On Almost Nonpositive $k$-Ricci Curvature

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Abstract
Motivated by the recent work of Chu–Lee–Tam on the nefness of canonical line bundle for compact Kähler manifolds with nonpositive $k$-Ricci curvature, we consider a natural notion of almost nonpositive $k$-Ricci curvature, which is weaker than the existence of a Kähler metric with nonpositive $k$-Ricci curvature. When $k = 1$, this is just the almost nonpositive holomorphic sectional curvature introduced by Zhang. We firstly give a lower bound for the existence time of the twisted Kähler-Ricci flow when there exists a Kähler metric with $k$-Ricci curvature bounded from above by a positive constant. As an application, we prove that a compact Kähler manifold of almost nonpositive $k$-Ricci curvature must have nef canonical line bundle.

Keywords Almost nonpositive $k$-Ricci curvature · Kähler manifolds · Nefness of canonical line bundle

Mathematics Subject Classification 53C55 · 32Q05 · 32Q15

1 Introduction and Statement of Result

In an attempt to generalize the hyperbolicity of Kobayashi to the $k$-hyperbolicity, Ni [9] introduced the concept of $k$-Ricci curvature. Given a compact Kähler manifold $(M^n, h)$ with Kähler form $\omega$ and Chern curvature tensor $R$, the $k$-Ricci curvature $Ric_k$ ($1 \leq k \leq n$) is defined as the Ricci curvature of the $k$-dimensional holomorphic subspaces of the holomorphic tangent bundle $T'M$. Clearly, $Ric_1$ is just the holomorphic sectional curvature $H(X)$ and $Ric_n$ coincides with the Ricci curvature $\text{Ric}$ of $M$. Hitchin [6] showed that $H$ and $\text{Ric}$ are independent to each other by an example.
There are many important results on a compact Kähler manifold with \( \text{Ric}_k \geq 0 \) or \( \text{Ric}_k \leq 0 \). It was proved by Yang [19] that a compact Kähler manifold with \( \text{Ric}_1 > 0 \) must be projective and rationally connected, confirming a conjecture of Yau [23]. In [10], Ni showed that it is also true if \( \text{Ric}_k > 0 \) for some \( 1 \leq k \leq n \). In their recent breakthrough [17], Wu–Yau confirmed a conjecture of Yau that a projective Kähler manifold with \( \text{Ric}_1 < 0 \) must have ample canonical line bundle. Tosatti and Yang [16] was able to drop the projectivity assumption in Wu–Yau theorem. They also proved that a compact Kähler manifold with \( \text{Ric}_1 \leq 0 \) must have nef canonical bundle. In a recent preprint [1], Chu–Lee–Tam proved that a compact Kähler manifold with \( \text{Ric}_k < 0 \) (\( \text{Ric} \leq 0 \)) have ample (nef) canonical bundle. Li et al. [8] also gave an alternate proof to the results of Chu–Lee–Tam. For more related works, we refer readers to [2, 4, 5, 7, 11–13, 15, 18, 20, 21].

As we all know, the ampleness of the canonical line bundle is equivalent to the existence of one Kähler metric with negative Ricci curvature and the nefness of the canonical bundle is defined by taking limits of a family of Kähler classes. Therefore, it may not be “best” choice to imply the nefness of the canonical line bundle from the nonpositivity of the \( k \)-Ricci curvature of one Kähler metric. A possible natural question is whether there is a condition in terms of \( k \)-Ricci curvature which is weaker than the existence of a Kähler metric with \( \text{Ric}_k \leq 0 \), but it can guarantee the nefness of the canonical line bundle. For the case of \( \text{Ric}_1 \leq 0 \), Zhang [24] was the first to consider this problem and he defined the concept of \textit{almost nonpositive holomorphic sectional curvature} (namely, almost nonpositive 1-Ricci curvature). He also proved that a compact Kähler manifold of almost nonpositive holomorphic sectional curvature has a nef canonical line bundle. Zhang and Zheng [25] also studied the compact Kähler manifolds with almost quasi-negative holomorphic sectional curvature. Motivated by the work of Zhang [24], we introduce a natural notion of \textit{almost nonpositive} \( k \)-\textit{Ricci curvature}.

Let us recall the concept of \( k \)-Ricci curvature on a compact Kähler manifold \((M^n, h)\) introduced in [9]. We denote Chern curvature tensor as \( R \). For a point \( p \in M \), let \( \Sigma \in T'_p M \) be a \( k \)-dimensional subspace. The \( k \)-Ricci curvature of the Kähler metric \( h \) on \( \Sigma \) is

\[
\text{Ric}_k^h(p, \Sigma)(X, \overline{Y}) = tr_h R(X, \overline{Y}, \cdot, \cdot),
\]

for \( X, Y \in \Sigma \) where the trace is taken with respect to \( h|_\Sigma \). For any \( k \)-dimensional subspace \( \Sigma \in T'_p M \) at any point \( p, X \in \Sigma \setminus \{0\} \). If the following inequality

\[
\frac{\text{Ric}_k^h(p, \Sigma)(X, \overline{X})}{|X|^2_h} \leq \lambda \quad (\geq \lambda)
\]

holds, we denote it as \( \text{Ric}_k^h \leq \lambda \ (\geq \lambda) \). we set

\[
\mu_h(k) = \sup_{x \in M} \left\{ \sup \text{Ric}_k^h|_x \right\},
\]
where it means the maximal value of the $k$-Ricci curvature of $h$ on the compact manifold $M$.

**Definition 1.1** Let $(M^n, \omega_0)$ be a compact Kähler manifold.

(a) Let $[\omega]$ be a Kähler class on $M$. We define the number $\mu_{[\omega]}(k)$ as follows:

$$
\mu_{[\omega]}(k) = \inf \{ \mu_{\omega'}(k) | \omega' \text{ is a Kähler metric in } [\omega] \}.
$$

(b) If there exist a sequence number $\varepsilon_i \searrow 0$ and a sequence of Kähler class $\alpha_i$ on $M$ such that $\mu_{\alpha_i}(k) \alpha_i < \varepsilon_i [\omega_0]$, we say that $M$ is of almost nonpositive $k$-Ricci curvature.

(c) If $\mu_{[\omega]}(k) = 0$, we say that the Kähler class $[\omega]$ is of almost nonpositive $k$-Ricci curvature.

In this note, we first prove the following property, which shows that $\mu_{[\omega]}(k)$ is well defined.

**Proposition 1.2** Let $(M^n, \omega_0)$ be a compact Kähler manifold. Let $\alpha$ be any Kähler class on $M$. For $1 \leq k \leq n$, we have $\mu_{\alpha}(k) > -\infty$.

**Remark 1.3** (1) Obviously, if $\mu_{\alpha}(k) = 0$, the $M$ must be of almost nonpositive $k$-Ricci curvature which means that the definition (b) is weaker than the definition (c). $\mu_{\alpha}(k) < 0$ if and only if there exists a Kähler metric $\omega' \in \alpha$ of negative $k$-Ricci curvature. Clearly, the condition $\mu_{\alpha}(k) = 0$ is by definition a condition weaker than the existence of a Kähler metric $\omega' \in \alpha$ of nonpositive $k$-Ricci curvature.

(2) We know that nefness is a positivity at the level of $(1, 1)$-classes, not $(1, 1)$-forms, so the definition of (b) seems reasonable where it is also a definition at the level of $(1, 1)$-classes. When $k = 1$, it is just the concept of almost nonpositive holomorphic sectional curvature. In [24], Zhang gave a lot of good properties and applications of the concept.

In [1], Chu–Lee–Tam used the twisted Kähler–Ricci flow to study the compact Kähler manifolds with nonpositive $k$-Ricci curvature. By employing this method, we obtain the following result.

**Theorem 1.4** A compact Kähler manifold $M^n$ of almost nonpositive $k$-Ricci curvature must have nef canonical line bundle.

In particular, from the above Theorem 1.4, we easily get the following result:

**Theorem 1.5** A compact Kähler manifold $M^n$ admitting a Kähler class $\alpha$ of almost nonpositive $k$-Ricci curvature must have nef canonical line bundle.

**Remark 1.6** Note that when $k = 1$, the above results was proved by Zhang [24]. Theorem 1.4 is also generalization of Chu–Lee–Tam’s result [1]. For $k = n$, $Ric_n$ is just the Chern Ricci curvature. If a Kähler class $\alpha$ is of almost nonpositive Ricci curvature, then there exists a sequence of Kähler metric $\omega_\varepsilon \in \alpha$ such that $Ric^{\omega_\varepsilon} < \varepsilon \omega_\varepsilon$ for any $\varepsilon > 0$, namely, $2\pi c_1(K_M) + \varepsilon \alpha > 0$. So the canonical line bundle must be nef.
To see Theorem 1.4, we give a useful proposition on a lower bound of the twisted Kähler–Ricci flow, which might have other applications.

**Proposition 1.7** Let \((M^n, \omega_0)\) be a compact Kähler manifold and \(\omega\) be a Kähler metric of the Kähler class \([\omega_0]\). For a fixed integer \(k\) with \(1 < k < n\), we assume that \(A\) is the maximal value of the \(k\)-Ricci curvature of \(\omega\) on \(M\). Set \(A > 0\). If there exists a positive constant \(\delta > 0\), such that \(\delta [\omega_0] + 2\pi c_1(K_M) > 0\). Then we can find a function \(\upsilon \in C^\infty(M)\) which satisfies

\[
\begin{align*}
\frac{\partial}{\partial \upsilon} \omega(t) &= -Ric(\omega(t)) - \eta \\
\omega(0) &= \hat{\omega}
\end{align*}
\]

exists a smooth solution on \(M \times [0, \frac{2n(k-1)}{(n-1)(2nA+(k-1)\delta)})\).

## 2 Proof of Proposition 1.2

Before proving Proposition 1.2, we give some algebraic estimates which are proved by Chu et al. [1]. They are useful in obtaining key estimates for the twisted Kähler–Ricci flow.

**Lemma 2.1** [1] Let \((M^n, h)\) be a compact Kähler manifold with \(Ric_k(X, \overline{X}) \leq -(k+1)\sigma |X|^2\), \(\sigma \in \mathbb{R}\). Then the following inequality holds

\[
(k-1)|X|^2 hRic(X, \overline{X}) + (n-k)R(X, \overline{X}, X, \overline{X}) \leq -(n-1)(k+1)\sigma |X|^2_h. \tag{2.1}
\]

Furthermore, by using the Royden’s trick [14], the following result holds.

**Lemma 2.2** [1] Let \((M^n, h)\) be a compact Kähler manifold with \(Ric_k(X, \overline{X}) \leq -(k+1)\sigma |X|^2\), \(\sigma \in \mathbb{R}\). If \(g\) is another Kähler metric, then the following inequality holds

\[
2g^{ij} g^{kl} R_{ijkl} \leq \frac{-(n-1)(k+1)\sigma}{n-k} \left( (trg h)^2 + |h|_g^2 \right) - \frac{k-1}{n-k} (trg h) \cdot (trg Ric) - \frac{k-1}{n-k} \langle h, Ric \rangle_g,
\]

where \(R\) is the Chern curvature tensor of \(h\) and \(Ric\) is the Ricci curvature of \(h\).

If the above Lemma 2.2 satisfies \(g = h\), it can also imply the relation on \(Ric\) and scalar curvature \(S\) under the assumption \(Ric_k(X, \overline{X}) \leq -(k+1)\sigma |X|^2\).

**Lemma 2.3** [1] Let \((M^n, h)\) be a compact Kähler manifold with \(Ric_k(X, \overline{X}) \leq -(k+1)\sigma |X|^2\), \(\sigma \in \mathbb{R}\) and \(1 < k \leq n\). Then we have

\[
(nk + n - k - 2)S \cdot h + nRic \leq -(n+1)(n-1)(k+1)\sigma h. \tag{2.3}
\]
Proof of Proposition 1.2  Since the case of $k = 1$ is proved by Zhang [24], we just consider the case of $1 < k \leq n$. Let $\omega \in \alpha$ is a Kähler metric. For any fixed $k$, we assume $\lambda_\omega = \frac{1}{k+1}\mu_\omega(k)$, then the $k$-Ricci curvature of $\omega$ satisfies $Ric^\omega_k \leq (k + 1)\lambda_\omega$. By the inequality (2.3), we have

\[(nk + n - k - 2)\int_M S^\omega \cdot \omega + nRic^\omega \leq n(n + 1)(n - 1)(k + 1)\lambda_\omega \omega, \tag{2.4}\]

and integrating the above inequality can be obtained

\[(nk + n - k - 2)\int_M S^\omega \omega^n + n\int_M Ric^\omega \wedge \omega^{n-1} \leq n(n + 1)(n - 1)(k + 1)\lambda_\omega \int_M \omega^n. \tag{2.5}\]

Note that

\[\int_M S^\omega \omega^n = \int_M nRic^\omega \wedge \omega^{n-1} = -2\pi n c_1(K_M) \cdot \omega^{n-1}. \tag{2.6}\]

Combining (2.5) and (2.6), we can get

\[\lambda_\omega \geq \frac{-2\pi c_1(K_M) \cdot \alpha^{n-1}}{(n + 1)\alpha^n}. \tag{2.7}\]

Therefore, we obtain

\[\mu_\omega(k) \geq \frac{-2\pi (k + 1)c_1(K_M) \cdot \alpha^{n-1}}{(n + 1)\alpha^n}, \tag{2.8}\]

and so

\[\mu_\alpha(k) \geq \frac{-2\pi (k + 1)c_1(K_M) \cdot \alpha^{n-1}}{(n + 1)\alpha^n}, \tag{2.9}\]

proving the Proposition 1.2. \qed

3 Proof of Proposition 1.7 and Theorem 1.4

Let $(M^n, \widehat{\omega})$ be a compact Kähler manifold. The twisted Kähler–Ricci flow running from $\widehat{\omega}$ satisfies the following equation:

\[
\begin{align*}
\partial_t \omega(t) &= -Ric(\omega(t)) - \eta \\
\omega(0) &= \widehat{\omega},
\end{align*}
\tag{3.1}
\]
where the $\eta$ is a closed real $(1,1)$ form. It is equivalent to the following Monge–Ampère type flow:

$$\begin{cases}
\partial_t \varphi = \log \left( \frac{\omega - t \text{Ric}(\omega) - t \eta + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi}{\omega^n} \right); \\
\varphi(0) = 0,
\end{cases}\quad (3.2)$$

Hence, if $\varphi$ is a smooth solution of Eq. (3.2) on $M \times [0, T)$, such that

$$\omega(t) = \omega - t \text{Ric}(\omega) - t \eta + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi > 0, \quad (3.3)$$

then $\omega(t) = \omega - t \text{Ric}(\omega) - t \eta + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi$ is also a solution of Eq. (3.1). If $\omega(t)$ satisfies Eq. (3.1), we can define

$$\varphi(t) = \int_0^t \log \frac{\omega(s)^n}{\omega^n} \, ds. \quad (3.4)$$

We can easily deduce that $\varphi(t)$ satisfies Eq. (3.2). The solution of the twisted Kähler–Ricci flow have a short-time existence (see [3]). For the convenience of proof, we need to state three useful lemmas [1] which are essentially the same as the case of Kähler-Ricci flow.

**Lemma 3.1** [1] Let $\omega(t)$ be a smooth solution to (3.1) on $M \times [0, T_0)$. If there is a positive constant $C > 0$ such that

$$C^{-1} \omega \leq \omega(t) \leq C \omega$$

on $M \times [0, T_0)$. Then there is $\varepsilon > 0$ such that $\omega(t)$ can be extended to $[0, T_0 + \varepsilon)$ which satisfies (3.1).

**Lemma 3.2** [1]

$$\begin{cases}
\left( \frac{\partial}{\partial t} - \Delta_{\omega(t)} \right) \dot{\varphi} = -tr_{\omega}(\text{Ric}(\omega) + \eta); \\
\left( \frac{\partial}{\partial t} - \Delta_{\omega(t)} \right) (t \dot{\varphi} - \varphi - nt) = -tr_{\omega} \omega,
\end{cases}$$

where $\omega = \omega(t)$ and $\dot{\varphi} = \partial_t \varphi$.

**Lemma 3.3** [1] Let $\omega(t)$ be a smooth solution to (3.1) on $M \times [0, T)$. Then the scalar curvature $S(\omega(t))$ satisfies

$$S(\omega(t)) + tr_{\omega} \eta \geq -\frac{n}{t + \sigma}$$
on $M \times [0, T)$ where $\sigma > 0$ and $\inf_M (S(\tilde{\omega}) + tr\tilde{\omega}\eta) \geq -n\sigma^{-1}$. Moreover,

$$\sup_M \log \frac{\det \omega(t)}{\det \tilde{\omega}} = \sup_M \phi(., t) \leq n \log \left( \frac{t + \sigma}{\sigma} \right).$$

We are now ready to prove Proposition 1.7 and Theorem 1.4.

**Proof of Proposition 1.7** For any fixed $k, 1 < k < n$, we can assume that $A = \mu\tilde{\omega}(k) > 0$. According to the hypothesis of Proposition 1.7, there exists a positive constant $\delta > 0$, such that $\delta[\omega_0] + 2\pi c_1(K_M) > 0$. Hence we can find a smooth function $\nu \in \mathcal{C}^\infty(M)$ which depends on $\tilde{\omega}, \hat{\omega} \in [\omega_0]$, such that

$$\delta\hat{\omega} - \text{Ric}(\hat{\omega}) + \sqrt{-1} \partial\bar{\partial}\nu > 0. \quad (3.5)$$

We assume that $\omega(t)$ is the twisted Kähler–Ricci flow running from $\hat{\omega}$ with $\eta = \frac{k-1}{2(n-k)}\sqrt{-1} \partial\bar{\partial}\nu$. Let $G = tr_{\omega(t)}\hat{\omega}$. To simplify notation we write the components of $\omega(t)$ as $g_{ij}$ and the components of $\hat{\omega}$ as $h_{ij}$. Then by the calculation of the parabolic Schwarz Lemma which is a parabolic version of the Schwarz lemma by Yau [22], we have

$$\left( \frac{\partial}{\partial t} - \Delta_g \right) \log G \leq \frac{1}{G} g^{ij} g^{kl} R_{ijkl}(h) + \frac{k-1}{2(n-k)} \frac{1}{G} g^{ij} g^{kl} h_{ij} v_{kl}. \quad (3.6)$$

Applying Lemma 2.2, we have

$$\frac{1}{G} g^{ij} g^{kl} R_{ijkl} \leq \frac{(n-1)A}{2(n-k)} \left( G + \frac{1}{G} |h|^2_g \right) - \frac{(k-1)}{2(n-k)} (tr_g \text{Ric}) - \frac{(k-1)}{2(n-k)} (tr_g \text{Ric}) g$$

$$= \frac{(n-1)A}{2(n-k)} \left( G + \frac{1}{G} |h|^2_g \right) - \frac{(k-1)}{(n-k)} (tr_g \text{Ric}) + \frac{(k-1)}{2(n-k)} \frac{1}{G} (G \cdot tr_g \text{Ric} - (h, \text{Ric})_g). \quad (3.7)$$

Choosing local coordinates such that $g_{ij} = \delta_{ij}, h_{ij} = h_{ij}\delta_{ij}$, then we also have

$$G \cdot tr_g \text{Ric} - (h, \text{Ric})_g = \sum_i \text{Ric}_{i7} \left( \sum_j h_{jj} - h_{i7} \right)$$

$$= \sum_i \left( \text{Ric}_{i7} \left( \sum_{j \neq i} h_{jj} \right) \right)$$

$$\leq \sum_i \left( \delta h_{i7} + v_{i7} \right) \left( \sum_j h_{jj} - h_{i7} \right)$$
\[ \leq \delta G^2 - \delta |h|_g^2 + G \cdot \Delta_g \nu - \langle \sqrt{-1} \partial \overline{\partial} \nu, h \rangle_g, \quad (3.8) \]

where we have used inequality (3.5) and \( g^{ij} g^{kl} h_{ij} h_{kl} = \langle \sqrt{-1} \partial \overline{\partial} \nu, h \rangle_g \). Hence, we have

\[
\left( \frac{\partial}{\partial t} - \Delta_g \right) \log G \leq \frac{(n-1)A}{2(n-k)} \left( G + \frac{1}{G} |h|_g^2 \right) - \frac{(k-1)}{(n-k)} (tr_g \text{Ric}) \\
+ \frac{(k-1)\delta}{2(n-k)} G - \frac{(k-1)\delta}{2(n-k)} |h|_g^2 + \frac{(k-1)}{(n-k)} \Delta_g \nu. \quad (3.9) \]

Since \( A > 0, \delta > 0, n|h|_g^2 \geq G^2 \) and \( |h|_g^2 \leq G^2 \), then it is easy to see

\[
\left( \frac{\partial}{\partial t} - \Delta_g \right) \log G \leq \frac{(n-1)A}{2n(n-k)} G + \frac{(n-1)(k-1)\delta}{2n(n-k)} G \\
+ \frac{(k-1)}{(n-k)} \Delta_g \nu - \frac{(k-1)}{(n-k)} (tr_g \text{Ric}). \quad (3.10) \]

Combining Lemma 3.2, we can get

\[
\left( \frac{\partial}{\partial t} - \Delta_g \right) \log G \leq \left( \frac{\partial}{\partial t} - \Delta_g \right) \left( -a \xi \right) + \frac{(k-1)}{(n-k)} \Delta_g \nu \\
+ \frac{(k-1)}{(n-k)} \left[ \left( \frac{\partial}{\partial t} - \Delta_g \right) \phi + \frac{(k-1)}{(n-k)} \Delta_g \phi \right] \\
= \left( \frac{\partial}{\partial t} - \Delta_g \right) \left[ -a \xi + \frac{(k-1)}{(n-k)} \phi - \frac{(k-1)}{(n-k)} \nu - \frac{(k-1)^2}{2(n-k)^2} \nu \right]. \quad (3.11) \]

where \( \xi = t \phi - \nu - nt \) and \( a = \frac{2n(n-1)A + (n-1)(k-1)\delta}{2n(n-k)} \). By the maximum principle arguments, we have

\[ \log G \leq C_1 + \left( \frac{(k-1)}{(n-k)} - at \right) \phi + a \phi + ant. \quad (3.12) \]

where \( C_1 \) depends on \( n, k, \sup_M |\nu| \) and \( \delta \). This, together with Lemma 3.3, implies

\[ C^{-1} \hat{\omega} \leq \omega(t) \leq C \hat{\omega} \quad (3.13) \]

where \( C \) is a positive constant and \( t < \min\{T_{\text{max}}, \frac{(k-1)}{a(n-k)}\} \). Combining with Lemma 3.1, we have

\[ T_{\text{max}} \geq \frac{(k-1)}{a(n-k)}. \]
Moreover, we conclude that

\[ T_{\max} \geq \frac{2n(k-1)}{(n-1)[2nA + (k-1)\delta]} \, . \]

This completes the proof of Proposition 1.7. \[\square\]

**Proof of Theorem 1.4** Since \( k = 1 \) is proved by Zhang [24], we only consider the case of \( 1 < k \leq n \).

(1) The case of \( k = n \). It means that \( M \) must be of almost nonpositive Ricci curvature. We can choose a family of Kähler classes \( \alpha_i \) and a fixed Kähler metric \( \omega_0 \), such that \( \mu_{\alpha_i}(n)\alpha_i < \frac{1}{i}[\omega_0], \, i = 1, 2, \ldots \) For any \( \alpha_i \), we can also find a fixed Kähler metric \( \omega_{\epsilon_i} \in \alpha_i \), such that

\[ \text{Ric}_{\omega_{\epsilon_i}} < \mu_{\alpha_i}(n)\omega_{\epsilon_i} + \frac{1}{i}\omega_0 \, . \]

and so

\[ 2\pi c_1(K_M) + \mu_{\alpha_i}(n)\alpha_i + \frac{1}{i}[\omega_0] > 0 \, . \] (3.14)

We may assume that \( \mu_{\alpha_i}(n) > 0 \). If some \( \mu_{\alpha_i}(n) \leq 0 \), we can easily get the result from the Remark 1.6. Since \( 0 < \mu_{\alpha_i}(n)\alpha_i < \frac{1}{i}[\omega_0] \), we have that

\[ c_1(K_M) = \lim_{i \to \infty} \left( c_1(K_M) + \frac{\mu_{\alpha_i}(n)\alpha_i}{2\pi} + \frac{1}{2\pi i}[\omega_0] \right) \, . \]

is nef.

(2) The case of \( 1 < k < n \).

First, we need to prove a fact: if we assume that \( \alpha \) is a Kähler class and \( \mu_{\alpha}(k) \geq 0 \). We define

\[ B_{\alpha} = \inf \{ b \in \mathbb{R} | 2\pi c_1(K_M) + b\alpha > 0 \} \, . \]

Then we have \( B_{\alpha} \leq \frac{4n}{k-1}\mu_{\alpha}(k) \).

Assume that \( B_{\alpha} > \frac{4n}{k-1}\mu_{\alpha}(k) \geq 0 \). We shall prove this fact via an argument by contradiction. We can choose that \( \delta = \frac{3}{2}B_{\alpha} > B_{\alpha} > 0 \). By the definition of \( B_{\alpha} \), we have

\[ 2\pi c_1(K_M) + \delta \alpha > 0 \, . \] (3.15)

For any small \( \epsilon > 0 \), there is a Kähler metric \( \omega_{\epsilon} \in \alpha \), such that

\[ \mu_{\omega_{\epsilon}}(k) < \mu_{\alpha}(k) + \epsilon \, . \]
By the inequality (3.15), we can find a smooth real function $\nu_\varepsilon$ on $M$, such that

$$-Ric(\omega_\varepsilon) + \sqrt{-1} \partial \overline{\partial} \nu_\varepsilon + \delta \omega_\varepsilon > 0.$$ 

Let $\eta_\varepsilon = \frac{k-1}{2(n-k)} \sqrt{-1} \partial \overline{\partial} \nu_\varepsilon$. Applying Proposition 1.7, we get that the twisted Kähler-Ricci flow running from $\omega_\varepsilon$, 

$$\left\{ \begin{array}{l}
\partial_t \omega_\varepsilon(t) = -Ric(\omega_\varepsilon(t)) - \eta_\varepsilon \\
\omega_\varepsilon(0) = \omega_\varepsilon,
\end{array} \right.$$ 

exists a smooth solution on $M \times [0, \frac{2n(k-1)}{(n-1)[2n(\mu_\alpha(k)+\varepsilon)+(k-1)\delta]})$. This, together with (3.3), implies

$$\omega_\varepsilon(t) = \omega_\varepsilon - tRic(\omega_\varepsilon) - t \frac{k-1}{2(n-k)} \sqrt{-1} \partial \overline{\partial} \nu_\varepsilon + \sqrt{-1} \partial \overline{\partial} \varphi > 0.$$ 

(3.16)

where $\varphi$ is a solution of (3.2). From the (3.16), it is easy to see

$$\frac{1}{t} \alpha + 2\pi c_1(K_M) > 0.$$ 

where $t \in [0, \frac{2n(k-1)}{(n-1)[2n(\mu_\alpha(k)+\varepsilon)+(k-1)\delta])}$. By the definition of $B_\alpha$, we have

$$B_\alpha \leq \frac{(n-1)[2n(\mu_\alpha(k)+\varepsilon)+(k-1)\delta]}{2n(k-1)}.$$ 

(3.17)

Since $\varepsilon$ is an arbitrary positive constant, we conclude that

$$B_\alpha < \frac{4n}{k-1} \mu_\alpha(k) \geq 0$$

and

$$\delta = \frac{3}{2} B_\alpha > B_\alpha > 0,$$

we have

$$B_\alpha < \frac{(n-1)[2n \cdot \frac{(k-1)B_\alpha}{4n} + (k-1) \cdot \frac{3B_\alpha}{2}]}{2n(k-1)}$$

$$= \left(1 - \frac{1}{n}\right) B_\alpha.$$ 

This is a contradiction. So this fact is true.

Now we are ready to prove Theorem 1.4 in case of $1 < k < n$. We can choose a family of Kähler classes $\alpha_i$ and a fixed Kähler metric $\omega_0$, such that $\mu_{\alpha_i}(k)\alpha_i < \frac{1}{2}[\omega_0]$, $i = 1, 2, \cdots$. If there is some $\mu_{\alpha_i}(k) = 0$, then by the above fact, the $K_M$ is nef. If there
exists some $\mu_{\alpha_i}(k) < 0$, this theorem has been proved. We assume that $\mu_{\alpha_i}(k) > 0$ for all $i$. Applying the above fact, we have

$$\frac{4n}{k-1} \mu_{\alpha_i}(k) \geq B_{\alpha_i}.$$  \hfill (3.18)

and so

$$2\pi c_1(K_M) + \frac{8n}{k-1} \mu_{\alpha_i}(k)\alpha_i > 0,$$  \hfill (3.19)

where it is a Kähler classes. Because of $0 < \mu_{\alpha_i}(k)\alpha_i < \frac{1}{i}[\omega_0]$, we have

$$c_1(K_M) = \lim_{i \to \infty} \left( c_1(K_M) + \frac{4n \mu_{\alpha_i}(k)\alpha_i}{\pi(k-1)} \right).$$

which implies that $K_M$ is nef. We have completed the proof Theorem 1.4. \hfill \square

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