Lewis-Riesenfeld quantization and SU(1,1) coherent states for 2D damped harmonic oscillator

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Abstract

In this paper we study a two-dimensional [2D] rotationally symmetric harmonic oscillator with time-dependent frictional force. At the classical level, we solve the equations of motion for a particular case of the time-dependent coefficient of friction. At the quantum level, we use the Lewis-Riesenfeld procedure of invariants to construct exact solutions for the corresponding time-dependent Schrödinger equations. The eigenfunctions obtained are in terms of the generalized Laguerre polynomials. By mean of the solutions we verify a generalization version of the Heisenberg’s uncertainty relation and derive the generators of the su(1,1) Lie algebra. We
construct the coherent states à la Barut-Girardello and à la Perelomov and respectively study their properties.

1 Introduction

The one-dimensional [1D] harmonic oscillator is one of the most simplest and fundamental classical as well as quantum system studied in the literature. However, the study of the two-dimensional [2D] harmonic oscillator in quantum mechanics for the case of the rotationally symmetric oscillator turns out to be interesting and less explored. In fact, it is more difficult to solve when the problem involves time-dependent parameters.

In the last few decades the problem of the time-dependent quantum systems has received a great interest since Lewis and Riesenfeld have introduced an excellent method of invariants to solve the time-dependent Schrödinger equation [1]. This method stimulated some interests in using the invariants for solving 1D and 2D time-dependent harmonic oscillators problems [2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22]. The 1D damped harmonic oscillator has been extensively studied in the literature [23,24,25,26,27], while its generalization in two-dimensions as far as we know is less explored. We discuss a system of two-non-interacting damped oscillators with equal time-dependent coefficients of friction and equal time-dependent frequencies.

In section 2, we study the system at the classical level and formulate the corresponding quantum system. We solve the classical equations of motion for a constant coefficient of friction and for some particular cases of frequencies.

In section 3 we use the Lewis-Riesenfeld’s method to construct the invariant operator $\hat{I}(t)$. The eigenvalues and the eigenfunctions of the invariants are calculated explicitly by operators methods, the key element being the introduction of an appropriate unitary operator. We derive then a conserved angular momentum $\hat{L}_z$ that is simultaneously commuting with the invariant operator $\hat{I}(t)$ and the Hamiltonian $\hat{H}(t)$. However both of the three operators cannot be simultaneously diagonalized at this stage of the problem.

In section 4 we introduce the helicity Fock basis in order to simultaneously diagonalize the operators $\hat{I}(t), \hat{H}(t)$ and $\hat{L}_z$. The rotationally symmetry of the system has been useful in determining an orthogonal basis of the Hilbert space for the procedure of the simultaneous diagonalization. Then we derive the exact solution of the Schrödinger equations in terms of generalized Laguerre polynomials.

In section 5 we use the eigenfunctions of the Hamiltonian to verify a generalization version of the Heisenberg’s uncertainty relations that is formulated following the standard arguments as follows: for the simultaneous measurement of two ob-
servables \( \hat{A} \) and \( \hat{B} \) in the states \(|\psi\rangle\), the uncertainty satisfy the inequality

\[
\Delta \hat{A} \Delta \hat{B} \geq \frac{\hbar}{2} \left| \langle \psi | [\hat{A}, \hat{B}] |\psi\rangle \right|,
\]

(1)

where \( \Delta \hat{A} \) and \( \Delta \hat{B} \) are respectively the dispersions defined as

\[
\Delta \hat{A} = \sqrt{\langle \psi | \hat{A}^2 |\psi\rangle - \langle \psi | \hat{A} |\psi\rangle^2}, \quad \Delta \hat{B} = \sqrt{\langle \psi | \hat{B}^2 |\psi\rangle - \langle \psi | \hat{B} |\psi\rangle^2}.
\]

(2)

Similar discussions can be also read in [10].

In section 6, we derive from the solution of the system the hidden generators of the \( su(1, 1) \) Lie algebra. We proceed by the factorization method as developed in [28, 29] to find the hidden symmetry of the system and derive from the eigenfunctions the related raising and lowering operators which generate the \( su(1, 1) \) Lie algebra.

In section 7, we discuss the \( SU(1, 1) \) coherent states à la Barut-Girardello [30] and à la Perelomov [31]. A brief story about these coherent states is that in 1926’s Schrödinger introduced for the first time in quantum mechanics the semiclassical states defined as the minimum uncertainty Gaussian states whose dynamics has maximum similarity to classical oscillator [32]. These states were rediscovered by Glauber in the framework of quantum optics in early 1960’s [33]. They are defined as eigenstates of the annihilation operator of harmonic oscillator and were obtained by action of the Weyl-Heisenberg operator on the ground state. These coherent states introduced by Glauber have inspired respectively Barut-Girardello [30] and Perelomov [31] in constructing the coherent states for \( SU(1, 1) \) Lie algebraic group through different approaches. The Barut-Girardello and the Perelomov coherent states gained lot of applications, for instance in the fields of quantum optics [34, 35], quantum computation [36, 37] and quantum mechanics [38, 39, 40].

The conclusion is given in section 8.

2 The Model

We consider in two-dimensional configuration space, the system of two non-interacting damped oscillators with equal time-dependent coefficients of friction and equal time-dependent frequencies. The equations of motion are given by

\[
\begin{align*}
\ddot{x}_1 + \eta(t) \dot{x}_1 + \omega^2(t) x_1 &= 0, \\
\ddot{x}_2 + \eta(t) \dot{x}_2 + \omega^2(t) x_2 &= 0,
\end{align*}
\]

(3)

where \( \eta(t) \) is the time-dependent coefficient of friction, \( \omega(t) \) is the time-dependent frequency and the dot represents time-derivative.
These equations of motion may be derived from the Lagrangian
\[
L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = f^{-1}(t) \left[ \frac{m}{2} (\dot{x}_1^2 + \dot{x}_2^2) - \frac{m\omega^2(t)}{2} (x_1^2 + x_2^2) \right],
\]
where \( f \) is an arbitrary function such that \( f(t) = e^{-\int_0^t \eta(t') \, dt'} \) or \( \eta(t) = -\frac{d}{dt} \ln f(t) \).
Let consider the transformation \( x' = R(\vartheta)x \), with \( R(\vartheta) = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix} \), \( \vartheta \in \mathbb{R} \).
This transformation preserves the invariance of the Lagrangian. This rotational invariance in the plane manifests the presence of the Noether charge which correspond to the angular-momentum of the system.

The canonical momentum associated with the variables \( x_1 \) and \( x_2 \) are
\[
\begin{align*}
p_1 &= \frac{\partial L}{\partial \dot{x}_1} = f^{-1}(t) m \dot{x}_1, \\
p_2 &= \frac{\partial L}{\partial \dot{x}_2} = f^{-1}(t) m \dot{x}_2.
\end{align*}
\]
The Hamiltonian is given by
\[
H(x_1, x_2, p_1, p_2, t) = \dot{x}_1 p_1 + \dot{x}_2 p_2 - L = \frac{f(t)}{2m} (p_1^2 + p_2^2) + f^{-1}(t) \frac{m\omega^2(t)}{2} (x_1^2 + x_2^2).
\]
We recover the 2D Hamiltonian that describes the dissipative system previously introduced in one dimension by Pedrosa [26, 27]. For \( f(t) = 1 \) and \( f(t) = \text{exp}(-\gamma t) \) with \( \omega(t) = \omega_0 \) where \( \gamma, \omega_0 \) are positive constants, the Hamiltonian [7] is respectively reduced to the ordinary 2D harmonic oscillator and the 2D Caldirola and Kanai Hamiltonian [11, 12].

Since we are in two dimensional configuration space, we can look for the solutions of the classical equations in the complex system by setting \( z = x_2 + ix_1 \). The classical equation of motion in term of the coordinate \( z \) is
\[
\ddot{z} + \eta(t) \dot{z} + \omega^2(t) z = 0.
\]
For \( \eta(t) = \gamma \) and \( \omega(t) = \omega_0 \), the equation (8) takes the form
\[
\ddot{z} + \gamma \dot{z} + \omega_0^2 z = 0,
\]
and the classical solutions are [8,3]
\[
z(t) = \begin{cases} 
  e^{-\frac{1}{2}\gamma t} \left[ A_1 \exp \left( \frac{1}{2} \gamma t \right) + A_2 \exp \left( -\frac{1}{2} \gamma t \right) \right] & \text{if } \tau^2 = \gamma^2 - 4 \omega_0^2 > 0, \\
  e^{-\frac{1}{2}\gamma t} \left[ A_1 \sin \left( \frac{1}{2} \gamma t \right) + A_2 \cos \left( \frac{1}{2} \gamma t \right) \right] & \text{if } \tau^2 = 4 \omega_0^2 - \gamma^2 > 0, \\
  e^{-\frac{1}{2}\gamma t} (A_1 + A_2) & \text{if } \gamma^2 = 4 \omega_0^2,
\end{cases}
\]
where $A_1$ and $A_2$ are constants.

For $\omega(t) = \omega_0 e^{-\frac{1}{2}\gamma t}$, the equation can be rewritten as follows

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 e^{-\gamma t} z = 0.$$  \hfill (11)

The solution is given by \cite{13}

$$\dot{z}(t) = e^{-\frac{1}{2}\gamma t} \left[ B_1 J_1 \left( \frac{2\omega_0}{\gamma} e^{-\frac{1}{2}\gamma t} \right) + B_2 Y_1 \left( \frac{2\omega_0}{\gamma} e^{-\frac{1}{2}\gamma t} \right) \right],$$  \hfill (12)

where $J_k$ and $Y_k$ are respectively Bessel functions of first and second kind, $B_1$ and $B_2$ are constants.

For $\omega(t) = \omega_0 e^{-\gamma t}$, the solution is known to be \cite{13}

$$z(t) = C_1 \cos \left( \frac{\omega_0 e^{-\gamma t}}{\gamma} \right) + C_2 \sin \left( \frac{\omega_0 e^{-\gamma t}}{\gamma} \right),$$  \hfill (13)

where $C_1$ and $C_2$ are constants.

At the quantum level, the corresponding Hamiltonian operator describing the system reads

$$\hat{H}(t) = \frac{f(t)}{2m} \left( \hat{p}_1^2 + \hat{p}_2^2 \right) + f^{-1}(t) \frac{m\omega_0^2(t)}{2} \left( \hat{x}_1^2 + \hat{x}_2^2 \right).$$  \hfill (14)

where the position operators $\hat{x}_1, \hat{x}_2$ and the momentum operators $\hat{p}_1, \hat{p}_2$ satisfy the canonical commutation relations

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = 0 = [\hat{p}_i, \hat{p}_j], \quad i, j = 1, 2.$$  \hfill (15)

To diagonalize this Hamiltonian, many methods can be considered to achieve this end \cite{1, 6, 44, 45, 46, 47, 48}. Among them, we have the Lewis-Riesenfeld method based on the construction of the Hermitian invariant operator \cite{1}.

### 3 Construction and eigensystems of the invariant operator

To construct the exact invariant operator for the quantum system described by the time-dependent Hamiltonian (14), we use the dynamic invariant method formulated by Lewis and Riesenfeld \cite{1}.

Now, we look for the invariant in the form

$$\hat{I}(t) = \alpha(t) \hat{J}_+ + \beta(t) \hat{J}_- + \delta(t) \hat{J}_0,$$  \hfill (16)
where \( \alpha, \beta, \delta \) are time-dependent real coefficients and \( J_+ = \frac{1}{2}(\hat{x}_1^2 + \hat{x}_2^2), \ J_- = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2), \ J_0 = \frac{1}{2}(\hat{x}_1\hat{p}_1 + \hat{p}_1\hat{x}_1 + \hat{x}_2\hat{p}_2 + \hat{p}_2\hat{x}_2) \) satisfy the following commutation relations

\[
[J_+, J_-] = i\hat{J}_0; \ [\hat{J}_0, J_\pm] = \pm 2i\hat{J}_\pm.
\]  

(17)

The Hamiltonian (14) is rewritten in term of the latter operators as follows

\[
\hat{H}(t) = \frac{f(t)}{m} J_- + f^{-1}(t)m\omega^2(t)\hat{J}_+.
\]  

(18)

To determine an explicit form of the Hermitian invariant (16), one solves the following equation

\[
\frac{d\hat{I}(t)}{dt} = \frac{\partial\hat{I}(t)}{\partial t} + \frac{1}{i} [\hat{I}(t), \hat{H}(t)] \equiv 0,
\]  

(19)

where \( \hbar = 1 \). By expansion of equating (19), we obtain the first-order linear differential equations for the unknown coefficient functions

\[
\dot{\alpha} - 2f^{-1}m\omega^2\delta = 0, \tag{20}
\]

\[
\dot{\beta} + 2f^{-1}m\delta = 0, \tag{21}
\]

\[
\dot{\delta} + f^{-1}m\omega^2\beta = 0. \tag{22}
\]

As in [1, 14], it is convenient to introduce another real function \( \rho(t) \)

\[
\beta(t) = \rho^2(t). \tag{23}
\]

For an arbitrary positive constant \( \nu \), the other coefficients are

\[
\delta(t) = -mf^{-1}\dot{\rho}\rho, \quad \alpha(t) = \frac{\nu^2}{\rho^2} + m^2 f^{-2}\dot{\rho}^2. \tag{24}
\]

Replacing (23), (24) in (16), the Hermitian invariant acquires the form

\[
\hat{I}(t) = \frac{1}{2} \left[ (mf^{-1}\dot{\rho}\hat{x}_1 - \rho\hat{p}_1)^2 + \frac{\nu^2}{\rho^2}\hat{x}_1^2 + (mf^{-1}\dot{\rho}\hat{x}_2 - \rho\hat{p}_2)^2 + \frac{\nu^2}{\rho^2}\hat{x}_2^2 \right], \tag{25}
\]

where the function \( \rho \) is the solution of the so-called Ermakov-Pinney equation [49]

\[
\ddot{\rho} + \eta\dot{\rho} + \omega^2\rho = \frac{\nu^2 f^2}{m^2 \rho^3}. \tag{26}
\]

Next we determine the spectrum of the invariant operator by solving the eigenvalue equation

\[
\hat{I}(t)\phi(x_1, x_2, t) = E\phi(x_1, x_2, t), \text{ with } E \equiv \text{constant}, \ \phi(x_1, x_2, t) \in \mathcal{H}, \tag{27}
\]
where $\mathcal{H}$ is the Hilbert space on which this operator is defined.

In order to solve equation (27) we introduce the unitary operator that is written as follows

$$
\hat{U} = \exp \left[ -\frac{im\hat{f}^{-1}\hat{p}}{2\rho} (\hat{x}_1^2 + \hat{x}_2^2) \right], \quad \hat{U}^\dagger \hat{U} = \hat{U} \hat{U}^\dagger = \mathbf{I}.
$$

(28)

Setting

$$
U \phi(x_1, x_2, t) = \phi'(x_1, x_2, t),
$$

(29)

and

$$
\hat{I}'(t) = \hat{U} \hat{I} \hat{U}^\dagger = \frac{1}{2} \left[ \rho^2 (\hat{p}_1^2 + \hat{p}_2^2) + \frac{k^2}{\rho^2} (\hat{x}_1^2 + \hat{x}_2^2) \right],
$$

(30)

it is easy to verify that

$$
\hat{I}'(t) \phi'(x_1, x_2, t) = E \phi'(x_1, x_2, t), \quad \text{with} \quad \phi'(x_1, x_2, t) \in \mathcal{H}.
$$

(31)

To achieve the diagonalization of equation (31) as clear as possible, we introduce the lowering and raising operators given by

$$
a_1' = \frac{1}{\sqrt{2\nu}} \left( \frac{\nu}{\rho} \hat{x}_1 + \nu \hat{p}_1 \right), \quad a_1'^\dagger = \frac{1}{\sqrt{2\nu}} \left( \frac{\nu}{\rho} \hat{x}_1 - \nu \hat{p}_1 \right),
$$

(32)

$$
a_2' = \frac{1}{\sqrt{2\nu}} \left( \frac{\nu}{\rho} \hat{x}_2 + \nu \hat{p}_2 \right), \quad a_2'^\dagger = \frac{1}{\sqrt{2\nu}} \left( \frac{\nu}{\rho} \hat{x}_2 - \nu \hat{p}_2 \right),
$$

(33)

which satisfy the following commutation relations

$$
[a_1', a_1'^\dagger] = I = [a_2', a_2'^\dagger], \quad [a_1', a_2'] = [a_1'^\dagger, a_2'^\dagger] = 0 = [a_1'^\dagger, a_2'].
$$

(34)

Let us consider any nonnegative integers $n_1, n_2$ and $|\phi'_{n_1, n_2}(t)\rangle$ the orthonormalized Fock space such as

$$
|\phi'_{n_1, n_2}(t)\rangle = \frac{1}{\sqrt{n_1!n_2!}} \left( a_1'^\dagger \right)^{n_1} \left( a_2'^\dagger \right)^{n_2} |\phi'_{0,0}(t)\rangle,
$$

(35)

$$
\langle \phi'_{n_1, n_2}(t) | \phi'_{n_1, n_2}(t) \rangle = \delta_{n_1, n_1} \delta_{n_2, n_2},
$$

(36)

with $|\phi'_{0,0}(t)\rangle$ is a normalized state annihilated by $a_1', a_2'$.

In order to determine the exact solution $\phi_{n_1, n_2}(x_1, x_2, t)$ of the invariant operator $I(t)$, we first express the ground state $|\phi_{0,0}(t)\rangle$ in the configuration space basis as follows

$$
\phi_{0,0}(x_1, x_2, t) = U^\dagger \langle x_1 | \phi'_{0}(t) \rangle \langle x_2 | \phi'_{0}(t) \rangle
$$

$$
= \left( \frac{\nu}{\pi \rho^2} \right)^{\frac{7}{2}} \exp \left[ \left( im \frac{1}{\rho} \hat{f} - \frac{\nu}{\rho^2} \right) \left( \frac{x_1^2 + x_2^2}{2} \right) \right].
$$

(37)
Then, the $n^{th}$ eigenfunction are obtained from (35) as

$$\phi_{n_1,n_2}(x_1,x_2,t) = U^\dagger \phi'_{n_1,n_2}(x_1,x_2,t)$$

$$= \frac{1}{\rho} \left( \frac{\nu}{2^{n_1+n_2} \pi n_1!n_2!} \right)^{\frac{1}{2}} H_{n_1} \left( x_1 \frac{\sqrt{\nu}}{\rho} \right) H_{n_2} \left( x_2 \frac{\sqrt{\nu}}{\rho} \right) \times \exp \left[ \left( imf^{-1} \frac{\hat{p}}{\rho} - \frac{\nu}{\rho^{2}} \right) \left( \frac{x_1^{2}}{2} + \frac{x_2^{2}}{2} \right) \right],$$

(38)

where $H_{n_1}$ and $H_{n_2}$ are the Hermite polynomials of order $n_1$ and $n_2$.

To obtain the eigenvalues $E_{n_1,n_2}$ of the invariant operator $\hat{I}(t)$, let us introduce a new pair of raising and lowing operators define as

$$a_{j} = U^\dagger a_{j}' U = \frac{1}{\sqrt{2\nu}} \left( mf^{-1} \hat{p} \hat{x}_{j} - \rho \hat{p}_{j} + i \frac{\nu}{\rho} x_{j} \right),$$

(39)

$$a_{j}^\dagger = U^\dagger a_{j}' U = \frac{1}{\sqrt{2\nu}} \left( mf^{-1} \hat{p} \hat{x}_{j} - \rho \hat{p}_{j} - i \frac{\nu}{\rho} x_{j} \right).$$

(40)

with $j = 1, 2$. In term of these operators the invariant operator $\hat{I}(t)$ takes the form

$$\hat{I}(t) = \nu \left( a_{1}^\dagger a_{1} + a_{2}^\dagger a_{2} + 1 \right).$$

(41)

The action of $a_{j}$ and $a_{j}^\dagger$ on $|\phi_{nj}(t)\rangle$ finds expression in

$$a_{j}^\dagger |\phi_{nj}(t)\rangle = \sqrt{n_{j} + 1} |\phi_{n_{j}+1}(t)\rangle,$$

(42)

$$a_{j} |\phi_{nj}(t)\rangle = \sqrt{n_{j}} |\phi_{n_{j}-1}(t)\rangle,$$

(43)

$$a_{j}^\dagger a_{j} |\phi_{nj}(t)\rangle = n_{j} |\phi_{nj}(t)\rangle.$$ 

(44)

Basing on these definitions, the invariant is diagonalized as follows

$$\hat{I}(t)|\phi_{n_1,n_2}(t)\rangle = \nu (n_1 + n_2 + 1) |\phi_{n_1,n_2}(t)\rangle.$$ 

(45)

Since the Hamiltonian of the system is time-dependent, the Schrödinger equation of the system is

$$i \frac{\partial}{\partial t} \psi(x_1,x_2,t) = \hat{H}(t)\psi(x_1,x_2,t), \quad \psi(x_1,x_2,t) \in \mathcal{H}$$

(46)

where the eigenfunction $\psi(x_1,x_2,t)$ is related to $\phi(x_1,x_2,t)$ by

$$\psi_{n_1,n_2}(x_1,x_2,t) = e^{i\theta_{n_1,n_2}(t)} \phi_{n_1,n_2}(x_1,x_2,t).$$ 

(47)

Inserting this equation in (46), one determines the phase function $\theta_{n_1,n_2}(t)$ in the form

$$\theta_{n_1,n_2}(t) = \int_{0}^{t} \langle \phi_{n_1,n_2}(t') | i \frac{\partial}{\partial t'} - \hat{H}(t') | \phi_{n_1,n_2}(t') \rangle dt'.$$ 

(48)
However, as we pointed out in the previous section, this system possesses a conserved angular-momentum

\[ \hat{L}_z = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1 = i(a_2^\dagger a_1 - a_1^\dagger a_2), \]  

which commutes with the invariant operator and with the Hamiltonian

\[ [\hat{L}_z, \hat{I}(t)] = 0, \quad [\hat{L}_z, \hat{H}(t)] = 0 \]  

Although the operator \( \hat{L}_z \) commutes with both \( \hat{I}(t) \) and \( \hat{H}(t) \), but the basis \( |\phi_{n_1,n_2}(t)\rangle \) cannot diagonalize them simultaneously. Therefore, it is convenient to find another basis of Hilbert space that diagonalizes these operators.

### 4 Eigensystems of the Hamiltonian operator

To recover the available eigenbasis of the invariant operator which can diagonalize simultaneously the invariant operator, the angular momentum and the Hamiltonian of the system, let us consider the helicity Fock algebra generators as follows

\[ a'_{\pm} = \frac{1}{\sqrt{2}} (a'_1 \pm ia'_2) , \quad a'^\dagger_{\pm} = \frac{1}{\sqrt{2}} \left( a'^\dagger_1 \mp ia'^\dagger_2 \right) , \quad (51) \]

with

\[ [a'_{\pm},a'^\dagger_{\mp}] = \mathbf{I}, \quad [a'_{\pm},a'^\dagger_{\pm}] = 0, \quad (52) \]

where \( a'_1, a'_2, a'^\dagger_1, a'^\dagger_2 \) are the ones in the previous equations. The associated helicity-like basis \( |\phi'_{n_+,n_-}(t)\rangle \) are defined as follows

\[ |\phi'_{n_+,n_-}(t)\rangle = \frac{1}{\sqrt{n_+!n_-!}} \left( a'^\dagger_+ \right)^{n_+} \left( a'^\dagger_- \right)^{n_-} |\phi'_{0,0}(t)\rangle , \quad (53) \]

\[ \langle \phi'_{n_+,n_-}(t)|\phi'_{m_+,m_-}(t)\rangle = \delta_{n_+,m_+} \delta_{n_-,m_-} , \quad (54) \]

with \( |\phi'_{0,0}(t)\rangle \) is a normalized state annihilated by \( a'_\pm \) as by \( a'_1, a'_2 \).

In order to find the exact expression of the joint eigenfunction of the invariant operator and the angular momentum, we introduce the polar coordinates through the following canonical transformation \( \hat{x}_1 = r \cos \alpha, \hat{x}_2 = r \sin \alpha, \hat{p}_1 = -i(\cos \alpha \partial_r - \frac{\sin \alpha}{r} \partial_\alpha) \) and \( \hat{p}_2 = -i(\sin \alpha \partial_r + \frac{\cos \alpha}{r} \partial_\alpha) \). In terms of these coordinates the operators in equation (51) can be written as

\[ a'^\dagger_{\pm} = \frac{1}{2} e^{\pm i\alpha} \left[ \left( \frac{\nu}{\rho} r - \rho \partial_r \right) \pm i \frac{\rho}{r} \partial_\alpha \right] , \quad (55) \]

\[ a'_{\pm} = \frac{1}{2} e^{\pm i\alpha} \left[ \left( \frac{\nu}{\rho} r + \rho \partial_r \right) \pm i \frac{\rho}{r} \partial_\alpha \right] . \quad (56) \]
From the relation \((53)\) we construct the eigenfunction for the invariant operator of the system according to \([50]\). One finds

\[
\phi_{n_+, n_-}(x_1, x_2, t) = U^t \phi'_{n_+, n_-}(x_1, x_2, t),
\]

that is

\[
\phi_{n_+, n_-}(x_1, x_2, t) = \left(-\right)^n \frac{\nu^{1+|\ell|}}{\rho^{1+|\ell|}} \sqrt{\frac{n!}{\sqrt{\pi}}} \Gamma(n + |\ell| + 1) e^{\left(\frac{mf^{-1} \rho^2}{\rho^2} \right)} \times L_n^{|\ell|} \left(\frac{\nu}{\rho^2} p^2\right) e^{i\ell\alpha},
\]

where \(\ell = n_+ - n_-\), \(n = \min(n_+, n_-) = \frac{1}{2}(n_+ + n_- - |\ell|)\), \(\Gamma(u)\) the Gamma function and \(L_n^{|\ell|}(u)\) the generalised Laguerre polynomials.

To obtain the expectation values of the operators \(\hat{I}(t), \hat{L}_z, \hat{H}(t)\) that are respectively \(\hat{E}_{n_+, n_-}, \hat{l}_{n_+, n_-}, \hat{e}_{n_+, n_-}\), we introduce a new pair of raising and lowering helicity operators define as

\[
a_\pm = U^t a_\pm U = \frac{1}{2\sqrt{\nu}} \left[ \left(mf^{-1} \hat{\rho} + \frac{f\nu}{\rho}\right) (\hat{x}_1 \pm i\hat{x}_2) - \rho (\hat{p}_1 \pm i\hat{p}_2) \right],
\]

\[
a_\pm^t = U^t a_\pm^t U = \frac{1}{2\sqrt{\nu}} \left[ \left(mf^{-1} \hat{\rho} - \frac{f\nu}{\rho}\right) (\hat{x}_1 \mp i\hat{x}_2) - \rho (\hat{p}_1 \mp i\hat{p}_2) \right].
\]

In term of these operators we have

\[
\hat{I}(t) = \nu \left(a_+^t a_+ + a_-^t a_- + 1\right),
\]

\[
\hat{L}_z = \left(a_-^t a_- - a_+^t a_+\right),
\]

\[
\hat{H}(t) = \frac{1}{2\nu} \left(mf^{-1} \rho^2 + \frac{f\nu^2}{m\rho^2} + m\omega^2 f^{-1} \rho^2\right) \left(a_+^t a_+ + a_-^t a_- + 1\right) + \frac{-mf^{-1} \hat{\rho} + \frac{f\nu}{\rho}}{2\nu} \frac{\rho^2}{2m\rho^2} - \frac{m\omega^2 f^{-1} \rho^2}{2\nu} a_- a_- + \frac{-mf^{-1} \hat{\rho} - \frac{f\nu}{\rho}}{2\nu} \frac{\rho^2}{2m\rho^2} - \frac{m\omega^2 f^{-1} \rho^2}{2\nu} a_+ a_+.
\]

The expectative values of the above operators read as

\[
E_{n_+} = \langle \phi_{n_+, n_-}(t) | \hat{I}(t) | \phi_{n_+, n_-}(t) \rangle = \nu (n_+ + n_- + 1),
\]

\[
l_{n_+} = \langle \phi_{n_+, n_-}(t) | \hat{L}_z | \phi_{n_+, n_-}(t) \rangle = n_- - n_+,
\]

\[
\mathcal{E}_{n_+} = \langle \phi_{n_+, n_-}(t) | \hat{H}(t) | \phi_{n_+, n_-}(t) \rangle = \frac{1}{2\nu} \left(mf^{-1} \rho^2 + \frac{f\nu^2}{m\rho^2} + m\omega^2 f^{-1} \rho^2\right).
\]
\[ \times (n_+ + n_- + 1), \quad (66) \]

where the action of \( a_\pm \) and \( a_\pm^\dagger \) on \( |\phi_{n\pm}(t)\rangle \) finds expression in
\[
a_\pm^\dagger |\phi_{n\pm,n\mp}(t)\rangle = \sqrt{n_\pm + 1} |\phi_{n\pm + 1,n\mp}(t)\rangle, \quad (67)
a_\pm |\phi_{n\pm,n\mp}(t)\rangle = \sqrt{n_\pm + 1} |\phi_{n\pm - 1,n\mp}(t)\rangle, \quad (68)
a_\pm^\dagger a_\pm |\phi_{n\pm,n\mp}(t)\rangle = n_\pm |\phi_{n\pm,n\mp}(t)\rangle. \quad (69)
\]

To determine the exact solution of the Schrödinger equation \([46]\), we have to find the exact expression of the phase function in equation \([48]\) such that
\[
\frac{d}{dt} \theta_{n_1,n_2}(t) = \langle \phi_{n_+,n_-}(t)|i\frac{\partial}{\partial t} - \hat{H}(t)|\phi_{n_+,n_-}(t)\rangle = \langle \phi_{n_0,0}(t)|i\frac{\partial}{\partial t} \phi_{n_0,0}(t) \rangle + \frac{1}{\sqrt{n_+!n_-!}} \times \langle \phi_{n_+,n_-}(t)|\frac{\partial}{\partial t} \left[ (a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-} \right] \phi_{n_0,0}(t) \rangle. \quad (70)
\]

Let us evaluate the following expression
\[
\langle \phi_{n_+,n_-}(t)|\frac{\partial}{\partial t} \phi_{n_+,n_-}(t) \rangle = \frac{1}{\sqrt{n_+!n_-!}} \langle \phi_{n_+,n_-}(t)|\frac{\partial}{\partial t} \left[ (a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-} \right] \phi_{n_0,0}(t) \rangle
\]
\[
= \langle \phi_{n_0,0}(t)|\frac{\partial}{\partial t} |\phi_{n_0,0}(t)\rangle + \frac{1}{\sqrt{n_+!n_-!}} \times \langle \phi_{n_+,n_-}(t)|\frac{\partial}{\partial t} \left[ (a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-} \right] |\phi_{n_0,0}(t)\rangle. \quad (71)
\]

We have
\[
\langle \phi_{n_0,0}(t)|\frac{\partial}{\partial t} |\phi_{n_0,0}(t)\rangle = \frac{imf^{-1}}{2\nu} (\hat{\rho} \rho + \hat{\rho} \rho - \rho^2), \quad (72)
\]
and
\[
\frac{1}{\sqrt{n_+!n_-!}} \langle \phi_{n_+,n_-}(t)|\frac{\partial}{\partial t} \left[ (a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-} \right] |\phi_{n_0,0}(t)\rangle = \frac{imf^{-1}}{2\nu} (\hat{\rho} \rho + \eta \hat{\rho} \rho - \rho^2)
\]
\[
\times (n_+ + n_-), \quad (73)
\]

where the expressions of \( \frac{\partial a_\pm^\dagger}{\partial t} \) and \( \frac{\partial a_\pm}{\partial t} \) in terms of \( a_\pm \) and \( a_\pm^\dagger \) are
\[
\frac{\partial a_\pm^\dagger}{\partial t} = \frac{1}{2\sqrt{\nu}} \left[ (mf^{-1}\hat{\rho} + mf^{-1}\hat{\rho} + iv\frac{\hat{\rho}}{\rho^2}) (\hat{x}_1 - i\hat{x}_2) - \hat{\rho}(\hat{p}_1 - i\hat{p}_2) \right]
\]
\[
= \frac{imf^{-1}}{2\nu} (\hat{\rho} \rho + \eta \hat{\rho} \rho - \rho^2) a_\mp + \left[ \frac{\hat{\rho}}{\rho} - \frac{imf^{-1}}{2\nu} (\hat{\rho} \rho + \eta \hat{\rho} - \rho^2) \right] a_\pm, \quad (74)
\]
\[
\frac{\partial a_\pm}{\partial t} = \frac{1}{2\sqrt{\nu}} \left[ (mf^{-1}\hat{\rho} + mf^{-1}\hat{\rho} + iv\frac{\hat{\rho}}{\rho^2}) (\hat{x}_1 + i\hat{x}_2) - \hat{\rho}(\hat{p}_1 + i\hat{p}_2) \right]
\]

\[
\frac{imf^{-1}}{2\nu} \left( \ddot{\rho} \rho + \eta \dot{\rho} \rho - \dot{\rho}^2 \right) a_+^* + \left[ \frac{\dot{\rho}}{\rho} - \frac{imf^{-1}}{2\nu} \left( \ddot{\rho} \rho + \eta \dot{\rho} - \dot{\rho}^2 \right) \right] a_+. \quad (75)
\]

We then find
\[
\langle \phi_{n,+},n_-(t) | \frac{\partial}{\partial t} | \phi_{n,+},n_-(t) \rangle = \frac{imf^{-1}}{2\nu} \left( \ddot{\rho} \rho + \eta \dot{\rho} \rho - \dot{\rho}^2 \right) (n_+ + n_- + 1) = \frac{imf^{-1}}{2\nu} \left( \frac{\nu^2 f^2}{m^2 \rho^2} - \omega^2 \rho^2 - \dot{\rho}^2 \right) (n_+ + n_- + 1). \quad (76)
\]

Finally, taking into account (66) and (76), we find that the phase function in (70) is given by
\[
\theta_{n,+},n_-(t) = -\frac{\nu}{2m} (n_+ + n_- + 1) \int_0^t f(t') \rho^2(t') \, dt'. \quad (77)
\]

Our result for \(\theta_{n,+},n_-(t)\) confirms the 2D case result of [26], slightly differs from the one calculated in [14] and largely differs from our previous result [51].

The solution of the Schrödinger equation is given by
\[
\psi_{n,\ell}(x_1, x_2, t) = (-)^n \frac{\left( \nu \right)^{\frac{1+|\ell|}{2}} \sqrt{\pi}}{\rho^{1+|\ell|} \sqrt{n!}} e^{i(\nu \rho \dot{\rho} - \frac{\nu}{\rho} \dot{\rho}^2)} \frac{n!}{\Gamma(n + |\ell| + 1)} e^{(imf^{-1} \rho \dot{\rho} - \frac{\nu}{\rho} \dot{\rho}^2)} x_{|\ell|} \rho \frac{e^{i\ell\alpha}}{e^{i\theta_{n,\ell}(t)}}. \quad (78)
\]

However, one can deduce from the Lagrangian (4) the usual kinetic momentum \(p_{kj}\) such as
\[
p_{kj} = \frac{\partial L}{\partial \dot{x}_j} = f(t)p_j, \quad j = 1, 2, \quad (79)
\]

where \(p_j\) the canonical momentum and \(p_{kj} = m\dot{x}_j\). The mechanical energy of the system in term of the Hamiltonian (7) reads as
\[
E_m = \frac{m}{2} \dot{x}_j^2 + \frac{m\omega^2(t)}{2} x_j^2 = f(t)H(t). \quad (80)
\]

As pointed out in the literature by several authors [52, 53, 54, 55, 56], the quantization of this dissipative system for particular value of the function \(f(t) = e^{-\gamma t}\) through a non-inertial canonical transformation, is unsatisfactory with the laws of quantum theory such that the zero-point of the expectation values of the energy instead of going to the quantum ground energy and the violation of the Heisenberg
uncertainty relations when one tends the time to infinity ($t \to \infty$). Therefore, the
expectation value of the mechanical energy (80) is given by
\[
\langle \psi_{n,\ell} | E_m | \psi_{n,\ell} \rangle = \frac{1}{2\nu} \left( m\rho^2 + \frac{f^2\nu^2}{m\rho^2} + m\omega^2\rho^2 \right) (2n + |\ell| + 1) \tag{81}
\]
and
\[
\lim_{t \to \infty} \langle \psi_{n,\ell} | E_m | \psi_{n,\ell} \rangle \neq 0, \quad \forall f \in \mathbb{R}. \tag{82}
\]
One infers that the problem of the zero-point energy caused by the use of the non-
inertial canonical transformation is raised up by this method of Lewis-Riesenfeld.
In the next section, let us check the validity of the generalized version of the
Heisenberg’s uncertainty relations.

5 Heisenberg’s uncertainty relations

To prove the validity of the generalized uncertainty relations [1] with $\hbar = 1$, we
start with the determination of the standard expectation values of the operators
\( \hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2 \) and \( \hat{p}_{k_j} \),
\[
\langle \psi_{n,\ell} | \hat{x}_1 | \psi_{n,\ell} \rangle = \langle \psi_{n,\ell} | \hat{x}_2 | \psi_{n,\ell} \rangle = 0, \quad \langle \psi_{n,\ell} | \hat{p}_1 | \psi_{n,\ell} \rangle = \langle \psi_{n,\ell} | \hat{p}_2 | \psi_{n,\ell} \rangle = 0, \tag{83}
\]
\[
\langle \psi_{n,\ell} | \hat{x}_1^2 | \psi_{n,\ell} \rangle = \langle \psi_{n,\ell} | \hat{x}_2^2 | \psi_{n,\ell} \rangle = \frac{\rho^2}{2\nu} (2n + |\ell| + 1), \tag{84}
\]
\[
\langle \psi_{n,\ell} | \hat{p}_1^2 | \psi_{n,\ell} \rangle = \langle \psi_{n,\ell} | \hat{p}_2^2 | \psi_{n,\ell} \rangle = (2n + |\ell| + 1) \left( \frac{m^2 f^2 \rho^2}{2\nu} + \frac{\nu}{2\rho^2} \right), \tag{85}
\]
\[
\langle \psi_{n,\ell} | [\hat{x}_1, \hat{p}_1] | \psi_{n,\ell} \rangle = \langle n, \ell | [\hat{x}_2, \hat{p}_2] | n, \ell \rangle = i, \tag{86}
\]
\[
\langle \psi_{n,\ell} | [\hat{x}_1, \hat{p}_{k_1}] | \psi_{n,\ell} \rangle = \langle n, \ell | [\hat{x}_2, \hat{p}_{k_2}] | n, \ell \rangle = i f(t). \tag{87}
\]
The dispersions of operators are computed to
\[
\Delta x_1 = \Delta x_2 = \sqrt{\frac{\rho^2}{2\nu} (2n + |\ell| + 1)}, \tag{88}
\]
\[
\Delta p_1 = \Delta p_2 = \sqrt{\frac{1}{2} (2n + |\ell| + 1) \left( \frac{m^2 f^2 \rho^2}{\nu} + \frac{\nu}{\rho^2} \right)}, \tag{89}
\]
\[
\Delta p_{k_1} = \Delta p_{k_2} = f(t) \sqrt{\frac{1}{2} (2n + |\ell| + 1) \left( \frac{m^2 f^2 \rho^2}{\nu} + \frac{\nu}{\rho^2} \right)}. \tag{90}
\]

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The Heisenberg uncertainty relations can be inferred

\[ \Delta x_1 \Delta p_1 = \Delta x_2 \Delta p_2 = \frac{1}{2} (2n + |\ell| + 1) \sqrt{1 + \frac{m^2 f - 2 \rho^2 \rho}{\nu^2}} \geq \frac{1}{2}, \]  
\[ \Delta x_1 \Delta p_{k_1} = \Delta x_2 \Delta p_{k_2} = \frac{f(t)}{2} (2n + |\ell| + 1) \sqrt{1 + \frac{m^2 f - 2 \rho^2 \rho}{\nu^2}} \geq \frac{f(t)}{2}, \]  
\[ \Delta x_1 \Delta x_2 = \frac{\rho^2}{2\nu} (2n + |\ell| + 1) \geq 0, \]  
\[ \Delta p_1 \Delta p_2 = (2n + |\ell| + 1) \left( \frac{m^2 f - 2 \rho^2 \rho}{2\nu} + \frac{\nu}{2\rho^2} \right) \geq 0, \]  
\[ \Delta p_{k_1} \Delta p_{k_2} = f^2(t) (2n + |\ell| + 1) \left( \frac{m^2 f - 2 \rho^2 \rho}{2\nu} + \frac{\nu}{2\rho^2} \right) \geq 0. \]  
\[ \text{(92)} \]  
\[ \text{(93)} \]  
\[ \text{(94)} \]  
\[ \text{(95)} \]  
\[ \text{(96)} \]  

These results are related to similar discussions in [10]. In the present case the uncertainty relations are satisfied except for the relation in equation (93). In fact this uncertainty relation may tend to zero if \( \lim_{t \to \infty} f(t) \to 0 \) (for instance \( f(t) = e^{-\gamma t} \)). At first sight this result appears violated the Heisenberg uncertainty relations, but as observed authors in [52, 53, 54, 55, 56], this result cannot be taken dissatisfied quantum mechanics theory, because the uncertainty relations hold only for the conjugate canonical operators \( \hat{x}_j \) and \( \hat{p}_j \). Accordingly, the Lewis-Riesenfeld approach removes all the major objections related to this model.

As we can also remark, with this approach, the determination of the spectrum allowed the introduction of the nonstationary discrete eigenbasis. Thus, to convert this spectrum into nonstationary continuous spectrum, it is useful to introduce a continuous basis in which the diagonalization is possible. In this sense, the coherent states are the best candidates to achieve this purpose. In the literature, various coherent states [57, 58, 59] are constructed for different Lie algebra. To construct the appropriate coherent states for this system whose eigenfunction is expressed in terms of the generalized Laguerre functions as in [61, 62, 63, 64, 65, 66], we factorise this eigenfunction to find the hidden symmetry of the system through the establishment of an appropriate Lie algebra.

6 The hidden dynamical Lie algebra

We construct in this section the raising and lowering operators from the Hamiltonian’s eigenfunction which generate the hidden Lie algebra. Since the eigenfunctions of the invariant operator and the Hamiltonian are expressed in terms of the generalized Laguerre functions \( L_n^\ell(u) \) with \( \ell > 0 \). It is important to review some useful properties related to this special function that will be used to generate the symmetry operators. Thus, the generalized Laguerre polynomials \( L_n^\ell(u) \) are
defined as \[ L_n^\ell(u) = \frac{1}{n!} e^u u^{-\ell} \frac{d^n}{du^n} (e^{-u} u^n). \] For \( \ell = 0 \), \( L_n^0(u) = L_n(u) \) and for \( n = 0 \), \( L_0^\ell(u) = 1 \). The generating functions corresponding to associated Laguerre polynomials are
\[
e^u z (1 - z)^{-\ell} = \sum_{n=0}^{\infty} L_n^\ell(u) z^n, \quad |z| < 1, (97)
\]
\[
J_\ell (2\sqrt{uz}) e^{uz} (uz)^{-\frac{\ell}{2}} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + \ell + 1)} L_n^\ell(u), (98)
\]
where the \( J_\kappa(x) \) is the ordinary Bessel function of \( \kappa \)-order.

The orthogonality relation is
\[
\int_0^\infty du e^{-u} u^\ell L_n^\ell(u) L_m^\ell(u) = \frac{\Gamma(n + 1 + \ell)}{n!} \delta_{nm}. (100)
\]

The generalised Laguerre polynomials satisfy the following differential equation
\[
\left[ u \frac{d^2}{du^2} + (\ell - u + 1) \frac{d}{du} + n \right] L_n^\ell(u) = 0, (101)
\]
and the recurrence relations
\[
(n + 1) L_{n+1}^\ell(u) - (2n + \ell + 1 - u) L_n^\ell(u) + (n + \ell) L_{n-1}^\ell(u) = 0, (102)
\]
\[
u \frac{d}{du} L_n^\ell(u) - n L_n^\ell(u) + (n + \ell) L_{n-1}^\ell(u) = 0. (103)
\]

With respect to the equations, we rewrite the eigenfunction of the invariant operator in equation (38) in the form
\[
\phi_n^\ell(u) = N(\rho, \alpha) \sqrt{\frac{n!}{\Gamma(n + \ell + 1)}} u^\ell e^{-\bar{\omega} u} L_n^\ell(u), (104)
\]
where \( u = \frac{\rho}{\bar{\omega}} v^2 \), \( N(\rho, \alpha) = (-)^n \sqrt{\frac{\rho}{\pi\bar{\omega}}} e^{i\alpha}, \quad \bar{\omega} = 1 - imf^{-1} \frac{\partial}{\partial \rho} \) and \( \Gamma(n) = (n-1)! \).

Basing on the recurrence relations (102) and (103), we obtain the following equations
\[
\left( -\frac{d}{du} + \frac{\ell}{2} + n - \frac{\bar{\omega}}{2} u \right) \phi_n^\ell(u) = \sqrt{n(n+\ell)} \phi_{n-1}^\ell(u), (105)
\]
\[
\left( \frac{d}{du} + \frac{\ell}{2} + n - \frac{\bar{\omega}}{2} u + 1 \right) \phi_n^\ell(u) = \sqrt{(n+1)(n+\ell+1)} \phi_{n+1}^\ell(u), (106)
\]
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where $\tilde{\omega} = 2 - \varpi$. For the sake of simplicity we define the raising operator $K_+$ and the lowering operator $K_-$ acting on the wave function $\phi_n^\ell(u)$ as

\begin{align}
K_- &= \left(-u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\varpi}{2} u\right), \quad (107) \\
K_+ &= \left(u \frac{d}{du} + \frac{\ell}{2} + n - \frac{\tilde{\omega}}{2} u + 1\right), \quad (108)
\end{align}

and hence obtain

\begin{align}
K_- \phi_n^\ell(u) &= \sqrt{n(n+\ell)} \phi_{n-1}^\ell(u), \\ 
K_+ \phi_n^\ell(u) &= \sqrt{(n+1)(n+\ell+1)} \phi_{n+1}^\ell(u).
\end{align}

(109) \hspace{1cm} (110)

By multiplying both sides of the latter equations by the factor $e^{i\theta_n^\ell(t)}$ we obtain

\begin{align}
K_- \psi_n^\ell(u) &= \sqrt{n(n+\ell)} \psi_{n-1}^\ell(u), \\ 
K_+ \psi_n^\ell(u) &= \sqrt{(n+1)(n+\ell+1)} \psi_{n+1}^\ell(u).
\end{align}

(111) \hspace{1cm} (112)

By successively applying $K_+$ on the ground state $\psi_0^\ell(u)$, we generate the eigenfunction $\psi_n^\ell(u)$ of the system as follows

\begin{align}
\psi_n^\ell(u) &= \sqrt{\frac{\Gamma(1+\ell)}{n!\Gamma(n+\ell+1)}} (K_+)^n \psi_0^\ell(u), \\
&= \sqrt{\frac{(2n+\ell+1)}{(2\pi)^\frac{1}{2}} e^{-\frac{\varpi^2}{4}}} u^\ell e^{i\theta_n^\ell(t)}. \\
&= N(\rho, \alpha) \sqrt{\frac{\Gamma(\ell+1)}{\Gamma(1+\ell)}} u^\ell e^{-\frac{\varpi}{2} u} e^{i\theta_n^\ell(t)}, \\
K_- \psi_0^\ell(u) &= 0.
\end{align}

(113) \hspace{1cm} (114) \hspace{1cm} (115) \hspace{1cm} (116)

One can also observe that the following relations are satisfied

\begin{align}
K_+K_- \psi_n^\ell(u) &= n(n+\ell) \psi_n^\ell(u), \\ 
K_+K_- \psi_n^\ell(u) &= (n+1)(n+\ell+1) \psi_n^\ell(u).
\end{align}

(117) \hspace{1cm} (118)

Now, to establish the dynamical Lie algebra associated with the ladder operators $K_\pm$, we calculate the commutator

\begin{align}
[K_-, K_+] \psi_n^\ell(u) &= (2n+\ell+1) \psi_n^\ell(u).
\end{align}

(119)

As a consequence, we can introduce the operator $K_0$ defined to satisfy

\begin{align}
K_0 \psi_n^\ell(u) &= \frac{1}{2} (2n+\ell+1) \psi_n^\ell(u).
\end{align}

(120)
The operators $K_±$ and $K_0$ satisfy the following commutation relations

$$[K_-, K_+] = 2K_0, \quad [K_0, K_±] = ±K_±,$$

which can be recognized as commutation relation of the generators of a non-compact Lie algebra $su(1,1)$. The corresponding Casimir operator for any irreducible representation is the identity times a number

$$K^2 = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) = \frac{1}{4}(\ell + 1)(\ell - 1).$$

It satisfies

$$[K^2, K_±] = 0 = [K^2, K_0].$$

Thus, a representation of $su(1,1)$ algebra is determined by the single real positive number $\ell$, called the Bargmann index. Now, with the properties of the generators $K_±$ and $K_0$ of the $su(1,1)$ algebra, we are in the position to construct the corresponding coherent states to this system.

7 SU(1,1) coherent states

We investigate in this section the $SU(1,1)$ coherent states by adopting Barut-Girardello [30] and Perelomov [31] approaches. We examin for each approach the resolution of identity and overlapping properties.

7.1 Barut-Girardello coherent states

7.1.1 Construction

Following the Barut and Girardello approach [30], $SU(1,1)$ coherent states are defined to be the eigenstates of the lowering generator $K_-$

$$K_- |\psi^\ell_z\rangle = z |\psi^\ell_z\rangle,$$

where $z$ is an arbitrary complex number Based on the completeness of the wavefunction such that $\sum_{n=0}^{\infty} |\psi^\ell_n\rangle \langle \psi^\ell_n| = I$, on can represent the coherent states $|\psi^\ell_z\rangle$ as follows

$$|\psi^\ell_z\rangle = \sum_{n=0}^{\infty} \langle \psi^\ell_n|\psi^\ell_z\rangle |\psi^\ell_n\rangle.$$ 

(125)

Acting the operator $K_-$ on the equation (125) and then, using the equations (124) and (111) we have the following result

$$\langle \psi^\ell_n|\psi^\ell_z\rangle = \frac{z}{\sqrt{n(n+\ell)}}(\psi^\ell_n|\psi^\ell_{n-1}|\psi^\ell_z).$$

(126)
After the recurrence procedure, the formal equation becomes
\[ \langle \psi_\ell^n | \psi_\ell^z \rangle = z^n \sqrt{\frac{\Gamma(1 + \ell)}{n!\Gamma(n + \ell + 1)}} \langle \psi_0^\ell | \psi_\ell^z \rangle. \] (127)

Referring to [60], the Gamma function is linked to the modified Bessel function \( I_\mu(x) \) of order \( \mu \) through the relation
\[ \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\Gamma(n + \mu + 1)} = I_\mu(2x). \] (128)

Therefrom, by setting \( x = z \) and \( \mu = \ell \), we deduce the Barut-Girardello coherent states as follows
\[ |\psi_\ell^z\rangle = \sqrt{|z|^{\ell}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(n + \ell + 1)}} |\psi_\ell^n\rangle, \] (129)
\[ \psi_\ell^z(u) = \frac{|z|^\ell N(\rho, \alpha)}{\sqrt{I_\ell(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + \ell + 1)} u^{\frac{\ell}{2}} e^{-\frac{\ell}{2}u} I_n^\ell(u) e^{i\theta_{n,\ell}(t)}. \] (130)

However, in term of the generating function [69], the Barut-Girardello coherent states can be written as follows
\[ \psi_\ell^z(u) = \left( \frac{z}{|z|} \right)^{-\frac{\ell}{2}} \frac{N(\rho, \alpha)e^{\frac{\ell}{2}|z|^2/2} u^{\ell/2} I_n^\ell(2\sqrt{uz})}{\sqrt{I_\ell(2|z|)}} e^{i\theta_{n,\ell}(t)}. \] (131)

### 7.1.2 Properties

It is well-known that the states [129] are normalized but not orthogonal and satisfy the resolution of identity. Thus, we can see that the scalar product of two coherent states does not vanish
\[ \langle \psi_\ell^z_1 | \psi_\ell^z_2 \rangle = \frac{I_\ell(2\sqrt{z_1^2 z_2})}{\sqrt{I_\ell(2|z_1|)}|I_\ell(2|z_2|)}. \] (132)

The overcompleteness relation reads as follows
\[ \int d\mu(z, \ell) |\psi_\ell^z\rangle \langle \psi_\ell^z| = \sum_{n=0}^{\infty} |\psi_\ell^n\rangle \langle \psi_\ell^n| = 1, \] (133)
with the measure
\[ d\mu(z, \ell) = \frac{2}{\pi} K_\ell(2|z|) I_\ell(2|z|) d^2 z, \] (134)
where \( d^2z = d(Rez)d(Imz) \) and \( K_v(x) \) is the \( v \)-order modified Bessel function of the second kind.

For arbitrary state \(|\Phi\rangle = \sum_{n=0}^{\infty} c_n |\psi^\ell_n\rangle \) in the Hilbert space, one can construct the analytic function \( f(z) \) such as

\[
f(z) = \sqrt{\frac{I_\ell(2|z|)}{|z|^\ell}} \langle \psi^\ell_z |\Phi\rangle = \sum_{m=0}^{\infty} \frac{c_m}{\sqrt{m!\Gamma(m + \ell + 1)}} z^m.
\] (135)

On the Barut-Girardello coherent states (129) one can explicitly express the state \(|\Phi\rangle\) as follows

\[
|\Phi\rangle = \int d\nu(z,\ell) \left( \frac{z^*}{|z|^\ell} \right)^{\frac{\ell}{2}} \sqrt{\frac{I_\ell(2|z|)}{|z|^\ell}} f(z)|\psi^\ell_z\rangle,
\] (136)

and we have

\[
||\Phi||^2 = \int d\mu(z,\ell) \frac{|z|^\ell}{I_\ell(2|z|)} |f(z)|^2 < \infty.
\] (137)

### 7.2 Perelomov coherent states

#### 7.2.1 Construction

In analogy to canonical coherent states construction, Perelomov \( SU(1,1) \) coherent states \(|\psi^\ell_\eta\rangle\) are obtained by acting the displacement operator \( S(\xi) \) on the ground state \(|\psi^\ell_0\rangle\) [31]

\[
|\psi^\ell_\eta\rangle = S(\xi) |\psi^\ell_0\rangle,
\]

where \( \xi \in \mathbb{C} \), such as \( \xi = -\frac{\theta}{2} e^{-i\varphi} \), with \(-\infty < \theta < +\infty\) and \( 0 \leq \varphi \leq 2\pi\).

Using Baker-Campbell-Haussdorf relation, we explicit the displacement operator as follows [68]

\[
S(\xi) = \exp(\xi K_+) \exp(\zeta K_0) \exp(-\xi^* K_-),
\] (139)

where \( \eta = -\tanh(\frac{\theta}{2}) e^{-i\varphi} \) and \( \zeta = -2 \ln \cosh |\xi| = \ln(1 - |\eta|^2) \). By using this normal form of the displacement operator (139), the standard Perelomov \( SU(1,1) \) coherent states are found to be

\[
|\psi^\ell_\eta\rangle = (1 - |\eta|^2)^{\ell+1} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n + \ell + 1)}{n!\Gamma(\ell + 1)}} \eta^n |\psi^\ell_n\rangle,
\] (140)

\[
\psi^\ell_\eta(u) = N(\rho,\alpha) \frac{(1 - |\eta|^2)^{\ell+1}}{\sqrt{\Gamma(\ell + 1)}} u^{\frac{\ell}{2}} e^{-\frac{\rho}{2} u} \sum_{n=0}^{\infty} \eta^n L^\ell_n(u) e^{i\theta_n,\ell(t)}.
\] (141)
In term of the generating function \( \langle 98 \rangle \), the Perelomov coherent states can be written as follows
\[
\psi^\ell_{\eta}(u) = N(\rho, \alpha) \frac{(1 - |\eta|^2)^{\ell+1}}{\sqrt{\Gamma(\ell + 1)}} u^\ell e^{-\frac{2}{\pi} \eta^2} \frac{e^{\frac{2\pi}{\eta} \cdot \sqrt{\eta \cdot (1 - \eta)}}}{\Gamma(\ell+1)}} e^{i\eta \cdot \ell(t)} , \quad \text{(142)}
\]

### 7.2.2 Properties

The Perelomov \( SU(1, 1) \) coherent states as the Barut-Girardello coherent states are normalized states but not orthogonal
\[
\langle \psi^\ell_{\eta_1} | \psi^\ell_{\eta_2} \rangle = \left[ (1 - |\eta_1|^2)(1 - |\eta_2|^2) \right]^{\ell+1/2} \left( 1 - \eta_1 \eta_2^* \right)^{-\ell-1}, \quad \text{(143)}
\]
and satisfy the completeness relation
\[
\int |\psi^\ell_{\eta}| \langle \psi^\ell_{\eta} | d\mu(\eta, \ell) = \sum_{n=0}^{\infty} |\psi^\ell_n| \langle \psi^\ell_n | = I, \quad \text{(144)}
\]
where the measure \( d\mu(\eta, \ell) = \frac{d\eta}{\pi (1 - |\eta|^2)^2} \).

As we noted for the Barut-Girardello coherent states, for any \( |\Psi\rangle = \sum_{n=0}^{\infty} c_n |\psi^\ell_n\rangle \) in the Hilbert space, one can construct an analytic function
\[
f(\eta) = (1 - |\eta|^2)^{-\ell-1} \langle \psi^\ell_{\eta} | \Psi \rangle = \sum_{n=0}^{\infty} c_n \sqrt{\frac{\Gamma(n + \ell + 1)}{n! \Gamma(\ell + 1)}} (\eta^*)^n. \quad \text{(145)}
\]
The expansion of \( |\Psi\rangle \) on the basis of coherent states \( \langle 140 \rangle \) can be written as
\[
|\Psi\rangle = \int d\mu(\eta, \ell)(1 - |\eta|^2)^{\ell+1} f(\eta) |\psi^\ell_{\eta}\rangle, \quad \text{(146)}
\]
\[
||\Psi||^2 = \int d\mu(\eta, \ell)(1 - |\eta|^2)^{\ell+1} |f(\eta)|^2 < \infty. \quad \text{(147)}
\]

### 8 Conclusion

In this paper we have investigated, the system of a nonrelativistic particle of mass \( m \) with time-dependent harmonic frequency \( \omega(t) \) in rotational symmetric in the plane under the influence of a time-dependent friction force. At the classical level we solved the equations of motion which describe three particular physical systems. At the quantum level, we used the Lewis-Riesenfeld’s method to construct the spectra of the invariant operator \( \hat{I}(t) \) and the Hamiltonian \( \hat{H}(t) \) on the helicity-like basis \( |\phi_{n\pm}(t)\rangle \). The configuration space wave functions of both operators are expressed in terms of the generalised Laguerre polynomials. This system
previously introduced in the one dimensional case [26, 27] as the generalization of the Kanai Hamiltonian [46] has been criticized for violating certain laws of quantum theory. Nevertheless, as many approaches of solution have been given to raise these controversies [52, 53, 54, 55, 56], we used the invariant method of Lewis-Riesenfeld to confirm the preservation of those laws by investigating the validity of the Heisenberg uncertainty relations and the expectation values of mechanical energy.

This model generalizes not only the 1D damped systems studied in the literature [26, 27] but also improves the technique of quantization of those model achieved in the framework of Lewis-Riesenfeld method [23, 24, 25, 26, 27]. By analogy with the work of Perdosa [27] who constructed the canonical coherent states for the 1 D case of this system, we constructed the system of SU(1, 1) coherent states based on the eigenfunction of the Hamiltonian. For these states the resolution of identity and some properties are examined. Referring to the original paper of Lewis-Riesenfeld [1], it would be also good to determine the transition amplitude connecting any initial state in the remote past to any final state in the remote future in the case of a constant frequency, we hope to report these aspects elsewhere.

Appendix

In this appendix we explicitly develop some intermediary calculations which allowed us to determine the expressions of operators $\hat{I}(t), \hat{L}_z, \hat{H}(t)$ of section 4 and the Heisenberg uncertainty relations of section 5.

\begin{equation}
\hat{a}_j = \hat{U}^\dagger a_j^\dagger \hat{U} = \frac{1}{\sqrt{2\nu}} \left( mf^{-1} \hat{\rho} \hat{x}_j - \rho \hat{p}_j + i \frac{\nu}{\rho} \hat{x}_j \right), \quad (148)
\end{equation}

\begin{equation}
\hat{a}_j^\dagger = \hat{U}^\dagger a_j^\dagger \hat{U} = \frac{1}{\sqrt{2\nu}} \left( mf^{-1} \hat{\rho} \hat{x}_j - \rho \hat{p}_j - i \frac{\nu}{\rho} \hat{x}_j \right). \quad (149)
\end{equation}

with $j = 1, 2$. Conversely,

\begin{equation}
\hat{x}_j = \sqrt{\nu} \left( a_j^\dagger - a_j \right), \quad \hat{p}_j = \sqrt{\nu} f^{-1} \left( a_j^\dagger + a_j \right) - \frac{\sqrt{\nu}}{2\rho} \left( a_j^\dagger + a_j \right). \quad (150)
\end{equation}

The helicity Fock algebra generators in terms of generators $a_j$ and $a_j^\dagger$ are given as follows

\begin{equation}
a_{\pm} = \frac{1}{\sqrt{2}} (a_1 \pm ia_2), \quad a_{\pm}^\dagger = \frac{1}{\sqrt{2}} \left( a_{1}^\dagger \mp ia_{2}^\dagger \right). \quad (151)
\end{equation}

The inverse relations are,

\begin{equation}
a_1 = \sqrt{2} (a_+ + a_-), \quad a_1^\dagger = \sqrt{2} \left( a_+^\dagger + a_-^\dagger \right),
\end{equation}

\begin{equation}
a_2 = \sqrt{2} (a_+ - a_-), \quad a_2^\dagger = \sqrt{2} \left( a_+^\dagger - a_-^\dagger \right),
\end{equation}

\begin{equation}
a_{\mp} = \frac{1}{\sqrt{2}} (a_1 \mp ia_2), \quad a_{\mp}^\dagger = \frac{1}{\sqrt{2}} \left( a_{1}^\dagger \pm ia_{2}^\dagger \right).
\end{equation}
\[ a_2 = -\frac{i}{\sqrt{2}} (a_+ - a_-), \quad a_2^\dagger = \frac{i}{\sqrt{2}} (a_+^\dagger - a_-^\dagger). \] (152)

In terms of helicity generators, the phase space operators read

\[ \hat{x}_1 = -\frac{i\rho}{2\sqrt{\nu}} \left( a_- - a_+^\dagger + a_+ - a_-^\dagger \right), \] (153)

\[ \hat{p}_1 = -\frac{imf^{-1}\dot{\rho}}{2\sqrt{\nu}} \left( a_- - a_+^\dagger + a_+ - a_-^\dagger \right) - \frac{\sqrt{\nu}}{2\rho} \left( a_- + a_+^\dagger + a_+ - a_-^\dagger \right), \] (154)

\[ \hat{x}_2 = \frac{\rho}{2\sqrt{\nu}} \left( a_- - a_+^\dagger - a_+ + a_-^\dagger \right), \] (155)

\[ \hat{p}_2 = \frac{mf^{-1}\dot{\rho}}{2\sqrt{\nu}} \left( a_- - a_+^\dagger - a_+ + a_-^\dagger \right) - i\frac{\sqrt{\nu}}{\rho} \left( a_- + a_+^\dagger - a_+ - a_-^\dagger \right). \] (156)

In particular

\[ \dot{x}_1 - i\dot{x}_2 = \frac{i\rho}{\sqrt{\nu}} \left( a_+^\dagger - a_- \right), \quad \dot{x}_1 + i\dot{x}_2 = \frac{i\rho}{\sqrt{\nu}} \left( a_- - a_+ \right), \] (157)

\[ \dot{p}_1 + i\dot{p}_2 = \frac{imf^{-1}\dot{\rho}}{\sqrt{\nu}} \left( a_+^\dagger - a_+ \right) - \frac{\sqrt{\nu}}{\rho} \left( a_-^\dagger + a_+ \right), \] (158)

\[ \dot{p}_1 - i\dot{p}_2 = \frac{imf^{-1}\dot{\rho}}{\sqrt{\nu}} \left( a_+^\dagger - a_- \right) - \frac{\sqrt{\nu}}{\rho} \left( a_+^\dagger + a_- \right). \] (159)

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