De Sitter space and perpetuum mobile

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Abstract

We give general arguments that any interacting non–conformal classical field theory in de Sitter space leads to the possibility of constructing a perpetuum mobile. The arguments are based on the observation that massive free falling particles can radiate other massive particles on the classical level as seen by the free falling observer. The intensity of the radiation process is non-zero even for particles with any finite mass, i.e. with a wavelength which is within the causal domain. Hence, we conclude that either de Sitter space can not exist eternally or that one can build a perpetuum mobile.
I. INTRODUCTION

It is generally believed that de Sitter space is stable at least on the classical level. The belief relies on the following arguments: (i) de Sitter space has a big isometry group \([1, 2]\); (ii) there are no exponentially growing linearized fluctuations over de Sitter space \([3]\).

The question is whether these arguments are sufficient to prove the classical stability of de Sitter space. In this note we argue that the answer on this question is negative. Indeed the situation changes if one turns on interactions. Let us ask the following question of a classical \([14]\) interacting field theory on de Sitter background: “Does an inertially moving charged particle in de Sitter space emit radiation or not?”. Because the space in question is conformally flat, we consider a field theory which is not conformally invariant, otherwise the behavior of fields is not much different from fields in Minkowski space.

Calculating the classical amplitude \([15]\) of the corresponding process and observing that it is not zero, we obtain an affirmative answer to the above question. Our point is that even in a classical field theory on de Sitter space, massive particles can radiate other massive particles with a wavelength within the causal domain.

Quantum particle production by another particle in de Sitter space was addressed in \([4–7]\). Classical electromagnetic radiation was considered in \([8, 9]\). On general physical grounds one can expect that the characteristic wavelength of the radiation of the massless fields and the wave creation length in such a process are of the order of the size of the cosmological horizon. This is one of the reasons why we consider radiation of massive particles with wavelengths smaller than the cosmological horizon. Our main point in this note is to show that there is a problem with the stability of de Sitter space even on the classical level – if the vacuum energy is truly fixed then one can create a perpetuum mobile.

II. FIELD THEORY IN DE SITTER SPACE

The \(D\)-dimensional de Sitter space is a hyperboloid \([1, 2, 10]\)

\[-z_0^2 + \sum_{i=1}^{D} z_i^2 = 1 \quad (1)\]

in \((D + 1)\)-dimensional Minkowski space. In this note we set the radius of de Sitter space to 1 and consider \(D \geq 3\). One can see explicitly that the isometry of de Sitter space is
the \( SO(D,1) \) symmetry group of the background \((D+1)\)–dimensional space. The presence of this large isometry is one of the arguments favoring the stability of de Sitter space — a space with a running vacuum energy would be less symmetric. The reason why such an argument is not sufficient is that in field theory it frequently happens that a less symmetric state is energetically more favorable than the more symmetric one. The classic example is the spontaneous symmetry breaking.

The specific induced metric on the hyperboloid, used below, is:

\[
\begin{align*}
ds^2 &= -dt^2 + \cosh^2 t \, d\Omega_{D-1}^2, \\
&= -dt^2 + \cosh^2 t \, d\Omega_{D-1}^2, \\
\end{align*}
\]

where \( d\Omega_{D-1}^2 \) is the metric on the unit \((D - 1)\)–dimensional sphere. These “global” coordinates cover the entire de Sitter space, and are those seen by inertial observers, since the coordinate time in this metric coincides with the proper time. All the results in this paper are those seen by an inertial observer.

Consider linearized fluctuations in de Sitter space. For simplicity and because we will use it below, we take a scalar field, \( \Phi \), with arbitrary mass, \( m \). The Klein–Gordon equation in the global coordinates is:

\[
\left( \partial_t^2 + (D-2) \tanh t \partial_t + m^2 - \frac{\Delta_{D-1}(\Omega)}{\cosh^2 t} \right) \Phi = 0, 
\]

(3)

\( \Delta_{D-1}(\Omega) \) is the Laplacian on the \((D - 1)\)–dimensional sphere.

Just by looking at the asymptotic form of (3) as \( t \to \pm \infty \), one can see that none of its solutions grows exponentially with time. Linearized fluctuations of the metric obey equations which are very similar to (3) with the zero mass. Hence, similar arguments lead to the conclusion that they do not grow exponentially with time and are not able to change the background de Sitter metric \( \mathbf{[3]} \). These arguments are valid on the linearized level, while we would like to consider an interacting theory.

To address our question we have to define what is meant by a particle in de Sitter space. Equation (3) is linear thus it seems that one could consider as particles any basis of solutions, because they obey the superposition principle. However, this is not quite correct. For example by taking an arbitrary basis of solutions of the Klein–Gordon equation (e.g. linear combination of positive and negative energy harmonics) in Minkowski space one can obtain radiation on mass–shell, which is known to be wrong. One must define particles as positive energy excitations over the Poincare invariant vacuum state to ensure that there
is no radiation on mass–shell in Minkowski space (i.e. no radiation from a particle moving with constant velocity). We clarify this observation below.

It is worth stressing here that one can use any basis of solutions of Eq. (3), but the resulting answer to the question posed in the introduction will be the same.

Using the separation of variables $\Phi_{j n}(t, \Omega) = \varphi_j(t) Y_{j n}(\Omega)$ one can find the basis of solutions of Eq. (3) \cite{1, 11}. Here $\Delta_{D-1}(\Omega) Y_{j n}(\Omega) = -j(j + D - 2) Y_{j n}(\Omega)$, and $n$ is the multi–index $(n_1, \ldots, n_{D-2})$.

The properties of these spherical harmonics $Y_{j n}(\Omega)$ are given in \cite{10}. The field $\varphi_j(t)$ obeys an equation following from Eq. (3). This equation has two distinguished complete sets of solutions: the in– and out–modes \cite{10, 11}. The complete set of in–modes is

$$\varphi_{\pm j}(t) \propto \cosh^j(t) e^{(j + \frac{D-1}{2} \mp i\mu)t} F\left(j + \frac{D-1}{2}, j + \frac{D-1}{2} \mp i\mu; 1 \mp i\mu; -e^{2t}\right)$$

where $\mu = \sqrt{m^2 - \left(\frac{D-1}{2}\right)^2}$ and $F(a, b; c; z)$ is the hypergeometric function. The solution (4) can be continued to the case when $m < (D - 1)/2$.

The reason for the name of these modes is that they behave at past infinity ($t \to -\infty$) as $\varphi_{\pm j} \to e^{(\frac{D-1}{2} \mp i\mu)t}$ and, hence, diagonalize the classical free Hamiltonian of the scalar field theory in this region of space–time. At future infinity ($t \to +\infty$) they behave as $\varphi_{\pm j} \to e^{-\frac{D-1}{2}t} (c_1 e^{\mp i\mu t} + c_2 e^{\pm i\mu t})$ with some non–zero (for even \cite{16} $D$) complex constants $c_1$ and $c_2$. Hence, for even $D$ at future infinity they do not diagonalize the free Hamiltonian.

At future infinity the free Hamiltonian is diagonalized by the out–modes $\bar{\varphi}_{\pm j}(t)$, which are related to the in–modes by $\varphi_{\pm j}(t) = (\bar{\varphi}_{\mp j}(-t))^*$.

The vitally important fact for our considerations below is that, unlike Minkowski and AdS spaces, there is no set of solutions of Eq. (3), which diagonalizes the classical free Hamiltonian in de Sitter space for all times. This is true in any dimension although for odd $D$ the same set of harmonics does diagonalize the free Hamiltonian both at past and future infinities.

III. WHAT IS MEANT BY CLASSICAL RADIATION

Although our arguments are general, for concreteness we consider equations of motion in a Yukawa type theory describing the interaction of a massive ($M$) fermion, $\Psi$, with a massive ($m$) boson, $\Phi$:
\[
\left[ \Delta(g) + m^2 \right] \Phi = \lambda \bar{\Psi} \Psi
\]

\[
\left[ \hat{D} + M \right] \Psi = \lambda \Phi \bar{\Psi}.
\] (5)

As well there is the complex conjugate equation for the fermion field. Here \(\Delta(g)\) is the d’Alembertian for the de Sitter metric, \(\hat{D}\) is the Dirac operator for the same metric, and \(\lambda\) is the interaction constant. The metric is taken to be non–dynamical. We have chosen a Yukawa type theory in order to avoid the possible counterarguments against \(\phi^3\) theory, which are based on the absence of a minimum for the potential energy for this theory.

First let us consider this field theory in Minkowski space. The simplest solution of the above system of equations is \(\Psi = 0\) and \(\Phi = 0\) (and a flat metric solution of the Einstein–Hilbert equations), which corresponds to empty space. If \(\lambda\) were zero one could excite the \(\Phi\) and \(\Psi\) fields to see that there is no exponentially growing mode in Minkowski or in de Sitter space.

We are interested here in what happens when one adiabatically turns on \(\lambda\) at past infinity and switches it off at the future infinity. Suppose we initial set \(\Phi\) to zero, but excite the field \(\Psi\) i.e. we add one mass–shell harmonic of this field on top of the de Sitter background. Then according to the system of equations (5) the solution for \(\Phi\) to leading order in \(\lambda\) is as follows

\[
\Phi(x) = \lambda \int d^D y G_R(x, y) \bar{\Psi}(y) \Psi(y).
\] (6)

Here \(G_R(x, y)\) is the retarded Green function for the massive scalar field. Its explicit form is not necessary for our further considerations. We just need to know that it can always be represented as

\[
G_R(x, y) = \theta(x_0 - y_0) \sum_\kappa \psi_\kappa(x) \psi^{*}_\kappa(y),
\] (7)

where \(\psi_\kappa(x)\) is the basis of the solutions (i.e. mass–shell harmonics) of the Klein–Gordon equation with proper normalization.

Now the mass–shell harmonics – i.e. solutions of (5) – in flat space for both bosons and fermions are proportional to the plane wave \(e^{i\vec{p}\cdot\vec{x}}\) and their time dependence is given by \(e^{-i\sqrt{\vec{p}^2 + \text{mass}^2} t}\) i.e. in Minkowski space the role of \(\kappa\) is played by the momentum, \(\vec{p}\).
Thus, our solution for the scalar field is given by

\[
\Phi(x) \propto \lambda \int d^{D-1}q \frac{e^{i q \cdot x}}{\sqrt{q^2 + m^2}} \int_{-\infty}^{X_0} dy_0 \int d^{D-1}y \ e^{i(p-q-k)\cdot y}.
\]  

(8)

Where we have explicitly substituted the Fourier expansion of the Green function (the plane waves for the mass–shell harmonics) and ignored the spinor pre–factors for the Fermi fields.

Now consider the last integral in this formula when \(x_0 \to +\infty\). It is just the classical amplitude for the radiation for this theory in Minkowski space. We will explain why this is so in a moment. It is proportional to:

\[
A \propto \int d^Dy \ e^{i(p-q-k)y} \propto \delta^{(D)}(p - q - k),
\]

(9)

In this formula \(p\) and \(k\) are the momenta of the fermion before and after the interaction process and \(q\) is the momentum of the emitted boson field. The \(\delta\)--function just imposes energy-momentum conservation at the vertex — \(p = k + q\). Note that in the classical theory the solution for \(\bar{\Psi}\) is just the complex conjugate of \(\Psi\) and, hence, \(p = k\), but we keep them more general.

All of the three momenta in the amplitude are on–shell, i.e. \(k^2 - M^2 = p^2 - M^2 = 0\) and \(q^2 - m^2 = 0\). Due to the latter relations the argument of the \(\delta\)-function is never zero. Hence, the amplitude is zero for all allowed \(p, k\), and \(q\). That means that if, in the past infinity, we excite one positive energy mass–shell harmonic of the fermion field it will not excite (emit) the boson field at future infinity. We will have just the single fermion plane wave excitation over Minkowski space. Note that it does not mean that on the Minkowski background the system \([5]\) can have single mass–shell wave solution for \(\Psi\) with \(\Phi\) set to zero: at intermediate times the \(\Phi\) field is not zero but it does vanish as \(x_0 \to +\infty\) (where in fact \(\lambda\) is adiabatically switched off).

One can show that the same thing happens at higher orders in \(\lambda\). Thus, as expected, that there is no classical radiation from a free floating particle in Minkowski space. Coming back to the discussion of the proper choice of vacuum and positive energy harmonics, let us stress that if one had chosen as the positive energy harmonics a linear combination of \(e^{-ipx}\) and \(e^{ipx}\) with various \(p\)'s instead of a single \(e^{-ipx}\), then one would have obtained a non–zero amplitude instead of (9), which is known to be physically incorrect. That is why we are careful with the choice of vacuum and positive harmonics.
Note that performing similar calculations in the same theory, but in the Einstein static space, $ds^2 = -dt^2 + R^2 d\Omega^2_{D-1}$, $R = \text{const}$., one observes \[5\] that an amplitude such as that in (9) is zero, similar to the Minkowski space case.

IV. RADIATION IN DE SITTER SPACE

Now we will show that the situation in de Sitter space is quite different. Again we would like to excite the theory with a single fermion mass–shell harmonic and see whether this does or does not lead to the excitation of the boson field.

We choose as the mass–shell harmonics the above mentioned in–solutions. However it is important to stress that whatever harmonics we choose the classical amplitude will always be non–zero and, hence, the excitation of the boson field will always occur. The fact that the amplitude given below does not vanish is closely related to the fact that one can not find a basis of harmonics in de Sitter space which diagonalizes the classical free Hamiltonian for all times.

Taking the same route as above from the equation (5) to (9) we obtain instead of (9) the following classical amplitude:

$$A \propto \int d\Omega Y_{j_1 n_1} (\Omega) Y^*_{j_2 n_2} (\Omega) Y^*_{j_3 n_3} (\Omega) \int_{-\infty}^{+\infty} dt \cosh^{D-1}(t) \times$$

$$\times \left[ \cosh^{j_1}(t) e^{(j_1+\frac{D-1}{2}+i\mu_1)t} F \left( j_1 + \frac{D-1}{2}, j_1 + \frac{D-1}{2} + i \mu_1; 1 + i \mu_1; -e^{2t} \right) \right] \times$$

$$\times \left[ \cosh^{j_2}(t) e^{(j_2+\frac{D-1}{2}-i\mu_1)t} F \left( j_2 + \frac{D-1}{2}, j_2 + \frac{D-1}{2} - i \mu_1; 1 - i \mu_1; -e^{2t} \right) \right] \times$$

$$\times \left[ \cosh^{j_3}(t) e^{(j_3+\frac{D-1}{2}-i\mu_2)t} F \left( j_3 + \frac{D-1}{2}, j_3 + \frac{D-1}{2} - i \mu_2; 1 - i \mu_2; -e^{2t} \right) \right] \right) \text{ (10)}$$

where \( \mu_1 = \sqrt{M^2 - (\frac{D-1}{2})^2} \), \( \mu_2 = \sqrt{m^2 - (\frac{D-1}{2})^2} \). Note that this amplitude is invariant under general covariance by construction.

If either mass, \( m \) or \( M \), is vanishing, the time integral in the amplitude (10) is divergent \[5\] (we shortly discuss the case of \( m = 0 \) in the concluding section). However, if we keep both masses \( M \) and \( m \) non–zero, this time integral is convergent \[5\], because the harmonics under the integral exponentially decay as $t \to \pm \infty$.

The simplest way to see that the amplitude (10) is not zero is to note that it is the
analytical continuation (from the $D$–dimensional sphere to $D$–dimensional de Sitter space) of the generalized $3j$ symbol on the $D$–dimensional sphere. Explicit numerical calculation of the time integral in (10) shows that it is not zero for $|j_2 - j_3| \leq j_1 \leq j_2 + j_3$ [5], i.e. where the $3j$ symbol in (10) is non–zero. The amplitude (10) is non–zero for any large, but finite mass. Note that the amplitude is non–zero even if $j_1 = j_2$, i.e. when $\bar{\Psi}$ is taken as just the complex conjugate of $\Psi$.

Hence, in (4) we have excited the field $\Phi$ at future infinity due to the presence of the mass-shell field $\Psi$ at the initial stage. Such a process is naturally identified as radiation — such a definition works universally for all types of space–times. If we take into account higher corrections in $\lambda$ then we observe the creation of many harmonics of the field $\Phi$.

The intensity of the formation of the concrete mode of the field $\Phi$ is proportional to the square modulus of the above defined classical amplitude. The process is indeed classical, because we are just studying the solution of the non–linear, classical equations of motion and do not take into account any quantum field theoretic effects.

V. CONCLUSIONS

We conclude that in a classical field theory in de Sitter space, massive particles can radiate other massive particles. The only force which acts on the particles is due to the gravitational background induced by the vacuum (“dark”) energy. It is the inertially moving observer which sees the radiation from the free floating bodies in de Sitter space, because the charged body and inertial observer accelerate away from each other along geodesics in de Sitter space. The important observation for our discussion is that the only force which acts both on the charged body and the inertial observer is the gravitational one, which is due to the vacuum energy.

One could also calculate the classical energy momentum flux of various massless fields from the corresponding free falling charges in de Sitter space [12]. It is straightforward to see that there is no electromagnetic radiation, because Maxwell’s theory is conformal and its classical physics in de Sitter space is not different from that in Minkowski space. Rather unexpectedly for us there is no gravitational radiation [12]. However, there is the radiation of massless minimally coupled field, $\phi$, which confirms the observations of [9]. But we disagree with the conclusions of the latter paper that the radiation will stop, when the
effective mass, $m - q \phi$ (where $q$ is the scalar charge) vanishes \cite{12}. It is declared in \cite{9} that the charged particle just disappears when the effective mass vanishes. We can find no reason why the effective mass can not be negative. Furthermore consider a particle charged both with respect to the scalar and electromagnetic field. Then, by virtue of electric charge conservation, this particle can not just disappear when its effective mass becomes zero.

One could ask as well the same question of classical radiation in AdS space. It seems that a free floating particle in AdS space will emit radiation as well. First, let us stress that AdS space, unlike de Sitter, has a globally defined time–like Killing vector. Hence, there is energy conservation in this space and one can diagonalize the free Hamiltonian for any field theory in AdS space for all times. In the corresponding basis the analog of the amplitude (10) for AdS space will vanish. Second, the answer to this question depends on the boundary conditions at spatial infinity of AdS space, because of the impossibility to define a Cauchy surface in such a background \cite{13}.

The radiation amplitude for de Sitter space given in equation (10) is small for realistic parameters, unless, as during the early stage of the Universe, the vacuum energy is huge. Thus the probability to create massive particles is small, but if de Sitter space is eternal, then one can build a perpetuum mobile and by living in such a space long enough extract any amount of energy from the radiating charges. For example, one can construct a box with walls which act as mirrors for the fermion and boson fields. Then one can put in such a box some amount of massive fermions. Once in a while these fermions will produce a massive scalar. By waiting long enough one can heat the box to any extent.

One might be tempted to think that if the back reaction on the radiation were taken into account then one would see that the system of non–linear equations (5) has some basis of stable soliton–like solutions for the fermions and bosons. Then fermions would not excite bosons. This is just equivalent to the statement that, while one can not diagonalize the free classical Hamiltonian, one can somehow find a soliton–like basis of solutions of the full interacting theory, in terms of which the full interacting Hamiltonian will be diagonalizable for all times. However, this statement, if it is true, is a non–trivial assertion which would have to be shown explicitly.

One could say that our conclusions are not surprising, since there is no energy conservation in de Sitter space, because of the absence of a globally defined time–like Killing vector and/or the absence of the corresponding Casimir operator of the isometry group, which is
responsible for the time translations in de Sitter space. But then one has to decide what is correct: either de Sitter space is not eternal and the vacuum energy will eventually go to zero or one can heat one's house for free.

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[14] We stress that the reason why we call our considerations classical is because in this paper we are just dealing with solutions of non-linear, classical wave equations.

[15] We explain in the main body of the text exactly what we mean by “classical radiation” and “classical amplitude”.

[16] If $D$ is odd, however, $c_2 = 0$ and it is believed that there is no ambiguity in the choice of the de Sitter invariant vacuum state [10].