A Local Limit Theorem for Robbins-Monro Procedure

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November 22, 2018

Abstract

The Robbins-Monro algorithm is a recursive, simulation-based stochastic procedure to approximate the zeros of a function that can be written as an expectation. It is known that under some technical assumptions, a Gaussian convergence can be established for the procedure. Here, we are interested in the local limit theorem, that is, quantifying this convergence on the density of the involved objects. The analysis relies on a parametrix technique for Markov chains converging to diffusions, where the drift is unbounded.

1 Introduction and Assumptions

This paper is devoted to the study of a Local Limit Theorem for a Robbins Monro procedure. These algorithms have first been introduced in [7] to approximate the solution of an equation

\[ h(\theta) = 0 \]

(1.1)

where \( h(\theta) \) can be written as an expectation. Since then, extensive literature have been published on the subject, but to the best of our knowledge, the local limit theorem has never been obtained. In this work, we limit ourselves to a simpler version of the one dimensional Robbins-Monro algorithms that already shows the technical difficulties we have to overcome in order to obtain the local limit theorem. We refer to the monograph from Benveniste Metivier and Priouret [1] or Nevelson and Khas’minskii [6] for a general presentation of these algorithms and a review of the literature.

We fix probability space \((\Omega, \mathcal{F}, P)\) where all the random variables we consider below are defined. Let \((\gamma_k)_{k \geq 0}\) be a decreasing time step that will be specified later, and \((\eta_k)_{k \geq 0}\) a collection of independent and identically distributed random variables. We define the following recursive procedure:

\[ \theta_{n+1} = \theta_n - \gamma_{n+1} \cdot \sigma(m(\theta_n) - \eta_{n+1}), \quad \theta_0 \in \mathbb{R}, \quad (1.1) \]

This algorithm is a special case of the general Robbins-Monro procedure which writes:

\[ \theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, \eta_{n+1}), \]

and where the innovations \( \eta_k \) can have a Markovian structure (instead of i.i.d. assumed here). Generally, this procedure is used to approximate the zeros of the function: \( h(\theta) = E[H(\theta, \eta)] \), where \( \eta \) has the same distribution as \( \eta_k \).

Even though the general theory extends to the case of multiple zeros, in this paper, we assume that \( h \) has only one zero, \( \theta^* \) (i.e. \( h(\theta^*) = 0 \)). The general assumption on the step sequence \((\gamma_k)_{k \geq 0}\) is usually the following:

\[ \sum_{k \geq 0} \gamma_k = +\infty, \quad \sum_{k \geq 0} \gamma_k^2 < +\infty. \]

Under these assumption, it can be shown that the convergence:

\[ \theta_n \xrightarrow{n \to +\infty} \theta^*, \]

holds almost surely. This convergence is exactly a Law of Large numbers in the case \( H(\theta, x) = \sigma(m(\theta) - x) \), and it is therefore natural to ask about a Central Limit Theorem. Following the procedure described in [1], we therefore look for a suitable renormalization for the process and investigate a convergence in Law.

In a general context, it can be shown that the procedure tends to follow the solution of the Ordinary Differential Equation (ODE):

\[ \frac{d}{dt} \bar{\theta}_t = h(\bar{\theta}_t), \quad \bar{\theta}_0 = \theta_0, \quad \text{where we recall that: } h(\theta) = E[H(\theta, \eta)], \quad (1.2) \]

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Thus, fluctuations of the algorithm are to be considered with respect to the solution \((\hat{\theta}_t)_{t \geq 0}\) of the ODE (1.2).

Precisely, we consider a shift in the indexation of the procedure that will allow us to consider \((\theta_n)_{n \geq 0}\) close to stationarity. Let \(N \in \mathbb{N}\), and consider a sequence \((\theta_{N,n})_{n \geq 0} = (\theta_{N+n})_{n \geq 0}\), of shifted Robbins-Monro algorithms. These algorithms satisfy the following recurrence equation:

\[
\begin{aligned}
\theta_{n+1}^N &= \theta_n^N - \gamma_{n+1}^N H(\theta_n^N, \eta_{n+1}^N) \\
\theta_0^N &\in \mathbb{R},
\end{aligned}
\]  

(1.3)

where \(\eta_{n+1}^N = \eta_{N+n+1}\), and \(\gamma_{n+1}^N = \gamma_{N+n+1}\). Set now:

\[t_0^N = 0, \ t_1^N = \gamma_1^N, \ t_2^N = \gamma_1^N + \gamma_2^N, \ldots, \ t_k^N = \gamma_1^N + \cdots + \gamma_k^N, \]

and for an arbitrary terminal time \(T > 0\), we set:

\[M(N) = \inf\{k \in \mathbb{N} : t_k^N \geq T\}.\]

We consider the re-normalized process:

\[U_t^N := \sum_{k=0}^{+\infty} \frac{\theta_k^N - \bar{\theta}_t}{\sqrt{\gamma_k^N}} 1_{\{t_k^N \leq t < t_{k+1}^N\}}.\]

(1.4)

Expressing the dynamics of this Markov chain we obtain (see Proposition 2.1 below):

\[U_{t_{k+1}^N}^N = U_{t_k^N}^N + \left(\alpha_k^N - \sigma \sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} \int_0^1 m'\left(\hat{\theta}_{t_k^N} + \delta U_{t_k^N}^N \sqrt{\gamma_k^N}\right) d\delta\right) U_{t_k^N}^N \gamma_{k+1}^N + \sqrt{\gamma_{k+1}^N} \sqrt{\gamma_k^N} \xi \cdot \xi + o(1),\]

where the \(o(1)\) is deterministic and goes to zero as \(N\) goes to infinity. Also, the step sequence \((\gamma_k^N)_{k \geq 1}\) can be chosen so that \(\alpha_k^N \rightarrow \frac{1}{2}\). We know from the literature that the process \((U_t^N)_{t \geq 0}\) converges weakly to \((X_t)_{t \geq 0}\) the solution of the SDE:

\[dX_t = \left(-am'(\hat{\theta}_t) + \frac{1}{2}\right) X_t dt + \sigma dW_t.\]

(1.5)

We point out that the coefficients of the limiting SDE can be seen as the point-wise limit of the coefficients in the dynamics of the Markov chain. We refer to Benveniste et al. [1] or Kushner and Yin [5] for the proof of this convergence. In this article, we are interested in quantifying the convergence of \((U_t^N)_{t \geq 0}\) towards \((X_t)_{t \geq 0}\), on the densities of the involved objects. However, as we will make clear below, the fact that the convergence is point-wise is problematic for our approach, also, we introduce a cut-off \(a_N \rightarrow +\infty\), and define:

\[F_N(t,z) = \left(\alpha_k^N - \sigma \sqrt{\frac{\gamma_k^N}{\gamma_{k+1}^N}} \int_0^1 m'\left(\hat{\theta}_t + \delta \text{sign}(z) \left(|z| \wedge a_N\right) \sqrt{\gamma_k^N}\right) d\delta\right) U_{t_k^N}^N \gamma_{k+1}^N + \sqrt{\gamma_{k+1}^N} \sqrt{\gamma_k^N} \xi \cdot \xi \]

where implicitly, \(k = k(t)\) is the index such that \(t_{k-1}^N \leq t < t_k^N\). This new function \(F_N\) is essentially the drift term in the dynamics of \((U_t^N)_{t_k^N \geq 0}\) above with the cut-off \(a_N\), and allows us to use the perturbation technique developed in Konakov Kozhina and Menozzi [4].

1.1 List of assumptions

We list here all the assumptions needed on our model.

A-1 (Smoothness condition) The function \(m(x)\) has four derivatives and these derivatives are bounded in a neighborhood of the image of \(\hat{\theta}_t\), for \(t \in [0,T]\), the solution of (1.2).

A-2 (Attractivity condition) We have the attractivity condition:

\[-\sigma m'(\theta^*) + \frac{1}{2} < 0.\]

A-3 (Condition on the partitions) The sequence \((\gamma_k)_{k \geq 0}\) is such that

\[\sum_{k \geq 1} \gamma_k = +\infty, \quad \sum_{k \geq 1} \gamma_k^2 < +\infty.\]
We denote $\gamma_k^N = \gamma_{k+N}$ and set:
\[
t_0^N = 0, \quad t_1^N = \gamma_1^N, \quad t_2^N = \gamma_1^N + \gamma_2^N, \ldots, \quad t_k^N = \gamma_1^N + \cdots + \gamma_k^N.
\]
and for an arbitrary terminal time $T > 0$, we denote:
\[
M(N) = \inf \{ k \in \mathbb{N} ; \ t_k^N > T \}.
\]
Finally, the sequence of partitions $t_1^N < \cdots < t_{M(N)}^N$ of the interval $[0, T]$ is chosen in such a way that, for sufficiently large $N$, there exists a constant $c > 1$, for all $0 \leq i \leq j \leq M(N)$:
\[
c^{-1} \leq \frac{\gamma_i^N}{\gamma_j^N} \leq c.
\]
\[\text{A-4 (Condition on the innovations)}\]
The innovations $\eta, \eta_1, \eta_2, \ldots$ are independent and identically distributed. We also assume that they have a common density we denote $\rho$ that satisfies:
- Centered:
  \[
  \int_R z \rho(z) dz = 0,
  \]
- Variance:
  \[
  \int_R z^2 \rho(z) dz = 1,
  \]
- Smoothness: There is an index $M > S + 1$, $S > 8$ such that for all $1 \leq \nu \leq 5$:
  \[
  \left| \frac{\partial^\nu}{\partial z^\nu} \rho(z) \right| \leq C \frac{1}{1 + |z|^M}, \quad C > 0.
  \]
\[\text{A-5 (Cut-off)}\]
We introduce a cut-off level $(a_N)_{N \geq 1}$, such that
\[
a_N \xrightarrow{N \to +\infty} +\infty, \quad a_N \sqrt{\gamma_0^N} \xrightarrow{N \to +\infty} 0.
\]
In the rest of this paper, these assumptions are always in force, except when explicitly stated otherwise.

**Example:** Let us mention that the sequence $\gamma_k = \frac{1}{k^{0.99}}$ satisfies these conditions.

**Remark 1.1.** In our specific case, it is an exercise to show that the algorithm (1.1) converges to $\theta^*$ almost surely and that the central limit theorem holds. One can for instance refer to the monograph by Benveniste, Métivier and Priouret [1], specifically Chapter 4 section 4.5 and check that the list of assumptions is verified in our case. Note also that the assumption that the innovations are centered and with variance 1 is not constraining due to the linear feature of our model.

Our main result is the following.

**Theorem 1.1.** Fix a terminal time $T$. There exists a sequence of stochastic processes $(V_t^N)_{t \geq 0}$ such that for some constant $C > 0$,
\[
P \left( \sup_{k \in [1, M(N)]} |U_{t_k^N}^N - V_{t_k^N}^N| > C \sqrt{\gamma_0^N} \right) \xrightarrow{N \to +\infty} 0.
\]
Moreover, denoting $p_N$ the density of $V_t^N$ and $p$ the density of the Gaussian diffusion (2.11) below, and $q_N$ the density of the cut-off diffusion $X_t^N$ defined in (3.1) below, there exist a constant $C > 0$, for all $t_k^N \leq T - \delta$, $\delta > 0$ for all $(x, y) \in K_x \times K_y$, with $K_x, K_y$ compact sets, and a constant $C$ depending on $T, \delta, K_x, K_y$ such that:
\[
|p_N - q_N)(t_k^N, T, x, y)| \leq C \sqrt{\gamma_0^N} \left( \frac{T - t_k^N}{1 + \frac{g(x) - y}{\sqrt{T - t_k^N}}} \right) s^{-T},
\]
and
\[
|p_N - q_N)(t_k^N, T, x, y)| \leq C a_N \sqrt{\gamma_0^N} g(x) - y, t_k^N, \theta_{t_k^N}(y) - x).
\]
where for all \((t, z) \in \mathbb{R}_+ \times \mathbb{R}\), \(g_c(t, z) := \frac{c}{\sqrt{t}} \exp \left( -c \frac{|z|^2}{2t} \right)\), and \(\theta_{1,T}(y)\) is the solution of the ODE:

\[
d \frac{d}{dt} z_t = \left( -\sigma m'(z_t) + \frac{1}{2} \right) z_t, \quad z_T = y,
\]

and \(\hat{\theta}_k^{\gamma,\eta}(y)\) is the backward Euler scheme:

\[
\begin{aligned}
x_k^{N,\gamma} &= x_k^{N,\eta} + F_N(t_k^{N,\gamma}, x_k^{N,\gamma}) - m_N(t_k^{N,\gamma}) \gamma_k^{N,\gamma}, \\
x_0^{N} &= y,
\end{aligned}
\]

where \(F_N\) is defined in (3.11) below. Also, \(a_N \to +\infty\) is a cut-off threshold such that \(a_N \sqrt{\gamma_0^{N,\eta}} \to 0\).

The multidimensional case and more general models will be considered in a separate publication. The proof of this result is based on an extension of two results by Konakov and Mammen [3], and Konakov Kozhina and Menozzi [1], and rely the parametrix expansions for the involved objects.

We decided to present only the linear case here for simplicity in the computations. This model is already interested in the algorithm:

\[
\theta_{n+1}^{N,\gamma} = \theta_n^{N,\gamma} + \gamma_{n+1} \cdot \sigma \left( m(\theta_n^{N,\gamma}) - \eta_{n+1} \right), \quad \theta_0^{N,\gamma} \in \mathbb{R},
\]

where \(a \in \mathbb{R}\), and the innovations \(\eta_k\) are i.i.d. From the general theory (see e.g. [1]), we know that the central limit theorem is obtained by considering the following renormalization:

\[
U_k^N = \sum_{k=0}^{\infty} \frac{\theta_k^N - \bar{\theta}_k^N}{\sqrt{\gamma_k^N}} 1_{\{t_k^N \leq t < t_{k+1}^N\}}.
\]

This is due to the fact that the algorithm tends to follow the solution of the ODE \(\frac{d}{dt} \bar{\theta}_t = h(\bar{\theta}_t)\).

**Proposition 2.1.** Set for all \(\theta \in \mathbb{R}\), \(\xi^\eta = H(\theta, \eta) - h(\theta)\), where \(\eta\) has the same distribution as the innovations \(\eta_k\). The Markov chain \((U_t^N)\) has the following dynamics:

\[
U_{t_{k+1}}^N = U_{t_k}^N + \left( \alpha_{t_k}^N - \sigma \sqrt{\frac{\gamma_{k}^N}{\gamma_{k+1}^N}} \int_0^1 m'(\bar{\theta}_{t_k}^N + \delta U_{t_k}^N \sqrt{\gamma_{k}^N} \, d\delta) \right) \gamma_{k+1}^{N,\gamma} \theta_{t_k}^{N,\gamma} + \sqrt{\gamma_{k+1}^N} \left( h(\bar{\theta}_{t_k}) - \frac{\theta_{t_k}^{N,\gamma} - \bar{\theta}_{t_k}^{N,\gamma}}{\gamma_{k+1}^{N,\gamma}} \right),
\]

where \(\sqrt{\gamma_{k+1}^N} \left( h(\bar{\theta}_{t_k}) - \frac{\theta_{t_k}^{N,\gamma} - \bar{\theta}_{t_k}^{N,\gamma}}{\gamma_{k+1}^{N,\gamma}} \right) \to 0\) and is deterministic. Also,

\[
\alpha_{t_k}^N = \frac{\sqrt{\gamma_k^N} - \sqrt{\gamma_{k+1}^N}}{(\gamma_{k+1}^{N,\gamma})^{1/2}} \to \alpha = \frac{1}{2}.
\]

**Proof.** We write its dynamics as a Markov chain. To alleviate the notations, we denote \(t_k\) and \(\eta_k\) instead of \(t_k^N\) and \(\eta_k^N\).
and $\eta^N_k$. We have:

$$U^N_{k+1} = \frac{\theta^N_{k+1} - \tilde{\theta}_{t_k+1}}{\sqrt{\gamma^N_{k+1}}}$$

$$= \frac{\theta^N_k + \gamma^N_{k+1} H(\theta^N_k, \eta_{k+1}) - \tilde{\theta}_{t_k+1}}{\sqrt{\gamma^N_{k+1}}}$$

$$= \frac{\theta^N_k - \tilde{\theta}_{t_k} \sqrt{\gamma^N_k} + \sqrt{\gamma^N_{k+1}} H(\theta^N_k, \eta_{k+1}) - \tilde{\theta}_{t_k+1} - \tilde{\theta}_{t_k}}{\sqrt{\gamma^N_{k+1}}}$$

$$= U^N_{t_k} \sqrt{\gamma^N_k} + \sqrt{\gamma^N_{k+1}} H(\theta_{t_k} + U^N_{t_k} \sqrt{\gamma^N_k}, \eta_{k+1}) \frac{\tilde{\theta}_{t_k+1} - \tilde{\theta}_{t_k}}{\sqrt{\gamma^N_{k+1}}}.$$ 

Now, we can write:

$$U^N_{t_k} \sqrt{\gamma^N_k} = U^N_{t_k} + U^N_{t_k} \sqrt{\gamma^N_k}$$

$$= U^N_{t_k} + U^N_{t_k} \sqrt{\gamma^N_k}$$

$$= U^N_{t_k} + \alpha^N_{t_k} U^N_{t_k} \sqrt{\gamma^N_{k+1}},$$

where we recall $\alpha^N_{t_k} = \frac{\sqrt{\gamma^N_k} - \sqrt{\gamma^N_{k+1}}}{(\gamma^N_{k+1})^{3/2}} \rightarrow \bar{\alpha} = \frac{1}{2}$ by assumption. The dynamics of $U^N$ becomes:

$$U^N_{k+1} = U^N_{t_k} + \alpha^N_{t_k} U^N_{t_k} \sqrt{\gamma^N_{k+1}} + \sqrt{\gamma^N_{k+1}} \left( H(\tilde{\theta}_{t_k} + U^N_{t_k} \sqrt{\gamma^N_k}, \eta_{k+1}) - H(\tilde{\theta}_{t_k}, \eta_{k+1}) \right)$$

$$+ \sqrt{\gamma^N_{k+1}} \left( H(\theta_{t_k}, \eta_{k+1}) - h(\theta_{t_k}) \right) + \sqrt{\gamma^N_{k+1}} \left( h(\theta_{t_k}) - \frac{\tilde{\theta}_{t_k+1} - \tilde{\theta}_{t_k}}{\gamma^N_{k+1}} \right).$$

Recalling that for all $\theta \in \mathbb{R}$, $\xi^0 = H(\theta, \eta) - h(\theta)$, we get:

$$U^N_{k+1} = U^N_{t_k} + \alpha^N_{t_k} U^N_{t_k} \sqrt{\gamma^N_{k+1}}$$

$$+ \sqrt{\gamma^N_{k+1}} \left( H(\tilde{\theta}_{t_k} + U^N_{t_k} \sqrt{\gamma^N_k}, \eta_{k+1}) - H(\tilde{\theta}_{t_k}, \eta_{k+1}) \right) + \sqrt{\gamma^N_{k+1}} \xi^0_{t_k}$$

$$+ \sqrt{\gamma^N_{k+1}} \left( h(\theta_{t_k}) - \frac{\tilde{\theta}_{t_k+1} - \tilde{\theta}_{t_k}}{\gamma^N_{k+1}} \right).$$

Now, we observe that:

$$\sqrt{\gamma^N_{k+1}} \left( h(\theta_{t_k}) - \frac{\tilde{\theta}_{t_k+1} - \tilde{\theta}_{t_k}}{\gamma^N_{k+1}} \right) \rightarrow 0.$$

Besides, we can use Taylor's formula:

$$H(\tilde{\theta}_{t_k} + U^N_{t_k} \sqrt{\gamma^N_k}, \eta_{k+1}) - H(\tilde{\theta}_{t_k}, \eta_{k+1}) = \left( \int_0^1 \partial_1 H(\tilde{\theta}_{t_k} + U^N_{t_k} \sqrt{\gamma^N_k} \delta, \eta_{k+1}) d\delta \right) U^N_{t_k} \sqrt{\gamma^N_k}.$$

Here, $\partial_1$ represents the derivative with respect to the first argument of $H(\theta, x)$.

Thus, we have the dynamics for $U^N$:

$$U^N_{t_k+1} = U^N_{t_k} + \left( \alpha^N_{t_k} + \sqrt{\gamma^N_{k+1}} \right) \int_0^1 \partial_1 H(\tilde{\theta}_{t_k} + U^N_{t_k} \sqrt{\gamma^N_k} \delta, \eta_{k+1}) d\delta \right) U^N_{t_k} \sqrt{\gamma^N_{k+1}}$$

$$+ \sqrt{\gamma^N_{k+1}} \xi^0_{t_k} + o(1),$$

where $o(1)$ is non random and goes to zero as $N$ goes to infinity. Let us point out that we can derive equation (2.7) for a general Robbins-Monro algorithm. However, the dependency in $\eta_{k+1}$ of the "drift term" can be a
problem. We can avoid it by limiting ourselves to the linear case. Indeed, plugging \( H(\theta, x) = -\sigma(m(\theta) - x) \), we have:

\[
H(\tilde{\theta}_k + U_t^N \sqrt{\gamma_k^N} \cdot \eta_{k+1}) - H(\tilde{\theta}_k, \eta_{k+1}) = -\sigma m(\tilde{\theta}_k + U_t^N \sqrt{\gamma_k^N} + \sigma m(\tilde{\theta}_k) = - \int_0^1 \sigma m'\left(\tilde{\theta}_k + \delta(U_t^N \sqrt{\gamma_k^N}) U_t^N \sqrt{\gamma_k^N} \delta\right). 
\]

Hence, the "drift term" does not depend on \( \eta_{k+1} \) anymore. Another consequence of our choice of the linear model is from the definition of \( \xi^\theta \). Observe that:

\[
\xi^\theta = H(\theta, \eta) - h(\theta) = \sigma(\eta - m(\theta)) - \sigma(\mathbb{E}(\eta) - m(\theta)) = \sigma(\eta - \mathbb{E}(\eta)). 
\]

Thus, in this case \( \xi_{t_k} := \xi_{k+1} \) are i.i.d., centered and with variance \( \sigma^2 \), and most importantly, does not depend on \( \tilde{\theta}_k \), and we recover the dynamics:

\[
U_{t_{k+1}}^N = U_{t_k}^N + \left(\alpha_{t_k} - \sigma \sqrt{\gamma_k^N} \eta_{k+1} \int_0^1 m'(\tilde{\theta}_k + \delta U_t^N \sqrt{\gamma_k^N} \delta \right) U_{t_k}^N \gamma_{k+1} + \sqrt{\gamma_{k+1}/\xi_{k+1} + o(1)}.
\]

In the next section, we use a Parametrization expansion to give an explicit expression of the density of the Markov chain \( U_t^N \). Building from equation (2.6), we can guess the expression for the limiting SDE. Summing (2.6) for \( k = 0 \) to \( n - 1 \), we get:

\[
U_{t_n}^N - U_{t_0}^N = \sum_{k=0}^{n-1} \left(\alpha_{t_k} - \sigma \sqrt{\gamma_k^N} \int_0^1 m'(\tilde{\theta}_k + \delta U_t^N \sqrt{\gamma_k^N} \delta \right) U_{t_k}^N \gamma_{k+1} + \sum_{k=0}^{n-1} \sqrt{\gamma_{k+1}/\xi_k + o(1)}.
\]

From the last equation, we can identify the limiting SDE, with heuristic arguments. We already discussed that \( \alpha_{t_k} \rightarrow \frac{1}{2} \) as \( N \) goes to infinity. Besides, for all \( x \in \mathbb{R} \) fixed, if \( t_k = t_k^N \rightarrow t \), we see that

\[
-\sigma m'\left(\tilde{\theta}_k + x \sqrt{\gamma_k^N} \right) \xrightarrow{N \rightarrow +\infty} -\sigma m'(\tilde{\theta}_k). \tag{2.8}
\]

Finally, the independent variables \( \xi_k \) converge to a Brownian motion because of the renormalisation. We can thus guess the limit to be equation (1.5) defined above:

\[
dX_t = \left(-\sigma m'(\tilde{\theta}_t) + \frac{1}{2} \right) X_t dt + \sigma dW_t. \tag{2.9}
\]

**Remark 2.1.** Observe that the general case where \( \xi = \xi^\theta \), we have to consider the sum of \( \sqrt{\gamma_k^N} \xi_{t_k} \) who do not have the same distribution. It can be shown that in this case, denoting \( R(\theta) = Var(\xi^\theta) \), the convergence holds:

\[
\sum_{k=0}^{n-1} \sqrt{\gamma_k^N} \xi_{t_k} \xrightarrow{N \rightarrow +\infty} \mathbb{E}(\int_0^s \sqrt{R(\theta_u)} dB_u)
\]

The actual proof for the convergence of \( U^N \) to \( X \) is given in Benveniste *et al.* [1], see Theorem 12, Chapter 4 (p.328). It relies on proving tightness for the law of \( U^N \), and uniqueness to the martingale problem associated with the limiting diffusion. Pointing out the major difficulties here, we see that for the limiting SDE, the drift is unbounded and the convergence of the drift (2.8) only holds for fixed \( x \in \mathbb{R} \).

To conclude this section, let us point out that since the limiting SDE is linear and the diffusion coefficient is constant, we can solve it explicitly by introducing the resolvent of the ODE \( \dot{\theta}_t = (-\alpha m'(\tilde{\theta}_t) + \frac{1}{2})x_t \). Indeed, following the notations in Theorem 1.1 let us denote \( \left(\theta_{s,t}(x)\right)_{s \geq t} \) the solution of this SDE such that \( x_t = x \).

Note that for an arbitrary Robbins Monro algorithm, from the general theory of ODEs, \( \theta_{s,t} \) is a matrix. Then, using Itô’s formula, we have that:

\[
X_{s,x}^t = \theta_{s,t}(x) + \int_t^s \theta_{s,u} \sigma dW_u.
\]
where we define:

\[ F(x) = \int_{-\infty}^{\infty} \chi_{C}(y-x) \, d\Pi(y), \]

In the rest of the paper, we will be led to consider \( F \). Consequently, if \( p(t, s, x, y) = \int_{dy} \mathbb{P}(X_s \in dy | X_t = x) \), then, \( p \) is a Gaussian density and the following density estimate holds:

\[
\frac{C_1}{\sqrt{s-t}} \exp\left(-\frac{C_1 |y-\theta_{s,t}(x)|^2}{s-t} \right) \leq p(t, s, x, y) \leq \frac{C_2}{\sqrt{s-t}} \exp\left(-\frac{C_2 |y-\theta_{s,t}(x)|^2}{s-t} \right),
\]

where \( C_1 \) and \( C_2 \) are positive constants. Nevertheless, we chose to write the density as a parametrix expansion, as it will be easier to compare it to the density of the Markov chain.

3 The Parametrix Setting

Fix a time horizon \( T \), and let us denote by \( p(t, s, x, z) \) the transition density of \((X_t)_{t \leq T}\). The goal is to quantify the difference between the transition density of the diffusion \((X_t)_{t \leq T}\) and the Markov chain \((U_{t_k}^N)_{k \in [1, N]}\)

However, the unbounded feature of the drifts in (2.6) and (1.5) and the fact that the convergence (2.8) above is point-wise is problematic. To deal with this, we introduce the following modified Markov chain. For all \( x \in \mathbb{R} \) fixed, we denote:

\[
G_N(t_k^N, x) = \left( \alpha_{N} + \sqrt{\gamma_N^{-1}} \int_0^1 m' \left( \theta_{t_k} + \delta x \sqrt{\gamma_N} \right) \, d\delta \right) \rightarrow N \rightarrow +\infty -am' \left( \theta_t \right) + \frac{1}{2}.
\]

We circle the problem of the point-wise convergence by changing the drift term:

\[
V_{t_k+1}^N = V_{t_k}^N + F_N(t_{k}^N, V_{t_k}^N) \gamma_k t_k + \gamma_k t_k + \xi_{k+1} + \sqrt{\gamma_k} \xi_{k+1},
\]

where we define:

\[
F_N(t_k^N, V_{t_k}^N) = \left( \alpha_{N} + \sqrt{\gamma_k} \int_0^1 m' \left( \theta_{t_k} + \delta \text{sign}(V_{t_k}^N) \left[ \left| V_{t_k}^N \right| \wedge a_N \right] \sqrt{\gamma_k} \right) \, d\delta \right).
\]

In the rest of the paper, we will be led to consider \( F_N \) for continuous time and arbitrary spatial point, writing:

\[
F_N(t, x) = \left( \alpha_{N} + \sqrt{\gamma_k} \int_0^1 m' \left( \hat{\theta}_t + \delta \text{sign}(x) \left[ \left| x \right| \wedge a_N \right] \sqrt{\gamma_k} \right) \, d\delta \right),
\]

where implicitly, \( k = k(t) \) is the index for which \( t_k^N \leq t < t_{k+1}^N \). Put simply, we consider \( F_N(u, x) = G_N(u, \text{sign}(x)(|x| \wedge a_N)) \) instead of \( G_N \) in the drift, and \( a_N \rightarrow +\infty \) for a sequence of positive integers \( a_N \rightarrow +\infty \) for a sequence of positive integers \( a_N \rightarrow +\infty \). Notice that as \( a_N \) grows bigger, the dynamics of (2.6) and (3.10) coincide, up to a deterministic term that goes to zero. Specifically, we have:

**Lemma 3.1.** The sequences \((V_{t_k}^N)_{t \geq 0}\) and \((U_{t_k}^N)_{t \geq 0}\) with \( U_{t_k}^N = V_{t_k}^N = x \), where \( x \) is chosen in a compact set \( K_x \) are close in the following sense:

\[
P\left( \sup_{k \in [\lfloor 1, M(t) \rfloor]} \left| U_{t_k}^N - V_{t_k}^N \right| > C \sqrt{\gamma_0} \bigg| N \rightarrow +\infty \right) \rightarrow 0,
\]

where \( C = C(T, K_x) > 0 \).

**Proof.** Firstly, we need to quantify the rate of convergence of the \( o(1) \) in (2.7). We define:

\[
\beta^N_{k+1} := \sqrt{\gamma_k} \left( \hat{\theta}_{t_k^N} - \frac{\hat{\theta}_{t_{k+1}}}{\gamma_k} \right),
\]

where \( (\hat{\theta}_t)_{t \geq 0} \) is the solution of the ODE:

\[
\frac{d}{dt} \hat{\theta}_t = \sigma \left( m(\hat{\theta}_t) - \mathbb{E}(\eta) \right), \quad \hat{\theta}_0 = \theta_0.
\]
Using Taylor’s formula, and the ODE to express the successive derivatives of \((\hat{\vartheta}_t)_{t \geq 0}\), we get:

\[
\frac{\vartheta_{t_{k+1}} - \vartheta_{t_k}}{\gamma_{k+1}} - h(\vartheta_{t_k}) = \frac{\sigma^2}{2} m'(\vartheta_{t_k})(\vartheta_{t_k} - E(\eta))\gamma_{k+1}^N
\]

\[+\frac{\sigma^2}{6} \left( m''(\vartheta_{t_k}) + \sigma m(\vartheta_{t_k})^2 \right) \left( m(\vartheta_{t_k} - E(\eta))(\gamma_{k+1})^2 + O((\gamma_{k+1})^3) \right)\]

Thus, we have:

\[
\beta_{k+1}^N = \frac{\sigma^2}{2} m'(\vartheta_{t_k})(\vartheta_{t_k} - E(\eta))(\gamma_{k+1})^2 + \frac{\sigma^2}{6} \left( m''(\vartheta_{t_k}) + \sigma m(\vartheta_{t_k})^2 \right) (m(\vartheta_{t_k}) - E(\eta))(\gamma_{k+1})^2 + O((\gamma_{k+1})^2)\]

Note that since the successive derivatives of \(m\) are bounded, we have:

\[
\sum_{k=0}^{M(N)} |\beta_{k+1}^N| \leq C \frac{\sigma^2}{\gamma_1} \int_0^T |m'(\theta_t)(m(\theta_t) - E(\eta))| dt \leq C \sqrt{\gamma_0}\]

(3.13)

Next, we introduce the exit time:

\[
\tau_{a_N} := \inf\{k \in [1, M(N)], |V_{k+1}^N| \geq a_N\}.
\]

We recall the dynamics of \((U_{k}^N)_{t \geq 0}\) and \((V_{k}^N)_{t \geq 0}\) are given in (2.6) and (3.10) respectively. Notice that by definition, for all \(k \leq \tau_{a_N}\), we have \(G_N(t_k^N, V_{k+1}^N) = F_N(t_k^N, V_{k+1}^N)\). In particular, since we start with \(U_{t_0}^N = V_{t_0}^N\), we have:

\[
U_{t_1}^N = U_{t_0}^N + G_N(t_0^N, U_{t_0}^N)U_{t_0}^N + \sqrt{\gamma_1} \xi_1 + \beta_1^N
\]

\[
= V_{t_0}^N + F_N(t_0^N, V_{t_0}^N)U_{t_0}^N + \sqrt{\gamma_1} \xi_1 + \beta_1^N = V_{t_1}^N + \beta_1^N.
\]

Besides, the derivative of \(x \mapsto F_N(t, x)\) is bounded by some constant \(L\) independent of \(t\) (see the proof of Lemma 3.6 below). We prove by induction the following:

\[
|U_{t_k}^N - V_{t_k}^N| \leq \prod_{i=1}^{k-1} (1 + L\gamma_{i+1}^N)(|\beta_i^N| + \cdots + |\beta_k^N|),
\]

(3.14)

where by definition, \(\prod_{i=1}^{k-1} = 1\). For \(k = 1\), it follows from the identity \(U_{t_1}^N = V_{t_1}^N + \beta_1^N\). Assume now that (3.14) holds for \(k \geq 1\). We write for \(k+1\):

\[
U_{t_k+1}^N = U_{t_k}^N + G_N(t_k^N, U_{t_k}^N)U_{t_k}^N \gamma_{k+1}^N + \sqrt{\gamma_{k+1}^N} \xi_k + \beta_{k+1}^N
\]

\[
= V_{t_k}^N + F_N(t_k^N, V_{t_k}^N)V_{t_k}^N \gamma_{k+1}^N + \sqrt{\gamma_{k+1}^N} \xi_k + \beta_{k+1}^N = V_{t_{k+1}}^N + \beta_{k+1}^N.
\]

where \(\beta_{k+1}^N = \sqrt{\gamma_{k+1}^N} \left( h(\vartheta_{t_k}) - \frac{\sigma^2}{6} \left( m''(\vartheta_{t_k}) + \sigma m(\vartheta_{t_k})^2 \right) \right)\). Taking the difference, we have:

\[
|U_{t_{k+1}}^N - V_{t_{k+1}}^N| \leq |U_{t_k}^N - V_{t_k}^N| + |G_N(t_k^N, U_{t_k}^N)U_{t_k}^N - G_N(t_k^N, V_{t_k}^N)V_{t_k}^N| \gamma_{k+1}^N + |\beta_{k+1}^N|
\]

Using the induction hypothesis, and the Lipschitz property of \(x \mapsto G_N(t, x)\), we have:

\[
|U_{t_{k+1}}^N - V_{t_{k+1}}^N| \leq \prod_{i=1}^{k-1} (1 + L\gamma_{i+1}^N)(|\beta_i^N| + \cdots + |\beta_k^N|) + |\beta_{k+1}^N|
\]

\[
\leq \prod_{i=1}^{k} (1 + L\gamma_{i+1}^N)(|\beta_i^N| + \cdots + |\beta_k^N|).
\]

Thus, we have obtained (3.14) for \(k \leq \tau_{a_N}\). Now, from (3.3) and (3.14), it follows that:

\[
\sup_{k \leq \tau_{a_N}} |U_{t_k}^N - V_{t_k}^N| \leq C \sqrt{\gamma_0^N} \left( (1 + L\gamma_{N}^N)^{\frac{1}{2}} \right)^{TL} \leq C \sqrt{\gamma_0^N}.
\]
and consequently, we have:

$$\mathbb{P} \left( \sup_{k \in [1, M(N)]} |U_{tk}^N - V_{tk}^N| > C \sqrt{\gamma_{0}^N} \right) \leq \mathbb{P} \left( \tau_{aN} \leq M(N) \right),$$

(3.15)

and it remains us to show that $\mathbb{P}(\tau_{aN} \leq M(N))$ goes to zero. Observe that by definition,

$$\mathbb{P} \left( \tau_{aN} \leq M(N) \right) = \mathbb{P} \left( \sup_{k \in [1, M(N)]} |V_{tk}^N| \geq a_N \right).$$

We claim that the Markov chains $(U_{tk}^N)_{k \in [1, M(N)]}$ and $(V_{tk}^N)_{k \in [1, M(N)]}$ have the same limiting diffusion $(X_t)_{t \in [0, T]}$, where:

$$dX_t = \left( -\sigma m'(\theta) + \frac{1}{2} \right) X_t dt + \sigma dW_t.$$ 

For $(U_{tk}^N)_{k \in [1, M(N)]}$ it is a well known fact, and we refer for instance to Benveniste Métivier and Priouret [1].

For $(V_{tk}^N)_{k \in [1, M(N)]}$, we rely on a theorem in Stroock and Varadhan [8]. According to the procedure described in [8], we define:

$$\Pi_{\gamma}(t, x, \Gamma) = \frac{1}{\gamma_{t}^{\frac{1}{2}}} \int_{\Gamma} \rho_{\xi} \left( \frac{y - x - F_N(t, x)x\gamma_{t}^{N}}{\sqrt{\gamma_{t}^{N}}} \right) dy,$$

where $\gamma_{t}^N = \gamma_{t}^N$ when $t_{k}^N \leq t < t_{k+1}^N$. Also, $\rho_{\xi}$ is the density of the innovation $(\xi_{i})_{i \in \mathbb{N}}$. Using this kernel we construct the probability measure concentrated on broken lines analogously to (2.2) on page 267 of [8].

Consider

$$a_{\gamma}(t, x) = \frac{1}{\gamma_{t}^{\frac{1}{2}}} \int_{\{|x-y| \leq 1\}} (y - x)^2 \Pi_{\gamma}(t, x, dy),$$

$$b_{\gamma}(t, x) = \frac{1}{\gamma_{t}^{\frac{1}{2}}} \int_{\{|x-y| \leq 1\}} (y - x) \Pi_{\gamma}(t, x, dy),$$

$$\Delta_{\gamma}(t, x) = \frac{1}{\gamma_{t}^{\frac{1}{2}}} \Pi_{\gamma}(t, x, \mathbb{R} \setminus B(x, \varepsilon)).$$

Changing variables to $v = (\gamma_{t}^{N})^{-\frac{1}{2}} (y - x - F_N(t, x)x\gamma_{t}^{N})$, we obtain:

$$a_{\gamma}(t, x) = \int_{\{|v+F_N(t,x)x\sqrt{\gamma_{t}^{N}}| \leq \sqrt{\gamma_{t}^{N}}\}} \left( \gamma_{t}^{N} v^2 + (\gamma_{t}^{N})^{\frac{1}{2}} v F_N(t, x)x + (F_N(t, x)x\gamma_{t}^{N})^2 \right) \rho_{\xi}(v) dv$$

For any fixed $R$ and for $|x| \leq R$, since $\gamma_{t}^{N} \to 0$ when $N \to +\infty$, the domain of integration $\{|v+F_N(t,x)x\sqrt{\gamma_{t}^{N}}| \leq \sqrt{\gamma_{t}^{N}}\}$ tends to the whole real line $\mathbb{R}$, uniformly for $t \in [0, T]$. Now, $\xi$ has zero mean and variance $\sigma$, so it’s clear that:

$$\lim_{N \to +\infty} \sup_{|x| \leq R, t \in [0, T]} |a_{\gamma}(t, x) - \sigma^2| = 0.$$  \hspace{1cm} (3.16)

Thus, condition (2.4) on page 268 of [8] holds true. Quite similarly, we obtain that the condition (2.5) on $b_{\gamma}$ also holds true:

$$\lim_{N \to +\infty} \sup_{|x| \leq R, t \in [0, T]} \left| b_{\gamma}(t, x) - \left( -\sigma m'(\bar{\theta}) + \frac{1}{2} \right) x \right| = 0.$$  \hspace{1cm} (3.17)

and it remains us to get condition (2.6). We have using Markov’s inequality on the third moment for $\xi$:

$$\lim_{N \to +\infty} \sup_{|x| \leq R, t \in [0, T]} \Delta_{\gamma}(t, x) = \lim_{N \to +\infty} \sup_{|x| \leq R, t \in [0, T]} \frac{1}{\gamma_{t}^{\frac{1}{2}}} \int_{\{|v+F_N(t,x)x\sqrt{\gamma_{t}^{N}}| \geq \sqrt{\gamma_{t}^{N}}\}} \rho_{\xi}(v) dv$$

$$\leq \lim_{N \to +\infty} \sup_{|x| \leq R, t \in [0, T]} \frac{1}{\gamma_{t}^{\frac{1}{2}}} \mathbb{P} \left( |\xi| \geq \frac{\varepsilon}{\sqrt{\gamma_{t}^{N}}} - CR \sqrt{\gamma_{t}^{N}} \right) = 0.$$ 

We need a bit stronger result, namely, with $\varepsilon = \varepsilon_{N} = (\gamma_{M(N)})^{\frac{1}{2}}$. Then, applying the Markov inequality for the 9th moment and proceeding as before, we get:

$$\lim_{N \to +\infty} \sup_{|x| \leq R, t \in [0, T]} \Delta_{\gamma}(t, x) \leq \lim_{N \to +\infty} \sup_{|x| \leq R, t \in [0, T]} \frac{1}{\gamma_{t}^{\frac{1}{2}}} \mathbb{P} \left( |\xi| \geq C (\gamma_{0}^{N})^{\frac{1}{2}} - CR \sqrt{\gamma_{t}^{N}} \right) \leq \lim_{N \to +\infty} \sup_{|x| \leq R, t \in [0, T]} \frac{1}{\gamma_{t}^{\frac{1}{2}}} \mathbb{P} \left( |\xi| \geq C (\gamma_{0}^{N})^{\frac{1}{2}} - CR \sqrt{\gamma_{t}^{N}} \right) = 0.$$
Now, we can mimic the proof of Lemma 11.2.1, page 268 of [S], to show that for each \( f \in C_{0}^{\infty}(\mathbb{R}) \),
\[
\frac{1}{\gamma_{t}^{N}} \int_{\mathbb{R}} \left( f(y) - f(x) \right) \Pi_{t}(y, t, dy) \rightarrow_{N \rightarrow +\infty} \frac{1}{2} \sigma^{2} f''(x) + \left( -\sigma m'(\theta_{t}) + \frac{1}{2} \right) x f'(x), \tag{3.18}
\]
uniformly on compact sets in \( \mathbb{R} \) and \( t \in [0, T] \). We now reproduce the proof of [S], in the case \( \varepsilon = \varepsilon_{N} \). Let us denote \( L_{\gamma} \) the operator with coefficients \( a_{\gamma} \) and \( b_{\gamma} \). Then, we have:
\[
\left| \frac{1}{\gamma_{t}^{N}} \int_{\mathbb{R}} \left( f(y) - f(x) \right) \Pi_{t}(y, t, dy) - L_{\gamma} f(x) \right| \leq \frac{1}{\gamma_{t}^{N}} \int_{\{|x-y|>1\}} |f(y) - f(x)| \Pi_{t}(y, t, dy)
\]
\[
+ C_{f} \int_{\{|x-y|\leq1\}} |y - x|^{3} \Pi_{t}(y, t, dy).
\]
It follows from (3.16) and (3.17) that \( L_{\gamma} f(x) \rightarrow L f(x) \) uniformly on compacts. Thus, the convergence (3.18) will follow if we show that uniformly on compact sets in \( \mathbb{R} \) and \( t \in [0, T] \):
\[
\frac{1}{\gamma_{t}^{N}} \int_{\{|x-y|>1\}} |f(y) - f(x)| \Pi_{t}(y, t, dy) \rightarrow 0 \quad \text{and} \quad \int_{\{|x-y|\leq1\}} |y - x|^{2} \Pi_{t}(y, t, dy) \rightarrow 0.
\]
But clearly, we have:
\[
\frac{1}{\gamma_{t}^{N}} \int_{\{|x-y|>1\}} |f(y) - f(x)| \Pi_{t}(y, t, dy) \leq 2\|f\|_{\infty} \Delta_{t}^{1}(t, x) \rightarrow_{N \rightarrow +\infty} 0,
\]
and
\[
\int_{\{|x-y|\leq1\}} |y - x|^{2} \Pi_{t}(y, t, dy) = \int_{\{|x-y|\leq\varepsilon_{N}\}} |y - x|^{2} \Pi_{t}(y, t, dy) \frac{\varepsilon_{N}^{3}}{\gamma_{t}^{N}} + \int_{\{|x-y|>\varepsilon_{N}\}} |y - x|^{2} \Pi_{t}(y, t, dy) \frac{\varepsilon_{N}^{3}}{\gamma_{t}^{N}} \leq \frac{\varepsilon_{N}^{3}}{\gamma_{t}^{N}} + \Delta_{t}^{\varepsilon_{N}}(t, x) \leq (\gamma_{M(N)})^{2} \Delta_{t}^{\varepsilon_{N}}(t, x) \rightarrow_{N \rightarrow +\infty} 0,
\]
uniformly on compacts in \( \mathbb{R} \) and \( t \in [0, T] \), and (3.18) follows. Now, since the martingale problem is well posed for all the considered processes (this follows e.g. from Theorems 6.3.4 and 10.2.2 in [S]), from the non-stationary version of Theorem 11.2.3, page 272 in [S], the weak convergence holds:
\[
\mathcal{L} \left( (V_{t}^{N})_{t \in \[1, M(N)\]} \right) \rightarrow_{N \rightarrow +\infty} \mathcal{L} \left( (X_{t})_{t \in [0, T]} \right).
\]
Now, observe that for all \( \varepsilon > 0 \), we can find \( K_{\varepsilon} \) such that
\[
P \left( \sup_{t \in [0, T]} |X_{t}| \geq K_{\varepsilon} \right) \leq \varepsilon.
\]
Consequently, we can write:
\[
P \left( \sup_{k \in \[1, M(N)\]} |V_{t}^{N} - V_{k}^{N}| \geq K_{\varepsilon} \right) \rightarrow_{N \rightarrow +\infty} P \left( \sup_{t \in [0, T]} |X_{t}| \geq K_{\varepsilon} \right) \leq \varepsilon
\]
Going back to the proximity of the Markov chains \( (U_{t}^{N})_{t \in \[1, M(N)\]} \) and \( (V_{t}^{N})_{t \in \[1, M(N)\]} \), from (3.19), we see that the last inequality implies that for all \( \varepsilon > 0 \):
\[
P \left( \sup_{k \in \[1, M(N)\]} |U_{t}^{N} - V_{k}^{N}| > C \sqrt{\frac{K_{\varepsilon}}{M(N)}} \right) \leq \varepsilon.
\]

**Remark 3.1.** We proved a version of Lemma 11.2.1 in Stroock and Varadhan [S] for the non-homogeneous case. Lemma 11.2.2 is based on Lemma 11.2.1 and uses Theorem 1.4.11. Theorem 1.4.11 is a general result about probability measures on metric spaces and not depending on any homogeneity. So, Lemma 11.2.2 may be proved analogously. Theorem 11.2.3 uses Lemma 11.2.2, Theorem 10.1.1 and Lemma 11.1.1. Note that Lemma 11.1.1 do not use homogeneity and Theorem 10.1.1 is proved for non-homogeneous case and, hence, can be used as it is.
Ideally, it would have been preferable to obtain a local limit theorem for the convergence of \((U_{t_k}^N)_{k \in [1, M(N)]}\) to \((X_t)_{t \in [0, T]}\), as exposed in Konakov et al. [4], as \(N \to +\infty\). However, in this case, the unbounded feature of the drift, and the fact that the convergence (28) happens for fixed \(x \in \mathbb{R}\) is problematic. Nevertheless, from the above estimate (3.12), we understand that it is justified to investigate a local limit theorem for \(V_{t_k}^N\) converging to \(X_t\), as the step size \(\gamma_k^N\) tends to zero. We denote \(p_N(t_j^N, t_i^N, x, z)\) the transition density of \(V_{t_k}^N:\)

\[
p_N(t_j^N, t_i^N, x, z)dz = \mathbb{P}\left(V_{t_i^N}^N \in dz | V_{t_j^N}^N = x\right).
\]

To prove Theorem 1.1, we introduce the intermediate diffusion equation:

\[
dX_t^N = F_N(t, X_t^N)X_t^N dt + \sigma dW_t.
\]

Observe that from now on, \(N\) is a fixed integer large enough so that the Markov chains \((U_{t_k}^N)_{t \leq T}\) and \((V_{t_k}^N)_{t \leq T}\) are close with great probability (cf equation (3.12)). We denote \(q_N\) the transition density of \((X_t^N)_{t \leq T}\). We split the difference:

\[
(p - p_N)(t, T, x, y) = (p - q_N)(t, T, x, y) + (q_N - p_N)(t, T, x, y),
\]

and we investigate the two contributions separately. Specifically, in Section 3.1 using a parametrix approach, we quantify the distance \(|p - q_N|\), and in Section 3.2, we turn to the Markov chains transition densities \(|q_N - p_N|\), whose analysis rely on a generalisation of a result of Konakov and Mammen [3] to the case of an unbounded drift.

3.1 The diffusive part

In this section, we prove the following Lemma:

**Lemma 3.2.** Assume \(a_N \sqrt{\gamma_0^N} \to 0\) as \(N \to +\infty\). Fix a time horizon \(T > 0\). There exists a constant \(C =: C(T) > 0\), for all \(t \leq T\), for all \((x, y) \in \mathbb{R} \times \mathbb{R}\),

\[
|p - q_N|(t, T, x, y) \leq C a_N \sqrt{\gamma_0^N} \Lambda_{T-t}(y) g_C(T - t, \theta_{t,T}(y) - x),
\]

where \(g_C(t, z)\) is defined in the statement of Theorem 1.1 and

\[
\Lambda_{T-t}(y) = 1 + \sqrt{T-t}(1 + |y|^2) + |y|e^{C(T-t)\sqrt{T-t}}\gamma_0^N |y|^2 \left(\sqrt{T-t} + T - t\right)(1 + |y|^2).
\]

**Remark 3.2.** For all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\), we will use the following notation throughout this paper:

\[
g_{t,N}(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right), \text{ or } g_C(t, x) = \frac{1}{C \sqrt{t}} \exp\left(-\frac{x^2}{Ct}\right),
\]

to denote a Gaussian-like density. We point out however that for any given \(C\), this function does not necessarily integrates to one. Moreover, we can write

\[
g_C(t, s, x, y) := g_C(s - t, y - x) = \frac{1}{C \sqrt{s-t}} e^{-|x-y|^2/(C(s-t))},
\]

and we have the following property: for all \(C_1, C_2 > 0\), there exists \(C_3, C > 0\) such that for all \(t, u, s \in \mathbb{R}_+\) and all \(x, y \in \mathbb{R}\),

\[
g_{C_3}(t, s, x, y) \leq C \int_{\mathbb{R}} g_{C_1}(t, u, x, z) g_{C_2}(u, s, z, y)dz.
\]

**Remark 3.3.** The presence of the term \(\Lambda_{T-t}(y)\) above comes from our approach. First, since we chose to work with an arbitrary time \(T < +\infty\), we have to keep track of the different exponents in time. Observe that if we fixed \(T < 1\), the expression above can be simplified significantly.

Secondly, the factor \(|y|^2\) can be absorbed in the Gaussian density, but the estimate we obtain is less precise, so we decided not to.

Thirdly, the only problematic contribution is \(e^{C(T-t)\sqrt{\gamma_0^N} |y|^2}\), and term stems from the control of the distance between the solutions of the two ODEs we consider (see Lemma 3.3 and 3.4). Nevertheless, since we assumed \(a_N \gamma_0^N \to 0\) as \(N\) goes to infinity, we do get the expected rate on the difference of the densities.

Finally, notice that if we set \(y\) in a compact set \(K_y\), the corrective term \(\Lambda_{T-t}(y)\) can be absorbed in some constant depending on \(K_y\), and we recover the usual rate of convergence.

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Proof. We recall here that:
\[ dX_t^N = F_N(t, X_t^N)X_t^N \, dt + \sigma dW_t, \]
where:
\[ F_N(t, x) = \left( \alpha_t^N - \sigma \sqrt{\gamma_t^N / k_{t+1}} \int_0^1 m' \left( \theta_t + \delta \text{sign}(x)(|x| \wedge a_N) \right) \sqrt{\gamma_t^N} \, d\theta \right) \]
\[ \rightarrow_{N \to +\infty} -am'(\bar{\theta}_t) + \frac{1}{2}. \]

We also recall that \( k = k(t) \) is the index for which \( t_{k+1}^N \leq t < t_k^N \). Thanks to the cut-off \( a_N \), this convergence is now uniform with respect to \( x \). Thus, we are now formally looking for a stability result for the density when the drift is not bounded. In other words, we need to control the distance between the transition density of \( (X_t^N)_{t \geq 0} \) and the transition density of the limiting SDE:
\[ dX_t = \left( -am'(\bar{\theta}_t) + \frac{1}{2} \right) X_t + \sigma dW_t. \]

This proof is inspired from Konakov et al. [4], in the case of an unbounded drift. To quantify the distance between \( p \) and \( q_N \), we rely on the parametrix representations. Since the drift coefficients are unbounded, we have to freeze along the solutions of the ODE. Fix a terminal time \( T > 0 \) and a terminal position \( y \in \mathbb{R} \). We consider \( \theta_{t,T}(y) \) and \( \theta_{t,T}^N(y) \) the backward flows respectively solutions of the following ODEs:
\[
\begin{cases}
\frac{\partial}{\partial t} z_t = (-am'(\bar{\theta}_t) + \frac{1}{2}) z_t, \\
\frac{\partial}{\partial T} \sigma W_z = F_N(t, z^N_t)z^N_t,
\end{cases}
\]
\[ z_T = y, \quad z_T^N = y. \]

To approximate \( (X_t)_{t \leq T} \), we consider the following frozen process:
\[ \tilde{X}^{T,y}_s = x + \int_s^T \left( -am'(\bar{\theta}_{u,T}(y)) + \frac{1}{2} \right) \theta_{u,T}(y) du + \sigma W_{s-t}. \]

This frozen process is a Gaussian process, and due to positivity of \( \sigma \), its density exists and we denote it by:
\[ \frac{d}{dz} \mathbb{P}(\tilde{X}^{T,y}_s \in dz| X^{T,y}_t = x) = \tilde{p}^{T,y}(t, s, x, z). \]

Observe that due to the choice of the freezing parameters we have the following identity:
\[ \tilde{p}^{T,y}(t, T, x, y) = \frac{1}{\sigma \sqrt{2\pi(T-t)}} \exp \left( -\frac{|\theta_{t,T}(y) - x|^2}{2\sigma^2(T-t)} \right). \]

From this point forward, we will use the following abuse of notation \( \tilde{p}(t, T, x, y) = \tilde{p}^{T,y}(t, T, x, y) \). Similarly, to approximate \( (X_t^N)_{t \leq T} \), we consider the following frozen processes:
\[ \tilde{X}^{T,y,N}_s = x + \int_s^T F_N(u, \theta_{u,T}^N(y)) \theta_{u,T}^N(y) du + \sigma W_{s-t}, \]

We denote its density by
\[ \frac{d}{dz} \mathbb{P}(\tilde{X}^{T,y,N}_s \in dz| X^{T,y,N}_t = x) = \tilde{q}^{T,y,N}_N(t, s, x, z), \]

and we once again have the identity:
\[ \tilde{q}^{T,y}_N(t, T, x, y) = \frac{1}{\sigma \sqrt{2\pi(T-t)}} \exp \left( -\frac{|\theta_{t,T}^N(y) - x|^2}{2\sigma^2(T-t)} \right). \]

Again, we will denote \( \tilde{q}_N(t, T, x, y) := \tilde{q}^{T,y}_N(t, T, x, y) \) for convenience. It is elementary to prove the convergence of the parametrix series in this specific case, especially since the diffusion coefficients are constants. We thus write
\[ (p - q_N)(t, T, x, y) = (\tilde{p} - \tilde{q}_N)(t, T, x, y) + \sum_{r=1}^{+\infty} \left( \tilde{p} \otimes H^{(r)} - \tilde{q}_N \otimes H^{(r)}_N \right)(t, T, x, y), \]
where \( f \otimes g(t, T, x, y) = \int_0^T \int_{R^d} f(t, u, x, z)g(u, T, z, y) du dz \) and

\[
H(t, T, x, y) = (L_t - \tilde{L}_t)\tilde{p}(t, T, x, y), \quad H_N(t, T, x, y) = (L_t^N - \tilde{L}_t^N)\tilde{q}_N(t, T, x, y),
\]

where \( L_t, \tilde{L}_t, L_t^N, \) and \( \tilde{L}_t^N \) represents the generators of \( (X_t)_{t \geq 0}, (\tilde{X}_t^{T,y})_{t \geq 0}, (X_t^N)_{t \geq 0} \) and \( (\tilde{X}_t^{T,y,N})_{t \geq 0} \) respectively. We will now proceed to give an estimate on each term in the above sum. We have the following Lemma:

**Lemma 3.3.** For all \( t \leq T \), there exists \( C > 0 \) such that for all \( x, y \in \mathbb{R} \), we have:

\[
|\tilde{p} - \tilde{q}_N| (t, T, x, y) | \leq C \sqrt{(T-t) \alpha N \sqrt{\gamma_0}} \exp(C(T-t)^2 \alpha^2 N \gamma_0 |y|^2) g_C(T-t, \lambda \theta, y) - x).
\]

**Proof.** We write using the finite increments theorem:

\[
(\tilde{p} - \tilde{q}_N)(t, T, x, y) = \frac{1}{\sigma \sqrt{2\pi(T-t)}} \exp \left( - \frac{|\theta_{t,T}(y) - x|^2}{2\sigma^2(T-t)} \right) - \exp \left( - \frac{|\theta_{t,N}(y) - x|^2}{2\sigma^2(T-t)} \right)
\]

\[
= \int_0^1 g^\sigma(t, T, \lambda \theta) + (1 - \lambda) \theta_{t,N} \theta_{t,N} - x) \theta_{t,T}(y) - \theta_{t,N}(y) d\lambda.
\]

Formally, since \( g^\sigma \) is a Gaussian density, it is known that its derivative yields a singularity in \((T-t)^{-1/2}\), thus, we see that to control the difference \(|\tilde{p} - \tilde{q}_N|\), we need a control of \(|\theta_{t,T}(y) - \theta_{t,N}(y)|\), which is done in the following Lemma:

**Lemma 3.4.** For all \( 0 \leq t \leq T \) there exits a constant \( C > 0 \) such that:

\[
|\theta_{t,T}(y) - \theta_{t,N}^N(y)| \leq C(T-t) \alpha N \sqrt{\gamma_0} |y|.
\]

Besides, for some positive \( C_T \), the following estimate holds:

\[
|\theta_{t,T}(y)| + |\theta_{t,N}^N(y)| \leq C_T |y|.
\]

Similarly, for all \( 0 \leq t_k^N \leq T \) there exits a constant \( C > 0 \) such that: we have:

\[
|\theta_{t_k^N,T}(y) - \theta_{t_k^N,N,T}(y)| \leq C(T-t_k^N) \alpha N \sqrt{\gamma_0} |y|.
\]

(3.20)

Besides, for some positive \( C_T \), the following estimate holds:

\[
|\theta_{t_k^N,N,T}(y)| \leq C_T |y|.
\]

The proof of this result relies on general results on stability for ODEs and we postpone this proof to the end of this section. We obtain that there exists \( C > 0 \) such that:

\[
|\tilde{p} - \tilde{q}_N| (t, T, x, y) \leq \int_0^1 |g^\sigma(T-t, \lambda \theta) \theta_{t,T}(y) + (1 - \lambda) \theta_{t,N}(y) - x) | |\theta_{t,T}(y) - \theta_{t,N}^N(y)| d\lambda
\]

\[
\leq C_T(T-t) \alpha N \sqrt{\gamma_0} |y| \int_0^1 d\lambda \frac{1}{\sqrt{T-t}} |g_C(T-t, \lambda \theta, y) + (1 - \lambda) \theta_{t,N}^N(y) - x)|.
\]

Now, we write using Young’s inequality:

\[
|\lambda \theta_{t,T}(y) + (1 - \lambda) \theta_{t,N}^N(y) - x|^2 = |\theta_{t,T}(y) - x + (1 - \lambda) [\theta_{t,N}^N(y) - \theta_{t,T}(y)]|^2
\]

\[
\geq \frac{1}{2} |\theta_{t,T}(y) - x|^2 - C^2(T-t)^2 \alpha^2 N \gamma_0 |y|^2.
\]

Plugging the above inequality into the Gaussian exponent yields:

\[
g_C(T-t, \lambda \theta_{t,T}(y) + (1 - \lambda) \theta_{t,N}^N(y) - x)
\]

\[
= \frac{1}{C \sqrt{2\pi(T-t)}} \exp \left( - \frac{|\lambda \theta_{t,T}(y) + (1 - \lambda) \theta_{t,N}(y) - x|^2}{2(T-t)C} \right)
\]

\[
\leq \frac{1}{C \sqrt{2\pi(T-t)}} \exp \left( - \frac{|\theta_{t,T}(y) - x|^2}{4C(T-t)} \right) \times \exp \left( C^2(T-t)^2 \alpha^2 N \gamma_0 |y|^2 \right)
\]

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Finally, we thus obtain the following upper bound. There are $C, C_1 > 0$ such that:

$$|\tilde{p} - \tilde{q}_N|(t, T, x, y) \leq C_1 \sqrt{(T - t)a_N \gamma_0} |y|^2 |y| \gamma_0 \gamma_0^N g_C(T - t, \theta_t, \theta_{t,T}(y) - x).$$

$\square$

**Remark 3.4** (Additional factor for the frozen densities).

We point out that the additional term in $|y|^2 e^{C(T - t)\gamma_0^N \gamma_0^N |y|^2}$ actually comes from the point-wise convergence (2.8). Observe that if the convergence (2.8) was actually uniform, from that point forward, we could have reproduced the proof of the stability result in the case of an unbounded drift. Also, we point out that the above proof also shows that in the argument of the Gaussian density $g_C(T - t, \cdot)$, we can put indifferently $\theta_t \theta_t(y)$ or $\tilde{\theta}_N(y)$ at the price of the additional factor $e^{C(T - t)\gamma_0^N \gamma_0^N |y|^2}$. That is, we actually have:

$$\frac{1}{2} e^{-\frac{2}{\gamma_0^N} \gamma_0^N |y|^2} g_N^2(T - t, \theta_{t,N}(y) - x) \leq g_C(T - t, \theta_t, \theta_{t,T}(y) - x)$$

$$\leq 2 e^{C(T - t)\gamma_0^N \gamma_0^N |y|^2} g_N^2(T - t, \theta_{t,N}(y) - x).$$

(3.21)

The rest of the proof thus consists in controlling the iterated convolutions with the additional factor $|y|^2 e^{C(T - t)\gamma_0^N \gamma_0^N |y|^2}$. Finally, since by assumption, $a_N \gamma_0^N \gamma_0^N \rightarrow 0$ as $N \rightarrow \infty$, we see that for large $N$ the correction becomes negligible.

We now turn to the estimate on the parametrix kernels. Observe that we can write:

$$\sum_{r=1}^{\infty} (\tilde{p} \otimes H^{(r)} - \tilde{q}_N \otimes H^{(r)})(t, T, x, y) = \tilde{p} \otimes \sum_{r=1}^{\infty} H^{(r)}(t, T, x, y) - \tilde{q}_N \otimes \sum_{r=1}^{\infty} H^{(r)}(t, T, x, y)$$

$$= \tilde{p} \otimes \Phi(t, T, x, y) - \tilde{q}_N \otimes \Phi_N(t, T, x, y),$$

where we wrote $H^{(r)} = H^{(r-1)} \otimes H$ and $H^{(r)} = H^{(r-1)} \otimes H_N$, with the convention that $H^{(0)}, H^{(0)} = Id$. Now, we can split as usual:

$$\tilde{p} \otimes \Phi(t, T, x, y) - \tilde{q}_N \otimes \Phi_N(t, T, x, y) = \tilde{p} \otimes (\Phi - \Phi_N)(t, T, x, y) + (\tilde{p} - \tilde{q}_N) \otimes \Phi_N(t, T, x, y).$$

We have the following lemma:

**Lemma 3.5.** For all $t \leq T$, there exits $C_T > 0$ such that for all $x, y \in \mathbb{R}$, the following estimates holds:

$$|\Phi(t, T, x, y) \leq C_T \frac{1}{\sqrt{T - t}} g_C(T - t, \theta_t, \theta_{t,T}(y) - x),$$

(3.22)

$$|\Phi_N(t, T, x, y) \leq C_T \frac{1}{\sqrt{T - t}} g_C(T - t, \theta_{t,N}(y) - x),$$

(3.23)

$$|\Phi_N - \Phi|(t, T, x, y) \leq C a_N \sqrt{\gamma_0^N \gamma_0^N} g_C(T - t, \theta_t, \theta_{t,T}(y) - x).$$

(3.24)

**Proof.** The first two estimates are a general estimate and derive from the controls on the parametrix kernels $H, H_N$. We give here some ideas of the proof. The two terms are treated similarly, we focus on the first estimate. First, observe that there exists a constant $C > 0$ such that we have the upper bound for the parametrix kernel:

$$|H|(t, T, x, y) \leq C_T \frac{1}{\sqrt{T - t}} g_C(T - t, \theta_t, \theta_{t,T}(y) - x).$$

We deduce by induction the following estimate:

$$|H^{(k)}|(t, T, x, y) \leq C_T (T - t)^{k-1} \prod_{r=1}^{k} B \left( \frac{r}{2}, \frac{1}{2} \right) g_C(T - t, \theta_{t,T}(y) - x),$$

where $B(\cdot, \cdot)$ denotes the Beta function. Notice that from the properties of the Beta function, we have:

$$\prod_{r=1}^{k} B \left( \frac{r}{2}, \frac{1}{2} \right) = \frac{\Gamma(1/2) + 1}{\Gamma \left( \frac{k+1}{2} \right)} = \sqrt{\frac{2}{\pi}}.$$
The announced estimate for $|\Phi|$ follows by taking the sum of the right hand side.

The third estimate is more involved, as we have to deal with the sensitivity in $N$. We start by obtaining a similar bound on $H - H_N$. We write:

\[
(H - H_N)(t, T, x, y) = (L_t - L_t^N)\tilde{p}(t, T, x, y) - (L_t^N - L_t^N)\tilde{q}_N(t, T, x, y)
= (L_t - L_t^N - L_t^N)\tilde{p}(t, T, x, y) + (L_t^N - L_t^N)(\tilde{p} - \tilde{q}_N)(t, T, x, y)
=: I + \|I\|,
\]

(3.25)

where we recall the expressions of the generators for a test function $\varphi$:

\[
L_t\varphi(x) = \left(-\sigma m'(\tilde{\theta}_t) + \frac{1}{2}\right)x \cdot \varphi'(x) + \frac{1}{2}\sigma^2 \varphi''(x),
\]

\[
\tilde{L}_t\varphi(x) = \left(-\sigma m'(\tilde{\theta}_t) + \frac{1}{2}\right)\theta_{t,T}(y) \cdot \varphi'(x) + \frac{1}{2}\sigma^2 \varphi''(x),
\]

\[
L_t^N\varphi(x) = F_N(t, x) \cdot \varphi'(x) + \frac{1}{2}\sigma^2 \varphi''(x),
\]

\[
\tilde{L}_t^N\varphi(x) = F_N(t, \tilde{\theta}_t^N(y)) \varphi'(x) + \frac{1}{2}\sigma^2 \varphi''(x).
\]

We will need the following estimate:

**Lemma 3.6.** There exists $C_T > 0$ such that:

\[
\left| \left(-\sigma m'(\tilde{\theta}_t) + \frac{1}{2}\right)(x - \theta_{t,T}(y)) - (xF_N(t, x) - F_N(t, \theta_{t,T}^N(y)))\theta_{t,T}^N(y) \right|
\leq C a_N \sqrt{\gamma_N} \left( |x - \theta_{t,T}(y)| + (T - t)(|y| + |y|^2) \right) \tag{3.26}
\]

**Proof.** We start the proof here by showing an auxiliary result, namely that $x \mapsto F_N(t, x)$ is Lipschitz (uniformly in $t \leq T$). The idea is to bound the first derivative. Note that we have to be careful here, since $F_N$ is not exactly differentiable at the cut-off point. We have:

\[
\left(F_N(t, x)x\right)' = F_N'(t, x)x + F_N(t, x).
\]

Now, since we assumed $m'$ to be bounded in a tubular neighborhood of $\tilde{\theta}_t$, we have that $F_N(t, x) \leq C$, for some constant $C > 0$. Thus, the derivative is bounded tubular neighborhood of $\theta_t$ when $F_N'(t, x)x$ is. Observe carefully now that:

\[
(F_N)'(t, x) = -\sigma \sqrt{\frac{\gamma_N}{k}} \int_0^1 m''(\tilde{\theta}_t + \text{sign}(x)(|x| \wedge a_N)\sqrt{\frac{1}{\gamma_N} \delta}) \sqrt{\gamma_N} \delta d\delta 1_{(|x| \leq a_N)}.
\]

(3.27)

Thus, thanks to our cut-off, we actually have:

- either $|x| \geq a_N$, and in which case $F_N'(t, x) = 0$,
- or $|x| \leq a_N$, but then, since $m''$ is bounded, we can write:

\[
|F_N'(t, x)x| \leq C|x| \leq C a_N \sqrt{\gamma_N}.
\]

Thus, in both cases, the derivative of $F_N(t, x)x$ is bounded and we can conclude that $x \mapsto F_N(t, x)x$ is Lipschitz.

We recall that $F_N$ is defined in 8.11, and we denote for convenience $A(\theta_t) = -\sigma m'(\theta_t) + \frac{1}{2}$. We write:

\[
\left| \left(-\sigma m'(\tilde{\theta}_t) + \frac{1}{2}\right)(x - \theta_{t,T}(y)) - (xF_N(t, x) - F_N(t, \theta_{t,T}^N(y)))\theta_{t,T}^N(y) \right|
\leq \left| A(\tilde{\theta}_t)(x - \theta_{t,T}(y)) - (xF_N(t, x) - F_N(t, \theta_{t,T}^N(y)))\theta_{t,T}(y) \right|
+ \left| F_N(t, \theta_{t,T}^N(y))\theta_{t,T}(y) - F_N(t, \theta_{t,T}(y))\theta_{t,T}(y) \right|
+ \left| F_N(t, \theta_{t,T}^N(y))\theta_{t,T}(y) - F_N(t, \theta_{t,T}(y))\theta_{t,T}(y) \right|
=: I_1 + I_2 + I_3.
\]

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We start with $I_2$, writing:

$$I_2 = \left| F_N(t, \theta_{t,T}^N) \left( \theta_{t,T}^N(y) - \theta_{t,T}(y) \right) \right| \leq |F_N(t, \theta_{t,T}^N(y))| \times |\theta_{t,T}^N(y) - \theta_{t,T}(y)|.$$ 

Now, since $m'$ is bounded in a tubular neighborhood of $\theta_t$, $|F_N(t, \theta_{t,T}^N(y))| \leq C$ and using Lemma 3.4 we obtain:

$$I_2 \leq C(T - t)a_N \sqrt{\gamma_0^N} |y|.$$ 

For $I_3$, we have:

$$I_3 = \left| \left( F_N(t, \theta_{t,T}^N(y)) - F_N(t, \theta_{t,T}(y)) \right) \theta_{t,T}(y) \right| \leq |F_N(t, \theta_{t,T}^N(y)) - F_N(t, \theta_{t,T}(y))| \times |\theta_{t,T}(y)|.$$ 

Since we assumed $m''$ to be bounded in a tubular neighborhood around $\hat{\theta}_t$, we have from Lemma 3.4 that $F_N''$ is bounded. We have to be careful at this point, as $x \wedge a_N$ is not smooth, only but piece-wise differentiable. Anyhow, the point that we can bound the derivative of $F_N$ still stands and we have directly the announced estimate for this term. Thus, $F_N(t, \cdot)$ is Lipschitz, so that:

$$I_3 \leq |F_N(t, \theta_{t,T}^N(y)) - F_N(t, \theta_{t,T}(y))| \times |\theta_{t,T}(y)| \leq C(T - t)a_N \sqrt{\gamma_0^N} |y|^2,$$

using again Lemma 3.4 and the definition of $\theta_{t,T}(y)$. Let us now turn to $I_1$ above. Set $\tilde{F}_N(t, x) = F_N(t, x)x$, we use the finite increment theorem:

$$\tilde{F}_N(t, x) - \tilde{F}_N(t, \theta_{t,T}(y)) = \int_0^1 \tilde{F}_N'(t, \lambda x + (1 - \lambda)\theta_{t,T}(y))(x - \theta_{t,T}(y))d\lambda.$$ 

We use the notation $F_N'$ to denote the derivative with respect to the $x$ variable to lighten the notations. Thus, we get for $I_1$:

$$I_1 \leq \left| \int_0^1 d\lambda \left( A(\hat{\theta}_t) - \tilde{F}_N'(t, \lambda x + (1 - \lambda)\theta_{t,T}(y)) \right) \right| \times |x - \theta_{t,T}(y)|.$$ 

Now, since $\tilde{F}_N'(t, x) = (F_N'(t, x)x + F_N(t, x))$, we write:

$$I_1 \leq \left| \int_0^1 d\lambda \left( A(\hat{\theta}_t) - F_N(t, \lambda x + (1 - \lambda)\theta_{t,T}(y)) \right) \right| \times |x - \theta_{t,T}(y)| + \left| \int_0^1 d\lambda (F_N'(t, \lambda x + (1 - \lambda)\theta_{t,T}(y))(\lambda x + (1 - \lambda)\theta_{t,T}(y)) \right| \times |x - \theta_{t,T}(y)|.$$ 

Using again the control above, we get that for all $x \in \mathbb{R}$, $|F_N'(t, x)x| \leq C a_N \sqrt{\gamma_0^N}$. Thus, we have:

$$\left| \int_0^1 d\lambda (F_N'(t, \lambda x + (1 - \lambda)\theta_{t,T}(y))(\lambda x + (1 - \lambda)\theta_{t,T}(y)) \right| \times |x - \theta_{t,T}(y)| \leq C a_N \sqrt{\gamma_0^N} |x - \theta_{t,T}(y)|.$$ 

Now, from estimate in the proof of Lemma 3.4 below, we know that for all $x \in \mathbb{R}$, 

$$\left| \left( F_N(t, x) - \left( -am^2(\hat{\theta}_t) + \frac{1}{2} \right) \right) \right| \leq C |x| \wedge a_N \sqrt{\gamma_0^N} \leq C a_N \sqrt{\gamma_0^N},$$

which gives for the first contribution above:

$$\left| \int_0^1 \left( A(\hat{\theta}_t) - F_N(t, \lambda x + (1 - \lambda)\theta_{t,T}(y)) \right) \times |x - \theta_{t,T}(y)| \right| \leq C a_N \sqrt{\gamma_0^N} |x - \theta_{t,T}(y)|.$$ 

Consequently, we obtain for this term:

$$I_1 \leq C a_N \sqrt{\gamma_0^N} |x - \theta_{t,T}(y)|.$$ 

Putting together the estimates for $I_1, I_2$ and $I_3$ proves (3.26). \qed
Returning to the control of (3.25), we start with $I$ above. In this case, we have:

$$\left| I \right| = \left| (L_t - \bar{L}_t - L^N_t + \bar{L}^N_t) \bar{p}(t, T, x, y) \right|$$

$$\leq \left| \nabla_x \bar{p}(t, T, x, y) \right| \left( -am'(\bar{t}) + \frac{1}{2} \right) (x - \theta_{t,T}(y)) - xF_N(t, x) + \theta^N_{t,T}(y)F_N(t, \theta^N_{t,T}(y)) \right|.$$

Then, we deduce the following estimate for $I$, using again estimate (3.26), that there exists $C > 0$ such that:

$$\left| I \right| \leq Ca_N \sqrt{\gamma^N_0} \left( |x - \theta_{t,T}(y)| + (T - t)(|y| + |y|^2) \right) \frac{1}{\sqrt{T - t}} g_C(T - t, \theta_{t,T}(y) - x).$$

Now, observe that there exists $C' > 0$ such that

$$\frac{|x - \theta_{t,T}(y)|}{\sqrt{T - t}} g_C(T - t, \theta_{t,T}(y) - x) \leq g_{C'}(T - t, \theta_{t,T}(y) - x).$$

Observe also that we have:

$$(T - t)(|y| + |y|^2) \frac{1}{\sqrt{T - t}} g_C(T - t, \theta_{t,T}(y) - x) = \sqrt{T - t}(|y| + |y|^2) g_C(T - t, \theta_{t,T}(y) - x)$$

Consequently, we obtain the following estimate for some $C > 0$:

$$\left| I \right| \leq Ca_N \sqrt{\gamma^N_0} \left( 1 + \sqrt{T - t}(|y| + |y|^2) \right) g_C(T - t, \theta_{t,T}(y) - x)$$

Let us now turn to $\mathcal{II}$ in (3.26) above. Then, we can bound:

$$\left| \mathcal{II} \right| := \left| (L^N_t - \bar{L}^N_t)(\bar{p} - \bar{q}_N)(t, T, x, y) \right|$$

$$= \left| \nabla_x (\bar{p} - \bar{q}_N)(t, T, x, y), F_N(t, x) - F_N(t, \theta^N_{t,T}(y)) \theta^N_{t,T}(y)) \right|$$

$$\leq \left| \nabla_x (\bar{p} - \bar{q}_N)(t, T, x, y) \right| \times |F_N(t, x) - F_N(t, \theta^N_{t,T}(y))\theta^N_{t,T}(y)|.$$

Using Lemma 3.16, we have:

$$|F_N(t, x) - F_N(t, \theta^N_{t,T}(y))\theta^N_{t,T}(y)|$$

$$\leq \left| (-am'(\bar{t}) + \frac{1}{2}) (x - \theta_{t,T}(y)) - (xF_N(t, x) - F_N(t, \theta^N_{t,T}(y))\theta^N_{t,T}(y)) \right|$$

$$+ \left| (-am'(\bar{t}) + \frac{1}{2}) (x - \theta_{t,T}(y)) \right|$$

$$\leq Ca_N \sqrt{\gamma^N_0} \left( |x - \theta_{t,T}(y)| + (T - t)(|y| + |y|^2) \right) + C|x - \theta_{t,T}(y)|.$$

Consequently, we have the estimate:

$$\left| \mathcal{II} \right| \leq \left| \nabla_x (\bar{p} - \bar{q}_N)(t, T, x, y) \right| Ca_N \sqrt{\gamma^N_0} \left( |x - \theta_{t,T}(y)| + (T - t)(|y| + |y|^2) \right)$$

$$+ C|x - \theta_{t,T}(y)||\nabla_x (\bar{p} - \bar{q}_N)(t, T, x, y)|.$$
Observe that we actually put $\theta_{t,T}^N(y)$ instead of $\theta_{t,T}(y)$ in the argument of the Gaussian density in order to match the contribution given by the difference of the generators $L_t^N - \bar{L}_t^N$. Consequently, we have the upper bound:

$$
|\mathcal{I}| \leq C(a_N \sqrt{\gamma_0^N})^2 g_C(T-t, \theta_{t,T}^N(y) - x) \left( |x - \theta_{t,T}(y)| + (T-t)(|y| + |y|^2) \right)
$$

$$
+ C|\mathcal{I}| \leq C(a_N \sqrt{\gamma_0^N})^2 g_C(T-t, \theta_{t,T}^N(y) - x)
$$

$$
\leq C(a_N \sqrt{\gamma_0^N})^2 g_C(T-t, \theta_{t,T}^N(y) - x) \left( \sqrt{T-t} + T - t + (T-t)(|y| + |y|^2) \right).
$$

Now, adding the estimates for $I$ and $\mathcal{I}$, we get:

$$
|H - H_N|(t,T,x,y) \leq C(a_N \sqrt{\gamma_0^N})^2 g_C(T-t, \theta_{t,T}(y) - x).
$$

To simplify the notations, let us define

$$
\Lambda_{T-t}^N(y) = 1 + \sqrt{T-t}(|y| + |y|^2) + |y|e^{C(T-t)\gamma_0^N} \left( \sqrt{T-t} + (T-t)(1 + |y|^2) \right),
$$

so that we have the estimate:

$$
(H - H_N)(t,T,x,y) \leq C(a_N \sqrt{\gamma_0^N})^2 \Lambda_{T-t}^N(y) g_C(T-t, \theta_{t,T}(y) - x). \tag{3.29}
$$

To obtain estimate (3.29), use estimate (3.29) to control the successive convolutions. We proceed prove the following estimate by induction.

$$
|H^{(k)} - H_N^{(k)}|(t,T,x,y) \leq C(a_N \sqrt{\gamma_0^N})^2 \Lambda_{T-t}^N(y) g_C(T-t, \theta_{t,T}(y) - x). \tag{3.30}
$$

For $k = 1$, this is estimate (3.29) and has already been established. Assume it holds for $k \geq 1$. We write:

$$
(H^{(k+1)} - H_N^{(k+1)})(t,T,x,y) = (H^{(k)} - H_N^{(k)}) \otimes H(t,T,x,y) + H^{(k)} \otimes (H - H_N)(t,T,x,y) = I + \mathcal{I}.
$$

For the contribution $I$ above, we use the induction hypothesis and we derive:

$$
|I| \leq C(a_N \sqrt{\gamma_0^N})^2 \prod_{r=1}^k B \left( r \frac{1}{2} \right) g_C(T-t, \theta_{t,T}(y) - x).
$$

We split the space the integration over $z$ as follows:

$$
D_1 = \{ z \in \mathbb{R} ; \Lambda_{u-t}^N(z - \theta_{t,T}(y)) \geq \Lambda_{T-t}^N(y) \}
$$

$$
D_2 = \{ z \in \mathbb{R} ; \Lambda_{u-t}^N(z - \theta_{t,T}(y)) < \Lambda_{T-t}^N(y) \}.
$$

Now, observe that when $z \in D_2$, we have:

$$
\Lambda_{u-t}^N(z) = \Lambda_{u-t}^N(z - \theta_{t,T}(y) + \theta_{t,T}(y)) \leq C_T \left( \Lambda_{u-t}^N(z - \theta_{t,T}(y)) + \Lambda_{T-t}^N(y) \right) \leq C_T \Lambda_{T-t}^N(y).
$$

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Thus, we can bound the $\Lambda_{n,t}^N(z)$ contribution by $\Lambda_{n,t}^N(y)$ and take it out of the integral:

$$|I_{|D_2|} \leq C_{a_N} \sqrt{\gamma_0} B_{\frac{r}{2}, \frac{1}{2}} \int_t^T (u-t)^{\frac{k-1}{2}} \int_{D_2} g_C(u-t, \theta_{t,u}(z) - x)$$

$$\times \frac{1}{\sqrt{T-u}} g_C(T-u, \theta_{u,T}(y)-z) dudz. $$

Now, we can bound the convolution of Gaussian densities:

$$\int_{D_2} g_C(u-t, \theta_{t,u}(z) - x) g_C(T-u, \theta_{u,T}(y)-z) dz \leq C g_C(T-t, \theta_{t,T}(y)-x),$$

and the time integral increase the Beta function by the right amount. For $z \in D_1$, observe that we have:

$$\Lambda_{n,t}^N(z) = \Lambda_{n,t}^N(z - \theta_{t,T}(y) + \theta_{t,T}(y)) \leq C_T \left( \Lambda_{n,t}^N(z - \theta_{t,T}(y)) + \Lambda_{n,t}^N(y) \right) \leq C_T \Lambda_{n,t}^N(z - \theta_{t,T}(y)).$$

Therefore:

$$|I_{|D_1|} \leq C_{a_N} \sqrt{\gamma_0} \prod_{r=1}^k B_{\frac{r}{2}, \frac{1}{2}} \int_t^T (u-t)^{\frac{k-1}{2}} \int_{D_2} g_C(u-t, \theta_{t,u}(z) - x)$$

$$\times \frac{1}{\sqrt{T-u}} g_C(T-u, \theta_{u,T}(y)-z) dudz. $$

The important thing to notice here is that up to a reversal of the flow, the same argument appears both in the Gaussian density and in $\Lambda_{n,t}^N(z - \theta_{t,T}(y))$. We now claim that the following estimate holds for $\gamma_0$ small enough:

$$\int \int g_C(u-t, \theta_{t,u}(z) - x) \Lambda_{n,t}^N(z - \theta_{t,T}(y)) \frac{dz}{T-u, \theta_{u,T}(y)-z} \leq g_C(T-t, \theta_{t,T}(y)-x). \quad (3.31)$$

Indeed, observe that

$$e^{\frac{C}{2} \gamma_0' |z - \theta_{u,T}(y)|^2} g_C(T-u, \theta_{u,T}(y)-z)$$

$$= e^{\frac{C}{2} \gamma_0' |z - \theta_{u,T}(y)|^2} \frac{1}{C \sqrt{2 \pi (T-u)}} \exp \left( - \frac{|z - \theta_{u,T}(y)|^2}{C(T-u)} \right)$$

$$= \frac{1}{C \sqrt{2 \pi (T-u)}} \exp \left( - \frac{|z - \theta_{u,T}(y)|^2}{C(T-u)} \left( \frac{1}{C(T-u)} - C(u-t) a_N^2 \gamma_0^N \right) \right)$$

$$\leq C g_C(T-u, \theta_{u,T}(y)-z),$$

since we assumed $a_N \sqrt{\gamma_0} \to 0$ when $N$ goes to infinity. Besides, for $m = 1, 2, 3$ we can bound up to a modification of $C$:

$$|z - \theta_{u,T}(y)|^m g_C(T-u, \theta_{u,T}(y)-z) \leq (T-u)^m/2 g_C(T-u, \theta_{u,T}(y)-z) \leq C_T g_C(T-u, \theta_{u,T}(y)-z).$$

Thus, the claim (3.31) follows from the convolution of Gaussian densities, and again, the integral in time increase the Beta function by the right amount. Thus, we have the required estimate on $I$. We now turn to $II$ above. Using the induction hypothesis for $k = 1$ and the estimate on $H_N^{(k)}$, we get:

$$II = C_T \int_t^T du \int \int g_C(u-t, \theta_{t,u}(z) - x)$$

$$\times \frac{1}{\sqrt{T-u}} \Lambda_{n,t}^N(y) g_C(T-u, \theta_{u,T}(y)-z).$$

In this case, we claim that:

$$\int \int g_C(u-t, \theta_{t,u}(z) - x) g_C(T-u, \theta_{u,T}(y)-z) \leq e^{C(T-t) a_N^2 \gamma_0^N |y|^2} g_C(T-t, \theta_{t,T}(y)-x) \quad (3.32)$$

which readily gives the announced estimate, up to a modification of the constant $C$. To prove the above inequality, we bound using (3.32):

$$g_C(T-u, \theta_{u,T}(y)-z) \leq 2 e^{C(T-t) a_N^2 \gamma_0^N |y|^2} g_C(T-u, \theta_{u,T}(y)-z).$$
Now, we use the fact that Gaussian densities are closed under convolution, i.e. the following holds:
\[
\int_{\mathbb{R}} dz g_C(u - t,\theta^N_{t,u}(z) - x)g_C(T - u,\theta^N_{u,T}(y) - z) \leq C g_C(T - t,\theta^N_{t,T}(y) - x).
\]
We thus get:
\[
\int_{\mathbb{R}} dz g_C(u - t,\theta^N_{t,u}(z) - x)g_C(T - u,\theta_{u,T}(y) - z) \leq e^{C(T-t)\sigma^2_N|y|^2}g_C(T - t,\theta^N_{t,T}(y) - x).
\]
We conclude using once again (3.21) to replace \(\theta^N_{t,T}(y)\) by \(\theta_{t,T}(y)\) in the above inequality. Finally, to get estimate (3.30) and complete the proof of Lemma 3.5, it remains to sum (3.30) for \(k \geq 1\).

To complete the proof of Lemma 3.2 we recall that
\[
(p - q_N)(t,T,x,y) = (\tilde{p} - \tilde{q}_N)(t,T,x,y) + \sum_{r=1}^{+\infty} \left( \tilde{p} \otimes H^{(r)} - \tilde{q}_N \otimes H^{(r)}_N \right)(t,T,x,y),
\]
thus, we can write:
\[
|p - q_N|(t,T,x,y) \leq |\tilde{p} - \tilde{q}_N|(t,T,x,y) + \tilde{p} \otimes |\Phi - \Phi_N|(t,T,x,y) + |\tilde{p} - \tilde{q}_N| \otimes |\Phi_N|(t,T,x,y) =: I + II + III.
\]
Now, the contribution \(I\) above clearly yields the announced estimate (see Lemma 3.3 above). For the contribution \(II\), we use Lemma 3.3 to estimate the difference of the frozen densities and Lemma 3.5 to control \(\Phi_N\). As discussed above, we bound:
\[
|z|^C(u-t)a^2\gamma^N_0|z|^2 \leq C(\theta^N_{t,T}(y) - z) + |y|e^{C(T-t)\sigma^2_N(\theta^N_{t,T}(y) - z)^2}e^{C(T-t)\sigma^2_N|y|^2},
\]
and use (3.21), and (3.31) to derive:
\[
|\tilde{p} - \tilde{q}_N| \otimes |\Phi_N|(t,T,x,y)
\leq C \int_{\mathbb{R}} \int_{\mathbb{R}} a_N \sqrt{\gamma} \sqrt{(u-t)}|z|e^{C(u-t)\sigma^2_N|z|^2}g_C(u-t,\theta_{t,u}(z) - x) \times \frac{1}{\sqrt{T-u}}g_C(T-u,\theta^N_{u,T}(y) - z)\, dz
\]
\[
\leq C a_N \sqrt{\gamma} \sqrt{(T-t)}e^{C(T-t)\sigma^2_N|y|^2} \int_{\mathbb{R}} \sqrt{u-t} \int_{\mathbb{R}} \theta^N_{t,T}(y) - z|e^{C(T-t)\sigma^2_N(\theta_{t,T}(y) - z)^2} \times \frac{1}{\sqrt{T-u}}g_C(T-u,\theta^N_{u,T}(y) - z)\, dz
\]
\[
+ C a_N \sqrt{\gamma} \sqrt{(T-t)}e^{C(T-t)\sigma^2_N|y|^2} \int_{\mathbb{R}} \sqrt{u-t} \int_{\mathbb{R}} e^{C(T-t)\sigma^2_N(\theta_{t,T}(y) - z)^2}g_C(u-t,\theta_{t,u}(z) - x) \times \frac{1}{\sqrt{T-u}}g_C(T-u,\theta^N_{u,T}(y) - z)\, dz
\]
Finally, using (3.32), we get:
\[
|\tilde{p} - \tilde{q}_N| \otimes |\Phi_N|(t,T,x,y)
\leq C a_N \sqrt{\gamma} \sqrt{(T-t)}e^{C(T-t)\sigma^2_N|y|^2}(T-t)\, B \left( \frac{3}{2}, \frac{1}{2} \right) g_C(T-t,\theta_{t,T}(y) - x)
\leq C a_N \sqrt{\gamma} \sqrt{(T-t)}\Lambda^N_{3-t}(y)(T-t)\, B \left( \frac{3}{2}, \frac{1}{2} \right) g_C(T-t,\theta_{t,T}(y) - x).
\]
Finally, for the contribution \(III\) above, we write using equation (3.24):
\[
\tilde{p} \otimes |\Phi - \Phi_N|(t,T,x,y)
\leq C \int_{\mathbb{R}} \int_{\mathbb{R}} a_N \sqrt{\gamma} \sqrt{(T-t)}\Lambda^N_{3-t}(y)g_C(T-u,\theta_{u,T}(y) - z)\, dz,
\]
which directly gives the announced control (see estimate (3.30)). Piecing estimates for \(I, II\) and \(III\) together yields the announced estimates. \(\square\)
Proof of Lemma [3.4]. We now turn to estimating $|\theta_{t,T}(y) - \theta_{t,T}^N(y)|$. To simplify the notations, we set $z_t^N = \theta_{t,T}^N(y)$, and $z_t = \theta_{t,T}(y)$. Observe first that the following estimate holds:

$$ |z_t^N| \leq C_T|y|. $$

Indeed, we have the following inequality:

$$ |z_t^N| \leq |y| + \int_t^T |F_N(t, z_u^N)||z_u^N|du. $$

Now, since for all $x \in \mathbb{R}$, the quantity $|F_N(t, x)|$ is bounded, using Gronwall’s Lemma, we readily derive:

$$ |z_t^N| \leq C|y|e^{K(T-t)} = C_T|y|. $$

Note that we have the representations:

$$ z_t^N = y - \int_t^T (-\sigma m'(\bar{\theta}_u) + \frac{1}{2}) z_u^N du - \int_t^T \left( F_N(u, z_u^N) - (-\sigma m'(\bar{\theta}_u) + \frac{1}{2}) \right) z_u^N du, $$

$$ z_t = y - \int_t^T (-\sigma m'(\bar{\theta}_u) + \frac{1}{2}) z_u du. $$

Thus, taking the difference yields:

$$ z_t^N - z_t = -\int_t^T \left( -\sigma m'(\bar{\theta}_u) + \frac{1}{2} \right) \left( z_u^N - z_u \right) du - \int_t^T \left( F_N(u, z_u^N) - (-\sigma m'(\bar{\theta}_u) + \frac{1}{2}) \right) z_u^N du. \tag{3.33} $$

Observe now that for all $x \in \mathbb{R}$, we can bound

$$ \left| \left( F_N(u, x) - (-\sigma m'(\bar{\theta}_u) + \frac{1}{2}) \right) \right| \leq C(|x| \wedge a_N) \sqrt{\gamma_0} \leq C a_N \sqrt{\gamma_0}. \tag{3.34} $$

Indeed, we write for all $x \in \mathbb{R}$:

$$ F_N(u, x) - (-\sigma m'(\bar{\theta}_u) + \frac{1}{2}) = \left( \alpha_u - \sigma \left( \sqrt{\gamma_k^N} \int_0^1 m' \left( \bar{\theta}_u + \delta \text{sign}(x) (|x| \wedge a_N) \sqrt{\gamma_k^N} \right) d\delta \right) - (-\sigma m'(\bar{\theta}_u) + \frac{1}{2}) 

= -\sigma \left( \sqrt{\gamma_k^N} \int_0^1 m' \left( \bar{\theta}_u + \delta \text{sign}(x) (|x| \wedge a_N) \sqrt{\gamma_k^N} \right) - m'(\bar{\theta}_u) d\delta \right) 

= -\sigma \left( \sqrt{\gamma_k^N} \int_0^1 m' \left( \bar{\theta}_u \right) + \left( \alpha_u - \frac{1}{2} \right) \right). $$

Since $m'$ is Lipschitz, we have

$$ |m' \left( \bar{\theta}_u + \delta \text{sign}(x) (|x| \wedge a_N) \sqrt{\gamma_k^N} \right) - m'(\bar{\theta}_u) | \leq C(|x| \wedge a_N) \sqrt{\gamma_k^N} \leq C a_N \sqrt{\gamma_0}. $$

On the other hand, we recall:

$$ \alpha_{t_k}^N = \sqrt{\gamma_k^N - \gamma_{k+1}^N} \xrightarrow{N \to +\infty} \bar{\alpha} = \frac{1}{2}, $$

thus, since $m'$ is bounded, we have:

$$ \left| -\sigma \left( \sqrt{\gamma_k^N \gamma_{k+1}^N} - 1 \right) m'(\bar{\theta}_u) + \left( \alpha_u - \frac{1}{2} \right) \right| $$

$$ \leq \left| -\sigma \left( \sqrt{\gamma_k^N \gamma_{k+1}^N} - 1 \right) m'(\bar{\theta}_u) \right| + \left| \sqrt{\gamma_k^N - \gamma_{k+1}^N} + \frac{1}{2} \right| $$

$$ \leq C \left| \sqrt{\gamma_k^N \gamma_{k+1}^N} - 1 \right| + \left| \sqrt{\gamma_k^N - \gamma_{k+1}^N} + \frac{1}{2} \right|. $$

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Thus, expanding $\sqrt{\frac{N}{\gamma_{k+1}}}^N$ and $\sqrt{\frac{N}{(\gamma_{k+1})^{1/2}}}$ we derive that:

$$-\sigma \left( \sqrt{\frac{\gamma_k}{\gamma_{k+1}}} - 1 \right) m'(\hat{\theta}_u) + \left( \alpha_u^N - \frac{1}{2} \right) \leq C \sqrt{\gamma_0^N}.$$  

Consequently, we have the upper bound:

$$|F_N(t, x) - \left(-\sigma m'({\hat{\theta}_u}) + \frac{1}{2}\right)| \leq C a_N \sqrt{\gamma_0^N}.$$  

Plugging this inequality in (3.33) yields:

$$|z_t^N - z_t| \leq \int_t^T |-\sigma m'({\hat{\theta}_u}) + \frac{1}{2}| \left| z_u^N - z_u \right| du$$

$$+ \int_t^T \left| F_N(t, z_u^N) - \left(-\sigma m'({\hat{\theta}_u}) + \frac{1}{2}\right) \right| |z_u^N| du$$

$$\leq \int_t^T |-\sigma m'({\hat{\theta}_u}) + \frac{1}{2}| \left| z_u^N - z_u \right| du + C \int_t^T a_N \sqrt{\gamma_0^N} |z_u^N| du$$

$$\leq \int_t^T |-\sigma m'({\hat{\theta}_u}) + \frac{1}{2}| \left| z_u^N - z_u \right| du + C(T - t) a_N \sqrt{\gamma_0^N} |y|,$$

where to get the last inequality, we recall that for all $u \leq T$, we have $|z_u^N| \leq C_T |y|$. Therefore, applying Gronwall’s Lemma yields:

$$|z_t^N - z_t| \leq C e^{C(T-t)} (T-t) a_N \sqrt{\gamma_0^N} |y| = C_T (T-t) a_N \sqrt{\gamma_0^N} |y|.$$  

This completes the proof for the continuous objects. The discrete part is obtained as a corollary of the previous result, writing:

$$|\theta_{t_k^N, T}(y) - \hat{\theta}_{t_k^N, T}(y)| \leq |\theta_{t_k^N, T}(y) - \hat{\theta}_{t_k^N, T}(y)| + |\hat{\theta}_{t_k^N, T}(y) - \hat{\theta}_{t_k^N, T}(y)|.$$  

The first contribution is estimated using the proved part of Lemma 3.3. The second contribution is nothing but the classical error for the Backward Euler scheme, which is well known to be estimated as:

$$|\theta_{t_k^N, T}(y) - \hat{\theta}_{t_k^N, T}(y)| \leq C_T \gamma_0^N |y|.$$  

We get the conclusion by putting the worst estimate. Besides, the fact that $|\hat{\theta}_{t_k^N, T}(y)| \leq C_T |y|$ can be obtained by induction, directly from the expression of $\hat{\theta}_{t_k^N, T}(y)$, since it satisfies the recursion:

$$\begin{cases} x_{t_{k+1}}^N = x_{t_k^N}^N + F(t_{k+1}^N, x_{t_k^N}^N) \gamma_{t_{k+1}}^N, \\
 x_{t_k^N}^N = y \end{cases}$$

Remark 3.5 (Importance of the cut-off in $F_N$). We point out that in the above proof, we actually proved the following inequality $\forall x \in \mathbb{R}$:

$$\left| \left( F_N(t, x) - \left(-\sigma m'({\hat{\theta}_u}) + \frac{1}{2}\right) \right) \right| \leq C |x| \sqrt{\gamma_0^N}.$$  

Without the cut-off, this would give the following bound $|z_t^N - z_t| \leq C_T (T-t) \sqrt{\gamma_0^N} |y|^2$, which cannot be handled with the arguments developed here.  

3.2 The Local Limit Theorem for the Markov Chain

In this section, we deal with the discretization of the diffusion introduced above. The Markov chains we define below are defined for times on the grid $\{t_k^N, k \in \llbracket 1, M(N) \rrbracket \}$. However, we fixed a time horizon $T > 0$ above and there is no reason why this time is on that grid. Nevertheless, we keep the notation $T$ where we actually mean $T_N$, the closest point of the grid on the left of $T$, to shorten the already long notations.
We investigate the distance between the transition densities of the Markov chain \((V_{t_k}^N)_{t_k \leq T}\) and the diffusion \((X_N^N)_{t \leq T}\). The analysis follows the arguments presented in Konakov and Mammen [3], but in the case of an unbounded drift coefficient. The strategy of the proof is the same, but due to the unbounded nature of the drift, some adjustments are necessary. We detail those below.

**Lemma 3.7.** Fix a time horizon \(T > 0\). There exists a positive integer \(S > 0\), such that for all \(t_k^N \leq T\), for all \((x, y) \in K_x \times K_y\), with \(K_x, K_y\) compact sets, and a constant \(C > 0\) depending on the compact sets such that:

\[
| (q_N - p_N)(t_k^N, T, x, y) | \leq C \sqrt[3]{\frac{1}{\gamma_{10}^N} (T - t_k^N)^{-\frac{1}{2}} \left(1 + \frac{| \gamma_{N,k}^{y,(y-z)} |}{\sqrt{T-t_k^N}} \right)^{\frac{3}{2}}}.
\]

The proof of this result is globally the same as for the diffusive part, in that we compare the two parametrix series. Just like for the diffusive case, to handle the unbounded drift, we freeze the coefficients along the solutions of the ODE associated. We keep the notations of the last paragraph. To approximate

\[
dX_t^N = F_N(t, X_t^N)X_t^N + \sigma dW_t,
\]

we considered the following frozen processes:

\[
\hat{X}_k^T, y : = x + \int_t^T F_N(u, \theta^N_{u,T}(y)) \theta^N_{u,T}(y) du + \sigma W_{s-t},
\]

and we once again have the identity:

\[
q_N^T(t, T, x, y) = g_{\sigma}(T - t, \theta^N_{t,T}(y) - x).
\]

Similarly, to approximate the Markov chain:

\[
V_{t_k+1}^N = V_{t_k}^N + F_N(t_k^N, V_{t_k}^N)V_{t_k}^N + \gamma_{k+1}^N \xi_{k+1},
\]

we consider \(\hat{\theta}^N_{t_k,T}(y)\) the backward Euler scheme for the ODE:

\[
x_{t_k}^N = x_{t_k}^N + F_N(t_{k+1}, x_{t_k}^N) x_{t_k}^N \gamma_{k+1}^N,
\]

and define the frozen Markov chain:

\[
\hat{V}_{t_k+1}^T, y : = \hat{V}_{t_k+1}^T, y + F_N(t_{k+1}, \hat{\theta}_{t_k+1,T}(y)) \hat{\theta}_{t_k+1,T}(y) \gamma_{k+1}^N + \gamma_{k+1}^N \xi_{k+1}.
\]

This Markov chain admits a transition density:

\[
\frac{d}{dz} \mathbb{P}(\hat{V}_{t_k}^T, y \in dz | \hat{V}_{t_k}^T, y = x) = \hat{p}_N^T(t_k^N, t_j^N, x, y).
\]

Observe that we can rewrite the frozen Markov chain:

\[
\hat{V}_{t_j}^T = x + \hat{\theta}_{t_j,T}(y) - \hat{\theta}_{t_j,T}(y) + \sum_{i=k}^{j-1} \gamma_{i+1}^N \xi_{i+1},
\]

consequently, once again, we have for \(t_j^N = T\):

\[
\hat{p}_N^T(t_k, T, x, y) = p_{SN} \left( \hat{\theta}_{t_j,T}(y) - x \right),
\]

if \(p_{SN}\) denotes the density of the sum i.i.d. variables \(\sum_{i=k}^{j-1} \sqrt{\gamma_{i+1}^N} \xi_{i+1}\). Defining now the one step generators, for a test function \(\varphi\):

\[
\mathcal{L}_N \varphi(t_j^N, t_j^N, x, y) = \frac{1}{t_{j+1}^N - t_j^N} \int_R \left( \varphi(t_{j+1}^N, t_j^N, x, y) - \varphi(t_{j+1}^N, t_j^N, x, y) \right) p_N(t_{j+1}^N, t_j^N, x, y) dz
\]

\[
= \frac{1}{t_{j+1}^N - t_j^N} \mathbb{E} \left( \varphi(t_{j+1}^N, t_j^N, V_{t_{j+1}^N}^N, y) - \varphi(t_{j+1}^N, t_j^N, V_{t_{j+1}^N}^N, y) \right) | \hat{V}_{t_{j+1}^N}^T, y = x)\right).
\]

\[
\hat{\mathcal{L}}_N \varphi(t_j^N, t_j^N, x, y) = \frac{1}{t_{j+1}^N - t_j^N} \int_R \left( \varphi(t_{j+1}^N, t_j^N, x, y) - \varphi(t_{j+1}^N, t_j^N, x, y) \right) \hat{p}_N^T(t_{j+1}^N, t_j^N, x, z) dz
\]

\[
= \frac{1}{t_{j+1}^N - t_j^N} \mathbb{E} \left( \varphi(t_{j+1}^N, t_j^N, \hat{V}_{t_{j+1}^N}^T, y) - \varphi(t_{j+1}^N, t_j^N, \hat{V}_{t_{j+1}^N}^T, y) \right) | \hat{V}_{t_{j+1}^N}^T, y = x)\right).
\]

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Using the Markov property repeatedly, we obtain the following representation for the density of the Markov Chain \((V^{N}_{t_k})_{t_k \leq T}\):

\[
p_N(t_k, T, x, y) = \sum_{r=0}^{N} \tilde{p}_N \otimes_N K_N^{[r]}(t_k, T, x, y),
\]

where we denote \(\otimes_N\) the discretized time-space convolution:

\[
f \otimes_N g(t_i^N, t_j^N, x, y) = \sum_{k=0}^{r-1} \gamma_{k+1}^N \int_{\mathbb{R}} dz f(t_i^N, t_k^N, x, z)g(t_k^N, t_j^N, z, y).
\]

We recall that \(\gamma_{N+1}^N = t_{k+1}^N - t_k^N\), so that the first sum can be seen as a discretized time integral. Also, we denoted \(K_N^{[r]} = K_N^{[r-1]} \otimes_N K_N^{[0]} = \text{Id}\), and

\[
K_N(t, T, x, y) = (\mathcal{L}_N - \tilde{\mathcal{L}}_N)\tilde{p}_N^{T,y}(t, T, x, y).
\]

From the results of the last section, we know that the parametrix series representation holds for the diffusion and so to estimate \(|q_N - p_N|\), we write:

\[
(q_N - p_N)(t_k, T, x, y) = \sum_{r=0}^{+\infty} \tilde{q}_N \otimes H_N^{[r]}(t_k, T, x, y) - \sum_{r=0}^{N} \tilde{p}_N \otimes_K K_N^{[r]}(t_k, T, x, y).
\]

Now, somehow have to compare the two expansions. To that end, we introduce a series of intermediate steps, summarized in the flowchart below.

Step 1: Replace \(\otimes\) with \(\otimes_N\): discretisation of the time integral

\[
\sum_{r=0}^{+\infty} \tilde{q}_N \otimes_H H_N^{[r]}(t_k, T, x, y)
\]

Step 2: Truncate the rest of the series: \(\sum_{r \geq N} |\tilde{q}_N \otimes H_N^{[r]}| \leq O(\gamma_0^N)g\)

\[
\sum_{r=0}^{N} \tilde{q}_N \otimes H_N^{[r]}(t_k, T, x, y)
\]

Step 3: Replace \(H_N\) with \(K_N + M_N\)

\[
\sum_{r=0}^{N} \tilde{q}_N \otimes (K_N + M_N)^{[r]}(t_k, T, x, y)
\]

Step 4: Replace \(\tilde{q}_N\) with \(\tilde{p}_N\): control via classical Edgeworth expansions

\[
\sum_{r=0}^{N} \tilde{p}_N \otimes (K_N + M_N)^{[r]}(t_k, T, x, y)
\]

Step 5: Replace \(K_N + M_N\) with \(K_N\)

\[
\sum_{r=0}^{N} \tilde{p}_N \otimes K_N^{[r]}(t_k, T, x, y) = p_N(t_k, T, x, y)
\]

Where we defined

\[
K_N(t_k, T, x, y) = (L_{t_k} - \tilde{L}_{t_k})\tilde{p}_N(t_k, T, x, y), \text{ and } M_N \text{ is a remainder (see (3.36))}
\]

Step one deals with the discretisation of the time integral and is proved exactly as in Konakov and Mammen [3]. Thus we omit this step here. Step two deals with the rest of the convergent parametrix series, and can be obtained as a corollary of the results of the last section. Step 4 is done using the classical Edgeworth expansions, and is a consequence of Theorem 19.3 in Battacharya and Rao [2], with the slight modification that the arguments in the densities involve the transport part (see Lemma 3.13 below). Finally, Steps 3 and 5 are very similar and are dealt with Lemma 5.8 below. Note that the proof of Step 5 is trivial once the controls of Lemma 5.8 are obtained. There are two main difficulties for these steps. The first one is to understand how the one-step generators \(\mathcal{L}_N\) and \(\tilde{\mathcal{L}}_N\) relate to the generators of the diffusions \(L_t\) and \(\tilde{L}_t\). The second difficulty is to control the iterated convolutions.
3.2.1 The discrete generators

Before we dive into the proof of steps 3 to 5, we take a few lines here to understand the discrete generators. To compute the one-step generator, we need an expression for the transition density of the Markov chain \( (V^N_{t_k})_{k\geq 1} \) for one step. Using the recurrence relation:

\[
\mathbb{P}(V^N_{t_{k+1}} \leq y | V^N_{t_k} = x) = \mathbb{P}(V^N_{t_k} + F_N(t^N_{t_k}, V^N_{t_k}) \gamma^N_{t_k+1} + \sqrt{\gamma^N_{t_k+1}} \xi_{k+1} \leq y | V^N_{t_k} = x) = \mathbb{P}(\xi_{k+1} \leq \frac{y - (x + F_N(t^N_{t_k}, x) \gamma^N_{t_k+1})}{\sqrt{\gamma^N_{t_k+1}}}).
\]

Consequently, we see that:

\[
p_N(t^N_k, t^N_{k+1}, x, y) = \frac{1}{\sqrt{\gamma^N_{k+1}}} \mathbb{P}_{t_k}\left( \frac{y - (x + F_N(t^N_k, x) \gamma^N_{k+1})}{\sqrt{\gamma^N_{k+1}}} \right).
\] (3.35)

Note that we can do the same computations \((\tilde{V}^T_{t^N_k})_{t^N_k \leq T}\), for the frozen Markov chain, to get the expression:

\[
\tilde{p}_N(t^N_k, t^N_{k+1}, x, y) = \frac{1}{\sqrt{\gamma^N_{k+1}}} \mathbb{P}_{t_k}\left( \frac{y - (x + F_N(t^N_k, x) \gamma^N_{k+1})}{\sqrt{\gamma^N_{k+1}}} \right).
\]

Next, when taking the difference \(\mathcal{L}_N - \tilde{\mathcal{L}}_N\), one can notice that we can rewrite:

\[
(\mathcal{L}_N - \tilde{\mathcal{L}}_N)\varphi(t^N_k, t^N_{k+1}, x, y) = \frac{1}{\gamma^N_{k+1}} \int \left( \varphi(t^N_{k+1}, t^N_{k+1}, x, y) - \varphi(t^N_{k+1}, t^N_k, x, y) \right) p_N(t_k, t^N_{k+1}, x, z) dz
\]

\[
- \frac{1}{\gamma^N_{k+1}} \int \left( \varphi(t^N_{k+1}, t^N_k, x, y) - \varphi(t^N_{k+1}, t^N_k, x, y) \right) \tilde{p}_N(t_k, t^N_{k+1}, x, z) dz
\]

\[
= \left( (\mathcal{L}_N - \tilde{\mathcal{L}}_N)_1 - (\mathcal{L}_N - \tilde{\mathcal{L}}_N)_2 \right) \varphi(t^N_k, t^N_{k+1}, x, y).
\]

That way, the arguments in the test function \(\varphi\) only change on the first spacial component, and we can expand \(\varphi(t^N_{k+1}, t^N_k, x, y) - \varphi(t^N_{k+1}, t^N_k, x, y)\) using Taylor’s Formula:

\[
\varphi(t^N_{k+1}, t^N_k, x, y) - \varphi(t^N_{k+1}, t^N_k, x, y) = \partial_x \varphi(t^N_{k+1}, t^N_k, x, y)(z - x) + \frac{1}{2} \partial^2_x \varphi(t^N_{k+1}, t^N_k, x, y)(z - x)^2 + \int_0^1 \partial^3_x \varphi(t^N_{k+1}, t^N_k, x, y + \lambda(z - x), y)(z - x)^3 \frac{(1 - \lambda)^2}{2} d\lambda.
\]

Now, for \((\mathcal{L}_N - \tilde{\mathcal{L}}_N)_1 \varphi(t^N_k, t^N_{k+1}, x, y)\), we integrate the expansion against \(p_N(t^N_k, t^N_{k+1}, x, z)\) to get:

\[
(\mathcal{L}_N - \tilde{\mathcal{L}}_N)_1 \varphi(t^N_k, t^N_{k+1}, x, y) = \frac{1}{\gamma^N_{k+1}} \int \left( \varphi(t^N_{k+1}, t^N_k, x, y) - \varphi(t^N_{k+1}, t^N_k, x, y) \right) p_N(t_k, t^N_{k+1}, x, z) dz
\]

\[
+ \frac{1}{2} \int_0^1 \partial^2_x \varphi(t^N_{k+1}, t^N_k, x, y)(z - x)^2 p_N(t^N_k, t^N_{k+1}, x, z) dz
\]

\[
+ \int_0^1 \partial^3_x \varphi(t^N_{k+1}, t^N_k, x, y + \lambda(z - x), y)(z - x)^3 \frac{(1 - \lambda)^2}{2} p_N(t^N_k, t^N_{k+1}, x, z) dz d\lambda
\]

Let us deal with the rest later. For the first two terms, the idea is to relate it to the moments of the innovations.
\( \xi / \), using in particular 3.35. We have:

\[
\frac{1}{\gamma_{k+1}} \int \partial_x \varphi(t_{k+1}^N, t_t^N, x, y)(z - x)p_N(t_k^N, t_{k+1}^N, x, z)dz
\]

\[
= \frac{1}{\gamma_{k+1}} \int \partial_x \varphi(t_{k+1}^N, t_t^N, x, y)(z - x) \frac{1}{\sqrt{\gamma_{k+1}}} p_{\epsilon} \left( \frac{z - (x + F_N(t_k^N, x)x\gamma_{k+1})}{\sqrt{\gamma_{k+1}}} \right) dz
\]

\[
= \frac{1}{\gamma_{k+1}} \int \partial_x \varphi(t_{k+1}^N, t_t^N, x, y) \left( \sqrt{\gamma_{k+1}} \tilde{z} + F_N(t_k^N, x)x\gamma_{k+1} \right) p_{\epsilon}(\tilde{z}) d\tilde{z}
\]

\[
= \partial_x \varphi(t_{k+1}^N, t_t^N, x, y) \cdot F_N(t_k^N, x)x,
\]

where we recall that the innovations \( \xi / \) have zero mean. Similarly, for the second term, we get:

\[
\frac{1}{\gamma_{k+1}} \int \frac{1}{2} \partial^2_x \varphi(t_{k+1}^N, t_t^N, x, y)(z - x)^2 p_N(t_k^N, t_{k+1}^N, x, z)dz
\]

\[
= \frac{1}{\gamma_{k+1}} \int \frac{1}{2} \partial^2_x \varphi(t_{k+1}^N, t_t^N, x, y)(z - x)^2 \frac{1}{\sqrt{\gamma_{k+1}}} p_{\epsilon} \left( \frac{z - (x + F_N(t_k^N, x)x\gamma_{k+1})}{\sqrt{\gamma_{k+1}}} \right) dz
\]

\[
= \frac{1}{\gamma_{k+1}} \int \frac{1}{2} \partial^2_x \varphi(t_{k+1}^N, t_t^N, x, y) \left( \sqrt{\gamma_{k+1}} \tilde{z} + F_N(t_k^N, x)x\gamma_{k+1} \right)^2 p_{\epsilon}(\tilde{z}) d\tilde{z}
\]

Expanding the square and using the fact that the innovations have zero mean and variance \( \sigma / \), we get:

\[
\frac{1}{\gamma_{k+1}} \int \frac{1}{2} \partial^2_x \varphi(t_{k+1}^N, t_t^N, x, y)(z - x)^2 p_N(t_k^N, t_{k+1}^N, x, z)dz
\]

\[
= \frac{\sigma^2}{2} \partial^2_x \varphi(t_{k+1}^N, t_t^N, x, y) + \gamma_{k+1} \frac{1}{2} \partial^2_x \varphi(t_{k+1}^N, t_t^N, x, y) \cdot (F_N(t_k^N, x)x)^2.
\]

The important remark here is that we recover the infinitesimal generator \( \mathcal{L} _N \) from the first two terms in the expansion of \( \varphi \). Finally, for the remainder in the expansion of \( \varphi \), we just change variables:

\[
\frac{1}{\gamma_{k+1}} \int \int \partial^2_x \varphi(t_{k+1}^N, t_t^N, x, y)(z - x)^{(1 - \lambda)^2}) d\lambda p_N(t_k^N, t_{k+1}^N, x, z)dz
\]

\[
= \frac{1}{\gamma_{k+1}} \int \int \partial^2_x \varphi(t_{k+1}^N, t_t^N, x, y)(z - x)^{(1 - \lambda)^2}) d\lambda
\]

\[
\times \frac{1}{\sqrt{\gamma_{k+1}}} p_{\epsilon} \left( \frac{z - (x + F_N(t_k^N, x)x\gamma_{k+1})}{\sqrt{\gamma_{k+1}}} \right) dz
\]

\[
= \int \int \partial^2_x \varphi \left( t_{k+1}^N, t_t^N, x + \lambda \left( \sqrt{\gamma_{k+1}} \tilde{z} + F_N(t_k^N, x)x\gamma_{k+1} \right), y \right) \left( \tilde{z} + F_N(t_k^N, x)x\sqrt{\gamma_{k+1}} \right)^3 d\lambda d\tilde{z}
\]

\[
\times (1 - \lambda)^2 d\lambda p_{\epsilon}(\tilde{z}) \sqrt{\gamma_{k+1}} d\tilde{z}
\]

The same computations can be done on \((\mathcal{L} _N - \tilde{\mathcal{L}} _N)_2\), for which we have the frozen density \( \tilde{p}_N / \) instead of \( p_N / \). In this case, we use the fact that

\[
\tilde{p}_N(t_k^N, t_{k+1}^N, x, z)dz = \frac{1}{\gamma_{k+1}} p_{\epsilon} \left( \frac{z - (x + F_N(t_k^N, \tilde{\theta}_{k+1}^N, T(y)) \tilde{\theta}_{k+1}^N, T(y) \gamma_{k+1}^N)}{\sqrt{\gamma_{k+1}}}, \right).
\]

Therefore we get:

\[
(\mathcal{L} _N - \tilde{\mathcal{L}} _N) \varphi(t_k^N, t_t^N, x, y)
\]

\[
= \left( L_N^\lambda(t_k^N, t_{k+1}^N) \varphi(t_k^N, t_t^N, x, y)ight.
\]

\[
+ \gamma_{k+1} \frac{1}{2} \partial^2_x \varphi(t_{k+1}^N, t_t^N, x, y) \cdot \left( (F_N(t_k^N, x)x)^2 - (F_N(t_k^N, \tilde{\theta}_{k+1}^N, T(y)) \tilde{\theta}_{k+1}^N, T(y))^2 \right)
\]

\[
+ R_\varphi(t_k^N, t_t^N, x, y),
\]

\[26\]
where we denote \( R_\varphi(t^N_k, t^N_l, x, y) \) the difference of the remainders. Let us obtain a nicer expression for the remainder. For the sake of clarity, define for a while

\[
\Psi(t, u, z) = \partial^2_x \varphi \left( t^N_k + t^N_l, x + \lambda \left( \sqrt{\gamma^N_{k+1} z + F_N(t^N_k, u) u^{\gamma^N_{k+1}}} \right), y \right) \left( z + F_N(t^N_k, u) u^{\gamma^N_{k+1}} \right)^3.
\]

We also drop the tilda in the integration variable, we have:

\[
R_\varphi(t^N_k, t^N_l, x, y) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_\lambda \Psi \left( t^N_k + t^N_l, x + \lambda \left( \sqrt{\gamma^N_{k+1} z + F_N(t^N_k, u) u^{\gamma^N_{k+1}}} \right), y \right) \left( z + F_N(t^N_k, u) u^{\gamma^N_{k+1}} \right)^3 \frac{(1 - \lambda)^2}{2} p_\lambda(z) \sqrt{\gamma^N_{k+1} z + F_N(t^N_k, u) u^{\gamma^N_{k+1}}} dz d\lambda
\]

Thus, using an additional Taylor formula:

\[
R_\varphi(t^N_k, t^N_l, x, y) = \sqrt{\gamma^N_{k+1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_\lambda \Psi \left( t^N_k + t^N_l, x + \lambda \left( \sqrt{\gamma^N_{k+1} z + F_N(t^N_k, u) u^{\gamma^N_{k+1}}} \right), y \right) \left( z + F_N(t^N_k, u) u^{\gamma^N_{k+1}} \right)^3 \frac{(1 - \lambda)^2}{2} p_\lambda(z) dz d\lambda d\mu
\]

It remains us to compute \( \partial_\mu \Psi \) and \( \partial_\lambda \Psi \):

\[
\partial_\mu \Psi = \partial_\mu \left\{ \partial^3_x \varphi \left( t^N_k + t^N_l, x + \lambda \left( \sqrt{\gamma^N_{k+1} z + F_N(t^N_k, u) u^{\gamma^N_{k+1}}} \right), y \right) \left( z + F_N(t^N_k, u) u^{\gamma^N_{k+1}} \right)^3 \right\} = \partial^3_x \varphi \left( t^N_k + t^N_l, x + \lambda \left( \sqrt{\gamma^N_{k+1} z + F_N(t^N_k, u) u^{\gamma^N_{k+1}}} \right), y \right) \times \lambda \gamma^N_{k+1} \left( \partial_\mu F_N(t^N_k, u) + F_N(t^N_k, u) \right) \left( z + F_N(t^N_k, u) u^{\gamma^N_{k+1}} \right)^3 + \partial^3_x \varphi \left( t^N_k + t^N_l, x + \lambda \left( \sqrt{\gamma^N_{k+1} z + F_N(t^N_k, u) u^{\gamma^N_{k+1}}} \right), y \right) \times 3 \left( z + F_N(t^N_k, u) u^{\gamma^N_{k+1}} \right) \left( \partial_\mu F_N(t^N_k, u) + F_N(t^N_k, u) \right)
\]

We plug this expression into the one for \( R_\varphi \). To lighten the notations, we denote for a while \( \mu x + (1 - \mu) \hat{\theta}^N_{k+1} \tau(y) = \Theta \). We get:

\[
I = \sqrt{\gamma^N_{k+1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \partial^3_x \varphi \left( t^N_k + t^N_l, x + \lambda \left( \sqrt{\gamma^N_{k+1} z + F_N(t^N_k, \Theta) u^{\gamma^N_{k+1}}} \right), y \right) \times \lambda \gamma^N_{k+1} \left( \partial_\mu F_N(t^N_k, \Theta) + F_N(t^N_k, \Theta) \right) \left( z + F_N(t^N_k, \Theta) u^{\gamma^N_{k+1}} \right)^3 + \partial^3_x \varphi \left( t^N_k + t^N_l, x + \lambda \left( \sqrt{\gamma^N_{k+1} z + F_N(t^N_k, \Theta) u^{\gamma^N_{k+1}}} \right), y \right) \times 3 \left( z + F_N(t^N_k, \Theta) u^{\gamma^N_{k+1}} \right) \left( \partial_\mu F_N(t^N_k, \Theta) + F_N(t^N_k, \Theta) \right) \left( z + F_N(t^N_k, \Theta) u^{\gamma^N_{k+1}} \right)^3 \frac{(1 - \lambda)^2}{2} p_\lambda(z) dz d\lambda d\mu.
\]
We now turn to $\mathcal{I}$ above. We have for $\partial_t \Psi$:

$$
\partial_t \Psi = \partial_x^2 \varphi \left( t_{k+1}^N, t_1^N, x + \lambda \left( \sqrt{\gamma_{k+1}^N} z + F_N(t, u) u \gamma_{k+1}^N \right), y \right) \\
\times \partial_t F_N(t, u) u \gamma_{k+1}^N \left( z + F_N(t, u) u \sqrt{\gamma_{k+1}^N} \right)^3 \\
+ 3 \partial_x^2 \varphi \left( t_{k+1}^N, t_1^N, x + \lambda \left( \sqrt{\gamma_{k+1}^N} z + F_N(t, u) u \gamma_{k+1}^N \right), y \right) \\
\times \left( z + F_N(t, u) u \sqrt{\gamma_{k+1}^N} \right)^2 \partial_t F_N(t, u) u \gamma_{k+1}^N.
$$

Again, plugging this value in $\mathcal{I}$ the definition of the remainder above yields:

$$
\mathcal{I} = -(\gamma_{k+1}^N)^{\frac{3}{2}} \int_0^1 \int_0^1 \frac{(1 - \lambda)^2}{2} dz \; d\lambda \; dp(z) \left[ \partial_x^4 \varphi \left( t_{k+1}^N, t_1^N, x + \lambda \left( \sqrt{\gamma_{k+1}^N} z + F_N(t, u) u \gamma_{k+1}^N \right), y \right) \right. \\
\times \partial_t F_N(t, u) u \gamma_{k+1}^N \left( z + F_N(t, u) u \sqrt{\gamma_{k+1}^N} \right)^3 \\
+ 3 \partial_x^2 \varphi \left( t_{k+1}^N, t_1^N, x + \lambda \left( \sqrt{\gamma_{k+1}^N} z + F_N(t, u) u \gamma_{k+1}^N \right), y \right) \\
\times \left( z + F_N(t, u) u \sqrt{\gamma_{k+1}^N} \right)^2 \partial_t F_N(t, u) u \gamma_{k+1}^N,
$$

with $t = \mu t_k^N + (1 - \mu) t_{k+1}^N$ and $u = \theta_{k+1, T}(y)$.

We keep this expression for the remainder. Observe that we use it substituting $\hat{p}_N, \hat{q}_N$ instead of $\varphi$. Obviously, a derivation yields a singularity, and the goal is to compensate the singularity. If we look closely at the expression above, we see that for the first contribution, a derivative of order 4 is taken. This leads to a singularity of order $(1/\gamma_{k+1}^N)^2$. One of those is compensated by the $\gamma_{k+1}^N$ in the expression above, and we can also compensate another $\sqrt{\gamma_{k+1}^N}$ by writing $x - \hat{\theta}_{k+1, T}(y)$ at the appropriate time scale (see the arguments exposed in the proof of Lemma 3.3. Notice that the last compensation is precisely due to our choice to freeze the coefficients along the Euler Scheme. This finally leads to a singularity of order $1/\sqrt{\gamma_{k+1}^N}$, which is summable.

Besides, this expression for the difference of the discrete generators will allow us to give an explicit expression for the remainder $M_N$ first introduced in Step 3 above. First, let us recall that:

$$
\mathcal{K}_N(t_k^N, t_j^N, x, y) = (\mathcal{L}_N - \hat{\mathcal{L}}_N) \hat{p}_N(t_k^N, t_j^N, x, y),
$$

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and:

$$
K_N(t_k^N, t_j^N, x, y) = \mathcal{L}^N_{t_k^N, t_j^N} \hat{p}_N(t_k^N, t_j^N, x, y).
$$

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Moreover, using the expression for the difference of the discrete generators $\mathcal{L}_N - \hat{\mathcal{L}}_N$, we have:

$$
\mathcal{K}_N(t_k^N, t_j^N, x, y) = K_N(t_{k+1}, t_1, x, y) + \frac{1}{2} \left\{ (\mathcal{F}_N(t_k^N, x)x)^2 - (\mathcal{F}_N(t_{k+1}, \hat{\theta}_{k+1, T}(y)) \hat{\theta}_{k+1, T}(y))^2 \right\} \\
+ \mathcal{R}_{\mathcal{F}_N}(t_k^N, t_1^N, x, y).
$$
Therefore, we have the explicit expression for the remainder term introduced in step 3. Namely, we set:

\[
M_N(t_k^N, t_{k+1}^N, x, y) = \gamma_{k+1} \frac{1}{2} \partial_x^2 \tilde{\phi}_N(t_{k+1}^N, t_k^N, x, y) \cdot \left( (F_N(t_k^N, x) x)^2 - (F_N(t_{k+1}^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) \tilde{\phi}_N(t_{k+1}^N, y))^2 \right) + R_{\tilde{\phi}_N}(t_k^N, t_{k+1}^N, x, y).
\] (3.36)

We now give some preliminary estimates on the kernels \(K_N, K_N\) and the remainder \(M_N\). The following estimates holds.

**Lemma 3.8.** For all \(0 \leq t_k^N \leq T\), all \(x, y \in \mathbb{R}\), there exists a constant \(C > 0\) such that:

\[
\left( |K_N| + |K_N| \right)(t_k^N, T, x, y) \leq C \frac{1 + a_N \sqrt{t_k^N}}{\sqrt{1 + \frac{1}{\sqrt{T - t_k^N}}}} \gamma_{k+1} \frac{1}{2} \partial_x^2 \tilde{\phi}_N(t_{k+1}^N, t_k^N, x, y) \cdot \left( (F_N(t_k^N, x) x)^2 - (F_N(t_{k+1}^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) \tilde{\phi}_N(t_{k+1}^N, y))^2 \right) + R_{\tilde{\phi}_N}(t_k^N, t_{k+1}^N, x, y).
\]

**Proof.** The control on \(K_N\) is obtained in a similar fashion as in the previous section, only we replace \(\tilde{\phi}_N\) with \(\hat{\phi}_N\), but the difference here is that the time parameter is not the same. Before diving in the proof of Lemma 3.8, we thus give an additional argument to deal with the different time parameters.

\[
\left( |K_N| + |K_N| \right)(t_k^N, T, x, y) \leq C \frac{1}{\sqrt{T - t_k^N}} \gamma_{k+1} \frac{1}{2} \partial_x^2 \tilde{\phi}_N(t_{k+1}^N, t_k^N, x, y) \cdot \left( (F_N(t_k^N, x) x)^2 - (F_N(t_{k+1}^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) \tilde{\phi}_N(t_{k+1}^N, y))^2 \right) + R_{\tilde{\phi}_N}(t_k^N, t_{k+1}^N, x, y).
\]

Besides, if \(y \in K\) where \(K\) is a compact set, then, there exists \(C\) depending on \(K\) such that:

\[
\left( |K_N| + |K_N| \right)(t_k^N, T, x, y) \leq C \frac{1}{\sqrt{T - t_k^N}} \gamma_{k+1} \frac{1}{2} \partial_x^2 \tilde{\phi}_N(t_{k+1}^N, t_k^N, x, y) \cdot \left( (F_N(t_k^N, x) x)^2 - (F_N(t_{k+1}^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) \tilde{\phi}_N(t_{k+1}^N, y))^2 \right) + R_{\tilde{\phi}_N}(t_k^N, t_{k+1}^N, x, y).
\]

To estimate the pre-factor, we split as follows:

\[
F_N(t_k^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) \tilde{\phi}_N(t_{k+1}^N, T, y) - F_N(t_k^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) \tilde{\phi}_N(t_{k+1}^N, T, y) = F_N(t_k^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) \tilde{\phi}_N(t_{k+1}^N, T, y) - F_N(t_k^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) \tilde{\phi}_N(t_{k+1}^N, T, y) + F_N(t_k^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) \tilde{\phi}_N(t_{k+1}^N, T, y) - F_N(t_k^N, \tilde{\phi}_N(t_{k}^N, T, y)) \tilde{\phi}_N(t_{k}^N, T, y) = I + II.
\]

For \(I\) above, we use the uniform (in time) Lipschitz property of \(x \mapsto F_N(t, x)\) to write:

\[
I \leq C |\tilde{\phi}_N(t_{k+1}^N, T, y) - \tilde{\phi}_N(t_{k}^N, T, y)| \leq C \gamma_{k+1} |y|,
\]

by definition of the Euler scheme. For the second term, we write:

\[
II \leq C \left| F_N(t_k^N, \tilde{\phi}_N(t_{k+1}^N, T, y)) - F_N(t_k^N, \tilde{\phi}_N(t_{k}^N, T, y)) \right| |y| \leq C \gamma_{k+1} |y|.
\]
Consequently, we have:
\[
F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y) - F_N(t_k^N, \hat{\theta}_k^N, T(y))\hat{\theta}_k^N, T(y) \leq \gamma_{k+1} |y|, \tag{3.37}
\]

Thus, the following estimate holds:
\[
(\hat{L}_{k+1}^N - \hat{L}_k^N) \varphi(x) \leq C\gamma_{k+1} |y| \varphi'(x). \tag{3.38}
\]

Now, replacing the test function with the corresponding density, we see that the coefficient \(\gamma_{k+1}^N\) can be used to compensate the singularity induced by the derivation.

The control on \(K_N\) is deduced from the one of \(K_N\) and from a control of the remainder \(M_N\), which is the main difficulty of this lemma. Recall \(M_N\) is made from two terms that we control separately. Set for a while
\[
M_N(t_k^N, T, x, y) = \frac{\gamma_{k+1}^N}{2} \partial^2_N \hat{\varphi}_N(t_{k+1}^N, T, x, y) \cdot \left((F_N(t_k^N, x)x)^2 - (F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y))^2\right),
\]
we claim that the following estimate holds:
\[
M_N(t_k^N, T, x, y) \leq \frac{C\gamma_{k+1}^N}{T - t_k^N} \left(\frac{|x - \theta_{k+1}^N(y)|}{\sqrt{T - t_k^N}}\right)^{s-2}(1 + |y|^2). \tag{3.39}
\]

To get to this control, notice that we have a second derivative on the density. From the control of Battaharya and Rao [2], we can deduce the following estimate:
\[
\left|\partial^2_N \hat{\varphi}_N(t_{k+1}^N, T, x, y)\right| \leq \frac{C}{T - t_k^N} \left(\frac{|x - \theta_{k+1}^N(y)|}{\sqrt{T - t_k^N}}\right)^{s-2}.
\]

See also estimate (3.15) in Konakov Mammen [3] for additional details. We now turn to the control of the multiplier term \((F_N(t_k^N, x)x)^2 - (F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y))^2\). Notice that we can write:
\[
(F_N(t_k^N, x)x)^2 - (F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y))^2
= \left(F_N(t_k^N, x)x + F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y)\right) \times \left(F_N(t_k^N, x) - F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y)\right).
\]

Now, from the controls of the previous section, specifically, the fact that \(x \mapsto F_n(t, x)x\) is Lipschitz, and estimate (3.37) above, we have:
\[
|F_N(t_k^N, x) - F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y)| \leq C\left(|x - \hat{\theta}_{k+1}^N, T(y)| + \gamma_{k+1} |y|\right).
\]

Now, for the second factor, we have:
\[
|F_N(t_k^N, x) + F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y)|
\leq |F_N(t_k^N, x)(x - \hat{\theta}_{k+1}^N, T(y))| + |F_N(t_k^N, x)\hat{\theta}_{k+1}^N, T(y)| + |F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y)|.
\]

Recall that we assumed \(F_N\) to be bounded, and bounding \(\hat{\theta}_{k+1}^N, T(y)\) using Lemma [5, 3] we can write:
\[
|F_N(t_k^N, x) + F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))\hat{\theta}_{k+1}^N, T(y)| \leq C_T\left(|x - \hat{\theta}_{k+1}^N, T(y)| + |y|\right).
\]

Combining the last two estimates, we get:
\[
\left|F_N(t_k^N, x)^2 - F_N(t_{k+1}^N, \hat{\theta}_{k+1}^N, T(y))^2\right| \hat{\theta}_{k+1}^N, T(y)^2 \right| \leq C\left(|x - \hat{\theta}_{k+1}^N, T(y)|^2 + |x - \hat{\theta}_{k+1}^N, T(y)||y| + \gamma_{k+1} |y|^2\right).
\]

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Thus, the estimate for \( M_{\gamma}^I \), the first term in the expression of \( M_N \):

\[
M_{\gamma}^I(t_k^N, T, x, y) \leq \frac{C_{\gamma_0}^N}{T - t_k^N} \left( |x - \hat{\theta}_{t_k^N, T}(y)|^2 + |x - \hat{\theta}_{t_k^N, T}(y)||y| + \gamma_{k+1}^N |y|^2 \right)
\]

\[
\times \left( \frac{1}{T - t_k^N} \right)^{-1/2} \left( \frac{1}{1 + \sqrt{T - t_k^N}} \right)^{S - 2}.
\]

Now, the idea is to pair \((T - t_k^N)^{-1}\) with \(|x - \hat{\theta}_{t_k^N, T}(y)|\) to cancel the singularity by losing a power in the polynomial estimate. Namely, we get:

\[
M_{\gamma}^I(t_k^N, T, x, y) \leq C\gamma_0^N \left( \frac{|x - \hat{\theta}_{t_k^N, T}(y)|^2}{T - t_k^N} + \frac{1}{\sqrt{T - t_k^N}} \frac{|x - \hat{\theta}_{t_k^N, T}(y)||y|}{\sqrt{T - t_k^N}} \right)
\]

Next, we have:

\[
\frac{1}{\sqrt{T - t_k^N}} \frac{|x - \hat{\theta}_{t_k^N, T}(y)||y|}{\sqrt{T - t_k^N}} \leq \frac{1}{\sqrt{T - t_k^N}} \frac{1}{1 + \sqrt{T - t_k^N}} |y|.
\]

Also, we have \( \frac{\gamma_{k+1}^N |y|^2}{T - t_k^N} \leq C|y|^2 \). To get the estimate for \( M_{\gamma}^I \), we bound it by the worst term appearing above, that is, we have:

\[
M_{\gamma}^I(t_k^N, T, x, y) \leq \frac{C_{\gamma_0}^N}{T - t_k^N} \left( \frac{(T - t_k^N)^{-1/2}}{1 + \sqrt{T - t_k^N}} \right) \left( \frac{1}{1 + \sqrt{T - t_k^N}} \right)^{S - 3} |y|.
\]

Observe that by choosing \( y \) in a compact set, we can remove the last multiplier \( 1 + |y|^2 \) up to a modification of the constant \( C \).

We now turn to \( R_{\tilde{\eta}N} \). Recall we denoted above \( \mu x + (1 - \mu)\hat{\theta}_{t_k^N, T}(y) = \Theta \). Set for a while

\[
R_{\tilde{\eta}N}^4(z, \mu) := \partial_2^4 \tilde{\eta}N \left( t_{k+1}^N, T, x + \lambda \left( \sqrt{\gamma_{k+1}^N} z + F_N(t_k^N, \Theta) \Theta_{t_k^N}^N \right), y \right)
\]

\[
\times \lambda \gamma_{k+1}^N \left( \partial u F_N(t_k^N, \Theta) \Theta + F_N(t_k^N, \Theta) \right)
\]

\[
\times \left( z + F_N(t_k^N, \Theta) \right)^3,
\]

and

\[
R_{\tilde{\eta}N}^3(z, \mu) := \partial_2^3 \tilde{\eta}N \left( t_{k+1}^N, T, x + \lambda \left( \sqrt{\gamma_{k+1}^N} z + F_N(t_k^N, \Theta) \Theta_{t_k^N}^N \right), y \right)
\]

\[
\times \left( z + F_N(t_k^N, \Theta) \right)^2
\]

\[
\sqrt{\gamma_{k+1}^N} \times \left( \partial u F_N(t_k^N, \Theta) \Theta + F_N(t_k^N, \Theta) \right),
\]

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so that in the remainder $R_{\tilde{p}_N}$, the term $I$ writes:

$$I = \sqrt{\gamma_{k+1}^N} \int_0^1 \frac{1}{2} \left( R_{\hat{p}_N}^4 + R_{\hat{p}_N}^3 \right) (z, \lambda, \mu) \times \frac{(1 - \lambda)^2}{2} \left( x - \bar{\hat{\theta}}_{k+1, \lambda, T}^N (y) \right) p_\xi (z) dz d\lambda \mu.$$ 

Notice that for clarity purposes, we denoted $R_{\hat{p}_N}^4$, the term corresponding to the fourth derivative, and $R_{\hat{p}_N}^3$, the one for the third derivative. We start with $R_{\hat{p}_N}^3$ which is the most singular term. For simplicity, let us denote for now

\[ u = (\bar{\hat{\theta}}_{k+1, \lambda, T}^N (y) - x) \left[ 1 + \lambda \mu \gamma_{k+1}^N F_N (t_k^N, \mu x + (1 - \mu) \bar{\hat{\theta}}_{k+1, \lambda, T}^N (y)) \right] \]

\[ v = \lambda \sqrt{\gamma_{k+1}^N} \gamma_{k+1}^N \lambda F_N (t_k^N, \mu x + (1 - \mu) \bar{\hat{\theta}}_{k+1, \lambda, T}^N (y)) \bar{\hat{\theta}}_{k+1, \lambda, T}^N (y) \]

We have:

$$\partial_{x}^3 \tilde{p}_N \left( t_k^N, T, x + \lambda \left( \sqrt{\gamma_{k+1}^N} \gamma_{k+1}^N \lambda F_N (t_k^N, \Theta \gamma_{k+1}^N) \right), y \right) \leq \frac{C}{(T - t_k^N)^2} \left( \frac{T - t_k^N}{1 + \sqrt{T - t_k^N}} \right)^{s - 2}.$$ 

Note that since $F_N$ is bounded, $|v| \leq C \left( |z| \sqrt{\gamma_{k+1}^N} \gamma_{k+1}^N |y| \right)$. Besides, for all $u, v \in \mathbb{R}$ with $|v| \leq \varepsilon$, we have:

$$\frac{1}{(1 + |u - v|)^{s - 2}} \leq \frac{C(\varepsilon, S)}{(1 + |u|)^{s - 2}}, \quad C(\varepsilon, S) = 2^S (1 + \varepsilon^S). \quad (3.40)$$

Using that estimate, we get:

$$\partial_{x}^3 \tilde{p}_N \left( t_k^N, T, x + \lambda \left( \sqrt{\gamma_{k+1}^N} \gamma_{k+1}^N \lambda F_N (t_k^N, \Theta \gamma_{k+1}^N) \right), y \right) \leq \frac{C}{(T - t_k^N)^2} \left( \frac{T - t_k^N}{1 + \sqrt{T - t_k^N}} \right)^{s - 2} \left( 1 + (\gamma_{k+1}^N)^{s - 2} \right) \left( |z|^{s - 2} + (\gamma_{k+1}^N)^{s - 2} |y|^{s - 2} \right).$$

Now, observe that $|u| \geq \frac{1}{2} |x - \bar{\hat{\theta}}_{k+1, \lambda, T}^N (y)|$ for $N$ sufficiently large. We get:

$$\partial_{x}^3 \tilde{p}_N \left( t_k^N, T, x + \lambda \left( \sqrt{\gamma_{k+1}^N} \gamma_{k+1}^N \lambda F_N (t_k^N, \Theta \gamma_{k+1}^N) \right), y \right) \leq \frac{C}{(T - t_k^N)^2} \left( \frac{T - t_k^N}{1 + \sqrt{T - t_k^N}} \right)^{s - 2} \left( 1 + (\gamma_{k+1}^N)^{s - 2} \right) \left( |z|^{s - 2} + (\gamma_{k+1}^N)^{s - 2} |y|^{s - 2} \right).$$

Now for the term multiplying the derivative, using estimate (3.24) and recalling $F_N$ is bounded, we have:

$$\left| \lambda \gamma_{k+1}^N \left[ \partial_x F_N (t_k^N, \Theta \gamma_{k+1}^N) + F_N (t_k^N, \Theta \gamma_{k+1}^N) \right] \right| \leq C \gamma_{k+1}^N \left[ a_N \sqrt{\gamma_{0}^N} |\Theta| + 1 \right] \left( |z| + \sqrt{\gamma_{k+1}^N} |\Theta| \right)^3 \leq C \gamma_{k+1}^N \left[ 1 + |z|^3 + a_N \sqrt{\gamma_{0}^N} |\Theta|^3 \right].$$

Note that we bounded $|\Theta| = |\mu x + (1 - \mu) \bar{\hat{\theta}}_{k+1, \lambda, T}^N (y)|$ by $C_T (|y| + |x - \bar{\hat{\theta}}_{k+1, \lambda, T}^N (y)|)$. Finally, this gives us the estimate for the fourth derivative term:

$$\frac{C}{(T - t_k^N)^2} \gamma_{k+1}^N \left( 1 + |z|^3 + a_N \sqrt{\gamma_{0}^N} |y|^3 + a_N \sqrt{\gamma_{0}^N} \sqrt{\gamma_{k+1}^N} |x - \bar{\hat{\theta}}_{k+1, \lambda, T}^N (y)|^3 \right) \left( 1 + \gamma_{k+1}^N \right)^{s - 2} |y|^{s - 2} \left( \frac{T - t_k^N}{1 + \sqrt{T - t_k^N}} \right)^{s - 2}.$$ 

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From the additional $\gamma_{k+1}^N$ in the numerator, we can cancel a singularity. Also, we expand and rearrange the terms to get the estimate:

$$R^4_{\tilde{p}_N}(z, \lambda, \mu) \leq \frac{C}{T - t_k^N} \left( \frac{(T - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|x - \hat{\theta}_{t_k^N,T}^N(y)|}{\sqrt{T - t_k^N}}} \right)^{3-6} \gamma_{k+1}^N \left( 1 + |z|^{S+2} + a_N \sqrt{\gamma_0^N |y|^{S+2}} \right).$$

Similarly, we get for the third derivative term:

$$R^3_{\tilde{p}_N}(z, \lambda, \mu) \leq \frac{C}{T - t_k^N} \left( \frac{(T - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|x - \hat{\theta}_{t_k^N,T}^N(y)|}{\sqrt{T - t_k^N}}} \right)^{3-6} \gamma_{k+1}^N \left( 1 + |z|^{S+2} + a_N \sqrt{\gamma_0^N |y|^{S+2}} \right).$$

To complete the estimation of the term $I$ in the remainder, we need to integrate the last estimate against the density $p_\xi$. Notice that thanks to our upper bound, we get rid of the integrals in $\mu$ and $\lambda$. We have:

$$I \leq \frac{C \sqrt{\gamma_{k+1}^N}}{T - t_k^N} \left( \frac{(T - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|x - \hat{\theta}_{t_k^N,T}^N(y)|}{\sqrt{T - t_k^N}}} \right)^{3-6} \int_R \left( 1 + |z|^{S+2} + a_N \sqrt{\gamma_0^N |y|^{S+2}} \right) \times |\hat{\theta}_{t_k^N,T}^N(y) - x| p_\xi(z) dz.$$

Again, we can cancel a singularity by matching it with $|\hat{\theta}_{t_k^N,T}^N(y) - x|$ to finally get the estimate:

$$I \leq \frac{C \sqrt{\gamma_{k+1}^N}}{T - t_k^N} \left( \frac{(T - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|x - \hat{\theta}_{t_k^N,T}^N(y)|}{\sqrt{T - t_k^N}}} \right)^{3-6} \gamma_{k+1}^N \left( 1 + a_N \sqrt{\gamma_0^N |y|^{S+2}} \right).$$

We can combine that last estimate with (3.39) to get to the estimate for $I$. Notice that since $a_N \to +\infty$, we have $\sqrt{\gamma_{k+1}^N} \leq a_N \sqrt{\gamma_0^N}$, which allows us to put the same estimate for both $I$ and $M_N^1$.

The same upper bound can be obtained quite similarly for $II$. Besides, as we pointed out above, substituting $\tilde{p}_N$ in (3.39), we can compensate the singularity to obtain a similar upper bound. Now, to conclude to the upper bound for $R_{\tilde{p}_N}(t_k^N, T, x, y)$, we recall the identity:

$$R_{\tilde{p}_N}(t_k^N, T, x, y) = I + II + (\tilde{L}_{t_k^N}^N - \tilde{L}_{t_k}^N) \tilde{p}_N(t_k^N, T, x, y).$$

Observe the announced estimate holds for each of the above contribution, and thus for the remainder.

As for $M_N$, the estimate follows from (3.36) and (3.39).

### 3.2.2 Proof of Step 3

This step consists in replacing $H_N$ with its approximation $K_N + M_N$. First, we split the sum:

$$\sum_{r=0}^{N} \tilde{q}_{N \otimes N} H_N^{(r)}(t_k^N, T, x, y) - \sum_{r=0}^{N} \tilde{q}_{N \otimes N} (K_N + M_N)^{(r)}(t_k^N, T, x, y)$$

$$= \sum_{r=0}^{N} \left( \tilde{q}_{N \otimes N} H_N^{(r)} - \tilde{q}_{N \otimes N} (H_N + M_N)^{(r)} \right)(t_k^N, T, x, y)$$

$$+ \sum_{r=0}^{N} \left( \tilde{q}_{N \otimes N} (H_N + M_N)^{(r)} - \tilde{q}_{N \otimes N} (K_N + M_N)^{(r)} \right)(t_k^N, T, x, y) =: I + II,$$

and to control $I$ and $II$, we proceed by induction. We prove the following result first.
Lemma 3.9. For all $0 \leq t_k^N \leq T - \delta$, $\forall \delta > 0$, all $x \in K_x, y \in K_y$, where $K_x, K_y$ are compact sets, for all $r \in \mathbb{N}$, there exists a constant $C > 0$ depending on the compacts $K_x, K_y$, such that:

$$
\left(\tilde{q}_N \otimes_N H_N^{[r]} - \bar{q}_N \otimes_N (H_N + M_N)^{[r]}\right)(t_k^N, T, x, y)
\leq CK^{r}(1 + (\sqrt{\gamma_0^N|x|})^{S-7})(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})\sqrt{\gamma_0^N(T - t_k^N)} + \\
\times \prod_{j=1}^{r+1} B\left(\frac{j}{2}, \frac{1}{2}\right) \left(\frac{(T - t_k^N)^{-\frac{1}{2}}}{\sqrt{\gamma_0^N}}\right)^{S-7},
$$

where

$$
K = \left(1 + \max_{x \in K_x, y \in K_y, t_k^N} \frac{|x - \bar{\theta}_{i,k}^N(y)|}{\sqrt{T - t_k^N}}\right)^{\frac{r+1}{2}} = K(K_x, K_y, T, \delta). \tag{3.42}
$$

Proof. We proceed by induction. First, for $r = 1$, we have:

$$
\tilde{q}_N \otimes_N H_N^{[1]} - \bar{q}_N \otimes_N (H_N + M_N)^{[1]} = \tilde{q}_N \otimes_N M_N.
$$

Thus, using Lemma 3.8, we write:

$$
\left|\tilde{q}_N \otimes_N M_N(t_k^N, T, x, y)\right| \leq \sum_{i=k}^{M(N)} \gamma_i^N \int_R dz g_C(t_i^N - t_k^N, x - \theta_{i,k}^N(z)) \frac{C\sqrt{\gamma_0^N}}{\sqrt{T - t_i^N}} \times \\
\times \left(\frac{(T - t_i^N)^{-\frac{1}{2}}}{\sqrt{T - t_i^N}}\right)^{S-7}(1 + a_N\sqrt{\gamma_0^N}|y|^{S+2})
$$

We have to estimate:

$$
I = \int_R dz \left(\frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\sqrt{\gamma_0^N}}\left(\frac{|x - \theta_{i,k}^N(z)|}{\sqrt{t_i^N - t_k^N}}\right)^{S-7}\left(\frac{|z - \bar{\theta}_{i,k}^N(y)|}{\sqrt{T - t_k^N}}\right)^{S-7}.
$$

We write:

$$
\theta_{i,k}^N(z) - x = \theta_{i,k}^N \left(z - \theta_{i,k}^N(x)\right).
$$

Hence, we have for some constant $C = C_T > 1$:

$$
C^{-1}|z - \theta_{i,k}^N(x)| \leq |\theta_{i,k}^N(z) - x| \leq C|z - \theta_{i,k}^N(x)|.
$$
Thus, we have:

\[
\int \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\sqrt{T - t_k^N}} \left( 1 + \frac{|x - \theta_{t_i^N,t_k^N}^N(x)|}{\sqrt{T - t_k^N}} \right)^{S-7} \leq C \frac{(T - t_k^N)^{-\frac{1}{2}}}{\sqrt{T - t_k^N}} \left( 1 + \frac{|z - \theta_{t_i^N,t_k^N}^N(y)|}{\sqrt{T - t_k^N}} \right)^{S-7} \]

Now, we can rewrite

\[
\frac{|\hat{\theta}_{t_i^N,t_k^N}^N(y) - \theta_{t_i^N,t_k^N}^N(x)|}{\sqrt{T - t_k^N}} = \left| \frac{\hat{\theta}_{t_i^N,t_k^N}^N(y) - \hat{\theta}_{t_i^N,t_k^N}^N(x)}{\sqrt{T - t_k^N}} \right| + \frac{\hat{\theta}_{t_i^N,t_k^N}^N(y) - \hat{\theta}_{t_i^N,t_k^N}^N(x)}{\sqrt{T - t_k^N}} (x - \hat{\theta}_{t_i^N,t_k^N}^N(x))
\]

Observe now that:

\[
\left| \frac{\hat{\theta}_{t_i^N,t_k^N}^N(y) - \hat{\theta}_{t_i^N,t_k^N}^N(x)}{\sqrt{T - t_k^N}} \right| \leq C_T \frac{|\theta_{t_i^N,t_k^N}^N(y) - \theta_{t_i^N,t_k^N}^N(x)|}{\sqrt{T - t_k^N}} \leq C_T \frac{\gamma_0^N}{\sqrt{T - t_k^N}} \leq C_T \sqrt{\gamma_0^N} |x|.
\]

Thus, using equation 33.40, we can write:

\[
\frac{(T - t_k^N)^{-\frac{1}{2}}}{\sqrt{T - t_k^N}} \left( 1 + \frac{|\hat{\theta}_{t_i^N,t_k^N}^N(y) - \theta_{t_i^N,t_k^N}^N(x)|}{\sqrt{T - t_k^N}} \right)^{S-7} \leq \frac{1 + (\sqrt{\gamma_0^N}|x|)^{S-7}(T - t_k^N)^{-\frac{1}{2}}}{\sqrt{T - t_k^N}} \left( 1 + \frac{|\hat{\theta}_{t_i^N,t_k^N}^N(y) - \hat{\theta}_{t_i^N,t_k^N}^N(x)}{\sqrt{T - t_k^N}} |y - x| \right)^{S-7}.
\]

Now, since we have for some \( C > 1 \):

\[
C^{-1} \frac{|\hat{\theta}_{t_i^N,y}^N(y) - x|}{\sqrt{T - t_k^N}} \leq \frac{|\hat{\theta}_{t_i^N,y}^N(y) - x|}{\sqrt{T - t_k^N}} \leq C \frac{|\hat{\theta}_{t_i^N,y}^N(y) - x|}{\sqrt{T - t_k^N}},
\]
we obtain the following estimate:

\[
\int_R dz \left( \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|x - \theta_{i_0}^N, t_i^N(z)|}{\sqrt{t_i^N - t_k^N}}} \right)^{S-\gamma} \left( \frac{(T - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|z - \theta_{i_0}^N, t_i^N, T(y)|}{\sqrt{T - t_k^N}}} \right)^{S-\gamma} \leq C \left( 1 + (\sqrt{\gamma_0 |x|})^{S-\gamma} \right) \left( \frac{(T - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|\theta_{i_0}^N, T(y) - x|}{\sqrt{T - t_k^N}}} \right)^{S-\gamma}.
\]

Observe that:

\[
\sum_{i=k}^{M(N)} \gamma_{i+1} \frac{C \sqrt{\gamma_0}}{\sqrt{T - t_i^N}} \leq C \sqrt{\gamma_0} \int_{h}^{T} \frac{1}{\sqrt{T - u}} du \leq C \sqrt{\gamma_0} \sqrt{T - t_k^N}.
\]

Thus, we have the estimate

\[
|q_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes_N (H_N + MN)^{[r]}| (t_i^N, T, x, y) \leq C \sqrt{\gamma_0} \sqrt{T - t_k^N} \left( \frac{(T - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|x - \theta_{i_0}^N, t_i^N, T(y)|}{\sqrt{T - t_k^N}}} \right)^{S-\gamma} (1 + a_N \sqrt{\gamma_0 |y|^{S+2}}) (1 + \sqrt{\gamma_0 |x|})^{S-\gamma},
\]

which is the required estimate for \( r = 1 \). Assume now that the estimate holds for some \( r \geq 1 \). We write:

\[
\left( q_N \otimes_N H_N^{[r+1]} - \tilde{q}_N \otimes_N (H_N + MN)^{[r+1]} \right) = \left( q_N \otimes_N H_N^{[r]} - \tilde{q}_N \otimes_N (H_N + MN)^{[r]} \right) \otimes_N (H_N + MN) - (q_N \otimes_N H_N^{[r]} \otimes_N M^N).
\]

Now, from the controls of Section 3 (specifically the proof of Lemma 5.5), and Lemma 5.8 we can estimate directly:

\[
\left| \left( q_N \otimes_N H_N^{[r]} \right) \otimes_N M_N(t_k^N, T, x, y) \right| \leq \sum_{i=k}^{M(N)} \gamma_{i+1} \int_R |q_N \otimes_N H_N^{[r]}|(t_i^N, t_k^N, x, z)|M_N|(t_i^N, T, z, y) dz
\]

\[
\leq C^r \sum_{i=k}^{M(N)} \gamma_{i+1} \int_R dz(t_i^N - t_k^N)^{\frac{1}{2}} \prod_{j=1}^{r+1} B \left( \frac{j}{2}, \frac{1}{2} \right) gC(t_i^N - t_k^N, \theta_{i_0}^N, t_i^N(z) - x) \times C \sqrt{\gamma_0} \left( \frac{(T - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|z - \theta_{i_0}^N, t_i^N, T(y)|}{\sqrt{T - t_k^N}}} \right)^{S-\gamma} (1 + a_N \sqrt{\gamma_0 |y|^{S+2}}).
\]

To estimate the last integral, we bound the Gaussian by a polynomial estimate, which admittedly is not the sharpest way to proceed:

\[
\left| \left( q_N \otimes_N H_N^{[r]} \right) \otimes_N M_N(t_k^N, T, x, y) \right| \leq C^r \sum_{i=k}^{M(N)} \gamma_{i+1}(t_i^N - t_k^N)^{\frac{1}{2}} \prod_{j=1}^{r+1} B \left( \frac{j}{2}, \frac{1}{2} \right) \times \int_R dz \left( \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{1 + \frac{|\theta_{i_0}^N, t_i^N, T(y) - x|}{\sqrt{t_i^N - t_k^N}}} \right)^{S-\gamma} \frac{C \sqrt{\gamma_0} (T - t_k^N)^{-\frac{1}{2}} (1 + a_N \sqrt{\gamma_0 |y|^{S+2}})}{(1 + \frac{|z - \theta_{i_0}^N, t_i^N, T(y)|}{\sqrt{T - t_k^N}})}.
\]

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Observe that we can bound the integral over $z$ as the convolution of two heavy tailed estimates. This yields the bound:

$$
\left| (\check{q}_N \otimes N H_N^{[r]}(t_k^N, T, x, y))\right| 
\leq C^r \sum_{i=k}^{M(N)} \gamma_i^N \frac{C \sqrt{\gamma_0}}{T - t_i^N} \sqrt{\gamma_0} (t_i^N - t_k^N)^{\frac{r+1}{2}} \prod_{j=1}^{r+1} B \left( \frac{j}{2}, \frac{1}{2} \right) \left( \frac{T - t_k^N}{{|x - \hat{\theta}_N^{(i)}(y)|}} \right)^{\frac{r+1}{2}} \left( 1 + \frac{|x - \hat{\theta}_N^{(i)}(y)|}{\sqrt{T - t_k^N}} \right)^{-\frac{1}{2}} 
\times (1 + a_N \sqrt{\gamma_0} |y|^{S+2}) (1 + \sqrt{\gamma_0} |x|^{S-7}).
$$

Comparing the sum to an integral yields:

$$
\sum_{i=k}^{M(N)} \gamma_i^N \frac{C \sqrt{\gamma_0}}{T - t_i^N} (t_i^N - t_k^N)^{\frac{r+1}{2}} \prod_{j=1}^{r+1} B \left( \frac{j}{2}, \frac{1}{2} \right) \leq C \sqrt{\gamma_0} (T - t_k^N)^{\frac{r+2}{2}} \prod_{j=1}^{r+2} B \left( \frac{j}{2}, \frac{1}{2} \right).
$$

Thus, we get the following estimate:

$$
\left| (\check{q}_N \otimes N H_N^{[r]}(t_k^N, T, x, y)) \right| 
\leq C^{r+1} \sqrt{\gamma_0} (T - t_k^N)^{\frac{r+2}{2}} \prod_{j=1}^{r+2} B \left( \frac{j}{2}, \frac{1}{2} \right) \left( T - t_k^N \right)^{-\frac{r+2}{2}} (1 + a_N \sqrt{\gamma_0} |y|^{S+2}) (1 + \sqrt{\gamma_0} |x|^{S-7}).
$$

Now, for $\left( \check{q}_N \otimes N H_N^{[r]} - \check{q}_N \otimes N (H_N + M_N)^{[r]} \right) \otimes N (H_N + M_N)$, we use the induction hypothesis and Lemma 3.8.

$$
\left| \left( \check{q}_N \otimes N H_N^{[r]} - \check{q}_N \otimes N (H_N + M_N)^{[r]} \right) \otimes N (H_N + M_N)(t_k^N, T, x, y) \right| 
\leq \sum_{i=k}^{M(N)} \gamma_i^N \int_\mathbb{R} \left| \check{q}_N \otimes N H_N^{[r]} - \check{q}_N \otimes N (H_N + M_N)^{[r]}(t_i^N, t_i^N, x, z) \right| |H_N + M_N(t_i^N, T, z, y)| dz
\leq C^r \left( 1 + \sqrt{\gamma_0} |x|^{S-7} \right) \sum_{i=k}^{M(N)} \gamma_i^N \int_\mathbb{R} \left( 1 + a_N \sqrt{\gamma_0} |y|^{S+2} \right) \sqrt{\gamma_0} (t_i^N - t_k^N)^{\frac{r+1}{2}} \prod_{j=1}^{r+2} B \left( \frac{j}{2}, \frac{1}{2} \right)
\times \left( \frac{|x - \hat{\theta}_N^{(i)}(y)|}{\sqrt{t_i^N - t_k^N}} \right)^{\frac{r+1}{2}} \left( 1 + \frac{|x - \hat{\theta}_N^{(i)}(y)|}{\sqrt{T - t_i^N}} \right)^{-\frac{r+1}{2}} dz.
$$

Notice that the term $1 + a_N \sqrt{\gamma_0} |y|^{S}$, coming from the unbounded feature of the drift, does not depend on $z$, and thus can be put in front of the integral. The corresponding term $1 + a_N \sqrt{\gamma_0} |z|^{S+2}$ is harder to deal with. Observe that we have the following:

$$
1 + a_N \sqrt{\gamma_0} |z|^{S+2} \leq C \max(1, T^{\frac{S+1}{2}}) \left( \frac{|x - \hat{\theta}_N^{(i)}(y)|}{\sqrt{t_i^N - t_k^N}} + 1 \right)^{\frac{S+1}{2}} \left( 1 + \frac{|z - \hat{\theta}_N^{(i)}(y)|}{\sqrt{T - t_i^N}} \right)^{\frac{S+1}{2}} + 1.
$$
where the constant $C > 0$ depends on the compacts $K_x, K_y$. Thus, plugging this estimate we get:

$$\left| \left( \tilde{q}_N \otimes_N H_N^{|r|} - \tilde{q}_N \otimes_N (H_N + M_N)^{|r|} \right) \otimes_N (H_N + M_N)(t_N^r, T, x, y) \right|$$

$$\leq C(1 + \sqrt{\frac{N}{70}} |x|^{S-7}) \sqrt{\frac{N}{70}} (1 + a_N \sqrt{\frac{N}{70}} |y|^{S+2}) \sum_{i=0}^{M(N)} \gamma_{i+1} (t_i^N - t_k^N)^{\frac{1}{2}}$$

$$\times \frac{1}{\sqrt{T - t_i^N}} \prod_{j=1}^{r+2} B \left( \frac{j}{2}, \frac{1}{2} \right) \int_R \left( \frac{|x - \tilde{\theta}_{i+1}^N(z)|}{t_i^N - t_k^N} + 1 \right)^{\frac{1}{2} + 1} \left( \frac{|z - \tilde{\theta}_{i+1}^N(y)|}{T - t_i^N} + 1 \right)^{\frac{1}{2} + 1}$$

$$\times \left( 1 + \frac{|x - \tilde{\theta}_{i+1}^N(z)|}{\sqrt{t_i^N - t_k^N}} \right)^{\frac{S-7}{2}} \left( 1 + \frac{|z - \tilde{\theta}_{i+1}^N(y)|}{\sqrt{T - t_i^N}} \right)^{\frac{S-7}{2}}$$

$$\leq C(1 + \sqrt{\frac{N}{70}} |x|^{S-7}) \sqrt{\frac{N}{70}} (1 + a_N \sqrt{\frac{N}{70}} |y|^{S+2})(T - t_i^N)^{\frac{1}{2} + 1} \prod_{j=1}^{r+3} B \left( \frac{j}{2}, \frac{1}{2} \right)$$

$$\times \int_R \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left( 1 + \frac{|x - \tilde{\theta}_{i+1}^N(z)|}{\sqrt{t_i^N - t_k^N}} \right)^{\frac{S-7}{2}} \left( 1 + \frac{|z - \tilde{\theta}_{i+1}^N(y)|}{\sqrt{T - t_i^N}} \right)^{\frac{S-7}{2}}} (T - t_i^N)^{-1/2} dz$$

By the convolution property of the polynomial densities, we have:

$$\int_R \frac{(t_i^N - t_k^N)^{-\frac{1}{2}}}{\left( 1 + \frac{|x - \tilde{\theta}_{i+1}^N(z)|}{\sqrt{t_i^N - t_k^N}} \right)^{\frac{S-7}{2}} \left( 1 + \frac{|z - \tilde{\theta}_{i+1}^N(y)|}{\sqrt{T - t_i^N}} \right)^{\frac{S-7}{2}}} (T - t_i^N)^{-1/2} dz$$

$$\leq C \left( \frac{T - t_i^N)^{-\frac{1}{2}}}{\left( 1 + \frac{|x - \tilde{\theta}_{i+1}^N(z)|}{\sqrt{t_i^N - t_k^N}} \right)^{\frac{S-7}{2}} \left( 1 + \frac{|z - \tilde{\theta}_{i+1}^N(y)|}{\sqrt{T - t_i^N}} \right)^{\frac{S-7}{2}}} \right)^{\frac{1}{2}} \times \left( \frac{|x - \tilde{\theta}_{i+1}^N(z)|}{\sqrt{T - t_i^N}} \right)^{\frac{1}{2} + 1}$$

$$\leq CK \left( \frac{|x - \tilde{\theta}_{i+1}^N(y)|}{\sqrt{T - t_i^N}} \right)^{-\frac{1}{2}}$$

Now, estimating the sum over $i$ by an integral yields the announced estimate. \qed

Now, to estimate $I$ in (5.31), we proceed again by induction. We prove the following Lemma:

**Lemma 3.10.** For all $0 \leq t_k^N \leq T - \delta$, $\delta > 0$, all $x \in K_x, y \in K_y$, for all $r \in \mathbb{N}$, there exists a constant $C > 0$ such that:

$$\left( \tilde{q}_N \otimes_N (H_N + M_N)^{|r|} - \tilde{q}_N \otimes_N (K_N + M_N)^{|r|} \right) (t_N^r, T, x, y)$$

$$\leq C' K^r (1 + a_N \sqrt{\frac{N}{70}} |y|^{S}) \sqrt{\frac{N}{70}} (T - t_i^N)^{\frac{1}{2}} \prod_{j=1}^{r} B \left( \frac{j}{2}, \frac{1}{2} \right) \frac{(T - t_i^N)^{-\frac{1}{2}}}{\left( \frac{|x - \tilde{\theta}_{i+1}^N(z)|}{\sqrt{T - t_i^N}} \right)^{\frac{S-7}{2}}}$$

where $K$ is defined in (3.42).
Proof. The proof uses similar techniques. For \( r = 1 \), notice that we have:

\[
\tilde{q}_N \otimes_N (H_N + M_N)^{[1]} - \tilde{q}_N \otimes_N (K_N + M_N)^{[1]} = \tilde{q}_N \otimes_N (H_N - K_N).
\]

We thus need a control of \( H_N - K_N \), which follows (by definition of \( H_N \) and \( K_N \)) from a control of \( \tilde{q}_N - \tilde{p}_N \). Specifically, in Step 4, we obtain an estimate on \( \tilde{q}_N - \tilde{p}_N \), based on Edgeworth expansions. We then need to "take the derivative of the expansion". Since is done with the same arguments than those exposed in [3], we only give the estimate here. We have:

\[
|H_N - K_N|(t^N_k, T, x, y) \leq C \sqrt{\gamma_0^N} \frac{(T - t^N_k)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \tilde{\theta}^N_{t^N_k T}(y)|}{\sqrt{T - t^N_k}}\right)^{s-3}}. 
\tag{3.43}
\]

Using this estimate, we can control the case \( r = 1 \) with the same arguments as those used above. Next, for \( r \geq 1 \), we write:

\[
\tilde{q}_N \otimes N (H_N + M_N)^{[r+1]} - \tilde{q}_N \otimes N (K_N + M_N)^{[r+1]} \\
= \left( \tilde{q}_N \otimes N (H_N + M_N)^{[r]} - \tilde{q}_N \otimes N (K_N + M_N)^{[r]} \right) \otimes N (K_N + M_N) \\
+ \tilde{q}_N \otimes N (H_N + M_N)^{[r]} \otimes N (H_N - K_N).
\]

For the first contribution, we can use Lemma 5.8 to estimate \( K_N + M_N \) and the induction hypothesis for \( \tilde{q}_N \otimes N (H_N + M_N)^{[r]} - \tilde{q}_N \otimes N (K_N + M_N)^{[r]} \). For the second contribution, we can obtain similarly to the control of \( \tilde{q}_N \otimes N H_N \) a control of \( \tilde{q}_N \otimes N (H_N + M_N) \), using Lemma 5.8. We then take the convolution of that estimate with the estimate 3.43 to get the announced estimate.

We can now conclude as for Step 3. Piecing together the estimates form Lemmas 3.9 and 3.10 we get the estimate:

**Proposition 3.11.** Let \( k \geq 1 \). For all \( 0 \leq t^N_k \leq T - \delta \), \( \delta > 0 \), all \( x \in K_x, y \in K_y \) compact sets, there exists a constant \( C > 0 \), such that:

\[
\left| \sum_{r=0}^{N} \tilde{q}_N \otimes N H^r_N \otimes N (K_N + M_N)^{[r]} \right| (t^N_k, T, x, y) \leq C_T (1 + \sqrt{\gamma_0^N |x|^{s-1}}) (1 + a_N \sqrt{\gamma_0^N |y|^{s}}) \sqrt{\gamma_0^N} \frac{(T - t^N_k)^{-\frac{1}{2}}}{\left(1 + \frac{|x - \tilde{\theta}^N_{t^N_k T}(y)|}{\sqrt{T - t^N_k}}\right)^{s-3}}.
\]

### 3.2.3 Proof of Step 4

For this step, we need to replace \( \tilde{q}_N \) with \( \tilde{p}_N \). Observe carefully now that for the frozen densities, we have:

\[
(\tilde{q}_N - \tilde{p}_N)(t, T, x, y) = g\sigma \left( T - t, \theta^N_{t T}(y) - x \right) - g\sigma \left( T - t, \tilde{\theta}^N_{t T}(y) - x \right) + g\sigma \left( T - t, \tilde{\theta}^N_{t T}(y) - x \right) - p_{SN} \left( \tilde{\theta}^N_{t T}(y) - x \right).
\tag{3.44}
\]

The first contribution can be handled with similar arguments than those developed in Section 3.1 (see Lemma 3.3 above). We have the following estimate:

**Lemma 3.12.** There exists a constant \( C, C' > 0 \), for all \( t \leq T \), for all \( (x, y) \in \mathbb{R} \times \mathbb{R} \),

\[
\left| g\sigma \left( T - t, \theta^N_{t T}(y) - x \right) - g\sigma \left( T - t, \theta^N_{t T}(y) - x \right) \right| \leq C \sqrt{(T - t)} a_N \sqrt{\gamma_0^N |y|^{2} |\sigma_C^{(T-t)}(\theta^N_{t T}(y) - x)|^2 |g_C(T - t, \theta^N_{t T}(y) - x)|}.
\]
Again, observe that we chose to put the transport of the differential equation \( \dot{x}_t = (-am'(\theta_t) + \frac{1}{2})x_t \) at the price of the additional exponential factor that goes to 1 when \( N \) goes to infinity. This is in order to get a homogeneous upper bound for the tails.

The second contribution in (3.44) can be controlled by Bhattacharya and Rao [2], and consists in controlling the convergence of the sum i.i.d. variables \( \sum_{i=k}^{M(N)-1} \xi_{i+1} \) to its Gaussian limit \( \sigma W_{T-t} \). The only difference is that the argument of the density is \( \dot{\theta}_{i,T}^N(y) - x \) instead of the usual \( y - x \). Up to minor modification in [3], we get the following Lemma:

**Lemma 3.13.** There exists \( S > 4 \), there exists a constant \( C > 0 \), for all \( t \leq T \), for all \( (x, y) \in \mathbb{R} \times \mathbb{R} \),

\[
|g_\sigma \left( T - t, \dot{\theta}_{i,T}^N(y) - x \right) - p_{SN} \left( \dot{\theta}_{i,T}^N(y) - x \right)| \leq C \sqrt{\frac{\gamma_0^N}{T-t}} \frac{1}{\left( 1 + \frac{|\dot{\theta}_{i,T}^N(y) - x|}{\sqrt{T-t}} \right)^{S-2}}.
\]

Combining Lemmas 3.12 and 3.13 we get the estimate:

\[
(q_N - \tilde{p}_N)(t, T, x, y) \leq C \sqrt{\frac{\gamma_0^N}{T-t}} |y| e^{C(T \gamma_0^N |y|^2) g_C(T-t, \dot{\theta}_{i,T}^N(y) - x)}
\]

\[
+ \sqrt{\frac{\gamma_0^N}{T-t}} \frac{1}{\left( 1 + \frac{|\dot{\theta}_{i,T}^N(y) - x|}{\sqrt{T-t}} \right)^{S-2}}.
\]

Besides, we can notice that:

\[
\sqrt{(T-t) a_N \sqrt{\gamma_0^N} |y| e^{C(T \gamma_0^N |y|^2)} \leq \sqrt{(T-t) a_N \sqrt{\gamma_0^N} |y| e^{C(T \gamma_0^N |y|^2)}} \leq a_N \sqrt{\gamma_0^N} e^{C(T \gamma_0^N |y|^2)}.
\]

To simplify the convolutions, notice also that we can use Lemma 3.14 to put the same argument in the Gaussian and in the polynomial estimate. Thus, we get the following estimate for the difference of the frozen densities:

\[
(q_N - \tilde{p}_N)(t_k, T, x, y) \leq C \left( a_N \sqrt{\gamma_0^N} e^{C(T \gamma_0^N |y|^2)} g_C(T-t_k, \dot{\theta}_{i,T}^N(y) - x) \right)
\]

\[
+ \sqrt{\frac{\gamma_0^N}{T-t_k}} \frac{1}{\left( 1 + \frac{|\dot{\theta}_{i,T}^N(y) - x|}{\sqrt{T-t_k}} \right)^{S-2}}. \tag{3.45}
\]

This takes care of replacing \( q_N \) with \( \tilde{p}_N \). To complete step 4, we need to point out that the estimate for \( q_N - \tilde{p}_N \) can be convoluted with \( (K_N + M_N)^{[r]} \). Specifically, we have the following lemma:

**Lemma 3.14.** For all \( 0 \leq t_k^N \leq T - \delta \), \( \delta > 0 \), for all \( (x, y) \in K_x \times K_y \), there exists a constant \( C > 0 \) depending on \( K_x, K_y, T \) and \( \delta \) such that for all \( N \):

\[
\left| \sum_{i=0}^{N} (q_N - \tilde{p}_N) \otimes_N (K_N + M_N)^{[r]} \right| \leq C \sqrt{\gamma_0^N} \frac{(T-t_k^N)^{\frac{r}{2}}}{\left( 1 + \frac{|x - \dot{\theta}_{i,T}^N(y)|}{\sqrt{T-t_k^N}} \right)^{S-\frac{r}{2}}}.
\]

**Proof.** We prove this estimate by induction. For \( k = 1 \), we have, define:

\[
I := C \gamma_0^N \sum_{i=k}^{M(N)-1} \frac{\gamma_{i+1}^N}{\sqrt{t_{i+1}^N - t_k^N}} \int_{\mathbb{R}} dz |z| e^{C(T \gamma_0^N |z|^2)} g_C(t_k^N - t_i^N, \dot{\theta}_{i,T}^N, t_i^N, t_k^N, t_i^N(z) - x) \times \frac{1 + a_N \sqrt{\gamma_0^N} |y|^{S+2}}{\sqrt{T-t_k^N}} \frac{1 + \frac{|z - \dot{\theta}_{i,T}^N(y)|}{\sqrt{T-t_k^N}}}{\left( 1 + \frac{z - \dot{\theta}_{i,T}^N(y)}{\sqrt{T-t_k^N}} \right)^{S-2}}.
\]

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We use the inequalities:

\[(\gamma_0^N | z|^2 \leq C \left[ (\gamma_0^N)^2 |x - 0\right] t_k, t_i'(N) (z)^2 + (\gamma_0^N | x|)^2 \right].

Now, for \(x, y\) such that \(\max(|x|, |y|) \leq \frac{1}{(a_N \sqrt{\gamma_0^N})^{\frac{1}{2}}}, we obtain:

\[e^{C^2(\gamma_0^N | z|^2)} \leq e^{(\gamma_0^N)^2 |x - 0\right] t_k, t_i'(N) (z)^2} \times e^{(\gamma_0^N | x|)^2} \leq C e^{(\gamma_0^N)^2 |x - 0\right] t_k, t_i'(N) (z)^2},

and

\[\sqrt{\gamma_0^N | z| \leq C(\sqrt{\gamma_0^N | x - 0\right] t_k, t_i'(N) (z)| + \sqrt{\gamma_0^N | x|) \leq C(1 + |x - 0\right] t_k, t_i'(N) (z))}\]

\[\leq C \max(1, \sqrt{T}) \left( 1 + \frac{|x - 0\right] t_k, t_i'(N) (z)|}{t_k - t_i^N} \right).

It follows from the above inequalities that

\[\sqrt{\gamma_0^N | z| e^{C(\gamma_0^N | z|^2) g_c(t_i^N - t_k, \hat{\theta}_k^N, \hat{t}_i^N (z) - x)} \leq C e^{C(\gamma_0^N)^2 |x - 0\right] t_k, t_i'(N) (z)|^2} g_c(t_i^N - t_k, \hat{\theta}_k^N, \hat{t}_i^N (z) - x) \leq C g_c(t_i^N - t_k, \hat{\theta}_k^N, \hat{t}_i^N (z) - x).

Now, we have:

\[I \leq C \sqrt{\gamma_0^N} \sum_{i=k}^{M(N)} \frac{\gamma_i^N}{\gamma_{i+1}^N - t_k} \int_R \frac{dz}{t_i - t_k - T} \frac{1}{T - t_i^{N-1}} \frac{\sqrt{T - t_i^{N-1}}}{\sqrt{1 + \frac{\frac{|z - 0\right] t_k, t_i'(N) (z)|}{\sqrt{T - t_i'}}}}^{S-\frac{3}{2}} \leq C \sqrt{\gamma_0^N} \left( 1 + a_N \sqrt{\gamma_0^N} |y|^{S+2} \right) \sum_{i=k}^{M(N)} \frac{\gamma_i^N}{t_i^{N-1} - t_k^{N-1}} \frac{1}{T - t_i^{N-1}} \frac{\sqrt{T - t_i^{N-1}}}{\sqrt{1 + \frac{\frac{|z - 0\right] t_k, t_i'(N) (z)|}{\sqrt{T - t_i'}}}}^{S-\frac{3}{2}} \leq C \sqrt{\gamma_0^N} \left( 1 + a_N \sqrt{\gamma_0^N} |y|^{S+2} \right) \left( 1 + \frac{\sqrt{|z - 0\right] t_k, t_i'(N) (z)|}}{\sqrt{T - t_i'}} \right)^{S-\frac{3}{2}} \frac{\sqrt{T - t_k^{N-1}}}{\sqrt{1 + \frac{\frac{|z - 0\right] t_k, t_i'(N) (z)|}{\sqrt{T - t_i'}}}}^{S-\frac{3}{2}}.

We obtain:

\[| (\hat{q}_N - \hat{p}_N) \otimes_N (K_N + M_N)(t_k, T, x, y) | \leq C \sqrt{\gamma_0^N} \left( 1 + a_N \sqrt{\gamma_0^N} |y|^{S+2} \right) \left( 1 + \frac{\sqrt{|z - 0\right] t_k, t_i'(N) (z)|}}{\sqrt{T - t_i'}} \right)^{S-\frac{3}{2}} \frac{\sqrt{T - t_k^{N-1}}}{\sqrt{1 + \frac{\frac{|z - 0\right] t_k, t_i'(N) (z)|}{\sqrt{T - t_i'}}}}^{S-\frac{3}{2}} \times \]
Suppose now that we have:

\[
\left| \left( \tilde{q}_N - \tilde{p}_N \right) \otimes_N \left( K_N + M_N \right) \right| (t_k, T, x, y) \leq C^r K^r \sqrt{\gamma_0^N} \left( 1 + a_N \sqrt{\gamma_0^N} \right)^{S+2} \left( 1 + \left( \sqrt{\gamma_0^N} |x| \right)^{S-\gamma} \right)
\]
\[
\times \left( T - t_k^N \right)^{\frac{\gamma - 1}{2}} \prod_{j=1}^{r} B \left( \frac{j}{2}, \frac{1}{2} \right) \frac{\sqrt{T - t_k^N}}{1 + \frac{|x - \tilde{p}_N^{N, \epsilon, i}(x)|}{\sqrt{t_i^N - t_k^N}}} \left( T - t_i^N \right)^{S-\gamma}.
\]

Then, we obtain

\[
\left| \left( \tilde{q}_N - \tilde{p}_N \right) \otimes_N \left( K_N + M_N \right) \right|^{(r+1)} (t_k, T, x, y) \leq C^r K^r \sqrt{\gamma_0^N} \left( 1 + a_N \sqrt{\gamma_0^N} \right)^{S+2} \left( 1 + \left( \sqrt{\gamma_0^N} |x| \right)^{S-\gamma} \right)
\]
\[
\times \sum_{i=k}^{M(N)} \gamma_{k+1}^N \int_R dz (1 + a_N |z|)^{S+2} \frac{\left( t_i^N - t_k^N \right)^{\frac{\gamma - 1}{2}}}{\left( 1 + \frac{|x - \tilde{p}_N^{N, \epsilon, i}(x)|}{\sqrt{t_i^N - t_k^N}} \right)} \frac{\sqrt{T - t_k^N}}{\left( 1 + \frac{|z - \tilde{p}_N^{N, \epsilon, i}(y)|}{\sqrt{T - t_i^N}} \right)} \left( T - t_i^N \right)^{S-\gamma}.
\]

Now, we use the inequality

\[
1 + a_N \sqrt{\gamma_0^N} |z|^{S+2} \leq C \max(1, T^{\frac{\gamma - 1}{2} + 1}) \left( 1 + \frac{|x - \tilde{p}_N^{N, \epsilon, i}(x)|}{\sqrt{t_i^N - t_k^N}} \right) \left( 1 + \frac{|z - \tilde{p}_N^{N, \epsilon, i}(y)|}{\sqrt{T - t_i^N}} \right).
\]

Substituting this estimate and convoluting the polynomial functions, we have:

\[
\int_R \frac{\sqrt{t_i^N - t_k^N}}{\left( 1 + \frac{|\tilde{p}_N^{N, \epsilon, i}(x) - z|}{\sqrt{t_i^N - t_k^N}} \right)^{\frac{\gamma - 1}{2}}} \frac{\sqrt{T - t_k^N}}{\left( 1 + \frac{|\tilde{p}_N^{N, \epsilon, i}(y) - z|}{\sqrt{T - t_k^N}} \right)^{\frac{\gamma - 1}{2}}} dz \leq \frac{\sqrt{T - t_k^N}}{\left( 1 + \frac{|\tilde{p}_N^{N, \epsilon, i}(y) - z|}{\sqrt{T - t_k^N}} \right)^{\frac{\gamma - 1}{2} - \gamma}},
\]

We can now recover the right power, writing:

\[
\frac{\sqrt{T - t_k^N}}{\left( 1 + \frac{|\tilde{p}_N^{N, \epsilon, i}(y) - z|}{\sqrt{T - t_k^N}} \right)^{\frac{\gamma - 1}{2}}} \cdot \frac{\sqrt{T - t_k^N}}{\left( 1 + \frac{|\tilde{p}_N^{N, \epsilon, i}(y) - z|}{\sqrt{T - t_k^N}} \right)^{\frac{\gamma - 1}{2}}} \leq \frac{\sqrt{T - t_k^N}}{\left( 1 + \frac{|\tilde{p}_N^{N, \epsilon, i}(y) - z|}{\sqrt{T - t_k^N}} \right)^{\frac{\gamma - 1}{2} - \gamma}} \leq CK \frac{\sqrt{T - t_k^N}}{\left( 1 + \frac{|\tilde{p}_N^{N, \epsilon, i}(y) - z|}{\sqrt{T - t_k^N}} \right)^{\frac{\gamma - 1}{2} - \gamma}},
\]

recalling that

\[
K = \left( 1 + \max_{x \in K_x, y \in K_y} \frac{|\tilde{p}_N^{N, \epsilon, i}(y) - x|}{\sqrt{T - t_k^N}} \right)^{\frac{\gamma - 1}{2} + 1}.
\]

Besides, comparing the sum to an integral, we get:

\[
\sum_{i=k}^{M(N)} \gamma_{k+1}^N (t_i^N - t_k^N) \frac{\sqrt{T - t_k^N}}{\left( T - t_i^N \right)^{\frac{\gamma - 1}{2}}} \leq \frac{\sqrt{T - t_k^N}}{\left( T - t_i^N \right)^{\frac{\gamma - 1}{2} - \gamma}} \leq \frac{B \left( T + \frac{1}{2}, \frac{1}{2} \right)}{2}.
\]
Finally, we obtain the desired expression:

\[
\begin{align*}
&\left| \tilde{q}_N - \tilde{p}_N \right| \circ N \left( K_N + M_N \right)^{r+1} (t_k, T, x, y) \leq C^{r+1} K^{r+1} \sqrt{\gamma^N_0} \left( 1 + a_N \sqrt{\gamma^N_0} \right) \left( 1 + (\sqrt{\gamma^N_0} |x|)^{S-7} \right) \\
&\times (T - t_k)^{r+1} \prod_{j=1}^{r+1} B \left( \frac{j}{2}, \frac{1}{2} \right) \sqrt{\frac{T - t_k}{T - t_N}} \left( 1 + \frac{|x - \hat{\theta}^N_{t_k,N}(y)|}{\sqrt{T - t_N}} \right)^{S-7}.
\end{align*}
\]

Summing up we obtain

\[
\sum_{r=0}^{N} \left| \tilde{q}_N - \tilde{p}_N \right| \circ N \left( K_N + M_N \right)^{|r|} (t_k, T, x, y) \leq C \gamma^N_0 \sqrt{\frac{T - t_k}{T - t_N}} \left( 1 + \frac{|x - \hat{\theta}^N_{t_k,N}(y)|}{\sqrt{T - t_N}} \right)^{S-7}.
\]

where the constant \( C \) depends on the compacts sets \( K_x, K_y \), and \( T \) but does not depends on \( N \).

\( \square \)

**Acknowledgments**

For the first author the study has been funded by the Russian Academic Excellence Project "5-100" and for the second author the study has been funded by the Russian Science Foundation (project n\(^{o}\)17-11-01098).

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