Quantum Fluctuations and Dynamical Chaos: An Effective Potential Approach

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We discuss the intimate connection between the chaotic dynamics of a classical field theory and the instability of the one-loop effective action of the associated quantum field theory. Using the example of massless scalar electrodynamics, we show how the radiatively induced spontaneous symmetry breaking stabilizes the vacuum state against chaos, and we speculate that monopole condensation can have the same effect in non-Abelian gauge theories.

I. INTRODUCTION

The effective potential \( \Gamma(\phi) \) describes the energy density of a quantum field theory under the constraint of the prescribed field expectation values \( \phi_\alpha \), assumed to be constant in space and time. It is well established that \( \Gamma \), as the Legendre transform of the generating functional of connected Green functions

\[
W[J_\alpha] = -i \ln \int [d\Phi] e^{-iS[\Phi]} - iJ_\alpha \Phi_\alpha,
\]

is always real and a convex function of the expectation values \( \phi_\alpha = \langle \Phi_\alpha \rangle \). On the other hand, it is a familiar property of many quantum field theories that the one-loop effective potential \( \Gamma^{(1)} \) has an imaginary part in regions where the tree-level potential \( U(\phi) \) is concave. A well-known example is the double-well quartic scalar potential

\[
U(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2
\]

in the region \( \phi^2 < v^2/3 \). The resolution of the apparent paradox is provided by the observation that the Gaussian approximation for the functional integral of the prescribed field expectation values \( \phi_\alpha \) fails when \( \phi^2 < v^2 \), and hence the naive loop expansion does not apply. A better approximation is obtained by a superposition of two Gaussians, centered at \( \Phi = \pm v \) and weighted so that \( \langle \Phi \rangle = \phi \). This procedure corresponds, effectively, to a Maxwell construction of the true effective potential.

The condition for the reality of the (naive) one-loop effective potential \( \Gamma^{(1)} \) is that the matrix of second derivatives of the bare potential

\[
M_{\alpha\beta}^2 = \frac{\partial^2 U}{\partial \phi_\alpha \partial \phi_\beta}
\]
does not have a negative eigenvalue. \( M_{\alpha\beta}^2 \) is the mass matrix for quantum fluctuations around the expectation values \( \phi_\alpha \) of the fields \( \Phi_\alpha \). If the matrix \( M^2 \) has a negative eigenvalue, there exists a direction in which fluctuations grow exponentially with time, indicating the instability of the classical field configuration \( \Phi_\alpha = \phi_\alpha \). The resulting imaginary part of the effective potential describes the decay rate (per unit space) of the unstable field configuration.

The matrix of second derivatives also plays a role in the classical field theory as the stability matrix for a classical field trajectory against small perturbations of the initial conditions. Here one looks at spatially homogeneous, but time dependent field configurations \( \phi_\alpha(t) \) which evolve according to the classical field equations. If the stability matrix \( M^2 \) has a negative eigenvalue, the trajectory \( \phi_\alpha(t) \) is exponentially sensitive to the initial conditions resulting, in general, in deterministic chaotic dynamics of the fields. Dynamical chaos of this type is ubiquitous in field theories with more than one dynamical field. Familiar examples are: scalar electrodynamics, Yang-Mills theories, and bilinearly coupled theories involving two scalar fields. Dynamical instability of the trajectories of classical field configurations finds its expression in the existence of positive Lyapunov exponents \( \lambda_\nu > 0 \). The presence of negative eigenvalues of the stability matrix \( M^2 \) is a well-known criterion (Toda-Brumer criterion) for the presence of the (local) dynamical instability expressed by the positive Lyapunov exponents. These considerations show that the dynamical chaos of classical fields and the complexity of the one-loop effective potential of quantum fields are closely connected.

The fact that the exact effective potential \( \Gamma \) is always real indicates that the classical chaos must be suppressed in the full quantum field dynamics. However, this implies that the Gaussian (loop) expansion must break down in these cases. As a result, the vacuum state of classically chaotic quantum fields must acquire a nontrivial structure. As we shall discuss below, the vacuum instabilities can sometimes be avoided by the mechanism of spontaneous symmetry breaking, e.g., in the case of gauge theories coupled to a scalar (Higgs) field. The emergence of a dynamical mass due to the vacuum expectation value of the scalar field suppresses chaos and makes the effective potential real in the vicinity of the vacuum state. In other cases, such as Yang-Mills fields, the vacuum must...
acquire a much more complex structure, and the instabilities are avoided by the confinement of elementary field excitations in the infrared domain.

In the next section, we discuss dynamical chaos of classical fields in the context of scalar electrodynamics (SED). In section III we study the instabilities of quantum fluctuations in SED and discuss the stabilizing mechanism of spontaneous symmetry breaking. Section IV is devoted to the investigation between the imaginary part of the effective potential in the loop expansion and dynamical chaos in SED. Section V contains some remarks concerning the application of these ideas to Yang-Mills theories.

II. CLASSICAL SCALAR ELECTRODYNAMICS

A. Equations of Motion

Let us first consider Classical Scalar Electrodynamics (SED) without self-interaction of the scalar field \( \phi \). Later we will introduce the self-interaction of the scalar field with and without spontaneous symmetry breaking (SSB) but it is convenient to neglect it at first, in order to understand the origin of dynamical chaos in SED.

The Lagrangian density for the system with bare scalar mass \( m_0^2 \) is given by

\[
L = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + (D_\mu \phi)^* (D^\mu \phi) - m_0^2 \phi^* \phi
\]

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( D_\mu = \partial_\mu + ieA_\mu \). The corresponding equations of motion are:

\[
(D_\mu D^\mu + m_0^2) \phi = 0
\]

\[
\partial_\mu F^{\mu \nu} = j^\nu
\]

with the conserved current

\[
j_\mu = -ie(\phi^* D^\mu \phi - \phi D^\mu \phi^*)
\]

\[
= -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) + 2e^2 \phi^* \phi A^\mu.
\]

In the following, we consider the case of spatially homogeneous classical fields \( A_\mu(t) \) and \( \phi(t) \) for which the study of chaos is extremely simplified. The assumption of spatial homogeneity is not Lorentz invariant, but any solution of the equations of motion obtained in this way can be boosted into another reference frame. It is easy to see that, in the gauge \( A_0 = 0 \), Gauss’ law implies that \( j^0 = 0 \) and the phase of the scalar field \( \phi = \frac{1}{\sqrt{2}} \text{e}^{i \alpha} \) is time independent. Furthermore, the spatial current vector takes on the simple form

\[
j^i = 2e^2 \rho^2 A^i.
\]

We thus arrive at the following system of equations for the real scalar fields \( \rho \) and the vector potential \( A_i \):

\[
\dot{\rho} + (m_0^2 + e^2 A^2) \rho = 0,
\]

\[
\dot{A}_i + e^2 \rho^2 A_i = 0,
\]

where a dot indicates a time derivative.

Assuming that only a single component of \( A_i \) is nonvanishing, this system is classically equivalent to the well-known two-dimensional dynamical system with quartic potential \( x^2 y^2 \) which exhibits a strong chaotic behavior. The \( x^2 y^2 \)-model appears in various contexts in science including chemistry, astronomy, astrophysics, cosmology and most interesting for us, in the homogeneous limit of the Yang-Mills equations [13] (see [15] for details and references).

B. Conditions for Dynamical Chaos

There are various methods for establishing the chaoticity of the system [14]. For us most useful is the Toda-Bruner criterion [11,12], which is based on the study of the stability matrix of the potential energy. Denoting the coordinates as \( q_i \) \( (i = 1, \ldots, n) \) the stability matrix for the potential \( U(q) \) is

\[
\begin{align*}
U''_{ik}(q) &= \frac{\partial^2 U}{\partial q_i \partial q_k} \quad (i, k = 1, 2, \ldots, n),
\end{align*}
\]

where we assume that the system is conservative with quadratic separable kinetic energy. If the determinant of this matrix (proportional to the Gaussian curvature of the potential) is negative, the system is chaotic. It is worthwhile remarking here that the use of this local hyperbolicity criterion as a condition of global instability requires some caution. However, for potentials like the \( x^2 y^2 \) potential, the Toda-Bruner criterion of chaoticity is quite effective, establishing a necessary condition for global chaos. Returning to our dynamical system [3], we write the corresponding potential

\[
U_{\rho A} = \frac{1}{2} m_0^2 \rho^2 + \frac{1}{2} e^2 \rho^2 A^2,
\]

and the stability matrix

\[
U''_{\rho A} = \begin{pmatrix}
m_0^2 & e^2 A^2 \\
e^2 \rho A & e^2 \rho^2
\end{pmatrix}
\]

which immediately gives the following condition for the onset of chaos:

\[
e^2 A^2 > e^2 A^2 = \frac{m_0^2}{3}\]

Note that \( A_{cr} \) is independent of \( \rho \), the amplitude of the scalar field. The corresponding minimal energy for the onset of chaos is

\[
E_{cr} = \frac{2}{3} m_0^2 \rho^2.
\]

At the classical minimum of the potential \( E_{cr} = 0 \).
We conclude that classical massless scalar electrodynamics is strongly chaotic in the long wavelength limit for any magnitude of the spatially homogeneous fields. Chaos begins at zero value of the energy of the system. The mass \( m_0 \) of the scalar field sets a threshold for chaos. As we will see later, the threshold is influenced by: i) the self-coupling \( \lambda \phi^4 \) of the scalar field and ii) the quantum corrections to the classical potential \([11]\), both increasing the threshold for chaos. Quantum corrections lead to the formation of a nontrivial minimum in the effective potential of SED and, thus, produce non-zero mass (in the case of massless classical SED) or increase the “classical” mass \( m_0 \). The self-interaction of the scalar field also increases the threshold for chaos (for \( \lambda > 0 \)).

C. Self-interacting SED

We begin with the study of the self-interaction of the scalar field at the classical level. Adding to the potential \([11]\) the quartic self-interaction \( \lambda \phi^4 \) we easily obtain the following modified conditions for the onset of chaos:

\[
e^2 A_{cr}^2 = \lambda \rho^2 + \frac{m_0^2}{3} \tag{15}
\]

and

\[
E_{cr} = \frac{3}{4} \lambda \rho^4 + \frac{2}{3} m_0^2 \rho^2. \tag{16}
\]

At the classical minimum \( \rho = 0 \), still \( E_{cr} = 0 \).

For the case of the spontaneously broken gauge symmetry at the classical level

\[
U_{\rho A} = \frac{1}{2} e^2 A^2 \rho^2 + \frac{\lambda}{4} (\rho^2 - v^2)^2. \tag{17}
\]

With the vacuum expectation value of the scalar field \( v \) and its mass \( m_s^2 = 2 \lambda v^2 \) we obtain the following critical magnitude of the gauge field potential \( A_{cr} \) necessary for the onset of chaos:

\[
e^2 A_{cr}^2 = \lambda \left( \rho^2 - \frac{v^2}{3} \right). \tag{18}
\]

At the minimum of the potential \([17]\), \( \rho = v \), this gives

\[
e^2 A_{cr}^2 = \frac{m_s^2}{3}. \tag{19}
\]

which coincides with the condition \([13]\) for the onset of chaos for the case of massive classical SED without self-interaction.

Substituting \( A_{cr}^2 \) into \([17]\) and minimizing \( U_{\rho A} \) with respect to \( \rho \) we find:

\[
\rho_{cr}^2 = \frac{4}{9} v^2 \tag{20}
\]

and

\[
e^2 A_{cr}^2 = \frac{\lambda v^2}{9} = \frac{m_s^2}{18} \tag{21}
\]

giving the minimal energy

\[
E_{cr} = \frac{11}{108} \lambda v^4 \tag{22}
\]

necessary for the onset of chaos \([17]\). Note that in the case of SSB the coupling constant \( \lambda \) is absorbed in the definition of the mass generated by SSB, whereas in the corresponding case without SSB but with \( m_0 \neq 0 \), \( \lambda \) enters in the condition \([15]\) along with \( m_0 \).

Concluding this section, we sum up the results obtained for classical SED:

- Classical massless SED is chaotic in its long wavelength limit at any magnitude of the fields. No threshold values exist for the scalar and vector fields and chaos begins at zero energy of the system.

- Introduction of a mass of the scalar field (directly or by SSB\(^2\)) sets threshold values of the fields beyond which the system is chaotic.

It is reasonable to expect that the inclusion of a spatial dependence of the fields, while not eliminating the general chaotic behavior of the system under study, will result in a more complicated and rich picture.

Finally, let us remark that spontaneously broken SED in its spatially homogeneous limit may serve as a model of the uniform superconductor in a homogeneous time dependent electrical field with its possible phase transitions\([4]\).

III. QUANTUM CORRECTIONS TO MASSLESS SCALAR ELECTRODYNAMICS

A. General Considerations

It is generally believed that quantum fluctuations lead to the suppression of the most characteristic manifesta-

\(^1\)The spatially uniform classical SED with its chaoticity and the corresponding quantum mechanical aspects leading to the suppression of chaos have been intensely studied by the Los Alamos group \([12,22]\).

\(^2\)There also exists a close analogy between this phenomenon and the elimination of the chaos in SU(2) Yang-Mills model by the Higgs condensate \([4]\) where, depending on the value of the Higgs condensate, one observes a phase transition of the type “order–disorder”.
tions of dynamical chaos. For the mechanical systems it is obvious: the discreteness of the phase space imposed by the quantum nature of the system suppresses or even eliminates the long-time random behavior of the classically chaotic systems that is characterized by the positivity of the Lyapunov exponents. Indeed, the non-stationary evolution of a quantum mechanical system which classically is chaotic (SED) is characterized by vanishing Lyapunov exponents.

For the field theory with its infinite number of degrees of freedom the situation is not so straightforward. It is well established that various field theories, among which are the spherically symmetric Yang-Mills equations, the Yang-Mills-Higgs equations in the interior of a 't Hooft-Polyakov monopole, and the equations of general relativity, exhibit dynamical chaos in the classical limit. Here, as in the case of the mechanical systems, the basic question of the competition and interference between the highly unstable classical fluctuations responsible for chaos and the quantum fluctuations of the interacting fields arises. Do the quantum fluctuations suppress the chaoticity of the classical field theory? Although practically all methods of the quantization of fields about chaotic classical solutions encounter the problem of instability there does not exist a proven way to avoid or circumvent this delicate problem (see, however, [27]).

In this paper, we also do not propose a general recipe for how to quantize a field theory that is chaotic in the classical limit, and confine ourselves to the loop expansion taking as a basis the chaotic classical theory. Our treatment is based on the notion of the effective potential. We consider here, as an example, mostly the massless SED since it is free from the well-known difficulties arising when, for the case of SSB, the new minimum lies far outside the validity of the one-loop approximation.

We write down the effective potential $\Gamma(\phi_c)$ as the minimal expectation value of the Hamiltonian $\langle \psi | H | \psi \rangle$ in the normalized state $|\psi\rangle$, wherein the field $\phi(x)$ has a given constant expectation value $\langle \psi | \phi(x) | \psi \rangle = \phi_c$. Since for the calculation of the $\Gamma(\phi_c)$ one must consider the space dependence of the fields as well as their time dependence, the phase of the scalar field cannot be eliminated in the $A_0 = 0$ gauge and, writing $\phi = \varphi_1 + i\varphi_2$, two real scalar fields $\varphi_1$ and $\varphi_2$ enter. But the effective potential can depend only on $|\phi|^2 = \varphi_1^2 + \varphi_2^2$, so one may take $\varphi_2 = 0$ and compute only graphs with $\varphi_1$-external lines [29]. The Landau gauge is most appropriate and economical here.

As it is known, the mass matrix computed from the classical potential enters into the expression of the effective potential for two interacting fields. The one-loop effective potential for massless SED can be written as

$$
\Gamma^{(1)}(\rho, A) = U_{\rho A} + \frac{\hbar}{64\pi^2} \text{tr} \left[ (U_{\rho A}'')^2 \ln \left( \frac{U_{\rho A}''}{\mu^2} \right) \right]
$$

where $U_{\rho A}$ is the classical potential [11] and the matrix $U_{\rho A''}$ is given by [12]. Here $\mu^2$ is the renormalization point.

From the definition of the effective potential, it is evident that the exact effective potential must be real. The approximate calculation of this quantity in the loop expansion leads to regions of complexity which are impossible to eliminate for the case of the classically chaotic system under study. However, as we discuss below, this does not imply that complexity makes the effective potential meaningless. We see from [23] that for a classically chaotic system characterized by (Toda-Brumer condition)

$$
\det \{U_{\rho A''}''\} = \left( \frac{\partial^2 U}{\partial A^2 \partial \rho^2} - \left( \frac{\partial^2 U}{\partial A \partial \rho} \right)^2 \right) < 0
$$

the one-loop effective potential becomes complex for almost all values of the fields $A, \rho$ and not only for some finite range of the fields as it occurs, e.g., in the case of SSB at the tree level for non-chaotic systems. Massless SED and the free Yang-Mills theory in the limit of homogeneous fields are such systems.

It is possible to say that the complexity of the loop-expanded effective potential is one more relic of the chaos of the initial classical theory. The imaginary part of the effective potential signals not only the instability of the field configuration, but it is a general consequence of the chaos of the classical system.

From these considerations one can understand why all efforts to eliminate the imaginary part of the one-loop effective potential for uniform chromomagnetic field of the pure Yang-Mills theory in Minkowski space were unsuccessful. Stable radiative corrections in Minkowski-space require a stable classical configuration. But such configurations (instantons) are known only in the Euclidean space-time. We postpone further considerations of this issue until section V, except for one related remark. The presence of the imaginary part in
the one-loop effective potential is intrinsically linked to
the asymptotic freedom of the non-Abelian gauge fields
\[21,22\]. It is worth noting that recently this unstable
mode was detected directly in Monte Carlo simulations
of the lattice gauge theory \[23\].

Below, we temporarily avoid the complications caused
by the imaginary part of the potential \[23\] considering
only the effect of the quantum corrections along the axis
\(A = 0\). This “projection” retains the picture of the SSB
on the axes \(A = 0\) and \(\rho = 0\), as a numerical
evaluation of the one-loop effective action in the \(\rho - A\)
plane shows.

### B. One-Loop Effective Potential

At this stage, we turn to consider the one-loop cor-
rected effective potential for massless SED with self-
interaction \(\lambda \phi^4\) of the scalar field. We begin with the case
without SSB at the classical level. Following \[28\], the
one-loop effective potential for massless SED with self-
interaction \(\lambda \phi^4\) is:

\[
\Gamma^{(1)}(\rho, A; M^2) = \frac{1}{2} e^2 A^2 \rho^2 + \lambda \frac{\rho^4}{4} \\
+ \frac{5 \rho^4}{32 \pi^2} \left( \lambda^2 + \frac{3 e^4}{10} \right) \left[ \ln \frac{\rho^2}{\mu^2} - \frac{25}{6} \right]
\]

(25)

The quantum corrections lead to a new minimum of
the potential at \(A = 0\) but \(\rho = \bar{\rho} \neq 0\) instead of \(A = \rho = 0\).
Implementing the standard procedure of dimensional
transmutation we write:

\[
\Gamma^{(1)}(\rho, A; \bar{\rho}) = \frac{1}{2} e^2 A^2 \rho^2 + \frac{5 \rho^4}{32} \left( \lambda^2 + \frac{3 e^4}{10} \right) \left[ \ln \frac{\rho^2}{\bar{\rho}^2} - \frac{1}{2} \right],
\]

(26)

where the \(\lambda^2 \rho^4\) term of the classical potential is absorbed
in the subtraction point of the logarithm.\(^6\)

The effective potential now has a minimum value

\[
E_0^{(1)}(\rho, A; \bar{\rho}) = \Gamma^{(1)}(\rho, A; \bar{\rho}) \bigg|_{\rho=\rho_0, A=0, \bar{\rho}=0} = -\frac{5 \rho^4}{64 \pi^2} \left( \lambda^2 + \frac{3 e^4}{10} \right),
\]

(27)

which lies below the classical vacuum \(E_0^{(0)} =
U_{\rho A|\rho=0, A=0} = 0\). The masses of the scalar boson and
photon are

\[
m_A^2 = \frac{\partial^2 \Gamma^{(1)}}{\partial \rho^2} \bigg|_{\rho=\rho_0, A=0, \bar{\rho}=0} = \delta m_\lambda^{(1)2} + \delta m_e^{(1)2}
\]

(28)

where

\[
\delta m_\lambda^{(1)2} = \frac{5 \lambda^2}{4 \pi^2} \rho^2, \quad \delta m_e^{(1)2} = \frac{3 e^4}{8 \pi^2} \rho^2
\]

(30)

are the mass quantum corrections to the classically mass-
less scalar boson generated by the scalar self-coupling and
scalar-photon coupling, respectively, as shown in Fig. 1.

We now consider \(29\) for the case of spatially uniform
fields \(\rho(t), A(t)\) and apply the Toda-Brumer criterion for
the onset of chaos:

\[
\det \left\{ \Gamma^{(1)}(\rho, A; \bar{\rho}) \right\} = 0.
\]

(31)

We obtain, from \(31\), the critical value of \(A\) beyond
which the chaos sets in:

\[
e^2 A_{cr}^2 = \frac{5 \rho^2}{8 \pi^2} \left( \lambda^2 + \frac{3 e^4}{10} \right) \left[ \ln \frac{\rho^2}{\bar{\rho}^2} + \frac{2}{5} \right].
\]

(32)

Taking \(32\) in the vicinity of the new minimum \(\rho = \bar{\rho}\),
where our equations are reliable, we arrive at the relation

\[
e^2 A_{cr}^2 = \frac{m_s^2}{3},
\]

(33)

where \(m_s\) is given by \(30\). This relation must be com-
pared with \(13\) and \(19\). The comparison shows that quantum corrections generate a finite threshold for the
onset of chaos in massless SED, which classically was
chaotic for an infinitesimal amplitude of the vector field.
Let us note here that \(13\) also agrees with the result
obtained when one includes the self-interaction \(\lambda \phi^4\) at
the classical level with \(m_0 \neq 0\), inserting the classical
minimum \(\rho = A = 0\) into \(13\).

Let us consider next the one-loop corrections for mass-
less SED with SSB at tree level. Adding to the classical
potential \(17\) the one-loop quantum corrections and
eliminating the renormalization scale \(\mu\) by the new min-
imum of the scalar field \(\bar{\rho}\), we obtain:

\[
\Gamma^{(1)}_{SSB}(\rho, A; v, \bar{\rho}) = \frac{1}{2} e^2 A^2 \rho^2 + \frac{\lambda}{4} \left( \rho^2 - v^2 \right)^2 \\
+ \left[ \ln \frac{\rho^2}{\bar{\rho}^2} - \frac{1}{2} \right]
\]

(34)

with

\[
a \equiv \lambda^2 + \frac{3 e^4}{10}, \quad b \equiv \frac{8 \pi^2}{5 a} \left( 1 - \frac{v^2}{\rho^2} \right).
\]

(35)

Following the same steps as above, we obtain the critical
values of \(A\) for the onset of chaos:

\[
e^2 A_{cr}^2 = \lambda \left( \rho^2 - \frac{v^2}{3} \right) + \frac{5 a}{8 \pi^2} \rho^2 \left[ \ln \frac{\rho^2}{\bar{\rho}^2} - \frac{2}{3} \right].
\]

(36)
Minimizing \( \Gamma^{(1)}_{\text{SSB}}(\rho, A_c; v, \bar{\rho}) \) with respect to \( \rho \) and inserting the tree level values for \( \bar{\rho} \) and \( \rho_{cr} \) into the terms describing the quantum corrections, we get:

\[
\lambda \rho_{cr}^2 \approx \frac{4}{9} \lambda v^2 + 0.05 \frac{\lambda^2 v^4}{6\pi^2} \left( \frac{\lambda^2 + 3e^4}{10} \right) v^2.
\]

Finally, we arrive at the minimal energy for the onset of chaos:

\[
E_{cr} \approx \frac{11}{108} \lambda v^4 - \frac{4}{81\pi^2} \left( \lambda^2 + \frac{3e^4}{10} \right) v^4.
\]

The sign of the second term in \([38]\) is not surprising since the shift of the vacuum energy for the case without SSB (see \([27]\)) is bigger than for the case with SSB.

Finally, a slightly philosophical remark. From the above results one may say that massless charged scalar particles do not exist in nature due to the unavoidable quantum fluctuations which give them mass, eliminating the chaotic behavior of the field theory in the presence of an infinitesimal electromagnetic field.

### C. Two-Loop Corrections

The situation is repeated for the case of the two-loop effective potential, which we describe briefly. After minimization we may write the effective potential \( \Gamma^{(2)} \) up to order \( e^6 \) (here we discard \( \lambda^2 \) against \( e^4 \)):

\[
\Gamma^{(2)}(\rho, A; \bar{\rho}) = \frac{1}{2} e^2 A^2 \rho^2 + \frac{3e^4}{64\pi^2} (1 + Ce^2) \rho^4 \left( \ln \frac{\rho^2}{\bar{\rho}^2} - \frac{1}{2} \right) + \frac{5e^2}{512\pi^2} \rho^4 \left( \frac{1}{\bar{\rho}^2} + \ln \frac{\rho^2}{\bar{\rho}^2} \right)
\]

where \( \rho = \bar{\rho}, A = 0 \) is the minimum of the potential. The constant \( C \) (of order of unity) is defined by the precise specification of the renormalization conditions.

Figure 2 shows the two-loop graph which gives the \( O(e^6) \) contribution in Landau gauge. Purely scalar loops enter at the higher order \( O(e^8) \). Again, as in \([24]\), the classical self-interaction is absorbed in the process of the dimensional transmutation. The minimal value of the two-loop effective potential is

\[
\Gamma^{(2)}\bigg|_{\rho = \bar{\rho}, A = 0} = -\frac{3e^4}{128\pi^2} (1 + Ce^2) \bar{\rho}^2.
\]

As for the case of the one-loop correction, we obtain the expression for the critical value of \( A \) beyond which the system is chaotic (at \( \rho = \bar{\rho} \)):

\[
3e^2 A_{cr}^2 = \frac{3e^4 \bar{\rho}^2}{8\pi^2} + \frac{15e^6 \bar{\rho}^2}{128\pi^2} \left[ \frac{2}{3} + \frac{16\pi^2}{5} C \right].
\]

\([23]\) can be written in the form analogous to \([33]\):
Thus, it is not surprising that the relation between the positive maximal Lyapunov exponents $\lambda_0$ characterizing classical chaos of the SU(2) and SU(3) gauge theories (without quarks) and the corresponding analytically calculated damping rates $\gamma$ in the thermalized system of gauge bosons $\lambda_0 = \gamma$ is found numerically on the lattice [34,35].

V. INSTABILITY OF THE EFFECTIVE POTENTIAL FOR NON-ABELIAN GAUGE THEORIES

We repeatedly emphasized in this paper the relation between the instability of the (one-loop) effective potential and the intrinsic chaoticity of the classical non-Abelian gauge theories in Minkowski space. To avoid this kind of instability, one needs to start from a stable classical configuration. Pure gauge theories (without fermions, thermal effects, or Higgs fields) do not possess such stable classical states in Minkowski space (see [3] for details on the sources of the stabilization of classical gluon fields). The problem in pure gauge theories is that it is not possible to generate a dynamical mass for the gauge bosons perturbatively, in contrast to theories involving scalar fields. There exists, of course, a non-perturbative mechanism of mass generation: confinement. This property of non-Abelian gauge theories avoids the existence of propagating massless modes of the gauge field; only color-singlet bound states exist as physical fluctuations around the vacuum state.

It is widely believed that confinement in non-Abelian gauge theories is due to the presence of a vacuum condensate of chromomagnetic monopoles [3]. A complete description of this mechanism is clearly beyond the reach of perturbation theory, because no classically stable magnetic monopole solutions exist in pure gauge theories. However, we will demonstrate below that the condition of the stability of the one-loop effective potential allows us to derive a lower limit on the density of magnetic monopoles in the gauge field vacuum.

It is well-known that the one-loop effective potential for a uniform chromomagnetic field is unstable [29]. From our point of view, developed in the present paper, it is the result of the chaoticity of the corresponding classical Yang-Mills theory. This raises the question, which mechanism can induce the stability of the effective potential in the infrared domain? The natural approach is to consider uniform non-Abelian flux tube configurations. The properties of various types of such tubes were studied at the classical level [29,38]. It was shown that the stability can be achieved only by confining the chromomagnetic flux, not only in the transverse direction, but also along the direction of the chromomagnetic field. This condition can be naturally realized if one ends the flux on sources of the magnetic field lines. These sources can be interpreted as magnetic monopoles.

For the stability of such configurations the length $L$ must be smaller than $L_0 \approx \pi/\sqrt{gH}$ to eliminate small momenta $k < \sqrt{gH}$ contributing to the instability. We can utilize this condition to estimate the density of magnetic monopoles in the QCD vacuum. The real part of the renormalized one-loop effective potential for a constant chromomagnetic field $H$ is

$$\Gamma^{(1)}(H) = \frac{1}{2} H^2 + \frac{11 N_c}{96 \pi^2} (gH)^2 \left[ \ln \frac{gH}{\mu^2} - \frac{1}{2} \right],$$

(46)

where $N_c$ is the number of colors and $\mu^2$ is the renormalization point. The minimum of the effective potential,

$$gH_0 = \mu^2 \exp \left( - \frac{48 \pi^2}{11 N_c g^2(\mu)} \right),$$

(47)

is a renormalization group invariant.

Now consider the corresponding magnetic flux in a region of length $L$. Gauss’ law states that the magnetic monopole charge density (per area) terminating the magnetic flux lines equals $H_0$. Because the elementary magnetic monopole charge in non-Abelian gauge theories is $4\pi/g$ [39], the required density of magnetic monopoles is

$$n_M = \frac{2 gH_0}{4\pi L_0},$$

(48)

where the factor 2 counts monopoles and antimonopoles. Inserting the condition $L \leq L_0$ for the vacuum stability, we obtain

$$n_M \geq \frac{gH_0}{2\pi L_0} = \left( \frac{gH_0}{2\pi} \right)^{3/2},$$

(49)

We can obtain a numerical estimate for the monopole density, if we identify $gH_0$ with the gluon condensate obtained in the QCD sum rule approach [40].

$$(gH_0)^2 = \langle g^2 F_{\mu\nu}^a F^{a\mu\nu} \rangle \approx 0.5 \text{ GeV}^4.$$

(50)

This yields a value for a monopole density:

$$n_M \approx 0.03 \text{ GeV}^3 \approx 4 \text{ fm}^{-3}.$$

(51)

Of course, the reliability of the one-loop approximation is questionable near the minimum of the effective potential, because the vacuum field $H_0$ is not strong. However, it is noteworthy that this very simple consideration provides a relation between the strength of the monopole
condensate and the strength of the gluon condensate in QCD, which can be tested by lattice calculations.

One can further develop this essentially perturbative picture of the chromomagnetic condensate, based on [16], by the extension of the electromagnetic duality principle to the weak-strong coupling duality [15] connecting the weakly coupled chromomagnetic phase with the strong coupling regime of the chromoelectric phase with its color confinement. This picture is in accord with Monte Carlo simulations [32] where the unstable modes in a chromomagnetic background were observed. These results support the idea that the QCD vacuum behaves in the continuum limit as a quasi-Abelian magnetic condensate with the properties of a dual superconductor.

VI. CONCLUSIONS

What lessons can we draw from the results presented here? First of all, we have shown that the onset of chaoticity of the classical fields in theories such as SED is delayed by the radiative corrections. In the case of massless SED, which is chaotic for all energies at the tree level, the radiative corrections introduce a threshold for the onset of chaos.

The classical chaoticity, in turn, leads to the instability of the corresponding effective potential, presumably, at any finite order of the loop expansion. This interdependence may explain the failure of numerous attempts to eliminate the Nielsen-Olesen instability [29] of the one-loop effective potential for a uniform chromomagnetic field in the pure Yang-Mills theory [36].

Since the true effective potential is known to be always a real and convex function of the field expectation values, the instabilities associated with deterministic chaos must be absent in the full quantum theory. Higher-order (non-Gaussian) quantum fluctuations provide the mechanism for this phenomenon. We have already discussed the double-well oscillator where the ground state can be approximated as the sum of two Gaussians.

The suppression of chaoticity can sometimes also be seen at the classical level, e.g., if one includes the anharmonicity stabilizing the inverted oscillator. Consider the equation

\[ \ddot{\varphi} - m^2 \varphi + \lambda \varphi^3 = 0. \]  

(52)

In the absence of the anharmonic term, the equation has a solution \( \varphi(t) \sim e^{mt} \) for zero energy, indicating exponential instability. If we include the \( \lambda \varphi^3 \) term, the zero-energy solution becomes \( \frac{8m^2}{\lambda} \frac{e^{\pm m(t-t_0)}}{1 + e^{\pm 2m(t-t_0)}} \),

\[ \varphi(t) = \sqrt{\frac{8m^2}{\lambda}} \frac{e^{\pm m(t-t_0)}}{1 + e^{\pm 2m(t-t_0)}}, \]  

(53)

which behaves as \( e^{-mt} \) for large times. Unfortunately, it is not known how to perform functional integrals in quantum field theory beyond the Gaussian approximation by analytical techniques.

This raises the question whether, in a given theory, it is possible to find a stable classical configuration in Minkowski space around which the theory can be quantized. Several mechanisms are known [15] which generate stable solutions (and hence eliminate chaos at low energies) in gauge theories: mass generation by the Higgs mechanism or topological effects, mass generation by medium polarization at finite temperature, and stabilization of fluctuations by external charges [13, 14]. Although none of these mechanisms directly applies to the QCD vacuum, the quark vacuum condensate may have a stabilizing effect. On a more general scope, the question of the possible stabilizing role of fermions in supersymmetric Yang-Mills theories arises. We hope to return to these issues in the future.

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