OBSTRUCTION OF $C_\infty$-ALGEBRA MODELS AND CHARACTERISTIC CLASSES

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Abstract. In this paper, we consider an obstruction-theoretical construction of characteristic classes of fiber bundles by simplicial method. We can get a certain obstruction class for a deformation of $C_\infty$-algebra models of fibers and a characteristic map from the exterior algebra of a vector space of derivations. Applying this construction for a surface bundle, we obtain the Euler class of a sphere bundle and the Morita-Miller-Mumford classes of a bundle with positive genus fiber.

1. Introduction

Our purpose of the paper is to construct characteristic classes of a smooth fiber bundle $X \to E \to B$ by obstruction theory for a certain simplicial bundle $Q \bullet (E) \to S \bullet (B)$ obtained from the original bundle. The base simplicial set $S \bullet (B)$ of the simplicial bundle $Q \bullet (E) \to S \bullet (B)$ is the simplicial set of singular simplices of $B$ and the $n$-th set $Q_n(E)\sigma$ of the fiber over an $n$-simplex $\sigma$ is the set of Chen’s formal homology connections [4, 5] on $\sigma^*E$. A formal homology connection on a manifold $X$ has rational homotopical information of $X$, which is equivalent to a minimal $C_\infty$-algebra model $f : (H, m) \to A$ of the reduced de Rham complex $A$ such that $m$ is a minimal $C_\infty$-algebra structure and the first term of $f$ induces the identity map on cohomologies (see [9]). The fiber of the bundle is the simplicial set $Q \bullet (X)$ of formal homology connections on $X \times \Delta^n$. This simplicial set is very close to the Maurer-Cartan simplicial set of the dgl $\hat{L}W \otimes A$, where $(\hat{L}W, \delta)$ is the dual of the bar-construction of the $C_\infty$-algebra $(H, m)$.

We introduce two versions of construction depending on whether the fiber $Q \bullet (X)$ is connected or not. The homotopy group of the Maurer-Cartan simplicial set is known in [7, 1, 2]. So the homotopy groups of $Q \bullet (X)$ can be also expressed as vector spaces by

$$\pi_n(Q \bullet (X), \tau) = H_n(Der(\hat{L}W), \delta)$$

for a formal homology connection $\tau = (\omega, \delta)$ on $X$.

In the case that $Q \bullet (X)$ is connected, under certain conditions, an obstruction class of existence of a partial section over the $n$-skeleton of $Q \bullet (E) \to S \bullet (B)$

$$o_n \in H^{n+1}(B; \Pi_n)$$

is obtained, where $\Pi_n$ is the local system of the $n$-th homotopy groups of fibers of $Q \bullet (E) \to S \bullet (B)$. Then we get the characteristic map

$$(\Lambda^p H_n(Der(\hat{L}W), \delta^*)^G \to H^{p(n+1)}(B; \mathbb{R})$$

for any $p \geq 1$. Here $G$ is the structure group of $E \to B$. As an application, this yields the Euler class of a sphere bundle.

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On the other hand, if $Q_\bullet(X)$ is not connected, the local system $\Pi_0$ of sets has a free and transitive action of a certain local system $\text{QIAut}(E)$ of groups. Since this group has a natural filtration, we get the graded Lie algebra $\text{gr}(\text{QIAut}(E))$. The fiber of $i$-th part can be identified with a certain vector space $\text{gr}_i(\hat{LW})$. Using this vector space in stead of the homotopy groups of $Q_\bullet(X)$, we can obtain the obstruction $\sigma(i) \in H^1(B; \text{gr}_i(\text{QIAut}(E)))$ and the characteristic map $(\Lambda^* \text{gr}_i(\text{QDer}(\hat{LW})))^* \to H^*(B; \mathbb{R})$ according to the stage $i$ of extension of a partial section. Applying for a surface bundle, the obstruction class for $i = 0$ corresponds the twisted Morita-Miller-Mumford class and the characteristic map gives the Morita-Miller-Mumford classes.

The paper is organized as follows. In Section 2 we define terms which we use in the paper. In Section 3 we review formal homology connections and investigate the simplicial set of these connections. In Section 4 we describe obstruction theory for simplicial sets by the form which is convenient to use. (The obstruction theory for simplicial sets are discussed in [3, 6].) In Section 5 we apply the discussion before the section for a smooth bundle and calculate for specific bundles.

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2. Preliminary

In this paper, all vector spaces are over the real number field $\mathbb{R}$. The standard $n$-simplex is described by

$$\Delta^n = \left\{ (t_i)_{i=0}^n; \sum_{i=0}^n t_i = 1 \right\}$$

and we fix its base point $(1, 0, \ldots, 0)$. Note that the base point is $\delta^n \cdots \delta^1(\Delta^n)$, where $\delta^i : \Delta^{n-1} \to \Delta^n$ is the $i$-th coface operator.

2.1. Graded vector space. Let $V$ be a $\mathbb{Z}$-graded vector space. We denote $V^i$ the subspace of elements of $V$ of cohomological degree $i$ and $V_i = V^{-i}$ the subspace of elements of homological degree $i$. Remark that the linear dual $V^* = \text{Hom}(V, \mathbb{R})$ of $V$ is graded by $(V^*)^i = \text{Hom}(V_i, \mathbb{R})$.

The $p$-fold suspension $V[p]$ of $V$ for an integer $p$ is defined by

$$V[p]^i := V^{i+p}$$

and elements of $V[p]^i$ are presented by $\sigma x$ for $x \in V^{i+p}$ using the symbol $\sigma$ of cohomological degree $-p$.

2.2. Graded Lie algebra. Let $W$ be a $\mathbb{Z}$-graded vector space. In this paper, $W$ is always homologically non-negatively graded. We denote by $LW$ the graded free Lie algebra generated by $W$, and by $\hat{LW}$ the completed free Lie algebra generated by $W$:

$$\hat{LW} := \lim_{\leftarrow \mathbb{n}} LW/\Gamma_n,$$

where $\{\Gamma_n\}_{n=1}^\infty$ is the lower central series of $LW$. The lower central series of $\hat{LW}$ is denoted by $\{\hat{\Gamma}_n\}_{n=1}^\infty$. 
We can also define $\hat{L}W$ as the primitive part of the completed tensor algebra $TW$. We often use the aspect.

2.3. Derivations. Let $(\hat{L}W, \delta)$ be a completed free dgl such that $\delta$ has the homological degree $-1$ and $\delta(W) \subset \hat{1}$. We consider the Lie algebra $\text{Der}(\hat{L}W)$ of Lie derivations on $\hat{L}W$. It is (completed) $\mathbb{Z}$-graded by

$$\text{Der}(\hat{L}W)_n := \{ D \in \text{Der}(\hat{L}W); \ D(W_i) \subset (\hat{L}W)_{n+i} \}.$$  

Then $(\text{Der}(\hat{L}W), \text{ad}(\delta))$ is a completed dgl.

The Lie algebra $\text{Der}(\hat{L}W)$ has the decreasing filtration defined by

$$\mathcal{D}^i := \text{Der}^{\geq i}(\hat{L}W) := \{ D \in \text{Der}(\hat{L}W); \ D(W) \subset \hat{1} \}.$$  

Then we have $\delta \in \mathcal{D}^1$ and $[\delta, \mathcal{D}^i] \subset \mathcal{D}^{i+1}$.

We introduce a filtration of the homology $\text{QDer}(\hat{L}W) = H_0(\text{Der}(\hat{L}W), \text{ad}(\delta))$ of $\text{Der}(\hat{L}W)$ by

$$\text{QDer}^{\geq 1}(\hat{L}W, \delta) := \text{Im}(Z_0(\mathcal{D}^i, \text{ad}(\delta)) \to \text{QDer}(\hat{L}W, \delta)).$$

**Definition 2.1.** The group $\text{IAut}(\hat{L}W, \delta)$ of automorphisms $f : (\hat{L}W, \delta) \to (\hat{L}W, \delta)$ such that $f : W \to \hat{L}W/\hat{1}_2 = W$ is the identity map $\text{id}_W$ has the bijection $\exp : \mathcal{D}^1_0 \to \text{IAut}(\hat{L}W, \delta)$ from $\mathcal{D}^1_0$, which is called the *exponential map*, defined by

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} \in \text{End}(\hat{L}W).$$

The map has the inverse map $\log : \text{IAut}(\hat{L}W, \delta) \to \mathcal{D}^1_0$. The product of the group $\text{IAut}(\hat{L}W, \delta)$ and the Lie bracket of $\mathcal{D}^1_0$ are related by the Baker-Campbell-Hausdorff formula.

We denote by $\text{QIAut}(\hat{L}W, \delta)$ the quotient group of $\text{IAut}(\hat{L}W, \delta)$ by the normal subgroup $\exp([\delta, \text{Der}(\hat{L}W)_1])$. Then we can also obtain the exponential map

$$\exp : \text{QDer}^+ (\hat{L}W, \delta) := \text{QDer}^{\geq 1}(\hat{L}W, \delta) \to \text{QIAut}(\hat{L}W, \delta)$$

and the filtration $\mathcal{F}^i$ of the group $\text{QIAut}(\hat{L}W, \delta)$ as the image of $\text{QDer}^{\geq i}(\hat{L}W, \delta)$.

If $\delta(W) \subset [W, W]$, we can define another grading of $\text{Der}(\hat{L}W)$ by

$$\text{Der}^i(\hat{L}W) := \{ D \in \text{Der}(\hat{L}W); \ D(W) \subset L^{i+1} \},$$

where $L^iW = LW \cap W^{\equiv i}$. Then $\delta$ has the degree 1 with respect to the grading. So we have the identification

$$\text{QDer}^i(\hat{L}W, \delta) := H_0^i(\text{Der}(\hat{L}W), \text{ad}(\delta)) \simeq \text{QDer}^{\geq i}(\hat{L}W, \delta)/\text{QDer}^{\geq i+1}(\hat{L}W, \delta),$$

where $i$ is the second grading of $\text{Der}(\hat{L}W)$.

2.4. Manifolds and fiber bundles. Throughout the paper, we consider a smooth fiber bundle $X \to E \to B$ whose fiber $X$ is a manifold with a base point, i.e., a fiber bundle with a section $B \to E$. We always suppose a manifold $X$ with base point is connected and has the finite-dimensional rational homology group, so we call such a manifold $X$ a **fiber manifold** for simplicity in the paper.

The structure group of the bundle, which is a subgroup of the diffeomorphism group $\text{Diff}(X)$, acts on the homology group of $X$. We call the image $G$ in the automorphism group of $H_*(X; \mathbb{R})$ the **homological structure group**.
2.5. **Formal homology connections and $C_\infty$-algebra models.** Let $X$ be a fiber manifold (with base point $\ast$). We denote the deRham complex on $X$ by $A^\bullet(X)$, the reduced deRham complex and cohomology by $A = \tilde{A}^\bullet(X) := \{ f \in A^0(X); f(\ast) = 0 \} \oplus A^\ast(X), \ H = \tilde{H}^\ast_{DR}(X)$ and the suspension of the reduced real homology by $W = \tilde{H}^\bullet_{\mathbb{R}}(X)[−1]$.

**Definition 2.2** (Chen [4, 5]). A formal homology connection on $X$ is a pair $(\omega, \delta)$ satisfying the following conditions:

(i) an $\hat{L}W$-coefficient differential form $\omega \in A \otimes \hat{L}W$ with cohomological degree 1 is described by

$$\int _{x_p} \omega _p = 1,$$

where $x_1, \ldots, x_n$ is a homogeneous basis of $W$, such that

(ii) a linear map $\delta : \hat{L}W \rightarrow \hat{L}W$ is a differential with homological degree $−1$ of $\hat{L}W$ such that

$$\delta(W) \subset \hat{Γ}_2.$$

(iii) the form $\omega$ is a Maurer-Cartan element of $(A \otimes \hat{L}W, d + \delta)$, i.e., the flatness condition $\delta \omega + d\omega + \frac{1}{2}[\omega, \omega] = 0$ holds. (Though the sign notation may be different from Chen’s original definition, they are equivalent.)

We call such a differential $\delta$ Chen’s differential of $X$. If $X$ is simply connected, we can replace the free Lie algebra $LW$ and its derivation $\delta : LW \rightarrow LW$ with $\hat{L}W$ and $\delta : \hat{L}W \rightarrow \hat{L}W$ respectively.

It is well-known that, given a formal homology connection on $X$, we can compute the real cohomology of the loop space $\Omega X$ [4, 5].

2.6. **$C_\infty$-algebra and formal homology connection.** We shall mention the relation between a formal homology connection and a $C_\infty$-algebra. For the concept of $C_\infty$-algebra, we refer to [8].

**Definition 2.3** ($C_\infty$-algebra). Let $A$ be a vector space and $m = \{m_i\}_{i=1}^\infty$ be a family of linear maps $m_i : A^\otimes i \rightarrow A$ with degree $2 − i$. The pair $(A, m)$ satisfying the following conditions is called a $C_\infty$-algebra:

- **(A_∞-relation)**

$$\sum_{k+i=1}^{k-1} \sum_{j=0}^{i-1} (-1)^{(j+1)(l+1)} m_k \circ (\text{id}_A^\otimes j \otimes m_l \otimes \text{id}_A^\otimes (i−j−l)) = 0$$

for $i \geq 1$, and

- **(commutativity)**

$$\sum_{\sigma \in \text{Sh}(j, i−j)} \epsilon \cdot m_i(a_{\sigma(1)}, \ldots, a_{\sigma(i)}) = 0$$

for $i > j > 0$ and homogeneous elements $a_1, \ldots, a_i \in A$, where $\text{Sh}(i, i−j)$ is the set of $(i, i−j)$-shuffles and $\epsilon$ is the Koszul sign.
If $m_1 = 0$, $(A, m)$ is called **minimal**. If higher products are all zero, i.e. $m_3 = m_4 = \cdots = 0$, $(A, m)$ can be regarded as differential graded commutative algebra (DGcA).

**Remark 2.4** (Bar construction of a $C_\infty$-algebra). Let $(A, m)$ be a $C_\infty$-algebra and $s : A \to A[1]$ be the suspension map. We denote the tensor coalgebra $T^c(A[1])$ generated by $A[1]$ by $BA$. It is a bialgebra by the tensor coproduct $\Delta$ and the shuffle product $\mu$. Defining the suspension of $m_i$ by $\bar{m}_i := s \circ m_i \circ (s^{-1})^{\otimes i}$ for all $i \geq 1$, then $\bar{m}_i : A[1]^{\otimes n} \to A[1]$ is degree 1 and satisfies the commutativity condition. Thus extending the unique coderivation $m_i : BA \to BA$ by the co-Leibniz rule $\Delta \circ m_i = (m_i \otimes id + id \otimes m_i) \circ \Delta$, then we have the Hopf derivation

$$m := \sum_{i=1}^{\infty} m_i.$$ 

Furthermore $m$ is a degree 1 codifferential, i.e. $m^2 = 0$, from the $A_\infty$-relations of $m$.

**Definition 2.5** ($C_\infty$-morphism). Let $(A, m)$ and $(A', m')$ be two $C_\infty$-algebras and $f = \{f_i\}_{i=1}^{\infty}$ be a family of linear maps $f_i : A^{\otimes i} \to A'$ with degree $1 - i$ satisfying the following conditions:

- **($A_\infty$-morphism)**

$$\sum_{\substack{k_1 + \cdots + k_l = k \geq 1 \geq 1,}} (-1)^{\sum_{j=1}^{l} k_j (l-j) + \sum_{\nu < \mu} k_{\nu} k_{\mu}} f_{k_1} \otimes \cdots \otimes f_{k_l}$$

$$= \sum_{s+1+t=i, \ s+l+i=k} (-1)^{1+k+(s+1)(l+1)} f_i \circ (id_A^{\otimes s} \otimes m_l \otimes id_A^{\otimes t})$$

for $k \geq 1$, and

- **(commutativity)**

$$\sum_{\sigma \in \text{Sh}(j,i-j)} \epsilon \cdot f_i(a_{\sigma(1)}, \cdots, a_{\sigma(i)}) = 0$$

for $i > j > 0$ and homogeneous elements $a_1, \ldots, a_i \in A$.

Then $f$ is called a $C_\infty$-morphism. If $f_1$ is a quasi-isomorphism, $f$ is called a $C_\infty$-quasi-morphism.

**Definition 2.6.** Given $C_\infty$-algebra $(A, m^A)$, a pair $f : (H, m) \to (A, m^A)$ of a $C_\infty$-algebra structure $m$ on the cohomology $H := H(A, m^A)$ and a $C_\infty$-quasi-isomorphism $f$ such that $f_1$ induces the identity map on the cohomology $H$ is called $C_\infty$-algebra model.

**Remark 2.7** (Bar construction of a $C_\infty$-morphism). Let $f : (A, m) \to (A', m')$ be a $C_\infty$-morphism. Defining the suspension of $f_i$ by $\bar{f}_i := s \circ f_i \circ (s^{-1})^{\otimes i} : A[1]^{\otimes i} \to A'[1]$ for all $i \geq 1$, then the degree of $\bar{f}_i$ is 0. Constructing the coalgebra map $BA \to BA'$

$$\bar{f} := \sum_{k=1}^{\infty} \sum_{\substack{i \geq 1, k_1 + \cdots + k_l = k \geq 1,}} \bar{f}_{k_1} \otimes \cdots \otimes \bar{f}_{k_l}$$

from maps $\bar{f}_n$, we have the equations

$$f \circ m = m' \circ f, \quad f \circ \mu = \mu \circ (f \otimes f).$$

So $f$ is a differential bialgebra map $(BA, m) \to (BA', m')$ between bar constructions.
According to [9], a formal homology connection \((\omega, \delta)\) on \(X\) is equivalent to a minimal \(C_\infty\)-algebra model \(f : (H, m) \to A\), i.e., a pair of a minimal \(C_\infty\)-algebra structure on \(H\) and a \(C_\infty\)-algebra morphism \(f : (H, m) \to A\) such that the first term \(f_1\) induces the identity on \(H\). It is verified as follows: put
\[
\omega = - \sum_{i_1, \ldots, i_k} (-1)^i \sigma^{-1} \tilde{f}_n(x^{i_1}, \ldots, x^{i_k}) x_{i_1} \cdots x_{i_k},
\]
where
\[
\epsilon = |x_{i_1}|(|x_{i_2}| + \cdots + |x_{i_k}|) + \cdots + |x_{i_k-1}||x_{i_k}|,
\]
\[
\tilde{f}_n = \sigma f_n(\sigma^{-1})^n : H[1]^n \to A[1],
\]
\(x^i\) is the dual basis of \(x_i\), and \(m\) is the bar-construction of \(m\). Then the differential \(\delta\) on the dual \((BH)^* = TW\) of the bar-construction \(BH\) can be restricted on \(\hat{LW}\) since \(\delta\) is a coderivation. So the pair \((\omega, \delta)\) is a formal homology connection on \(X\). Conversely we can recover \(f : (H, m) \to A\) from \((\omega, \delta)\). Note that the condition that \(f\) is an \(A_\infty\)-morphism corresponds to the flatness.

3. Formal homology connections

3.1. The simplicial set of formal homology connections. Let \((X, *)\) be a fiber manifold. The set of formal homology connections on \(X\) is denoted by \(Q_0(X)\).

We define the simplicial deRham dga \(A_* = \{A_n\}_{n=0}^\infty\) on \(X\) by
\[
A_n := A^n(X \times \Delta^n).
\]
Its face maps and degeneracy maps are induced by the coface maps and codegeneracy maps of the cosimplicial space \(\Delta^* = \{\Delta^n\}_{n=0}^\infty\).

The family \(Q_*(X) = \{Q_n(X) := Q_0(X \times \Delta^n)\}_{n=0}^\infty\) of sets is a simplicial set by the induced structure by \(A_*.\) Given a Chen’s differential \(\delta\) on \(X\), the set of formal homology connections \((\omega, \delta)\) on \(X \times \Delta^n\) is denoted by \(Q_n(X, \delta)\). Then \(Q_*(X, \delta)\) is also a simplicial set. We denote the set of Maurer-Cartan elements of \((A_n \otimes \hat{LW}, d + \delta)\) by \(MC_n(X, \delta)\). We obtain the simplicial set \(MC_*(X, \delta)\), and then \(Q_*(X, \delta)\) is a subsimplicial set of \(MC_*(X, \delta)\).

**Lemma 3.1.** For any \(n\)-th simplicial Maurer-Cartan element \(\alpha \in MC_n(X, \delta)\), if \(\partial_i \alpha \in Q_{n-i}(X)\) for some \(0 \leq i \leq n\), then \(\alpha \in Q_n(X, \delta)\).

**Proof.** Regarding \(\alpha\) as a \(C_\infty\)-map \(f : H \to A_n\), \(f_1 : H \to H(A_n)\) is the identity map since \(\partial_i\) for any \(i\) gives the standard identification by \(H^n(X \times \Delta^n) \simeq H^n(X \times \Delta^{n-1})\) and \(\partial_i f_1 : H \to H(A_{n-1})\) is the identity map under the assumption.

Since the simplicial set \(MC(X, \delta)\) is a Kan complex (proved in Section 4 of [7]), the following lemma is obtained immediately from Lemma 3.1:

**Lemma 3.2.** The simplicial set \(Q_*(X)\) is a Kan complex. Furthermore the map induced by the inclusion
\[
\pi_0(Q_*(X, \delta)) \to \pi_0(MC_*(X, \delta))
\]
is injective, and the map
\[
\pi_n(Q_*(X, \delta), \tau) \to \pi_n(MC_*(X, \delta), \tau)
\]
for \(\tau \in Q_0(X, \delta)\) and \(n \geq 1\) is an isomorphism.
Theorem 3.3. The homotopy groups of the simplicial set \( Q_\bullet(X) \) are described by
\[
\pi_n(Q_\bullet(X), \tau) \simeq H_n(Der(\hat{LW}, \delta))
\]
for \( n \geq 1 \) and a formal homology connection \( \tau = (\omega, \delta) \) on \( X \), where \( H_1(Der(LW), \delta) \) is equipped with the Baker-Campell-Hausdorff product of \( H_0(A \otimes \hat{LW}) \).

Proof. From Proposition 5.4 and Theorem 5.5 in \([1]\), we have
\[
F = \pi_n(Q_\bullet(X), \tau) \simeq H_{n-1}(A \otimes \hat{LW}, d + \delta + [\omega, -]).
\]

We shall prove the suspension of \((A \otimes \hat{LW}, d + \delta + [\omega, -])\) and the chain complex \( \text{Der}_F(BH, BA) \) of Hopf derivations over the bar-construction \( F : BH \to BA \) of the \( C_\infty \)-morphism corresponding to \( \tau \) are isomorphic. Here the differential \( \mathcal{D} \) of the latter complex is defined by
\[
\mathcal{D}(D) = m^A \circ D - (-1)^D D \circ m,
\]
where \( m^A \) and \( m \) are the bar-constructions of \( C_\infty \)-structures of \( A \) and \( (H, m) \) respectively.

Through the natural identification \( \hat{T}W = (BH)^* \), consider the linear isomorphism \( \Phi : A[1] \otimes \hat{LW} \to \text{Der}_F(BH, BA) \subset \text{Hom}(BH, A[1]) \) defined by
\[
\Phi(\alpha \otimes f)(x) = f(x)\alpha
\]
for \( x \in BH \). Here the differential on \( A[1] \otimes \hat{LW} \) is equal to \( \sigma(d + \delta + [\omega, -])\sigma^{-1} \).

Then, using \( F = \Phi(\sigma \omega) \), we have
\[
\begin{align*}
\Phi(\sigma(d + \delta + [\omega, -])\sigma^{-1}(\alpha \otimes f))(x) &= d\sigma f(x) + (-1)^{\alpha+1} \sigma \delta f(x) + \sigma [\omega, \sigma^{-1} \alpha \otimes f](x) \\
&= d\sigma f(x) - (-1)^{\alpha+1} \sigma \delta f(x) + m^A_\ast \circ (F \otimes \Phi(\alpha f))(x) + m^A_\ast \circ (\Phi(\alpha f) \otimes F)(x) \\
&= \mathcal{D}(\Phi(\alpha f))(x).
\end{align*}
\]
Thus the map \( \Phi \) is a chain isomorphism.

On the other hand, the map
\[
F \circ : (\text{Der}(\hat{LW}), \text{ad}(\delta)) = (\text{Der}(BH), \text{ad}(m)) \to (\text{Der}_F(BH, BA), \mathcal{D})
\]
is a quasi-isomorphism because \( F \) is a quasi-isomorphism. So we get the isomorphism
\[
H_{n-1}(A \otimes \hat{LW}, d + \delta + [\omega, -]) \simeq H_n(\text{Der}(\hat{LW}), \text{ad}(\delta)).
\]

The set \( \pi_0(Q_\bullet(X)) \) of connected components can be identified with the set of homotopy classes of \( C_\infty \)-morphisms \( f : (H, m) \to A \) such that \( f_1 \) induces the identity map on \( H \). The group \( \text{QIAut}(H, m) \) of \( C_\infty \)-automorphisms \( f : (H, m) \to (H, m) \) such that \( f_1 = \text{id}_H \) acts on the right on \( \pi_0(Q_\bullet(X, \delta)) \) freely and transitively (details in \([11]\)).

3.2. The simplicial bundle of formal homology connections. Let \( X \to E \to B \) be a smooth fiber bundle. In the section, we shall define the simplicial bundle of formal homology connections on fibers corresponding to a smooth bundle.

Definition 3.4. We define the simplicial bundle \( Q_\bullet(E) \to S_\bullet(B) \) over the simplicial set \( S_\bullet(B) \) of regular simplices as follows:

- the fiber over an \( n \)-simplex \( \sigma \in S_n(B) \) is \( Q_n(E)_\sigma := Q_0(\sigma^* E) \), and
the face maps and the degeneracy maps are the induced maps
\( Q_n(E)_{\sigma} \to Q_{n-1}(E)_{\partial,\sigma} \) and \( Q_n(E)_{\sigma} \to Q_{n+1}(E)_{s,\sigma} \) by the coface maps and the codegeneracy maps of \( \Delta^\bullet \) respectively.

We can check that \( Q_\bullet(E) \to S_\bullet(B) \) is a bundle of simplicial sets in the sense of May [15].

**Proposition 3.5.** The simplicial map \( Q_\bullet(E) \to S_\bullet(B) \) is a simplicial bundle with fiber \( Q_\bullet(X) \).

**Proof.** For an \( n \)-simplex \( \sigma \in S_n(B) \) and a trivialization \( \varphi_\sigma : \Delta^n \times X \simeq \sigma^*E \), we obtain the trivialization \( \hat{\varphi}_{\sigma,P} : \Delta^i \times X \simeq \sigma(P)^*E \) for \( P \in \Delta[n]_i \) by the diagram

\[
\begin{array}{ccc}
\Delta^n \times X & \xrightarrow{\varphi_\sigma} & \sigma^*E \\
\uparrow_{f_P \times \text{id}_X} & & \uparrow \\
\Delta^i \times X & \xrightarrow{\varphi_{\sigma,P}} & \sigma(P)^*E
\end{array}
\]

regarding \( \sigma \) as a simplicial map \( \sigma : \Delta[n] \to S_\bullet(B) \). Here the map \( f_P : \Delta^i \to \Delta^n \) is the induced map \( P : \Delta[i] \to \Delta[n] \).

Then we obtain the simplicial trivialization

\[
\hat{\varphi}_{\sigma} : \sigma^*Q_\bullet(E) \simeq \Delta[n] \times Q_\bullet(X)
\]

by \( (P, \alpha) \mapsto (P, \varphi_{\sigma,P}^*\alpha) \), where

\[
\sigma^*Q_\bullet(E) = \{(P, \alpha) \in \Delta[n]_i \times Q_0(\sigma(P)^*E)\}.
\]

\[\square\]

We consider to fix a Chen’s differential on fibers.

**Definition 3.6.** Fix a Chen’s differential \( \delta \in \text{Der}(\hat{L}W)_{-1} \) of \( X \) is \( G \)-invariant with respect to the action of the homological structure group \( G \) on \( \text{Der}(\hat{L}W) \) (induced by the action on \( W \)). Then it gives the section \( \hat{\delta} \) of the bundle

\[
\mathcal{D}(E) \to B,
\]

where \( \mathcal{D}(E)_b := \{ \text{Chen’s differential of } E_b \} \) for \( b \in B \). We call \( \hat{\delta} \) a section of Chen’s differentials. Given this, we can consider the simplicial bundle \( Q_\bullet(E, \hat{\delta}) \to S_\bullet(B) \) defined by

\[
Q_n(E, \hat{\delta})_{\sigma} := Q_0(\sigma^*E, \hat{\delta}(\sigma))
\]

for \( \sigma \in S_n(B) \). Here \( \hat{\delta}(\sigma) \) is the Chen’s differential of \( \sigma^*E \) defined by \( \hat{\delta}(\sigma_0) \) through the isomorphism \( H(\sigma^*E) \simeq H(E_{\sigma_0}) \). Here \( \sigma_0 = \partial_1 \cdots \partial_n \sigma \) is the image of the base point of \( \Delta^n \).

For example, if \( X \) is formal, the differential \( \delta \) corresponding to the cohomology ring structure of \( X \) is \( \text{Diff}(X) \)-invariant.

4. Obstruction theory

Obstruction theory for simplicial sets is studied in [3, 6]. In Section 4.1 and 4.2, we shall review a part of them and rewrite obstruction theory as in Steenrod [19] for simplicial sets in order to fit our use briefly. In Section 4.2, we introduce obstruction classes to extend a section over the 0-skeleton stepwisely.
4.1. Local system. We shall define cohomology with local coefficients briefly. We can see definitions in this subsection in [3, 6].

**Definition 4.1.** Let $\mathcal{X}$ be a Kan complex. We define the fundamental groupoid $\Pi_1(\mathcal{X})$ of $\mathcal{X}$ such that the set of objects is $\mathcal{X}_0$ and the set of morphisms from $x$ to $y$ is the set of homotopy classes of $\gamma \in \mathcal{X}_1$ satisfying $\partial_0 \gamma = x$ and $\partial_1 \gamma = y$. A covariant functor $\Pi_1(\mathcal{X}) \to \text{Ab}$ is called a **local system** on $\mathcal{X}$. Here $\text{Ab}$ is the category of abelian groups.

Let $E \to B$ be a Kan simplicial bundle with $n$-simple fiber $\mathcal{X}$, i.e., $\mathcal{X}$ is a Kan complex and $\pi_1(\mathcal{X}, x)$ acts on $\pi_n(\mathcal{X}, x)$ trivially.

**Definition 4.2.** We define the local system $\Pi_n(E/B)$ on $B$ as follows: for a vertex $v \in B_0$,

$$\Pi_n(E/B)_v := \pi_n(v^*E).$$

Note that we need not to choose a base point of $v^*E$ because it is $n$-simple. For a path $\gamma \in B_1$ such that $v_0 = \partial_1 \gamma$ and $v_1 = \partial_0 \gamma$, take a trivialization $\varphi_\gamma : \Delta[1] \times v_0^*E \simeq v_1^*E$

such that

$$\Delta[1] \times v_0^*E \xrightarrow{\varphi_\gamma} \gamma^*E$$

Here $\delta^i : \Delta[0] \to \Delta[1]$ is the coface maps. Then we have the isomorphism $g_\gamma : v_0^*E \to v_1^*E$, which is called **holonomy** along $\gamma$, defined by

$$\Delta[1] \times v_0^*E \xrightarrow{\varphi_\gamma} \gamma^*E$$

So we put

$$\Pi_n(E/B)(\gamma) := (g_\gamma^{-1})_* : \pi_n(v_1^*E) \to \pi_n(v_0^*E).$$

We can prove that it is depend on only the homotopy class of $\gamma$ since $E \to B$ is Kan fibration. In fact, for another path $\gamma'$ homotopic to $\gamma$ by a homotopy $\sigma \in B_2$, there exists a homotopy $h$ satisfying the commutative diagram

$$\Lambda^2[2] \times v_0^*E \xrightarrow{\varphi_\gamma \cup \varphi_{\gamma'}} \sigma^*E$$

by Theorem 7.8 in [15]. Here $\Lambda^2[2]$ is the $(2, 2)$-horn.

The cochain complex and the cohomology with local coefficients are defined as follows.
Definition 4.3. Let $X$ be a Kan complex, $A$ a subsimplicial set of $X$, and $M : \Pi_1(X) \to \text{Ab}$ a local system on $X$. We define the cochain complex with coefficient $M$ by

$$C^n(X, A; M) := \left\{ c : X_n \to \coprod_{v \in X_0} M(v); \ c(x) \in M(x_0), \ c|A = 0 \right\} ,$$

where $x_0 = \partial_1 \cdots \partial_n x$, and its normalized version by

$$N^n(X, A; M) := \bigcap_{i=0}^n \text{Ker}(s_i^* : C^n(X, A; M) \to C^{n-1}(X, A; M)).$$

The differential $\delta : C^n(X, A; M) \to C^{n+1}(X, A; M)$ is defined by

$$\delta c(x) = M(x_0) c(\partial_0 x) - c(\partial_1 x) + \cdots + (-1)^{n+1} c(\partial_{n+1} x),$$

where $x_0 = \partial_2 \cdots \partial_n x$. Its cohomology is denoted by $H^n(X, A; M)$.

4.2. Obstruction cocycles and difference cochains. Let $A$ be a subsimplicial set of $B$. We call a simplicial map $s$ satisfying the following diagram an $n$-partial section relative to $A$:

$$\begin{array}{ccc}
E & \xrightarrow{s} & \mathcal{E} \\
\downarrow & & \downarrow \\
\text{sk}_n(B) \cup A & \xrightarrow{s} & B
\end{array}$$

Given an $n$-partial section $s : \text{sk}_n(B) \cup A \to \mathcal{E}$ relative to $A$, we shall construct the obstruction cocycle of $s$

$$c(s) \in N^{n+1}(B, A; \Pi_n(\mathcal{E}/B))$$

to extend a partial section $\text{sk}_{n+1}(B) \cup A \to \mathcal{E}$ as follows: for an $(n+1)$-simplex $\sigma \in B_{n+1}$, we get the induced section $s_{\sigma}$ such that

$$\begin{array}{ccc}
\sigma^* \mathcal{E} & \xrightarrow{s_{\sigma}} & \mathcal{E} \\
\downarrow & & \downarrow \\
\text{sk}_n(\Delta[n+1]) & \xrightarrow{\text{sk}_n(\sigma)} & \text{sk}_n(B)
\end{array}$$

So we put

$$c(s)(\sigma) := g_{\sigma}^{-1}[s_{\sigma}] \in \pi_n(\sigma^* \mathcal{E}),$$

where $g_{\sigma} : \pi_n(\sigma^* \mathcal{E}) \to \pi_n(\sigma^* \mathcal{E})$ is an isomorphism induced by the inclusion $\sigma^* \mathcal{E} \to \sigma^* \mathcal{E}$.

Proposition 4.4. The cochain $c(s)$ is a cocycle.

Proof. For an $(n+2)$-simplex $\sigma \in B_{n+2}$, we have

$$\begin{array}{ccc}
(\partial_1 \sigma)^* \mathcal{E} & \xrightarrow{s_{(\partial_1 \sigma)^* \mathcal{E}}} & \sigma^* \mathcal{E} & \xrightarrow{s_{\sigma}} & \mathcal{E} \\
\downarrow & & \downarrow & & \downarrow \\
\text{sk}_n(\Delta[n+1]) & \xrightarrow{\text{sk}_n(\sigma)} & \text{sk}_n(\Delta[n+2]) & \xrightarrow{\text{sk}_n(\sigma)} & \text{sk}_n(B)
\end{array}$$
Thus we obtain using the relation \( \sigma \) for 0. Here note that \( [sk]_{\partial} \) imply the equations
\[
\sigma_{\partial}^{-1} \sigma_{0}^{-1} \sigma_{\partial} = g_{\partial, \tau}^{-1} (s_{\tau})_{\ast} [sk_{n} (\partial^{\ast})], \quad g_{\partial_{0}}^{-1} g_{\partial_{1}}^{-1} [s_{\partial_{0}} \sigma] = g_{\partial}^{-1} (s_{\partial})_{\ast} [sk_{n} (\partial^{0})].
\]
Here note that \( [sk_{n} (\partial^{\ast})] \in \pi_{n}(sk_{n}(\Delta[n + 2]), 0) \) and \( [sk_{n} (\partial^{0})] \in \pi_{n}(sk_{n}(\Delta[n + 2]), 1) \). Thus we obtain
\[
(\delta c(s))(\sigma) = g_{\partial}^{-1} (s_{\partial})_{\ast} \left( (\sigma_{01})_{\ast} [sk_{n} (\partial^{0})] + \sum_{\ast = 0}^{(-1)^{i}} [sk_{n} (\partial^{i})] \right) = 0,
\]
using the relation \( (\sigma_{01})_{\ast} [sk_{n} (\partial^{0})] + \sum_{\ast = 0}^{(-1)^{i}} [sk_{n} (\partial^{i})] = 0 \) in \( \pi_{n}(sk_{n}(\Delta[n + 2]), 0) \).

We shall define the difference cochain for n-partial sections \( s_{0}, s_{1} : sk_{n}(B) \to E \) and a fiberwise homotopy \( h : sk_{n-1}(B) \times \Delta[1] \to E \times \Delta[1] \) between their restriction on \( sk_{n-1}(B) \). Gluing these maps, we have the map
\[
\tilde{h}^{\square} : (sk_{n}(B) \times sk_{0}(\Delta[1])) \cup (sk_{n-1}(B) \times \Delta[1]) \to E \times \Delta[1].
\]
We consider the obstruction cocycle
\[
c(h^{\square}) \in N^{n+1}(sk_{n}(B) \times \Delta[1], (sk_{n}(B) \times sk_{0}(\Delta[1])) \cup (sk_{n-1}(B) \times \Delta[1]); \Pi_{n}^{\square}),
\]
where \( \Pi_{n}^{\square} = \Pi_{n}(E \times \Delta[1]/B \times \Delta[1]) \). Note that faces of non-degenerate simplices of \( sk_{n}(B) \times \Delta[1] \) are in \( (sk_{n}(B) \times sk_{0}(\Delta[1])) \cup (sk_{n-1}(B) \times \Delta[1]) \). Through the Eilenberg-Zilber map
\[
\times : N_{n}(B) \otimes N_{1}(\Delta[1]) \to N_{n+1}(sk_{n}(B) \times \Delta[1], (sk_{n}(B) \times sk_{0}(\Delta[1])) \cup (sk_{n-1}(B) \times \Delta[1])),
\]
we can define the cochain \( d(s_{0}, h, s_{1}) \in N^{n}(B; \Pi_{n}(E/B)) \) by
\[
d(s_{0}, h, s_{1})(\sigma) := (-1)^{n} c(h^{\square})(\sigma \times I)
\]
for \( \sigma \in B_{n} \). Here I is the unique non-degenerate simplex in \( \Delta[1] \).

**Proposition 4.5.** The cochain \( d(s_{0}, h, s_{1}) \) satisfies
\[
\delta d(s_{0}, h, s_{1}) = c(s_{1}) - c(s_{0}).
\]

**Proof.** It is proved by the equations
\[
\delta d(s_{0}, h, s_{1})(\sigma) = g_{\sigma_{0}}^{-1} d(s_{0}, h, s_{1})(\partial_{0} \sigma) + \sum_{\ast = 0}^{(-1)^{i}} d(s_{0}, h, s_{1})(\partial_{i} \sigma)
\]
\[
= (-1)^{n} g_{\sigma_{0}}^{-1} c(h^{\square})(\partial_{0} \sigma \otimes I) + \sum_{\ast = 0}^{(-1)^{n+1} c(h^{\square})(\partial_{i} \sigma \otimes I)}
\]
\[
= c(h^{\square})(\sigma \otimes I) - \delta c(h^{\square})(\sigma \otimes I)
\]
\[
= c(s_{1}) - c(s_{0}).
\]

The next two propositions hold in the same way as in obstruction theory [19].
Proposition 4.6. An $n$-partial section $s : sk_n(B) \to E$ extends to an $(n+1)$-partial section $sk_{n+1}(B) \to E$ if and only if $c(s) = 0$.

Proposition 4.7. For $n$-partial sections $s, s' : sk_n(B) \to E$, if obstruction cocycles $c(s)$ and $c(s')$ are cohomologous, there is a homotopy between $s|sk_{n-1}(B)$ and $s'|sk_{n-1}(B)$.

Suppose a fiber $\mathcal{X}$ of a Kan fiber bundle $E \to B$ is $(n-1)$-connected (and $\pi_1(\mathcal{X}, x)$ is abelian if $n = 1$). Then we can get an $n$-partial section $s : sk_n(B) \to E$. If we get another $n$-partial section $s'$, these is a homotopy between $s|sk_{n-1}(B)$ and $s'|sk_{n-1}(B)$. So we obtain an invariant

$$\sigma_n(E) := [c(s)] \in H^{n+1}(B; \Pi_n(E/B)).$$

It is called the obstruction class of $E \to B$.

4.3. Obstruction for $n = 0$. We consider an extension of a 0-partial section under the following situation: for a simplicial bundle $E \to B$, suppose that the local system $\Pi_0(E/B)$ of sets has a free and transitive right action of a local system $G$ of groups on $B$.

At first, we define the non-abelian obstruction class of a 0-partial section. For that, we remark the definition of the non-abelian cohomology with values in a local system of non-abelian groups. Here “non-abelian cohomology” is in the sense of [10].

Definition 4.8. Let $X$ be a simplicial set and $G$ a local system of groups on $X$. Define the (non-abelian) cochain complex of $X$ with coefficient $G$

$$C^n(X; G) := \left\{ c : X_n \to \prod_{v \in X_0} G(v); \ c(x) \in G(x_0) \right\}$$

for $0 \leq n \leq 2$ and the following datum:

(i) the affine action $\varphi$ of $C^0(X; G)$ on $C^1(X; G)$:

$$(\varphi(f)c)(\gamma) = f(\partial_1 \gamma)c(\gamma)(G(\gamma)^{-1}f(\partial_0 \gamma)^{-1})$$

for $f \in C^0(X; G)$ and $c \in C^1(X; G)$,

(ii) the action $\psi$ of $C^0(X; G)$ on $C^2(X; G)$:

$$(\psi(f)c)(\sigma) = Ad(G(\partial_2 \sigma)^{-1}f(\partial_0 \partial_2 \sigma))(c(\sigma)),$$

(iii) the map $\delta : C^1(X; G) \to C^2(X; G)$ satisfying $\delta(1) = 1$ and $\delta(\varphi(f)c) = \psi(f)c$ for $f \in C^0(X; G)$ and $c \in C^1(X; G)$:

$$\delta c(\sigma) = (G(\partial_2 \sigma)^{-1}c(\partial_0 \sigma))c(\partial_1 \sigma)^{-1}c(\partial_2 \sigma)$$

for $c \in C^1(X; G)$ and $\sigma \in X_2$.

The we get the 0-th cohomology group

$$H^0(X; G) := \ker(C^0(X; G) \to \text{Aut}(C^1(X; G)) \times C^1(X; G) \to C^1(X; G))$$

and the 1-st cohomology set

$$H^1(X; G) := \delta^{-1}(1)/C^0(X; G).$$
Given a 0-partial section \( s : \text{sk}_0(\mathcal{B}) \to \mathcal{E} \), put
\[
c(s)(\gamma) = [s(\partial_1 \gamma)]^{-1}(\Pi_0(\gamma)^{-1}[s(\partial_0 \gamma)]) \in \mathcal{G}_{\gamma_0}
\]
for \( \gamma \in \mathcal{B}_1 \), i.e., \( c(s)(\gamma) \in \mathcal{G}_{\gamma_0} \) is the unique element satisfying
\[
[s(\partial_1 \gamma)]c(s)(\gamma) = \Pi_0(\gamma)^{-1}[s(\partial_0 \gamma)].
\]
By definition, \( c(s) \in C^1(\mathcal{B}; \mathcal{G}) \) is a cocycle. For another section \( s' : \text{sk}_0(\mathcal{B}) \to \mathcal{E} \), if we can get \( f \in C^0(\mathcal{B}; \mathcal{G}) \) uniquely such that
\[
s'(x) = s(x)f(x)
\]
for \( x \in X_0 \), then \( c(s') = \varphi(f)c(s) \) holds. We denote \( f \) by \( d(s,s') \) as in Section 4.2. Especially the cohomology class
\[
o_0(\mathcal{E}) := [c(s)] \in H^1(\mathcal{B}; \mathcal{G})
\]
is independent of a choice of a 0-partial section \( s : \text{sk}_0(\mathcal{B}) \to \mathcal{E} \). As with usual obstructions, \( o_0(\mathcal{E}) = 1 \) if and only if there is a 1-partial section \( \text{sk}_1(\mathcal{B}) \to \mathcal{E} \). It follows from the following proposition:

**Proposition 4.9.** If \( o_0(\mathcal{E}) = 1 \), there exists a 0-partial section \( s : \text{sk}_0(\mathcal{B}) \to \mathcal{E} \) such that \( c(s) = 1 \).

**Proof.** If \( [c(s)] = 1 \), there exists \( f \in C^0(\mathcal{B}; \mathcal{G}) \) such that \( c(s) = \varphi(f)(1) \). So replacing \( s \) with \( sf^{-1} \), we get the proposition. \( \square \)

The non-abelian obstruction \( o_0(\mathcal{E}) \) is hard to deal with, we shall replace a certain abelian cocycle using a filtration \( \{\mathcal{F}_i\mathcal{G}_i\}_{i=1}^\infty \) of \( \mathcal{G} \) such that
\[
\mathcal{G}_b = \mathcal{F}_1\mathcal{G}_b \supset \mathcal{F}_2\mathcal{G}_b \supset \cdots,
\]
\[
[\mathcal{F}_i\mathcal{G}_b, \mathcal{F}_j\mathcal{G}_b] \subset \mathcal{F}_{i+j}\mathcal{G}_b
\]
for \( b \in \mathcal{B}_0 \), and the map \( \mathcal{G}(\gamma) \) for \( \gamma \in \mathcal{B}_1 \) preserves the filtration. Given such a filtration, we can consider the local system of Lie algebras
\[
\text{gr} \mathcal{G} := \bigoplus_{i=1}^\infty \text{gr}_i \mathcal{G} := \bigoplus_{i=1}^\infty \mathcal{F}_i \mathcal{G} / \mathcal{F}_{i+1} \mathcal{G}.
\]

If the image of \( c(s) \) to \( C^1(\mathcal{B}; \text{gr}_i \mathcal{G}) \) is trivial, i.e., \( c(s)(\gamma) \in \mathcal{F}_i \mathcal{G}_{\gamma_0} \) for \( \gamma \in \mathcal{B}_1 \), we get its image \( c_i(s) \) to the (abelian) chain complex \( C^1(\mathcal{B}; \text{gr}_i \mathcal{G}) \). For another partial section \( \text{sk}_0(\mathcal{B}) \to \mathcal{E} \) satisfying the same condition, we can also get the image \( d_i(s,s') \) of \( d(s,s') \) to \( C^1(\mathcal{B}; \text{gr}_i \mathcal{G}) \). Then it satisfies the equation
\[
c_i(s') - c_i(s) = \delta d_i(s,s').
\]
It means \( \phi^{(i)}(\mathcal{E}) := [c_i(s)] \in H^1(\mathcal{B}; \text{gr}_i \mathcal{G}) \) is obtained uniquely.

**Proposition 4.10.** If \( \phi^{(i)}(\mathcal{E}) \) is defined and trivial, there exists a partial section \( s : \text{sk}_0(\mathcal{B}) \to \mathcal{E} \) such that \( c(s)(\gamma) \in \mathcal{F}_{i+1} \mathcal{G}_{\gamma_0} \) for \( \gamma \in \mathcal{B}_1 \).

**Proof.** Supposing \( \phi^{(i)}(\mathcal{E}) = [c_i(s)] = 1 \), we have \( 1 = [c(s)] \in H^1(\mathcal{B}; \mathcal{G}/\mathcal{F}_{i+1} \mathcal{G}) \). Then there exists a 0-partial section \( s' : \text{sk}_0(\mathcal{B}) \to \mathcal{E} \) such that \( c(s') = 1 \in C^1(\mathcal{B}; \mathcal{G}/\mathcal{F}_{i+1} \mathcal{G}) \). This section satisfies the required condition. \( \square \)
5. Obstruction of the bundles of formal homology connections

Let $E \to B$ be a smooth fiber bundle with homological structure group $G$ and fiber $X$. Fix a $G$-invariant Chen’s differential $\delta$ on $LW$, where $W = H_*(X; \mathbb{R})[-1]$.

5.1. Connected cases. Suppose $Q\text{Der}^+(\hat{L}W, \delta) = 0$ and $H_n(\text{Der}(\hat{L}W), \delta) = 0$ for $n > i > 0$. In addition, suppose, if $n = 1$, $H_1(\text{Der}(\hat{L}W), \delta) \simeq H_0(\hat{L}W \otimes A, d + \delta + [\tau, -])$ is abelian with respect to the Baker-Campbell-Hausdorff product. Then we get the obstruction class of the simplicial bundle $Q \to S\text{Der}(\hat{L}W, \delta) \to S_2(B)$

$$\mathfrak{o} = o_n(Q_\bullet(E, \delta)) \in H^{n+1}(B; \Pi_n),$$

where $\Pi_n = \Pi_n(Q_\bullet(E, \delta)/S_2(B))$, and the characteristic maps of a fiber bundle $E \to B$

$$(\Lambda^p H_n(\text{Der}(\hat{L}W), \delta)^*)^G \to H^{p(n+1)}(B; \mathbb{R})$$

by $\psi \mapsto \psi(o, \ldots, o)$ for $p \geq 1$.

5.2. Example of a sphere bundle. We consider the sphere bundle $S^2 \to E = S^3 \times_{S^1} S^2 \to S^2$ associated to the Hopf fibration $S^1 \to S^3 \to S^2$, where $U(1) = S^1$ acts on $S^2 = \mathbb{C} \cup \{\infty\}$ by rotations. Since the action of $S^1$ on $S^2$ has two fixed points $0$ and $\infty$, this fiber bundle has a section $S^2 \to S^3 \times_{S^1} S^2$ defined by $[b] \mapsto [b, \infty]$. We fix the section.

Denote the volume form on the fiber $S^2 = \mathbb{C} \cup \{\infty\}$ by

$$\nu = \frac{\sqrt{-1}}{2\pi} \frac{dwd\bar{w}}{1 + |w|^2}$$

and the desuspension of the fundamental form by $x \in W = H_2(S^2)[-1]$. Then a dgl model of $S^2$ is given by

$$LW = L(x) \ (|x| = 1), \ \delta x = 0$$

and its Lie algebra of derivations

$$\text{Der}(LW) = \left\langle x \frac{\partial}{\partial x}, [x, x] \frac{\partial}{\partial x} \right\rangle.$$ 

Note that

$$H_1(\text{Der}(LW), \delta) = \text{Der}(LW)_1 = \left\langle [x, x] \frac{\partial}{\partial x} \right\rangle.$$ 

For simplicity, we restrict the bundle $Q_\bullet(E) \to S_\bullet(S^2)$ to the Kan complex defined by

$$K_n = \{ (\Delta^n, sk_1 \Delta^n) \to (S^2, \infty) \} \subset S_n(S^2).$$

If $n \leq 1$, $K_n$ is described by

$$K_0 = \{p_\infty\}, \ K_1 = \{\gamma_\infty\},$$

where $p_\infty : \Delta^0 \to S^2$ and $\gamma_\infty : \Delta^1 \to S^2$ are constant maps to the point $\infty$. We put $Q_\bullet := Q_\bullet(E)|_{K_\bullet}$.

We use the map $\rho : D^2 \to S^2$ defined by

$$\rho(z) = \begin{cases} 
2z/(1 - |z|^2) & (|z| < 1) \\
\infty & (|z| = 1), 
\end{cases}$$
regarding $D^2 = \{ z \in \mathbb{C}; |z| \leq 1 \} \subset \mathbb{C}$, and trivializations $\varphi_\rho : D^2 \times S^2 \to \rho^*E$ defined by
$$\varphi_\rho(z, w) = \left( z, \left[ \frac{2z}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2} \right], w \right).$$
Choose an orientation-preserving diffeomorphism $h : \Delta^2/(\partial_1 \Delta^2 \cup \partial_2 \Delta^2) \to D^2$ such that
$$\Delta^1 \overset{\delta}{\to} \Delta^2/(\partial_1 \Delta^2 \cup \partial_2 \Delta^2) \overset{h}{\to} D^2$$
is given by $t \mapsto e^{2\pi \sqrt{-1} t}$. Then we get the 2-simplex in $K,$
$$\sigma : \Delta^2 \to \Delta^2/(\partial_1 \Delta^2 \cup \partial_2 \Delta^2) \overset{\delta}{\to} D^2 \overset{\rho}{\to} S^2$$
and the trivialization $\varphi_\sigma : \Delta^2 \times S^2 \simeq \sigma^*E$ induced by $\varphi_\rho$. The restriction $g : \Delta^1 \times S^2 \to \gamma_\infty E = \Delta^1 \times E_\infty \simeq \Delta^1 \times S^2$ of $\varphi_\sigma$ on $\partial_0 \Delta^2$ is described by
$$g(t, w) = (t, \varphi_0^{-1}((-2\pi \sqrt{-1} t, 0), w)) = (t, e^{2\pi \sqrt{-1} t}w).$$

The partial section $s : sk_1 K \to \mathcal{Q}$ is defined as follows:
$$s(p_2) := v_0 x \in \mathcal{Q}_0(E)_{p_2}, \quad s(\gamma) := v_1 x \in \mathcal{Q}_1(E)_{\gamma_{\infty}},$$
where $v_0 := (\varphi_0^{-1})*v \in A^2(E_\infty)$ and $v_1 := (\varphi_0^{-1})*v \in A^2(\gamma_{\infty} E)$ if the trivialization $\varphi_0 : S^2 \cong \rho^*_\infty E = E_\infty$ and $\varphi_1 : \Delta^1 \times S^2 \cong \gamma_{\infty}E = \Delta^1 \times E_\infty$ are described by
$$\varphi_1(t, w) = (t, [(1, 0), w]), \quad \varphi_0(w) = [(1, 0), w].$$
Since $[s_\sigma] = [v_1 x] \in \pi_1(\sigma^*Q\bullet(E), v_0 x)$, we have
$$c(s)(\sigma) = g^*[s_\sigma] = g^*[v_1 x] = [g^*(v_1 x)] \in \pi_1(\mathcal{Q}\bullet(S^2), vx)$$
under the identification $\varphi_\sigma^1 : \pi_1(\mathcal{Q}\bullet(E_\infty), v_0 x) \simeq \pi_1(\mathcal{Q}\bullet(S^2), vx)$. Calculating directly, we get
$$g^*(v) = v + \xi dt,$$
where
$$\xi = -\frac{\bar{w}dw + wd\bar{w}}{(1 + |w|^2)^2} = df, \quad f(w) = 1 - \frac{1}{2} \frac{1}{1 + |w|^2}.$$ 
Then putting
$$\Xi = t_1 \xi dt_2 - t_2 \xi dt_1 + 2f dt_1 dt_2,$$
this satisfies the equation
$$(v + \Xi)^2 = 2v \Xi = 4fv dt_1 dt_2 = -4fv dt_0 dt_2 = -4(fv(t_0 dt_2 - t_2 dt_0)).$$
So we obtain the formal homology connection $\alpha = (v + \Xi)x - 4fv(t_0 dt_2 - t_2 dt_0)[x, x] \in \mathcal{Q}_2(S^2)$ satisfying
$$\partial_2 \alpha = (v + \Xi dt_0)x, \quad \partial_1 \alpha = vx + 4fvdz_0[x, x], \quad \partial_2 \alpha = vx.$$
Therefore the equation
$$[g^*(v_1 x)] = [(v + \Xi dt_0)x] = [vx + 4fvdz_0[x, x]] \in \pi_1(\mathcal{Q}\bullet(S^2), vx)$$
holds. Furthermore
$$\int_{S^2} 4fv = \int_{S^2} \sqrt{-1} \frac{dwd\bar{w}}{1 + |w|^2} = \frac{1}{\pi} \int_0^\infty 2rdr \left( \frac{1}{1 + r^2} \right)^3 \int_0^{2\pi} d\theta = 2 \int_0^{\infty} \frac{dx}{(1 + x)^3} = 1$$
means that the deRham cohomology class $[4f v] \in H^2(S^2)$ is non-trivial. According to Theorem 4.10 of [1], we have $c(s)(\sigma) \neq 0$ and
$$\phi = [c(s)] \neq 0 \in H^2(K; H_1(\text{Der}(LW))).$$
Finally evaluating the class with the dual basis $\nu$ of $[x,x] \partial / \partial x \in \text{Der}(LW)_1$, we get the non-trivial characteristic class

$$\nu(\alpha) \in H^2(K) = H^2(S^2),$$

which is the Euler class of the sphere bundle $E \to S^2$ (see [16]).

5.3. Non-connected cases. If $Q\text{Der}^+(\hat{LW}, \delta) \neq 0$, we can apply the construction in Section 4.3. Putting $\Pi_0 = \Pi_0(\mathcal{Q}(E, \delta)/\mathcal{S}_b(B))$, we have the identification

$$\Pi_0(b) = \{C_\infty\text{-algebra map } (H(E_b), m_b) \to A(E_b) \text{ s.t. } (f_1)_* = \text{id}_H \}/(C_\infty\text{-homotopic}),$$

where $m_b$ is the $C_\infty$-algebra structure on $H$ corresponding to $\tilde{\delta}(b)$. According to the homotopy theory of $C_\infty$-algebras, the group $\text{QIAut}(H(E_b), m_b)$ of homotopy classes of $C_\infty$-automorphisms $(H(E_b), m_b) \to (H(E_b), m_b)$ such that $f_1 = \text{id}_{H(E_b)}$ acts on the set $\Pi_0(b)$ on the right freely and transitively.

The local system $\text{QIAut}(E)$ of groups is defined by

$$\text{QIAut}(E)_b := \text{QIAut}(H(E_b), m_b), \quad \gamma_b(f) := (g_{\gamma})^{-1} \circ f \circ (g_{\gamma})^*$$

for $b \in B$, $\gamma \in S_1(B)$ and $f \in \text{QIAut}(E_{\gamma(0)})$, where $g_{\gamma} : E_{\gamma(0)} \to E_{\gamma(1)}$ is the holonomy along $\gamma$. Then we get the non-abelian obstruction class

$$\alpha_0 = \alpha_0(Q_{\bullet}(E)) \in H^1(B; \text{QIAut}(E))$$

in Section 4.3.

Furthermore we have the filtration $\{\text{QIAut}^{\geq i}(E)\}_{i=1}^\infty$ of $\text{QIAut}(E)$ defined in Section 2.3. By the observations in Section 2.3, there exists the identification as local system of vector spaces

$$\text{gr}_i(\text{QIAut}(E)) \simeq \text{gr}_i(\text{QDer}^+(E)),$$

where the local system $\text{QDer}^+(E)$ of Lie algebras is defined in the same way as $\text{QIAut}(E)$. Here note that $\text{gr}(\text{QDer}^+(E))$ is defined similarly to $\text{gr}(\text{QIAut}(E))$ using its filtration.

Suppose we get the obstruction class $\alpha_i \neq 0 \in H^1(B; \text{gr}_i(\text{QDer}^+(E)))$ with respect to the filtration. In the same way as in Section 4.2, the characteristic map

$$(\Lambda^* \text{gr}_i(\text{QDer}^+(\hat{LW}, \delta))^*)^G \to H^*(B; \mathbb{R})$$

is obtained.

Especially, if $X$ is formal and $\delta$ corresponds to the product of the cohomology $H$ of $X$, we obtain the characteristic map

$$(\Lambda^* \text{QDer}^i(\hat{LW}, \delta))^*^G \to H^*(B; \mathbb{R}).$$

We shall show a relation between the characteristic map constructed in [11] and the construction above. By discussions in [11], given a metric of the fiber bundle $E \to B$, we have the map $s : B \to Q_0(E)$: for $b \in B$, the metric on $E_b$ gives a Hodge decomposition of $E_b$, so let $s(b)$ be the $C_\infty$-minimal model defined by the Hodge decomposition. Composing the natural projection $Q_0(E) \to \mathcal{D}(E)$ with $s$, we get a section of Chen’s differential $\tilde{\delta}$

**Theorem 5.1.** Let $X$ be a pointed oriented closed manifold and $E \to B$ be a smooth bundle with section and metric. Suppose the metric gives a section $\tilde{\delta}$ of
Chen’s differentials corresponding to a $G$-invariant Chen’s differential $\delta$ of $X$. Then we have the commutative diagram of chain complexes

$$
\begin{array}{ccc}
C^\bullet_{CE}(\text{QDer}^+(\hat{L}W, \delta))^G & \xrightarrow{\Phi} & A^\bullet(B) \\
\downarrow \Phi_1 & & \downarrow f \\
(\Lambda^\bullet \text{gr}_1(\text{QDer}^+(\hat{L}W, \delta))^G & \xrightarrow{\Phi_1} & C^\bullet(B; \mathbb{R}),
\end{array}
$$

where the first row map $\Phi$ is the characteristic map in [11], the second row $\Phi_1$ is the characteristic map defined by

$$
\Phi_1(\zeta)(\gamma) = \zeta(c_1(s)(\gamma), \ldots, c_1(s)(\gamma))
$$

for $\zeta \in (\Lambda^p \text{gr}_1(\text{QDer}^+(\hat{L}W, \delta))^G$ and $\gamma \in S_1(B)$, the first column is the natural projection and the second column $f$ is the deRham map.

**Proof.** Take a base point $*$ of $B$ and put the universal covering of $B$

$$
\tilde{B} = \{ \gamma : [0, 1] \rightarrow B; \gamma(0) = * \} / \text{(homotopy preserving boundary)}.
$$

We identify the fiber $E_*$ on $*$ with the typical fiber $X$.

The smooth map $\mu : \tilde{B} \rightarrow Q(X, \delta)$ from the universal cover $\tilde{B}$ of $B$ to the moduli space $Q(X, \delta) := \pi_0(Q_\bullet(X, \delta))$ of $C_{\infty}$-algebra models of $X$ is defined by

$$
\mu([\gamma]) = g_\gamma^{-1} \cdot [s(\gamma(1))].
$$

Here $g_\gamma : E_* \rightarrow E_{\gamma(1)}$ is the holonomy along $\gamma$. Pull-backing the right-invariant Maurer-Cartan form defined by the right-action of $\text{QIAut}(\hat{L}W, \delta)$ on $Q(X, \delta)$

$$
\eta \in A^1(Q(X, \delta); \text{QDer}^+(\hat{L}W, \delta)),
$$

we get the flat connection

$$
\eta_\mu := \mu^* \eta \in A^1(\tilde{B}; \text{QDer}^+(\hat{L}W, \delta)).
$$

On the other hand, we can regard $s$ as the 0-partial section $s : \text{sk}_0(S_\bullet(B)) \rightarrow Q_\bullet(E)$. Its non-abelian obstruction cocycle is described by

$$
c(s)(\gamma) = [s(\gamma(0))]^{-1} g_\gamma^{-1} [s(\gamma(1))] = g_l(\mu([l])^{-1} \mu([\gamma[l]])),
$$

where $l$ is a path from $*$ to $\gamma(0)$ and a path $\tilde{\gamma} : [0, 1] \rightarrow \tilde{B}$ is the lift of $\gamma$ such that $\tilde{\gamma}(0) = [l]$. The map $\Psi : Q(X, \delta) \rightarrow Q(X, \delta)$ defined by $\Psi(\alpha) = \mu([l])^{-1} \alpha$ satisfies the differential equation $d\Psi = \Psi \eta$. Thus, solving the equation over the path $\mu \tilde{\gamma}$, we have

$$
\mu([l])^{-1} \mu([\gamma[l]]) = \Psi(\mu \tilde{\gamma}(1)) = \sum \int_{\mu \tilde{\gamma}} \eta \cdots \eta.
$$

Therefore we get the description using iterated integrals

$$
c(s)(\gamma) = g_l \cdot \sum \int_{\mu \tilde{\gamma}} \eta \cdots \eta = g_l \cdot \sum \int_{\gamma} \eta_\mu \cdots \eta_\mu.
$$

Its projection to $\text{gr}_1(\text{QDer}^+(\hat{L}W, \delta))$ is equal to $c_1(s)(\gamma) = g_l \int \eta_\mu$ and

$$
\int \Phi(\xi) = \int \xi(\eta_\mu, \ldots, \eta_\mu) = \tilde{\xi} \left( \int \eta_\mu, \ldots, \int \eta_\mu \right) = \Phi_1(\tilde{\xi}) \in C^p(\tilde{B})
$$

for $\xi \in C^p_{CE}(\text{QDer}^+(\hat{L}W, \delta))^G$, where $\tilde{\xi}$ is the projection of $\xi$. Since the element is $\pi_1(B, *)$-invariant, we can regard it as element in $C^p(B)$. 

$\square$
Furthermore, if \( c_1(s) = \cdots = c_{g−1}(s) = 0 \), we get the (cocycle-level) characteristic map \( \Phi_i : (A^\bullet \text{gr}_i(Q\text{Der}^+(\hat{L}W, δ))^G) \to C^\bullet(B; \mathbb{R}) \) defined by
\[
\Phi_i(\zeta(\gamma)) = (\zeta(c_1(s)(\gamma)), \ldots, \zeta(c_g(s)(\gamma)))
\]
for \( \zeta \in (\Lambda^p \text{gr}_i(Q\text{Der}^+(\hat{L}W, δ))^G \) and \( \gamma \in S_1(B) \) since \( \eta_p \in A^1(B; \text{QDer}^{\geq 1}(\hat{L}W, δ)) \). Then the same commutative diagram holds. So the construction above using obstructions is the “leading term” of the characteristic map obtained in [11]
\[
\Phi : C^\bullet_C(\text{QDer}^+(\hat{L}W, δ))^G \to A^\bullet(B).
\]

5.4. **Example of surface bundles.** We consider the case of \( X = \Sigma_g \), which is the closed oriented surface with genus \( g \geq 2 \). This is a formal manifold, so we can put
\[
\delta = \frac{\partial}{\partial v},
\]
where \( v \in W_1 \) is the fundamental form of \( \Sigma_g \) and \( \omega \in [W_0, W_0] \) is the intersection form, i.e., \( \omega = \sum_{i=1}^g [x^i, y^i] \) for a symplectic basis \( \{x^i, y^i\} \) of \( W_0 \) with respect to the intersection form of \( \omega \).

5.4.1. **The first obstruction for surface bundles.** For an oriented surface bundle (with section), its homologically structure group is in the symplectic group \( \text{Sp}(W_0) \) of \( W_0 \).

**Proposition 5.2.** We have the identification as \( \text{Sp}(W_0) \)-vector space
\[
Q\text{Der}^1(\hat{L}W, δ) \simeq \Lambda^3 W_0.
\]

**Proof.** An element \( D \in \text{Der}^1(\hat{L}W)_0 \) is described by the form
\[
D = D_0 + [v, z] \frac{\partial}{\partial v}
\]
for \( D_0 \in \text{Der}^1(LW_0) \) and \( z \in W_0 \). Then we can calculate the image by \( \text{ad}(\delta) \):
\[
[\delta, D] = -D_0(\omega) + [\omega, z] \frac{\partial}{\partial v}.
\]
So, \( D \) is in the kernel if and only if \( D_0(\omega) \in (\omega) \), where \( (\omega) \) is the Lie ideal in \( LW_0 \) generated by \( \omega \). This condition is equivalent to the condition: \( D_0 \) induces a derivation on \( LW_0/\omega \).

On the other hand, an element \( P \in \text{Der}^0(\hat{L}W)_1 \) is described by
\[
P = \sum b_i v \frac{\partial}{\partial x_i}
\]
for \( b_i \in \mathbb{R} \), where \( \{x_i\}_{i=1}^{2g} \) is a basis of \( W_0 \). Its image of \( \text{ad}(\delta) \) is
\[
[\delta, P] = \sum b_i \omega \frac{\partial}{\partial x_i} - P(\omega) \frac{\partial}{\partial v}.
\]

Since we can prove \([v, W_0] = \{P(\omega); P \in \text{Der}^0(\hat{L}W)_1\} \) by direct calculus, for any \( D \in \text{Der}^1(\hat{L}W)_0 \), there exists \( P \in \text{Der}^0(\hat{L}W)_1 \) such that
\[
D_P := D + [\delta, P] \in \text{Der}^1(LW_0).
\]
Furthermore, for another \( P' \in \text{Der}^0(\hat{L}W)_1 \) such that \( D_{P'} = D + [\delta, P'] \in \text{Der}^1(LW_0) \), their difference \([\delta, P - P'] \) is in \( \text{Hom}(W_0, \mathbb{R}) \subset \text{Der}^1(LW_0) \). So if \( D \) is in the kernel, \( D_P \) and \( D_{P'} \) induce the same derivation on \( LW_0/\omega \). Therefore we get the isomorphism
\[
Q\text{Der}^1(\hat{L}W, δ)_0 \simeq \text{Der}^1(LW_0/\omega).
\]
According to [17], we have the isomorphism $\text{Der}^1(LW_0/(\omega)) \simeq \Lambda^3W_0$. □

By the proposition above, for a oriented surface bundle $E \to B$ with section, we get the obstruction class

$$\phi^{(1)} = \phi^{(1)}(Q(E, \hat{\delta})/S_\bullet(B)) \in H^1(B; \Lambda^3W_0(E)).$$

Here $\Lambda^3W_0(E)$ is the local system of vector spaces such that

$$\Lambda^3W_0(E)(b) = \Lambda^3\tilde{H}_1(E_b; \mathbb{R})[-1].$$

This local system is defined in the same way as $\text{QIAut}(E)$ and $\text{QDer}^+(E)$. Then we also get the characteristic map

$$(\Lambda^\bullet(\Lambda^3W_0))^\text{Sp}(W_0) \to H^\bullet(B; \mathbb{R}).$$

### 5.4.2. Twisted Morita-Miller-Mumford class.

We shall show that the obstruction $\phi^{(1)}$ can be regarded as one of the twisted Morita-Miller-Mumford classes. For the purpose, we use notations as follows:

- the mapping class group $\mathcal{M}_{g,*}$ of the oriented closed surface $\Sigma_g$ with a base point,
- the space $\text{Met}_g$ of Riemannian metrics which has constant curvature $-1$ on $\Sigma_g$,
- the Teichmüller space $\mathcal{T}_{g,*}$, which is the orbit space of $\text{Met}_g$ by the action of the group $\text{Diff}_0(\Sigma_g, *)$ of diffeomorphisms of $(\Sigma_g, *)$ isotopic to identity,
- the moduli space $\mathcal{M}_{g,*} = \mathcal{T}_{g,*}/\mathcal{M}_{g,*}$ of Riemann surfaces with a base point, and
- the universal family $\mathcal{C}_{g,*} = \text{Met}_g \times_{\text{Diff}(\Sigma_g, *)} \Sigma_g$ of Riemann surfaces with a base point.

Applying the construction in Section 5.4.1 for the “universal surface bundle” $\mathcal{C}_{g,*} \to \mathcal{M}_{g,*}$, we get the obstruction

$$\phi^{(1)} \in H^1(\mathcal{M}_{g,*}; \Lambda^3W_0(\mathcal{C}_{g,*})).$$

**Theorem 5.3.** The obstruction class $\phi^{(1)}$ is equal to the minus of the twisted Morita-Miller-Mumford class

$$-m_{0,3} \in H^1(\mathcal{M}_{g,*}; \Lambda^3W_0).$$

**Proof.** Take the canonical metric of $\mathcal{C}_{g,*} \to \mathcal{M}_{g,*}$. According to the proof of Theorem 5.1, we have $\mu : \mathcal{T}_{g,*} \to Q(\Sigma_g, \delta)$ and the cocycle

$$c_1(\gamma) = \int_\gamma \eta_\mu = \int_\gamma \eta_1,$$

where $\eta_1$ is the QDer$^1$-part of $\eta_\mu$. So by the same discussion in [12], the cohomology class $\phi^{(1)} = [c_1(s)]$ is equal to the twisted Morita-Miller-Mumford class in $H^1(\mathcal{M}_{g,*}; \Lambda^3W_0)$. (The discussion is also used in Section 4 of [14].) □

So the obtained characteristic map

$$(\Lambda^\bullet(\Lambda^3W_0))^\text{Sp}(W_0) \to H^\bullet(\mathcal{M}_{g,*}; \mathbb{R}) = H^\bullet(\mathcal{M}_{g,*}; \mathbb{R})$$

gives Morita-Miller-Mumford classes by the result of [13].
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