Steep Decay Phase Shaped by the Curvature Effect. I. Flux Evolution

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Abstract

The curvature effect may be responsible for the steep decay phase observed in gamma-ray bursts. To test the curvature effect with observations, the zero time point t₀ adopted to plot the observer time and flux on a logarithmic scale should be appropriately selected. In practice, however, the true t₀ cannot be directly constrained from the data. Thus, we move t₀ to a certain time in the steep decay phase, which can be easily identified. In this situation, we derive an analytical formula to describe the flux evolution of the steep decay phase. The analytical formula reads as \( F_\nu \propto (1 + t_{obs}/\hat{t}_c)^{-\alpha} \), with \( \alpha(t_{obs}) = 2 + \int_0^{\mathcal{t}_c} \beta(\tau)d[\log(1 + \tau/\hat{t}_c)]/\log(1 + t_{obs}/\hat{t}_c) \), where \( F_\nu \) is the flux observed at frequency \( \nu \), \( t_{obs} \) is the observer time by setting \( t_0 \) at a certain time in the steep decay phase, \( \beta \) is the spectral index estimated around \( \nu \), and \( \hat{t}_c \) is the decay timescale of the phase with \( t_{obs} \geq 0 \). We test the analytical formula with the data from numerical calculations. It is found that the analytical formula presents a good estimate of the evolution of the flux shaped by the curvature effect. Our analytical formula can be used to confront the curvature effect with observations and estimate the decay timescale of the steep decay phase.

Key words: gamma-ray burst: general

1. Introduction

Gamma-ray bursts (GRBs) are the most powerful explosive events in the universe. They are always traced by the Burst Alert Telescope (BAT) in the \( \gamma \)-ray energy bands (Barthelmy et al. 2005a). This phase is the so-called \( \gamma \)-ray prompt emission, which can last from 10 milliseconds to several minutes, and even longer (Kouveliotou et al. 1993; Gendre et al. 2013; Stratta et al. 2013; Virgili et al. 2013; Levan et al. 2014; Zhang et al. 2014). Following the \( \gamma \)-ray prompt emission is a long-lived afterglow emission, which emits mainly at longer wavelengths, such as X-ray, optical, and radio. The observations of the Swift satellite reveal that the light curve of X-ray afterglow emission is composed of five components (Nousek et al. 2006; O’Brien et al. 2006; Zhang et al. 2006, 2007). The first of these components is the initial steep decay phase, which appears at around \( 10^{2-3} \) s after the burst trigger (Cusumano et al. 2006; O’Brien et al. 2006; Vaughan et al. 2006). By extrapolating the prompt \( \gamma \)-ray light curve to the X-ray band, it is found that the initial steep decay phase observed can connect smoothly to this extrapolated X-ray light curve. Thus, it is believed that the initial steep decay phase may be the “tail” of the prompt emission (Barthelmy et al. 2005b; Liang et al. 2006; O’Brien et al. 2006). Besides the prompt emission phase, the steep decay is also observed in the decay phase of flares (e.g., Jia et al. 2016; Mu et al. 2016; Uhm & Zhang 2016). The behavior of the steep decay phase is our focus in this work.

For the steep decay phase, the temporal decay index \( \alpha \) of the observed flux is typically \( \sim 3-5 \). Moreover, the value of \( \alpha \) is found to be correlated with the spectral index \( \beta \). This led to the development of the “curvature effect” model, which plays an important role in shaping the flux decline in the steep decay phases (Liang et al. 2006; Wu et al. 2006; Yamazaki et al. 2006; Zhang et al. 2006). When emission in a spherical relativistic jet ceases/decays abruptly, the observed flux is controlled by high latitude emission in the jet shell. In this situation, the photons from the higher latitude would be observed later and have a lower Doppler factor. Then, the observed flux would progressively decrease. For an intrinsic spectrum described by a single power-law form (i.e., \( F' \propto \nu^{-\beta} \), with \( \beta = \) constant in the comoving frame of the jet shell), the relation between \( \alpha \) and \( \beta \) due to the curvature effect is given as (see Uhm & Zhang 2015 for details; Kumar & Panaitescu 2000; Dermer 2004; Dyks et al. 2005)

\[
\alpha = 2 + \beta. \tag{1}
\]

As shown in Nousek et al. (2006), the above relation is in rough agreement with the data on the steep decay phase of some Swift bursts. Adopting a time-averaged \( \beta \) in the steep decay phases, Liang et al. (2006) finds that Equation (1) is generally valid.

To test Equation (1) with observations, the zero time point \( t_0 \) is usually employed (Liang et al. 2006; Zhang et al. 2006; Uhm & Zhang 2016). If one wants to find the relation given in Equation (1) based on the observational data, the time \( t_0 \) for the steep decay phase should be appropriately selected. This is because the light curves of GRBs are plotted on a logarithmic scale for both the observer time and the flux in order to find the decay slope \( \alpha \). A different reference time \( t_0 \) adopted to plot the light curves would affect the obtained value of \( \alpha \). For a spherical relativistic jet moving with a constant Lorentz factor, Equation (1) can be found by setting \( t_0 \) at the observed time of
jet (Uhm & Zhang 2015). In practice, however, the true \( t_0 \) cannot be directly constrained from the data. There are three reasons: (1) the radiation of the jet in GRBs always begins at the radius \( r_0 \gg 0 \) rather than at \( r_0 = 0 \); (2) the detector misses the initial portion of jet radiation due to detector sensitivity; and (3) the initial portion of the jet radiation may be buried under the background (Uhm & Zhang 2016). One practical way to test the curvature effect model may be to move \( t_0 \) to a certain time in the steep decay phase that can be easily identified in the GRB light curves. Since the setting of \( t_0 \) in this situation is not physically motivated, the \( \alpha - \beta \) law and even the flux evolution pattern naturally deviate from the standard law as shown in Equation (15) of Uhm & Zhang (2015). Then, we try to derive the analytical formula to describe the flux evolution in this situation.

This paper is organized as follows. Since our analytical formula for flux evolution will be tested with data from numerical calculations, the numerical procedures in our numerical calculations are presented in Section 2. Moving \( t_0 \) to a certain time in the steep decay phase, the analytical formula of the flux evolution is presented and tested in Sections 3 and 4, respectively. Our conclusions are summarized in Section 5.

(I): \( H'(x) = x^{-\beta_0 - k \log(x)} \)

(II): \( H'(x) = x^{\alpha' + 1}[1 + g x^{(\alpha' - k)w}]^{-1/w} \)

(III): \( H'(x) = \begin{cases} x^{\alpha' + 1} \exp(-x), & x \leq (\alpha' - \beta') \\ (\alpha' - \beta')^{\alpha' - \beta} \exp(\beta' - \alpha')x^{\beta' + 1}, & x \geq (\alpha' - \beta') \end{cases} \)

(IV): \( H'(x) = x^2 \{ \exp(x) - 1 \}^{-1} \)

2. Procedures for Simulating Jet Emission

The curvature effect is a combination of the time delay and the Doppler shifting of the intrinsic spectrum for high latitude emission with respect to the line of sight. Then, the arrival time of photons and Doppler shifting of the intrinsic spectrum should be prescribed. For an expanding spherical thin jet shell, the shell is assumed to be located at a radius \( r \) at time \( t \), where the value of \( r \) is measured with respect to the jet base. In addition, we discuss a spherical thin jet shell radiating from radius \( r_0 \) to \( r \). Thus, the arrival time for photons from an emitter in the jet shell to the observer is

\[
\tau_{\text{obs}} = \left[ \int_{r_0}^r \frac{1 - \beta_{\text{jet}}(l)}{c \beta_{\text{jet}}(l)} \frac{dl}{c} + \frac{r(1 - \cos \theta)}{c} \right](1 + z),
\]

where the emitter is located at \((r, \theta)\), \( c \beta_{\text{jet}}(l) = c dr/dt \) is the velocity of the jet shell at radius \( r = l \), \( c \) is the light velocity, \( \theta \) is the polar angle of the emitter with respect to the line of sight in spherical coordinates (the origin of the coordinates is at the jet base), and \( z \) is the redshift of the explosion producing the jet shell. In Equation (2), \( \tau_{\text{obs}} = 0 \) is set at the observed time for the first photon, which is from the emitter located at \( r = r_0 \) and \( \theta = 0 \). In addition, we define

\[
\tau_{\text{obs}} = (1 + z) \int_{r_0}^r \left[ 1 - \beta_{\text{jet}}(l) \right] \frac{dl}{c \beta_{\text{jet}}(l)}.
\]

which is the observed time for the first photon from the jet shell located at \( r \).

The radiation of an electron is always discussed with relativistic electrons (\( \gamma_e \gg 1 \)) and strong magnetic field. With these two ingredients, photons are produced by a synchrotron process and can be scattered to higher energy by an inverse Compton process. In our work, the shape of the radiation spectrum, rather than the detailed radiation processes, is important. To simplify the problem, the radiation spectrum of an electron with \( \gamma_e \) is assumed to be (e.g., Uhm & Zhang 2015)

\[
P'_0(\nu') = P'_0 H'(\nu'/\nu'_0),
\]

where \( P'_0 \) describes the spectral power in the jet shell comoving frame and \( \nu'_0 \) is the characteristic radiation frequency. The values of \( P'_0 \) and \( \nu'_0 \) may be related to \( \gamma_e \) and thus may evolve with time. It should be noted that the above description of the radiation spectrum follows that of Uhm & Zhang (2015). For the analytical formula of \( H'(x) \), we study the following four cases:

\[
(I): H'(x) = x^{-\beta_0 - k \log(x)},
\]

\[
(II): H'(x) = x^{\alpha' + 1}[1 + g x^{(\alpha' - k)w}]^{-1/w},
\]

\[
(III): H'(x) = \begin{cases} x^{\alpha' + 1} \exp(-x), & x \leq (\alpha' - \beta') \\ (\alpha' - \beta')^{\alpha' - \beta} \exp(\beta' - \alpha')x^{\beta' + 1}, & x \geq (\alpha' - \beta') \end{cases}
\]

\[
(IV): H'(x) = x^2 \{ \exp(x) - 1 \}^{-1},
\]
complication in modeling the jet dynamic for the photosphere emission, however, we only discuss Case (IV) for an extreme fast cooling thin shell in this work (see Section 4.1). It should be noted that the photospheric surface is not spherical (Pe’er 2008; Beloborodov 2011; see Deng & Zhang 2014 and references therein).

In our numerical calculations, the radiation of the jet shell at time \( t \) is modeled with a number of emitters randomly distributed in the jet shell. The number of relativistic electrons \( n'_e \) is the same for different emitters. Thus, the total radiation power from an emitter in the comoving frame is \( n'_e P'_0 H(\nu'/\nu'_0) \), which is the same for different emitters. For a relativistic moving jet with a Lorentz factor \( \Gamma \), the comoving emission frequency \( \nu' \) is boosted to \( \nu = D \nu' \) in the observer’s frame. Here, \( D \) is the Doppler factor described as

\[
D = [\Gamma (1 - \beta \cos \theta)]^{-1}.
\]

During the shell’s expansion for \( \delta t \) (~0), the observed spectral energy \( \delta U \) from an emitter into a solid angle \( \delta \Omega \) in the direction of the observer is given as (Uhm & Zhang 2015)

\[
\delta U_t(t_{\text{obs}}) = (D^2 \delta \Omega) \left[ \frac{\Gamma^2}{4 \pi n'_e P'_0} H \left( \frac{\nu (1 + z)}{D \nu'_0} \right) \right],
\]

where the emission of electrons is assumed to be isotropic in the jet shell comoving frame (c.f. Beniamini & Granot 2016; Geng et al. 2017).

The procedures for obtaining the observed flux is as follows. First, an expanding jet is modeled with a series of jet shells at radius \( r_0 \), \( r_1 = r_0 + \beta \Gamma (r_0)c \delta t, r_2 = r_1 + \beta \Gamma (r_1)c \delta t, \ldots \), \( r_n = r_{n-1} + \beta \Gamma (r_{n-1})c \delta t, \) \ldots appearing at the time \( t = 0, \delta t, 2\delta t, \ldots \), \( \delta t \), with velocity \( \beta \Gamma (r_0), \beta \Gamma (r_1), \beta \Gamma (r_2), \) \ldots, \( \beta \Gamma (r_n), \) \ldots, respectively. During the shell’s expansion over \( \delta t \), the shell moves from \( r_{n-1} \) to \( r_n \) with the same radiation behavior as emitters. Second, we produce \( N \) emitters centered at \( (r_n, \theta, \varphi) \) in spherical coordinates, where the value of \( \cos \theta \) and \( \varphi \) are randomly picked up from the linear space of \( \cos \theta_{\text{jet}}, 1 \) and \( [0, 2\pi] \), respectively. Here, \( \theta_{\text{jet}} \) is the half-opening angle of the jet. The observed spectral energy from an emitter during the shell’s expansion from \( r_{n-1} \) to \( r_n \) is calculated with Equation (7). By discretizing the observer time \( t_{\text{obs}} \) into a series of time intervals, i.e., \( [0, \delta t_{\text{obs}}], [\delta t_{\text{obs}}, 2\delta t_{\text{obs}}], \ldots, \), \( (k - 1)\delta t_{\text{obs}}, k\delta t_{\text{obs}}), \ldots \), we can find the total observed spectral energy

\[
U_t = \sum_{(k-1)\delta t_{\text{obs}}<t_{\text{obs}}<k\delta t_{\text{obs}}} \delta U_t(t_{\text{obs}}) \tag{8}
\]

in the time interval \( [(k - 1)\delta t_{\text{obs}}, k\delta t_{\text{obs}}] \) based on Equations (2) and (7). Here, \( t_{\text{obs}} = \sum_{i=0}^{n} [(1 - \beta \Gamma (r_i)c \delta t) \delta t_{\text{obs}} r_{n-1} = 0 \) are used. Then, the observed flux at the time \( (k - 2)/2 \delta t_{\text{obs}} \) is

\[
F_t = \frac{U_t}{D^2 \delta t_{\text{obs}} \delta \Omega},
\]

where \( D_L \) is the luminosity distance of the jet shell with respect to the observer. In our numerical calculations, the jet shell is assumed to begin radiating at radius \( r_0 = 10^{19} \) cm with a Lorentz factor \( \Gamma (r_0) = \Gamma_0 = 300 \). The values of \( \nu'_0 = 1 \) keV \((1 + z) \), \( N \gg 1 \), \( \theta_{\text{jet}} \gg 1/\Gamma_0 \), \( \delta t \ll t_{c,r} \), and \( \delta t_{\text{obs}} = 0.005t_{c,r} \) are adopted and remained constants in the numerical calculation, where \( t_{c,r} = r (1 + z)/\Gamma^2 c \). The total duration of our produced light curves is set to be \( 50t_{c,r} \). Then, the obtained data would be significantly large. To reduce the file size of our figures, we only plot the data in the time interval with \( k \) satisfying \( (k - 1)\delta t_{\text{obs}} < 1.1^{w} \times 0.01t_{c,r} + t_0 < k\delta t_{\text{obs}} \), where \( m \geq 0 \) is any integer and \( t_0 \) is the observer time set for \( t_{\text{obs}} = 0 \) (see Section 3). The light curves in these figures are consistent with those plotted based on all of the data from our numerical calculations.

### 3. Analytical Formula of the Flux Evolution

In this section, the analytical formula of the flux evolution is obtained by analyzing the radiation from an extreme fast cooling thin shell (EFCS). For this situation, we assume the radiation behavior of the jet shell to be unchanged during the shell’s expansion time \( \delta t \sim 0 \). Then, we have \( r = r_0 + c\beta \Gamma \delta t \sim r_0 \), and Equation (2) can be reduced to

\[
t_{\text{obs}} = (r/c)(1 - \cos \theta)/(1 + z),
\]

which describes the delay time of photons from \( r, \theta \) with respect to those from \( r, \theta = 0 \). It reveals that \( t_{\text{obs}} = 0 \) is the beginning of the phase shaped by the curvature effect for flux from an EFCS. That is to say, \( t_{\text{obs}} \) is the observer time by setting the zero time point \( t_0 \) at the beginning of the steep decay phase. Then, our obtained analytical formula for the flux evolution in the steep decay phase, i.e., Equation (18), describes the flux evolution by setting \( t_0 \) (i.e., \( t_{\text{obs}} = 0 \)) at the beginning of the phase shaped by the shell curvature effect. With \( \Gamma \gg 1, D \) can be reduced to

\[
D \approx \left[ \Gamma - \Gamma \left( \frac{1 - \cos \theta}{1 - 1/2}\right)^{1/2} \right]^{-1},
\]

or

\[
D \approx \frac{2\Gamma}{1 + t_{\text{obs}}/t_{c,r}},
\]

where \( t_{c,r} \) is the characteristic timescale of the shell curvature effect at the radius \( r \),

\[
t_{c,r} = r (1 + z)/(2\Gamma c).
\]

The difference between \( D \) and \( 2\Gamma/(1 + t_{\text{obs}}/t_{c,r}) \) can be neglected for a significantly large value of \( \Gamma \). Then, we would like to use \( D = 2\Gamma/(1 + t_{\text{obs}}/t_{c,r}) \) in our analysis.

For the observed time interval \( \delta t_{\text{obs}} \), the observed total number of emitters is \( N[f(\cos \theta)]/(1 - \cos \theta_{\text{jet}}) \), with \( f(\cos \theta) = c\beta \Gamma \delta t_{\text{obs}}/r (1 + z) \) derived based on Equation (10). Then, the observed flux at the time \( t_{\text{obs}} \) is

\[
F_t = \frac{\delta U_t(t_{\text{obs}})N[f(\cos \theta)]/(1 - \cos \theta_{\text{jet}})}{D^2 \delta t_{\text{obs}} \delta \Omega},
\]

or

\[
F_t = AD^2 H(\nu (1 + z)/D \nu'_0),
\]

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where $A = n_\text{e}^\prime n_\text{e}^\prime N \text{e} c \delta t/[4\pi D_\text{L}^2 \Gamma \tau (1 - \cos \theta_{\text{jett}}) (1 + z)]$ is a constant for an EFCS. The method used to derive Equation (15) is from Uhm & Zhang (2015). As shown in Uhm & Zhang (2015), Equation (15) can also be derived using another method. The reader can refer to the above paper for details. For a constant observed frequency $\nu = Dv^\prime/(1 + z)$, the observed flux $F_\nu$ in Case (I) can be described as

$$F_\nu = F_\nu \left( \frac{\nu}{\nu_0} \right)^{-\beta} = F_\nu \left( \frac{\nu}{\nu_0} \right)^{-\beta_0 - k \log(\nu/\nu_0) - 2k \log(1 + t_{\text{obs}}/t_c)} ,$$

with

$$F_\nu = F_\nu \left( 1 + \frac{t_{\text{obs}}}{t_c} \right)^{-2 - \beta_0 - k \log(1 + t_{\text{obs}}/t_c)} ,$$

where $\nu_0 = D_0^\prime n_\text{e}^\prime/(1 + z)$, $D_0 = 2\Gamma$ is the Doppler factor of the emitter observed at $t_{\text{obs}} = 0$, and $F_\nu$ is the observed flux at time $t_{\text{obs}} = 0$ and frequency $\nu_0$. With Equations (16) and (17), the evolution of the flux from an EFCS can be described as

$$F_\nu = F_{\nu,0} \left( 1 + \frac{t_{\text{obs}}}{t_c} \right)^{-\alpha(t_{\text{obs}})} ,$$

where $F_{\nu,0}$ is the flux observed at $t_{\text{obs}} = 0$ and $t_c$ is the decay timescale for the phase shaped by the shell curvature effect. The main ingredients of Equation (18) are the value of $t_c$ and the temporal decay index $\alpha(t_{\text{obs}})$. For the value of $\alpha$, it is always associated with the spectral index $\beta$. Based on the discussion in Appendix A, we have

$$\alpha(t_{\text{obs}}) = 2 + \frac{1}{\log(1 + t_{\text{obs}}/t_c)} \times \int_0^{\log(1 + t_{\text{obs}}/t_c)} \beta(\tau) d[\log(1 + \tau/t_c)] .$$

It should be noted that for the flux from an EFCS located at $r_0$, $t_{\text{obs}} = 0$ is the beginning of the phase shaped by the curvature effect. Then, Equation (18) with $\alpha$ evolving as Equation (19) describes the flux evolution by setting $t_0$ at the beginning of the steep decay phase.

In practice, one may set $t_0$ at a certain time in the steep decay phase rather than at the beginning of the steep decay phase. We take $t_0^\prime(>0)$ as the time difference of $t_0$ with respect to the beginning of the steep decay phase. By defining $\tilde{t}_{\text{obs}} = t_{\text{obs}} - t_0^\prime$, Equation (12) is reduced to

$$D(\tilde{t}_{\text{obs}}) = \frac{2\Gamma}{1 + t_0^\prime/t_c} \frac{1}{1 + \tilde{t}_{\text{obs}}/t_c} = \frac{D_0}{1 + \tilde{t}_{\text{obs}}/\tilde{t}_c} ,$$

where $D_0 = 2\Gamma/(1 + t_0^\prime/t_c)$ is the Doppler factor of the emitter observed at $t_{\text{obs}} = 0$, and $\tilde{t}_{\text{obs}} = t_c + t_0^\prime$ is adopted. Using the same process to derive Equations (18) and (19), one can have

$$F_\nu(\tilde{t}_{\text{obs}}) = F_{\nu,0} \left( 1 + \frac{\tilde{t}_{\text{obs}}}{\tilde{t}_c} \right)^{-\alpha(\tilde{t}_{\text{obs}})} ,$$

with

$$\alpha(\tilde{t}_{\text{obs}}) = 2 + \int_0^{\log(1 + \tilde{t}_{\text{obs}}/\tilde{t}_c)} \beta(\tau) d[\log(1 + \tau/\tilde{t}_c)] .$$

This reveals that Equation (15) describes the flux evolution in the steep decay phase shaped by an EFCS with Case (I). It should be noted that Equations (23)–(28) is also applicable for $t_0^\prime = 0$ (i.e., $t_0 > 0$).

We test Equations (21) and (23) in Figure 1, which shows the evolution of the flux (upper panels) and spectral indexes (lower panels) for an EFCS with Case (I). Here, $\beta_0 = 0$ and $k = 0.5$ are adopted in the numerical calculations. The red “×” and violet “○” represent the data for the observed photon.
energy \( h\nu = 300 \text{ keV} \) and \( 900 \text{ keV} \), respectively. In addition, \( t_0 = 0 \) s and \( 10t_{cr,0} \) are adopted in the left and right panels, respectively. For comparison, we plot Equations (21) and (23) with red (violet) solid lines in the upper and lower panels for \( h\nu = 300 \text{ keV} \) (900 keV), respectively. Here, the values of \( \tilde{E}_{\nu,0} \) and \( \beta_0 \) are estimated based on the data at \( \tilde{t}_{\text{obs}} = 0 \), and \( \tilde{t}_c = t_{cr,0} \) (11\( t_{cr,0} \)) is adopted in the left (right) panels. From this figure, one can find that Equation (23) describes well the spectral evolution for the radiation from an EFCS with Case (I). Moreover, Equation (21) can describe the flux evolution in the steep decay phase. It should be noted that the value of \( \nu_f \) is almost constant for \( \tilde{t}_{\text{obs}}/\tilde{t}_c \ll 1 \). This behavior can be found in our figures and could not be read as \( \alpha = 0 \).

In reality, the intrinsic radiation spectrum may be similar to that of Case (II) or (III). Then, we study the radiation behavior of an EFCS with Case (II) or (III). The evolution of the flux and \( \beta \) are shown in Figure 2, where \( \alpha' = -1 \) and \( \beta' = -2.3 \) are adopted. In this figure, the black “+,” red “×,” and violet “○” represent the data for \( h\nu = 100 \text{ keV} \), 300 keV, and 900 keV, respectively. In this figure and afterwards, the light curves for \( h\nu = 300 \text{ keV} \) (900 keV) are shifted by dividing by 1.5 (3) in the plot for clarity. The upper part of this figure shows the data with \( t_0 = 0 \), the lower part shows the data with \( t_0 = t_{cr,0} \), the left panels show the data from the EFCS with Case (II), and the right panels show the data from the EFCS with Case (III). For comparison, we show Equation (21) with solid lines in each panels, where the black, red, and violet solid lines are for the observed photon energies \( h\nu = 100 \text{ keV} \), 300 keV, and 900 keV, respectively. In addition, \( \tilde{t}_c = t_{cr,0} \) and \( 2t_{cr,0} \) are adopted for the upper and lower parts, respectively. It can be found that the solid lines are well consistent with the numerical calculations’ data. We also study the radiation of an EFCS with Case (IV). The results are shown in Figure 3, the symbols and lines of which have the same meaning as those in Figure 2. The flux is plotted with \( t_0 = 0 \) s and \( t_{cr,0} \) in the left and right panels, respectively. Equation (21) is shown with solid lines, where \( \tilde{t}_c = t_{cr,0} \) and \( 2t_{cr,0} \) are adopted in the left and right panels, respectively. It can be seen that the solid lines are well consistent with the data from the numerical calculations.

Thus, we can conclude that for the radiation of an EFCS, Equation (21) can present a good estimate of the flux evolution in the phase shaped by the shell curvature effect.

4.2. Real Situation for a Thin Shell

In this subsection, Equation (21) is tested with a thin shell radiating from radius \( r_0 \) to \( r_s = 2r_0 \). Since the Lorentz factor of the jet shell may be related to \( r \) (such as Uhm & Zhang 2015, 2016), we assume

\[
\Gamma = \Gamma_0 \left( \frac{r}{r_0} \right)^\gamma, \tag{29}
\]

where \( s > 0 \) (< 0) represents an accelerating (decelerating) jet. In addition, it is assumed that \( n'_{c,0} \) increases with time \( t' \) in the comoving frame of the jet shell, i.e., \( n'_{c,0} = n'_{c,0}t' \), and \( t' = 0 \) is set at the radius \( r_0 \). Since the light curves are normalized by the peak flux in our focus phase, the exact value of the constant \( n'_{c,0} \) does not matter in our work. Based on Equations (3) and (29), the observed time \( \tilde{t}_{\text{obs,stop}} = \tilde{t}_{\text{obs,stop}} \) of the jet shell stopping radiation are \( \tilde{t}_{\text{obs,stop}} = 2.33t_{cr,0}, t_{cr,0} \), and \( 0.5t_{cr,0} \) for \( s = -1, 0, \) and 1, respectively.

In Figure 4, we show the flux evolution for Case (I) with \( k = 0 \) and \( \tilde{\beta}_0 = 1.3 \), where \( s = -1, 0, \) and 1 are adopted in the left, middle, and right panels, respectively. The gray “+,” blue “×,” and red “○” represent the data by setting \( t_0 = 0 \), \( t_0 = 0 \), and \( t_0 = 5t_{cr,0} \), respectively. Here, \( t_p \) is the peak time of the observed flux and \( t_p = 2.32t_{cr,0}, 1.01t_{cr,0}, \) and \( 0.50t_{cr,0} \).
are found for $s = -1, 0, \text{ and } 1$, respectively. Comparing $t_p$ with $t_{\text{obs,stop}}$, $t_p$ in our light curves is the observed time of the jet shell stopping radiation. Thus, the phase with $t_{\text{obs}} \geq t_p$ is dominated by the shell curvature effect. For the phase with $t_{\text{obs}} \geq t_p$, the spectral index $\beta(t_{\text{obs}}) = 1.3$ is found. Then, we fit the flux plotted by the blue “+” with Equation (21) and obtain $\alpha = 3.3$, which is shown by the blue solid lines in this figure. The value of $t_c = 6.41 t_{c,r}^0$, $1.96 t_{c,r}^0$, and $0.69 t_{c,r}^0$ are reported from our fittings for $s = -1, 0, \text{ and } 1$, respectively. By comparing the solid lines with the data, it is found that Equation (21) can present a better estimate of the flux evolution. We also fit the flux plotted by the red “×” with Equation (21) and obtain $\alpha = 3.3$, which is shown by the red lines in this figure. The values of $t_c = 9.26 t_{c,r}^0 \approx (5t_{c,r0} - t_p) + 6.41 t_{c,r0}$, $6.13 t_{c,r0} \approx (5t_{c,r0} - t_p) + 1.96 t_{c,r0}$, and $5.39 t_{c,r0} \approx (5t_{c,r0} - t_p) + 0.69 t_{c,r0}$ are reported from our fittings for $s = -1, 0, \text{ and } 1$, respectively.

Figure 2. Evolutions of the flux (upper panels) and spectral indexes (lower panels) of an EFCS with Case (II) (left panels) or (III) (right panels). The black “+,” red “×,” and violet “◦” represent the data for the observed photon energy $h\nu = 100$ keV, 300 keV, and 900 keV, respectively. The black, red, and violet lines represent the flux following Equation (21) for $h\nu = 100$ keV, 300 keV, and 900 keV respectively, where $t_c = t_{c,r}^0$ and $t_c = 2t_{c,r}^0$ are adopted in the upper panels and lower panels, respectively. The light curves for $h\nu = 300$ keV (900 keV) are shifted by dividing 1.5 (3) in the plot for clarity.
According to Equation (21), the decay timescale $\tilde{t}_c$ in this situation (i.e., $t_0 = 5t_{c, r_0}$) would be larger than that found in the situation with $t_0 = t_p$ by $t^0_0 = 5t_{c, r_0} - t_p$. The reported value of $\tilde{t}_c$ in the situation with $t_0 = 5t_{c, r_0}$ is consistent with Equation (21). Then, Equation (21) can be used to describe the flux evolution in the steep decay phase for a radiating thin shell with Case (I).

Figures 5 and 6 show the flux and spectral evolution for Cases (II) and (III) with $\alpha' = -1$ and $\beta' = -2.3$, respectively. The upper, middle, and lower parts show the light curves with $t_0 = 0$, $t_p$, and $t_p + 2t_{c, r_0}$, respectively, where the values of $t_p = 2.35t_{c, r_0}$, $1.01t_{c, r_0}$, and $0.50t_{c, r_0}$ are found in our numerical calculations with $s = -1$, $s = 0$, and $s = 1$, respectively. The meaning of the symbols and solid lines in Figures 5 and 6 are the same as those in Figure 2. Here, the value of $\tilde{t}_c = 6.41t_{c, r_0}$ (8.41$t_{c, r_0}$), 1.96$t_{c, r_0}$ (3.96$t_{c, r_0}$), and 0.69$t_{c, r_0}$ (2.69$t_{c, r_0}$), which are found in the numerical calculations with Case (I), are used for $s = -1$, 0, and 1 with $t_0 = t_p, (t_0 = t_p + 2t_{c, r_0})$, respectively. From Figures 5 and 6, one can find that Equation (21) can present a better estimate of the flux evolution in the steep decay phase. In Figures 7 and 8, we study the applicability of Equation (21) for Case (II) and (III) with $t_0 = t_p$ and different $(\alpha', \beta')$, i.e., $(\alpha' = -0.7, \beta' = -2)$, $(0.7, -2.6)$, $(-1, -2)$, and $(-1, -2.6)$. The meaning of the symbols and lines are the same as those in Figure 2, and the values of $\tilde{t}_c = 6.41t_{c, r_0}$, 1.96$t_{c, r_0}$, and 0.69$t_{c, r_0}$ are used to plot the solid lines following Equation (21) for $s = -1$, 0, and 1, respectively. It is found that the solid lines present a better estimate of the flux evolution in the steep decay phase.

**Figure 3.** Evolutions of the flux (upper panels) and spectral indexes (lower panels) of an EFCS with Case (IV). The meanings of the symbols and solid lines are the same as those in Figure 2, where the value of $t_0 = 0$, $\tilde{t}_c = t_{c, r_0}$, and $t_0 = t_{c, r_0}$ are adopted in the left and right panels, respectively.

**Figure 4.** Flux evolution of an expanding thin jet shell with Case (I) and $k = 0$. The gray “•,” blue “+,” and red “×” represent the data by setting $t_0 = 0$, $t_p$ and $5t_{c, r_0}$, respectively. The blue and red lines are the best-fitting results for the flux evolution with Equation (21) and $\alpha = 3.3$. The meanings of the symbols and solid lines are the same as those in Figure 2.
Thus, we can conclude that Equation (21) is applicable to describe the flux evolution in the phase shaped by the shell curvature effect.

5. Conclusions

For the radiation from a relativistic expanding spherical shell, the curvature effect may play an important role in shaping the flux.
evolution in the steep decay phase. In this work, we study in detail the steep decay phase shaped by the shell curvature effect. We move the zero time point $t_0$ to a certain time in the steep decay phase and derive an analytical formula to describe the flux evolution in the steep decay phase. Our obtained analytical formula is given as $F_\nu \propto (1 + \tilde{t}_{\text{obs}}/\tilde{t})^{-\alpha}$, with $\alpha (\tilde{t}_{\text{obs}}) = 2 + \int_0^{\log (1 + \tilde{t}_{\text{obs}}/\tilde{t})} \beta (\tau) d[\log (1 + \tau/\tilde{t})]/\log (1 + \tilde{t}_{\text{obs}}/\tilde{t})$, where

Figure 6. Flux and spectral evolution of a real situation for Case (III). The meanings of the symbols and solid lines are the same as those in Figure 5.
$F_\nu$ is the observed flux at a constant observed frequency $\nu$, $\bar{t}_{\text{obs}}$ is the observer time by setting $t_0$ at a certain time in the steep decay phase, $\beta$ is the spectral index estimated around $\nu$, and $\bar{t}_c$ is the decay timescale of the phase with $\bar{t}_{\text{obs}} \geq 0$. We test our analytical formula with numerical calculations. It is found that our analytical formula can present a good estimate of the flux evolution in the

Figure 7. Flux and spectral evolution of a real situation for Case (II) with different values of $\alpha'$ and $\beta'$. The meanings of the symbols and solid lines are the same as those in Figure 5.
phase shaped by the curvature effect. Our analytical formula can be used to test the curvature effect with observations and estimate the decay timescale $\bar{t}_c$ of the steep decay phase.

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Appendix A
Relation of $\alpha$ and $\beta$

The value of $\alpha$ is always associated with the spectral index $\beta$. This can be found in Equation (17) with $k = 0$. For this situation, we can find $\beta = \beta_0^s$ and thus $\alpha = 2 + \beta$. Then, Equation (18) with $t_{\text{obs}} \geq t_c$ can be reduced to $F \propto t_{\text{obs}}^{-2-\beta}$, which has been extensively discussed in previous works (e.g., Fenimore et al. 1996; Kumar & Panaitescu 2000; Dermer 2004; Dyks et al. 2005). However, what would be the relation between $\alpha$ and $\beta$ if $\beta$ varies with the observer time $t_{\text{obs}}$? In our work, the spectral index $\beta$ around $\nu$ is estimated using

$$\beta = - \frac{\log(F_{\nu_{\text{obs}}} - \log(F_{\nu_{1.1}})}{\log(1.1\nu) - \log(\nu/1.1)}.$$  (30)

It should be noted that $\beta$ is different from the spectral index $\beta_{0.3-10\text{keV}}$, which is estimated based on the 0.3–10 keV observations of the X-ray telescope on board the Swift mission. Please see Lin et al. (2017) for the discussion about the $\beta_{0.3-10\text{keV}}$ evolution in the steep decay phase. The value of $\beta_{0.3-10\text{keV}}$ may approximate to the value of $\beta$ estimated around $\nu = 1.7$ keV/$h$, where $h$ is Planck’s constant. Equation (30) reveals that the value of $-\beta$ is the spectral slope of the intrinsic spectrum (in the space of $\log \nu' - \log H_{\nu'}$) around the frequency $\nu' = \nu/D$. Then, we have

$$\log H_{\nu'/D_0} - \log H_{\nu'/D} = \int_{\log(\nu/D_0)}^{\log(\nu/D)} \beta d[\log(\nu/D)].$$  (31)

With Equations (15), (12), and (18) and $t_c = t_{c,r}$, we have

$$\alpha (t_{\text{obs}}) = 2 + \frac{1}{\log(1 + t_{\text{obs}}/t_c)} \times \int_{t_c}^{t_{\text{obs}}} \beta d[\log(1 + \nu/t_0)].$$  (32)

This is the relationship of $\alpha$ and $\beta$. For $\beta(t_{\text{obs}}) = constant$, the relation $\alpha = 2 + \beta$ can be found from Equation (32).

Appendix B
Discussion About the Value of $\bar{t}_c$

In this section, we discuss the value of $\bar{t}_c$ under the situation where $t_0$ is set at a certain time in the steep decay phase. Based on the analysis in Section 3, $\bar{t}_c = t_{c,r} + t_0^0$ can be found for an EFCS located at $r$. However, the value of $\bar{t}_c$ for a jet shell radiating from $r_0$ to $r$ depends on the behavior of the jet’s dynamics and radiation. In this section, we discuss the situation in Section 4.2 with Case (I) and $k = 0$.

As shown in Section 2, an expanding jet in our numerical calculations is modeled with a series of jet shells at radius $r_0$, $r_1 = r_0 + \beta_0 c dt$, $r_2 = r_1 + \beta_0 c dt$, $r_3 = r_2 + \beta_0 c dt$, $\cdots$, $r_n = r_{n-1} + \beta_0 c dt$, $\cdots$, appearing at the time $t = 0$, $dt$, $2dt$, $\cdots$, $n dt$, $\cdots$ with velocity $\beta_0 (r_0)$, $\beta_0 (r_1)$, $\beta_0 (r_2)$, $\cdots$, $\beta_0 (r_n)$, $\cdots$, respectively. During the shell’s expansion over $dt$, the shell moves from $r_{n-1}$ to $r_n$ with the same radiation behavior as emitters. That is to say, the radiation of our jet can be regarded as the radiation from a series of EFCSs appearing with different observer times $t_{\text{obs}}$, $t_{c,r_0}$, and $t_{c,r_n}$. Then, the observed flux $F_{\nu}(t_{\text{obs}})$ can be described by

$$F_{\nu}(t_{\text{obs}}) = \sum_{\bar{t}_c} F_{\nu}(\nu, t_{\text{obs}}).$$  (33)
Figure 8. Flux and spectral evolution of a real situation for Case (III) with different values of $\alpha'$ and $\beta'$. The meanings of the symbols and solid lines are the same as those in Figure 5.
Different values of which describes the flux evolution in the steep decay phase. Different values of \( s \) may form different values of \( \bar{t}_{\text{obs}} \).

For \( s = 0 \), the value of \( t_{c,s} - 2s \bar{t}_{\text{obs},s} + t_0 \) is the same for different \( F_{v,s} \). That is to say, for a different \( r_n \), the dependence of \( F_{v,s}(\bar{t}_{\text{obs}}) \) on \( \bar{t}_{\text{obs}} \) is the same except for the difference in \( F_{v,s,0} \). Then, one can find \( \bar{t}_c = t_{c,s} + t_0 \), which is consistent with those found by fitting the flux data in the middle panel of Figure 4. For \( s = 0 \), the value of \( t_{c,s} - 2s \bar{t}_{\text{obs},s} + t_0 \) depends on the value of \( \bar{t}_{\text{obs},s} \). For \( s = -1 \), the decay timescale of \( F_{v,s} \) increases with \( \bar{t}_{\text{obs},s} \). The maximum and minimum decay timescales of \( F_{v,s} \) are \( t_{c,s} + t_0 - t_p \) and \( t_{c,s} + t_0 \), respectively. Here, \( \bar{t}_{\text{obs},s} = t_p \) is found in our numerical calculations. Based on Equation (37), one can find the relation 3.32\( t_{c,s} < \bar{t}_c < 8 t_{c,s} \) if \( t_0 = t_p \) is set, where \( t_{c,s} = 8 t_{c,s} \) is estimated based on Equation (29) and \( s = -1 \). The reported result by fitting the data in the left panel of Figure 4, i.e., \( \bar{t}_c = 6.41 t_{c,s} \) for \( t_0 = t_p \), is consistent with the above analysis.

For \( s = 1 \), the flux decay timescale would decrease with increasing \( \bar{t}_{\text{obs},s} \). Then, the maximum and minimum decay timescales of \( F_{v,s} \) are \( t_{c,s} + t_0 \) and \( t_{c,s} + t_0 - t_p \), respectively. According to Equation (37), one should find the relation of 0.5\( t_{c,s} < \bar{t}_c < 1.50 t_{c,s} \), if \( t_0 = t_p \) is set, where \( t_{c,s} = 0.5 t_{c,s} \) is estimated based on Equation (29) and \( s = 1 \). The fitting result from the right panel of Figure 4, i.e., \( \bar{t}_c = 0.69 t_{c,s} \) for \( t_0 = t_p \), also confirms our analysis. Since \( \bar{t}_c = 6.41 t_{c,s} \) and \( \bar{t}_c = 0.69 t_{c,s} \) for \( s = -1 \) and \( s = 1 \), this shows the steep decay phase in our numerical calculations is dominated by the emission from the EFCS located at \( \sim r_c \). Then, we use the decay timescale found in Case (I) with \( k = 0 \), i.e., 6.41\( t_{c,s} \) for \( \bar{t}_c = 0.69 t_{c,s} \), and 0.69\( t_{c,s} \), to discuss the situations with Cases (II) or (III).

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