On the modification of phase structure of black D6 branes in canonical ensemble and its origin

J. X. Lu\textsuperscript{a}, Jun Ouyang\textsuperscript{a} and Shibaji Roy\textsuperscript{b}

\textsuperscript{a} Interdisciplinary Center for Theoretical Study  
University of Science and Technology of China, Hefei, Anhui 230026, China

\textsuperscript{b} Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta-700 064, India

Abstract

It is well-known that charged black D\textsubscript{p}-branes of type II string theory share a universal phase structure of van der Waals-Maxwell liquid-gas type except D5- and D6-branes. Interestingly, the phase structure of D5- and D6-branes can be changed to the universal form with the inclusion of particular delocalized charged lower dimensional branes. For D5-branes one needs to introduce delocalized D1-branes and for D6-branes one needs to introduce delocalized D0-branes to obtain the universal structure. In a previous paper [JHEP 04 (2013) 100] one of us JXL with R. Wei has studied the phase structure of black D6-branes with the introduction of delocalized D0-branes in a special case when their charges are equal and the dilaton charge vanishes. In this paper we look at the phase structure of black D6/D0 system with the generic values of the parameters, which makes the analysis more involved but the structure more rich. We also provide reasons why the respective modifications of the phase structures to the universal form for the black D5- and D6-branes occur when specific delocalized lower dimensional branes are introduced.
1 Introduction

It is quite well-known that charged AdS black holes give rise to an interesting thermodynamic phase structure isomorphic to the van der Waals-Maxwell liquid-gas system [1, 2]. The interests in AdS black hole stem from the fact that they are thermodynamically stable and hence are suitable to study equilibrium thermodynamics [3]. Moreover, by AdS/CFT correspondence, they are holographically dual to finite temperature field theories [4] and indeed the above mentioned phase structure in the field theory has similarities with catastrophe theory [2]. However, it has been noted before that the large part of this phase structure including the van der Waals-Maxwell liquid-gas type is not unique to the AdS black holes only, but appear in suitably stabilized dS as well as asymptotically flat space charged black holes also [5, 6]. Such universal structure for the charged black holes with different asymptopia suggests that holography might be at work not just for AdS space, but for dS as well as flat space.

Motivated by this, in [7] the phase structure of suitably stabilized, flat, charged black \( p \)-brane solutions in arbitrary dimensions was analyzed and surprisingly it was found that they also have very similar phase structure as that of the black holes and in particular, has the van der Waals-Maxwell liquid-gas type structure when the charge of the black brane is below certain non-zero critical value. However, this happens only when the dimensions of the space transverse to \( p \)-brane satisfy \( \hat{d} + 2 = D - p - 1 > 4 \), where \( D \) is the total space-time dimensions. When \( D = 10 \), i.e., for string theory branes this implies that all the charged black Dp-branes with \( p < 5 \) share the same universal phase structure as the charged black holes in AdS/dS/flat space, but the phase structures of D5- and D6-branes differ. It was found in [8, 9], that this difference in phase structure can be removed if one adds specific delocalized charged lower dimensional branes to the system. So, for example, D5-branes restore the same universal phase structure if one adds delocalized D1-branes to the system, on the other hand, D6-branes restore the universal phase structure if one adds delocalized D0-branes to the system. Note that for D5-branes adding other lower dimensional branes, namely, the delocalized charged D3-branes do not help producing the universal phase structure. Similarly, for D6-branes, the other charged lower dimensional branes, namely, the delocalized D4- or D2-branes do not help even though the D6/D2 system belongs to the same class of Dp/D(p - 4) as the D5/D1 system. This shows that in order to obtain the universal phase structure from D5- or D6-branes it is not \textit{a priori} clear which charged lower dimensional branes one should include to the system if at all they can bring about this change. Also it should be emphasized that the inclusion of lower dimensional branes does not automatically imply that they
will make the necessary change in phase structure, as one might think so since the lower (< 5) dimensional branes themselves have universal phase structure. In fact one can check that the delocalized (in the other four D5-brane worldvolume spatial directions) charged D1-branes and delocalized (in the six D6-brane worldvolume spatial directions) charged D0-branes individually have the same phase structure as D5-branes and D6-branes respectively[^8][^9]. Thus when they are combined to form bound states, it is their interactions with each other which makes the necessary change in the phase structure possible.

The phase structure of black D5/D1 system has been studied with all its generalities in[^8]. The black D6/D0 system with the associated phase structure has been studied in[^9]. As the parameter space of the latter system is quite complicated to analyze in general, only a special case has been considered which enabled the authors to show the universal phase structure, postponing the discussion of the general case as well as the reason behind the appearance of this universal structure to a later publication. It is this task that we undertake in this paper. In the previous publication, the charges of the D6-branes and the D0-branes were chosen to be equal, which gives the vanishing dilaton charge from the parameter relation[^4]. However, in this paper we look at the general case which makes the solution of the parameter space far more complicated and the phase structure which has the expected universal form becomes more rich than before. We also provide possible reasons why the additions of delocalized D1-branes in D5-branes and D0-branes in D6-branes change qualitatively the thermodynamic phase structure of D5- and D6-branes to have the universal form. In the former case, it is the addition of extra degrees of freedom or the change in entropy while in the latter case, it is the repulsive nature of interaction between the constituent branes which makes the necessary change in the thermodynamic phase structure to take the universal form.

This paper is organized as follows. In section 2, we discuss the general charged black D6/D0 bound state solution in Euclidean signature and describe the general parameter space for which there exists a regular horizon such that a meaningful thermodynamics can be given. The corresponding thermodynamics and the phase structure are described in section 3. We provide reasons for the appearance of the universal phase structure of van der Waals-Maxwell liquid-gas type in section 4. Finally we give our concluding remarks in section 5.

[^4]: The other two solutions give naked singularities and therefore are not relevant for thermodynamical consideration.
2 D6/D0 bound state and the parameter space

In this section we write the spherically symmetric, time independent, electrically charged black D6/D0 bound state solution in Euclidean signature for the purpose of studying thermodynamics and phase structure\cite{10, 11}. As we will see the solution contains three independent parameters: the mass and the charges of D6-branes and D0-branes. We will argue that the parameters can not take arbitrary values as naked singularities can develop in general. We will determine the region of the parameter space for which there exists a regular horizon. The D6/D0 solution is given as

\begin{equation}
 ds^2 = \frac{F}{8} B^{-\frac{7}{8}} dt^2 + \left(\frac{B}{A}\right)^{\frac{1}{8}} \sum_{i=1}^{6} dx_i^2 + A^\frac{7}{8} B^{\frac{1}{8}} \left( F^{-1} d\rho^2 + \rho^2 d\Omega_2^2 \right)
\end{equation}

where the metric in (1) is given in the Einstein frame and the various functions appearing in the metric are defined as,

\begin{align}
 F(\rho) &= \left( 1 - \frac{\rho_+}{\rho} \right) \left( 1 - \frac{\rho_-}{\rho} \right), \\
 A(\rho) &= \left( 1 - \frac{\rho A_+}{\rho} \right) \left( 1 - \frac{\rho A_-}{\rho} \right), \\
 B(\rho) &= \left( 1 - \frac{\rho B_+}{\rho} \right) \left( 1 - \frac{\rho B_-}{\rho} \right),
\end{align}

where we here use the configuration given in \cite{10} with some modifications. The magnetic part of the one-form given there has been changed to an electric seven-form. In $D=10$, D0-branes and D6-branes are electric-magnetic dual to each other. Also we change the sign of the charge parameters $Q$ and $P$ there and assume without any loss of generality $Q > 0$, $P > 0$ for convenience. Further, we correct a typo in the electric one-form potential given there by replacing the dilaton charge $\Sigma$ by $\Sigma/\sqrt{3}$.

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with
\[ \rho_{\pm} = M \pm \sqrt{M^2 + \Sigma^2 - \frac{P^2}{4} - \frac{Q^2}{4}}, \]
\[ \rho_{A\pm} = \frac{\Sigma}{\sqrt{3}} \pm \sqrt{\frac{P^2\Sigma/2}{\Sigma - \sqrt{3}M}}, \]
\[ \rho_{B\pm} = -\frac{\Sigma}{\sqrt{3}} \pm \sqrt{\frac{Q^2\Sigma/2}{\Sigma + \sqrt{3}M}}. \] (3)

In (3), there are four parameters, namely, the mass parameter, \( M \), the delocalized D0-brane charge parameter, \( Q \), the D6-brane charge parameter, \( P \), and the dilaton charge parameter, \( \Sigma \). However, not all of them are independent and in fact, dilaton charge parameter \( \Sigma \) is related to \( M \), \( P \) and \( Q \) by the relation,
\[ \frac{8}{3} \Sigma = \frac{Q^2}{\Sigma + \sqrt{3}M} + \frac{P^2}{\Sigma - \sqrt{3}M}, \] (4)
leaving only three of them independent. As noted in \cite{10}, under the electric-magnetic duality, the parameters of the solution transform as \( Q \leftrightarrow P \), \( \Sigma \leftrightarrow -\Sigma \) and \( M \leftrightarrow M \).

Also in (1), \( A_{[1]} \) and \( A_{[7]} \) are the electric one-form and 6-form to which D0-branes and D6-branes couple and give the corresponding charges \( Q \) and \( P \) respectively. The form fields are chosen to vanish at \( \rho_{+} \) so that they are well-defined in the local inertial frame and \( \phi_0 \) is the asymptotic value of the dilaton.

Note that the solution (1) given in terms of three parameters \( M \), \( Q \) and \( P \) is not necessarily physical as for generic values of these parameters it can have naked singularity. We will see in this section that for some restricted region of the parameter space we can indeed have physical solution with well-defined horizon which in turn will be suitable for studying thermodynamics and the associated phase structure. Also, in addition to \( P, Q > 0 \), we will assume by duality symmetry that \( \Sigma \geq 0 \) without loss of generality. \( \Sigma < 0 \) branch can be obtained from \( \Sigma > 0 \) simply by exchanging \( Q \leftrightarrow P \). The three quantities, which will be useful for showing the existence of a regular horizon, are \( \rho_{+}, \rho_{A+} \) and \( \rho_{B+} \) and are given in terms of the parameters of the solution as,
\[ \rho_{+} = M + \sqrt{M^2 + \Sigma^2 - \frac{P^2}{4} - \frac{Q^2}{4}}, \]
\[ \rho_{A+} = \frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{P^2\Sigma/2}{\Sigma - \sqrt{3}M}}, \]
\[ \rho_{B+} = -\frac{\Sigma}{\sqrt{3}} + \sqrt{\frac{Q^2\Sigma/2}{\Sigma + \sqrt{3}M}}. \] (5)

Actually \( \rho = \rho_{+} \) is the horizon as long as it is greater than both \( \rho_{A+} \) and \( \rho_{B+} \). Let

\[ ^6 \text{Note that the metric (1) has curvature singularities at both } \rho = \rho_{A+} \text{ and } \rho = \rho_{B+}. \]
us first assume that $\Sigma \geq \sqrt{3}M$. Now it can be easily checked from (5) that with this $\rho_{A+} > \rho_{B+}$. So, in order to get a horizon at $\rho_+$, we must have $\rho_+ > \rho_{A+}$. However, using their expressions from (3) we find that this condition can not be satisfied. Thus if $\Sigma \geq \sqrt{3}M$, the solution has a naked singularity at $\rho = \rho_{A+}$. Therefore, in order to have a horizon (if it at all exists) we must take $0 < \Sigma < \sqrt{3}M$. Note that we have excluded $\Sigma = 0$ case since it corresponds to (from (4)) $Q^2 = P^2$, which has been considered in [9]. Now as $\Sigma < \sqrt{3}M$, we can see from (3) that $\rho_{A\pm}$ are both imaginary and therefore do not play any role in determining whether there exists a horizon. We, therefore, must demand $\rho_+ > \rho_{B+}$ in order to have a well-defined horizon. Note that $\rho_{B+} > 0$ which can be verified using (4). Using the form of $\rho_+$ and $\rho_{B+}$ from (5) and after some algebraic manipulation and further using (4) the condition $\rho_+ > \rho_{B+}$ gives,

$$\sqrt{M^2 + \Sigma^2 - P^2/4 - Q^2/4} > \frac{\sqrt{3}}{4} \frac{P^2}{\sqrt{3}M - \Sigma} - \frac{1}{\sqrt{3}} \left( \sqrt{3}M - \Sigma \right).$$  \hspace{1cm} (6)

It can be easily checked that if the rhs of (6) is positive, i.e., if $\sqrt{3}M - \Sigma < \frac{\sqrt{3}}{2}P$, then (6) implies $\Sigma < 0$ when (4) is used. This is a contradiction to our assumption that $\Sigma > 0$. Therefore we must have $\sqrt{3}M - \Sigma > \frac{\sqrt{3}}{2}P$, or in other words, the rhs of (6) must be negative. So, to summarize, in order to have a well-defined horizon we must have at least,

$$\Sigma > 0, \quad \text{and} \quad \sqrt{3}M - \Sigma > \frac{\sqrt{3}}{2}P \quad \Rightarrow \quad \sqrt{3}M > \Sigma + \frac{\sqrt{3}}{2}P \quad (7)$$

along with (4).

We will see that the condition (4) and the positivity of the quantity inside the square-root of the expression of $\rho_+$ given in (5) will put more restrictions on $\Sigma$ in terms of $P$ and $Q$ in order to have well-defined horizon. Let us first look at the condition (4). Defining $X = \sqrt{3}M > 0$ we rewrite it as,

$$X^2 + \frac{3(P^2 - Q^2)}{8\Sigma} X + \frac{3}{8} (Q^2 + P^2) - \Sigma^2 = 0,$$  \hspace{1cm} (8)

from which we solve $X$ to get

$$X_{\pm} = \frac{3(Q^2 - P^2)}{16\Sigma} \pm \frac{1}{2} \sqrt{\frac{9(Q^2 - P^2)^2}{64\Sigma^2} + 4\Sigma^2 - \frac{3}{2}(Q^2 + P^2)}.$$  \hspace{1cm} (9)

Let us now check the condition (7), i.e., $X_+ > \Sigma + (\sqrt{3}/2)P$. Using $X_+$ given in (9) we get,

$$\frac{3(Q^2 - P^2)}{8\Sigma} + \sqrt{\frac{9(Q^2 - P^2)^2}{64\Sigma^2} + 4\Sigma^2 - \frac{3}{2}(Q^2 + P^2)} > 2\Sigma + \sqrt{3}P,$$

or,

$$\frac{9(Q^2 - P^2)^2}{64\Sigma^2} + 4\Sigma^2 - \frac{3}{2}(Q^2 + P^2) > \left( 2\Sigma + \sqrt{3}P - \frac{3(Q^2 - P^2)}{8\Sigma} \right)^2,$$  \hspace{1cm} (10)
In writing the second inequality in (10), we have assumed $2\Sigma + \sqrt{3}P \geq 3(\frac{Q^2 - P^2}{8\Sigma})$, which is certainly true if $Q^2 < P^2$. However we note that the second inequality in (10) leads to a contradiction since it gives $2\Sigma + \sqrt{3}P < 3(\frac{Q^2 - P^2}{8\Sigma})$. Therefore we must have $2\Sigma + \sqrt{3}P < 3(\frac{Q^2 - P^2}{8\Sigma})$. This not only implies $Q > P$ but also ensures that the quantity inside the square root of the expression of $X_\pm$ given in (9) is positive definite. From this condition we have

$$\left(\Sigma + \frac{\sqrt{3}}{4}(Q + P)\right)\left(\Sigma - \frac{\sqrt{3}}{4}(Q - P)\right) < 0,$$

which gives a restriction on $\Sigma$ as

$$\Sigma < \frac{\sqrt{3}}{4}(Q - P).$$

(12)

For $X_-$ it can be easily checked that the condition $X_- > \Sigma + (\sqrt{3}/2)P$ is contradictory with $2\Sigma + \sqrt{3}P < 3(\frac{Q^2 - P^2}{8\Sigma})$ and therefore, $X_-$ is not a valid solution for our discussion. In summary, so far we find that the solution (1) has a horizon, i.e., $\rho_+ > \rho_B$, if

$$\sqrt{3}M = \frac{3(Q^2 - P^2)}{16\Sigma} + \frac{1}{2} \sqrt{\frac{9(Q^2 - P^2)^2}{64\Sigma^2} + 4\Sigma^2 - \frac{3}{2}(Q^2 + P^2)},

0 < \Sigma < \frac{\sqrt{3}}{4}(Q - P), \quad Q > P.$$

(13)

We may think that we have fixed the parameter space, but this is not quite true. We have to consider, as we mentioned before, one more condition coming from the quantity inside the square root of $\rho_+$ given in (5) which must be positive semi-definite.

Therefore, from the expression of $\rho_+$ given in (5) we have,

$$M^2 \geq \frac{P^2 + Q^2}{4} - \Sigma^2$$

(14)

Using the expression for $M$ given in (13) the above relation reduces to,

$$\frac{3(Q^2 - P^2)}{16\Sigma} + \frac{1}{2} \sqrt{\frac{9(Q^2 - P^2)^2}{64\Sigma^2} + 4\Sigma^2 - \frac{3}{2}(Q^2 + P^2)} \geq \sqrt{3}\sqrt{\frac{P^2 + Q^2}{4} - \Sigma^2}.$$

(15)

From the above it is clear that (15) will be automatically satisfied if we have

$$\frac{\sqrt{3}Q^2 - P^2}{16 \Sigma} \geq \sqrt{\frac{P^2 + Q^2}{4} - \Sigma^2}.$$

(16)

From this, we will determine the condition for $\Sigma$. Eq. (16) can be simplified as,

$$\Sigma^4 - \frac{P^2 + Q^2}{4} \Sigma^2 + \frac{3(Q^2 - P^2)^2}{16^2} \geq 0,$$

(17)
which gives

$$(\Sigma^2 - \Sigma_+^2)(\Sigma^2 - \Sigma_-^2) \geq 0,$$  \hspace{1cm} (18)

where

$$\Sigma_\pm^2 = \frac{P^2 + Q^2}{8} \pm \frac{1}{8} \sqrt{\frac{P^4 + Q^4 + 14P^2Q^2}{4}}.$$  \hspace{1cm} (19)

From (18) we have either $\Sigma^2 \geq \Sigma_+^2$ or $\Sigma^2 \leq \Sigma_-^2$. We also need $\Sigma < \sqrt{3}(Q - P)/4$ from (13). But it can be easily shown that $\Sigma_+ > \sqrt{3}(Q - P)/4$ and so is not relevant, however, $\Sigma_- < \sqrt{3}(Q - P)/4$ and therefore, $\Sigma_-$ sets a new bound on $\Sigma$, i.e., $\Sigma < \Sigma_-$. However, this is not the complete story. We still need to consider the case

$$\frac{\sqrt{3}Q^2 - P^2}{16 \Sigma} < \sqrt{\frac{P^2 + Q^2}{4} - \Sigma^2}.$$  \hspace{1cm} (20)

such that the inequality (15) holds. This can give further restrictions on $\Sigma$. We rewrite (15) as,

$$\sqrt{\frac{9(Q^2 - P^2)^2}{64\Sigma^2} + 4\Sigma^2 - \frac{3}{2}(Q^2 + P^2)} \geq 2\sqrt{3}\sqrt{\frac{P^2 + Q^2}{4} - \Sigma^2 - \frac{3(Q^2 - P^2)}{8\Sigma}} > 0.$$  \hspace{1cm} (21)

Now squaring both sides and doing some algebraic manipulations, we get,

$$\frac{9}{32}(Q^2 + P^2) - \Sigma^2 \leq \frac{3\sqrt{3}(Q^2 - P^2)}{32\Sigma} \sqrt{\frac{Q^2 + P^2}{4} - \Sigma^2} - \Sigma^2.$$  \hspace{1cm} (22)

Now let us define a dimensionless variable $y = 4\Sigma^2/(P^2 + Q^2)$, then in terms of $y$ (22) can be rewritten as,

$$y^3 - \frac{9}{4}y^2 + \frac{81 + 27k^2}{64}y - \frac{27k^2}{64} \leq 0,$$  \hspace{1cm} (23)

where

$$k = \frac{Q^2 - P^2}{Q^2 + P^2} < 1.$$  \hspace{1cm} (24)

The left side of the above inequality (23) can actually be factorized and it can be written as

$$\left[y - \frac{3}{4}\left(1 - \frac{A}{2}\right)\right] \left[y - \frac{3}{4}\left(1 + \frac{A}{4}\right)\right] + \frac{3^3}{16^2}(1 - k^2)^{\frac{2}{3}} \left[(1 + k)^{\frac{2}{3}} - (1 - k)^{\frac{2}{3}}\right] \leq 0,$$  \hspace{1cm} (25)

where

$$A = (1 - k^2)^{1/3}[1 + k^{1/3} + (1 - k)^{1/3}] < 2.$$  \hspace{1cm} (26)

We, therefore, have

$$y \leq \frac{3}{4}\left(1 - \frac{A}{2}\right),$$  \hspace{1cm} (27)
which gives, after plugging the definition for $y$,

\[ \Sigma \leq \Sigma_0 = \frac{\sqrt{3}}{4} (Q - P) \left[ 1 - \frac{(QP)^{2/3}}{(Q^{2/3} + P^{2/3} + Q^{1/3}P^{1/3})^2} \right]^{1/2} < \frac{\sqrt{3}}{4} (Q - P). \quad (28) \]

Now in order to show that $\Sigma_0$ is the correct bound, we need to show $\Sigma_0 > \Sigma_-$, where $\Sigma_-$ is given in (19). For this let us compare the expressions for $\Sigma^2_0$ from (28) and $\Sigma^2_-$ from (19). They have the forms,

\[
\Sigma_0^2 = \frac{3}{16} \left[ (Q - P)^2 - (QP)^{2/3} (Q^{1/3} - P^{1/3})^2 \right],
\]

\[
\Sigma_-^2 = \frac{Q^2 + P^2}{8} - \frac{1}{16} \sqrt{P^4 + Q^4 + 14P^2Q^2} < \frac{(Q - P)^2}{8} \quad (29)
\]

So, all we need to show is $\Sigma_0^2 > (Q - P)^2/8$. Now substituting the form of $\Sigma_0$ from (28) this condition leads to

\[
(Q^{1/3} - P^{1/3})^2 + 3(QP)^{1/3} > \sqrt{3}(QP)^{1/3} \quad (30)
\]

which obviously holds true and therefore this shows that $\Sigma_0 > \Sigma_-$. So, finally we conclude that for the existence of a sensible horizon for the D6/D0 black brane bound state solution we must have

\[
\sqrt{3}M = \frac{3(Q^2 - P^2)}{16\Sigma} + \frac{1}{2} \sqrt{\frac{9(Q^2 - P^2)^2}{64\Sigma^2} + 4\Sigma^2 - \frac{3}{2}(Q^2 + P^2)}, \quad (31)
\]

with

\[
0 < \Sigma \leq \Sigma_0, \quad (32)
\]

and $Q > P$, where, $\Sigma_0$ is given in (28). We can extend the above range of $\Sigma$ to $-\Sigma_0 \leq \Sigma < 0$ if we require $P > Q$. For our purpose, we will focus on the $Q > P$ branch in what follows.

### 3 The general phase structure of D6/D0

In this section we will analyze the phase structure of black D6/D0 system with generic charges with the parameters $M$ and $\Sigma$ satisfying the condition given in (31) and (32) for the existence of a well-defined horizon. Since this system is asymptotically flat, we need to stabilize it by placing it in a cavity following [12, 7] and in this paper we will analyze the phase structure in canonical ensemble which will be specified later on. All we need to know is the form of the local temperature (or the inverse of the local temperature to
be precise) of the system at the location of the wall of the cavity, which can be obtained
from the black D6/D0 metric in Euclidean signature as given in (1) by demanding the
absence of conical singularity at the horizon. We will express the inverse of the local
temperature at the given location as a function of the horizon radius only and therefore,
we need to express the other parameters, namely, $M$ and $\Sigma$ also in terms of the horizon
radius. However, for this system and from our past experience [9], we know that $\rho$ is not
a good coordinate for this purpose and we will define a new radial coordinate by,

$$r = \rho + a,$$

(33)

where $a$ is a parameter to be determined later. From now on we will assume $Q > P$ and
$0 < \Sigma \leq \Sigma_0$. From (33) and using the new radial coordinate $r$, we have $r_+ = \rho_+ + a$ and
$r_- = \rho_- + a$ where $r_+$ defines the location of horizon and using these two we have

$$r_+ r_- = \rho_+ \rho_- + (\rho_+ + \rho_-)a + a^2 = \frac{P^2 + Q^2}{4} - \Sigma^2 + 2Ma + a^2,$$

(34)

where in writing the second equality we have used the form of $\rho_\pm$ as given in (3). Now
since we know from [9] that $r_+ r_- = Q^2$, when $P = 0$ (also from [9] that $r_+ r_- = P^2$ when
$Q = 0$), so we generalize it to the present case as,

$$r_+ r_- = P^2 + Q^2,$$

(35)

which can be used to determine $r_-$ in terms of $r_+$. (33) along with (34) fixes the parameter $a$ as,

$$a = -M + \sqrt{M^2 + \Sigma^2 + \frac{3}{4}(P^2 + Q^2)},$$

(36)

where we have used only the plus sign in front of the square root since this reduces to the
correct form when $P = 0$. We thus find

$$r_+ = \rho_+ + a = \sqrt{M^2 + \Sigma^2 - \frac{P^2 + Q^2}{4}} + \sqrt{M^2 + \Sigma^2 + \frac{3}{4}(P^2 + Q^2)},$$

(37)

which can be further simplified to give,

$$\frac{1}{4} \left( r_+ - \frac{P^2 + Q^2}{r_+} \right)^2 = M^2 + \Sigma^2 - \frac{P^2 + Q^2}{4}.$$

(38)

Note that our intention here is to express $\Sigma$ and $M$ in terms of the horizon radius $r_+$ and
for this purpose we will use the equation (38). To eliminate $M^2$ from this equation, we
first have from (4)

$$M^2 = \frac{\Sigma^2}{3} + \frac{Q^2 - P^2 \sqrt{3} M \Sigma}{8 \Sigma^2} - \frac{P^2 + Q^2}{8},$$

(39)
we then use (31) to obtain $\sqrt{3}M\Sigma$ and substitute it in the above (39) to obtain $M^2$ in terms of $\Sigma$ and the known charges $P$ and $Q$. Using this expression of $M^2$ in (38) and after some algebraic manipulation we obtain

$$\left(\frac{32}{3}\right)^3 \Sigma^6 - 2 \left(\frac{32}{3}\right)^2 G(r_+)\Sigma^4 + \frac{32}{3} G^2(r_+) \left(1 + 3 \frac{(Q^2 - P^2)^2}{G^2(r_+)}\right) \Sigma^2$$

$$-4(Q^2 - P^2)^2 G(r_+) \left(1 - \frac{Q^2 + P^2}{G(r_+)}\right) = 0, \quad (40)$$

where we have defined $G(r_+) = 2 \left(r_+ - \frac{P^2 + Q^2}{r_+}\right)^2 + 3(P^2 + Q^2)$. Note that (40) is an equation involving $\Sigma$ and $r_+$, whose explicit solution $\Sigma(r_+)$ is what we want. For this purpose we further define the following quantities

$$Y = \frac{32 \Sigma^2}{3 G}, \quad d = \frac{Q^2 - P^2}{G} < \frac{1}{3}, \quad c = \frac{P^2 + Q^2}{G} < \frac{1}{3}, \quad (41)$$

and rewrite (40) as,

$$Y^3 - 2Y^2 + (1 + 3d^2)Y - 4d^2(1 - c) = 0. \quad (42)$$

This is a cubic equation and has three roots in general. We should of course take only the real roots. However, as we will see that even for the real roots only not all of them are allowed. From the definition of $Y$ and to have a well-defined horizon, we conclude that the allowed solution must be such that

$$Y = \frac{32 \Sigma^2}{3 G} < \frac{32}{3} \frac{\frac{3}{16}(Q - P)^2}{2 \left(r_+ - \frac{Q^2 + P^2}{r_+}\right)^2 + 3(P^2 + Q^2)} < \frac{2}{3}, \quad (43)$$

where we have used (28) and the definition of $G(r_+)$ as given before. Thus we conclude that the allowed values of $Y = 32\Sigma^2/(3G)$ must be less than $2/3$.

The equation for $Y$, i.e. (42) can be solved and we get the three solutions as follows,

$$Y_1 = \frac{2}{3} - \frac{(C + \sqrt{C^2 - D})^{1/3} + (C - \sqrt{C^2 - D})^{1/3}}{3},$$

$$Y_2 = \frac{2}{3} + \frac{1}{6} \left[ (1 + i\sqrt{3}) \left(C + \sqrt{C^2 - D}\right)^{1/3} + (1 - i\sqrt{3}) \left(C - \sqrt{C^2 - D}\right)^{1/3} \right],$$

$$Y_3 = \frac{2}{3} + \frac{1}{6} \left[ (1 - i\sqrt{3}) \left(C + \sqrt{C^2 - D}\right)^{1/3} + (1 + i\sqrt{3}) \left(C - \sqrt{C^2 - D}\right)^{1/3} \right], \quad (44)$$

where

$$C = 1 - 27d^2(1 - 2c), \quad D = (1 - 9d^2)^3. \quad (45)$$
with \( c \) and \( d \) as given in (41). Note that when \( C^2 > D \), we have only one real positive root \( Y_1 \). The other two roots \( Y_2 \) and \( Y_3 \) are complex conjugate to each other and must be discarded. Since \( Y_1 < 2/3 \), it is an allowed solution. On the other hand when \( C^2 < D \), all three roots are real and positive. In this case let us define \( C = R \cos \theta \) and \( \sqrt{D - C^2} = R \sin \theta \), where \( R^2 = D = (1 - 9d^2)^3 \) and \( \cos \theta = C/\sqrt{D} \) which lies between 0 and 1 and so, \( \theta \) lies between 0 and \( \pi/2 \). With these (44) can be written as,

\[
Y_1 = \frac{2}{3} - \frac{R^{1/3} e^{i\theta/3} + e^{-i\theta/3}}{3} = \frac{2}{3} \left( 1 - R^{1/3} \cos \theta/3 \right) > 0,
\]

\[
Y_2 = \frac{2}{3} + \frac{R^{1/3} e^{i(\pi + \theta)/3} + e^{-i(\pi + \theta)/3}}{3} = \frac{2}{3} \left( 1 + R^{1/3} \cos(\pi + \theta)/3 \right) > \frac{2}{3},
\]

\[
Y_3 = \frac{2}{3} + \frac{R^{1/3} e^{-i(\pi - \theta)/3} + e^{i(\pi - \theta)/3}}{3} = \frac{2}{3} \left( 1 + R^{1/3} \cos(\pi - \theta)/3 \right) > \frac{2}{3}.
\]

Note that since \( \theta < \pi/2 \), \( \cos(\pi \pm \theta)/3 \) > 0 and therefore both \( Y_2 \) and \( Y_3 \) are greater than \( 2/3 \) and therefore should be discarded. However, \( Y_1 < 2/3 \) and this is the only allowed solution. Thus we obtain that no matter whether \( C^2 > D \) or \( C^2 < D \), \( Y_1 \) is the only allowed solution. We then write,

\[
Y = \frac{32 \Sigma^2}{3G} = \frac{2}{3} - \frac{(C + \sqrt{C^2 - D})^{1/3} + (C - \sqrt{C^2 - D})^{1/3}}{3},
\]

where \( C \) and \( D \) are as given in (45) and \( c \) and \( d \) in the expression of \( C, D \) are as given in (41). Also \( G(r_+) \) is a function of \( r_+ \) and is given right after (40). Eq.(47) therefore uniquely determines \( \Sigma \) in terms of \( r_+ \). Further, \( M \) can also be expressed in terms of \( r_+ \) using (38) and (39) as

\[
M = \frac{\Sigma}{\sqrt{3} (Q^2 - P^2)} \left( G(r_+) - \frac{32}{3} \Sigma^2 \right),
\]

once we have \( \Sigma \) in terms of \( r_+ \). Using (38), (40) and the above, we have

\[
a = \frac{\Sigma}{\sqrt{3} (Q^2 - P^2)} \left( \frac{32}{3} \Sigma^2 - G(r_+) \right) + \frac{1}{2} \left( r_+ + \frac{Q^2 + P^2}{r_+} \right),
\]

which will be useful later on.

Once we express \( M \) and \( \Sigma \) in terms of \( r_+ \), we can express the entire solution (1) in terms of this single parameter \( r_+ \) (note that \( P \) and \( Q \) are fixed charges and therefore don’t vary). To do this we replace \( \rho \) by \( r - a \), where \( a \) is given in (36). Then the functions
$F(\rho), A(\rho)$ and $B(\rho)$ given in (2) can be expressed in terms of $r$ as,

$$F(\rho) = \triangle_+ \triangle_- \left(1 - \frac{a}{r}\right)^{-2},$$

$$A(\rho) = \left(1 - \frac{r_{A+}}{r}\right) \left(1 - \frac{r_{A-}}{r}\right) \left(1 - \frac{a}{r}\right)^{-2} \equiv A(r) \left(1 - \frac{a}{r}\right)^{-2},$$

$$B(\rho) = \left(1 - \frac{r_{B+}}{r}\right) \left(1 - \frac{r_{B-}}{r}\right) \left(1 - \frac{a}{r}\right)^{-2} \equiv B(r) \left(1 - \frac{a}{r}\right)^{-2},$$

(50)

where we have defined, as usual,

$$\triangle_{\pm} = 1 - \frac{r_{\pm}}{r},$$

(51)

and

$$r_{A\pm} = a + \rho_{A\pm} = a + \sum_{\pm} \frac{\sqrt{2} \Sigma}{\sqrt{3}} \pm \frac{\sqrt{2} \Sigma}{\sqrt{3}} \left(\frac{1 - \sqrt{3} a \pm \Sigma \sqrt{3} a}{\sqrt{3}}\right),$$

$$r_{B\pm} = a + \rho_{B\pm} = a - \sum_{\pm} \frac{\sqrt{2} \Sigma}{\sqrt{3}} \pm \frac{\sqrt{2} \Sigma}{\sqrt{3}} \left(\frac{1 - \sqrt{3} a \pm \Sigma \sqrt{3} a}{\sqrt{3}}\right).$$

(52)

In terms of the new radial coordinate $r$, the configuration (1) is

$$ds^2 = \frac{\triangle_+ \triangle_-}{A(r)^{3/8} B(r)^{5/8}} dt^2 + \left(\frac{B(r)}{A(r)}\right)^{\frac{3}{4}} \sum_{i=1}^{6} dx_i^2 + A^2(r) B^{1/2}(r) \left(\frac{dr^2}{\triangle_+ \triangle_-} + r^2 d\Omega_2^2\right),$$

$$A_{[1]} = i e^{-3\phi_0/4} Q \left[1 - \frac{\sqrt{3} a + \Sigma}{\sqrt{3} r} - \frac{1 - \sqrt{3} a + \Sigma}{\sqrt{3} r A(r)}\right] dt,$$

$$A_{[7]} = i e^{3\phi_0/4} P \left[1 - \frac{\sqrt{3} a - \Sigma}{\sqrt{3} r B(r)} - \frac{1 - \sqrt{3} a - \Sigma}{\sqrt{3} r B(r)}\right] dt \wedge dx^1 \wedge \cdots \wedge dx^7,$$

$$e^{2(\phi - \phi_0)} = \left(\frac{B(r)}{A(r)}\right)^{3/2}.$$

(53)

Assuming that this configuration has a well-defined horizon at $r = r_+$, the metric can be made free of conical singularity at the horizon if the Euclidean time 't' is compact with periodicity

$$\beta^* = \frac{4 \pi r_+^2 \sqrt{A(r_+) B(r_+)} }{r_+ - r_-}.$$

(54)

This is the inverse of the temperature of the black D6/D0 system at infinity. The inverse of the local temperature at a given $r$ which is important for the analysis of the phase structure is given as,

$$\beta(r) = \sqrt{\frac{A(r_+) B(r_+)}{A^{1/8}(r) B^{7/8}(r)}} \frac{4 \pi r_+ (\triangle_+ \triangle_-)^{1/2}}{1 - \frac{r}{r_+}}.$$

(55)
As mentioned in \[7, 9\], we should use physical radius $\bar{r} = \sqrt{A^{7/8}(r)B^{1/8}(r)} r$ instead of the coordinate radius $r$ and also the physical parameters $\bar{r}_\pm = \sqrt{A^{7/8}(r)B^{1/8}(r)} r_\pm$ at a given $r$. Note that with these, $\Delta_\pm(r) = \Delta_\pm(\bar{r})$. For other related parameters, their physical correspondences should also be used accordingly. For examples, given $r_+ r_- = Q^2 + P^2$ from (35), the physical $\bar{Q} = \sqrt{A^{7/8}(r)B^{1/8}(r)} Q$ and so is for $\bar{P}$. Now in terms of the physical coordinate the inverse of the local temperature (55) at the given radius $\bar{r}$ takes the form,

$$\beta(\bar{r}) = \sqrt{\frac{A(\bar{r}_+ B(\bar{r}_+) 4\pi \bar{r}_+ (\Delta_+^{\bar{r}}) \Delta_-^{\bar{r}})^{1/2}}{1 - \frac{r_+}{\bar{r}_+}}}.$$  (56)

To study the equilibrium thermodynamics \[13\] in canonical ensemble, as mentioned in the beginning of this section, the allowed configuration must be placed in a cavity with fixed radius $\bar{r} = \bar{r}_B > \bar{r}_+$. The other quantities which are held fixed are the cavity temperature, $1/\bar{\beta}$, the physical periodicity of each $x^i$, for $i = 1, 2, \ldots, 6$, the dilaton value $\bar{\phi}$ on the surface of the cavity (at $\bar{r} = \bar{r}_B$) and the charges enclosed in the cavity $\bar{P}, \bar{Q}$. In equilibrium these values are taken to be equal to the corresponding values of the allowed configuration enclosed in the cavity. Note that the usual asymptotic value of dilaton $\phi_0$ is not fixed but is expressed in terms of the fixed $\bar{\phi}$ via (53) as $e^{\phi_0} = e^{\bar{\phi}}(A(\bar{r}_B)B(\bar{r}_B))^{3/4}$ where we have set $\phi(\bar{r}_B) = \bar{\phi}$. In what follows, we use a ‘bar’ above the symbol to denote the corresponding physical or fixed parameter.

In the canonical ensemble, the stability analysis can be performed using the Helmholtz free energy $F$ of the system under consideration which, to the leading order, is given as $F = I_E/\bar{\beta}$ with $I_E$, the Euclidean action \[13\]. We actually ask the question: in the canonical ensemble, i.e., with fixed $\bar{r}_B, \bar{Q}, \bar{P}, \bar{\phi}, \bar{\beta}$, what are the thermodynamically stable phases that can exist for the charged black D6/D0 in the cavity? Note that in the canonical ensemble, the only variable for this system is the horizon size $\bar{r}_+$ and so the local minimum of $F$ with respect to $\bar{r}_+$ will determine the local stability of the system. With $\bar{\beta}$ fixed, this can in turn be determined from the local minimum of $I_E$ with respect to $\bar{r}_+$.

Following our previous works \[7, 8, 9\], the Euclidean action $I_E$ for the charged black D6/D0 configuration (53) in the canonical ensemble, as specified above, can be explicitly computed and its so-called reduced form $\tilde{I}_E$, which is actually relevant for the above
mentioned stability analysis, is given as

\[ I_E = \frac{2\kappa^2 I_E}{4\pi \Omega_2 \bar{V}_6 \bar{r}_B^2} \]

\[ = b \left[ \frac{q_+^2 - q_-^2}{16 (\Delta_+ \Delta_-)^{1/2}} \left( \frac{A_-(\bar{r}_+) - A_-(\bar{r}_B)}{xA(\bar{r}_+)} - A_+(\bar{r}_B) \right) + \frac{7(q_+^2 + q_-^2)}{16 (\Delta_+ \Delta_-)^{1/2}} \left( \frac{A_+(\bar{r}_+) - A_+(\bar{r}_B)}{xB(\bar{r}_+) - B(\bar{r}_B)} \right) \right] + 4 - 2 \left( \frac{\Delta_+ \Delta_-}{\Delta_-} \right)^{1/2} - \frac{1}{4} \left( \frac{7A_+(\bar{r}_B) + A_-(\bar{r}_B)}{A(\bar{r}_B)} \right) \right] - x^2 \sqrt{A(\bar{r}_+)B(\bar{r}_+)} A(\bar{r}_+)B(\bar{r}_B), \]

(57)

Here \( \Omega_n \) denotes the volume of a unit \( n \)-sphere, the physical volume \( \bar{V}_6 \) is related to the coordinate volume \( V_6^* \equiv \int dx^1 dx^2 \cdots dx^6 \) via \( \bar{V}_6 = (B(\bar{r}_B)/A(\bar{r}_B))^{3/8} V_6^* \) (as can be seen from the metric given in (53)), and \( \kappa \) is a constant with \( 1/(2\kappa^2) \) appearing in front of the Hilbert-Einstein action in canonical frame but not containing the asymptotic string coupling \( g_s = e^{\phi_0} \). Also in the above, as usual for simplicity, we introduce the so called reduced quantities at the fixed radius \( \bar{r} = \bar{r}_B \) by the relations,

\[ x \equiv \frac{\bar{r}_+}{\bar{r}_B} < 1, \quad \bar{b} \equiv \frac{\bar{b}}{4\pi \bar{r}_B^2}, \quad q_+ \equiv \frac{Q_+}{\bar{r}_B} < x, \quad q_- \equiv \frac{Q_-}{\bar{r}_B} < q_+, \]

(58)

with \( Q_+^2 = Q^2 + \bar{P}^2, \quad Q_-^2 = Q^2 - \bar{P}^2 \) (assuming \( Q > P \)).\(^7\) In (57), we also define

\[ A_\pm(\bar{r}) = 1 - \frac{\sqrt{3\bar{a} \pm \Sigma}}{\sqrt{3\bar{r}}}. \]

(59)

In terms of these reduced quantities, the functions \( A(\bar{r}), A(\bar{r}_+), B(\bar{r}) \) and \( B(\bar{r}_+) \) can be written as,

\[ A(\bar{r}_B) = A_\pm^2(\bar{r}_B) + \frac{q_+^2}{4} - \frac{2}{3} \left( \frac{\Sigma}{\bar{r}_B} \right)^2 + \frac{2}{3q_-} \left[ \frac{32}{3} \left( \frac{\Sigma}{\bar{r}_B} \right)^2 - g(x) \right] \left( \frac{\Sigma}{\bar{r}_B} \right)^2, \]

\[ x^2 A(\bar{r}_+) = \left( x - \frac{\sqrt{3\bar{a} + \Sigma}}{\sqrt{3\bar{r}_B}} \right)^2 + \frac{q_-^2}{4} - \frac{2}{3} \left( \frac{\Sigma}{\bar{r}_B} \right)^2 + \frac{2}{3q_-} \left[ \frac{32}{3} \left( \frac{\Sigma}{\bar{r}_B} \right)^2 - g(x) \right] \left( \frac{\Sigma}{\bar{r}_B} \right)^2, \]

\[ B(\bar{r}_B) = A_\pm^2 - \frac{q_+^2}{4} - \frac{2}{3} \left( \frac{\Sigma}{\bar{r}_B} \right)^2 - \frac{2}{3q_-} \left[ \frac{32}{3} \left( \frac{\Sigma}{\bar{r}_B} \right)^2 - g(x) \right] \left( \frac{\Sigma}{\bar{r}_B} \right)^2, \]

\[ x^2 B(\bar{r}_+) = \left( x - \frac{\sqrt{3\bar{a} - \Sigma}}{\sqrt{3\bar{r}_B}} \right)^2 - \frac{q_-^2}{4} - \frac{2}{3} \left( \frac{\Sigma}{\bar{r}_B} \right)^2 - \frac{2}{3q_-} \left[ \frac{32}{3} \left( \frac{\Sigma}{\bar{r}_B} \right)^2 - g(x) \right] \left( \frac{\Sigma}{\bar{r}_B} \right)^2, \]

(60)

\(^7\)As mentioned earlier, we assume \( Q > P \) in our discussion and \( Q < P \) case can be obtained from the \( Q > P \) by the duality following the discussion after eq. (4).
where \( g(x) = 2x^2 \left( 1 - \frac{q^2}{x^2} \right)^2 + 3q^2 \).

Note that \( I_E = \beta E - S \), where \( E \) is the internal energy of the system and \( S \) is the entropy. In terms of the reduced Euclidean action and the other reduced quantities, we have \( \tilde{I}_E(x; q_+, q_-) = \tilde{b} \tilde{E}_{q_+, q_-}(x) - \tilde{S}_{q_+, q_-}(x) \). By comparing this with (57), one can read off both \( \tilde{E}_{q_+, q_-}(x) \) and \( \tilde{S}_{q_+, q_-}(x) \), explicitly. As stressed earlier, in the canonical ensemble, both \( q_+ \), \( q_- \) are fixed, the only variable is the reduced horizon size \( x \) and so varying \( \tilde{I}_E \) with respect to \( x \) we get,

\[
\frac{d\tilde{I}_E}{dx} = \frac{d\tilde{E}_{q_+, q_-}(x)}{dx} (\tilde{b} - b_{q_+, q_-}(x)) ,
\]

where

\[
b_{q_+, q_-}(x) \equiv \frac{dS_{q_+, q_-}(x)/dx}{dE_{q_+, q_-}(x)/dx} .
\]

\( \tilde{I}_E \) will be extremum if \( d\tilde{I}_E/dx = 0 \) and this implies from (61), \( b_{q_+, q_-}(\bar{x}) = \tilde{b} \) which is nothing but the condition for the thermal equilibrium of the charged black system, with a horizon size \( x = \bar{x} \) determined by this equation, with the fixed reduced temperature \( 1/\tilde{b} \) of the cavity. At \( x = \bar{x} \), we further have from (61)

\[
\left. \frac{d^2\tilde{I}_E}{dx^2} \right|_{x=\bar{x}} = - \left. \frac{d\tilde{E}_{q_+, q_-}(x)}{dx} \right|_{x=\bar{x}} \left. \frac{db_{q_+, q_-}(x)}{dx} \right|_{x=\bar{x}} .
\]

Now since \( E_{q_+, q_-}(x) \) is a monotonically increasing function of \( x \) for \( 0 < x < 1 \), the minimum of \( \tilde{I}_E \) implies that the slope of \( b_{q_+, q_-}(x) \) at \( x = \bar{x} \) must be negative. So the behavior of the function \( b_{q_+, q_-}(x) \) is crucial in determining the underlying phase structure. The explicit expression of \( b_{q_+, q_-}(x) \) can be obtained as described above, but the computation is quite lengthy and as expected it came out to be \( \beta(\bar{r}) \) given in (56) at \( \bar{r} = \bar{r}_B \) and expressed in terms of the reduced quantities. It is given as

\[
b_{q_+, q_-}(x) = \sqrt{\frac{A(\bar{r}_+) B(\bar{r}_+)}{A(\bar{r}_B) B(\bar{r}_B)}} x (1 - x)^{1/2} \left( 1 - \frac{q^2}{x^2} \right)^{1/2} \left( 1 - \frac{q^2}{x^2} \right)^{-1} ,
\]

where functions \( A(\bar{r}_+), B(\bar{r}_+) \) and \( A(\bar{r}_B), B(\bar{r}_B) \) are given in (50).

From our experience \([7, 8, 9]\), we know that the existence of the universal van der Waals-Maxwell liquid-gas type phase structure depends crucially on whether the \( b_{q_+, q_-}(x) \) blows up at \( x \to q_+ \), i.e., the extremal limit. For this we need to examine the behaviors

\[
\begin{align*}
\frac{d\beta(\bar{r})}{d\bar{r}} &\equiv \frac{\beta(\bar{r}_B)}{4\pi\bar{r}_B} \\
\end{align*}
\]
of $A(\tilde{r}_B)$, $A(\tilde{r}_+)$, $B(\tilde{r}_B)$ and $B(\tilde{r}_+)$. When $q_+ = q_-$, i.e., $P = 0$, we can obtain from (47), $\Sigma/\tilde{r}_B = \sqrt{3} q_+/(4x)$ and from there we obtain $\bar{a}/\tilde{r}_B = 3q_+^2/(4x)$. We then have from (60),

$$A(\tilde{r}_B) = \left(1 - \frac{q_+^2}{x}\right)^2, \quad B(\tilde{r}_B) = 1 - \frac{q_+^2}{x}, \quad A(\tilde{r}_+) = \left(1 - \frac{q_+^2}{x^2}\right)^2, \quad B(\tilde{r}_+) = 1 - \frac{q_+^2}{x^2}. \quad (65)$$

Substituting these into (64), we get

$$b_{q_+,q_+}(x) = x(1 - x)^{1/2} \left(1 - \frac{q_+^2}{x}\right)^{-1} \left(1 - \frac{q_+^2}{x^2}\right)^{1/2} \quad (66)$$

This is precisely the result obtained in [7] for charged black D6-branes when $D = 10$. Note here that the structure of the inverse of the reduced local temperature (66) for D6-branes is different (it is actually regular as $x \to q_+$) from the structure obtained for D6/D0 system (it blows up in the extremal limit) for a special case with $Q = P$ in [9]. We now look at the case when $q_- < q_+$. For this let us find the expressions for $\bar{\Sigma}/\tilde{r}_B$ and $\bar{a}/\tilde{r}_B$ first. From the solution of $Y$ in (47) we find,

$$\frac{\Sigma}{\tilde{r}_B} = \frac{1}{4} g^4(x) \left[1 - \frac{\left(C + \sqrt{C^2 - D}\right)^4}{2} + \frac{\left(C - \sqrt{C^2 - D}\right)^4}{2} + \frac{\left(D + \sqrt{D^2 - C}\right)^4}{2} + \frac{\left(D - \sqrt{D^2 - C}\right)^4}{2}\right] \frac{\Sigma}{\tilde{r}_B}. \quad (67)$$

From (49), we have

$$\frac{\bar{a}}{\tilde{r}_B} = \frac{1}{2} x \left[1 + \frac{q_+^2}{x^2}\right] + \frac{\sqrt{3}}{q} \left[\frac{32}{9} \frac{\Sigma^2}{\tilde{r}_B^2} - \frac{1}{3} g(x) \right] \frac{\Sigma}{\tilde{r}_B}. \quad (68)$$

Note that the parameters $c$ and $d$ given in (41) can now be written as $c(x) = q_+^2/g(x)$ and $d(x) = q_+^2/g(x)$ and therefore, as $x \to q_+$, $g(x) \to 3q_+^2$ and so $c(x)$, $d(x)$ as well as $C(x)$, $D(x)$ (given in (45)) go to,

$$c(x) \to \frac{1}{3}, \quad d(x) \to \frac{q_+^2}{3q_+^2}, \quad C(x) \to 1 - \frac{q_+^4}{q_+^4}, \quad D(x) \to \left(1 - \frac{q_+^4}{q_+^4}\right)^3 \quad (69)$$

Now substituting these in (67) and in (68), we find that $\bar{\Sigma}/\tilde{r}_B \approx q_+(q_-/q_+)^2/6$ and $\bar{a}/\tilde{r}_B \approx q_+/2$, both are regular as $x \to q_+$. Using (60), we have then

$$A(\tilde{r}) \approx \left(1 - \frac{q_+}{2}\right)^2 + \frac{q_+^2}{12}, \quad B(\tilde{r}) \approx \left(1 - \frac{q_+}{2}\right)^2 - \frac{q_+^2}{12} > 0, \quad A(\tilde{r}_+) = B(\tilde{r}_+) \approx \frac{1}{4}, \quad (70)$$

which are all regular as $x \to q_+$. From these, we have

$$\sqrt{\frac{A(\tilde{r}_+)B(\tilde{r}_+)}{A(\tilde{r})B(\tilde{r})}} \approx \frac{1}{4 \left[1 - \frac{q_+^4}{12}\right]^{1/2}}, \quad (71)$$
which is also regular. Thus the singular structure of $b_{q_+, q_-}(x)$ given in (64) for $q_+ > q_-$ as $x \to q_+$ is the same as the $Q = P$ case studied previously in [9]. Therefore, the phase structure essentially remains the same as in $Q = P$ case, however, we expect the phase structure to be much richer here (since the first square root factor in (64) will change the details of the phase structure), similar to that of D5/D1 system. But, unlike D5/D1 system, we are unable to give a full analytic analysis of the underlying phase structure in particular, the critical phenomenon here, because of the complicated dependence of $b_{q_+, q_-}(x)$ on $x$ as well as $q_+, q_-$ (as given in (64), with $A(\bar{r}_B), B(\bar{r}_B), A(\bar{r}_+)$ and $B(\bar{r}_+)$ given in (60)). However, we can still say something about the critical charge $(q_{c+}, q_{c-})$ in the present case with reference to $q_c = \sqrt{5} - 2 \approx 0.24$ in the case of $Q = P$ (or $q_- = 0$) given in [9]. For each $q_- \neq 0$ with $0 < q_- < q_+$, we expect the corresponding critical charge $q_{c+} > q_c = \sqrt{5} - 2$ for the following reason. To understand this, let us denote the first square root factor in (64) as $w_{q_+, q_-}(x)$ and the remaining as $b_{q_+}(x)$, we can then

---

Footnote: Note that for $Q = P$ case $A(\bar{r}_B) = B(\bar{r}_B) = A(\bar{r}_+) = B(\bar{r}_+) = 1$ and so, in that case the inverse of the reduced temperature has the form given in (64) without the first square root factor.
Figure 2: The behavior of $w_{q^+, q^-}(x)$ vs $x$ for a given $q_- = 0.25$ and three different $q^+ = 0.30, 0.50, 0.90$, respectively.

We rewrite

$$b_{q^+, q^-}(x) = w_{q^+, q^-}(x)b_{q^+}(x), \quad (72)$$

Note that for $q_- = 0$, $b_{q^+, q^-}(x) = b_{q^+}(x)$ since in that case $w_{q^+, q^-}(x) = 1$.

We also know from [60] that for $q_- \neq 0$, $w_{q^+, q^-}(x \to 1) \to 1$ and $w_{q^+, q^-}(x \to q^+) < 1$. Actually $w_{q^+, q^-}(x)$ is a monotonically increasing function of $x$ for $q^+ < x < 1$. We give two figures with different pairs of $(q^+, q^-)$ values for showing this. From Figures 1 and 2, we see that $w_{q^+, q^-}(x \to 1) \to 1$ and close to $x = 1$, this function is more sensitive to $q^+$ values while close to $x = q^+$, it is sensitive to both $q^+$ and $q^-$ values. For $q_- = 0$, $b_{q^+}(x)$ gives the corresponding critical charge $q_c = \sqrt{5} - 2$ which is determined by requiring that both its first and second derivatives vanish [9]. For this critical $q_c$, we also have a critical reduced horizon size $x_c = 5 - 2\sqrt{5}$ [9], and if $x$ is close to $x_c$, we have $b_{q^+}(x > x_c) = b_{q^-}(x < x_c)$ up to $O((x - x_c)^3)$. Now for $q^+ = q_c$ and $q^- \neq 0$, we must have $b_{q^+, q^-}(x < x_c) = w_{q^+, q^-}(x < x_c)b_{q^+}(x < x_c) < b_{q^+, q^-}(x > x_c) = w_{q^+, q^-}(x > x_c)b_{q^+}(x > q_c)$ since $w_{q^+}(x < x_c) < w_{q^+}(x > x_c)$.

\[\text{Note that for } q^+ = 0.50 \text{ or } 0.90, \text{ the valid region of } x \text{ is } q^+ < x < 1, \text{ but in these two figures we extend } x \text{ in the lower end to } x = 0.30. \text{ Even for } 0.30 < x < 1, \text{ the corresponding } w_{q^+, q^-}(x) \text{ is still an increase function of } x.\]
Figure 3: The behavior of $b_{q_+,q_-}(x)$ vs $x$ for a given $q_+ = 0.20$ and three different $q_- = 0.01, 0.10, 0.19$, respectively.

$w_{q_+,q_-}(x > x_c)$ even though we still have $b_{q_+}(x > x_c) \approx b_{q_+}(x < x_c)$ in the sense described above. Given our experience about the van der Waals-Maxwell liquid-gas type phase structure, we infer that $q_+ = q_c = \sqrt{5} - 2$ is less than the actual critical charge $q_{+c}$ for the present system since otherwise we should have $b_{q_+,q_-}(x < x_c) \geq b_{q_+,q_-}(x > x_c)$. In other words, the critical charge $q_{+c} > q_c = \sqrt{5} - 2 \approx 0.24$ in the case of $q_{-c} \neq 0$. In the following, we give a few figures to show this and also indicate how the phase structure depends on both $q_+$ and $q_-$. For the $b_{q_+,q_-}(x)$ behavior, we consider a small $q_-$ corresponding to $Q \gg P$, a characteristic value of $q_-$, corresponding to $Q > P$, and a $q_- \ll q_+$, corresponding to $P \gg 0$. From Figures 4 and 5, we see that the corresponding critical charge $q_{+c}$ falls between 0.30 and 0.50, which is indeed consistent with what we discussed above about $q_{+c} > q_c = \sqrt{5} - 2$. Figure 5 indicates that the influence of $q_-$ on the behavior of $b_{q_+,q_-}(x)$ becomes less important even for $x$ close to the end $x = q_+$ when $q_+ > q_{+c}$. Here we give three more figures (Figures 6, 7 and 8), each of which is now for a given value of $q_-$ and three different values of $q_+$, to illustrate what has been commented.
Figure 4: The behavior of $b_{q^+,q^-}(x)$ vs $x$ for a given $q^+ = 0.30$ and three different $q^- = 0.02, 0.15, 0.28$, respectively.

Figure 5: The behavior of $b_{q^+,q^-}(x)$ vs $x$ for a given $q^+ = 0.50$ and three different $q^- = 0.05, 0.30, 0.45$, respectively.
on the critical charge $q_{+c}$.

We point out that similar to the D5/D1 system [8], here also the charge $q_+$ and $q_-$ span a two-dimensional region bounded by $q_+ = q_-; q_- = 0$, (with $0 \leq q_+ \leq 1$) and $q_+ = 1$ (with $0 \leq q_- \leq 1$), as shown in Figure 9. In this Figure, we also draw a characteristic critical line, determined by the vanishing of the first and the second derivatives of $b_{q_+, q_-}(x)$ with respect to $x$. As mentioned above, the complicated expression of $b_{q_+, q_-}(x)$ makes it impossible for us to give an analytic analysis of this critical line, unlike the case of D5/D1 system [8]. As already discussed, this critical line starts at $q_{+c} = q_c = \sqrt{5} - 2, q_{-c} = 0$, and once $q_{-c} > 0, q_{+c} > q_c = \sqrt{5} - 2$, but the end point cannot be determined analytically since $q_+ = q_-$ can never even be a pseudo ‘critical point’ since this corresponds to $P = 0$ and the corresponding system, similar to the chargeless case, has no van der Waals-Maxwell liquid-gas type phase structure[9]. Our numerical attempts indicate that the end point is around $q_+ = 0.52$ with a $q_-$ very close to this value, but not reaching the $q_- = q_+$ line. Figure 10 gives an indication of this for $(q_{+c}, q_{-c}) = (0.520000000, 0.519999999)$. From this, one can see that the critical size $x_c$ should fall between 0.52 and 0.64.

\[11\]In the sense that the first and the second derivatives of $b_{q_+, q_-}(x)$ vanish similar to the D5/D1 system even though this point is not a true critical point.
Figure 7: The behavior of $b_{q^+,q^-}(x)$ vs $x$ for a given $q^- = 0.10$ and three different $q^+ = 0.20, 0.30, 0.40$, respectively.

Figure 8: The behavior of $b_{q^+,q^-}(x)$ vs $x$ for a given $q^- = 0.18$ and three different $q^+ = 0.20, 0.30, 0.40$, respectively.
critical line

Figure 9: The two-dimensional region of allowed reduced charge $q_+$ and $q_-$. 

This critical line separates the $(q_+, q_-)$-region into two parts, the small one on the left and the large one on the right, as shown in Figure 9. For each given pair of $(q_+, q_-)$ with $q_- < q_+$ in the left part, $b_{q_+, q_-}(x)$ has a minimum $b_{\text{min}}$ and a maximum $b_{\text{max}}$ in the region $q_+ < x < 1$, occurring at $x_{\text{min}}$ and $x_{\text{max}}$, respectively. If the given $\bar{b}$ on the surface of the cavity falls between $b_{\text{min}}$ and $b_{\text{max}}$, then $\bar{b} = b_{q_+, q_-}(\bar{x})$ gives three solutions $x_1 < x_2 < x_3$, which can be easily understood from, for example, Figure 3. Only at $x_1$ and $x_3$, the corresponding slope of $b_{q_+, q_-}(x)$ is negative, giving the local minima of the free energy. For this given pair of $(q_+, q_-)$, there exists a unique $b_t$ in the range $b_{\text{min}} < b_t < b_{\text{max}}$ such that the free energy at $x_1$ and that at $x_3$ (now $x_1$ and $x_3$ are determined from $b_t = b_{q_+, q_-}(\bar{x})$), are equal. Therefore these two phases, one with the (reduced) horizon size $x_1$ and the other with size $x_3$, can coexist and the phase transition between the two is a first order one since it involves an entropy change (note that the entropy for each phase is determined by its horizon size). One expects that like in D5/D1 system, $b_t$ is a function of both $q_+$ and $q_-$ and therefore spans a first order transition surface (two-dimensional), ending on a one-dimensional critical line, rather than a first order transition line, ending on a critical point as for charged black $p$-brane with $p < 5$. For $\bar{b} > b_t$, following the analysis given in [7], we know that the phase with the smaller (reduced) horizon size $x_1$ has the lower free energy, therefore the stable phase, while for $\bar{b} < b_t$, the phase with the larger (reduced) horizon size $x_3$ is the stable one. In other words, the smaller stable black D6/D0 is like the liquid phase while the larger one is like the gas phase. For a given pair of $(q_+, q_-)$ with $q_- < q_+$ in the right part, for each given $\bar{b}$ we have a unique solution $\bar{x}$ from $\bar{b} = b_{q_+, q_-}(\bar{x})$ and the slope of $b_{q_+, q_-}(x)$ at this $\bar{x}$ is always negative, as can be seen, for example, from Figure 5, the corresponding free energy is the lowest and therefore the phase is stable.

Now the reader might wonder why adding charge to the uncharged black configuration
Figure 10: The critical behavior of $b_{q_+, q_-}(x)$ vs $x$ for a given pair of $(q_+, q_-) = (0.520000000, 0.519999999)$.

(Schwarzschild black hole or black $p$-branes with $p < 5$) or adding particular delocalized charged lower dimensional branes to the original branes (for D5 or D6 branes) can modify the usual Hawking-Page type phase structure to the van der Waals-Maxwell liquid-gas type? This is what we try to address in the next section.

### 4 Origin of the phase structure modification

One key observation for the qualitative change of phase structure from the uncharged black configuration to the charged one is the appearance of the divergent behavior of the reduced inverse temperature at one end $b(x \to q) \to \infty$, while the condition at the other end $b(x \to 1) \to 0$ remains the same (note that 1 and $q$ are the upper and the lower end points of the variable $x$ respectively). The limit $x \to q$ is actually the extremal limit and so, we can use the extremal black holes/branes to understand the reason behind the qualitative change of phase structure, a great simplification. Before we address the black holes/branes, let us understand the usual van der Waals liquid-gas phase structure
For a given \( T \)

\[
0 = \frac{a}{kT}
\]

\[ p = \frac{kT v - a}{v^2}, \] (74)

whose behavior is drawn in Figure 11. This is quite similar to the \( b(x) \) vs \( x \) diagram of uncharged black holes/branes. When we turn on the repulsive interaction, i.e., \( b \neq 0 \), we have the usual van der Waals-Maxwell liquid-gas structure. The exact same thing happens when we add charge to the uncharged black hole (in other words, in this case we add the repulsive interaction due to the added charge to the original gravitational attractive interaction due to mass), giving also the van der Waals-Maxwell liquid-gas type phase structure. This seems to suggest that the van der Waals liquid-gas type phase structure

We caution the reader not to confuse the van der Waals parameter \( a \) and \( b \) used here with the same parameter used for describing the D6/D0 system in the earlier sections.
is the result of competition between the attractive and the repulsive interactions and is independent of whether the underlying system is a liquid-gas system or a gravitational system.

This also does seem to help us understand the phase structure of the charged black $p$-brane systems. For example, adding the delocalized charged $D(p-2)$-branes to the charged black $Dp$-branes does not change the phase structure of the original $Dp$-branes since in the extremal limit, the interaction between the delocalized $D(p-2)$-branes and $Dp$-branes is attractive\footnote{For interactions between branes with different dimensionalities, see for example, \cite{14}.}. However, adding the delocalized charged $D0$-branes to the original $D6$-branes increases the repulsive interaction and therefore changes the phase structure from something similar to the chargeless case to the van der Waals-Maxwell liquid-gas type. For $D5$-branes this picture does not resolve the puzzle, namely, we know that in the extremal limit there is no interaction between the delocalized $D1$-branes and $D5$-branes, but the phase structure still qualitatively changes to have the van der Waals-Maxwell liquid-gas type when we add delocalized $D1$-branes to $D5$-branes.

This hints at the fact that having the additional repulsive interaction is not the complete story. In addition to providing repulsive interaction, adding charge or additional delocalized charged lower dimensional branes can also increase the degeneracy or the entropy of the underlying system. Note that in canonical ensemble the underlying phase structure is determined by the Helmholtz free energy, which consists of two parts, the internal energy and the entropy. Therefore it is natural to expect that entropy also has a role to play in addition to what has been mentioned about the nature of interactions. Let us examine the origin of the divergent behavior mentioned earlier, which is the key to the underlying phase structure, in detail.

First let us focus on the van der Waals isotherm. We have

\begin{align}
E &= \frac{3}{2}NkT - \frac{aN}{v}, \quad S = Nk \left[ \ln \left( \frac{(v-b)T^{3/2}}{\Phi} \right) + \frac{5}{2} \right], \\
p &= -\frac{1}{N} \left( \frac{\partial F}{\partial v} \right)_{T,N} = \frac{T}{N} \left( \frac{\partial S}{\partial v} \right)_{T,N} - \frac{1}{N} \left( \frac{\partial E}{\partial v} \right)_{T,N}, \\
&= \frac{kT}{v-b} - \frac{a}{v^2},
\end{align}

(75)

where $F, E$ and $S$ are the free energy, the internal energy and the entropy(Note that $\Phi$ is just a constant), respectively, with $F = E - TS$. From the above it is clear that the divergence $p \to \infty$ as $v \to b$ actually originates from the entropy. When $b = 0$, given that $v \geq v_0 = a/kT$ (see Figure 11), both the internal energy and the entropy of the system are finite when $v \to v_0$, so do $(\partial S/\partial v)_{T,N}$ and $(\partial E/\partial x)_{T,N}$. However, when
we turn on $b$, i.e., $b \neq 0$, the entropy blows up when $v \rightarrow b$ while the internal energy essentially remains unchanged (except that we need to replace the lower end limit $v \rightarrow v_0$ by $v \rightarrow b$). The same is true for $(\partial S/\partial v)_{T,N}$ and $(\partial E/\partial x)_{T,N}$, respectively. In other words, the appearance of phase structure of van der Waals-Maxwell liquid-gas is due to the dramatic change of entropy when $v \rightarrow b$ (with non-zero repulsive interaction $b$). So, the repulsive core of molecules or atoms has more dramatic influence on the entropy than on the internal energy.

Let us see what happens for the black holes. Here we have the following expressions for the so-called reduced internal energy, reduced entropy and reduced inverse temperature, for example, from \cite{6}

\[
\tilde{E} = 4 \left[ 1 - \sqrt{1 - x} \left( 1 - \frac{q^2}{x^2} \right) \right], \quad \tilde{S} = x^2,
\]

\[
b_q(x) = \frac{(\partial \tilde{S}/\partial x)_q}{(\partial \tilde{E}/\partial x)_q} = \frac{x(1 - x)^{1/2} \left( 1 - \frac{q^2}{x^2} \right)^{1/2}}{1 - \frac{q^2}{x^2}}.
\]

Here we have denoted the reduced inverse temperature with a subscript $q$ to indicate that this is a charged case and for chargeless case $q$ should be put to zero. It is clear from \cite{6} that the divergence of $b_q(x)$ as $x \rightarrow q$ is due to the fact that $(\partial \tilde{E}/\partial x)_q$ vanishes and $(\partial \tilde{S}/\partial x)_q$ remains finite in this limit. This is quite different from the previous case where the divergence of $p$ was due to the blowing up of $dS/dv$ as $v \rightarrow b$. Note that here both the entropy and the internal energy change in the same way, i.e., from zero in the chargeless case to a finite value in the charged case, in the respective lower end limit, i.e., $x \rightarrow 0$ (for the chargeless case) or $x \rightarrow q$ (for the charged case). However, their rate with respect to $x$ changes in the opposite way. For the entropy, the rate changes from zero to a positive finite value in the above respective lower end limit while for the internal energy the corresponding rate changes from a positive finite value to zero in the same respective lower end limit. Such a change of rate for either entropy or internal energy is due to the addition of charge since adding charge not only gives rise to the repulsive interaction, but also to the increase of degrees of freedom of the system, therefore, the entropy. So the vanishing of $(\partial \tilde{E}/\partial x)_q$ in the limit $x \rightarrow q$ is due to the addition of charge and is mostly responsible for the blowing up of $b_q(x)$ in this same limit, therefore for the underlying phase structure (given that, the non-vanishing of $(\partial \tilde{S}/\partial x)_q$ in this same limit is also important). So the reason for the underlying phase structure in the present case (where the rate of entropy is finite) is quite opposite to the van der Waals isotherm (where

\footnote{Our definition for either $\tilde{E}$ or $\tilde{S}$ differs from \cite{6} by a factor of 4.}
the rate of entropy blows up) we discussed earlier.

Now let us move on to the black $p$-brane case and see what happens there. For simple charged black $p$-branes, we have

$$\tilde{E}(x) = 2 \left[ (8 - p) - \frac{7 - p}{2} \sqrt{\Delta_+ \Delta_-} - \frac{9 - p}{2} \sqrt{\Delta_+ \Delta_-} \right],$$

$$\tilde{S}(x) = x^{1/2} \left( 1 - \frac{\Delta_+}{\Delta_-} \right)^{\frac{x - p}{x^{1/2} - p}},$$

$$b_q(x) = \frac{\left( \partial \tilde{S} / \partial x \right)_q}{\left( \partial \tilde{E} / \partial x \right)_q} = \frac{x^{1/2}}{7 - p} \sqrt{\frac{\Delta_+}{\Delta_-}} \left( 1 - \frac{\Delta_+}{\Delta_-} \right)^{\frac{p - x}{x^{1/2} - p}},$$

(77)

where $q < x < 1$ and

$$\Delta_+ = 1 - x, \quad \Delta_- = 1 - \frac{q^2}{x}. \quad (78)$$

Notice that the reduced entropy vanishes in the lower end limit either in the chargeless case ($x \to 0$) or the charged case ($x \to q$) for $p \leq 7$. Further, $(\partial \tilde{S}(x)/\partial x)_q$ vanishes in the chargeless case for all $p \leq 7$ in the $x \to 0$ limit but it blows up in the charged case only for $p < 5$, becomes a finite value for $p = 5$, and vanishes again for $p = 6$ in the extremal limit $x \to q$. The internal energy itself changes from zero in the chargeless case to a positive finite value in the charged case in its respective lower end limit and $(\partial \tilde{E}/\partial x)_q$ is always positively finite in either case in the corresponding extremal limit for $p \leq 7$. So the divergent behavior of $b_q(x)$ as $x \to q$ is once again due to the blowing up of $(\partial \tilde{S}/\partial x)_q$ for $p < 5$ and this divergent rate of entropy is responsible for the underlying phase structure. In other words, $p < 5$ systems behave much like the van der Waals isotherm in phase structure, we discussed.

Let us consider the special case of $p = 5$. Previous study [7] showed that when 5-brane is charged the phase structure is essentially of the same type as the chargeless case without a van der Waals-Maxwell liquid-gas structure, even though there are three different sub-structures, analogous to the $p < 5$ cases. Further study [8] demonstrated that this phase structure can be qualitatively modified to a van der Waals-Maxwell liquid-gas type by adding delocalized charged D1-branes to the black charged D5-branes. As discussed previously, since in the extremal limit $x \to q_5$, there is no interaction between D1-branes and D5-branes, the divergent behavior of $b_{q_1,q_5}(x)$ must come from the blowing up of $(\partial \tilde{S}/\partial x)_{q_1,q_5}$ in this limit. This can be understood as the addition of delocalized charged D1-branes increases the degeneracy of underlying system, therefore the entropy. Let us examine to see if this is indeed the case in detail. For the D1/D5 system, we
have

\[
\tilde{E}(x) = 2 \left[ 3 + \sqrt{\frac{\Delta_+}{\Delta_-}} (1 - G_1^{-1}) - 2 \sqrt{\frac{\Delta_+}{\Delta_-} - \sqrt{\Delta_+ \Delta_-}} \right],
\]

\[
\tilde{S}(x) = x^{1/2} \left( 1 - \frac{\Delta_+}{\Delta_-} \right) \left[ 1 + \frac{1 - G_1^{-1}}{\Delta_+ - 1} \right]^{1/2},
\]

\[
b_{q_1,q_5}(x) = \frac{\left( \partial \tilde{S}/\partial x \right)_{q_1,q_5}}{\left( \partial \tilde{E}/\partial x \right)_{q_1,q_5}} = \frac{x^{1/2}}{2} \left( \frac{\Delta_+}{\Delta_-} \right)^{1/2} \left[ 1 + \frac{1 - G_1^{-1}}{\Delta_+ - 1} \right]^{1/2},
\]

where now \( q_5 < x < 1 \), and

\[
\frac{\Delta_+}{\Delta_-} = \frac{1 - x}{1 - q_5^2/x},
\]

\[
1 - G_1^{-1} = \frac{1}{2} \left[ \sqrt{\left( \frac{\Delta_-}{\Delta_+} - 1 \right)^2 + 4q_1^2 \frac{\Delta_-}{\Delta_+} - \left( \frac{\Delta_-}{\Delta_+} - 1 \right)} \right].
\]

From the above, we have \( b_{q_1,q_5}(x) \to \infty \) as \( x \to q_5 \). Note that \( \tilde{S} \) continues to vanish in the extremal limit \( x \to q_5 \) but \( \left( \partial \tilde{S}/\partial x \right)_{q_1,q_5} \) blows up in the same limit. Both the reduced internal energy \( \tilde{E} \) and its rate \( \left( \partial \tilde{E}/\partial x \right)_{q_1,q_5} \) are non-zero finite in the same limit. So the divergent behavior of \( b_{q_1,q_5}(x) \) in the limit \( x \to q_5 \) is indeed due to the blowing up of \( \left( \partial \tilde{S}/\partial x \right)_{q_1,q_5} \) in the same limit, also much like the van der Waals isotherm in phase structure, as anticipated.

Finally, let us consider the special case of \( p = 6 \). As shown in [7], when charge is added to black D6-branes, the resulting phase structure of charged black D6-branes remains the same as its chargeless counterpart (except that we need to replace the zero of the lower end of \( x \) by finite \( q \)). It was also shown in [8] and discussed in [9] as well in the previous sections in this paper that this phase structure cannot be modified to the van der Waals-Maxwell liquid-gas type by adding either delocalized charged D4- or D2-branes except by adding the delocalized charged D0-branes. We demonstrated in the previous sections that the phase structure for a general D6/D0 system is essentially the same as that of the special case when D0-brane charge \( Q \) is set equal to D6-brane charge \( P \) [9]. For this reason, for simplicity, we, in what follows, just use this special case to uncover the reason behind such a change of phase structure. For D6/D0 system with \( q_0 = q_6 = q \) (here we
are using the reduced charges of D0- and D6-branes), we have

\[
\tilde{E}(x) = 4 \left[ 1 - \sqrt{(1 - x) \left(1 - \frac{q^2}{x}\right)} \right],
\]

\[
\tilde{S}(x) = x^2, \quad (q < x < 1)
\]

\[
b_q(x) = \left( \frac{\partial \tilde{S}}{\partial x} \right)_q = \frac{x(1 - x)^{1/2} \left(1 - \frac{q^2}{x}\right)^{1/2}}{1 - \frac{q^2}{x^2}},
\]

(81)

where now \(b_q(x) \to \infty\) as \(x \to q\). If we compare this case with the charged black hole discussed earlier in (76), we find that we have exactly the same \(\tilde{E}, \tilde{S}, b_q(x)\) in both cases. This is not surprising since it is well-known that when we dimensionally reduce this system to \(D = 4\), we end up precisely with the \(D = 4\) charged black hole. So we expect that the discussion given there applies here, too. In other words, the qualitative change of phase structure is due to the added ‘repulsive interaction’. The deep reason behind this can also be understood from string/M theory since we know that the interaction between D0- and D6-branes are repulsive and adding delocalized D0-branes to the charged black D6-branes precisely adds this repulsive interaction to the system making the qualitative change of phase structure possible.

With the above analysis, we understand the underlying reason for the appearance of van der Waals-Maxwell liquid-gas type phase structure in various cases. The key to this is to scrutinize what causes the divergent behavior of the local function, the inverse temperature \(b_q(x)\), for the various black systems in the extremal limit \(x \to q\). Since we consider canonical ensemble, the thermodynamical function of interest is the Helmholtz free energy \(F = E(x) - TS(x)\), where \(E(x)\) and \(S(x)\) are the internal energy and the entropy and \(T\) is the fixed temperature of the cavity. So, it is the rate of change of entropy and the internal energy with respect to \(x\) which are responsible for the divergent behavior of \(b_q(x)\) in the extremal limit \(x \to q\) and not the entropy and internal energy themselves. When \(q = 0\), \(E(x), S(x)\) and \(b_q(x)\) all vanish in the limit \(x \to 0\). This has to be true given the physical context of chargeless black system. For \(q = 0\), the black system has just mass and therefore the interaction is only attractive. In string/M theory context, we know that the system has equal number of branes and antibranes and the net interaction has to be attractive. However, when a non-zero charge \(q\) is added, actually two ingredients are added to the system: one is the repulsive interaction (in addition to already existing attractive one due to mass), and the other is the increase in the degeneracy, therefore, the entropy (since adding charge means adding additional degrees of freedom). This is particularly obvious in the context of string/M theory. These
two new ingredients brought to the system when charge is added are what is needed for modifying the phase structure, since the phase structure is determined by the free energy or in turn by the internal energy and the entropy. In string/M theory since there exists various kinds of branes, there are various ways to add these two ingredients to the already existing system. So, for example, we can add charges to the chargeless branes to provide both the repulsive interaction and the additional entropy or add different kind of branes to provide more repulsive interaction (as in the case of adding D0-branes to D6-branes) or add different kind of branes to increase the entropy (as in the case of adding D1-branes to D5-branes) of the system. This is precisely what we have tried and succeeded for D5- and D6-branes. Addition of particular delocalized branes makes $b_q(x)$ divergent in $x \to q$ limit by either blowing up of $dS(x)/dx$ or making $dE(x)/dx$ vanish or both.

5 Conclusion

To conclude, in this paper we have studied the charged black D6/D0 bound state configuration of type IIA supergravity and its thermodynamic phase structure with all generality. The phase structure of the same system has been studied before but only in a special case when the charges associated with D6-branes and D0-branes are equal and that associated with the dilaton is zero. But here we have considered all the parameters of the solution to take generic values. In general the solution is characterized by three independent parameters. We have argued that the solution is not well-defined in the entire parameter space. There are naked singularities in certain region of the parameter space. We have given general arguments to show that when we restrict ourselves to certain other region of the parameter space then only the D6/D0 solution has well-defined horizon and is suitable for studying thermodynamics. We have studied the equilibrium thermodynamics and the phase structure of the general black D6/D0 solution in the canonical ensemble. For this purpose, we have computed the Euclidean action, the form of the so-called reduced inverse temperature in a suitable coordinate and expressed this inverse temperature in terms of a single parameter $x$ (the reduced horizon radius of the black D6/D0 solution). We argued that the phase structure which is governed by the singularity structure of the reduced inverse temperature as $x \to q$, is similar to the special case studied before. But here the analysis is much more involved and the phase structure is richer than that of the special case. This shows that it is a general feature (not a consequence of the special case) that when charged delocalized D0-branes are added to charged D6-branes, the phase structure of D6-branes get qualitatively changed and takes the universal form (as for other $Dp$-branes with $p < 5$) which has van der Waals-Maxwell liquid-gas type structure. We have
tried to unravel the reasons why such a drastic change in phase structure occurs when charges and/or other branes are added to the existing system. We have shown in a case by case basis that adding charge and/or other branes actually adds either the repulsive interaction or the additional degrees of freedom, i.e., entropy to the system. These two ingredients are actually causing the qualitative change of phase structure to the universal form in various cases.

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