Remark on nondegeneracy of simple abelian varieties with many endomorphisms

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Abstract

We investigate a relationship between nondegeneracy of a simple abelian variety \( A \) over an algebraic closure of \( \mathbb{Q} \) and of its reduction \( A_0 \). We prove that under some assumptions, nondegeneracy of \( A \) implies nondegeneracy of \( A_0 \).

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Keywords: abelian variety, the Hodge conjecture, the Tate conjecture

Introduction

Let \( A \) be an abelian variety over an algebraically closure \( \mathbb{Q}^{\text{alg}} \) of \( \mathbb{Q} \) in \( \mathbb{C} \). In this paper, we say that \( A \) is an abelian variety with many endomorphisms if the reduced degree of the \( \mathbb{Q} \)-algebra \( \text{End}^0(A) := \text{End}(A) \otimes \mathbb{Q} \) is equal to \( 2 \dim A \). This condition is equivalent to that \( A \) is of CM-type. An abelian variety \( A \) over \( \mathbb{Q}^{\text{alg}} \) is said to be nondegenerate if all the Hodge classes (see §1) on \( A \) are generated by divisor classes in the Hodge ring of \( A \). If \( A \) is nondegenerate, then the Hodge conjecture holds for \( A \). We know that products of elliptic curves over \( \mathbb{C} \) are nondegenerate (Tate [19], Murasaki [11], Imai [2], Murty [12]). However, there are examples which is degenerate but the Hodge conjecture holds (cf. [1] [16]). For other known results on the Hodge conjecture for abelian varieties, we refer to Gordon’s article in Lewis’s book [7, Appendix B].

\[ \text{The reduced degree of } \text{End}^0(A) \text{ is always } \leq 2 \dim A. \]
Let $p$ be a prime number. Let $F$ be an algebraic closure of a finite field $F_p$ with $p$-elements. Let $\ell$ be a prime number different from $p$. An abelian variety $A_0$ over $F$ is said to be nondegenerate if all the $\ell$-adic Tate classes (see §1) on $A_0$ are generated by divisor classes in the $\ell$-adic étale cohomology ring of $A_0$. If $A_0$ is nondegenerate, then the Tate conjecture holds for $A_0$. Spiess [17] proved that products of elliptic curves over finite fields are nondegenerate. For certain abelian varieties over finite fields, nondegeneracy is known by Lenstra-Zarhin [6], Zarhin [24], Kowaloski [4]. However, there are examples which is not nondegenerate but the Tate conjecture holds ([10, Example 1.8]). For other known results on the Tate conjecture, we refer to [22].

Milne [9, Theorem] proved that if the Hodge conjecture holds for all CM abelian varieties over $\mathbb{C}$, then the Tate conjecture holds for all abelian varieties over the algebraic closure of a finite field. He furthermore studied a relationship between the Hodge conjecture for an abelian variety $A$ with many endomorphisms over $\mathbb{Q}^{\text{alg}}$ and the Tate conjecture for the reduction $A_0/F$ of $A$ at a prime $w$ of $\mathbb{Q}^{\text{alg}}$ dividing $p$ (see Theorem [12]). Here, we note that by a result of Serre–Tate [13, Theorem 6], one can consider the reduction of $A$. However a relationship between nondegeneracy of $A/\mathbb{Q}^{\text{alg}}$ and of $A_0/F$ is not clear. In this paper, we investigate a relationship between nondegeneracy of certain simple abelian variety with many endomorphisms over $\mathbb{Q}^{\text{alg}}$ and of its reduction. The following theorem is our main result.

**Theorem 0.1.** Let $A$ be a simple abelian variety with many endomorphisms over $\mathbb{Q}^{\text{alg}}$.

1. Assume that the CM-field $\text{End}^0(A)$ is a abelian extension of $\mathbb{Q}$. If all powers of $A$ are nondegenerate, then for any prime $w$ of $\mathbb{Q}^{\text{alg}}$, all powers of a simple factor of the reduction of $A$ at $w$ are nondegenerate.

2. Let $w$ be a prime of $\mathbb{Q}^{\text{alg}}$. Let $A_0$ be the reduction of $A$ at $w$. Assume that the restriction of $w$ to the Galois closure of the CM-field $\text{End}^0(A)$ is unramified over $\mathbb{Q}$ and its absolute degree is one.

   (a) If the Hodge conjecture holds for all powers of $A$, then the Tate conjecture holds for all powers of $A_0$.

   (b) All powers of $A$ are nondegenerate if and only if all powers of $A_0$ are nondegenerate.

Statement (2) of the theorem is almost a corollary of a result of Milne. We prove this theorem, using a result of Milne (Theorems [11] and [12]) and a necessary and sufficient condition for nondegeneracy (Theorem [13]). The key (Proposition [2.1]) is to compare the conditions of nondegeneracy over $\mathbb{C}$ and $F$ by a result of Shimura–Taniyama on the prime ideal decomposition of Frobenius endomorphism.
This paper is organized as follows: In section 1, we recall Milne’s results on the Hodge conjecture and the Tate conjecture. We also recall a necessary and sufficient condition for nondegeneracy of certain simple abelian varieties (Theorem 1.6, Theorem 1.8). In section 2, we prove a key proposition (Proposition 2.1) for our main result. Using the key proposition and results mentioned in section 1, we give a proof of Theorem 0.1. In the last section, using a result of Aoki we give an example of a degenerate simple abelian variety over \( \mathbb{F} \) for which the Tate conjecture holds.

**Notation.**

For an abelian variety \( A \) with many endomorphisms over an algebraically closed field \( k \), \( \text{End}^0(A) \) denotes \( \text{End}_k(A) \otimes \mathbb{Q} \), and \( C(A) \) denotes the center of \( \text{End}_k^0(A) \).

For a finite étale \( \mathbb{Q} \)-algebra \( E \), \( \Sigma_E := \text{Hom}(E, \mathbb{Q}^{\text{alg}}) \). If \( E \) is a field Galois over \( \mathbb{Q} \), we identify \( \Sigma_E \) with the Galois group \( \text{Gal}(E/\mathbb{Q}) \).

For a finite set \( S \), \( \mathbb{Z}^S \) denotes the set of functions \( f : S \to \mathbb{Z} \).

An affine algebraic group is of multiplicative type if it is commutative and its identity component is a torus. For such a group \( W \) over \( \mathbb{Q} \), \( \chi(W) := \text{Hom}(W_{\mathbb{Q}^{\text{alg}}}, \mathbb{G}_m) \) denotes the group of characters of \( W \).

For a finite étale \( \mathbb{Q} \)-algebra \( E \), \( (\mathbb{G}_m)_{E/\mathbb{Q}} \) denotes the Weil restriction \( \text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m) \) which is characterized by \( \chi((\mathbb{G}_m)_{E/\mathbb{Q}}) = \mathbb{Z}^{\Sigma_E} \).

\section{The Hodge conjecture and the Tate conjecture for abelian varieties}

We first recall a statement of conjectures.

**The Hodge conjecture.** Let \( A \) be an abelian variety of dimension \( g \) over \( \mathbb{C} \). By \( H^i_B(A, \mathbb{Q}) \) we denote the Betti cohomology of \( A \). For each integer \( i \) with \( 0 \leq i \leq g \), we define the space of the Hodge classes of degree \( i \) on \( A \) as follows:

\[
H^i_B(A, \mathbb{Q}) \cap H^i(A, \Omega^i).
\]

We know that the image of a cycle map is contained in the space of the Hodge classes. A Hodge class is said to be algebraic if it belongs to the image of cycle map.

**Conjecture.** All Hodge classes on \( A \) are algebraic.

By the Lefschetz–Hodge theorem, all the Hodge classes of degree one are generated by divisor classes. Therefore \( A \) is nondegenerate if and only if all the Hodge classes on \( A \) are generated by the Hodge classes of degree one.
The Tate conjecture. Let \( \mathbb{F}_p \) be a finite field with \( p \)-elements and let \( \mathbb{F} \) be the algebraic closure of \( \mathbb{F}_p \). Let \( A_1 \) be an abelian variety of dimension \( g \) over a finite subfield \( \mathbb{F}_q \) of \( \mathbb{F} \). Let \( A_0 \) be the abelian variety \( A_1 \otimes_{\mathbb{F}_q} \mathbb{F} \) over \( \mathbb{F} \). By \( H^{2i}(A_0, \mathbb{Q}_\ell(i)) \) we denote the \( \ell \)-adic étale cohomology group of \( A_0 \). For each integer \( i \) with \( 0 \leq i \leq g \), we define the space of the \( \ell \)-adic Tate classes of degree \( i \) on \( A_0 \) as follows:

\[
\lim_{L/\mathbb{F}_q} H^{2i}(A_0, \mathbb{Q}_\ell(i))^{\text{Gal}(\mathbb{F}/L)}.
\]

Here \( L/\mathbb{F}_q \) runs over all finite extensions of \( \mathbb{F}_q \). We know that the image of the \( \ell \)-adic étale cycle map is contained in the space of the Tate classes. A Tate class is said to be algebraic if it belongs to the image of the \( \ell \)-adic étale cycle map.

Conjecture. All Tate classes on \( A_0 \) are algebraic.

This is conjectured by Tate [19, Conjecture 1]. By a result of Tate [20], we know that for any abelian variety \( A_0 \), all the Tate classes of degree one are generated by divisor classes on \( A_0 \). Therefore \( A_0 \) is nondegenerate if and only if all the Tate classes on \( A_0 \) are generated by the Tate classes of degree one.

1.1 Necessary and sufficient condition

Let \( A \) be an abelian variety over an algebraically closed field \( k \) such that the reduced degree of \( \text{End}^0(A) \) is \( 2 \dim A \). In this case, \( A \) is said to have many endomorphisms. There are important algebraic groups of multiplicative type \( L(A), M(A), MT(A) \) and \( P(A) \) over \( \mathbb{Q} \) attached to \( A \). Using these groups, Milne gave a necessary and sufficient condition for the Hodge conjecture and the Tate conjecture for abelian varieties with many endomorphisms.

Theorem 1.1 (Milne [10, p. 14, Theorem]). (1) Let \( A \) be an abelian variety with many endomorphisms over an algebraically closed field \( k \) of characteristic zero. Then \( MT(A) \subset M(A) \subset L(A) \), and

(i) the Hodge conjecture holds for all powers of \( A \) if and only if \( MT(A) = M(A) \);

(ii) all powers of \( A \) are nondegenerate if and only if \( MT(A) = L(A) \).

(2) Let \( A_0 \) be an abelian variety over \( \mathbb{F} \). Then \( P(A_0) \subset M(A_0) \subset L(A_0) \), and

(i) the Tate conjecture holds for all powers of \( A_0 \) if and only if \( P(A_0) = M(A_0) \);
(ii) all powers of $A_0$ are nondegenerate if and only if $P(A_0) = L(A_0)$.

For the relationship between the Hodge conjecture and the Tate conjecture, Milne proved the following:

**Theorem 1.2** (Milne [10]). Let $A$ be an abelian variety with many endomorphisms over $\mathbb{Q}^{\text{alg}}$ and let $A_0$ be the reduction of $A$ at a prime of $\mathbb{Q}^{\text{alg}}$. If the Hodge conjecture holds for all powers of $A$ and

$$P(A_0) = L(A_0) \cap MT(A) \quad \text{(intersection inside $L(A)$)},$$

then the Tate conjecture holds for all powers of $A_0$.

In the rest of this subsection, we briefly recall the definitions of the groups $L$, $M$, $MT$ and $P$ associated to an abelian variety $A$ over $k$ (For more detail, see [8], [9] and [10]), and we recall a necessary and sufficient condition for nondegeneracy for certain simple abelian varieties (Theorem 1.6, Theorem 1.8).

Let $A$ be an abelian variety with many endomorphisms over an algebraically closed field $k$. Put $E := \text{End}^0(A)$. Let $C(A)$ be the center of $E$. A polarization $\lambda : A \to A^\vee$ of $A$ determines an involution of $E$ which stabilizes $C(A)$. The restriction of the involution to $C(A)$ is independent of the choice of $\lambda$. By $\dagger$, we denote this restriction to $C(A)$.

**Definition 1.3** ([8, 4.3, 4.4], [9, p. 52–53], [10, A.3]). The Lefschetz group $L(A)$ of $A$ is the algebraic group over $\mathbb{Q}$ such that

$$L(A)(R) = \{ \alpha \in (C(A) \otimes R)^\times \mid \alpha \alpha^\dagger \in R^\times \}$$

for all $\mathbb{Q}$-algebras $R$.

In case that $k = \mathbb{C}$, we can describe $L(A)$ as a subgroup of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ in terms of characters as follows ([10, A.7]): $L(A)$ is a subgroup of $(\mathbb{G}_m)_{E/\mathbb{Q}}$ whose character group is

$$\mathbb{Z}^{\Sigma_E}$$

$$\{ g \in \mathbb{Z}^{\Sigma_E} \mid g = \iota g \text{ and } \sum g(\sigma) = 0 \} .$$

Here $\iota g$ is a function sending an element $\sigma$ of $\Sigma_E$ to $g(\iota \sigma)$, and $\sum g(\sigma)$ denotes $\sum_{\sigma \in \Sigma_E} g(\sigma)$.

In case that $k = \mathbb{F}$, $L(A)$ is a subgroup of $(\mathbb{G}_m)_{C(A)/\mathbb{Q}}$ whose character group is

$$\mathbb{Z}^{\Sigma_{C(A)}}$$

$$\{ g \in \mathbb{Z}^{\Sigma_{C(A)}} \mid g = \iota g \text{ and } \sum g(\sigma) = 0 \}.$$
Definition 1.4. Jannsen [3] proved that the category of motives generated by abelian varieties over \( F \) with the algebraic cycles modulo numerical equivalence as the correspondences is Tannakian. The group \( M(A) \) is defined as the fundamental group of the Tannakian subcategory of this category generated by \( A \) and the Tate object.

Definition 1.5. When the characteristic of \( k \) is zero, the Mumford–Tate group \( MT(A) \) is defined to be the largest algebraic subgroup of \( L(A) \) fixing the Hodge classes on all powers of \( A \).

When \( A \) is simple and the characteristic of \( k \) is zero, we describe a condition for which a character of \( L(A) \) is trivial on \( MT(A) \). To give the condition, we introduced notion of CM-type.

Let \( E \) be a CM-algebra. A subset \( \Phi \) of \( \Sigma_E \) is called CM-type of \( E \) if \( \Sigma_E = \Phi \cup i\Phi \) and \( \Phi \cap i\Phi = \emptyset \). Here \( i \) is complex conjugation on \( \mathbb{C} \).

When \( E = \text{End}^0(A) \), the action of \( E \) on \( \Gamma(A, \Omega^1) \) defines a CM-type of \( E \).

Now assume that \( A \) is simple. Let \( \Phi \) be the CM-type of the CM-field \( \text{End}^0(A) \). A character \( g \) of \( L(A) \) is trivial on \( MT(A) \) if and only if

\[
\sum_{\sigma \in \Phi} g(\tau \circ \sigma) = 0
\]

for all \( \tau \in \text{Gal}(\mathbb{Q}^{\text{alg}}/\mathbb{Q}) \).

For nondegeneracy of certain abelian varieties, the following result is known:

Theorem 1.6. Let \( A \) be a simple abelian variety with many endomorphisms over \( \mathbb{Q}^{\text{alg}} \). Let \( \Phi \) be the CM-type of the CM-field \( E := \text{End}^0(A) \) defined by the action of \( E \) on \( \Gamma(A, \Omega^1) \). Assume that \( E \) is a abelian extension over \( \mathbb{Q} \) with its Galois group \( G \). Then all powers of \( A \) are nondegenerate if and only if

\[
\sum_{\sigma \in \Phi} \chi(\sigma) \neq 0
\]

for any character \( \chi \) of \( G \) such that \( \chi(i) = -1 \).

For a proof of the theorem, see [5].

Definition 1.7 ([9 §4], [10 A.7]). Let \( k \) be the algebraic closure \( \mathbb{F} \) of a finite filed \( \mathbb{F}_p \). Let \( A_1 \) be a model of \( A \) and let \( \pi_1 \) be the Frobenius endomorphism of \( A_1 \). Then the group \( P(A) \) is the smallest algebraic subgroup of \( L(A) \) containing some power of \( \pi_1 \). It is independent of the choice of \( A_1 \).
When $A/\mathbb{F}$ is simple, we describe a condition for which a character of $L(A)$ is trivial on $P(A)$. To give the condition, we introduce some notion about Weil numbers.

A **Weil $q$-number** of weight $i$ is an algebraic number $\alpha$ such that $q^{N}\alpha$ is an algebraic integer for some $N$ and the complex absolute value $|\sigma(\alpha)|$ is $q^{i/2}$, for all embeddings $\sigma : \mathbb{Q}[\alpha] \to \mathbb{C}$. We know that $\pi_1$ is a Weil $q$-number of weight one. Then $\pi_1$ is a unit at all primes of $\mathbb{Q}[\pi_1]$ not dividing $p$. We define the **slope function** $s_{\pi_1}$ of $\pi_1$ as follows: for any prime $p$ dividing $p$ of a field containing $\pi_1$,

$$s_{\pi_1}(p) = \frac{\text{ord}_p(\pi_1)}{\text{ord}_p(q)}.$$  \hfill (1.4)

The slope function determines a Weil $q$-number up to a root of unity. From the definition of Weil numbers, $s_{\pi_1}(p) + s_{\pi_1}(\iota p) = 1$.

We define a **Weil germ** to be an equivalent class of Weil numbers. For a Weil germ $\pi$, the slope function of $\pi$ are the slope function (see (1.4)) of any representative of $\pi$.

Now assume that $A/\mathbb{F}$ is simple. Let $\pi_A$ denote the germ represented by $\pi_1$. Milne’s result on the character of $P(A)$ is the following ([10, A.7]): let $g$ be a character of $L(A)$. Then $g$ is trivial on $P(A)$ if and only if

$$\sum_{\sigma \in \Sigma_{C(A)}} g(\sigma) s_{\pi_A}(p) = 0$$  \hfill (1.5)

for all primes $p$ dividing $p$ of a field containing all conjugates $\sigma(\pi_1)$.

Using Theorem 1.1 and (1.2) (1.5), we obtain the following:

**Theorem 1.8** ([18]). Let $A_0$ be a simple abelian variety over $\mathbb{F}$. Assume that $C(A_0)$ is abelian extension of $\mathbb{Q}$ with its Galois group $G_0$. Let $p$ be a prime of $C(A_0)$ dividing $p$. Then any power of $A_0$ is nondegenerate if and only if

$$\sum_{\sigma \in G_0} s_{\pi}(\sigma p) \chi(\sigma) \neq 0$$

for any character $\chi$ of $G_0$ such that $\chi(\iota) = -1$.

This is an analogous result to Theorem 1.6 for simple abelian varieties over $\mathbb{F}$.

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2Let $\pi$ be a Weil $p^{\ell}$-number and let $\pi'$ be a Weil $p^{\ell'}$-number. We say $\pi$ and $\pi'$ are **equivalent** if $\pi' = \pi^{\ell} \cdot \zeta$ for some root of unity $\zeta$. 
2 Proof of the main theorem

We prove Theorem 0.1 using the theorems mention in the previous section. We first fix the notation:

\[ A \] a simple abelian variety with many endomorphisms over \( \mathbb{Q}^{\text{alg}} \)
\[ w \] a prime of \( \mathbb{Q}^{\text{alg}} \) dividing \( p \)
\[ A_0 \] a simple factor of the reduction of \( A \) at \( w \)
\[ E \] the CM-field \( \text{End}^0(A) \)
\[ \Phi \] the CM-type of \( E \)
\[ \varphi \] the characteristic function of \( \Phi \)
\[ E_0 \] the center of \( \text{End}^0(A_0) \) \( \quad \) (\( E_0 \) is a subfield of \( E \))
\[ \pi \] the Weil germ attached to \( A_0 \)
\[ K/\mathbb{Q} \] a finite Galois extension which include all conjugate of \( E \)
\[ G := \text{Gal}(K/\mathbb{Q}) \]
\[ p \] the restriction of \( w \) to \( K \)
\[ G_p \] the decomposition group of \( p \) in \( K \)

The following proposition is a key in a proof of our main result.

**Proposition 2.1.** Let the notation as above. Then for any \( \sigma \in \Sigma_{E_0} \) and any \( \tau \in G \),

\[ s_{\sigma \pi}(\tau p) = \frac{1}{|G_p|} \sum_{h \in G_p} \varphi(h \tau^{-1} \circ \sigma). \]

**Proof.** We identify \( \Sigma_E \) with \( \text{Hom}_\mathbb{Q}(E, K) \). Let \( \sigma \in \Sigma_{E_0} \) and \( \tau \in G \). Let \( \tilde{\sigma} \in G \) be a lift of \( \sigma \). Since \( E_0 \) is equal to the smallest subfield of \( \mathbb{Q}^{\text{alg}} \) containing a representative of \( \pi \), we have \( s_{\sigma \pi}(\tau p) = s_{\tilde{\sigma} \pi}(\tau p) \). Therefore we may fix the lift \( \tilde{\sigma} \in G \) for each \( \sigma \in \Sigma_{E_0} \). Then we have \( s_{\tilde{\sigma} \pi}(\tau p) = s_{\pi}(\tilde{\sigma}^{-1} \tau p) \).

By a theorem of Shimura–Taniyama (see Tate [21, Lemma 5]), \( s_{\pi}(\tilde{\sigma}^{-1} \tau p) \) is given as follows

\[ s_{\pi}(\tilde{\sigma}^{-1} \tau p) = \frac{|\Phi(\tilde{\sigma}^{-1} \tau p)|}{|\Sigma_E(\tilde{\sigma}^{-1} \tau p)|} \]

where

\[ \Sigma_E(\tilde{\sigma}^{-1} \tau p) := \{ f \in \Sigma_E \mid \tilde{\sigma}^{-1} \tau p = f^{-1} p \ldots (\ast) \} \]
\[ \Phi(\tilde{\sigma}^{-1} \tau p) := \Phi \cap \Sigma_E(\tilde{\sigma}^{-1} \tau p) \].
Here \((\ast)\) means that for any \(x \in E\),
\[
v_{\pi}(\tilde{\sigma}(x)) = v_{\pi}(f(x)).
\]

Now we consider condition \((\ast)\). Let \(\tilde{f} \in G\) be a lift of \(f\). Then we have the following equivalences

\[
\begin{align*}
\text{condition } (\ast) & \iff \tilde{\sigma}^{-1}\tau p \text{ and } \tilde{f}^{-1} p \text{ lie over the same prime of } E \\
& \iff \tilde{\sigma}^{-1}\tau p = \eta \tilde{f}^{-1} p \text{ for some } \eta \in \text{Gal}(K/E) \\
& \iff \tilde{f} \eta^{-1} \tilde{\sigma}^{-1}\tau \in G_p \text{ for some } \eta \in \text{Gal}(K/E) \\
& \iff \tilde{f} \in G_p \tilde{\sigma}^{-1}\pi \text{Gal}(K/E)
\end{align*}
\]

If \(\tilde{f} \in G_p \tilde{\sigma} \text{Gal}(K/E)\), then any lifts of \(f\) are also in \(G_p \tilde{\sigma} \text{Gal}(K/E)\). Hence the property that \(\tilde{f}\) belongs to \(G_p \tilde{\sigma} \text{Gal}(K/E)\) is independent of the choice of the lift of \(f\).

For \(h_1, h_2 \in G_p\) and \(\eta_1, \eta_2 \in \text{Gal}(K/E)\), we have
\[
(h_1 \tau^{-1} \tilde{\sigma} \eta_1)^{-1} (h_2 \tau^{-1} \tilde{\sigma} \eta_2) \in \text{Gal}(K/E) \iff \tilde{\sigma}^{-1}\tau h_1^{-1} h_2 \tau^{-1} \tilde{\sigma} \in \text{Gal}(K/E) \iff h_1^{-1} h_2 \in \text{Gal}(K/\tau^{-1} \tilde{\sigma}(E)).
\]

From the above argument, we have
\[
|\Sigma_E(\tilde{\sigma}^{-1}\tau p)| = |G_p/\text{Gal}(K/\tau^{-1} \tilde{\sigma}(E))|
= \frac{|G_p|}{|G_p \cap \text{Gal}(K/\tau^{-1} \tilde{\sigma}(E))|}
\]

Next we calculate \(|\Phi(\tilde{\sigma}^{-1}\tau p)|\). Since \(h \tau^{-1} \tilde{\sigma}(x) = \tau^{-1} \tilde{\sigma}(x)\) for any \(h \in \text{Gal}(K/\tau \tilde{\sigma}(E))\) and for any \(x \in E\), we have
\[
\varphi(h \tau^{-1} \circ \sigma) = \varphi(\tau^{-1} \circ \sigma).
\]

Therefore we have
\[
|\Phi(\tilde{\sigma}^{-1}\tau p)| = \frac{1}{|G_p \cap \text{Gal}(K/\tau^{-1} \tilde{\sigma}(E))|} \sum_{h \in G_p} \varphi(h \tau^{-1} \circ \sigma).
\]

Hence we have
\[
s_{\sigma \pi}(\tau p) = \frac{1}{|G_p|} \sum_{h \in G_p} \varphi(h \tau^{-1} \circ \sigma).
\]
Before starting the proof of (1) of Theorem 0.1, we prepare some notation. By the assumption that $E$ is abelian over $\mathbb{Q}$, we may take $K = E$. We write $G$ for the Galois group of $E/\mathbb{Q}$ and $G_0$ for the Galois group of $E_0/\mathbb{Q}$. We identify $\Sigma_E$ (resp. $\Sigma_{E_0}$) with $G$ (resp. $G_0$). Then there is an exact sequence of finite abelian groups

$$1 \rightarrow G_1 \rightarrow G \rightarrow G_0 \rightarrow 1,$$

where $G_1 = \text{Gal}(E/E_0)$.

We define the subgroup $\hat{G}^-$ of the character group of $G$ as follows:

$$\hat{G}^- := \{ \chi : G \rightarrow \mathbb{C}^\times \mid \chi(\iota) = -1 \}.$$

Here $\iota \in G$ is the complex conjugation. Similarly to $\hat{G}^-$, we define the subgroup $\hat{G}_0^-$ of the character group of $G_0$. Since $G_0$ is a quotient group of $G$, we consider $\hat{G}_0^-$ as the subgroup of $\hat{G}^-:

$$\hat{G}_0^- = \{ \chi \in \hat{G}^- \mid \chi(G_1) = 1 \}.$$

**Remark 2.2.** Since $p$ is completely decomposed in $E_0$ (cf. [18, Proposition 3.5]), the decomposition group $G_p$ of $p$ is contained in $G_1$. If $p$ is unramified and its absolute degree is one, then $E = E_0$ and hence $G = G_0$.

**Proof of (1) of Theorem 0.1.** Let $p_0$ be the prime $p \cap E_0$ of $E_0$. By Proposition 2.1 for any $\chi \in \hat{G}_0^-$ we have

$$\sum_{\sigma \in G_0} s_\pi(\sigma p_0) \chi(\sigma) = \frac{1}{|G_1|} \sum_{\sigma \in G} s_\pi(\sigma p) \chi(\sigma)
= \frac{1}{|G_1|} \sum_{\sigma \in G} \frac{1}{|G_p|} \sum_{h \in G_p} \varphi(h \sigma^{-1}) \chi(\sigma)
= \frac{1}{|G_1|} \cdot \frac{|G_p|}{|G_1|} \sum_{h \in G_p} \sum_{\sigma \in G} \varphi(h \sigma^{-1}) \chi(\sigma)
= \frac{1}{|G_1|} \sum_{\sigma \in G} \varphi(\sigma^{-1}) \chi(\sigma)
= \frac{1}{|G_1|} \sum_{\sigma \in \Phi} \tilde{\chi}(\sigma).$$

From this, we obtain that for any $\chi \in \hat{G}_0^- \subset \hat{G}^-$,

$$\sum_{\sigma \in G_0} s_\pi(\sigma p_0) \chi(\sigma) \neq 0 \text{ if and only if } \sum_{\sigma \in \Phi} \tilde{\chi}(\sigma) \neq 0.$$

Therefore the assertion follows from Theorem 1.6 and Theorem 1.8. \qed
Proof of (2) of Theorem 0.1. To prove the assertion, by Theorem 1.1 and Theorem 1.2 it suffices to show that $L(A) = L(A_0)$ and $MT(A) = P(A_0)$.

We first show that $L(A) = L(A_0)$. By the assumption that $p$ is unramified and its absolute degree is one, we obtain that $E = E_0$ from a result of Shimura–Taniyama [14, p. 100, Theorem 2]. By the description (1.1) (1.2) of the character group of $L(A)$ and $L(A_0)$, we have $L(A) = L(A_0)$.

Next we show that $MT(A) = P(A_0)$. We easily see that the condition (1.3) of triviality on $MT(A)$ of a character of $L(A)$ is described in terms of the characteristic function $\varphi$ of the CM-type Φ as follows: for all $\tau \in \text{Gal}(Q_{\text{alg}}/Q)$,

$$\sum_{\sigma \in \Sigma_E} \varphi(\tau^{-1} \circ \sigma) g(\sigma) = 0.$$  \hfill (2.1)

On the other hand, the assumption on $p$ implies that $G_p = 1$. Therefore, by Proposition 2.1, we obtain that for all $\tau \in \text{Gal}(Q_{\text{alg}}/Q)$,

$$s_{\sigma \pi}(\tau p) = \varphi(\tau^{-1} \circ \sigma).$$

From this equation and the equality $E = E_0$, the condition (1.3) of triviality on $P(A_0)$ of a character of $L(A)$ is described as follows: for all $\tau \in \text{Gal}(Q_{\text{alg}}/Q)$,

$$\sum_{\sigma \in \Sigma_{E_0}} g(\sigma) s_{\sigma \pi}(\tau p) = \sum_{\sigma \in \Sigma_E} \varphi(\tau^{-1} \circ \sigma) g(\sigma) = 0.$$  \hfill (2.2)

Since $L(A) = L(A_0)$ and conditions (2.1) (2.2) are coincide, we obtain that $MT(A) = P(A_0)$. This completes the proof. \qed

3 Example

From Theorem 1.1 and a result of Aoki [1] on CM abelian varieties of Fermat type, we obtain examples of a simple degenerate abelian variety $A_0$ over $\mathbb{F}$ for which the Tate conjecture holds. Here we give a such example.

Let $m = 27$ and let $\alpha = (1, 9, 17)$. Here $\alpha$ is an element of the set $A^1_m$ defined as follows:

$$A^1_m := \{ \alpha = (a_0, a_1, a_2) \in (\mathbb{Z}/m\mathbb{Z})^3 \mid a_i \not\equiv 0 \pmod{m}, a_0 + a_1 + a_2 \equiv 0 \pmod{m} \}.$$

We define a subset $\Phi_\alpha$ of $\mathbb{Z}/m\mathbb{Z}$ as

$$\Phi_\alpha := \{ t \in \mathbb{Z}/m\mathbb{Z} \mid \langle ta_0 \rangle + \langle ta_1 \rangle + \langle ta_2 \rangle = m \}$$

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where for any \( c \in \mathbb{Z}/m\mathbb{Z} \) we denote by \( \langle c \rangle \) the least natural number such that \( \langle c \rangle \equiv c \mod m \). Then let \( A = A_\alpha \) be a simple abelian variety with CM-type \((\mathbb{Q}(\mu_m), \Phi_\alpha)\). Then by a result of Aoki [1, Theorem 2.1], \( A \) is degenerate and the Hodge conjecture holds for all powers of \( A \).

On the other hand, let \( A_0 \) be a simple factor of the reduction of \( A \) at a prime \( w \) of \( \mathbb{Q}_{\text{alg}} \) dividing a prime \( p \). By Theorem 0.1 (2), we see that if \( p \equiv 1 \mod m \), then \( A_0 \) is degenerate and the Tate conjecture holds for all powers of \( A_0 \). Furthermore, using Theorem 1.8, one can see that if \( p^9 \equiv 1 \mod m \), then all powers of \( A_0 \) are nondegenerate.

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