A generalization of Colmez-Greenberg-Stevens formula

Bingyong Xie
Department of Mathematics, East China Normal University, Shanghai, China
byxie@math.ecnu.edu.cn

Abstract

In this paper we study the derivatives of Frobenius and the derivatives of Hodge-Tate weights for families of Galois representations with triangulations. We give a generalization of the Fontaine-Mazur $L$-invariant and use it to build a formula which is a generalization of the Colmez-Greenberg-Stevens formula.

Key words: Frobenius, Hodge-Tate weight, $L$-invariant.
MSC(2010) classification: 11F80, 11F85.

1 Introduction

In their remarkable paper [10], Mazur, Tate and Teitelbaum proposed a conjectural formula for the derivative at $s = 1$ of the $p$-adic $L$-function of an elliptic curve $E$ over $\mathbb{Q}$ when $p$ is a prime of split multiplicative reduction. An important quantity in this formula is the so called $L$-invariant, namely $L(E) = \log_p(q_E)/v_p(q_E)$ where $q_E \in \mathbb{Q}_p^\times$ is the Tate period for $E$. This conjectural formula was proved by Greenberg and Stevens [8] using Hida’s families. Indeed, for the weight 2 newform $f$ attached to $E$, there exists a family of $p$-adic ordinary Hecke eigenforms containing $f$. A key formula they proved is

$$L(E) = -2 \frac{\alpha'(f)}{\alpha(f)}$$ (1.1)

where $\alpha$ is the function of $U_p$-eigenvalues of the eigenforms in the Hida family. On the other hand, they showed that $-2 \frac{\alpha'(f)}{\alpha(f)}$ is equal to $\frac{L'(f,1)}{L(f,1)}$. Combining these two facts they obtained the conjectural formula.

In this paper we will focus on (1.1) which was later generalized by Colmez [6] to the non-ordinary setting. We state Colmez’s result below.

**Theorem 1.1.** ([6]) Suppose that, at each closed point $z$ of $\text{Max}(S)$ one of the Hodge-Tate weight of $\mathcal{V}_z$ is 0, and there exists $\alpha \in S$ such that $(B_{\text{cris},S} \hat{\otimes} S \mathcal{V})^{G_{\mathbb{Q}_p}}$ is locally free of rank 1 over $S$. Suppose $z_0$ is a closed point of $\text{Max}(S)$ such that $\mathcal{V}_{z_0}$ is semistable with Hodge-Tate weights $1, 0$ and $k \geq 1$.

This paper is supported by Science and Technology Commission of Shanghai Municipality (grant no. 13dz2260400) and the National Natural Science Foundation of China (grant no. 11671137).

In this paper, the Hodge-Tate weights are defined to be minus the generalized eigenvalues of Sen’s operators. In particular the Hodge-Tate weight of the cyclotomic character $\chi_{\text{cyc}}$ is $-1$. 

1
Then the differential
\[
\frac{d\alpha}{\alpha} - \frac{1}{2} L d\kappa + \frac{1}{2} d\delta
\]
is zero at \(z_0\), where \(L\) is the Fontaine-Mazur \(L\)-invariant of \(V_{z_0}\).

See [6] for the precise meanings of \(\kappa\) and \(\delta\). Roughly speaking, \(d\delta\) is the derivative of Frobenius, and \(d\kappa\) is the derivative of Hodge-Tate weights.

The condition that \(\left(\mathcal{B}_{\text{cris}}(S), S^Z/V\right)^{\text{G}_p}\) is locally free of rank 1 over \(S\) in Theorem 1.1 is equivalent to that \(V\) admits a triangulation [5]. So, Theorem 1.1 means that the derivatives of Frobenius and the derivatives of Hodge-Tate weights of a family of 2-dimensional representations of \(\text{G}_p\) with a triangulation satisfy a non-trivial relation at each semistable (but non-crystalline) point.

Colmez’s theorem was generalized by Zhang [14] for families of 2-dimensional Galois representations of \(K\) (\(K\) a finite extension of \(\mathbb{Q}_p\)) and Pottharst [12] who considered families of (not necessarily étale) \((\phi, \Gamma)\)-modules of rank 2 instead of families of 2-dimensional Galois representations.

In this paper we give a generalization of Colmez’s theorem which includes the above generalizations as special cases.

Fix a finite extension \(K\) of \(\mathbb{Q}_p\). What we work with is a family of \(K\)-\(B\)-pair (called \(S\)-\(B\)-pair in our context) that is locally triangulable. We will provide conditions for Fontaine-Mazur \(L\)-invariant to be defined. Note that, the \(L\)-invariant is now a vector with component number equal to \([K: \mathbb{Q}_p]\).

**Theorem 1.2.** Let \(W\) be an \(S\)-\(B\)-pair that is semistable at a point \(z \in \text{Max}(S)\). Suppose that \(W\) is locally triangulable at \(z\) with the local triangulation parameters \((\delta_1, \cdots, \delta_n)\). Assume that for \(D_z\), the filtered \(\mathcal{E}-(\phi, N)\)-module attached to \(W_z\), the Fontaine-Mazur \(L\)-invariant \(L_{s, t}\) (see Definition 6.5) can be defined for \(s, t \in \{1, 2, \cdots, n\}\). Then
\[
\frac{1}{[K: \mathbb{Q}_p]} \left( \frac{d\delta_i(p)}{\delta_i(p)} - \frac{d\delta_s(p)}{\delta_s(p)} \right) + L_{s, t} \cdot (d\bar{\omega}(\delta_i) - d\bar{\omega}(\delta_s)) = 0.
\]

Here, \(\bar{\omega}(\delta_i)\) is the Hodge-Tate weight of the character \(\delta_i\).

In [13] we proved Theorem 1.2 for a special case, where we consider the case of \(K = \mathbb{Q}_p\) and demand that the Frobenius is semisimple at \(z\). The motivation and some potential applications of our theorem was also discussed in [13].

Our paper is organized as follows. In Section 2 we recall the theory of \(B\)-pairs built by Berger. Then in Section 3 we extend a part of this theory to families of \(B\)-pairs, and discuss the relation between triangulations of semistable \(B\)-pairs and refinements of their associated filtered \((\phi, N)\)-modules. In Section 4 we compare cohomology groups of \((\phi, \Gamma)\)-modules and those of \(B\)-pairs, and then attach a 1-cocycle to each infinitesimal deformation of a \(B\)-pair. In Section 5 we use the reciprocity law to build an auxiliary formula for \(L\)-invariants. The \(L\)-invariant is defined in Section 6. In Section 7 we prove a formula called “projection vanishing property” for the above 1-cocycle. Finally in Section 8 we use the auxiliary formula in Section 5 and the projection vanishing property to deduce Theorem 1.2.

**Notations**

Let \(K\) be a finite extension of \(\mathbb{Q}_p\), \(G_K\) the absolute Galois group \(\text{Gal}(\overline{K}/K)\). Let \(K_0\) be the maximal absolutely unramified subfield of \(K\). Let \(G_{K_0}^{\text{ab}}\) denote the maximal abelian quotient of \(G_K\).
Let $\chi_{\text{cyc}}$ be the cyclotomic character of $G_K$, $H_K$ the kernel of $\chi_{\text{cyc}}$ and $\Gamma_K$ the quotient $G_K/H_K$. Then $\chi_{\text{cyc}}$ induces an isomorphism from $\Gamma_K$ onto an open subgroup of $\mathbb{Z}_p^\times$.

Let $E$ be a finite extension of $K$ such that all embeddings of $K$ into an algebraic closure of $E$ are contained in $E$, $\text{Emb}(K, E)$ the set of embeddings of $K$ into $E$. We consider $E$ as a coefficient field and let $G_K$ acts trivially on $E$.

Let $\text{rec}_K$ be the reciprocity map of local class field theory such that $\text{rec}_K(\pi_K)$ is a lifting of the inverse of $q$th power Frobenius of $k$, where $\pi_K$ is a uniformizing element of $K$ and $k$ is the residue field of $K$ with cardinal number $q$. Note that the image of $\text{rec}_K$ coincides with the image of the Weil group $W_K \subset G_K$ by the quotient map $G_K \to G_K^{ab}$. Let $\text{rec}_K^{-1} : W_K \to K^\times$ be the converse map of $\text{rec}_K$.

## 2 \((\varphi, \Gamma_K)\)-modules and $B$-pairs

### 2.1 Fontaine’s rings

We recall the construction of Fontaine’s period rings. Please consult [7, 2] for more details.

Let $C_p$ be a completed algebraic closure of $Q_p$, with valuation subring $\mathfrak{o}_{C_p}$ and $p$-adic valuation $v_p$ normalized such that $v_p(p) = 1$.

Let $\mathcal{E}$ be \[ \{(x^{(i)})_{i \geq 0} \mid x^{(i)} \in C_p, \ (x^{(i+1)})^p = x^{(i)} \ \forall \ i \in \mathbb{N}\}, \] and let $\mathcal{E}^+$ be the subset of $\mathcal{E}$ such that $x^{(0)} \in \mathfrak{o}_{C_p}$. If $x, y \in \mathcal{E}$, we define $x + y$ and $xy$ by

\[ (x + y)^{(i)} = \lim_{j \to \infty} (x^{(i+j)} + y^{(i+j)})^p^j, \quad (xy)^{(i)} = x^{(i)} y^{(i)}. \]

Then $\mathcal{E}$ is a field of characteristic $p$. Define a function $v_{\mathcal{E}} : \mathcal{E} \to \mathbb{R} \cup \{+\infty\}$ by putting $v_{\mathcal{E}}((x^{(n)})) = v_p(x^{(0)})$. This is a valuation for which $\mathcal{E}$ is complete and $\mathcal{E}^+$ is the ring of integers in $\mathcal{E}$. If we let $\varepsilon = (\varepsilon^{(n)})$ be an element of $\mathcal{E}^+$ with $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$, then $\mathcal{E}$ is a completed algebraic closure of $F_p((\varepsilon - 1))$. Put $\omega = [\varepsilon] - 1$. Let $\bar{p}$ be an element of $\mathcal{E}$ such that $\bar{p}^{(0)} = p$.

Let $\mathcal{A}^+$ be the ring $\mathcal{W}(\mathcal{E}^+)$ of Witt vectors with coefficients in $\mathcal{E}^+$, $\mathcal{A}$ the ring of Witt vectors $\mathcal{W}(\mathcal{E})$, and $\mathcal{B}^+ = \mathcal{A}[1/p]$. The map

\[ \theta : \mathcal{B}^+ \to C_p, \quad \sum_{n \gg \infty} p^n x_k \mapsto \sum_{n \gg \infty} p^n x_k^{(0)} \]

is surjective. Let $\mathcal{B}_{\text{dr}}^+$ be the $\ker(\theta)$-adic completion of $\mathcal{B}^+$. Then $t_{\text{cyc}} = \log [\varepsilon]$ is an element of $\mathcal{B}_{\text{dr}}^+$, and put $\mathcal{B}_{\text{dr}} = \mathcal{B}_{\text{dr}}^+[1/t_{\text{cyc}}]$. There is a filtration $\text{Fil}^\bullet$ on $\mathcal{B}_{\text{dr}}$ such that $\text{Fil}^i \mathcal{B}_{\text{dr}} = \bigoplus_{j \geq 1} \mathcal{B}_{\text{dr}}^+ t_{\text{cyc}}^j$.

Let $\mathcal{B}_{\text{max}}$ be the subring of $\mathcal{B}^+$ consisting of elements of the form \[ \sum_{n \gg 0} b_n([\bar{p}]/p)^n \], where $b_n \in \mathcal{B}^+$ and $b_n \to 0$ when $n \to +\infty$. Put $\mathcal{B}_{\text{max}} = \mathcal{B}_{\text{max}}^+[1/t_{\text{cyc}}];$ $\mathcal{B}_{\text{max}}$ is equipped with a $\varphi$-action.

Put $\mathcal{B}_{\log} = \mathcal{B}_{\text{max}}[\log [\varepsilon]]; \mathcal{B}_{\log}$ is equipped with a $\varphi$-action and a monodromy $N; \mathcal{B}_{\log}^{N=0} = \mathcal{B}_{\text{max}}; \mathcal{B}_{\log}$ is a subring of $\mathcal{B}_{\text{dr}}$. Put $\mathcal{B}_e = \mathcal{B}_{\text{max}}^1$. We have the following fundamental exact sequence

\[ 0 \to Q_p \to \mathcal{B}_e \to \mathcal{B}_{\text{dr}} \to \mathcal{B}_{\text{dr}}^+ \to 0. \]

If $r$ and $s$ are two elements in $\mathbb{N}[1/p] \cup \{+\infty\}$, we put $A^{[r,s]} = A^+ \left( \frac{p}{[\bar{\omega}]^{s/r}} \right)$ and $\mathcal{B}^{[r,s]} = \mathcal{A}^{[r,s]}[1/p]$ with the convention that $p/[\bar{\omega}^{+\infty}] = 1/[\bar{\omega}]$ and $[\bar{\omega}^{+\infty}]/p = 0$. We equip these rings with
the $p$-adic topology. There are natural continuous $G_K$-actions on $\tilde{A}_{[r,s]}$ and $\tilde{B}_{[r,s]}$. Frobenius induces isomorphisms $\phi : \tilde{A}_{[r,s]} \rightarrow \tilde{A}_{[pr,ps]}$ and $\phi : \tilde{B}_{[r,s]} \rightarrow \tilde{B}_{[pr,ps]}$. If $r \leq r_0 \leq s_0 \leq s$, then we have the $G_K$-equivariant injective natural map $\tilde{A}_{[r,s]} \hookrightarrow \tilde{A}_{[r_0,s_0]}$. For $r > 0$ we put $\tilde{B}_{\text{rig}} = \bigcap_{s \in [r, +\infty)} \tilde{B}_{[r,s]}$ (equipped with certain Frechet topology) and $\tilde{\Gamma}_{\text{rig}} = \bigcup_{r > 0} \tilde{B}_{\text{rig}}^{1,r}$ (equipped with the inductive limit topology). Frobenius induces isomorphisms $\phi : \tilde{B}_{\text{rig}}^{1,r} \rightarrow \tilde{B}_{\text{rig}}^{1,pr}$ and $\phi : \tilde{B}_{\text{rig}}^{1} \rightarrow \tilde{B}_{\text{rig}}^{1}$.

Put

$$A_{K^0} = \left\{ \sum_{k \geq -\infty} a_k \omega^k \mid a_k \in \mathcal{O}_{K^0}, a_k \rightarrow 0 \text{ when } k \rightarrow -\infty \right\}$$

and $B_{K^0} = A_{K^0}[1/p]$. Here $K^0$ is the maximal absolutely unramified subfield of $K = K(\mu_p)$. Then $A_{K^0}$ is a complete discrete valuation ring with $p$ as a prime element, and $B_{K^0}$ is the fractional field of $A_{K^0}$. The $G_K$-action and $\phi$ preserve $A_{K^0}^+$: $\varphi(\omega) = (1 + \omega)^p - 1$ and $g(\omega) = (1 + \omega)^{\kappa_{\text{cycl}}} - 1$.

Let $A$ be the $p$-adic completion of the maximal unramified extension of $A_{K^0}$ in $A$, $B$ its fractional field. Then $\phi$ and the $G_K$-action preserve $A$ and $B$.

We put $B_K = B_{\text{rig}}^{1}$ and $B_{K}^{1,r} = B_K \cap B_{\text{rig}}^{1,r}$. Let $B_{\text{rig},K}^{1,r}$ be the Frechet completion of $B_{K}^{1,r}$ for the topology induced from that on $B_{\text{rig}}^{1,r}$, and put $B_{\text{rig},K}^{1} = \bigcup_{r > 0} B_{\text{rig},K}^{1,r}$ equipped with the inductive limit topology. Frobenius induces injections $B_{\text{rig},K}^{1,r} \hookrightarrow B_{\text{rig},K}^{1,pr}$ and $B_{\text{rig},K}^{1} \hookrightarrow B_{\text{rig},K}^{1}$; there are continuous $\Gamma_K$-actions on $B_{\text{rig},K}^{1,r}$ and $B_{\text{rig},K}^{1}$.

We end this subsection by the definition of $E$-$(\varphi, \Gamma_K)$-modules [11].

**Definition 2.1.** An $E$-$(\varphi, \Gamma_K)$-module is a finite $B_{\text{rig},K}^{1} \otimes \mathbb{Q}_p$ $E$-module $M$ equipped with a Frobenius semilinear action $\varphi_M$ and a continuous semilinear $\Gamma_K$-action such that $M$ is free as a $B_{\text{rig},K}^{1}$-module, that $\text{id}_{B_{\text{rig},K}^{1}} \otimes \varphi_M : B_{\text{rig},K}^{1} \otimes \varphi_{B_{\text{rig},K}^{1}} \rightarrow M$ is an isomorphism, and that $\varphi_M$ and the $\Gamma_K$-action commute with each other.

By [11, Lemma 1.30] if $M$ is an $E$-$(\varphi, \Gamma_K)$-module, then $M$ is free over $B_{\text{rig},K}^{1} \otimes \mathbb{Q}_p E$.

### 2.2 $B$-pairs

We recall the theory of $E$-$B$-pairs [3, 11].

Put $B_{e,E} = B_{e} \otimes \mathbb{Q}_p E$, $B_{\text{dr},E}^{+} = B_{\text{dr}}^{+} \otimes \mathbb{Q}_p E$ and $B_{\text{dr},E} = B_{\text{dr}} \otimes \mathbb{Q}_p E$. We extend the $G_K$-actions $E$-linearly to these rings.

**Definition 2.2.** An $E$-$B$-pair of $G_K$ is a couple $W = (W_e, W_{\text{dr}}^{+})$ such that

- $W_e$ is a finite $B_{e,E}$-module with a continuous semilinear action $G_K$-action which is free as a $B_{e}$-module,
- $W_{\text{dr}}^{+} \subset W_{\text{dr}} = B_{\text{dr}} \otimes B_e$, $W_e$ is a $G_K$-stable $B_{\text{dr},E}^{+}$-lattice.

By [11, Remark 1.3] $W_e$ is free over $B_{e,E}$ and $W_{\text{dr}}^{+}$ is free over $B_{\text{dr},E}^{+}$.

If $V$ is an $E$-representation of $G_K$, then $W(V) = (B_{e,E} \otimes E V, B_{\text{dr},E}^{+} \otimes E V)$ is an $E$-$B$-pair, called the $E$-pair associated to $V$.

If $S$ is a Banach $E$-algebra, we can define $S$-$B$-pairs similarly; to each $S$-representation $V$ of $G_K$ is associated an $S$-$B$-pair $W(V) = (B_{e,E} \otimes E V, B_{\text{dr},E}^{+} \otimes E V)$. 

4
If $W_1 = (W_{1,e}, W_{1,dr}^+)$ and $W_2 = (W_{2,e}, W_{2,dr}^+)$ are two $E$-$B$-pairs, we define $W_1 \otimes W_2$ to be

$$(W_{1,e} \otimes W_{2,e}, W_{1,dr}^+ \otimes W_{2,dr}^+)_{B_{+},E}.$$ 

Here, $W_{1,e} \otimes W_{2,e}$ is equipped with the diagonal $G_K$-action, and $W_{1,dr}^+ \otimes B_{+}^{dr,E} \otimes W_{2,dr}^+$ is naturally considered as a $G_K$-stable $B_{+}^{dr,E}$-lattice of

$$B_{dr} \otimes B_{+} (W_{1,e} \otimes W_{2,e}) = W_{1,dr} \otimes W_{2,dr},$$

where $W_{1,dr} = B_{dr} \otimes B_{+} W_{1,e}$ and $W_{2,dr} = B_{dr} \otimes B_{+} W_{2,e}$.

If $W = (W_e, W_{dr}^+)$ is an $E$-$B$-pair with $W_{dr} = B_{dr} \otimes B_{+} W_e$, we define the dual of $W$ to be $W^* = (W_e^*, W_{dr}^{*+})$, where $W_e^*$ is Hom$_{B_{dr}}(W_{e}, B_{+})$ equipped with the natural $G_K$-action, and $W_{dr}^{*+}$ is the $G_K$-stable lattice of $B_{dr} \otimes B_{+} W_e^* \cong$ Hom$_{B_{dr}}(W_{dr}, B_{dr})$ defined by

$$\{\ell \in \text{Hom}_{B_{dr}}(W_{dr}, B_{dr}) : \ell(x) \in B_{dr}^+ \text{ for all } x \in W_{dr}^+\}.$$ 

The relation between $(\varphi, \Gamma_K)$-modules and $B$-pairs is built by Berger [3]. We recall Berger's construction below.

Let $M$ be a $(\varphi, \Gamma_K)$-module of rank $d$ over the Robba ring $B_{rig}^+$. Berger [3] showed that

$$W_e(M) := (\tilde{B}_{rig}^+ [1/t] \otimes B_{rig,K}^+ M)^{\varphi = 1}$$

is a free $B_{+}$-module of rank $d$ and equipped with a continuous semilinear $G_K$-action.

For sufficiently large $r_0 > 0$ we can take a unique $\Gamma_K$-stable finite free $B_{rig,K}^+$-submodule $M' \subset M$ such that

$$B_{rig,K}^+ \otimes B_{rig,K}^+ M' = M$$

and

$$\text{id}_{B_{rig,K}^+} \otimes \varphi_M : B_{rig,K}^+ \otimes B_{rig,K}^+ M' \xrightarrow{\sim} M^{pr}$$

for any $r \geq r_0$. Berger [3] showed that the $B_{dr}^+$-module

$$W_{dr}^+(M) := B_{dr}^+ \otimes_{in} B_{rig,K}^+(p^{-1})^{n-1} M^{(p-1)p^{n-1}}$$

is independent of any $n$ such that $(p - 1)p^{n-1} \geq r_0$, and showed that there is a canonical $G_K$-equivariant isomorphism $B_{dr} \otimes B_{+} W_e(M) \xrightarrow{\sim} B_{dr} \otimes B_{dr}^+ W_{dr}^+(M)$.

Put $W(M) = (W_e(M), W_{dr}^+(M))$. This is an $E$-$B$-pair of rank $d = \text{rank}_{B_{rig,K}^+} M$.

The following is a variant version of Berger’s result [3, Theorem 2.2.7].

**Proposition 2.3.** [11, Theorem 1.36] The functor $M \mapsto W(M)$ is an exact functor and this gives an equivalence of categories between the category of $E$-$(\varphi, \Gamma_K)$-modules and the category of $E$-$B$-pairs of $G_K$.

**Proposition 2.4.** The functor $M \mapsto W(M)$ respects the tensor products and duals.
Proof. Let $M_1$ and $M_2$ be two $E'(\varphi, \Gamma_K)$-modules. By taking $\varphi$-invariants, the isomorphism

$$(\tilde{B}^1_{\text{rig}}[1/t] \otimes \tilde{B}^1_{\text{rig}, K} M_1) \otimes \tilde{B}^1_{\text{rig}, \mathbb{Q}_p} E[1/t] \otimes \tilde{B}^1_{\text{rig}, K} M_2) \cong \tilde{B}^1_{\text{rig}}[1/t] \otimes \tilde{B}^1_{\text{rig}, K} (M_1 \otimes M_2)$$

induces a $G_K$-equivariant injective map

$$W_e(M_1) \otimes_{\mathbb{B}_{\text{rig}, K}} W_e(M_2) \to W_e(M_1 \otimes M_2).$$

Here, $M_1 \otimes M_2$ denotes the $E'(\varphi, \Gamma_K)$-module $M_1 \otimes \tilde{B}^1_{\text{rig}, K} \otimes \tilde{B}^1_{\text{rig}, K} M_2$. Comparing dimensions and using [11, Lemma 1.10] we see that this map is in fact an isomorphism. From the above Berger’s construction we see that the natural map

$$W_{dR}^+(M_1) \otimes B_{\text{rig}}^+ \otimes G_{\mathbb{Q}_p} E W_{dR}^+(M_2) \to W_{dR}^+(M_1 \otimes M_2)$$

is an isomorphism. This proves that the functor $M \mapsto W(M)$ respects tensor products. The proof of that it respects duals is similar.

2.3 Semistable $E$-$B$-pairs

Definition 2.5. An $E'(\varphi, N)$-module over $K$ is a $K_0 \otimes_{\mathbb{Q}_p} E$-module $D$ with a $\varphi \otimes 1$-semilinear isomorphism $\varphi_D : D \to D$, and a $K_0 \otimes_{\mathbb{Q}_p} E$-linear map $N_D : D \to D$ such that $N_D \varphi_D = p \varphi_D N_D$. A filtered $E'(\varphi, N)$-module over $K$ is an $E'(\varphi, N)$-module with an exhaustive $\mathbb{Z}$-indexed descending filtration $\text{Fil}^*$ on $K \otimes_{K_0} D$.

We have an isomorphism of rings

$$K \otimes_{\mathbb{Q}_p} E \cong \bigoplus_{\tau \in \text{Emb}(K, E)} E_{\tau}, \quad a \otimes b \mapsto (\tau(a)b)_{\tau},$$

where $E_{\tau}$ is a copy of $E$ for each $\tau \in \text{Emb}(K, E)$. Let $e_{\tau}$ be the unity of $E_{\tau}$. Then $1 = \sum e_{\tau}$. Put $D_{\tau} = e_{\tau}(K \otimes_{K_0} D)$. Then $K \otimes_{K_0} D = \bigoplus_{\tau \in \text{Emb}(K, E)} D_{\tau}$. Let $\text{Fil}_{\tau}$ denote the induced filtration on $D_{\tau}$.

Definition 2.6. Let $W = (W_e, W_{dR}^+)$ be an $E$-$B$-pair. We define $D_{\text{cris}}(W) = (B_{\text{max}} \otimes B_e W_e)^{G_K}$, $D_{\text{st}}(W) = (B_{\log} \otimes B_e W_e)^{G_K}$ and $D_{\text{dR}}(W) = (B_{\text{dR}} \otimes B_e W_e)^{G_K}$. Then we have $\dim_{K_0}(D_{\tau}(W)) \leq \text{rank}_{B_e W_e}$ for $? = \text{cris, st}$, and $\dim_K(D_{\text{dR}}(W)) \leq \text{rank}_{B_e W_e}$. We say that $W$ is crystalline (resp. semistable) if $\dim_{K_0}(D_{\tau}(W)) = \text{rank}_{B_e W_e}$ for $? = \text{cris}$ (resp. st).

If $W$ is a semistable $E$-$B$-pair, we attach to $W$ a filtered $E'(\varphi, N)$-module as follows. The underlying $E'(\varphi, N)$-module is $D_{\text{st}}(W)$; the filtration on $D_{\text{dR}}(W) = K \otimes_{K_0} D_{\text{st}}(W)$ is given by $\text{Fil}^* D_{\text{dR}}(W) = t^i W_{dR}^+ \cap D_{\text{dR}}(W)$.

Proposition 2.7. (a) The functor $W \mapsto D_{\text{st}}(W)$ realizes an equivalence of categories between the category of semistable $E$-$B$-pairs of $G_K$ and the category of filtered $E'(\varphi, N)$-modules over $K$.

(b) If $W_1$ and $W_2$ are semistable, then so is $W_1 \otimes W_2$.

(c) The functor $W \mapsto D_{\text{st}}(W)$ respects the tensor products and duals.
(d) If 
\[
0 \longrightarrow W_1 \longrightarrow W \longrightarrow W_2 \longrightarrow 0
\]
is a short exact sequence of $E$-$B$-pairs, and $W$ is semistable, then $W_1$ and $W_2$ are semistable.

(e) The functor $W \mapsto D_{\text{st}}(W)$ is exact.

**Proof.** Assertion (a) follows from [3, Proposition 2.3.4]. See also [11, Theorem 1.18 (2)].

Let $W_1$ and $W_2$ be two $E$-$B$-pairs. The isomorphism
\[(B_{\log} \otimes_{B_{\log} \otimes Q_{p_E}} (B_{\log} \otimes_{B_{\log} \otimes Q_{p_E}} W_2) \sim B_{\log} \otimes_{B_{\text{st}}} (W_1 \otimes W_2)\]
induces an injective map
\[D_{\text{st}}(W_1) \otimes_{K_{\text{st}} \otimes Q_{p_E}} D_{\text{st}}(W_2) \to D_{\text{st}}(W_1 \otimes W_2). \tag{2.2}\]
When $W_1$ and $W_2$ are semistable, the dimension of the source over $K_0$ is $\frac{\text{rank}_{B_{\log}} W_1 \cdot \text{rank}_{B_{\log}} W_2}{[E : Q_{p_E}]}$. The dimension of the target over $K_0$ is always equal to or less than $\text{rank}_{B_{\log}} (W_1 \otimes W_2) = \frac{\text{rank}_{B_{\log}} W_1 \cdot \text{rank}_{B_{\log}} W_2}{[E : Q_{p_E}]}$. Hence, (2.2) is an isomorphism, and so $W_1 \otimes W_2$ is semistable. This proves (b). Similarly, the isomorphism
\[(B_{\text{dR}} \otimes_{B_{\text{st}}} W_1) \otimes_{B_{\text{dR}} \otimes Q_{p_E}} (B_{\text{dR}} \otimes_{B_{\text{st}}} W_2) \sim B_{\text{dR}} \otimes_{B_{\text{st}}} (W_1 \otimes W_2) \tag{2.3}\]
induces an isomorphism
\[D_{\text{dR}}(W_1) \otimes_{K_{\text{dR}} \otimes Q_{p_E}} D_{\text{dR}}(W_2) \to D_{\text{dR}}(W_1 \otimes W_2).\]

Via the isomorphism (2.3) the filtration on $(B_{\text{dR}} \otimes_{B_{\text{st}}} W_1) \otimes_{B_{\text{dR}} \otimes Q_{p_E}} (B_{\text{dR}} \otimes_{B_{\text{st}}} W_2)$ coincides with that on $B_{\text{dR}} \otimes_{B_{\text{st}}} (W_1 \otimes W_2)$. Therefore, the filtration on $D_{\text{dR}}(W_1) \otimes_{K_{\text{dR}} \otimes Q_{p_E}} D_{\text{dR}}(W_2)$ and that on $D_{\text{dR}}(W_1 \otimes W_2)$ coincide. Indeed, they are the restrictions of the filtrations on $(B_{\text{dR}} \otimes_{B_{\text{st}}} W_1) \otimes_{B_{\text{dR}} \otimes Q_{p_E}} (B_{\text{dR}} \otimes_{B_{\text{st}}} W_2)$ and $B_{\text{dR}} \otimes_{B_{\text{st}}} (W_1 \otimes W_2)$ respectively. Similarly we can show that $W \mapsto D_{\text{st}}(W)$ respects duals. This proves (c).

For (d) we have the following exact sequence
\[0 \longrightarrow D_{\text{st}}(W_1) \longrightarrow D_{\text{st}}(W) \longrightarrow D_{\text{st}}(W_2) \tag{2.4}\]
So (d) follows from a dimension argument. Furthermore, when $W$ is semistable, $D_{\text{st}}(W) \to D_{\text{st}}(W_2)$ is surjective. For any $i \in \mathbb{Z}$ we write $d_i(W)$ for $\text{dim}_K \text{Fil}^i D_{\text{st}}(W)$. As the maps in the exact sequence (2.4) respect filtrations, we have $d_i(W) \leq d_i(W_1) + d_i(W_2)$. Similarly, we have $d_{1-i}(W^*) \leq d_{1-i}(W_1^*) + d_{1-i}(W_2^*)$. As $W \mapsto D_{\text{st}}(W)$ respects duals, we have $d_i(W) = \text{dim}_K (D_{\text{dR}}(W)) - d_{1-i}(W^*)$. Then
\[
d_i(W) = \text{dim}_K (D_{\text{dR}}(W)) - d_{1-i}(W^*) \\
\geq (\text{dim}_K (D_{\text{dR}}(W_1)) - d_{1-i}(W_1^*)) + \text{dim}_K (D_{\text{dR}}(W_2)) - d_{1-i}(W_2^*) \\
= d_i(W_1) + d_i(W_2).
\]
Thus we must have $d_i(W) = d_i(W_1) + d_i(W_2)$ for all $i \in \mathbb{Z}$. In other words, the maps in (2.4) are strict for the filtrations, which shows (e). \qed
By [3, Proposition 2.3.4] the quasi-inverse of the functor $D_{st}$ is given by

$$D_B(D) = ((K_0 \otimes_{K_0} D)^{\varphi = 1, N = 0}, \text{Fil}^0(K_0 \otimes_{K_0} D)).$$  \hspace{1cm} (2.5)$$

For a filtered $E-(\varphi, N)$-module $D$ we put

$$X_{\log}(D) = (K_0 \otimes_{K_0} D)^{\varphi = 1, N = 0}$$

and

$$X_{dR}(D) = (B_{dR} \otimes_{K_0} D)/\text{Fil}^0(B_{dR} \otimes_{K_0} D).$$

If $D_B(D) = (W_e, W^+_{dR})$, then $X_{\log}(D) = W_e$ and $X_{dR}(D) = (B_{dR} \otimes_{K_0} D)/W^+_{dR}$.

3. **$S$-$B$-pairs of rank 1 and triangulations**

3.1 **$S$-$B$-pairs of rank 1**

Let $S$ be a Banach $E$-algebra.

For any $a \in S^\times$ we define a filtered $S$-$\varphi$-module $D_a$ as follows. As a $K_0 \otimes_{Q_p} S$-module,

$$D_a = K_0 \otimes_{Q_p} S = \oplus_{i : K_0 \otimes_{Q_p} S} e_i,$$

the $\varphi \otimes 1$-semilinear action $\varphi$ on $D_a$ satisfies

$$\varphi(e_i) = e_{\varphi-1}, \varphi(e_{\varphi^{-1}}) = e_{\varphi^2 - 2}, \cdots, \varphi(e_{\varphi^{i-1}}) = \varphi^{i-1}.$$

the descending filtration on $D_{a,K} = K_0 \otimes_{Q_p} S$ is given by $\text{Fil}^0 D_{a,K} = D_{a,K}$ and $\text{Fil}^1 D_{a,K} = 0$.

**Lemma 3.1.** If $a \in S$ satisfies that $a - 1$ is topologically nilpotent, then there exists a unit $u_0 \in B_{\text{max}} \otimes_{K_0} S$ such that $\varphi^{[K_0 : Q_p]}(u_0) = au_0$. Consequently

$$\{x \in B_{\text{max}} \otimes_{K_0} S : \varphi^{[K_0 : Q_p]}(x) = ax\} = (B_{e, K_0} \otimes_{K_0} S)u_0.$$

**Proof.** Let $Q_{p}^{ur}$ be the completed unramified extension of $Q_p$. Then there exists an inclusion $Q_{p}^{ur} \hookrightarrow B_{\text{max}}$ that is compatible with $\varphi$.

As $\varphi^{[K_0 : Q_p]} - 1$ is surjective on $Q_{p}^{ur}$, there exists a sequence $c_0 = 1, c_1, \cdots$ of elements in $Q_{p}^{ur}$ such that

$$(\varphi^{[K_0 : Q_p]} - 1)c_i = c_{i+1}$$

for $i \geq 1$. The image of $c_i$ by the map

$$Q_{p}^{ur} \hookrightarrow B_{\text{max}} \rightarrow B_{\text{max}} \otimes_{K_0} S$$

is again denoted by $c_i$. Put

$$u_0 = \sum_{i=0}^{\infty} c_i(a - 1)^i.$$ 

Then $u_0$ is a unit and we have $\varphi^{[K_0 : Q_p]}(u_0) = au_0$. \hfill $\square$

**Proposition 3.2.** If $a \in S$ satisfies that $a - 1$ is topologically nilpotent, then $D_B(D_a)$ is an $S$-$B$-pair of rank 1. Here $D_B$ is the functor defined by (2.5).
Proof. For each \( z \in \mathbf{B}_{\text{max}} \otimes_{\mathbf{Q}_p} D_{\alpha} \) we write \( z = \sum c_\tau e_\tau \) with \( c_\tau \in \mathbf{B}_{\text{max}} \otimes_{\mathbf{K}_0} S \). Then \( \varphi(z) = z \) if and only if \( \varphi(c_\tau) = c_{\varphi^{-1}} \tau(i = 1, \cdots, [K_0 : \mathbf{Q}_p]) \) and \( \varphi^{[K_0 : \mathbf{Q}_p]}(c_{\text{id}}) = ac_{\text{id}}. \) Our assertion follows from Lemma 3.1. \( \square \)

For any \( a \in S^\times \), let \( \delta_a : K^\times \rightarrow S^\times \) denote the character such that \( \delta_a(\pi_K) = a \) and \( \delta_a|_{S^\times_{\mathbf{K}_0}} = 1. \)

Remark 3.3. In the case of \( S = E \), for any \( u \in E^\times \), \( D_E(D_{\alpha}) \) coincides with the \( E-B \)-pair \( W(\delta_a) \) defined in [11] (see [11, §1.4]). From now on the base change of \( W(\delta_a) \) from \( E \) to \( S \) is again denoted by \( W(\delta_a) \).

Let \( \delta : K^\times \rightarrow S^\times \) be a continuous character such that \( \delta(\pi_K) = au \), where \( u \in E^\times \) and \( a \in S \) satisfies that \( a - 1 \) is topologically nilpotent. We call such a character a good character. Let \( W_{\alpha} \) be the resulting \( S-B \)-pair in Proposition 3.2. Let \( \delta' \) be the unitary continuous character \( K^\times \rightarrow E^\times \) such that \( \delta'|_{S^\times_{\mathbf{K}_0}} = \delta|_{S^\times_{\mathbf{K}_0}} \) and \( \delta'(\pi_K) = 1 \). By local class field theory, this induces a continuous character \( \overline{\delta}' : G_K \rightarrow S^\times \) such that \( \overline{\delta}' \circ \text{rec}_K = \delta' \). Then we put

\[
W(\delta) = W(S(\overline{\delta}')) \otimes W(\delta_a) \otimes W_{\alpha},
\]

where \( W(S(\overline{\delta}'')) \) is the \( S-B \)-pair attached to the Galois representation \( S(\overline{\delta}') \).

If \( \delta \) is a continuous character \( \delta : K^\times \rightarrow S^\times \), we write \( \log(\delta) \) for the logarithmic of \( \delta|_{S^\times_{\mathbf{K}_0}} \), which is a \( \mathbf{Z}_p \)-linear homomorphism \( \log(\delta) : K \rightarrow S \).

For any \( \tau \in \text{Emb}(K,E) \) we use the same notation \( \tau \) to denote the composition of \( \tau : K \hookrightarrow E \) and \( E \twoheadrightarrow S \). Then \( \{ \tau : K \rightarrow S \} \) is a basis of \( \text{Hom}_{\mathbf{Z}_p}(E,S) \) over \( S \). Write \( \log(\delta) = \sum k_\tau \tau, \ k_\tau \in S \). We call \( (k_\tau) \) the weight vector of \( \delta \) and denote it by \( \tilde{w}(\delta) \). We use \( w_\tau(\delta) \) to denote \( k_\tau \).

Remark 3.4. Let \( S \) be an affinoid algebra over \( E \). For any continuous character \( \delta : K^\times \rightarrow S^\times \) and any point \( z_0 \in \text{Max}(S) \), there exists an affinoid neighborhood \( U = \text{Max}(S') \) of \( z_0 \) in \( \text{Max}(S) \) such that the restriction of \( \delta \) to \( U \) is good.

Lemma 3.5. Let \( \delta \) be a character of \( K^\times \) with values in \( S = E[\mathbf{Z}]/(\mathbf{Z}^2) \), \( \delta \) the character of \( K^\times \) with values in \( E \) obtained from \( \delta \) modulo \( (Z) \). Write \( \delta = \delta_S(1 + Z_\epsilon) \), where \( \delta_S \) is the character \( K^\times \twoheadrightarrow E^\times \hookrightarrow S^\times \). Let \( \epsilon' \) be the additive character of \( G_K \) such that \( \epsilon' \circ \text{rec}_K(p) = 0 \) and \( \epsilon' \circ \text{rec}_K|_{S^\times_{\mathbf{K}_0}} = \epsilon|_{S^\times_{\mathbf{K}_0}} \).

Assume that \( W(\overline{\delta}) \) is crystalline and \( \varphi^{[K_0 : \mathbf{Q}_p]} \) acts on \( D_{\text{cris}}(W(\overline{\delta})) \) by \( \alpha \). Then there is a nonzero element

\[
x \in (\mathbf{B}_{\text{max},E} \otimes_{\mathbf{B}_{\epsilon,E}} W(\delta_e))^{\varphi([K_0 : \mathbf{Q}_p]) = \alpha(1 + Z_\epsilon(\pi_K)(p)):G_K=(1+Z\epsilon')}\]

whose reduction modulo \( Z \) is a basis of \( D_{\text{st}}(W(\overline{\delta})) \) over \( K \otimes_{\mathbf{Q}_p} E \).

Proof. This follows from the fact that \( W(\delta) = W(\delta_S) \otimes W_{d_1 + 2 \epsilon_{p}(\pi_K)(p)} \otimes W(1 + Z_\epsilon') \). \( \square \)

3.2 Triangulations and refinements

Now let \( S \) be an affinoid algebra over \( E \). For any open affinoid subset \( U \) of \( S \) and any \( S-B \)-pair \( W \) let \( W_U \) denote the restriction to \( U \) of \( W \).

Definition 3.6. Let \( W \) be an \( S-B \)-pair of rank \( n \), \( z_0 \) a point of \( \text{Max}(S) \). If there is
• an affinoid neighborhood $U = \text{Max}(S_U)$ of $z_0$,
• a strictly increasing filtration
  \[ \{0\} = \text{Fil}_0 W_U \subset \text{Fil}_1 W_U \subset \cdots \subset \text{Fil}_n W_U = W_U \]
of saturated free sub-$S_U$-$B$-pairs, and
• $n$ good continuous characters $\delta_i : \mathbb{Q}_p^\times \to S_U^\times$
such that for any $i = 1, \cdots, n$,
  \[ \text{Fil}_i W_U / \text{Fil}_{i-1} W_U \simeq W(\delta_i), \]
we say that $W$ is \textit{locally triangulable at} $z_0$; we call $\text{Fil}_\bullet$ a \textit{local triangulation} of $W$ at $z_0$, and call $(\delta_1, \cdots, \delta_n)$ the \textit{local triangulation parameters} attached to $\text{Fil}_\bullet$.

Please consult [6, 4] for more knowledge on triangulations.

To discuss the relation between triangulations and refinements, we restrict ourselves to the case of $S = E$.

Let $D$ be a filtered $E$-$(\varphi, N)$-module of rank $n$. The operator $\varphi^{[K_0:Q_p]}$ on $D$ is $K_0 \otimes \mathbb{Q}_p E$-linear. We assume that the eigenvalues of $\varphi^{[K_0:Q_p]} : D \to D$ are all in $K_0 \otimes \mathbb{Q}_p E$, i.e. there exists a basis of $D$ over $K_0 \otimes \mathbb{Q}_p E$ such the matrix of $\varphi^{[K_0:Q_p]}$ with respect to this basis is upper-triangular.

Following Mazur [9] we define a \textit{refinement} of $D$ to be a filtration on $D$
  \[ 0 = F_0 D \subset F_1 D \subset \cdots \subset F_n D = D \]

by $E$-subspaces stable by $\varphi D$ and $N D$, such that each factor $\text{gr}^F_i D = F_i D / F_{i-1} D$ $(i = 1, \cdots, n)$ is of rank 1 over $K_0 \otimes \mathbb{Q}_p E$. Any refinement fixes an ordering $\alpha_1, \cdots, \alpha_n$ of eigenvalues of $\varphi^{[K_0:Q_p]}$
and an ordering $\vec{k}_1, \cdots, \vec{k}_n$ of Hodge-Tate weights of $K \otimes K_0 D$ taken with multiplicities such that the eigenvalue of $\varphi^{[K_0:Q_p]}$ on $\text{gr}^F_i D$ is $\alpha_i$ and the Hodge-Tate weight of $\text{gr}^F_i D$ is $\vec{k}_i$.

We have the following analogue of [1, Proposition 1.3.2].

**Proposition 3.7.** Let $W$ be a semistable $E$-$B$-pair, $D = D_{\text{st}}(W)$.

(a) The equivalence of categories between the category of semistable $E$-$B$-pairs and the category of filtered $E$-$(\varphi, N)$-modules induces a bijection between the set of triangulations on $W$ and the set of refinements on $D$.

(b) If $(\text{Fil}_i W)$ is a triangulation of $W$ with triangulation parameters $(\delta_1, \cdots, \delta_n)$ that correspond to a refinement $F_\bullet D$ of $D$ with the ordering of Hodge-Tate weights being $\vec{k}_1, \cdots, \vec{k}_n$, then $\delta_i = \bar{\delta}_i \prod_{\tau \in \text{Emb}(K, E)} \tau(\varphi)^{k_i, \tau}$, where $\bar{\delta}_i$ is a smooth character.

**Proof.** Assertion (a) follows from the fact that $D_{\text{st}}$ is an exact. Assertion (b) follows from [11, Lemma 4.1].

### 4 Cohomology Theory

#### 4.1 Cohomology of $(\varphi, \Gamma_K)$-modules and cohomology of $B$-pairs

Let $M$ be a $(\varphi, \Gamma_K)$-module. Assume that $\Gamma_K$ has a topological generator $\gamma$. Define the cohomology $H^*_M(M)$ by the complex $C^*(M)$ defined by

\[
C^0(M) = M \xrightarrow{(\gamma^{-1}, \varphi^{-1})} C^1(M) = M \oplus M \to C^2(M) = M,
\]

10
where the map \( C^1(M) \to C^2(M) \) is given by \((x, y) \mapsto (\varphi - 1)x - (\gamma - 1)y\). Denote the kernel of \( C^1(M) \to C^2(M) \) by \( Z^1(M) \).

There is a one-to-one correspondence between \( H^1(M) \) and the set of extensions of \( M_0 \) by \( M \) in the category of \((\varphi, \Gamma_K)\)-modules, where \( M_0 = B^1_{\text{rig}, K} e_0 \) is the trivial \((\varphi, \Gamma_K)\)-module with \( \varphi(e_0) = \gamma(e_0) = e_0 \). Let \( \hat{M} \) be an extension of \( M_0 \) by \( M \), and let \( \hat{e} \) be any lifting of \( e_0 \) in \( \hat{M} \). Then the element in \( H^1(M) \) corresponding to the extension \( \hat{M} \) is the class of \((\gamma - 1)\hat{e}, (\varphi - 1)\hat{e}) \in Z^1(M) \).

In [11] Nakamura introduced a cohomology for \( B \)-pairs and use it to compute the cohomology of \((\varphi, \Gamma_K)\)-modules.

If \( W = (W_e, W_{dR}^+) \) is an \( E-B \)-pair, let \( C^*(W) \) be the complex of \( G_K \)-modules defined by

\[
C^0(W) := W_e \to C^1(W) := W_{dR}/W_{dR}^+.
\]

Here, \( W_e \to W_{dR}/W_{dR}^+ \) is the natural map.

**Definition 4.1.** Let \( W = (W_e, W_{dR}^+) \) be an \( E-B \)-pair. We define the Galois cohomology of \( W \) by

\[
H^i_B(W) := H^i(G_K, C^*(W)).
\]

By definition there is a long exact sequence

\[
\cdots \to H^i_B(W) \to H^i(G_K, W_e) \to H^i(G_K, W_{dR}/W_{dR}^+) \to \cdots. \tag{4.1}
\]

For a \( G_K \)-module \( M \) put \( C^0(M) = M \) and let \( C^i(M) \) be the space of continuous functions from \((G_K)^{*1}\) to \( M \). Let \( \delta_0 : C^0(M) \to C^1(M) \) be the map \( x \mapsto (g \mapsto g(x) - x) \) and let \( \delta_1 : C^1(M) \to C^2(M) \) be the map \( f \mapsto ((g_1, g_2) \mapsto f(g_1 g_2) - f(g_1) - g_1 f(g_2)) \).

Nakamura [11] showed that \( H^1_B(W) \) is isomorphic to \( \ker(\tilde{\delta}_1)/\text{im}(\tilde{\delta}_0) \), where \( \tilde{\delta}_0 \) and \( \tilde{\delta}_1 \) are defined by

\[
\tilde{\delta}_0 : C^0(W_e) \oplus C^0(W_{dR}^+) \to C^1(W_e) \oplus C^1(W_{dR}^+) \oplus C^0(W_{dR}^+) : (x, y) \mapsto (\delta_0(x), \delta_0(y), x - y),
\]

\[
\tilde{\delta}_1 : C^1(W_e) \oplus C^1(W_{dR}^+) \oplus C^0(W_{dR}^+) \to C^2(W_e) \oplus C^2(W_{dR}^+) \oplus C^1(W_{dR}^+) : (f_1, f_2, x) \mapsto (\delta_1(f_1), \delta_1(f_2), f_1 - f_2 - \delta_0(x)).
\]

The map \( H^1_B(W) \to H^1(G_K, W_e) \) is induced by the forgetful map

\[
C^1(W_e) \oplus C^1(W_{dR}^+) \oplus C^0(W_{dR}^+) \to C^1(W_e).
\]

There is a one-to-one correspondence between \( H^1(G_K, W) \) and the set of extensions of \( W_0 \) by \( W \) in the category of \( E-B \)-pairs. Here, \( W_0 = (B_e \otimes_{\mathbb{Q}_p} E, B^+_{dR} \otimes_{\mathbb{Q}_p} E) \) is the trivial \( E-B \)-pair. Let \( \bar{W} = (\bar{W}_e, \bar{W}_{dR}^+) \) be an extension of \( W_0 \) by \( W \). Let \( (\bar{w}_e, \bar{w}_{dR}^+) \) be a lifting in \( \bar{W} \) of \((1, 1) \in W_0 \). Then the element in \( H^1_B(W) \) corresponding to the extension \( \bar{W} \) is just the class of \(((\sigma \mapsto (\sigma - 1)\bar{w}_e), (\sigma \mapsto (\sigma - 1)\bar{w}_{dR}^+), \bar{w}_e - \bar{w}_{dR}^+)) \in \ker(\tilde{\delta}_1) \).

By Proposition 2.3 there is a one-to-one correspondence between \( \text{Ext}(M_0, M) \) and \( \text{Ext}(W_0, W(M)) \).

It induces a natural isomorphism

\[
i_M : H^1_B(M) \to H^1_B(W(M)).
\]
4.2 1-cocycles from infinitesimal deformations

Let $S$ be the $E$-algebra $E[Z]/(Z^2)$, $\tilde{M}$ an $S$-$(\varphi, \Gamma_K)$-module. Let $\{e_1, \ldots, e_n\}$ be an $S$-basis of $M$, $\{e_1^*, \ldots, e_n^*\}$ the dual basis of $M^*$. Put $M = M \otimes S E$ and $M^* = M^* \otimes S E$. Let $e_{i,z}$ denote $e_i \mod Z$ and $e_{i,z}^*$ denote $e_i^* \mod Z$. Then $\{e_{1,z}, \ldots, e_{n,z}\}$ is an $E$-basis of $M$, and $\{e_{1,z}^*, \ldots, e_{n,z}^*\}$ is an $E$-basis of $M^*$.

The matrices of $\varphi$ and $\gamma$ with respect to $\{e_1, \ldots, e_n\}$ are denote by $\hat{A}_\varphi$ and $\hat{A}_\gamma$ respectively, so that $\varphi(e_j) = \sum_i (\hat{A}_\varphi)_{ij} e_i$ and $\gamma(e_j) = \sum_i (\hat{A}_\gamma)_{ij} e_i$. Write $\hat{A}_\varphi = (I_n + ZU_{e})A_{\varphi}$ and $\hat{A}_\gamma = (I_n + ZU_{\gamma})A_{\gamma}$. Put $c_{\varphi}(\tilde{M}) = \sum_{i,j} (U_{\varphi})_{ij} e_{i,z}^* \otimes e_{i,z} - (U_{\varphi})_{ij} e_{i,z}^* \otimes e_{i,z}$. Write $\mathbf{D}_B(\tilde{M}) = (\tilde{W}_e, \tilde{W}^+_d)$. $\mathbf{D}_B(M) = W$ and $\mathbf{D}_B(M^*) = W^*$.

Let $f_1, \ldots, f_n$ be a basis of $\tilde{W}_e$ over $B_{e,E}$, and let $g_1, \ldots, g_n$ be a basis of $\tilde{W}^+_d$ over $B^+_{dR,E}$. We write the matrix of $\sigma \in G_K$ with respect to the basis $\{f_1, \ldots, f_n\}$ by $(I_n + ZU_{e,\sigma})A_{e,\sigma}$, and the matrix of $\sigma$ with respect to the basis $\{g_1, \ldots, g_n\}$ by $(I_n + ZU^+_{dR,\sigma})A^+_{dR,\sigma}$. Here, $U_{e,\sigma} \in M_n(B_{e,E}), U^+_{dR,\sigma} \in M_n(B^+_{dR,E}), A_{e,\sigma} \in \text{GL}_n(B_{e,E}),$ and $A^+_{dR,\sigma} \in \text{GL}_n(B^+_{dR,E})$.

Write $(f_1, \ldots, f_n) = (g_1, \ldots, g_n)(I_n + ZU_{dR})A_{dR}$ and put

$$c_B(\tilde{M}) = \left( (\sigma \mapsto \sum_{i,j} (U_{e,\sigma})_{ij} f_{i,z}^* \otimes f_{i,z}), (\sigma \mapsto \sum_{i,j} (U^+_{dR,\sigma})_{ij} g_{j,z}^* \otimes g_{j,z}), \sum_{i,j} (U_{dR})_{ij} g_{j,z}^* \otimes g_{j,z} \right).$$

**Proposition 4.2.** (a) $c_{\varphi}(\tilde{M})$ is in $Z^1(M^* \otimes M)$.

(b) $c_B(\tilde{M})$ is in $\ker(\delta_{1,W^* \otimes W})$.

(c) We have $i_M([c_{\varphi}(\tilde{M})]) = [c_B(\tilde{M})]$.

**Proof.** It is easy to verify (a) and (b).

Put $\tilde{M}_S = M^* \otimes E S$. We consider $\tilde{M}_S \otimes S \tilde{M}$ as an extension of $M^* \otimes E M$ by itself, and form the following commutative diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & M^* \otimes E M & \longrightarrow & M^* \otimes S \tilde{M} & \longrightarrow & M^* \otimes E M & \longrightarrow & 0,
\end{array}$$

where the vertical map $M_0 \rightarrow M^* \otimes E M$ is given by $1 \mapsto \sum_{i=1}^n e_{i,z}^* \otimes e_{i,z}$, which does not depend of the choice of the basis $\{e_1, \ldots, e_n\}$. Pulling back $\tilde{M}_S \otimes S \tilde{M}$ via $M_0 \rightarrow M^* \otimes E M$ we obtain an extension of $M_0$ by $M^* \otimes E \tilde{M}$. Let $\mathcal{M}$ denote the resulting extension. Then $\mathcal{M}$ is a sub-$E$-$B$-pair of $M^*_S \otimes S \tilde{M}$. Put $\mathbf{D}_B(\mathcal{M}) = (\tilde{W}_e, \tilde{W}^+_{dR})$. 


A lifting of 1 in \( \mathcal{W}_e \) is \( \sum_j f^*_j, z \otimes f_j \), and a lifting of 1 in \( \mathcal{W}^+_{\text{dir}} \) is \( \sum g^*_j, z \otimes g_j \). We have

\[
(\sigma - 1) \sum_j f^*_j, z \otimes f_j = \sigma(f^*_1, z, \ldots, f^*_n, z) \otimes (f_1 \ldots f_n) - (f^*_1, z, \ldots, f^*_n, z) \otimes (f_1 \ldots f_n)
\]

\[
= (f^*_1, z, \ldots, f^*_n, z)(A^t_{e,\sigma})^{-1} \otimes A^t_{e,\sigma}(1 + z U^t_{e,\sigma})
\]

\[
= (f^*_1, z, \ldots, f^*_n, z) \otimes U^t_{e,\sigma} z
\]

Similarly,

\[
(\sigma - 1) \sum_j g^*_j, z \otimes g_j = (g^*_1, z, \ldots, g^*_n, z) \otimes (U^t_{\text{dir},\sigma})^t z
\]

and

\[
\sum_j f^*_j, z \otimes f_j - \sum_j g^*_j, z \otimes g_j = (g^*_1, z, \ldots, g^*_n, z) \otimes U^t_{\text{dir},\sigma} z
\]

Hence the element in \( H^1_B(D_B(M^* \otimes_E M)) \) attached to the extension \( D_B(M) \) is \([c_B(\tilde{M})]\).

A similar computation shows that the element in \( H^1_B(M^* \otimes_E M) \) attached to the extension \( M \) is \([c_{\Phi \Gamma}(M)]\). Now (c) follows. \( \square \)

5 The reciprocity law and an application

5.1 Reciprocity law

In [14, Section 2] using local class field theory Zhang precisely described the perfect pairing

\[
H^1(G_K, E) \times H^1(G_K, E(1)) \rightarrow H^2(G_K, E(1)).
\]

We recall it below.

The Kummer theory gives us a canonical isomorphism so called the Kummer map

\[
\lim_n (K^\times/(K^\times)^p)^n \otimes_{\mathbb{Z}_p} E \rightarrow H^1(G_K, E(1))
\]

\[
\sum_i \alpha_i \otimes a_i \mapsto \sum_i \alpha_i[(\alpha_i)].
\]
Here $(\alpha)$ is the 1-cocycle such that 
\[
g\left(\frac{r^\alpha}{\alpha}\right) = \varepsilon_n^{(a_\phi)}
\]
for $\alpha \in K^\times$ and $g \in G_K$, where $(r^\alpha)^p = r^\sqrt[1/p]{\alpha}$. Combining the Kummer map and the exponent map 
\[\exp : \rho K \to K^\times\]
and extending it by linearity we obtain an embedding from $K \otimes_{Q_p} E$ to $H^1(G_K, E(1))$, again denoted by $\exp$. Then we have 
\[H^1(G_K, E(1)) = \exp(K \otimes_{Q_p} E) \oplus E \cdot [(p)].\]

Let $\text{Hom}(G_K, E)$ be the group of additive characters of $G_K$ with values in $E$. As the action of $G_K$ on $E$ is trivial, $H^1(G_K, E)$ is naturally isomorphic to $\text{Hom}(G_K, E)$. Let $\psi_0 : G_K \to E$ be the additive character that vanishes on the inertial subgroup of $G_K$ and maps the geometrical Frobenius to $[K_0 : \mathbb{Q}_p]$. For any $\tau \in \text{Emb}(K, E)$ let $\psi_\tau$ be the composition $\tau \circ \log \circ \text{rec}_{K}^{-1}$, where $\log$ is normalized such that $\log(p) = 0$. Then the space $E(E) \otimes_{Q_p} E$ is an $E$-basis of $H^1(G_K, E)$.

**Lemma 5.1.** [14, Proposition 2.1] The cup product of $a_0 \psi_0 + \sum_{\tau \in \text{Emb}(K, E)} a_\tau \psi_\tau$ $(a_0, a_\tau \in E)$ and $b_0([p]) + \exp(b)$ $(b_0 \in E, b \in K \otimes_{Q_p} E)$ is 
\[
(a_0 b_0 - \text{tr}_{K/Q_p}((a_\tau) \cdot b) \left(\psi_0 \cup [(p)]\right)).
\]
Here, $(a_\tau)$ is considered as an element in $K \otimes_{Q_p} E$ via the isomorphism (2.1).

**Lemma 5.2.** For $\lambda_0, \lambda_\tau \in E$ $(\tau \in \text{Emb}(K, E))$, the extension of $E$ (as a trivial $G_K$-module) by $E$ corresponding to the cocycle $\lambda_0 \psi_0 + \sum_{\tau \in \text{Emb}(K, E)} \lambda_\tau \psi_\tau$ is de Rham if and only if $\lambda_\tau = 0$ for each $\tau$.

**Proof.** By [11, Lemma 4.3], the subspace of extensions of $E$ by $E$ that are de Rham is 1-dimensional, and so consists of those corresponding to the cocycles $\lambda_0 \psi_0$ $(\lambda_0 \in E)$.

### 5.2 An auxiliary formula

Let $\mathcal{L} = (\mathcal{L}_\sigma)_{\sigma : K \to E}$ be a vector. We consider $\mathcal{L}$ as an element of $K \otimes_{Q_p} E$ via the isomorphism (2.1).

Let $D$ be a filtered $E$-$(\varphi, N)$-module: the underlying $E$-$(\varphi, N)$-module $D$ is a $(K_0 \otimes_{Q_p} E)$-module with a basis $\{f_1, f_2, f_3\}$ such that 
\[
\varphi^{[K_0 : \mathbb{Q}_p]} f_1 = p^{-[K_0 : \mathbb{Q}_p]} f_1, \quad \varphi^{[K_0 : \mathbb{Q}_p]} f_2 = f_2, \quad \varphi^{[K_0 : \mathbb{Q}_p]} f_3 = f_3,
\]
and 
\[N(f_1) = 0, \quad N(f_2) = -f_1, \quad N(f_3) = f_1;\]
the filtration on 
\[K \otimes_{K_0} D = (K \otimes_{Q_p} E) f_1 + (K \otimes_{Q_p} E) f_2 + (K \otimes_{Q_p} E) f_3\]

Since the character $\psi_\tau$ of the Weil group $W_K$ sends any lifting of the $q$th power Frobenius to 0, it can be extended to a character of $G_K$ which is again denoted by $\psi_\tau$. 

14
satisfies
\[ \text{Fil}^i D = \begin{cases} (K \otimes \mathbb{Q}_p E)(f_2 - \tilde{L}f_1) \oplus (K \otimes \mathbb{Q}_p E)(f_3 + \tilde{L}f_1) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases} \]

Let \( \pi_i \) be the projection map
\[ X_{\log}(D) \to B_{\log,E}, \quad \sum_{j=1}^3 a_j f_j \mapsto a_i. \]

**Lemma 5.3.** Let \( c : G_K \to X_{\log}(D) \) be a 1-cocycle whose class in \( H^1(G_K, X_{\log}(D)) \) belongs to \( \ker(H^1(G_K, X_{\log}(D)) \to H^1(G_K, X_{dR}(D))) \). Then there exist \( \gamma_{2,0}, \gamma_{2,\tau}, \gamma_{3,0}, \gamma_{3,\tau} \in E \) (\( \tau \in \text{Emb}(K, E) \)) such that
\[ \pi_2(c) = \gamma_{2,0} \psi_0 + \sum_{\tau \in \text{Emb}(K, E)} \gamma_{2,\tau} \psi_{\tau} \]
and
\[ \pi_3(c) = \gamma_{3,0} \psi_0 + \sum_{\tau \in \text{Emb}(K, E)} \gamma_{3,\tau} \psi_{\tau}. \]
Furthermore,
\[ \gamma_{2,0} - \gamma_{3,0} = \sum_{\tau \in \text{Emb}(K, E)} L_{\tau}(\gamma_{2,\tau} - \gamma_{3,\tau}). \]

In our proof of Lemma 5.3 we need the following

**Lemma 5.4.** Let \( D \) be an \( E-(\varphi, N) \)-module. If \( \text{Fil}_1 \) and \( \text{Fil}_2 \) are two filtrations on \( K \otimes_{K_0} D \) such that \( \text{Fil}_1^0(K \otimes_{K_0} D) = \text{Fil}_2^0(K \otimes_{K_0} D) \), then the kernel of
\[ H^1(G_K, X_{\log}(D)) \to H^1(G_K, X_{dR}(D, \text{Fil}_1)) \]
coincides with the kernel of
\[ H^1(G_K, X_{\log}(D)) \to H^1(G_K, X_{dR}(D, \text{Fil}_2)). \]

**Proof.** The proof is similar to that of [13, Proposition 2.5] \( \square \)

**Proof of Lemma 5.3.** The argument is similar to the proof of [13, Lemma 5.1]. We only give a sketch.

Write \( c_{\sigma} = \lambda_{1,\sigma} f_1 + \lambda_{2,\sigma} f_2 + \lambda_{3,\sigma} f_3 \). As \( c \) takes values in \( X_{\log}(D) \), we have \( \lambda_{2,\sigma}, \lambda_{3,\sigma} \in E \). This ensures the existence of \( \gamma_{2,0}, \gamma_{2,\tau}, \gamma_{3,0}, \gamma_{3,\tau} \).

Let \( \text{Fil} \) be the filtration on \( D \) such that \( \text{Fil}^{-1} D = D \) and \( \text{Fil}^i D = \text{Fil}^i D \) if \( i \geq 0 \). Then \( (D, \text{Fil}) \) is admissible. Let \( V \) be the semistable \( E \)-representation of \( G_K \) attached to \( D_V = (D, \text{Fil}) \). By Lemma 5.4, \( [c] \) is in the kernel of \( H^1(G_K, X_{\log}(D_V)) \to H^1(G_K, X_{dR}(D_V)) \) and so there exists a 1-cocycle \( c^{(1)} : G_K \to V \) such that the image of \( [c^{(1)}] \) by \( H^1(G_K, V) \to H^1(G_K, X_{\log}(D_V)) \) is \([c] \).
We form the following commutative diagram

\[
\begin{array}{ccccccccc}
V' & \downarrow & V & \downarrow & T & \downarrow & 0 \\
0 & \rightarrow & V_0 & \rightarrow & V & \rightarrow & V_1 & \rightarrow & T_1 & \rightarrow & 0 \\
\end{array}
\]

with the horizontal lines being exact, where \(V_0\) (resp. \(V'\)) is the subrepresentation of \(V\) corresponding to the filtered \(E-(\varphi, N)\)-submodule of \(D_V\) generated by \(f_1\) (resp. by \(f_2 + f_3\)) which is admissible.

From (5.1) we obtain the following commutative diagram

\[
\begin{array}{ccccccc}
H^1(G_K, V) & \rightarrow & \pi_{V,V_1} & \rightarrow & H^2(G_K, V_0) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(G_K, V_1) & \rightarrow & H^1(G_K, T_1) & \rightarrow & H^2(G_K, V_0), \\
\end{array}
\]

where the horizontal lines are exact.

Write \(c^{(2)}\) for the 1-cocycle \(\varphi \colon V \rightarrow T \rightarrow T_1\). By a simple computation we obtain

\[
[c^{(2)}] = \left[ (\gamma_{2,0} - \gamma_{3,0})\psi_0 + \sum_{\tau \in \text{Emb}(K,E)} (\gamma_{2,\tau} - \gamma_{3,\tau})\psi_\tau \right] \bar{f}_2,
\]

where \(\bar{f}_2\) is the image of \(f_2 \in V\) in \(T_1\). Note that \(T_1\) is isomorphic to \(E\), and \(V_0\) is isomorphic to \(E(1)\).

By [14, Lemma 5.5], as an extension of \(E\) by \(E(1)\), \(V_1\) corresponds to the element \([\langle p \rangle] + \exp(\vec{L})\). Now Lemma 5.1 yields our second assertion.

\section{L-invariants}

Let \(D\) be a filtered \(E-(\varphi, N)\)-module of rank \(n\). Fix a refinement \(\mathcal{F}\) of \(D\). Then \(\mathcal{F}\) fixes an ordering \(\alpha_1, \ldots, \alpha_n\) of the eigenvalues of \(\varphi\) on \([K_0 \otimes \mathbb{Q}_p]\) and an ordering \(\vec{k}_1, \ldots, \vec{k}_n\) of the Hodge-Tate weights.

\subsection{The operator \(N_\mathcal{F}\)}

The operator \(\varphi\) induces a \(K_0 \otimes \mathbb{Q}_p\) \(E\)-semilinear operator \(\varphi_\mathcal{F}\) on \(\text{gr}^\mathcal{F}_D = \bigoplus_{i=1}^n \mathcal{F}_i D/\mathcal{F}_{i-1} D\).

We define a \(K_0 \otimes \mathbb{Q}_p\) \(E\)-linear operator \(N_\mathcal{F}\) on \(\text{gr}^\mathcal{F}_D\). The definition is similar to the one defined in [13], so we omit some details.

For any \(i \in \{1, \ldots, n\}\), if \(N(\mathcal{F}_i D) = N(\mathcal{F}_{i-1} D)\), we demand that \(N_\mathcal{F}\) maps \(\text{gr}_i^\mathcal{F} D\) to zero.

Now we assume that \(N(\mathcal{F}_i D) \supseteq N(\mathcal{F}_{i-1} D)\). Let \(j\) be the minimal integer such that

\[
N(\mathcal{F}_i D) \subseteq N(\mathcal{F}_{i-1} D) + \mathcal{F}_j D.
\]
Proposition 6.1. $N(F_{i-1}D) \cap F_jD = N(F_{i-1}D) \cap F_jD$.

Proof. Note that $F_jD$, $F_{j-1}D$, $N(F_{i-1}D)$ and $F_jD + N(F_{i-1}D)$ are stable by $\varphi$. Thus $(N(F_{i-1}D) + F_jD)/(N(F_{i-1}D) + F_jD)$ is a $\varphi$-module, and so must be free over $K_0 \otimes_{Q_p} E$. Hence the map

$$F_jD/F_{j-1}D \to (N(F_{i-1}D) + F_jD)/(N(F_{i-1}D) + F_jD)$$

(6.1) is an isomorphism. It follows that $N(F_{i-1}D) \cap F_jD = N(F_{i-1}D) \cap F_jD$.

The operator $N$ induces a $K_0 \otimes_{Q_p} E$-linear map

$$F_jD/F_{j-1}D \to (N(F_{i-1}D) + F_jD)/(N(F_{i-1}D) + F_jD).$$

We define the map $N_F : \operatorname{gr}_F^j D \to \operatorname{gr}_F^j D$ to be the composition of this map and the inverse of (6.1).

Finally we extend $N_F$ to the whole $\operatorname{gr}_F^j D$ by $K_0 \otimes_{Q_p} E$-linearity. Note that $N_F \varphi_F = p \varphi_F N_F$. By definition, for any $i$ we have either $N(\operatorname{gr}_F^j D) = 0$ or $N(\operatorname{gr}_F^j D) = \operatorname{gr}_F^j D$ for some $j$.

Definition 6.2. For $j \in \{1, \cdots, n-1\}$ we say that $j$ is marked (or a marked index) for $F$ if there is some $i \in \{2, \cdots, n\}$ such that $N_F(\operatorname{gr}_F^j D) = \operatorname{gr}_F^j D$.

Note that $i$ and $j$ in the above definition are determined by each other. We write $i = t_F(j)$ and $j = s_F(i)$.

Proposition 6.3. The following two assertions are equivalent:

(a) $s$ is marked and $t = t_F(s)$.

(b) $N F_{t-1}D \cap F_sD = NF_{t-1}D \cap F_{s-1}D$ and $N F_{t}D \cap F_sD \supseteq NF_{t}D \cap F_{s-1}D$.

Proof. We have already seen that, if (a) holds, then (b) holds. Conversely, we assume that (b) holds. Then $NF_{t}D \cap F_sD \supseteq NF_{t-1}D \cap F_sD$. Thus $NF_{t}D \supseteq NF_{t-1}D$.

We show that $NF_{t}D \not\subseteq NF_{t-1}D + F_{s-1}D$. If it is not true, then there exists $y \in F_{t}D \setminus F_{t-1}D$ which is a lifting of a basis of $\operatorname{gr}_F^j D$ over $K_0 \otimes_{Q_p} E$ such that $N(y) \in F_{t-1}D$. For any $z \in F_{t}D$, write $z = w + \lambda y$ with $w \in F_{t-1}D$ and $\lambda \in K_0 \otimes_{Q_p} E$. If $N(z)$ is in $F_{t}D$, then $N(w)$ is also in $F_{t}D$. But $NF_{t-1}D \cap F_{t}D = NF_{t-1}D \cap F_{s-1}D$. Thus $N(w)$ is in $F_{t-1}D$, which implies that $N(z) = N(w) + \lambda N(y)$ is also in $F_{s-1}D$. So, $NF_{t}D \cap F_sD = NF_{t}D \cap F_{s-1}D$, a contradiction.

From $NF_{t}D \cap F_sD \supseteq NF_{t-1}D \cap F_sD$ we see that there is $x \in F_{t}D \setminus F_{t-1}D$ such that $N(x) \in F_{t}D$. We must have $NF_{t}D \subseteq NF_{t-1}D + F_{t}D$. Otherwise, let $j$ be the smallest integer such that $NF_{t}D \subseteq NF_{t-1}D + F_{t}D$ and assume that $j > s$. Then $NF_{t}(x + F_{t-1}D) = 0$, which contradicts the fact that $N_F : \operatorname{gr}_F^j D \to \operatorname{gr}_F^j D$ is an isomorphism.

6.2 Strongly marked indices and $L$-invariants

Assume that $s$ is marked for $F$ and $t = t_F(s)$. We consider the decompositions

$$F_tD/F_{s-1}D = (K_0 \otimes_{Q_p} E) \cdot e_s \oplus L \oplus (K_0 \otimes_{Q_p} E) e_t$$

that satisfy the following conditions:

- $F_{t}(F_tD/F_{s-1}D) = (K_0 \otimes_{Q_p} E)e_s$ and $F_{t-s}(F_tD/F_{s-1}D) = (K_0 \otimes_{Q_p} E)e_s \oplus L$, where $F$ is the refinement on $F_tD/F_{s-1}D$ induced by $F$.

- Both $L$ and $(K_0 \otimes_{Q_p} E)e_s \oplus (K_0 \otimes_{Q_p} E)e_t$ are stable by $\varphi$ and $N$: $\varphi|_{K_0 \otimes_{Q_p} E}(e_t) = \alpha_t e_t$ and $N(e_s) = e_s$.

Such a decomposition is called an $s$-decomposition.
Definition 6.5. If there exists a perfect $\mathcal{F}_t D/\mathcal{F}_{s-1} D = E\bar{e}_s \oplus L \oplus E\bar{e}_t$.

There is a natural isomorphism $E\bar{e}_s \oplus E\bar{e}_t \rightarrow (\mathcal{F}_t D/\mathcal{F}_{s-1} D)/L$ of $(\varphi, N)$-modules. Usually the filtration on the filtered $E\mathcal{E}$-($\varphi, N$)-submodule $E\bar{e}_s \oplus E\bar{e}_t$ and that on $(\mathcal{F}_t D/\mathcal{F}_{s-1} D)/L$ are different.

When these two filtrations satisfy certain compatible condition, we say the decomposition dec is perfect. Precisely, we say that dec is perfect if for any $\tau : K \rightarrow E$ we have $k_{s, \tau} < k_{t, \tau}$, and if there exist $k'_{s, \tau}, k'_{t, \tau}$ and $\mathcal{L}_{\text{dec}, \tau} \in E$ satisfying $k_{s, \tau} \leq k'_{s, \tau} < k'_{t, \tau} \leq k_{t, \tau}$ such that the following conditions hold.

- The filtration on the filtered $E\mathcal{E}$($\varphi, N$)-submodule $E\bar{e}_s \oplus E\bar{e}_t$ satisfies

$$\text{Fil}_s^i E\bar{e}_s \oplus E\bar{e}_t = \begin{cases} E\bar{e}_{s, \tau} \oplus E\bar{e}_{t, \tau} & \text{if } i \leq k_{s, \tau}, \\ E(\bar{e}_{i, \tau} + \mathcal{L}_{\text{dec}, \tau} \bar{e}_{s, \tau}) & \text{if } k_{s, \tau} < i \leq k'_{s, \tau}, \\ 0 & \text{if } i > k'_{s, \tau}, \end{cases}$$

- The filtration on the quotient of $\mathcal{F}_t D/\mathcal{F}_{s-1} D$ by $L$ satisfies

$$\text{Fil}_s^i \mathcal{F}_t D/\mathcal{F}_{s-1} D = \begin{cases} E\bar{e}_{s, \tau} \oplus E\bar{e}_{t, \tau} & \text{if } i \leq k'_{s, \tau}, \\ E(\bar{e}_{i, \tau} + \mathcal{L}_{\text{dec}, \tau} \bar{e}_{s, \tau}) & \text{if } k'_{s, \tau} < i \leq k_{t, \tau}, \\ 0 & \text{if } i > k_{t, \tau}, \end{cases}$$

where the images of $\bar{e}_s$ and $\bar{e}_t$ in $\mathcal{F}_t D/\mathcal{F}_{s-1} D$ are again denoted by $\bar{e}_s$ and $\bar{e}_t$.

**Remark 6.4.** $s$-decompositions may be not exist. However, if $\varphi$ is semisimple, then $s$-decompositions always exist (see [13]).

Let dec denote an $s$-decomposition $\mathcal{F}_t D/\mathcal{F}_{s-1} D = E\bar{e}_s \oplus L \oplus E\bar{e}_t$.

**Definition 6.5.** If there exists a perfect $s$-decomposition, we say that $s$ is strongly marked (or a strongly marked index). In this case we attached to each pair $(s, t)$ with $t = t_{F}(s)$ an invariant $\bar{L}_{F, s, t} = (\mathcal{L}_{\text{dec}, \tau})_{\tau}$, where dec is a perfect $s$-decomposition. Proposition 6.6 below tells us that $\bar{L}_{F, s, t}$ is independent of the choice of perfect $s$-decompositions. We call $\bar{L}_{F, s, t}$ the Fontaine-Mazur $\mathcal{L}$-invariant associated to $(F, s, t)$, and denote $\mathcal{L}_{\text{dec}, \tau}$ by $\bar{L}_{F, s, t, \tau}$.

In the case of $t = s + 1$, $s$ is strongly marked if and only if $k_{s, \tau} < k_{t, \tau}$ for all $\tau$.

**Proposition 6.6.** If dec$_1$ and dec$_2$ are two perfect $s$-decompositions, then $\mathcal{L}_{\text{dec}_1, \tau} = \mathcal{L}_{\text{dec}_2, \tau}$ for any $\tau$.

**Proof.** The argument is similar to the proof of [13, Proposition 4.9].

Let $D^*$ be the filtered $E\mathcal{E}$($\varphi, N$)-module that is the dual of $D$. Let $\bar{F}$ be the refinement on $D^*$ such that $\bar{F}_i D^* := (\mathcal{F}_{n-i} D)^\perp = \{ y \in D^* : \langle y, x \rangle = 0 \text{ for all } x \in \mathcal{F}_{n-i} D \}$.

We call $\bar{F}$ the dual refinement of $\mathcal{F}$.

If $L \subset M$ are submodules of $D$, then $M^\perp \subset L^\perp$. The pairing $\langle \cdot, \cdot \rangle : L^\perp \times M$ induces a non-degenerate pairing on $L^\perp / M^\perp \times M / L$, so that we can identify $L^\perp / M^\perp$ with the dual of $M / L$ naturally. In particular, $\text{gr}_{F}^\mathcal{E} D^*$ is naturally isomorphic to the dual of $\text{gr}_{\mathcal{F}}^\mathcal{E} D^*$. Thus $\text{gr}_{F}^\mathcal{E} D^*$ is naturally isomorphic to the dual of $\text{gr}_{\mathcal{F}}^\mathcal{E} D$.

**Proposition 6.7.** (a) $N_{\bar{F}}$ is dual to $-N_{\bar{F}}$.

(b) $s$ is marked for $\mathcal{F}$ if and only if $n+1-t_{\bar{F}}(s)$ is marked for $\bar{F}$.

(c) $s$ is strongly marked for $\mathcal{F}$ if and only if $n+1-t_{\bar{F}}(s)$ is strongly marked for $\bar{F}$.

**Proof.** The proof of (a) is similar to that of [13, Proposition 4.14]. The proof of (b) is similar to that of [13, Proposition 4.13]. The proof of (c) is similar to that of [13, Proposition 4.15 (a)].
7 Projection vanishing property

Put $S = E[Z]/(Z^{2})$. Let $z$ be the closed point defined by the maximal ideal $(Z)$ of $S$.

Let $W = (W_{z}, W_{z}^{\vee})$ be an $S$-$B$-pair. Let $\{w_{1}, \ldots, w_{n}\}$ be a $B_{z,S}$-basis of $W_{z}$. Suppose that $W$ admits a triangulation $Fil_{\ast}$. Let $(\delta_{1}, \ldots, \delta_{n})$ be the corresponding triangulation parameters. Then for each $i = 1, \ldots, n$ there exists a continuous additive character $\epsilon_{i}$ of $K^{\times}$ with values in $E$ such that $\delta_{i} = \delta_{i,z}(1 + Z\epsilon_{i})$.

Suppose that $W_{z}$, the evaluation of $W$ at $z$, is semistable, and let $D_{z}$ be the filtered $E$-$(\varphi, N)$-module attached to $W_{z}$. Let $\mathcal{F}$ be the refinement of $D_{z}$ corresponding to the induced triangulation of $W_{z}$, and let $\{\epsilon_{1,z}, \epsilon_{2,z}, \ldots, \epsilon_{n,z}\}$ be a $(K_{0} \otimes_{Q_{p}} E)$-basis of $D_{z}$ that is compatible with $\mathcal{F}$ i.e. $\mathcal{F}D = (K_{0} \otimes_{Q_{p}} E)\epsilon_{1,z} \oplus \cdots \oplus (K_{0} \otimes_{Q_{p}} E)\epsilon_{n,z}$. Let $\alpha_{i,z} \in E$ be such that $\varphi^{[K_{0}:Q_{p}]}(\epsilon_{i,z}) = \alpha_{i,z}\epsilon_{i,z}$ mod $\mathcal{F}_{i-1}$.

Let $x_{ij} \in B_{log,E}$ ($i, j = 1, \ldots, n$) be such that

$$e_{i,z} = x_{1i}w_{1,z} + \cdots + x_{ni}w_{n,z}. \quad (7.1)$$

Then $X = (x_{ij})$ is in $GL_{n}(B_{log,E})$. Write the matrix of $\sigma \in G_{K}$ with respect to the basis $\{w_{1}, \ldots, w_{n}\}$ by $(I_{n} + ZU_{e}\varphi)A_{e}\varphi$. As $\epsilon_{1,z}, \ldots, \epsilon_{n,z}$ are fixed by $G_{K}$, we have $X^{-1}A_{e}\varphi(X) = I_{n}$ for all $\sigma \in G_{K}$.

For $i = 1, \ldots, n$ put $e_{i} = x_{1i}w_{1,z} + \cdots + x_{ni}w_{n,z}$. Then $\{e_{1}, \ldots, e_{n}\}$ is a basis of $B_{log,S} \otimes_{S} W_{e}$ over $B_{log,S}$.

**Lemma 7.1.** If $T$ is the matrix of $\varphi_{D_{e}}$ for the basis $\{e_{1,z}, \ldots, e_{n,z}\}$, then $T$ is also the matrix of $\varphi_{B_{log,S} \otimes_{S} W_{e}}$ for the basis $\{e_{1}, \ldots, e_{n}\}$.

**Proof.** The assertion follows from the definition of $\{e_{1}, \ldots, e_{n}\}$ and the fact that $w_{1,z}, \ldots, w_{n,z}, w_{1}, \ldots, w_{n}$ are fixed by $\varphi$. \hfill $\square$

In Section 4.1 we attach to $W$ an element $c_{B}(W)$ in $H^{1}_{B}(W_{z}^{\vee} \otimes W_{z})$. Consider the composition

$$H^{1}_{B}(W_{z}^{\vee} \otimes W_{z}) \to H^{1}(G_{K}, W_{c,z}^{\vee} \otimes_{B_{z,E}} W_{c,z}) \to H^{1}(G_{K}, B_{log,E} \otimes_{E} (D_{e}^{\ast} \otimes D_{z})).$$

As the matrix of $\sigma \in G_{K}$ for the basis $\{e_{1}, \ldots, e_{n}\}$ is $I_{n} + ZX^{-1}U_{e}\varphi X$, from the discussion in Section 4 we see that the image of $c_{B}$ in $H^{1}(G_{K}, B_{log,E} \otimes_{E} (D_{e}^{\ast} \otimes D_{z}))$ is the class of the 1-cocycle

$$(U_{e}\varphi)_{ij}w_{j,z}^{\ast} \otimes w_{i,z} = (X^{-1}U_{e}\varphi X)_{ij}e_{j,z}^{\ast} \otimes e_{i,z}.$$ 

Let $\pi_{h\ell}$ be the projection

$$B_{log,E} \otimes_{E} (D_{e}^{\ast} \otimes D_{z}) \to B_{log,E}, \quad \sum_{j,i} b_{ji}e_{j,z}^{\ast} \otimes e_{i,z} \mapsto b_{h\ell}. \quad (7.2)$$

For $h = 1, \ldots, n$, let $\epsilon_{h}$ be the additive character of $G_{K}$ such that $\epsilon_{h} \circ \text{rec}_{K}(p) = 0$ and $\epsilon_{h} \circ \text{rec}_{K}|_{\alpha_{K}} = \epsilon_{h}|_{\alpha_{K}}$.

**Theorem 7.2.** (a) For any pair of integers $(h, \ell)$ such that $h < \ell$ we have $\pi_{h\ell}([c]) = 0$.

(b) For any $h = 1, \ldots, n$, $\pi_{h,h}([c])$ coincides with the image of $[\epsilon_{h}^{\ast}]$ in $H^{1}(G_{K}, B_{log,E})$.

We call (a) the projection vanishing property.
Proof. The filtered $E$-$(\varphi,N)$-module attached to $W_z/\text{Fil}_{h-1}W_z$ is $D_z/F_{h-1}D_z$. We denote the image of $e_{\ell,z}$ ($\ell \geq h$) in $D_z/F_{h-1}D_z$ again by $e_{\ell,z}$.

Let $\delta'_{h}$ be the character of $G_K$ such that $\delta'_{h} = 1 + Z\epsilon'_{h}$. By Lemma 3.5 there exists an element

$$x \in (\mathcal{B}_{\text{max},E} \otimes \mathcal{B}_{E,E} (W/\text{Fil}_{h-1}W)_e)^{G_K = \delta'_{h},\sigma[X]}\oplus_{\alpha_{\ell,z} = 1 + Z\epsilon'_{h}(1 + Z\epsilon_{h}(\pi_{k})e_{h}(p))}$$

whose image in $D_z/F_{h-1}D_z$ is $e_{h,z}$. Write $x = e_h + Z \sum_{\ell \geq h} \lambda_{\ell} e_{\ell}$ with $\lambda_{\ell} \in \mathcal{B}_{\log,E}$.

As the matrix of $\sigma \in G_K$ for the basis $\{e_1, \cdots, e_n\}$ is $I_n + Z X^{-1} U_{e,\sigma} X$, we have

$$[1 + Z\epsilon'_{h}(\sigma)]x = [1 + Z\epsilon'_{h}(\sigma)](e_h + Z \sum_{\ell \geq h} \lambda_{\ell} e_{\ell})$$

$$= \sigma(x) = e_h + Z \sum_{\ell \geq h} (X^{-1} U_{e,\sigma} X)_{\ell h} e_{\ell} + Z \sum_{\ell \geq h} \sigma(\lambda_{\ell}) e_{\ell}.$$

For $\ell > h$, comparing the coefficients of $e_{\ell}$ we obtain

$$(X^{-1} U_{e,\sigma} X)_{\ell h} = (1 - \sigma) \lambda_{\ell},$$

which shows (a). Similarly, comparing coefficients of $e_h$ we obtain

$$(X^{-1} U_{e,\sigma} X)_{hh} - \epsilon'_{h}(\sigma) = (1 - \sigma) \lambda_{h}, \quad (7.3)$$

which implies (b). \hfill \Box

8 The proof of Theorem 1.2

We will need the following lemmas.

Lemma 8.1. The inclusion $E \hookrightarrow \mathcal{B}_{E,E}$ induces an isomorphism

$$H^1(G_K, E) \xrightarrow{\sim} \ker(N : H^1(G_K, \mathcal{B}_{E,E}) \to H^1(G_K, \mathcal{B}_{\log,E})).$$

Proof. The proof is identical to that of [13, Corollary 1.4]. \hfill \Box

Lemma 8.2. The map $N : \mathcal{B}_{\log,E}^{p = 1} \to \mathcal{B}_{\log,E}^{p = 1}$ is surjective.

Proof. The proof is identical to that of [13, Lemma 1.2]. \hfill \Box

For the proof of Theorem 1.2 we may assume that $S = E[Z]/(Z^2)$, and $z$ is the closed point defined by the maximal ideal $(Z)$. Let $W$ be as in Theorem 1.2. Replacing $W$ by the $E$-$B$-pair $\mathcal{F}_W/\mathcal{F}_{s-1}W$ and replacing $\mathcal{F}$ by the induced refinement on $\mathcal{F}_W/\mathcal{F}_{s-1}W$, we may assume that $s = 1$ and $t = n = \text{rank}_{\mathcal{B}_{E,E}}(W_e)$. Let $e_{1,z}, e_{2,z}, \cdots, e_{n,z}$ be a $K_0 \otimes_{\mathcal{Q}_E} E$-basis of $D_z$ such that

$$(K_0 \otimes_{\mathcal{Q}_E} E)e_{1,z} \oplus L \bigoplus_{i=2}^{n-1}(K_0 \otimes_{\mathcal{Q}_E} E)e_{i,z} \quad (8.1)$$

with $L = \oplus_{i=2}^{n-1}(K_0 \otimes_{\mathcal{Q}_E} E)e_{i,z}$ a perfect 1-decomposition of $D_z$ for $\mathcal{F}$ (see §6.2 for the meaning of perfect decompositions). Let $e_{1,z}^*, e_{2,z}^*, \cdots, e_{n,z}^*$ be the dual basis of $D_z^*$ over $K_0 \otimes_{\mathcal{Q}_E} E$. 20
Let $D_1$ be the quotient of $D_2$ by $L$. $D_2$ is the quotient of $D_2^*$ by $\bigoplus_{i=2}^{n-1} (K_0 \otimes_{Q_p} E)c_{i,z}^*$. Put $\mathcal{D} = D_2^* \otimes D_1$. The images of $e_{1,z}$ and $e_{n,z}$ in $D_1$ are again denoted by $e_{1,z}$ and $e_{n,z}$, and the images of $e_{1,z}^*$ and $e_{n,z}^*$ in $D_2^*$ are again denoted by $e_{1,z}^*$ and $e_{n,z}^*$ respectively. So $e_{1,z}^* \otimes e_{1,z}, e_{1,z}^* \otimes e_{n,z}, e_{n,z}^* \otimes e_{1,z}, e_{n,z}^* \otimes e_{n,z}$ form a $K_0 \otimes_{Q_p} E$-basis of $\mathcal{D}$. Let $\mathcal{D}_0$ be the filtered $E$-$\varphi, N$-submodule of $\mathcal{D}$ with a $K_0 \otimes_{Q_p} E$-basis $\{e_{1,z}^* \otimes e_{1,z}, e_{n,z}^* \otimes e_{1,z}, e_{n,z}^* \otimes e_{n,z}\}$. Let $\mathcal{W} = (\mathcal{W}, \mathcal{W}^+_d)$ (resp. $\mathcal{W}_0$) be the $E$-$B$-pair attached to $\mathcal{D}$ (resp. $\mathcal{D}_0$). Note that

$$\varphi_{[K_0:Q_p]}(e_{1,z}^* \otimes e_{1,z}) = e_{1,z}^* \otimes e_{1,z}, \varphi_{[K_0:Q_p]}(e_{n,z}^* \otimes e_{n,z}) = e_{n,z}^* \otimes e_{n,z},$$

and

$$-N(e_{1,z}^* \otimes e_{1,z}) = N(e_{n,z}^* \otimes e_{n,z}) = e_{n,z}^* \otimes e_{1,z}, N(e_{n,z}^* \otimes e_{1,z}) = 0.$$

Let $\tilde{\mathcal{L}}_x = \tilde{\mathcal{L}}_{x,t}$ be the $L$-invariant defined in Definition 6.5. As (8.1) is a prefect decomposition, we have

$$\text{Fil}^0(K \otimes_{K_0} \mathcal{D}) = Ee_{n,z}^* \otimes (e_{n,z} + \tilde{\mathcal{L}}_x e_{1,z}) \oplus E(e_{1,z}^* - \tilde{\mathcal{L}}_x e_{n,z}) \otimes e_{1,z},$$

and

$$\text{Fil}^0(K \otimes_{K_0} \mathcal{D}_0) = Ee_{n,z}^* \otimes (e_{n,z} + \tilde{\mathcal{L}}_x e_{1,z}) \oplus E(e_{1,z}^* - \tilde{\mathcal{L}}_x e_{n,z}) \otimes e_{1,z}.$$

Consider $W$ as an infinitesimal deformation of $W_z$. In Section 4.2 we attach to this infinitesimal deformation an element $c_B(W)$ in $H^1_b(W_z \otimes W_z)$. Let $[c]$ be the image of $c_B(W)$ by the composition

$$H^1_b(W_z \otimes W_z) \rightarrow H^1(G_K, W_z \otimes_{B_{e,e}} W, \mathcal{W}) \rightarrow H^1(G_K, B_{log,E} \otimes_{K_0 \otimes_{Q_p} E} (D_z^* \otimes D_z)),$$

and choose a 1-cocycle $c$ representing $[c]$. Write $c$ in the form

$$c = \sum_{j, i} c_{j,i} e_{j,z}^* \otimes e_{i,z}$$

with $c_{j,i}$ being a 1-cocycle of $G_K$ with values in $B_{log,E}$. By the projection vanishing property (Theorem 7.2 (a)) we have $[c_{1,n}] = 0$.

**Lemma 8.3.** There exist $\xi_1, \xi_n \in B_{e,e}$ and $\gamma_{1,0}, \gamma_{1,\tau}, \gamma_{n,0}, \gamma_{n,\tau}$ ($\tau \in \text{Emb}(K,E)$) such that

$$c_{1,1}(\sigma) = (\sigma - 1)\xi_1 + \gamma_{1,0}\psi_0(\sigma) + \sum_{\tau \in \text{Emb}(K,E)} \gamma_{1,\tau}\psi_\tau(\sigma)$$

and

$$c_{n,n}(\sigma) = (\sigma - 1)\xi_n + \gamma_{n,0}\psi_0(\sigma) + \sum_{\tau \in \text{Emb}(K,E)} \gamma_{n,\tau}\psi_\tau(\sigma)$$

for any $\sigma \in G_K$.

**Proof.** Let $\tilde{c}_B$ be the image of $c_B$ in $H^1_b(\mathcal{W})$, and let $\tilde{c}$ be the 1-cocycle

$$\tilde{c} = \sum_{j, i \in \{1, n\}} c_{j,i} e_{j,z}^* \otimes e_{i,z}$$

21
of $G_K$ with values in $B_{\log,E} \otimes_{K_0 \otimes Q_p,E} \mathcal{D}$. Then the image of $\bar{c}_B$ in

$$H^1(G_K, B_{\log,E} \otimes_{K_0 \otimes Q_p,E} \mathcal{D})$$

is $[\bar{c}]$. 

Note that $\bar{c}$ has values in $\mathcal{H}_c = (B_{\log,E} \otimes_{K_0 \otimes Q_p,E} \mathcal{D})_{\varphi=1,N=0}$. So, in particular $c_{1,1}$ and $c_{n,n}$ have values in $B_{e,E}$. As $N \bar{c} = 0$, we have

$$N(c_{n,1}) = c_{1,1} - c_{n,n}, \quad -N(c_{1,1}) = N(c_{n,n}) = c_{1,n}.$$ 

As $[c_{1,n}] = 0$, the statement follows from Lemma 8.1.

Write $\delta_i = \delta_{i,z}(1 + Z\epsilon_i)$. Let $\epsilon'_i$ be the additive character of $G_K$ with values in $E$ such that $\epsilon'_i \circ \text{rec}_K(p) = 0$ and $\epsilon'_i \circ \text{rec}_K|_{\sigma_K} = \epsilon_i|_{\sigma_K}$. Then there are $\epsilon_{i,\tau}$ ($\tau \in \text{Emb}(K,E)$) such that $\epsilon_i = \sum_{\tau \in \text{Emb}(K,E)} \epsilon_{i,\tau} \tau$.

**Lemma 8.4.** For $h = 1, n$ we have $[K_0 : Q_p] \gamma_{h,0} = -v_p(\pi_K) \epsilon_h(p)$ and $\gamma_{h,\tau} = \epsilon_{h,\tau}$.

**Proof.** We keep to use notations in the proof of Theorem 7.2. By (7.3) and Lemma 8.3 we have

$$\begin{align*}
(\sigma - 1)(\lambda_h) &= -(X^{-1}U_x X)_{hh} + \sum_{\tau \in \text{Emb}(K,E)} \epsilon_{h,\tau} \psi_\tau(\sigma) \\
&= -(\sigma - 1)\xi_h - \gamma_{h,0} \psi_0(\sigma) + \sum_{\tau \in \text{Emb}(K,E)} (\epsilon_{h,\tau} - \gamma_{h,\tau}) \psi_\tau(\sigma).
\end{align*}$$

Note that there exists $\omega \in W(F_p)$ such that $\varphi(\omega) - \omega = 1$, where $W(F_p)$ is the ring of Witt vectors with coefficients in the algebraic closure of $F_p$. Then $(\sigma - 1)\omega = \psi_0(\sigma)$. Hence

$$\sum_{\tau \in \text{Emb}(K,E)} (\epsilon_{h,\tau} - \gamma_{h,\tau}) \psi_\tau(\sigma) = (\sigma - 1)(\lambda_h + \xi_h + \gamma_{h,0} \omega).$$

In other words, the cocycle $\sum_{\tau \in \text{Emb}(K,E)} (\epsilon_{h,\tau} - \gamma_{h,\tau}) \psi_\tau(\sigma)$ is de Rham. By Lemma 5.2 we have $\gamma_{h,\tau} = \epsilon_{h,\tau}$ and $\lambda_h + \xi_h + \gamma_{h,0} \omega \in E$. Then

$$\varphi([K_0 : Q_p] - 1) \lambda_h = -(\varphi - 1)\xi_h - \gamma_{h,0} (\varphi|[K_0 : Q_p] - 1) \omega = -[K_0 : Q_p] \gamma_{h,0}. \quad (8.2)$$

By our choice of the basis $\{\epsilon_{1,z}, \ldots, \epsilon_{n,z}\}$, $Y_1 = \oplus_{i=2}^n Z \epsilon_{i,z}$ is stable by $\varphi$. Put $Y_n = 0$. Let $x$ be as in the proof of Theorem 7.2. By Lemma 7.1 we have $\varphi|[K_0 : Q_p] \epsilon_{h,z} = \alpha_{h,z} \epsilon_{h,z}$. Thus for $h = 1, n$ we have

$$\varphi|[K_0 : Q_p](x) = (1 + Z \varphi|[K_0 : Q_p](\lambda_h)) \alpha_{h,z} \epsilon_{h} \quad (\text{mod } Y_h).$$

On the other hand,

$$\begin{align*}
\varphi|[K_0 : Q_p](x) &= (1 + Z v_p(\pi_K) \epsilon_h(p)) \alpha_{h,z} x \\
&= (1 + Z v_p(\pi_K) \epsilon_h(p)) \alpha_{h,z} (1 + Z \lambda_h) \epsilon_{h} \quad (\text{mod } Y_h).
\end{align*}$$

Hence we obtain

$$\varphi|[K_0 : Q_p] - 1) \lambda_h = v_p(\pi_K) \epsilon_h(p). \quad (8.3)$$
By (8.2) and (8.3) we have

\[ [K_0 : \mathbb{Q}_p] g_{n,0} = -(\varphi |_{K_0 : \mathbb{Q}_p} - 1) \lambda_h = -v_p(\pi_K) \epsilon_h(p), \]

as wanted. \( \Box \)

By Lemma 8.2 there exists some \( y \in B_{\log,E}^{\geq p} \) such that \( N(y) = \xi_1 - \xi_h \). Let \( \bar{\epsilon}' \) be the 1-cocycle of \( G_K \) with values in \( B_{\log,E} \otimes_{K_0 \otimes \mathbb{Q}_p} \mathcal{D}_0 \) such that

\[ \bar{\epsilon}' = c_{1,1}' e_{1,2} \otimes e_{1,3} + c_{n,1}' e_{n,2} \otimes e_{n,3} + c_{n,1}' e_{n,2} \otimes e_{1,3} \]

with

\[ c_{1,1}' = \gamma_{1,0} \psi_0 + \sum_{\tau \in \text{Emb}(K,E)} \gamma_{1,\tau} \psi_\tau, \quad c_{n,1}' = \gamma_{n,0} \psi_0 + \sum_{\tau \in \text{Emb}(K,E)} \gamma_{n,\tau} \psi_\tau \]

and

\[ c_{n,1}'(\sigma) = c_{n,1}'(\sigma) - (\sigma - 1)y, \quad \sigma \in G_K. \]

It is easy to check that \( \varphi(\bar{\epsilon}') = \bar{\epsilon}' \) and \( N(\bar{\epsilon}') = 0 \). Hence \( \bar{\epsilon}' \) is a 1-cocycle of \( G_K \) with values in \( X_{\log}(\mathcal{D}_0) \).

**Proposition 8.5.** The image of \([\bar{\epsilon}']\) in \( H^1(G_K, X_{\log}(\mathcal{D}_0)) \) belongs to the kernel of

\[ H^1(G_K, X_{\log}(\mathcal{D}_0)) \to H^1(G_K, X_{\text{dR}}(\mathcal{D}_0)). \]

**Proof.** Consider the following commutative diagram

\[
\begin{array}{c}
H^1(G_K, X_{\log}(\mathcal{D}_0)) \longrightarrow H^1(G_K, X_{\text{dR}}(\mathcal{D}_0)) \\
\downarrow \quad \quad \quad \downarrow \\
H^1(G_K, X_{\log}(\mathcal{D})) \longrightarrow H^1(G_K, X_{\text{dR}}(\mathcal{D})).
\end{array}
\]

The right vertical arrow in the above diagram is injective (see [13, Corollary 2.4]). So we only need to show that the image of \([\bar{\epsilon}']\) in \( H^1(G_K, X_{\text{dR}}(\mathcal{D})) \) is zero. Note that

\[ [\bar{\epsilon}'] = [\bar{\epsilon}'] - [c_{1,n} e_{1,2} \otimes e_{n,3}] = -[c_{1,n} e_{1,2} \otimes e_{n,3}] \]

in \( H^1(G_K, X_{\text{dR}}(\mathcal{D})) \). As the image of \([c_{1,n}]\) in \( H^1(G_K, B_{\log,E}) \) is zero, so is its image in \( H^1(G_K, B_{\text{dR,E}}/\text{Fil}^f B_{\text{dR,E}}) \), where \( f \) is the smallest integer such that \( e_{1,2} \otimes e_{n,3} \in \text{Fil}^{-f} \mathcal{D}_K \). Hence, the image of \([\bar{\epsilon}']\) in \( H^1(G_K, X_{\text{dR}}(\mathcal{D})) \) is zero. \( \Box \)

Now, applying Lemma 5.3 to \( \mathcal{D}_0 \) with \( f_1 = e_{1,2} \otimes e_{1,3}, f_2 = e_{1,2} \otimes e_{1,3} \) and \( f_3 = e_{n,3} \otimes e_{n,3} \), we get

\[ \gamma_{n,0} - \gamma_{1,0} = \sum_{\tau \in \text{Emb}(K,E)} \mathcal{L}_\tau(\gamma_{n,\tau} - \gamma_{1,\tau}). \]

Hence, by Lemma 8.4 we have

\[
\frac{v_p(\pi_K)}{[K_0 : \mathbb{Q}_p]}(\epsilon_n(p) - \epsilon_1(p)) + \sum_{\tau \in \text{Emb}(K,E)} \mathcal{L}_\tau(\epsilon_{n,\tau} - \epsilon_{1,\tau}) = 0. \]

23
As \( \frac{d\delta_n(p)}{d\epsilon(p)} = \epsilon_h(p)dZ \) and \( d\vec{w}(\epsilon_h) = (\epsilon_h, \tau) = 0 \), we obtain

\[
\frac{1}{[K : Q_p]} \left( \frac{d\delta_n(p)}{\delta_n(p)} - \frac{d\delta_1(p)}{\delta_1(p)} \right) + \mathcal{L}_F \cdot (d\vec{w}(\delta_n) - d\vec{w}(\delta_1)) = 0,
\]
as desired. This finishes the proof of Theorem 1.2.

References

[1] D. Benois, *A generalization of Greenberg’s \( \mathcal{L} \)-invariant*. Amer. J. Math. 133 (2011), 1573-1632.

[2] L. Berger, *Représentations p-adiques et équations différentielles*. Invent. Math. 148 (2002), 219-284.

[3] L. Berger, *Construction de \( (\varphi, \Gamma) \)-modules: représentations p-adiques et B-paires*. Algebra Number Theory 2 (2008), 91-120.

[4] L. Berger, *Trianguline representations*. Bull. Lond. Math. Soc. 43 (2011), no. 4, 619-635.

[5] P. Colmez, *Représentations triangulines de dimension 2*. Asterisque 319 (2008), 213-258.

[6] P. Colmez, *Invariants \( \mathcal{L} \) et dérivées de valeurs propres de Frobenius*. Astérisque 331 (2010), 13-28.

[7] J.-M. Fontaine, *Le corps des périodes p-adiques*. Astérisque 223 (1994), 59-111.

[8] R. Greenberg, G. Steven, *p-adic L-functions and p-adic periods of modular forms*. Invent. Math. 111 (1993), 407-447.

[9] B. Mazur, *The theme of p-adic variation*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence 2000, 433-459.

[10] B. Mazur, J. Tate, J. Teitelbaum, *On p-adic analogs of the conjectures of Birch and Swinnerton-Dyer*. Invent. Math. 84 (1986), 1-48.

[11] K. Nakamura, *Classification of two-dimensional split trianguline representations of p-adic fields*. Compos. Math. 145 (2009), 865-914.

[12] J. Pottharst, *The \( \mathcal{L} \)-invariant, the dual \( \mathcal{L} \)-invariant, and families*. Ann. Math. Qué. 40 (2016), 159-165.

[13] B. Xie, *Derivatives of Frobenius and Derivatives of Hodge-Tate weights*. To appear in Acta Mathematica Sinica, a special issue on Arithmetic Algebraic Geometry, edited by Fu, Liu, Tian and Xu.

[14] Y. Zhang, *\( \mathcal{L} \)-invariants and logarithmic derivatives of eigenvalues of Frobenius*. Science China Mathematics 57 (2014), 1587-1604.