On the blow-up of solutions for the unstable sixth order parabolic equation

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Received 3 May, 2012

Abstract. We study the universal blow-up of sixth-order parabolic thin film equation with the initial boundary conditions. We prove that the problem in finite time blow-up will happen, if the initial datum 

\[ u_0 \in C^{6+\alpha}(\Omega) \] with 

\[ \int_{\Omega} (H(u_0) + \frac{1}{2}|\Delta u_0|^2) \, dx \geq 0. \]

And then, we get some nondegeneracy results on blow-up for this problem.

2000 Mathematics Subject Classification: 35K55; 35K90; 76A20

Keywords: blow-up, nondegeneracy, sixth order parabolic equation

1. INTRODUCTION

In this paper, we consider the following initial boundary problem of sixth-order equation

\[
\begin{aligned}
&u_t - \Delta (\Delta^2 u - |u|^{p-1}u) = 0, \quad \text{in } \Omega \times (0, T), \\
&u = \Delta u = \Delta^2 u = 0, \quad \text{on } \partial \Omega \times [0, T), \\
&u = u_0, \quad \text{in } \Omega \times \{0\},
\end{aligned}
\]  

(1.1)

where \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( p > 1 \).

During the past years, only a few works have been devoted to the sixth-order parabolic equation [1,4,5,7].

Recently, Evans, Galaktionov and King [4,5] considered the sixth-order thin film equation containing an unstable (backward parabolic) second-order term

\[ \frac{\partial u}{\partial t} = \text{div} \left[ |u|^n \nabla \Delta^2 u \right] - \Delta (|u|^{p-1}u), n > 0, p > 1. \]

By a formal matched expansion technique, they show that, for the first critical exponent \( p = p_0 = n + 1 + \frac{2}{N} \) for \( n \in (0, \frac{2}{3}) \), where \( N \) is the space dimension, the free-boundary problem with zero-height, zero-contact-angle, zero-moment, and zero-flux conditions at the interface admits a countable set of continuous branches of radially symmetric self-similar blow-up solutions \( u_k(x,t) = (T-t)^{-\frac{N}{4n+2}} f_k(y), \)

\[ y = \frac{x}{(T-t)^{\frac{1}{4n+2}}}, \]

where \( T > 0 \) is the blow-up time.
In fact, when \( n = 0 \), the equation (1.1) is obtained. In this paper we study the universal blow-up and some nondegeneracy results on blow-up of the equation (1.1). Our method about universal finite time blow-up is similar to that of Elliott and Zheng [3] which treats the blow-up problem for Cahn-Hilliard equation. We can show that if the initial datum \( u_0 \in C^{6+\alpha}(\mathbb{R}) \) with \( -\int_{\mathbb{R}} \left( H(u_0) + \frac{1}{2} |\Delta u_0|^2 \right) dx \geq 0 \), then the solution to the above problem (1.1) should blow up in finite time.

We also establish some nondegeneracy results on the blow-up of the problem. We mainly follow the purpose of Giga and Kohn [6] and Cheng and Zheng [2]. More accurately, there is a constant \( \beta > 0 \), depending on \( n, p \) and the constant in the estimates of the fundamental solution to \( u_t - \Delta^3 u = 0 \) (see (3.1) below), such that if \( u \) is a solution of the equation

\[
  u_t - \Delta (\Delta^2 u - |u|^{p-1} u) = 0, \quad \text{on} \quad Q_r = B_r(a) \times [t_1 - r^6, t_1),
\]

where \( 1 < p < 3, a \in \mathbb{R}^n, t_1 \in \mathbb{R} \) and \( 0 < r \leq 1 \), and if

\[
  |u(x,t)| \leq \varepsilon (t_1 - t)^{-\frac{p}{2(p-1)}} \quad \text{for all} \quad (x,t) \in Q_r,
\]

then \( u \) does not blow up at \( (a, t_1) \).

The following sections include our main results. In Section 2, we establish universal finite time blow-up. Section 3 is devoted to the nondegeneracy results on the blow-up.

### 2. Universal Finite Time Blow-up

**Theorem 1.** Assume \( u_0 \in C^{6+\alpha}(\mathbb{R}) \) with \( \int_{\mathbb{R}} \left( H(u_0) + \frac{1}{2} |\Delta u_0|^2 \right) dx \geq 0 \). Then the solution of the problem (1.1) must blow up at a finite time, namely, for some \( T > 0 \)

\[
  \lim_{t \to T} \|u(t)\| = +\infty,
\]

where \( H(u) = -\frac{|u|^{p+1}}{p+1} \).

**Proof.** Let

\[
  F(t) = \int_{\Omega} \left( H(u) + \frac{1}{2} |\Delta u|^2 \right) dx,
\]

then

\[
  \frac{dF(t)}{dt} = \int_{\Omega} \left( -|u|^{p-1} u \varphi(u) u_t + \frac{1}{2} \Delta u \Delta u_t \right) dx
\]

\[
  = \int_{\Omega} \left( -|u|^{p-1} u + \frac{1}{2} \Delta^2 u \right) u_t dx
\]

\[
  = -\int_{\Omega} |\nabla (|u|^{p-1} u + \frac{1}{2} \Delta^2 u)|^2 dx \leq 0.
\]

So

\[
  2 \int_{\Omega} H(u) dx - 2F(0) \leq -\|\Delta u\|^2, \quad (2.1)
\]
where

\[ F(0) = \int_{\Omega} \left( H(u_0) + \frac{1}{2} |\Delta u_0|^2 \right) \, dx. \]

Let \( \phi \) be the unique solution to

\[
\begin{cases}
\Delta \phi = u, & \text{in } \Omega, \\
\nabla \phi = 0, & \text{on } \partial \Omega.
\end{cases}
\]

It is easy to get that

\[
\|\nabla \phi\|^2 \leq C \|\Delta \phi\|^2 \leq C \|u\|^2. \tag{2.2}
\]

Now multiplying (1.1) by \( \phi \) and integrating with respect to \( x \), we obtain

\[
\frac{d}{dt}\|\nabla \phi\|^2 \geq -2 \int_{\Omega} \phi(u)u \, dx - 2 \|\Delta u\|^2 \, dx
\]

\[
\geq 4 \int_{\Omega} H(u) \, dx - 4 F(0) - 2 \int_{\Omega} \varphi(u)u \, dx
\]

\[
= \int_{\Omega} \left( 2 - \frac{4}{p+1} \right) |u|^{p+1} \, dx - 4 F(0)
\]

\[
\geq \frac{2(p-1)}{p+1} \left( \int_{\Omega} u^2 \, dx \right)^{\frac{p+1}{2}} - 4 F(0). \tag{2.3}
\]

Combining (2.2), (2.3) and \( -F(0) \geq 0 \), we have

\[
\frac{d}{dt}\|\nabla \phi\|^2 \geq \frac{2C(p-1)}{p+1} \|\nabla \phi\|^{p+1}. \tag{2.4}
\]

Let \( y(t) = \|\nabla \phi\|^2 \) with \( t \in [0, T) \), then

\[
y'(t) \geq \gamma (y(t))^{\frac{p+1}{2}}, \tag{2.5}
\]

where \( \gamma = \frac{2C(p-1)}{p+1} \). A direct integration of (2.5) then yields

\[
y^{\frac{p+1}{2}}(t) \geq \frac{1}{y^{\frac{1-p}{2}}(0) - \frac{p-1}{2} \gamma t}.
\]

It turns out that the solution of the problem (1.1) will blow up in finite time. The proof of this theorem is completed. \( \square \)

3. Nondegeneracy results on the blow-up

Let \( \Gamma(x,t) \) be the fundamental solution to \( u_t - \Delta^3 u = 0 \). According to [8], we have the follow inequalities:

\[
|D_t^\mu D_x^\nu \Gamma(x,t)| \leq C \, t^{-\frac{1}{2}(\alpha+6\mu+\nu)} \exp \left\{ -\omega \left( \frac{|x|}{t^\frac{1}{2}} \right)^\frac{\delta}{\nu} \right\}, \quad t > 0, \tag{3.1}
\]

where \( C > 0, \omega > 0 \) are constants, and \( \mu, \nu \) are nonnegative integers.
Our purpose in this section is to have some nondegeneracy results on the blow-up. We state that the solution \( u(x,t) \) to blows up at \((a,t_1)\) if it is not locally bounded nearby, i.e., if there is a sequence \( \{(x_k, \tau_k)\} \subset \Omega \times [0,t_1) \) with \((x_k, \tau_k) \to (a,t_1)\) as \( k \to \infty \) such that \( |u(x_k, \tau_k)| \to \infty \).

**Theorem 2.** There is a constant \( \varepsilon > 0 \), depending on \( n \), \( p \) and the constant in \((3.1)\), such that if \( u \) is a solution of the equation
\[
 u_t - \Delta (\Delta^2 u - |u|^{p-1} u) = 0, \quad \text{on} \quad Q_r = B_r(a) \times [t_1-r^6, t_1),
\]
where \( 1 < p < 3, a \in \mathbb{R}^n, t_1 \in \mathbb{R} \) and \( 0 < r \leq 1 \), and if
\[
 |u(x,t)| \leq \varepsilon (t_1-t)^{-\frac{2}{n(n-1)}} \quad \text{for all} \quad (x,t) \in Q_r, \tag{3.2}
\]
then \( u \) does not blow up at \((a,t_1)\).

Next, we introduce the two lemma which will be used in the article and whose proofs can be found in [2] and [6].

**Lemma 1.** For \( 0 < a < 1, \theta > 0 \), and \( 0 < h < 1 \), the integral
\[
 I(h) = \int_h^1 (s-h)^{-a} s^{-\theta} ds,
\]
satisfies
\[
\begin{align*}
(1) \quad I(h) & \leq \left( \frac{1}{1-a} + \frac{1}{a+\theta-1} \right) \quad \text{if} \quad a + \theta > 1, \\
(2) \quad I(h) & \leq \frac{1}{1-a} + |\log h| \quad \text{if} \quad a + \theta = 1, \\
(3) \quad I(h) & \leq \frac{1}{1-a-\theta} \quad \text{if} \quad a + \theta < 1.
\end{align*}
\]

**Lemma 2.** If \( y(t), r(t) \) and \( q(t) \) are continuous functions defined on \([t_0,t_1]\), such that \( y(t) \leq y_0 + \int_{t_0}^t y(s) r(s) ds + \int_{t_0}^t q(s) ds \), \( t_0 \leq t \leq t_1 \), and \( r(t) \geq 0 \) on \([t_0,t_1]\), then
\[
 y(t) \leq \exp \left\{ \int_{t_0}^t r(\tau) d\tau \right\} \left[ y_0 + \int_{t_0}^t q(\tau) \exp \left\{ -\int_{t_0}^t r(\sigma) d\sigma \right\} d\tau \right].
\]

Then, we began to prove the main Theorem 2.

**Proof.** Without loss of generality, we may assume \( a = 0 \) and \( t_1 = 0 \). By scaling, it is sufficient to consider the case \( r = 1 \). In the fact, if \( u \) satisfies the assumptions of the theorem with \( r < 1 \), then \( u_r(x,t) = r^{\frac{4}{n-1}} u(rx, r^6 t) \) satisfies them with \( r = 1 \) (using the same \( \varepsilon \)), and clearly \( u_r \) blow up at \((0,0)\) if \( u \) does.

Let \( \phi \) be a smooth function supported on \( B_1(0) \) such that \( \phi \equiv 1 \) on \( B_1^2(0) \) and \( 0 \leq \phi \leq 1 \). Consider \( \omega = \phi u \); then \( \omega_t - \Delta^3 \omega = g \) where
\[ g = -2\nabla \Delta^2 u \nabla \phi - \Delta^2 u \Delta \phi - \Delta (u \Delta^2 \phi + 4\nabla \Delta u \nabla \phi + 6\Delta u \Delta^2 \phi + 4\nabla \Delta u \Delta \phi) - \phi \Delta (|u|^{p-1} u) \]

The semigroup representation formula for \( \omega \) gives that
\[ \omega(t) = e^{(t+1) \Delta^3} \omega(-1) + \int_{-1}^{t} e^{(t-s) \Delta^3} g(s) ds \quad \text{for} \quad -1 \leq t < 0, \quad (3.3) \]
where \( e^{t \Delta^3} \) is the semigroup associated with the equation \( u_t - \Delta^3 u = 0 \) in \( \mathbb{R}^n \), i.e.,
\[ (e^{t \Delta^3} h)(x) = \int_{\mathbb{R}^n} \Gamma(x - y, t) h(y) dy. \]
Noticing that \( \int_{\mathbb{R}^n} \Gamma(x - y, t) dy = 1 \). It follows that
\[ \|e^{t \Delta^3} h\| \leq \|h\|_{\infty}. \quad (3.4) \]
The (3.1) implies that
\[ |(e^{t \Delta^3} D_i h)(x)| = \left| \int_{\mathbb{R}^n} \Gamma(x - y, t) D_i h(y) dy \right| \]
\[ = \left| \int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} \Gamma(x - y, t) h(y) dy \right| \leq C t^{-\frac{1}{2}} \|h\|_{\infty}, \quad \forall i = 1, 2, \ldots, n. \]
So, we get that
\[ \|e^{t \Delta^3} D_i h\|_{\infty} \leq C t^{-\frac{1}{2}} \|h\|_{\infty}, \quad \|e^{t \Delta^3} D_{ij} h\|_{\infty} \leq C t^{-\frac{3}{2}} \|h\|_{\infty}, \]
\[ \|e^{t \Delta^3} D_{ijk} h\|_{\infty} \leq C t^{-\frac{2}{2}} \|h\|_{\infty}, \quad \|e^{t \Delta^3} D_{ijkm} h\|_{\infty} \leq C t^{-\frac{5}{2}} \|h\|_{\infty}. \quad (3.5) \]
where \( i, j, k, m, q \in \{1, 2, \ldots, n\} \).

Now let \( g = g_1 + g_2 \), where \( g_2 = -\phi \Delta (|u|^{p-1} u) \). As above, we estimate
\[ \left| \int_{-1}^{t} e^{(t-s) \Delta^3} g_2(s) ds \right| \]
\[ \leq \int_{-1}^{t} \left| \int_{\mathbb{R}^n} \Delta(\phi \Gamma(x - y, t - s))(|u|^{p-1} u)(y, s) dy \right| ds \]
\[ \leq \int_{-1}^{t} \left| \int_{\mathbb{R}^n} \Delta(\Gamma(x - y, t - s) \Delta \phi + 2\nabla \Gamma(x - y, t - s) \cdot \nabla \phi)(|u|^{p-1} u) dy \right| ds \]
\[ + \int_{-1}^{t} \left| \int_{\mathbb{R}^n} (\Gamma(x - y, t - s) \Delta \phi + 2\nabla \Gamma(x - y, t - s) \cdot \nabla \phi)(|u|^{p-1} u) dy \right| ds \]
\[ \leq C \int_{-1}^{t} (t-s)^{-\frac{1}{2}} \|\phi u^p\|_{\infty}(s) ds + C \int_{-1}^{t} \|\Delta \phi u^p\|_{\infty}(s) ds \]
\[ + C \int_{-1}^{t} (t-s)^{-\frac{1}{6}} \| \nabla \phi u^P \|_{\infty}(s) ds \]
\[ \leq C \int_{-1}^{t} (t-s)^{-\frac{1}{6}} \| u \|_{\infty}^{-\frac{1}{6}} \| \omega \|_{\infty}(s) ds + C \int_{-1}^{t} \| u^P \|_{\infty}(s) ds \]
\[ + C \int_{-1}^{t} (t-s)^{-\frac{1}{6}} \| u \|_{\infty}(s) ds \]
\[ \leq C e^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\frac{2}{3}} \| \omega \|_{\infty}(s) ds + C e^{p} \int_{-1}^{t} (-s)^{-\frac{2p}{3p-11}} ds \]
\[ + C e^{p} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\frac{2p}{3p-11}} ds. \] (3.6)

Due to our assumption.

On the other hand, it is found similarly that
\[
\left| \int_{-1}^{t} e^{(t-s)A} g(s) ds \right| = \int_{-1}^{t} \int_{\mathbb{R}^n} \Gamma(x-y,t-s)(-2\nabla^2 u \nabla \phi - \Delta^2 u \Delta \phi \\
- \Delta(\Delta \Delta \phi + 4\nabla \Delta u \nabla \phi + 6\Delta u \Delta \phi + 4\nabla u \Delta \phi)) dy ds \leq C \int_{-1}^{t} (t-s)^{-\frac{1}{6}} \| u \|_{\infty}(s) ds \]
\[ \leq C e^{p} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\frac{2p}{3p-11}} ds. \] (3.7)

By (3.2)-(3.4), (3.6) and (3.7), we get that for \(-1 \leq t < 0,\)
\[ \| \omega(t) \|_{\infty} \leq \varepsilon + C e^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\frac{2}{3}} \| \omega \|_{\infty}(s) ds \]
\[ + C e^{p} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\frac{2p}{3p-11}} ds + C \int_{-1}^{t} (t-s)^{-\frac{5}{6}} (-s)^{-\frac{2}{3p-11}} ds \]
\[ \leq \varepsilon + C e^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\frac{2}{3p-11}} ds. \] (3.8)

Due to \(1 < p < 3\) and Lemma (1).

Let \(y(t) = \| \omega(t) \|_{\infty};\) therefore
\[ y(t) \leq \varepsilon + C e(-t)^{-\frac{1}{6}} (-s)^{-\frac{2}{3p-11}} + C e^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\frac{2}{3p-11}} y(s) ds. \] (3.9)

Define \(f(t) = \chi_{[-1,0]}(t)y(t), \forall t < 0.\) We introduce a special maximal function on \((-\infty, 0):\)
\[ (Mf)(t) = \sup_{r > 0} \frac{1}{r} \int_{t-r}^{t} |f(s)| ds, \quad \forall t \in (-\infty, 0). \]
Now $\forall r > 0,$
\[
\int_{-1}^{t} (t - s)^{-\frac{1}{3}} (-s)^{-\frac{2}{3}} y(s) ds = \int_{-\infty}^{t} (t - s)^{-\frac{1}{3}} (-s)^{-\frac{2}{3}} f(s) ds = \int_{t-r}^{t} (t - s)^{-\frac{1}{3}} (-s)^{-\frac{2}{3}} f(s) ds + \int_{-\infty}^{t-r} (t - s)^{-\frac{1}{3}} (-s)^{-\frac{2}{3}} f(s) ds = I_1 + I_2.
\]
We compute these two integrals, respectively.
\[
I_1 \leq (-t)^{-\frac{1}{3}} \int_{t-r}^{t} (t - s)^{-\frac{1}{3}} f(s) ds
\leq (-t)^{-\frac{1}{3}} \sum_{k=0}^{\infty} \int_{t-\frac{r}{2k+1}}^{t} (t - s)^{-\frac{1}{3}} f(s) ds
\leq (-t)^{-\frac{1}{3}} \sum_{k=0}^{\infty} \left( \frac{r}{2k+1} \right)^{\frac{1}{3}} \int_{t-\frac{r}{2k+1}}^{t} f(s) ds
\leq (-t)^{-\frac{1}{3}} \sum_{k=0}^{\infty} \left( \frac{1}{2k+1} \right)^{\frac{1}{3}} r \left( Mf \right)(t)
= Cr^{\frac{2}{3}} (-t)^{-\frac{1}{3}} (Mf)(t),
\]
and
\[
I_2 \leq r^{-\frac{1}{3}} \int_{-\infty}^{t-r} (-s)^{-\frac{2}{3}} f(s) ds
\leq r^{-\frac{1}{3}} \int_{-\infty}^{t} (-s)^{-\frac{2}{3}} f(s) ds = r^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s) ds.
\]
Then,
\[
f(t) \leq \varepsilon + C\varepsilon(-t)^{\frac{1}{6} - \frac{2}{3(2\alpha-1)}} + C\varepsilon^{p-1} \left[ r^{\frac{2}{3}} (-t)^{-\frac{2}{3}} (Mf)(t) + r^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s) ds \right],
\]
for all $r > 0$ and $t \in (-\infty, 0)$.

Let
\[
r = \frac{\int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s) ds}{(-t)^{-\frac{2}{3}} (Mf)(t)},
\]
so we have
\[
f(t) \leq \varepsilon + C\varepsilon(-t)^{\frac{1}{6} - \frac{2}{3(2\alpha-1)}} + C\varepsilon^{p-1} \left( (-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s) ds \right)^{\frac{2}{3}} ((Mf)(t))^{\frac{1}{3}}
\leq \varepsilon + C\varepsilon(-t)^{\frac{1}{6} - \frac{2}{3(2\alpha-1)}} + C\varepsilon^{p-1} (-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s) ds
+ C\varepsilon^{p-1} (Mf)(t).
\]
(3.10)
If we define
\[ g(t) = (-t)^{-\frac{1}{2}} \int_{-1}^{t} (-s)^{-\frac{3}{2}} f(s) ds, \]
then
\[ g'(t) = (-t)^{-1}\left[ \frac{1}{3} (-t)^{-\frac{1}{2}} \int_{-1}^{t} (-s)^{-\frac{3}{2}} f(s) + f(t) \right] \geq 0. \]
Hence \( g(t) \) is increasing in \((\infty, 0)\).

Then we get
\[
\max_{-1 \leq \tau \leq t} f(\tau) \leq \varepsilon + C_\varepsilon (-t)^{\frac{1}{2} - \frac{2}{3(p-3)}} + C_\varepsilon^{p-1} g(t) + C_\varepsilon^{p-1} \max_{-1 \leq \tau \leq t} (Mf)(\tau), \quad \forall t \in [-1, 0), \tag{3.11}
\]
where we have used \( \frac{1}{6} - \frac{2}{3(p-3)} < 0 \) since \( 1 < p < 3 \).

Clearly, \( \max_{-1 \leq \tau \leq t} (Mf)(\tau) \leq \max_{-1 \leq \tau \leq t} f(\tau) \) by our definition of the maximal function. Therefore (3.11) implies that for any \(-1 \leq t < 0\),
\[
\max_{-1 \leq \tau \leq t} f(\tau) \leq \frac{1}{1 - C_\varepsilon^{p-1}} \left[ \varepsilon + C_\varepsilon (-t)^{\frac{1}{2} - \frac{2}{3(p-3)}} + C_\varepsilon^{p-1} (-t)^{-\frac{1}{2}} \int_{-1}^{t} (-s)^{-\frac{3}{2}} f(s) ds \right],
\]
provided that \( C_\varepsilon^{p-1} < 1 \). Especially,
\[
f(t) \leq \frac{1}{1 - C_\varepsilon^{p-1}} \left[ \varepsilon + C_\varepsilon (-t)^{\frac{1}{2} - \frac{2}{3(p-3)}} + C_\varepsilon^{p-1} (-t)^{-\frac{1}{2}} \int_{-1}^{t} (-s)^{-\frac{3}{2}} f(s) ds \right]
\forall t \in [-1, 0).
\]

Then for \( \varepsilon > 0 \) small enough, we obtain
\[
(-t)^{\frac{1}{2}} f(t) \leq 2 \left[ \varepsilon + C_\varepsilon (-t)^{\frac{1}{2} - \frac{2}{3(p-3)}} + C_\varepsilon^{p-1} \int_{-1}^{t} (-s)^{-\frac{1}{2}} f(s) ds \right]
\forall t \in [-1, 0).
\]

Define \( h(t) = (-t)^{\frac{1}{2}} f(t) \); then
\[
h(t) \leq 2\varepsilon + 2C_\varepsilon (-t)^{\frac{1}{2} - \frac{2}{3(p-3)}} + 2C_\varepsilon^{p-1} \int_{-1}^{t} (-s)^{-1} h(s) ds, \tag{3.12}
\]
Applying Lemma(2), we have
\[
h(t) \leq (-t)^{-2C_\varepsilon^{p-1}} \left[ 2\varepsilon + C(p, e) (-t)^{\frac{1}{2} - \frac{2}{3(p-3)}} + 2C_\varepsilon^{p-1} \right]
\leq 2\varepsilon (-t)^{2C_\varepsilon^{p-1}} + C(p, e) (-t)^{\frac{1}{2} - \frac{2}{3(p-3)}}, \quad \forall t \in [-1, 0).
\]

Then \( f(t) \leq 2\varepsilon (-t)^{\frac{1}{2} - 2C_\varepsilon^{p-1}} + C(p, e) (-t)^{\frac{1}{2} - \frac{2}{3(p-3)}}, \forall t \in [-1, 0) \), or
\[
y(t) \leq 2\varepsilon (-t)^{-\frac{3}{2} - 2C_\varepsilon^{p-1}} + C(p, e) (-t)^{\frac{1}{2} - \frac{2}{3(p-3)}}, \quad \forall t \in [-1, 0). \tag{3.13}
\]
Choose \( \varepsilon > 0 \) small enough that \( \frac{1}{3} + 2C \varepsilon^{p-1} < \frac{2}{3(p-1)} \) which is possible since \( 1 < p < 3 \). Define \( \alpha = \max \left\{ \frac{1}{3}, 2C \varepsilon^{p-1}, \frac{2}{3(p-1)} - \frac{1}{6} \right\} \leq \frac{2}{3(p-1)} \), it is easy to find that \( \alpha > \frac{1}{3} \); then (3.13) implies \( y(t) \leq C(p, \varepsilon) \varepsilon(-t)^{-\alpha}, \forall t \in [-1, 0) \). Hence

\[
|u(x, t)| \leq C(p, \varepsilon) \varepsilon(-t)^{-\alpha}, \quad \forall (x, t) \in B_{\frac{1}{2}}(0) \times [-1, 0).
\]  

(3.14)

Now let \( \tilde{\phi} \) be a function supported on \( B_{\frac{1}{2}}(\varepsilon) \) with \( \tilde{\phi} \equiv 1 \) on \( B_{\frac{1}{4}}(0) \) and \( 0 \leq \tilde{\phi} \leq 1 \), and define \( \tilde{\omega} = \tilde{\phi}u \); then we go back to (3.6)-(3.8) and we have that

\[
\|	ilde{\omega}(t)\|_\infty \leq \varepsilon + C \int_{-1}^{t} (t-s)^{-\frac{1}{3}} \|u\|_\infty^{p-1} \|	ilde{\omega}\|_\infty ds + C \int_{-1}^{t} \|u\|_\infty^p ds
\]

\[
+ C \int_{-1}^{t} (t-s)^{-\frac{1}{3}} \|u\|_\infty^{p-1} ds + C \varepsilon \int_{-1}^{t} (t-s)^{-\frac{1}{3}} \|u\|_\infty ds
\]

\[
\leq \varepsilon + C \varepsilon^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\alpha(p-1)} (-s)^{-\alpha} ds + C \varepsilon^p \int_{-1}^{t} (-s)^{-\alpha p} ds
\]

\[
+ C \varepsilon^p \int_{-1}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\alpha} ds + C \varepsilon \int_{-1}^{t} (t-s)^{-\frac{1}{3}} \|u\|_\infty ds
\]

\[
\leq \varepsilon + C \varepsilon^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\alpha p} ds + C \varepsilon^p \int_{-1}^{t} (-s)^{-\alpha p} ds + C \varepsilon \int_{-1}^{t} (t-s)^{-\frac{1}{3}} \|u\|_\infty ds
\]

\[
+ C \varepsilon^p \int_{-1}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\alpha} ds + C \varepsilon \int_{-1}^{t} (t-s)^{-\frac{1}{3}} \|u\|_\infty ds
\]

(3.15)

due to (3.14).

Since \( \frac{1}{3} < \alpha < \frac{2}{3(p-1)} \), we get

\[
\frac{5}{6} - \alpha p > \frac{2}{3} - \alpha p > \frac{1}{6} - \alpha.
\]

Hence by Lemma(1), we obtain

\[
\|	ilde{\omega}(t)\|_\infty \leq \varepsilon + C \varepsilon^{p-1} + C \varepsilon^p (-t)^{\frac{1}{3} - \alpha} \leq (2 + C \varepsilon^{p-1})(-t)^{\frac{1}{3} - \alpha}, \quad \forall t \in [-1, 0).
\]

Which means, for small \( \varepsilon > 0 \),

\[
|u(x, t)| \leq (2 + C \varepsilon^{p-1})(-t)^{\frac{1}{3} - \alpha}, \quad \forall (x, t) \in B_{r_0}(0) \times [-1, 0).
\]  

(3.16)

Iterating the argument finitely many times we can get that there is a number \( 0 < r_0 < \frac{1}{4} \) such that

\[
|u(x, t)| \leq K(-t)^{-\frac{1}{6p}}, \quad \forall (x, t) \in B_{r_0}(0) \times [-1, 0),
\]  

(3.17)

where \( K \) is constant.
Next, we choose another cut-off function \( \hat{\phi} \) supported on \( B_{r_0} \) such that \( \hat{\phi} \equiv 1 \) on \( B_{2r_0} \) and define \( \phi = \hat{\phi} u \). Going back to (3.15) and applying Lemma (1), we have

\[
\| \hat{\phi}(t) \|_{\infty} \leq \varepsilon + C \int_{-1}^{t} (t-s)^{-\frac{1}{2}} \| u \|_{\infty}^{p-1} \| \hat{\phi} \|_{\infty} ds + C \int_{-1}^{t} \| u \|_{\infty}^{p} ds + C \varepsilon \int_{-1}^{t} (t-s)^{-\frac{1}{2}} \| u \|_{\infty} ds
\]

\[
+ C \int_{-1}^{t} (t-s)^{-\frac{1}{2}} \| u \|_{\infty}^{p} ds + C \varepsilon \int_{-1}^{t} (t-s)^{-\frac{1}{2}} \| u \|_{\infty} ds
\]

\[
\leq \varepsilon + CK^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{2}} \| (s) \|_{\hat{\phi}}^{p} ds + CK^{p} \int_{-1}^{t} (t-s)^{-\frac{1}{2}} \| u \|_{\infty} ds
\]

\[
+ CK^{p} \int_{-1}^{t} (t-s)^{-\frac{1}{2}} \| (s) \|_{\hat{\phi}}^{p} ds + CK \int_{-1}^{t} (t-s)^{-\frac{1}{2}} \| u \|_{\infty} ds
\]

\[
\leq \varepsilon + CK^{p-1},
\]

which means that \( |u(x,t)| \leq C \) in \( B_{2r_0} \times [-1,0] \). This completes the proof of the theorem.

\[\Box\]

Using the same argument, we can easily draw the following conclusion.

**Theorem 3.** Suppose \( p \geq 3 \), then for any \( \varepsilon \in (0, \frac{2}{3(p-1)}) \), there is a constant \( \varepsilon > 0 \), depending on \( n \), \( p \) and the constant in (3.1), such that if \( u \) is a solution of the equation

\[ u_t - \Delta (\Delta^2 u - |u|^{p-1} u) = 0, \quad \text{on} \quad Q_r = B_r(a) \times [t_1 - r^6, t_1] \]

where \( a \in \mathbb{R}^n, t_1 \in \mathbb{R} \) and \( 0 < r \leq 1 \), and if

\[ |u(x,t)| \leq \varepsilon (t_1 - t)^{-\frac{2}{3(p-1)}} \quad \text{for all} \quad (x,t) \in Q_r, \]

then \( u \) does not blow up at \( (a,t_1) \).

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