The b-coloring of infinite graphs

Qing Yang, Shuang liang Tian*, Man Liu

Mathematics and computer institute, Northwest Minzu University, Lanzhou, Gansu 730030, China

*Corresponding author’s e-mail: sl-tian@163.com

Abstract. We consider the b-coloring number of infinite graphs. We prove that the parameters of the infinite square, triangular and hexagonal lattices are 5, 7 and 4 on the plane. We also obtain that the parameters of infinite square lattices and its induction graph of \( n \)-dimensional are \( 2n+1 \) and \( n^2+n+1 \).

1. Introduction

The proper \( k \)-vertex coloring number \( \chi(G) \) of a graph \( G \) is the minimum \( k \) such that \( V \) has a partition \( V_1, V_2, ..., V_k \) into independent sets. On the basis of define of the proper \( k \)-vertex coloring number, the achromatic coloring has been introduced by Garey et al. [1]. The achromatic number \( \psi(G) \) of a graph \( G \) is the maximum \( k \) such that \( V \) has a partition \( V_1, V_2, ..., V_k \) into independent sets, the union of no pair of which is dependent. In 1999, Robert et al. [2] showed that \( \psi(G) \) can be viewed as the maximum over all minimal elements of a partial order defined on the set of all coloring of \( G \).

Similarly, Robert et al. [2] put forward concept of a b-coloring. And they proved that determining the b-chromatic number is NP-hard for genera graph, and the b-chromatic number of trees is \( m(T) \) or \( m(T) - 1 \) (This metric was upper bounded by the largest integer \( m(G) \) for which \( G \) has at least \( m(G) \) vertices with degree at least \( m(G) - 1 \)).

The theory of b-chromatic index attracted many researchers. In 2015 Victor et al. [3] proved that computing the b-chromatic index of a graph \( G \) is NP-hard, even complexity of the problem restricted to trees, more specifically, they solved the problem for caterpillars graphs. In 2015, Campos et al. [4] proved that every graph with girth at least 7 has b-chromatic number at least \( m(G) - 1 \). In 2002, Mouider and Maheo [5] proved the determination of two lower bounds for the b-chromatic number of the Cartesian product of two graphs. Marko and Iztok determined lower bound for the b-chromatic number of the Lexicographic product (see [6]). In 2108, Chuan and Mike [7] showed that \( K_m \square K_n \) has a upper bound of b-chromatic number, and give different approaches that come close to this bound. We also consider Cartesian powers of general graphs, and show that the Cartesian product of \( d \) graphs each with b-chromatic number \( n \) is at least \( d(n-1)+1 \).

Let \( G \) be a simple graph; the vertex-set of \( G \), denoted by \( V(G) \); the edge-set of \( G \), denoted by \( E(G) \); the maximum degree of \( G \), denoted by \( \Delta(G) \); denoted by \([k] = \{0, 1, \ldots, k-1\} \), \((x)_k = x \mod k \).

Definition 1[8] For a graph \( G \), suppose that vertices of \( G \) are ordered \( v_1, v_2, ..., v_n \) so that \( d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n) \). Then the \( m \)-degree, \( m(G) \), of \( G \) is defined by
Definition 2 Let $G$ be a simple graph. A b-coloring is a proper coloring of the vertices of $G$ that has a central set. The b-chromatic number, denoted $\varphi(G)$, is the largest number of colors in any b-chromatic of $G$.

Lemma 1\(^2\) For any graph $G$, then $\varphi(G) \leq m(G)$.

We have straightforward bounds for $\varphi(G)$. For any graph $G$, then $\chi(G) \leq \varphi(G) \leq \Delta(G) + 1$.

The Cartesian (or box) product of any two graphs $G$ and $H$, denoted by $G \square H$. The vertex set of $G \square H$ is the Cartesian product $V(G) \times V(H)$ of $G$ and $H$. There is an edge between two vertices of $G \square H$ if and only if they are adjacent in exactly one coordinate and agree in the other (see [8]).

In this paper, the neighbor sum distinguishing edge coloring of the infinite square, hexagonal and triangular lattices on the plane, and the infinite square lattices and this induction graph of $n$-dimensional are studied.

The following definition is about the infinite lattices on the plane.

Definition 3\(^3\) Let $P_n$ be a path of infinite order. The infinite square lattices on the plane, $L$, is define by

$$L = P_\infty \square P_\infty.$$  

Then for any two vertices $(x, y)$ and $(x', y')$ are adjacent in $L$ if and only if $x = x'$ and $|y - y'| = 1$, or $y = y'$ and $|x - x'| = 1$. For any two vertices $(x, y)$ and $(x', y')$ are adjacent in $H$ if and only if $y = y'$ and $|x - x'| = 1$, or $x = x'$ and $(x + y)_2 = 1$, $|y - y'| = 1$. For any two vertices $(x, y)$ and $(x', y')$ are adjacent in $T$, if and only if $x = x'$ and $|y - y'| = 1$, or $y = y'$ and $|x - x'| = 1$, or $x - x' = 1$ and $y - y' = 1$, or $x' - x = 1$ and $y' - y = 1$.

The following definitions are about the infinite square lattices and this induction graph of n-dimensional.

Definition 4 Let $P_\infty$ be a path of infinite order. The infinite square lattices of n-dimensional, $L_n$, is define by

$$L_n = P_\infty \square P_\infty \square \ldots \square P_\infty \square P_\infty.$$  

Then for any two vertices $u = (x_1, x_2, \ldots, x_n)$ and $v = (x'_1, x'_2, \ldots, x'_n)$ are adjacent in $L_n$ if and only if $v = (x_1, x_2, \ldots, x_i \pm 1, \ldots, x_n)$, where $i \in \{1, 2, \ldots, n\}$. For any two vertices $u = (x_1, x_2, \ldots, x_n)$ and $v = (x'_1, x'_2, \ldots, x'_n)$ are adjacent in $L_n$ if and only if $v = (x_1, x_2, \ldots, x_i \pm 1, \ldots, x_n)$ or $v = (x_1, x_2, \ldots, x_i - 1, \ldots, x_m - 1, \ldots, x_n)$ or $v = (x_1, x_2, \ldots, x_j + 1, \ldots, x_m + 1, \ldots, x_n)$.

where $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, n - 1\}$, $m \in \{2, 3, \ldots, n\}$ and $j < m$.

2. Main results and proofs

The following theorem is about the b-coloring of $L$.

Theorem 1 $\varphi(L) = 5$.

Proof The degree of any vertex of $L$ is 4, then $\Delta(L) = 4$. It is known by Lemma 2, that $\varphi(L) \leq \Delta(L) + 1 = 5$. Now we prove that $\varphi(L) \geq 5$.

Each vertex $u$ of $L$ is defined by its coordinates, i.e., $u = (x, y)$. Let us define the following coloring: each vertex $u = (x, y)$ is assigned color $\sigma(x, y) = (x + 2y)_3$.

Clearly, this coloring uses no more than 5 colors.

Let us first show that this is a proper coloring. For this, assume that two neighbors $u$ and $v$ are...
assigned the same color. Form the definition of $L$, then $u = (x, y)$ and $v = (x \pm y, y)$ or $v = (x, y \pm 1)$, by definition of $\sigma(u)$ and $\sigma(v)$ we have

$$(x + 2y)_5 = (x + 1 + 2y)_5 \text{ or } (x + 2y)_5 = (x + 2y \pm 1)_5.$$  

Thus we end up with $0 = (\pm j)_5$. However, $j \in \{1, 2\}$ hence this is impossible.

Now let us prove that this is a b-chromatic index. For every vertex $u = (x, y)$ of $L$ have four incident vertices, by definition of $\sigma(x, y)$, then $\sigma(y_i) = (x + 2y + i)_5$, where $i = 1, 2, 3, 4$. If $i \neq j$, then $\sigma(y_i) \neq \sigma(y_j)$, where $j = 1, 2, 3, 4$. The colors appearing on incident vertex are different about coloring of $\sigma$. Then $\sigma$ is the b-coloring of $L$, then $\varphi(L) \geq 5$.

**Corollary 1** $\varphi(T_r) = 7$ and $\varphi(H) = 4$.

The proof is similar to proof of theorem 1. There is only one different for the construction of b-coloring when each vertex $u = (x, y)$ of $T_r$ (or $H$) is assigned color $\sigma(x, y) = (x + 2y)_7$ (or $\sigma(x, y) = (x + 2y)_4$).

The following theorem is about the b-chromatic index of $L_n$.

**Theorem 2** $\varphi(L_n) = 2n + 1$.

**Proof** The degree of any vertex of $L_n$ is $2n$; then $\Delta(L_n) = 2n$. It is known by Lemma 2, that $\varphi(L_n) \leq \Delta(L_n) + 1 = 2n + 1$. Now we prove that $\varphi(L_n) \geq 2n + 1$.

Each vertex $u$ of $L_n$ is defined by its coordinates, i.e., $u = (x_1, x_2, \ldots, x_n)$. Let us define the following coloring: each vertex $u = (x_1, x_2, \ldots, x_n)$ is assigned color

$$\sigma(u) = \left(\sum_{i=1}^{n} i x_i\right)_{2n+1}.$$  

Clearly, this coloring uses no more than $2n + 1$ colors.

Let us first show that this is a proper coloring. For this, assume that two neighbors $u$ and $v$ are assigned the same color. Assume also that the coordinates of $u$ and $v$ differ on the $i$th dimension. Since $u = (x_1, x_2, \ldots, x_n)$ and $v = (x_1, x_2, \ldots, x_j \pm x_j, \ldots, x_n)$, by definition of $\sigma(u)$ and $\sigma(v)$ we have

$$(jx_j + \sum_{i=1, i \neq j}^{n} i x_i)_{2n+1} = (j(x_j \pm 1) + \sum_{i=1, i \neq j}^{n} i x_i)_{2n+1}.$$  

Thus we end up with $0 = (\pm j)_{2n+1}$. However, $j \in \{1, n\}$ hence this is impossible.

Now let us prove that this is a b-chromatic index. For every vertex $u = (x, y)$ of $L_n$ have $2n$ incident vertices, by definition of $\sigma(x, y)$, then $\sigma(y_i) = (x + 2y + i)_{2n+1}$, where $i = 1, 2, \cdots 2n$. If $i \neq j$, then $\sigma(y_i) \neq \sigma(y_j)$, where $j = 1, 2, \cdots 2n$. The colors appearing on incident vertex are different about coloring of $\sigma$. Then $\sigma$ is the b-coloring of $L_n$, then $\varphi(L_n) \geq 2n + 1$.

The following theorem is about the b-coloring of $\tilde{L}_n$.

**Theorem 3** $\varphi(\tilde{L}_n) = n^2 + n + 1$.

**Proof** The degree of any vertex of $\tilde{L}_n$ is $n^2 + n$, then $\Delta(\tilde{L}_n) = n^2 + n$. It is known by le, that $\varphi(\tilde{L}_n) \leq \Delta(\tilde{L}_n) + 1 = n^2 + n + 1$. Now we prove that $\varphi(\tilde{L}_n) \geq n^2 + n + 1$.

Each vertex $u$ of $\tilde{L}_n$ is defined by its coordinates, i.e., $u = (x_1, x_2, \ldots, x_n)$. Let us define the following coloring: each vertex $u = (x_1, x_2, \ldots, x_n)$ is assigned color

$$\sigma(u) = \left(\sum_{i=1}^{n} i x_i\right)_{n^2 + n + 1}.$$  

Clearly, this coloring uses no more than $n^2 + n + 1$ colors.

Let us first show that this is a proper coloring. For this, assume that two neighbours $u$ and $v$ are
assigned the same color. Assume also that the coordinates of \( u \) and \( v \) differ on the \( j \)th dimension. Since \( u = (x_1, x_2, \ldots, x_n) \) and \( v = (x_1, x_2, \ldots, x_j \pm 1, \ldots, x_n) \). Or assume also that the coordinates of \( u \) and \( v \) differ on the \( j \)th and \( m \)th dimension. Since \( u = (x_1, x_2, \ldots, x_n) \) and \( v = (x_1, x_2, \ldots, x_j - 1, \ldots, x_m - 1, \ldots, x_n) \) or \( v = (x_1, x_2, \ldots, x_j + 1, \ldots, x_m + 1, \ldots, x_n) \). By definition of \( \sigma(u) \) and \( \sigma(v) \) we have

\[
(j x_j + \sum_{i \neq j}^n i x_i) \mod x^{j - \text{dim}} = (j(x_j \pm 1) + \sum_{i \neq j}^n i x_i) \mod x^{j - \text{dim}}, \text{ or }
\]

\[
(j x_j + m x_m + \sum_{i \neq j}^n i x_i) \mod x^{j - \text{dim}} = (j(x_j - 1) + m(x_m - 1) + \sum_{i \neq j}^n i x_i) \mod x^{j - \text{dim}}, \text{ or }
\]

\[
(j x_j + m x_m + \sum_{i \neq j}^n i x_i) \mod x^{j - \text{dim}} = (j(x_j + 1) + m(x_m + 1) + \sum_{i \neq j}^n i x_i) \mod x^{j - \text{dim}}.
\]

Thus we end up with \( 0 = (\pm j) \mod x^{j - \text{dim}} \) or \( 0 = (\pm j + m) \mod x^{j - \text{dim}} \). However, \( j \in [1, n-1], m \in [2, n] \) and \( j > m \) hence this is impossible.

Now let us prove that this is a b-chromatic index. For every vertex \( u = (x, y) \) of \( \tilde{L}_n \) have \( n^2 + n \) incident vertices, by definition of \( \sigma(x, y) \), then \( \sigma(v_j) = (x + 2y + i) \mod n^2 + n + 1 \), where \( i = 1, 2, \ldots, n^2 + n + 1 \). If \( i \neq j \), then \( \sigma(v_i) \neq \sigma(v_j) \), where \( j = 1, 2, \ldots, n^2 + n + 1 \). The colors appearing on incident vertex are different about coloring of \( \sigma \). Then \( \sigma \) is the b-coloring of \( \tilde{L}_n \), then \( \varphi(\tilde{L}_n) \geq n^2 + n + 1 \).

3. Conclusion

For the b-coloring of the infinite graphs on the plane square, triangular and hexagonal lattices, we obtained the b-coloring number of the infinite square, triangular and hexagonal lattices as Theorem 1 and Corollary 1. Similarly, the b-coloring of the infinite graphs on the \( n \)-dimensional, we obtained the parameters of infinite square lattices and its induction graph as Theorem 2 and Theorem 3. This paper only considers the b-coloring of the common lattices graphs, and can also other lattices graphs. infinite square, triangular and hexagonal lattices

Acknowledgments

This work was financially supported by Key Laboratory of Streaming Data Computing Technologies and Applications, State Key Discipline of Civil Affairs Commission (Applied Mathematics), Gansu Key Discipline (Mathematics) and Innovative Team Subsidize of Northwest Minzu University.

References

[1] Gary M R, Johnson D S. Computers and intractability[M]. Freeman, 1979.
[2] Irving R W, Manlove D F. The b-chromatic number of a graph[J]. Discrete Applied Mathematics, 1999, 91(1-3):127-141.
[3] Victor A. Campos, Carlos V. Lima, Nicolas A. Martins, et. al. The b-chromatic index of graphs[J]. Discrete Mathematics, 2015, 338(11):2072-2079.
[4] Campos V, Lima C, Silva A. Graphs of girth at least 7 have high b-chromatic number [J]. European Journal of Combinatorics, 2015, 48:154-164.
[5] Koutour M, Maryvonne O. Some bounds for the b-chromatic number of a graph[J]. Discrete Mathematics, 2002, 256(1):267-277.
[6] Jakovac M, Peterin I. On the b-chromatic number of some graph products[J]. Studia Scientiarum Mathematicarum Hungarica, 2012, 49(49):156-169.
[7] Guo C, Newman M. On the b-chromatic number of cartesian products[J]. Discrete Applied Mathematics, 2018.
[8] Bondy J A, Murtywritten U S R. Graph Theory with Applications[M]. North Holland, 1976.
[9] Bondy B A, Murty U S R. Graph Theory[M]. Springer London, 2008.