ON THE EINSTEIN CONDITION FOR LORENTZIAN 3-MANIFOLDS

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Abstract. It is well known that in Lorentzian geometry there are no compact spherical space forms; in dimension 3, this means there are no closed Einstein 3-manifolds with positive Einstein constant. We generalize this fact here, by proving that there are also no closed Lorentzian 3-manifolds \((M, g)\) whose Ricci tensor satisfies
\[
\text{Ric} = fg + (f - \lambda)T^\flat \otimes T^\flat,
\]
for any unit timelike vector field \(T\), any positive constant \(\lambda\), and any function \(f\) that never takes the values 0, \(\lambda\). (Observe that this reduces to the positive Einstein case when \(f = \lambda\).) We show that there is no such obstruction if \(\lambda\) is negative. Finally, the “borderline” case \(\lambda = 0\) is also examined: we show that if \(\lambda = 0\), then \((M, g)\) must be isometric to \((S^1 \times N, -dt^2 \oplus h)\) with \((N, h)\) a Riemannian manifold.

1. Introduction

The goal of this paper is to generalize the nonexistence of positive Lorentzian Einstein metrics in dimension 3, by proving the following

Theorem. Let \(M\) be a closed 3-manifold, \(\lambda > 0\) a constant, and \(f\) a smooth function that never takes the values 0, \(\lambda\). Then there is no Lorentzian metric \(g\) on \(M\) whose Ricci tensor satisfies
\[
\text{Ric} = fg + (f - \lambda)T^\flat \otimes T^\flat,
\]
for any unit timelike vector field \(T\). If (1) holds with \(\lambda = 0\), then \((M, g)\) is isometric to \((S^1 \times N, -dt^2 \oplus h)\) with \((N, h)\) a Riemannian manifold.

The following remarks help to set this result in context:

1. Although our Theorem does not cover the case \(f = \lambda\), nevertheless this would reduce (1) to the Einstein condition \(\text{Ric} = \lambda g\) with positive Einstein constant. In dimension 3, this is equivalent to \((M, g)\) having constant positive sectional curvature—and it is well known that there are no such (closed) Lorentzian manifolds. (Its proof came in two stages: [CM62] showed that there are no geodesically complete such manifolds, in any dimension; [Kli96] then showed that every closed constant curvature Lorentzian manifold must be geodesically complete, with the flat case having already been established in [Car89]. See [Lun15] for a comprehensive account.) It is in this sense that (1) generalizes the nonexistence of positive Einstein metrics; but more than that, it also shows
that certain so-called “quasi-Einstein” metrics are also impossible. Recall that a (semi-Riemannian) metric $g$ is quasi-Einstein if there is a vector field $X$, a constant $\mu$, and a constant $m > 0$ such that
\[
\text{Ric} = \mu g - \frac{1}{2} \mathcal{L}_X g + \frac{1}{m} X^\flat \otimes X^\flat; \tag{2}
\]
see, e.g., [Lim10]. Now observe that if $X$ is a unit timelike Killing vector field (i.e., $\mathcal{L}_X g = 0$), then (1) provides global obstructions to the existence of certain quasi-Einstein metrics (2).

2. The condition imposed on $\lambda$ deserves commentary. When $\lambda > 0$, then the endomorphism of the normal bundle $T^\perp$ given by (5) below will have, presuming (1) holds, real eigenvalues, and this leads directly to the existence of a distinguished frame on $(M, g)$ that is crucial to our proof. This will not be the case if $\lambda < 0$ — nor should this be surprising: there are in fact many examples of Lorentzian Einstein metrics with negative Einstein constant (see, e.g., [KR85] and [Gol85]), so that a result like (1) should not be expected when $\lambda < 0$. Indeed, as we show in Section 2 below, the Lorentzian metric $g_L$ on $S^3$ defined via the standard (round) Riemannian metric $\hat{g}$ by
\[
g_L := \hat{g} - 2\hat{g}(T, \cdot) \otimes \hat{g}(T, \cdot), \tag{3}
\]
where $T$ is the Hopf Killing vector field, satisfies (1) with $f = 8, \lambda = -2$:
\[
\text{Ric}_L = 8g_L + (8 - (2)) T^\flat \otimes T^\flat. \tag{4}
\]
(Here $T^\flat = g_L(T, \cdot) = -\hat{g}(T, \cdot)$.) Finally, the nowhere vanishing condition on $f$ arises for the following reason: our Theorem is obtained by showing that if (1) holds, then there necessarily exists a vector field $Z$ on $M$ and a smooth function $\psi$ such that $Z(\psi) = f$, which is impossible on a compact manifold when $f$ is nowhere vanishing — or else has just one zero.

3. As all compact 3-manifolds have Euler characteristic zero, there is no topological obstruction to $M$ admitting a Lorentzian metric (see, e.g., [O’N83]). Furthermore, (3) is but one instance of the following fact: every Lorentzian metric $g_L$ with a unit timelike vector field $T$ is necessarily of the form
\[
g = g_R - 2g_R(T, \cdot) \otimes g_R(T, \cdot)
\]
for some Riemannian metric $g_R$ on $M$ with $g_R(T, T) = 1$. In other words, there are many candidates for $g$ and $T$ on any closed 3-manifold $M$; nevertheless, by our Theorem, they will all fail to satisfy (1).

4. Our proof uses the three-dimensional version of the Newman-Penrose formalism [NP62], which we outline in Section 2. This is a frame technique which has by now appeared in many guises in dimensions 3 and 4, both Lorentzian and Riemannian; see, e.g., [HMP87, SW14, NTC15, AMT19, BS18]. Indeed, our proof, which appears in Section 4, is a Lorentzian conversion of two Riemannian results in [AMT19]; Section 4 will make their precise relationship clear. The principal virtue of the
Newman-Penrose formalism is that it converts second-order differential equations involving curvature into first-order differential equations involving properties of the flow of a privileged vector field, in our case the vector field \( T \) in (1). Doing so can, in certain settings, simplify the analysis considerably — our Theorem, it turns out, is precisely such a setting.

5. Observe that if (1) were to hold, then \( \lambda, f, f \) would be the eigenvalues of the Ricci operator \( \hat{\text{Ric}} : TM \rightarrow TM \), the smooth bundle endomorphism whose action \( X \mapsto \hat{\text{Ric}}(X) \) is the unique vector satisfying
\[
g(\hat{\text{Ric}}(X), Y) = \text{Ric}(X, Y) \quad \text{for all } Y \in TM.
\]

Much work has been done in locally classifying Lorentzian 3-manifolds (i.e., on \( \mathbb{R}^3 \)) according to the possible eigenvalues of \( \hat{\text{Ric}} \), in particular when these eigenvalues are constants; see [Bue97a, Bue97b, CK09]. Although we make no use of \( \hat{\text{Ric}} \) explicitly — in the Lorentzian setting, \( \hat{\text{Ric}} \) is not, in general, diagonalizable — nevertheless it provides another way to appreciate the generalization achieved in our Theorem, since the Einstein case \( \text{Ric} = \lambda g \) corresponds to the eigenvalues \( \lambda, \lambda, \lambda \).

2. **The Newman-Penrose formalism for Lorentzian 3-manifolds**

Let \( T \) be a smooth unit length timelike vector field defined in an open subset of a Lorentzian 3-manifold \( (M, g) \) without boundary, so that \( \nabla_v T \perp T \) for all vectors \( v \) (\( \nabla \) is the Levi-Civita connection, and we adopt the index \( -++ \) for the Lorentzian metric \( g \)). Let \( X \) and \( Y \) be two smooth spacelike vector fields such that \( \{T, X, Y\} \) is a local orthonormal frame, where by “timelike” and “spacelike” we mean simply that
\[
g(T, T) = -1 \quad , \quad g(X, X) = g(Y, Y) = 1.
\]

For such a \( T \), there is an endomorphism \( D \) defined on the normal bundle \( T^\perp \subset TM \):
\[
D : T^\perp \rightarrow T^\perp \quad , \quad v \mapsto \nabla_v T.
\]
(5)

As is well known, the matrix of \( D \) with respect to the frame \( \{T, X, Y\} \),
\[
D = \begin{pmatrix}
g(\nabla_X T, X) & g(\nabla_Y T, X) \\
g(\nabla_X T, Y) & g(\nabla_Y T, Y)
\end{pmatrix}
\]
carries three crucial pieces of information associated to the flow of \( T \):

1. The divergence of \( T \), denoted \( \text{div} T \), is the trace of \( D \).
2. By Frobenius’s theorem, \( T^\perp \) is integrable if and only if the off-diagonal elements of (8) satisfy
\[
\omega := g(T, [X, Y]) = g(\nabla_Y T, X) - g(\nabla_X T, Y) = 0.
\]
(6)

Since \( \omega^2 \) equals the determinant of the anti-symmetric part of \( D \) (see (8) below), \( \omega \) is invariant up to sign, called the twist of \( T \). We say that the flow of \( T \) is twist-free if \( \omega = 0 \).
3. The third piece of information is the shear $\sigma$ of $T$; it is given by the trace-free symmetric part of $D$, whose components $\sigma_1, \sigma_2$ we combine here into a complex-valued quantity, for reasons that will become clear below:

$$\sigma := \frac{1}{2} g(\nabla_Y T, Y) - g(\nabla_X T, X) \left/ \sigma_1 \right. + i \frac{1}{2} g(\nabla_Y T, X) + g(\nabla_X T, Y) \left/ \sigma_2 \right..$$

(7)

Although $\sigma$ itself is not invariant, its magnitude $|\sigma|^2$ is: by (8) below, it is minus the determinant of the trace-free symmetric of $D$. We say that the flow of $T$ is shear-free if $\sigma = 0$. As with being twist-free, being shear-free is a frame-independent statement. In terms of $\text{div} T$, $\omega$, and $\sigma$, $D$ is

$$D = \left( \frac{1}{2} \text{div} T - \sigma_1, \sigma_2 + \frac{\omega}{2} \right) \left/ \frac{1}{2} \text{div} T + \sigma_1 \right..$$

(8)

Note that (5) has been applied to arbitrary $n$-dimensional Lorentzian manifolds, yielding integral inequalities via Bochner’s technique, in particular when $D$ is skew-symmetric, or (in dimension 3) when the vector field of interest is spacelike, in [RS96] and [RS98].

We now present the Newman-Penrose formalism for Lorentzian 3-manifolds, which is well known; see [HMP87], and more recently, [NTC15]. Our presentation here parallels the three-dimensional Riemannian treatment to be found in [AMT19], and is meant to fix notation; in what follows, any sign changes that arise due to the Lorentzian index, as compared to the Riemannian case in [AMT19], are indicated by red text. Let $\{T, X, Y\}$ be as above and define the complex-valued quantities

$$m := \frac{1}{\sqrt{2}} (X - iY), \quad \overline{m} := \frac{1}{\sqrt{2}} (X + iY).$$

Henceforth we work with the complex frame $\{T, m, \overline{m}\}$, for which only $g(T, T) = -1$ and $g(m, \overline{m}) = 1$ are nonzero. The following quantities associated to this complex triad play a central role in all that follows.

**Definition.** The spin coefficients of the complex frame $\{T, m, \overline{m}\}$ are the complex-valued functions

$$\kappa = -g(\nabla_T T, m), \quad \rho = -g(\nabla_m T, m), \quad \sigma = -g(\nabla_m T, m), \quad \varepsilon = g(\nabla_T m, \overline{m}), \quad \beta = g(\nabla_m m, \overline{m}).$$

Note that the flow of $T$ is geodesic, $\nabla_T T = 0$, if and only if $\kappa = 0$; that $\varepsilon$ is purely imaginary; that $\sigma$ is actually the complex shear (7); and finally that the spin coefficient $\rho$ is given by

$$-2\rho = \text{div} T + i \omega.$$  

(9)
It is clear that $\kappa, \rho, \sigma$ directly represent the three geometric properties of the flow of $T$ discussed above. In terms of the five spin coefficients in Definition 2, the covariant derivatives of $\{T, m, \overline{m}\}$ are given by

\[
\begin{align*}
\nabla_T T &= -\kappa m - \kappa \overline{m}, \\
\nabla_m T &= -\rho m - \sigma \overline{m}, \\
\nabla_T m &= -\kappa T + \epsilon m, \\
\nabla_m m &= -\sigma T + \beta m, \\
\nabla_m \overline{m} &= -\bar{\rho} T - \bar{\beta} \overline{m},
\end{align*}
\]

while their Lie brackets are

\[
[T, m] = -\kappa T + (\epsilon + \bar{\rho}) m + \sigma \overline{m}, \\
[m, \overline{m}] = - (\bar{\rho} - \rho) T + \bar{\beta} m - \beta \overline{m}.
\]

All other covariant derivatives, as well as the Lie bracket $[T, \overline{m}]$, are obtained by complex conjugation. Now onto curvature; begin by observing that the Riemann and Ricci tensors

\[
\begin{align*}
R(u, v, w, z) &= g(\nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w, z), \\
\Ric(\cdot, \cdot) &= -R(T, \cdot, \cdot, T) + R(m, \cdot, \cdot, \overline{m}) + R(m, \cdot, \cdot, m),
\end{align*}
\]

satisfy the following relationships in the complex frame $\{T, m, \overline{m}\}$:

\[
\begin{align*}
\Ric(m, m) &= +R(T, m, T, m), \\
\Ric(T, T) &= -2R(T, m, T, \overline{m}), \\
\Ric(T, m) &= -R(T, m, m, \overline{m}), \\
\Ric(m, \overline{m}) &= -\frac{1}{2} \Ric(T, T) - R(\overline{m}, m, m, \overline{m}).
\end{align*}
\]

Using these, and expressing the Riemann tensor as

\[
R(u, v, w, z) = u g(\nabla_u w, z) - g(\nabla_v w, \nabla_u z) - v g(\nabla_u w, z) + g(\nabla_u w, \nabla_v z) - g(\nabla_{[u,v]} w, z),
\]

leads to the following equations; along with (14) and (15) below, play the crucial role in the Newman-Penrose formalism:

\[
\begin{align*}
T(\rho) - \overline{m}(\kappa) &= -|\kappa|^2 + |\sigma|^2 + \rho^2 + \kappa \bar{\beta} + \frac{1}{2} \Ric(T, T), \\
T(\sigma) - m(\kappa) &= -\kappa^2 + 2\sigma \epsilon + \sigma (\rho + \bar{\rho}) - \kappa \beta - \Ric(m, m), \\
m(\rho) - \overline{m}(\sigma) &= 2\sigma \bar{\beta} - (\bar{\rho} - \rho) \kappa + \Ric(T, m), \\
T(\beta) - m(\epsilon) &= \sigma (\kappa - \bar{\beta}) + \kappa (-\epsilon - \bar{\rho}) + \beta (\epsilon + \bar{\rho}) - \Ric(T, m), \\
m(\bar{\beta}) + \overline{m}(\beta) &= -|\sigma|^2 + |\rho|^2 - 2|\beta|^2 - (\rho - \bar{\rho}) \epsilon - \Ric(m, \overline{m}) - \frac{1}{2} \Ric(T, T).
\end{align*}
\]
Finally, there are two nontrivial differential Bianchi identities,

$$\nabla_T R(T, m, m, \overline{m}) + (\nabla_m R)(T, m, \overline{m}, T) + (\nabla_{\overline{m}} R)(T, m, T, m) = 0,$$

$$\nabla_T R(\overline{m}, m, m, \overline{m}) + (\nabla_m R)(\overline{m}, m, \overline{m}, T) + (\nabla_{\overline{m}} R)(\overline{m}, m, T, m) = 0,$$

which, in terms of spin coefficients, take the following forms:

$$T(Ric(T, m)) - \frac{1}{2} m(Ric(T, T)) - \overline{m}(Ric(m, m)) =$$

$$- \kappa (+ Ric(T, T) + Ric(m, \overline{m})) + (\epsilon + 2\rho + \overline{\rho}) Ric(T, m)$$

$$+ \sigma Ric(T, \overline{m}) - (\overline{\kappa} - 2\overline{\beta}) Ric(m, m)$$

and

$$m(Ric(T, \overline{m})) + \overline{m}(Ric(T, m)) - T(Ric(m, \overline{m}) + (1/2)Ric(T, T)) =$$

$$- (\rho + \overline{\rho})(Ric(T, T) + Ric(m, m)) - \overline{\sigma}Ric(m, m) - \sigma Ric(\overline{m}, \overline{m})$$

$$- (-2\overline{\kappa} + \overline{\beta}) Ric(T, m) - (-2\kappa + \beta) Ric(T, \overline{m}).$$

E.g., to derive (15), expand the second differential Bianchi identity, beginning with its first term:

$$\nabla_T R(\overline{m}, m, m, \overline{m}) = T(R(\overline{m}, m, m, \overline{m})) - R(\nabla_T \overline{m}, m, m, \overline{m})$$

$$- R(\overline{m}, \nabla_T m, m, \overline{m}) - R(\overline{m}, m, \nabla_T \overline{m}, \overline{m}) - R(\overline{m}, m, m, \nabla_T \overline{m}).$$

In terms of spin coefficients and the Ricci tensor, each term is

$$T(R(\overline{m}, m, m, \overline{m})) = -T(Ric(m, \overline{m}) + \frac{1}{2} Ric(T, T)),$$

$$R(\nabla_T \overline{m}, m, m, \overline{m}) = -\kappa \left( R(T, m, m, \overline{m}) + \overline{\epsilon} R(\overline{m}, m, m, \overline{m}) \right),$$

$$-Ric(T, m) - \frac{1}{2} Ric(T, T),$$

$$-\overline{\kappa} R(\overline{m}, m, m, \overline{m}) + \overline{\epsilon} R(\overline{m}, m, m, \overline{m})$$

$$= R(\overline{m}, m, m, \nabla_T \overline{m}),$$

$$R(\overline{m}, \nabla_T m, m, \overline{m}) = -\kappa \left( R(\overline{m}, T, m, \overline{m}) + \epsilon R(\overline{m}, m, m, \overline{m}) \right),$$

$$-Ric(T, \overline{m}) - \frac{1}{2} Ric(T, T),$$

$$= R(\overline{m}, m, \nabla_T m, \overline{m}),$$

Thus the term $$\nabla_T R(\overline{m}, m, m, \overline{m})$$ simplifies to

$$-T(Ric(m, \overline{m}) + (1/2)Ric(T, T)) - 2\kappa Ric(T, \overline{m}) - 2\overline{\kappa} Ric(T, m),$$

where two terms cancel because $$\epsilon + \overline{\epsilon} = 0$$. Repeating this process on the remaining terms in the second differential Bianchi identity yields (15); the first differential Bianchi identity (14) is similarly derived. This concludes the derivation of the Newman-Penrose formalism for Lorentzian 3-manifolds. As a first, minor application, let us verify that
Lemma. On $S^3$, the Lorentzian metric $g_L$ given by (3) satisfies (4).

Proof. Let $\hat{g}$ denote the standard (round) metric on $S^3$. Then on $\mathbb{R}^4 = \{(x^1, y^1, x^2, y^2)\}$, the restriction to $(S^3, \hat{g})$ of the vector field

$$T = \sum_{i=1}^{2} (-y^i \partial_{x^i} + x^i \partial_{y^i})$$

is a unit-length Killing vector field, tangent to the Hopf fibration. With respect to the Lorentzian metric $g_L$ in (3),

$$g_L := \hat{g} - 2\hat{g}(T, \cdot) \otimes \hat{g}(T, \cdot),$$

(16)

the Koszul formula shows that $T$ remains a Killing vector field of unit length, but now a timelike one: $g_L(T, T) = -1$. As a consequence, and using the Koszul formula again, any local $g_L$-orthonormal frame $\{T, X, Y\}$ will have covariant derivatives

$$\nabla_L^T X = -\nabla^o X, \quad \nabla_L^Y T = -\nabla^o Y, \quad \nabla_L^T X = \nabla_L^T Y = 2\nabla^o T, \quad \nabla^o b = \nabla^o b$$

for all other $a, b \in \{T, X, Y\}$, where $\nabla^L$ is the Levi-Civita connection of $g_L$ and $\nabla^o$ that of $\hat{g}$ (for more general formulae for Lorentzian metrics of the form (16), consult [Ole14]). It now remains to compute the Ricci tensor, but in fact (10)-(12) will simplify the computation. This is because any unit timelike Killing vector field $T$ on a Lorentzian 3-manifold will satisfy

$$\kappa = \rho + \bar{\rho} = \sigma = 0$$

with respect to any complex frame $\{T, m, \bar{m}\}$ (in fact in dimension 3, such a vector field is completely determined by these equations). Inserting these into (10), its real part simplifies to

$$\text{Ric}_L(T, T) = \frac{\omega^2}{2},$$

(17)

where the twist $\omega^2_L$ satisfies

$$\omega^2 = \frac{1}{6} g_L([X, Y])^2 = \frac{16}{16} \hat{g}(T, [X, Y])^2 = \omega^2 = 4.$$ (That $\omega^2 = 4$ follows from the Einstein condition $\text{Ric}_{\hat{g}} = 2\hat{g}$, together with the Riemannian version of (10), obtained from (10) by changing each (red) “−” to “+”; doing so leads to $\text{Ric}_{\hat{g}}(T, T) = \frac{\omega^2}{2}$ as in (17).) Next, (11) and (12) simplify, respectively, to

$$\text{Ric}_L(m, m) = 0, \quad \text{Ric}_L(T, m) = 0,$$

where in the latter equality we’ve used the fact that $m(\rho) = 0$ because $\omega^2_L = 4$ is a constant. All in all, we thus have that

$$\text{Ric}_L(T, T) = 2, \quad \text{Ric}_L(X, X) = \text{Ric}_L(Y, Y),$$
and $\text{Ric}_L(X,Y) = \text{Ric}_L(T,X) = \text{Ric}_L(T,Y) = 0$. The final step in the proof is to show that $\text{Ric}_L(X,X) = \text{Ric}_L(X,X)$ is in fact a global constant, which indeed it is, after one expands

$$\text{Ric}_L(X,X) = -\frac{1}{7} R_L(T,X,X,T) + R_L(Y,X,X,Y) = 8,$$

where the computations of the curvature tensor components are straightforwardly carried out, using both the Einstein condition $\text{Ric}_\bar{g} = 2\bar{g}$ and the fact that $T$ is a unit Killing vector field in both $(S^3, \bar{g})$ and $(S^3, g_L)$. That $g_L$ satisfies (4) now follows. 

\[ \square \]

3. Evolution equations for divergence, twist, and shear

We first need to gather some information regarding the flow of $T$; the first-order differential equations appearing here are, essentially, “what the Newman-Penrose formalism is good for.”

**Proposition 1.** Let $(M,g)$ be a Lorentzian 3-manifold whose Ricci tensor satisfies

$$\text{Ric} = fg + (f - \mu)T^b \otimes T^b,$$

for some unit timelike vector field $T$, constant $\mu$, and smooth function $f$ which never takes the value $\mu$. Then $T$ has geodesic flow, and its divergence, twist, and shear satisfy the following differential equations:

$$T(\text{div} T) = \frac{\omega^2}{2} - 2|\sigma|^2 - \frac{1}{2}(\text{div} T)^2 + \mu, \quad (18)$$

$$T(\omega^2) = -2(\text{div} T) \omega^2, \quad (19)$$

$$T(|\sigma|^2) = -2(\text{div} T) |\sigma|^2. \quad (20)$$

Furthermore, $f$ satisfies

$$T(f - \mu) = -(\text{div} T)(f - \mu), \quad (21)$$

and, recalling (8), the function $H := \det D - \frac{\mu^2}{2}$ satisfies

$$T(\text{div} T) = 2H - (\text{div} T)^2 + 2\mu, \quad (22)$$

$$T(H) = -(\text{div} T)H. \quad (23)$$

**Proof.** Let $\{T,X,Y\}$ be an orthonormal frame, with $X,Y$ possibly only locally defined, and let $\{T,m,\overline{m}\}$ be the corresponding complex frame (recall that $\text{div} T, \omega^2, \text{and} |\sigma|^2$ are globally defined, frame-independent quantities). Then the Ricci tensor in this complex frame satisfies

$$\text{Ric}(T,T) = -\mu, \quad \text{Ric}(m,\overline{m}) = f, \quad \frac{1}{2}(\text{Ric}(X,X) + \text{Ric}(Y,Y))$$
with all other components vanishing. That $T$ has geodesic flow, $\nabla_T T = 0$, now follows from the differential Bianchi identity (14), which reduces to

$$\kappa \left( \text{Ric}(m, m) + \text{Ric}(T, T) \right) = 0 \quad \Rightarrow \quad \kappa = 0.$$  

Since $\kappa = 0$ if and only if $\nabla_T T = 0$, this proves the geodesic flow of $T$. Next, (18) and (19) are the real and imaginary parts of (10), respectively, after setting $\kappa = 0$ therein. With $\kappa = \text{Ric}(m, m) = 0$, (11) also simplifies, to $T(\sigma) = 2\sigma \varepsilon + \sigma(\rho + \bar{\rho})$, from which (20) follows because $|\sigma|^2 = \sigma \bar{\sigma}$ and $\varepsilon + \bar{\varepsilon} = 0$. The second differential Bianchi identity (15) yields

$$-T(f - \mu/2) = (\text{div} T)(-\mu + f),$$

which is (21), since $T(f - \mu/2) = T(f - \mu)$. Finally, as

$$\det D = \frac{\omega^2}{4} - |\sigma|^2 + \frac{(\text{div} T)^2}{4},$$

(22) and (23) both follow from (18)-(20).

An immediate consequence of these evolution equations is the following

**Corollary.** Assume the hypotheses of Proposition 1. If $\mu \geq 0$ and $M$ is closed, then $T$ is also divergence-free.

**Proof.** By Proposition 1, $T$ has geodesic flow; because $M$ is closed, this flow is complete. We now consider the cases $\mu > 0$ and $\mu = 0$ separately:

1. $\mu > 0$: To show that $\text{div} T = 0$, we will use (23), by showing that $H$ is in fact a nonzero constant on $M$. Indeed, suppose that $H(p) = 0$ at some point $p \in M$, and let $\gamma^{(p)}(t)$ be the (complete) integral curve of $T$ starting at $p$. By (23), the function $H \circ \gamma^{(p)} : \mathbb{R} \to \mathbb{R}$ is identically zero; by (22), $\theta(t) := (\text{div} T \circ \gamma^{(p)})(t)$ satisfies

$$\frac{d\theta}{dt} = -\theta^2 + 2\mu. \quad (24)$$

With $\mu > 0$, this has complete solutions $\theta(t) = \sqrt{2\mu} \tanh(\sqrt{2\mu} t + c)$ and $\theta(t) = \pm \sqrt{2\mu}$. But in fact all of these solutions are impermissible, as can be seen by restricting (21) to $\gamma^{(p)}$. Indeed, substituting $\theta(t) = \pm \sqrt{2\mu}$ into (21) yields the solutions

$$f(t) - \mu = (f(0) - \mu)e^{\mp \sqrt{2\mu} t}.$$  

Likewise, the solution $\theta(t) = \sqrt{2\mu} \tanh(\sqrt{2\mu} t + c)$, when it is inserted into (21), yields the solution

$$f(t) - \mu = (f(0) - \mu) \text{sech}(\sqrt{2\mu} t + c).$$

But in either case, the right-hand side goes to zero whereas the left-hand side is bounded away from zero (recall that $M$ is closed and $f$
never takes the value $\mu$). Thus $H$ must be nowhere vanishing on $M$, in which case, consider $1/H$:

$$T(1/H) \overset{(23)}{=} \frac{\text{div} T}{H} \quad \Rightarrow \quad T(T(1/H)) \overset{(23)}{=} 2 + \frac{2\mu}{H}. \quad (25)$$

The latter equation has solution

$$(1/H)(t) = -\frac{1}{\mu} + c_1 e^{2\sqrt{\mu} t} + c_2 e^{-2\sqrt{\mu} t},$$

but unless $c_1 = c_2 = 0$, this solution is unbounded. We therefore conclude that the function $H$ is a nonzero constant on $M$, which immediately implies that $\text{div} T = 0$ by (23).

2. $\mu = 0$: Once again, suppose that $H(p) = 0$, so that $H \circ \gamma^{(p)} : \mathbb{R} \rightarrow \mathbb{R}$ is identically zero, in which case (22) now becomes

$$\frac{d\theta}{dt} = -\theta^2.$$

This has $\theta(t) = 0$ as its only complete solution. If $H(p) \neq 0$, so that $H \circ \gamma^{(p)} : \mathbb{R} \rightarrow \mathbb{R}$ is nowhere vanishing, then applying (25) along $\gamma^{(p)}$ yields $(1/H)''(t) = 2$, hence

$$(1/H)(t) = t^2 + c_1 t + c_2 \quad \Rightarrow \quad \theta(t) = \frac{2t + c_1}{t^2 + c_1 t + c_2},$$

with $c_1^2 < 4c_2^2$ to ensure that $(1/H)(t)$ is nowhere vanishing. But then (21) would yield

$$f(t) = \frac{c_3}{t^2 + c_1 t + c_2},$$

which is impossible because $f$, never taking the value $\mu = 0$, must be bounded away from 0 on closed $M$. We conclude that $H$ must be the zero function, and so $\text{div} T = 0$ once again.

This completes the proof. \qed

(As an aside, it is instructive to consider what happens when $\mu < 0$. In this case, if $H \circ \gamma^{(p)} : \mathbb{R} \rightarrow \mathbb{R}$ is identically zero, then

$$\frac{d\theta}{dt} = -\theta^2 + 2\mu$$

has no complete solutions. Thus $H$ must be nowhere vanishing on $M$, but this time (25) yields

$$(1/H)(t) = -\frac{1}{\mu} + c_1 \sin(\sqrt{2|\mu|} t) + c_2 \cos(\sqrt{2|\mu|} t)$$

and (21)

$$f(t) = \mu + \frac{c_3}{-\frac{1}{\mu} + c_1 \sin(\sqrt{2|\mu|} t) + c_2 \cos(\sqrt{2|\mu|} t)}.$$ But both of these are well behaved for appropriate constants, so there is no contradiction.) In any case, when $\mu > 0$, then a distinguished orthonormal frame appears, which plays a crucial role in the proof of our Theorem:
Proposition 2. Assume the hypotheses of Proposition 1. If \( \mu > 0 \) and \( M \) is closed and simply connected, then there exists a global orthonormal frame \( \{ T, X, Y \} \) with respect to which the spin coefficients \( \kappa, \rho, \beta, \sigma \) take the form

\[
\kappa = \rho = \beta = 0, \quad \sigma = -\sqrt{\frac{\mu}{2} + i \frac{\omega}{2}}. \tag{26}
\]

In particular, \( T \) is geodesic, divergence-free, and twist-free.

Proof. By Proposition 1, \( T \) has geodesic flow, so that \( \kappa = 0 \); by its Corollary, \( T \) is also divergence-free, so that \( \rho = -i\frac{\omega}{2} \) (recall (9)). Now we show the existence of a local orthonormal frame \( \{ T, X, Y \} \) with respect to which the shear \( \sigma \) takes the form (26). Begin by observing that when \( \kappa = \text{div } T = 0 \), (18) reduces to

\[
2|\sigma|^2 - \frac{\omega^2}{2} = \mu.
\]

This in turn implies that the endomorphism \( D: T^\perp \rightarrow T^\perp \), whose matrix is given by (8), has the two distinct eigenvalues \( \pm \sqrt{\mu/2} \) (note that \( D \) is self-adjoint with respect to the induced (Riemannian) metric \( g|_{T^\perp} \) on \( T^\perp \) if and only if \( \omega \) vanishes). Therefore, consider the respective kernels of the two bundle endomorphisms \( D \pm \sqrt{\mu/2} I: T^\perp \rightarrow T^\perp \); if \( M \) is simply connected, then these line bundles have smooth nowhere vanishing global sections \( X, Y_1 \),

\[
D(X) = \sqrt{\mu/2} X, \quad D(Y_1) = -\sqrt{\mu/2} Y_1,
\]

which we can take to have unit length, and which are both spacelike because they are orthogonal to \( T \). If necessary, modify \( Y_1 \) so that it is orthogonal to \( X \), by defining

\[
Y := -g(X, Y_1) X + Y_1
\]

and normalizing \( Y \) to have unit length. We now claim that the global orthonormal frame \( \{ T, X, Y \} \) has shear \( \sigma \) given by (26); indeed, substituting (8) into \( D(X) = \sqrt{\mu/2} X \) yields

\[
\begin{pmatrix}
-\sigma_1 \\
\sigma_2 - \frac{\omega}{2}
\end{pmatrix} = \sqrt{\mu/2} \begin{pmatrix}
1 \\
0
\end{pmatrix} \Rightarrow \sigma_1 = -\sqrt{\mu/2}, \quad \sigma_2 = \frac{\omega}{2}, \tag{27}
\]

completing the proof. \( \square \)

4. Proof of Theorem

Armed with Propositions 1, 2, and our Corollary above, we now prove our Theorem, starting with the \( \lambda > 0 \) case:

Proof of \( \lambda > 0 \) case of Theorem. (Once the sign changes (in red) are accounted for, this proof and Proposition 2 above are identical to that of Theorem 1 in [AMT19].) Suppose a closed Lorentzian 3-manifold \( (M, g) \) exists satisfying (1), with \( \lambda > 0 \). By the Corollary to Proposition 1, \( \text{div } T = 0 \).
Now assume that $M$ is simply connected; then by Proposition 2, there exists a global orthonormal frame $\{T, X, Y\}$ satisfying

$$
\kappa = 0, \quad \rho = -i\frac{\omega}{2}, \quad \sigma = -\sqrt{\frac{\lambda}{2}} + i\frac{\omega}{2}.
$$

Using this information, we now show the existence of a vector field $Z$ and a smooth function $\psi$ on $M$ satisfying

$$
Z(\psi) = f,
$$

which is impossible, as $f$ never takes the value 0 and $M$ is closed. This will be shown by inserting (28) into (11), (12), and (13). Indeed, doing so yields, in order,

$$
T(\sigma) \overset{(11)}{=} 2\sigma \varepsilon \quad \Rightarrow \quad \sigma \varepsilon = 0 \quad \Rightarrow \quad \varepsilon = 0,
$$

where $T(\sigma) = 0$ because $\text{div} T = 0$, hence $T(\omega^2) = 0$ in (19); next, because

$$
\beta = \frac{1}{\sqrt{2}} \left( g(\nabla_Y X, Y) - i g(\nabla_X Y, X) \right) = \frac{1}{\sqrt{2}} (\text{div} X - i \text{div} Y) \quad \text{(since $\nabla_T T = 0$)},
$$

$$
\frac{m(\rho) - \overline{m}(\sigma)}{-\sqrt{2} \chi(\omega)} \overset{(12)}{=} 2\sigma \bar{\beta} \quad \Rightarrow \quad \left\{ \begin{array}{l}
\frac{\sqrt{2} \lambda}{\text{div} X} + \omega \text{div} Y = 0, \\
\frac{\sqrt{2} \lambda}{\text{div} Y} - \omega \text{div} X = X(\omega).
\end{array} \right.
$$

Finally, (13), when simplified using $\varepsilon = 0$, yields

$$
X(\text{div} X) \overset{(30)}{=} -\frac{1}{\sqrt{2} \lambda} X(\omega) \text{div} Y - \frac{\omega}{\sqrt{2} \lambda} X(\text{div} Y) \\
\overset{(30)}{=} -(\text{div} Y)^2 + \frac{\omega}{\sqrt{2} \lambda} (\text{div} X)(\text{div} Y) - \frac{\omega}{\sqrt{2} \lambda} X(\text{div} Y),
$$

so that (31) reduces, finally, to

$$
\left( \frac{\omega}{\sqrt{2} \lambda} X - Y \right) (\text{div} Y) = f.
$$

With $Z := \frac{\omega}{\sqrt{2} \lambda} X - Y$ and $\psi := \text{div} Y$, this is precisely (29). This proves the Theorem in the case when $M$ is simply connected. If $M$ is not simply connected, then pass to its simply connected universal cover $(\tilde{M}, \tilde{g})$; it is locally isometric to $(M, g)$ via the projection $\pi: \tilde{M} \rightarrow M$, and therefore its Ricci tensor $\tilde{\text{Ric}}$ will satisfy (1), with $f \circ \pi$ in place of $f$, and with $\tilde{T}$ the
(complete) lift of $T$. Repeating step-by-step our argument on $(\tilde{M}, \tilde{g})$, we arrive once again at (32). Although $\tilde{M}$ need not be compact, a contradiction is still obtained because $f \circ \pi$ is bounded away from zero, because $\text{div} Y$ is also bounded (since $d\pi(Y)$ is well defined up to sign, $|\text{div} d\pi(Y)|$ is continuous on $M$), and because $Z$ is complete on $\tilde{M}$. This completes the proof. □

Proof of $\lambda = 0$ case of Theorem. (This proof parallels that of Theorem 3 in [AMT19], but generalizes it: due to our Corollary above, the function $f$ is not assumed to be constant, as it was in [AMT19]; furthermore, (36) below is a necessary step that was not required in [AMT19].) When $\lambda = 0$, \begin{equation}
|\sigma|^2 - \frac{\omega^2}{4} = 0 \implies D \text{ has eigenvalues 0, 0.} \tag{33}
\end{equation}
As there are no longer two distinct eigenvalues, we cannot call upon Proposition 2; instead, we prove that if (33) holds then $T$ must be parallel. Doing so will then allow us to draw two conclusions: first, that $(M, g)$ must be geodesically complete, by [RS95]; second, that the universal cover of $(M, g)$ must be isometric to $(\mathbb{R} \times \tilde{N}, -dt^2 \oplus \tilde{h})$ for some Riemannian 2-manifold $(\tilde{N}, \tilde{h})$, by the de Rham Decomposition Theorem for Lorentzian manifolds [SW12]. To begin with, observe that $T$ being parallel is equivalent to the condition \[
\kappa = \rho = \sigma = 0.
\]
By Proposition 1 and its Corollary, $\nabla_T T = \text{div} T = 0$, so that we need only show that $\omega^2 = 0$; we’ll do this by showing that the open set \[
U = \{ p \in M : \omega^2(p) \neq 0 \}
\]
is empty. Assume for the moment that $U$ is simply connected. Then over $U$, $D$ has constant rank 1 (recall (8)), so that its kernel is a line bundle over $U$; as the latter is simply connected, this kernel has a nowhere vanishing section $X$ on $U$. Now let $\{T, X, Y\}$ be an orthonormal frame, with $Y$ perhaps only locally defined in $U$. Then the analogue of (27) is now \[
\left( \begin{array}{c}
-\sigma_1 \\
\sigma_2 - \frac{\omega}{2}
\end{array} \right) = \left( \begin{array}{c}
0 \\
0
\end{array} \right) \implies \sigma_1 = 0 \quad , \quad \sigma_2 = \frac{\omega}{2},
\]
and thus the analogue of (26) is $\rho = -i\frac{\omega}{2}$ and $\sigma = i\frac{\omega}{2}$. Proceeding as in the proof of the $\lambda > 0$ case above, (30) becomes \begin{equation}
\omega(\text{div} Y) = 0 \quad , \quad X(\omega) = -\omega(\text{div} X). \tag{34}
\end{equation}
Now $\text{div} Y = 0$ because $\omega$ is nowhere vanishing in $U$, in which case (31) reduces to \begin{equation}
X(\text{div} X) = -(\text{div} X)^2 - f. \tag{35}
\end{equation}
If the flow of $X$ was complete in $U$, then we would obtain a contradiction because (35) has no complete solutions when $f > 0$. Our task is therefore done if we can prove that the flow of $X$ is complete. Thus, let $\gamma : [0, b) \rightarrow U$ be an integral curve of $X$ that is maximally extended to the right, and
suppose that $b < \infty$ (the case $(-b,0]$ will follow from this one by letting $X \to -X$, which leaves (34) and (35) unaltered). To begin with, there is a sequence $t_n \to b$ such that \{\gamma(t_n)\} converges to some $q \in M \setminus U$ (if $q$ were in $U$, then the integral curve $\gamma$ would be extendible, contradicting our assumption that it was maximally extended; see, e.g., [O’N83, Lemma 56]). Let us give a proof of this that will also suffice should we need to pass to the universal cover of $M$ below: consider the Riemannian metric $h$ on $M$

\[ h := g + 2T^9 \otimes T^9. \] 

(36)

Since $M$ is closed, $h$ is complete; as $X$ has $h$-unit length,
\[ d_h(\gamma(0), \gamma(t)) \leq L_h(\gamma|_{[0,t]}) = t, \]

where $d_h$ is the Riemannian distance associated to $h$. This implies that
\[ \gamma([0,b]) \subseteq \{p \in M : d_h(\gamma(0),p) \leq b\}. \]

By the completeness of $(M,h)$, the latter set is compact, hence any sequence \{\gamma(t_n)\} with $t_n \to b$ has a convergent subsequence; cf. [CS08, Proposition 3.4]. Now, set $\theta(t) := (\text{div} X \circ \gamma)(t)$ and $\omega^2(t) := (\omega^2 \circ \gamma)(t)$; in particular, observe that $\lim_{n \to \infty} \omega^2(t_n) = 0$ because $q \not\in U$. By (34),
\[ \omega^2(t) = \omega_0^2 e^{\int_0^t \theta(s) ds} \text{ for all } t \in [0,b). \]

By (35), $\theta(t)$ is nonincreasing ($f > 0$), in which case $-2 \int_0^t \theta(s) ds \geq -2\theta_0 t \geq -2\theta_0 b$ for all $t \in [0,b)$. But then
\[ \omega^2(t) \geq w_0^2 e^{-2\theta_0 b} > 0 \quad \Rightarrow \quad \lim_{n \to \infty} w^2(t_n) > 0, \]

a contradiction. Thus we must have $b = \infty$; this proves that $U$, if simply connected, must be empty. If $U$ is not simply connected, then pass to the universal covers of $(U, g|_U) \subset (M,g)$ and repeat the argument (with $\omega^2 \circ \pi$, and noting that the lift of the Riemannian metric (36) will be complete), noting that any integral curve of $T$ starting in $U$ stays in $U$, because $T(\omega^2) = 0$ via (19). This completes the proof that the universal cover of $(M,g)$ is isometric to $(\mathbb{R} \times \tilde{N},-dt^2 \oplus \tilde{h})$, in which case $(M,g)$ itself is isometric to $(S^1 \times N, -dt^2 \oplus h)$ with $(N,h)$ a Riemannian 2-manifold. This establishes the $\lambda = 0$ case of the Theorem. \hfill $\square$

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