Abstract—A private information retrieval (PIR) protocol guarantees that a user can privately retrieve files stored in a database without revealing any information about the identity of the requested file. Existing information-theoretic PIR protocols ensure strong privacy, i.e., zero information leakage to the server, but at the cost of high download. In this work, we present weakly-private information retrieval (WPIR) schemes that trade off strong privacy to improve the download cost when the database is stored on a single server, i.e., for the single server scenario. We study the tradeoff between the download cost and information leakage in terms of mutual information (MI) and maximal leakage (MaxL) privacy metrics. By relating the WPIR problem to rate-distortion theory, the download-leakage function, which is defined as the minimum required download cost of all single-server WPIR schemes for a given level of information leakage, is introduced. By characterizing the download-leakage function for the MI and MaxL metrics, the capacity of single-server WPIR can be fully characterized.

I. INTRODUCTION

User privacy is becoming increasingly important both socially and politically in today’s modern age of information (as demonstrated, for instance, by the European Union’s General Data Protection Regulation). As such, private information retrieval (PIR), introduced by Chor et al. in [1], has gained traction in the information theory community. In PIR, a user can retrieve a file from a database without revealing its identity to the servers storing it. From an information-theoretic perspective, the file size is typically much larger than the size of the queries to all servers. Therefore rather than accounting for both the upload and the download cost, as is usually done in the computer science community, here efficiency is measured in terms of the download cost. More precisely, efficiency is measured in terms of PIR rate, which is the ratio between the requested file size and the total number of symbols downloaded. The supremum of the PIR rates of all possible schemes is called the PIR capacity.

PIR was first addressed in the information theory literature by Shah et al. [2], while the tradeoff between storage overhead and PIR rate was first considered in [3]. Shortly after, Sun and Jafar [4] characterized the PIR capacity for the classical PIR model of replicated servers. Since then the concept of PIR has been extended to several relevant scenarios: maximum distance separable (MDS) coded servers [5], [6], arbitrary linear coded servers [7]–[9], colluding servers [5], [7],[9]–[13], robust PIR [10], PIR with Byzantine servers [14], optimal upload cost [15], access complexity in PIR [16], single-server PIR with private side information [17], and PIR with secure storage [18].

Weakly-private information retrieval (WPIR) [19]–[21] is an interesting extension of the original PIR problem as it allows for improvements in the download cost at the expense of some information leakage to the servers on the identity of the requested file. In particular, [19], [21] consider WPIR with a privacy metric related to differential privacy, while [20] considers mutual information (MI) and worst-case information leakage [22]. In contrast to WPIR, where the information leakage to the servers is considered, recently, the authors of [23] studied the information leakage of the undesired files to the user in PIR systems. In distributed storage systems (DSSs), for several applications, it is more realistic to assume that all servers can collude against the user. In the single server scenario, it can be shown that all files stored in the database need to be downloaded to guarantee strong privacy [24]. This implies that the PIR rate tends to zero as the number of stored files increases. In [17], the authors introduce PIR schemes that improve the download cost by leveraging on the assumption that the user has some prior side information on the content of the database.

In this paper, we relax the condition of perfect privacy in the single server setting. In similar lines to [20], we show that by relaxing the strong privacy requirement, the download cost can be improved. Unlike [20], in addition to the MI metric, we consider the maximum leakage (MaxL) privacy metric [25], [26], which is the most robust information-theoretic metric for information leakage yet known. In particular, we establish a connection between the single-server WPIR problem and the well-established rate-distortion theory in information theory, which provides fundamental insights to describe the optimal tradeoff between the download cost and the allowed information leakage. The primary contribution of this work is to characterize the capacity (the inverse of minimum download cost) of single-server WPIR when the information leakage to the server is measured in terms of MI or MaxL. Furthermore, we propose a simple novel single-server WPIR scheme that achieves the WPIR capacity for both the MI and MaxL privacy.
metrics.

The remainder of this paper is structured as follows. Section II presents the notation, definitions, and the problem formulation. In Section III, we introduce the download-leakage function of single-server WPIR, which is defined as the minimum achievable download cost for a given information leakage constraint. Moreover, we discuss some properties of the function when the leakage is measured in terms of the MI or MaxL metrics. In Section IV, a basic solution for single-server WPIR is presented in which the files are partitioned into several partitions. In Section V, we give a closed-form expression for the single-server WPIR capacity for both the MI and MaxL metrics. A capacity-achieving WPIR scheme is proposed in Section VI. The converse result on the minimum download cost for the MI metric is provided in Section VII, while that of the MaxL metric is given in Section VIII. Finally, Section IX concludes the paper.

II. Preliminaries and Problem Statement

A. Notation

We denote by \( \mathbb{N} \) the set of all positive integers, \([a] \triangleq \{1, 2, \ldots, a\}\), and \([a : b] \triangleq \{a, a+1, \ldots, b\}\) for \(a, b \in \{0\} \cup \mathbb{N}\) and \(a \leq b\). The set of nonnegative real numbers is denoted by \( \mathbb{R}_+ \). Vectors are denoted by bold letters, matrices by sans serif capital letters, and sets by calligraphic uppercase letters, e.g., \(x\), \(X\), and \(X'\), respectively. In general, vectors are represented as row vectors throughout the paper. We use uppercase letters for random variables (RVs) (either scalar or vector), e.g., \(X\) or \(X'\). For a given index set \(S\), we write \(X^S\) to represent \(\{X^{(m)} : m \in S\}\). \(X \perp Y\) means that the two RVs \(X\) and \(Y\) are independent. \((\cdot)^T\) denotes the transpose of its argument. The Hamming weight of a vector \(x\) is denoted by \(w_H(x)\), while its support will be denoted by \(\chi(x)\). \(E_X[\cdot]\) and \(E_{P_X}[\cdot]\) denote expectation with respect to the RV \(X\) and distribution \(P_X\), respectively. \(H(X), H(P_X),\) or \(H(p_1, \ldots, p_{|X|})\) represents the entropy of \(X\), where \(P_X(z) = (p_1, \ldots, p_{|X|})\) denotes the distribution of the RV \(X\). \(I(x : y)\) denotes the MI between \(X\) and \(Y\). The Galois field with \(q\)-ary units, we have \(H(X^{(m)}) = \beta\), \(\forall m \in [M]\). A user wishes to efficiently and privately retrieve one of the \(M\) files. Similar to the detailed mathematical description in [20], the requested file index \(M\) can be seen as a RV that is uniformly distributed over \([M]\). We give the following definition for a single-server WPIR scheme.

Definition 1. An \(M\)-file WPIR scheme \(\mathcal{C}\) for a single server storing \(M\) files consists of:

- A random strategy \(S\), whose alphabet is \(S\).
- A query function
  \[ \phi : \{1, \ldots, M\} \times S \rightarrow Q \]
  that generates a query \(Q = \phi(M, S)\) with alphabet \(Q\).
- The query \(Q\) is sent to the server.
- An answer function
  \[ \varphi : Q \times GF(q)^{\beta M} \rightarrow A^{\beta L} \]
  that returns the answer \(A = \varphi(Q, X^{[M]}_A)\) back to the user, with download symbol alphabet \(A\).
- Here, \(L = L(Q)\) is the normalized length of the answer, which is a function of the query \(Q\).
- A privacy leakage metric
  \[ \rho^{(\cdot)} : \{1, \ldots, M\} \times Q \rightarrow \mathbb{R}_+ \]
  that measures the amount of leaked information of the identity of the requested file to the server by observing the generated query \(Q\), where the superscript “(\cdot)” indicates the corresponding used metric.

Furthermore, the scheme should allow a user to retrieve the requested file from the answer, the query, and the index of the requested file. In other words, this scheme must satisfy the condition of perfect retrievability,

\[ H(X^{(M)} | A, Q, M) = 0. \tag{1} \]

We remark that a PIR scheme is equivalent to a WPIR scheme for which no information leakage is allowed.

C. Metrics of Information Leakage

Given a single-server WPIR scheme for \(M\) files, its designed query conditional probability mass function (PMF) given the index \(M\) of the requested file, \(P_{Q|M}\), can be seen as a privacy mechanism (a randomized mapping). The server receives the random outcome \(Q\) of the privacy mechanism \(P_{Q|M}\), and is curious about the index \(M\) of the requested file. The information leakage of a WPIR scheme is then measured with respect to its corresponding privacy mechanism \(P_{Q|M}\).

In this paper, we focus on two commonly-used information-theoretic measures, namely MI and MaxL. For the former, the information leakage is quantified by the min-entropy (MinE) measure discussed in the computer science literature, see, e.g., [25]. The initial uncertainty about the RV \(M\), quantified by the MinE measure, is defined as

\[ H_{\infty}(M) \triangleq -\log_2 \max_{m \in [M]} P_M(m). \]

The second privacy metric, MaxL, can be defined based on the min-entropy (MinE) measure discussed in the computer science literature, see, e.g., [25]. The initial uncertainty about the RV \(M\), quantified by the MinE measure, is defined as

\[ H_{\alpha}(M) \triangleq \frac{1}{1 - \alpha} \log_2 \left( \sum_{m \in [M]} P_M(m)^{\alpha} \right), \alpha > 0 \text{ with } \alpha \neq 1. \]
It is known that \( H_\infty(M) \) converges to \( H_\infty(M) \) as \( \alpha \to \infty \). The \textit{conditional MinE} measure \( H_\infty(M|Q) \) of \( M \) given \( Q \),
\[
H_\infty(M|Q) \equiv -\log_2 \sum_{q \in Q} P_Q(q) \max_{m \in [M]} P_{M|Q}(m|q),
\]
quantifies the remaining uncertainty about \( M \) given \( Q \). Thus, the MinE \( P \) \textit{information leakage} \( l_\infty(M;Q) \) for the privacy mechanism \( P_{Q|M} \) is defined as the difference between the initial uncertainty \( H_\infty(M) \) and the remaining uncertainty \( H_\infty(M|Q) \), i.e.,
\[
l_\infty(M;Q) \equiv H_\infty(M) - H_\infty(M|Q).
\]

It is worth mentioning that since we assume that \( M \) is uniformly distributed, the MinE information leakage and the MaxL privacy metric can be shown to be equivalent [26], [28]. We review a closed-form expression of the MaxL metric below.

**Proposition 1** ([28, Th. 1], [26, Th. 1]).
\[
\text{MaxL}(M;Q) \equiv \log_2 \sum_{q \in Q} \max_{m \in [M]} P_{Q|M}(q|m) = l_\infty(M;Q).
\]

Henceforth, we will use the closed-form expression in (2) as the MaxL privacy metric for \( P_{Q|M} \). We denote by
\[
\rho^{(\text{MaxL})}(M,Q) \equiv \text{MaxL}(M;Q)
\]
the privacy metric of a WPIR scheme under the MaxL privacy metric. It is also worth mentioning that there is a relation between MaxL and \textit{differential privacy} [29], see, e.g., [28, Th. 3].

The following lemma summarizes some useful properties for both the MI and MaxL privacy metrics.

**Lemma 1** (Data processing inequalities [26, Lem. 1, Cor. 1]). For any joint distribution \( P_{X,Y} \),
1) if the RVs \( X,Y \), and \( Z \) form a Markov chain, then
\[
\begin{align*}
I(X;Z) &\leq \min\{I(X;Y),I(Y;Z)\}, \\
\text{MaxL}(X;Z) &\leq \min\{\text{MaxL}(X;Y),\text{MaxL}(Y;Z)\}.
\end{align*}
\]
2) Consider a fixed distribution \( P_X \). Then, both \( I(X;Y) \) and \( q^{\text{MaxL}(X;Y)} \) are convex functions in \( P_{Y|X} \).

Throughout the paper, the information leakage metric of a WPIR scheme \( \mathcal{C} \) is denoted by \( \rho^{(\mathcal{C})}(\cdot) \). Moreover, since \( P_M \) is fixed, we will also simply write the leakage measure \( \rho^{(\mathcal{C})}(\cdot,\cdot) \), the mutual information \( I(\cdot,\cdot) \), and the maximal leakage MaxL(\cdot,\cdot) as a function of the desired query distribution \( P_{Q|M} \), i.e.,
\[
\rho^{(\mathcal{C})}(P_{Q|M}) = \rho^{(\mathcal{C})}(M,Q). \quad I(P_{Q|M}) = I(M;Q), \quad \text{MaxL}(P_{Q|M}) = \text{MaxL}(M;Q).
\]

**D. Download Cost and Rate for a Single-Server WPIR Scheme**

The download cost of a single-server WPIR scheme \( \mathcal{C} \), denoted by \( \text{D}(\mathcal{C}) \), is defined as the expected length of the returned answer for the retrieval of a single file,
\[
\text{D}(\mathcal{C}) \equiv \mathbb{E}_Q[L(Q)].
\]
Accordingly, the WPIR rate is defined as \( \text{R}(\mathcal{C}) \equiv \text{D}(\mathcal{C})^{-1} \).

Intuitively, a smaller download cost can be achieved if we allow a higher level of information leakage. In this paper, our goal is to characterize the optimal tradeoff between the download cost and the allowed information leakage with respect to a privacy metric. We start with the following definition of an achievable download-leakage pair.

**Definition 2.** Consider a single server that stores \( M \) files. A download-leakage pair \((D,\rho)\) is said to be achievable in terms of the information leakage metric \( \rho^{(\cdot)} \) if there exists a WPIR scheme \( \mathcal{C} \) such that \( \mathbb{E}_Q[L(Q)] \leq D \) and \( \rho^{(\mathcal{C})}(\mathcal{C}) \leq \rho \). The download-leakage region is the set of all achievable download-leakage pairs \((D,\rho)\).

**Remark 1.** By Definition 2, it is clear that if the pair \((D,\rho)\) is achievable, then the pair \((D',\rho')\) with \( D' \geq D \) and \( \rho' \geq \rho \) is also achievable.

**III. CHARACTERIZATION OF THE OPTIMAL DOWNLOAD-LEAKAGE TRADEOFF**

Consider a single-server WPIR scheme, where the leakage is measured by \( \rho^{(M)} \) or \( \rho^{(\text{MaxL})} \). The minimum achievable download cost for a given leakage constraint \( \rho \) can be formulated by the optimization problem
\[
\begin{align*}
\text{minimize} & \quad \mathbb{E}_{P_Q}[L(Q)] \\
\text{subject to} & \quad P_{Q|M} \in P_{\text{ret}}, \quad (3a) \\
& \quad \rho^{(\cdot)}(P_{Q|M}) \leq \rho, \quad (3b)
\end{align*}
\]
where \( P_{\text{ret}} \) is defined as the set of all PMFs that satisfy (1).

We remark that since \( P_M \) is assumed to be a uniform distribution, the minimization is taken over the set of all conditional distributions \( P_{Q|M} \) such that (3a) and (3b) are satisfied. For convenience, since we know that a designed conditional distribution \( P_{Q|M} \) of a WPIR scheme always satisfies (3a), throughout this paper we will assume that any \( P_{Q|M} \in P_{\text{ret}} \).

**A. The Download-Leakage Function for Single-Server WPIR**

To characterize the optimal achievable pairs of download cost and information leakage, we define two functions that describe the boundary of the download-leakage region.

**Definition 3.** The download-leakage function \( \text{D}^{(\cdot)}(\rho) \) for a single-server WPIR is the minimum of all possible download costs \( D \) for a given information leakage constraint \( \rho \) such that \((D,\rho)\) is achievable, i.e.,
\[
\text{D}^{(\cdot)}(\rho) \equiv \min_{P_{Q|M} : \rho^{(\cdot)}(P_{Q|M}) \leq \rho} \mathbb{E}_{P_Q}[L(Q)].
\]
Accordingly, the single-server WPIR capacity is \( C^{(\cdot)}(\rho) \equiv [\text{D}^{(\cdot)}(\rho)]^{-1} \).

It is known that the single-server PIR capacity is \( C_M \equiv \frac{1}{N} [\text{H}^L]. \quad [24]. \)

Naturally, we can also determine the optimal download-leakage region by interchanging the roles of the download cost and information leakage.
Definition 4. The leakage-download function \( \rho^{(1)}(D) \) for single-server WPIR is the minimum of all possible information leakages \( \rho \) for a given download cost \( D \) such that \( (D, \rho) \) is achievable.

Lemma 2. The MI download-leakage function

\[
D^{(\text{MI})}(\rho) = \min_{P_{Q|M}: \text{MI}(P_{Q|M}) \leq \rho} \mathbb{E}_{P_{Q}|L(Q)}
\]

is convex in \( \rho \), while the MaxL download-leakage function

\[
D^{(\text{MaxL})}(\rho) = \min_{P_{Q|M}: \text{MaxL}(P_{Q|M}) \leq \rho} \mathbb{E}_{P_{Q}|L(Q)}
\]

is not a convex function, but \( D^{(\text{MaxL})}((\log_2(\rho)))\) is convex in \( \rho \).

Proof: We first prove it for MI leakage. Assume that \( D^{(\text{MI})}(\rho_1) = \mathbb{E}_{P_{Q}|L(Q)} \) and \( D^{(\text{MI})}(\rho_2) = \mathbb{E}_{P_{Q}|L(Q)} \) are achieved by the distributions \( P_{Q|M} \) and \( P_{Q|M} \), respectively, where \( l(M; Q_1^M) \leq \rho_1 \) and \( l(M; Q_2^M) \leq \rho_2 \). Let \( P_{Q|M} \) be the distribution

\[
P_{Q|M} = (1 - \lambda)P_{Q_1|M} + \lambda P_{Q_2|M}.
\]

Clearly, \( P_{Q|M} \in \mathcal{P}_{\text{ret}} \).

Observe that since \( l(M; Q) \) is convex in \( P_{Q|M} \), it follows that \( l((1 - \lambda)P_{Q_1|M} + \lambda P_{Q_2|M}) \leq (1 - \lambda)l(P_{Q_1|M}) + \lambda l(P_{Q_2|M}) \leq (1 - \lambda)\rho_1 + \lambda \rho_2 \), which implies that \( P_{Q|M} \) is an element of \( \{P_{Q|M}: l(M; Q) \leq (1 - \lambda)\rho_1 + \lambda \rho_2 \} \).

Thus, by definition we get

\[
D^{(\text{MI})}((1 - \lambda)\rho_1 + \lambda \rho_2) = \min_{P_{Q|M}: l(P_{Q|M}) \leq (1 - \lambda)\rho_1 + \lambda \rho_2} \mathbb{E}_{P_{Q}|L(Q)} \\
\leq \mathbb{E}_{P_{Q}}[L(Q)] \\
= (1 - \lambda)\mathbb{E}_{P_{Q_1}|L(Q)} \mathbb{E}_{P_{Q_2}|L(Q)} + \lambda \mathbb{E}_{P_{Q_2}|L(Q)} \\
= (1 - \lambda)D^{(\text{MI})}(\rho_1) + \lambda D^{(\text{MI})}(\rho_2),
\]

where (4) follows directly from the definition of \( P_{Q|M} \). This shows that \( D^{(\text{MI})}(\rho) \) is convex in \( \rho \) for the MI metric. The proof for the MaxL metric is analogous, since \( 2^{\text{MaxL}(P_{Q|M})} \) is convex in \( P_{Q|M} \).

From Remark 1 we can see that the convexity of the download-leakage function is very useful, since it can help to describe the download-leakage region if some achievable pairs are known. This observation can be summarized in the following corollary.

Corollary 1. Assume that both pairs \( (D_1, \rho_1) \) and \( (D_2, \rho_2) \) are achievable. Then, for any \( \lambda \in [0, 1] \), the pair \( (D_\lambda = (1 - \lambda)D_1 + \lambda D_2, \rho_\lambda = (1 - \lambda)\rho_1 + \lambda \rho_2) \) is achievable under MI leakage, while the pair \( (D_\lambda = (1 - \lambda)D_1 + \lambda D_2, \rho_\lambda = \log_2 [(1 - \lambda)2^{\rho_1} + \lambda 2^{\rho_2})] \) is achievable for MaxL.

B. Connection to Rate-Distortion Theory

The celebrated rate-distortion theory of Shannon and Kolmogorov (see, e.g., [30, Ch. 9], [31, Ch. 10], and references therein) determines the minimum source compression rate required to reproduce any source sequence under a fidelity constraint, which is provided through a distortion measure between the source sequence and the reconstructed sequence. Consider an information source sequence with independent and identically distributed components according to \( P_X \) and a distortion measure \( d(x, \hat{x}) \) between the source sequence \( x \) and the reconstructed sequence \( \hat{x} \). The optimal rate-distortion region is characterized by means of the rate-distortion function, which is defined as the minimum achievable compression rate \( \mathbb{I}(X; \hat{X}) \) under a given constraint on the average distortion \( \mathbb{E}_{P_X}[d(X, \hat{X})] \), where \( \hat{X} \) represents the reconstructed source.

One important observation from Definition 1 is that, given the normalized length of the answer function \( L(Q) \) of a WPIR scheme, the download cost can be expressed as \( \mathbb{E}_{P_X}[L(Q)] = \mathbb{E}_{P_{Q|M}[L(M, Q)]]} \), where \( L(m, Q) = \mathbb{L}(Q), \forall m \in [M] \). Thus, in terms of the MI privacy metric, the leakage-download function can be related to the rate-distortion function, where the leakage and the download cost play similar roles as the compression rate and the average distortion, respectively. We defer the detailed discussion to Section VII where we will utilize results from the rate-distortion theory to characterize the optimal leakage-download tradeoff for single-server WPIR.

IV. BASIC PARTITION WPIR SCHEME

In [20], a simple construction of a WPIR scheme was proposed based on partitioning. The database is first partitioned into \( \eta \) equally-sized partitions, each consisting of \( M_\eta \) files, where \( M_\eta = M/\eta \in \mathbb{N} \). If there exists a viable \( M_\eta \)-file WPIR scheme, the user can apply the \( M_\eta \)-file WPIR scheme as a subscheme on each partition, and retrieve a file from the corresponding partition.

The partition \( M \)-file WPIR scheme is formally described as follows. Assume that the requested file \( X^{(m)} \) belongs to the \( j \)-th partition, where \( j \in [\eta] \). Then, the query \( Q \) is constructed as

\[
Q = (\tilde{Q}, j) \in \tilde{Q} \times [\eta],
\]

where \( \tilde{Q} \) is the query of an existing \( M_\eta \)-file WPIR scheme. The following theorem states the achievable rate-leakage pairs of the partition scheme.

Theorem 1. Consider a single server that stores \( M \) files and let \( M_\eta = M/\eta \in \mathbb{N}, \eta \in \mathbb{N} \). Assume that an \( M_\eta \)-file WPIR scheme \( \mathcal{C} \) with achievable pair \( (\tilde{R}, \tilde{\rho}) \) exists. Then, the pair

\[
(R(\mathcal{C}), \rho^{(1)}(\mathcal{C})) = (\tilde{R}, \tilde{\rho} + \log_2 \eta)
\]

is achievable by the \( M \)-file partition scheme \( \mathcal{C} \) constructed from \( \mathcal{C} \) as described in (5).

Proof: We prove the theorem for the MaxL privacy metric, while the proof for the MI metric will be provided in the extended version of [20]. We refer the requested file index \( M \) by the pair \((M, j)\), where \( M \) represents the requested file index in the \( j \)-th partition, \( j \in [\eta] \). Hence, from (2), we have

\[
2^{\text{MaxL}(M; Q)} = \sum_{q \in Q} \max_{m \in [M]} P_{Q|M}(q|m) \\
= \sum_{j \in [\eta]} \sum_{q \in Q} \max_{m \in [M]} P_{Q|M}(q|m)
\]
Fig. 1. (a) The capacity $C^r(\tilde{\rho})$ for a small number of files $M = 2, 3, 4$ with privacy metrics $\rho^{(\text{MI})}$ and $\rho^{(\text{MaxL})}$. (b) The capacity $C^r(\tilde{\rho})$ for a large number of files $M = 10, 50$ with privacy metrics $\rho^{(\text{MI})}$ and $\rho^{(\text{MaxL})}$. The dark green circles mark the achievable rate-leakage pairs of $C^r$-basic.

$$
\rho^{(\text{MaxL})} = 2 \max_{\rho \in [0,1]} \rho \left[ 1 - \frac{\log_2 \rho - \frac{w}{M}}{\log_2 M} \right]^{-1},
$$
for $1 - \frac{\log_2 w}{\log_2 M} \leq \tilde{\rho} \leq 1 - \frac{\log_2 (w - 1)}{\log_2 M}$, $w \in [2: M]$. (8)

The following theorem states the single-server WPIR capacity for the MaxL privacy metric.

**Theorem 4.** For a single server that stores $M$ files, the single-server WPIR capacity under both the MI metric $\rho^{(\text{MI})}$ and the MaxL metric $\rho^{(\text{MaxL})}$ is upperbounded as

$$
C^r(\tilde{\rho}) \leq C_{UB}(\tilde{\rho}) \leq \frac{1}{M^{1-\tilde{\rho}}}, \quad 0 \leq \tilde{\rho} \leq 1.
$$

**Proof:** Since it is easy to show that the capacity for MI leakage is larger than or equal to the capacity for MaxL (see Fig. 1), we only need to prove that the inverse of (8) is lowerbounded by $M^{1-\tilde{\rho}}$. Observe that $M^{1-\tilde{\rho}}$ is a convex function of $\tilde{\rho}$ and each point $(w, 1 - \frac{\log_2 w}{\log_2 M})$, $w \in [M]$, lies on the curve described by that function, i.e., $M^{1-\tilde{\rho}} = w$ for $\tilde{\rho} = 1 - \frac{\log_2 w}{\log_2 M}$. Thus, by the convexity of $M^{1-\tilde{\rho}}$, we have $M^{1-\tilde{\rho}} \leq \left[ C^r(\tilde{\rho}) \right]^{-1}$, where the inequality follows since
Fig. 2. The capacity $C_{\text{UB}}(\bar{\rho})$ and its upper bound $C_{\text{UB}}(\bar{\rho})$ for a number of files $M = 10^2, 10^3, 10^4, 10^6$ with privacy metrics $\rho^{(M)}$ and $\rho^{(M\lambda)}$.

The inverse of (8) can be seen as a convex combination of $1 - \frac{\log_2 w}{\log_2 M}$ and $1 - \frac{\log_2 (w-1)}{\log_2 M}$, $w \in [M]$.

In Fig. 2, the capacity $C_{\text{UB}}(\bar{\rho})$ and the upper bound $C_{\text{UB}}(\bar{\rho})$ are plotted for $M = 10^2, 10^3, 10^4, 10^6$, which illustrates the asymptotic behavior of $C_{\text{UB}}(\bar{\rho})$ as $M$ tends to infinity. It can be shown that as $M$ tends to infinity, the asymptotic capacity is equal to

$$C_{\text{UB}}(\bar{\rho}) = \begin{cases} 0 & \text{if } 0 \leq \bar{\rho} < 1, \\ 1 & \text{if } \bar{\rho} = 1. \end{cases}$$

This indicates that the asymptotic capacity is still equal to zero, unless the server exactly knows the index of the requested file.

VI. Achievability

Throughout this section, for simplicity, we set $\beta = 1$, i.e., $H(X^{(m)}) = 1$, $\forall m \in [M]$. In fact, our proposed single-server WPIR capacity-achieving scheme works for an arbitrary file size $\beta$, which indicates that subpacketization does not improve the performance of single-server WPIR. We will later show that this scheme is optimal for both the MI and MaxL privacy metrics.

A. Motivating Example 1: $M = 2$ Files

Assume that the single server stores $M = 2$ files, $X_1^{(1)}$ and $X_1^{(2)}$. We design the queries and answers via a conditional distribution $P_{Q_1|M}$ defined as:

| $Q_1$ | $P_{Q_1|M}(q | 1)$ | $P_{Q_1|M}(q | 2)$ | $A_1$ | $P_{Q_1}(q)$ |
|-------|--------------------|--------------------|-------|---------------|
| (1, 0) | $\lambda$ | 0 | $X_1^{(1)}$ | $\frac{\lambda}{2}$ |
| (0, 1) | 0 | $\lambda$ | $X_1^{(2)}$ | $\frac{\lambda}{2}$ |
| (1, 1) | $1 - \lambda$ | $1 - \lambda$ | $\{X_1^{(1)}, X_1^{(2)}\}$ | $1 - \lambda$ |

where $0 \leq \lambda \leq 1$. It can easily be verified that the perfect retrievability condition of (1) is satisfied. The corresponding download cost is a function of $\lambda$ and is given by $D_{\lambda} = 2(1 - \lambda) + \lambda$.

The MI leakage is $l(M; Q) = H(\frac{\lambda}{2}, \frac{\lambda}{2}, 1 - \lambda) - H(\lambda, 1 - \lambda) = \lambda$ and the MaxL is $\text{MaxL}(M; Q) = \log_2 (1 + \lambda)$. One can easily verify that this matches the download-leakage function for $M = 2$ for both the MI and MaxL metrics.

The following remarks can be made from the above WPIR scheme. From Corollary 1, it follows that the pair $(\lambda, \lambda)$ is also achievable.

B. Motivating Example 2: $M = 3$ Files

Before describing the achievable scheme in detail for the general case of $M$ files, we present another example for $M = 3$. Based on the remarks made for the case $M = 2$, we generalize the WPIR scheme by considering three achievable download-leakage pairs $(D_1, \rho_1) = (1, \log_2 3)$, $(D_2, \rho_2) = (2, \log_2 \frac{3}{2})$, and $(D_3, \rho_3) = (3, \log_2 \frac{3}{2})$. In terms of the MI or MaxL privacy metrics, one can check that the pair $(D_w, \rho_w)$ is achievable.

The retrievability condition in (1) is satisfied for any convex combination of the three PMFs.

Now, construct two conditional query distributions as follows,

$$P_{Q_{2,3}|M} = (1 - \lambda_1)P_{Q_2,3|M} + \lambda_1 P_{Q_1|M},$$

$$P_{Q_{2,3}|M} = (1 - \lambda_2)P_{Q_2,3|M} + \lambda_2 P_{Q_2,3|M},$$

where $0 \leq \lambda_1, \lambda_2 \leq 1$.

The retrievability condition can be easily verified for the above WPIR scheme. From Corollary 1, it follows that the WPIR scheme achieves the download cost

$$D_{\lambda_1} = (1 - \lambda_1)D_2 + \lambda_1 D_1 = 2 - \lambda_1,$$
\(D_{\lambda_2} = (1 - \lambda_2)D_3 + \lambda_2D_2 = 3 - \lambda_2, 0 \leq \lambda_2 \leq 1,\)

the MI leakage
\[
\rho^{(M)} = \begin{cases} 
(1 - \lambda_1) \log_2 \frac{3}{2} + \lambda_1 \log_2 \frac{3}{2}, & 0 \leq \lambda_1 \leq 1, \\
(1 - \lambda_2) \log_2 \frac{3}{2} + \lambda_2 \log_2 \frac{3}{2}, & 0 \leq \lambda_2 \leq 1,
\end{cases}
\]

and the MaxL
\[
\rho^{(MaxL)} = \begin{cases} 
\log_2 ((1 - \lambda_1) \frac{3}{2} + \lambda_1 \frac{3}{2}), & 0 \leq \lambda_1 \leq 1, \\
\log_2 ((1 - \lambda_2) \frac{3}{2} + \lambda_2 \frac{3}{2}), & 0 \leq \lambda_2 \leq 1.
\end{cases}
\]

In terms of the MI or MaxL privacy metrics, it can be verified that the download cost corresponds to the single-server WPIR capacity for \(M = 3\). Note that for \(M > 2\), the capacity is a piecewise continuous function (see Fig. 1(a)).

C. Arbitrary Number of Files \(M\)

We describe the achievable scheme for the general case of \(M\) files. From Corollary 1, it follows that it is sufficient to show that the download-leakage pairs
\[
(D_w, \rho_w) = \left(w, \log_2 \frac{M}{w}\right), \quad w \in [M],
\]

are achievable.

1) Query Generation: Consider \(M\) random queries \(Q_w, w \in [M],\) whose alphabet is \(Q_w \triangleq \{q = (q_1, \ldots, q_M) \in \{0,1\}^M : \forall m(q) = w\}.\) Each query \(q \in Q_w\) sent to the server is generated by the conditional PMF
\[
P_{Q_w|M}(q|m) = \begin{cases} 
\frac{1}{(w-1)} & \text{if } m \in \chi(q), \\
0 & \text{otherwise}.
\end{cases}
\]

This is a valid query design, since for each \(m \in [M],\) we have
\[
\sum_{q \in Q_w} P_{Q_w|M}(q|m) = \sum_{q \in Q_w : m \in \chi(q)} P_{Q_w|M}(q|m) = 1.
\]

2) Answer Construction: The answer function \(\varphi\) maps the query \(q \in Q_w\) onto \(A = \varphi(q, X^{[M]}) = X^q(q).\) The answer length is \(L(q) = w.\)

3) Download Cost and Information Leakage: Clearly, the download cost is equal to \(E_{P_{Q_w}}[L(Q_w)] = w.\) The MI leakage is
\[
\rho^{(M)} = I(M; Q_w) = H(M) - H(M|Q_w) = \log_2 M - \log_2 w = \log_2 \frac{M}{w}
\]

and the MaxL is
\[
\rho^{(MaxL)}(P_{Q_w|M}) = \log_2 \sum_{q \in Q_w} \max_{m \in [M]} \frac{1}{M-1} \left(\frac{M}{w-1}\right) = \log_2 \frac{M}{w},
\]

This completes the achievability proof of Theorems 2 and 3.

We remark that the presented capacity-achieving WPIR scheme can be seen as a generalization of the basic WPIR scheme \(\varphi^{basic}\). If \(M/\eta = w \in N,\) since \((D, \tilde{\rho}) = (w, 0)\) is achievable for a \(w\)-file single-server PIR scheme, from Theorem 1 it follows that the pair \((D(\varphi^{basic}), \rho^{(basic)}) = (w, \log_2 w)\) is also achievable by \(\varphi^{basic}\) for both the MI and MaxL privacy metrics.

VII. CONVERSE OF THEOREM 2

Consider a single-server WPIR scheme where the leakage at the server is measured by \(I(M; Q).\) A general converse (upper bound) for Theorem 2 can be derived from the download-leakage function of a given leakage constraint \(\rho,\) or equivalently, from the leakage-download function of a given download cost constraint \(D.\) Here, we will use a known lower bound on the rate-distortion function to prove the converse by focusing on the leakage-download function. Similar to (3), the leakage-download function can be formulated by the convex optimization problem
\[
\begin{align*}
\text{minimize} & \quad I(M; Q) \quad (9a) \\
\text{subject to} & \quad E_{P_Q}[L(Q)] \leq D. \quad (9b)
\end{align*}
\]

The minimization is taken over the set of all conditional distributions \(P_{Q|M}\) such that (9b) is satisfied, namely the set
\[
F_D \triangleq \left\{ P_{Q|M} : \sum_q P_{Q|M}(q)L(q) = \sum_q \sum_{m \in [M]} P_{Q|M}(q|m)L(m, q) \leq D \right\}.
\]

The proof consists of two parts. First, we show a lower bound to (9) by defining a new RV that is a function of \(Q.\) To facilitate the exposition, we introduce the following notation.

- For any subset \(\mathcal{M} \subseteq [M],\) we define \(\hat{Q}^\mathcal{M}\) to be the set of queries that are designed to recover the files \(X^{(m)}, m \in \mathcal{M},\) i.e., \(\hat{Q}^\mathcal{M} \triangleq \{q \in Q : H(X^{(m)}|A, Q = q, M = m) = 0, \forall m \in \mathcal{M}\}.\) Furthermore, define
\[
Q^\mathcal{M} = \hat{Q}^\mathcal{M} \bigcap \left( \bigcup_{\mathcal{M} \subseteq [M], \mathcal{M} \neq \mathcal{M}} \hat{Q}^{\mathcal{M}} \right).
\]

The set \(Q^\mathcal{M}\) contains all queries that are designed to recover all files in \(\mathcal{M},\) but no more. Note that if \(q \in Q^\mathcal{M}\) but \(m \notin \mathcal{M},\) then \(P_{Q|M}(q|m) = 0.\)

- A binary length-\(M\) indicator vector \(1_M = (u_1, \ldots, u_M)\) of a subset \(\mathcal{M} \subseteq [M]\) is defined as
\[
u_m = \begin{cases} 
1 & \text{if } m \in \mathcal{M}, \\
0 & \text{otherwise}.
\end{cases}
\]

\(Q\) is a RV that is induced by conditional distribution \(P_{Q|M}.\) Let us further define a new RV \(U\) as
\[
U(q) = 1_M, \quad \text{for } q \in Q^\mathcal{M}, \quad \forall \mathcal{M} \subseteq [M]. \quad (10)
\]

Note that the mapping in (10) is well-defined since the query sets \(Q^\mathcal{M}\) are disjoint and their union is equal \(Q,\) i.e., they constitute a partition of \(Q.\) This leads to the conditional PMF
\[
P_{U|M}(u|m) = \begin{cases} 
\sum_{q \in Q^\mathcal{M}} P_{Q|M}(q|m) & \text{if } m \in \chi(u), \\
0 & \text{otherwise},
\end{cases} \quad \text{for } P_{U|M} \in P_{ret} \text{ (the set of all PMFs that satisfy (1))}.
\]
We then design a WPIR scheme where the queries are generated according to $P_{U|M}$ and the normalized answer-length function is constructed for any $u \in \{0,1\}^M$ by

$$L(m,u) = \begin{cases} w_H(u) & \text{if } m \in \chi(u), \\ \infty & \text{otherwise,} \end{cases}$$

where an infinite length for a given file index $m$ and query $u$ indicates that the $m$-th file is never retrieved by the designed query $u$.

Accordingly, we define the set

$$\mathcal{P}_D = \left\{ P_{Q|M} : \sum_{u \in \{0,1\}^M} \sum_{q \in Q^{\chi(u)}} \sum_{m \in [M]} P_M(m)P_{Q|M}(q|m)L(m,u) \leq D \right\}.$$  

Given an arbitrary $P_{Q|M} \in \mathcal{F}_D$, since

$$\begin{align*}
\sum_{u \in \{0,1\}^M} \sum_{m \in [M]} P_M(m)P_{U|M}(u|m)L(m,u) &= \sum_{u \in \{0,1\}^M} \sum_{m \in [M]} P_M(m) \sum_{q \in Q^{\chi(u)}} P_{Q|M}(q|m)L(m,u) \\
&\leq \sum_{u \in \{0,1\}^M} \sum_{q \in Q^{\chi(u)}} \sum_{m \in [M]} P_M(m)P_{Q|M}(q|m)L(m,q)
\end{align*}$$

(12)

where (12) holds because for any $q \in Q^{\chi(u)}$, the length of the normalized answer should be larger than the total size of the retrieved file, i.e., it satisfies $L(m,q) \geq w_H(u)$. It follows that $P_{Q|M}$ also lies in $\mathcal{P}_D$, and hence $\mathcal{F}_D \subseteq \mathcal{P}_D$. The intuition behind this fact is that the function $L(m,u)$ defines the minimum required lengths of answers for a valid single-server WPIR scheme, thus we have more choices of conditional PMFs $P_{Q|M}$ in $\mathcal{P}_D$.

To simplify the notation, in the following we use $p(q|m)$ and $p(u|m)$ to denote the conditional PMFs $P_{Q|M}$ and $P_{U|M}$, respectively. If we take the minimization over $\mathcal{P}_D$, which is a candidate set that is larger than $\mathcal{F}_D$, we have

$$\begin{align*}
\min_{p(q|m) \in \mathcal{F}_D} I(M;Q) &\geq \min_{p(q|m) \in \mathcal{P}_D} I(M;Q) \\
&\geq \min_{p(u|m) : I(L(M,U)) \leq D} I(M;U),
\end{align*}$$

(13)

where (13) follows directly from the data processing inequality. Therefore, a lower bound to the original leakage-download function is given.

In the second part of the proof, we show that (13) admits a closed-form expression by using a useful result from rate-distortion theory, a lower bound on the rate-distortion function.$^1$ This lower bound on the rate-distortion function is adapted to (13), and is re-stated as follows.

$^1$The proof of this lower bound is based on the Karush–Kuhn–Tucker (KKT) optimality conditions, see the details in [30], [31].

**Lemma 3** ([30, Th. 9.4.1], [31, Th. 10.19]). Define

$$\rho(\mu)(D) \triangleq \min_{p(u|m) : I(L(M,U)) \leq D} I(M;U),$$

where $D \in [M]$. Then,

$$\rho(\mu)(D) \geq H(M) + \sum_{m \in [M]} P_M(m) \log_2 \nu_m - \lambda D$$

(14)

for an arbitrary choice of $\lambda > 0$ and for any $\nu_m, m \in [M]$, satisfying

$$\sum_{m \in [M]} \nu_m 2^{-\lambda L(m,u)} \leq 1, \quad u \in \{0,1\}^M.$$  

(15)

We remark again that the length function $L(m,u)$ is equal to $w_H(u)$ for all $m \in \chi(u)$. By using the condition in (15) for each $u$ with $w_H(u) = 1$, we obtain

$$\sum_{m \in [M]} \nu_m 2^{-\lambda L(m,u)} = \sum_{m \in \chi(u)} \nu_m 2^{-\lambda w_H(u)} = \nu_m 2^{-\lambda w_H(u)} \leq 1, \quad m \in \chi(u).$$

This implies that for any $m \in [M], \nu_m 2^{-\lambda w_H(u)} \leq 1$, and hence by symmetry, we can simply assume that

$$\nu_m = \nu, \quad \forall m.$$  

Next, we apply (15) for all $u \in \{0,1\}^M$:

$$\nu \cdot 2^{-1 \lambda} \leq 1 \quad \text{if } w_H(u) = 1,$$

$$2\nu \cdot 2^{-2 \lambda} \leq 1 \quad \text{if } w_H(u) = 2,$$

$$3\nu \cdot 2^{-3 \lambda} \leq 1 \quad \text{if } w_H(u) = 3,$$

$$\vdots$$

$$M\nu \cdot 2^{-M \lambda} \leq 1 \quad \text{if } w_H(u) = M.$$

From the above conditions, we obtain

$$\log_2 \nu \leq \begin{cases} \lambda & \text{if } \lambda > \log_2 \frac{M}{M-1}, \\
2\lambda - \log_2 2 & \text{if } \log_2 \frac{M}{M-1} < \lambda \leq \log_2 \frac{2}{2}, \\
3\lambda - \log_2 3 & \text{if } \log_2 \frac{2}{2} < \lambda \leq \log_2 \frac{3}{3}, \\
\vdots & \text{if } \log_2 \frac{M}{M-1} \leq \lambda \leq \log_2 \frac{M}{M-1}.
\end{cases}$$

Now, taking

$$(\log_2 \nu, \lambda) = \left( w \lambda - \log_2 w, \log_2 \frac{w}{w-1} \right), \quad w \in [2 : M],$$

and substituting this in (14) with $P_M(m) = \frac{1}{M}$, we have

$$\rho(\mu)(D) \geq \log_2 M + \left( w \log_2 \frac{w}{w-1} - \log_2 w \right) - \left( \log_2 \frac{w}{w-1} \right) D$$

$$= \log_2 \frac{M}{w-1} - \left( \log_2 \frac{w}{w-1} \right) (D - (w - 1)),$$

(16)

for $w \in [2 : M]$.

Here, (16) is a linear function of $D$ with slope $-\lambda = -\log_2 \frac{w}{w-1}$, which is strictly increasing in $w \in [2 : M]$. 
Therefore, the best lower bound for \( \rho_{LB}^{(MI)}(D) \) is the piecewise function:

\[
\rho_{LB}^{(MI)}(D) = \begin{cases} 
\log_2 \frac{M}{1} - \left(\log_2 \frac{2}{1}\right)(D - 1) & \text{if } 1 \leq D \leq 2, \\
\log_2 \frac{M}{2} - \left(\log_2 \frac{2}{2}\right)(D - 2) & \text{if } 2 < D \leq 3, \\
\vdots & \\
\log_2 \frac{M}{M-1} - \left(\log_2 \frac{M}{M-1}\right)(D - (M - 1)) & \text{if } M - 1 < D \leq M.
\end{cases}
\]

See also the pictorial illustration in Fig. 3. For instance, the red line \( \log_2 \frac{M}{w-1} - \left(\log_2 \frac{w}{w-1}\right)(D - (w - 1)) \) going through the points \((w - 1, \log_2 \frac{M}{w-1})\) and \((w, \log_2 \frac{M}{w})\) has the largest function values over the interval \([w - 1, w]\).

It can be shown that the pair \((\rho, D^{(MI)}(\rho))\) of Theorem 2 lies on \(\rho_{LB}^{(MI)}(D)\) (details omitted for brevity), which completes the converse proof.

**VIII. CONVERSE OF THEOREM 3**

Following an argumentation similar to the first part of the converse proof of Theorem 2, since the MaxL metric also satisfies the data processing inequality from Lemma 2, the leakage-download function for the MaxL privacy metric can be bounded from below by

\[
\rho^{(\text{MaxL})}(D) \triangleq \min_{p(u|m): \mathbb{E}[L(M,U)] \leq D} \text{MaxL}(M;U). \quad (17)
\]

This in itself is not a convex minimization problem. In this proof, we derive a lower bound to (17) directly. To make the problem tractable, we use the fact that \(2^{\text{MaxL}(M;U)}\) is convex in \(P_{U|M}\) (see Lemma 1).

We know from (2) that maximizing the objective function \(\text{MaxL}(M;U)\) is equivalent to maximizing the function

\[
2^{\text{MaxL}(M;U)} = \sum_{u \in \{0,1\}^M} \max_{m \in [M]} p(u|m).
\]

Moreover, from (11) we know that

\[
p(u|m) = 0, \quad \text{if } u \in \{0,1\}^M, \ m \not\in \chi(u).
\]

Thus, (17) can be re-written as the convex minimization problem

\[
\begin{align*}
\text{minimize} & \quad \sum_{u \in \{0,1\}^M} \max_{m \in \chi(u)} p(u|m) \\
\text{subject to} & \quad \sum_{u \in \{0,1\}^M, m \in [M]} P_M(m)p(u|m)L(m, u) \leq D, \\
& \quad \sum_{u \in \{0,1\}^M} p(u|m) = 1, \quad \forall \ m \in [M]. \quad (18b)
\end{align*}
\]

Furthermore, using the fact that

\[
\sum_{m \in \chi(u)} p(u|m) \leq w_H(u) \max_{m \in \chi(u)} p(u|m), \quad \forall \ u \in \{0,1\}^M,
\]

and

\[
p(1_m|m) = 1 - \sum_{u : \ m \in \chi(u)} p(u|m), \quad \forall \ m \in [M], \quad (19)
\]

the objective function (18a) becomes\(^2\)

\[
\sum_{u} \max_{m \in \chi(u)} p(u|m) = \sum_{u} p(u|\chi(u)) + \sum_{u : \ w_H(u) > 1} \max_{m \in \chi(u)} p(u|m)
\]

\[
= \sum_{m \in [M]} \left[1 - \sum_{u : \ w_H(u) > 1} p(u|m)\right]
\]

\[
+ \sum_{u : \ w_H(u) > 1} \max_{m \in \chi(u)} p(u|m)
\]

\[
\geq M - \sum_{m \in [M]} \sum_{u : \ w_H(u) > 1} p(u|m)
\]

\[
+ \sum_{u : \ w_H(u) > 1} \sum_{m \in \chi(u)} \left(1 - \frac{1}{w_H(u)}\right) p(u|m), \quad (20)
\]

where (20) holds since by expanding the double summation, one can see that

\[
\sum_{m \in [M]} \sum_{u : \ w_H(u) > 1} p(u|m) = \sum_{u} \sum_{m \in \chi(u)} p(u|m).
\]

Similarly, by substituting (19) into the download cost constraint (18b), we get

\[
\sum_{u} \sum_{m \in \chi(u)} p(u|m)L(m, u) = \sum_{u : \ w_H(u) = 1} \sum_{m \in \chi(u)} p(u|m) \cdot 1
\]

\(^\text{2}\)In the following, the ranges of the summations and also the explicit summation variable are sometimes omitted as they are clear from the context.
\[ p(u|m)w_H(u) \]

\[ M - \sum_{u_H(u) > 1 \in \chi(u)} p(u|m) + \sum_{u_H(u) > 1 \in \chi(u)} w_H(u) \sum_{m \in \chi(u)} p(u|m) \]

\[ M + \sum_{w_H(u) > 1 \in \chi(u)} (w_H(u) - 1) \left[ \sum_{m \in \chi(u)} p(u|m) \right] \leq MD. \]

Next, define \( y_w \triangleq \sum_{u : u_H(u) = w} \sum_{m \in \chi(u)} p(u|m) \) for \( w \in [2 : M] \). Because

\[ \sum_{u : u_H(u) = w} (w_H(u) - 1) \sum_{m \in \chi(u)} p(u|m) = \sum_{w=2}^{M} (w - 1) \sum_{u : u_H(u) = w} \sum_{m \in \chi(u)} p(u|m), \]

it can be shown that a lower bound to (18) can be computed from the linear programming (LP) formulation

\[
\text{minimize } \rho_{LB}^{\text{MaxL}}(D) \triangleq M - \sum_{w=2}^{M} \left[ 1 - \frac{1}{w} \right] y_w \quad (21a) \\
\text{subject to } \sum_{w=2}^{M} y_w \leq M, \quad (21b) \\
\sum_{w=2}^{M} (w - 1) \cdot y_w \leq M(D - 1), \quad (21c)
\]

with variables \( y_w, w \in [2 : M] \).

We convert the inequalities in the constraints (21b) and (21c) to equalities by introducing variables \( z_1 \) and \( z_2 \). Thus, we have the constraints as

\[ \sum_{w=2}^{M} y_w + z_1 = M, \]

\[ \sum_{w=2}^{M} (w - 1) \cdot y_w + z_2 = M(D - 1). \]

Now, we further define \( c_w \triangleq 1 - \frac{1}{w} \) for \( w \in [2 : M] \), and set \( c_{M+1} = c_{M+2} = 0 \). In matrix form, we can write the objective function of (21) as

\[ M - c[y_2, \ldots, y_M, z_1, z_2]^T = M - \sum_{w=2}^{M} \left( 1 - \frac{1}{w} \right) y_w \]

and the constraints as \( A[y, z]^T = b^T \), where

\[ A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ 1 & 2 & \cdots & M - 1 & 0 & 1 \end{pmatrix}, \quad b^T = \begin{pmatrix} M \\ M(D - 1) \end{pmatrix}. \]

This LP problem can be solved using the simplex method [32]. The corresponding simplex tableau of (21) is shown in Table II.

Consider a linear system of \( m \) equations and \( n \) variables. Then, a basic solution is a trivial solution to the system of equations where a subset of \( n - m \) variables are zero, and the \( m \) nonzero variables are referred to as the basic variables, while the remaining variables are known as the nonbasic variables. The simplex algorithm is based on the following idea. Given a basic solution for the constraints in an LP, one can arrive at the optimal solution to the LP by sequentially (and selectively) converting nonbasic variables to basic variables.

To solve (21) using the simplex method, we start with a basic solution, where \( z_1 = M, \) \( z_2 = M(D - 1), \) and \( y_w = 0, w \in [2 : M] \). Then, we arrive to the optimal solution by following the cases below.

**Case 1.** \( 1 \leq D \leq 2; \) The variable \( y_2 \) is chosen to be basic. Since \( \arg\min_{y_2} \{ \frac{b_1}{y_2} : a_{1,2} > 0 \} = 2, \) set \( a_{2,2} = 1 \) to be the pivot and update all other rows by pivoting operations. We obtain the new simplex tableau as follows.

**Case 2.** \( w - 1 < D \leq w, w \in [3 : M]; \) We first choose \( y_{w-1} \) to be basic. Since in this case \( \arg\min_{y_{w-1}} \{ \frac{b_{1,2}}{a_{i,2}} : a_{i,2} > 0 \} = 1, \) the pivot is \( a_{1,2} = 1. \) The updated simplex tableau from Table II is given in Table III(a). Secondly, we choose \( y_w \) to be basic. Again, because \( \arg\min_{y_w} \{ \frac{b_{1,2}}{a_{i,2}} : a_{i,2} > 0 \} = 2, \) this means that \( a_{2,2} \) is the pivot. The updated simplex tableau via pivoting is shown in Table III(b). From Table III(b), we see that the relative cost coefficient for \( y'_{w'} \) with \( w', w \in [3 : M], w' \neq w - 1, w, \) is

\[
\frac{1}{w'} - \frac{1}{w - 1} + (w' - 1) \left[ \frac{1}{w - 1} - \frac{1}{w} \right].
\]
Performing some simple calculations, we have
\[
\frac{1}{w'} - \frac{1}{w-1} + (w' + 1 - w) \left( \frac{1}{w-1} - \frac{1}{w} \right) > 0,
\]
either for \( w' > (w - 1) \) or \( w' < w \). Moreover, it is easy to see that the relative cost coefficients for \( z_1 \) and \( z_2 \) are
\[
\frac{1}{w-1} - (w - 2) \left( \frac{1}{w-1} - \frac{1}{w} \right) = \frac{w - 2}{w} > 0
\]
and \( \frac{1}{w-1} - \frac{1}{w} > 0 \), respectively. This implies that the current basic feasible solution is optimal.

Finally, we summarize the optimal solutions and the minimum values of (21) as follows:
\[
y_w' = M(D - 1),
\]
\[
y_w'' = 0, \quad w'' \neq 2, \quad \text{where } w'' \in [3 : M],
\]
\[
y_w'' = M(w - D), \quad \text{if } 1 \leq D \leq 2,
\]
\[
y_w' = M(D - (w - 1)), \quad \text{if } w - 1 \leq D \leq w, \quad w \in [3 : M],
\]
\[
y_w'' = 0, \quad w' \neq w - 1, w, \quad \text{where } w' \in [3 : M],
\]
\[
y_w' = M(D - (w - 1)), \quad \text{if } w - 1 \leq D \leq w, \quad w \in [3 : M],
\]

It can be shown that the pair \( (2^\rho, D^{(MaxL)}(\rho)) \) of Theorem 3 lies on \( 2^{\rho_{LB}}(D) \) (details omitted for brevity), which completes the converse proof.

### IX. Conclusion

We considered relaxing the perfect privacy condition in single-server PIR and presented a scheme for the considered weakly-private scenario, referred to as WPIR. In doing so, we showed that one can trade privacy to gain in terms of download cost. Furthermore, we characterized the information leaked using two different metrics: MI and MaxL. The latter is known to be a more robust metric to measure information leakage. Finally, we derived the single-server WPIR capacity for both the MI and MaxL metrics, and showed that the proposed protocol is capacity-achieving. As a final note, we drew the connection between WPIR and rate-distortion theory.

An interesting direction for future work is the derivation of fundamental bounds on other performance metrics like the upload cost and the access complexity for the single server scenario.

### References

[1] B. Chor, O. Goldreich, E. Kushilevitz, and M. Sudan, “Private information retrieval,” in Proc. 36th Annu. IEEE Symp. Found. Comp. Sci. (FOCS), Milwaukee, WI, USA, Oct. 23–25, 1995, pp. 41–50.

[2] N. B. Shah, K. V. Rashmi, and K. Ramchandran, “One extra bit of download ensures perfectly private information retrieval,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Honolulu, HI, USA, Jun. 29 – Jul. 4, 2014, pp. 856–860.

[3] T. H. Chan, S.-W. Ho, and H. Yamamoto, “Private information retrieval for coded storage,” in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Hong Kong, China, Jun. 14–19, 2015, pp. 2842–2846.

[4] H. Sun and S. A. Jafar, “The capacity of private information retrieval,” IEEE Trans. Inf. Theory, vol. 63, no. 7, pp. 4075–4088, Jul. 2017.

[5] R. Tajeddine, O. W. Gnilke, and S. El Rouayheb, “Private information retrieval from MDS coded data in distributed storage systems,” IEEE Trans. Inf. Theory, vol. 64, no. 11, pp. 7081–7093, Nov. 2018.

[6] K. Banawan and S. Ulukus, “The capacity of private information retrieval from coded databases,” IEEE Trans. Inf. Theory, vol. 64, no. 3, pp. 1945–1956, Mar. 2018.
