Fuzzy Torus and q-Deformed Lie Algebra

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Abstract

It will be shown that the defining relations for fuzzy torus and deformed (squashed) sphere proposed by J. Arnlind, et al (hep-th/0602290) (ABHHS) can be rewritten as a new algebra which contains q-deformed commutators. The quantum parameter $q$ ($|q| = 1$) is a function of $\hbar$. It is shown that the $q \to 1$ limit of the algebra with the parameter $\mu < 0$ describes fuzzy $S^2$ and that the squashed $S^2$ with $q \neq 1$ and $\mu < 0$ can be regarded as a new kind of quantum $S^2$. Throughout the paper the value of the invariant of the algebra, which defines the constraint for the surfaces, is not restricted to be 1. This allows the parameter $q$ to be treated as independent of $N$ (the dimension of the representation) and $\mu$. It was shown by ABHHS that there are two types of representations for the algebra, “string solution” and “loop solution”. The “loop solution” exists only for $q$ a root of unity ($q^N = 1$) and contains undetermined parameters. The ’string solution’ exists for generic values of $q$ ($q^N \neq 1$). In this paper we will explicitly construct the representation of the q-deformed algebra for generic values of $q$ ($q^N \neq 1$) and it is shown that the allowed range of the value of $q + q^{-1}$ must be restricted for each fixed $N$.

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Among fuzzy compact manifolds fuzzy sphere (S^2) is the simplest and well-studied one. The representation of fuzzy S^2 is classified according to the unitary irreducible representation of SU(2) Lie algebra and the number of independent functions on fuzzy S^2 is finite. The functions in the spin j representation of fuzzy S^2 are mapped onto (2j+1) \times (2j+1) matrices and the number of independent functions is (2j + 1)^2. The non-commutative product (star product) of functions is one to one correspondence with the product of the corresponding matrices. This property is also true for other fuzzy compact manifolds. The representation of fuzzy CP^2 corresponds to a series of unitary irreducible representations of SU(3) and so forth.

A torus is a compact manifold, but there has been no satisfactory formalism for fuzzy torus (T^{2n}) in which one can define a star product such that the algebra of only a finite number of independent functions is closed.

Recently, in an interesting paper a prescription for constructing polynomial relations among non-commutative coordinates of compact fuzzy Riemann surfaces was proposed and the case of fuzzy T^2 was worked out explicitly. A fuzzy T^2 algebra was defined by the eqs.

\[ [X, Y] = i\hbar Z, \]
\[ [Y, Z] = i\hbar \{X, X^2 + Y^2 - \mu\}, \]
\[ [Z, X] = i\hbar \{Y, X^2 + Y^2 - \mu\}, \]
\[ (X^2 + Y^2 - \mu)^2 + Z^2 = 1 \]

Here X, Y, Z are hermitian matrices representing non-commutative coordinates and [A, B] = AB - BA, {A, B} = AB + BA. \( \hbar, \mu \) are real parameters. For \( \mu > 1 \) this algebra describes a fuzzy T^2 and for \( -1 < \mu < 1 \) a deformed (squashed) fuzzy S^2. Surprisingly, although the relations are complicated, these satisfy Jacobi identities, finite dimensional representations exist and were constructed explicitly in.

It was shown that

\[ C = (X^2 + Y^2 - \mu)^2 + Z^2 \]

commutes with X, Y, Z. So this is a multiple of an identity. This value does not necessarily be equal to 1 as in. Therefore we will set \( C = c \mathbf{1} \) in this paper and let c take an arbitrary positive value. Then the relations.
describe fuzzy $T^2$ for $\mu > \sqrt{c}$ and fuzzy squashed $S^2$ for $-\sqrt{c} < \mu < \sqrt{c}$. Actually the value of $c$ will be determined later in terms of $\hbar$, $\mu$ and the dimension $N$ of the representation.

In [5] the parameter $\hbar$ was introduced as a quantization parameter for replacing Poisson brackets by commutators. In this paper it will be regarded as a deformation parameter for deforming a Lie algebra into a $q$-deformed Lie algebra. It will be shown in this paper that when $\mu < 0$, the $\hbar \to 0$ limit of the squashed $S^2$ is the ordinary fuzzy $S^2$.

The relations (1)-(3), are not a Lie algebra and the right-hand sides are complicated. In the case of usual fuzzy $S^2$ it is natural to embed the sphere in a flat $\mathbb{R}^3$ in an SO(3) invariant way because of the SO(3) (SU(2)) symmetry of the fuzzy $S^2$ algebra and the constraint $X^2 + Y^2 + Z^2 = R^2 1$. In the case of (1)-(4) without symmetry principles except for rotations in $X - Y$ plane it is not clear how the $T^2$ and squashed $S^2$ are embedded in 3-dim space. The construction of field theories on these fuzzy manifolds is not straightforward and there will be ambiguities, even if one knows the solutions representing fuzzy configurations. If there exists symmetry, then it can be used as a guiding principle for constructing field theories on fuzzy manifolds.

One of the purposes of the present paper is to show that the algebra for fuzzy $T^2$ (1)-(3) can be recombined into that which has the structure of a $q$-deformed Lie algebra. We will then derive irreducible $N$-dim representations of this algebra for generic values of $q$ ($q^N \neq 1$) ("string solution"). In this analysis $q$ defined by (13) below is treated as an independent $q$-deformation parameter. According to the sign of $\mu$ the range for the value of $q + q^{-1}$ is restricted for each value of $N$. Then the value of $c$ (5) will be determined. The condition on $q + q^{-1}$ for describing fuzzy $T^2$ is also obtained. For the representation which exists only for $q^N = 1$ ("loop solution") see note added.

Let us derive the deformed Lie algebra. As in [8] we introduce $W \equiv X + iY$ and hermitian matrices $D$, $\tilde{D}$ by

$$D \equiv WW^\dagger, \quad \tilde{D} \equiv W^\dagger W.$$  \hspace{1cm} (6)

In the complex notation the algebra (1)-(3) is written as

$$[W, W^\dagger] = 2\hbar Z,$$ \hspace{1cm} (7)

$$[Z, W] = \hbar \{W, \varphi\}.$$ \hspace{1cm} (8)
\[ [Z, W^\dagger] = -\hbar \{W^\dagger, \varphi\}, \quad (9) \]

where the matrix \( \varphi \) is defined by

\[ \varphi \equiv W^\dagger W + \hbar Z - \mu = WW^\dagger - \hbar Z - \mu \quad (10) \]

and it commutes with \( Z \). By using (8) and (10) one obtains

\[ [Z, W] = h (W(W^\dagger W + \hbar Z - \mu) + (WW^\dagger - \hbar Z - \mu)W) \]
\[ = 2h(WW^\dagger W - \mu W) - \hbar^2 [Z, W] \]
\[ = \frac{2h}{1 + h^2} (WW^\dagger W - \mu W) \quad (11) \]

One can eliminate \( Z \) by using (7). By expanding the commutators one obtains

the eq. (8)

\[ 2\alpha WW^\dagger W - W^2 W^\dagger - W^\dagger W^2 = -\frac{4h^2 \mu}{1 + h^2} W \quad (12) \]

Here \( \alpha = (1 - h^2)/(1 + h^2) \).

Now let us define a new parameter \( q \) by \( q + q^{-1} = 2\alpha \). Because \(-1 < \alpha \leq 1\), \( q \) is a complex number

\[ q = \alpha \pm i\sqrt{1 - \alpha^2} = (1 - h^2 \pm 2i\hbar)/(1 + h^2) \quad (13) \]

and its absolute value is \( |q| = 1 \). Now we will use (6). Because \( WW^\dagger W \) can be expressed in two ways, \( DW \) or \( W\tilde{D} \), we have from eq (12)

\[ (qDW + \frac{1}{q} W\tilde{D}) - WD - \tilde{D}W = -\frac{4h^2 \mu}{1 + h^2} W \quad (14) \]

Defining a q-deformed commutator

\[ [ A, B ]_q \equiv qAB - BA, \quad (15) \]

we can put this into the following form.

\[ [ D - \frac{1}{q} \tilde{D}, W ]_q = -\frac{4h^2 \mu}{1 + h^2} W \quad (16) \]

Similarly from eq (9) we also obtain the eq.

\[ \frac{1}{q} [ D - q\tilde{D}, W^\dagger ]_q = \frac{4h^2 \mu}{1 + h^2} W^\dagger \quad (17) \]
Further, we note that if we interchange $q$ and $q^{-1}$ in eq (14), we obtain another set of eqs.

$$\frac{1}{q} \left[ W, D - q\tilde{D} \right]_q = \frac{4\hbar^2 \mu}{1 + \hbar^2} W;$$

(18)

$$\left[ W^\dagger, D - \frac{1}{q}\tilde{D} \right]_q = -\frac{4\hbar^2 \mu}{1 + \hbar^2} W^\dagger$$

(19)

We also consider the relations among $W, W^\dagger, D, \tilde{D}$.

$$[W, W^\dagger]_q = qD - \tilde{D},$$

(20)

$$[W^\dagger, W]_q = q\tilde{D} - D,$$

(21)

$$[D, \tilde{D}] = 0$$

(22)

Note that the commutator in (22) is the ordinary one. The derivation of these eqs is straightforward due to (6). The last eq was proved in [8]. It is easy to show that the algebra (17)-(21) can be derived from (16)-(22). If $q \neq \pm 1$, which we assume, from (20) and (21) one can derive $D = WW^\dagger$ and $\tilde{D} = W^\dagger W$. By substituting these eqs into (14) and using a definition $Z = (D - \tilde{D})/2\hbar$, we obtain (11). Then it is straightforward to derive (17)-(21).

Now we adopt the eqs (16)-(22) as the defining algebra for the fuzzy $T^2$. This algebra must be supplemented with two constraint eqs.

$$D + \tilde{D} = WW^\dagger + W^\dagger W$$

(23)

and a quadratic one

$$c1 = \frac{1}{4}(D + \tilde{D} - 2\mu)^2 + \frac{1}{4\hbar^2}(D - \tilde{D})^2$$

$$= \frac{1}{4}(D + \tilde{D})^2 - \mu(D + \tilde{D}) + \mu^2 + \frac{1}{4\hbar^2}(D - \tilde{D})^2$$

$$= -\frac{q}{2(q-1)^2} (KK^\dagger + K^\dagger K) - \mu(WW^\dagger + W^\dagger W) + \mu^2,$$

(24)

which is derived from the invariant (5). Here $c$ is a positive number and $K \equiv D - q\tilde{D}, K^\dagger \equiv D - q^{-1}\tilde{D}$. Eqs $X^2 + Y^2 = (WW^\dagger + W^\dagger W)/2 = (D + \tilde{D})/2$ and $Z = (D - \tilde{D})/2\hbar$ are used to transform (5) into a quadratic form. Actually, the first constraint (23) can be derived from (20) and (21) as explained above.

The right-hand sides of (16)-(22) are at most linear in $W, W^\dagger, D, \tilde{D}$ and this algebra has a structure of a q-deformed Lie algebra. This algebra is similar.
to the q-deformed SU(2) Lie algebra for the quantum $S^2$. Actually, our algebra is a q-analog of SU(2) (or SU(1,1)) Lie algebra with ‘doubled’ Cartan subalgebra ($D$, $\tilde{D}$). There may exist a quantum group symmetry associated with this new q-deformed Lie algebra.

There is a similarlity to the case of quantum $S^2$. It is known that the $q \to 1$ limit of quantum $S^2$ is the ordinary fuzzy $S^2$. To discuss $q \to 1$ limit in the present case it is better to rescale the matrices as $(X, Y, Z) \to (\hbar X', \hbar Y', \hbar Z')$. Then the relations (1)-(3) take the form.

$$[X', Y'] = iZ',$$
$$[Y', Z'] = i \left\{ X', \hbar^2 (X'^2 + Y'^2) - \mu \right\},$$
$$[Z', X'] = i \left\{ Y', \hbar^2 (X'^2 + Y'^2) - \mu \right\},$$

(25)

In the $\hbar \to 0$ limit this reduces to SU(2) or SU(1,1) algebra according to the sign of $\mu$. Therefore when $\mu < 0$, the classical limit $\hbar \to 0$ can be taken and fuzzy $S^2$ is obtained. When $\mu > 0$, the limiting algebra does not describe fuzzy $S^2$ nor $T^2$ but hyperboloid.

Under the above rescaling the q-deformed Lie algebra takes the form.

$$[D' - \frac{1}{q} \tilde{D}', W']_q = -\frac{(q + 1)^2 \mu}{q} W',$$
$$[D' - q \tilde{D}', W'^\dagger]_q = (q + 1)^2 \mu W'^\dagger,$$
$$[W', D' - q \tilde{D}']_q = (q + 1)^2 \mu W',$$
$$[W'^\dagger, D' - \frac{1}{q} \tilde{D}']_q = -\frac{(q + 1)^2 \mu}{q} W'^\dagger,$$
$$[W', W'^\dagger]_q = qD' - \tilde{D'},$$
$$[W'^\dagger, W']_q = q\tilde{D}' - D',$$
$$[D', \tilde{D}'] = 0,$$
$$D' + \tilde{D}' = W'W'^\dagger + W'^\dagger W',$$
$$c\mathbf{1} = \frac{1}{4}(D' + \tilde{D}' - 2\mu)^2 + \frac{1}{4\hbar^2}(D' - \tilde{D}')^2$$

(26)

Here we set $(W, W'^\dagger, D, \tilde{D}) \to (hW', hW'^\dagger, hD', h\tilde{D}')$. If we let $q \to 1$, the algebra becomes

$$[D' - \tilde{D}', W'] = -4\mu W',$$

$^1$SU(2) for $\mu < 0$ and SU(1,1) for $\mu > 0$ after trivial rescaling of $X'$, $Y'$, $Z'$. 


\[ [D' - \tilde{D}', W^\dagger] = 4\mu W'^\dagger, \]
\[ [W', W^\dagger] = D' - \tilde{D}', \]
\[ [D', \tilde{D}] = 0 \quad (27) \]

In addition, the \( O(q-1) \) terms of the first eq of (26) yield
\[ D'W' - W'\tilde{D}' = 0. \quad (28) \]

This last equation determines the matrix \( D' + \tilde{D}' \) decoupled from (27). The algebra (27) except for the last eq is an SU(2) (or SU(1,1)) Lie algebra according to the sign of \( \mu \). Therefore for \( q \neq 1 \) and \( \mu < 0 \) the algebra (26) will describe a q-deformation of the ordinary fuzzy sphere. We note that the algebra (26) also makes sense for real values of \( q \). This case, however, will not be considered in this paper.

Now the N-dimensional irreducible, unitary representation of the algebra (26) will be presented. This corresponds to 'string solution' in [8] and exists for generic values of \( q \). Although the representations of (11)-(14) were obtained in [8], some relations among \( q \) (\( h \)), \( N \), \( \mu \) were assumed there. This is because the invariant \( c \) was set to 1 in (4). In this paper we will take \( c \) as a free parameter to be determined later and \( q \) will be treated as independent.

We will take \( D', \tilde{D}' \) to be diagonal. As in the case of classical SU(2) algebra the matrix elements of \( W', W'^\dagger \) are assumed to be
\[ W_{mn}' = a_m \delta_{m+1,n}, \]
\[ (W'^\dagger)_{mn} = a_{m-1} \delta_{m,n+1}. \quad (m, n = 1, 2, \ldots, N) \quad (29) \]

Here we set \( a_0 = a_N = 0 \). By (6) the diagonal components of \( D', \tilde{D}' \) are given by
\[ D'_n = |a_n|^2, \quad \tilde{D}'_n = |a_{n-1}|^2. \quad (30) \]

Then the first relations in (26) yield the following equation.
\[ |a_{n+1}|^2 - (q + q^{-1}) |a_n|^2 + |a_{n-1}|^2 = \frac{(q + 1)^2}{q} \mu \quad (31) \]

This equation is solved with the initial condition \( a_0 = 0 \).
\[ |a_n|^2 = \frac{q(q^n - q^{-n})}{q^2 - 1} |a_1|^2 + \frac{(q + 1)(q^n + q^{-n+1} - q - 1)}{(q - 1)^2} \mu \quad (32) \]
Then the condition $a_N = 0$ determines $a_1$. When $\mu \neq 0$, the result for $a_n$ is

$$|a_n|^2 = \frac{(q + 1)^2(q^n - q^N)(q^n - 1)}{(q - 1)^2q^n(q^N + 1)} \mu. \quad (33)$$

The case $\mu = 0$ requires care and the solution exists only if $q^{2N} = 1$, and is given by

$$|a_n|^2 = \frac{q^n - q^{-n}}{q - q^{-1}} |a_1|^2. \quad (34)$$

Note that in the above solution (33) no relation among $q$, $N$ and $\mu$ is assumed. The condition $|a_n|^2 \geq 0$, however, restricts the allowed values of $N$ for fixed $q$. For given $N$ this condition in turn restricts the range of $q + q^{-1}$.

Let us present some examples.

When $N = 2$,

$$|a_1|^2 = -\frac{q + q^{-1} + 2}{q + q^{-1}} \mu. \quad (35)$$

Because $-2 \leq q + q^{-1} \leq 2$ for $|q| = 1$, the allowed value of $q + q^{-1}$ is $-2 \leq q + q^{-1} < 0$ for $\mu > 0$, and $0 < q + q^{-1} \leq 2$ for $\mu < 0$.

When $N = 3$,

$$|a_1|^2 = |a_2|^2 = -\frac{q + q^{-1} + 2}{q + q^{-1} - 1} \mu. \quad (36)$$

In this case $-2 \leq q + q^{-1} < 1$ for $\mu > 0$, and $1 < q + q^{-1} \leq 2$ for $\mu < 0$.

When $N = 4$,

$$|a_1|^2 = |a_3|^2 = -\frac{(q + q^{-1} + 2)(q + q^{-1} + 1)}{(q + q^{-1})^2 - 2} \mu,$n

$$|a_2|^2 = |a_4|^2 = -\frac{(q + q^{-1} + 2)^2}{(q + q^{-1})^2 - 2} \mu. \quad (37)$$

So $-1 \leq q + q^{-1} < \sqrt{2}$ for $\mu > 0$ and $\sqrt{2} < q + q^{-1} \leq 2$ for $\mu < 0$. In all the above cases the $q \to 1$ limit is allowed only for $\mu < 0$. This is in accord with the above comment on this limit. When $\mu = 0$, the solution which satisfy $|a_n|^2 > 0$ is $q = \exp(\pm \frac{\pi i}{N})$ and $|a_n|^2 = |a_1|^2 \sin \frac{n\pi}{N} / \sin \frac{\pi}{N}$.

Finally, the invariant $c$ (for $\mu \neq 0$) is obtained as

$$c = \frac{\hbar^4}{4} \left( |a_n|^2 + |a_{n-1}|^2 - \frac{2\mu}{\hbar^2} \right)^2 + \frac{\hbar^2}{4} \left( |a_n|^2 - |a_{n-1}|^2 \right)^2$$

$$= \mu^2 \frac{q + q^{-1} + 2}{q^N + q^{-N} + 2}. \quad (38)$$

The final result does not depend on $n$, as it should.
As mentioned at the beginning of this paper fuzzy $T^2$ is realized for $\mu > 0$ and $c < \mu^2$. This last equation yields a condition on the range of $q + q^{-1}$. For example, when $N = 2$, the condition $q + q^{-1} < q^2 + q^{-2}$ is satisfied for $q + q^{-1} < -1$. Combining this with the condition for $\mu > 0$ obtained above, we obtain the allowed range for fuzzy $T^2$, $-2 \leq q + q^{-1} < -1$. When $N = 3$, the allowed range for fuzzy $T^2$ is $-2 \leq q + q^{-1} < 0$. When $N = 4$, it is $-1 \leq q + q^{-1} < (\sqrt{5} - 1)/2$.

In the classical limit $q \to 1$ the invariant $c$ (38) approaches $\mu^2$. The condition (5) on the rescaled matrices $X', Y', Z'$ becomes in this limit

$$- 2\mu(X'^2 + Y'^2) + Z'^2 = \lim_{\hbar \to 0} \frac{c - \mu^2}{\hbar^2} = \mu^2 (N^2 - 1).$$

When we set $N = 2j + 1$, the right-hand side becomes $j(j + 1) (2\mu)^2$. For $\mu < 0$ after trivial rescaling of $X', Y', Z'$, this coincides with the condition for fuzzy $S^2$ in the spin $j$ representation.

To summarize it has been shown that the fuzzy $T^2$ algebra presented in [8] can be rearranged into a different algebra (26), which takes the form of a q-deformed Lie algebra. This may make it possible to treat fuzzy $T^2$ by arguments based on symmetry principles. There is a barrier at $\mu = 0$. The algebra for $\mu < 0$ is SU(2)-like but that for $\mu > 0$ is SU(1,1)-like. The classical $q \to 1$ limit of the squashed $S^2$ with $\mu < 0$ is fuzzy $S^2$, but the classical limit of the fuzzy torus which belongs to $\mu > 0$ does not exist. For $q$ not a root of unity irreducible $N$-dimensional representations (“string representation”) of the q-deformed Lie algebra (26) are obtained and the allowed range for $q + q^{-1}$ which depends on the sign of $\mu$ is obtained for some values of $N$. The allowed range for $q + q^{-1}$ which describes fuzzy $T^2$ is also obtained. It is not clear whether a quantum group associated with the deformed Lie algebra (26) exists. If it does, for further investigation and construction of field theories on fuzzy $T^2$ it will be necessary to study the representations of the quantum group.

Note added

In addition to the representation (26), (33) there also exits for $q$ a root of unity ($q^N = 1$) a solution called “loop solution” [8] with

$$W'_{mn} = a_m \delta_{m+1,n} \mod N.\quad (40)$$
$a_m$ is periodic; $a_{m+N} = a_m$. In [8] it was discussed that this 'loop solution' may be important. In this note this representation for $q^N = 1$ will be briefly discussed.

As will be shown soon this solution is not uniquely determined by the algebra \[11\]-\[14\] or \[20\]. It contains two free parameters. So this solution does not determine fuzzy $T^2$ or $S^2$ completely. This may bring about a problem when the matrix model for these surfaces is formulated. The number of the classical solutions to the eqs of motion in the matrix model will be continuously infinite with the solutions depending on the free parameters. So throughout the main text of this paper we discussed only the representations with generic values of $q$.

With the ansatz \[40\] for $W'$ the recursion relation for $a_n$ is still given by \[31\]. Defining $b_n$ by

$$b_n = |a_n|^2 + \frac{(q + 1)^2}{(q - 1)^2} \mu$$

and solving the relation $b_{n+1} - qb_n = q^{-1}(b_n - qb_{n-1})$, we obtain

$$b_n = \frac{q^{1-n} - q^{1+n}}{1 - q^2} b_1 + \frac{q^n - q^{-2-n}}{1 - q^2} b_0. \quad \text{(42)}$$

It will be assumed that $q^2 \neq 1$. The periodicity conditions $b_N = b_0$, $b_{N+1} = b_1$ lead to

$$\begin{align*}
(q^N - q^{2-N} + q^2 - 1) b_0 + (q^{1-N} - q^{1+N}) b_1 &= 0, \\
(q^{N+1} - q^{1-N}) b_0 + (q^{-N} - q^{N+2} + q^2 - 1) b_1 &= 0
\end{align*} \quad \text{(43)}$$

The determinant of the coefficient matrix vanishes only for $q^N = 1$. When $q^N \neq 1$, these eqs have only the solution $b_0 = b_1 = 0$ and by \[12\] all $b_n = 0$. This gives a collapsed ($Z = 0$) surface. Instead when $q^N = 1$, $b_n$ and hence $a_n$ is expressed in terms of $a_0$, $a_1$ by \[11\], \[12\].

$$|a_n|^2 = \frac{q^{1-n} - q^{1+n}}{1 - q^2} |a_1|^2 + \frac{q^n - q^{-2-n}}{1 - q^2} |a_0|^2 + \mu \frac{q + 1}{(q - 1)^2} (q^n + q^{1-n} - 1 - q) \quad \text{(44)}$$

The two parameters $a_0$, $a_1$ are left undetermined. The "loop solution" of [8] contains a free parameter $\beta$. (eq(5.22) of this reference)
We will remark on the “boundary” between representations for torus and sphere. In this paper the invariant $c$ (24) is not set to 1 but treated as a parameter to be determined by the representation matrices. The invariant $c$ (24) for this “loop solution” (44) is given by

$$c = -\frac{q(1-q)^2}{(1+q)^4} \left\{ |a_0|^4 + |a_1|^4 - (q + q^{-1})|a_0|^2|a_1|^2 - \mu q^{-1}(q + 1)^2(|a_0|^2 + |a_1|^2) \right\} + \mu^2$$

(45)

For $c > \mu^2$ and $\mu > 0$ one obtains fuzzy $T^2$ and for $c < \mu^2$ fuzzy $S^2$. The boundary is determined by $c - \mu^2 = 0$ and is a quadratic curve on the ($|a_0|^2$, $|a_1|^2$) plane. There are some special points on this curve. At $a_0 = a_1 = 0$ the reduction of the dimensionality of the representation ($N \to N - 1$) takes place and this was argued to be a singularity. One can show there also exist points that correspond to $a_M = a_{M+1} = 0$, ($M = 1, 2, ..., N - 1$) on this curve. Yet other points on the curve are nonsingular.

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