Character analogues of certain Hardy-Berndt sums

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Abstract

In this paper we consider transformation formulas for
\[ B(z, s : \chi) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \chi(m) \chi(2n + 1) (2n + 1)^{s-1} e^{\pi i m (2n+1)z/k}. \]

We derive reciprocity theorems for the sums arising in these transformation formulas and investigate certain properties of them. With the help of the character analogues of the Euler–Maclaurin summation formula we establish integral representations for the Hardy-Berndt character sums \( s_{3,p}(d, c : \chi) \) and \( s_{4,p}(d, c : \chi) \).

Keywords: Dedekind sums, Hardy-Berndt sums, Bernoulli polynomials, Euler-Maclaurin formula.

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1 Introduction

Berndt [5] and Goldberg [13] derived transformation formulas for the logarithms of the classical theta functions \( \theta_j(z), j = 2, 3, 4 \). In these formulas six different arithmetic sums arise that are known as Berndt’s arithmetic sums or Hardy sums. Goldberg [13] show that these sums also arise in the theory of \( r_m(n) \), the number of representations of \( n \) as a sum of \( m \) integral squares and in the study of the Fourier coefficients of the reciprocals of \( \theta_j(z), j = 2, 3, 4 \). Three of these sums, which we call Hardy-Berndt sums, are defined for \( c > 0 \) by

\[ S(d, c) = \sum_{n=1}^{c-1} (-1)^{n+1+[dn/c]}, \quad s_3(d, c) = \sum_{n=1}^{c-1} (-1)^n \overline{B}_1 \left( \frac{dn}{c} \right), \quad s_4(d, c) = \sum_{n=1}^{c-1} (-1)^{[dn/c]}, \]

where \( \overline{B}_p(x) \) are the Bernoulli functions (see Section 2) and \( [x] \) denotes the greatest integer not exceeding \( x \).

Analogous to Dedekind sums these sums also obey reciprocity formulas. For instance, for coprime positive integers \( d \) and \( c \) we have [5, 13]:

\[ S(d, c) + S(c, d) = 1, \quad \text{if } d + c \text{ is odd}, \]

\[ 2s_3(d, c) - s_4(c, d) = 1 - \frac{d}{c}, \quad \text{if } c \text{ is odd} \]

and [18]

\[ s_4(c, d) + s_4(d, c) \equiv -1 + cd \pmod{8}. \]

We note that various properties of these sums have been investigated by many authors, see [2, 5, 6, 7, 13, 16, 17, 18, 20, 21, 22, 23], and several generalizations have been studied in [8, 9, 12, 15, 19].
A character analogue of classical Dedekind sum, called as Dedekind character sum, appears in the transformation formula of a generalized Eisenstein series $G(z, s; r_1, r_2)$ (see (6) below) associated to a non-principle primitive character $\chi$ of modulus $k$ defined by Berndt in [3]. This sum is defined by

$$s(d, c : \chi) = \sum_{n=1}^{ck} \chi(n) \overline{B}_{1, \chi}(\frac{dn}{c}) \overline{B}_1(\frac{n}{ck})$$

and possesses the reciprocity formula

$$s(c, d : \chi) + s(d, c : \chi) = B_{1, \chi} B_{1, \overline{\chi}},$$

whenever $d$ and $c$ are coprime positive integers, and either $c$ or $d \equiv 0 \pmod{k}$ ([3]). Here $\overline{B}_{p, \chi}(x)$ are the generalized Bernoulli functions (see (5) below) and $B_{p, \chi} = \overline{B}_{p, \chi}(0)$. The sum $s(d, c : \chi)$ is generalized by

$$s_p(d, c : \chi) = \sum_{n=1}^{ck} \chi(n) \overline{B}_{p, \chi}(\frac{dn}{c}) \overline{B}_1(\frac{n}{ck})$$

and corresponding reciprocity formula is established ([11]).

To the writers’ knowledge generalizations of Hardy-Berndt sums, in the sense of $s_p(d, c : \chi)$, have not been studied. To introduce such generalization, originated by Berndt’s paper [3] and the fact

$$\log \theta_4(z) = -2 \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{\pi im(2n+1)z},$$

we set the function $B(z, s; \chi)$ to be

$$B(z, s; \chi) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \chi(m) \chi(2n+1) (2n+1)^{s-1} e^{\pi im(2n+1)z/k}$$

for $Im(z) > 0$ and for all $s$.

The objective of this paper is to obtain transformation formulas for $B(z, s; \chi)$ and investigate certain properties of the sums that arise in these formulas. These are generalizations, containing characters and generalized Bernoulli functions, of the sums $S(d, c)$, $s_3(d, c)$, $s_4(d, c)$ and the sums considered in [8]. We will show that these sums satisfy reciprocity formulas in the sense of (1) and (2). We also give integral representations for them.

A brief plan of the paper is as follows: Section 2 is the preliminary section where we give definitions and terminology needed. In Section 3 we state and prove main theorems concerning transformation formulas for $B(z, s; \chi)$. In Section 4 we give definitions of the character analogues of Hardy-Berndt sums and prove corresponding reciprocity theorems. We present some additional results in Section 5, in particular we derive several interesting formulas relating these sums by employing fixed points of a modular transformation. In the final section we apply the character analogues of Euler–Maclaurin summation formula to give an alternative proof of one of the reciprocity formula.

## 2 Preliminaries

Throughout this paper $\chi$ denotes a non-principal primitive character of modulus $k$. The letter $p$ always denotes positive integer. We use the modular transformation $(az + b) / (cz + d)$ where $a$, $b$, $c$ and $d$ are integers with $ad - bc = 1$ and $c > 0$. The upper half-plane $\{x + iy : y > 0\}$ will be denoted by $\mathbb{H}$ and the upper quarter-plane
\( \{ x + iy : x > -\frac{d}{2}, y > 0 \} \) by \( K \). We use the notation \( \{ x \} \) for the fractional part of \( x \). Unless otherwise stated, we assume that the branch of the argument is defined by \( -\pi \leq \arg z < \pi \).

The Bernoulli polynomials \( B_n(x) \) are defined by means of the generating function

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}
\]

and \( B_n(0) = B_n \) are the Bernoulli numbers with \( B_0 = 1 \), \( B_1 = -1/2 \) and \( B_{2n-1} \left( \frac{1}{2} \right) = B_{2n+1} = 0 \) for \( n \geq 1 \).

The Bernoulli functions \( \overline{B}_n(x) \) are defined by

\[
\overline{B}_n(x) = B_n(\{x\}), \quad n > 1,
\]

\[
\overline{B}_1(x) = \begin{cases} 0, & x \text{ integer}, \\ B_1(\{x\}), & \text{otherwise} \end{cases}
\]

and satisfy Raabe theorem for any \( x \)

\[
\sum_{j=0}^{r-1} \overline{B}_n \left( x + \frac{j}{r} \right) = r^{1-n} \overline{B}_n (rx).
\]

(3)

\( \overline{E}_n(x) \) are the Euler functions and we are only interested in the following property ([10, Eq. (4.5)])

\[
r^{n-1} \sum_{j=0}^{r-1} (-1)^j \overline{E}_n \left( \frac{x + j}{r} \right) = -\frac{n}{2} \overline{E}_{n-1} (x)
\]

(4)

for even \( r \) and any \( x \).

\( \overline{B}_{n, \chi}(x) \) are the generalized Bernoulli functions defined by Berndt [4]. We will often use the following property that can confer as a definition

\[
\overline{B}_{n, \chi}(x) = k^{n-1} \sum_{j=0}^{k-1} \chi(j) \overline{B}_n \left( \frac{j+x}{k} \right), \quad n \geq 1.
\]

(5)

The Gauss sum \( G(z, \chi) \) is defined by

\[
G(z, \chi) = \sum_{m=0}^{k-1} \chi(m)e^{2\pi imz/k}.
\]

We put \( G(1, \chi) = G(\chi) \). If \( n \) is an integer, then [1, p. 168]

\[
G(n, \chi) = \overline{\chi}(n)G(\chi).
\]

Let \( r_1 \) and \( r_2 \) be arbitrary real numbers. For \( z \in \mathbb{H} \) and \( Re(s) > 2 \), Berndt [3] defines \( G(z, s : \chi : r_1, r_2) \) as

\[
G(z, s : \chi : r_1, r_2) = \sum_{m,n := -\infty}^{\infty} \frac{\chi(m)\overline{\chi}(n)}{(m + r_1)z + n + r_2}^s,
\]

where the dash means that the possible pair \( m = -r_1, n = -r_2 \) is omitted from the summation. Extending the definition of \( \chi \) to the set of all real numbers by defining \( \chi(r) = 0 \) if \( r \) is not an integer, it is shown that

\[
G(z, s : \chi : r_1, r_2) = \frac{G(\chi) (-2\pi i/k)^s}{\Gamma(s)} \left\{ A(z, s : \chi : r_1, r_2) + e^{\pi is} A(z, s : \chi : -r_1, -r_2) \right\}
\]

\[
+ \chi (-r_1) \left\{ L(s, \chi, r_2) + \chi (-1) e^{\pi is} L(s, \overline{\chi}, -r_2) \right\},
\]

(6)
where

\[ L(s, \chi, \alpha) = \sum_{m > -a}^{\infty} \chi(m) (m + \alpha)^s, \ \alpha \ \text{real and} \ \text{Re}(s) > 1 \]

and

\[ A(z, s : \chi ; r_1, r_2) = \sum_{m > -r_1}^{\infty} \chi(m) \sum_{n=1}^{\infty} \chi(n) n^{s-1} e^{2\pi i n ((m+r_1)z + r_2)/k}. \]

For \( r_1 = r_2 = 0 \) we will use the notations \( G(z, s : \chi) = G(z, s : \chi ; 0, 0) \) and \( A(z, s : \chi) = A(z, s : \chi ; 0, 0) \).

Also, for \( r_1 = r_2 = 0 \), (6) reduces to

\[ \Gamma(s) G(z, s : \chi) = G(\overline{\chi}) \left( -\frac{2\pi i}{k} \right)^s H(z, s : \chi) \]

where \( H(z, s : \chi) = (1 + e^{\pi i s}) A(z, s : \chi) \).

The following lemma due to Lewittes [14, Lemma 1].

**Lemma 1** Let \( A, B, C \) and \( D \) be real with \( A \) and \( B \) not both zero and \( C > 0 \). Then for \( z \in \mathbb{H} \),

\[ \arg \left( (Az + B) / (Cz + D) \right) = \arg (Az + B) - \arg (Cz + D) + 2\pi l, \]

where \( l \) is independent of \( z \) and \( l = \begin{cases} 1, & A \leq 0 \text{ and } AD - BC > 0, \\ 0, & \text{otherwise}. \end{cases} \)

In accordance with the subject of this study we present Berndt’s transformation formulas for \( r_1 = r_2 = 0 \) (see [19, Theorem 2] for a generalization).

**Theorem 2** [3, Theorem 2] Let \( Tz = (az + b) / (cz + d) \). Suppose first that \( a \equiv d \equiv 0 \text{(mod} k) \). Then for \( z \in \mathbb{K} \) and all \( s \),

\[ (cz + d)^{-s} \Gamma(s) G(Tz, s : \chi) \]

\[ = \overline{\chi}(b) \chi(c) \Gamma(s) G(z, s : \chi) + \overline{\chi}(b) \chi(c) e^{-\pi i s} \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \chi(\mu c + j) \chi \left( \left\lfloor \frac{dj}{c} \right\rfloor + \nu \right) f(z, s : c, d), \]

where

\[ f(z, s : c, d) = \int_{C} e^{-(\mu c + j)(cz + d)u/c} e^{(\nu + \left\lfloor \frac{dj}{c} \right\rfloor)u} e^{2\pi i u} \frac{u^{s-1} du}{e^{ku} - 1}, \]

(7)

where \( C \) is a loop beginning at \( +\infty \), proceeding in the upper half-plane, encircling the origin in the positive direction so that \( u = 0 \) is the only zero of \( (e^{-(cz + d)u/c} - 1) (e^{ku} - 1) \) lying “inside” the loop, and then returning to \( +\infty \) in the lower half-plane. Here we choose the branch of \( u^s \) with \( 0 < \arg u < 2\pi \).

Secondly, if \( b \equiv c \equiv 0 \text{(mod} k) \), we have for \( z \in \mathbb{K} \) and all \( s \),

\[ (cz + d)^{-s} \Gamma(s) G(Tz, s : \chi) \]

\[ = \overline{\chi}(a) \chi(d) \Gamma(s) G(z, s : \chi) + \overline{\chi}(a) \chi(d) e^{-\pi i s} \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \chi(j) \chi \left( \left\lfloor \frac{dj}{c} \right\rfloor + d\mu - \nu \right) f(z, s : c, d). \]
3 Transformation Formulas

In the sequel, unless otherwise stated, we assume that $k$ is odd.

Put $B'(z, s : \chi) = (1 + e^{\pi is}) B(z, s : \chi)$. We then use the relation

$$B(z, s : \chi) = A\left(\frac{z}{2}, s : \chi\right) - \chi(2) 2^{s-1} A(z, s : \chi)$$  \hspace{1cm} (8)

in order to achieve transformation formulas for $B'(z, s : \chi)$.

**Theorem 3** Let $Tz = (az + b) / (cz + d)$ with $b$ is even. If $a \equiv d \equiv 0(\text{mod } k)$, then for $z \in \mathbb{K}$ and all $s$

$$(cz + d)^{-s} G(\chi) B'(Tz, s : \chi) = \mathcal{X}\left(\frac{b}{2}\right) \chi(2c) G(\chi) B'(z, s : \chi)$$

$$+ \mathcal{X}\left(\frac{b}{2}\right) \chi(2c) \left(\begin{array}{c} - \frac{k}{2\pi i} \\ - \frac{k}{2\pi i} \end{array}\right) e^{-\pi is} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \sum_{j=1}^{2c} \chi(2c\mu + j) \chi\left(\left\lfloor \frac{dj}{2c}\right\rfloor - \nu \right) f\left(\frac{z}{2}, s : 2c, d\right)$$

$$- \chi(2) 2^{s-1} \sum_{j=1}^{c} \chi(j) \chi\left(\left\lfloor \frac{dj}{2c}\right\rfloor + d\mu - \nu \right) f(z, s : c, d)$$  \hspace{1cm} (9)

If $b \equiv c \equiv 0(\text{mod } k)$, then for $z \in \mathbb{K}$ and all $s$

$$(cz + d)^{-s} G(\chi) B'(Tz, s : \chi) = \mathcal{X}(a) \chi(d) G(\chi) B'(z, s : \chi)$$

$$+ \mathcal{X}(a) \chi(d) \left(\begin{array}{c} - \frac{k}{2\pi i} \\ - \frac{k}{2\pi i} \end{array}\right) e^{-\pi is} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \sum_{j=1}^{2c} \chi(j) \chi\left(\left\lfloor \frac{dj}{2c}\right\rfloor + d\mu - \nu \right) f\left(\frac{z}{2}, s : 2c, d\right)$$

$$- \chi(2) 2^{s-1} \sum_{j=1}^{c} \chi(j) \chi\left(\left\lfloor \frac{dj}{2c}\right\rfloor + d\mu - \nu \right) f(z, s : c, d)$$  \hspace{1cm} (10)

where $f(\frac{z}{2}, s : 2c, d)$ and $f(z, s : c, d)$ are given by (7).

**Proof.** Let $b$ be even and consider $Uz = (az + \frac{b}{2}) / (2cz + d)$. Since $U(z/2) = \frac{1}{2} T(z)$ we have by (8)

$$(cz + d)^{-s} B'(Tz, s : \chi) = \left(2c\frac{z}{2} + d\right)^{-s} H(U(z/2), s : \chi) - \chi(2) 2^{s-1} (cz + d)^{-s} H(Tz, s : \chi)$$

Applying Theorem 2 to the right-hand side of equality we get the desired results.  

This theorem may be simplified for nonpositive integer value of $s$ as in the following.

**Corollary 4** Let $p$ be odd and $Tz = (az + b) / (cz + d)$ with $b$ is even. If $a \equiv d \equiv 0(\text{mod } k)$, then for $z \in \mathbb{H}$

$$(cz + d)^{p-1} G(\chi) B'(Tz, 1 - p : \chi) = \mathcal{X}(\frac{b}{2}) \chi(2c) G(\chi) B'(z, 1 - p : \chi) + \left(\frac{2\pi i}{p+1}\right)^p \chi(-1) g_1(c, d; z, p; \chi)$$  \hspace{1cm} (11)

If $b \equiv c \equiv 0(\text{mod } k)$, then for $z \in \mathbb{H}$

$$(cz + d)^{p-1} G(\chi) B'(Tz, 1 - p : \chi) = \mathcal{X}(a) \chi(d) G(\chi) B'(z, 1 - p : \chi) + \left(\frac{2\pi i}{p+1}\right)^p \chi(-1) g_1(c, d; z, p; \chi)$$  \hspace{1cm} (12)
where
\[ g_1(c, d; z, p; \chi) = - \sum_{m=1}^{p} \frac{(p+1)}{m} (-(cz+d))^{m-1} k^{m-p} \frac{m}{2m} \sum_{n=1}^{c(k-1,m,c)} \chi(n) \mathcal{B}_{p+1-m} \left( \frac{dn}{2c} \right) \mathcal{B}_{m-1} \left( \frac{n}{ck} \right). \quad (13) \]

**Proof.** For \( s = 1 - p \) in Theorem 3, by residue theorem, we have
\[ f(z, 1-p : c, d) = \frac{2\pi ik^{p-1}}{(p+1)!} \sum_{m=0}^{p} \left( \frac{p+1}{m} \right) (-(cz+d))^{m-1} B_{p+1-m} \left( \frac{\nu + \{dj/c\}}{k} \right) B_{m} \left( \frac{\mu c + j}{ck} \right). \quad (14) \]
Let \( b \) be even and \( a \equiv d \equiv 0(\text{mod } k) \). Substituting (14) in (9) and by the fact that the sum over \( \mu \) is zero for \( m = 0 \) and the sum over \( \nu \) is zero for \( m = p + 1 \), we have
\[ (cz+d)^{p-1} G(\chi)B'(Vz, 1-p : \chi) = \chi \left( \frac{b}{2} \right) \chi(2c)G(\chi)B'(z, 1-p : \chi) \]
\[ + \chi \left( \frac{b}{2} \right) \chi(2c) \left( \frac{2(2c)^p}{(p+1)!} \right) \sum_{m=1}^{p} \left( \frac{p+1}{m} \right) (-(cz+d))^{m-1} \]
\[ \times \left\{ \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{2c} \chi(2\mu c + j) \chi \left( \left[ \frac{dj}{2c} \right] - \nu \right) B_{p+1-m} \left( \frac{\nu + \{dj/2c\}}{k} \right) B_{m} \left( \frac{2\mu c + j}{2ck} \right) \right\}. \quad (15) \]

We first evaluate the triple sum in (15). Observe that the triple sum is unchanged if we replace \( B_{p+1-m} \left( \frac{\nu + \{dj/2c\}}{k} \right) \) by \( \mathcal{B}_{p+1-m} \left( \frac{\nu + \{dj/2c\}}{k} \right) \) since \( B_{p+1-m} \left( \frac{\nu + \{dj/2c\}}{k} \right) = \mathcal{B}_{p+1-m} \left( \frac{\nu + \{dj/2c\}}{k} \right) \) when \( 0 < \frac{\nu + \{dj/2c\}}{k} < 1 \), and \( \chi(d) = 0 \) when \( \frac{\nu + \{dj/2c\}}{k} = 0 \) \( (d \equiv 0(\text{mod } k)) \). By the same reason the triple sum is unchanged when \( B_{m} \left( \frac{2\mu c + j}{2ck} \right) \) is replaced by \( \mathcal{B}_{m} \left( \frac{2\mu c + j}{2ck} \right) \). By (5),
\[ \chi \left( -1 \right) k^{m-p} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{2c} \chi(2\mu c + j) \mathcal{B}_{m} \left( \frac{2\mu c + j}{2ck} \right) \mathcal{B}_{p+1-m} \left( \frac{\nu + \{dj\}}{k} \right) \]
\[ = \chi \left( -1 \right) k^{m-p} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{2c} \chi(2\mu c + j) \mathcal{B}_{p+1-m} \left( \frac{\nu + \{dj\}}{k} \right) \mathcal{B}_{m} \left( \frac{2\mu c + j}{2ck} \right) \]
Put \( 2\mu c + j = n \), where \( 1 \leq n \leq 2ck \) in the right side of equality above. Since \( \mathcal{B}_{p,\chi}(-x) = (-1)^{p} \chi(-1) \mathcal{B}_{p,\chi}(x) \) and \( \mathcal{B}_{p,\chi}(x+k) = \mathcal{B}_{p,\chi}(x) \), (15) becomes
\[ \chi \left( -1 \right) k^{m-p} \sum_{n=1}^{2ck} \chi(n) \mathcal{B}_{p+1-m} \left( \frac{dn}{2c} \right) \mathcal{B}_{m} \left( \frac{n}{2ck} \right) \]
\[ = \chi \left( -1 \right) k^{m-p} \sum_{n=1}^{ck} \chi(n) \mathcal{B}_{p+1-m} \left( \frac{dn}{2c} \right) 2\mathcal{B}_{m} \left( \frac{n}{2ck} \right). \quad (17) \]
Secondly, (16) can be evaluated as
\[ \chi \left( -1 \right) k^{m-p} \chi(2) 2^{-p} \sum_{n=1}^{ck} \chi(n) \mathcal{B}_{p+1-m} \left( \frac{dn}{c} \right) \mathcal{B}_{m} \left( \frac{n}{ck} \right). \quad (19) \]
It can be seen from (5) and (3) that
\[
\chi(r)p^{1-m}B_{m,x}(rx) = \sum_{j=0}^{r-1} B_{m,x} \left( x + \frac{jk}{r} \right)
\]
for \((r, k) = 1\). Thus, in the light of (20), (19) becomes
\[
\chi(-1)k^{m-p} \sum_{n=1}^{ck} \chi(n)B_{p+1-m,x} \left( \frac{dn}{2c} \right) 2^{1-m}B_m \left( \frac{n}{ck} \right).
\]
Then, utilizing (3) and (4), (18) and (21) yield (11) for \(z \in \mathbb{K}\). Since the functions \(g_1(c, d; z, p)\) and \(B'(z, 1 - p : \chi)\) are analytic on \(\mathbb{H}\), (11) is valid for all \(z \in \mathbb{H}\) by analytic continuation.

The proof for \(b \equiv c \equiv 0(\text{mod } k)\) is analogous. 

By taking \(Vz = T(z + k) = (az + b + ak) / (cz + d + ck)\) with \(a\) and \(b\) odd, instead of \(Tz = (az + b) / (cz + d)\) in Corollary 4 we obtain the following result which involves a different sum.

**Corollary 5** Let \(p\) be odd and let \(Vz = (az + b + ak) / (cz + d + ck)\) with \(a\) and \(b\) are odd. If \(a \equiv d \equiv 0(\text{mod } k)\), then for \(z \in \mathbb{H}\)
\[
(cz + d + ck)^{p-1}G(\chi)B'(Vz, 1 - p : \chi)
= \chi \left( \frac{b + ak}{2} \right) \chi(2c) \left\{ G(\chi)B'(z, 1 - p : \chi) + \frac{(2\pi i)^p \chi(-1)}{(p+1)!} g_1(c, d + ck; z, p; \chi) \right\}.
\]
If \(b \equiv c \equiv 0(\text{mod } k)\), then for \(z \in \mathbb{H}\)
\[
(cz + d + ck)^{p-1}G(\chi)B'(Vz, 1 - p : \chi)
= \chi(a)\chi(d)G(\chi)B'(z, 1 - p : \chi) + \chi(a)\chi(d) \frac{(2\pi i)^p \chi(-1)}{(p+1)!} g_1(c, d + ck; z, p; \chi),
\]
where
\[
g_1(c, d + ck; z, p; \chi)
= - \sum_{m=1}^{p} \left( \frac{p+1}{m} \right) (cz + d + ck)^{m-1} k^{m-p} \frac{m}{2^m} \sum_{n=1}^{ck} \chi(n)B_{p+1-m,x} \left( \frac{(d + ck)n}{2c} \right) E_{m-1} \left( \frac{n}{ck} \right).
\]
We note that this result shows that the sum \(S(d, c)\) can arise in the formula for \(\log \theta_4 \left( \frac{az + b + c}{cz + d + ec} \right)\), where \(a\) and \(b\) are odd.

We conclude this section by assuming that \(a\) is even instead of \(b\) in Theorem 3.

**Theorem 6** Let \(Tz = (az + b) / (cz + d)\) with \(a\) is even. If \(a \equiv d \equiv 0(\text{mod } k)\), then for \(z \in \mathbb{K}\) and \(s \in \mathbb{C}\)
\[
2^{1-s}(cz + d)^sG(\chi)B'(Tz, s : \chi)
= \chi(b)\chi(c)G(\chi) \left\{ 2H(2z, s : \chi) - \chi(2)H(z, s : \chi) \right\}
+ \chi(b)\chi(c) \left( - \frac{k}{2\pi i} \right)^s e^{-\pi is} \sum_{c-1}^{k-1} \sum_{j=1}^{k-1} \sum_{\mu=0}^{k-1} \chi(c \mu + j)
\times \left\{ 2\chi \left( \left[ \frac{2d}{c} \right] - \nu \right) f(2z, s : c, 2d) - \chi(2)\chi \left( \left[ \frac{d}{c} \right] - \nu \right) f(z, s : c, d) \right\}.
\]
If \( b \equiv c \equiv \bar{0}\text{mod} k \), then for \( z \in \mathbb{K} \) and \( s \in \mathbb{C} \)

\[
2^{1-s} (cz + d)^{-s} G \left( \overline{\chi} \right) B' \left( Tz, s : \chi \right)
= \overline{\chi} \left( \frac{a}{2} \right) \chi(2d)G(\overline{\chi}) \left\{ 2H \left( 2z, s : \chi \right) - \overline{\chi}(2)H \left( z, s : \chi \right) \right\}
+ \overline{\chi} \left( \frac{a}{2} \right) \chi(2d) \left( -\frac{k}{2 \pi i} \right)^s e^{-\pi is} \sum_{j=1}^{\infty} \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \chi(j) \times \left\{ 2\overline{\chi} \left( \frac{2d\nu}{c} \right) + 2d\mu - \nu \right\} f(2z, s : c, 2d) - \overline{\chi}(2)\overline{\chi} \left( \frac{d\nu}{c} \right) + d\mu - \nu \right\} f(z, s : c, d) \right\}.
\] (26)

Proof. Let \( a \) be even and \( Wz = \left( \frac{a}{2}z + b \right)/(cz + 2d) \). Since \( W \left( 2z \right) = \frac{1}{4}T(z) \) we have by (8)

\[
(cz + d)^{-s} B'(\overline{T}z, s : \chi) = 2^s (2cz + 2d)^{-s} H(W \left( 2z \right), s : \chi) - \chi(2) 2^{-s} (cz + d)^{-s} H(Tz, s : \chi).
\]

Again applying Theorem 2 to the right-hand side we get the desired results. \( \blacksquare \)

**Corollary 7** Let \( p \) be odd and \( a \) be even. If \( a \equiv d \equiv \bar{0}\text{mod} k \), then for \( z \in \mathbb{H} \)

\[
2^p (cz + d)^{p-1} G \left( \overline{\chi} \right) B' \left( Tz, 1 - p : \chi \right)
\]

\[
eq \overline{\chi}(b)\chi(c)G(\chi) \left\{ 2H \left( 2z, 1 - p : \chi \right) - \chi(2)H \left( z, 1 - p : \chi \right) \right\} + \overline{\chi}(b)\chi(c) \left( 2\pi i \right)^p \frac{\chi(-2)}{p+1} g_2(c, d; z, p; \overline{\chi}).
\] (27)

If \( b \equiv c \equiv \bar{0}\text{mod} k \), then for \( z \in \mathbb{H} \)

\[
2^p (cz + d)^{p-1} G \left( \overline{\chi} \right) B' \left( Tz, 1 - p : \chi \right)
\]

\[
eq \overline{\chi} \left( \frac{a}{2} \right) \chi(2d)G(\overline{\chi}) \left\{ 2H \left( 2z, 1 - p : \chi \right) - \overline{\chi}(2)H \left( z, 1 - p : \chi \right) \right\} + \overline{\chi} \left( \frac{a}{2} \right) \chi(2d) \left( 2\pi i \right)^p \frac{\chi(-2)}{p+1} g_2(c, d; z, p; \chi),
\]

where

\[
g_2(c, d; z, p; \chi) = - \sum_{m=1}^{p} \binom{p+1}{m} \left( - (cz + d) \right)^{m-1} \frac{m}{2} \sum_{n=1}^{ck} (-1)^n \chi(n) B_{p+1-m, \chi} \left( \frac{n}{c} \right) \mathcal{E}_{m-1} \left( \frac{n}{ck} \right).
\] (29)

**Proof.** Let \( a \) be even and \( a \equiv d \equiv \bar{0}\text{mod} k \). Since

\[
f(z, 1 - p : c, d) = \frac{2\pi ik^{p-1}}{(p+1)!} \sum_{m=0}^{p+1} \binom{p+1}{m} \left( - (cz + d) \right)^{m-1} B_{p+1-m} \left( \frac{\nu + (dj/c)}{k} \right) B_m \left( \frac{\nu + dj/c}{k} \right),
\]

(25) becomes

\[
2^p (cz + d)^{p-1} G \left( \overline{\chi} \right) B' \left( Tz, 1 - p : \chi \right)
\]

\[
eq \overline{\chi}(b)\chi(c)G(\chi) \left\{ 2H \left( 2z, 1 - p : \chi \right) - \chi(2)H \left( z, 1 - p : \chi \right) \right\}
+ \overline{\chi}(b)\chi(c) \left( 2\pi i \right)^p \frac{\chi(-2)}{p+1} g_2(c, d; z, p; \chi)
\]

\[
\times \left\{ 2^m \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \chi(c\mu + j) \chi \left( \frac{2d\nu}{c} \right) - \nu \right\} B_{p+1-m} \left( \frac{\nu + (2dj/c)}{k} \right) B_m \left( \frac{\nu + dj/c}{k} \right)
\]

\[
- \chi(2) \sum_{j=1}^{c} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{k-1} \chi(c\mu + j) \chi \left( \frac{d\nu}{c} \right) - \nu \right\} B_{p+1-m} \left( \frac{\nu + (dj/c)}{k} \right) B_m \left( \frac{\nu + dj/c}{k} \right) \right\}.
\] (30)
Note that the above sums over \( \mu \) and \( \nu \) are zero for \( m = 0 \) and \( m = p + 1 \), respectively. Similar to the evaluation of (15), (30) can be evaluated as

\[
2^m \chi(-1) k^{m-p} \sum_{n=1}^{ck} \chi(n) B_{p+1-m} \left( \frac{2dn}{c} \right) B_m \left( \frac{n}{ck} \right)
\]

and (31) can be evaluated as

\[
\chi(2) \chi(-1) k^{m-p} \sum_{n=1}^{ck} \chi(n) B_{p+1-m} \left( \frac{dn}{c} \right) B_m \left( \frac{n}{ck} \right)
\]

\[
= \chi(2) \chi(-1) k^{m-p} \sum_{n=1}^{ck} \chi(2n) B_{p+1-m} \left( \frac{2dn}{c} \right) B_m \left( \frac{2n}{ck} \right)
\]

since \( 2n \) runs through a complete residue system \((\text{mod } ck)\) as \( n \) does, for \( (2,ck) = 1 \). Substituting (32), (34) in (30), (31), respectively, and using (3), (4) we have

\[
\chi(-1) k^{m-p} 2^{m-1} \sum_{n=1}^{ck} \chi(n) B_{p+1-m} \left( \frac{2dn}{c} \right) \left\{ B_m \left( \frac{n}{ck} \right) - B_m \left( \frac{n}{ck} + \frac{1}{2} \right) \right\}
\]

\[
= -\chi(-2)^{m-1} k^{m-p} \left( \sum_{n=1}^{ck} \chi(n) B_{p+1-m} \left( \frac{2nd}{c} \right) B_m \left( \frac{2n}{ck} \right) \right)
\]

\[
= -\chi(-2)^{m-1} k^{m-p} \left( \sum_{n=1}^{ck} \chi(n) B_{p+1-m} \left( \frac{2nd}{c} \right) B_m \left( \frac{2n}{ck} \right) \right)
\]

\[
= -\chi(-2)^{m-1} k^{m-p} \sum_{n=0}^{ck-1} \chi(2n+1) B_{p+1-m} \left( \frac{d(2n+1)}{c} \right) E_m \left( \frac{2n+1}{ck} \right)
\]

\[
= -\chi(-2)^{m-1} k^{m-p} \sum_{n=0}^{ck-1} \chi(2n+1) B_{p+1-m} \left( \frac{d(2n+1)}{c} \right) E_m \left( \frac{2n+1}{ck} \right)
\]

This gives (27) for \( z \in \mathbb{K} \). Then, by analytic continuation, (22) is valid for all \( z \in \mathbb{H} \). \( \blacksquare \)

### 4 Reciprocity Theorems

In this section we prove some reciprocity theorems. The next result can be viewed as the reciprocity formula for the function \( g_1(d, c + dk; z, p; \chi) \) given by (24).

**Theorem 8** Let \( p \) and \((d + c)\) be odd with \( c, d > 0 \) and \((c, d) = 1 \). If \( d \equiv 0 \pmod{k} \), then

\[
g_1(d, -c - dk; z, p; \chi) - \chi(-4) (c - k)^{p-1} g_1(c, d + ck; V_1(z), p; \chi) = g_1(1, -k; z, p; \chi),
\]

if \( c \equiv 0 \pmod{k} \), then

\[
g_1(d, -c - dk; z, p; \chi) - \chi(-4) (c - k)^{p-1} g_1(c, d + ck; V_1(z), p; \chi) = g_1(1, -k; z, p; \chi),
\]

where \( V_1(z) = (-kz + k^2 - 1) / (z - k) \).
Proof. Let \( d + c \) be odd and \( V(z) = (az + b + ak)/(cz + d + ck) \), where \( a \) and \( b \) are odd. Further let
\( V^*(z) = (bz - a - bk) / (dz - c - dk) \) and \( V_1(z) = (-kz + k^2 - 1) / (z - k) \).
Suppose \( a = d \equiv 0 \pmod{k} \). By replacing \( z \) by \( V_1(z) \) in (22) we get
\[
\left( \frac{dz - c - dk}{z - k} \right)^{p-1} G(\chi) B'(V^*(z), 1 - p : \chi) = \overline{\chi} \frac{b + ak}{2} (2c) G(\chi) B'(V_1(z), 1 - p : \chi) + \overline{\chi} \frac{b + ak}{2} (2c) G(\chi) \left( \frac{2\pi i}{p(1)!} \chi^{-1} \right) g_1(c, d + ck; V_i(z), p; \overline{\chi}). \tag{37}
\]
By applying \( V^*(z) \) to (23) we deduce that
\[
(z - k)^{p-1} G(\chi) B'(V_1(z), 1 - p : \chi) = \overline{\chi} (b) (c) G(\chi) B'(z, 1 - p : \chi) + \overline{\chi} (b) (c) \left( \frac{2\pi i}{p(1)!} \chi^{-1} \right) g_1(d, -c - dk; z, p; \chi). \tag{38}
\]
To determine \( B'(V_1(z), 1 - p : \chi) \) we replace \( V(z) \) by \( V_1(z) \) and \( \chi \) by \( \overline{\chi} \) in (22) to obtain
\[
(z - k)^{p-1} G(\chi) B'(V_1(z), 1 - p : \chi) = \overline{\chi} \left( \frac{k^2 - 1}{2} \right) (2c) G(\chi) B'(z, 1 - p : \chi) + \overline{\chi} \left( \frac{k^2 - 1}{2} \right) (2c) \left( \frac{2\pi i}{p(1)!} \chi^{-1} \right) g_1(1, -k; z, p; \chi). \tag{39}
\]
Combining (37), (38) and (39) we see that
\[
\overline{\chi} (b) (c) \left( g_1(d, -c - dk; z, p; \chi) - \chi^{-1} (z - k)^{p-1} g_1(c, d + ck; V_1(z), p; \overline{\chi}) \right) = \overline{\chi} (b) (c) g_1(1, -k; z, p; \chi).
\]
This gives (35) since \( \overline{\chi} (b) \neq 0 \) and \( \chi (c) \neq 0 \) for \( (b, k) = (c, k) = 1 \).

To prove (36), first replace \( z \) by \( V_1(z) \) in (23) and apply \( V^*(z) \) to (22), and then replace \( V(z) \) by \( V_1(z) \) in (22).

The theorem above may be simplified for the special value of \( z \). In particular, we consider \( z = (c + dk) / d \) and let \( d \equiv 0 \pmod{k} \). We first calculate \( g_1(1, -k; z, p; \chi) \). For this, we use (17) and (19) instead of (24) and find that
\[
g_1(1, -k; z, p; \chi) = \sum_{m=1}^{p} \binom{p+1}{m} (k - z)^{m-1} k^{m-p} \times \left( \sum_{n=1}^{2k} \chi(n) p_{p+1-m, \chi} \left( \frac{-kn}{2} \right) p_m \left( \frac{n}{2k} \right) - \chi(2)^{2} p_{k^{1-m} B_{p+1-m, \chi} B_{m, \chi}} \right).
\]
The sum over \( n \) may be written as
\[
\sum_{n \text{ odd}} + \sum_{n \text{ even}} \chi(n) p_{p+1-m, \chi} \left( \frac{-kn}{2} \right) p_m \left( \frac{n}{2k} \right) = p_{p+1-m, \chi} \left( \frac{-k}{2} \right) \left( \sum_{n=1}^{2k} - \sum_{n \text{ even}} \right) + \sum_{n \text{ even}},
\]
where
\[
p_{p+1-m, \chi} \left( \frac{-k}{2} \right) = p_{p+1-m, \chi} \left( \frac{k}{2} \right) = (\chi(2)^{2} p_{k^{1-m} B_{p+1-m, \chi}}, \sum_{n \text{ even}} = \sum_{n=1}^{k} \chi(2n) p_m \left( \frac{2n}{2k} \right) - \chi(2)^{k^{1-m} B_{m, \chi}}
\]
and
\[
\sum_{n=1}^{2^k} \chi(n) B_m \left( \frac{n}{2^k} \right) = \sum_{n=1}^{k} \chi(n) \left\{ B_m \left( \frac{n}{2^k} \right) + B_{m+1} \left( \frac{n}{2^k} + \frac{1}{2} \right) \right\} = 2^{1-m} k^{1-m} B_{m, \chi},
\]
by (20), (5) and (3). Thus,
\[
g_1(1, -k; \frac{c + dk}{d}, p; \chi) = -k^{1-p} \sum_{m=1}^{p} \left( \frac{p + 1}{m} \right) \left( -\frac{c}{d} \right)^{m-1} \left( \chi(2)2^m - 1 \right) \left( 2^{-p}\chi(2) - 2^{1-m} \right) B_{p+1-m, \chi} B_{m, \chi}.
\]
(40)
The functions \( g_1(c, d + ck; -\frac{kz+k^2-1}{z-k}, p; \chi) \) and \( g_1(d, -c - dk; z, p; \chi) \) become
\[
g_1(c, d + ck; -\frac{d - ck}{c}, p; \chi) = -\frac{p + 1}{2kp^{-1}} \sum_{n=1}^{ck} \chi(n) B_{p, \chi} \left( \frac{d + ck}{2c}n \right)
\]
and
\[
g_1(d, -c - dk; \frac{c + dk}{d}, p; \chi) = \frac{p + 1}{2kp^{-1}} \sum_{n=1}^{dk} \chi(n) B_{p, \chi} \left( \frac{c + dk}{2d}n \right)
\]
by the fact \( B_{p, \chi} (-x) = (-1)^p \chi(-1) B_{p, \chi} (x) \).

**Definition 9** The Hardy-Berndt character sum \( S_p(d, c : \chi) \) is defined for \( c > 0 \) by
\[
S_p(d, c : \chi) = \sum_{n=1}^{ck} \chi(n) B_{p, \chi} \left( \frac{d + ck}{2c}n \right).
\]
Using (40), (41) and (42) in (35) we have proved the following reciprocity formula for \( d \equiv 0(\text{mod } k) \). The proof for \( c \equiv 0(\text{mod } k) \) follows from (36).

**Corollary 10** Let \( p \) be odd and let \( d \) and \( c \) be coprime positive integers with \( (d + c) \) is odd. If either \( c \) or \( d \equiv 0(\text{mod } k) \), then
\[
\chi(-1) (p + 1) \{ cd^p \xi_p(c, d : \chi) + d^p \chi(4) S_p(d, c : \chi) \}
\]
\[
= \sum_{m=1}^{p} \left( \frac{p + 1}{m} \right) \left( -\frac{c}{d} \right)^{m-1} \left( \chi(2)2^m - 1 \right) \left( 2^{-p}\chi(2) - 2^{1-m} \right) c^m d^{p+1-m} B_{p+1-m, \chi} B_{m, \chi}.
\]
Corollary 4 do not give reciprocity formula for the function \( g_1(d, c; z, p; \chi) \) in the sense of Theorem 8 because of the restriction on \( b \) (\( b \) is even). By the similar restriction on \( a \) in Corollary 7 we do not have reciprocity formula for the function \( g_2(d, c; z, p; \chi) \). However, we have the following result.

**Theorem 11** Let \( p \) and \( d \) be odd with \( c, d > 0 \) and \( (c, d) = 1 \). If \( d \equiv 0(\text{mod } k) \), then
\[
\chi(-2) g_2(d, -c; z, p; \chi) - 2^{p+1} g_1(c, d; \frac{-1}{z}, p; \chi) = \chi(-2) g_2(1, 0; z, p; \chi),
\]

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if \( c \equiv 0 \pmod{k} \), then
\[
\chi(-2)g_2(d, -c; z, p; \chi) = 2^p z^{p-1} g_1(c, d; \frac{-1}{z}, p, \chi) = \chi(-2)g_2(1, 0; z, p; \chi).
\]

**Proof.** Let \( d \) be odd and \( T(z) = (az + b) / (cz + d) \), where \( b \) is even. Let \( T^*(z) = (bz - a) / (dz - c) \) and \( T_1(z) = -1/z \).

Suppose \( a \equiv d \equiv 0 \pmod{k} \). By replacing \( z \) by \( T_1(z) \) in (11) we get
\[
\left( \frac{dz - c}{z} \right)^{p-1} G(\chi) B'(T^*(z), 1 - p : \chi)
\]
\[
= \chi(-2c) G(\chi) \left\{ 2H(2z, 1 - p : \chi) - \chi(2)H(z, 1 - p : \chi) \right\}
\]
\[
+ \chi(-2c) \left( 2\pi i \right)^p \frac{\pi(-2)}{(p+1)!} g_2(d, -c; z, p; \chi).
\]

Applying \( T^*(z) \) to (28) gives
\[
2^p (dz - c)^{p-1} G(\chi) B'(T^*(z), 1 - p : \chi)
\]
\[
= \chi(-1) G(\chi) \left\{ 2H(2z, 1 - p : \chi) - \chi(2)H(z, 1 - p : \chi) \right\}
\]
\[
+ \chi(-1) (2\pi i)^p \frac{\pi(-2)}{(p+1)!} g_2(1, 0; z, p; \chi).
\]

Combining (43), (44) and (45) deduces that
\[
\chi(-2c) \left\{ \chi(-2)g_2(d, -c; z, p; \chi) - 2^p z^{p-1} g_1(c, d; T_1(z), p; \chi) \right\} = \chi(-2c) \chi(1) g_2(1, 0; z, p; \chi).
\]

This completes the proof for \( a \equiv d \equiv 0 \pmod{k} \) since \( \chi(b/2) \neq 0 \) and \( \chi(-2c) \neq 0 \) for \( (b, k) = (2c, k) = 1 \).

The proof for \( b \equiv c \equiv 0 \pmod{k} \) is similar. ■

This theorem is simplified by setting \( z = c/d \). To calculate \( g_2(1, 0; c/d, p; \chi) \), we use (32) and (33) instead of (29), and find that
\[
\chi(-2) g_2 \left( 1, 0; \frac{c}{d}, p; \chi \right)
\]
\[
= \chi(-1) \sum_{m=1}^{p} \left( \frac{p+1}{m} \right) \left( \frac{-c}{d} \right)^{m-1} \left\{ 2^m - \chi(2) \right\} \left( \frac{n}{k} \right) B_{p+1-m, \chi} \left( 0 \right) B_m \left( \frac{n}{k} \right).
\]

We also have
\[
g_1(c, d; -d/c, p; \chi) = \frac{p+1}{2kp-1} \sum_{n=1}^{ck} \chi(n) B_{p, \chi} \left( \frac{dn}{2c} \right),
\]
\[
g_2(d, -c; c/d, p; \chi) = \frac{p+1}{2kp-1} \sum_{n=1}^{dk} \chi(n) B_{p, \chi} \left( \frac{-cn}{d} \right)
\]
\[
= \chi(-1) \frac{p+1}{2kp-1} \sum_{n=1}^{dk} \chi(n) B_{p, \chi} \left( \frac{cn}{d} \right).
\]
Definition 12 The Hardy-Berndt character sums $s_{3,p}(d,c : \chi)$ and $s_{4,p}(d,c : \chi)$ are defined for $c > 0$ by

$$s_{3,p}(d,c : \chi) = \sum_{n=1}^{ck} (-1)^n \chi(n) \overline{B}_{p,\chi}\left(\frac{dn}{c}\right),$$

$$s_{4,p}(d,c : \chi) = \sum_{n=1}^{ck} \chi(n) \overline{B}_{p,\chi}\left(\frac{dn}{2c}\right).$$

Using (47), (48) and (46) in Theorem 11 we have proved the following reciprocity formula for $s_{3,p}(d,c : \chi)$ and $s_{4,p}(d,c : \chi)$.

Corollary 13 Let $p$ be odd and let $d$ and $c$ be coprime positive integers with $d$ is odd. If either $c$ or $d \equiv 0 \pmod{k}$, then

$$\chi(-1)(p+1)\{ (2c)^p s_{4,p}(d,c : \chi) + cd^p \chi(2)s_{3,p}(c,d : \chi) \}$$

$$= 2 \sum_{m=1}^{p} \left( \binom{p+1}{m} \right) (-1)^m \left( \chi(2) - 2^m \right) c^m d^{p+1-m} B_{p+1-m,\chi} B_{m,\chi}.$$

5 Further Results

For $c > 0$ and $1 \leq m \leq p$ we define

$$S_{p+1-m,m}(d,c : \chi) = -\frac{m}{2^m} \sum_{n=1}^{ck} \chi(n) \overline{B}_{p+1-m,\chi}\left(\frac{(d+ck)n}{2c}\right) \overline{E}_{m-1}\left(\frac{n}{ck}\right),$$

$$s_{4,p+1-m,m}(d,c : \chi) = -\frac{m}{2^m} \sum_{n=1}^{ck} \chi(n) \overline{B}_{p+1-m,\chi}\left(\frac{dn}{2c}\right) \overline{E}_{m-1}\left(\frac{n}{ck}\right).$$

Note that we have

$$S_{p}(d,c : \chi) = -2S_{p,1}(d,c : \chi) \quad \text{and} \quad s_{4,p}(d,c : \chi) = -2s_{4,p,1}(d,c : \chi).$$

The fixed points of a modular transformation $Tz = (az+b)/(cz+d)$ are $z_{1,2} = \left( a - d \pm \sqrt{(a+d)^2 - 4} \right)/2c$.

Then $z = \left( a - d + \sqrt{(a+d)^2 - 4} \right)/2c$ is in the upper half-plane $\Leftrightarrow |a+d| < 2$.

Following Goldberg ([13, Ch. 7]), we employ fixed points of a modular transformation to deduce certain properties of $s_{4,p+1-m,m}(d,c : \chi)$ and $S_{p+1-m,m}(d,c : \chi)$.

Theorem 14 Let $d$, $c$ and $k$ be odd integers and let $c \equiv 0 \pmod{k}$ and $d^2 \equiv -1 \pmod{c}$. If $p \equiv \chi(-1) \pmod{4}$, then

$$\sum_{m=1}^{p} \left( \binom{p+1}{m} \right) (-i)^m k^m s_{4,p+1-m,m}(d,c : \chi) = 0.$$ 

If $p \equiv -\chi(-1) \pmod{4}$, then

$$4\chi(-1) G(\chi) B(z_0, 1-p : \chi) = -\frac{(2\pi i)^p}{(p+1)!} \sum_{m=1}^{p} \left( \binom{p+1}{m} \right) (-i)^{m-1} k^{m-p} s_{4,p+1-m,m}(d,c : \chi),$$

where $z_0 = (a+d)/c$. 

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**Proof.** Let \( a = -d \) and \( d^2 \equiv -1 \pmod{c} \). Then, there exist an even integer \( b \) such that \( -d^2 - bc = ad - bc = 1 \). Hence, \( Tz = (az + b) / (cz + d) \) is a modular transformation and \( z_0 = (-d + i)/c \) is a fixed point of \( Tz \) in the upper half-plane. Therefore, by setting \( a = -d \) and \( z = z_0 = (-d + i)/c \) in (12) we see that

\[
\frac{(2\pi i)^p}{(p+1)!} g_1(c, d; z_0, p; \chi) = \left( (i)^{p-1} - \chi(-1) \right) G(\chi) B' \left( z_0, 1 - p : \chi \right).
\]

It follows from this and (13) that if \( p \equiv \chi(-1) \pmod{4} \), then

\[
\sum_{m=1}^{p} \binom{p+1}{m} (-i)^{m-1} k^{m-p} s_{4,p+1-m,m}(d, c : \chi) = 0
\]

and if \( p \equiv -\chi(-1) \pmod{4} \), then

\[
4\chi(-1) G(\chi) B \left( z_0, 1 - p : \chi \right) = -\frac{(2\pi i)^p}{(p+1)!} \sum_{m=1}^{p} \binom{p+1}{m} (-i)^{m-1} k^{m-p} s_{4,p+1-m,m}(d, c : \chi).
\]

\( \square \)

From (11), similar result to that of Theorem 14 can be obtained for \( d \equiv 0 \pmod{k} \) with real \( \chi \). Moreover, from Theorem 14, it is seen that

\[
2G(\chi) B \left( z_0, 0 : \chi \right) = \pi is_{4,1,1}(d, c : \chi)
\]

for \( p = 1 = -\chi(-1) \), and \( s_{4,1,1}(d, c : \chi) = 0 \) for \( p = 1 = \chi(-1) \). In addition, if \( \chi \) is real, then \( s_{4,2,2}(d, c : \chi) = 0 \) and \( s_{4,3,1}(d, c : \chi) = k^2 s_{4,1,3}(d, c : \chi) \) for \( p = 3 \) and \( \chi(-1) = -1 \).

The next theorem follows from (23) by using \( V(z) = (az + b + ak) / (cz + d + ck) \) with \( a \) and \( b \) are odd, and \( a = -d - ck \).

**Theorem 15** Let \((d + c)\) be odd and let \( c \equiv 0 \pmod{k} \) and \( d^2 \equiv -1 \pmod{c} \). If \( p \equiv \chi(-1) \pmod{4} \), then

\[
\sum_{m=1}^{p} \binom{p+1}{m} (-i)^{m} k^{m} S_{p+1-m,m}(d, c : \chi) = 0.
\]

If \( p \equiv -\chi(-1) \pmod{4} \), then

\[
-4\chi(-1) G(\chi) B \left( z_1, 1 - p : \chi \right) = \frac{(2\pi i)^p}{(p+1)!} \sum_{m=1}^{p} \binom{p+1}{m} (-i)^{m-1} k^{m-p} S_{p+1-m,m}(d, c : \chi),
\]

where \( z_1 = (-d - ck + i)/c \).

In the sequel we will frequently use the following lemma [11, Lemma 5.5]. We state it here in the following form for \( x = y = 0 \).

**Lemma 16** Let \((c, d) = 1 \) with \( c > 0 \). Let \( \chi \) is a primitive character of modulus \( k \), where \( k \) is a prime number if \((k, cd) = 1 \) and \( p \) is even, otherwise \( k \) is an arbitrary integer. Then

\[
\sum_{n=1}^{ck-1} \chi(n) B_{p, \chi}(\frac{dn}{c}) = e^{-p} \chi(c) \chi(-d) \frac{B_{p}(-1)}{B_{p}(0)}.
\]

We want to show that the reciprocity formulas given by Corollaries 10 and 13 are also valid when \((c, d) > 1 \). For this we need the following.
Proposition 17 Let \( p \) be odd and \( c > 0 \) with \( (c, d) = 1 \). Then, for positive integer \( q \), we have if \( (d + c) \) is odd, \[ S_p(qd, qc : \chi) = \begin{cases} S_p(d, c : \chi), & q \text{ odd,} \\ 0, & q \text{ even,} \end{cases} \]
if \( c \) is odd, \[ s_{3,p}(qd, qc : \chi) = \begin{cases} s_{3,p}(d, c : \chi), & q \text{ odd,} \\ 0, & q \text{ even,} \end{cases} \]
if \( d \) is odd, \[ s_{4,p}(qd, qc : \chi) = \begin{cases} s_{4,p}(d, c : \chi), & q \text{ odd,} \\ 0, & q \text{ even.} \end{cases} \]

Proof. Let \( d \) be odd. By the definition of \( s_{4,p}(d, c : \chi) \), we have
\[
s_{4,p}(qd, qc : \chi) = \sum_{n=1}^{q} (\chi(n)\overline{B}_{p,\chi}(dn/c))
= \sum_{n=1}^{q} \sum_{m=0}^{n-1} \chi(n)\overline{B}_{p,\chi}(dn/c + km) = \sum_{n=1}^{q-1} \chi(n)\sum_{m=0}^{n-1} \overline{B}_{p,\chi}(dn/c + km).
\]

It follows from \((20)\) that
\[
\sum_{m=0}^{q-1} \overline{B}_{p,\chi}(dn/c + km) = \begin{cases} \frac{q-1}{2} \chi(2)2^{1-r}\overline{B}_{p,\chi}(dn/c) + \overline{B}_{p,\chi}(dn/c), & q \text{ odd,} \\ \frac{q}{2} \chi(2)2^{1-r}\overline{B}_{p,\chi}(dn/c), & q \text{ even.} \end{cases}
\]

Thus, by Lemma 16 we have
\[
s_{4,p}(qd, qc : \chi) = \begin{cases} s_{4,p}(d, c : \chi), & q \text{ odd,} \\ 0, & q \text{ even.} \end{cases} \]

Let \( c \) be odd. We have
\[
s_{3,p}(qd, qc : \chi) = \sum_{n=1}^{q} (-1)^n \chi(n)\overline{B}_{p,\chi}(dn/c)
= \sum_{n=1}^{q} \sum_{m=0}^{n-1} (-1)^{n+cm} \chi(n)\overline{B}_{p,\chi}(dn/c + km) = s_{3,p}(d, c : \chi) \sum_{m=0}^{q-1} (-1)^m,
\]
which completes the proof.

Now from Proposition 17 and Corollaries 10 and 13, we arrive at the following reciprocity formulas for \((c, d) > 1\).

Corollary 18 Let \( p \) be odd and let \( c, d > 0 \) with \((d + c)\) odd, \((c, d) = q \text{ odd and } (q, k) = 1\). If either \( c \) or \( d \equiv 0 (mod \ k) \), then
\[
\chi(-1)(p + 1) \{ cd^p S_p(c, d : \chi) + dc^p \chi(4) S_p(d, c : \chi) \}
= \sum_{m=1}^{p} \left( \frac{p + 1}{m} \right) (-1)^m (\chi(2)2^m - 1) (\chi(2)2^{1-p} - 2^{2-m}) c^m d^{p+1-m} B_{p+1-m,\chi} B_{m,\chi}.
\]
Corollary 19 Let $p$ be odd and let $c, d > 0$ with $d$ odd, $(c, d) = q$ odd and $(q, k) = 1$. If either $c$ or $d \equiv 0 \pmod{k}$, then

$$\chi(-1) (p + 1) \{d (2c)^p s_{4,p}(d, c : \chi) + cd^p \overline{\chi}(2)s_{3,p}(c, d : \chi)\}$$

$$= 2 \sum_{m=1}^{p+1} \left(\frac{p+1}{m}\right) \left(-1\right)^m \left(\overline{\chi}(2) - 2^m\right) \left(c^m d^{p+1-m}\right) B_{p+1-m, \chi} B_{m, \overline{\chi}}.$$

So far we assume that $(d + c)$ and $p$, $c$ and $p$, $d$ and $p$ are odd for the sums $S_p(d, c : \chi)$, $s_{3,p}(d, c : \chi)$ and $s_{4,p}(d, c : \chi)$, respectively. For the remaining three cases these sums may be evaluated as in the following:

Proposition 20 Let $k$ be as in Lemma 16 and odd. For $c > 0$ with $(c, d) = 1$, we have

$$(i) \ S_p(d, c : \chi) = \begin{cases} 0, & \text{if } p \text{ odd and } (d + c) \text{ even}, \\ \chi(2) \lambda, & \text{if } p \text{ and } (d + c) \text{ even}, \\ 2^{-p}\chi(2) \lambda, & \text{if } p \text{ even and } (d + c) \text{ odd}, \end{cases}$$

where $\lambda = c^{1-p}(c)\overline{\chi}(-d)(k^p - 1)B_p$.

$$(ii) \ s_{3,p}(d, c : \chi) = \begin{cases} 0, & \text{if } c + p \text{ odd}, \\ (2^p - 1) \lambda, & \text{if } p \text{ and } c \text{ even}. \end{cases}$$

$$(iii) \ s_{4,p}(d, c : \chi) = \begin{cases} 0, & \text{if } p \text{ odd and } d \text{ even}, \\ \chi(2) \lambda, & \text{if } p \text{ and } d \text{ even}, \\ 2^{-p}\chi(2) \lambda, & \text{if } p \text{ even and } d \text{ odd}. \end{cases}$$

Proof. Let $(d + c)$ be even. Then, $(d + ck)$ is even and

$$S_p(d, c : \chi) = \sum_{n=1}^{ck} \chi(n)\overline{B}_{p, \chi} \left(\frac{(d + kc)/2}{c^2}n\right) = (-1)^p \sum_{n=1}^{ck} \chi(n)\overline{B}_{p, \chi} \left(\frac{(d + kc)/2}{c^2}n\right).$$

by the identity $\overline{B}_{p, \chi}(-x) = (-1)^p \overline{\chi}(-1)\overline{B}_{p, \chi}(x)$. Hence, from Lemma 16, we get $S_p(d, c : \chi) = 0$ for odd $p$, and $S_p(d, c : \chi) = \chi(2) \lambda$ for even $p$.

Let $p$ be even and $(d + c)$ be odd. We have

$$S_p(d, c : \chi) = \sum_{n=1}^{ck} \chi(n)\overline{B}_{p, \chi} \left(\frac{(d + kc)/2}{c^2} + \frac{(d + kc)/2}{c} + \frac{1}{2}\right).$$

Upon the use of (20) we deduce

$$2S_p(d, c : \chi) = \sum_{n=1}^{ck} \chi(n) \left(\overline{B}_{p, \chi} \left(\frac{(d + kc)/2}{c^2}\right) + \overline{B}_{p, \chi} \left(\frac{(d + kc)/2}{c} + \frac{1}{2}\right)\right)$$

$$= 21^{-p}\chi(2) \sum_{n=1}^{ck} \chi(n)\overline{B}_{p, \chi} \left(\frac{4n}{c}\right).$$

Thus, using Lemma 16 we complete the proof for $S_p(d, c : \chi)$. \qed
6 Integral Representations

In [4], Berndt derived the character analogue of the Euler–Maclaurin summation formula. We apply this formula to generalized Bernoulli function to obtain identities involving integrals for \( s_{3,p} (d, c : \chi) \) and \( s_{4,p} (d, c : \chi) \). These identities lead an alternative proof of Corollary 13 which is given in the end of this section.

Berndt’s formula is presented here in the following form.

**Theorem 21** [4, Theorem 4.1] Let \( k \) be an arbitrary integer and \( f \in C^{(l+1)} [\alpha, \beta] \), \( -\infty < \alpha < \beta < \infty \). Then

\[
\sum_{\alpha \leq n \leq \beta} \chi(n) f(n) = \chi(-1) \sum_{j=0}^{l} \frac{(-1)^{j+1}}{(j+1)!} \left\{ B_{j+1, \chi}(\beta) f^{(j)}(\beta) - B_{j+1, \chi}(\alpha) f^{(j)}(\alpha) \right\} \\
+ \chi(-1) \frac{(-1)^{l}}{(l+1)!} \int_{\alpha}^{\beta} B_{l+1, \chi}(u) f^{(l+1)}(u) \, du,
\]

where the dash indicates that if \( n = \alpha \) or \( n = \beta \), only \( \frac{1}{2} \chi(\alpha) f(\alpha) \) or \( \frac{1}{2} \chi(\beta) f(\beta) \), respectively, is counted.

Let \( f(x) = B_{p,\chi}(xy) \), \( y \in \mathbb{R} \), and \( c \) be a positive integer. The property

\[
\frac{d}{dx} B_{m, \chi}(x) = m B_{m-1, \chi}(x), \ m \geq 2
\]

([4, Corollary 3.3]) entails that

\[
\frac{d^j}{dx^j} f(x) = \frac{d^j}{dx^j} B_{p, \chi}(xy) = y^j \frac{p!}{(p-j)!} B_{p-j, \chi}(xy)
\]

for \( 0 \leq j \leq p-1 \) and \( f \in C^{(p-1)} [\alpha, \beta] \).

We consider the following three cases;

1) \( y = b/c, \alpha = 0 \) and \( \beta = ck \),

2) \( y = b/(2c), \alpha = 0 \) and \( \beta = ck \),

3) \( y = 2b/c, \alpha = 0 \) and \( \beta = ck/2 \),

where \( c > 0 \), separately.

For \( \alpha = 0, \beta = ck \) and \( 1 \leq l+1 \leq p-1 \), Theorem 21 can be written as

\[
\sum_{n=1}^{ck} \chi(n) B_{p, \chi}(ny) = \chi(-1) \frac{p+1}{p+1} \sum_{j=0}^{l} \frac{p+1}{j+1} (-1)^{j+1} y^j \left\{ B_{p-j, \chi}(cky) - B_{p-j, \chi} \right\} B_{j+1, \chi} \\
- \chi(-1) (-y)^{l+1} \left( \frac{p}{l+1} \right) \int_{0}^{ck} B_{l+1, \chi}(u) B_{p-l-1, \chi} \left( \frac{bu}{c} \right) \, du. 
\]  
\[ (49) \]

1) Let \( y = b/c, \alpha = 0 \) and \( \beta = ck \). Then (49) becomes

\[
\sum_{n=1}^{ck} \chi(n) B_{p, \chi} \left( \frac{bn}{c} \right) = -\chi(-1) \left( \frac{p}{l+1} \right) \left( \frac{b}{c} \right)^{l+1} \int_{0}^{ck} B_{l+1, \chi}(u) B_{p-l-1, \chi} \left( \frac{bu}{c} \right) \, du. 
\]  
\[ (50) \]

• If \( b = c \), combining (50) and Lemma 16 with \( d = c = 1 \) we deduce that

\[
\int_{0}^{k} B_{r, \chi}(u) B_{m, \chi}(u) \, du = (-1)^{r-1} \frac{r!m!}{(m+r)!} \left( k^{m+r} - 1 \right) B_{m+r}
\]

for \( k \) as in Lemma 16 (where \( l+1 = r, p-r = m \)).
Remark 22 The equation above stated as

\[
\int_0^k \overline{B}_{r,\chi}(u) \overline{B}_{m,\chi}(u) \, du = (-1)^{m-1} \frac{m!r!}{(m+r)!} k^{m+r} B_{m+r}, \quad m, r \geq 1
\]

in \[4, \text{Proposition 6.6}\].

- If \( b = ck \), it follows from the fact \( \sum_{n=0}^{ck} \chi(n) = 0 \) and (50) that

\[
\int_0^k \overline{B}_{l+1,\chi}(u) \overline{B}_{p-l-1,\chi}(ku) \, du = 0.
\]

- Now assume that \((b, c) = 1\). Then, it follows from Lemma 16 and (50) that

\[
\int_0^k \overline{B}_{l+1,\chi}(cu) \overline{B}_{p-l-1,\chi}(bu) \, du = \frac{(-1)^l \ell^{l+p-1}}{(l+1)!} \chi(c) \chi(b) (k^{p-1} - 1) B_p
\]

for \(k\) as in Lemma 16.

- Let \((b, c) = q\) and put \( c = qc_1, b = qb_1 \). From the equation above, we have

\[
\int_0^k \overline{B}_{l+1,\chi}(cu) \overline{B}_{p-l-1,\chi}(bu) \, du = \int_0^k \overline{B}_{l+1,\chi}(c_1 u) \overline{B}_{p-l-1,\chi}(b_1 u) \, du
\]

\[
= q^p \frac{(-1)^l \ell^{l+p-1}}{(l+1)!} \chi \left( \frac{c}{q} \right) \chi \left( \frac{b}{q} \right) (k^{p-1} - 1) B_p
\]

for \(k\) as in Lemma 16 with replacing \(c\) by \(c_1\) and \(b\) by \(b_1\).

II) Let \(k\) and \(b\) be odd and consider \( y = b/2c \) with \((b, c) = 1\). From (20), for odd \( b\) we have

\[
\overline{B}_{p-j,\chi} \left( \frac{bk}{2} \right) = \overline{B}_{p-j,\chi} \left( \frac{k}{2} \right) = \{2^{j+1-p} \chi(2) - 1\} B_{p-j,\chi}.
\]

Therefore, (49) becomes

\[
s_{4,p}(b, c : \chi) = \sum_{n=1}^{ck} \chi(n) \overline{B}_{p,\chi} \left( \frac{b}{2c} n \right)
\]

\[
= -2^{1-p} \chi(\frac{-1}{p+1}) \sum_{m=p-l}^{p} \left( \frac{p+1}{m} \right) \left( -\frac{b}{c} \right)^{p-m} \chi(2) - 2^m \} B_{m,\chi} B_{p+1-m,\chi}
\]

\[
- \chi(\frac{-1}{c}) \left( \frac{p}{l+1} \right) \left( -\frac{b}{2c} \right)^{l+1} \int_0^k \overline{B}_{l+1,\chi}(cu) \overline{B}_{p-l-1,\chi}(\frac{b}{2} u) \, du
\]

by setting \(j = p - m\).

For \(l = 0\) we have the following integral representation:

\[
\chi(-1) s_{4,p}(b, c : \chi) = \frac{pb}{2} \int_0^k \overline{B}_{1,\chi}(cu) \overline{B}_{p-1,\chi} \left( \frac{b}{2} u \right) \, du - \{2^{1-p} \chi(2) - 2\} B_{p,\chi} B_{1,\chi}.
\]
• If $p$ is even, it is seen from (52) and Proposition 20 that

$$c \left( \frac{p}{l+1} \right) \left( -\frac{b}{2c} \right)^{l+1} \int_{0}^{k} \mathcal{B}_{l+1,\chi}(cu) \mathcal{B}_{p-l-1,\chi} \left( \frac{b}{2} u \right) du$$

$$= -2^{-p} c^{1-p} \chi(2c) \chi(b) (k^p - 1) B_p - \sum_{m=p-l}^{p} \left( \frac{p+1}{m} \right) \left( -\frac{b}{c} \right)^{p-m} \{ \chi(2) - 2^m \} \mathcal{B}_{m,\chi} \mathcal{B}_{p+1-m,\chi}$$

for $k$ as in Lemma 16.

• Let $p$ be odd and put $l + 1 = p - 1$ in (52). Then,

$$\chi(-1) \left( \frac{p}{l+1} \right) b \left( \frac{2c}{l+1} \right)^{l+1} s_{4,p} (b, c : \chi)$$

$$= 2 \sum_{m=2}^{p} \left( \frac{p+1}{m} \right) (-1)^m \chi \{ \chi(2) - 2^m \} \mathcal{B}_{m,\chi} \mathcal{B}_{p+1-m,\chi}$$

$$- 2 c b p (p+1) c \int_{0}^{k} \mathcal{B}_{p-l,\chi}(cu) \mathcal{B}_{1,\chi} \left( \frac{b}{2} u \right) du. \quad (54)$$

III) Let $k$ and $c$ be odd and consider $y = 2b/c$, $\alpha = 0$, $\beta = ck/2$ with $(b, c) = 1$. Then, from Theorem 21 and Eq. (51) we have

$$\sum_{0 \leq n \leq ck/2} \chi(n) \mathcal{B}_{p,\chi} \left( \frac{2b}{c} n \right) = \sum_{n=1}^{\frac{(ck-1)}{2}} \chi(n) \mathcal{B}_{p,\chi} \left( \frac{2b}{c} n \right)$$

$$= -\frac{\chi(-1)}{p+1} \sum_{j=1}^{l+1} \left( \frac{p+1}{j} \right) \left( -\frac{b}{c} \right)^{j-1} \{ \chi(2) - 2^j \} \mathcal{B}_{j,\chi} \mathcal{B}_{p+1-j,\chi}$$

$$- \chi(-1) \left( -\frac{2b}{c} \right)^{l+1} \left( \frac{p}{l+1} \right) \frac{c}{2} \int_{0}^{k} \mathcal{B}_{l+1,\chi} \left( \frac{c}{2} u \right) \mathcal{B}_{p-l-1,\chi} (bu) du. \quad (55)$$

Now consider the sum $s_{3,p} (b, c : \chi)$.

$$s_{3,p} (b, c : \chi) = \sum_{n=1}^{ck} (-1)^n \chi(n) \mathcal{B}_{p,\chi} \left( \frac{bn}{c} \right)$$

$$= 2 \chi(2) \sum_{n=1}^{\frac{ck}{2}} \chi(n) \mathcal{B}_{p,\chi} \left( \frac{2bn}{c} \right) - \sum_{n=1}^{\frac{ck}{2}} \chi(n) \mathcal{B}_{p,\chi} \left( \frac{bn}{c} \right). \quad (56)$$

• If $p$ is even, then from (55), Proposition 20 and Lemma 16

$$c \left( \frac{-2b}{c} \right)^{l+1} \left( \frac{p}{l+1} \right) \int_{0}^{k} \mathcal{B}_{l+1,\chi} \left( \frac{c}{2} u \right) \mathcal{B}_{p-l-1,\chi} (bu) du$$

$$= -c^{1-p} \chi(c) \chi(2b) (k^p - 1) B_p - \sum_{j=1}^{\frac{p+1}{2}} \left( \frac{p+1}{j} \right) \left( -\frac{b}{c} \right)^{j-1} \{ \chi(2) - 2^j \} \mathcal{B}_{j,\chi} \mathcal{B}_{p+1-j,\chi}$$

for $k$ as in Lemma 16.
Let \( p \) be odd. Put \( l + 1 = p - 1 \) in (55). Then, (55), (56) and Lemma 16 yield

\[
\chi(-2)(p+1)bcps_{3,p}(b,c) = 2\sum_{j=1}^{p-1}\binom{p+1}{j}(-1)^j b^j c^{p+1-j}\{\chi(2) - 2^j\} B_j \chi B_{p+1-j} - 2^{p-1}b^p c^2 p(p+1) \int_0^k \overline{B}_{p-1} \chi \left( \frac{c}{2} u \right) \overline{B}_{1,\chi} (bu) \, du.
\] (57)

Putting \( l = 0 \) in (55) gives an integral representation for \( s_{3,p}(b,c) \):

\[
\frac{1}{2} \chi(-2)s_{3,p}(b,c) = bp \int_0^k \overline{B}_{1,\chi} \left( \frac{c}{2} u \right) \overline{B}_{p-1,\chi} (bu) \, du - \{\chi(2) - 2\} B_1 \chi B_{p,\chi}.
\] (58)

Combining (57) and (53) (or (54) and (58)) we arrive at the reciprocity formula

\[
\chi(-1)(p+1)\{c(2b)^p s_{4,p}(c,b) + \chi(2)bcps_{3,p}(b,c)\} = 2\sum_{j=1}^{p+1}\binom{p+1}{j}(-1)^j b^j c^{p+1-j}\{\chi(2) - 2^j\} B_j \chi B_{p+1-j} - \{\chi(2) - 2\} B_1 \chi B_{p,\chi}
\]
given by Corollary 13 without the restriction \( c \) or \( b \equiv 0(\text{mod } k) \).

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