Explicit Construction of the Inverse of an Analytic Real Function: Some Applications

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Abstract: In this paper, we introduce a general procedure to construct the Taylor series development of the inverse of an analytical function; in other words, given \( y = f(x) \), we provide the power series that defines its inverse \( x = h_f(y) \). We apply the obtained results to solve nonlinear equations in an analytic way, and generalize Catalan and Fuss–Catalan numbers.

Keywords: inverse functions; Taylor series; Taylor Remainder; nonlinear equations; Catalan numbers; Fuss–Catalan numbers

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1. Introduction

In this paper, we have taken as a basis the previous works [1,2], in which the inverse of a polynomial function is constructed, with the aim of generalizing the methods developed there to any analytic function. That is to say: given an analytic function around the point \( x_0 \):

\[
f(x) = f(x_0) + \sum_{p=1}^{\infty} \frac{f^{(p)}(x_0)}{p!} (x - x_0)^p
\]

where \( f^{(p)}(x) \) is the \( p \)-th derivative of \( f(x) \), throughout these lines, we construction the function, \( x = h_f(y) \), with \( y = f(x) \) and, therefore, \( x_0 = h_f(y_0) \) and \( y_0 = f(x_0) \), such that:

\[
h_f(y) = h_f(y_0) + \sum_{p=1}^{\infty} \frac{h_f^{(p)}(y_0)}{p!} (y - y_0)^p
\]

To accomplish this task we have organized this article as follows:

- In Section 2, background theory is presented.
- In Section 3, the successive derivatives of \( h_f \) are computed in an explicit way.
- In Section 4, a bound for \( |h_f^{(p)}(y_0)| \) (see (1)) is found.
- In Section 5, we study the convergence and establish the radius of convergence and Taylor Remainder of series (1).
- In the next two sections, we introduce some applications for solving nonlinear equations in an analytic way, and to generalize the Catalan and Fuss–Catalan numbers.
In the last section, we present our conclusions. We recall that Catalan numbers sequence is defined as:
\[
C_0 = 1; \quad C_n = \frac{1}{n + 1} \binom{2n}{n} = \frac{(2n)!}{n! \cdot (n+1)!}; \quad n = 1, 2, \ldots
\]  
and that they satisfy the recursive formula:
\[
C_n = \sum_{i_1+i_2=n-1} C_{i_1} C_{i_2}; \quad n \geq 1. \tag{3}
\]

Catalan numbers appeared for the first time in the book *Quick Methods for Accurate Values of Circle Segments*, by Ming Antu (1692–1763), a Chinese mathematician. In this book, he provides some trigonometric equalities and power series, in which Catalan numbers are involved.

Nicolas Fuss (1755–1826) introduced, in his paper of 1791 (see [3]), the Fuss–Catalan numbers, as:
\[
C^m_n = 1; \quad C^m_n = \frac{1}{(m-1)n+1} \binom{m+n}{n} = \frac{(m+n)!}{((m-1)n+1)! \cdot n!} \tag{4}
\]
fixed \(m \geq 2\); with \(n = 1, 2, \cdots\). Notice that, for \(m = 2\), they coincide with Catalan numbers. Furthermore, he provided a generalization of (3):
\[
C^m_n = \sum_{i_1+i_2+\cdots+i_m=n-1} C_{i_1}^m \cdots C_{i_m}^m; \quad \text{fixed } m \geq 2; \quad \forall n \geq 1. \tag{5}
\]

From that time forward, throughout mathematical history, Catalan numbers have made important contributions. We list some of them that we have chosen in an arbitrary manner, by way of illustration:

- Development in power series of the function \(f(x) = \sqrt{1-4x}\) (Euler (1707–1783)).
- The ballot problem (Statistics and Probability), introduced for the first time in 1887 by Joseph Bertrand (1822–1900), see [4].
- In [5], the reader can find more than 200 practical combinatorial interpretations of Catalan numbers.
- Binary trees (Graph Theory), see [6].
- Lattice path theory (Graph Theory), see [7].

Throughout this paper, all the necessary computational tasks have been performed with the program Wolfram Mathematica 11.2.0.0.

2. Some Recent Results

In this section, we review some outcomes, previously published by the authors, (see [1,2]), which will be used in the following sections. Such a summary has been written in detail for the sake of clarity and the self-developed reading of these lines. In fact, for our goal, we will only need formulas (19)–(21).

We posed the functional equation:
\[
Q(x_2 \cdots, x_p, h(x_2, \cdots, x_p)) = x_p h^p(x) + \cdots + x_2 h^2(x) - h(x) + 1 = 0 \tag{6}
\]
where \(p > 1\), \(x = (x_2, x_3, \cdots, x_p) \in \mathbb{R}^{p-1}\) and \(h : \mathbb{R}^{p-1} \rightarrow \mathbb{R}\) is the unknown to solve.

We proved that, if \(h(x)\) is a solution of (6), then it satisfies the first order partial derivative equation:
\[
1 + (2x_2 - 1)h(x) + (3x_3 + 4x_2^2 - x_2)h_{x_2}(x) + (4x_4 + 6x_2x_3 - 2x_3)h_{x_3}(x) + \cdots + (px_p + 2(p-1)x_{p-1} - (p-2)x_{p-1})h_{x_{p-1}}(x) + (2px_2x_p - (p-1)x_p)h_{x_p}(x) = 0 \tag{7}
\]
for all \(x\), such that \(\frac{\partial Q}{\partial h}(x, h(x)) \neq 0\).
From (7), we showed that, if \( h(x) \) is a solution of Equation (6), then the equality:

\[
\frac{\partial^{q_2 + \cdots + q_p} h(x)}{\partial x_2^{q_2} \cdots \partial x_p^{q_p}}(0, \ldots, 0) = \frac{(2q_2 + 3q_3 + \cdots + pq_p)!}{(q_2 + 2q_3 + \cdots + (p - 1)q_p + 1)!}
\]

holds, with \( q_2 + \cdots + q_p = n \) and \( q_2, ..., q_p \) non-negative integers, where \( \frac{\partial^{q_2 + \cdots + q_p} h(x)}{\partial x_2^{q_2} \cdots \partial x_p^{q_p}}(0, \ldots, 0) \) is the \( n \)-th partial derivative of \( h \) with respect to \( x_2, \ldots, q_p \) times with respect to \( x_p \).

Therefore, we can express the function \( h(x) \) as:

\[
h(x) = \sum_{n=0}^{\infty} \sum_{q_2 + \cdots + q_p = n} C_{q_2, \ldots, q_p} x_2^{q_2} \cdots x_p^{q_p}
\]

where

\[
C_{q_2, \ldots, q_p} = \frac{(2q_2 + 3q_3 + \cdots + pq_p)!}{(q_2 + 2q_3 + \cdots + (p - 1)q_p + 1)! q_2! \cdots q_p!}
\]

Consider the polynomial function of degree \( p \), given by:

\[
y = P(x) = a_0 + a_1 x + \cdots + a_p x^p \quad (a_1, \ldots, a_p \neq 0)
\]

with \( x, a_i \in \mathbb{R}, 0 \leq i \leq p \), and the functions \( X_i, 2 \leq i \leq p \):

\[
X_i(y) = \frac{(a_0 - y)^{i-1} a_i}{(-a_1)^i}
\]

then, the series:

\[
f_P(y) = \frac{a_0 - y}{-a_1} h(X_2(y), \ldots, X_p(y))
\]

is the inverse function of \( P(x) \), if it makes sense. Taking into account (9), by substituting functions (12) in (13), we obtain:

\[
f_P(y) = \frac{a_0 - y}{-a_1} \sum_{n=0}^{\infty} \sum_{q_2 + \cdots + q_p = n} C_{q_2, \ldots, q_p} \left( \frac{(a_0 - y)a_2}{(-a_1)^2} \right)^{q_2} \cdots \left( \frac{(a_0 - y)^{p-1} a_p}{(-a_1)^p} \right)^{q_p}
\]

\[
= \frac{y - a_0}{a_1} \sum_{n=0}^{\infty} \sum_{q_2 + \cdots + (p - 1)q_p = n_1} (-1)^{n_1} C_{q_2, \ldots, q_p} \left( \frac{a_2}{(-a_1)^2} \right)^{q_2} \cdots \left( \frac{a_p}{(-a_1)^p} \right)^{q_p} (y - a_0)^{n_1}.
\]

Next, by making the subscripts and superscripts change: \( n_1 = q_2 + 2q_3 + \cdots + (p - 1)q_p \), the terms of series (14) are rearranged in the form:

\[
f_P(y) = \frac{y - a_0}{a_1} \sum_{n_1=0}^{\infty} \sum_{q_2 + \cdots + (p - 1)q_p = n_1} (-1)^{n_1} C_{q_2, \ldots, q_p} \left( \frac{a_2}{(-a_1)^2} \right)^{q_2} \cdots \left( \frac{a_p}{(-a_1)^p} \right)^{q_p} (y - a_0)^{n_1}.
\]

Series (14) and (15) are absolutely convergent in a neighborhood of \( a_0 \), \( V_{a_0} \), given by:

\[
V_{a_0} = \left\{ y \in \mathbb{R}; \frac{p^2}{p - 1} \left| \frac{(a_0 - y)a_2}{a_1^2} \right| + \cdots + \frac{p^p}{(p - 1)^{p-1}} \left| \frac{(a_0 - y)^{p-1} a_p}{a_1^p} \right| < 1 \right\}
\]

that is, obviously, not empty. From now on, we will denote the inverse function of \( y = P(x) \) as \( x = h_P(y) \).

Therefore, if the inequality:

\[
\frac{p^2}{p - 1} \left| \frac{a_0 a_2}{a_1^2} \right| + \frac{p^3}{(p - 1)^2} \left| \frac{a_2^2 a_3}{a_1^3} \right| + \cdots + \frac{p^p}{(p - 1)^{p-1}} \left| \frac{a_0^{p-1} a_p}{a_1^p} \right| < 1
\]
holds then, the series:
\[
    r = \frac{a_0}{-a_1} \sum_{n=0}^{\infty} \sum_{q_2 + \cdots + q_p = n} C_{q_2 \cdots q_p} \left( \frac{a_0 a_2}{(-a_1)^2} \right)^{q_2} \cdots \left( \frac{a_0^{p-1} a_p}{(-a_1)^p} \right)^{q_p}
\]
(18)
is absolutely convergent to the root of \( P(x) \), \( r \), closest to the coordinate origin.

Finally, we consider the Taylor series of the polynomial, \( P(x) \), around the point \( x_0 \), with \( P'(x_0) \neq 0 \), which is:
\[
    y = P(x) = P(x_0) + P'(x_0)(x - x_0) + \frac{P''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{P^{(p)}(x_0)}{p!}(x - x_0)^p
\]
with \( y_0 = P(x_0) \). Then, one gets:
\[
    h_p(y) = x_0 + \frac{y - y_0}{P'(x_0)} \sum_{n=1}^{\infty} \sum_{q_2 + \cdots + (p-1)q_p = n_1} (-1)^{n_1} C_{q_2 \cdots q_p} \left( \frac{P''(x_0)}{2!(-P'(x_0))^2} \right)^{q_2} \cdots \left( \frac{P^{(p)}(x_0)}{p!(-P'(x_0))^p} \right)^{q_p} (y - y_0)^{n_1}
\]
(20)
\[
    = x_0 + \frac{1}{P'(x_0)} \sum_{n_1=0}^{\infty} \sum_{q_2 + \cdots + (p-1)q_p = n_1} (-1)^{n_1} C_{q_2 \cdots q_p} \left( \frac{P''(x_0)}{2!(-P'(x_0))^2} \right)^{q_2} \cdots \left( \frac{P^{(p)}(x_0)}{p!(-P'(x_0))^p} \right)^{q_p} (y - y_0)^{n_1+1}
\]
h_p(y) being the inverse function of (19) in \( V_{y_0} \):
\[
    V_{y_0} = \left\{ y \in \mathbb{R}; \quad \frac{p^2}{p - 1} \left| \frac{y_0 - y}{P'(x_0)^2} \right| + \cdots + \frac{p^p}{(p - 1)^{p-1}} \left| \frac{y_0 - y}{P'(x_0)^p} \right| < 1 \right\}
\]
(21)
according to (16), with \( y_0 = P(x_0) \).

Once we have seen the background on which this paper is based, we introduce the original contributions of this article in the coming sections.

3. Definition of the Function \( h_f \) and Calculation of Its Derivatives

**Theorem 1.** Let \( \Omega_{\mathbb{R}} \subset \mathbb{R} \) be an open set. Consider the function \( f : \Omega_{\mathbb{R}} \rightarrow \mathbb{R} \). Assume that \( f \) is analytic in the open ball, \( B_{\mathbb{R}}(x_0, R) \subset \Omega_{\mathbb{R}}, \) \( R \) being its radius of convergence, and \( f'(x_0) \neq 0 \).

Let \( z = P(x) \) be the Taylor Polynomial of degree \( p \) of the function \( f \), that is to say:
\[
    z = P(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(p)}(x_0)}{p!}(x - x_0)^p.
\]
(22)

If we express the inverse functions of \( f(x) \) and \( P(x) \), around the point \( y_0 = f(x_0) = P(x_0) \) as \( h_f(y) \) and \( h_P(z) \), respectively, then the next equality holds \( \forall n \geq 1 \) \( (n \leq p) \):
\[
    h_f^{(n)}(y_0) = h_P^{(n)}(y_0).
\]
(23)

**Proof.** By hypothesis, the series:
\[
    y = f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(p)}(x_0)}{p!}(x - x_0)^p + \cdots
\]
(24)
is absolutely convergent in the closed ball:

$$\overline{B}_\mathbb{R}(x_0, \rho) = \{x \in \mathbb{R}; |x - x_0| \leq \rho < R\};$$

with \( \rho > 0 \)

and \( f'(x_0) = P'(x_0) \), with \( f'(x_0) \neq 0 \) so, from the Inverse Function Theorem, there is a closed neighborhood of \( y_0 = P(x_0) = f(x_0) \):

$$\overline{B}_\mathbb{R}(y_0, \rho_1) = \{y \in \mathbb{R}; |y - y_0| \leq \rho_1\};$$

with \( \rho_1 > 0 \)

where \( h_f(y) \) and \( h_P(y) \) exist and they are bijections.

In order to simplify the notation, we suppose, without loss of generality, that \( \rho \) and \( \rho_1 \) satisfy the properties \( \overline{B}_\mathbb{R}(y_0, \rho_1) \subset f(\overline{B}_\mathbb{R}(x_0, \rho)) \) and \( \overline{B}_\mathbb{R}(y_0, \rho_1) \subset P(\overline{B}_\mathbb{R}(x_0, \rho)) \).

By substituting in (24) \( x \) by \( h_f(y) \), where \( y \) is the only \( y \in \overline{B}_\mathbb{R}(y_0, \rho_1) \), such that there is an only \( x \in \overline{B}_\mathbb{R}(x_0, \rho) \) with \( y = f(x) \), then (24) turns out to be:

$$y = f(x_0) + f'(x_0)(h_f(y) - x_0) + \frac{f''(x_0)}{2!}(h_f(y) - x_0)^2 + \cdots + \frac{f^{(p)}(x_0)}{p!}(h_f(y) - x_0)^p + \cdots (25)$$

In a similar way, by substituting in (22) \( x \) by \( h_P(z) \), where \( z \) is the only \( z \in \overline{B}_\mathbb{R}(y_0, \rho_1) \), such that there is an only \( x \in \overline{B}_\mathbb{R}(x_0, \rho) \) with \( z = P(x) \), then (22) becomes:

$$z = f(x_0) + f'(x_0)(h_P(z) - x_0) + \cdots + \frac{f^{(p)}(x_0)}{p!}(h_P(z) - x_0)^p. (26)$$

To prove (23), we use the induction method. For \( n = 1 \), on the one hand, we consider the Taylor Polynomial (26) of degree 1:

$$z = f(x_0) + f'(x_0)(h_P(z) - x_0). (27)$$

We compute the first derivative of both sides of (27) with respect to \( z \) at the point \( z = y_0 \), obtaining:

$$1 = f'(x_0)h_P'(y_0) \implies h_P'(y_0) = \frac{1}{f'(x_0)}. (28)$$

On the other hand, we compute the first derivative of both sides of (25) with respect to \( y \) at the point \( y = y_0 \), obtaining:

$$y' = \left[ f(x_0) + f'(x_0)(h_f(y) - x_0) + \frac{f''(x_0)}{2!}(h_f(y) - x_0)^2 + \cdots + \frac{f^{(p)}(x_0)}{p!}(h_f(y) - x_0)^p + \cdots \right]'_{y = y_0}. (29)$$

Note that:

$$\left[ \frac{f^{(p)}(x_0)}{p!}(h_f(y) - x_0)^p \right]^{(k)}_{y = y_0} = \sum_{\substack{q_0 + \cdots + q_k = k \\ q_0 + \cdots + q_k = p}} \frac{P!}{q_0! \cdots q_k!} \left[ (h_f(y) - x_0)^{q_0} \right]^{(q_0)} \left[ (h_f(y) - x_0)^{(q_1)} \right]^{(q_1)} \cdots \left[ (h_f(y) - x_0)^{(q_k)} \right]^{(q_k)} (30)$$

where:
• \( q_0 \) is the number of times that \((h_f(y) - x_0)\) is repeated as a factor in its corresponding summands of (30).

• \( q_1 \) plays the same role with respect to the expression \((h_f(y) - x_0)^{(1)}\).

\[ \ldots \]

• The same thing can be said with respect to \( q_k \), related to the expression \((h_f(y) - x_0)^{(k)}\).

Each term of (30) satisfies:

\[
q_1 + q_2 + \cdots + q_k \leq q_1 + 2q_2 + \cdots + kq_k = k
\]

\[
q_0 + q_1 + q_2 + \cdots + q_k = p > k
\]

so \( q_0 > 0 \) in all of them. Therefore, the factor \((h_f(y) - x_0)^{q_0}\) appears in each term of (30). As \((h_f(y) - x_0)^{q_0}\) equals zero at the point \( y = y_0 \), (29) follows. Hence, (28) comes to be:

\[
1 = f'(x_0)h'_f(y_0) \implies h'_f(y_0) = \frac{1}{f'(x_0)}
\]

and the result is true for \( n = 1 \).

Suppose now that the result is true for \( n = r - 1 \).

We consider the polynomial (26) with \( p = r \). On the one side, we compute the \( r \)-th derivative of both sides of (26) with respect to \( z \) at the point \( z = y_0 \), obtaining:

\[
0 = \left[ f(x_0) + f'(x_0)(h_p(z) - x_0) + \cdots + \frac{f^{(r)}(x_0)}{r!} (h_p(z) - x_0)^r \right]_{z = y_0}^{(r)}.
\]

By applying differentiation rules to (31), we get a polynomial, \( P_1 \), in the variables \( f'(x_0), \ldots, f^{(r)}(x_0), h'_p(y_0), \ldots, h_p^{(r)}(y_0) \), that is to say:

\[
0 = P_1 \left( f'(x_0), \ldots, f^{(r)}(x_0), h'_p(y_0), \ldots, h_p^{(r)}(y_0) \right).
\]

Observe that, due to differentiation rules (30), the exponent of \( h_p^{(r)}(y_0) \) as a variable of the polynomial function, \( P_1 \), is equal to 1 and its coefficient is different to zero. Indeed, the coefficients and exponents corresponding to \( r \)-th derivative of \( h_f \) are:

• From \( [f'(x_0)]^{(r)} \), 0.
• From \( [f'(x_0)(h_p(y) - x_0)]^{(r)} \), \( f'(x_0) \left[ (h_p(y) - x_0)^{(r)} \right] \frac{1}{y = y_0} = f'(x_0)h_p^{(r)}(y_0) \).
• From \( \left[ \frac{f''(x_0)}{2!} (h_p(y) - x_0)^2 \right]^{(r)} \), \( f''(x_0) \left[ (h_p(y) - x_0)^{(r)} \right] \frac{1}{(h_p(y) - x_0)^{(0)}} \frac{1}{y = y_0} = 0.
• From \( \left[ \frac{f'''(x_0)}{3!} (h_p(y) - x_0)^3 \right]^{(r)} \), \( \frac{f'''(x_0)}{2!} \left[ (h_p(y) - x_0)^{(r)} \right] \frac{1}{(h_p(y) - x_0)^{(0)}} \frac{1}{y = y_0} = 0.
• \ldots
• From \( \left[ \frac{f^{(r)}(x_0)}{r!} (h_p(y) - x_0)^r \right]^{(r)} \), \( \frac{f^{(r)}(x_0)}{(r - 1)!} \left[ (h_p(y) - x_0)^{(r)} \right] \frac{1}{(h_p(y) - x_0)^{(0)}} \frac{1}{y = y_0} = 0.

Therefore, the coefficient of \( h_p^{(r)}(y_0) \) is \( f'(x_0) \neq 0 \) and its exponent, 1. Thus, solving for \( h_p^{(r)}(y_0) \) in (32) makes sense and we arrive at the only solution:

\[
h_p^{(r)}(y_0) = R_1 \left( f'(x_0), \ldots, f^{(r)}(x_0), h'_p(y_0), \ldots, h_p^{(r-1)}(y_0) \right)
\]

where \( R_1 \) is a rational function.
On the other hand, we compute the \( r \)-th derivative of both sides of (25) with respect to \( y \) at the point \( y = y_0 \), obtaining:

\[
0 = \left[ f(x_0) + f'(x_0)(h_f(y) - x_0) + \cdots + \frac{f^{(r)}(x_0)}{r!} (h_f(y) - x_0)^r \right]^{(r)}_{y = y_0}
\]

\[
+ \left[ \frac{f^{(r+1)}(x_0)}{(r+1)!} (h_f(y) - x_0)^{r+1} + \cdots \right]^{(r)}_{y = y_0}.
\]

From the \((r+1)\)-th term of the second side of (33) onward, the \( r \)-th derivative at the point \( y = y_0 \) equals zero, due to (29).

By applying again differentiation rules to the first \( r \) terms of the second side of (33), we obtain the same polynomial function as in (32), in other words:

\[
0 = P_1 \left( f'(x_0), \cdots, f^{(r)}(x_0), h_f'(y_0), \cdots, h_f^{(r)}(y_0) \right)
\]

by induction hypothesis, we conclude that:

\[
0 = P_1 \left( f'(x_0), \cdots, f^{(r)}(x_0), h_p'(y_0), \cdots, h_p^{(r-1)}(y_0), h_f^{(r)}(y_0) \right).
\]

Therefore:

\[
h_f^{(r)}(y_0) = R_1 \left( f'(x_0), \cdots, f^{(r)}(x_0), h_p'(y_0), \cdots, h_p^{(r-1)}(y_0) \right)
\]

and the result follows. \( \square \)

**Remark 1.** Theorem 1 may have been approached by applying Bell polynomials and the Faà di Bruno formula (see [8,9]), in the following way:

Bell polynomials are introduced as:

\[
Y_n(x_1, \ldots, x_{n-k+1}) = \sum_{k=1}^{n} B_{n,k}(x_1, \ldots, x_{n-k+1}); \quad \forall n \geq 1
\]

where \( B_{n,k}(x_1, \ldots, x_{n-k+1}) \) are the partial Bell polynomials, given by:

\[
B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{q_1+2q_2+3q_3+\ldots+nq_n=n}^{q_1+q_2+q_3+\ldots=k} \frac{n!}{q_1! \cdots q_n!} \left( \frac{x_1}{1!} \right)^{q_1} \left( \frac{x_2}{2!} \right)^{q_2} \cdots
\]

with \( q_1, q_2, \ldots \) non-negative integer numbers. As \( q_1 + 2q_2 + 3q_3 + \ldots = n \Rightarrow q_i = 0; \quad \forall i > n, \) and

\[
\begin{cases}
q_1 + 2q_2 + 3q_3 + \ldots + nq_n = n \\
q_1 + q_2 + q_3 + \ldots + q_n = k
\end{cases} \quad \Rightarrow \begin{cases}
q_2 + 2q_3 + \ldots + (n-1)q_n = n-k \\
q_1 + q_2 + q_3 + \ldots + q_n = k
\end{cases} \quad \Rightarrow q_{n-k+1} = \cdots = q_n = 0
\]

then, we can rewrite:

\[
B_{n,k}(x_1, \ldots, x_n) = \sum_{q_1+2q_2+3q_3+\ldots+nq_n=n}^{q_1+q_2+q_3+\ldots+q_n=n} \frac{n!}{q_1! \cdots q_n!} \left( \frac{x_1}{1!} \right)^{q_1} \left( \frac{x_2}{2!} \right)^{q_2} \cdots \left( \frac{x_n}{n!} \right)^{q_n}
\]

(34)

taking \( q^0 = 1 \), if necessary.

For \( n = 1 \), \( h_f^{(1)} = \frac{1}{f'(y)}, \) where \( h_f \) is the inverse function of \( f \).
For $n > 1$, we use Faà di Bruno formula, obtaining:

$$0 = \sum_{k=1}^{n} h_f^{(k)} B_{n,k}(f^{(1)}, f^{(2)}, \ldots, f^{(n)})$$

where $B_{n,k}(f^{(1)}, f^{(2)}, \ldots, f^{(n)})$ are the partial Bell polynomials, with $k = 1, 2, \ldots, n$. Substituting $B_{n,k}$ according to (34), we arrive at:

$$0 = \sum_{k=1}^{n} h_f^{(k)} \sum_{q_1+2q_2+\ldots+nq_n=n \atop q_1+q_2+\ldots+q_n=k} \frac{n!}{q_1! \ldots q_n!} \left( \frac{f^{(1)}}{1!} \right)^{q_1} \left( \frac{f^{(2)}}{2!} \right)^{q_2} \cdots \left( \frac{f^{(n)}}{n!} \right)^{q_n}. \tag{35}$$

We remove the last term, $B_{n,n}$, of the sum (35). As:

$$\{(q_1, q_2, \ldots, q_n); q_1 + 2q_2 + \ldots + nq_n = n \text{ and } q_1 + q_2 + \ldots + q_n = n\} = \{(n, 0, \ldots, 0)\}$$

we obtain:

$$0 = \sum_{k=1}^{n-1} h_f^{(k)} \sum_{q_1+2q_2+\ldots+nq_n=n \atop q_1+q_2+\ldots+q_n=k} \frac{n!}{q_1! \ldots q_n!} \left( \frac{f^{(1)}}{1!} \right)^{q_1} \left( \frac{f^{(2)}}{2!} \right)^{q_2} \cdots \left( \frac{f^{(n)}}{n!} \right)^{q_n} + h_f^{(n)} (f^{(1)})^n. \tag{36}$$

In addition, solving for $h_f^{(n)}$, we get:

$$h_f^{(n)} = -\sum_{k=1}^{n-1} h_f^{(k)} \sum_{q_1+2q_2+\ldots+nq_n=n \atop q_1+q_2+\ldots+q_n=k} \frac{n!}{q_1! \ldots q_n!} \left( \frac{f^{(2)}}{2! (f^{(1)})^2} \right)^{q_2} \cdots \left( \frac{f^{(n)}}{n! (f^{(1)})^n} \right)^{q_n}. \tag{36}$$

The calculation of $h_f^{(n)}$ requires the reiterative substitution of $h_f^{(k)}$; $k = 1, \ldots, n-1$ in formula (36). Moreover, the general expression of each $h_f^{(k)}$ has the same structure and complexity as (36) itself. For this reason, we were facing great difficulties and we opted for the process described in Section 2, which we considered more affordable, providing, furthermore, a new alternative.

**Corollary 1.** Let $\Omega_R \subset \mathbb{R}$ be an open set. Consider the function $f : \Omega_R \to \mathbb{R}$. Assume that $f$ is analytic in a neighborhood of each point $x \in \Omega_R$, with $f'(x) \neq 0$. If $h_f(y)$ is the inverse function of $f$ around the point $y = f(x)$, then the $(p+1)$-th derivative of $h_f$ at the point, $y$, divided by $(p+1)!$ is given by:

$$A_p(x) = \frac{1}{f'(x)} \sum_{q_1+2q_3+\ldots+pq_{p+1}=p} (-1)^p C_{q_2-qp+1} \left( \frac{f''(x)}{2! (-f'(x))^2} \right)^{q_2} \left( \frac{f'''(x)}{3! (-f'(x))^3} \right)^{q_3} \cdots \left( \frac{f^{(p+1)}(x)}{(p+1)! (-f'(x))^{p+1}} \right)^{q_{p+1}}. \tag{37}$$

**Proof.** The result follows from Theorem 1 and formulas (19)–(21). Indeed:

$$\frac{f^{(p+1)}(x)}{(p+1)!} = \frac{1}{f'(x)} \sum_{q_1+2q_3+\ldots+pq_{p+1}=p} (-1)^p C_{q_2-qp+1} \left( \frac{f''(x)}{2! (-f'(x))^2} \right)^{q_2} \left( \frac{f'''(x)}{3! (-f'(x))^3} \right)^{q_3} \cdots \left( \frac{f^{(p+1)}(x)}{(p+1)! (-f'(x))^{p+1}} \right)^{q_{p+1}} = A_p(x).$$
In other words:
\[ f^{(p+1)}(x) = (p + 1)! A_p(x) . \]

\[ \square \]

**Remark 2.** Consider the set, \( S_p \), of subscripts and superscripts of the definition of \( A_p \), see (37), given by:

\[ S_p = \{ q_2, q_3, \ldots, q_{p+1}; q_2, q_3, \ldots, q_{p+1}; \text{non-negative integers}; q_2 + 2q_3 + \cdots + pq_{p+1} = p \} . \]

As the number \( p \) increases, so too does the run time of the computer code for calculating \( A_p \), disproportionately. As an illustration, we show them until \( p = 10 \):

\[
\begin{align*}
S_1 & = \{1\} \\
S_2 & = \{\{2,0\}, \{0,1\}\} \\
S_3 & = \{\{0,0,1\}, \{1,1,0\}, \{3,0,0\}\} \\
S_4 & = \{\{0,0,0,1\}, \{0,2,0,0\}, \{1,0,1,0\}, \{2,1,0,0\}, \{4,0,0,0\}\} \\
\vdots \\
S_{10} & = \{\{0,0,0,0,0,0,0,0,0,1\}, \{0,0,0,0,2,0,0,0,0,0\}, \{0,0,0,0,1,0,1,0,0,0\}, \{0,0,0,1,0,0,0,1,0,0\}, \ldots \}.
\end{align*}
\]

For this reason, we provide the next Corollary as an alternative.

**Corollary 2.** As in Corollary 1, the successive \( A_p(x) \), \( p \geq 0 \), with \( x = h_f(y) \), are given by:

\[ A_0(x) = \frac{1}{f'(x)}; \quad A_p(x) = \frac{A'_{p-1}(x)}{p+1} \frac{1}{f'(x)}; \quad p \geq 1. \]

**Proof.** From Corollary 1, we have that:

\[ A_{p-1}(x) = \frac{h_f^{(p)}(y)}{p!}; \quad y = f(x); \quad p \geq 1. \]

Thus:

\[ A'_{p-1}(x) = \frac{h_f^{(p+1)}(y)}{p!} f'(x) = \frac{(p + 1) h_f^{(p+1)}(y)}{(p + 1)!} f'(x) \quad \implies \quad A'_{p-1}(x) = (p+1) A_p(x) f'(x) \]

and the result follows. \( \square \)
4. Calculation of an Upper Bound for the Derivatives of $h_f$

Jacques Hadamard said: “The shortest path between two truths in the real domain passes through the complex domain”.

We find this quote very valid for the ensuing paragraphs, in which we accomplish the task of bounding formula (37).

**Remark 3.** Mean Value Theorem is not true in general in the complex field. Nevertheless, it can be applied under some conditions that we next establish, in agreement with [10]. Let $\Omega_C \subset \mathbb{C}$ be an open and convex set and $G : \Omega_C \rightarrow \mathbb{C}$, an holomorphic function. Consider $w_1$, $w_2 \in \Omega_C$.

Then, there exist $a$, $b \in L$, where $L$ is the segment with endpoints $w_1$ and $w_2$, such that:

$$Re(G'(a)) = Re\left(\frac{G(w_2) - G(w_1)}{w_2 - w_1}\right); \quad Im(G'(b)) = Im\left(\frac{G(w_2) - G(w_1)}{w_2 - w_1}\right).$$

Hence:

$$G(w_2) - G(w_1) = \left(Re(G'(a)) + i Im(G'(b))\right)(w_2 - w_1). \quad (38)$$

**Theorem 2.** Under the same hypothesis as Theorem 1, let $B_C(x_0, R)$ be an open ball in the complex field, whose restriction to $\mathbb{R}$ is just $B_R(x_0, R)$ (introduced in Theorem 1) and let $F : B_C(x_0, R) \rightarrow \mathbb{C}$ be the complex function, given by:

$$s = F(w) = f(x_0) + f'(x_0)(w - x_0) + \frac{f''(x_0)}{2!}(w - x_0)^2 + \ldots + \frac{f^{(p)}(x_0)}{p!}(w - x_0)^p + \ldots$$

Then, there exists a real number $M > 0$ (defined below) such that:

$$\left|h_f^{(p)}(y_0)\right| = \left|h_f^{(p)}(y_0)\right| \leq \frac{p! \left(\frac{x_0 + \rho_2}{\rho_2 M}\right)^p}{\rho_2}, \quad \forall \ p \geq 1; \quad \text{with} \quad 0 < \rho_2 < R.$$

**Remark 4.** Observe that the restriction of $F$ to $\mathbb{R}$ is equal to the function $f$, in agreement with (24).

**Proof.** Obviously, $F$ is also an analytic function in the complex field, with the same radius of convergence as $f$, $R$. Consequently, $F$ is also an holomorphic function.

$F'(x_0) = f'(x_0) \neq 0$; therefore, from the Inverse Function Theorem for holomorphic functions, we can find two closed balls, $\overline{B}_C(x_0, \rho_2) \subset B_C(x_0, R)$ and $\overline{B}_C(y_0, \rho_3)$ and a closed neighborhood, $\overline{V}_{x_0}$, such that there exists the inverse function of $F$, $h_F$ and both functions:

$$F : \overline{V}_{x_0} \subset \overline{B}_C(x_0, \rho_2) \rightarrow \overline{B}_C(y_0, \rho_3); \quad h_F : \overline{B}_C(y_0, \rho_3) \rightarrow \overline{V}_{x_0} \quad (39)$$

are bijections.

As $h_F$ is an holomorphic function in the closed ball $\overline{B}_C(y_0, \rho_3)$, we can apply the Cauchy Integral Formula, in the following form:

$$h_F(y_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{h_F(s)}{s - y_0} ds; \quad \gamma = \{s \in \mathbb{C}; \ s = y_0 + \rho_3 e^{it}; \ t \in [0, 2\pi]\}. \quad (40)$$

By differentiating (40) with respect to $y_0$, we obtain the Cauchy Differentiation Formula for the function $h_F$, in the following manner:

$$h_F^{(p)}(y_0) = \frac{p!}{2\pi i} \int_{\gamma} \frac{h_F(s)}{(s - y_0)^{p+1}} ds. \quad (41)$$
If \( s \in \overline{B}_C(y_0, \rho_3) \), then the next inequalities hold:

\[
|h_F(s)| - |h_F(y_0)| \leq |h_F(s) - h_F(y_0)| = |w - x_0| \leq \rho_2 \implies |h_F(s)| \leq |x_0| + \rho_2
\]  

(42)

since \( w \in V_{x_0} \) and, in correspondence with (38), if \( s \in \gamma \):

\[
\rho_3 = |s - y_0| = |F(w) - F(x_0)| = |Re(F'(a)) + i Im(F'(b))| |w - x_0|
\]  

(43)

for some \( a, b \in L \), where \( L \subset \overline{B}_C(x_0, \rho_2) \) is the segment with endpoints \( w \) and \( x_0 \). If we indicate \( w, a, b \) and \( F \) as \( w = c + id, a = a_1 + i a_2, b = b_1 + i b_2 \) and \( F(c + id) = u(c,d) + i \nu(c,d) \), \( u(c,d), \nu(c,d) \) being functions from \( \mathbb{R}^2 \) into \( \mathbb{R} \) that represent the real and imaginary parts of \( F \), respectively, then (43) turns out to be:

\[
\rho_3 = |s - y_0| = |\frac{\partial u}{\partial c}(a_1, a_2) + i \frac{\partial \nu}{\partial c}(b_1, b_2)| |w - x_0| = \sqrt{\left(\frac{\partial u}{\partial c}(a_1, a_2)\right)^2 + \left(\frac{\partial \nu}{\partial c}(b_1, b_2)\right)^2} |w - x_0| .
\]  

(44)

If:

\[
M_1 \leq \min \left\{ \left( \frac{\partial u}{\partial c}(c,d) \right)^2 ; c + id \in \overline{B}_C(x_0, \rho_2) \right\}
\]

\[
M_2 \leq \min \left\{ \left( \frac{\partial \nu}{\partial c}(c,d) \right)^2 ; c + id \in \overline{B}_C(x_0, \rho_2) \right\}
\]  

(45)

and if we take \( M \) such that \( 0 < M \leq \sqrt{M_1 + M_2} \) then, in consonance with the hypothesis of the theorem, we can choose \( \rho_3 \) between the values given by the inequalities:

\[
\rho_2 M \leq \rho_3 = |s - y_0| \leq \rho_2 N
\]  

(46)

where, in agreement with (44), \( N = \max \left\{ \sqrt{\left( \frac{\partial u}{\partial c}(c,d) \right)^2 + \left( \frac{\partial \nu}{\partial c}(c,d) \right)^2} ; c + id \in \overline{B}_C(x_0, \rho_2) \right\} .
\]

As a consequence of all of this, using (41), (42) and (46), we obtain Cauchy’s Estimate for the successive derivatives of \( h_F \) as follows:

\[
\left| h_F^{(p)}(y_0) \right| \leq \frac{p!}{2\pi} \int_{\gamma} \frac{|h_F(s)|}{|s - y_0|^{p+1}} ds \leq \frac{p!}{2\pi} \frac{|x_0| + \rho_2}{\rho_3^{p+1}} 2\pi \rho_3
\]

\[
= \frac{p! (|x_0| + \rho_2)}{\rho_3^p} \leq \frac{p! (|x_0| + \rho_2)}{(\rho_2 M)^p}; \forall p \geq 1.
\]  

(47)

In addition, the result follows. \( \square \)

5. Taylor Series and Taylor Remainder of the Function \( h_F \)

**Theorem 3.** If the same hypothesis as Theorems 1 and 2 hold, then \( h_F(y) \) is an analytic function, around the point \( y = y_0 \), with a radius of convergence of, at least, \( \rho_2 M. \)

**Proof.** Once we have calculated and bounded the successive derivatives of \( h_F \) in Theorems 1 and 2, respectively, we are now in position for introducing its corresponding Taylor series:

\[
h_F(y) = x_0 + \frac{1}{f'(x_0)}(y - y_0) + \sum_{p=1}^{\infty} A_p(x_0)(y - y_0)^{p+1}.
\]  

(48)
Theorem 4. Under the same hypothesis as Theorems 1 and 2, then the values when estimating an upper bound, and we have rounded down for a lower bound.

Proof. The error in evaluating $h_f(y)$ by its Taylor Polynomial of degree $p$ is:

$$R_{p+1} = \left| h_f(y) - \left( x_0 + \sum_{n=0}^{p-1} A_n(x_0)(y - y_0)^n \right) \right| = \left| \sum_{n=p}^{\infty} A_n(x_0)(y - y_0)^n \right|.$$  \hspace{1cm} (52)

Taking into account (50), then (52) becomes:

$$R_{p+1} \leq \sum_{n=p}^{\infty} \frac{|x_0| + \rho_2}{(\rho_2 M)^{n+1}} |y - y_0|^{n+1} = (|x_0| + \rho_2) \sum_{n=p}^{\infty} \frac{y - y_0}{\rho_2 M}.$$  \hspace{1cm} (53)

Therefore:

$$R_{p+1} \leq (|x_0| + \rho_2) \frac{|y - y_0|}{\rho_2 M} \frac{1}{1 - \frac{|y - y_0|}{\rho_2 M}}.$$  \hspace{1cm} (54)

The result follows. $\square$

In the following paragraphs, we introduce an example, in detail, of the construction of the inverse function, $h_f$. In order to ensure the veracity of the inequalities, we have rounded up the numerical values when estimating an upper bound, and we have rounded down for a lower bound.

Example 1. Find the Taylor series of the inverse function of $y = f(x) = e^x + x + 1$, $h_f(y)$, around the point $y_0 = f(0) = 2$, compute the power series that defines $h_f(1.9)$, in an analytic way and, finally, approximate its numerical value with an error lower or equal to 0.001, using the Taylor Remainder.
Step 1: Definition of $F(w)$ and $F'(w)$, according to Theorem 2:

$$f(x) = e^x + x + 1; \quad F(w) = e^w + w + 1; \quad \text{with } w = c + id$$

$$F(w) = u(c,d) + i v(c,d) = e^c \cos(d) + c + 1 + i (e^c \sin(d) + d)$$

$$F'(w) = \frac{\partial u}{\partial c}(c,d) + i \left( \frac{\partial v}{\partial c}(c,d) \right) = 1 + e^c \cos(d) + i e^c \sin(d).$$

Step 2: Construction of the closed ball $\mathcal{B}_C(x_0, \rho_2)$.

We look for a square $S_q = [x_0 - \rho_2, x_0 + \rho_2] \times [-\rho_2, \rho_2]$ to contain $\mathcal{B}_C(x_0, \rho_2)$ and in which either $1 + e^c \cos(d)$ or $e^c \sin(d)$ does not change their signs. If we take $x_0 = 0$ and $\rho_2 = 3$, due to the fact that in the interval $[-3, 3]$, $e^c$ is increasing, $\cos(d)$ reaches its minimum and maximum values at the points $d = 3$ and $d = 0$, respectively, and $\sin(d)$ at the points $d = -\pi/2$ and $d = \pi/2$, respectively, we can say that:

$$0.950711 \leq 1 + e^{-3} \cos(3) \leq 1 + e^c \cos(d) \leq 1 + e^3 \cos(0) \leq 21.085537 \quad (53)$$

and, therefore, $F'(w) \neq 0$ for all $w \in \mathcal{B}_C(x_0, \rho_2) \subset S_q$.

Step 3: Calculation of $M_1$, $M_2$, and $M$.

For the same reasons as in Step 2, we arrive at:

$$-20.085537 \leq e^3 \sin \left( -\frac{\pi}{2} \right) \leq e^c \sin(d) \leq e^3 \sin \left( \frac{\pi}{2} \right) \leq 20.085537. \quad (54)$$

According to (45) and in agreement with (53) and (54), $M_1$, $M_2$, and $M$ are given by:

$$M_1 = 0.950711^2; \quad M_2 = 0; \quad \sqrt{M_1 + M_2} = 0.950711 \geq M = 0.95.$$

Step 4: Calculation of the radius of convergence and the Taylor series of $h_f$ around the point $y_0 = f(x_0) = 2$.

As a consequence of all of this, we conclude that the inverse function of $f$ according to (48) is:

$$h_f(y) = 0 + \sum_{p=0}^{\infty} A_p(0) \; (y - 2)^{p+1}$$

and has a radius of convergence of, at least, $\rho_2 \; M = 2.85$.

The exact value of $h_f(1.9)$ is given by the power series:

$$h_f(1.9) = 0 + \sum_{p=0}^{\infty} A_p(0) \; (1.9 - 2)^{p+1}$$

that is well defined, since:

$$2 - 2.85 \leq 1.9 \leq 2 + 2.85$$

Step 5: Numerical approximation of $h_f(1.9)$.

For approximating the numerical value of the irrational number $h_f(1.9)$, with an error lower or equal to 0.001, we need to give numerical values to the number $p$ in (51). In Table 1, we resume the performed calculations with such a purpose. Its first row indicates the values of $p$, the second one shows the corresponding values of $R_{p+1}$.

| $p$ | 1  | 2  | 3  | 4  |
|-----|----|----|----|----|
| $R_{p+1}$ | 0.003827 | 0.000135 | 0.000004 | 0.00000016 |
For $0 \leq p \leq 1$, $R_{p+1}$ does not attain the required value. For $p = 2$, we obtain $R_3 \leq 0.000135 < 0.001$.

For $p \geq 3$, the error decreases more and more.

Therefore:

$$h_f(1.9) \approx 0 + \frac{1}{p} A_p(0) (1.9 - 2)^{p+1} \approx -0.050625$$

and:

$$-0.05076 \leq -0.050625 - 0.000135 \leq h_f(1.9) \leq -0.050625 + 0.000135 \leq -0.05049.$$

6. Applications I: Resolution of Nonlinear Equations

**Corollary 3.** Under the same hypothesis as Theorem 3, consider the closed interval:

$$[a, b] = \begin{cases} 
[h_f(y_0 - \rho_2 M), h_f(y_0 + \rho_2 M)]; & \text{if } h_f(y_0 - \rho_2 M) < h_f(y_0 + \rho_2 M) \\
[h_f(y_0 + \rho_2 M), h_f(y_0 - \rho_2 M)]; & \text{if } h_f(y_0 - \rho_2 M) > h_f(y_0 + \rho_2 M)
\end{cases}$$

Then, there is one root and only one root, $r \in [a, b]$ of the function $y = f(x)$ if, and only if, $0 \in [y_0 - \rho_2 M, y_0 + \rho_2 M]$.

**Proof.** Throughout the proof of Theorem 2, formula (39) brings to light that the function $F : \nabla x_0 \subset \overline{B}_C(x_0, \rho_2) \rightarrow \overline{B}_C(y_0, \rho_2)$ is a bijection: As $f$ is the restriction of $F$ to the real numbers then, in agreement with (46), $\rho_3 > \rho_2 M$, so $f : [a, b] \rightarrow [y_0 - \rho_2 M, y_0 + \rho_2 M]$ is a bijection too. Therefore, if $r \in [a, b]$, then $f(r) = 0 \in [y_0 - \rho_2 M, y_0 + \rho_2 M]$. Conversely, if $0 \in [y_0 - \rho_2 M, y_0 + \rho_2 M]$, then $h_f(0) = r \in [a, b]$, and the result follows. □

**Example 2.** Find a root of the function $y = f(x) = e^x + x + 1$, in an analytic way, through the series $h_f(0)$, and approximate its numerical value with an error lower or equal to 0.001.

With the same values as in the previous example for the parameters $x_0, y_0, M$, and $\rho_2$, the series $h_f(0)$ makes sense, since $2 - 2.85 \leq 0 \leq 2 + 2.85$, so the exact value of the required root is given by:

$$h_f(0) = 0 + \sum_{p=0}^{\infty} A_p(0)(-2)^{p+1}.$$

With these values of the parameters, we need to take $p = 26$, at least, to approximate $h_f(0)$ with the required precision, in consonance with Table 2. For $p \geq 27$, the error decreases more and more.

**Table 2. Values of $p$ and $R_{p+1}$.**

| $p$   | 25       | 26       | 27       | 28       |
|-------|----------|----------|----------|----------|
| $R_{p+1}$ | 0.001008 | 0.000708 | 0.000496 | 0.000348 |

Then, the radius of convergence of the series $h_f$ around the point $y_0$ is, at least, $\rho_2 M = 2.85$, so $h_f(0)$ is well defined and $R_{27} \leq 0.000708 < 0.001$. Hence:

$$h_f(0) \approx 0 + \sum_{p=0}^{25} A_p(0)(-2)^{p+1} \approx -1.278464$$

with:

$$-1.279172 \leq -1.278464 - 0.000708 \leq h_f(0) \leq -1.278464 + 0.000708 \leq -1.277756.$$
Remark 5. In the previous example, the parameters $x_0$, $M$, and $\rho_2$ needed to find the required root are given in the solution of Example 1, but it may be questioned, in general, how to get such values. In the following example, we attempt to find a possible solution to this problem.

Example 3. Calculate the parameters $x_0$, $M$, and $\rho_2$ in order to find a root of $y = f(x) = 2 \cos(x) - \frac{1}{2} x$ in the interval $[0, \pi]$, if it exists.

We solve the problem in the following steps:

Step 1: Definition of $F(w)$ and $F'(w)$.

\[
F(w) = u(c,d) + i\, v(c,d) = 2\cos(c)\cosh(d) - \frac{1}{2}c + i\left(-2\sin(c)\sinh(d) - \frac{1}{2}d\right)
\]

\[
F'(w) = \frac{\partial u}{\partial c}(c,d) + i\left(\frac{\partial v}{\partial c}(c,d)\right) = -\frac{1}{2} - 2\cosh(d)\sin(c) + i\left(-2\cos(c)\sinh(d)\right).
\]

Step 2: Setting of the initial parameters $\rho_2$, $x_0$, and $M$.

As said in Example 1, the main idea is to look for rectangles, $S_q = [x_0 - \rho_2, x_0 + \rho_2] \times [-\rho_2, \rho_2]$, where the closed ball, $\overline{B}_C(x_0, \rho_2)$, is contained, and, in addition, either $-1/2 - 2\cosh(d)\sin(c)$ or $-2\cos(c)\sinh(d)$ do not change their signs.

With such a purpose, we fix $\rho_2 = \pi/2$ as the half of the length of the interval and $x_0$ as its middle point. Therefore, $S_q$ is left as $[0, \pi] \times [-\pi/2, \pi/2]$ and $x_0 = \pi/2$.

We bound both partial derivatives in $S_q$, obtaining:

\[
-\frac{1}{2} - 2\cosh(d)\sin(c) \leq -\frac{1}{2} - 2\cosh(0)\sin(0) \leq -\frac{1}{2}.
\]

It is easy to see that $-2\cos(c)\sinh(d)$ changes its sign in $S_q$, since $-2\cos(c)\sinh(-\rho_2)$ and $-2\cos(c)\sinh(\rho_2)$ take values with contrary signs. Thus, from now on, we are only going to pay attention to the expression $-\frac{1}{2} - 2\cosh(d)\sin(c)$.

Thus, the real part of the derivative of $F(w)$ is not positive and, furthermore,

\[
f(x_0 - \rho_2) \cdot f(x_0 + \rho_2) < 0.
\]

Then, there is only one root in $(x_0 - \rho_2, x_0 + \rho_2)$.

Under these conditions, we can compute $M = 0.5$, in agreement with (45), the radius of convergence of $h_f$, $\rho_2 = \pi/4$ in consonance with Theorem 3 and the interval $[y_0 - \rho_2, M, y_0 + \rho_2] = [-1.570796, 0]$.

As $0 \notin (-1.570796, 0)$, we look for a better $\rho_2$, $M$, and $x_0$ (for practical purposes, we consider $0 \notin (-1.570796, 0)$ instead of $0 \in [-1.570796, 0]$, in order to improve the convergence of the series $h_f(0)$).

Step 3: Improvement of $\rho_2$, $M$, and $x_0$.

We fix $\rho_2 = \pi/3$. In other words, the third of the interval $[0, \pi]$, and we define the rectangles:

- $S_{q_1} = [0, 0 + 2\rho_2] \times \left[-\frac{\pi}{3}, \frac{2\pi}{3}\right] = \left[0, \frac{2\pi}{3}\right] \times \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, with $x_0 = \frac{\pi}{3}$.
- $S_{q_2} = [0 + \rho_2, 0 + 3\rho_2] \times \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] = \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \times \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, with $x_0 = \frac{2\pi}{3}$.

In such a way that $S_{q_1} \cup S_{q_2} = [0, \pi] \times \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

We bound $-1/2 - 2\cosh(d)\sin(c)$ in $S_{q_1}$:

\[
-\frac{1}{2} - 2\cosh(d)\sin(c) \leq -\frac{1}{2} - 2\cosh(0)\sin(0) \leq -\frac{1}{2}.
\]
Then, there is only one root in \((x_0 - \rho_2, x_0 + \rho_2)\).

Under these conditions, we can compute \(M = 0.5\), the radius of convergence of \(h_f, \rho_2 M = \pi/6\)
and the interval \([y_0 - \rho_2 M, y_0 + \rho_2 M] = [-0.047198, 1]\).

As \(0 \in (-0.047198, 1)\), we take \(\rho_2 = \pi/3, M = 0.5, \text{and } x_0 = \pi/3.\)

See a resume of this process in Table 3.

| \(n\) | Interval | \(\rho_2\) | \(\text{Sign of } \frac{\partial u}{\partial c}\) | \(\text{Sign of } f(E_1) \cdot f(E_2)\) | \(M\) | \(x_0\) | \(y_0 - \rho_2 M\) | \(y_0 + \rho_2 M\) |
|------|----------|------------|----------------|----------------|------|--------|----------------|----------------|
| 2    | \([0, \pi]\) | \(\pi/2\) | - | - | 0.5 | \(\pi/2\) | -1.570796 | 0 |
| 3    | \([0, \frac{2\pi}{3}]\) | \(\pi/3\) | - | - | 0.5 | \(\pi/3\) | -0.047198 | 1 |

**Example 4.** Find a root in the interval \([0, \pi]\) of the function \(y = f(x) = 2 \cos(x) - \frac{1}{2^x}\), in an analytic way, through the series \(h_f(0)\), and approximate its numerical value with an error lower or equal to 0.001.

With the same values as in the previous example for the parameters \(x_0 = \pi/3, \rho_2 = \pi/3\) and \(M = 0.5, y_0 = f(x_0) = 0.476401\) the series \(h_f(0)\) makes sense, since \(-0.047198 \leq 0 \leq 1\), so the exact value of the required root is given by:

\[
h_f(0) = \frac{\pi}{3} + \sum_{p=0}^{\infty} A_p \left(\frac{\pi}{3}\right) (-0.476401)^{p+1}.
\]

With these values of the parameters, we need to take \(p = 106\), at least, to approximate \(h_f(0)\) with the required precision. Thus, in order to get a better result, we repeat Step 3 of the Example 3 for \(\rho_2 = \pi/3\) and \(\rho_2 = \pi/4\) (see the outcomes in Table 4).

| \(n\) | Interval | \(\rho_2\) | \(\text{Sign of } \frac{\partial u}{\partial c}\) | \(\text{Sign of } f(E_1) \cdot f(E_2)\) | \(M\) | \(x_0\) | \(y_0 - \rho_2 M\) | \(y_0 + \rho_2 M\) |
|------|----------|------------|----------------|----------------|------|--------|----------------|----------------|
| 2    | \([0, \pi]\) | \(\pi/2\) | - | - | 0.5 | \(\pi/2\) | -1.570796 | 0 |
| 3    | \([0, \frac{2\pi}{3}]\) | \(\pi/3\) | - | - | 0.5 | \(\pi/3\) | -0.047198 | 1 |
| 4    | \([\pi, \frac{3\pi}{4}]\) | \(\pi/4\) | - | - | 0.5 | \(\pi/4\) | 0.628815 | 1.414214 |
| 4    | \([\pi, \frac{3\pi}{2}]\) | \(\pi/4\) | - | - | 0.5 | \(\pi/4\) | 2.199612 | 1.414214 |

In consonance with Table 4, we make a new choice: \(x_0 = \pi/2, y_0 = -0.785398, \rho_2 = \pi/4, M = 1.91, \text{and } p = 13\) (see Table 5). For \(p \geq 14\), the error decreases more and more.
Table 5. Values of \( p \) and \( R_{p+1} \).

| \( p \) | 12 | 13 | 14 | 15 |
|-------|----|----|----|----|
| \( R_{p+1} \) | 0.001098 | 0.000576 | 0.000301 | 0.000157 |

Then, the radius of convergence of the series \( h_f \) around the point \( y_0 \) is, at least, \( \rho_2 \ M = 1.5 \), so \( h_f(0) \) is well defined and \( R_{14} \leq 0.000576 < 0.001 \). Hence:

\[
h_f(0) \approx \frac{\pi}{2} + \sum_{p=0}^{12} A_p \left( \frac{\pi}{2} \right) (-0.785398)^{p+1} \approx 1.252353
\]

with:

\[
1.251777 \leq 1.252353 - 0.000576 \leq h_f(0) \leq 1.252353 + 0.000576 \leq 1.252929.
\]

**Example 5.** Find a root in the interval \([0.5, 1.3]\) of the function \( y = f(x) = \cos(x) - x^3 \), in an analytic way, through the series \( h_f(0) \), and approximate its numerical value with an error lower or equal to 0.001.

**Step 1:** Definition of \( F(w) \) and \( F'(w) \), according to Theorem 2.

\[
F(w) = \cos(w) - w^3; \ w = c + id
\]

\[
F(w) = \cos(c) \cosh(d) - c^3 + 3cd^2 + i (-\sin(c) \sinh(d) - 3c^2d + d^3)
\]

\[
F'(w) = \frac{\partial}{\partial c}(c, d) + i \left( \frac{\partial}{\partial c}(c, d) \right) = -\sin(c) \cosh(d) - 3c^2 + 3d^2 + i (-\cos(c) \sinh(d) - 6cd).
\]

**Step 2:** Construction of the closed ball \( \mathbb{B}_C(x_0, \rho_2) \).

Proceeding as in Example 3, we take \( x_0 = 0.9 \) and \( \rho_2 = 0.4 \), due to the fact that in the interval \([0.5, 1.3]\):

- The function \( \cos(c) \) reaches its minimum and maximum values at the points \( c = 1.3 \) and \( c = 0.5 \), respectively.
- The function \( \sin(c) \) reaches its minimum and maximum values at the points \( c = 0.5 \) and \( c = 1.3 \), respectively.
- The function \( \sinh(d) \), at the points \( d = -0.4 \) and \( d = 0.4 \), respectively.
- The function \( \cosh(d) \), at the points \( d = 0 \) and \( d = 0.4 \), respectively.

Then, we can say that:

\[
-\sin(c) \cosh(d) - 3c^2 + 3d^2 \leq -\sin(0.5) \cosh(0) - 3 \cdot 0.5^2 + 3 \cdot 0.4^2 \leq -0.749426 \tag{55}
\]

for all \( w \in \mathbb{B}_C(0.9, 0.4) \) and, therefore, \( F'(w) \neq 0 \) in \( \mathbb{B}_C(0.9, 0.4) \) as required.

**Step 3:** Calculation of \( M_1, M_2, \) and \( M \).

For the same reasons as in Step 2, we arrive at:

\[
-3.480469 \leq -\cos(0.5) \sinh(0.4) - 6 \cdot 0.4 \cdot 1.3 \leq -\cos(c) \sinh(d) - 6dc \leq -\cos(0.5) \sinh(-0.4) - 6 (-0.4) 1.3 \leq 3.480469. \tag{56}
\]

According to (45) and in agreement with (56) and (55), \( M_1, M_2, \) and \( M \) are given by:

\[
M_1 = (-0.749426)^2; \ M_2 = 0; \ \sqrt{M_1 + M_2} = 0.749426 \geq M = 0.74.
\]

**Step 4:** Calculation of the radius of convergence, and the Taylor series of \( h_f \) around the point \( y_0 = f(x_0) = -0.10739 \).
As a consequence of all of this, we conclude that the inverse function of \( f \) is:

\[
h_f(y) = 0.9 + \sum_{p=0}^{\infty} A_p(0.9) \left( y + 0.10739 \right)^{p+1}
\]

and has a radius of convergence of, at least, \( \rho^2 M = 0.3 \).

The exact value of \( h_f(0) \) is given by the power series:

\[
h_f(0) = 0.9 + \sum_{p=0}^{\infty} A_p(0.9) \left( 0.10739 \right)^{p+1}
\]

that is well defined, since:

\[-0.10739 - 0.3 \leq 0 \leq -0.10739 + 0.3\]

**Step 5: Numerical approximation of \( h_f(0) \).**

We choose \( p = 7 \), according to the Table 6. For \( p \geq 8 \), the error decreases more and more.

| Table 6. Values of \( p \) and \( R_{p+1} \). |
|---|---|---|---|
| \( p \) | 6 | 7 | 8 | 9 |
| \( R_{p+1} \) | 0.001688 | 0.000613 | 0.000222 | 0.000081 |

Then, \( R_8 \leq 0.000613 < 0.001 \).

Hence:

\[
h_f(0) \approx 0.9 + 6 \sum_{p=0}^{\infty} A_p(0.9) \left( 0.10739 \right)^{p+1} \approx 0.865474
\]

with:

\[
0.864861 \leq 0.865474 - 0.000613 \leq h_f(0) \leq 0.865474 + 0.000613 = 0.866087.
\]

**7. Applications II: A Generalization of Catalan and Fuss–Catalan Numbers**

In this section, we are going to analyze the relations between Catalan and Fuss–Catalan numbers and the inverse functions of polynomials.

In agreement with (14), the inverse function of \( y = f(x) = a_0 + a_1 x + a_2 x^2 \), with \( a_1, a_2 \neq 0 \), is defined as:

\[
x = h_f(y) = \frac{1}{a_1} \sum_{q_2=0}^{\infty} (-1)^{q_2} C_{q_2} \left( \frac{a_2}{a_1} \right)^{q_2} \left( y - a_0 \right)^{q_2+1}
\]

being:

\[
C_{q_2} = \frac{(2q_2)!}{(q_2 + 1)! q_2!}; \quad q_2 = 0, 1, 2, ...
\]

the sequence of Catalan numbers, in concordance with (2).

Taking into account (16), series (57) is absolutely convergent if \( y \in V_{a_0} \) that in this case is defined as:

\[
V_{a_0} = \left\{ y \in \mathbb{R}; \quad 4 \left| \frac{(y - a_0)a_2}{a_1^2} \right| < 1 \right\}.
\]

Choosing \( a_0, a_1, \) and \( a_2 \) in such a way that \( 0 \in V_{a_0} \), from (57), we get:

\[
0 = a_0 + a_1 h_f(0) + a_2 h_{f^2}(0) \implies h_f(0) = -\frac{a_0}{a_1} - \frac{a_2}{a_1} h_{f^2}(0).
\]

\[
(58)
\]
As:

\[ h_f(0) = -\frac{a_0}{a_1} - \frac{a_0}{a_1} \sum_{q_2=1}^{\infty} \frac{(2q_2)!}{q_2!(q_2+1)!} \sum_{p=0}^{\infty} \frac{(2i_1)!}{i_1!i_2!(i_1+1)!(i_2+1)!} \left( \frac{a_0}{a_1} \right)^{i_1+i_2} \]

and

\[ h_f^2(0) = -\frac{a_0}{a_1} - \frac{a_0}{a_1} \sum_{q_2=1}^{\infty} \frac{(2q_2)!}{q_2!(q_2+1)!} \sum_{p=0}^{\infty} \frac{(2i_1)!}{i_1!i_2!(i_1+1)!(i_2+1)!} \left( \frac{a_0}{a_1} \right)^{i_1+i_2} \]

we obtain:

\[ i_1, i_2 \text{ being non-negative integer numbers; then, substituting } h_f(0) \text{ and } h_f^2(0) \text{ in the second formula of (58), we obtain:} \]

\[ -\frac{a_0}{a_1} - \frac{a_0}{a_1} \sum_{q_2=1}^{\infty} \frac{(2q_2)!}{q_2!(q_2+1)!} \left( \frac{a_0}{a_1} \right)^{q_2} \sum_{p=0}^{\infty} \frac{(2i_1)!}{i_1!i_2!(i_1+1)!(i_2+1)!} \left( \frac{a_0}{a_1} \right)^{i_1+i_2} \]

We deduce from this that:

\[ \sum_{q_2=1}^{\infty} \frac{(2q_2)!}{q_2!(q_2+1)!} \left( \frac{a_0}{a_1} \right)^{q_2} \sum_{p=0}^{\infty} \frac{(2i_1)!}{i_1!i_2!(i_1+1)!(i_2+1)!} \left( \frac{a_0}{a_1} \right)^{i_1+i_2} = \sum_{p=0}^{\infty} \frac{(2i_1)!}{i_1!i_2!(i_1+1)!(i_2+1)!} \left( \frac{a_0}{a_1} \right)^{i_1+i_2} \]

(59)

By equating the terms with the same degree of the series of both sides of (59) and making \( q_2 = i_1 + i_2 + 1 \geq 1; i_1 + i_2 = q_2 - 1 \), we arrive at:

\[ \frac{(2q_2)!}{q_2!(q_2+1)!} = \sum_{i_1+i_2=q_2-1} \frac{(2i_1)!}{i_1!i_2!(i_1+1)!(i_2+1)!}; \forall q_2 \geq 1. \]

In this way, we have provided an alternative proof of the recursive relation, given by (3).

Equivalently, the inverse function of \( y = f(x) = a_0 + a_1 x + a_m x^m \), with \( a_1, a_m \neq 0, m > 2 \), is:

\[ x = h_f(y) = \frac{1}{a_1} \sum_{q_m=0}^{\infty} (-1)^{(m-1)q_m} C_{q_m} \left( \frac{a_m}{-a_1} \right)^{q_m} (y - a_0)^{(m-1)q_m+1} \]

(60)

where, for \( n = q_m \), we have that:

\[ C_n^m = C_{q_m} = \frac{(m q_m)!}{((m-1)q_m+1)! q_m!}; q_m = 0, 1, 2, ... \]

that is, the Fuss–Catalan numbers sequence, in consonance with (4).

In agreement with (16), series (60) is absolutely convergent if \( y \in V_{a_0} \), that in this case is defined as:

\[ V_{a_0} = \left\{ y \in \mathbb{R}; \left| \frac{m^m}{(m-1)^{m-1}} \right| \left| \frac{(y - a_0)^{m-1}a_m}{a_1^m} \right| < 1 \right\} \]

Choosing \( a_0, a_1, \) and \( a_m \) in such a way that \( 0 \in V_{a_0} \), from (60), we get:

\[ 0 = a_0 + a_1 h_f(0) + a_m h_f^m(0) \Rightarrow h_f(0) = \frac{-a_0}{a_1} \frac{a_m}{a_1} h_f^m(0). \]

(61)
As:

\[ h_f(0) = \frac{a_0}{a_1} \sum_{q_m=1}^{\infty} C_{q_m} \left( \frac{a_m a_0^{m-1}}{(-a_1)^m} \right)^{i_1} \]

\[ h_f^m(0) = \left( -\frac{a_0}{a_1} \right)^m \sum_{i_1+i_2+\ldots+i_m=p} C_{i_1} C_{i_2} \cdots C_{i_m} \left( \frac{a_m a_0^{m-1}}{(-a_1)^m} \right)^{i_1 \cdots +i_m} \]

\( i_1, i_2, \ldots, i_m \) being non-negative integers numbers; then, substituting \( h_f(0) \) and \( h_f^m(0) \) in the second formula of (61), we arrive at:

\[ = \frac{a_0}{a_1} \sum_{q_m=1}^{\infty} C_{q_m} \left( \frac{a_m a_0^{m-1}}{(-a_1)^m} \right)^{q_m} \]

\[ = \left( -\frac{a_0}{a_1} \right)^m \sum_{p=0}^{\infty} \sum_{i_1+i_2+\ldots+i_m=p} C_{i_1} \cdots C_{i_m} \left( \frac{a_m a_0^{m-1}}{(-a_1)^m} \right)^{i_1 \cdots +i_m} \]

We deduce from this that:

\[ \sum_{q_m=1}^{\infty} C_{q_m} \left( \frac{a_m a_0^{m-1}}{(-a_1)^m} \right)^{q_m} = \sum_{p=0}^{\infty} \sum_{i_1+i_2+\ldots+i_m=p} C_{i_1} \cdots C_{i_m} \left( \frac{a_m a_0^{m-1}}{(-a_1)^m} \right)^{i_1 \cdots +i_m+1} \]

By equating the terms with the same degree of the series of both sides of (62) and making \( q_m = i_1 + \cdots + i_m + 1 \geq 1 \), we arrive at:

\[ C_{q_m} = \sum_{i_1+i_2+\ldots+i_m=q_m-1} C_{i_1} \cdots C_{i_m}, \quad \forall q_m \geq 1. \]

That is the well known recursive formula (5), for which we have provided a new proof.

In this framework, we provide an original theorem.

**Theorem 5.** The numbers \( C_{q_2, \ldots, q_p} \), introduced in (10), are a generalization of Catalan and Fuss–Catalan numbers and they hold the recursive sequence:

\[ C_{q_2, \ldots, q_p} = \sum_{i_1+i_2+\ldots+i_p=q_2-1} C_{i_1} C_{i_3} \cdots C_{i_p} + \sum_{i_1+i_2+\ldots+i_p=q_3-1} C_{i_1} C_{i_3} \cdots C_{i_p} + \sum_{i_1+i_2+\ldots+i_p=q_4-1} C_{i_1} C_{i_3} \cdots C_{i_p} + \sum_{i_1+i_2+\ldots+i_p=q_p-1} C_{i_1} C_{i_3} \cdots C_{i_p} \]

\[ + \cdots + \sum_{i_1+i_2+\ldots+i_p=q_p} C_{i_1} C_{i_3} \cdots C_{i_p} \]

which is a generalization of (3) and (5).

Formula (63) is true for all \( n \geq 1 \) and for all \( p \geq 2 \), with \( q_2 + q_3 + \ldots + q_p = n \). If \( q_2 = 0 \); then, the first summand of the second side of (63) does not exist, so we can take it as zero in (63); equally, if \( q_3 = 0 \), then the second summand of the second side of (63) does not exist, so we can take it as zero; and so on for the following \( q_4, \ldots, q_p \).

**Proof.** Without loss of generality, we prove the Theorem only for the case \( C_{q_3, \ldots, q_p} \), since its generalization follows exactly the same process and, in this way, the development of the reasoning gains enough clarity, due to the complexity of the superscripts and subscripts.
From (10), it is easy to check that $C_{q20\ldots0} = C_{q7}$ (Catalan numbers) and that $C_{0\ldots0q4s\ldots0} = C_{q4s}$ (Fuss–Catalan numbers).

Taking into account (14), the inverse function of $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ ($a_1, a_3 \neq 0$) is given by:

$$h_f(y) = \frac{1}{a_1} \sum_{p=0}^{\infty} \sum_{q_2+q_3=p} (-1)^{q_2+2q_3} \frac{a_2}{a_1^2} C_{q2q3} \left( \frac{a_2}{a_1^2} \right)^{q_2} \left( \frac{a_3}{(-a_1)^3} \right)^{q_3} (y - a_0)^{q_2+2q_3+1}$$

(64)

with

$$C_{q2q3} = \frac{(2q_2 + 3q_3)!}{q_2! q_3! (q_2 + 2q_3 + 1)!}$$

according to (10).

In agreement with (16), series (64) is absolutely convergent if $y \in V_{a_0}$ that in this case is defined as:

$$V_{a_0} = \left\{ y \in \mathbb{R} \mid \frac{9}{2} \left| \frac{(y - a_0)a_2}{a_1^2} + \frac{27}{4} \left| \frac{(y - a_0)^2a_3}{a_1^3} \right| \right| < 1 \right\}.$$

Choosing $a_0, a_1, a_2,$ and $a_3$ in such a way that $0 \in V_{a_0}$ from (64), we get:

$$0 = a_0 + a_1 h_f(0) + a_2 h_f^2(0) + a_3 h_f^3(0) \Rightarrow h_f(0) = -\frac{a_0}{a_1} - \frac{a_2}{a_1} h_f^2(0) - \frac{a_3}{a_1} h_f^3(0).$$

(65)

We define:

$$m_0 = \frac{a_2 a_0}{a_1^2} ; n_0 = \frac{a_3 a_2^2}{(-a_1)^3} \text{ and } A_n = \sum_{q_2+q_3=n} C_{q2q3} m_0^{q_2} n_0^{q_3}.$$

From (64), we can say that:

$$h_f(0) = \frac{a_0}{-a_1} \sum_{p=0}^{\infty} \sum_{q_2+q_3=p} C_{q2q3} m_0^{q_2} n_0^{q_3} = \frac{a_0}{-a_1} \sum_{p=0}^{\infty} A_p$$

$$h_f^2(0) = \frac{a_0^2}{a_1^2} \sum_{p=0}^{\infty} \sum_{i_1+i_2=p} A_{i_1} A_{i_2}; \text{ with } i_1, i_2 \text{ non-negative integers}$$

$$h_f^3(0) = \frac{a_0^3}{(-a_1)^3} \sum_{p=0}^{\infty} \sum_{j_1+j_2+j_3=p} A_{j_1} A_{j_2} A_{j_3}; \text{ with } j_1, j_2, j_3 \text{ non-negative integers}$$

then, substituting $h_f(0), h_f^2(0),$ and $h_f^3(0)$ in the second equation of (65) and cancelling the common factor $a_0 / (-a_1)$, we obtain:

$$\sum_{p=0}^{\infty} A_p = 1 + m_0 \sum_{p=0}^{\infty} \sum_{i_1+i_2=p} A_{i_1} A_{i_2} + n_0 \sum_{p=0}^{\infty} \sum_{j_1+j_2+j_3=p} A_{j_1} A_{j_2} A_{j_3}. $$

(66)

The second summand of the second side of (66) becomes:

$$m_0 \sum_{p=0}^{\infty} \sum_{i_1+i_2=p} \left( \sum_{r_2+r_3=i_2} C_{r_2r_3} m_0^{r_2} n_0^{r_3} \right) \left( \sum_{s_2+s_3=i_3} C_{s_2s_3} m_0^{s_2} n_0^{s_3} \right).$$

From Cauchy product rule:

$$m_0 \sum_{p=0}^{\infty} \sum_{i_1+i_2=p} \sum_{r_2+r_3=i_2} C_{r_2r_3} m_0^{r_2} n_0^{r_3} = \sum_{p=0}^{\infty} \sum_{i_1+i_2=p} m_0^{i_1+i_2+p} \sum_{r_2+r_3=i_2} C_{r_2r_3} m_0^{r_2} n_0^{r_3}$$

(67)
with $i', i'_2$ non-negative integers. The third summand of the second side of (66) turns out to be:

$$n_0 \sum_{p=0}^{\infty} \sum_{j_1+j_2+j_3=p} \left( \sum_{t_2+t_3=j_1} C_{t_2t_3} m_0^{i_1} n_0^{j_2} \right) \left( \sum_{u_2+u_3=j_2} C_{u_2u_3} m_0^{i_2} n_0^{j_3} \right) \left( \sum_{v_2+v_3=j_3} C_{v_2v_3} m_0^{i_3} n_0^{j_3} \right).$$

Again, from Cauchy product rule:

$$\sum_{p=0}^{\infty} \sum_{i'_1+i'_2=p} \sum_{t_2+u_2+v_2=i'_1} C_{t_2} C_{u_2} C_{v_2} m_0^{i_1} n_0^{i_2+1}$$

(68)

with $i', i'_2$ non-negative integers. If we make the change $i'_1 + 1 = q_2$ and $i'_2 + 1 = q_3$, from (66)–(68), as $m_0$ and $n_0$ are arbitrary numbers, we have that:

$$C_{q_2q_3} = \sum_{r_2+r_3=q_2-1} C_{r_2r_3} C_{q_2q_3} + \sum_{t_2+u_2+v_2=q_3} C_{t_2u_2} C_{u_2v_2} C_{v_2q_3}$$

(69)

for $C_{q_2q_3} \neq C_{q_0}$. If $q_2 = 0$, then $i'_1 = -1$ and its corresponding term in the development of $h_0^2(0)$ does not exist, in agreement with (67), so we can take the first summand of the second side of (69) as zero. The same thing happens if $q_3 = 0$, then $i'_2 = -1$ and its corresponding term of the development of $h_0^2(0)$ does not exist, in correspondence with (68), so we can take the second summand of the second side of (69) as zero too.

Finally, it is easy to check that, for $q_3 = 0$, (69) coincides with (3) and that for $q_2 = 0$, with (5).

The result follows. \qed

To close this section, we provide a procedure to generate combinatorial identities by comparing the Taylor development of a function, $f$, and its inverse $h_f$. To give an example, consider the function $y = e^x$ and its inverse, $x = \log(y)$. From (37), with $x = 0$ and $y = e^0 = 1$, we have that:

$$\log^{(p+1)}(1) = (-1)^p p! = A_p(0)(p+1)!$$

$$= \sum_{q_2+q_3+\cdots+q_p+1=p} (-1)^p C_{q_2+q_3+\cdots+q_p+1} \left( \frac{1}{2!} (-1)^2 \right)^{q_2} \cdots \left( \frac{1}{(p+1)!} (-1)^{p+1} \right)^{q_p+1}.$$

Taking into account that:

$$(-1)^p (-1)^2 q_2+q_3+\cdots+(p+1) q_p+1 = (-1)^p (-1)^{q_2+\cdots+q_p+1} = (-1)^{q_2+\cdots+q_p+1}$$

we arrive at the amazing relation:

$$\frac{(-1)^p}{p+1} = \sum_{q_2+q_3+\cdots+q_p+1=p} (-1)^{q_2+\cdots+q_p+1} \frac{C_{q_2+q_3+\cdots+q_p+1}}{(2!)^q_2 (3!)^q_3 \cdots ((p+1)!)^q_{p+1}}.$$

8. Conclusions

As known, due to the Inverse Function Theorem, given an analytic real function, $f(x)$, and a point, $x_0$, with $f'(x_0) \neq 0$, there is a neighborhood of $x_0$, $V_{x_0}$, in such a manner that the inverse function of $f(x)$ is well defined in $f(V_{x_0})$. In other words, we know about its existence, but, with respect to its explicit formulation, in the general case, nothing has yet been established. In this context, throughout this paper, we have provided a general procedure to construct the inverse function, $x = h_f(y)$, of an arbitrary analytic real function, $f(x)$. 
We have addressed this problem by developing the Taylor series of \( h_f(y) \), in the same way as the most important functions in real analysis (\( e^x, \log(x), \sin(x), \cos(x), \ldots \)) are defined. With this aim in mind, we have found a general formula to calculate the \( n \)-th derivative of \( x = h_f(y) \) as a function of the derivatives of \( y = f(x) \).

Just as it happens, for example, with \( e^3 \) or \( \sin(2) \), for practical purposes, the numerical values of the inverse function, \( h_f(y) \), need to be expressed with a prefixed number of digits of accuracy. We have faced this question by elaborating a formulation of the Taylor Remainder, valid for any inverse function, \( h_f(y) \).

We have shown, through several examples, how the inverse function, \( x = h_f(y) \), can be used to solve, in an analytic, not numeric, way the nonlinear equation \( f(x) = 0 \). Indeed, the series \( h_f(0) \) gives an exact solution of the equation \( f(x) = 0 \), in an analytic manner and, providing a numeric value, with a predetermined accuracy, to the number \( h_f(0) \) (the already obtained solution), it is not the same as finding it by numerical methods.

Finally, we have obtained a new expansion of the Catalan and Fuss–Catalan numbers that we hope can be used in future research in the field of Graph Theory.

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