A note on passing from a quasi-symmetric function expansion to a Schur function expansion of a symmetric function

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Shortly after the Egge, Loehr and Warrington paper [2] became available Jeff Remmel presented the contents in his topics course. I happened to be in the audience. During Remmel’s presentation it occurred to me that their result implied that when a symmetric function has been given an expansion in terms of the Gessel fundamentals indexed by compositions then a schur function expansion can be obtained by replacing each Gessel fundamental by a Schur function indexed by the same composition. After the lecture I devised the direct proof of this result given in this paper. Upon reading my write up Jeff discovered an error in my involution and corrected it. Jeff wrote this paper after we encountered a great deal of scepticism about this interpretation of the Egge, Loehr and Warrington result. The paper remained in my files for several years. After Jeff’s passing I decided that this contribution of Jeff should be recorded. The only addition to Jeff exposition I have inserted is some applications of the Egge, Loehr and Warrington result that have been made under my direction and under the direction of Jeff Remmel.

These are listed at the end of this manuscript. They include the work of Emily Sergel [12], the work of Dun Qiu [10] and work of Austin Roberts [11].

Abstract

Egge, Loehr and Warrington gave in [2] a combinatorial formula that permits to convert the expansion of a symmetric function, homogeneous of degree \( n \), in terms of Gessel’s fundamental quasi-symmetric functions into an expansion in terms of Schur functions. Surprisingly the Egge, Loehr and Warrington result may be shown to be simply equivalent to replacing the Gessel fundamental by a Schur function indexed by the same composition. In this paper we give a direct proof of the validity of this replacement. This interpretation of the result in [2] has already been successfully applied to Schur positivity problems.

1 Preliminaries

We say that a sequence of positive integers \( \alpha = (\alpha_1, \ldots, \alpha_k) \) is a composition of \( m \) into \( k \) parts if \( \sum_{i=1}^{k}\alpha_i = m \). If, in addition, \( \alpha_1 \geq \ldots \geq \alpha_k \), then we say that \( \alpha \) is a partition of \( m \). We say that a sequence of non-negative integers \( \gamma = (\gamma_1, \ldots, \gamma_\ell) \) is a weak composition of \( m \) into \( \ell \) parts if \( \sum_{i=1}^{\ell}\gamma_i = m \). Thus the difference between compositions and weak compositions is that 0 parts are allowed in weak compositions. We shall write \( \lambda \vdash m \) to denote that \( \lambda \) is a partition of \( m \), \( \alpha \vdash m \) to denote that \( \alpha \) is a composition of \( m \), and \( \gamma \vdash_w m \) to denote that \( \gamma \) is a weak composition of \( m \). Let \( S_n \) denote the symmetric group.

Suppose that \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is a weak composition of \( n \) into \( n \) parts. We let \( X = (x_1, \ldots, x_n) \) and

\[
\Delta_{\gamma}(X) = \det ||x_i^{\gamma_j+n-j}|| = \sum_{\sigma \in S_n} \text{sgn}(\sigma)x_1^{\gamma_1+n-1}\cdots x_n^{\gamma_n+n-n}.
\]
We let $\Delta(X) = \det ||x_i^{n-j}||$ be the Vandermonde determinant. Then the Schur function $s_\gamma(X)$ is defined to be

$$s_\gamma(X) = \frac{\Delta_\gamma(X)}{\Delta(X)}.$$  

(1)

It is well known that for any such weak composition $\gamma$, either $s_\gamma(X) = 0$ or there is a partition $\lambda$ of $n$ such that $s_\gamma(X) = \pm s_\lambda(X)$. In fact, there is a well-known straightening relation which allows to prove that fact. Namely, if $\gamma_{i+1} > 0$, then

$$s(\gamma_1, \ldots, \gamma_i, \gamma_{i+1}, \ldots, \gamma_n)(X) = -s(\gamma_1, \ldots, \gamma_{i+1}-1, \gamma_i+1, \ldots, \gamma_n)(X).$$  

(2)

See [9].

Suppose $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition of $n$ with $k$ parts. We associate a subset $S(\alpha)$ of $\{1, \ldots, n-1\}$ with $\alpha$ by setting

$$S(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}.$$ 

We let $\bar{\alpha}$ be the weak composition of $n$ with $n$ parts by adding a sequence of $n-k$ 0’s at the end of $\alpha$. For example, if $\alpha = (2, 3, 2, 1)$, then $S(\alpha) = \{2, 5, 7\}$ and $\bar{\alpha} = (2, 3, 2, 1, 0, 0, 0, 0)$. Gessel [3] introduced a fundamental quasisymmetric function associated with each composition $\alpha$ which is defined by

$$F_\alpha(X) = \sum_{\substack{i \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n \ni \in S(\alpha) \rightarrow a_i < a_{i+1}}} x_{a_1}x_{a_2}\cdots x_{a_n}.$$  

(3)

The $F_\alpha(X)$’s as $\alpha$ ranges over the compositions of $k$ are a basis for the space of quasisymmetric functions $Q_k(x_1, \ldots, x_n)$ of degree $k$.

There are many examples in the literature where one can give a combinatorial description of the coefficients that arise in the expansion of important symmetric functions in terms of the fundamental quasisymmetric functions where one does not have a combinatorial interpretation of the coefficients in terms of the Schur functions. For example, Haglund, Haiman, and Loehr [5] gave a combinatorial description of the coefficients that arise in expanding the modified Macdonald polynomials $H_\mu(x_1, \ldots, x_n; q, t)$ as a sum of fundamental quasi-symmetric functions. Similarly, Loehr and Warrington gave a combinatorial description of the coefficients that arise in expanding the plethysm of two Schur functions in terms of fundamental quasisymmetric functions [8]. The shuffle conjecture of Haglund, Haiman, Loehr, Remmel, and Uylanov [6] provides a conjectured combinatorial description of the expansion of the Frobenius image of the character generating function of the space of diagonal harmonics in terms parking functions weighted by fundamental quasisymmetric functions [7]. In the last few years, there have been several refinements and extension of the shuffle conjecture where we have a similar situation, see [7], [4], and [1]. In all of these cases, we have no combinatorial description of the coefficients that arise in the Schur function expansion of these symmetric functions.

In a remarkable and important paper, Egge, Loehr and Warrington [2] gave a combinatorial description of how to start with the expansion of a symmetric function $P(X)$, which is homogeneous of degree $n$, in terms of fundamental quasisymmetric functions

$$P(X) = \sum_{\alpha \vdash n} a_\alpha F_\alpha(X)$$

and transform it into an expansion in terms of Schur functions

$$P(X) = \sum_{\lambda \vdash n} b_\lambda s_\lambda(X).$$

The purpose of this note is to elucidate a simple but important consequence of their result. That is, we shall prove the following theorem.
Theorem 1. Suppose that $P(X)$ is a symmetric function which is homogenous of degree $n$ and

$$P(X) = \sum_{\alpha \vdash n} a_{\alpha} F_{\alpha}(X).$$  \hfill (4)

Then

$$P(X) = \sum_{\alpha \vdash n} a_{\alpha} s_{\alpha}(X).$$  \hfill (5)

Thus to obtain the Schur function of $P(X)$, one simply has to replace each $F_{\alpha}(X)$ by $s_{\alpha}(X)$ and then straighten the resulting Schur functions.

As we shall see the proof of Theorem 1 is much simpler than the original proof in [2].
2 Proof of Theorem 1

We start with the basic fact that if $A_n$ is the polynomial operator

$$A_n = \sum_{\sigma \in S_n} sgn(\sigma)\sigma,$$

where for any monomial $x_1^{a_1} \cdots x_n^{a_n}$ and $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, $\sigma(x_1^{a_1} \cdots x_n^{a_n}) = x_{\sigma_1}^{a_1} \cdots x_{\sigma_n}^{a_n}$, then for any symmetric function $f(X)$ we have,

$$f(X) = \frac{1}{\Delta(X)} A_n f(X) x^{\delta_n}$$

where $x^{\delta_n} = \prod_{i=1}^{n} x_i^{n-i}$. This is an immediate consequence of the determinantal expansion

$$\Delta(X) = \sum_{\sigma \in S_n} sgn(\sigma)\sigma(x^{\delta_n})$$

and the fact that for any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$,

$$f(X) = f(x_{\sigma_1}, \ldots, x_{\sigma_n}).$$

Thus to prove Theorem 1 we need only prove that for each composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of $n$,

$$s_{\alpha}(X) = \frac{1}{\Delta(X)} A_n F_\alpha(X)x^{\delta_n} = \sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n} \frac{1}{\Delta(X)} A_n x_{a_1} x_{a_2} \cdots x_{a_n} x^{\delta_n}.$$

We consider the following involution $I$ of the monomials that appear on the right-hand side of (7). First $I$ has one fixed point, namely, the monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Given any other monomial $x^u = x_{a_1} x_{a_2} \cdots x_{a_n} x^{\delta_n}$ which appears on the right-hand side of (7), look for the $s = s(u) < k$ such that

$$x^u = x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} x_{s+1}^{b_{s+1}} \cdots x_{r+s}^{b_{s+r}} \left( \prod_{i=s+r+1}^{n} x_i^{b_i} \right) x^{\delta_n}$$

where $r = r(u) \geq 2$, $b_{s+1} + \cdots + b_{s+r} = \alpha_{s+1}$, and $b_{s+r} > 0$. For example, if $\alpha = (2, 3, 3)$, then $(a_1, \ldots, a_5) = (1, 1, 2, 2, 3, 5)$, then $x^u = x_{a_1} x_{a_2} x_{a_3} x_{a_4} x_{a_5} = x_1^2 x_2^3 x_3 x_4^2 x_5^6 x_6^0 x_7^0 \cdots x_9^0$ so that $s(u) = 2$, $r(u) = 3$, and $x_{s+1} x_{s+2} \cdots x_{s+r} = x_3 x_4 x_5^2$.

Then we let

$$I \left( x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} x_{s+1}^{b_{s+1}} \cdots x_{r+s}^{b_{s+r}} \left( \prod_{i=s+r+1}^{n} x_i^{b_i} \right) x^{\delta_n} \right) =
\left. \left( x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} x_{s+1}^{b_{s+1}} \cdots x_{s+r}^{b_{s+r}} \left( \prod_{i=s+r+1}^{n} x_i^{b_i} \right) x^{\delta_n} \right) \right|_{i=r+r+1}^{-1} =
\left( x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s} x_{s+1}^{b_{s+1}} \cdots x_{s+r-1}^{b_{s+r-1}} \left( \prod_{i=s+r+1}^{n} x_i^{b_i} \right) x^{\delta_n} \right) x^{\delta_n}.$$
then
\[
\frac{1}{\Delta(X)} A_n x^v = \frac{1}{\Delta(X)} A_n \left( x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_s^{\alpha_s} x_{s+1}^{b_{s+1}} \cdots x_{s+r-1}^{b_{s+r-1}} \left( \prod_{t>r} x_{s+r} \right)^{\delta_n} \right) = s_{(\gamma_1, \ldots, \gamma_r-1, \gamma_r-1+1, \ldots, \gamma_n)}(X)
\]
so that by (2), these two terms cancel each other.

Thus I shows that the right-hand side of (7) reduces to
\[
\frac{1}{\Delta(X)} A_n x_1^{\alpha_1} \cdots x_k^{\alpha_k} x^\delta_n = s_{\tilde{\alpha}}(X)
\]
which is what we wanted to prove.

### 3 Some applications

After discovering the present interpretation of the Egge-Loehr-Warrington result. Some efforts were directed towards identifying the surviving terms after the replacement of a Gessel fundamental by a compositional indexed Schur function. The first successful use of this kind of the Egge-Loehr-Warrington result was obtained by Emily Sergel in [12]. Encouraged by Sergel’s success Dun Qiu and Jeff Remmel, in a truly remarkable paper [10], were able to prove Schur positivity for a wider variety of Rational Parking function modules.

We will next describe a specific example were our attempts led to a conjecture with measurable success. Let us recall that in [5] Haglund, Haiman and Loehr derive the Lascoux-Schutzenberger charge result from their combinatorial proof the Haglund formula. Since their work consisted in showing that co-charge came out of the Haglund’s \(inv_\mu\) statistic it was compelling to see if co-charge could be bypassed altogether. This led to the following computer experimentation.

The point of departure is the identity
\[
\bar{H}_\mu[X; t] = \bar{H}_\mu[X; 0, t]
\]
expressing a modified Hall-Littlewood polynomial in terms of the modified Macdonald polynomial. Now, in the present context Haglund’s formula may be written in the form
\[
\bar{H}_\mu[X; q, t] = \sum_{\sigma \in S_n} t^{maj_\mu(\sigma)} q^{inv_\mu(\sigma)} F_{pides(\sigma)}[X]
\]
where the French Ferrer’s diagram of \(\mu\) is filled by \(\sigma\) in the reading order, that is by rows from left to right and from top to bottom. The statistic “\(maj_\mu(\sigma)\)” is simply the sum of the major indexes of the column of \(\mu\) read from top to bottom, “\(inv_\mu(\sigma)\)” counts the number of counterclockwise triplets and “\(pides(\sigma)\)” gives the composition of the descent set of the inverse of \(\sigma\). Thus 3.1 reduces this identity to
\[
\bar{H}_\mu[X; 0, t] = \sum_{\sigma \in S_n; inv_\mu(\sigma) = 0} t^{maj_\mu(\sigma)} F_{pides(\sigma)}[X]
\]
Now in an unpublished algorithm Loehr and Warrington show how to construct \(inv_\mu(\sigma) = 0\) fillings. Their algorithm is based on the fact that, for \(k = l(\mu)\), it suffices to choose the decomposition
\[
T_1 + T_2 + \cdots + T_k = \{1, 2, \ldots, n\} \quad \text{ (with } |T_i| = \mu_i \text{)}
\]
of the entries of \(\sigma\) to be placed in the rows of \(\mu\). In fact, once the first row of \(\mu\) is filled by the elements of \(T_1\) in increasing order, then the \(inv_\mu(\sigma) = 0\) condition recursively forces the order in which row \(i\) must be filled by the elements of \(T_i\).
This given, 3.3 may be rewritten as
\[ \tilde{H}_\mu[X;0,t] = \sum_{\sigma;T_1+T_2+\cdots+T_k=[1,n]} t^{maj_\mu(\sigma)} F_{spides(\sigma)}[X] \]
and our interpretation of the Egge-Loehr-Warrington result gives
\[ \tilde{H}_\mu[X;0,t] = \sum_{\sigma;T_1+T_2+\cdots+T_k=[1,n]} t^{maj_\mu(\sigma)} s_{spides(\sigma)}[X]. \]  

Now it is well known that we have three alternatives
\[ s_{spides(\sigma)}[X] = \begin{cases} 
0 & \text{if } spides(\sigma) \text{ straightens to } 0 \\
-s_\lambda(\sigma) & \text{if } spides(\sigma) \text{ straightens to } -s_\lambda(\sigma) \\
s_\lambda(\sigma) & \text{if } spides(\sigma) \text{ straightens to } s_\lambda(\sigma)
\end{cases} \]

The parking functions that produce the first alternative do not contribute to the sum. Due to the Schur positivity of the left hand side of 3.4, the parking functions that produce the second alternative must cancel out with exactly one of the parking functions that produces the third alternative with exactly the same \( maj_\mu(\sigma) \) statistic. The resulting sum is over a subset of the original parking functions. An a priori identification of the leftovers would deliver the Schur function expansion of the modified Hall-Littlewood polynomial \( \tilde{H}_\mu[X;0,t] \).

This given, what initially felt as a wild guess, was the conjecture that the leftovers are the inv\(_\mu(\sigma) = 0 \) fillings that produce the third alternative and that in addition the Schensted row insertion of \( \sigma \) results in a pair of standard tableaux of shape \( \lambda(\sigma) \).

The resulting computer data revealed the astonishing fact that the “leftover” according to this simple criterion yielded the correct Schur expansion of \( \tilde{H}_\mu[X;0,t] \) up to partition of 9 excluding the partition \([3,3,3]\). But even in that case the Schur expansion was only short one term. the existence of this counter example discouraged further experimentations. But given the size of the counter example one could be left with the idea that a suitable \( \mu \)-variant of the Schensted algorithm may correctly identify the leftovers without exceptions.

This particular study of the consequences of Haglund’s formula entered a new chapter as a result of a poster of Austin Roberts in the Paris FPSAC of 2013. This poster exhibited a similar experiment involving the unrestricted Haglund Formula. The Roberts experiment revealed that the Schur expansion of \( \tilde{H}_\mu[X;q,t] \) could be obtained from Schensted correspondence provided \( \mu \) did not contain \([3,3,3]\) and another partition. This circumstance prompted the first named author to ask Roberts to see if in the case of the modified Hall-Littlewood \( \tilde{H}_\mu[X;0,t] \) the only obstruction to the use of Schensted to obtain the Schur expansion was containment of \([3,3,3]\). It turned out Roberts succeeded not only in proving this fact but also showed in [11] how the conjectured algorithm had to be modified to yield the correct answer without exceptions.

Very recently, we received from Ira Gessel a manuscript with a new proof of the Egge-Loehr-Warrington result obtained by constructing an involution that proves the validity of our replacement for the Schur basis.
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