Analysis of the triaxial, strongly deformed bands in odd-odd nucleus $^{164}$Lu with the tops-on-top model

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The top-on-top model with angular-momentum-dependent moments of inertia is extended to the tops-on-top model for an odd-odd nucleus, where one proton and one neutron in each single-$j$ orbital are coupled to the triaxial rotor. For a pure rotor case, an explicit algebraic formula for the triaxial, strongly deformed (TSD) band levels is given, and its stability problem is discussed. Both positive and negative parity TSD bands are well reproduced by taking account of attenuation factors in the Coriolis interaction and the proton–neutron interaction in the recoil term. Difference in quantum numbers between the yrast and yrare TSD bands is confirmed by direct estimation of spin alignments. The electromagnetic transition rates of $B(M1)$ are much reduced because of the different sign of g-factors in comparison with the odd-$A$ case, while $B(E2)$ are in the same order.

1. Introduction

It has been demonstrated that the top-on-top model with angular-momentum-dependent moments of inertia works quite well in describing triaxial, strongly deformed (TSD) bands in odd-$A$ nuclei of Lu isotopes $^{154,156,158}$Lu and $^{160,162}$Ta. It reproduces not only the excitation energies relative to the reference, i.e., $E^* - a I (I + 1)$ with $a = 0.0075$ MeV, but also the in-band and out-of-band electromagnetic transition rates among TSD band levels, i.e., $B(E2)_{\text{in}}, B(E2)_{\text{out}}, B(M1)_{\text{out}}$, and their ratios $B(E2)_{\text{out}}/B(E2)_{\text{in}}$ and $B(M1)_{\text{out}}/B(E2)_{\text{in}}$. Evolution of intrinsic structure, which is mainly due to the decrease of pairing gap, is simulated by a gradual increase of the core moments of inertia as a function of total angular momentum $I$, i.e., the angular-momentum-dependent moments of inertia.

In the algebraic treatment of the model Hamiltonian for odd-$A$ nuclei, Holstein–Primakoff (HP) transformation was applied to both $\mathbf{I}$ and the single-particle angular momentum $\mathbf{j}$, and the bosonized Hamiltonian $H_B$ was obtained by retaining up to the next-to-leading order in the HP boson expansion in a way consistent with $D_2$ invariance or Bohr symmetry [5]. Then a quantum state is realized as a function of $\mathbf{I}$ and $\mathbf{j}$ due to the wobbling motion of the triaxial rotor and the precession of $\mathbf{j}$ due to single-particle motion in an anisotropic field. Then, $H_B$ is brought to diagonal form in terms of quasibosons ($\hat{\alpha}, \hat{\beta}$), which are related to the original HP bosons ($\hat{a}, \hat{b}$) through the boson Bogoliubov transformation. Two kinds of number operators $\hat{n}_\alpha = \hat{\alpha}^+ \hat{\alpha}$ and $\hat{n}_\beta = \hat{\beta}^+ \hat{\beta}$ describe the precession of $\mathbf{R}$ and that of $\mathbf{j}$, respectively. It is noticeable that the algebraic solution approximates with reasonable accuracy the exact solution which is obtained...
by diagonalizing $H$ [1,2]. The top-on-top model includes the higher-order contribution from the HP boson expansion. The precession of the rotor is affected by the interaction with the valence nucleon through Coriolis terms, so that it deviates from the original picture of “the wobbling” as proposed by Bohr and Mottelson [6].

In the present paper, an extension of the top-on-top model is attempted to describe an odd-odd nucleus, where two single particles are coupled to the triaxial rotor with angular-momentum-dependent moments of inertia. We name this model the “tops-on-top” model. As an example, we apply this model to $^{164}$Lu where spins and parities have been assigned to three TSD bands [7,8]. In Sect. 2, we derive the algebraic formula for the triaxially deformed rotor Hamiltonian ($H_{\text{rot}}$), which is expressed in terms of the wobbling quantum number $n_{\alpha}$ and two precession quantum numbers $n_{\beta}$ and $n_{\gamma}$. In Sect. 3, the total Hamiltonian including the triaxially deformed single-particle potentials is numerically diagonalized to get the rotational levels, $B(E2)$ and $B(M1)$ values, and spin alignment for $^{164}$Lu. Section 4 provides the conclusion and presents some discussion.

2. Formalism

2.1. Tops-on-top model

We start from the Hamiltonian with two non-interacting particles, i.e., one proton in the $j_1$ orbital and one neutron in the $j_2$ orbital, coupled to the triaxially deformed core:

$$H = H_{\text{rot}} + H_{\text{sp}}(\vec{j}_1) + H_{\text{sp}}(\vec{j}_2),$$

with

$$H_{\text{rot}} = \sum_{k=x,y,z} A_k (I_k - j_1k - j_2k)^2,$$

$$H_{\text{sp}}(\vec{j}) = \frac{V_j}{j(j+1)} \left[ \cos \gamma (3j_x^2 - j_y^2) - \sqrt{3} \sin \gamma (j_x^2 - j_y^2) \right],$$

where $A_k = 1/(2J_k)$, and $\vec{j}$ stands for either $\vec{j}_1$ or $\vec{j}_2$. Here we remark that $H_{\text{sp}}(\vec{j})$ is derived from the deformed Nilsson potential [9] by applying the Wigner–Eckart theorem for the single-$j$ case. Similarly to the odd-$A$ case [1–4], we introduce the angular-momentum-dependent rigid-body moments of inertia:

$$J_k = \frac{J(I)}{1 + (\frac{3J}{16\pi})^{1/2} \beta_2} \left[ 1 - \left( \frac{5}{4\pi} \right)^{1/2} \beta_2 \cos \left( \gamma + \frac{2\pi k}{3} \right) \right], \quad J(I) = J_0 \frac{I - c_1}{I + c_2},$$

where $k = 1, 2, 3$ correspond to $x, y, z$, and $\beta_2$ and $\gamma$ are the deformation parameters. The maximum moment of inertia is about the $x$-axis and the relation $J_x \geq J_y \geq J_z$ holds in the range of $0 \leq \gamma \leq \pi/3$. Our formula is applicable to the high spin and highly excited region, i.e., the TSD band region. The parameters $c_1$ and $c_2$ are so chosen that $J(I)$ increases monotonically with increasing $I$, and goes to $J_0$ as $I \to \infty$. In other words, $c_1 + c_2$ must be positive as a coefficient in the expansion of $J(I)$ into a series of $1/I$.

The main reason for choosing rigid-body moments of inertia is as follows. Both rigid-body moments of inertia and the oscillator strength $\omega_k$ in the deformed Nilsson potential [9] are derived from the radius $R_k = R_0 [1 + \sqrt{5/(4\pi)} \beta_2 \cos (\gamma + 2\pi k/3)]$. As a consequence, the same physical situation for $J_k$ and $\omega_k$ is repeated periodically in $\gamma$ with a span of $2\pi/3$. In contrast, hydrodynamical moments of inertia change their roles in every span of $\pi/3$, and do not coincide with the behavior of $\omega_k$ even if $\gamma$ is changed to $-\gamma$. In addition, the energy level sequence of TSD bands in odd-$A$ nuclei cannot be reproduced in terms of hydrodynamical moments of inertia [3].
For the choice of $x$-axis as a quantization axis, a complete set of the $D_2$-invariant basis \cite{5} is given by

\[
\left\{ \sqrt{\frac{2I+1}{16\pi^2}} D^I_{MK}(\theta) \phi^{j_1}_{\Omega_1} \phi^{j_2}_{\Omega_2} + (-1)^{I-j_1-j_2} D^I_{M-K}(\theta) \phi^{-j_1}_{-\Omega_1} \phi^{-j_2}_{-\Omega_2} \right\};
\]

\[K - \Omega_1 - \Omega_2 = \text{even}, \quad \Omega_1 > 0, \]

where $K$, $\Omega_1$, and $\Omega_2$ denote eigenvalues of $I_x$, $j_{1x}$, and $j_{2x}$, respectively. The number of independent bases is given by $(2I+1)(2j_1+1)(2j_2+1)/4$. The wave functions $\phi^{j_1}_{\Omega_1}$ and $\phi^{j_2}_{\Omega_2}$ stand for the single-particle states, and $D^I_{MK}(\theta)$ is Wigner $D$-function. The magnitude $R$ of the rotor angular momentum $\vec{R} = \vec{I} - \vec{j}_1 - \vec{j}_2$ is restricted to $R = |I - j|, |I - j| + 1, \ldots, I + j - 1, I + j$, with $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$. If we define $R = I - j_1 - j_2 + n_\beta + n_\gamma$ with integers $n_\beta$ and $n_\gamma$, the two integers $n_\beta$ and $n_\gamma$ range as $n_\beta = 0, 1, 2, \ldots, 2j_1 - 1, 2j_1$ and $n_\gamma = 0, 1, 2, \ldots, 2j_2 - 1, 2j_2$, so that $R$ exhausts all the possible integral values defined above. Since $R_x$ runs from $R$ to $-R$, and $R_x = I_x - j_{1x} - j_{2x} = K - \Omega_1 - \Omega_2 = \text{even}$, a non-negative integer $n_a$ defined by the relation $R_x = R - n_a$ ranges as

\[n_a = 0, 2, 4, \ldots, 2R, \quad \text{for } R = \text{even},
\]

\[n_a = 1, 3, 5, \ldots, 2R - 1, \quad \text{for } R = \text{odd}.
\]

We note that each basis in Eq. (4) is an eigenstate of $H$ for the case of $V_{j_1} = V_{j_2} = 0$ and $\gamma = \pi/3$ ($A_y = A_z$).

We assume that the two particles occupy different orbitals or different isospins, so that $[j_{ik}, j_{k'}] = 0$ for $k$ or $k' = x, y, z$. The commutation relations satisfied among three angular momenta $\vec{I}$, $\vec{j}_1$, and $\vec{j}_2$ are $[\vec{I}, \vec{j}_1] = [\vec{I}, \vec{j}_2] = [\vec{j}_1, \vec{j}_2] = 0$, and $[I_x, I_{k'}] = -i \delta_{k'x}$ and $[j_{k}, j_{k'}] = i \delta_{kk'}$ for $\vec{j}_1$ or $\vec{j}_2$. These are realized in terms of three kinds of Holstein–Primakoff (HP) boson operators $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$, and $\hat{c}, \hat{c}^\dagger$ as follows:

\[I_+ = \hat{I}_+ = I_y + iI_z = -\hat{a}^\dagger \sqrt{2I - \hat{n}_a}, \quad I_x = I - \hat{n}_a,
\]

\[j_{1y} = j_{1y} = j_{1y} + ij_{1z} = \sqrt{2j_1 - \hat{n}_b} \hat{b}, \quad j_{1x} = j_1 - \hat{n}_b,
\]

\[j_{2y} = j_{2y} + ij_{2z} = \sqrt{2j_2 - \hat{n}_c} \hat{c}, \quad j_{2x} = j_2 - \hat{n}_c,
\]

where $\hat{n}_a = \hat{a}^\dagger \hat{a}$, $\hat{n}_b = \hat{b}^\dagger \hat{b}$, and $\hat{n}_c = \hat{c}^\dagger \hat{c}$. In applying these HP transformations to Eq. (1), we expand $\sqrt{2I - \hat{n}_a}$, $\sqrt{2j_1 - \hat{n}_b}$, and $\sqrt{2j_2 - \hat{n}_c}$ into series in $\hat{n}_a/(2I)$, $\hat{n}_b/(2j_1)$, and $\hat{n}_c/(2j_2)$, and retain up to the next-to-leading order, i.e., up to the second order in these quantities.

2.2. \textit{Approximate solution}

We consider the states of $I > j_1 + j_2$ for the case of $V_{j_1} = V_{j_2} = 0$. The diagonalization of $H_{\text{rot}}$ is achieved by the unitary transformation to quasibosons ($\hat{a}, \hat{b}, \hat{c}$):

\[
\begin{pmatrix}
\hat{a} \\
\hat{b} \\
\hat{c}
\end{pmatrix}
= K

\begin{pmatrix}
\hat{a} \\
\hat{b} \\
\hat{c}
\end{pmatrix}
+ M

\begin{pmatrix}
\hat{a}^\dagger \\
\hat{b}^\dagger \\
\hat{c}^\dagger
\end{pmatrix}
.
\]

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Two $3 \times 3$ matrices are given by

$$K = \begin{pmatrix}
    f_1 \eta_+ & -f_1 \eta_- & -f_2 \eta_-
n_1 & f_1 g_1 & f_1 g_2
    0 & -g_2 & g_1
\end{pmatrix}, \quad (8)$$

$$M = \begin{pmatrix}
    -f_1 \eta_- & f_1 \eta_+ & f_2 \eta_+
n_{12} & 0 & 0
    0 & 0 & 0
\end{pmatrix}, \quad (9)$$

where

$$f_1 = \sqrt{\frac{I}{I - j_1 - j_2}}, \quad f_1 = \sqrt{\frac{j_1}{I - j_1 - j_2}}, \quad f_2 = \sqrt{\frac{j_2}{I - j_1 - j_2}},$$

$$f_{12} = \sqrt{\frac{j_1 + j_2}{I - j_1 - j_2}}, \quad g_1 = \sqrt{\frac{j_1}{j_1 + j_2}}, \quad g_2 = \sqrt{\frac{j_2}{j_1 + j_2}}, \quad (10)$$

and

$$\eta_\pm = \left\{ \frac{1}{\text{sign}(q - p)} \right\}^{1/2} \left[ \frac{1}{2} \left( \frac{p + q}{2\sqrt{pq}} \pm 1 \right) \right]^{1/2}, \quad (11)$$

with $p \equiv A_y - A_x, q = A_z - A_x$. Applying an algebraic procedure employed in Ref. [1], we derive an approximate formula for the eigenvalue of $H_{rot}$:

$$E_{rot}(I, n_\alpha, n_\beta, n_\gamma) = A_x R(R + 1) - \frac{p^2}{2} n_\alpha^2 + \left( 2 R \sqrt{pq} + \sqrt{pq} - \frac{p + q}{2} \right) \left( n_\alpha + \frac{1}{2} \right),$$

$$R = I - j_1 - j_2 + n_\beta + n_\gamma, \quad (12)$$

where $n_\alpha, n_\beta,$ and $n_\gamma$ are the eigenvalues of number operators $\hat{n}_\alpha = \hat{\alpha}^\dagger \hat{\alpha}, \hat{n}_\beta = \hat{\beta}^\dagger \hat{\beta},$ and $\hat{n}_\gamma = \hat{\gamma}^\dagger \hat{\gamma},$ respectively. In the symmetric limit of $A_y = A_z$ where $\eta_+ = 1$ and $\eta_- = 0$, $E_{rot}$ takes the well-known form of $A_x R(R + 1) + (A_x - A_z)(R - n_\alpha)^2$. Therefore, $n_\alpha$ describes the precession of the rotor angular momentum $\vec{R} = \vec{I} - \vec{j}_1 - \vec{j}_2,$ i.e., wobbling quantum number [6], and $n_\beta + n_\gamma$ describes the precession of $\vec{j}_1 + \vec{j}_2.$ Even in the symmetric limit, $\hat{\alpha}, \hat{\beta},$ and $\hat{\gamma}$ do not coincide with $\hat{\alpha}, \hat{\beta},$ and $\hat{\gamma}$ respectively, but the relation $\hat{n}_\alpha - \hat{n}_\beta - \hat{n}_\gamma = \hat{n}_\alpha - \hat{n}_\beta - \hat{n}_\gamma$ holds. Thus, we have

$$R_x = K - \Omega_1 - \Omega_2 = I - j_1 - j_2 - n_\alpha + n_\beta + n_\gamma = I - j_1 - j_2 - n_\alpha + n_\beta + n_\gamma. \quad (13)$$

This fact implies that all the bases given by Eq. (4) are exhausted by changing integer values of $(n_\alpha, n_\beta, n_\gamma)$ under the condition of Eq. (5).

Since the rotor Hamiltonian $H_{rot}$ depends only on the angular momentum $\vec{R} (= \vec{I} - \vec{j}_1 - \vec{j}_2)$, physical states of a rotating body are specified by quantum numbers $R$ and $R_x$. Due to D$_2$ invariance, the states with $R_x = 0$ and $R = \text{odd}$ are excluded. Therefore, the spin of the band-head level $I_{bh}$ ($\geq j_1 + j_2$) is determined by $R_x = 0$ for band levels belonging to $R = \text{even}$, and by $R_x = 2$ for those belonging to $R = \text{odd}$, i.e.,

$$I_{bh} = \begin{cases}
    j_1 + j_2 + n_\alpha - n_\beta - n_\gamma & \text{for } R = \text{even}, \\
    j_1 + j_2 + n_\alpha - n_\beta - n_\gamma + 2 & \text{for } R = \text{odd}.
\end{cases} \quad (14)$$

From Eqs. (12) to (14), precession quantum numbers $n_\beta$ and $n_\gamma$ appear only in the combination of $n_p = n_\beta + n_\gamma$. Each rotational level is characterized by a set of quantum numbers $(I, n_\alpha, n_p)$, and the degeneracy of the level is given by $n_p + 1$. As for the state of $I = j_1 + j_2$, its energy level is
Fig. 1. Comparison of the energy levels of odd spin $I (13 \leq I \leq 31)$ between the boson model (solid lines) and the exact result (dashed lines). Quantum numbers and the degeneracy of levels are given by $(n_{\alpha}, n_p)_{n_p+1}$ below each rotational band. Angular momentum values are assigned to the lowest two levels and the highest level in the right-hand side of each band. The parameter set is given in the text.

Fig. 2. Comparison of the energy levels of even spin $I (14 \leq I \leq 32)$ between the boson model (solid lines) and the exact result (dashed lines). The definition of the symbols and the parameter set are as in Fig. 1.

well defined by Eq. (12), and reproduces the exact result with good accuracy, while the coefficients $f_1$, $f_2$, and $f_5$ diverge in the limit of $I \to j_1 + j_2$. Therefore, algebraic wave functions are not applicable to the electromagnetic transitions involving low-lying levels with $I = j_1 + j_2$.

In order to examine the accuracy of the boson model formula given by Eq. (12), we compare the energy eigenvalues predicted by this formula with those obtained from the exact diagonalization of $H_{\text{tot}}$ for the set of parameters $J_0 = 82.0 \text{ MeV}^{-1}$, $c_1 = c_2 = 0$, $\beta_2 = 0.38$, $\gamma = 17^\circ$, and $j_1 = j_2 = 13/2$. In both Figs. 1 (odd spin) and 2 (even spin), boson model levels are denoted by solid lines, while exact results by dashed lines. In both figures, the quantum numbers and the degeneracy are denoted by $(n_{\alpha}, n_p)_{n_p+1}$ at the bottom. The spin values of the levels are assigned to two low-lying levels and the highest level in the right-hand side of each rotational band. We see that the coincidence between the boson model levels and the exact ones is remarkable, and it becomes more accurate with increasing $I$. 
Here, we consider the stability of the wobbling motion in the rotor Hamiltonian $H_{\text{rot}}$, which is described only by the rotor angular momentum $\tilde{R}$ satisfying the commutation relation $[R_k, R'_k] = -iR_k \times R'_k$. We apply the HP transformation to $\tilde{R}$,

$$
R_+ = R_+^\dagger = R_y + iR_z = -\hat{d}^\dagger \sqrt{2R - \hat{n}_d} \simeq -\sqrt{2R} \left( \hat{d}^\dagger - \frac{\hat{d}^\dagger \hat{d} \hat{d} \hat{d}}{2R} \right).
$$

$$
R_x = R - \hat{n}_d, \quad \text{with} \quad \hat{n}_d = \hat{d}^\dagger \hat{d}.
$$

(15)

Then, in the lowest-order approximation we get

$$
H_{\text{rot}} = \sum_{k=x,y,z} A_k R_k^2 \simeq A_x R (R + 1)
+ \frac{R}{2} \left( \hat{d}^\dagger \hat{d} \right) \begin{pmatrix} A_y + A_z - 2A_x & A_y - A_z \\
-(A_y - A_z) & -(A_y + A_z - 2A_x) - \omega \end{pmatrix} \left( \hat{d}^\dagger \hat{d} \right).
$$

(16)

Diagonalization of the second term is equivalent to finding the eigenvalues $\omega$ of the $2 \times 2$ matrix. Taking account of the metric arising from the boson commutator, we get the eigenvalue equation:

$$
\begin{vmatrix}
A_y + A_z - 2A_x - \omega & A_y - A_z \\
-(A_y - A_z) & -(A_y + A_z - 2A_x) - \omega
\end{vmatrix} = \omega^2 - 4(A_y - A_x)(A_z - A_x) = 0.
$$

(17)

If $A_x$ is the largest or the smallest, the eigenvalue $\omega$ is real. However, if $A_x$ is an intermediate value between $A_y$ and $A_z$, $\omega^2$ becomes negative. Thus, we conclude that wobbling motion of $\tilde{R}$ about the principal axis possessing intermediate value of the moment of inertia (intermediate axis) does not exist. This conclusion is supported by the well-known phenomenon in classical mechanics [10] that the rotational axis cannot be confined to move around the intermediate axis. Since the HP transformation in Eq. (15) generates a small deviation from the rotational state, $\omega^2 < 0$ implies instability of the rotation about the intermediate axis without any additional stabilization mechanism. As for the TSD bands in odd-$A$ nuclei, the same stability problem was studied with inclusion of the single-particle potentials [1].

### 2.3. Approximate transition rates

Now, let us turn our attention to the dominant linking transition rates among the TSD bands. In the algebraic treatment, eigenstates of the bosonized Hamiltonian $H_B$ are expressed in terms of quasiboson numbers $n_\alpha, n_\beta$, and $n_\gamma$ together with $I_1$, $j_1$, and $j_2$. We need the transformation coefficients between two boson Fock spaces, i.e., the one which is generated on the quasivacuum $|0\rangle_\alpha$ for quasibosons ($\hat{a}, \hat{b}, \hat{c}$) and the other on the original vacuum $|0\rangle_\alpha$ for HP bosons ($\hat{a}, \hat{b}, \hat{c}$). We consider the overlap between $|n_\alpha n_\beta n_\gamma, I_1 j_1 j_2\rangle = (n_\alpha n_\beta n_\gamma)^{-1/2} \langle \hat{a}^\dagger \hat{b}^\dagger \hat{c}^\dagger \rangle_{n_\alpha} \langle \hat{\beta}^\dagger \hat{\gamma}^\dagger \rangle_{n_\beta} |0\rangle_\alpha$ and $|n_\alpha n_\beta n_\gamma, I_1 j_1 j_2\rangle = (n_\alpha n_\beta n_\gamma)^{-1/2} \langle \hat{\alpha}^\dagger \hat{\beta}^\dagger \hat{\gamma}^\dagger \rangle_{n_\alpha} \langle \hat{\alpha}^\dagger \hat{\beta}^\dagger \hat{\gamma}^\dagger \rangle_{n_\beta} |0\rangle_\alpha$:

$$
G_{n_\alpha n_\beta n_\gamma: n_\alpha n_\beta n_\gamma} = \frac{a(0) \hat{\alpha}^\dagger \hat{\beta}^\dagger \hat{\gamma}^\dagger \langle \hat{\alpha}^\dagger \hat{\beta}^\dagger \hat{\gamma}^\dagger \rangle_{n_\alpha} \langle \hat{\beta}^\dagger \hat{\gamma}^\dagger \rangle_{n_\beta} |0\rangle_\alpha}{N^{1/2}}
= \frac{a(0) |0\rangle_\alpha}{N^{1/2}} a(0) \hat{\alpha}^\dagger \hat{\beta}^\dagger \hat{\gamma}^\dagger \langle \hat{\alpha}^\dagger \hat{\beta}^\dagger \hat{\gamma}^\dagger \rangle_{n_\alpha} \langle \hat{\beta}^\dagger \hat{\gamma}^\dagger \rangle_{n_\beta} |0\rangle_\alpha.
$$

(18)

with $N = n_\alpha n_\beta n_\gamma n_\alpha n_\beta n_\gamma$ and $(\hat{\alpha}^\dagger) \equiv 1/\langle 0|a(0)|0\rangle_\alpha$. In this expression $n_\alpha = I - K$, $n_\beta = j_1 - \Omega_1$, and $n_\gamma = j_2 - \Omega_2$ stand for the eigenvalues of $\hat{n}_\alpha, \hat{n}_\beta$, and $\hat{n}_\gamma$, respectively. Such a set of the coefficients in Eq. (18) is calculated by applying the extended form of the generalized Wick
Theorem [1,11]. In the lowest-order approximation with $V_{j_1} = V_{j_2} = 0$ [3,4], $G_{000:000}$ and $G_{100:100}$ become
\[ G_{000:000} = \left( \frac{I - j_1 - j_2}{I} \right)^{1/2} \frac{1}{\eta_{1+}^{1/2}}, \quad G_{100:100} = \frac{I - j_1 - j_2}{I} \frac{1}{\eta_{1+}^{3/2}}. \] (19)

As is shown in the previous papers [1–4], the most dominant transitions from the band with quantum numbers $(n_\alpha, n_\beta, n_\gamma) = (1, 0, 0)$ to the band with $(0,0,0)$ are described in terms of $G_{000:000}$ and $G_{100:100}$ as
\[ B(E2)_{\text{out}} = B(E2; I, 100 \rightarrow I - 1, 000) \simeq \frac{5\varepsilon^2}{16\pi} \frac{3}{I} (Q_0' G_{000:000} G_{100:100})^2 \]
\[ \simeq \frac{5\varepsilon^2}{16\pi} \frac{3}{I} \left( \frac{I - j_1 - j_2}{I} \right)^3 \frac{1}{\eta_{1+}^3}, \]
\[ B(M1)_{\text{out}} = B(M1; I, 100 \rightarrow I - 1, 000) \simeq \frac{3\mu_N^2}{4\pi} \frac{(j_1 g_1^{\text{eff}} + j_2 g_2^{\text{eff}})^2}{I} (G_{000:000} G_{100:100})^2 \]
\[ \simeq \frac{3\mu_N^2}{4\pi} \frac{(j_1 g_1^{\text{eff}} + j_2 g_2^{\text{eff}})^2}{I} \left( \frac{I - j_1 - j_2}{I} \right)^3 \frac{1}{\eta_{1+}^4}, \] (20)

where $Q_0' = -Q_0 (1 + 3/\tan \gamma)/2$ and $g_2^{\text{eff}} = g_{\ell \ell} - g_R + a_{jk} (g_{sk} - g_{sk})$. Here, $Q_0$ is the intrinsic quadrupole moment, $g_\ell$ the orbital g-factor, $g_s$ the spin g-factor, $g_R$ the effective g-factor for the rotational motion. The value of $a_{jk}$ is $1/(2\sqrt{2})$ for $jk = \ell \ell + 1/2$ and $-1/(2\sqrt{2})$ for $jk = \ell \ell - 1/2$, and $k = 1$ stands for a valence proton and 2 for a valence neutron.

In the odd-A case these transition rates are expressed in terms of a factor $(G_{000:000} G_{100:100})^2 \propto [(I - j)/I]^3$ [3,4]. The value of $j_1 + j_2$ is almost 13 due to the alignment of two particles both in the $i_{13/2}$ orbital for an odd-odd nucleus, while the value of $j$ in $i_{13/2}$ is 6.5 for an odd-A nucleus. Then, the factor $(I - j_1 - j_2)^3$ is smaller for the odd-odd case than the odd-A case, so that it makes the observation of the other partner of the TSD band in an odd-odd nucleus difficult [4]. This conclusion is valid only when the values of $Q_0$, $\eta_{1+}^2$, and $j_1 g_1^{\text{eff}} + j_2 g_2^{\text{eff}}$ do not much differ from those in odd-A. Taking the ratios of $B(E2)_{\text{out}}$ and $B(M1)_{\text{out}}$ to $B(E2)_{\text{in}} = B(E2; I, 100 \rightarrow I - 2, 100) \simeq 5\varepsilon^2/(16\pi) (Q_0')^2 G_{100:100}^4$ with $Q_0' = Q_0 (\sqrt{3} - \tan \gamma)/(2\sqrt{2})$, we get
\[ \frac{B(E2)_{\text{out}}}{B(E2)_{\text{in}}} \simeq \frac{6}{I} \tan^2 \left( \frac{\gamma + \pi}{6} \right) \left( \frac{G_{000:000}}{G_{100:100}} \right)^2 \sim \frac{6\eta_{1+}^2}{I - j_1 - j_2} \tan^2 \left( \frac{\gamma + \pi}{6} \right), \]
\[ \frac{B(M1)_{\text{out}}}{B(E2)_{\text{in}}} \simeq \frac{3}{5} \left[ \frac{2(j_1 g_1^{\text{eff}} + j_2 g_2^{\text{eff}})}{Q_0'} \right]^2 \frac{1}{I} \left( \frac{G_{000:000}}{G_{100:100}} \right)^2 \left( \frac{\mu_N}{e} \right)^2 \]
\[ \sim \frac{3}{5} \left[ \frac{2(j_1 g_1^{\text{eff}} + j_2 g_2^{\text{eff}})}{Q_0'} \right]^2 \frac{\eta_{1+}^2}{I - j_1 - j_2} \left( \frac{\mu_N}{e} \right)^2. \] (21)

Due to the factor $1/(I - j_1 - j_2)$ these ratios are more or less enhanced in comparison with the odd-A case (see Eq. (60) in Ref. [1]), as long as $Q_0'$, $\eta_{1+}^2$, and $j_1 g_1^{\text{eff}} + j_2 g_2^{\text{eff}}$ do not much differ from those in the odd-A case.

Now our interest is in the total Hamiltonian $H$ given by Eq. (1) with single-particle potentials. Therefore, a general 6 × 6 linear boson transformation connecting two sets of boson operators $(\hat{a}^\dagger, \hat{b}^\dagger, \hat{c}^\dagger, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and $(\hat{a}^\dagger, \hat{b}^\dagger, \hat{c}^\dagger, \hat{\alpha}, \hat{\beta}, \hat{\gamma})$ must be applied to the Hamiltonian, which is already expanded by taking into account the next-to-leading order. In addition, it is a hard task to solve the eigenvalue problem for a 6 × 6 matrix as an extension of the 4 × 4 matrix given by Eq. (23) in Ref. [1]. Therefore, we directly diagonalize $H$ within a given model space in the next section.
3. Application to $^{164}$Lu

3.1. Interaction strengths

We perform the exact diagonalization of $H$ in Eq. (1) by using Lanczos method [12]. Two TSD bands with positive parity (TSD3 and TSD2) and one TSD band with negative parity (TSD1) are identified in $^{164}$Lu [8]. In the beginning, TSD2 was assumed to be of negative parity [7] (hence its name), but it was later confirmed to be of positive parity [8].

We refer to the level scheme in Ref. [8] where the band-head level of $I = 13^+$ in the TSD3 band is connected by the linking transition to the $12^+$ level going down to the $8^+$ level. Therefore, we estimate the excitation energy of the $13^+$ level in the TSD3 band as 2.014 MeV relative to the latter $8^+$ level. In Figs. 4–7, $E^*$ in the ordinates stands for the excitation energy relative to this $8^+$ level. All the theoretical excitation energies are adjusted only by a common constant which shifts the $13^+$ level energy in the TSD3 band to the experimental one.

The proton single-particle orbital is $i_{13/2}$, the same as in the other odd-$A$ Lu isotopes, but the neutron single-particle orbital is not yet definite. References [7,8] suggest that the neutron orbital is also $i_{13/2}$ for the TSD3 band, as TSD3 starts from $13^+$ state, and the $h_{9/2}$ orbital for the TSD1 band, though TSD1 starts from the $14^−$ state. We adopt the same deformation of $β_2 = 0.38$ and $γ = 17^\circ$ as in the neighboring isotope $^{163}$Lu for all the orbitals ($πi_{13/2}, νi_{13/2}, νf_{15/2}$, and $νh_{9/2}$) which will be employed in the subsequent analysis.

As for the strength of single-particle potential $V_j$, we consult with the oscillator strength of the original deformed Nilsson potential, i.e., $V_j = 0.95β_2h\omega_0(r^2)β_2,γ/6$, where an average value $(r^2)β_2,γ$ is calculated from the radial part of the single-particle wave function for a given deformation of $β_2$ and $γ$. Assuming $(r^2)β_2,γ$ by the spherical value $N + 3/2$, we estimate $V_j$ as 2.9 MeV for $N = 5$, 3.4 MeV for $N = 6$, and 3.8 MeV for $N = 7$. For odd-$A$ cases, we took somewhat smaller values, $V_j = 2.3$ MeV in Ref. [1,2] and 2.6 MeV in Ref. [3], which do not differ much from the above estimation. In the present paper we assume $V_{i_{13/2}} = V_{j_{2}} = 2.3$ MeV, similar to the odd-$A$ case.

Since we have tried to use a common set of parameters for both positive and negative parity bands [4], we compare two cases, i.e., Case 1 where we assume the $vi_{13/2}$ orbital for TSD3 and $vf_{15/2}$ for TSD1 bands, and Case 2 with $vh_{9/2}$ for TSD3 and $vh_{9/2}$ for TSD1.

In analysing the zigzag bands in odd-$N$ nuclei based on the particle-rotor model, the attenuation factor should be introduced in the Coriolis interaction [13,14]. Based on the renormalization group method, we have shown that the effect of truncating the basic Fock space must be compensated by scaling the coupling constant, and the attenuation factor in the Coriolis interaction is a typical example [15]. In addition, we have to consider the possibility that $vi_{13/2}$ and $vh_{9/2}$ are partially filled in this deformation with $N = 93$. In the calculation assuming one neutron in a single-$j$ shell, the dominant contribution to $E^* - aI(I + 1)$ comes from the lowest level with $j_κ \sim j$. In contrast, in the calculation assuming a partially filled single-$j$ shell, wherein an unpaired valence neutron is above the Fermi level, its contribution to the Coriolis term becomes smaller due to smaller alignment of $j_κ$. Thus we need an additional reduction factor in the Coriolis term. In such a case our attenuation factor for the Coriolis term includes this reduction, too. We adopt the attenuation factor of $r_1 = 0.55$ for Case 1 and $r_1 = 0.65$ for Case 2 in the Coriolis term of $−\sum_{k=x,y,z} I_k(j_k + j_κ)/J_k$.

There is another problem in Case 1, because both proton and neutron are confined to the same $i_{13/2}$ orbital. Then, the interaction between the proton and neutron becomes important, although our model Hamiltonian has no such correlation as an additional interaction. We need to introduce another attenuation factor $r_2 = 0.55$ to the proton–neutron interaction $\sum_{k=x,y,z} j_1k j_κ/J_k$ in the recoil term.
Employing the parameter set of Case 1, we calculate energy eigenvalue $E_I$ for positive parity TSD3 and TSD2 bands in $^{164}$Lu. Theoretical values with both attenuations are shown by filled squares connected by solid lines; those with attenuation factor only in the Coriolis term by open triangles connected by solid lines; and those without attenuation by open squares connected by solid lines. The vertical axis is in units of MeV. The proton orbital is $i_{13/2}$ and the neutron orbital $i_{13/2}$. The parameter set is Case 1 (see the text).

In summary, the parameter set in Case 1 is given by $V_{j1} = V_{j2} = 2.3$ MeV, $J_0 = 82.0$ MeV$^{-1}$, $c_1 = 8.0$, and $c_2 = 41.0$, together with the attenuation factor of $r_1 = r_2 = 0.55$ in $-r_1 \sum_{k=x,y,z} I_k (j_{1k} + j_{2k})/J_k$ and $r_2 \sum_{k=x,y,z} j_{1k} j_{2k}/J_k$. Configuration of the positive parity band (TSD3 and TSD2) is assumed to be $\pi_{i_{13/2}} \otimes v_{j_{15/2}}$, and of the negative parity band (TSD1 and TSD1-2) $\pi_{i_{13/2}} \otimes v_{j_{15/2}}$. The parameter set in Case 2 is given by $V_{j1} = V_{j2} = 2.3$ MeV, $J_0 = 87.7$ MeV$^{-1}$, $c_1 = 8.5$, and $c_2 = 47.3$, together with the attenuation factor of $r_1 = 0.65$ only in $-r_1 \sum_{k=x,y,z} I_k (j_{1k} + j_{2k})/J_k$ (i.e. $r_2 = 1.0$). Configuration of the positive parity band is assumed to be $\pi_{i_{13/2}} \otimes v_{j_{9/2}}$ and of the negative parity band $\pi_{i_{13/2}} \otimes v_{j_{9/2}}$.

In order to study the effect of the attenuation factors, we show in Fig. 3 three cases for the positive parity TSD bands with the Case 1 parameter set: (i) with both attenuation factors ($r_1 = r_2 = 0.55$), denoted by filled squares; (ii) only with the Coriolis attenuation factor ($r_1 = 0.55$, $r_2 = 1.0$), denoted by open triangles; and (iii) without attenuation factors ($r_1 = r_2 = 1.0$), denoted by open squares. We observe that the difference between cases (i) and (iii) is quite large, while that between (i) and (ii) is small. This indicates that the attenuation effect is more important in the Coriolis term than in the p-n interaction in the recoil term. This situation is seen from a rough estimation of the ratio between the Coriolis term and recoil term as follows. Since $I_x \sim I$ and $j_{1x} \sim j_{2x} \sim j$, we have $A_x I_x (j_{1x} + j_{2x})/(A_x j_{1x} j_{2x}) \sim 2I/j$, which increases with increasing $I$. Thus, the attenuation in the Coriolis term plays a dominant role in controlling the TSD energy.

### 3.2. Energy scheme of TSD bands

Employing the parameter set of Case 1, we calculate energy eigenvalue $E_I$ and show $E^* - aI(I+1)$ with $a = 0.0075$ for TSD3 and TSD2 in Fig. 4, and for TSD1 in Fig. 5. Only the band-head energy of TSD3 is adjusted to experimental value. While $E^* - aI(I+1)$ at $I$ around the ending of the TSD2 band are not in good agreement with experimental data, theoretical values reproduce both the TSD3 band with $(n_\alpha = 0, n_\beta = 0, n_\gamma = 0)$ and the TSD2 band with $(1,0,0)$ quite well over all regions. The assignment of $(n_\alpha, n_\beta, n_\gamma)$ to the TSD bands is inferred from the calculated alignments in the next subsection.
Fig. 4. Comparison between the experimental and the theoretical energy levels $E^* - aI(I+1)$ as functions of angular momentum $I$ for positive parity TSD3 and TSD2 bands in $^{164}$Lu. The vertical axis is in units of MeV. Theoretical values are shown as filled squares connected by solid lines, with experimental values as open triangles connected by solid lines. The proton orbital is $i_{13/2}$ and the neutron orbital $i_{13/2}$. The parameter set is Case 1 (see the text). The experimental data are from Ref. [8].

The $20^+$ level in TSD2 decays to four $18^+$ levels [8]. According to Ref. [8], we choose the $18^+$ level 0.566 MeV down from the $20^+$ level in Fig. 4. However, if we take the $18^+$ level 0.536 MeV down from $20^+$, which is originally chosen as a member of the TSD2 band by Ref. [7], the experimental band-head energy in Fig. 4 goes up by 0.029 MeV, in better agreement with the theoretical band-head energy of TSD2. With the same parameter set of Case 1, we show theoretical results for the negative parity band of TSD1 with $(0,0,0)$ and its partner band TSD1-2 with $(1,0,0)$ in Fig. 5. There is no experimental data for TSD1-2 band levels with odd $I$ and negative parity. Instead, Ref. [8] identifies the other sequence of levels along the band X2 which forks from TSD1 at $42^-$.

Similarly, we show the theoretical level scheme with Case 2 in Fig. 6 ($\nu g_{9/2}$) for TSD3 and TSD2, and Fig. 7 ($\nu h_{9/2}$) for TSD1. Again, only the band-head energy of TSD3 is adjusted to the experimental level of $13^+$. The agreement of the theoretical levels for TSD3 and TSD2 with the experimental data seems to be better than for Case 1. However, as seen in Fig. 7, the band with odd spin sequence 19, 21, ... (TSD1-2) appears lower than the one with even spin sequence 16, 18, ... (TSD1). As
Fig. 6. Comparison between the experimental and the theoretical energy levels \( E^* - aI(I + 1) \) as functions of angular momentum \( I \) for positive parity TSD3 and its partner TSD2 bands in \(^{164}\text{Lu}\). The proton orbital is \( i_{13/2} \) and the neutron orbital is \( g_{9/2} \). The parameter set is Case 2. The experimental data are from Ref. [8].

Fig. 7. Comparison between the experimental and the theoretical energy levels \( E^* - aI(I + 1) \) as functions of angular momentum \( I \) for negative parity TSD1 and its partner TSD1-2 bands in \(^{164}\text{Lu}\). The proton orbital is \( i_{13/2} \) and the neutron orbital is \( h_{9/2} \). The parameter set is Case 2. The experimental data are from Ref. [8].

\[ I - j_1 - j_2 = I - 11, \] the odd spin sequence of TSD1-2 has precession quantum numbers (0,0,0) and the even spin sequence of TSD1 has (1,0,0) because of the \( D_2 \) symmetry—see Eq. (5). The experimental levels (open triangles) belong to TSD1. If the band-head energy of TSD1 is shifted to the experimental level of \( I = 16^- \) in Fig. 7, the theoretical curve for TSD1 agrees quite well with the experimental level sequence continued to the X2 band. If we proceed with the calculation for the configuration \( \pi i_{13/2} \otimes \nu j_{15/2} \) with the parameter set of Case 2, the gradient of \( E^* - aI(I + 1) \) turns out to be much steeper than the experimental curve. Since no lower band like TSD1-2 is observed experimentally, and because we want to describe both positive and negative parity bands with a common set of parameters, \( \nu h_{9/2} \) is not a good candidate for the TSD1 band. From now on we concentrate on the theoretical results based on Case 1.

3.3. Alignment of spins
In order to clarify the character of each band, we investigate the alignment of spins in the TSD bands. In Fig. 8 (positive parity TSD2 and TSD3 bands) and Fig. 9 (negative parity TSD1 and TSD1-2 bands), we show the calculated alignment of \( \langle I_{1x}^2 \rangle^{1/2} \), \( \langle R_{1x}^2 \rangle^{1/2} \), and \( \langle j_{kx}^2 \rangle^{1/2} \) for \( k = 1(\pi), 2(\nu) \). In Fig. 8, as both proton and neutron occupy the same \( i_{13/2} \) orbital, so that \( \langle j_{1x}^2 \rangle^{1/2} = \langle j_{2x}^2 \rangle^{1/2} \), we
Fig. 8. The alignments of $I_x$, $R_x$, and $j_{1x}$ for two positive parity TSD bands as functions of $I$. Solid lines correspond to TSD3, dashed lines to TSD2. The alignment of $j_{1x}$ comes from the $\pi_{13/2}$ orbital. As for $j_{2x}$, see the text.

Fig. 9. The alignments of $I_x$, $R_x$, $j_{1x}$, and $j_{2x}$ for two negative parity TSD bands as functions of $I$. Solid lines correspond to TSD1, dashed lines to TSD1-2. The alignment of $j_{2x}$ comes from the $\nu_{15/2}$ orbital, and $j_{1x}$ from the $\pi_{13/2}$ orbital.

show only $\langle j_{1x}^2 \rangle^{1/2}$. Here, the state $|\rangle$ stands for the eigenstate of $H$ belonging to the eigenvalue $E_I$. Comparing solid lines (TSD3) with dashed lines (TSD2) for $\langle I_x^2 \rangle^{1/2}$ and $\langle R_x^2 \rangle^{1/2}$, we recognize almost one unit difference between TSD3 and TSD2 independent of $I$. This confirms the one-unit difference in $n_\alpha$ between the yrare and yrast TSD bands. As seen in Fig. 8, the fact that $\langle j_{1x}^2 \rangle^{1/2} = \langle j_{2x}^2 \rangle^{1/2} \sim 13/2$ for both the TSD3 and TSD2 bands demonstrates full alignment of $\vec{j}_1$ and $\vec{j}_2$ to the $x$ direction, indicating $n_\beta = n_\gamma = 0$ for both yrast and yrare TSD bands. Similarly, in Fig. 9, almost one unit difference is also seen between TSD1 and TSD1-2 in $\langle I_x^2 \rangle^{1/2}$ and $\langle R_x^2 \rangle^{1/2}$. Calculated alignments give $\langle j_{2x}^2 \rangle^{1/2} \sim 15/2$ and $\langle j_{1x}^2 \rangle^{1/2} \sim 13/2$. This demonstrates that spins of single particles in $\nu_{15/2}$ and $\pi_{13/2}$ orbitals are fully aligned. Both Figs. 8 and 9 demonstrate that the yrast TSD (TSD3 and TSD1) band has quantum number $(0, 0, 0)$ and the yrare TSD (TSD2 and TSD1-2) band $(1, 0, 0)$. 


Fig. 10. Angular-momentum dependence of triaxial moments of inertia. The parameter set is given in the text.

Fig. 11. Comparison of kinematic moments of inertia $J^{(1)}$ with experimental ones for positive parity TSD3 and TSD2 bands as functions of $I$. Solid lines correspond to theoretical values, dashed lines to experimental ones [8]. Both theoretical and experimental values are deduced from energy $E_I$.

3.4. Moments of inertia

In Fig. 10, we show the angular-momentum dependence of triaxial moments of inertia calculated from Eq. (3) with parameters in Case 1. They are smooth functions of $I$ showing upward convexity. Kinematic and dynamic moments of inertia, $J^{(1)}$ and $J^{(2)}$, which are developed for the nucleus with axially symmetric deformation [16], cannot be directly related to the inertial tensor for a triaxially deformed nucleus. However, their detailed behavior assessed from experimental data can be available for the test of theoretical band levels $E_I$, since they are defined in terms of the first and the second differences of $E_I$ with respect to $I$, i.e.,

$$J^{(1)} = \frac{2I - 1}{E_I - E_{I-2}}, \quad J^{(2)} = \frac{4}{E_{I-2} + E_{I+2} - 2E_I}. \quad (22)$$

Theoretical $J^{(1)}$ and $J^{(2)}$ for positive parity bands are compared with experimental ones in Figs. 11 and 12, while theoretical ones for negative parity bands are compared with experimental ones in Figs. 13 and 14, respectively. Throughout Figs. 11–14, the solid lines correspond to theoretical values and the dashed lines to experimental ones. The difference between the yrast (TSD3 and TSD1) and yrare (TSD2 and TSD1-2) bands is very small both in theoretical $J^{(1)}$ and $J^{(2)}$. For the case of $V_1 = V_2 = 0$, this fact is confirmed by employing the energy expression given by Eq. (12), if the
Fig. 12. Comparison of dynamic moments of inertia $J^{(2)}$ with experimental ones for positive parity TSD3 and TSD2 bands as functions of $I$. Solid lines correspond to theoretical values, dashed lines to experimental ones [8]. Both theoretical and experimental values are deduced from energy $E_I$.

Fig. 13. Comparison of kinematic moments of inertia $J^{(1)}$ with experimental ones for negative parity TSD1 and TSD1-2 bands as functions of $I$. Solid lines correspond to theoretical values, dashed lines to experimental ones [8] which include the X2 band from $42^−$. Both theoretical and experimental values are deduced from energy $E_I$.

$I$-dependence of the moments of inertia is disregarded for simplicity:

$$J^{(1)}_{\text{yrast}} - J^{(1)}_{\text{yrare}} = \frac{pq(I + 1/2)}{8A^2_x(I + 1/2 - j_1 - j_2 + \sqrt{pq/(2A_x)})(I + 1/2 - j_1 - j_2 + 3\sqrt{pq/(4A_x)})},$$

$$J^{(2)}_{\text{yrast}} - J^{(2)}_{\text{yrare}} = 0. \tag{23}$$

As can be seen in Figs. 11 and 13, the tops-on-top model reproduces the kinematic moments of inertia $J^{(1)}$ quite well. However, as seen in Figs. 12 and 14, this model is not enough to explain the detailed behavior of the dynamic moment of inertia $J^{(2)}$, though it reproduces the average value, especially in the middle of the band. In Fig. 14, the divergence of the experimental value of $J^{(2)}$ around $42^−$ comes from the band crossing between TSD1 and X2. Such a band crossing is out of the scope of the present tops-on-top model, but should be an object of a self-consistent microscopic many-$j$ approach. Similarly, the complicated behavior of the experimental $J^{(2)}$ seen in Fig. 12 suggests strong correlations of TSD2 with other bands which are not yet observed. We note that, if we choose the $18^+$ level 0.536 MeV down from the $20^+$ level [7] instead of 0.566 MeV [8], $J^{(2)}$ becomes 63.5 MeV$^{-1}$ at $20^+$ and the abnormal behavior in the starting point of $J^{(2)}$ disappears in Fig. 12.
and compared with experimental data [17]. It can be seen that the theoretical transition rates agree with experimental ones within the error bars.

Next, we discuss the electromagnetic transitions. All the theoretical results in Table 1 are obtained by the exact diagonalization of the total Hamiltonian. We employ bare values of g-factors, i.e., $g_{\ell_1} = 1$, $g_{\ell_2} = 0$, $g_{s_1} = 3.906$, $g_{s_2} = -2.678$, and $g_R = Z/A$. As for $Q_0$, we employ $7b$ [8]. In addition, we take into account the same quenching factor 0.5 for all g values, as adopted in $^{163}\text{Lu}$ [17]. In Table 1, we compare $B(E2)_{\text{out}}$, $B(E2)_{\text{in}}$, and $B(M1)_{\text{out}}$ for $^{164}\text{Lu}$ and with those for $^{163}\text{Lu}$ [1,2]. $B(E2)_{\text{out}}$ and $B(E2)_{\text{in}}$ values are given in units of $(eb)^2$, and $B(M1)_{\text{out}}$ in units of $(\mu_N^2)^2$, with $\mu_N = e\hbar/(2Mc)$. The theoretical transition rates for $^{163}\text{Lu}$ are calculated with the same parameter set as in Refs. [1–3], and compared with experimental data [17]. It can be seen that the theoretical transition rates agree with experimental ones within the error bars.

We temporarily consider the range of $I$ from 23.5 to 31.5 in $^{163}\text{Lu}$ corresponding to the range from 24 to 32 in $^{164}\text{Lu}$. As seen in Table 1, $B(M1)_{\text{out}}$ in $^{164}\text{Lu}$ is much reduced in comparison with $^{163}\text{Lu}$ because of cancellation due to the different sign of $g_s$ between proton and neutron. As for

**3.5. Electromagnetic transitions**

Next, we discuss the electromagnetic transitions. All the theoretical results in Table 1 are obtained by the exact diagonalization of the total Hamiltonian. We employ bare values of g-factors, i.e., $g_{\ell_1} = 1$, $g_{\ell_2} = 0$, $g_{s_1} = 3.906$, $g_{s_2} = -2.678$, and $g_R = Z/A$. As for $Q_0$, we employ 7b [8]. In addition, we take into account the same quenching factor 0.5 for all g values, as adopted in $^{163}\text{Lu}$ [17]. In Table 1, we compare $B(E2)_{\text{out}}$, $B(E2)_{\text{in}}$, and $B(M1)_{\text{out}}$ for $^{164}\text{Lu}$ and with those for $^{163}\text{Lu}$ [1,2]. $B(E2)_{\text{out}}$ and $B(E2)_{\text{in}}$ values are given in units of $(eb)^2$, and $B(M1)_{\text{out}}$ in units of $(\mu_N^2)^2$, with $\mu_N = e\hbar/(2Mc)$. The theoretical transition rates for $^{163}\text{Lu}$ are calculated with the same parameter set as in Refs. [1–3], and compared with experimental data [17]. It can be seen that the theoretical transition rates agree with experimental ones within the error bars.

We temporarily consider the range of $I$ from 23.5 to 31.5 in $^{163}\text{Lu}$ corresponding to the range from 24 to 32 in $^{164}\text{Lu}$. As seen in Table 1, $B(M1)_{\text{out}}$ in $^{164}\text{Lu}$ is much reduced in comparison with $^{163}\text{Lu}$ because of cancellation due to the different sign of $g_s$ between proton and neutron. As for

**Table 1.** Comparison of the $B(E2)_{\text{out}}$, $B(E2)_{\text{in}}$, and $B(M1)_{\text{out}}$ for $^{164}\text{Lu}$ with those for $^{163}\text{Lu}$, together with the experimental data for $^{163}\text{Lu}$ [17]. The first column gives the initial angular momentum $I$, the second column $B(E2)_{\text{out}}$, the third column $B(E2)_{\text{in}}$, and the fourth column $B(M1)_{\text{out}}$ for $^{164}\text{Lu}$. The fifth column gives the initial angular momentum $I$, the sixth column $B(E2)_{\text{out}}$, the seventh column $B(E2)_{\text{in}}$, and the eighth column $B(M1)_{\text{out}}$ for $^{163}\text{Lu}$. The ninth column is for $B(E2)_{\text{out}}$, the tenth column for $B(E2)_{\text{in}}$, and the eleventh column for $B(M1)_{\text{out}}$. The values of $B(E2)_{\text{out}}$ and $B(E2)_{\text{in}}$ are in units of $(eb)^2$, and $B(M1)_{\text{out}}$ in units of $\mu_N^2$.

| $^{164}\text{Lu}$ | $^{163}\text{Lu}$ | $^{163}\text{Lu}$ (exp.) |
|------------------|------------------|------------------|
| $I$ $B(E2)_{\text{out}}$ $B(E2)_{\text{in}}$ $B(M1)_{\text{out}}$ | $I$ $B(E2)_{\text{out}}$ $B(E2)_{\text{in}}$ $B(M1)_{\text{out}}$ | $B(E2)_{\text{out}}$ $B(E2)_{\text{in}}$ $B(M1)_{\text{out}}$ |
| 24 0.254 1.141 0.0013 | 23.5 0.528 2.385 0.024 | 0.54$^{+0.13}_{-0.11}$ 2.56$^{+0.57}_{-0.44}$ 0.017$^{+0.006}_{-0.005}$ |
| 26 0.235 1.150 0.0012 | 25.5 0.489 2.404 0.022 | 0.54$^{+0.09}_{-0.08}$ 2.67$^{+0.41}_{-0.33}$ 0.017$^{+0.005}_{-0.005}$ |
| 28 0.219 1.159 0.0010 | 27.5 0.455 2.418 0.019 | 0.70$^{+0.18}_{-0.15}$ 2.81$^{+0.53}_{-0.41}$ 0.024$^{+0.008}_{-0.007}$ |
| 30 0.205 1.165 0.0010 | 29.5 0.425 2.431 0.017 | 0.65$^{+0.34}_{-0.26}$ 2.19$^{+0.94}_{-0.65}$ 0.023$^{+0.013}_{-0.011}$ |
| 32 0.192 1.171 0.0008 | 31.5 0.399 2.442 0.016 | 0.66$^{+0.29}_{-0.24}$ 2.25$^{+0.75}_{-0.48}$ 0.024$^{+0.012}_{-0.010}$ |
\[ B(E2)_{\text{out}} \text{ and } B(E2)_{\text{in}}, \] the values in \(^{164}\text{Lu}\) are reduced by nearly a half compared with those in \(^{163}\text{Lu}\), as we adopted \(Q_0 = 10\) in \(^{163}\text{Lu}\) \(^{[1,2]}\), while 7 \(b\) in \(^{164}\text{Lu}\) \(^{[8]}\). There is no difference in the ratio \(B(E2)_{\text{out}}/B(E2)_{\text{in}}\). However, the linking transitions from TSD2 to TSD3 are not yet observed.

From Table 1, we learn that the ratio of \(B(M1)_{\text{out}}/B(E2)_{\text{out}}\) in \(^{164}\text{Lu}\) decreases very slowly with increasing \(I\), but is almost constant at 0.005 \((\mu_N/e_b)^2\). The constancy of this ratio is calculated from \(B(M1)_{\text{out}}\) and \(B(E2)_{\text{out}}\) in Eq. (20), which gives 0.007 \((\mu_N/e_b)^2\) without single-particle potentials. The Case 2 parameter set also produces similar results for \(B(E2)_{\text{out}}, B(E2)_{\text{in}},\) and \(B(M1)_{\text{out}}\).

4. Conclusion and discussion

Our attempt in the present paper is to foresee properties of the TSD bands in an odd-odd nucleus \(^{164}\text{Lu}\) based on a theoretical model which is as simple as possible. Calculation is carried out with the “tops-on-top” model, which is essentially the particle-rotor model with valence proton and neutron each in a definite single-\(j\) orbital coupled to the rotor with angular-momentum-dependent rigid-body moments of inertia.

As for the pure rotor case without single-particle potentials, it is confirmed that the algebraic energy level formula works quite well. The stable wobbling of the rotor angular momentum \(\vec{R}\) is confined to move around the principal axis with the largest or the smallest moment of inertia. Non-existence of the wobbling motion of \(\vec{R}\) around the intermediate axis implies the instability of the rotational motion around this axis.

The angular-momentum dependence simulates the collapse of pairing correlation in the rotating core. In order to explain both energy level schemes for positive and negative parity TSD bands \(^{[8]}\), we chose the Case 1 parameter set. As for the positive parity bands, an assumption that a valence neutron occupies the \(i_{13/2}\) orbital is favorable, while for the negative parity bands the \(j_{15/2}\) orbital is favorable in \(^{164}\text{Lu}\). The X2 band seems to be a natural extension of the TSD1 band from the \(I = 42^-\) state.

We have confirmed that theoretical alignments \(\langle I_x^2 \rangle^{1/2}\) and \(\langle R_x^2 \rangle^{1/2}\) give a stable difference by one unit between two TSD bands characterized by the precession (or wobbling) quantum number \(n_\alpha = 1\) and 0. The degeneracy of the calculated alignments \(\langle j_{1x}^2 \rangle^{1/2}\) and \(\langle j_{2x}^2 \rangle^{1/2}\) between yrare and yrast TSD bands indicates precession quantum numbers \(n_\beta = n_\gamma = 0\) for both TSD bands.

In the present paper, we have not performed specific analysis for partial occupation in the \(vi_{13/2}\) shell. Instead, we draw a physical picture for this case that the aligned spin of an unpaired neutron produces the main contribution to energy through the reduced Coriolis term, and the other paired neutrons participate in the rotating core. However, for a detailed description of partial occupation, the quasiparticle picture is necessary where quasiparticles in the \(vi_{13/2}\) shell correlate with those in the rotating core through the pairing interaction within a framework of microscopic theory.

From the energy eigenvalues \(E_I\), kinematic and dynamic moments of inertia are deduced. The theoretical kinematic moment of inertia \(J^{(1)}\) well reproduces the experimental one, but the detailed behavior of dynamic moments of inertia \(J^{(2)}\), for example the divergence coming from the band crossing, is out of the scope of our macroscopic model. Our single-particle space is too small to discuss such a band crossing mechanism.

Finally, we have calculated electromagnetic transition rates for in-band and out-of-band \(E2\) and \(M1\) transitions. \(B(M1)_{\text{out}}\) is much reduced due to cancellation by the different signs of proton and neutron gyromagnetic factors. The ratio of \(B(E2)_{\text{out}}/B(E2)_{\text{in}}\) is in the same order as that of the odd-\(A\) case.
Without angular-momentum dependence of moments of inertia, experimental levels of TSD bands cannot be reproduced. It is desirable to derive the functional dependence of the moments of inertia on angular momentum from the microscopic formalism.

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