GLOBAL EXISTENCE AND BOUNDEDNESS IN A CHEMOREPULSION SYSTEM WITH SUPERLINEAR DIFFUSION

MARC EL FREITAG*
Institut für Mathematik, Universität Paderborn
33098 Paderborn, Germany
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Abstract. In a bounded domain $\Omega \subset \mathbb{R}^n$, where $n \geq 3$, we consider the quasilinear parabolic-parabolic Keller-Segel system

\[
\begin{align*}
    u_t &= \nabla \cdot (D(u) \nabla u + u \nabla v) \quad \text{in } \Omega \times (0, \infty), \\
    v_t &= \Delta v - v + u \quad \text{in } \Omega \times (0, \infty),
\end{align*}
\]

with homogeneous Neumann boundary conditions. We will find that the condition $D(u) \geq Cu^{m-1}$ suffices to prove the uniqueness and global existence of solutions along with their boundedness if $D(0) > 0$ and $m > 1 + \frac{(n-2)(n-1)}{n^2}$ which is a very different result from what we know about the same system with nonnegative sensitivity. In the case of degenerate diffusion ($D(0) = 0$) and for the same parameters, locally bounded global weak solutions will be established.

1. Introduction. The system considered in this work is, in its most general form, represented by

\[
\begin{align*}
    u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v) \quad \text{in } \Omega \times (0, \infty), \\
    v_t &= \Delta v - v + u \quad \text{in } \Omega \times (0, \infty),
\end{align*}
\]

for a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with smooth boundary, homogeneous Neumann boundary conditions and prescribed initial data. The biological phenomenon this class of systems of differential equations is used to describe is called chemotaxis, the movement of cells which is controlled by a chemical substance produced by the same cells. The most basic setting, proposed by Keller and Segel in 1970 ([16]), considers $D \equiv 1$ and $S(u) = u$ and even there the detection of blow-up, the question whether it arises in finite or infinite time or, on the other hand, verifying the global existence and boundedness of solutions are nontrivial tasks. The only exception is the case $n = 1$ where no blow-up occurs at all ([26]). The behaviour in two dimensions is more intricate: If the initial mass $\int_\Omega u_0$ is less than $4\pi$, then every solution is bounded according to [10] and [24]. In higher dimensions a smallness condition on $\|u_0\|_{L^2(\Omega)} + \|v_0\|_{W^{1,\infty}(\Omega)}$ has been proven by [3] to be sufficient to guarantee the...
same. On the other hand, if the initial data is large, solutions can be found which explode either in finite or infinite time ([14]). In order to gain additional knowledge one can specify \( \Omega \) to be a ball with radially symmetric functions. Then for \( n = 2 \) and \( \int_\Omega u_0 < 8\pi \) solutions are global and bounded ([24]) while for initial data above this threshold [11] and [23] have detected blow-up in finite time. According to [34], for \( n \geq 3 \) no such threshold for the initial mass exists and blow-up solutions can be constructed whenever \( \int_\Omega u_0 > 0 \). There are many potential choices for \( D \) and \( S \), for a vast overview [1] is recommended. One approach is to link both functions via some \( Q \in C^2([0, \infty)) \) which describes the probability of a cell at \((x, t)\) to find space nearby and the relations \( D(u) = Q(u) - uQ'(u) \) and \( S(u) = uQ'(u) \). [12] considers a decreasing function with decay at large densities as the best fit while in [37] an overview of hydrodynamic approaches or those involving cellular Potts models is given. To incorporate additional properties of the biological reality, one or both of the functions are often equipped with a signal-dependency: [33], [13] and [20] write \( S(u, v) \) to model saturation effects or activation thresholds for the cross-diffusion. Similar ideas for \( D \) are discussed in [8], [21], [29] and [30]. We now want to consider a more specific system, namely

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (D(u) \nabla u + u \nabla v) & \text{in } \Omega \times (0, \infty), \\
\frac{\partial v}{\partial t} &= \Delta v - v + u & \text{in } \Omega \times (0, \infty), \\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial n} = 0 & \text{on } \partial \Omega \times (0, \infty), \\
u(\cdot, 0) &= u_0, \ v(\cdot, 0) &= v_0 & \text{in } \Omega.
\end{align*}
\]

(S)

where \( D(u) \geq C_D u^{m-1} \) with some \( C_D > 0 \) and \( m \geq 1 \) and we point out the crucial difference the changed sign in the first equation will make for \( n \geq 3 \). With \( D \) as here and the additional condition \( D(0) > 0 \) as well as \( S(u) = u \),

\[
m > 1 + \frac{n-2}{n}
\]

has been identified as a crucial relation in \((S_0)\): if it holds, then [32] shows the global existence and boundedness of solutions while for \( m \) below this threshold (see [35]) the following holds true: If \( \Omega = B_R(0) \subset \mathbb{R}^n \), where \( n \geq 2 \), then for any \( M > 0 \) we find \( T \in (0, \infty) \) and a radially symmetric solution \((u, v)\) in \( \Omega \times (0, T) \) such that \( \int_\Omega u \equiv M \) but \( u \) is unbounded in \( \Omega \times (0, T) \). Such results can be extended (see [32], [17] and [4] as well as [27] and [15]) to more general choices of \( D \) and \( S \) as \( \frac{S(s)}{D(s)} \geq C s^\alpha \) for some \( C > 0 \), \( \alpha > \frac{2}{n} \) and all \( s \geq 1 \). For \( D(u) = (u + 1)^{m-1} \) and \( S(u) = (u + 1)^{\kappa - 1} \), [5] and [6] have determined the point of time at which blow-up occurs: If both \( m \geq 1 \) and \( \kappa \geq 1 \), then blow-up happens in finite time. On the other hand, if \( m < \kappa + \frac{n-2}{n} \) and \( \kappa < \frac{m}{2} - \frac{n-2}{2m} \), then solutions exist globally but we have \( \limsup_{t \to \infty} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \). The positive sensitivity in \((S)\), resulting in repulsion instead of attraction, promotes global existence and boundedness of solutions, especially for \( m = 1 \) we already have relevant results: If \( n = 2 \), global solutions and their boundedness have been established while for \( n \in \{3, 4\} \) locally bounded global weak solutions have been found (both in [7]). Additionally we also have a result for nonlinear sensitivity: [31] has found uniform-in-time bounds for classical solutions and convergence to the average of the initial mass. In this work we want to consider the case of superlinear diffusion and to achieve a condition similar to \((m_0)\). To this end we begin by proving
Theorem 1.1. We assume that we have some positive constants $C_D$ and $C'_D$ as well as some $M \geq m > 1 + \frac{(n-2)(n-1)}{n}$. Then for any $D \in C^1([0,\infty))$ with
\[ C_D(s+1)^{m-1} \leq D(s) \leq C'_D(s+1)^{M-1} \]
for all $s \geq 0$. Then for any $u_0 \in C^0(\Omega)$ and $v_0 \in C^1(\Omega)$ the system $(S)$ has a unique classical solution. This solution is global and bounded in the sense that there is some $C > 0$ with
\[ \|u(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, \infty). \]
Moreover, for the case of degenerate diffusion, this result can be used to detect global weak solutions that are locally bounded using the fact that $D$ vanishing at zero influences the construction of solutions but not the size of the bounds derived in this work:

Theorem 1.2. We assume that we have some positive constants $C_D$ and $C'_D$ as well as some $M \geq m > 1 + \frac{(n-2)(n-1)}{n}$. Then for any $D \in C^1([0,\infty))$ with
\[ C_D(s+1)^{m-1} \leq D(s) \leq C'_D(s+1)^{M-1} \]
for all $s \geq 0$. Then for any $u_0 \in L^1(\Omega)$ and $v_0 \in L^1(\Omega)$ we find a locally bounded global weak solution to $(S_0)$ in the sense of definition 4.1.

Note that theorems 1.1 and 1.2 could be modified to incorporate the case $n = 2$ with the expected condition $M \geq m > 1$. However, this is the same bound as the one for the chemotaxis case discussed in [32] and one can easily check that the proof therein covers the two-dimensional version of our system as well – only in higher dimensions do we see the effect of the different sign in the first equation of $(S)$. Alternatively, very simple changes to lemma 3.4 can be made to include $n = 2$, but since nothing substantial could be gained this way, for the sake of clarity and readability we are only considering $n \geq 3$.

2. Helpful lemmata. Before we present our findings, let us collect some known results that have been established before and also verify general estimates that will help with other upcoming proofs. Of central importance to us are the well-known Gagliardo-Nirenberg inequality (see [25] for example) and a variant for fractional Sobolev spaces (see Section III in [2]).

Lemma 2.1 (Gagliardo-Nirenberg inequality). Let $p \in (1, \infty)$, $q \in (1, \infty)$ with $q < \frac{np}{(n-p)+}$ for $p \leq n$ and $r \in (0, p)$. Then there are some $C > 0$ and a constant $a \in (0, 1)$ with
\[ \|w\|_{L^p(\Omega)} \leq C \|\nabla w\|_{L^q(\Omega)}^a \|w\|_{L^r(\Omega)}^{1-a} + C \|w\|_{L^r(\Omega)} \]
for all $w \in C^1(\Omega)$ and $a$ is given by
\[ -\frac{n}{p} = \left(1 - \frac{n}{q}\right)a - \frac{n}{r}(1-a). \]

Lemma 2.2 (Gagliardo-Nirenberg inequality for fractional Sobolev spaces). For fixed $r \in \left(0, \frac{1}{2}\right)$ there are $C > 0$ and $a \in (0, 1)$ as in Lemma 2.1 such that
\[ \|w\|_{W^{r,\frac{1}{2}+2}(\Omega)} \leq C \|\nabla w\|_{L^2(\Omega)}^a \|w\|_{L^2(\Omega)}^{1-a} + C \|w\|_{L^2(\Omega)} \]
holds for any $w \in C^1(\Omega)$. 

Next and in preparation for the discussion of the case of degenerate diffusion we cite a version of the Aubin-Lions lemma which can be found as a corollary in chapter 8 of [28].

**Lemma 2.3** (Aubin-Lions lemma). For $T > 0$ and three Banach spaces $X \xrightarrow{\text{cpt.}} Y$ let $F$ be a bounded set of $L^p((0,T); X)$-functions. If furthermore the set of derivatives $\frac{\partial F}{\partial t} := \{ \frac{\partial f}{\partial t} : f \in F \}$ is bounded in $L^1((0,T); Y)$ (for $1 \leq p < \infty$) or in $L^r((0,T); Y)$ for some $r > 1$ (in the case $p = \infty$), then $F$ is relatively compact in $L^p((0,T); B)$ or $C^0((0,T); B)$ respectively.

Furthermore, there is a simple consequence of Young’s inequality that will help us in utilising Hölder’s inequality. It can in fact be proven by consecutively employing Young’s inequality multiple times.

**Lemma 2.4.** For $\beta, \gamma \in (0,1)$ with $\beta + \gamma < 1$ and any $\varepsilon > 0$ we find some $C > 0$ such that

$$(1 + a^\beta)(1 + b^\gamma) \leq \varepsilon(a + b) + C$$

holds for all positive numbers $a$ and $b$.

Finally, we want to use integration by parts to tackle certain integrals involving the second component of a solution to $(S)$, but for nonconvex domains $\frac{\partial w}{\partial \nu}\big|_{\partial \Omega} = 0$ is not sufficient to deduce nonpositivity of $\frac{\partial |\nabla w|^q}{\partial \nu}$ on the boundary of $\Omega$. Hence, the following lemma will be introduced.

**Lemma 2.5.** Let $q > 1$. Then there is $C > 0$ such that

$$\int_\Omega |\nabla w|^{2q-2} \Delta |\nabla w|^2 \leq -\frac{q-1}{q^2} \int_\Omega |\nabla |\nabla w|^q|^2 + C$$

holds for every $w \in C^2(\Omega)$ with $\frac{\partial w}{\partial \nu}\big|_{\partial \Omega} = 0$.

**Proof.** According to [22] there is $C_1 > 0$ such that $\frac{\partial |\nabla w|^q}{\partial \nu}\big|_{\partial \Omega} \leq C_1 |\nabla w|^2$ holds for all $w \in C^2(\Omega)$ with $\frac{\partial w}{\partial \nu}\big|_{\partial \Omega} = 0$ and an application of the classical Gagliardo-Nirenberg inequality 2.1 provides us with some $a \in (0,1)$ and $C_2 > 0$ such that

$$\int_\Omega |\nabla w|^{2q} \leq C_2 + C_2 \left( \int_\Omega |\nabla |\nabla w|^q|^2 \right)^a.$$

Fixing $r \in (0, \frac{1}{2})$ and using the embedding $W^{r+\frac{1}{2},2}(\Omega) \hookrightarrow L^2(\partial \Omega)$, the fractional version 2.2 of the Gagliardo-Nirenberg inequality achieves

$$\int_{\partial \Omega} |\nabla w|^{2q} = \|\nabla w|^q\|_{L^2(\partial \Omega)}^2 \leq C_3 \|\nabla w|^q\|_{W^{r+\frac{1}{2},2}(\Omega)}^2 \leq C_4 \|\nabla |\nabla w|^q\|_{L^2(\Omega)} \|\nabla w|^q\|_{L^2(\Omega)}^{2(1-b)} + C_4 \|\nabla w|^q\|_{L^2(\Omega)}^2$$

for some $b \in (0,1)$ and positive constants $C_3$ and $C_4$. Combining these two estimates with Young’s inequality gives us some $C_5 > 0$ and

$$\int_{\partial \Omega} |\nabla w|^{2q-2} \frac{\partial |\nabla w|^2}{\partial \nu} \leq \frac{3q-1}{q^2} \int_\Omega |\nabla |\nabla w|^q|^2 + C_5.$$

The claim then follows upon integration by parts. \(\square\)
3. Uniform boundedness of functions solving \((S)\). In this part we will prove that any classical solution \((u, v)\) to our system \((S)\) is uniformly bounded during its entire existence time. Furthermore, the bounds we find do not depend on \(D(0)\) which enables us to utilise these results in an upcoming approximation process.

Under the overarching condition

\[ m > 1 + \frac{(n-2)(n-1)}{n^2}, \quad (m) \]

which is less strict than \((m_0)\), we shall assume that we have been given some time \(T \in (0, \infty)\) and a pair of classical solutions to \((S)\) in \(\Omega \times (0, T)\). Without stating so in every lemma, none of the arising constants will depend on \(T\). The main result of this section accordingly reads as follows:

**Lemma 3.1.** Let \(K > 0\), \(C_D > 0\) and \(m > 1 + \frac{(n-2)(n-1)}{n^2}\). Then there is a positive constant \(C\) such that for every \(D \in C^1([0, \infty))\) with

\[ D(s) \geq C_D s^{m-1} \text{ for all } s \geq 0 \]

and for all choices \(u_0 \in C^0(\overline{\Omega})\) and \(v_0 \in C^1(\overline{\Omega})\) with

\[ \|u_0\|_{L^\infty(\Omega)} + \|v_0\|_{W^{1, \infty}(\Omega)} \leq K \]

as well as any \(T \in (0, \infty)\) and any classical solution \((u, v)\) of \((S)\) in \(\Omega \times (0, T)\) we have

\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \]

for all \(t \in (0, T)\).

When discussing chemotaxis systems, initial steps for the regularity of any solution often consist of proving \(L^1\)-boundedness and then using \(L^p\)-\(L^q\)-estimates to show \(v \in W^{1, q}(\Omega)\) for any \(q \in (1, \frac{n}{n-1})\). Here however, the small difference in the first equation enables us to go even further as was first shown by \([7]\).

**Lemma 3.2.** There is a constant \(C > 0\) such that

\[ \|u(\cdot, t)\|_{L^1(\Omega)} \leq C \text{ for all } t \in (0, T) \]

and

\[ \|\nabla v(\cdot, t)\|_{L^2(\Omega)} \leq C \text{ for all } t \in (0, T). \]

**Proof.** While the first part can be seen upon computing \(\frac{d}{dt} \int_\Omega u = 0\), the second half follows from \(x \ln x \geq -\frac{1}{e}\) for \(x > 0\) and

\[
\frac{d}{dt} \left( \int_\Omega u \ln u + \frac{1}{2} \int_\Omega |\nabla v|^2 \right) = \int_\Omega (1 + \ln u) u_t + \int_\Omega \nabla v \nabla v_t
\]

\[ = -\int_\Omega \frac{\nabla u}{u} (D(u) \nabla u + u \nabla v) \]

\[ - \int_\Omega |\Delta v|^2 - \int_\Omega |\nabla v|^2 + \int_\Omega \nabla u \cdot \nabla v \]

\[ = -\int_\Omega \frac{D(u)}{u} |\nabla u|^2 - \int_\Omega |\Delta v|^2 - \int_\Omega |\nabla v|^2 \]

which holds for all \(t \in (0, T)\). \(\Box\)
It is our goal to prove uniform boundedness of both components of any solution \((u, v)\) to (S) and the next step on this way is concerned with a higher regularity for \(u\). To prepare for this we prove

**Lemma 3.3.** For every \(p > \max \{1, m - 1\}\) and \(q > 1\) there are positive constants \(C_1\) and \(C_2\) such that

\[
\frac{d}{dt} \left( \int_\Omega u^p + \int_\Omega |\nabla v|^{2q} \right) + C_1 \int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 + C_1 \int_\Omega |\nabla v|^2 \leq C_2 \int_\Omega u^{-m+p+1} |\nabla v|^2 + C_2 \int_\Omega u^{2} |\nabla v|^{2q-2}
\]

holds for all \(t \in (0, T)\).

**Proof.** Using our estimate for \(D_i\), integration by parts and Young’s inequality as well as lemma 3.2, we find positive constants \(C_1\) and \(C_2\) with

\[
\frac{d}{dt} \int_\Omega u^p \leq -C_1 \int_\Omega |\nabla u^{\frac{m+p-1}{2}}|^2 + C_2 \int_\Omega u^{-m+p+1} |\nabla v|^2
\]

for all \(t \in (0, T)\). Due to

\[
\Delta |\nabla v|^2 = 2 \nabla v \cdot \nabla \Delta v + 2 |D^2 v|^2
\]

we have

\[
\frac{\partial}{\partial t} |\nabla v|^2 = 2 \nabla v \cdot \nabla \Delta v - 2|\nabla v|^2 + 2 \nabla u \cdot \nabla v = \Delta |\nabla v|^2 - 2|\nabla v|^2 + 2 \nabla u \cdot \nabla v
\]

and therefore, together with some \(C_3 > 0\) given by lemma 2.5,

\[
\frac{1}{q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} \leq -\frac{q-1}{q^2} \int_\Omega |\nabla v|^2 + C_3
\]

\[
-2 \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 - 2 \int_\Omega |\nabla v|^{2q} + 2 \int_\Omega |\nabla v|^{2q-2} \nabla u \cdot \nabla v
\]

holds for all \(t \in (0, T)\). Utilising \(|\Delta v|^2 \leq n |D^2 v|^2\), integration by parts and Young’s inequality we see

\[
2 \int_\Omega |\nabla v|^{2q-2} \nabla u \cdot \nabla v \leq \frac{q-1}{2} \int_\Omega |\nabla v|^{2q-4} |\nabla |\nabla v||^2
\]

\[
+ \left( 2(q-1) + \frac{n}{2} \right) \int_\Omega u^2 |\nabla v|^{2q-2} + 2 \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2
\]

for all \(t \in (0, T)\). Adding this to the previous result completes the proof.

In the next lemma, which is also the source of our restrictions for \(m\) and \(n\), we will fix several parameters which combined with the previous result will show that \(u\) belongs to \(L^p(\Omega)\) for any finite \(p\).

**Lemma 3.4.** For every \(\hat{p} > 1\) and every \(\hat{q} > 1\) there are \(p \geq \hat{p}\) and \(q \geq \hat{q}\) as well as \(\theta > 1\) and \(\mu > 1\) such that

\[
\frac{1}{-m+p+1} < \theta \leq \frac{n}{n-2} \frac{m+p-1}{-m+p+1}, \quad (\theta_1)
\]

\[
1 < \theta' \leq \frac{m+1}{n-2}
\]

and

\[
\frac{n}{2} \mu \leq \frac{n}{n-2} \frac{m+p-1}{2}, \quad (\mu_1)
\]
\[ \frac{1}{q - 1} < \mu' \leq \frac{n}{n - 2} \frac{q}{q - 1}, \quad (\mu_2) \]

as well as

\[ \frac{-m + p + 1 - \frac{1}{\sigma}}{1 - \frac{n}{2} + \frac{n(m + p - 1)}{2}} + \frac{1}{\sigma} + \frac{m}{2} < \frac{1}{n} \quad (\sigma_1) \]

and

\[ \frac{2 - \frac{1}{\mu}}{1 - \frac{n}{2} + \frac{n(m + p - 1)}{2}} + \frac{2 - 1}{\mu} < \frac{2}{n} \quad (\sigma_2) \]

hold for \( \theta' := \frac{\theta}{\sigma - 1} \) and \( \mu' := \frac{\mu}{\mu - 1} \).

**Proof.** In the case that \( \frac{n - 2}{n} \frac{m + \frac{1}{4} - 3}{2m + \frac{2}{n} - 3} > 1 \) we increase \( \hat{q} \) to \( q_1 \) with

\[ \frac{nq}{n(q - 1) + 2} \leq \frac{n - 2}{n} \frac{1}{2m + \frac{2}{n} - 3} \]

dividing for all \( q \geq q_1 \). Since \( (m) \) guarantees that

\[ \frac{n - 2}{n} \frac{1}{2m + \frac{2}{n} - 3} < \frac{n}{n - 2}, \]

we can pick some \( \theta \in \left( \frac{n - 2}{n} \frac{1}{2m + \frac{2}{n} - 3}, \frac{n}{n - 2} \right) \) and the conditions \( (\theta_1) \) and \( (\theta_2) \) hold for all \( p \geq p_1 := \max \{ m, \hat{p} \} \) and \( q \geq q_1 \).

In the converse case we choose \( q_1 \geq \hat{q} \) such that

\[ \frac{nq}{n(q - 1) + 2} < \frac{n}{n - 2} \]

holds for all \( q \geq q_1 \) and conclude as before with some \( \theta \in \left( \frac{nq}{n(q - 1) + 2}, \frac{n}{n - 2} \right) \).

Setting \( p_2 := \max \{ p_1, n - 1 - m \} \) we fix \( \mu \in \left( \frac{n}{2}, \frac{n}{n - 2} \frac{m + p_2 - 1}{2} \right) \) and accordingly we have \( (\mu_1) \) as well as \( (\mu_2) \) for any choices of \( p \geq p_2 \) and \( q \geq q_2 := \max \{ q_1, 2 \} \).

Following the definition \( p_3 := \max \{ p_1, n + 1 \} \), for \( p \geq p_3 \) we consider

\[ \hat{q}(p) := \frac{n - 2}{n} + \frac{2 + 1 - \frac{1}{\mu}}{2 - \frac{1}{\mu}} \left( \frac{2}{n} + m + p - 2 \right) \]

and, possibly after another adjustment, we identify \( p_4 \geq p_3 \) such that \( \hat{q}(p) > q_2 \) holds for every \( p \geq p_4 \). Now, for \( p \geq p_1 \) and \( q \in (q_2, \hat{q}(p)) \) we consider

\[ f(p, q) := \frac{-m + p + 1 - \frac{1}{\sigma}}{1 - \frac{n}{2} + \frac{n(m + p - 1)}{2}} + \frac{1}{\sigma} + \frac{m}{2} \]

and

\[ g(p, q) := \frac{2 - \frac{1}{\mu}}{1 - \frac{n}{2} + \frac{n(m + p - 1)}{2}} + \frac{q - 2 + \frac{1}{\mu}}{1 - \frac{n}{2} + \frac{n\mu}{2}}. \]

For fixed \( p \) we quickly see

\[ \frac{\partial g}{\partial q}(p, q) = \frac{1 - \frac{n}{2} + \frac{n\mu}{2} - \frac{n}{2} \left( q - 2 + \frac{1}{\mu} \right)}{\left( 1 - \frac{n}{2} + \frac{n(m + p - 1)}{2} \right)^2} = \frac{1 + \frac{n}{2} \left( 1 - \frac{1}{\mu} \right)}{\left( 1 - \frac{n}{2} + \frac{n(m + p - 1)}{2} \right)^2} > 0 \]
which, together with
\[ g(p, \tilde{q}(p)) = \frac{2 - \frac{1}{\mu}}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2}} + \frac{n-2}{1 - \frac{n}{2} + \frac{n-2}{2} + \left( \frac{\theta + 1 - \frac{1}{2}}{\mu} \right) \left( 1 - \frac{\theta}{2} + \frac{n(m+p-1)}{2} \right)} = \frac{2}{n}, \]
shows \((\sigma_2)\) for \(p \geq p_4\) and \(q \in (q_2, \tilde{q}(p))\). On the other hand, for \(p \geq p_4\) we see
\[ f(p, \tilde{q}(p)) = \frac{-m + p + 1 - \frac{1}{\theta}}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2}} + \frac{\frac{1}{\theta}}{1 - \frac{n}{2} + \frac{n-2}{2} + \left( \frac{\theta + 1 - \frac{1}{2}}{\mu} \right) \left( 1 - \frac{\theta}{2} + \frac{n(m+p-1)}{2} \right)} = \frac{-m + p + 1 + \frac{1}{\theta} \left( \frac{1}{2} - \frac{1}{\mu} \right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2}}, \]
which tends to \(\frac{2}{n}\) as \(p \to \infty\). The derivative of the latter term with respect to \(p\) is
\[ 1 - \frac{n}{2} + \frac{n(m+p-1)}{2} - \frac{n}{2} \left( -m + p + 1 + \frac{1}{\theta} \left( \frac{1}{2} - \frac{1}{\mu} \right) \right) = \frac{2m + \frac{2}{n} - 3 - \frac{1}{\theta} \left( \frac{1}{2} - \frac{1}{\mu} \right)}{2} \left( 1 - \frac{n}{2} + \frac{n(m+p-1)}{2} \right)^2 \]
and due to our additional condition \(\theta > \frac{n-2}{n} \frac{1}{2m+\frac{1}{2} - \frac{1}{2}}\) which we introduced at the beginning of the proof as well as a rough estimate for \(\mu > \frac{n}{2}\) it is positive. Therefore picking any \(p \geq p_4\) and some large \(q \in (q_2, \tilde{q}(p))\) ensures that both \((\sigma_1)\) and \((\sigma_2)\) hold while the other four properties remain intact. 

As announced, we now use these parameters together with lemma 3.3 in order to obtain a useful regularity result for \(u\).

**Lemma 3.5.** For any \(p > 1\) there is \(C > 0\) such that
\[ \|u(\cdot, t)\|_{L^p(\Omega)} \leq C \]
holds for all \(t \in (0, T)\).

**Proof.** Without loss of generality we claim that \(p > m - 1\) is at least as large as \(p\) in lemma 3.4 and use \(q\), \(\theta\) and \(\mu\) as provided by the same lemma, furthermore we fix \(\theta' := \frac{1}{1+\frac{1}{2}}\) and \(\mu' := \frac{1}{1+\frac{1}{2}}\). Applying Hölder’s inequality to the right-hand side of the result of lemma 3.3 shows why we want to use the Gagliardo-Nirenberg inequality to prove the existence of some \(C > 0\) with
\[ \left( \int_\Omega u^{-(m+p+1)\theta} \right)^{\frac{1}{\theta}} \leq C + C \left( \int_\Omega \left| \nabla u^{\frac{m+p-1}{2}} \right|^2 \right)^{\frac{1}{2}} \left( \frac{m+p+1-\frac{1}{2}}{2} \right)^{\frac{1}{m+p+1-\frac{1}{2}}}. \]
and
\[
\left( \int_{\Omega} u^{2u} \right)^{\frac{1}{u}} \leq C + C \left( \int_{\Omega} \left| \nabla u^{\frac{m+p-1}{4}} \right|^2 \right)^{\frac{2}{m+p-1}} \left( \int_{\Omega} \left| \nabla u^{\frac{m+p-1}{4}} \right|^2 \right)^{\frac{2}{m+p-1}}
\]
as well as
\[
\left( \int_{\Omega} |\nabla v|^{2q'} \right)^{\frac{1}{q'}} \leq C + C \left( \int_{\Omega} |\nabla v|^q \right)^{\frac{2}{q}} \left( \int_{\Omega} |\nabla v|^q \right)^{\frac{2}{q}}
\]
and
\[
\left( \int_{\Omega} |\nabla v|^{2(q-1)\mu'} \right)^{\frac{1}{\mu'}} \leq C + C \left( \int_{\Omega} |\nabla v|^q \right)^{\frac{2}{q}} \left( \int_{\Omega} |\nabla v|^q \right)^{\frac{2}{q}}
\]
for all \( t \in (0, T) \). Firstly we have some \( C_1 > 0 \) with
\[
\left( \int_{\Omega} u^{(-m+p+1)\theta} \right)^{\frac{1}{\theta}} = \left\| u^{\frac{m+p-1}{4}} \right\|_{L^{\frac{2(m+p+1)}{m+p-1}}(\Omega)} \left\| u^{\frac{m+p-1}{4}} \right\|_{L^{\frac{2(m+p+1)}{m+p-1}}(\Omega)}
\]
\[
\leq C_1 \left\| \nabla u^{\frac{m+p-1}{4}} \right\|_{L^\infty(\Omega)} \left\| u^{\frac{m+p-1}{4}} \right\|_{L^{\frac{2(m+p+1)}{m+p-1}}(\Omega)} + C_1 \left\| u^{\frac{m+p-1}{4}} \right\|_{L^{\frac{2(m+p+1)}{m+p-1}}(\Omega)}
\]
where we have used (\( \theta_1 \)) and where \( a \) is given by
\[
-\frac{n(m-p-1)}{2(-m+p+1)\theta} = \left( 1 - \frac{n}{2} \right) a - \frac{n(m+p-1)}{2}(1-a)
\]
which leads to
\[
a = \frac{n(m+p-1)}{2(-m+p+1)} 1 - \frac{n}{2} + \frac{n(m+p-1)}{2}.
\]
Since (\( \mu_1 \)), we also have
\[
\left( \int_{\Omega} u^{2\mu} \right)^{\frac{1}{\mu}} = \left\| u^{\frac{m+p-1}{4}} \right\|_{L^{\frac{m+p-1}{4u}}(\Omega)} \left\| u^{\frac{m+p-1}{4u}} \right\|_{L^{\frac{m+p-1}{4u}}(\Omega)}
\]
\[
\leq C_1 \left\| \nabla u^{\frac{m+p-1}{4}} \right\|_{L^\infty(\Omega)} \left\| u^{\frac{m+p-1}{4}} \right\|_{L^{\frac{4(1-b)}{m+p-1}}(\Omega)} + C_1 \left\| u^{\frac{m+p-1}{4}} \right\|_{L^{\frac{4(1-b)}{m+p-1}}(\Omega)}
\]
with \( b = \frac{n(m+p-1)}{4} \left( \frac{2}{1-\frac{n}{2}} + \frac{n(m+p-1)}{2} \right) \). With respect to the integrals containing \( v \) we use (\( \theta_2 \)) to find
\[
\left( \int_{\Omega} |\nabla v|^{2q'} \right)^{\frac{1}{q'}} = \left\| |\nabla v|^q \right\|_{L^{\frac{2q}{q'}}(\Omega)} \left\| |\nabla v|^q \right\|_{L^{\frac{2q}{q'}}(\Omega)}
\]
\[
\leq C_1 \left\| |\nabla v|^q \right\|_{L^{\frac{2q}{q}}(\Omega)} \left\| |\nabla v|^q \right\|_{L^{\frac{2q}{q}}(\Omega)} + \left\| |\nabla v|^q \right\|_{L^{\frac{2q}{q}}(\Omega)}
\]
with \( c = \frac{na}{2} 1 - \frac{n}{2} \) and
\[
\left( \int_{\Omega} |\nabla v|^{2(q-1)\mu'} \right)^{\frac{1}{\mu'}} = \left\| |\nabla v|^q \right\|_{L^{\frac{2(q-1)\mu'}{q}}(\Omega)} \left\| |\nabla v|^q \right\|_{L^{\frac{2(q-1)\mu'}{q}}(\Omega)}
\]
\[
\leq C_1 \left\| |\nabla v|^q \right\|_{L^{\frac{2(q-1)\mu'}{q}}(\Omega)} \left\| |\nabla v|^q \right\|_{L^{\frac{2(q-1)\mu'}{q}}(\Omega)} + \left\| |\nabla v|^q \right\|_{L^{\frac{2(q-1)\mu'}{q}}(\Omega)}
\]
Lemma 3.6. There is a positive constant $C$ with

$$\|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \text{ for all } t \in (0, T).$$

Proof. We write

$$v(\cdot, t) = e^{t(\Delta^{-1})} v_0 + \int_0^t e^{(t-s)(\Delta^{-1})} u(\cdot, s) \, ds$$

for $t \in (0, T)$ and fix some $p > n$. By lemma 3.5 we find $C_1 > 0$ with

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \text{ for all } t \in (0, T).$$

with $d = \frac{n a}{2(q - 1) \cdot \frac{q - 1}{q - 1} + \frac{n a}{2}}$, which utilises $(\mu_2)$. In a next step we use the boundedness of $\|u(\cdot, t)\|_{L^1(\Omega)}$ and $\|\nabla v(\cdot, t)\|_{L^2(\Omega)}$ as provided by lemma 3.2, so that lemma 2.4 gives us some positive constant $C_2$ with

$$\frac{d}{dt} \left( \int_\Omega u^p + \int_\Omega |\nabla v|^{2q} \right) + C_2 \int_\Omega \left| \nabla u \right|^{\frac{m + p - 1}{2}} + C_2 \int_\Omega |\nabla \nabla|^2 q \leq \frac{1}{C_2}$$

for all $t \in (0, T)$. Similarly to before, the Gagliardo-Nirenberg inequality allows for the comparison of the occurent terms. We see

$$\int_\Omega u^p \leq C_1 \left\| \nabla u \right\|_{L^{\frac{2p}{m + p - 1}}(\Omega)}^{\frac{m + p - 1}{2}} \left\| u \right\|_{L^{\frac{2(1 - q)p}{m + p - 1}}(\Omega)}^{2(1 - q)p} + C_1 \left\| u \right\|_{L^{\frac{2p}{m + p - 1}}(\Omega)}^{\frac{m + p - 1}{2}}$$

for all $t \in (0, T)$ and this gives us some positive constants $C$ and $C_4$ and the ordinary differential inequality

$$\dot{y}(t) + C_4 y(t) \leq \frac{1}{C_4}$$

for $y(t) := \int_\Omega u^p(\cdot, t) + \int_\Omega |\nabla v(\cdot, t)|^{2q}$, a comparison argument shows

$$y(t) \leq \max \left\{ \int_\Omega u^p_0 + \int_\Omega |\nabla v_0|^{2q}, C_4^{\frac{2}{q}} \right\}$$

for all $t \in (0, T)$ which completes the proof. \qed

From this and $L^p$-$L^q$-estimates we can deduce uniform boundedness of $v$ and its gradient similarly to results in section 3 in [9]:

Lemma 3.6. There is a positive constant $C$ with

$$\|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \text{ for all } t \in (0, T).$$

Proof. We write

$$v(\cdot, t) = e^{t(\Delta^{-1})} v_0 + \int_0^t e^{(t-s)(\Delta^{-1})} u(\cdot, s) \, ds$$

for $t \in (0, T)$ and fix some $p > n$. By lemma 3.5 we find $C_1 > 0$ with

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \text{ for all } t \in (0, T)$$
and obviously there is $C_2 > 0$ such that
\[ \|v_0\|_{W^{1,\infty}(\Omega)} \leq C_2. \]
For the heat equation we also know that positive constants $C_3$ and $C_4$ with
\[ \|e^{\tau A} \varphi\|_{L^\infty(\Omega)} \leq C_3 \|\varphi\|_{L^\infty(\Omega)} \]
for all $\tau > 0$ and for every $\varphi \in L^\infty(\Omega)$ and
\[ \|\nabla e^{\tau A} \varphi\|_{L^\infty(\Omega)} \leq C_3 \|\varphi\|_{W^{1,\infty}(\Omega)} \]
for all $\tau > 0$ and for every $\varphi \in W^{1,\infty}(\Omega)$ exist. As a final preparation we use $L^p$-$L^q$ estimates as in lemma 1.3 in [36] to find $C_5 > 0$ and $C_6 > 0$ with
\[ \|e^{\tau A} \varphi\|_{L^p(\Omega)} \leq C_5 \left(1 + \tau^{-\frac{n}{p'}}\right) \|\varphi\|_{L^p(\Omega)} \]
for all $\tau > 0$ and every $\varphi \in L^p(\Omega)$ and
\[ \|\nabla e^{\tau A} \varphi\|_{L^p(\Omega)} \leq C_6 \left(1 + \tau^{-\frac{1}{2}-\frac{n}{p'}}\right) \|\varphi\|_{L^p(\Omega)} \]
for all $\tau > 0$ and every $\varphi \in L^p(\Omega)$. Together with the representation above we therefore see
\[ \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq e^{-t} \|e^{t A} v_0\|_{L^\infty(\Omega)} + \int_0^t e^{-(t-s)} \left\|e^{(t-s) A} u(\cdot, s)\right\|_{L^\infty(\Omega)} ds \]
\[ \leq C_2 C_3 + C_1 C_5 \int_0^\infty e^{-\tau} \left(1 + \tau^{-\frac{n}{p'}}\right) d\tau \]
for every $t \in (0, T)$ and the integral on the right-hand side is finite. We can estimate similarly for the norm of the derivative of $v$:
\[ \|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \leq e^{-t} \|\nabla e^{t A} v_0\|_{L^\infty(\Omega)} + \int_0^t e^{-(t-s)} \left\|\nabla e^{(t-s) A} u(\cdot, s)\right\|_{L^\infty(\Omega)} ds \]
\[ \leq C_2 C_4 + C_1 C_6 \int_0^\infty e^{-\tau} \left(1 + \tau^{-\frac{1}{2}-\frac{n}{p'}}\right) d\tau \]
holds for all $t \in (0, T)$ and here, too, the right-hand side is a positive number. \(\square\)

With these regularity results for $u$ and $v$ achieved, we are now able to prove even $u \in L^\infty(\Omega \times (0, T))$ using an estimate that basically removes the dependence on $p$ in the $L^p$-norms of $u$.

**Lemma 3.7.** There is a positive constant $C$ with
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T). \]

**Proof.** Lemma 3.5 gives us some $p > n + 2$ and $C_1 > 0$ with
\[ \|u(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \text{ for all } t \in (0, T) \]
and due to lemma 3.6 we have some positive $C_2$ with
\[ \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_2 \text{ for all } t \in (0, T) \]
Together with lemma A.1 from [32] we conclude as claimed. \(\square\)

**Proof of Lemma 3.1.** We combine the two previous lemmata 3.7 and 3.6 to find the asserted estimate. \(\square\)
4. Existence of solutions and conclusions. The crucial question concerning the existence and uniqueness of solutions is whether $D$ vanishes at $u = 0$ or not. In the case of nondegenerate diffusion we will detect the existence of classical solutions and the previous section proves their boundedness. Using an approximation process, this will also result in us finding weak solutions for systems where $D(0) = 0$ and for this it is crucial that the bounds from before do not depend on the precise value of $D$ at $u = 0$.

4.1. Definition of weak solutions and proofs of the theorems. After the results of the previous section and together with some technical assumptions, the global existence of classical solutions in the nondegenerate case can be proven relatively quickly:

Proof of theorem 1.1. Using standard arguments (namely from [18]) we find a local solution to $(S)$ in $\Omega \times (0, T_{\text{max}})$ for some $T_{\text{max}} \in (0, \infty]$. We also see that either $T_{\text{max}} = \infty$ or

$$
\limsup_{t \uparrow T_{\text{max}}} \left( \|u(\cdot, t)\|_{L^p(\Omega)} + \|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \right) = \infty.
$$

Lemma 3.1 shows that the second alternative cannot occur.

Using an approximation process, this allows us to obtain a result for degenerate diffusion as well. Let us begin by defining an appropriate solution concept:

Definition 4.1. Let $D(s) := \int_0^s D(\sigma) \, d\sigma$ for $s \geq 0$. By a locally bounded global weak solution to $(S)$ we mean a pair of functions $(u, v) : \Omega \times [0, \infty) \rightarrow \mathbb{R}^2$ with the regularity

$$
u \in L^\infty_{\text{loc}}([0, \infty); L^\infty(\Omega)),$$

$$D(u) \in L^2_{\text{loc}}([0, \infty); L^2(\Omega))$$

and

$$v \in L^\infty_{\text{loc}}([0, \infty); W^{1, \infty}(\Omega))$$

which solves

$$
- \int_0^\infty \int_\Omega u \varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) = - \int_0^\infty \int_\Omega \nabla D(u) \nabla \varphi - \int_0^\infty \int_\Omega u \nabla v \nabla \varphi
$$

and

$$
- \int_0^\infty \int_\Omega v \varphi_t - \int_\Omega v_0 \varphi(\cdot, 0) = - \int_0^\infty \int_\Omega \nabla v \nabla \varphi - \int_0^\infty \int_\Omega v \varphi + \int_0^\infty \int_\Omega u \varphi
$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$.

To prove the existence of such a solution we will use $\varepsilon \in (0, 1)$ and the function $D_\varepsilon := D(\cdot + \varepsilon)$ to approximate $D$. Clearly, upon an appropriate discussion of the initial data, this choice allows for the employment of theorem 1.1 since

$$C_D \varepsilon^{m-1} (s + 1)^{m-1} \leq D(s) \leq C'_D (s + 1)^{M-1}$$

holds for all $s \geq 0$. While this may seem to couple the estimates to $\varepsilon \in (0, 1)$, which is now a necessary part of such a lower bound for the diffusion, the proofs only rely on $D_\varepsilon(s) \geq C_D \varepsilon^{m-1}$ which is valid independently of $\varepsilon \in (0, 1)$.

As a basis for all following steps we want to fix the used approximations and their properties.
Lemma 4.2. Let \( C_D > 0, \ C_D' > 0 \) as well as \( M \geq m > 1 + \frac{(n-2)(n-1)}{n^2} \) and \( D \in C^1([0, \infty)) \) with
\[
C_D s^{m-1} \leq D(s) \leq C_D'(s + 1)^{M-1}
\]
for all \( s \geq 0 \). Furthermore, let nonnegative \( u_0 \in L^1(\Omega) \) as well as \( v_0 \in L^1(\Omega) \) and for \( \varepsilon \in (0, 1) \) define \( D_\varepsilon := D(\cdot + \varepsilon) \). Then we have
\[
C_D \varepsilon^{m-1}(s + 1)^{M-1} \leq D_\varepsilon(s) \leq C_D' 2^{M-1}(s + 1)^{M-1}
\]
and
\[
D_\varepsilon(s) \geq C_D s^{m-1}
\]
for all \( s \geq 0 \). Additionally there are \( K > 0 \) and two sequences of functions, \((u_{0\varepsilon})_{\varepsilon \in (0, 1)} \subset C^0(\Omega)\) and \((v_{0\varepsilon})_{\varepsilon \in (0, 1)} \subset C^1(\Omega)\), such that
\[
\|u_{0\varepsilon}\|_{L^\infty(\Omega)} + \|v_{0\varepsilon}\|_{W^{1, \infty}(\Omega)} \leq K
\]
for every \( \varepsilon \in (0, 1) \) as well as
\[
\|u_0 - u_{0\varepsilon}\|_{L^1(\Omega)} + \|v_0 - v_{0\varepsilon}\|_{L^1(\Omega)} \to 0
\]
as \( \varepsilon \to 0 \).

Proof. The estimates for \( D_\varepsilon \) are an immediate consequence of the properties given to \( D \) and the rest is a matter of choosing a helpful approximation.

Having fixed these quantities, we now consider a slightly different system than before. Aside from the adapted initial data ensuring sufficient regularity we also change the first equation in such a way that the diffusion is no longer degenerate. The resulting system is the following:
\[
\begin{aligned}
  u_{\varepsilon t} &= \nabla \cdot (D_\varepsilon(u_\varepsilon) \nabla u_\varepsilon + u_\varepsilon \nabla v_\varepsilon) \quad \text{in } \Omega \times (0, \infty), \\
  v_{\varepsilon t} &= \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon \quad \text{in } \Omega \times (0, \infty), \\
  \frac{\partial u_\varepsilon}{\partial \nu} &= \frac{\partial v_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
  u_\varepsilon(0, \cdot) &= u_{0\varepsilon}, \quad v_\varepsilon(\cdot, 0) = v_{0\varepsilon} \quad \text{in } \Omega.
\end{aligned}
\]

This system meets all the requirements we have previously seen in deriving classical solutions with the helpful properties of uniqueness and global existence. Additionally, since all \( D_\varepsilon \) share the quality \( D_\varepsilon(s) \geq C_D s^{m-1} \) for every \( s \geq 0 \), we find a common upper bound for the family of approximating solutions:

Lemma 4.3. For the quantities from lemma 4.2 and every \( \varepsilon \in (0, 1) \) the system \((S_\varepsilon)\) has a unique classical solution \((u_\varepsilon, v_\varepsilon)\) that is global and there is \( C > 0 \) with
\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C
\]
for every \( \varepsilon \in (0, 1) \) and every \( t \in (0, \infty) \).

Proof. Firstly, theorem 1.1 gives us unique classical solutions along with their global existence. The parameter-independent estimate \( D_\varepsilon(s) \geq C_D s^{m-1} \) for all \( s \geq 0 \) and every \( \varepsilon \in (0, 1) \) then guarantees the uniform boundedness together with theorem 3.1 and thereby finishes the proof.

These solutions to the approximate problems \((S_\varepsilon)\) will be shown to converge to solutions of the actual system \((S)\) in a suitable fashion. In preparation of the discussion of this convergence we will now find and fix several bounds that will enable us to start a process where we repeatedly choose subsequences along which
Lemma 4.4. Given $T \in (0, \infty)$ there is $C_T > 0$ with the following property: For any $\varepsilon \in (0, 1)$ and the corresponding solution $(u_\varepsilon, v_\varepsilon)$ to the approximate problem we have

$$\|u_\varepsilon\|_{L^\infty(\Omega \times (0,T))} \leq C_T,$$

$$\|v_\varepsilon\|_{L^\infty((0,T);W^{1,\infty}(\Omega))} \leq C_T$$

and

$$\|D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon\|_{L^2(\Omega \times (0,T))} \leq C_T$$

as well as

$$\|\nabla u_\varepsilon^{m-1}\|_{L^2(\Omega \times (0,T))} \leq C_T;$$

$$\int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) u_\varepsilon^{m-3} |\nabla u_\varepsilon|^2 \leq C_T,$$

$$\|v_\varepsilon\|_{L^2((0,T);(W^{1,1}(\Omega))')} \leq C_T$$

and

$$\|(u_\varepsilon^{m-1})\|_{L^1((0,T);(W^{0,1}(\Omega))')} \leq C_T.$$

Proof. The first two statements correspond to lemma 4.3 and have already been proven. For the next claim we define

$$\overline{D}_\varepsilon(u) := \int_0^u D_\varepsilon(\sigma) \, d\sigma := \int_0^u \int_0^\sigma D(\tau) \, d\tau \, ds$$

and rewrite the first differential equation in $(S_\varepsilon)$ as

$$u_\varepsilon t = \Delta \overline{D}_\varepsilon(u_\varepsilon) + \nabla \cdot (u_\varepsilon \nabla v_\varepsilon)$$

to which clearly $(u_\varepsilon, v_\varepsilon)$ remains a solution. Testing this with $\overline{D}_\varepsilon(u_\varepsilon)$ and using Young’s inequality gives us

$$\int_0^T \int_\Omega (\overline{D}_\varepsilon(u_\varepsilon))_t \, d\sigma := \int_0^T \int_\Omega \nabla \overline{D}_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \, d\sigma = \int_0^T \int_\Omega \nabla \overline{D}_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \, d\sigma \leq \frac{1}{2} \int_0^T \int_\Omega |\nabla \overline{D}_\varepsilon(u_\varepsilon)|^2 + \frac{1}{2} \int_0^T \int_\Omega u_\varepsilon^2 |
\n\nand all of these quantities are bounded independently of $\varepsilon \in (0,1)$. Next, from the first two claims we have some $C_1 > 0$ with $\|\overline{D}_\varepsilon(u_\varepsilon)\|_{L^2((0,T);(W^{0,1}(\Omega))')} \leq 3C_1 \|\overline{D}_\varepsilon(u_\varepsilon)\|_{W^{1,1}(\Omega)}$ for any $\varphi \in C_0^\infty(\Omega)$ for all $\varepsilon \in (0,1)$. Therefore, we have already found a bound for $\|v_\varepsilon\|_{L^2((0,T);(W^{0,1}(\Omega))')}$. Now let there be $C_2 > 0$ such that for any $\Phi \in W^{1,1+n}(\Omega)$ with $\|\Phi\|_{W^{1,1+n}(\Omega)} \leq 1$ we have $\|\Phi\|_{L^\infty(\Omega \times (0,T))} \leq C_2$ and such that for any $\Phi \in L^\infty((0,T);W^{1,1+n}(\Omega))$ with $\|\Phi\|_{L^\infty((0,T);W^{1,1+n}(\Omega))} \leq 1$ we have $\|\Phi\|_{W^{1,2}(\Omega \times (0,T))} \leq C_2$ as well as the estimate $\|\Phi\|_{W^{1,1}(\Omega \times (0,T))} \leq C_2$. u and v converge in a certain sense. Here and in the subsequent proof we follow ideas from [19].
Setting $X := L^1((0, T); (W^{1,n+1}_0(\Omega))^*)$ we see $X^* = L^\infty((0, T); W^{1,n+1}_0(\Omega))$ and thus any $\varphi \in X^*$ with $||\varphi||_{X^*} \leq 1$ gives us

$$\frac{1}{m-1} \left| \int_0^T \int_\Omega (u^{m-1}_\varepsilon)_t \varphi \right| = \left| \int_0^T \int_\Omega u^{m-2}_\varepsilon u_{\varepsilon t} \varphi \right|$$

$$\leq \left| \int_0^T \int_\Omega u^{m-2}_\varepsilon \varphi \cdot (D_\varepsilon(u_\varepsilon)\nabla u_\varepsilon) \right|$$

$$+ \left| \int_0^T \int_\Omega u^{m-2}_\varepsilon \varphi \cdot (u_\varepsilon \nabla v_\varepsilon) \right|$$

$$\leq |m - 2| \left| \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon)u^{m-3}_\varepsilon \varphi |\nabla u_\varepsilon|^2 \right|$$

$$+ \left| \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon)u^{m-2}_\varepsilon \nabla u_\varepsilon \nabla \varphi \right|$$

$$+ \left| \int_0^T \int_\Omega u^{m-2}_\varepsilon \varphi \nabla u_\varepsilon \nabla v_\varepsilon \right|$$

$$+ \left| \int_0^T \int_\Omega u^{m-2}_\varepsilon \nabla v_\varepsilon \nabla \varphi \right|$$

$$= I_1 + I_2 + I_3 + I_4$$

for all $\varepsilon \in (0, 1)$. Firstly we have

$$I_1 \leq |m - 2|C_2 \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon)u^{m-3}_\varepsilon |\nabla u_\varepsilon|^2$$

and

$$I_2 \leq \frac{1}{2} \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon)u^{m-3}_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{2} \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon)u^{m-1}_\varepsilon |\nabla \varphi|^2$$

$$\leq \frac{1}{2} \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon)u^{m-3}_\varepsilon |\nabla u_\varepsilon|^2 + 2^{M-2}C_Dc_2 (1 + C_1)^{m+M-2}.$$
Together with an elementary bound for \( x \mapsto x \log x \), this shows that the for this case relevant quantities
\[
\int_0^T \int_\Omega \frac{D_x(u_\varepsilon)}{u_\varepsilon} |\nabla u_\varepsilon|^2
\]
and
\[
\int_0^T \int_\Omega |\nabla u_\varepsilon|^2
\]
are bounded. If conversely \( m \neq 2 \), then an almost identical computation for \( \frac{\pi}{\Omega} \int_0^m \int_\Omega |\nabla u_\varepsilon|^2 \) proves the equivalent of the two estimates above. Therefore, for any admissible \( m \) all four claims hold true.

We can now prove the existence of a weak solution \((u, v)\) to \((S)\) by taking a zero sequence \((\varepsilon_k)_{k \in \mathbb{N}} \subset (0, 1)\) and solutions \((u_{\varepsilon_k}, v_{\varepsilon_k})\) to the approximating problems \((S_\varepsilon)\) for \( \varepsilon = \varepsilon_k \) and letting \( k \to \infty \).

**Proof of theorem 1.2.** For \( \varepsilon \in (0, 1) \) we again use \( \overline{D}_x(s) := \int_s^0 D(\sigma) \, d\sigma \) and by lemma 4.3 we have a uniquely determined global classical solution to \((S_\varepsilon)\) so that there can be no confusion concerning the functions we are dealing with in the upcoming proof.

We begin by proving the following claim: For every \( k \in \mathbb{N} \) there is a zero sequence \((\varepsilon_{k,l})_{l \in \mathbb{N}} \subset (0, 1)\) which for \( k \in \mathbb{N} \) is a subsequence of \((\varepsilon_{k-1,l})_{l \in \mathbb{N}}\) and for which we have the following convergences as \( l \to 0 \):

- \( u_{\varepsilon_{k,l}} \) converges a.e. in \( \Omega \times (0, k) \) and in \( L^1(\Omega \times (0, k))\),
- \( \overline{D}_{\varepsilon_{k,l}}(u_{\varepsilon_{k,l}}) \) converges weakly in \( L^2((0, k); W^{1,2}_0(\Omega))\),
- \( v_{\varepsilon_{k,l}} \) converges uniformly in \( \Omega \times (0, k) \) and
- \( \nabla v_{\varepsilon_{k,l}} \) converges weakly* in \( L^\infty(\Omega \times (0, k))\).

We start with an arbitrary monotonic zero sequence \((\varepsilon_{0,l})_{l \in \mathbb{N}} \subset (0, 1)\) and so we can assume that for some \( k \in \mathbb{N} \) we have a sequence \((\varepsilon_{k-1,l})_{l \in \mathbb{N}}\) with the desired properties. Thanks to lemma 4.4 we find \( C_1(k) > 0 \) with

\[
\begin{align*}
\|u_{\varepsilon_{k-1,l}}\|_{L^\infty((0,k)\times(0,k))} & \leq C_1(k) \quad \forall l \in \mathbb{N}, \\
\|v_{\varepsilon_{k-1,l}}\|_{L^\infty((0,k);W^{1,\infty}(\Omega))} & \leq C_1(k) \quad \forall l \in \mathbb{N}, \\
\|D_{\varepsilon_{k-1,l}}(u_{\varepsilon_{k-1,l}})\nabla u_{\varepsilon_{k-1,l}}\|_{L^2((0,k)\times(0,k))} & \leq C_1(k) \quad \forall l \in \mathbb{N}, \\
\|\nabla u_{\varepsilon_{k-1,l}}^m\|_{L^2((0,k)\times(0,k))} & \leq C_1(k) \quad \forall l \in \mathbb{N}, \\
\|(v_{\varepsilon_{k-1,l}})^i\|_{L^2((0,k);W^{1,2}_0(\Omega))}^* & \leq C_1(k) \quad \forall l \in \mathbb{N} \text{ and} \\
\|(u_{\varepsilon_{k-1,l}}^{-1})^i\|_{L^1((0,k);W^{1,n+1}_0(\Omega))}^* & \leq C_1(k) \quad \forall l \in \mathbb{N}.
\end{align*}
\]

The abbreviation \( d_k := \sup_{\varepsilon \in (0, 1)} \sup_{0 \leq s \leq C_1(k)} D_x(s) \leq C_D (C_1(k) + 2)^M - 1 \) shows

\[
\overline{D}_{\varepsilon_{k-1,l}}(u_{\varepsilon_{k-1,l}}(x,t)) \leq \int_0^{C_1(k)} d_k = C_1(k)d_k \quad \forall l \in \mathbb{N}
\]

and so we find \( C_2(k) > 0 \) with

\[
\|\overline{D}_{\varepsilon_{k-1,l}}(u_{\varepsilon_{k-1,l}})\|_{L^2((0,k);W^{1,2}_0(\Omega))} \leq C_2(k) \quad \forall l \in \mathbb{N}.
\]
Therefore, we may select a first subsequence \((\varepsilon_{k,l}^{(1)})_{l \in \mathbb{N}} \subseteq (\varepsilon_{k-1,l})_{l \in \mathbb{N}}\) along which 
\[
(\mathcal{D}_\varepsilon(u_{\varepsilon}))_{\varepsilon \in (0,1)}
\]
converges weakly in \(L^2((0,k);W^{1,2}(\Omega))\). Next we find \(C_3(k) > 0\) as an upper bound in 
\[
\left\| u_{k,l}^{m-1} \right\|_{L^2((0,k);W^{1,2}(\Omega))} \leq C_3(k) \quad \forall l \in \mathbb{N}
\]
and note the boundedness of the derivative \((u_{k,l}^{m-1})_{l \in \mathbb{N}}\) in \(L^1((0,k);\left(W^{1,n+1}_0(\Omega)\right)^*)\). Recalling \(W^{1,2}(\Omega) \subset L^2(\Omega) \hookrightarrow \left(W^{1,n+1}_0(\Omega)\right)^*\) we invoke lemma 2.3 to find a subsequence \((\varepsilon_{k,l}^{(2)})_{l \in \mathbb{N}} \subseteq (\varepsilon_{k,l}^{(1)})_{l \in \mathbb{N}}\) such that \((u_{k,l}^{m-1})_{l \in \mathbb{N}}\) converges in \(L^2(\Omega \times (0,k))\) for \(l \to \infty\). Since \(m > 1\), the mapping \([0,\infty) \ni y \mapsto y^{\frac{m-1}{m}}\) is continuous and so we have a sequence \((\varepsilon_{k,l}^{(3)})_{l \in \mathbb{N}} \subseteq (\varepsilon_{k,l}^{(2)})_{l \in \mathbb{N}}\) giving us the convergence of 
\[
(\varepsilon_{k,l}^{(3)})_{l \in \mathbb{N}}
\]
almost everywhere in \(\Omega \times (0,k)\) (instead of convergence only for its \((m-1)\)-st power). Thanks to Lebesgue’s dominated convergence theorem and the constant bound we also have convergence with respect to \(\left\| \cdot \right\|_{L^1(\Omega \times (0,k))}\). Applying the same lemma 2.3 to \(W^{1,\infty}(\Omega) \supseteq L^2(\Omega) \hookrightarrow \left(W^{1,1}_0(\Omega)\right)^*\) gives us another refinement 
\[
(\varepsilon_{k,l}^{(4)})_{l \in \mathbb{N}} \subseteq (\varepsilon_{k,l}^{(3)})_{l \in \mathbb{N}}
\]
converging uniformly in \(\Omega \times (0,k)\) while a final subsequence \((\varepsilon_{k,l}^{(5)})_{l \in \mathbb{N}} \subseteq (\varepsilon_{k,l}^{(4)})_{l \in \mathbb{N}}\) secures the weak*-convergence of 
\[
(\nabla v_{\varepsilon_{k,l}}^{(5)})_{l \in \mathbb{N}}
\]
in \(L^\infty(\Omega \times (0,k))\) due to the boundedness 
\[
\left\| \nabla v_{\varepsilon_{k,l}}^{(5)} \right\|_{L^\infty(\Omega \times (0,k))} \leq C_1(k) \quad \forall l \in \mathbb{N}.
\]
This completes the induction and setting \((\varepsilon_k)_{k \in \mathbb{N}} := (\varepsilon_{k,k})_{k \in \mathbb{N}}\) we find functions \(u, v, z : \Omega \to \mathbb{R}\) and \(\zeta : \Omega \to \mathbb{R}^n\) with 
\[
\begin{align*}
u_{\varepsilon_k} & \to u \quad \text{in } L^1_{\text{loc}}([0,\infty);L^1(\Omega)) \quad \text{and a.e. in } \Omega \times (0,\infty), \\
u_{\varepsilon_k} & \to v \quad \text{in } L^\infty_{\text{loc}}([0,\infty);C^0(\Omega)), \\
abla u_{\varepsilon_k} & \to \zeta \quad \text{in } L^2_{\text{loc}}([0,\infty);W^{1,2}(\Omega)) \quad \text{and} \\
abla v_{\varepsilon_k} & \to \zeta \quad \text{in } L^\infty_{\text{loc}}([0,\infty);L^\infty(\Omega))
\end{align*}
\]
as \(k \to \infty\). From \(u_{\varepsilon_k} + \varepsilon_k \to u\) a.e. in \(\Omega \times (0,\infty)\) as \(k \to \infty\) and the continuity of \(\mathcal{D}\) we see 
\[
\mathcal{D}_{\varepsilon_k}(u_{\varepsilon_k}) = \mathcal{D}_{\varepsilon_k}(u_{\varepsilon_k} + \varepsilon_k) - \mathcal{D}_{\varepsilon_k}(\varepsilon_k) \to \mathcal{D}(u)
\]
as \( k \to \infty \). Therefore we already know \( z = \overline{D}(u) \) while the second and fourth row combine to show \( \zeta = \nabla v \). Additionally the local boundedness of \( \|u_{\varepsilon_k}\|_{L^\infty(\Omega \times [0,k))} \) and its convergence ensure \( u \in \text{Lip}(0,\infty;L^\infty(\Omega)) \).

Thus we have proven that \( (u_{\varepsilon_k},v_{\varepsilon_k}) \) converges to a solution of \( (S) \) as \( k \to \infty \):

For any \( \varphi \in C^0(\Omega \times [0,\infty)) \) and any \( k \in \mathbb{N} \) we have

\[
-\int_0^\infty \int_\Omega u_{\varepsilon_k} \varphi_t - \int_\Omega u_{\varepsilon_k} \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla D_{\varepsilon_k}(u_{\varepsilon_k}) \nabla \varphi - \int_0^\infty \int_\Omega u_{\varepsilon_k} \nabla v_{\varepsilon_k} \nabla \varphi
\]

and

\[
-\int_0^\infty \int_\Omega v_{\varepsilon_k} \varphi_t - \int_\Omega v_{\varepsilon_k} \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v_{\varepsilon_k} \nabla \varphi - \int_0^\infty \int_\Omega v_{\varepsilon_k} \varphi + \int_0^\infty \int_\Omega u_{\varepsilon_k} \varphi
\]

and together with the convergences established above and those of the initial data the proof is complete. \( \square \)

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E-mail address: marcelf@math.uni-paderborn.de