Interaction between a drift and a fractional power of a Laplacian in semi-group theory

Rémi Léandre
Laboratoire de Mathématiques. Université de Franche-Comté. Route de Gray. 25030.
Besançon. FRANCE
E-mail: Remi.Leandre@univ-fcomte.fr

Abstract. We give an interaction between a drift and a fractional power of a degenerated Laplacian such that the involved semi-group has a density by using the Malliavin Calculus for boundary processes translated by ourself in semi-group theory in [1].

1. Introduction and statement of the main theorem
We consider $m$ vector fields on $\mathbb{R}^d$ with bounded derivatives at each order $X_1, ..., X_m$ and the diffusion generator

$$L = \frac{\partial}{\partial s} + \frac{1}{2} \sum_{i>0} (X_i)^2$$

(1)
on $\mathbb{R}^{d+1}$. We could add a drift in (1), but it is done to simplify the proof. We consider a vector field on $\mathbb{R}^d D$ with bounded derivatives at each order. Bismut [2] considers the generator

$$A = D - \frac{1}{2} \sqrt{-2L}$$

(2)

For the theory of fractional powers of Laplacian, we refer to the book of Yosida [3]. Let us recall quickly its definition. $L$ generates a semi-group $P_s$ acting on bounded continuous functions $f$ on $\mathbb{R}^{d+1}$:

$$\frac{\partial}{\partial t} P_t f(s, x) = LP_t(s, x)$$

(3)

Then

$$\sqrt{-2L} = C \int_0^\infty s^{-3/2} (P_s - I) ds$$

(4)

$A$ generates a Markovian semi-group $\exp[tA]$ acting on continuous functions $f$ on $\mathbb{R}^{1+d}$:

$$\frac{\partial}{\partial t} \exp[tA] f(s, x) = A \exp[tA] f(s, x)$$

(5)

There is a stochastic representation of this semi-group (See [2]). Let $(B_1, ..., B_m, z_t)$ be a Brownian motion on $\mathbb{R}^{m+1}$ starting from the origin. Let $L_t$ be the local time associated to
\( z_t \) ([4]) and \( A_t \) its right inverse process. We introduce the stochastic differential equation in Stratonovitch sense issued from \( x \):

\[
dx_t = \sum_{i=1}^{m} X_i(x_t) dB_i + D(x_t) dL_t
\]  

(6)

We consider the subordinated process \((x_{A_t}, A_t + s)\). Unlike \( x_t \), this process is not continuous but is still a Markov process. We have the main relation

\[
\exp[tA]f(s, x) = E[f(A_t + s, x_{A_t})]
\]  

(7)

This paper follows the probabilistic intuition which comes from this stochastic representation of the semi-group. But in [2] and [5], only stochastic differential equations appear which explain that we can expulse the probabilistic language of [2] and [5].

The natural question is to know if the semi-group has an heat-kernel:

\[
\exp[tA]f(s, x) = \int_{\mathbb{R}^{d+1}} q_t(s, x, s', x') f(s', x') ds' dx'
\]  

(8)

This problem was solved by Bismut by using the Malliavin Calculus and a stochastic representation of it ([2]) in the elliptic case. We applied Bismut’s technic to state an Hoermander theorem for fractional powers of Laplacians in [5] We have translated Malliavin Calculus of Bismut for Boundary processes in semi-group theory in [1] and state a regularity result for the semi-group associated to \( A \) in the elliptic case. We do now an Hoermander type hypothesis. We put

\[
G_1 Y = Y
\]  

(9)

\[
G_l Y = \bigcup_{i \geq 0} \bigcup_{Z \in G_{l-1}} ([Z, X_i]) \cup G_{l-1} Z
\]  

(10)

We put:

\[
E_l = \bigcup_{j \leq l} \bigcup_{i > 0} (G_j X_i)
\]  

(11)

The following theorem was proved in [5] by using the Calculus of Boundary Process of Bismut. We prove it again by using the Malliavin Calculus of Bismut type in semi-group theory of [1]:

**Theorem 1** Let us suppose that the uniform Hoermander’s hypothesis is checked:

\[
\inf_{x \in \mathbb{R}^d, \|f\|=1} \sum_{Y \in E_{l=1}} <Y, f>^2 + <D, Y>, f>^2 > C > 0
\]  

(12)

Then the heat-kernel on \( \mathbb{R}^{d+1} \) \( q_t(0, x, s, y) \) exists.

**Remark:** It should be possible to show that \((s, y) \rightarrow q_t(0, x, s, y)\) is smooth.

**Remark:** It is possible to replace (12) by the general hypothesis (3.5) of [5].

### 2. The main ingredient of the proof

Let \( E_d = \mathbb{R}^{1+d} \times G_d \times M_d \) where \( G_d \) denotes the set of invertible matrices on \( \mathbb{R}^d \) and \( M_d \) the set of symmetric matrices on \( \mathbb{R}^d \). \((s, x, U, V)\) is the generic element of \( E_d \). \( V \) is called the Malliavin matrix.

On \( E_d \) we consider the vector fields:

\[
\hat{D} = (0, D, DD(x)U, 0)
\]  

(13)

\[
\hat{X}_i = (0, X_i, DX_i(x)U, 0)
\]  

(14)
\[ \hat{Y} = (0, 0, 0, \sum_{i=1}^{m} < U^{-1} X_i, >^2) \] (15)

We consider the Malliavin generator \( \hat{L} \) on \( E_d \):

\[ \hat{L} = \frac{\partial}{\partial s} + \frac{1}{2} \sum_{i=1}^{m} (\hat{X}_i)^2 + \hat{Y} \] (16)

and the square root associated \( \sqrt{-L} \). This semi-group on this bigger space is defined according the line of (5).

We consider \( \hat{A} = \hat{D} - 1/2 \sqrt{-2\hat{L}} \) (17) and the Malliavin semi-group \( \exp[t\hat{A}] \).

An adaptation of one of main result of Léandre [1] is the following:

**Theorem 2** Let us suppose that the Malliavin condition is checked: for all \( p \in N \), all \( s > 0 \)

\[ \exp[t\hat{A}][V^{-p}1_{[0,s]}](0, x, I, 0) < \infty \] (18)

then

\[ \exp[tA]f(0, x) = \int_{R^{1+d}} f(s, y)q_t(s, y)dsdy \] (19)

where \( q_t(s, y) \geq 0 \).

**Remark:** The proof follows the proof of Theorem 2.1 of [1]. Following the general strategy of the Malliavin Calculus, it is enough to show the theorem to get integration by parts formulas. If \( f \) is with compact support

\[ |\exp[tA](df)(s, x)| \leq C\|f\|_{\infty} \] (20)

where \( C \) depends only from the support of \( f \) and \( \|f\|_{\infty} \) denotes the supremum norm of \( f \). For that, we integrate by parts under the underlying diffusion \( P_s \) and the Brownian motions \( B_i \) as in part 3 of [1]. This allows to remove the space derivatives of \( f \). In order to remove the time derivatives in \( df(s, x) \), we integrate by parts on the subordinators \( A_t \) as it was done in part 4 of [1]. The main difference with [1] is that \( \hat{D} \) appears when we take the variation of the subordinated semi-group. It is the only change in the abstract theorem of part 5 of [1]. When we get this abstract theorem, the drift \( D \) will appear another time in the inversion of the Malliavin matrix \( V \).

### 3. Inversion of the Malliavin matrix in semi-group theory

Let be

\[ F_l(x, U, \xi) = \sum_{Y \in G_l} < U^{-1} Y(x), \xi >^2 \] (21)

where \( \xi \) is of modulus one. A simple adaptation of Lemma 3 of [6] shows:

**Lemma 3** Let us suppose that

\[ \exp[t_0\hat{A}][1_{[0,s_0]}; F_l(I, ., \xi) > C\|U\|^m](0, x_0, U_0, 0) > C > 0 \] (22)

for all \( x_0 \in R^d \), \( \|U_0\| < t_0^{-\epsilon} \) for a small \( \epsilon \) and some positive \( \beta \). Then (22) remains true on an interval of length \( t_0^{\beta} \) for another \( \beta \).
Since \( \hat{A} \) is Markovian, \( \exp(t\hat{A}) \) is represented by a stochastic process \( X_t \) following the same line of the representation of \( \exp(tA) \) by a stochastic process. \( X_t \) is a Markov jump process. It has a Levy measure [7]. The main remark is the following: if the Levy measure of a jump process is enough concentrated in small jumps, there are a lot of small jumps. We get:

**Definition 4** If \( f \) is a function from \( R^+ \) into \( R^+ \), we consider the Levy measure associated with the Malliavin matrix where \( \xi \) is of norm 1:

\[
\mu_\xi(f) = C \int_0^{s_0} \frac{ds}{s^{3/2}} P_s[f(V'((\xi)) - V(\xi))](0, x, U, V)
\]  

(23)

We recall (See Lemma 3 of [6]):

**Lemma 5** Let us suppose that \( F_1(x, U, \xi) \geq \rho \) for \( |U| + |U^{-1}| < \rho^{-\epsilon} \) for some small \( \epsilon \). Then \( \mu_\xi[z > \rho^\alpha] \geq C \rho^\beta \) for some positive \( \alpha \) and some negative \( \beta \).

The theorem will follow as in the proof of Theorem 1 (38), (39), (40) in [6] if we show the next proposition:

**Proposition 6** Let us suppose that \( |U| + |U^{-1}| < \rho^{-\epsilon} \) for some small \( \epsilon \). There exists \( \alpha \) such that

\[
\exp[\rho^\alpha \hat{A}][F_1(x, U, \xi) \geq \rho; 1_{[0,s_0]}](0, x, U, 0) > C > 0
\]  

(24)

We can state an analog of Lemma 2 of [6]:

**Lemma 7** Let us suppose that

\[
\exp[\rho^\alpha \hat{A}][F_1(x, U, \xi) \geq \rho^\beta; 1_{[0,s_0]}](0, x_0, U_0, 0) > C > 0
\]  

(25)

where \( |U_0| + |U_0^{-1}| < \rho^{-\epsilon} \) for some small \( \epsilon \). Then (25) remains true for \( l - 1 \) for others \( \alpha \) and \( \beta \).

By using this lemma, it is enough to show the following proposition in order to show Proposition 6:

**Proposition 8** If we take \( l_0 + 2 \), (25) is checked if \( |U| + |U^{-1}| < \rho^{-\epsilon} \) for some small \( \epsilon \).

**Proof of proposition 8** Let us suppose that

\[
F_{l_0}(x_0, U_0, \xi) \leq \rho^\beta
\]  

(26)

and

\[
F_{l_0+2} \leq \rho^{\beta_1}
\]  

(27)

By Hypothesis (12), we can find a \( Y \in E_{l_0} \) such that

\[
< [D, Y](x_0), \xi > > C > 0
\]  

(28)

We choose \( C > 0 \) to simplify the exposition and we choose \( \epsilon = 0 \) in order to simplify the exposition of the proof.

We remark that if \( u < t \) and if \( \gamma < 1/2 \) that

\[
\exp[u \hat{A}][1_{[\gamma, \infty]}](0, x_0, U_0, 0) \leq Ct^\gamma
\]  

(29)

for some \( r > 0 \). So it is enough to estimate

\[
\exp[u \hat{A}][F_{l_0}(\cdot, \cdot, \xi) \geq \rho^\alpha; 1_{[0,\gamma]}](0, x_0, U_0, 0)
\]  

(30)

for some well chosen \( \alpha \). We put

\[
G(x, U, \xi) = < U^{-1}Y, \xi > g\left(\frac{F_{l_0+2}(x, U, \xi)}{\rho^{\beta_1}}\right)
\]  

(31)
where \( g \) is a smooth function from \( R^+ \) into \([0, 1]\) equals to 1 in a neighborhood of 0 and to 0 in a neighborhood of the infinity.

Let us suppose that
\[
|x - x_0| + |U - U_0| < C \rho^{\beta/2}
\] (32)

Let us estimate for \( s' \leq t^{\gamma} \)
\[
\sqrt{-\hat{L}[G(.,.,\xi)1_{[0,t^{\gamma}]})](s', x, U, 0)} \leq \rho^{\beta/2} t^{\gamma/2} \rho - 2\beta t^{2\gamma/2}
\] (33)

For that we look at
\[
f(s) = \hat{P}_s[G(.,.,\xi)1_{[0,t^{\gamma}]})](s', x, U, 0)
\] (34)

where \( \hat{P}_s \) is the semi-group generated by \( \hat{L} \).
\[
f'(s) = \hat{P}_s[\hat{L}[G(x', U', \xi)1_{[0,t^{\gamma}]})](s', x, U, 0)
\] (35)

We distinguish if
\[
|x - x'| + |U - U'| < C \rho^{\beta/2}
\] (36)
or not. If yes
\[
\hat{L}[G(x', U', \xi)1_{[0,t^{\gamma}]})] \leq \rho^{\beta/2} 1_{[0,t^{\gamma}]}
\] (37)

If not we remark that
\[
\hat{P}_s||x - x'| + |U - U'| \geq \rho^{\beta/2})(s', x, U, 0) \leq C \rho^{-\beta}
\] (38)

In conclusion, we deduce that
\[
|\sqrt{-\hat{L}[G(.,.,\xi)1_{[0,t^{\gamma}]})](s', x, U, 0)}| \leq \rho^{\beta/2} t^{\gamma/2} + \rho^{-\beta} \rho - 2\beta t^{2\gamma/2}
\] (39)

Let us consider the case
\[
|x - x_0| + |U - U_0| > C \rho^{\beta/2}
\] (40)

In such a case
\[
|f'(s)| \leq C \rho^{-2\beta} 1_{[0,t^{\gamma}]}
\] (41)

Therefore
\[
|\sqrt{-\hat{L}[G(.,.,\xi)1_{[0,t^{\gamma}]})](s', x, U, 0)}| \leq C \rho^{-\beta} t^{\gamma/2} 1_{[0,t^{\gamma}]}
\] (42)

On the other hand
\[
|\sqrt{-\hat{L}[U' - U_0]^2} + |x' - x_0|^2; 1_{[0,t^{\gamma}]})](s', x, U, 0)|
\]
\[
\leq A1_{[0,t^{\gamma}]}) \int_0^{t^{\gamma}} ds \frac{s}{s^{3/2}} \leq C t^{\gamma/2} 1_{[0,t^{\gamma}]}
\] (43)

This shows that
\[
exp[u\hat{A}][U - U_0]^2 + |x - x_0|^2; 1_{[0,t^{\gamma}]})](0, x_0, U_0, 0) \leq C u t^{\gamma/2}
\] (44)

Therefore
\[
exp[u\hat{A}][U - U_0] + |x - x_0| > \rho^{\beta/2} 1_{[0,t^{\gamma}]})](0, x_0, U_0, 0) \leq ut^{\gamma/2} \rho^{-\beta}
\] (45)

By putting all together, we deduce that if \( \gamma < 1/2 \)
\[
|\exp[u\hat{A}][\sqrt{-\hat{L}[G(.,.,\xi)1_{[0,t^{\gamma}]})]}](0, x_0, U_0, 0)|
\]
\[
C(\rho^{\beta/2} t^{\gamma/2} + \rho^{-\beta} \rho - 2\beta t^{2\gamma/2} + ut^{\gamma/2} \rho^{-\beta})
\] (46)
Let us now estimate
\[
\exp[u\hat{A}][DG(x, U, \xi)1_{[0,t\gamma]}](0, x_0, U_0, 0)
\] (47)
We suppose first of all that
\[
|x - x_0| + |U - U_0| < C \rho^{\beta/2}
\] (48)
In such a case we have a lower bound in \(C > 0\) of the expression. If the previous inequality is not checked we have an estimate by using the previous considerations in \(C_{\beta/2} \rho - \beta \rho_1 - \beta\) By using the semi-group property for \(\exp[u\hat{A}]\) we deduce that if \(u < t\gamma\)
\[
g'(u) = \frac{\partial}{\partial u} \exp[u\hat{A}][G(., ., \xi)1_{[0,t\gamma]}](0, x_0, U_0, 0) \geq C - C't^{3\gamma/2}\rho^{-\beta_1}\rho^{-\beta} - C(\rho^{\beta/2}t^{\gamma/2} + \rho^{-\beta}t^{2\beta_1}t^{3\gamma/2})
\] (49)
We distinguish if
\[
|g(0)| < C \rho^{3\beta}
\] (50)
or not. If (50) is not checked, we can apply Lemma (7). If not we have \(\beta_1 < \beta\). We deduce that
\[
g(t) \geq Ct - C\rho^{3\beta} - C\rho^{-\beta}_1\rho^{-\beta}t^{1+3\gamma/2} - C\rho^{\beta/2}t^{1+\gamma/2}
\] (51)
We choose \(t = C_1\rho^3\) for a big \(C_1\). From (51), we deduce that
\[
g(C_1\rho^3) \geq C\rho^\beta - C\rho^{-\beta}_1\rho^{(1+3\gamma/2)}
\] (52)
to choose \(\gamma\) close from 1/2 and \(\beta_1\) very small in order to deduce that
\[
g(C_1\rho^3) \geq C\rho^\beta
\] (53)
We deduce that
\[
\exp[C_1\rho^3 \hat{A}][<U^{-1}Y(x), \xi >> C \rho^\beta](0, x_0, U_0) > C > 0
\] (54)
Therefore the result holds.
\(\Diamond\).

Remark: The principle of the proof is very simple: we establish a criterium in order to show that the Levy measure associated to \(V(\xi)\) is very concentrated in small jumps. If the Levy measure is very concentrated in small jumps, there are a lot of small jumps which obliges that \(V(\xi)\) to be not very small. If this criterium is not satisfied, another criterium will obliged it to be satisfied. In [6], this criterium comes from the interaction between two Levy measures. Here it come from the interaction between a Levy measure and the drift \(D\).

References
[1] Léandre R 2011, *Inter. Jour. of Diff. Equ.* article ID 575383.
[2] Bismut J.M. 1983 *Annales Scientifiques E.N.S.* 17 507.
[3] Yosida K 1977 *Functional analysis* (Berlin: Springer).
[4] Ikeda N and Watanabe S 1981 *Stochastic differential equations and diffusions processes* (Amsterdam: North-Holland)
[5] Léandre R 1984 *Une extension du théorème de Hoermander a divers processus de sauts* (PHD Thesis, Besançon, France: Université de Franche-Comté).
[6] Léandre R 2012 XII *International Carpathian control conference* (Podbanske), (IEEE ISBN 978-1-4577-1866-3; IEEE: Xplore), p 421.
[7] Ishikawa Y 2013 *Stochastic Calculus of variations for jump processes* (Berlin: De Gruyter)