HÖLDER ESTIMATES AND ASYMPTOTIC BEHAVIORS AT INFINITY OF SOLUTIONS OF SOME DEGENERATE ELLIPTIC EQUATIONS IN HALF SPACES

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Abstract. In this paper we investigate the asymptotic behaviors at infinity of viscosity solutions of a kind of degenerate elliptic equations, which are linearized equations of a class of degenerate Monge-Ampère equations. Meanwhile, the Hölder estimates up to the boundary will be obtained by using the rescaling method, and a Liouville type result on some kind of Baouendi-Grushin type operator was deduced as a byproduct.

1. Introduction

In this paper we investigate the asymptotic behavior at infinity of viscosity solutions of the following degenerate non-divergence elliptic equations

\[(1.1) \quad Lu = x_n^{2\alpha} \sum_{i,j=1}^{n-1} a_{ij}(x) D_{ij}u(x) + 2x_n^\alpha \sum_{i=1}^{n-1} a_{in}(x) D_{in}u(x) + D_{nn}u(x) = 0 \quad \text{in } \mathbb{R}^n_+ \backslash B_1^+,
\]

where \( n \geq 2, \alpha > 0, \mathbb{R}^n_+ = \mathbb{R}^n \cap \{x_n > 0\}, B_1^+ = \mathbb{R}^n_+ \cap \{|x| < 1\}. \) Note that (1.1) relates to the linearization of the following degenerate Monge-Ampère equation

\[(1.2) \quad \det D^2u = f(x)x_n^{2\alpha} \quad \text{on } \{x_n > 0\},
\]

where \( \alpha > 0, \) and \( f(x) \) is positive and continuous.

To ensure the ellipticity of operator \( L, \) it is always assumed that \( a_{ij}(x), a_{in}(x) \in C(\mathbb{R}^n_+) \) \((i, j = 1, \cdots, n-1)\) and there exist constants \( 0 < \lambda \leq \Lambda < \infty \) such that for any \( \xi \in \mathbb{R}^{n-1}, \)

\[(1.3) \quad \lambda|\xi|^2 \leq \xi^T \sum_{i,j=1}^{n-1} a_{ij}(x)\xi \leq \Lambda|\xi|^2, \quad \forall \ x \in \mathbb{R}^n_+,
\]

and for some \( 0 < \delta < 1, \)

\[(1.4) \quad 1 - \lambda^{-1} \sum_{i=1}^{n-1} ||a_{in}||^2_{L^\infty(\mathbb{R}^n_+)} > \delta.
\]

In this paper, solutions always indicate viscosity solutions (cf.[3, 4, 9, 14] for definition).

As for \( \alpha = 0, \) by (1.3) and (1.4), \( L \) is uniformly elliptic. The asymptotic behavior at infinity was considered in [10]. Note that the crucial key to obtain the asymptotic behavior is the boundary Hölder estimates, which is classical for uniformly elliptic equations (cf.[3, 7, 9]).

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As for $\alpha > 0$, When $a_{ij} \equiv 1$ and $a_{in} \equiv 0$ ($i, j \leq n - 1$), $L$ is a Baouendi-Grushin type operator, which was introduced by [1] and [8], respectively. There have been extensive works on the studies of the Baouendi-Grushin type operators (cf.[2, 5, 6, 11, 13, 15] and references therein). When $a_{ij}$ satisfies (1.3), Le and Savin [12] obtained the boundary Schauder estimates for solutions of the degenerate elliptic equation

$$x_n^\alpha \sum_{i,j=1}^{n-1} a_{ij}(x) D_{ij} u(x) + D_{nn} u(x) = x_n^\alpha f(x) \quad \text{in } B_1^+$$

with $\alpha > 0$.

In this paper, we would like to study the asymptotic behavior at infinity of solutions of (1.1) with the coefficients satisfying (1.3) and (1.4). The asymptotic behavior at infinity of solutions of (1.1) is a key point to obtain the asymptotic behavior at infinity of solutions of (1.2) provided that $f(x)$ tends to some positive constant at infinity with proper decay rate.

By the similar rescaling method in [12], we establish the Hölder estimates up to the flat boundary of solutions of (1.1) with the coefficients satisfying (1.3) and (1.4).

**Theorem 1.1.** Let $u \in C(\overline{B_1^+})$ be a solution of

$$\begin{cases}
Lu(x) = 0 & \text{in } B_1^+ \\
u(x) = 0 & \text{on } B_1 \cap \{x_n = 0\},
\end{cases}$$

(1.5)

where $L$ is given by (1.1) with the coefficients satisfying (1.3) and (1.4). Then $u \in C^{\frac{1}{1+n}}(\overline{B_1^+})$.

As we all known (cf. [6]), the function

$$d(x) = d(x', x_n) = \left\{ |x'|^2 + \frac{1}{(\alpha + 1)^2} x_n^{2(\alpha + 1)} \right\}^{\frac{1}{2(\alpha + 1)}}$$

(1.6)

is the nature gauge associated with the Baouendi-Grushin type operator

$$\mathcal{L} = x_n^{2\alpha} \Delta_{x'} + \partial_{x_n x_n}^2 \quad \text{in } \mathbb{R}^n_+.$$  

That is,

$$\mathcal{L}(d(x)^Q) = 0 \quad \text{in } \mathbb{R}^n_+,$$

(1.7)

where $Q = (\alpha + 1)(n - 1) + 1$, and see Section 3 below for details.

Theorem 1.1 together with Harnack inequalities and the comparison principle yields our main theorem as follows.

**Theorem 1.2.** Let $u \in C^1(\mathbb{R}^n_+ \setminus B_1^+)$ be a solution of

$$\begin{cases}
Lu = 0 & \text{in } \mathbb{R}^n_+ \setminus B_1^+ \\
u = 0 & \text{on } \{x_n = 0, |x| \geq 1\},
\end{cases}$$

(1.8)

where $L$ is given by (1.1) with the coefficients satisfying (1.3) and (1.4), and for some $s > 0$,

$$|a_{ij}(x) - \delta_{ij}| + |a_{in}(x)| \leq d(x)^{-s} \quad \text{in } \mathbb{R}^n_+ \setminus B_1^+, \quad i, j < n.$$  

(1.9)

Assume that $|u| \leq 1$ on $\partial B_1 \cap \{x_n > 0\}$, $|Du| \leq 1$ in $\mathbb{R}^n_+ \setminus B_1^+$ and $|Du| \to 0$ as $|x| \to \infty$. Then

$$|u(x)| \leq \frac{C x_n}{d(x)^Q} \quad \text{in } \mathbb{R}^n_+ \setminus B_{R}^+,$$

(1.10)
where \( d(x) \) is as in \((1.6)\), \( C > 0 \) and \( R \geq 1 \) depend only on \( \alpha, \delta, s \) and \( n \).

**Remark 1.3.** If \( \alpha = 0 \), Theorem 1.2 still holds (cf.\([10]\)).

By Theorem 1.2 and comparison principle, we have the following Liouville type theorem.

**Theorem 1.4.** Let \( u \in C^1(\mathbb{R}_+^n) \) be a solution of
\[
\begin{cases}
  \mathcal{L}u = 0 & \text{in } \mathbb{R}_+^n, \\
  u = 0 & \text{on } \{x_n = 0\},
\end{cases}
\]
where \( \mathcal{L} \) was given by \((1.7)\). If \( |Du| \to 0 \) as \( |x| \to \infty \). Then \( u(x) \) must be zero.

Note that the proof of Theorem 1.4 is obviously, and thus we omit it here.

This paper is organized as follows. In Section 2, we first claim that \( L \) is uniformly elliptic in interior, and then the interior Hölder estimates is clear. Secondly, we show the boundary Hölder estimates, which can be approached by the interior Hölder estimates via rescaling. In Section 3, a supersolution is constructed according to the fundamental solution of one Baouendi-Grushin type operator in the upper half space. Then it together with the Hölder estimates up to the flat boundary implies that Theorem 1.2 holds.

## 2. Proof of Theorem 1.1

Firstly, by \((1.3)\) and \((1.4)\), we show that \( L \) is elliptic in \( \overline{B}_1^+ \) and uniformly elliptic if \( x_n > \varepsilon_0 \) with any fixed \( \varepsilon_0 > 0 \). Precisely,

**Lemma 2.1.** Let the coefficients of \( L \) in \((1.1)\) satisfy \((1.3)\) and \((1.4)\). Then \( L \) is elliptic in \( \overline{B}_1^+ \). Furthermore, for any fixed \( \varepsilon_0 > 0 \), \( L \) is uniformly elliptic in \( \overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\} \).

**Remark 2.2.** Lemma 2.1 is standard, and one can see its proof in Appendix below.

To show the Hölder estimates up to the boundary, we need to give some notions as follows (cf.\([12]\)).

**Definition 2.3.** We define a distance \( d_\alpha \) between point \( y \) and point \( z \) by
\[
d_\alpha(y, z) := |y' - z'| + |y_n^{1+\alpha} - z_n^{1+\alpha}|.
\]

Observe that the relation between \( d_\alpha \) and the Euclidean distance satisfies:
\[
c|y - z|^{1+\alpha} \leq d_\alpha(y, z) \leq C|y - z|
\]
and
\[
d_\alpha(y, z) \sim |y - z| \quad \text{if } y, z \in \overline{B}_1^+ \cap \left\{x_n \geq \frac{1}{8}\right\}.
\]

For any \( h > 0 \) and any \( \tilde{x} \in \mathbb{R}^n \), we denote
\[
E_h(\tilde{x}) = \left\{x \in \mathbb{R}^n : |x' - \tilde{x}'|^2 + |x_n - \tilde{x}_n|^{2(1+\alpha)} < h\right\},
\]
and
\[
F_h = \text{diag} \left(h^{\frac{1}{2}}, h^{\frac{1}{2}}, \ldots, h^{\frac{1}{2}}, h^{\frac{1}{2(1+\alpha)}}\right).
\]
For simplicity, we always denote
\[ E_h = E_h(0) = \left\{ x \in \mathbb{R}^n : |x'|^2 + |x_n|^{2(1+\alpha)} < h \right\}; \quad E^+_h = E_h \cap \{ x_n > 0 \}. \]

A simple calculation gives
\[ F_h E_{\alpha'} \left( \frac{1}{2} e_n \right) = E_{\alpha'h} \left( \frac{1}{2} h^{\frac{1}{2(1+\alpha)}} e_n \right), \quad F_h E^+_h = E^+_h, \]
where
\[ e_n = (0, \cdots, 0, 1), \quad \alpha' = 4^{-2(1+\alpha)}. \]

Note that (1.1) and \( d_\alpha \) keep their forms under the transformation \( x \to F_h x \). Precisely, let
\[ \tilde{u}(x) = u(F_h x), \quad x \in E_1, \]
and then it solves
\[ \tilde{L}\tilde{u} = x_n^{2\alpha} \sum_{i,j=1}^{n-1} \tilde{a}_{ij}(x) D_{ij} \tilde{u}(x) + x_n^\alpha \sum_{i=1}^{n-1} 2\tilde{a}_{in}(x) D_{in} \tilde{u}(x) + D_{nn} \tilde{u}(x) = 0 \]
with
\[ \tilde{a}_{ij}(x) = a_{ij}(F_h x), \quad \tilde{a}_{in}(x) = a_{in}(F_h x), \quad i, j \leq n - 1, \]
and
\[ d_\alpha(y, z) = h^{-\frac{\alpha}{2}} d_\alpha(F_h y, F_h z). \]

If function \( w \) is \( \gamma \)-Hölder continuous in \( \Omega \subset \overline{B}_1^+ \) with respect to \( d_\alpha \), we write
\[ w \in C^\gamma_\alpha(\overline{\Omega}) \]
and define
\[ [w]_{C^\gamma_\alpha(\overline{\Omega})} = \sup_{y, z \in \overline{\Omega}, y \neq z} \frac{|w(y) - w(z)|}{(d_\alpha(y, z))^{\gamma}}, \quad ||w||_{C^\gamma_\alpha(\overline{\Omega})} = ||w||_{L^{\infty}(\overline{\Omega})} + [w]_{C^\gamma_\alpha(\overline{\Omega})}. \]

Next we show Theorem 1.1.

Proof of Theorem 1.1.

We divided this into two case: \( u \in C^{\frac{1}{1+\alpha}}(\overline{B}_1^+ \cap \{ x_n > \frac{1}{8} \}) \) and \( u \in C^{\frac{1}{1+\alpha}}(\overline{B}_1^+ \cap \{ x_n \leq \frac{1}{8} \}). \)

**Case 1.** \( x \in \overline{B}_1^+ \cap \{ x_n > \frac{1}{8} \}. \)

By Lemma 2.1, \( L \) is uniformly elliptic in \( \overline{B}_1^+ \cap \{ x_n > \frac{1}{8} \} \). Applying the classical Hölder estimates to \( u \), there exists \( C > 0 \), depending only on \( \lambda, \Lambda, \alpha, \delta, n \) and \( ||u||_{L^{\infty}} \), such that
\[ [u]_{C^{\frac{1}{1+\alpha}}(\overline{E}_{\alpha'}(\frac{1}{4} x_n))} \leq C ||u||_{L^{\infty}} \leq C. \]

**Case 2.** \( x \in \overline{B}_1^+ \cap \{ x_n \leq \frac{1}{8} \}. \) We show this case by four steps as the following.

**Step 1.** There exists some \( C > 0 \), depending only on \( \lambda, \Lambda, \alpha, \delta, n \) and \( ||u||_{L^{\infty}} \), such that
\[ |u(x)| \leq C x_n \quad \text{in} \quad B_\frac{1}{4}. \]

It only need to show that for any \( x_0 \in \{ x_n = 0, |x'| < \frac{1}{4} \}, \)
\[ |u(x_0, x_n)| \leq C x_n. \]
Let $$\bar{u}(x) = Cx_n + B|x'| - x_0'|^2 - \frac{C}{2}x_n^{2+\alpha}$$ with $$B = 16||u||_{L^\infty}.$$ One can choose $$C > 0,$$ depending only on $$\Lambda,$$ $$\alpha,$$ $$n$$ and $$||u||_{L^\infty},$$ such that

$$L\bar{u} \leq 0$$ in $$B_1^+,$$ 
\[
\bar{u} \geq ||u||_{L^\infty} \geq u \quad \text{on } \partial B_1^+,
\]

if taking

$$2(n-1)\Lambda B - (2 + \alpha)(1 + \alpha)C/2 \leq 0 \quad \text{and} \quad \frac{C}{2}x_n + B|x'| - x_0'|^2 > ||u||_{L^\infty} \quad \text{on } \partial B_1^+.$$ 

Therefore, (2.10) and the comparison principle (cf.[14, Theorem 6]) yield Step 1.

Step 2. For any fixed $$h \in (0, 1),$$

$$[u]_{C^{1+\alpha}_{E_{\alpha', h} \left( \frac{1}{2}h \frac{1}{2(1+\alpha)}e_n \right)}} \leq C.$$ 

(2.11)

In fact, let $$\tilde{u}$$ be as in (2.5), and then $$\tilde{u}$$ solves (2.6) in $$B_1^+.$$ By (2.5) and (2.9), we have

$$\tilde{u} \leq Ch^{\frac{1}{2(1+\alpha)}} \quad \text{in } B_1^+.$$ 
(2.12)

Similar to Case 1, applying the classical Hölder estimates to $$\tilde{u}$$ in $$E_{\alpha', h} \left( \frac{1}{2}e_n \right),$$ we have

$$[\tilde{u}]_{C^{1+\alpha}_{E_{\alpha', \frac{1}{2}e_n}}} \leq C||\tilde{u}||_{L^\infty(B_1^+)} \leq Ch^{\frac{1}{2(1+\alpha)}}.$$ 

By (2.2), we see

$$[\tilde{u}]_{C^{1+\alpha}_{E_{\alpha', h} \left( \frac{1}{2}e_n \right)}} \leq Ch^{\frac{1}{2(1+\alpha)}}.$$ 

This together with (2.4), (2.5) and (2.8) yields (2.11), since

$$\frac{|\tilde{u}(y) - \tilde{u}(z)|}{(d_\alpha(y, z))^{1+\alpha}} = \frac{|u(F_{\alpha}y) - u(F_{\alpha}z)|}{h^{\frac{1}{2(1+\alpha)}}(d_\alpha(F_{\alpha}y, F_{\alpha}z))^{1+\alpha}}.$$ 

Step 3. We prove that $$u \in C^{1+\alpha}_{\alpha, e_n}$$ at 0 along $$e_n$$ direction, i.e.,

$$\sup_{0<h<1} \left| u \left( \frac{1}{2}h^{\frac{1}{2(1+\alpha)}}e_n \right) - u(0) \right| \left( \frac{1}{2}h^{\frac{1}{2(1+\alpha)}} \right)^{1+\alpha} \leq C$$

for some $$C > 0$$ depending only on $$\lambda,$$ $$\Lambda,$$ $$\alpha$$ and $$n.$$ It suffices to prove that

$$\left| u \left( \frac{1}{2}h^{\frac{1}{2(1+\alpha)}}e_n \right) - u(0) \right| \leq Ch^{\frac{1}{2(1+\alpha)}},$$

where $$C > 0$$ independents on $$h.$$ Indeed, Step 2 yields that for any $$k = 1, 2, \cdots ,$$

$$\left| u \left( \frac{1}{2^k}h^{\frac{1}{2(1+\alpha)}}e_n \right) - u \left( \frac{1}{2^{k+1}}h^{\frac{1}{2(1+\alpha)}}e_n \right) \right| \leq C2^{-k-1}h^{\frac{1}{2(1+\alpha)}},$$

These imply that
\[
\left| u \left( \frac{1}{2^k h^{2(1+\alpha)}} e_n \right) - u(0) \right| \leq \sum_{k=1}^{\infty} \left| u \left( \frac{1}{2^k h^{2(1+\alpha)}} e_n \right) - u \left( \frac{1}{2^{k+1} h^{2(1+\alpha)}} e_n \right) \right| \leq \sum_{k=1}^{\infty} C 2^{-k-1} \leq C.
\]

Therefore, \( u \in C^{1+\alpha}_x \) at 0 along \( e_n \) direction.

**Step 4.** We show **Case 2.**

Similar to **Step 3**, we have that \( u \in C^{1+\alpha}_x \) at any \( x \in B^{+}_1 \cap \{ x_n = 0 \} \) along \( e_n \) direction.

Let \( y, z, x \in B^{+}_1 \cap \{ x_n \leq \frac{1}{8} \} \) and denote by \( y_n, z_n \) the \( n \)th component of \( y \) and \( z \), respectively.

If \( z \in E_{2^{-2(1+\alpha)}} y_n^{2(1+\alpha)} (y_n) \) or \( y \in E_{2^{-2(1+\alpha)}} z_n^{2(1+\alpha)} (z_n) \), by (2.11) in **Step 2**, we are done.

Otherwise, \( z \notin E_{2^{-2(1+\alpha)}} y_n^{2(1+\alpha)} (y_n) \) and \( y \notin E_{2^{-2(1+\alpha)}} z_n^{2(1+\alpha)} (z_n) \), and then we have

\[
|y - z|^2 \geq \max \left\{ 2^{-2(1+\alpha)} y_n^{2(1+\alpha)}, 2^{-2(1+\alpha)} z_n^{2(1+\alpha)} \right\}.
\]

By **Step 3** and the boundary value condition, we get

\[
|u(y) - u(z)| \leq |u(y) - u(y', 0)| + |u(y', 0) - u(z', 0)| + |u(z', 0) - u(z)| 
\leq C |y_n| + C |z_n| \leq C |x - y|^{1+\alpha} \quad \text{(by (2.13)).}
\]

It follows that \( u \in C^{1+\alpha}_x \left( B^{+}_1 \cap \{ x_n \leq \frac{1}{8} \} \right) \).

Therefore, **Case 1** and **Case 2** finish the proof of Theorem 1.1. \( \square \)

3. Proof of Theorem 1.2

In this section we divide the proof of Theorem 1.2 into two steps as the following. In fact, Subsection 3.1 gives the convergence at infinity of the solutions in Theorem 1.2, and then Subsection 3.2 shows its asymptotic behavior at infinity with decay rate.

Recall that the symbols \( F_h, E_h \) and \( E^+_h \) are defined in Section 2.

3.1. The convergence at infinity. In the subsection we apply the Hölder estimates up to the flat boundary to show that the solution in Theorem 1.2 converges at infinity.

Hereinafter, we say some constant is universal if it depends only on \( \lambda, \Lambda, \alpha, \delta \) and \( n \). The universal constant may change from line to line if necessary.

A straightforward corollary of the boundary Hölder estimates is the following.

**Corollary 3.1.** Let \( u \in C(E^+_{4R} \setminus \overline{E^+_R}) \) be a solution of

\[
\begin{aligned}
Lu &= 0 \quad \text{in} \ E^+_{4R} \setminus \overline{E^+_R}, \\
u &\leq 1 \quad \text{on} \ \partial (E^+_{4R} \setminus \overline{E^+_R}) \cap \{ x_n > 0 \}, \\
u &\leq \frac{1}{2} \quad \text{on} \ \partial (E^+_{4R} \setminus \overline{E^+_R}) \cap \{ x_n = 0 \},
\end{aligned}
\]

where \( L \) is given by (1.1) with coefficients satisfying (1.3) and (1.4) in \( E^+_{4R} \setminus \overline{E^+_R} \) for some \( R > 0 \). Then there exists universal constant \( c_0 > 0 \) such that

\[
u(x) \leq 1 - c_0 \quad \text{on} \ \partial E^+_2 \cap \{ x_n \geq 0 \}.
\]
Proof. It only need to prove this corollary with \( u(x) = \frac{1}{2} \) on \( \partial (E_{4R}^+ \setminus \overline{E}_R^+) \cap \{ x_n = 0 \} \). Otherwise, one can consider a supersolution \( v \) with \( v(x) = \frac{1}{2} \) on \( \partial (E_{4R}^+ \setminus \overline{E}_R^+) \cap \{ x_n = 0 \} \), and if it holds for \( v \), by the comparison principle, so does for \( u \).

Let
\[
\hat{u}(x) = u(F_R x), \quad x \in E_4^+ \setminus \overline{E}_1^+.
\]
By the definitions of \( F_R \) and \( E_4^+ \) in Section 2, we have \( F_R (E_4^+ \setminus \overline{E}_1^+) = E_{4R}^+ \setminus \overline{E}_R^+ \). Then
\[
\begin{aligned}
\bar{L} \hat{u} &= 0 \quad \text{in } E_4^+ \setminus \overline{E}_1^+, \\
\hat{u} &\leq 1 \quad \text{on } \partial (E_4^+ \setminus \overline{E}_1^+) \cap \{ x_n > 0 \}, \\
\hat{u} &= \frac{1}{2} \quad \text{on } \partial (E_4^+ \setminus \overline{E}_1^+) \cap \{ x_n = 0 \},
\end{aligned}
\]
where \( \bar{L} \) is given by (2.6). Clearly, coefficients of \( \bar{L} \) also satisfy (1.3) and (1.4) in \( E_4^+ \setminus \overline{E}_1^+ \). Then by the third equality in (3.2) and Theorem 1.1, there exists a universal \( 0 < \tau \leq 1 \) such that
\[
\hat{u}(x) \leq \frac{2}{3} \quad \text{on } \partial E_2 \cap \{ 0 \leq x_n \leq \tau \}.
\]
By the comparison principle, we have \( \hat{u} \leq 1 \) in \( E_4^+ \setminus \overline{E}_1^+ \). \( 1 - \hat{u} \) satisfies
\[
\bar{L}(1 - \hat{u}) = 0 \quad \text{in } E_4^+ \setminus \overline{E}_1^+.
\]
Then by the interior Harnack inequality to \( 1 - \hat{u} \), there exists a universal \( C \geq 1 \) such that
\[
C \inf_{\partial E_2 \cap \{ x_n \geq \tau \}} (1 - \hat{u}) \geq \sup_{\partial E_2 \cap \{ x_n \geq \tau \}} (1 - \hat{u}) \geq \sup_{\partial E_2 \cap \{ x_n = \tau \}} (1 - \hat{u}) \geq \frac{1}{3}.
\]
This implies
\[
\hat{u}(x) \leq 1 - \frac{1}{3C} \quad \text{on } \partial E_2 \cap \{ x_n \geq \tau \}.
\]
This corollary follows from the definition of \( \hat{u} \), (3.3) and (3.4), if we take \( c_0 = \frac{1}{3C} \). \( \square \)

Applying Corollary 3.1, we have a convergence result as follows.

**Theorem 3.2.** Let \( u \in C(\mathbb{R}_+^n \setminus \overline{E}_1^+) \) be a solution of
\[
Lu = 0 \quad \text{in } \mathbb{R}_+^n \setminus \overline{E}_1^+,
\]
where \( L \) is given by (1.1) with the coefficients satisfying (1.3) and (1.4) in \( \mathbb{R}_+^n \setminus \overline{E}_1^+ \). If

- \( |u| \leq 1 \) on \( \partial E_1 \cap \{ x_n > 0 \} \cup \{ x_n = 0, |x| \geq 1 \} \),
- \( u(x', 0) \to \beta \) as \( |x'| \to \infty \),
- \( |Du(x)| \to 0 \) as \( |x| \to \infty \).

Then \( u(x) \to \beta \) as \( |x| \to \infty \).

**Proof.** The proof of this theorem is divided into two steps as follows.

**Step 1.** \( |u| \leq 1 \) in \( \mathbb{R}_+^n \setminus \overline{E}_1^+ \).

For any \( \varepsilon > 0 \), since \( |Du| \to 0 \) as \( |x| \to \infty \), there exists \( R_\varepsilon \geq 1 \) such that
\[
|Du| \leq \varepsilon \quad \text{in } \mathbb{R}_+^n \setminus Q_{R_\varepsilon}^+,
\]
where \( Q_{R_\varepsilon}^+ := \{ (x', x_n) : |x'| < R_\varepsilon, 0 < x_n < R_\varepsilon \} \) is a cylinder.
By $|u| \leq 1$ on $\{x_n = 0, |x| \geq 1\}$, (3.5) and Newton-Leibniz formula, we have

$$|u(x)| \leq 1 + 2\varepsilon x_n \quad \text{on } \partial Q^+_R \cap \{x_n > 0\}.$$ 

Since $|u| \leq 1$ on $(\partial E \cap \{x_n > 0\}) \cup \{x_n = 0, |x| \geq 1\}$, we get

$$|u(x)| \leq 1 + 2\varepsilon x_n \quad \text{on } \partial Q^+_R \setminus \partial E_1^+.$$ 

Obviously, $1 + 2\varepsilon x_n$ solves (1.1) in $Q^+_R \setminus \partial E_1^+$. Then by the comparison principle, we get

$$|u(x)| \leq 1 + 2\varepsilon x_n \quad \text{in } Q^+_R \setminus \partial E_1^+.$$ 

Letting $\varepsilon \to 0$, it completes the proof of this step.

**Step 2.** $u(x) \to \beta$ as $|x| \to \infty$.

Without loss of generality, we suppose that $\beta = 0$. Otherwise, we consider $\frac{u(x) - \beta}{1 + |x|}$.

Now we argue by contradiction. If this step is not true, by Step 1, $u$ has finite superior limit $\bar{u} > 0$ or inferior limit $\underline{u} < 0$ at infinity. It suffices to assume $\bar{u} > 0$.

By the definition of $\bar{u}$ and $u(x', 0) \to \beta$ as $|x'| \to \infty$, there exists large $R_1 \geq 1$ such that for all $R \geq R_1$,

$$u(x) \leq \left(1 + \frac{c_0}{2}\right) \bar{u} \quad \text{in } \mathbb{R}^n_+ \setminus \partial E_1^+$$

and

$$u(x', 0) \leq \frac{1}{2} \left(1 + \frac{c_0}{2}\right) \bar{u} \quad \text{if } |x'| \geq R,$$

where $c_0$ is given by Corollary 3.1. Then applying Corollary 3.1 to $\frac{u(x)}{(1 + |x|) \bar{u}}$ in $E_1^+ \setminus \partial E_2^+$, we get for all $R \geq R_1$,

$$u(x) \leq (1 - c_0) \left(1 + \frac{c_0}{2}\right) \bar{u} \leq \left(1 - \frac{c_0}{2}\right) \bar{u} \quad \text{on } \partial E_2^+ \cap \{x_n \geq 0\}.$$ 

This implies

$$u(x) \leq \left(1 - \frac{c_0}{2}\right) \bar{u} \quad \text{in } \mathbb{R}^n_+ \setminus \partial E_{2R_1},$$

which reaches a contradiction. □

Theorem 3.2 implies the following corollary immediately.

**Corollary 3.3.** Let $u$ be as in Theorem 1.2. Then

$$u(x) \to 0 \quad \text{as } |x| \to \infty.$$ 

**Proof.** It’s obvious and thus we omit it here. □

### 3.2. The asymptotic behavior at infinity

In this subsection we obtain the asymptotic behavior at infinity of solutions in Theorem 1.2, through constructing a barrier function.

To get the barrier function, we first give a solution of

$$\mathcal{L}u := x_n^{2\alpha} \sum_{i=1}^{n-1} D_i u(x) + D_{nn} u(x) = 0 \quad \text{in } \mathbb{R}^n_+.$$ 

See [6] for more details on the Baouendi-Grushin type operator. Let

$$w(x', x_n) = \frac{x_n}{(x'^2 + \beta x_n^{2+2\alpha})^\gamma},$$

where

$$\mathcal{L}w := x_n^{2\alpha} \sum_{i=1}^{n-1} D_i w(x') + D_{nn} w(x') = 0 \quad \text{in } \mathbb{R}^n_+.$$
where \( \beta = \frac{1}{(1+\alpha)^2} \), \( \gamma = \frac{n-1}{2} + \frac{1}{2(1+\alpha)} \).

Obviously,

\[
w(x) \sim \frac{x_n}{d(x)^q},
\]

where \( d(x) \) and \( Q \) are given by (1.6).

Simple calculations deduce that

\[
D_{iw} = -\frac{2\gamma x_ix_n}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}}, \quad i < n;
\]

\[
D_{njw} = -\frac{2\gamma x_n x_j}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} + \frac{4\gamma(\gamma+1)x_i x_j x_n}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+2}}, \quad i, j < n;
\]

\[
D_{inw} = -\frac{2\gamma x_i}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} + \frac{2\gamma \beta (2 + 2\alpha) x_i x_n^{2+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+2}}, \quad i < n;
\]

\[
\begin{align*}
D_{nnw} &= -\frac{\gamma \beta (2 + 2\alpha) x_n^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} - \frac{\gamma \beta (2 + 2\alpha)^2 x_n^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} \\
&\quad + \frac{\gamma(\gamma+1)\beta^2 (2 + 2\alpha)^2 x_n^{3+4\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+2}}.
\end{align*}
\]

Then

\[
\mathcal{L}w = -\frac{2\gamma(n-1)x_n^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} + \frac{4\gamma(\gamma+1)|x'|^2 x_n^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+2}} - \frac{\gamma \beta (2 + 2\alpha) x_n^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} \\
- \frac{\gamma \beta (2 + 2\alpha)^2 x_n^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} + \frac{\gamma(\gamma+1)\beta^2 (2 + 2\alpha)^2 x_n^{3+4\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+2}}
\]

\[
\begin{align*}
&\mathcal{L}w = \frac{-2\gamma(n-1) - \gamma \beta (2 + 2\alpha)(3 + 2\alpha) |x'|^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} + \frac{4\gamma(\gamma+1)|x'|^2 + \beta^2 (1 + \alpha)^2 x_n^{2+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+2}} \\
&\quad + \frac{\gamma(\gamma+1)|x'|^2 |x'|^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} + \frac{4\gamma(\gamma+1)\beta^2 (2 + 2\alpha)^2 x_n^{3+4\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+2}}
\end{align*}
\]

\[
\begin{align*}
\mathcal{L}w &= \frac{2\gamma\{-(n-1) - (1 + \alpha)^{-1}(3 + 2\alpha) + 2(\gamma + 1)\} x_n^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} \\
&= \frac{2\gamma\{n - 1 - (1 + \alpha)^{-1} + 2\gamma\} x_n^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}}
\end{align*}
\]

where \( \mathcal{L} \) is given by (3.6).

Using \( w \), we can construct a supersolution of (1.1) as follows.
Lemma 3.4. Let $L$ be given by (1.1) with coefficients satisfying (1.3), (1.4) and (1.9). Then for any $\rho \in \left(0, \min \left\{ \frac{s}{n-1}, 1 \right\} \right)$, there exists $R_0 \geq 1$ depending only on $\rho$, $s$, $\alpha$ and $n$ such that

\begin{equation}
L(w - w^{1+\rho}) \leq 0 \quad \text{in} \quad \mathbb{R}_+^{n} \setminus \overline{E}^+_{R_0},
\end{equation}

Proof. For $i, j < n,$

\begin{equation}
|D_{ij}(w^{1+\rho})| = |(1 + \rho)w^\rho D_{ij}w + \rho(1 + \rho)w^{\rho-1}D_i w D_j w|
\end{equation}

\begin{align*}
\leq & (1 + \rho)w^\rho \left\{ \frac{2\gamma x_n}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} + \frac{4\gamma(\gamma + 1)|x'|^2 x_n}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+2}} \right\} \\
& + \rho(1 + \rho)w^{\rho-1}\frac{4\gamma|x'|^2 x_n^2}{(|x'|^2 + \beta x_n^{2+2\alpha})^{2(\gamma+1)}} \\
\leq & \frac{C(\rho, \alpha, n)w^\rho x_n}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} + \frac{C(\rho, \alpha, n)w^{\rho-1}x_n^2}{(|x'|^2 + \beta x_n^{2+2\alpha})^{2\gamma+1}}
\end{align*}

and

\begin{equation}
|D_{in}(w^{1+\rho})| = |\rho(1 + \rho)w^{\rho-1}D_i w D_n w + (1 + \rho)w^\rho D_n w|
\end{equation}

\begin{align*}
\leq & \frac{2\gamma \rho(1 + \rho)w^{\rho-1}|x'| x_n}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} \left\{ \frac{1}{(|x'|^2 + \beta x_n^{2+2\alpha})^\gamma} + \frac{\gamma(2 + 2\alpha)x_n^{2+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} \right\} \\
& + (1 + \rho)w^\rho \left\{ \frac{2\gamma|x'|}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}} + \frac{2\gamma(2 + 2\alpha)|x'| x_n^{2+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+2}} \right\} \\
\leq & \frac{C(\rho, \alpha, n)w^{\rho-1}|x'| x_n}{(|x'|^2 + \beta x_n^{2+2\alpha})^{2\gamma+1}} + \frac{C(\rho, \alpha, n)w^\rho |x'|}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1}}
\end{align*}

where $C(\rho, \alpha, n)$ is positive, depending only on $\rho$, $\alpha$ and $n$, and may change from line to line.

Thus,

\begin{align*}
\mathcal{L}(w^{1+\rho}) = & \sum_{i=1}^{n-1} (1 + \rho)w^\rho D_i w + \rho(1 + \rho)w^{\rho-1}D_i w D_i w + (1 + \rho)w^\rho D_n w \\
& + \rho(1 + \rho)w^{\rho-1}(D_n w)^2 \\
= & \sum_{i=1}^{n-1} \rho(1 + \rho)w^{\rho-1} \sum_{i=1}^{n-1} (D_i w)^2 + \rho(1 + \rho)w^{\rho-1}(D_n w)^2
\end{align*}
\[ L (w^{1+\rho}) \geq \mathcal{L} (w^{1+\rho}) - \sum_{i,j=1}^{n-1} |a_{ij}(x) - \delta_{ij}| |D_{ij}(w^{1+\rho})| x_n^{2\alpha} - \sum_{i=1}^{n-1} |a_{in}(x)||D_{in}(w^{1+\rho})| \\
\geq \frac{\rho(1 + \rho)\upsilon^{\rho - 1}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{2\gamma}} - \frac{C(\rho, \alpha, n)\upsilon^{\rho - 1}x_n^{2+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{2\gamma+1}} \\
- \left(|x'| + x_n^{1+\alpha}\right)^{-s} C(\rho, \alpha, n)\upsilon^{\rho - 1}|x_n| \\
\geq \frac{\rho(1 + \rho)\upsilon^{\rho - 1}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{2\gamma}} - \frac{C(\rho, \alpha, n)\upsilon^{\rho - 1}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{2\gamma+\frac{s}{2}}} - \frac{C(\rho, \alpha, n)\upsilon^{\rho - 1}}{|x'|^2 + \beta x_n^{2+2\alpha} - \frac{s}{2} - x_n^{1+\alpha}} \\
\geq \frac{1}{2}\rho(1 + \rho)\upsilon^{\rho - 1} \left| x_n \right|^2 \left| x_n \right|^{2\gamma} \text{ in } \mathbb{R}_+^{n} \setminus E_{R_0}
\]

for some \( R_0 \geq 1 \) depending only on \( \rho, s, \alpha \) and \( n \). Similarly,

\[ L w \leq \mathcal{L} w + \sum_{i,j=1}^{n-1} |a_{ij}(x) - \delta_{ij}| |D_{ij} w| x_n^{2\alpha} + \sum_{i=1}^{n-1} |a_{in}(x)||D_{in} w| \]
Above two inequalities imply that for some \( C > \partial E \)

\[
\frac{C(\rho, \alpha, n)x^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1+\frac{\rho}{2}}} + \frac{C(\rho, \alpha, n)|x'|}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1+\frac{\rho}{2}}} \leq \frac{C(\rho, \alpha, n)x^{1+2\alpha}}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1+\frac{\rho}{2}}} + \frac{C(\rho, \alpha, n)|x'|}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1+\frac{\rho}{2}}} \leq \frac{C(\rho, \alpha, n)}{(|x'|^2 + \beta x_n^{2+2\alpha})^{\gamma+1+\frac{\rho}{2}}}.
\]

Since \( \rho \in \left(0, \min \left\{ \frac{s}{n-1}, 1 \right\} \right) \), we get

\[
(3.14) \quad \frac{w^\rho - 1}{(|x'|^2 + \beta x_n^{2+2\alpha})^{2\gamma}} = \frac{x^\rho - 1}{(|x'|^2 + \beta x_n^{2+2\alpha})^{2\gamma+\gamma(\rho-1)}} \geq \left( |x'|^2 + \beta x_n^{2+2\alpha} \right)^{-2\gamma-\gamma(\rho-1)-\frac{1-\rho}{2(1+\alpha)}}.
\]

(3.15) \quad \left( -2\gamma - \gamma(\rho-1) - \frac{1-\rho}{2(1+\alpha)} \right) + \left( \gamma + 1 + \frac{s}{2} - \frac{1+2\alpha}{2(1+\alpha)} \right) = -\frac{n-1}{2} + \rho + \frac{s}{2} > 0.

By (3.12), (3.13), (3.14) and (3.15), we have

\[
L(w - w^{1+\rho}) \leq 0 \quad \text{in } \mathbb{R}^n_+ \setminus \overline{E}^+_R
\]

for larger \( R_0 \geq 1 \) depending only on \( \rho, s, \alpha \) and \( n \).

\[ \square \]

**Proof of Theorem 1.2.**

By Lemma 3.4, for any fixed \( \rho \in \left(0, \min \left\{ \frac{s}{n-1}, 1 \right\} \right) \), there exists \( R > 1 \) depending only on \( s, \alpha \) and \( n \) such that

\[
L(w - w^{1+\rho}) \leq 0 \quad \text{in } \mathbb{R}^n_+ \setminus \overline{E}^+_R.
\]

By \( u(x) = 0 \) on \( \{x_n = 0\} \), \( |Du(x)| \leq 1 \) in \( \mathbb{R}^n_+ \setminus \overline{E}^+_1 \) and Newton-Leibniz formula, we get

\[
|u(x)| \leq 2x_n \quad \text{on } \partial E_R \cap \{x_n \geq 0\}.
\]

On \( \partial E_R \cap \{x_n \geq 0\} \), it is clear that

\[
w - w^{1+\rho} = w(1 - w^\rho) \geq c(R, \alpha, n)x_n.
\]

Above two inequalities imply that for some \( C > 0 \) depending only on \( s, \delta, \alpha \) and \( n \),

\[
|u(x)| \leq C(w - w^{1+\rho}), \quad \text{on } \partial E_R \cap \{x_n \geq 0\}.
\]

For any \( \varepsilon > 0 \), by Corollary 3.3, there exists \( R_\varepsilon > R \) such that

\[
|u(x)| \leq \varepsilon, \quad x \in \partial E_{R_\varepsilon} \cap \{x_n \geq 0\}.
\]

It follows from (3.16), (3.17) and \( u(x) = 0 \) on \( (E_{R_\varepsilon} \setminus \overline{E}_R) \cap \{x_n = 0\} \) that

\[
|u(x)| \leq C(w - w^{1+\rho}) + \varepsilon \quad \text{on } \partial (E^+_{R_\varepsilon} \setminus \overline{E}^+_R).
\]
By the comparison principle,
\[ |u(x)| \leq C(w - w^{1+\rho}) + \varepsilon \quad \text{in } E_R^+ \setminus E_R. \]
Letting \( \varepsilon \to 0 \), it follows (1.10).

\[ \square \]

### 4. Appendix

In this section, Lemma 2.1 will be proved as follows.

**Proof.** Denote
\[ A'(x) = \begin{pmatrix} a_{11}(x) & \cdots & a_{1,n-1}(x) \\ \vdots & \ddots & \vdots \\ a_{n-1,1}(x) & \cdots & a_{n-1,n-1}(x) \end{pmatrix} \]
and
\[ \tilde{A}(x) = \begin{pmatrix} A'(x)x_1^{2\alpha} & a_{1,n}(x)x_n^\alpha \\ \vdots & \ddots \\ a_{n,1}(x)x_n^\alpha & \cdots & a_{n,n-1}(x)x_n^\alpha \end{pmatrix}, \]
where \( a_{ij}(x) \) and \( a_{in}(x) \) are given by (1.1).

It suffices to show that eigenvalues of \( \tilde{A}(x) \) are positive in \( \overline{B}_1^+ \) and have uniformly bound (depending on the fixed number \( \varepsilon_0 \)) in \( \overline{B}_1^+ \cap \{ x_n \geq \varepsilon_0 \} \).

If \( A'(x) \) have eigenvalues \( \lambda_1(x), \cdots, \lambda_{n-1}(x) \), by (1.3), we get \( \lambda \leq \lambda_i(x) \leq \Lambda, i = 1, 2, \cdots, n-1 \). It’s clear that there exists a orthogonal matrix \( P'_{(n-1)\times(n-1)} \) such that
\[ (P')^T A' P' = \text{diag}\{ \lambda_1(x), \cdots, \lambda_{n-1}(x) \}. \]

Observe that eigenvalues of \( \tilde{A}(x) \) are that of the following matrix
\[ B(x) := P^T A P = \begin{pmatrix} \lambda_1(x)x_n^{2\alpha} & \tilde{a}_{1,n}(x)x_n^\alpha \\ \vdots & \ddots \\ \tilde{a}_{n,1}(x)x_n^\alpha & \cdots & \tilde{a}_{n,n-1}(x)x_n^\alpha \end{pmatrix}, \]
with
\[ \tilde{a}_{i,n}(x) = \sum_{j=1}^{n-1} P'_{ij} a_{j,n}(x), \quad i = 1, \cdots, n-1; \quad P = \begin{pmatrix} P' & 0 \\ 0 & 1 \end{pmatrix}. \]

Thus, we only need to show that all eigenvalues of \( B(x) \) are positive in \( \overline{B}_1^+ \) and have uniformly bound in \( \overline{B}_1^+ \cap \{ x_n \geq \varepsilon_0 \} \). Let
\[ e_i = (0, \cdots, 0, 1, 0, \cdots, 0), \quad i \leq n, \]
we have
\[ e_i^T B(x) e_i = \lambda_i x_n^{2\alpha}, \quad e_i^T B(x) e_n = a_{in} x_n^\alpha, \quad i \leq n - 1; \quad e_n^T B(x) e_n = 1. \]
For any $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, there exists a unique consequence $\{b_i\}_{i=1}^n$ such that

$$\xi = \sum_{i=1}^n b_i e_i, \quad \sum_{i=1}^n b_i^2 = 1.$$ 

Then

$$\xi^T B(x) \xi = \sum_{i=1}^n (b_i e_i)^T B(x) (b_i e_i) = \sum_{i=1}^n b_i^2 e_i^T B(x) e_i$$

$$= \sum_{i=1}^{n-1} \lambda_i b_i^2 x_n^2 + \sum_{i=1}^{n-1} 2b_i b_n \bar{a}_{i,n} x_n^\alpha + b_n^2$$

$$\geq \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + \sum_{i=1}^{n-1} 2b_i b_n \bar{a}_{i,n} x_n^\alpha + b_n^2 \quad \text{(by (1.3)).}$$

Applying Cauchy’s inequality to $2b_i b_n \bar{a}_{i,n} x_n^\alpha$, we have that for any $\tau \in (0, 1)$,

$$\left| \sum_{i=1}^{n-1} 2b_i b_n \bar{a}_{i,n} x_n^\alpha \right| \leq \tau \sum_{i=1}^{n-1} \left( \lambda \frac{1}{\tau} b_i x_n^\alpha \right)^2 + \tau^{-1} \sum_{i=1}^{n-1} \left( \lambda^{-\frac{1}{\tau}} b_i \bar{a}_{i,n} \right)^2$$

$$= \tau \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + \tau^{-1} b_n^2 \lambda^{-1} \sum_{i=1}^{n-1} \bar{a}_{i,n}^2.$$ 

Therefore, for any $\tau \in (1 - \delta, 1)$,

$$\xi^T B(x) \xi \geq \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + b_n^2 - \tau \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 - \tau^{-1} b_n^2 \lambda^{-1} \sum_{i=1}^{n-1} \bar{a}_{i,n}^2$$

$$\geq (1 - \tau) \lambda x_n^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + b_n^2 \{1 - \tau^{-1}(1 - \delta)\} \quad \text{(by (1.4)),}$$

which implies that $L$ is elliptic in $\overline{B}_1^+$. And then if $\{x_n \geq \varepsilon_0\}$,

$$\xi^T A(x) \xi \geq (1 - \tau) \lambda \varepsilon_0^{2\alpha} \sum_{i=1}^{n-1} b_i^2 + b_n^2 \{1 - \tau^{-1}(1 - \delta)\} \geq \min \{(1 - \tau) \lambda \varepsilon_0^{2\alpha}, 1 - \tau^{-1}(1 - \delta)\}.$$ 

In particular, taking $\tau = 1 - \frac{1}{2} \delta$, we have that for any $x \in \overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\}$,

$$\xi^T A(x) \xi \geq \min \left\{ \frac{1}{2} \delta \varepsilon_0^{2\alpha}, 1 - \left(1 - \frac{1}{2} \delta \right)^{-1}(1 - \delta) \right\} > 0.$$ 

Therefore, eigenvalues of $B(x)$ have uniformly below bound in $\overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\}$.

Similarly, one can also obtain the uniformly upper bound of eigenvalues of $B(x)$ in $\overline{B}_1^+ \cap \{x_n \geq \varepsilon_0\}$. \(\square\)
References

[1] M.S. Baouendi, *Sur une classe d’opérateurs elliptiques dégénérant au bord*, C. R. Acad. Sci. Paris Sér. A-B 262 (1966), A337–A340. MR 0194744

[2] W. Bauer, K. Furutani, and C. Iwasaki, *Fundamental solution of a higher step Grushin type operator*, Adv. Math. 271 (2015), 188–234. MR 3291861

[3] L.A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995. MR 1351007

[4] M.G. Crandall, H. Ishii, and P.L. Lions, *User’s guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) 27 (1992), no. 1, 1–67. MR 1118699

[5] B. Franchi and E. Lanconelli, *Hölder regularity theorem for a class of linear nonuniformly elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 10 (1983), no. 4, 523–541. MR 753153

[6] N. Garofalo, *Unique continuation for a class of elliptic operators which degenerate on a manifold of arbitrary codimension*, J. Differential Equations 104 (1993), no. 1, 117–146.

[7] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364

[8] V.V. Grushin, *A certain class of hypoelliptic operators*, Mat. Sb. (N.S.) 83 (125) (1970), 456–473. MR 0279436

[9] Q. Han and F.H. Lin, *Elliptic partial differential equations*, second ed., Courant Lecture Notes in Mathematics, vol. 1, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2011. MR 2777537

[10] X.B. Jia, D.S. Li, and Z.S. Li, *Asymptotic behavior at infinity of solutions of Monge-Ampère equations in half spaces*, J. Differential Equations 269 (2020), no. 1, 326–348. MR 4081524

[11] I. Kombe, *Nonlinear degenerate parabolic equations for Baouendi-Grushin operators*, Math. Nachr. 279 (2006), no. 7, 756–773. MR 2246410

[12] N.Q. Le and O. Savin, *Schauder estimates for degenerate Monge-Ampère equations and smoothness of the eigenfunctions*, Invent. Math. 207 (2017), no. 1, 389–423. MR 3592760

[13] R. Monti and D. Morbidelli, *Kelvin transform for Grushin operators and critical semilinear equations*, Duke Math. J. 131 (2006), no. 1, 167–202. MR 2219239

[14] M. Ramaswamy and S. Ramaswamy, *Maximum principles for viscosity subsolutions of some second order linear operators and some consequences*, Nonlinear Anal. 26 (1996), no. 3, 415–428. MR 1359223

[15] D.W. Robinson and A. Sikora, *Analysis of degenerate elliptic operators of Grushin type*, Math. Z. 260 (2008), no. 3, 475–508. MR 2434466

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