Relations in the modulo 2 homology of framed disks algebras

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Abstract

We study in this work the homology structure of spaces having an action of the framed disks operad. In particular, we compute the relations between Kudo-Araki operations and generalized Batalin-Vilkovisky operators. As an application, we complete the computations of the modulo 2 homology of $\Omega^2 S^3$ as a Batalin-Vilkovisky algebra, and give some evidence for a classical conjecture about the modulo 2 Hurewicz homomorphism of the infinite loop space of the sphere spectrum.

1 Introduction

1.1 Summary of results

Let $X$ be a well pointed space having the homotopy type of a CW-complex, and consider the iterated loop space $\Omega^n X$ for $n \geq 2$, and let $\mathbb{K}$ be a ground coefficient field for the singular homology functor $H_*$. The natural action of the little $n$-disks operad $D_n$ on $\Omega^n X$ provides $H_* \Omega^n X$ with the structure of a $H_* D_n$-algebra. F. Cohen [3] has shown that this gives $H_* \Omega^n X$ the structure of an $e_n$-algebra (see Definition A.1 in the appendix). Over the field of rational numbers $\mathbb{Q}$, these two notions essentially coincide.

Suppose now that $X$ is equipped with a (continuous) pointed action of $SO(n)$, which means that $SO(n)$ acts on $X$ through pointed maps. Getzler [7], Salvatore and Wahl [19] have noticed that the little $n$-disks operad $D_n$ is an $SO(n)$-operad which acts in the category of $SO(n)$-spaces on $\Omega^n X$, or equivalently that the framed $n$-disks operad $fD_n$ acts on $\Omega^n X$. Therefore $H_* \Omega^n X$ is an $H_* SO(n)$-algebra over the $H_* SO(n)$-operad $H_* D_n$, or in other words $H_* \Omega^n X$ is an algebra over the operad $H_* fD_n$. One goal of this paper is to provide some explicit computations of this structure, particularly the case of $H_* \Omega^2 S^3$ left unachieved in [5].

Obviously, the structure of $H_* fD_n$-algebra is the structure of an $e_n$-algebra together with the structure of $H_* SO(n)$-module which satisfy some compatibility relations between them. Restricting to the field of rational numbers $\mathbb{Q}$ for $n = 2$, Getzler [7] has shown that $H_* (fD_2; \mathbb{Q})$-algebras are exactly $BV_2$-algebras (i.e. Batalin-Vilkovisky algebras, see Appendix A). Over fields of positive characteristic, $H_* (fD_2; \mathbb{K})$-algebras are still $BV_2$-algebras, but carry Kudo-Araki operations as well. More generally, the structure of $H_* fD_n$-algebras over the field $\mathbb{Q}$ was explored in [19, 5]. We will study in this work this structure for $\mathbb{K} = \mathbb{F}_2$.

As an example of a typical relation (Corollary 4.5) we get that the Batalin-Vilkovisky operator $\Delta$ and the Kudo-Araki operation $Q_1$ in the homology of a two fold loop space satisfy for all $x$ in $H_* \Omega^2 X$

$$\Delta(Q_1 x) = \{\Delta(x), x\} + \Delta(x) \star \Delta(x) = \Delta(x \star (\Delta x))$$

where $\star$ denotes the Pontryagin product.

As explained above, this relation yields a complete computation of $H^* \Omega^2 S^3$ as a $BV$-algebra.
The action of the Batalin-Vilkovisky operator $BV$ on $H_*(\Omega^2 S^3; \mathbb{F}_2)$ is given on monomials $u_{i_1}^{\ell_1} u_{i_2}^{\ell_2} \cdots u_{i_n}^{\ell_n}$ with $\ell_j > 0$ by

$$BV(u_{i_1}^{\ell_1} u_{i_2}^{\ell_2} \cdots u_{i_n}^{\ell_n}) = \sum_{j=1}^{n} \ell_j u_{i_1}^{\ell_1} u_{i_2}^{\ell_2} \cdots u_{i_j}^{\ell_j-1} \cdots u_{i_n}^{\ell_n}.$$  

Recall from [19] that an $(n-1)$-connected group-like $fD_n$-algebra is weakly equivalent to an $fD_n$-algebra to an $n$-fold loop space on a certain pointed $SO(n)$-space. It is therefore a mild restriction to think of $fD_n$-algebras as $n$-fold loop spaces on pointed $SO(n)$-spaces, although our results hold for general $fD_n$-algebras. In [5, section 3], we explained why for computational purposes, it is further a small restriction to focus on $fD_n$-algebras that are $n$-fold loop spaces $\Omega^n X$ over a space $X$ with trivial $SO(n)$-action. In this case, the global action of $SO(n)$ on $\Omega^n X$ is not trivial, and in fact the action of $H_*SO(n)$ on $H_*\Omega^n X$ is in this case nicely related to the $J$-homomorphism. This relation was central to [5, section 5]. This fact and the computation in [5, section 6] (see also [6]) will serve as basis for the computations in Section 6 of the present work. We prove:

**Theorem 6.5.** All classes in $\pi_{n,0}QS^0$ divisible by the image of the $J$-homomorphism are annihilated by the mod 2 Hurewicz homomorphism, except the Hopf maps and their composition squares (which are not annihilated by the Hurewicz homomorphism). In other words, Conjecture [6.7] holds on the ideal generated by the image of the $J$-homomorphism.

In presenting this result on the Hurewicz homomorphism for $QS^0$, we do not claim originality. Although we are not aware of any published reference, it is difficult to believe that such a result is not known to any expert. Rather, these computations are meant to advertise that group operations (the group being in our case $SO(n)$) on spaces might help to analyze the Hurewicz homomorphism. More work in this direction appears in [4]. We also mention that it is possible to push a little further the idea of analyzing the Hurewicz homomorphism for $QS^0$ by higher composition methods (i.e. Toda brackets).

**1.2 Organisation of the paper**

From now on we assume that $\mathbb{K} = \mathbb{F}_2$. After the necessary recollections on Kudo-Araki operations (Section 2), we explore the structure of $H_*(fD_n)$-algebras (Theorem 3.3, 3.6, and 20). The $H_*(fD_n)$-structure allows one to produce new formulas about composition products with elements coming from the $J$-homomorphism in the homology of loop spaces. We then compute the action of $H_*(SO(2))$ on $H_*(\Omega^2 S^k)$ for all $k \geq 1$ (Theorem 5.3, 5.4, and 5.8), which was begun in [5]. In Section 6, we show how the methods of [5] and of the present paper apply to give some information about the Hurewicz homomorphism for $QS^0$, the infinite loop space associated to the stable homotopy of spheres (Theorem 6.5).

**1.3 Notations and conventions**

Recall that we assume all topological spaces to have the homotopy type of a CW-complex. If $X$ is a topological space, the singular homology groups are denoted by $H_*X$ and are always taken with $\mathbb{Z}/2\mathbb{Z}$-coefficients; $D_X : X \to X \times X$ is the diagonal map of $X$; we usually, alleviate the notation by omitting $X$ from it. For any continuous map $\Psi$, we let $\Psi_*$ be its induced map in homology. The map $\mathcal{H} : \pi_* X \to H_*X$ is the *modulo 2 Hurewicz homomorphism*.

We fix a non zero positive integer $n > 0$. The bottom and top homology classes in $H_*S^n$ are denoted respectively by $b_0$ and $b_n$. $S^n$ equipped with the antipodal action $T$ becomes a $\mathbb{Z}/2\mathbb{Z}$-space. We provide
$S^n$ with its usual $\mathbb{Z}/2\mathbb{Z}$ equivariant cell structure (with two antipodal cells $e^i$ and $Te^i$ in each dimension $0 \leq i \leq n$).

Given a cell complex $X$ (i.e. a CW-complex together with a specific cell decomposition), we let $C^\text{cell}_* X$ be the associated cellular chain complex. In particular $C^\text{cell}_* S^n$ is the cellular chain complex of $S^n$ with respect to the above classical $\mathbb{Z}/2\mathbb{Z}$-cell structure, and is denoted by $W_*$, and is the standard periodic resolution of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{F}_2$ truncated at level $n$:

$$(W_n)_i = \mathbb{F}_2 e^i \oplus \mathbb{F}_2 Te^i , \text{ for } i \leq n$$

and $d(e^i) = e^{i-1} + Te^{i-1}$, and $(W_i) = 0$ for $i > n$.

The operad of little $n$-disks consists of the symmetric sequence of spaces $(D_n(i))_{i \geq 0}$ of TD-embeddings (i.e. affine embeddings that are compositions of translations and dilatations) of the disjoint union of $i$-little $n$-disks in the interior of a standard $n$-disc, whose images are pairwise disjoint, equipped with the suitable topology. The symmetric group on $i$ letters acts on $D_n(i)$ in the obvious way, and $(D_n(i))_{i \geq 0}$ forms a topological operad by “putting big disks in small embedded discs”. Any $n$-fold loop spaces is an algebra over this operad, and it happens that any connected algebra over the little $n$-disks operad is actually equivalent as a $D_n$-algebras to an $n$-fold loop space. The reference for this material is the classical [15].

Recall that the space of little 2-disks in $D^n$ is homotopy equivalent to $S^{n-1}$, equivariantly with respect to the actions of $\mathbb{Z}/2\mathbb{Z}$ and $SO(n)$.

The little $n$-disks operad is a sub-operad of the framed little $n$-disks operad $fD_n$ which consists of the semi-direct product of $D_n$ with the linear group $SO(n)$ (see [19]).

The symbol $\delta^i_j$ for $i, j$ integers is the Kronecker symbol, whose value is 1 if $i = j$ and 0 otherwise.

If $X$ is an $H$-space, we let $*$ denote the Pontryagin product associated to the $H$-space structure on $H_\ast X$.

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2 Recollection on Kudo-Araki operations

2.1 Extended products and their homology

2.1.1 Extended products.

The classical Kudo-Araki operations form a set of modulo 2 homology operations for $D_n$-algebras [10, 11]. We give here a short recollection.

Let $n > 0$. $\mathbb{Z}/2\mathbb{Z}$ acts on $S^n$ by the antipodal action, and on $X \times X$ by exchanging factors. Let $E_n(X) = S^n \times_{\mathbb{Z}/2\mathbb{Z}} (X \times X)$. We have a fiber bundle with structural group $\mathbb{Z}/2\mathbb{Z}$

$$X \times X \longrightarrow E_n(X) \longrightarrow \mathbb{R}P^n$$

and a Serre spectral sequence

$$H^w_\ast (\mathbb{R}P^n, H_\ast \mathbb{R}P^{n-2}) \Rightarrow H_\ast (E_n(X))$$
where \( H^\text{tw} \) stands for homology with twisted coefficients. It is classical (see for instance [14, 17, 21]) that this spectral sequence collapses at the \( E_2 \) term. Hence there is a vector space isomorphism

\[
H_*E_n(X) \cong H^\text{tw}_*(\mathbb{R}P^n, H_*X^\otimes 2).
\]

There are classical isomorphisms [14, lemma 1.1]:

\[
H^\text{tw}_*(\mathbb{R}P^n, H_*(X \times X)) = H_*(C^\text{cell}_* S^n \otimes_{\mathbb{Z}/2\mathbb{Z}} H_*(X^\times 2))
\]

\[
= H_*(W_n \otimes_{\mathbb{Z}/2\mathbb{Z}} H_*(X^\times 2))
\]

\[
= H_*(W_n \otimes_{\mathbb{Z}/2\mathbb{Z}} H_*(X)^\otimes 2)
\]

More generally, let \( A \) be a free \( \mathbb{Z}/2\mathbb{Z} \)-CW-complex and let \( X \) be any space. There is a fibration sequence as before:

\[
X \times X \to A \times_{\mathbb{Z}/2\mathbb{Z}} (X \times X) \to (A)_{\mathbb{Z}/2\mathbb{Z}}
\]

We have isomorphisms:

\[
H^\text{tw}_*((A)_{\mathbb{Z}/2\mathbb{Z}}, H_*(X \times X)) \cong H_*(C^\text{cell}_* A \otimes_{\mathbb{Z}/2\mathbb{Z}} H_*(X^\times 2))
\]

\[
= H_*(C^\text{cell}_* A \otimes_{\mathbb{Z}/2\mathbb{Z}} H_*(X)^\otimes 2)
\]

These isomorphisms are natural in the variable \( A \) with respect to equivariant cellular maps and in \( X \) with respect to all maps.

Furthermore, if \((g_i)_{i \in I}\) is a basis (in particular the indexing set \( I \) is totally ordered) for \( H_*(X) \), a basis for the twisted homology \( H^\text{tw}_*(\mathbb{R}P^n, H_*X^\otimes 2) \cong H_*E_n(X) \) is given by the list of homology classes

\[
[e^i \otimes g_i \otimes g_j] \ , \ i < j ; \ [e^n \otimes g_i \otimes g_j] \ , \ i < j ; \ [e^k \otimes g_i \otimes g_i] \ , \ 0 \leq k \leq n
\]

for \( i, j \in I \).

### 2.1.2 Operations on the homology of \( D_{n+1} \)-algebras.

Let \( X \) be a \( D_{n+1} \)-algebra; \( X \) is most typically an \( (n+1) \)-fold loop space. There is an operadic action map:

\[
\theta : S^n \times X \times X \to X
\]

which factorizes as

\[
\tilde{\theta} : E_n(X) \to X.
\]

We define:

\[
\theta_*(b_0 \otimes x \otimes y) = \theta_*([e^0 \otimes x \otimes y]) = x \ast y \tag{4}
\]

\[
\tilde{\theta}_*([e^i \otimes x \otimes x]) = Q_i x \text{ for } 0 \leq i \leq n
\]

\[
\theta_*(b_n \otimes x \otimes y) = \tilde{\theta}(e^n \otimes x \otimes y) = \{x, y\}
\]

We have used the following classical computation of the map \( \zeta_* \) induced by the quotient map

\[
\zeta : S^n \times X \times X \to E_n(X)
\]

\[
\zeta(b_0 \otimes g_i \otimes g_j) = [e^0 \otimes g_i \otimes g_j] \ , \ \text{for } i \neq j \tag{5}
\]

\[
\zeta(b_n \otimes g_i \otimes g_j) = [e^n \otimes g_i \otimes g_j] \ , \ \text{for } i \neq j
\]
and $\zeta_*$ is zero otherwise.

Elements of the form $[e_i \otimes x \otimes x]$ are called Kudo-Araki elements, those of the form $[e_0 \otimes x \otimes y]$ are called Pontryagin elements, and those of the form $[e^n \otimes x \otimes y]$ for $x \neq y$ are called brackets. Of course $*$ is the Pontryagin product, $\{-,-\}$ is the Browder bracket, and the operators $Q_i$ are the Kudo-Araki operations. One checks that this definition and others are equivalent.

The natural quotient map $S^n \longrightarrow S^n \lor S^n$ obtained by collapsing the $(n-1)$-skeleton is $\mathbb{Z}/2\mathbb{Z}$-equivariant and induces a natural quotient map $\xi : E_n(X) \longrightarrow S^n \wedge (X \times X)$ that will be crucial to us in order to elucidate the structure of $H_*(fD_{n+1})$-algebras. We will need to know that this quotient map satisfies [17, part I, chap 3, p. 37-39]:

$$\xi_*(e_0 \otimes x \otimes y) = 0 \quad (6)$$
$$\xi_*(e_i \otimes x \otimes x) = 0 \text{ for } i \leq n-1$$
$$\xi_*(e_n \otimes x \otimes x) = b_n \otimes x \otimes x$$
$$\xi_*(e_n \otimes x \otimes y) = b_n \otimes x \otimes y + b_n \otimes y \otimes x$$

Observe that in [17], the author works with cohomology; we leave it to the reader to make the necessary changes to get this result.

**Remark 2.1** One usually defines ‘upper index’ Kudo-Araki operations by

$$Q^i x := Q_{i-|x|} x$$

In this way, the operation $Q^i$ simply raises the degree by $i$. These reindexed Kudo-Araki operations satisfy unstability: $Q^i X = 0$ if $i < |x|$.

### 2.2 Properties of the Browder bracket and the Kudo-Araki operations

The following holds for all little $(n+1)$-disks algebras, hence in particular for framed little $(n+1)$-disks algebras, i.e. algebras over $fD_{n+1}$.

**Proposition 2.2** [3, Theorem 1.3 (5) p. 218] On the homology of a $D_{n+1}$-algebra $X$, the operations $Q_i$ are additive for $i < n$ and $Q_n$ is quadratic with respect to the Browder bracket, which means that for all $x, y \in H^*X$

$$Q_n(x + y) = Q_n(x) + Q_n(y) + \{x, y\} \quad (7)$$

**Corollary 2.3** The Kudo-Araki operations determine Browder brackets in the modulo 2 homology of $D_{n+1}$-spaces.

In order to elucidate the structure of $H_*(fD_{n+1})$-algebra at the prime 2, it is therefore crucial to understand the relation between the $H_*SO(n+1)$-action with Pontryagin products and Kudo-Araki operations. We do this in Section 4. A direct consequence of Proposition 2.2 is:

**Remark 2.4** [3, Theorem 1.2 (3) p. 215] In the modulo 2 homology of a little $(n+1)$-disks algebra $X$, there is a relation for all $x \in H_*X$:

$$\{x, x\} = 0 \quad (8)$$

**Remark 2.5** Another fact worth pointing out is the fact that the Browder bracket on the homology of an $n$-fold loop space vanishes as soon as the $n$-fold loop space structure extends to an $(n+1)$-fold loop space structure. In this case, a Browder bracket associated to the $(n+1)$-fold loop space structure is defined and the latter does not need to vanish.
3 The structure of the modulo 2 homology of $H_* f\mathcal{D}_n$-algebras

Let $n \geq 2$ and let $Z$ be an $f\mathcal{D}_n$-algebra. In this section, we study the relations between the $H_* SO(n)$ action on $H_* Z$ due to the $SO(n)$-action coming from $f\mathcal{D}_n$ and the mod 2 Kudo-Araki operations due to the action of the operad of little $n$-disks in the operad $f\mathcal{D}_n$. We first need to recall some elementary facts about the homology of $SO(n)$ in order to proceed to the computations.

3.1 Recollection on the homology of orthogonal groups

We consider the reflection map $\kappa: \mathbb{R}P^{n-1} \to SO(n)$, that sends a line to the reflection about its orthogonal hyperplane, then composed with a fixed reflection (in order to get an orientation preserving linear transformation). Recall that

$$H_* SO(n) = \Lambda(d_1, d_2, \ldots, d_{n-1})$$

where $d_i := \kappa_* e_i$ is the image of the unique non-zero class $e_i \in H_i \mathbb{R}P^{n-1}$ by $\kappa_*$. In particular, the diagonal applied on the generators is the Cartan diagonal $D_*:

$$D_* d_k = \sum_{i+j=k} d_i \otimes d_j .$$

A reference for this classical fact is [1].

**Definition 3.1** Let $Z$ be an $f\mathcal{D}_n$-space. In what follows, we denote the action of $d_i \in H_* Z$ on a class $x$ by $\Delta_i x$. The natural operators $\Delta_i$ are called higher BV operators.

The action of $g \in H_*(SO(n))$ on $b_0 \in H_0(S^{n-1})$ is denoted by $g \cdot b_0$. In particular, for $n = 2$, $\Delta_1$ is the Batalin-Vilkovisky operator on the homology of 2-fold loops on $SO(2)$-spaces [7], simply denoted by $\Delta$ or $BV$.

For $n > 0$, consider the natural (unpointed) action of $SO(n)$ on $S^{n-1}$. For $g \in H_* SO(n)$ and $x \in H_* S^n$, we denote by $g.x$ the image of $g \otimes x \in H_* S^{n-1}$ by the map induced in homology by the natural action. Recall that $b_0 \in H_0 S^{n-1}$ is a generator. The element $g.b_0$ is none but $H_* (ev)(g)$ where $ev: SO(n) \to S^{n-1}$ is the evaluation of the action at the point $(0, \cdots, 0, 1)$ of $S^{n-1}$. This can classically be computed as the edge homomorphism in the Serre spectral sequence of the fibration sequence $SO(n-1) \to SO(n) \to S^{n-1}$. We get:

$$e_{n-1} \cdot b_0 = b_{n-1}$$

(9)

As indicated in [8] p. 291-299, $SO(n)$ possesses a nice cell structure compatible with the reflection map $\kappa$. The modulo 2 cellular chain complex $C^\text{cell}_*(SO(n))$ is the differential graded bialgebra $\Lambda(d^1, d^2, \ldots, d^{n-1})$ with zero differential and Cartan diagonal on the generators $d^i$.

There is a chain equivalence

$$C^\text{cell}_*(SO(n) \times S^{n-1}) \cong C^\text{cell}_*(SO(n)) \otimes C^\text{cell}_*(S^{n-1})$$

and we will need a cellular approximation of the action $SO(n) \times S^{n-1} \to S^{n-1}$ and its effect

$$C^\text{cell}_*(SO(n)) \otimes C^\text{cell}_*(S^{n-1}) \to C^\text{cell}_*(S^{n-1}) .$$

on cellular chains.
Lemma 3.2 There is a cellular approximation of the action $SO(n) \times S^{n-1} \to S^{n-1}$ which induces on cellular chains the map
\[ C^\text{cell}_*(SO(n)) \otimes C^\text{cell}_*(S^{n-1}) \to C^\text{cell}_*(S^{n-1}) \]
given by

- $d^i.e^j = 0$ except if $j = 0$ or $i = 0$,
- $d^0.e^i = e^i$ for $i \geq 0$,
- $d^i.e^0 = 0$ for $i > 0$,
- $d^i.Te^j = T(d^i.e^j)$ for $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$.

where the action of the generator $d^i \in C^\text{cell}_*SO(n)$ on the generator $e^j \in C^\text{cell}_*S^{n-1}$ is denoted by $d^i.e^j$.

NB: $d^i$ is a generator in the chain complex $C^\text{cell}_*SO(n)$, while $d_i$ is the induced homology class in $H_*(SO(n))$.

Proof. We prove the lemma by induction on $n \geq 2$. We begin by the construction of a nice cellular approximation of the action for each $n$.

We first observe that there is a commutative diagram for $k \geq n$
\[
\begin{array}{ccc}
SO(n) \times S^{n-1} & \longrightarrow & S^{n-1} \\
\downarrow & & \downarrow \\
SO(k) \times S^{k-1} & \longrightarrow & S^{k-1}
\end{array}
\]

where the vertical maps are the usual inclusions, which are cellular.

Beginning with $n = 2$, where $SO(n) = S^1$ and the action of $SO(n)$ on $S^1$ is the action by translation of $S^1$ on itself, we construct inductively for $n \geq 2$ compatible cellular approximations in the sense that Diagram (10) commutes, and such that the restriction of the action $SO(n) \times S^{n-1} \to S^{n-1}$ to $\{d^0\} \times S^{n-1}$ is the identity. This follows by a simple application of the cellular approximation theorem for CW-pairs.

As our cellular approximations restrict to the identity on $SO(n) \times S^{n-1} \to S^{n-1}$, we immediately derive that
\[ d^i.e^i = e^i \]
for $0 \leq i \leq n-1$ and all $n \geq 2$ at the cellular chains level.

Now assume that Lemma 3.2 holds for some $n \geq 2$, we want to derive that it holds for $(n+1)$ as well. As we have chosen compatible cellular approximations of the action, we have for $i, j \leq n - 1$ that $d^i.e^j = 0$ if neither $i \neq 0$ nor $j \neq 0$. Moreover
\[ d^i.e^0 = e^i + Te^i, \quad i > 0; \quad d^0.e^i = e^i \]
for $i \leq n - 1$. By degree reasons we have for $i, j > 0$
\[ d^i.e^n = d^n.e^j = 0 \]
It remains thus to check:
\[ d^n.e^0 = e^n + Te^n. \]
But by Formula (9), we must have $d^n.e^0 = e^n + Te^n$ because $e^n + Te^n$ is the unique $n$-cycle that represents the top class of $H^*S^n$. 

7
The last point of Lemma 3.2 follows from similar considerations.

We still have to check the case \( n = 2 \) as initial input of the induction process. As already noticed, in this case \( SO(n) = S^1 \), and the action of \( SO(n) \) on \( S^1 \) is the action by translation of \( S^1 \) on itself. The effect of the action in homology is simply given by the algebra structure of \( H^* S^n = \Lambda(b_1) = \Lambda(e^1) \), and we have \( d_1.b_0 = b_1 \) in homology, which forces at the cellular chains level \( d_1.e^0 = e^1 + T.e^1 \) (but this is actually obvious in this case at the topological level). This settles the case \( n = 2 \).

### 3.2 The fundamental diagram

Let \( Z \) be an \( fD_n \)-space, i.e. a space with an action of the operad \( fD_n \) of framed little \( n \)-disks. Then \( Z \) is in particular a pointed \( SO(n) \)-space with an action of the little \( n \)-disks operad, such that the following diagram commutes:

\[
\begin{array}{ccc}
SO(n) \times S^{n-1} \times Z \times Z & \xrightarrow{a^D} & S^{n-1} \times Z \times Z \\
\downarrow{\text{id} \times \theta} & & \downarrow{\theta} \\
SO(n) \times Z & \xrightarrow{a} & Z
\end{array}
\]  

(11)

Here \( \theta \) is the action of little \( n \)-disks, \( a \) is the action of \( SO(n) \) on \( Z \), and \( a^D \) is the diagonal action of \( SO(n) \), where the action of \( SO(n) \) on \( S^{n-1} \) is the natural one (i.e. the restriction of the action of the natural action of \( SO(n) \) on \( \mathbb{R}^n \) to its unit sphere \( S^{n-1} \)). We emphasize again that we can use, instead of the space of 2 small disjoint little \( n \)-disks in a big \( n \)-disk, the equivariantly homotopy equivalent space \( S^{n-1} \).

We have to translate the commutation of Diagram (11) in terms of relations between operations coming from the \( H_* SO(n) \) action (the \( \Delta_i \)'s) and the operations from the loop structure (Pontryagin product, Browder bracket, Kudo-Araki operations). The relations involving Kudo-Araki operations are (of course) a little more difficult to derive. We therefore postpone this calculation to the Section 4.

### 3.3 Higher BV operations and Pontryagin products

We now compute the behavior of the \( H_* SO(n) \) action on \( H_* Z \) with respect to Pontryagin products in \( H^* Z \). The computation is:

\[
\Delta_i(x \ast y) = a_*(d_i \otimes \theta_*(e_0 \otimes x \otimes y))
\]

\[
= a_*(\text{Id}_{H_*(SO(n))} \otimes \theta_*)(d_i \otimes (e_0 \otimes x \otimes y))
\]

\[
= \theta_*(\sum_{\alpha+\gamma=i} d_\alpha.e_0 \otimes d_\gamma.x \otimes d_\gamma.y)
\]

\[
= \delta^i_{n-1}\theta_*(e_{n-i} \otimes x \otimes y) + \theta_*(\sum_{\epsilon+\gamma=i} e_\epsilon \otimes d_\epsilon.x \otimes d_\gamma.y)
\]

\[
= \delta^i_{n-1}\{x,y\} + \theta_*(e_0 \otimes \sum_{\epsilon+\gamma=i} d_\epsilon.x \otimes d_\gamma.y)
\]

\[
= \delta^i_{n-1}\{x,y\} + \sum_{\epsilon+\gamma=i} (\Delta_\epsilon x \ast \Delta_\gamma y)
\]

Hence we recover (see [19]):
Theorem 3.3 Let \( Z \) be an \( fD_n \)-algebra and \( x, y \) be elements in the modulo 2 cohomology of \( Z \). Then, for \( i \leq n - 1 \):
\[
\Delta_i(x \ast y) = \delta_{n-1}^i \{ x, y \} + \sum_{i + j = i} \Delta_i x \ast \Delta_j y .
\]

We notice that \( H_* SO(n) \) has a primitive element \( p_i \) in every odd degree \( i \). Moreover, \( p_i \) is equal to \( d_i \) up to decomposable elements. For \( i \) odd, let \( \Delta_{p_i} \) be the operator associated to the primitive class \( p_i \). For example
\[
p_1 = d_1, \quad p_3 = d_3 + d_2 * d_1
\]

hence
\[
\Delta_{p_1} = \Delta_1, \quad \Delta_{p_3} = \Delta_3 + \Delta_2 \Delta_1 = \Delta_3 + \Delta_1 \Delta_2
\]

Using the same method as in the proof of Theorem 3.3, we see that for \( i \) odd with \( i < n - 1 \), then \( \Delta_{p_i} \) is a derivation with respect to the Pontryagin product, while for \( n \) even, \( \Delta_{p_{n-1}} \) is a \( BV_n \) operator inducing the natural Gerstenhaber algebra structure of the cohomology.

Theorem 3.4 Let \( Z \) be an \( fD_n \)-algebra and \( x, y \) be elements in the modulo 2 cohomology of \( Z \). For any odd integer \( i \) with \( i \leq n - 1 \):
\[
\Delta_{p_i}(x \ast y) = \delta_{n-1}^i \{ x, y \} + (\Delta_{p_i} x) y + x (\Delta_{p_i} y) .
\]

For \( i = n - 1 \) (that is, for \( n \) is even), this is the modulo 2 expression of the relations defining a \( BV_n \)-algebra (see Definition A.2). We get:

Corollary 3.5 Let \( Z \) be an \( fD_n \)-algebra. For \( i \leq n - 1 \), \( i \) odd, let \( \Delta_{p_i} \) be the natural operator on \( H_* Z \) obtained as the action of the primitive element \( p_i \in H_* SO(n) \) on \( H_* Z \). Then for \( i < n - 1 \), the \( \Delta_{p_i} \) operator is a derivation with respect to the Pontryagin product. If \( n \) is even, \( \Delta_{p_{n-1}} \) endows \( H_* Z \) with the structure of a \( BV_n \)-algebra.

3.4 Higher BV operations and Browder brackets

We now compute the behavior of the \( H_* SO(n) \) action on \( H_* Z \) with respect to Browder brackets. The computation is essentially the same as above. Let \( x, y \in H_* Z \)
\[
\Delta_i(\{x, y\}) = a_i(d_i \otimes \theta_i(e_{n-1} \otimes x \otimes y))
\]
\[
= (\text{Id}_{H_* SO(n)} \otimes \theta_i)(d_i \otimes e_{n-1} \otimes x \otimes y)
\]
\[
= \theta_i \left( \sum_{i + j + \gamma = i} d_i e_{n-1} \otimes d_i x \otimes d_j y \right)
\]
\[
= \sum_{i + j + \gamma = i} e_{n-1} \otimes d_i x \otimes d_j y
\]

Hence we get:

Theorem 3.6 Let \( \Omega^n X \) be an \( n \)-fold loop space and \( x, y \) be elements in the modulo 2 cohomology of \( \Omega^n X \). For \( i \leq n - 1 \):
\[
\Delta_i(\{x, y\}) = \sum_{i + j = i} \{\Delta_i x, \Delta_j y\} .
\]

4 Higher BV operators and Kudo-Araki operations

Let \( n \geq 2 \). We now compute the relation between \( \Delta_i \) and \( Q_j \) operations, for \( i, j \leq n - 1 \). Let \( Z \) be any algebra over the framed \( n \)-disks operad \( f\mathcal{D}_n \). Typically, \( Z = \Omega^n X \) is an \( n \)-fold loop space.
4.1 Decomposition of the $SO(n)$-action

The topological group $SO(n)$ acts on $S^{n-1}$ in the standard fashion, and acts diagonally on products of $SO(n)$-spaces. Thus we obtain a natural action of $SO(n)$ on $S^{n-1} \times Z \times Z$, and this action is denoted by $a^D$. We break down this action into two actions of $SO(n)$, the first one is an action $a_1$ on $S^{n-1}$, and one by $a_2$ on $Z \times Z$. In other words, the diagonal action $a^D$ of $SO(2)$ decomposes as:

$$SO(n) \times S^{n-1} \times Z^2 \xrightarrow{D \times Id} SO(n)^2 \times S^{n-1} \times Z^2 \xrightarrow{1 \times a_1} SO(n) \times S^{n-1} \times Z^2 \xrightarrow{a_2} S^{n-1} \times Z \times Z$$ (17)

These two actions $a_1$ and $a_2$ are $\mathbb{Z}/2\mathbb{Z}$-equivariant. That is, Diagram (17) is a diagram of $\mathbb{Z}/2\mathbb{Z}$-spaces. Hence the two actions $a_1$ and $a_2$ pass to the quotient $E_{n-1}(Z)$, on which they induce corresponding actions (denoted by $\bar{a}_1$ and $\bar{a}_2$). We therefore get a commutative diagram:

We have a decomposition of the diagonal action $a^D$ as:

$$SO(n) \times E_{n-1}(Z) \longrightarrow SO(n)^2 \times E_{n-1}(Z) \xrightarrow{1 \times \bar{a}_1} SO(n) \times E_{n-1}(Z) \xrightarrow{\bar{a}_2} E_{n-1}(Z)$$

By the commutation of Diagram (11) and Diagram (18), we only need to compute

$$\Delta_i Q_j x = \bar{a}_2_*(1 \otimes (\bar{a}_1)_*)(1 \otimes D_\ast)(d_i \otimes [e^j \otimes x \otimes x])$$ (19)

hence the main issue is to compute $(\bar{a}_2)_*$ and $(\bar{a}_1)_*$.

4.2 Calculation of $(\bar{a}_1)_*$

The result is as follows:

**Proposition 4.1** The following formula hold:

- For $x \neq y$,
  
  $$(\bar{a}_1)_*\left(d_i \otimes [e^j \otimes x \otimes y]\right) = [d_i \ast e^j \otimes x \otimes y]$$
  
  where $d_i \ast e^j = 0$ except in the cases $d_0 \ast e^i = e^i$, $d_j \ast e^0 = e^j$ and $d_{n-1} \ast e^0 = e^{n-1}$.

- For $x = y$,
  
  $$(\bar{a}_1)_*\left(d_i \otimes [e^j \otimes x \otimes x]\right) = [d_i \ast e^j \otimes x \otimes x]$$
  
  with $d_i \ast e^j = 0$ except in the case $d_0 \ast e^i = e^i$.

**Proof.** We first notice that the formula for $(\bar{a}_1)_*(d_i \otimes [e^0 \otimes x \otimes y])$ holds because such classes come from $H_*(S^{n-1} \times Z \times Z)$, and we know how the action of $H_\ast SO(n)$ is at this level (Formula (20)), as well as the map $H_\ast(S^{n-1} \times Z \times Z \longrightarrow E_{n-1}(Z))$ (Formula (24)). In the same way the formula for $(\bar{a}_1)_*(d_i \otimes [e_{n-1} \otimes x \otimes y])$ for $x \neq y$ must hold.

In general, we can not argue this way to compute $(\bar{a}_1)_*(d_i \otimes [e^j \otimes x \otimes x])$ for $0 < i < n$. Instead, we rely on the naturality of the isomorphisms (3). Let $\tilde{x}$ be a cycle representing $x$, and recall $d^i$ is a generator.
in the cellular complex of $SO(n)$ corresponding to the cohomology class $d_i$. The class $d_i \otimes [e^j \otimes x \otimes x]$ is represented by the cycle $d^i \otimes e^j \otimes \tilde{x} \otimes \tilde{x}$. At the level of chains, $(a_1)_*(d^i \otimes e^j \otimes \tilde{x} \otimes \tilde{x}) = d^i.e^j \otimes \tilde{x} \otimes \tilde{x}$, which by the Lemma 3.2 is zero except if $j = 0$. In this case, we compute

$$(a_1)_*(d^i \otimes e^0 \otimes \tilde{x} \otimes \tilde{x}) = d^i.e^0 \otimes \tilde{x} \otimes \tilde{x} = (e^i + Te^i) \otimes \tilde{x} \otimes \tilde{x}$$

But now

$$[(e^i + Te^i) \otimes x \otimes x] = [e^i \otimes x \otimes x] + [Te^i \otimes x \otimes x] = [e^i \otimes x \otimes x] + [e^i \otimes x \otimes x] = 0$$

in $H_*(E_{n-1}(Z))$. □

4.3 Calculation of $(\bar{a}_2)_*$

We can decompose $a_2$ a little further. The map $\bar{a}_2$ is the bottom composition in the commutative diagram:

$$
\begin{array}{ccc}
SO(n) \times S^{n-1} \times Z^2 & \longrightarrow & S^{n-1} \times (SO(n) \times Z)^2 \\
\downarrow & & \downarrow \\
SO(n) \times (S^{n-1} \times_{Z/2Z} Z^2) & \longrightarrow & S^{n-1} \times_{Z/2Z} (SO(n) \times Z)^2 \\
\end{array}
$$

We can compute $\psi_*$ easily:

$$\psi([e^j \otimes (s_1 \otimes x) \otimes (s_2 \otimes y)]) = [e^j \otimes s_1.x \otimes s_2.y]. \quad (20)$$

This relies as above, on the naturality in $H_*Z$ of the Serre spectral of the fiber sequence:

$$Z \times Z \longrightarrow E_{n-1}(Z) \longrightarrow \mathbb{R}P^{n-1}$$

at the $E_2$-term, together with the naturality explained in Section 2.1.1.

4.4 Calculation of $\varphi_*$

The result.

The result for the computation of $\varphi_*$ is as follows.

Lemma 4.2 For all $0 \leq r \leq n-1$, $\alpha \in H^*SO(n)$, $x \in H^*Z$

$$\varphi_*(\alpha \otimes [e^r \otimes x \otimes x]) = \sum_{i, r + 2i - |\alpha| \leq n-1} [e^{r+2i-|\alpha|} \otimes (Sq^i_* \alpha \otimes x) \otimes (Sq^i_* \alpha \otimes x)]$$

$$+ \delta_{n-1} \sum_{|\alpha'| < |\alpha|} [e^{n-1} \otimes (\alpha' \otimes y) \otimes (\alpha' \otimes y)]$$

(21)

where $D_*(\alpha) = \sum \alpha' \otimes \alpha''$ and $D$ is the diagonal of $SO(n)$.

We will proceed as follows: the formula up to a defect term appears already elsewhere as we shall briefly recall; we then compute this defect term using the map $\xi$ of Section 2.
Related results in the literature and the method of calculation.

Our method to compute $\varphi_*$ is to a certain extent classical. Very close computations occur for instance in [3] p. 363, Theorem 3.2. (ii) or [3] p. 11-13. We consider the diagonal map

$$\varphi : X \times (S^{n-1} \times \mathbb{Z}/2^2) \to S^{n-1} \times \mathbb{Z}/2^2 (X \times Y)^2$$

induced by the natural map

$$(sh) \circ (D \times Id_{S^{n-1} \times Y^2}) : X \times S^{n-1} \times Y^2 \to S^{n-1} \times (X \times Y)^2$$

where $D$ is the diagonal of $X$ and $sh$ is the obvious shuffling of factors in the product $X \times X \times S^{n-1} \times Y \times Y$. We are interested in the special case $X = SO(n)$. We will actually compute the map $\varphi_* : H_* (Y) \otimes H_* E_{n-1} X \to H_* E_{n-1} (X \times Y)$ in general. According to [3] Lemma 4.2 (i), p. 368, Theorem 3.2:

**Lemma 4.3** There is a formula, for $x \in H_* X$ and $y \in H_* Y$:

$$\varphi_*(x \otimes [e^r \otimes y \otimes y]) = \sum_{i,r+2l-|x| \leq n-1} [e^{r+2l-|x|} \otimes (Sq^i_* x \otimes y) \otimes (Sq^j_* x \otimes y)] + \delta^{n-1}_r \Gamma_r$$

(22)

where $\Gamma_r$ is in the image of map induced in homology by the quotient map $S^{n-1} \times (X \times Y)^2 \to S^{n-1} \times \mathbb{Z}/2^2 (X \times Y)^2$.

We warn the reader that our notations differ from those of the reference, and moreover that the formula [3] Lemma 4.2 (i), p. 368, Theorem 3.2 has a small typo in the indices. The fact that $\Gamma_r$ is zero for $r < n-1$ is not part of [3] Lemma 4.2 (i), p. 368, Theorem 3.2. This can be seen as follows. We can apply the homotopy to the commutative diagram:

\[ \begin{array}{ccc}
X \times E_r (Y) & \xrightarrow{\varphi} & E_r (X \times Y) \\
\downarrow{\text{id} \times \xi} & & \downarrow{\xi} \\
X \times (S^r \wedge X \times Y) & \xrightarrow{\varphi_*} & E_{r+1} (X \times Y) \\
\downarrow{\text{id} \times \xi} & & \downarrow{\xi} \\
X \times (S^{n-1} \wedge X \times Y) & \xrightarrow{\varphi_*} & E_{n-1} (X \times Y)
\end{array} \]

where the maps labelled $\xi$ are induced by natural inclusions. The element $x \otimes [e^r \otimes y \otimes y]$ comes from the homology of $X \times E_r (Y)$ hence the element $\Gamma_r$ comes from some element $\Gamma'_r$ in the homology of $E_r (X \times Y)$. Brackets are detected by the map $\xi$ (Formulas [5], in the sense that $\xi_*$ is injective on the subspace generated by bracket elements. We see that $\Gamma'_* \equiv 0$ if and only if the image of $\Gamma'_r$ by the map

$$\langle \xi \rangle_* : H_* E_r (X \times Y) \to H_* S^r \wedge [(X \times Y) \times (X \times Y)] \to S^{n-1} \wedge [(X \times Y) \times (X \times Y)]$$

is non zero. But the map $\iota_*$ is zero on the image of $\xi_*$. This means that $\Gamma'_r$ is zero for $r < n-1$, and for $r = n-1$:

$$\Gamma := \Gamma_{n-1} = \sum [e^{n-1} \otimes x_1 \otimes y_1 \otimes x_2 \otimes y_2]$$

We set

$$G = \sum_{i,r+2l-|x| \leq n-1} [e^{r+2l-|x|} \otimes (Sq^i_* x \otimes y) \otimes (Sq^j_* x \otimes y)]$$

So that $\varphi_*(x \otimes [e^{n-1} \otimes y \otimes y]) = G + \Gamma$. 

12
Computation of $\Gamma$.

There is a commutative diagram

$$
\begin{array}{ccc}
X \times E_{n-1}(Y) & \xrightarrow{\varphi} & E_{n-1}(X \times Y) \\
\downarrow{id \times \xi} & & \downarrow{\xi} \\
X \times [S^{n-1} \wedge (Y \times Y)] & \xrightarrow{\tilde{D}} & S^{n-1} \wedge [(X \times Y) \times (X \times Y)]
\end{array}
$$

Here $\xi$ is as in Section 2.1 and $\tilde{D}$ is induced by the diagonal on $X$ and shuffling the factors. We compute that

$$
\xi_* \varphi_*(x \otimes [e^{n-1} \otimes y \otimes y]) = \xi_*(G + \Gamma) = (\tilde{D})_* (id \otimes \xi_*)(x \otimes [e^{n-1} \otimes y \otimes y])
$$

But

$$
(\tilde{D})_* (id \otimes \xi_*)(x \otimes [e^{n-1} \otimes y \otimes y]) = (\tilde{D})_* (x \otimes b_{n-1} \otimes y \otimes y) = \sum b_{n-1} \otimes x' \otimes y \otimes x'' \otimes y
$$

where $D_* x = \sum x' \otimes x''$. If we set $D_* x = \sum x' \otimes x'' = \sum_{x' = x^2} x' \otimes x'' + \sum_{x' \neq x^2} x' \otimes x''$ then if $|x|$ is even

$$
S_{\xi_*}[x]/2 (x) = \sum_{x' = x^2} x' \otimes x''
$$

because $S_{\xi_*}[x]/2$ is dual to the cup square and the diagonal $D_*$ is dual to the cup product. In other words

$$
(\tilde{D})_* (id \otimes \xi_*)(x \otimes [e^{n-1} \otimes y \otimes y]) = \sum_{x' = x^2} e_{n-1} \otimes (S_{\xi_*}[x]/2 x \otimes y) \otimes (S_{\xi_*}[x]/2 x \otimes y)
$$

$$
+ \sum_{x' \neq x^2} e_{n-1} \otimes x' \otimes y \otimes x'' \otimes y
$$

where the first term has to considered as zero if $x$ has odd degree. We compute $\xi_*(G)$ is non zero if and only if $x$ has even degree (Formula (6)), in which case we have

$$
\xi_*(G) = \sum_{i, r, r+2i-|x| \leq n-1} [e^{r+2i-|x|} \otimes (S_{\xi_*}[x]/2 x \otimes y) \otimes (S_{\xi_*}[x]/2 x \otimes y)]
$$

$$
= e_{n-1} \otimes (S_{\xi_*}[x]/2 x \otimes y) \otimes (S_{\xi_*}[x]/2 x \otimes y)
$$

$$
= b_{n-1} \otimes (S_{\xi_*}[x]/2 x \otimes y) \otimes (S_{\xi_*}[x]/2 x \otimes y)
$$

So it follows that

$$
\xi_*(\Gamma) = \sum_{x' \neq x^2} b_{n-1} \otimes x' \otimes y \otimes x'' \otimes y
$$

In view of the formulas for $\xi_*(\Gamma)$, we must have

$$
\Gamma = \sum_{|x'| < |x''|} [e^{n-1} \otimes x' \otimes y \otimes x'' \otimes y].
$$
4.5 Higher BV Relations

Let \( j > 0 \). Putting together the previous calculations, we get:

\[
\Delta_i Q_j x = \tilde{\delta}_i (\tilde{a}_2)_* (1 \otimes (\tilde{a}_1)_*) (D_\ast \otimes 1) (d_i \otimes [e^j \otimes x \otimes x]) = \tilde{\delta}_i (\tilde{a}_2)_* (1 \otimes (\tilde{a}_1)_*) (\sum_{m+i=i} (d_m \otimes d_i) \otimes [e^j \otimes x \otimes x])
\]

\[
= \tilde{\delta}_i (\tilde{a}_2)_* (\sum_{m+i=i} d_m \otimes [d_i, e^j \otimes x \otimes x])
\]

\[
= \tilde{\delta}_i (\tilde{a}_2)_* (d_i \otimes [e^j \otimes x \otimes x]) = \tilde{\delta}_i (\psi_\ast \varphi_\ast) (d_i \otimes [e^j \otimes x \otimes x])
\]

\[
= \tilde{\delta}_i \psi_\ast \left( \sum_{k,j+2k-1 \leq n} [e^{j+2k-i} \otimes (Sq^k_\ast d_i \otimes x) \otimes (Sq^k_\ast d_i \otimes x)] + \delta_{2j}^{n-1} [e_{n-1} \otimes \sum_{a+b=i, a < b} (d_\alpha \otimes x) \otimes (d_\beta \otimes x)] \right)
\]

Recall that \( a \) is the action of \( SO(n) \) on \( Z \), and by \( \Delta_i x \) the element \( a_\ast (d_i \otimes x) \). If we let \( Sq^k_\ast \Delta_i x \) denote \( a_\ast (Sq^k_\ast d_i \otimes x) \), we get:

\[
\Delta_i Q_j x = \sum_{k,j+2k-1 \leq n} Q_{j+2k-1} (Sq^k_\ast \Delta_i x) + \delta_{2j}^{n-1} \sum_{a+b=i, a < b} \{ \Delta_\alpha x, \Delta_\beta x \}
\]

The action of the Steenrod algebra in our case is well known:

\[
Sq^k_\ast d_i = C_i^{n-k} d_{i-k} \text{ for } 2k \geq i.
\]

This comes from the action of the Steenrod algebra on \( H_\ast \mathbb{R}P^n \). Here \( C_i^{n-k} \) denotes the usual binomial coefficient \( k!/(i-k)!(i-k)! \) reduced modulo 2.

**Theorem 4.4** The following formula holds in the homology of \( fD_n \)-spaces:

\[
\Delta_i Q_j x = \sum_{k,j+2k-1 \leq n, 2k \geq i} C_i^{n-k} Q_{j+2k-1} (\Delta_{i-k} x) + \delta_{2j}^{n-1} \sum_{a+b=i, a < b} \{ \Delta_\alpha x, \Delta_\beta x \}
\]

In particular, if \( \Delta_i \) does not support any Steenrod operations, for example if we consider the operation associated to the primitive \( p_{2i-1} \in H_\ast SO(n) \), we get:

\[
\Delta_{p_{2i-1}} Q_j x = Q_{j-p_{2i-1}} \Delta_{p_{2i-1}} x + \delta_{p_{2i-1} - 1}^{n-1} \{ x, \Delta_{p_{2i-1}} x \}
\]

We insist again that these formulas hold for any \( fD_n \)-algebra. By specializing the result to the case \( n = 2, i = j = 1 \), we obtain:

**Corollary 4.5** Let \( \Omega^2 X \) be a 2-fold loop space, and \( x \) be an element in \( H_\ast \Omega^2 X \)

\[
\Delta(Q_1 x) = \{ \Delta(x), x \} + \Delta(x) \ast \Delta(x) = \Delta(x \ast (\Delta x)).
\]

Where \( \Delta := \Delta_1 = BV \) is the Batalin-Vilkovisky operator. Here we have considered \( X \) as a trivial \( SO(2) \)-space. We insist again that such a formula holds in the \textit{modulo} 2 homology of any \( fD_2 \) algebra.

In particular if \( Y \) is an infinite loop space, all the Browder brackets on \( H_\ast Y \) are trivial, all Kudo-Araki operations are defined, and there is an action of the infinite special orthogonal group \( SO \). For example, in the case of infinite loop spaces, we have:

\[
\Delta_i Q_j x = Q_j \Delta_{i/2} x \text{ for } 2i > j.
\]
Here $\Delta_{i/2}$ is zero if $i$ is odd. For example $\Delta_{2p+1}Q_0$ is trivial. In other words, the odd generators of $H_*SO$ act trivially on the Pontryagin squares in the homology of infinite loop spaces. More generally, as $H_*SO(n) = \Lambda(d_1, \ldots, d_{n-1})$, any homogeneous element of odd degree is a sum of homogeneous monomials, each of them having some $d_i$ factor with $i$ odd. Hence, each such monomial act trivially on Pontryagin squares.

**Corollary 4.6** Let $Y$ be an infinite loop space. The action of $H_{\text{odd}}SO$ on $H_*Y$ is trivial on Pontryagin squares.

**Remark 4.7** We have noticed in previous work that the action of $SO(n)$ on an $n$-fold loop space can be interpreted in terms of $\odot$-products and of the $J$-homomorphism [5, section 5]. Formulas relating the circle product with elements coming from $H_*Q_1S^0$ (whom the $J$-homomorphism factors through) and Kudo-Araki operations coming from the loop structure are already known (see for example Formula (2.2)). These formulas come from the structure of $E_\infty$-module over the $E_\infty$-ring $QS^0$. Albeit less general (as this works only for elements coming from the $J$-homomorphism), the new set of formulas given above is new and quite simple. It relies on the structure of $fS_n$-space. It should be in principle possible to get our formulas from Formula (2.2) directly, by using an explicit description of the generators of the homology image of $J$. This looks however quite tedious. We will go a little further in this direction in Section 6.

5 Calculations of the Batalin-Vilkovisky structures of $H_*(\Omega^2S^k)$ for $k \geq 2$

In this section, we accomplish the promised computations. As before, the notation $*$ stands for Pontryagin products.

5.1 Classical splittings

We denote by $\eta : S^3 \to S^2$, $\nu : S^7 \to S^4$, and $\sigma : S^{15} \to S^8$ the classical maps of Hopf invariant 1. Let $a \in \pi_3SO(2)$, $b \in \pi_3SO(4)$ and $c \in \pi_7SO(8)$ be generators of the $\mathbb{Z}$-summands that map to the Hopf maps in the corresponding degrees under the $J$-homomorphism. We denote the corresponding stable elements in the homotopy of $SO$ and $QS^0$ in the same way.

**Lemma 5.1** Let $F \to E \to B$ be a homotopy fiber sequence with $F$, $E$ and $B$ path-connected. Suppose that there is a map $s : B \to E$ such that the composite $p \circ s$ is a homotopy equivalence and that $E$ is an $H$-space with multiplication $\mu$. Then the composite $\mu \circ (j \times s) : F \times B \xrightarrow{\sim} E$ is a homotopy equivalence. In particular $j$ admits a retract up to homotopy.

Let $f = \eta, \nu$ or $\sigma : S^{2d-1} \to S^d$, $d \in \{2, 4, 8\}$, be a Hopf fibration. Let $\partial : \Omega S^d \to S^{d-1}$ be the connecting homomorphism. Let $\iota : S^{d-1} \to \Omega S^d$ be the adjoint of the identity map of $S^d$. Since $\pi_{d-1}(\partial)$ is an isomorphism, $\pi_{d-1}(\iota)$ maps the generator $\iota$ to $\pm \iota d_{S^d}$. Therefore up to signs, $\iota$ is a section up to homotopy of $\partial$. By applying Lemma 5.1 to the homotopy fiber sequence $\Omega S^{2d-1} \xrightarrow{\Omega f} \Omega S^d \xrightarrow{\partial} S^{d-1}$, we obtain that the loop sum $\iota * \Omega f : S^{d-1} \times \Omega S^{2d-1} \xrightarrow{\sim} \Omega S^d$ is a homotopy equivalence and in particular $\Omega f : \Omega S^{2d-1} \to \Omega S^d$ has a retract up to homotopy. As a consequence, by looping once again, one obtains splittings:

\[
\begin{align*}
\Omega^2S^2 & \cong \Omega S^1 \times \Omega^2S^3 \\
\Omega^4S^4 & \cong \Omega^3S^3 \times \Omega^4S^7 \\
\Omega^8S^8 & \cong \Omega^7S^7 \times \Omega^8S^{15}.
\end{align*}
\]
And we obtain that the 4-fold loop map $\Omega^4 \nu : \Omega^4 S^7 \to \Omega^4 S^4$ and the 8-fold loop map $\Omega^8 \sigma : \Omega^8 S^{15} \to \Omega^8 S^8$ are injective in homology. In the case of $\eta$, since $\Omega^2 S^1 \to \Omega^2 S^3$ is a fibration with path connected base and contractible fiber, we have that the two fold loop map $\Omega^2 \eta : \Omega^2 S^3 \to \Omega^2 S^2$ is a homotopy equivalence.

Let $ad_d : \pi_+ d(X) \xrightarrow{\sim} \pi_+(\Omega^d X)$ denote the adjunction map. We have the following commuting diagram

$$
\begin{array}{ccc}
\pi_{2d-1} S^{2d-1} & \xrightarrow{ad_d} & \pi_{d-1} \Omega^d S^{2d-1} \\
\pi_{2d-1}(f) & \xrightarrow{ad_d} & \pi_{d-1}(\Omega^d f) \\
\pi_{2d-1} S^d & \xrightarrow{ad_d} & \pi_{d-1} \Omega^d S^d \\
\pi_{2d-1}(f) & \xrightarrow{ad_d} & \pi_{d-1}(\Omega^d f) \\
\end{array}
$$

where $J$ is the classical $J$-homomorphism and $J$ is defined, for example, as in [5, section 5]. Recall also that $H$ is the (modulo 2) Hurewicz homomorphism. Since $H_{d-1}(\Omega^d f)$ is injective, $H \circ ad_d \circ \pi_{2d-1}(f)$ is also injective and so

$$H \circ ad_d \circ \pi_{2d-1}(f)(id_{S^2}) = H \circ ad_d(f) \neq 0,$$

where $ad_d(f)$ is detected by the Hurewicz homomorphism for $\Omega^d S^4$. Since $J(a) = \eta$ and $H \circ ad_d(\eta) = 0$, it follows that $H(a) = 0$. Similarly $H(b) \neq 0$ and $H(c) \neq 0$. We have finally proved:

**Lemma 5.2** The Hopf maps $\eta \in \pi_1 \Omega^2 S^2$, $\nu \in \pi_3 \Omega^4 S^4$, and $\sigma \in \pi_7 \Omega^8 S^8$ as well as their preimages $a \in \pi_1 SO(2)$, $b \in \pi_3 SO(4)$, $c \in \pi_7 SO(8)$ under the $J$-homomorphism are detected by the Hurewicz homomorphism. This holds unstably as well as stably.

The stable case follows for instance from the unstable one by using the work of Milgram [10], where it is shown that the mod 2 homology of $SO$ injects in the mod 2 homology of $QS^0$ via the $J$-homomorphism.

### 5.2 Homology of $\Omega^2 S^3$ and $\Omega^2 S^2$ as BV-algebras

The homology of $\Omega^2 S^3$, as a Pontryagin algebra, is polynomial on generators $u_n$ of degree $2^n - 1$, $n \geq 1$, where $u_n = Q_1(u_{n-1})$ (see Cohen’s work in [3]). We first notice that $u_1$ is the bottom non trivial class in positive degrees, and as such, must be in the image of the Hurewicz homomorphism, according to the Hurewicz theorem. But $\pi_1 \Omega^2 S^3$ is infinite cyclic generated by $\epsilon = ad_2(Id_{S^3})$ the adjoint of the identity of $S^3$.

Since $u_1 = H \epsilon = H \circ ad_2(id_{S^3})$, thanks to part 1) of [5, corollary 5.7],

$$BV(u_1) = H(ad_2(\Sigma \eta))$$

where $ad_2(\Sigma \eta) \in \pi_2(\Omega^2 S^3)$ is the adjoint of suspension of the Hopf map $\Sigma \eta \in \pi_2 S^3$. We can conclude that $H(ad(\Sigma \eta))$ has to be non zero, because the two-fold loop equivalence $\Omega^2 S^3 \to \Omega_0^2 S^2$ takes $ad(\Sigma \eta)$ to $\Sigma \eta \circ \eta$ which is well known to be spherical (because it is a map of Kervaire invariant one and such elements are known to be spherical, see Remark $\bf{[2]}$). We will nevertheless give a direct argument to see that $ad(\Sigma \eta)$ is detected by the Hurewicz homomorphism. We want to show that the Hurewicz homomorphism for $\Omega^2 S^3$ is non trivial in degree 2. This can be seen as follows. We know that $\pi_2 \Omega^2 S^3 \cong \mathbb{Z}$ and $\pi_2 \Omega^2 S^2 \cong \mathbb{Z}/2\mathbb{Z}$ generated by $ad_2(\Sigma \eta)$.

Let $p : S^3 \to K(\mathbb{Z}, 3)$ represent the generator for the third integral cohomology group of $S^3$, $\pi_3 K(\mathbb{Z}, 3) \cong H^3(S^3, \mathbb{Z})$. Let $S^3(3)$ be the homotopy fiber of $p$ and let $j : S^3(3) \to S^3$ be the fiber
inclusion. Let \( \iota : S^1 \to \Omega^2 S^3 \) the adjoint of the identity of \( S^3 \). Since \( S^3(3) \) is 3-connected, \( \pi_1(\Omega^2 p) \) is an isomorphism and so maps \( \iota \) to \( \pm \text{id}_{S^1} \). Therefore by applying Lemma 5.1 to the homotopy fiber sequence \( \Omega^2 S^3(3) \xrightarrow{\Omega^2 j} \Omega^2 S^1 \xrightarrow{\Omega^2 p} \Omega^2 K(\mathbb{Z}, 3) \simeq S^1 \), we obtain that \( \Omega^2 j : \Omega^2 S^3(3) \to \Omega^2 S^3 \) has a retract up to homotopy and so is injective in homology. Since \( \pi_4(S^1) = \pi_4(S^3) = 0 \), \( \pi_2(\Omega^2 j) : \pi_2(\Omega^2 S^3(3)) \xrightarrow{\simeq} \pi_2(\Omega^2 S^3) \simeq \mathbb{Z}/2\mathbb{Z} \) is an isomorphism. Since \( \Omega^2 S^3(3) \) is simply connected, the Hurewicz homomorphism \( \mathcal{H} : \pi_2(\Omega^2 S^3(3)) \xrightarrow{\sim} H_2(\Omega^2 S^3(3)) \) is an isomorphism. Since \( H_2(\Omega^2 j) \) is also an isomorphism, the Hurewicz homomorphism

\[
\mathcal{H} : \pi_2(\Omega^2 S^3) = \mathbb{Z}/2\mathbb{Z}.\text{ad}_2(\Sigma_{\eta}) \xrightarrow{\sim} H_2(\Omega^2 S^3) = \mathbb{Z}/2\mathbb{Z}.u_1^2
\]

is an isomorphism. Hence \( \mathcal{H}(\text{ad}_2(\Sigma_{\eta})) = u_1^2 \) and so \( BV(u_1) = u_1^2 \). We refer to the appendix in [5] for a different derivation of this result.

As next step, we compute using Corollary 5.3 that

\[
BV(u_2) = BV(Q_1u_1) = \{BV(u_1), u_1\} + BV(u_1) \ast BV(u_1) = (u_1)^4.
\]

because all Browder brackets vanish in \( H_2(\Omega^2 S^3) \): \( S^3 \) is a Lie group, hence a loop space and therefore \( \Omega^2 S^3 \) is in fact a 3-fold loop space (Recall that the Browder bracket is an obstruction to extend the 2-fold loop structure to a 3-fold one, see Remark 2.3. We thus have \( BV(u_2) = (u_1)^4 \), and more generally, for \( n > 1 \), we get by induction:

\[
BV(u_n) = BV(Q_1u_{n-1}) = \{BV(u_{n-1}), u_1\} + (BV(u_{n-1}))^2 = u_1^n.
\]

As all brackets are trivial, \( BV \) is a derivation for the Pontryagin product, and we obtain:

**Theorem 5.3** The action of the Batalin-Vilkovisky operator \( BV \) on \( H_*(\Omega^2 S^3, \mathbb{F}_2) = \mathbb{F}_2[u_i; i \geq 1] \) is given on Pontryagin monomials \( u_1^{\ell_1} \ast u_2^{\ell_2} \ast \ldots \ast u_n^{\ell_n} \) with \( \ell_j > 0 \) by

\[
BV(u_1^{\ell_1} \ast u_2^{\ell_2} \ast \ldots \ast u_n^{\ell_n}) = \sum_{j=1}^{n} \ell_j u_1^{2\ell_j} \ast u_1^{\ell_1} \ast u_2^{\ell_2} \ast \ldots \ast u_{j-1}^{\ell_{j-1}} \ast \ldots \ast u_n^{\ell_n}.
\]

(32)

where the coefficient \( \ell_j \) in the sum is of course reduced \( \text{mod } 2 \).

Thanks to the two-fold loop equivalence \( H_*\Omega^2 S^3 \cong H_*\Omega^2 S^3 \), we can compute the \( BV \) structure of \( H_*\Omega^2 S^3 \). Indeed, let \([i]\) be the zero dimensional class in \( H_*\Omega^2 S^2 \) corresponding to the degree \( i \) component. All elements of \( H_*\Omega^2 X \) are uniquely of the form \( x \ast [i] \) with \( x \in H_*\Omega^2 S^2 \). Hence there is an isomorphism of algebras (for the Pontryagin product)

\[
H_*\Omega^2 S^3 \cong \mathbb{F}_2[\mathbb{Z}] \ast H_*\Omega^2 S^2.
\]

We have to compute \( BV([i]) \) and Browder brackets of the form \( \{[i], u_1^{\ell_1} \ast u_2^{\ell_2} \ast \ldots \ast u_n^{\ell_n}\} \): the action of \( BV \) on a general element \( [i] \otimes u_1^{\ell_1} \ast u_2^{\ell_2} \ast \ldots \ast u_n^{\ell_n} \) follows by the Batalin-Vilkovisky Formula (39). We begin with the following lemma, which follows from the discussion in [5] section 5:

**Lemma 5.4** For any pointed topological space \( X \) and for any element \( g \) in \( H_*(\Omega^2 X) \), the following formula holds:

\[
BV(g) = g \circ (u_1 \ast [1])
\]

where \( \circ \) is the map induced in homology by the composition action:

\[
\Omega^2 S^2 \times \Omega^2 X \to \Omega^2 X.
\]
In particular, in $H_*\Omega^2S^2$:

$$BV([1]) = [1] \circ (u_1 \ast [1]) = (u_1 \ast [1])$$

and more generally:

**Lemma 5.5** For any $i \in \mathbb{Z}$, the following formula holds

$$BV([i]) = i(u_1 \ast [i]) .$$

Indeed,

$$BV([2i]) = BV([i] \ast [i]) = 2BV[i] = 0$$

while

$$BV([2i + 1]) = BV([1]) \ast [2i] = u_1 \ast [2i + 1]$$

by Formula (39).

**Lemma 5.6** All Browder brackets are trivial in $H_*\Omega^2S^2$.

**Proof.** Firstly, all brackets are trivial on the subobject $H_*\Omega^3S^2 \cong H_*\Omega^2S^3$ as shown above. We have:

$$\{[1], u_1\} = BV([1] \ast u_1) + BV[1] \ast u_1 + [1] \ast BV(u_1) = BV(BV[1]) + u_1 \ast u_1 + [1] + u_1 \ast [1]$$

$$= BV^2([1]) + 2(u_1 \ast u_1 \ast [1]) = 0 .$$

By induction on $i$, the brackets $\{[i], u_1\}$ are trivial for all $i \in \mathbb{Z}$. Now, there is a general formula in the homology of two fold loop spaces [3, Theorem 1.3 (4) p. 218]

$$\{x, Q_1y\} = \{\{x, y\}, y\}$$

hence

$$\{[1], u_2\} = \{[1], Q_1u_1\} = \{\{[1], u_1\}, u_1\} = 0$$

and more generally, using Formula (39) inductively we get for $u_{n+1} = Q_1u_n$ that $\{[1], u_{n+1}\} = 0$. We conclude with the Formula (39) that all brackets are trivial in $H^\ast\Omega^2S^2$.

Gathering all these calculations, we obtain:

**Theorem 5.7** The BV structure on $H_*\Omega^2S^2 \cong \mathbb{F}_2[\mathbb{Z}] \otimes H_*\Omega^2S^3$ is given by restriction on $H_*\Omega^2S^3$ (theorem above) by the on the component $[0] \otimes H_*\Omega^2S^3$. For $i \neq 0$ and $f \in H_*\Omega^2S^3$, the action of the BV operator is given by:

$$BV([i] \ast f) = BV([i]) \ast f + [i] \ast BV(f)$$

with $BV([i]) = [i] \ast u_1$ if $i$ is odd and zero otherwise.
5.3 The BV-structure of $H_*\Omega^2S^{k+2}$ for $k > 1$

The homology of $\Omega^2S^{k+2}$ is, as a Pontryagin algebra, the polynomial algebra $F_2[u_n, n \geq 1]$ on generators $u_n = Q_1u_{n-1}$ of degree $2^{n-1}(k + 1) - 1$ (see Cohen’s work in \[3\]). By Corollary 4.3 for $n > 1$,

$$BV(u_n) = BV(Q_1u_{n-1}) = \{BV(u_{n-1}), u_{n-1}\}.$$ 

Since $k > 1$, $BV(u_1) = 0$, for degree reasons. Hence by induction, $BV(u_n) = 0$ for all $n \geq 1$.

We are going now to see that all the Browder brackets vanish in $H_*\Omega^2S^{k+2}$. According to \[3\] Theorem 1.2 (3) p. 215, \{x, x\} = 0, and according to the Formula \[36\]. Therefore \{u_n, u_n\} = 0 and \{u_{n+1}, u_n\} = \{u_n, u_n\}. So by induction, \{u_{n+i}, u_n\} = 0. By anti-commutativity of the Browder bracket, for all $m, n \geq 1$, \{u_m, u_n\} = 0. Since the Browder brackets are trivial on the generators, using the Poisson relation, all the Browder brackets are trivial on $H_*\Omega^2S^{k+2}$. Therefore by \[36\], the operator $BV$ is a derivation with respect to the Pontryagin product. It follows that $BV$ is the trivial operator.

**Theorem 5.8** The operator $BV$ acts trivially on the module 2 homology of $\Omega^2S^{2+k}$ for $k > 1$.

6 Some calculations of the mod 2 Hurewicz homomorphism for $QS^0$

We wish to describe a method to get information on the mod 2 Hurewicz homomorphism for $QS^0$. We recall that this map is conjecturally described by the conjecture stated \[18\] under the name of Curtis-Madsen conjecture.

**Conjecture 6.1** \[18\] chap. 1] Let $f \in \pi_\ast QS^0$ be any stable map of positive degree. Then $f$ is not in the kernel of the mod 2 Hurewicz homomorphism if and only if $f$ has Hopf or Kervaire invariant one.

**Remark 6.2** It is well known, in connection with the Kervaire invariant problem, that maps of Kervaire invariant one (when they do exist) are detected by the Hurewicz homomorphism (see for instance \[3\]). The composition squares of the Hopf maps are maps of Kervaire invariant one, and we will recover in the following that they are detected by the Hurewicz homomorphism by direct calculations.

There is for each $n > 0$ a natural inclusion $SO(n) \rightarrow \Omega^nS^n$ by compactifying the natural action of $SO(n)$ on $\mathbb{R}^n$. These inclusions are compatible as $n$ grows, and stabilize to a map $\Theta : SO \rightarrow Q_1S^0$. We notice that the effect of

- the $\sigma$-product on $H_*Q_1S^0$,
- the map $H_*\Theta : H_*SO \rightarrow H_*Q_1S^0$,
- and the map $J : \pi_\ast SO \rightarrow \pi_\ast QS^0$,

are completely known by \[12\]. Let us only recall what is relevant for our purposes. First the map $H_*\Theta : H_*SO \rightarrow H_*Q_1S^0$ is injective and takes the Pontryagin product in $H_*SO$ to the circle product (the product in homology induced by the topological monoid structure given by composition of loops). Finally we point out that $H_*QS^0$ is a polynomial algebra for the Pontryagin loop sum.

We notice that we already know that the following elements $\eta, \nu, \sigma,$ and $\eta^2$ are spherical detected by the Hurewicz homomorphism. Once again, it is possible to provide a direct argument for this fact in the special case of $\nu^2$ and $\sigma^2$. Let us check the case of $\nu^2$, as the case of $\sigma^2$ is quite similar. One has:

$$H(\nu^2) = H(\nu)H(\nu) = \delta_{\nu^2}(H(\nu)) = \delta_{\nu^2}(\theta * [-1])$$  \[36\]
where $\theta = H_*(\Theta)(H(b))$. The equality \[24\] expresses the fact that the action of $SO(n)$ on loops spaces corresponds to the composition (see \[25\] Theorem 5.6). The equality \[35\] corresponds to the fact that $H(\nu)$ is non zero and must be primitive in $H_*SO(n)$ (as anything lying in the image of $H$), hence its action is that of $\Delta_p$. Finally the equation \[30\] is the observation that $\nu = J(a)$ is in the image of the $J$-homomorphism, that decomposes as $J = (- * [-1])\Theta$. $\Delta_p$ is a derivation for the Pontryagin product, and we have
\[
\Delta_p \theta [-1] = \Delta_p \theta [-1] + \theta \Delta_p[-1] .
\]
But $\Delta_p$ is the composition product with $\theta$, hence
\[
\Delta_p(\theta [-1]) = (\theta \circ \theta) [-1] + \theta \star (\theta \circ [-1]) = \theta \star \chi \theta = \theta \star \theta \star [-2] = (\theta \star [-1])^2
\]
because $H_*SO$ is an exterior algebra. Here $\chi$ is the antipode of the Hopf algebra $H_*QS^0$. In particular, $H(\nu^2)$ is non zero. One shows in the same way that $\sigma^2$ is detected by the Hurewicz homomorphism, consistently with Remark \[6.2\].

Another observation that one can do is:
\[
H(\eta^3) = H(\eta).H(\eta^2) = \Delta_4(H(\eta)H(\eta)) = (BV)^2(H(\eta)) = 0 .
\]
Because $BV$ has order 2, as well as any higher $BV$-operator because $H_*SO(n)$ is an exterior algebra. This work in the same way for $\nu$ and $\sigma$, and we deduce:
\[
H(\nu^3) = H(\sigma^3) = 0 .
\]
Alternatively, one could argue as follows to show for instance that $H(\eta^3)$ vanishes. The element $H(\eta)$ is $J_*H(\eta(a))$ where $a$ was the generator of $\pi_1SO$. Hence $H(a)$ is a homogeneous polynomial of odd degree in $H_*SO$. But from Corollary \[4.6\] we know that the action of odd degree homogeneous elements in $H_*SO$ on Pontryagin squares is trivial, hence:
\[
H(\eta^3) = \Delta_4(H(\eta^2)) = \Delta_4(\theta \star [-1])^2 = 0 .
\]
This works all the same for $\nu^3$ and $\sigma^3$.

To go further, we need one more result, which the reader might draw from the information in \[1\].

**Proposition 6.3** The mod 2 Hurewicz homomorphism for $SO$ annihilates all classes except those yielding an element of Hopf invariant one.

We note that we have already proved (Lemma \[5.2\]) that the classes $a$, $b$, and $c$ are detected by the Hurewicz homomorphism for $SO$. The fact that the rest of the homotopy is not follows from a simple computation using Bott periodicity and the information in \[1\].

By a result of Novikov \[17\] if $\alpha$ is in $\text{Im}(J)_1$ and $\beta$ is in $\Theta_j$ with $j < 2i$ then $\alpha\beta$ is again in $\text{Im}(J)$.

Here $\Theta_j$ classically denotes the subgroup of the $j$th stable homotopy group $\pi^S_*$ of the sphere spectrum that consists in elements representable by homotopy spheres. The classical works on exotic spheres show that the quotient of the stable stem $\pi^S_*$ by $\Theta_i$ has at most order two, and non triviality of the cokernel happens only in degrees of the form $2n - 2$. Hence from the above Proposition \[6.3\] we could already deduce particular cases of Theorem \[6.3\] which is however much more general.

**Definition 6.4** We say that an element $\theta \in \pi_*QS^0 \cong \pi_*^S$ is divisible by $x \in \pi_*QS^0 \cong \pi_*^S$ if $\theta = x\theta'$ for some $\theta'$.

Hence the set of elements divisible by those of a fixed subset of $\pi_*^S$ is the ideal generated by this subset. Now we can state a result that fits perfectly with the Curtis-Madsen conjecture recalled in \[6.1\]
Theorem 6.5 All classes in \( \pi_\ast QS^0 \) divisible by the image of the \( J \)-homomorphism are annihilated by the mod 2 Hurewicz homomorphism, except the Hopf maps and their composition squares which are not annihilated. In other words, Conjecture 6.1 holds on the ideal generated by the image of the \( J \)-homomorphism.

Before proceeding to the proof, one might wonder how strong this result is. Are there many elements that are not in the image of the \( J \)-homomorphism, but divisible by the image of the \( J \)-homomorphism? The general problem of describing the behaviour of the multiplication by elements in the image of the \( J \)-homomorphism in \( \pi_\ast S^0 \) is probably very difficult. Nevertheless, one can explicitly construct some infinite families of elements in the stable homotopy groups of spheres that are divisible by the image of the \( J \) homomorphism, but are not in the image of the \( J \)-homomorphism. This can be done for instance by using the spectrum of topological modular forms. In [2], a computation of the homotopy groups of this spectrum is produced. We see that

- the spectrum of topological modular forms \( \text{tmf} \) is a ring spectrum,
- the unit map \( S^0 \rightarrow \text{tmf} \) detects the Hopf maps \( \eta \) and \( \nu \),
- the homotopy groups of \( \text{tmf} \) have a periodicity of order 192,
- there are periodic families of chromatic filtration 2 that are detected by \( S^0 \rightarrow \text{tmf} \), and whose image support a non trivial multiplication by \( \eta \) or \( \nu \) in the homotopy of \( \text{tmf} \).

So, each of these families provides infinitely many elements divisible by the image of the \( J \)-homomorphism but not in the image of the \( J \)-homomorphism. A concrete example is given by the periodic family generated by \( \eta \kappa \), which is divisible by \( \eta \) (see [9, p. 24]).

Proof. As we noticed before, the Hopf maps \( \eta \), \( \nu \) and \( \sigma \), and their composition squares are detected by the Hurewicz homomorphism (Lemma 5.2), while for \( n > 2 \), \( \eta^n \), \( \nu^n \) and \( \sigma^n \) are all annihilated by the Hurewicz homomorphism (observe that in fact these elements are zero for \( n > 0 \)). Assume now that \( \varphi = \psi \theta \), where \( \psi \) is an element of the image of the \( J \)-homomorphism which is not a Hopf map. Then we have

\[
\mathcal{H}(\varphi) = \mathcal{H}(\psi \theta) = \mathcal{H}(\psi) \circ \mathcal{H}(\theta).
\]

But Proposition 6.3 states that \( \mathcal{H}(\psi) \) is trivial if \( \psi \) is not a Hopf map. It remains to prove that \( \mathcal{H}(\psi \theta) = 0 \) as soon as \( \psi \) is one of the Hopf maps. The first case is \( \psi = \eta \). We have:

\[
\mathcal{H}(\eta \theta) = \mathcal{H}(\eta) \circ \mathcal{H}(\theta) = \Delta_1(\mathcal{H}(\theta)).
\]

This follows from [5] Recall that \( H_* QS^0 \) is generated by Kudo-Araki operations applied to the element [1] and shifting components by \( - \ast [i] \) for all \( i \). In particular \( \mathcal{H}(\eta) \) being non zero in \( H_1 Q_0 S^0 \), we have:

\[
\mathcal{H}(\eta) = Q_1([1]) \ast [-2] = Q_1^1([1]) \ast [-2]
\]

Indeed, the only non zero element of degree one in \( H_* QS^0 \) are of the form \( Q_1^1([1]) \ast [i] \). \( Q_1^1([1]) \) belongs to \( H_* Q_2 S^0 = H_* Q_0 S^0 \ast [2] \subset H^* QS^0 \), and the Hurewicz homomorphism lands in \( H^* Q_0 S^0 \subset H_* QS^0 \) as \( Q_0 S^0 \) is the component of the basepoint.

According to [13] theorem 6.18, there is a formula

\[
Q^i [1] \circ a = \sum_k Q^{i+k} (S^k a)
\]

(37)
where $Sq^k_*$ are dual to the Steenrod operations. In our case, we obtain:

$$ \mathcal{H}(\eta \theta) = \sum_k Q^{1+k} Sq^k_*(\mathcal{H}(\theta)) .$$

On the other hand, all Steenrod operations of strictly positive degree vanish on spherical elements, hence $\mathcal{H}(\eta \theta) = Q^1(\mathcal{H}(\theta))$. We conclude that by instability (see Remark 2.1) that $\mathcal{H}(\eta \theta) = 0$ if $\theta$ has degree strictly greater than 1. On the other hand in degree 1, the only possible non-trivial element is $\theta = \eta$, and we already know that $\eta \eta$ is detected by the Hurewicz homomorphism.

Wellington [20, remark 5.8, p. 47] provides formula for $\mathcal{H}(\nu)$ and $\mathcal{H}(\sigma)$ in terms of Kudo-Araki operations applied to [1]. Formula (37) generalizes to the formula (proposition 1.6 of May’s article *The homology of \( E_\infty \)-ring spaces* in [3]):

$$ Q^ix \circ a = \sum_k Q^{i+k}(x Sq^k_* a) .$$

This, together with an instability argument yields:

**Lemma 6.6** If the degree of $\theta$ is strictly bigger than 7, $\mathcal{H}(\nu \theta)$ and $\mathcal{H}(\sigma \theta)$ are trivial.

In degree less than 7, a system of generators for $\pi_*^S$ is given by $\{ \eta, \eta^2, \nu, \nu^2, \sigma \}$. These cases have already been taken care of. This finishes the proof.

### A Two definitions

We recall first the definition of Gerstenhaber algebras.

**Definition A.1** An $e_n$-algebra is a commutative graded algebra $A$ equipped with a linear map $\{-, -\} : A \otimes A \to A$ of degree $n-1$ such that:

a) the bracket $\{-, -\}$ gives $A$ the structure of graded Lie algebra of degree $n-1$. This means that for each $a, b, c \in A$,

$$ \{a, b\} = (-1)^{|a||n-1| + |b||n-1|} \{b, a\} \quad \text{and} \quad \{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a||n-1| + |b||n-1|} \{b, \{a, c\}\} .$$

b) the product and the Lie bracket satisfy the Poisson relation:

$$ \{a, bc\} = \{a, b\} c + (-1)^{|a|+|n-1|} b \{a, c\} .$$

We also recall the definition of a $BV_n$-algebra.

**Definition A.2** [17, Def 5.2] A $BV_n$-algebra $A$ is an $e_n$-algebra with a linear endomorphism $BV : A \to A$ of degree $n-1$ such that $BV \circ BV = 0$ and for each $a, b \in A$,

$$ \{a, b\} = (-1)^{|a|} \left( BV(ab) - (BVa)b - (-1)^{|a|} a (BVb) \right) .$$  \[(39)\]

The bracket measures the deviation of the operator $BV$ from being a derivation with respect to the product.
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