LAVAURS ALGORITHM FOR CUBIC SYMMETRIC POLYNOMIALS

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Abstract. To investigate the degree $d$ connectedness locus, Thurston studied $\sigma_d$-invariant laminations, where $\sigma_d$ is the $d$-tupling map on the unit circle, and built a topological model for the space of quadratic polynomials $f_c(z) = z^2 + c$. In the same spirit, we consider the space of all cubic symmetric polynomials $f_\lambda(z) = z^3 + \lambda^2 z$ in three articles. In the first one we construct the lamination $C_3 CL$ together with the induced factor space $S/C_3 CL$ of the unit circle $S$. As will be verified in the third paper, $S/C_3 CL$ is a monotone model of the cubic symmetric connectedness locus, i.e., the space of all cubic symmetric polynomials with connected Julia sets. In the present paper, the second in the series, we develop an algorithm for constructing $C_3 CL$ analogous to the Lavaurs algorithm for constructing a combinatorial model $M_2^{comb}$ of the Mandelbrot set $M_2$.

1. Introduction

We use standard notation ($\mathbb{R}, \mathbb{C}$ for the real/complex numbers, $\mathbb{D}$ for the unit disk centered at the origin, etc). The Riemann sphere is denoted by $\hat{\mathbb{C}}$. The boundary (in $\mathbb{C}$) of a set $X \subset \mathbb{C}$ is denoted by $\text{Bd}(X)$. We consider only complex polynomials $P$; for such a $P$, let $J_P$ be its Julia set and $K_P$ be its filled Julia set. A chord is a closed straight line segment with endpoints on the unit circle $S = \text{Bd}(\mathbb{D})$.

The connectedness locus $M_d$ is the space of polynomials of degree $d$, up to affine conjugacy, with connected Julia sets. A fundamental problem is to understand the structure of $M_d$. Major progress has been made for $d = 2$ but much less is known for $d > 2$. Thurston [Thu85] introduced geodesic invariant laminations to provide a combinatorial model $M_2^{comb}$ of the Mandelbrot set $M_2$.
model for \( \mathcal{M}_2 \). A lamination \( \mathcal{L} \) is a compact set of chords, called leaves, that are pairwise disjoint in \( \mathbb{D} \) (equivalently, do not cross).

More precisely, Thurston constructs a lamination \( QML \) whose leaves tag all invariant quadratic laminations (for \( d \geq 2 \), a lamination is invariant if it is invariant under the map \( \sigma_d(z) = z^d \) restricted to \( \mathbb{S} \), see Definition 2.4). It can be shown that the quotient space \( \mathcal{M}_2^{Comb} = \mathbb{S}/QML \) is a monotone image of \( \text{Bd}(\mathcal{M}_2) \) (conjecturally, this map is a homeomorphism), cf. [Thu85]. No such models exist for \( d > 2 \).

A natural next object of study is \( \mathcal{M}_3 \), and some slices of \( \mathcal{M}_3 \) have already been considered. In [BOSTV1] we construct the lamination \( C_{s,CL} \) (this stands for cubic symmetric comajor lamination) together with the induced factor space \( \mathbb{S}/C_{s,CL} \) of the unit circle \( \mathbb{S} \). In [BOSTV3] we verify that \( \mathbb{S}/C_{s,CL} \) is a monotone model of the cubic symmetric connected locus, i.e. the space \( \mathcal{M}_{3,s} \) of symmetric cubic polynomials \( P(z) = z^3 + \lambda^2 z \) with connected Julia sets.

To understand the structure of \( C_{s,CL} \) and to be able to obtain suitable pictures of this space in this paper we provide an algorithm for constructing a dense set of leaves in \( C_{s,CL} \) (see Figure 1).

2. Laminations: classical definitions

2.1. Invariant laminations. The reader is referred to [Mil06, Thu85] for basic notions of complex polynomial dynamics on \( \mathbb{C} \), including Fatou and Julia sets, external rays, landing etc.

Identify \( \mathbb{S} \) with \( \mathbb{R}/\mathbb{Z} \) and define the map \( \sigma_d : \mathbb{S} \to \mathbb{S} \) for \( d \geq 2 \) as \( \sigma_d(z) = dz \mod 1 \); clearly, \( \sigma_d \) is locally one-to-one on \( \mathbb{S} \). A complex
polynomial $P$ with locally connected Julia set $J_P$ gives rise to an equivalence relation $\sim_P$ on $S$ so that $x \sim_P y$ if and only if the external rays of arguments $x$ and $y$ land at the same point of $J_P$. Equivalence classes of $\sim_P$ have pairwise disjoint convex hulls. The topological Julia set $S/\sim_P = J_{\sim_P}$ is homeomorphic to $J_P$, and the topological polynomial $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$, induced by $\sigma_d$, is topologically conjugate to $P|_{J_P}$.

For a closed set $A \subset S$ we denote its convex hull by $\text{CH}(A)$. An edge of $\text{CH}(A)$ is a chord of $S$ contained in the boundary of $\text{CH}(A)$. Given points $a, b \in S$, let $(a, b)$ be the positively oriented arc in $S$ from $a$ to $b$ and $ab$ be the chord with endpoints $a$ and $b$.

**Definition 2.1.** A lamination $L$ is a set of chords in the closed unit disk $\mathbb{D}$, called leaves of $L$, if it satisfies the following conditions:

- (L1) leaves of $L$ do not cross;
- (L2) the set $L^* = \bigcup_{\ell \in L} \ell$ is closed.

If (L2) is not assumed then $L$ is called a prelamination.

A degenerate leaf is a point of $S$. Given a leaf $\ell = \overline{ab} \in L$, let $\sigma_d(\ell)$ be the chord with endpoints $\sigma_d(a)$ and $\sigma_d(b)$; then $\ell$ is called a pullback of $\sigma_d(\ell)$. If $a \neq b$ but $\sigma_d(a) = \sigma_d(b)$, call $\ell$ a critical leaf. Let $\sigma_d^* : L^* \to \mathbb{D}$ be the linear extension of $\sigma_d$ over all the leaves in $L$. Then $\sigma_d^*$ is continuous and $\sigma_d^*$ is one-to-one on any non-critical leaf. If $L$ includes all points of $S$ as degenerate leaves, then $L^*$ is a continuum.

**Definition 2.2 (Gap).** A gap $G$ of a lamination $L$ is the closure of a component of $\mathbb{D} \setminus L^*$; its boundary leaves are called edges (of the gap).

If $G$ is a leaf/gap of $L$, then $G$ equals the convex hull of $G \cap S$. If $G$ is a leaf or a gap of $L$ we let $\sigma_d(G)$ be the convex hull of $\sigma_d(G \cap S)$. Notice that $\text{Bd}(G) \cap S = G \cap S$. Points of $G \cap S$ are called the vertices of $G$. A gap $G$ is called infinite (finite) if and only if $G \cap S$ is infinite (finite). A gap $G$ is called triangular gap if $G \cap S$ consists of three points.

**Definition 2.3.** Let $L$ be a lamination. The equivalence relation $\sim_L$ induced by $L$ is defined by declaring that $x \sim_L y$ if and only if there exists a finite concatenation of leaves of $L$ joining $x$ to $y$.

**Definition 2.4 (Invariant (pre)laminations).** A (pre)lamination $L$ is $(\sigma_d)$-invariant if,

- (D1) $L$ is forward invariant. For each $\ell \in L$ either $\sigma_d(\ell) \in L$ or $\sigma_d(\ell)$ is a point in $S$ and
- (D2) $L$ is backward invariant.

1. For each $\ell \in L$ there exists a leaf $\ell' \in L$ such that $\sigma_d(\ell') = \ell$. 

(2) For each \( \ell \in L \) such that \( \sigma_d(\ell) \) is a non-degenerate leaf, there exists \( d \) disjoint leaves \( \ell_1, \ldots, \ell_d \) in \( L \) such that \( \ell = \ell_1 \) and \( \sigma_d(\ell_i) = \sigma_d(\ell) \) for all \( i \).

**Definition 2.5** (q-lamination). A \( \sigma_d \)-invariant lamination \( L \) is called a \( q \)-lamination if \( x \sim_L y \) implies that \( x \) and \( y \) are vertices of the same finite gap or leaf. The convex hulls of \( \sim_L \)-classes are also called \( \sim_L \)-sets or \( L \)-sets.

**Remark 2.6.** It readily follows from the definition of a \( q \)-lamination that at most two leaves of a \( q \)-lamination can share an endpoint.

**Definition 2.7** (Siblings). Two chords are called siblings if they have the same image. Any \( d \) disjoint chords with the same non-degenerate image are called a sibling collection.

**Definition 2.8** (Monotone Map). Let \( X, Y \) be topological spaces and \( f : X \to Y \) be continuous. Then \( f \) is said to be monotone if \( f^{-1}(y) \) is connected for each \( y \in Y \). It is known that if \( f \) is monotone and \( X \) is a continuum then \( f^{-1}(Z) \) is connected for every connected \( Z \subset f(X) \).

**Definition 2.9** (Gap-invariance). A lamination \( L \) is gap invariant if for each gap \( G \), the set \( \sigma_d(G) \) is a gap, or a leaf, or a single point. In the first case we also require that \( \sigma^*_d|_{\text{Bd}(G)} : \text{Bd}(G) \to \text{Bd}(\sigma_d(G)) \) maps as the composition of a monotone map and a covering map to the boundary of the image gap, with positive orientation (i.e., as you move through the vertices of \( G \) in clockwise direction around \( \text{Bd}(G) \), their corresponding images in \( \sigma_d(G) \) must also be aligned clockwise in \( \text{Bd}(\sigma_d(G)) \)).

**Definition 2.10** (Degree). The degree of the map \( \sigma^*_d|_{\text{Bd}(G)} : \text{Bd}(G) \to \text{Bd}(\sigma_d(G)) \), or of the gap \( G \), is defined as the number of components of \( (\sigma^*_d)^{-1}(x) \) in \( \text{Bd}(G) \), for any \( x \in \text{Bd}(\sigma_d(G)) \). In other words, if every leaf of \( \sigma_d(G) \) has \( k \) disjoint pre-image leaves in \( G \), then the degree of the map \( \sigma^*_d \) is \( k \). A gap \( G \) is called critical gap if \( k > 1 \).

The following results are proved in [BMOV13].

**Theorem 2.11.** Every \( (\sigma_d) \)-invariant lamination is gap invariant.

**Theorem 2.12.** The closure of an invariant prelamination is an invariant lamination. The space of all \( \sigma_d \)-invariant laminations is compact.

### 3. Parameter lamination \( C_sCL \): preliminaries

This section describes results of [BOSTV1]. From now on normalize the circle so that its length is 1; the length of arcs and angles are
measured accordingly. Given a chord $\ell = \overrightarrow{ab}$ denote by $-\ell$ the chord obtained by rotating $\ell$ by the angle $\frac{1}{2}$. Define the length $\|\overrightarrow{ab}\|$ of a chord $\overrightarrow{ab}$ as the shorter of the lengths of the arcs in $S = \mathbb{R}/\mathbb{Z}$ with the endpoints $a$ and $b$. The maximum length of a chord is $\frac{1}{2}$. Divide leaves into three categories by their length.

**Definition 3.1.** A short leaf is a leaf $\ell$ such that $0 < \|\ell\| < \frac{1}{6}$; a medium leaf is a leaf $\ell$ such that $\frac{1}{6} \leq \|\ell\| < \frac{1}{3}$ and a long leaf is a leaf $\ell$ such that $\frac{1}{3} < \|\ell\| \leq \frac{1}{2}$.

Critical leaves are exactly leaves of length $\frac{1}{3}$. A leaf $\ell$ is long/medium if $\|\ell\| \geq \frac{1}{6}$. A non-critical leaf $\ell$ of $L$ has siblings (Definition 2.7). A non-critical long/medium leaf $\ell$ has a long/medium sibling $\hat{\ell}$, and there is a unique component $C^\circ(\ell)$ of $D \setminus (\ell \cup \hat{\ell})$ whose boundary contains $\ell \cup \hat{\ell}$.

**Lemma 3.2** (Lemma 3.4 [BOSTV1]). The possibilities for leaves in a sibling collection are

- (sss): all leaves are short;
- (mmm): all leaves are medium;
- (sml): one leaf is short, one medium, and one long.

A sibling collection is completely determined by its type and one leaf.

These are general facts; let us now become more specific.

**Definition 3.3** (Cubic symmetric lamination). A $\sigma_3$-invariant lamination $L$ is called a cubic symmetric lamination if:

(D3) for each $\ell \in L$ we have $-\ell \in L$.

Unless otherwise stated, let $L$ be a cubic symmetric lamination.

**Definition 3.4.** Suppose that $\ell = \overrightarrow{ab}$ is a non-critical chord which is not a diameter and the arc $(a, b)$ is shorter than the arc $(b, a)$. Denote the chord $(a + \frac{1}{3})(b - \frac{1}{3})$ by $\ell'$ and the chord $(a + \frac{2}{3})(b - \frac{2}{3})$ by $\ell''$.

As $\sigma_3(\ell') = \sigma_3(\ell'') = \sigma_3(\ell)$, $\{\ell, \ell', \ell''\}$ is a sibling collection. For a long/medium non-critical leaf $\ell \in L$, it follows that $\ell'$ is long/medium and $\ell''$ is short; if, moreover, $\ell \in L$ (recall that $L$ is a symmetric lamination), its sibling collection is $\{\ell, \ell', \ell''\}$ (all other possibilities lead to crossings with $\ell$ or $-\ell$). So, for $L$ a sibling collection of type (mmm) is impossible.

**Definition 3.5.** Given two chords $\ell, \hat{\ell}$ that do not cross let $S(\ell, \hat{\ell})$ be a component of $D \setminus (\ell \cup \hat{\ell})$ with boundary containing $\ell$ and $\hat{\ell}$; call $S(\ell, \hat{\ell})$ the strip between $\ell$ and $\hat{\ell}$. 

The above notation is convenient when dealing with laminations.

**Definition 3.6** (Short strips). For a sibling collection \(\{\ell, \ell', \ell''\}\) of type (sml), with \(\ell\) and \(\ell'\) long/medium, let \(C(\ell) = S(\ell, \ell')\), (the short leaf \(\ell''\) cannot lie in \(C(\ell)\)). The set \(C(\ell)\) has two boundary circle arcs of length \(|\frac{1}{3} - \|\ell\||\) (and so does \(-C(\ell)\)). Given a long/medium chord \(\ell \in \mathcal{L}\), call the region \(SH(\ell) = C(\ell) \cup -C(\ell)\) the short strips (of \(\ell\)) and each of \(C(\ell)\) and \(-C(\ell)\) a short strip (of \(\ell\)). Call \(|\frac{1}{3} - \|\ell\|| = w(\mathcal{C}(\ell)) = w(\mathcal{SH}(\ell))\) the width of \(C(\ell)\) (or of \(-C(\ell)\), or of \(\mathcal{SH}(\ell)\)). Note that \(-C(\ell) = C(-\ell)\).

**Definition 3.7.** A leaf \(\ell\) is closer to criticality than a leaf \(\ell'\) if \(\|\ell\|\) is closer to \(\frac{1}{3}\) than \(\|\ell'\|\). A chord \(\ell\) is closest to criticality (in a family of chords \(\mathcal{A}\)) if its length is the closest to criticality among lengths of chords from \(\mathcal{A}\).

The next two facts established in [BOSTV1] are similar to important results proven in [Thu85]. The first one is somewhat technical.

**Proposition 3.8** (Lemma 3.7 [BOSTV1]). If \(\ell \in \mathcal{L}\), \(\|\ell\| > \frac{1}{6}\), and \(k \in \mathbb{N}\) is minimal such that \(\ell_k = \sigma^k_3(\ell)\) intersects the interior of \(\mathcal{SH}(\ell)\), then \(\|\ell_k\| > \frac{1}{7}\), and \(\ell_k\) is closer to criticality than \(\ell\). A leaf \(\ell\) that is the closest to criticality in its forward orbit is medium/long, and no forward image of \(\ell\) enters the interior of \(\mathcal{SH}(\ell)\).

Proposition 3.8 implies Theorem 3.9.

**Theorem 3.9** (Theorem 3.8 [BOSTV1]). Let \(\mathcal{L}\) be a symmetric lamination and \(G\) be a gap of \(\mathcal{L}\). Then \(G\) is preperiodic unless an eventual forward image of \(G\) is a leaf or a point.

Call a finite periodic gap of \(\mathcal{L}\) a periodic polygon.

**Lemma 3.10** (Lemma 4.5 [BOSTV1]). Let \(G\) be a periodic polygon, and let \(g\) be the first return map of \(G\). One of the following is true.

(a) The first return map \(g\) acts on the sides of \(G\) transitively as a rational rotation.

(b) The edges of \(G\) form two disjoint periodic orbits, and \(G\) eventually maps to the gap \(-G\). If \(\ell\) and \(\ell'\) are two adjacent edges of \(G\), then the leaf \(\ell\) eventually maps to the edge \(-\ell'\) of \(-G\).

**Definition 3.11.** If case (a) from Lemma 3.10 holds, we call a gap \(G\) a 1-rotational gap. If case (b) from Lemma 3.10 holds we call such a gap a 2-rotational gap.

If \(c\) is a short chord, then there are two long/medium chords with the same image as \(c\). We will denote them by \(M_c\) and \(M'_c\). Also, denote
by $Q_c$ the convex hull of $M_c \cup M'_c$. This applies in the degenerate case, too: if $c \in S$ is just a point, then $M_c = M'_c = Q_c$ is a critical leaf $\ell$ disjoint from $c$ such that $\sigma_3(c) = \sigma_3(M_c)$.

**Definition 3.12 (Major).** A leaf $M \in \mathcal{L}$ closest to criticality in $\mathcal{L}$ is called a major of $\mathcal{L}$.

If $M$ is a major of $\mathcal{L}$, then the medium/long sibling $M'$ of $M$ is also a major of $\mathcal{L}$, as well as the leaves $-M$ and $-M'$. A lamination has either exactly 4 non-critical majors or 2 critical majors.

**Definition 3.13 (Comajor).** The short siblings of major leaves of $\mathcal{L}$ are called comajors; we also say that they form a comajor pair. A pair of symmetric chords is called a symmetric pair. If the chords are degenerate, their symmetric pair is called degenerate, too.

A symmetric lamination has a symmetric pair of comajors $\{c, -c\}$.

**Definition 3.14 (Minor).** Images of majors (equivalently, comajors) are called minors of a symmetric lamination. Similarly to comajors, every symmetric lamination has two symmetric minors $\{m, -m\}$.

Critical majors of a lamination have no siblings, and we define degenerate comajors and minors as corresponding points on $S$. If majors $M$ and $-M$ are non-critical, then there is a critical gap, say, $G$ with edges $M$ and $M'$, and a critical gap $-G$ with edges $-M$ and $-M'$.

**Lemma 3.15 (Lemma 5.4 [BOSTV1]).** Let $\{m, -m\}$ be the minors of $\mathcal{L}$, and let $\ell$ be a leaf of $\mathcal{L}$. Then no forward image of $\ell$ is shorter than $\min(||\ell||, ||m||)$.

**Definition 3.16.** For a family $\mathcal{A}$ of chords, $\ell$ is a two sided limit leaf of $\mathcal{A}$ if $\ell$ is approximated by chords of $\mathcal{A}$ from both sides.

**Lemma 3.17 (Lemma 5.5 [BOSTV1]).** Let $c$ be a comajor and $M$ be a major of $\mathcal{L}$ such that $\sigma_3(c) = \sigma_3(M)$.

1. If $c$ is non-degenerate, then one of the following holds:
   a. the endpoints of $c$ are both strictly preperiodic with the same preperiod and period;
   b. the endpoints of $c$ are both not preperiodic, and $c$ is approximated from both sides by leaves of $\mathcal{L}$ that have no common endpoints with $c$.

2. If $M$ is non-critical, then its endpoints are either both periodic or both strictly preperiodic with the same preperiod and period, or both not preperiodic.

In particular, a non-degenerate comajor is not periodic.
Comajors can be described more explicitly.

**Definition 3.18** (Legal pairs, Definition 5.6 [BOSTV1]). Let a symmetric pair \( \{c, -c\} \) be either degenerate or satisfy the following:

(a) no two iterated forward images of \( c \) and \( -c \) cross, and
(b) no forward image of \( c \) crosses the interior of \( \text{SH}(c) \).

Then \( \{c, -c\} \) is said to be a legal pair.

We will also need an important concept of a pullback of a set.

**Definition 3.19** (Pullbacks, Definition 5.7 [BOSTV1]). Suppose that a family \( \mathcal{A} \) of chords is given and \( \ell \) is a chord. A pullback chord of \( \ell \) generated by \( \mathcal{A} \) is a chord \( \ell' \) such that \( \sigma_3(\ell') = \ell \) such that \( \ell' \) does not cross chords from \( \mathcal{A} \). An iterated pullback chord of \( \ell \) generated by \( \mathcal{A} \) is a pullback chord of an (iterated) pullback chord of \( \ell \).

**Lemma 3.20** (Lemma 5.8 [BOSTV1]). The only two symmetric laminations with comajors of length \( \frac{1}{6} \) have two critical Fatou gaps and are as follows.

1. \( \mathcal{L}_1 \) has the comajor pair \( (\frac{11}{12}, \frac{7}{12}) \). The gap \( U'_1 \) is invariant; \( U'_1 \cap \mathbb{S} \) consists of all \( \gamma \in \mathbb{S} \) such that \( \sigma_3^n(\gamma) \in [0, \frac{1}{2}] \). The gap \( U''_1 \) is invariant; \( U'_1 \cap \mathbb{S} \) consists of all \( \gamma \in \mathbb{S} \) such that \( \sigma_3^n(\gamma) \in [\frac{1}{2}, 0] \). The gaps \( U'_1, U''_1 \) share an edge \( 0_{\frac{1}{2}} \); their edges are the appropriate pullbacks of \( 0_{\frac{1}{2}} \) that never separate \( \frac{11}{12}, \frac{7}{12}, \frac{5}{12}, \frac{1}{12}, \frac{3}{4}, \frac{1}{4} \).

2. \( \mathcal{L}_2 \) has the comajor pair \( (\frac{11}{12}, \frac{1}{12}, \frac{5}{12}, \frac{7}{12}) \). The gaps \( U''_2, U''_2 \) form an orbit and the set \( (U'_2 \cup U''_2) \cap \mathbb{S} \) consists of all \( \gamma \in \mathbb{S} \) such that \( \sigma_3^n(\gamma) \in [\frac{1}{12}, \frac{5}{12}] \cup [\frac{7}{12}, \frac{11}{12}] \). The gaps \( U'_2, U''_2 \) share an edge \( 0_{\frac{1}{2}} \); their edges are the appropriate pullbacks of \( \frac{1}{4} \) that never separate \( \frac{11}{12}, \frac{7}{12}, \frac{5}{12} \) and \( \frac{1}{4} \).

Though the laminations from Lemma 3.20 are not the pullback laminations described below, knowing them allows us to consider only legal pairs with comajors of length less than \( \frac{1}{6} \) and streamline the proofs.

**Construction of a symmetric pullback lamination \( \mathcal{L}(c) \) for a legal pair \( \{c, -c\} \).**

**Degenerate case.** For \( c \in \mathbb{S} \), let \( \pm \ell = \pm M_\ell \). (call \( \ell, -\ell \) and their pullbacks “leaves” even though we apply this term to existing laminations, and we are only constructing one). Consider two cases.

(a) If \( \ell \) and \( -\ell \) do not have periodic endpoints, then the family of all iterated pullbacks of \( \ell, -\ell \) generated by \( \ell, -\ell \) is denoted by \( \mathcal{L} \).

(b) Suppose that \( \ell \) and \( -\ell \) have periodic endpoints \( p \) and \( -p \). Then there are two similar cases. First, the orbits of \( p \) and \( -p \) may be distinct
(and hence disjoint). Then iterated pullbacks of \( \ell \) generated by \( \ell \), \(-\ell\) are well-defined (unique) until the \( n \)-th step (\( n \) equals the period of \( p \) and the period of \(-p\)), when there are two iterated pullbacks of \( \ell \) that have a common endpoint \( x \) and share other endpoints with \( \ell \). Two other iterated pullbacks of \( \ell \) have a common endpoint \( y \neq 0 \) and share other endpoints with \( \ell \). These four iterated pullbacks of \( \ell \) form a collapsing quadrilateral \( Q \) with diagonal \( \ell \); moreover, \( \sigma_3(x) = \sigma_3(y) \) and \( \sigma_3^n(x) = \sigma_3^n(y) = z \) is the non-periodic endpoint of \( \ell \). Evidently, \( \sigma_3(Q) = \sigma_3(p)\sigma_3(x) \) is the \((n-1)\)-st iterated pullback of \( \ell \). Then in the pullback lamination \( L(c) \) that we are defining we postulate the choice of only the short pullbacks among the above listed iterated pullbacks of \( \ell \). So, only two short edges of \( Q \) are included in the set of pullbacks \( \mathcal{C}_c \). A similar situation holds for \(-\ell\) and its iterated pullbacks.

In general, the choice of pullbacks of the already constructed leaf \( \hat{\ell} \) is ambiguous only if \( \hat{\ell} \) has an endpoint \( \sigma_3(\pm\ell) \). In this case we always choose short pullbacks of \( \hat{\ell} \). Evidently, this defines a set \( \mathcal{C}_c \) of chords in a unique way.

We claim that \( \mathcal{C}_c \) is an invariant prelamination. To show that \( \mathcal{C}_c \) is a prelamination we need to show that its leaves do not cross. Suppose otherwise and choose the minimal \( n \) such that \( \hat{\ell} \) and \( \ell \) are pullbacks of \( \ell \) or \(-\ell\) under at most the \( n \)-th iterate of \( \sigma_3 \) that cross. By construction, \( \hat{\ell}, \ell \) are not critical. Hence their images \( \sigma_3(\hat{\ell}), \sigma_3(\ell) \) are not degenerate and do not cross. It is only possible if \( \hat{\ell}, \ell \) come out of the endpoints of a critical leaf of \( L \). We may assume that \( \|\hat{\ell}\| \geq \frac{1}{6} \) (if \( \hat{\ell} \) and \( \ell \) are shorter than \( \frac{1}{6} \) then they cannot cross). However by construction this is impossible. Hence \( \mathcal{C}_c \) is a prelamination. The claim that \( \mathcal{C}_c \) is invariant is straightforward; its verification is left to the reader. By Theorem 2.12, the closure of \( \mathcal{C}_c \) is an invariant lamination denoted \( L(c) \). Moreover, by construction \( \mathcal{C}_c \) is symmetric (this can be easily proven using induction on the number of steps in the process of pulling back \( \ell \) and \(-\ell\)). Hence \( L(c) \) is a symmetric invariant lamination.

**Non-degenerate case.** As in the degenerate case, we will talk about leaves even though we are still constructing a lamination. By Lemma 3.20, we may assume that \( |c| < \frac{1}{6} \). Set \( \pm M = \pm M_c, \pm Q = \pm Q_c \). If \( d \) is an iterated forward image of \( c \) or \(-c\), then, by Definition 3.18(b), it cannot intersect the interior \( Q \) or \(-Q \). Consider the set of leaves \( D \) formed by the edges of \( \pm Q \) and \( \bigcup_{m=0}^{\infty} \{ \sigma_3^m(c), \sigma_3^m(-c) \} \). It follows that leaves of \( D \) do not cross among themselves. The idea is to construct pullbacks of leaves of \( D \) in a step-by-step fashion and show that this results in an invariant prelamination \( \mathcal{C}_c \) as in the degenerate case.
More precisely, we proceed by induction. Set \( D = C^0_c \). Construct sets of leaves \( C^{n+1}_c \) by collecting pullbacks of leaves of \( C^0_c \) generated by \( Q \) and \( -Q \) (the step of induction is based upon Definition 3.18 and Definition 3.19). The claim is that except for the property (D2)(1) from Definition 2.4 (a part of what it means for a lamination to be backward invariant), the set \( C^n_c \) has all the properties of invariant laminations listed in Definition 2.4. Let us verify this property for \( C^1_c \). Let \( \ell \in C^1_c \). Then \( \sigma_3(\ell) \in D \), so property (D1) from Definition 2.4 is satisfied. Property (D2)(2) is, evidently, satisfied for edges of \( Q \) and \( -Q \). If \( \ell \) is not an edge of \( \pm Q \), then, since leaves \( \pm \sigma_3(Q) = \sigma_3(\pm c) \) do not cross \( \sigma(\ell) \), and since on the closure of each component of \( S \setminus (Q \cup -Q) \) the map is one-to-one, then \( \ell \) will have two sibling leaves in \( C^1_c \) as desired. Literally the same argument works for \( \ell \in C^{n+1}_c \) and proves that each set \( C^{n+1}_c \) has properties (D1) and (D2)(2) from Definition 2.4. This implies that \( \bigcup_{i \geq 0} C^i_c = C_c \) has all properties from Definition 2.4 and is, therefore, an invariant prelamination. By Theorem 2.12, its closure \( L(c) \) is an invariant lamination.

The lamination \( L(c) \) is called the pullback lamination (of \( c \)); we often use \( c \) as the argument, instead of the less discriminatory \( \{ c, -c \} \).

**Lemma 3.21** (Lemma 5.9 [BOSTV1]). A legal pair \( \{ c, -c \} \) is the co-major pair of the lamination \( L(c) \). A symmetric pair \( \{ c, -c \} \) is a co-major pair if and only if it is legal.

Theorem 3.22 summarizes the main results of [BOSTV1]. Co-periodic comajors are defined as preperiodic comajors of preperiod 1. The name is due to the fact that a co-periodic major is a sibling of a periodic major.

**Theorem 3.22.** The set of all comajors of cubic symmetric laminations is a q-lamination. Co-periodic comajors are disjoint from all other comajors.

Based upon this theorem we define the main object of our interest.

**Definition 3.23.** All comajors of cubic symmetric laminations form a lamination \( C_{sCL} \) called the Cubic symmetric Comajor Lamination.

The following useful notation is justified by Theorem 3.22.

**Definition 3.24.** For a non-diameter chord \( n = \overline{ab} \), the smaller of the two arcs into which \( n \) divides \( S \), is denoted by \( H(n) \). Denote the closed subset of \( \overline{D} \) bounded by \( n \) and \( H(n) \) by \( R(n) \). Given two comajors \( m \) and \( n \), write \( m \prec n \) if \( m \subset R(n) \), and say that \( m \) is under \( n \).
Lemma 3.25 (Lemma 5.14 [BOSTV1]). Let \( \{c, -c\} \) and \( \{d, -d\} \) be legal pairs, where \( c \) is degenerate and \( c \prec d \). Suppose that \( c \) is not an endpoint of \( d \), or \( \sigma_3(c) \) is not periodic. Then \( d \in \mathcal{L}(c) \). In addition, the following holds.

1. Majors \( D, D' \) of \( \mathcal{L}(d) \) are leaves of \( \mathcal{L}(c) \) unless \( \mathcal{L}(c) \) has two finite gaps \( G, G' \) that contain \( D, D' \) as their diagonals, share a critical leaf \( M \) of \( \mathcal{L}(c) \) as a common edge, and are such that \( \sigma_3(G) = \sigma_3(G') \) is a preperiodic gap.
2. If majors of \( \mathcal{L}(d) \) are leaves of \( \mathcal{L}(c) \) and \( \ell \in \mathcal{L}(d) \) is a leaf that never maps to a short side of a collapsing quadrilateral of \( \mathcal{L}(d) \), then \( \ell \in \mathcal{L}(c) \).

4. Fatou conjecture on density of hyperbolicity

Co-periodic comajors correspond to periodic majors. In Section 4 we associate them with q-laminations with periodic Fatou gaps of degree greater than 1 and show that these are dense.

Definition 4.1. If a symmetric lamination \( \mathcal{L} \) has a periodic Fatou gap of degree greater than 1 (i.e., if it has properties listed in Lemma 4.6), then \( \mathcal{L} \) is called hyperbolic.

We need a result of [BMOV13]. Recall that, as in Definition 2.3, a lamination \( \mathcal{L} \) generates an equivalence relation \( \sim \) on \( S \) by declaring that \( a \sim b \) if and only if a finite concatenation of leaves of \( \mathcal{L} \) connects points \( a \in S \) and \( b \in S \).

Definition 4.2 (Proper lamination, Definition 4.1 [BMOV13]). Two leaves with a common endpoint \( v \) and the same image which is a leaf (and not a point) are said to form a critical wedge (the point \( v \) then is said to be its vertex). A lamination \( \mathcal{L} \) is proper if it contains no critical leaf with periodic endpoint and no critical wedge with periodic vertex.

Proper laminations generate laminational equivalence relations.

Theorem 4.3 (Theorem 4.9 [BMOV13]). Let \( \mathcal{L} \) be a proper invariant lamination. Then \( \sim \) is an invariant laminational equivalence relation.

We also need a nice result due to Jan Kiwi [Kiw02].

Theorem 4.4 ([Kiw02]). Let \( \mathcal{L} \) be a \( \sigma_d \)-invariant lamination. Then any infinite gap of \( \mathcal{L} \) is (pre)periodic. For any finite periodic gap \( G \) of \( \mathcal{L} \) its vertices belong to at most \( d - 1 \) distinct cycles except when \( G \) is a fixed return \( d \)-gon. In particular, a cubic lamination cannot have a fixed return \( n \)-gon for \( n > 3 \). Moreover, if all images of a \( k \)-gon \( G \) with \( k > d \) have at least \( d + 1 \) vertices then \( G \) is preperiodic.
Finally, here is an important claim.

**Corollary 4.5** (Corollary 4.8 [BOSTV1]). If $E$ is a preperiodic polygon of a symmetric lamination such that $E$ is not precritical, then no diagonal of $E$ can be a leaf of a symmetric lamination.

Let us now describe laminations related to co-periodic comajors.

**Lemma 4.6.** Let $L$ be a symmetric lamination with a periodic Fatou gap of degree greater than 1. Then $L$ has two critical Fatou gaps of degree greater than 1. Moreover, $L$ is a q-lamination.

*Proof.* Because of the symmetry, a hyperbolic symmetric lamination $L$ has two critical Fatou gaps of degree greater than 1. These gaps either belong to the same cycle of Fatou gaps, or belong to two distinct cycles of Fatou gaps. Moreover, by Theorem 4.3 the equivalence $\sim_L$ is laminational. We claim that $L$ coincides with the q-lamination $\hat{L}$ generated by $\sim_L$. We need to show that any leaf of $L$ is a leaf of $\hat{L}$.

In general, edges of a Fatou gap $U$ may form a finite concatenation in which case $U$ is not a gap of the corresponding q-lamination (by definition, in the q-lamination we add one more leaf to the concatenation to make it into a finite gap; this extra leaf will be an edge of a new, smaller Fatou gap of the q-lamination). However this cannot happen in our case: if it did it would yield a symmetric q-lamination with fixed return finite gaps contradicting Lemma 3.10. Hence the Fatou gaps of $L$ are gaps of $\hat{L}$. Otherwise, if $\ell \in L$ is not a leaf of $\hat{L}$ then $\ell$ must be a diagonal of a finite gap $G$ of $\hat{L}$. However by Corollary 4.5 this is impossible. Hence $L = \hat{L}$ is a q-lamination as desired. \qed

Hyperbolic laminations are constructed in Theorem 4.7.

**Theorem 4.7.** A preperiodic point $q \in S$ of preperiod 1 and period $k$ is an endpoint of a non-degenerate co-periodic comajor $c$ of period $k$ of a cubic symmetric lamination. Take the short edges of $\pm Q_c$, and remove their backward orbits from $L(c)$. Then the resulting lamination $\hat{L}(c)$ is a hyperbolic q-lamination with comajor pair $\{c, -c\}$.

*Proof.* Let $\ell = M_q = \overline{x_0 p}$ be the critical leaf with $\sigma_3(\ell) = \sigma_3(q)$ and $k$-periodic endpoint $p$. Consider the pullback lamination $L(q)$. Let $G$ be the central symmetric gap or leaf of $L(q)$ located between $\ell$ and $-\ell$. Then $G$ contains the origin and has leaves $\pm M$ closest to criticality. Clearly, the short siblings $\pm d$ of leaves $\pm M$ form a legal pair. Hence if $\ell$ shares an endpoint with $M$, then, by Lemma 3.21 we can set $c = d$. Assume now that leaves $\pm \ell$ are disjoint from $\pm M$.


If the orbits of $p$ and $-p$ are disjoint, let $n = k$. Otherwise $k = 2n$ for
some $n$, $\sigma_3^n(p) = -p$ and $\sigma_3^n(-p) = p$. We will assume in the rest of
the proof that $k = n$, the case when $k = 2n$ is similar. Consider the strip
$S$ between $M$ and $M'$. If $s = \overline{x_0x_1}$ is the short pullback of $\ell$ included
in $\mathcal{L}(q)$ by the construction, then $\sigma_3^k(s) = \overline{x_0p}$. Hence there is another
leaf $\overline{x_1x_2}$ such that $\sigma_3^k(\overline{x_1x_2}) = \overline{x_0x_1}$. The leaf $\overline{x_1x_2}$ is short as if $\overline{x_1x_2}$
is long/medium, then its $k$-th image $s$ is short and non-disjoint from
the interior of its short strips, contradicting Lemma 3.8. Repeating
this, we get a concatenation $A$ of pullbacks of $\ell$ under powers of $\sigma_3^k$; $A$
consists of short leaves of $\mathcal{L}(q)$, begins with $\ell \cup \overline{x_0x_1} \cup \overline{x_1x_2}$, converges
to a point $t \in \mathbb{S}$ of period $k$, and points $x_0, x_1, \ldots$ belong to the short
circular arc $I$ that bounds $S$ and does not contain $p$. Since $t$ and $p$
belong to distinct circle arcs on the boundary of $S$, then $t \neq p$.

Clearly, an infinite periodic gap $U$ of $\mathcal{L}(q)$ contains $A$ in its boundary,
and there is a gap $U'$ with the same image as $U$ that shares an edge
$\ell$ with $U$. Consider the chord $\overline{pt}$; it is periodic of period $k$, and there
is another chord $\overline{x_0\ell'}$ with the same image as $\overline{pt}$. The chord $\overline{pt}$ is
compatible with $\mathcal{L}(q)$ because by construction its images stay inside
images of $U$ and never cross leaves of $\mathcal{L}(q)$. Moreover, the iterated
images of $\overline{pt}$ do not cross as for this to happen some leaves from the
concatenation $A$ must cross, and this is not the case. We claim that
then $\overline{pt}$ never enters the strip between itself and $\overline{x_0\ell'}$. Indeed, if it does
then, by Lemma 3.8 it will have to cross $\ell$, a contradiction. Likewise,
images of $\overline{pt}$ never cross $-\ell$. By definition this implies that the short
sibling $\overline{py}$ of $\overline{pt}$, together with $-\overline{pq}$, forms a legal pair. Thus, $\overline{pq} = c$
is a comajor of a symmetric lamination as desired.

The leaf $\sigma_3(c) = \sigma_3(\overline{pt})$ is an $k$-periodic leaf of $\mathcal{L}(q)$. By Proposition
3.8, the leaf $\sigma_3^k(c) = \overline{pt}$ is a major of $\mathcal{L}(c)$. Let $\bar{x}$ and $\bar{y}$ be the two short
edges of $Q_c = Q$. Removing them and their backward orbits from $\mathcal{L}(c)$
yields the family of chords $\hat{\mathcal{L}}$; we claim that $\hat{\mathcal{L}}$ is an invariant lamination,
eto. Indeed, by definition $\mathcal{L}(c)$ has two quadrilaterals $X$ and $Y$
attached to $Q$ at $\bar{x}$ and $\bar{y}$, respectively. This implies that both $\bar{x}$ and $\bar{y}$
are isolated in $\mathcal{L}(c)$. So, $\hat{\mathcal{L}}$ is obtained by removing a countable family
of isolated leaves from $\mathcal{L}$; hence, $\hat{\mathcal{L}}$ is closed. The other properties
of invariant laminations for $\hat{\mathcal{L}}$ are immediate. E.g., we need to verify
that any non-critical chord of $\hat{\mathcal{L}}$ can be included in a sibling collection.
The only problematic case is that of $c$ (or $\overline{pt}$, or $\overline{x_0\ell'}$), however $c$, $\overline{pt}$,
and $\overline{x_0\ell'}$ themselves form a sibling collection. Thus, $\hat{\mathcal{L}}$ is an invariant
lamination. Evidently, $\hat{\mathcal{L}}$ is symmetric.

Consider the gap $U$ of $\hat{\mathcal{L}}$ with $U \supset Q$. Countably many pullbacks of $Q$
are consecutively attached to one another and contained in $U$. Hence
$U$ is an infinite periodic gap that maps forward 2-to-1, that is, $U$ is a Fatou gap of degree two. By definition, $\widehat{\mathcal{L}}$ is hyperbolic. Moreover, by the construction $c$ remains a leaf of $\widehat{\mathcal{L}}$. Hence $\{c, -c\}$ is the comajor pair of $\widehat{\mathcal{L}}$. □

We now consider preperiodic points of preperiod greater than 1 or periodic points (by Lemma 3.17, there are no non-degenerate periodic comajors). Recall that a dendrite is a locally connected continuum that contains no Jordan curves. A q-lamination with no infinite gaps gives rise to a topological Julia set which is a dendrite; we call such q-laminations dendritic (see [BOPT17, BOPT19]). We will also need Theorem 2.19 from [BOSTVI]. This theorem coincides with Lemma 2.31 of [BOPT20] except for two extra claims proven in [BOSTVI].

**Theorem 4.8** (Lemma 2.31[BOPT20], Theorem 2.19[BOSTVI]).
Let $G$ be an infinite $n$-periodic gap and $K = \text{Bd}(G)$. Then $\sigma^n_{\alpha}|_K$ is the composition of a covering map and a monotone map of $K$. If $\sigma^n_{\alpha}|_K$ is of degree one, then either statement (1) or statement (2) below holds.

1. The gap $G$ has countably many vertices, finitely many of which are periodic and the rest are preperiodic. All non-periodic edges of $G$ are (pre)critical and isolated. There is a critical edge with a periodic endpoint among edges of gaps from the orbit of $G$.

2. The map $\sigma^n_{\alpha}|_K$ is monotonically semi-conjugate to an irrational circle rotation so that each fiber is a finite concatenation of (pre)critical edges of $G$. Thus, there are critical leaves (edges of some images of $G$) with non-preperiodic endpoints.

In particular, if all critical sets of a lamination are non-degenerate finite polygons then the lamination has no infinite gaps.

Consider now the preperiodic case of preperiod greater than 1.

**Lemma 4.9.** If $x \in \mathbb{S}$ is preperiodic of preperiod $n > 1$ then there exists a symmetric dendritic q-lamination $\widehat{\mathcal{L}}$ with finite critical preperiodic sets $\pm G$ of preperiod $n$ and a gap/leaf $T \neq \pm G$ of $\widehat{\mathcal{L}}$ with $\sigma_3(T) = \sigma_3(G)$ and $x \in T$. Moreover,

1. if $T$ is degenerate, then there are no non-degenerate comajors containing $x$,
2. if $T$ is a non-degenerate leaf, then $T$ is a comajor containing $x$,
3. if $T$ is a gap, then the edges of $T$ with endpoint $x$ are comajors containing $x$.

The lamination $\widehat{\mathcal{L}}$ coincides with the family of limit leaves of iterated pullbacks of critical leaves $\pm M_x$ of $\mathcal{L}(x)$. All edges of $T$ are comajors that are limits of comajors disjoint from $T$. 
Proof. Set $\mathcal{L}(x) = \mathcal{L}$, $\ell = M_x$. We claim that $\mathcal{L}$ has no infinite gaps. Indeed, if $U$ is an infinite gap of $\mathcal{L}$, then by Theorem 4.1 an eventual image $V$ of $U$ is periodic. Moreover, no gap of the orbit of $V$ is critical as $\mathcal{L}$ has two critical leaves $\pm \ell$ and hence no gap of $\mathcal{L}$ can map onto its image $k$-to-1 with $k > 1$. Thus, $V$ is periodic of degree 1. By Theorem 4.8 we may assume that $V$ has a critical edge with a periodic endpoint or with both non-preperiodic endpoints. Since neither $\ell$ nor $-\ell$ is like that, then all gaps of $\mathcal{L}$ are finite.

By Theorem 4.3 the equivalence relation $\sim_{\mathcal{L}}$ is laminational. Let $\hat{\mathcal{L}}$ be the q-lamination generated by $\sim_{\mathcal{L}}$. All gaps of $\hat{\mathcal{L}}$ are finite (if $\hat{W}$ is an infinite gap of $\hat{\mathcal{L}}$ then by the construction no leaf of $\mathcal{L}$ can be inside $\hat{W}$, and so there is an infinite gap $W$ of $\mathcal{L}$ containing $\hat{W}$, a contradiction). Hence the topological Julia set $J_{\sim_{\mathcal{L}}}$ is a dendrite, and there are no isolated leaves in $\hat{\mathcal{L}}$. Clearly, $\hat{\mathcal{L}}$ is symmetric, with critical sets $G \supset \ell$, $-G \supset -\ell$, and there is a $\hat{\mathcal{L}}$-set $T$ with $\sigma_3(T) = \sigma_3(G)$.

In order to prove claims (1) — (3) of the lemma, assume first that $T = \{x\}$ is a singleton. Then $\hat{\mathcal{L}}$ has critical leaves $\pm G = \pm \ell$. Suppose that there is a sequence of $\hat{\mathcal{L}}$-gaps $H_i$ that converges to $\ell$. By Theorem 3.9 all of them are (pre)periodic. We may assume that $H_1 = H$ has an edge $c$ that separates the interior of $H$ from $\ell$, with endpoints close to the endpoints of $\ell$. We may follow the orbit of $H$ and $c$ and choose the closest to criticality iterated image $d$ of $c$ (it is always possible since the orbit of $c$ is finite and $c$ never maps to $\pm \ell$). By Proposition 3.8 the leaf $d$ never enters its short strips. Hence the short sibling $d''$ of $d$, together with $-d''$, forms a legal pair. Evidently, $d''$ separates a short circle arc containing $x$ from the rest of the circle. Since by Theorem 3.22 comajors form a q-lamination, non-degenerate comajors cannot contain $x$ as claimed.

If there are no gaps located close to $\ell$ then, since $\sigma_3$-periodic points are dense in $\mathbb{S}$, we can choose a sequence of periodic leaves converging to $\ell$, and repeat for them the above argument. So, the case when $T = \{x\}$ is a singleton is considered. If $T$ is a leaf/gap, then it is easy to check that any leaf on the boundary of $T$ is legal as desired.

Let us prove the next to the last claim of the lemma. Take a leaf of $\mathcal{L}(c)$ which is the limit of a sequence of pullbacks of $\pm M_x$. Each such pullback is contained in a pullback of a critical set of $\hat{\mathcal{L}}$. Hence their limit is the limit of a sequence of leaves of $\hat{\mathcal{L}}$, that is itself a leaf of $\hat{\mathcal{L}}$. On the other hand, by definition $\hat{\mathcal{L}} \subset \mathcal{L}(x)$. Hence if there is a leaf $\ell \in \hat{\mathcal{L}}$ which is not the limit of a sequence of pullbacks of $\pm M_x$, then $\ell$ is a pullback of $\pm M_x$. We may assume that, say, $M_x$ is a leaf of $\hat{\mathcal{L}}$. 
but is not the limit of pullbacks of \(\pm M_x\). Then there must exist two gaps of \(\hat{\mathcal{L}}\) sharing \(M_x\) as an edge which is impossible for the dendritic lamination \(\hat{\mathcal{L}}\) in which these two gaps will have to be merged into one.

The last claim of the lemma follows from the construction and the fact that all leaves of \(T\) are limits of comajors disjoint from them proven in Lemma 6.8 in [BOSTV1] and stated in this paper as Lemma 5.5. \(\square\)

**Definition 4.10.** A preperiodic comajor \(c\) of preperiod greater than 1 or a periodic comajor (necessarily degenerate) is called a Misiurewicz comajor, and any symmetric lamination with a Misiurewicz comajor pair is said to be a Misiurewicz symmetric lamination.

We will need the following lemmas.

**Lemma 4.11 (Lemma 6.7 [BOSTV1]).** Let \(c \in C_sCL\) be a non-degenerate comajor such that \(\sigma_3(c)\) is not periodic. If there exists a sequence of leaves \(c_i \in \mathcal{L}(c)\) with \(c < c_i\) and \(c_i \to c\), then \(c\) is the limit of co-periodic comajors \(\hat{c}_j \in \mathcal{L}\) with \(c < \hat{c}_j\) for all \(j\).

**Lemma 4.12 (Lemma 6.2 [BOSTV1]).** Let \(c \in C_sCL\) be a non-degenerate comajor. If \(\ell \in \mathcal{L}(c), \ell < c\) and \(\|\ell\| > \frac{\|c\|}{3}\), then \(\ell \in C_sCL\). In particular, if \(c_i \in \mathcal{L}(c), c_i < c\) and \(c_i \to c\), then \(c_n \in C_sCL\) for sufficiently large \(n\).

We are ready to prove the density of hyperbolicity (Fatou conjecture) for symmetric laminations.

**Theorem 4.13.** Co-periodic comajors are dense in \(C_sCL\).

**Proof.** Consider a non-degenerate comajor \(c \in C_sCL\) that is not co-periodic. We have two cases here.

(a) There is a sequence of leaves \(c_i \in \mathcal{L}(c)\) with \(c < c_i\) and \(c_i \to c\). Then, by Lemma 4.11, the comajor \(c\) is the limit of co-periodic comajors \(\hat{c}_j\) such that \(c < \hat{c}_j\).

(b) A sequence of leaves \(c_i \in \mathcal{L}(c)\) converging to \(c\) with \(c < c_i\) does not exist. Then \(c\) is an edge of a gap \(G\) of \(\mathcal{L}(c)\) with all vertices of \(G\) outside of \(H(c)\). The lamination \(\mathcal{L}(c)\) has critical quadrilaterals \(\pm Q_c = \pm Q\). If \(\sigma_3(c)\) eventually maps to an edge of \(Q\), then this edge is periodic which shows that \(c\) is co-periodic, a contradiction with our assumption. Hence \(\sigma_3(c)\) never maps to an edge of \(Q\), and, therefore, \(G\) never maps to a leaf or point. By Theorem 3.9, this implies that \(G\) and \(c\) are preperiodic of preperiod greater than 1 (recall that \(c\) is not periodic by Lemma 3.17).

We claim that all edges of \(G\) are comajors. Properties of laminations imply that there are two gaps, \(L\) and \(R\), attached to \(Q_c\) at the
appropriate majors of $L(c)$ and such that $\sigma_3(L) = \sigma_3(R) = \sigma_3(G)$. Now, choose among the edges of $G$ the edge $\ell$ with the greatest length. Then, clearly, $G \cap S \subset H(\ell)$. Set $M = M_\ell, M' = M'_\ell$. Then $M$ (or $M'$) cannot enter the strip $S$ between $M$ and $M'$ as otherwise, by Proposition 3.8, their images would have to cross edges of $L, R,$ or $Q_c$. This implies that in fact any edge $d$ of $G$ is a comajor because $\{d, -d\}$ is legal.

It follows now that this is exactly the situation described in Lemma 4.9 and that $L(c)$ gives rise to a laminational equivalence relation $\sim_{L(c)}$ which, in turn, gives rise to a dendritic q-lamination $\hat{L}$ such that $G$ is a gap of $\hat{L}$ (the last claim follows, e.g., from the fact that, by Theorem 3.22, comajors form a q-lamination). Since there are no isolated leaves in $L$, the comajor $c$ is approximated by uncountably many leaves $\hat{\ell}$ of $\hat{L}$ such that $\hat{\ell} \prec c$. By Lemma 4.12, we may assume that all these leaves of $\hat{L}$ are comajors. Now, choose a sequence of them that converge to $c$ and satisfy the conditions of case (a) of this proof. By (a) these leaves are all limits of co-periodic comajors, hence so is $c$ as desired. □

5. L-algorithm

In this section, we provide an algorithm for constructing all co-periodic comajor leaves. By Theorem 4.13, they are dense in $C_sCL$, hence this renders the entire $C_sCL$. The algorithm is similar to the famous Lavaurs algorithm for Thurston’s Quadratic Minor Lamination QML [Lav86, Lav89] (see [Sou21, BBS21] for an extension of this algorithm to the degree $d$ unicritical case). We call it the L-algorithm.

5.1. Preliminaries.

Lemma 5.1 (Lemma 6.1 [BOSTV1]). A co-periodic comajor leaf is disjoint from all other comajors in $C_sCL$.

The following is Definition 6.4 from [BOSTV1].

Definition 5.2. Let $\ell$ be a leaf of a symmetric lamination $L$ and $k > 0$ be such that $\sigma_3^k(\ell) \neq \ell$ (in particular, the leaf $\ell$ is not a diameter). If the leaf $\sigma_3^k(\ell)$ is under $\ell$, then we say that the leaf $\ell$ moves in by $\sigma_3^k$; if $\sigma_3^k(\ell)$ is not under $\ell$, then we say that the leaf $\ell$ moves out by $\sigma_3^k$. If two leaves $\ell$ and $\hat{\ell}$ with $\ell \prec \hat{\ell}$ of the same lamination both move in or both move out by the map $\sigma_3^k$, then we say that the leaves move in the same direction. If one of the leaves $\ell, \hat{\ell}$ moves in and the other moves out, then we say that the leaves move in the opposite directions. There are two ways of moving in the opposite directions: if $\ell$ moves out and
\(\ell\) moves in, we say they move towards each other; if \(\ell\) moves in and \(\hat{\ell}\) moves out, we say that they move away from each other.

The strip \(S(\ell, \hat{\ell})\) between non-crossing chords \(\ell, \hat{\ell}\) was introduced in Definition 3.5.

**Lemma 5.3** (Lemma 6.5 \[BOSTV1\]). Let \(\hat{\ell} \neq \ell\) be non-periodic leaves of a symmetric lamination \(L\) with \(\hat{\ell} > \ell\). Given an integer \(k > 0\), let \(h : S \to S\) be either the map \(\sigma^k_3\) or the map \(-\sigma^k_3\). Suppose that the leaves \(\ell\) and \(\hat{\ell}\) move towards each other by the map \(h\) and neither the leaves \(\ell\) and \(\hat{\ell}\), nor any leaf separating them, can eventually map into a leaf (including degenerate) with both endpoints in one of the boundary arcs of the strip \(S(\ell, \hat{\ell})\). Then there exists a \(\sigma_3\)-periodic leaf \(y \in L\) that separates \(\ell\) and \(\hat{\ell}\).

For the notion of two-sided limit leaves, see Definition 3.16.

**Corollary 5.4** (Corollary 6.7 \[BOSTV1\]). Every not eventually periodic comajor \(c\) is a two sided limit leaf in the Cubic Symmetric Comajor Lamination \(C_{sCL}\).

**Lemma 5.5** (Lemma 6.8 \[BOSTV1\]). A non-degenerate preperiodic comajor \(c\) of preperiod at least 2 is a two sided limit leaf of \(C_{sCL}\) or an edge of a finite gap \(H\) of \(C_{sCL}\) all of whose edges are limits of comajors of \(C_{sCL}\) disjoint from \(H\).

**Definition 5.6.** We say a gap \(G\) weakly separates two leaves \(\ell_1\) and \(\ell_2\) if \(\ell_1 \setminus G\) and \(\ell_2 \setminus G\) are nonempty sets in two different components of \(D \setminus G\). Similarly we say a leaf \(\ell\) weakly separates two leaves \(\ell_1\) and \(\ell_2\) if \(\ell_1 \setminus \ell\) and \(\ell_2 \setminus \ell\) are nonempty sets in two different components of \(D \setminus \ell\).

**Lemma 5.7.** Let \(\ell' \neq \ell\) be two leaves in a cubic symmetric lamination \(L\) such that \(\ell < \ell'\). Suppose that:

1. the leaves \(\ell\) and \(\ell'\) move away from each other under \(\sigma^k_3\),
2. no leaf weakly separating \(\ell\) and \(\ell'\) maps to a critical chord of \(S\) under the map \(\sigma^i_3\) for \(i < k\).

Then, there exists a periodic leaf \(y = ab\) with \(\sigma^k_3(a) = a, \sigma^k_3(b) = b\) that weakly separates \(\ell\) and \(\ell'\).

**Proof.** A gap \(G\) of \(L\) with edges \(\ell, \ell'\) does not exist as otherwise the gap \(\sigma^k_3(G)\) would strictly cover the gap \(G\). Hence the family of leaves \(C \subseteq L\) that consists of \(\ell, \ell'\), and the leaves that weakly separate \(\ell\) and \(\ell'\) has at least one leaf that weakly separates \(\ell\) and \(\ell'\). Clearly, \(C\) is closed.
Let $A$ be the set of leaves of $C$ that move in under $\sigma_3^k$ such that for every leaf $m \in A$, if a leaf $n$ weakly separates $\ell$ and $m$, then $n$ also moves in under the map $\sigma_3^k$. So, all the leaves in $A$ move in under $\sigma_3^k$. Then the closure $\overline{A}$ of $A$ (with respect to the Hausdorff metric) is a family of leaves, too; let $y \in \overline{A}$ be the leaf of $\overline{A}$ farthest from $\ell$ (i.e., every leaf in $\overline{A} \setminus \{\ell, y\}$ weakly separates $\ell$ from $y$). By continuity, either $y \in A$, or $\sigma_3^k(y) = y$. We claim that $\sigma_3^k(y) = y$. Indeed, suppose that $y$ moves in under $\sigma_3^k$. There are two cases. First, it can be that $y$ is approximated by leaves with endpoints outside $H(y)$ (see Definition 3.24). However, this contradicts the choice of $y$. Second, $y$ can be an edge of a gap $G$ with vertices outside of $H(y)$ while all vertices of $\sigma_3^k(G)$ belong to $H(y)$. If now $\hat{\ell}$ is the edge of $G$ with $y \prec \hat{\ell}$, then $\hat{\ell} \in A$, a contradiction.

Thus, $y = \overline{ab} = \sigma_3^k(y)$. We claim that $\sigma_3^k$ fixes the endpoints of the leaf $y$. Assume that $\sigma_3^k$ flips $y$, and consider cases. If $y$ is a two sided limit leaf and $t \in A$ is close to $y$, then the leaf $t$ would move out under $\sigma_3^k$, a contradiction. If $y$ is an edge of a gap $G$, then $y$ is an edge of the gap $G' = \sigma_3^k(G)$, the gaps $G$ and $G'$ are on both sides of the leaf $y$, and $\sigma_3^k$ maps one gap to the other. Hence there is an edge $t \prec y$ of $G$ or $G'$ that belongs to $A$ but moves out under $\sigma_3^k$, a contradiction. Finally, $\sigma_3^k(y) = y$ is non-degenerate. 

**Lemma 5.8** (Lemma 6.3 [BOSTV1]). Let $L$ be a cubic symmetric lamination with comajor pair $\{c, -c\}$. Suppose that a short leaf $\ell_s \in L$ with $c \prec \ell_s$ is such that the leaf $\ell_m = \sigma_3(\ell_s)$ never maps under $\pm \ell_m$. Then, there exists a cubic symmetric lamination $\mathcal{L}(\ell_s)$ with comajor pair $\{\ell_s, -\ell_s\}$.

### 5.2. The description of the L-algorithm.

According to [Mil93, Mil09], cubic polynomials with *Fatou domains whose first return map is of degree 4* are said to be of type B (Bi-transitive) and cubic polynomials with *two cycles of Fatou domains* are said to be of type D (Disjoint); in the latter case first return maps on periodic Fatou domains are, evidently, of degree 2. We classify co-periodic comajors of in the similar fashion below. Recall that, by Theorem 4.7, co-periodic comajors $c$ generate hyperbolic $q$-laminations $\hat{\mathcal{L}}(c)$.

The nature of cubic symmetric laminations gives rise to two notions describing two types of periodic points and related (pre)periodic objects. We give a general definition that applies to all of them.

**Definition 5.9** (Type B and type D). A $2n$-periodic point $x$ of $\sigma_3$ such that $\sigma_3^n(x) = -x$, is said to be of type B. All other periodic points of $\sigma_3$ are said to be of type D. A periodic leaf of a symmetric lamination
is of type B if its endpoints are of type B, and of type D otherwise. A co-periodic leaf of a symmetric lamination is of type B if its image is a periodic leaf of type B, and of type D otherwise.

Lemma 5.10 (Corollary 3.7 [BMOV13]). Suppose that $\ell$ and $\hat{\ell}$ are two leaves of a $\sigma_d$-invariant lamination that share an endpoint and have non-degenerate distinct images. Then the orientation of the triple of their endpoints is preserved under the map $\sigma_d$.

To justify Definition 5.9 we need the next lemma.

Lemma 5.11. A periodic leaf of a symmetric lamination $\mathcal{L}$ cannot have endpoints of type B and type D.

Proof. Suppose that $\ell = xy$ is a periodic leaf of $\mathcal{L}$ such that $x$ is of type B while $y$ is of type D. Then $x$ is of period $2n$ and $\sigma_3^n(x) = -x$. It follows that $y$ is also of period $2n$ but $\sigma_3^n(y) \neq -y$. Since $(-x)(\sigma_3^n(y)) = \sigma_3^n(\ell)$ is a leaf of $\mathcal{L}$, then the leaf $x(-\sigma_3^n(y))$ is a leaf of $\mathcal{L}$, too. Thus, the cone of leaves $\ell = xy$ and $x(-\sigma_3^n(y))$ is mapped by $\sigma_3^n$ to the cone of leaves $(-x)(\sigma_3^n(y))$ and $(-x)(-y)$. However it is easy to see that the orientation of the triple $(y, x, -\sigma_3^n(y))$ is opposite to the orientation of the triple $(\sigma_3^n(y), -x, -y)$. This contradicts Lemma 5.10 and completes the proof. $\square$

Evidently, the $\sigma_3$-image of an object of type B (D) is an object of the same type; co-periodic comajors can be either of type B or of type D. Also, Definition 5.9 allows us to talk about majors, comajors, and minors of types B or D. In the type B case a periodic major $M = \overline{ab}$ eventually maps to $-M$ so that $a$ and $b$ of $M$ eventually map to the $-a$ and $-b$, respectively. In the type D case, the orbits of majors are disjoint. Thus, if a co-periodic comajor $c$ is of type B, then the lamination $\hat{\mathcal{L}}(c)$ from Theorem 4.7 has a pair of symmetric Fatou gaps whose first return map is of degree 4; if $c$ is of type D then $\hat{\mathcal{L}}(c)$ has a pair of symmetric Fatou gaps whose first return map is of degree 2.

Definition 5.12. A periodic point (leaf) of type B and period $2n$ is said to be of block period $n$. A periodic point (leaf) of type D and period $n$ is said to be of block period $n$. A co-periodic leaf is said to be of block period $n$ if its image is of block period $n$.

In [BOSTV1] we considered the map $\tau$ that rotates the unit disk by 180 degrees. If $\mathcal{L}$ is a cubic symmetric lamination, then $\tau$ acts on leaves and gaps of $\mathcal{L}$. We will also interchangeably use the notation $-\ell$ for $\tau(\ell)$ and $-G$ for $\tau(G)$ where $\ell$ is a leaf of $\mathcal{L}$ and $G$ is a gap of $\mathcal{L}$.
Define the map \( g_j = \tau \circ \sigma_3^j : \mathcal{L} \to \mathcal{L} \) for some \( j \). Lemma 5.13 is similar to Lemma 5.7. We state it without proof.

**Lemma 5.13.** Let \( \ell' \neq \ell \) be two leaves in a cubic symmetric lamination \( \mathcal{L} \) such that \( \ell \prec \ell' \). Suppose that:

(i) the leaves \( \ell \) and \( \ell' \) move away from each other under \( g_k \),

(ii) no leaf weakly separating \( \ell \) and \( \ell' \) maps to a critical chord of \( S \) under the map \( g_i \) for \( i < k \).

Then, there exists a periodic leaf \( y \) of period 1 under the map \( g_k \) that weakly separates \( \ell \) and \( \ell' \).

The next lemma deals with dynamics of comajors.

**Lemma 5.14.** Suppose that \( c' \prec c \) are distinct co-periodic comajors that are leaves of a lamination \( \mathcal{L} \). Then there is no finite gap \( H \) of \( \mathcal{L} \) such that both \( c' \) and \( c \) are edges of \( H \). In particular, comajors of type \( B \) or \( D \).

**Proof.** The leaves \( m' = \sigma_3(c') \prec m = \sigma_3(c) \) are periodic. By way of contradiction assume that both are edges of a periodic gap \( \sigma_3(H) = G \) of \( \mathcal{L} \). Then their endpoints stay in the same circular order along their periodic orbits. By Lemma 3.10, if \( G \) is 1-rotational, then the leaf \( m \) will eventually map to the leaf \( m' \), and if \( G \) is 2-rotational, then the leaf \( m \) will eventually map to the leaf \( -m' \), in either case contradicting that \( m \) is the shortest leaf in its orbit (see Lemma 3.15).

Now, the main theorem needed for the L-algorithm is as follows.

**Theorem 5.15.** Suppose that co-periodic comajors \( c \) and \( c' \) have the following properties:

(i) \( c' \prec c \),

(ii) both \( c \) and \( c' \) are either of type \( B \) or type \( D \), and

(iii) \( c \) and \( c' \) have the same block period \( n \).

Then there exists a co-periodic comajor \( d \) with \( c' \prec d \prec c \) such that \( d \) is of block period \( j < n \).

**Proof.** Choose a preperiodic point \( p \) of preperiod bigger than 1 and period bigger than \( n \) in the arc \( H(c') \). By Lemma 4.9, there exists a cubic symmetric dendritic q-lamination \( \mathcal{L} \) with a pair of finite critical gaps/leaves \( \{\Delta, -\Delta\} \) such that \( \sigma_3(p) \in \sigma_3(\Delta) \) (i.e., the critical leaves \( \pm \ell \) of \( \mathcal{L}(p) \) are contained in the critical sets \( \Delta \) and \( -\Delta \)), iterated preimages of \( \pm \ell \) converge to all sides of \( \Delta \) and \( -\Delta \), so that pullbacks of the critical sets are dense in \( \mathcal{L} \), and \( c \) and \( c' \) are leaves of \( \mathcal{L} \). The leaves \( m = \sigma_3(c) \) and \( m' = \sigma_3(c') \) are periodic and such that \( m' \prec m \). Since preimages of \( \pm \Delta \) are dense in \( \mathcal{L} \), then it follows from Lemma 5.14.
that for a minimal $k$, the sets $\Delta$ or $-\Delta$ separates $\sigma^k_3(m)$ and $\sigma^k_3(m')$. Consider cases.

(i): comajors $c$ and $c'$ are of type $D$. Then the periodic orbits of $m$ and $-m$ (and also $m'$ and $-m'$) are disjoint and have period $n$. We claim that $k \neq n - 1$. If $k = n - 1$, then $\sigma^{k-1}_3(m)$ and $\sigma^{k-1}_3(m')$ are long/medium siblings of $c$ and $c'$, respectively. Hence they must be separated by $\Delta$. The circular order of the four endpoints of $m$ and $m'$ is preserved in the leaves $\sigma^{n-1}_3(m)$ and $\sigma^{n-1}_3(m')$, but when $\sigma_3$ is applied one more time, exactly one of the leaves $\sigma^{n-1}_3(m)$ and $\sigma^{n-1}_3(m')$ flips because of the critical gap between them. Hence the order among the endpoints of $\sigma^k_3(m) = m$ and $\sigma^k_3(m') = m'$ cannot be the same as the order among the endpoints of $m$ and $m'$, which is absurd. Thus, $0 < k < n - 1$.

![Figure 2](image-url)

**Figure 2.** Cubic symmetric lamination $\mathcal{L}$ with its gaps $\Delta, -\Delta$ and $\Delta^*$ separating the leaves $m$ and $m'$ illustrating the proof of case (i).

(a): it is $\Delta$ that separates the leaves $\sigma^k_3(m)$ and $\sigma^k_3(m')$. Since the leaves and gaps separating $m$ and $m'$ map one-to-one under $\sigma^k_3$, there is a set $\Delta^*$ separating $m$ and $m'$ with $\sigma^k_3(\Delta^*) = \Delta$ and $\sigma^{k+1}_3(\Delta^*) = \sigma_3(\Delta) \prec m'$. Let $\ell^*$ be the side of $\Delta^*$ that separates $m$ and $m'$ and is closest to the leaf $m$. Then $\ell^*$ moves in under the map $\sigma^{k+1}_3$. On the other hand, the leaf $\sigma^{k+1}_3(m)$ is neither under the leaf $m$ nor under the leaf $-m$ because the minor is the shortest leaf in its orbit. Hence the leaves $m$ and $\ell^*$ move away from each other under the map $\sigma^{k+1}_3$. 
Let us verify condition (2) from Lemma 5.7. Note that $\sigma_3^k(\ell^*) = M$ is a major of $L$. For $i \leq k$, the map $\sigma_3^i$ takes the leaves separating $\ell^*$ and $m$ in the strip $S(\ell^*, m)$ one-to-one to the leaves separating $M$ and $\sigma_3^i(m)$ in the strip $S(M, \sigma_3^i(m))$. As there are no critical chords of $S$ in $S(M, \sigma_3^i(m))$, no leaf separating $\ell^*$ and $m$ maps to a critical chord of $S$ under the map $\sigma_3^i$ for $i \leq k$. Hence, by Lemma 5.7 there is a periodic leaf $y \in L$ of period $k + 1 < n$ separating $m$ and $\ell^*$.

Let $C$ be the collection of the leaves separating $m$ and $m'$. Let $C_1$ be the collection of all $\sigma_3$-periodic leaves in $C$ of period smaller than $n$. Let $C_2$ be the collection of all fixed leaves under the maps $g_k = -\sigma_3^k$, $0 < k < n$ in $C$; we associate the minimal such $k$ with all leaves from $C_2$. Since $y \in C_1$, then $C_1 \neq \emptyset$, but $C_2$ could be empty.

Let $y_1$ be a leaf of the least period $j_1 \leq k + 1 < n$ in $C_1$. Choose $y_1$ to be the closest to $m$ among leaves of $C_1$ of period $j_1$. Similarly, choose a $-\sigma_3^{j_2}$-fixed leaf $y_2$ in $C_2$ such that $j_2$ is the smallest possible; choose $y_2$ to be the closest to $m$ among $-\sigma_3^{j_2}$-fixed leaves in $C_2$. If $j_1 \leq j_2$, then we claim that the leaf $d$ which is the short pullback of $y_1$ in $L$ is the desired comajor of block period $j = j_1 < n$ (recall that $y_1$ is located between the minors $m$ and $m'$). By Lemma 5.8, it suffices to prove that the leaf $y_1$ neither maps under itself nor under the leaf $-y_1$ under the map $\sigma_3^i$ where $i$ can be any block period smaller than $j_1$.

(1) If $y_1$ maps under itself under $\sigma_3^i$, for some $i < j_1$, then the leaves $y_1$ and $m$ move away from each other under $\sigma_3^i$. By Lemma 5.7, there is a $\sigma_3$-periodic leaf $y_1'$ of period $i < j_1$ separating $m$ and $y_1$; a contradiction with the minimality of $j_1$.

(2) If $y_1$ maps under $-y_1$ under $\sigma_3^i$ for some $i < j_1$, then the leaf $g_i(y_1)$ is under the leaf $y_1$. Now, the leaves $y_1$ and $m$ move away from each other under $g_i = -\sigma_3^i$. By Lemma 5.13, there is a $-\sigma_3^i$-fixed leaf $y_1'$ that separates $m$ and $y_1$. Clearly $y_1'$ separates $m$ and $m'$, too. Then $i < j_1 \leq j_2$ is the block period associated with $y_1$, contradicting the choice of $j_2$.

Thus, the short pullback $d$ of $y_1$ in $L$ is the desired comajor of block period $j = j_1 < n$. Similarly if $j_2 < j_1$, then we obtain that the short pullback $d$ of $y_2$ in $L$ is the desired comajor of block period $j = j_2 < n$.

(b): it is $-\Delta$ that separates the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$, not $\Delta$. We use the arguments from case (a) and find a gap $\Delta^*$ with $\sigma_3^k(\Delta^*) = -\Delta$ separating $m$ and $m'$. Then we have the gap $\sigma_3^{k+1}(\Delta^*)$ going under the leaf $-m'$. The only difference in the arguments is that we use Lemma 5.13 first to find a leaf $y$ separating $m$ and $m'$ such that $g_{k+1}(y) = y$. Thus, the collection $C_2$ is non-empty here whereas collection $C_1$ could...
be empty. The rest of the argument follows exactly as before and we end up with a comajor \( d \) between \( c \) and \( c' \) of a block period \( j < n \).

(ii): comajors \( c \) and \( c' \) are of type B. The leaves \( m \) and \( m' \) are now periodic of period \( p = 2n \) and have symmetric orbits (the orbits of \( m \) and \( -m \) are the same). Similarly, the orbits of the leaves \( m' \) and \( -m' \) are the same as well. In this case, the proof is very similar to that of case (i).

First, we show that there exists an integer \( k \) with \( 0 < k < n - 1 \) such that \( \Delta \) or \( -\Delta \) separates the leaves \( \sigma_3^k(m) \) and \( \sigma_3^k(m') \). Indeed, let \( k \) be the smallest integer between 0 and \( p = 2n \) such that the leaves \( \sigma_3^k(m) \) and \( \sigma_3^k(m') \) are separated by a critical gap/leaf. As the orbits of both the leaves \( m \) and \( m' \) are symmetric, the strips formed by the leaves \( \sigma_3^k(m) \) and \( \sigma_3^i(m') \) where \( 0 < i \leq n - 1 \) are symmetric to the strips formed by the leaves \( \sigma_3^i(m) \) and \( \sigma_3^i(m') \) where \( n \leq r < 2n \). It follows that, for the first time, the separation by one of the critical gaps/leaves \( \Delta \) and \( -\Delta \) happens during the first half of the cycle, i.e., \( 0 < k \leq n - 1 \).

To see that \( k \) cannot be equal to \( n - 1 \), assume the contrary. Since \( \sigma_3^n(m) = -m \) and \( \sigma_3^n(m') = -m' \), the leaves \( \sigma_3^{n-1}(m) \) and \( \sigma_3^{n-1}(m') \) must be long/medium siblings of \( -c \) and \( -c' \), respectively. Hence they
are separated by $-\Delta$. The circular order of the four endpoints of $m$ and $m'$ is preserved in the leaves $\sigma_3^{n-1}(m)$ and $\sigma_3^n(m')$, and exactly one of them flips under the next iteration because of a critical gap between them. Without loss of generality, assume that the leaf $\sigma_3^{n-1}(m)$ flips its endpoints when it maps to the leaf $-m = \sigma_3^n(m)$. This contradicts the fact that the endpoints of $m$ form two symmetric cycles rather than one cycle of period $4n$. Thus, $0 < k < n - 1$. We have two subcases here similar to case (i).

(a): it is $\Delta$ that separates the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$. Then, following the similar arguments as in case(i) part(a), we find a comajor $d$ of block period $j < k + 1 = n$ separating the leaves $c$ and $c'$.

(b): it is $-\Delta$ that separates the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$, not $\Delta$. Then, using similar arguments to case(i) part(b), we find a comajor $d$ of block period $j < k + 1 = n$ separating the leaves $c$ and $c'$.

□

Theorem 5.15 allows us to describe an algorithm for finding co-periodic cubic comajors similar to the original Lavaurs algorithm [Lav86, Lav89] for finding periodic quadratic minors. We call this algorithm the L-algorithm.

L-algorithm

1) Draw co-periodic comajors of block period 1. It is easy to verify that type D co-periodic comajors of period 1 are $\frac{11}{6}$ and $\frac{25}{3}$. Similarly, type B co-periodic comajors of block period 1 are $\frac{5}{12}$ and $\frac{11}{12}$.

We proceed by induction. Suppose that all preperiodic comajors of block periods from 1 to $k$ (inclusively) have been drawn. Denote the family of them by $F_k$. Consider a component $A$ of $D \setminus \bigcup_{\ell \in F_k} \ell$. Then there are two cases.

(a) Suppose that there is a comajor $\ell_0$ such that all points of $A$ are located under $\ell_0$. Then there may be several comajors $\ell_1, \ldots, \ell_s$ located under $\ell_0$ with endpoints in $A \cap S$ (this collection of comajors may be empty). Consider the set $B = \{b_1 < \cdots < b_i\}$ of preperiodic points of type B of preperiod 1 and period $k + 1$ that belong to $A \cap S$. These points (if any) must be connected to create several comajors. By Lemma 5.1, these comajors are pairwise disjoint. By Theorem 5.15 no two comajors from that collection can be located so that one of them is under the other one. Hence $t = 2r$ is even and the comajors in question are $\overline{b_1b_2}, \ldots, \overline{b_{2r-1}b_{2r}}$. We can also consider the set $D$ of preperiodic points of type D of preperiod 1 and period $k + 1$ that belong to $A \cap S$. These points should be connected similar to how points from $B$ were connected, i.e. consecutively.
Do this for all components $A$ for which there is a comajor $\ell_0$ such that all points of $A$ are located under $\ell$.

(b) There is exactly one component $C$ of $\mathbb{D}\setminus\bigcup_{\ell\in\mathcal{F}_n}\ell$ for which there is no comajor $\ell_0$ with all points of $A$ located under $\ell$. This is the “central” component left after the closures of all components described in (a) are removed from $\mathbb{D}$. Evidently, this component contains the center of $\mathbb{D}$ and is unique.

As before, let $B$ be the set of preperiodic points of type B of preperiod 1 and block period $k+1$ that belong to $C\cap S$. However, unlike before let us divide $B$ into four subsets: $B^1 = B \cap (\frac{1}{12}, \frac{1}{6})$, $B^2 = B \cap (\frac{1}{3}, \frac{5}{12})$, $B_3 = B \cap (\frac{7}{12}, \frac{2}{3})$, and $B_4 = B \cap (\frac{5}{6}, \frac{11}{12})$. Since comajors are short, a comajor cannot connect two points from two distinct $B$-sets. Hence, as in case (a), comajors connect points from $B$ consecutively within $B$-sets. If, e.g., $B_1 = \{b_1 < \cdots < b_t\}$, then, as in (a), $t = 2r$ is even, and the corresponding comajors are $b_1b_2$, $\ldots$, $b_{2r-1}b_{2r}$. Points of type D that belong to $\partial C$ should be treated similarly.

Thus, L-algorithm for cubic symmetric laminations is as follows. First, we make step 1 and draw the comajors $\frac{1}{12}, \frac{1}{12}, \frac{5}{12}, \frac{7}{12},$ and $\frac{25}{36}$. Then on each next step, say, $k+1$, we first plot all type B points of preperiod 1 and period $k+1$ and connect them consecutively, starting from the smallest positive angle. Then we plot all type D points of preperiod 1 and period $k+1$ and connect them consecutively, too, starting from the smallest positive angle.

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