One loop superstring effective actions and $d = 4$ supergravity

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Abstract

We review our recent work on the existence of a new independent $\mathcal{R}^4$ term, at one loop, in the type IIA and heterotic effective actions, after reduction to four dimensions, besides the usual square of the Bel-Robinson tensor. We discuss its supersymmetrization.
1 $R^4$ terms in ten dimensions

The superstring $\alpha'^3$ effective actions contain two independent terms $X, Z$ which involve only the fourth power of the Weyl tensor, given by

$$X := t_8 t_8 W^4, \quad Z := - \varepsilon_{10} \varepsilon_{10} W^4. \quad (1)$$

For the heterotic string two other $R^4$ terms $Y_1$ and $Y_2$ appear at order $\alpha'^3$ \cite{1,2,3}:

$$Y_1 := t_8 (\text{tr} W^2)^2, \quad Y_2 := t_8 \text{tr} W^4 = \frac{X}{24} + \frac{Y_1}{4}. \quad (2)$$

Each $t_8$ tensor has eight free spacetime indices, acting in four two-index antisymmetric tensors as defined in \cite{4,5}.

The effective action of type IIB theory must be written, because of its well known $\text{SL}(2, \mathbb{Z})$ invariance, as a product of a single linear combination of order $\alpha'^3$ invariants and an overall function of $\Omega = C^0 + i e^{-\phi}$, $C^0$ being the axion and $\phi$ the dilaton. The $R^4$ terms of this effective action are given in the string frame by

$$\frac{1}{\sqrt{-g}} \mathcal{L}_{\text{IIB}} \bigg|_{\alpha'} = - e^{-2\phi} \alpha'^3 \frac{\zeta(3)}{3 \times 2^{10}} \left( X - \frac{1}{8} Z \right) - \alpha'^3 \frac{1}{3 \times 2^{16} \pi^5} \left( X - \frac{1}{8} Z \right). \quad (3)$$

The corresponding $\alpha'^3$ action of type IIA superstrings has a relative "-" sign flip in the one loop term \cite{6}, because of the different chirality properties of type IIA and type IIB theories, which reflects on the relative GSO projection between the left and right movers:

$$\frac{1}{\sqrt{-g}} \mathcal{L}_{\text{IIA}} \bigg|_{\alpha'} = - e^{-2\phi} \alpha'^3 \frac{\zeta(3)}{3 \times 2^{10}} \left( X - \frac{1}{8} Z \right) - \alpha'^3 \frac{1}{3 \times 2^{16} \pi^5} \left( X + \frac{1}{8} Z \right). \quad (4)$$

Heterotic string theories in $d = 10$ have $\mathcal{N} = 1$ supersymmetry, which allows corrections already at order $\alpha'$, including $R^2$ terms. These corrections come both from three and four graviton scattering amplitudes and anomaly cancellation terms (the Green-Schwarz mechanism). Up to order $\alpha'^3$, the terms from this effective action which involve only the Weyl tensor are given in the string frame by

$$\frac{1}{\sqrt{-g}} \mathcal{L}_{\text{heterotic}} \bigg|_{\alpha' + \alpha'^3} = e^{-2\phi} \left[ \frac{1}{16} \alpha' \text{tr} R^2 + \frac{1}{2^9} \alpha'^3 Y_1 - \frac{\zeta(3)}{3 \times 2^{10}} \alpha'^3 \left( X - \frac{1}{8} Z \right) \right] - \alpha'^3 \frac{1}{3 \times 2^{14} \pi^5} (Y_1 + 4Y_2). \quad (5)$$

Next we will take these terms reduced to four dimensions, in the Einstein frame, in order to consider them in the context of supergravity.
2 \( \mathcal{R}^4 \) terms in four dimensions

In \( d = 4 \), the Weyl tensor can be decomposed in its self-dual and ant-self-dual parts:

\[
\mathcal{W}_{\mu\nu\rho\sigma} = \mathcal{W}_{\mu\nu\rho\sigma}^+ + \mathcal{W}_{\mu\nu\rho\sigma}^-.
\]

\[
\mathcal{W}_{\mu\nu\rho\sigma}^\pm := \frac{1}{2} \left( \mathcal{W}_{\mu\nu\rho\sigma} \pm \frac{i}{2} \varepsilon_{\mu\nu}{}^\lambda \mathcal{W}_{\lambda\tau\rho\sigma} \right).
\] (6)

The totally symmetric Bel-Robinson tensor is given by

\[
\mathcal{W}_{\mu\nu\rho\sigma}^+ \mathcal{W}_{\mu\nu\rho\sigma}^- \mathcal{W}_{\mu\nu\rho\sigma}^\pm.
\]

In the van der Warden notation, using spinorial indices, to \( \mathcal{W}_{\mu\nu\rho\sigma}^+ \) and \( \mathcal{W}_{\mu\nu\rho\sigma}^- \) correspond the totally symmetric \( \mathcal{W}_{ABCD} \). \( \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}} \) being given by (in the notation of [7])

\[
\mathcal{W}_{ABCD} := -\frac{1}{8} \mathcal{W}_{\mu\nu\rho\sigma}^+ \sigma^{\mu\nu}_{AB} \sigma^{\rho\sigma}_{CD}, \quad \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}} := -\frac{1}{8} \mathcal{W}_{\mu\nu\rho\sigma}^- \sigma^{\mu\nu}_{\dot{A}\dot{B}} \sigma^{\rho\sigma}_{\dot{C}\dot{D}}.
\]

The decomposition (6) is written as

\[
\mathcal{W}_{A\dot{A}B\dot{B}C\dot{C}D\dot{D}} = -2 \varepsilon_{AB} \varepsilon_{\dot{C}\dot{D}} \mathcal{W}_{ABCD} - 2 \varepsilon_{AB} \varepsilon_{\dot{C}\dot{D}} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}};
\]

the Bel-Robinson tensor is simply given by \( \mathcal{W}_{ABCD} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}} \).

In four dimensions, there are only two independent real scalar polynomials made from four powers of the Weyl tensor [8], given by

\[
\mathcal{W}^2_+ \mathcal{W}^2_- = \mathcal{W}^{ABCD}_+ \mathcal{W}^{ABCD}_- \mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}}_+ \mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}}_-.
\] (7)

\[
\mathcal{W}^4_+ + \mathcal{W}^4_- = (\mathcal{W}^{ABCD}_+ \mathcal{W}^{ABCD}_-)^2 + (\mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}}_+ \mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}}_-)^2.
\] (8)

In particular, the terms \( X, Z, Y_1, Y_2 \), when computed directly in \( d = 4 \) (i.e. expanded only in terms of the Weyl tensor and restricting the sums over contracted indices to four dimensions), should be expressed in terms of them. The details of the calculation can be seen in [9]: \( X - \frac{1}{8} Z \) is the only combination of \( X \) and \( Z \) which in \( d = 4 \) does not contain \( \mathcal{W}^2 \), i.e. which contains only the square of the Bel-Robinson tensor \( \mathcal{W}^2 \); \( Y_1 \) (but not \( Y_2 \)) is also only expressed in terms of \( \mathcal{W}^2 \). We then write the effective actions (3), (4), (5) in four dimensions, in the Einstein frame (considering only terms which are simply powers of the Weyl tensor, without any other fields except their couplings
to the dilaton, and introducing the $d = 4$ gravitational coupling constant $\kappa$):

$$\frac{\kappa^2}{\sqrt{-g}} L_{\text{IIB}} \bigg|_{\mathcal{R}^4} = -\frac{\zeta(3)}{32} e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 - \frac{1}{2^{11} \pi^5} e^{-4\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2,$$  

(9)

$$\frac{\kappa^2}{\sqrt{-g}} L_{\text{IA}} \bigg|_{\mathcal{R}^4} = -\frac{\zeta(3)}{32} e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2$$

$$- \frac{1}{2^{12} \pi^5} e^{-4\phi} \alpha'^3 \left[ (\mathcal{W}_+^4 + \mathcal{W}_-^4) + 224 \mathcal{W}_+^2 \mathcal{W}_-^2 \right],$$

(10)

$$\frac{\kappa^2}{\sqrt{-g}} L_{\text{het}} \bigg|_{\mathcal{R}^2 + \mathcal{R}^4} = -\frac{1}{16} e^{-2\phi} \alpha' \left( \mathcal{W}_+^2 + \mathcal{W}_-^2 \right) + \frac{1}{64} \left( 1 - 2\zeta(3) \right) e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2$$

$$- \frac{1}{3 \times 2^{12} \pi^5} e^{-4\phi} \alpha'^3 \left[ (\mathcal{W}_+^4 + \mathcal{W}_-^4) + 20 \mathcal{W}_+^2 \mathcal{W}_-^2 \right].$$

(11)

These are only the moduli-independent $\mathcal{R}^4$ terms. Strictly speaking not even these terms are moduli-independent, since they are all multiplied by the volume of the compactification manifold, a factor we omitted for simplicity. But they are always present, no matter which compactification is taken. The complete action, for every different compactification manifold, includes many other moduli-dependent terms which we do not consider here: we are mostly interested in a $T^6$ compactification.

### 3 $\mathcal{R}^4$ terms and four-dimensional supergravity

We are interested in the full supersymmetric completion of $\mathcal{R}^4$ terms in $d = 4$. In general each superinvariant consists of a leading bosonic term and its supersymmetric completion, given by a series of terms with fermions.

The supersymmetrization of the square of the Bel-Robinson tensor $\mathcal{W}_+^2 \mathcal{W}_-^2$ has been known for a long time, in simple [10, 11] and extended [12, 13] four dimensional supergravity. For the term $\mathcal{W}_+^4 + \mathcal{W}_-^4$, which appears at one string loop in the type IIA and heterotic effective actions (10) and (11), there is a "no-go theorem", which goes as follows [14]: for a polynomial $I(\mathcal{W})$ of the Weyl tensor to be supersymmetrizable, each one of its terms must contain equal powers of $\mathcal{W}_{\mu\nu\rho\sigma}^+$ and $\mathcal{W}_{\mu\nu\rho\sigma}^-$. The whole polynomial must then vanish when either $\mathcal{W}_{\mu\nu\rho\sigma}^+$ or $\mathcal{W}_{\mu\nu\rho\sigma}^-$ do.

The derivation of this result is based on $\mathcal{N} = 1$ chirality arguments, which require equal powers of the different chiralities of the gravitino in each term of a superinvariant. The rest follows from the supersymmetric completion. That is why the only exception to this result is $\mathcal{W}_+^2 + \mathcal{W}_-^2$, which appears in (11): this term is part of the $d = 4$ Gauss-Bonnet topological invariant (it can be made equal to it with suitable
field redefinitions). This term plays no role in the dynamics and it is automatically supersymmetric; its supersymmetric completion is 0 and therefore does not involve the gravitino.

The derivation of [14] has been obtained using $\mathcal{N} = 1$ supergravity, whose supersymmetry algebra is a subalgebra of $\mathcal{N} > 1$. Therefore, it should remain valid for extended supergravity too. But one must keep in mind the assumptions which were made, namely the preservation by the supersymmetry transformations of $R$-symmetry which, for $\mathcal{N} = 1$, corresponds to U(1) and is equivalent to chirality. In extended supergravity theories $R$–symmetry is a global internal U($\mathcal{N}$) symmetry, which generalizes (and contains) U(1) from $\mathcal{N} = 1$.

Preservation of chirality is true for pure $\mathcal{N} = 1$ supergravity, but to this theory and to most of the extended supergravity theories one may add matter couplings and extra terms which violate U(1) $R$-symmetry and yet can be made supersymmetric, inducing corrections to the supersymmetry transformation laws which do not preserve U(1) $R$-symmetry. That was the procedure taken in [9], through the superspace lagrangian

$$\mathcal{L} = \frac{1}{4\kappa^2} \int \epsilon \left[ \left( \nabla^2 + \frac{1}{3} R \right) \left( \Omega (\Phi, \bar{\Phi}) + \alpha^\alpha \left( b\Phi (\nabla^2 W)^2 + \bar{b}\Phi \left( \nabla^2 W^2 \right)^2 \right) \right] - 8 P (\Phi)] d^2 \theta + \text{h.c.} \right. \right. \right. (12)$$

$\epsilon$ is the chiral density; $\nabla^2 + \frac{1}{3} R$ is the chiral projector; $\Phi$ is a chiral superfield;

$$K \left( \Phi, \overline{\Phi} \right) = - \frac{3}{\kappa^2} \ln \left(- \frac{\Omega (\Phi, \overline{\Phi})}{3} \right), \quad \Omega \left( \Phi, \overline{\Phi} \right) = - 3 + \Phi \overline{\Phi} + c\Phi + \bar{c}\overline{\Phi}$$

is a Kähler potential and

$$P (\Phi) = d + a\Phi + \frac{1}{2} m\Phi^2 + \frac{1}{3} g\Phi^3$$

is a superpotential. $W_{ABC}$ is the chiral $\mathcal{N} = 1$ superfield such that, at the linearized level, $\nabla D W_{ABCD} = W_{ABCD} + \ldots$; $W_+^{-} + W_+^{+}$ is proportional to $(\nabla^2 W^2)^2 + \text{h.c.}$ This term appears in the supersymmetric lagrangian (12) after eliminating of the auxiliary fields $F = - \frac{1}{2} \nabla^2 \Phi$ and $\overline{F}$.

A similar procedure may be taken in $\mathcal{N} = 2$ supergravity, since there exist $\mathcal{N} = 2$ chiral superfields which must be Lorentz and SU(2) scalars but can have an arbitrary U(1) weight, allowing for supersymmetric U(1) breaking couplings.

Such a result should be more difficult to achieve for $\mathcal{N} \geq 3$, because there are no generic chiral multiplets. But for $3 \leq \mathcal{N} \leq 6$ there are still matter multiplets.
which one can couple to the Weyl multiplet. Those couplings could eventually (but not necessarily) break $U(1) R$-symmetry and lead to the supersymmetrization of $(8)$.

An even more complicated problem is the $\mathcal{N} = 8$ supersymmetrization of $(8)$. The reason is the much more restrictive character of $\mathcal{N} = 8$ supergravity, compared to lower $\mathcal{N}$. Besides, its multiplet is unique, which means there are no extra matter couplings one can take in this theory. Plus, in this case the $R$-symmetry group is $SU(8)$ and not $U(8)$: the extra $U(1)$ factor, which in $\mathcal{N} = 2$ could be identified with the remnant $\mathcal{N} = 1 R$-symmetry and, if broken, eventually turn the supersymmetrization of $(8)$ possible, does not exist. Apparently there is no way to circumvent in $\mathcal{N} = 8$ the result from [14]. In order to supersymmetrize $(8)$ in this case one should then explore the different possibilities which were not considered in [14]. Since that article only deals with the term $(8)$ by itself, in [15] we considered extra couplings to it and only then tried to supersymmetrize. This procedure is very natural, taking into account the scalar couplings that multiply $(8)$ in the actions $(10)$, $(11)$.

We therefore considered the linearized superfield expressions which, when expressed in terms of $x$–space fields, would result in $(8)$, multiplied by some scalar fields. We did not obtain any expression which was supersymmetric, not even with nonlinear supersymmetric transformations. Therefore we cannot expect $(8)$ to emerge from the nonlinear completion of some (necessarily $\alpha'^3$) linearized superinvariant. One must really understand the full $\alpha'$-corrections to the Bianchi identities. Since these corrections are necessarily nonlinear, this means one cannot supersymmetrize $(8)$ at the linearized level at all. That never happened for any of the previously known higher-order terms, which all had its linearized superinvariant.

The main obstruction to this supersymmetrization is that, as we argued in [15], $(8)$ is not compatible with the full $R$–symmetry group $SU(8)$. Indeed only the local symmetry group of the moduli space of compactified string theories (for type II superstrings on $\mathbb{T}^6$, $SU(4) \otimes SU(4)$) should be preserved by the four dimensional perturbative string corrections. Most probably, $(8)$ only has this later symmetry. If that is the case, in order to supersymmetrize this term besides the supergravity multiplet one must also consider $U$–duality multiplets, with massive string states and nonperturbative states. These would be the contributions we were missing.

But in conventional extended superspace one cannot simply write down a superinvariant that does not preserve the $SU(\mathcal{N}) R$–symmetry, which is part of the structure group. One can only consider higher order corrections to the Bianchi identities preserving $SU(\mathcal{N})$, which would not be able to supersymmetrize $(8)$. $\mathcal{N} = 8$ supersymmetrization of this term would then be impossible.

The fact that one cannot supersymmetrize in $\mathcal{N} = 8$ a term which string theory requires to be supersymmetric, together with the fact that one needs to consider non-
perturbative states (from $U$–duality multiplets) in order to understand a perturbative contribution may be seen as indirect evidence that $\mathcal{N} = 8$ supergravity is indeed in the swampland, as proposed in [16]. This topic deserves further study.

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