LARGE WEYL SUMS AND HAUSDORFF DIMENSION

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Abstract. We obtain the exact value of the Hausdorff dimension of the set of coefficients of Gauss sums which for a given $\alpha \in (1/2, 1)$ achieve the order at least $N^\alpha$ for infinitely many sum lengths $N$. For Weyl sums with polynomials of degree $d \geq 3$ we obtain a new upper bound on the Hausdorff dimension of the set of polynomial coefficients corresponding to large values of Weyl sums. Our methods also work for monomial sums, match the previously known lower bounds, just giving exact value for the corresponding Hausdorff dimension when $\alpha$ is close to 1. We also obtain a nearly tight bound in a similar question with arbitrary integer sequences of polynomial growth.

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1. Introduction

1.1. Set-up and motivation. For $x = (x_1, \ldots, x_d) \in T_d$ where

$$T_d = [0, 1]^d$$

is the $d$-dimensional unit cube, we define the Weyl sums of length $N$ as

$$S_d(x; N) = \sum_{n=1}^{N} e(x_1n + \ldots + x_dn^d)$$
where $e(z) = \exp(2\pi iz)$. These sums were originally introduced by Weyl to study equidistribution of fractional parts of polynomials and later find their applications to the circle method and Riemann zeta function.

For many applications of Weyl sums, the key problem is to estimate the size of the sum $S_d(x; N)$. There are often three kinds of estimates of Weyl sums, namely individual bounds, mean value bounds and almost all bounds. Despite more than a century since these sums were introduced, their behaviour for individual values of $x$ is not well understood, see $[10, 11, 32]$.

Much more is known about the average behaviour of $S_d(x; N)$. The recent advances of Bourgain, Demeter and Guth $[9]$ (for $d \geq 4$) and Wooley $[39]$ (for $d = 3$) (see also $[40]$) towards the optimal form Vinogradov mean value theorem imply the estimate

$$N^{s(d)} \leq \int_{T_d} |S_d(x; N)|^{2s(d)} \, dx \leq N^{s(d)+o(1)},$$

where

$$s(d) = d(d+1)/2$$

and is best possible up to $o(1)$ in the exponent of $N$.

We study exceptional sets of $x \in T_d$, which generate abnormally large Weyl sums $S_d(x; N)$.

The first results concerning the almost all behaviour of Weyl sums are due to Hardy and Littlewood $[27]$ who have estimated the following special sums

$$G(x, N) = \sum_{n=1}^{N} e\left(xn^2\right),$$

in terms of the continued fraction expansion of $x$. Among other things Hardy and Littlewood $[27]$ proved that for almost all $x \in \mathbb{T}$,

$$|G(x, N)| \leq N^{1/2+o(1)}, \quad \text{as} \quad N \to \infty.$$  

Their idea has been expanded upon by Fiedler, Jurkat and Körner $[25, \text{Theorem 2}]$ who give the following optimal lower and upper bounds. Suppose that $\{f(n)\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive numbers. Then for almost all $x \in \mathbb{T}$ one has

$$(1.1) \quad \lim_{N \to \infty} \frac{|G(x, N)|}{\sqrt{N} f(N)} < \infty \iff \sum_{n=1}^{\infty} \frac{1}{nf(n)^4} < \infty.$$
For the sums $S_2(x; N)$, $x \in T_2$, Fedotov and Klopp [24] have given a similar result, however adding the term $e(x_1 n)$ leads to more cancellations in the sums $S_2(x; N)$. Suppose that $\{g(n)\}_{n=1}^\infty$ is a non-decreasing sequence of positive numbers. Then for almost all $x \in T_2$ one has

$$\lim_{N \to \infty} \frac{|S_2(x; N)|}{\sqrt{Ng(\ln N)}} < \infty \iff \sum_{n=1}^\infty \frac{1}{g(n)^6} < \infty.$$  

(1.2)

It is natural to expect that analogues of (1.1), (1.2) hold for Weyl sums $S_d(x; N)$ with any $d \geq 3$, however this question seems to be still open. However, we have the following nearly sharp bounds. For $d \geq 3$, Chen and Shparlinski [14,16] have shown in two different ways in [14, Appendix A], and [16, Theorem 2.1] that for almost all $x \in T_d$ one has

$$|S_d(x; N)| \leq N^{1/2+o(1)} \quad \text{as} \quad N \to \infty.$$  

(1.3)

Recently, Chen, Kerr, Maynard and Shparlinski [13, Theorem 2.3] have shown that the exponent $1/2$ is optimal, that is, there exists a constant $c > 0$ such that for almost all $x \in T_d$, the inequality

$$|S_d(x; N)| \geq cN^{1/2}$$  

(1.4) holds for infinitely many $N$.

This motivates our study of the “the exceptional sets” of Weyl sums. Precisely, for $1/2 < \alpha < 1$ define

$$\mathcal{E}_{d,\alpha} = \{x \in T_d : |S_d(x; N)| \geq N^{\alpha} \text{ for infinitely many } N \in \mathbb{N}\}.$$  

Chen and Shparlinski [14, Theorem 1.3] show that for any $d \geq 2$ and $1/2 < \alpha < 1$ the set $\mathcal{E}_{d,\alpha}$ is of second category in the sense of Baire, and the proof of [14, Theorem 1.3] implies that the set $\mathcal{E}_{d,\alpha}$ is a dense subset of $T_d$. Therefore, the Minkowski dimension (or box dimension) of $\mathcal{E}_{d,\alpha}$ is $d$. See [23] for more details on the Minkowski dimension.

The above results (1.2) and (1.3) imply that $\mathcal{E}_{d,\alpha}$ is of zero Lebesgue measure for all $d \geq 2$ and any $\alpha \in (1/2, 1)$. For sets of Lebesgue measure zero, it is common to use the Hausdorff dimension to describe their size and structure, and we are going to estimate the Hausdorff dimension of the set $\mathcal{E}_{d,\alpha}$ for any $d \geq 2$ and any $\alpha \in (1/2, 1)$. We first recall the formal definition of Hausdorff dimension, and we refer to [23,33] for more details.
Definition 1.1. The Hausdorff dimension of a set $F \subseteq \mathbb{R}^d$ is defined as

$$\dim F = \inf \left\{ s > 0 : \forall \varepsilon > 0, \exists \{U_i\}_{i=1}^{\infty}, U_i \subseteq \mathbb{R}^d, \right.$$ 

such that $F \subseteq \bigcup_{i=1}^{\infty} U_i$ and $\sum_{i=1}^{\infty} (\text{diam} U_i)^s < \varepsilon$,

$$\left. \right\},$$

where

$$\text{diam} U = \sup\{\|u - v\| : u, v \in U\}$$

and $\|w\|$ is the Euclidean norm in $\mathbb{R}^d$.

We remark that we could also define the set $E_{d,\alpha}$ for $\alpha \in (0, 1/2]$. However, by (1.4) the set $E_{d,\alpha}$ is of full Lebesgue measure. This, by [17, Theorem 2.5] and the definition of the Hausdorff dimension, is enough to conclude that

$$\dim E_{d,\alpha} = d.$$ 

For an integer $d \geq 2$ and real $\alpha \in (1/2, 1)$ some explicit upper and lower bounds on $\dim E_{d,\alpha}$ have been given in [14–16]. In particular, for any $\alpha \in (1/2, 1)$, there are explicit functions $l(d, \alpha), u(d, \alpha)$ such that

$$0 < l(d, \alpha) \leq \dim E_{d,\alpha} \leq u(d, \alpha) < d.$$ 

We show more details in the following. For $d \geq 2$, let

$$\kappa_d = \max_{\nu=1,...,d} \min \left\{ \frac{1}{2\nu^2}, \frac{1}{2d - \nu} \right\}.$$ 

For each $1/2 < \alpha < 1$ and any cube $Q \subseteq T_d$ we have the following lower bounds of $\dim E_{d,\alpha}$:

(i) for $d = 2$,

$$\dim E_{2,\alpha} \cap Q \geq 3(1 - \alpha)/2;$$ 

(ii) for $d \geq 3$,

$$\dim E_{d,\alpha} \cap Q \geq 2\kappa_d(1 - \alpha).$$ 

For the upper bound of $\dim E_{d,\alpha}$ with $d \geq 2$ and $1/2 < \alpha < 1$, we have

$$\dim E_{d,\alpha} \leq u(d, \alpha),$$

where

$$u(d, \alpha) = \min_{k=0,...,d-1} \frac{(2d^2 + 4d)(1 - \alpha) + k(k + 1)}{4 - 2\alpha + 2k}.$$ 

It is not hard to show $u(d, \alpha) < d$ for any $\alpha \in (1/2, 1)$.
In fact for $\alpha \to 1$ the behaviour of $\dim E_{d,\alpha}$ is understood reasonably well as a combination of the above mentioned lower and upper bounds, implies that there are positive constants $c_1(d), c_2(d)$ such that

$$c_1(d) \leq \liminf_{\alpha \to 1} (1 - \alpha)^{-1} \dim E_{d,\alpha} \leq \limsup_{\alpha \to 1} (1 - \alpha)^{-1} \dim E_{d,\alpha} \leq c_2(d).$$

For $\alpha \in (1/2, 1)$, some heuristic arguments have been given in [13] towards the following:

**Conjecture 1.2.** For any $\alpha \in (1/2, 1)$, the set $E_{d,\alpha}$ is of Hausdorff dimension

$$\dim E_{d,\alpha} = \min_{j=1, \ldots, d} \frac{d + 1 + j\vartheta_j - \sum_{i=1}^j \vartheta_i}{1 + \vartheta_j},$$

where

$$\vartheta_i = \frac{i}{2(1 - \alpha)} - 1, \quad i = 1, \ldots, d.$$

1.2. **New results and methods.** In this paper, we confirm the Conjecture 1.2 for $d = 2$, and we obtain new upper bounds of $\dim E_{d,\alpha}$ when $d \geq 3$ and $\alpha$ is close to 1. Moreover, we also consider the following one parametric family of exponential sums. Namely, for a real sequence $f(n), n \in \mathbb{N}$, $x \in \mathbb{T}$ and $N \in \mathbb{N}$ we denote

$$(1.5) \quad V_f(x; N) = \sum_{n=1}^N e\left(x f(n)\right).$$

Chen and Shparlinski [16, Corollary 2.2] shows that for any polynomial $f \in \mathbb{Z}[X]$ with $\deg f \geq 2$ we have for almost all $x \in \mathbb{T}$,

$$|V_f(x; N)| \leq N^{1/2+o(1)} \quad \text{as } N \to \infty.$$  

Similarly to the definition of $E_{d,\alpha}$, for $\alpha \in (1/2, 1)$ we define the set

$$\mathcal{F}_{f,\alpha} = \{x \in \mathbb{T} : |V_f(x; N)| \geq N^\alpha \text{ for infinitely many } N \in \mathbb{N}\}.$$  

Perhaps the most interesting sums of this type are sums with monomials $xn^d$, in which case we denote this special quantity by $\mathcal{F}_{d,\alpha}$, that is,

$$\mathcal{F}_{d,\alpha} = \left\{x \in \mathbb{T} : \left|\sum_{n=1}^N e\left(x n^d\right)\right| \geq N^\alpha \text{ for infinitely many } N \in \mathbb{N}\right\}.$$  

Some lower bounds of $\dim \mathcal{F}_{d,\alpha}$ have been obtained in [14, Theorem 1.7]. In particular, [14, Theorem 1.7] implies that for $\alpha \in (1/2, 1)$

$$(1.6) \quad \dim \mathcal{F}_{2,\alpha} \geq 2(1 - \alpha),$$
and for \( d \geq 3 \) and \( \alpha \in [d/(d + 2), 1) \),

\[
(1.7) \quad \dim \mathcal{F}_{d,\alpha} \geq \left(1 + \frac{1}{d}\right)(1 - \alpha).
\]

Some heuristic arguments have been given in [13, Section 8], suggesting that in a certain range of \( \alpha \) we may have

\[
\dim \mathcal{F}_{d,\alpha} = 4(1 - \alpha)/d,
\]

which is consistent with (1.6) and (1.7) for \( d = 2 \) and \( d = 3 \).

Moreover, for \( \alpha \in (0, 1/2) \) the set \( \mathcal{F}_{d,\alpha} \) is of positive Lebesgue measure (see [13] for more details), and hence,

\[
\dim \mathcal{F}_{d,\alpha} = 1.
\]

It is very likely that for \( f \in \mathbb{Z}[X] \) the bounds (1.6) and (1.7) can be extended to the sets \( \mathcal{F}_{f,\alpha} \).

To obtain these results we develop two different approaches:

- For \( \alpha \) close to 1, we employ the classification of Baker [4, 5] in the form given in [6].
- For smaller values of \( \alpha \) (which means that \( \alpha \) is close to 1/2), and also for sums \( V_f(x;N) \) with non-polynomial functions when the above classification is not available, we link Hausdorff dimension of the sets \( \mathcal{E}_{d,\alpha} \) and \( \mathcal{F}_{f,\alpha} \) to various mean value theorems.

The above arguments are complemented by the use of the Frostman Lemma (see [23, Corollary 4.12]) and the Gál–Koksma Theorem [26, Theorem 4].

1.3. Notation. Throughout the paper, the notations \( U = O(V) \), \( U \ll V \) and \( V \gg U \) are equivalent to \( |U| \leq cV \) for some positive constant \( c \), which throughout the paper may depend on the degree \( d \), the growth rate \( \tau \) and occasionally on the small real positive parameter \( \varepsilon \) and the real parameter \( t \).

For any quantity \( V > 1 \) we write \( U = V^{o(1)} \) (as \( V \to \infty \)) to indicate a function of \( V \) which satisfies \( V^{-\varepsilon} \leq |U| \leq V^{\varepsilon} \) for any \( \varepsilon > 0 \), provided that \( V \) is large enough. One additional advantage of using \( V^{o(1)} \) is that it absorbs \( \log V \) and other similar quantities without changing the whole expression.

For \( m \in \mathbb{N} \), we write \([m]\) to denote the set \( \{0, 1, \ldots, m - 1\} \).

2. Results for sets of very large sums

2.1. Multiparametric families of Gauss sums and Weyl sums. Here we confirm the Conjecture 1.2 for \( d = 2 \), that is, for the Gauss
sums
\[ G(x; N) = \sum_{n=1}^{N} e(x_1n + x_2n^2), \]
we improve the previous upper and lower bounds of [14–16], and obtain the exact value of the Hausdorff dimension of \( \mathcal{E}_{2,\alpha} \).

**Theorem 2.1.** For any \( \alpha \in (1/2, 1) \) we have
\[
\dim \mathcal{E}_{2,\alpha} = \begin{cases} 
7/2 - 3\alpha & \text{if } 5/6 \geq \alpha > 1/2, \\
6(1 - \alpha) & \text{if } 1 > \alpha > 5/6.
\end{cases}
\]

For \( d \geq 3 \), by applying the same idea as in the proof of Theorem 2.1 and combining some other new ideas, we obtain the following upper bound, which improves the previous bound of [15, Theorem 1.1] when \( \alpha \) is close to 1. For \( d \geq 3 \) we always write
\[
D = \min\{2^{d-1}, 2d(d-1)\}.
\]

**Theorem 2.2.** For \( d \geq 3 \) and any \( \alpha \in (1 - 1/D, 1) \), where \( D \) is given by (2.1), we have
\[
\dim \mathcal{E}_{d,\alpha} \leq \min_{h=1,\ldots,d} \frac{(d^2 + 1)(1 - \alpha)}{h} + \frac{h - 1}{2}.
\]

### 2.2. One parametric families of Weyl sums with real polynomials.

We now obtain upper bounds on \( \dim \mathcal{F}_{f,\alpha} \), which in the case of monomial sums and large values of \( \alpha \) matches the lower bounds (1.6) and (1.7). Depending on using different methods, we divide the results on \( \dim \mathcal{F}_{f,\alpha} \) into two subsections.

**Theorem 2.3.** Let \( f \in \mathbb{R}[X] \) be a polynomial of degree \( d \). For any \( \alpha \in (1 - 1/D, 1) \), where \( D \) is given by (2.1), we have
\[
\dim \mathcal{F}_{f,\alpha} \leq \begin{cases} 
2(1 - \alpha) & \text{if } d = 2, \\
(1 + 1/d)(1 - \alpha) & \text{if } d \geq 3.
\end{cases}
\]

Combining Theorem 2.3 with (1.6) and (1.7) and noticing that for \( d \geq 2 \),
\[
\frac{d}{d + 2} \leq 1 - \frac{1}{D},
\]
we obtain the following exact values in the case of monomial sums.

**Corollary 2.4.** For any \( \alpha \in (1 - 1/D, 1) \), where \( D \) is given by (2.1), we have
\[
\dim \mathcal{F}_{d,\alpha} = \begin{cases} 
2(1 - \alpha) & \text{if } d = 2, \\
(1 + 1/d)(1 - \alpha) & \text{if } d \geq 3.
\end{cases}
\]
As we have mentioned, we believe that in the case $f \in \mathbb{Z}[X]$ the lower bounds (1.6) and (1.7) and thus Corollary 2.4, can be extended to $\dim \mathcal{F}_{f,\alpha}$.

We observe that Theorem 2.3 can be applied to estimate $\dim \mathcal{E}_{d,\alpha} \cap \mathcal{L}$ where $\mathcal{L}$ is a straight line in $\mathbb{R}^d$ passing through the origin. Precisely, let $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$, $\mathbf{v} \neq 0$, and

$$\mathcal{L}_\mathbf{v} = \{\lambda \mathbf{v} : \lambda \in T\}.$$  

For $\mathbf{x} \in \mathcal{L}_\mathbf{v}$ for some $\lambda \in T$ we have

$$S_d(\mathbf{x}; N) = \sum_{n=1}^{N} \mathbf{e}(\lambda f(n)),$$

where $f(n) = v_1 n + \ldots + v_d n^d$. It follows that $\dim (\mathcal{E}_{d,\alpha} \cap \mathcal{L}_\mathbf{v}) \leq \dim \mathcal{F}_{f,\alpha}$, and by Theorem 2.3 we derive the following explicit bound.

**Corollary 2.5.** Let $\mathbf{v} = (v_1, \ldots, v_k, 0, \ldots, 0) \in \mathbb{R}^d$ with $v_k \neq 0$ and $v_j = 0$ when $k < j \leq d$. Let

$$D_k = \min\{2^{k-1}, 2k(k-1)\}.$$  

Then for any $\alpha \in (1 - 1/D_k, 1)$ we have

$$\dim \mathcal{E}_{d,\alpha} \cap \mathcal{L}_\mathbf{v} \leq \begin{cases} 2(1 - \alpha) & \text{if } k = 2, \\ (1 + 1/k) (1 - \alpha) & \text{if } k \geq 3. \end{cases}$$

We remark that Corollary 2.4 implies that the bounds of Corollary 2.5 is sharp in general when $\alpha$ is close to 1. This follows by choosing $\mathbf{v} = (v_1, \ldots, v_d)$ such that $v_k = 1$ and $v_j = 0$ when $j \neq k$. Moreover, motivated from the research on Diophantine approximation on manifolds (see, for instance [28, Chapter 9]), we pose the following general question.

**Question 2.6.** Given an “interesting” surface $\Gamma \subseteq \mathbb{R}^d$, for example, an algebraic hypersurface or a smooth analytic surface of given curvature, determine the Hausdorff dimension of the intersection $\mathcal{E}_{d,\alpha} \cap \Gamma$.

We note that upper bounds on the means values of Weyl sums along various surfaces have been given in [17, 19].

**2.3. Comparison.** We observe that Theorem 2.2 improves the upper bound

$$(2.2) \quad \dim \mathcal{E}_{d,\alpha} \leq \min_{k=0, \ldots, d-1} \frac{(2d^2 + 4d)(1 - \alpha) + k(k + 1)}{4 - 2\alpha + 2k}.$$
of [15, Theorem 1.1] for the range \( \alpha \in (1 - 1/D, 1) \). Indeed, consider the functions

\[
F(h, \beta) = \frac{(d^2 + 1)\beta}{h} + \frac{h - 1}{2}, \quad G(k, \beta) = \frac{(2d^2 + 4d)\beta + k(k + 1)}{2 + 2\beta + 2k}
\]

and note that it is enough to verify that for \( \beta \in (0, 1/D) \) and \( h = 1, \ldots, d \) we have

\[
(2.3) \quad F(h, \beta) < G(h-1, \beta).
\]

Clearly, the inequality (2.3) is equivalent to

\[
\frac{(d^2 + 1)\beta}{h} + \frac{h - 1}{2} < \frac{(2d^2 + 4d)\beta + h(h - 1)}{2\beta + 2h}
\]

or

\[
(2(d^2 + 1)\beta + h(h - 1)) (\beta + h) < ((2d^2 + 4d)\beta + h(h - 1)) h.
\]

Simplifying we obtain an equivalent inequality

\[
2(d^2 + 1)\beta^2 < \beta (4dh - h(h + 1))
\]

and finally

\[
2(d^2 + 1)\beta < 4dh - h(h + 1).
\]

Since \( \beta < 1/D \) and the right hand side of above inequality is increasing with \( h \in [1, d] \) it is only enough to check that

\[
2(d^2 + 1)/D < 4d - 2,
\]

which can be numerically verified for \( 3 \leq d \leq 7 \) and established via elementary calculus for \( d \geq 8 \). This shows that (2.3) holds and thus Theorem 2.2 is stronger than (2.2) for any \( d \geq 3 \) and all admissible values of \( \alpha \).

Clearly, when \( \alpha \) is close to 1 then the choice of \( h = 1 \) in Theorem 2.2 is optimal. However sometimes other choices of \( h \) give better result. For example, if \( \alpha \in (1 - 1/D, 1 - 1/(d^2 + 1)) \) then choice of \( h = 2 \) is better than \( h = 1 \). The above range is non-empty provided that \( d^2 + 1 \geq D \), and this happens when \( 3 \leq d \leq 6 \) only. Moreover, for larger \( d \) (say \( d \geq 7 \)) and \( \alpha \in (1 - 1/D, 1) \) the value \( \alpha \) is quite near the value 1, thus we may take \( h = 1 \) only in Theorem 2.2 when \( d \geq 7 \). Although for our applications here the values \( h \geq 3 \) are never used, we present the argument in full generality as we believe it can be used to study Weyl sums with other polynomials.
2.4. Ideas behind the proofs. We concentrate on the ideas in the proof of Theorem 2.1. Before doing this, we remark that a similar argument to the proof of the upper bound on \( \dim \mathcal{E}_{2,\alpha} \) also implies an upper bound on \( \dim \mathcal{E}_{d,\alpha} \) and \( \dim \mathcal{F}_{f,\alpha} \) in Theorems 2.2 and 2.3. However, the idea for obtaining the lower bound on \( \dim \mathcal{E}_{2,\alpha} \) does not work for \( d \geq 3 \). The main reason is that we do not have a version of Lemma 7.1 when \( d \geq 3 \) and in fact for a prime \( p \equiv 2 \pmod{3} \) there are many vectors \((a, b, c) \in \mathbb{Z}^3\) such that

\[
\sum_{n=1}^{p} e_p(an + bn^2 + cn^3) = 0,
\]

see [14, Remark 2.8] for more details.

**Upper bound:** Our argument is based on a combination of the Frostman Lemma (see [23, Corollary 4.12]) and the Gál–Koksma Theorem [26, Theorem 4], which are presented in Section 4.

For any \( t < \dim \mathcal{E}_{2,\alpha} \), by the Frostman Lemma (see [23, Corollary 4.12] or Lemma 4.1 below), there exists a Radon measure \( \mu \) on \( \mathcal{E}_{2,\alpha} \) with

\[
\mu(\mathcal{E}_{2,\alpha}) > 0 \quad \text{and} \quad \mu(B(x, r)) \ll r^{t}
\]

for all \( x \) and \( r > 0 \). Applying the description of Baker [4, Theorem 3] of the structure of large Gauss sums, we obtain the following type \( L^\rho \) bound: for any \( M, N \in \mathbb{N} \) we have

\[
\int_{\mathbb{T}^2} \left| \sum_{n=M+1}^{M+N} e(x_1n + x_2n^2) \right|^\rho d\mu(x) \lesssim N^{s_1 + o(1)}(N + M)^{s_2},
\]

where the exponents \( s_1 \) and \( s_2 \) depend on \( \rho \) and \( t \). By a result of Gál and Koksma [26, Theorem 4], we obtain that for almost all \( x = (x_1, x_2) \) with respect to \( \mu \)

\[
\sum_{n=1}^{N} e(x_1n + x_2n^2) = o \left( N^{(s_1 + s_2)/\rho} \right), \quad N \to \infty.
\]

Since \( \mu(\mathcal{E}_{2,\alpha}) > 0 \), there is a set of \((x_1, x_2)\) of positive \( \mu \)-measure such that

\[
\left| \sum_{n=1}^{N} e(x_1n + x_2n^2) \right| \geq N^\alpha
\]

for infinitely many \( N \in \mathbb{N} \). It follows that

\[
\alpha \leq (s_1 + s_2)/\rho.
\]
By taking the concrete parameters we obtain
\[ t < \min\{1/2 + 3(1 - \alpha), 6(1 - \alpha)\}. \]

Note that this inequality holds for any \( t < \dim E_{2,\alpha} \). Thus we obtain
\[ \dim E_{2,\alpha} \leq \min\{1/2 + 3(1 - \alpha), 6(1 - \alpha)\}, \]
which yields the desired upper bound.

Proofs of Theorems 2.2 and 2.3 follow a similar idea, albeit in a more technically involved way.

**Lower bound:** we make the heuristic argument of [13, Section 8] rigorous for the case \( d = 2 \). In brief, Gauss sums are large at rational points and their small neighbourhood, and we know the size of them from Diophantine analysis. Indeed, first note that the Gauss sums are large at rational points, for instance for any \( a, b, p \) where \( p \) is a prime number, \( (p, b) = 1 \) we have
\[
\left| \sum_{n=1}^{p} e_p(an + bn^2) \right| = \sqrt{p},
\]
where \( e_p(z) = \exp(2\pi iz/p) \), and hence, by periodicity, for suitably large \( N \) we have
\[
\left| \sum_{n=1}^{N} e_p(an + bn^2) \right| \approx \frac{N}{\sqrt{p}} \approx N^\alpha.
\]

Here \( z \approx Z \) means that \( Z/C \leq z \leq CZ \) for some absolute positive constant \( C \). By the continuity of the Gauss sums, we have
\[
\left| \sum_{n=1}^{N} e(xn + yn^2) \right| \approx N^\alpha,
\]
provided that
\[
(2.4) \quad \left| x - \frac{a}{p} \right| < p^{-\frac{1}{2(1-\alpha)}}, \quad \left| y - \frac{b}{p} \right| < p^{-\frac{1}{2(1-\alpha)}}.
\]

It follows that if \((x, y)\) satisfies (2.4) for infinitely many \(a, b, p\) then \((x, y) \in E_{2,\alpha}\), and we denote the collection of these \((x, y)\) by \(W_\alpha\). Thus \(W_\alpha \subseteq E_{2,\alpha}\). By Rynne [36, Theorem 1], for \( \alpha \in (1/2, 1) \) we have
\[
\dim W_\alpha = \min \{1/2 + 3(1 - \alpha), 6(1 - \alpha)\},
\]
and thus yields the desired lower bound.
3. Results for sets of exponential sums of arbitrary size

3.1. One parametric families of Weyl sums with integer polynomials. Theorem 2.3 says nothing for dim $F_{f,\alpha}$ when $\alpha \in (1/2, 1 - 1/D)$. Our next result fills that gap, and in particular for any polynomial $f \in \mathbb{Z}[X]$ with degree $d \geq 2$ and any $\alpha \in (1/2, 1)$ we have

$$\dim F_{f,\alpha} < 1.$$  

(3.1)

In fact, Theorem 3.5 below implies that for any real polynomial $f \in \mathbb{R}[X]$ with degree $d \geq 2$ the nontrivial bound (3.1) still holds.

**Theorem 3.1.** Let $f \in \mathbb{Z}[x]$ be a polynomial with degree $d \geq 2$. For any real $\alpha \in (1/2, 1)$, we have

$$\dim F_{f,\alpha} \leq \min\{U_1(d, \alpha), U_2(d, \alpha)\},$$

where

$$U_1(d, \alpha) = \min_{r=1, \ldots, d} \frac{d + 1 - \alpha + 2^r(1 - \alpha) - r}{d + 1 - \alpha},$$

$$U_2(d, \alpha) = \min_{r=1, \ldots, d} \frac{d + 1 - \alpha + r(r + 1)(1 - \alpha) - r}{d + 1 - \alpha}.$$

Now we extract some easier upper bounds for $\dim F_{f,\alpha}$. Taking $r = 2$ and $r = d$ in the definition of $U_1(d, \alpha)$, we obtain

$$U_1(d, \alpha) \leq \min \left\{ \frac{d + 3 - 5\alpha}{d + 1 - \alpha}, \frac{(2^d + 1)(1 - \alpha)}{d + 1 - \alpha} \right\}.$$

Furthermore, taking $r = d$ in the definition of $U_2(d, \alpha)$ we obtain

$$U_2(d, \alpha) \leq \frac{(d^2 + d + 1)(1 - \alpha)}{d + 1 - \alpha}.$$

We formulate the following corollary.

**Corollary 3.2.** Let $f \in \mathbb{Z}[x]$ be a polynomial with degree $d \geq 2$. For any real $\alpha \in (1/2, 1)$, we have

$$\dim F_{f,\alpha} \leq \min \left\{ \frac{d + 3 - 5\alpha}{d + 1 - \alpha}, \frac{(2^d + 1)(1 - \alpha)}{d + 1 - \alpha}, \frac{(d^2 + d + 1)(1 - \alpha)}{d + 1 - \alpha} \right\}.$$

Corollary 3.2 implies that for any $f \in \mathbb{Z}[x]$ be a polynomial with degree $d \geq 2$ and any $\alpha \in (1/2, 1)$ we have

$$\dim F_{f,\alpha} < 1.$$

Furthermore, for monomials we have yet another bound.
Theorem 3.3. Let \( d \geq 2 \). For any real \( \alpha \in (1/2, 1) \), we have

\[
\dim \mathcal{F}_{d, \alpha} \leq \frac{(1 + s_0)(1 - \alpha)}{d + 1 - \alpha},
\]

where

\[
s_0 = d(d - 1) + \min_{r=1,\ldots,d} \frac{2d + (r - 1)(r - 2)}{r}.
\]

Furthermore, for “polynomial-like” sequences, such in the special case of Piatetski-Shapiro sequences \( f(n) = \lfloor n^\tau \rfloor \) we have the following result.

Theorem 3.4. Let \( f(n) = \lfloor n^\tau \rfloor \) for some \( \tau \geq 1 \). For any \( \alpha \in (1/2, 1) \), we have

\[
\dim \mathcal{F}_{f, \alpha} \leq \begin{cases} 
1 - \frac{4\alpha + \tau - 1}{\tau + 1 - \alpha} & \text{if } \tau < 2, \\
1 - \frac{4\alpha - 2}{\tau + 1 - \alpha} & \text{if } \tau \geq 2.
\end{cases}
\]

Note that for any \( \alpha \in (1/2, 1) \) Theorem 3.4 provides nontrivial upper bound, that is, \( \dim \mathcal{F}_{f, \alpha} < 1 \), in a wide range of parameters \( \alpha \) and \( \tau \), for instance, when \( 2 > \tau > 4 - 4\alpha \).

3.2. One parametric families of exponential sums with arbitrary sequences. Let \( f(n) \) be a real sequence. We extend the definition of exponential sums \( V_f(x; N) \) in (1.5) and of the sets \( \mathcal{F}_{f, \alpha} \) to exponential sums with an arbitrary real sequence \( f(n), n = 1, 2, \ldots \).

First we consider sequences with a given rate of their growth on average. Namely, we assume that there exists a real number \( \tau > 0 \) such that for all large enough \( N \) we have

\[
\frac{1}{N} \sum_{n=1}^{N} |f(n)| \ll N^\tau.
\]

Theorem 3.5. Let \( f(n) \) be a real sequence such that \( |f(n) - f(m)| \gg 1 \) for all \( n \neq m \) and the sequence \( f(n) \) satisfies (3.2) for some \( \tau > 0 \). For any real \( \alpha \in (1/2, 1) \), we have

\[
\dim \mathcal{F}_{f, \alpha} \leq 1 - \frac{2\alpha - 1}{\tau + 1 - \alpha}.
\]

Clearly for any \( \alpha \in (1/2, 1) \) and \( \tau > 0 \) we have

\[
1 - \frac{2\alpha - 1}{\tau + 1 - \alpha} < 1.
\]
Theorem 3.6. Let $f(n) \in \mathbb{Z}$ be a strictly convex integer sequence that satisfies (3.2) for some $\tau > 0$. Then we have

$$\dim \mathcal{F}_{f, \alpha} \leq \min\{U_1(\tau, \alpha), U_2(\tau, \alpha)\},$$

where

$$U_1(\tau, \alpha) = \frac{\tau + 45/13 - 5\alpha}{\tau - \alpha},$$

$$U_2(\tau, \alpha) = \inf_{k \geq 3, k \in \mathbb{N}} \frac{\tau - \alpha + 2k - 1 + 2^{-k+1} - 2k\alpha}{\tau + 1 - \alpha}.$$

Next we consider sequences with a restriction on the growth of individual terms rather than on average as in (3.2).

Theorem 3.7. Let $f(n)$ be a sequence of strictly increasing sequence of natural numbers with $f(n) = O(n^\tau)$ for some $\tau \geq 1$. For any $\alpha \in (1/2, 1)$, we have

$$\dim \mathcal{F}_{f, \alpha} \leq 1 - \frac{2\alpha - 1}{\tau}.$$

We note that the upper bound of Theorem 3.7 is nearly sharp when $\alpha \to 1$. In fact we consider the following set of exponential sums with an even more stringent condition. Namely, for $c > 0$, we define the set

$$(3.3) \quad \mathcal{G}_{f,c} = \{ x \in T : |V_f(x; N)| \geq cN \text{ for infinitely many } N \in \mathbb{N} \}.$$

Here we mention some related work on $\mathcal{G}_{f,c}$. Suppose further that $f(n) \in \mathbb{N}$ for all $n$ with $f(n) = O(n^\tau)$ for some $\tau \geq 1$. First Salem [37] and then Erdős and Taylor [22] have shown that the set of $x \in T$ such that the sequence

$$xf(n), \ n = 1, 2, \ldots, \text{ is not uniformly distributed } \pmod{1}$$

has Hausdorff dimension at most $1-1/\tau$. The result has been extended to arbitrary real sequences $f(n) = O(n^\tau)$ by Baker [1]. Moreover, for each $p \geq 1$, Ruzsa [35] exhibits an integer sequence $f(n) = O(n^\tau)$ to show that the above upper bound $1-1/\tau$ is attained. The above results are related to the set $\mathcal{G}_{f,c}$ by using the Weyl criterion (see [20, Section 1.2.1]) and the countable stability of Hausdorff dimension (see [23, Section 2.2]). More precisely, we recall that the countable stability of Hausdorff dimension asserts that for any sequence of sets $\mathcal{F}_i$ we have

$$\dim \bigcup_{i \in \mathbb{N}} \mathcal{F}_i = \sup_{i \in \mathbb{N}} \dim \mathcal{F}_i.$$
It follows from the above result of Baker [1], Erdős and Taylor [22] and Salem [37] that for a sequence \( f(n) = O(n^\tau) \) with \( \tau \geq 1 \) and any \( c > 0 \) we have
\[
(3.4) \quad \dim \mathcal{G}_{f,c} \leq 1 - 1/\tau.
\]
Moreover, the result of Ruzsa [35] implies that for any \( \tau \geq 1 \) and any \( \varepsilon > 0 \) there exists a sequence \( f(n) = O(n^\tau) \) such that
\[
(3.5) \quad \dim \mathcal{G}_{f,c} \geq 1 - 1/\tau - \varepsilon.
\]
By combining with other ideas, we could remove the \( \varepsilon \) of (3.5), and obtain the following.

**Theorem 3.8.** For any \( \tau \geq 1 \) there exits a strictly increasing sequence of natural numbers \( f(n) \) with \( f(n) = O(n^\tau) \) such that for some \( c > 0 \), we have
\[
\dim \mathcal{G}_{f,c} = 1 - 1/\tau.
\]

Let \( f(n) = O(n^\tau), \tau \geq 1 \) be a monotone increasing real sequence such that \( f(n + 1) - f(n) \gg 1 \). Baker, Coatney and Harman [7, Theorem 1] show that the set of \( x \in T_d \) such that the sequence
\[
f(n)x, \ n = 1, 2, \ldots, \text{ is not uniformly distributed (mod 1)}
\]
is of Hausdorff dimension at most \( d - 1/\tau \). Moreover, [7, Theorem 2] shows that the bound \( d - 1/\tau \) is sharp. By adapting the method of the proof of Theorem 3.1 (which is same as in the proofs of Theorems 3.3, 3.4, 3.5 and 3.6), we obtain the following result where \( f(n) \) is a sequence of matrices.

For a \( d \times d \) matrix \( A \) we use \( \|A\| \) to denote its operator norm, that is,
\[
\|A\| = \sup \{ \|xA\| : x \in \mathbb{R}^d, \|x\| = 1 \}
\]
where, as before, \( \|w\| \) denotes the Euclidean norm in \( \mathbb{R}^d \).

**Theorem 3.9.** Let \( S = (A_n)_{n \in \mathbb{N}} \) be a sequence of \( d \times d \) integer matrices such that for some \( \tau \geq 1/d \) and for all \( N \in \mathbb{N} \) we have
\[
(3.6) \quad \frac{1}{N} \sum_{n=1}^{N} \|A_n\| \ll N^\tau.
\]
Moreover for any \( n \neq m \) the matrix \( A_n - A_m \) is invertible. Let \( \mathcal{E}_S \) be the collection of point \( x \in T_d \) such that the sequence
\[
xA_n, \ n = 1, 2, \ldots, \text{ is not uniformly distributed (mod 1)}.
\]
Then we have
\[
\dim \mathcal{E}_S \leq d - 1/\tau.
\]
We remark that the reason of taking \( \tau \geq 1/d \) is making the estimate (12.1) meaningful. Moreover, we claim that the bound of Theorem 3.9 is sharp in general, and this follows by using the aforementioned [7, Theorem 2]. More precisely, [7, Theorem 2] shows that for any \( \tau \geq 1 \), there exists an integer sequence \( f(n) = O(n^\tau) \) such that the set of \( x \in T_d \) for which

\[ xf(n), \ n = 1, 2, \ldots, \text{ is not uniformly distributed (mod 1)} \]

is of Hausdorff dimension \( d - 1/\tau \). For example, for each \( n \in \mathbb{N} \) let \( A_n \) be a diagonal matrix with the same diagonal elements \( f(n) \), then this sequence \( A_n \) attains the above upper bound \( d - 1/\tau \) which implies the claim above.

3.3. Comparison. Clearly Theorem 3.7 applies to polynomials \( f \in \mathbb{R}[X] \) with \( \tau = d \) and thus complements Theorem 2.3. In particular, as we have mentioned we see that for any real polynomial \( f \in \mathbb{R}[X] \) with degree \( d \geq 2 \) the nontrivial bound (3.1) still holds.

We remark that Theorem 3.5 still hold under a relaxed condition, that is \( |f(n) - f(m)| \gg 1 \) for all \( n \neq m \geq N_0 \) for any constant \( N_0 \), and thus also applies to polynomials \( f \in \mathbb{R}[X] \) with \( \tau = d \), however the corresponding upper bounds implied by Theorems 3.7 and 3.5 satisfy

\[ 1 - \frac{2\alpha - 1}{d} < 1 - \frac{2\alpha - 1}{d + 1 - \alpha}. \]

Finally the lower bound in Theorem 3.8 is based on an idea of Ruzsa [35].

3.4. Ideas behind the proofs. Results of Section 3 are all based on various mean values theorems, continuity of exponential sums and on the completion technique in the style used in [15,16,18]. We roughly show that how their arguments imply the upper bounds of \( \dim \mathcal{F}_{f,\alpha} \). For obtaining the upper bound of \( \dim \mathcal{F}_{f,\alpha} \), we find some intervals to cover the set \( \mathcal{F}_{f,\alpha} \). We collect these intervals by using the continuity of the sums

\[ V_f(x; N) = \sum_{n=1}^{N} e(xf(n)), \]

that is if \( |V_f(x; N)| \geq N^\alpha \) then \( |V_f(y; N)| \geq N^\alpha/2 \) when \( y \) belongs to some small neighbourhood of \( x \). Moreover, the mean value bounds of \( V_f(x; N) \) control the cardinality of above chosen intervals, which yields the desired upper bounds. Thus for obtaining better bounds, we have to know how small neighbourhood of above \( x \) explicitly, and we need various mean values theorems as well. For technical reasons
(completion technique), we in fact use an auxiliary exponential sums $W_f(x; N)$, which is given by (10.1) to ‘control’ the size of $V_f(x; N)$.

More precisely, to establish Theorem 3.1, 3.5 and 3.6, we combine Lemma 10.10 with mean values theorems collected in Section 10.1. To prove Theorem 3.7 we use the mean value bound from Lemma 11.1, which comes from [2], and, as for results in Section 2, on the Frostman Lemma and Gál–Koksma Theorem, see Lemmas 4.1 and 4.3, respectively.

4. Frostman Lemma and Gál–Koksma Theorem

4.1. Frostman Lemma. For a real $s \geq 0$ and a set $F \subseteq \mathbb{R}^d$ denote

$$H_s(\mathcal{F}) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam} U_i)^s : \mathcal{F} \subseteq \bigcup_{i=1}^{\infty} U_i, \right. \left. U_i \subseteq \mathbb{R}^d \text{ and diam}(U_i) \leq \delta, i \in \mathbb{N} \right\},$$

and the $s$-dimensional Hausdorff measure of the set $\mathcal{F}$ is defined as

$$H^s(\mathcal{F}) = \lim_{\delta \to 0} H^s_\delta(\mathcal{F}).$$

Moreover, alternatively the Hausdorff dimension of $\mathcal{F}$ can also be defined as

$$\dim \mathcal{F} = \inf \{ s \geq 0 : H^s(\mathcal{F}) = 0 \} = \sup \{ s \geq 0 : H^s(\mathcal{F}) = \infty \}.$$ 

We also need the following result, which is known as the Frostman Lemma, see, for instance, [23, Corollary 4.12].

**Lemma 4.1.** Let $\mathcal{F} \subseteq \mathbb{R}^d$ be a Borel set with $0 < H^s(\mathcal{F}) \leq \infty$. Then there exists a compact set $\mathcal{E} \subseteq \mathcal{F}$ such that $0 < H^s(\mathcal{E}) < \infty$ and

$$H^s(\mathcal{E} \cap B(x, r)) \ll r^s$$

for all $x$ and $r > 0$.

We remark that for our application of Lemma 4.1 we take $\mu$ to be the restriction of the $s$-Hausdorff measure $H^s$ on $\mathcal{E}$, that is for any $\mathcal{A} \subseteq \mathbb{R}^d$,

$$\mu(\mathcal{A}) = H^s(\mathcal{E} \cap \mathcal{A}).$$

It follows that $\mu$ is a Radon measure such that $\mu(\mathcal{E}) > 0$ and

$$\mu(B(x, r)) \ll r^s$$

for all $x$ and $r > 0$. 
4.2. **Gál–Koksma Theorem.** We first recall the following result of Gál and Koksma [26, Theorem 4], which shows that mean value bounds imply almost all bounds with respect to the same measure. Note that [26] treats the case of the Lebesgue measure on a set $S$ rather than an arbitrary Radon measure $\mu$, however the proof goes through without change.

**Lemma 4.2.** Let $\mu$ be a Radon measure on $\mathbb{T}_d$ and let $f_1(x), f_2(x), \ldots$ be a sequence of Borel measurable function on $\mathbb{R}^d$. Suppose that we have the bound, for some $\rho > 0$ and for all $M \geq 0, N \geq 1$,

$$\int_{\mathbb{R}^d} \left| \sum_{n=M+1}^{M+N} f_n(x) \right|^\rho d\mu(x) \leq C \Psi(N) \Phi(M, N), \quad \Phi(M, N) \geq 1,$$

where $C$ is an absolutely constant, $\Psi(N)$ and $\Phi(M, N)$ are some positive functions, and $\Psi(N)/N^{1+\gamma}$ is nondecreasing for some positive $\gamma$. Let $\psi(N) > 0$ be nondecreasing and

$$\psi(2^n) \geq \Phi(0, 2^n) + \sum_{\lambda=1}^{n} 2^{(\lambda-n)(1+\beta)} \sum_{k=0}^{2^{n-\lambda}-1} \Phi(2^n + k 2^\lambda, 2^{\lambda-1}),$$

where $0 < \beta < \gamma$ is a constant. Let $\chi(N)$ be a positive nondecreasing function with

$$\sum_{n=1}^{\infty} (N \chi(N))^{-1} < \infty.$$

Then for almost all $x$ with respect to $\mu$, we have

$$\sum_{n=1}^{N} f_n(x) = o(\Psi(N) \psi(N) \chi(N))^{1/\rho}, \quad N \to \infty.$$

For convenience of our application, we formulate the following particular case of Lemma 4.2.

**Lemma 4.3.** Let $\mu$ be a Radon measure on $\mathbb{T}_d$ and let $f_1(x), f_2(x), \ldots$ be a sequence of Borel measurable function on $\mathbb{R}^d$. Suppose that we have the bound, for some $\rho > 0$ and for all $M \geq 0, N \geq 1$,

$$\int_{\mathbb{R}^d} \left| \sum_{n=M+1}^{M+N} f_n(x) \right|^\rho d\mu(x) \leq N^{s_1+o(1)} (M + N)^{s_2}$$
for some constants $s_1 > 1$ and $s_2 > 0$. Then for almost all $x \in \mathbb{R}^d$ with respect to $\mu$ we have

$$\left| \sum_{n=1}^{N} f_n(x) \right| \leq N^{(s_1+s_2)/\rho + o(1)}, \quad N \to \infty.$$  

Proof. We show that $\psi(N) = CN^{s_2}$ satisfies (4.1) when $C$ is some large constant. To see this, we note that for $1 \leq \lambda \leq n$, we have

$$\sum_{k=0}^{2^n-\lambda-1} (2^n + k2^\lambda + 2^{\lambda-1})^{s_2} \ll 2^{n-\lambda}2^{\lambda s_2},$$  

and for some small $\beta > 0$,

$$\sum_{\lambda=1}^{n} 2^{(\lambda-n)(1+\beta)}2^{\lambda}2^{\lambda s_2} = \sum_{\lambda=1}^{n} 2^{-(n-\lambda)\beta}2^{n s_2} \ll 2^{n s_2}.$$  

Moreover, for any $\varepsilon > 0$ the function $\chi(N) = N^{\varepsilon}$ satisfies (4.2). By Lemma 4.2 we obtain that for almost all $x \in \mathbb{R}^d$ with respect to $\mu$ we have

$$\left| \sum_{n=1}^{N} f_n(x) \right| \leq N^{(s_1+s_2+\varepsilon)/\rho + o(1)}, \quad N \to \infty.$$  

By the arbitrary choice of $\varepsilon > 0$ we obtain the desired bound. \qed

5. Bounds of Gauss sums and Weyl sums

5.1. Structure of large Gauss sums. The following result of Baker [4, Theorem 3] and [5, Theorem 4] describes the structure of large Gauss sums.

For each $m \in \mathbb{N}$ recalling that $[m] = \{0, 1, \ldots, m-1\}$.

Lemma 5.1. We fix some $\varepsilon > 0$, and suppose that for a real $A > N^{1/2+\varepsilon}$, we have $|G(x_1, x_2; N)| \geq A$ for some $(x_1, x_2) \in \mathbb{R}^2$. Then there exist integers $q, a_1, a_2$ such that

$$1 \leq q \leq (NA^{-1})^2 N^{o(1)},$$  

and for $i = 1, 2$ we have

$$\left| x_i - \frac{a_i}{q} \right| \leq (NA^{-1})^2 q^{-1} N^{-i+o(1)}.$$  

We now use Lemma 5.1 to describe the structure of large sums $G(x_1, x_2; M, N)$. 
Lemma 5.2. We fix some \( \varepsilon > 0 \), and suppose that for a real \( A > N^{1/2+\varepsilon} \), we have \( |G(x_1, x_2; M, N)| \geq A \) for some \( (x_1, x_2) \in T_2 \). Then there exist integers \( q, b_1, b_2 \) such that
\[
1 \leq q \leq \left( NA^{-1}\right)^2 N^{o(1)}, \quad 0 \leq b_1, b_2 \leq q,
\]
and
\[
\left| x_1 - \frac{b_1}{q} \right| \leq (M + N)q^{-1}A^{-2}N^{o(1)},
\]
\[
\left| x_2 - \frac{b_2}{q} \right| \leq q^{-1}A^{-2}N^{o(1)}.
\]

Proof. Elementary arithmetic shows that
\[
|G(x_1, x_2; M, N)| = |G(x_1 + 2Mx_2, x_2; N)|.
\]
By Lemma 5.1 there exist \( q, a_1, a_2 \) such that
\[
\left| x_1 + 2Mx_2 - \frac{a_1}{q} \right| \leq q^{-1}A^{-2}N^{1+o(1)},
\]
\[
\left| x_2 - \frac{a_2}{q} \right| \leq q^{-1}A^{-2}N^{o(1)}.
\]

Clearly \( A^{-2}N^{o(1)} < 1 \), we conclude that \( 0 \leq a_2 \leq q \), and we take \( b_2 = a_2 \).

We now search for an integer \( b_1 \) with the desired property. From (5.1) we obtain
\[
\left| x_1 - \frac{a_1 - 2Ma_2}{q} \right| \leq (M + N)q^{-1}A^{-2}N^{o(1)}.
\]

Suppose that \( (M + N)A^{-2}N^{o(1)} \leq 1 \), then by (5.2) we conclude that
\[
0 \leq a_1 - 2Ma_2 \leq q,
\]
and we take \( b_1 = a_1 - 2Ma_2 \). Suppose to the contrary that
\[
(M + N)A^{-2}N^{o(1)} > 1,
\]
then trivially there exists \( b_1 \) with \( 0 \leq b_1 \leq q \) and such that
\[
\left| x_1 - \frac{b_1}{q} \right| \leq q^{-1} \leq (M + N)q^{-1}A^{-2}N^{o(1)},
\]
which finishes the proof. \( \Box \)
Remark 5.3. For large enough $M$ the bound
\[ |x_1 - \frac{b_1}{q}| \leq (M + N)q^{-1}A^{-2}N^{o(1)} \]
of Lemma 5.1 would be trivial. Thus we may add $1/q$ term, that is\[ |x_1 - \frac{b_1}{q}| \leq \min\{(M + N)q^{-1}A^{-2}N^{o(1)}, 1/q\}.
\]
However, the bound $(M + N)q^{-1}A^{-2}N^{o(1)}$ is sufficient for our applications, and hence we use this bound only.

5.2. Frequency of large Gauss sums. Let $\mu$ be a Radon measure on $\mathbb{T}_2$ such that
\[ \mu(B(x, r)) \ll r^t \]
holds for some $t > 0$ and for all $x \in \mathbb{T}_2$ and $r > 0$. Let $\mathcal{R}$ be a rectangle with side length $a < b$, then we have\[ \mu(\mathcal{R}) \ll \min\{ba^{t-1}, b^t\}.
\]
Indeed, we can either include $\mathcal{R}$ in a ball of radius $O(b)$ or cover it by $O(b/a)$ balls of radius $a$.

Lemma 5.4. Let $\mu$ be a Radon measure satisfying (5.3) and $A$ be a real number with $1 \leq A \leq N$. For fixed $M, N \in \mathbb{N}$ denote\[ \mathcal{E}_B = \{x \in \mathbb{T}_2 : |G(x; M, N)| \geq B\}.
\]
Then we have\[ \mu(\mathcal{E}_B) \leq N^{6-2t+o(1)}B^{-6} \min\{M + N, (M + N)^t\}.
\]
Proof. Denote\[ Q = (NB^{-1})^2N^{o(1)}.
\]
From Lemma 5.1 we conclude that\[ \{x \in \mathbb{T}_2 : |G(x; M, N)| \geq B\} \subseteq \bigcup_{q \leq Q} \bigcup_{(a_1, a_2) \in [q]^2} \mathcal{R}_{q,b_1,b_2},
\]
where $\mathcal{R}_{q,b_1,b_2}$ is a rectangle with side lengths\[ (M + N)B^{-2}q^{-1}N^{o(1)} \quad \text{and} \quad B^{-2}q^{-1}N^{o(1)}.
\]
For $Z \geq 1$, we write $z \sim Z$ to denote that $Z/2 < z \leq Z$. Denote\[ \delta_z = B^{-2}z^{-1}.
\]
Taking a dyadic partition of the interval $[1, Q]$, we see that there is a number $1 \leq Z \leq Q$ such that
\[
\mu(E_B) \leq N^{o(1)} \sum_{q \sim Z} \sum_{(a_1, a_2) \in [q]^2} \mu(\mathcal{R}_{q, a_1, a_2}) \\
\leq N^{o(1)} Z^3 \min\{(M + N)\delta_Z^t, (M + N)^t \delta_Z^t\} \\
\leq N^{o(1)} \min\{I, J\},
\]
where
\[
I = Z^3(M + N)\delta_Z^t \leq Z^{3-t} B^{-2t}(M + N) \leq N^{6-2t} B^{-6}(M + N),
\]
and
\[
J = Z^3(M + N)^t \delta_Z^t \leq N^{6-2t} B^{-6}(M + N)^t,
\]
which finishes the proof. \qed

5.3. **Bounds of Weyl sums.** Corresponding to the bounds of Gauss sums, we have the following estimation for Weyl sums which is needed for the proof of Theorem 2.2.

An integer number $n$ is called

- $r$-th power free if any prime number $p | n$ satisfies $p^r \nmid n$;
- $r$-th power full if any prime number $p | n$ satisfies $p^r | n$.

We note that 1 is both cube free and cube full.

For any integer $i \geq 2$ it is convenient to denote

\[
\mathcal{P}_i = \{n \in \mathbb{N} : n \text{ is } i\text{-th power full}\} \quad \text{and} \quad \mathcal{P}_i(x) = \mathcal{P}_i \cap [1, x].
\]

The classical result of Erdős and Szekeres [21] gives an asymptotic formula for the cardinality of $\mathcal{P}_i(x)$ which we present here in a very relaxed form as the upper bound

\[
\# \mathcal{P}_i(x) \ll x^{1/i}.
\]

The following Lemma 5.5 comes from [6, Lemma 2.7].

**Lemma 5.5.** We fix $d \geq 3$, some $\varepsilon > 0$, and suppose that for a real $A > N^{1-1/D+\varepsilon}$, where $D$ is given by (2.1), we have $|S_d(x; N)| \geq A$ for some $x \in \mathbb{R}^d$. Then there exist positive integers $q_2, \ldots, q_d$ with $\gcd(q_i, q_j) = 1$, $2 \leq i < j \leq d$, and such that

(i) $q_2$ is cube free,
(ii) $q_i$ is $i$-th power full but $(i+1)$-th power free when $3 \leq i \leq d-1$,
(iii) $q_d$ is $d$-th power full,
and

\[ \prod_{i=2}^{d} q_i^{1/i} \leq N^{1+o(1)} A^{-1} \]

and integers \( b_1, \ldots, b_d \) such that

\[ \left| x_j - \frac{b_j}{q_2 \cdots q_d} \right| \leq (NA^{-1})^d N^{-j+o(1)} \prod_{i=2}^{d} q_i^{-d/i}, \quad j = 1, \ldots, d. \]

We now need a version of Lemma 5.5 for the sums

\[ S_d(x; M, N) = \sum_{n=1}^{N} e \left( x_1(M + n) + \ldots + x_d(M + n)^d \right) \]

over arbitrary intervals.

**Lemma 5.6.** We fix \( d \geq 3 \) and some \( \varepsilon > 0 \). Let \( M \geq 0 \) and \( N \geq 1 \). Suppose \( x \in T_d \) and

\[ |S_d(x; M, N)| \geq B \geq N^{1-1/D+\varepsilon}, \]

where \( D \) is given by (2.1). Then there exists \( q = q_2q_3 \ldots q_d \) with \( \gcd(q_i, q_j) = 1 \), \( 2 \leq i < j \leq d \), and such that

(i) \( q_2 \) is cube free,

(ii) \( q_i \) is \( i \)-th power full but \((i+1)\)-th power free when \( 3 \leq i \leq d-1 \),

(iii) \( q_d \) is \( d \)-th power full,

and

\[ q_2^{1/2} q_3^{1/3} \ldots q_d^{1/d} \leq N^{1+o(1)} B^{-1}, \]

and there are \( a_1, \ldots, a_d \in [q] \) such that

\[ x_k = \frac{a_k}{q} + O \left( (M + N)^{d-k} r \right), \quad k = 1, \ldots, d, \]

where

\[ r = N^{o(1)} B^{-d} \prod_{i=2}^{d} q_i^{-d/i}. \]

**Proof.** The coefficient of \( n^k \) in \( (M + n)x_1 + \ldots + (M + n)^d x_d \) is

\[ y_k = \sum_{j=k}^{d} \binom{j}{k} M^{j-k} x_j. \]
Note that $y_d = x_d$. It follows that
\[
\left| \sum_{n=1}^{N} e \left( (M + n) x_1 + \ldots + (M + n)^d x_d \right) \right| = \left| \sum_{n=1}^{N} e \left( y_1 n + \ldots + y_d n^d \right) \right| .
\]

Then we have the following equivalence
\[
|S_d(x; M, N)| \geq B \iff |S_d(y; N)| \geq B.
\]

By Lemma 5.5 there exist positive integers $q_2 \ldots q_d$ with the above mentioned properties (i), (ii), (iii) and
\[
\prod_{i=2}^{d} q_i^{1/i} \leq N^{1 + o(1)} A^{-1},
\]
and integers $b_1, \ldots, b_d$ such that
\[
\left| y_j - \frac{b_j}{q_2 \ldots q_d} \right| \leq N^{d-j} r, \quad j = 1, \ldots, d,
\]
where $r$ is given by (5.6).

We now going to show (5.5) holds by induction. First note that since $x_d = y_d$, we have the bound
\[
x_d = \frac{b_d}{q_2 \ldots q_d} + O(r).
\]

Suppose that (5.5) hold for any $k + 1 \leq j \leq d$, that is, there exist $a_j$, $k + 1 \leq j \leq d$, such that
\[
(5.9) \quad x_j = \frac{a_j}{q} + O \left( (M + N)^{d-j} r \right), \quad j = k + 1, \ldots, d.
\]

Applying (5.7), (5.8), we derive that
\[
\left| x_k + \sum_{j=k+1}^{d} \binom{j}{k} M^{j-k} x_j - \frac{b_k}{q_2 \ldots q_d} \right| \leq N^{d-k} r.
\]

Combining with (5.9) we conclude that there exists $a_k$ such that
\[
\left| x_k - \frac{a_k}{q_2 \ldots q_d} \right| \leq N^{d-k} r + O \left( \sum_{j=k+1}^{d} M^{j-k} (M + N)^{d-j} r \right)
\]
\[
\ll (M + N)^{d-k} r,
\]
from which we obtain (5.5) by induction.
Applying similar argument as in the proof of Lemma 5.2, we can always find \(a_1, a_2, \ldots, a_d \in [q_2 \ldots q_d]\) such that the desired property hold. \(\square\)

Suppose that \(\mathcal{R}\) is a rectangle with side lengths
\[K^{d-1}\delta \geq K^{d-2}\delta \geq \ldots \geq K\delta \geq \delta\]
for some constants \(K \geq 1, \delta > 0\). Then, by elementary geometric argument, for any integer \(1 \leq h \leq d\) we can cover \(\mathcal{R}\) by \(O(K^{h(h-1)/2})\) cubes with side length \(K^{d-h}\delta\). Furthermore, suppose that \(\mu\) is a Radon measure satisfying (5.3), then we conclude that
\[\mu(\mathcal{R}) \ll K^{h(h-1)/2} (K^{d-h}\delta)^t.\]

**Lemma 5.7.** We fix \(d \geq 3\) and some \(\varepsilon > 0\). Let \(\mu\) be a Radon measure satisfying (5.3) and let \(B\) be a real number with \(N^{1-1/D+\varepsilon} \leq B \leq N\), where \(D\) is given by (2.1). For fixed \(M, N \in \mathbb{N}\) denote
\[\mathcal{E}_B = \{x \in T_d : |S_d(x; M, N)| \geq B\}.

Then for any integer \(h\) with \(1 \leq h \leq d\) we have
\[\mu(\mathcal{E}_B) \leq B^{-d^2-1} N^{d^2+1-dt+o(1)} (M + N)^{(d-h)t+h(h-1)/2}.

**Proof.** Denote
\[Q = (NB^{-1})^d,\]
and fix some \(\eta > 0\). By Lemma 5.6, we conclude that
\[\mathcal{E}_B \subseteq \bigcup_{(q_2, \ldots, q_d) \in \Omega} \bigcup_{a \in [q_2 \ldots q_d]^d} \mathcal{R}_{q_2, \ldots, q_d, a},\]
where, slightly relaxing the conditions of Lemma 5.6, we can take
\[\Omega = \left\{(q_2, \ldots, q_d) \in \mathbb{N}^{d-1} : q_j \in \mathcal{Q}_j, 3 \leq j \leq d, \prod_{j=2}^{d} q_j^{1/j} \leq CQ^{1/d} N^\eta\right\}\]
and \(\mathcal{R}_{q_2, \ldots, q_d, a}\) is a rectangle with side lengths
\[(M + N)^{d-1} r_{q_2, \ldots, q_d} \geq \ldots \geq (M + N) r_{q_2, \ldots, q_d} \geq r_{q_2, \ldots, q_d},\]
with some
\[r_{q_2, \ldots, q_d} = N^{o(1)} B^{-d} \prod_{j=2}^{d} q_j^{-d/j}.\]
Let $1 \leq h \leq d$ be an integer. Combining (5.12) with the bound (5.10), we obtain
\[ \mu(\mathcal{E}_B) \ll \sum_{(q_2, \ldots, q_d) \in \Omega} (q_2 \ldots q_d)^{d-h} r_{q_2, \ldots, q_d}^t (M + N)^{(d-h)t + h(h-1)/2}. \]

Covering $\Omega$ by $O((\log N)^{d-1})$ dyadic boxes, we see that there are some integers $Q_2, \ldots, Q_d \geq 1$ with
\[ d \prod_{j=2}^d Q_j^{1/j} \ll Q^{1/d} N^\eta \]

such that
\[ \mu(\mathcal{E}_B) \ll (\log N)^{d-1} (M + N)^{(d-h)t + h(h-1)/2} \times \sum_{q_2 \sim Q_2, \ldots, q_d \sim Q_d} (q_2 \ldots q_d)^d r_{q_2, \ldots, q_d}^t \]
\[ = N^{o(1)} B^{-dt} (M + N)^{(d-h)t + h(h-1)/2} \times \sum_{q_2 \sim Q_2, \ldots, q_d \sim Q_d} \prod_{j=2}^d q_j^{-dt/j} \]
\[ = N^{o(1)} B^{-dt} (M + N)^{(d-h)t + h(h-1)/2} \times Q_2^{-dt/2} \prod_{j=3}^d \left( Q_j^{d-dt/j} \# \mathcal{D}_j(Q_j) \right). \]

Recalling (5.4) we derive
\[ \mu(\mathcal{E}_B) \leq N^{o(1)} B^{-dt} (M + N)^{(d-h)t + h(h-1)/2} \prod_{j=2}^d Q_j^{\alpha_j}, \]

where
\[ \alpha_2 = d + 1 - dt/2, \quad \text{and} \quad \alpha_j = d + 1/j - dt/j, \quad 3 \leq j \leq d. \]

Observe that for $j = 2, \ldots, d$, we have
\[ \alpha_j \leq d \alpha_d / j, \]

which for $j \geq 3$ is obvious from
\[ j \alpha_j = dj + 1 - dt \leq d^2 + 1 - dt = d \alpha_d \]

and for $j = 2$ from
\[ 2 \alpha_2 = 2d + 2 - dt \leq d^2 + 1 - dt \]
since \(d \geq 3\).

Therefore, in view of (5.16), recalling (5.14), we obtain

\[
\prod_{j=2}^{d} Q_{j}^{\alpha_{j}} \leq \left( \prod_{j=2}^{d} Q_{j}^{1/3} \right)^{d\alpha_{d}} \ll Q_{d}^{\alpha_{d} N^{d\alpha_{d}}}.
\]

Then combining this with (5.15) we obtain

\[
\mu(E_{B}) \leq N \circledast \left( \begin{array}{c}
1 \\text{if } d = 2, \\
N^{d+1-\varepsilon}B^{-d-1} \quad \text{if } d \geq 3
\end{array} \right).
\]

Recalling the choice of \(Q\) in (5.11), since \(\eta > 0\) is arbitrary we obtain the desired bound. \(\Box\)

We need the following analogue of Lemma 5.7.

**Lemma 5.8.** We fix some \(\varepsilon > 0\). Let \(f \in \mathbb{R}[X]\) be a polynomial of degree \(d\). Let \(\mu\) be a Radon measure satisfying (5.3) and let \(B\) be a real number with \(N^{1-1/D+\varepsilon} \leq B \leq N\), where \(D\) is given by (2.1). For fixed \(M,N \in \mathbb{N}\) denote

\[E_{B} = \{ x \in T : |V_{f}(x; M, N)| \geq B \}.
\]

Then

\[
\mu(E_{B}) \leq N \circledast \left\{ \begin{array}{c}
N^{4-2t}B^{-4} \quad \text{if } d = 2, \\
N^{d+1-\varepsilon}B^{-d-1} \quad \text{if } d \geq 3.
\end{array} \right.
\]

**Proof.** We proceed as in the proof of Lemma 5.7. In particular, we fix some \(\eta > 0\) and define \(Q\) by (5.11).

Suppose that

\[f(n) = \beta_{0} + \beta_{1}n + \ldots + \beta_{d}n^{d},\]

where \(\beta_{i} \in \mathbb{R}, 0 \leq i \leq d\) and \(\beta_{d} \neq 0\). Since the leading coefficient of \(f(n + M)\) coincides with that of \(f(n)\), that is \(\beta_{d}\), we see that

\[
\left| \sum_{n=1}^{N} e\left(xf(n + M)\right) \right| = \left| \sum_{n=1}^{N} e\left(y_{1}n + \ldots + y_{d}n^{d}\right) \right|,
\]

where \(y_{i}, 0 \leq i \leq d\) depend on \(M, x\) and in particular \(y_{d} = x\beta_{d}\). It follows that if \(|V_{f}(x; M, N)| \geq B\) for some \(x \in T\) then

\[
\left| \sum_{n=1}^{N} e\left(y_{1}n + \ldots + y_{d}n^{d}\right) \right| \geq B,
\]

where \(y_{d} = \beta_{d}x\). By Lemma 5.5 there are \(q_{2}, q_{3}, \ldots, q_{d}\), which satisfy the conditions of Lemma 5.5 and some integer \(b\) such that

\[
\left| \beta_{d}x - \frac{b}{q_{2} \ldots q_{d}} \right| \leq N \circledast B^{-d} \prod_{i=2}^{d} q_{i}^{-d/i},
\]
which is equivalent to

\begin{equation}
|x - \frac{b}{q_2 \cdots q_d \beta_d}| \leq N^{o(1)} B^{-d} \prod_{i=2}^{d} q_i^{-d/i}.
\end{equation}

Since \(|\beta_d x| \leq |\beta_d|\), we derive that

\begin{align*}
\frac{|b|}{q_2 \cdots q_d} &\leq |\beta_d| + N^{o(1)} B^{-d} \prod_{i=2}^{d} q_i^{-d/i},
\end{align*}

and thus

\begin{equation}
|b| \leq 2|\beta_d| q_2 \cdots q_d
\end{equation}

provided that \(N\) is large enough. It follows that for large enough \(N\) we have

\begin{equation}
\mathcal{E}_B \subseteq \bigcup_{(q_2, \ldots, q_d) \in \Omega} \bigcup_{b \in \mathbb{Z} \atop |b| \leq 2|\beta_d| q_2 \cdots q_d} \mathcal{I}_{q_2, \ldots, q_d, b},
\end{equation}

where, as in the proof of Lemma 5.7, the set \(\Omega\) is given by (5.13) and \(\mathcal{I}_{q_2, \ldots, q_d, b}\) is an interval of length

\begin{align*}
|\mathcal{I}_{q_2, \ldots, q_d, b}| &\leq N^{o(1)} B^{-d} \prod_{j=2}^{d} q_j^{-d/j}.
\end{align*}

Hence we derive from (5.3), (5.17), (5.18) and (5.19), that

\begin{align*}
\mu(\mathcal{E}_B) &\leq N^{o(1)} B^{-dt} \sum_{(q_2, \ldots, q_d) \in \Omega} \left( \prod_{j=2}^{d} q_j \prod_{j=2}^{d} q_j^{-d/j} \right)^t \\
&\leq N^{o(1)} B^{-dt} \sum_{(q_2, \ldots, q_d) \in \Omega} \prod_{j=2}^{d} q_j^{1-dt/j}.
\end{align*}

Again as in the proof of Lemma 5.7, covering \(\Omega\) by \(O\left((\log N)^{d-1}\right)\) dyadic boxes, we see that that there are some integers \(Q_2, \ldots, Q_d \geq 1\)
with (5.14) such that

\[
\mu(E_B) \leq N^{o(1)} B^{-dt} \sum_{q_2 \sim Q_2, \ldots, q_d \sim Q_d, q_3 \in \mathcal{F}_3(Q_3), \ldots, q_d \in \mathcal{F}_d(Q_d)} \prod_{j=2}^{d} q_j^{1-dt/j}
\]

\[
\leq N^{o(1)} B^{-dt} \prod_{j=2}^{d} Q_j^{1-dt/j} \sum_{q_2 \sim Q_2, \ldots, q_d \sim Q_d, q_3 \in \mathcal{F}_3(Q_3), \ldots, q_d \in \mathcal{F}_d(Q_d)} 1
\]

\[
\leq N^{o(1)} B^{-dt} Q_2^{2-dt/2} \prod_{j=3}^{d} \left( Q_j^{1-dt/j} \# \mathcal{F}_j(Q_j) \right).
\]

Thus, by (5.4), we have

\[(5.20) \quad \mu(E_B) \leq N^{o(1)} B^{-dt} Q_2^{2-dt/2} \prod_{j=3}^{d} Q_j^{1-(dt-1)/j}.\]

Denote

\[\prod_{j=2}^{d} Q_j^{1/j} = R.\]

Then we can rewrite (5.20) as

\[(5.21) \quad \mu(E_B) \leq N^{o(1)} B^{-dt} R^{-dt+1} Q_2^{3/2} \prod_{j=3}^{d} Q_j.\]

If \(d = 2\) then \(Q_2 = R^2\) and (5.21) becomes

\[\mu(E_B) \leq N^{o(1)} B^{-2t} R^{4-2t}.\]

Using \(R \ll Q_1^{1/2} N^\eta\) and recalling the definition of \(Q\) in (5.11) we obtain,

\[(5.22) \quad \mu(E_B) \leq N^{4-2t+\eta(4-2t)+o(1)} B^{-4}.\]

Now let \(d \geq 3\), then trivially

\[\prod_{j=3}^{d} Q_j \leq \prod_{j=3}^{d} Q_j^{d/j} = \left( RQ_2^{1/2} \right)^d.\]

Hence we derive from (5.21) that

\[
\mu(E_B) \leq N^{o(1)} B^{-dt} R^{d-dt+1} Q_2^{3/2-d/2} \leq N^{o(1)} B^{-dt} R^{d-dt+1}.
\]
Since $t \leq 1$ we have $d - dt + 1 > 0$. Therefore, using $R \ll Q^{1/d}N^\eta$ we obtain

$$
\mu(\mathcal{E}_B) \leq N^{\eta(2d-dt+1)} B^{-dt} (NB^{-1})^{d-dt+1} = N^{d-dt+1+\eta(d-dt+1)} B^{-d-1}.
$$

(5.23)

Since $\eta$ is arbitrary, from (5.22) and (5.23) we derive the desired result. \qed

6. Proof of the upper bound of Theorem 2.1

6.1. Mean values of Gauss sums. We need the following mean value estimate for Gauss sums with respect to an arbitrary Radon measure, which is interesting in its own right.

**Lemma 6.1.** Let $\mu$ be a Radon measure on $T_2$ such that

$$
\mu(B(x, r)) \ll r^t
$$

holds for some $t > 0$ and for all $x \in T_2$ and $r > 0$. Then for all $M, N$ we have

$$
\int_{T_2} |G(x; M, N)|^6 \, d\mu(x) \leq N^{6-2t+o(1)} (M + N)^{\min\{1, t\}}.
$$

**Proof.** Let us fix some $\varepsilon > 0$. Denote

$$
K = N^{1/2+\varepsilon}.
$$

By a dyadic partition argument, there exits $B \in [K, N]$ such that

$$
\int_{T_2} |G(x; M, N)|^6 \, d\mu(x) \leq K^6 \mu(T_2) + B^6 \mu(\mathcal{E}_B) N^{o(1)}.
$$

(6.1)

By Lemma 5.4 we have

$$
\mu(\mathcal{E}_B) \leq N^{6-2t+o(1)} B^{-6} (M + N)^{\min\{1, t\}}.
$$

which after substitution in (6.1) implies

$$
\int_{T_2} |G(x; M, N)|^6 \, d\mu(x) \leq N^{3 + 6\varepsilon} + N^{6-2t+o(1)} (M + N)^{\min\{1, t\}}.
$$

Since $t \leq 2$ and $\varepsilon$ is arbitrary, the result now follows. \qed
6.2. **Concluding the proof.** We now turn to the proof of the upper bound of Theorem 2.1. Let $t \in (0, \dim \mathcal{E}_{2,\alpha})$. Then $\mathcal{E}_{2,\alpha}$ has infinite $\mathcal{H}^t$-measure. By Lemma 4.1, there exists a Radon measure $\mu$ on $T_2$ with
\[
\mu(\mathcal{E}_{2,\alpha}) > 0 \quad \text{and} \quad \mu(B(x, r)) \ll r^t
\]
for all $x \in T_2$ and $r > 0$. Taking the function
\[
f_n(x_1, x_2) = e(x_1 n + x_2 n^2)
\]
and applying Lemmas 4.3 and 6.1, we immediately derive that for almost all $(x_1, x_2) \in T_2$ with respect to $\mu$,
\[
|G(x_1, x_2; N)| \leq N^{1-t/3+\min\{1/6, t/6\}+o(1)}.
\]
Since $\mu(\mathcal{E}_{2,\alpha}) > 0$, there is a set of $(x_1, x_2) \in T_2$ of positive $\mu$-measure such that
\[
|G(x_1, x_2; N)| \geq N^{t/3}
\]
for infinitely many $N \in \mathbb{N}$ Combining with (6.2) we derive
\[
\alpha \leq 1 - t/3 + \min\{1/6, t/6\},
\]
which implies
\[
t \leq \min\{1/2 + 3(1 - \alpha), 6(1 - \alpha)\}.
\]
Since this holds for any $t < \dim \mathcal{E}_{2,\alpha}$, we conclude that
\[
\dim \mathcal{E}_{2,\alpha} \leq \min\{1/2 + 3(1 - \alpha), 6(1 - \alpha)\},
\]
which yields the desired upper bound.

7. **Proof of the lower bound of Theorem 2.1**

7.1. **Large values of Gauss sums.** The main purpose of this subsection is to show Lemma 7.4. We start from recalling the following property of Gaussian sums, see [31, Equation (1.55)].

**Lemma 7.1.** Let $p \geq 3$ and $a, b \in \mathbb{Z}_p$ with $b \neq 0$, then
\[
\left| \sum_{n=0}^{p-1} e_p(an + bn^2) \right| = \sqrt{p}.
\]

Using the Gauss bound together with the standard completion technique, see [31, Sections 11.11 and 12.2] we also immediately obtain:

**Lemma 7.2.** For any prime $p$ and any $a \in \mathbb{F}_p \setminus \{0\}$ we have
\[
\max_{1 \leq M, N \leq p} \left| \sum_{M+1 \leq n \leq M+N} e_p(an^2) \right| \ll \sqrt{p} \log p.
\]

The continuity of Gauss sums yields the following result.
Lemma 7.3. For $N \gg p \log p$ we have

$$|G(x_1, x_2; N) - G(a/p, b/p; N)| \ll N p^{-1/2} (|x_1 - a/p| N + |x_2 - b/p| N^2).$$

Proof. We use the following version of summation by parts. Let $a_n$ be a sequence and for each $t \geq 1$ denote

$$A(t) = \sum_{1 \leq n \leq t} a_n.$$

Let $\psi : [1, N] \to \mathbb{C}$ be a continuously differentiable function. Then

$$\sum_{n=1}^{N} a_n \psi(n) = A(N) \psi(N) - \int_{1}^{N} A(t) \psi'(t) dt.$$

Let $\delta_1 = x - a/p$, $\delta_2 = x_2 - b/p$. Then define

$$\Delta = G(x_1, x_2; N) - G(a/p, b/p; N)$$

(7.1)

$$= \sum_{n=1}^{N} e(na/p + n^2b/p) \left( e(\delta_1 n + \delta_2 n^2) - 1 \right).$$

For an integer $M$ with $1 \leq M \leq N$, we split the sum $G(a/p, b/p; M)$ into $O(N/p)$ complete sums and at most one incomplete sum, applying Lemmas 7.1 and 7.2, we derive

$$\max_{1 \leq M \leq N} |G(a/p, b/p; M)| \ll N p^{-1/2} + p^{1/2} \log p.$$

Hence, applying to the sum in (7.1) summation by parts with $a_n = e(na/p + n^2b/p)$ and $\psi(t) = e(\delta_1 t + \delta_2 t^2) - 1$, we derive that

$$\Delta \ll \max_{1 \leq M \leq N} |G(a/p, b/p; M)| (|\delta_1| N + |\delta_2| N^2)$$

$$\ll \left( N p^{-1/2} + p^{1/2} \log p \right) (|\delta_1| N + |\delta_2| N^2)$$

$$\ll N p^{-1/2} (|\delta_1| N + |\delta_2| N^2),$$

which finishes the proof. \qed

From Lemma 7.1 and Lemma 7.3 we obtain the following, which is the main purpose of this subsection.

Lemma 7.4. We fix $\alpha \in (1/2, 1)$. Let $p \geq 3$ and $a, b \in \mathbb{Z}_p$ with $b \neq 0$. Let $N$ be the smallest number such that $p | N$ and

$$N \geq p^{\frac{1}{2\alpha + 1}}.$$
Then there exists a sufficiently small number $\eta > 0$ such that for any $(x_1, x_2) \in T_2$ with
\[
\left| x_1 - \frac{a}{p} \right| < \eta p^{-\frac{1}{2(1-\alpha)}} \quad \text{and} \quad \left| x_2 - \frac{b}{p} \right| < \eta p^{-\frac{1}{1-\alpha}}
\]
we have
\[
G(x_1, x_2; N) \gg N^\alpha.
\]

**Proof.** Recalling that $z \approx Z$ means $Z/C \leq z \leq CZ$ for some absolute positive constant $C$. First note that the choice of $N$ implies
\[
N \approx p^{\frac{1}{2(1-\alpha)}} \quad \text{and} \quad Np^{-1/2} \approx N^\alpha.
\]
By Lemma 7.3 we have
\[
|G(x_1, x_2; N) - G(a/p, b/p; N)| \ll \eta N^\alpha.
\]
Lemma 7.1 implies $G(a/p, b/p; N) \approx N^\alpha$. Therefore, we obtain the desired bound by choosing a sufficiently small $\eta$. $\square$

**7.2. Simultaneous Diophantine approximations.** Let $\vartheta = (\vartheta_1, \ldots, \vartheta_d)$ be a vector of positive real numbers and let $Q$ an arbitrary set of positive integers. Without losing generality, assuming that $\vartheta_1 \leq \ldots \leq \vartheta_d$.

We denote by $W_{Q, \vartheta}$ be the collection of points $(x_1, \ldots, x_d) \in T_d$ for which there are infinitely many $q \in Q$ such that
\[
\|qx_i\| < q^{-\vartheta_i}, \quad i = 1, \ldots, d,
\]
where $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$. Denote
\[
\nu(\Omega) = \inf \left\{ \nu \in \mathbb{R} : \sum_{q \in \Omega} q^{-\nu} < \infty \right\}.
\]
We need the following result of Rynne [36, Theorem 1].

**Lemma 7.5.** Suppose that $\vartheta_1 + \ldots + \vartheta_d \geq \nu(\Omega)$, then we have
\[
\dim W_{Q, \vartheta} = \min_{1 \leq j \leq d} \frac{d + \nu(\Omega) + j \vartheta_j - \sum_{i=1}^{j} \vartheta_i}{1 + \vartheta_j}.
\]
For $d = 2$ and $\nu(\Omega) = 1$ we have the following.
\[
\dim W_{Q, \vartheta} = \min \left\{ \frac{3}{1 + \vartheta_1}, \frac{3 + \vartheta_2 - \vartheta_1}{1 + \vartheta_2} \right\}.
\]

We now turn to the proof of the lower bound of Theorem 2.1. Indeed this follows by combining Lemma 7.4 and Lemma 7.5. Let $\Omega$ be the collection of prime numbers. Clearly we have $\nu(\Omega) = 1$. By Lemma 7.4, for any $\varepsilon > 0$ we obtain
\[
W_{Q, \vartheta} \subseteq E_{2, \alpha - \varepsilon}
\]
with
\[ \vartheta_1 = \frac{1}{2(1 - \alpha)} - 1 \quad \text{and} \quad \vartheta_2 = \frac{1}{1 - \alpha} - 1. \]

Since \( \alpha \in (1/2, 1) \), we have \( \vartheta_1 + \vartheta_2 \geq 1 \). Thus by Lemma 7.5 we obtain
\[ \dim W_{\Omega, \alpha} = \min \{1/2 + 3(1 - \alpha), 6(1 - \alpha)\}. \]

It follows that
\[ \dim E_{2, \alpha - \varepsilon} \geq \min \{1/2 + 3(1 - \alpha), 6(1 - \alpha)\}. \]

Since \( \varepsilon > 0 \) is arbitrary, we obtain the desired bound.

8. Proof of Theorem 2.2

8.1. Mean values of Weyl sums. We need the following mean value estimate of Weyl sums with respect to a general measure.

**Lemma 8.1.** Let \( \mu \) be a Radon measure on \( T_d, d \geq 3 \) such that
\[ \mu(B(x, r)) \ll r^t \]
holds for some \( t > 0 \) and for all \( x \in T_d \) and \( r > 0 \). Then for any integer \( h \) with \( 1 \leq h \leq d \) we have
\[ \int_{T_d} |S_d(x; M, N)|^{d^2+1} d\mu(x) \leq N^{(1-1/D)(d^2+1)+o(1)}\]
\[ + N^{d^2+1-dt+o(1)}(M+N)^{(d-h)t+h(h-1)/2}, \]
where \( D \) is given by (2.1).

**Proof.** Let us fix some \( \varepsilon > 0 \). Denote
\[ K = N^{1-1/D+\varepsilon}. \]

Similar to the proof of Lemma 6.1, taking a dyadic partition of the interval \( [K, N] \), there exists a number \( B \in [K, N] \) such that
\[ \int_{T_d} |S_d(x; M, N)|^{d^2+1} d\mu(x) \leq K^{d^2+1} + N^{o(1)}B^{d^2+1}\mu(\{x \in T_d : B \leq S_d(x; M, N) \leq 2B\}). \]

Combining with Lemma 5.7 we obtain
\[ \int_{T_d} |S_d(x; M, N)|^{d^2+1} d\mu(x) \leq N^{(1-1/D)(d^2+1)+\varepsilon(d^2+1)} + N^{o(1)}N^{d^2+1-dt}(M+N)^{(d-h)t+h(h-1)/2}. \]

By the arbitrary choice of \( \varepsilon > 0 \) we obtain the desired bound. \( \square \)
8.2. **Concluding the proof.** We now turn to the proof of Theorem 2.2. Let \( t \in (0, \dim \mathcal{E}_{d,\alpha}) \). Then we see that \( \mathcal{E}_{d,\alpha} \) has infinite \( \mathcal{H}^t \)-measure. By Lemma 4.1, there exists a Radon measure \( \mu \) on \( T_d \) with
\[
\mu(\mathcal{E}_{2,\alpha}) > 0 \quad \text{and} \quad \mu(B(x, r)) \ll r^t
\]
for all \( x \in T_d \) and \( r > 0 \).

Let \( 1 \leq h \leq d \) be an integer. There are two cases to consider.

**Case 1.** Suppose that the ‘total exponent’
\[
d^2 + 1 - dt + (d - h)t + h(h - 1)/2 = d^2 + 1 - ht + h(h - 1)/2
\]
in the second term in the bound of Lemma 8.1 is at least as large as the exponent of the first term, that is,
\[
d^2 + 1 - ht + h(h - 1)/2 \geq (1 - 1/D)(d^2 + 1).
\]
Since \( t < \dim \mathcal{E}_{d,\alpha} \leq d \), we have \( d^2 + 1 - dt > 1 \), and thus by Lemma 4.3 we derive that for almost all \( x \in T_d \) with respect to \( \mu \),
\[
|S_d(x; N)| \leq N d^2 + 1 - dt + o(1).
\]
Since \( \mu(\mathcal{E}_{d,\alpha}) > 0 \), there is a set of \( x \in T_d \) of positive \( \mu \)-measure such that
\[
|S_d(x; N)| \geq N^\alpha
\]
for infinitely many \( N \in \mathbb{N} \). Combining with (8.1) we derive
\[
\alpha \leq \frac{d^2 + 1 - ht + h(h - 1)/2}{d^2 + 1},
\]
which implies
\[
t \leq \frac{(d^2 + 1)(1 - \alpha)}{h} + \frac{h - 1}{2}.
\]

**Case 2.** Suppose that \( d^2 + 1 - th + h(h - 1)/2 < (1 - 1/D)(d^2 + 1) \). Then Lemma 8.1 implies that
\[
|S_d(x; N)| \leq N^{d^2 + 1 - dt + o(1)}(M + N)^{(1-1/D)(d^2 + 1)-(d^2 + 1 - dt)}.
\]
By Lemma 4.3 we conclude that for almost all \( x \in T \) with respect to \( \mu \),
\[
|S_d(x; N)| \leq N^{1-1/D + o(1)}.
\]
Then applying the similar argument to **Case 1**, we obtain
\[
\alpha \leq 1 - 1/D,
\]
which contradicts our assumption that \( \alpha \in (1 - 1/D, 1) \). Thus we are in **Case 1** and we have (8.3) for any integer \( 1 \leq h \leq d \). Since (8.3) holds for any \( t < \dim \mathcal{E}_{d,\alpha} \), we obtain the desired upper bound.
9. Proof of Theorem 2.3

9.1. One-dimensional mean values of Weyl sums. For the proof of Theorem 2.3, similarly to the proofs of the upper bounds of Theorems 2.1 and 2.2, we see from Lemma 4.1 that it is sufficient to prove the following mean value bounds.

We start with quadratic polynomials.

Lemma 9.1. Let \( f \in \mathbb{R}[X] \) be a polynomial of degree \( d = 2 \). Let \( \mu \) be a Radon measure on \( T \) such that
\[
\mu(B(x, r)) \ll r^t
\]
holds for some \( t \in (0, 1) \) and for all \( x \in T \) and \( r > 0 \). Then for all \( M, N \) we have
\[
\int_T |V_f(x; M, N)|^4 d\mu(x) \leq N^{4(1-t/2)+o(1)}.
\]

Proof. Let us fix some \( \varepsilon > 0 \). Denote
\[
K = N^{1/2+\varepsilon}.
\]
Similar to the proofs of Lemmas 6.1 and 8.1, taking a dyadic partition of the interval \([K, N]\), there exists a number \( B \in [K, N] \) such that
\[
\int_T |V_f(x; M, N)|^4 d\mu(x) \leq K^4 + N^{o(1)} B^4 \mu(E_B).
\]
Hence by Lemma 5.8 applied with \( d = 2 \), we have
\[
\int_T |V_f(x; M, N)|^4 d\mu(x) \leq K^4 + N^{4-2t+o(1)}.
\]
Since \( t \leq 1 \) and \( \varepsilon \) is arbitrary, the result now follows. \( \square \)

For polynomials of higher degree we have a similar bound.

Lemma 9.2. Let \( f \in \mathbb{R}[X] \) be a polynomial of degree \( d \geq 3 \). Let \( \mu \) be a Radon measure on \( T \) such that
\[
\mu(B(x, r)) \ll r^t
\]
holds for some \( t \in (0, 1) \) and for all \( x \in T \) and \( r > 0 \). Then for all \( M, N \) we have
\[
\int_T |V_f(x; M, N)|^{d+1} d\mu(x) \leq N^{d+1-(d+1)/D+o(1)} + N^{d+1-dt+o(1)},
\]
where \( D \) is given by (2.1).
Proof. Let us fix some $\varepsilon > 0$. Denote
\[ K = N^{1-1/D+\varepsilon}. \]
Then, similarly to the above, by Lemma 5.8 with $d \geq 3$ there exists $K \leq B \leq N$ such that
\[ \int_T |V_f(x;M,N)|^{d+1} d\mu(x) \leq K^{d+1} + N^{\omega(1)} B^{d+1} \mu(E_B). \]
By Lemma 5.8, we have
\[ \int_T |V_f(x;M,N)|^{d+1} d\mu(x) \leq K^{d+1} + N^{d-dt+1+o(1)}. \]
Since $\varepsilon$ is arbitrary, the result now follows. \qed

9.2. Concluding the proof. Similarly to the proofs of the upper bounds of Theorems 2.1 and 2.2, Lemmas 9.1 and 9.2 combined with Lemma 4.3 imply the desired upper bound of Theorem 2.3.

In particular, for $d = 2$ the proof is a full analogue of that of the upper bound of Theorem 2.1 where we use Lemma 9.1 in an appropriate place instead of Lemma 6.1.

For $d \geq 3$, as in the proofs of Theorem 2.2, we consider two cases
\[ t \leq \frac{d+1}{dD} \quad \text{and} \quad t > \frac{d+1}{dD}. \]
Now, by Lemma 4.3, in the first case, similarly to (8.2), we derive
\[ \alpha \leq \frac{d-dt+1}{d+1} \]
which gives the desired bound, while in the second case we obtain (8.4), which contradicts the assumption $\alpha \in (1-1/D,1)$.

10. Proofs of Theorems 3.1, 3.3, 3.4, 3.5 and 3.6

10.1. Mean values of exponential polynomials. For a real sequences $f(n)$ we define the sums
\[ W_f(x;N) = \sum_{h=-N}^{N} \frac{1}{|h|+1} \left| \sum_{n=1}^{N} e(hn/N + xf(n)) \right|. \]
Then a special form of [16, Lemma 3.2] implies for $x \in T$ and $1 \leq M \leq N$ we have
\[ V_f(x;M) \ll W_f(x;N), \]
where $W_f(x;N)$ is given by (10.1).

Our method is based on mean value estimates on the sums $W_f(x;N)$. However our next result shows that for integer-valued sequences the
mean value of $W_f(x; N)$ is controlled by the mean value of $V_f(x; N)$ provided that the exponent is some even integer. It follows by using a similar argument to the proof of [18, Lemma 2.4]. We give a proof here for completeness.

**Lemma 10.1.** Let $f \in \mathbb{Z}[x]$ be an integer sequence such that for some even number $s > 0$ and some real $t > 0$ one has

$$\int_T |V_f(x; N)|^s dx \ll N^t,$$

then we have

$$\int_T W_f(x; N)^s dx \ll N^t (\log N)^s.$$

**Proof.** Write

$$W_f(x; N) = \sum_{h=-N}^{N} \left( \frac{1}{|h| + 1} \right)^{1-1/s} \left( \frac{1}{|h| + 1} \right)^{1/s} \times \left| \sum_{n=1}^{N} e\left( h n / N + x f(n) \right) \right|.$$

Applying the Hölder inequality, we obtain

$$W_f(x; N)^s \ll (\log N)^{s-1} \sum_{h=-N}^{N} \frac{1}{1 + |h|} \times \left| \sum_{n=1}^{N} e\left( h n / N + x f(n) \right) \right|^s.$$

(10.3)

For any $h$ and $N$ and even number $s$, opening the integral and applying the orthogonal property of $e(x)$, we derive

$$\int_T \left| \sum_{n=1}^{N} e\left( h n / N + x f(n) \right) \right|^s dx
= \# \left\{ (n_1, \ldots, n_s) : 1 \leq n_i \leq N, \sum_{i=1}^{s/2} (f(n_i) - f(n_{s/2+i}) = 0 \right\}
= \int_T \left| \sum_{n=1}^{N} e\left( x f(n) \right) \right|^s dx.$$

Combining with (10.3) we obtain the desired bound. $\square$
Observe that Lemma 10.1 implies that for integer-values sequences, we only need to estimate the moments of the sums $V_f(x; N)$ rather than of $W_f(x; N)$.

We now recall some mean value estimates on the sums $V_f(x; N)$ in (1.5) when $f \in \mathbb{Z}[x]$ is a polynomial. We first recall the following result of Hua [29], see also [40, Section 14].

**Lemma 10.2.** Let $f \in \mathbb{Z}[x]$ be a polynomial with degree $d \geq 2$, then for each natural number $1 \leq r \leq d$,

$$
\int_T |V_f(x; N)|^{2r} \, dx \leq N^{2^r - r + o(1)}.
$$

Wooley [40, Corollary 14.2] (see also [8, Theorem 10]) obtains the following better bound when $r$ is large.

**Lemma 10.3.** Let $f \in \mathbb{Z}[x]$ be a polynomial with degree $d \geq 2$, then for each natural number $1 \leq r \leq d$,

$$
\int_T |V_f(x; N)|^{(r+1)} \, dx \leq N^{r^2 + o(1)}.
$$

Furthermore, for the case of monomials Wooley [40, Corollary 14.7] gives a stronger result.

**Lemma 10.4.** Let $d \geq 2$ and

$$
s_0 = d(d - 1) + \min_{r=1,\ldots,d} \frac{2d + (r - 1)(r - 2)}{r}.
$$

Then

$$
\int_T \left| \sum_{n=1}^{N} e\left(xn^d\right) \right|^{s_0} \, dx \leq N^{s_0 - d + o(1)}.
$$

We now turn to mean value theorems for sums with arbitrary sequences.

In particular, we have the following simple bound on the second moment of the sums $W_f(x; N)$, defined by (1.5) with well-spaced sequences.

**Lemma 10.5.** Let $f(n)$ be a real sequence such that $f(n) - f(m) \gg 1$ for all $m \neq n$. Then for any interval $\mathcal{I}$ we have

$$
\int_{\mathcal{I}} W_f(x; N)^2 \, dx \ll N(\log N)^3,
$$

where $W_f(x; N)$ is given by (10.1).
Proof. By the Cauchy inequality, we obtain
\[ W_f(x; N)^2 \ll \log N \sum_{h=-N}^{N} \frac{1}{|h|+1} \left| \sum_{n=1}^{N} e\left(\frac{hn}{N} + xf(n)\right) \right|^2 \]
\[ \ll \log N \sum_{h=-N}^{N} \frac{1}{|h|+1} (N + \Sigma(x)) , \]
where
\[ \Sigma(x) = \sum_{1 \leq n \neq m \leq N} e(h(n - m)/N + x(f(n) - f(m)). \]

The condition \(|f(n) - f(m)| \gg 1\) implies that the values \(f(1), \ldots, f(N)\) are separated from each by a unit interval, and thus so are the values \(f(1) - \zeta, \ldots, f(N) - \zeta\) for any real \(\zeta\). In particular
\[ \sum_{n=1}^{N} \frac{1}{|f(n) - \zeta|} \ll \log N. \]

Therefore,
\[ \int_I |\Sigma(x)| \, dx \ll \sum_{1 \leq n \neq m \leq N} \left| \int_I e(x(f(n) - f(m)) \, dx \right| \]
\[ \ll \sum_{1 \leq n \neq m \leq N} \frac{1}{|f(n) - f(m)|} \ll N \log N. \]

Combining with (10.1) we obtain the desired bound. \qed

Suppose that \(f(n) \in \mathbb{N}\) is a strictly convex sequence, Iosevich, Konyagin, Rudnev, and Ten [30, Equation (1.13)] (general even number \(s\)) and Shkredov [38, Theorem 1.1] (\(s = 4\)) gives the following bounds.

Lemma 10.6. Suppose that \(f(n) \in \mathbb{N}\) is a strictly convex sequence, then
\[ \int_T |V_f(x; N)|^4 \, dx \ll N^{2^1/13 + o(1)} \]
and for any even number \(s \geq 6\) we have
\[ \int_T |V_f(x; N)|^s \, dx \ll N^{s-2 + 2^{1-s}/2}. \]

We note that for sequences satisfying stronger conditions than convexity stronger versions of Lemma 10.6 are known, see [12].

We now observe that the result of Robert and Sargos [34, Theorem 2] implies the following bound.
Lemma 10.7. Let \( f(n) = \lfloor n^{\tau} \rfloor \) for some \( \tau \geq 1 \).

\[
\int_T |V_f(x; N)|^4 dx \ll N^{2+o(1)} + N^{4-\tau+o(1)}.
\]

10.2. Continuity of exponential polynomials. The following is a special form of [15, Lemma 2.4].

Lemma 10.8. Let \( f(n) \) be a real sequence that satisfies (3.2) for some constant \( \tau > 0 \). Let \( 0 < \alpha < 1 \) and let \( \varepsilon > 0 \) be sufficiently small. If \( W_f(x; N) \geq N^\alpha \) for some \( x \in T \), then

\[
W_f(y; N) \geq N^\alpha / 2
\]

holds for any \( y \in (x - \zeta, x + \zeta) \) provided that \( N \) is large enough and

\[
0 < \zeta \leq N^{\alpha-\tau-1-\varepsilon}.
\]

Proof. Note that for any \( x, y \in \mathbb{R} \) and any \( h, N \) we have

\[
e^{(hn/N + xf(n))} - e^{(hn/N + yf(n))} \ll |x - y||f(n)|.
\]

Thus we obtain

\[
|W_f(x; N) - W_f(y; N)| \ll |x - y| \log N \sum_{n=1}^{N} |f(n)|,
\]

which yields the desired result. \( \square \)

Lemma 10.9. Let \( f(n) \) be a real sequence that satisfies (3.2) for some constant \( \tau > 0 \). Suppose that

\[
\int_T W_f(x; N)^s dx \leq N^{t+o(1)}.
\]

Then

\[
\{x \in T : W_f(x; N) \geq N^\alpha\} \subseteq \bigcup_{I \in \mathcal{I}_N} I,
\]

where \( \mathcal{I}_N \) is a collection of intervals with equal length such that \( |I| \leq N^{\alpha-\tau-1-\varepsilon} \) for each \( I \in \mathcal{I}_N \) and of cardinality

\[
\#\mathcal{I}_N \leq N^{t-s\alpha+\tau+1-\alpha+2\varepsilon}
\]

provided that \( N \) is sufficiently large.

Proof. Let

\[
\zeta = 1/\lceil N^{\tau+1+\varepsilon-\alpha} \rceil.
\]

We divide \( T \) into \( \zeta^{-1} \) intervals of the type \( [k\zeta, (k+1)\zeta] \) with \( k = 0, 1, \ldots, \zeta^{-1} - 1 \). Let \( \mathcal{D}_N \) be the collection of these intervals and

\[
\mathcal{I}_N = \{I \in \mathcal{D}_N : \exists x \in I \text{ such that } W_f(x; N) \geq N^\alpha\}.
\]
Lemma 10.8 implies that for each $I \in \mathcal{I}_N$,

$$W_f(x; N) \geq N^\alpha / 2, \quad \forall x \in I.$$  

It follows that

$$(\# \mathcal{I}_N) \zeta N^\alpha \ll \int_T W_f(x; N)^s dx \leq N^{t+o(1)},$$

which gives the desired result.  

10.3. Mean values and Hausdorff dimension for polynomially growing sequences. We have the following general result about the upper bound on $\dim F_{f, \alpha}$.

**Lemma 10.10.** Let $f(n)$ be a real sequence that satisfies (3.2) for some constant $\tau > 0$. Suppose that there are positive constants $s, t$ such that

$$\int_T |W_f(x; N)|^s dx \leq N^{t+o(1)}.$$  

Then

$$\dim F_{f, \alpha} \leq \frac{\tau + 1 - \alpha + t - s\alpha}{\tau + 1 - \alpha}.$$  

**Proof.** For each $N \in \mathbb{N}$ denote

$$B_N = \{x \in T : W_f(x; N) \geq N^\alpha\}.$$  

Let $N_i = 2^i$, $i \in \mathbb{N}$ and $\eta > 0$. Applying (10.2) we obtain

$$F_{f, \alpha + \eta} \subseteq \bigcap_{q=1}^\infty \bigcup_{i=q}^\infty B_{N_i}.$$  

Indeed, let $x \in F_{f, \alpha + \eta}$ and suppose to the contrary that

$$x \notin \bigcap_{q=1}^\infty \bigcup_{i=q}^\infty B_{N_i}.$$  

Then $W_f(x; N_i) < N_i^\alpha$ holds for all large $N_i$. For any $N$ there exists a number $i \in \mathbb{N}$ such that $N_i \leq N < N_{i+1}$, and for large enough $N$ by Lemma 10.9 we have

$$V_f(x; N) \ll W_f(x; N_{i+1}) \ll N_{i+1}^\alpha \ll N^\alpha,$$  

which contradicts our assumption.

For each $N_i$ by Lemma 10.9 we obtain

$$B_{N_i} \subseteq \bigcup_{I \in \mathcal{I}_N} \mathcal{I}_N,$$

where $|I| \leq N^{\alpha - \tau - 1 - \varepsilon}$ for each $I \in \mathcal{I}_N$ and $\# \mathcal{I}_N \leq N_i^{t-s\alpha+\tau+1-\alpha+2\varepsilon}$. 
From the definition of the Hausdorff dimension, using the above notation, we have the following inequality

\[(10.4) \dim \mathcal{F}_{f,\alpha+\eta} \leq \inf \left\{ \nu > 0 : \sum_{i=1}^{\infty} \sum_{I \in \mathcal{I}_{N_i}} |I|^{\nu} < \infty \right\} .\]

Note that

\[\sum_{i=1}^{\infty} \sum_{I \in \mathcal{I}_{N_i}} |I|^{\nu} \ll \sum_{i=1}^{\infty} N_i^{t-s\alpha+\tau+1-\alpha+2\varepsilon} N_i^{(\alpha-\tau-1)\nu},\]

thus to make the series convergegent it is sufficient to have

\[\nu > \frac{t-s\alpha+\tau+1-\alpha+2\varepsilon}{\tau+1-\alpha} .\]

Combining with (10.4) and the arbitrary choice of \(\varepsilon > 0\) we obtain

\[\dim \mathcal{F}_{f,\alpha+\eta} \leq \frac{\tau+1-\alpha+t-s\alpha}{\tau+1-\alpha} .\]

Since this holds for any \(\eta > 0\), we obtain the desired bound. \[\square\]

10.4. **Concluding the proofs.** Combining Lemma 10.10 (and taking \(\tau = d\) for polynomial sequences) with

- Lemmas 10.1, 10.2 and 10.3, we obtain Theorem 3.1;
- Lemmas 10.1 and 10.4, we obtain Theorem 3.3;
- Lemmas 10.1 and 10.7, we obtain Theorem 3.4;
- Lemma 10.5, we obtain Theorem 3.5;
- Lemmas 10.1 and 10.6, we obtain Theorem 3.6.

11. **Proofs of Theorems 3.7 and 3.8**

11.1. **Proof of Theorem 3.7.** The upper bound of Theorem 3.7 follows by applying Lemmas 4.1 and 4.3 and the following mean value bound of Baker [2, Equation (18)].

**Lemma 11.1.** Let \(f(n)\) be a sequence of natural numbers with \(f(n) = O(n^\tau)\) for some real number \(\tau > 0\). Let \(\mu\) be a Radon measure on \(T\) such that

\[\mu(B(x, r)) \ll r^t\]

holds for some \(t \in (0, 1)\) and for all \(x \in T\) and \(r > 0\). Then for all \(M, N\) we have

\[\int_T |V_f(x; M, N)|^2 d\mu(x) \ll N(M + N)^{(1-t)} .\]
11.2. **Proof of Theorem 3.8.** By (3.4) the upper bound holds always, and hence we now prove the lower bound. For any $\tau \geq 1$, Ruzsa [35] defines a strictly increasing sequence of natural numbers $g(n)$ with $g(n) = O(n^\tau)$, a constant $0 < \ell_0 < 1$ (depending on $\tau$) and a set $\mathcal{G} \subseteq \mathbb{T}$ with $\dim \mathcal{G} = 1 - 1/\tau$, having the following property.

If $x \in \mathcal{G}$ then there are infinitely many $N$ and corresponding intervals $\mathcal{I}(N, x) \subseteq \mathbb{T}$ of length $\ell_0$ such that

$$(11.1) \quad \sum_{n=1}^{N} \{xg(n)\} \in \mathcal{I}(N, x) 1 \geq 2\ell_0 N,$$

where as usual $\{u\}$ denotes the fractional part of a real $u$.

Fix an integer

$$(11.2) \quad H > 4/\ell_0.$$

By [3, Lemma 2.7], there is a trigonometric polynomial

$$\Psi_{N, x}(y) = C_0 + \sum_{0 < |k| \leq H} C_k(N, x) e(ky),$$

depending on $N$ and $x$ with

$$(11.3) \quad C_0 = \ell_0 + \frac{1}{H+1}$$

and such that

$$(11.4) \quad \Psi_{N, x}(y) \begin{cases} 1 & \text{if } y \in \mathcal{I}(N, x), \\ 0 & \text{otherwise.} \end{cases}$$

Note that since $\Psi_{N, x}(y) \geq 0$, for any $k \neq 0$ we have

$$|C_k(N, x)| = \left| \int_{\mathbb{T}} \Psi_{N, x}(y) e(-ky) dy \right| \leq \int_{\mathbb{T}} \Psi_{N, x}(y) dy = C_0,$$

and hence

$$(11.5) \quad |C_k(N, x)| \leq C_0, \quad 0 < |k| \leq H.$$

Let $x \in \mathcal{G}$. Then by (11.1) and (11.4) we have

$$\sum_{n=1}^{N} \Psi_{N, x}(xg(n)) \geq 2\ell_0 N.$$

Thus

$$C_0 N + \sum_{0 < |k| \leq H} |C_k(N, x)| \left| \sum_{n=1}^{N} e(kxg(n)) \right| \geq 2\ell_0 N.$$
Recalling (11.2) and (11.3) we see that \(2\ell_0 - C_0 \geq \ell_0 / 2\). Hence
\[
\sum_{0 < |k| \leq H} |C_k(N, x)| \left| \sum_{n=1}^{N} e(kxg(n)) \right| \geq \frac{\ell_0}{2N},
\]
which together with (11.5) implies that there exists a number \(k \in \{\pm 1, \ldots, \pm H\}\) such that
\[
\left| \sum_{n=1}^{N} e(kxg(n)) \right| \geq \frac{\ell_0}{4HC_0} N.
\]
(we note that the number \(k\) depends on \(x\) and \(N\)).

Since for any \(x \in \mathcal{G}\) and \(N\) such that (11.1) holds there are finite choices of \(k\), we conclude that for any \(x \in \mathcal{G}\) there exists a number \(k \in \{\pm 1, \ldots, \pm H\}\) such that for infinitely many \(N\) we have
\[
\left| \sum_{n=1}^{N} e(kxg(n)) \right| \geq \frac{\ell_0}{4HC_0} N.
\]
That is, with \(f_k(n) = |k|g(n)\) we have
\[
\mathcal{G} \subseteq \bigcup_{k=1}^{H} \mathcal{G}_{f_k,c},
\]
where \(c = \ell_0/(4HC_0)\) and \(\mathcal{G}_{f_k,c}\) is given by (3.3). Therefore, for some \(k \in \{1, \ldots, H\}\) we have
\[
\dim \mathcal{G}_{f_k,c} \geq \dim \mathcal{G} \geq 1 - 1/\tau.
\]
Together with the upper bound (3.4) this finishes the proof.

12. Proof of Theorem 3.9

12.1. Preliminaries. Let \(\mathcal{S} = (A_n)_{n \in \mathbb{N}}\) be a sequence of \(d \times d\) matrix. For any \(h \in \mathbb{R}^d\) (which treat as a column vector) and \(N \in \mathbb{N}\) let
\[
V_{\mathcal{S}, h}(x; N) = \sum_{n=1}^{N} e(\langle xA_n, h \rangle).
\]
where \(\langle y, z \rangle\) denotes the standard scalar product.

**Lemma 12.1.** Let \(\mathcal{S} = (A_n)_{n \in \mathbb{N}}\) be a sequence of \(d \times d\) integer matrix such that \((A_n - A_m)\) is invertible if \(n \neq m\). Then for any \(h \in \mathbb{Z}^d \setminus \{0\}\) and \(N \in \mathbb{N}\) we have
\[
\int_{\mathbb{T}^d} |V_{\mathcal{S}, h}(x; N)|^2 \, dx = N.
\]
Proof. Opening the square we have
\[
\int_{T_d} \left| \sum_{n=1}^{N} e(\langle x A_n, h \rangle) \right|^2 dx = \sum_{1 \leq n, m \leq N} \int_{T_d} e(\langle x, (A_n - A_m)h \rangle) dx = N + \sum_{1 \leq n \neq m \leq N} \int_{T_d} e(\langle x, (A_n - A_m)h \rangle) dx.
\]
By our condition that \( A_n - A_m \) is invertible when \( n \neq m \), we conclude that \( (A_n - A_m)h \) is a non-zero integer vector, and hence for \( n \neq m \) we have
\[
\int_{T_d} e(\langle x, (A_n - A_m)h \rangle) dx = 0,
\]
which yields the desired identity. \( \square \)

We have the following analogy of Lemma 10.8.

**Lemma 12.2.** Let \( S = (A_n)_{n \in \mathbb{N}} \) be a sequence of \( d \times d \) matrices that satisfies (3.6) for some \( \tau \geq 1/d \) and let \( h \in \mathbb{R}^d, c \in (0, 1) \). Then there exists \( \varepsilon > 0 \) such that if \( |V_{S,h}(x; N)| \geq cN \) for some \( x \in T_d \) then
\[
|V_{S,h}(x; N)| \geq cN/2
\]
holds for any \( y \in B(x, \varepsilon N^{-\tau}) \), where \( B(x, r) \) denotes the ball of \( T_d \) centered at \( x \) and of radius \( r \).

**Proof.** For any \( n \in \mathbb{N}, h \in \mathbb{R}^d \) and \( x, y \in T_d \) we have
\[
e(\langle x A_n, h \rangle) - e(\langle y A_n, h \rangle) \ll \langle x A_n, h \rangle - \langle y A_n, h \rangle = \langle (x - y) A_n, h \rangle \ll \|A_n\|\|h\|\|x - y\|.
\]
It follows that
\[
V_{S,h}(x; N) - V_{S,h}(y; N) \ll \|x - y\| N^{\tau+1} \|h\|,
\]
which yields the desired bound. \( \square \)

**Lemma 12.3.** Let \( S = (A_n)_{n \in \mathbb{N}} \) be a sequence of \( d \times d \) matrix that satisfies (3.6) for some \( \tau \geq 1/d \) and let \( h \in \mathbb{R}^d, c \in (0, 1) \). Then there exists \( \varepsilon > 0 \) such that
\[
\{x \in T_d : |V_{S,h}(x; N)| \geq cN\} \subseteq \bigcup_{Q \in \mathcal{Q}_N} Q,
\]
where \( \mathcal{Q}_N \) is a certain collection of equal cubes with the side lengths \( 1/ \lfloor N^{\tau \varepsilon^{-1}} \rfloor \) and
\[
\#\mathcal{Q}_N \ll N^{d\tau - 1},
\]
with...
where the implied constant depends on \( \varepsilon \).

**Proof.** Divide \( T_d \) into \( \zeta^{-d} \) interior disjoint equal cubes in a natural way such that each cube has side length \( \zeta = 1/\lceil N^\tau \varepsilon^{-1} \rceil \), and let \( D_n \) be a collection of these cubes. Let
\[
Q_N = \{ Q \in D_n : \exists x \in Q, \text{ such that } |V_{S,h}(x; N)| \geq cN \}.
\]

It is sufficient to show that \( Q_N \) satisfies (12.1). For any \( Q \in Q_N \) by Lemma 12.2 we have
\[
|V_{S,h}(x; N)| \geq cN/2 \text{ for all } x \in Q.
\]

Hence
\[
N^2 \#Q_N N^{-\tau} \varepsilon^d \leq \int_{Q_N} |V_{S,h}(x; N)|^2 \, dx \leq \int_{T_d} |V_{S,h}(x; N)|^2 \, dx.
\]

Combining with the mean value bound Theorem 12.1 we derive
\[
N^2 N^{-\tau} \#Q_N \ll N,
\]
which implies the desired bound. \( \square \)

We remark that the condition \( \tau \geq 1/d \) in Lemma 12.3 comes from the inequality (12.1).

### 12.2. Concluding the proof

We now turn to the proof of Theorem 3.9. Let \( S = (A_n)_{n \in \mathbb{N}} \) satisfy the condition of Theorem 3.9. For \( c > 0 \) and \( h \in \mathbb{R}^d \) define
\[
G_{S,h,c} = \{ x \in T_d : |V_{S,h}(x; N)| \geq cN \text{ for infinitely many } N \in \mathbb{N} \}.
\]

By using the Weyl criterion (see [20, Section 1.2.1]) and the countable stability of Hausdorff dimension (see [23, Section 2.2]), it is sufficient to prove that for any \( c > 0 \) and any non-zero vector \( h \in \mathbb{Z}^d \) one has
\[
\dim G_{S,h,c} \leq d - 1/\tau.
\]

For \( N \in \mathbb{N} \) denote
\[
B_N = \{ x \in T_d : V_{S,h}(x; N) \geq cN/2 \}.
\]

Let \( \beta > 1 \) and \( N_i = i^\beta \). Then we have
\[
G_{S,h,c} \subseteq \bigcap_{k=1}^\infty \bigcup_{i=k}^\infty B_{N_i}.
\]

Indeed let \( x \in G_{S,h,c} \) and suppose to the contrary that for all large enough \( N_i \) we have
\[
|V_{S,h}(x; N_i)| < cN_i/2.
\]

For any large \( N \) there is \( i \in \mathbb{N} \) such that \( N_i \leq N < N_{i+1} \). Observe that \( N_{i+1} - N_i = O(i^{\beta-1}) \) for all \( i \in \mathbb{N} \), and
\[
|V_{S,h}(x; N)| \leq |V_{S,h}(x; N_{i})| + N_{i+1} - N_i 
\leq cN_i/2 + O(i^{\beta-1}) \leq 2cN/3
\]
provided that \( N \) is large enough, which contradicts our assumption that \( x \in \mathcal{G}_{S,h,c} \).

Let \( \varepsilon > 0 \) be the same on as in Lemma 12.3. For each \( N_i \) by Lemma 12.3 we obtain

\[ B_{N_i} \subseteq \bigcup_{Q \in Q_{N_i}} Q, \]

where each \( Q \in Q_{N_i} \) has side length \( 1/\lfloor N^r\varepsilon^{-1} \rfloor \) and \( \#Q_{N_i} \ll N_i^{dr-1} \).

From the definition of Hausdorff dimension we obtain

\[
\dim \mathcal{G}_{S,h,c} \leq \inf \left\{ \nu > 0 : \sum_{i=1}^{\infty} \sum_{Q \in Q_{N_i}} N_i^{-\tau \nu} < \infty \right\}. 
\]

(12.2)

Note that

\[
\sum_{i=1}^{\infty} \sum_{Q \in Q_{N_i}} N_i^{-\tau \nu} \leq \sum_{i=1}^{\infty} N_i^{dr-1} N_i^{-\tau \nu},
\]

thus the series is convergent provided

\[ \beta(d \tau - 1 - \tau \nu) < -1, \]

which is equivalent to

\[ \nu > d - \frac{1}{\tau} + \frac{1}{\beta \tau}. \]

Combining with (12.2) we obtain

\[ \dim \mathcal{G}_{S,h,c} \leq d - \frac{1}{\tau} + \frac{1}{\beta \tau}, \]

and by the arbitrary choice of \( \beta > 1 \) we obtain the desired bound.

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