Two-parametric extension of $h$-deformation of $GL(1|1)$

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Abstract. The two-parametric quantum deformation of the algebra of coordinate functions on the supergroup $GL(1|1)$ via a contraction of $GL_{p,q}(1|1)$ is presented. Related differential calculus on the quantum superplane is introduced.

Mathematics Subject Classifications (1991): 16S80, 81R50.

Key words: supergroup, $q$-deformation, $h$-deformation, differential calculus.

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1. Introduction

Recently matrix groups like GL(2), GL(1|1), etc., were generalized in two ways called the $q$-deformation and the $h$-deformation. One of them [$q$-deformation] is based on a deformation of the algebra of functions on the groups generated by coordinate functions $t^{i\ j}$ which normally commute.

In the $q$-deformation of matrix groups [1,2], the commutation relations satisfied between the coordinate functions on the groups are determined by a matrix $R_q$ so that the functions do not commute but satisfy the equation

$$R_q(T \otimes T) = (I \otimes T)(T \otimes I)R_q,$$

or an endomorphism of the space with non-commuting coordinates may be used [3]. In this equation the elements of $R_q$ are complex numbers but the matrix $T = (t^{i\ j})$ is formed by generally non-commuting elements of an algebra.

The second type of deformation, called the $h$-deformation, is a new class of quantum deformations of matrix groups. The $h$-deformation is obtained as a contraction from the $q$-deformation of matrix groups and it has been intensively studied by many authors [4-12].

The purpose of this paper is to present the two-parametric extension of the $h$-deformation of the simplest supergroup GL(1|1). The single parameter $h$-deformation of GL(1|1), $GL_h(1|1)$, was introduced by Dabrowski and Parashar [13]. An interesting feature of the two-parameter deformation is that both the deformation parameters are anticommuting grassmann numbers.

The paper is organized as follows. In section 2 we give some notations and useful formulas which will be used in this work. In the following section we present the two-parameter deformation of GL(1|1) as related to superplanes. A two parameter R-matrix which deforms the supergroup GL(1|1) is introduced in section 4. In section 5 we construct the differential calculus on the quantum superplane on which the two-parameter quantum supergroup acts.

2. Review of $GL_{p,q}(1|1)$

We know that the supergroup GL(1|1) can be deformed by assuming that the linear transformations in GL(1|1) are invariant under the action of the quantum superplane and its dual [3].

In this paper we denote $(p, q)$-deformed objects by primed quantities. Unprimed quantities represent transformed coordinates.
Consider the Manin quantum superplane \( A_p \) and its dual \( A^*_q \). The quantum superplane \( A_p \) is generated by coordinates \( x' \) and \( \xi' \), and the commutation rules
\[
x' \xi' - p \xi' x' = 0, \quad \xi'^2 = 0.
\]
The quantum (dual) superplane \( A^*_q \) is generated by coordinates \( \eta' \) and \( y' \), and the commutation rules
\[
\eta'^2 = 0, \quad \eta' y' - q^{-1} y' \eta' = 0.
\]
Taking 
\[
T' = \begin{pmatrix} a' & \beta' \\ \gamma' & d' \end{pmatrix}
\]
as a supermatrix in \( GL(1|1) \), we demand that the relations (1), (2) are preserved under the action of \( T' \) on the quantum superplane and its dual. Under some assumption one obtains the following \((p, q)\)-commutation relations
\[
a' \beta' = q \beta' a', \quad a' \gamma' = p \gamma' a', \quad \beta'^2 = 0,
\]
\[
d' \beta' = q \beta' d', \quad d' \gamma' = p \gamma' d', \quad \gamma'^2 = 0,
\]
\[
\beta' \gamma' + pq^{-1} \gamma' \beta' = 0, \quad a' d' = d' a' + (p - q^{-1}) \gamma' \beta'.
\]
These relations will be used in section 3.

Above relations are equivalent to the equation
\[
R_{p,q} T'_1 T'_2 = T'_2 T'_1 R_{p,q}
\]
where \( T'_1 = T' \otimes I \), \( T'_2 = I \otimes T' \) and
\[
R_{p,q} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & qp^{-1} & 0 & 0 \\ 0 & 0 & q - p^{-1} & 1 \\ 0 & 0 & 0 & p^{-1} \end{pmatrix}.
\]
Here we employ the convenient grading notation
\[
(T_1)^{ij}_{kl} = (T \otimes I)^{ij}_{kl} = (-1)^{k(i+l)} T^i_{kl} \delta^j_l, \quad \delta^i_l = 1 \text{ if } i = l, 0 \text{ otherwise},
\]
\[
(T_2)^{ij}_{kl} = (I \otimes T)^{ij}_{kl} = (-1)^{i(j+l)} T^j_{kl} \delta^i_l.
\]

3. The two-parametric deformation of \( GL(1|1) \)

We introduce new coordinates \( x \) and \( \xi \) by
\[
U = g_{h_1}^{-1} U', \quad U' = \begin{pmatrix} x' \\ \xi' \end{pmatrix}
\]
where
\[ g_{h_1} = \begin{pmatrix} 1 & 0 \\ f_1 & 1 \end{pmatrix}, \quad f_1 = \frac{h_1}{p-1} \] (9)

Here the deformation parameter \( h_1 \), in contrast to the usual situation, is an odd (grassmann) number which has the following properties
\[ h_1^2 = 0 \quad \text{and} \quad h_1 \xi = -\xi h_1. \] (10)

Now, in the limit \( p \to 1 \) we get the following exchange relations
\[ x\xi = \xi x + h_1 x^2, \quad \xi^2 = -h_1 x\xi. \] (11)

These relations define a new deformation, which we called the \( h_1 \)-deformation, of the algebra of functions on the Manin superplane generated by \( x \) and \( \xi \), and we denote it by \( A_{h_1} \).

Let us now consider other (dual) coordinates \( \eta \) and \( y \) with
\[ V = g_{h_2}^{-1}V', \quad V' = \begin{pmatrix} \eta' \\ y' \end{pmatrix} \] (12)
where
\[ g_{h_2} = \begin{pmatrix} 1 & f_2 \\ 0 & 1 \end{pmatrix}, \quad f_2 = \frac{h_2}{q-1} \] (13)

Again, the deformation parameter \( h_2 \) is an odd (grassmann) number and it has the following properties
\[ h_2^2 = 0 \quad \text{and} \quad h_2 \eta = -\eta h_2. \] (14)

Next, taking the \( q \to 1 \) limit we obtain the following relations, which define the dual \( h_2 \)-superplane \( A_{h_2}^* \) as generated by \( \eta \) and \( y \) with the commutation rules
\[ \eta^2 = -h_2 \eta y, \quad \eta y = y\eta - h_2 y^2. \] (15)

Note that in order to obtain the superplane \( A_{h_1} \) and its dual \( A_{h_2}^* \), we introduced above, two matrices \( g_{h_1} \) and \( g_{h_2} \). Of course, this result could be obtained by using a single matrix \( g = g_{h_1}g_{h_2} \). But in that case the required steps are rather complicated and tedious. However, in section 4, in order to obtain an R-matrix we shall take \( g = g_{h_1}g_{h_2} \).

We now consider the linear transformations
\[ T : A_{h_1} \to A_{h_1} \quad \text{and} \quad T : A_{h_2}^* \to A_{h_2}^*. \] (16)

Then, we define the corresponding \( (h_1, h_2) \)-deformation of the supergroup \( GL(1|1) \) as a quantum matrix supergroup \( GL_{h_1,h_2}(1|1) \) generated by \( a, \beta, \gamma, d \) which satisfy the following \( (h_1, h_2) \)-commutation relations
\[ a\beta = \beta a - h_2(a^2 - \beta \gamma - ad), \quad d\beta = \beta d + h_2(d^2 + \beta \gamma - da), \]

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\[ a\beta = \beta a - h_2(a^2 - \beta \gamma - ad), \quad d\beta = \beta d + h_2(d^2 + \beta \gamma - da), \]
\[ a\gamma = \gamma a + h_1(a^2 + \gamma \beta - ad), \quad d\gamma = \gamma d - h_1(d^2 - \gamma \beta - da), \]
\[ \beta^2 = h_2\beta(a - d), \quad \gamma^2 = h_1\gamma(d - a), \]
\[ \beta\gamma = -\gamma\beta + (h_1\beta - h_2\gamma)(d - a), \]
\[ ad = da + (h_1\beta + h_2\gamma)(a - d) - h_1h_2(a^2 - 2da + d^2) \]
provided that \( \beta \) and \( \gamma \) anticommute with \( \xi, \eta, h_1 \) and \( h_2 \), and
\[ h_1h_2 = -h_2h_1. \]  
(18)

One can see that when \( h_2 = 0 \), these relations go back to those of \( \text{GL}_h(1|1) \) in Ref. 13.

Alternatively, the relations (17) can be obtained using the following similarity transformation which was used first in [12]:
\[ T' = gTg^{-1} \]  
(19)
where in our case
\[ g = g_{h_1}g_{h_2}. \]  
(20)
To do this, we use the relations (3) and then take the limits \( p \to 1, q \to 1 \).

We denote by \( \mathcal{A}_{h_1,h_2} \) the algebra generated by the elements \( a, \beta, \gamma, d \) with the relations (17). The algebra \( \mathcal{A}_{h_1,h_2} \) is a (graded) Hopf algebra with the usual co-product
\[ \Delta(t^i_j) = t^i_k \otimes t^k_j \]  
(21)
(sum over repeated indices), co-unit
\[ \varepsilon(t^i_j) = \delta^i_j, \]  
(22)
and the antipode (co-inverse), which is the same as in [13],
\[ T^{-1} = \begin{pmatrix} a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1} & -a^{-1}\beta d^{-1} \\ -d^{-1}\gamma a^{-1} & d^{-1} + d^{-1}\gamma a^{-1}\beta d^{-1} \end{pmatrix}, \]  
(23)
provided that the formal inverses \( a^{-1} \) and \( d^{-1} \) exist.

The quantum superdeterminant of \( T \) is defined as, like that in the quantum supergroup \( \text{GL}_h(1|1) \),
\[ D_{h_1,h_2} = ad^{-1} - \beta d^{-1}\gamma d^{-1} \]  
(24)
which is independent of the relations (17). The equation
\[ ad^{-1} - \beta d^{-1}\gamma d^{-1} = d^{-1}a - d^{-1}\beta d^{-1}\gamma \]  
(25)
is also valid, however the proof is rather lengthly but straightforward. It can be checked using the relations
\[ d^{-1} \beta = \beta d^{-1} - h_2(1 - ad^{-1} + d^{-1} \beta \gamma d^{-1}), \]
\[ d^{-1} \gamma = \gamma d^{-1} + h_1(1 - ad^{-1} - d^{-1} \beta \gamma d^{-1}), \]
\[ ad^{-1} = d^{-1} a + h_1 d^{-1} \beta (1 - ad^{-1}) + h_2(1 - d^{-1} a) \gamma d^{-1}, \]
\[ \gamma d^{-1} \gamma = 0, \quad h_1 \beta d^{-1} \gamma \beta = -h_1 h_2 \beta \gamma (ad^{-1} - 1). \]

It can be also verified that \( D_{h_1, h_2} \) commutes with all matrix elements of \( T \) (and \( h_1, h_2 \)), that is, \( D_{h_1, h_2} \) belongs to the centre of the algebra \( TD = DT \).

Moreover, it can be checked that \( D_{h_1, h_2} \) has the multiplicative property
\[ \Delta(D_{h_1, h_2}) = D_{h_1, h_2} \otimes D_{h_1, h_2}. \]  
(26)

We close this section with the following note. If we set \( h_1 h_2 = 0 \) then only the last term in the last relation in (17) vanishes. Essentially, we can eliminate the factor \( h_1 h_2 \) from the last equation in (17). Indeed some algebra gives
\[ ad = da + h_1 \beta (a - d) + h_2 (a - d) \gamma. \]  
(27)

Therefore the factor \( h_1 h_2 \) does not appear in any of the relations in (17). Thus we can demand that \( h_1 h_2 = 0 \). In this situation, the relations (17) can be easily obtained from (19). This issue will be used in the following two sections.

4. R-matrix for \( \text{GL}_{h_1, h_2}(1|1) \)

We shall obtain an R-matrix for the quantum supergroup \( \text{GL}_{h_1, h_2}(1|1) \) from the R-matrix of \( \text{GL}_{p,q}(1|1) \). We know that the associative algebra (3) is equivalent to
\[ R_{p,q} T'_1 T'_2 = T'_2 T'_1 R_{p,q} \]  
[see equ.s (4)-(7)]. Now substituting (19) into (4) and defining the R-matrix \( R_{h_1, h_2} \) as
\[ R_{h_1, h_2} = \lim_{p \to 1} \lim_{q \to 1} (g \otimes g)^{-1} R_{p,q} (g \otimes g), \]  
(28)
where the matrix \( g \) is given by (20), we get the following R-matrix \( R_{h_1, h_2} \)
\[ R_{h_1, h_2} = \begin{pmatrix} 1 & -h_1 & h_2 & 0 \\ -h_1 & 1 & -h_1 h_2 & h_2 \\ h_1 & -h_1 h_2 & 1 & h_2 \\ 0 & h_1 & h_1 & 1 + h_1 h_2 \end{pmatrix} \]  
(29)
which gives the \((h_1, h_2)\)-deformed algebra of functions on \(\text{GL}_{h_1, h_2}(1|1)\) with the equation
\[
R_{h_1, h_2} T_1 T_2 = T_2 T_1 R_{h_1, h_2}.
\] (30)

Note that the above \(R\)-matrix for the special case \(h_2 = 0\) coincides with the one in Ref. 13. For the special case \(h_1 = 0 = h_2\), the above \(R\)-matrix becomes the unit matrix. Also \(\hat{R}_{h_1, h_2} = PR_{h_1, h_2}\), where \(P\) is the super permutation matrix, satisfies
\[
\hat{R}_{h_1, h_2}^2 = I,
\]
and thus it has two eigenvalues \(\pm 1\).

If we set \(h_1 h_2 = 0\) in (29) then the matrix \(R_{h_1, h_2}\) in (29) can be decomposed in the form
\[
R_{h_1, h_2} = R_{h_1} R_{h_2}
\] (31)
where
\[
R_{h_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -h_1 & 1 & 0 & 0 \\ h_1 & 0 & 1 & 0 \\ 0 & h_1 & h_1 & 1 \end{pmatrix}, \quad R_{h_2} = \begin{pmatrix} 1 & -h_2 & h_2 & 0 \\ 0 & 1 & 0 & h_2 \\ 0 & 0 & 1 & h_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\] (32)
Here the matrix \(R_{h_1}\) coincides with the \(R_h\) matrix of [13].

It can be checked that these matrices both satisfy the graded Yang-Baxter equation
\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \quad R \in \{ R_{h_1}, R_{h_2} \}
\] (33)
where
\[
(R_{12})^{abc}_{\text{def}} = R^{ab}_{\text{de}} \delta^c_f, \\
(R_{13})^{abc}_{\text{def}} = (-1)^{b(c+f)} R^{ac}_{\text{df}} \delta^b_e, \\
(R_{23})^{abc}_{\text{def}} = (-1)^{a(b+c+e+f)} R^{bc}_{\text{ef}} \delta^a_d.
\] (34)
Also, the matrix \(R_{h_2}\) obeys the ungraded Yang-Baxter equation.

If we set \(\hat{R} = PR\) then one can show that they satisfy the graded braid equation
\[
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}, \quad \hat{R} \in \{ \hat{R}_{h_1}, \hat{R}_{h_2} \}
\] (35)
where \(\hat{R}_{h_1} = PR_{h_1}\) and \(\hat{R}_{h_2} = PR_{h_2}\) with the grading again given by (34). Note that the matrix \(R_{h_2}\) does not satisfy the ungraded braid equation.
5. Differential Calculus on the Quantum Superplane

It is well known, through the work of Woronowicz [16], that quantum groups provide a concrete example of non-commutative differential geometry. Wess and Zumino [17] developed a differential calculus on the quantum hyperplane covariant with respect to quantum groups. They have shown that one can define a consistent differential calculus on the non-commutative space of the quantum hyperplane.

In this section we shall construct a covariant differential calculus on the quantum superplane. Before discussing the differential calculus, we note the following. Let us consider the following dual (exterior) superplane $\Lambda^q$ as generated by $\varphi'$ and $u'$ with the relations

$$\varphi'^2 = 0, \quad \varphi'u' + q^{-1}u'\varphi' = 0. \quad (36)$$

We define

$$\hat{V} = g_{h_2}^{-1}\hat{V}' \quad (37)$$

where the matrix $g_{h_2}$ is given by (13). Then we get the following relations

$$\varphi^2 = h_2\varphi u, \quad w\varphi + \varphi u = -h_2u^2 \quad (38)$$

under the assumption

$$h_2u = -uh_2. \quad (39)$$

These relations define a deformation of the (exterior) algebra of functions on the dual (exterior) superplane generated by $\varphi$ and $u$, and we denote it by $\Lambda_{h_2}$.

Now one can check that the transformations

$$T : A_{h_1} \rightarrow A_{h_1}, \quad T : \Lambda_{h_2} \rightarrow \Lambda_{h_2} \quad (40)$$

define a two-parameter deformation of the supergroup GL(1|1), that is, they give the $(h_1, h_2)$- commutation relations in (17).

We now pass to the differential calculus on the superplane. Consider the coordinates $x$ and $\xi$, belonging to the associative algebra $A_{h_1}$ which satisfy the commutation relations

$$U^i U^j = \left(\hat{R}_{h_1}\right)^{ij}_{kl} U^k U^l, \quad \hat{R}_{h_1} = PR_{h_1} \quad (41)$$

where $U = (x, \xi)^T$. These relations are equivalent to (11). Similarly, one can express the relations (38) in the form

$$\hat{V}^i \hat{V}^j = -\left(\hat{R}_{h_2}\right)^{ij}_{kl} \hat{V}^k \hat{V}^l. \quad (42)$$
Note that the relations (41) and (42) can be also expressed using the $R_{h_1,h_2}$ matrix in (29) provided it is permuted by the super permutation matrix $P$.

Denoting the partial derivatives with

$$\partial_i = \frac{\partial}{\partial U^i}, \quad \hat{\partial}_i = \frac{\partial}{\partial \hat{V}^i},$$

one arrives at

$$\partial_j U^i = \delta^i_j + \left( \hat{R}_{h_1,h_2} \right)^{ik}_{jl} U^l \partial_k$$

and

$$\hat{\partial}_j \hat{V}^i = \left( \hat{R}_{h_1,h_2} \right)^{ik}_{jl} \hat{V}^l \hat{\partial}_k.$$  

Then we set up the relations between the coordinates of $U$, $\hat{V}$ and their partial derivatives as follows: [Here, for simplicity we assumed that $h_1 h_2 = 0$]

$$\partial_x x = 1 + x \partial_x + h_1 x \partial_\xi - h_2 \xi \partial_x, \quad \partial_x \xi = \xi \partial_x - h_1 (x \partial_x + \xi \partial_\xi),$$

$$\partial_\xi x = x \partial_\xi + h_2 (x \partial_x + \xi \partial_\xi), \quad \partial_\xi \xi = 1 - \xi \partial_\xi - h_1 x \partial_\xi + h_2 \xi \partial_x,$$

and from (45)

$$\partial_\phi \phi = \phi \partial_\phi + h_1 \phi \partial_u - h_2 u \partial_\phi, \quad \partial_\phi u = u \partial_\phi - h_1 (\phi \partial_\phi + u \partial_u),$$

$$\partial_u \phi = \phi \partial_u + h_2 (\phi \partial_\phi + u \partial_u), \quad \partial_u u = - u \partial_u - h_1 \phi \partial_u + h_2 u \partial_\phi.$$  

The relations between the coordinates $U$ and $\hat{V}$ are

$$U^i \hat{V}^j = \left( \hat{R}_{h_1,h_2} \right)^{ij}_{kl} \hat{V}^k U^l,$$  

which read

$$x \phi = \phi x + h_2 (u x - \phi \xi), \quad x u = u x + h_1 \phi x + h_2 u \xi,$$

$$\xi \phi = \phi \xi - h_1 \phi x + h_2 u \xi, \quad \xi u = - u \xi - h_1 (\phi \xi + u x).$$

Finally one gets from the following equation

$$\partial_i \partial_j = \left( \hat{R}_{h_1,h_2} \right)^{ik}_{ji} \partial_k \partial_l$$

which are the commutation relations among the partial derivatives:

$$\partial_x \partial_\xi = \partial_\xi \partial_x - h_2 \partial_x^2,$$

$$\partial_\xi^2 = h_2 \partial_\xi \partial_x.$$  

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One of the interesting problems is to construct $U_{h_1,h_2}(gl(1|1))$. Work on this issue is in progress.

Acknowledgement

This work was supported in part by T. B. T. A. K. the Turkish Scientific and Technical Research Council.

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