The fractional and mixed-fractional CEV model

Axel A. Araneda
Frankfurt Institute for Advanced Studies
60438 Frankfurt am Main, Germany.

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Abstract
The continuous observation of the financial markets has identified some ‘stylized facts’ which
challenge the conventional assumptions, promoting the born of new approaches. On one hand, the
long range dependence has been faced replacing the traditional Gauss-Wiener process (Brownian
motion), characterized by their stationary independent increments, by a fractional version. On the
other hand, the local volatility CEV model, is an efficient formulation which address the ‘Leverage
effect’ and the smile-skew phenomena. In this paper, these two insights are merge and the fractional
and mixed-fractional extension of the Constant Elasticity of Variance model are developed. Using
the fractional versions of both the Itô’s calculus and the Fokker-Planck equation, the transition
probability density function of the asset price is obtained as a solution of a non-stationary Feller
process with time-varying coefficients; and the analytical valuation formula for the European Call
Option is provided. Besides, the greeks are computed and compared with the standard case.

Keywords: fBM, mfBm, CEV, fractional Fokker-Planck, fractional Itô’s calculus, Feller’s process

1 Introduction
One of the most important insight in financial mathematics has been the Black-Scholes model [1], which
uses a Geometric Brownian motion (GBM), describing the returns of the asset prices as a regular diffusion
process and arriving to an analytical formulæfor a Vanilla European option.

However, some “Stylized facts” in the financial markets aren’t agree with the assumptions using in the
Black-Scholes model. One of this findings is the long range dependence [2–5]. This, has motivated the
creation of a fractal version of the Black-Scholes model [6, 7], based on fractional Brownian motion [8–10].
Hu & Øskendal [11] and Necula [12] arrives to an analytical formula for the European Call option for
the fractional Black-Scholes case, using Wick-Itô calculus [13, 14]. However at the fractional framework,
the arbitrage possibilities aren’t totally omitted [15–17]. Addressing it fact, Cheridito [18] introduces
the mixed-fractional Brownian motion (see further mathematical details at refs. [19, 20]). This kind of
models ensure the absence of arbitrage opportunities [21 22] and also a pricing formula for European
type contracts could be obtained [23 24].

On the other hand, and come back to shortcomings of the original Black-Scholes model, the
homo-scedasticity assumption is not consistent with other empirical facts as the volatility smile-ske
[25 28] and leverage effect [29 32]. The former is the change in the implied volatility pattern in function of
the strike or maturity of an option. The latter is understood as the inverse relationship between the volatility
and the price. In this context, a very popular formulation is the Constant Elasticity of Variance (CEV)
model developed by Cox [33 34], which faces the heteroscedasticity and the leverage effect modeling
the volatility in function of the asset price level. The model also deal with the volatility skew-smile
phenomena [35 37]. Despite that the CEV model consider only one more parameter (elasticity) than the
Black-Scholes model, the latter is outperformed by the CEV in both prices and option pricing performance
[38 43]. Another plus point to the use of the CEV model is existence of a closed form formula for an
European vanilla option. The original Cox’s work derives the Call price in terms of summations of
the incomplete gamma function, but later Schroder [44] put it in a compact expression in function of the
non-central \( \chi^2 \) distribution.

Given the previous statement, the aim of this paper is to merge the local volatility approach and
the fractional calculus, extending the CEV model under classical Brownian motion to the fractional and

\( ^* \text{Email: araneda@fias.uni-frankfurt.de, Tel.:+49 69 798 47501} \)

\( ^{\dagger} \text{a.k.a. persistence, ‘Memory effect’ or ‘Joseph effect’} \)
mixed-fractional cases. Previously, the fractional CEV case has been addressed previously in the literature [45], proposing an European Call formula in terms of the standard complementary gamma distribution function, similar to the Cox’s result, but without an explicit evaluation of the added terms. At this time, for the fractional CEV, the European Call price is derived in a compact and explicit way, in terms of the non-central-chi-squared distribution and the M-Whittaker function, following the Schroder scheme and using a time-varying coefficients’ version of the Feller’s diffusion problem. Besides, in a similar way, a pricing formula for the mixed-fractional CEV model is studied. Also, the convergence of the fractional CEV pricing to the fractional Black-Scholes case is showed and the greeks of the models are computed and compared with the standard CEV case.

The paper outline is the following. First, the CEV model is revisited. Later, the fractional extension to classical CEV model is analyzed, and the pricing formula for an European Call option is derived. After that, a mixed fractional structure is proposed, arriving to the related European Call pricing formula. At next, the computation and analysis of the greeks is performed. Finally, the main conclusions are displayed.

2 The CEV model

Under the risk neutral measure, the asset price $S$ follow the next stochastic differential equation at the constant elasticity of variance model:

$$dS = rSdt + \sigma S^{\frac{\alpha}{2}}dB_t$$

where $r$, $\sigma$ and $\alpha$ are the constant parameters of the model, with $\sigma > 0$ and $\alpha \in [0, 2]$. $B_t$ is a standard Brownian motion, such that $dB_t \sim N(0, dt)$. In the limit case, when $\alpha \to 2$, the CEV turns into the Black-Scholes model.

Applying the following change of variable:

$$x(S,t) = S^{2-\alpha}$$

and by the Itô’s Lemma, the Eq. 1 becomes:

$$dx = (2 - \alpha) \left[ rx + \frac{1}{2} (1 - \alpha) \sigma^2 \right] dt + (2 - \alpha) \sigma \sqrt{x} dB_t$$

Let $P(x_T, T|x_0, 0)$, the transition probability function which rules the evolution of $x$ from $x(0) = x_0$ to $x(T) = x_T$, and $T > 0$. Then, $P$ evolves according to the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[ (2 - \alpha)^2 \sigma^2 x P \right] - \frac{\partial}{\partial x} \left[ (2 - \alpha) \left( rx + \frac{1}{2} (1 - \alpha) \sigma^2 \right) P \right]$$

This type of PDE was solved by Feller [46] and as shown in detail at ref. [47]. The eq. 3 can be solved matching it with the famous Feller’s lemma [46] which is summarized at next.

**Feller’s lemma.** Let $u = u(x, t)$, and $a, b, c$ constants, with $a > 0$ and $t > 0$. The solution of the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} \left[ a x u \right] - \frac{\partial}{\partial x} \left[ (bx + c) u \right]$$

conditional to

$$u(x, 0) = \delta(x - x_0)$$
is given by

\[ u(x,t|x_0,0) = \frac{b}{a(e^b-1)} \left( \frac{xe^{-bt}}{x_0} \right)^{c-a} \exp \left[ -\frac{b(x+x_0e^{bt})}{a(e^b-1)} \right] I_{1-c/a} \left[ \frac{2b}{a(1-e^{-bt})} \sqrt{e^{-bt}x_0x} \right] \]

where \( I_k(x) \) is the modified Bessel function of the first kind of order \( k \).

Proof. See refs. [46, 47].

Thus, is easy to show that, if we setting \( a = (2-\alpha)^2 \sigma^2/2 \), \( b = r(2-\alpha) \) and \( c = (2-\alpha)(1-\alpha)\sigma^2/2 \), by the Feller’s Lemma, the transition probability distribution from \( x(0) = x_0 \) to \( x(T) = x_T \), is given by:

\[
P(x_T, T|x_0,0) = \frac{2r}{\sigma^2 (2-\alpha) [e^{r(2-\alpha)T} - 1]} \left( \frac{x_T}{x_0} e^{-r(2-\alpha)T} \right) \frac{1}{2^{2-\alpha}} \exp \left[ \frac{-2r(x+x_0e^{r(2-\alpha)T})}{\sigma^2 (2-\alpha) [e^{r(2-\alpha)T} - 1]} \right] \times I_{1/(2-\alpha)} \left[ \frac{4r}{\sigma^2 (2-\alpha) (1-e^{-r(2-\alpha)T})} \sqrt{e^{-r(2-\alpha)T}x_0x} \right] (4)
\]

Come back to the original variables and reordering terms, the density for \( S_T \) given \( S_0 \) is equal to [44]:

\[
P(S_T, T|S_0,0) = P(x_T, T|x_0,0) \frac{\partial x_T}{\partial S_T} = (2-\alpha)k \frac{1}{2^{2-\alpha}} (yw^{1-2\alpha})^{\frac{1}{1-2\alpha}} e^{-y^{-w}} I_{1/(2-\alpha)} (2\sqrt{yw}) (5)
\]

where:

\[
k = \frac{2r}{\sigma^2 (2-\alpha) [e^{r(2-\alpha)T} - 1]}, (6)
\]

\[
y = kS_0^{2-\alpha} e^{r(2-\alpha)T}, (7)
\]

\[
w = kS_T^{2-\alpha}. (8)
\]

Later, the value of a Call option at time \( t = 0 \), with maturity \( T \) and exercise price \( E \), is computed by the Feynman-Kac formula:

\[
C(S,0) = e^{-rT} \int_{-\infty}^{\infty} \max \{S_T - E, 0\} \ P(S_T, T|S_0,0) \ dS_T
\]

\[
= e^{-rT} \int_{E}^{\infty} (S_T - E) \ P(S_T, T|S_0,0) \ dS_T
\]

\[
= e^{-rT} \int_{z}^{\infty} \left[ \left( \frac{w}{k} \right)^{\frac{1}{1-\alpha}} - E \right] (2-\alpha)k \frac{1}{2^{2-\alpha}} (yw^{1-2\alpha})^{\frac{1}{1-2\alpha}} e^{-y^{-w}}
\times I_{1/(2-\alpha)} (2\sqrt{yw}) \left[ \frac{1}{2-\alpha} (kw^{1-\alpha})^{-\frac{1}{1-\alpha}} \right] \ dw
\]

\[
= e^{-rT} \int_{z}^{\infty} \left( \frac{w}{k} \right)^{\frac{1}{1-\alpha}} \left( \frac{y}{w} \right)^{\frac{1}{1-\alpha}} e^{-y^{-w}} I_{1/(2-\alpha)} (2\sqrt{yw}) \ dw
\]

\[
- e^{-rT} \int_{z}^{\infty} E \left( \frac{y}{w} \right)^{\frac{1}{1-\alpha}} e^{-y^{-w}} I_{1/(2-\alpha)} (2\sqrt{yw}) \ dw
\]

\[
= C_1 - C_2 (9)
\]
where \( z = kE^{2-\alpha} \). As noted by Schroder [44] the arguments of both integrals are the pdfs of the non-central chi squared distributions with \( \nu \) degrees of freedom and non-centrality parameter \( \lambda \); noted by \( \chi^2_{\nu}(\lambda) \), defined as:

\[
P_{\chi^2_{\nu}(\lambda)}(l) = \left( \frac{x}{\lambda} \right)^{\nu-2} e^{-(x+\lambda)/2} I_{\nu} \left( \sqrt{x\lambda} \right)
\]

\[
= f(l; \nu, \lambda)
\]

(10)

Back to the pricing equation, the first integral is developed as:

\[
C_1 = e^{-rT} \int_{z}^{\infty} k^{-1} \frac{1}{(wy)^{\nu-\alpha}} e^{-y-w} I_{1/(2-\alpha)} (2\sqrt{yw}) \, dw
\]

\[
= e^{-rT} \int_{z}^{\infty} \frac{y}{k} \frac{1}{(w/y)^{\nu-\alpha}} e^{-y-w} I_{1/(2-\alpha)} (2\sqrt{yw}) \, dw
\]

\[
= e^{-rT} \int_{z}^{\infty} (S_0 e^{rT}) (w/y)^{\nu-\alpha} e^{-y-w} I_{1/(2-\alpha)} (2\sqrt{yw}) \, dw
\]

\[
= S_0 \int_{z}^{\infty} \left( \frac{2w}{2y} \right)^{\nu-\alpha} e^{-(2y+2w)/2} I_{2+\frac{2}{2-\alpha}} \left( \sqrt{(2y)(2w)} \right) \, dw
\]

\[
= S_0 \int_{z}^{\infty} f \left( 2w, 2 + \frac{2}{2 - \alpha}, 2y \right) \, dw
\]

While the second one:

\[
C_2 = E e^{-rT} \int_{z}^{\infty} \left[ \frac{2y}{2w} \right]^{\nu-\alpha} e^{-(2y-2w)/2} I_{2+\frac{2}{2-\alpha}} \left( \sqrt{(2y)(2w)} \right) \, dw
\]

\[
= E e^{-rT} \int_{z}^{\infty} f \left( 2y, 2 + \frac{2}{2 - \alpha}, 2w \right) \, dw
\]

Called \( Q \) to the complementary distribution function of \( \chi^2_{\nu}(\lambda) \):

\[
\int_{m}^{\infty} f(l; \nu, \lambda) \, dl = Q(m, \nu, \lambda)
\]

and using the following identity[44]:

\[
\int_{m}^{\infty} f(2l; 2\nu, 2\lambda) \, d\lambda = 1 - Q(2l; 2\nu - 2, 2m)
\]

the call formula can be wrote as:

\[
C(S, 0) = S_0 Q \left( 2z; 2 + \frac{2}{2 - \alpha}, 2y \right) - E e^{-rT} \left[ 1 - Q \left( 2y; \frac{2}{2 - \alpha}, 2z \right) \right]
\]

(11)
As we pointed previously, the Black-Scholes model could be treated as a limit case of the CEV model, \( a \to 2 \). For to show the convergence of the solution given in \( 11 \) to the Black-Scholes case, we use the following result for the complementary distribution function \( Q \), based on the central limit theorem \( 48 \):

\[
Q(m, \nu, \lambda) \approx Q_N \left( \frac{m - (\nu + \lambda)}{\sqrt{2(\nu + 2\lambda)}} \right), \quad \text{as } \nu \to \infty
\]  

(12)

where \( Q_N(\cdot) \) is the standard normal complementary density function.

Thus, the first complementary function of the eq. \( 11 \) when \( \alpha \to 2 \), can be computed as:

\[
\lim_{a \to 2} Q \left( 2z; 2 + \frac{2}{2 - \alpha}, 2y \right) = Q_N \left[ \lim_{a \to 2} \frac{2z - 2 - \frac{2}{2 - \alpha} - 2y}{\sqrt{2 \left( 2 + \frac{2}{2 - \alpha} + 4y \right)}} \right]
\[
= Q_N \left[ \lim_{a \to 2} \frac{2rE^{2-\alpha} - 2rS_0^{2-\alpha}e^{rT(2-\alpha)} + \sigma^2 (3 - \alpha) \left( e^{rT(2-\alpha)} - 1 \right)}{\sigma^2 (2 - \alpha) \left( e^{rT(2-\alpha)} - 1 \right)} \right]
\]

\[
\times \frac{1}{\sqrt{4rS_0^{2-\alpha}e^{rT(2-\alpha)} + \sigma^2 (3 - \alpha) \left( e^{rT(2-\alpha)} - 1 \right)}}
\]

\[
= Q_N \left[ \ln \left( \frac{S_0}{E} \right) + (r + \frac{1}{2}\sigma^2) T \right] \frac{1}{\sigma \sqrt{T}}
\]

\[
= Q_N (-d_1)
\]

(13)

and for the symmetry of the normal function, we have that:

\[
Q_N (-d_1) = N (d_1)
\]

being \( N(\cdot) \) the standard normal cumulative density.

The calculus of second \( Q \) function of the eq. \( 11 \) when the degrees of freedom tends to infinity, is:

\[
\lim_{a \to 2} Q \left( 2y; \frac{2}{2 - \alpha}, 2z \right) = Q_N \left[ \lim_{a \to 2} \frac{2y - \frac{2}{2 - \alpha} - 2x}{\sqrt{2 \left( \frac{2}{2 - \alpha} + 4x \right)}} \right]
\]

\[
= Q_N \left[ \lim_{a \to 2} \frac{2rS_0^{2-\alpha}e^{rT(2-\alpha)} - 2rE^{2-\alpha} - 2\sigma^2 (e^{rT(2-\alpha)} - 1)}{\sigma^2 (2 - \alpha) \left( e^{rT(2-\alpha)} - 1 \right)} \right]
\]

\[
\times \frac{1}{\sqrt{4rE^{2-\alpha} + \sigma^2 (e^{rT(2-\alpha)} - 1)}}
\]

\[
= Q_N \left[ \ln \left( \frac{S_0}{E} \right) + (r - \frac{1}{2}\sigma^2) T \right] \frac{1}{\sigma \sqrt{T}}
\]

\[
= Q_N (d_2)
\]

(14)

and for the identity between the cumulative and complementary function:
Later, at the limit \( \alpha \to 2 \), the European Call pricing of the CEV model converges to:

\[
\lim_{\alpha \to 2^-} C(S, 0) = S_0 N(d_1) - E e^{-rT} N(d_2)
\]

which is the classical Black-Scholes formula provided in [1].

3 A fractional CEV model

Now, for to address the ‘Joseph effect’ at the CEV environment, the standard Brownian motion of the eq. 1 is replaced by a fractional one:

\[
dS = rSdt + \sigma S^{\alpha/2} dB_H
\]

where \( B_H \) is a fractional Brownian motion with Hurst exponent \( H > 1/2 \).

Considering the shift of coordinates defined in the eq. 2, the fractional Itô’s formula [51, 52] yields to:

\[
dx = (2 - \alpha) \left[ rx + Ht^{2H-1} (1 - \alpha) \sigma^2 \right] dt + (2 - \alpha) \sigma \sqrt{x} dB_t
\]

Then, the fractional Fokker-Planck equation [53–55] related to the stochastic process defined in the eq. 16 is given by:

\[
\frac{\partial P_H}{\partial t} = \frac{\partial^2}{\partial x^2} \left[ Ht^{2H-1} (2 - \alpha)^2 \sigma^2 x P \right] - \frac{\partial}{\partial x} \left[ (2 - \alpha) \left( rx + Ht^{2H-1} (1 - \alpha) \sigma^2 \right) P \right]
\]

Unfortunately, the relation 16 can’t be solved using the Feller’s lemma because the coefficient are time-dependent (i.e., non-constant). However, Masoliver [56] provides an interesting approach for the non-stationary Feller process when the coefficients are time-dependent, and the main useful result for us is provided in the following statement:

Feller’s lemma with time-varying coefficients. Let \( u = u(x, \tau) \), \( A = A(\tau) \), \( C = C(\tau) \) and \( \theta \) constant defined by

\[
\theta = \frac{C(\tau)}{A(\tau)}
\]

The solution of the parabolic equation

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2 Following [13], a fBM is a Gaussian process which fulfills (for \( 0 < H < 1 \); \( t, s \geq 0 \)):

i) \( \mathbb{E} \left( B^H_t \right) = 0 \)

ii) \( \mathbb{E} \left( B^H_t \cdot B^H_s \right) = \frac{1}{2} \left\{ |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right\} \)

Then, for \( H > 1/2 \), the autocorrelation function of \( B_H \) is positive and decays hyperbolically in function of the lags, i.e., long range dependency: \( \sum_{n=1}^{\infty} \mathbb{E} \left[ B^H_t \cdot (B^H_{t+n} - B^H_n) \right] = \infty \).

3 Analogously to their classical counterpart, by the fractional Girsanov theorem (see [11, 12, 49]), the eq. 15 is wrote under the risk-neutral \( \mathbb{Q} \)–measure with drift \( r \), where \( B^H_t \) is a \( \mathbb{Q} \)–fractional Brownian motion.

4 The fractional Brownian motion is not a semi-martingale for \( H \neq 1/2 \); i.e., there is not an equivalent martingale measure. As pointed in [20], and despite the non-martingale condition, the \( \mathbb{Q} \)–expected discounted value is equal to the current value.
\[
\frac{\partial u}{\partial \tau} = \frac{\partial^2}{\partial x^2} [Axu] + \frac{\partial}{\partial x} [(x - C) u]
\]

conditional to

\[u(x, 0) = \delta(x - x_0)\]

is given by

\[u(x, \tau | x_0, 0) = \frac{1}{\phi(\tau)} \left( \frac{xe^\tau}{x_0} \right)^{\theta - 1} \frac{2}{2} \exp \left[ -\frac{(x + x_0e^{-\tau})}{\phi(\tau)} \right] I_{1-\theta} \left[ \frac{2}{\phi(\tau)} \sqrt{e^\tau x_0} \right]
\]

where \(I_k(x)\) is the modified Bessel function of the first kind of order \(k\) and

\[\phi(\tau) = \int_0^\tau A(\tau - s)e^{-s}ds\]

**Proof.** See ref. [56].

So, if we use the previous definitions of \(a, b, c\) (cf. page 3), and set:

\[\tau = -bt\]
\[A(\tau) = -a \frac{2H}{b} \left( -\frac{\tau}{b} \right)^{2H-1}\]
\[C(\tau) = -c \frac{2H}{b} \left( -\frac{\tau}{b} \right)^{2H-1}\]

the eq. 17 transform into:

\[
\frac{\partial P_H}{\partial \tau} = \frac{\partial^2}{\partial x^2} [A(\tau)xP] + \frac{\partial}{\partial x} [(x + C(\tau)) P]
\]

and \(\theta = c/a\), which is constant.

Then [21] can be solved by the *Feller’s lemma with time-varying coefficients*. Indeed:

\[P_H (x, \tau | x_0, 0) = \frac{1}{\phi(\tau)} \left( \frac{xe^\tau}{x_0} \right)^{c/a} \exp \left[ -\frac{(x + x_0e^{-\tau})}{\phi(\tau)} \right] I_{1-c/a} \left[ \frac{2}{\phi(\tau)} \sqrt{e^\tau x_0} \right]
\]

where:

\[\phi(\tau) = -\frac{a}{b} \int_0^\tau 2H \left( \frac{\tau - s}{b} \right)^{2H-1}e^{-s}ds = -\frac{a}{2H + 1} \left( -\frac{\tau}{b} \right)^{2H} \left[ 2H + 1 + e^{-1/2} (-\tau)^{-H} M_{H,H+1/2} (\tau) \right]
\]

and \(M_{\kappa,\upsilon} (l)\) is the M-Whittaker function [61–63], which could be expressed in terms of the \(M\) confluent hypergeometric Kummer’s function:

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5 The Whittaker (or the confluent hypergeometric) function appear in the solution of other related problems that involve the CEV process, see for example [57–60], among others.

7
$M_{n, v}(l) = l^{v+1/2}e^{-l/2}M\left(v - \frac{1}{2}, 1 + 2v; l\right)$

Now, solving for the original time-coordinate, at time $t = T$, we have:

$$P_H(x, T|x_0, 0) = \frac{1}{\phi(T)} \left(\frac{xe^{-bT}}{x_0}\right) \frac{c-a}{2a} \exp\left[-\frac{(x + x_0e^{bT})}{\phi(T)}\right] I_{1-c/a} \left[\frac{2}{\phi(T)} e^{bT}x_0x\right]$$

with

$$\phi(T) = \frac{a}{2H+1} T^{2H} \left[2H + 1 + e^{2bT} (bt)^{-H} M_{H,H+1/2} (bt)\right]$$

Later, moving to the original frame of reference $(S, t)$, and replacing the values for $a, b$ and $c$, the probability density function of $S(T) = S_T$, $T > 0$, given $S(0) = S_0$ is:

$$P_H(S_T|S_0, 0) = P_H(x_T, T|x_0, 0) \frac{\partial x_T}{\partial S_T} = (2 - \alpha) k_H^{-1} (yw^{1-2\alpha})^{\frac{1}{2(2-\alpha)}} e^{y-w} I_{1/(2-\alpha)} (2\sqrt{yw})$$

being:

$$k_H = [\phi(T)]^{-1},$$
$$y_H = k_H S_0^{2-\alpha} e^{(2-\alpha)T},$$
$$w_H = k_H S_0^{2-\alpha}$$

The transition probabilities $P$ (eq. [5]) and $P_H$ (eq. [22]) differ only by the terms $k$ and $k_H$. For the particular case $H = 1/2$, these terms are equal and yields to $P = P_H|_{H=1/2}$.

After that, an European option may be computed taking expectations of the discounted payoff (see appendix A). Fixing $z_H(l) = k_H(l)E^{2-\alpha}$, and following the development given from the eq. [6] to the eq. [11] the pricing for an European call pricing, in the fractal CEV, becomes:

$$C_H(S_0, 0) = S_0 Q \left(2z_H; 2 + \frac{2}{2 - \alpha}, 2y_H\right) - Ee^{-rT} \left[1 - Q \left(2y_H; \frac{2}{2 - \alpha}, 2z_H\right)\right]$$

Using the derivatives, asymptotics and recurrence properties of both $M_{n, v}(l)$ and $M$ [83], an interesting result from [26] is obtained computing the limit case $\alpha \to 2$. By [12], we get:

$$M_{1,2,1}(l) = \frac{2e^{-l/2} [e^l - (l + 1)]}{\sqrt{l}} \Rightarrow e^{l/2} (l)^{-1/2} M_{1,2,1}(l) = \frac{2}{l} [e^l - (l + 1)]$$
$$\Rightarrow \frac{1}{2b} \left[2 + e^{l/2} (l)^{-1/2} M_{1,2,1}(0)\right] = \frac{1}{b} (e^l - 1)$$
$$\Rightarrow \phi(t)|_{H=1/2} = \frac{a}{b} (e^{bT} - 1)$$
$$\Rightarrow k_H|_{H=1/2} = \frac{b}{a} (e^{bT} - 1) = k$$

$^6$Mainly, we use:
\[
\lim_{\alpha \to 2^-} Q \left( 2z_H; 2 - \frac{2}{2 - \alpha}, 2y_H \right) = Q_N \left[ \lim_{\alpha \to 2^-} \frac{2z_H - 2 - \frac{2}{2 - \alpha} - 2y_H}{\sqrt{2 \left( 2 + \frac{2}{2 - \alpha} + 4y_H \right)}} \right]
\]
\[
= Q_N \left[ \lim_{\alpha \to 2^-} \frac{E^{2 - \alpha \text{ alpha}} - \phi(T) \left( 1 + \frac{1}{2 - \alpha} \right) - S_0^{2 - \alpha \text{ alpha}}(2 - \alpha)T}{\sqrt{\phi(T) \left( 1 + \frac{1}{2 - \alpha} \right) + 2S_0^{2 - \alpha \text{ alpha}}(2 - \alpha)T}} \right]
\]
\[
= Q_N \left[ -\ln \left( \frac{S_0}{2} \right) + rT + \frac{1}{2} \sigma^2 T^2 \right]
\]
\[
= Q_N \left( -d_1^H \right)
\]
\[
= N \left( d_1^H \right) \quad (27)
\]

\[
\lim_{\alpha \to 2^-} Q \left( 2y_H; \frac{2}{2 - \alpha}, 2z_H \right) = Q_N \left[ \lim_{\alpha \to 2^-} \frac{2y_H - \frac{2}{2 - \alpha} - 2z_H}{\sqrt{\frac{2}{2 - \alpha} + 4z_H}} \right]
\]
\[
= Q_N \left[ \lim_{\alpha \to 2^-} \frac{\sqrt{2 - \alpha}}{\sqrt{2 - \alpha}} \frac{S_0^{2 - \alpha \text{ alpha}}(2 - \alpha)T - E^{2 - \alpha \text{ alpha}} - \phi(T) \left( \frac{1}{2 - \alpha} \right)}{\sqrt{\phi(T) \left( 1 + \frac{1}{2 - \alpha} \right) + 2S_0^{2 - \alpha \text{ alpha}}(2 - \alpha)T}} \right]
\]
\[
= Q_N \left[ -\ln \left( \frac{S_0}{2} \right) + rT - \frac{1}{2} \sigma^2 T^2 \right]
\]
\[
= Q_N \left( d_2^H \right)
\]
\[
= 1 - N \left( d_2^H \right) \quad (28)
\]

and replacing in \[26\] we arrive to the fractional Black-Scholes formula \[\[11, 12\]:

\[
C_H (S, 0) = S_0 N \left( d_1^H \right) - e^{-rT} N \left( d_2^H \right)
\]

Then, the convergence of the CEV to the Black-Scholes model, in the limit case \(\alpha \to 2\), remains in the fractional scheme.

The Fig. 1 plot the values of the fractional CEV formula for \(\sigma\) equal to 15% (blue) and 30% (red), and three values of \(H=\{0.5, 0.7, 0.9\}\) considering different maturities. The solid lines draw the case \(H = 1/2\), which correspond to the classical CEV pricing. The semi-solid and dotted lines show the pricing using \(H = 0.7\) and \(H = 0.9\), respectively. The fractional CEV retain the property of be a monotonically increasing function of the elasticity parameter. Also, is a rising function of \(\sigma\) and \(T\). For expiration times below the year \(\ln(1)\), the option price fall if \(H\) moves to 1. In the opposite way, for \(T > 1\) (cases \[1c, 1d\]), the prices grow if \(H\) rise in the interval \([1/2, 1]\). When \(\alpha = 2\), the fractional CEV pricing transform into the fractional Black-Scholes price.

- \(\lim_{\alpha \to 0} M_{\alpha, \nu} (l) = 0\)
- \(\frac{\partial}{\partial l} M_{\alpha, \nu} (l) = \left( \frac{1}{2} - \frac{\alpha}{2} \right) M_{\alpha, \nu} (l) + l^{-1} \left( \frac{1}{2} + \kappa + \nu \right) M_{\alpha + 1, \nu} (l)\)
- \(M_{\alpha, \alpha + 1/2} (l) = M_{\alpha, \alpha + 1/2} (l) + l^{\alpha + 1} e^{-l/2}\)
Figure 1: Fractional CEV formula for an European Call with $\sigma = \{10\%; 30\%\}$ and $H = \{0.5; 0.7; 0.9\}$ and different maturities in function of the elasticity parameter, fixing $S_0 = E = 100$ and $r = 5\%$. 
4 A mixed-fractional CEV model

A mixed-Brownian motion, is defined as a linear combination between both an standard and an independent fractional Brownian motions:

\[ M^{\beta,\gamma,H}_t = \beta B_t + \gamma B^H_t; \quad \beta \geq 0, \gamma \geq 0, \]

Then, for to extent the CEV model to the mixed-fractional case, the (fractional) Brownian motion which drives the (fractional) CEV model is replaced by \( M^{H,\beta}_t \):

\[ dS = rS dt + \sigma S^{\alpha/2} dM^{\beta,\gamma,H}_t \]

Is clear that if \((\beta, \gamma) = (0, 1)\) we recover the fractional case studied in the previous section. Also, if \(\beta = 1\) and \(\gamma = 0\), the eq. 29 describes the classical CEV model (Sec. 2).

Analogous to the previous cases, the transformation \(2\) and the fractional Itô’s lemma goes the eq. 29 to:

\[ dx = (2 - \alpha) \left[ rx + \left( \frac{1}{2} \beta + \gamma H t^{2H-1} \right) (1 - \alpha) \sigma^2 \right] dt + (2 - \alpha) \sigma \sqrt{\frac{e^{-\tau}}{x}} (\beta dB_t + \gamma dB^H_t) \]

Since that \( B_t = B^{1/2}_t \), the fractional Fokker-Planck equation for the above process is:

\[ \frac{\partial P_M}{\partial \tau} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[ (\beta + 2 \gamma H t^{2H-1}) (2 - \alpha)^2 \sigma^2 x P \right] - \frac{\partial}{\partial x} \left\{ (2 - \alpha) \left[ rx + (\beta + 2 \gamma H t^{2H-1}) \left( \frac{1 - \alpha}{2} \right) \sigma^2 \right] P \right\} \]

Setting:

\[ A'(\tau) = \gamma A(\tau) - \frac{a}{\beta} \]
\[ C'(\tau) = \gamma C(\tau) - \frac{a}{\beta} \]
\[ \theta' = \frac{C'(\tau)}{A'(\tau)} \]

the relation 30 becomes:

\[ \frac{\partial P_M}{\partial \tau} = \frac{\partial^2}{\partial x^2} [A'(\tau) x P] + \frac{\partial}{\partial x} \left\{ (2 - \alpha) [x - C'(\tau)] P \right\} \]

where \( A, C \) are given by the eqs. 18-20.

Given that \( \theta' = \theta \) (constant), a solution for 31 is obtained through the Feller’s lemma with time-varying coefficients:

\[ P_M (x, \tau | x_0, 0) = \frac{1}{\phi(\tau)} \left( \frac{xe^{\gamma}}{x_0} \right)^{2a/\sigma} \exp \left[ -\frac{(x + x_0 e^{-\gamma})}{\phi(\tau)} \right] I_{1-c/a} \left[ \frac{2}{\phi(\tau)} \sqrt{e^{-\tau} x_0 x} \right] \]

and

\[ \phi(\tau) = -\frac{a}{\beta} \int_0^\tau \gamma \left[ 2H \left( \frac{s - \tau}{b} \right)^{2H-1} - \beta \right] e^{-s} ds \]
\[ = \gamma \frac{a}{2H + 1} \left( \frac{\tau}{b} \right)^{2H} \left[ 2H + 1 + e^{-\frac{\tau}{b}} (-\tau)^{-H} M_{H,H+1/2} (\tau) \right] + \beta \frac{a}{b} (e^{-\tau} - 1) \]
Later, the transition probability density function for $S_T$ condition to $S_0$ under a mixed-fractional regime is given by:

$$P_M(S_T, T|S_0, 0) = (2 - \alpha)k_M^{\frac{2}{1-2\alpha}}(yw^{1-2\alpha})^{\frac{1}{1-2\alpha}}e^{-yw}I_{1/(2-\alpha)}(2\sqrt{yw})$$

being:

$$k_M = [\phi'(T)]^{-1},$$
$$y_M = k_M S_0^{2-\alpha}e^{\phi(2-\alpha)T},$$
$$w_M = k_M S_T^{2-\alpha}$$

with,

$$\phi'(T) = \frac{a}{2H + 1}T^{2H}\left[2H + 1 + e^{2bt} (bt)^{-H} M_{H,H+1/2}(bt)\right] + \beta \frac{a}{b} (e^{-\tau} - 1)$$

$$= \gamma\phi(T) + \beta \frac{a}{b} (e^{-\tau} - 1)$$

Using the argument provided in the previous sections, the European Call price, at time $t=0$ under the mixed-fractional CEV framework is computed by:

$$C_M(S_0, 0) = S_0 Q\left(2z_M; 2 + \frac{2}{2 - \alpha}, 2y_M\right) - Ee^{-rT}\left[1 - Q\left(2y_M; \frac{2}{2 - \alpha}, 2z_M\right)\right]$$

where $z_H(t) = k_H(t)E^{2-\alpha}$.

The eq. 35 turns into the pure fractional CEV case (eq. 26) if $(\beta, \gamma) = (1, 0)$, and becomes the classical CEV model if $(\beta, \gamma) = (1, 0)$ or $(\beta, \gamma, H) = (0, 1, 1/2)$.

Using the results computed in the eqs. 13, 14, 27 and 28; at the limit case $\alpha \to 2$, the sol. 35 yields to:

$$\lim_{\alpha \to 2^{-}} C_M(S_0, 0) = S_0N\left(d_1^M\right) - Ee^{-rT}N\left(d_2^M\right)$$

with

$$d_1^M = \frac{\ln \left(\frac{S_0}{F} + rT + \frac{1}{2}\beta\sigma^2 T + \frac{1}{2}\gamma \sigma^2 T^{2H} \right) }{\sigma \sqrt{\beta T + \gamma T^{2H}}}$$

$$d_2^M = \frac{\ln \left(\frac{S_0}{F} + rT - \frac{1}{2}\beta\sigma^2 T - \frac{1}{2}\gamma \sigma^2 T^{2H} \right) }{\sigma \sqrt{\beta T + \gamma T^{2H}}}$$

which is the mixed-fractional Black-Scholes pricing formula (for instance, see [24] with $\beta = 1$ and $\gamma = 1$); keeping the convergence between the CEV and Black-Scholes.

The Fig. 2 display the value for an European Call under the mixed-fractional CEV model, setting the pair of coefficients $\beta$ and $\gamma$ as $(\beta, \gamma) = (1, 1)$ and $(\beta, \gamma) = (0, 1)$ (blue and red respectively), with the aim of to compare the mixed fractional and pure fractional cases. We consider two maturities $T=0.5$ and $T=1.5$ and $H \in \{0.5; 0.7; 0.9\}$. In both subplots the classical CEV pricing is represented by the red-solid-line. We can observe that the mixed-fractional price is higher than the both classical and fractional price. As in the pure fractional case, for $T<1$ the price decrease if $H$ tends to one, and increase for $T>1$ and $H$ moves to one.
5 Greeks

For to analyze the sensitivities of the pricing formula in function of the parameters of the model, we compare the greeks of both classical, fractional and mixed-fractional CEV models.

The most common sensitivities are related to the price, maturity, volatility and interest rate. We use the results given in ref. [64] for \( \Delta \), \( \Gamma \), \( \nu \), \( \Theta \) and \( \rho \) greeks under the classical CEV model and here we carefully extend it to the fractional cases.

5.1 Delta

\[
\Delta = \frac{\partial C}{\partial S} = Q \left( 2z, 2 + \frac{2}{2 - \alpha}, 2y \right) + \frac{2y (2 - \alpha)}{S} \left[ S f \left( 2z; 4 + \frac{2}{2 - \beta}, 2y \right) - E e^{-rT} f \left( 2y; \frac{2}{2 - \beta}, 2z \right) \right] \tag{36}
\]

where \( f \) is the non-central-\( \chi^2 \) PDF defined in [10].

For the fractional case, we have:

\[
\Delta_H = \frac{\partial C_H}{\partial S} = Q \left( 2z_H, 2 + \frac{2}{2 - \alpha}, 2y_H \right) + \frac{2y_H (2 - \alpha)}{S} \left[ S f \left( 2z_H; 4 + \frac{2}{2 - \beta}, 2y_H \right) - E e^{-rT} f \left( 2y_H; \frac{2}{2 - \beta}, 2z_H \right) \right] \tag{37}
\]

and in the mixed-fractional:

\[
\Delta_M = \frac{\partial C_M}{\partial S} = Q \left( 2z_M, 2 + \frac{2}{2 - \alpha}, 2y_M \right) + \frac{2y_M (2 - \alpha)}{S} \left[ S f \left( 2z_M; 4 + \frac{2}{2 - \beta}, 2y_M \right) - E e^{-rT} f \left( 2y_M; \frac{2}{2 - \beta}, 2z_M \right) \right] \tag{38}
\]

The charts at the Figure 3 show the behavior of the \( \Delta, \Delta_H \) and \( \Delta_M \) varying the spot price (3a) and elasticity (3b) for maturities below and above the unity. The solid blue line correspond the classical CEV model.
Figure 3: Delta for fractional (blue) and mixed-fractional (red) models setting $E=100$, $\sigma = 20\%$, $r = 5\%$ and $\gamma = \beta = 1$.

5.2 Gamma

The eqs. 39, 40 and 41 provides the Gamma sensitivity for standard, fractional and mixed-fractional CEV, respectively, plotting it at Fig. 4.

\[
\Gamma = \frac{\partial^2 C}{\partial S^2} \\
= \frac{\partial}{\partial S} \Delta \\
= \frac{2y(2-\alpha)^2}{S} \left\{ \left[ \frac{3-\alpha}{2-\alpha} - y \right] f\left(2z; 4 + \frac{2}{2-\beta}, 2y\right) + y f\left(2z; 6 + \frac{2}{2-\beta}, 2y\right) \right\} \\
+ \frac{2y(2-\alpha)^2}{S^2} E e^{-rT} \left[ y f\left(2y; 2 + \frac{2}{2-\beta}, 2z\right) - z f\left(2y; 2 + \frac{2}{2-\beta}, 2z\right) \right] \\
\]

(39)

\[
\Gamma_H = \frac{\partial^2 C_H}{\partial S^2} \\
= \frac{2y_H(2-\alpha)^2}{S} \left\{ \left[ \frac{3-\alpha}{2-\alpha} - y_H \right] f\left(2z_H; 4 + \frac{2}{2-\beta}, 2y_H\right) + y_H f\left(2z_H; 6 + \frac{2}{2-\beta}, 2y_H\right) \right\} \\
+ \frac{2y_H(2-\alpha)^2}{S^2} E e^{-rT} \left[ y_H f\left(2y_H; 2 + \frac{2}{2-\beta}, 2z_H\right) - z_H f\left(2y_H; 2 + \frac{2}{2-\beta}, 2z_H\right) \right] \\
\]

(40)

\[
\Gamma_M = \frac{\partial^2 C_M}{\partial S^2} \\
= \frac{2y_M(2-\alpha)^2}{S} \left\{ \left[ \frac{3-\alpha}{2-\alpha} - y_M \right] f\left(2z_M; 4 + \frac{2}{2-\beta}, 2y_M\right) + y_M f\left(2z_M; 6 + \frac{2}{2-\beta}, 2y_M\right) \right\} \\
+ \frac{2y_M(2-\alpha)^2}{S^2} E e^{-rT} \left[ y_M f\left(2y_M; 2 + \frac{2}{2-\beta}, 2z_M\right) - z_M f\left(2y_M; 2 + \frac{2}{2-\beta}, 2y_M\right) \right] \\
\]

(41)
Figure 4: Gamma for fractional (blue) and mixed-fractional (red) models setting $E=100$, $\sigma = 20\%$, $r = 5\%$ and $\gamma = \beta = 1$.

5.3 Vega

The partial derivative with respect to the volatility is called Vega (commonly represented by the greek $\nu$). In the CEV model, the volatility is a function of both $S$ and the parameter $\sigma$, defined by $\sigma^2 = \sigma S^{2-\alpha}$. Then the Vega for the CEV model is:

$$\nu = \frac{\partial C}{\partial \sigma}; \quad \sigma^2 = \sigma^2 S^{2-\alpha}$$

$$= \frac{\partial C}{\partial \sigma} \frac{\partial (2y)}{\partial k} \left\{ \frac{-2k}{\sigma} S^{2-\alpha} \right\}$$

$$= \left\{ \frac{\partial C}{\partial (2y)} \frac{\partial (2y)}{\partial k} \right\} \left\{ \frac{-2k}{\sigma} S^{2-\alpha} \right\}$$

$$= \frac{-4y}{\sigma} \left\{ Sf \left( 2z, 4 + \frac{2}{2-2\alpha}, 2y \right) - Ee^{-rT}f \left( 2y, \frac{2}{2-\alpha}, 2z \right) \right\}$$ (42)

For the fractional and mixed-fractional models, we have:

$$\nu_H = \frac{\partial C_H}{\partial \sigma}$$

$$= \left\{ \frac{\partial C_H}{\partial (2y_H)} \frac{\partial (2y_H)}{\partial k_H} \right\} \left\{ \frac{-2k_H}{\sigma} S^{2-\alpha} \right\}$$

$$= \frac{-4y_H}{\sigma} \left\{ Sf \left( 2z_H, 4 + \frac{2}{2-2\alpha}, 2y_H \right) - Ee^{-rT}f \left( 2y_H, \frac{2}{2-\alpha}, 2z_H \right) \right\}$$ (43)

$$\nu_M = \frac{\partial C_M}{\partial \sigma}$$

$$= \left\{ \frac{\partial C_M}{\partial (2y_M)} \frac{\partial (2y_M)}{\partial k_M} \right\} \left\{ \frac{-2k_M}{\sigma} S^{2-\alpha} \right\}$$

$$= \frac{-4y_M}{\sigma} \left\{ Sf \left( 2z_M, 4 + \frac{2}{2-2\alpha}, 2y_M \right) - Ee^{-rT}f \left( 2y_M, \frac{2}{2-\alpha}, 2z_M \right) \right\}$$ (44)

The graphics in the Fig. 5 display the Vega for the fractional and mixed-fractional models. The standard CEV Vega is plotted by the solid-blue line.
5.4 Theta

The change rate of the option price with respect to the maturity $T$ is noted by the greek $\Theta$, and is computed for the CEV model at the eq. 45, for the fractional CEV at eq. 46 and for the mixed-fractional CEV at eq. 47. The Fig. 6 draw the shape of $\Theta_H$ and $\Theta_M$ under different values of $T$ (6a) and alpha (6b).

\[ \Theta = \frac{\partial C}{\partial T} \]
\[ = S \frac{\partial Q(2y, 2 + \frac{2}{2 - \alpha}, 2y) \partial(2y)}{\partial(2y)} + rE^{-rT} \left[1 - Q(2y, 2/(2 - \alpha), 2\sigma)\right] \]
\[ + E^{-rT} \frac{\partial Q(2y, 2/(2 - \alpha), \sigma)}{\partial(2y)} \frac{\partial(2y)}{\partial T} \]
\[ = -2yr (2 - \alpha) \frac{\sigma^2}{\phi(T)} \left[ Sf \left(2, 4 + \frac{2}{2 - \alpha}, 2y\right) - E^{-rT} f \left(2, \frac{2}{2 - \alpha}, 2\sigma\right)\right] \]
\[ + rE^{-rT} \left[1 - Q(2y, 2/(2 - \alpha), 2\sigma)\right] \] (45)

\[ \Theta_H = \frac{\partial C_H}{\partial T} \]
\[ = S \frac{\partial Q(2y, 2 + \frac{2}{2 - \alpha}, 2y_H) \partial(2y_H)}{\partial(2y_H)} + rE^{-rT} \left[1 - Q(2y_H, 2/(2 - \alpha), 2z_H)\right] \]
\[ + E^{-rT} \frac{\partial Q(2y_H, 2/(2 - \alpha), 2z_H)}{\partial(2y_H)} \frac{\partial(2y_H)}{\partial T} \]
\[ = -2y_HHT^{2H-1} \frac{\alpha^2 \sigma^2}{\phi(T)} \left[ Sf \left(2z_H, 4 + \frac{2}{2 - \alpha}, 2y_H\right) - E^{-rT} f \left(2y_H, \frac{2}{2 - \alpha}, 2z_H\right)\right] \]
\[ + rE^{-rT} \left[1 - Q(2y_H, 2/(2 - \alpha), 2z_H)\right] \] (46)
Figure 6: Theta for fractional (blue) and mixed-fractional (red) models setting $S = E = 100$, $r = 5\%$, $\sigma = 20\%$ and $\gamma = \beta = 1$. 

\begin{align*}
\Theta_M &= \frac{\partial C_M}{\partial T} \\
&= -2y_M \frac{\sigma^2 (2-\alpha)^2}{\phi'(T)} \left[ \gamma HT^{2H-1} + \frac{\beta}{2} \right] \left[ S f \left( 2z_M, 4 + \frac{2}{2-\alpha}, 2y_M \right) \\
&- E e^{-rT} f \left( 2y_M, \frac{2}{2-\alpha}, 2z_M \right) \right] + T E e^{-rT} \left[ 1 - Q \left( 2y_M, 2/(2-\alpha), 2z_M \right) \right] 
\end{align*}

(47)

5.5 Rho

Finally, the sensitivity respect to the risk-free-interest rate, for the CEV model and its extentions (fractional and mixed-fractional), are shown at the Fig. 7 and explicitly computed at the eqs. 48-49-50.

\begin{align*}
\rho &= \frac{\partial C}{\partial r} \\
&= S \frac{\partial Q (2z, 2 + 2/(2-\alpha), 2y) \partial(2y)}{\partial r} + T E e^{-rT} \left[ 1 - Q \left( 2y, 2/(2-\alpha), 2z \right) \right] \\
&+ E e^{-rT} \frac{\partial Q (2y, 2/(2-\alpha), 2z) \partial(2y)}{\partial T} \\
&= 2y \left[ \frac{1}{r} - \frac{(2-\alpha) T}{e^{r(2-\alpha)T} - 1} \right] \left[ S f \left( 2z, 4 + \frac{2}{2-\alpha}, 2y \right) - E e^{-rT} \left( 2y, \frac{2}{2-\alpha}, 2z \right) \right] \\
&+ T E e^{-rT} \left[ 1 - Q \left( 2y, 2/(2-\alpha), 2z \right) \right] 
\end{align*}

(48)
Figure 7: Rho for fractional (blue) and mixed-fractional (red) models setting $S = E=100$, $\sigma = 20\%$ and $\gamma = \beta = 1$. 

(a) $r \in [0.01, 0.2]$ and $\alpha = 1.5$

(b) $r = 5\%$ and $\alpha \in [1, 2]$
and mixed-fractional models, could be interpreted as a generalization of the classical CEV and Black-
Scholes model. Besides, the Greeks are computed showing its behavior under different values of the Hurst
exponent and considering maturities lower and greater than one.
Since the added terms on the call formula in both fractional models, in relation to the classical CEV, doesn’t have a dependency on the strike price, the fractional and mixed-fractional CEV keep the capability
for to address the smile-skew issue.

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A  Risk-neutral pricing in the fractional CEV model

As pointed in [11, 12], the fractional Clark-Ocone theorem and quasi-expectations are used for to price a derivative under fractional Brownian motion. Here, we use the derivation of Hu and Øksendal [11] for to obtain a risk-neutral pricing at time $t = 0$, but using a fractional CEV approach instead a fractional GBM.

Since the market is complete, at time $t$, a derivative $F$ is replicated by:

$$ F(t) = a(t)B(t) + b(t)S(t) $$

where $a$ and $b$ are weights, $B$ is a money bank account (bond) which pays a continuously composed interest rate $r$ (risk-less interest rate; i.e, $dB(t) = rB(t)dt$) and $S$ is ruled by the eq. [15]

Later:

$$ dF(t) = a(t)dB(t) + b(t)dS(t) $$

$$ = a(t)rB(t)dt + b(t)\left[rS(t)dt + \sigma(S(t))^{\alpha/2}dB_H^t\right] $$

$$ = r\left[a(t)B(t) + b(t)S(t)\right]dt + b(t)\sigma(S(t))^{\alpha/2}dB_H^t $$

$$ = rF(t)dt + b(t)\sigma S(t)dB_H^t $$

(52)

Multiplying [52] by $e^{-rt}$ and integrating it from zero to $t$, we get:

$$ e^{-rt}F(t) = F(0) + \int_0^t e^{-rt}b(t)\sigma(S(t))^{\alpha/2}dB_H^t $$

(53)

On the other hand, the Clark-Ocone theorem for standard Brownian motions is given by [65]:

$$ G(t) = E[G(T)] + \int_0^T D_tE[G(T)|\mathcal{F}_t]dB_t $$

22
where $D_t$ is the Malliavin derivative and $\mathcal{F}_t$ is the natural filtration of Brownian motion. The fractional extension of the eq. is provided by refs. [11, 66]:

$$G(t) = \mathbb{E}[G(T)] + \int_0^T D_t \mathbb{E}[G(t)|\mathcal{F}_t^H] \, dB_t^H$$

(54)

being $\mathcal{F}_t^H$ the $\sigma$-algebra generated by $B_s^H$, $s \leq t$. Put $G(t) = e^{-rt}F(t)$ in (54)

$$e^{-rt}F(t) = \mathbb{E}[e^{-rt}F(T)] + \int_0^T D_t \mathbb{E}[e^{-rt}F(t)|\mathcal{F}_t^H] \, dB_t^H$$

(55)

Comparing the expressions 53-55 we arrive to the completeness of the market by:

$$D_t \mathbb{E}[F(t)|\mathcal{F}_t^H] = b(t)\sigma (S(t))^{\alpha/2}$$

and

$$F(0) = \mathbb{E}[e^{-rt}F(T)]$$

(56)

Then, at the initial the price of a derivative is computed as the discounted price of the expected value, as in the classical model driven by a Brownian motion.

For the mixed-fractional case, the extension is straightforward and is shown in [23, 67] for the mixed GBM.