Rational exponential sums over the divisor function

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Abstract
We consider a problem posed by Shparlinski, of giving nontrivial bounds for rational exponential sums over the arithmetic function $\tau(n)$, counting the number of divisors of $n$. This is done using some ideas of Sathe concerning the distribution in residue classes of the function $\omega(n)$, counting the number of prime factors of $n$, to bring the problem into a form where, for general modulus, we may apply a bound of Bourgain concerning exponential sums over subgroups of finite abelian groups and for prime modulus some results of Korobov and Shkredov.

1 Introduction
We consider a problem posed by Shparlinski [19, Problem 3.27] of bounding rational exponential sums over the divisor function. More specifically, for integers $a, m$ with $(a, m) = 1$ and $m$ odd we consider the sums

$$T_{a,m}(N) = \sum_{n=1}^{N} e_m(a\tau(n)),$$

where $e_m(z) = e^{2\pi iz/m}$ and $\tau(n) = \sum_{d|n} 1$ counts the number of divisors of $n$. Arithmetic properties of the divisor function have been considered in a number of works, see for example [5, 6, 9, 13], although we are concerned mainly with congruence properties of the divisor function, which have also been considered in [4, 15, 16]. Exponential sums over some other arithmetic functions have been considered in [1, 2]. Our first step in bounding the sums (1) is to give a sharper version of a result of Sathe [16, Lemma 1] concerning the distribution of the function.
\[ \omega(n) \text{ in residue classes, where } \omega(n) \text{ counts the number of distinct prime factors of } n. \] This allows us to reduce the problem of bounding (1) to bounding sums of the form

\[ S_m(r) = \sum_{n=1}^{t} e_m(r2^n), \quad (2) \]

where \( t \) denotes the order of 2 (mod \( m \)) and we may not necessarily have \( (r, m) = 1 \). For arbitrary \( m \) we deal with these sums using a bound of Bourgain [3] and when \( m \) is prime we obtain sharper bounds using results of Korobov [12] when the order of 2 (mod \( m \)) is not to small. For smaller values we use results of Shkredov [18], which are based on previous results of Heath-Brown and Konyagin [10].

### 2 Notation

We use the notation \( f(x) \ll g(x) \) and \( f(x) = O(g(x)) \) to mean there exists some absolute constant \( C \) such that \( f(x) \leq C g(x) \) and we use \( f(x) = o(g(x)) \) to mean that \( f(x) \leq \varepsilon g(x) \) for any \( \varepsilon > 0 \) and sufficiently large \( x \).

If \( p|n \) and \( \theta \) is the largest power of \( p \) dividing \( n \), we write \( p^\theta||n \). We let \( S \) denote the set of all square-free integers, \( M_m \) the set of integers which are perfect \( m \)-th powers, \( Q_m \) the set of integers \( n \), such that if \( p^\theta||n \) then \( 2 \leq \theta \leq m - 1 \) and \( K \) the set of integers \( n \) such that if \( p^\theta||n \) then \( \theta \geq 2 \).

Given an arbitrary set of integers \( A \), we let \( A(x) \) count the number of integers in \( A \) less than \( x \), so that

\[ Q_m(x) \leq K(x) \ll x^{1/2}, \]

hence the sums

\[ H(r, m) = \sum_{\tau(q) \equiv r \pmod{m}} h(q) \frac{1}{q}, \quad h(q) = \prod_{p | q} \left( 1 + \frac{1}{p} \right)^{-1}, \quad (3) \]

converge. We let \( \zeta(s) \) denote the Riemann-zeta function,

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1, \]

and \( \Gamma(s) \) the Gamma function,

\[ \Gamma(s) = \int_{0}^{\infty} x^{s-1} e^{-x} dx, \quad \Re(s) > 0. \]

For odd integer \( m \) we let \( t \) denote the order of 2 (mod \( m \)) and define

\[ \alpha_t = 1 - \cos(2\pi/t). \quad (4) \]
3 Main results

Theorem 1. Suppose $m$ is odd and sufficiently large. Then with notation as in (1), (2), (3) and (4) we have

$$T_{a,m}(N) = \frac{\zeta(m)}{t} \frac{6}{\pi^2} \left( \sum_{r=0}^{m-1} H(r, m) S_m(ar) \right) N + O(tN(\log N)^{-\alpha_1}).$$

When $m = p$ is prime we use a different approach to save an extra power of $\log N$ in the asymptotic formula above, although our bound is worse in the $t$ aspect.

Theorem 2. Suppose $p > 2$ is prime, then

$$T_{a,m}(N) = \frac{\zeta(p)}{t} \frac{6}{\pi^2} \left( \sum_{r=0}^{p-1} H(r, p) S_p(ar) \right) N + O(pN(\log N)^{-(\alpha_1+1)}).$$

Combining Theorem 1 with the main result from [3] we obtain a bound which is nontrivial for $N \geq e^{ct^{1/\alpha_1}}$ for some fixed constant $c$.

Theorem 3. Suppose $m$ is odd and sufficiently large, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $t > m^\varepsilon$ then

$$\max_{(a,m) = \frac{1}{2}} |T_{a,m}(N)| \ll \left( \frac{1}{m^\delta} + t(\log N)^{-\alpha_1} \right) N.$$

Combining Theorem 2 with results from [12] and [18] we get,

Theorem 4. Suppose $p > 2$ is prime and let

$$A(t) = \begin{cases} 
    p^{1/8}t^{-7/18}(\log p)^{7/6}, & t \leq p^{1/2}, \\
    p^{1/4}t^{-23/36}(\log p)^{7/6}, & p^{1/2} < t \leq p^{3/5}(\log p)^{-6/5}, \\
    p^{1/6}t^{-1/2}(\log p)^{4/3}, & p^{3/5} < t \leq p^{2/3}(\log p)^{-2/3}, \\
    p^{1/2}t^{-1}(\log p), & t > p^{2/3}(\log p)^{-2/3}, 
\end{cases}$$

then we have

$$\max_{(a,p) = \frac{1}{2}} |T_{a,p}(N)| \ll (A(t) + p(\log N)^{-(\alpha_1+1)}) N.$$
4 Preliminary results

We use the decomposition of integers as in [16].

**Lemma 5.** For integer $m$, any $n \in \mathbb{N}$ may be written uniquely in the form

$$n = sqk$$

with $s \in S$, $q \in \mathcal{Q}_m$, $k \in \mathcal{M}_m$ and $\gcd(q, s) = 1$. For such a representation, we have

$$\tau(n) \equiv \tau(s)\tau(q) \pmod{m}.$$  

**Proof.** We first fix an integer $m$. Given any integer $n$, let $n = p_1^{\alpha_1} \ldots p_j^{\alpha_j}$ be the prime factorisation of $n$, so that

$$\tau(n) = (\alpha_1 + 1) \ldots (\alpha_j + 1).$$  

(5)

Let $\beta_i$ be the remainder when $\alpha_i$ is divided by $m$. Then for some $k \in \mathcal{M}_m$ we have

$$n = kp_1^{\beta_1} \ldots b_j^{\beta_j} = k \prod_{\beta_i = 1} p_i^{\beta_i} \prod_{\beta_i \neq 1} p_i^{\beta_i} = ksq,$$

with $s \in S$, $q \in \mathcal{Q}_m$ and $\gcd(q, s) = 1$. Finally, we have from (5)

$$\tau(n) \equiv (\beta_1 + 1) \ldots (\beta_j + 1) \equiv \tau(qs) \equiv \tau(q)\tau(s) \pmod{m},$$

since $\gcd(q, s) = 1$. 

Given integer $k$, we let $\omega(k)$ denote the number of distinct prime factors of $k$. The proof of the following Lemma is well known (see [7] and references therein for sharper results and generalizations). We provide a standard proof.

**Lemma 6.** Suppose $q$ is squarefree and let $A_q(X)$ count the number of integers $n \leq X$ such that any prime dividing $n$ also divides $q$, then

$$A_q(X) \ll \frac{1}{\omega(q)!} (\log X + 2\omega(q)^{1/2} \log q)^{\omega(q)}.$$

**Proof.** Suppose $p_1, \ldots p_N$ are the distinct primes dividing $q$. Let $\langle \ldots \rangle$ denote the standard inner product on $\mathbb{R}^N$, $||.||$ the Euclidian norm and let $\mathbb{R}^N_+ \subset \mathbb{R}^N$ be the set of all points with nonnegative coordinates. Let $P = (\log p_1, \ldots, \log p_N)$ and

$$P(Y) = \{x \in \mathbb{R}^N_+ : \langle x, P \rangle \leq Y\},$$

so that

$$A_q(X) = \#(\mathbb{Z}^N \cap P(\log X)).$$  

(6)
Let $\mathcal{C}$ denote the set of cubes of the form

$$[j_1, j_1 + 1] \times \cdots \times [j_N, j_N + 1], \quad j_1, \ldots, j_N \in \mathbb{Z},$$

which intersect $\mathcal{P}(\log X)$, so that by (5) we have

$$A_q(X) \leq \# \mathcal{C}. \quad (7)$$

Suppose $\mathcal{B} \in \mathcal{C}$, then for some $a \in \mathbb{R}^N$ independent of $\mathcal{B}$ we have

$$\mathcal{B} \subset \mathcal{P}(\log X + 2 N^{1/2}||P||) + a. \quad (8)$$

Since choosing $x_0 \in \mathcal{B} \cap \mathcal{P}(\log X)$ we may write any $x \in \mathcal{B}$ as $x = x_0 + x'$ with $||x'|| \leq N^{1/2}$. Hence by the Cauchy-Schwarz inequality and the assumption $x_0 \in \mathcal{B}$ we have

$$\langle x, P \rangle = \langle x_0, P \rangle + \langle x', P \rangle \leq \log X + N^{1/2}||P||,$$

$$\langle x, P \rangle \geq -||x'|| \geq -N^{1/2}||P||,$$

so that (5) holds with $a = -(N^{1/2}||P||, \ldots, N^{1/2}||P||)$. Hence from (7)

$$A_q(X) \leq \# \mathcal{C} \leq \int_{\mathcal{B} \subset \mathcal{P}(\log X + 2 N^{1/2}||P||)} 1 \, dx = \frac{(\log X + 2 N^{1/2}||P||)^N}{N! \log p_1 \cdots \log p_N},$$

and the result follows since $N = \omega(q)$ and $||P|| \leq \log p_1 + \cdots + \log p_N = \log q$ since $q$ is squarefree.

We use the following result of Selberg [17], for related and more precise results see [20] II.6.

**Lemma 7.** For any $z \in \mathbb{C}$,

$$\sum_{n \leq x, n \in \mathcal{S}} z^{\omega(n)} = G(z)x(\log x)^{z-1} + O(x(\log x)^{\Re(z)-2}),$$

with

$$G(z) = \frac{1}{\Gamma(z)} \prod_p \left(1 + \frac{z}{p}\right) \left(1 - \frac{1}{p}\right)^z,$$

and the implied constant is uniform for all $|z| = 1$.

We combine Lemma 6 and Lemma 7 to get a sharper version of [16] Lemma 1.
Lemma 8. For integers $q, r, t$ let

$$M(x, q, r, t) = \# \{ n \leq x : n \in S, \, \omega(n) \equiv r \pmod{t}, \, (n, q) = 1 \}.$$

Then for $x \geq q$ we have

$$M(x, q, r, t) = \frac{6h(q)}{\pi^2 t} x + O \left( \frac{x^{1/2}(e^4 \log x)^{\omega(q)}}{q} \right) + O \left( x (\log x)^{-a} \log \log q \right).$$

Proof. Suppose first $q$ is squarefree and let

$$S(a, x) = \sum_{n \leq x \atop n \in S} e_t(a \omega(n)),$$

and

$$S_1(a, q, x) = \sum_{n \leq x \atop n \in S \atop (n, q) = 1} e_t(a \omega(n)).$$

Since the numbers $e_m(a \omega(n))$ with $(n, q) = 1$ and $n \in S$ are the coefficients of the Dirichlet series

$$\prod_{p \nmid q} \left( 1 + \frac{e_t(a)}{p^s} \right) = \prod_{p \mid q} \left( 1 + \frac{e_t(a)}{p^s} \right) \prod_{p} \left( 1 + \frac{e_t(a)}{p^s} \right),$$

we let the numbers $a_n$ and $b_n$ be defined by

$$\prod_{p \nmid q} \left( 1 + \frac{e_t(a)}{p^s} \right) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

$$\prod_{p} \left( 1 + \frac{e_t(a)}{p^s} \right) = \sum_{n=1}^{\infty} \frac{b_n}{n^s},$$

so that

$$S(a, x) = \sum_{n \leq x} b_n,$$

and

$$S_1(a, q, x) = \sum_{n \leq x} \sum_{d_1 d_2 = n} b_{d_1} a_{d_2} = \sum_{n \leq x} a_n S(a, x/n).$$
Consider when $a \neq 0$, by Lemma 7

$$\sum_{n \leq x} a_n S(a, x/n) = G(e_1(a)) x \sum_{n \leq x} \frac{a_n}{n} (\log (x/n))^{\epsilon_1(a)-1}$$

$$+ O \left( \sum_{n \leq x} |a_n| \frac{x}{n} (\log x/n)^{\cos(2\pi/t)-2} \right)$$

$$\ll x (\log x)^{-1-\cos(2\pi/t)} \sum_{n \leq x} \frac{|a_n|}{n}$$

$$\ll x (\log x)^{-\alpha t} \prod_{p|q} \left( 1 - \frac{1}{p} \right)^{-1},$$

and since

$$\prod_{p|q} \left( 1 - \frac{1}{p} \right)^{-1} = \frac{q}{\phi(q)} \ll \log \log q,$$

where $\phi$ is Euler’s totient function, we get

$$S_1(a, q, x) \ll x (\log x)^{-\alpha t} \log \log q. \quad (9)$$

For $a = 0$, by [8, Theorem 334] 

$$S_1(0, q, x) = \sum_{n \leq x} a_n S(0, x/n)$$

$$= \frac{6x}{\pi^2} \sum_{n \leq x} \frac{a_n}{n} + O \left( x^{1/2} \sum_{n \leq x} \frac{|a_n|}{n^{1/2}} \right)$$

$$= \frac{6x}{\pi^2} \prod_{p|q} \left( 1 + \frac{1}{p} \right)^{-1} + O \left( x \sum_{n \geq x} \frac{|a_n|}{n} \right) + O \left( x^{1/2} \sum_{n \leq x} \frac{|a_n|}{n^{1/2}} \right).$$

For the first error term, with notation as in Lemma 6 we have

$$\sum_{n \leq t} |a_n| = A_q(t),$$

so that

$$\sum_{n \geq x} \frac{|a_n|}{n} \ll \int_x^\infty \frac{A_q(t)}{t^2} dt$$

$$\ll \frac{1}{\omega(q)!} \int_x^\infty \frac{\log t + 2\omega(q)^{1/2} \log q}{t^2} dt,$$

$$\ll \frac{1}{\omega(q)} \int_x^\infty \frac{\log t + 2\omega(q)^{1/2} \log q}{t^2} dt, \quad (10)$$
and
\[
\int_x^\infty \frac{(\log t + 2\omega(q)^{1/2} \log q)^{\omega(q)}}{t^2} \, dt = \sum_{n=0}^{\omega(q)} \binom{\omega(q)}{n} \frac{(2\omega(q)^{1/2} \log q)^{\omega(q)-n}}{(\omega(q)^{1/2} \log q)^{\omega(q)}} \int_x^\infty \frac{(\log t)^{n}}{t^2} \, dt.
\]

(11)

The integral
\[
\int_x^\infty \frac{(\log t)^{n}}{t^2} \, dt,
\]

is the \(n\)-th derivative of the function
\[
H(z) = \int_x^\infty t^{z-2} \, dz = \frac{x^{z-1}}{1-z},
\]
evaluated at \(z = 0\). Hence by Cauchy's Theorem, letting \(\gamma \subset \mathbb{C}\) be the circle centered at 0 with radius \(1/\log x\) we have
\[
\int_x^\infty \frac{(\log t)^{n}}{t^2} \, dt = \frac{n!}{2\pi i} \int_\gamma \frac{x^{z-1}}{1-z} \frac{1}{z^{n+1}} \, dz \ll \frac{n!(\log x)^n}{x}.
\]

Hence by (10) and (11)
\[
\sum_{n \geq x} \frac{|a_n|}{n} \ll \frac{1}{x} \frac{(2\omega(q)^{1/2} \log q)^{\omega(q)}}{\omega(q)!} \sum_{n=0}^{\omega(q)} \binom{\omega(q)}{n} n! \left(\frac{\log x}{(\omega(q)^{1/2} \log q)}\right)^n,
\]

and by Stirling's formula \cite[Equation B.26]{[14]}
\[
\sum_{n=0}^{\omega(q)} \binom{\omega(q)}{n} n! \left(\frac{\log x}{(\omega(q)^{1/2} \log q)}\right)^n \ll \sum_{n=0}^{\omega(q)} \binom{\omega(q)}{n} n^{1/2} \left(\frac{n}{e}\right)^n \left(\frac{\log x}{(\omega(q)^{1/2} \log q)}\right)^n
\]
\[
\leq \omega(q)^{1/2} \sum_{n=0}^{\omega(q)} \binom{\omega(q)}{n} \left(\frac{\omega(q)^{1/2} \log x}{e \log q}\right)^n
\]
\[
\ll \omega(q)^{1/2} \left(\frac{\omega(q)^{1/2} \log x}{e \log q} + 1\right)^{\omega(q)},
\]

so that
\[
\sum_{n \geq x} \frac{|a_n|}{n} \ll \frac{1}{x} \frac{\omega(q)^{1/2} (2\omega(q)^{1/2} \log q)^{\omega(q)}}{\omega(q)!} \left(\frac{\omega(q)^{1/2} \log x}{e \log q} + 1\right)^{\omega(q)}.
\]
By another application of Stirling’s formula,

\[ \sum_{n \geq x} \frac{|a_n|}{n} \ll \frac{2^{\omega(q)} x^\omega(q)}{(\omega(q)^{1/2})^\omega(q)} \left( \frac{\omega(q)^{1/2} \log x}{\omega(q)} + 1 \right) \]

\[ \ll \frac{2^{\omega(q)} x}{\log x + e^{\omega(q)^{1/2}} x^{\omega(q)}} \]

\[ \ll 2^{\omega(q)} \left( 1 + \frac{3}{\omega(q)^{1/2}} \frac{(\log x)^{\omega(q)}}{x} \right) \]

\[ \ll 2^{\omega(q)} e^{3 \omega(q)^{1/2}} \frac{(\log x)^{\omega(q)}}{x} \ll e^{4 \omega(q)} \frac{(\log x)^{\omega(q)}}{x}, \]

which gives

\[ S_1(0, q, x) = \frac{6h(q)}{\pi^2} x + O \left( (e^4 \log x)^{\omega(q)} \right) + O \left( x^{1/2} \sum_{n \leq x} \frac{|a_n|}{n^{1/2}} \right). \]

For the last term,

\[ \sum_{n \leq x} \frac{|a_n|}{n^{1/2}} \leq \prod_{p \nmid q} (1 - p^{-1/2})^{-1} \leq \prod_{p \nmid q} (e^4 \log x) = (e^4 \log x)^{\omega(q)}, \]

so that

\[ S_1(0, q, x) = \frac{6h(q)}{\pi^2} x + O \left( x^{1/2} (e^4 \log x)^{\omega(q)} \right). \]  \hspace{1cm} (12)

Since

\[ M(x, q, r, t) = \frac{1}{t} \sum_{a=0}^{t-1} e(t(-ar)S_1(a, q, x)) \]

\[ = \frac{1}{t} S_1(0, q, x) + \frac{1}{t} \sum_{a=1}^{t-1} e(t(-ar)S_1(a, q, x)), \]

we have from (9) and (12)

\[ M(x, q, r, t) = \frac{6h(q)}{\pi^2 t} x + O \left( x^{1/2} (e^4 \log x)^{\omega(q)} \right) + O \left( x(\log x)^{-\omega_1} \log \log q \right). \]

If \( q \) is not squarefree, repeating the above argument with \( q \) replaced by its squarefree part gives the general case since the error term is increasing with \( q \). \hfill \Box

For complex \( s \) we write \( s = \sigma + it \) with both \( \sigma \) and \( t \) real.
Lemma 9. Let $m$ be odd and $\chi$ a multiplicative character (mod $m$). Let

$$L(s, \chi, \tau) = \sum_{n=1}^{\infty} \frac{\chi(\tau(n))}{n^s},$$

then for $\sigma > 1$ we have

$$L(s, \chi, \tau) = \zeta(s)^{\chi(2)}F(s, \chi),$$

with $F(1, \chi) \neq 0$ and

$$F(s, \chi) = \sum_{n=1}^{\infty} \frac{b(\chi, n)}{n^s},$$

for some constants $b(\chi, n)$ satisfying

$$\sum_{n=1}^{\infty} \frac{|b(\chi, n)|(\log n)^3}{n} = O(1),$$

uniformly over all characters $\chi$.

Proof. Since both $\chi$ and $\tau$ are multiplicative we have for $\sigma > 1$,

$$L(s, \chi, \tau) = \prod_p \left(1 + \sum_{n=1}^{\infty} \frac{\chi(\tau(p^n))}{p^{ns}}\right)$$

$$= \zeta(s)^{\chi(2)}F(s, \chi),$$

with

$$F(s, \chi) = \prod_p \left(1 + \sum_{n=1}^{\infty} \frac{\chi(n+1)}{p^{ns}}\right) \left(1 - \frac{1}{p^s}\right)^{\chi(2)}.$$

We have

$$F(s, \chi) = \prod_p \left(1 - \frac{\chi(2)}{p^s}\right) \left(1 + \frac{\chi(2)}{p^s} + \sum_{n=2}^{\infty} \frac{\chi(n+1)}{p^{ns}}\right) \times$$

$$\prod_p \left(1 - \frac{\chi(2)}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{\chi(2)}$$

$$= F_1(s, \chi)F_2(s, \chi),$$
where
\[ F_1(s, \chi) = \prod_p \left( 1 - \frac{\chi(2)}{p} \right) \left( 1 + \frac{\chi(2)}{p} + \sum_{n=2}^{\infty} \frac{\chi(n+1)}{p^{ns}} \right) \]
\[ = \prod_p \left( 1 + \sum_{n=2}^{\infty} \frac{\chi(n+1) - \chi(2n)}{p^{ns}} \right), \quad (13) \]
and
\[ F_2(s, \chi) = \prod_p \left( 1 - \frac{\chi(2)}{p} \right)^{-1} \left( 1 - \frac{1}{p^s} \right)^{\chi(2)}. \]

Considering \( F_2(s, \chi) \), we have for \( \sigma > 1 \)
\[ \log F_2(s, \chi) = \sum_p \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\chi(2n)}{p^{ns}} - \frac{\chi(2)}{p^{ns}} \right) \]
\[ = \sum_p \sum_{n=2}^{\infty} \frac{\chi(2n) - \chi(2)}{n} \frac{1}{p^{ns}}. \quad (14) \]

In the equations (13) and (14), the product and the series converge absolutely on \( \sigma = 1 \) so that \( F(1, \chi) \neq 0 \). Also since \( |\chi(j) - \chi(k)| \leq 2 \) for all integers \( k, j \) we see that the coefficients \( b(\chi, n) \) in
\[ F(s, \chi) = \sum_{n=1}^{\infty} \frac{b(\chi, n)}{n^s}, \]
satisfy
\[ |b(\chi, n)| \leq c_n, \]
where the numbers \( c_n \) are defined by
\[ \prod_p \left( 1 + \sum_{n=2}^{\infty} \frac{2}{p^{ns}} \right) \exp \left( \sum_p \sum_{n=2}^{\infty} \frac{2}{n} \frac{1}{p^{ns}} \right) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}. \]
The function defined by the above formula converges uniformly in any halfplane \( \sigma \geq \sigma_0 > 1/2 \), so that
\[ \sum_{n \leq X} c_n = O(X^{1/2 + \epsilon}) \]
and the last statement of the Lemma follows by partial summation. \( \square \)
The following is Theorem 7.18.

Lemma 10. Suppose for each complex $z$ we have a sequence $(b_z(n))_{n=1}^\infty$ such that the sum
\[ \sum_{n=1}^\infty \frac{|b_z(n)|(\log n)^{2R+1}}{n}, \]
is uniformly bounded for $|z| \leq R$ and for $\sigma \geq 1$ let
\[ F(s, z) = \sum_{n=1}^\infty \frac{b_z(n)}{n^s}. \]

Suppose for $\sigma > 1$ we have
\[ \zeta(s) F(s, z) = \sum_{n=1}^\infty \frac{a_z(n)}{n^s}, \]
for some $a_z(n)$ and let $S_z(x) = \sum_{n \leq x} a_z(n)$. Then for $x \geq 2$, uniformly over all $|z| \leq R$,
\[ S_z(x) = \frac{F(1, z)}{\Gamma(z)} x (\log x)^{z-1} + O(x (\log x)^{\Re(z)-2}). \]

Combining Lemma 9 and Lemma 10 gives

Lemma 11. For integer $m$ let $\chi$ be a multiplicative character $\pmod{m}$ and let
\[ G(\chi) = \frac{1}{\Gamma(\chi(2))} \prod_p \left( \sum_{n=0}^\infty \frac{\chi(n+1)}{p^n} \right) \left( 1 - \frac{1}{p} \right)^{\chi(2)}. \]

Then uniformly over all characters $\chi$,
\[ \sum_{n \leq x} \chi(\tau(n)) = G(\chi) x (\log x)^{\chi(2)-1} + O(x (\log x)^{\Re(\chi(2))-2}). \quad (15) \]

Lemma 12. For any integer $m$,
\[ \sum_{\substack{q \in \mathcal{K} \atop \tau(q) \equiv 0 \pmod{m}}} \frac{1}{q} \ll \frac{1}{m \log 2/2}, \]
and if $p$ is prime
\[ \sum_{\substack{q \in \mathcal{K} \atop \tau(q) \equiv 0 \pmod{p}}} \frac{1}{q} \ll \frac{1}{2 p/2}. \]
Proof. Suppose $\tau(q) \equiv 0 \pmod{m}$ and let $q = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the prime factorization of $q$, so that

$$\tau(q) = (\alpha_1 + 1) \cdots (\alpha_k + 1) \geq m.$$  

By the arithmetic-geometric mean inequality,

$$q \geq 2^{(\alpha_1+1)+\cdots+(\alpha_k+1) - k} \geq 2^k\tau(q)^{1/k} - 1 \geq 2^{k(m^{1/k} - 1)} \geq 2^{\log m} = m^{\log 2},$$

and since $K(x) \ll x^{1/2}$ we get

$$\sum_{q \in K, \tau(q) \equiv 0 \pmod{m}} \frac{1}{q} \leq \sum_{q \in K, q \geq m^{\log 2}} \frac{1}{q} \ll \int_{m^{\log 2}}^\infty \frac{K(x)}{x^2} \, dx \ll \frac{1}{m^{\log 2/2}}.$$

Suppose $p$ is prime, if $\tau(n) \equiv 0 \pmod{p}$ then $n \geq 2^{p-1}$. As before we get

$$\sum_{q \in K, \tau(q) \equiv 0 \pmod{p}} \frac{1}{q} \ll \frac{1}{2^{p/2}}.$$

\[\square\]

5 Proof of Theorem 1

By Lemma 5 we have

$$\sum_{n=1}^{N} e_m(a\tau(n)) = \sum_{k \in M_m} \sum_{q \in Q_m} \sum_{s \in S} \frac{e_m(a\tau(kqs))}{1}$$

$$= \sum_{k \in M_m} \sum_{q \in Q_m} \sum_{s \in S, \text{gcd}(s,q) = 1} e_m(a\tau(q)s)$$

$$= \sum_{k \in M_m} \sum_{q \in Q_m} \sum_{s \in S, \text{gcd}(s,q) = 1} e_m(a\tau(q)2^{\omega(s)}).$$
Let $K = N^{1/2}$ and grouping together values of $2^{\omega(s)}$ in the same residue class (mod $m$) we have,

$$
\sum_{n=1}^{N} e_m(a\tau(n)) = \sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{q \leq K/k} \sum_{r=1}^{t} M(N/qk, q, r, t)e_m(a\tau(q)2^r)
$$

$$
+ \sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{K/k < q \leq N/k} \sum_{r=1}^{t} M(N/qk, q, r, t)e_m(a\tau(q)2^r).
$$

By choice of $K$, we have $N/qk \geq q$ when $q \leq K/k$. Hence we may apply Lemma 8 to the first sum above,

$$
\sum_{n=1}^{N} e_m(a\tau(n)) = \frac{6}{\pi^2} N \sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{q \leq N/k} \sum_{r=1}^{t} \frac{h(q)}{qk} e_m(a\tau(q)2^r)
$$

$$
+ O \left( \sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{k \leq N} \sum_{q \leq K/k} \frac{Nt \log \log q}{qk} (\log (N/qk))^{-\alpha} \right) + O \left( \sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{K/k < q \leq N/k} \frac{tN}{qk} \right) 
$$

$$
+ O \left( \sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{k \leq N} \sum_{q \leq K/k} e^4 \log (N/kq)^{\omega(q)} \left( \frac{N}{kq} \right)^{1/2} \right).
$$

Considering the first two error terms,

$$
\sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{k \leq N} \sum_{K/k < q \leq N/k} \frac{tN}{qk} \ll tN \int_{K}^{N} \frac{K(x)}{x^2} \, dx \ll \frac{tN}{K^{1/2}}, \quad (16)
$$

and since the sum

$$
\sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{k \leq N} \sum_{q \leq K/k} \log \log q \frac{1}{qk},
$$

is bounded uniformly in $m$ as $K, N \to \infty$, we get

$$
\sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{k \leq N} \frac{Nt \log q}{qk} (\log (N/qk))^{-\alpha} \ll Nt \log (N/K)^{-\alpha}, \quad \sum_{k \in M_m} \sum_{q \in \mathbb{Q}_m} \sum_{k \leq N} \frac{\log \log q}{qk}
$$

$$
\ll Nt (\log (N/K))^{-\alpha}. \quad (17)
$$
For the last term,
\[
\sum_{k \in M, k \leq N} \sum_{q \in Q, q \leq K/k} (e^4 \log (N/kq))^{\omega(q)} \left( \frac{N}{kq} \right)^{1/2} \leq N^{2/3} \sum_{n \in K, n \leq N} \left( \frac{1}{n} \right)^{2/3} (e^4 \log N)^{\omega(n)},
\]
and since
\[
\omega(n) \leq (1 + o(1)) \frac{\log n}{\log \log n},
\]
we get
\[
N^{2/3} \sum_{k \in M, k \leq N} \sum_{q \in Q, q \leq K/k} (e^4 \log N)^{\omega(q)} \left( \frac{1}{kq} \right)^{2/3} \leq N^{5/6 + o(1)} \sum_{n \in K, n \leq N} \left( \frac{1}{n} \right)^{2/3} (e^4 \log N)^{5/6)^{\omega(n)}.}
\]

We may bound the sum on the right by noting
\[
\sum_{n \in K, n \leq N} \left( \frac{1}{n} \right)^{2/3} (e^4 \log N)^{5/6)^{\omega(n)} \leq \prod_p \left( 1 + e^4 \log N)^{5/6} \sum_{k=2}^{\infty} \frac{1}{p^{2k/3}} \right),
\]
taking logarithms we see that
\[
\log \left( \prod_p \left( 1 + e^4 \log N)^{5/6} \sum_{k=2}^{\infty} \frac{1}{p^{2k/3}} \right) \right) = \sum_p \log \left( 1 + \frac{e^4 \log N)^{5/6}}{p^{2/3} - p^{2/3}} \right) \leq e^4 \log N)^{5/6} \sum_p \frac{1}{p^{2/3} p^{2/3} - 1} \ll (\log N)^{5/6},
\]
hence we have for some absolute constant c
\[
\sum_{k \in M, k \leq N} \sum_{q \in Q, q \leq K/k} (e^4 \log (N/kq))^{\omega(q)} \left( \frac{N}{kq} \right)^{1/2} \ll N^{5/6 + o(1)} e^{c \log N)^{5/6}}. \tag{18}
\]

Combining (16), (17) and (18) gives
\[
\sum_{n=1}^{N} e_m(a \tau(n)) = \frac{6}{\pi^2} N t \sum_{k \in M, k \leq N} \sum_{q \in Q, q \leq N/k} h(q) \frac{1}{q} \sum_{r=1}^{t} e_m(a \tau(q)2^r) + O \left( N t (\log(N/K))^{-\alpha} \right) + O \left( \frac{t N}{K^{1/2}} \right) + O \left( N^{5/6 + o(1)} e^{c \log N)^{5/6}} \right).
\]
Recalling the choice of $K$ we get
\[
\sum_{n=1}^{N} e_m(a\tau(n)) = \frac{6}{\pi^2} t \sum_{k \in M_m, k \leq N} \frac{1}{k} \sum_{q \in Q_m, q \leq N/k} h(q) \sum_{r=1}^{t} e_m(a\tau(q)2^r) + O \left( N t (\log N)^{-\alpha_t} \right).
\]

(19)

For the main term,
\[
\sum_{k \in M_m} \frac{1}{k} \sum_{q \in Q_m, q \leq N/k} h(q) \sum_{r=1}^{t} e_m(a\tau(q)2^r) = \sum_{k \in M_m} \frac{1}{k} \sum_{q \in Q_m} h(q) \sum_{r=1}^{t} e_m(a\tau(q)2^r)
\]
\[+ O \left( t \sum_{k \in M_m} \frac{1}{k} \left( \frac{k}{N} \right)^{1/2} \right) \]
\[= \sum_{q \in Q_m} h(q) \sum_{r=1}^{t} e_m(a\tau(q)2^r)
\]
\[+ O \left( \frac{t}{N^{1/2}} \right) \]
\[= \zeta(m) \sum_{q \in Q_m} h(q) \sum_{r=1}^{t} e_m(a\tau(q)2^r) + O \left( \frac{t}{N^{1/2}} \right). \]

Hence we have
\[
\sum_{n=1}^{N} e_m(a\tau(n)) = \frac{\zeta(m)}{t} \frac{6}{\pi^2} \left( \sum_{(r \mod m)} H(r, m) S_m(ar) \right) N + O \left( tN(\log N)^{-\alpha_t} \right).
\]

6 Proof of Theorem 2

Let
\[
C(p, r, N) = \# \{ n \leq N : \tau(n) \equiv r \pmod{p} \},
\]
so that
\[
\sum_{n=1}^{N} e_p(a\tau(n)) = \sum_{r=0}^{p-1} C(p, r, N)e_m(an). \tag{20}
\]
Suppose \((r, p) = 1\), using orthogonality of characters and Lemma \([\text{II}1]\),

\[
C(p, r, N) = \frac{1}{p - 1} \sum_{n=1}^{N} \sum_{\chi \equiv 0 \pmod{p}} \overline{\chi}(r)\chi(\tau(n))
\]

\[
= \frac{1}{p - 1} \sum_{\chi(2)=1} \overline{\chi}(r)G(\chi)N + \frac{1}{p - 1} \sum_{\chi(2)\neq 1} \overline{\chi}(r)G(\chi)N(\log N)^{\chi(2)-1}
\]

\[+ O\left(N(\log N)^{-\alpha+1}\right),
\]

and for \(r = 0\), we have by another applications of Lemma \([\text{II}1]\)

\[
C(p, 0, N) = \sum_{n=1}^{N} (1 - \chi_0(\tau(n))) = C_p N + O\left(N(\log N)^{-2}\right),
\]

for some constant \(C_p\). Hence from \([20]\)

\[
\sum_{n=1}^{N} e_p(a\tau(n)) = C_p N + O\left(p N(\log N)^{-\alpha+1}\right)
\]

\[+ \frac{N}{p - 1} \sum_{r=1}^{p-1} \left( \sum_{\chi(2)=1} \overline{\chi}(r)G(\chi)e_p(ar) + \sum_{\chi(2)\neq 1} \overline{\chi}(r)e_p(ar)G(\chi)(\log N)^{\chi(2)-1} \right)
\]

\[= A_p N + \frac{N}{p - 1} \sum_{\chi(2)\neq 1} G(\chi)(\log N)^{\chi(2)-1} \sum_{r=1}^{p-1} \overline{\chi}(r)e_p(ar)
\]

\[+ O\left(p N(\log N)^{-\alpha+1}\right),
\]

for some constant \(A_p\). If \(\chi(2) \neq 1\) then we have

\[
\left|\sum_{r=1}^{p-1} \overline{\chi}(r)e_p(ar)\right| = p^{1/2},
\]

so that

\[
\sum_{n=1}^{N} e_p(a\tau(n)) = A_p N + O\left(p^{1/2} N(\log N)^{-\alpha}\right) + O\left(p N(\log N)^{-\alpha+1}\right)
\]

\[= A_p N + O\left(p N(\log N)^{-\alpha+1}\right),
\]

since if

\[
p^{1/2} N(\log N)^{-\alpha} \leq p N(\log N)^{-\alpha+1} \quad \text{then} \quad N \leq p N(\log N)^{-(\alpha+1)}.
\]
Finally, comparing (21) with the leading term in the asymptotic formula from Theorem 1, we see that

\[ A_p = \frac{\zeta(p)}{t} \frac{6}{\pi^2} \left( \sum_{r=0}^{p-1} H(r, p) S_p(ar) \right). \]

### 7 Proof of Theorem 3

Considering the main term in Theorem 1,

\[
\left| \sum_{r=0}^{m-1} H(r, m) S_m(ar) \right| \leq \sum_{d|m} \sum_{\substack{r=0 \atop \gcd(r, m) = d}}^{m-1} H(r, m) S_m(ar) \\
\leq \sum_{d|m} \left( \sum_{\substack{r=0 \atop \gcd(r, m) = d}}^{m-1} H(r, m) \right) \max_{\gcd(\lambda, m) = d} |S_m(\lambda)|.
\]

Writing \( c = \log 2/2 \), by Lemma 12

\[
\sum_{\substack{r=0 \atop \gcd(r, m) = d}}^{m-1} H(r, m) = \sum_{\substack{r=0 \atop \gcd(r, m) = d}}^{m-1} \sum_{q \in \mathbb{Q}_m} \frac{h(q)}{q} \\
\leq \sum_{q \in K} \frac{1}{q} \ll \frac{1}{dc^2},
\]

so that

\[
\sum_{r=0}^{m-1} H(r, m) S_m(ar) \ll \sum_{d|m} \frac{1}{dc} \max_{\gcd(\lambda, m) = d} |S_m(\lambda)|. \tag{22}
\]

Suppose \( \gcd(\lambda, m) = d \), so that \( \lambda = d\lambda' \) and \( m = dm' \) for some \( \lambda' \) and \( m' \) with \( \gcd(\lambda', m') = 1 \). Let \( t_d \) denote the order of 2 (mod \( m' \)). Then we have

\[
S_m(\lambda) = \sum_{n=1}^{t} e_m(\lambda 2^n) = \frac{t_d}{t_d} \sum_{n=1}^{t_d} e_{m'}(\lambda' 2^n).
\]

By the main result of [3], if \( t_d \geq (m/d)^{\varepsilon} \) then for some \( \delta > 0 \),

\[
S_m(\lambda) \ll \left( \frac{d}{m} \right)^{\delta} t. \tag{23}
\]
Suppose $t \geq m^{\epsilon}$; then since
\[ t_d \geq \frac{t}{d} \geq \frac{m^{\epsilon/2}}{d} \cdot m^{\epsilon/2}, \]
if $d \leq m^{\epsilon/2}$ then we have $t_d \geq (m/d)^{\epsilon/2}$. Hence by (23)
\[ S_m(\lambda) \ll \left( \frac{d}{m} \right)^{\delta} t \ll \frac{t}{m^{\epsilon_0}}. \]

Hence by (22), for some $\delta_1 > 0$
\[
\sum_{r=0}^{m-1} H(r, m) S_m(ar) \ll \sum_{d|m, \delta \leq m^{\epsilon/2}} \frac{1}{d^{c}} \max_{\gcd(\lambda, m)=d} |S_m(\lambda)| + \sum_{d|m, \delta \geq m^{\epsilon/2}} \frac{1}{d^{c}} \max_{\gcd(\lambda, m)=d} |S_m(\lambda)|
\]
\[
\ll \sum_{d|m, \delta \leq m^{\epsilon/2}} \frac{t}{m^{\delta_1}} + \sum_{d|m, \delta \geq m^{\epsilon/2}} \frac{1}{m^{\delta_1}} = \tau(m) \frac{t}{m^{\delta_1}}, \tag{24}
\]
and the result follows combining (24) with Theorem 1.

### 8 Proof of Theorem 4

By Lemma 12
\[
\sum_{r=0}^{p-1} H(r, p) S_p(ar, t) = \sum_{r=1}^{p-1} H(r, p) S_p(ar) + t H(0, p)
\]
\[
\ll \frac{t}{2p^{1/2}} + \left( \sum_{r=1}^{p-1} H(r, p) \right) \max_{\gcd(\lambda, p)=1} |S_p(\lambda)|
\]
\[
\ll \frac{t}{2p^{1/2}} + \max_{\gcd(\lambda, p)=1} |S_p(\lambda)|.
\]

In [11] it is shown the following bound is a consequence of [12] and [18],
\[
\max_{\gcd(\lambda, p)=1} |S_p(\lambda)| \ll \begin{cases} 
p^{1/8} t^{22/36} (\log p)^{7/6}, & t \leq p^{1/2}, 
p^{1/4} t^{13/36} (\log p)^{7/6}, & p^{1/2} < t \leq p^{3/5} (\log p)^{-6/5}, 
p^{1/6} t^{1/2} (\log p)^{4/3}, & p^{3/5} < t \leq p^{2/3} (\log p)^{-2/3}, 
p^{1/2} \log p, & t > p^{2/3} (\log p)^{-2/3}, \end{cases}
\]
and the result follows by Theorem 2.
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