x, y^{(1)}, \ldots, y^{(k-1)}, p), \quad \forall a \in \mathbb{R}^+.
\]
Applying the Theorem 8.4.1 we have:

**Theorem 10.4.1** The Legendre mapping \( \varphi \) transforms the canonical \( k \)-spray of the Finsler space \( F^{(k)}n \):
\[
S = y^{(1)i} \frac{\partial}{\partial x^i} + \cdots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k+1)G^i \frac{\partial}{\partial y^{(k)i}}
\]
with the coefficients
\[
(k+1)G^i(x, y^{(1)}, \ldots, y^{(k)}) = \frac{1}{2} a^{ij} \left\{ \Gamma^j_i \left( \frac{\partial F^2}{\partial y^{(k)j}} \right) - \frac{\partial F^2}{\partial y^{(k-1)j}} \right\}
\]
in the \( \xi \)-dual \( k \)-spray
\[
S^*_{\xi} = y^{(1)i} \frac{\partial}{\partial x^i} + \cdots + (k-1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-2)i}} + \\
+ k \xi^i(x, y^{(1)}, \ldots, y^{(k-1)}, p) \frac{\partial}{\partial y^{(k-1)i}} + \eta_i(x, y^{(1)}, \ldots, y^{(k-1)}, p) \frac{\partial}{\partial p_i}
\]
with \( \xi^i \) from (10.4.3) and
\[
\eta_i = -a_{is} \frac{\partial \xi^s}{\partial y^{(1)r}} y^{(1)r} + \cdots + \\
+(k-1) \frac{\partial \xi^s}{\partial y^{(k-1)r}} \xi^r + (k+1)G^s(x, y^{(1)}, \ldots, y^{(k-1)}, \xi)
\]

**Proof:** By means of theorem 8.4.1, the \( k \)-spray \( S \), which is 2 -homogeneous on the fibres of \( T^k M \), is transformed in the dual \( k \)-semispray \( S^*_{\xi} \) from (10.4.6), (10.4.7). We must prove that \( S^*_{\xi} \) is 2-homogeneous on the fibres of \( T^k M \). But every term of \( S^*_{\xi} \) is 2-homogeneous on the fibres of \( T^k M \). Thus, \( S^*_{\xi} \) is a dual \( k \)-spray.
Theorem 10.4.2 The dual $k$-spray $S^*_\xi$ from (10.4.6) determines a local nonlinear connection $N^*$ on the manifold $\tilde{T}^*kM$, which depends on the Finsler space of order $k$, $F^{(k)n}$, and $N^*$ has the following dual coefficients

$$
M^{*}_{(1) i j} = -\frac{\partial \xi^i}{\partial y^{(1)} j}, \ldots, M^{*}_{(k-1) i j} = -\frac{\partial \xi^i}{\partial y^{(k-1)} j},
$$

$$
N^*_{ij} = \frac{\delta \eta_i}{\delta y^{(1)} j},
$$

where the operators $\frac{\delta}{\delta y^{(1)} j}$ are constructed by means of the coefficients $M^{*}_{(1) i j}, \ldots, M^{*}_{(k-1) i j}$ from the dual coefficients (10.4.8).

Indeed, this theorem is just the Theorem 8.4.2 applied to Finsler spaces of order $k$.

We remark the following property of homogeneity of the coefficients $M^*$, $N^*$:

Proposition 10.4.1 The coefficients $M^{*}_{(1) i j}, \ldots, M^{*}_{(k-1) i j}, N^*_{ij}$ are homogeneous on the fibres of $T^*kM$ of degree $k-1, \ldots, 1, k$, respectively.

Indeed, $\xi^i$ being $k$-homogeneous and $\eta_i$ from (10.4.7) being $k+1$ homogeneous, by means of (10.4.8) the property follows.

Now, let us consider $\tilde{N}$ a fixed nonlinear connection on the submanifold $T^{k-1}M$ of $T^*kM = T^{k-1}M \times_MT^*M$. We assume that the coefficients $\tilde{M}^{(1) i j}, \ldots, \tilde{M}^{(k-1) i j}$ are homogeneous of degree 1, ..., $k-1$ on the fibres of $T^{k-1}M$.

Thus, the Legendre mapping $\varphi: u = (x, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}) \in T^kM \rightarrow u^* = (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^*kM$ transforms the Liouville $d$-vector field $z^{(k)i}$ at the point $u$:

$$
(10.4.9) \quad k z^{(k)i} = k y^{(k)i} + (k - 1) M^{(1) i m}_{(k-1) m} y^{(k-1)m} + \cdots + M^{(1) i m}_{(k-1) m} y^{(1)m}
$$

in the $d$-vector field $\dot{z}^{(k)i}$ at the point $u^*$. The vector field $\dot{z}^{(k)i}$ is:

$$
(10.4.9a) \quad k \dot{z}^{(k)i} = k \xi^i + (k - 1) M^{(1) i m}_{(k-1) m} y^{(k-1)m} + \cdots + M^{(1) i m}_{(k-1) m} y^{(1)m}.
$$

Of course, $\dot{z}^{(k)i}$ is $k$-homogeneous on the fibres of $T^*kM$. Consider the function

$$
(10.4.10) \quad K^2(x, y^{(1)}, \ldots, y^{(k-1)}, p) = 2p_t \dot{z}^{(k)i} - F^2(x, y^{(1)}, \ldots, y^{(k-1)}, \xi(x, y^{(1)}, \ldots, y^{(k-1)}, p)).
$$
Theorem 10.4.3 We have:

1° The pair \((M, K^2(x, y^{(1)}, \ldots, y^{(k-1)}, p))\) is a Hamilton space. Its fundamental function \(K^2\) is \(2k\)-homogeneous on the fibres of \(T^kM\).

2° The fundamental tensor of the space \((M, K^2)\) is positively defined and it does not depend on the apriori given nonlinear connection \(\mathcal{N}\). It is given by:

\[
g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = \delta^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p).\]

The proof follows from Theorem 8.4.3 in which \(H = K^2\).

We shall say that \((M, K^2)\) is a Cartan spaces of order \(k\), dual (via Legendre transformation) to the Lagrange space \(L^{(k)n} = (M, F^2)\) associated to the Finsler space \(F^{(k)n} = (M, F)\).

The inverse problem: being given a Cartan space of order \(k\), \(C^{(k)n} = (M, K(x, y^{(1)}, \ldots, y^{(k-1)}, p))\) let us determine its dual as a Finsler space of order \(k\), \(F^{(k)n} = (M, F(x, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}))\). We follow the theory that has been done in the section 5, ch. 8.

Let \(\mathcal{N}\) be an apriori given nonlinear connection on \(T^{k-1}M\), with the dual coefficients \(\left(\overline{\mathcal{N}}^i, \overline{\mathcal{N}}^j\right)\) and the mapping

\[
\xi^*: u^* = (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^{*k}M \rightarrow u = (x, y^{(1)}, \ldots, y^{(k-1)}, y^{(k)}) \in T^kM
\]
defined by

\[
g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = \delta^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p),
\]

where \(\xi^i\) is expressed from the formula

\[
k\xi^i + (k-1)\overline{\mathcal{N}}^i = y^{(k-1)i} + \cdots + \overline{\mathcal{N}}^i = \frac{k}{2} \partial_i K^2.
\]

Theorem 10.4.4 The mapping \(\xi^*\) is a local diffeomorphism which preserves the fibres of \(T^kM\) and \(T^kM\).

Indeed, by means of theorem 7.5.1, the Jacobian of \(\xi^*\) is \(\det |g^{ij}|\), \(g^{ij}\) being the fundamental tensor of the Cartan space \(C^{(k)n}\). q.e.d.

The formulas (10.4.11) and (10.4.12) imply:

\[
\nabla^i \xi^j = g^{ij},
\]

\[
z^{(k)i}(x, y^{(1)}, \ldots, y^{(k-1)}, \xi^*(u^*)) = \frac{1}{2} \partial^i K^2(x, y^{(1)}, \ldots, y^{(k-1)}, \varphi^*).
\]

where \(z^{(k)i}\) is the Liouville vector field (10.4.9).

Let \(F^2 = \xi^*(K^2)\) be the Lagrangian

\[
F^2(x, y^{(1)}, \ldots, y^{(k)}) = 2p_i z^{(k)i}(x, y^{(1)}, \ldots, y^{(k-1)}, \xi^*) - K^2(x, y^{(1)}, y^{(k-1)}, \varphi^*).\]
where \( \varphi^* \) is the local inverse of \( \xi^* \).

Theorem 8.5.2 allows to state:

**Theorem 10.4.5** The pair \((M, F)\), with \( F \) from (10.4.14), has the properties:

1. It is a Finsler space, having the fundamental function \( F^2 \), \( 2k \)-homogeneous on the fibres of \( T^k M \).
2. Its fundamental tensor field is given by
   \[
   a_{ij}(u) = g_{ij}(x, y^{(1)}, y^{(2)}, \ldots, y^{(k-1)}, \varphi^*(u)).
   \]

So, the space \((M, F)\) is called the Finsler space of order \( k \) dual to the Cartan space of order \( k \), \( C^k_n = (M, K) \).

From (10.4.14) we deduce

\[
(10.4.15) \quad p_i = \frac{1}{2} \frac{\partial F^2}{\partial y^{(k)i}}, \quad p_i = \varphi^*_i(x, y^{(1)}, \ldots, y^{(k)}).
\]

Therefore, by means of Theorem 8.5.3, we have:

**Theorem 10.4.6** The Legendre transformation determined by the Lagrange space \( L^k_n = (M, F^2) \), with \( F^2 \) from (10.4.14) is defined by the local diffeomorphism \( \varphi^* \), the inverse of the local diffeomorphism \( \xi^* \) and \( K^2 = \varphi^*(F^2) \) is given by

\[
K^2(x, y^{(1)}, \ldots, y^{(k-1)}, p) = 2p_iz^{(k)}(x, y^{(1)}, \ldots, y^{(k-1)}, \xi^*) - F^2(x, y^{(1)}, \ldots, y^{(k-1)}, \xi^*).
\]

**Remark 10.4.1** It follows \( \varphi = \varphi^* \) and \( \varphi^{-1} = \xi^* \). Therefore, locally, we get \( K^2 = \varphi^*(\xi^*(K^2)) \) and \( F^2 = \xi^*(\varphi^*(F^2)) \).

### 10.5 Canonical Nonlinear Connection. \( N \)-Linear Connections

As we know, from section 6, Ch. 8 and from Theorem 10.4.6 we can determine a nonlinear connection \( N^* \) of the Cartan space \( C^k_n = (M, K) \) by means of an apriori given nonlinear connection \( \tilde{N} \) on the manifold \( T^{k-1}M \). We construct a bundle morphism and determine the space \( L^k_n = (M, F^2) \), with \( F^2 = \xi^*(K^2) \).

The Legendre transformation \( \varphi^* = \xi^{*-1} \) (with \( \varphi^*_i = \frac{1}{2} \frac{\partial F^2}{\partial y^{(k)i}} \)) transforms the \( k \)-spray \( S_{k}^\xi \) of \( L^k_n \) in the dual \( k \)-spray \( S_{k}^\xi^* \):

\[
(10.5.1) \quad S_{k}^\xi^* = y^{(1)i} \frac{\partial}{\partial x^i} + \cdots + (k-1)y^{(k-1)i} \frac{\partial}{\partial y^{(k-2)i}} + k\xi^{*i} \frac{\partial}{\partial y^{(k-1)i}} + y_i \frac{\partial}{\partial p_i}.
\]
Theorem 10.5.1 The following systems of functions

\[(10.5.2)\]
\[M^{s_i}_{(1)j} = -\frac{\partial \xi^{s_i}}{\partial y^{(1)j}}, \ldots, M^{s_i}_{(k-1)j} = -\frac{\partial \xi^{s_i}}{\partial y^{(k-1)j}},\]

and

\[(10.5.2a)\]
\[N^*_{ij} = \frac{\delta \eta_i}{\delta y^{(1)j}},\]

where the operators \(\delta \frac{\partial}{\partial y^{(1)j}}\) are constructed by means of (10.5.2), give the dual coefficients of a nonlinear connection \(N^*\) which depends on the fundamental function \(K\) of Cartan space \(C^{(k)n} = (M, K)\) and on the apriori given nonlinear connection \(N\) of the manifold \(T^{k-1}M\).

Indeed, this is a particular case of the Theorem 8.6.1.

The nonlinear connection \(N^*\) is called canonical for the Cartan space \(C^{(k)n} = (M, K)\).

In the following we denote \(N^*\) by \(N\) and consider the adapted basis and adapted cobasis determined by \(N\) and by the vertical distribution \(W_k:\)

\[(10.5.3)\]
\[
\left\{ \delta \frac{\partial}{\partial x^i}, \delta \frac{\partial}{\partial y^{(1)i}}, \ldots, \delta \frac{\partial}{\partial y^{(k-1)i}}, \frac{\partial}{\partial p_i} = \bar{\partial}^i \right\}
\]

and

\[(10.5.3a)\]
\[
\left\{ dx^i, \delta y^{(1)i}, \ldots, \delta y^{(k-1)i}, \delta p_i \right\}.
\]

Let \(D\) be a \(N\)-linear connection with the coefficients \(D\Gamma(N) = (H^i_{jh}, C^i_{jh}, C^j_{ih}), (\alpha = 1, ..., k-1).\)

The fundamental tensor field \(g^{ij}\) of \(C^{(k)n}\) is absolute parallel with respect to \(D\) if

\[(10.5.4)\]
\[g^{ij}_h = 0, g^{ij}_{(\alpha)h} = 0, (\alpha = 1, ..., k-1), \ g^{ij}_h = 0.\]

Assume that the equations (10.5.4) hold. Then \(D\) is called the metrical \(N\)-linear connection.

In the case when \(h^-\), \(v^-\), and \(w_k\)-torsions of \(D\) vanish, \(D\) is called canonical metrical \(N\)-linear connection of the Cartan space \(C^{(k)n}\).

Theorem 8.7.1 leads to the following important result:

Theorem 10.5.2 \(\theta\) There exists an unique canonical metrical \(N\)-linear connection \(D\) of the space \(C^{(k)n} = (M, K)\). Its coefficients are given by the gener-
The Cartan Spaces of Order $k$

alized Christoffel symbols:

$$H_{jh}^i = \frac{1}{2} g^{is} \left( \frac{\delta g_{sh}}{\delta x^i} + \frac{\delta g_{js}}{\delta x^h} - \frac{\delta g_{jh}}{\delta x^s} \right),$$

(10.5.5)

$$C_{(\alpha)jh}^i = \frac{1}{2} g^{is} \left( \frac{\delta g_{sh}}{\delta y^{(\alpha)j}} + \frac{\delta g_{js}}{\delta y^{(\alpha)h}} - \frac{\delta g_{jh}}{\delta y^{(\alpha)s}} \right) \ (\alpha = 1, \ldots, k - 1),$$

$$C_{jh}^i = \frac{1}{2} g_{is} \left( \frac{\partial g^{sh}}{\partial p_j} + \frac{\partial g^{js}}{\partial p_h} - \frac{\partial g^{jh}}{\partial p_s} \right).$$

This connection depends only on the canonical nonlinear connection $N$ and on the fundamental function $K$ of the Cartan space of order $k$, $C^{(k)n}$.

The coefficients $(H_{jh}^i, C_{(\alpha)jh}^i, \ldots, C_{(k-1)jh}^i, C_{jh}^i)$ are homogeneous on the fibres of bundle $T^*kM$ of degree: 0, $-1$, ..., $1-k$, $-k$, respectively.

Corollary 10.5.1 The following identities hold

$$\mathcal{L}_{k^{-1}}^{\Gamma + kC^s} H_{jh}^i = 0,$$

(10.5.6)

$$\mathcal{L}_{k^{-1}}^{\Gamma + kC^s(\alpha)jh} C_{(\alpha)jh}^i = -\alpha C_{(\alpha)jh}^i \ (\alpha = 1, \ldots, k - 1),$$

$$\mathcal{L}_{k^{-1}}^{\Gamma + kC^s} C_{jh}^i = -kC_{jh}^i.$$

Corollary 10.5.2 We have

$$C_{jh}^i = g_{is} C^{sjh}.$$

Consider the $d$-Liouville vector fields of the space $C^{(k)n}$: $z_{(1)i}, \ldots, z_{(k-1)i}$, (cf. (6.2.7), (6.2.7')). They are homogeneous on the fibres of $T^*kM$ of degree $1, 2, \ldots, k - 1$, respectively.

The deflection tensor fields of the canonical metrical $N$-linear connection $D$ are given by (see ch. 7, section 5):

$$\left( \begin{array}{c}
(\alpha)
\delta_i^z \\
\delta_j^z
\end{array} \right) = \left( \begin{array}{c}
(\alpha)
\delta_i^z \\
\delta_j^z
\end{array} \right) + \left( \begin{array}{c}
(\alpha)
\frac{\delta z_i}{\delta x^j} \\
\frac{\delta z_j}{\delta x^i}
\end{array} \right) m H_{mj}^i,$$

(10.5.8)

$$\left( \begin{array}{c}
(\alpha\beta)
\delta_i^z \\
\delta_j^z
\end{array} \right) = \left( \begin{array}{c}
(\alpha\beta)
\delta_i^z \\
\delta_j^z
\end{array} \right) + \left( \begin{array}{c}
(\alpha\beta)
\frac{\delta z_i}{\delta y^{(\beta)j}} \\
\frac{\delta z_j}{\delta y^{(\beta)i}}
\end{array} \right) m C_{(\beta)mj}^i,$$

$$\left( \begin{array}{c}
\alpha
\delta_i^z \\
\delta_j^z
\end{array} \right) = \left( \begin{array}{c}
\alpha
\delta_i^z \\
\delta_j^z
\end{array} \right) + \left( \begin{array}{c}
\alpha
\frac{\delta z_i}{\delta p_j} \\
\frac{\delta z_j}{\delta p_i}
\end{array} \right) m C_{ij}^m, \ (\alpha, \beta = 1, \ldots, k - 1)$$

and

$$\Delta_{ij} = p_{ij} = -m H_{ij}^m,$$

(10.5.8a)

$$\beta^i_{ij} = p_i | j = -m C_{ij}^m, \ (\alpha = 1, \ldots, k - 1),$$

$$\beta^i_j = p_i^j = \delta_i^j - m C_{ij}^m.$$
The degrees of homogeneity on the fibres of $T^{*k}M$ for these tensor fields are easily determined.

The deflection tensors (10.5.8) and (10.5.8a) lead to important identities for the canonical metrical $N$-linear connection $D$, derived from the Ricci identities applied to the Liouville $d$-vector fields $z^{(1)i}$, ..., $z^{(k-1)i}$.

Indeed, Theorem 7.6.2 and 7.6.3 give us:

**Theorem 10.5.3** The canonical metrical $N$-linear connection of Cartan space $C^{(k)n}$ satisfies the following identities:

\[
\begin{align*}
\frac{(\alpha)}{D}^{i} j h - \frac{(\alpha)}{D}^{i} h j & = z^{(\alpha)s} R_{s}^{i} j h - \sum_{\beta=1}^{k-1} \left\{ \frac{(\alpha \beta)}{D}^{i} s R_{s}^{i} j h \right\} + \frac{(\alpha)}{D}^{i} s R_{s}^{j h}, \\
\frac{(\alpha)}{D}^{i} \left( \frac{(\beta)}{D} \right) j h - \frac{(\alpha)}{D}^{i} h j & = z^{(\alpha)s} P_{s}^{i} j h - \frac{(\alpha)}{D}^{i} s C_{s}^{i} j h - \frac{(\alpha)}{D}^{i} s H_{s}^{j h} - \\
& - \sum_{\gamma=1}^{k-1} \left\{ \frac{(\alpha \gamma)}{D}^{i} s B_{s}^{i} j h \right\} - \frac{(\alpha)}{D}^{i} s B_{s}^{j h}, \\
\frac{(\alpha)}{D}^{i} \left( \frac{(\beta)}{D} \right) h j - \frac{(\alpha)}{D}^{i} j h & = z^{(\alpha)s} P_{s}^{i} h j - \frac{(\alpha)}{D}^{i} s C_{j}^{i h} - \frac{(\alpha)}{D}^{i} s H_{s}^{h j} - \\
& - \sum_{\gamma=1}^{k-1} \left\{ \frac{(\alpha \gamma)}{D}^{i} s B_{s}^{i} h j \right\} - \frac{(\alpha)}{D}^{i} s B_{s}^{j h}, \\
\frac{(\alpha)}{D}^{i} \left( \frac{(\gamma)}{D} \right) j h - \frac{(\alpha)}{D}^{i} h j & = z^{(\alpha)s} S_{s}^{i} j h - \frac{(\alpha)}{D}^{i} s C_{s}^{i} h j - \frac{(\alpha)}{D}^{i} s C_{s}^{j h} - \\
& - \sum_{\sigma=1}^{k-1} \left\{ \frac{(\alpha \sigma)}{D}^{i} s B_{s}^{i} j h \right\} - \frac{(\alpha)}{D}^{i} s C_{s}^{j h}, \\
\frac{(\alpha)}{D}^{i} \frac{j h}{D} - \frac{(\alpha)}{D}^{i} \frac{h j}{D} & = z^{(\alpha)s} S_{s}^{i} j h - \sum_{\gamma=1}^{k-1} \left\{ \frac{(\alpha \gamma)}{D}^{i} s B_{s}^{i} j h \right\} - \frac{(\alpha)}{D}^{i} s B_{s}^{j h}, \\
\frac{(\alpha)}{D}^{i j} h - \frac{(\alpha)}{D}^{i j} h j & = \alpha^{s} R_{s}^{i} j h - \sum_{\gamma=1}^{k-1} \left\{ \frac{(\alpha \gamma)}{D}^{i} s R_{s}^{i} j h \right\} - \frac{(\alpha)}{D}^{i} s R_{s}^{j h}, \\
\Delta_{i j} \left( \frac{(\beta)}{D} \right) h - \frac{\Delta_{i h j}}{D} & = -p_{s} R_{i}^{s} j h - \sum_{\gamma=1}^{k-1} \left\{ \frac{(\beta)}{D} \right\}_{s} R_{s}^{i} j h - \frac{(\beta)}{D}^{s} R_{s}^{j h}, \\
\Delta_{i j} \left( \frac{(\beta)}{D} \right) h - \Delta_{i h j} \frac{\beta_{i j}}{D} & = -p_{s} P_{s}^{i} j h - \Delta_{i s} C_{s}^{i h} - \frac{(\beta)}{D}^{s} H_{s}^{j h} - \sum_{\gamma=1}^{k-1} \left\{ \frac{(\beta)}{D} \right\}_{s} B_{s}^{i} j h - \frac{(\beta)}{D}^{s} B_{s}^{j h}, \\
\Delta_{i j} \frac{h}{D} - \frac{\beta_{i j}^{h}}{D} & = -p_{s} P_{s}^{i} j h - \Delta_{i s} C_{j}^{i h} - \frac{\beta^{s}}{D} H_{s}^{h j} - \sum_{\gamma=1}^{k-1} \left\{ \frac{(\gamma)}{D} \right\}_{s} B_{s}^{i} j h - \frac{(\gamma)}{D}^{s} B_{s}^{j h}, \\
\Delta_{i j} \frac{h}{D} - \frac{\Delta_{i h j}}{D} \frac{h}{D} & = -p_{s} P_{s}^{i} j h - \Delta_{i s} C_{j}^{i h} - \frac{\beta^{s}}{D} H_{s}^{h j} - \sum_{\gamma=1}^{k-1} \left\{ \frac{(\gamma)}{D} \right\}_{s} B_{s}^{i} j h - \frac{(\gamma)}{D}^{s} B_{s}^{j h}.
\end{align*}
\]
The Cartan Spaces of Order $k$

(10.5.9a)

$$
\begin{align*}
\beta_{ij}^h - \beta_{ih}^j = -p_s S_{(\beta\gamma)^s}^i j_h - \beta_{is}^h C^{s}_{(\gamma) j} - \\
- \sum_{\sigma=1}^{k-1} \left( \beta_{is}^{(\sigma)} C_{(\beta\gamma)^s}^i j_h - \beta_{is}^{(\sigma)} B_{s i j} \right),
\end{align*}
$$

The Ricci identities and Bianchi identities of the canonical metrical $N$-linear connection of the Cartan space $C^{(k)n}$ can be written using the corresponding identities of the $N$-linear connections of $T^*kM$, given in the chapter 6.

Applying the Ricci identities to the fundamental tensor $g_{ij}$ of the space $C^{(k)n}$, with respect to the canonical metrical $N$-linear connection, we obtain the identities (7.6.8).

So, we have:

**Theorem 10.5.4** In a Cartan space $C^{(k)n}$, with respect to the canonical metrical $N$-linear connection, the following identities hold:

$$
\begin{align*}
g^{i j} R_{s}^{ h m} + g^{i s} R_{h m}^{ j} = 0, \\
g^{i j} P_{(\alpha)^s}^{ i h m} + g^{i s} P_{(\alpha)^s}^{ j h m} = 0, \quad (\alpha = 1, \ldots, k-1), \\
g^{i j} S_{s}^{ ih m} + g^{i s} S_{s}^{ jh m} = 0.
\end{align*}
$$

### 10.6 Parallelism of Vector Fields in Cartan Space $C^{(k)n}$

Consider a Cartan space of order $k$, $C^{(k)n} = (M, K)$, endowed with the canonical metrical $N$-linear connection $CT(N) = (H^{j m}, C^{(i)}_{j m}, C^{(i)m})$, $(\alpha = 1, \ldots, k-1)$, the coefficients being given by (10.5.5). The local vector fields $\left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(\alpha)i}}, \frac{\delta}{\delta p_i} \right)$, $(\alpha = 1, \ldots, k-1)$ determine an adapted basis and

$$
(dx^i, \delta y^{(\alpha)i}, \delta p_i), \quad (\alpha = 1, \ldots, k-1)
$$

is its dual basis.

Along a smooth parametrized curve $\gamma : I \to T^*kM$, having the image in a domain of a local chart:

$$
(10.6.1) \quad x^i = x^i(t), \quad y^{(\alpha)i} = y^{(\alpha)i}(t), \quad p_i = p_i(t), \quad t \in I, \quad (\alpha = 1, \ldots, k-1)
$$
the tangent vector field $\dot{\gamma}$ is given by (7.7.2), i.e.

\begin{equation}
\dot{\gamma} = \frac{dx^i}{dt} \delta + \sum_{\alpha=1}^{k-1} \frac{\delta y^{(\alpha)i}}{dt} \delta + \frac{\delta p_i}{dt} \partial^i,
\end{equation}

where

\begin{equation}
\begin{aligned}
\frac{\delta y^{(\alpha)i}}{dt} &= \frac{dy^{(\alpha)i}}{dt} + M^{(1)}_{(\alpha)j} \frac{dy^{(\alpha-1)j}}{dt} + \cdots + M^{(k-1)}_{(\alpha)j} \frac{dy^{(1)j}}{dt} + M^{1}_{(\alpha)j} \frac{dx^j}{dt},
\end{aligned}
\end{equation}

\begin{equation}
\frac{\delta p_i}{dt} = \frac{dp_i}{dt} - N^j_{ji} \frac{dx^j}{dt}, \quad (\alpha = 1, \ldots, k-1).
\end{equation}

Consider the 1-forms of metrical canonical $N$-linear connection $\Gamma(N)$:

\begin{equation}
\omega^i_j = H^i_{js} dx^s + \sum_{\alpha=1}^{k-1} C^i_{(\alpha)s} \delta y^{(\alpha)i} + C^i_{js} \delta p.
\end{equation}

Then the vector field $X \in \mathcal{X}(T^*kM)$:

\begin{equation}
X = X^i \frac{\delta}{\delta x^i} + \sum_{\alpha=1}^{k-1} X^{(\alpha)i} \frac{\delta}{\delta y^{(\alpha)i}} + X_i \partial^i
\end{equation}

has the covariant differential along curve $\gamma$:

\begin{equation}
\frac{DX}{dt} = \left( \frac{d X^i}{dt} + X^s \omega^i_s \right) \frac{\delta}{\delta x^i} + \sum_{\alpha=1}^{k-1} \left( \frac{d X^{(\alpha)i}}{dt} + X^{(\alpha)j} \omega^i_s \frac{\delta}{\delta y^{(\alpha)i}} \right) + \left( \frac{d X_i}{dt} - X_s \omega_s^i \right) \partial^i.
\end{equation}

Theorem 7.7.1 takes the form:

**Theorem 10.6.1** The vector $X$ from (10.6.5) is parallel along curve $\gamma$ if and only if $(X^i, X^{(\alpha)i}, X_i)$ are the solutions of the system of differential equations:

\begin{equation}
\begin{aligned}
\frac{d X^i}{dt} + X^s \omega^i_s &= 0, \\
\frac{d X^{(\alpha)i}}{dt} + X^{(\alpha)j} \omega^i_s &= 0, \quad (\alpha = 1, \ldots, k-1), \\
\frac{d X_i}{dt} - X_s \omega_s^i &= 0.
\end{aligned}
\end{equation}
We obtain, also

**Theorem 10.6.2** The Cartan spaces \( C^{(k)n} \) is with absolute parallelism of vectors, with respect to \( CΓ(N) \), if and only if all curvature \( d \)-tensors of \( CΓ(N) \) vanish.

In the case \( \dot{X} = \dot{\gamma} \), the equation \( \frac{D\dot{\gamma}}{dt} = 0 \) says that \( \gamma \) is an autoparallel curve of \( C^{(k)n} \) with respect to \( CΓ(N) \).

¿From Theorem 7.7.3 it follows:

**Theorem 10.6.3** A smooth parametrized curve \( \gamma \), (10.6.1), is an autoparallel curve of the Cartan space \( C^{(k)n} \), endowed with metrical canonical \( N \)-linear connection \( CΓ(N) \), if and only if the following system of differential equations is verified:

\[
\begin{align*}
\frac{d^2 x^i}{dt^2} + \frac{dx^i_s}{dt} \omega^j_s &= 0, \\
\frac{d}{dt} \frac{dy^{(\alpha)i}}{dt} + \frac{dy^{(\alpha)s}}{dt} \omega^j_s &= 0, \\
(\alpha = 1, ..., k - 1), \\
\frac{d}{dt} \frac{dp_i}{dt} - \frac{dp_i_s}{dt} \omega^j_s &= 0.
\end{align*}
\]

(10.6.8)

As we know from section 7, ch. 7, a curve \( \gamma \) is horizontal if \( \dot{\gamma} = \dot{\gamma}^H \), i.e.

\[
\begin{align*}
x^i &= x^i(t), \\
\frac{dy^{(\alpha)i}}{dt} &= 0, (\alpha = 1, ..., k - 1), \\
\frac{dp_i}{dt} &= 0, t \in I.
\end{align*}
\]

(10.6.9)

Therefore, taking into account the definitions of horizontal paths, \( v_\alpha \)-paths and \( w_k \)-paths (§7, ch. 6), we obtain:

**Theorem 10.6.4** The Cartan space \( C^{(k)n} \) endowed with the metrical canonical \( N \)-linear connection \( CΓ(N) \) has the following properties:

a. The horizontal paths are characterized by the system of differential equations:

\[
\frac{d^2 x^i}{dt^2} + H_j^n \frac{dx^j}{dt} \frac{dx^h}{dt} = 0, \quad \frac{dy^{(\alpha)i}}{dt} = 0, \quad \frac{dp_i}{dt} = 0.
\]

(10.6.10)

b. The \( v_\alpha \)-paths at the point \( x = x_0 \) are characterized by the system of differential equations:

\[
\begin{align*}
\frac{dx^i}{dt} &= 0, \quad \frac{dy^{(\beta)i}}{dt} = 0, (\beta \neq \alpha), \\
\frac{dp_i}{dt} &= 0, \\
\frac{d}{dt} \frac{dy^{(\alpha)i}}{dt} + C^{(\alpha)s}_{(\alpha)j} \frac{dy^{(\alpha)s}}{dt} \frac{dy^{(\alpha)j}}{dt} &= 0.
\end{align*}
\]

(10.6.11)
c. The \( w_k \)-paths at the point \( x = x_0 \) are characterized by the system of differential equations:

\[
\frac{dx^i}{dt} = 0, \quad \frac{dy^{(\alpha)i}}{dt} = 0, \quad (\alpha = 1, \ldots, k - 1),
\]

\[
\frac{d^2 p_i}{dt^2} - C_{ijs}^i (x,0,\ldots,0,p) \frac{dp_j}{dt} \frac{dp_s}{dt} = 0.
\]

10.7 Structure Equations of Metrical Canonical \( N \)-Connection

In a Cartan space of order \( k \), \( C^{(k)n} = (M,K) \), endowed with the metrical canonical \( N \)-linear connection \( C\Gamma(N) \), lemma 7.8.1 (ch. 7) holds:

The following object fields

\[
d(dx^i) - dx^m \wedge \omega^i_m, \quad d(\delta y^{(\alpha)i}) - dy^{(\alpha)m} \wedge \omega^i_m, \quad (\alpha = 1, \ldots, k - 1),
\]

\[
d(\delta p_i) + \delta p_m \wedge \omega^i_m
\]

are \( d \)-vector fields and

\[
d\omega^i_j - \omega^m_j \wedge \omega^i_m
\]

is a \( d \)-tensor field of type \((1,1)\).

Consequently, we have:

**Theorem 10.7.1** A Cartan space of order \( k \), \( C^{(k)n} = (M,K) \), has the following structure equations of the metrical canonical \( N \)-linear connection \( C\Gamma(N) \):

\[
d(dx^i) - dx^m \wedge \omega^i_m = - \Omega^i_i,
\]

\[
d(\delta y^{(\alpha)i}) - dy^{(\alpha)m} \wedge \omega^i_m = - \Omega^i_i, \quad (\alpha = 1, \ldots, k - 1),
\]

\[
d(\delta p_i) + \delta p_m \wedge \omega^i_m = -\Omega_i.
\]

and

\[
d\omega^i_j - \omega^m_j \wedge \omega^i_m = -\Omega^i_j,
\]
The Cartan Spaces of Order k

(10.7.3)

\[ (0) \quad \Omega_i = dx^j \wedge \left( k \sum_{\alpha=1}^{\alpha} C^i_{\alpha j m} \delta y^{(\alpha)m} + C^i_{j m} \delta p_m \right), \]

\[ (\alpha) \quad \Omega_i = dx^j \wedge \left( \sum_{\gamma=1}^{\gamma} \delta y^{(\gamma)j} \wedge \left( \sum_{\alpha=1}^{\alpha} C^i_{\alpha j m} \delta y^{(\gamma)m} + C^i_{j m} \delta p_m \right) \right), \quad (\alpha = 1, \ldots, k-1), \]

\[ \Omega_i = dx^j \wedge \left( \sum_{\gamma=1}^{\gamma} \delta y^{(\gamma)j} \wedge \left( \sum_{\alpha=1}^{\alpha} C^i_{\alpha j m} \delta y^{(\gamma)m} + C^i_{j m} \delta p_m \right) \right), \]

and where the 2-forms of curvature are

(10.7.4)

\[ \Omega_j^i = \frac{1}{2} P_{jhm}^i dx^h \wedge dx^m + \sum_{\gamma=1}^{\gamma} \delta y^{(\gamma)j} \wedge \left( \sum_{\alpha=1}^{\alpha} \sum_{\beta=1}^{\beta} S^i_{\alpha \beta j h m} \delta y^{(\gamma)h} \wedge \delta y^{(\beta)m} \right) \]

where, according to §5, ch. 6, \( P^i_{\alpha,0} \), \( P^i_{\alpha,1} \), \ldots, \( P^i_{\alpha,\alpha-1} \) are as follows:

(10.7.5)

\[ P^i_{\alpha,0} = d M^i_{\alpha j} - N_m^m d M^i_{\alpha j m}, \]

\[ P^i_{\alpha,1} = d M^i_{\alpha-1 j} - N_m^m d M^i_{\alpha-1 j m}, \]

\[ \ldots \]

Of course this theorem is important for the theory of metrical canonical connection. Also, it allows to determine the Bianchi identities of the spaces \( C^{(k)n} \).

We can use the previous results in a theory of Cartan subspaces of order \( k \) in the Cartan space \( C^{(k)n} \), cf. Ch.9.
10.8 Riemannian Almost Contact Structure of the Space $C^{(k)n}$

Consider a Cartan space of order $k$, $C^{(k)n} = (M, K(x, y^{(1)}, \ldots, y^{(k-1)}, p))$ and its canonical nonlinear connection $N$.

The adapted basis $(\delta x^i, \delta y^{(1)}_i, \ldots, \delta y^{(k-1)}_i, \delta p_i)$, $\delta x^i = \frac{\partial}{\partial x^i}$ and its dual basis $(\delta x^i, \delta y^{(1)}_i, \ldots, \delta y^{(k-1)}_i, \delta p_i)$, where $\delta x^i = dx^i$ are well determined by $N$. As we know from section 1, the fundamental tensor of $C^{(k)n}$ is:

\[(10.8.1) \quad g^{ij} = \frac{1}{2} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} K^2.\]

Taking into account the associated Hamilton space $H^{(k)n} = (M, K^2)$ of the space $C^{(k)n}$, we can study the Riemannian almost contact structure of the Cartan space of order $k$ by means of the corresponding structure of the space $H^{(k)n} = (M, K^2)$. Such that, using the covariant tensor $g^{ij}$ of the fundamental tensor $g^{ij}$ we define the tensor

\[(10.8.2) \quad \nabla_G = g_{ij} dx^i \otimes dx^j + \sum_{\alpha=1}^{k-1} g_{ij} \delta y^{(\alpha)i} \otimes \delta y^{(\alpha)j} + g^{ij} \delta p_i \otimes \delta p_j,\]

$\nabla_G$ is the $N$-lift of the fundamental tensor $g^{ij}$ of the space $C^{(k)n}$. Since $g^{ij}$ is positively defined on $T^*kM$ and $N$ is given on $T^*kM$, it follows:

**Theorem 10.8.1** $\nabla_G$ is a Riemannian structure on the manifold $T^*kM$ determined only by the fundamental tensor $g^{ij}$ of the Cartan space $C^{(k)n}$ and by the canonical nonlinear connection $N$.

$\nabla_G$ The distributions $N_0$, $N_1$, $\ldots$, $N_{k-2}$, $V_{k-1}$, $W_k$ are mutual orthogonal with respect to $\nabla_G$.

**Proposition 10.8.1** The tensor $\nabla_G$ is not homogeneous on the fibres of the bundle $T^*kM$.

Indeed, the first term in $\nabla_G$ is 0-homogeneous, the second term is 2-homogeneous, $\ldots$, the last term is 2$k$-homogeneous. So, the whole $\nabla_G$ is not homogeneous.

Let us consider the following invariants:

\[(10.8.3) \quad K_1^2 = g_{ij} z^{(1)i} z^{(1)j}, \quad K_2^2 = g_{ij} z^{(2)i} z^{(2)j}, \ldots, \quad K_{k-1}^2 = g_{ij} z^{(k-1)i} z^{(k-1)j}, \quad K_0^2 = g^{ij} p_i p_j,\]

where $z^{(1)i}$, $\ldots$, $z^{(k)i}$ are the Liouville vector fields determined by the canonical nonlinear connection of space $C^{(k)n}$.
Of course, all invariants \( K_0^2, K_1^2, ..., K_{k-1}^2 \) are positive.

Thus we can construct a new Riemannian structure on \( \tilde{T}^k M \):

\[
(10.8.4) \quad \mathcal{G}^k = g_{ij} dx^i \otimes dx^j + \sum_{\alpha=1}^{k-1} \frac{1}{K_0} g_{ij} \delta y^{(\alpha)i} \otimes \delta y^{(\alpha)j} + \frac{1}{K_0^2} g^{ij} \delta p_i \otimes \delta p_j.
\]

**Theorem 10.8.2** \( \mathcal{G}^k \) is a Riemannian structure on the manifold \( \tilde{T}^k M \) determined only by \( g^{ij} \) and \( N \).

1. \( \mathcal{G}^k \) is \( 0 \)-homogeneous on the fibres of the bundle \( T^k M \).
2. The distributions \( N_0, N_1, ..., N_{k-2}, V_{k-1}, W_k \) are mutual orthogonal with respect to \( \mathcal{G} \).

The proof follows without difficulties.

The Riemannian structure \( \mathcal{G} \) is of the form

\[
(10.8.5) \quad \mathcal{G} = \mathcal{G}^H + \mathcal{G}^V + \cdots + \mathcal{G}^{V_{k-1}} + \mathcal{G}^W
\]

with

\[
(10.8.6) \quad \mathcal{G}^H = g_{ij} dx^i \otimes dx^j, \quad \mathcal{G}^V = g_{ij} \delta y^{(1)i} \otimes \delta y^{(1)j}, ..., \quad \mathcal{G}^{V_{k-1}} = g_{ij} \delta y^{(k-1)i} \otimes \delta y^{(k-1)j}, \quad \mathcal{G}^W = g^{ij} \delta p_i \otimes \delta p_j.
\]

The tensors \( \mathcal{G}^H, \mathcal{G}^V, (\alpha = 1, ..., k-1) \) and \( \mathcal{G}^W \) are \( d \)-tensor fields on the manifold \( T^k M \).

Exactly as in the chapter 8 we can prove:

**Theorem 10.8.3** The \( d \)-tensor fields \( \mathcal{G}^H, \mathcal{G}^V, (\alpha = 1, ..., k-1) \) and \( \mathcal{G}^W \) are covariant constant with respect to the canonical metrical \( N \)-connection of the Cartan space of order \( k \), \( C^{(k)n} \).

Now, let us consider the natural almost \((k-1)n\)-contact structure \( \mathcal{F} \) determined by the canonical nonlinear connection \( N \). It is defined by (6.6.3):

\[
(10.8.7) \quad \mathcal{F} \left( \delta \frac{\partial}{\partial x^i} \right) = - \frac{\partial}{\partial y^{(k-1)i}} \delta x^i, \quad \mathcal{F} \left( \frac{\partial}{\partial y^{(k-1)i}} \right) = \delta \frac{\partial}{\partial x^i}, \quad \mathcal{F} \left( \frac{\partial}{\partial \delta y^{(\alpha)i}} \right) = 0, \quad (\alpha = 1, k-1), \quad \mathcal{F} \left( \frac{\partial}{\partial \delta p_i} \right) = 0.
\]

\( \mathcal{F} \) is a tensor field of type \((1, 1)\) and in adapted basis it is expressed by (3.5.4).

The condition of normality of \( \mathcal{F} \) is given by the equation (6.6.5).
But the pair of structures \((\mathcal{G}, \mathcal{F})\) is a Riemannian almost \((k - 1)n\) contact structure on \(T^{*k}M\) determined only by \(N\) and by the fundamental function \(K\) of the Cartan space \(C^{(k)n}\). So, we have:

**Theorem 10.8.4** On the manifold \(T^{*k}M\) there exists a natural Riemannian almost \((k - 1)n\)-contact structure \((\mathcal{G}, \mathcal{F})\), determined only by the canonical nonlinear connection \(N\) and the fundamental function \(K\) of the Cartan space \(C^{(k)n} = (M, K)\).

The canonical nonlinear connection \(N\) and the fundamental tensor \(g^{ij}\) of the Cartan space \(C^{(k)n}\) determine the almost \((k - 1)n\)-contact structure \(\mathcal{F}\):

\[
\mathcal{F}\left( \frac{\delta}{\delta x^i} \right) = -g_{ij} \frac{\partial}{\partial p_j}, \quad \mathcal{F}\left( \frac{\delta^i}{\delta x^j} \right) = 0, \quad (\alpha = 1, k - 1), \quad \mathcal{F}\left( \frac{\delta}{\delta p_i} \right) = g^{ij} \frac{\delta}{\delta x^j}.
\]

Theorem 6.7.1, reads:

**Theorem 10.8.5** The structure \(\mathcal{F}\) of a Cartan space \(C^{(k)n}\) has the following properties:

\[
\mathcal{F} = -g_{ij} \frac{\delta}{\delta p_j} \otimes dx^i + g^{ij} \frac{\delta}{\delta x^i} \otimes \delta p_j,
\]

\[
\text{Ker} \mathcal{F} = N_1 \oplus \cdots \oplus N_{k-1}, \quad \text{Im} \mathcal{F} = N_0 \oplus W_k,
\]

\[
\text{rank} \mathcal{F} = 2n,
\]

\[
\mathcal{F}^3 + \mathcal{F} = 0.
\]

Consequently, \(\mathcal{F}\) is an almost \((k - 1)n\)-contact structure on the manifold \(\widetilde{T^{*k}M}\).

We have, also:

**Theorem 10.8.6** For a Cartan space \(C^{(k)n}\) the following properties hold:

1. The pair \((\mathcal{G}, \mathcal{F})\) is a Riemannian almost \((k - 1)n\)-contact structure determined only by the canonical nonlinear connection \(N\) and the fundamental tensor \(g^{ij}\).

2. The associated 2-form is

\[
\theta = \delta p_i \wedge dx^i
\]

and if the coefficients \(N_{ij}\) of \(N\) are symmetric then \(\theta\) is the canonical presymplectic structure

\[
\theta = dp_i \wedge dx^i.
\]

Concluding, the space \((\tilde{T^{*k}M}, \mathcal{G}, \mathcal{F})\) is the geometrical model of the Cartan space of order \(k, C^{(k)n}\).
Chapter 11

Generalized Hamilton and Cartan Spaces of Order $k$. Applications to Hamiltonian Relativistic Optics

On the total space of the dual bundle $(T^*kM, \pi^*, M)$ there exist some geometrical structures defined by a general $d$-tensor field $g^{ij}(x, y^{(1)}, ..., y^{(k-1)}, p)$, symmetric and nondegenerate, which are suggested by the Relativistic Optics. Generally the $d$-tensor $g^{ij}$ is not the metric tensor of a Hamilton space $H^{(k)n}$ or of a Cartan space $C^{(k)n}$. In this case the pair $GH^{(k)n} = (M, g^{ij}(x, y^{(1)}, ..., y^{(k-1)}, p))$ defines a 'Generalized Hamilton space of order $k$'. If $g^{ij}$ is 0-homogeneous on the fibres of the bundle $T^*kM$ we say that the pair $GC^{(k)n}$ is a 'Generalized Cartan space of order $k$'.

We study, in this chapter, the geometry of these spaces and apply it to the theory of Hamiltonian Relativistic Optics.

11.1 The Space $GH^{(k)n}$

Definition 11.1.1 A generalized Hamilton space of order $k$ is a pair $GH^{(k)n} = (M, g^{ij}(x, y^{(1)}, ..., y^{(k-1)}, p))$, where

1. $g^{ij}$ is a $d$-tensor field of type $(2, 0)$, symmetric and nondegenerate on the manifold $\widetilde{T^*kM}$:

\begin{equation}
\text{rank} \|g^{ij}\| = n
\end{equation}

2. The quadratic form $g^{ij}X_iX_j$ has a constant signature on $\widetilde{T^*kM}$.

217
The tensor $g^{ij}$ is called fundamental for the space $GH^{(k)n}$.

In the case when the base manifold $M$ is paracompact then $\tilde{T}^k M$ is paracompact. Thus on $\tilde{T}^k M$ there exists the $d$-tensor $g^{ij}$, with the property that $GH^{(k)n} = (M, g^{ij})$ is a generalized Hamilton space of order $k$.

If $g^{ij}$ is positively defined on $\tilde{T}^k M$, then the conditions (11.1.1) is verified.

**Definition 11.1.2** The space $GH^{(k)n} = (M, g^{ij})$ is called reducible to a Hamilton space of order $k$, if there exists an Hamiltonian $H(x, y^{(1)}, \ldots, y^{(k-1)}, p)$ such that the following equality holds:

\[(11.1.2) \quad g^{ij} = \frac{1}{2} \partial^i \partial^j H\]

Let us consider the $d$-tensor field:

\[(11.1.3) \quad C^{ijh} = -\frac{1}{2} \partial^i \partial^j \partial^h g^{ij}\]

We have:

**Proposition 11.1.1** A necessary condition for a generalized Hamilton space $GH^{(k)n} = (M, g^{ij})$ be reducible to a Hamilton space of order $k$ is that the $d$-tensor field $C^{ijh}$ be totally symmetric.

Indeed, if (11.1.2) holds, then $C^{ijh} = -\frac{1}{4} \partial^i \partial^j \partial^h H$ is totally symmetric.

**Example 1.**

\[\text{(1)} \quad \text{Let } g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = \gamma^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}) \text{ be a } d\text{-tensor on the } T^k M \text{ determined by the fundamental tensor } \gamma^{ij} \text{ of a Finsler space of order } k-1.\]

The space $GH^{(k)n} = (M, g^{ij})$ is reducible to the Hamilton space $H^{(k)n} = (M, g^{ij} \partial^i \partial^j)$.

\[\text{(2)} \quad \text{Consider the fundamental tensor } \gamma^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}) \text{ of a Finsler space of order } k-1, F^{(k-1)n} = (M, y^{(1)}, \ldots, y^{(k-1)}). \text{ The } d\text{-tensor field}\]

\[(11.1.4) \quad g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = e^{-2\sigma(x, y^{(1)}, \ldots, y^{(k-1)}, p)} \gamma^{ij}(x, y^{(1)}, \ldots, y^{(k-1)})\]

with the property $\sigma \in F(T^k M)$ and $\gamma^{ij}$ is a fundamental tensor of $F^{(k-1)n}$, determines a generalized Hamilton space of order $k$, $GH^{(k)n} = (M, g^{ij})$.

This space is reducible to a Hamilton space $H^{(k)n}$ only if $\partial^i \sigma = 0$.

Let $g_{ij}$ be the covariant tensor of the fundamental tensor $g^{ij}$ of the space $GH^{(k)n} = (M, g^{ij})$. Then we have:

\[(11.1.5) \quad g_{ih} g^{hj} = \delta^j_i\]

Also, we consider the $d$-tensor field
Generalized Hamilton and Cartan Spaces of Order \( k \)

\[ C^j_i = -\frac{1}{2}g_{is}(\partial^j g^{sh} + \partial^j g^{js} - \partial^s g^{jh}) \]

Evidently:

\[ S^j_i = C^j_i - C^i_j = 0. \]

If the space \( GH^{(k)n} \) is reducible to a Hamilton space then:

\[ C^j_i = -g_{is}C^{shj}. \]

The \( d \)-tensor field \( C^j_i \) are the coefficients of the \( w_k \)-covariant derivatives.

Indeed, we have:

\[ g^{ij} = \partial^i g^{ij} + C^i_s g^{sj} + C^j_s g^{is} = 0. \]

### 11.2 Metrical \( N \)-Linear Connections

If we consider a generalized Hamilton space of order \( k \), \( GH^{(k)n} = (M, g^{ij}) \), in general we cannot determine a nonlinear connection only by means of the fundamental tensor \( g^{ij} \).

But there are some particular cases when this is possible. For instance, in examples 1° and 2° we can consider the canonical nonlinear connection \( N \) of the Finsler space \( F^{(k-1)n} = (M, F(x, y^{(1)}, ..., y^{(k-1)}) \) with the coefficients \( (N^i_j, ... N^i_j) \). Then the system of functions \( (N^i_j, ... N^i_j, N_{ij}) \) with \( N_{ij} = \delta^{(1)} \)

\[ \frac{\delta}{\delta y^{(1)}_j}(N^i_j p_k), \text{ if } N^i_j(x, y^{(1)}), \text{ does not depend on } y^{(2)i}, ..., y^{(k-1)i} \]

determines a nonlinear connection only by means of the fundamental tensor \( g^{ij} \) of the space.

Now, let us consider an apriori fixed nonlinear connection \( N \), with the coefficients \( (N^i_j, ... N^i_j, N_{ij}) \) on the manifold \( T^{*k}M \). We will study the geometry of the space \( GH^{(k)n} \) endowed with the nonlinear connection \( N \).

The adapted basis to the direct decomposition \( (6.2.9) \),

\[ (\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)}}, ..., \frac{\delta}{\delta y^{(k-1)}}, \frac{\partial}{\partial p_i}) \]

is expressed in \( (6.2.10) \) and the adapted cobasis \( (dx^i, \delta y^{(1)}_i, ..., \delta y^{(k-1)}_i, \delta p_i) \) is written in \( (6.3.2) \).

Now, as usual, we can prove:

**Theorem 11.2.1** 1) A generalized Hamilton space of order \( k \) endowed with a nonlinear connection \( N \), has an unique \( N \)-linear connection \( CT(N) = (H^j_i, C^j_i, ..., C^j_i, C^j_i) \) satisfying the following axioms:

- \[ (H^j_i, C^j_i, ..., C^j_i, C^j_i) \]
1° The nonlinear connection $N$ is apriori given.

2° $g^{ij}|_h = 0$, $g^{ij}^{(\alpha)}|_h = 0$, ($\alpha = 1, ..., k - 1$), $g^{ij}|^h = 0$.

3° $T^j_i|^h = 0$, $S_i^j|^h = 0$, ($\alpha = 1, ..., k - 1$), $S_i^j|^h = 0$.

2) The coefficients of $\Gamma(N)$ are the following generalized Christoffel symbols:

\[
H_{ij}^k = \frac{1}{2}g^{is}(\delta g_{sk} + \delta g_{is} - \delta g_{ik}) ,
\]

\[
C_{ij}^k(\alpha) = \frac{1}{2}g^{is}(\delta g_{sk}^{(\alpha)} + \delta g_{is}^{(\alpha)} - \delta g_{ik}^{(\alpha)}) , (\alpha = 1, ..., k - 1) ,
\]

\[
C_{ij}^k = -\frac{1}{2}g^{is}(\partial^j g_{sk}^{i} + \partial^h g_{is}^{j} - \partial^k g_{is}^{j}) .
\]

The previous connection $\Gamma(N)$ is called canonical metrical $N$-connection.

More general, one proves

**Theorem 11.2.2** 1) A generalized Hamilton space $GH^{(k)n}$, endowed with a nonlinear connection $N$, has an unique $N$-linear connection $\Gamma(N)$ ($\alpha = 1, ..., k - 1$) satisfying the axioms:

1° $N$ is apriori given on $\tilde{T}^k M$.

2° $h$-, $v_\alpha$- and $w_k$- covariant derivation of $g^{ij}$ vanish:

\[
g^{ij}|_h = 0, g^{ij}^{(\alpha)}|_h = 0, g^{ij}|^h = 0,
\]

3° The skewsymmetric tensors of torsion

\[
\tilde{T}^i_{jh} = \tilde{H}^i_{jh} - \tilde{H}^h_{ij}, \tilde{S}^i_{jh} = \tilde{C}^i_{jh} - \tilde{C}^h_{ij}, (\alpha = 1, ..., k - 1), \tilde{S}^j_i = \tilde{C}^j_i - \tilde{C}^h_i
\]

are apriori given.

2) This connection has the following coefficients:

\[
\tilde{H}^i_{jh} = H^i_{jh} + \frac{1}{2}g^{is}(g_{sr}\tilde{T}^r_{jh} - g_{jr}\tilde{T}^r_{sh} + g_{hr}\tilde{T}^r_{js}) ,
\]

\[
\tilde{C}^i_{jh}(\alpha) = C^i_{jh}(\alpha) + \frac{1}{2}g^{is}(g_{sr}\tilde{S}^r_{jh}(\alpha) - g_{jr}\tilde{S}^r_{sh}(\alpha) + g_{hr}\tilde{S}^r_{js}(\alpha)) , (\alpha = 1, ..., k - 1) ,
\]

\[
\tilde{C}^h_i = C^h_i - \frac{1}{2}g^{is}(g^h_s\tilde{S}^r_{ih} - g^h_r\tilde{S}^r_{ih} + g^h_s\tilde{S}^r_{ih}) ,
\]

where $(H^i_{jh}, C^i_{jh}(\alpha), C^h_i^{(1)}, ..., C^i_{jh}(k)$ are the coefficients of the canonical metrical $N$-connection $\Gamma(N)$. 
If we are interested on the all \( N \)-linear connections which verify the equations (11.2.2), we can prove:

**Theorem 11.2.3** In a generalized Hamilton space \( GH^{(k)n} = (M, g^{ij}) \) the set of all \( N \)-linear connections \( \mathcal{D} \Gamma(N) \) which satisfy the equations (11.2.2) is given by

\[
\mathcal{T}_{jh} = H^j_{jh} + \Omega^{is}_{rj} X^r_{sh},
\]

(11.2.3)

\[
\mathcal{C}_{jh} = C_{jh} + \Omega^{is}_{rj} X^r_{sh}, \quad (\alpha = 1, \ldots, k - 1)
\]

where \( \Omega^{is}_{rj} = \frac{1}{2} (\delta^i_r \delta^s_j - g_{rj} g^{is}) \) are Obata’s operators and \( CT(N) = (H^j_{jh}, C_{jh}, \ldots, (1)) \)

\( C_{jh} \) is the canonical metrical \( N \)-connection and \( X^r_{sh}, X^r_{sh} \), \( (\alpha = 1, \ldots, k - 1) \), \( X^r_s \) are arbitrary \( d \)-tensor fields.

**Corollary 11.2.1** The mappings \( \mathcal{D} \Gamma(N) \to \mathcal{D} \Gamma(N) \) determined by (11.2.3), together with the composition of these mappings is an abelian group.

Now, we can repeat all considerations from the section 7 of the chapter 8. So, we have:

**Proposition 11.2.1** The curvature \( d \)-tensor fields of the canonical metrical \( N \)-connection \( CT(N) \), (11.2.1) satisfy the identities (7.6.8).

**Proposition 11.2.2** The tensors of deflection of \( CT(N) \), (11.2.1) satisfy the identities (7.6.9), with \( T^i_{jk} = 0 \), \( S^i_{jh} = 0 \), \( S^i_{jh} = 0 \), \( (\alpha = 1, \ldots, k - 1) \).

Let \( \omega^i_j \) be the 1-forms connection of \( CT(N) \),

\[
\omega^i_j = H^i_{j} dx^i + \sum_{\alpha=1}^{k-1} C^i_{j\alpha} \delta y^{(\alpha)} + C^i_j \delta p_s
\]

(11.2.4)

and a curve \( \gamma : t \in I \to \gamma(t) \in T^*k M \) expressed by (7.7.1).

Thus, we have (cf. Th. 7.7.1):

**Theorem 11.2.4** In a space \( GH^{(k)n} \) endowed with the canonical metrical \( N \)-connection \( CT(N) \) a vector field \( X = X_{(0)i} \frac{\delta}{\delta x^i} + \ldots + X_{(k-1)i} \frac{\delta}{\delta y^{(k-1)i}} + X_i \frac{\partial}{\partial p_i} \)
is parallel along curve \( \gamma \) if and only if \((X^{(0)}i, \ldots, X^{(k-1)}i, X_i)\) are the solutions of the system of differential equations

\[
\frac{dX^{(0)}i}{dt} + X^{(0)}s \omega^i_s = 0, \ldots, \frac{dX^{(k-1)}i}{dt} + X^{(k-1)}s \omega^i_s = 0, \quad \frac{dX_i}{dt} - X_s \omega^i_s = 0.
\]

**Theorem 11.2.5** The space \( GH^{(k)n} \) endowed with the \( N \)-linear connection \( C^\Gamma(N) \) is with absolute parallelism of vectors if and only if all curvature \( d \)-tensor of \( C^\Gamma(N) \) vanish.

A smooth parametrized curve \( \gamma : t \in I \rightarrow (x(t), y^{(1)}(t), \ldots, y^{(k-1)}(t), p(t)) \in T^kM \) is an autoparallel curve of \( C^\Gamma(N) \) if \( D_\gamma \gamma = 0 \).

Thus, applying Theorem 7.7.3 we have:

**Theorem 11.2.6** The curve \( \gamma : I \rightarrow T^kM \) is autoparallel for the space \( GH^{(k)n} \), with respect to the canonical \( N \)-linear connection \( C^\Gamma(N) \) if and only if the following system of differential equations is verified:

\[
\frac{d^2x^i}{dt^2} + \frac{dx^s \omega^i_s}{dt} = 0,
\]

\[
\frac{d}{dt} \left( \frac{\delta y^{(\alpha)}i}{dt} \right) + \frac{\delta y^{(\alpha)}s \omega^i_s}{dt} = 0, (\alpha = 1, \ldots, k-1),
\]

\[
\frac{d}{dt} \left( \frac{\delta p_i}{dt} \right) - \frac{\delta p_s \omega^i_s}{dt} = 0.
\]

Recall that \( \gamma \) is a horizontal curve if and only if:

\[
x^i = x^i(t), \quad \frac{\delta p_i}{dt} = 0, \quad \frac{\delta y^{(\alpha)}i}{dt} = 0, (\alpha = 1, \ldots, k-1).
\]

Therefore, we have:

**Theorem 11.2.7** For a generalized Hamilton space of order \( k \), \( GH^{(k)n} \) endowed with the canonical metrical \( N \)-connection \( C^\Gamma(N) \) the following properties hold:

1. The horizontal paths are characterized by the system of differential equations:

\[
\frac{d^2x^i}{dt^2} + H^i_j \frac{dx^j}{dt} \frac{dx^h}{dt} = 0, \quad \frac{\delta y^{(\alpha)}i}{dt} = 0, \quad \frac{\delta p_i}{dt} = 0, (\alpha = 1, \ldots, k-1).
\]

2. The \( v_\alpha \)-paths in a point \( x_0 \in M \) are characterized by

\[
x^i = x_0^i, \quad \frac{\delta y^{(\beta)}i}{dt} = 0, (\beta \neq \alpha), \quad \frac{dp_i}{dt} = 0,
\]
\[ \frac{d}{dt} \left( \frac{\delta y^{(\alpha) i}}{dt} \right) + C_{sj}^{(\alpha) i j} \frac{dy^{(\alpha) j}}{dt} = 0 \]

3° The \( w_k \)-path are characterized by
\[ \frac{dx^i}{dt} = 0, \quad \frac{dy^{(1) i}}{dt} = \ldots = \frac{dy^{(k-1) i}}{dt} = 0, \]
\[ \frac{d^2 p_i}{dt^2} - C^{jm}_{i}(x, y^{(1)}, \ldots, y^{(k-1)}) \frac{dp_j}{dt} \frac{dp_m}{dt} = 0. \]

Finally, we remark:

The structure equations of the canonical metrical \( N \)-connection \( C \Gamma(N) \) of the space \( GH^{(k)n} \) are given by Theorem 7.8.1.

11.3 Hamiltonian Relativistic Optics

In the book: 'The Geometry of Higher Order Lagrange Spaces' - Kluwer FTPH, Vol. 82 is given a generalized Lagrange metric, formula (10.5.6) of the Relativistic Optics of order \( k \). It is rather complicated. In the dual spaces \( T^*^k M \) it can be introduced much more simple.

Consider a semidefined Finsler space of order \( k - 1 \), \( F^{(k-1)} M = (M, F(x, y^{(1)}, \ldots, y^{(k-1)})) \) and \( \gamma_{ij}(x, y^{(1)}, \ldots, y^{(k-1)}) \) its fundamental tensor field.

The projection \( \pi^k_{k-1} : (x, y^{(1)}, \ldots, y^{(k-1)}, p) \in T^k M \rightarrow (x, y^{(1)}, \ldots, y^{(k-1)}) \in T^{k-1} M \) allows to consider \( d \)-tensor \( \gamma_{ij} \circ \pi^k_{k-1} \) on \( T^k M \). It will be denoted by \( \gamma_{ij} \), too. Its contravariant \( \gamma^{ij} \) will be considered defined on the manifold \( \widetilde{T^k M} \).

Let us consider a differentiable function \( n \) on \( T^k M \) with the property \( n > 1 \). It will be called a refractive index.

Some notations:

\[
\begin{align*}
\vee p^i &= \gamma^{ij} p_j, & p_i &= \gamma_{ij} p^j, & \vee p^i &= \gamma^{ij} p_j p_j = \|p\|^2, \\
\frac{1}{n(x, y^{(1)}, \ldots, y^{(k-1)}, p)} &= u(x, y^{(1)}, \ldots, y^{(k-1)}, p).
\end{align*}
\]

Now, we define on \( \widetilde{T^k M} \) the \( d \)-tensor field

\[
g^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}, p) = \\
= \gamma^{ij}(x, y^{(1)}, \ldots, y^{(k-1)}) + \left( 1 - \frac{1}{n^2(x, y^{(1)}, \ldots, y^{(k-1)}, p)} \right) \vee p^i \vee p^j
\]
\textbf{Theorem 11.3.1} \( f^* g^{ij} \) is a symmetric \( d \)-tensor field of type \((2,0)\) on the manifold \( T^*kM \).

\( 2^o \) \text{rank} \( g^{ij} \) = \( n \).

Indeed, \( 1^o \) \( g^{ij} \) is a sum of two symmetric \( d \)-tensor of type \((2,0)\).

\( 2^o \) Consider the \( d \)-tensor

\begin{equation}
(11.3.3) \quad g^{ij} = \gamma^{ij} - \frac{1}{a^2} \left( 1 - \frac{1}{n^2} \right) p_ip_j,
\end{equation}

where

\begin{equation}
(11.3.3a) \quad a = 1 + \left( 1 - \frac{1}{n^2} \right) \|p\|^2.
\end{equation}

It is easy to verify the following equality:

\begin{equation}
(11.3.4) \quad g^{ih}g_{hj} = \delta^i_j.
\end{equation}

Consequently the pair \( GH^{(k)n} = (M, g^{ij}) \) is a generalized Hamilton space of order \( k \).

\textbf{Theorem 11.3.2} The space \( GH^{(k)n} = (M, g^{ij}) \) is not reducible to a Hamilton space of order \( k \).

\textbf{Proof.} The tensor field \( C^{ijk} \) from (11.1.3) is as follows

\begin{equation}
-C^{ijh} = \partial^h \sigma \left( \hat{p}^j \hat{p}^i + \hat{p}^i \hat{p}^j \right) + \gamma^{ih} \gamma^{jh} \gamma^{jk}, \quad \sigma = 1 - u^2
\end{equation}

If we assume that \( C^{ijh} = C^{ihj} \), we obtain

\begin{equation}
(*) \quad \left( \partial^h \sigma \hat{p}^j - \partial^j \sigma \hat{p}^h \right) \gamma^{ih} \gamma^{jh} \gamma^{jk} = 0
\end{equation}

Contracting by \( p_i \) we have

\begin{equation}
\hat{p}^j \gamma^{ih} p^i - \hat{p}^i \gamma^{ij} p^h = 0.
\end{equation}

Substituting in (*) we get \( \gamma^{ih} \gamma^{j} p^i - \gamma^{ij} p^h = 0 \). A new contraction with \( \gamma^{ih} \) leads to \((n - 1) p^j = 0 \). Consequently \( p^j = 0 \), i.e. \( p_j = 0 \). But this is impossible and the assumption we made is false. \textit{q.e.d.}

The space \( GH^{(k)n} \) will be called the generalized Hamiltonian space of order \( k \) of the Hamiltonian Relativistic Optics.
Let us consider a local $d$-vector field $V^i(x)$ and a local $d$-covector field $\eta_i(x)$ on the manifold $M$. It is not difficult to see that the mapping

$$S_{V,\eta}: M \rightarrow T^{*k}M \quad \text{defined locally by}$$

$$x^i = x^{i'},$$

$$(11.3.5) \quad y^{(1)i} = V^i(x), \ldots, y^{(k-1)i} = \frac{1}{(k-2)!} \frac{d^{k-2}V^i}{dt^{k-2}},$$

$$p_i = \eta_i(x)$$

is a cross-section of the canonical projection $\pi^{*k}: T^{*k}M \rightarrow M$. It follows that $S_{V,\eta}(M)$ is a local embedding of $M$ in the manifold $\tilde{T}^{*k}M$.

The restriction of the fundamental tensor $g^{ij}$ of the space $GH^{(k)}n$ to $S_{V,\eta}$ will be called the Synge metric of the Hamiltonian Relativistic Optics. The restriction of function $n(x, y^{(1)}, \ldots, y^{(k-1)}, p)$ to $S_{V,\eta}(M)$ is the refractive index of the dispersive optic medium:

$$(M, V(x), \eta(x), n(x, V(x), \ldots, \frac{1}{(k-2)!} \frac{d^{k-2}V}{dt^{k-2}}, \eta(x)).$$

Therefore, we say that the geometry of the generalized Hamilton space of order $k$, $GH^{(k)}n = (M, g^{ij})$, with the fundamental tensor $g^{ij}$ in (11.3.2) is the geometrical theory of the previous optic medium, endowed with Synge metric.

If the refractive index $n$ depend on $x \in M$ only, then the optic medium is called nondispersive.

Let us consider the canonical nonlinear connection $\tilde{N}$ with the dual coefficients $M^i_j^{(1)}, \ldots, M^i_j^{(k-1)}$ of the Finsler space $F^{(k-1)}n = (M, F)$. For simplicity we assume that $M^i_j^{(1)} = M^i_j(x, y^{(1)})$, (see ch. 8, §8). In this case $(M^i_j^{(1)}, \ldots, M^i_j^{(k-1)}, N_{ij})$, with:

$$(11.3.6) \quad N_{ij} = \frac{\delta}{\delta y^{(1)i}} (M^j_p^{(1)}) p_h$$

define a nonlinear connection $N$ of the space $GH^{(k)}n = (M, g^{ij})$ determined only by the fundamental tensor $g^{ij}$.

Let $(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \ldots, \frac{\delta}{\delta y^{(k-1)i}}, \frac{\delta}{\delta p_i})$ and $(dx^i, \delta y^{(1)i}, \ldots, \delta y^{(k-1)i}, \delta p_i)$ be the adapted local basis and adapted local cobasis corresponding to the nonlinear connection $N$.

We can determine the canonical metrical $N$-linear connection $\Gamma(N)$ of the space $GH^{(k)}n$ starting from the expressions (11.3.2) and (11.3.3) for the fundamental tensor $g^{ij}$ and its covariant tensor field $g_{ij}$ and applying the usual techniques of calculus.
The connections $N$ and $\Gamma \Gamma(N)$ allow to study the geometry of the space $GH^{(k)n}$. Taking the restriction of its main geometrical object field to the section $S_{V,\gamma}(M)$ we obtain the main notions and properties of the Hamiltonian Relativistic Optics.

Evidently, there are some particular cases, as:

a) the nondispersive media

b) the Finsler space $F^{(k-1)n}$ is the prolongation of order $k-1$ of a Finsler space $F^n = (M, F(x, y^{(1)}))$.

Let us consider the absolute energy of the space $GH^{(k)n}$, [115]:

\begin{equation}
E = g^{ij} p_i p_j
\end{equation}

(11.3.7)

Taking into account the expression (11.3.2) of $g^{ij}$ and the function $a$ from (11.3.2a), we get:

\begin{equation}
E = a \|p\|^2
\end{equation}

(11.3.7a)

Consequently, we can say that $E(x, y^{(1)}, ..., y^{(k-1)}, p)$ is a differentiable Hamiltonian uniquely determined by the fundamental tensor $g^{ij}$ of the space $GH^{(k)n}$. It allows to determine the Hamilton - Jacobi equations of the space. These equations are given by (5.1.17), (5.1.17'). The energies of order $k-1$, ..., 1 are expressed in (5.3.1) with $H = E$ and the law of conservation is mentioned in Theorem 5.3.2. Also, Theorem 5.5.3 gives the N\"other symmetries for the Hamiltonian $H = E$ from (11.3.7).

11.4 The Metrical Almost Contact Structure of the Space $GH^{(k)n}$

The generalized Hamilton space of order $k$, $GH^{(k)n} = (M, g^{ij})$, endowed with an a priori given nonlinear connection $N$ determines a metrical almost contact structure on the manifold $T^*kM$.

The $N$-lift of the fundamental tensor field $g^{ij}$ is

\begin{equation}
\mathcal{V} g = g_{ij} dx^i \otimes dx^j + \sum_{\alpha=1}^{k-1} g_{ij} \delta y^{(\alpha)i} \otimes \delta y^{(\alpha)j} + g^{ij} \delta p_i \otimes \delta p_j
\end{equation}

(11.4.1)

Evidently:

1° $\mathcal{V} g$ is a pseudo-Riemannian structure on $T^*kM$.

2° The distributions $N_0, N_1, ..., N_{k-1}, V_{k-1}$ and $W_k$ are mutual orthogonal with respect to $\mathcal{V} g$.

3° $\mathcal{G}^H = g_{ij} dx^i \otimes dx^j$, $\mathcal{G}^V_\alpha = g_{ij} \delta y^{(\alpha)i} \otimes \delta y^{(\alpha)j}$, ($\alpha = 1, ..., k-1$),

\begin{equation}
\mathcal{V} g = g_{ij} dx^i \otimes dx^j + \sum_{\alpha=1}^{k-1} g_{ij} \delta y^{(\alpha)i} \otimes \delta y^{(\alpha)j} + g^{ij} \delta p_i \otimes \delta p_j
\end{equation}
$G^W_k = g^{ij} \delta p_i \otimes \delta p_j$ are $d$-tensor fields.

4° $G^H, G^V_\alpha, (\alpha = 1, \ldots, k-1)$ and $G^W_k$ are covariant constant with respect to the canonical metrical $N$ -connection $C\Gamma(N)$.

The geometrical object fields $N$ and $g^{ij}$ determine an almost contact $(k-1)n$-structure $\mathcal{F}$, (given in (6.8.2)):

\begin{equation}
\mathcal{F}(\delta_x) = -g^{ij} \frac{\delta}{\delta p_j} \otimes dx_i + g^{ij} \frac{\delta}{\delta x^i} \otimes \delta p_j.
\end{equation}

Theorem 6.8.1, from ch. 6 is valid:

**Theorem 11.4.1** 1° The structure $\mathcal{F}$ is defined only by $N$ and $g^{ij}$.

2° $\mathcal{F}$ is the following tensor field of type (11.1.1) on $T^*kM$:

\begin{equation}
\mathcal{F} = -g^{ij} \frac{\delta}{\delta p_j} \otimes dx_i + g^{ij} \frac{\delta}{\delta x^i} \otimes \delta p_j.
\end{equation}

3° $\ker \mathcal{F} = N_1 \oplus \ldots \oplus N_{k-1}$, $\text{im } \mathcal{F} = N \oplus W_k$.

4° $\text{rank } \mathcal{F} = 2n$.

5° $\mathcal{F}^3 + \mathcal{F} = 0$.

Consequently $\mathcal{F}$ is an almost $(k-1)n$-contact structure on the manifold $\widetilde{T^*kM}$.

The condition of normality of the structure $\mathcal{F}$ is given by (see (6.6.5)):

\begin{equation}
\mathcal{N}(X,Y) + \sum_{i=1}^{n} \left[ \sum_{\alpha=1}^{k-1} d(\delta g^{(\alpha)ij})(X,Y) + d(\delta p_i)(X,Y) \right] = 0, \forall X,Y \in \mathcal{X}(T^*kM)
\end{equation}

where $\mathcal{N}$ is the Nijenjuis tensor of $\mathcal{F}$.

Theorem 8.9.4 is valid for spaces $GH^{(k)n}$.

**Theorem 11.4.2** For any generalized Hamilton space of order $k$, $GH^{(k)n} = (M, g^{ij})$ endowed with a nonlinear connection $N$ the following properties hold.

1° The pair $(\mathcal{G}, \mathcal{F})$ is a pseudo-Riemannian almost $(k-1)n$-contact structure determined only by $N$ and $g^{ij}$.

2° The associated 2-form is

$$\theta = \delta p_i \land dx^i$$

3° If the coefficients $N_{ij}$ of $N$ are symmetric, then

$$\theta = dp_i \land dx^i$$
is the canonical presymplectic structure on the manifold $\tilde{T}^{*k}M$.

The conditions of normality of the structure $\check{\mathcal{F}}$ is expressed by (11.4.4).

Finally, taking into account $G$ from (11.4.1) and $F$ from (11.4.3) it follows.

**Theorem 11.4.3** With respect to the canonical metrical connection $CT(N)$ of the space $GH^{(k)n}$ we have

$$D_X \check{G} = 0, D_X \check{F} = 0$$

As such the geometry of the pseudo-Riemannian almost $(k - 1)n$-contact space $(T^{*k}M, \check{G}, \check{F})$ can be studied by means of the canonical metrical $N$-linear connection $CT(N)$ of the generalized Hamilton space of order $k$, $GH^{(k)n}$.

### 11.5 Generalized Cartan Space of Order $k$

**Definition 11.5.1** A generalized Cartan space of order $k$, is a Generalized Hamilton space of order $k$, $GH^{(k)n} = (M, g_{ij})$ in which the fundamental tensor $g^{ij}$ satisfies the axioms:

1° $g^{ij}$ is positively defined on $\tilde{T}^{*k}M$.

2° $g^{ij}$ is $0$-homogeneous on the fibres of the dual bundle $(T^{*k}M, \tilde{\pi}^*k, M)$.

We denote by $GC^{(k)n} = (M, g^{ij})$ a generalized Cartan space of order $k$.

From the axiom 2° it follows

**Proposition 11.5.1** The following identities hold:

1° $g^{ij}$ being $0$-homogeneous, we have

$$L_{\Gamma^{-1}} + kC, g^{ij} = 0$$

or, developed:

$$(11.5.1a) \quad g^{(1)i} \frac{\partial g^{jh}}{\partial y^{(1)i}} + \ldots + (k - 1)g^{(k-1)i} \frac{\partial g^{jh}}{\partial y^{(k-1)i}} + kp_i \partial^i g^{jh} = 0$$

2° The absolute energy

$$(11.5.2) \quad \mathcal{E} = g^{ij} p_i p_j$$

is $2k$-homogeneous on the fibres of $T^{*k}M$.

3° $L_{\Gamma^{-1}} + kC, \mathcal{E} = 2k \mathcal{E}$.

4° $L_{\Gamma^{-1}} + kC, C^{ijh} = -kC^{ijh}$.

**Example 2.** Let $g^{ij}$ be the fundamental tensor of the Cartan space $C^{(k)n}$ and $\sigma \in F(T^{*k}M)$ with the properties:
a) $\sigma$ is 0-homogeneous;
b) $\partial^i \sigma$ nonvanishes.

The pair $GC^{(k)n} = e^{-2\sigma} \tilde{g}$ is a generalized Cartan space of order $k$.

In particular we can consider

$$\sigma = \frac{p_i y^{(1)i} \sqrt{\tilde{g}}}{\sqrt{\tilde{g}} p_i p_j \sqrt{g_{ij} y^{(1)i} y^{(1)j}}}.$$

The previous example shows the existence of the spaces $GC^{(k)n}$ are not reducible to a Cartan space of order $k$.

Let $N^*$ be a nonlinear connection on $T^*kM$, having the coefficients $(M^*_{ji}^{(1)}, ..., M^*_{ji}^{(k-1)}, N_{ij})$ homogeneous of degree $k-1, ..., 1, k$ respectively.

**Theorem 11.5.1** There exists an unique canonical metrical $N^*$-connection of the space $GC^{(k)n}$. Its coefficients are given by (9.5.5).

Now, the geometry of $GC^{(k)n}$ can be studied like the geometry of $GH^{(k)n}$ spaces.

The $N^*$-lift to $\tilde{T}^*kM$ of the fundamental tensor $g^{ij}$ of $GC^{(k)n}$, given by (11.3.2) is not homogeneous on the fibres of the bundle $T^*kM$. Therefore, taking into account the following Hamiltonians:

$$\mathcal{E}_1 = g_{ij}^{(1)i(j)} z^i z^j, \mathcal{E}_2 = g_{ij}^{(2)i(j)} z^i z^j, ..., \mathcal{E}_{k-1} = g_{ij}^{(k-1)i(j)} z^i z^j, \mathcal{E} = g^{ij} p_i p_j,$$

we can define the following tensor field:

$$\mathcal{G} = g_{ij} dx^i \otimes dx^j + \sum_{\alpha=1}^{k-1} \frac{1}{\mathcal{E}_{\alpha}} g_{ij} \delta y^{(\alpha)i} \otimes \delta y^{(\alpha)j} + \frac{1}{\mathcal{E}} g^{ij} \delta p_i \otimes \delta p_j.$$

Evidently, $\mathcal{E} > 0$.

**Theorem 11.5.2** $\mathcal{G}$ is a Riemannian structure on $T^*kM$, determined only by the fundamental tensor $g^{ij}$ of the space $GC^{(k)n}$ and the nonlinear connection $N^*$.

Consider the $\mathcal{F}(T^*kM)$-linear mapping $\mathcal{F}: \mathcal{X}(T^*kM) \to \mathcal{X}(T^*kM)$ defined by:

$$\mathcal{F} \left( \frac{\delta}{\delta x^i} \right) = -\mathcal{E} g_{ij} \frac{\delta}{\delta p_j}, \mathcal{F} \left( \frac{\delta}{\delta y^{(\alpha)i}} \right) = 0, (\alpha = 1, ..., k-1), \mathcal{F} \left( \frac{\delta}{\delta p_i} \right) = \frac{g^{ij}}{\mathcal{E}} \frac{\delta}{\delta x^j}$$

We obtain:
Theorem 11.5.3 We have:

1° \( F \) is a tensor field on \( T^*kM \) of type \( (1,1) \).

2° \( F \) is expressed in the adapted basis by

\[
F = -\mathcal{E} g_{ij} \frac{\delta}{\delta p_j} \otimes dx^i + \frac{g^{ij}}{\mathcal{E}} \frac{\delta}{\delta x^i} \otimes \delta p_j
\]

3° \( \text{rank } F = 2n \)

4° \( F^3 + \frac{\partial}{\partial t} F = 0 \)

5° \( F \) is determined only by \( g^{ij} \) and \( \mathcal{N}^* \).

Theorem 11.5.4 For a generalized Cartan spaces of order \( k, GC^{(k)n} = (M, g^{ij}) \) endowed with a nonlinear connection \( \mathcal{N}^* \) the following properties hold:

1° The pair \((\mathcal{G}, F)\) is a Riemannian almost \((k-1)n\)-contact structure determined only by \( \mathcal{N}^* \) and \( g^{ij} \).

2° The associated 2-form is

\[
\theta = \mathcal{E} g_{ij} \delta p_i \wedge dx^j
\]

The proofs are made by usual methods.

Concluding, the space \((\mathcal{T}^*kM, \mathcal{G}, F)\) is the geometrical model for the generalized Cartan space of order \( k \).
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Index

Absolute energies 272
Accelerations of order \( k \) 2
Adapted
basis 12, 121
cobasis 14, 123
Almost
complex structure
contact structure 139, 204
Hermitian structure 70
Kählerian structure 70
product structure 138
symplectic structure
Autoparallel curves 18, 131
Berwald
connections 165
spaces 160
Bianchi identities 31, 166
Bundle
differentiable 2, 72
\( k \)-osculator 2, 72
of higher order accelerations 2
dual 72
Canonical
spray 44, 60
semispray 43
\( k \)-semispray 43
metrical connections 46, 246
nonlinear connections 44, 194
\( N \)-linear connections 46, 195
Cartan
nonlinear connection 61
metrical connection 67
spaces 233
spaces of higher order 233
Christoffel
symbols 19, 117
generalized symbols 46, 197
Coefficients
of a nonlinear connections 44
of a \( N \)-linear connection 48, 155
dual of a nonlinear connection 44, 195
primal of a nonlinear connection 44, 195
Connections
nonlinear 11, 195
\( N \)-metrical 46, 197
Levi-Civita
Berwald 165
Covariant
\( h \)- and \( v \)- derivatives 25
\( h \)- and \( v_\alpha \)- derivatives 25, 157
\( h \)- and \( w_\alpha \)- derivatives 157
differential 24
Craig-Synge equation 59
Curvature of an \( N \)-linear connection 29, 152
d-vectors 22, 145
d-tensors 22, 145
distinguished tensors 22, 145
Deflection tensors 26
Direct decomposition 12, 120
Distributions 13
horizontal 11
vertical 6, 76
Dual
coefficients 15, 124
semisprays 113
of \( T^k M \) 72
Electrodinamics 45, 201

242
# Index

Energy
- of higher order 39, 99
- absolute 272
Embeeding 212
Euler-Lagrange equations 36

Finsler
- metrics 56
- spaces 55
Fundamental
- functions 56, 179
- tensors 56, 179

Gauss-Weingarten formulas 226
Gauss-Codazzi equations 230
General Relativity 266
Generalized
- Lagrange spaces 49
- Hamilton spaces 259
- Cartan spaces 272
Geodesics 58

Hamilton-Jacobi equations 42, 96
Hamilton-Jacobi-Ostrogradski equations 103
Hamiltonian system of higher order 186
Hamiltonian space of electrodinamics 201
Hamilton
- space 179
- vector field 77
- 1-form 77
Higher order
- Hamilton space 179
- Lagrange space 42
Integral of action 36, 91

Jacobi-Ostrogradski momenta 42, 101
Jacobi method 42

Lagrangian
- differentiable 34
- of higher order 34
- regular 35
Lagrange
- space 42
- space of higher order 42
Law of conservation 99
Legendre
- mapping 187, 240
- transformation 187, 240
Lie derivative 54
Lift
- homogeneous 275
- horizontal 11, 118
- N-lift 50, 118
Liouville
- vector fields 6, 77
- d-vector fields 127
- 1-form 78
Main invariants 35
N-linear connection 46, 147
Nonlinear connection 11, 118
Poisson structure 83, 184
Presymplectic structure 78, 144
Projector 118
- horizontal 118
- vertical 188
Relativistic Optics 266
Ricci identities 161

Sections
- in $T^kM$ 3
- in $T^{*k}M$ 72
Structure equations 174, 252
Spaces
- Finsler 55
- Hamilton 179
- Randers
Subspaces in Hamilton Spaces 212
Symplectic structure 184
Torsions 26, 151
Variational problem
- in Lagrange spaces 36
- in Hamilton spaces 91