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SEQUENCES WITH CONSTANT NUMBER OF RETURN WORDS
LUBOMÍRA BALKOVÁ, EDITA PELANTOVÁ, AND WOLFGANG STEINER

Abstract. An infinite word has the property $R_m$ if every factor has exactly $m$ return words. Vuillon showed that $R_2$ characterizes Sturmian words. We prove that a word satisfies $R_m$ if its complexity function is $(m-1)n+1$ and if it contains no weak bispecial factor. These conditions are necessary for $m=3$, whereas for $m=4$ the complexity function need not be $3n+1$. New examples of words satisfying $R_m$ are given by words related to digital expansions in real bases.

1. Introduction

Recently, return words have been intensively studied in (symbolic) dynamical systems, combinatorics on words and number theory. Roughly speaking, for a given factor $w$ of an infinite word $u$, a return word of $w$ is a word between two successive occurrences of the factor $w$. This can be seen as a symbolic version of the first return map in a dynamical system. This notion was introduced by Durand [5] to give a nice characterization of primitive substitutive sequences. A slightly different notion of return words was used by Ferenczi, Mauduit and Nogueira [8].

Sturmian words are aperiodic words over a bilateral alphabet with the lowest possible factor complexity; they were defined by Morse and Hedlund [11]. Using return words, Vuillon [13] found a new equivalent definition of Sturmian words. He showed that an infinite word $u$ over a bilateral alphabet is Sturmian if and only if any factor of $u$ has exactly two return words. A short proof of this fact is given in Section 5.

A natural generalization of Sturmian words to $m$-letter alphabets is constituted by infinite words with every factor having exactly $m$ return words. This property is called $R_m$. It covers other generalizations of Sturmian words: Justin and Vuillon [10] proved that Arnoux-Rauzy words of order $m$ satisfy $R_m$, Vuillon [14] proved this property for words coding regular $m$-interval exchange transformations.

The factor complexity, i.e., the number of different factors of length $n$, of the two classes of words with property $R_m$ in the preceding paragraph is $(m-1)n+1$ for all $n \geq 0$. Vuillon [14] observed that this condition is not sufficient to describe words satisfying $R_m$, $m \geq 3$: the fixed point of a certain recoding of the Chacon substitution, which has complexity $2n+1$ by Ferenczi [6], has factors with more than 3 return words.

A deeper inspection of the two classes of words with property $R_m$ shows that not only the first difference of complexity is constant, but also that the bilateral order of every factor (see Cassaigne [4] and Section 4) is zero. We show that this condition is indeed sufficient to have the property $R_m$, and provide a less known class of words satisfying this condition. If a word satisfies $R_3$, then we can show that no factor is weak bispecial, i.e., no factor has negative bilateral order. Therefore the words with $R_3$ are characterized by complexity $2n+1$ and the absence of weak bispecial factors.

In Section 6.1, we provide a word satisfying $R_4$ with an even number of factors of every positive length (containing infinitely many weak bispecial factors). Therefore words satisfying $R_m$ do not necessarily have complexity $(m-1)n+1$, and it is an open question whether there exists a nice characterization of words satisfying $R_m$ for $m \geq 4$.

We conclude the article by exhibiting a large class of purely substitutive words satisfying $R_m$. Every word in this class codes the sequence of distances between consecutive $\beta$-integers for some real number $\beta > 1$.

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In this article we focus only on the number of return words corresponding to a given factor of an infinite word. We do not study the ordering of return words in the infinite word, i.e., we do not study derivated sequences (see [3] for the precise definition). Let us just mention here that a derivated sequence of a word with property $R_m$ is again a word satisfying $R_m$. A description of derivated sequences of Sturmian words can be found in [3].

2. Basic definitions

An alphabet $A$ is a finite set of symbols called letters. A (possibly empty) concatenation of letters is a word. The set $A^*$ of all finite words provided with the operation of concatenation is a free monoid. The length of a word $w$ is denoted by $|w|$. A finite word $w$ is called a factor (or subword) of the (finite or right infinite) word $u$ if there exist a finite word $v$ and a word $v'$ such that $u = vvv'$. The word $w$ is a prefix of $u$ if $v$ is the empty word. Analogously, $w$ is a suffix of $u$ if $v'$ is the empty word. A concatenation of $k$ words $w$ will be denoted by $w^k$.

The language $L(u)$ is the set of all factors of the word $u$, and $L_n(u)$ is the set of all factors of $u$ of length $n$. Let $w$ be a factor of an infinite word $u$ and let $a, b \in A$. If $aw$ is a factor of $u$, then we call $a$ a left extension of $w$. Analogously, we call $b$ a right extension of $w$ if $wb \in L(u)$. We will denote by $E_l(w)$ the set of all left extensions of $w$, and by $E_r(w)$ the set of right extensions. A factor $w$ is left special if $|E_l(w)| \geq 2$, right special if $|E_r(w)| \geq 2$ and bispecial if $w$ is both left special and right special.

Let $w$ be a factor of an infinite word $u = u_0u_1 \cdots$ (with $u_j \in A$), $|w| = \ell$. An integer $j$ is called an occurrence of $w$ in $u$ if $u_ju_{j+1} \cdots u_{j+\ell-1} = w$. Let $j, k$, $j < k$, be successive occurrences of $w$. Then $u_ju_{j+1} \cdots u_{k-1}$ is a return word of $w$. The set of all return words of $w$ is denoted by $R(w)$, 

$R(w) = \{ u_ju_{j+1} \cdots u_{k-1} \mid j, k \text{ being successive occurrences of } w \text{ in } u \}$. 

If $v$ is a return word of $w$, then the word $avw$ is called complete return word.

An infinite word is recurrent if any of its factors occurs infinitely often or, equivalently, if any of its factors occurs at least twice. It is uniformly recurrent if, for any $n \in \mathbb{N}$, every sufficiently long factor contains all factors of length $n$. It is not difficult to see that a recurrent word on a finite alphabet is uniformly recurrent if and only if the set of return words of any factor is finite.

The variability of local configurations in $u$ is expressed by the factor complexity function (or simply complexity) $C(n) = \#L_n(u)$. It is well known that a word $u$ is aperiodic if and only if $C(n) \geq n + 1$ for all $n \in \mathbb{N}$. Infinite aperiodic words with the minimal complexity $C(n) = n + 1$ for all $n \in \mathbb{N}$ are called Sturmian words. These words have been studied extensively, and several equivalent definitions of Sturmian words can be found in Berstel [3].

3. Simple facts for return words

3.1. Restriction to bispecial factors. If a factor $w$ is not right special, i.e., if it has a unique right extension $b \in A$, then the sets of occurrences of $w$ and $wb$ coincide, and 

$R(w) = R(wb)$. 

If a factor $w$ has a unique left extension $a \in A$, then $j \geq 1$ is an occurrence of $w$ in the infinite word $u$ if and only if $j - 1$ is an occurrence of $bw$. This statement does not hold for $j = 0$. Nevertheless, if $u$ is a recurrent infinite word, then the set of return words of $w$ stays the same no matter whether we include the return word corresponding to the prefix $w$ of $u$ or not. Consequently, we have 

$R(aw) = aR(w)a^{-1} = \{ ava^{-1} \mid v \in R(w) \}$, 

where $ava^{-1}$ means that the word $v$ is prolonged to the left by the letter $a$ and it is shortened from the right by erasing the letter $a$ (which is always the suffix of $v$ for $v \in R(w)$).

For an aperiodic uniformly recurrent infinite word $u$, each factor $w$ can be extended to the left and to the right to a bispecial factor. To describe the cardinality and the structure of $R(w)$ for arbitrary $w$, it suffices therefore to consider bispecial factors $w$. 

3.2. Tree of return words. It is convenient to consider a tree (or trie) constructed in the following way: Label the root with a factor \( w \), and attach \( \#E_r(w) \) children, with labels \( wb, b \in E_r(w) \). Repeat this recursively with every node labeled by \( v \), except if \( w \) is a suffix of \( v \). If \( u \) is uniformly recurrent, then this algorithm stops, and it is easy to see that the labels of the leaves of this tree are exactly the complete return words of \( w \). Therefore we have

\[
\#R(w) = \#\{\text{leaves}\} = 1 + \sum_{\text{non-leaves } v} (\#E_r(v) - 1).
\]

In particular, if \( w \) is the unique right special factor of its length, then \( \#R(w) = \#E_r(w) \).

A similar construction can be done with left extensions, yielding similar formulae. Since we can restrict our attention to bispecial factors \( w \) by Section 3.1, we obtain the following proposition.

**Proposition 3.1.** Let \( u \) be a recurrent word and \( m \in \mathbb{N} \). Suppose that for every \( n \in \mathbb{N} \) at least one of the following conditions is satisfied:

- There is a unique left special factor \( w \in L_n(u) \), and \( \#E_\ell(w) = m \).
- There is a unique right special factor \( w \in L_n(u) \), and \( \#E_r(w) = m \).  

Then \( u \) satisfies property \( R_m \), i.e., every factor has exactly \( m \) return words.

Recall that Arnoux-Rauzy words of order \( m \) are defined as uniformly recurrent infinite words which have for every \( n \in \mathbb{N} \) exactly one right special factor \( w \) of length \( n \) with \( \#E_r(w) = m \) and exactly one left special factor \( w \) of length \( n \) with \( \#E_\ell(w) = m \). They are also called strict episturmian words. It is easy to see that Sturmian words are recurrent, and we obtain the following corollary to Proposition 3.1.

**Corollary 3.2.** Arnoux-Rauzy words of order \( m \) satisfy \( R_m \), in particular Sturmian words satisfy \( R_2 \).

4. Sufficient conditions for property \( R_m \)

This section is devoted to sufficient conditions for a word \( u \) having the property \( R_m \), but we mention first two evident necessary conditions.

The alphabet \( \mathcal{A} \) of \( u \) must have \( m \) letters since the occurrences of the empty word are all integers \( n \geq 0 \), and its return words are therefore all letters \( u_n \). Furthermore, \( u \) must be uniformly recurrent since every factor has a return word and only finitely many of them.

An important role in our further considerations is played by weak bispecial factors.

**Definition 4.1.** A factor \( w \) of a recurrent word is weak bispecial if \( B(w) < 0 \), where

\[
B(w) = \#\{awb \in L(u) \mid a, b \in \mathcal{A}\} - \#E_\ell(w) - \#E_r(w) + 1
\]

is the bilateral order of \( w \).

Since \( \#\{awb \in L(u) \mid a, b \in \mathcal{A}\} = \sum_{a \in E_\ell(w)} \#E_r(aw) = \sum_{b \in E_r(w)} \#E_\ell(wb) \), the inequality \( B(w) < 0 \) is equivalent to

\[
\sum_{a \in E_\ell(w)} (\#E_r(aw) - 1) < \#E_r(w) - 1
\]
and to
\[
\sum_{b \in E_r(w)} (\#E_r(wb) - 1) < \#E_r(w) - 1.
\]

The bilateral order was defined by Cassaigne \[4\] in order to calculate the second complexity difference. If we set \(\Delta C(n) = C(n + 1) - C(n)\), then we have
\[
\Delta C(n) = \sum_{w \in E_n(u)} (\#E_r(w) - 1) = \sum_{w \in E_n(u)} (\#E_r(w) - 1)
\]
and therefore
\[
\Delta C(n + 1) - \Delta C(n) = \sum_{w \in E_n(u)} \sum_{a \in E_r(w)} (\#E_r(aw) - 1) - \sum_{w \in E_n(u)} (\#E_r(w) - 1)
\]
\[
= \sum_{w \in E_n(u)} \left( \#\{a \in b \in E(w) \mid a, b \in A \} - \#E_r(w) - \#E_r(w) + 1 \right) = \sum_{w \in E_n(u)} B(w).
\]

If \(B(w) = 0\) for all factors \(w\), then the first complexity difference is constant. If no factor is weak bispecial, then \(\Delta C(n)\) is non-decreasing. Since \(\Delta C(0) = \#A - 1\) and \(\#A = m\), we obtain the following lemma.

**Lemma 4.2.** If \(u\) satisfies \(R_m\) and no factor is weak bispecial, then \(\Delta C(n) \geq m - 1\) for all \(n \geq 0\).

The number of return words can be bounded by the following lemmas.

**Lemma 4.3.** If \(u\) is a uniformly recurrent word with no weak bispecial factor, then
\[
\#R(w) \geq 1 + \Delta C(|w|)
\]
for every factor \(w \in E(u)\).

**Proof.** Let \(w \in E(u)\) and denote by \(v_1, v_2, \ldots, v_r\) the right special factors of length \(|w|\). Since no factor is weak bispecial and \(w\) is uniformly recurrent, every \(v_j\) can be extended to the left without decreasing the total amount of “right branching” until \(w\) is reached. More precisely, we have (mutually different) right special factors \(v_j^{(1)}, v_j^{(2)}, \ldots, v_j^{(s_j)}\) with suffix \(v_j\), prefix \(w\) and no other occurrence of \(w\) such that \(\#E_r(v_j^{(1)}) - 1 \leq \sum_{i=1}^{s_j} (\#E_r(v_j^{(i)}) - 1)\). Since all \(v_j^{(i)}\) are nodes in the tree of return words and \(v_j^{(i)} \neq v_j^{(i')}\) if \((j, i) \neq (j', i')\), we can use (1) and obtain
\[
\#R(w) \geq 1 + \sum_{j=1}^{r} \sum_{i=1}^{s_j} (\#E_r(v_j^{(i)}) - 1) \geq 1 + \sum_{j=1}^{r} (\#E_r(v_j) - 1) = 1 + \Delta C(|w|).
\]

**Lemma 4.4.** If \(u\) has no weak bispecial factor and \(\Delta C(n) < m\) for all \(n \geq 0\), then
\[
\#R(w) \leq m
\]
for every factor \(w \in E(u)\).

**Proof.** Let \(v_1, v_2, \ldots, v_r\) denote the right special factors which are labels of non-leave nodes in the tree of return words of \(w\), and \(n = \max_{1 \leq i \leq r} |v_i|\). Since no bispecial factor is weak, every \(v_j\) can be extended to the left to factors of length \(n\) without decreasing the total amount of “right branching”. More precisely, we have (mutually different) right special factors \(v_j^{(1)}, v_j^{(2)}, \ldots, v_j^{(s_j)}\) of length \(n\) with suffix \(v_j\) such that \(\#E_r(v_j) - 1 \leq \sum_{i=1}^{s_j} (\#E_r(v_j^{(i)}) - 1)\). Since \(w\) occurs in \(v_j\) only as prefix, no \(v_j\) can be a proper suffix of \(v_j\). Hence we have \(v_j^{(i)} \neq v_j^{(i')}\) if \((j, i) \neq (j', i')\) and
\[
\#R(w) = 1 + \sum_{j=1}^{r} (\#E_r(v_j) - 1) \leq 1 + \sum_{j=1}^{r} \sum_{i=1}^{s_j} (\#E_r(v_j^{(i)}) - 1) \leq 1 + \Delta C(n) \leq m.
\]

For words with no weak bispecial factors, these three lemmas give a very simple characterization of the property \(R_m\).

**Theorem 4.5.** If \(u\) is a uniformly recurrent word with no weak bispecial factor, then it satisfies \(R_m\) if and only if \(C(n) = (m - 1)n + 1\) for all \(n \geq 0\).
5. Properties $R_2$ and $R_3$

For $m = 2$ and $m = 3$, we can completely characterize the words with property $R_m$.

**Definition 5.1.** Let $v$ be a return word of $w \in \mathcal{L}(u)$. We say that return word $v$ starts with $b$ if $wb$ is a prefix of the complete return word $vw$ and that it ends with $a$ if $aw$ is a suffix of $vw$.

A right special factor $w$ is called **maximal right special** if $w$ is not a proper suffix of any right special factor, i.e., $\sum_{w \in \mathcal{E}_r(w)} (#\mathcal{E}_r(aw) - 1) = 0$. Any maximal right special factor is therefore weak bispecial.

**Lemma 5.2.** If $w \in \mathcal{L}(u)$ is a maximal right special factor such that for any $b \in \mathcal{E}_r(w)$ there exists a unique return word $v \in \mathcal{R}(w)$ starting with $b$, then $u$ is eventually periodic.

**Proof.** Denote the return words of $w$ by $v_1, v_2, \ldots, v_r$, where, w.l.o.g., $v_j$ starts with $b_j$, ends with $a_j$ and $b_{j+1}$ is the only letter in $\mathcal{E}_r(a_jw)$ for $1 \leq j < r$. Then $b_1$ is the only letter in $\mathcal{E}_r(a_1w)$ and $u = p(v_1v_2\ldots v_r)\infty$ for some prefix $p$. \(\square\)

**Corollary 5.3.** If $u$ satisfies $R_2$, then it has no maximal right special factor.

**Proof.** Assume that $w$ is a maximal right special factor. Then the two return words of $w$ have different starting letters, hence $u$ is eventually periodic by Lemma 5.2 and $#\mathcal{R}(wu) = 1$. \(\square\)

On a binary alphabet, the notions “weak bispecial” and “maximal right special” coincide. Therefore Corollaries 3.2, 5.3 and Lemma 4.3 provide a short proof of the following theorem.

**Theorem 5.4** (Vuillon [13]). An infinite word $u$ satisfies $R_2$ if and only if it is Sturmian.

For words with property $R_3$, we need the following lemma.

**Lemma 5.5.** Let $w$ be a weak bispecial factor with a unique $a \in \mathcal{E}_r(w)$ such that more than one return word of $w$ starts with a letter in $\mathcal{E}_r(aw)$, then $\#\mathcal{R}(aw) < \#\mathcal{R}(aw)$.

**Proof.** Any return word of $aw$ has the form $av_1v_2\ldots v_r a^{-1}$ for some $r \geq 1$ and $v_j \in \mathcal{R}(w)$, $1 \leq j \leq r$. If $v_1$ ends with $a$, then $r = 1$. If $v_1$ ends with $a'$, then the assumption of the lemma implies that there is a unique return word of $w$ starting with a letter in $\mathcal{E}_r(a'w)$ (and $\#\mathcal{E}_r(a'w) = 1$). Therefore $v_2$ and inductively the sequence of words $v_2, \ldots, v_r$ are completely determined by the choice of $v_1$. This implies that $\#\mathcal{R}(aw)$ equals the number of return words of $w$ starting with a letter in $\#\mathcal{E}_r(aw)$. Since $w$ is weak bispecial, we have $\#\mathcal{E}_r(aw) < \#\mathcal{E}_r(w)$ and thus $\#\mathcal{R}(aw) < \#\mathcal{R}(w)$. \(\square\)

**Remark.** There are two cases for Lemma 5.5. Either $aw$ is right special or there is more than one return word of $w$ starting with the unique right extension of $aw$.

**Corollary 5.6.** If $u$ satisfies $R_3$, then it has no weak bispecial factor.

**Proof.** Assume that $w$ is a weak bispecial factor.

If $\#\mathcal{E}_r(w) = 3$, then every return word of $w$ starts with a different letter in $\mathcal{E}_r(w)$. Since at most for one $a \in \mathcal{E}_r(w)$, the factor $aw$ is right special, we obtain a contradiction to $R_3$ by Lemma 5.2 or 5.3.

If $\#\mathcal{E}_r(w) = 2$, then $\mathcal{E}_r(aw) = \{b\}$ and $\mathcal{E}_r(a'w) = \{b'\}$. Since, w.l.o.g., two return words of $w$ start with $b$ and one starts with $b'$, we obtain a contradiction to $R_3$ by Lemma 5.3. \(\square\)

By combining Corollary 5.6 and Theorem 1.5, we obtain the following theorem.

**Theorem 5.7.** A uniformly recurrent word $u$ satisfies $R_3$ if and only if $C(n) = 2n + 1$ for all $n \geq 0$ and $u$ has no weak bispecial factor.

**Remarks.**

- The theorem remains true if “weak bispecial” is replaced by “maximal right special”: If $\Delta C(n) = 2$ for all $n \geq 0$, then every factor $w$ with $\#\mathcal{E}_r(w) = 3$ is the unique right special factor of its length, and it cannot be weak bispecial. If $\#\mathcal{E}_r(w) = 2$, then the two notions coincide.
• By symmetry, “weak bispecial” can be replaced by “maximal left special”.
• The condition on weak bispecial factors cannot be omitted. Ferenczi showed that the fixed point \(\sigma^\infty(1)\) of the substitution given by \(\sigma: 1 \mapsto 12, 2 \mapsto 312, 3 \mapsto 3312\), a recoding of the Chacon substitution, has complexity \(2n + 1\) and it contains weak bispecial factors.

### 6. Property \(R_4\)

#### 6.1. A word with complexity \(\neq 3n + 1\)

The following proposition shows that \(C(n)\) need not be \((m − 1)n + 1\) for all \(n \geq 0\) if \(u\) satisfies \(R_m\).

**Proposition 6.1.** Define the substitution \(\sigma\) by

\[
\sigma : 1 \mapsto 13231 \\
2 \mapsto 13231424131 \\
3 \mapsto 42324131424 \\
4 \mapsto 42324
\]

Then the fixed point \(\sigma^\infty(1)\) satisfies \(R_4\).

**Proof.** By Section 3.1, it is sufficient to consider bispecial factors of \(u = \sigma^\infty(1)\). The factors of length 2 are \(\mathcal{L}_2(u) = \{13, 14, 23, 24, 31, 32, 41, 42\}\). For the bispecial factors 1, 2, 23, 2413, the return words are easily determined:

\[
\mathcal{R}(1) = \{13, 1323, 1424, 142324\} \\
\mathcal{R}(2) = \{23, 2314, 2413, 241314\} \\
\mathcal{R}(23) = \{2314, 2314241314, 232413, 232413142413\} \\
\mathcal{R}(2413) = \{241314, 24131423, 24132314, 2413231424\}
\]

The language of \(u\) is closed under the morphism \(\varphi\) defined by \(\varphi : 1 \mapsto 4, 2 \mapsto 3\), since \(\sigma \varphi(w) = \varphi \sigma(w)\) for all factors \(w\). Therefore we have \(\mathcal{R}(\varphi(w)) = \varphi(\mathcal{R}(w))\).

The only factors of the form \(a1b\), \(a, b \in A\) are 314 and 413, hence 1 is a weak bispecial factor, and 1, 4 are the only bispecial factors with prefix or suffix 1 or 4. Similarly, 23 and 32 are weak bispecial factors and no other bispecial factor has prefix or suffix 23 or 32.

The return words of the weak bispecial factor 2413142 are factors of \(\sigma(v)\), with a factor \(v\) of length \(|v| \geq 2\) having prefix 2 or 3, suffix 2 or 3 and no other occurrence of 2 and 3. Since the only possibilities for \(v\) are 23, 2413, 32, 3142, we obtain

\[
\mathcal{R}(2413142) = \{24131423, 2413142324131423, 24131424133214, 2413142413231424132413\}.
\]

All remaining bispecial factors \(w\) have prefix 24132 or 31423 and suffix 23142 or 32413, and therefore a decomposition \(w = t \sigma(v) t'\) with \(t \in \{24, 31\}\), \(t' \in \{132414, 4232413\}\) and a unique bispecial factor \(v\). If \(v\) is empty, then we have w.l.o.g. \(w = 241323142\) and

\[
\mathcal{R}(w) = \{2413231423, 2413231423241314, 2413231423413243, 2413231423413243241324\}.
\]

If \(v\) is not empty, then the uniqueness of \(v\) implies that the set of complete return words of \(w\) is \(t \sigma(\mathcal{R}(v) v) t'\). Since \(v\) is shorter than \(w\), we obtain inductively that all bispecial factors have exactly 4 return words.

#### 6.2. Weak bispecial factors

The preceding example shows that weak bispecial factors cannot be excluded in words \(u\) satisfying \(R_4\). Nevertheless, we can show that the existence of a weak bispecial factor imposes strong restrictions on the structure of the word \(u\).

**Lemma 6.2.** Let \(w\) be a weak bispecial factor of a word \(u\) satisfying \(R_4\). Then there exist factors \(w_1, w_2 \in Aw \cup wA\) and \(v_1, v_2, v_3, v_4\) such that

\[
(2) \quad \mathcal{R}(w_1) = \{v_1v_3, v_1v_4, v_2v_3, v_2v_4\} \quad \text{and} \quad \mathcal{R}(w_2) = \{v_3v_1, v_3v_2, v_4v_1, v_4v_2\}.
\]

**Proof.** Let \(w\) be a weak bispecial factor. Since the problem is symmetric, assume, w.l.o.g., \(\#\mathcal{L}(w) \geq \#\mathcal{L}(\sigma(\cdot))\). We have three different situations:
• \( \#E_r'(w) = 2 \): We have \( E_r'(aww) = 1 \) for all \( a \in E_r'(w) \). For both \( b \in E_r'(w) \), there must be more than one return word of \( w \) starting with \( b \) by Lemma 5.2. Therefore, there exists two return words of \( w \) starting with each \( b \). Let \( w_1 = wb_1, w_2 = wb_2 \).

• \( \#E_r'(w) = 3 \): There exists a unique \( b \in E_r'(w) \) such that two return words of \( w \) start with \( b \). By Lemma 5.3, there exists therefore some \( a \in E_r'(w) \) such that \( \#E_r'(aw) \geq 2 \). Since \( w \) is weak bispecial with \( \#E_r'(w) = 3 \), we have \( \#E_r'(aw) = 2 \) and \( E_r'(a'w) = 1 \) for all other \( a' \in E_r'(w) \). Let \( w_1 = aw, w_2 = wb \).

• \( \#E_r'(w) = 4 \): For every \( b \in E_r'(w) \), there is a unique return word of \( w \) starting with \( b \). By Lemma 5.3, we have \( a_1, a_2 \in E_r'(w) \) with \( \#E_r'(a_1w) = 2 \). Let \( w_1 = a_1w, w_2 = a_2w \).

Consider "complete return words of the set \( \{w_1, w_2\} \)": words which have either \( w_1 \) or \( w_2 \) as prefix, either \( w_1 \) or \( w_2 \) as suffix, and no other occurrence of \( w_1 \) and \( w_2 \). By the definitions of \( w_1 \) and \( w_2 \), there are exactly two such words \( v_1w_{i_1}, v_2w_{i_2} \) with prefix \( w_1 \) and two words \( v_3w_{i_3}, v_4w_{i_4} \) with prefix \( w_2 \).

If \( i_1 = i_2 = 2 \) and \( i_3 = i_4 = 1 \), then \( R_4 \) implies that (4) holds.

If \( i_1 = i_2 = 1 \), then \( w_1 \) has only the two return words \( v_1, v_2 \). If \( i_2 = i_3 = i_4 = 1 \), then the return words of \( w_1 \) are \( v_1, v_3, v_4, v_2 \). Similarly, \( i_3 = i_4 = 2 \) and \( i_1 = i_2 = i_3 = 2 \) are not possible.

The only remaining case is \( i_1 = i_4 = 1, i_2 = i_3 = 2 \). Then the return words of \( w_1 \) are \( v_1 \) and \( v_2v_3v_4, i \in \{1, 2, 3\} \), \( 0 \leq t_1 < t_2 < t_3 \). The return words of \( w_2 \) are \( v_3 \) and \( v_4v_1^{i_1}v_2, i \in \{1, 2, 3\} \), \( 0 \leq s_1 < s_2 < s_3 \).

The return words of \( v_2v_3 \) are therefore of the form \( v_2v_3(v_1^iv_4v_1^{i_4}) \). Let \( S_1 \) be the set of these 4 pairs \((r_i, s_i)\). Similarly, \( S_2 \) be the set of the 4 pairs \((s_i, r_j)\) such that \( v_4v_1^{i_1}v_2v_3^{i_3} \) is a return word of \( v_3w_1 \).

We show that there must be some \( i \in \{1, 2, 3\} \) such that \( (r_i, s_2) \in S_1 \) and \( (r_i, s_3) \in S_1 \), by considering the return words of \( v_1^iv_2v_3w_1 \) and \( v_1^iv_2v_3w_1 \). The return words of \( v_1^iv_2v_3w_2 \) are of the form \( v_1^iv_2v_3v_4v_1^{i_1}v_2v_3^{i_3} \) with \( t \in \{v_3^2v_1^{i_1}v_2v_3^{i_3}, v_1^iv_2v_3^{i_3}, v_3^2v_1^{i_1}v_2v_3^{i_3}\} \), \( i \in \{1, 2, 3\} \) and \( j \in \{2, 3\} \). For these \( t \) and \( r_1, v_2v_3v_4v_1^{i_1}v_2v_3^{i_3} \) is a return word of \( v_3^2w_1 \). If there was no \( r_1 \) with \( (r_i, s_2) \in S_1 \) and \( (r_i, s_3) \in S_1 \), then these words would provide 4 different return words of \( v_3^2w_1 \), with contradicts \( R_4 \) since \( v_1 \) is another return word.

Similarly, we must have some \( i \in \{1, 2, 3\} \) such that \( (s_i, r_2) \in S_2 \) and \( (s_i, r_3) \in S_2 \). By considering the return words of \( v_1^iv_2w_1 \) and \( v_4v_1^{i_1}w_1 \), we obtain as well the existence of some \( i \in \{1, 2, 3\} \) such that \( (s_2, r_i) \in S_2 \) and \( (s_3, r_i) \in S_2 \). Finally, we must also have some \( i \in \{1, 2, 3\} \) such that \( (s_2, r_i) \in S_2 \) and \( (s_3, r_i) \in S_2 \).

Putting everything together, we have two possibilities for \( S_1 \). Either it contains \( (r_1, s_1) \) and no other \( (r_i, s_j) \) with \( i = 1 \) and \( j = 1 \), or \( S_1 = \{(r_1, s_2), (r_1, s_3), (r_2, s_1), (r_3, s_1)\} \). Analogously, \( S_2 \) contains \( (s_1, r_j) \) and no other \( (s_i, r_j) \) with \( i = 1 \) and \( j = 1 \), or \( S_2 = \{(s_1, r_2), (s_1, r_3), (s_2, r_1), (s_3, r_1)\} \).

If \( (r_1, s_1) \in S_1 \) and \( (s_1, r_1) \in S_2 \), then \( v_2v_3v_4w_1 \) has only one return word, \( v_2v_3v_4v_1^{i_1} \). If \( (s_1, r_1) \notin S_1 \) and \( (s_1, r_1) \notin S_2 \), then \( v_2v_3v_4w_1 \) has only two return words, \( v_2v_3v_4v_1^{i_1} \) and \( v_2v_3v_4v_1^{i_1} \). If \( (r_1, s_1) \in S_1 \) and \( (s_1, r_1) \notin S_2 \), then the return words of \( v_2v_3v_4w_1 \) are of the form \( v_2v_3v_4v_1^{i_1}v_2v_3^{i_3}v_4v_1^{i_4} \) with \( (r_i, s_j) \in S_1 \setminus \{(r_1, s_1)\} \), thus there are only three words. Similarly, \( v_4v_1^{i_1}v_2v_3 \) has only three return words if \( (r_1, s_1) \notin S_1 \) and \( (s_1, r_1) \in S_2 \).

This shows that \( i_1 = i_4 = 1, i_2 = i_3 = 2 \) is impossible, and the lemma is proved.

7. Words associated with \( \beta \)-integers

In this section, we describe a new class of infinite words with the property \( R_m \). The language of these words is not necessarily closed under reversal.

Consider the fixed point \( u = \sigma^\infty(0) \) of a primitive substitution of the form

\[
\sigma : \\
0 &\mapsto 0^11 \\
1 &\mapsto 0^{i_2}2 \\
\vdots \\
m - 2 &\mapsto 0^{i_{m-1}}(m-1) \\
m - 1 &\mapsto 0^{i_m}
\]

(3)
for some integers $m \geq 2$, $t_1, t_m \geq 1$ and $t_2, \ldots, t_{m-1} \geq 0$. The incidence matrix of $\sigma$ is a companion matrix of the polynomial $x^m - t_1 x^{m-1} - \cdots - t_m$. Let $\beta > 1$ be the dominant root of this polynomial (the Perron-Frobenius eigenvalue of the matrix). If

$$t_1 \cdots t_m < t_1 \cdots t_n$$

for all $j \in \{2, \ldots, m\}$,

where $\preceq$ denotes the lexicographic ordering, then $\sigma$ is a $\beta$-substitution and $\beta$ is a simple Parry number. It is easy to see that $u$ codes in this case the sequence of distances between consecutive nonnegative $\beta$-integers

$$\mathbb{Z}_\beta^+ = \left\{ \sum_{j=0}^{\infty} x_j \beta^j \mid J \geq 0, x_j \in \mathbb{Z}, x_j \geq 0, x_j \cdots x_0 < t_1 \cdots t_m \text{ for all } j, 0 \leq j \leq J \right\},$$

and a letter $k$ corresponds to the distance $t_{k+1}/\beta + \cdots + t_m/\beta^{m-k}$. (0 corresponds to distance 1.)

**Remark.** The most prominent example of a $\beta$-substitution is the Fibonacci substitution ($m = 2, t_1 = t_2 = 1$), where $\beta$ is the golden mean.

It is not difficult to show that all prefixes of $u$ are left special factors, with all $m$ letters being left extensions (see e.g. Frougny, Masáková and Pelantová [1]). For every factor $w$, the tree of return words constructed by the left extensions (see Section 3.2) contains therefore a node with $m$ children, the shortest prefix of $u$ having $w$ as suffix. The word $u$ is uniformly recurrent since all fixed points of primitive substitutions have this property (Queffélec [2]). Therefore every factor $w$ has at least $m$ return words. If there exists a left special factor which is not a prefix of $u$, then this factor has more than $m$ return words. By Proposition 3.1 we obtain the following proposition.

**Proposition 7.1.** If $u = \sigma^\infty(0)$ for some substitution $\sigma$ of the form (3), then it satisfies $R_m$ if and only if $C(n) = (m-1)n + 1$ for all $n \geq 0$.

Bernat, Masáková and Pelantová [2] characterized the fixed points of $\beta$-substitutions satisfying $\Delta C(n) = m - 1$ for all $n \geq 0$. The techniques of their proof can also be used to construct non-prefix left special factors if $\sigma$ is a substitution of the form (3) which is not a $\beta$-substitution, and their conditions can be reformulated as in the following corollary.

**Corollary 7.2.** If $u = \sigma^\infty(0)$ for some substitution $\sigma$ of the form (3), then it has the property $R_m$ if and only if

- $t_m = 1$
- $t_j \cdots t_{m-1}t_1 \cdots t_{j-1} \preceq t_1 \cdots t_{m-1}$ for all $j \in \{2, \ldots, m-1\}$.

Note that the language of $u$ is closed under reversal if and only if $t_1 = t_2 = \cdots = t_{m-1}$. In this case, $u$ is an Arnoux-Rauzy word of order $m$.

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