Exact Quantum-Statistical Dynamics of Time-Dependent Generalized Oscillators

Sang Pyo Kim
Department of Physics, Kunsan National University, Kunsan 573-701, Korea

Don N. Page
CIAR Cosmology Program, Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1

(Dated: December 30, 2021)

Using linear invariant operators in a constructive way we find the most general thermal density operator and Wigner function for time-dependent generalized oscillators. The general Wigner function has five free parameters and describes the thermal Wigner function about a classical trajectory, whose eccentricity determines the squeezing of the initial vacuum.

PACS numbers: PACS numbers: 03.65.Ta, 05.30.-d, 42.50.Dv, 03.65.Fd

A quantum system of time-dependent oscillators has been a continuing issue of interest since the advent of quantum mechanics. Paul trap is one of such oscillators, which has a time-periodic frequency. Recently geometric phase has been studied for time-dependent quantum oscillators. Various methods have been applied to time-dependent quantum oscillators in many areas. Agarwal and Kumar, and Aliaga et al. studied statistical properties of time-dependent oscillators. Also the density matrix and density operator for time-dependent oscillators were studied in Refs. [5, 6].

On the other hand, Lewis and Riesenfeld introduced a method to find the exact quantum states for the time-dependent Schrödinger equation. In particular, for time-dependent oscillators they found a quadratic invariant operator, satisfying the quantum Liouville-von Neumann equation

\[ i\hbar \frac{\partial}{\partial t} \hat{I}(t) + [\hat{I}(t), \hat{H}(t)] = 0, \]

where X, Y and Z explicitly depend on time. Lewis and Riesenfeld have shown that the invariant operator satisfying the quantum Liouville-von Neumann equation

\[ \hat{H}(t) = \frac{X(t)}{2} \hat{p}^2 + \frac{Y(t)}{2} (\hat{p} \hat{q} + \hat{q} \hat{p}) + \frac{Z(t)}{2} \hat{q}^2, \]

provides the exact quantum states of the time-dependent Schrödinger equation as its eigenstates up to time-dependent phase factors. Following Ref. [3] we introduce a pair of linear invariant operators

\[ \hat{a}_u(t) = \frac{i}{\sqrt{\hbar}} \left[ u^*(t) \hat{p} - \frac{1}{X(t)} [\hat{a}^*(t) - Y(t) u^*(t)] \hat{q} \right], \]

\[ \hat{a}^+_u(t) = -\frac{i}{\sqrt{\hbar}} \left[ u(t) \hat{p} - \frac{1}{X(t)} [\hat{a}(t) - Y(t) u(t)] \hat{q} \right], \]

where u is a complex solution to the classical equation of motion

\[ \frac{d}{dt} \left( \frac{\dot{u}}{X} \right) + \left[ XZ - Y^2 + \frac{XY - XY}{X} \right] \left( \frac{u}{X} \right) = 0. \]

with overdots denoting the derivative with respect to t. Normalizing the complex solution to satisfy the Wronskian condition

\[ \text{Wr}\{u^*, u\} = \frac{1}{X} (u \dot{u}^* - u^* \dot{u}) = i, \]

one can make the invariant operators satisfy the standard commutation relation

\[ [\hat{a}_u(t), \hat{a}^+_u(t)] = 1. \]

Another complex solution v to Eq. (4), which can be expressed as a linear superposition of u:

\[ v(t) = \mu^* u(t) - \nu^* u^*(t), \]
for complex constants $\mu$ and $\nu$ given by
\[ \mu = i \text{Wr}\{u, v^*\}, \quad \nu = i \text{Wr}\{u^*, v^*\}, \]
leads to another set of the invariant operators $\hat{a}_v$ and $\hat{a}^*_v$. The Wronskian condition on $v$
\[ \text{Wr}\{v^*, v\} = i \Leftrightarrow |\mu|^2 - |\nu|^2 = 1, \]
also guarantees the standard commutation relation
\[ [\hat{a}_v(t), \hat{a}^*_v(t)] = 1. \]

In fact, these two sets of invariant operators are related through the Bogoliubov transformation
\[ \hat{a}_v(t) = \hat{S}^{-1}(t)\hat{a}_u(t)\hat{S}(t), \quad \hat{a}^*_v(t) = \hat{S}^{-1}(t)\hat{a}^*_u(t)\hat{S}(t), \]
by the squeezing operator
\[ \hat{S}(t) = e^{i\theta_v\hat{a}^*_v} \exp\left[ \frac{1}{2} e^{i(\theta_v - \theta_u)} \cosh^{-1} |\mu|\hat{a}^*_u - \text{H.c.} \right], \]
where $\mu = |\mu| e^{i\theta_u}, \nu = |\nu| e^{i\theta_v}$. We may use the freedom in choosing the overall constant phase of $\hat{a}_v$, which is not physically important, to fix the phase $\theta_v = 0$. Thus there are only two parameters $|\mu|$ and $\theta_v$ or a complex constant $\nu$, i.e. $|\nu|$ and $\theta_v$.

The most general, quadratic, Hermitian invariant operator constructed from the pair $\hat{a}_v$ and $\hat{a}^*_v$ takes the form
\[ \hat{I}(t) = \frac{A}{2} \hat{a}^2_v(t) + \frac{B}{2} [\hat{a}^*_v(t)\hat{a}_v(t) + \hat{a}_v(t)\hat{a}^*_v(t)] + \frac{A^*}{2} \hat{a}^2_v(t) + D\hat{a}^*_v(t) + D^*\hat{a}_v(t) + E, \]
where $A$ and $D$ are complex constants, and $B$ and $E$ are real constants. By choosing $\mu$ and $\nu$, i.e. $u$ such that
\[ A\mu^2 + 2B\mu^*\nu + A^*\nu^2 = 0, \]
the invariant operator (7) can be written in the canonical form
\[ \hat{I}_u(t) = \hbar\omega_0\hat{a}^*_u(t)\hat{a}_u(t) + \delta\hat{a}^*_u(t) + \delta^*\hat{a}_u(t) + \epsilon. \]

Hence this implies that by allowing all the complex $u$’s satisfying both Eqs. (4) and (5) the invariant operator (7) is general enough for our purpose. From now on we shall work on the Fock bases $\hat{a}_u$ and $\hat{a}^*_u$ for all the complex $u$’s and drop the subscript $u$.

Since the invariant operator (7) satisfies Eq. (2), we use it to define the density operator
\[ \hat{\rho}(t) = \frac{1}{Z} e^{-\beta \hat{I}_u(t)}. \]

Here $\beta$ is a free parameter that may be identified with the inverse temperature of the system in equilibrium, and $Z = \text{Tr}(e^{-\beta \hat{I}_u})$. The density operator has five free parameters, i.e. $\beta$ or $\omega_0$, a complex constant $\delta$, which is related with the classical position $q_c$ and momentum $p_c$ as will be shown below, and $|\mu|$ and $\theta_v$ in choosing $u$. By introducing the displacement operator
\[ \hat{D}(z) = e^{-\beta\hat{u}(t) + \beta^*\hat{u}^*(t)}, \]
with $z = -\delta/(\hbar\omega_0), \epsilon = |\delta|^2/(\hbar\omega_0)$, we write the density operator as
\[ \hat{\rho}(t) = \hat{D}^\dagger(z)\hat{\rho}_T(t)\hat{D}(z), \]
where
\[ \hat{\rho}_T(t) = \frac{1}{Z_T} e^{-\beta\hbar\omega_0\hat{a}^*(t)\hat{a}(t)}, \]
is a thermal density operator. It follows that $Z = Z_T$ due to the unitary transformation (8). The coherent state, defined as $\hat{a}(t)|z, t\rangle = z|z, t\rangle$, is also given by
\[ |z, t\rangle = \hat{D}^\dagger(z)|0, t\rangle, \]
where $|0, t\rangle$ is the vacuum state that is annihilated by $\hat{a}(t)$. The position and momentum expectation value with respect to the coherent state is
\[ \langle z, t|\hat{q}|z, t\rangle = \sqrt{\beta}(uz + u^*z^*) \equiv q_c, \]
\[ \langle z, t|\hat{p}|z, t\rangle = -\frac{Y}{X} q_c + \frac{\sqrt{\beta}}{X}(uz + u^*z^*) \equiv p_c. \]
The $q_c$ and $p_c$ satisfy the classical Hamilton equations
\[ \dot{q}_c = Xp_c + Yq_c, \]
\[ \dot{p}_c = -Yu_c - Zq_c. \]

Now, from the definition of the thermal expectation value
\[ \langle \hat{O} \rangle = \text{Tr}[\hat{O}\hat{\rho}(t)] = \text{Tr}[\hat{D}(z)\hat{O}\hat{D}^\dagger(z)\hat{\rho}_T], \]
we find the expectation value of position and momentum operators
\[ \langle \hat{q} \rangle = q_c, \quad \langle \hat{p} \rangle = p_c, \]
and that of quadratic operators

\[
\langle \hat{q}^2 \rangle = \frac{\hbar u^* u (1 + 2\bar{n}) + q_c^2}{} , \\
\langle \hat{p}^2 \rangle = \frac{\hbar}{X^2} \left[ (\hat{u}^* - Y u^*)(\hat{u} - Y u) (1 + 2\bar{n}) + p_c^2 \right] , \\
\frac{1}{2} \langle \hat{q} \hat{p} + \hat{p} \hat{q} \rangle = \frac{\hbar}{2X} \left[ (\hat{u}^* - Y u^*)u + (\hat{u} - Y u) u^* \right] \\
\times (1 + 2\bar{n}) + q_c p_c .
\]

where

\[\bar{n} = \frac{1}{e^{\beta \hbar \omega_0} - 1}\]

is the mean number density of Bose-Einstein distribution. The vacuum result is obtained by taking the limit \( \beta \to \infty \). It is worth noting that the dispersion of position and momentum around the classical trajectory \((q_c, p_c)\) is entirely determined by the thermal one: \(\langle \hat{q} - q_c \rangle^2 = \langle \hat{q}^2 \rangle_T\) and \(\langle \hat{p} - p_c \rangle^2 = \langle \hat{p}^2 \rangle_T\), where \(\langle \hat{O} \rangle_T = \text{Tr}[\hat{O} \rho_T(t)]\).

Using \(\hat{D}(z) = e^{ip \cdot q_c/2 \hbar} e^{-ip \cdot q'/\hbar} e^{-ip \cdot \hat{q}/\hbar}\) we find the coordinate representation of the displacement operator

\[
\langle \hat{q} | \hat{D}(z) | q \rangle = e^{ip \cdot q_c/2 \hbar} e^{-ip \cdot q'/\hbar} \delta(q + q_c - q'),
\]

and that of its Hermitian conjugate, \(\langle \hat{q}' | \hat{D}^\dagger(z) | \hat{q} \rangle = \langle \hat{q}' | \hat{D}(z) | q \rangle^*\). Hence the density matrix is given by

\[
\rho(q, q') = \langle \hat{q} | \hat{\rho}(t) | \hat{q}' \rangle = \langle \hat{q} | \hat{D}(z) | \hat{q} \rangle \int dq_1 | q_1 \rangle \langle q_1 | \hat{\rho}_T \int dq_2 | q_2 \rangle \langle q_2 | \hat{D}(z) | \hat{q} \rangle
\]  \[= e^{ip \cdot (q - q')/\hbar} \rho_T(q - q_c, q' - q_c, t) ,
\]

where \(\rho_T\) is the density matrix for the thermal state given, for instance, in Ref. [3]. Finally, the Wigner function is given by

\[
P(q, p) = \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} dy \rho(q - y, q + y) e^{2ipy/\hbar}
\]  \[= P_T(q - q_c, p - p_c) ,
\]

where \(P_T\) is the Wigner function for the thermal state:

\[
P_T(q, p) = \frac{\text{tanh}(\beta \hbar \omega_0/2)}{\pi \hbar} \exp \left[ - \frac{2 \text{tanh}(\beta \hbar \omega_0/2)}{\hbar \omega_0} \mathcal{H}_E \right] ,
\]

\[
\mathcal{H}_E(q, p) = \frac{\omega_0 u^* u}{2 \hbar} \left( p - \frac{d \ln(u^* u)^{1/2}}{dt} q \right)^2 + \frac{\omega_0}{4u^* u} q^2 .
\]

The Wigner function [12] also has five parameters: \(q_c, p_c, \beta, (\omega_0, \mu, \theta_\mu)\).

The Wigner functions \(P\) and \(P_T\) and their vacuum limit are positive definite in contrast with those for excited states that may take negative values in some region of phase space [11]. Hence \(P\) and \(P_T\) may be used as a distribution of phase space for the quantum evolution. For instance, the harmonic oscillator with \(X = 1/X, Y = 0\) and \(Z = m_0 \omega_0^2\), has the solution

\[u(t) = e^{-i\omega_t t} \sqrt{2m_0 \omega_0} \text{ which recovers the well-known Wigner function with } \mathcal{H}_E(q, p) = H(q, p) .\]

In general, \(\mathcal{H}_E\) depicts an ellipse centered at the origin, which can be written in the canonical form

\[\mathcal{H}_E = \left( \lambda_+ \hat{p}^2 + \lambda_- \hat{q}^2 \right) \times \frac{\omega_0}{2} ,\]

\[\lambda_\pm = u^* u + \frac{1}{4u^* u} \left( \frac{d \ln(u^* u)^{1/2}}{dt} \right)^2 \pm \left[ \left( u^* u + \frac{1}{4u^* u} \left( \frac{d \ln(u^* u)^{1/2}}{dt} \right)^2 - 1 \right) ^{1/2} ,\]

where \((\hat{q}, \hat{p})\) are new phase-space coordinates rotated with the angle

\[\tan[2\theta(t)] = \frac{2(u^* u/X) [d \ln(u^* u)^{1/2}/dt]}{u^* u + (1/4u^* u) - (u^* u/X^2) [d \ln(u^* u)^{1/2}/dt]^2} .\]

Since \(\lambda_+ \lambda_- = 1\), the area of the ellipse does not depend on the solution \(u\). Therefore, as shown in Fig. 1, the contour of the Wigner function [12] follows an elliptic orbit with a constant area whose center \((q_c, p_c)\) in turn moves on a classical trajectory and principal axes \((\hat{q}, \hat{p})\) rotate with the angle \(\theta(t)\). The shape of the ellipse somehow determines the squeezing of the initial vacuum [12].

Now we introduce a geometric measure for the squeezing of the initial vacuum and particle production in terms of the eccentricity of the ellipse. For that purpose we consider the exactly solvable oscillator [3]

\[X = \frac{1}{m}, \quad Y = 0, \quad Z = m[\omega_1^2 - \omega_0^2 \text{tanh}(t/\tau)] .\]

The oscillator has an asymptotic frequency \(\omega_i = (\omega_0^2 + \omega_0^2)^{1/2}\) at \(t = -\infty\) and \(\omega_f = (\omega_0^2 - \omega_0^2)^{1/2}\) at \(t = \infty\). As \(\tau\) is an interval for the frequency change, the adiabatic change is prescribed by the condition \(\tau \gg 1\). The solution that has the correct asymptotic form \(u = e^{-i\omega_i t}/\sqrt{2m\omega_i}\) at \(t = -\infty\) is given by

\[u(t) = \frac{e^{-i\omega_i t}}{\sqrt{2m\omega_i}} F_1(-t^2/2(\omega_i + \omega_f), -t^2/2(\omega_i - \omega_f); 1 - i\tau \omega_i; -e^{2t/\tau}) ,\]

where \(F_1\) is the hypergeometric function. At \(t = -\infty\) we obtain \(\mathcal{H}_E = p^2/(2m) + m\omega_i^2 \omega_f^2/2 = H_i\). Whereas, at \(t = \infty\) the solution has another asymptotic form

\[u(t) = \frac{1}{\sqrt{2m\omega_i}} [\alpha_e^{-i\omega_i t} + \alpha^*_e e^{i\omega_i t}] ,\]

where

\[\alpha_\pm(\tau) = \frac{\Gamma(1 - i\omega_i \tau) \Gamma(\mp i\omega_f \tau)}{\Gamma(1 - i\omega_i (\omega_i \mp \omega_f)) \Gamma(\mp i\omega_i (\omega_i \mp \omega_f))} .\]

For the adiabatic limit \(\tau \to \infty\), \(|\alpha_-| \to 0\) and \(u(t = \infty) = \alpha_+ e^{-i\omega_i t}/\sqrt{2m\omega_i}\). As \(|\alpha_+(\tau = \infty)| = \sqrt{\omega_i/\omega_f},\)
In terms of the eccentricity \( e \) with regard to the equilibrium, we consider the harmonic oscillator and thereby the amount of particle production.

Thus we show that the geometric shape determined by the eccentricity of the ellipse measures the squeezing of the vacuum state and thereby the amount of particle production.

As an illustrative but nontrivial application of our general Wigner function, we consider the harmonic oscillator with

\[
X = \frac{1}{m}, \quad Y = 0, \quad Z = m \omega_0^2,
\]

which has the most general solution

\[
u(t) = \frac{1}{\sqrt{2m\omega_0}} [\mu e^{-i\omega_0 t} + \nu e^{i\omega_0 t}], \quad (\nu \neq 0).
\]

As shown in Fig. 1, the contour of the Wigner function moves on a small elliptical orbit about another elliptical orbit for the classical trajectory

\[
q_c = q_0 \cos(\omega_0 t + \varphi_0), \quad p_c = -m \omega_0 q_0 \sin(\omega_0 t + \varphi_0),
\]

where

\[
q_0 = \sqrt{\frac{2\hbar}{m\omega_0}} |\mu z + \nu^* z^*|, \quad e^{-i\varphi_0} = \frac{\mu z + \nu^* z^*}{|\mu z + \nu^* z^*|}.
\]

In fact, the contour is similar to the epicycle of an elliptical orbit moving on another elliptical orbit. Now the eccentricity \( e(t) = \sqrt{\min \{\lambda_+ (t)/\max \{\lambda_\ominus (t)\}\} } \) and the rotation angle \( \theta(t) \) of the small elliptical orbit explicitly depend on time through the magnitude of the complex solution

\[
u^* u = |\mu|^2 + |\nu|^2 + \mu^* \nu e^{-2i\omega_0 t} + \mu^* \nu e^{2i\omega_0 t}.
\]

Hence the density matrix and Wigner function for \( \nu \neq 0 \) describe various kinds of nontrivial quantum states beyond the standard static ones.

In summary, we showed that the most general, quadratic, Hermitian invariant operator (3) can be written in the canonical form (5) by suitably choosing a complex solution \( u \) to the classical equation of motion (4). Then the general thermal density operator takes the form (5), which is nothing but the displaced state of the thermal state (4). The density matrix (11) and Wigner function (12), the main results of this paper, are the thermal ones shifted by a classical solution. The general Wigner function for thermal state has five free parameters, which characterize the classical configuration \((q_c, p_c)\) together with \( \beta \) (or \( \omega_0 \)) for thermal equilibrium, \( |\mu| \) (or \( |\nu| \)) for the squeezing of the initial vacuum, and \( \theta_\nu \). Also it is positive definite and describes a distribution of phase space for quantum evolution. Further, we showed that the contour of the Wigner function depicts an elliptical orbit centered at a classical trajectory. Our general Wigner function provides a nontrivial time-dependent Wigner function even for a time-independent harmonic oscillator, whose contour describes an analog of epicycle of an elliptical orbit moving about another elliptical orbit.

Acknowledgments

The work of S.P.K. was supported by the Korea Research Foundation under Grant No. 2000-015-DP0080 and the work of D.N.P. by the Natural Sciences and Engineering Council of Canada.
[1] L.S. Brown, Phys. Rev. Lett. 66, 527 (1991).
[2] C. Jarzynski, Phys. Rev. Lett. 74, 1264 (1995); Y.C. Ge and M.S. Child, ibid. 78, 2507 (1997); G.G. de Polavieja, ibid. 81, 1 (1998); J. Lie, B. Hu, and B. Li, ibid. 81, 1749 (1998); D.-Y. Song, ibid. 85, 1141 (2000).
[3] H. Dekker, Phys. Rep. 80, 1 (1981); C.I. Um, K.H. Yeon, and T.F. George, ibid. 362, 63 (2002).
[4] G.S. Agarwal and S.A. Kumar, Phys. Rev. Lett. 67, 3665 (1991); J. Aliaga, G. Crespo, and A.N. Proto, ibid. 70, 434 (1993).
[5] O. ´Eboli, R. Jackiw, S.-Y. Pi, Phys. Rev. D 37, 3557 (1988).
[6] S.P. Kim and C.H. Lee, Phys. Rev. D 62, 125020 (2000).
[7] H.R. Lewis, Jr. and W.B. Riesenfeld, J. Math. Phys. (N.Y.) 10, 1458 (1969).
[8] S.P. Kim and D.N. Page, Phys. Rev. A 64, 012104 (2001).
[9] J.M. Cerveró and J. D. Lejarreta, J. Phys. A 22, L663 (1989); X.-C. Gao, J.-B. Xu, and T.-Z. Qian, Ann. Phys. 204, 235 (1990).
[10] H.P. Yuen, Phys. Rev. A 13, 2226 (1976).
[11] S.P. Kim and C.H. Lee, Phys. Rev. D 65, 045013 (2002).
[12] Y.S. Kim and M.E. Noz, Phase Space Picture of Quantum Mechanics (World Scientific, Singapore, 1991).
[13] N.D. Birrel and P.C.W. Davies, Quantum Fields in Curved Space (Cambridge University Press, Cambridge, England, 1982).
FIG. 1: The contour of the Wigner function $P$. The ellipse centered at the origin describes the classical trajectory, a periodic motion, and those centered on the ellipse correspond to a thermal state. The $\varphi$ is the polar angle of the classical position $(q_c, p_c)$. 