Elementary computation of the stable reduction of the Drinfeld modular curve $X(\pi^2)$

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Abstract

In [W3][Theorem 1.1], Jared Weinstein proves that, in the stable reduction of the Lubin-Tate space $X(\pi^n)$, all irreducible components admit a purely inseparable map to one of the following four curves; the projective line, the curve with the Artin-Schreier affine model $a^q - a = t^2$, the Deligne-Lusztig curve with affine model $x^q y - xy^q = 1$ and the curve with affine model $a^q - a = t^{q+1}$. To prove this, he uses non-abelian Lubin-Tate theory for $GL_2$ in [W3][Theorem 3.3] and the Bushnell-Kutzko type theory. In this paper, we precisely determine defining equations of all irreducible components in the stable reduction of the Drinfeld modular curve of level 2. Our method is purely local, explicit and elementary with using blow-up. As a corollary, we determine the inertia action and $GL_2$-action on each components in the stable reduction of $X(\pi^2)$ explicitly.

1 Introduction

Let $F$ be a non-archimedean local field with uniformizer $\pi$. Let $C$ be the completion of a fixed algebraic closure of $F$. By a model for a scheme $X$ over $F$, we mean a scheme $\mathcal{X}$ over the ring of integers $\mathcal{O}_F$ of $F$ such that $X \simeq \mathcal{X} \otimes_{\mathcal{O}_F} F$. When a curve $C$ over $F$ does not have a model with good reduction over $\mathcal{O}_F$, it may have the “next best thing,” i.e., a stable model. The stable model is unique up to isomorphism if it exists, and it does over the ring of integers in some finite extension of $F$, as long as the genus of the curve is at least 2, which is proved by Deligne and Mumford in [DM]. Moreover, if $\mathcal{C}$ is a stable model for $C$ over $\mathcal{O}_F$, and $F \subset E \subset C$, then $\mathcal{C} \otimes_{\mathcal{O}_F} \mathcal{O}_E$ is a stable model for $C \otimes_F E$ over $\mathcal{O}_E$. The special fiber of any stable model for $C$ is called the stable reduction.

The stable models of $X_0(p)$ and $X_0(p^2)$ were previously known, due to works of J-I. Igusa and Deligne-Rapoport [DR] Section 7.6, and B. Edixhoven [E] Theorem 2.1.2 or [E2] respectively. In [CM], R. Coleman and K. McMurdy calculated the stable reduction of $X_0(p^3)$, using the notion of stable coverings of a rigid-analytic curve by basic wide opens. In loc. cit., they use the Woods Hole theory in [WH] and the Gross-Hopkins theory in [GH] to deduce the defining equations of all irreducible components in the stable reduction of $X_0(p^3)$. They also determine
the stable reduction of $X_0(Np^3)$ with $(N, p) = 1$ and compute the inertia action on the stable model of $X_0(p^3)$ in [CM2]. See [CM] Introduction for other prior results regarding the stable models of modular curves at prime power levels. In [T], we compute the stable reduction of the modular curve $X_0(p^3)$ on the basis of the Coleman-McMurdy’s work. Similarly as [CM], we actually construct a stable covering of $X_0(p^3)$ by basic wide opens. To compute the reduction of irreducible components in $X_0(p^3)$, we use the Kronecker polynomial.

Let $\mathcal{X}(\pi^n)$ denote the Lubin-Tate space. This space is a rigid-analytic deformation space with Drinfeld $\pi^n$-level structure of a one-dimensional formal $\mathcal{O}_F$-module of height $h$ over the residue field $F_q$ of $F$. Using the type theory of Bushnell-Kutzko and Deligne-Carayol’s non-abelian Lubin-Tate theory for $GL_2$ in [W3][Theorem 3.3], J. Weinstein determines the stable model of $\mathcal{X}(\pi^n)$ ($h = 2$), up to purely inseparable map in [W3][Theorem 1.1]. More precisely, he proves that all irreducible components admit a purely inseparable map to one of the following four curves; 1. the projective line $\mathbb{P}^1$ 2. the curve with Artin-Schreier affine model $a^q - a = s^2$ 3. Deligne-Lusztig curve with affine model $x^q y - xy^q = 1$ 4. the curve with affine model $a^q - a = t^{q+1}$.

In this paper, we calculate precisely defining equations of all irreducible components in the stable reduction of the Lubin-Tate space $\mathcal{X}(\pi^2)$ (char $F = p > 0$) on the basis of the ideas in [CM] and in [T]. Similarly as [T], our method in this paper is purely local, very explicit and elementary with using blow-up. Techniques and ideas in loc. cit. can be applied to the LubinTate space $\mathcal{X}(\pi^2)$. The $\pi$-multiplication of the universal formal $\mathcal{O}_F$-module $\mathcal{F}^{univ}$ over an open unit ball $\mathcal{X}(1)$ has the following simple form, if we choose an isomorphism $\mathcal{X}(1) \simeq B(1) \ni u$ appropriately,

$$[\pi]_{\mathcal{F}^{univ}}(X) = X^{q^2} + u X^q + \pi X.$$ This fact is crucial for our computation. By using this equation, we can directly compute the reduction of all irreducible components in the stable reduction of the Lubin-Tate space $\mathcal{X}(\pi^2)$. We define several rigid analytic subspaces of $\mathcal{X}(\pi^2)$, which we denote by $Y_{2,2}, Y_{3,1}$ and $Z_{1,1}$. We compute the reduction of these spaces. Irreducible components in the stable reduction of the Lubin-Tate space $\mathcal{X}(\pi^2)$ consist of the reduction of these spaces. We briefly introduce definitions of these spaces. See subsection 3.1 for the precise definition. See notation below for the rigid analytic notations. Let $p_2 : \mathcal{X}(\pi^2) \rightarrow \mathcal{X}(1)$ be the natural forgetful map. We set as follows

$$Y_{2,2} := p_2^{-1}(B[p^{-\frac{1}{2}}]), Y_{3,1} := p_2^{-1}(C[p^{-\frac{1}{2}}]), Z_{1,1} := p_2^{-1}(C[p^{-\frac{3}{2}}]).$$

Let $\mathcal{X}_{LT}(\pi^n)$ be the zero-dimensional Lubin-Tate space and LT the universal formal group of height 1 over $\mathcal{X}_{LT}(1)$. Let $F_0$ be the completion of the maximal unramified extension of $F$ in a fixed algebraic closure. Furthermore, let $F_n = F_0(\mathcal{X}_{LT}(\pi^n))$ denote the classical Lubin-Tate extension. Then, it is well-known that the base change $\mathcal{X}(\pi^n) \times_F F_n$ has $q^{n-1}(q - 1)$ connected components. Hence, we write $\mathcal{X}(\pi^n) \times_F F_n = \bigsqcup_{n \in \mathcal{X}_{LT}(\pi^n)(F_n)} \mathcal{X}^{\pi_n}(\pi^n)$. The connected component $\mathcal{X}^{\pi_n}(\pi^n)$ is defined by the following equation

$$\mu_n(X_n, Y_n) = \pi_n$$

where $\mu_n$ is called the Moore determinant. See subsection 2.4 for more detail. We set $Y_{a,b}^{\pi_n} := Y_{a,b} \cap \mathcal{X}^{\pi_n}(\pi^n)$ and $Z_{a,b}^{\pi_n} := Z_{a,b} \cap \mathcal{X}^{\pi_n}(\pi^n)$.

In the following, we explain defining equations of the reduction of these spaces $Y_{2,2}^{\pi_n}, Y_{3,1}^{\pi_n}$ and $Z_{1,1}^{\pi_n}$. The reduction of $Y_{2,2}^{\pi_n}$ is defined by the following equations

$$x^q y - xy^q = 1, Z_{1,1}^{\pi_n} = X^{q^3} y - xy^{q^3}.$$
This affine curve has $q(q^2-1)$ singular points at $(x, y)$ with $x = \zeta y, \zeta \in \mathbb{F}_q^\times \setminus \mathbb{F}_p^\times$ and $y^{q+1} = \frac{1}{\zeta - \zeta}$. We analyze the residue classes of these singular points. Then, by blowing up the singular points, we find $q(q^2-1)$ irreducible components defined by $a^q - a = t^{q^2+1}$. Similar phenomenon is observed in the stable reduction of the modular curve $X_0(p^2)$ in [4, Section 4].

The reduction of the space $Y_{3,1}^{p^2}$ has $(q+1)$ connected components and each component is defined by $x^q y - x y^q = 1, w^a = y$. Let $Y_{3,1, \zeta}$ denote a connected component of $Y_{3,1}^{p^2}$.

The reduction of $Z_{1,1}^{p^2}$ has $(q+1)$ connected components and each component is defined by the following equation

$$Z^q = X^{q^2-1} + \frac{1}{X^{q^2-1}}$$

with genus 0. Let $Z_{1,1, \zeta}$ denote a connected component of the reduction $Z_{1,1}^{p^2}$. This affine curve has $2(q^2-1)$ singular points at $X = \zeta, \zeta \in \mu_{2(q^2-1)}$. By analyzing the residue classes of these singular points, we find $2(q^2 -1)$ irreducible components defined by $a^q - a = s^2$. Similar phenomenon is observed in the stable reduction of the modular curve $X_0(p^2)$ in [CM]. We also compute the inertia action and $GL_2$-action on the reduction of the spaces $Y_{3,2}^{p^2}, Y_{3,1}^{p^2}$ and $Z_{3,1}^{p^2}$ explicitly. These computations of the defining equations of all irreducible components in the stable reduction of $X(\pi^2)$ are done in Section 4.

In section 5, we analyze the whole spaces $X(\pi)$ and $X(\pi^2)$. More precisely, we prove that the complements $X(\pi) \setminus Y_{1,1}$ and $X(\pi^2) \setminus (Z_{1,1} \cup Y_{2,2} \cup Y_{3,1})$ are disjoint unions of annuli. Hence, we conclude that the wide open space $X(\pi)$ is a basic wide open space. On the other hand, the space $X(\pi^2)$ is not basic wide open. In subsection 5.3, we actually construct a stable covering of the wide open space $X(\pi^2)$ on the basis of the idea of Coleman-Mcmurdy [CM, Section 9]. Furthermore, we give intersection multiplicities in the stable reduction of the Lubin-Tate space $X(\pi^2)$ in subsection 5.4.

We explain a shape of the stable reduction of the Lubin-Tate space $X(\pi^2)$. Let $Y_{2,2}$ be the projective completion of the affine curve $Y_{2,2}^{p^2}$. Then, the complement $Y_{2,2, \zeta}$ consists of $(q+1)$ closed points. The projective curve $Y_{2,2, \zeta}$ meets the projective completion $Y_{1,1, \zeta}$ of $(q+1)$ affine curves $Z_{1,1, \zeta}$ at each infinity. The complement $Y_{3,1, \zeta}$ of $Y_{1,1, \zeta}$ consists of two closed points. The projective curve $Y_{3,1, \zeta}$ meets the projective completion $Y_{3,1, \zeta}$ of $(q+1)$ affine curves $Y_{3,1, \zeta}$ at each infinity. The curve $Y_{3,1, \zeta}$ meets the Igusa curve $Ig(p^2)$ at each infinity. Since the affine curve $Y_{3,1, \zeta}$ has $(q+1)$ infinity points, there exist $(q+1)$ Igusa curves $Ig(p^2)$ in the stable reduction of the Lubin-Tate space $X(\pi^2)$.

We are greatly inspired by the works of Coleman-McCallum in [CW] on the stable reduction of the quotient of the Fermat curve and of Coleman-McMurdy on the stable reduction of the modular curve $X_0(p^2)$ in [CM]. We are also inspired by the work of T. Yoshida [Y] and J. Weinstein [W3]. We would like to thank Professor T. Saito and A. Abbes for helpful comments on this work and encouragements. We would like to thank S. Yasuda and S. Kondo for their interest on our work and stimulating discussions.

**Notation.** Let $\pi$ be a uniformizer of $F$. We fix some $\pi$-adic notation. We let $\mathcal{C}$ be the completion of a fixed algebraic closure of $F$, with integer ring $\mathbb{R}$ and with $\mathfrak{m}_\mathbb{R}$ the maximal ideal of $\mathbb{R}$. Let $\nu$ denote the unique valuation on $\mathbb{C}$ with $\nu(p) = 1$, $|\cdot|$ the absolute value given by $|x| = p^{-\nu(x)}$ and $\mathbb{R} = [\mathcal{C}_p] = p^{\mathbb{Q}}$. Throughout the paper, we let $F$ be a complete subfield of $\mathbb{C}$ with ring of integers $R_F$ and residue field $F_F$. For $r \in \mathbb{R}$, we let $B_F[r]$ and $B_F(r)$ denote the closed and open disk over $F$ of radius $r$ around $0$, i.e. the rigid spaces over $F$ whose $\mathbb{C}$-valued points are $\{x \in \mathcal{C} : |x| \leq r\}$ and $\{x \in \mathcal{C} : |x| < r\}$ respectively. If $r, s \in \mathbb{R}$ and $r \leq s$, let $A_F[r, s]$ and $A_F(r, s)$ be the rigid spaces over $F$ whose $\mathbb{C}$-valued points are $\{x \in \mathcal{C} : r \leq |x| \leq s\}$ and $\{x \in \mathcal{C} : r < |x| < s\}$, which we call closed annuli and open annuli. By the width of such an
annulus, we mean \( \log_p(s/r) \). A closed annuli of width 0 will be called a circle, which we will also denote the circle, \( A_F[s,s] \), by \( C_F[s] \).

## 2 Preliminaries ([W2] and [CM2])

### 2.1 definition of formal modules

We begin with the definitions of formal \( \mathcal{O}_F \)-modules.

**Definition 2.1.** Let \( R \) be a commutative \( \mathcal{O}_F \)-algebra, with structure map \( i : \mathcal{O}_F \rightarrow R \). A formal one-dimensional \( \mathcal{O}_F \)-module \( \mathcal{F} \) is a power series \( \mathcal{F}(X,Y) = X+Y+\cdots \in R[[X,Y]] \) which is commutative, associative, admits 0 as an identity, together with a power series \( [a]_{\mathcal{F}}(X) \in R[[X]] \) for each \( a \in \mathcal{O}_F \) satisfying \( [a]_{\mathcal{F}}(X) \equiv i(a)X \mod X^2 \) and \( \mathcal{F}([a]_{\mathcal{F}}(X),[a]_{\mathcal{F}}(Y)) = [a]_{\mathcal{F}}(\mathcal{F}(X,Y)) \).

The addition law on a formal \( \mathcal{O}_F \)-module \( \mathcal{F} \) will usually be written \( X +_{\mathcal{F}} Y \). If \( R \) is a \( k \)-algebra, we either have \( [\pi]_{\mathcal{F}}(X) = 0 \) or else \( [\pi]_{\mathcal{F}}(X) = f(X^q) \) for some power series \( f(X) \) with \( f'(0) \neq 0 \). In the latter case, we say \( \mathcal{F} \) has height \( h \) over \( R \). Let \( \Sigma \) be a one-dimensional formal \( \mathcal{O}_F \)-module over \( \hat{k} \) of height \( h \). The functor of deformations of \( \Sigma \) to complete local Noetherian \( \hat{\mathcal{O}}_F \)-algebra is reresentable by a universal deformation \( \mathcal{F}^{\text{univ}} \) over an algebra \( A \) which is isomorphic to the power series ring \( \hat{\mathcal{O}}_F[[u_1,\ldots,u_{h-1}]] \) in \( (h-1) \) variables, cf [Dr]. That is , if \( A \) is a complete local \( \hat{\mathcal{O}}_F \)-algebra with maximal ideal \( P \), then, the isomorphism classes of deformations of \( \Sigma \) to \( A \) are given exactly by specializing each \( u_i \) to an element of \( P \) in \( \mathcal{F}^{\text{univ}} \).

### 2.2 The universal deformation in the equal characteristic case

Assume \( \text{char} F = p > 0 \), so that \( F = k((\pi)) \) is the field of Laurent series over \( k \) in one variable, with \( \mathcal{O}_F = k[[\pi]] \), then, a model for \( \Sigma \) is given by the simple rules

\[
X +_{\Sigma} Y = X + Y, [\zeta]_{\Sigma}(X) = \zeta X, \zeta \in k, [\pi]_{\Sigma}(X) = X^{q^h}.
\]

The universal deformation of \( \mathcal{F}^{\text{univ}} \) also has a simple model over \( A \simeq \hat{\mathcal{O}}_F[[u_1,\ldots,u_{h-1}]] \):

\[
X +_{\mathcal{F}^{\text{univ}}} Y = X + Y
\]

\[
[\zeta]_{\mathcal{F}^{\text{univ}}}(X) = \zeta X, \zeta \in k
\]

\[
[\pi]_{\mathcal{F}^{\text{univ}}}(X) = \pi X + u_1 X^q + \cdots + u_{h-1} X^{q^{h-1}} + X^{q^h}.
\] (2.1)

### 2.3 Moduli of deformations with level structure

Let \( A \) be a complete local \( \mathcal{O}_F \) with maximal ideal \( M \), and let \( \mathcal{F} \) be a one-dimensional \( \mathcal{O}_F \)-module over \( A \), and let \( h > 1 \) be the height of \( \mathcal{F} \otimes A/M \).

**Definition 2.2.** Let \( n > 1 \). A Drinfeld level \( \pi^n \)-structure on \( \mathcal{F} \) is an \( \mathcal{O}_F \)-module homomorphism

\[
\phi : (\pi^{-n} \mathcal{O}_F / \mathcal{O}_F)^h \rightarrow M
\]
for which the relation
\[ \prod_{z \in \pi F \oplus \pi F \in \pi} (X - \varphi(x)) = |\pi| F(X) \]
holds in \( A[[X]] \). If \( \varphi \) is a Drinfeld level \( \pi^n \)-structure, the image of \( \varphi \) of the standard basis elements \((\pi^n, 0, 0, \ldots, 0), \ldots, (0, 0, \pi^n)\) of \((\pi^n \pi F \oplus \pi F \in \pi)^h\) form a Drinfeld basis of \( F[\pi^n] \).

Fix a formal \( \pi F \)-module \( \Sigma \) of height \( h \) over \( k \). Let \( A \) be a noetherian local \( \hat{\pi F} \)-algebra such that the structure morphism \( \hat{\pi F} \rightarrow A \) induces an isomorphism between residue fields. A deformation of \( \Sigma \) with level \( \pi^n \)-structure over \( A \) is a triple \((F, \eta, \varphi)\) where \( \eta : F \otimes k \cong \Sigma \) is an isomorphisms of \( \pi F \)-modules over \( k \) and \( \varphi \) is a Drinfeld level \( \pi^n \)-structure on \( F \).

**Proposition 2.3.** (\( \text{[DR]} \)) The functor which assigns to each \( A \) as above the set of deformations of \( \Sigma \) with Drinfeld level \( \pi^n \)-structure over \( A \) is represented by a regular local ring \( A(\pi^n) \) of dimension \( h - 1 \) over \( \hat{\pi F} \). Let \( X_1^{(n)}, \ldots, X_h^{(n)} \) be the corresponding Drinfeld basis for \( F_{univ}[\pi^n] \). Then, these elements form a set of regular parameters for \( A(\pi^n) \).

There is a finite injection of \( \hat{\pi F} \)-algebras \([\pi]_u : A(\pi^n) \hookrightarrow A(\pi^{n+1})\) corresponding to the obvious degeneration map of functors. We therefore may consider \( A(\pi^n) \) as a subalgebra of \( A(\pi^{n+1}) \), with the equation \([\pi]_u(X_1^{(n)}) = X_1^{(n+1)}\) holding in \( A(\pi^{n+1}) \). Let \( X(\pi^n) = \text{Spf} A(\pi^n) \), so that \( X(\pi^n) \rightarrow \text{Spf} \hat{\pi F} \) is formally smooth of relative dimension \( h - 1 \). Let \( \mathcal{X}(\pi^n) \) be the generic fiber of \( X(\pi^n) \); then \( \mathcal{X}(\pi^n) \) is a rigid analytic variety. The coordinates \( X_1^{(n)} \) are then analytic functions on \( \mathcal{X}(\pi^n) \) with values in the open unit disc. We have that \( \mathcal{X}(1) \) is the rigid analytic open unit polydisc of dimension \( h - 1 \). The group \( \text{GL}_h(\pi F \pi F \otimes \pi F) \) acts on the right on \( \mathcal{X}(\pi^n) \) and on the left on \( A(\pi^n) \). The degeneration map \( X(\pi^n) \rightarrow \mathcal{X}(1) \) is Galois with group \( \text{GL}_h(\pi F \pi F \otimes \pi F) \). For an element \( M \in \text{GL}_h(\pi F \pi F \otimes \pi F) \) and an analytic function \( f \) on \( \mathcal{X}(\pi^n) \), we write \( M(f) \) for the translated function \( z \mapsto f(zM) \). When \( f \) happens to be one of the parameters \( X_i^{(n)} \), there is a natural definition of \( M(X_i^{(n)}) \) when \( M \in M_h(\pi F \pi F \otimes \pi F) \) is an arbitrary matrix: if \( M = (a_{ij}) \), then
\[ M(X_1^{(n)}) = [a_{1j}]_{F_{univ}} X_1^{(n)} + [a_{2j}]_{F_{univ}} X_2^{(n)} + \cdots + [a_{kj}]_{F_{univ}} X_k^{(n)}. \]

### 2.4 Determinants

First, we briefly recall the determinant of level \( \pi \)-structures restricted to the case \( h = 2 \) from [W2 Section 3]. Define the polynomial in 2 variables
\[ \mu(X_1, Y_1) = X_1^h Y_1 - X_1 Y_1^h \in k[X_1, Y_1]. \]

This polynomial is \( k \)-linear alternating form, known as the Moore determinant. Secondly, we recall a determinant of structure of higher level again restricted to the case \( h = 2 \) from [W2 Section 3.3]. Now let \( n \geq 1 \), and suppose \( X_n, Y_n \) are sections of \( F_{univ}[\pi^n] \). We simply write \([\pi^n]_u(X)\) for \([\pi^n]_{F_{univ}}(X)\). We define the form \( \mu_n \)
\[ \mu_n(X_n, Y_n) = \sum_{(a_1, a_2)} \mu([\pi^n]_u(X_n), [\pi^{a_2}](Y_n)), \]
where the sum runs over pairs of integers \((a_1, a_2)\) with \( 1 \leq a_i \leq n \) whose sum is \( n \). This is \( k \)-alternating in \( X_n, Y_n \). It is proved that \( \mu_n \) is \( \pi F \)-linear in [W2 Proposition 3.7].

Let \( \text{LT} \) be a one-dimensional formal \( \pi F \)-module over \( \hat{\pi F} \) for which \( \text{LT} \otimes \bar{k} \) has height one. Let \( F_0 = \hat{F} \), and for \( n \geq 1 \), let \( F_n = F_0(\text{LT}[\pi^n]) \) be the classical Lubin-Tate extension.
Finally, let $\mathcal{X}_{LT}(\pi^n)$ be the zero-dimensional space of deformations of $LT \otimes \hat{k}$ with Drinfeld $\pi^n$-structure, so that $\mathcal{X}_{LT}(\pi^n)(F_n)$ is the set of bases for $LT[\pi^n](F_n)$ as a free $(\mathcal{O}_F/\pi^n\mathcal{O}_F)$-module of rank one.

For the remainder of the paper, LT will denote the formal $\mathcal{O}_F$-module over $\hat{\mathcal{O}}_F^\ur$ with operations

$$X +_{LT} Y = X + Y$$
$$[\alpha]_{LT}(X) = \alpha X, \alpha \in k$$
$$[\pi]_{LT}(X) = \pi X + (-1)^{h-1}X^q.$$  

We introduce the following theorem proved in $[W2, \text{Theorem 3.2}].$

**Theorem 2.4. ($[W2, \text{Theorem 3.2}]$)** Assume $\text{char } F = p > 0$. For each $n \geq 1$, there exists a morphism

$$\mu_n : \mathcal{F}^\text{univ}[\pi^n]^h \rightarrow \text{LT}[\pi^n] \otimes \mathcal{A}$$

of group schemes over $\mathcal{A} \simeq \hat{\mathcal{O}}_F^\ur[[u_1, \ldots, u_h-1]]$ which is $\mathcal{O}_F$-multilinear and alternating and which satisfies the following properties:

1. The maps $\mu_n$ are compatible in the sense that
   $$[\pi]_{LT}(\mu_n(X_1, \ldots, X_h)) = \mu_{n-1}([\pi]_u(X_1), \ldots, [\pi]_u(X_h))$$
   for $n \geq 2$.
2. If $X_1, \ldots, X_h$ are sections of $\mathcal{F}^\text{univ}[\pi^n]$ over an $\mathcal{A}$-algebra $R$ which form a Drinfeld level $\pi^n$-structure, then $\mu_n(X_1, ..., X_h)$ is a Drinfeld level $\pi$ structure for $\text{LT}[\pi^n] \otimes \mathcal{A}$.

The base change $\mathcal{X}(\pi^n) \times_F F_n$ has $q^{n-1}(q - 1)$ connected components and write $\mathcal{X}(\pi^n) \times_F F_n = \bigsqcup_{\pi_n \in \mathcal{X}_{LT}(\pi^n)(F_n)} \mathcal{X}_{\pi_n}(\pi^n)$. Then, each connected component $\mathcal{X}_{\pi_n}(\pi^n)$ is defined by the equation

$$\mu_n(X_n, Y_n) = \pi_n$$

by the above theorem. See $[W3, \text{subsection 3.6}]$ and $[S1, \text{Theorem 4.4}]$ for more detail on geometrically connected components of the Lubin-Tate space $\mathcal{X}(\pi^n)$.

### 2.5 Action of Inertia

We will recall the action of inertia on the stable model of a curve over $\mathbb{C}$ from $[CM2, \text{Section 6}].$

If $Y/F$ is a curve, and $\mathcal{Y}$ its stable model over $\mathbb{C}$, there is a homomorphism $w_Y$

$$w_Y : I_F := \text{Aut}_{\text{cont}}(\mathbb{C}/\mathbb{F}^\ur) \rightarrow \text{Aut}(\mathcal{Y}).$$

It is characterized by the fact that for each $P \in Y(\mathbb{C})$ and $\sigma \in I_F,$

$$\mathcal{P}^{\sigma} = w_Y(\sigma)(\mathcal{P}).$$  \hspace{1cm} (2.2)

We have something similar if $Y$ is a reduced affinoid over $F.$ Namely, we have a homomorphism $w_Y : I_F \rightarrow \text{Aut}(\mathcal{Y}_\mathbb{C})$ characterized by $[CM2].$ This follows from the fact that $I_F$ preserves $(\mathcal{Y}_\mathbb{C})^0$ (power bounded elements of $A(\mathcal{Y}_\mathbb{C})$) and $A(\mathcal{Y}_\mathbb{C})^\circ$ (topologically nilpotent elements of $A(\mathcal{Y}_\mathbb{C})$). Moreover, inertia action behaves well with respect to morphisms in the following sense.

**Lemma 2.5.** ($[CM2, \text{Lemma 6.1}]$)** If $f : X \rightarrow Y$ is a morphism of reduced affinoids over $F$ and $\sigma \in I_F,$ then $w_Y(\sigma) \circ f = f \circ w_X(\sigma).$  

6
3 Several subspaces in $\mathcal{X}(\pi^n)$

Throughout the remainder of the paper, we fix the following notations and assumptions. Let $F$ be a non-archimedean local field of equal characteristic with residue field $\mathcal{O}_F$. Let $\mathcal{X}(\pi^n)$ be the Lubin-Tate space over $\mathcal{O}_F$. Let $\mathcal{X}(\pi^n)$ be the universal formal $\mathcal{O}_F$-module over $\mathcal{X}(1)$. We assume $h = 2$. We fix an identification $\mathcal{X} \cong \hat{O}_F[[u]]$ or $\mathcal{X}(1) \cong B(1) \ni u$ such that $[\pi]_u(X) := [\pi]_{F \rightarrow \mathcal{O}_F}(X) = X^q + uX + \pi X$ as in [21]. The set of $\mathcal{C}$-valued points $\mathcal{X}(\pi^n)(\mathcal{C})$ is identified with the following

$$\{(u, X_n, Y_n) \in C^{3} | v(u) > 0, \mu_n(X_n, Y_n) \neq 0, [\pi^n]_u(X_n) = [\pi^n]_u(Y_n) = 0\}.$$ 

Let $\pi_n \in \mathcal{X}_{1,1}(\pi^n)(F_n)$. Then, the set of $\mathcal{C}$-valued points $\mathcal{X}^{\pi_n}(\pi^n)(\mathcal{C})$ is identified with the following

$$\{(u, X_n, Y_n) \in C^{3} | v(u) > 0, \mu_n(X_n, Y_n) = \pi_n, [\pi^n]_u(X_n) = [\pi^n]_u(Y_n) = 0\}.$$ 

We write $[\pi^i]_u(X_n) = X_{n-i}, [\pi^i]_u(Y_n) = Y_{n-i}$ for $0 \leq i \leq n - 1$.

3.1 Subspaces $Y_{a,b}$ and $Z_{a,b}$ in $\mathcal{X}(\pi^n)$

In this subsection, we define several subspaces $Y_{a,b}$ and $Z_{a,b}$ of the Lubin-Tate space $\mathcal{X}(\pi^n)$. We expect that the reduction of these spaces plays a fundamental role in the stable reduction of the Drinfeld modular curve. Actually, the reduction of the spaces $Y_{1,1}, Y_{2,2}$ and $Z_{1,1}$ becomes irreducible components of the Lubin-Tate space $\mathcal{X}(\pi^2)$.

Let $n \geq 1$ be a positive integer. Let $p_n : \mathcal{X}(\pi^n) \rightarrow \mathcal{X}(1); (F, \eta, \phi) \mapsto (F, \eta)$ be the natural forgetful map. Let $\mathcal{X}_0$ be a closed disc $B[\mathcal{O}_F^{-\frac{1}{n}}] \subset \mathcal{X}(1) \cong B(1)$. This is called the “too-supersingular locus.” For $(F, \eta) \in \mathcal{X}_0$, it is known that the formal group $F$ has no canonical subgroup. We define a subspace $Y_{2n-m, m} \subset \mathcal{X}(\pi^n) (1 \leq m \leq n)$ as follows:

$$Y_{n, n} := p_n^{-1}(\mathcal{X}_0) \subset \mathcal{X}(\pi^n),$$

$$Y_{2n-m, m} := p_n^{-1}(\mathcal{C}[p^{-\frac{1}{q^{m-1}(q+1)}}]) \subset \mathcal{X}(\pi^n) (1 \leq m \leq n - 1).$$

Let $n \geq 2$ be a positive integer. For $1 \leq m \leq n - 1$, we define subspaces $Z_{2(n-1)-m, m} \subset \mathcal{X}(\pi^n)$ as follows

$$Z_{2(n-1)-m, m} := p_n^{-1}(\mathcal{C}[p^{-\frac{1}{q^{m-1}(q+1)}}]) \subset \mathcal{X}(\pi^n) (1 \leq m \leq n - 1).$$

For a subspace $X \subset \mathcal{X}(\pi^n)$ and $\pi_n \in \mathcal{X}_{1,1}(\pi^n)(F_n)$, we set $X^{\pi_n} := X \cap \mathcal{X}^{\pi_n}(\pi^n)$.

3.2 Subspaces of the spaces $Y_{3,1}$ and $Z_{1,1}$

To compute the reduction of the spaces $Y_{3,1}$ and $Z_{1,1}$, we decompose these spaces to disjoint unions of several subspaces of them. Let $(u, X_2, Y_2) \in \mathcal{X}(\pi^2)$. We define several subspaces of $Y_{3,1}$ and $Z_{1,1}$ by conditions of valuations of parameters $X_i, Y_i (i = 1, 2)$. Recall that we have $v(u) = \frac{1}{q + 1}$ and $v(u) = 1/2$ on the spaces $Y_{3,1}$ and $Z_{1,1}$ respectively.

**Definition 3.1.** 1. We define a subspace $(u, X_2, Y_2) \in Y_{3,1,i} \subset Y_{3,1}$ by the following conditions:

$$v(X_1) = \frac{q}{q^2 - 1}, v(X_2) = \frac{1}{q(q^2 - 1)}, v(Y_1) = \frac{1}{q(q^2 - 1)}, v(Y_2) = \frac{1}{q(q^2 - 1)}.$$


2. We define a subspace \((u, X_2, Y_2) \in Y_{3,1,e_1} \subset Y_{3,1}\) by the following condition; \((u, X_2, Y_2) \in Y_{3,1,e_1}\) is equivalent to \((u, Y_2, X_2) \in Y_{3,1,e_1}\).

3. We define a subspace \((u, X_2, Y_2) \in Y_{3,1,c} \subset Y_{3,1}\) by the following conditions:

\[
v(X_1) = v(Y_1) = \frac{1}{q(q^2 - 1)}, v(X_2) = v(Y_2) = \frac{1}{q^3(q^2 - 1)}.
\]

4. We define a subspace \((u, X_2, Y_2) \in Z_{1,1,e_1} \subset Z_{1,1}\) by the following conditions;

\[
v(X_1) = \frac{1}{2(q - 1)}, v(X_2) = \frac{1}{2q^2(q - 1)}, v(Y_1) = \frac{1}{2q(q - 1)}, v(Y_2) = \frac{1}{2q^3(q - 1)}.
\]

5. We define a subspace \((u, X_2, Y_2) \in Z_{1,1,e_1} \subset Z_{1,1}\) by the following condition; \((u, X_2, Y_2) \in Z_{1,1,e_1}\) is equivalent to \((u, Y_2, X_2) \in Z_{1,1,e_1}\).

6. We define a subspace \((u, X_2, Y_2) \in Z_{1,1,c} \subset Z_{1,1}\) by the following conditions;

\[
v(X_1) = v(Y_1) = \frac{1}{2(q - 1)}, v(X_2) = v(Y_2) = \frac{1}{2q^3(q - 1)}.
\]

**Lemma 3.2.** 1. The space \(Y_{3,1}\) has the following description

\[Y_{3,1} = Y_{3,1,e_1} \bigsqcup Y_{3,1,e_1'} \bigsqcup Y_{3,1,c}.
\]

2. The space \(Z_{1,1}\) has the following description

\[Z_{1,1} = Z_{1,1,e_1} \bigsqcup Z_{1,1,e_1'} \bigsqcup Z_{1,1,c}.
\]

4 Reduction of the spaces \(Y_{a,b}\) \((a, b \geq 1, a + b = 2, 4)\) and \(Z_{1,1}\)

In this section, we compute the reduction of the spaces \(Y_{1,1} \subset \mathcal{X}(\pi)\) and \(Y_{3,1}, Y_{2,2}, Z_{1,1} \subset \mathcal{X}(\pi^2)\) by only using blow-up. Irreducible components of the stable reduction of \(\mathcal{X}(\pi^2)\) consist of the reduction of these spaces. See also Introduction for the defining equations of the reduction of these spaces. We also determine the inertia action on the reduction of these spaces. Furthermore, we also describe the GL\(_2\)-action on the stable reduction of \(\mathcal{X}(\pi^2)\). Similar computation is also found in [1] Sections 3 and 4].

4.1 Calculation of the reduction of the space \(Y_{1,1} \subset \mathcal{X}(\pi)\)

We compute the reduction of the space \(Y_{1,1}\). The reduction of the space is the Deligne-Lusztig curve as in the lemma below. It is well-known that the Deligne-Lusztig curve with affine model \(X^a Y - XY^b = 1\) appears in the stable reduction of the (Drinfeld) modular curve \(X(\pi)\). This fact is also deduced from the Katz-Mazur model in [KM]. In this subsection, for the convenience of a reader, we write down a computation of this component. See also [Y] Proposition 6.15], [W2] and [W3] Theorem 3.9].

Let \(\pi_1 \in X_{LT}(\pi)(F_1)\). Then, we have \(v(\pi_1) = 1/(q - 1)\). Let \((u, X_1, Y_1) \in X(\pi)\). Recall that the space \(Y_{1,1}\) is defined by the following conditions; \(v(u) \geq \frac{1}{q^{2r-1}}, v(X_1) = v(Y_1) = \frac{1}{q^{2r-1}}\). We
choose an element $\alpha$ such that $\alpha^{q+1} = \pi_1$. Then, we have $v(\alpha) = \frac{1}{q^2 - 1}$. We consider an equation of $Y_{1,1}^{\pi_1}$

$$\mu(X_1, Y_1) = X_1^q Y_1 - X_1 Y_1^q = \pi_1.$$  \hspace{1cm} (4.1)

We change variables as follows $X_1 = \alpha x, Y_1 = \alpha y$. Substituting them to the above equality \hspace{1cm} (4.1) and dividing it by $\pi_1$, we acquire the following

$$x^q y - x y^q = 1.$$

Therefore, we obtain the following lemma.

**Lemma 4.1.** The reduction of the space $Y_{1,1}^{\pi_1}$ is defined by the following equation

$$x^q y - x y^q = 1.$$

This curve is called the Deligne-Lusztig curve for $\text{SL}_2(\mathbb{F}_q)$. The genus of the curve is equal to $q(q - 1)/2$.

**Lemma 4.2.** Let $\sigma \in I_F$ be an element fixing $\pi_1$. We write $\sigma(\alpha) = \zeta \alpha$ with $\zeta \in \mu(q+1)$. Then, the element $\sigma \in I_F$ acts on the reduction $Y_{1,1}^{\pi_1}$ as follows

$$\sigma : \bar{Y}_{1,1}^{\pi_1} \rightarrow \bar{Y}_{1,1}^{\pi_1}; (x, y) \mapsto (\zeta^{-1} x, \zeta^{-1} y).$$

**Lemma 4.3.** Let $g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{O}_F/\pi \mathbb{O}_F)$. Then, the element $g$ acts on the reduction of $Y_{1,1}^{\pi_1}$ as follows

$$g : \bar{Y}_{1,1}^{\pi_1} \rightarrow \bar{Y}_{1,1}^{\pi_1}; (x, y) \mapsto (\bar{a} x + \bar{c} y, \bar{b} x + \bar{d} y).$$

### 4.2 Computation of the reduction of the space $Y_{3,1,e_1} \subset \mathcal{X}(\pi^2)$

In this subsection, we compute the reduction of the space $Y_{3,1,e_1}$. We prove that the reduction $Y_{3,1,e_1}^{\pi_2}$ is defined by the Deligne-Lusztig equation $-x^q y^q + x y^q = 1$ in lemma below.

If $v(f - g) > \alpha$ with $\alpha \in \mathbb{Q}_{\geq 0}$, we write $f \equiv g$ (mod $\alpha+$). Let $(u, X_2, Y_2) \in Y_{3,1,e_1}$. First, recall that the space $Y_{3,1,e_1}$ is defined by the following conditions;

$$v(u) = \frac{1}{q + 1}, v(X_1) = \frac{q}{q^2 - 1}, v(X_2) = \frac{1}{q(q^2 - 1)}, v(Y_1) = \frac{1}{q(q^2 - 1)}, v(Y_2) = \frac{1}{q^3(q^2 - 1)}.$$

Let $\pi_2 \in X_{17}(\pi^2)(F_2)$. We choose an element $\alpha$ such that $\alpha^{q^2(q+1)} = \pi_2$ with $v(\alpha) = 1/q^3(q^2 - 1)$. We consider a defining equation of $Y_{3,1,e_1}^{\pi_2}$

$$\mu_2(X_2, Y_2) = X_1 Y_2^q - X_1 Y_2 - X_2^q Y_1 + X_2 Y_1^q = \pi_2.$$  \hspace{1cm} (4.2)

We change variables as follows $X_1 = \alpha^{q^2} x_1, Y_1 = \alpha^{q^2} y_1, X_2 = \alpha^{q^2} x, Y_2 = \alpha y$. Substituting them to the equality \hspace{1cm} (4.2) and dividing it by $\pi_1$, we acquire the following

$$-x^q y_1 + x y^q = 1.$$

Hence, this induces the following $-x^q y^q + x y^q = 1$, because we have $y_1 = y^q$ by $|\pi|_u(Y_2) = Y_1$. Therefore, we have obtained the following

**Lemma 4.4.** The reduction of the space $Y_{3,3,e_1}^{\pi_2}$ is defined by the following equation

$$-x^q y^q + x y^q = 1.$$
Lemma 4.5. Let $\sigma \in I_F$ be an element fixing $\pi_2$. We write $\sigma(\alpha) = \zeta \alpha$ with $\zeta \in \mu_{q^2(q+1)}$. Then, the element $\sigma \in I_F$ acts on the reduction $Y^2_{3,1,c_i}$ as follows

$$\sigma : Y^2_{3,1,c_i} \longrightarrow Y^2_{3,1,c_i}; (x, y) \mapsto (\zeta^{-q^2}x, \zeta^{-1}y).$$

Remark 4.6. By the definition \ref{Y31}, the space $Y^2_{3,1,c_i}$ has the same reduction as the one of the space $Y^2_{3,1,c_i}$ by swapping $X_1$ for $Y_1$ $(i = 1, 2)$.

4.3 Calculation of the reduction of the space $Y_{3,1,c} \subset X(\pi^2)$

In this subsection, we compute the reduction of the space $Y^2_{3,1,c}$. We prove that the reduction of the space $Y^2_{3,1,c}$ has $(q - 1)$ connected components and each component is defined by the same equation $-x^q y^3 + xy^3 = 1$ as the one of the space $Y^2_{3,1,c_i}$.

Let $\pi_2$ be as in the previous subsection. Let $(u, X_2, Y_2) \in Y^2_{3,1,c}$. Recall that the space $Y^2_{3,1,c}$ is defined by the following conditions:

$$v(u) = \frac{1}{q + 1}, v(X_1) = v(Y_1) = \frac{1}{q(q^2 - 1)}; v(X_2) = v(Y_2) = \frac{1}{q^3(q^2 - 1)}.$$

We choose an element $\alpha$ such that $\alpha^{q^2(q+1)} = \pi_2$ with $v(\alpha) = 1/q^3(q^2 - 1)$. Then, we change variables as follows $X_1 = \alpha^q x_1, Y_1 = \alpha^q y_1, X_2 = \alpha x, Y_2 = \alpha y$. Substituting them to the equality \ref{Y31} and dividing it by $\alpha^{q^2(q+1)}$, we acquire the following equality

$$(x_1 y^q - x^q y_1) - \gamma(x_1^q y - x y_1^q) = \gamma^{q/(q-1)} (4.3)$$

where we set $\gamma := \alpha^{(q-1)/q^2}$. Since we have the following congruence $x_1 \equiv x^q, \gamma \equiv y^q$ modulo $(1/q^2)+$, the equality \ref{Y31} induces the following congruence

$$(x^q y - xy y^q) - \gamma(x^q y - xy y^q) \equiv \gamma^{q/(q-1)} \pmod{(1/q^2)+}. (4.4)$$

We change variables as follows $a = x/y, t = 1/y$ with $a$ an invertible function. Furthermore, we set $Z := \frac{a^{q+1}}{a^q - 1}$. Then, the above congruence \ref{Y31} has the following form

$$Z^q - \gamma \left(sZ^q + \frac{Z^q}{s^{(q-1)}} + \frac{Z}{s^q}\right) \equiv \gamma^{q/(q-1)} \pmod{(1/q^2)+} (4.5)$$

where we set $s := t^{q^2-1}$. By this congruence, we have $v(Z) = 1/q^2$. Now, we change a variable as follows $Z = \gamma^{1/(q-1)}z$. Substituting this to \ref{Y31}, and dividing this by $\gamma^{q/(q-1)}$, we acquire the following congruence

$$z^q - \frac{z}{s^q} = 1 \pmod{0+}. (4.6)$$

We change variables as follows $x := -zt^q, y = 1/t$. Then, the equation \ref{Y31} has the following form

$$-x^q y^q + xy^3 = 1.$$ 

Note that $a$ is an invertible function. By $Z \equiv 0 \pmod{0+}$, we acquire $a \in \mathbb{F}_q^\times$. Hence, we have obtained the following

Proposition 4.7. The reduction of the space $Y^2_{3,1,c}$ is a disjoint union of $(q - 1)$ curves defined by the following equation

$$-x^q y^q + xy^3 = 1.$$
Let \( \{Y_{3,1,c,\zeta}^2\}_{\zeta \in \mathbb{F}_q^*} \) be connected components of the reduction of \( Y_{3,1,c}^{2} \). We compute the inertia action on the reduction of the space \( Y_{3,1,c}^{2} \).

**Lemma 4.8.** Let \( \sigma \in I_F \) be an element fixing \( \pi_2 \). We write \( \sigma(\alpha) = \zeta \alpha \) with \( \zeta \in \mu_{q^2(q+1)} \). Then, the element \( \sigma \in I_F \) acts on the reduction \( Y_{3,1,c,\zeta}^{2} \) as follows

\[
\sigma : Y_{3,1,c,\zeta}^{2} \rightarrow Y_{3,1,c,\zeta'}(X,Y) \mapsto (\zeta'^2 X, \zeta'^{-1} Y).
\]

**Proof.** Note that \( x^\sigma = \zeta^{-1} x, y^\sigma = \zeta^{-1} y \). Hence, we acquire \( a^\sigma = a, t^\sigma = \zeta t \) and \( z^\sigma = z \). Therefore, the required assertion follows. \( \square \)

We describe the action of \( \text{SL}_2(O_F/\pi^2 O_F) \) on the reduction of the space \( Y_{3,1,c}^{2} \).

**Lemma 4.9.** Let \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(O_F/\pi^2 O_F) \).

1. If \( c, d \) are units, \( Y_{3,1,c,\zeta}^{2} \) goes to \( Y_{3,1,c,\zeta'}^{2} \) by the action of \( g \). Moreover, \( g \) acts as follows;

\[
g : Y_{3,1,c,\zeta}^{2} \rightarrow Y_{3,1,c,\zeta'}^{2} ; (x,y) \mapsto \left( \frac{x}{d}, \frac{\bar{d}y}{c} \right).
\]

2. If \( c \) is a unit and \( d \) is divisible by \( \pi \), \( Y_{3,1,c,\zeta}^{2} \) goes to \( Y_{3,1,c,\zeta}^{2} \) by the action of \( g \). Further, the element \( g \) acts as follows;

\[
g : Y_{3,1,c,\zeta}^{2} \rightarrow Y_{3,1,c,\zeta'}^{2} ; (x,y) \mapsto (\bar{c}x, \bar{d}y + \frac{c}{\pi} y^\sigma).
\]

3. If \( c \) is divisible by \( \pi \) and \( d \) is a unit, \( Y_{3,1,c,\zeta}^{2} \) is stable under the action of \( g \). Further, \( g \) acts as follows;

\[
g : Y_{3,1,c,\zeta}^{2} \rightarrow Y_{3,1,c,\zeta}^{2} ; (x,y) \mapsto (\bar{a}x + \frac{c}{\pi} y^\sigma, \bar{d}y).
\]

**4.4 Calculation of the reduction of the space \( Y_{2,2} \subset \mathcal{X}(\pi^2) \)**

In this subsection, we compute the reduction of the space \( Y_{2,2} \). We prove that the reduction of the space \( Y_{2,2}^{2} \) is defined by the following equations;

\[ x^q y - xy^q = 1, Z^q = x^q y - xy^q. \]

This affine curve has \( q(q^2 - 1) \) singular points at \( (x, y) \) with \( x = \zeta y, \zeta \in \mathbb{F}_q^* \setminus \mathbb{F}_q^* \) and \( y^q+1 = \frac{1}{\zeta^q-\zeta} \).

Let \( \pi_2 \) be as in the previous subsection. Let \( (u, X_2, Y_2) \in Y_{2,2} \). We recall that \( Y_{2,2}^{2} \) is defined by the following conditions;

\[ v(u) \geq \frac{q}{q+1}, v(X_1) = v(Y_1) = \frac{1}{q^2-1}, v(X_2) = v(Y_2) = \frac{1}{q^2(q^2-1)}. \]

We choose an element \( \alpha \) such that \( \alpha^{q^2(q+1)} = \pi_2 \).

Now, we consider the equality (4.2). Then, we change variables as follows \( X_1 = \alpha^{q^2} x_1, X_2 = \alpha x_2, Y_1 = \alpha^{q^2} y_1, Y_2 = \alpha y_1 \). Substituting them to the equality (4.2) and dividing it by \( \pi_2 \), we acquire the following

\[ (x_1 y_2^q - x_2^q y_1) - \gamma (x_1^q y_2 - y_1^q x_2) = 1 \text{ (mod } 1/q+). \] (4.7)
where we set \( \gamma := \alpha^{(q-1)(q^2-1)} \). Since we have \( y_1^{q^2} \equiv y_2^{q^2} \pmod{(1/q)^+} \), \( x_1^{q^2} \equiv x_2^{q^2} \pmod{(1/q)^+} \), (4.7) induces the following congruence

\[
(x_2^q y_2 - x_2 y_2^q)^\gamma - \gamma (x_2^q y_2 - y_2^q x_2) = 1 \pmod{(1/q)^+}.
\] (4.8)

In the following, we simply write \( x, y \) for \( x_2, y_2 \). Now, we introduce a new parameter \( Z \) as follows

\[
x^q y - xy^q = 1 + \gamma_1 Z \text{ where the element } \gamma_1 \text{ satisfies } \gamma_1^q = \gamma.
\]

Substituting \( x^q y - xy^q = 1 + \gamma_1 Z \) to the above congruence (4.8), and dividing it by \( \gamma \), we obtain the following congruence

\[
Z^q \equiv x^q y - xy^q \pmod{(1/q)^2}.
\] (4.9)

Hence, we obtain the following proposition.

**Proposition 4.10.** The reduction of the space \( Y_{2,2}^2 \) is defined by the following equations

\[
x^q y - xy^q = 1, Z^q = x^q y - xy^q.
\]

In particular, this curve is an affine curve of genus \( q(q-1)/2 \). This curve has singular points at \( x = \zeta y \) with \( \zeta \in \mathbb{F}_q^\times \backslash \mathbb{F}_q^\times \) and \( y^{q+1} = \frac{1}{\zeta^\alpha} \). Hence, this curve has \( q(q^2-1) \) singular points.

**Lemma 4.11.** Let \( \sigma \in I_F \) be an element fixing \( \pi_2 \). Let \( \alpha_1 \) be an element such that \( \alpha_1^q = \alpha \). We write \( \sigma(\alpha_1) = \zeta \alpha_1 \) with \( \zeta \in \mu_{q^2(q+1)} \). Then, the element \( \sigma \in I_F \) acts on the reduction \( \overline{\mathbb{Y}}_{2,2}^2 \) as follows

\[
\sigma : \overline{\mathbb{Y}}_{2,2}^2 \to \overline{\mathbb{Y}}_{2,2}^2; (x, y, Z) \mapsto (\zeta^{-q} x, \zeta^{-q} y, Z).
\]

**Lemma 4.12.** Let \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(O_F/\pi^2O_F) \). For an element \( a \in O_F/\pi^2O_F \), we denote by \( \tilde{a} \) the image of \( a \) by the canonical map \( O_F/\pi^2O_F \to O_F/\pi O_F \). Then, the element \( g \) acts on the reduction of \( \overline{\mathbb{Y}}_{2,2}^2 \) as follows

\[
g : \overline{\mathbb{Y}}_{2,2}^2 \to \overline{\mathbb{Y}}_{2,2}^2; (x, y, Z) \mapsto (\tilde{a} x + \tilde{b} y + \tilde{c} z, \tilde{d} y, Z).
\]

### 4.5 Analysis of singular residue classes in \( Y_{2,2} \)

In this subsection, we analyze the singular residue classes of the space \( Y_{2,2}^2 \). We find \( q(q^2-1) \) irreducible components defined by \( a^q - a = t^{q+1} \) which attach to the curve in Proposition 4.10 at each singular point. In \([1]\), we prove that, for each supersingular point, there exist \( (p+1) \) components defined by \( a^p - a = t^{p+1} \) in the stable reduction of the modular curve \( X_0(p^4) \). A computation in this subsection is very similar to the one in \([1]\) subsection 4.4.

We keep the same notation as in the previous subsection. We change variables as follows

\[
a := x/y, t := 1/y.
\]

Then, we have the following congruences by the computations in the previous subsection

\[
a^q - a = t^{q+1}(1 + \gamma_1 Z)
\] (4.10)

\[
Z^q \equiv \frac{a^q - a}{t^{q+1}} \pmod{(1/q^2)+}.
\] (4.11)

We set \( s := t^{q-1} \). Then, the congruence (4.11) has the following form under the variables \( (Z, s) \)

\[
Z^q \equiv \frac{(s+1)^q}{s^{q-1}} + \frac{1 + \gamma_1 Z}{s^q} \pmod{(1/q^2)+}.
\] (4.12)
We set \( s + 1 := s_1 \) and consider a locus \( v(s_1) = 1/q^2(q + 1) \). We can easily check that the following congruence holds on the term in the right hand side of the congruence (4.12):

\[
\frac{(s + 1)^q}{s^{q-1}} + \frac{1 + \gamma_1 Z}{s^q} \equiv -1 - \gamma_1 Z - s_1^{q+1} \pmod{(1/q^2)^+}. 
\]

Hence, (4.11) is written as follows

\[
Z^q \equiv -1 - \gamma_1 Z - s_1^{q+1} \pmod{(1/q^2)^+}. 
\]

We choose an element \( \gamma_0 \) such that \( \gamma_0^3 + 1 + \gamma_1 \gamma_0 = 0 \). Further, we choose elements \( \beta, \beta_1 \) such that \( \beta^{q-1} = -\gamma_1, \beta_1^{q+1} = \beta^q \). Then, we have \( v(\beta) = 1/q^3 \) and \( v(\beta_1) = 1/q^2(q + 1) \).

We change variables as follows

\[
Z = \gamma_0 + \beta a, s_1 = \beta_1 s_2.
\]

Substituting them to the congruence (4.13) and dividing it by \( \beta^q \), we acquire the following by the definitions of \( \gamma_0, \beta, \beta_1 \)

\[
a^q - a = s_2^{q+1} \pmod{0+}.
\]

Hence, we have proved the following proposition.

**Proposition 4.13.** In the reduction of the space \( Y_{2,2}^{q^2} \), there exist \( q(q^2 - 1) \) irreducible components defined by \( a^q - a = t^{q+1} \), which attach to the curve in Proposition 4.10 at each singular point.

Let \( \{ D_\zeta \} \) be the underlying affinoid of the singular residue classes of \( Y_{2,2}^{q^2} \).

**Lemma 4.14.** Let \( \sigma \in I_F \) be an element fixing \( \pi_2 \). Then, the element \( \sigma \in I_F \) acts on the reduction \( D_\zeta \) as follows

\[
\sigma : D_\zeta \to D_\zeta; (a, s_2) \mapsto (\frac{\gamma_0 - \sigma(\gamma_0)}{\sigma(\beta)}, \frac{\beta}{\sigma(\beta)}a, \frac{\beta_1}{\sigma(\beta_1)}s_2)
\]

with \( \frac{\gamma_0 - \sigma(\gamma_0)}{\sigma(\beta)} \in \mathbb{F}_{q^2}, \frac{\beta}{\sigma(\beta)} \in \mathbb{F}_q^\times \) and \( \left( \frac{\beta_1}{\sigma(\beta_1)} \right)^{q+1} = \left( \frac{\beta}{\sigma(\beta)} \right)^{\gamma_1} \).

### 4.6 Calculation of the reduction of the space \( Z_{1,1,e_1} \subset X(\pi^2) \)

We will compute the reduction of the space \( Z_{1,1,e_1} \). We prove that the reduction of the space \( Z_{1,1,e_1}^{q^2} \) is defined by the following equation

\[
Z^q = Xq^{q-1} + \frac{1}{X^{q-1}}.
\]

This affine curve with genus 0 has \( 2(q^2 - 1) \) singular points at \( X = \zeta, \zeta \in \mu_{2(q^2 - 1)} \). Similar phenomenon is already observed in the defining equation of "bridging component" \( Z_{1,1}^{q^2} \) in the stable reduction of the modular curve \( X_0(p^3) \) found by Coleman-McMurdy in [CM] Proposition 8.2. See also [P] [Proposition 3.1].

Let \( \pi_2 \) be as in the previous subsection. Let \((u, X_2, Y_2) \in Z_{1,1,e_1} \). Now, we recall that the space \( Z_{1,1,e_1} \) is defined by the following conditions:

\[
v(u) = 1/2, v(X_1) = \frac{1}{2(q - 1)}, v(X_2) = \frac{1}{2q^2(q - 1)}, v(Y_1) = \frac{1}{2q(q - 1)}, v(Y_2) = \frac{1}{2q^3(q - 1)}.
\]
We choose an element $\alpha$ such that $\alpha^{2q^2} = \pi_2$. Then, we have $v(\alpha) = 1/2q^3(q-1)$.

We change variables as follows $X_1 = \alpha^{q^2}x_1$, $X_2 = \alpha^{q}x$, $Y_1 = \alpha^{q^2}y_1$, $Y_2 = \alpha y$. Substituting them to (4.12) and dividing it by $\pi_2$, we acquire the following congruence

$$-x^qy_1 + \gamma(x_1y^q + xy_1^q) = 1 \pmod{(1/2q^3)}$$

(4.14)

where we set $\gamma := \alpha^{q(q-1)^2}$. We have $v(\gamma) = (q-1)/2q^2$. Since we have $y_1 \equiv y_2^q$, $x_1 \equiv x_2^q$ modulo $(1/2q^3)$, the congruence (4.14) induces the following congruence

$$-x^qy^q + \gamma(x^q y^q + xy^q) = 1 \pmod{(1/2q^3)}.$$  

(4.15)

Then, we introduce a new parameter $Z$ as follows $1 + xy^q = \gamma_1Z$ where $\gamma_1$ satisfies $\gamma_1^q = \gamma$. By substituting $1 + xy^q = \gamma_1Z$ to (4.15) and dividing it by $\gamma$, we acquire the following congruence

$$
\left( Z + \frac{1}{y^{q^2}} + y^{q^2-1}\right)^q \equiv \gamma_1 y^{q(q^2-1)}Z \pmod{(1/2q^3)}.
\tag{4.16}
$$

Again, we introduce a new parameter $Z_1$ as follows

$$Z + \frac{1}{y^{q^2-1}} + y^{q^2-1} = \gamma_2y^{-q^2-1}Z_1$$

(4.17)

where we choose an element $\gamma_2$ such that $\gamma_2^q = \gamma_1$. Substituting (4.17) to (4.16) and dividing it by $y^{q^2(q^2-1)}/\gamma_1$, the following congruence holds $Z_1^q = Z \pmod{(1/2q^3)}$. Furthermore, by substituting this to (4.17), we obtain the following congruence

$$Z_1^q + \frac{1}{y^{q^2-1}} + y^{q^2-1} \equiv \gamma_2y^{q^2-1}Z_1 \pmod{(1/2q^3)}.$$  

(4.18)

Hence, we have obtained the following proposition.

**Proposition 4.15.** The reduction of the space $\mathbb{Z}^{x_2}_{1,1,e_1}$ is defined by the following equation

$$Z_1^q + \frac{1}{y^{q^2-1}} + y^{q^2-1} = 0.$$  

In particular, this curve is an affine curve of genus 0. This curve has singular points at $y = \zeta$ with $\zeta \in \mu_{2(q^2-1)}$.

**Remark 4.16.** By the definition (3.1), the space $\mathbb{Z}^{x_2}_{1,1,e_1}$ has the same reduction as the one of the space $\mathbb{Z}^{x_2}_{1,1,e_1}$ by swapping $X_i$ for $Y_i$ ($i = 1, 2$).

**Lemma 4.17.** Let $\sigma \in I_F$ be an element fixing $\pi_2$. We write $\sigma(\alpha) = \zeta \alpha$ with $\zeta \in \mu_{2q^2}$. Then, the element $\sigma \in I_F$ acts on the reduction $\mathbb{Z}^{x_2}_{1,1,e_1}$ as follows

$$\sigma: \mathbb{Z}^{x_2}_{1,1,e_1} \longrightarrow \mathbb{Z}^{x_2}_{1,1,e_1}; (y, Z) \mapsto (\zeta^{-1}y, Z).$$

4.7 Analysis of the singular residue classes of $\mathbb{Z}^{x_2}_{1,1,e_1}$

In this subsection, we analyze the singular residue classes in $\mathbb{Z}^{x_2}_{1,1,e_1}$. Then, we find $2(q^2-1)$ irreducible components defined by the Artin-Schreier equation $a^q - a = t^2$. A calculation in this subsection is very similar to the one in [1][subsection 3.2].
We keep the same notation as in the previous subsection. We set $s := y^{q^2 - 1}$. We recall the following congruence (4.19)

$$Z_1^q + \frac{1}{s} + \left(\frac{s}{1 - \gamma_2 Z_1}\right) \equiv 0 \pmod{(1/2q^3)+}.$$  

We set $F(s, Z_1) := Z_1^q + \frac{1}{s} + \left(\frac{s}{1 - \gamma_2 Z_1}\right)$.

We choose an elements $\gamma_0$ such that $\gamma_0^q + 2(1 + \gamma_2 \gamma_0)^{1/2} = 0$. We set $s_0 := (1 + \gamma_2 \gamma_0)^{1/2}$. Then, we have $\partial_s F(s_0, \gamma_0) = 0, F(s_0, \gamma_0) = 0$. Furthermore, we choose elements $\beta, \beta_1$ such that $\beta^{q-1} = \gamma_2 s_0, \beta_1^q = -\beta^q/s_0^q$. Note that $v(\beta) = 1/2q^3, v(\beta_1) = 1/4q^3$.

We change variables as follows

$$Z_1 = \gamma_0 + \beta a, s = s_0 + \beta_1 s_1.$$

By substituting these to (4.19), and dividing it by $\beta^q$, we acquire the following, $a^q - a \equiv s_1^2 \pmod{0+}$. Hence, we have obtained the following proposition.

**Proposition 4.18.** In the reduction of the space $Z^2_{1,1,c_1}$, there exist $2(q^2 - 1)$ irreducible components defined by $a^q - a = t^2$.

Let $D_\zeta$ be the underlying affinoid of the singular residue classes in $Z^2_{1,1,c_1}$.

**Lemma 4.19.** Let $\sigma \in I_F$ be an element fixing $\pi_2$. Then, the element $\sigma \in I_F$ acts on the reduction $D_\zeta$ as follows

$$\mathfrak{D}_\zeta \rightarrow \mathfrak{D}_\zeta; (a, s_1) \mapsto \left(\frac{\gamma_0 - \sigma(\gamma_0)}{\sigma(\beta)}, \frac{\beta}{\sigma(\beta)} a, \frac{\beta_1}{\sigma(\beta_1)} s_1\right)$$

with $(\frac{\gamma_0 - \sigma(\gamma_0)}{\sigma(\beta)}) \in \mathbb{F}_q, (\frac{\beta}{\sigma(\beta)}) \in \mathbb{F}_q^\times$ and $(\frac{\beta_1}{\sigma(\beta_1)})^2 = (\frac{\beta}{\sigma(\beta)})$.

### 4.8 Calculation of the reduction of the space $Z_{1,1,c} \subset \mathcal{X}(\pi^2)$

In this subsection, we compute the reduction of the space $Z_{1,1,c}$. The reduction of the space $Z^2_{1,1,c}$ has $(q - 1)$ connected components and each component is defined by $Z^q = X^{q^2 - 1} + (1/X)^{q^2 - 1}$ as the reduction $Z^2_{1,1,c}$.

Let $\pi_2$ be as in the previous subsection. Let $(u, X_2, Y_2) \in Z^2_{1,1,c}$. Recall that the space $Z^2_{1,1,c}$ is defined by the following conditions;

$$v(u) = 1/2, v(X_1) = v(Y_1) = \frac{1}{2q(q - 1)}, v(X_2) = v(Y_2) = \frac{1}{2q^3(q - 1)}.$$

We choose an element $\alpha$ such that $\alpha^{2q^2} = \pi_2$ with $v(\alpha) = 1/2q^3(q - 1)$. We change variables as follows $X_1 = \alpha^q x_1, Y_1 = \alpha^q y_1, X_2 = \alpha x, Y_2 = \alpha y$. Substituting them to (4.12), and dividing this by $\alpha^{q+1}$, we acquire the following

$$(x_1 y^q - x^q y_1)^q - \gamma^{q+1}(x_1 y^q - x y^q) = \gamma^{q/(q-1)}$$

where we set $\gamma := \alpha^{(q-1)^2}$. Then, we have $v(\gamma) = (q - 1)/2q^3$. Since we have $x_1 \equiv x^{q^2}, y_1 \equiv y^{q^2}$ modulo $(q + 1)/2q^2 +$, the equality (4.20) induces the following congruence

$$(x^q y - y^q)^q - \gamma^{q+1}(x^q y - y^q) = \gamma^{q/(q-1)} \pmod{\left(\frac{q + 1}{2q^2} \right)^+}.$$  

(4.21)
We put
\[ Z := x^q y - x y^q \]  
Further, we set \( a := x/y, t := 1/y \). Substituting (4.22) to (4.21), we acquire the following
\[ Z^q - \gamma^{q+1} \left( \frac{t^{q^2-1} Z^q}{t(q+1)(q^2-1)} + \frac{Z}{t(q-1)(q^2-1)} \right) \equiv \gamma^{q/(q-1)} \pmod{\frac{q+1}{2q^2}}. \]  
(4.23)

We set \( Z = \gamma^{1/(q-1)} + \frac{\gamma^{q/(q-1)}}{t^{q^2-1}} + Z_1 \). Substituting this to (4.23), we obtain the following congruence
\[ Z_1^q - \gamma^{q+1} \frac{Z_1}{t(q^2-1)} \equiv \gamma^{(q^2+q-1)/(q-1)} \left( \frac{1}{t(q+1)(q^2-1)} + \frac{1}{t(q-1)(q^2-1)} \right) \pmod{\frac{q+1}{2q^2}}. \]  
(4.24)

We choose an element \( \gamma_1 \) such that \( \gamma_1^q = \gamma \). We change a variable as follows \( Z_1 = \gamma_1^{(q^2+q-1)/(q-1)} \epsilon Z_1 \). Substituting this to (4.21) and dividing it by \( \gamma^{(q^2+q-1)/(q-1)} \), we acquire the following
\[ Z^q = \frac{1}{t^{q^2-1}} + t^{q^2-1} + \gamma_1 \frac{Z}{t^{q^2-1}} \pmod{1/2q^3}. \]  
(4.25)

Hence, we have obtained the following

**Proposition 4.20.** The reduction of the space \( Z_{1,1,c}^{\pi} \) is a disjoint union of \( (q-1) \) curves defined by the following equation
\[ Z^q = \frac{1}{t^{q^2-1}} + t^{q^2-1}. \]

Furthermore, there exist \( 2(q^2 - 1) \) irreducible components defined by \( a^q - a = s^2 \) which attach to the above curve at each singular point.

**Proof.** The required assertion follows from (4.25) and the computation in the previous subsection. \( \square \)

We describe the action of \( SL_2(O_F/\pi^2 O_F) \) on the reduction of the space \( Z_{1,1,e_i}^{\pi} \).  

**Lemma 4.21.** Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(O_F/\pi^2 O_F) \). For an element \( a \in O_F/\pi^2 O_F \), we denote by \( \bar{a} \) by the image of \( a \) by the canonical map \( O_F/\pi^2 O_F \to O_F/\pi O_F \).

1. If \( c, d \) are units, \( Z_{1,1,e_i}^{\pi} \) goes to \( Z_{1,1,e_i}^{\pi} \). The element \( g \) acts as follows;
   \[ g : Z_{1,1,e_i}^{\pi} \to Z_{1,1,e_i}^{\pi} ; (Z, y) \mapsto (Z, \bar{dy}) \]

2. If \( c \) is a unit and \( d \) is divisible by \( \pi \), \( Z_{1,1,e_i}^{\pi} \) goes to \( Z_{1,1,e_i}^{\pi} \). The element \( g \) acts as follows;
   \[ g : Z_{1,1,e_i}^{\pi} \to Z_{1,1,e_i}^{\pi} ; (Z, y) \mapsto (Z, \bar{cx}) \]

3. If \( c \) is divisible by \( \pi \) and \( d \) is a unit, \( Z_{1,1,e_i}^{\pi} \) is stable by \( g \). The element \( g \) acts as follows;
   \[ g : Z_{1,1,e_i}^{\pi} \to Z_{1,1,e_i}^{\pi} ; (Z, y) \mapsto (Z, \bar{dy}) \]
5 Analysis of the spaces $\mathcal{X}(\pi)$ and $\mathcal{X}(\pi^2)$

In this section, we analyze the spaces $\mathcal{X}(\pi)$ and $\mathcal{X}(\pi^2)$. In the stable reduction of the spaces $\mathcal{X}(\pi^i)$ ($i = 1, 2$), nothing interesting happens except for the reduction of $Y_{1,1}, Y_{3,1}, Y_{2,2}$ and $Z_{1,1}$. More precisely, we prove that the complements $\mathcal{X}(\pi) \backslash Y_{1,1}$ and $\mathcal{X}(\pi^2) \backslash (Z_{1,1} \cup Y_{2,2} \cup Y_{3,1})$ are disjoint unions of annuli. We prove it in subsections 5.1 and 5.2. In other words, we prove that the inverse image by $p_2 : \mathcal{X}(\pi^2) \to \mathcal{X}(1)$ of every circle $C[p^{-\nu}] \subset \mathcal{X}(1)$ with $\alpha \in \left((0, \frac{1}{q}) \cap \mathbb{Q}\right) \backslash \left\{\frac{1}{q}, \frac{1}{q+1}\right\}$ is a disjoint union of circles. As a result of the analysis in subsection 5.2, we will construct a stable covering of the wide open space $\mathcal{X}(\pi^2)$ by defining basic wide open subspaces in $\mathcal{X}(\pi^2)$ which contain the spaces $Y_{3,1}, Y_{2,2}$ and $Z_{1,1}$ in subsection 5.3. In subsection 5.4, we give intersection multiplicities for stable reduction of the space $\mathcal{X}(\pi^2)$. The notion of wide open spaces and stable coverings is due to R. Coleman, for example see [CM, Section 2].

5.1 Analysis of the space $\mathcal{X}(\pi)$

We analyze the space $\mathcal{X}(\pi)$. We prove that the complement $\mathcal{X}(\pi) \backslash Y_{1,1}$ is a disjoint union of annuli. Hence, we conclude that the space $\mathcal{X}(\pi)$ is a basic wide open space. This fact is well-known, but we write down a calculation for the convenience of a reader.

**Lemma 5.1.** We consider the following equality

$$[\pi]_u(X) = X^q + uX^q + \pi X = 0.$$  

Then, we have the following

1. If $v(X) > 1/(q^2 - 1)$, we have $v(X) = (1 - v(u))/(q - 1)$ and $v(u) < q/(q + 1)$.
2. If $v(X) = 1/(q^2 - 1)$, we have $v(u) \geq q/(q + 1)$.
3. If $v(X) < 1/(q^2 - 1)$, we have $v(X) = v(u)/q(q - 1)$ and $v(u) < q/(q + 1)$.

Let $X, Y$ be $\pi$-torsion points of the universal formal group $F^{\text{univ}}$.

**Lemma 5.2.** Let $(u, X, Y) \in \mathcal{X}(\pi)$. A case 1 for $X$ and 1 for $Y$ in Lemma 5.1 does not occur.

**Proof.** We consider the equality $X^qY - XY^q = \pi_1$. The valuation of the left hand side is larger than $(q + 1)(1 - v(u))/(q - 1)$. By $v(u) < q/(q + 1)$, we acquire $(q + 1)(1 - v(u))/(q - 1) > v(\pi_1) = 1/(q - 1)$. Hence, this case does not happen. \[\square\]

Let $1 \leq a, b \leq 3$ be positive integers. Let $W_{a,b}$ denote a subspace of $(u, X, Y) \in \mathcal{X}(\pi)$ defined by the conditions $a$ for $X$ in Lemma 5.1 and $b$ for $Y$ in Lemma 5.1. Let $W_{a,b}$ except for $(a, b) = (1, 3), (3, 1), (2, 2), (3, 3)$ is empty by Lemmas 5.1 and 5.2. Furthermore, we note that the space $W_{2,2}$ is equal to $Y_{1,1}$. Recall that, for a subspace $X$ and $\pi_n \in \mathcal{X}^\pi_n(\pi^n)(F_n)$, we write $X^{\pi_n}$ for the intersection $X \cap \mathcal{X}^\pi_n(\pi^n)$.

**Lemma 5.3.** The spaces $W_{1,3}^{\pi_1}$ and $W_{3,1}^{\pi_1}$ are annuli of width $1/(q^2 - 1)$.

**Proof.** We prove the assertion only for the space $W_{1,3}$. By $v(u) < q/(q + 1)$, we acquire $v(Y) < v(X)$. We have $v(XY^q) = 1/(q - 1)$. Hence, we acquire the following $XY^q = \pi_1 +$ higher terms. Here, the valuation of the higher terms is greater than $1/(q - 1)$. Therefore, the required assertion follows. \[\square\]

**Lemma 5.4.** The space $W_{3,1}^{\pi_1}$ is a disjoint union of $(q - 1)$ annuli with width $1/(q^2 - 1)$.
Proof. Consider the equality $X^aY - XY^a = \pi_1$. We set $X = \zeta Y + Z$ with $\zeta \in \mathbb{F}_q^\times$ and $v(Y) < v(Z)$. Then, the equality $X^aY - XY^a = \pi_1$ has the following form $-Y^aZ = \pi_1 + \text{higher terms with } v(Z) = \frac{1-v(u)}{q-1} > v(Y)$. Here the valuation of the higher terms is greater than $1/(q-1)$. Hence, $X$ and $Z$ are written with respect to $Y$. Thereby, we have proved the required assertion.

By Lemmas [5.3 and 5.4] we know that the space $\mathcal{X}(\pi)$ is a basic wide open space.

**Corollary 5.5.** The complement $\mathcal{X}(\pi)\backslash Y_{1,1}$ is a union of annuli $W_{1,3} \cup W_{3,1} \cup W_{3,3}$. In particular, $\mathcal{X}(\pi)\backslash Y_{1,1}$ is a disjoint union of annuli. In other words, the space $\mathcal{X}(\pi)$ is a basic wide open space.

**Proof.** The required assertion follows from Lemmas [5.3 and 5.4].

### 5.2 Analysis of the space $\mathcal{X}(\pi^2)$

In this subsection, we analyze the space $\mathcal{X}(\pi^2)$. To do so, we define several subspaces of $\mathcal{X}(\pi^2)$ and prove that the subspaces are merely disjoint union of annuli. Finally, we prove that the complement $\mathcal{X}(\pi^2)\backslash (Y_{3,1} \cup Z_{1,1} \cup Y_{2,2})$ is a disjoint union of annuli. Propositions [5.6 and 5.7] play a key role to show that a covering $\mathcal{C}(\pi^2)$, which will be constructed in the next subsection, is actually a stable covering.

Let $(u, X_2, Y_2) \in W_1$ denote a subspace defined by the following conditions;

$$0 < v(u) < \frac{1}{q + 1}, v(X_1) = \frac{1-v(u)}{q-1}, v(X_2) = \frac{1-v(u)}{q(q-1)}, v(Y_1) = \frac{v(u)}{q(q-1)}, v(Y_2) = \frac{v(u)}{q^3(q-1)}.$$ 

Let $(u, X_2, Y_2) \in W_2$ denote a subspace defined by the following conditions;

$$0 < v(u) < \frac{1}{q + 1}, v(X_1) = \frac{1-v(u)}{q-1}, v(X_2) = \frac{v(u)}{q(q-1)}, v(Y_1) = \frac{v(u)}{q(q-1)}, v(Y_2) = \frac{v(u)}{q^3(q-1)}.$$ 

Let $(u, X_2, Y_2) \in W_3$ denote a subspace defined by the following conditions;

$$\frac{1}{q+1} < v(u) < \frac{q}{q+1}, v(X_1) = \frac{1-v(u)}{q-1}, v(X_2) = \frac{1-v(u)}{q^2(q-1)}, v(Y_1) = \frac{v(u)}{q(q-1)}, v(Y_2) = \frac{v(u)}{q^3(q-1)}.$$ 

For $1 \leq i \leq 3$, let $W_i^v \subset \mathcal{X}(\pi^2)$ be a subspace defined by the following condition; $(u, X_2, Y_2) \in W_i^v$ is equivalent to $(u, Y_2, X_2) \in W_i$. Let $U_1$ (resp. $U_2$ resp. $U_3$) be a subspace defined by the following conditions;

$$v(X_1) = v(Y_1) = \frac{v(u)}{q(q-1)}, v(X_2) = v(Y_2) = \frac{v(u)}{q^3(q-1)}$$

and $0 < v(u) < \frac{1}{q+1}$, (resp. $\frac{1}{q+1} < v(u) < \frac{1}{q}$, resp. $\frac{1}{q} < v(u) < \frac{q}{q+1}$.)

**Proposition 5.6.** Let the notation be as above. Then, we have the followings

1. The spaces $W_1^{\pi^2}$ and $W_2^{\pi^2}$ are annuli of width $1/q^3(q^2-1)$.
2. The spaces $W_2^{\pi^2}$ and $W_3^{\pi^2}$ are disjoint unions of $(q-1)$ annuli of width $1/q^3(q^2-1)$.
3. The spaces $W_3^{\pi^2} \backslash Z_{1,1,\pi^2}$ and $W_3^{\pi^2} \backslash Z_{4,1,\pi^2}$ are disjoint unions of two annuli of width $1/2q^4(q+1)$. 

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Proof. We prove the assertion 1. Note that $v(X_2 Y_1^q) = 1/q(q - 1)$. We consider the equality (4.2). Then, we acquire the following $X_2 Y_1^q = \pi_2 + \text{higher terms}$. Here, the valuation of the higher terms is greater than $1/q(q - 1)$. Hence, the space $W_1(C)$ is isomorphic to
\[ \{ Y_2 \in C | 0 < v(Y_2) < \frac{1}{q^3(q^2 - 1)} \}. \]
Therefore, the required assertion follows.

We prove the assertion 2. Note that $v(X_2 Y_1^q) = v(X_2 Y_1^q) < 1/q(q - 1)$. By the equality (4.2), the following holds $X_2 = \zeta Y_1 + \text{higher terms}$ with some $\zeta \in F_q^*$. Here, the valuation of the higher terms is greater than $v(X_2)$. Therefore, the space $W_2$ splits to $(q - 1)$ components, and the $C$-valued point of each component is identified with
\[ \{ Y_2 \in C | 0 < v(Y_2) < \frac{1}{q^3(q^2 - 1)} \}. \]
Hence, the required assertion follows.

We prove the assertion 3. Consider the equality (4.2)
\[ X_1 Y_2^q - X_1^q Y_2 - X_1^q Y_1 + X_2 Y_1^q = \pi_2. \] (5.1)
Note that the valuation of the term $X_1^q Y_1$ is smallest among the terms in the left hand side of the equality (5.1). Note that $v(X_2 Y_1^q) < v(X_1 Y_2^q)$ is equivalent to $v(u) < 1/2$. Moreover, note that $v(X_2^q Y_2) > v(X_2 Y_2) - v(Y_2^{q(q^2 - 1)})$ is equivalent to $v(u) < \frac{1}{q+1}$.

By (5.1) and $[\pi]_{u}(X_2) = X_1, [\pi]_{u}(Y_2) = Y_1$, we acquire the following equality
\[ X_2^q Y_2^q - X_2^q Y_2 - X_1^q Y_2 = -\pi_2 + X_2 Y_2^{q^2} + \text{higher terms} \] (5.2)
where the valuation of the higher terms is greater than $\min\{ (v(X_2)/q) + v(Y_2^{q(q^2 - 1)}) - v(Y_2^q) \}$. Note that $(v(X_2)/q) + v(Y_2^{q(q^2 - 1)}) < q(v(X_2) + (q^3 + 1)v(Y_2))$ is equivalent to $v(u) < 1/2$.

We introduce a new parameter $Z$ as follows
\[ X_2 Y_2^q - X_2^q Y_2 = -\pi_2^{1/q} + Y_2^{q^2} Z. \] (5.3)
Substituting this to (5.2) and dividing it by $Y_2^{q^2}$, we acquire the following
\[ Z^{q^2} = X_2 + \text{higher terms} \] (5.4)
where the valuation of the higher terms is greater than $\min\{ v(Y_2^{q^2} Z), v(Z^{q^2} Y_2) \} - v(Y_2^q)$. By substituting (5.1) to (5.3), the following equality holds
\[ (Z Y_2)^{q^2} = -\pi_2^{1/q} + Y_2^{q^2} Z + Z^{q^2} Y_2 + \text{higher terms} \] (5.5)
where the valuation of the higher terms is greater than $\min\{ v(Y_2^{q^2} Z), v(Z^{q^2} Y_2) \}$. Note that $v(Y_2^{q^2} Z) = v(Z^{q^2} Y_2)$ is equivalent to $v(u) > 1/2$.

Now, we consider a case $\frac{1}{q+1} < v(u) < \frac{1}{2}$. Again, we introduce a new parameter $Z_1$ as follows
\[ Z Y_2 = -\pi_2^{1/q} + Y_2^{q^2} Z_1. \]
Substituting this to (5.5) and dividing it by $Y_2^{q^2}$, we obtain the following equality $Z_1^{q^2} = Z + \text{higher terms}$. Here, the valuation of the higher terms is greater than $v(Z)$. Hence, the
parameters $Z, X_2, Y_2$ are written with respect to $Z_1$. Note that $v(Z_1) = \frac{1-v(u)}{q(q-1)}$. Hence, the required assertion follows.

Secondly, we consider a case $\frac{1}{q+1} < v(u) < \frac{q}{q+1}$. We set $ZY_2 = -\pi^2_2/q^2 + Z_2Y_1$. Then, substituting this to (5.8) and dividing it by $Z^q$, we acquire the following equality $Z^q = Y_2 +$ higher terms. Here the valuation of the higher terms is greater than $v(Y_2)$. Hence, the required assertion follows.

**Proposition 5.7.** Let the notation be as above. Then, we have the followings

1. The space $U_1^2$ is a disjoint union of $q(q-1)$ annuli of width $1/q^3(y^2 - 1)$.
2. The spaces $U_2^2$ and $U_3^2$ are disjoint unions of $(q-1)$ annuli of width $1/2q^3(q + 1)$.

**Proof.** We prove the assertion 1. Recall that we have $0 < v(u) < \frac{1}{q+1}$ on $U_1$. Consider the equality (4.2). Since we have $[\pi]_u(X_2) = X_1, [\pi]_u(Y_2) = Y_1$, we acquire the following

$$(X_2^qY_2 - X_2Y_2^q) - (X_2qY_2 - X_2Y_2^q) = \text{higher terms}$$

(5.6)

where the valuation of the higher terms is greater than $v(Y_2^{q(q+1)})$. Then, we set $a := X_2/Y_2$ with $v(a) = 0$ and $z := a^q - a$. By substituting these to the equality (5.6), and dividing it by $Y_2^{q(q+1)}$, we acquire the following

$$z^g - Y_2^{(q-1)(q^2 - 1)} = \text{higher terms}$$

where the valuation of the higher terms is greater than $v(Y_2^{q(q+1)})$. Therefore, we find $v(z) = v(Y_2^{q(q+1)}) > 0$ and this equation splits to $q$-equations. By $v(z) > 0$, we acquire $a \in \mathbb{F}_q^\times$. Therefore, the required assertion follows.

We prove the assertion 2. Recall that we have $\frac{1}{q+1} < v(u) < \frac{q}{q+1}$, $v(X_1) = v(Y_1) = \frac{v(u)}{q(q-1)}$, and $v(X_2) = v(Y_2) = \frac{v(u)}{q(q-1)}$ on $U_2 \cup U_3$. We consider the equality (4.2). Since we have $[\pi]_u(X_2) = X_1, [\pi]_u(Y_2) = Y_1$, we obtain the following equality

$$(X_2^qY_2 - X_2Y_2^q) - (X_2^qY_2 - X_2Y_2^q) = \pi_2 + \text{higher terms}$$

(5.7)

where the valuation of the higher terms is greater than $\min\{v(Y_2^{q(q+1)}), v(Y_2^{q(q+1)}), v(Y_2^{q(q+1)}), v(Y_2^{q(q+1)}) \}$. We set $a := X_2/Y_2$ with $v(a) = 0$ and $z := a^q - a$. Then, the equality (5.7) induces the following

$$Y_2^{q(q+1)}z^g - Y_2^{q(q+1)}(z^g + z) = \pi_2 + \text{higher terms}$$

(5.8)

where the valuation of the higher terms is greater than $\min\{v(Y_2^{q(q+1)}), v(Y_2^{q(q+1)}), v(Y_2^{q(q+1)}), v(Y_2^{q(q+1)}) \}$. Note that $v(z) = \frac{1}{q(q-1)} - (q + 1)v(Y_2) > 0$. We set $Y_2^{q+1}z = \pi_2a + Y_2^{q-1} \pi_2^a q^2 + z_1$. By substituting this to (5.8), the equality (5.8) has the following form

$$z_1^q - Y_2^{q-1}(q^2 - 1)\pi_2^q - Y_2^{q+1}(q^2 - 1)\pi_2^2 q^2 - Y_2^{q(q+1)}z_1 = 0 + \text{higher terms}$$

(5.9)

where the valuation of the higher terms is greater than $\min\{v(Y_2^{q(q+1)}), v(Y_2^{q(q+1)}), v(Y_2^{q(q+1)}) \}$. Note that $v(Y_2^{q(q+1)(q^2 - 1)}\pi_2) < v(Y_2^{q(q+1)(q^2 - 1)}\pi_2^2 q^2)$ is equivalent to $1/2 < v(u)$.

First, we consider a case $1/(q+1) < v(u) < 1/2$. In this case, we have $v(z_1^q) = v(Y_2^{q(q+1)(q^2 - 1)}\pi_2^2 q^2)$ by (5.9). By $v(u) > \frac{1}{q+1}$, we obtain $v(z_1^q) < v(Y_2^{q(q-1)}z_1)$. Then, By setting $Z := \frac{\pi_2^2 q^2 Y_2^{q-1} z_1}{z_1}$,
we acquire the following $Y_2 = Z^q + \text{higher terms}$ by \cite{[8,9]}. Here, the valuation of the higher terms is greater than $v(Y_2)$. Further, we have $a \in \mathbb{F}_q^\times$. Therefore, the required assertion follows.

Secondly, we consider the other case $1/2 < v(u) < q/(q + 1)$. In this case, we have $v(z_1^2) = v(Y_2^{q(1-q^2-1)}\pi_2)$ by \cite{[8,9]}. By $v(u) > 1/2$, we acquire $v(z_1^2) < v(Y_2^{q(q^2-1)/2})$. By setting $Z := \frac{Y^{q^2-1}}{\pi_2}$, we obtain $Y_2 = Z^q + \text{higher terms}$ by \cite{[8,9]}. Here, the valuation of the higher terms is greater than $v(Y_2)$. Furthermore, we have $a \in \mathbb{F}_q^\times$. Hence, the required assertion follows.

\textbf{Corollary 5.8.} The complement $\mathcal{X}(\pi^2) \backslash (Y_{3,1} \cup Z_{1,1} \cup Y_{2,2})$ is a disjoint union of annuli.

\textbf{Proof.} The required assertion follows from Propositions \cite{[8,9]} and \cite{[10, subsection 5.2]}. \hfill \Box

\section{5.3 Stable covering of $\mathcal{X}(\pi^2)$}

In this subsection, we construct the stable covering of the wide open space $\mathcal{X}(\pi^2)$. The space $\mathcal{X}(\pi^2)$ is not basic wide open. We construct a covering $\{V_i\}_{i \in I}$ of $\mathcal{X}(\pi^2)$ with the piece $V_i$ a basic wide open space. Namely, we prove that all intersections $V_i \cap V_j = \emptyset$ are annuli. We prove that, if $i, j, k \in I$ are different from each other, the intersection $V_i \cap V_j \cap V_k$ is empty. Furthermore, we show that the space $V_i$ contains an underlying affinoid $Z_{V_i}$ and the complement $V_i \backslash Z_{V_i}$ is equal to a disjoint union of annuli $\bigcup_{j \in I} (V_i \cap V_j)$. Similar constructions of stable coverings of modular curves are found in [Section 9 \cite{CM} and \cite{11} subsection 5.2]. See \cite{CM} Section 2 or \cite{W3} section 2.3 for (basic) wide open spaces and stable coverings.

We define several subspaces of $\mathcal{X}(\pi^2)$. Let $(u, X_2, Y_2) \in \mathcal{X}(\pi^2)$. Let $V$ be a subspace defined by the following condition $v(u) > 1/2$. This space contains $Y_{2,2}$. Let $T$ be the set of the singular residue classes in $Y_{2,2}$. For $T \in T$, let $X_T \subset T$ be the underlying affinoid. We set

$$V_1 := V \backslash \bigcup_{T \in T} X_T.$$

In this subsection, we write $V_{e_1}'$ for $W_3$ in the previous subsection. This space contains the space $Z_{1,1,e_1}$. Let $S_{e_1}$ be the set of the singular residue classes in $Z_{1,1,e_1}$. For $S \in S_{e_1}$, let $X_S \subset S_{e_1}$ be the underlying affinoid. We put

$$V_{2,e_1} := V_{e_1}' \backslash \bigcup_{S \in S_{e_1}} X_S.$$

Let $(u, X_2, Y_2) \in V_{3,e_1}$ be a subspace defined by the following conditions;

$$0 < v(u) < 1/2, v(X_1) = \frac{1 - v(u)}{q - 1}, v(Y_1) = \frac{v(u)}{q(q - 1)}.$$ 

Then, the space $V_{3,e_1}$ contains the space $Y_{3,1,e_1}$. Furthermore, for $i = 2, 3$, let $V_{i,e_1}'$ be a subspace defined by the following condition; $(u, X_2, Y_2) \in V_{i,e_1}'$ is equivalent to $(u, Y_2, X_2) \in V_{i,e_1}$. Let $(u, X_2, Y_2) \in V_{3,e}$ (resp. $V_{2,e}$) be a subspace defined by the following conditions;

$$v(X_1) = v(Y_1) = \frac{v(u)}{q(q - 1)}, v(X_2) = v(Y_2) = \frac{v(u)}{q^3(q - 1)}.$$
and $0 < v(u) < 1/2$. (resp. $\frac{1}{q+1} < v(u) < \frac{q}{q+1}$.) Let $\mathcal{S}_c$ be a set of the singular residue classes of the space $\mathbb{Z}_{1,1,c}$. For $S \in \mathcal{S}_c$, let $X_S \subset S$ denote the underlying affinoid subdomain. Then, we set

$$V_{2,c} := V'_{2,c} \bigcup_{S \in \mathcal{S}_c} X_S.$$

Let $\mathcal{C}(\pi^2)$ be a covering of $\mathcal{X}(\pi^2)$ consists of

$$\{V_1, V_{i,e_1}, V_{i,e_2}, V_{i,e_3} \}_{i=2,3} \cup \{X_{T}, X_{S_1}, X_{S_2} \}_{T \in T, S_1 \in S_{e_1}, S_2 \in S_c}.$$

**Proposition 5.9.** Let the notation be as above. Then, the covering $\mathcal{C}(\pi^2)$ is a stable covering of $\mathcal{X}(\pi^2)$.

**Proof.** Note that the disjoint union $(V_1 \cap V'_{2,e_1} \cap \mathcal{V}_{3,1,e_1}^\pi) \cup (V_{2,e_1} \cap \mathcal{V}_{3,1,e_1}^\pi)$ is equal to $W_{3,1,e_1}^\pi$. Hence, the intersections $V_1 \cap V_{2,e_1}$ and $V'_{2,e_1} \cap \mathcal{V}_{3,1,e_1}^\pi$ are disjoint unions of annuli by Proposition 5.6.3. The intersection $V_{2,c} \cap V_1$ is equal to $U_3$ and, hence the intersection is a disjoint union of annuli by Proposition 5.7.2. The intersection $V_{2,c} \cap V_{3,c}$ is equal to $U_2$ and, hence is a disjoint union of annuli by Proposition 5.7.1. The complements $V_{3,e_1} \cap \mathcal{V}_{3,1,e_1}^\pi$ and $V_{3,c} \cap \mathcal{V}_{3,1,c}$ are disjoint union of annuli by Propositions 5.3,1,2 and 5.7.1. Hence, the required assertion follows.

We explain a shape of the stable reduction of the Lubin-Tate space $\mathcal{X}(\pi^2)$ as already mentioned in Introduction. Let $\overline{Y}_{2,2}$ be the projective completion of the affine curve $\overline{Y}_{2,2}$. Then, the complement $\overline{Y}_{2,2} \setminus \overline{Y}_{2,2}^\pi$ consist of $(q+1)$ closed points. The projective curve $\overline{Y}_{2,2}$ meets the projective completion $\{\overline{Z}_{1,1,\zeta}^\pi\}$ of $(q+1)$ affine curves $\{\overline{Z}_{1,1,\zeta}\}$ at each infinity. The complement $\overline{Z}_{1,1,\zeta} \setminus \overline{Z}_{1,1,\zeta}^\pi$ consists of two closed points. The projective curve $\overline{Z}_{1,1,\zeta}$ meets the projective completion $\overline{Y}_{3,1,\zeta}$ of $(q+1)$ affine curves $\overline{Y}_{3,1,\zeta}$ at each infinity. The curve $\overline{Y}_{3,1,\zeta}$ meets the Igusa curve $\text{Ig}(p^2)$ at each infinity. Since the affine curve $\overline{Y}_{3,1,\zeta}$ has $(q+1)$ infinity points, there exist $q(q+1)$ Igusa curves $\text{Ig}(p^2)$ in the stable reduction of the Lubin-Tate space $\mathcal{X}^\pi(\pi^2)$.

### 5.4 Intersection Data

We include the intersection multiplicities in $\mathcal{X}(\pi^2)$ below in Table 1. These numbers have been obtained via a rigid analytic reformulation. Let $C$ be a projective smooth curve over a non-archimedean local field $F$. We assume that $C$ admits a semi-stable model $\mathcal{C}$ over some extension $E/F$. Suppose that $X$ and $Y$ are irreducible components of $C$, and that they intersect in an ordinary double point $P$. Then, $\text{red}^{-1}(P)$ is an annulus, say with width $w(P)$. Let $e_p(E)$ denote the ramification index of $E/F$. In this case, the intersection multiplicity of $X$ and $Y$ at $P$ can be found by

$$M_{E}(P) := e_p(E) \cdot w(P).$$

Note that while intersection multiplicity depends on $E$, the width makes sense even over $C$, which in some sense makes width a more natural invariant from the purely geometric perspective as mentioned in [CM, Section 9.1]. Now, for our calculation of $M_{E}(P)$ on $\mathcal{X}(\pi)$, we take $e_p(E) = (q^2 - 1)$. Then, we have $\text{Ig}(p), \text{Y}_{1,1}) = 1$. Now, for our calculation of $M_{E}(P)$ on $\mathcal{X}(\pi^2)$, we take $e_p(E) = q^4(q^2 - 1)$.  

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\[
\begin{array}{|c|c|c|c|c|c|}
\hline
P (l_6(p^2), \overline{Y}_{5,1,c_j}) & (l_6(p^2), \overline{Z}_{5,1,c_j}) & (l_6(p^2), \overline{Y}_{5,2,c_j}) & (l_6(p^2), \overline{Z}_{5,2,c_j}) & (l_6(p^2), \overline{Y}_{5,2,c_j}) \\
\hline
w(P) & \frac{1}{q^2(q^2-1)} & \frac{1}{2q^2(q+1)} & \frac{1}{2q^2(q+1)} & \frac{1}{4q^2} & \frac{1}{q^2(p+1)} \\
\hline
M_E(P) & q & \frac{q-1}{2} & \frac{q-1}{2} & \frac{q^2-1}{4} & q(q-1) \\
\hline
\end{array}
\]

Table 1: Intersection Multiplicity Data for \( \mathcal{X}(\pi^2) \)

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