Quantum and classical phase transitions in double-layer quantum Hall ferromagnets

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Abstract

We consider the problem of quantum and classical phase transitions in double-layer quantum Hall systems at $\nu = 1/m$ ($m$ odd integers) from a long-wavelength statistical mechanics viewpoint. We derive an explicit mapping of the long-wavelength Lagrangian of the quantum Hall system into that of a three-dimensional isotropic classical $XY$ model whose coupling constant depends on the quantum fluctuation in the original quantum Hall Hamiltonian. Universal properties of the quantum phase transition at the critical layer separation are completely determined by this mapping. The dependence of the Kosterlitz-Thouless transition temperature on layer separation, including quantum fluctuation effects, is approximately obtained by simple finite-size scaling analyses.

73.40.Hm, 73.20.Dx, 75.30.Kz
Low-dimensional electron systems exhibit a richer variety of physical properties than their higher-dimensional counterparts due to enhanced interaction effects. For a two-dimensional electron gas in a perpendicular magnetic field, the interaction effects are especially important because of Landau level quantization. When electrons are entirely restricted to the lowest Landau level by a strong magnetic field, electron-electron interaction completely dominates the properties of the system as the electron kinetic energy is quenched to an unimportant constant. One of the most interesting phenomena in these strongly correlated electron systems is the quantum Hall effect (QHE), which has attracted a great deal of experimental and theoretical interest.\(^1\) In recent years, a lot of attention has been directed to quantum Hall systems in double-layer structures where electrons are confined to two parallel planes separated by a distance comparable to the in-plane inter-electron distance. With the introduction of this layer degree of freedom, many qualitatively new effects due entirely to interlayer electron correlations appear.\(^2\)–\(^9\) These new features include QHE phases with various spontaneously-broken symmetries, such as the interlayer coherent state\(^3\) at \(\nu = 1/m\) (\(m\) odd integers) and the canted antiferromagnetic state\(^7\) at \(\nu = 2\), where interesting phase transitions both at zero and finite temperatures may occur. Thus, multi-component quantum Hall systems provide a suitable platform for studying various quantum phase transitions and their crossover behaviors.\(^3\) In this paper, we consider the quantum phase transition at \(d = d_\text{c}\) and the Kosterlitz-Thouless transition at \(d < d_\text{c}\) in double-layer systems at \(\nu = 1/m\), where \(d\) is the layer separation. Our consideration is based on an explicit mapping of the long-wavelength Lagrangian of the quantum Hall system into that of a three-dimensional (3D) isotropic classical XY model whose coupling constant \(g\) depends on the quantum fluctuation terms of the original Hamiltonian. The mapping shows unambiguously that the quantum phase transition at \(d_\text{c}\) is in the same universality class as that of a 3D XY-model transition at its critical coupling constant \(g_\text{c}\). The dependence of the Kosterlitz-Thouless transition temperature on layer separation is approximately obtained by a straightforward finite-size scaling analysis around the quantum critical point. In this way, both quantum and classical phase transitions in this problem are described in terms of the known properties of
To be specific, we restrict ourselves to $\nu = 1$ (i.e. $m = 1$), where various energy scales can be determined in the Hartree-Fock approximation. Our results, however, apply qualitatively to the general case of $\nu = 1/m$ with $m$ an odd integer. There has been a lot of work on the $\nu = 1$ quantum Hall system. At large layer separations, where the interlayer Coulomb interaction is negligible, the double-layer system is effectively a pair of decoupled half-filled single layers which exhibit no QHE. At small layer separations, the interlayer Coulomb interaction is almost as important as the intralayer interaction. All electrons are in the symmetric state where interlayer and intralayer electron correlations are treated on an equal footing. At $\nu = 1$, the electrons form a filled band and exhibit the QHE. The QHE phase at small $d$ and the non-QHE phase at large $d$ are separated by a continuous transition at $d = d_c$. For convenience, we introduce a pseudospin variable $S$ to describe the layer degree of freedom, where $S_z = \pm 1/2$ represent electron occupation of the right or left layers, respectively, and $S_x = \pm 1/2$ represent electron occupation of the symmetric or antisymmetric subbands, respectively. The transition between the QHE and non-QHE phases at $d = d_c$ may be viewed as a magnetization transition: The non-QHE phase at $d > d_c$ corresponds to a pseudospin disordered phase and the QHE phase at $d < d_c$ corresponds to a pseudospin magnetization in $\hat{x}$-direction. (Note that we assume that physical spins of the electrons are completely polarized by the applied magnetic field and are not relevant variables at all.) Even though, there has been a lot of work on the $\nu = 1$ system, most theoretical efforts, however, were directed towards the understanding of the properties in the QHE phase. These studies usually ignore quantum fluctuations, and hence shed no light on the nature of the quantum phase transition at $d_c$. In fact, many quantities obtained in this way, such as the pseudospin stiffness, the magnetization, and the susceptibility, do not show any sign of a phase transition at all. The present work is concerned solely with the phase transition, so it is essential that we include quantum fluctuations. Our goal is accomplished by a mapping of the low energy physics of the $\nu = 1$ system into that of a 3D classical XY model. Although, the basic ideas involved here have largely been known...
in the literature\textsuperscript{34} we think it is still very useful to explicitly carry out the derivations and put these ideas into a concrete basis so that more sophisticated calculations may start from here.

For the purpose of discussing the spontaneous pseudospin magnetization, we prohibit interlayer tunneling. (The tunneling acts like a Zeeman term in the pseudospin space.) This is not an unreasonable restriction, as the interlayer tunneling can be made very small in real semiconductor samples. The Hamiltonian of the $\nu = 1$ double-layer quantum Hall system is

$$H = \frac{1}{2} \sum_{ij} \sum_{\alpha_1 \alpha_2} \frac{1}{\Omega} \sum_q V_{ij}(q) e^{-q^2 l_o^2/2} e^{i q_x (\alpha_1 - \alpha_2) l_o^2} \times C_{i \alpha_1 + q_y}^\dagger C_{j \alpha_2}^\dagger C_{j \alpha_2 + q_y} C_{i \alpha_1}, \quad (1)$$

where $\Omega$ is the area of the sample, $l_o$ is the magnetic length, and $C_{i \alpha}$ annihilates an electron in the lowest Landau level in layer $i$ ($i = 1, 2$) and with the intra-Landau level index $\alpha$. The interaction potentials are $V_{ij} = 2 \pi e^2 / \epsilon q$ for $i = j$ and $V_{ij} = V_{11} e^{-qd}$ for $i \neq j$. At $d = 0$, $H$ has a $SU(2)$ symmetry in the pseudospin space. When $d > 0$, only a $U(1)$ symmetry is left because $H = H(S_z)$. Since $V_{11} > V_{12}$ for $d > 0$, it is always energetically favorable to maintain equal occupations of electrons in the two layers. Therefore, $H$ describes an easy-plane pseudospin magnetism with possible spontaneous magnetization only in the $\hat{x}$-$\hat{y}$ plane. Since $[H, S_{x,y}] \neq 0$, the QHE phase, which has a pseudospin ordering in the $\hat{x}$-$\hat{y}$ plane, is subjected to quantum fluctuations.

In the QHE phase, charge excitations\textsuperscript{34} are gapped, so the relevant degrees of freedom at low temperatures involve only neutral pseudospin excitations. Then, the partition function can be expressed in terms of a coherent state path integral over pseudospin configurations

$$Z = \int_{|\bm{m}|=1} D\bm{m} e^{-S_E(\bm{m})}, \quad (2)$$

where $\bm{m}$ is a unit vector representing the orientation of the pseudospin at position $(\mathbf{r}, \tau)$. The Euclidean action is

$$S_E(\bm{m}) = \int d^2 r \int_0^{L_x} d\tau \left( E(\bm{m}) - \frac{i \nu}{4 \pi l_o^2} \mathbf{A} \cdot \partial_r \bm{m} \right), \quad (3)$$
where $L_{\tau} = 1/k_B T$, the inverse of temperature, is the system size in the (imaginary) time-direction. We shall first consider the case of zero temperature, so $L_{\tau} = \infty$. The vector potential $A$ accounts for the Berry phase accumulated under time evolution of the pseudospins: $\epsilon_{ijk} \partial A_k / \partial m_j = m_i$. In the long-wavelength limit, the energy functional $E(m)$ has been obtained from the microscopic Hamiltonian of Eqn. (1)

$$E(m) = \beta m m_z^2 + \frac{\rho_A}{2} (\nabla m_z)^2 + \frac{\rho_E}{2} \left[ (\nabla m_x)^2 + (\nabla m_y)^2 \right]$$

$$E(m) \approx \beta_m m_z^2 + \frac{\rho_E}{2} (\nabla \phi)^2,$$

where $\beta_m$ and $\rho^{A(E)}$ are constants which will be given below. These terms have clear physical meanings: The gradient terms intend to maintain a pseudospin ordering: an exchange-induced pseudospin stiffness; $\beta_m m_z^2$ intends to suppress pseudospin polarization in $\hat{z}$-direction, i.e., it intends to maintain equal occupations of electrons in the two layers to minimize Coulomb interaction energy. In arriving at Eqn. (3), we have parameterized $m = (\sqrt{1 - m_z^2} \cos \phi, \sqrt{1 - m_z^2} \sin \phi, m_z)$, neglected $(\rho_A/2)(\nabla m_z)^2$ in comparison with $\beta_m (m_z)^2$ under the long-wavelength approximation, and assumed $|m_z| \ll 1$ because of the suppression of pseudospin polarization in $\hat{z}$-direction at finite layer separations. The $\beta_m m_z^2$ term in Eqn. (3) carries the quantum fluctuation effects: It is proportional to the expectation value of $(\partial / \partial \phi)^2$, which clearly does not commute with $\phi$. This term is, for example, equivalent to the charging energy in Josephson junction arrays.

At $\nu = 1$, we have

$$\beta_m = \frac{\nu}{16\pi^2 l_o^2} \int_0^\infty q^2 V_{11}(q) \left[ d - (1 - e^{-qd})/q \right] e^{-q^2 l_o^2/2} dq,$$

$$\rho_E = \frac{\nu l_o^2}{32\pi^2} \int_0^\infty q^3 V_{11}(q) e^{-qd} e^{-q^2 l_o^2/2} dq,$$

with $\rho^A = \rho^E(d = 0)$. $\rho^E$ decreases as the layer separation increases: It is the exchange-induced pseudospin stiffness associated with the interlayer phase coherence in the $\nu = 1$ system. Contributions to $\beta_m$ come largely from the static charging energy of the double-layer electronic system. $\beta_m$ increases monotonically as a function of $d$ with $\beta_m(d = 0) = 0$, which represents the increased suppression of pseudospin polarization in $\hat{z}$-direction as $d$ gets
larger. For $d$ not too small, we may think that the low temperature physics is completely dominated by the $\phi$-fluctuations and we may integrate out the $m_z$ degree of freedom:

$$Z = \int_{|m|=1} Dm \ e^{-S_E(m)} = \int D\phi \ e^{-S_{\phi}(\phi)}, \quad (7)$$

with

$$e^{-S_{\phi}(\phi)} = \int Dm_z \ e^{-S_E(m)} = \int Dm_z \ e^{-\beta m_z^2 + \frac{\nu}{\pi \l_0^2} (\nabla \phi)^2 - \frac{i}{4\pi \l_0^2} \mathbf{A} \cdot \partial \mathbf{m}}. \quad (8)$$

The integration over $m_z$ is straightforward: $\phi$ and $m_z$ are coupled only through the Berry phase term which, under a suitable gauge choice, is $\mathbf{A} \cdot \partial \mathbf{m} = -m_z \partial \mathbf{r} \phi$. Up to an irrelevant constant, we obtain

$$S_{\phi}(\phi) = \frac{1}{g} \int d^2r dx_0 \left[ (\nabla \phi)^2 + \left( \frac{\partial}{\partial x_0} \phi \right)^2 \right], \quad (9)$$

where the coupling constant is

$$g = \frac{8\sqrt{2\pi \l_0^2}}{\nu} \sqrt{\frac{\beta_m}{\rho E}}, \quad (10)$$

and the time-dimension has been rescaled by

$$\tau = \frac{\nu}{4\sqrt{2\pi \l_0^2} \sqrt{\beta_m \rho E} x_0}. \quad (11)$$

The action in Eqn. $(9)$, $S_{\phi}$, is exactly that of a 3D classical $XY$ system, which is known to have a phase transition at $g = g_c = k_{3D}/\sqrt{2\pi \l_0^2}/\nu$, where $k_{3D}$ is the dimensionless critical coupling constant. Since $g$ increases monotonically as a function of $d$ with $g(d=0) = 0$, the critical coupling constant $g_c$ thus corresponds to a critical layer separation $d_c$ given by $g(d_c) = g_c$, which is

$$\left( \sqrt{\frac{\l_0^2 \beta_m}{\rho E}} \right)_{d=d_c} = \frac{1}{8k_{3D} \sqrt{\frac{\nu}{\pi}}} \quad (12)$$

We have succeeded in mapping the long-wavelength physics of the quantum Hall system into that of an isotropic 3D classical $XY$ model. The result shows that there is an easy-plane
pseudospin order-disorder transition at $g_c$, which is associated with the QHE to non-QHE quantum phase transition at $d_c$. Universal properties of this quantum phase transition are therefore the same as those of a 3D classical $XY$ model, which are well known. The fact that $g \propto \sqrt{\beta_m}$ suggests that the phase transition at $d_c$ is driven by quantum fluctuations. Note, however, that the value of the critical layer separation $d_c$, which is not a universal quantity, may not be accurately given by this approach. In practice, one may treat $g_c$, or $k_{3D}$, as an adjustable parameter to make the value of $d_c$ given by Eqn. (12) match that found in experiments.

It is known that there is a linear mode in the QHE phase, which is associated with the pseudospin-channel superfluidity. This neutral superfluid mode can be obtained easily in the present formalism: A simple examination of Eqn. (9) gives

$$\omega(q) = \frac{4\pi l_o^2}{\nu} \sqrt{2 \beta_m \rho E q}, \quad (13)$$

which agrees completely with earlier results. This mode is linear because the effective action $S_{\text{eff}}$ in Eqn. (9) is isotropic in the three-dimensional $(\tau, r)$-space.

The zero-temperature easy-plane pseudospin order at $d < d_c$ persists, in the form of a quasi-long-range order, up to a critical temperature. This finite temperature transition is a Kosterlitz-Thouless transition, since the system is effectively two-dimensional at $T > 0$. The critical temperature of the phase transition should depend on $d$ and vanish at $d = d_c$.

For simplicity, let us first consider the case where the coupling constant $g$ in Eqn. (3) is small so that the easy-plane pseudospin order can persist up to relatively high temperatures where $L_0 = (4\pi \sqrt{2} l_o^2 / \nu) \sqrt{\beta_m \rho E} L_r$, the system size in $\hat{x}_0$-direction, is small. We may neglect the time dependence of $\phi$, i.e., let $\phi(x_0, r) = \phi(r)$, and approximate Eqn. (3) as

$$S_{\text{eff}}(\phi) = \frac{L_0}{g} \int d^2 r (\nabla \phi)^2, \quad \text{for} \quad L_0 \to 0. \quad (14)$$

It becomes a well-studied two-dimensional $XY$ model, for which a Kosterlitz-Thouless transition occurs at

$$(L_0)_c \approx \frac{g}{\pi}, \quad \text{for small} \quad g, \quad (15)$$
where \((L_0)_c = L_0(T_c)\). The critical temperature determined in this way is \(k_B T = (\pi/2)\rho E\), a result obtained earlier.\(^3\) This result neglected the time dependence in \(\phi\) and hence excluded quantum fluctuations. It is valid only for small \(g\) and fails completely at \(g \to g_c\), where the pseudospin order is destroyed by quantum fluctuations even at zero temperature. Correction to the critical temperature from the quantum fluctuations at \(g \to g_c\) can be analyzed by finite-size scaling arguments. For small \(T\) and \((g - g_c)\), the free-energy density of the system is proportional to a universal function \(f(L_0/\xi)\), where \(\xi = \xi_o(1 - g/g_c)^{-\nu}\), with \(\nu = 2/3\), is the correlation length of a 3D XY model.\(^4\) Finite temperature phase transition, which corresponds to a singularity point \(u^*\) in \(f(u)\), occurs at \((L_0)_c/\xi = u^*\), i.e., at \((L_0)_c = u^*\xi_o(1 - g/g_c)^{-\nu}\). The coefficient \((u^*\xi_o)\) can be fixed by the limit of Eqn. (15). These considerations give the critical temperature of the Kosterlitz-Thouless transition in the \(\nu = 1\) double-layer system as

\[
k_B T_c = \frac{\pi}{2} \rho E (1 - g/g_c)^{2/3},
\]

for \(0 < (1 - g/g_c) << 1\). This simple relationship should be experimentally verifiable.

Combining the descriptions of both the zero temperature quantum phase transition at \(d = d_c\) and the finite temperature Kosterlitz-Thouless transition at \(d < d_c\), a phase diagram for the \(\nu = 1\) double-layer system can be constructed and is shown in Fig. 1. It is clear from the figure that the QHE phase exists only for \(d < d_c\) and \(T < T_c\). Experimental evidence for both the quantum phase transition at \(d = d_c\) and the Kosterlitz-Thouless transition at \(d < d_c\) has been observed.\(^{6,8}\) The simple result of \(T_c\) in Fig. 1, i.e., given by Eqn. (16), also provides a useful check for any microscopic calculations of the critical temperature. We mention that the phase diagram given here does not apply in the limit of \(d \to 0\) where out-of-plane pseudospin fluctuations become important. This limit is, however, not experimentally accessible and is not considered here.

In summary, we have discussed both the zero temperature quantum phase transition at \(d = d_c\) and the finite temperature Kosterlitz-Thouless transition at \(d < d_c\) in a \(\nu = 1\) double-layer system by an explicit mapping of the long-wavelength Lagrangian of the quantum Hall
system into that of a 3D classical XY model. The effects of quantum fluctuations are included naturally in our treatment. This approach gives a simple and unified description of the quantum and classical phase transitions in terms of the known properties of the 3D XY model. In particular, it enables an approximate description of the effect of quantum fluctuations on the Kosterlitz-Thouless transition temperature by finite-size scaling analyses. The results presented here are not quantitatively accurate, but they correctly capture the essential physics of the phase transitions in double-layer quantum Hall systems at $\nu = 1/m$ ($m$ odd integers).

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FIGURES

FIG. 1. Phase diagram of double-layer quantum Hall systems at $\nu = 1$, where $T$ is temperature and $g$ is the coupling constant. The finite temperature phase boundary is given by Eqn. (16).
Fig. 1 L. Zheng

\[ \frac{\kappa_B T}{\rho^E} \]

\[ \nu = 1 \]

\[ \text{QHE} \quad \text{non-QHE} \]

\[ (d < d_c \quad T < T_c) \]

\[ g/g_c \]