Plancherel formula for $\text{GL}_n(F) \backslash \text{GL}_n(E)$ and applications to the Ichino-Ikeda and formal degree conjectures for unitary groups

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Abstract

We establish an explicit Plancherel decomposition for $\text{GL}_n(F) \backslash \text{GL}_n(E)$ where $E/F$ is a quadratic extension of local fields of characteristic zero by making use of a local functional equation for Asai $\gamma$-factors. We also give two applications of this Plancherel formula: first to the global Ichino-Ikeda conjecture for unitary groups by completing a comparison between local relative characters that was left open by W. Zhang [Zh3] and secondly to the Hiraga-Ichino-Ikeda conjecture on formal degrees [HII] in the case of unitary groups.

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1 Introduction

Let $E/F$ be a quadratic extension of local fields of characteristic 0. In this paper, we develop an explicit Plancherel decomposition for the symmetric space $GL_n(F)\backslash GL_n(E)$. The proof is, at least superficially, similar to the factorization of global Flicker-Rallis periods (see [Fli], [GJR §2] and [Zh3 §3.2]) and moreover the result is in remarkable agreement with general conjectures of Sakellaridis-Venkatesh [SV] on the spectrum of general spherical varieties.

Let us mention here that Plancherel decompositions of symmetric spaces have already been thoroughly studied in the literature. In particular, most relevant for us is the work of Harinck [Har] giving another explicit Plancherel formula for the case at hand when $E/F =$
\( \mathbb{C}/\mathbb{R} \) (and more generally for symmetric spaces of the form \( G(\mathbb{R}) \backslash G(\mathbb{C}) \)) and also the work of Delorme [Del2] (following what was done by Sakellaridis-Venkatesh [SV] for split spherical varieties) on \( p \)-adic symmetric spaces (which is less explicit since it consider as a black box the "discrete spectrum" of certain Levi varieties). However, the present work is rather orthogonal to those and uses heavily particular features of the pair \( (\text{GL}_n(E), \text{GL}_n(F)) \) allowing to express the relevant \( L^2 \)-inner products as residues of certain families of local Zeta integrals (similarly, the Flicker-Rallis period is usually studied by considering it as the residue of a global family of Zeta integrals).

We then give two applications of this Plancherel formula: first to the global Ichino-Ikeda conjecture for unitary groups by completing a comparison between local relative characters that was left open by W. Zhang [Zh3] and secondly to the Hiraga-Ichino-Ikeda conjecture on formal degrees for unitary groups [HII]. Besides the Plancherel formula, the main tools to derive these applications are certain local analogs of the Jacquet-Rallis trace formulae completed with a certain comparison of "relatively unipotent" orbital integrals. Actually, this relationship between the Plancherel formula for \( \text{GL}_n(F) \backslash \text{GL}_n(E) \) and local Jacquet-Rallis trace formulae was already investigated in the case \( n = 2 \) by Ioan Filip in his PhD thesis [Fil].

In the rest of this introduction, we give more details on the main results.

### 1.1 Plancherel formula for \( \text{GL}_n(F) \backslash \text{GL}_n(E) \)

We now state the explicit Plancherel formula we obtain for the symmetric space \( Y_n = \text{GL}_n(F) \backslash \text{GL}_n(E) \). Let \( U(n) \) be a quasi-split unitary group of rank \( n \) (with respect to the extension \( E/F \)), \( \text{Temp}(U(n))/\text{stab} \) the set of all tempered \( L \)-packets of \( U(n) \), \( \text{Temp}(\text{GL}_n(E)) \) the set of all isomorphism classes of tempered irreducible representations of \( \text{GL}_n(E) \) and \( BC_n : \text{Temp}(U(n))/\text{stab} \to \text{Temp}(\text{GL}_n(E)) \) the stable base-change map if \( n \) is odd, unstable base-change map if \( n \) is even (this last case requires a choice, which however is unimportant for what follows).

Let us emphasize here that the local Langlands correspondence for unitary groups is now fully known by [Mok], [KMSW] and thus combining it with the local correspondence for \( \text{GL}_n \) ([HT], [Hen], [Sch]), we see that the preceding sentence makes perfect sense. Actually, we could have written our Plancherel formula without appealing to the local Langlands correspondence since only the image of \( BC_n \) matters and this can be described in a purely representation-theoretic way using local Asai \( L \)-functions (moreover, as we will see, this is more or less exactly how it shows up in the computations). Nevertheless, we prefer to write everything using this correspondence since we find the resulting formulations more suggestive and moreover this translation will play an important role for the applications (to be described in the second part of this introduction).

A first weak version of our Plancherel formula can be stated as follows.

**Theorem 1**  There exists an isomorphism of unitary representations

\[
L^2(Y_n) \cong \int_{\text{Temp}(U(n))/\text{stab}} \mathcal{H}_\sigma \, d\sigma
\]
where $\sigma \in \text{Temp}(U(n))/\text{stab} \mapsto \mathcal{H}_\sigma$ is a measurable field of unitary representations of $\text{GL}_n(E)$ with $\mathcal{H}_\sigma \simeq BC_n(\sigma)$ for every $\sigma \in \text{Temp}(U(n))/\text{stab}$ and the measure $d\sigma$ is in then natural class of measures on $\text{Temp}(U(n))/\text{stab}$.

Notice that as a direct corollary of this result, we can describe explicitly the discrete spectrum of $L^2(Y_n)$: the irreducible unitary representations $\pi$ of $\text{GL}_n(E)$ embedding continuously into $L^2(Y_n)$ are precisely the base-change (stable or unstable as before) of discrete series of $U(n)$. Notice that those representations were already shown to embed in $L^2(Y_n)$ by Jerrod Smith [Smith]. Moreover, the above decomposition confirms in the case of the symmetric space $Y_n$ a general conjecture of Sakellaridis-Venkatesh [SV, Conjecture 16.5.1] on Plancherel decompositions of spherical varieties.

To state our Plancherel formula more precisely, we need to introduce more notation. Let $\mathcal{S}($GL$_n(E))$ be the Schwartz space of $\text{GL}_n(E)$ i.e. the space of complex valued functions on $\text{GL}_n(E)$ which are locally constant and compactly supported if $E$ is $p$-adic or which are $C^\infty$ and "rapidly decreasing with all their derivatives" in some suitable sense if $E$ is Archimedean (we refer the reader to Section 2.4 for a precise definition; a similar space of functions is defined on the set of $F$-points of any smooth algebraic variety over $F$). For $\pi \in \text{Temp}($GL$_n(E))$, we define a positive semi-definite hermitian form $\langle ..,.. \rangle_{Y_n,\pi}$ on $\mathcal{S}($GL$_n(E))$ by

$$\langle f_1, f_2 \rangle_{Y_n,\pi} := \sum_{W \in \mathcal{B}(\pi, \psi_n)} \beta(\pi(f_1)W) \overline{\beta(\pi(f_2)W)}.$$ 

Here, $\psi$ is a non-trivial additive character of $E$ trivial on $F$, $\psi_n$ the corresponding generic character of the standard maximal unipotent subgroup $N_n(E)$ of $\text{GL}_n(E)$, $\mathcal{B}(\pi, \psi_n)$ is a suitable orthonormal basis of the Whittaker model $W(\pi, \psi_n)$ for the natural scalar product on the associated Kirillov model (i.e. $L^2$-scalar product on $N_n(E)\backslash P_n(E)$ where $P_n$ is the mirabolic subgroup of $\text{GL}_n$) and $\beta$ is the linear form given by

$$\beta(W) := \int_{N_n(F) \backslash P_n(F)} W(p) dp.$$ 

We equip $\text{Temp}(U(n))/\text{stab}$ with a natural and canonical measure $d\sigma$ which is locally given by a Haar measure on certain groups of unramified characters (see Section 2.7). Using a nontrivial additive character $\psi'$ of $F$, we can also define normalized Haar measures on $\text{GL}_n(E)$ and $\text{GL}_n(F)$ (see Section 2.3) hence an invariant measure on $Y_n$. To $\sigma \in \text{Temp}(U(n))/\text{stab}$ we associate its adjoint $\gamma$-factor $\gamma(s, \sigma, \text{Ad}, \psi')$ as well as a certain finite group $S_\sigma$ which is just the centralizer of its Langlands parameter when $\sigma$ is a discrete series (for the general case cf. [Pra]). Then, the explicit Plancherel formula we prove for $Y_n$ reads as follows (cf. Theorem 4.2.2).

---

1Strictly speaking, the general conjectures of [SV] do not apply to $Y_n$ since in loc. cit. the group acting $G$ (which here is $R_{E/F} \text{GL}_{n,E}$) was assumed to be split over $F$. However, in light of another conjecture of Jacquet (see [Pra]) there is a natural way to extrapolate the conjecture of Sakellaridis-Venkatesh to the case at hand by considering the "$L$-group" of $Y_n$ to be $^L Y_n = ^L U(n)$ and the morphism $^L Y_n \to ^L G$ to be given by base-change (stable or unstable as before).
Theorem 2 For every $\sigma \in \text{Temp}(U(n))/\text{stab}$, the hermitian form $(\cdot,\cdot)_{Y_n,BC_n(\sigma)}$ factorizes through the natural projection $S(\text{GL}_n(E)) \to S(Y_n)$. Moreover, assuming that all Haar measures have been normalized using the additive character $\psi$, for every functions $\varphi_1, \varphi_2 \in S(Y_n)$ we have

$$(\varphi_1, \varphi_2)_{L^2(Y_n)} = \int_{\text{Temp}(U(n))/\text{stab}} (\varphi_1, \varphi_2)_{Y_n,BC_n(\sigma)} \frac{\gamma^*(0, \sigma, \text{Ad}, \psi')}{|S_\sigma|} d\sigma$$

where the left-hand side denotes the $L^2$-inner product on $Y_n$, the right-hand side is absolutely convergent and we have set

$$\gamma^*(0, \sigma, \text{Ad}, \psi') = (\zeta_p(s) n_\sigma \gamma(s, \sigma, \text{Ad}, \psi'))_{s=0},$$

$n_\sigma$ being the order of the zero of $\gamma(s, \sigma, \text{Ad}, \psi')$ at $s = 0$.

Although we will not explain this in any details, the above explicit version of the Plancherel decomposition of $Y_n$ is pleasantly aligned with certain speculations of Sakellaridis-Venkatesh [SV] §17 on factorization of global automorphic periods. The main reason being that the linear form $\beta$ also appears in the factorization of the global Flicker-Rallis periods (we refer the reader to [GJR] §2 and [Zl3] §3.2 for a precise statement) and, as we will see in the second part of this introduction, the quotient $\frac{|\gamma^*(0,\sigma,\text{Ad},\psi')|}{|S_\sigma|}$ equals on the nose (again if we normalize measures correctly using the character $\psi'$) the Plancherel densities of unitary groups. In fact, the proof of Theorem 2 is quite similar to the global computations leading to the factorization of Flicker-Rallis periods. Let us explain the main steps. Let $\varphi_1, \varphi_2 \in S(Y_n)$ and choose (arbitrarily) functions $f_1, f_2 \in S(\text{GL}_n(E))$ such that $\varphi_i(x) = \int_{\text{GL}_n(F)} f_i(\gamma x) d\gamma$ for $i = 1, 2$. Then, simple manipulations show that

$$(\varphi_1, \varphi_2)_{L^2(Y_n)} = \int_{\text{GL}_n(F)} f(h) dh$$

where $f = f_2 * f_1^\vee$ (here as usual $f_1^\vee$ denotes the function $g \mapsto f_1(g^{-1})$). Therefore, we are essentially reduced to finding a spectral expansion for the linear form $f \in S(\text{GL}_n(E)) \mapsto \int_{\text{GL}_n(F)} f(h) dh$. The first step is then, by some "local unfolding", to prove an identity (see Proposition 4.3.1)

$$\int_{\text{GL}_n(F)} f(h) dh = C_1 \int_{N_n(F) \backslash P_n(F)} \int_{N_n(F) \backslash \text{GL}_n(F)} W_f(p, h) d\mu_{\text{GL}_n(E)}(\pi)$$

where $W_f$ is a certain Whittaker function associated to $f$ analog to the global Whittaker function associated to cusp forms (see Section 2.14 for the precise definition of $W_f$) and $C_1$ is a certain constant depending on the choice of the character $\psi$. Let $W_f = \int_{\text{Temp}(\text{GL}_n(E))} W_f(\pi) d\mu_{\text{GL}_n(E)}(\pi)$ be the spectral decomposition of the Whittaker function $W_f$ (that is the Plancherel decomposition for Whittaker functions which according to [SV] §6.3] can be deduced from the Plancherel formula for the group; see also Section 2.14 of this
paper where we make this slightly more precise), where $d\mu_{\text{GL}_n(E)}(\pi)$ denotes the Plancherel measure for $\text{GL}_n(E)$. Then, we would like to use this expansion of $W_f$ to get a spectral decomposition for the right-hand side of \[1.1.1\]. Unfortunately, the resulting expression is not absolutely convergent and in particular we cannot switch the spectral integral with the integral over $N_n(F) \backslash \text{GL}_n(F)$. To circumvent this difficulty, we will express the inner integral

$$
\int_{N_n(F) \backslash \text{GL}_n(F)} W_f(p, h) dh = \int_{Z_n(F)N_n(F) \backslash \text{GL}_n(F)} W_{\tilde{f}}(p, h) dh
$$

where $\tilde{f} = \int Z_n(F) f(z) dz$ ($Z_n(F)$ denoting the center of $\text{GL}_n(F)$) as the residue at $s = 0$ of some Zeta integral $Z(s, W_{\tilde{f}}(p,.), \phi)$ associated to an auxiliary test function $\phi \in \mathcal{S}(F^n)$ (see Lemma 2.16.1 ii) for a precise statement). The important point is that the formation of this Zeta integral "commutes" with the spectral decomposition of $W_{\tilde{f}}$ (i.e. the resulting expression is absolutely convergent) when $\Re(s) > 0$ so that in this range we can write

$$
Z(s, W_{\tilde{f}}(p,.), \phi) = \int_{\text{Temp}(GL_n(E))} Z(s, W_{f,\pi}(p,.), \phi) d\mu_{\text{GL}_n(E)}(\pi)
$$

where we have set $\text{GL}_n(E) = Z_n(F) \backslash \text{GL}_n(E)$. Then, using a local functional equation for the above Zeta integrals in terms of Asai $\gamma$-factors $\gamma(s, \pi, As, \psi')$ (see Section 2.16), this can be rewritten as an expression of the form

$$
(1.1.2) \int_{\text{Temp}(\text{GL}_n(E))} \Phi_\pi(s) \gamma(s, \pi, As, \psi')^{-1} d\mu_{\text{GL}_n(E)}(\pi)
$$

where now the family of function $s \mapsto \Phi_\pi$ is analytic at $s = 0$. Thus, we end up with the problem of computing the residue at $s = 0$ of the above distribution. This can be achieved by a direct computation using an explicit formula for the Plancherel measure $d\mu_{\text{GL}_n(E)}(\pi)$ in terms of the adjoint $\gamma$-factor of $\pi$ which is essentially due to Harish-Chandra [H-C2] in the Archimedean case and follows from work of Shahidi [Sha] and Silberger-Zink [SZ] in the $p$-adic case (see [HII] for a convenient uniform reformulation of all these results). Although the computation is not very enlightening (it is done in Part 3 of this paper), the final result is very neat: we get, up to an explicit constant, the integral of the function $\Phi = \Phi_0$ against the push-forward by base-change of the measure $\gamma^*(0, \sigma, \lambda \delta, \psi') d\sigma$ on $\text{Temp}(U(n))/\text{stab}$ (see Proposition 3.4.1). Together with the previous steps, this then very easily implies Theorem 2.

1.2 Applications to the Ichino-Ikeda and formal degree conjectures for unitary groups

In Part 5 of this paper we give two applications of the Plancherel formula of Theorem 2. Namely, we establish the formal degree conjecture of Hiraga-Ichino-Ikeda [HII] for unitary groups as well as a certain comparison of local relative characters left open by W. Zhang.
Conjecture 4.4] and which has application to the so-called Ichino-Ikeda conjecture for unitary groups [Ha]. We now state these two results in turn.

Let \( E \) be a (finite dimensional) hermitian space over \( E \), \( U(V) \) the corresponding unitary group and \( \mu^*_U(V)(\pi) \) the “Plancherel density" for \( U(V)(F) \) i.e. the unique function on the tempered dual \( \text{Temp}(U(V)) \) of \( U(V)(F) \) such that the Plancherel measure for \( U(V)(F) \) is given by \( d\mu_{U(V)}(\pi) = \mu^*_U(V)(\pi)d\pi \) where \( d\pi \) is a certain natural and canonical measure on \( \text{Temp}(U(V)) \) (see Section 2.7). Notice that the Plancherel measure depends on some Haar measure on \( U(V)(F) \) for which, as before, there is a natural choice depending only on the nontrivial additive character \( \psi' \) of \( F \). Therefore, \( \mu^*_U(V)(\pi) \) also depends on \( \psi' \). With these notation we prove:

**Theorem 3** We have

\[
\mu^*_U(V)(\pi) = \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|}
\]

for almost all \( \pi \in \text{Temp}(U(V)) \) where \( \gamma^*(0, \pi, \text{Ad}, \psi') \) and \( S_\pi \) are defined as before. In particular, for every discrete series \( \pi \) of \( U(V)(F) \) we have

\[
d(\pi) = \frac{|\gamma(0, \pi, \text{Ad}, \psi')|}{|S_\pi|}
\]

where \( d(\pi) \) denotes the formal degree of \( \pi \) and \( S_\pi \) is the centralizer of the Langlands parameter of \( \pi \).

The second part of the above theorem is precisely [HII Conjecture 1.4] for unitary groups. Also, although we will not use it, the first part can be deduced from the second part using Langlands’ normalization of standard intertwining operators (which is known for unitary groups see [Mok Proposition 3.3.1], [KMSW, Lemma 2.2.3]) the formal degree conjecture for \( \text{GL}_n(E) \) (which is also known by work of Silberger-Zink, see [HII Theorem 3.1]) and the description of the Plancherel measure as in [Wald]. We also remark that Theorem 3 is not new when \( F \) is Archimedean since in that case it can be deduced from Harish-Chandra’s explicit formula for the Plancherel measure [HC2] (see [HII Proposition 2.1] for the translation, at least for discrete series).

To state the second application, we need to introduce more notation. Let \( (V, h) \) be a \( n \)-dimensional hermitian space over \( E \) and set \( H = U(V), G = U(V) \times U(V') \) where \( V' = V \oplus Ev_0 \) is equipped with the hermitian form given by \( h'(v_1 + \lambda v_0, v_2 + \mu v_0) = h(v_1, v_2) + \lambda \mu c \) for all \( v_1, v_2 \in V \) and \( \lambda, \mu \in E \) (in this paper, we denote by \( c \) the non-trivial \( F \)-automorphism of \( E \)). We consider \( H \) as a subgroup of \( G \) through the natural diagonal embedding \( H \hookrightarrow G \). We also define \( G' = R_{E/F} \text{GL}_{n,E} \times R_{E/F} \text{GL}_{n+1,E} \) (here, as elsewhere in the paper, \( R_{E/F} \) stands for Weil’s restriction of scalars from \( E \) to \( F \)) and its two subgroups \( H_1 = R_{E/F} \text{GL}_{n,E} \) (diagonally embedded) and \( H_2 = \text{GL}_{n,F} \times \text{GL}_{n+1,F} \). Then, Jacquet and Rallis [JR] have defined a notion of transfer between functions \( f \in S(G(F)) \) and \( f' \in S(G'(F)) \). This transfer is itself the byproduct of a natural injective correspondence between regular semisimple orbits (or double cosets) [Zha2 Lemma 2.3]

\[
(1.2.1) \quad H(F)\backslash G(F)/H(F) \hookrightarrow H_1(F)\backslash G'_\text{rs}(F)/H_2(F).
\]
where $G'_{rs}$ (resp. $G_{rs}$) denotes the open subset of elements $\gamma \in G'$ (resp. $\delta \in G$) whose double coset $H_1 \gamma H_2$ (resp. $H \delta H$) is closed and with a trivial stabilizer in $H_1 \times H_2$ (resp. in $H \times H$). For $(f, f') \in \mathcal{S}(G(F)) \times \mathcal{S}(G'(F))$ and $(\delta, \gamma) \in G'_{rs}(F) \times G''_{rs}(F)$ we define as usual (relative) orbital integrals $O(\delta, f)$ and $O_\eta(\gamma, f')$ (the second one being twisted by a certain quadratic character $\eta$ of $H_2(F)$; see Section 5.2). Then, we say that $f$ and $f'$ match (or that they are transfer of each other) if we have an identity

$$\Omega(\gamma)O_\eta(\gamma, f') = O(\delta, f)$$

whenever $\gamma$ and $\delta$ correspond to each other by the correspondence \ref{12.1} and where $\Omega(\gamma)$ is a certain transfer factor (which has an explicit and elementary definition, see Section 5.2).

It is one of the main achievement of W. Zhang \cite{Zh1} that in the $p$-adic case every function $f \in \mathcal{S}(G(F))$ admits a transfer $f' \in \mathcal{S}(G'(F))$ and conversely. For a partial result in that direction in the Archimedean case, which is however sufficient for applications, we refer the reader to \cite{Xue} or Section 5.2 of this paper.

We define as in \cite{Zh3} relative characters $f \in \mathcal{S}(G(F)) \mapsto J_\pi(f)$ and $f' \in \mathcal{S}(G'(F)) \mapsto I_\Pi(f')$ for every tempered irreducible representations $\pi \in \text{Temp}(G)$ and $\Pi \in \text{Temp}(G')$ (we warn the reader here that our convention for $J_\pi$ and $I_\Pi$ is slightly different from the one in \cite{Zh3}, in particular we have replaced $\pi$ and $\Pi$ by their contragredient and moreover we don’t use the same scalar product as in loc. cit. to normalize $I_\Pi$; these changes are nevertheless minor and don’t affect the global applications: see remark below). By its very definition, the family of relative characters $\pi \mapsto J_\pi$ is supported on the set $\text{Temp}_H(G)$ of $H$-distinguished tempered representations (i.e. the one admitting a nonzero continuous $H(F)$-invariant form).

The second main theorem of Part 5 can now be stated as follows (see Theorem 5.4.1).

**Theorem 4** Let $f \in \mathcal{S}(G(F))$ and $f' \in \mathcal{S}(G'(F))$ be matching functions. Then, for every $\pi \in \text{Temp}_H(G)$, we have

$$\kappa_V J_\pi(f) = I_{BC(\pi)}(f')$$

where $BC(\pi)$ denotes the stable base-change of $\pi$ and $\kappa_V$ is an explicit constant which depends only on $V$ and the normalization of transfer factors.

**Remark 1** • The above result was already proved in \cite{Beu2} when $F$ is $p$-adic. The proof we give here, although using similar tools, differs at some crucial points from loc. cit. and moreover has the good feature, at least to the author’s taste, of treating uniformly the Archimedean and non-Archimedean case. In particular, in this paper we make no use at all of the result of W. Zhang \cite{Zh3} on truncated local expansion for the relative characters $I_\Pi$. We need however to import from \cite{Zh1} the existence of smooth transfer as well as its compatibility with Fourier transform at the Lie algebra level. Furthermore, although it might not be so transparent, we need the Jacquet-Rallis fundamental lemma proved by Yun and Gordon \cite{Yu} in order to derive the “weak comparison” of relative characters given in Proposition 5.7.1 from a global comparison of (simple) Jacquet-Rallis trace formulae.
The above theorem confirms [Zh3, Conjecture 4.4]. Actually, as already pointed, our normalization for relative characters is not the same thus explaining why our constant $\kappa_V$ differs from loc. cit. However, this discrepancy is not completely afforded by the change of normalization since in [Zh3] the constant up to which “Fourier transform commutes with transfer” was not exactly the correct one and here we used the one computed by Chaudouard in [Chau]. For the global applications however (see below) this difference is inessential: all that matters is that if $V$ is a hermitian space relative to a quadratic extension $k'/k$ of number fields then the product of the local constants $\prod_v \kappa_{V_v}$ is one.

As explained in [Zh3] and [Beu2], Theorem 4 has direct application to the Ichino-Ikeda conjecture for unitary groups. Namely, from [Zh3, Theorem 4.3] and [Beu2, Theorem 3.5.1] and the above theorem we deduce:

**Theorem 5** Let $k'/k$ be a quadratic extension of number fields, $V$ an hermitian space over $k'$ and define the groups $H \subset G$ as above. Let $\pi = \otimes'_v \pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$ ($\mathbb{A}$ being the ring of adèles of $k$) satisfying:

- For every place $v$ of $k$, the representation $\pi_v$ is tempered;
- There exists a non-Archimedean place $v$ of $k$ with $BC(\pi_v)$ supercuspidal (e.g. if $v$ splits in $k'$ and $\pi_v$ is itself supercuspidal).

Then, the Ichino-Ikeda conjecture as stated in [Ha, Conjecture 1.2] or [Zh3, Conjecture 1.1] is true for $\pi$.

**Remark 2** Of course, the above theorem also includes previous works of many other authors including W. Zhang, Z. Yun and J. Gordon as well as H. Xue. We have mainly stated it for convenience to the reader that would like to get the most up-to-date result on the Ichino-Ikeda conjecture. Moreover, the assumption that $BC(\pi_v)$ is supercuspidal for at least one place $v$ (which is the last one that needs to be removed) originates from the use of some simple version of Jacquet-Rallis trace formulae. To drop this assumption, one would need complete spectral decompositions of these trace formulae which is work in progress by Chaudouard and Zydor (see [Zyd] and [CZ] for partial progress in that direction).

Theorems 3 and 4 are proved together by comparing certain distributions on $G(F)$ and $G'(F)$. Namely, we will compare both local analogs of the aforementioned Jacquet-Rallis trace formulae as well as certain “relatively unipotent” orbital integrals. To be more specific, we discuss the two comparisons in turn.

First, the local analog of the unitary Jacquet-Rallis trace formula (which was already used in [Beu2]) is a certain $H(F) \times H(F)$-invariant sesquilinear form $J(\cdot,\cdot)$ on $S(G(F))$ admitting the following “geometric” and “spectral” expansions (see Section 5.3.1):

$$\int_{H(F) \backslash G \times H(F) / H(F)} O(\delta, f_1) \overline{O(\delta, f_2)} d\delta = J(f_1, f_2) = \int_{\text{Temp}(G)} \pi J_\pi(f_1) \overline{J_\pi(f_2)} d\mu_G(\pi)$$
where \( d\delta \) is a certain natural measure on the set of regular semi-simple orbits and \( d\mu_G(\pi) \) denotes the Plancherel measure of \( G(F) \). Similarly, the local analog of the linear Jacquet-Rallis trace formula is a certain \( H_1(F) \times (H_2(F), \eta) \)-equivariant sesquilinear form \( I(\ldots) \) on \( \mathcal{S}(G'(F)) \) admitting the following “geometric" and “spectral" expansions (see Section 5.5.2):

\[
(1.2.3) \quad \int_{H_1(F) \backslash G_\infty(F) / H_2(F)} O_\eta(\gamma, f'_1) O_\eta(\gamma, f'_2) d\gamma = I(f'_1, f'_2)
\]

\[
= C_2 \int_{\text{Temp}(G_{qs}) / \text{stab}} I_{BC(\pi)}(f'_1) I_{BC(\pi)}(f'_2) \frac{|\gamma^\pi(0, \pi, \text{Ad}, \psi')|}{|S_{\pi}|} d\pi
\]

where \( d\gamma \) is a certain natural measure on the set of regular semi-simple orbits, \( G_{qs} \) is a quasi-split inner form of \( G \), \( \text{Temp}(G_{qs}) / \text{stab} \) denotes the set of tempered \( L \)-packets for \( G_{qs} \) and \( C_2 \) is a certain positive constant depending on the choice of \( \psi \). Here we remark that the proof of \([1.2.2]\) doesn’t pose any analytical difficulty and is rather easy and direct whereas to get the spectral side of \([1.2.3]\) we need to use the explicit Plancherel formula of Theorem 2. When \( f_k \) match \( f'_k \) for \( k = 1, 2 \) the left-hand sides of \([1.2.2]\) and \([1.2.3]\) are easily seen to be equal (this is not quite correct, as we need first to sum \([1.2.2]\) over some pure inner forms but we will ignore this issue in the introduction) and we deduce from this the equality of the right-hand sides. This is the first comparison that we will use.

Next, we consider the “trivial" orbital integral \( O(1, f) = \int_{H(F)} f(h) dh \) on \( \mathcal{S}(G(F)) \) for which we have the spectral expansion (see \([5.6.1]\))

\[
(1.2.4) \quad O(1, f) = \int_{\text{Temp}(G)} J_\pi(f) d\mu_G(\pi)
\]

as well as a certain “regularized" regular unipotent orbital integral \( f' \mapsto O_+(f') \) on \( \mathcal{S}(G'(F)) \) for which we have the spectral expansion (see \([5.6.3]\))

\[
(1.2.5) \quad \gamma O_+(f') = C_3 \int_{\text{Temp}(G_{qs}) / \text{stab}} I_{BC(\pi)}(f) \frac{|\gamma^\pi(0, \pi, \text{Ad}, \psi')|}{|S_{\pi}|} d\pi
\]

where \( C_3 \) is again a positive constant depending on \( \psi \) and \( \gamma \) a certain product of abelian local \( \gamma \)-factors. Incidentally (or not), the same regularized orbital integral appears in the truncated local expansions of W. Zhang \([Zh3]\) for the relative characters \( I_\Pi \). Once again, the identity \([1.2.4]\) comes almost for free and doesn’t require much hard work whereas \([1.2.5]\) is a consequence of (some form of) the explicit Plancherel formula of Theorem 2. Then, using analogs of \([1.2.4]\) and \([1.2.5]\) for Lie algebras and the fact that Fourier transform commutes, up to a constant, with transfer (\([Zha1], [Xue]\)), we can show that if \( f \in \mathcal{S}(G(F)) \) and \( f' \in \mathcal{S}(G'(F)) \) match then \( O(1, f) \) and \( O_+(f') \) are equal up to some (absolute and explicit) constant (once again the correct statement involves summing \([1.2.4]\) over some set of pure inner forms). This, together with \([1.2.3]\) and \([1.2.5]\) implies a second spectral identity between matching functions.

Theorem 4 and Theorem 5 (or rather an analog of Theorem 5 pertaining to the Plancherel measure of \( G \) restricted to \( \text{Temp}_H(G) \)) can then be easily deduced from these two comparisons combined with some "weak" identity of relative characters (Proposition 5.7.1). We refer the reader to Section 5.7 for all the details of the proof.
1.3 Outline of the paper

We now briefly describe the content of the paper. Part 2 is devoted to fixing notation and numerous normalizations. We also collect there various results from the literature which will be needed in the sequel in particular regarding local base-change for unitary groups (Section 2.10), local $\gamma$-factors (Section 2.12), the Plancherel formula for the group and for Whittaker functions (Sections 2.13 and 2.14) and the local functional equation for Asai $\gamma$-factors of Rankin-Selberg type (Section 2.16). Part 3 is the most technical of the paper and is concerned with computing explicitly poles of analytic families of distributions like 1.1.2. As we said, the final result is very neat but, unfortunately, the author wasn’t able to find a conceptual way to derive it, therefore we have essentially reduced everything to certain computations in $\mathbb{R}^n$. In Part 4, we establish the explicit Plancherel formula of Theorem 2. This is a rather easy task as the adequate preliminary results have been obtained in Part 3. In the last part of this paper, Part 5, we prove Theorems 3 and 4 following the outline given above. Finally, we have added one appendix at the end of the paper containing the proof of a rather technical result related to the Harish-Chandra Plancherel formula and for which the author was unable to find a proper reference in the literature.

1.4 Acknowledgment

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2 Preliminaries

2.1 General notation

Here is a list of notation that will be used throughout in this paper:

- $E/F$ is a quadratic extension of local fields of characteristic zero (either Archimedean or non-Archimedean), we denote by $c$ the non-trivial $F$-automorphism of $E$, $\text{Tr}_E/F : E \to F$ the trace and by $\eta_{E/F} : F^\times \to \{\pm 1\}$ the quadratic character associated to $E/F$ by local class field theory. Also, in the non-Archimedean case we write $\mathcal{O}_F$ and $\mathcal{O}_E$ for the ring of integers of $F$ and $E$ respectively.

- We fix a character $\eta'$ of $E^\times$ extending $\eta_{E/F}$ as well as nontrivial additive characters $\psi'$ and $\psi$ of $F$ and $E$ respectively with $\psi$ trivial on $F$. We denote by $\tau$ the unique element in $E$ of trace zero such that $\psi(z) = \psi'(\text{Tr}_{E/F}(\tau z))$ for all $z \in E$. We will also denote by $\psi'_E$ the character $z \mapsto \psi'(\text{Tr}_{E/F}(z))$ of $E$. 


• $W_F$ and $W_E$ are the Weil groups of $F$ and $E$ respectively, $q_F$, $q_E$ the cardinality of the residue fields of $F$ and $E$ in the $p$-adic case whereas in the Archimedean case we set $q_F = q_E = e^{1/2}$.

• We fix an algebraic closure $\overline{F}$ of $F$ containing $E$ and for every finite extension $K$ of $F$ we denote by $|.|_K$ the normalized absolute value on $K$. Most of the time we will simply write $|.|$ for $|.|_F$.

• If $X$ is an algebraic variety over $F$ and $K/F$ a finite extension, we denote by $X_K$ the variety obtained by base-change from $F$ to $K$. In the other direction, if $X$ is an algebraic variety over $K$, we denote by $R_{K/F}X$ the Weil’s restriction of scalar of $X$ from $K$ to $F$.

• Unless otherwise specified, all algebraic varieties will be tacitly assumed to be defined over $F$.

• If $X$ is an algebraic variety over $F$, we will use freely the notion of norms on $X(F)$ as defined by Kottwitz [Kott, Sect. 18]. We always denote by $\|\|_X$ such a norm and set $\sigma_X = \log(2 + \|\|_X)$ for the associated log-norm. We refer the reader to [Ben1] §1.2 for more details on these notions.

• For $K$ a compact group, we denote by $\hat{K}$ the set of all isomorphism classes of its irreducible representations. Moreover, if $V$ is a representation of $K$ for each $\rho \in \hat{K}$ we denote by $V[\rho]$ its $\rho$-isotypic component. Similarly, if $A$ is a locally compact abelian group, $\hat{A}$ stands for its dual group.

• $\Re(z)$ and $\Im(z)$ will stand for the real and imaginary part respectively of a complex number $z$.

• $\mathcal{H} = \{s \in \mathbb{C} \mid \Re(s) > 0\}$ and $\lim_{s \to 0^+}$ means that we take the limit as $s$ goes to 0 from $\mathcal{H}$.

• A sentence like “$f(x) \ll g(x)$ for all $x \in X$" means that there exists $C > 0$ such that $f(x) \leq Cg(x)$ for all $x \in X$. Also, we write $f(x) \sim g(x)$ when $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

• Lie algebras of algebraic groups will always be denoted by the corresponding gothic letter (e.g. $\mathfrak{g}$ for $G$ or $\mathfrak{h}$ for $H$).

• For each integer $n \geq 1$, $\mathfrak{S}_n$ stands for the group of permutations of $\{1, \ldots, n\}$.

• A holomorphic function $\varphi : \mathbb{C}^n \to \mathbb{C}$ will be said to be of moderate growth in vertical strips together with all its derivatives if for all $a, b \in \mathbb{R}$ and every holomorphic differential operator with constant coefficients $D$ on $\mathbb{C}^n$ there exists $N \geq 1$ such that

$$|(D\varphi)(z_1, \ldots, z_n)| \ll (1 + |\Im(z_1)|)^N \ldots (1 + |\Im(z_n)|)^N$$
for all \((z_1, \ldots, z_n) \in \mathbb{C}^n\) with \(a < \Re(z_1), \ldots, \Re(z_n) < b\). Similarly, a meromorphic function \(\varphi : \mathbb{C} \to \mathbb{C}\) will be said to be of \textit{moderate growth on vertical strips away from its poles together with all its derivatives} if for all \(a, b \in \mathbb{R}\) there exists \(R \in \mathbb{C}(T)\) and for every \(n \geq 0\) there exists \(N \geq 1\) and such that

\[
\left| \left( \frac{d}{dz} \right)^n (R(z) \varphi(z)) \right| \ll (1 + |\Im(z)|)^N
\]

for all \(z \in \mathbb{C}\) with \(a < \Re(z) < b\).

- We also say that a smooth function \(\varphi : \mathbb{R}^n \to \mathbb{C}\) is of \textit{moderate growth together with all its derivatives} if for every differential operator with constant coefficients \(D\) on \(\mathbb{R}^n\) there exists \(N \geq 1\) such that

\[
|D\varphi(x_1, \ldots, x_n)| \ll (1 + |x_1|)^N \ldots (1 + |x_n|)^N
\]

for all \((x_1, \ldots, x_n) \in \mathbb{R}^n\).

- For each \(n \geq 1\), we denote by \((e_1, \ldots, e_n)\) the standard basis of \(F^n\).

- If \(f\) is a function on a group \(G\), we set \(f^\vee(g) = f(g^{-1})\) for all \(g \in G\).

- If a group \(G\) acts on a set \(X\) on the right (resp. on the left), we shall denote by \(R\) (resp. \(L\)) the right (resp. left) regular action of \(G\) on functions on \(X\). This action usually extends to some space of functions on \(G\). If moreover \(G\) is a Lie group, \(X\) is a smooth manifold and the action is differentiable, we denote by the same letter the resulting action of the Lie algebra \(\mathfrak{g}\) of \(G\) and also of its enveloping algebra. If \(G\) is a topological group equipped with a Haar measure and the action is on the right and continuous, for every function \(f\) and \(\varphi\) on \(G\) and \(X\) respectively we set (whenever it makes sense)

\[
(\varphi \star f)(x) = \int_G \varphi(xg^{-1})f(g)dg, \quad x \in X.
\]

Notice that \(\varphi \star f = R(f^\vee)\varphi\).

### 2.2 Groups

Let \(G\) be a connected reductive group over \(F\). We assume that \(G\) is fixed until the end of Section 2.14. We denote by \(Z_G\) the center of \(G\) and \(A_G\) the maximal split torus in \(Z_G\). Let \(X^*(G)\) be the group of algebraic characters of \(G\). We set

\[
\mathcal{A}_G^* = X^*(G) \otimes \mathbb{R} = X^*(A_G) \otimes \mathbb{R}, \quad \mathcal{A}_{G, \mathbb{C}}^* = \mathcal{A}_G^* \otimes_{\mathbb{R}} \mathbb{C}
\]

and

\[
\mathcal{A}_G = \text{Hom}(X^*(G), \mathbb{R}), \quad \mathcal{A}_{G, \mathbb{C}} = \mathcal{A}_G \otimes_{\mathbb{R}} \mathbb{C}
\]
Let \( \langle ., . \rangle \) be the natural pairing between \( \mathcal{A}_{G,F}^* \) and \( \mathcal{A}_{G,C} \). We define a morphism \( H_G : G(F) \to \mathcal{A}_G \) by \( \langle \chi, H_G(g) \rangle = \log|\chi(g)| \) for all \( \chi \in X^*(G) \) and \( g \in G(F) \). As usual a sentence like “Let \( P = MU \) be a parabolic subgroup of \( G^* \)” means that \( P \) is a parabolic subgroup of \( G \) defined over \( F \), with unipotent radical \( U \) and \( M \) is a Levi component of \( P \). A Levi subgroup of \( G \) means a Levi component of a parabolic subgroup. If \( M \) is a Levi subgroup, we denote by \( \mathcal{P}(M) \) the set of all parabolic subgroups with Levi component \( M \) and we write \( W(G, M) = \text{Norm}_{G(F)}(M)/M(F) \) for its associated Weyl group. If \( P = MU \) is a parabolic subgroup, we denote by \( \delta_P \) the modular character of \( P(F) \) and by \( H_P : P(F) \to \mathcal{A}_M \) the composition of \( H_M \) with the projection from \( P(F) \) to \( M(F) \). If moreover a maximal compact subgroup \( K \) of \( G(F) \), which is special in the \( p \)-adic case, has been fixed (so that \( G(F) = P(F)K \) by the Iwasawa decomposition) we extend \( H_P \) to \( G(F) \) by setting \( H_P(muk) = H_M(m) \) for every \( (m, u, k) \in M(F) \times U(F) \times K \).

When the group \( G \) is understood from the context, we will simply write \( \| . \|_G \) and \( \sigma_G \).

In the Archimedean case, we denote by \( \mathcal{U}(g) \) the enveloping algebra of the complexified Lie algebra of \( G(F) \) and by \( \mathcal{Z}(g) \) its center.

For every integer \( n \geq 1 \), we write \( G_n \) for the algebraic group \( \text{GL}_n \) (say defined over \( \mathbb{Z} \)). We denote by \( Z_n, B_n, A_n \) and \( N_n \) the subgroups of scalar, resp. upper triangular, resp. diagonal, resp. unipotent upper triangular matrices in \( G_n \). For \( g \in G_n \), we write \( g \) for its transpose and \( g_{ij} \), \( 1 \leq i, j \leq n \), for its entries. Also, for \( a \in A_n \) we simply write \( a \) for \( a_{ii} \) (\( 1 \leq i \leq n \)). We denote by \( \delta_n \) and \( \delta_{n,E} \) the modular characters of \( B_n(F) \) and \( B_n(E) \) respectively. We will always consider \( G_n \) as a subgroup of \( G_{n+1} \) through the upper left corner embedding i.e. \( g \in G_n \mapsto \begin{pmatrix} g & \ 1 \\ \ 1 \\ \end{pmatrix} \in G_{n+1} \). We let \( P_n \) be the mirabolic subgroup (that is the set of elements \( g \in G_n \) with last row \((0, \ldots, 0, 1)) \), \( U_n \) be the unipotent radical of \( P_n \) (so that \( P_n = G_{n-1}U_n \)) and \( N_n' \) be the derived subgroup of \( N_n \) (i.e. the subgroup of \( u \in N_n \) such that \( u_{i,i+1} = 0 \) for \( 1 \leq i \leq n-1 \)). The natural extension of \( c \) to \( G_n(E) \) will be written as \( g \mapsto g^c \) and for every \( g \in G_n(E) \) we set \( g^* = (g^{-1})^c \). We also let \( K_n \) and \( K_n,E \) be the standard maximal compact subgroups of \( G_n(F) \) and \( G_n(E) \) i.e. \( K_n = G_n(O_F) \), \( K_{n,E} = G_n(O_E) \) in the \( p \)-adic case whereas \( K_n = \{ g \in G_n(F) \ | \ g^cg = I_n \} \) and \( K_{n,E} = \{ g \in G_n(E) \ | \ g^cg = I_n \} \) in the Archimedean case. We will denote by \( \psi'_n : N_n(F) \to \mathbb{S}^1 \) and \( \psi_n : N_n(E) \to \mathbb{S}^1 \) the characters defined by

\[
\psi'_n(u) = \psi'\left((-1)^n \sum_{i=1}^{n-1} u_{i,i+1}\right) \quad \text{and} \quad \psi_n(u) = \psi\left((-1)^n \sum_{i=1}^{n-1} u_{i,i+1}\right)
\]

Most of the time, we will consider \( G_n(E) \) as the group of \( F \)-points of \( R_{E/F}G_{n,E} \) (so that all constructions involving the \( F \)-points of a reductive group over \( F \) can be applied to \( G_n(E) \)).

We define

\[
G_n(E) := G_n(E)/Z_n(F)
\]

and we will always consider this group as the set of \( F \)-points of \( (R_{E/F}G_{n,E})/Z_{n,F} \).

Recall that we have fixed a character \( \eta' : E^* \to \mathbb{C}^* \) such that \( \eta'|_{F^*} = \eta_{E/F} \). We define for every integer \( k \geq 1 \) characters \( \eta_k \) and \( \eta_k' \) of \( G_n(F) \) and \( G_n(E) \) respectively by \( \eta_k(h) = \)
Hence, if $\eta_{E/F}(h)^{k+1}$ and $\eta_k'(g) = \eta'(\det g)$ if $k$ is even, 1 if $k$ is odd. Notice that the restriction of $\eta_k'$ to $G_n(F)$ is equal to $\eta_k$.

Whenever $\chi$ is a character of $F^\times$, for $h \in G_n(F)$ we will usually write $\chi(h)$ for $\chi(\det h)$. This in particular will be applied to the character $\eta_{E/F}$.

By a hermitian space we will always mean a finite dimensional $E$-vector space $V$ equipped with a non-degenerate hermitian form $h$ (our convention will be to take $h$ to be linear in the first variable). If $V$ is such a hermitian space, we denote by $U(V)$ the corresponding unitary group (thought as an algebraic group defined over $F$). For every $n \geq 1$, we also define $U(n)$ as the unitary group of the standard quasi-split hermitian form on $E^n$ i.e. for every $F$-algebra $R$ we have

$$U(n)(R) = \{ g \in G_n(R \otimes_F E) \mid \forall g w_n g = w_n \} \text{ where } w_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$ 

If $G = U(V)$ is a unitary groups its Levi subgroups all have the form

$$M = R_{E/F}G_{n_1,E} \times \ldots \times R_{E/F}G_{n_k,E} \times U(W)$$

where $n_1, \ldots, n_k$ are positive integers and $W$ is a nondegenerate subspace of $V$ such that $\dim(V) = \dim(W) + n_1 + \ldots + n_k$. For such a Levi subgroup there is a unique identification $A_M \simeq \mathbb{R}^k$ such that $H_M(g_1, \ldots, g_k, \gamma) = (\log(|\det g_1|_E), \ldots, \log(|\det g_k|_E))$ for every $(g_1, \ldots, g_k, \gamma) \in M(F)$. There is also a natural identification of $W(G, M)$ with the subgroup of permutations $w \in \mathfrak{S}_{2k}$ preserving the partition $\{\{i, 2k + 1 - i\} \mid 1 \leq i \leq k\}$ and such that $n_{w(i)} = n_i$ for each $1 \leq i \leq k$ where we have set $n_j = n_{2k+1-j}$ for $k+1 \leq j \leq 2k$. This identification is uniquely characterized by the following: for every $w \in W(G, M)$ there exists a representatives $\tilde{w} \in \text{Norm}_{G(F)}(M)$ such that $\tilde{w}^{-1}(g_1, \ldots, g_k, \gamma)\tilde{w} = (g_{w(1)}, \ldots, g_{w(k)}, \gamma)$ for all $(g_1, \ldots, g_k, \gamma) \in M(F)$ where we have set $g_j = g_{2k+1-j}^*$ for $k+1 \leq j \leq 2k$.

**2.3 LF spaces, completed tensor products**

In this paper, by a topological vector space (TVS) we always mean a Hausdorff locally convex topological vector space over $\mathbb{C}$. We refer the reader to [Beu1, Appendix A] for a Précis on the basic notions of vector-valued integrals as well as smooth and analytic functions in TVS that we shall use extensively in this paper. If $E$ is a TVS, we will denote by $E'$ its continuous dual equipped with the strong topology (i.e. the topology of uniform convergence on bounded subsets).

Recall that a LF space is a topological vector space $V$ that can be written as the direct limit of an at most countable family of Fréchet spaces. All topological vector spaces considered in this paper will be LF spaces. We recall here two basic properties of LF spaces which will be used thoroughly. First, since LF spaces are barreled [Hi, Corollary 3 of Proposition 33.2] they satisfy the uniform boundedness principle (aka Banach-Steinhaus theorem). Hence, if $V$ is an LF space, $W$ any topological vector space and $T_n : V \rightarrow W$ a sequence
of continuous linear maps converging pointwise to \( T : V \to W \), then \( T \) is continuous and \( T_n \) converges to \( T \) uniformly on compact subsets of \( V \). Secondly, LF spaces also satisfy the closed graph theorem [Gr, Theorem 4.B]: a linear map \( T : V \to W \) between LF spaces is continuous if and only if its graph is closed.

The following terminology will be at some points useful. We will call a LF space \( V \) strict if it can be written as the increasing union \( V = \bigcup_{n \geq 1} V_n \) of a sequence of closed Fréchet subspaces. In this case any bounded subset of \( V \) is contained in one of the \( V_n \) (see [Gr, Intro §IV]). Then, we define a tame LF space to be a LF space \( V \) together with a presentation \( V = \lim_{d \to \infty} V_d \) where each \( V_d \) is a strict LF space and the homomorphisms \( V_d \to V_{d+1} \) are continuous injections. We will usually abbreviate this by just writing \( V = \bigcup_{d \geq 1} V_d \) (with the understanding that each \( V_d \) carries its own topology which is not necessarily the one induced from \( V \)). If \( V = \bigcup_{d \geq 1} V_d \) and \( W = \bigcup_{d \geq 1} W_d \) are two tame LF spaces then a continuous linear map \( T : V \to W \) will be said to be tame if there exists integers \( a,b \geq 1 \) such that for every \( d \geq 1 \), \( T \) induces a continuous linear map \( V_d \to W_{ad+b} \).

Let \( V \) and \( W \) be topological vector spaces. We denote by \( V \hat{\otimes} W \) the completed projective tensor product of \( V \) and \( W \) (see [Tr, Chapter 43]). It is a complete topological vector space containing \( V \otimes W \) as a dense subspace and satisfying the following universal property ([Tr, Proposition 43.4]): for every complete topological vector space \( U \), restriction to \( V \otimes W \) induces an (algebraic) isomorphism

\[
\text{Hom}_{\text{cont}}(V \hat{\otimes} W, U) \cong \text{Bil}_{\text{cont}}(V \times W, U)
\]

where the left (resp. right) hand side denotes the space of continuous linear (resp. bilinear) maps from \( V \hat{\otimes} W \) (resp. from \( V \times W \)) to \( U \). In particular, if \( V \) is complete we have a canonical isomorphism \( V \hat{\otimes} \mathbb{C} \cong V \) and for every continuous linear form \( \lambda : W \to \mathbb{C} \), we get a continuous morphism \( \text{Id} \hat{\otimes} \lambda : V \hat{\otimes} W \to V \). The completed projective tensor product is associative (cf. [Gr] §1.1.4) and for any finite family \( V_1, \ldots, V_n \) of topological vector spaces we shall denote by \( V_1 \hat{\otimes} \ldots \hat{\otimes} V_n \) or \( \bigotimes_{1 \leq i \leq n} V_i \) their completed projective tensor product (in any order). The following fact will be quite useful:

(2.3.2) For \( V, W \) Fréchet spaces and \( U \) a complete topological vector space, a sequence of continuous linear maps \( T_n : V \hat{\otimes} W \to U \) that converges point-wise on \( V \otimes W \) actually converges point-wise everywhere to a continuous linear map \( T : V \hat{\otimes} W \to U \).

Indeed, if we let, for all \( n \), \( B_n : V \times W \to U \) be the continuous bilinear map corresponding to \( T_n \) via (2.3.1) then, by the uniform boundedness principle, there exists a separately continuous bilinear map \( B : V \times W \to U \) such that the sequence \( (B_n)_{n \geq 0} \) converges point-wise to \( B \). As \( V \) and \( W \) are Fréchet spaces, \( B \) is continuous and so corresponds to a continuous linear map \( T : V \hat{\otimes} W \to U \) via (2.3.1) again. We have that \( (T_n)_{n \geq 0} \) converges to \( T \) on the dense subspace \( V \otimes W \) of \( V \hat{\otimes} W \) and therefore (by continuity) the convergence holds point-wise everywhere.

By an easy induction, we deduce a property analog to (2.3.2) for the projective completed tensor product of any finite family of Fréchet spaces that we shall also use without further notice.
2.4 Space of functions

For $X$ a smooth variety over $F$, we will denote by $\mathcal{S}(X(F))$ the Schwartz space of $X$. In the $p$-adic case, it is just the space of all locally constant and compactly supported (complex-valued) functions. In the case where $F = \mathbb{R}$, it is the Schwartz space in the sense of Aizenbud and Gourevitch \cite{AG} i.e. the space smooth functions which are “rapidly decreasing together with all their derivatives”. When $X = G$ and $F = \mathbb{R}$, a function $f$ belongs to $\mathcal{S}(G(F))$ if and only if for all $u, v \in \mathcal{U}(g)$ and $N \geq 1$ we have

$$|(R(u)L(v)f)(g)| \ll \|g\|^{-N}, \quad g \in G(F)$$

On the other hand, for $V$ a finite dimensional $F$-vector space $\mathcal{S}(V)$ coincides with the usual Schwartz space. In all cases, $\mathcal{S}(X(F))$ has a natural topology making it a Fréchet space when $F = \mathbb{R}$ and a LF space when $F$ is $p$-adic (in this case we just endow $\mathcal{S}(X(F))$, which is of countable dimension, with its finest locally convex topology). Moreover, if $V$ is a finite dimensional $F$-vector space with a decomposition $V = W \oplus W'$ then there is a natural isomorphism

$$(2.4.1) \quad \mathcal{S}(W) \otimes \mathcal{S}(W') \simeq \mathcal{S}(V)$$

induced from the linear map $\varphi \otimes \psi \mapsto (w + w' \mapsto \varphi(w)\psi(w'))$ (see \cite[Theorem 51.6]{tt}). More generally, for any finite decomposition $V = \bigoplus_{1 \leq i \leq n} W_i$ there is a natural isomorphism

$$(2.4.2) \quad \bigotimes_{1 \leq i \leq n} \mathcal{S}(W_i) \simeq \mathcal{S}(V).$$

Let $\Xi^G$ be the spherical Xi function of Harish-Chandra for $G$ (see \cite[§II.1]{wald2}, \cite{var}). It depends on the choice of a maximal compact subgroup of $G(F)$ but two such choices yield equivalent functions and we will only use $\Xi^G$ for estimates purposes. The space of tempered functions $\mathcal{C}^w(G(F))$ on $G(F)$ is defined as follows. If $F$ is $p$-adic, it is the space of functions $f : G(F) \to \mathbb{C}$ which are biinvariant by a compact-open subgroup and satisfies an inequality

$$(2.4.3) \quad |f(g)| \ll \Xi^G(g)\sigma(g)^d, \quad \forall g \in G(F)$$

for some $d \geq 1$. In the Archimedean case, it is the space of smooth functions $f : G(F) \to \mathbb{C}$ for which there exists $d \geq 1$ such that for all $u, v \in \mathcal{U}(g)$ we have an inequality

$$(2.4.4) \quad |(R(u)L(v)f)(g)| \ll \Xi^G(g)\sigma(g)^d, \quad \forall g \in G(F)$$

For every integer $d \geq 1$ we denote by $\mathcal{C}^w_d(G(F))$ the subspace of functions $f \in \mathcal{C}^w(G(F))$ satisfying \ref{2.4.3} in the $p$-adic case (resp. \ref{2.4.4} for all $u, v \in \mathcal{U}(g)$ in the Archimedean case). Then $\mathcal{C}^w_d(G(F))$ has a natural topology making it a strict LF space (and even a Fréchet space in the Archimedean case). Moreover we have $\mathcal{C}^w(G(F)) = \bigcup_{d \geq 1} \mathcal{C}^w_d(G(F))$ so that $\mathcal{C}^w(G(F))$
is naturally equipped with a structure of tame LF space (in the sense of Section 2.3). Note that \( \mathcal{S}(G(F)) \) is dense in \( C^w(G(F)) \) for this topology.

Assume now that \( G \) is quasi-split, let \( B = TN \) be a Borel subgroup with unipotent radical \( N \) and \( \xi : N(F) \to \mathbb{S}^1 \) a non-degenerate (aka generic) character. We define \( \mathcal{S}(N(F)\backslash G(F), \xi) \) as the space of functions \( f : G(F) \to \mathbb{C} \) satisfying \( f(u g) = \xi(u) f(g) \) for every \( (u, g) \in N(F) \times G(F) \) and which are: locally constant and compactly supported modulo \( N(F) \) in the \( p \)-adic case, smooth and satisfying an inequality

\[
|\langle R(u) f \rangle | \ll \|g\|_{N \backslash G}^{-N} \quad \forall \, g \in N(F) \backslash G(F)
\]

for every \( N \geq 1 \) and \( u \in \mathcal{U}(g) \) in the Archimedean case. Once again, \( \mathcal{S}(N(F)\backslash G(F), \xi) \) has a natural topology making it a Fréchet space when \( F = \mathbb{R} \) and a LF space when \( F \) is \( p \)-adic. Fixing a maximal compact subgroup \( K \) of \( G(F) \) and using the Iwasawa decomposition \( G(F) = N(F)T(F)K \) we define a function \( \Xi^{N,G} \) on \( N(F)\backslash G(F) \) by

\[
\Xi^{N,G}(tk) = \delta_B(t)^{1/2}, \quad t \in T(F), k \in K
\]

Then, we define the space of tempered functions \( C^w(N(F)\backslash G(F), \xi) \) as the space of functions \( f : G(F) \to \mathbb{C} \) such that \( f(u g) = \xi(u) f(g) \) for every \( (u, g) \in N(F) \times G(F) \) which are right invariant by a compact-open subgroup in the \( p \)-adic case, smooth in the Archimedean case and satisfying the following growth condition: if \( F \) is \( p \)-adic, we ask that there exists \( d > 0 \) such that

\[
|\langle R(u) f \rangle | \ll \Xi^{N,G}(g) \sigma_{N \backslash G}(g)^d, \quad \forall \, g \in N(F) \backslash G(F)
\]

if \( F \) is Archimedean, we ask that there exists \( d > 0 \) such that for every \( u \in \mathcal{U}(g) \) we have

\[
|\langle R(u) f \rangle | \ll \Xi^{N,G}(g) \sigma_{N \backslash G}(g)^d, \quad \forall \, g \in N(F) \backslash G(F)
\]

We define as before for every integer \( d \geq 1 \) the subspace \( C^w_d(N(F)\backslash G(F), \xi) \) which has the structure of a strict LF (even Fréchet in the Archimedean case) space and we make \( C^w(N(F)\backslash G(F), \xi) \) into a tame LF space by writing it as

\[
C^w(N(F)\backslash G(F), \xi) = \bigcup_{d \geq 1} C^w_d(N(F)\backslash G(F), \xi).
\]

Consider two quasi-split groups \( G_1, G_2 \) with maximal unipotent subgroups \( N_1, N_2 \) and generic characters on their \( F \)-points \( \xi_1, \xi_2 \) respectively. Then, we have the following easy lemma which will be used without further notice in this paper.

**Lemma-Definition 2.4.1** Let \( \lambda \) be a continuous linear form on \( C^w(N_1(F)\backslash G_1(F), \xi_1) \). Then, the linear map

\[
C^w(N_1(F)\backslash G_1(F) \times N_2(F)\backslash G_2(F), \xi_1 \boxtimes \xi_2) \to C^w(N_2(F)\backslash G_2(F), \xi_2)
\]

\[
W \mapsto (g \mapsto \lambda(W(\cdot, g)))
\]

is well-defined, continuous (even tame) and will be denoted by \( \hat{\lambda} \boxtimes \text{Id} \). For \( \lambda_1 \) and \( \lambda_2 \) two continuous linear forms on \( C^w(N_1(F)\backslash G_1(F), \xi_1) \) and \( C^w(N_2(F)\backslash G_2(F), \xi_2) \) respectively we have \( \lambda_2 \circ (\lambda_1 \boxtimes \text{Id}) = \lambda_1 \circ (\text{Id} \boxtimes \lambda_2) \) and the resulting linear form will be denoted by \( \lambda_1 \boxtimes \lambda_2 \).
Proof: We explain the Archimedean case since the $p$-adic case is similar and easier. For the first part, it suffices to notice that for every $d > 0$ and $W \in C^u(N_1(F)\backslash G_1(F) \times N_2(F)\backslash G_2(F), \xi_1 \boxtimes \xi_2)$ the function

$$F : g \in G_2(F) \mapsto W(., g) \in C^u_d(N_1(F)\backslash G_1(F), \xi_1)$$

has the following properties:

- it is smooth;
- $F(ug) = \xi_2(u)F(g)$ for every $g \in G_2(F)$ and $u \in N_2(F)$;
- for every $u \in \mathcal{U}(g)$ the set

$$\{\Xi^{N_2\backslash G_2}(g)^{-1}\sigma_{N\backslash G}(g)^{-d}(R(u)F)(g) \mid g \in G(F)\}$$

is bounded in $C^w_d(N_1(F)\backslash G_1(F), \xi_1)$.

Therefore the same holds for the composition of $F$ with $\lambda$ showing that $\lambda \boxtimes \text{Id}$ is well-defined. The continuity follows from the closed graph theorem. Finally $\lambda_2 \circ (\lambda_1 \boxtimes \text{Id}) = \lambda_1 \circ (\text{Id} \boxtimes \lambda_2)$ since these two continuous linear forms agree on the dense subspace $C^w_d(N_1(F)\backslash G_1(F), \xi_1) \otimes C^w_d(N_2(F)\backslash G_2(F), \xi_2)$. ■

**Remark 2.4.2** The notation introduced in the previous lemma might seem to enter in conflict with the one introduced in Section 2.3. This can be justified a posteriori as follows: let $C^w_d(N_1(F)\backslash G_1(F), \xi_1) \otimes_C C^w_d(N_2(F)\backslash G_2(F), \xi_2)$ be the completed $\epsilon$-tensor product ([Beu3, Definition 4.3.5]). There are natural continuous linear maps with dense images

$$(2.4.5) \quad C^w_d(N_1(F)\backslash G_1(F), \xi_1) \otimes_C C^w_d(N_2(F)\backslash G_2(F), \xi_2) \to C^w_d(N_1(F)\backslash G_1(F), \xi_1) \otimes_C C^w_d(N_2(F)\backslash G_2(F), \xi_2)$$

$$(2.4.6) \quad C^w(N_1(F)\backslash G_1(F) \times N_2(F)\backslash G_2(F), \xi_1 \boxtimes \xi_2) \to C^w_d(N_1(F)\backslash G_1(F), \xi_1) \otimes C^w_d(N_2(F)\backslash G_2(F), \xi_2)$$

and the linear map $\lambda \boxtimes \text{Id} : C^w_d(N_1(F)\backslash G_1(F), \xi_1) \otimes C^w_d(N_2(F)\backslash G_2(F), \xi_2) \to C^w_d(N_2(F)\backslash G_2(F), \xi_2)$ defined in Section 2.3 factorizes through 2.4.5. The linear map defined by the above lemma is then the composition of this factorized map with 2.4.6. Finally, we remark that the spaces $C^w(N_k(F)\backslash G_k(F), \xi_k)$, $k = 1, 2$, are probably complete and nuclear (but the author didn’t attempt to prove it) in which case both 2.4.3 and 2.4.6 would be topological isomorphisms.

The next lemma is probably standard and can be proved in much the same way as [Beu3, Lemma 2.4.3].

**Lemma 2.4.3** Let $W \in C^w(N_n(E)\backslash G_n(E), \psi_n)$. Then, for all $N \geq 1$ we have

$$|W(ak)| \ll \prod_{i=1}^{n-1} (1 + \left| \frac{a_i}{a_{i+1}} \right|_E)^{-N} \delta_{n,E}(a)^{1/2} \sigma(a)^d, \quad a \in A_n(E), k \in K_{n,E}.$$
This will be many times combined with the following basic convergence result for estimates purpose and so we record it here.

**Lemma 2.4.4** There exists \( N \geq 1 \) such that for all \( s \in \mathcal{H} \) and \( d > 0 \) the integral

\[
\int_{A_n(F)} \prod_{i=1}^{n-1} \left( 1 + \left| \frac{a_i}{a_{i+1}} \right| \right)^{-N} (1 + |a_n|)^{-N} \sigma(a)^d |\det a|^s \, da
\]

converges absolutely.

### 2.5 Measures

We equip \( F \) with the Haar measure \( dx = d\psi x \) which is autodual with respect to \( \psi' \). For any \( n \geq 1 \), \( F^n \) will be equipped with the \( n \)-fold product of this measure. We endow \( F^x \) with the Haar measure \( d^x x = \frac{dx}{|x|_F} \). Similarly, we equip \( E \) with the Haar measure \( dz = d\psi_E z \) autodual with respect to \( \psi'_E = \psi' \circ \text{Tr}_{E/F} \) and \( E^x \) with the measure \( d^x z = \frac{dz}{|z|_E} \).

Let \( X \) be a smooth variety over \( F \). Then to any volume form \( \omega \) on \( X \) (i.e., a differential form of maximal degree) we can associate a measure \( |\omega|_{\psi'} \) on \( V(F) \) as follows. Let \( x \in V(F) \) and choose local coordinates \( x_1, \ldots, x_n \) around \( x \). Then, there exists a function \( f \) on some open neighborhood of \( x \) such that on that neighborhood \( \omega = f(x_1, \ldots, x_n) dx_1 \wedge \ldots \wedge dx_n \). Then, we define \( |\omega|_{\psi'} \) on that neighborhood to be \( |f(x_1, \ldots, x_n)|^{1/[K:F]} d\psi' x_1 \ldots d\psi' x_n \). These locally defined measures can be glued together to give a global measure \( |\omega|_{\psi'} \). This standard construction actually extends to any volume form \( \omega \) on \( X_{\mathfrak{F}} \). Indeed, choose a finite extension \( K \) of \( F \) such that \( \omega \) is defined over \( X_K \). Then, in local coordinates \( \omega \) can be written as \( \omega = f(x_1, \ldots, x_n) dx_1 \wedge \ldots \wedge dx_n \) where this time \( f \) takes values in \( K \) and we define \( |\omega|_{\psi'} \) locally as \( |f(x_1, \ldots, x_n)|^{1/[K:F]} d\psi' x_1 \ldots d\psi' x_n \). Once again, we can glue to get a global measure.

Let \( G \) be a connected reductive group over \( F \). We equip \( G(F) \) with a canonical measure \( dg = d\psi g \) as follows. Let \( G_{\mathbb{Z}} \) be the split reductive group over \( \mathbb{Z} \) such that \( G_{\mathbb{Z},\mathfrak{F}} = G_{\mathbb{Z}} \times_{\mathbb{Z}} \mathfrak{F} \approx G_{\mathfrak{F}} \). We fix such an isomorphism \( \alpha \). Let \( \omega_{G_{\mathbb{Z}}} \) be a generator of the (free of rank 1) \( \mathbb{Z} \)-module of \( G_{\mathbb{Z}} \)-invariant volume form on \( G_{\mathbb{Z}} \) and \( \omega_{G_{\mathbb{Z}},\mathfrak{F}} \) be its base change to \( G_{\mathbb{Z},\mathfrak{F}} \). We set \( \omega_{G_{\mathfrak{F}}} := \alpha_*(\omega_{G_{\mathbb{Z}},\mathfrak{F}}) \). Then, \( \omega_{G_{\mathfrak{F}}} \) is a \( G_{\mathfrak{F}} \)-invariant volume form on \( G_{\mathfrak{F}} \) which depends on the various choices only up to a root of unity so that the associated Haar measure \( dg = |\omega_{G_{\mathfrak{F}}}|_{\psi} \) is independent of all choices (except of course \( \psi' \)). We also equip \( g(F) \) with the Haar measure \( dX = d\psi X \) defined by \( dX := |\omega_{G_{\mathfrak{F}},1}|_{\psi} \) where \( \omega_{G_{\mathfrak{F}},1} \) denotes the value of \( \omega_{G_{\mathfrak{F}}} \) at 1 (a differential form of maximal degree on \( g \)). Notice that in the case where \( G = \mathbb{G}_m \), the measure so defined coincides with the one we have already fixed on \( F^x \). More generally, we obtain on \( G_n(F) \) the measure \( dg = |\det g|^{-n} \prod_{i,j} dg_{i,j} \) (where \( dg_{i,j} \) corresponds to the \( \psi' \)-autodual Haar measure on \( F \)) and on \( G_n(E) \) the measure \( dg = |\det g|_E^{-n} \prod_{i,j} dg_{i,j} \) (where this time \( dg_{i,j} \) corresponds to the \( \psi'_E \)-autodual Haar measure on \( E \)).

The above construction can actually be applied to any algebraic linear group \( G \) over \( F \) once we chose a model over \( \mathbb{Z} \) for \( G_{\mathfrak{F}} \) (if it exists). For example, we will apply this to \( G = N_n, U_n \) and \( N'_n \) (see Section 222) with their obvious models over \( \mathbb{Z} \) thus equipping the
unipotent groups $N_n(F)$, $U_n(F)$, $N'_n(F)$ as well as their Lie algebras with Haar measures. These Haar measures are of course very easy to describe. For example on $N_n(F)$ we get the measure $du = \prod_{1 \leq i < j \leq n} du_{i,j}$. We will also apply this construction to $N_n(E)$, $U_n(E)$, $N'_n(E)$ either by considering these groups as the $F$-points of $R_{E/F}N_{n,E}$, $R_{E/F}U_{n,E}$ and $R_{E/F}N'_{n,E}$ and using the natural isomorphisms $(R_{E/F}H_E)_F \simeq H_F \times H_F$ ($H = N_n, U_n, N'_n$) or by doing the same as before with $F$ replaced by $E$ and $\psi'$ replaced by $\psi'_E$. Finally, the same can be applied to endow the Borel subgroups $B_n(F)$, $B_n(E)$ and the mirabolic subgroups $P_n(F)$, $P_n(E)$ with right Haar measures (to be denoted simply by $db$ and $dp$ respectively) using the natural model of $B_n$ and $P_n$ over $\mathbb{Z}$. Notice that the Haar measure so obtained on $P_n(F)$ is, through the decomposition $P_n(F) = U_n(F) \rtimes G_{n-1}(F)$, the product of the Haar measures on $U_n(F)$ and $G_{n-1}(F)$. We also equip $P_n(F) \setminus G_n(F)$ with the “twisted” measure obtained as the quotient of the Haar measure $dg$ with the (left) Haar measure $|\det p|^{-1}dp$ on $P_n(F)$: then the integral $\int_{P_n(F) \setminus G_n(F)} dg$ is a linear form on the space $C_c(P_n(F) \setminus G_n(F), |\det|)$ of functions $f : G_n(F) \to \mathbb{C}$ satisfying $f(pg) = |\det p|f(g)$ for every $(p, g) \in P_n(F) \times G_n(F)$ and which are of compact support modulo $P_n(F)$. We have the usual integration formula

$$\int_{G_n(F)} f(g)dg = \int_{P_n(F) \setminus G_n(F)} \int_{P_n(F)} f(pg)|\det p|^{-1}dpdg, \quad f \in \mathcal{S}(G_n(F)).$$

Moreover the measure $|\det g|dg$ on $P_n(F) \setminus G_n(F)$ (which is now a true measure, although not invariant) can be identified with the Haar measure $dx_1 \ldots dx_n$ on $F^n$ through the isomorphism $P_n(F) \setminus G_n(F) \simeq F^n \setminus \{0\}$, $g \mapsto e_n g$. Using this, we readily check the following Fourier inversion formula

$$\int_{P_{n-1}(F) \setminus G_{n-1}(F)} \int_{U_n(E)} \varphi(v)\psi_n(hv_h^{-1})^{-1}dv|\det h|dh = |\tau_{E/F}^{(n-1)/2}\int_{U_n(F)} \varphi(v)dv, \quad \varphi \in \mathcal{S}(U_n(E))$$

(of course the left-hand side is only convergent as an iterated integral). Here we recall that $\tau \in E^\times$ is of trace zero and $\psi$ is given by $\psi(z) = \psi'(\text{Tr}_{E/F}(\tau z))$.

If $G = A$ is a split torus, then we endow $iA^* := X^*(A) \otimes i\mathbb{R}$ with the unique Haar measure giving the quotient $iA^*_F := iA^*/(2\pi i/\log(q_F))X^*(A)$ volume 1 (recall that by convention $q_F = e^{1/2}$ in the Archimedean case). Let $\hat{A}(F)$ be the unitary dual of $A(F)$ and $d\psi'\chi$ be the Haar measure on $\hat{A}(F)$ dual to the Haar measure we have just fixed on $A(F)$. Set

$$d\chi := \gamma^*(0, 1_F, \psi')^{-\dim(A)}d\psi'\chi$$

where $\gamma^*(0, 1_F, \psi')$ is the “regularized” value at 0 of $\gamma(s, 1_F, \psi')$ as defined in Section 2.12 below. Then, $d\chi$ is a measure on $\hat{A}(F)$ which is independent of the choice of $\psi'$ and moreover for this measure the local isomorphism

$$iA^* \to \hat{A}(F)$$

$$\chi \otimes \lambda \mapsto (a \mapsto |\chi(a)|^{\lambda_F})$$

is locally measure preserving.
2.6 Representations

In this paper, all representations will be tacitly assumed to be on complex vector spaces. By a representation of $G(F)$ we will always mean a smooth representation of finite length. Here smooth has the usual meaning in the $p$-adic case (i.e. every vector has an open stabilizer) whereas in the Archimedean case it means a smooth admissible Fréchet representation of moderate growth in the sense of Casselman-Wallach [Cas, Wall2 Sect. 11] or, which is the same, an admissible SF representation in the sense of [BK]. We shall always abuse notation and denote by the same letter a representation and the space on which it acts. In the Archimedean case this space is always coming with a topology (it is a Fréchet space) whereas in the $p$-adic case it will sometimes be convenient, in order to make uniform statements, to equip this space with its finest locally convex topology (it then becomes an LF space).

Let $\pi$ be a representation of $G(F)$. We denote by $\pi^\vee$ the contragredient representation (aka smooth dual) and by $\langle \cdot, \cdot \rangle$ the natural pairing between $\pi$ and $\pi^\vee$. In the Archimedean case, $\pi^\vee$ can be described as the space of linear forms on $\pi$ which are continuous with respect to any $G(F)$-continuous norm on $\pi$ together with the natural $G(F)$-action on it (a norm on $\pi$ is said to be $G(F)$-continuous if the action of $G(F)$ on $\pi$ is continuous for this norm). If $G = R_{E/F}G_{n,E}$ so that $G(F) = G_n(E)$ then we write $\pi^c$ for the composition of $\pi$ with the automorphism of $G_n(E)$ induced by $c$ and we set

$$\pi^* = (\pi^\vee)^c$$

For $\chi$ a continuous character of $G(F)$ we write $\pi \otimes \chi$ for the twist of $\pi$ by $\chi$ and for $\lambda \in \mathcal{A}_{G,\mathbb{C}}^*$ by $\pi_{\lambda}$ for the twist of $\pi$ by the character $g \mapsto e^{\langle \lambda, H_{G}(g) \rangle}$. When $G = G_n$, we will also write $\pi_x = \pi \otimes |\det|^x$ for all $x \in \mathbb{C}$.

If $G_1, G_2$ are two reductive groups over $F$ and $\pi_1, \pi_2$ are representations of $G_1(F), G_2(F)$ respectively, we denote by $\pi_1 \boxtimes \pi_2$ the tensor product representation of $G_1(F) \times G_2(F)$ where in the Archimedean case this representation is realized on the completed projective tensor product of $\pi_1$ and $\pi_2$.

For $\pi$ a representation of $G(F)$, we let $\text{End}_{\mathcal{X}}(\pi)$ be the space of endomorphisms $T : \pi \to \pi$ which are biinvariant by a compact-open subgroup in the $p$-adic case and which are continuous with respect to any $G(F)$-continuous norm in the Archimedean case. Then the natural map $\pi^\vee \otimes \pi \to \text{End}_{\mathcal{X}}(\pi)$ extends to an isomorphism of $G(F) \times G(F)$-representations $\pi^\vee \boxtimes \pi \simeq \text{End}_{\mathcal{X}}(\pi)$. The canonical pairing $\pi^\vee \otimes \pi \to \mathbb{C}$ extends continuously to $\text{End}_{\mathcal{X}}(\pi)$ and we shall denote this extension by $T \mapsto \text{Trace}(T)$. Moreover, for every function $f \in \mathcal{S}(G(F))$ the expression

$$\pi(f)v = \int_{G(F)} f(g)\pi(g)v\,dg$$

converges absolutely in (the space of) $\pi$ for all $v \in \pi$ and defines an operator $\pi(f) \in \text{End}_{\mathcal{X}}(\pi)$.

For $P = MU$ a parabolic subgroup of $G$ and $\sigma$ a representation of $M(F)$ we write $i_P^G(\sigma)$ for the corresponding normalized parabolically induced representation: it is the right regular representation of $G(F)$ on the space of smooth functions $\epsilon : G(F) \to \sigma$ satisfying $\epsilon(mug) = \delta_P(m)^{1/2} \sigma(m)\epsilon(g)$ for all $(m, u, g) \in M(F) \times U(F) \times G(F)$. When $G = G_n$ and
$M$ is of the form $M = G_{n_1} \times \ldots \times G_{n_k}$ we write
\[ \tau_1 \times \ldots \times \tau_k \]
for $i_P^\sigma(\tau_1 \boxtimes \ldots \boxtimes \tau_k)$ where $\tau_i$ is a representation of $G_{n_i}(F)$ for every $1 \leq i \leq k$. Similarly when $G = U(V)$ for $V$ a hermitian space and $M$ is of the form $M = R_{E/F}G_{n_1,E} \times \ldots \times R_{E/F}G_{n_k,E} \times U(W)$ for $W$ a non-degenerate subspace of $V$, we write
\[ \tau_1 \times \ldots \times \tau_k \times \sigma_0 \]
for $i_P^\sigma(\tau_1 \boxtimes \ldots \boxtimes \tau_k \boxtimes \sigma_0)$ where $\tau_i$ is a representation of $G_{n_i}(E)$ for every $1 \leq i \leq k$ and $\sigma_0$ a representation of $U(W)(F)$.

We denote by $\text{Irr}(G)$ the set of all isomorphism classes of irreducible representations of $G(F)$. Of course in the Archimedean case irreducible should be understood as topologically irreducible. For $\pi \in \text{Irr}(G)$, we denote by $\omega_\pi : Z_G(F) \to \mathbb{C}^\times$ its central character and in the Archimedean case by $\chi_\pi : \mathcal{Z}(g) \to \mathbb{C}$ its infinitesimal character.

Let $\text{Temp}(G) \subset \text{Irr}(G)$ be the subset of tempered irreducible representations and $\Pi_2(G) \subset \text{Temp}(G)$ the further subset of square-integrable representations. For every $\pi \in \Pi_2(G)$ we define its formal degree $d(\pi)$ by the relation
\[
\int_{G(F)/A_G(F)} \langle \pi(g)v_1, v_1' \rangle \langle v_2, \pi^\vee(g)v_2' \rangle dg = \frac{\langle v_1, v_2' \rangle \langle v_2, v_1' \rangle}{d(\pi)}, \quad v_1, v_2 \in \pi, v_1', v_2' \in \pi^\vee
\]
Let $\text{Temp}_{\text{ind}}(G)$ be the set of isomorphism classes of representations of the form $i_P^\sigma(\sigma)$ where $P = MU$ is a parabolic subgroup of $G$ and $\sigma \in \Pi_2(M)$. The isomorphism class of $i_P^\sigma(\sigma)$ is independent of $P \in \mathcal{P}(M)$ and we shall thus write it as $i_M^\sigma(\sigma)$. These representations are always semi-simple and in fact even unitarizable. According to Harish-Chandra, for $M, M'$ two Levi subgroups of $G$ and $\sigma \in \Pi_2(M), \sigma' \in \Pi_2(M')$ the two representations $i_M^\sigma(\sigma)$ and $i_{M'}^\sigma(\sigma')$ have a constituent in common if and only if they are isomorphic and this happens precisely when there exists $g \in G(F)$ such that $gMg^{-1} = M', g\sigma g^{-1} \simeq \sigma'$. Moreover, every representation of $\text{Temp}(G)$ embeds in a unique representation of $\text{Temp}_{\text{ind}}(G)$ thus yielding a map
\[ \text{Temp}(G) \to \text{Temp}_{\text{ind}}(G). \]
When $G = G_n$, the representations $i_M^\sigma(\sigma)$ ($M$ a Levi subgroup and $\sigma \in \Pi_2(M)$) are all irreducible so that $\text{Temp}(G) = \text{Temp}_{\text{ind}}(G)$.

For $\pi \in \text{Temp}(G)$, we have the following (see [Beulla (2.2.5)]):

(2.6.1) The assignment $T \mapsto (g \in G(F) \to \text{Trace}(\pi(g)T))$ defines a continuous linear map $\text{End}_\mathbb{C}(\pi) \to C_0^w(G(F))$. In particular for every $(v, v^\vee) \in \pi \times \pi^\vee$, the function $g \mapsto \langle \pi(g)v, v^\vee \rangle$ belongs to $C_0^w(G(F))$ and the bilinear map $\pi \times \pi^\vee \to C_0^w(G(F)), (v, v^\vee) \mapsto (g \mapsto \langle \pi(g)v, v^\vee \rangle)$, is continuous.

For $M$ a Levi subgroup of $G$ and $\sigma \in \Pi_2(M)$ we set
\[ W(G, \sigma) = \{ w \in W(G, M) \mid w\sigma \simeq \sigma \} \]
23
Then the map \( \lambda \in iA^*_M \mapsto i^G_M(\sigma_\lambda) \in \text{Temp}_{\text{ind}}(G) \) is \( W(G, \sigma) \)-invariant and moreover there exists a unique topology on \( \text{Temp}_{\text{ind}}(G) \) such that the induced maps \( iA^*_M/W(G, \sigma) \to \text{Temp}_{\text{ind}}(G) \) (for all \( M \) and \( \sigma \)) are local isomorphisms near 0. When \( F = \mathbb{R} \) and the central character of \( \sigma \) is trivial on the connected component of \( A_M(F) \) the map \( iA^*_M/W(G, \sigma) \to \text{Temp}_{\text{ind}}(G) \) actually induces an isomorphism between \( iA^*_M/W(G, \sigma) \) and a connected component of \( \text{Temp}_{\text{ind}}(G) \). In general, connected components of \( \text{Temp}_{\text{ind}}(G) \) are always of the form

\[
\mathcal{O} = \{ i^G_M(\sigma_\lambda) \mid \lambda \in iA^*_M \}
\]

for some Levi \( M \) and \( \sigma \in \Pi_2(M) \) and in the \( p \)-adic case these components are all compacts. For \( V \) a topological vector space, we will say that a function \( f: \text{Temp}_{\text{ind}}(G) \to V \) is smooth if for every Levi subgroup \( M \) of \( G \) and every \( \sigma \in \Pi_2(M) \) the function \( \lambda \in iA^*_M \mapsto f(i^G_M(\sigma_\lambda)) \in V \) is smooth.

Assume one moment that \( G = U(V) \) for some hermitian space \( V \) over \( E \). Let \( M \) be a Levi subgroup of \( G \) of the form

\[
M = R_{E/F}G_{n_1,E} \times \ldots \times R_{E/F}G_{n_k,E} \times U(W)
\]

where \( W \subset V \) is a nondegenerate subspace and let

\[
\sigma = \tau_1 \boxtimes \ldots \boxtimes \tau_k \boxtimes \sigma_0
\]

be a square-integrable representation of \( M(F) \) where \( \tau_i \in \Pi_2(G_{n_i}(E)) \) for all \( 1 \leq i \leq k \) and \( \sigma_0 \in \Pi_2(U(W)) \). Recall that the group \( W(G, M) \) can be identified with the subgroup of permutations \( w \in \mathfrak{S}_{2k} \) preserving the partition \( \{ \{i, 2k + 1 - i\} \mid 1 \leq i \leq k \} \) and such that \( n_{w(i)} = n_i \) for all \( 1 \leq i \leq 2k \) where we have set \( n_i = n_{2k+1-i} \) for every \( k + 1 \leq i \leq 2k \). Then, using this identification \( W(G, \sigma) \) is the subgroup of element \( w \in W(G, M) \) such that \( \tau_{w(i)} = \tau_i \) for all \( 1 \leq i \leq 2k \) where we have set \( \tau_i = \tau_{2k+1-i} \) for every \( k + 1 \leq i \leq 2k \).

Assume that \( F = \mathbb{R} \). We define a norm \( \pi \mapsto N(\pi) \) on \( \text{Temp}(G) \) as in [Beull §2.2] that is: fix a maximal torus \( T \subset G \) and fix a \( W(G_C, T_C) \)-invariant norm on \( \mathfrak{t}(\mathbb{C})^* \), then identifying the infinitesimal character \( \chi_\pi \) with an element of \( \mathfrak{t}(\mathbb{C})^*/W(G_C, T_C) \) by the Harish-Chandra isomorphism we set

\[
N(\pi) = 1 + \| \chi_\pi \|.
\]

Since \( N(\pi) \) only depends on the infinitesimal character \( \chi_\pi \) it also makes sense for \( \pi \in \text{Temp}_{\text{ind}}(G) \). The same definition also makes sense for disconnected groups and will in particular be applied to maximal compact subgroups of \( G(\mathbb{R}) \). Note that there exists \( z \in \mathcal{Z}(\mathfrak{g}) \) such that

\[
(2.6.2) \quad N(\pi) \ll \chi_\pi(z), \quad \pi \in \text{Temp}_{\text{ind}}(G)
\]

Indeed if \( z_1, \ldots, z_n \) is a generating family of homogeneous elements of \( \mathcal{Z}(\mathfrak{g}) \) then we can take \( z = 1 + \sum_{i=1}^n z_i z_i^* \) where \( u \mapsto u^* \) denotes the conjugate-linear antiautomorphism of \( \mathcal{U}(\mathfrak{g}) \) sending every \( X \in \mathfrak{g}(F) \) to \(-X\).
2.7 Spectral measures

We define two measures $d_{\psi}\pi$ and $d\pi$ on $\text{Temp}_{\text{ind}}(G)$ as follows. Let $M$ be a Levi subgroup of $G$. Then, we define $d_{\psi}\sigma$ (resp. $d\sigma$) to be the unique Borel measure on $\Pi_2(M)$ for which the local isomorphism

\begin{equation}
\Pi_2(M) \rightarrow A_M(F)
\end{equation}

\[ \sigma \mapsto \omega_\sigma |_{A_M(F)} \]

is locally measure preserving when $A_M(F)$ is equipped with the Haar measure $d\psi\chi$ (resp. $d\chi$). The induction map

\begin{equation}
i^G_M : \Pi_2(M) \rightarrow \text{Temp}_{\text{ind}}(G)
\end{equation}

\[ \sigma \mapsto i^G_M(\sigma) \]

is quasi-finite and proper with image the union of certain connected components of $\text{Temp}_{\text{ind}}(G)$. The restrictions of $d_{\psi}\pi$ and $d\pi$ to this image are then defined to be

\[ |W_{p,G,M,q}|^{-1} i^G_M d_{\psi}\sigma \text{ and } |W_{p,G,M,q}|^{-1} i^G_M d\sigma \]

respectively where $i^G_M d_{\psi}\sigma$ and $i^G_M d\sigma$ stands for the push-forward of the measures $d_{\psi}\sigma$ and $d\sigma$ respectively.

Near a representation $\pi_0 \in \text{Temp}_{\text{ind}}(G)$ the measure $d\pi$ can be described more explicitly as follows. Let $M$ be a Levi subgroup of $G$ and $\sigma \in \Pi_2(M)$ such that $\pi_0 \simeq i^G_M(\sigma)$. Let $\mathcal{V}$ be a sufficiently small $W(G,\sigma)$-invariant open neighborhood of 0 in $iA^*_M$ such that the map $\lambda \in iA^*_M \rightarrow \pi_\lambda := i^G_M(\sigma_\lambda)$ induces a topological isomorphism between $\mathcal{V}/W(G,\sigma)$ and an open neighborhood $\mathcal{U}$ of $\pi_0$ in $\text{Temp}_{\text{ind}}(G)$. Then, we have the integration formula

\begin{equation}
\int_{\mathcal{U}} \varphi(\pi)d\pi = \frac{1}{|W(G,\sigma)|} \int_{\mathcal{V}} \varphi(\pi_\lambda)d\lambda, \quad \varphi \in C_c(\mathcal{U})
\end{equation}

As said before, when $F = \mathbb{R}$ and the central character of $\sigma$ is trivial on the connected component of $A_M(F)$ (which we can always be arranged up to twisting $\sigma$ by an unramified character) we can take $\mathcal{V} = iA^*_M$ and $\mathcal{U} = O$ the connected component of $\pi_0$ in $\text{Temp}_{\text{ind}}(G)$.

Finally, we have the following basic estimates:

\begin{equation}
\text{(2.7.4) There exists } k \geq 1 \text{ such that the integral}
\end{equation}

\[ \int_{\text{Temp}_{\text{ind}}(G)} N(\pi)^{-k}d\pi \]

converges.

2.8 Whittaker models

Assume that $G$ is quasi-split, let $B = TN$ be a Borel subgroup of $G$ and $\xi : N(F) \rightarrow S^1$ be a generic character. Recall that $\pi \in \text{Irr}(G)$ is said to be $(N,\xi)$-generic if there exists a nonzero continuous linear form $\lambda : \pi \rightarrow \mathbb{C}$ satisfying $\lambda \circ \pi(u) = \xi(u)\lambda$ for all $u \in N(F)$. Such
a linear form, which we call a \((N, \xi)\)-Whittaker functional, is always unique up to a scalar ([CHM, Theorem 9.2], [Rod, Théorème 2]). If \(\pi\) is \((N, \xi)\)-generic, we denote by \(\mathcal{W}(\pi, \xi)\) its Whittaker model i.e. the space of all functions of the form \(g \in G(F) \mapsto \lambda(\pi(g)v)\) where \(\lambda\) is a nonzero \((N, \xi)\)-Whittaker functional on \(\pi\). Since the uniqueness of Whittaker functionals still holds for representations parabolically induced from irreducible ones ([CHM, Theorem 9.1], [Rod, Théorème 4]), all of these definitions trivially extend to any \(\pi \in \text{Temp}_{\text{ind}}(G)\). The following lemma is [Wall2, Lemma 15.7.3] in the Archimedean case and follows from [LM, Theorem 3.1] in the \(p\)-adic case (the second part of the lemma is automatic by the closed graph theorem).

**Lemma 2.8.1** Let \(\pi \in \text{Temp}_{\text{ind}}(G)\) which is \((N, \xi)\)-generic. Then, we have \(\mathcal{W}(\pi, \xi) \subset \mathcal{C}^w(N(F)\backslash G(F), \xi)\). Moreover if \(v \mapsto W_v\) is an isomorphism between \(\pi\) and its Whittaker model, the linear map \(v \mapsto W_v \in \mathcal{C}^w(N(F)\backslash G(F), \xi)\) is continuous.

Assume now that \(G = R_{E/F}G_n\). We define a scalar product \((.,.)^{\text{Whitt}}\) on \(\mathcal{C}^w(N_n(E)\backslash G_n(E), \psi_n)\) by

\[
(W, W')^{\text{Whitt}} = \int_{N_n(E)\backslash P_n(E)} W(p)\overline{W'(p)} dp, \quad W, W' \in \mathcal{C}^w(N_n(E)\backslash G_n(E), \psi_n)
\]

This integral is easily seen to be absolutely convergent by Lemma 2.4.3 and Lemma 2.4.4 and we have the following result which is due to Bernstein [Ber2] in the \(p\)-adic case and independently Baruch [Bar] and Jacquet [Jac, Proposition 5] in the Archimedean case.

**Theorem 2.8.2** For \(\pi \in \text{Temp}(G_n(E))\), the restriction of \((.,.)^{\text{Whitt}}\) to \(\mathcal{W}(\pi, \psi_n)\) is \(G_n(E)\)-invariant.

### 2.9 Space of Schwartz functions on \(\text{Temp}_{\text{ind}}(G)\)

Let \(V\) be a Fréchet space or a strict LF space then we define \(\mathcal{S}(\text{Temp}_{\text{ind}}(G), V)\) as the space of functions \(f: \text{Temp}_{\text{ind}}(G) \to V\) such that

- \(f\) is smooth (in the sense of Section 2.6);
- In the Archimedean case, \(f\) and all its derivatives are of rapid decay: for every Levi subgroup \(M\) of \(G\), every \(k \geq 1\), every continuous semi-norm \(\nu\) on \(V\) and every \(D \in \text{Sym}^\ast(A_M^*, \mathcal{C})\) that we see as a differential operator with constant coefficients on \(iA_M^*\) we have
  \[
  \nu \left( D(\lambda \in iA_M^* \mapsto f(i_M^G(\sigma\lambda)))_{\lambda=0} \right) \ll N(i_M^G(\sigma))^{-k}
  \]
  for all \(\sigma \in \Pi_2(M)\).
- In the \(p\)-adic case: \(f\) has compact support.
When $V$ is Fréchet, $\mathcal{S}(\text{Temp}^{\text{ind}}(G), V)$ has a natural structure of Fréchet space whereas if $V = \bigcup_{n \geq 1} V_n$ with $(V_n)_{n \geq 1}$ an increasing sequence of closed Fréchet subspaces then we have (as a bounded subset of $V$ is included in one of the $V_n$ and Fréchet spaces are complete)
\[
\mathcal{S}(\text{Temp}^{\text{ind}}(G), V) = \lim_n \mathcal{S}(\text{Temp}^{\text{ind}}(G), V_n)
\]
thus making $\mathcal{S}(\text{Temp}^{\text{ind}}(G), V)$ into a LF space. For any integer $k \geq 1$, we define similarly $S^{(k)}(\text{Temp}^{\text{ind}}(G), V)$ by only asking that for all Levi subgroup $M$ of $G$ and $\sigma \in \Pi_2(M)$ the map $\lambda \in \mathfrak{a}_M \mapsto f(i_M^G(\sigma_\lambda)) \in V$ to be $C^k$ plus the same condition of compact support (in the $p$-adic case) or of rapid decay for derivatives up to order $k$ (in the Archimedean case). Once again, $S^{(k)}(\text{Temp}^{\text{ind}}(G), V)$ is a naturally a LF (resp. Fréchet) space when $V$ is a strict LF (resp. Fréchet) space.

More generally, let $V = \bigcup_{d \geq 1} V_d$ be a tame LF space. Then, we define $\mathcal{S}(\text{Temp}^{\text{ind}}(G), V)$ as the space of functions $f : \text{Temp}^{\text{ind}}(G) \to V$ such that there exist $a, b \geq 1$ with $f \in S^{(k)}(\text{Temp}^{\text{ind}}(G), V_{ad})$ for all $k \geq 1$. With this definition $\mathcal{S}(\text{Temp}^{\text{ind}}(G), V)$ has a natural structure of LF space and moreover if $T : V \to W$ is a tame continuous linear map between tame LF spaces, the composition $f \mapsto T \circ f$ maps $\mathcal{S}(\text{Temp}^{\text{ind}}(G), V)$ into $\mathcal{S}(\text{Temp}^{\text{ind}}(G), W)$ and induces a continuous mapping $\mathcal{S}(\text{Temp}^{\text{ind}}(G), V) \to \mathcal{S}(\text{Temp}^{\text{ind}}(G), W)$. In this paper we will only consider the cases where $V = \mathbb{C}$, $V = C^u(G(F))$ or $C^w(N(F) \backslash G(F), \xi)$. In the first case, we will simply set $\mathcal{S}(\text{Temp}^{\text{ind}}(G)) = \mathcal{S}(\text{Temp}^{\text{ind}}(G), \mathbb{C})$ and we will denote by $\mathcal{S}_c(\text{Temp}^{\text{ind}}(G))$ the subspace of functions $f \in \mathcal{S}(\text{Temp}^{\text{ind}}(G))$ which are supported in a finite number of connected components of $\text{Temp}^{\text{ind}}(G)$ (in the $p$-adic case we have $\mathcal{S}_c(\text{Temp}^{\text{ind}}(G)) = \mathcal{S}(\text{Temp}^{\text{ind}}(G)))$. Finally, given the definition of the topology on $C^u_d(G(F))$, the following criterion is clear.

**Lemma 2.9.1** Let $f : \text{Temp}^{\text{ind}}(G) \to C^u(G(F))$ be a smooth function which is of compact support in the $p$-adic case. Then $f$ belongs to $\mathcal{S}(\text{Temp}^{\text{ind}}(G), C^u(G(F)))$ if and only if it satisfies the following condition: there exist $a, b \geq 1$ such that for every Levi subgroup $M$ of $G$, every $D \in \text{Sym}^*(\mathfrak{a}_M^*, \mathbb{C})$ and also every $k \geq 1$ and $u, v \in \mathcal{U}(\mathfrak{g})$ in the Archimedean case we have
\[
|D(\lambda \mapsto f(\pi_\lambda)(g))_{\lambda = 0}| \ll \Xi^G(g)\sigma(g)^{a_{\text{deg}}(D)+b} \text{ in the $p$-adic case,}
\]
\[
|D(\lambda \mapsto (R(u)L(v)f(\pi_\lambda))(g))_{\lambda = 0}| \ll N(\pi)^{-k}\Xi^G(g)\sigma(g)^{a_{\text{deg}}(D)+b} \text{ in the Archimedean case}
\]
for all $\sigma \in \Pi_2(M)$ and $g \in G(F)$ where we have set $\pi_\lambda = i_M^G(\sigma_\lambda)$, $\pi = \pi_0$ and $\text{deg}(D)$ denotes the degree of $D$.

### 2.10 Local Langlands correspondences and base-change

Let $W_F$ be the Weil group of $F$ and
\[
W'_F = \begin{cases} 
W_F \times SL_2(\mathbb{C}) & \text{if } F \text{ is } p \text{-adic} \\
W_F & \text{if } F \text{ is Archimedean}
\end{cases}
\]
the Weil-Deligne group of $F$. 

For $G$ a connected reductive group over $F$, we will denote by $^L G = G^\vee \rtimes W_F$ the Weil form of the $L$-group of $G$ where $G^\vee$ denotes the Langlands dual of $G$ (a connected reductive complex group) and the action of $W_F$ on $G^\vee$ is by pinned automorphisms. We will actually abuse this notation slightly and denote by $^L G_n(E)$ and $^L G_n(E)$ the $L$-groups of the natural connected reductive groups over $F$ of which $G_n(E)$ and $G_n(E)$ are the sets of $F$-points. Thus, we have

$$^L G_n(E) = (GL_n(\mathbb{C}) \times GL_n(\mathbb{C})) \rtimes W_F$$

where the action of $W_F$ on $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ is given by

$$\sigma \cdot (g_1, g_2) = \begin{cases} (g_2, g_1) & \text{if } \sigma \in W_F \backslash W_E \\ (g_1, g_2) & \text{otherwise} \end{cases}$$

and $^L G_n(E)$ is the subgroup of elements $(g_1, g_2, \sigma) \in ^L G_n(E)$ such that $\det(g_1) \det(g_2) = 1$. If $V$ is a hermitian space of dimension $n$ over $E$, we have $^L U(V) = GL_n(\mathbb{C}) \rtimes W_F$ where the action of $W_F$ on $GL_n(\mathbb{C})$ is given by

$$\sigma \cdot g = \begin{cases} J_n^t g J_n^{-1} & \text{if } \sigma \in W_F \backslash W_E \\ g & \text{otherwise} \end{cases}$$

with $J_n = \begin{pmatrix} 1 \\ & \ddots \\ & & 1 \\ & & & (-1)^{n-1} \end{pmatrix}$.

Recall that a Langlands parameter for $G$ (or by abuse of language for $G(F)$) is a $G^\vee$-conjugacy class of continuous homomorphisms: $\varphi : W_F^\vee \to ^L G$ which are algebraic when restricted to $SL_2(\mathbb{C})$ (in the $p$-adic case), sending $W_F$ to semi-simple elements and commuting with the natural projections $W_F^\vee \to W_F$, $^L G \to W_F$. A Langlands parameter $\varphi$ is tempered (resp. discrete) if the projection of $\varphi(W_F)$ in $G^\vee$ is bounded (resp. if it is tempered and the centralizer of $\varphi(W_F)$ in $G^\vee$ is finite modulo $Z(G^\vee)^{W_F}$). We shall denote by $\Phi(G)$, resp. $\Phi_{\text{temp}}(G)$, resp. $\Phi_2(G)$, the set of all, resp. all tempered, resp. all discrete, Langlands parameters for $G$.

The local Langlands correspondence postulates the existence of a finite-to-one map $\text{Irr}(G) \to \Phi(G)$, $\pi \mapsto \varphi_\pi$ satisfying certain properties and whose fibers are usually called $L$-packets (of $G$). This correspondence has been established in some cases: for any real reductive group by Langlands [La] and in the $p$-adic case for $G_n(F)$ by Harris-Taylor [HT], Henniart [Hen] and Scholze [Sch] and more recently for unitary groups $U(V)$ by Mok [Mok] and Kaletha-Minguez-Shin-White [KMSW] (following the work of Arthur [Art3] for split orthogonal and symplectic groups). These correspondences naturally extend to product of such groups, including in particular all Levi subgroups of groups in the previous list, as well as groups obtained by extension of scalars or quotient by a split central torus thus including $G_n(E)$ and $G_n(E)$ in our list. The basic properties of these correspondences that we shall use in this paper are listed below (where $M$ denotes a Levi subgroup of $G$):

- It sends $\text{Temp}(G)$ (resp. $\Pi_2(G)$) to $\Phi_{\text{temp}}(G)$ (resp. $\Phi_2(G)$). Moreover the map $\text{Temp}(G) \to \Phi_{\text{temp}}(G)$ factorizes through the quotient $\text{Temp}_{\text{ind}}(G)$;
• If $G$ is quasi-split then $\text{Irr}(G) \to \Phi(G)$, $\text{Temp}_{\text{ind}}(G) \to \Phi_{\text{temp}}(G)$ and $\Pi_2(G) \to \Phi_2(G)$ are all surjective (in particular this applies to $U(n)$) and these are even bijections for $G_n(F)$, $G_n(E) \text{ and } G_{n}(E)$.

• For $\sigma \in \text{Temp}(M(F))$ and $\pi$ a constituent of $i^G_M(\sigma)$ we have $\varphi_\pi = \iota \circ \varphi_\sigma$ where $\iota : L^M \to L^G$ is the natural embedding between $L$-groups. Conversely, if $\varphi_\pi = \iota \circ \varphi_M$ where $\varphi_M \in \Phi_2(M)$ then there exists $\sigma \in \Pi_2(M)$ with $\varphi_\sigma = \varphi_M$ such that $\pi \mapsto i^G_M(\sigma)$.

• For $\pi \in \text{Irr}(G)$ and $\lambda \in A^*_{G,C} = X^*(G) \otimes \mathbb{C}$, we have $\varphi_{\pi,\lambda} = \varphi_\lambda |^\lambda$ where $|.|^\lambda$ is the character $W^*_F \to (Z(G^\vee)^W_F)^0 = X^*(G) \otimes \mathbb{C}^\times$ obtained by composing the absolute value $|.|$ on $W^*_F$ with the character $t \in \mathbb{R}_+^* \to t^\lambda \in X^*(G) \otimes \mathbb{C}^\times$.

• If $G = G_1 \times G_2$ and $\pi = \pi_1 \boxtimes \pi_2 \in \text{Irr}(G)$ then the Langlands parameter $\varphi$ of $\pi$ is the product in a suitable sense of the Langlands parameters $\varphi_1$ and $\varphi_2$ of $\pi_1$ and $\pi_2$.

We will denote by $\text{Temp}(G)/\text{stab}$ the quotient of $\text{Temp}(G)$ by the relation $\pi \sim_{\text{stab}} \pi' \iff \varphi_\pi = \varphi_{\pi'}$ (i.e. the set of tempered $L$-packets for $G$). This notation will actually only be used for unitary groups, as for other groups considered in this paper (general linear groups and their variants) we have $\text{Temp}(G)/\text{stab} = \text{Temp}(G)$. Note that we have natural surjections $\text{Temp}(G) \to \text{Temp}(G)/\text{stab}$ and $\text{Temp}_{\text{ind}}(G) \to \text{Temp}(G)/\text{stab}$ (by the first point above). If $G_{qs}$ is a quasi-split inner form of $G$ (e.g. $G$ is a unitary groups of rank $n$ and $G_{qs} = U(n)$), by the second point above we have a natural identification $\text{Temp}(G_{qs})/\text{stab} = \Phi_{\text{temp}}(G_{qs})$ and since $G$ and $G_{qs}$ share the same $L$-group $\Phi_{\text{temp}}(G_{qs}) = \Phi_{\text{temp}}(G)$ and the Langlands correspondence gives a map $\text{Temp}_{\text{ind}}(G) \to \text{Temp}(G_{qs})/\text{stab}$.

**Lemma 2.10.1** Assume that $G$ is a product of unitary groups. Then, there exists a unique topology on $\text{Temp}(G_{qs})/\text{stab}$ such that $\text{Temp}_{\text{ind}}(G_{qs}) \to \text{Temp}(G_{qs})/\text{stab}$ is a local isomorphism and for every connected component $\mathcal{O} \subset \text{Temp}_{\text{ind}}(G)$ the map $\mathcal{O} \to \text{Temp}(G_{qs})/\text{stab}$ induces an isomorphism between $\mathcal{O}$ and a connected component of $\text{Temp}(G_{qs})/\text{stab}$.

**Proof:** As the Langlands correspondence is compatible with products (cf. last point above), we may assume that $G = U(V)$ for $V$ a hermitian space of dimension $n$ over $E$ and $G_{qs} = U(n)$. Let $\pi \in \text{Temp}_{\text{ind}}(G)$ and $\mathcal{O} \subset \text{Temp}_{\text{ind}}(G)$ be its connected component. We can write $\pi = i^G_M(\sigma)$ where $M$ is a Levi subgroup of the form $M = R_{E/F}G_{n_1} \times \ldots \times R_{E/F}G_{n_k} \times U(W)$, for a certain non-degenerate subspace $W \subset V$ of dimension $m$, and $\sigma = \sigma_1 \ldots \sigma_k \sigma_0 \in \Pi_2(M)$. Let $\pi' \in \text{Temp}_{\text{ind}}(U(n))$ be such that $\varphi_{\pi'} = \varphi_\pi$ and $\mathcal{O'} \subset \text{Temp}_{\text{ind}}(U(n))$ be its connected component. Then, by the compatibility of LLC with parabolic induction (third point above) and with products, we have $\pi' = i^U_{U(n)}(\sigma')$ where $L$ is a Levi subgroup (of $U(n)$) of the form $L = R_{E/F}G_{n_1} \times \ldots \times R_{E/F}G_{n_k} \times U(m)$ and $\sigma' = \sigma_1 \ldots \sigma_k \sigma_0' \in \Pi_2(U(m))$. There exist identifications $i^A_M \simeq (i^{R(F)})^k \simeq i^A_L$ such that $\sigma_\lambda := i^A_M(\sigma_\lambda) = \tau_1 \ldots \tau_k \lambda \times \sigma_0$ and $\pi'_\lambda := i^U_{U(n)}(\sigma'_\lambda) = \tau_1 \lambda_1 \times \ldots \times \tau_k \lambda_k \times \sigma_0'$ for every $\lambda \in (i^R)^k$. By the compatibility of LLC with unramified products (fourth point above) and with parabolic induction we see that $\varphi_{\pi_\lambda} = \varphi_{\pi'_\lambda}$ for every $\lambda \in (i^R)^k$. Moreover, by the precise descriptions of $W(G, \sigma)$ and $W(U(n), \sigma')$ given
in Section 2.6 there is an isomorphism \( W(G, \sigma) \simeq W(U(n), \sigma') \) compatible with the previous identification. Therefore, we have a commutative diagram

\[
\begin{array}{c}
\overset{i \mathcal{A}_M^* / W(G, \sigma)}{\longrightarrow} \quad \overset{i \mathcal{A}_L^* / W(U(n), \sigma')}{\longrightarrow} \\
\overset{\text{Temp}(G_{qs}) / \text{stab}}{\text{O}} \quad \overset{\text{O'}}{\text{Temp}(G_{qs}) / \text{stab}}
\end{array}
\]

where the two vertical arrows, given by \( \lambda \mapsto \pi_\lambda \) and \( \lambda \mapsto \pi'_\lambda \), are surjective and local isomorphisms near 0 (by definition of the topologies on \( \text{Temp}_{\text{ind}}(G) \) and \( \text{Temp}_{\text{ind}}(U(n)) \)).

Using the above diagram for \( G = G_{qs} \) and \( \pi, \pi' \) any two tempered representations lying in the same \( L \)-packet, we conclude that \( \text{Temp}(G_{qs}) / \text{stab} \) has a unique topology such that \( \text{Temp}_{\text{ind}}(G_{qs}) \to \text{Temp}(G_{qs}) / \text{stab} \) is a local isomorphism. Moreover, the same diagram shows that for any \( G \) the map \( \text{Temp}_{\text{ind}}(G) \to \text{Temp}(G_{qs}) / \text{stab} \) is a local isomorphism and that the image of any two connected components \( \mathcal{O}_1, \mathcal{O}_2 \subseteq \text{Temp}_{\text{ind}}(G) \) in \( \text{Temp}(G_{qs}) / \text{stab} \) are either disjoint or identical.

Thus, to conclude the proof it only remains to show that \( \mathcal{O} \to \text{Temp}(U(n)) / \text{stab} \) is injective. If \( \varphi_{\pi_\lambda} = \varphi_{\pi_\mu} \) for some \( \lambda, \mu \in i \mathcal{A}_M^* \), by the compatibility of LLC with parabolic induction and products we have \( \pi_\mu \simeq \tau_{1,\lambda_1} \times \ldots \times \tau_{k,\lambda_k} \rtimes \sigma_0'' \) for some \( \sigma_0'' \in \Pi_2(U(W)) \). According to Harish-Chandra, this implies that the representations \( \tau_{1,\mu_1} \rtimes \ldots \rtimes \tau_{k,\mu_k} \rtimes \sigma_0 \) and \( \tau_{1,\lambda_1} \rtimes \ldots \rtimes \tau_{k,\lambda_k} \rtimes \sigma_0'' \) of \( M(F) \) are conjugated under \( W(G, M) \) hence \( \sigma_0'' = \sigma_0 \) and finally \( \pi_\mu = \pi_\lambda \).

Still assuming that \( G \) is a product of unitary groups, by the previous lemma there exists a unique Borel measure on \( \text{Temp}(G_{qs}) / \text{stab} \) for which the map \( \text{Temp}_{\text{ind}}(G_{qs}) \to \text{Temp}(G_{qs}) / \text{stab} \) is locally measure preserving where we equip \( \text{Temp}_{\text{ind}}(G_{qs}) \) with the measure \( d\pi \) defined in Section 2.6 and we shall also denote this measure by \( d\pi \). Arguing similarly as in the proof of previous lemma, we see that the map \( \text{Temp}_{\text{ind}}(G) \to \text{Temp}(G_{qs}) / \text{stab} \) is also locally measure preserving. Therefore, we have the integration formulas

\[
(2.10.1) \quad \int_{\text{Temp}_{\text{ind}}(G)} f(\pi)d\pi = \int_{\text{Temp}(G_{qs}) / \text{stab}} \sum_{\varphi_\pi' \in \text{Temp}_{\text{ind}}(G)} f(\pi')d\pi, \quad f \in C_c(\text{Temp}_{\text{ind}}(G))
\]

\[
(2.10.2) \quad \int_{U / \text{stab}} \varphi(\pi)d\pi = \frac{1}{|W(G, \sigma)|} \int_{V} \varphi(\pi_\lambda)d\lambda, \quad \varphi \in C_c(U / \text{stab})
\]

where \( \pi, \sigma, U \) and \( V \) are as in 2.7.3 and \( U / \text{stab} \) denotes the image of \( U \) in \( \text{Temp}(G_{qs}) / \text{stab} \) (which by Lemma 2.10.1 is isomorphic to \( U \)).

For \( V \) be a hermitian space of dimension \( n \) over \( E \) we define a morphism \( BC : U(V) \to L^*G_n(E) \) called base-change by \( (g, \sigma) \mapsto (g, J_n^tg^{-1}J_n, \sigma) \). This induces a map between set
of Langlands parameters $\Phi_{\text{temp}}(U(V)) \to \Phi_{\text{temp}}(G_n(E))$ and thus composing with the Langlands correspondence a map $\text{Temp}(U(V))/\text{stab} \to \text{Temp}(G_n(E))$ which we shall also denote by $BC$. This map is injective and has its image contained in the subset of representations $\pi \in \text{Temp}(G_n(E))$ satisfying $\pi \simeq \pi^\ast$. The base-change map satisfies the following basic properties:

(2.10.3) $\omega_{BC(\sigma)}(z) = \omega_{\sigma}(z/z^\ast)$ for all $\sigma \in \text{Temp}(U(V))$ and all $z \in E^\times$.

(2.10.4) $BC(\tau \times \sigma) = \tau \times BC(\sigma)$ for all $\tau \in \text{Temp}(G_m(E))$ and $\sigma \in \text{Temp}(U(W))$ where $W$ is a certain hermitian space with $\dim(W) + 2m = n$.

At some point it will be convenient to consider certain twists of this base-change map. Recall that we have defined a character $\eta_n$ of $G_n(E)$ in Section 2.2. We set $BC_n(\sigma) = BC(\sigma) \otimes \eta_n$ for all $\sigma \in \text{Temp}(U(V))/\text{stab}$. When $n$ is odd $BC_n = BC$ and when $n$ is even $BC_n$ is usually called unstable base-change. The following property characterizes the image of $BC_n$ from a quasi-split unitary group:

(2.10.5) Let $\pi \in \text{Temp}(G_n(E))$. Then, $\pi$ belongs to the image of $BC_n : \text{Temp}(U(n)) \to \text{Temp}(G_n(E))$ if and only if it can be written as

$$\pi = \left( \prod_{i=1}^k \tau_i \times \tau_i^\ast \right) \times \prod_{j=1}^\ell \mu_j$$

where for every $1 \leq i \leq k$, $\tau_i \in \Pi_2(G_{n_i}(E))$ for some $n_i \geq 1$ and for every $1 \leq j \leq \ell$, $\mu_j \in BC_{m_j}(\text{Temp}(U(m_j))) \cap \Pi_2(G_{m_j}(E))$ for some $m_j \geq 1$.

### 2.11 Groups of centralizers

Let $G$ be a general linear groups or one of its variant (e.g. such that $G(F) = G_n(E)$ or $G_n(E)$) or a product of unitary groups. Let $\pi \in \text{Temp}(G)$ and choose a Levi subgroup $M$ of $G$ and $\sigma \in \Pi_2(M)$ such that $\pi \leftrightarrow i_M^G(\sigma)$. We set $S_\pi := S_{\varphi_\sigma}$ where $\varphi_\sigma : W_F^\vee \to L^M$ is the Langlands parameter of $\sigma$ and $S_{\varphi_\sigma}$ denotes the centralizer of $\varphi_\sigma$ in $(M/A_M)^\vee$ (which is naturally a subgroup of $M^\vee$). We will actually not need the precise definition of this group. All that matter is its cardinality which can be explicitly computed using the following short list of properties that $S_\pi$ satisfies (all of them being standard or very easy):

(2.11.1) For $\pi \in \text{Temp}(U(n))$ (resp. $\pi \in \text{Temp}(G_n(E))$) which embeds in $\pi_1 \times \ldots \times \pi_t \times \sigma_0$ (resp. $\pi_1 \times \ldots \times \pi_t$) where for every $1 \leq i \leq t$, $\pi_i \in \Pi_2(G_{n_i}(E))$ for some $n_i \geq 1$ and $\sigma_0 \in \Pi_2(U(m))$ for some $m \geq 0$, we have

$S_\pi \simeq S_{\pi_1} \times \ldots \times S_{\pi_t} \times S_{\sigma_0}$ (resp. $S_\pi \simeq S_{\pi_1} \times \ldots \times S_{\pi_t}$).

(2.11.2) $S_\pi \simeq \mathbb{Z}/2n\mathbb{Z}$ for every $\pi \in \Pi_2(G_n(E))$.

(2.11.3) For $\sigma \in \Pi_2(U(n))$, we have $S_\sigma \simeq (\mathbb{Z}/2\mathbb{Z})^k$ where $k$ is such that $BC(\sigma) \simeq \pi_1 \times \ldots \times \pi_k$ for some $\pi_i \in \Pi_2(G_{n_i}(E))$ (1 ≤ $i$ ≤ $k$).
Finally, we note that, for $V$ a hermitian space, the assignment $\pi \in \temp(U(V)) \mapsto S_\pi$ factorizes through $\temp_{\ind}(U(V))$ and even through $\temp(U(V))/\stab$ so that it can as well been considered as an assignment on these former sets.

### 2.12 Local $\gamma$-factors

Let $\varphi : W'_F \to \GL(M)$ be a continuous semi-simple and algebraic when restricted to $SL_2(\mathbb{C})$ finite dimensional complex representation of $W'_F$. We associate to $\varphi$ a local $L$-factor $L(s, \varphi)$ and a local $\epsilon$-factor $\epsilon(s, \varphi, \psi')$ as in [Ta, §3] and [Gr] §2.2. In the $p$-adic case, $L(s, \varphi)$ is of the form $P(q_F^{-s})$ where $P \in \mathbb{C}[T]$ is such that $P(0) = 1$ whereas $\epsilon(s, \varphi, \psi')$ is of the form $c q^{n(s-1/2)}$ where $n \in \mathbb{Z}$ and $c = \epsilon(1/2, \varphi, \psi') \in \mathbb{C}^\times$. In the Archimedean case, $L(s, \varphi)$ is a product of functions of the form $\pi^{-(s+s_0)/2} \Gamma((s + s_0)/2)$ for some $s_0 \in \mathbb{C}$ and $\epsilon(s, \varphi, \psi')$ is of the form $c q^{s-1/2}$ where $Q \in \mathbb{R}_+$ and $c = \epsilon(1/2, \varphi, \psi') \in \mathbb{C}^\times$. When $\varphi = 1_F$ is the trivial one-dimensional representation of $W'_F$, we will also write $\zeta_F(s)$ for $L(s, 1_F)$. We have

$$\zeta_F(s) = \begin{cases} (1 - q_F^{-s})^{-1} & \text{if } F \text{ is } p\text{-adic} \\ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) & \text{if } F = \mathbb{R} \end{cases}$$

In both cases we have $\zeta_F(s) \sim_{s \to 0} (s \log(q_F))^{-1}$ (where we recall that we set $q_\mathbb{R} = e^{1/2}$).

Returning to the general case, we define the local $\gamma$-factor associated to $\varphi$ as

$$\gamma(s, \varphi, \psi') = \epsilon(s, \varphi, \psi') \frac{L(1 - s, \varphi^\vee)}{L(s, \varphi)}$$

where $\varphi^\vee$ stands for the contragredient of $\varphi$. These factors are additive: for every (semi-simple, continuous, algebraic) complex representations $\varphi, \varphi'$ of $W'_F$ we have

\begin{equation}
(2.12.1) \quad \gamma(s, \varphi \oplus \varphi', \psi') = \gamma(s, \varphi, \psi') \gamma(s, \varphi', \psi')
\end{equation}

Of course, we can define similarly $\gamma$-factors $\gamma(s, \varphi, \psi'_E)$ for any (semi-simple, continuous and algebraic) complex representation of $W'_F$ where $\psi'_E$ is the additive character of $E$ defined by $\psi'_K(z) = \psi'(Tr_{E/F}(z))$. Set

$$\lambda_{E/F}(\psi') = \gamma\left(\frac{1}{2}, \eta_{E/F}, \psi'\right) = \epsilon\left(\frac{1}{2}, \eta_{E/F}, \psi'\right)$$

where we consider $\eta_{E/F}$ as a character of $W'_F$ through local class field theory. Then, $\lambda_{E/F}(\psi')$ is a fourth root of unity and if $\varphi$ is a complex representation of $W'_F$ of dimension $n$, by the inductivity of $\gamma$-factors in degree 0 ([Fa, Theorem 3.4.1]) we have

\begin{equation}
(2.12.2) \quad \gamma(s, \varphi, \psi'_E) = \lambda_{E/F}(\psi')^{-n} \gamma(s, \Ind_{W'_F}^{W'_E} \varphi, \psi')
\end{equation}

where $\Ind_{W'_F}^{W'_E} \varphi$ denotes the representation induced by $\varphi$ from $W'_F$ to $W'_E$.

The following property of Archimedean $\gamma$-factors will be used repeatedly:
Moreover, if $\varphi$ is tempered, meaning that $\varphi(W_F)$ is bounded in $GL(M)$, the local $\gamma$-factor $\gamma(s, \varphi, \psi')$ does not vanish in $H$ and has no pole in $-\mathcal{H}$. For such a $\varphi$, we will set
\[
\gamma^{*}(0, \varphi, \psi') = \lim_{s \to 0} \zeta(s) n_{\varphi}^{s} \gamma(s, \varphi, \psi')
\]
where $n_{\varphi}$ is the order of the zero of $\gamma(s, \varphi, \psi')$ at $s = 0$.

Assuming that the local Langlands correspondence is known for $G$, for every continuous complex representation $r : L G \to GL(M)$ which is semi-simple and bounded when restricted to $W_F$ and algebraic on $G^\vee$ and every $\pi \in \text{Irr}(G)$ we set $L(s, \pi, r) = L(s, r \circ \varphi_\pi)$ and $\gamma(s, \pi, r, \psi') = \gamma(s, r \circ \varphi_\pi, \psi')$. For $\pi \in \text{Temp}(G)$, the representation $r \circ \varphi_\pi$ is tempered and therefore
(2.12.4) $\gamma(s, \pi, r, \psi')$ does not vanish in $H$ and has no pole in $-\mathcal{H}$.

Still for $\pi \in \text{Temp}(G)$, we set (as before)
\[
\gamma^{*}(0, \pi, r, \psi') = \lim_{s \to 0} \zeta(s) n_{\pi, r}^{s} \gamma(s, \pi, r, \psi')
\]
where $n_{\pi, r}$ is the order of the zero of $\gamma(s, \pi, r, \psi')$ at $s = 0$. We will need the following result which is probably well-known but for which, in lack of a proper reference, we provide a quick proof.

**Lemma 2.12.1** Assume that $F = \mathbb{R}$. Then for all $s \in H$ there exists $k \geq 1$ such that
\[
|\gamma^{*}(0, \pi, r, \psi')| \ll N(\pi)^{k} \quad \text{and} \quad |\gamma(s, \pi, r, \psi')^{-1}| \ll N(\pi)^{k}
\]
for all $\pi \in \text{Temp}(G)$.

**Proof:** Since for every Levi subgroup $M$ of $G$ the functions $\sigma \in \Pi_2(M) \to N(\iota_M^{\sigma})$, $\sigma \in \Pi_2(M) \to N(\sigma)$ are equivalent and $\gamma(s, \iota_M^{\sigma}(\sigma), r, \psi') = \gamma(s, \sigma, r \circ \iota_M, \psi')$ where $\iota_M : L M \to L G$ is the natural embedding of $L$-groups, we are immediately reduced to prove a similar statement for discrete series. Thus, we assume that $G(\mathbb{R})$ admits discrete series. Let $T \subset G$ be a maximal elliptic torus, $B_C \subset G_C$ a Borel subgroup containing $T_C$ and $\rho$ half the sum of the roots of $t_C$ in $b_C$. Then, there exists an embedding of $L$-groups $\iota : L T \to L G$ (here it is important to consider Weil forms of $L$-groups) such that for every $\pi \in \Pi_2(G)$ there exists $\chi \in \hat{T}(\mathbb{R})$ with $\varphi_\pi = \iota \circ \varphi_\chi$ and the equality of infinitesimal characters $\chi_\pi = d\chi + \rho$ in $t(\mathbb{C})^* / W(G_C, T_C)$ where we have identified $\chi_\pi$ with an element of the latter set through Harish-Chandra isomorphism. In particular, it follows that $\gamma(s, \pi, r, \psi') = \gamma(s, \chi, r \circ \iota, \psi')$ and $N(\pi) \sim N(\chi)$ so that we are ultimately reduced to the case where $G = T$ is a torus. Moreover, since $r$ is semi-simple, we may assume that $r$ is irreducible. It is then of dimension one or two. Assume first that $\dim(r) = 2$. Then, $r$ is induced from a character $\mu$ of $T^\vee \times W_C$.
and for every \( \chi \in \hat{T}(\mathbb{R}) \) the representation \( r \circ \varphi_{\chi} \) is induced from the representation \( \mu \circ \varphi_{\chi_C} \) of \( W_{\mathbb{C}} \) where \( \chi_C \in \hat{T}(\mathbb{C}) \) is the composition of \( \chi \) with the norm map \( T(\mathbb{C}) \to T(\mathbb{R}) \) hence

\[
\gamma(s, \chi, r, \psi') = \lambda_{\mathbb{C}/\mathbb{R}}(\psi') \gamma(s, \chi_C, \mu, \psi'_{\mathbb{C}})
\]

Furthermore, the restriction of \( \mu \) to \( T^\chi \) determines a cocharacter \( G_m, \mathbb{C} \to T(\mathbb{C}) \) hence a morphism \( \mu_1 : W_{\mathbb{C}} = \hat{\mathbb{C}}^\times \to T(\mathbb{C}) \) and denoting by \( \mu_2 \) the restriction of \( \mu \) to \( W_{\mathbb{C}} \) (a unitary character) we have

\[
\gamma(s, \chi, \mu, \psi'_{\mathbb{C}}) = \gamma(s, (\chi_C \circ \mu_1)\mu_2, \psi'_{\mathbb{C}}), \quad \chi \in \hat{T}(\mathbb{R})
\]

As \( N((\chi_C \circ \mu_1)\mu_2) \ll N(\chi) \) for all \( \chi \in \hat{T}(\mathbb{R}) \), in this case the lemma is a consequence of the inequalities \( |\gamma^*(0, \nu, \psi'_{\mathbb{C}})| \ll N(\nu) \) and \( |\gamma(s, \nu, \psi'_{\mathbb{C}})|^{-1} \ll N(\nu)^{R(2s-1)} \) for all \( \nu \in \hat{W}_{\mathbb{C}} = \text{Temp}(\mathbb{C}^\times) \) which can be checked directly using standard properties of the Gamma function. Indeed for all \( \nu \in \hat{\mathbb{C}}^\times \), \( d\nu : \mathbb{C} \to i\mathbb{R} \) is of the form \( z \mapsto z(\nu)z - z(\nu)z \) for an unique \( z(\nu) \in \mathbb{C} \) and we have

\[
|\gamma(s, \nu, \psi'_{\mathbb{C}})| = \begin{cases} 
C(\psi'_{\mathbb{C}})^{R(s-1/2)} \left| \frac{\Gamma(1-\nu+az(\nu))}{\Gamma(s+a)} \right| & \text{if } z(\nu) \in \mathcal{H} \\
C(\psi'_{\mathbb{C}})^{R(s-1/2)} \left| \frac{\Gamma(1-\nu-az(\nu))}{\Gamma(s-a)} \right| & \text{if } z(\nu) \in -\mathcal{H}
\end{cases}
\]

where \( C(\psi'_{\mathbb{C}}) \in \mathbb{R}^+_\mathbb{C} \) only depends on \( \psi'_{\mathbb{C}} \). As \( N(\nu) \sim |z(\nu)| \) for all \( \nu \in \hat{\mathbb{C}}^\times \), the claimed inequalities are simple consequences of the fact that

\[
\left| \frac{\Gamma(a+z)}{\Gamma(b+z)} \right| \ll |z|^{R(a-b)}
\]

for all \( z \in \mathcal{H} \), \( a \) in a fixed compact subset of \( \mathcal{H} \) and \( b \) in a fixed compact subset of \( \mathbb{C} \) which itself can be deduced from Stirling’s asymptotic formula (cf. [WW] [§13.6]).

Assume now that \( \dim(r) = 1 \). Then, the restriction of \( r \) to \( T^\chi \) comes from a cocharacter \( G_m, \mathbb{R} \to T(\mathbb{C}) \) inducing a morphism \( \mu_1 : W_{ab}^\mathbb{R} = \hat{\mathbb{C}}^\times \to T(\mathbb{R}) \) and denoting by \( \mu_2 \) the restriction of \( r \) to \( W_{\mathbb{R}} \) we have

\[
\gamma(s, \chi, r, \psi') = \gamma(s, (\chi \circ \mu_1)\mu_2, \psi'), \quad \chi \in \hat{T}(\mathbb{R})
\]

As \( N((\chi \circ \mu_1)\mu_2) \ll N(\chi) \) for all \( \chi \in \hat{T}(\mathbb{R}) \), in this case the lemma is a consequence of the inequalities \( |\gamma^*(0, \nu, \psi'_{\mathbb{C}})| \ll N(\nu)^{1/2} \) and \( |\gamma(s, \nu, \psi'_{\mathbb{C}})|^{-1} \ll N(\nu)^{R(s-1/2)} \) for all \( \nu \in \hat{W}_{ab}^\mathbb{R} = \text{Temp}(\mathbb{R}^\times) \) which can again be checked directly using standard properties of the Gamma function. ■

In this paper, we will only consider \( \gamma \)-factors associated to the following representations of \( L \)-groups:
• $r = \text{Ad}$ is the adjoint representation of $^L G$ on $\text{Lie}(G)$. This notation will be used for all groups encountered in this paper with the exception of $G_n(E)$ where we will denote this representation by $\overline{\text{Ad}}$ (in order to distinguish it from the adjoint representation of $^L G_n(E)$). Note that we have
\begin{equation}
(2.12.5) \quad \gamma(s, \pi, \text{Ad}, \psi') = \gamma(s, 1_F, \psi') \gamma(s, \pi, \overline{\text{Ad}}, \psi')
\end{equation}
for all $\pi \in \text{Irr}(G_n(E))$ (the first $\gamma$-factor being defined by viewing $\pi$ as a representation of $G_n(E)$).

• The tensor product representation $r$ of $^L (G_n(E) \times G_m(E))$ ($m, n \in \mathbb{N}^*$). It is the induced to $^L (G_n(E) \times G_m(E))$ of the representation $((g_1, g_2), (g_3, g_4), \sigma) \mapsto g_1 \otimes g_2 \in \text{GL}_{mn}(\mathbb{C})$ of the index two subgroup $(G_n(E)^{\gamma} \times G_m(E)^{\gamma}) \times W_E$. In this case, for all $(\pi, \pi') \in \text{Irr}(G_n(E)) \times \text{Irr}(G_m(E))$ we will write $\gamma(s, \pi \times \pi', \psi')$ for $\gamma(s, \pi \boxtimes \pi', r, \psi')$ (this should not be confused with the standard $\gamma$-factor of the representation $\pi \times \pi' \in \text{Irr}(G_{m+n}(E))$, but we will never consider standard $\gamma$-factors in this paper).

• $r = \text{As : } ^L G_n(E) \to \text{GL}(\mathbb{C}^n \otimes \mathbb{C}^n)$ the Asai representation which is defined by $\text{As}(g_1, g_2) = g_1 \otimes g_2$ for $g_1, g_2 \in \text{GL}_n(\mathbb{C})$ and $\text{As}(\sigma) = \iota$ if $\sigma \in W_F \setminus W_E$ where $\iota \in \text{GL}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is given by $\iota(v \otimes w) = w \otimes v$.

The corresponding $\gamma$-factors satisfy the following properties (where we omit the additive character $\psi'$ since it is fixed throughout in this paper)
\begin{equation}
(2.12.6) \quad \gamma(s, \pi.|_{E^\oplus}, \text{Ad}) = \gamma(s, \pi, \text{Ad}), \quad (\pi \in \text{Irr}(G_n(E)), x \in \mathbb{C}).
\end{equation}

\begin{equation}
(2.12.7) \quad \gamma(s, \pi_1 \times \pi_2, \text{Ad}) = \gamma(s, \pi_1, \text{Ad})\gamma(s, \pi_2, \text{Ad})\gamma(s, \pi_1 \times \pi_2) = \gamma(s, \pi_1^\gamma \times \pi_2),
((\pi_1, \pi_2) \in \text{Irr}(G_n(E)) \times \text{Irr}(G_m(E))).
\end{equation}

\begin{equation}
(2.12.8) \quad \gamma(s, \pi, \text{Ad}) \text{ has a simple zero at } s = 0 \text{ for every } \pi \in \Pi_2(G_n(E)).
\end{equation}

\begin{equation}
(2.12.9) \quad \gamma(s, \pi_1 \times (\pi_2 \times \pi_3)) = \gamma(s, \pi_1 \times \pi_2)\gamma(s, \pi_1 \times \pi_3)
((\pi_1, \pi_2, \pi_3) \in \text{Temp}(G_n(E)) \times \text{Temp}(G_m(E)) \times \text{Temp}(G_k(E)))
\end{equation}
(1 here $\pi_2 \times \pi_3$ denotes the parabolic induction of $\pi_2 \boxtimes \pi_3$ to $G_{m+k}(E)$).
(2.12.11) For every \((\pi_1, \pi_2) \in \Pi_2(G_n(E)) \times \Pi_2(G_m(E)), \gamma(s, \pi_1 \times \pi_2)\) has at most a simple zero at \(s = 0\) and moreover
\[
\gamma(0, \pi_1 \times \pi_2) = 0 \iff \pi_1 \simeq \pi_2^\vee.
\]

(2.12.12) \[\gamma(s, \pi |_{\mathcal{E}}, \text{As}) = \gamma(s + 2x, \pi, \text{As}), \quad (\pi \in \text{Irr}(G_n(E)), x \in \mathbb{C}).\]

(2.12.13) \[\gamma(s, \pi_1 \times \pi_2, \text{As}) = \gamma(s, \pi_1, \text{As})\gamma(s, \pi_2, \text{As})\gamma(s, \pi_1 \times \pi_2^\vee), \quad (\pi_1, \pi_2) \in \text{Irr}(G_n(E)) \times \text{Irr}(G_m(E)).\]

(2.12.14) For every \(\pi \in \Pi_2(G_n(E)), \gamma(s, \pi, \text{As})\) has at most a simple zero at \(s = 0\) and moreover
\[
\gamma(0, \pi, \text{As}) = 0 \iff \pi \in BC_n(\text{Temp}(U(n))).
\]

(2.12.15) \[\gamma(s, \sigma, \text{Ad}) = \frac{\gamma(s, BC_n(\sigma), \text{Ad})}{\gamma(s, BC_n(\sigma), \text{As})}, \quad (\sigma \in \text{Temp}(U(V)), n = \text{dim}(V)).\]

2.13 Harish-Chandra Plancherel formula

Let \(f \in \mathcal{S}(G(F))\). For all \(\pi \in \text{Temp}_{\text{ind}}(G)\) we define a function \(f_\pi\) by
\[
f_\pi(g) = \text{Trace}(\pi(g)\pi(f^\vee)), \quad g \in G(F)
\]
where \(f^\vee(x) = f(x^{-1})\). Recall that in Section 2.4 we have introduced the space \(\mathcal{S}(\text{Temp}_{\text{ind}}(G), C^w(G(F)))\) of Schwartz functions \(\text{Temp}_{\text{ind}}(G) \to C^w(G(F))\). The following is probably well-known but in lack of a proper reference we provide a proof in Appendix A.

**Proposition 2.13.1** For all \(\pi \in \text{Temp}_{\text{ind}}(G)\) we have \(f_\pi \in C^w(G(F))\), the map \(\pi \mapsto f_\pi\) belongs to \(\mathcal{S}(\text{Temp}_{\text{ind}}(G), C^w(G(F)))\) and moreover the linear map
\[
f \in \mathcal{S}(G(F)) \mapsto (\pi \mapsto f_\pi) \in \mathcal{S}(\text{Temp}_{\text{ind}}(G), C^w(G(F)))
\]
is continuous.

According to Harish-Chandra (see \[\text{[H-C2]}\], \[\text{Wald1]}\)), there exists a (necessarily unique) Borel measure \(d\mu_G\) on \(\text{Temp}(G(F))\) such that
\[
(2.13.1) \quad f(g) = \int_{\text{Temp}(G(F))} f_\pi(g)d\mu_G(\pi)
\]

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for all \( f \in \mathcal{S}(G(F)) \) and \( g \in G(F) \). Moreover, \( d\mu_G(\pi) = \mu_G(\pi)d_{\psi'}\pi \) where \( d_{\psi'}\pi \) is the measure on \( \text{Temp}_{\text{ind}}(G) \) defined in Section 2.3 and for \( \pi = \mathfrak{i}^G_M(\sigma) \), where \( M \) is a Levi subgroup of \( G \) and \( \sigma \in \Pi_2(M) \), we have
\[
\mu_G(\pi) = d(\sigma)j(\sigma)^{-1}
\]
where \( d(\sigma) \) is the formal degree of \( \tau \) (as defined in Section 2.6) and \( j(\sigma) \) is a certain product of standard intertwining operators that we won’t need to describe precisely here. When \( \pi \) is a discrete series we simply have \( \mu_G(\pi) = d(\pi) \). In the Archimedean case the function \( \pi \mapsto \mu_G(\pi) \) is of moderate growth i.e. there exists \( k > 0 \) such that
\[
(2.13.2) \quad \mu_G(\pi) \ll N(\pi)^k, \quad \pi \in \text{Temp}_{\text{ind}}(G)
\]
Combining this with Proposition 2.13.1 we see that the integral \( \int_{\text{Temp}_{\text{ind}}(G)} f_\pi d\mu_G(\pi) \) makes sense in \( \mathcal{C}^w(G(F)) \) and by 2.13.1 it follows that
\[
(2.13.3) \quad f = \int_{\text{Temp}_{\text{ind}}(G)} f_\pi d\mu_G(\pi) \text{ in } \mathcal{C}^w(G(F)).
\]
Recall that in Section 2.11 we have associated to any \( \pi \in \text{Temp}(G_n(E)) \) a finite abelian group \( S_\pi \). By 2.11.1 and 2.11.2 we immediately see that if \( \pi \) is induced from a discrete series of a Levi of the form \( M = R_{E/F}G_{n_1,E} \times \ldots \times R_{E/F}G_{n_k,E} \) then
\[
(2.13.4) \quad |S_\pi| = 2^{k_n}n_1 \ldots n_k
\]
Finally recall that we have set \( \overline{G_n(E)} := G_n(E)/Z_n(F) \) (i.e. the group of \( F \)-points of the quotient \( (R_{E/F}G_{n,E})/Z_{n,F} \)). The following formula for \( d\mu_{\overline{G_n(E)}} \) is essentially due to Harish-Chandra [HC2] in the Archimedean case and Shahidi [Sha1] and Silberger-Zink [SZ] in the \( p \)-adic case. Actually, this also incorporates a reformulation of the result of Silberger-Zink due to Hiraga-Ichino-Ikeda [HII]. Because our normalization of measures does not compare obviously to theirs (in particular because we have normalized everything by considering \( G_n(E) \) as an \( F \)-group) we provide a short proof to explain the relation.

**Proposition 2.13.2** We have the following equality
\[
d\mu_{\overline{G_n(E)}}(\pi) = \lambda_{E/F}(\psi')^{-n^2}\gamma^*(0, \pi, \text{Ad}, \psi')/|S_\pi| d\pi
\]
of measures on \( \text{Temp}(\overline{G_n(E)}) \).

**Proof:** By Fourier inversion on \( Z_n(F) \) it is easy to see that
\[
\mu_{\overline{G_n(E)}}(\pi) = \mu_{G_n(E)}(\pi), \quad \pi \in \text{Temp}(\overline{G_n(E)})
\]
Thus in fact it suffices to show that
\[
(2.13.5) \quad d\mu_{G_n(E)}(\pi) = \lambda_{E/F}(\psi')^{-n^2}\omega(-1)^{n-1}\gamma^*(0, \pi, \text{Ad}, \psi')/|S_\pi| d\pi, \quad \pi \in \text{Temp}(G_n(E))
\]
The Plancherel measure for $G_n(E)$ is completely computed by results of Shahidi [Sha1] and Silberger-Zink [SZ] in the $p$-adic case and by Harish-Chandra [H-C2] in the Archimedean case but to state their results it is actually better to consider $G_n(E)$ as the group of $E$-points of $G_n$ (and not as the group of $F$-points of $R_E/F G_{n,E}$ as in other parts of this paper). Using the character $\psi_E = \psi' \circ Tr_{E/F}$ we can construct as before a measure on $G_n(E)$, which coincides with the one we already fixed, and a measure $d_{\psi_E'} \pi$ on $\text{Temp}(G_n(E))$, which does not coincide with $d_{\psi_\pi} \pi$. Then, we have

\[ d_{\mu_Gn(E)}(\pi) = \mu_{G_n(E)}^E(\pi) \circ d_{\psi_E'} \pi \]

where $\mu_{G_n(E)}^E(\pi)$ is the Plancherel density for $G_n(E)$ considered as an $E$-group. For $\pi = i_M^{G_n(E)}(\sigma)$ where $M$ is a Levi subgroup of $G_{n,E}$ and $\sigma \in \Pi_2(M(E))$ we have

\[ \mu_{G_n(E)}^E(\pi) = dE(\sigma) \cdot j(\sigma)^{-1} \]

where $j(\sigma)$ is the same factor as before and $dE(\sigma)$ stands for the formal degree of $\sigma$ when $M(E)$ is considered as an $E$-group (i.e. by integrating product of matrix coefficients on $M(E)/A_M(E)$ and not, assuming that $M$ is defined over $F$, over $M(E)/A_M(F)$). Assume that $M$ is of the form

\[ M \simeq G_{n,1,E} \times \ldots \times G_{n_k,E} \]

and that

\[ \sigma = \tau_1 \times \ldots \times \tau_k \]

where $\tau_i \in \Pi_2(G_{n_i}(E))$ for all $1 \leq i \leq k$. Then, by a result of Shahidi [Sha1], [Sha2] we have

\[ j(\sigma)^{-1} = \left( \prod_{i=1}^{k} \omega_{\tau_i}(-1)^{n-n_i} \right) \gamma(0, \sigma, \text{Ad}_{G_n/M}, \psi_E') \]

where $\text{Ad}_{G_n/M}$ denotes the adjoint representation of $M^\vee$ on $\text{Lie}(G_{n,E})/\text{Lie}(M^\vee)$. On the other hand, by a result of Silberger-Zink [SZ] in the $p$-adic case and Harish-Chandra [H-C2, Lemma 23.1] in the Archimedean case as reformulated by Hiraga-Ichino-Ikeda [HII] and reproved and refined by Ichino-Lapid-Mao [ILM, Theorem 2.1] we have

\[ dE(\sigma) = \left( \prod_{i=1}^{k} \omega_{\tau_i}(-1)^{n_i-1} \right) \gamma(0, \sigma, \text{Ad}_{M/A_M}, \psi_E') \]

where $\text{Ad}_{M/A_M}$ stands for the adjoint representation of $M^\vee$ on $M^\vee/A_M^\vee$. On the other hand, on the connected component of $\pi$ we have

\[ d_{\psi_E'} \pi = \gamma^*(0, 1_E, \psi_E')^k d^E \pi \]

where $d^E \pi$ is the analog of the measure $d\pi$ by viewing $G_n(E)$ as an $E$-group and

\[ \gamma^*(0, 1_E, \psi_E') = \lim_{s \to 0^+} \zeta_E(s) \gamma(s, 1_E, \psi_E') = \left( \frac{\log(q_E)}{\log(q_{EF})} \right) \lim_{s \to 0^+} \zeta_E(s) \gamma(s, 1_E, \psi_E') \]

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Finally by the description \[2.7.3\] of the measure \(d\pi\) and its analog for \(d^E\pi\) we easily check that

\[
d^E\pi = 2^{-k} \frac{\log(q_F)}{\log(q_E)}^k d\pi
\]

(the factor \(2^k\) stems from the fact that \(|\cdot|_E = |\cdot|_F^2\) on \(F^\times\)). Combining \[2.13.6\] \[2.13.7\] \[2.13.8\] \[2.13.9\] \[2.13.10\] \[2.13.11\] and \[2.13.12\] we obtain

\[
d\mu_{G_n(E)}(\pi) = \omega_\pi(-1)^{n-1} \gamma(0, \sigma, \text{Ad}_{G_n/M}, \psi_E') \frac{\gamma(0, \sigma, \text{Ad}_{M/A_M}, \psi_E')}{2^k \sqrt{n_1 \cdots n_k}} \lim_{s \to 0^+} \zeta_F(s)^k \gamma(s, 1_E, \psi_E') d\pi
\]

Now, the adjoint representation of \(M^\vee\) on \(\text{Lie}(G_n)^\vee\) is the direct sum of \(\text{Ad}_{G_n/M}, \text{Ad}_{M/A_M}\) and \(k\) copies of the trivial representation so that by additivity of \(\gamma\)-factors \[2.12.1\] and their inductivity in degree 0 \[2.12.2\]

\[
\gamma(s, \sigma, \text{Ad}_{G_n/M}, \psi_E') \gamma(s, \sigma, \text{Ad}_{M/A_M}, \psi_E') \gamma(s, 1_E, \psi_E')^k = \gamma(s, \pi, \text{Ad}, \psi') = \lambda_{E/F}(\psi')^{-n^2} \gamma(s, \pi, \text{Ad}, \psi')
\]

Together with \[2.13.4\] this gives precisely \[2.13.3\] and ends the proof of the proposition. ■

### 2.14 Plancherel formula for Whittaker functions

In this section we make a little bit more precise the Plancherel formula for Whittaker functions proved by Sakellaridis and Venkatesh in [SV §6.3] (in particular, our discussion includes real groups).

Assume that \(G\) is quasi-split, let \(B = TN\) be a Borel subgroup with unipotent radical \(N\) and \(\xi : N(F) \to S^1\) a generic character. For any \(f \in \mathcal{S}(G(F))\) we define a function \(W_f\) on \(G(F) \times G(F)\) by

\[
W_f(g_1, g_2) = \int_{N(F)} f(g_1^{-1} u g_2) \xi(u)^{-1} du, \quad g_1, g_2 \in G(F)
\]

Clearly, we have \(W_f(u_1 g_1, u_2 g_2) = \xi(u_1)^{-1} \xi(u_2) W_f(g_1, g_2)\) for every \((g_1, g_2) \in G(F)^2\) and every \((u_1, u_2) \in N(F)^2\). Moreover, \(W_f(g, \cdot) \in \mathcal{S}(N(F) \backslash G(F), \xi)\) for all \(g \in G(F)\).

Recall from [Ben1 §7.1], that the functional \(f \in \mathcal{S}(G(F)) \mapsto \int_{N(F)} f(u) \xi(u)^{-1} du\) extends (necessarily uniquely) by continuity to a functional \(\mathcal{C}^w(G(F)) \to \mathbb{C}\) to be denoted by

\[
f \mapsto \int_{N(F)}^* f(u) \xi(u)^{-1} du
\]

(actually in [Ben1] only the case of unitary groups is considered but the proof is easily seen to work in general). This allows to extend the definition of \(W_f\) to every \(f \in \mathcal{C}^w(G(F))\) by

\[
W_f(g_1, g_2) = \int_{N(F)}^* f(g_1^{-1} u g_2) \xi(u)^{-1} du, \quad g_1, g_2 \in G(F)
\]

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Note that $\xi^{-1} \boxtimes \xi$ is a non-degenerate character on the $F$-points of the maximal unipotent subgroup $N \times N$ of $G \times G$. Thus, following the general construction of Section 2.4, there is a corresponding space of tempered functions $C_w^\w(N(F) \times N(F) \setminus G(F) \times G(F), \xi^{-1} \boxtimes \xi)$. Recall that both $C_w^\w(G(F))$ and $C_w^\w(N(F) \times N(F) \setminus G(F) \times G(F), \xi^{-1} \boxtimes \xi)$ are equipped with structure of tame LF spaces in the sense of Section 2.3.

**Lemma 2.14.1** For all $f \in C_w^\w(G(F))$ we have $W_f \in C_w^\w(N(F) \times N(F) \setminus G(F) \times G(F), \xi^{-1} \boxtimes \xi)$ and moreover the linear map

$$f \in C_w^\w(G(F)) \mapsto W_f \in C_w^\w(N(F) \times N(F) \setminus G(F) \times G(F), \xi^{-1} \boxtimes \xi)$$

is continuous and tame.

**Proof:** By the closed graph theorem it suffices to show the existence of $b \geq 1$ such that $f \in C_w^\w(G(F))$ implies $W_f \in C_w^\w(N(F) \times N(F) \setminus G(F) \times G(F), \xi^{-1} \boxtimes \xi)$. By Dixmier-Malliavin [DM], any $f \in C_w^\w(G(F))$ is a finite sum of functions of the form $\varphi_1 \ast f' \ast \varphi_2$ where $f' \in C_w^\w(G(F))$ and $\varphi_1, \varphi_2 \in C_c^\infty(G(F))$. Thus, we may as well assume that $f$ is of this form.

From the continuity of the linear form $f \mapsto \int_{N(F)}^b f(u)\xi(u)^{-1}du$ it follows that

$$W_{\varphi_1 \ast f' \ast \varphi_2} = W_{f'} \ast (\varphi_1^\vee \otimes \varphi_2)$$

where $\varphi_1^\vee \otimes \varphi_2 \in C_c^\infty(G(F) \times G(F))$ is defined by $\varphi_1^\vee \otimes \varphi_2(g_1, g_2) = \varphi_1(g_1^{-1})\varphi(g_2)$. Therefore the function $W_{f'}$ is smooth and moreover in the Archimedean case $R(u_1 \otimes u_2)W_{f'} = W_{f'} \ast (R(u_1)\varphi_1^\vee \otimes R(u_2)\varphi_2)$ for all $u_1, u_2 \in U(g)$. It thus suffices to check the existence of $b \geq 1$ such that for every $d \geq 1$ and $f \in C_w^\w(G(F))$ we have

$$|W_f(g_1, g_2)| \ll \Xi(G)(g_1)^d \Xi(G)(g_2)^d \sigma_{N, G}(g_1)^{d+b} \sigma_{N, G}(g_2)^{d+b}, \quad g_1, g_2 \in G(F).$$

Such an inequality is proved in [Beu1, Lemma 7.3.1(ii)] in the case where $G$ is a unitary group and with $d + b$ replaced by an unspecified $d' > 0$. Looking closer into the proof of loc. cit. we see that it works verbatim for any reductive group and moreover that we can take $d'$ to be $d + b$ where $b$ is such that $\Xi^d(t) \ll \delta^b(t)^{\frac{1}{2}}\sigma(t)^b$ for all $t \in T(F)$ (see in particular Claim (7.3.8) of loc. cit.) thus proving the lemma.

Let $f \in S(G(F))$. For every $\pi \in \text{Temp}_{\text{ind}}(G)$, we set $W_{f, \pi} := W_{f^\pi}$ where $f^\pi \in C_w^\w(G(F))$ is defined as in Section 2.13.

Let $\pi \in \text{Temp}_{\text{ind}}(G)$. Then the linear form

$$T \in \text{End}_G(\pi) \ni \pi^\vee \boxtimes \pi \mapsto \int_{N(F)}^b \text{Trace}(\pi(u)T)\xi(u)^{-1}du$$

is well-defined and continuous by 2.6.1 and is obviously a $(N \times N, \xi^{-1} \boxtimes \xi)$-Whittaker functional. Therefore, if $\pi$ is not $(N \times N, \xi^{-1} \boxtimes \xi)$-generic we have $W_{f, \pi} = 0$ for every $f \in S(G(F))$ whereas if $\pi$ is $(N \times N, \xi^{-1} \boxtimes \xi)$-generic we have

$$(2.14.1) \quad W_{f, \pi} \in \mathcal{W}(\pi^\vee \boxtimes \pi, \xi^{-1} \boxtimes \xi), \quad \forall f \in S(G(F))$$

The following proposition is a direct consequence of the previous lemma, of Proposition 2.13.1 and of the Plancherel formula 2.13.3.
**Proposition 2.14.2** The map \( \pi \in \text{Temp}_{\text{ind}}(G) \mapsto W_{f,\pi} \) belongs to
\[
S(\text{Temp}(G(F)), C^w(N(F) \times N(F) \setminus G(F) \times G(F), \xi^{-1} \boxtimes \xi))
\]
and the linear map
\[
S(G(F)) \rightarrow S(\text{Temp}(G(F)), C^w(N(F) \times N(F) \setminus G(F) \times G(F), \xi^{-1} \boxtimes \xi))
\]
\[
f \mapsto (\pi \mapsto W_{f,\pi})
\]
is continuous. Moreover, for every \( f \in S(G(F)) \) we have the following equality
\[
W_f = \int_{\text{Temp}_{\text{ind}}(G)} W_{f,\pi} d\mu_G(\pi)
\]
in \( C^w(N(F) \times N(F) \setminus G(F) \times G(F), \xi^{-1} \boxtimes \xi) \).

Assume now that \( G = R_{\mathbb{E}/F}G_n \) and recall that in Section 2.8 we have defined a scalar product \((.,.)^{\text{Whitt}}\) on \( C^w(N_n(E) \setminus G_n(E), \psi_n) \). The next proposition is [LM2, Lemma 4.4]. Notice that the extra factor \(|\tau|_E^{-n(n-1)/2}\) is a consequence of our choice of measures which are normalized using the character \( \psi'_E = \psi' \circ \text{Tr}_{E/F} \) rather than \( \psi \). Also, that the left-hand side of the proposition coincides with what is denoted \([W,W']^{\psi_n}\) in loc. cit. follows from Proposition 2.3 and Proposition 2.11 of loc. cit. in the \( p \)-adic case (this last proposition actually shows that \( \int_{N_n(E)} \psi_n(u)^{-1} du \) extends continuously to the bigger space \( C^\infty(G_n(E)) \) of all smooth functions on \( G_n(E) \)) and from the alternative definition of \((.,.)^{\psi_n}\) given in the middle of p.465 of loc. cit. in the Archimedean case.

**Proposition 2.14.3** For every \( W, W' \in C^w(N_n(E) \setminus G_n(E), \psi_n) \) we have
\[
\int_{N_n(E)} \langle R(u)W, W' \rangle^{\text{Whitt}} \psi_n(u)^{-1} du = |\tau|_E^{-n(n-1)/2} W(1)\overline{W'(1)}.
\]

Let \( f \in S(G_n(E)) \) and \( \pi \in \text{Temp}(G_n(E)) \). By the above construction, we associate to \( f \) and \( \pi \) a function \( W_{f,\pi} \in C^w(N_n(E) \setminus G_n(E) \times N_n(E) \setminus G_n(E), \psi_n^{-1} \boxtimes \psi_n) \). The \( G_n(E) \)-invariant scalar product \((.,.)^{\text{Whitt}}\) on \( W(\pi, \psi_n) \) induces an isomorphism
\[
\text{End}_\mathbb{C}(\pi) \simeq \pi' \hat{\otimes} \pi \simeq W(\pi, \psi_n) \hat{\otimes} W(\pi, \psi_n)
\]
where \( W(\pi, \psi_n) \) denotes the complex conjugate of \( W(\pi, \psi_n) \). Let \( B(\pi, \psi_n) \) be an orthonormal basis of (the Hilbert completion of) \( W(\pi, \psi_n) \) for \((.,.)^{\text{Whitt}}\) obtained by taking the union of orthonormal basis for \( W(\pi, \psi_n)[\delta] \) for every \( \delta \in \overline{K_{n,E}} \). Then, through the above identification, we have
\[
(2.14.2) \quad \pi(f) = \sum_{W \in B(\pi, \psi_n)} \overline{W} \otimes R(f)W
\]
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where the sum converges absolutely in $\text{End}_\mathbb{Z}(\pi)$ (notice that this sum is actually finite in the $p$-adic case). From this, Proposition 2.14.3 and 2.6.1 we deduce (recall that $f_\pi(g) = \text{Trace}(\pi(g)\pi(f'))$)

\[(2.14.3)\quad W_{f,\pi} = |\tau|_p^{-\frac{n(n-1)}{2}} \sum_{W \in \mathfrak{R}(\pi, \psi_n)} W \otimes R(f') W\]

where the sum converges absolutely in $C^u(N_n(E) \backslash G_n(E) \times N_n(E) \backslash G_n(E), \psi_n^{-1} \boxtimes \psi_n)$.

### 2.15 The linear form $\beta$

Let $n \geq 1$. For every $W \in C^u(N_n(E) \backslash G_n(E), \psi_n)$, we set

$$\beta(W) = \int_{N_n(F) \backslash P_n(F)} W(p) dp \quad \text{and} \quad \beta_n(W) = \int_{N_n(F) \backslash P_n(F)} W(p) \eta_{E/F}(\det h)^{n-1} dp.$$ 

**Lemma 2.15.1** The integral defining $\beta(W)$, $\beta_n(W)$ are absolutely convergent and $\beta, \beta_n$ are continuous linear forms on $C^u(N_n(E) \backslash G_n(E), \psi_n)$.

**Proof:** By Lemma 2.4.3 and the Iwasawa decomposition $P_n(F) = N_n(F)A_{n-1}(F)K_{n-1}$, it suffices to show for all $d > 0$ the convergence for $N \geq 1$ sufficiently large of

$$\int_{A_{n-1}(F)} \prod_{i=1}^{n-2} (1 + \left| \frac{a_i}{a_{i+1}} \right|)^{-N} (1 + |a_{n-1}|)^{-N} \delta_{n,E}(a)^{1/2} \sigma(a)^d \delta_{n-1}(a)^{-1} da$$

As $\delta_{n,E}(a)^{1/2} \delta_{n-1}(a)^{-1} = \delta_n(a) \delta_{n-1}(a)^{-1} = |\det a|$ this follows from Lemma 2.4.4.

Let $\sigma \in \text{Temp}(U(n))$ and set $\pi = BC_n(\sigma)$. Then, by slightly reformulating the main results of [Har] and [Mat] we have that $\pi$ is $G_n(F)$-distinguished (i.e. it admits a nonzero continuous $G_n(F)$-invariant linear form). Hence, by [LM3] (2) p.263 there exists a sign $c_1(\sigma) \in \{ \pm 1 \}$ such that, setting $\hat{W}(g) = W(w_n g^{-1})$ for every $g \in G_n(E)$, we have

\[(2.15.1)\quad \beta(\hat{W}) = c_1(\sigma) \beta(W)\]

for all $W \in \mathcal{W}(\pi, \psi_n)$.

### 2.16 Rankin-Selberg local functional equation for Asai $\gamma$-factors

For every $W \in C^u(N_n(E) \backslash G_n(E), \psi_n)$, $\phi \in \mathcal{S}(F^n)$ and $s \in \mathcal{H}$ set

$$Z(s, W, \phi) := \int_{N_n(F) \backslash G_n(F)} W(h) \phi(e_n h) |\det h|^s dh$$

where $e_n = (0, \ldots, 0, 1) \in F^n$. For all $\phi \in \mathcal{S}(F^n)$ we shall also denote by

$$\hat{\phi}(y) = \int_{F^n} \phi(x) \psi'(y^t x) dx, \quad y \in F^n$$

the Fourier transform of $\phi$ (recall that $dx$ denotes the autodual measure with respect to $\psi'$).
Lemma 2.16.1 (i) For $s \in \mathcal{H}$, the integral defining $Z(s, W, \phi)$ is absolutely convergent and defines a continuous linear form on $C^w(N_n(E) \backslash G_n(E), \psi_n)$. Moreover, for all $\phi \in \mathcal{S}(F)$ the map $s \in \mathcal{H} \mapsto Z(s, \phi) \in C^w(N_n(E) \backslash G_n(E), \psi_n)$ is analytic.

(ii) For all $W \in \mathcal{S}(Z_n(F)N_n(E) \backslash G_n(E), \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ we have

$$\lim_{s \to 0^+} n\gamma(s, 1_F, \psi')Z(s, W, \phi) = \phi(0) \int_{Z_n(F)N_n(F) \backslash G_n(F)} W(h)dh.$$ 

Proof:

(i) Let $\phi \in \mathcal{S}(F^n)$. By Lemma 2.4.3 and the Iwasawa decomposition $G_n(F) = N_n(F)A_n(F)K_n$, it suffices to show that for all $d > 0$ there exists $N \geq 1$ such that the following integral converges uniformly on compact subsets of $\mathcal{H}$

$$\int_{A_n(F)} \prod_{i=1}^{n-1} (1 + |a_i|)^{-N} \delta_n(a)_{1/2} |\phi(e_n a)| \delta_n(a)^{-1} |\det a|^{|R(s)}da.$$ 

Since $\delta_n(a)^{1/2} = \delta_n(a)$ and $|\phi(e_n a)| \ll (1 + |a_n|)^{-N}$ for all $a \in A_n(F)$, this follows from Lemma 2.4.4.

(ii) Let $W \in \mathcal{S}(Z_n(F)N_n(E) \backslash G_n(E), \psi_n)$ and $\phi \in \mathcal{S}(F^n)$. Then, again by the Iwasawa decomposition, for all $s \in \mathcal{H}$ we have

$$\gamma(ns, 1_F, \psi')Z(s, W, \phi) = \int_{A_{n-1}(F)} \int_{K_n} W(ak) \gamma(ns, 1_F, \psi') \int_{Z_n(F)} \phi(e_n z k) |\det z|^s |dzdk| |\det a|^s \delta_n(a)^{-1}da.$$ 

Set $\phi_k(x) = \phi(xe_n k)$ for all $x \in F$ and $k \in K$. Then $\phi_k \in \mathcal{S}(F)$ and by Tate’s thesis for every $k \in K_n$ we have

$$\gamma(ns, 1_F, \psi') \int_{Z_n(F)} \phi(e_n z k) |\det z|^s |dz = \int_F \hat{\phi}_k(x) |x|^{-ns}dx$$ 

for all $s \in \mathcal{H}$ sufficiently close to 0. Since $\{\phi_k \mid k \in K_n\}$ is a bounded subset of $\mathcal{S}(F^n)$ so does $\{\hat{\phi}_k \mid k \in K_n\}$ and therefore $(k, s) \mapsto \int_F \hat{\phi}_k(x) |x|^{-ns}dx$ is uniformly bounded for $s$ in a neighborhood of 0. By dominant convergence we deduce that

$$\lim_{s \to 0^+} \gamma(ns, 1_F, \psi')Z(s, W, \phi) = \int_{A_{n-1}(F)} \int_{K_n} W(ak) \int_F \hat{\phi}_k(x) |x|^s |dxdk| |\det a|^s \delta_n(a)^{-1}da$$

$$= \phi(0) \int_{A_{n-1}(F) \times K_n} W(ak) \delta_n(a)^{-1} |da| = \phi(0) \int_{Z_n(F)N_n(F) \backslash G_n(F)} W(h)dh.$$ 

As $\gamma(ns, 1_F, \psi') \sim s \to 0 n\gamma(s, 1_F, \psi')$ this ends the proof of the lemma. ■
Let $\pi \in \text{Temp}(G_n(E))$. Then we have $\mathcal{W}(\pi, \psi_n) \subset \mathcal{C}^w(N_n(E) \backslash G_n(E), \psi_n)$ and therefore for $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ the expression $Z(s, W, \phi)$ is well-defined for all $s \in \mathcal{H}$. Recall that for $W \in \mathcal{W}(\pi, \psi_n)$, $\hat{W}$ denotes the function defined by $\hat{W}(g) = W(w_n g^{-1})$ for all $g \in G_n(E)$. Note that $\hat{W} \in \mathcal{W}(\pi^\vee, \psi_n^{-1})$ and that up to replacing $\psi$ by $\psi^{-1}$ the previous lemma ensures that $Z(s, \hat{W}, \phi)$ is well-defined for all $s \in \mathcal{H}$ and $\phi \in \mathcal{S}(F^n)$. Finally, recall that $\phi \in \mathcal{S}(F^n) \mapsto \hat{\phi} \in \mathcal{S}(F^n)$ denotes the Fourier transform with respect to $\psi'$ and the corresponding autodual measure. The following result is [Ben3, Theorem 3.4.1].

**Theorem 2.16.2** For every $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$ and $s \in \mathcal{H}$ with $\Re(s) < 1$, we have

$$Z(1-s, \widehat{W}, \hat{\phi}) = \omega_\pi(\tau)^{n-1} |\tau|^{\frac{n(n-1)}{2}(s-1/2)} \lambda_{E/F}(\psi')^{-\frac{n(n-1)}{2}} \gamma(s, \pi, As, \psi') Z(s, W, \phi)$$

Recall that we have associated in the last section to any $\sigma \in \text{Temp}(U(n))$ a sign $c_1(\sigma)$ satisfying 2.15.1. We will need the following.

**Lemma 2.16.3** Let $\sigma \in \text{Temp}(U(n))$ and set $\pi = BC_n(\sigma)$. Then, we have

$$Z(1, \widehat{W}, \hat{\phi}) = \phi(0) c_1(\sigma) \beta(W)$$

for all $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$.

**Proof:** Let $W \in \mathcal{W}(\pi, \psi_n)$ and $\phi \in \mathcal{S}(F^n)$. We have

$$Z(1, \widehat{W}, \hat{\phi}) = \int_{\mathcal{P}_n(F) \backslash G_n(F)} \int_{\mathcal{P}_n(F)} \hat{W}(ph) \hat{d}\phi(e_n h) |\det h| dh = \int_{\mathcal{P}_n(F) \backslash G_n(F)} \beta(R(h)\hat{W}) \phi(e_n h) |\det h| dh.$$ 

By 2.15.1 the linear form $W \in \mathcal{W}(\pi, \psi_n) \mapsto \beta(W)$ is invariant by $\mathcal{P}_n(F)$ and its transpose hence by $G_n(F)$. Therefore,

$$Z(1, \widehat{W}, \hat{\phi}) = \beta(\hat{W}) \int_{\mathcal{P}_n(F) \backslash G_n(F)} \hat{\phi}(e_n h) |\det h| dh = \beta(\hat{W}) \int_{F^n} \hat{\phi}(x) dx = \phi(0) \beta(\hat{W})$$

and a new appeal to 2.15.1 ends the proof of the lemma. ■

3 Computation of certain spectral distributions

3.1 Principal values and poles of certain distributions

Set

$$PV(\int_{i\mathbb{R}} \frac{\varphi(x)}{x} dx) := \lim_{\varepsilon \to 0^+} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x} dx$$

for every $\varphi \in \mathcal{S}(i\mathbb{R})$ (the limits is well-known, and actually easily seen, to exist). We record the following standard formula (see [GS, Eq. 4.4(6)])

$$\lim_{s \to 0^+} \int_{i\mathbb{R}} \frac{\varphi(x)}{x + s} dx = PV(\int_{i\mathbb{R}} \frac{\varphi(x)}{x} dx) + \pi \varphi(0), \quad \varphi \in \mathcal{S}(i\mathbb{R}).$$

The next two results will be needed in the proof of Proposition 3.3.1.
Lemma 3.1.1  Let $W \subset V$ be real vector spaces and $\lambda_1, \ldots, \lambda_k$ be (real) linear forms on $V$ whose restrictions to $W$ are linearly independent. Fix Haar measures on $iW$ and $iV$. Let $\varphi \in \mathcal{S}(iV)$ and set

$$\varphi_s(v) = \int_{iW} \frac{\varphi(v + w)}{\prod_{i=1}^k (\lambda_i(v + w) + s)} dw$$

for all $v \in iV/iW$ and $s \in \mathcal{H}$. Then, $\varphi_s \in \mathcal{S}(iV/iW)$ for all $s \in \mathcal{H}$ and the map $s \mapsto \varphi_s$ extends (uniquely) to a continuous map $s \in \mathcal{H} \cup \{0\} \mapsto \varphi_s \in \mathcal{S}(iV/iW)$. Moreover, if the function $v \mapsto \frac{\varphi(v)}{\prod_{i=1}^k \lambda_i(v)}$ belongs to $\mathcal{S}(iV)$ we have

$$\varphi_0(0) = \int_{iW} \frac{\varphi(w)}{\prod_{i=1}^k (\lambda_i(w))} dw.$$ 

Proof: Clearly, $\varphi_s \in \mathcal{S}(iV/iW)$ and the linear map $\varphi \in \mathcal{S}(iV) \mapsto \varphi_s \in \mathcal{S}(iV/iW)$ is continuous for every $s \in \mathcal{H}$. Let $X$ be a complement subspace of $W$ in $V$ on which $\lambda_1, \ldots, \lambda_k$ are trivial. As $\mathcal{S}(iV) = \mathcal{S}(iX) \otimes \mathcal{S}(iW)$ (see 2.4.1), by 2.3.2 for the first part of the lemma it suffices to show that for every $\varphi \in \mathcal{S}(iW)$ the limit

$$\lim_{s \to 0^+} \int_{iW} \frac{\varphi(w)}{\prod_{i=1}^k (\lambda_i(w) + s)} dw$$

exists. For this, without loss of generality, we may assume that $W = \mathbb{R}^n$ and $\lambda_i(x_1, \ldots, x_n) = x_i$ for every $1 \leq i \leq k$. Then, as $\mathcal{S}(i\mathbb{R}^n) \simeq \bigotimes_{1 \leq j \leq n} \mathcal{S}(i\mathbb{R})$, by 2.3.2 again we may further assume that $\varphi(x_1, \ldots, x_n) = \varphi_1(x_1) \ldots \varphi_n(x_n)$ for some $\varphi_1, \ldots, \varphi_n \in \mathcal{S}(i\mathbb{R})$. We are then reduced to the case $n = k = 1$ where the limit exists by 3.1.1. The second part of the lemma is an easy consequence of dominated convergence. □

Proposition 3.1.2  Let $n \geq 1$ and $h_1, \ldots, h_n > 0$. Let $(i\mathbb{R}^n)_0$ be the subspace of vectors $(x_1, \ldots, x_n) \in i\mathbb{R}^n$ with $x_1 + \ldots + x_n = 0$ and equip it with the unique Haar measure such that if $i\mathbb{R}^n$ is endowed with the Lebesgue measure then the quotient measure on $i\mathbb{R}^n/(i\mathbb{R}^n)_0 \simeq i\mathbb{R}$ (where the isomorphism is given by $(x_1, \ldots, x_n) \mapsto x_1 + \ldots + x_n$) is the Lebesgue measure on $i\mathbb{R}$. Then, we have

$$\lim_{s \to 0^+} \int_{(i\mathbb{R}^n)_0} \frac{\varphi(x)}{\prod_{i=1}^n (\frac{x_i}{h_i} + s)} d\tilde{x} = \frac{\prod_{i=1}^n h_i}{\sum_{i=1}^n h_i} (2\pi)^{n-1} \varphi(0)$$

for every $\varphi \in \mathcal{S}(i\mathbb{R}^n)$. 

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Proof: First, since
\[
\int_{(\mathbb{R}^n)_0} \frac{\varphi(x)}{\prod_{i=1}^n (x_i + s_i)} dx = \prod_{i=1}^n h_i \int_{(\mathbb{R}^n)_0} \frac{\varphi(x)}{\prod_{i=1}^n (x_i + h_i s_i)} dx
\]
for every \(s \in \mathcal{H}\), it suffices to prove
\[
(3.1.2) \quad \lim_{s \to 0^+} \int_{\mathbb{R}^n} \frac{\varphi(x)}{(x + h_1 s)(-x + h_2 s)} dx = \frac{2\pi}{h_1 + h_2} \varphi(0,0), \quad \varphi \in \mathcal{S}(i\mathbb{R}^2).
\]

Fix \(\varphi \in \mathcal{S}(i\mathbb{R}^2)\). Then,
\[
(h_1 + h_2) \int_{\mathbb{R}^n} \frac{\varphi(x, -x)}{(x + h_1 s)(-x + h_2 s)} dx = \int_{\mathbb{R}^n} \frac{\varphi(x, -x)}{x + h_1 s} dx + \int_{\mathbb{R}^n} \frac{\varphi(x, -x)}{-x + h_2 s} dx
\]
for every \(s \in \mathcal{H}\) and by \(3.1.1\)
\[
\lim_{s \to 0^+} \int_{\mathbb{R}^n} \frac{\varphi(x, -x)}{x + h_1 s} dx = PV \left( \int_{\mathbb{R}^n} \frac{\varphi(x, -x)}{x} dx \right) + \pi \varphi(0,0),
\]
\[
\lim_{s \to 0^+} \int_{\mathbb{R}^n} \frac{\varphi(x, -x)}{-x + h_2 s} dx = PV \left( \int_{\mathbb{R}^n} \frac{\varphi(x, -x)}{x} dx \right) + \pi \varphi(0,0)
\]
\[
= -PV \left( \int_{\mathbb{R}^n} \frac{\varphi(x, -x)}{x} dx \right) + \pi \varphi(0,0).
\]
The equality \(3.1.3\) follows.

We now treat the general case, assuming the result is known for \(n - 1\). Since \(\mathcal{S}(i\mathbb{R}^n) = \mathcal{S}(i\mathbb{R}^{n-1}) \otimes \mathcal{S}(i\mathbb{R})\), by \(2.3.2\) we may assume that there exist \(\varphi' \in \mathcal{S}(i\mathbb{R}^{n-1})\) and \(\varphi'' \in \mathcal{S}(i\mathbb{R})\) such that
\[
\varphi(x_1, \ldots, x_n) = \varphi'(x_1, \ldots, x_{n-1}) \varphi''(x_n), \quad (x_1, \ldots, x_n) \in i\mathbb{R}^n
\]
For all \(a \in i\mathbb{R}\), set
\[
(i\mathbb{R}^{n-1})_a := \{ (x_1, \ldots, x_{n-1}) \in i\mathbb{R}^{n-1} \mid x_1 + \ldots + x_{n-1} = a \}.
\]
We equip this affine subspace with the invariant measure transferred from the one on \((i\mathbb{R}^{n-1})_0\). Then, we have
\[
(3.1.4) \quad \int_{(i\mathbb{R}^n)_0} \frac{\varphi(x)}{\prod_{i=1}^n (x_i + h_i s_i)} dx = \int_{i\mathbb{R}} \int_{(i\mathbb{R}^{n-1})_a} \frac{\varphi'(x)}{\prod_{i=1}^{n-1} (x_i + h_i s_i)} dx \frac{\varphi''(-a)}{-a + h_n s} da
\]
for every \( s \in \mathcal{H} \). Set
\[
\varphi_s'(a) := (a + (\sum_{i=1}^{n-1} h_i)s) \int_{(\mathbb{R}^{n-1})_a} \frac{\varphi'(x)}{\prod_{i=1}^{n-1} (x_i + h_i s)} \, d\mathbf{x}
\]
for every \( a \in \mathbb{R} \) and \( s \in \mathcal{H} \). Noticing that
\[
\varphi_s'(a) = \sum_{i=1}^{n-1} \int_{(\mathbb{R}^{n-1})_a} \frac{\varphi'(x)}{\prod_{1 \leq j \leq n-1, j \neq i} (x_i + h_i s)} \, d\mathbf{x}
\]
and since for all \( 1 \leq i \leq n-1 \) the linear forms \( x \mapsto x_i \) for \( 1 \leq j \leq n-1 \) and \( j \neq i \) have linearly independent restrictions to \( \mathbb{R}^{n-1} \), by Lemma 3.1.1 we see that the family \( s \mapsto \varphi_s' \) extends to a continuous map \( s \in \mathcal{H} \cup \{0\} \mapsto \varphi_s' \in S(\mathbb{R}) \). Moreover, by the induction hypothesis we have \( \varphi_0'(0) = (2\pi)^{n-2} \varphi'(0) \). By 3.1.4 we have
\[
\int_{(\mathbb{R}^{n-1})_0} \frac{\varphi'(x)}{\prod_{i=1}^{n-1} (x_i + h_i s)} \, d\mathbf{x} = \int_{\mathbb{R}^n} \frac{\varphi_s'(a)\varphi''(-a)}{(a + (\sum_{i=1}^{n-1} h_i)s)(-a + h_n s)} \, da.
\]
Finally, by the \( n = 2 \) case that we already treated and the uniform boundedness principle, we get
\[
\lim_{s \to 0^+} s \int_{\mathbb{R}} \frac{\varphi_s'(a)\varphi''(-a)}{(a + (\sum_{i=1}^{n-1} h_i)s)(-a + h_n s)} \, da = \frac{2\pi}{\sum_{i=1}^{n} h_i} \varphi_0'(0)\varphi''(0) = \frac{(2\pi)^{n-1}}{\sum_{i=1}^{m} h_i} \varphi'(0)\varphi''(0)
\]
As \( \varphi'(0)\varphi''(0) = \varphi(0) \), this finishes the proof of 3.1.2 and thus of the proposition by induction.

\[\square\]

### 3.2 Some polynomial identities

**Proposition 3.2.1** (i) Let \( m \geq n \) be two nonnegative integers and define \( P_{m,n}, S_{m,n} \in \mathbb{Q}(X_1, \ldots, X_m, X_1^*, \ldots, X_n^*) \) by
\[
P_{m,n} := \prod_{1 \leq j < k \leq m} \frac{(X_j - X_k)}{\prod_{1 \leq j, k \leq n} (X_j + X_k^*)} \prod_{1 \leq j \leq m, 1 \leq k \leq n} (X_j - X_k)
\]
and
\[
S_{m,n} := \prod_{1 \leq j < k \leq n} \frac{(X_j^* - X_k^*)}{\prod_{1 \leq j \leq m, 1 \leq k \leq n} (X_j + X_k^*)} \prod_{1 \leq j \leq m, n+1 \leq k < j \leq m} (X_j - X_k)
\]

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Let \((w, P) \mapsto w \cdot P\) denote the natural action of \(W = \mathfrak{S}_m \times \mathfrak{S}_n\) on \(\mathbb{Q}(X_1, \ldots, X_m, X_1^*, \ldots, X_n^*)\) where \(\mathfrak{S}_m\) acts by permutation of the variables \(X_1, \ldots, X_m\) whereas \(\mathfrak{S}_n\) acts by permutation of the variables \(X_1^*, \ldots, X_n^*\). Let \(W' := \mathfrak{S}_{m-n} \times \mathfrak{S}_n^{\text{diag}}\) be the subgroup of elements \((\sigma_1, \sigma_2) \in W\) such that \(\sigma_2^{-1}\sigma_1\) fixes \(\{1, \ldots, n\}\) point-wise. Then, we have the identity

\[
P_{m,n} = \frac{1}{|W'|} \sum_{w \in W} w \cdot S_{m,n}.
\]

Moreover, set

\[
P_{m,n}^* := \left( \prod_{1 \leq k \leq n} (X_k + X_k^*) \right) P_{m,n}, \quad S_{m,n}^* := \left( \prod_{1 \leq k \leq n} (X_k + X_k^*) \right) S_{m,n}
\]

and

\[
P_{m,n,s}(\underline{x}) := P_{m,n}(x_1 + \frac{s}{2}, \ldots, x_m + \frac{s}{2}, x_1^* + \frac{s}{2}, \ldots, x_n^* + \frac{s}{2}) \quad \text{and} \quad S_{m,n,s}(\underline{x}) := S_{m,n}(x_1 + \frac{s}{2}, \ldots, x_m + \frac{s}{2}, x_1^* + \frac{s}{2}, \ldots, x_n^* + \frac{s}{2})
\]

for every \(s \in \mathcal{H}\) and \(\underline{x} = (x_1, \ldots, x_m, x_1^*, \ldots, x_n^*) \in i\mathbb{R}^m \times i\mathbb{R}^n\). Finally, let \(V\) be the subspace \(\{\underline{x} \in i\mathbb{R}^m \times i\mathbb{R}^n \mid x_k = -x_k^*, \forall 1 \leq k \leq n\}\) of \(i\mathbb{R}^m \times i\mathbb{R}^n\). Then, the function \(\underline{x} \in V \mapsto P_{m,n}^*(\underline{x})\) (which is a priori only well-defined on a Zariski open subset) extends to a polynomial function on \(V\) and we have

\[
\lim_{s \to 0^+} s^n P_{m,n,s}(\underline{x}) = \lim_{s \to 0^+} s^n S_{m,n,s}(\underline{x}) = P_{m,n}^*(\underline{x}) = S_{m,n}^*(\underline{x})
\]

for almost all \(\underline{x} \in V\).

(ii) Let \(p\) be a positive integer and define \(Q_p, T_p \in \mathbb{Q}(Y_1, \ldots, Y_p)\) by

\[
Q_p := \frac{\prod_{1 \leq j \neq k \leq p} (Y_j - Y_k)}{\prod_{1 \leq j < k \leq p} (Y_j + Y_k)}
\]

and

\[
T_p := (-1)^{\varepsilon} \frac{\prod_{1 \leq j < k \leq p} (Y_j - Y_k) \prod_{1 \leq k \leq \left[\frac{p}{2}\right]} (Y_{p+1-k} - Y_k)}{\prod_{1 \leq k \leq \left[\frac{p}{2}\right]} (Y_k + Y_{p+1-k})}
\]

where \(\varepsilon := \frac{1}{2} \left( \frac{p(p-1)}{2} - \left\lfloor \frac{p}{2} \right\rfloor \right)\). Let \((w, P) \mapsto w \cdot P\) denote the natural action of \(W := \mathfrak{S}_p\) on \(\mathbb{Q}(Y_1, \ldots, Y_p)\) and \(W' := \mathfrak{S}_{\left[\frac{p}{2}\right]} \times (\mathbb{Z}/2\mathbb{Z})^{\left\lfloor \frac{p}{2} \right\rfloor}\) be the subgroup of \(W\) preserving the partition \(\{\ell, p+1-\ell\mid 1 \leq \ell \leq \left[\frac{p}{2}\right]\}\) of \(\{1, \ldots, p\}\). Then, we have the identity

\[
(3.2.3) \quad Q_p = \frac{1}{|W'|} \sum_{w \in W} w \cdot T_p.
\]
Moreover, set

\[ Q_p^* := \left( \prod_{1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor} (Y_k + Y_{p+1-k}) \right) Q_p, \quad T_p^* := \left( \prod_{1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor} (Y_k + Y_{p+1-k}) \right) T_p \]

and

\[ Q_{p,s}(y) := Q_p(y_1 + \frac{s}{2}, \ldots, y_p + \frac{s}{2}), \quad T_{p,s}(y) := T_p(y_1 + \frac{s}{2}, \ldots, y_p + \frac{s}{2}) \]

for every \( s \in \mathcal{H} \) and \( y = (y_1, \ldots, y_p) \in \mathbb{Z}^p \). Finally, let \( V \) be the subspace \( \{ y \in \mathbb{Z}^p \mid y_k = -y_{p+1-k}, \; \forall 1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor \} \) of \( \mathbb{Z}^p \). Then, the function \( y \in V \mapsto Q_p^*(y) \) (which is a priori only well-defined on a Zariski open subset) extends to a polynomial function on \( V \) and we have

\[ \lim_{s \to 0^+} s^{|\frac{q}{2}|} Q_{p,s}(y) = \lim_{s \to 0^+} s^{|\frac{q}{2}|} T_{p,s}(y) = Q_p^*(y) = T_p^*(y) \]

for almost all \( y \in V \).

**(iii)** Let \( q \) be a positive integer and define \( R_q, U_q \in \mathbb{Q}(Z_1, \ldots, Z_q) \) by

\[ R_q := \prod_{1 \leq j \neq k \leq q} \frac{(Z_j - Z_k)}{(Z_j + Z_k)} \]

and

\[ U_q := (-1)^\epsilon \prod_{1 \leq j < k \leq q} \frac{(Z_j - Z_k)}{\prod_{1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor} (Z_k + Z_{q+1-k}) \prod_{1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor} 2Z_k} \]

where \( \epsilon := \frac{1}{2} \left( (\frac{q(q-1)}{2} - \frac{q}{2}) \right) \). Let \( (w, P) \mapsto w \cdot P \) denote the natural action of \( \mathbb{Z} \) on \( \mathbb{Q}(Z_1, \ldots, Z_q) \) and \( \mathbb{Z}^* := \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^{\frac{q}{2}} \) be the subgroup of \( \mathbb{Z} \) preserving the partition \( \{ \ell, q + 1 - \ell \mid 1 \leq \ell \leq \left\lfloor \frac{q}{2} \right\rfloor \} \) of \( \{ 1, \ldots, q \} \). Then, we have the identity

\[ R_q = \frac{1}{|W'|} \sum_{w \in W} w \cdot U_q. \]

Moreover, set

\[ R_q^* := \left( \prod_{1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor} (Z_k + Z_{q+1-k}) \right) R_q, \quad U_q^* := \left( \prod_{1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor} (Z_k + Z_{q+1-k}) \right) U_q \]

and

\[ R_{q,s}(z) := R_q(z_1 + \frac{s}{2}, \ldots, z_q + \frac{s}{2}), \quad U_{q,s}(z) := U_q(z_1 + \frac{s}{2}, \ldots, z_q + \frac{s}{2}) \]
for every $s \in \mathcal{H}$ and $\mathbf{z} = (z_1, \ldots, z_q) \in i\mathbb{R}^q$. Finally, let $V$ be the subspace $\{ \mathbf{z} \in i\mathbb{R}^q \mid z_k = -z_{q+1-k}, \ \forall \ 1 \leq k \leq \left\lfloor \frac{q}{2} \right\rfloor \}$ of $i\mathbb{R}^q$. Then, the function $\mathbf{z} \in V \mapsto R_q^*(\mathbf{z})$ and $\mathbf{z} \in V \mapsto U_q^*(\mathbf{z})$ (which are a priori only well-defined on a Zariski open subset) extends to polynomial functions on $V$ and we have

$$\lim_{s \to 0^+} s^{|\mathbf{z}^2|} R_{q,s}(\mathbf{z}) = \lim_{s \to 0^+} s^{|\mathbf{z}^2|} U_{q,s}(\mathbf{z}) = R_q^*(\mathbf{z}) = U_q^*(\mathbf{z})$$

for almost all $\mathbf{z} \in V$.

**Proof:**

(i) Set

$$\Delta := \prod_{1 \leq j < k \leq m} (X_j - X_k), \quad \Delta^* := \prod_{1 \leq j < k \leq n} (X_j^* - X_k^*)$$

and $\text{sgn} : W \to \{\pm 1\}$ be the product of the sign characters on $\mathfrak{S}_m$ and $\mathfrak{S}_n$. Then,

$$w \cdot (\Delta \Delta^*) = \text{sgn}(w) \Delta \Delta^*$$

for every $w \in W$. Hence,

$$\sum_{w \in W} w \cdot S_{m,n} = \Delta \Delta^* \sum_{w \in W} \text{sgn}(w) w \cdot \left( \prod_{1 \leq k \leq n} (X_k + X_k^*) \frac{\prod_{1 \leq k \leq n} (X_j - X_k)}{\prod_{1 \leq k \leq n} (X_j^* - X_k^*)} \right)$$

and $\Delta \Delta^* \prod_{1 \leq j \leq m} (X_j + X_j^*)$ where $Q \in \mathbb{Q}[X_1, \ldots, X_m, X_1^*, \ldots, X_n^*]$ is homogeneous of total degree

$$\frac{(m-n)(m-n-1)}{2} - n + mn = \frac{m(m-1)}{2} + \frac{n(n-1)}{2} = \deg(\Delta) + \deg(\Delta^*)$$

and satisfies $w \cdot Q = \text{sgn}(w) Q$ for every $w \in W$. It follows that $Q$ is a scalar multiple of $\Delta \Delta^*$ and therefore there exists $c \in \mathbb{Q}$ such that

$$cP_{m,n} = \frac{1}{|W'|} \sum_{w \in W} w \cdot S_{m,n}$$

Now, noticing that $W'$ is the stabilizer of $S_{m,n}$ in $W$ and that for every $w \in W \setminus W'$ the restriction of

$$\left( \prod_{1 \leq k \leq n} (X_k + X_k^*) \right) w \cdot S_{m,n}$$

to $V$ vanishes almost everywhere, we also have

$$cP_{m,n}(\mathbf{z}) = S_{m,n}(\mathbf{z})$$
for almost all $\underline{x} \in V$. On the other hand, it is clear that
\[
\lim_{s \to 0^+} s^n P_{m,n,s}(\underline{x}) = P^*_{m,n}(\underline{x}) \quad \text{and} \quad \lim_{s \to 0^+} s^n S_{m,n,s}(\underline{x}) = S^*_{m,n}(\underline{x})
\]
for almost all $\underline{x} \in V$. Hence, to get (3.2.1) and (3.2.2) it only remains to prove that $P^*_{m,n}(\underline{x}) = S^*_{m,n}(\underline{x})$ almost everywhere on $V$. For a generic point $\underline{x} = (x_1, \ldots, x_m, x^*_1, \ldots, x^*_n) \in V$, we have
\[
P^*_{m,n}(\underline{x}) = S^*_{m,n}(\underline{x}) - \frac{\prod_{1 \leq j < k \leq n} (x^*_k - x^*_j) \prod_{1 \leq j < k \leq n} (x_j - x_k) \prod_{1 \leq k \leq n} (x_j + x^*_k)}{\prod_{1 \leq j \leq m} \prod_{1 \leq k \leq n, j \neq k} (x^*_k + x_j) \prod_{1 \leq j \leq m} \prod_{1 \leq k \leq n} (x^*_j + x^*_k) \prod_{1 \leq j \leq m} \prod_{1 \leq k \leq n, j \neq k} (x^*_j + x^*_k)\prod_{1 \leq j \leq m} \prod_{1 \leq k \leq n} (x_j + x^*_k)}
\]
and the claim follows.

(ii) This time we set $\Delta := \prod_{1 \leq j < k \leq p} (Y_j - Y_k)$ and let $\text{sgn} : W \to \{\pm 1\}$ be the sign character. Then, $w \cdot \Delta = \text{sgn}(w)\Delta$ for every $w \in W$. Therefore,
\[
\sum_{w \in W} w \cdot T_p = (-1)^p \Delta \sum_{w \in W} \text{sgn}(w)w \cdot \left( \frac{\prod_{1 \leq k \leq \lfloor \frac{p}{2} \rfloor} (Y_{p+1-k} - Y_k) \prod_{1 \leq k \leq \lfloor \frac{p}{2} \rfloor} (Y_k + Y_{p+1-k})}{\prod_{1 \leq k \leq \lfloor \frac{p}{2} \rfloor} (Y_k + Y_{p+1-k})} \right)
\]
can be written as $\Delta \frac{P}{(Y_j + Y_k)}$ where $P \in \mathbb{Q}[Y_1, \ldots, Y_p]$ is homogeneous of degree $\left\lfloor \frac{p}{2} \right\rfloor + \frac{p(p-1)}{2} = \deg(\Delta)$ and satisfies $w \cdot P = \text{sgn}(w)P$ for every $w \in W$. It follows that $P$ is a scalar multiple of $\Delta$ and thus there exists $c \in \mathbb{Q}$ such that
\[
cQ_p = \frac{1}{|W'|} \sum_{w \in W} w \cdot T_p
\]
Noticing that $W'$ is the stabilizer of $T_p$ in $W$ and that for all $w \in W \setminus W'$ the restriction of
\[
\left( \prod_{1 \leq k \leq \lfloor \frac{p}{2} \rfloor} (Y_k + Y_{p+1-k}) \right) w \cdot T_p
\]
to $V$ vanishes almost everywhere, we also have
\[ cQ_p^*(y) = T_p^*(y) \]
for almost all $y \in V$. On the other hand, it is clear that
\[ \lim_{s \to 0^+} s^{\frac{\theta}{2}} |Q_{p,s}(y)| = Q_p^*(y) \] and \[ \lim_{s \to 0^+} s^{\frac{\theta}{2}} |T_{p,s}(y)| = T_p^*(y) \]
for almost all $y \in V$. Hence, to get 3.2.3 and 3.2.4 it only remains to prove that $Q_p^*(y) = T_p^*(y)$ almost everywhere on $V$. For a generic point $y = (y_1, \ldots, y_p) \in V$, we have
\[
Q_p^*(y) = (-1)^{\epsilon} T_p^*(y) \frac{\prod_{1 \leq k < j \leq p} (y_j - y_k) \prod_{j > p+1-k} (y_j + y_k)}{\prod_{1 \leq j < k \leq p \atop k \neq p+1-j} (y_j + y_k) \prod_{1 \leq j < k \leq p \atop k > p+1-j} (y_j - y_k) \prod_{j > p+1-k} (y_j + y_k)}
\]
and the claim follows.

(iii) Notice that
\[
R_q(Z_1, \ldots, Z_q) = \frac{Q_q(Z_1, \ldots, Z_q)}{\prod_{1 \leq j \leq q} 2Z_j}
\]
Therefore, as $\prod_{1 \leq j \leq q} 2Z_j$ is $W$-invariant, by (ii) we have
\[
R_q(Z_1, \ldots, Z_q) = \frac{1}{|W'|} \sum_{w \in W} w \cdot \left( \frac{T_q(Z_1, \ldots, Z_q)}{\prod_{1 \leq j \leq q} 2Z_j} \right)
\]
Moreover,
\[
\frac{T_q(Z_1, \ldots, Z_q)}{\prod_{1 \leq j \leq q} 2Z_j} = (-1)^{\epsilon} \left( \prod_{1 \leq j \leq \left\lfloor \frac{q}{2} \right\rfloor} \frac{Z_{q+1-j} - Z_j}{4Z_j Z_{q+1-j}} \right) \times \prod_{1 \leq j \leq \left\lceil \frac{q}{2} \right\rceil} \Delta \left( Z_j + Z_{q+1-j} \right)
\]
where we have set \( \Delta := \prod_{1 \leq j < k \leq q} (Z_j - Z_k) \) and

\[
(3.2.10) \quad \prod_{1 \leq j \leq \lfloor \frac{q}{2} \rfloor} \frac{Z_{q+1-j} - Z_j}{4Z_j Z_{q+1-j}} = 2^{-\lfloor \frac{q}{2} \rfloor} \prod_{1 \leq j \leq \lfloor \frac{q}{2} \rfloor} \left( \frac{1}{2Z_j} - s_j \cdot \frac{1}{2Z_j} \right)
\]

\[
= 2^{-\lfloor \frac{q}{2} \rfloor} \sum_{s \in (\mathbb{Z}/2\mathbb{Z}) \lfloor \frac{q}{2} \rfloor} \text{sgn}(s) s \cdot \left( \prod_{1 \leq j \leq \lfloor \frac{q}{2} \rfloor} \frac{1}{2Z_j} \right)
\]

where \( s_j \) denotes the transposition \((j, q+1-j)\), we have identified \((\mathbb{Z}/2\mathbb{Z}) \lfloor \frac{q}{2} \rfloor\) with the subgroup of \( W \) generated by the \( s_j \) for \( 1 \leq j \leq \lfloor \frac{q}{2} \rfloor \) and \( \text{sgn} : W \to \{ \pm 1 \} \) denotes the sign character. Since

\[
s \cdot \left( \prod_{1 \leq j \leq \lfloor \frac{q}{2} \rfloor} \frac{\Delta}{(Z_j + Z_{q+1-j})} \right) = \text{sgn}(s) \prod_{1 \leq j \leq \lfloor \frac{q}{2} \rfloor} \frac{\Delta}{(Z_j + Z_{q+1-j})}
\]

for every \( s \in (\mathbb{Z}/2\mathbb{Z}) \lfloor \frac{q}{2} \rfloor \), we obtain from \(3.2.9\) and \(3.2.10\) that

\[
\frac{T_q(Z_1, \ldots, Z_q)}{\prod_{1 \leq j \leq q} 2Z_j} = 2^{-\lfloor \frac{q}{2} \rfloor} \sum_{s \in (\mathbb{Z}/2\mathbb{Z}) \lfloor \frac{q}{2} \rfloor} (-1)^s \cdot s \cdot \left( \sum_{s \in (\mathbb{Z}/2\mathbb{Z}) \lfloor \frac{q}{2} \rfloor} \frac{\Delta}{(Z_j + Z_{q+1-j})} \prod_{1 \leq j \leq \lfloor \frac{q}{2} \rfloor} \frac{1}{2Z_j} \right)
\]

\[
= 2^{-\lfloor \frac{q}{2} \rfloor} \sum_{s \in (\mathbb{Z}/2\mathbb{Z}) \lfloor \frac{q}{2} \rfloor} s \cdot U_q
\]

Hence, by \(3.2.8\)

\[
R_q(Z_1, \ldots, Z_q) = \frac{2^{-\lfloor \frac{q}{2} \rfloor}}{|W|} \sum_{w \in W} \sum_{s \in (\mathbb{Z}/2\mathbb{Z}) \lfloor \frac{q}{2} \rfloor} w s \cdot U_q = \frac{1}{|W|} \sum_{w \in W} w \cdot U_q
\]

and this proves \(3.2.5\).

Once again, it is obvious that

\[
\lim_{s \to 0^+} s^{\lfloor \frac{q}{2} \rfloor} R_q(s(Z)) = R_q^*(Z) \quad \text{and} \quad \lim_{s \to 0^+} s^{\lfloor \frac{q}{2} \rfloor} U_q(s(Z)) = U_q^*(Z)
\]

for almost all \( Z \in V \). By \(3.2.7\) we get

\[
R_q^*(Z_1, \ldots, Z_q) = \frac{Q_q^*(Z_1, \ldots, Z_q)}{\prod_{1 \leq j \leq \lfloor \frac{q}{2} \rfloor} 4Z_j Z_{q+1-j}}
\]

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Similarly, we have

$$T_q^*(Z_1, \ldots, Z_q) = \left( \prod_{1 \leq j \leq \frac{q}{2}} 2Z_j(Z_{q+1-j} - Z_j) \right) U_q^*(Z_1, \ldots, Z_q)$$

Therefore, by (ii) we obtain

$$R_q^*(Z) = \left( \prod_{1 \leq j \leq \frac{q}{2}} \frac{2z_j(z_{q+1-j} - z_j)}{4z_jz_{q+1-j}} \right) U_q^*(Z) = U_q^*(Z)$$

for almost all $z \in V$ and $3.2.6$ follows. We show similarly that

$$U_q^*(Z) = (-1)^e \prod_{1 \leq j < k \leq q \atop k \neq q+1-j} (z_j - z_k)$$

for almost all $z \in V$ and consequently $U_q^*(Z)$ extends to a polynomial function on $V$. ■

### 3.3 Explicit computation of certain residual distributions

In this section, we fix the following data:

- $r, s, t \in \mathbb{N}$ three nonnegative integers;
- $(m_i, n_i) \in \mathbb{N}^* \times \mathbb{N}$ such that $m_i \geq n_i$ and $d_i \in \mathbb{N}^*$ for all $1 \leq i \leq r$;
- $p_j \in \mathbb{N}^*$ and $e_j \in \mathbb{N}^*$ for all $1 \leq j \leq s$;
- $q_k \in \mathbb{N}^*$ and $f_k \in \mathbb{N}^*$ for all $1 \leq k \leq t$;

and we set

$$\mathcal{A} := \prod_{1 \leq i \leq r} (i\mathbb{R})^{m_i} \times (i\mathbb{R})^{n_i} \times \prod_{1 \leq j \leq s} (i\mathbb{R})^{p_j} \times \prod_{1 \leq k \leq t} (i\mathbb{R})^{q_k}.$$  

For any $\lambda \in \mathcal{A}$, we will write $\underline{x}_i(\lambda)$ ($1 \leq i \leq r$), $\underline{y}_j(\lambda)$ ($1 \leq j \leq s$) and $\underline{z}_k(\lambda)$ ($1 \leq k \leq t$) for the projections of $\lambda$ onto $(i\mathbb{R})^{m_i} \times (i\mathbb{R})^{n_i}$, $(i\mathbb{R})^{p_j}$ and $(i\mathbb{R})^{q_k}$ respectively. We will further write the coordinate of these vectors as follows:

$$\underline{x}_i(\lambda) = (x_{i,1}(\lambda), \ldots, x_{i,m_i}(\lambda), x_{i,1}^*(\lambda), \ldots, x_{i,n_i}(\lambda)) \in (i\mathbb{R})^{m_i} \times (i\mathbb{R})^{n_i}.$$
and we equip $A$ with the product of the Lebesgue measure on $i\mathbb{R}$. Let $\Sigma : A \to i\mathbb{R}$ be the linear form given by

$$\Sigma(\lambda) = \sum_{i=1}^{r} \left( \sum_{\ell=1}^{m_i} x_{i,\ell}(\lambda) + \sum_{\ell=1}^{n_i} x_{i,\ell}^*(\lambda) \right) + \sum_{j=1}^{s} \sum_{\ell=1}^{p_j} y_{j,\ell}(\lambda) + \sum_{k=1}^{t} \sum_{\ell=1}^{q_k} z_{k,\ell}(\lambda)$$

and $A_0 := \text{Ker}(\Sigma)$. We equip $A_0$ with the unique Haar measure such that the quotient measure on $A/A_0 \simeq i\mathbb{R}$ (the isomorphism being induced by $\Sigma$) is the Lebesgue measure.

For all $1 \leq i \leq r$ (resp. $1 \leq i \leq r$, $1 \leq j \leq s$ and $1 \leq k \leq t$), we let $\mathcal{S}_{m_i}$ (resp. $\mathcal{S}_{n_i}$, $\mathcal{S}_{p_j}$ and $\mathcal{S}_{q_k}$) act on $(i\mathbb{R})^{m_i}$ (resp. $(i\mathbb{R})^{n_i}$, $(i\mathbb{R})^{p_j}$ and $(i\mathbb{R})^{q_k}$) by permutation of the coordinates and we set

$$W := \prod_{i=1}^{r} (\mathcal{S}_{m_i} \times \mathcal{S}_{n_i}) \times \prod_{j=1}^{s} \mathcal{S}_{p_j} \times \prod_{k=1}^{t} \mathcal{S}_{q_k}$$

Then $W$ acts on $A$ by the product of the previous actions and therefore also on the Schwartz space $S(A)$. Denote by $S(A)^W$ the subspace of $W$-invariant functions. Using the notation of Proposition 3.2.1 for every $s \in \mathcal{H}$ we define a distribution $D_s \in S(A)^W$ by

$$D_s(\varphi) := \int_{A_0} \varphi(\lambda) \prod_{i=1}^{r} P_{m_i,n_i,s}(x_i(\lambda)/d_i) \times \prod_{j=1}^{s} Q_{p_j,s}(y_j(\lambda)/e_j) \times \prod_{k=1}^{t} R_{q_k,s}(z_k(\lambda)/f_k) d\lambda, \quad \varphi \in S(A).$$

Let $A'$ be the subspace of $A$ defined by the relations

- $x_{i,\ell}(\lambda) + x_{i,\ell}^*(\lambda) = 0$ for every $1 \leq i \leq r$ and $1 \leq \ell \leq n_i$;
- $y_{j,\ell}(\lambda) + y_{j,p_j+1-\ell}(\lambda) = 0$ for every $1 \leq j \leq s$ and $1 \leq \ell \leq \lfloor p_j/2 \rfloor$;
- $z_{k,\ell}(\lambda) + z_{k,q_k+1-\ell}(\lambda) = 0$ for every $1 \leq k \leq t$ and $1 \leq \ell \leq \lfloor q_k/2 \rfloor$.

The map $\lambda \mapsto \left( (x_{i,\ell}(\lambda))_{1 \leq i \leq r, 1 \leq \ell \leq n_i}, (y_{j,\ell}(\lambda))_{1 \leq j \leq s, 1 \leq \ell \leq \lfloor p_j/2 \rfloor}, (z_{k,\ell}(\lambda))_{1 \leq k \leq t, 1 \leq \ell \leq \lfloor q_k/2 \rfloor} \right)$ induces an isomorphism

$$A' \simeq \prod_{i=1}^{r} (i\mathbb{R})^{n_i} \times \prod_{j=1}^{s} \left( (i\mathbb{R})^{\lfloor p_j/2 \rfloor} \right) \times \prod_{k=1}^{t} \left( (i\mathbb{R})^{\lfloor q_k/2 \rfloor} \right)$$

and we equip $A'$ with the measure which transfer to the Lebesgue measure via this isomorphism. Define the following subgroup of $W$:

$$W' := \prod_{i=1}^{r} (\mathcal{S}_{n_i}^{\text{diag}} \times \mathcal{S}_{m_i-n_i}) \times \prod_{j=1}^{s} \left( \mathcal{S}_{\lfloor p_j/2 \rfloor} \times (\mathbb{Z}/2\mathbb{Z})^{\lfloor p_j/2 \rfloor} \right) \times \prod_{k=1}^{t} \left( \mathcal{S}_{\lfloor q_k/2 \rfloor} \times (\mathbb{Z}/2\mathbb{Z})^{\lfloor q_k/2 \rfloor} \right)$$
where for all integers \( m \geq n \) and \( p, \mathcal{S}_n^\text{diag} \times \mathcal{S}_{m-n} \) denotes the subgroup of elements \((\sigma, \tau) \in \mathcal{S}_n \times \mathcal{S}_m\) such that \( \tau^{-1} \sigma \) fixes \( \{1, \ldots, n\} \) point-wise and we identify \( \mathcal{S}_n^{\mathbb{Z}/2\mathbb{Z}} \) with the subgroup of \( \mathcal{S}_p \) preserving the partition \( \{ \ell, p+1-\ell \mid 1 \leq \ell \leq \lceil \frac{p}{2} \rceil \} \) of \( \{1, \ldots, p\} \). It is easy to check that \( \mathcal{A}' \) is \( W' \)-invariant. Finally, we define a distribution \( D' \in \mathcal{S}(\mathcal{A}') \) by

\[
D'(\varphi) := \frac{D}{n} (2\pi)^{-1} \int_{\mathcal{A}'} \varphi(\mu) \prod_{i=1}^r P_{m_i,n_i} s^N \prod_{i=1}^r P_{m_i,n_i,\hat{a}}(\frac{X_i(\mu)}{d_i}) \times \prod_{j=1}^s Q_{p_j,s} \frac{Y_j(\mu)}{e_j} \times \prod_{k=1}^t R_{q_k,s} \frac{Z_k(\mu)}{f_k} d\mu
\]

for all \( \varphi \in \mathcal{S}(\mathcal{A}) \), where

- \( n := \sum_{i=1}^r (n_i + m_i) d_i + \sum_{j=1}^s p_j e_j + \sum_{k=1}^t q_k f_k \);
- \( D := \prod_{i=1}^r d_i^{m_i} \times \prod_{j=1}^s e_j^{p_j} \times \prod_{k=1}^t f_k^{q_k} \);
- \( c := |\{1 \leq k \leq t \mid q_k \equiv 1 [2]\}| \);
- \( N := \sum_{i=1}^r n_i + \sum_{j=1}^s \left| \frac{p_j}{2} \right| + \sum_{k=1}^t \left| \frac{q_k}{2} \right| \).

Note that by Proposition 3.2.1 we have

\[
\lim_{s \to 0^+} s^N \prod_{i=1}^r P_{m_i,n_i,\hat{a}}(\frac{X_i(\mu)}{d_i}) \times \prod_{j=1}^s Q_{p_j,s} \frac{Y_j(\mu)}{e_j} \times \prod_{k=1}^t R_{q_k,s} \frac{Z_k(\mu)}{f_k} = 
\prod_{i=1}^r S_{m_i,\hat{a}}(\frac{X_i(\mu)}{d_i}) \times \prod_{j=1}^s T_{p_j}(\frac{Y_j(\mu)}{e_j}) \times \prod_{k=1}^t U_{q_k}(\frac{Z_k(\mu)}{f_k})
\]

for almost all \( \mu \in \mathcal{A}' \) and that this extends to a polynomial function on \( \mathcal{A}' \) so that the distribution \( D' \) is well-defined.

**Proposition 3.3.1** For every \( \varphi \in \mathcal{S}(\mathcal{A})^W \), we have

\[
\lim_{s \to 0^+} s D_s(\varphi) = \begin{cases} \frac{|W|}{|W'|} D'(\varphi) & \text{if } m_i = n_i \text{ for all } 1 \leq i \leq r \text{ and } p_j \text{ is even for all } 1 \leq j \leq s, \\
0 & \text{otherwise.} \end{cases}
\]

**Proof:** Let \( \varphi \in \mathcal{S}(\mathcal{A})^W \). By Proposition 3.2.1, we have

\[
\prod_{i=1}^r P_{m_i,\hat{a}}(\frac{X_i(\lambda)}{d_i}) \times \prod_{j=1}^s Q_{p_j}(\frac{Y_j(\lambda)}{e_j}) \times \prod_{k=1}^t R_{q_k}(\frac{Z_k(\lambda)}{f_k}) = 
\frac{1}{|W'|} \sum_{w \in W} \prod_{i=1}^r S_{m_i,\hat{a}}(\frac{X_i(w\lambda)}{d_i}) \times \prod_{j=1}^s T_{p_j}(\frac{Y_j(w\lambda)}{e_j}) \times \prod_{k=1}^t U_{q_k}(\frac{Z_k(w\lambda)}{f_k})
\]

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for every \( \lambda \in \mathcal{A} \) and \( s \in \mathcal{H} \). Therefore, as \( \varphi \) is \( W \)-invariant, setting
\[
\tilde{D}_s(\varphi) = \int_{\mathcal{A}_0} \varphi(\lambda) \prod_{i=1}^{r} S_{m_i,n_i,s} \left( \frac{X_i(\lambda)}{d_i} \right) \times \prod_{j=1}^{s} T_{p_j,s} \left( \frac{Y_j(\lambda)}{e_j} \right) \times \prod_{k=1}^{t} \frac{\psi_k(\lambda)}{f_k} d\lambda, \quad s \in \mathcal{H}
\]
we need to show that
\[
\lim_{s \to 0^+} s \tilde{D}_s(\varphi) = \begin{cases} D'(\varphi) & \text{if } m_i = n_i \text{ for all } 1 \leq i \leq r \text{ and } p_j \text{ is even for all } 1 \leq j \leq s \\ 0 & \text{otherwise.} \end{cases}
\]
Set
\[
\psi(\lambda) = \varphi(\lambda) \prod_{i=1}^{r} S_{m_i,n_i} \left( \frac{X_i(\lambda)}{d_i} \right) \prod_{j=1}^{s} T_{p_j} \left( \frac{Y_j(\lambda)}{e_j} \right) \prod_{k=1}^{t} \frac{\psi_k(\lambda)}{f_k} \left( \prod_{1 \leq \ell \leq \left[ \frac{n_k}{2} \right]} \frac{2z_{k,\ell}(\lambda)}{f_k} \right), \quad \lambda \in \mathcal{A}
\]
Notice that \( \psi \) being the product of \( \varphi \) with a polynomial function on \( \mathcal{A} \) we have \( \psi \in \mathcal{S}(\mathcal{A}) \). Moreover,
\[
\tilde{D}_s(\varphi) = \int_{\mathcal{A}_0} \psi(\lambda) \prod_{i=1}^{r} \prod_{1 \leq \ell \leq n_i} \left( \frac{x_{i,\ell}(\lambda) + x_{i,\ell}(\lambda)}{d_i} + s \right)^{-1} \prod_{j=1}^{s} \frac{y_{j,p_j+1-\ell}(\lambda)}{e_j} + s \right)^{-1} \prod_{1 \leq \ell \leq \left[ \frac{n_k}{2} \right]} \frac{2z_{k,\ell}(\lambda)}{f_k} + s \right)^{-1} \prod_{1 \leq \ell \leq \left[ \frac{n_k}{2} \right]} \frac{2z_{k,\ell}(\lambda)}{f_k} + s \right)^{-1} d\lambda.
\]
Let \( \mathcal{A}_0' := \mathcal{A}' \cap \mathcal{A}_0 \). Fixing a Haar measure on \( \mathcal{A}_0' \) which coincides with the one we fixed on \( \mathcal{A}' \) if \( \mathcal{A}_0' = \mathcal{A}' \), we define
\[
\psi_s(\lambda) = \int_{\mathcal{A}_0} \psi(\lambda + \mu) \prod_{1 \leq \ell \leq \left[ \frac{n_k}{2} \right]} \left( \frac{2z_{k,\ell}(\lambda + \mu)}{f_k} + s \right)^{-1} d\mu
\]
for every \( s \in \mathcal{H} \) and \( \lambda \in \mathcal{A}_0 / \mathcal{A}_0' \). Then, equipping \( \mathcal{A}_0 / \mathcal{A}_0' \) with the quotient measure, we have
\[
\tilde{D}_s(\varphi) = \int_{\mathcal{A}_0 / \mathcal{A}_0'} \psi_s(\lambda) \prod_{i=1}^{r} \prod_{1 \leq \ell \leq n_i} \left( \frac{x_{i,\ell}(\lambda) + x_{i,\ell}(\lambda)}{d_i} + s \right)^{-1} \prod_{j=1}^{s} \frac{y_{j,p_j+1-\ell}(\lambda)}{e_j} + s \right)^{-1} \prod_{k=1}^{t} \prod_{1 \leq \ell \leq \left[ \frac{n_k}{2} \right]} \frac{z_{k,\ell}(\lambda) + z_{k,\ell}(\lambda)}{f_k} + s \right)^{-1} d\lambda
\]
for every \( s \in \mathcal{H} \).
We readily check that the linear forms \( \lambda \mapsto z_{k,\ell}(\lambda) \) \((1 \leq k \leq t, 1 \leq \ell \leq \left\lfloor \frac{q}{2} \right\rfloor)\) are linearly independent on \( \mathcal{A}'_0 \). Hence, by Lemma 3.1.1, the family \( s \mapsto \psi_s \) extends to a continuous map \( \mathcal{H} \cup \{0\} \rightarrow \mathcal{S}((\mathcal{A}'_0)'/\mathcal{A}_0)\). If there exists \( 1 \leq i \leq r \) such that \( m_i \neq n_i \) or \( 1 \leq j \leq s \) such that \( p_j \not\in 2\mathbb{N} \), then the linear forms \( x_{i,\ell}(\cdot) + x_{i,\ell}^*(\cdot) \) \((1 \leq i \leq r, 1 \leq \ell \leq \frac{p_i}{2})\) and \( z_{k,\ell}(\cdot) + z_{k,\ell+1}(\cdot) \) \((1 \leq k \leq t, 1 \leq \ell \leq \left\lfloor \frac{q}{2} \right\rfloor)\) restricted to \( \mathcal{A}_0 \) are linearly independent. Hence, by 3.3.3 Lemma 3.1.1 and the uniform boundedness principle, \( \tilde{D}_s(\varphi) \) admits a limit as \( s \rightarrow 0^+ \) and thus \( \lim_{s \rightarrow 0^+} s \tilde{D}_s(\varphi) = 0 \). This shows 3.3.2 in this case.

Assume from now on that \( m_i = n_i \) for all \( 1 \leq i \leq r \) and \( p_j \) is even for all \( 1 \leq j \leq s \). Then, we have \( \mathcal{A}'_0 = \mathcal{A}' \) and moreover the map sending \( \lambda \in \mathcal{A}_0 / \mathcal{A}' \) to

\[
\left( \left( x_{i,\ell}(\lambda) + x_{i,\ell}^*(\lambda) \right)_{1 \leq i \leq r, 1 \leq \ell \leq \frac{p_i}{2}} , (y_{j,\ell}(\lambda) + y_{j,p_j+1-\ell}(\lambda))_{1 \leq j \leq s, 1 \leq \ell \leq \frac{q}{2}} , (z_{k,\ell}(\lambda) + z_{k,q_k+1-\ell}(\lambda))_{1 \leq k \leq t, 1 \leq \ell \leq \left\lfloor \frac{q}{2} \right\rfloor , q_k \equiv 1[2]} \right)
\]

induces an isomorphism of vector spaces

\[ \mathcal{A}_0 / \mathcal{A}' \cong (i\mathbb{R}^N)_0 \]

which sends the measure on \( \mathcal{A}_0 / \mathcal{A}' \) to the measure on \( (i\mathbb{R}^N)_0 \) appearing in Proposition 3.1.2. Thus, by 3.3.3 viewing \( \psi_s \) as a function on \( (i\mathbb{R}^N)_0 \) via this isomorphism, we have

\[
\tilde{D}_s(\varphi) = \int_{(i\mathbb{R}^N)_0} \psi_s(t_1, \ldots, t_N) dt
\]

where the sequence \((h_{\ell})_{1 \leq \ell \leq N}\) is the concatenation of the sequences \((d_i)_{1 \leq i \leq r}, (e_j)_{1 \leq j \leq s}, (f_k)_{1 \leq k \leq t, \frac{p_i}{2} \leq \ell \leq 1[2]}\) and \((\ell_{k})_{1 \leq k \leq t, \frac{p_i}{2} \leq \ell \leq \left\lfloor \frac{q}{2} \right\rfloor} \). Hence, by Proposition 3.1.2 and the uniform boundedness principle, we get

\[
(3.3.4) \quad \lim_{s \rightarrow 0^+} s \tilde{D}_s(\varphi) = \lim_{s \rightarrow 0^+} \int_{(i\mathbb{R}^N)_0} \psi_s(t_1, \ldots, t_N) dt = D \frac{(2\pi)^{N-1}}{n} \psi_0(0)
\]

where the last equality follows from a painless computation. Finally, by the last part of Lemma 3.1.1 and since the function

\[
\mu \in \mathcal{A}' \mapsto \psi(\mu) \prod_{k=1}^{t} \prod_{1 \leq \ell \leq \left\lfloor \frac{q}{2} \right\rfloor} \left( \frac{2z_{k,\ell}(\mu)}{f_k} \right)^{-1} = \varphi(\mu) \prod_{i=1}^{r} S_{m_i,n_i}(\cdot) \left( \frac{X_i(\mu)}{d_i} \right) \prod_{j=1}^{s} T_{p_j} \left( \frac{Y_j(\mu)}{e_j} \right) \prod_{k=1}^{t} U_{q_k} \left( \frac{Z_k(\mu)}{f_k} \right),
\]

being the product of a Schwartz function by a polynomial (by Proposition 3.2.1), belongs to \( \mathcal{S}(\mathcal{A}') \), we have

\[
\psi_0(0) = \int_{\mathcal{A}'} \varphi(\mu) \prod_{i=1}^{r} S_{m_i,n_i}(\cdot) \left( \frac{X_i(\mu)}{d_i} \right) \prod_{j=1}^{s} T_{p_j} \left( \frac{Y_j(\mu)}{e_j} \right) \prod_{k=1}^{t} U_{q_k} \left( \frac{Z_k(\mu)}{f_k} \right) d\mu
\]
Combining this with 3.3.1 and 3.3.4 we obtain 3.3.2 and this ends the proof of the proposition.

\section{3.4 A spectral limit}

Recall that we have set $BC_n(\sigma) = BC(\sigma) \otimes \eta_n'$ for every $\sigma \in \text{Temp}(U(n))$ (see Section 2.10) and that $S_c(\text{Temp}(G_n(E)))$ denotes the space of functions $\Phi \in S(\text{Temp}(G_n(E)))$ which are supported on a finite number of connected components (see Section 2.9).

**Proposition 3.4.1** For every $\Phi \in S_c(\text{Temp}(G_n(E)))$, we have

\begin{equation}
\lim_{s \to 0^+} n\gamma(s, 1_F, \psi') \int_{\text{Temp}(G_n(E))} \Phi(\pi) \gamma(s, \pi, As, \psi')^{-1} d\mu_{G_n(E)}(\pi) = \\
\lambda_{E/F}(\psi')^{-n^2} \int_{\text{Temp}(U(n))/\text{stab}} \Phi(BC_n(\sigma)) \frac{\gamma^s(0, \sigma, \text{Ad}, \psi')}{|S_\sigma|} d\sigma
\end{equation}

where the right-hand side is absolutely convergent and so does the left hand side for any $s \in \mathcal{H}$.

Proof: The convergence of both sides of (3.4.1) for $s \in \mathcal{H}$ follows directly from Lemma 2.12.1 and 2.13.2 and 2.7.4. By Proposition 2.13.2 we have

\begin{equation}
\int_{\text{Temp}(G_n(E))} \Phi(\pi) \gamma(s, \pi, As, \psi')^{-1} d\mu_{G_n(E)}(\pi) = \lambda_{E/F}(\psi')^{-n^2} \int_{\text{Temp}(G_n(E))} \Phi(\pi) \frac{\gamma^s(0, \pi, \text{Ad}, \psi')}{|S_\pi|} \gamma(s, \pi, As, \psi') d\pi
\end{equation}

for every $s \in \mathcal{H}$. Fix $\pi \in \text{Temp}(G_n(E))$. We can write it as

\[ \pi = \prod_{i=1}^{r} \tau_i^{m_i} \times (\tau_i^*)^{m_i} \times \prod_{j=1}^{s} \mu_j^{p_j} \times \prod_{k=1}^{t} v_k^{q_k} \]

(Recall that $\tau^*$ stands for $(\tau^c \gamma^s)$ where

- For all $1 \leq i \leq r$, $\tau_i \in \Pi_2(G_{d_i}(E))$ for some $d_i \in \mathbb{N}^*$ is such that $\tau_i \neq \tau_i^*$ and $m_i, n_i \in \mathbb{N}^*$ are such that $m_i \geq n_i$. Moreover, $\tau_i \neq \tau_j$ and $\tau_i \neq \tau_j^*$ for all $1 \leq i < j \leq r$.

- For all $1 \leq j \leq s$, $\mu_j \in \Pi_2(G_{e_j}(E))$ for some $e_j \in \mathbb{N}^*$ is such that $\mu_j \approx \mu_j^*$ but $\gamma(0, \mu_j, As, \psi') \neq 0$ and $p_j \in \mathbb{N}^*$. Moreover, $\mu_i \neq \mu_j$ for all $1 \leq i < j \leq s$.

- For all $1 \leq k \leq t$, $\nu_k \in \Pi_2(G_{f_k}(E))$ for some $f_k \in \mathbb{N}^*$ is such that $\nu_k \approx \nu_k^*$ and $\gamma(0, \nu_k, As, \psi') = 0$ and $q_k \in \mathbb{N}^*$. Moreover, $\nu_i \neq \nu_j$ for all $1 \leq i < j \leq t$.}

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In other words, we have written $\pi = \overline{t_{M(F)}}(\Pi)$ where

$$M(F) = \left( \prod_{i=1}^{r} G_{d_i}(E)^{m_i+n_i} \times \prod_{j=1}^{s} G_{e_j}(E)^{p_j} \times \prod_{k=1}^{t} G_{f_k}(E)^{q_k} \right) / Z_n(F)$$

is a Levi subgroup of $G_n(E)$ and $\Pi$ is a certain square-integrable representation of $M(F)$.

In the Archimedean case (i.e. when $F = \mathbb{R}$), by linearity we may assume that $\Phi$ is supported in the connected component $O \subset \text{Temp}(G_n(E))$ of $\pi$. Moreover, up to twisting $\Pi$ we may also assume that the restriction of the central character of $\Pi$ to $A_M(F)$ has finite order.

In the $p$-adic case, using a partition of unity, we may assume that $\Phi$ is supported in a small enough neighborhood $U \subset \text{Temp}(G_n(E))$ of $\pi$.

Let $A_0$ be as in Section 3.3. Then, there exists a unique isomorphism of vector spaces

(3.4.3) $A_0 \simeq i A_M^*$

which when composed with the map $\lambda \in i A_M^* \mapsto \pi_\lambda := \overline{t_{M(F)}}(\Pi_\lambda)$ becomes (with the notation of Section 3.3)

(3.4.4)

$$\lambda \in A_0 \mapsto \pi_\lambda := \left( \prod_{i=1}^{r} \prod_{\ell=1}^{m_i} \nu_{\xi_i,\ell}(\lambda) \right) \times \left( \prod_{j=1}^{s} \prod_{\ell=1}^{n_j} \mu_{\gamma_j,\ell}(\lambda) \right) \times \left( \prod_{k=1}^{t} \prod_{\ell=1}^{q_k} \nu_{\kappa_k,\ell}(\lambda) \right)$$

Define $W$ as in Section 3.3. Then, there exists an isomorphism $W \simeq W(G_n(E), \Pi)$ such that $W$ transports the action of $W$ on $A_0$ to the action of $W(G_n(E), \Pi)$ on $i A_M^*$. In the $p$-adic case, up to shrinking $U$, we may assume, which we do in what follows, that there exists a small open neighborhood $V \subset A_0$ of 0 such that $\lambda \mapsto \pi_\lambda$ induces a topological isomorphism $V/W \simeq U$. The inverse image by 3.4.3 of $iX^*(A_M)$ is $1/2 \Lambda_0$ where $\Lambda_0$ denotes the intersection of $A_0$ with the lattice

$$\Lambda := \prod_{1 \leq i \leq r} (i \mathbb{Z})^{m_i} \times \prod_{1 \leq j \leq s} (i \mathbb{Z})^{n_j} \times \prod_{1 \leq k \leq t} (i \mathbb{Z})^{q_k}$$

of $A$ (the factor $1/2$ stemming from the fact that $|x|_E = |x|^2_F$ for every $x \in F^\times$). Recall that we have fixed a Haar measure on $A_0$ in Section 3.3. By definition of this Haar measure, we have $\text{vol}(A_0/\Lambda_0) = 1$. Hence, by the definition of the Haar measure on $i A_M^*$ (see Section 2.5) the isomorphism 3.4.3 sends this Haar measure to $\left( \frac{\pi}{\log(qF)} \right)^{1-S}$ times the Haar measure on $A_0$ where $S = \dim(A_0) + 1 = \sum_{i=1}^{r} m_i + n_i + \sum_{j=1}^{s} p_j + \sum_{k=1}^{t} q_k$.
Therefore, by \(3.4.2\) and \(2.7.3\) we have
\[
(3.4.5) \quad \int_{\text{Temp}(G_n(E))} \Phi(\pi)\gamma(s, \pi, As, \psi')^{-1} d\mu_{G_n(E)}(\pi) = \\
\frac{\lambda_{E/F}(\psi')^{-n^2}}{|W|} \left( \frac{\pi}{\log(q_F)} \right)^{1-S} \int_{\mathcal{A}_0} \varphi(\lambda) \frac{\gamma^s(0, \pi\lambda, Ad, \psi')}{|S_{\pi\lambda}|} d\lambda
\]
where we have set \(\varphi(\lambda) = \Phi(\pi\lambda)\) in the Archimedean case and
\[
\varphi(\lambda) = \begin{cases} 
\Phi(\pi\lambda) & \text{if } \lambda \in \mathcal{V} \\
0 & \text{otherwise}
\end{cases}
\]
in the \(p\)-adic case. Notice that in both cases we have \(\varphi \in \mathcal{S}(\mathcal{A}_0)^W\).

By \(2.11.1, 2.11.2\) and \(2.11.3\) we readily check that
\[
(3.4.6) \quad |S_{\pi\lambda}| = 2^S P
\]
for every \(\lambda \in \mathcal{A}_0\) where we have set \(P = \prod_{i=1}^r d_i^{m_i+n_i} \prod_{j=1}^s e_j^{p_j} \prod_{k=1}^t f_k^{q_k}\).

From \(2.12.3, 2.12.4, 2.12.6, 2.12.7, 2.12.9, 2.12.10\) and \(2.12.11\) we infer that there exists a function \(F \in C^\infty(\mathcal{A}_0)^W\) which is of moderate growth together with all its derivatives in the Archimedean case such that
\[
(3.4.7) \quad \gamma^s(0, \pi\lambda, Ad, \psi') = \left( \prod_{i=1}^r \prod_{1 \leq \ell \neq \ell' \leq m_i} \frac{x_{i,\ell}(\lambda) - x_{i,\ell'}(\lambda)}{d_i} \right) \prod_{1 \leq \ell \neq \ell' \leq n_i} \left( \frac{x_{i,\ell}(\lambda) - x_{i,\ell'}(\lambda)}{d_i} \right) \\
\times \left( \prod_{j=1}^s \prod_{1 \leq \ell \neq \ell' \leq p_j} \frac{y_{j,\ell}(\lambda) - y_{j,\ell'}(\lambda)}{e_j} \right) \times \left( \prod_{k=1}^t \prod_{1 \leq \ell \neq \ell' \leq q_k} \frac{z_{k,\ell}(\lambda) - z_{k,\ell'}(\lambda)}{f_k} \right) F(\lambda)
\]
for almost all \(\lambda \in \mathcal{A}_0\).

Set \(\mathcal{A}_C := \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C} = \prod_{i=1}^r \mathbb{C}^{m_i+n_i} \prod_{j=1}^s \mathbb{C}^{p_j} \prod_{k=1}^t \mathbb{C}^{q_k} \simeq \mathbb{C}^S\) and embed \(\mathbb{C}\) in \(\mathcal{A}_C\) diagonally. From \(2.12.3, 2.12.4, 2.12.9, 2.12.10, 2.12.11, 2.12.12, 2.12.13\) and \(2.12.14\) we infer similarly that, up to shrinking \(\mathcal{U}\) in the \(p\)-adic case, there exists a \(W\)-invariant meromorphic function \(G\) on \(\mathcal{A}_C\) whose polar divisors are affine subspaces not meeting \((-\epsilon + H)^S\) for some \(\epsilon > 0\) and which is of moderate growth on vertical strips with all its derivatives there in the Archimedean case, resp. whose polar divisors are affine subspaces disjoint from \((H \cup \{0\})^S + \mathcal{V}\) in the \(p\)-adic case, such that
\[
(3.4.8) \quad \gamma(s, \pi\lambda, As, \psi')^{-1} = \left( \prod_{i=1}^r \prod_{1 \leq \ell \leq m_i} \left( s + \frac{x_{i,\ell}(\lambda)}{d_i} \right)^{-1} \right) \times \\
\left( \prod_{j=1}^s \prod_{1 \leq \ell \leq p_j} \left( s + \frac{y_{j,\ell}(\lambda)}{e_j} \right)^{-1} \right) \times \left( \prod_{k=1}^t \prod_{1 \leq \ell \leq q_k} \left( s + \frac{z_{k,\ell}(\lambda)}{f_k} \right)^{-1} \right) G\left(\frac{s}{2} + \lambda\right)
\]
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for all $\lambda \in \mathcal{A}_0$ and $s \in \mathcal{H}$.

From 3.4.7 and 3.4.8 it follows that, with the notation of Section 3.3, there exists a continuous family $s \in \mathcal{H} \cup \{0\} \mapsto \varphi_s \in \mathcal{S}(\mathcal{A}_0)^W$ such that

\[(3.4.9) \quad \varphi(\lambda) \gamma^*(0, \pi, \lambda, d, \psi') = \varphi(s(\lambda) \prod_{i=1}^r P_{m_j, n_i, s}(\frac{X_i(\lambda)}{d}) \times \prod_{j=1}^s Q_{p_j, s}(\frac{Y_j(\lambda)}{e_j}) \times \prod_{k=1}^t R_{q_k, s}(\frac{Z_k(\lambda)}{f_k})\]

for all $s \in \mathcal{H}$ and almost all $\lambda \in \mathcal{A}_0$. Therefore, using again notation from Section 3.3 by 3.4.5 3.4.6 and since $s$ is odd, we have

\[(3.4.10) \quad n \gamma(s, 1_F, \psi') \int_{\text{Temp}(G_{n}(E))} \Phi(\pi) \gamma(s, \pi, \lambda, d, \psi')^{-1} d\mu_{G_{n}(E)}(\pi) \sim_{s \to 0}^+ \]

Assume first that $\pi$ cannot be written as $BC_n(\sigma)$ for some $\sigma \in \text{Temp}(U(n))$. This means that (by 2.10.5 and 2.12.14) there exists $1 \leq i \leq r$ such that $m_i \neq n_i$ or there exists $1 \leq j \leq s$ such that $p_j$ is odd. In the Archimedean case, by the assumption that the central character of $\Pi$ restricted to $A_M(F)$ has finite order, it follows that $BC_n(\text{Temp}(U(n)))$ does not meet $\mathcal{O}$ at all. In the $p$-adic case on the other hand, up to shrinking $\mathcal{U}$ if necessary, we may assume that $BC_n(\text{Temp}(U(n))) \cap \mathcal{U} = \emptyset$. Then, in both cases, the right-hand side of 3.4.11 turns out to be just zero whereas by 3.4.10 and Proposition 3.3.4 so does the left-hand side. This proves the proposition in this case.

Assume now that there exists $\sigma \in \text{Temp}(U(n))/\text{stab}$ such that $\pi = BC_n(\sigma)$. Then $m_i = n_i$ for all $1 \leq i \leq r$, $p_j$ is even for all $1 \leq j \leq s$ and we can write

$$\sigma = \left[ \prod_{i=1}^r \mu_i^{m_i} \times \mu_j^{p_j} \times \prod_{k=1}^t \nu_k^{q_k} \right] \times \sigma_0$$

where $\sigma_0$ is a discrete series of some $U(m)$ with

$$BC_m(\sigma_0) = \times_{1 \leq k \leq l, \text{mod } 2} \nu_k.$$
which when composed with the map \( \mu \in i A^*_L \mapsto \sigma_\mu := i^{U(n)}_{L(F)}(\Sigma_\mu) \) becomes

\[
\mu \in A' \mapsto \sigma_\mu = \left[ \prod_{i=1}^{r} m_i \tau_i \otimes |\det_E|_{d_i} \times \prod_{j=1}^{s} \mu_j \otimes |\det_E|_{e_j} \times \prod_{k=1}^{t} \nu_k \otimes |\det_E|_{l_k} \right] \times \sigma_0
\]

By 2.10.4 and 2.10.5 in the Archimedean case, for every \( \lambda \in A_0 \) we have \( \pi_\lambda \in BC_n(\text{Temp}(U(n))) \) if and only if \( \lambda \in A' \) in which case \( \pi_\lambda = BC_n(\sigma_\lambda) \). Similarly, in the \( p \)-adic case and up to shrinking \( \mathcal{U} \) if necessary, for every \( \lambda \in \mathcal{V} \) we have \( \pi_\lambda \in BC_n(\text{Temp}(U(n))) \) if and only if \( \lambda \in A' \) in which case we again have \( \pi_\lambda = BC_n(\sigma_\lambda) \). It follows that the function \( \Phi \circ BC_n \) on \( \text{Temp}(U(n))/\text{stab} \) is supported in the connected component \( \{ \sigma_\mu \mid \mu \in A' \} \) of \( \sigma \) in the Archimedean case and even in the open neighborhood \( \{ \sigma_\mu \mid \mu \in A' \cap \mathcal{V} \} \) of \( \sigma \) in the \( p \)-adic case. We check easily, as before, that the isomorphism 3.4.11 sends the Haar measure on \( i A^*_L \) to \( \left( \frac{\pi}{\log(q_F)} \right)^{N-S} \) times the Haar measure on \( A' \) that was defined in Section 3.3 where \( N = S - \dim(A') = \sum_{i=1}^{r} n_i + \sum_{j=1}^{s} \frac{p_j}{2} + \sum_{k=1}^{t} \frac{q_k}{2} \) (using again notation from Section 3.3).

Moreover, using the precise description of \( W(U(n), \Sigma) \) given in Section 2.6 we see that there exists an isomorphism \( W' \simeq W(U(n), \Sigma) \) (where \( W' \) is defined as in Section 3.3) such that 3.4.11 transports the action of \( W' \) on \( A' \) to the action of \( W(U(n), \Sigma) \) on \( i A^*_L \). Therefore by 2.10.2 assuming \( \mathcal{U} \) sufficiently small, we have

\[
\int_{\text{Temp}(U(n))/\text{stab}} \Phi(BC_n(\sigma)) \left( \gamma^*(0, \sigma, \text{Ad}, \psi') / |S_\sigma| \right) d\sigma = \left( \frac{\pi}{\log(q_F)} \right)^{N-S} |W'|^{-1} \int_{A'} \varphi(\mu) \left( \gamma^*(0, \sigma_\mu, \text{Ad}, \psi') / |S_{\sigma_\mu}| \right) d\mu
\]

Moreover, using 2.11.1 2.11.2 and 2.11.3 we readily check that

\[
|S_{\sigma_\mu}| = 2^{c + s - N} \frac{P}{D}
\]

for all \( \mu \in A' \) where \( c = |\{1 \leq k \leq t \mid q_k \equiv 1 \mod 2\}| \) and \( D = \prod_{i=1}^{r} d_i^{m_i} \times \prod_{j=1}^{s} e_j^{p_j} \times \prod_{k=1}^{t} f_k^{q_k/2} \)

(same notation as in Section 3.3) so that \( c + S - N = \sum_{i=1}^{r} m_i + \sum_{j=1}^{s} \frac{p_j}{2} + \sum_{k=1}^{t} \frac{q_k}{2} \) and \( \frac{P}{D} = \prod_{i=1}^{r} d_i^{m_i} \times \prod_{j=1}^{s} e_j^{p_j} \times \prod_{k=1}^{t} f_k^{q_k/2} \). Consequently, 3.4.12 can be rewritten

\[
\int_{\text{Temp}(U(n))/\text{stab}} \Phi(BC_n(\sigma)) \left( \gamma^*(0, \sigma, \text{Ad}, \psi') / |S_\sigma| \right) d\sigma = \left( \frac{2\pi}{\log(q_F)} \right)^{N-S} \frac{2^{-c} D}{P} \int_{A'} \varphi(\mu) \gamma^*(0, \sigma_\mu, \text{Ad}, \psi') d\mu
\]
On the other hand, by Proposition 3.3.1 and the uniform boundedness principle we have

\[
\lim_{s \to 0^+} n \gamma(s, 1_F, \psi') \int_{\text{Temp}(G_n(E))} \Phi(\pi) \gamma(s, \pi, As, \psi')^{-1} d\mu_{G_n(E)}(\pi) = \\
\frac{\lambda_{E/F}(\psi')^{-n^2}}{|W'|} \left( \frac{2\pi}{\log(q_F)} \right)^{-S} (2\pi)^{N-2-c} \frac{D}{P} \gamma^*(0, 1_F, \psi') \times \\
\lim_{s \to 0^+} s^N \varphi_s(\mu) \prod_{i=1}^r P_{m_i,n_i,s}(\frac{X(\mu)}{d_i}) \times \prod_{j=1}^s Q_{p_j,s}(\frac{\gamma(\mu)}{e_j}) \times \prod_{k=1}^t R_{q_k,s}(\frac{\psi_k(\mu)}{f_k}) d\mu
\]

From 3.4.9 and the fact that \( \zeta_F(s) \sim_{s \to 0} (s \log(q_F))^{-1} \) this can be rewritten as

\[
\lim_{s \to 0^+} n \gamma(s, 1_F, \psi') \int_{\text{Temp}(G_n(E))} \Phi(\pi) \gamma(s, \pi, As, \psi')^{-1} d\mu_{G_n(E)}(\pi) = \\
\frac{\lambda_{E/F}(\psi')^{-n^2}}{|W'|} \left( \frac{2\pi}{\log(q_F)} \right)^{N-S} 2^{-c} \frac{D}{P} \gamma^*(0, 1_F, \psi') \int_{A'} \varphi(\mu) \lim_{s \to 0^+} \zeta_F(s)^{-N} \gamma^*(0, \pi, Ad, \psi') d\mu
\]

From 3.4.8 it is easy to infer that \( s \to \gamma(s, \pi, As, \psi') \) has a zero of order \( N \) at \( s = 0 \) for almost all \( \mu \in A' \). Therefore by 2.12.3 and 2.12.15 the above equality is equivalent to

\[
\lim_{s \to 0^+} n \gamma(s, 1_F, \psi') \int_{\text{Temp}(G_n(E))} \Phi(\pi) \gamma(s, \pi, As, \psi')^{-1} d\mu_{G_n(E)}(\pi) = \\
\frac{\lambda_{E/F}(\psi')^{-n^2}}{|W'|} \left( \frac{2\pi}{\log(q_F)} \right)^{N-S} 2^{-c} \frac{D}{P} \int_{A'} \varphi(\mu) \gamma^*(0, \sigma, Ad, \psi') d\mu
\]

Comparing this with 3.4.13 we get the identity of the proposition. ■

### 3.5 A corollary to Proposition 3.4.1

For every \( \sigma \in \text{Temp}(U(n))/\text{stab} \), set

\[
c(\sigma) := \lambda_{E/F}(\psi')^{-\frac{n(n+1)}{2}} c_1(\sigma) \omega_\sigma(-1)^{1-n} \eta_{E/F}(-1)^{\frac{n(n-1)}{2}}
\]

where \( c_1(\sigma) \) is the constant defined by 2.1.1. Notice that \( c(\sigma) \) is just a certain root of unity.

Recall from Section 2.1.3 the continuous linear form \( \beta : C^w(N_n(E) \backslash G_n(E), \psi_n) \to \mathbb{C} \) and also that in Section 2.1.4 we have associated to any function \( f \in S(G_n(E)) \) and \( \pi \in \text{Temp}(G_n(E)) \) a function \( W_{f, \pi} \in C^w(N_n(E) \times N_n(E) \backslash G_n(E) \times G_n(E), \psi_n^{-1} \boxtimes \psi_n) \). In particular, we have \( W_{f, \pi}(g, .) \in C^w(N_n(E) \backslash G_n(E), \psi_n) \) for all \( g \in G_n(E) \).

**Corollary 3.5.1** For every \( f \in S(G_n(E)) \) and \( g \in G_n(E) \), we have

\[
(3.5.1) \quad \int_{N_n(F) \backslash G_n(F)} W_f(g, h) dh = \\
|\tau|_E^{\frac{n(n-1)}{2}} \int_{\text{Temp}(U(n))/\text{stab}} \beta(W_{f, BC_n(\sigma)}(g, .))(\gamma^*(0, \sigma, Ad, \psi') c(\sigma)) d\sigma
\]

where the right-hand side is an absolutely convergent integral.
Proof: First note that up to replacing \( f \) by \( L(g)f \) we may assume that \( g = 1 \). By Lemma 2.12.1 and Proposition 2.14.2, the right-hand side of (3.5.1) is convergent and defines a continuous linear form on \( S(G_n(E)) \). It is easy to see that the same holds for the left-hand side. Therefore, it suffices to check (3.5.1) for the dense subspace of \( f \in S(G_n(E)) \) whose Fourier transform \( \pi \mapsto f_\pi \) is supported in a finite number of components of \( \text{Temp}(G_n(E)) \) (e.g. take \( f \) to be \( K \)-finite for \( K \) a maximal compact subgroup; of course this condition is automatic in the \( p \)-adic case). Let \( f \in S(G_n(E)) \) be such a function and define \( \tilde{f} \in S(G_n(E)) \) by

\[
\tilde{f}(g) = \int_{Z_n(F)} f(zg)dz.
\]

Then, we have

\[
\int_{N_n(F)/G_n(F)} W_f(1,h)dh = \int_{Z_n(F)N_n(F)/G_n(F)} W_{\tilde{f}}(1,h)dh.
\]

Choose \( \phi \in S(F^n) \) such that \( \phi(0) = 1 \). Since \( \tilde{f}_\pi = f_\pi \) for every \( \pi \in \text{Temp}(G_n(E)) \), by Lemma 2.16.1(i)-(ii) and Proposition 2.14.2, we have

\[
\int_{Z_n(F)N_n(F)/G_n(F)} W_{\tilde{f}}(1,h)dh = \lim_{s \to 0^+} n\gamma(s,1_F,\psi')Z(s,W_{\tilde{f}}(1,\cdot),\phi)
\]

\[
= \lim_{s \to 0^+} n\gamma(s,1_F,\psi') \int_{\text{Temp}(G_n(E))} Z(s,W_{f,\pi}(1,\cdot),\phi) d\mu_{G_n(E)}(\pi).
\]

By 2.14.1, applying the functional equation of Theorems 2.16.2 this becomes

\[
(3.5.2) \quad \int_{N_n(F)/G_n(F)} W_f(1,h)dh = \lim_{s \to 0^+} n\gamma(s,1_F,\psi')\left|\frac{1}{2}-s\right|^{\frac{n(n-1)}{2}} \lambda_{E/F}(\psi')^{\frac{n(n-1)}{2}} \times
\]

\[
\int_{\text{Temp}(G_n(E))} Z(1-s,W_{f,\pi}(1,\cdot),\hat{\phi}) \omega_\pi(2\pi) \gamma_\pi(s,\pi,As,\psi')^{-1} d\mu_{G_n(E)}(\pi)
\]

For \( s \) sufficiently close to 0, set \( \Phi_\pi(\pi) = Z(1-s,W_{f,\pi}(1,\cdot),\hat{\phi}) \omega_\pi(2\pi) \gamma_\pi(s,\pi,As,\psi')^{-1} d\mu_{G_n(E)}(\pi) \). Then, by Proposition 2.14.2 (Lemma 2.16.1(i) and the assumption made on \( f \), we see that \( s \mapsto \Phi_\pi \) is an analytic map into \( S_c(\text{Temp}(G_n(E))) \). Therefore, by Proposition 3.4.1 (and the uniform boundedness principle), we have

\[
\lim_{s \to 0^+} n\gamma(s,1_F,\psi')\left|\frac{1}{2}-s\right|^{\frac{n(n-1)}{2}} \lambda_{E/F}(\psi')^{\frac{n(n-1)}{2}} \times
\]

\[
\int_{\text{Temp}(G_n(E))} Z(1-s,W_{f,BC_n(\pi)}(1,\cdot),\hat{\phi}) \omega_{BC_n(\pi)}(\tau)^{-1} \gamma_\pi(s,\pi,As,\psi')^{-1} d\mu_{G_n(E)}(\pi)
\]

\[
= \lambda_{E/F}(\psi')^{-\frac{n(n+1)}{2}} \left|\frac{1}{2}\right|^{\frac{n(n-1)}{2}} \int_{\text{Temp}(U(n))/\text{stab}} Z(1-s,W_{f,BC_n(\pi)}(1,\cdot),\hat{\phi}) \omega_{BC_n(\pi)}(\tau)^{-1} \gamma_\pi(s,\pi,As,\psi')^{-1} \frac{d\sigma}{|S_\sigma|}
\]

By 2.10.3 we have

\[
\omega_{BC_n(\pi)}(\tau)^{-1} = \omega_{BC(\sigma)}(\tau)^{-1} \eta_{\pi h}(\tau)^{-1} = \omega_{\pi}(-1)^{n-1} \eta_{E/F}(-1)^{\frac{n(n-1)^2}{2}}
\]

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whereas by Lemma 2.16.3 and the fact that \( \phi(0) = 1 \), we have
\[
Z(1, \widehat{W}_{f,B\mathbb{C}^n}(\sigma)(1,.), \widehat{\phi}) = c_1(\sigma)\beta(W_{f,B\mathbb{C}^n}(\sigma)(1,.)).
\]
Combined with 3.5.2 and 3.5.3 this gives 3.5.1 and ends the proof of the corollary. \( \square \)

4 A Plancherel formula for \( G_n(F)\bigm/ G_n(E) \)

4.1 Précis on Plancherel decompositions

Let \( G \) be a reductive algebraic group over \( F \) which acts regularly on the right of a smooth \( F \)-algebraic variety \( X \). For every \( \pi \in \text{Irr}(G(F)) \), we denote by \( S(X(F))_{\pi} \) the \( \pi \)-isotypic maximal quotient of \( S(X(F)) \) i.e. the quotient of \( S(X(F)) \) by the intersection of kernels of all continuous \( G(F) \)-equivariant linear maps \( S(X(F)) \rightarrow \pi \). Let \( \text{Irr}_{\text{unit}}(G) \subset \text{Irr}(G) \) be the subset of unitarizable irreducible representations (i.e. the one admitting a continuous \( G(F) \)-invariant scalar product) that we equip with the Fell topology (\cite[Sect. 18.1]{Dix}). Assume moreover that \( X(F) \) is equipped with a \( G(F) \)-invariant measure. We denote by \( L^2(X(F)) \) the Hilbert space of square-integrable functions for this measure and by \( (.,.)_{L^2(X(F))} \) the corresponding inner product. The right regular action induces a unitary representation of \( G(F) \) on \( L^2(X(F)) \). Since \( G(F) \) is a postliminal locally compact group (\cite[13.11.12]{Dix}, \cite{Ber1}), by \cite[Theorem 8.6.6]{Dix} there exists a unique (in a suitable sense) direct integral decomposition of \( L^2(X(F)) \)
\[
L^2(X(F)) = \bigoplus_{\text{Irr}_{\text{unit}}(G)} \mathcal{H}_\pi \, d\mu_X(\pi).
\]

Here \( \mu_X \) is a Borel measure on \( \text{Irr}_{\text{unit}}(G) \), and \( \pi \mapsto \mathcal{H}_\pi \) is a measurable field of continuous unitary representations of \( G(F) \) (in the sense of \cite[Sect. 18.7]{Dix}) with \( \mathcal{H}_\pi \) isomorphic for \( \mu_X \)-almost all \( \pi \) to an Hilbert direct sum of copies of (an Hilbert completion of) \( \pi \). Such a decomposition is usually called a Plancherel decomposition for \( L^2(X(F)) \).

According to Gelfand-Kostyuchenko and Bernstein (see \cite{Ber3} and also \cite[Sect. 3.3]{Li}), a Plancherel decomposition for \( L^2(X(F)) \) is equivalent to the following set of data:

- A Borel measure \( \mu_X \) on \( \text{Irr}_{\text{unit}}(G) \);
- For \( \mu_X \)-almost all \( \pi \in \text{Irr}_{\text{unit}}(G) \), a continuous hermitian form \( (.,.)_{X,\pi} \) on \( S(G(F)) \) which is positive semi-definite (i.e. \( (\varphi,\varphi)_{X,\pi} \geq 0 \) for every \( \varphi \in S(X(F)) \)); from now on we will call such an hermitian form a positive semi-definite scalar product, \( G(F) \)-invariant and factorizing through the quotient \( S(X(F))_{\pi} \),

such that the following condition is satisfied: for every \( \varphi_1,\varphi_2 \in S(X(F)) \), the function \( \pi \mapsto (\varphi_1,\varphi_2)_{X,\pi} \) is \( \mu_X \)-integrable and we have
\[
(\varphi_1,\varphi_2)_{L^2(X(F))} = \int_{\text{Irr}_{\text{unit}}(G)} (\varphi_1,\varphi_2)_{X,\pi} \, d\mu_X(\pi).
\]
It is under this form that we will describe the Plancherel decomposition for $L^2(G_n(F)\backslash G_n(E))$ in the next section.

Actually, we don’t even need to assume that the hermitian forms $(\ldots)_{X,\pi}$ are positive definite i.e. once [4,1.1] is satisfied then the forms $(\ldots)_{X,\pi}$ are automatically positive semi-definite for $\mu_X$-almost all $\pi$. This is the content of [SV, Proposition 6.1.1] in the p-adic case and actually the proof of loc. cit. works verbatim in the Archimedean case provided we replace each occurrence of “the Bernstein center” in it by “Arthur’s multiplier algebra” ([Art2], [Del]). We will actually need this result in a slightly different form but for which the proof of loc. cit. is still valid and we content ourself to state it referring however the interested reader to Lemma 5.7.2 for a similar kind of argument.

**Proposition 4.1.1** Assume given

- A Borel measure $\mu_X$ on $\operatorname{Irr}_{\text{unit}}(G)$;

- For $\mu_X$-almost all $\pi \in \operatorname{Irr}_{\text{unit}}(G)$, a continuous sesquilinear form $(\ldots)_{X,\pi}$ on $\mathcal{S}(G(F))$ which is $G(F)$-invariant and factorizes through the quotient $\mathcal{S}(X(F))_\pi$

such that for every $\varphi_1, \varphi_2 \in \mathcal{S}(X(F))$, the function $\pi \mapsto (\varphi_1, \varphi_2)_{X,\pi}$ is $\mu_X$-integrable and [4,1.1] is satisfied. Then, for $\mu_X$-almost all $\pi$ the form $(\ldots)_{X,\pi}$ is a positive semi-definite scalar product so that the data $(\mu_X, (\ldots)_{X,\pi})$ induce a Plancherel decomposition for $L^2(X(F))$.

Finally we remark that the restriction of the Fell topology to $\operatorname{Temp}(G_n(E)) \subset \operatorname{Irr}_{\text{unit}}(G_n(E))$ coincides with the natural topology on $\operatorname{Temp}(G_n(E))$ that was described in Section 2.6. Here is a quick proof: let $(\pi_n)_{n}$ be a sequence in $\operatorname{Temp}(G_n(E))$ converging to $\pi \in \operatorname{Temp}(G_n(E))$ for the topology defined in Section 2.6. Then, there exists a Levi subgroup $M \subseteq G_n$, $\sigma \in \Pi_2(M(E))$ and a sequence $(\chi_n)_{n}$ of unramified characters of $M(E)$ converging to the trivial character such that $\pi_n = i^{G_n(E)}_{M(E)}(\sigma \otimes \chi_n)$ for $n$ large enough and $\pi = i^{G_n(E)}_{M(E)}(\sigma)$. Identifying each $\pi_n$ and $\pi$ with a compact model $i^{K_p}_{K_p}(\sigma)$ (where $K = K_{n,E}$, $P \in \mathcal{P}(M)$ and $K' = K \cap P(E)$) and equipping this last space with the usual $K$-invariant scalar product we have $(\pi_n(g)e, e) \rightarrow (\pi(g)e, e)$ uniformly on compact subsets for every $e \in i^{K_p}_{K_p}(\sigma)$. Therefore, by definition of the Fell topology, $\pi_n \rightarrow \pi$ in $\operatorname{Irr}_{\text{unit}}(G_n(E))$. Conversely, if $(\pi_n)_{n}$ is a sequence of tempered representations converging to $\pi \in \operatorname{Temp}(G_n(E))$ for the Fell topology then by [Tad, Theorem 2.2(i)] and [BDix] the infinitesimal character $\chi_{\pi_n}$ of $\pi_n$ converges point-wise to $\chi_{\pi}$ where in the p-adic case by “infinitesimal character" we mean the corresponding character of the Bernstein center. However, the map sending $\pi \in \operatorname{Temp}(G_n(E))$ to $\chi_{\pi}$ is proper (for the topology of Section 2.6 on $\operatorname{Temp}(G_n(E))$ and the topology of point-wise convergence on the set of all infinitesimal characters): this follows from [Wald, Théorème VIII.1.2] in the p-adic case and [H-C1, Corollary 35 p.81] in the Archimedean case. Since the sequence $(\pi_n)_{n}$ can only have $\pi$ as a limit point in $\operatorname{Temp}(G_n(E))$ (by what we just saw) this shows that $\pi_n \rightarrow \pi$ in $\operatorname{Temp}(G_n(E))$ and thus the claim follows.
4.2 The theorem

Let \( n \geq 1 \) and set \( Y_n = G_n(F) \backslash G_n(E) \) that we equip with the quotient measure. There is a surjection \( S(G_n(E)) \to S(Y_n) \) given by

\[
f \mapsto \varphi_f(x) = \int_{G_n(F)} f(hx)dh
\]

which identifies \( S(Y_n) \) with the space of \( G_n(F) \)-coinvariants of \( S(G_n(E)) \) for the left regular action (see [AL Corollary D.5]). For \( \pi \in \text{Temp}(G_n(E)) \) and \( f_1, f_2 \in S(G_n(E)) \), we define

\[
(f_1, f_2)_{Y_n, \pi} = \sum_{W \in \mathcal{B}(\pi, \psi_n)} \beta(R(f_1)W)\overline{\beta(R(f_2)W)}
\]

where \( \mathcal{B}(\pi, \psi_n) \) is an orthonormal basis of \( W(\pi, \psi_n) \) as in Section 2.14 and \( \beta \) is the linear form defined in Section 2.15.

Lemma 4.2.1 \((f_1, f_2)_{Y_n, \pi} \) is defined by an absolutely convergent expression and does not depend on the choice of the basis \( \mathcal{B}(\pi, \psi_n) \). The function \( \pi \in \text{Temp}(G_n(E)) \mapsto (f_1, f_2)_{Y_n, \pi} \) belongs to \( S(\text{Temp}(G_n(E))) \) and the sesquilinear map

\[
S(G_n(E)) \times S(G_n(E)) \to S(\text{Temp}(G_n(E)))
\]

\[
(f_1, f_2) \mapsto (\pi \mapsto (f_1, f_2)_{Y_n, \pi})
\]

is continuous. Moreover, \((.,.)_{Y_n, \pi} \) is a continuous right \( G_n(E) \)-invariant positive semi-definite scalar product on \( S(G_n(E)) \) and if \( \pi \in BC_n(\text{Temp}(U(n))) \) it factorizes in both variables through the quotient \( S(Y_n)_{\pi'} \).

Proof: Clearly, \((.,.)_{Y_n, \pi} \), if well-defined, is positive semi-definite. By 2.14.3 we have

\[
W_{\overline{f_2}}f_1, \pi = R_1(\overline{f_2})W_{f_1}, \pi = |\tau|^{\frac{n(n-1)}{2}} \sum_{W \in \mathcal{B}(\pi, \psi_n)} \overline{R(f_2)W} \otimes R(f_1)W
\]

the sum being absolutely convergent in \( C^w(N_n(E) \backslash G_n(E) \times N_n(E) \backslash G_n(E), \psi_n^{-1} \otimes \psi_n) \) (here we have denoted by \( R_1(\overline{f_2})W_{f_1}, \pi \) the right regular action of \( \overline{f_2} \) on the first variable of \( W_{f_1}, \pi \)). Applying the continuous functional \( \beta \hat{\otimes} \beta \) to this decomposition we get that \((f_1, f_2)_{Y_n, \pi} \) is indeed well-defined and

\[
(f_1, f_2)_{Y_n, \pi} = |\tau|^{\frac{n(n-1)}{2}}(\beta \hat{\otimes} \beta)(W_{\overline{f_2}}f_1, \pi)
\]

This already shows that \((f_1, f_2)_{Y_n, \pi} \) does not depend on the choice of the basis \( \mathcal{B}(\pi, \psi_n) \) and that it is right \( G_n(E) \)-invariant. Moreover, from Proposition 2.14.2 and the continuity of convolution, we also deduce that the function \( \pi \in \text{Temp}(G_n(E)) \mapsto (f_1, f_2)_{Y_n, \pi} \) belongs to \( S(\text{Temp}(G_n(E))) \) and that the sesquilinear map

\[
S(G_n(E)) \times S(G_n(E)) \to S(\text{Temp}(G_n(E)))
\]
Recall that in Section 2.14 we have associated to any $(f_1, f_2) \mapsto (\pi \in \text{Temp}(G_n(E)) \mapsto (f_1, f_2)_{\gamma_{\pi}}$ is continuous. This implies in particular that $(\ldots)_{\gamma_{\pi}}$ is continuous. Since $(f_1, f_2)_{\gamma_{\pi}}$ only depends on $\pi(f_1)$ and $\pi(f_2)$, we have that $(\ldots)_{\gamma_{\pi}}$ factorizes through the maximal $\pi^\gamma$-isotypic quotient of $S(G_n(E))$ (for the right $G_n(E)$-action). Finally, by [2.15.1], if $\pi \in BC_n(\text{Temp}(U(n)))$, the form $(\ldots)_{\gamma_{\pi}}$ is invariant for the left regular action by $P_n(F)$ and its transpose hence by $G_n(F)$ so that it factorizes through $\mathcal{S}(Y_n)$, hence through $\mathcal{S}(Y_n)_{\pi^\gamma}$, in both variables. ■

By the lemma, for every $\pi \in BC_n(\text{Temp}(U(n)))$, $(\ldots)_{\gamma_{\pi}}$ induces a positive semi-definite scalar product on $\mathcal{S}(Y_n)$ that we shall denote the same way. We can now state the main theorem of this chapter.

**Theorem 4.2.2** For every $\varphi_1, \varphi_2 \in \mathcal{S}(Y_n)$, we have

$$(\varphi_1, \varphi_2)_{L^2(Y_n)} = \int_{\text{Temp}(U(n))/\text{stab}} (\varphi_1, \varphi_2)_{Y_n, \text{BC}_n(\sigma)} \frac{|\gamma^*(0, \sigma, \text{Ad}, \psi')|}{|\mathcal{S}_{\sigma}|} d\sigma$$

where the right-hand side is an absolutely convergent expression. Moreover we have

$$c(\sigma) \gamma^*(0, \sigma, \text{Ad}, \psi') = |\gamma^*(0, \sigma, \text{Ad}, \psi')|$$

for every $\sigma \in \text{Temp}(U(n))$, where $c(\sigma)$ is the constant defined in Section 3.3.

First, the convergence of the right-hand side of the proposition follows directly from Lemma 2.12.1, Lemma 4.2.1 and 2.7.4.

Let $\varphi_1, \varphi_2 \in \mathcal{S}(Y_n)$ and choose $f_1, f_2 \in \mathcal{S}(G_n(E))$ such that $\varphi_i = \varphi_{f_i}$ for $i = 1, 2$. Then, we have

$$(\varphi_1, \varphi_2)_{L^2(Y_n)} = \int_{Y_n} \int_{G_n(F) \times G_n(F)} f_1(h_1 x) f_2(h_2 x) d h_1 d h_2 d x = \int_{G_n(F)} f(h) d h$$

where we have set $f = f_2 \ast f_1^\gamma$. Moreover, by [4.2.1] we also have

$$(\varphi_1, \varphi_2)_{Y_n, \pi} = |\tau|^{n(n-1)/2}(\beta \hat{\otimes} \beta)(W_{f, \pi})$$

for every $\pi \in BC_n(\text{Temp}(U(n)))$.

### 4.3 A local unfolding identity

Recall that in Section 2.14 we have associated to any $f \in \mathcal{S}(G_n(E))$ a function $W_f \in \mathcal{C}^w(N_n(E) \setminus G_n(E) \times N_n(E) \setminus G_n(E), \psi_n^{-1} \boxtimes \psi_n)$.

**Proposition 4.3.1** For every $f \in \mathcal{S}(G_n(E))$, we have

$$\int_{G_n(F)} f(h) d h = |\tau|^{n(n-1)/4} \int_{N_n(F) \setminus P_n(F)} \int_{N_n(F) \setminus G_n(F)} W_f(p, h) d h d p$$

where the right-hand side is given by an absolutely convergent expression.
Proof: First, we check the absolute convergence of the right-hand side. Let $\nu$ and $r$ be the functions on $G_n(E)$ defined by $\nu(g) = |\det g|_E^{-1/4}$ and $r(g) = (1 + \|e_n g\|_E^{1/2})$ where $e_n = (0, \ldots, 0, 1)$ and $\|\cdot\|$ denotes the following norm on $E^n$: $\langle x_1, \ldots, x_n \rangle \mapsto \max(|x_1|_E, \ldots, |x_n|_E)$ in the $p$-adic case, $(x_1, \ldots, x_n) \mapsto (|x_1|^2_E + \ldots + |x_n|^2_E)^{1/2}$ in the Archimedean case. Then, for every integer $N \geq 1$, $r^n \nu f \in S(G_n(E))$ and moreover we easily check that

$$W_{r^n \nu f}(a_{n-1}k_{n-1}, a_n k_n) = (1 + |a_{n,n}|)^N |\det a_{n-1}^{-1} a_n|^{-1/2} W_f(a_{n-1}k_{n-1}, a_n k_n)$$

for every $(a_{n-1}, a_n, k_{n-1}, k_n) \in A_{n-1}(F) \times A_n(F) \times K_{n-1} \times K_n$. Therefore by Lemma 2.4.3 (or rather its obvious analog for $G_n(E) \times G_n(E)$) applied to $W_{r^n \nu f}$ and the Iwasawa decomposition $G_n(F) = N_n(F) A_n(F) K_n$, $P_n(F) = N_n(F) A_{n-1}(F) K_{n-1}$, it suffices to show the existence of $N \geq 1$ such that for every $d > 0$ the following expression converges:

$$\int_{A_n(F) \times A_{n-1}(F)} (1 + |a_{n,n}|)^{-N} |\det a_{n-1}^{-1} a_n|^{1/2} \prod_{i=1}^{n-1} \left( 1 + \frac{|a_{n-1,i}|}{a_{n-1,i+1}} \right)^{-N} \prod_{i=1}^{n-1} \left( 1 + \frac{|a_{n,i}|}{a_{n,i+1}} \right)^{-N} \delta_{n,E}(a_{n-1})^{1/2} \delta_{n,E}(a_{n-1})^{1/2} \sigma(a_{n-1})^{d} \sigma_1(a_{n-1})^{-1} d a_n da_{n-1}$$

$$= \int_{A_n(F)} (1 + |a_{n,n}|)^{-N} \prod_{i=1}^{n-1} \left( 1 + \frac{|a_{n,i}|}{a_{n,i+1}} \right)^{-N} \sigma(a_{n-1})^{d} |\det a_{n-1}|^{1/2} d a_n$$

$$\times \int_{A_{n-1}(F)} (1 + |a_{n-1,n-1}|)^{-N} \prod_{i=1}^{n-2} \left( 1 + \frac{|a_{n-1,i}|}{a_{n-1,i+1}} \right)^{-N} \sigma(a_{n-1})^{d} |\det a_{n-1}|^{1/2} d a_{n-1}.$$ 

But, by Lemma 2.4.4 both integrals above are convergent for $N \gg 1$ (and any $d$).

We now show the identity of the proposition by induction on $n$. For $n = 1$, this equality is a tautology. Assume from now on that $n \geq 2$ and the result is known for smaller values of $n$. Then, we can write

$$W_f(p, h) dh dp = \int_{P_n(F) \times G_n(F)} W_f(p'p, h'h) |\det(p'h')|^{-1}dh dp' dp dh$$

for every $g \in G_{n-1}(E)$. Then, $\tilde{f} \in S(G_{n-1}(E))$ and we have

$$W_f(p'p, h'h) = W_f(p', h') = \int_{U_{n-1}(E)} \int_{U_n(E)} f'(p'p u'h') \psi_n(u^{-1}) du \psi_n(u)^{-1} dv$$

$$= |\det p'|^2 \int_{U_{n-1}(E)} \int_{U_n(E)} f'(vp'u'h') \psi_n(v^{-1}) du \psi_n(u)^{-1} dv$$

$$= |\det(p'h')| \int_{U_{n-1}(E)} \tilde{f}(p'p u'h') \psi_n(v) du \psi_n(u)^{-1} dv$$

$$= |\det(p'h')| W_{\tilde{f}}(\epsilon_{n-1} p', \epsilon_{n-1} h').$$
for every \((p', h') \in P_{n-1}(F) \times G_{n-1}(F)\) where we have set
\[
\epsilon_{n-1} = \begin{pmatrix}
(-1)^{n-2} \\
\vdots \\
1
\end{pmatrix} \in A_{n-1}(F) \cap P_{n-1}(F).
\]

By the induction hypothesis, we thus get
\[
\int_{N_{n-1}(F) \setminus P_{n-1}(F)} \int_{N_{n-1}(F) \setminus G_{n-1}(F)} W_f(p', h') \det(p'h')^{-1} dh' dp' = \int_{N_{n-1}(F) \setminus P_{n-1}(F)} \int_{N_{n-1}(F) \setminus G_{n-1}(F)} W_f(p', h') dh' dp'.
\]
\[
= |\tau|_{(n-1)(n-2)/4}^{(n-1)(n-2)/4} \int_{G_{n-1}(F)} \int_{P_{n-1}(F) \setminus G_{n-1}(F)} f(p'h') \psi_n(v) \det p^{-1} dh' dp.'
\]

Combining this with (4.3.1) we obtain
(4.3.2)
\[
|\tau|_{E}^{(n-1)(n-2)/4} \int_{N_{n}(F)} \int_{P_{n}(F) \setminus G_{n}(F)} \int_{U_n(F)} W_f(p, h) dh dp.
\]

Let \(h \in G_{n}(F)\) and define \(\varphi \in \mathcal{S}(U_n(E))\) by \(\varphi(v) := \int_{G_{n-1}(F)} (R(h)f)(v') \det h'^{-1} dh'.\) Then, by (2.5.1) we have
\[
\int_{P_{n-1}(F) \setminus U_n(F)} \int_{G_{n-1}(F)} \int_{U_n(F)} |\det h'|-1 f(p'h') \psi_n(pvp^{-1}) \det p dp dh.
\]
\[
= \int_{P_{n-1}(F) \setminus G_{n-1}(F)} \int_{U_n(F)} \varphi(v) \psi_n(h'\psi_n^{-1}) \det h' dh.
\]
\[
= |\tau|_{E}^{(n-1)/2} \int_{U_n(F)} \varphi(v) dv = |\tau|_{E}^{(n-1)/2} \int_{U_n(F)} \int_{G_{n-1}(F)} f(p'h') \det h'^{-1} dh' dv.
\]

Plugging this into (4.3.2) we obtain
\[
\int_{N_{n}(F) \setminus P_{n}(F)} \int_{N_{n}(F) \setminus G_{n}(F)} W_f(p, h) dh dp.
\]
\[
= |\tau|_{E}^{(n-1)/4} \int_{P_{n}(F) \setminus G_{n}(F)} \int_{U_n(F)} \int_{G_{n-1}(F)} f(p'h') \det h'^{-1} dh' dv dh.
\]
\[
= |\tau|_{E}^{(n-1)/4} \int_{G_{n}(F)} f(h) dh.
\]
4.4 End of the proof of Theorem 4.2.2

Let \( \varphi_1, \varphi_2 \in \mathcal{S}(Y_n) \) and \( f_1, f_2, f \in \mathcal{S}(G_n(E)) \) be as in the end of Section 4.2. By Proposition 4.3.1 and Corollary 3.5.1 we have

\[
\begin{align*}
\int_{G_n(F)} f(h) dh &= \left| \beta \right|_E^{n(n-1)/2} \int_{N_n(F) \backslash P_n(F)} \int_{\text{Temp}(U(n))/\text{stab}} \beta(W_{f,BC_n(\sigma)}(p, \cdot)) \frac{\gamma^*(0, \sigma, \text{Ad}, \psi')}{|S_\sigma|} c(\sigma) d\sigma dp \\
&= \left| \beta \right|_E^{n(n-1)/2} \int_{\text{Temp}(U(n))/\text{stab}} (\beta \hat{\otimes} \beta)(W_{f,BC_n(\sigma)}) \frac{\gamma^*(0, \sigma, \text{Ad}, \psi')}{|S_\sigma|} c(\sigma) d\sigma \\
&= \int_{\text{Temp}(U(n))/\text{stab}} (\varphi_1, \varphi_2)_{Y_n,BC_n(\sigma)} \frac{\gamma^*(0, \sigma, \text{Ad}, \psi')}{|S_\sigma|} c(\sigma) d\sigma
\end{align*}
\]

From this, Lemma 4.2.1 and Proposition 4.1.1, we deduce that the right-hand side of (4.4.1) is an absolutely convergent expression and therefore

\[
\int_{G_n(F)} f(h) dh = \left| \beta \right|_E^{n(n-1)/2} \int_{\text{Temp}(U(n))/\text{stab}} (\varphi_1, \varphi_2)_{Y_n,BC_n(\sigma)} \frac{\gamma^*(0, \sigma, \text{Ad}, \psi')}{|S_\sigma|} c(\sigma) d\sigma
\]

where in the last equality we have used 4.2.3. Together with 4.2.2 this shows

\[
(\varphi_1, \varphi_2)_{L^2(Y_n)} = \int_{\text{Temp}(U(n))/\text{stab}} (\varphi_1, \varphi_2)_{Y_n,BC_n(\sigma)} \frac{\gamma^*(0, \sigma, \text{Ad}, \psi')}{|S_\sigma|} c(\sigma) d\sigma
\]

From this, Lemma 4.2.1 and Proposition 4.1.1 we deduce that \( c(\sigma) \gamma^*(0, \sigma, \text{Ad}, \psi') = |\gamma^*(0, \sigma, \text{Ad}, \psi')| \) for almost all \( \sigma \in \text{Temp}(U(n))/\text{stab} \), hence for all. Given this, the above identity is precisely the content of Theorem 4.2.2. ■

5 Applications to the Ichino-Ikeda and formal degree conjectures for unitary groups

5.1 Notation, matching of orbits

Fix an integer \( n \geq 1 \). In this chapter we will consider the following groups and subgroups:
• \( G' = R_{E/F}G_{n,E} \times R_{E/F}G_{n+1,E} \) with its two subgroups \( H_1 = R_{E/F}G_{n,E} \) (diagonally embedded) and \( H_2 = G_{n,F} \times G_{n+1,F} \). We also equip \( H_2(F) = G_n(F) \times G_{n+1}(F) \) with the character \( \eta = \eta_n \boxtimes \eta_{n+1} \) where we recall that \( \eta_k \) stands for the character of \( G_k(F) \) given by \( \eta_k(g) = \eta_{E/F}(\det g)^k \).

• For \( V \) a hermitian space of dimension \( n \) (with underlying hermitian form \( h \)), we set \( H^V = U(V) \) and \( G^V = U(V) \times U(V') \) where \( V' = V \oplus E v_0 \) is equipped with the hermitian form \( h' \) given by \( h'(v_1 + \lambda v_0, v_2 + \mu v_0) = h(v_1, v_2) + \lambda \mu \) for all \( v_1, v_2 \in V \) and \( \lambda, \mu \in E \). We consider \( H^V \) as a subgroup of \( G^V \) through the natural diagonal embedding \( H^V \hookrightarrow G^V \).

We let \( H_1 \times H_2 \) (resp. \( H^V \times H^V \)) act on \( G' \) (resp. \( G^V \)) by \((h_1, h_2) \cdot g = h_1 g h_2^{-1}\). A geometric point \( g \in G' \) (resp. \( g \in G^V \)) is said to be regular semi-simple if its stabilizer in \( H_1 \times H_2 \) (resp. \( H^V \times H^V \)) for this action is trivial and its orbit \( H_1 g H_2 \) (resp. \( H^V g H^V \)) is Zariski closed. We denote by \( G'_{rs} \subset G' \) and \( G^V_{rs} \subset G^V \) the open subsets of regular semi-simple elements. These are not empty (\([Zha2, \S 2.1]\)) and the actions of \( H_1 \times H_2 \) and \( H^V \times H^V \) on \( G'_{rs} \) and \( G^V_{rs} \) respectively are free. Let \( \mathcal{B} \) and \( \mathcal{B}^V \) be the geometric quotients \( H_1 G'/H_2 \) and \( H^V G^V/H^V \) respectively. Geometric points in \( \mathcal{B} \) and \( \mathcal{B}^V \) correspond bijectively to closed geometric orbits in \( G' \) and \( G^V \) respectively. Let \( \mathcal{B}_{rs} \subset \mathcal{B} \) and \( \mathcal{B}^V_{rs} \subset \mathcal{B}^V \) be the open subsets corresponding to regular semi-simple elements (or orbits) i.e. \( \mathcal{B}_{rs} = H_1 \backslash G'_{rs}/H_2 \) and \( \mathcal{B}^V_{rs} = H^V \backslash G^V_{rs}/H^V \). Then, there is a natural isomorphism \( \mathcal{B} \cong \mathcal{B}^V \) which restricts to \( \mathcal{B}_{rs} \cong \mathcal{B}^V_{rs} \) (\([CZ, \text{Lemme 15.1.4.1}]\)). Moreover, when taking \( F \)-points this isomorphism induces a bijection \( (\ref{eq:isomorphism}) \)

\[
(5.1.1) \quad H_1(F)\backslash G'_{rs}(F)/H_2(F) \cong \bigsqcup_V H^V(F)\backslash G^V_{rs}(F)/H^V(F)
\]

where the disjoint union of the right-hand side runs over a set of representatives of isomorphism classes of hermitian spaces of dimension \( n \). Two regular semi-simple elements \( \gamma \in G'_{rs}(F) \) and \( \delta \in G^V_{rs}(F) \) whose orbits correspond to each other by the above bijection will be said to match and we will abbreviate this by the notation \( \gamma \leftrightarrow \delta \).

Since the action is free, the quotient of the (restriction to \( G'_{rs}(F) \) of the) Haar measure on \( G'(F) \) by the Haar measure on \( H_1(F) \times H_2(F) \) defines a measure on \( H_1(F)\backslash G'_{rs}(F)/H_2(F) \) that we shall denote by \( d\gamma \). Similarly for every hermitian space \( V \) of dimension \( n \), the quotient of the Haar measure on \( G^V(F) \) by the Haar measure on \( H^V(F) \times H^V(F) \) gives rise to a measure \( d\delta = d^V\delta \) on \( H^V(F)\backslash G^V_{rs}(F)/H^V(F) \).

**Lemma 5.1.1** The bijection \((5.1.1)\) is measure preserving, i.e. it sends the measure \( d\gamma \) to the sum of the measures \( d^V\delta \).

Proof: Let \( V \) be a hermitian space of dimension \( n \). Then, by our normalization of measures (see Section \( 2.3 \)), \( d\gamma = |\omega'|_\psi \) and \( d^V\delta = |\omega^V|_\psi \) where \( \omega' \) and \( \omega^V \) are certain volume forms on \( \mathcal{B}_{rs,F} = \mathcal{B}^V_{rs,F} \). More precisely, \( \omega^V \) is obtained as the quotient of the volume form \( \omega_{G^V_{rs,F}} \) by \( \omega_{H^V_{rs,F}} \times H^V_{rs,F} \) fixed in Section \( 2.3 \) whereas \( \omega' \) is the quotient of \( \omega_{G'_{rs,F}} \) by \( \omega_{H_1,F} \times H_2,F \). Obviously, it
suffices to show that $\omega'$ and $\omega^V$ are equal up to a sign. We have $B_{rs,F} = H_{1,F}\backslash G'_{rs,F}/H_{2,F}$ and using the $\mathcal{F}$-algebra isomorphism $E \otimes_{\mathcal{F}} \mathcal{F} \simeq \mathcal{F} \times \mathcal{F}, \, x \otimes y \mapsto (xy, x^c y)$ we get identifications

$$H_{1,F} \simeq G_{n,F} \times G_{n,F}, \quad G'_F \simeq (G_{n,F} \times G_{n,F}) \times (G_{n+1,F} \times G_{n+1,F}), \quad H_{2,F} \simeq G_{n,F} \times G_{n+1,F}$$

such that the inclusion $H_{2,F} \subset G'_F$ is the product of the diagonal embeddings $G_{k,F} \hookrightarrow G_{k,F} \times G_{k,F}$ for $k = n, n + 1$ whereas the inclusion $H_{1,F} \subset G'_F$ is the identity in the first component and the natural embedding $G_{n,F} \times G_{n,F} \hookrightarrow G_{n+1,F} \times G_{n+1,F}$ in the second. Similarly, using the same $\mathcal{F}$-algebra isomorphism and a linear bijection $V \simeq E^n$ that we extend to $V' \simeq E^{n+1}$ by sending $v_0$ to $(0, \ldots, 0, 1)$ we get identifications

$$H^V_{F} \simeq G_{n,F}, \quad G^V_F \simeq G_{n,F} \times G_{n+1,F}$$

such that the inclusion $H^V_{F} \subset G^V_F$ is the identity in the first component and the natural embedding $G_{n,F} \hookrightarrow G_{n+1,F}$ in the second. Using these identifications, we get an isomorphism

(5.1.2) \[ H_{1,F} \simeq H^V_{F} \times H^V_{F} \]

and the projection onto the first components yields another isomorphism

(5.1.3) \[ G'_F/H_{2,F} \simeq G^V_F \]

which is $G'_F \simeq G^V_F \times G^V_F$-equivariant thus inducing an identification $B_F = H_{1,F}\backslash G'_F/H_{2,F} \simeq H^V_{F}\backslash G^V_F/H^V_{F} = B^V_F$. This last isomorphism is by its very definition the same as before (base-changed to $\mathcal{F}$). Moreover, both 5.1.2 and 5.1.3 obviously extends to the natural split forms over $\mathbb{Z}$ of all the groups under consideration. Therefore (and since the group of outer automorphisms of $G_n$ has order 2), 5.1.2 sends $\omega_{H_{1,F}}$ to $\pm \omega_{H^V_{F} \times H^V_{F}}$ and 5.1.3 sends the quotient of the volume form $\omega_{G'_F}$ by $\omega_{H_{2,F}}$ to $\pm \omega_{G^V_F}$. From this, it immediately follows that the isomorphism $B_{rs,F} \simeq B^V_{rs,F}$ sends $\omega'$ to $\pm \omega^V$. $\blacksquare$

At some point we will need to “linearize” certain expressions and thus we introduce the following extra notation:

- Let $S$ be the symmetric space $S = \{ g \in R_{E/F}G_{n+1} \mid gg^c = 1 \}$ and $\mathfrak{s}$ be its tangent space at the identity i.e. $\mathfrak{s} = \{ Y \in R_{E/F} \mathfrak{gl}_{n+1} \mid Y + Y^c = 0 \}$.

- For every hermitian space $V$ of dimension $n$, let $u^V = u(V')$ be the Lie algebra of $U(V')$.

We let $G_n$ (resp. $U(V)$) act on $\mathfrak{s}$ (resp. $u^V$) by conjugation. As before, a geometric point $X \in \mathfrak{s}$ (resp. $X \in u^V$) is said to be regular semi-simple if its stabilizer in $G_n$ (resp. $U(V)$) for this action is trivial and the corresponding orbit is closed. We denote by $\mathfrak{s}_{rs} \subset \mathfrak{s}$ and $u^V_{rs} \subset u^V$ the open subsets of regular semi-simple elements. These are not empty ([JR, §3 & 4]) and the actions of $G_n$ and $U(V)$ on $\mathfrak{s}_{rs}$ and $u^V_{rs}$ respectively are free. Moreover, there is also a natural isomorphism between geometric quotients $\mathfrak{s}/G_n \simeq u^V/U(V)$ which restricts

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to $\mathfrak{s}_{rs}/G_n \simeq \mathfrak{u}^V_{rs}/U(V)$ (see [Chau, Proposition 2.2.2.1]) and gives rise, when taking $F$-points, to a bijection ([Chau, Lemmes 2.1.5.1, 2.1.5.3 & Proposition 2.2.4.1])

\begin{equation}
\mathfrak{s}_{rs}(F)/G_n(F) \simeq \bigsqcup_V \mathfrak{u}^V_{rs}(F)/U(V)(F)
\end{equation}

where the disjoint union of the right-hand side runs, once again, over a set of representatives of isomorphism classes of hermitian spaces of dimension $n$. Two regular semi-simple elements $Y \in \mathfrak{s}_{rs}(F)$ and $X \in \mathfrak{u}^V_{rs}(F)$ whose orbits correspond by the above bijection will be said to match and we will abbreviate this by $Y \leftrightarrow X$.

There is the following $G_n$-equivariant isomorphism

$$
\nu : R_{E/F}G_{n+1}/G_{n+1} \rightarrow S
$$

$$
g \mapsto g(g^{-1})^c.
$$

We use $\nu$ to transfer the measure on $G_{n+1}(E)/G_{n+1}(F)$ to a measure on $S(F)$ which in turn induces a Haar measure on $\mathfrak{s}(F)$ by taking its fiber at 1 (this makes sense since the measure on $S(F)$ thus obtained is in the natural class of measures on this $F$-analytic manifold). Similarly, the Haar measure on $U(V')(F)$ induces one on $u^V(F)$. Now, since the actions are free, the quotient of the measure on $\mathfrak{s}(F)$ (resp. $u^V(F)$) by the Haar measure on $G_n(F)$ (resp. $U(V)(F)$) defines a measure on $\mathfrak{s}_{rs}(F)/G_n(F)$ (resp. $\mathfrak{u}^V_{rs}(F)/U(V)(F)$) that we shall denote by $dY$ (resp. $dX = d^V X$). By essentially the same arguments as for Lemma 5.1.1 we have:

**Lemma 5.1.2** The bijection (5.1.4) is measure preserving, i.e. it sends the measure $dY$ to the sum of the measures $d^V X$.

Let $c : s \rightarrow S$ (resp. $c^V : U^V \rightarrow U(V')$) be the rational isomorphism (henceforth called Cayley map) sending $X$ to $\frac{1+X/2}{1-X/2}$. Note that $c$ (resp. $c^V$) is $G_n$-equivariant (resp. $U(V)$-equivariant). By abuse of notation we will also write $c$ for $c^V$, hoping that it will not create any confusion for the attentive reader. We fix once and for all a small open neighborhood $U$ of 0 in $(\mathfrak{s}/G_n)(F) = (\mathfrak{u}^V/U(V))(F)$ on the inverse image of which (both in $\mathfrak{s}(F)$ and $\mathfrak{u}^V(F)$ for every hermitian space $V$ of dimension $n$) $c$ is well-defined as well as a cut-off function $\alpha \in C^\infty_c(U)$ (here we remark that the “base” $\mathfrak{s}/G_n$ is smooth, and even an affine space, see [Chau, Proposition 2.1.5.2]) such that $\alpha = 1$ on some neighborhood of 0. This allows to define two applications $\Phi' \in S(S(F)) \rightarrow \Phi'_2 \in S(\mathfrak{s}(F))$ and $\Phi^V \in S(U(V')(F)) \rightarrow \Phi^V_2 \in S(u^V(F))$ by

$$
\Phi'_2(Y) = \begin{cases} 
\eta(\det(1 - Y))^{-n} \alpha(Y) \Phi'(c(Y)) & \text{if } Y \in U \\
0 & \text{otherwise}
\end{cases}
$$

and

$$
\Phi^V_2(X) = \begin{cases} 
\alpha(X) \Phi^V(c(X)) & \text{if } X \in U \\
0 & \text{otherwise}
\end{cases}
$$

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for every $Y \in \mathfrak{s}(F)$ and $X \in \mathfrak{u}^V(F)$. The presence of the extra factor $\eta'(\det(1 - Y))^{-n}$ is justified a posteriori by 5.2.1. Note that in the $p$-adic case, up to shrinking $\mathcal{U}$ we can make this factor identically 1 on $\mathcal{U}$.

We also define two applications $f' \in \mathcal{S}(G'(F)) \mapsto \tilde{f}' \in \mathcal{S}(\mathcal{S}(F))$ and $f^V \in \mathcal{S}(G^V(F)) \mapsto \tilde{f}^V \in \mathcal{S}(U(V')(F))$ by

$$
\tilde{f}^V(g) = \int_{H^V(F)} f(h(1, g))dh, \quad g \in U(V')(F)
$$

and

$$
\tilde{f}'(s) = \int_{H_1(F) \times G_{n+1}(F)} f'(h_1(1, \nu^{-1}(s)h_{n+1}))\eta'_{n+1}(\nu^{-1}(s)h_{n+1})dh_{n+1}dh_1, \quad s \in S(F)
$$

where $\nu$ is the isomorphism $G_{n+1}(E)/G_{n+1}(F) \simeq S(F)$ defined above and we recall that $\eta'_{n+1}$ is a character of $G_{n+1}(E)$ extending the character $\eta_{n+1}$ of $G_{n+1}(F)$ (see 2.2).

We shall denote the composition of the two maps $f' \mapsto \tilde{f}'$ and $\Phi' \mapsto \Phi'_{\tilde{f}'}$ (resp. $f^V \mapsto \tilde{f}^V$ and $\Phi^V \mapsto \Phi^V_{\tilde{f}'}$) by $f' \in \mathcal{S}(G'(F)) \mapsto \tilde{f}' \in \mathcal{S}(\mathcal{S}(F))$ (resp. $f^V \in \mathcal{S}(G^V(F)) \mapsto \tilde{f}^V \in \mathcal{S}(\mathcal{S}(F))$).

We define two non-degenerate $V$- and $U(V')(F)$-invariant symmetric bilinear forms $\langle \cdot, \cdot \rangle : \mathfrak{s}(F) \times \mathfrak{s}(F) \to F$ and $\langle \cdot, \cdot \rangle : \mathfrak{u}^V(F) \times \mathfrak{u}^V(F) \to F$ by

$$
\langle X, Y \rangle = \text{Trace}(XY)
$$

for all $X, Y \in \mathfrak{s}(F)$ or $X, Y \in \mathfrak{u}^V(F)$. This allows to define Fourier transforms $\varphi' \in \mathcal{S}(\mathfrak{s}(F)) \mapsto \mathcal{F} \varphi' \in \mathcal{S}(\mathcal{S}(F))$ and $\varphi^V \in \mathcal{S}(\mathfrak{u}^V(F)) \mapsto \mathcal{F} \varphi^V \in \mathcal{S}(\mathcal{S}(F))$ by

$$
\mathcal{F} \varphi'(Y) = \int_{\mathfrak{s}(F)} \varphi'(Y')\psi'(\langle Y', Y \rangle)dY' \quad \text{(resp.} \quad \mathcal{F} \varphi^V(X) = \int_{\mathfrak{u}^V(F)} \varphi^V(X')\psi'(\langle X', X \rangle)dX' \text{)}
$$

for every $Y \in \mathfrak{s}(F)$ and $X \in \mathfrak{u}^V(F)$. By our choice of Haar measures, we have $\mathcal{F}(\mathcal{F} \varphi')(Y) = \varphi'(-Y)$ and $\mathcal{F}(\mathcal{F} \varphi^V)(X) = \varphi^V(-X)$ for every $\varphi' \in \mathcal{S}(\mathfrak{s}(F))$ and $\varphi^V \in \mathcal{S}(\mathfrak{u}^V(F))$.

Finally, we fix once an for all a set $V$ of representatives of hermitian spaces of dimension $n$ and we set $G = \bigsqcup_{V \in V} G^V$, $\mathfrak{u} = \bigsqcup_{V \in V} \mathfrak{u}^V$ so that

$$
\mathcal{S}(G(F)) = \bigoplus_{V \in V} \mathcal{S}(G^V(F)) \quad \text{and} \quad \mathcal{S}(\mathfrak{u}(F)) = \bigoplus_{V \in V} \mathcal{S}(\mathfrak{u}^V(F)).
$$

We extend the maps $f^V \mapsto \tilde{f}^V$ by linearity to

$$
\mathcal{S}(G(F)) \to \mathcal{S}(\mathfrak{u}(F))
$$

$$
f = (f^V)_V \mapsto \tilde{f} = (\tilde{f}^V)_V.
$$

We could also extend the Fourier transforms $\varphi^V \mapsto \mathcal{F} \varphi^V$ by linearity to $\mathcal{S}(\mathfrak{u}(F))$ but it turns out to be a better choice to use the following convention (see in particular Theorem 5.2.2)

$$
\mathcal{F} \varphi = (\eta_{E/F}(\text{disc}(V))^n \mathcal{F} \varphi^V)_V
$$

for every $\varphi = (\varphi^V)_V \in \mathcal{S}(\mathfrak{u}(F))$ where $\text{disc}(V)$ stands for the discriminant of the hermitian space $V$ (that is the determinant of the matrix representing the hermitian form, in any basis, seen as an element of $F^×/N(E^×)$).
5.2 Relative orbital integrals and matching of functions

We keep the notation of the previous section. Let $V \in \mathcal{V}$. For $f' \in \mathcal{S}(G'(F))$ (resp. $f^V \in \mathcal{S}(G^V(F))$) and $\gamma \in G'_{rs}(F)$ (resp. $\delta \in G^V_{rs}(F)$), we define a relative orbital integral $O_\eta(\gamma, f')$ (resp. $O(\delta, f^V)$) by

$$O_\eta(\gamma, f') = \int_{H_1(F) \times H_2(F)} f'(h_1 \gamma h_2) \eta(h_2) dh_2 dh_1$$

(resp. $O(\delta, f^V) = \int_{H^V(F) \times H^V(F)} f^V(h_1 \delta h_2) dh_2 dh_1$)

Similarly, for $\varphi' \in \mathcal{S}(s(F))$ (resp. $\varphi^V \in \mathcal{S}(u^V(F))$) and $Y \in s_{rs}(F)$ (resp. $X \in u^V_{rs}(F)$) we define a relative orbital integral

$$O_\eta(Y, \varphi') = \int_{G_n(F)} \varphi'(hYh^{-1}) \eta_E/F(h) dh$$

(resp. $O(X, \varphi^V) = \int_{U(V)(F)} \varphi^V(hXh^{-1}) dh$)

We define transfer factors

$$\Omega : G'_{rs}(F) \to \mathbb{C}^\times \text{ and } \omega : s_{rs}(F) \to \mathbb{C}^\times$$

by

$$\omega(Y) = \eta'(\det(e_{n+1}, e_{n+1}Y, \ldots, e_{n+1}Y^n)), \quad Y \in s_{rs}(F)$$

and

$$\Omega((g_n, g_{n+1})) = \eta'(g_n^{-1}g_{n+1})^{-n}\eta'(\det(e_{n+1}, e_{n+1}s, \ldots, e_{n+1}s^n)), \quad (g_n, g_{n+1}) \in G'_{rs}(F)$$

where $e_{n+1} = (0, \ldots, 0, 1)$ and in the last equality we have used the notation $s = \nu(g_n^{-1}g_{n+1})$ for simplicity.

We say that two functions $f = (f^V)_V \in \mathcal{S}(G(F))$ and $f' \in \mathcal{S}(G'(F))$ match or that they are transfer of each other if

$$\Omega(\gamma)O_\eta(\gamma, f') = O(\delta, f^V)$$

for every $V \in \mathcal{V}$ and every pair $(\gamma, \delta) \in G'_{rs}(F) \times G^V_{rs}(F)$ with matching orbits. Similarly, we say that two functions $\varphi = (\varphi^V)_V \in \mathcal{S}(u(F))$ and $\varphi' \in \mathcal{S}(s(F))$ match or that they are transfer of each other if

$$\omega(Y)O_\eta(Y, \varphi') = O(X, \varphi^V)$$

for every $V \in \mathcal{V}$ and every pair $(Y, X) \in s_{rs}(F) \times u^V_{rs}(F)$ with matching orbits.

The following follows easily from the definitions and a painless computation:

(5.2.1) If $f' \in \mathcal{S}(G'(F))$ and $f \in \mathcal{S}(G(F))$ match then $\tilde{f}'_{\tilde{z}}$ and $\tilde{f}_{\tilde{z}}$ match.

Finally, we recall the following deep results from [Zha1] and [Xue].

**Theorem 5.2.1 (Zhang, Xue)**

(i) Assume that $F$ is a $p$-adic field. Then, for every $f \in \mathcal{S}(G(F))$ (resp. $\varphi \in \mathcal{S}(u(F))$) there exists $f' \in \mathcal{S}(G'(F))$ (resp. $\varphi' \in \mathcal{S}(s(F))$) such that $f$ and $f'$ match (resp. $\varphi$ and $\varphi'$ match). Conversely, for every $f' \in \mathcal{S}(G'(F))$ (resp. $\varphi' \in \mathcal{S}(s(F))$) there exists $f \in \mathcal{S}(G(F))$ (resp. $\varphi \in \mathcal{S}(u(F))$) such that $f$ and $f'$ match (resp. $\varphi$ and $\varphi'$ match).
(ii) Assume that \( F \) is Archimedean. Then, there exists dense subspaces \( \mathcal{S}(G(F))_{\text{trans}} \subset \mathcal{S}(G(F)), \mathcal{S}(G(F))_{\text{trans}} \subset \mathcal{S}(G(F)), \mathcal{S}(u(F))_{\text{trans}} \subset \mathcal{S}(u(F)) \) and \( \mathcal{S}(s(F))_{\text{trans}} \subset \mathcal{S}(s(F)) \) satisfying the following: for every \( f \in \mathcal{S}(G(F))_{\text{trans}} \) (resp. \( \varphi \in \mathcal{S}(u(F))_{\text{trans}} \) there exists \( f' \in \mathcal{S}(G(F)) \) (resp. \( \varphi' \in \mathcal{S}(s(F))_{\text{trans}} \)) such that \( f \) and \( f' \) match (resp. \( \varphi \) and \( \varphi' \) match) and conversely, for every \( f' \in \mathcal{S}(G(F))_{\text{trans}} \) (resp. \( \varphi' \in \mathcal{S}(s(F))_{\text{trans}} \)) there exists \( f \in \mathcal{S}(G(F)) \) (resp. \( \varphi \in \mathcal{S}(u(F)) \)) such that \( f \) and \( f' \) match (resp. \( \varphi \) and \( \varphi' \) match).

**Theorem 5.2.2 (Zhang, Xue)** Let \( \varphi \in \mathcal{S}(u(F)) \) and \( \varphi' \in \mathcal{S}(s(F)) \). Then, if \( \varphi \) and \( \varphi' \) match so do \( \eta_{E/F}(-1)^{n(n+1)/2} \lambda_{E/F}(\psi')^{n(n+1)/2} \mathcal{F} \varphi \) and \( \mathcal{F} \varphi' \) (where the Fourier transform \( \mathcal{F} \varphi \) of \( \varphi \) is as defined in the end of Section 5.1).

**Remark 5.2.3** We remark that the constant \( \eta_{E/F}(-1)^{n(n+1)/2} \lambda_{E/F}(\psi')^{n(n+1)/2} \) appearing in the above theorem is not exactly the same as the one we can extract from the computations of [Zha1, Sect. 4]. We refer the reader to [Chau, Theorem 3.4.2.1] for a precise determination of this constant (which fits precisely the one given above) in the p-adic case. In the Archimedean case, the above theorem corresponds precisely what is stated in [Xue, §9].

### 5.3 Relative characters

Let \( V \in \mathcal{V} \) and \( \pi \in \text{ Temp}(G^V) \). Then, for every \( f \in \mathcal{S}(G^V(F)) \) we set

\[
J_\pi(f) = \int_{H^V(F)} \text{Trace}(\pi(h)\pi(f^\vee))dh = \int_{H^V(F)} f_\pi(h)dh
\]

the integral being absolutely convergent by the following lemma.

**Lemma 5.3.1** For every \( f \in \mathcal{C}^w(G^V(F)) \) the integral

\[
\int_{H^V(F)} f(h)dh
\]

converges absolutely and defines a continuous linear form on \( \mathcal{C}^w(G^V(F)) \).

**Proof:** This follows easily from [Beu1 Lemma 6.5.1(i)]. □

Set \( N' = R_{E/F}N_n \times R_{E/F}N_{n+1} \) (a maximal unipotent subgroup of \( G' \)) and \( \psi_{N'} = \psi_n \square \psi_{n+1} \) (a generic character of \( N'(F) \)). We define continuous linear forms

\[
\beta' : \mathcal{C}^w(N'(F) \backslash G'(F), \psi_{N'}) \rightarrow \mathbb{C}
\]

by

\[
\beta'(W) = \int_{N_2(F) \backslash P_2(F)} W(p)\eta(p)dp
\]

and

\[
\lambda(W) = \int_{N_1(F) \backslash H_1(F)} W(h_1)dh_1, \quad W \in \mathcal{W}(\Pi, \psi)
\]
where $P_2 = H_2 \cap P'$, $N_2 = H_2 \cap N'$ and $N_1 = H_1 \cap N'$. Note that $\beta' = \beta_n \hat{\otimes} \beta_{n+1}$ where $\beta_n$ and $\beta_{n+1}$ are defined in Section 2.15. That $\lambda$ is absolutely convergent and continuous is a consequence of the following lemma.

**Lemma 5.3.2**  
(i) For every $W \in \mathcal{C}^w(N'(F)\backslash G'(F), \psi_{N'})$ the integral

$$\lambda(W) = \int_{N_1(F)\backslash H_1(F)} W(h_1)dh_1$$

is absolutely convergent and defines a continuous linear form on $\mathcal{C}^w(N'(F)\backslash G'(F), \psi_{N'})$.

(ii) For every $\phi \in \mathcal{C}^w(G'(F))$ the integral

$$\int_{H_1(F)} \phi(h_1)dh_1$$

is absolutely convergent and defines a continuous linear form on $\mathcal{C}^w(G'(F))$.

**Proof:**

(i) The proof is completely similar to the proof of Lemma 2.15, and left to the reader.

(ii) By definition of $\phi \in \mathcal{C}^w(G'(F))$ it suffices to show that for every $d > 0$ the integral

$$\int_{H_1(F)} \Xi_{G'}(h_1)\sigma(h_1)^d dh_1$$

converges. This is the content of [Wald2, 4.1(3)] in the $p$-adic case but the proof works verbatim in the Archimedean case.

Let $\Pi \in \text{Temp}(G')$. We can write $\Pi = \Pi_n \hat{\otimes} \Pi_{n+1}$ where $\Pi_k \in \text{Temp}(G_k(E))$, $k = n, n+1$. Let $\mathcal{W}(\Pi, \psi) = \mathcal{W}(\Pi_n, \psi_n) \hat{\otimes} \mathcal{W}(\Pi_{n+1}, \psi_{n+1})$ be the Whittaker model of $\Pi$ with respect to the character $\psi_{N'}$. We equip $\mathcal{W}(\Pi, \psi)$ with the $G'(F)$-invariant scalar product (see Section 2.8)

$$(W, W')^{\text{Whitt}}_{G'} = \int_{N'(F)\backslash P'(F)} W(p)\overline{W'(p)}dp, \ W, W' \in \mathcal{W}(\Pi, \psi)$$

where $P' = R_{E/F} P_n \times R_{E/F} P_{n+1}$. Let $\mathcal{B}(\Pi, \psi)$ be an orthonormal basis of (the Hilbert completion of) $\mathcal{W}(\Pi, \psi)$ for this scalar product obtained by taking the union of orthonormal basis for $\mathcal{W}(\Pi, \psi)[\delta]$ for every $\delta \in \hat{K}'$ where $K' = K_{n,E} \times K_{n+1,E}$. Then, for every $f \in \mathcal{S}(G'(F))$ we set

$$I_{\Pi}(f) = \sum_{W \in \mathcal{B}(\Pi, \psi)} \overline{\lambda(W)}\beta'(\Pi(f')W).$$

We also define on $\mathcal{S}(G'(F))$ the following positive semi-definite scalar product

$$(f_1, f_2)_{\mathcal{X}_2,\Pi} = \sum_{W \in \mathcal{B}(\Pi, \psi)} \beta'(\Pi(f_1')W)\overline{\beta'(\Pi(f_2')W), \ f_1, f_2 \in \mathcal{S}(G'(F))}$$

where as usual $f_k'(g) = f_k(g^{-1})$ for $k = 1, 2$. 

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Proposition 5.3.3 The expressions defining $I_{\Pi}(f)$ and $(f_1, f_2)_{X_2, \Pi}$ are absolutely convergent and do not depend on the choice of the basis $B(\Pi, \psi)$. The functions

$$\Pi \in \text{Temp}(G'(F)) \mapsto I_{\Pi}(f) \text{ and } \Pi \in \text{Temp}(G'(F)) \mapsto (f_1, f_2)_{X_2, \Pi}$$

belong to $S(\text{Temp}(G'(F)))$ and the linear (resp. sesquilinear) map

$$f \in S(G'(F)) \mapsto (\Pi \mapsto I_{\Pi}(f)) \in S(\text{Temp}(G'))$$

(resp. $(f_1, f_2) \in S(G'(F))^2 \mapsto (\Pi \mapsto (f_1, f_2)_{X_2, \Pi}) \in S(\text{Temp}(G'))$)

is continuous. Moreover, we have

$$\int_{H_1(F)} (L(h_1)f_1, f_2)_{X_2, \Pi} dh_1 = |\tau|_E^{-n(n-1)/2} I_{\Pi}(f_1)I_{\Pi}(f_2)$$

for every $f_1, f_2 \in S(G'(F))$.

Proof: The proof of the first part of the proposition is completely similar to the proof of the first part of Lemma 4.2.1. We prove the last part (i.e. identity 5.3.1). Let $f_1, f_2 \in S(G'(F))$. Then, we have (at least formally)

$$\int_{H_1(F)} (L(h_1)f_1, f_2)_{X_2, \Pi} dh_1 = \int_{H_1(F)} \sum_{W \in B(\Pi, \psi)} \beta'(\Pi(f_1^\gamma)\Pi(h_1)W)\beta'(\Pi(f_2^\gamma)W) dh_1$$

$$= \int_{H_1(F)} \sum_{W \in B(\Pi, \psi)} \sum_{W' \in B(\Pi, \psi)} (\Pi(h_1)W, W')_{G'}^\text{Whitt} \beta'(\Pi(f_1^\gamma)W')\beta'(\Pi(f_2^\gamma)W) dh_1$$

We claim

$$\int_{H_1(F)} (L(h_1)f_1, f_2)_{X_2, \Pi} dh_1 = \int_{H_1(F)} \sum_{W \in B(\Pi, \psi)} \beta'(\Pi(f_1^\gamma)W)W'$$

and

$$\sum_{W' \in B(\Pi, \psi)} \beta'(\Pi(f_2^\gamma)W)W'$$

converge absolutely in $W(\Pi, \psi)$ whereas by 2.6.1, the sesquilinear map

$$W(\Pi, \psi) \times W(\Pi, \psi) \rightarrow C^w(G'(F))$$

$$(W, W') \mapsto (g \mapsto (R(g)W, W')_{G'}^\text{Whitt})$$

is continuous. It follows that the series of functions

$$g \in G'(F) \mapsto \sum_{W \in B(\Pi, \psi)} \sum_{W' \in B(\Pi, \psi)} (\Pi(g)W, W')_{G'}^\text{Whitt} \beta'(\Pi(f_1^\gamma)W')\beta'(\Pi(f_2^\gamma)W)$$

is actually finite and the result follows from 2.6.1.
converges absolutely in $\mathcal{C}^w(G'(F))$ and the claim now follows from Lemma $5.3.2$ (ii).

By $5.3.2$ we can write
\[
\int_{H_1(F)} (L(h_1)f_1, f_2)_{X_2,\Pi} dh_1 = \sum_{W \in \mathcal{B}(\Pi, \psi)} \sum_{W' \in \mathcal{B}(\Pi, \psi)} \int_{H_1(F)} (\Pi(h_1)W, W')^{Whitt}_{G'} dh_1 \beta'(\Pi(f_1)W')\beta'(\Pi(f_2)W)
\]

Assume one moment proved the following:

(5.3.3) For every $W, W' \in \mathcal{W}(\Pi, \psi)$ we have
\[
\int_{H_1(F)} (\Pi(h_1)W, W')^{Whitt}_{G'} dh_1 = |\tau|_E^{n(n-1)/2} \lambda(W)\lambda(W')
\]

Then, we would get
\[
\int_{H_1(F)} (L(h_1)f_1, f_2)_{X_2,\Pi} dh_1 = |\tau|_E^{n(n-1)/2} \sum_{W \in \mathcal{B}(\Pi, \psi)} \sum_{W' \in \mathcal{B}(\Pi, \psi)} \lambda(W')\lambda(W)\beta'(\Pi(f_1)W')\beta'(\Pi(f_2)W)
\]
\[
= |\tau|_E^{n(n-1)/2} I_{\Pi}(f_1)I_{\Pi}(f_2)
\]

hence the result.

It only remains to prove $5.3.3$ Since, by $2.6.1$ and Lemma $5.3.2$ (ii) again, the sesquilinear form
\[
(W, W') \in \mathcal{W}(\Pi, \psi)^2 \mapsto \int_{H_1(F)} (\Pi(h_1)W, W')^{Whitt}_{G'} dh_1
\]
is continuous, we just need to show $5.3.3$ when $W = W_n \otimes W_{n+1}$ and $W' = W_n' \otimes W_{n+1}'$ where $W_k, W_k' \in \mathcal{W}(\Pi_k, \psi_k), k = n, n+1$. Then, returning to the definitions, we have
\[
\int_{H_1(F)} (\Pi(h_1)W, W')^{Whitt}_{G'} dh_1 = \int_{G_n(E)} \int_{N_n(E)\setminus G_n(E)} (R(h)W_n, W_n')^{Whitt}_{G'} W_{n+1}(gh)W_{n+1}'(g) dg dh
\]
where $(., .)^{Whitt}$ is the scalar products on $\mathcal{C}^w(N_n(E)\setminus G_n(E), \psi_n)$ defined in Section $2.8$. Then, by formal manipulations we get
\[
\int_{H_1(F)} (\Pi(h_1)W, W')^{Whitt}_{G'} dh_1 = \int_{G_n(E)} \int_{N_n(E)\setminus G_n(E)} (R(h)W_n, W_n')^{Whitt}_{G'} W_{n+1}(gh)W_{n+1}'(g) dg dh
\]
\[
= \int_{N_n(E)\setminus G_n(E)} \int_{N_n(E)\setminus G_n(E)} (R(uh)W_n, R(g)W_n')^{Whitt}_{G'} W_{n+1}(h)W_{n+1}'(g) dg dh
\]
\[
= \int_{N_n(E)\setminus G_n(E)} \int_{N_n(E)\setminus G_n(E)} \int_{N_n(E)} (R(uh)W_n, R(g)W_n')^{Whitt}_{G'} W_{n+1}(h)W_{n+1}'(g) du \psi_n(u)^{-1} dh W_{n+1}(h)W_{n+1}'(g) dg dh
\]
\[
= |\tau|_E^{n(n-1)/2} \int_{N_n(E)\setminus G_n(E)} \int_{N_n(E)\setminus G_n(E)} W_n(h)W_n(g) W_{n+1}(h)W_{n+1}'(g)dg dh = |\tau|_E^{n(n-1)/2} \lambda(W')\lambda(W')
\]

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where in the fourth equality we have used Proposition 2.14.3. If these formal manipulations were justified this would prove 5.3.3. Unfortunately, the above expression is certainly not absolutely convergent in general. However, by Lemma 2.6.1 the function
\[ \phi : g \in G_n(E) \mapsto (R(g)W_n, W'_n)^{\text{Whitt}} \]
begins to \( C^w(G_n(E)) \) and we recall that to every function \( \phi \in C^w(G_n(E)) \) we have associated in Section 2.14 a function
\[ W_\phi \in C^w(N_n(E) \backslash G_n(E) \times N_n(E) \backslash G_n(E), \psi_n^{-1} \boxtimes \psi_n). \]
Then that the result of the above formal manipulations is indeed correct follows from:

\[ (5.3.4) \text{ For every } \phi \in C^w(G_n(E)) \text{ we have } \]
\[ \int_{G_n(E)} \phi(h)(R(h)W_{n+1}, W'_{n+1})^{\text{Whitt}} dh = \int_{(N_n(E) \backslash G_n(E))^2} W_\phi(g, h)W_{n+1}(h)W'_{n+1}(g)dh dg \]

By Lemma 2.6.1, Lemma 2.14.1 and Lemma 5.3.2 both sides of 5.3.4 are absolutely convergent and define continuous linear forms on \( C^w(G_n(E)) \). Therefore, it suffices to check the equality for \( \phi \in \mathcal{S}(G_n(E)) \) where the same formal manipulations as before are now justified due to the absolute convergence of the relevant expressions. This shows 5.3.4 and ends the proof of the proposition. ■

### 5.4 Statement of the main theorems

Let \( V \in \mathcal{V} \). Recall that a representation \( \pi \in \text{Temp}(G^V) \) is said to be \( H^V \)-distinguished if there exists a non-zero (continuous) \( H^V(F) \)-invariant linear form on (the space of) \( \pi \). By [Beu1, Theorem 7.2.1], \( \pi \) is \( H^V \)-distinguished if and only if the relative character \( J_\pi \) is not identically zero. We denote by \( \text{Temp}_{H^V}(G^V) \) the subset of irreducible \( H^V \)-distinguished tempered representations of \( G^V(F) \). By [Beu1, Corollary 7.6.1], \( \text{Temp}_{H^V}(G^V) \) is a union of connected components of \( \text{Temp}(G^V) \).

The first main theorem of this chapter is the following one which, as explained in the introduction, has direct applications to the global Ichino-Ikeda conjecture for unitary groups.

**Theorem 5.4.1** Let \( f = (f^V)_V \in \mathcal{S}(G(F)) \) and \( f' \in \mathcal{S}(G'(F)) \) be matching functions. Then, for every \( V \in \mathcal{V} \) and every \( \pi \in \text{Temp}_{H^V}(G^V) \), we have
\[ \kappa_V J_\pi(f^V) = I_{BC(\pi)}(f') \]
where
\[ \kappa_V = \left( \eta'((-1)^{n+1})\lambda_{E/F}(\psi') \right)^{n(n+1)/2} \left| \tau \right|_{E/F}^{n(n-1)/2} \eta_{E/F}(\text{disc}(V))^n. \]
Remark 5.4.2 The constant that appears in the theorem above differs slightly from Conjecture 4.4. For this, we offer the following explanation. First our normalization of the relative characters $J_π$ and $I_Π$ is not the same as in loc. cit. since we have replaced the representations $π$ and $Π$ by their contragredient and we have used the Whittaker model of $Π$ with respect to $ψ_n$ rather than $ψ_E$. This last point explain the discrepancy for the powers of $|τ|_E$. The difference for the exponents of $η_{E/F}(-1)$ and $η_{E/F}(\text{disc}(V))$ seems for its part to originate from the precise computation of the constant up to which “Fourier transform and transfer commute” (see Remark 5.2.3).

To state the second main result of this chapter, we introduce the following notation: for $G$ a connected reductive group over $F$, we let $π \mapsto \mu^*_G(π)$ be the function (or density) such that $d\mu_G(π) = \mu^*_G(π)dπ$. Note that by definition of $dπ$, $\mu^*_G(π)$ differs from $\mu_G(π)$ (defined in Section 2.13) by an integral power of $γ_0(0, I_F, ψ')$.

**Theorem 5.4.3** We have

$$\mu^*_G(π) = \frac{|γ^*(0, π, \text{Ad}, ψ')|}{|S_π|}$$

for almost all $π \in \text{Temp}_{H^V}(G^V)$.

We note the following interesting corollary which is a particular case of a general conjecture of Hiraga-Ichino-Ikeda ([HII]).

**Corollary 5.4.4** For every hermitian space $W$ over $E$, we have

$$\mu^*_U(σ) = \frac{|γ^*(0, σ, \text{Ad}, ψ')|}{|S_σ|}$$

for almost all $σ \in \text{Temp}(U(W))$. In particular, for every $σ \in Π_2(U(W))$ we have the following formula for its formal degree:

$$d(σ) = \frac{|γ(0, σ, \text{Ad}, ψ')|}{|S_σ|}.$$
such that $\sigma_0(f_\sigma) \neq 0$. This readily implies that $J_{\sigma_0 \boxtimes \sigma} \neq 0$ so that $\sigma_0 \boxtimes \sigma \in \Temp_{H^V}(G^V)$. Therefore, the connected component of $\sigma_0$ has the desired property.

Let $O_0$ be as in 5.4.1. Then, by Theorem 5.4.3 we have

$$\mu^*_U(V)(\sigma_0)\mu^*_U(V')(\sigma) = \mu^*_G(V)(\sigma_0 \boxtimes \sigma) = \frac{|\gamma^*(0, \sigma_0, \Ad, \psi')|}{|S_{\sigma_0}|} \frac{|\gamma^*(0, \sigma, \Ad, \psi')|}{|S_{\sigma}|}$$

for almost all $(\sigma_0, \sigma) \in O_0 \times O$. By the induction hypothesis, we also have $\mu^*_U(V)(\sigma_0) = \frac{|\gamma^*(0, \sigma_0, \Ad, \psi')|}{|S_{\sigma_0}|}$ for almost all $\sigma_0 \in O_0$ and moreover this term is almost everywhere nonzero. Therefore

$$\mu^*_U(V')(\sigma) = \frac{|\gamma^*(0, \sigma, \Ad, \psi')|}{|S_{\sigma}|}$$

for almost all $\sigma \in O$ and since $O$ was arbitrary the same holds for almost all $\sigma \in \Temp(U(W))$.

5.5 Local Jacquet-Rallis trace formulas

5.5.1 The unitary case

Let $V \in \mathcal{V}$ and let $f_1, f_2 \in S(G^V(F))$. Consider the following expression

$$J(f_1, f_2) = \int_{H^V(F)} \int_{H^V(F)} \int_{G^V(F)} f_1(h_1 g h_2) \overline{f_2(g)} dg dh_2 dh_1$$

which is absolutely convergent by [Zha] Lemma A.4. By definition of the measure on $H^V(F) \backslash G^V_{rs}(F) / H^V(F)$ and since the complement of $G^V_{rs}(F)$ in $G^V(F)$ is of measure 0 we have

$$J(f_1, f_2) = \int_{H^V(F) \backslash G^V_{rs}(F) / H^V(F)} O(\delta, f_1) \overline{O(\delta, f_2)} d\delta. \quad (5.5.1)$$

On the other hand by [Ben] Lemma 7.2.2 (v)], we also have

$$J(f_1, f_2) = \int_{\Temp_{ind}(G^V)} J_\pi(f_1) \overline{J_\pi(f_2)} d\mu_{G^V}(\pi). \quad (5.5.2)$$

where we have extended the definition of $J_\pi$ to $\pi \in \Temp_{ind}(G^V)$ by linearity. Notice that the above integral is absolutely convergent by 2.13.3 and Lemma 5.3.1. Combining 5.5.1 and 5.5.2 we get

$$\int_{H^V(F) \backslash G^V_{rs}(F) / H^V(F)} O(\delta, f_1) \overline{O(\delta, f_2)} d\delta = \int_{\Temp_{ind}(G^V)} J_\pi(f_1) \overline{J_\pi(f_2)} d\mu_{G^V}(\pi). \quad (5.5.3)$$
5.5.2 The linear case

Let \( f_1, f_2 \in \mathcal{S}(G'(F)) \). Consider the following expression

\[
I(f_1, f_2) = \int_{H_1(F)} \int_{H_2(F)} \int_{G'(F)} f_1(h_1g_2) \overline{f_2(g)} d\eta(h_2) dh_2 dh_1.
\]

We claim

\[(5.5.4)\] The above expression is absolutely convergent and defines a continuous sesquilinear form on \( \mathcal{S}(G'(F)) \).

Proof: For every \( d > 0 \) the above integral is, up to continuous norms in \( f_1 \) and \( f_2 \), bounded by

\[
\int_{H_1(F)} \int_{H_2(F)} \int_{G'(F)} \Xi^G(h_1g_2) \Xi^G(g) \sigma(h_1g_2)^{-d} \sigma(g)^{-2d} d\eta(h_2) dh_2 dh_1
\]

hence by

\[
\int_{H_1(F)} \int_{H_2(F)} \int_{G'(F)} \Xi^G(h_1g_2) \Xi^G(g) \sigma(h_1g_2)^{-d} \sigma(h_2)^{-d} dh_2 \sigma(h_1)^d dh_1.
\]

Assuming (as we may) that the log-norm \( \sigma \) is \( K' \)-bi-invariant this last expression equals

\[
\int_{H_1(F)} \int_{H_2(F)} \int_{G'(F)} \int_{K' \times K'} \Xi^G(h_1k_1gk_2h_2) dk_2 dk_1 \Xi^G(g) \sigma(h_1gk_2h_2)^{-d} d\eta(h_2)^{-d} dh_2 \sigma(h_1)^d dh_1
\]

and by the "doubling principle" ([Wald1, Lemme II.1.3], [Var, Proposition 16(iii) p.329]) we have

\[
\int_{K' \times K'} \Xi^G(h_1k_1gk_2h_2) dk_2 dk_1 = \Xi^G(h_1) \Xi^G(g) \Xi^G(h_2).
\]

To conclude, it suffices to remark that for every \( d > 0 \) the integral

\[
\int_{H_1(F)} \Xi^G(h_1) \sigma(h_1)^d dh_1
\]

is convergent by [Wald2, 4.1(3)] whereas for \( d > 0 \) sufficiently large the two integrals

\[
\int_{G'(F)} \Xi^G(g)^2 \sigma(g)^{-d} dg \quad \text{and} \quad \int_{H_2(F)} \Xi^G(h_2) \sigma(h_2)^{-d} dh_2
\]

are convergent by [Wald1, Lemme II.1.5] and [Var, Proposition 31 p.340] noting that \( \Xi^G_{|H_2} \ll (\Xi^G_{|H_2})^2 \sigma^{d'} \) for some \( d' > 0 \). 

By \ref{5.5.4} the definition of the measure on \( H_1(F) \setminus G'_{rs}(F)/H_2(F) \) and the fact that the complement of \( G'_{rs}(F) \) in \( G'(F) \) is of measure 0 we have

\[(5.5.5)\]

\[
I(f_1, f_2) = \int_{H_1(F)} \int_{G'_{rs}(F)/H_2(F)} O_\eta(\gamma, f_1) \overline{O_\eta(\gamma, f_2)} d\gamma.
\]
In order to get a "spectral" expression for \( I(f_1, f_2) \) we will have to use Theorem [4.2.2] in a slightly disguised form. There exists a unique \( V_0 \in \mathcal{V} \) such that \( G_{V_0} \) is quasi-split and we set \( G_{qs} = G_{V_0} \). Let

\[
BC : \text{Temp}(G_{qs})/\text{stab} \to \text{Temp}(G')
\]

be the "tensor product" of the two base-change maps \( \text{Temp}(U(V_0))/\text{stab} \to \text{Temp}(G_n(E)) \) and \( \text{Temp}(U(V'_0))/\text{stab} \to \text{Temp}(G_{n+1}(E)) \).

**Proposition 5.5.1** For every \( f_1, f_2 \in S(G'(F)) \) we have

\[
\int_{H_2(F)} \int_{G'(F)} f_1(gh_2)f_2(g)d\eta(h_2)dh_2 = \int_{\text{Temp}(G_{qs})/\text{stab}} (f_1, f_2)_{X_2, BC(\pi)} \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi
\]

where the right-hand side is absolutely convergent.

**Proof:** By Proposition 5.3.3 together with Lemma 2.12.1 and 2.7.4 we see that the right-hand side is absolutely convergent and defines a continuous sesquilinear form on \( S(G'(F)) \). Obviously, so does the left-hand side and therefore it suffices to establish the proposition when \( f_1, f_2 \) belong to the dense subspace \( S(G_n(E)) \otimes S(G_{n+1}(E)) \). Thus, we assume that \( f_k = f_{k,n} \otimes f_{k,n+1} \) where \( f_{k,n} \in S(G_n(E)) \) and \( f_{k,n+1} \in S(G_{n+1}(E)) \) for \( k = 1, 2 \). Set \( f'_{k,l}(g) = f_{k,l}(g^{-1})\eta_l(g^{-1}) \) for \( k = 1, 2 \) and \( l = n, n+1 \) where \( \eta_l \) is the character of \( G_l(E) \) defined in Section 2.2 which extends \( \eta_l \). Then, the left-hand side of the proposition can be rewritten as

\[
(5.5.6) \quad \prod_{l=n, \ldots, n+1} \int_{Y_l} \varphi_1(x)\varphi_2(x)dx
\]

where we have set \( \varphi_{k,l}(x) = \int_{G_l(F)} f'_{k,l}(hx)dh \) and \( Y_l = G_l(F)/G_l(E) \) for all \( k \in \{1, 2\} \), \( l \in \{n, n+1\} \).

On the other hand, for \( \Pi = \Pi_n \otimes \Pi_{n+1} \in \text{Temp}(G') \), if we denote by \( \mathcal{B}(\Pi_n, \psi) \) and \( \mathcal{B}(\Pi_{n+1}, \psi_{n+1}) \) orthonormal basis of \( \mathcal{W}(\Pi_n, \psi) \) and \( \mathcal{W}(\Pi_{n+1}, \psi_{n+1}) \) as in Section 2.14 we have (by definition and Proposition 5.3.3)

\[
(f_1, f_2)_{X_2, \Pi} = \prod_{l=n, \ldots, n+1} \sum_{W \in \mathcal{B}(\Pi_l, \psi_l)} \beta_l(R(f'_{1,l})W)\overline{\beta_l(R(f'_{2,l})W)}
\]

Since \( W \in \mathcal{W}(\Pi_l, \psi_l) \mapsto W' := \eta_lW \in \mathcal{W}(\Pi_l \otimes \eta_l', \psi) \) is an isomorphism preserving the scalar products and \( \beta_l(R(f'_{k,l})W) = \beta(R(f'_{k,l})W) \) for \( k = 1, 2 \) and \( l = n, n+1 \), we see that the above expression equals

\[
\prod_{l=n, \ldots, n+1} \langle f'_{1,l}, f'_{2,l} \rangle_{Y_l, \Pi_l \otimes \eta_l'}
\]

where \( (\ldots)_{Y_l, \Pi_l \otimes \eta_l'} \), \( l = n, n+1 \), are the positive semi-definite scalar products defined in Section 4.2. Moreover, for \( \pi = \pi_n \otimes \pi_{n+1} \in \text{Temp}(G_{qs}) \), we have

\[
S_\pi \simeq S_{\pi_n} \otimes S_{\pi_{n+1}}
\]

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\[ \gamma^*(0, \pi, \text{Ad}, \psi') = \gamma^*(0, \pi_n, \text{Ad}, \psi') \gamma^*(0, \pi_{n+1}, \text{Ad}, \psi') \]

All in all, we conclude that the right-hand side of the proposition equals

\[ \prod_{l=n, n+1} \int_{\text{Temp}(U(l))/\text{stab}} (\varphi'_{1,l}, \varphi'_{2,l}) \gamma_{\text{BC}}(\pi) \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi \]

where we recall that \( BC_{1}(\pi) = BC(\pi) \otimes \eta' \) for \( l = n, n + 1 \). The equality of (5.5.6) and (5.5.7) now follows directly from Theorem 4.2.2.

By Proposition 5.5.1 we have

\[ (5.5.8) \quad I(f_1, f_2) = \int_{H_1(F)} \int_{\text{Temp}(G_{\text{qs}})/\text{stab}} (L(h_1)f_1, f_2)_{X_2, BC(\pi)} \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi dh_1. \]

We claim

\[ (5.5.9) \quad \text{The above expression is absolutely convergent.} \]

Indeed, by [CHH, Theorem 2] in the \( p \)-adic case, [Sim, Theorem 1.2] in the Archimedean case, and since for every \( \Pi \in \text{Temp}(G') \) the function \( g \in G'(F) \mapsto (L(g)f_1, f_2)_{X_2, \Pi} \) is a matrix coefficient of \( \Pi^\vee \), there exist \( f_1', f_2' \in S(G'(F)) \) such that

\[ |(L(g)f_1, f_2)_{X_2, \Pi}| \leq \|f_1'||_{X_2, \Pi}\|f_2'||_{X_2, \Pi} \Xi_{G'}(g) \]

for every \( \Pi \in \text{Temp}(G') \) and \( g \in G'(F) \) where we have set \( \|f_k'||_{X_2, \Pi} = (f_k', f_k')_{X_2, \Pi}^{1/2} \) for \( k = 1, 2 \). By Proposition 5.5.1 and Cauchy-Schwartz inequality we see that the integral

\[ \int_{\text{Temp}(G_{\text{qs}})/\text{stab}} \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi \]

is convergent. Therefore, the expression (5.5.8) where we replace the integrand by its absolute value is bounded up to a constant by

\[ \int_{H_1(F)} \Xi_{G'}(h_1) dh_1. \]

This last expression is convergent by [Wald, 4.1(3)] therefore proving (5.5.9).

By 5.5.1 and Proposition 5.3.3 we obtain

\[ I(f_1, f_2) = |\tau|^{-n(n-1)/2} \int_{\text{Temp}(G_{\text{qs}})/\text{stab}} I_{BC(\pi)}(f_1) I_{BC(\pi)}(f_2) \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi \]

Combined with (5.5.5) this gives the identity

\[ \int_{H_1(F) \backslash G'_\text{qs}(F)/H_2(F)} O_{\eta}(\gamma, f_1) O_{\eta}(\gamma, f_2) d\gamma = \]

\[ |\tau|^{-n(n-1)/2} \int_{\text{Temp}(G_{\text{qs}})/\text{stab}} I_{BC(\pi)}(f_1) I_{BC(\pi)}(f_2) \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi \]

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5.6 Spectral expansions of certain unipotent relative orbital integrals

5.6.1 The unitary case

Let \( V \in \mathcal{V} \). For each \( f \in \mathcal{S}(G^V(F)) \) we set

\[ O(1, f) = \int_{H^V(F)} f(h)dh \]

By Lemma 5.3.1 we have

\[ \int_{\mathrm{Tempol}(G^V)} J_\pi(f)\mu_{G^V}(\pi) \]

the integral being absolutely convergent. On the other hand, by definition of \( \tilde{f}_z \), we have

\[ O(1, f) = \tilde{f}_z(0) \]

By Fourier inversion and the choice of the measure on \( u_{rs}(F)/U(V)(F) \) this can be rewritten as

\[ O(1, f) = \int_{u^V(F)} F\tilde{f}_z(X)dX = \int_{u_{rs}(F)/U(V)(F)} F\tilde{f}_z dX \]

5.6.2 The linear case Lie algebra version

Set

\[
\begin{pmatrix}
0 & \tau^{-1} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \tau^{-1} & 0 \\
0 & \ldots & \ldots & 0 & 0
\end{pmatrix}
\in \mathfrak{s}(F),
\quad
\xi_+ = (-1)^n
\begin{pmatrix}
0 & \ldots & \ldots & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & \tau & 0
\end{pmatrix}
\in \mathfrak{s}(F)
\]

\[ \xi_- = \tau^2 \xi_+ = (-1)^n \]

Let \( \varphi \in \mathcal{S}(\mathfrak{s}(F)) \). For \( s \in \mathbb{C} \) we define

\[ O_s(\xi_+, \varphi) = \int_{G_{u}(F)} \varphi(h\xi_+ h^{-1})|\det h|^s\eta_{E/F}(h)dh \]

whenever the integral is convergent. Note that the formula defining \( \omega(Y) \) for \( Y \in \mathfrak{s}_{rs}(F) \) still makes sense for \( \xi_- \) and that we have

\[ \omega(\xi_-) = \eta'(((-1)^{n-1} \tau)^{n(n+1)/2} \]

The goal of this section is to show the following:
Proposition 5.6.1 For \( \Re(s) > 1 - 1/n \), \( O_s(\xi_+, \varphi) \) is defined by an absolutely convergent expression. Moreover, the function \( s \mapsto O_s(\xi_+, \varphi) \) admits a meromorphic continuation to \( \mathbb{C} \) with no pole at \( s = 0 \) and setting \( O(\xi_+, \varphi) = O_0(\xi_+, \varphi) \), we have

\[
\gamma \omega(\xi_+)O(\xi_+, \varphi) = \int_{\mathfrak{s}(F)} \omega(Y)(\mathcal{F} \varphi)(Y)dY = \int_{\mathfrak{s}(E)/G_n(F)} \omega(Y)O_n(Y, \mathcal{F} \varphi)dY
\]

where

\[
\gamma = \prod_{k=1}^{n} \gamma(1 - k, \eta_{E/F}^k, \psi')
\]

Remark 5.6.2 The factor \( \gamma \) is non-zero: this boils down to the fact that \( L(s, \eta_{E/F}^k) \) has no pole at \( s = 1 - k \) for every \( 1 \leq k \leq n \). This last fact is easy to check in the p-adic case whereas in the Archimedean case it follows from the fact that \( L(s, \eta_{E/F}^k) \) has the same poles as \( \Gamma(\frac{s+1}{2}) \) if \( k \) is odd, \( \Gamma(\frac{s}{2}) \) if \( k \) is even.

For the proof of Proposition 5.6.1 we need some preparations. Let \( N_S \) be the image of \( R_{E/F}N_{n+1,E} \) by \( \nu_k \) and \( \mathfrak{n}_S = \mathfrak{s} \cap R_{E/F} \mathfrak{n}_{n+1,E} \) be its tangent space at the origin. Then, \( \nu \) induces isomorphisms \( N_S(F) \cong N_{n+1}(E)/N_{n+1}(F) \) and \( \mathfrak{n}_S(F) \cong \mathfrak{n}_{n+1}(E)/\mathfrak{n}_{n+1}(F) \) and we equip \( N_S(F) \), \( \mathfrak{n}_S(F) \) with the quotient measures. Set

\[
I^1(\varphi, s) = |\tau|_{E}^{\dim(N_S)/2} \int_{G_n(F)/N_{n}(F)} \int_{\mathfrak{n}_S(F)} \varphi(hYh^{-1})\psi'(\langle \xi_-, Y \rangle)dY|\det h|^{s}\eta_{E/F}(h)dh
\]

for every \( s \in \mathbb{C} \) for which this expression makes sense. First we show:

Lemma 5.6.3 The expression defining \( O_s(\xi_+, \varphi) \) converges absolutely for \( \Re(s) > 1 - 1/n \) and the expression defining \( I^1(\varphi, s) \) converges as an iterated integral for \( \Re(s) < 1 \). Moreover, \( O_s(\xi_+, \varphi) \) and \( I^1(\varphi, s) \) are holomorphic functions in their region of convergence and in the range \( 1 - 1/n < \Re(s) < 1 \) we have

\[
I^1(\varphi, s) = \prod_{k=1}^{n} \gamma(ks - (k - 1), \eta_{E/F}^k, \psi')O_s(\xi_+, \varphi)
\]

Recall that \( N'_{n+1} \) denotes the derived subgroup of \( N_{n+1} \). Let \( N'_S \) be the image of \( R_{E/F}N'_{n+1,E} \) by \( \nu \) and \( \mathfrak{n}'_S \) be its tangent space at the origin. As before, \( \nu \) induces an isomorphism \( \mathfrak{n}'_S(F) \cong \mathfrak{n}'_{n+1}(E)/\mathfrak{n}'_{n+1}(F) \) and we equip this space with the quotient of the Haar measures we fixed on \( \mathfrak{n}'_{n+1}(E) \) and \( \mathfrak{n}'_{n+1}(F) \) (see Section 2.5). As an intermediate step for the proof of Lemma 5.6.3 we need the following lemma:

Lemma 5.6.4 For every \( f \in S(\mathfrak{n}_S(F)) \), we have

\[
\int_{\mathfrak{n}_S(F)} f(h\xi_+h^{-1})dh = |\tau|_{E}^{\dim(N'_S)/2} \int_{\mathfrak{n}'_S(F)} f(\xi_+ + Y)dY
\]
Proof: The isomorphism \( n_{n+1}(F) \simeq n_S(F) \), \( Y \mapsto (-1)^n \tau^{-1} Y \), sends \( \xi'_+ = (-1)^n \tau \xi_+ \) and the Haar measure on \( n'_{n+1}(F) \) to \( |\tau|_{E}^{\dim(N_S)/2} \) times the Haar measure on \( n'_S(F) \). Therefore, it suffices to show that

\[
N_n(F) \to \xi'_+ + n'_{n+1}(F)
\]

\[
h \mapsto h\xi'_+ h^{-1}
\]

is an isomorphism preserving measures. Given the definition of the Haar measures on \( N_n(F) \) and \( n'_{n+1}(F) \), it even suffices to show that

\[
\iota_n : N_n \to \xi'_+ + n'_{n+1}
\]

\[
h \mapsto h\xi'_+ h^{-1}
\]

is an isomorphism over \( \mathbb{Z} \). This last statement is easy to show by induction on \( n \), noting that

\[
\iota_n(vu) = \begin{pmatrix}
v_{1,n} \\
v_{n-1}(u)v^{-1} \\
\vdots \\
v_{n-1,n} \\
0 & \ldots & 0 & 0
\end{pmatrix}
\]

for every \((u,v) \in N_{n-1} \times U_n\). ■

Proof (Lemma 5.6.3): By Lemma 5.6.4 and the Iwasawa decomposition \( G_n(F) = K_n A_n(F) N_n(F) \), provided everything is absolutely convergent, we have

\[
|\tau|_{E}^{\dim(N_S)/2} O_n(\xi_+, \varphi) = \int_{G_n(F)/N_n(F)} \int_{n'_S(F)} \varphi(h(\xi_+ + Y)h^{-1})dY|\det h|^s \eta_{E/F}(h)dh
\]

\[
= \int_{K_n} \int_{A_n(F)} \int_{n'_S(F)} \varphi(ka(\xi_+ + Y)a^{-1}k^{-1})dY|\det a|^s \eta_{E/F}(a)\delta_n(a)d\eta_{E/F}(k)dk
\]

\[
= \int_{K_n} \int_{A_n(F)} \int_{n'_S(F)} \varphi(k(a\xi_+ a^{-1} + Y)k^{-1})dY|\det a|^s \eta_{E/F}(a)\delta_n(a)\delta_{n_S}(a)^{-1}d\eta_{E/F}(k)dk
\]

for every \( s \in \mathbb{C} \) where we have set \( \delta_{n_S}(a) = |\det(Ad(a)|_{n_S})| \). Set

\[
f_{\varphi}(x_1, \ldots, x_n) := \int_{K_n} \int_{n'_S(F)} \varphi(k(\xi_+(x_1, \ldots, x_n) + Y)k^{-1})dY\eta_{E/F}(k)dk
\]

for every \((x_1, \ldots, x_n) \in F^n\) where by definition

\[
\xi_+(x_1, \ldots, x_n) := (-1)^n \begin{pmatrix}
x_1 \tau^{-1} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & x_n \tau^{-1}
\end{pmatrix}
\]

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Then \( f_\varphi \) is a Schwartz function on \( F^n \) and by [5.6.4] we have (provided everything converges)

\[
|\tau|^{-\dim(N'_S)/2} O_s(\xi_+, \varphi) = \int_{A_n(F)} f_\varphi(a_1/a_2, \ldots, a_{n-1}/a_n, a_n) |\det a|^s \eta_{E/F}(a) \delta_n(a) \delta_{n'_S}(a)^{-1} da.
\]

A direct and painless computation shows that \( \delta_n(a) \delta_{n'_S}(a)^{-1} = |a_2 \ldots a_n|^{-1} \) and therefore the above expression is equal to

\[
\int_{(F^\times)^n} f_\varphi(a_1/a_2, \ldots, a_{n-1}/a_n, a_n) \eta_{E/F}(a_1 \ldots a_n)|a_1|^s |a_2 \ldots a_n|^{s-1} d^\times a_1 \ldots d^\times a_n.
\]

By the change of variables \( a_1 \mapsto a_1 a_2, \ldots, a_{n-1} \mapsto a_{n-1} a_n \) this can further be rewritten

\[
(5.6.5) \quad |\tau|^{-\dim(N'_S)/2} O_s(\xi_+, \varphi) = \int_{(F^\times)^n} f_\varphi(a_1, \ldots, a_n) \prod_{k=1}^{n} \eta_{E/F}(a_k)^k |a_k|^{ks-(k-1)} d^\times a_1 \ldots d^\times a_n
\]

which is clearly a convergent integral when \( \Re(s) > 1 - 1/n \) showing the convergence of \( O_s(\xi_+, \varphi) \) in this range (indeed, as we easily see by the same computations where we replace the integrand by its absolute value).

By similar manipulations, and noticing that the isomorphism \( F^n \simeq n_S(F)/n'_S(F) \), \( (x_1, \ldots, x_n) \mapsto \xi_+(x_1, \ldots, x_n) \), sends the Haar measure on \( F^n \) to \( |\tau|^{-\dim(N'_S)-\dim(N_S)/2} \) times the Haar measure on \( n_S(F)/n'_S(F) \), we have

\[
|\tau|^{-\dim(N'_S)/2} I^1(\varphi, s) = \int_{A_n(F)} \int_{F^n} f_\varphi(x_1, \ldots, x_n) \psi'(\frac{a_2}{a_1} x_1 + \ldots + \frac{a_n}{a_{n-1}} x_{n-1} + a_n^{-1} x_n) dx_1 \ldots dx_n \delta_{n+1}(a)^{-1} \delta_n(a) |\det a|^s \eta_{E/F}(a) da
\]

\[
= \int_{(F^\times)^n} \hat{f}_\varphi(\frac{a_2}{a_1}, \ldots, \frac{a_n}{a_{n-1}}, a_n^{-1}) |a_1 \ldots a_n|^{s-1} \eta_{E/F}(a_1 \ldots a_n) d^\times a_1 \ldots d^\times a_n
\]

where we recall that \( f \in \mathcal{S}(F^n) \mapsto \hat{f}_\varphi \in \mathcal{S}(F^n) \) denotes the Fourier transform for \( \psi' \) and the corresponding autodual measure. By the same change of variables as before, this becomes

\[
(5.6.6) \quad |\tau|^{-\dim(N'_S)/2} I^1(\varphi, s) = \int_{(F^\times)^n} \hat{f}_\varphi(a_1^{-1}, \ldots, a_n^{-1}) \prod_{k=1}^{n} \eta_{E/F}(a_k)^k |a_k|^{k-1} d^\times a_1 \ldots d^\times a_n
\]

\[
= \int_{(F^\times)^n} \hat{f}_\varphi(a_1, \ldots, a_n) \prod_{k=1}^{n} \eta_{E/F}(a_k)^k |a_k|^{-k} d^\times a_1 \ldots d^\times a_n.
\]

This shows that when \( \Re(s) < 1 \), \( I^1(\varphi, s) \) is defined by a convergent expression. Moreover the identity of the lemma follows from [5.6.3] [5.6.6] and Tate’s thesis. \( \blacksquare \)

**Proof of Proposition 5.6.1** By Lemma 5.6.3 and Remark 5.6.2 we already know that \( s \mapsto O_s(\xi_+, \varphi) \) has meromorphic continuation to \( \mathbb{C} \) with no pole at \( s = 0 \) and that

\[
(5.6.7) \quad |\tau|^{\dim(N_S)/2} \gamma O(\xi_+, \varphi) = \int_{G_n(F)/N_n(F)} \int_{n_S(F)} \varphi(hYh^{-1}) \psi'((\xi_-, Y)) dY \eta_{E/F}(h) dh.
\]
Let $B_S$ be the image of $R_{E/F}B_{n+1}$ by $\nu$, $b_S$ its tangent space at the origin. We equip as before $b_S(F)$ with the transfer of the quotient measure on $b_{n+1}(E)/b_{n+1}(F)$ through $\nu$. Then, by Fourier inversion we have
\[
\int_{b_S(F)} f(Y)\psi'(\langle \xi_- , Y \rangle) dY = \int_{b_S(F)} \mathcal{F} f(\xi_- + Y) dY
\]
for every $f \in S(S(F))$. Together with (5.6.7) this gives
\[
|\tau|^E_{\dim(N_S)/2} \gamma O(\xi_+, \varphi) = \int_{G_n(F)/N_n(F)} \int_{b_S(F)} \mathcal{F} \varphi(h(\xi_- + Y)h^{-1}) dY \eta_{E/F}(h) dh.
\]
Noticing that $\omega(h(\xi_- + Y)h^{-1}) = \eta_{E/F}(h)\omega(\xi_-)$ for every $h \in G_n(F)$ and $Y \in b_S(F)$, the above can be rewritten as
\[
|\tau|^E_{\dim(N_S)/2} \gamma \omega(\xi_-) O(\xi_+, \varphi) = \int_{G_n(F)/N_n(F)} \int_{b_S(F)} \mathcal{F} \varphi(h(\xi_- + Y)h^{-1})\omega(h(\xi_- + Y)h^{-1}) dY dh.
\]
Therefore, the proposition would follow if we can show:

(5.6.8) For every $f \in L^1(S(F))$, we have the integration formula
\[
\int_{S(F)} f(Y) dY = |\tau|^E_{\dim(N_S)/2} \int_{G_n(F)/N_n(F)} \int_{b_S(F)} f(h(\xi_- + Y)h^{-1}) dY dh.
\]
As in the proof of Lemma 5.6.4 we are easily reduced to the same statement with $S$ and $b_S$ replaced by $\mathfrak{g}l_{n+1}$ and $b_{n+1}$, $\xi_-$ replaced by $\xi'_- = (-1)^n\tau^{-1}\xi_-$ and without the factor $|\tau|^E_{\dim(N_S)/2}$. Then, given the definition of our Haar measures, it suffices to show that the morphism
\[
G_n \times N_n (\xi'_- + b_{n+1}) \to \mathfrak{g}l_{n+1}
\]
\[
(h,Y) \mapsto hYh^{-1}
\]
is an open immersion over $\mathbb{Z}$ where we have denoted by $G_n \times N_n (\xi'_- + b_{n+1})$ the quotient of $G_n \times (\xi'_- + b_{n+1})$ by the free $N_n$-action given by $(h,Y) \cdot u = (hu,u^{-1}Yu)$. Set
\[
\mathfrak{g}l^r_{n+1} = \{ Y \in \mathfrak{g}l_{n+1} \mid \det(e_{n+1},e_{n+1}Y,\ldots,e_{n+1}Y^n) \neq 0 \}
\]
Then $\mathfrak{g}l^r_{n+1}$ is an open subscheme of $\mathfrak{g}l_{n+1}$ and clearly the above morphism factorizes through it. We claim that the induced map
\[
G_n \times N_n (\xi'_- + b_{n+1}) \to \mathfrak{g}l^r_{n+1}
\]
is an isomorphism. Indeed, it suffices to show that for every commutative ring $R$ the map $G_n(R) \times N_n(R) (\xi'_- + b_{n+1}(R)) \to \mathfrak{g}l^r_{n+1}(R)$ is a bijection. This in turn amounts to establishing the two following facts:
• For every $Y \in \mathfrak{g}_{n+1}^c(R)$, there exists $h \in G_n(R)$ such that $h^{-1}Yh \in \xi'_- + b_{n+1}(R)$ i.e. such that $e_{n+1}Y^k h \in e_{n+1-k} + \langle e_{n+2-k}, \ldots, e_{n+1} \rangle_R$ for every $1 \leq k \leq n$ (here we have denoted by $(e_1, \ldots, e_{n+1})$ the standard basis of $R^{n+1}$ and $\langle S \rangle_R$ stands for the $R$-submodule generated by $S$);

• For every $Y \in \xi'_- + b_{n+1}(R)$ and $h \in G_n(R)$, if $hYh^{-1} \in \xi'_- + b_{n+1}(R)$ then $h \in N_n(R)$.

For the first point, denoting by $\mathfrak{z}$ the image of $x \in R^{n+1}$ in $\mathfrak{r} = \mathfrak{r}^{n+1}/\mathfrak{r}e_{n+1}$, it suffices to choose for $h$ the unique element of $G_n(R)$ sending the basis $(e_{n+1}Y, \ldots, e_{n+1}Y^n)$ of $R^n$ to the basis $(\mathfrak{z}, \ldots, \mathfrak{z})$. The second point follows by noticing that if $hYh^{-1} \in \xi'_- + b_{n+1}(R)$ then $h$ preserves the submodule $V_k = \langle e_{n+1-k}, \ldots, e_{n+1} \rangle_R$ and acts trivially on the quotient $V_k/V_{k-1}$ for every $1 \leq k \leq n$ and this readily implies that $h \in N_n(R)$. ■

### 5.6.3 The linear case group version

Let $f \in \mathcal{S}(G'(F))$. We set

$$O_+(f) = O(\xi_+, \tilde{f}_z).$$

We also recall that by the general construction of Section 2.14, we associate to $f$ a function

$$W_f \in C^w(N'(F)\backslash G'(F) \times N'(F)\backslash G'(F), \psi_{N'}^{-1} \boxtimes \psi_{N'}).$$

**Lemma 5.6.5** We have the equality

$$\gamma O_+(f) = |T|^{\dim(N_S)/2} \int_{N_1(F)\backslash H_1(F)} \int_{N_2(F)\backslash H_2(F)} W_f(h_1, h_2)\eta(h_2)dh_2dh_1$$

where $\gamma$ is the same constant as in Proposition 5.6.1 and the right-hand side is absolutely convergent.

**Proof:** That the right hand side is absolutely convergent follows from Lemma 2.15.1 and Lemma 5.3.2. Unfolding the definitions, we have

$$\int_{N_1(F)\backslash H_1(F)} \int_{N_2(F)\backslash H_2(F)} W_f(h_1, h_2)\eta(h_2)dh_2dh_1 =$$

$$\int_{N_n(F)\backslash G_n(F)} \int_{N_{n+1}(F)\backslash G_{n+1}(F)} \int_{N_{n+1}(F)\backslash G_{n+1}(F)} \int_{N_n(E)\backslash G_n(E)} \int_{N_{n+1}(E)\times N_{n+1}(E)} f(g_n^{-1}u_nh_n, g_n^{-1}u_{n+1}h_{n+1})$$

$$\psi_n(u_n)^{-1}\psi_{n+1}(u_{n+1})^{-1}du_n du_{n+1} dg_n_1 \eta_{n+1}(h_n) dh_{n+1} \eta_n(h_n) dh_n$$

By the change of variable $u_{n+1} \mapsto u_n u_{n+1}$ and merging the integrals over $N_n(E)\backslash G_n(E)$ and $N_{n}(E)$ this becomes

$$\int_{N_n(F)\backslash G_n(F)} \int_{N_{n+1}(F)\backslash G_{n+1}(F)} \int_{G_n(E)} \int_{N_{n+1}(E)} f(g_nh_n, g_n uh_{n+1})\psi_{n+1}(u)^{-1}du$$

$$dg_n \eta_{n+1}(h_n) dh_{n+1} \eta_n(h_n) dh_n$$

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Note that the triple inner integral is absolutely convergent (this follows from the fact that $H_1 \times R_{E/F} N_{n+1} \times N_{n+1} G_{n+1} \to G'$, $(h_1, u, h_{n+1}) \mapsto h_1(1, uh_{n+1})$, is a closed embedding). Therefore, by breaking the integral over $N_{n+1}(E)$ into one over $N_{n+1}(F)$ followed by one over $N_{n+1}(E)/N_{n+1}(F)$ and the change of variable $g_n \mapsto g_n h_n^{-1}$ we see that the above expression equals

$$\int_{N_{n}(F)\backslash G_{n}(F)\times N_{n+1}(E)/N_{n+1}(F)\times G_{n+1}(F)} f(g_n, g_n h_n^{-1} u h_{n+1})\eta_{n+1}'(h_n^{-1} u h_{n+1}) dh_n$$

By definition of $\tilde{f}$ and of the Haar measure on $N_S(F)$, this last expression can be rewritten as

$$\int_{G_n(F)\backslash N_n(F)\times N_S(F)} \tilde{f}(hvh^{-1})\psi_S(v)^{-1} dv_{E/F}(h) dh$$

where we have denoted by $\psi_S$ the (unique) factorization of $\psi_{n+1}$ through $\nu : N_{n+1}(E) \to N_S(F)$. Finally, noting that the Cayley map $c$ induces a $G_n(F)$-equivariant isomorphism between $n_S(F)$ and $N_S(F)$ preserving measures and sending the character $Y \mapsto \psi'((\xi, Y))$ to $\psi_S^{-1}$, we obtain that

$$\int_{N_1(F)\backslash H_1(F)\times N_2(F)\times H_2(F)} W_f(h_1, h_2)\eta(h_2) dh_2 dh_1 =$$

$$\int_{G_n(F)\backslash N_n(F)\times n_2(F)} \tilde{f}(hYh^{-1})\psi'(\langle \xi, Y \rangle) dY \eta_{E/F}(h) dh$$

The lemma now follows readily from Lemma 5.6.3.

Recall that $G_{qs}$ stands for the unique quasi-split group of the form $G^{V_0}$ where $V_0 \in \mathcal{V}$.

**Proposition 5.6.6** We have

$$\gamma O_+(f) = |\tau|^{n(n-1)/4} \int_{\text{Temp}(G_{qs})/\text{stab}} I_{BC}(\pi)(f) \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi$$

where the right-hand side is absolutely convergent.

Proof: That the right-hand side is absolutely convergent and defines a continuous linear form on $\mathcal{S}(G'(F))$ follows from Proposition 5.3.3 together with Lemma 2.12.1 and 2.7.4. On the other hand, by Lemma 5.6.5, Lemma 5.3.2 and Lemma 2.15.1 the left hand side also defines a continuous linear form on $\mathcal{S}(G'(F))$. Therefore, it suffices to establish the proposition when $f = f_n \otimes f_{n+1}$ where $f_k \in \mathcal{S}(G_k(E))$ for $k = n, n+1$. Then, by Corollary 3.5.1, applied to the functions $f'_k = f_k \eta_k'$ and Theorem 4.2.2, we readily see that (see the proof of Proposition 3.5.1 for a similar argument)

$$\int_{N_2(F)\times H_2(F)} W_f(g, h_2)\eta(h_2) dh_2$$

$$= |\tau|^{\frac{n(n-1)}{4} + \frac{n(n+1)}{4}} \int_{\text{Temp}(G_{qs})/\text{stab}} \beta'(W_f, BC(\pi)(g, .)) \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi$$

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for every $g \in G'(F)$. From this, Lemma 5.6.5, Proposition 2.14.2 and Lemma 5.3.2, it follows that
\[
\gamma O_+(f) = |\tau|_E^{n(n-1)/4 + n(n+1)/4} \int_{N_1(F)\backslash H_1(F)} \int_{\text{Temp}(G_{q\pi})/\text{stab}} \beta'(W_{f,BC}(\pi)(h_2, \cdot)) \frac{|\gamma^*(0, \pi, \text{Ad}, \psi')|}{|S_\pi|} d\pi d\tau
\]
Moreover, by definition of $I_\Pi(f)$ and 2.14.3, we have
\[
(\lambda \hat{\otimes} \beta')(W_{f,\Pi}) = |\tau|_E^{n(n-1)/4 + n(n+1)/4} I_\Pi(f)
\]
for every $\Pi \in \text{Temp}(G')$ and the identity of the proposition follows since
\[
\frac{n(n-1)}{4} + \frac{n(n+1)}{4} + \frac{\dim(N_S)}{2} - \frac{n(n-1)}{2} - \frac{n(n+1)}{2} = -\frac{n(n-1)}{4}.
\]

\section{5.7 Proof of Theorems 5.4.1 and 5.4.3}

\subsection{5.7.1 Weak comparison of relative characters}

We recall the following result from [Beu2, Proposition 4.2.1].

\begin{proposition}
For every $V \in \mathcal{V}$ and every $\pi \in \text{Temp}_{H^V}(G^V)$, there exists $\kappa(\pi) \in \mathbb{C}^\times$ such that
\[
J_\pi(f^V) = \kappa(\pi) I_{BC(\pi)}(f')
\]
for all matching functions $f = (f^V)_V \in S(G(F))$ and $f' \in S(G'(F))$. Moreover, the function $\pi \in \text{Temp}_{H^V}(G^V) \mapsto \kappa(\pi)$ is continuous.
\end{proposition}

Here we remark that in loc. cit. only the $p$-adic case was considered. However, the proof extends readily to the Archimedean case the main points being that the globalization result [Beu2, Proposition 3.6.1] (which was borrowed from [ILM]) still holds for Archimedean places. Looking closer into the proof we see that everything works equally well in the Archimedean situation except that the proof of [Beu2, Lemma 3.6.2] has to be slightly modified. Indeed, rather than appealing to Mœglin-Tadic’s classification of discrete series for classical $p$-adic groups, we should use the description by Harish-Chandra of discrete series for real reductive groups (and in particular of their infinitesimal character). Except from this modification, the proof of [Beu2, Proposition 3.6.1] also works for Archimedean places.
5.7.2 Two comparisons

Let \( f_1, f_2 \in \mathcal{S}(G(F)) \) and \( f'_1, f'_2 \in \mathcal{S}(G'(F)) \) such that \( f_k \) and \( f'_k \) match for \( k = 1, 2 \). Since the transfer factors have absolute value 1, by Lemma 5.1.1 we have

\[
\sum_{V \in \mathcal{V}} \int_{H(V(F))G(V(F))/H(V(F))} O(\delta, f_1 V)O(\overline{\delta}, f'_2 V)d\delta = \int_{H_1(F)G_1(F)/H_2(F)} O_{\eta}(\gamma, f_1 \overline{\eta(\gamma, f'_2 V)}d\gamma
\]

By 5.5.3 and 5.5.10 this gives

\[
\sum_{V \in \mathcal{V}} \int_{\text{Temp}(G^V)_{\text{ind}}(G^V)} J_{\pi}(f_1 V)J_{\pi}(f'_2 V)d\mu_{G^V}(\pi) =
\]

\[
|\tau|_E^{-n(n-1)/2} \int_{\text{Temp}(G_{\eta V})/\text{stab}} I_{BC(\pi)}(f'_1 V)I_{BC(\pi)}(f'_2 V)\left|\gamma^*(0, \pi, \text{Ad}, \psi')\right| |S_{\pi}| d\pi
\]

Using Proposition 5.7.1 and 2.10.1 this can be rewritten

\[
(5.7.1) \quad \int_{\text{Temp}(G_{\eta V})/\text{stab}} \left( \sum_{V \in \mathcal{V}} \sum_{\pi' \sim_{\text{stab}} \pi} |\kappa(\pi')|^2 \mu_{G^V}^*(\pi') \right) I_{BC(\pi)}(f'_1 V)I_{BC(\pi)}(f'_2 V)d\pi =
\]

\[
|\tau|_E^{-n(n-1)/2} \int_{\text{Temp}(G_{\eta V})/\text{stab}} I_{BC(\pi)}(f'_1 V)I_{BC(\pi)}(f'_2 V)\left|\gamma^*(0, \pi, \text{Ad}, \psi')\right| |S_{\pi}| d\pi
\]

where \( \pi' \sim_{\text{stab}} \pi \) means that \( \pi' \) and \( \pi \) share the same Langlands parameter. We remark that both sides of the above identity are absolutely convergent by 5.5.9 and the convergence of 5.5.2.

Now, let \( f \in \mathcal{S}(G(F)) \) and \( f' \in \mathcal{S}(G'(F)) \) be matching functions. By 5.2.1 and Theorem 5.2.2 the functions \( (\epsilon_V \mathcal{F} \tilde{f}_E V) \) and \( \mathcal{F} \tilde{f}'_E \) match where we have set

\[
(5.7.2) \quad \epsilon_V = \eta_{E/F}(-1)^{n(n+1)/2} \lambda_{E/F}(\psi')^{n(n+1)/2} \eta_{E/F}(\text{disc} V)^n, \quad V \in \mathcal{V}.
\]

Therefore, by Lemma 5.1.2 we have

\[
\sum_{V \in \mathcal{V}} \epsilon_V \int_{u(V(F))/U(V(F))} O(X, \mathcal{F} \tilde{f}_E V)dX = \int_{s_{\eta V}(F)/G_\eta(V)} \omega(Y)O_{\eta}(Y, \mathcal{F} \tilde{f}_E V) dY
\]

By 5.6.2 and Proposition 5.6.1 this implies

\[
\sum_{V \in \mathcal{V}} \epsilon_V O(1, f^V) = \gamma_{\omega}(\xi_+)O(\xi_+, \tilde{f}'_E) = \gamma_{\omega}(\xi_-)O_+(f')
\]

Then, by 5.6.1 and Proposition 5.6.6 this gives

\[
\sum_{V \in \mathcal{V}} \epsilon_V \int_{\text{Temp}(G^V)} J_{\pi}(f^V) d\mu_{G^V}(\pi) = \omega(\xi_-)|\tau|_E^{-n(n-1)/2} \int_{\text{Temp}(G_{\eta V})/\text{stab}} I_{BC(\pi)}(f')\left|\gamma^*(0, \pi, \text{Ad}, \psi')\right| |S_{\pi}| d\pi
\]

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Finally, using Proposition 5.7.1 and 2.10.4 this can be rewritten

\[ (5.7.3) \quad \int_{\text{Temp}(G_{qs})/\text{stab}} \left( \sum_{V \in \mathcal{V}} c_V \sum_{\pi' \in \text{Temp}(G_{qs})/\text{stab}} \kappa(\pi') \mu_{G_{qs}}^\pi(\pi') \right) I_{BC}(\pi)(f')d\pi = \]

\[ \omega(\xi_\tau)|\tau|^{-n(n-1)/4} \int_{\text{Temp}(G_{qs})/\text{stab}} I_{BC}(\pi)(f') \left| \gamma^\pi(0, \pi, \text{Ad}, \psi') \right| / |S_\pi| d\pi \]

Notice that both sides of the above identity are absolutely convergent by Proposition 5.6.6 and the convergence of 5.6.1.

### 5.7.3 How to separate each spectral contribution

In order to finish the proofs of Theorems 5.4.1 and 5.4.3 we need to separate each spectral contribution in 5.7.1 and 5.7.3 (i.e. to deduce from these identities similar equalities for each \( \pi \in \text{Temp}(G_{qs})/\text{stab} \)). As usual, the argument ultimately rests upon the Stone-Weierstrass theorem by using some multipliers algebra. It is actually quite standard in the \( p \)-adic case (see e.g. [SV, Proof of Proposition 6.1.1]) but is slightly more subtle in the Archimedean case since using the center of the enveloping algebra as a substitute for the Bernstein center is not enough. Therefore, we explain carefully the proof here.

Consider the following general situation: \( G \) is a connected reductive group over \( F \), \( \mu \) is a Borel measure on the set \( \text{Irr}_\text{unit}(G) \) of unitary (or rather unitarizable) irreducible representations of \( G(F) \) (equipped with the Fell topology) and we are given for \( \mu \)-almost every \( \Pi \in \text{Irr}_\text{unit}(G) \) a continuous linear form \( L_\Pi : \mathcal{S}(G(F)) \rightarrow \mathbb{C} \) which factorizes through the map \( f \mapsto \Pi^\vee(f) \). In the applications we have in mind, we will take \( G = G' \), for \( \mu \) the pushforward of the measure \( |\gamma^\pi(0, \pi, \text{Ad}, \psi')| / |S_\pi| d\pi \) to \( \text{Temp}(G_{qs})/\text{stab} \) by \( BC \) and for \( L_\Pi \) a certain multiple of \( I_{\Pi} \). We assume moreover that the following holds:

- For every \( f \in \mathcal{S}(G(F)) \), the function \( \Pi \in \text{Irr}_\text{unit}(G) \mapsto L_\Pi(f) \) is \( \mu \)-integrable and we have

\[ \int_{\text{Irr}_\text{unit}(G)} L_\Pi(f) \mu(\Pi) = 0. \]

**Lemma 5.7.2** Under the above assumptions, we have \( L_\Pi = 0 \) for \( \mu \)-almost all \( \Pi \in \text{Irr}_\text{unit}(G) \).

**Proof:** (the proof is inspired from [SV, Proof of Proposition 6.1.1]) Since \( \mathcal{S}(G(F)) \) is separable (it is even of countable dimension in the \( p \)-adic case), up to multiplying \( \mu \) by

\[ \Pi \mapsto \sum_n \frac{|L_\Pi(f_n)|}{n^2\|L(f_n)\|_{L^1(\mu)}} \]

where \( (f_n)_{n \geq 1} \) is a dense sequence in \( \mathcal{S}(G(F)) \), \( \|L(f)\|_{L^1(\mu)} := \int_{\text{Irr}_\text{unit}(G)} |L_\Pi(f)| \mu(\Pi) \) and dividing the family \( \Pi \mapsto L_\Pi \) by the same function, we may assume that \( \mu \) is finite. Let \( \mathcal{Z}(G) \)
denote the Bernstein center in the $p$-adic case or the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ in the Archimedean case. Then, we have a continuous map with finite fibers

$$ p : \text{Irr}_{\text{unit}}(G) \to \overline{Z}(G) $$

which associates to $\Pi$ its “infinitesimal character” $\chi_{\Pi}$. Let $X$ be the image of this map and $\overline{\mu}$ be the push-forward of $\mu$ to $X$. Then, by the disintegration of measures, there exists a measurable family $\chi \in X \mapsto \mu_{\chi}$ of finite measures on $\text{Irr}_{\text{unit}}(G)$ such that for $\overline{\mu}$-almost every $\chi \in X$, $\mu_{\chi}$ is supported on the finite set $p^{-1}(\chi)$ and moreover

$$ \int_{\text{Irr}_{\text{unit}}(G)} f(\Pi)\mu(\Pi) = \int_X \sum_{\Pi \in p^{-1}(\chi)} f(\Pi)\mu_{\chi}(\Pi)\overline{\mu}(\chi) $$

for every $\mu$-integrable function $f$. By the hypothesis, we therefore have

$$ \int_X \sum_{\Pi \in p^{-1}(\chi)} L_{\Pi}(f)\mu_{\chi}(\Pi)\overline{\mu}(\chi) = 0 \tag{5.7.4} $$

for every $f \in \mathcal{S}(G(F))$. Let $K \subset G(F)$ be a maximal compact subgroup and $\mathcal{H}(G(F)) \subset \mathcal{S}(G(F))$ be the corresponding “Hecke algebra” of $G(F)$, that is the space of smooth compactly supported and bi-$K$-finite functions on $G(F)$ (of course in the $p$-adic case we simply have $\mathcal{H}(G(F)) = \mathcal{S}(G(F))$). We will now use the existence of a suitable “algebra of multipliers” $\mathcal{M}(G)$ on $\mathcal{H}(G(F))$ i.e. an algebra of endomorphisms $z$ of $\mathcal{H}(G(F))$ for which there exists a function on $\overline{Z}(G)$ (to be denoted by the same letter) such that $\Pi^\vee (zf) = z(\chi_{\Pi})\Pi^\vee (f)$ for every $\Pi \in \text{Irr}(G)$ and $f \in \mathcal{H}(G(F))$. Assuming the existence of such a multiplier algebra, by applying $\overline{\text{5.7.4}}$ to $zf$ (and by the hypothesis made on $L_{\Pi}$) we get

$$ \int_X z(\chi) \sum_{\Pi \in p^{-1}(\chi)} L_{\Pi}(f)\mu_{\chi}(\Pi)\overline{\mu}(\chi) = 0 \tag{5.7.5} $$

for every $f \in \mathcal{H}(G(F))$ and $z \in \mathcal{M}(G)$. For each finite set $S \in \hat{K}$, let $X_S$ be the set of infinitesimal characters of representations $\Pi \in \text{Irr}_{\text{unit}}(G)$ which are generated by their $\rho$-isotypic component for some $\rho \in S$. Assume that the following holds: for every $z \in \mathcal{M}(G)$ and $S \subset \hat{K}$, the function $\chi \in X_S \mapsto z(\chi)$ converges to zero at infinity and the algebra of functions $\{ z|_{X_S} \mid z \in \mathcal{M}(G) \}$ is stable by complex conjugation, separates points and does not vanish identically anywhere. Then, by the Stone-Weierstrass theorem applied to the one-point compactification of $X_S$, this algebra is dense in the Banach space $C_0(X_S)$ of continuous functions on $X_S$ tending to zero at infinity. Since for each $f \in \mathcal{H}(G(F))$ the function $\chi \mapsto \sum_{\Pi \in p^{-1}(\chi)} L_{\Pi}(f)\mu_{\chi}(\Pi)$ is supported on $X_S$ for some $S$ (again by the hypothesis made on $L_{\Pi}$), this together with $\overline{\text{5.7.5}}$ implies that for every $f \in \mathcal{H}(G(F))$

$$ \sum_{\Pi \in p^{-1}(\chi)} L_{\Pi}(f)\mu_{\chi}(\Pi) = 0 $$

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for $\overline{\mu}$-almost every $\chi \in X$. As $\mathcal{H}(G(F))$ contains a sequence which is dense in $\mathcal{S}(G(F))$, by continuity of the linear forms $L_\Pi$, the above equality actually holds for $\overline{\mu}$-almost every $\chi \in X$ and every $f \in \mathcal{S}(G(F))$. Since the map

$$\mathcal{S}(G(F)) \to \bigoplus_{\Pi \in p^{-1}(\chi)} \text{End}_Z(\Pi^\vee), \ f \mapsto (\Pi^\vee(f))_\Pi$$

is surjective (indeed, the image is a closed and $G(F) \times G(F)$-invariant subspaces, the $G(F) \times G(F)$-representations $\text{End}_Z(\Pi^\vee)$ are irreducible and pairwise non-equivalent and for each $\Pi$ the map $f \in \mathcal{S}(G(F)) \mapsto \Pi^\vee(f) \in \text{End}_Z(\Pi^\vee)$ is surjective), by the hypothesis made on linear forms $L_\Pi$ this implies $\mu_\chi(\Pi)L_\Pi = 0$ for $\overline{\mu}$-almost every $\chi \in X$ and every $\Pi \in p^{-1}(\chi)$ i.e. $L_\Pi = 0$ for $\mu$-almost every $\Pi \in \text{Irr}_{\text{unit}}(G)$.

Thus, it only remains to show the existence of an algebra of multipliers satisfying the required assumption. Notice that if $f \mapsto zf$ is a multiplier then so is $f \mapsto z^*f := (zf^*)^*$ (where we recall that $f^*(g) = \overline{f(g^{-1})}$) and that $z^*(\chi_\Pi) = z(\chi_\Pi)$ whenever $\Pi$ is a unitary representation. Therefore, we only need to find an algebra of multipliers tending to 0 at infinity on $X_S$ for every $S \subset \hat{K}$ and separating points (including infinity). In the $p$-adic case, we just take $\mathcal{M}(G) = \mathcal{Z}(G)$ the Bernstein center. Clearly, it separates points and does not vanish identically anywhere on $\mathcal{Z}(G)$. Since for each $S \subset \hat{K}$, $X_S$ is compact (\cite[Theorem 2.5]{Tad}) this algebra has all the desired properties. In the Archimedean case, we will use a certain subalgebra of Arthur’s algebra of multipliers (\cite{Art2, Del}). To be more precise, we need to introduce more notation. Let $T \subset G$ be a maximal torus. Harish-Chandra’s isomorphism gives an identification

$$\mathcal{Z}(G) = t(\mathbb{C})^*/W$$

where $W = W(G_{\mathbb{C}}, T_{\mathbb{C}})$. Let $t_\mathbb{R} \subset t(\mathbb{C})$ be the $\mathbb{R}$-points of the split form of $t$. Then $W$ preserves $t_\mathbb{R}$ and fixing a Haar measure on $t_\mathbb{R}$ we define a Fourier transform $\varphi \in C_c^\infty(t_\mathbb{R})^W \mapsto \hat{\varphi} \in C(t(\mathbb{C})^*/W)$ by

$$\hat{\varphi}(\lambda) = \int_{t_\mathbb{R}} \varphi(X)e^{\lambda(X)}dX.$$

By \cite[Theorem 4.2]{Art2}, \cite[Theorem 3]{Del} there exists an algebra of multipliers $\mathcal{M}(G)$ whose associated set of functions on $\mathcal{Z}(G)$ is precisely $C_c^\infty(t_\mathbb{R})^W$. This algebra has all the desired properties the only non-trivial point being that $z$ tends to 0 at infinity on $X_S$ for any $z \in \mathcal{M}(G)$ and $S \subset \hat{K}$ but this follows from the fact that $X_S$ has compact image in $it_\mathbb{R}^s \setminus t(\mathbb{C})^*/W$ (see \cite[Theorem 5.2]{BW} and \cite[Corollary 7.7.3]{Wall}) and usual properties of the Fourier transform.

### 5.7.4 End of the proof

First we show that \ref{5.7.1} holds for every $f'_1, f'_2 \in \mathcal{S}(G'(F))$, both sides being absolutely convergent. In the $p$-adic case, this already follows from the computations of \ref{5.7.2} since every function in $\mathcal{S}(G'(F))$ admits a transfer to $\mathcal{S}(G(F))$ (Theorem \ref{5.2.1} (i)). In the
Archimedean case, by Theorem 5.2.4 (ii), we know at least that it holds for \(f_1', f_2'\) in a certain dense subspace \(\mathcal{S}(G'(F))_{\text{trans}}\) of \(\mathcal{S}(G'(F))\). Therefore, it suffices to show that both side of 5.7.1 are absolutely convergent for every \(f_1', f_2' \in \mathcal{S}(G'(F))\) and that they define continuous sesquilinear forms on \(\mathcal{S}(G'(F))\). For the right-hand side, this follows readily from Proposition 5.3.3 together with Lemma 2.12.1 and 2.7.4. By Cauchy-Schwartz inequality, it then suffices to prove that the left-hand side is always less or equal to the right-hand side whenever \(f_1' = f_2'\). That it is indeed the case is a consequence of Fatou’s lemma together with the fact that the linear forms \(\Pi, \Pi \in \text{Temp}(G')\), are continuous.

Thus, we can now apply Lemma 5.7.2 to 5.7.1 giving us the identity

\[
\left( \sum_{V \in \mathcal{V}} \sum_{\pi' \sim \text{stab} \pi \atop \pi' \in \text{Temp}_{H^V}(G^V)} |\kappa(\pi')|^2 \mu_{G^V}^*(\pi') \right) I_{BC(\pi)}(f_1') I_{BC(\pi)}(f_2') = \frac{|\gamma(0, \pi, \text{Ad}, \psi')|}{|S_{\pi}|} |\pi|^{-n(n-1)/2} I_{BC(\pi)}(f_1') I_{BC(\pi)}(f_2')
\]

for almost every \(\pi \in \text{Temp}(G_{qs})/\text{stab}\) and every \(f_1', f_2' \in \mathcal{S}(G'(F))\). By the local Gan-Gross-Prasad conjecture ([Ben1, Theorem 12.4.1]), for every \(\pi \in \text{Temp}(G_{qs})/\text{stab}\) there exists exactly one \(V \in \mathcal{V}\) and one representation \(\pi' \in \text{Temp}_{H^V}(G^V)\) such that \(\pi' \sim \text{stab} \pi\). Therefore, as the linear form \(I_{BC(\pi)}\) is non-zero (this can for example be deduced from Proposition 5.7.1 since \(J_\pi \neq 0 \iff \pi \in \text{Temp}_{H^V}(G^V)\)) the above identity can be rewritten

\[
|\kappa(\pi')|^2 \mu_{G^V}^*(\pi') = |\pi|^{-n(n-1)/2} \frac{|\gamma(0, \pi, \text{Ad}, \psi')|}{|S_{\pi}|}
\]

for almost every \(\pi \in \bigcup_{V \in \mathcal{V}} \text{Temp}_{H^V}(G^V)\). As a first consequence, multiplying both sides by \(\mu_{G^V}^*(\pi)\) and using 2.13.2 together with Lemma 2.12.1, we see that in the Archimedean case there exists \(k > 0\) such that

\[
|\kappa(\pi)| |\mu_{G^V}^*(\pi)| \ll N(\pi)^k
\]

for almost every \(\pi \in \bigcup_{V \in \mathcal{V}} \text{Temp}_{H^V}(G^V)\).

We now show that 5.7.3 holds for every \(f' \in \mathcal{S}(G'(F))\). Once again, in the \(p\)-adic case there is nothing to say and in the Archimedean case it suffices to show that both sides are absolutely convergent and define continuous linear forms on \(\mathcal{S}(G'(F))\). But this follows readily from Proposition 5.3.3 together with 5.7.7 (for the left-hand side), Lemma 2.12.1 (for the right-hand side) and 2.7.4.

Thus, we can apply Lemma 5.7.2 to 5.7.3 and using the non-vanishing of \(I_{BC(\pi)}\) this gives

\[
\sum_{V \in \mathcal{V}} \sum_{\pi' \sim \text{stab} \pi \atop \pi' \in \text{Temp}_{H^V}(G^V)} \kappa(\pi') \mu_{G^V}^*(\pi') = \omega(\xi_-) |\pi|^{-n(n-1)/2} \frac{|\gamma(0, \pi, \text{Ad}, \psi')|}{|S_{\pi}|}
\]

\[
\sum_{V \in \mathcal{V}} \sum_{\pi' \sim \text{stab} \pi \atop \pi' \in \text{Temp}_{H^V}(G^V)} \kappa(\pi') \mu_{G^V}^*(\pi') = \omega(\xi_-) |\pi|^{-n(n-1)/2} \frac{|\gamma(0, \pi, \text{Ad}, \psi')|}{|S_{\pi}|}
\]

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for almost every $\pi \in \text{Temp}(G_{q_0})/\text{stab}$. Applying again the local Gan-Gross-Prasad conjecture, this can be rewritten as

\[
\epsilon_V \kappa(\pi) \mu^*_{G^V}(\pi) = \omega(\xi_-)|\tau|^{-n(n-1)/4} \frac{|\gamma^*(0, \pi, \text{Ad, } \psi')|}{|S_\pi|}
\]

for every $V \in \mathcal{V}$ and almost every $\pi \in \text{Temp}_{H^V}(G^V)$.

Squaring the module of $5.7.8$ and dividing it by $5.7.6$, we obtain (as $\mu^*_{G^V}(\pi) \geq 0$ for every $\pi$)

\[
\mu^*_{G^V}(\pi) = \frac{|\gamma^*(0, \pi, \text{Ad, } \psi')|}{|S_\pi|}
\]

for every $V \in \mathcal{V}$ and almost every $\pi \in \text{Temp}_{H^V}(G^V)$ thus proving Theorem 5.4.3. Moreover, dividing $5.7.8$ by the above identity gives

\[
\epsilon_V \kappa(\pi) = \omega(\xi_-)|\tau|^{-n(n-1)/4}
\]

By 5.6.3 and 5.7.2 we easily check that $\kappa_V = \epsilon_V \omega(\xi_-) |\tau|^{n(n-1)/4}$ (where $\kappa_V$ is the constant defined in the statement of Theorem 5.4.1) and therefore we have

\[
\kappa(\pi) = \kappa_V^{-1}
\]

for every $V \in \mathcal{V}$ and almost every $\pi \in \text{Temp}_{H^V}(G^V)$. By continuity of $\pi \mapsto \kappa(\pi)$ (Proposition 5.7.1) this last equality is true for every $\pi \in \text{Temp}_{H^V}(G^V)$ and this proves Theorem 5.4.1.

A Proof of Proposition 2.13.1

First, we recall the following elementary lemma (see [CHH Proposition]).

**Lemma A.0.1 (Sobolev lemma for compact groups)** Let $H$ be a Hilbert space, $K$ a compact group and $C(K, H)$ the space of continuous functions from $K$ to $H$. We consider $C(K, H)$ as a representation of $K$ through the right regular action and for $\rho \in \hat{K}$ we denote by $C(K, H)[\rho]$ the $\rho$-isotypic component. Then, for every $\rho \in \hat{K}$ and $\varphi \in C(K, H)[\rho]$ we have

\[
\sup_{k \in K} \|\varphi(k)\| \leq \dim(\rho) \left( \int_K \|\varphi(k)\|^2 dk \right)^{1/2}.
\]

Now we proceed to the proof of Proposition 2.13.1.

That $f_\pi \in \mathcal{C}^w(G(F))$ for every $\pi \in \text{Temp}_{\text{ind}}(G)$ follows from 2.6.1. Moreover, if we can show that $\pi \mapsto f_\pi$ belongs to $\mathcal{S}(\text{Temp}_{\text{ind}}(G), \mathcal{C}^w(G(F)))$ for every $f \in \mathcal{S}(G(F))$ then the linear map

\[
f \in \mathcal{S}(G(F)) \mapsto (\pi \mapsto f_\pi) \in \mathcal{S}(\text{Temp}_{\text{ind}}(G), \mathcal{C}^w(G(F)))
\]

would automatically be continuous by the closed graph theorem since for all $\pi \in \text{Temp}_{\text{ind}}(G)$ and $g \in G(F)$ the linear form $f \in \mathcal{S}(G(F)) \mapsto f_\pi(g)$ is continuous. Thus it only remains to show that $(\pi \mapsto f_\pi) \in \mathcal{S}(\text{Temp}_{\text{ind}}(G), \mathcal{C}^w(G(F)))$ for every $f \in \mathcal{S}(G(F))$.  

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Let \( f \in \mathcal{S}(G(F)) \). The function \( \pi \in \text{Temp}_{\text{ind}}(G) \mapsto f_\pi \in C^\infty(G(F)) \) is smooth by [Beu1, Lemma 2.3.1(ii)] together with [Art1 §3] in the Archimedean case and [Wald1, Proposition VII.1.3] in the \( p \)-adic case. Moreover, in the \( p \)-adic case the function \( \pi \mapsto f_\pi \) is compactly supported by [Wald1, Théorème VIII.1.2]. Therefore, we only need to check that \( \pi \mapsto f_\pi \) satisfies the condition of Lemma 2.9.1. We will concentrate on the Archimedean case, the \( p \)-adic case being actually simpler. In this case, we will show the following: for every Levi subgroup \( M \) of \( G, D \in \text{Sym}^*(\mathcal{A}_{M,C}) \), \( u, v \in \mathcal{U}(g) \) and \( k \geq 1 \) we have

\[
(A.0.1) \quad \quad N(\pi)^k |D(\lambda \mapsto (R(u)L(v)f_{\pi,\lambda}(g)))_{\lambda=0}| < \Xi^G(g)\sigma(g)^{\deg(D)}
\]

for all \( \tau \in \Pi_2(M) \) and \( g \in G(F) \) where we have set \( \pi_\lambda = i\mathcal{G}_M(\tau,\lambda) \) and \( \pi = \pi_0 \) for \( \lambda \in i\mathcal{A}_{M}^* \).

As \( R(u)L(v)f_\pi = (R(u)L(v)f)_\pi \), up to replacing \( f \) by \( R(u)L(v)f \) we only need to establish \( A.0.1 \) when \( u = v = 1 \). Assume proved the slightly weaker inequality (for any \( D \in \text{Sym}^*(\mathcal{A}_{M,C}) \))

\[
(A.0.2) \quad \quad |D(\lambda \mapsto f_{\pi,\lambda}(g))_{\lambda=0}| < \Xi^G(g)\sigma(g)^{\deg(D)}, \quad \tau \in \Pi_2(M), g \in G(F).
\]

Then, we will show that \( A.0.1 \) holds for any \( k \geq 1 \) (and \( u = v = 1 \)). We do this by induction on \( \deg(D) \). Let \( z \in \mathcal{Z}(g) \) be such that [2.6.2] is satisfied. Then we may as well assume that \( N(\pi) = \chi_\pi(z) \). Since \( (z^k f)_\pi = \chi_\pi(z)^k f_\pi \) for every \( \pi \in \text{Temp}_{\text{ind}}(G) \) the result in degree 0 just follows by replacing \( f \) by \( z^k f \) in \( A.0.2 \) for every \( k \geq 1 \). In the general case the difference between

\[
(A.0.3) \quad D(\lambda \mapsto (z^k f_{\pi,\lambda}(g))_{\lambda=0}) = D(\lambda \mapsto \chi_\pi(z)^k f_{\pi,\lambda}(g))_{\lambda=0}
\]

and

\[
\chi_\pi(z)^k D(\lambda \mapsto (z^k f_{\pi,\lambda}(g))_{\lambda=0}
\]

can be written as a finite sum

\[
\sum_{i=1}^{n} D_i(\lambda \mapsto \chi_{\pi,\lambda}(z)^k)_{\lambda=0} D'_i(\lambda \mapsto (z^k f_{\pi,\lambda}(g))_{\lambda=0}
\]

where \( D_i, D'_i \in \text{Sym}^*(\mathcal{A}_{M,C}) \) for all \( 1 \leq i \leq n \) are of degree strictly less than \( D \). Since the terms \( D_i(\lambda \mapsto \chi_{\pi,\lambda}(z)^k)_{\lambda=0} \) are all essentially bounded by a power of \( N(\pi) \), the above sum can be controlled by the induction hypothesis whereas \( A.0.3 \) is controlled by \( A.0.2 \) applied to \( z^k f \). This shows \( A.0.1 \) assuming \( A.0.2 \).

Fix \( D \in \text{Sym}^*(\mathcal{A}_{M,C}) \), a parabolic subgroup \( P \) with Levi \( M \), a maximal compact subgroup \( K \) of \( G(F) \) in good position relative to \( M \) and set \( K_M = K \cap M \). Then, by restriction to \( K \) we can identify \( \pi_\lambda = i_P^G(\tau,\lambda) \) with \( \pi_K = i_K^{K_M}(\tau|_{K_M}) \) as a \( K \)-representation for every \( \tau \in \Pi_2(M) \) and \( \lambda \in i\mathcal{A}_{M}^* \). Choosing an invariant scalar product \( (.,.) \) on \( \tau \), we endow \( \pi_K \) with the \( K \)-invariant scalar product

\[
(e, e') = \int_{K} (e(k), e'(k))dk.
\]

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Finally, choose for any $\rho \in \hat{K}$ an orthonormal basis $\mathcal{B}_r(\rho)$ of the $\rho$-isotypic component $\pi_K[\rho]$ of $\pi_K$. Then, we have

\[(A.0.4) \quad f_{\pi_\lambda}(g) = \sum_{\rho \in K} \sum_{e \in \mathcal{B}_r(\rho)} (\pi_\lambda(g)\pi_\lambda(f^\vee)e, e)\]

for all $\tau \in \Pi_2(M)$, $\lambda \in i \mathcal{A}_M^*$ and $g \in G(F)$. Assume now that we can show the existence of $r > 0$ such that

\[(A.0.5) \quad |D(\lambda \mapsto (\pi_\lambda(g)\pi_\lambda(f^\vee)e, e))_{\lambda=0}| < N(\rho)^r \Xi^G(g)\sigma(g)^{\deg(D)}\]

for all $\tau \in \Pi_2(M)$, $g \in G(F)$, $\rho \in \hat{K}$ and $e \in \mathcal{B}_r(\rho)$. Let $z_K \in \mathcal{Z}(t)$ be such that $2.6.2$ is satisfied for $G(F) = K$. Then we may as well assume that $N(\rho) = \rho(z_K)$ for all $\rho \in \hat{K}$ and hence up to replacing $f$ by $L(z_K)^{k+r} f$ in $[A.0.5]$ we obtain the same inequality with $N(\rho)^r$ replaced by $N(\rho)^{-k}$. Therefore, by $[A.0.4]$ we would get for any $k > 0$ an inequality

\[|D(\lambda \mapsto f_{\pi_\lambda(g)})_{\lambda=0}| < \Xi^G(g)\sigma(g)^{\deg(D)} \sum_{\rho \in K} \dim(\pi_K[\rho])N(\rho)^k, \quad \tau \in \Pi_2(M), \ g \in G(F).\]

As for $k$ large enough the sum $\sum_{\rho \in K} \dim(\pi_K[\rho])N(\rho)^k$ converges and is bounded independently of $\pi$ (see e.g. [Ben1, (2.2.2)]). For such a $k$ the above inequality implies $[A.0.2]$.

Thus, it only remains to establish $[A.0.3]$. Actually, it suffices to show the existence of $r > 0$ such that

\[(A.0.6) \quad |D(\lambda \mapsto (\pi_\lambda(g)e, e))_{\lambda=0}| < N(\rho)^r \Xi^G(g)\sigma(g)^{\deg(D)}\]

for all $\tau \in \Pi_2(M)$, $g \in G(F)$, $\rho \in \hat{K}$ and $e \in \mathcal{B}_r(\rho)$. Indeed, if this is the case we would get

\[
\begin{align*}
|D(\lambda \mapsto (\pi_\lambda(g)\pi_\lambda(f^\vee)e, e))_{\lambda=0}| &= \left|D(\lambda \mapsto \int_G f^\vee(\gamma)(\pi_\lambda(g\gamma)e, e)d\gamma)_{\lambda=0}\right| \\
&= \left|\int_G f^\vee(\gamma)D(\lambda \mapsto (\pi_\lambda(g\gamma)e, e))_{\lambda=0}d\gamma\right| < N(\rho)^r \int_G |f^\vee(\gamma)|\Xi^G(g\gamma)\sigma(g\gamma)^{\deg(D)}d\gamma \\
&< N(\rho)^r \sigma(g)^{\deg(D)} \int_{G \in K} \sup_{K \in K} |f^\vee(k\gamma)| \Xi^G(gk\gamma)dk\sigma(g^\gamma)^{\deg(D)}d\gamma \\
&= N(\rho)^r \Xi^G(g)\sigma(g)^{\deg(D)} \int_{G \in K} \sup_{K \in K} |f^\vee(k\gamma)|\Xi^G(g\gamma)\sigma(g^\gamma)^{\deg(D)}d\gamma
\end{align*}
\]

for all $\tau \in \Pi_2(M)$, $g \in G(F)$, $\rho \in \hat{K}$ and $e \in \mathcal{B}_r(\rho)$, where the differentiation under the integral sign is justified by the absolute convergence of the resulting expression and in the last line we have used the well-known ‘doubling formula’ $\int_{K} \Xi^G(gk\gamma)dk = \Xi^G(g)\Xi^G(\gamma)$ (see [Var], Proposition 16.(iii), p.329).

Fix a decomposition $\gamma = m_P(\gamma)u_P(\gamma)k_P(\gamma)$ for every $\gamma \in G(F)$ where $m_P(\gamma) \in M(F)$, $u_P(\gamma) \in U_P(F)$ and $k_P(\gamma) \in K$ and set $H_P(\gamma) := H_M(m_P(\gamma))$. Then to prove $[A.0.6]$ we first note that

\[(\pi_\lambda(g)e, e) = \int_K \delta_P(m_P(kg))^{1/2}e^{\langle \lambda, H_P(kg) \rangle}(\tau(m_P(kg))e(k_P(kg)), e(k))dk\]

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so that (once again differentiation under the integral is easily justified)

\[ D (\lambda \mapsto (\pi_\lambda(g)e,e))_{\lambda=0} = \int_K \delta_p(m_P(kg))^{1/2}D(H_P(kg))(\tau(m_P(kg))e(k_P(kg)),e(k))dk \]

for all \( \tau \in \Pi_2(M) \), \( e \in \pi_K \) and \( g \in G(F) \) where when we write \( D(H_P(kg)) \) we consider \( D \) as a polynomial function on \( \mathcal{A}_M \). Clearly \( |D(H_P(kg))| \ll \sigma(g)^{\deg(D)} \) for all \( g \in G(F) \) and \( k \in K \) and therefore

(A.0.7)

\[ |D(\lambda \mapsto (\pi_\lambda(g)e,e))_{\lambda=0}| \ll \sigma(g)^{\deg(D)} \int_K \delta_p(m_P(kg))^{1/2}|(\tau(m_P(kg))e(k_P(kg)),e(k))|dk \]

for all \( \tau \in \Pi_2(M) \), \( e \in \pi_K \) and \( g \in G(F) \). By \cite{CHH} Theorem 2, we have

\[ |(\tau(m)v,v')| \ll \dim(\tau(K_M)v)^{1/2} \dim(\tau(K_M)v')^{1/2} \Xi^M(m)\|v\|\|v'| \]

for all \( \tau \in \Pi_2(M) \), all \( v, v' \in \tau \) and all \( m \in M(F) \). Notice that \( \dim(\tau(K_M)e(k)) \leq \dim(\pi(K)e) \) for all \( \tau \in \Pi_2(M) \) and \( e \in \pi_K \) and that by a new application of \cite{CHH} Theorem 2 \( \dim(\pi(K)e) \leq \dim(\rho)^2 \) if \( e \in \pi_K[\rho] \). Combining this with \cite{A.0.7} we obtain

\[ |D(\lambda \mapsto (\pi_\lambda(g)e,e))_{\lambda=0}| \ll \sigma(g)^{\deg(D)} \dim(\rho)^2 \int_K \delta_p(m_P(kg))^{1/2} \Xi^M(m_P(kg))\|e(k_P(kg))\|\|e(k)\|dk \]

\[ \leq \sigma(g)^{\deg(D)} \dim(\rho)^2 \sup_{k \in \hat{K}} \|e(k)\|^2 \int_K \delta_p(m_P(kg))^{1/2} \Xi^M(m_P(kg))dk \]

\[ \leq \sigma(g)^{\deg(D)} \dim(\rho)^2 \Xi^G(g)\|e\| \]

for all \( \tau \in \Pi_2(M) \), \( \rho \in \hat{K} \), \( e \in \pi_K[\rho] \) and \( g \in G(F) \) where in the last inequality we have used \cite{Var} Proposition 16(iv) p.329 and Lemma \cite{A.0.1}. Since \( \|e\| = 1 \) and there exists \( n \geq 1 \) such that \( \dim(\rho) \leq N(\rho)^n \) \cite{Wall} p.291], this gives \cite{A.0.6} and ends the proof of the Proposition 2.13.1.

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