CYCLIC COHOMOLOGY OF CERTAIN NUCLEAR FRÉCHET AND DF ALGEBRAS

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Abstract. We give explicit formulae for the continuous Hochschild and cyclic homology and cohomology of certain $\otimes$-algebras. We use well-developed homological techniques together with some niceties of the theory of locally convex spaces to generalize the results known in the case of Banach algebras and their inverse limits to wider classes of topological algebras. To this end we show that, for a continuous morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ of complexes of complete nuclear $DF$-spaces, the isomorphism of cohomology groups $H^n(\varphi) : H^n(\mathcal{X}) \to H^n(\mathcal{Y})$ is automatically topological. The continuous cyclic-type homology and cohomology are described up to topological isomorphism for the following classes of biprojective $\otimes$-algebras: the tensor algebra $E \otimes F$ generated by the duality $(E, F, \langle \cdot, \cdot \rangle)$ for nuclear Fréchet spaces $E$ and $F$ or for nuclear $DF$-spaces $E$ and $F$; nuclear biprojective Köthe algebras $\lambda(P)$ which are Fréchet spaces or $DF$-spaces; the algebra of distributions $E^*(G)$ on a compact Lie group $G$.

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1. Introduction

Cyclic cohomology groups of topological algebras play an essential role in noncommutative geometry [2]. There has been a number of papers addressing the calculation of cyclic-type continuous homology and cohomology groups of some Banach, $C^*$- and topological algebras; see, e.g., [2, 8, 14, 17, 18, 20, 32]. However, it remains difficult to describe these groups explicitly for many topological algebras. To compute the continuous Hochschild and cyclic cohomology groups of Fréchet algebras one has to deal with complexes of complete $DF$-spaces. Here, in addition to presenting known homological techniques we also supply technical enhancements that permit the necessary generalization of results known in the case of Banach algebras and their inverse limits to wider classes of topological algebras notably to those that occur in noncommutative geometry.

The category of Banach spaces has the useful property that it is closed under passage to dual spaces. Fréchet spaces do not have this property: the strong dual...
of a Fréchet space is a complete \( DF \)-space. \( DF \)-spaces have the awkward feature that their closed subspaces need not be \( DF \)-spaces. However, closed subspaces of complete \textit{nuclear} \( DF \)-spaces are again \( DF \)-spaces \cite[Proposition 5.1.7]{21}.

In Section 3 we use the strongest known results on the open mapping theorem to give sufficient conditions on topological spaces \( E \) and \( F \) to imply that any continuous linear operator \( T \) from \( E^* \) onto \( F^* \) is open. This allows us to prove the following results. In Lemma \( \text{3.6} \) we show that, for a continuous morphism \( \varphi : \mathcal{X} \to \mathcal{Y} \) of complexes of complete nuclear \( DF \)-spaces, the isomorphism of cohomology groups \( H^n(\varphi) : H^n(\mathcal{X}) \to H^n(\mathcal{Y}) \) is automatically topological.

We use this fact to describe explicitly up to \textit{topological} isomorphism the continuous Hochschild and cyclic cohomology groups of nuclear \( \hat{\otimes} \)-algebras \( \mathcal{A} \) which are Fréchet spaces or \( DF \)-spaces and have trivial Hochschild homology \( \mathcal{H}H_n(\mathcal{A}) \) for all \( n \geq 1 \) (Theorem \( \text{4.3} \)). In Proposition \( \text{4.4} \) under the same condition on \( \mathcal{H}H_n(\mathcal{A}) \), we give explicit formulae, up to isomorphism of linear spaces, for continuous cyclic-type homology of \( \mathcal{A} \) in a more general category of underlying spaces.

In Theorem \( \text{6.8} \) the continuous cyclic-type homology and cohomology groups are described up to topological isomorphism for the following classes of biprojective \( \hat{\otimes} \)-algebras: the tensor algebra \( E \hat{\otimes} F \) generated by the duality \( (E, F, \langle \cdot, \cdot \rangle) \) for nuclear Fréchet spaces or for nuclear complete \( DF \)-spaces \( E \) and \( F \); nuclear biprojective Fréchet Köthe algebras \( \lambda(P) \); nuclear biprojective Köthe algebras \( \lambda(P)^* \) which are \( DF \)-spaces; the algebra of distributions \( \mathcal{E}^*(G) \) and the algebra of smooth functions \( \mathcal{E}(G) \) on a compact Lie group \( G \).

2. Definitions and notation

We recall some notation and terminology used in homology and in the theory of topological algebras. Homological theory can be found in any relevant textbook, for instance, Loday \cite{16} for the pure algebraic case and Helemskii \cite{7} for the continuous case.

Throughout the paper \( \hat{\otimes} \) is the projective tensor product of complete locally convex spaces, by \( X \hat{\otimes}^n \) we mean the \( n \)-fold projective tensor power \( X \hat{\otimes} \cdots \hat{\otimes} X \) of \( X \), and \( \text{id} \) denotes the identity operator.

We use the notation \( \mathcal{B}an, \mathcal{F}r \) and \( \mathcal{L}CS \) for the categories whose objects are Banach spaces, Fréchet spaces and complete Hausdorff locally convex spaces respectively, and whose morphisms in all cases are continuous linear operators. For topological homology theory it is important to find a suitable category for the underlying spaces of the algebras and modules. In \cite{7} Helemskii constructed homology theory for the following categories \( \Phi \) of underlying spaces, for which he used the notation \( (\Phi, \hat{\otimes}) \).

**Definition 2.1.** (\cite[Section II.5]{7}) A suitable category \textit{for underlying spaces of the algebras and modules} is an arbitrary complete subcategory \( \Phi \) of \( \mathcal{L}CS \) having the following properties:

(i) if \( \Phi \) contains a space, it also contains all those spaces topologically isomorphic to it;
(ii) if \( \Phi \) contains a space, it also contains any of its closed subspaces and the completion of any its Hausdorff quotient spaces;

(iii) \( \Phi \) contains the direct sum and the projective tensor product of any pair of its spaces;

(iv) \( \Phi \) contains \( \mathbb{C} \).

Besides \( \mathcal{B}an, \mathcal{F}r \) and \( \mathcal{L}CS \) important examples of suitable categories \( \Phi \) are the categories of complete nuclear spaces \([31, \text{Proposition 50.1}]\), nuclear Fréchet spaces and complete nuclear \( DF \)-spaces. As to the above properties for the category of complete nuclear \( DF \)-spaces, recall the following results. By \([12, \text{Theorem 15.6.2}]\), if \( E \) and \( F \) are complete \( DF \)-spaces, then \( E \hat{\otimes} F \) is a complete \( DF \)-space. By \([21, \text{Proposition 5.1.7}]\), a closed linear subspace of a complete nuclear \( DF \)-space is also a complete nuclear \( DF \)-space. By \([21, \text{Proposition 5.1.8}]\), each quotient space of a complete nuclear \( DF \)-space by a closed linear subspace is also a complete nuclear \( DF \)-space.

By definition a \( \hat{\otimes} \)-algebra is a complete Hausdorff locally convex algebra with jointly continuous multiplication. A left \( \hat{\otimes} \)-module \( X \) over a \( \hat{\otimes} \)-algebra \( A \) is a complete Hausdorff locally convex space \( X \) together with the structure of a left \( A \)-module such that the map \( A \times X \to X, (a, x) \mapsto a \cdot x \) is jointly continuous. For a \( \hat{\otimes} \)-algebra \( A \), \( \hat{\otimes}_A \) is the projective tensor product of left and right \( A \)-\( \hat{\otimes} \)-modules (see \([6], [7, \text{II.4.1}]\)). The category of left \( A \)-\( \hat{\otimes} \)-modules is denoted by \( A \)-mod and the category of \( A \)-\( \hat{\otimes} \)-bimodules is denoted by \( A \)-mod-A.

Let \( K \) be one of the above categories. A \textit{chain complex} \( X_\sim \) in the category \( K \) is a sequence of \( X_n \in K \) and morphisms \( d_n \)

\[
\cdots \leftarrow X_n \xleftarrow{d_n} X_{n+1} \xleftarrow{d_{n+1}} X_{n+2} \leftarrow \cdots
\]

such that \( d_n \circ d_{n+1} = 0 \) for every \( n \). The \textit{homology groups} of \( X_\sim \) are defined by

\[
H_n(X_\sim) = \text{Ker } d_{n-1} / \text{Im } d_n.
\]

A continuous morphism of chain complexes \( \psi_\sim : X_\sim \to P_\sim \) induces a continuous linear operator \( H_n(\psi_\sim) : H_n(X_\sim) \to H_n(P_\sim) \) \([9, \text{Definition 0.4.22}]\).

If \( E \) is a topological vector space \( E^* \) denotes its dual space of continuous linear functionals. Throughout the paper, \( E^* \) will always be equipped with the strong topology unless otherwise stated. The \textit{strong topology} is defined on \( E^* \) by taking as a basis of neighbourhoods of 0 the family of polars \( V^0 \) of all bounded subsets \( V \) of \( E \); see \([31, \text{II.19.2}]\).

For any \( \hat{\otimes} \)-algebra \( A \), not necessarily unital, \( A_+ \) is the \( \hat{\otimes} \)-algebra obtained by adjoining an identity to \( A \). For a \( \hat{\otimes} \)-algebra \( A \), the algebra \( A^e = A_+ \hat{\otimes} A_+^{op} \) is called the \textit{enveloping algebra} of \( A \), where \( A_+^{op} \) is the \textit{opposite algebra} of \( A_+ \) with multiplication \( a \cdot b = ba \).

A complex of \( A \)-\( \hat{\otimes} \)-modules and their morphisms is called \textit{admissible} if it splits as a complex in \( \mathcal{L}CS \) \([7, \text{III.1.11}]\). A module \( Y \in A \)-mod is called \textit{flat} if for any admissible complex \( X \) of right \( A \)-\( \hat{\otimes} \)-modules the complex \( X \hat{\otimes}_A Y \) is exact. A module \( Y \in A \)-mod-A is called \textit{flat} if for any admissible complex \( X \) of \( A \)-\( \hat{\otimes} \)-bimodules the
complex $X \hat{\otimes} \mathcal{A}^e Y$ is exact. For $Y, X \in \mathcal{A}$-mod-$\mathcal{A}$, we shall denote by $\text{Tor}_n^\mathcal{A}(X, Y)$ the $n$th homology of the complex $X \hat{\otimes} \mathcal{A}^e \mathcal{P}$, where $0 \leftarrow Y \leftarrow \mathcal{P}$ is a projective resolution of $Y$ in $\mathcal{A}$-mod-$\mathcal{A}$, [7, Definition III.4.23].

It is well known that the strong dual of a Fréchet space is a complete $DF$-space and that nuclear Fréchet spaces and complete nuclear $DF$-spaces are reflexive [21, Theorem 4.4.12]. Moreover, the correspondence $E \leftrightarrow E^*$ establishes a one-to-one relation between nuclear Fréchet spaces and complete nuclear $DF$-spaces [21, Theorem 4.4.13]. $DF$-spaces were introduced by A. Grothendieck in [5].

Further we shall need the following technical result which extends a result of Johnson for the Banach case [11, Corollary 1.3].

Proposition 2.2. Let $(\mathcal{X}, d)$ be a chain complex of
(a) Fréchet spaces and continuous linear operators, or
(b) complete nuclear $DF$-spaces and continuous linear operators,
and let $N \in \mathbb{N}$. Then the following statements are equivalent:
(i) $H_n(\mathcal{X}, d) = \{0\}$ for all $n \geq N$ and $H_{N-1}(\mathcal{X}, d)$ is Hausdorff;
(ii) $H^n(\mathcal{X}^*, d^*) = \{0\}$ for all $n \geq N$.

Proof. Recall that $H_n(\mathcal{X}, d) = \text{Ker} \ d_{n-1}/\text{Im} \ d_n$ and $H^n(\mathcal{X}^*, d^*) = \text{Ker} \ d^*_n/\text{Im} \ d^*_{n-1}$. Let $L$ be the closure of $\text{Im} \ d_{N-1}$ in $X_{N-1}$. Consider the following commutative diagram

$$
\begin{array}{c}
\begin{array}{c}
0 \leftarrow L \leftarrow X_N \xleftarrow{d_N} X_{N+1} \xleftarrow{d_{N+1}} \ldots \\
\downarrow i \quad \sqrt[d_{N-1}]{d_N} \\
X_{N-1}
\end{array}
\end{array}
$$

(1)

in which $i$ is the natural inclusion and $j$ is a corestriction of $d_{N-1}$. The dual commutative diagram is the following

$$
\begin{array}{c}
\begin{array}{c}
0 \rightarrow L^* \rightarrow X^*_N \xrightarrow{d^*_N} X^*_{N+1} \xrightarrow{d^*_{N+1}} \ldots \\
\uparrow i^* \quad \sqrt[d^*_{N-1}]{d^*_N} \\
X^*_{N-1}
\end{array}
\end{array}
$$

(2)

It is clear that $H_{N-1}(\mathcal{X}, d)$ is Hausdorff if and only if $j$ is surjective. Since $i$ is injective, condition (i) is equivalent to the exactness of diagram (1). On the other hand, by the Hahn-Banach theorem, $i^*$ is surjective. Thus condition (ii) is equivalent to the exactness of diagram (2).

In the case of Fréchet spaces, by [18, Lemma 2.3], the exactness of the complex (1) is equivalent to the exactness of the complex (2).

In the case of complete nuclear $DF$-spaces, by [21, Proposition 5.1.7], $L$ is the strong dual of a nuclear Fréchet space. By [21, Theorem 4.4.12], complete nuclear $DF$-spaces are reflexive, and therefore the complex (1) is the dual of the complex (2) of nuclear Fréchet spaces and continuous linear operators. By [18, Lemma 2.3], the exactness of the complex (1) is equivalent to the exactness of the complex (2). The proposition is proved. □
3. The open mapping theorem in complete nuclear DF-spaces

It is known that there exist closed linear subspaces of DF-spaces that are not DF-spaces. For nuclear spaces, however, we have the following.

Lemma 3.1. [21 Proposition 5.1.7] Each closed linear subspace $F$ of the strong dual $E^*$ of a nuclear Fréchet space $E$ is also the strong dual of a nuclear Fréchet space.

In a locally convex space a subset is called a barrel if it is absolutely convex, absorbent and closed. Every locally convex space has a neighbourhood base consisting of barrels. A locally convex space is called a barrelled space if every barrel is a neighbourhood [26]. By [26 Theorem IV.1.2], every Fréchet space is barrelled. By [26 Corollary IV.3.1], a Hausdorff locally convex space is reflexive if and only if it is barrelled and every bounded set is contained in a weakly compact set. Thus the strong dual of a nuclear Fréchet space is barrelled. For a generalization of the open mapping theorem to locally convex spaces, V. Pták introduced the notion of $B$-completeness in [24]. A subspace $Q$ of $E^*$ is said to be almost closed if, for each neighbourhood $U$ of 0 in $E$, $Q \cap U^0$ is closed in the relative weak* topology $\sigma(E^*, E)$ on $U^0$. A locally convex space $E$ is said to be $B$-complete or fully complete if each almost closed subspace of $E^*$ is closed in the weak* topology $\sigma(E^*, E)$.

Theorem 3.2. [24]. Let $E$ be a $B$-complete locally convex space and $F$ be a barrelled locally convex space. Then a continuous linear operator $f$ of $E$ onto $F$ is open.

Recall [10, Theorem 4.1.1] that a locally convex space $E$ is $B$-complete if and only if each linear continuous and almost open mapping $f$ of $E$ onto any locally convex space $F$ is open. By [10 Proposition 4.1.3], every Fréchet space is $B$-complete.

Theorem 3.3. Let $E$ be a semi-reflexive metrizable barrelled space, $F$ be a Hausdorff reflexive locally convex space and let $E^*$ and $F^*$ be the strong duals of $E$ and $F$ respectively. Then a continuous linear operator $T$ of $E^*$ onto $F^*$ is open.

Proof. By [10 Theorem 6.5.10] and by [10 Corollary 6.2.1], the strong dual $E^*$ of a semi-reflexive metrizable barrelled space $E$ is $B$-complete. By [26 Corollary IV.3.2], if a Hausdorff locally convex space is reflexive, so is its dual under the strong topology. By [26 Corollary IV.3.1], a Hausdorff reflexive locally convex space is barrelled. Hence $F^*$ is a barrelled locally convex space. Therefore, by Theorem 3.2 $T$ is open.

Corollary 3.4. Let $E$ and $F$ be nuclear Fréchet spaces and let $E^*$ and $F^*$ be the strong duals of $E$ and $F$ respectively. Then a continuous linear operator $T$ of $E^*$ onto $F^*$ is open.

For a continuous morphism of chain complexes $\psi : \mathcal{X} \to \mathcal{P}$ in $Fr$, a surjective map $H_n(\psi) : H_n(\mathcal{X}) \to H_n(\mathcal{Y})$ is automatically open, see [7 Lemma 0.5.9]. To get the corresponding result for dual complexes of Fréchet spaces one has to assume nuclearity.
Lemma 3.5. Let \( (\mathcal{X}, d_\mathcal{X}) \) and \( (\mathcal{Y}, d_\mathcal{Y}) \) be chain complexes of nuclear Fréchet spaces and continuous linear operators and let \( (\mathcal{X}^*, d_\mathcal{X}^*) \) and \( (\mathcal{Y}^*, d_\mathcal{Y}^*) \) be their strong dual complexes. Let \( \varphi : \mathcal{X}^* \to \mathcal{Y}^* \) be a continuous morphism of complexes. Suppose that
\[
\varphi_* = H^n(\varphi) : H^n(\mathcal{X}^*, d_\mathcal{X}^*) \to H^n(\mathcal{Y}^*, d_\mathcal{Y}^*)
\]
is surjective. Then \( \varphi_* \) is open.

Proof. Let \( \sigma_{\mathcal{Y}*} : \text{Ker} \, (d_{\mathcal{Y}})_n \to H^n(\mathcal{Y}^*, d_\mathcal{Y}^*) \) be the quotient map. Consider the map
\[
\psi : \text{Ker} \, (d_\mathcal{X}^*)_n \oplus Y_{n-1}^* \to \text{Ker} \, (d_{\mathcal{Y}})_n \subset Y_n^*
\]
given by \((x, y) \mapsto \varphi_n(x) + (d_{\mathcal{Y}})_n^{-1}(y)\).

By Lemma 3.1, \( \text{Ker} \, (d_\mathcal{X}^*)_n \) and \( \text{Ker} \, (d_{\mathcal{Y}})_n \) are the strong duals of nuclear Fréchet spaces and hence are barrelled. By [10, Theorem 6.5.10] and [10, Corollary 6.2.1], the strong dual of a semi-reflexive metrizable barrelled space is \( B \)-complete. Thus \( \text{Ker} \, (d_\mathcal{X}^*)_n, Y_{n-1}^* \) and
\[
\text{Ker} \, (d_\mathcal{X}^*)_n \oplus Y_{n-1}^* \cong [(\text{Ker} \, (d_\mathcal{X}^*)_n)^* \oplus Y_{n-1}]^*
\]
is \( B \)-complete. By assumption \( \varphi_* \) maps \( H^n(\mathcal{X}^*, d_\mathcal{X}^*) \) onto \( H^n(\mathcal{Y}^*, d_\mathcal{Y}^*) \), which implies that \( \psi \) is a surjective linear continuous operator from the \( B \)-complete locally convex space \( \text{Ker} \, (d_\mathcal{X}^*)_n \oplus Y_{n-1}^* \) to the barrelled locally convex space \( \text{Ker} \, (d_{\mathcal{Y}})_n \). Therefore, by Theorem 3.2 \( \psi \) is open. Consider the diagram
\[
\begin{array}{ccc}
\text{Ker} \, (d_\mathcal{X}^*)_n \oplus Y_{n-1}^* & \xrightarrow{j} & \text{Ker} \, (d_{\mathcal{Y}})_n \\
\downarrow \psi & & \downarrow \varphi_* \\
\text{Ker} \, (d_{\mathcal{Y}})_n & \xrightarrow{\sigma_{\mathcal{Y}*}} & H^n(\mathcal{Y}^*, d_\mathcal{Y}^*)
\end{array}
\]
(3)
in which \( j \) is a projection onto a direct summand and \( \sigma_{\mathcal{X}*}, \sigma_{\mathcal{Y}*-} \) and \( \sigma_{\mathcal{Y}*-} \) are the natural quotient maps. Obviously this diagram is commutative. Note that the projection \( j \) and quotient maps \( \sigma_{\mathcal{X}*}, \sigma_{\mathcal{Y}*-} \) are open. As \( \psi \) is also an open map, so is \( \sigma_{\mathcal{Y}*} \circ \psi = \varphi_* \circ \sigma_{\mathcal{Y}*} \circ j \). Since \( \sigma_{\mathcal{X}*} \circ j \) is continuous, \( \varphi_* \) is open. \( \square \)

Corollary 3.6. Let \( (\mathcal{X}, d_\mathcal{X}) \) and \( (\mathcal{Y}, d_\mathcal{Y}) \) be cochain complexes of complete nuclear DF-spaces and continuous linear operators, and let \( \varphi : \mathcal{X} \to \mathcal{Y} \) be a continuous morphism of complexes. Suppose that \( \varphi_* = H^n(\varphi) : H^n(\mathcal{X}, d_\mathcal{X}) \to H^n(\mathcal{Y}, d_\mathcal{Y}) \) is surjective. Then \( \varphi_* \) is open.

Proof. By [21] Theorem 4.4.13], \( (\mathcal{X}, d_\mathcal{X}) \) and \( (\mathcal{Y}, d_\mathcal{Y}) \) are strong duals of chain complexes \( (\mathcal{X}^*, d_\mathcal{X}^*) \) and \( (\mathcal{Y}^*, d_\mathcal{Y}^*) \) of nuclear Fréchet spaces and continuous operators. The result follows from Lemma 3.5. \( \square \)

4. **Cyclic and Hochschild cohomology of some \( \hat{\otimes} \)-algebras**

One can consult the books by Loday [16] or Connes [2] on cyclic-type homological theory.

Let \( \mathcal{A} \) be a \( \hat{\otimes} \)-algebra and let \( X \) be an \( \mathcal{A} \)-\( \hat{\otimes} \)-bimodule. We assume here that the category of underlying spaces \( \Phi \) has the properties from Definition 2.1. Let us recall the definition of the standard homological chain complex \( C_\sim(\mathcal{A}, X) \). For \( n \geq 0 \), let
$C_n(A, X)$ denote the projective tensor product $X \hat{\otimes} A^{\hat{n}}$. The elements of $C_n(A, X)$ are called $n$-chains. Let the differential $d_n : C_{n+1} \to C_n$ be given by

$$d_n(x \otimes a_1 \otimes \ldots \otimes a_{n+1}) = x \cdot a_1 \otimes \ldots \otimes a_{n+1} +$$

$$\sum_{k=1}^{n} (-1)^k (x \otimes a_1 \otimes \ldots \otimes a_1 a_{k+1} \otimes \ldots \otimes a_{n+1}) + (-1)^{n+1} (a_{n+1} \cdot x \otimes a_1 \otimes \ldots \otimes a_n)$$

with $d_{-1}$ the null map. The homology groups of this complex $H_n(C_n(A, X))$ are called the continuous Hochschild homology groups of $A$ with coefficients in $X$ and denoted by $H_n(A, X)$ [7, Definition II.5.28]. We also consider the cohomology groups $H^n((C_n(A, X))^\ast)$ of the dual complex $(C_n(A, X))^\ast$ with the strong dual topology. For Banach algebras $A$, $H^n((C_n(A, X))^\ast)$ is topologically isomorphic to the Hochschild cohomology $H^n(A, X^\ast)$ of $A$ with coefficients in the dual $A$-bimodule $X^\ast$ [7, Definition I.3.2 and Proposition II.5.27]. The weak bidimension of a Fréchet algebra $A$ is

$$db_wA = \inf\{n : H^{n+1}(C_n(A, X))^\ast = \{0\}\} \text{ for all Fréchet } A\text{-bimodules } X.$$}

The continuous bar and ‘naive’ Hochschild homology of a $\hat{\otimes}$-algebra $A$ are defined respectively as

$$H^\text{bar}_*(A) = H_*(C(A), b') \text{ and } H^\text{naive}_*(A) = H_*(C(A), b),$$

where $C_n(A) = A^{\hat{n}} \hat{\otimes} A^{\hat{n+1}}$, and the differentials $b, b'$ are given by

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$

and

$$b(a_0 \otimes \cdots \otimes a_n) = b'(a_0 \otimes \cdots \otimes a_n) + (-1)^n (a_n a_0 \otimes \cdots \otimes a_{n-1}).$$

Note that $H^\text{naive}_*(A)$ is just another way of writing $H_*(A, A)$, the continuous homology of $A$ with coefficients in $A$, as described in [7, 11].

There is a powerful method based on mixed complexes for the study of the cyclic-type homology groups; see papers by C. Kassel [13], J. Cuntz and D. Quillen [4] and J. Cuntz [3]. We shall present this method for the category $\mathcal{LCS}$ of locally convex spaces and continuous linear operators; see [1] for the category of Fréchet spaces. A mixed complex $(\mathcal{M}, b, B)$ in the category $\mathcal{LCS}$ is a family $\mathcal{M} = \{M_n\}_{n \geq 0}$ of locally convex spaces $M_n$ equipped with continuous linear operators $b_n : M_n \to M_{n-1}$ and $B_n : M_n \to M_{n+1}$, which satisfy the identities $b^2 = bB + BB = B^2 = 0$. We assume that in degree zero the differential $b$ is identically equal to zero. We arrange the
mixed complex \((\mathcal{M}, b, B)\) in the double complex

\[
\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
& b & b & b \\
M_2 & \leftarrow & M_1 & \leftarrow M_0 \\
& b & b & b \\
M_1 & \leftarrow & M_0 \\
& b & b \\
M_0 \\
\end{array}
\]  

(4)

There are three types of homology theory that can be naturally associated with a mixed complex. The Hochschild homology \(H^b_*(\mathcal{M})\) of \((\mathcal{M}, b, B)\) is the homology of the chain complex \((\mathcal{M}, b)\), that is,

\[
H^b_n(\mathcal{M}) = H^b_n(\mathcal{M}, b) = \text{Ker} \{b_n : M_n \rightarrow M_{n-1}\} / \text{Im} \{b_{n+1} : M_{n+1} \rightarrow M_n\}.
\]

To define the cyclic homology of \((\mathcal{M}, b, B)\), let us denote by \(B_c\mathcal{M}\) the total complex of the above double complex, that is,

\[
\cdots \rightarrow (B_c\mathcal{M})_n \xrightarrow{b+B} (B_c\mathcal{M})_{n-1} \rightarrow \cdots \rightarrow (B_c\mathcal{M})_0 \rightarrow 0,
\]

where the spaces

\[
(B_c\mathcal{M})_0 = M_0, \ldots, (B_c\mathcal{M})_{2k-1} = M_1 \oplus M_3 \oplus \cdots \oplus M_{2k-1}
\]

and

\[
(B_c\mathcal{M})_{2k} = M_0 \oplus M_2 \oplus \cdots \oplus M_{2k}
\]

are equipped with the product topology, and the continuous linear operators \(b+B\) are defined by

\[
(b+B)(y_0, \ldots, y_{2k}) = (by_2 + B y_0, \ldots, by_{2k} + B y_{2k-2})
\]

and

\[
(b+B)(y_1, \ldots, y_{2k+1}) = (by_1, \ldots, by_{2k+1} + B y_{2k-1}).
\]

The cyclic homology of \((\mathcal{M}, b, B)\) is defined to be \(H^*(B_c\mathcal{M}, b+B)\). It is denoted by \(H_c^*(\mathcal{M}, b, B)\).

The periodic cyclic homology of \((\mathcal{M}, b, B)\) is defined in terms of the complex

\[
\cdots \rightarrow (B_{p\mathcal{M}})_{ev} \xrightarrow{b+B} (B_{p\mathcal{M}})_{odd} \rightarrow (B_{p\mathcal{M}})_{ev} \rightarrow (B_{p\mathcal{M}})_{odd} \rightarrow \cdots,
\]

where even/odd chains are elements of the product spaces

\[
(B_{p\mathcal{M}})_{ev} = \prod_{n \geq 0} M_{2n} \quad \text{and} \quad (B_{p\mathcal{M}})_{odd} = \prod_{n \geq 0} M_{2n+1},
\]

respectively. The spaces \((B_{p\mathcal{M}})_{ev/odd}\) are locally convex spaces with respect to the product topology \([15\text{ Section } 18.3.(5)]\). The continuous differential \(b+B\) is defined as an obvious extension of the above. The periodic cyclic homology of \((\mathcal{M}, b, B)\) is \(H^\nu_p(\mathcal{M}, b, B) = H^\nu_*(B_{p\mathcal{M}}, b+B)\), where \(\nu \in \mathbb{Z}/2\mathbb{Z}\).

There are also three types of cyclic cohomology theory associated with the mixed complex, obtained when one replaces the chain complex of locally convex spaces
Then, for any even integer $N$ Proposition 4.2 that isomorphisms of (co)homology groups are automatically topological.

Consider the mixed complex $(\tilde{\Omega}A_+, \tilde{b}, \tilde{B})$, where $\tilde{b} = \left( \begin{array}{cc} b & 1 - \lambda \\ 0 & -b' \end{array} \right)$, and $\tilde{B} = \left( \begin{array}{cc} 0 & 0 \\ N & 0 \end{array} \right)$, where $\lambda(a_1 \otimes \cdots \otimes a_n) = (-1)^{n-1}(a_n \otimes a_1 \otimes \cdots \otimes a_{n-1})$ and $N = \text{id} + \lambda + \cdots + \lambda^{n-1}$ [16 1.4.5]. The continuous Hochschild homology of $A$, the continuous cyclic homology of $A$ and the continuous periodic cyclic homology of $A$ are defined by

$$\mathcal{H}_*(A) = H^b_*(\tilde{\Omega}A_+, \tilde{b}, \tilde{B}), \quad \mathcal{H}_c(A) = H^c_*(\tilde{\Omega}A_+, \tilde{b}, \tilde{B})$$

and

$$\mathcal{H}_p(A) = H^p_*(\tilde{\Omega}A_+, \tilde{b}, \tilde{B})$$

where $H^b_*$, $H^c_*$ and $H^p_*$ are Hochschild homology, cyclic homology and periodic cyclic homology of the mixed complex $(\tilde{\Omega}A_+, \tilde{b}, \tilde{B})$ in LCS, see [17].

There is also a cyclic cohomology theory associated with a complete locally convex algebra $A$, obtained when one replaces the chain complexes of $A$ by their dual complexes of strong dual spaces.

**Lemma 4.1.** (i) Let $A$ be a [nuclear] Fréchet algebra. Then the following complexes $(C(A), b)$, $(\tilde{\Omega}A_+, \tilde{b})$, $(B_c\tilde{\Omega}A_+, \tilde{b} + \tilde{B})$ and $(B_p\tilde{\Omega}A_+, \tilde{b} + \tilde{B})$ are complexes of [nuclear] Fréchet spaces and continuous linear operators.

(ii) Let $A$ be a [nuclear] $\hat{\otimes}$-algebra which is a DF-space. Then the following complexes $(C(A), b)$, $(\tilde{\Omega}A_+, \tilde{b})$, and $(B_c\tilde{\Omega}A_+, \tilde{b} + \tilde{B})$ are complexes of [nuclear] complete DF-spaces and continuous linear operators, and $(B_p\tilde{\Omega}A_+, \tilde{b} + \tilde{B})$ is a complex of [nuclear] complete locally convex spaces and continuous linear operators, but it is not a DF-space in general.

**Proof.** It is well known that Fréchet spaces are closed under countable cartesian products and projective tensor product [31]; nuclear locally convex spaces are closed under cartesian products, countable direct sums and projective tensor product [12 Corollary 21.2.3]; complete DF-spaces are closed under countable direct sums, projective tensor product, but not under infinite cartesian products [12, Theorem 12.4.8 and Theorem 15.6.2].

Propositions 4.2 and 4.3 below are proved by the author in [17, 18] and show the equivalence between the continuous cyclic (co)homology of $A$ and the continuous periodic cyclic (co)homology of $A$ when $A$ has trivial continuous Hochschild (co)homology $HH_n(A)$ for all $n \geq N$ for some integer $N$. Here we add in these statements certain topological conditions on the algebra which allow us to show that isomorphisms of (co)homology groups are automatically topological.

**Proposition 4.2.** [17, Proposition 3.2] Let $A$ be a complete locally convex algebra. Then, for any even integer $N$, say $N = 2K$, and the following assertions, we have $(i)_N \Rightarrow (ii)_N \Rightarrow (iii)_N \Rightarrow (ii)_{N+1}$ and $(ii)_N \Rightarrow (iv)_N$: 
(i) \( H_{n}^{\text{naive}}(A) = \{0\} \) for all \( n \geq N \) and \( H_{n}^{\text{bar}}(A) = \{0\} \) for all \( n \geq N - 1 \);
(ii) \( HH_{n}(A) = \{0\} \) for all \( n \geq N \);
(iii) for all \( k \geq K \), up to isomorphism of linear spaces,
\[ HC_{2k}(A) = HC_{N}(A) \quad \text{and} \quad HC_{2k+1}(A) = HC_{N-1}(A); \]
(iv) up to isomorphism of linear spaces, \( HP_{0}(A) = HC_{N}(A) \) and \( HP_{1}(A) = HC_{N-1}(A) \).

For Fréchet algebras the isomorphisms in (iii) \( N \) and (iv) \( N \) are automatically topological. For a nuclear \( \hat{\otimes} \)-algebra \( A \) which is a DF-space the isomorphisms in (iii) \( N \) are automatically topological.

**Proof.** A proof of the statement is given in [17, Proposition 3.2]. Here we add a part on the automatic continuity of the isomorphisms. In view of the proofs of [17, Propositions 2.1 and 3.2] it is easy to see that isomorphisms of homology groups in (iii) \( N \) and (iv) \( N \) are induced by continuous morphisms of complexes. Note that by Lemma 4.1 for a Fréchet algebra \( A \), the following complexes \((B_{\bar{\Omega}}A_{+}, \bar{b} + \bar{B})\) and \((B_{p}\bar{\Omega}A_{+}, \bar{b} + \bar{B})\) are complexes of Fréchet spaces and continuous linear operators. Thus, for Fréchet algebras, by [7, Lemma 0.5.9], isomorphisms of homology groups are topological.

By Lemma 4.1 for a nuclear \( \hat{\otimes} \)-algebra \( A \) which is a DF-space, the following complex \((B_{\bar{\Omega}}A_{+}, \bar{b} + \bar{B})\) is a complex of nuclear complete DF-spaces and continuous linear operators. By Corollary 3.6 for complete nuclear DF-spaces the isomorphisms for homology groups in (iii) \( N \) are also topological.

**Proposition 4.3.** [18, Proposition 3.1] Let \( A \) be a complete locally convex algebra. Then, for any even integer \( N \), say \( N = 2K \), and the following assertions, we have (i) \( N \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (iv) \):
(i) \( H_{n}^{\text{naive}}(A) = \{0\} \) for all \( n \geq N \) and \( H_{n}^{\text{bar}}(A) = \{0\} \) for all \( n \geq N - 1 \);
(ii) \( HH_{n}(A) = \{0\} \) for all \( n \geq N \);
(iii) for all \( k \geq K \), up to isomorphism of linear spaces, \( HC_{2k}(A) = HC_{N}(A) \) and \( HC_{2k+1}(A) = HC_{N-1}(A) \);
(iv) up to isomorphism of linear spaces, \( HP_{0}(A) = HC_{N}(A) \) and \( HP_{1}(A) = HC_{N-1}(A) \).

For nuclear Fréchet algebras the isomorphisms in (iii) \( N \) and (iv) \( N \) are topological isomorphisms. For a nuclear \( \hat{\otimes} \)-algebra \( A \) which is a DF-space the isomorphisms in (iii) \( N \) are topological isomorphisms.

**Proof.** We need to add to the proof of [18, Proposition 3.1] the following part on automatic continuity. In view of the proof of [18, Proposition 3.1] it is easy to see that the isomorphisms of cohomology groups in (iii) \( N \) and (iv) \( N \) are induced by continuous morphisms of complexes.

For nuclear Fréchet algebras, by Lemma 4.1, the complexes \((B_{\bar{\Omega}}A_{+})^{\ast}, \bar{b}^{\ast} + \bar{B}^{\ast}\) and \((B_{p}\bar{\Omega}A_{+})^{\ast}, \bar{b}^{\ast} + \bar{B}^{\ast}\) are complexes of strong duals of nuclear Fréchet spaces. By Lemma 3.5 the isomorphisms of cohomology groups in (iii) \( N \) and (iv) \( N \) are topological.
For a nuclear Fréchet space which is a $DF$-space, by Lemma 4.1 and by Theorem 4.4.13, the chain complex $(\mathcal{B},\Omega A_+)$ is the strong dual of a complex of nuclear Fréchet spaces. By Theorem 4.4.12, complete nuclear Fréchet spaces are reflexive. Therefore, $(\mathcal{B},\Omega A_+, b^\ast, \hat{B}^\ast)$ is a complex of nuclear Fréchet spaces. Thus, by Lemma 0.5.9, the isomorphisms of cohomology groups in (iii) are topological. □

The space of continuous traces on a topological algebra $A$ is denoted by $A^\text{tr}$, that is,

$$A^\text{tr} = \{ f \in A^* : f(ab) = f(ba) \text{ for all } a, b \in A \}.$$ 

The closure in $A$ of the linear span of elements of the form $\{ab - ba : a, b \in A\}$ is denoted by $[A, A]$. Recall that $b_0 : A \hat{\otimes} A \to A$ is uniquely determined by $a \otimes b \mapsto ab - ba$.

**Proposition 4.4.** Let $A$ be in $\Phi$ and be a $\hat{\otimes}$-algebra.

(i) Suppose that the continuous cohomology groups $\mathcal{H}_n^\text{naive}(A) = \{0\}$ for all $n \geq 1$ and $\mathcal{H}_n^\text{bar}(A) = \{0\}$ for all $n \geq 0$. Then, up to isomorphism of linear spaces,

$$\mathcal{H}_n(A) = \{0\} \text{ for all } n \geq 1 \text{ and } \mathcal{H}_0(A) = A/\text{Im } b_0;$$

(5) $\mathcal{H}_2(A) = A/\text{Im } b_0$ and $\mathcal{H}_2(A) = \{0\}$ for all $\ell \geq 0$; $\mathcal{H}_0(A) = A/\text{Im } b_0$ and $\mathcal{H}_1(A) = \{0\}$.

(ii) Suppose that the continuous cohomology groups $\mathcal{H}_n^\text{naive}(A) = \{0\}$ for all $n \geq 1$ and $\mathcal{H}_n^\text{bar}(A) = \{0\}$ for all $n \geq 0$. Then, up to isomorphism of linear spaces,

$$\mathcal{H}_n^\text{naive}(A) = \{0\} \text{ for all } n \geq 1 \text{ and } \mathcal{H}_0(A) = A^\text{tr};$$

(6) $\mathcal{H}_n^\text{naive}(A) = A^\text{tr}$ and $\mathcal{H}_n^\text{naive}(A) = \{0\}$ for all $\ell \geq 0$; $\mathcal{H}_0(A) = A^\text{tr}$ and $\mathcal{H}_1(A) = \{0\}$.

**Proof.** (i). One can see that $\mathcal{H}_n^\text{bar}(A) = \{0\}$ for all $n \geq 0$ implies that $\mathcal{H}_n(A) = \mathcal{H}_n^\text{naive}(A)$ for all $n \geq 0$,

see [17, Section 3]. Note that by definition of the 'naive' Hochschild homology of $A$, $\mathcal{H}_0^\text{naive}(A) = A/\text{Im } b_0$. Therefore, $\mathcal{H}_n(A) = \{0\}$ for all $n \geq 1$ and $\mathcal{H}_0(A) = A/\text{Im } b_0$.

From the exactness of the long Connes-Tsygan sequence of continuous homology it follows that

$$\mathcal{H}_0(A) = \mathcal{H}_0^\text{naive}(A) = A/\text{Im } b_0 \text{ and } \mathcal{H}_1(A) = \{0\}.$$ 

The rest of Statement (i) follows from Proposition 4.2

(ii) It is known that $\mathcal{H}_n^\text{bar}(A) = \{0\}$ for all $n \geq 0$, implies $\mathcal{H}_n(A) = \mathcal{H}_n^\text{naive}(A)$ for all $n \geq 0$. By definition of the 'naive' Hochschild cohomology of $A$, $\mathcal{H}_0^\text{naive}(A) = A^\text{tr}$. Thus $\mathcal{H}_0(A) = \{0\}$ for all $n \geq 1$ and $\mathcal{H}_0(A) = A^\text{tr}$.

From the exactness of the long Connes-Tsygan sequence of continuous cohomology it follows that $\mathcal{H}_0(A) = \mathcal{H}_0^\text{naive}(A) = A^\text{tr}$ and $\mathcal{H}_1(A) = \{0\}$. The rest of Statement (ii) follows from Proposition 4.3. □
5. Cyclic-type cohomology of biflat \(\hat{A}\)-algebras

Recall that a \(\hat{A}\)-algebra \(A\) is said to be biflat if it is flat in the category of \(A\)-\(\hat{A}\)-bimodules [7 Def. 7.2.5]. A \(\hat{A}\)-algebra \(A\) is said to be biprojective if it is projective in the category of \(A\)-\(\hat{A}\)-bimodules [7 Def. 4.5.1]. By [7 Proposition 4.5.6], a \(\hat{A}\)-algebra \(A\) is biprojective if and only if there exists an \(A\)-\(\hat{A}\)-bimodule morphism \(\rho_A : A \rightarrow A\hat{\otimes} A\) such that \(\pi_A \circ \rho_A = \text{id}_A\), where \(\pi_A\) is the canonical morphism \(\pi_A : A\hat{\otimes} A \rightarrow A\), \(a_1 \otimes a_2 \mapsto a_1a_2\). It can be proved that any biprojective \(\hat{A}\)-algebra is biflat and \(A = \overline{A^2} = \text{Im } \pi_A\) [7 Proposition 4.5.4]. Here \(\underline{A}^2\) is the closure of the linear span of the set \(\{a_1 \cdot a_2 : a_1, a_2 \in A\}\) in \(A\). A \(\hat{A}\)-algebra \(A\) is said to be contractible if \(A_+\) is projective in the category of \(A\)-\(\hat{A}\)-bimodules. A \(\hat{A}\)-algebra \(A\) is contractible if and only if \(A\) is biprojective and has an identity [7 Def. 4.5.8]. For biflat Banach algebras \(A\), Helemskii proved \(A = \overline{A^2} = \text{Im } \pi_A\) [7 Proposition 7.2.6] and gave the description of the cyclic homology \(\mathcal{H}_C\) and cohomology \(\mathcal{H}C^*\) groups of \(A\) in [8]. Later the author generalized Helemskii’s result to inverse limits of biflat Banach algebras [17 Theorem 6.2] and to locally convex strict inductive limits of amenable Banach algebras [18, Corollary 4.9].

**Proposition 5.1.** Let \(A\) be in \(\Phi\) and be a biflat \(\hat{A}\)-algebra such that \(A = \text{Im } \pi_A\); in particular, let \(A \in \Phi\) be a biprojective \(\hat{A}\)-algebra. Then

(i) \(\mathcal{H}^n_{\text{naive}}(A) = \{0\}\) for all \(n \geq 1\), \(\mathcal{H}^0_{\text{naive}}(A) = A/\text{Im } b_0\) and \(\mathcal{H}^n_{\text{bar}}(A) = \{0\}\) for all \(n \geq 0\);

(ii) for the homology groups \(\mathcal{H}_H, \mathcal{H}_C, \mathcal{H}H_C\) of \(A\) we have the isomorphisms of linear spaces (5).

*If, furthermore, \(A\) is a Fréchet space or \(A\) is a nuclear DF-space, then \(\mathcal{H}^0_{\text{naive}}(A) = A/\{A, A\}^\perp\) is Hausdorff, and, for a biflat \(A\), \(A = \overline{A^2}\) implies that \(A = \text{Im } \pi_A\).*

**Proof.** By [7 Theorem 3.4.25], up to topological isomorphism, the homology groups

\[\mathcal{H}^n_{\text{naive}}(A) = \mathcal{H}^n(A, A) = \text{Tor}^A_n(A, A_+)\]

for all \(n \geq 0\). Since \(A\) is biflat, by [7 Proposition 7.1.2], \(\mathcal{H}^n_{\text{naive}}(A) = \{0\}\) for all \(n \geq 1\).

By [7 Theorem 3.4.26], up to topological isomorphism, the homology groups

\[\mathcal{H}^n_{\text{bar}}(A) = \mathcal{H}^{n+1}(A, C) = \text{Tor}^{A}_{n+1}(C, C)\]

for all \(n \geq 0\), where \(C\) is the trivial \(A\)-bimodule. Note that, for the trivial \(A\)-bimodule \(C\), there is a flat resolution

\[0 \leftarrow C \leftarrow A_+ \leftarrow A \leftarrow 0\]

in the category of left or right \(A\)-\(\hat{A}\)-modules. By [7 Theorem 3.4.28], \(\mathcal{H}^n_{\text{bar}}(A) = \{0\}\) for all \(n \geq 1\). By assumption, \(A = \text{Im } \pi_A\), hence \(\mathcal{H}^0_{\text{bar}}(A) = A/\text{Im } \pi_A = \{0\}\). Thus the conditions of Proposition 4.4 (i) are satisfied.

In the categories of Fréchet spaces and complete nuclear DF-spaces, the open mapping theorem holds – see Corollary 3.4 for DF-spaces. Thus, by [7 Propositions 3.3.5 and 7.1.2], up to topological isomorphism, \(\mathcal{H}^0_{\text{naive}}(A) = \text{Tor}^A_0(A, A_+)\) is
Hausdorff. Since $A$ is biflat, by [7, Proposition 7.1.2], $\text{Tor}_0^A(C, A)$ is also Hausdorff. By [7, Proposition 3.4.27], $\overline{A^2} = \text{Im} \ \pi_A$. 

A $\otimes$-algebra $A$ is amenable if $A_+$ is a flat $A$-$\otimes$-bimodule. For a Fréchet algebra $A$ amenability is equivalent to the following: for all Fréchet $A$-bimodules $X$, $\mathcal{H}_0(A, X)$ is Hausdorff and $\mathcal{H}_n(A, X) = \{0\}$ for all $n \geq 1$. Recall that an amenable Banach algebra $A$ is biflat and has a bounded approximate identity [7, Theorem VII.2.20].

**Lemma 5.2.** Let $A$ be an amenable $\otimes$-algebra which is a Fréchet space or a nuclear $DF$-space. Then $\mathcal{H}_n^{\text{naive}}(A) = \{0\}$ for all $n \geq 1$, $\mathcal{H}_0^{\text{naive}}(A) = A/[A, A]$ and $\mathcal{H}_n^{\text{bar}}(A) = \{0\}$ for all $n \geq 0$.

**Proof.** In the categories of Fréchet spaces and complete nuclear $DF$-spaces, the open mapping theorem holds. Therefore, by [7, Theorem III.4.25 and Proposition 7.1.2], up to topological isomorphism, for the trivial $A$-bimodule $C$,

$$H_n^{\text{bar}}(A) = \mathcal{H}_{n+1}(A, C) = \text{Tor}_{n+1}^A(C, A_+) = \{0\}$$

for all $n \geq 0$;

$$H_n^{\text{naive}}(A) = \mathcal{H}_n(A, A) = \text{Tor}_n^A(A, A_+) = \{0\}$$

for all $n \geq 1$ and $\mathcal{H}_0^{\text{naive}}(A) = \text{Tor}_0^A(A, A_+)$ is Hausdorff, that is, $\mathcal{H}_0^{\text{naive}}(A) = A/[A, A]$. 

**Theorem 5.3.** Let $A$ be a $\otimes$-algebra which is a Fréchet space or a nuclear $DF$-space. Suppose that the continuous homology groups $\mathcal{H}_n^{\text{naive}}(A) = \{0\}$ for all $n \geq 1$, $\mathcal{H}_0^{\text{naive}}(A)$ is Hausdorff and $\mathcal{H}_n^{\text{bar}}(A) = \{0\}$ for all $n \geq 0$. In particular, asssume that $A$ is a biflat algebra such that $A = \overline{A^2}$ or $A$ is amenable. Then

(i) up to topological isomorphism,

$$\mathcal{H}_n(A) = \{0\} \quad \text{for all } n \geq 1 \text{ and } \mathcal{H}_0(A) = A/[A, A];$$

$$\mathcal{H}_\ell^0(A) = A/[A, A] \text{ and } \mathcal{H}_\ell^1(A) = \{0\} \quad \text{for all } \ell \geq 0;$$

(ii) up to topological isomorphism for Fréchet algebras and up to isomorphism of linear spaces for nuclear $DF$-algebras,

$$\mathcal{H}_0(A) = A/[A, A] \text{ and } \mathcal{H}_1(A) = \{0\};$$

(iii)

$$\mathcal{H}_n^{\text{naive}}(A) = \{0\} \quad \text{for all } n \geq 1;$$

$$\mathcal{H}_n^{\text{bar}}(A) = \{0\}; \quad \text{for all } n \geq 0;$$

(iv) up to topological isomorphism for nuclear Fréchet algebras and nuclear $DF$-algebras and up to isomorphism of linear spaces for Fréchet algebras,

$$\mathcal{H}_n^0(A) = \{0\} \quad \text{for all } n \geq 1 \text{ and } \mathcal{H}_0^0(A) = A^{tr};$$

$$\mathcal{H}_\ell^0(A) = A^{tr} \text{ and } \mathcal{H}_\ell^1(A) = \{0\} \quad \text{for all } \ell \geq 0;$$

(v) up to topological isomorphism for nuclear Fréchet algebras and up to isomorphism of linear spaces for Fréchet algebras and for nuclear $DF$-algebras,

$$\mathcal{H}_0(A) = A^{tr} \text{ and } \mathcal{H}_1(A) = \{0\}. $$
Proof. In view of Proposition 5.1 and Lemma 5.2, a biflat algebra $A$ such that $A = A^2$ and an amenable $A$ satisfy the conditions of the theorem.

By Proposition 2.2, firstly, $\mathcal{H}_n^\text{bar}(A) = \{0\}$ for all $n \geq 0$ if and only if $\mathcal{H}_n^\text{naive}(A) = \{0\}$ for all $n \geq 1$ if and only if $\mathcal{H}_n^\text{naive}(A) = \{0\}$ for all $n \geq 1$ and $\mathcal{H}_0^\text{naive}(A)$ is Hausdorff.

By Proposition 4.4, we have isomorphisms of linear spaces in (i) – (v). In Propositions 4.2 and 4.3 we show also when the above isomorphisms are automatically topological.

□

Remark 5.4. Recall that, for a biflat Banach algebra $A$, $db_{w}A \leq 2$ [27, Theorem 6]. By [14, Theorem 5.2], for a Banach algebra $A$ of a finite weak bidimension $db_{w}A$, we have isomorphisms between the entire cyclic cohomology and the periodic cyclic cohomology of $A$, $HE^0(A) = HP^0(A) = A^\text{tr}$ and $HE^1(A) = HP^1(A) = \{0\}$. The entire cyclic cohomology $HE^k(A)$ of $A$ for $k = 0, 1$ are defined in [2, IV.7]. In [25, Theorem 6.1] M. Puschnigg extended M. Khalkhali’s result on the isomorphism $HE^k(A) = HP^k(A)$ for $k = 0, 1$ from Banach algebras to some Fréchet algebras.

The following statement shows that the above theorems give the explicit description of cyclic type homology and cohomology of the projective tensor product of two biprojective $\hat{\otimes}$-algebras.

Proposition 5.5. Let $B$ and $C$ be biprojective $\hat{\otimes}$-algebras. Then the projective tensor product $A = B \hat{\otimes} C$ is a biprojective $\hat{\otimes}$-algebra.

Proof. Since $B$ is biprojective, there is a morphism of $B$-$\hat{\otimes}$-bimodules $\rho_B : B \to B \hat{\otimes} B$ such that $\pi_B \circ \rho_B = \text{id}_B$. A similar statement is valid for $C$. Let $i$ be the topological isomorphism

$$i : (B \hat{\otimes} B) \hat{\otimes} (C \hat{\otimes} C) \to (B \hat{\otimes} C) \hat{\otimes} (B \hat{\otimes} C)$$

given by $(b_1 \otimes b_2) \otimes (c_1 \otimes c_2) \mapsto (b_1 \otimes c_1) \otimes (b_2 \otimes c_2)$. Note that $\pi_{B \hat{\otimes} C} = (\pi_B \hat{\otimes} \pi_C) \circ i^{-1}$.

It is routine to check that

$$\rho_{B \hat{\otimes} C} : B \hat{\otimes} C \to (B \hat{\otimes} C) \hat{\otimes} (B \hat{\otimes} C)$$

defined by $\rho_{B \hat{\otimes} C} = i \circ (\rho_B \hat{\otimes} \rho_C)$ is a morphism of $B \hat{\otimes} C$-$\hat{\otimes}$-bimodules and $\pi_{B \hat{\otimes} C} \circ \rho_{B \hat{\otimes} C} = \text{id}_{B \hat{\otimes} C}$. □

Remark 5.6. For amenable Banach algebras $B$ and $C$, B. E. Johnson showed that the Banach algebra $A = B \hat{\otimes} C$ is amenable [11]. By [19, Proposition 5.4], for a biflat Banach algebra $A$, each closed two-sided ideal $I$ with bounded approximate identity is amenable and the quotient algebra $A/I$ is biflat. Thus the explicit description of cyclic type homology and cohomology of such $I$ and $A/I$ is also given in Theorem 5.3. One can find a number of examples of biflat and simplicially trivial Banach and $C^*$-algebras in [17, Example 4.6, 4.9].
6. Applications to the cyclic-type cohomology of biprojective $\hat{\otimes}$-algebras

In this section we present examples of nuclear biprojective $\hat{\otimes}$-algebras which are Fréchet spaces or $DF$-spaces and the continuous cyclic-type homology and cohomology of these algebras.

**Example 6.1.** Let $G$ be a compact Lie group and let $\mathcal{E}(G)$ be the nuclear Fréchet algebra of smooth functions on $G$ with the convolution product. It was shown by Yu.V. Selivanov that $A = \mathcal{E}(G)$ is biprojective [29].

Let $\mathcal{E}^*(G)$ be the strong dual to $\mathcal{E}(G)$, so that $\mathcal{E}^*(G)$ is a complete nuclear $DF$-space. This is a $\hat{\otimes}$-algebra with respect to convolution multiplication: for $f, g \in \mathcal{E}^*(G)$ and $x \in \mathcal{E}(G)$, $< f * g, x > = < f, y >$, where $y \in \mathcal{E}(G)$ is defined by $y(s) = < g, x_s >$, $s \in G$ and $x_s(t) = x(s^{-1}t)$, $t \in G$. J.L. Taylor proved that the algebra of distributions $\mathcal{E}^*(G)$ on a compact Lie group $G$ is contractible [30].

**Example 6.2.** Let $(E, F)$ be a pair of complete Hausdorff locally convex spaces endowed with a jointly continuous bilinear form $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{C}$ that is not identically zero. The space $A = E \hat{\otimes} F$ is a $\hat{\otimes}$-algebra with respect to the multiplication defined by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = (x_2, y_1)x_1 \otimes y_2, \quad x_i \in E, \quad y_i \in F.$$ 

Yu.V. Selivanov proved that this algebra is biprojective and usually non unital [28, 29]. More exactly, if $A = E \hat{\otimes} F$ has a left or right identity, then $E$ or $F$ respectively is finite-dimensional. If the form $\langle \cdot, \cdot \rangle$ is nondegenerate, then $A = E \hat{\otimes} F$ is called the tensor algebra generated by the duality $(E, F, \langle \cdot, \cdot \rangle)$.

In particular, if $E$ is a Banach space with the approximation property, then the algebra $A = E \hat{\otimes} E^*$ is isomorphic to the algebra $\mathcal{N}(E)$ of nuclear operators on $E$ [7 II.2.5].

**6.1. Köthe sequence algebras.** The following results on Köthe algebras can be found in A. Yu. Pirkovskii’s papers [22, 23].

A set $P$ of nonnegative real-valued sequences $p = (p_i)_{i \in \mathbb{N}}$ is called a Köthe set if the following axioms are satisfied:

(P1) for every $i \in \mathbb{N}$ there is $p \in P$ such that $p_i > 0$;

(P2) for every $p, q \in P$ there is $r \in P$ such that $\max\{p_i, q_i\} \leq r_i$ for all $i \in \mathbb{N}$.

Suppose, in addition, the following condition is satisfied:

(P3) for every $p \in P$ there exist $q \in P$ and a constant $C > 0$ such that $p_i \leq Cq_i^2$ for all $i \in \mathbb{N}$.

For any Köthe set $P$ which satisfies (P3), the Köthe space

$$\lambda(P) = \{ x = (x_n) \in \mathbb{C}^\mathbb{N} : \| x \|_p = \sum_n |x_n|p_n < \infty \text{ for all } p \in P \}$$

is a complete locally convex space with the topology determined by the family of seminorms $\{ \| x \|_p : p \in P \}$ and a $\hat{\otimes}$-algebra with pointwise multiplication. The $\hat{\otimes}$-algebras $\lambda(P)$ are called Köthe algebras.

By [21] and [6], for a Köthe set, $\lambda(P)$ is nuclear if and only if

(P4) for every $p \in P$ there exist $q \in P$ and $\xi \in \ell^1$ such that $p_i \leq \xi_iq_i$ for all $i \in \mathbb{N}$.
By [22] Theorem 3.5, \( \lambda(P) \) is biprojective if and only if 
(P5) for every \( p \in P \) there exist \( q \in P \) and a constant \( M > 0 \) such that \( p^2_i \leq Mq_i \)
for all \( i \in \mathbb{N} \).

The algebra \( \lambda(P) \) is unital if and only if \( \sum p_n < \infty \) for every \( p \in P \).

**Example 6.3.** Fix a real number \( 1 \leq R \leq \infty \) and a nondecreasing sequence \( \alpha = (\alpha_i) \) of positive numbers with \( \lim_{i \to \infty} \alpha_i = \infty \). The power series space
\[
\Lambda_R(\alpha) = \{ x = (x_n) \in \mathbb{C}^\mathbb{N} : \|x\|_r = \sum_n |x_n|r^{\alpha_n} < \infty \text{ for all } 0 < r < R \}
\]
is a Fréchet Köthe algebra with pointwise multiplication. The topology of \( \Lambda_R(\alpha) \) is determined by a countable family of seminorms \( \{\|x\|_{r_k} : k \in \mathbb{N} \} \) where \( \{r_k\} \) is an arbitrary increasing sequence converging to \( R \).

By [22, Corollary 3.3], \( \Lambda_R(\alpha) \) is biprojective if and only if \( R = 1 \) or \( R = \infty \).

By the Grothendieck-Pietsch criterion, \( \Lambda_R(\alpha) \) is nuclear if and only if for \( \lim_n \frac{\log n}{\alpha_n} = 0 \) for \( R < \infty \) and \( \lim_n \frac{\log n}{\alpha_n} < \infty \) for \( R = \infty \), see [22] Example 3.4].

The algebra \( \Lambda_R((n)) \) is topologically isomorphic to the algebra of functions holomorphic on the open disc of radius \( R \), endowed with Hadamard product, that is, with “co-ordinatewise” product of the Taylor expansions of holomorphic functions.

**Example 6.4.** The algebra \( \mathcal{H}(\mathbb{C}) \cong \Lambda_\infty((n)) \) of entire functions, endowed with the Hadamard product, is a biprojective nuclear Fréchet algebra [23].

**Example 6.5.** The algebra \( \mathcal{H}(\mathbb{D}_1) \cong \Lambda_1((n)) \) of functions holomorphic on the open unit disc, endowed with the Hadamard product, is a biprojective nuclear Fréchet algebra. Moreover it is contractible, since the function \( z \mapsto (1 - z)^{-1} \) is an identity for \( \mathcal{H}(\mathbb{D}_1) \) [23].

For any Köthe space \( \lambda(P) \) the dual space \( \lambda(P)^* \) can be canonically identified with
\[
\{ (y_n) \in \mathbb{C}^\mathbb{N} : \exists p \in P \text{ and } C > 0 \text{ such that } |y_n| \leq C p_n \text{ for all } n \in \mathbb{N} \}.
\]
It is shown in [23] that, for a biprojective Köthe algebra \( \lambda(P) \), \( \lambda(P)^* \) is a sequence algebra with pointwise multiplication.

The algebra \( \lambda(P)^* \) is unital if and only if there exists \( p \in P \) such that \( \inf_i p_i > 0 \).

**Example 6.6.** The nuclear Fréchet algebra of rapidly decreasing sequences
\[
s = \{ x = (x_n) \in \mathbb{C}^\mathbb{N} : \|x\|_k = \sum_n |x_n|n^k < \infty \text{ for all } k \in \mathbb{N} \}
\]
is a biprojective Köthe algebra [22]. The algebra \( s \) is topologically isomorphic to \( \Lambda_\infty(\alpha) \) with \( \alpha_n = \log n \) [23]. The nuclear Köthe \( \hat{\otimes} \)-algebra \( s^* \) of sequences of polynomial growth is contractible [30].

**Example 6.7.** [23, Section 4.2] Let \( P \) be a Köthe set such that \( p_i \geq 1 \) for all \( p \in P \) and all \( n \in \mathbb{N} \). Then the formula \( (a, b) = \sum_i a_i b_i \) defines a jointly continuous, nondegenerate bilinear form on \( \lambda(P) \times \lambda(P) \). Thus \( M(P) = \lambda(P) \hat{\otimes} \lambda(P) \) can be considered as the tensor algebra generated by the duality \( (\lambda(P), \lambda(P), \langle \cdot , \cdot \rangle) \), and so is biprojective. There is a canonical isomorphism between \( M(P) \) and the algebra...
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λ($P$ $×$ $P$) of $\mathbb{N} × \mathbb{N}$ complex matrices $(a_{ij})_{i,j} \in \mathbb{N} × \mathbb{N}$ satisfying the condition $\|a\|_p = \sum_{i,j} |a_{ij}|p_i p_j < \infty$ for all $p \in \mathbb{N}$ with the usual matrix multiplication.

In particular, for $P = \{ (n^k)_{n \in \mathbb{N}} : k = 0, 1, \ldots \}$, we obtain the biprojective nuclear Fréchet algebra $\mathbb{R} = s \hat{\otimes} s$ of “smooth compact operators” consisting of $\mathbb{N} \times \mathbb{N}$ complex matrices $(a_{ij})$ with rapidly decreasing matrix entries. Here $s$ is from Example 6.6.

**Theorem 6.8.** Let $\mathcal{A}$ be a $\hat{\otimes}$-algebra belonging to one of the following classes:

(i) $\mathcal{A} = \mathcal{E}(G)$ or $\mathcal{A} = \mathcal{E}^*(G)$ for a compact Lie group $G$;

(ii) $\mathcal{A} = \mathcal{E} \otimes F$, the tensor algebra generated by the duality $(\mathcal{E}, F, \langle \cdot, \cdot \rangle)$ for nuclear Fréchet spaces $\mathcal{E}$ and $F$ (e.g., $\mathcal{R} = s \hat{\otimes} s$) or for nuclear complete $DF$-spaces $\mathcal{E}$ and $F$;

(iii) Fréchet Kôthe algebras $\mathcal{A} = \lambda(P)$ such that the Kôthe set $P$ satisfies (P3), (P4) and (P5); in particular, $\Lambda_1(\alpha)$ such that $\lim_n \log_n \alpha_n < \infty$. (e.g., $\mathcal{H}(\mathcal{D}_1)$, $\mathcal{H}(\mathcal{C})$).

(iv) Kôthe algebras $\mathcal{A} = \lambda(P)^*$ which are the strong duals of $\lambda(P)$ from (iii).

(v) the projective tensor product $\mathcal{A} = \mathcal{B} \hat{\otimes} \mathcal{C}$ of biprojective nuclear $\hat{\otimes}$-algebras $\mathcal{B}$ and $\mathcal{C}$ which are Fréchet spaces or $DF$-spaces; in particular, $\mathcal{A} = \mathcal{E}(G) \hat{\otimes} \mathcal{R}$.

Then, up to topological isomorphism,

$$\mathcal{H}_n^{\text{naive}}(\mathcal{A}) = \{0\} \quad \text{for all } n \geq 1 \quad \text{and} \quad \mathcal{H}_0^{\text{naive}}(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}];$$

$$\mathcal{H}_n^{\text{bar}}(\mathcal{A}) = \{0\} \quad \text{for all } n \geq 0;$$

$$\mathcal{H}_n(\mathcal{A}) = \{0\} \quad \text{for all } n \geq 1 \quad \text{and} \quad \mathcal{H}_0(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}];$$

$$\mathcal{H}_{2\ell}^{\text{naive}}(\mathcal{A}) = \{0\} \quad \text{for all } \ell \geq 0;$$

$$\mathcal{H}_{2\ell+1}(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}] \quad \text{and} \quad \mathcal{H}^{2\ell+1}(\mathcal{A}) = \{0\} \quad \text{for all } \ell \geq 0;$$

$$\mathcal{H}^{2\ell}(\mathcal{A}) = \mathcal{A}^{\text{tr}} \quad \text{and} \quad \mathcal{H}^{2\ell+1}(\mathcal{A}) = \{0\} \quad \text{for all } \ell \geq 0;$$

and, up to topological isomorphism for Fréchet algebras and up to isomorphism of linear spaces for $DF$-algebras,

$$\mathcal{H}^{0}(\mathcal{A}) = \mathcal{A}/[\mathcal{A}, \mathcal{A}] \quad \text{and} \quad \mathcal{H}^{1}(\mathcal{A}) = \{0\};$$

$$\mathcal{H}^{0}(\mathcal{A}) = \mathcal{A}^{\text{tr}} \quad \text{and} \quad \mathcal{H}^{1}(\mathcal{A}) = \{0\}.$$

**Proof.** We have mentioned above that the algebras in (i)-(iii) and (v) are biprojective and nuclear. By [23, Corollary 3.10], for any nuclear biprojective Fréchet Kôthe algebra $\lambda(P)$, the strong dual $\lambda(P)^*$ is a nuclear, biprojective Kôthe $\hat{\otimes}$-algebra which is a $DF$-space. For nuclear Fréchet algebras and for nuclear $DF$-algebras, the conditions of Theorem 5.3 are satisfied. Therefore, for the homology and cohomology groups $\mathcal{H}_n$ and $\mathcal{H}^n$ of $\mathcal{A}$ we have the topological isomorphisms (7) and (10). For the periodic cyclic homology and cohomology groups $\mathcal{H}_n$ of $\mathcal{A}$, for Fréchet algebras, we have topological isomorphisms and, for nuclear $DF$-algebras, isomorphisms of linear spaces (8) and (11). It is obvious that, for commutative algebras, $\mathcal{A}^{\text{tr}} = \mathcal{A}^*$ and $\mathcal{A}/[\mathcal{A}, \mathcal{A}] = \mathcal{A}$. □
The cyclic-type homology and cohomology of \( \mathcal{E}(G) \) for a compact Lie group \( G \) were calculated in [20].

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