Grothendieck’s Classification of Holomorphic Bundles over the Riemann Sphere

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Abstract

In this paper we look at Grothendieck’s work on classifying holomorphic bundles over \( \mathbb{P}^1(\mathbb{C}) \). The paper is divided into 4 parts. The first and second part we build up the necessary background to talk about vector bundles, sheaves, cohomology, etc. The main result of the 3\(^{rd}\) chapter is the classification of holomorphic vector bundles over \( \mathbb{P}^1(\mathbb{C}) \). In the 4\(^{th}\) chapter we introduce principal \( G \)-bundles and some of the theory behind them and finish off by proving Grothendieck’s theorem in full generality. The goal is a (mostly) self-contained proof of Grothendieck’s result accessible to someone who has taken differential geometry.
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Chapter 1

Complex Manifolds and Vector Bundles

1.1 Complex Manifolds

Definition 1 (Complex Manifold). We say a manifold $M$ is a complex manifold if each of the charts, $\phi_\alpha$, map from an open subset $U_\alpha$ to an open subset of $V_\alpha \subset \mathbb{C}^n$ and the transition maps $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ are biholomorphisms (bijective holomorphisms with a holomorphic inverse) as maps from $\phi_\alpha(U_\alpha \cap U_\beta)$ to $\phi_\beta(U_\alpha \cap U_\beta)$.

We say that the (complex) dimension of the manifold over $\mathbb{C}$ is $n$. A Riemannian surface is a manifold in the special case that the dimension is 1.

Definition 2 (Projective Spaces). We define the $n$ dimensional projective space over $\mathbb{C}$, $\mathbb{P}(\mathbb{C})^n$, as the set of equivalence classes of non-zero vectors in $v \in \mathbb{C}^{n+1}$ under the equivalence $v \sim \lambda v$ for $\lambda \in \mathbb{C}$.

Proposition 1. The $n$ dimensional projective space is indeed an $n$-dimensional complex manifold.

Proof. Let $[z_1, \cdots, z_{n+1}]$ ($z_i \in \mathbb{C}$, not all zero) be the equivalence class corresponding to $(z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1}$. Let $U_i$ be the set of equivalence classes with $z_i$ non-zero. Then $U_i$ cover $\mathbb{P}(\mathbb{C})^n$. Let $\phi_i : U_i \to \mathbb{C}^n$ by $[z_1, \cdots, z_{n+1}] \mapsto (\frac{z_1}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_{n+1}}{z_i})$.

One immediately sees that $\phi_i$ is well defined and if $z_i, z_j \neq 0$ then $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a biholomorphism. \qed
Theorem 1. A holomorphic function, $f$, on a compact Riemann surface is constant.

Proof. We have that $|f(p)|$ is maximal for some $p$. Take a chart around $p$ to a neighbourhood of 0. Then the composition of $f$ with the chart is maximal at 0, contradicting the maximum modulus principle.

The Riemann sphere is defined to be the Riemann surface $\mathbb{P}^1(\mathbb{C})$.

1.2 Vector Bundles

Let $M$ be a manifold, we say a (real) vector bundle $V$ over $M$ is pair of a manifold and projection map $(V, \pi)$ with $\pi : V \to M$ so that for every $p \in M$, $\pi^{-1}(p)$ is a $\mathbb{R}$-vector space we have that there is some open $U$ around $p$, and a homeomorphism $\varphi_U$ with $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$. We say that the rank of the vector bundle $V$ (over $\mathbb{R}$) is $k$.

We can extend this definition to holomorphic vector bundles over a complex manifold in the following way:

Definition 3. Let $M$ be a complex manifold, we say a pair $(E, \pi)$ over $M$ with rank $k$ is a holomorphic vector bundle if for every $p \in M$, $\pi^{-1}(p)$ is a $\mathbb{C}$-vector space and there is an open subset $U$ of $M$ with a biholomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k$. Equivalently, we can require the transition maps to $\mathbb{C}$ be linear isomorphisms:

$$\text{proj}(\varphi_U \circ \varphi_V^{-1})(U \cap V) : \mathbb{C}^l \to \mathbb{C}^l$$

Furthermore, we say a vector bundle is a line bundle if it has rank 1 and we say a (complex) vector bundle $E$ is trivial if it is isomorphic to $\mathbb{C}^k \times M$. Note that, locally, every bundle is trivial.

Definition 4. A section of a vector bundle $V$ is a continuous map $\sigma : M \to V$ so that $\pi \circ \sigma = 1_M$. The vector space of all sections on $V$ over $M$ is denoted by $\Gamma(M, V)$. If the vector bundle $E$ is holomorphic and the map $\sigma$ is holomorphic we say it is a holomorphic section and denote the corresponding vector space $H^0(M, \mathcal{O}(E))$.

The reason we use this notation will become clear later on.

Proposition 2. Let $E_1$ and $E_2$ be 2 holomorphic vector bundles over a complex manifold $M$ of rank $k,l$. Then we can define the vector bundles $E_1^*$, $\det(E_1)$, $E_1 \oplus E_2$, and $E_1 \otimes E_2$. 

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Proof. Let $U, V$ be sufficiently small open sets around $p$. Let $\phi_1, \phi_2$ be 2 corresponding charts for $E_1$ and similarly $\varphi_1, \varphi_2$ for $E_2$. Let $T_{12}$ and $T_{12}$ be the linear transition maps for $E_1$ and $E_2$. Then we define the transition charts for $E_1^*, \det(E_1), E_1 \oplus E_2,$ and $E_1 \otimes E_2$:

- $T_{12}^*$
- $\det(T_{12})$
- $T_{12} \oplus T_{12}$
- $T_{12} \otimes T_{12}$

These are then invertible and linear and the vector bundles have rank $k, 1, k + l,$ and $kl$ respectively.

Definition 5. Let $E_1$ and $E_2$ be 2 holomorphic vector bundles over a complex manifold $M$. Suppose we have an invertible map $f$ so that the following diagram commutes and the restriction, $f|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \pi^{-2}(p)$ is linear.

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\pi_1 & \downarrow & \pi_2 \\
M & & 
\end{array}
\]

We say that $E_1$ and $E_2$ are isomorphic. Similarly, if there is an injection from $E_1$ to $E_2$ then we say $E_1$ is a subbundle of $E_2$.

Theorem 2. Let $M$ be a complex manifold and consider a short exact sequence of vector bundles over $M$:

\[
0 \to E_1 \xrightarrow{p} E \xrightarrow{q} E_2 \to 0
\]

This sequence splits, that is, $E \cong E_1 \oplus E_2$.

Proof. We first construct an inner product over $E$.

Let $U_\alpha$ cover $M$ with $E$ trivial over each $U_\alpha$. Let $\rho_\alpha$ be a corresponding partition of unity. We can choose an inner product $\langle \cdot, \cdot \rangle_\alpha$ on each $E|_{U_\alpha}$. Extend each inner product to be 0 outside $E|_{U_\alpha}$. Now consider the inner product given by:

\[
\langle \cdot, \cdot \rangle = \sum_\alpha \rho_\alpha \langle \cdot, \cdot \rangle_\alpha
\]
This is defined on all of $E$.

Under this inner product we can write $E = (p(E_1))^\perp \oplus (p(E_1))$. Note that $p(E_1) \cong E_1$ by injectivity. We also have the restriction of $q|_{(\ker q)^\perp}$ from $(\ker q)^\perp \to E_2$ is surjective (by exactness) and injective as $q(x) = q(y)$ means $x - y \in \ker(q)$. But $E_2 \cong (\ker q)^\perp = (p(E_1))^\perp$ and so the sequence splits.

**Lemma 1.** Let $L$ be a line bundle on a complex manifold $M$. Then $L$ is trivial if and only if there is some nowhere 0 section on $L$.

**Proof.** Suppose $L$ is the trivial bundle. Let $x \in M$. Then the section sending $x \mapsto (x,1)$ is nowhere 0.

Suppose we have a nowhere section $\sigma$, sending $x \mapsto (x,\sigma(x))$. Then consider the isomorphism $f : M \times \mathbb{C} \to L$ by $(x,c) \mapsto (x,c\sigma(x))$. \hfill $\square$

**Lemma 2.** Let $S$ be a Riemann surface and let $E$ be a vector bundle. Then $E \cong E' \oplus I_{\text{rank}E-1}$ for some line bundle $E'$.

**Proof.** We note that if the rank of $E$ is at least 2 then there is a section $\sigma$ that is non-zero everywhere by perturbing a section (not identically 0) locally around its zeroes.

Take the line bundle parametrized by $\sigma$, $L \cong I_1$. We can then split $E$ as:

$$0 \to I_1 \to E \to E_1$$

Is short exact for some $E_1$. We then do the same for $E_1$, inductively, so $E \cong E' \oplus I_{\text{rank}E-1}$. \hfill $\square$

Once we have the notion of a degree of a line bundle we will be able to show 2 line bundles are isomorphic if and only if their degree is the same. This, combined with the above, gives us that $E_1 \cong E_2$ if and only if they have the same rank and same degree.
Chapter 2

Sheaves And Cohomology

2.1 Sheaves

Definition 6. Let \( X \) be a topological space. For every open \( U \subseteq X \) we associate an abelian group \( \mathcal{F}(U) \) so that:

1. \( \mathcal{F}(\emptyset) = 0 \)
2. If \( V \subseteq U \) there is a group morphism \( \rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V) \)
3. \( \rho_{U,U} = 1 \)
4. If \( W \subseteq V \subseteq U \) then \( \rho_{U,W} = \rho_{V,W} \circ \rho_{U,V} \)

We say that \( \mathcal{F} \) is a presheaf. We can write \( \rho_{U,V}(f) \) as \( f|_V^U \).

Definition 7. Let \( \mathcal{F} \) be a presheaf on \( X \). Let \( U \) be open with open cover \( \{U_i\} \). \( \mathcal{F} \) is a sheaf if we have:

1. If \( s \in \mathcal{F}(U) \) with \( \rho_{U,U_i}(s) = 0 \) for all \( i \), then \( s = 0 \)
2. If \( s_i \in \mathcal{F}(U_i) \) with (for any \( i, j \)):
   \[
   \rho_{U_i \cap U_j}(s_i) = \rho_{U_i \cap U_j}(s_j)
   \]
   then there is \( s \in \mathcal{F}(U) \) so that \( \rho_{U,U_i}(s) = s_i \)
Proposition 3. Let $k \in \mathbb{Z}$. Let $E$ be a vector bundle over a complex manifold $M$. Let $X$ be a Riemann surface with $x \in X$ and $L$ be a line bundle over $X$. Then the following are sheaves:

- $\mathcal{O}(E)$ where to each $U \subset M$ we associate the abelian group (under pointwise multiplication) $H^0(E, U)$ with the maps $\rho_{U,V}$ being restrictions.
- $\mathcal{O}_M$, where to each $U \subset M$ we associate the abelian group of holomorphic functions $f : U \to \mathbb{C}$
- $\mathcal{O}_M^*$, where to each $U \subset M$ we associate the abelian group of holomorphic functions $f : U \to \mathbb{C}^*$
- $\mathcal{O}_X(-kx)$, where to each $U \subset X$ we associate the abelian group of holomorphic functions $f : U \to \mathbb{C}$ that vanish at $x$ with multiplicity $k$,
- $\mathcal{O}_X(kx)$ where to each $U \subset X$ we associate the abelian group of holomorphic functions $f : U \to \mathbb{C}$ that that have a pole of order $k$ at $x$.
- $L(-x)$, where $U$ is associated to the holomorphic sections of $L|_U$, vanishing at $x$.
- $\mathbb{C}_x$, the skyscraper sheave, with $U$ associated to $\mathbb{C}$ if $x \in U$ and 0, otherwise.

2.2 Cech Cohomology

Let $\mathcal{U} = \{U_\alpha\}$ cover a complex manifold $X$ (for $\alpha$ in some index set $I$) and let $\mathcal{F}$ be a sheaf on $X$.

Definition 8. Let

$$C^i = \prod_{\alpha_1, \ldots, \alpha_i \in I} \mathcal{F}(\cap_{k=1}^i U_{\alpha_k})$$

Let $d_i : C^i \to C^{i+1}$ via (taking the product of the maps over the indices):

$$f_{\{\alpha_1, \ldots, \alpha_i\}} \mapsto \sum_{k} (-1)^k f_{\cap_{j \neq k} U_{\alpha_j}}$$

This gives rise to the ech complex:

$$C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \ldots$$
Exercise 1. One can see this forms a complex by verifying that $d_{i+1} \circ d_i = 0$

Definition 9. We define the $p^{th}$ ech cohomology group by taking the quotient group:

$$H^p(X, \mathcal{U}, \mathcal{F}) = \frac{\ker(d_p)}{\operatorname{Im}(d_{p+1})}$$

One may wonder to what extent does the cohomology depend on the open cover. It turns out that by a result due to Leray, beyond the scope of this paper, we can choose sufficiently refined coverings so the cohomology doesn’t change.

Note 1. Our use of the notation $H^0(X, \mathcal{O}(E))$ before is justified as $\ker d_0 = H^0(X, \mathcal{O}(E))$

Lemma 3 (Induced Long Exact Sequences). Suppose we have a short exact sequence of sheaves:

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

That is, for any open set $U$, the functors at $U$ to the category of abelian groups form a short exact sequence.

Then there is an induced long exact sequence of cohomology groups:

$$0 \rightarrow H^0(X, \mathcal{E}) \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{G}) \rightarrow H^1(X, \mathcal{E}) \cdots$$

Proof. The proof is an easy but tedious application of Snake lemma twice. We will not, however, prove it here. \qed
Chapter 3

Line Bundles over \( \mathbb{P}^1(\mathbb{C}) \) and Grothendieck’s Theorem for Vector Bundles

3.1 The degree map

For the rest of the paper we can fix \( p \in X \). We now turn our attention over to line bundles over \( \mathbb{P}^1(\mathbb{C}) \). We proved earlier in the paper that any vector bundle \( E \) over \( X \) can be written as \( L \oplus I_m \) where \( L \) is a line bundle. When are 2 line bundles isomorphic?

We leave the following proposition as an exercise.

**Proposition 4.** Let \( E, F, G, H \) be vector bundles. If

\[
0 \to E \to F \to G \to 0
\]

is short exact then:

\[
0 \to E \otimes H \to F \otimes H \to G \otimes H \to 0
\]

Is short exact as well. Furthermore, if \( H^1(X, G^* \otimes E) = 0 \), then \( F \cong E \otimes G \).

**Definition 10.** Consider the short exact sequence:

\[
0 \to \mathcal{O}_X(\mathbb{Z}) \to \mathcal{O}_X \xrightarrow{2\pi i f} \mathcal{O}_X(\mathbb{C}^*) \to 0
\]
And the induced long exact sequence:

\[ \cdots H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{\text{deg}} H^2(X, \mathcal{O}_X(\mathbb{Z})) \to H^2(X, \mathcal{O}_X) \cdots \]

The degree map is then defined to be \( \text{deg} \).

**Lemma 4.** The degree map is a bijection.

**Proof.** By Leray \( H^1(X, \mathcal{O}_X) \cong 0 \) and by a theorem of Grothendieck’s \( H^2(X, \mathcal{O}_X) \cong 0 \). By exactness, the degree map is a bijection.

Furthermore, by Poincaré duality, \( H^2(X, \mathcal{O}_X(\mathbb{Z})) \cong H_0(X, \mathcal{O}_X(\mathbb{Z})) \cong \mathbb{Z} \).

### 3.2 The Classification of Vector Bundles

**Proposition 5.** \( H^1(X, \mathcal{O}_X^*) \) is the set of line bundles on \( X \) (up to isomorphism).

**Proof.** Let \( L \) be a line bundle. Choose a cover fine enough so that \( L \) is trivial on each intersection. Let \( \phi_j^{-1} \circ \phi_i \) be the transitions, then \( \ker d_1 \) is precisely the set of \( \phi_j^{-1} \circ \phi_i \), as \( (\phi_j^{-1} \circ \phi_i)^{-1} = \phi_i^{-1} \circ \phi_j \). Let \( \varphi_i \) be another trivialization of \( L \). Then \( \varphi_i^{-1} \circ \varphi_j^{-1} \circ \phi_j \circ f \cdot g \) where \( f, g \) are biholomorphic maps on \( U_i \cap U_j \). Note that \( \text{Im}(d_0) \) is the set of maps that can be written as \( fg^{-1} \) for some \( f : U_i \to \mathbb{C}^* \), \( g : U_j \to \mathbb{C}^* \). So up to \( \text{Im}(d_0) \), line bundles are unique elements of \( \ker d_1 \). The conclusion follows.

**Proposition 6.** \( H^1(X, \mathbb{C}_p) = 0 \).

**Proof.** We want to show \( \ker(d_1) = 0 \) so it suffices to check that \( d_0(C_0) = 0 \). Take any refinement with only one open set, \( U_1 \) containing \( p \). Let \( c \in \mathcal{F}(U_1) \), then \( d_0(c) = 0 \).

We also need the following 2 lemmas:

**Lemma 5.** \( \dim H^0(X, \mathcal{O}(m)) = m + 1 \) if \( m \geq 0 \).

**Proof.** There are some charts \( \phi_0 \) on \( U_1 \) around 0 and some chart \( \phi_1 \) on \( U_2 \) so that \( U_1 \cap U_2 \neq \emptyset \) of that set so that the transition function is \( \frac{1}{z^m} \).

It follows that the image of any section under the charts must have Laurent expansion:

\[
\frac{1}{z^m} \sum_{k=0}^{m} \alpha_i z^i
\]
Note that if \( m < 0 \), we only have the 0 section.

**Lemma 6.** For any vector bundle \( E \) of rank \( k \) over \( X \) we can find some \( O(n) \) so that \( E \otimes O(n) \) has a holomorphic section that is not everywhere 0.

**Proof.** Let \( n > \dim H^1(X, E) \). We have a section \( \sigma \) that only vanishes at \( p \). This yields the following short exact sequence:

\[
0 \to E \to E \otimes O(n) \xrightarrow{\sigma(p)^n} \mathbb{C}^n_p \to 0
\]

This induces a long exact sequence with the sum of alternating dimensions being 0, so we now have

\[
\dim H^0(X, E \otimes O(n)) = \dim H^1(X, E \otimes O(n)) + \dim H^0(\mathbb{C}_p^{nk}) + \dim H^0(X, E) - \dim H^1(X, E)
\geq nk - \dim H^1(X, E).
\]

And so \( \dim H^0(X, E \otimes O(n)) \geq 1 \).

\[\square\]

Note that this implies we can let \( n \) be so that \( \dim H^0(X, E \otimes O(n - 1)) = 0 \) but \( \dim H^0(X, E \otimes O(n)) > 0 \) (as \( \dim H^0(X, E \otimes O(n - 1)) < \dim H^0(X, E \otimes O(n)) \)).

We are finally ready to prove Grothendieck’s classification of vector bundles.

**Theorem 3** (Grothendieck). Let \( E \) be a rank \( k \) vector bundle over \( X \). Then:

\[
E \cong \bigoplus_{i=1}^{k} O(d_i)
\]

**Proof.** Let \( O(n) \) be as above for \( p \in X \), arbitrary. We can then take a holomorphic section \( \sigma \) that never vanishes (If it did vanish at \( p \) then \( \sigma \sigma_p^{-1} \in E \otimes O(m - 1) \) wouldn’t) and thus find a trivial subbundle, \( L \) of \( E \otimes O(n) \). We let \( Q \) be the quotient bundle of \( L \) and \( E \) and suppose by induction that it decomposes as \( Q = \bigoplus_{i=1}^{k-1} O(b_i) \).

Note by Riemann-Roch, \( \dim H^1(X, O(-1)) = 0 \).
So we have the following 2 exact sequences (after tensoring with $\mathcal{O}(-1)$):

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}(E \otimes \mathcal{O}(n - 1)) \rightarrow \mathcal{O}(Q(-1)) \rightarrow 0$$

$$0 \rightarrow H^0(X, \mathcal{O}(Q(-1))) \rightarrow 0$$

So $b_i \leq 0$.

Note by Riemann-Roch, $\dim H^1(X, \mathcal{O}(-b_i)) = 0$. We now calculate:

$$H^1(X, \mathcal{O}^*) = H^1(X, \bigoplus_{i=1}^{k-1} \mathcal{O}(-b_i)) = 0$$

Now consider again

$$0 \rightarrow L \rightarrow E \otimes \mathcal{O}(m) \xrightarrow{\alpha} Q \rightarrow 0$$

Tensoring by $Q^*$:

$$0 \rightarrow \mathcal{O}(Q^*) \rightarrow \mathcal{O}(\text{Hom}(Q, E \otimes \mathcal{O}(m))) \rightarrow \mathcal{O}(\text{Hom}(Q, Q)) \rightarrow 0$$

The induced cohomology has the following surjection:

$$H^0(X, \text{Hom}(Q, E \otimes \mathcal{O}(m))) \rightarrow H^0(X, \text{Hom}(Q, Q)) \rightarrow 0$$

Thus there is some $\beta : Q \rightarrow E \otimes \mathcal{O}(m)$ so that $\alpha \circ \beta = id_Q$, and by splitting lemma

$$E \otimes \mathcal{O}(m) \cong L \oplus Q.$$}

Tensoring

$$E \cong \mathcal{O}(-m) \oplus \bigoplus_{i=1}^{k-1} \mathcal{O}(-m + b_i),$$

as required.

\[\square\]
Chapter 4

Principal Bundles

4.1 Preliminaries

Let $X$ be a Riemann Surface.

**Definition 1.** A fiber bundle over $X$ is a triple $(E,F,\pi)$ with $E$ and $F$ being topologies so that:

- $\pi : E \to X$ is a surjection.
- For every $x \in X$ there is an open set $x \in U$ and a chart $\phi : \pi^{-1}(U) \to U \times F$ so that $\text{proj}_U \circ \phi(q) = \pi(q)$ for $q \in \pi^{-1}(U)$

We say that $F$ is the fiber, $E$ is the total space and $\pi$ is the projection.

**Definition 2.** Let $G$ be a group. A principal $G$-Bundle, $P$, is a fiber bundle with $G$ as its fiber. We also require a continuous right $G$-action on $P$ that is free and transitive.

We mainly concern ourselves with $G$ being a Lie-group.

**Definition 3.** Let $(P,\pi)$ be a principal $G$-bundle over $X$. Let $\rho$ be a continuous action on the space of homeomorphisms of a topology $F$. Let $\rho$ be the right action given by the $(p,f)g = (pg,\rho(g^{-1}f))$. We say the associated bundle is $(P \times_\rho F,\pi_\rho)$ where:

- $P \times_\rho F = P \times F/\sim$ where the equivalence classes are given by $[pg,f] = [p,\rho(g)f]$
• $\pi_p[p, f] = \pi(p)$

**Definition 4.** Let $H$ be a subgroup of $G$. We say $P$ has a reduction to $H$, if there is a non-zero section in $P \times_G G/H$.

**Definition 5.** Let $\rho$ be a representation of $G$ into $GL(V)$. We define $P \times_G \rho : P \times_G G \to P \times_G GL(V)$.

**Definition 6.** Let $G$ be a connected compact Lie group. Let $T$ be a maximal torus and $N$, its normalizer. Then we define the Weyl group to be $N/T$.

**Definition 7.** We say a Lie group is reductive if its Lie algebra is reductive. We say a Lie algebra is reductive if it can be written as a direct sum of a semi-simple algebra and its center.

From this point on we let $G$ be a compact Lie group, let $G_0$ be the connected component at the identity and let $\mathfrak{g}$ be its Lie-algebra. We let $H, N, W$ and $\mathfrak{h}$ be a Cartan subgroup, normalizer of a Cartan subgroup, Weyl group and the lie subalgebra of the Cartan subgroup. Let let $ad$ be the adjoint representation of $G$.

Let $P$ be a holomorphic principal $G$-bundle and $E = P \times_G ad$.

Finally, let $H^1(X, \mathcal{O}_X(G))$ be the set of holomorphic $G$-bundles over $X$.

**Theorem 4** (Grothendieck’s Theorem for the Orthogonal Case). A vector bundle $V$ has an orthogonal form if and only if it is isomorphic to its dual.

We do not prove this theorem within this paper.

**Lemma 7.** Suppose we have a holomorphic section $s$ in $E$ and there is a fiber $E_a$ so that $s(a)$ is a regular element of the lie algebra of $E_a$. Then for any $x$, $s(x)$ is regular in $E_x$.

**Proof.** The coefficients of the polynomial defining $ads(x)$ must be constant as they are holomorphic functions, by compactness of $X$. Thus $s(x)$ is a regular element everywhere. \hfill $\square$

**Lemma 8.** Suppose we have a section $s$ in $E$. Then we have a section in $P \times_G G/N$.

**Proof.** By the maximal torus theorem, any 2 Cartan subgroups are conjugate. The kernel of the action on any particular maximal torus $T$ is $N(T)$. It follows that $G/N$ is the set of Cartan subalgebras. The section given by sending $s(x)$ to its corresponding subalgebra gives a section in $P \times_G G/N$. \hfill $\square$

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Lemma 9. Suppose we have a section in $P \times_G G/N$. Then we have a section in $P \times_G G/T$.

Proof. We first prove the Weyl group is discrete. For any torus $T$ of rank $n$ we have the following short exact sequence:

$$0 \to \mathbb{Z}^n \to \mathbb{R}^n \to T \to 0$$

It follows that $\text{Gal}(K^{\text{sep}}/K)(T) \subset \text{GL}_n(\mathbb{Z})$ which is discrete.

We then have the sequence:

$$0 \to P \times_G W \to P \times_G G/T \to P \times_G G/N \to 0$$

Since $X$ is simply connected $P \times_G W$ is trivial and we have the desired.

Definition 8. We define the Killing form as:

$$B(x,y) = \text{tr}(\text{ad}(x)\text{ad}(y)) \text{ for } x,y \in \mathfrak{g}.$$ 

It has a few key properties that we will use. Namely:

- That the Killing form of a nilpotent algebra is everywhere 0.
- A Lie algebra is Semi-simple iff the Killing form is non-degenerate over the algebra
- 2 ideals of a Lie algebra have no intersections then they are orthogonal with respect to the Killing form.

Suppose $G$ is a compact reductive Lie group. Writing $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s}$ for the abelian and semi-simple parts respectively induces a decomposition of each of the fibers $E_x = E_x^1 \oplus E_x^s$. It suffices to show we can find a regular element in the semi-simple part.

Now let $G$ be a compact semi-simple Lie group. Let $E_k$ be the vector subfibers of $E$ with meromorphic sections of degree at least $k$. Notice that $[E_i, E_j] \subset E_{i+j}$ by counting degrees. This implies that elements of $E_1$ are $ad_{\mathfrak{g}}$-nilpotent. Let the sub-algebra defined by $E_1$ be $\mathfrak{g}_1$ and we now have an orthogonal fiber $E_0$, by the Killing form. Let the orthogonal sub-algebra under the Killing form be $\mathfrak{g}_0$.

Lemma 10. There is a section in $P \times_G \text{ad}$ that is regular at some point.

Proof. Consider the Cartan subalgebras of $\mathfrak{g}_0$. Choose a regular element. Since $\mathfrak{g}_0$ is orthogonal to $\mathfrak{g}_1$, lift it to a global section.

We need 1 more lemma before we are finally ready to prove Grothendieck’s theorem.

Lemma 11. If $G$ is a reductive connected Lie group. There is some finite subgroup, $z$, so that $G/z$ is the product of an abelian and semisimple group.
4.2 Grothendieck’s Theorem

**Theorem 5.** Classification of Principle Bundles on $\mathbb{P}^1(\mathbb{C})$ Let $G$ be a reductive connected Lie group. The map:

$$H^1(X, O_X(H))/W \to H^1(X, O_X(G))$$

Is a bijection.

**Proof.** We have seen the surjectivity of it above. Consider the commutative diagram:

$$
\begin{array}{ccc}
H^1(X, O_X(H)) & \longrightarrow & H^1(X, O_X(G)) \\
\downarrow & & \downarrow \\
H^1(X, O_X(H/z)) & \longrightarrow & H^1(X, O_X(G/z))
\end{array}
$$

Suppose $\alpha, \beta \in H^1(X, O_X(H))$ are mapped to the same image in $H^1(X, O_X(G))$. Looking at the diagram, it is clear that they must have the same image in $H^1(X, O_X(G/z))$ or $H^1(X, O_X(H/z))$. In the first, by 3rd isomorphism theorem we have that $\alpha$ and $\beta$ are in the same equivalence class when taking the quotient with the Weyl group: $(N/z)/(H/z) = W$. In the second case we have a contradiction as $H^1(X, z) = 0$ by $z$ being finite and $X$ being connected inducing a bijection in the first cohomology groups $H^1(X, H)$ and $H^1(X, H/z)$. 

\[\square\]
Chapter 5

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