Estimates on modulation spaces for Schrödinger operators with time-dependent sub-linear vector potentials

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Abstract
In this paper, we give estimates of the solutions to Schrödinger equation on modulation spaces with vector potential of sub-linear growth.

1 Introduction
In this paper, we consider the following initial value problem for the Schrödinger equations with magnetic vector potentials of sub-linear growth
\begin{align}
&i\partial_t u(t, x) + \frac{1}{2} (\nabla - ia(t, x))^2 u(t, x) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R}^n,
\end{align}
and give estimates for the solutions in modulation spaces by the initial data, where \(i = \sqrt{-1}\), \(u(t, x)\) is a complex-valued unknown function of \((t, x) \in \mathbb{R} \times \mathbb{R}^n\), \(u_0(x)\) is a complex-valued given function of \(x \in \mathbb{R}^n\), \(\partial_t u = \partial u/\partial t\), \(\partial_{x_j} u = \partial u/\partial x_j \ (j = 1, \ldots, n)\) and \(\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})\).

In [10], the second author has shown estimates of the solutions of (1) on modulation spaces in the case that \(a(t, x) = A(t)x\) with real symmetric matrices \(A(t)\). Here we shall show estimates on modulation spaces in the case that vector potential \(a(t, x)\) depends on \(t\) and \(x\) more generally but is sub-linear in \(x\), which satisfies the following Assumption 1.1.

Our basic strategy in previous papers [5] and [10] is to estimate the integral equation on the phase space by the wave packet transform and to apply Gronwall’s inequality. However, this strategy is not applicable in the present case since we have to estimate the canonical momentum of classical particle \(\xi(t)\), which arises in the integral equation and comes from the first order differential term \(-ia(t, x) \cdot \nabla\).

Assumption 1.1. For \(j = 1, \ldots, n\), \(j\)-th component \(a_j(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\) of \(a(t, x)\) is \(C^\infty\)-function with respect to \(t, x\) and satisfies that there exists \(\rho < 1\) such that for any \(\alpha \in \mathbb{Z}^n_+\), there exists a constant \(C_\alpha > 0\) satisfying
\[\max_{1 \leq j \leq n} |\partial_{x_j}^\alpha a_j(t, x)| \leq C_\alpha \langle x \rangle^{\rho - |\alpha|}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n\]
where, \(\langle x \rangle = (1 + |x|^2)^{1/2}\).

Let \(S(\mathbb{R}^n)\) be the space of rapidly decreasing smooth functions on \(\mathbb{R}^n\) and \(S'(\mathbb{R}^n)\) be its dual space.

Definition 1.2 (Wave packet transform). Let \(\varphi \in S(\mathbb{R}^n) \setminus \{0\}\) and \(f \in S'(\mathbb{R}^n)\). The wave packet transform \(W_{\varphi} f\) of \(f\) with the basic wave packet \(\varphi\) is defined by
\[W_{\varphi} f(x, \xi) = \int_{\mathbb{R}^n} \overline{\varphi(y - x)} e^{-iy \cdot \xi} f(y) dy, \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.\]
Definition 1.3 (Modulation space). Let $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and $1 \leq p, q \leq \infty$. We define the modulation spaces $M^p_q(\mathbb{R}^n)$ as follows.

$$M^p_q(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \|f\|_{M^p_q} = \left\| W_{\varphi} f(x, \xi) \right\|_{L^p_x L^q_\xi} < \infty \right\}. $$

Our purpose in this study is to estimate the solution of the initial value problem (1) on modulation spaces. The following theorem is our main result.

Theorem 1.4. Let $1 \leq p \leq \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, $T > 0$ and $u(t, x)$ be the solution of (1) in $C(\mathbb{R}; L^2(\mathbb{R}^n))$ for $u_0 \in \mathcal{S}(\mathbb{R}^n)$. If $a$ satisfies the Assumption 1.1, then there exists $C_T > 0$ such that

$$\|u(t, \cdot)\|_{M^p_q} \leq C_T \|u_0\|_{M^p_q}$$

for all $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and $t \in [-T, T]$.

Remark 1.5. The estimate (3) is also valid in the case that the vector potential $a(t, x) = A(t)x + a_0(t, x)$ with $A(t)$ being a real symmetric matrix-valued smooth function and $a_0(t, x)$ satisfying Assumption 1.1, which follows from Theorem 1.4 and the previous result [10].

We denote the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ by $\hat{f}$ or $\mathcal{F}f$ and the inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ by $\mathcal{F}^{-1}f$. We denote the Schrödinger propagator of a free particle by $e^{it\Delta/2}$, which is defined by

$$\left( e^{it\frac{\Delta}{2}} f \right)(x) = \mathcal{F}^{-1}_{\xi \rightarrow x} \left[ e^{-it\frac{\xi^2}{2}} \hat{f}(\xi) \right](x), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

For simplicity, we use the notations

$$W_{\varphi(t, \cdot)} u(t, x, \xi) = W_{\varphi(t, \cdot)} [u(t, \cdot)](x, \xi) = \int_{\mathbb{R}^n} \varphi(t, y - x) u(t, y) e^{-iy \cdot \xi} dy$$

and $\|u(t)\|_{M^p_q(\mathbb{R}^n)} = \left\| W_{\varphi(t, \cdot)} u(t, x, \xi) \right\|_{L^p_x L^q_\xi}$.

Schrödinger equations with time-independent magnetic potential $a(t, x) = a(x)$ have been investigated by B. Simon [11], T. Kato [7], and so on. In [11], B. Simon showed the essentially self-adjointness of $H_0 = -\nabla^2 - ia(x)^2$ on $C^\infty_0$ when $\text{div} a = 0$, $a \in L^q_{\text{loc}}(\mathbb{R}^n)$ with $q > \text{max}(n, 4)$. In [7], T. Kato relaxed some conditions of $a(x)$ for $H_0$ to be essentially self-adjoint on $C^\infty_0$ stated in [11]. In H. Leinfelder and C. G. Simader’s work [9], they proved the existence and uniqueness of the $L^2$-solution to the equation (1) with $a(x)$ under the more general assumption which allows $a(x)$ to be in $L^q_{\text{loc}}(\mathbb{R}^n)$ and its derivative to be in $L^2_{\text{loc}}(\mathbb{R}^n)$.

When the magnetic potential depends on time, this problem becomes more difficult and delicate. K. Yajima, in the work of [13], proved the existence and uniqueness of the $L^2$-solution to the equation (1) and $L^p$-smoothing property of the unitary propagator $\{U(t, s)\}_{t, s \in \mathbb{R}}$ of (1) assuming that growth of $a(t, x)$ and $\partial_t a(t, x)$ are equal to first degree polynomial at infinity; i.e. $|a(t, x)| + |\partial_t a(t, x)| \sim |x|$.

From these results, the equation (1) can be solved on $L^2(\mathbb{R}^n)$, while we cannot expect to solve this equation on $L^p(\mathbb{R}^n)$ for $p \neq 2$. On the contrary, there are many works on existence of solutions to the following Schrödinger equations with scholar potentials $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ in modulation spaces.

$$\begin{cases}
i \partial_t u(t, x) + \frac{i}{2} \Delta u(t, x) = V(t, x) u(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n.
\end{cases}$$
In the work of A. Bényi, K. Gröchenig, K. A. Okoudjou and L. G. Rogers [1], it is shown that the Schrödinger group \( e^{it|\nabla|^\alpha} \) for \( 0 \leq \alpha \leq 2 \) is bounded on \( M^{p,q}(\mathbb{R}^n) \) and this implies that the free Schrödinger equation (i.e. equation (1) with \( V(t,x) \equiv 0 \)) can be solved in \( M^{p,q}(\mathbb{R}^n) \) for \( 1 \leq p, q \leq \infty \). B. Wang and H. Hudzik showed that the global well-posedness of the nonlinear Schrödinger equation with power type nonlinearity by using the dispersive estimate for the free Schrödinger equation in \( M^{p,q}(\mathbb{R}^n) \), \( \|u(t,\cdot)\|_{M^{p,q}} \leq C(1+|t|)^{-n(1/2-1/p)}\|u_0\|_{M^{p,q}} \), see [2]. In the work of [3], the solution to the free Schrödinger equation or Schrödinger equation with the harmonic oscillator preserve the norm of \( M^{p,q}(\mathbb{R}^n) \), \( \|u(t,\cdot)\|_{M^{p,q}} = \|u_0\|_{M^{p,q}} \). In the case of time-dependent potential, the estimate of the solution to equation (4) with quadratic or sub-quadratic potential on \( M^{p,q}(\mathbb{R}^n) \), \( \|u(t,\cdot)\|_{M^{p,q}} \leq C_T \|u_0\|_{M^{p,q}} \) is obtained in [2] and [4].

This paper is organized as follows. In Section 2, we introduce terminology and preliminaries. We will give the representation of solution of (1) by wave packet transform and introduce some lemmas on characteristics corresponding to (1). In Section 3 we will prove the Theorem 1.4.

### 2 Preliminaries

In this section, we prepare several lemmas for the proof of Theorem 1.4. We firstly remark the following properties of modulation spaces (For more details and proofs, see §4 and §6 in [3]).

**Lemma 2.1.** Let \( 1 \leq p, q, p_1, p_2, q_2 \leq \infty \). Then

(i) \( M^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^n) \) for \( p_1 \leq p_2, q_1 \leq q_2 \).

(ii) \( M^{p,q}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow M^{p',q}(\mathbb{R}^n) \) for \( 1 \leq q_1 \leq \min(p,p') \) and \( q_2 \geq \max(p,p') \) with \( 1/p + 1/p' = 1 \). In particular, \( M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \) holds.

(iii) \( S(\mathbb{R}^n) \) is dense in \( M^{p,q}(\mathbb{R}^n) \) for \( 1 \leq p, q < \infty \).

(iv) \( M^{p,q}(\mathbb{R}^n) \) is a Banach space with norm \( \|\cdot\|_{M^{p,q}} \).

(v) The definition of \( M^{p,q}(\mathbb{R}^n) \) is independent of the choice of the basic wave packet \( \varphi \). More precisely, for any \( \varphi, \psi \in S(\mathbb{R}^n) \setminus \{0\} \), the norm \( \|\cdot\|_{M^{p,q}} \) is equivalent to the norm \( \|\cdot\|_{M^{p,q}} \).

**Definition 2.2** (Inverse wave packet transform). Let \( \varphi \in S(\mathbb{R}^n) \setminus \{0\} \) and \( F \in S'(\mathbb{R}^{2n}) \), we define the adjoint operator \( W_{\varphi}^* \) of \( W_{\varphi} \) by

\[
W_{\varphi}^*[F(y,\xi)](x) = \int_{\mathbb{R}^{2n}} \varphi(x-y)e^{i(x \cdot \xi)}F(y,\xi)dyd\xi, \quad x \in \mathbb{R}^n,
\]

where \( d\xi = (2\pi)^{-n}d\xi \).

Then, for \( f \in S'(\mathbb{R}^n) \), the following inversion formula holds (see [4] Corollary 11.2.7);

\[
(5) \quad f(x) = \frac{1}{\|\varphi\|_{L^2}^2} W_{\varphi}^*[W_{\varphi}f](x).
\]

For the proof of Theorem 1.4, we reduce (1) to a first-order partial differential equation in \( \mathbb{R}^{2n} \) by using the wave packet transform. Using formula

\[
\frac{1}{2}(\nabla - ia)^2 u = \frac{1}{2}\Delta u - \frac{1}{2}a^2 u - i \left( \frac{1}{2}(\nabla \cdot a) + a \cdot \nabla \right) u,
\]
we have

\[
W_{\varphi(t, \cdot)} \left[ i \partial_t u + \frac{1}{2} \Delta u - \frac{1}{2} a^2 u \right] (t, x, \xi)
= \left( i \partial_t + i \xi \cdot \nabla_x - \frac{\left| \xi \right|^2}{2} \right)
- \frac{1}{2} a^2 (t, x) - i \nabla_x a^2 (t, x) \cdot \nabla \xi + \frac{1}{2} \nabla_x a^2 (t, x) \cdot x \right) W_{\varphi(t, \cdot)} u(t, x, \xi)
+ W(\omega_0)_{c(t, \cdot)} u(t, x, \xi) + R_1 u(t, x, \xi),
\]

where

\[
R_1 u(t, x, \xi) = -\frac{1}{2} \sum_{j, k, l=1}^n \int_{t, x, \xi} \psi(t, y - x) R_{j, k, l}^1(t, y, x) u(t, y) e^{-iy \cdot \xi} dy,
\]

\[
R_{j, k, l}^1(t, y, x) = \int_0^1 (\partial^2_{x \theta} a_j)(t, x + \theta(y - x))(1 - \theta) d\theta(y_k - x_k)(y_l - x_l).
\]

By integration by parts, we have

\[
W_{\varphi(t, \cdot)} \left[ -i \left( \frac{1}{2} (\nabla \cdot a) + a \cdot \nabla \right) \right] (t, x, \xi)
= \left( \frac{i}{2} (\nabla \cdot a)(t, x) - i a(t, x) \cdot \nabla_x + \xi \cdot a(t, x) + \nabla_x (\xi \cdot a(t, x)) \cdot (i \nabla \xi - x) \right)
\times W_{\varphi(t, \cdot)} u(t, x, \xi) + R_2 u(t, x, \xi) + R_3 u(t, x, \xi),
\]

where

\[
R_2 u(t, x, \xi) = \sum_{j, k=1}^n \int \left( R_{j, k}^{2, 1}(t, y, x) \varphi(t, y - x) + R_{j, k}^{2, 2}(t, y, x) \partial_k \varphi(t, y - x) \right) u(t, y) e^{-iy \cdot \xi} dy,
\]

\[
R_3 u(t, x, \xi) = \sum_{j, k, l=1}^n \int \varphi_{j, k, l}(t, y - x) \xi_j u(t, y) R_{j, k, l}^1(t, y, x) e^{-iy \cdot \xi} dy,
\]

\[
R_{j, k, l}^{2, 1}(t, y, x) = \int_0^1 (\partial^2_{x \theta} a_j)(t, x + \theta(y - x))(1 - \theta) d\theta(y_k - x_k),
\]

\[
R_{j, k, l}^{2, 2}(t, y, x) = \int_0^1 (\partial_{x \theta} a_k)(t, x + \theta(y - x))(1 - \theta) d\theta(y_l - x_l),
\]

\[
\varphi_{j, k, l}(t, y - x) = (y_l - x_l)(y_k - x_k) \varphi(t, y - x).
\]

Taking \( \varphi(t, x) = e^{it \Delta/2} \varphi_0(x) \) for \( \varphi_0(x) \in \mathcal{S}(\mathbb{R}^n) \) and combining (6)-(7), we transform (1) into

\[
\left\{ \begin{array}{l}
(i \partial_t + i \nabla_x H(t, x, \xi) \cdot \nabla_x - i \nabla_x H(t, x, \xi) \cdot \nabla \xi \\
+ b(t, x, \xi)) W_{\varphi(t, \cdot)} u(t, x, \xi) = Ru(t, x, \xi),
\end{array} \right.
\]

\[
W_{\varphi(0, \cdot)} u(0, x, \xi) = W_{\varphi_0} u_0(x, \xi),
\]

where

\[
H(t, x, \xi) = \frac{1}{2} \left| \xi - a(t, x) \right|^2,
\]
\[ h(t, x, \xi) = -H(t, x, \xi) + \nabla_x H(t, x, \xi) \cdot x + \frac{i}{2} \nabla_x \cdot a(t, x), \]
\[ R_u(t, x, \xi) = -R_1 u(t, x, \xi) - R_2 u(t, x, \xi) - R_3 u(t, x, \xi). \]

By the method of characteristics, we obtain the following lemma.

**Lemma 2.3.** For \( t \in \mathbb{R} \) and \( x, \xi \in \mathbb{R}^n \), we define \( x(s) = x(s; t, x, \xi) \) and \( \xi(s) = \xi(s; t, x, \xi) \) as the solutions of

\[
\begin{aligned}
\dot{x}(s) &= \nabla_\xi H(s, x(s), \xi(s)), \quad x(t) = x, \\
\dot{\xi}(s) &= -\nabla_x H(s, x(s), \xi(s)), \quad \xi(t) = \xi.
\end{aligned}
\]

Then the solution \( u(t, x) \) of (11) satisfies the integral equation

\[
W_{\varphi(t, \cdot)} u(t, x, \xi) = e^{-i \int_0^t h(s, x(s), \xi(s)) \, ds} \left( W_{\varphi(0, \cdot)} u_0(x(0), \xi(0)) \right) - i \int_0^t e^{i \int_0^t h(s, x(s), \xi(s)) \, ds} R_u(\tau, x(\tau; t, x, \xi), \xi(\tau; t, x, \xi)) \, d\tau.
\]

Taking the \( L^\infty(\mathbb{R}^n_x) \times L^\infty(\mathbb{R}^n_x) \) norm on both sides of (11), we have the following integral inequality

\[
\| W_{\varphi(t, \cdot)} u(t, x, \xi) \|_{L^\infty_x L^\infty_\xi} \leq C_1 \left( \| W_{\varphi(0, \cdot)} u_0(x(0), \xi(0)) \|_{L^\infty_x L^\infty_\xi} \right) + C_1 \sum_{j=1}^3 \left( \int_0^T |R_j u(\tau, x(\tau; t, x, \xi), \xi(\tau; t, x, \xi))| \, d\tau \right)_{L^\infty_x L^\infty_\xi}
\]

where \( C_1 = \sup_{\tau \in [0, T]} \| e^{i \int_0^\tau h(s, x(s), \xi(s)) \, ds} \|_{L^\infty_x L^\infty_\xi} \).

We get the estimate of the first term and remainder term with \( j = 1, 2 \) on the right hand side of the above by the same way discussed in [10] or [5].

**Lemma 2.4.** Let \( T > 0 \), \( \varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \) and \( N \in \mathbb{N} \) be \( N \geq n + 1 \). Then

\[
\| W_{\varphi(t, \cdot)} u_0(x(0), \xi(0)) \|_{L^\infty_x L^\infty_\xi} = \| W_{\varphi_0} u_0(x, \xi) \|_{L^\infty_x L^\infty_\xi},
\]
\[
\| R_j u(\tau, x(\tau; t, x, \xi), \xi(\tau; t, x, \xi)) \|_{L^\infty_x L^\infty_\xi} \leq C_2 \| W_{\varphi(t, \cdot)} u(\tau, x, \xi) \|_{L^\infty_x L^\infty_\xi}, \quad \tau \in [0, T], \; j = 1, 2
\]

hold where

\[
C_2 = \sup_{\tau \in [0, T]} \| (\eta \cdot \nabla)^{-2N} \|_{L^2_x} \sum_{k,l=1}^n \sum_{\beta_1, \beta_2 \in \mathbb{Z}_+^n} \| (\hat{\varphi}^\beta_1 \hat{u}_{k,l}(\tau)) \|_{L^1} \| (\hat{\varphi}^\beta_2 \varphi(\tau)) \|_{L^1}.
\]

We only give an outline of the proof for the reader’s convenience. For more detail, see §4 in [5] or §3 in [10].

**Outline of the proof.** The change of variables yields the first equality since the Jacobian of \((x(s), \xi(s))\) with respect to \((x, \xi)\) is constant 1 for any \( s, t \in \mathbb{R} \) and \( x, \xi \in \mathbb{R}^n \). We only treat \( R_1 u \). Applying the inversion formula (4) and integration by parts for 2N times with 2N > n, we have

\[
\| R_1 u(\tau, x(\tau), \xi(\tau)) \|_{L^\infty_x L^\infty_\xi}
\]
Thus we have the second inequality by Assumption 1.1 and Housdorff-Young's inequality.

Lemma 2.5. Let $\delta > 0$. Then there exist $T_0 \in (0, 1)$ and a constant $C_\delta > 0$ such that for any $T \in (0, T_0)$, $t \in \mathbb{R}$ and $x, \xi \in \mathbb{R}^n$, it holds that
\[
\int_0^T \langle \xi(t); x(t), \xi(t) \rangle \leq C_\delta (1 + T).
\]
The above estimate of characteristics (8) is the key for the proof of our result. The lemma is proved by putting $q(t) = x(t)$, $v(t) = \xi(t) - a(t, x(t))$ in the proof of Lemma 2.1 in [14].

3 Proof of Theorem 1.4

In this section, we prove Theorem 1.4 by the following 4 steps:

(I) To show (3) for $p = 2$.

(II) To show (3) for $p = \infty$.

(III) To show (3) for $2 < p < \infty$ by usual complex interpolation.

(IV) To show (3) for $1 \leq p < 2$ by duality argument.

(I) From $L^2$ conservation law and Lemma 2.1 (ii), we get
\begin{equation}
\|u(t)\|_{M^2; T_0^2} \leq C_{\varphi_0} \|u(t)\|_{L^2(\mathbb{R}^n)} = C_{\varphi_0} \|u_0\|_{L^2(\mathbb{R}^n)} \leq C_{\varphi_0} \|u_0\|_{M^2; T_0^2}.
\end{equation}
(II) For the proof of the case $p = \infty$, we prepare the following key lemma, which will be proved at the end of the section.

Lemma 3.1. Under the assumption in Theorem 1.4, there exist a sufficiently large $\lambda \geq 1$ and constant $C_T > 0$ such that
\begin{equation}
\|u(t)\|_{M_{\varphi_\lambda(t)}^\infty} \leq C_T \|u_0\|_{M_{\varphi_\lambda(0)}^\infty}
\end{equation}
holds for any $t \in [0, T]$, where $\varphi_\lambda(x) = \lambda^{n/2} \varphi(\lambda x)$, $\varphi_\lambda(t, x) = e^{it\Delta/2} \varphi_\lambda(x)$.

By virtue of the above lemma and Lemma 2.1 (v), the estimate (3) with $p = \infty$ is obtained as
\begin{equation}
\|u(t)\|_{M_{\varphi_\lambda(t)}^\infty} \leq C_{\varphi_0} \|u(t)\|_{M_{\varphi_\lambda(t)}^\infty} \leq C_{\varphi_0, T} \|u_0\|_{M_{\varphi_\lambda(0)}^\infty} \leq C_{\varphi_0, T} \|u_0\|_{M_{\varphi_0}^\infty}.
\end{equation}
(III) For the case that $2 \leq p \leq \infty$, we get (3) by using (11), (13) and Riesz-Thorin's interpolation theorem.

(IV) Suppose that $u(t, x)$ satisfies the same assumption stated in Theorem 1.4 and (3) holds for
where $\{\}$

By virtue of (9), the solution of (14) is represented by the wave packet transform $p$ of $N(16)$

$$\frac{1}{2} (\nabla - ia(s, x))^2 u(s, x) = 0, \quad (s, x) \in \mathbb{R} \times \mathbb{R}^n,$$

$$u(t, x) = u_0(x), \quad x \in \mathbb{R}^n.$$

For $\phi \in L^{p'}(\mathbb{R}^n)$, we have

$$\left| (W_{\psi(t)} u(t), \phi)_{L^2(\mathbb{R}^n)} \right| = \| \phi \|_{L^2(\mathbb{R}^n)}^2 \left| (W_{\psi(t)} U(t, 0)[W_{\phi^*} W_{\phi^0} u_0], \phi)_{L^2(\mathbb{R}^n)} \right|$$

$$\leq C \left| (W_{\phi^0} u_0, W_{\phi^0} U(0, t) W_{\psi(t)} \phi)_{L^2(\mathbb{R}^n)} \right|$$

$$\leq C \| W_{\phi^0} u_0 \|_{L^{p'}(\mathbb{R}^n)} \| W_{\phi^0} U(0, t) W_{\psi(t)} \phi \|_{L^{p'}(\mathbb{R}^n)}^2,$$

which shows

$$\| W_{\psi(t)} u(t) \|_{L^r L^r} = \sup_{\phi \in L^{p'}(\mathbb{R}^n)} \left| (W_{\psi(t)} u(t), \phi)_{L^2(\mathbb{R}^n)} \right| \| \phi \|_{L^{p'} L^{p''}}$$

$$\leq C \| W_{\phi^0} u_0 \|_{L^{p'}(\mathbb{R}^n)} \| \phi \|_{L^{p'} L^{p''}}^2 \sup_{\phi \neq 0} \left| (W_{\phi^0} U(0, t) W_{\psi(t)} \phi)_{L^2(\mathbb{R}^n)} \right| \| \phi \|_{L^{p'}(\mathbb{R}^n)} \| \phi \|_{L^{p''}(\mathbb{R}^n)}.$$

Hence it suffices to prove that

$$\| W_{\phi^0} U(0, t) W_{\psi(t)} \phi \|_{L^{p'}(\mathbb{R}^n)} \leq C_T \| \phi \|_{L^{p'} L^{p''}}.$$

By virtue of (13), the solution of (14) is represented by the wave packet transform $W_{\psi(s)\star}(s, x, \xi)$

$$W_{\psi(s)\star}(s, x, \xi) = e^{-i \int^s_0 (\tau, x(\tau), \xi(\tau)) d\tau} W_{\psi(t)} u_0(x(s; s, x, \xi), \xi(s; s, x, \xi))$$

$$- i \int^s_0 e^{i \int^\tau_0 (\tau, x(\tau'), \xi(\tau')) d\tau'} Ru(\tau, x(s; s, x, \xi), \xi(s; s, x, \xi)) d\tau',$$

where

$$\begin{cases}
\dot{x}(\tau) = \nabla_{\xi} H(\tau, x(\tau), \xi(\tau)),
\xi(s) = \xi,
\dot{\xi}(\tau) = -\nabla_{\xi} H(\tau, x(\tau), \xi(\tau)),
\end{cases}$$

and $\varphi(s) = e^{is\Delta/2} \varphi_0$. Thus we have for $2 \leq p' \leq \infty$,

$$\| W_{\phi^0}(s) U(s, t) u_0 \|_{L^{p'} L^{p''}} \leq C_T \| W_{\psi(t)} u_0 \|_{L^{p'} L^{p''}}, \quad s \in [t - T, t + T].$$

Putting $u_0(x) = W_{\phi^0}(\phi(x))$ and $s = 0$ into (16), we get (15) by virtue of

$$\| W_{\psi(t)} W_{\phi^0}(\phi) \|_{L^{p'} L^{p''}}$$

$$\leq \sum_{|\alpha| + |\beta| \leq 2N} \| \langle \xi \rangle^{-2N} \| \| \partial^\alpha_x \varphi(t) \|_{L^1} \| \partial^\beta_x \varphi(t) \|_{L^1} \| \phi \|_{L^{p'} L^{p''}}$$

where $N \geq n + 1$, which completes the proof.
Proof of Lemma 3.1. It suffices to show that there exist constants $T_0 = T_0(\lambda, n, \rho, \phi_0 > 0$ and $C = C(n, \rho, \phi_0) > 0$ such that

\begin{equation}
\left\| \int_0^{T_0} |R_3(\tau, x(\tau), \xi(\tau))| d\tau \right\|_{L^\infty_t L^\infty_x} \leq C \lambda^{-1} \sup_{\tau \in [0, T_0]} \| W_{\varphi^\lambda(\tau)} u(\tau, x, \xi) \|_{L^\infty_t L^\infty_x}.
\end{equation}

Indeed, (17) and Lemma 2.4 yield

\begin{align*}
\| W_{\varphi^\lambda(\tau)} u(t, x, \xi) \|_{L^\infty_t L^\infty_x} &\leq C_1 \| W_{\varphi^\lambda(\tau)} w_0(x, \xi) \|_{L^\infty_t L^\infty_x} \\
&+ C_1 (2C_2 T_0 + C \lambda^{-1}) \sup_{\tau \in [0, T_0]} \| W_{\varphi^\lambda(\tau)} u(\tau, x, \xi) \|_{L^\infty_t L^\infty_x}.
\end{align*}

Thus the inequality (12) holds for $t \in [0, T_0]$ if we take $\lambda$ and $T_0$ satisfying small as $C_1 (2C_2 T_0 + C \lambda^{-1}) < 1/2$ and taking $\sup_{\tau \in [0, T_0]}$ on both sides of the above. For general $T > 0$, we obtain the conclusion by applying the estimate (12), iteratively.

Now we will show (17). We prove (17) in the case that $n$ is odd. For even $n$, we can show by using

\begin{equation}
\frac{(1 - \Delta_y)^{n/2}}{\langle \eta \rangle^{n}} e^{iy \eta} = \sum_{j=1}^{n} \frac{\eta_j}{\langle \eta \rangle^{n}} (1 - \Delta_y)^{n/2} (-i \partial_y) e^{iy \eta}
\end{equation}

instead of

\begin{equation}
\frac{(1 - \Delta_y)^{N}}{\langle \eta \rangle^{2N}} e^{iy \eta} = e^{iy \eta}
\end{equation}

in the following proof. Using the inversion formula (5), integration by parts and applying the formula (13) with $2N = n + 1$, we have

\begin{align*}
\int_0^{T} |R_3(\tau, x(\tau), \xi(\tau))| d\tau &= \sum_{j,k,l=1}^{n} \sum_{|\alpha_1| + |\alpha_2| + |\alpha_3| \leq n+1} C_{\alpha_1, \alpha_2, \alpha_3} \int_0^{T} \int \int \frac{\partial^{\alpha_1}_{\xi} \varphi_{j,k,l}^\lambda(\tau, y - x(\tau), \xi(\tau), y - \xi(\tau))}{\langle \eta - \xi(\tau) \rangle^{n+1}} \langle \eta \rangle^{n+1} W_{\varphi^\lambda(\tau)} u(\tau, x, \xi) dydzd\eta d\tau.
\end{align*}

Here, we only estimate the right hand side for $\alpha_3 = 0$, fixed $\alpha_1, \alpha_2$ satisfying $|\alpha_1| + |\alpha_2| = n + 1$ and fixed $j, k, l \in \{1, ..., n\}$. In the case that $\alpha_3 \neq 0$, it can be obtained in the same way. From Assumption (14) $|\xi_j(\tau) R_{j,k,l}(\tau, y, x(\tau))|$ can be estimated as

\begin{align*}
| \xi_j(\tau) R_{j,k,l}(\tau, y, x(\tau)) | &\leq |\xi_j(\tau)| \int_0^{1} |(\partial_{x_3, x_j}(\tau, x(\tau) + \theta(y - x(\tau)))| d\theta \\
&\leq C |\xi(\tau)| \int_0^{1} (x(\tau) + \theta(y - x(\tau)))^{n-2} d\theta \\
&\leq C \frac{\| \xi(\tau) \| (y - x(\tau))^{2-\rho}}{(x(\tau))^{2-\rho}}.
\end{align*}
Hence we have
\[
\left\| \int_0^T \left\| \int \int \int \frac{\partial_{y_1}^{\alpha_1} \varphi_{k,l}^\lambda(\tau, y - x(\tau)) \partial_{y_2}^{\alpha_2} \varphi^\lambda(\tau, y - z)}{e^{\tau_0 - \xi(\tau)}} e^{w_0 (\tau - \xi(\tau))} W^{\varphi^\lambda(\tau)} u(\tau, z, \eta) dy dz d\eta d\tau \right\|_{L^2_x L^\infty_t} \leq C \int_0^T \left( \int \int |\partial_{y_1}^{\alpha_1} \varphi_{k,l}^\lambda(\tau, y - x(\tau))| |\partial_{y_2}^{\alpha_2} \varphi^\lambda(\tau, y - z)| \right.
\]
\[
\times (y - x(\tau))^{2 - \rho} \left( \frac{\xi(\tau)}{\langle x(\tau) \rangle^{2 - \rho}} \right) \left( \frac{W^{\varphi^\lambda(\tau)} u(\tau, z, \eta)}{(y - \xi(\tau))^{n + 1}} \right) dy dz d\tau \right\|_{L^\infty_x L^2_t} \leq CC \rho (1 + T) \]
\[
\times \sup_{\tau \in [0, T]} \left\| (y)^{2 + |\alpha_1| - \rho} \partial_{y_1}^{\alpha_1} \varphi_{k,l}^\lambda(\tau, y) \right\|_{L^1_y} \left\| \partial_{y_2}^{\alpha_2} \varphi^\lambda(\tau) \right\|_{L^1_y} \left\| W^{\varphi^\lambda(\tau)} u(\tau) \right\|_{L^2_x L^\infty_t}.
\]
The last inequality of the above follows from Lemma 2.5 by taking sufficiently small \( T \in (0, 1) \) since \( \rho - 2 < -1 \). Hence Hausdorff-Young’s inequality yields
\[
\left\| \int_0^T \left\| R_3(\tau, x(\tau), \xi(\tau)) \right\|_t \right\|_{L^\infty_x L^2_t} \leq 2CC \rho \sum_{j,k,l=1}^n \sum_{\alpha_1, \alpha_2, \alpha_3} \left\| (\eta)^{-n+1} \right\|_{L^1_\eta} \times \sup_{\tau \in [0, T]} \left\| (y)^{2 + |\alpha_1| - \rho} \partial_{y_1}^{\alpha_1} \varphi_{k,l}^\lambda(\tau, y) \right\|_{L^1_y} \left\| \partial_{y_2}^{\alpha_2} \varphi^\lambda(\tau) \right\|_{L^1_y} \left\| W^{\varphi^\lambda(\tau)} u(\tau) \right\|_{L^2_x L^\infty_t}.
\]
The proof is done if we show
\[
\left\| (y)^{2 + |\alpha_1| - \rho} \partial_{y_1}^{\alpha_1} \varphi_{k,l}^\lambda(\tau, y) \right\|_{L^1_y} \left\| \partial_{y_2}^{\alpha_2} \varphi^\lambda(\tau) \right\|_{L^1_y} \leq C_{n, \varphi^\lambda} \lambda^{-1}.
\]
Since \( \partial^\alpha e^{i\tau \Delta/2} = e^{i\tau \Delta/2} \partial^\alpha \) and \( y_k e^{i\tau \Delta/2} = e^{i\tau \Delta/2} (y_k - i\tau \partial_{y_k}) \) hold, we have
\[
\begin{align*}
\partial_{y_1}^{\alpha_1} \varphi^\lambda(\tau, y) &= \lambda^{\alpha_1} (\partial_{y_1}^{\alpha_1})^\lambda(\tau, y), \\
\varphi_{k,l}^\lambda(\tau, y) &= y_k y_l \varphi^\lambda(t, x) \\
&= \lambda^{-2} ((\varphi_{k,l})^\lambda(\tau, y) \\
&\quad - 2i\lambda^2 \tau (\partial_{y_k} \varphi_{k,l})^\lambda(\tau, y) - (\lambda^2 \tau)^2 (\partial_{y_k \partial_{y_l}} \varphi_{k,l}^\lambda(\tau, y)), \\
(y)^{2N} \varphi^\lambda(\tau, y) &= \sum_{|\alpha_1| + |\alpha_2| \leq 2N} (-i\lambda^2)^{|\alpha_1|} (\partial_{y_1}^{\alpha_1} \varphi_{k,l}^\lambda(\tau, y)),
\end{align*}
\]
for \( \alpha \in \mathbb{Z}^n, N \in \mathbb{N} \) and \( k, l = 1, \ldots, n \).
For $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$, $\tau \in [0, T]$ and $N \geq n + 1$, it holds that

$$\|\phi^\lambda(\tau)\|_{L^1} \leq C_{n, N, \phi_0} \lambda^{-n/2} \sum_{j=0}^{2N} (\lambda^2 T)^j.$$  

Indeed, using integration by parts and formula (13), we have

$$\|\phi^\lambda(\tau)\|_{L^1} = \int \left| \int e^{iy' \cdot \xi} e^{-i\tau |\xi|^2/2} \lambda^{-n/2} \phi_0(\xi/\lambda) \hat{d} \xi \right| \, dy'$$

$$= \lambda^{n/2} \int \left| \int e^{iy' \cdot \xi} e^{-i\lambda^2 \tau |\xi|^2/2} \phi_0(\xi) \hat{d} \xi \right| \, dy'$$

$$= \lambda^{-n/2} \int \left| \int e^{iy \cdot \xi} e^{-i\lambda^2 \tau |\xi|^2/2} \phi_0(\xi) \hat{d} \xi \right| \, dy$$

$$\leq \lambda^{-n/2} C_N \sum_{|\beta_1| + |\beta_2| \leq 2N} (\lambda^2 \tau)^{|\beta_1|} \int \frac{dy}{(y)^{2N}} \int \xi^{|\beta_1|} \partial^{|\beta_2|} \phi_0(\xi) \hat{d} \xi$$

$$\leq C_{n, N, \phi_0} \sum_{|\beta_1| + |\beta_2| \leq 2N} \|\xi^{|\beta_1|} \partial^{|\beta_2|} \phi_0(\xi)\|_{L^1} \lambda^{-n/2} \sum_{j=0}^{2N} (\lambda^2 T)^j.$$  

Combining (20)–(23) and taking $T_0 \in (0, 1)$ with $\lambda^2 T_0 < 1/2$, we have for $N \geq n + 1$,

$$\sup_{\tau \in [0, T_0]} \| (y)^{2+|\alpha_1|} \partial^{|\alpha_1|} \varphi_{k,l}(\tau) \|_{L^1_y} \| \partial^{|\alpha_2|} \varphi^\lambda(\tau) \|_{L^1_y}$$

$$\leq \lambda^{-2} \sup_{\tau \in [0, T_0]} \| (y)^{2N} \partial^{|\alpha_1|} \varphi_{k,l}(\tau) \|_{L^1_y} + 2 \lambda^2 T_0 \| (y)^{2N} \partial^{|\alpha_2|} \varphi(\tau) \|_{L^1_y}$$

$$\leq \lambda^{-|\alpha_1|} \sum_{|\beta_1| + |\beta_2| \leq 2N} (\lambda T_0)^{|\beta_1|} \sup_{\tau \in [0, T_0]} \left( \left| \int \partial^{|\beta_2|} y^{|\beta_1|} \varphi_{k,l} \hat{d} \xi \right| \right) \left( \left| \int \partial^{|\beta_2|} y^{|\beta_1|} \phi(\tau) \hat{d} \xi \right| \right) \lambda^{-|\alpha_1|} \| \partial^{|\alpha_2|} \varphi(\tau) \|_{L^1_y}$$

$$\leq C_{n, N, \phi_0} \lambda^{-|\alpha_1|} \lambda^{2} \sum_{|\beta_1| \leq 2N} (\lambda T_0)^{|\beta_1|} \left( \lambda^{-n/2} \sum_{j=0}^{2N} (\lambda^2 T)^j \right)^2 \lambda^{2} \sum_{j=0}^{2N} (\lambda^2 T)^j$$

$$\leq C_n \lambda^{-1} \left( \sum_{k=0}^{\infty} (\lambda^2 T_0)^k \right)^3 \leq 8 C_n \lambda^{-1}.$$  

The constant $C_2$ in Lemma 23, with $\varphi(\tau) = \varphi^\lambda(\tau)$ can be estimated as

$$C_2 \leq 4 C_n \lambda^{-1}.$$
by using the above estimates. Hence we get (17) as
\[
\left\| \int_0^{T_0} |R_3(\tau,x(\tau),\xi(\tau))| \, d\tau \right\|_{L^\infty L^\infty_x L^\infty_\xi} \\
\leq 2C_\rho 8C_\rho^\alpha \lambda^{-1} \left( \sum_{j,k,l=1}^n \sum_{\alpha_1,\alpha_2,\alpha_3} C_\alpha \left\| (\eta)^{-(n+1)} \right\|_{L^1_\eta} \right) \sup_{\tau \in [0,T_0]} \left\| W_{\varphi^\lambda(\tau)} u(\tau) \right\|_{L^\infty_x L^\infty_\xi} \\
\leq 16C_\rho 8C_\rho^\alpha \lambda^{-1} \sup_{\tau \in [0,T_0]} \left\| W_{\varphi^\lambda(\tau)} u(\tau) \right\|_{L^\infty_x L^\infty_\xi},
\]
which completes the proof.

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