MULTIPLE WEIGHT RIESZ AND FOURIER TRANSFORMS
IN BILATERAL ANISOTROPIC GRAND LEBESGUE SPACES

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Abstract. In this short article we introduce so-called anisotropic (weight) Grand
Lebesgue Spaces (more exactly, Grand Lebesgue-Riesz Spaces), which are general-
ization of the classical Lebesgue-Riesz Spaces and ordinary Grand Lebesgue Spaces,
and investigate the boundedness of weight Riesz potential and weight Fourier trans-
forms in this spaces.
We construct also several examples to show the exactness of offered estimations.

Key words and phrases: Grand and ordinary Lebesgue Spaces (GLS), Fourier
weight transform, Orlicz and other rearrangement invariant (r.i.) spaces, weight
(generalized) Riesz’s potential (transform), Pitt-Beckner-Okikiolu (PBO) inequality,
equivalent norms, upper and lower estimations, dilation method, slowly varying
function.

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1 Introduction. Notations. Problem Statement.

We will describe now using rearrangement invariant Banach spaces of measurable
functions.

1. Classical Lebesgue-Riesz Spaces.

Let \((X, A, \mu)\) be measurable space with sigma-finite non - trivial measure \(\mu\).
For the measurable real valued function \(f(x), x \in X, f : X \to R\) the symbol
\(|f|_p = |f|_p(X, \mu)\) will denote the usually \(L_p\) norm:
\(|f|_p = |f|L_p(X, \mu) = \left( \int_X |f(x)|^p \mu(dx) \right)^{1/p}, \ p \geq 1;\)
\[ L_p(X, \mu) = L_p = \{ f : f : X \rightarrow R, |f|_p < \infty \}. \]

We will use further only the case when \( X \) is classical whole Euclidean space \( X = \mathbb{R}^d \) or some its measurable subset with positive measure and \( \mu \) is ordinary Lebesgue measure.

2. Grand Lebesgue Spaces (GLS).

We recall in this section for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [11], [12], [13], [17], [18], [20], [25], [30], [31], etc. appears the so-called Grand Lebesgue Spaces \( \text{GLS} = G(\psi) = G\psi = G(\psi; A, B), A, B = \text{const}, A \geq 1, A < B \leq \infty, \) spaces consisting on all the measurable functions \( f : X \rightarrow R \) with finite norms

\[ ||f||_{G(\psi)} \overset{\text{def}}{=} \sup_{p \in (A, B)} [||f||_p/\psi(p)]. \]

Here \( \psi(\cdot) \) is some continuous positive on the open interval \((A, B)\) function such that

\[ \inf_{p \in (A, B)} \psi(p) > 0, \ \psi(p) = \infty, \ p \notin (A, B). \]

We will denote

\[ \text{supp}(\psi) \overset{\text{def}}{=} (A, B) = \{ p : \psi(p) < \infty, \}. \]

The set of all \( \psi \) functions with support \( \text{supp}(\psi) = (A, B) \) will be denoted by \( \Psi(A, B) \).

This spaces are rearrangement invariant, see [6], and are used, for example, in the theory of probability [20], [30], [31]; theory of Partial Differential Equations [12], [18]; functional analysis [13], [17], [25], [31]; theory of Fourier series [30], [34], [36], [37], [38], [39]; theory of martingales [31], mathematical statistics [34], [35], [36], [37], [38], [39]; theory of approximation [44] etc.

Notice that in the case when \( \psi(\cdot) \in \Psi(A, \infty) \) and a function \( p \rightarrow p \cdot \log \psi(p) \) is convex, then the space \( G\psi \) coincides with some exponential Orlicz space.

Conversely, if \( B < \infty \), then the space \( G\psi(A, B) \) does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

**Remark 1.1** If we introduce the discontinuous function

\[ \psi_r(p) = 1, \ p = r; \psi_r(p) = \infty, \ p \neq r, \ p, r \in (A, B) \]

and define formally \( C/\infty = 0, \ C = \text{const} \in \mathbb{R}^1 \), then the norm in the space \( G(\psi_r) \) coincides with the \( L_r \) norm:

\[ ||f||_{G(\psi_r)} = |f|_r. \]
Thus, the Grand Lebesgue Spaces are direct generalization of the classical exponential Orlicz’s spaces and Lebesgue spaces $L^r$.

**Remark 1.2** The function $ψ(·)$ may be generated as follows. Let $ξ = ξ(x)$ be some measurable function: $ξ : X → R$ such that $∃(A, B) : 1 ≤ A < B ≤ ∞$, $∀p ∈ (A, B)$ $|ξ|^p < ∞$. Then we can choose

$$ψ(p) = ψξ(p) = |ξ|^p.$$ 

Analogously let $ξ(t, ·) = ξ(t, x)$, $t ∈ T$, $T$ is arbitrary set, be some family $F = \{ξ(t, ·)\}$ of the measurable functions: $∀t ∈ T ξ(t, ·) : X → R$ such that $∃(A, B) : 1 ≤ A < B ≤ ∞$, $\sup t ∈ T |ξ(t, ·)|^p < ∞$. Then we can choose

$$ψ_F(p) = sup t ∈ T |ξ(t, ·)|^p.$$ 

The function $ψ_F(p)$ may be called as a natural function for the family $F$. This method was used in the probability theory, more exactly, in the theory of random fields, see [30].

### 3. Anisotropic Lebesgue-Riesz spaces.

We recall here the definition of the so-called anisotropic Lebesgue (Lebesgue-Riesz) spaces. More detail information about these spaces see in the books of Besov O.V., Ilin V.P., Nikolskii S.M. [7], chapter 16,17; Leoni G. [24], chapter 11; using for us theory of operators interpolation in this spaces see in [7], chapter 17,18.

Let $(X_j, A_j, µ_j)$ be measurable spaces with sigma-finite non-trivial measures $µ_j$. Let also $p = (p_1, p_2, ..., p_l)$ be $l$-dimensional vector such that $1 ≤ p_j ≤ ∞$. Recall that the anisotropic Lebesgue space $L_p$ consists on all the total measurable real valued function $f = f(x_1, x_2, ..., x_l) = f(⃗x)$

$$f : \otimes^l_{j=1} X_j → R$$

with finite norm $|f|_p ≔$

$$\left(\int_{X_1} \mu_1(dx_1) \left(\int_{X_{l-1}} \mu_{l-1}(dx_{l-1}) \cdots \left(\int_{X_1} |f(⃗x)|^{p_1} \mu(dx_1)\right)^{p_2/p_1} \cdots \right)^{1/p_l}\right).$$

Note that in general case $|f|_{p_1, p_2} ≠ |f|_{p_2, p_1}$, but $|f|_{p, p} = |f|_p$.

Observe also that if $f(x_1, x_2) = g_1(x_1) \cdot g_2(x_2)$ (condition of factorization), then $|f|_{p_1, p_2} = |g_1|_{p_1} \cdot |g_2|_{p_2}$, (formula of factorization).

### 4. Anisotropic Grand Lebesgue-Riesz spaces.

Let $Q$ be convex (bounded or not) subset of the set $\otimes^l_{j=1}[1, ∞]$. Let $ψ = ψ(⃗p)$ be continuous in an interior $Q^0$ of the set $Q$ strictly positive function such that
\[
\inf_{\vec{p} \in Q^0} \psi(\vec{p}) > 0; \quad \inf_{\vec{p} \notin Q^0} \psi(\vec{p}) = \infty.
\]

We denote the set all of such a functions as \( \Psi(Q) \).

The Anisotropic Grand Lebesgue Spaces \( AGLS = AGLS(\psi) \) space consists on all the measurable functions

\[
f : \otimes_{j=1}^l X_j \to \mathbb{R}
\]

with finite (mixed) norms

\[
||f||_{AG\psi} = \sup_{\vec{p} \in Q^0} \left[ \frac{|f|_{\vec{p}}}{\psi(\vec{p})} \right].
\]

An application into the theory of multiple Fourier transform of these spaces see in articles [3] and [41], where are considered some problems of boundedness of singular operators in (weight) Grand Lebesgue Spaces and Anisotropic Grand Lebesgue Spaces. We intend to generalize some results obtained in [3], [41].

5. Description of considered operators.

5A. Weight Riesz’s ordinary operator. Recall that the operator, more exactly, the family of operators \( I_{\alpha,\beta,\gamma}[f](x) \), \( x \in \mathbb{R}^d \) of a view

\[
u(x) = I_{\alpha,\beta,\gamma}[f](x) = I_{d,\alpha,\beta,\gamma}[f](x) = I[f](x) = |x|^{-\beta} \int_{\mathbb{R}^d} f(y) \frac{|y|^{-\alpha} \, dy}{|x - y|^{\gamma}};
\] (1.5A.1)

is said to be weight Riesz’s operator; here \( \gamma = \text{const}, \alpha, \beta = \text{const} \geq 0, \alpha + \gamma < d \) and \( |x| \) denotes ordinary Euclidean norm of the vector \( x \).

We refer used in this article results from [42]. Let us define the function \( q = q(p) \) as follows:

\[
d + \frac{d}{q} = \frac{d}{p} + (\alpha + \beta + \gamma). \quad (1.5A.2)
\]

We will denote the set of all such a values \((p, q)\) as \( G(d; \alpha, \beta, \gamma) \) or for simplicity \( G = G(d; \alpha, \beta, \gamma) \).

Further we will suppose in this subsection that \((p, q) \in G(d; \alpha, \beta, \gamma) = G\).

We denote also

\[
p_- := \frac{d}{d - \alpha}, \quad p_+ := \frac{d}{d - \alpha - \gamma};
\]

and correspondingly

\[
q_- := \frac{d}{\beta + \gamma}, \quad q_+ := \frac{d}{\beta};
\]

where in the case \( \beta = 0 \Rightarrow q_+ := +\infty; \)

\[
\kappa \overset{\text{def}}{=} \kappa(\alpha, \beta, \gamma) := (\alpha + \beta + \gamma)/d.
\]
There exist a positive finite coefficient \( K_{RD;\alpha,\beta,\gamma}(p) \), \( p \in (p_-, p_+) \) for which

\[
|I_{\alpha,\beta,\gamma}[f]|_q \leq K_{RD;\alpha,\beta,\gamma}(p),
\]

where by definition

\[
K_{RD;\alpha,\beta,\gamma}(p) = \sup_{p \in (p_-, p_+)} \sup_{0 < |f| < \infty} \left[ \frac{|I_{\alpha,\beta,\gamma}[f]|_q(p)}{|f|_p} \right].
\]

It is known that the coefficient \( K_{RD;\alpha,\beta,\gamma}(p) \) is bilateral bounded:

\[
\frac{C_1(d; \alpha, \beta, \gamma)}{[(p - p_+)(p_+ - p)]^\gamma} \leq K_R(d; \alpha, \beta, \gamma) \leq \frac{C_2(d; \alpha, \beta, \gamma)}{[(p - p_+)(p_+ - p)]^\gamma}. \tag{1.5A.5}
\]

Notice that this estimation may be obtained also by the direct computation from the recent article of Cruz-Uribe D., Moen K. [10].

\textbf{5B. Weight Fourier transform.}

We define alike to the book Okikiolu [29], p. 313-314, see also [2] the so-called double weight Fourier transform \( F_{\alpha,\beta}[f](x) \), \( x \in \mathbb{R}^d \) by the following way:

\[
F_{\alpha,\beta}[f](x) = (2\pi)^{-d/2}|x|^\alpha \int_{\mathbb{R}^d} |y|^\beta f(y) e^{ixy} \, dy, \tag{1.5B.1}
\]

\[
F[f](x) = F_{0,0}[f](x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) e^{ixy} \, dy,
\]

where \( xy \) denotes the inner product of the vectors \( x, y : xy = \sum_k x_k y_k \).

Let us introduce the following important conditions in the domain of parameters \((p, q)\):

\[
p_0 := d/(d - \beta), \ p_1 := \infty, \ q_0 := 1, \ q_1 := d/\alpha > 1; \ \beta - \alpha = d - d(1/p + 1/q), \tag{1.5B.2}
\]

\[
p > p_0, \ 1 \leq q < q_1. \tag{1.5B.3}
\]

which defined uniquely the continuous function \( q = q(p) \).

The inequality of a view (relative the Lebesgue measures in whole space \( \mathbb{R}^d \))

\[
|F_{\alpha,\beta}[f]|_q \leq K_{PBO}(p) \ |f|_p \tag{1.5B.4}
\]

is said to be (generalized) weight Pitt - Beckner - Okikiolu (PBO) inequality.

We will understood as customary as the value \( K_{PBO}(p) \) its minimal value:

\[
K_{PBO}(p) = \sup_{f \neq 0, |f| < \infty} \left[ \frac{|F_{\alpha,\beta}[f]|_q}{|f|_p} \right]. \tag{1.5B.5}
\]
It is known [41] that

\[ C_1(\alpha, \beta, d) \left[ \frac{p}{p - p_0} \right]^{(\alpha + \beta)/d} \leq K_{PBO}(p) \leq C_2(\alpha, \beta, d) \left[ \frac{p}{p - p_0} \right]^{\max(1, (\alpha + \beta)/d)}, \]

(1.5B.6)

0 < C_{1,2}(\alpha, \beta, d) < \infty.

The right hand side of bilateral estimate (1.5B.6) may be obtained after some computations from the articles [4], [5], [2], [9], [15], [19]. The lower estimate is obtained in [41].

See also many publications about this problem [21], [22], [23], [27], [28], [45], [46], [49], [51], [54], [56] etc.

The paper is organized as follows. In the next section we investigate multiple weight Riesz and Fourier transforms and ground the neediness of introducing of anisotropic spaces. In the third section we obtain the conditions for boundedness of the multiple weight Riesz and Fourier transforms in anisotropic Grand Lebesgue-Riesz spaces. The fourth section is devoted to the multiple Riesz potential in bounded domains. The last section contains some slight generalizations and concluding remarks.

We use symbols \( C(X,Y), C(p,q; \psi), \) etc., to denote positive finite constants along with parameters they depend on, or at least dependence on which is essential in our study. To distinguish between two different constants depending on the same parameters we will additionally enumerate them, like \( C_1(X,Y) \) and \( C_2(X,Y) \).

\[ \square \]

2 Multiple weight Riesz and Fourier transforms.

Neediness of anisotropic spaces.

We consider in this section the weight multidimensional (vector): \( l \geq 2 \) generalization of weight Riesz’s \( L_p \) estimation and PBO inequality.

In this section \( x = \bar{x} \in \mathbb{R}^d \) be \( d- \) dimensional vector, \( d = 1, 2, ... \) which consists on the \( l \) subvectors \( x_j, j = 1, 2, ..., l \):

\[ x = (x_1, x_2, ..., x_l), \ x_j = \bar{x}_j \in \mathbb{R}^{m_j}, \dim(x_j) = m_j \geq 1; \]

\[ \alpha = \vec{\alpha} = \{ \alpha_1, \alpha_2, \ldots, \alpha_l \}, \beta = \vec{\beta} = \{ \beta_1, \beta_2, \ldots, \beta_l \} \]

and \( \gamma = \vec{\gamma} = \{ \gamma_1, \gamma_2, \ldots, \gamma_l \} \) be three fixed \( l- \) dimensional numerical vectors.

Note that in the case of weight multiple Riesz potential the third vector \( \vec{\gamma} \) is absent.
We denote as ordinary
\[
|x|^{-\alpha} = |\vec{x}|^{-\vec{\alpha}} = \prod_{j=1}^l x_j^{-\alpha_j}, \quad |y|^\beta = |\vec{y}|^{\vec{\beta}} = \prod_{j=1}^l y_j^{\beta_j}, \quad |z|^\gamma = |\vec{z}|^{\vec{\gamma}} = \prod_{j=1}^l z_j^{\gamma_j}.
\] (2.1.0)

Obviously, \(\sum_j m_j = d\).

**A. Weight multiple Riesz potential.**

Let \(f, f: R^d \to R\) be (total) measurable function. Let \(\gamma_j = \text{const}, \alpha_j, \beta_j = \text{const} \geq 0, \alpha_j + \gamma_j < m_j, \ j = 1, 2, \ldots, l.\)

We define the following
\[
m_j \left(1 + \frac{1}{q_j} - \frac{1}{p_j}\right) = \alpha_j + \beta_j + \gamma_j,
\] (2.1.1)
i.e. \((p_j, q_j) \in G(m_j; \alpha_j, \beta_j, \gamma_j) =: G_j\).

Further we will suppose in this subsection that \((p_j, q_j) \in G(m_j; \alpha_j, \beta_j, \gamma_j) =: G_j\).

We denote also as before
\[
p_{-}^{(j)} := \frac{m_j}{m_j - \alpha_j}, \quad p_{+}^{(j)} := \frac{m_j}{m_j - \alpha_j - \gamma_j};
\] (2.1.2)
and correspondingly
\[
q_{-}^{(j)} := \frac{m_j}{\beta_j + \gamma_j}, \quad q_{+}^{(j)} := \frac{m_j}{\beta_j},
\] (2.1.2a)
where in the case \(\beta_j = 0 \Rightarrow q_{+}^{(j)} := +\infty;\)

\[
\kappa_j \overset{\text{def}}{=} \kappa_j(\alpha_j, \beta_j, \gamma_j) := (\alpha_j + \beta_j + \gamma_j)/m_j.
\] (2.1.3)

We define the following *multiple* weight Riesz (linear) potential:
\[
u(\vec{x}) = I_{\vec{\alpha}, \vec{\beta}, \vec{\gamma}}[f](\vec{x}) = I_{m; \vec{\alpha}, \vec{\beta}, \vec{\gamma}}[f](\vec{x}) = \otimes_{j=1}^l I_{m_j; \alpha_j, \beta_j, \gamma_j}[f](\vec{x}) =
\]
\[
\int_{R^m} \frac{|x_1|^{-\beta_1} |y_1|^{-\alpha_1} \ dy_1}{|x_1 - y_1|^{\gamma_1}} \left[ \int_{R^m} \frac{|x_2|^{-\beta_2} |y_2|^{-\alpha_2} \ dy_2}{|x_2 - y_2|^{\gamma_2}} \left[ \ldots \left( \int_{R^m} \frac{|x_l|^{-\beta_l} |y_l|^{-\alpha_l} \ f(\vec{y}) \ dy_l}{|x_l - y_l|^{\gamma_l}} \right) \right]\right].
\] (2.1.4)

Note that our definition different on the ones in the articles [16], [21], [26], [47] etc.

More general case, namely when the weight function is regular varying will be considered further.

**Theorem 2.1.**

The conditions (2.1.1) are necessary and sufficient for the existence non-trivial coefficient \(K_{m; \vec{\alpha}, \vec{\beta}, \vec{\gamma}}(\vec{p})\) for the following estimate:
\[
|I_{m; \vec{\alpha}, \vec{\beta}, \vec{\gamma}}[f](\vec{\cdot})|_q \leq K_{m; \vec{\alpha}, \vec{\beta}, \vec{\gamma}}(\vec{p}) \ |f(\vec{\cdot})|_{\vec{p}},
\] (2.1.5)
and under this conditions for the minimal value of coefficient $K_{\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p})$ holds the following inequality:

$$\frac{C_3(\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma})}{\prod_{j=1}^l [(p_j^+(\vec{p})) - p_j^-(\vec{p})]^\kappa_j} \leq K_{\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \leq \frac{C_4(\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma})}{\prod_{j=1}^l [(p_j^+(\vec{p})) - p_j^-(\vec{p})]^\kappa_j},$$

(2.1.6)

if for all the values $j \Rightarrow p_j \in \left(p^+_\vec{p}, p^-_\vec{p}\right)$ and $K_{\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) = \infty$ in other case.

Recall that the values $\{p_j\}$ are variable, in contradiction to the values $\{p^+_\vec{p}, p^-_\vec{p}\}$.

**Proof.** We can and will suppose without loss of generality $f \in S(R^d)$, $f \neq 0$, where $S(R^d)$ denotes Schwartz class of functions in whole space $R^d$. We use the multidimensional generalization of the so-called dilation method, see [55], [54], chapter 3.

Indeed, let for simplicity $l = 2$. Let $f \in S(R^d)$, $d = m_1 + m_2$, $f \neq 0$ be any function for which the inequality (2.1.5) there holds. Let $\lambda_1, \lambda_2 = \text{const} > 0$ be arbitrary (independent) numbers. A multidimensional dilation operator $T_{\lambda_1,\lambda_2}[f]$ may be defined as follows:

$$T_{\lambda_1,\lambda_2}[f](x_1, x_2) = T_{\lambda_1,\lambda_2}[f](x_1, x_2) = f(\lambda_1 x_1, \lambda_2 x_2).$$

Evidently $T_{\lambda_1,\lambda_2}[f] \in S(R^d)$.

We have consequently:

$$| T_{\lambda_1,\lambda_2}[f] |_{\vec{p}} = \lambda_1^{-m_1/p_1} \lambda_2^{-m_2/p_2} | f |_{\vec{p}};$$

$$I_{\vec{\alpha},\vec{\beta},\vec{\gamma}}[T_{\lambda_1,\lambda_2}[f]](x_1/\lambda_1, x_2/\lambda_2) = \lambda_1^{\alpha_1+\beta_1+\gamma_1-m_1} \lambda_2^{\alpha_2+\beta_2+\gamma_2-m_2} T_{\lambda_1,\lambda_2}[I_{\vec{\alpha},\vec{\beta},\vec{\gamma}}[f]](x_1, x_2);$$

$$I_{\vec{\alpha},\vec{\beta},\vec{\gamma}}[T_{\lambda_1,\lambda_2}[f]](x_1, x_2) = \lambda_1^{\alpha_1+\beta_1+\gamma_1-m_1} \lambda_2^{\alpha_2+\beta_2+\gamma_2-m_2} T_{\lambda_1,\lambda_2}[I_{\vec{\alpha},\vec{\beta},\vec{\gamma}}[f]](x_1, x_2);$$

$$| I_{\vec{\alpha},\vec{\beta},\vec{\gamma}}[T_{\lambda_1,\lambda_2}[f]](x_1, x_2) |_{\vec{q}} = \lambda_1^{\alpha_1+\beta_1+\gamma_1-m_1/q_1} \lambda_2^{\alpha_2+\beta_2+\gamma_2-m_2/q_2} | I_{\vec{\alpha},\vec{\beta},\vec{\gamma}}[f] |_{\vec{q}}.$$

Substituting into (2.1.5) we obtain

$$C_1 \lambda_1^{\alpha_1+\beta_1+\gamma_1-m_1/q_1} \lambda_2^{\alpha_2+\beta_2+\gamma_2-m_2/q_2} \leq C_2 \lambda_1^{-m_1/p_1} \lambda_2^{-m_2/p_2}.$$

Since the values $\lambda_1, \lambda_2$ are arbitrary positive, we obtain the equalities (2.1.1). We refer also the reader to the article [41] for details.

It remains to obtain the estimations (2.1.6). First of all we will obtain the upper estimate.
It is sufficient again to consider only the "two-dimensional" case \( l = 2 \). Namely, define the function \( u = u(x_1, x_2) \), \( x_1 \in R^{m_1}, x_2 \in R^{m_2} \) as follows:

\[
 u(x_1, x_2) = \otimes_{j=1}^2 I_{m_j; \alpha_j, \beta_j, \gamma_j} [f](\vec{x}) =
\]

\[
\int_{R^{m_1}} \frac{|x_1|^{-\beta_1} |y_1|^{-\alpha_1}}{|x_1 - y_1|^{\gamma_1}} dy_1 \left[ \int_{R^{m_2}} \frac{|x_2|^{-\beta_2} |y_2|^{-\alpha_2} f(y_1, y_2)}{|x_2 - y_2|^{\gamma_2}} dy_2 \right] = I_{m_1; \alpha, \beta, \gamma}(g)(x_1, x_2),
\]

where

\[
g(x_1, x_2) = \int_{R^{m_2}} \frac{|x_2|^{-\beta_2} |y_2|^{-\alpha_2} f(y_1, y_2)}{|x_2 - y_2|^{\gamma_2}} dy_2.
\]

We have taking the \( L(q_1) \) norm on the variable \( x_1 \) and using the one-dimensional version for Riesz’s potential

\[
|u(\cdot, x_2)|_{L_{q_1, R_1^{m_1}}} \leq \frac{C_2(m_1; \alpha_1, \beta_1, \gamma_1)}{[(p_1 - p^{(1)}_\beta)(p^{(1)}_\beta - p_1)]^{\kappa_1}} \cdot |g(\cdot, \cdot)|_{L_{p_1, R_1^{m_2}}}.
\]

We use the triangle inequality for the \( L_{p_1} \) norm, denoting \( h(y_2) = |f(y_1, y_2)|_{p_1, R_1^{m_1}} \):

\[
|g(\cdot, \cdot)|_{L_{p_1, R_1^{m_2}}} \leq \int_{R^{m_2}} \frac{|x_2|^{-\beta_2} |y_2|^{-\alpha_2} \cdot |f(y_1, y_2)|_{p_1, R_1^{m_2}}}{|x_2 - y_2|^{\gamma_2}} dy_2 =
\]

\[
\int_{R^{m_2}} \frac{|x_2|^{-\beta_2} |y_2|^{-\alpha_2} h(y_2)}{|x_2 - y_2|^{\gamma_2}} dy_2 = I_{m_2; \alpha_2, \beta_2, \gamma_2}[h](x_2),
\]

therefore

\[
|u(\cdot, x_2)|_{L_{q_1, R_1^{m_1}}} \leq \frac{C_2(m_1; \alpha_1, \beta_1, \gamma_1)}{[(p_1 - p^{(1)}_\beta)(p^{(1)}_\beta - p_1)]^{\kappa_1}} \times I_{m_2; \alpha_2, \beta_2, \gamma_2}[h](x_2).
\]

We have taking the \( L(q_2) \) norm on the variable \( x_2 \) and using again the one-dimensional version for Riesz’s potential

\[
|u(\cdot, \cdot)|_{L_{q_1, R_1^{m_1}} L_{q_2, R_2^{m_2}}} \leq \frac{C_2(m_1; \alpha_1, \beta_1, \gamma_1)}{[(p_1 - p^{(1)}_\beta)(p^{(1)}_\beta - p_1)]^{\kappa_1}} \times \frac{C_2(m_2; \alpha_2, \beta_2, \gamma_2)}{[(p_2 - p^{(2)}_\beta)(p^{(2)}_\beta - p_2)]^{\kappa_2}} \cdot |h|_{p_2, R_2^{m_2}}
\]

or equally

\[
|u(\cdot)|_{\bar{q}} \leq \frac{\prod_{j=1}^l C_2(m_j; \alpha_j, \beta_j, \gamma_j)}{\prod_{j=1}^l [(p^{(j)}_\beta - p_j)(p_j - p_j^{(j)})]^{\kappa_j}} \cdot |f|_{\bar{p}}, \quad (2.1.7)
\]

which is equivalent to the right-hand side inequality of theorem 2.1 with

\[
C_4(\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}) = \prod_{j=1}^l C_2(m_j; \alpha_j, \beta_j, \gamma_j). \quad (2.1.8)
\]
The lower estimation for $|I_{\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}}[f](\vec{v})|_{\vec{q}}$ may be obtained by means of consideration of factorized function of a view

$$f^{(0)}(\vec{x}) = \prod_{j=1}^{l} f_{m_j}(x_j), \quad (2.1.9)$$

and analogously

$$C_3(\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}) = \prod_{j=1}^{l} C_1(m_j; \alpha_j, \beta_j, \gamma_j). \quad (2.1.10)$$

More exactly, the equalities (2.1.8) and (2.1.10) imply that the constants $C_3(\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma})$ and $C_4(\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma})$ in (2.1.6) may be estimates as follows:

$$C_3(\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}) \geq \prod_{j=1}^{l} C_1(m_j; \alpha_j, \beta_j, \gamma_j), \quad (2.1.11)$$

$$C_4(\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}) \leq \prod_{j=1}^{l} C_2(m_j; \alpha_j, \beta_j, \gamma_j). \quad (2.1.12)$$

The necessity of relations (2.1.1) provided by means of the so-called dilation method, see [54], [55] alike the one-dimensional case [42].

**B. Weight Fourier transforms.**

We consider in this subsection the multidimensional (vector): $l \geq 2$ generalization of PBO inequality. Namely, we investigate the inequality of a view

$$| F_{\alpha,\beta}[f](\vec{y}) |_{\vec{q}} \leq K_{d,\vec{\alpha},\vec{\beta}}(\vec{p}) | f(\vec{x}) |_{\vec{p}} \quad (2.2.1)$$

or for simplicity

$$| y|^{-\alpha} F(y) |_{q} \leq K_{d,\alpha,\beta}(p) | x|^{\beta} f(x) |_{p}. \quad (2.2.2)$$

Let us impose the following constrain:

$$1 < p_j < \infty, 0 \leq \alpha_j < m_j/q_j, 0 \leq \beta_j < m_j/p'_j \quad (2.2.3)$$

or equally $p_j > m_j/(m_j - \beta_j)$;

$$[p'_j \overset{def}{=} p/(p-1)], \quad \beta_j - \alpha_j = m_j(1 - 1/p_j - 1/q_j). \quad (2.2.4)$$

**Theorem 2.2.**

1. The conditions (2.2.3) and (2.2.4) are necessary and sufficient for the existence and finiteness of the constant $K_{d,\vec{\alpha},\vec{\beta}}(\vec{p})$ for the inequality (2.2.1.)

2. If the conditions (2.2.3) and (2.2.4) are satisfied, then the sharp (minimal) value of the coefficient $K_{d,\vec{\alpha},\vec{\beta}}(\vec{p})$ satisfies the inequalities
\[ C_1(d, \vec{\alpha}, \vec{\beta}) \prod_{j=1}^{l} \left[ \frac{p_j}{p_j - m_j/(m_j - \beta_j)} \right]^{(\alpha_j + \beta_j)/m_j} \leq K_{d,\vec{\alpha},\vec{\beta}}(\vec{p}) \leq C_2(d, \vec{\alpha}, \vec{\beta}) \prod_{j=1}^{l} \left[ \frac{p_j}{p_j - m_j/(m_j - \beta_j)} \right]^{\max(1, (\alpha_j + \beta_j)/m_j)}. \]  

(2.2.5)

**Proof** of the second proposition is at the same as the proof of the second proposition of theorem 2.1 and may be omitted. It based on the one-dimensional case \( l = 1 \) and contained particularly in the articles [41], [42].

The first assertion of theorem 2.2 may be proved again by the dilation method. Indeed, we have in the case \( l = 1 \) and for \( \lambda = \text{const} > 0 \):

\[ |T_\lambda[f]|_p = \lambda^{-d/p} |f|_p; \quad F_{\alpha,\beta}[T_\lambda f](x/\lambda) = \lambda^{-d+\alpha-\beta} F_{\alpha,\beta}[f](x); \]

\[ F_{\alpha,\beta}[T_\lambda f](x) = \lambda^{-d+\alpha-\beta} F_{\alpha,\beta}[f](\lambda x); \]

\[ |F_{\alpha,\beta}[T_\lambda f](\cdot)|_q = \lambda^{-d+\alpha-\beta-d/q} |F_{\alpha,\beta}[f](\cdot)|_q; \]

therefore

\[ -d + \alpha - \beta - d/q = -d/p, \quad \alpha - \beta = d \left(1 + \frac{1}{q} - \frac{1}{p}\right). \]

In the multidimensional case we construct the ”counter-example” functions as follows:

\[ f(\vec{x}) = \prod_{j=1}^{l} g_j(x_j), \quad 0 \neq g_j(\cdot) \in S(R^{m_j}), \]

and obviously

\[ \alpha_j - \beta_j = m_j \left(1 + \frac{1}{q_j} - \frac{1}{p_j}\right). \]

**Remark 2.2.1.** It is proved also in [41] that the classical multidimensional PBO inequality, i.e. in the ordinary Lebesgue spaces \( L_p \), see e.g. [29], p. 313-315; [14] etc. may be obtained by virtue of equality \( |f|_{p,p} = |f|_p \) from theorem 2.2 as a particular case iff

\[ \frac{\beta_j - \alpha_j}{m_j} = \text{const}, \quad j = 1, 2, \ldots, l. \]

**Remark 2.2.2.** The particular case \( \vec{\alpha} = \vec{\beta} = 0 \) was considered by Benedek A. and Panzone R. in the year 1961 [3]; see also [14].
3 Multiple weight Riesz and Fourier transforms in anisotropic Grand Lebesgue-Riesz spaces.

0. Let $Q$ be appropriate for concrete considered problem: Riesz potential (R) or Fourier transform (F) convex (bounded or not) subset of the set $\otimes_{j=1}^{n} [1, \infty]$. Let $\psi = \psi(\vec{p})$ be continuous in an interior $Q^0$ of the set $Q$ strictly positive function such that

$$\inf_{\vec{p} \in Q^0} \psi(\vec{p}) > 0; \inf_{\vec{p} \notin Q^0} \psi(\vec{p}) = \infty.$$  

Let $f(\vec{x}) = f(x)$ be some function from the space $AG\psi$. We denote for both the considered problem the introduced before correspondent function $\vec{q} = \vec{q}(\vec{p})$ and denote the inverse function by $\vec{p} = \vec{p}(\vec{q})$.

**Problem R.** Define a new function

$$\nu_R(\vec{q}) = \psi(\vec{p}(\vec{q})) \cdot K_{\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}(\vec{p}(\vec{q})).$$

**Theorem 3.1.**

$$||I_{\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}[f](\vec{q})||_{AG\nu_R} \leq 1 \cdot ||f||_{AG\psi}, \quad (3.1)$$

where the constant "1" is the best possible.

**Problem F.** Define a new function

$$\nu_F(\vec{q}) = \psi(\vec{p}(\vec{q})) \cdot K_{\vec{d}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}(\vec{p}(\vec{q})).$$

**Theorem 3.2.**

$$||F_{\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}[f](\vec{q})||_{AG\nu_F} \leq 1 \cdot ||f||_{AG\psi}, \quad (3.2)$$

where the constant "1" is the best possible.

**Proofs.** It is sufficient to prove theorem 3.1; the second proposition (3.2) provided analogously.

Let $f(\cdot) \in AG\psi$; we can suppose without loss of generality $||f||_{AG\psi} = 1$. This imply that

$$|f|_{\vec{p}} \leq \psi(\vec{p}).$$

We have using the result of theorem 2.1

$$|u(\cdot)|_{\vec{q}} \leq K_{\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}(\vec{p}) \cdot \psi(\vec{p}) \leq K_{\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}(\vec{p}) \cdot \psi(\vec{p}) \cdot ||f||_{AG\psi}. \quad (3.3)$$
As long as the variable $\vec{p}$ is uniquely defined monotonic function on $\vec{q}$, the inequality (3.2) is equivalent to the assertion of theorem 3.1.

The exactness of this estimation is proved in one-dimensional $l = 1$ general case in the article [41]; the multidimensional case $l \geq 2$ provided analogously.

\[\Box\]

4 Multiple Riesz potential in bounded domains.

We consider in this subsection the truncated Riesz’s operator of a view

\[u^{(B)} = u^{(B)}(x) = I_{\alpha,\beta,\gamma}^{(B)}[f](x) = I_{d;\alpha,\beta,\gamma}^{(B)}[f](x) = |x|^{-\beta} \int_{B} \frac{f(y) |y|^{-\alpha} dy}{|x-y|^{\gamma}}, \quad (4.1)\]

where $B$ is open bounded domain in $R^d$ contained the origin and such that

\[0 < \inf_{x \in \partial B} |x| \leq \sup_{x \in \partial B} |x| < \infty, \quad (4.2)\]

$\partial B$ denotes boundary of the set $B$.

For instance, we can assume that the set $B$ is unit ball in the space $R^d$ with the center in origin and with finite positive radii:

\[B = \{x, \ x \in R^d, \ |x| < r\}, \ r = \text{const} \in (0, \infty).\]

**Definition 4.1** We will call the subsets $\{B\}$ satisfying formulated conditions (4.2) as **interior** domains. Notation: $B \in I(R^d)$.

In contradiction, the set $D$ in the space $R^d$ is said to be **exterior** domain if by definition the open complement of $D$: $(R^d \setminus D)^c$ is **interior** domain. Notation: $D \in E(R^d)$.

It is proved in fact in the article [41] that in our notations and under our assertions if $B \in I(R^d)$, then there is a positive finite constant $K_R^{(B)}(d;\alpha,\beta,\gamma)$ for which

\[|I_{\alpha,\beta,\gamma}^{(B)}[f]|_q \leq K_R^{(B)}_{d;\alpha,\beta,\gamma}(p) |f|_p, \ p \in (1,p_+), \quad (4.3)\]

where by definition

\[K_R^{(B)}_{d;\alpha,\beta,\gamma}(p) = \sup_{p \in (1,p_+)} \sup_{0 < |f| < \infty} \left[ \frac{|I_{\alpha,\beta,\gamma}^{(B)}[f]|_q(p)}{|f|_p} \right].\]

It is known [41] that the coefficient $K_R^{(B)}(d;\alpha,\beta,\gamma)$ is under formulated before conditions bilateral bounded:
\[
\frac{C(1)(B(R; d; \alpha, \beta, \gamma))}{|p_+ - p|^\kappa} \leq K_{K;\alpha,\beta,\gamma}(p) \leq \frac{C(2)(B(R; d; \alpha, \beta, \gamma))}{|p_+ - p|^\kappa}.
\]

(4.4)

We define as in section 2 the following multiple weight Riesz (linear) potential in bounded domain \( B \):

\[
u^{(B)}(\vec{x}) = \nu_{\vec{\alpha},\vec{\beta},\vec{\gamma}}^{(B)}[f](\vec{x}) = \nu_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}^{(B)}[f](\vec{x}) = \text{\bigotimes}_{j=1}^l \nu_{\vec{m}_j;\alpha_j,\beta_j,\gamma_j}[f](\vec{x}) = \int_B \frac{|x_1|^{\beta_1} |y_1|^{\alpha_1} \, dy_1}{|x_1 - y_1|^{\gamma_1}} \left[ \int_B \left| \frac{x_2|^{\beta_2} |y_2|^{\alpha_2} \, dy_2}{|x_2 - y_2|^{\gamma_2}} \left[ \cdots \left[ \int_B \frac{|x_l|^{\beta_l} |y_l|^{\alpha_l} \, dy_l}{|x_l - y_l|^{\gamma_l}} \right] \right] \right].
\]

(4.5)

The following result based on the estimation (4.4) is obtained analogously to the proof of theorem 2.1.

**Theorem 4.1a.**

Let \( B \in I(R^d) \). The conditions (2.1.1) are necessary and sufficient for the existence of non-trivial coefficient \( K^{(B)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \) for the following estimate:

\[
|\nu_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}^{(B)}[f](\vec{x})|_q \leq K^{(B)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \ |f(\cdot)|_{\vec{p}},
\]

where for the minimal value of coefficient \( K^{(B)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \) holds the following inequality:

\[
\frac{C_3^{(B)}(\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma})}{\prod_{j=1}^l \left( p_j^{\beta_j} - p_j \right)^{\alpha_j} \gamma_j} \leq K^{(B)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \leq \frac{C_4^{(B)}(\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma})}{\prod_{j=1}^l \left( p_j^{\beta_j} - p_j \right)^{\alpha_j} \gamma_j},
\]

(4.7)

if for all the values \( j \Rightarrow p_j \in (1, p_j^{(j)}) \) and \( K^{(B)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) = \infty \) in other case.

For the exterior domain \( D \) we conclude:

**Theorem 4.1b.**

Let \( D \in E(R^d) \). The conditions (2.1.1) are necessary and sufficient for the existence of non-trivial coefficient \( K^{(D)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \) for the following estimate:

\[
|\nu_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}^{(D)}[f](\vec{x})|_q \leq K^{(D)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \ |f(\cdot)|_{\vec{p}},
\]

where for the minimal value of coefficient \( K^{(D)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \) holds the following inequality:

\[
\frac{C_3^{(D)}(\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma})}{\prod_{j=1}^l \left( p_j^{\beta_j} - p_j \right)^{\alpha_j} \gamma_j} \leq K^{(D)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \leq \frac{C_4^{(D)}(\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma})}{\prod_{j=1}^l \left( p_j^{\beta_j} - p_j \right)^{\alpha_j} \gamma_j},
\]

(4.9)

if for all the values \( j \Rightarrow p_j \in (p_j^{(j)}, \infty) \) and \( K^{(D)}_{\vec{m};\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) = \infty \) in other case. \( \square \)
5 Concluding remarks.

A. Generalization on the regular varying weights.

We consider in this subsection some generalization on the weight Riesz’s potential (and further on the Fourier weight operator) of a view

\[ I_{d,\alpha,\delta}^{(S)} f(x) = \int_{\mathbb{R}^d} \frac{f(y) |\log |x-y||^\delta S(|\log |x-y||)}{|x-y|^{d-\alpha}} \, dy, \] (5.1A.0)

where \( \alpha = \text{const} \in (0, d), \, \delta = \text{const} > 0, \, p \in (1, d/\alpha), \) \( p \) is the following function of variable \( q : \)

\[ p = p(q) = \frac{dq}{d+\alpha q}, \] (5.1A.1)

and conversely \( q = q(p), \) \( \vec{q} = \vec{q}(\vec{p}); \) \( S(z) \) is a \textit{slowly varying} as \( z \to \infty \) continuous positive function:

\[ \forall x > 0 \Rightarrow \lim_{z \to \infty} S(xz)/S(z) = 1. \]

We refer the reader to the book of Seneta [52] to the using further facts about regular and slowly varying functions.

It is known, see [41] that under some simple conditions

\[ |I_{d,\alpha,\delta}^{(S)} f|_q \leq \frac{C(S) |f|_p}{(|p-1)|d/\alpha-p|^{1-\alpha/d}}, \, p \in (1, d/\alpha). \] (5.1A.2)

and the last inequality is exact up to multiplicative constant.

We define as in section 2 the following \textit{multiple} Riesz (linear) potential with regular varying weight

\[ z^{(S)}(\vec{x}) = I^{(S)}_{\vec{m},\vec{\alpha},\vec{\delta}}[f](\vec{x}) = I^{(S)}_{m_j,\alpha_j,\delta_j}[f](\vec{x}) = \otimes_{j=1}^{l} I^{(S)}_{m_j,\alpha_j,\delta_j}[f](\vec{x}). \] (5.1A.3)

where \( \alpha_j = \text{const} \in (0, m_j), \, \delta_j = \text{const} > 0, \, p_j \in (1, m_j/\alpha_j), \) \( p_j \) is the following function of variable \( q_j : \)

\[ p_j = p_j(q_j) = \frac{m_j q_j}{m_j + \alpha_j q_j} \] (5.1A.4)

and conversely \( q_j = q_j(p_j), \) \( \vec{q} = \vec{q}(\vec{p}); \) \( \{S_j(z)\} \) are slowly varying as \( z \to \infty \) continuous positive functions.

The following result based on the estimation (5.1A.2) is obtained analogously the proof of theorem 2.1.

**Theorem 5.1A.1.**

The conditions (5.1A.1) are necessary and sufficient for the existence non-trivial coefficient \( K^{(S)}_{m,\vec{m},\vec{\alpha},\vec{\delta}}(\vec{p}) \) for the following estimate:
\[ |I_{m,\alpha,\delta}^{(s)}[f]|_{q} \leq K_{m,\alpha,\delta}^{(s)}(\vec{p}) \ |f(\cdot)|_{\vec{p}}. \]  

(5.1A.4)

where for the minimal value of coefficient \( K_{m,\alpha,\delta}^{(s)}(\vec{p}) \) holds the following inequality:

\[
\frac{C_{3}^{(s)}(m,\alpha,\delta)}{\prod_{j=1}^{l} [(p_j - 1) (m_j/\alpha_j - p_j)]^{1+\delta_j-\alpha_j/m_j}] \leq K_{m,\alpha,\delta}^{(s)}(\vec{p}) \leq \frac{C_{4}^{(s)}(m,\alpha,\delta)}{\prod_{j=1}^{l} [(p_j - 1) (m_j/\alpha_j - p_j)]^{1+\delta_j-\alpha_j/m_j}] .
\]  

(5.1A.5)

if for all the values \( j \Rightarrow p_j \in (1, m_j/\alpha_j) \) and \( K_{m,\alpha,\delta}^{(s)}(\vec{p}) = \infty \) in other case.

Analogously may be provided the multiple regular varying weight generalization of PBO inequality on the Anisotropic Lebesgue-Riesz spaces; ordinary case is investigated in [41].

Let \( L_j = L_j(z), M_j = M_j(z), j = 1, 2, \ldots, l, z \in (0, \infty) \) be a family of slowly varying simultaneously as \( z \to 0 \) and as \( z \to \infty \) continuous positive functions:

\[
\lim_{z \to 0} \frac{L_j(xz)}{L_j(z)} = \lim_{z \to \infty} \frac{L_j(xz)}{L_j(z)} = 1;
\]

\[
\lim_{z \to 0} \frac{M_j(xz)}{M_j(z)} = \lim_{z \to \infty} \frac{M_j(xz)}{M_j(z)} = 1.
\]

Let us consider the following inequality:

\[
\left| |\vec{y}|^{-\vec{\alpha}} \prod_{j=1}^{l} L_j(|y_j|) F[f](\vec{y}) \right| \leq K_{L,M,m,\alpha,\beta}(\vec{p}) \left| |\vec{x}|^{\vec{\beta}} \prod_{j=1}^{l} M_j(|x_j|) f(\vec{x}) \right|_{\vec{p}}. \]  

(5.2A.1)

Let us impose the following restrictions:

\[
1 < p_j \leq q_j < \infty, 0 \leq \alpha_j < m_j/q_j, 0 \leq \beta_j < m_j/p_j';
\]  

(5.2A.2)

or equally \( p_j > m_j/(m_j - \beta_j) \);

\[
\beta_j - \alpha_j = m_j(1 - 1/p_j - 1/q_j);
\]  

(5.2A.3)

\[
M_j(z) \approx L_j(1/z) \iff 0 < \inf_{z} \frac{M_j(z)}{L_j(1/z)} \leq \sup_{z} \frac{M_j(z)}{L_j(1/z)} < \infty.
\]  

(5.2A.4)

**Theorem 5.2A.1.**

1. The conditions (5.2A.2), (5.2A.3) and (5.2A.4) are necessary and sufficient for the existence and finiteness of the constant \( K_{L,M,m,\alpha,\beta}(\vec{p}) \) for the inequality (5.2A.1).

2. If the conditions (5.2A.2), (5.2A.3) and (5.2A.4) are satisfied, then the sharp (minimal) value of the coefficient \( K_{L,M,m,\alpha,\beta}(\vec{p}) \) satisfies the inequalities
\[ C_1(\vec{L}, \vec{M}, \vec{m}, \vec{\alpha}, \vec{\beta}) \prod_{j=1}^{l} \left[ \frac{p_j}{p_j - m_j/(m_j - \beta_j)} \right]^{(\alpha_j + \beta_j)/m_j} \leq K_{\vec{L}, \vec{M}, \vec{m}, \vec{\alpha}, \vec{\beta}}(\vec{p}) \leq C_2(\vec{L}, \vec{M}, \vec{m}, \vec{\alpha}, \vec{\beta}) \prod_{j=1}^{l} \left[ \frac{p_j}{p_j - m_j/(m_j - \beta_j)} \right]^{\max(1, (\alpha_j + \beta_j)/m_j)}. \tag{5.2A.5} \]

**Proof** is at the same as the proof of theorem 2.1 and may be omitted. It based on the one-dimensional case \( l = 1 \) and contained particularly in the articles [41], [42].

**B. Composed Riesz and Fourier operators.**

Let the set \( J := [1, 2, \ldots, d] \) consists in two non-empty disjoint subsets (partition): \( J = J(R) \cup J(F) \), \( J(R) \cap J(F) = \emptyset \). We define the following composed linear Riesz-Fourier weight operator \( (R = \text{Riesz}, \ F = \text{Fourier}) \) as follows:

\[ W_{\vec{a}, \vec{\beta}, \vec{\gamma}}[f](\vec{x}) \overset{\text{def}}{=} \left( \otimes_{j \in J(R)} I_{\alpha_j, \beta_j, \gamma_j} \right) \otimes \left( \otimes_{j \in J(F)} F_{\alpha_j, \beta_j, \gamma_j} \right) [f](\vec{x}). \tag{5.1B.1} \]

We investigate in this pilcrow the inequality of a view

\[ | W_{\vec{a}, \vec{\beta}, \vec{\gamma}}[f] |_{\vec{q}} \leq K_{RF}(\vec{p}) |f|_{\vec{p}}. \tag{5.1B.2} \]

We impose the following conditions:

1. \( j \in J(F) \Rightarrow \)

\[ 1 < p_j \leq q_j < \infty, 0 \leq \alpha_j < m_j/q_j, 0 \leq \beta_j < m_j/p_j', \]

\[ \beta_j - \alpha_j = m_j(1 - 1/p_j - 1/q_j). \tag{5.1B.3} \]

2. \( j \in J(R) \Rightarrow \)

\[ \gamma_j = \text{const}, \alpha_j, \beta_j = \text{const} \geq 0, \alpha_j + \gamma_j < m_j, \]

\[ m_j + \frac{m_j}{q_j} = \frac{m_j}{p_j} + (\alpha_j + \beta_j + \gamma_j). \tag{5.1B.4} \]

The relations \( (5.1B.3) \) and \( (5.1B.4) \) define uniquely determined functions \( \vec{p} = p(\vec{q}) \) or conversely functions \( \vec{q} = q(\vec{p}) \).

**Proposition 5.1B.1.**

\( \alpha. \) The conditions \( (5.1B.3) \) and \( (5.1B.4) \) are necessary and sufficient for finiteness of the coefficient \( K_{RF}(\vec{p}) \) in \( (5.1B.2) \).
β. If this conditions are satisfied, then the minimal value of coefficient $K_{RF}(\tilde{p})$ is bilateral bounded as follows:

$$C_5(\tilde{m}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \frac{\prod_{j \in J(F)}(p_j / (p_j - m_j / (m_j - \beta_j)))^{-(\alpha_j + \beta_j - \gamma_j) / m_j}}{\prod_{j \in J(R)}((p_j - p_{-j})^{(p_{-j} - p_j)} \gamma_j)} \leq$$

$$K_{RF}(\tilde{p}) \leq C_6(\tilde{m}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \frac{\prod_{j \in J(F)}(p_j / (p_j - m_j / (m_j - \beta_j)))^{-\max(1, (\alpha_j + \beta_j - \gamma_j) / m_j)}}{\prod_{j \in J(R)}((p_j - p_{-j})^{(p_{-j} - p_j)} \gamma_j)}.$$  \hspace{1cm} (5.1B.5)

This assertion may be simple obtained by synthesis of theorems 2.1 and 2.2.

C. Mixture Riesz and Fourier operators.

The following (linear) operator

$$G_{\alpha, \beta, \gamma}[f](x) = |x|^{-\beta} \int_{R^d} e^{ixy} \frac{|y|^{-\alpha} f(y) \, dy}{|x - y|^\gamma}, \quad x, y \in R^d$$  \hspace{1cm} (5.1C.1)

is said to be "one-dimensional", i.e. $l = 1$ mixture Riesz and Fourier operator.

We consider also in this subsection its multidimensional $l \geq 2$ generalization

$$v(x) = G_{\alpha, \beta, \gamma}[f](x) = G_{\tilde{m}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}}[f](x) = \bigotimes_{j=1}^l G_{m_j, \alpha_j, \beta_j, \gamma_j}[f](x) =$$

$$\int_{R^{m_1}} e^{ix_1y_1} \frac{|x_1|^{-\beta_1} |y_1|^{-\alpha_1} \, dy_1}{|x_1 - y_1|^\gamma_1} \left[ \int_{R^{m_2}} e^{ix_2y_2} \frac{|x_2|^{-\beta_2} |y_2|^{-\alpha_2} \, dy_2}{|x_2 - y_2|^\gamma_2} \ldots \right. \left. \ldots \right.$$

$$\left[ \int_{R^{m_l}} e^{ix_ly_l} \frac{|x_l|^{-\beta_l} |y_l|^{-\alpha_l} f(\tilde{y}) \, dy_l}{|x_l - y_l|^\gamma_l} \right].$$  \hspace{1cm} (5.1C.2)

The inequality of a view (relative the Lebesgue measures in whole space $R^d$)

$$|G_{d, \alpha, \beta, \gamma}[f]|_q \leq K_{GPBO}(p) \ |f|_p$$  \hspace{1cm} (5.1C.3)

is said to be generalized (G) weight Pitt - Beckner - Okikiolu (GPBO) inequality.

We will understood as customary as the variable $K_{GPBO}(p)$ its minimal value:

$$K_{GPBO}(p) = \sup_{f \neq 0, |f|_p < \infty} \left[ \frac{|G_{d, \alpha, \beta, \gamma}[f]|_q}{|f|_p} \right].$$

Denote

$$\tilde{p}_{-} = \frac{d}{d - \beta}, \quad \tilde{q}_{+} = \frac{d}{\alpha - \gamma}, \quad \tilde{p}_{+} = \infty, \quad \tilde{q}_{-} = 1,$$  \hspace{1cm} (5.1C.4)

and suppose

$$\beta + \gamma - \alpha = d \left( 1 - \frac{1}{p} - \frac{1}{q} \right),$$
\[ \alpha - \gamma > 0, \gamma < d, \alpha + \beta > \gamma, \]
\[ p \in (\bar{p}_-, \bar{p}_+) \iff q \in (\bar{q}_-, \bar{q}_+), p < q. \]  \tag{5.1C.5} \]

**Proposition 5.1C.1.**

\[ C_1(d, \alpha, \beta, \gamma) \left( \frac{p}{p - \bar{p}_-} \right)^{(\alpha + \beta - \gamma)/d} \leq K_{GPBO}(p) \leq C_2(d, \alpha, \beta, \gamma) \max(1, (\alpha + \beta - \gamma)/d), 0 < C_{1,2}(d; \alpha, \beta, \gamma) < \infty. \]  \tag{5.1C.6} \]

The right hand side of this bilateral estimate may be obtained after some computations from the articles [19], [51], [21], [50], [57]. The opposite estimate may be obtained by means of at the same counterexample as in article [41].

Note that both the possibilities in the power in the right hand side in (5.1C.6): ”1” and 
\[(\alpha + \beta - \gamma)/d\]  are attainable.

We consider in this subsection the multidimensional (vector): \( l \geq 2 \) generalization of GPBO inequality.

Namely, we investigate the inequality of a view

\[ |G_{\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}[f](\vec{\cdot})|_{\vec{q}} \leq K_{\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}(\vec{p}) \, |f|_{\vec{p}}, \]

where as usually

\[ K_{\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}(\vec{p}) \overset{\text{def}}{=} \sup_{0 < |f|_{\vec{p}} < \infty} \left[ \frac{|G_{\vec{m}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}}[f](\vec{\cdot})|_{\vec{q}}}{|f|_{\vec{p}}} \right]. \]  \tag{5.1C.7} \]

Denote as before

\[ \bar{p}^{(j)} = \frac{m_j}{m_j - \beta_j}, \quad \bar{q}^{(j)} = \frac{m_j}{\alpha_j - \gamma_j}, \quad \bar{p}^+_+ = \infty, \quad \bar{q}^+_+ = 1, \]  \tag{5.1C.8} \]

and suppose

\[ \beta_j + \gamma_j - \alpha_j = m_j \left( 1 - \frac{1}{p_j} - \frac{1}{q_j} \right), \]

\[ \alpha_j - \gamma_j > 0, \gamma_j < m_j, \alpha_j + \beta_j > \gamma_j, \]

\[ p_j \in (\bar{p}^{(j)}, \bar{p}^+_+) \iff q_j \in (\bar{q}^{(j)}, \bar{q}^+_+), p_j \leq q_j, \]  \tag{5.1C.9} \]

so that in equality (5.1C.7) the vector \( \vec{q} \) is uniquely defined function on the vector \( \vec{p} : \vec{q} = \vec{q}(\vec{p}) \).
Theorem 5.1C.1.

1. The conditions (5.1C.9) are necessary and sufficient for finiteness of the coefficient $K_{\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p})$.

2. If the conditions (5.1C.9) are satisfied, then

$$C_1(\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}) \prod_{j=1}^{l} \left[ \frac{p_j}{p_j - \tilde{p}_j} \right]^{(\alpha_j+\beta_j-\gamma_j)/m_j} \leq K_{\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}}(\vec{p}) \leq C_2(\vec{m},\vec{\alpha},\vec{\beta},\vec{\gamma}) \prod_{j=1}^{l} \left[ \frac{p_j}{p_j - \tilde{p}_j} \right]^{\max(1,(\alpha_j+\beta_j-\gamma_j)/m_j)}.$$  \hspace{1cm} (5.1C.10)

Sketch of proof. The necessity of the conditions (5.1C.9) may be grounded as before by means of multidimensional dilation method; see e.g. [41]. The estimates (5.1C.10) may be proved as in the theorem 2.1 on the basis the one-dimensional version.

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