A Connection between Fermionic Strings and Quantum Gravity States – A Loop Space Approach

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Abstract

We present physical arguments-based on Loop Space representations for Dirac/Klein Gordon determinants (Triviality quantum decoherence of quantum chromodynamics in the presence of an external strong white-noise electromagnetic field DQ8944 - Phys. Rev. D) that some suitable Fermionic String Ising Models at the critical point and defined on the space-time base manifold $M \subset \mathbb{R}^3$ are formal quantum states of the gravitational field when quantized in the Ashtekar-Sen connection canonical formalism. These results complements our previous Loop Space studies on the subject (Random surface representation for Einstein quantum gravity Phys. Rev. 52D, 6941(1995))

1 Introduction

The dynamical formulation of Einstein General Relativity in terms of a new set of complex $SU(2)$ coordinates has opened new perspectives in the general problem of quantization of the gravitation field by non-perturbative means. The new set of dynamical variables proposed by Ashtekar are the projection of the tetrads (the so called triads) on the three-dimensional base manifold $M$ of our cylindrical space-time $M \times \mathbb{R}$ added with the four-dimensional spin connection for the left-handed spinor again restricted to the embedded space-time base manifold $M$ (the Ashtekar-Sen $SU(2)$ connections) [1] and paralleling successful procedure used to quantize canonically pure three-dimensional gravity [2].

The fundamental result obtained with this approach is related to the fact that it is possible to canonically quantizes the Einstein classical action in the same way one canonically quantizes others quantum fields [3]. As a consequence, the governing Schrödinger-Wheeler-Dewitt dynamical equations which emerges in such gravity gauge field parametrization supports exactly highly non-trivial prospective explicitely (regularized) functional solutions [4].

In this paper we intend to present in Section 7.3 a Loop Space-Path integral supporting the fact that the formal continuum limit of a $3D$ Ising model, a Quantum Fermionic String
on the space-time base manifold $M$, is a (formal operatorial) solution of the Wheeler-De Witt equation in the above mentioned Ashtekar-Sen parametrization of the Gravitation Einstein field. We present too a propose of ours on a Loop geometrodynamical representation for a kind of $\lambda \phi^4$ third-quantized geometrodynamical field theory of Einstein Gravitation in terms of Ashtekar-Sen gauge fields.

2 The Loop Space Approach for Quantum Gravity

Let us start our analysis by writing the governing wave equations in the following operatorial ordered form [5].

\[
\hat{C}[A]\psi[A] = \delta^{jk} \frac{\delta^2}{\delta A^i_\mu(x) \delta A^j_\nu(x)} \left\{ [F^k_{\mu\nu}(A(x))\psi[A]] \right\} = 0 ; \quad (1)
\]

\[
\hat{C}_\mu[A]\psi[A] = \frac{\delta}{\delta A^\xi_\mu(x)} \left\{ F^\xi_{\mu\nu}(A(x))\psi[A] \right\} = 0 ; \quad (2)
\]

\[
Q_\mu = D_i \left\{ \frac{\delta}{\delta A^i_\mu(x)} \psi[A] \right\} = 0 ; \quad (3)
\]

where we have considered in the usual operatorial-functional derivative form the Hamiltonian, diffeomorphism and Gauss law constraints respectively implemented in a functional space of quantum gravitational states formed by wave functions $\psi[A]$ [1].

At this point we come to the usefulness of possessing linear-functional field equations by considering explicitly functional solutions for the set eq.(7.1)-eq.(7.3).

Let us therefore, consider the space of bosonic loops with a marked point $x \in M$ and the associated Gauge invariant Wilson Loop defined by a given Ashtekar gauge field configuration $A^i_\mu(x)$.

\[
W[C_{xx}] = Tr \left( P_{SU(2)} \left\{ \exp i \oint_{C_{xx}} A^i_\mu(X(\sigma)dX_\mu(\sigma) \right\} \right) ; \quad (4)
\]

here the bosonic loop $C_{xx}$ is explicitly parametrized by a continuous (in general everywhere non-differentiable) periodic function $X_\mu(\sigma) = X_\mu(\sigma + T)$ and such that $X_\mu(T) = X_\mu(0) = x_\mu$ (see refs.[6]–[7]).

Following refs [4], one shows that eq.(7.4) satisfies the diffeomorphism constraint, namely

\[
\left( \frac{\delta}{\delta A^\xi_\mu(x)} \left[ \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \varepsilon^{irs} A^r_\nu A^s_\mu \right](x) \right) W[C_{xx}]
\]

\[
+ F^i_{\mu\nu}(A(x)) \left( \frac{\delta}{\delta A^\xi_\mu(x)} W[C_{xx}] \right)
\]

\[
= 2 \times \left[ (\partial_\mu^x + \varepsilon^{irs} \delta^s \mu \partial_\nu A^r_\mu(x)) W[C_{xx}] \right.
\]

\[
+ P \left\{ F^i_{\mu\nu}(A(X(0)) \cdot X^\nu(0)W[C_{x(0)x(0)}] \right\} \]

\[
= 2 \times \left( \partial_\mu^x + \frac{\delta}{\delta X_\mu(0)} \right) W[C_{xx}] = 0 ; \quad (5)
\]
where we have the Migdal usual derivative relation for the marked Wilson Loop – note that
the loop orientability is responsible for the minus signal on the Wilson Loop marked point
derivative [6].

\[- \partial^x\mu \mathcal{W}[C_{xx}] = \lim_{\sigma \to 0} \left\{ \frac{\delta}{\delta X_\mu(\sigma)} \mathcal{W}[C_{xx}] \right\} \; ; \quad (6)\]

If one had used the usual Smolin factor ordering as given below instead of that of Gambini-
Pullin eq.(7.1)-eq.(7.3), one could not satisfy in a straightforward way the diffeomorphism
constraint

\[
F^i_{\mu\nu}(A(x)) \frac{\delta}{\delta A^i_\mu(x)} \mathcal{W}[C_{xx}]
= P \left\{ F^i_{\mu\nu}(A(X(0))) X^\nu(0) \mathcal{W}[C_{xx}] \right\}
= P \left\{ \bar{X}\mu(0) \mathcal{W}[C_{xx}] \right\} \neq 0 ; \quad (7)
\]

Note that we have assumed the validity of the Lorentz dynamical equation for the loops
\(X_\mu(\sigma) \ (0 \leq \sigma \leq T)\) on the last line of eq.(7.7).

Also, the Schrödinger-Wheeler-De Witt equation is solved by the marked point Wilson Loop
within the same functional derivative procedure. Firstly, we note that the Smolin and Gambini-
Pullin operator ordering coincides in the realm of the Wheeler-DeWitt equation. Namely:

\[
\varepsilon^{ijk} \frac{\delta^2}{\delta A^i_\mu(x) \delta A^j_\nu(x)} \left\{ F^k_{\mu\nu}(A(x)) \mathcal{W}[C_{xx}] \right\}
= \left( \varepsilon^{ijk} \frac{\delta^2}{\delta A^i_\mu(x) \delta A^j_\nu(x)} F^k_{\mu\nu}(A(x)) \right) \mathcal{W}[C_{xx}]
+ \varepsilon^{ijk} \left( \frac{\delta}{\delta A^i_\mu(x)} F^k_{\mu\nu}(A(x)) \right) \left( \frac{\delta \mathcal{W}[C_{xx}]}{\delta A^j_\nu(x)} \right)
+ \varepsilon^{ijk} \left( \frac{\delta}{\delta A^j_\nu(x)} F^k_{\mu\nu}(A(x)) \right) \left( \frac{\delta \mathcal{W}[C_{xx}]}{\delta A^i_\mu(x)} \right)
+ \varepsilon^{ijk} F^k_{\mu\nu}(A(x)) \left( \frac{\delta^2}{\delta A^i_\mu(x) \delta A^j_\nu(x)} \mathcal{W}[C_{xx}] \right)
= 0 + 0 + 0 + \varepsilon^{ijk} F^k_{\mu\nu}(A(x)) \frac{\delta^2 \mathcal{W}[C_{xx}]}{\delta A^i_\mu(x) \delta A^j_\nu(x)} ; \quad (8)
\]

An important step should be implemented at this point of our analysis and related to a loop
regularization process. We propose to consider a weak form of the Wheeler-DeWitt operatorial
equation as expressed below

\[
\hat{C}(\varepsilon)[A] = \int_M dx dy \delta(\varepsilon)(x - y) \varepsilon^{ijk} F^k_{\mu\nu}(A(x)) \left( \frac{\delta^2}{\delta A^i_\mu(x) \delta A^j_\nu(y)} \right) ; \quad (9)
\]
here $\delta(x-y)$ is a $C^\infty(M)$ regularization of the delta function on the space-time base manifold $M$. Rigorously, one should consider eq.(7.9) in each local chart of $M$ with the usual induced volume associated to the flat metric of $R^4$. Note that the validity of eq.(7.9) (at least locally) comes from the supposed cylindrical topology of our (Euclidean) space-time.

Proceeding as usual one gets the following result

$$
\hat{C}(\varepsilon)[A]W[C_{xx}]
= \int_M dx \int_M dy \delta(\varepsilon)(x-y) \left\{ \oint_{C_{xx}} \delta(x - X(\sigma))dX^\mu(\sigma) \oint_{C_{xx}} \delta(y - X(\sigma'))dX^\nu(\sigma') \right.
\times \text{Tr}_{SU(2)} P \left\{ F_{\mu\nu}^k(A(X(\sigma)))\varepsilon^{ijk}W[C_X(0),X(\sigma')] \right.$$}

$$
\left. \lambda^i W[C_{X(\sigma')}X(\sigma')]\lambda^j W[C_X(\sigma)X(T)] \right\} 
= \oint_{C_{xx}} \oint_{C_{xx}} \delta(\varepsilon)(X(\sigma) - X(\sigma'))dX^\mu(\sigma)dX^\nu(\sigma')
\times \text{Tr}_{SU(2)} P \left\{ F_{\mu\nu}^k(A(X(\sigma)))\varepsilon^{ijk}W[C_X(0),X(\sigma')] \right.$$}

$$
\left. \lambda^i W[C_{X(\sigma')}X(\sigma')]\lambda^j W[C_X(\sigma)X(T)] \right\} = 0 ;
$$

As a consequence, one should expect that the cut-off removing $\varepsilon \to 0$ will not be a difficult technical problem in the case of everywhere self-intersecting Brownian loops $C_{xx}$ [7]. Note that in the case of trivial self-intersections $\sigma = \sigma'$, the validity of eq.(7.10) comes directly from the fact that $dX^\mu(\sigma)dX^\nu(\sigma)$ is a symmetric tensor on the spatial indexes $(\mu, \nu)$ and $F_{\mu\nu}(A(X(\sigma))$ is an antisymmetric tensor with respect to these same indexes. In the general case of smooth paths with non-trivial self-intersection [1], one should makes the loop restrictive hypothesis of the $(\mu, \nu)$ symmetry of the complete bosonic loop space object $\delta(\varepsilon)(X(\sigma)-X(\sigma'))dX^\mu(\sigma)dX^\nu(\sigma')$ [8,9], otherwise we can not obviously satisfy the Wheeler-DeWitt equation – a common non-trivial fact in the Literature of Wilson Loop as formal quantum states defined by smooth $C^\infty$ – differentiable paths! [1].

At this point we remark that all the governing equations of the theory eq(7.1)-eq(7.3) are linear. As a consequence one can sum up over all closed Brownian loops $C_{xx}$ (with a fixed back-ground metric) in the following way (see second ref. on [6]).

$$
\Omega[A^\mu_i] = - \int_M d^3 x_\mu \int_0^\infty \frac{dT}{T} \int_{X_\mu(0)=x_{\mu}(T)=x_{\mu}} D^F [X_\mu(\sigma)] e^{-\frac{T}{2} \int_0^T (X_\mu(\sigma))^2 d\sigma} W[C_{xx}]
= \lg \det[\nabla_A \nabla_A^*] ;
$$

where one can see naturally the appearance of the functional determinant of Gauged-Klein-Gordon operator as a result of this loop sum.

At this point we introduce Fermionic Loops – an alternative procedure –, which do not have non-trivial spatial self-intersections on $R^3$ – and representing now closed path – trajectories of $SU(2)$ Fermionic particles on the Wilson Loop eq.(7.4) [8].

Here the Fermionic closed loop $C_{xx}^F$ is described by a fermionic (Grassmanian) vector position $X_{\mu}^{(F)}(\sigma, \theta) = X_\mu(\sigma) + i \theta \psi_\mu(\sigma)$, with $X_\mu(\sigma)$ the ordinary periodic (bosonic) position coordinate
and $\psi_\mu(\sigma)$ Grassman variables associated to intrinsic spin loop coordinates. The Fermionic Gauge-Invariant Wilson Loop is given as

$$W[X_\mu^{(F)}(\sigma, \theta)] = \Tr_{SU(2)} \left\{ P \left[ \exp \left( \int_0^T d\sigma \int d\theta A_\mu(X_\mu^{(F)}(\sigma, \theta)) \left( \frac{\partial}{\partial \sigma} + i\theta \frac{\partial}{\partial \sigma} \right) X_\mu^{(F)}(\sigma, \theta) \right) \right] \right\};$$

we get as a result the following expression:

$$\hat{C}[^{(\varepsilon)}_i][A] W[C^{F}_{xx}] = \oint C^{F}_{xx} d\sigma d\theta \oint C^{F}_{xx} d\sigma' d\theta' \Tr_{SU(2)} P \left\{ F^{\mu}_{\nu} \left( A(X^{(F)}(\sigma, \theta)) \right) W \left[ C^{F}_{X(0)X(\sigma')} \right] \lambda W \left[ C^{F}_{X(\sigma')X(T)} \right] \right\};$$

By proceeding analogously as in the bosonic loop case, we obtain as a (formal) operatorial quantum state of Gravity, the functional determinant of the Dirac Operator on $M$ (with a fixed back-ground metric associated to the embedding of $M$ on $R^4$ which is not relevant in our study!) as another formal Einstein gravitation quantum state to be used in the analysis which follows [11], an important result by itself.

$$\Omega[A^i_\mu(x)] = \lg \det[D(A) D^*(A)];$$

We note that the others constraints eq.(7.2)-eq.(7.3) are satisfied in a straightforward manner in the same way one verifies them for the Bosonic Loop case [(eq.(7.5)-eq.(7.6)) and note the explicitly gauge invariance of the Fermionic Wilson Loop [8]].

Let us present our proposed Loop Space argument that one can obtain the continuum version of Ising models on $M$ from the quantum gravity state $3D$ fermionic determinant eq.(7.12).

In order to see this formal connection let us consider an ensemble of continuous surfaces $\Sigma$ on $M$ and the restriction of the Ashtekar-Sen $SU(2)$ connection to each surface $\Sigma$. Since the Ashtekar-Sen connection is the $M$-restriction of the four-dimensional left-handed spin connection, one can see that the $\Sigma$-restricted quantum gravity state can be re-written as a fermionic path-integral of covariant two-dimensional fermions now defined on the surface $\Sigma$, namely (see section 7.2)

$$(Z^{(n)}[A^i_\mu]) = \exp \left( \hat{\Omega}_\Sigma[A^i_\mu(x)] \right) = \int \frac{d^{(cov)}[\Sigma^\mu(\xi, \sigma)] d^{(cov)}[\psi^{(n)}(\xi, \sigma)\Sigma^\mu(\xi, \sigma)]}{2\pi \alpha'} \exp \left( -\frac{1}{2} \int d\xi d\sigma \left( \sqrt{g} \delta_{ab} \partial_a \Sigma^\mu(\xi, \sigma) \right) \right) \times \exp \left( -\frac{1}{2} \int d\xi d\sigma \left( \sqrt{g} \psi^{(n)}(\gamma \tilde{\nabla}_a) \psi^{(n)}(\xi, \sigma) \right) \right);$$

The main point of our argument on the connection of the string theory eq.(7.13) and the Ising model on $M$ is basically related to the fact that the two-dimensional spin connection on
the 2D-fermionic action eq.(7.13) is exactly given by the restriction of the four-dimensional spin connection to the surface \(\Sigma\) or – in an equivalently geometrical way – the restriction of the three-dimensional Ashtekar-Sen connection to the surface \(\Sigma\)!

Let us now give a Loop Space argument that the string theory eq.(7.13) represents a 3D Ising model at a formal replica limit on the geometrical fermionic degrees \(\{\psi_\mu(n), \bar{\psi}_\mu(n)\}_{1 \leq n \leq N}\). This can easily be seen by integrating out these geometrical fermion fields, writing the resulting surface two-dimensional determinant in terms of closed loops \(\{C_L(t), L = 1, 2; C_L(t) \in \Sigma\}\) on the string world-sheet \(\Sigma\) by using the replica limit together with a surface proper-time representation for 2D fermion determinant \[9\]

\[
\lim_{N \to 0} \left( \frac{Z^{(N)}[A_\mu] - 1}{N} \right) = \log \det[\tilde{\nabla}_a]
\]

\[
= \sum_{\{C_L(t)\}} \left[ \text{Tr}_{SU(2)} \exp \left( i \int_{C_L} \omega_L(C^L)dC^L \right) \right]; \quad (14)
\]

At this point one verifies that the Wilson Loop on the string surface as given by eq.(7.14) and defined by the two-dimensional spin connection \(\omega_L\) coincides with the Ising model sign factor of Sedrakyan and Kavalov \[9\] which is expected to underlying the continuum string representation of the partition functional of the three-dimensional Ising model on a regular lattice in \(R^3\) at the critical point, namely

\[
Z_{\text{ising}}[\beta \to \beta_{\text{crit}}] =
\lim_{\beta \to \beta_{\text{crit}}} \left\{ (cosh \beta)^N \sum_{\{\Sigma\}CZ^3} \left\{ \exp \left[ -A(\tilde{\Sigma}) \ln \left( \frac{1}{\tanh \beta} \right) \right] \right\} \Phi[\tilde{C} (\tilde{\Sigma})] \right\}; \quad (15)
\]

where the sum in the above written equation is defined over the set of all closed two-dimensional lattice surfaces \(\tilde{\Sigma}CZ^3\) with a weight given by the (lattice) area of \(\tilde{\Sigma}\); \(N\) is the number of the plaquettes, \(\beta = J/kT\) denotes the ratio of the Ising hope parameter and the temperature. The presence of the Ising weight \(\Phi[\tilde{C} (\tilde{\Sigma})]\) inside the partition functional expression eq.(7.15) is the well-known sign factor defined on the manifold of the lines of self-intersection \(\tilde{C}(\tilde{\Sigma})\) appearing on the surface \(\tilde{\Sigma}\) with the explicitly Polyakov-Sedrakyan-Kavalov expression \(\Phi[\tilde{C}(\tilde{\Sigma})] = \exp\{i\pi \text{length}[\tilde{C}(\tilde{\Sigma})]\}\).

As a consequence of the above made remarks, one can see that at the replica limit of \(N \to 0\) eq.(7.14) should be expected to coincide at the critical point of the partition functional eq.(7.15), since the phase factor inside eq.(7.14) is the continuum version of the Ising model factor \(\Phi[\tilde{C}(\tilde{\Sigma})]\) \[9\].

This completes the exposition of our Loop Space argument that critical Ising models on \(M\) may be relevant quantum states to understand the new physics of quantum gravity when parametrized by the Gauge field-like connections of Ashtekar-Sen.

All the above made analysis would be a mathematical rigorous proof if one had a mathematical result that Fermionic Loops (Grassmanian Wiener Trajectories) do not have non-trivial space-time self-intersections on eq.(7.11c).
On the other hand this formal mathematical fact about the non existence of non-trivial self-intersection fermionic paths that leads naturally to the triviality of the Thirring model (a “$\lambda \varphi^4$” – Fermion Field Theory!) in space-times with dimension greater than 2. Finally, let us comment that it is expected that the Ashtekar-Sen connections defining the above studied quantum gravity states are distributional objects with a functional measure given by a $\sigma$-model like path integral with a scalar intrinsic field $E(x, t)$ on $M \times \mathbb{R}$, the geometrodynamical analogous of the $\sigma$-dimensional manifold particle covariant Brink-Howe-Polyakov path integral, namely

$$
d\mu[A_i] = \prod_{x \in M} \prod_{t \in [0, \infty]} \left[d(A_i^j(x, t)d(E(x, t))) \right]$$

$$\times \exp \left\{ -\frac{1}{16\pi G} \int_0^\infty dt \int_M d^3x (E(x, t))^{-1} \right.$$

$$\times \left[ \left( \frac{\partial}{\partial t} A_{i,\mu} M^{\mu,\nu j}[A] \frac{\partial}{\partial t} A_{j,\nu} \right)(x, t) \right] \right\}$$

$$\times \exp \left\{ -\mu \int_0^\infty dt \int_M d^3x E(x, t) \right\} ; \quad (16)$$

where the invariant metric on the Wheeler-DeWitt super space of Ashtekar-Sen connections is given explicitly by

$$M^{\mu,\nu j}[A] = (b(A))^{-1} (J^{\mu j} - J^{\mu j} J^{\nu i})(A) ; \quad (17)$$

with

$$J^{\mu a}(A) = \frac{1}{2} \varepsilon^{\mu \alpha \rho} F_{\alpha \rho}^a(A) ; \quad (18)$$

and

$$b(A) = \det(J(A)^{\mu a}) ; \quad (19)$$

Work on the averaged, Wilson Loop eq.(7.4) with the functional path space measure eq.(7.16) – expected to be relevant to analyze the matter interaction with Quantum Gravity is presented in next section.

### 3 The Wheeler - De Witt Geometrodynamical Propagator

The starting point in Wheeler-De Witt Geometrodynamics is the Probability Amplitude for metrics propagation in a cylindrical Space Time $R^3 \times [0, T]$, the so called Wheeler Universe

$$G[g^{IN}; g^{OUT}] = \int_{3g^{IN}}^{3g^{OUT}} d\mu[h_{\mu\nu}] \exp[-S(h_{\mu\nu})] \quad (20)$$

where the integration over the four metrics Functional Space on the cylinder $R^3 \times [0, T]$ is implemented with the Boundary conditions that the metric field $h_{\mu\nu}(x, t)$ induces on the Cylinder
Boundaries the Classically Observed metrics $^3g^{IN}(x)$ and $^3g^{OUT}(x)$ respectively. The Covariant Functional measure averaged with the Einstein $S(h_{\mu\nu}) = \int_{R^3 \times [0,T]} d^3x dt (\sqrt{g} R(g))$ is given explicitly in ref.4.

Unfortunately the use of eq.(7.20) in terms of metrics variables is difficulted by the “Conformal Factor Problem” in the Euclidean Framework. In order to overcome such difficulty I follow section 7.2 by using from the begining, the Astekar Variables to describe the Gometrodynamical Propagation.

Let me thus, consider Einstein Gravitation Theory Parametrized by the $SU(2)$ Three-Dimensional Astekar - Sen connection $A^a_{\mu}(x, t)$ associated to the Projected Spin Connection on the Space - Time Three - Dimensional Boundaries.

\begin{align*}
A^a_{\mu, IN}(x) &= -i \omega^{0a}_{\mu}(x, 0) + \frac{1}{2} \epsilon^{a}_{bi} \omega^{bi}_{\mu}(x, 0) \\
A^a_{\mu, OUT}(x) &= -i \omega^{0a}_{\mu}(x, T) + \frac{1}{2} \epsilon^{a}_{bi} \omega^{bi}_{\mu}(x, T)
\end{align*}

(21)

(22)

An appropriate action on the Functional Space of Astekar-Sen connections is proposed by myself to be given explicitly by a slight modification of that proposed in chapter 1 of my book “Methods of Bosonic and Fermionic Path Integrals Representations”. My proposed action is given by a covariant $\sigma$-model like Path Integral with a scalar intrinsic field $E(x, t)$ on $R^3 \times [0,T]$. Here $\mu^2$ denotes a scalar “mass” parameter which my be vanishing (massless Wheeler-Universes).

\begin{equation}
S_{\mu^2}[A^a_{\mu}(x, t), E(x, t)] = \frac{1}{16\pi G} \int_0^T dt \int_{R^3} d^3x (E(x, t))^{-1} \left[ \left( \frac{\partial}{\partial t} A_{a,\mu} \right) G^{\mu a,\nu b}[A] \left( \frac{\partial}{\partial t} A_{b,\nu} \right) + \mu^2 \int_0^T dt \int_{R^3} d^3xE(x, t), \right]
\end{equation}

where the invariant metric on the Wheeler-de Witt superspace of Astekar connections is given by

\begin{equation}
G^{\mu a,\nu b}[A] = (b(A))^{-1}(J^{\mu a} J^{\nu b} - J^{\mu b} J^{\nu a})(A),
\end{equation}

with

\begin{equation}
J^{\mu a}(A) = \frac{1}{2} \epsilon^{\mu a\rho} F_{a\rho}(A)
\end{equation}

and

\begin{equation}
b(A) = \det(J(A))_{\mu,\alpha}.
\end{equation}

My proposed quantum geometrodynamical propagator will be given now by the following formal path integral:

\begin{equation}
G[A^{IN}, A^{OUT}] = \int_{A^{a}_{\mu}(x, T) = A^{a}_{\mu, OUT}(x)}^{A^{a}_{\mu}(x, 0) = A^{a}_{\mu, IN}(x)} dINV (A^a_{\mu}(x, t)) \times \int \left( \prod_{(x,t) \in R^3 \times [0,T]} (dE(x, t)) \right) \exp(-S_{\mu^2}[A^a_{\mu}; E])
\end{equation}

(27)
where the invariant functional measure over the Astekar-Sen connections is given by the invariant functional metric

\[ dS^2_{\text{INV}} = \int_{R^3 \times [0,T]} d^3 x dt [(\delta A_{\mu,a}) G^{\mu a,\nu b}[A](\delta A_{\nu,b})](x,t). \quad (28) \]

In order to show that the geometrodynamical propagator equation (7.27) satisfies the Wheeler-de Witt equation, I follow our procedure to deduce the functional wave equations from geometrical path integrals by exploiting the effective functional translation invariance on the functional space of the scalar intrinsic metrics \( E(x,t) \) at the boundary \( t \to 0^+ \). As a consequence, we have that the propagator equation (7.27) satisfies the Wheeler-de Witt equation with the “mass” parameter \( \mu^2 \).

\[
\varepsilon_{abc} F^c_{\mu \nu}(A^{IN})(x) \frac{\delta^2}{\delta A^{b,IN}_\mu(x) \delta A^{b,IN}_\nu(x)} G(A^{IN}; A^{OUT}) = -\mu^2 G(A^{IN}; A^{OUT}) + \delta^4[F](A^{IN,a} - A^{OUT,a}), \quad (29)
\]

where we have used the Euclidean commutation relation

\[
\left[ \left( \frac{G^{\mu a,\nu b}[A]}{E} \times \frac{\partial}{\partial t A_{\nu,b}} \right) (x,t); A_{\mu,a}(x',t) \right] = \delta^4(x - x'). \quad (30)
\]

It is instructive to remark that the classical canonical momentum written in eq. (7.30) is given by the Schrödinger functional representation in the euclidean quantum-mechanical equation (7.29)

\[
\Pi^{\mu a}(x) = \frac{\delta}{\delta A^{a,IN}_\mu(x)} \quad (31)
\]

It is worth pointing out that the usual covariant Polyakov path integral for Klein-Gordon particles may be considered as the 0-dimensional reduction of the geometrodynamical propagator equation (7.27).

At this point we remark that by fixing the gauge \( E(x,t) = \frac{E}{\mu^2} \), with \( \mu^2 \) the “mass” parameter, we arrive at the analogous proper-time Schwinger representation for this geometrodynamical quantum gravity propagator

\[
G_E[A^{IN}, A^{OUT}] = \int_0^\infty dt e^{-(Et)} \times \int d^{INV}(A^{a}_\mu) \exp(-S[A^{a}_\mu(x,t)]). \quad (32)
\]

where \( E = (E, \mu^2) \times vol(R^3) \) is the renormalized mass parameter in the Schwinger Proper-Time representation.

In the next we will use the proper-time-dependent propagator given below as usually is done in the Symanzik’s loop space approach for quantum field theories to write a third-quantized theory for gravitation Einstein theory in terms of Astekar-Sen variables.

\[
G[A^{IN}, A^{OUT}; T] = \int_{A^{a}_\mu(x,T) = A^{a,OUT}_\mu} d^{INV}(A^{a}_\mu) \times \\
\times \exp \left\{ -\frac{1}{16\pi G} \int_0^T dt \int_{R^3} \left[ \left( \frac{\partial}{\partial t A_{\alpha},} \times G^{\mu a,\nu b}[A] \times \left( \frac{\partial}{\partial t A_{\nu,b}} \right) \right](x,t) \right\}. \quad (33)
\]
Unfortunately exactly solutions for eq. (7.29) with \( \mu^2 \neq 0 \) or eq. (7.14) were not found yet. However its \( \sigma \)-like structure and \( SU(2) \) Gauge Invariance may afford to truncated approximate solutions as usually done for the Wheeler-De Witt equations by means of the Mini-Super Space Ansatz. Finally let me comment on the introduction of a Quantized Matter Field represented by a massless field \( \phi(x, t) \) on the Space Time.

By considering the effect of the introduction of this quantized field as a fluctuation on the Geometrodynamical Propagator eq. (7.27) one should consider the following functional representing the interaction of this massless quantized matter and the Astekar-Sen connection as one can easily see by making \( E(x, t) \) variations

\[
S_{INT}[A^a_{\mu}, E, \varphi] = \int_0^T dt \int_{\mathbb{R}^3} d^3 x \left\{ \left[ \varphi \left( -\frac{\partial}{\partial t} \left( E \frac{\partial}{\partial t} \right) \right) \varphi \right] (x, t) + \left[ \varphi \left( \frac{1}{E} G^{a,\mu,bF}[A] \partial_{\rho} A_{b,\rho} \times G^{\mu,\sigma,aF}[A] \cdot \frac{\partial}{\partial t} A_{\sigma,\mu} \right) \partial_{\nu} \varphi \right] (x, t) \right\}
\] (34)

Now the effect on integrating at the scalar matter field in eq. (7.33) is the appearance of the further effective action to be added on the \( \sigma \)-like action of our Proposed Geometrodynamical Propagator.

\[
S_{EFF}[A^a_{\mu}, E, T] = -\frac{1}{2} \log \det_F \left\{ -\frac{\partial}{\partial t} \left( E \frac{\partial}{\partial t} \right) + \partial_{\mu} \left( \frac{1}{E} G^{a,\mu,bF}[A] \partial_{\rho} A_{b,\rho} \times G^{\mu,\sigma,aF}[A] \cdot \frac{\partial}{\partial t} A_{\sigma,\mu} \right) \partial_{\nu} \right\}
\] (35)

The coupling with (Weyl) Fermionic Matter is straightforward and leading to the Left-Handed Fermionic Functional determinant in the presence of the Astekar-Sen connection \( A^a_{\mu}(x, t) \).

The joint probability for the massless field propagator in the presence of a fluctuating
Let me start the analysis by considering the generating functional of the following geometrodynamical field path integral as the simplest generalization for quantum gravity of a similar 4-dimensional geometry parametrized by the Astekar-Sen connection is given by

\[ G[A^{IN}_{\mu}, A^{OUT}_{\mu}; \langle \varphi(x_1, t_1)\varphi(x_2, t_2)\rangle] = \int_{A^{IN}_{\mu}(x, +\infty) = A^{OUT}_{\mu}} d^{3}x \exp \left\{ -\frac{1}{16\pi G} \int_{-\infty}^{+\infty} dt \int_{R^3} d^3x \times \left( \frac{1}{E(x, t)} \times \left( \frac{\partial}{\partial t} A_{\alpha, \mu} \right) G^{\mu a, b \rho}[A] \left( \frac{\partial}{\partial t} A_{\mu, b} \right) (x, t) + \mu^2 \int_{-\infty}^{+\infty} dt \int_{R^3} d^3x E(x, t) \right) \right\} \times \times \delta \left[ -\frac{\partial}{\partial t} E \frac{\partial}{\partial t} \right] + \partial_{\mu} \left( \frac{1}{E} G^{\mu a, b \rho}[A] \frac{\partial}{\partial t} A_{\mu, b} \times G^{\mu a, \sigma}[A] \frac{\partial}{\partial t} A_{\sigma, \mu} \right) \partial_{\nu} \times \lim_{J(x, t) \to 0} \delta J(x_1, t_1) \delta J(x, t) \times \exp \left\{ -\frac{1}{2} \int_{-\infty}^{+\infty} dt dt' \int_{R^3} d^3x d^3y \times \left\{ E J_{abc} e^{\mu \rho \nu} \left( G^{\mu a, \nu \rho}[A] \frac{\partial}{\partial t} A_{\nu, \rho} \times G^{\mu a, \nu \rho}[A] \frac{\partial}{\partial t} A_{\nu, \rho} \times G^{\mu a, \nu \rho}[A] \frac{\partial}{\partial t} A_{\nu, \rho} \right) \right\} (x, t) \times \left\{ -\frac{\partial}{\partial t} E \frac{\partial}{\partial t} \right] \left( \frac{\partial}{\partial t} A_{\mu, b} \times G^{\mu a, \sigma}[A] \frac{\partial}{\partial t} A_{\sigma, \mu} \right) \partial_{\nu} \right\}^{-1} ((x, t), (y, t)) \times \left\{ E J_{abc} e^{\mu \rho \nu} \left( G^{\mu a, \nu \rho}[A] \frac{\partial}{\partial t} A_{\nu, \rho} \times G^{\mu a, \nu \rho}[A] \frac{\partial}{\partial t} A_{\nu, \rho} \times G^{\mu a, \nu \rho}[A] \frac{\partial}{\partial t} A_{\nu, \rho} \right) \right\} (y, t') \right\} \right\} \right\} \right\} \] (36)

4 A $\lambda \phi^4$ Geometrodynamical Field Theory for Quantum Gravity

Let me start the analysis by considering the generating functional of the following geometrodynamical field path integral as the simplest generalization for quantum gravity of a similar well-defined quantum field theory path integral of strings and particles [6]

\[ Z[J(\phi)] = \int D^{F}(\phi[A]) \times \exp \left\{ -\int d\nu(A)\phi[A] \times \left( \int d^3x \left( \varepsilon_{abc} F_{\mu \nu}^c(A) \frac{\delta^2}{\delta A_{\mu} \delta A_{\nu}} \right) (x) \right) \phi[A] \right\} \times \exp \left\{ -\lambda \int d^3x d^3y \int d\nu(A) d\nu(\bar{A})(\phi^2[A(x)](\phi^2[\bar{A}(y)]) \times \delta^{(3)}(A_\mu(x) - \bar{A}_\mu(y)) \right\} \times \exp \left\{ -\int d\nu(A) J(\phi[A]) \right\}. \] (37)

The notation is as follows: i) The quantum gravity third-quantized field is given by a functional $\phi[A]$ defined over the space of all Astekar-Sen connections configurations $M = \{ A^{a}_{\mu}(x), x \in \mathbb{R}^3 \}$. The sum over the functional space $M$ is defined by the gauge and diffeomorphism invariant and topological non-trivial path integral of a Chern-Simons field theory
on the Astekar-Sen connections

\[ d\nu(A) = \int \left( \prod_{x \in \mathbb{R}^3} dA^a_\mu(x) \right) \times \exp \left\{ - \int d^3x (A \wedge dA + \frac{2}{3} A \wedge A \wedge A)(x) \right\}. \] (38)

ii) The third quantized functional measure in eq. (7.37) is given formally by the usual Feynman product measure

\[ D^F(\phi[A]) = \prod_{A \in M} d\phi[A]. \] (39)

iii) The \(\lambda \phi^4\)-like interaction vertex is given by a self-avoiding geometrodynamical interaction among the Astekar-Sen field configurations in the extrinsic space \(\mathbb{R}^3\)

\[ \lambda \sum_{a=1}^3 \delta^{(3)}(A^a_\mu(x) - \bar{A}^a_\mu(y)). \] (40)

The proposed interaction vertex was defined in such a way that it allows the replacement of the Four Universe interaction in eq.(7.37) by an independent interaction of each Astekar-Sen connection with an extrinsic triplet of Gaussian stochastic field \(W^a(x)\) followed by an average over \(W^a\). A similar procedure is well known in the many-body and many-random surface path integral quantum field theory. So, we can write eq.(7.37) in the following form

\[ Z[J(A)] = \left\langle \int D^F(\phi[A]) \times \exp \left\{ - \int d\nu(A) \left[ \phi[A] \left( L(A) - \right. \right. \\
- i\lambda \int d^3x \left( \sum_{a=1}^3 W^a(A^a_\mu) \right) \phi[A] + J(A)\phi[A] \left. \right) \right] \right\rangle_W. \] (41)

Here, \(W^a(A^a)\) means the external \(a\)-component of the triplet of the external stochastic field \(\{W^a\}\) projected on the Astekar-Sen connection \(\{A^a_\mu\}\), namely

\[ W^a(A^a) = W^a(A^a_1(x), A^a_2(x), A^a_3(x)), \] (42)

and has the white-noise stochastic correlation function

\[ \langle W^a(x^\mu)W^b(y^\nu) \rangle = \delta^{(3)}(x^\mu - y^\nu)\delta^{ab}. \] (43)

The \(L(A)\) operator on the functional space of the universe field is the Wheeler-de Witt operator defining the quadratic action in eq. (7.41).

In the free case \(\lambda = 0\). The third-quantized gravitation path integral equation (7.37) is exactly soluble with the following generating functional:

\[ \frac{Z[J(A)]}{Z[0]} = \exp \left\{ + \frac{1}{2} \int d\nu(A)d\nu(\bar{A})J(A) \left( \int d^3x \in_{abc} F^c_{\mu\nu}(A) \frac{\delta^2}{\delta A^a_\mu \delta A^b_\nu} \right)^{-1} (A, \bar{A})J(\bar{A}) \right\}. \] (44)
Here the functional inverse of the Wheeler-de Witt operator is given explicitly by the geometrodynamical propagator equation (7.32) with $\tilde{E} = 0$

$$
\left( \int d^3 x \in_a e F^c_{\mu \nu} (A) \frac{\delta^2}{\delta A^a_\mu \delta A^b_\nu} \right)^{-1} (A, \bar{A}) = \int_0^\infty dT G [A, \bar{A}, T]. \quad (45)
$$

In order to reformulate the third-quantized gravitation field theory as a dynamics of self-avoiding geometrodynamical propagators, we evaluate formally the Gaussian $\phi [A]$ functional path integral in eq.(7.37) with the following result

$$
Z [J(A)] = \langle \frac{-1}{4} \det \left[ \int_{R^3} d^3 x \in_a e F^c_{\mu \nu} (A) \frac{\delta^2}{\delta A^a_\mu \delta A^b_\nu} + i \lambda \left( \sum_{a=1}^3 W^a (A^a_a (x)) \right) \right] \times

\exp \left\{ + \frac{1}{2} \int d\nu (A) d\nu (\bar{A}) \times J(A) \left[ \int_{R^3} d^3 x \in_a e F^c_{\mu \nu} (A) \frac{\delta^2}{\delta A^a_\mu \delta A^b_\nu} + i \lambda \left( \sum_{a=1}^3 W^a (A^a_a (x)) \right) \right] \right\} \right\} \quad (46)
$$

Let us follow our previous studies implemented for particles and strings in previous chapters by defining the functional determinant of the Wheeler-de Witt operator by the proper-time technique

$$
\frac{1}{2} \log \det \left[ L(A) + i \lambda \int_{R^3} d^3 x \left( \sum_{a=1}^3 W^a (A^a_a (x)) \right) \right] =

\int_0^{\infty} \frac{dT}{T} \left\{ \int d\nu (A) d\nu (\bar{A}) \delta (F) (A - \bar{A}) \times

\exp \left\{ - T \left( L(A) + i \lambda \int_{R^3} d^3 x \left( \sum_{a=1}^3 W^a (A^a_a (x)) \right) \right) \right\} \right\}. \quad (47)
$$

with the geometrodynamical propagator (see eq.(7.33) in the presence of the extrinsic potential $\{W^a_\mu (x)\}$ which is given explicitly by the path integral below

$$
\langle A | \exp \left[ - T (L(A)) + i \lambda \int_{R^3} d^3 x \left( \sum_{a=1}^3 W^a (A^a_a (x)) \right) \right] | \bar{A} \rangle =

\int d^{3N} V [B^a_\mu (x, t)] \exp \left\{ - \frac{1}{16 \pi G} \int_0^T dt \int_{R^3} d^3 x \right.

\left[ \left( \frac{\partial}{\partial t} B_{a, \mu} \right) \times G^{\mu a, \nu b} [B] \left( \frac{\partial}{\partial t} B_{b, \nu} \right) \right] (x, t) \right\} \times

\exp \left[ - i \lambda \int_0^T dt \int_{R^3} d^3 x \left( \sum_{a=1}^3 W^a (B^a_\mu (x, t)) \right) \right]. \quad (48)
$$
By substituting eq.(7.48) and eq.(7.47) into eq.(7.46) and making a loop expansion of the functional determinant, we obtain eq.(7.37) as a theory of an ensemble of geometrodynamical propagators interacting with the extrinsic Gaussian stochastic field \( \{W^a(x)\} \). The Gaussian average \( \langle \rangle_w \) may be straightforwardly evaluated at each loop expansion producing the self-avoiding interaction among the geometrodynamical propagators (the Wheeler quantum universes) and leading to the picture of joining and splitting of these Wheeler Universes as necessary for the description of the Universe in its Space-Time Third Quantized form picture of Wheeler. For instance, by neglecting the functional determinant in eq.(7.46)we have the following expression for the geometrodynamical third quantized propagator:

\[
\langle \Phi[A^{a,IN}_\mu] \Phi[A^{a,OUT}_\mu] \rangle^{(0)} = \\
\int_0^\infty dT \times \int_{B^a_\mu(x,0)=A;B^a_\mu(x,T)=A} d^{1NV}B^a_\mu(x,t) \times \\
\times \exp \left\{ -\frac{1}{16\pi G} \int_0^T dt \int_{R^3} d^3x \left[ \left( \frac{\partial}{\partial t} B_{a,\mu} \right) G^{\mu a,\nu b}[B] \left( \frac{\partial}{\partial t} B_{\nu,b} \right) \right] (x,t) \right\} \times \\
\times \exp \left\{ -\frac{\lambda^2}{2} \int_0^T dt \int_0^T dt' \int_{R^3} d^3x d^3y \left( \sum_{a=1}^3 \delta^{(3)}(B^a_\mu(x,t) - (B^a_\mu(y,t'))) \right) \right\}. \tag{49}
\]

Next corrections will involve self-avoiding interactions among different Wheeler Universes associated to different Astekar-Sen connections associated to different Geometrodynamical Propagators appearing from the functional determinant loop expansion equation (7.47).

Finally, I comment that calculations will be done successfully only if one is able to handle correctly the Gometrodynamical Propagator eq.(7.33) on eq.(7.36) and, thus, proceed to generalized for this Quantum gravity case the analogous framework used in the Theory of Random Lines and Surfaces.
In this short appendix we call the reader attention that there are (formal) states satisfying the Wheeler-DeWitt equation (7.1), the diffeomorphism constraint eq.(7.2), but not the gauge-invariant Gauss law eq.(7.3).

For instance, the non-gauge invariant “mass term” wave functional below

$$M[A] = \exp \left\{ -\frac{1}{2} \int_M d^3 x A^i_\mu (\delta^{i1} \delta^{j1} \delta^{\mu\nu}) A^j_\nu \right\}; \quad (A1)$$

satisfies the Wheeler-DeWitt equation, since

$$\left( \varepsilon^{ijk} F^k_{\mu\nu}(A)(x) \frac{\delta^2}{\delta A^i_\mu(x) \delta A^j_\nu(y)} M[A] \right) \sim \varepsilon^{11k}(\ldots) = 0; \quad (A2)$$

and the diffeomorphism constraint

$$\frac{\delta}{\delta A^i_\mu(x)} \left( F^i_{\mu\nu}(A) M[A] \right) = \partial^x M[A]$$

$$+ F^i_{\mu\nu}[A] M[A] \left( -\frac{1}{2} A^1_\mu(x) \right) \delta^{i1} \delta^{\mu\nu}$$

$$= 0 - \frac{1}{2} A^1_\mu(x) \cdot F_{\mu\nu}(A) = 0; \quad (A3)$$

At this point and closely related to the above made remark it is worth call the reader attention that the 3D-fermionic functional determinant with a mass term still satisfies the Wheeler-DeWitt and the diffeomorphism constraint. However, at the limit of large mass $m \to \infty$, one can see the appearance of a complete cut-off dependent mass term like eq.(A1), added with Chern-Simon terms and higher order terms of the strenght field $F_{\alpha\beta}(A(x))$ in the full quantum state [12]. As a result one can argue that this “fermion classical limit” of large mass may be equivalent to the appearance of a dynamical cosmological constant, if one neglects the gauge-violating quantum induced mass term [1].
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