Fixed-point Quantum Search for Different Phase Shifts

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Abstract

Grover recently presented the fixed-point search algorithm. In this letter, we study the fixed-point search algorithm obtained by replacing equal phase shifts of $\pi/3$ by different phase shifts.

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1 Introduction

Grover’s search algorithm and fixed-point search algorithm are both used to find a desired state from an unsorted database. His original algorithm consists of inversion of the amplitude in the desired state and inversion-about-average operation\textsuperscript{1}. In \textsuperscript{2}, Grover presented a general algorithm: $Q = -I_\gamma U^{-1} I_\tau U$, where $U$ is any unitary operation, $U^{-1}$ is the adjoint of $U$, $I_\gamma = I - 2|\gamma\rangle\langle\gamma|$, $I_\tau = I - 2|\tau\rangle\langle\tau|$, $|\gamma\rangle$ is an initial state and $|\tau\rangle$ is a desired state. When $U^{-1} = U = W$, where $W$ is the Walsh-Hadamard transformation, and $|\gamma\rangle = |0\rangle$, the general algorithm becomes the original algorithm. Grover showed that the desired state can be found with certainty after $O(\sqrt{N})$ applications of $Q$ to the initial state $|\gamma\rangle$. Long extended Grover’s algorithm\textsuperscript{3}. Long’s algorithm is $Q = -I_\gamma U^{-1} I_\tau U$, where $I_\gamma = I - (-e^{i\theta} + 1)|\gamma\rangle\langle\gamma|$, $I_\tau = I - (-e^{i\phi} + 1)|\tau\rangle\langle\tau|$. When $\theta = \phi = \pi$, Long’s algorithm becomes Grover’s general algorithm. Li et al. proposed that $U^{-1}$ in Long’s algorithm can be replaced by any unitary operation $V$\textsuperscript{4}\textsuperscript{5}. In \textsuperscript{6}, Galindo et al. gave a family of Grover’s quantum searching algorithms. In \textsuperscript{7}, Pang et al. generalized Grover’s algorithm and applied their algorithm to image compression.

Grover’s quantum search algorithm can be considered as a rotation of the state vectors in two-dimensional Hilbert space generated by the initial ($s$) and target ($t$) vectors\textsuperscript{2}. The amplitude of the target state increases monotonically towards its maximum and decreases monotonically after reaching the maximum\textsuperscript{5}. As mentioned in \textsuperscript{5}, unless we stop as soon as it reaches the target state, it will drift away. However, the number of iteration steps to find the target state is not an integer\textsuperscript{5}. It means that we either stop near the target state or we drift away.

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Grover presented the new algorithm by replacing the selective inversions by selective phase shifts of $\pi/3$ [8]. The new algorithm converges to the target state irrespective of the number of iterations. In [9], an algorithm for obtaining fixed points in iterative quantum transformations was given and it was illustrated that the algorithm has much better average-case behavior. In [10], Boyer et al. described an algorithm that succeeds with probability approaching to 1.

The quantum search algorithm with different phase rotation angles were discussed in [3] [4] [5] [11] [12]. In this letter, we study the fixed-point search algorithm obtained by replacing equal phase shifts of $\pi/3$ by different phase shifts and show that the deviation for different phase shifts is smaller than for equal phase shifts. However, the smallest average deviation does not occur at different phase shifts. For the definition of the average deviation, see section 4.1 of this letter. It is well known that the smaller the deviation is, the more rapidly the algorithm converges to the desired item. Let $U_{ts}$ be the amplitude of reaching the target state $|t\rangle$ by applying $U$ to the start state $|s\rangle$ and $\|U_{ts}\|^2 = 1 - \epsilon$. For any range $(\beta, \alpha)$ of $\epsilon$, we argue that $\alpha$ and $\beta$ determine the phase shifts at which the smallest average deviation occurs.

This paper is organized as follows. In section 2, we introduce the fixed-point search algorithm with different phase shifts and derive the deviation. Section 3 is used to find different phase shifts for small deviation. Section 3 is devoted to study the smallest average deviation.

## 2 The algorithm with different phase shifts

### 2.1 Replacing equal phase shifts of $\pi/3$ by different phase shifts

Grover presented the fixed-point search algorithm by replacing the selective inversions by selective phase shifts of $\pi/3$ [8]. He described the transformation $UR_sU^+R_tU$ applied to the start state $|s\rangle$, where

$$
\begin{align*}
R_s &= I - (1 - e^{i\pi/3})|s\rangle\langle s|, \\
R_t &= I - (1 - e^{i\theta/3})|t\rangle\langle t|,
\end{align*}
$$

(1)

$t$ stands for the target state.

We consider the transformation $UR_sU^+R_tU$ applied to the start state $|s\rangle$, where

$$
\begin{align*}
R_s &= I - (1 - e^{i\varphi})|s\rangle\langle s|, \\
R_t &= I - (1 - e^{i\theta})|t\rangle\langle t|.
\end{align*}
$$

(2)

$\theta$ and $\varphi$ are called selective phase shifts. It is enough to consider $\theta$ and $\varphi$ to be in $[0, \pi]$. When $\theta = \varphi = \pi/3$, one recovers Grover’s fixed-point search algorithm.
above. In [13], the fixed-point search algorithm with two equal phase shifts was discussed.

We compute

$$U R_s U^+ R_t U |s\rangle = (e^{i\phi} + (1 - e^{i\phi})(1 - e^{i\theta}) \| U_{ts} \|^2) U |s\rangle - (1 - e^{i\theta}) U_{ts} |t\rangle.$$  

By the definition of the deviation in [8], let $D(\theta, \varphi)$ be the deviation of this superposition from the state $|t\rangle$. Then we can derive

$$D(\theta, \varphi) = (1 - \| U_{ts} \|^2) \| e^{i\phi} + (1 - e^{i\phi})(1 - e^{i\theta}) \| U_{ts} \|^2 \| e^{i\phi} + (1 - e^{i\phi})(1 - e^{i\theta}) \| U_{ts} \|^2. \tag{3}$$

### 2.2 Simplifying the deviation for different phase shifts

Grover let $\| U_{ts} \|^2 = 1 - \epsilon$, where $0 < \epsilon < 1$. As indicated in [2], $\| U_{ts} \|$ is very small and almost $1/\sqrt{N}$, where $N$ is the size of the database. For example, $\epsilon = 1 - 1/N > \frac{1}{4}$ when $N > 4$. Therefore $\epsilon$ is close to 1.

Substituting $\| U_{ts} \|^2 = 1 - \epsilon$ and reducing the second term of $D(\theta, \varphi)$ in (3), we obtain

$$D(\theta, \varphi) = \epsilon (1 - 8(1 - \epsilon) \sin(\theta/2) \sin(\varphi/2) \cos((\theta - \varphi)/2) + 16(1 - \epsilon)^2 \sin^2(\theta/2) \sin^2(\varphi/2)). \tag{4}$$

See Fig.1 for $D(\theta, \varphi)$. When $\theta = \varphi$, (4) becomes

$$D(\theta, \theta) = \epsilon (4(1 - \epsilon) \sin^2(\theta/2) - 1)^2. \tag{5}$$

When $\theta = \varphi = \pi/3$ are chosen as phase shifts, the deviation $D(\pi/3, \pi/3) = \epsilon^3[8]$.

### 2.3 The deviation does not vanish at different phase shifts

$D(\theta, \varphi)$ in (4) is rewritten as follows.

$$D(\theta, \varphi) = \epsilon [(4(1 - \epsilon) \sin(\theta/2) \sin(\varphi/2) - \cos((\theta - \varphi)/2))^2 + \sin^2((\theta - \varphi)/2)]. \tag{6}$$

From (6), it is straightforward that $D(\theta, \varphi) = 0$ if and only if $\theta = \varphi$ and $\cos \theta = 1 - \frac{1}{2(1 - \epsilon)}$, where $\epsilon \leq 3/4$. It means that the deviation vanishes only when two phase shifts are equal. Therefore if $\epsilon$ is known and $\epsilon \leq 3/4$, then we choose $\theta = \arccos(1 - \frac{1}{2(1 - \epsilon)})$ as two equal phase shifts to make the deviation vanish. However, as pointed out before, $\epsilon$ is always close to 1.
3 Different phase shifts for small deviation

From (4) and (5) we calculate
\[
D(\theta, \varphi) - D(\theta, \theta) =
8\epsilon(1 - \epsilon) \sin(\theta/2) \sin((\theta - \varphi)/2)[\epsilon \cos(\varphi/2) + (1 - \epsilon) \cos((2\theta + \varphi)/2)].
\] (7)

Let us reduce (7) as follows.
\[
\epsilon \cos(\varphi/2) + (1 - \epsilon) \cos((2\theta + \varphi)/2) \text{ in (7)} =
\epsilon(\cos(\varphi/2) - \cos((2\theta + \varphi)/2)) + \cos((2\theta + \varphi)/2) = 2\epsilon \sin((\theta + \varphi)/2) \sin(\theta/2) + \cos((2\theta + \varphi)/2).
\] Then (7) is rewritten as follows.
\[
D(\theta, \varphi) - D(\theta, \theta) =
8\epsilon(1 - \epsilon) \sin(\theta/2) \sin((\theta - \varphi)/2)[2\epsilon \sin((\theta + \varphi)/2) \sin(\theta/2) + \cos((2\theta + \varphi)/2)].
\] (8)

Following (8), we have the following results.

Result 1.
The deviation for different phase shifts \( \theta \) and \( \varphi \) is smaller than the deviation for equal phase shifts \( \theta \), i.e., \( D(\theta, \varphi) < D(\theta, \theta) \), if \( 0 < \theta < \varphi \) and \( \epsilon > -\cos((2\theta + \varphi)/2)/(2 \sin((\theta + \varphi)/2) \sin(\theta/2)) \text{ or } 0 \leq \varphi < \theta \) and \( \epsilon < -\cos((2\theta + \varphi)/2)/(2 \sin((\theta + \varphi)/2) \sin(\theta/2)) \).

Result 2.
\( D(\theta, \varphi) < D(\theta, \theta) \) whenever \( 0 < \theta < \varphi \) and \( 0 < (2\theta + \varphi) < \pi \).

The following example follows result 1 immediately.

Example 1. When \( \theta = \pi/3 \) and \( \pi/3 < \varphi < \pi \), \( D(\pi/3, \varphi) < D(\pi/3, \pi/3) \) for \( \epsilon > -\cos(\pi/3 + \varphi/2)/\sin(\pi/6 + \varphi/2) \).

It means that the deviation for one phase shift of \( \pi/3 \) and one larger phase shift is smaller than for two equal phase shifts of \( \pi/3 \).

Example 2. \( D(\theta, \pi/2) < D(\theta, \theta) \) when \( 0 < \theta < \pi/2 \) and \( \epsilon \) is large.

The proof is as follows.

When \( 0 < \theta \leq \pi/4 \), from result 2 \( D(\theta, \pi/2) < D(\theta, \theta) \) for any \( \epsilon \) in \( (0, 1) \).

When \( \pi/4 < \theta < \pi/2 \), from result 1 \( D(\theta, \pi/2) < D(\theta, \theta) \) for \( \epsilon > (\sin \theta - \cos \theta)/(2(\sin(\theta/2) + \cos(\theta/2)) \sin(\theta/2)). \)

This example can also be verified by computing \( D(\theta, \pi/2) \) and reducing \( D(\theta, \pi/2) - D(\theta, \theta) \) as follows.
\[
D(\theta, \pi/2) - D(\theta, \theta) = 4\epsilon(1 - \epsilon) \sin(\theta/2) \sin(\theta/2) - \cos(\theta/2) + 2(1 - \epsilon) \sin(\theta/2) \cos \theta.
\] It is not hard to see that \( D(\theta, \pi/2) < D(\theta, \theta) \) if \( \epsilon > 1 - [(\cos(\theta/2) - \sin(\theta/2))/(2 \sin(\theta/2) \cos \theta)]. \)
4 The phase shifts for the smallest average deviation

4.1 The definition of the average deviation

Usually, people pay much attention to the average-case behavior of an algorithm besides the worst-case behavior. For Grover’s fixed-point search, Tulsi et al. studied the average-case behavior in [9]. As indicated in [2], $||U_t||$ is very small, i.e., $\epsilon$ is very large and close to 1. Unfortunately, we don’t know the exact value of $\epsilon$. However, it may be possible to know an range of $\epsilon$. From (4), it is clear that the deviation is a function of $\epsilon$. Let $\epsilon$ lie in the range $(\beta, \alpha)$. Intuitively, the deviation varies as $\epsilon$ does in $(\beta, \alpha)$. What is the average value of the deviation? It seems essential and significant to define the average deviation in the present letter.

Assume that $\epsilon$ is in the range $(\beta, \alpha)$, where $0 \leq \beta < \alpha \leq 1$. In terms of mean-value theorem for integrals, the average value $\bar{D}(\theta, \phi)$ over the range $(\beta, \alpha)$ of deviation $D(\theta, \phi)$ is defined and calculated as follows.

$$\bar{D}(\theta, \phi) = \frac{1}{\alpha - \beta} \int_{\beta}^{\alpha} D(\theta, \phi) \, d\epsilon = \frac{1}{\alpha - \beta} \left[ \frac{1}{2}(\alpha^2 - \beta^2) + A \sin(\phi/2) \sin(\theta/2) \cos((\theta - \phi)/2) + B \sin^2(\theta/2) \sin^2(\phi/2) \right],$$

(9)

where $A = -\frac{4}{3}(\alpha^2(3 - 2\alpha) - \beta^2(3 - 2\beta))$ and $B = -\frac{4}{3}((1 - \alpha)^3(3\alpha + 1) - (1 - \beta)^3(3\beta + 1))$.

We argue $A < 0$, $B > 0$ and $A/B < -1/2$ in Appendix A.

For example, when $\theta = \phi = \pi/3$, the average deviation $\bar{D}(\pi/3, \pi/3) = (\alpha + \beta)(\alpha^2 + \beta^2)/4$.

4.2 The phase shifts for the smallest average deviation

Apparently, the average deviation $\bar{D}(\theta, \phi)$ in (9) is a function of phase shifts $\theta$ and $\phi$. It is natural to ask what phase shifts can make the average deviation $\bar{D}(\theta, \phi)$ the smallest. For this purpose, we need to find the minimum of the average deviation $\bar{D}(\theta, \phi)$.

To find the extremes of the average deviation $\bar{D}(\theta, \phi)$, we compute the following partial derivatives:

$$\partial \bar{D}(\theta, \phi)/\partial \theta = 1/(2(\alpha - \beta)) \sin(\phi/2) (A \cos((2\theta - \phi)/2) + B \sin \theta \sin(\phi/2)),$$

(10)

$$\partial \bar{D}(\theta, \phi)/\partial \phi = 1/(2(\alpha - \beta)) \sin(\theta/2) (A \cos((2\phi - \theta)/2) + B \sin \phi \sin(\theta/2)).$$

(11)
The extremes of the average deviation $\bar{D}(\theta, \varphi)$ are $(\alpha + \beta)/2$ at the extreme points of $\bar{D}(\theta, \varphi)$: \{0, 0\}, \{0, \pi\} and \{\pi, 0\}. Let $A\cos((2\theta - \varphi)/2) + B\sin \theta \sin(\varphi/2) = 0$ in (10) and $A\cos((2\varphi - \theta)/2) + B\sin \varphi \sin(\theta/2) = 0$ in (11). Then we derive the extreme points $\theta = \varphi$. Letting $\theta = \varphi$, $\bar{D}(\theta, \varphi)$ becomes

$$\bar{D}(\theta, \theta) = ((\alpha + \beta)^2)/2 + A\sin^2(\theta/2) + B\sin^4(\theta/2))/(\alpha + \beta).$$

After rewriting,

$$\bar{D}(\theta, \theta) = \frac{B}{4(\alpha - \beta)}(1 + \frac{A}{B} - \cos \theta)^2 + \frac{\alpha + \beta}{2} - \frac{A^2}{4B(\alpha - \beta)}.$$ \hspace{1cm} (13)

Let us prove that the smallest average deviation occurs at $\theta = \phi$.

### 4.2.1 The smallest average deviation for $\alpha + \beta \geq 1$

For large $\epsilon$, the range $(\beta, \alpha)$ of $\epsilon$ may satisfy $\alpha + \beta \geq 1$. When $\alpha + \beta \geq 1$, $A/B \leq -1$. See (3) in Appendix A. In (5) of Appendix A, we demonstrate when $\alpha + \beta \geq 1$, the average deviation $\bar{D}(\theta, \varphi) \leq (\alpha + \beta)/2$. It implies that the smallest average deviation occurs at equal phase shifts. From (13), let us find the smallest average deviation as follows.

Case 1. $-2 \leq A/B \leq -1$

When $\theta = \arccos(1 + A/B)$ is chosen as equal phase shifts, $\bar{D}(\theta, \theta)$ reaches its minimum $(\alpha + \beta)/2 - A^2/(4B(\alpha - \beta))$, which is also the minimum of $\bar{D}(\theta, \varphi)$. So, $\arccos(1 + A/B)$, which is in $[\pi/2, \pi]$, is called the smallest average deviation point.

Example 3. Let $\beta = 0$ and $\alpha = 1$. Then $A = -4/3$, $B = 4/3$ and $1 + A/B = 0$. $\bar{D}(\theta, \theta)$ is calculated as $\bar{D}(\theta, \theta) = \cos^2 \theta/3 + 1/6$. Straightforwardly, $\pi/2$ is the smallest average deviation point at which $\bar{D}(\theta, \theta)$ reaches its minimum 1/6. See Fig. 2.

Case 2. $A/B < -2$

$\bar{D}(\theta, \theta)$ decreases as $\theta$ increases from 0 to $\pi$ and reaches its minimum $\frac{\alpha + \beta}{2} + 4(\alpha - 1 + \beta)(\alpha^2 - \alpha + \beta^2 - \beta)$ at $\theta = \pi$, which is also the minimum of $\bar{D}(\theta, \phi)$.

Example 4. Let $\beta = 3/4$ and $\alpha = 1$. Then $A = -5/24$, $B = 13/192$, $1 + A/B = -\frac{27}{25}$. $\bar{D}(\theta, \theta)$ becomes $(13/192)(27/13 + \cos \theta)^2 + 73/312$ and $\bar{D}(\theta, \theta)$ reaches its minimum $\frac{5}{16}$ at $\theta = \pi$. See Fig. 3.

### 4.2.2 The smallest average deviation for $\alpha + \beta < 1$

When $\alpha + \beta < 1$, $-1 < A/B < -1/2$. Then it can be shown that when $\theta = \arccos(1 + A/B)$, $\bar{D}(\theta, \theta)$ reaches its minimum $(\alpha + \beta)/2 - A^2/(4B(\alpha - \beta))$, which is less than $(\alpha + \beta)/2$, and when $\theta = \pi$, $\bar{D}(\theta, \theta)$ reaches its maximum $\frac{\alpha + \beta}{2} + 4(\alpha - 1 + \beta)(\alpha^2 - \alpha + \beta^2 - \beta)$, which is greater than $(\alpha + \beta)/2$.

As stated already, the extremes of $\bar{D}(\theta, \phi)$ at \{0, 0\}, \{0, \pi\} and \{\pi, 0\} are $(\alpha + \beta)/2$. Consequently, when $\alpha + \beta < 1$, the smallest average deviation occurs
at $\theta = \phi = \arccos(1 + A/B)$, which is in $(\pi/3, \pi/2)$, and the maximal average deviation occurs at $\theta = \phi = \pi$.

Remark:

Conclusively, for any range $(\beta, \alpha)$ of $\epsilon$, $\alpha$ and $\beta$ determine the phase shifts at which the smallest average deviation occurs, the smallest average deviation points are greater than $\pi/3$ and the smallest average deviations are smaller than the average deviation $D(\pi/3, \pi/3)$ for equal phase shifts of $\pi/3$.

5 Summary

In this letter, we demonstrate the possibility of the fixed-point quantum search algorithm with two different phase shifts. Intuitively, not only there are more choices for phase shifts to adjust an algorithm for future physical realization, but also we can find some different phase shifts for small deviation. Thus, more loose constraint opens a door for more feasible or robust realization. In this letter, We also show that the smallest average deviation can be obtained by choosing the following equal phase shifts. Let $(\beta, \alpha)$ be the range of $\epsilon$. Then if $A/B \geq -2$, then $\arccos(1 + A/B)$ is chosen as equal phase shifts. Otherwise, the closer to $\pi$ the equal phase shifts are, the better.

Appendix A

1. The proof of $B > 0$

We can factor $B$ as follows.

$$B = -\frac{1}{3}((1 - \alpha)^3(3\alpha + 1) - (1 - \beta)^3(3\beta + 1)) = \frac{1}{3}((\alpha - \beta)D$$

where $D = 3\alpha^3 - 8\alpha^2 + 3\beta\alpha^2 + 6\alpha - 8\beta\alpha + 3\beta^2\alpha + 6\beta - 8\beta^2 + 3\beta^3$. So, we need only to show that $D > 0$. $D$ can be rewritten as $(\alpha + \beta)[3(\alpha + \beta)^2 - 8(\alpha + \beta) + 6] - \alpha\beta[6(\alpha + \beta) - 8]$. There are two cases. Case 1: $6(\alpha + \beta) - 8 < 0$. Clearly $D > 0$ since $3(\alpha + \beta)^2 - 8(\alpha + \beta) + 6 > 0$. Case 2: $6(\alpha + \beta) - 8 > 0$. Also, $D > 0$ since $(\alpha + \beta)[3(\alpha + \beta)^2 - 8(\alpha + \beta) + 6] > (\alpha + \beta)^2(6(\alpha + \beta) - 8)/4 > \alpha\beta[6(\alpha + \beta) - 8]$.

The proof was given by Mr. P.Y. Sun.

2. The proof of $A/B < -1/2$

We demonstrate $B > 0$ in Appendix A. Hence, it is easy to see $A/B < -1/2$ if and only if $2A + B < 0$. By factoring, $2A + B = \frac{4}{3}(\alpha - \beta)E$ where $E = 3\alpha^3 - 4\alpha^2 + 3\beta\alpha^2 - 4\beta\alpha + 3\beta^2\alpha - 4\beta^2 + 3\beta^3$. We only need to argue $E < 0$.

Letting $\alpha = \beta + \gamma$, where $0 < \gamma \leq 1$,

$$E = 12\beta^3 + 18\beta^2\gamma + 12\beta\gamma^2 + 3\gamma^3 - 12\beta^2 - 12\beta\gamma - 4\gamma^2$$

$$= (12\beta^3 + 12\beta^2\gamma - 12\beta^2) + (6\beta^2\gamma + 12\beta\gamma^2 - 12\beta\gamma) + (3\gamma^3 - 4\gamma^2)$$

$$= 12\beta^2(\beta + \gamma - 1) + 6\beta\gamma(\beta + 2\gamma - 2) + \gamma^2(3\gamma - 4)$$

$$= 12\beta^2(\alpha - 1) + 6\beta\gamma(\alpha + \gamma - 2) + \gamma^2(3\gamma - 4) < 0.$$
By factoring, we obtain $A = \frac{1}{4} (\alpha - \beta) C$, where $C = 2(\alpha^2 + \alpha \beta + \beta^2) - 3(\alpha + \beta)$. By letting $\alpha = \beta + \gamma$, where $0 < \gamma < 1$, $C = 6\alpha \beta + 2\gamma^2 - (6\beta + 3\gamma)$. It is easy to know $C < 0$ since $6\alpha \beta < 6\beta$ and $2\gamma^2 < 3\gamma$.

5. The proof of $\bar{D}(\theta, \varphi) \leq (\alpha + \beta)/2$ when $\alpha + \beta \geq 1$

When $\alpha + \beta \geq 1$, we show that $A/B \leq -1$ in (3) of this appendix. To show $\bar{D}(\theta, \varphi) \leq (\alpha + \beta)/2$, from (I) we only need to show $A \sin(\theta/2) \sin(\varphi/2) \cos((\theta - \varphi)/2) + B \sin^2(\theta/2) \sin^2(\varphi/2) \leq 0$. Let us argue this as follows.

$$A \sin(\theta/2) \sin(\varphi/2) \cos((\theta - \varphi)/2) + B \sin^2(\theta/2) \sin^2(\varphi/2) = B \sin(\theta/2) \sin(\varphi/2) \cos((\theta - \varphi)/2) \left( \frac{A}{B} + \frac{\sin(\theta/2) \sin(\varphi/2)}{\cos((\theta - \varphi)/2)} \right).$$

Next we compute $\frac{A}{B} + \frac{\sin(\theta/2) \sin(\varphi/2)}{\cos((\theta - \varphi)/2)} = \frac{A}{B} + \frac{1}{2} - \frac{1}{2} \frac{\cos(\theta + \varphi)/2}{\cos((\theta - \varphi)/2)}$. Since $\frac{A}{B} \leq -1$, $\frac{A}{B} + \frac{1}{2} - \frac{1}{2} \frac{\cos(\theta + \varphi)/2}{\cos((\theta - \varphi)/2)} \leq -1 \frac{1}{2} - \frac{1}{2} \frac{\cos(\theta + \varphi)/2}{\cos((\theta - \varphi)/2)} = -\frac{1}{2} \left( 1 + \frac{\cos(\theta + \varphi)/2}{\cos((\theta - \varphi)/2)} \right) = -\frac{1}{2} \left( 1 + \frac{\cos(\theta/2) \cos(\varphi/2)}{\cos((\theta - \varphi)/2)} \right) \leq 0.$

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