Fermion Bag Approach for Massive Thirring Model at Finite Density

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We consider the 2+1 dimensional massive Thirring model with one flavor at finite density. Two numerical methods, fermion bag approach and complex Langevin dynamics, are used to calculate the chiral condensate and fermion density of this model. The numerical results obtained by fermion bag approach are compared with those obtained by complex Langevin dynamics. They are also compared with those obtained under phase quenched approximation. We show that in some range of fermion coupling strength and chemical potential the sign problem in fermion bag approach is mild, while it becomes severe for the complex Langevin dynamics.

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I. INTRODUCTION

The sign problem remains one of the biggest challenges in many fields, e.g., polymer field theory in condensed matter physics [1], lattice field theory in high energy physics. The usual sampling methods, e.g., Langevin dynamics and Monte Carlo method, fail for the sign problem due to the high oscillation of complex action, where the Boltzmann factor can not be regarded as the probability density. Because of the introduction of fields necessary to decouple repulsive interaction between monomer, the sign problem can not be avoided for polymer field theory [2]. For the lattice field theory in high energy physics, three reasons will always lead to the complex action: (1) grand partition function with finite density; (2) fermion systems; (3) topological terms in the action.

To overcome the sign problem, the complex Langevin (CL) dynamics, which is obtained from the complexification of the Langevin dynamics, was used. The CL is rather successful in XY model [3], Bose gas [4], Thirring model [5], Abelian and Non-Abelian lattice gauge model [6], QCD model [7], and its simplified model including one link U(1) model, one link SU(3) model, QCD model in the heavy mass limit [8], one link SU(N) model [9], SU(3) spin model [10], Polyakov chain model [11]. It was also applied to quantum fields in nonequilibrium [12] and in real time [13][14]. For some range of chemical potential and large fluctuation, the complex Langevin may fail, e.g., the XY model at finite chemical potential for large fluctuation) [15] and in the Thirring model in 0+1 dimension [16]. Unfortunately from early studies of complex Langevin evolutions [17][18][19] until this day, the convergence properties of complex Langevin equations are not well understood. Recently Aarts etc. provided a criterion for checking the correctness of the complex Langevin dynamics [20]. The recent discussion about complex Langevin dynamics can be found in Ref. [21][22][23][24][25][26][27][28][29][30].

Since the partition function is always real, it is possible to find suitable variables to represent this partition function with real action. This is called the dual variable method. It is successfully applied to many models, including Bose gas [31], SU(3) spin model [32], U(1) and Z(3) gauge Higgs lattice theory [33], massive lattice Schwinger model [34], O(3), O(N) and CP(N-1) model [35][36][37][38], fermion bag approach [39], 4-fermion lattice theory, including massless Thirring model [40], Gross-Neveu model [41], Yukawa model [42], Non-Abelian Yang-Mills model [43][44], and its coupling with fermion field [45], lattice chiral model, and Sigma model [46]. For the recent progress of solving the sign problem for the nonrelativistic fermion systems, see Ref. [47][48][49][50][51][52][53].

For the fermion systems, the dual method is called fermion bag approach [39]. This numerical method not only overcome the sign problem for model with small chemical potential, but also a high computational efficiency is achieved for the small or large interaction between fermions. We study the 2+1 dimensional massive Thirring model at finite density, cf. [54], which can be regards as the effective theories of high temperature superconductors and graphene, see e.g., references given in [55]. We have studied this model at finite density in 0+1 dimension and compared the complex Langevin dynamics and fermion bag approach [56]. In this paper we continue to compare the complex Langevin dynamics and the fermion bag approach for the massive Thirring model at finite density in 2+1 dimension.

The arrangement of the paper is as follows. In section II the Fermion bag approach for Thirring model is presented and the chiral condensate and fermion density are obtained. In section III the complex Langevin dynamics is given...
for this model by introducing bosonic variable. In section IV the chiral condensate and fermion density are calculated by these two methods and are compared with each other. Conclusions are given in section V.

II. THIRRING MODEL

The lattice partition function for the massive Thirring model at the finite density in \(d\) dimensional lattice \(\Lambda = \{ x = (x_0, \cdots, x_{d-1}), x_i = 0, \cdots, N - 1, i = 0, \cdots, d-1 \} \) with even \(N\) reads

\[
Z = \int d\bar{\psi}d\psi e^{-S}
\]

where \(d\bar{\psi}d\psi = \prod_{x \in \Lambda} d\bar{\psi}(x)d\psi(x)\) is the measure of the Grassmann fields \(\psi = \{ \psi(x) \}_{x \in \Lambda}\) and \(\bar{\psi} = \{ \bar{\psi}(x) \}_{x \in \Lambda}\). We adopt anti-periodic condition for \(\psi\) and \(\bar{\psi}\) in \(x_0\) direction and periodic condition in the other directions

\[
\psi(x + kN\hat{\alpha}) = (-1)^{k\delta_{\alpha,0}}\psi(x), \bar{\psi}(x + kN\hat{\alpha}) = (-1)^{k\delta_{\alpha,0}}\bar{\psi}(x), x \in \Lambda, k = 0, \pm 1, \cdots,
\]

where \(\hat{\alpha}\) denotes the unit vector in \(\alpha\) direction. The action \(S\) in (1) is

\[
S = \sum_{x,y \in \Lambda} \bar{\psi}(x)D_{x,y}\psi(y) - U \sum_{x \in \Lambda, \alpha = 0, \cdots, d-1} \bar{\psi}(x)\psi(x + \hat{\alpha})\psi(x + \hat{\alpha})
\]

with nonnegative coupling constant \(U\) between fermions. The fermion matrix \(D\), which depends on the fermion mass \(m\) and chemical potential \(\mu\), is given by

\[
D_{x,y} = D(\mu, m)_{x,y} = \sum_{\alpha = 0, \cdots, d-1} \frac{\eta_{\alpha}}{2} (e^{\mu\delta_{\alpha,0}} s_{x,\alpha}^+ \delta_{x+y,\alpha} - e^{-\mu\delta_{\alpha,0}} s_{x,\alpha}^- \delta_{x,y,x+y,\alpha}) + m\delta_{x,y}
\]

(4) with staggered phase factor \(\eta_{x,0} = 1, \eta_{x,\alpha} = (-1)^{x_{\alpha+\cdots+x_{\alpha-1}}}, \alpha = 1, \cdots, d-1\), satisfying \(\eta_{x+\hat{\alpha},\alpha} = \eta_{x,\alpha}\). The boundary condition for \(\bar{\psi}\) and \(\psi\) are accounted for by the sign functions \(s^+\) and \(s^-\)

\[
s_{x,\alpha}^+ = \begin{cases} -1 & \text{if } \alpha = 0 \text{ and } x_0 = N - 1 \\ 1 & \text{Otherwise} \end{cases}, s_{x,\alpha}^- = \begin{cases} -1 & \text{if } \alpha = 0 \text{ and } x_0 = 0 \\ 1 & \text{Otherwise} \end{cases}
\]

(5) with periodic extension for \(s^+\) and \(s^-\) with respect to \(x\) for any direction \(\alpha\). These two sign functions satisfies \(s_{x,0}^+ = s_{x+\hat{\alpha},0}^-\) for any lattice \(x\) and any direction \(\alpha\).

The fermion matrix has two kind of symmetries with respect to \(\mu\) and \(m\) which leads to the symmetry of the determinant \(\det D\) (Note that \(N\) is even)

\[
D(\mu, m)_{x,y} = -D(-\mu, -m)_{y,x} \implies \det D(\mu, m) = \det D(-\mu, -m)
\]

(6) \(\varepsilon_x D(\mu, m)_{x,y} = -D(\mu, -m)_{x,y} \implies \det D(\mu, m) = \det D(\mu, -m)\)

(7) where \(\varepsilon_x = (-1)^{x_0+\cdots+x_{d-1}}\) is the parity of site \(x\). Thus it is sufficient to study the massive Thirring model for \(\mu \geq 0\) and \(m \geq 0\).

The fermion bag approach for the Thirring model is based on the high temperature expansion of the interacting term

\[
\exp \left( U \sum_{x \in \Lambda, \alpha = 0, \cdots, d-1} \bar{\psi}(x)\psi(x)\bar{\psi}(x + \hat{\alpha})\psi(x + \hat{\alpha}) \right)
= \prod_{x \in \Lambda, \alpha = 0, \cdots, d-1, \varepsilon_x = 0} \sum_{k_{\varepsilon_x,\alpha} = 0}^{1} (U \bar{\psi}(x)\psi(x)\bar{\psi}(x + \hat{\alpha})\psi(x + \hat{\alpha}))^{k_{\varepsilon_x,\alpha}}
\]

(8)
Inserting this expansion into the partition function in (11), one has an expansion of $Z$ with respect to $U$

$$Z = \sum_{k=(k_{x,a})} U^k C(x_1, \ldots, x_{2j})$$

(9)

where the summation is taken over all configuration $k$ with $k_{x,a} = 0, 1$ for all two neighboring sites $(x, x + \hat{a})$ and $\sum_{x=0}^{x=1} k_{x,a}$ must be 0 or 1 for all site $x$. If $k_{x,a} = 1$, we say there is a bond connecting $x$ and $x + \hat{a}$; otherwise, there are no bonds connecting them. For a given configuration $k$, for example, there are $j$ bonds $(x_1, x_2), \ldots, (x_{2j-1}, x_{2j})$ connecting $2j$ different sites, and the weight in (9) depending on these $2j$ different sites $\{x_i\}_{i=1}^{2j}$ is

$$C(x_1, \ldots, x_{2j}) = \int d\tilde{\psi}d\psi \exp \left(-\sum_{x,y} \tilde{\psi}(x)D_{x,y}\psi(y)\right)\tilde{\psi}(x_1)\psi(x_1)\cdots\tilde{\psi}(x_{2j})\psi(x_{2j})$$

$$= \det D \det G(\{x_1, \ldots, x_{2j}\}) = \det D(\{\{x_1, \ldots, x_{2j}\}\})$$

(10)

where $G(\{x_1, \ldots, x_{2j}\})$ is a $(2j) \times (2j)$ matrix of propagators between $2j$ sites $x_i$, $i = 1, \ldots, 2j$, whose matrix element are $G(\{x_1, \ldots, x_{2j}\})_{i,j} = D^{-1}_{x_i,x_j}$, $i, j = 1, \ldots, 2j$. The matrix $G(\{x_1, \ldots, x_{2j}\})$ depends on the order of $\{x_1, \ldots, x_{2j}\}$, but its determinant does not. $D(\{\{x_1, \ldots, x_{2j}\}\})$ is the $(N^d - 2j) \times (N^d - 2j)$ matrix which is obtained by deleting rows and columns corresponding to sites $x_1, \ldots, x_{2j}$. The first equality in (10) holds due to the basic Gaussian integration for the Grassmann variables $\{\tilde{\psi}, \psi\}$. In the second equality of (10), we expand the exponential and then integrating the Grassmann variables $\{\tilde{\psi}, \psi\}$. The average number of bonds depends on the interaction strength $U$ between fermions. If $U$ is small, there are few bonds between two neighboring sites, we use $G(\{x_1, \ldots, x_{2j}\})$ to calculate $C(x_1, \ldots, x_{2j})$; Otherwise, $U$ is large and there are many occupied bonds between neighboring sites and thus $D(\{\{x_1, \ldots, x_{2j}\}\})$ is used to calculate $C(x_1, \ldots, x_{2j})$. For any number of different sites $\{x_i\}_{i=1}^{n}$, the function $C = C(x_1, \ldots, x_n; D(\mu, m))$, depends on the fermion matrix $D(\mu, m)$.

Because of the symmetry (4) and (7) of $D$, the function $C$ for any different sites $\{x_i\}_{i=1}^{n}$ have the symmetry (APPENDIX A)

$$C(x_1, \ldots, x_n; D(\mu, m)) = (-1)^n C(x_1, \ldots, x_n; D(\mu, -m)) = C(x_1, \ldots, x_n; D(\mu, m)) = (-1)^n C(x_1, \ldots, x_n; D(\mu, -m))$$

(11)

for any real number $\mu$ and $m$. According to the representation of the partition function in (11), where $n = 2j$ is even, the weight $C$ becomes nonnegative for any $\mu$ and $m$ if $C(x_1, \ldots, x_{2j}; D(\mu, m))$ is nonnegative for any even number of sites $(x_1, \ldots, x_n)$ and for any nonnegative $\mu$ and nonnegative $m$. Unfortunately $C$ is not always positive and thus the sign problem still exist. But we want to justify that the sign problem in the representation of (11) is rather mild.

If $d = 1$, we can prove that for any $\mu > 0$ and $m > 0$, $C(x_1, \ldots, x_n; D(\mu, m)) > 0$ for any number of different sites $\{x_i\}_{i=1}^{n}$ (APPENDIX B). If $\mu = 0$ and $m = 0$, the fermion matrix $D$ is real and anti-Hermitian and thus its eigenvalues comes in complex conjugate pairs with vanishing real part. If $\mu = 0$ and $m > 0$, the determinant of $D$ is positive. $D(\{x_1, \ldots, x_{2j}\}) (\mu = 0$ and $m = 0)$ is also real and anti-Hermitian since the rows and columns corresponding to these sites are deleted. Thus if $\mu = 0$ and $m > 0$, $C(x_1, \ldots, x_n; D(\mu, m)) > 0$ for any configuration $k$ and any dimension $d$. The numerical test shows that the function $C$ in (11) for $d > 1$, is always positive for any configuration $k$ if $\mu > 0$ is close to zero. When $\mu$ is increased, $C$ may be negative for some configurations. The left figure of FIG. 1 shows that the frequency of negative $C$ for two dimensional Thirring model is rather small, which is less than 0.1.

For the three dimensional Thirring model, the frequency of negative $C$ becomes larger (close to 0.35 when $\mu = 2$). Moreover, when $\mu$ is increased, the frequency of negative $C$ also becomes larger. In fact, our simulation shows that this frequency is zero when $\mu \leq 1.3$ for both two and three dimensional Thirring model. Thus the presentation of the partition function in (11) avoid the sign problem at least for small chemical potential.

The chiral condensate is

$$\langle \bar{\psi}\psi \rangle = \frac{1}{N^d} \frac{\partial \ln Z}{\partial m} = \frac{1}{N^d} \langle \frac{\partial m C(x_1, \ldots, x_{2j})}{C(x_1, \ldots, x_{2j})} \rangle$$

(12)

where the average is taken with respect to the weight of the partition function (11). Similar to the calculation of $C$ in (11), the ratio $\partial m C / C$ have two formulae

$$\frac{\partial m C(x_1, \ldots, x_{2j})}{C(x_1, \ldots, x_{2j})} = \sum_{x \neq x_1, \ldots, x_{2j}} \frac{\det G(\{x, x_1, \ldots, x_{2j}\})}{\det G(\{x_1, \ldots, x_{2j}\})}$$

$$= \sum_{x \neq x_1, \ldots, x_{2j}} \frac{\det D(\{x, x_1, \ldots, x_{2j}\})}{\det D(\{x_1, \ldots, x_{2j}\})}$$

(13)
The ratio between the determinant of submatrix $G$ can be obtained by

$$\frac{\det G(\{x, x_1, \cdots, x_{2j}\})}{\det G(\{x_1, \cdots, x_{2j}\})} = G([x]) - G(x, \text{occu\_sites})G(\{x_1, \cdots, x_{2j}\})^{-1}G(\text{occu\_sites}, x)$$

(14)

where $\text{occu\_sites} = (x_1, \cdots, x_{2j})$ denotes $2j$ occupied sites, $G(x, \text{occu\_sites})$ is a row vector with $2j$ components, $D_{x_i^{-1}, x_i}$, $i = 1, \cdots, 2j$. The column vector $G(\text{occu\_sites}, x)$ is the transpose of $G(x, \text{occu\_sites})$. The ratio between the determinant of submatrix $D$ is

$$\frac{\det D(\{x, x_1, \cdots, x_{2j}\})}{\det D(\{x_1, \cdots, x_{2j}\})} = D_{\text{Inv}, x, x}$$

(15)

where $D_{\text{Inv}, x, x}$ is the diagonal element of $D(\{x_1, \cdots, x_{2j}\})^{-1}$ corresponding to site $x \neq x_1, \cdots, x_{2j}$.

Similarly, the fermion density is

$$\langle n \rangle = \frac{1}{N^d} \frac{\partial \ln Z}{\partial \mu} = \frac{1}{N^d} \left( \frac{\partial_{\mu} C(x_1, \cdots, x_{2j})}{C(x_1, \cdots, x_{2j})} \right)$$

(16)

can also be calculated.

The Monte Carlo algorithm based on the partition function in (40) can be found in Ref. [40]. We adopt the following three steps to update the current configuration. Assume that the current configuration $k$ has $n_b$ bonds

$$C = ([x_1, x_2], \cdots, [x_{2n_b-1}, x_{2n_b}])$$

Try to delete a bond, e.g. $[x_{2n_b-1}, x_{2n_b}]$ from the current configuration $C$ to be

$$C' = ([x_1, x_2], \cdots, [x_{2n_b-3}, x_{2n_b-2}])$$

According to the detailed balance

$$W(C)P_{\text{try}}(C \rightarrow C')P_{\text{acc}}(C \rightarrow C') = W(C')P_{\text{try}}(C' \rightarrow C)P_{\text{acc}}(C' \rightarrow C)$$

(17)

where $W(C)$ and $W(C')$ are the weight in the partition function (9) for the configuration $C$ and $C'$, respectively. The try probability from $C(C')$ to $C'(C')$ are

$$P_{\text{try}}(C \rightarrow C') = \frac{1}{n_b}, \quad P_{\text{try}}(C' \rightarrow C) = \frac{1}{n_f}$$

respectively. Here $n_f$ is the number of bonds which can be created from the configuration $C'$. Thus accept probability from $C$ to $C'$ is

$$P_{\text{acc}}(C \rightarrow C') = \frac{n_b W(C')}{n_f W(C)}$$
Try to add a bond, e.g. \([x_{2n_b+1}, x_{2n_b+2}]\) from the current configuration \(C\) to be
\[
C' = ([x_1, x_2], \ldots, [x_{2n_b-1}, x_{2n_b}], [x_{2n_b+1}, x_{2n_b+2}])
\]
The detailed balance is Eq. 17 where
\[
P_{\text{try}}(C \to C') = \frac{1}{n_f}, \quad P_{\text{try}}(C' \to C) = \frac{1}{n_b + 1}
\]
Here \(n_f\) is the number of bonds which can be created from the configuration \(C\). Thus the accept probability from \(C\) to \(C'\) is
\[
P_{\text{acc}}(C \to C') = \frac{n_f}{n_b + 1} \frac{W(C')}{W(C)}
\]
Try to delete a bond, e.g. \([x_{2n_b-1}, x_{2n_b}]\) from the current configuration \(C\) and then add a bond, e.g., \([y_{2n_b-1}, y_{2n_b}]\)
\[
C' = ([x_1, x_2], \ldots, [x_{2n_b-3}, x_{2n_b-2}], [y_{2n_b-1}, y_{2n_b}])
\]
In the detailed balance 17,
\[
P_{\text{try}}(C \to C') = P_{\text{try}}(C' \to C) = \frac{1}{n_f n_b}
\]
Here \(n_f\) is the number of bonds which can be created from the configuration \(C\) where \([x_{2n_b-1}, x_{2n_b}]\) is deleted. Thus the accept probability to move a bond is
\[
P_{\text{acc}}(C \to C') = \frac{W(C')}{W(C)}
\]

### III. COMPLEX LANGEVIN DYNAMICS

The expansion of Eq. 8 can also be written as an integral of bosonic variables \(A_\alpha(x)\) by Hubbard-Stratonovich transformation
\[
\exp \left( \int \frac{1}{2\pi i} \sum_{x,\alpha=0,\ldots,d-1} \bar{\psi}(x) \psi(x) \bar{\psi}(x + \hat{\alpha}) \psi(x + \hat{\alpha}) \right)
\]
\[
= \prod_{x,\alpha} \left( \frac{1}{2\pi i} \right)^{1/2} \int \prod_{x,\alpha} dA_\alpha(x) \exp \left( -\frac{1}{8U} \sum_{x,\alpha} A_\alpha^2(x) \right) \exp \left( -\sum_{x,y} \tilde{\psi}(x) \sum_{\alpha} \frac{i}{2} \left( B_{x,\alpha} A_\alpha(x) \delta_{x+\hat{\alpha},y} + C_{x-\hat{\alpha},\alpha} A_\alpha(y) \delta_{x,y+\hat{\alpha}} \right) \psi(y) \right)
\]
for any two bosonic fields \(B_{x,\alpha}\) and \(C_{x,\alpha}\) satisfying \(B_{x,\alpha} C_{x,\alpha} = 1\).

Choosing
\[
B_{x,\alpha} = e^{\mu \delta_{\alpha,0}} \eta_{x,\alpha}, \quad C_{x,\alpha} = e^{-\mu \delta_{\alpha,0}} \eta_{x,\alpha}
\]
and inserting Eq. 9 to the partition function \(Z\) in Eq. 1 and integrating the Grassmann fields \(\psi, \bar{\psi}\), one has
\[
Z = \int \prod_{x,\alpha} dA_\alpha(x) \exp \left( -\frac{1}{8U} \sum_{x,\alpha} A_\alpha^2(x) \right) \det K = \int \prod_{x,\alpha} dA_\alpha(x) e^{-S_{\text{eff}}}
\]
where we omitted the factor \(\prod_{x,\alpha} \left( \frac{1}{2\pi i} \right)^{1/2} \). The matrix \(K\) depends on \(A\)
\[
K_{x,y} = \sum_{\alpha=0,\ldots,d-1} \frac{\eta_{x,\alpha}}{2} \left( (s^+_{x,\alpha} + iA_\alpha(x)) e^{\mu \delta_{\alpha,0}} \delta_{x+\hat{\alpha},y} - (s^-_{x,\alpha} - iA_\alpha(y)) e^{-\mu \delta_{\alpha,0}} \delta_{x,y+\hat{\alpha}} \right) + m \delta_{x,y}
\]
which is complex, although the fields $A$ is real. The matrix $K$ is reduced to be $D$ in (1) if $A_\alpha$ vanishes. The effective action in (20) is

$$S_{\text{eff}} = \frac{1}{8U} \sum_{x,\alpha} A_\alpha^2(x) - \ln \det K$$

(22)

The complex Langevin dynamics reads

$$A_\alpha(x, \Theta + \Delta \Theta) = A_\alpha(x, \Theta) - \Delta t \frac{\partial S_{\text{eff}}}{\partial A_\alpha(x, \Theta)} + \sqrt{2 \Delta t} \eta_\alpha(x, \Theta)$$

(23)

where $\Theta$ denotes the discrete complex Langevin time, $\Delta \Theta$ is the time step. The real white noise $\eta_{x,\Theta}$ satisfies

$$\langle \eta_\alpha(x, \Theta) \eta_{\alpha'}(x', \Theta') \rangle = \delta_{\alpha,\alpha'} \delta_{x,x'} \delta_{\Theta,\Theta'}$$

The drift force can be written as

$$- \frac{\partial S_{\text{eff}}}{\partial A_\alpha(x)} = - \frac{1}{4U} A_\alpha(x) + \text{Tr} \left( K^{-1} \frac{\partial K}{\partial A_\alpha(x)} \right)$$

(24)

The chiral condensate in (12) is written as

$$\langle \bar{\psi} \psi \rangle = \frac{1}{N^d} \langle \text{Tr}(K^{-1}) \rangle$$

(25)

and the fermion density in (16) reads

$$\langle n \rangle = \frac{1}{N^d} \langle \text{Tr}(K^{-1} \frac{\partial K}{\partial \mu}) \rangle$$

(26)

where the average is taken with respect to weight $e^{-S_{\text{eff}}}$. Note that

$$\text{Tr}(K^{-1} \frac{\partial K}{\partial \mu}) = \sum_{x,y} K^{-1}_{x,y} \left( e^\mu \left( s_{x,0} + i A_0(x) \right) \delta_{x+0,y} + \frac{e^{-\mu}}{2} \left( s_{x,0} - i A_0(y) \right) \delta_{x,y+0} \right)$$

If we can choose instead of (19)

$$B_{x,\alpha} = e^{\mu \delta_{\alpha,0}} \eta_{x,\alpha} s_{x,\alpha}^+, \quad C_{x,\alpha} = e^{-\mu \delta_{\alpha,0}} \eta_{x,\alpha} s_{x+\hat{a},\alpha}^-$$

satisfying $B_{x,\alpha} C_{x,\alpha} = 1$, the partition function $Z$ can also be written as Eq. (20), where the matrix $K$ is replaced by

$$\hat{K}_{x,y} = \sum_{\alpha=0,\ldots,d-1} \frac{\eta_{x,\alpha}}{2} \left( s_{x,\alpha}^+ (1 + i A_0(x)) e^{\mu \delta_{\alpha,0}} \delta_{x+\hat{a},y} - s_{x,\alpha}^- (1 - i A_0(y)) e^{-\mu \delta_{\alpha,0}} \delta_{x,y+\hat{a}} \right) + m \delta_{x,y}$$

(27)

IV. SIMULATION RESULTS

The implementation of fermion bag approach and complex Langevin dynamics can be found in [56]. We use the $\Gamma$ method to estimate the error for the samples in each Monte Carlo simulation or complex Langevin dynamics [58]. The following simulation results are given for one and three dimensional Thirring model with fixed $N = 8$, $m = 1$ but with different coupling strength $U$ and different chemical potential $0 \leq \mu \leq 2$.

FIG.2 and FIG.3 shows the comparison of the chiral condensate and fermion density obtained by fermion bag approach (FB) and by complex Langevin dynamics (CL) for different chemical potential $\mu$ and coupling strength $U$. Both these averages agree with each other very well by these two numerical methods. The statistic error are almost invisible in FIG.2 and FIG.3. When the coupling strength $U$ is increasing, e.g., $U = 0.25$, the chiral condensate and fermion density obtained by FB and by CL are quite different for the intermediate values of chemical potential $\mu$, as shown in FIG.4. One reason of this difference is related to the severeness of the sign problem, which can be
FIG. 2: Comparison of chiral condensate and fermion density obtained by complex Langevin dynamics and by fermion bag approach. In fermion bag approach, the sampling starts after $1 \times 10^6$ Monte Carlo step and finished after $1 \times 10^7$ steps. Two subsequent samples are separated by $10 \times 3 \times N^3$ Monte Carlo steps. In complex Langevin dynamics, $\Delta \Theta = 0.001$, the equilibrium Langevin time $t_{eq} = 20$, the sampling end time $t_{end} = 40$. Two subsequent samples are separated by 10 complex Langevin steps.

FIG. 3: Parameters are the same with those in FIG. 2 except $U$.

FIG. 4: Parameters are the same with those in FIG. 2 except $U$. FB-Fermion Bag, CL-Complex Langevin, pq-phase quenched.
measured by the phase \( \langle e^{i\phi}_{pq} \rangle = Z/Z_{pq} \). The sign problem is rather severe for CL, while it is still mild for FB if \( 1 \leq \mu \leq 2 \). We thus compare the results obtained by these two numerical methods under the phase quenched approximation (APPENDIX D). The chiral condensate and fermion density agree with each other for FB and this method under phase quenched approximation (FB(pq)). While these agreement can not be achieved for CL and the complex Langevin dynamics under phase quenched approximation (CL(pq)). The severity of the sign problem by both approaches is shown in FIG.4. Because the determinant \( \det(K) \) of \( K \) becomes too large if \( \mu > 1.6 \), we just calculate the phase \( \langle e^{i\phi}_{pq} \rangle \) by CL for \( 0 \leq \mu \leq 1.6 \). We also calculate this phase by FB for different \( U \) and different chemical potential \( 0 \leq \mu \leq 2 \). For \( U = 0.25 \), the phase \( \langle e^{i\phi}_{pq} \rangle \) is almost very close to 1 for FB in \( 0 \leq \mu \leq 2 \). Thus the sign problem is almost overcome and this can explain why the results obtained by FB agrees with those obtained under the quenched approximation (FB(pq)) (See FIG.4). The sign problem for FB becomes severe during the intermediate range of chemical potential, as shown for \( U = 0.25, 1.0, 2.0 \). Moreover this range shift to larger chemical potential if \( U \) is increased. For CL with \( U = 0.25 \), the phase drops very fast to zero if \( \mu > 0.6 \) and is very close to zero for \( \mu \geq 1 \). This also explain the difference between those obtained by CL and by CL(pq) in FIG.4.

As shown in FIG.5 the (real part of) phase drops rapidly in the intermediate value of \( \mu \) (0.6 \( \leq \mu \leq 1.2 \)) for CL (\( U=0.25(\text{CL}) \)). Although the statistical error of the chiral condensate and fermion density in this range of \( \mu \) is larger than those for \( \mu < 0.6 \) or \( \mu > 1.2 \), the statistical error in the whole range of \( \mu \) is almost invisible in FIG.4. In FIG.6 we compared the chiral condensate obtained by CL with and by CL with the exact result for one dimensional Thirring model with the same parameters \( 5 \). The chiral condensate obtained by CL agrees with the exact result in the whole range of \( \mu \), while the chiral condensate by CL is slightly smaller than the exact result in the intermediate value of \( \mu \), where the phase \( Z/Z_{pq} \) drops rapidly from 1 to 0 for CL. These results is quite similar with those in the left figure of FIG.4, where the chiral condensate obtained by CL is smaller than those obtained by FB in the intermediate value of \( \mu \). The statistical error for CL in left figure of FIG.6 is smaller than those in FIG.5.

Our calculation for the chiral condensate and fermion density in one and three dimensional Thirring model at finite density by CL quantitatively agree with those obtained by Pawlowski etc. \[10 \]. Compared with the lower figure of Figure 3 in Ref.\[5 \], where \( \beta = 1 = 1/(4U) \), i.e., \( U = 0.25 \), our result by CL is more close to those obtained by FB in the left figure of FIG.4. When \( \mu = 1 \), the chiral condensate are 0.305 \pm 0.0027 by CL and 0.356 \pm 0.0011 by FB, respectively in the left figure of FIG.4 while it is 0.25 in lower figure of Figure 3 of Ref.\[5 \]. The statistical error is also almost invisible in Figure 3 of Ref.\[5 \] in the intermediate value of \( \mu \) where the phase drops rapidly in this range as shown in Figure 4 of Ref.\[5 \]. We can also compare the chiral condensate of one dimensional Thirring model in FIG.4 with Figure 5(b) in Ref.\[10 \]. Our result in FIG.5 by CL is better than those in Figure 5(b) of \[10 \]. For example, at \( \mu = 1 \), the chiral condensate obtained by CL is 0.27 \pm 0.03 and the exact is 0.293 in FIG.5 while it is 0.14 \pm 0.023 in Figure 5(b) in Ref.\[10 \]. Moreover the statistical error in Figure 5(b) of Ref.\[10 \] are larger than those (e.g., Figure 3 in Ref.\[10 \]) in three dimensional Thirring model at finite density, which is quite similar to the statistical error in our calculation by CL for one and three dimensional Thirring model.

The discussion above shows that the difference of chiral condensate obtained by CL and by FB in the intermediate value of \( \mu \) is definitely related to the fast decay of real part of phase \( \langle e^{i\phi}_{pq} \rangle \), i.e., the severity of the sign problem, although the statistical error is small as shown in FIG.4. According to Ref.\[10 \] the quantity

\[
\langle LO \rangle \equiv \left\langle \sum_{x,\mu} \frac{d}{dA_\mu(x)} \left( \frac{dS_{\text{eff}}}{dA_\mu(x)} \right) \frac{d}{dA_\mu(x)} O(A) \right\rangle
\]

(28)

should vanish for any holomorphic function \( O(A) \) if CL works. We choose the observable (the chiral condensate) \( O(A) = -\frac{1}{2N} \text{Tr}(K^{-1}) \) for \( \mu = 1, m = 1 \) and \( N = 8 \). In the one dimensional case, \( \langle LO \rangle \) is 0.0137 \pm 0.00708 if \( U = 0.0025 \) and becomes \(-5.88 \pm 7.33 \) if \( U = 0.16 \). In the three dimensional case, \( \langle LO \rangle \) is \(-1.207 \pm 0.0025 \) if \( U = 0.0025 \), \( 61.6 \pm 64.02 \) if \( U = 0.1 \) and \(-206.3 \pm 420.7 \) if \( U = 0.25 \). Thus \( \langle LO \rangle \) becomes large if \( U \) is increased and the chiral condensate by CL for \( \mu = 1 \) in the left figure of FIG.4 is not reliable.

Finally we also compared the chiral condensate obtained by FB and by CL for one dimensional Thirring model with parameters \( U = 10, m = 1 \) and \( N = 8 \) (Figure 5 in \[10 \]). FB recover the exact result for large coupling strength \( U = 10 \) for the chemical potential \( 0 \leq \mu \leq 2 \) while the result obtained by CL is totally wrong. This is because there is no sign problem in FB in one dimensional Thirring model, while the sign problem is very severe in CL.

In the heavy fermion limit

\[
m \to \infty, \quad \mu \to \infty, \quad \zeta \equiv (2m)^{-1} e^{i\theta} \text{ fixed}
\]

the exact solution is known \[3 \], which does not depend on \( U \) (APPENDIX C). FIG.4 shows the comparison between the condensate calculated by FB and by CL with the exact solution for different coupling strength \( U \) in this limit. The results obtained by FB agree with the exact result for the different coupling strength \( U \). The results obtained by CL agree with the exact result only when \( U \) is small, e.g., \( U = 1/12 \). When \( U \) is increased, e.g., \( U = 0.25 \), the chiral condensate obtained by CL is less than the exact result in the intermediate range of chemical potential \( 4.8 \leq \mu \leq 5.6 \), which was also found in Ref.\[3 \].
FIG. 5: The real part of phase $\langle e^{i\varphi} \rangle_{pq}$ obtained by fermion bag approach for $U = 0.25, 1.0, 2.0, 5.0$ and by complex Langevin dynamics for $U = 0.25$ (U=0.25(CL)).

FIG. 6: Comparison between Fermion bag approach, complex Langevin dynamics and exact solution for one dimensional Thirring model (See Ref. 56).

FIG. 7: The chiral condensate in the heavy quark limit for coupling strength $U = 0.25$ (Left) and $U = 1/12 = 0.08333$ (Right). Parameters are the same with those in FIG. 2 except $U$ and $m$. 
V. CONCLUSIONS

The three dimensional massive Thirring model at finite density are solved by two numerical methods: fermion bag approach and complex Langevin dynamics. Two average quantities, chiral condensate and fermion density, are calculated and are compared by these numerical methods. If the fermion coupling strength $U$ is small, these averages obtained by fermion bag approach agree with those obtained by complex Langevin dynamics. When $U$ and chemical potential are increasing, the sign problem for complex Langevin becomes severe, the results obtained by complex Langevin dynamics are quite different with those obtained under the phase quenched approximation. For the parameters, where the sign problem becomes severe for complex Langevin dynamics, the sign problem for the fermion bag approach is still mild and thus the result obtained by fermion bag approach are reliable for these model parameters. Moreover, in the heavy quark limit, the fermion bag approach can recover the exact result for large coupling strength $U$, while the complex Langevin dynamics just recover the exact result for small coupling strength $U$. I believe that these advantages of the fermion bag approach over complex Langevin dynamics can be checked for the other interacting fermion systems with finite density, e.g., Gross-Neveu model, Yukawa model, etc.

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For any different sites \( \{x_i\}_{i=1}^n \),

\[
C(x_1, \cdots, x_n; D(\mu, m)) = \int d\bar{\psi}d\psi \exp \left( -\sum_{x,y} \bar{\psi}(x)D(\mu,m)_{x,y}\psi(y) \right) \bar{\psi}(x_1)\psi(x_1) \cdots \bar{\psi}(x_n)\psi(x_n)
\]

\[
= \int d\bar{\chi}d\chi \exp \left( -\sum_{x,y} \chi(x)D(\mu,m)_{x,y}\bar{\chi}(y) \right) \chi(x_1)\bar{\chi}(x_1) \cdots \chi(x_n)\bar{\chi}(x_n), \quad \bar{\psi} \to \chi, \quad \psi \to \bar{\chi}
\]

\[
= \int d\bar{\chi}d\chi \exp \left( \sum_{x,y} \chi(x)D(-\mu,-m)_{x,y}\bar{\chi}(y) \right) \chi(x_1)\bar{\chi}(x_1) \cdots \chi(x_n)\bar{\chi}(x_n) \quad \text{by} \quad (\ref{eq:6})
\]

\[
= (-1)^n \int d\bar{\chi}d\chi \exp \left( -\sum_{x,y} \bar{\chi}(x)D(-\mu,-m)_{x,y}\chi(y) \right) \bar{\chi}(x_1)\chi(x_1) \cdots \bar{\chi}(x_n)\chi(x_n)
\]

\[
= (-1)^n C(x_1, \cdots, x_n; D(-\mu,-m)) \quad (\text{A1})
\]

In the second equality we used \( d\bar{\psi}d\psi = d\chi d\bar{\chi} = d\bar{\chi}d\chi \) since there are even number of sites. By the symmetry \( \bar{\chi} \) of \( D \),

\[
C(x_1, \cdots, x_n; D(\mu, m)) = \int d\bar{\psi}d\psi \exp \left( \sum_{x,y} \bar{\psi}(x)\varepsilon_x D(\mu,-m)_{x,y}\varepsilon_y\psi(y) \right) \bar{\psi}(x_1)\psi(x_1) \cdots \bar{\psi}(x_n)\psi(x_n)
\]

\[
= (-1)^n \int d\bar{\chi}d\chi \exp \left( -\sum_{x,y} \bar{\chi}(x)D(\mu,-m)_{x,y}\chi(y) \right) \bar{\chi}(x_1)\chi(x_1) \cdots \bar{\chi}(x_n)\chi(x_n)
\]

\[
= (-1)^n C(x_1, \cdots, x_n; D(\mu, -m)) \quad (\text{A2})
\]

where in the second equality we used \( \bar{\psi}(x) = -\varepsilon_x \bar{\chi}(x), \psi(x) = \varepsilon_x \chi(x) \), and thus \( d\bar{\psi}d\psi = d\bar{\chi}d\chi \) due to even number of sites. Combing \( \text{A1} \) and \( \text{A2} \), we obtain \( \text{11} \).
Appendix B: There is no sign problem for $d = 1$

If $d = 1$, the $N \times N$ (even $N$) fermion matrix is

$$D = D(\mu, m) = \begin{pmatrix} -\frac{\mu}{2} & \frac{\mu}{2} & \frac{\mu}{2} & \cdots & \frac{\mu}{2} \\ -\frac{\mu}{2} & -m & \frac{\mu}{2} & \cdots & \frac{\mu}{2} \\ -\frac{\mu}{2} & \frac{\mu}{2} & -m & \cdots & \frac{\mu}{2} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ -\frac{\mu}{2} & \frac{\mu}{2} & \frac{\mu}{2} & \cdots & -m \end{pmatrix}_{N \times N}.$$ 

According to a formula of the determinant [59], the determinant of $D$ is

$$\det D = \frac{e^{N\mu}}{2^N} + \frac{e^{-N\mu}}{2^N} + \text{Tr}(T)$$

where the $2 \times 2$ transfer matrix $T$ is $T = \begin{pmatrix} m & \frac{1}{2} \\ 1 & 0 \end{pmatrix}^N$. Obviously, $\det D > 0$ for any $\mu > 0$ and $m > 0$. Choose $n$ different indices, $1 \leq i_1 < \cdots < i_n \leq N$ and delete $n$ rows and columns corresponding to these $n$ indices from $D$ to obtain $\tilde{D}$. We want to prove that $(N - n) \times (N - n)$ matrix $\tilde{D}$ satisfies $\det \tilde{D} > 0$. This holds because the structure of $\tilde{D}$ is the same with $D$ and thus the determinant of $\tilde{D}$ can be calculated [59], which must be positive. For example, $N = 10$, $n = 2$, $i_1 = 4$, $i_2 = 7$,

$$\tilde{D} = \begin{pmatrix} * & * & | & | & | & | & | & | & * \\ * & * & | & | & | & | & | & | & * \\ | & | & | & | & | & | & | & | \\ | & | & | & | & | & | & | & | \\ | & | & | & | & | & | & | & | \\ * & | & | & | & | & | & | & * \\ | & | & | & | & | & | & | & * \\ | & | & | & | & | & | & | & * \\ * & | & | & | & | & | & | & * \\ | & | & | & | & | & | & | & * \end{pmatrix}$$

Since $C(x_1, \cdots, x_{d-1})$ can be presented by the determinant of the submatrix of $D$, which is nonnegative, the sign problem is avoided for $d = 1$.

Appendix C: Heavy quark limit

Introducing notations $X = (x_1, \cdots, x_{d-1})$, $Y = (y_1, \cdots, y_{d-1})$. The matrix element of $K$ in [27] can be written as

$$\tilde{K}_{(t,X),(t,Y)} = (B_t)_{X,Y}$$

$$= \sum_{\alpha=1,\cdots,d-1} \eta_{x,\alpha} \frac{1}{2} \left( (1 + iA_\alpha(x))\delta_{x,\alpha,y} - (1 - iA_\alpha(y))\delta_{y,\alpha} \right) + m\delta_{x,y}, \quad t = 0, \cdots, N - 1$$

$$\tilde{K}_{(t,X),(t+1,Y)} = s_{x,0}^+ \frac{e^{\mu}}{2} (C_t)_{X,Y}, \quad (C_t)_{X,Y} \equiv (1 + iA_0(x))\delta_{X,Y}, \quad t = 0, \cdots, N - 1$$

$$\tilde{K}_{(t,X),(t-1,Y)} = -s_{x,0}^- \frac{e^{-\mu}}{2} (C_{t-1})_{X,Y}, \quad t = 0, \cdots, N - 1$$
The matrix $\tilde{K}$ is a $N^d \times N^d$ matrix

$$
\tilde{K} = \begin{pmatrix}
B_0 & \frac{\mu}{2}C_0 & \cdots & \frac{\mu}{2}C_{N-1} \\
-\frac{\mu}{2}C_0 & B_1 & \cdots & \frac{\mu}{2}C_{N-2} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\mu}{2}C_1 & \cdots & -\frac{\mu}{2}C_{N-3} & B_{N-2} \\
-\frac{\mu}{2}C_2 & \cdots & -\frac{\mu}{2}C_{N-4} & \cdots & B_{N-1} \\
\end{pmatrix}
$$

In the heavy quark limit

$$m \to \infty, \quad \mu \to \infty, \quad \zeta \equiv (2m)^{-1}e^\mu \text{ fixed}$$

The matrix $\tilde{K}$ becomes

$$(2m)^{-1}\tilde{K}_{x,y} = \zeta(s_{x,0}^+ + iA_0(x))\delta_{x+\hat{a},y} + \delta_{x,y}$$

$$= \begin{pmatrix}
I & \zeta C_0 & 0 \\
0 & I & \zeta C_1 \\
& & \ddots & \ddots & \ddots \\
& & & I & \zeta C_{N-2} \\
& & & 0 & I \\
\end{pmatrix}
$$

The determinant of $\tilde{K}$ satisfies

$$\frac{1}{(2m)^{N^d}} \det \tilde{K} = \det(I + \zeta C_{N-1}C_0 \cdots C_{N-2}) = \prod_X (1 + \xi P_X)$$

where $\xi = \zeta^N$ and $P_X = \prod_t (1 + A_0(t, X))$ is the Polyakov loop starting and ending at the space point $X$. The partition function $Z$ in $[20]$ reads

$$Z = (2m)^{N^d} \int \prod_{x,\alpha} dA_0(x) \exp \left( -\frac{1}{8U} \sum_{x,\alpha} A_0^2(x) \right) \prod_X (1 + \xi P_X)$$

$$= (2m)^{N^d} \left( \frac{1}{2\pi U} \right)^{(d-1)N^d} \int \prod_x dA_0(x) \exp \left( -\frac{1}{8U} \sum_x A_0^2(x) \right) \prod_X (1 + \xi P_X)$$

$$= (2m)^{N^d} \left( \frac{1}{2\pi U} \right)^{(d-1)N^d} \prod_X \int \prod_t dA_0(t, X) \exp \left( -\frac{1}{8U} \sum_t A_0^2(t, X) \right) (1 + \xi P_X)$$

$$= (2m)^{N^d} \left( \frac{1}{2\pi U} \right)^{\frac{d-N^d}{2}} (1 + \xi)^{N^d-1}$$

where in the last equality we used

$$\int \prod_t dA_0(t, X) \exp \left( -\frac{1}{8U} \sum_t A_0^2(t, X) \right) (1 + \xi P_X)$$

$$= \left( \frac{1}{2\pi U} \right)^{\frac{N^d}{2}} + \xi \int \prod_t dA_0(t, X) \exp \left( -\frac{1}{8U} \sum_t A_0^2(t, X) \right) P_X$$

$$= \left( \frac{1}{2\pi U} \right)^{\frac{N^d}{2}} + \xi \int \prod_t dA_0(t, X) \exp \left( -\frac{1}{8U} A_0^2(t, X) \right) (1 + A_0(t, X))$$

$$= \left( \frac{1}{2\pi U} \right)^{\frac{N^d}{2}} + \xi \left( \frac{1}{2\pi U} \right)^{\frac{N^d}{2}}$$

In the heavy quark limit, the chiral condensate and fermion density are

$$\langle \bar{\psi} \psi \rangle = \frac{1}{m(1 + \xi)} , \quad \langle n \rangle = \frac{1}{1 + \xi}$$

respectively, which does not depend on $U$ [3].
**Appendix D: Phase quenched approximation**

The phase quenched approximation to (20) is to replace $\det K$ by its module $|\det K| = \sqrt{\det(KK^\dagger)}$

$$Z_{pq} = \int \prod_{x,\alpha} dA_\alpha(x) \exp \left( -\frac{1}{8U} \sum_{x,\alpha} A^2_\alpha(x) \right) |\det K| = \int \prod_{x,\alpha} dA_\alpha(x) e^{-S_{pq,eff}} \quad (D1)$$

where the effective action is

$$S_{pq,eff} = \frac{1}{8U} \sum_{x,\alpha} A^2_\alpha(x) - \frac{1}{2} \ln \det(KK^\dagger) \quad (D2)$$

Since

$$\frac{\partial}{\partial A_\alpha(x)} \frac{1}{2} \ln \det(KK^\dagger) = \frac{1}{2} \Tr \left( (KK^\dagger)^{-1} \frac{\partial}{\partial A_\alpha(x)} (KK^\dagger) \right)$$

$$= \frac{1}{2} \Tr \left( (KK^\dagger)^{-1} \left[ \frac{\partial K}{\partial A_\alpha(x)} K^\dagger + K \frac{\partial K^\dagger}{\partial A_\alpha(x)} \right] \right)$$

$$= \frac{1}{2} \Tr \left( K^{-1} \frac{\partial K}{\partial A_\alpha(x)} + K^{-1} \frac{\partial K^\dagger}{\partial A_\alpha(x)} \right)$$

$$= \Re \left[ \Tr \left( K^{-1} \frac{\partial K}{\partial A_\alpha(x)} \right) \right]$$

the drift force is

$$-\frac{\partial S_{pq,eff}}{\partial A_\alpha(x)} = -\frac{1}{4U} A_\alpha(x) + \Re \left[ \Tr \left( K^{-1} \frac{\partial K}{\partial A_\alpha(x)} \right) \right]$$

which is just the real part of the complex drift force in (24). This is because the effective action (D2) in the quenched approximation is taken to be the real part of the complex effective action in (22)

$$e^{-S} = e^{-S_{pq,eff}} e^{i\varphi}, \quad e^{i\varphi} = \frac{\det K}{|\det K|} \quad \iff \quad \Re(\ln \det K) = \frac{1}{2} \ln \det(KK^\dagger)$$

Here the logarithm $\ln$ is understood to be the principal value of the logarithm.

The chiral condensate and fermion density in (23) is replaced by

$$\langle \bar{\psi} \psi \rangle = \frac{1}{N^d} \frac{\partial}{\partial m} \ln Z_{pq} = \frac{1}{N^d} \langle \Re \Tr (K^{-1}) \rangle_{pq}, \quad \langle n \rangle = \frac{1}{N^d} \frac{\partial}{\partial \mu} \ln Z_{pq} = \frac{1}{N^d} \langle \Re \Tr (K^{-1} \frac{\partial K}{\partial \mu}) \rangle_{pq}$$

where the average is taken with respect to the weight of partition function in (D1). The average phase factor in the phase-quenched theory $\langle e^{i\varphi} \rangle_{pq} = Z/Z_{pq}$ indicates the severeness of the sign problem in the thermodynamic limit.

Since the real function $C$ may be negative, the phase quenched approximation of (10) is

$$Z_{pq} = \sum_{k=(k_x,\alpha)} U^j |C(x_1, \cdots, x_{2j})| \quad (D3)$$

The chiral condensate and fermion density under this quenched phase approximation are

$$\langle \bar{\psi} \psi \rangle = \frac{1}{N^d} \frac{\partial}{\partial m} \ln Z_{pq} = \frac{1}{N^d} \langle \frac{\partial m C}{C} \rangle_{pq}, \quad \langle n \rangle = \frac{1}{N^d} \frac{\partial}{\partial \mu} \ln Z_{pq} = \frac{1}{N^d} \langle \frac{\partial \mu C}{C} \rangle_{pq}$$

respectively. The average phase factor is

$$\langle e^{i\varphi} \rangle_{pq} = \langle \frac{C}{|C|} \rangle_{pq} = \frac{\sum_{k=(k_x,\alpha)} U^j C(x_1, \cdots, x_{2j})}{\sum_{k=(k_x,\alpha)} U^j |C(x_1, \cdots, x_{2j})|}$$

Here the average $\langle \cdot \rangle_{pq}$ is taken with respect to the partition function $Z_{pq}$ in (D3).