On $\omega C$-Continuous Functions

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Abstract. We devoted this research to study the $c$-continuous functions and we have provided other kinds of that functions. The relations between those functions have been studied, many examples and facts have been submitted to support the results that we found in our work.

Key words: continuous, $c$-continuous, $\omega c$-continuous, $\omega^*c$-continuous, $\omega^{**}c$-continuous.

1- Introduction

In (1970) the researchers (Karl R. Gentry and Hughes B. Hoyle) presented the concept of $c$-continuous functions and defined it as (A function $f$ from a space $X$ into a space $Y$ is called $c$-continuous function at a point $s$ in $X$ if for any open subset $M$ in $Y$ with $f(s)$ inside it and $M^c$ is compact subset of $Y$, we can locate a subset $K$ whose an open set in $X$, $s$ is admist it and its image inner $M$. When $f$ is $c$-continuous during each point in $X$, then it is $c$-continuous on $X$ [1]). They reached the conclusion that (A function $f$ from a space $X$ into another $Y$ is said to be $c$-continuous if and only if the reverse image of each open set $K$ in $Y$ with compact complement is an open set in $X$). In (1973) the researchers (Paul E. Long and Michael D. Hendrix) [2] expanded of the study of $c$-continuous functions, and they introduced many of results about this subject such as (Let $f$ be a function from a space $X$ into a space $Y$, $f$ is $c$-continuous if and only if the inverse image of each closed and compact set $M$ in $Y$ is closed set in $X$). Also, in (2010) the researchers (Hadi Jaber Mustafa, Sajda Kwadhem Muhammad and Neeran Tahir Al-Khafa) [3] presented the notion of $c\alpha$-continuous functions (=A function $f$ from the space $X$ into the space $Y$ is $c\alpha$-continuous if and only if for any point $s$ in $X$ and any open set $M$ in $Y$ having the image of $s$ and its complement is $c\alpha$-compact, we can determine an open set $K$ with $s$ inside it and its image in $M$). In this our work we used the $\omega$-open sets which introduced by Hdeib [4] (= If for each element $s$ belong to a subset $U$ of a space $X$, we can find an open set $K$ in $X$ including $s$ and $K - U$ is countable, then $U$ is $\omega$-open set) in the definition of $(\omega, \omega^*, \omega^{**})$ $c$-continuous functions, where we exhibited the relation between these functions. By $X$ we mean a topological space, A function $f$ is not necessarily continuous unless we state.

1- On $\omega C$-Continuous functions

We noted in the introduction about the definition of $c$-continuous functions, as an example on it, we introduce the following.

(3-1) Example

\[ I_{\mathcal{R}_{\omega}}(\mathcal{R}_{\omega}, \mathcal{I}_{\omega}) \rightarrow (\mathcal{R}_{\omega}, \mathcal{I}_{\omega}) \] is $c$-continuous function.

The next remark appears in [1] without proof, we will submit its proof for completeness.
(3-2) Remark

As long as a function \( f \) is continuous then it is c-continuous, but the converse is not true. Since if \( f \) is continuous function, so for any point \( \xi \) in the domain and every open set \( M \) in the co-domain containing the image of \( \xi \) and has compact complement, since \( f \) is continuous then there is an open set \( K \) in the domain with \( \xi \) belong to it and its image existing in \( M \), so \( f \) is c-continuous.

(3-3) Example

Envisage the function \( f \) with domain consist of the set of all real numbers with any topology defined on it and co-domain consist of the same set with the indiscrete topology, this function is c-continuous. Also, it is continuous function.

The converse of the above remark is incorrect as the following example shows.

(3-4) Example

Give \( f \) to be a function from \((\mathcal{R}, J_{\omega})\) into the same space defined as

\[
f(x) = \begin{cases} 
\frac{1}{x-1} & \text{if } x \neq 1 \\
\frac{1}{4} & \text{if } x = 1
\end{cases}
\]

Then \( f \) is not continuous at \( x=1 \), but continuous at any other point in the domain, since if we take \( M=(0, 1) \) as an open set in the co-domain containing \( f(1) = \frac{1}{3} \), while there is no proper set in \((\mathcal{R}, J_{\omega})\) involving 1 and its image situated in \( M \). It is continuous at any point differ from1. Now to show that \( f \) is c-continuous at 1, since for any open set \( M \) in the co-domain including \( f(1) = \frac{1}{3} \) with \( M^C \) is compact, there is an open set \( U=(0.9, 1.1) \) encompass1 and \( f(U) \subseteq M \). The example is valid on all points of domain.

Now we will offer a weak form of c-continuous function by using \( \omega \)-open sets.

(3-5) Definition

A function \( f \) from a space \( X \) into another \( Y \) is \( \omega \)-c-continuous function if for any element \( \xi \) in \( X \) and any set \( M \) in \( Y \) whom open, includes \( f(\xi) \) and its complement is compact, there is an \( \omega \)-open set in \( X \) say \( K \) with \( \xi \) lies in it and its image inner \( M \).

(3-6) Example

The identity function from a space consist of any finite set with the excluded point topology \( J_{E} \) into the same space, where \( J_{E} = \{ U \subseteq X \text{ with } x_{0} \notin U \text{ for some } x_{0} \in X \} \cup \{ \emptyset \} \), is \( \omega \)-c-continuous function.

We got the next remark from [5] without proof, we will proof it for wholeness.

(3-7) Remark

Whenever a subset \( K \) of a space \( X \) is open then it is \( \omega \)-open. But the contrary is not true, since we can determine an open set \( G=K \) in \( X \) for each an item \( \xi \) in \( K \) such that \( G \) containing \( \xi \) and \( G \) without \( K \) is empty set, so \( K \) is \( \omega \)-open set.

As an example to the adverse we proffer the following.
(3-8) Example

Any set in the space \((Z, J^E_1)\) is \(\omega\)-open set but not open.

(3-9) Definition [6]

A point \(s\) in a subset \(H\) of a space \(X\) is \(\omega\)-interior point for \(H\) if and only if we can locate an \(\omega\)-open set \(K\) in \(X\) containing \(s\) and contained in \(H\).

We found the following lemma in [7] without proof, we proof it for perfection.

(3-10) Lemma

Every interior point is \(\omega\)-interior point. But the converse is not true in general, since if \(G\) is a subset of \(X\) and \(s\) is an interior point for \(G\), then there is an open set \(K\) in \(X\) involving \(s\) and lies in \(G\), but \(K\) is \(\omega\)-open (by remark (3-7)), so \(s\) is \(\omega\)-interior point.

The next example shows the contrast for this lemma.

(3-11) Example

Any point in \((X, J^E_1)\) where \(X\) is finite, is \(\omega\)-interior point but not interior point.

In [5] we found the next lemma without proof, we extract its prove from theorem in [18].

(3-12) Lemma

The infinite union of \(\omega\)-open sets is \(\omega\)-open.

Proof

Let \(\{K_l \mid l \in I\}\) be a collection of \(\omega\)-open subsets of a space \(X\) and let \(s\) be any point in the infinite union of the sets \(K_l\), \(l \in I\); then \(s\) belongs to \(K_j\) for some \(j\) in \(I\). This implies that there is an open set \(G_j\) in \(X\) with \(s\) in it, such that \(G_j\) without \(K_j\) is countable. Since \(G_j = \bigcup_{l \in I} K_l \subseteq G_j - K_j\), then \(G_j - \bigcup_{l \in I} K_l\) is countable (every subset of countable set is countable), thus \(\bigcup_{l \in I} K_l\) is \(\omega\)-open.

In [6] we found the next proposition without proof, we will introduce its proof for completeness.

(3-13) Proposition

Take \(G\) as a subset of a space \(X\), then it is called \(\omega\)-open set if and only if any point in \(G\) is an \(\omega\)-interior point.

Proof

Suppose \(G\) is \(\omega\)-open set, in view of every set includes all its points and situated in its self, so any point in \(G\) is \(\omega\)-interior point. Conversely, since \(G = \bigcup_{s \in G} \{s\}\), for any \(s\) in \(G\) prescribe an \(\omega\)-open set \(K\) in \(X\) with \(s\) belong \(K\) to \(k\) which is a subset of \(G\) (owing to any point in \(G\) is \(\omega\)-interior point), then \(G = \bigcup_{s \in G} \{K_s, k\}, s \in G\). So \(G\) is \(\omega\)-open set (by lemma (3-12)).

(3-14) Example

Any set in the indiscrete space \((Z, J^E_1)\) is \(\omega\)-open set, where every point in it is an \(\omega\)-interior point to it.
(3-15) Remark

Whenever a function $f$ is c-continuous then it is $\omega$c-continuous, since when $f$ is c-continuous function from a space $X$ into a space $Y$, hence for any point $s$ in $X$ and for any open set $M$ in $Y$ possess the image of $s$ and $M^c$ is compact set, we can specify an open set $K$ in $X$ with $s$ in it and its image existing in $M$. $K$ is $\omega$-open set (by remark (3-7)) containing $s$ and its image inner $M$, so $f$ is $\omega$ - continuous.

The above remark is un reversible as the following example shows.

(3-16) Example

Ponder the identity function $I_Z$ from the indiscrete space $(Z, J^i_Z)$ into the space $(Z, J^i_Z)$ differs from the domain space, is $\omega$c-continuous function but neither continuous nor c-continuous.

(3-17) Definition

In definition (3-5) if $M$ is $\omega$-open subset in $Y$ and $K$ is open set in $X$, then $f$ is $\omega^c$c-continuous function which is a stronger form of c-continuous function.

(3-18) Example

The function $f: (\mathcal{R}, J^c_U)\rightarrow(\mathcal{R}, J^c)$ is $\omega^c$-c-continuous, continuous and c-continuous function.

(3-19) Definition

In definition (3-5) if each of $M$ and $K$ are $\omega$-open subsets of $Y$ and $X$ respectively, $f$ is $\omega^{**}$c-continuous function. There is no relation between this kind and c-continuous function.

(3-20) Example

The function $f$ from the space $(X, J^c_X)$ where $X$ is finite set, into any space $(Y, J)$ is $\omega^{**}$c-continuous.

(3-21) Theorem

Imagin the function $f$ from a space $X$ into a space $Y$, the following phrases are valent.

1- $f$ is $\omega$c-continuous function.
2- The inverse image of each open set in $Y$ with compact complement is an $\omega$-open set in $X$.

Proof

Suppose $f$ is $\omega$c-continuous function and $M$ is an open set in $Y$ with $M^c$ is compact. Put the point $s$ belongs to the inverse image of $M$, then the image of $s$ is in $M$ whose open set in $Y$ and $M^c$ is compact. So we can choose an $\omega$-open set $K$ in $X$ containing $s$ and its image inner $M$ (because $f$ is $\omega$c-continuous), and by taking the inverse image we get $K \subseteq f^{-1}(f(K)) \subseteq f^{-1}(M)$, therefore $s$ is an $\omega$- interior point to $f^{-1}(M)$ which lead us to $f^{-1}(M)$ is $\omega$-open set in $X$ (by proposition (3-13)). Conversely, imagine $M$ is an open set in $Y$ includes the image of a point $s$ in $X$ and $M^c$ is compact. That means $f(s) \in M$, by taking the inverse image for both sides we get $s \in f^{-1}(M)$ which is $\omega$-open set in $X$ (by hypothesis), and its image is a subset of $M$, set $K=f^{-1}(M)$ we reach the result.
(3.22) Example

\( l_\mathcal{R} : (\mathcal{R}, \mathcal{J}_c) \rightarrow (\mathcal{R}, \mathcal{J}_c) \) is \( \omega_c \)-continuous function.

(3.23) Proposition

Give \( X, Y \) are any spaces and \( f \) is a function from \( X \) into \( Y \), the following expressions are equipollent

1. \( f \) is \( \omega^* \)-continuous function.
2. The inverse image of every \( \omega \)-open set \( M \) in \( Y \) with \( M^c \) is compact, is open set in \( X \).

Proof

Propose \( f \) as \( \omega^* \)-continuous function and select \( M \) as an \( \omega \)-open set in \( Y \) with \( M^c \) is compact and let \( s \) be any point belong to the inverse image of \( M \), then the image of \( s \) is included in \( M \) whose \( \omega \)-open set in \( Y \) and its complement is compact, hence we can determine an open set \( K \) in \( X \) containing \( s \) and its image inner \( M \) (since \( f \) is \( \omega^* \)-continuous), by taking the inverse image for \( f(K) \subseteq M \) we obtain \( K \subseteq f^{-1}(f(K)) \subseteq f^{-1}(M) \), so \( s \) is an interior point to \( f^{-1}(M) \) and then the last one is open set in \( X \). Conversely, choose \( M \) to be an \( \omega^* \)-open set in \( Y \) containing the image of a point \( s \) in \( X \) and its complement is compact, that means \( f(s) \in M \) and by applying the inverse image we have, \( s \) belongs to the inverse image of \( M \) which is open set in \( X \) (by hypothesis), but \( f(f^{-1}(M)) \subseteq M \), place \( f^{-1}(M)=K \), then the proof is complete.

(3.24) Example

The function \( f:(\mathcal{R}, \mathcal{J}_c) \rightarrow (\mathcal{R}, \mathcal{J}) \) is \( \omega^* \)-continuous function.

(3.25) Proposition

Take \( f:X \rightarrow Y \), where \( X, Y \) are any spaces, then the next statements are synonymous

1. \( f \) is \( \omega^{**} \)-continuous function.
2. the converse image of any \( \omega \)-open set \( M \) in \( Y \) with \( M^c \) is compact, is an \( \omega \)-open subset of \( X \).

Proof

Ponder the function \( f \) as \( \omega^{**} \)-continuous, set \( M \) as \( \omega \)-open set in \( Y \) with compact complement and \( s \) as any member in the converse image of \( M \), so \( f(s) \) belongs to \( M \) who is \( \omega \)-open set in \( Y \) and its complement is compact, hence we can specify an \( \omega \)-open set \( K \) in \( X \) including \( s \) and its image inside \( M \) (owing to \( f \) is \( \omega^{**} \)-continuous), now if we take the converse image for \( f(K) \subseteq M \) the result is \( K \subseteq f^{-1}(f(K)) \subseteq f^{-1}(M) \), so \( s \) is an \( \omega \)-interior point to \( f^{-1}(M) \), which implies \( f^{-1}(M) \) is \( \omega \)-open set in \( X \). Conversely, take \( M \) as an \( \omega \)-open set in \( Y \) with compact complement and has \( f(s) \) in it, where \( s \) is any member in \( X \), apply the converse image for \( f(s) \subseteq M \) we have \( s = f^{-1}(f(s)) \in f^{-1}(M) \) whom \( \omega \)-open set in \( X \) (by hypothesis), the image of \( f^{-1}(M) \) is a subset of \( M \), put \( K=f^{-1}(M) \). Then we get the conclusion.

(3.26) Example

Suppose \( l_\mathcal{R} \) is a function from the space \( (\mathcal{R}, \mathcal{J}_c) \) into the same space, it is \( \omega^{**} \)-continuous function.
(3-27) Definition [8], [9]

Conceive \( \chi \) as a point in a space \( X \) and \( M \) be a subset of \( X \), whenever each open set \( G \) in \( X \) containing \( \chi \) has uncountable intersection with \( M \), then we say that \( \chi \) is a condensation point of \( M \).

(3-28) Example

If we Ponder the set of all real numbers with indiscrete topology and take the set of irrational numbers as a subset of \( \mathcal{R} \), each point in \( \mathcal{R} \) is a condensation point for \( \mathcal{O}_c \).

(3-29) Definition [10]

In case a subset \( M \) of a space \( X \) contains all its condensation points, so it is \( \omega \)-closed set.

(3-30) Example

In the previous example \( \mathcal{O}_c \) is not \( \omega \)-closed set in \( \mathcal{R} \), while any set in \( (\mathcal{Z}, \mathcal{J}_c) \) is \( \omega \)-closed set.

We got the next remark in [11] without proof, we will introduce its proof here.

(3-31) Remark

Every closed set \( M \) in a space \( X \) is \( \omega \)-closed, but the converse is not true since if we take \( \chi \) does not belong to \( M \) so its situated in \( M^c \) which is open set in \( X \), and since the intersection between \( M \) and \( M^c \) is countable so \( \chi \) is not condensation point to \( M \), and then \( M \) contains all its condensation points, therefore \( M \) is \( \omega \)-closed set. For example about the reverse, the set of all natural numbers \( \mathbb{N} \) is an \( \omega \)-closed subset of the space \( (\mathcal{Z}, \mathcal{J}_c) \), but not closed.

(3-32) Theorem

The function \( f \) from a space \( X \) into a space \( Y \) is \( \omega \) (\( \omega^* \), \( \omega^{**} \))c-continuous function if and only if for any closed (\( \omega \)-closed) and compact set \( M \) in \( Y \), \( f^{-1}(M) \) is \( \omega \)-closed (closed, \( \omega \)-closed) set in \( X \).

Proof

Suppose \( f \) is \( \omega c \)-continuous, also suppose \( M \) is closed and compact set in \( Y \). So \( M^c \) is open set in \( Y \) and its complement is compact, the inverse image of \( M^c \) is \( \omega \)-open set in \( X \) (since \( f \) is \( \omega c \)-continuous), but it is equal to the complement of the inverse image to \( M \), then \( (f^{-1}(M))^c = f^{-1}(M) \) is \( \omega \)-closed set in \( X \). Conversely, let \( K \) be any open subset of \( Y \) and its complement is compact set, hence the inverse image of \( K^c \) which is equal to the complement of the inverse image to \( K \) is \( \omega \)-closed set in \( X \) (by hypothesis), then \( (f^{-1}(K))^c = f^{-1}(K) \) is \( \omega \)-open set in \( X \), therefore \( f \) is \( \omega c \)-continuous. By the same way we can proof the rest of possibilities. The next example satisfying the first and the third parts of the previous theorem.

(3-33) Example

A function \( f \) from the included point topology \( J_i \) with any non-empty countable set \( X \) into any space \( (Y, J) \), where \( J_i = \{U \subseteq X, \chi_n \in U \text{ for some } \chi_n \in X \} \cup \{\emptyset\} \), is \( \omega (\omega^* \omega^{**})c \)-continuous function.

(3-34) Example

A function \( f \) from the space \( X(\neq \emptyset) \) with any topology defined on it into the space \( Y(\neq \emptyset) \) with any topology defined on it, such that \( f(\chi) = c \) for any \( \chi \) in \( X \), it is \( \omega^* c \)-continuous function.
The following diagram is useful.

(3-35) Definition [12]

Let $X$ be a space and $a, b$ are two non-equal points in $X$, if we can find two $\omega$-open sets $U$ and $V$ such that $U$ including $a$ but not $b$ and $V$ including $b$ but not $a$, then $X$ is $\omega I_1$-space.

(3-36) Example

The discrete space $(\mathcal{R}, J_D)$ is $\omega I_1$-space.

(3-37) Remark [13]

Any $I_1$-space is $\omega I_1$-space, since if $X$ is $I_1$-space, then for any two non-equal points $a, b$ in $X$, there are two open sets $G$ and $W$, where $G$ comprising $a$ but not $b$ and $W$ comprising $b$ but not $a$, but $G$ and $W$ are also $\omega$-open sets (by remark (3-7)), so $X$ is $\omega I_1$-space. This remark is not reversible, for example, the excluded point space $(X, J_E)$ with $X$ is finite, is $\omega I_1$-space but not $I_1$-space.

(3-38) Proposition

If for each member $x$ in a space $X$, $\{x\}$ is $\omega$-closed subset of $X$, then the space $X$ is $\omega I_1$-space.

Proof

Let $x, y$ any points in $X$ with $x \neq y$, since $\{x\}$ is $\omega$-closed subset of $X$ (by hypothesis), so $X-\{x\}$ is $\omega$-open subset of $X$ containing $y$ but not $x$, also $X-\{y\}$ is $\omega$-open set in $X$ containing $x$ but not $y$, so $X$ is $\omega I_1$-space.

(3-39) Example

Every singleton in the cofinite space $(\mathcal{R}, J_C)$ which is $\omega I_1$-space, is $\omega$-closed set.
(3-40) Remark

The contrary of proposition (3-38) is not true in general. As example, the included point space \((X, J_X)\), where \(X\) is uncountable non-empty set, it is \(\omega I_1\)-space but not every singleton in it is \(\omega\)-closed set.

(3-41) Theorem

Suppose \(f: X \rightarrow Y\) is one-one and \(\omega (\omega^\ast, \omega^{++})c\)-continuous function where \(X\) and \(Y\) are two spaces, if \(Y\) is \(I_1(\omega I_1, \omega I_1)\)-space, then \(X\) is \(\omega I_1(I_1, \omega I_1)\)-space.

Proof

According to \(Y\) is \(I_1\)-space, then \({f(q)}\) is closed set in \(Y\) for any \(q\) belong to \(X\), and it is compact (since it is finite) with open complement in \(Y\), so \(Y-{f(q)}\) is open in \(Y\) and its complement is closed and compact, so its reversal image is equal to the complement of \({q}\) in \(X\) and it is \(\omega\)-open (since \(f\) is \(\omega c\)-continuous function), hence \({q}\) is \(\omega\)-closed in \(X\), therefore \(X\) is \(\omega I_1\)-space (by proposition (3-38)). The other two parts of the theorem can be prove by the same way.

(3-42) Example

1- Meditate \(f:(X, J_E) \rightarrow (Y, J)\) where \(X\) is countable non-empty set, it is \(\omega c\)-continuous function and attains the first part of the theorem.
2- The function \(f\) from the discrete space \((X, J_E)\) into any space \((X, J)\) is \(\omega^* c\)-continuous and Achieves the second part of the above theorem.
3- The function \(f\) from the space \((X, J_X)\) into any space \((Y, J_Y)\), where \(X = \{1, 2, 3\}\) and \(J_X = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}\) satisfies the third part of the theorem.

(3-43) Definition

Whenever for each pair of disjoint closed sets \(A\) and \(B\) in a space \(X\) we can pinpoint two disjoint \(\omega\)-open sets \(U\) and \(V\) in \(X\) with \(A \subseteq U\) and \(B \subseteq V\), then \(X\) is an \(\omega\)-normal space.

(3-44) Definition

In the previous definition if \(A, B\) are \(\omega\)-closed sets and \(U, V\) are open sets, then \(X\) is \(\omega^*\)-normal space.

The following definition appears in [14] as \(\omega\)-normal

(3-45) Definition

In the previous definition if \(U, V\) are \(\omega\)-open sets, then \(X\) is \(\omega^{++}\)-normal space.

The following diagram is useful.
When the image of each open set under a function $f$ defined from a space $X$ into a space $Y$ is $\omega$-open set, then $f$ is $\omega$-open function. For example, a function from any space into the excluded point topology on countable set.

(3-47) Definition

As long as the image of each $\omega$-open set under a function $f$ defined from a space $X$ into a space $Y$, is open ($\omega$-open) set then $f$ is $\omega^*(\omega^*)$-open function. As an example about the first type is the function from any space into the discrete space on any set, and for the second type, a function from any space into the indiscrete space on countable set.

Now we introduce the following scheme.
(3-48) **Definition** [16]

A space $X$ is $\omega I_{\mathcal{L}}$-space if and only if for any two non-equal points $a, b$ in $X$, we can specify disjoint $\omega$-open sets $U, V$ in $X$ with $U$ contains $a$ and $V$ contains $b$. For example, the included point topological space $(X, J_1)$, where $X$ is finite set.

(3-49) **Theorem**

Suppose a function $f$ from a space $X$ into a space $Y$ is $\omega c$-continuous, open ($\omega$-open, $\omega^*$-open, $\omega^{**}$-open) and surjective function, whenever $X$ is $\omega^*$-normal space and $Y$ is $T_4$-space, then $Y$ is also $I_{\mathcal{L}}(\omega I_{\mathcal{L}}, I_{\mathcal{L}}, \omega I_{\mathcal{L}})$-space.

**Proof**

Take $y_1, y_2$ as two distinct points in $Y$, so $\{y_1\}$ and $\{y_2\}$ are closed and compact subsets of $Y$, by theorem (3-29) $f^{-1}(\{y_1\})$ and $f^{-1}(\{y_2\})$ are $\omega$-closed sets in $X$, then we can appoint two disjoint open sets $U_1, U_2$ with $f^{-1}(\{y_1\})$ is a subset of $U_1$, and $f^{-1}(\{y_2\})$ is a subset of $U_2$ (because of $X$ is $\omega^*$-normal space). Since $f$ is open function, then $f(f^{-1}(\{y_1\})) \subseteq f(U_1)$ and $f(f^{-1}(\{y_2\})) \subseteq f(U_2)$, where $f(U_1), f(U_2)$ are open subsets in $Y$, and $f(U_1) \cap f(U_2) = f(U_1 \cap U_2) = f(\emptyset) = \emptyset$, therefore $Y$ is $I_{\mathcal{L}}$-space.

We can prove the following proposition by the same way.

(3-50) **Proposition**

Let $f$ be $\omega^*(\omega^{**})c$-continuous and $\omega^{**}(\omega)$-open function from the $\omega^{**}(\omega^*)$-normal space $X$ onto the $\omega I_{\mathcal{L}}(I_{\mathcal{L}})$-space $Y$, then $Y$ is $\omega I_{\mathcal{L}}$-space.

(3-51) **Definition** [17]

In case the reverse image of any open set $V$ in $Y$ under the effect of the function $f$ from $X$ into $Y$ is $\omega$-open set in $X$, then $f$ is $\omega$-continuous function.

(3-52) **Example**

Suppose $f$ is a function from the included point space $(Z, J_1)$ into the same space is $\omega$-continuous function.

(3-53) **Definition**

In the previous definition if $V$ is $\omega$-open set in $Y$ and its reverse image is open ($\omega$-open) set in $X$, then $f$ is $\omega^*(\omega^{**})$-continuous function.

(3-54) **Example**

1- The function $I_{\mathcal{L}}$ from the discrete space $(Z, J_{\mathcal{L}})$ into the cofinite space $(Z, J_{\mathcal{C}})$ is $\omega^*$-continuous function.

2- The function $f$ from the excluded point space $(Z, J_{\mathcal{E}})$ into the same space is $\omega^{**}$-continuous function.

(3-55) **Theorem**

A function $f$ from the space $(X, J)$ into the space $(Y, \sigma)$, and the collection of open sets with compact complement in $\sigma$ is a base for the topology $\sigma$ on $Y$, then $f$ is $\omega c$-continuous if and only if the
function $f_*$ from the space $(X,J)$ into the space $(Y,\mathcal{A})$ is $\omega_c$-continuous function, also the function $t$ from the space $(Y,\mathcal{A})$ into the space $(Y,\mathcal{A})$ is continuous function, and the function $t^{-1}$ from the space $(Y,\mathcal{A})$ into the space $(Y,\mathcal{A})$ is $\omega_c$-continuous function. Where $f_*(x)=f(x)$ for each $x$ in $X$.

\[ f \quad (X,J) \quad (Y,\mathcal{A}) \]

\[ t^{-1} \quad t \\
\]

\[ f_* \quad (Y,\mathcal{A}) \]

\textbf{Proof}

Suppose $f$ is $\omega_c$-continuous function, to prove $f_*$ is $\omega_c$-continuous function, let $K$ be an open set in $(Y,\mathcal{A})$, so its complement is compact (by hypothesis), since the open sets with compact complement in $\mathcal{A}$ is a base for $\mathcal{A}$, so $\mathcal{A} \subseteq \mathcal{A}$ and then $K$ is open set in $(Y,\mathcal{A})$, $f^{-1}(K)$ is $\omega$-open set in $X$ (since $f$ is $\omega$-continuous function). Since $f(x)=f_*(x)$ for each $x$ in $X$, then $f_*^{-1}$ is $\omega$-open set in $X$, and this implies $f_*$ is $\omega_c$-continuous function. It is clear that the function $t$ is continuous, since if $K$ is an open set in $(Y,\mathcal{A})$ then it is an open set in $(Y,\mathcal{A})$ (since $\mathcal{A} \subseteq \mathcal{A}$), but $t^{-1}(K)=K$, so $t$ is continuous function. Also, the function $t^{-1}$ is $\omega_c$-continuous function since if $K$ is an open set in $(Y,\mathcal{A})$ with compact complement, then it is an open set in $(Y,\mathcal{A})$ (by hypothesis) and then it is $\omega$-open set in $(Y,\mathcal{A})$ (by remark (3.7)), but $(t^{-1})^{-1}(K)=t(K)=K$ so $t^{-1}$ is $\omega_c$-continuous function. Conversely, let $f_*$ be $\omega_c$-continuous function, to prove $f$ is $\omega_c$-continuous function too. Take $K$ as an open set with compact complement in $(Y,\mathcal{A})$ so it is a subset of $(Y,\mathcal{A})$ (since the collection of all open sets with compact complement in $(Y,\mathcal{A})$ forms a base for $(Y,\mathcal{A})$), since $f_*$ is $\omega_c$-continuous function, so the inverse image of $K$ is $\omega$-open set in $X$, but $f_*^{-1}(K) = f^{-1}(K)$ (because $f_*(x)=f(x)$ for each $x$ in $X$), therefore $f$ is $\omega_c$-continuous function.

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