Tensor Analysis with

*n*-Mode Generalized Difference Subspace

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Abstract

The increasing use of multiple sensors requires more efficient methods to represent and classify multi-dimensional data, since these applications produce a large amount of data, demanding modern techniques for data processing. Considering these observations, we present in this paper a new method for multi-dimensional data classification which relies on two premises: 1) multi-dimensional data are usually represented by tensors, due to benefits from multilinear algebra and the established tensor factorization methods; and 2) this kind of data can be described by a subspace lying within a vector space. Subspace representation has been consistently employed for pattern-set recognition, and its tensor representation counterpart is also available in the literature. However, traditional methods do not employ discriminative information of the tensors, which degrades the classification accuracy. In this scenario, generalized difference subspace (GDS) may provide an enhanced subspace representation by reducing data redundancy and revealing discriminative structures. Since GDS is not able to directly handle tensor data, we propose a new projection called \emph{n}-mode GDS, which efficiently handles tensor data. In addition, \emph{n}-mode Fisher score is introduced as a class separability index and an improved metric based on the geodesic distance is provided to measure the similarity between tensor data. To confirm the advantages of the proposed method, we address the problem of representing and classifying tensor data for gesture and action recognition. The experimental results have shown that the proposed approach outperforms methods commonly used in the literature without adopting pre-trained models or transfer learning.

\textbf{Keywords:} Action recognition; Tensor data classification; Generalized difference subspace; \emph{n}-mode singular value decomposition.

Abbreviations: DCC: Discriminative Canonical Correlation; GDS: Generalized Difference Subspace; TCCA: Tensor Canonical Correlation Analysis; MPCA: Multilinear Principal Component Analysis; MLDA: Multilinear Linear Discriminant Analysis; PGM: Product Grassmann Manifold; TB: Tangent Bundle; DNN: Deep Neural Networks; MSM: Mutual Subspace Method; CNN: Convolutional Neural Network; SVD: Single Value Decomposition; DS: Difference Subspace; MDS: Multi Dimensional Scaling.
1. Introduction

Many applications make use of multi-dimensional data, such as multiple-view image recognition and video analysis. Due to the increasing data density produced by sensors, improved techniques are required to process this kind of data. In this scenario, tensors, which can be defined as a generalization of matrices, present a suitable model for such data representation, since tensors allow a natural representation of multi-dimensional data. For instance, video data is intuitively described by its correlated images over the time axis. Through vectorization and concatenation of the video data pixels, it is possible to produce a representation that describes the data as a matrix or a vector. Then, this vectorized representation can be exploited to train a classification model to build a machine learning model. However, this representation does not provide an intuitive or a natural pattern representation. Besides, the vectorization procedure may degrade the spatio-temporal relationship between pixels of a video tensor data, causing information loss (Zhang et al., 2015a; Khokher et al., 2018).

Applications that benefit from tensorial representation include high-resolution video analysis, hyperspectral image classification, medical image analysis, gene expression representation and recommendation systems (Hore et al., 2016; Sun et al., 2018; Jiang et al., 2018; Torkamani-Azar et al., 2015; Li et al., 2015; Fan et al., 2018; Zhang et al., 2017b; Yin et al., 2017; Taneja & Arora, 2018; Morise et al., 2019). For example, bioelectrical time signals (Motrenko & Strijov, 2018; Zhang et al., 2017a) are usually obtained by sensors based on differential amplifiers, which record the signal difference between two electrodes connected to the skin, where the signal difference changes over time. Many sensors can be used to cover a wider area. Thus, this data acquisition produces a massive number of time-varying signals, where not only the temporal correlation but also the spatial structure between the collected signals should be exploited. In addition, recently, self-driving cars equipped with multiple sensors have been producing a large amount of data, which requires efficient representation (Daniel et al., 2017; Luckow et al., 2015). Finally, the use of non-efficient methods to handle high-dimensional data can compromise the associated hardware cost.

Currently, training deep neural network (DNN) architectures from scratch is not feasible to handle 3-mode or higher mode tensors when datasets present a small number of samples. More precisely, training a deep network requires large datasets due to its large number of trainable parameters. Also, the computational complexity of training a DNN architecture may increase exponentially according to the number of modes, which requires efficient representation (Daniel et al., 2017; Luckow et al., 2015). Finally, the use of non-efficient methods to handle high-dimensional data can compromise the associated hardware cost.
Figure 1: Illustration of the unfolding procedure of a 3-mode tensor. The unfolding of the 3-mode tensor $A$ produces 3 sets of matrices $X_{(1)}$, $X_{(2)}$ and $X_{(3)}$.

requiring more data and computational resources, restricting the range of applications. To overcome this problem, we propose in this paper a method whose complexity grows linearly according to the number of tensor modes, making the proposed method an alternative for tensor data classification.

The order of a tensor is related to the number of its dimensions, also known as ways or modes (Kolda & Bader, 2009). Tensor unfolding is a procedure that reorganizes the tensor data in such a way that permits the analysis of each mode separately, possibly revealing correlations which were not immediately observed. For example, a video data may provide a 3-mode tensor, consisting of 2 spatial modes and a temporal one. In this context, in terms of tensor unfolding, the 3-mode tensor can be represented by 3 subspaces, where each subspace is computed from one of the tensor unfolded modes. This tensor unfolding procedure is shown in Figure 1. The tensor unfolding maneuver is relevant for the interpretability of the modes, as it is in the case of medical image analysis (Johansen-Berg & Behrens, 2013; Madabhushi & Lee, 2016).

In computer vision, subspaces are systematically employed to express pattern-set data. A pattern-set can be defined as a collection of observations that share a particular label. To solve pattern-set recognition problems, the Mutual Subspace Method (MSM) was introduced (Ikeuchi, 2014; Maeda, 2010). The central hypothesis of MSM is that a pattern-set produces a compact cluster in the high-dimensional vector space, which can be efficiently represented by a subspace computed by eigen-decomposition, for instance. In MSM, the similarity between the subspaces is computed by the canonical angles (Björck & Golub, 1973; Hotelling, 1992). Advantages of using MSM include its extremely compact representation and its robustness to noise. For example, it is reasonable that 20% of the basis vectors generated by eigen-decomposition can efficiently represent 90% of a particular pattern-set. Product Grassmann Manifolds (PGM) is one example of the use of subspaces to represent tensor data, as in action recognition problems (Lui et al., 2010; Lui, 2012b). In this case, PGM extracts subspaces from tensor data and represents them as a point on the product space of 3 Grassmann manifolds, where each subspace corresponds to a point on one of the Grassmann manifolds. It is worth noting that PGM is capable of handling more than 3 modes, although the application of action recognition requires only 3 modes. Then, the classification is performed based on the chordal distance (Golub et al., 1996; Stewart & Sun, 1990) on the product manifold. It is known that the chordal distance on a product manifold is equivalent to the Cartesian product of geodesics.
from the manifolds (Lui, 2012a; Harandi et al., 2015). By incorporating the chordal
distance on the product manifold, we may express the relation between the subspaces
of all available tensor modes in a unified design. The nearest neighbor classifier using
the chordal distance on a Grassmann manifold is equivalent to the MSM. According
to this theoretical relation, PGM and MSM are equivalent in regarding pattern-set
representation. Although MSM has been established as a standard framework in the
research field of pattern-set recognition, solving many practical problems, its discriminant
ability is known to be insufficient, since each class subspace is created without reflecting
the between-class relationship. Therefore, PGM inherits the main disadvantage of MSM:
absence of a discriminative mechanism.

Fukui et al. (2006) proposed the concept of the difference between two subspaces (DS),
which minimizes data redundancy while extracting suitable features for classification.
Then, further refinement of this method was introduced to handle multiple class subspaces
by using the Generalized Difference Subspace (GDS). GDS was then employed to solve
several pattern-set related problems, such as face and handshape classification. More
precisely, GDS projection acts as a feature extractor for MSM. Since GDS represents
the difference among class subspaces, the GDS projection can increase the angles among
the class subspaces toward orthogonal status. As a result, GDS projection operates
as a quasi-orthogonalization process, which is a practical feature extraction for any
subspace-based method. These operations allow the generation of discriminative features,
overcoming the limitations of MSM. Despite its useful properties, GDS has not been yet
employed in tensor data applications since, in such applications, the ordering relationship
between the patterns must be preserved. As GDS is based on the eigen-decomposition
of the pattern-sets, the temporal relationship between the patterns is usually lost.

In this paper, we introduce the $n$-mode GDS projection, which is able to extract
discriminative information from tensor data and to provide suitable subspaces for tensor
data classification. Under this formulation, we can efficiently express tensor data as
a point on a product manifold (Lui et al., 2010; Lui, 2012b), simplifying the tensorial
data representation, as well as inheriting the main characteristics of GDS. In order to
evaluate the quasi-orthogonality properties of the proposed method, we develop a new
separability index based on the Fisher score. Since the Fisher score is not able to handle
tensorial data, we redefine the traditional Fisher score formulation to handle tensor data.
Once the $n$-mode GDS is embodied into the product manifold, we can represent the
relationship between all modes of a tensor in a unified design. Besides, we can go further
and evaluate each mode separately, providing information to create a flexible measure
of similarity. This measure of similarity is developed as weighted geodesic distance. In
summary, the main contributions of this work are as follows:

1. We propose a novel tensor data representation called $n$-mode GDS.
2. We incorporate the $n$-mode GDS projection on the conventional product manifold,
   providing a tensor classification framework.
3. We optimize the proposed $n$-mode GDS projection on the product manifold space
   through a redefined Fisher score designed for tensor data.
4. We introduce an improved version of the geodesic distance, which incorporates the importance of each tensor mode for classification.

We have evaluated the proposed approach on five video datasets containing human actions and compared its results with the results achieved by other state-of-the-art approaches. The experimental results have shown that the n-mode GDS outperforms conventional subspace-based methods on action recognition in terms of accuracy. Moreover, the proposed n-mode GDS does not require pre-training, which is an advantage in several applications where pre-trained models are scarce.

This paper is organized as follows. Section 2 introduces the proposed method. Then, experimental results are provided in Section 3. Finally, Section 4 summarizes the main conclusions derived from this study and suggests possible future work.

2. Proposed Method

In this section, we first introduce the tensor matching problem. From this formulation, we show the procedure to extract subspaces from tensor data. Then, we present the GDS projection to provide discriminative properties. After that, we describe the formulation that encloses the geodesic distance and its improved version to compute the similarity between tensors. Finally, we present the Fischer score for n-mode subspaces. We use the following notations in this paper. Scalars are denoted by lower case letters and sets are denoted by boldface uppercase letters. Calligraphic letters will be assigned to tensors. Given a matrix $A \in \mathbb{R}^{w \times h}$, $A^\top \in \mathbb{R}^{h \times w}$ denotes its transpose.

2.1. Problem Formulation

Multi-dimensional data is usually represented by a set of modes (n-mode tensor) in order to reduce computational complexity. This procedure has the immediate advantage of allowing parallel processing. Besides, the n-mode tensor representation permits the examination of the correlations among the various factors inherent in each mode.

Given two n-mode tensors $A$ and $B$, we can formulate the tensor matching problem by two steps. First, we create a convenient representation, where $A$ and $B$ can be expressed in a compact and informative manner. Second, we establish a mechanism to produce a reliable measure of similarity between these representations, allowing the comparison of $A$ and $B$.

2.2. Tensor Representation by Subspaces

The tensors $A$ and $B$ present distinct properties in each mode. For instance, in the case of video data, where $n = 3$, we have two spatial modes and a temporal one. Thus, each mode must be analyzed independently, according to its factors. To simplify this procedure, we employ the unfolding process. We denote by $X = \{X_i\}_{i=1}^n$ the set of unfolded images corresponding to the mode-1, mode-2 and mode-3 unfolding of $A$, respectively. The same procedure is conducted on the tensor $B$, resulting in $Y = \{Y_i\}_{i=1}^n$. 

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Eigen-decomposition can be exploited to derive a set of eigenvectors for each element of $X$ and $Y$. It is expected that the eigenvectors associated to the largest eigenvalues of each element of $X$ and $Y$ accurately represent their elements in terms of variance maximization (Jolliffe, 2011). After selecting these eigenvectors, we obtain the following sets $U_X = \{U_i\}_{i=1}^n$ and $U_Y = \{U_i\}_{i=1}^n$, respectively. Since each mode expresses a distinct factor, it is reasonable to expect that each set of eigenvectors has a different distribution and property, requiring a disjointed analysis to represent them accurately. For example, some modes may require more eigenvectors for their representation than the others.

Now that we have $U_X$ and $U_Y$, which span the $n$-mode subspaces $P = \{P_i\}_{i=1}^n$ and $Q = \{Q_i\}_{i=1}^n$, we can employ a mechanism to extract more discriminative information from $A$ and $B$. According to the literature, subspaces are very efficient tools for representing small and medium sized datasets. However, this representation may not be ideal for classification, since the subspaces are calculated independently, without learning correlations between the distinct classes distributions. At this point, we may employ some of the available techniques to enhance the subspace representation (Fukui & Maki, 2015; Fukui & Yamaguchi, 2007; Kawahara et al., 2007; Tan et al., 2018) to create a set of subspaces $D = \{D_i\}_{i=1}^n$, whereby projecting the sets $P$ and $Q$, we obtain suitable subspaces for classification. In our investigations, we adopt GDS (Fukui & Maki, 2015), since it provides a reasonable balance between robustness and computational complexity considering that it is mainly based on eigen-decomposition. Once we have projected the $n$-mode subspaces $P$ and $Q$ onto $D$, we obtain the sets $\hat{P}$ and $\hat{Q}$. After we select the similarity function, we have the essential components to represent and measure the similarity between $A$ and $B$. The following sections present the details to compute $D$.

2.3. Generating $n$-mode Subspaces

In order to compute the $n$-mode subspaces from a tensor data, we employ the $n$-mode singular value decomposition ($n$-mode SVD) (Kolda & Bader, 2009; De Lathauwer et al., 2000). The $n$-mode SVD provides the means to extract basis vectors from unfolded
tensors through the use of the traditional SVD. Given a $n$-mode tensor $\mathbf{A}$, the objective of $n$-mode SVD is to derive a set of orthogonal basis vectors $\mathbf{U} = \{U_{ij}\}_{i=1}^{n}$ and a core tensor $\mathbf{S}$. By using $n$-mode SVD, every such tensor can be decomposed as follows:

$$\mathbf{A} = \mathbf{S} \times U_1 \times U_2 \times \ldots \times U_n,$$

(1)

where the core tensor $\mathbf{S}$ includes information regarding the various mode matrices $U_1, U_2, \ldots, U_n$ and each mode matrix $U_i$ contains the orthonormal vectors spanning the column space of the matrix $X_i$, which is the result of the tensor $\mathbf{A}$ unfolding. This decomposition provides flexibility, since it gives the tools to analyze each tensor factor independently. Besides, by employing this strategy, we preserve the computational complexity of SVD, since $n$-mode SVD can be implemented by a series of $n$ SVD decompositions. It is important to note that the employed collection of matrices $\mathbf{X}$ do not satisfy the zero expectation condition, (i.e., $E(X_i) \neq 0$, $\forall X_i \in \mathbf{X}$), contrasting with the originally proposed $n$-mode SVD (Kolda & Bader, 2009; De Lathauwer et al., 2000).

Previous studies indicate that $\mathbf{S}$ contains rich information regarding the relation between the set $\mathbf{U}$ and can be exploited in classification and reconstructions methods (Savas & Eldén, 2007; Zhang et al., 2015b). In spite of its importance, we employ the average canonical angle to describe the relationship between the $n$-mode subspaces, which does not require $\mathbf{S}$.

2.4. Selecting the $n$-mode Subspace Dimensions

One of the benefits of using subspaces to represent tensors is that it provides a useful mechanism to define the compactness ratio, i.e., how much information of the patterns of a particular mode should be preserved to maintain the trade-off between data contribution concerning variance and subspace dimension. Given one of the tensor unfolded mode $X$, Equation (2) defines the proportion of the basis vectors employed to describe $X$ compactly (Etemad & Chellappa, 1998; Itoh et al., 2016):

$$\mu(K) \leq 100\% \times \frac{\sum_{k=1}^{K} \lambda_k}{\sum_{k=1}^{R} \lambda_k}.$$  (2)

where $K$ is the number of selected basis vectors that span a subspace $P$, $\lambda_k$ is the $k$-th eigenvalue of $X$ and $R = \text{rank}(P)$. The function $\mu(\cdot)$ controls the trade-off between the compactness ratio of $X$ and its amount of accumulated energy in the first $k$ eigenvectors. This parameter depends on the complexity of the linear correlations of each tensor mode and is also application-dependent. For example, when we have a high correlation between the matrices of a particular mode, or it presents repeated exemplars, its subspace representation will display a compact shape, i.e., its first eigenvectors associated to the first eigenvalues will explain most of the data.

2.5. Generating the $n$-mode GDS Projection

In a $m$-class classification problem, $\mathbf{P} = \{P_{ij}\}_{i,j=1}^{n,m}$ denotes the set of all $n$-mode subspaces spanned by $\mathbf{U} = \{U_{ij}\}_{i,j=1}^{n,m}$. Then, we can now develop the $n$-mode GDS
projection $D_i = \{D_{ij}\}_{i=1}^n$ that act on $P$, to extract discriminative information. Since each mode subspace reflects a particular factor, it is essential to handle each one independently and compute a model that reveals hidden discriminative structures. In traditional GDS, this procedure is performed through the removal of the overlapping components that represent the intersection between the subspaces. In mathematical terms, the GDS projection can be described as the extension of the difference vector between two vectors in a multi-dimensional space.

By discarding the components that express the intersection between subspaces, GDS entirely consists of the required elements for classification (Fukui & Maki, 2015; Tan et al., 2018). Therefore, by projecting the subspaces onto the $n$-mode GDS, we expect to extract suitable information for tensor classification. Figure 2 shows the advantages of using the $n$-mode GDS projection on the PGM. The Euclidean distance neglects the manifold surface, which may result in information loss. On the other hand, the geodesic distance exploits the manifold surface, where the $n$-mode GDS projection improves the distance between the different $n$-mode subspace classes. In order to compute the $n$-mode GDS, we compute the sum of the projection matrices of each $i$-mode subspace as follows:

$$G_i = \frac{1}{m} \sum_{j=1}^{m} U_{ij} U_{ij}^T, \quad \text{for } 1 \leq i \leq n. \tag{3}$$

Since $G_i$ has information regarding all class subspaces in a particular mode, it is beneficial to decompose it to exploit discriminative elements. Applying eigen-decomposition to $G_i$, we obtain:

$$G_i = V_i \Sigma_i V_i^T, \quad \text{for } 1 \leq i \leq n, \tag{4}$$

where the columns in $V_i = \{\phi_1, \phi_2, \ldots, \phi_{R_i}\}$ are the normalized eigenvectors of $G_i$, and $\Sigma_i$ is the diagonal matrix with corresponding eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_{R_i}\}$ in descending order, where $R_i = \text{rank}(G_i)$. The $n$-mode GDS projection discards the first few eigenvectors of $G_i$ with large eigenvalues and retains only the last few eigenvectors of $G_i$ with small eigenvalues. Thus, the $n$-mode GDS provides the difference information between $n$-mode class subspaces. Therefore, we can define $D_i = \{\phi_{\alpha_i}, \ldots, \phi_{\beta_i}\}$, where $\alpha_i < \beta_i \leq R_i$. The $n$-mode GDS dimension is defined by maximizing the mean canonical angles between $n$-mode class subspaces.

2.6. Projecting the $n$-mode Subspaces onto the $n$-mode GDS

Once $D_i = \{D_{ij}\}_{i=1}^n$ is computed, we can extract more discriminative structures from $P = \{P_{ij}\}_{i,j=1}^{n,m}$. According to Fukui et al. (2006) and Fukui & Maki (2015), this procedure can be achieved by conducting two different approaches. The first approach involves projecting subspaces onto a discriminative space, then orthogonalizing the projected subspaces by using the Gram-Schmidt orthogonalization. The second procedure includes projecting subspaces onto a discriminative space directly, then applying SVD to generate the projected subspaces. Fukui et al. (2006) and Fukui & Maki (2015) established that these two procedures are algebraically equivalent. In this work, we employ the first
Figure 3: Conceptual figure of the $n$-mode GDS projection. First, we unfold the $3$-mode tensor $A$ and compute its subspaces from the unfolded modes. Then, we project the subspaces onto the $n$-mode GDS. The product of manifolds can be exploited to represent the projected subspaces. Finally, the chordal distance $\rho(A, B)$ determines the similarity between tensors $A$ and $B$.

procedure, which is consistent with the conventional method. Therefore, the procedure to compute $\hat{P} = \{\hat{P}_{ij}\}^{n,m}_{i,j=1}$ is:

$$\hat{P}_{ij} = \text{orth} \left(D_i^T P_{ij}\right),$$

where the $\text{orth}(\cdot)$ operator denotes the orthogonalization and normalization of a set of vectors by using the Gram-Schmidt orthogonalization.

2.7. Representing the $n$-mode Subspaces $\hat{P}$ on the Product Manifold

We introduce the product manifold to describe $\hat{P}$ into a single manifold. This manifold consists of the product of the projected $n$-mode subspaces onto the $n$-mode GDS. In traditional PGM, however, the subspaces are generated directly from the tensors by employing $n$-mode SVD. Although this procedure presents a convenient representation for the tensors, the generated subspaces may not be ideal for classification. In contrast, we project $P$ onto $D$ before applying the product manifold. Our objective is to achieve more efficient subspaces for classification. Therefore, given a set of manifolds $\mathcal{M} = \{M_i\}_{i=1}^n$ composed by $\hat{P}$, Equation (6) describes the product manifold:

$$M_D = (\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n),$$

where $\times$ denotes the Cartesian product, $M_i$ is a $i$-mode manifold and $\hat{P}_i \in M_i$. It is worth noting that the manifold topology of $M_D$ is equivalent to the product topology (Absil et al., 2009). The advantage of using $M_D$ is that it provides a combined topological space associated with $\hat{P}$. For illustration, in gesture and action recognition problems, where we handle 3-mode tensors, we can replace the tensor representation by elements on a product manifold. Therefore, a tensor data can be regarded as a point on the
product manifold $M_D$. Another benefit of employing $M_D$ to represent tensor data is that it provides the means to work directly with geodesics through the use of the geodesic distance. The geodesic distance between two points is the length of the geodesic path, which is the shortest path between the points that lie on the surface of the manifold. Besides, the geodesic distance presents a more accurate similarity between two points on the product manifold, since it exploits the surface of the manifold \cite{yang2008}. Figure 2 illustrates the geodesic distance.

2.8. Fisher score for n-mode Subspaces

In this section, we introduce the Fisher score for tensorial class separability index. Traditionally, the Fisher score $F(\Psi)$ of a transformation matrix $\Psi$ can be defined as the ratio of two variables: $F(\Psi) = F^b/F^w$, where $F^b$ and $F^w$ are the inter-class and intra-class variability, respectively. Therefore, a high Fisher score ensures high inter-class and low intra-class variability. We extend the Fisher score to evaluate subspace separability by re-defining the $F^b$ and $F^w$ scores as follows:

$$F^b = \frac{1}{m} \sum_{j=1}^{m} \text{Sim}(\hat{P}_j^{\mu}, \hat{P}^{\mu}),$$  \hspace{1cm} (7)

$$F^w = \frac{1}{v} \sum_{j=1}^{m} \sum_{k=1}^{m_j} \text{Sim}(\hat{P}_{jk}, \hat{P}_k^{\mu}),$$  \hspace{1cm} (8)

where $\hat{P}_j^{\mu}$ stands for the Karcher mean of the $j$-class subspace, $\hat{P}^{\mu}$ is the Karcher mean of the $\hat{P}_j^{\mu}$ subspaces, $m$ is the number of subspaces in a particular subspace class and $v = mm_j$, where $m_j$ is the number of subspaces in the $j$-class. Finally, Sim(·, ·) is a function that measures the similarity between subspaces. Since $F(\cdot)$ is not defined to handle $n$-mode subspaces, we adapt the Fisher score to the $n$-mode case as the average of the Fisher scores in each mode as follows:

$$F^b_n = \frac{1}{n} \sum_{i=1}^{n} F^b_i,$$  \hspace{1cm} (9)

$$F^w_n = \frac{1}{n} \sum_{i=1}^{n} F^w_i.$$  \hspace{1cm} (10)

Then, the $F_n(\Phi) = F^b_n/F^w_n$ score is the class separability index for $n$-mode subspaces, where $\Phi = \{\Phi_i\}_{i=1}^{n}$ is a set of transformation matrices. The introduced $n$-mode Fisher score will be used as an evaluation tool to select the optimal dimension of $D$.

2.9. Weighted Geodesic Distance

We employ the average of the canonical angles to compare the subspaces of the different modes. A practical technique to compute the canonical angles between two
subspaces $P$ and $Q$ is by computing the eigenvalues of the product of their basis vectors. Therefore, given $U_P$ and $U_Q$, which span the subspaces $P$ and $Q$, Equation (11) computes the canonical correlations between $P$ and $Q$.

$$U_P^TU_Q = U\Sigma V^T. \tag{11}$$

where the eigenvalues matrix $\Sigma$ provides the canonical correlations between the principal angles of $U_P$ and $U_Q$ and can be exploited to compute the canonical angles, since $\Sigma = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_K)$. Then, the canonical angles $\{\theta_k\}_{k=1}^K$ can be computed using the inverse cosine of $\Sigma$, as $\{\theta_k = \cos^{-1}(\lambda_k)\}_{k=1}^K$. Finally, the average canonical angle $\bar{\theta}$ between $P$ and $Q$ is defined as $\bar{\theta} = \frac{1}{K} \sum_{k=1}^K \theta_k$, where $K \leq \min(\text{rank}(U_P), \text{rank}(U_Q))$.

Once obtained the average canonical angles of all the available modes $\bar{\theta} = \{\bar{\theta}_i\}_{i=1}^n$, we can introduce the weighted geodesic distance based on the product manifold, which is defined as:

$$\rho(A, B) = \left(\sum_{i=1}^n (w_i \bar{\theta}_i)^2\right)^{1/2}, \tag{12}$$

where we estimate $w_i$ by using the Fisher score since each mode will provide a different separability index reflecting the importance of each mode in regarding classification:

$$w_i = \frac{F(D_i)}{\sum_{i=1}^n F(D_i)}, \tag{13}$$

where $F(D_i)$ is the Fisher score for the projection matrix $D_i$. It is worth noting that when $w_i = 1$, this distance on the Cartesian product is regarded as the product manifold distance. The geodesics in the product manifold $M_D$ are just the products of geodesics in the factor manifolds $M = \{M_i\}_{i=1}^n$. By employing $w_i$, we can exploit the importance of each factor manifold, which can improve the classification accuracy.

In Equation (12), we must minimize $\rho(\cdot, \cdot)$ when tensors $A$ and $B$ are observations of the same class. Otherwise, when $A$ and $B$ represent distinct classes, $\rho(\cdot, \cdot)$ should return high values. Since $\rho(\cdot, \cdot)$ is essentially proportional to the average of the canonical angles between $P$ and $Q$, its optimization process requires only the proper selection of $\alpha_i$, $\beta_i$ and $w_i$. We can achieve quasi-orthogonality between the $n$-mode subspaces by generating the appropriate GDS projection $D$, hence, enlarging its geodesic distance on the product manifold. Formally, we can obtain $D$ as follows:

$$D = \arg \max F_n(V'), \tag{14}$$

where $V' = \{V'_i\}_{i=1}^n$ and $V'_i = \{\phi_{\alpha_i}, \ldots, \phi_{\beta_i}\}$ is the eigenvector subset of $V$ obtained by the Equation (4). In the next section, we provide experimental results that support our claim.
3. Experimental Results

In this section, we evaluate the $n$-mode GDS projection to show its advantages over tensor-based methods for action and gesture recognition problems. First, we present the datasets and the experimental protocol employed. Next, we provide the visualization of the difference between the $n$-mode subspaces. After that, we evaluate the model discriminability by using the $n$-mode Fisher score. Then, we compare the proposed approach with related methods. The tensor modes are evaluated independently and simultaneously, followed by comparison with related methods. Finally, feature extraction techniques are employed on $n$-mode GDS projection framework and comparison with the state-of-the-art is provided.

3.1. Datasets and Experimental Protocol

Our experiments were conducted using five datasets. The KTH (Schuldt et al., 2004) is a video dataset containing six types of human actions (walking, jogging, running, boxing, hand waving and hand clapping) executed by 25 subjects in four different scenarios: outdoors, outdoors with scale variation, outdoors with different clothes and indoors. This dataset consists of 2 391 sequences, where all videos were recorded over homogeneous backgrounds with a static camera (in most sequences, but hard shadows are present) with a temporal resolution of 25 frames per second. In addition, there are significant variations in length and viewpoint. Also, the videos were down sampled to $160 \times 120$ pixels and have a length of four seconds in average. The videos are divided with respect to the subjects into a training set (8 persons), a validation set (8 persons) and a test set (9 persons).

We employ the Cambridge Gesture dataset (Kim & Cipolla, 2007) containing 900 video sequences of nine hand gesture classes. Each gesture class consists of 100 video sequences, and the video was collected from five different illumination sets designated as set1, set2, set3, set4, and set5. The set5 is used for the training and the other four sets are employed as test sets.

The UCF-101 (Soomro et al., 2012) dataset is a large action recognition dataset which comprises 13 320 YouTube video clips of 101 action categories, separated into five categories: Human-Object Interaction, Body-Motion Only, Human-Human Interaction, Playing Musical Instruments, Sports. Most of the videos are related to actions performed in sports. The videos duration varies from 2 to 15 seconds, with 25 frames per second.

The HMDB-51 (Kuehne et al., 2011) dataset contains 6 766 video clips, where 3 570 are employed for training and 1 530 for testing. This dataset presents 51 classes that were obtained from multiple sources, including movies, YouTube and Google. Both UCF-101 and HMDB-51 datasets present 3 (training, testing)-folds, and we report the average accuracy of the three testing splits. Due to their complexity, in our experiments, we resize UCF-101 and HMDB-51 videos to $340 \times 256$. Compared to UCF-101, the videos in HMDB-51 are more difficult, since they present the complexity of real-world actions. The performance of these two datasets is calculated using the average accuracy.
The Osaka University Kinect Action Dataset (Mansur et al., 2013) contains 10 actions performed by eight subjects and collected by Osaka University. Action types consist of jumping jack type 1, jumping jack type 2, jumping on both legs, jumping on right leg, jumping on left leg, running, walking, side jumps, skipping on left leg, and skipping on right leg. The videos were down sampled from $320 \times 240$ to $160 \times 120$ pixels and have a length varying from 2 to 4 seconds. In addition to the RGB data, this dataset provides depth and skeleton data. During the data recording process, illumination conditions and background had few changes. In our experiments, we employed the depth information to segment the foreground from the background pixels. For evaluation purposes, we use the leave-one-out cross-validation scheme.

3.2. Visualization of the $n$-mode GDS Projection

In this experiment, we aim to show the visual differences between the images of the unfolded tensors, the basis vectors, the $n$-mode sum subspace, and the $n$-mode principal subspace. The $n$-mode GDS projection is computed using two classes of the KTH dataset (boxing and waving).

![Figure 4](image1.png)  
**Figure 4:** Unfolded tensors of the classes boxing and waving of the KTH dataset.

![Figure 5](image2.png)  
**Figure 5:** Basis vectors of the unfolded tensors.

![Figure 6](image3.png)  
**Figure 6:** The $n$-mode principal space and the $n$-mode GDS basis vectors.

Figure 4 displays the images of the unfolded tensors. Figures 4(a), 4(b) and 4(c) show the first 3 frames of the 1-, 2- and 3-mode unfolding of the class boxing. Figures 4(d), 4(e) and 4(f) show the first 3 frames of the 1-, 2- and 3-mode unfolding of the class waving. Then, Figure 5 presents the first 3 basis vectors of the respective unfolded tensors. Finally, Figure 6 provides the principal subspace and the difference subspace, which are the decomposition results of the sum subspace. Figures 6(a), 6(b) and 6(c) present the common structures contained in both unfolded tensors. These structures provide low
discriminative information since projecting different class subspaces will result in closer subspaces and thus decreasing the canonical angles between them. On the other hand, Figures 6(d), 6(e) and 6(f) provide the difference between the components contained in both unfolded tensors. In mathematical terms, the $n$-mode GDS projection represents the linear combination of the difference between the samples of a particular tensor mode. This linear combination preserves the discriminative structures in the form of a subspace, where projecting similar subspaces representing different classes will result in enlarged canonical angles.

3.3. Evaluating the $n$-mode GDS Projection Separability Using the $n$-mode Fisher Score

In this section, we evaluate the separability of the $n$-mode subspaces using both the Multi Dimensional Scaling (MDS) and the proposed $n$-mode Fisher score on the Osaka dataset. The MDS enables the visualization of the similarity between the $n$-mode subspaces by projecting the pairwise canonical angles among the subspaces onto an abstract space. In this experiment, the video data from the Osaka Kinect dataset were pre-processed by using a people detector, and the detected patches were cropped. Considering that the cropped patches present different sizes, we also normalized them to $30 \times 90$. Later, linear interpolation was employed to normalize the number of frames to compound the $30 \times 90 \times 30$ tensors. We denote the modes obtained by tensor unfolding as follows: 1-mode denotes the unfolding in the temporal $t$-axis direction, 2-mode represents the unfolding of the tensor in the spatial $y$-axis direction, and the 3-mode is described by unfolding in the spatial $x$-axis direction.

The proposed $n$-mode GDS projection is compared to PGM obtained using MSM on the unfolded modes. When we compare MSM to the proposed method, we observe that, even though MSM may operate directly on tensors, it works with only one of the modes. Thus, the relationship between MSM, GDS, PGM and $n$-mode GDS is as follows. MSM executes the pattern-set matching using only one of the available tensor modes. GDS executes the same strategy as MSM, however, employing a discriminative space to improve its classification accuracy. PGM, in turn, applies the same approach as MSM but operating in all available tensor modes. Lastly, $n$-mode GDS utilizes a discriminative space produced by GDS, but in each separate mode and executes the fusion of those subspaces through the product of manifolds. Therefore, by performing this experiment, we can evaluate whether it is worth using the three modes, a specific combination (e.g., 1-mode and 3-mode) or only one of the modes.

MSM maximized its accuracy when the dimensions of the subspaces were set to 15, 10, and 12 for the 1-, 2- and 3-mode unfolding, respectively. The number of canonical angles that results in the best accuracy is 5 for the 2- and 3-mode unfolding and 7 for the 1-mode unfolding. Using a 10-fold cross-validation scheme, MSM obtained the reasonable accuracy of 74.30% on the 1-mode unfolding, followed by 71.60% and 62.90% on the 2- and 3-mode unfolding, respectively. By using the same set of parameters, the PGM achieved 77.67% of accuracy. Accordingly, the number of basis vectors required to express the videos is very low compared to the original amount of data contained in the dataset. Moreover, the accuracy attained by MSM on each separated mode supports the
Table 1: The accuracy and the $n$-mode Fisher score (in parenthesis) for the MSM and GDS subspaces.

| method | 1-mode     | 2-mode     | 3-mode     |
|--------|------------|------------|------------|
| MSM    | 74.30% (0.57) | 71.60% (0.41) | 62.90% (0.46) |
| GDS    | 81.10% (0.62) | 76.50% (0.49) | 65.20% (0.51) |

idea that the modes should be exploited in an unified framework and a discriminative mechanism should be employed.

Figure 7: 3D scatter plots of Osaka Kinect dataset using MSM. (a) 3−mode, (b) 2−mode and (c) 1−mode unfolding are represented using MSM. PGM is shown on (d).

Figure 8: 3D scatter plots of Osaka Kinect dataset using GDS. (a) 3−mode, (b) 2−mode and (c) 1−mode unfolding are represented using GDS. $n$-mode GDS is shown on (d).

Figure 7 shows the MDS plots of the subspaces obtained using MSM. This figure suggests that the distance of the subspaces changes according to its mode unfolding, confirming that each mode may contribute differently to the classification accuracy. The 1−mode unfolding shows more compact clusters than the others, suggesting that the 1−mode provides slightly more robust features for classification. Figure 7(d) displays the arrangement of the PGM. Compared to the 1−mode unfolding, the PGM presents a lower intra-class separability, although the clusters of the different classes appear closer, suggesting that the fusion schema adopted by PGM does not take into consideration the relationship between the different classes.

It is worth mentioning that PGM presents reasonable results in this dataset. Although the Osaka Kinect dataset provides only ten classes, it provides a challenge for PGM,
since the inter-class distance of some classes are very low due to the similar actions contained in the dataset. For instance, the classes jumping with the right leg and jump with the left leg present subspaces with a high amount of structural overlap, since just a small part of the subject body differentiates the classes. Taking into account that this discriminative element may not be present in the subspaces, MSM and PGM are not able to correctly represent these cases.

In terms of the $n$-mode GDS projection, it attained its best accuracy when the dimensions of the subspaces were set to 17, 12, and 15 for the 1-, 2- and 3-mode unfolding, respectively. The number of canonical angles that results in the best accuracy is 5 for all the unfolding modes. Using a 10-fold cross-validation scheme, the $n$-mode GDS projection achieved 81.10% of accuracy on the 1-mode unfolding, followed by 76.50% and 65.20% on the 2- and 3-mode unfolding respectively. By employing the same set of parameters, the $n$-mode GDS using all available modes achieved 83.30% of accuracy.

Figure 8 presents the MDS plots of the subspaces using the $n$-mode GDS projection. The clusters formed by the introduction of the GDS on the unfolded tensors act by reducing the intra-class distance, creating more recognizable clusters. Figure 8(d) presents the relation between the $n$-mode subspaces when discriminative structures are available. Visually, it is possible to observe that the proposed method provides a lower intra-class distance, while improving the inter-class distance. It is worth mentioning that the increase in inter-class distance seems to occur in all classes, although very similar classes (such as jumping on both legs, jumping on right leg and jumping on left leg) still present much overlap, which is expected. Since PGM has no mechanism to extract discriminative structures, it relies on the data distribution itself, depending on the structural differences between the tensor subspaces. When two $n$-mode subspaces representing different classes present high overlap, the similarity between them is high, and the subspace accuracy decreases. On the other hand, $n$-mode GDS can detect and remove the common components existing in similar classes, exposing only the relevant components for classification. In algebraic terms, $n$-mode GDS enlarges the canonical angles between similar classes, since the common structures between the subspaces are removed. This observation is also supported by the computed $n$-mode Fisher score of MSM and GDS, which are listed in Table 1.

3.4. Evaluating Tensor Modes

In the following experiments, we use KTH and Cambridge gesture datasets. Here, we evaluate the tensor mode independently and in combinations to determine their contribution regarding the recognition rate. In this experiment, we compare our proposed framework to MSM, GDS, and PGM. These methods employ the concept of subspaces and canonical angles, which may provide an objective interpretation of each mode subspace. In video data, since each mode assigns to a different factor, each subspace will have a distinct contribution in the classification. In addition, as advocated in the literature of feature subset selection, two subsets of attributes which do not operate adequately independently may perform very well when employed in combination (Bermingham et al., 2015).
Following the experimental protocol of Kim & Cipolla (2009), in KTH and Cambridge gesture datasets, the video data were resized into $20 \times 20 \times 20$. For the subspaces dimension, $K$ was chosen to encode 90% of the variance in the original data. Tables 2 and 3 list the accuracy results achieved by MSM and GDS on different modes using Cambridge and KTH datasets respectively. When 1-mode unfolding is employed, both MSM and GDS achieved the highest scores. As expected, GDS outperformed MSM in both scenarios.

Table 2: The average accuracy and standard deviation of different modes and combinations using the Cambridge dataset.

| approach     | 1-mode       | 2-mode       | 3-mode       | 4-mode       |
|--------------|--------------|--------------|--------------|--------------|
| MSM          | 64.55 ± 4.9  | 40.21 ± 5.9  | 56.29 ± 5.3  | -            |
| GDS          | 78.29 ± 3.7  | 51.50 ± 4.3  | 67.24 ± 4.1  | -            |
| approach     | a-mode       | b-mode       | c-mode       | d-mode       |
| PGM (Lui, 2012b) | 68.35 ± 2.2  | 79.23 ± 2.2  | 61.67 ± 2.4  | 88.13 ± 2.1  |
| n-mode GDS   | 74.17 ± 2.2  | 88.76 ± 2.1  | 71.33 ± 2.3  | 93.51 ± 2.1  |
| n-mode wGDS  | 74.49 ± 2.1  | 89.11 ± 2.0  | 71.48 ± 2.3  | 94.25 ± 1.9  |

Table 3: The average accuracy and standard deviation of different modes and combinations using the KTH dataset.

| approach     | 1-mode       | 2-mode       | 3-mode       | 4-mode       |
|--------------|--------------|--------------|--------------|--------------|
| MSM          | 83.03 ± 3.5  | 67.12 ± 4.3  | 71.37 ± 3.9  | -            |
| GDS          | 91.51 ± 1.9  | 70.78 ± 1.5  | 83.45 ± 2.1  | -            |
| approach     | a-mode       | b-mode       | c-mode       | d-mode       |
| PGM (Lui, 2012b) | 80.15 ± 2.8  | 85.73 ± 2.6  | 74.56 ± 2.9  | 96.17 ± 1.7  |
| n-mode GDS   | 83.84 ± 2.1  | 91.28 ± 1.9  | 78.54 ± 2.1  | 97.33 ± 1.6  |
| n-mode wGDS  | 84.16 ± 2.1  | 91.67 ± 1.9  | 78.34 ± 2.1  | 98.64 ± 1.4  |

Tables 2 and 3 also show the results of PGM, n-mode GDS and n-mode wGDS (weighted GDS) when the modes are combined, where the $a$-, $b$-, $c$- and $d$-mode are the combinations 1-2, 1-3, 2-3 and 1-2-3, respectively. The results show that mode combinations improve the accuracy of all methods, indicating that the time information is a decisive discriminative information. In addition, the best results are obtained when all the available modes are employed. The weighed geodesic distance strategy demonstrated to be efficient, validating the strategy of using weights at each geodesic distance.

3.5. Comparison with Related Methods

For this comparison, we employ the following traditional methods: Discriminative Canonical Correlation (DCC), Generalized Difference Subspace (GDS), tensor Canonical Correlation Analysis (TCCA), Multilinear Principal Component Analysis (MPCA), Multilinear Linear Discriminant Analysis (MLDA), Product Grassmann Manifold (PGM) and Tangent Bundle (TB). These methods are established in classification problems.
involving tensorial data and operate on subspace representations to classify tensor data. According to Table 4, PGM and TB consistently produce competitive results. PGM produces reliable results compared to supervised methods, such as DCC and GDS. This observation is an indication of the advantages that the unfolding process employed by the tensor representation can provide.

Table 4: Cambridge and KTH datasets evaluation.

| method                  | Cambridge (Kim & Cipolla, 2007) | KTH (Schuldt et al., 2004) |
|-------------------------|---------------------------------|---------------------------|
| DCC (Kim et al., 2007)  | 76%                             | 90%                       |
| GDS (Fukui & Maki, 2015)| 78%                             | 91%                       |
| TCCA (Kim & Cipolla, 2009)| 82%                           | 95%                       |
| MPCA (Deng et al., 2005)| 42%                             | 67%                       |
| MLDA (Yan et al., 2006) | 43%                             | 71%                       |
| PGM (Lui, 2012b)       | 88%                             | 96%                       |
| TB (Lui, 2011)         | 91%                             | 96%                       |
| n-mode GDS             | 93%                             | 97%                       |
| n-mode wGDS            | **94%**                         | **98%**                   |

3.6. Comparison with Existing Methods using Handcrafted Features

In this experiment we evaluate the proposed method with state-of-the-art handcrafted features and compare it with deep learning related methods. For this comparison, we employ the 3D Convolutional Neural Network (C3D) (Ji et al., 2012), Two-Stream Network (Simonyan & Zisserman, 2014) and Two-Stream Network I3D (Carreira & Zisserman, 2017). The C3D is equipped with spatio-temporal three-dimensional kernels, improving the performance levels in the field of action recognition. Differently, the Two-Stream Network learns different types of features which are combined for action classification. In this network, a spatial-CNN is trained to extract appearance features using RGB images, while a temporal-CNN is trained using optical flow to extract the motion pattern. The streams features are then concatenated to represent actions in realistic videos. Although deep learning approaches have made significant advances in videos related tasks, handcrafted features produce competitive results compared to the state-of-the-art on many standard action recognition tasks (Lan et al., 2015; Hoai & Zisserman, 2014). These solutions are usually based on improved variations of HOG and HOF, for instance, improved Dense Trajectories (iDT) (Wang & Schmid, 2013).

The proposed n-mode GDS can benefits from handcrafted features since its n-mode subspace representation allows the use of other types of features other than raw features. In this section, we investigate the combination of the n-mode GDS and handcrafted features, connecting the best of both strategies via a single hybrid tensor classification architecture. To evaluate the synergy between n-mode GDS and handcrafted features, we use HOG, HOF, MBH, and iDT. These handcrafted features present different characteristics that will be exploited in the proposed method. HOG is able to extract the local appearance and shape of objects by using local intensity gradients. In this
Table 5: The average accuracy of $n$-mode GDS and deep learning approaches.

| approaches                                      | datasets |            |            |
|------------------------------------------------|----------|------------|------------|
| C3D (Ji et al., 2012)                          | HMDB-51  | 51.9       | 85.4       |
| Two-stream (Simonyan & Zisserman, 2014)        | UCF-101  | 59.4       | 88.3       |
| Two-stream I3D (Carreira & Zisserman, 2017)    |          | **80.7**   | **98.0**   |
| $n$-GDS                                        |          | 45.1       | 73.6       |
| $n$-wGDS                                       |          | 46.6       | 75.7       |
| $n$-wGDS + HOG                                 |          | 48.3       | 77.1       |
| $n$-wGDS + HOF                                 |          | 51.8       | 80.2       |
| $n$-wGDS + MBH                                 |          | 53.5       | 82.7       |
| $n$-wGDS + iDT                                 |          | 55.7       | 83.9       |

In this experiment, the HOG features replace the 3-mode unfolding, since this mode comprises the appearance of the actions. HOF is a popular handcrafted feature that can accurately obtain the motion information from videos. It is similar to HOG, however, HOF includes optical flow data across the frames, preserving temporal information. In this experiment, the HOF descriptor replace both 1- and 2-mode unfolding. Finally, the MBH descriptor works by extracting the derivatives of the horizontal and vertical components of the optical flow. MBH preserves the relative motion between pixels and represents the gradient of the optical flow. The camera motion is removed, and information about changes in the motion boundaries is preserved. As a result, MBH is robust to camera motion and provides discriminative features for action recognition.

In the last experiment, we use the improved trajectory features (iDT), which is a state-of-the-art handcrafted descriptor proposed by (Wang & Schmid, 2013) for human action recognition. This robust descriptor employs HOG, HOF, and MBH, followed by dimensionality reduction and Fisher vector encoding. Since the above descriptor is based on a 1-dimensional histogram representation of individual features (HOG, HOF, MBH), they directly model values of given features and can be directly employed as subspaces in the proposed framework. Table 5 lists the results of I3D, C3D and Two-Stream Convolutional Networks, as well as the results attained by the proposed method and its combination with handcrafted features commonly used in the literature. According to the results, the $n$-mode GDS enhances its results when equipped with the weighted geodesic distance while extracting features from the modes improves its accuracy even further. The use of HOG features as the appearance mode improves the $n$-mode GDS accuracy in about 2% in both datasets, confirming that extracting appearance features is beneficial for the proposed method.

While I3D provides the best results in these experiments, the training required for this deep neural network is preventive for more specific applications. For instance, I3D is pre-trained on ImageNet and Kinetics Human Action dataset. On the other hand, the proposed method does not rely on pre-training, making use of handcrafted features,
which covers a broader range of applications. The use of HOF and MBH increased the $n$-mode GDS accuracy in about 7%. These features provide temporal information, which is an advantage comparing to HOG features. Also, the discriminative approach, incorporated with the weighting strategy employed by the proposed method, presents a competitive result for the $n$-mode GDS. By employing all the available descriptors, $n$-mode GDS projection produces competitive results comparing to C3D and Two-Stream Convolutional Networks, confirming the effectiveness of the proposed method.

In this experiment, we can see the effects of the $w$ over the $n$-mode GDS performance. When the weight approach based on the $n$-mode Fisher score is employed, the proposed method improves 1.5% and 2% in both datasets. The weights computed by the $n$-mode Fisher score generated 0.33, 0.30 and 0.35 for the 1-, 2- and the 3-mode subspace, respectively. The approach shows to be very efficient mostly because it enables a direct estimation for the importance of each mode directly, without the use of an exhaustive search. When HOG features are employed to replace the raw images of the 3-mode subspace features, the $n$-mode Fisher score estimates the weights to 0.32, 0.29 and 0.38 for the 1-, 2- and the 3-mode subspace, respectively. This new set of weights increases the importance of the tensor mode, which uses the handcrafted features, implying the importance of these features for the tensorial class separability and consequently to the classification accuracy. This behavior is present in the other experiments, where the tensor mode employing handcrafted features is reported with higher weights than the other modes. These observations confirm the hypothesis that interpreting the spatial-temporal modes across the manifolds is useful, and how to preserve a balance between these modes is essential to improve the accuracy of the proposed method.

4. Conclusion and Future Directions

In this paper, we propose a tensor representation and classification method based on Product Grassmann manifold. By exploiting the geometry metric on the product manifold, the proposed $n$-mode GDS projection based on the subspace learning method obtains a discriminative model on the Product Grassmann Manifold. The $n$-mode Fisher score is also proposed to evaluate the $n$-mode subspace separability of the new model. In addition, we introduce the weighted geodesic distance into the proposed model.

The proposed method was investigated in action and gesture recognition problems. The high performance in the classification experiments conducted on different video datasets indicates that the new model is well suitable for representing high dimensional data and revealing intrinsic subspaces structures underlying data. In future work, we will focus on investigating different metrics of the Product Grassmann Manifold and test the proposed methods on larger scale complex videos. Another research direction would be to extend this framework to take into account the nonlinear nature of the data distribution. For instance, employing a kernel approach to handle nonlinear patterns (Yan et al., 2019; Gao et al., 2019).
Availability of Data and Material

KTH (Schuldt et al., 2004), Cambridge Gesture (Kim & Cipolla, 2007), HMDB-51 (Kuehne et al., 2011), UCF-101 (Soomro et al., 2012) and The Osaka University Kinect Action Dataset (Mansur et al., 2013) are used for training and testing the proposed method. These datasets are freely available. The source code of the proposed method will also be made available in the final version of the paper.

Competing Interests

The authors declare that they have no competing interests.

Credit Authorship Contribution Statement

Bernardo B. Gatto: Conceptualization, Methodology, Software, Validation, Formal Analysis, Investigation, Data Curation, Writing - Original Draft, Visualization, Funding Acquisition. Eulanda M. dos Santos, Alessandro L. Koerich and Kazuhiro Fukui: Writing - Review & Editing, Supervision. Waldir S. S. Júnior: Supervision.

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