Loop quantization of the supersymmetric two-dimensional BF model

Clisthenis P Constantinidis, Ruan Couto, Ivan Morales and Olivier Piguet

Departamento de Física, Universidade Federal do Espírito Santo (UFES), Vitória, ES, Brazil

E-mail: cpconstantinidis@pq.cnpq.br, ruan.giacomini@gmail.com, mblivan@gmail.com and opiguet@yahoo.com

Received 9 March 2012, in final form 29 May 2012
Published 2 July 2012
Online at stacks.iop.org/CQG/29/155008

Abstract
In this paper, we consider the quantization of the 2D BF model coupled to topological matter. Guided by the rigid supersymmetry, this system can be viewed as a super-BF model, where the field content is expressed in terms of superfields. A canonical analysis is done and the constraints are then implemented at the quantum level in order to construct the Hilbert space of the theory under the perspective of loop quantum gravity methods.

PACS numbers: 03.70.+k, 11.10.-z, 04.60.-m, 11.15.-q, 11.30.Pb

1. Introduction

It is well known that the BF model in two-dimensional (2D) spacetime for the gauge group \( SO(1,2) \) is equivalent to the Jackiw–Teitelboim model for 2D gravity with a cosmological constant \([1–5]\), and in order to develop methods of quantization for gravity, it has been used as a useful laboratory \([6–11]\). Leitgeb et al \([12]\) have proposed an enlargement of the 2D BF model, in which it is coupled to vector and scalar fields. The action is

\[ S = \frac{1}{2} \int d^2x \epsilon^{\mu\nu} \mathcal{F}_{\mu\nu} + \int d^2x \epsilon^{\mu\nu}(D_\mu B_\nu)\psi^i, \]

where \( \phi^i \) is a scalar field, \( \mathcal{F}_{\mu\nu} \) is the curvature associated with the gauge field \( A^i \), \( \psi^i \) is another scalar field and \( B_{\mu} \) is a vector field. \( D_\mu \) is the covariant derivative. The index \( i = 1, 2, 3 \) labels the basis of the Lie algebra of the gauge group, taken in an anti-Hermitian basis, \([T_i, T_j] = \epsilon_{ij}^k T_k\). In this paper, we consider the gauge group \( SU(2) \), corresponding to Riemannian gravity with a positive cosmological constant. All fields are valued in the Lie algebra \( su(2) \). Such a field \( \psi \) is a matrix \( \psi = \psi^i T_i \).

1 Present address: Physics Department, Federal University of Viçosa—UFV, Viçosa, MG, Brazil.
Our purpose here is to perform the quantization of this model under the perspective of loop quantum gravity. In order to do this, we consider the canonical structure of the theory, obtain the first-class constraints and impose them at the quantum level for the construction of the Hilbert space. But we still explore another symmetry of action (1.1), namely a rigid supersymmetry present in the model, which will guide us through the construction of the quantum theory.

2. The supersymmetric BF model

In the present context, \( N = 1 \) supersymmetry transformations are generated by a unique nilpotent operator \( Q \): \( Q^2 = 0 \). \( N = 1 \) superfields read

\[
\phi(x, \theta) = \phi_0(x) + \theta \phi_1(x),
\]

where \( x = (x^\mu, \mu = 0, 1) \) are the spacetime manifold coordinates—denoted below as \((t, x)\)—and \( \theta \), with \( \theta^2 = 0 \), is the (unique) Grassman superspace coordinate. By definition, a superfield transforms infinitesimally under supersymmetry as

\[
Q \phi = \frac{\partial}{\partial \theta} \phi,
\]
or, in components,

\[
Q \phi_0 = \phi_1, \quad Q \phi_1 = 0.
\]

Let us introduce the superfield extensions of the fundamental fields present in the usual 2D BF model (whose components are the fields present in the Leitgeb–Schweda–Zerrouki action (1.1)):

\[
\Phi = \psi + \theta \phi, \quad A = A + \theta B,
\]

where \( \Phi \) is an odd parity scalar superfield and \( A \) is the even parity superconnection. The basic fields transform under supersymmetry as two doublets:

\[
Q A = B, \quad Q B = 0, \quad \text{and} \quad Q \psi = \phi, \quad Q \phi = 0.
\]

It is easy to check that the Leitgeb–Schweda–Zerrouki action is invariant under these supersymmetry transformations. This invariance is still more obvious from definition (2.2) for the superspace action

\[
S_T[\Phi, A] := \text{Tr} \int d\theta \Phi \mathcal{F}[A],
\]

which is equivalent to (1.1), with the difference that here we are considering the fields \( B \) and \( \psi \) with odd parity. \( \mathcal{F} \) is the supercurvature of the superconnection \( A \):

\[
\mathcal{F} = dA + \frac{1}{2}[A, A] = F + \theta \mathcal{F},
\]

with \( F \) being the usual Yang–Mills curvature and \( \mathcal{F} \) its supersymmetry partner, and, thus, an odd quantity:

\[
F = dA + \frac{1}{2}[A, A], \quad \mathcal{F} = DB,
\]

where \( D \) is the covariant derivative, \( D = d + [A, \cdot] \), and \( d = dx^\mu \partial_\mu \) is the usual spacetime exterior derivative. The trace symbol \( \text{Tr} \) is taken to be the Killing form of the \( su(2) \) algebra. Integration in \( \theta \) is defined by the Berezin integral [13, 14], which in the present case amounts to the definition

\[
\int d\theta \cdots = \frac{\partial}{\partial \theta} \cdots, \quad \text{or} \quad \int d\theta 1 = 0, \quad \int d\theta \theta = 1.
\]

2 This supersymmetry seemed unnoted by the authors of [12].

3 We consider even and odd parities in order to distinguish the fields, and other objects, of bosonic and fermionic nature, respectively.

4 The brackets \([ \cdot, \cdot ]\) are graduated commutators, i.e. an anticommutator if both its arguments are odd, and a commutator otherwise.
Varying action (2.5), we have
\[
\delta S_T = \text{Tr} \int d\theta (\delta \Phi \delta \Phi + \Phi \delta \Phi) = \text{Tr} \int d\theta (\delta \Phi \delta \Phi + \delta A \delta \Phi).
\]
\[
= \text{Tr} \int (\delta \phi F - \delta \psi \vec{F} - \delta A (-\delta \phi - [A, \phi] + [B, \psi]) + \delta B (D\psi + [A, \psi])),
\]
where \(D := d + [A, \cdot] + [B, \cdot]\) is the covariant derivative for the superconnection. From (2.8), we obtain the equations of motion
\[
\frac{\delta S_T}{\delta \phi} = F = 0, \quad \frac{\delta S_T}{\delta A} = d\phi + [A, \phi] - [B, \psi] = D\phi - [B, \psi] = 0,
\]
\[
\frac{\delta S_T}{\delta \psi} = \vec{F} = 0, \quad \frac{\delta S_T}{\delta B} = d\psi + [A, \psi] = D\psi = 0.
\]

2.1. The gauge group

In order to treat the graded structure of the superfields, we also consider the ‘supergauge group’ \(G\) of elements:

\[
\mathcal{G}(x) = \mathcal{G}(\alpha, \beta) := e^{G(x)} = e^{\alpha(x) + \theta \beta(x)},
\]
where the parameters \(\alpha\) and \(\beta\) are evaluated in \(su(2)\). Observe that \(\alpha\) is even while the quantity \(\beta\) is odd, the latter being the supersymmetry transform (2.2) of the former: \(\beta = Q\alpha\). When expanded in \(\theta\), this expression can be written as

\[
\mathcal{G}(\alpha, \beta) := g(\alpha) + \theta \beta \triangleright g(\alpha)
\]

with

\[
g(\alpha) := e^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}
\]

being an element of \(SU(2)\), and the quantity

\[
\beta \triangleright g(\alpha) := \beta^i \frac{\partial}{\partial \alpha^i} g(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} \alpha^{n-k} \beta \alpha^{k-1}
\]

is defined as the insertion of the Grassmannian parameter \(\beta\) into \(g\). It is the supersymmetry transform of the group element \(g(\alpha)\),

\[
\beta \triangleright g(\alpha) = Q g(\alpha).
\]

There exists an inverse element \(\mathcal{G}^{-1}\) given by

\[
\mathcal{G}^{-1}(\alpha, \beta) := e^{-\alpha} = e^{-\alpha + \theta \beta} = g^{-1}(\alpha) + \theta \beta \triangleright g^{-1}(\alpha)
\]

with

\[
\beta \triangleright g^{-1}(\alpha) := \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} \alpha^{n-k} \beta \alpha^{k-1}.
\]

This insertion enjoys the following property:

\[
(\beta \triangleright g) g^{-1} = -g(\beta \triangleright g^{-1}).
\]

Under left and right multiplication of a group element \(g\),

\[
g' = g g^{-1}, \quad \text{with} \quad g = g(\alpha), \quad g_k = g(\alpha_k), \quad \beta_k = Q\alpha_k, \quad k = 1, 2,
\]

the \(\beta\)-insertion transforms as

\[
(\beta \triangleright g') = g_1 (\beta \triangleright g) g_2^{-1} + (\beta_1 \triangleright g_1) g_2^{-1} - g_1 g_2^{-1} (\beta_2 \triangleright g_2) g_2^{-1}.
\]
2.2. Symmetries

As in the usual BF theory, one verifies that action (2.5) is invariant under the following finite supergauge transformations: for the superconnection $A$,

$$A(x) \rightarrow A'(x) = G(x) dG^{-1}(x) + G(x) A(x) G(x)^{-1},$$

(2.19)

from which we identify

$$A'(x) = g(x) d g^{-1}(x) + g(x) A(x) g^{-1}(x)$$

(2.20)

$$B'(x) = g(B + D(g^{-1}(\beta \triangleright g))) g^{-1},$$

(2.21)

where $D$ is the covariant derivative:

$$D(g^{-1}(\beta \triangleright g)) = d(g^{-1}(\beta \triangleright g)) + Ag^{-1}(\beta \triangleright g) + g^{-1}(\beta \triangleright g)A,$$

(2.22)

and for the supersymmetric extension $\Phi$ of the scalar field,

$$\Phi(x) \rightarrow \Phi'(x) = G(x) \Phi(x) G^{-1}(x),$$

(2.23)

from which one reads

$$\phi'(x) = g(x) \phi(x) g^{-1}(x)$$

(2.24)

$$\psi'(x) = g(x) \left( \psi(x) + \theta \left[ \psi(x), g^{-1}(\beta \triangleright g)(x) \right] \right) g^{-1}(x).$$

(2.25)

In order to obtain the infinitesimal supergauge transformations, we take $\Omega(x)$ small in (2.10), and from (2.19) and (2.23), we obtain

$$\delta_{\Omega} A = -D\Omega,$$

(2.26)

$$\delta_{\Omega} \Phi = -[\Phi, \Omega].$$

(2.27)

Considering the splitting in even and odd quantities yields the following infinitesimal gauge transformations, which are of two kinds:

- gauge transformations of type $\alpha$, which involve the even parameter:
  $$\delta_{\alpha} A = -D\alpha, \quad \delta_{\alpha} \phi = [\alpha, \phi], \quad \delta_{\alpha} B = [\alpha, B], \quad \delta_{\alpha} \psi = [\alpha, \psi],$$
  (2.28)

- gauge transformations of type $\beta$, associated with the odd parameter:
  $$\delta_{\beta} A = 0, \quad \delta_{\beta} \phi = [\beta, \psi], \quad \delta_{\beta} B = D\beta, \quad \delta_{\beta} \psi = 0.$$
  (2.29)

These gauge transformations are the same as the ones of the Leitgeb–Schweda–Zerrouki model, given by action (1.1). The difference is that the transformations of parameter $\beta$ are now interpreted as local supersymmetry transformations, with $\beta$ being odd now.

Diffeomorphisms are also symmetries of the BF model. Considering a vector field $v$, infinitesimal diffeomorphism transformations, $\delta x^\mu = v^\mu(x)$, are given by the Lie derivative

$$\mathcal{L}_v \Phi = i_v d\Phi, \quad \mathcal{L}_v A = (i_v d + di_v) A,$$

(2.30)

where $i_v$ is the interior derivative in the direction of the vector $v$. We can easily check that these infinitesimal diffeomorphisms can be expressed as

$$\mathcal{L}_v \Phi = i_v \delta S_T / \delta B + \theta_i \delta S_T / \delta A - \delta (i_v A), \quad \mathcal{L}_v A = i_v \delta S_T / \delta \phi + \theta_i \delta S_T / \delta \psi - \delta (i_v A).$$

(2.31)

from which one realizes that they are related to gauge symmetries—if we consider $i_v A = v^\mu A_\mu$ as the parameter of the infinitesimal transformation—modulo equations of motion.
3. Canonical analysis

In order to proceed with the canonical analysis, we consider the two-dimensional spacetime manifold foliated as $\mathcal{M} = \mathbb{R} \times \Sigma$, with coordinates $x^\mu = (t, x)$, where $t$ and $x$ are time and space coordinates, respectively. Action (2.5) can be written as

$$S_T = \int_{\mathbb{R}} dt \int_\Sigma dx \, d\theta \Phi_i (\partial_t A'_i - \partial_x A'_i + f_{jk} A'_j A'_k)$$

$$= \int_{\mathbb{R}} dt \int_\Sigma dx \, \mathcal{L}_T,$$

with

$$\mathcal{L}_T = \phi_i (\partial_t A'_i - \partial_x A'_i + f_{jk} A'_j A'_k) - \psi_i (\partial_t B'_i - \partial_x B'_i + f_{jk} (A'_j B'_k - A'_k B'_j)),$$

where in the second equality we have integrated over the Grassmannian parameter $\theta$, using the Berezin integral.

Before proceeding with the analysis, we define the generalization of the Poisson brackets for the case where we have fields with even and odd parities [15], so we define, for any quantities $M$ and $N$,

$$\{ M, N \} := \sum_A \int dx (-1)^{|M||Q_A|} \left( \frac{\delta M}{\delta Q_A(x)} \frac{\delta N}{\delta P_A(x)} - (-1)^{|Q_A|} \frac{\delta M}{\delta P_A(x)} \frac{\delta N}{\delta Q_A(x)} \right),$$

where $Q_A$ and $P_A$ are generalized configuration variables and their conjugate momenta. In particular, the basic non-vanishing brackets are

$$\{ Q_A(x), P^B(y) \} = (-1)^{|Q_A||P_B|} \delta^B_\gamma \delta(x - y).$$

Taking $A'_i$, $B'_i$ as configuration variables, we read from actions ((3.1), (3.2)) that $\phi_i$ and $\psi_i$ can be identified with the conjugate momenta of $A'_i$ and $B'_i$, respectively.

Thus $A_i$, $B_i$ have non-vanishing Poisson brackets with $\phi$ and $\psi$, respectively,

$$\{ A'_i(x), \phi_j(y) \} = \delta^i_j \delta(x - y) = - \{ \phi_j(y), A'_i(x) \},$$

$$\{ B'_i(x), \psi_j(y) \} = - \delta^i_j \delta(x - y) = \{ \psi_j(y), B'_i(x) \}.$$

On the other hand, the conjugate momenta $\pi^{(A_i)}_i$ and $\pi^{(B_i)}_i$ of $A'_i$ and $B'_i$ turn out to vanish: in the Dirac–Bergmann canonical formalism [16], these are interpreted as primary constraints

$$\pi^{(A_i)}_i(x) \approx 0, \quad \pi^{(B_i)}_i(x) \approx 0,$$

where $\approx$ means a weak equality, i.e. an equality which will be fulfilled only after all Poisson algebra manipulations are done. For consistency under the time evolution generated by the canonical Hamiltonian, these constraints produce secondary constraints. The canonical Hamiltonian, obtained from the Lagrangian (3.2) by a Legendre transform, reads

$$H = - \int dx \left( A'_i(x) \mathcal{G}_i(x) + B'_i(x) \mathcal{S}_i(x) \right),$$

with

$$\mathcal{G}_i(x) := D_t \phi_i(x) + [B_i(x), \psi(x)], \quad \mathcal{S}_i(x) := D_x \psi_i(x),$$

and the secondary constraints are the terms proportional to $A_i$ and $B_i$ in the Hamiltonian (which is thus itself a constraint, as expected in a background-independent theory):

$$\mathcal{G}_i(x) \approx 0, \quad \mathcal{S}_i(x) \approx 0.$$

The parity function, denoted by $| |$ is defined as $|M| = 0 (1)$ if $M$ is even (odd).
These constraints are first class, i.e. they have weakly vanishing Poisson brackets. Indeed, let us use the smeared form of these constraints for more clarity:
\[
\mathcal{G}(\alpha) := \int dx' \alpha'(x) \mathcal{G}(x), \quad \mathcal{S}(\beta) := \int dx' \beta'(x) \mathcal{S}(x),
\]
where \(\alpha'\) and \(\beta'\) are smooth test functions. \(\mathcal{S}\) and \(\mathcal{G}\) form a supersymmetry doublet: \(Q\mathcal{S} = \mathcal{G}\). Putting \(\mathcal{S}(x)\) and \(\mathcal{G}(x)\) together into a superfield \(\mathcal{S}(x, \theta) = \mathcal{S}(x) + \theta \mathcal{G}(x)\), and writing, with \(\Omega\) as given in (2.10),
\[
\mathcal{S}(\Omega) = \int dx \int d\theta \mathcal{J}(x, \theta) \mathcal{S}(x, \theta) = \mathcal{G}(\alpha) + \mathcal{S}(\beta) \approx 0,
\]
one checks that the latter forms a closed Poisson algebra:
\[
\{\mathcal{S}(\Omega_1), \mathcal{S}(\Omega_2) = \mathcal{S}([\Omega_1, \Omega_2])\}.
\]
Being thus first class, \(\mathcal{S}\) is the infinitesimal generator of gauge transformations, namely of the supertransformations (2.19), (2.23):
\[
[\mathcal{S}(\Omega), \mathcal{A}_i(x, \theta)] = \partial_i \Omega(x, \theta) + [\mathcal{A}_i(x, \theta), \Omega(x, \theta)],
\]
\[
[\mathcal{S}(\Omega), \Phi(x, \theta)] = [\Phi(x, \theta), \Omega(x, \theta)] = 0.
\]
In components, this yields the algebra
\[
[\mathcal{G}(\alpha), \mathcal{G}(\alpha')] = \mathcal{G}([\alpha', \alpha]), \quad [\mathcal{G}(\alpha), \mathcal{S}(\beta)] = \mathcal{S}([\alpha, \beta]), \quad [\mathcal{S}(\beta), \mathcal{S}(\beta')] = 0,
\]
and the infinitesimal form of the gauge transformations of types \(\alpha\) and \(\beta\) given in subsection 2.2,
\[
[\mathcal{G}(\alpha), \mathcal{A}_i'(x)] = D_i \alpha'(x), \quad [\mathcal{G}(\alpha), \phi_i(x)] = -[\alpha(x), \phi_i(x)],
\]
\[
[\mathcal{G}(\alpha), \mathcal{B}_i'(x)] = -[\alpha(x), \mathcal{B}_i(x)], \quad [\mathcal{G}(\alpha), \psi_i(x)] = -[\alpha(x), \psi_i(x)],
\]
\[
[\mathcal{S}(\beta), \mathcal{A}_i'(x)] = 0, \quad [\mathcal{S}(\beta), \phi_i(x)] = -[\beta(x), \phi_i(x)],
\]
\[
[\mathcal{S}(\beta), \mathcal{B}_i'(x)] = D_i \beta', \quad [\mathcal{S}(\beta), \psi_i(x)] = 0.
\]

**Classical observables**

Once we know all the transformation rules for the holonomies and the insertions, we search for the gauge invariant quantities of the theory, the observables. Considering the topology of \(\Sigma\) being \(S_1\), it is well known that for the two-dimensional BF model one observable (gauge invariant quantity) is the Wilson loop, \(W_0\), [8, 11], the trace of the holonomy trough a closed path—which here coincides with space \(S_1\) itself. In this model we construct another observable, \(W_1\), which is the trace of the insertion of \(B\) in the holonomy along the same closed path. Observe that \(W_0\) is an even quantity and \(W_1\) is odd. We write them as
\[
W_0 := \text{Tr}(h[A]), \quad W_1 := \text{Tr}(B [B > h[A]]).
\]
Their gauge invariance follows from (4.10) and (4.12). We note that there are no gauge invariant multilinear \(B\)-insertion of order \(\geq 2\) in \(B\). Other invariant quantities are
\[
L_0 := \text{Tr}(\psi \phi) = \psi \phi, \quad L_1 := \text{Tr}(\phi^2) = \phi \phi_i.
\]
\(L_1\) is present in the two-dimensional BF model [8, 11]. One observes that these four observables form two supersymmetry doublets:
\[
QW_0 = W_1, \quad QW_1 = 0, \quad QL_0 = L_1, \quad QL_1 = 0.
\]
The first two can be taken as the basis for the wave functionals which will describe the quantum space of the model.
4. Loop quantization

We now begin with the construction of the Hilbert space of the theory, in which the basic elements are wave functionals of the type $\Psi[A] := \Psi[A, B]$. We thus choose as coordinates the connection $A$ and the vector field $B$, which fix the polarization as

$$\hat{A}_k(x) \Psi[A, B] := A_k(x) \Psi[A, B], \quad \hat{\phi}(x) \Psi[A, B] := i\hbar \frac{\delta}{\delta A_k} \Psi[A, B],$$

$$\hat{B}_k(x) \Psi[A, B] := B_k(x) \Psi[A, B], \quad \hat{\psi}(x) \Psi[A, B] := i\hbar \frac{\delta}{\delta B_k} \Psi[A, B],$$

and the brackets (3.4) are promoted to (graded) commutators; the non-vanishing ones are

$$[\hat{A}_i(x), \hat{\phi}_j(y)]_- = i\hbar \delta^i_j \delta(x - y), \quad [\hat{B}_i(x), \hat{\psi}_j(y)]_+ = -i\hbar \delta^i_j \delta(x - y). \quad (4.1)$$

4.1. Superholonomies

Following the steps of loop quantum gravity, in order to achieve a well-defined Hilbert space, we shall write the wave functional in terms of the holonomies of the connection and of their $B$-insertions—to be defined hereafter—instead of the local fields $A$ and $B$ themselves. The constraints are then imposed in order to select the vectors belonging to the physical Hilbert space of the theory. Holonomies are convenient variables when one imposes the Gauss constraint, once they are endowed with transformation properties which permit to construct gauge-invariant quantities in a relatively simple way.

Guided by the (rigid) supersymmetry of the model, we first construct the superholonomy for the superconnection $\mathcal{A}$, through a path $\gamma$ on the manifold $\Sigma$, as follows:

$$\mathbf{H}_\gamma[A] := \mathbf{h}_\gamma[A, B] := \mathcal{P} e^{-\int_\gamma A} = \mathcal{P} e^{-\int_\gamma (A + \theta B)}. \quad (4.2)$$

Expanding in powers of $\theta$, and rearranging terms, this expression can be written as

$$\mathbf{H}_\gamma[A, B] := h_\gamma[A] - \theta B \triangleright h_\gamma[A], \quad (4.3)$$

where $h_\gamma[A]$ is the usual holonomy for the connection $A$ through $\gamma$ on $\Sigma$:

$$h_\gamma[A] = P e^{-\int_\gamma A} \quad (4.4)$$

and $B \triangleright h_\gamma[A]$ is the insertion of the Grassmannian field $B$ in the holonomy through the curve $\gamma$ parametrized by $s$, with $s_0 < s < s_f$, which reads

$$B \triangleright h_\gamma[A](s_f, s_0) := \int_{s_0}^{s_f} ds h[A](s, s) B(s) h[A](s, s_0). \quad (4.5)$$

Making use of the notion of superfields, the superholonomy $\mathbf{H}_\gamma[A, B]$ can be written as

$$\mathbf{H}_\gamma[A, B] := h_\gamma[A] + \theta Q h_\gamma[A], \quad (4.6)$$

where $Q$ is the supersymmetry generator (2.2), and we consequently have

$$Q h_\gamma[A] = -B \triangleright h_\gamma[A]. \quad (4.7)$$

With these considerations in mind, let us now introduce some useful relations present in this formalism. Consider a composed path on $\Sigma$, given by $\gamma = \gamma_2 \circ \gamma_1$. The holonomy $h_\gamma[A](s_f, s_0)$ satisfies the following property:

$$h_\gamma[A](s_f, s_0) = h_{\gamma_2}[A](s_f, s) h_{\gamma_1}[A](s, s_0), \quad (4.8)$$
where $s_0$ and $s_f$ are the initial and final points of the path $\gamma$ and $s$ is the end point of $\gamma_1$ and initial point of $\gamma_2$. The insertion of the field $B$ in the holonomy, i.e. the quantity $Q h_{\gamma_2}[A](s_f, s_0)$, satisfies the following one:

$$Q (h_{\gamma_2}[A](s_f, s_0)) = Q h_{\gamma_2}[A](s_f, s) h_{\gamma_1}[A](s, s_0) + h_{\gamma_2}[A](s_f, s) Q h_{\gamma_1}[A](s, s_0).$$  \hfill (4.9)

Under gauge transformations, holonomies transform as follows:

$$h_{\gamma_2}[A](s_f, s_0) = g(s_f) h_{\gamma_2}[A](s_f, s_0) g^{-1}(s_0),$$  \hfill (4.10)

and in a similar fashion the supergauge transformation for the superconnection is given by

$$H_{\gamma_2}[A](s_f, s_0) := G(s_f) H_{\gamma_2}[A](s_f, s_0) G^{-1}(s_0).$$  \hfill (4.11)

where $G$ is parametrized as in (2.11). Differentiating this expression in $\theta$ and using (2.15), (4.7), (4.6) and (4.10), we obtain the form of the gauge transformations for the $B$-insertion,

$$(B \triangleright h_{\gamma_2}[A](s_f, s_0)) = g(s_f) B \triangleright h_{\gamma_2}[A](s_f, s_0) g^{-1}(s_0) + (B \triangleright g) (s_f) h_{\gamma_2}[A](s_f, s_0) g^{-1}(s_0) - g(s_f) h_{\gamma_2}[A](s_f, s_0) (g^{-1}(B \triangleright g) g^{-1}(s_0)).$$  \hfill (4.12)

**N.B.** This transformation rule has the same form as that of the $\beta$-insertion given in (2.18), but with $g_1$, $g_2$ replaced by $g(s_f)$, $g(s_0)$, $g$ with $h_{\gamma_2}[A]$ and $\beta \triangleright g$ with the $B$-insertion $B \triangleright h_{\gamma_2}[A]$.

### 4.2. Hilbert space

The Hilbert space of the theory will be constructed on the basis of wave functionals of the form

$$\Psi[A, B] := f(h[A], B \triangleright h[A]),$$  \hfill (4.13)

which are the so-called cylindrical functions. Their arguments are the holonomies $h[A]$ and the $B$-insertions $B \triangleright h[A]$, given in (4.4) and (4.5), respectively, with $\gamma$ being a curve in the space $S_1$. The set of such functionals forms the vector space $\text{Cyl}$.

The physical state vectors are then obtained by the imposition of the constraints (3.7). Thus, the physical wave functionals will be gauge invariant, hence given by functions of the gauge invariant quantities $W_0$ and $W_1$ given by (3.15)

$$\Psi[A, B] = f(W_0[A], W_1[A, B]) = \psi(h[A], B \triangleright h[A]).$$  \hfill (4.14)

The second equality defines the wave functional as a function $\psi$ of the holonomies and their $B$-insertions, i.e. a function on the supergauge group $G$.

Due to $W_1$ being an anticommuting odd number, $W_1^2 = 0$, the function $f$ in (4.14) expands in $W_1$ as

$$f(W_0[A], W_1[A, B]) = a(W_0) + W_1 b(W_0).$$  \hfill (4.15)

Choosing a primitive $\hat{b}$ of the function $b$, $b = \hat{b}'$, we can rewrite the last equation as

$$f(W_0[A], W_1[A, B]) = a(W_0[A]) + W_1[A, B] \hat{b}(W_0[A]) = a(W_0[A]) + Q \hat{b}(W_0[A]),$$  \hfill (4.16)

which shows that the space of supergauge-invariant functionals splits into singlet and doublet representations of the rigid supersymmetry. The singlets are the constant functions.

We also conclude from (4.15) that we have two types of wave functionals:

even: $\Psi_s[A, B] = f_s(W_0[A]) = \psi_s(h[A]),$

odd: $\Psi_o[A, B] = Q f_o(W_0[A]) = \text{Tr}(B \triangleright h[A]) \psi_o(h[A]) = W_1 \psi_o(h[A]).$  \hfill (4.17)

The internal product of two state vectors (4.14) will be defined by an integral on the supergauge group $G$ of elements $\tilde{G}$:

$$\langle \Psi_1 | \Psi_2 \rangle = \int_G d\mu(\tilde{G}) \left( \psi_1(\tilde{G})^* \psi_2(\tilde{G}) \right),$$  \hfill (4.18)

where $d\mu$ is an invariant measure on $G$, a generalization of the usual Haar measure we are going to give now.
4.2.1. Integration measure. In order to define an internal product, we need to define an integration measure on the configuration space, whose points are holonomies of $A$- and $B$-insertions. For this purpose, we will now construct an invariant integration measure on the supergauge group $G$ defined in subsection 2.1:

$$\int_G d\mu(G) f(G) = \int_G d^3 \alpha d^3 \beta \rho(\alpha, \beta) F(\alpha, \beta), \quad (4.19)$$

where $d^3 \beta = \partial \beta_1 \partial \beta_2 \partial \beta_3$ is the Berezin integration measure over the odd parameters of the group $F(\alpha, \beta) = f(G(\alpha, \beta))$, and $\rho(\alpha, \beta)$ is a weight which must be chosen such that the integral be invariant under left and right multiplication:

$$\int_G d\mu(G) f(G) = \int_G d\mu(G_1) f(G_1), \quad \forall G_1 \in G. \quad (4.20)$$

This will ensure the supergauge invariance of the scalar product.

In terms of the parametrization $(\alpha, \beta)$ defined by (2.10), the product $G' = G_1 G$ writes

$$\alpha'^i = p(\alpha_1, \alpha), \quad \beta'^i = Q p(\alpha_1, \alpha) = (\beta_1^j \partial \alpha_1^j + \beta_1^j \partial \alpha_1^j) p(\alpha_1, \alpha), \quad (4.21)$$

where $p(\alpha_1, \alpha)$ is the product law, in terms of the $\alpha$ parametrization, for the bosonic part of the supergauge group, which is here $SU(2)$. Thus, the invariance condition (we restrict ourselves here to the left invariance)

$$\int_G d^3 \alpha d^3 \beta \rho(\alpha, \beta) F(\alpha, \beta) = \int_G d^3 \alpha d^3 \beta \rho(\alpha', \beta') F(\alpha', \beta'), \quad (4.22)$$

implies, thanks to the super–Jacobian of the change of integration variables $(\alpha, \beta) \rightarrow (\alpha', \beta')$ on the right-hand side being equal to 1, the invariance condition

$$\rho(\alpha, \beta) = \rho(\alpha', \beta') \quad (4.23)$$

for the integration weight.

The $\beta$-dependence of the function $F(\alpha, \beta)$ may have any one of the four following forms:

$$F_0(\alpha), \quad F_i(\alpha)\beta^i, \quad \frac{1}{2} F_{ij}(\alpha)\beta^i\beta^j, \quad \text{or} \quad \frac{1}{3!} F_{ijk}(\alpha)\beta^i\beta^j\beta^k. \quad (4.24)$$

Each of these expressions must be integrated with one of the following weight functions $\rho(\alpha, \beta)$, respectively:

$$\frac{1}{3!} \rho_{ijk}(\alpha)\beta^i\beta^j\beta^k, \quad \text{or} \quad \frac{1}{2} \rho_{ij}(\alpha)\beta^i\beta^j, \quad \rho_i(\alpha)\beta^i, \quad \text{or} \quad \rho_0(\alpha). \quad (4.25)$$

The integrals read, after the Berezin integration $\int d^3 \beta \beta^i\beta^j\beta^k = \epsilon^{ijk}$ has been performed,

$$\frac{1}{3!} \epsilon^{ijk} \int d^3 \alpha \rho_{ijk}(\alpha) F_0(\alpha) = \int d^3 \alpha \tilde{\rho}(\alpha) F_0(\alpha),$$

$$\frac{1}{2} \epsilon^{ijk} \int d^3 \alpha \rho_{ij}(\alpha) F_i(\alpha) = \int d^3 \alpha \tilde{\rho}^i(\alpha) F_i(\alpha),$$

$$\frac{1}{2} \epsilon^{ijk} \int d^3 \alpha \rho_{ij}(\alpha) F_i(\alpha) = \frac{1}{2} \int d^3 \alpha \tilde{\rho}^i(\alpha) F_i(\alpha),$$

$$\frac{1}{3!} \epsilon^{ijk} \int d^3 \alpha \rho_0(\alpha) F_{ijk} = \frac{1}{3!} \int d^3 \alpha \tilde{\rho}^{ijk}(\alpha) F_{ijk}(\alpha). \quad (4.26)$$
Writing the invariance condition (4.23) for each of the weight functions (4.25), one obtains terms in $\beta$ and $\beta_1$. Those in $\beta_1$ are irrelevant since they are of lower order in $\beta$ and thus do not contribute to the Berezin integrals. This finally yields the conditions

$$
\tilde{\rho}(\alpha') = \tilde{\rho}(\alpha) \text{det} \frac{\partial \alpha}{\partial \alpha'},
$$

$$
\tilde{\beta}^i(\alpha') = \tilde{\beta}^m(\alpha) \frac{\partial \alpha^i}{\partial \alpha^m} \text{det} \frac{\partial \alpha}{\partial \alpha'},
$$

$$
\tilde{\beta}^{ij}(\alpha') = \tilde{\beta}^{\mu m}(\alpha) \frac{\partial \alpha^i}{\partial \alpha^\mu} \frac{\partial \alpha^j}{\partial \alpha^m} \text{det} \frac{\partial \alpha}{\partial \alpha'},
$$

$$
\tilde{\beta}^{ijk}(\alpha') = \tilde{\beta}^{\mu np}(\alpha) \frac{\partial \alpha^i}{\partial \alpha^\mu} \frac{\partial \alpha^j}{\partial \alpha^n} \frac{\partial \alpha^k}{\partial \alpha^p} \text{det} \frac{\partial \alpha}{\partial \alpha'}.
$$

(4.27)

Since the wave functionals we consider are linear in the $B$-insertions, it suffices, for defining an internal product, to restrict the remainder of the discussion to the terms of at most order 2 in $\beta$. Thus we are interested to find solutions for $\tilde{\rho}$ corresponding to the first three lines of (4.26) and (4.27). The first of conditions (4.27) means that $\tilde{\rho}$ transforms like the Haar measure of the bosonic part of the supergauge group, and can thus be identified with:

$$
\tilde{\rho}(\alpha) = \rho_H(\alpha).
$$

(4.28)

The second condition implies that $\partial_\alpha \tilde{\beta}^i$ transforms like the Haar measure $\tilde{\rho}$; hence $\tilde{\beta}^i$ can be identified with a solution of

$$
\partial_\alpha \tilde{\beta}^i(\alpha) = \rho_H(\alpha).
$$

(4.29)

The third condition implies that the divergence $\partial_\alpha \tilde{\beta}^{ij}$ transforms like $\tilde{\rho}^{ij}$; hence, we are tempted to identify it with the latter. However, this does not work because this would imply the vanishing of $\rho_H$ as can be seen taking the divergence of both terms and observing that $\partial_\alpha \partial_\alpha \tilde{\beta}^{ij}$ identically vanishes due to the antisymmetry of $\tilde{\rho}^{ij}$. We need another solution for the latter, obeying the third of the invariance conditions (4.27). But we do not need to know it explicitly. Let us suppose that we have such a solution for $\tilde{\rho}^{ij}$. As we shall see in subsection 4.2.2, equation (4.34), supergauge invariant wavefunctions are either functions of $\alpha$, or linear in $\beta$, of the form $\beta^i \partial_\alpha f(\alpha)$. Thus we only need to define integrals of the restricted form

$$
\int d^3 \alpha \tilde{\beta}^{ij}(\alpha) \partial_\alpha f_1(\alpha) \partial_\alpha f_2(\alpha),
$$

(4.30)

which, after partial integrations, is equal to

$$
-\int d^3 \alpha \partial_\alpha \tilde{\beta}^{ij}(\alpha) \partial_\alpha f_1(\alpha) f_2(\alpha) = \int d^3 \alpha \partial_\alpha \tilde{\beta}^{ij}(\alpha) f_1(\alpha) \partial_\alpha f_2(\alpha).
$$

This suggests to define the integral (4.30) by substituting $\partial_\alpha \tilde{\beta}^{ij}$ with the weight function $\tilde{\rho}^{ij}$, solution of (4.29)—which has the correct transformation property to make the integral invariant:

$$
\int d^3 \alpha \tilde{\rho}^{ij}(\alpha) \partial_\alpha f_1(\alpha) \partial_\alpha f_2(\alpha) := \frac{1}{2} \int d^3 \alpha \tilde{\rho}^{ij}(\alpha) (f_1(\alpha) \partial_\alpha f_2(\alpha) - \partial_\alpha f_1(\alpha) f_2(\alpha)).
$$

(4.31)

Recapitulating, the integration weights we shall need are the Haar measure $\rho_H$ and the weight $\tilde{\rho}^{ij}$ solution of (4.29).

For the present gauge group $SU(2)$ with the parametrization (2.12), these two functions explicitly read

$$
\rho_H(\alpha) = \frac{4 \sin^2(r/2)}{r^2},
$$

(4.32)

$$
\tilde{\rho}^{ij}(\alpha) = \frac{\alpha^i}{r^2} (r - \sin r),
$$

(4.33)
with \( r^2 = (\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2 \). We have imposed the normalization condition 
\[
\int d^3 \alpha \, d^3 \beta \, \rho(\alpha, \beta) = 1,
\]
which is non-trivial only for the first one.

### 4.2.2. Internal product.

A Hermitian internal product of two supergauge invariant state vectors \( \Psi_1 \) and \( \Psi_2 \) belonging to Cyl, given as in (4.14) with functions \( \psi_1 \) and \( \psi_2 \) of the holonomies \( h[A] \) and \( B \triangleright h[A] \) along the space \( S_1 \) will now be defined with the help of the invariant integration measure (4.19) we have constructed:

\[
\langle \Psi_1 | \Psi_2 \rangle = \int_G d\mu(\mathcal{G}) \, (\psi_1(\mathcal{G}))^* \psi_2(\mathcal{G}) = \int_G d^3 \alpha \, d^3 \beta \, \rho(\alpha, \beta) \, (F_1(\alpha, \beta))^* F_2(\alpha, \beta),
\]

where * means complex conjugation. For an even vector \( \Psi_+ \), the corresponding function \( F \) is equal to a function \( f_+ \langle \text{Tr} g(\alpha) \rangle \), and for an odd vector, the corresponding function is of the form

\[
F(\alpha, \beta) = Q f_- \langle \text{Tr} g(\alpha) \rangle = \beta^i \partial_{\alpha^i} f_- \langle \text{Tr} g(\alpha) \rangle.
\]

We have thus the following three internal product formulas, obtained using (4.26) and (4.28), (4.31) and the notation \( \chi = \text{Tr} g \):

\[
\langle \Psi_1_+ | \Psi_2_+ \rangle = \int d^3 \alpha \rho_+(\alpha) \, (f_+(\chi(\alpha)))^* f_2_+(\chi(\alpha)),
\]

\[
\langle \Psi_1_+ | \Psi_2_- \rangle = \int d^3 \alpha \rho_+(\alpha) \, (f_+(\chi(\alpha)))^* \partial_{\alpha^i} f_2_-(\chi(\alpha)),
\]

\[
\langle \Psi_1_- | \Psi_2_- \rangle = i \int d^3 \alpha \rho_+(\alpha) \, (\partial_{\alpha^i} f_2_-(\chi(\alpha)))^* \partial_{\alpha^i} f_2_-(\chi(\alpha))
\]

\[
= i \int d^3 \alpha \rho_+(\alpha) \, (f_2_-(\chi(\alpha)))^* \partial_{\alpha^i} f_2_-(\chi(\alpha)) - (\partial_{\alpha^i} f_2_-(\chi(\alpha)))^* f_2_-(\chi(\alpha)),
\]

the last equality following from (4.31).

Using the supersymmetry singlet and doublet structure of the state vector space, we can take as a basis the state vectors \( | j+ \rangle \) (even) and \( | j- \rangle \) (odd) defined by

\[
\langle A | j+ \rangle = \text{Tr} R^{(j)}(h[A]), \quad \langle A, B | j- \rangle = Q \text{Tr} R^{(j)}(h[A]),
\]

where \( R^{(j)} \) is the spin \( j \) representation matrix of \( g \in SU(2) \), \( j = 0, \frac{1}{2}, 1, \ldots \). The even part of the basis is just the spin network basis of the loop quantization of the bosonic 2DBF model [8]. Observe that the only supersymmetry singlet is the null spin state \( | 0 \rangle \).

### 4.2.3. Observables.

The quantization of the observables \( W_0, W_1, (3.15), \) and \( L_0, L_1, (3.16) \), is rather straightforward. The first two act by multiplication (there are mapping cylindrical functions to cylindrical functions):

\[
\hat{W}_{0,1} \psi[A, B] = W_{0,1} \psi[A, B].
\]

Let us calculate the action of \( L_0 \) and \( L_1 \) on the basis vectors (4.35). When acting on wave functionals, they are represented by the operators

\[
\hat{L}_0(x) = -\hbar^2 \frac{\delta}{\delta A(x)} \cdot \frac{\delta}{\delta B(x)}, \quad \hat{L}_1(x) = -\hbar^2 \frac{\delta}{\delta A(x)} \cdot \frac{\delta}{\delta A(x)},
\]

and the supersymmetry generator \( Q \) by

\[
\hat{Q} = \int dx B(x) \cdot \frac{\delta}{\delta A(x)}.
\]

These operators obey the (anti-)commutation relations

\[
[\hat{L}_0, \hat{Q}]_+ = \hat{L}_1, \quad [\hat{L}_1, \hat{Q}]_- = 0.
\]
From these (anti-)commutation relations and the result [8, 17]

\[ \delta \frac{\delta}{\delta A(x)} \cdot \delta \frac{\delta}{\delta A(x)} R^{ij}(h[A]) = -j(j + 1)R^{ij}(h[A]), \]

one easily shows that

\[
\begin{align*}
\hat{L}_0 |j+\rangle &= 0, \\
\hat{L}_0 |j-\rangle &= \hbar^2 j(j + 1)|j+\rangle, \\
\hat{L}_1 |j+\rangle &= \hbar^2 j(j + 1)|j+\rangle, \\
\hat{L}_1 |j-\rangle &= \hbar^2 j(j + 1)|j-\rangle.
\end{align*}
\]

(4.36)

One sees that each supersymmetry doublet \(|j+\rangle, |j-\rangle\) is an eigenvector of the bosonic observable \(\hat{L}_1\) with the same eigenvalue \(\hbar^2 j(j + 1)\), whereas the fermionic observable \(\hat{L}_0\) plays the role of a step operator.

5. Concluding remarks

After having recognized that the model of [12] had a rigid supersymmetry whose effect is to promote the full gauge symmetry of this model to a supergauge symmetry group \(G\), the main step was to construct an invariant measure of integration on \(G\). This was achieved for a restricted class of integrants, suitable for the purpose of defining an internal product.

We have thus obtained a full quantization of the \(N = 1\) supersymmetric extension of the \(SU(2)\) BF-model in two dimensions, with a basis of the physical space given by a supersymmetry singlet \(|0\rangle\) and doublets indexed by half-integer spin numbers \(j \geq \frac{1}{2}\). We have also obtained explicit expressions for the action of the observables on the state vectors.

Acknowledgments

Work of CPC, IM and OP was supported in part by the Conselho Nacional de Desenvolvimento Científico e Tecnológico, CNPq (Brazil) and by the PRONEX project no. 35885149/2006 from FAPES, CNPq (Brazil). Work of RC was supported by CAPES (Brazil). IM was also supported by the Fundação de Amparo à Pesquisa do Espírito Santo, FAPES.

References

[1] Jackiw R 1984 Liouville field theory: A two-dimensional model for gravity? Quantum Theory of Gravity: Essays in Honor of the 60th Birthday of Bryce S. DeWitt ed S M Christensen (Bristol: Adam Hilgar Ltd) pp 403–20
[2] Jackiw R 1985 Lower dimensional gravity Nucl. Phys. B 252 343–56
[3] Jackiw R 1995 Two lectures on two-dimensional gravity arXiv:gr-qc/9511048
[4] Teitelboim C 1983 Gravitation and Hamiltonian structure in two space-time dimensions Phys. Lett. B 126 41–5
[5] Teitelboim C 1984 The Hamiltonian structure of two-dimensional space-time and its relation with conformal anomaly Quantum Theory of Gravity: Essays in Honor of the 60th Birthday of Bryce S. DeWitt ed S M Christensen (Bristol: Adam Hilgar Ltd) pp 327–44
[6] Isler K and Trugenberger C A 1989 A gauge theory of two-dimensional quantum gravity Phys. Rev. Lett. 63 834–6
[7] Fukuyama T and Kamimura K 1985 Gauge theory of two-dimensional gravity Phys. Lett. B 160 259
[8] Livine E R, Perez A and Rovelli C 2003 2D manifold-independent spinfoam theory Class. Quantum Grav. 20 4425
[9] Bautista L I M 2007 Formalismo Hamiltoniano do Modelo de Jackiw–Teitelboim no Calibre temporal http://www.cce.ufes.br/pgfis/Dissertações/D-Luis
[10] Constantinidis C P, Lourenco J A, Morales I, Piguet O and Rios A 2008 Canonical analysis of the Jackiw–Teitelboim model in the temporal gauge: I. The classical theory Class. Quantum Grav. 25 125003
[11] Constantinidis C P, Piguet O and Perez A 2009 Quantization of the Jackiw–Teitelboim model Phys. Rev. D 79 084007
[12] Leitgeb R, Schweda M and Zerrouki H 1999 Finiteness of 2-d topological BF theory with matter coupling Nucl. Phys. B 542 425 (arXiv:hep-th/9904204)

[13] Berezin F A, Kirillov A A and Leites D (ed) 1987 Introduction to Superanalysis (Dordrecht: Reidel)

[14] Cornwell J F 1989 Group Theory In Physics: Supersymmetries and Infinite Dimensional Algebras vol 3 (New York: Academic)

[15] Henneaux M and Teitelboim C 1992 Quantization of Gauge Systems (Princeton, NJ: Princeton University Press)

[16] Dirac P A 1964 Lecture in Quantum Mechanics (New York: Belfer Graduate School of Science, Yeshiva University)

[17] Rovelli C 2004 Quantum Gravity (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)