Configuration Space Integrals and Invariants for 3-Manifolds and Knots

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1. Introduction

In this paper we give a brief description of the way proposed in [4] of associating invariants of both 3-dimensional rational homology spheres (r.h.s.) and knots in r.h.s.’s to certain combinations of trivalent diagrams. In addition, we discuss the relation between this construction and Kontsevich’s proposal [8].

The same diagrams appear in the LMO invariant [9] for 3-manifolds, and it would be very interesting to know if there exists any relationship between the two approaches in the case of r.h.s.’s.

The reason for restricting here to r.h.s.’s is quite technical, as will be clear in Sect. 4. Until that point, without any loss of generality we can assume M to be any connected, compact, closed, oriented 3-manifold.

Our construction yields the invariants in terms of integrals over a suitable compactification of the configuration space of points on M. More precisely, the number of points corresponds to the number of vertices in the trivalent diagram, and the integrand is obtained by associating to each edge in the diagram a certain 2-form that represents the integral kernel of an “inverse of the exterior derivative \(d\).”

One reason for constructing invariants in terms of “\(d^{-1}\)” comes from perturbative Witten–Chern–Simons theory [11]. (More precisely, one should invert the covariant derivative with respect to a flat connection; so the present construction is related to the trivial-connection contribution.)

Another reason, which is perhaps more transparent to topologists, relies on the definition of the linking number of two curves in \(\mathbb{R}^3\) as...
the intersection number of one curve with any surface cobounding the
other. Thus, linking number may be defined in terms of an inverse of
the boundary operator \( \partial \). If one wants to represent the linking number
with an integral formula (viz., Gauss’s formula), then one must consider
the Poincaré duals of curves and apply “\(d^{-1} \)”.

The exterior derivative \( d \) is neither injective nor surjective. Thus,
to invert it, one must restrict it to the complement of its kernel and
invert it on its image. Notice that one needs an explicit choice of the
complement of the kernel, and this introduces an element of arbitrar-
iness in the construction. Actually, our main task will be to prove that
the invariants we define are really independent of this arbitrary choice.

A general way of defining the inverse of \( d \) is by introducing a parametrix, i.e., a
linear operator on \( \Omega^*(M) \) that decreases by one the form
degree and satisfies the following equation:

\[
d \circ P + P \circ d = I - S,
\]

where \( I \) is the identity operator and \( S \) is a suitable smooth operator
such that (1) has a solution.

The definition of the parametrix is far from unique. For the choice
of \( S \) is to a large extent arbitrary and, moreover, given a solution \( P \),
we get another solution of the form \( P + d \circ Q - Q \circ d \) for any linear
operator \( Q \) that decreases by two the form degree. These ambiguities
reflect the ambiguities in defining an inverse of \( d \).

**Remark 1.1.** One possible choice for the parametrix is the Rie-
mannian parametrix \( P_g \), which is based on the choice of a Riemannian
metric \( g \):

\[
P_g = d^* \circ (\Delta + \pi_{\text{Harm}})^{-1}, \quad \alpha \in \Omega^*(M).
\]

where \( * \) is the Hodge-* operator, \( \Delta = d^*d + dd^* \) is the Laplace operator,
and \( \pi_{\text{Harm}} \) is the projection to harmonic forms. \( P_g \) satisfies (1) with
\( S = \pi_{\text{Harm}} \).

The main point is now to define an integral kernel (of course not
unique) for \( P \): i.e., we want to represent \( P \) as a convolution. In the
language of differential topology, convolution can be written as

\[
P \alpha = -\pi_2(\hat{\eta} \, \pi_1^* \alpha), \quad \alpha \in \Omega^*(M),
\]

where \( \pi_1 \) and \( \pi_2 \) are the projections from \( M \times M \) to either copy of \( M \),
and \( \hat{\eta} \) is the integral kernel for \( P \) (wedges will be understood through-
out). By dimensional reasons it is clear that \( \hat{\eta} \) must be a 2-form on
\( M \times M \). Because of the identity operator in (1), it is also clear that
\( \hat{\eta} \) cannot be a smooth form. This is clearly a problem since we want
to use \( \hat{\eta} \) to define smooth invariants for the manifold \( M \). The solution
consists in replacing $M \times M$ with a suitable compactification $C_2(M)$ of the configuration space.

The ambiguities in the definition of the parametrix imply that the form $\hat{\eta}$ is not unique. Thus, the main point will be to prove that the invariants are independent of the choices involved in the construction of $\hat{\eta}$. The plan of the construction is then the following:

1. One constructs a form $\hat{\eta}$ to represent the integral kernel of a parametrix with a prescription on its behavior on the boundary of $C_2(M)$.
2. One introduces the unit interval $I$ as a space of parameters to take care of the arbitrary choices involved in this construction.
3. One associates to each trivalent diagram with $n$ vertices a function on $I$ by integrating pullbacks of the form $\hat{\eta}$ over a suitable compactification $C_n(M)$ (to be defined in Sect. 3) of the configuration space of $n$ points on $M$. An invariant will then be a constant function on $I$.
4. If the integrand form is closed, then the differential of the corresponding function on $I$ is determined only by contributions on the boundary of $C_n(M)$. One shows then that if $M$ is a r.h.s., it is possible to associate a closed non-trivial integrand to each trivalent diagram.
5. Because of the prescribed behavior of $\hat{\eta}$ on the boundary, one can cancel the boundary contributions by summing up appropriate combinations of diagrams.

We discuss points 1. and 2. in Sect. 3 and points 3., 4. and 5. in Sect. 4.

More concisely, the final statement is that to certain combinations (cocycles) of trivalent diagrams it is possible to associate a well-defined element of $H^{3n}(C_n(M), \partial C_n(M))$, where $M$ is a r.h.s. The invariant for $M$ is then obtained by comparing this element with the unit generator.

As we will see, the form $\hat{\eta}$ is not closed, and this accounts for the complications in point 4. This is due to the fact that the cohomology of $M$ is not trivial. There are however cases when one can obtain a closed form; viz.:

1. One can introduce a flat bundle $E$ over $M$ and consider the relevant covariant derivative instead of the exterior derivative. If the bundle is non trivial, it may happen that the complex $H^*(M; E)$ is acyclic. In this case, one can construct a covariantly closed form to represent the inverse of the covariant derivative.
2. If $M$ is a r.h.s., one can remove one point, thus obtaining a rational homology disc and, consequently, a closed form $\hat{\eta}$. 
Case 1. was studied by Axelrod and Singer \([3]\) in the Riemannian framework of eq. \([2]\) (with \(\Delta\) the covariant Laplace operator and, by hypothesis, \(\pi_{\text{Harm}} = 0\)). For a more general treatment of this case, s. \([3]\). Notice, however, that this approach does not apply in general: e.g., it does not even work for \(S^3\) whose only flat connection is trivial.

Case 2. was proposed by Kontsevich \([8]\). A realization of this proposal for the simplest invariant—known as the \(\Theta\)-invariant—was then studied by Taubes \([10]\). He also proved his version of the \(\Theta\)-invariant to be trivial on integral homology spheres. This rules out any relationship with the LMO invariants which predict the \(\Theta\)-invariant on integral homology spheres to be the Casson invariant.

In Sect. 5 we will apply Kontsevich’s proposal to the invariants constructed in \([4]\) (and Sect. 4), and will compare our result with Taubes’s.

**Remark 1.2.** The above construction can be used to define invariants for knots in a r.h.s. as well, s. \([4]\) (and Sect. 4).

Also in this case the construction is simplified if one gets a closed form \(\hat{\eta}\). This happens, e.g., when \(M = \mathbb{R}^3\) \([6,1]\), or when \(M = \Sigma \times [0,1]\) with \(\Sigma\) a connected, compact, closed, oriented 2-manifold \([2]\).

Another possibility is an approach à la Kontsevich when \(M\) is a r.h.s., s. Sect. 5.

**2. A compactification of configuration spaces**

In this section we assume \(M\) to be a \(d\)-dimensional connected, compact, closed, oriented manifold.

The (open) configuration space \(C_0^n(M)\) of \(n\) points in \(M\) is obtained by removing all diagonals from the Cartesian product \(M^n\).

A \(C^\infty\)-compactification for these spaces was proposed in \([3]\) (generalizing the algebraic compactification of \([7]\)). This compactification is obtained by taking the closure

\[
C_n(M) \equiv \bar{C_0^n(M)} \subset M^n \times \prod_{S \subseteq \{1,2,...,n\}, \mid S \mid \geq 2} BL(M^S, \Delta_S),
\]

where \(BL(M^S, \Delta_S)\) denotes the differential-geometric blowup obtained by replacing the principal diagonal \(\Delta_S\) in \(M^S\) with its unit normal bundle \(N(\Delta_S)/\mathbb{R}^+\). Notice that \(\Delta_S\) is diffeomorphic to \(M\) and that

\[
N(\Delta_S) \simeq TM^{\oplus S}/\text{global translations}.
\]

Thus, the boundary of \(BL(M^S, \Delta_S)\) is a bundle over \(M\) associated to the tangent bundle.

Because of all the blowups, the spaces \(C_n(M)\) turn out to be manifolds with corners (\(C_2(M)\) is simply a manifold with boundary). The
codimension-one components of the boundary of \( C_n(M) \) are labeled by subsets of \( \{1, \ldots, n\} \). By permuting the factors, we can always put ourselves in the case when this subset is \( \{1, \ldots, k\} \), \( 2 \leq k \leq n \). We will denote by \( S_{n,k} \subset \partial C_n(M) \) a face of this kind. Then we have the following functorial description of \( S_{n,k} \):

\[
S_{n,k} \cong (\pi_1)^{-1} \hat{C}_k(TM) \longrightarrow \hat{C}_k(TM)
\]

(3)

\[
\begin{array}{ccc}
C_{n-k+1}(M) & \longrightarrow & M \\
\downarrow & & \downarrow \\
\pi_1
\end{array}
\]

Here \( \pi_1 \) is the projection onto the first copy of \( M \) (i.e., where the first \( k \) points have collapsed) and \( \hat{C}_k(TM) \) is a bundle associated to the tangent bundle of \( M \) whose fiber \( \hat{C}_k(\mathbb{R}^d) \), \( d = \dim M \), is obtained from \( (\mathbb{R}^d)^k/G \)—\( G \) being the group of global translations and scalings—by blowing up all diagonals. More precisely,

\[
\hat{C}_k(\mathbb{R}^d) = \overline{C_k^G(\mathbb{R}^d)/G} \subset \left( (\mathbb{R}^d)^k \times \prod_{S \subseteq \{1,2,\ldots,k\} \atop |S| \geq 2} \text{Bl}( (\mathbb{R}^d)^S, \Delta_S) \right) / G.
\]

Notice that each \( \hat{C}_k(\mathbb{R}^d) \) is a compact manifold with corners. In the simplest case we have \( \hat{C}_2(\mathbb{R}^d) = S^{d-1} \).

The diagonal action of \( SO(d) \) on \( (\mathbb{R}^d)^k \) descends to \( \hat{C}_k(\mathbb{R}^d) \). If we choose a Riemannian metric on \( M \), we can write

\[
\hat{C}_k(TM) = OM \times_{SO(d)} \hat{C}_k(\mathbb{R}^d),
\]

where \( OM \) is the orthonormal frame bundle of \( TM \). In particular, when \( n = 2 \), we have

\[
C_2(M) = \text{Bl}(M \times M, \Delta),
\]

and \( \partial C_2(M) \simeq S(TM) = OM \times_{SO(d)} S^{d-1} \).

3. The construction of a parametrix

In this section we assume that \( M \) is any connected, compact, closed, oriented 3-manifold.

We start considering the following commutative diagram:

\[
\begin{array}{ccc}
\partial C_2(M) & \xrightarrow{\iota} & C_2(M) \\
\pi & & \pi \\
\Delta & \xrightarrow{\iota \Delta} & M \times M
\end{array}
\]
Then we define the involution $T$ that exchanges the factors in $M \times M$. By abuse of notation, we will denote by $T$ also the corresponding involution on $C_2(M)$ and on its boundary $\partial C_2(M)$. On the latter $T$ acts as the antipodal map on the fiber crossed with the identity on the base. We will denote by $H^*_\pm$ the $+$ and $-$ eigenspaces of $T$ in the cohomology of any of the above spaces.

We will denote by $\chi_\Delta \in \Omega^3(M \times M)$ a representative of the Poincaré dual of the diagonal $\Delta$. Since $[\chi_\Delta] \in H^2_+(M \times M)$, there is really no loss of generality in choosing an odd representative.

On the sphere bundle $\partial C_2(M) \to \Delta$, one can introduce a global angular form $\eta$; i.e., a form $\eta \in \Omega^2(\partial C_2(M))$ with the following properties:

1. the restriction of $\eta$ to each fiber is a generator of the cohomology of the fiber;
2. $d\eta = -\pi^{\partial_0} e$, where $e$ is a representative of the Euler class.

Since $M$ is 3-dimensional, the Euler class is trivial. Moreover, since $H^2_+(S^2) = 0$, we may choose the global angular form to be odd. Since $T$ acts as the identity on the base, 2. is then replaced by $d\eta = 0$. We have the following

**Proposition 3.1.** Given an odd global angular form $\eta$ and an odd representative $\chi_\Delta$ of the Poincaré dual of the diagonal $\Delta$ in $M \times M$, there exists a form $\hat{\eta} \in \Omega^2(C_2(M))$ with the following properties:

\[
\begin{align*}
(4a) & \quad d\hat{\eta} = \pi^* \chi_\Delta, \\
(4b) & \quad \iota_{\partial_0} \hat{\eta} = -\eta, \\
(4c) & \quad T^* \hat{\eta} = -\hat{\eta}.
\end{align*}
\]

This is a simple generalization of the analogous proposition in [4] for the case when $M$ is a r.h.s.

**Proof.** Let $U$ be a tubular neighborhood of $\Delta$ in $M \times M$, and $\tilde{U} = \pi^{-1}U$ its preimage in $C_2(M)$. Then $\tilde{U}$ has the structure of $\partial C_2(M) \times [0, 1]$. Let us denote by $\partial_0 \tilde{U} = \partial C_2(M)$ and by $\partial_1 \tilde{U} = \pi^{-1}\partial U$ the two boundary components of $\tilde{U}$.

Let $\rho$ be a function on $\tilde{U}$ which is constant and equal to $-1$ in a neighborhood of $\partial_0 \tilde{U}$ and is constant and equal to $0$ in a neighborhood of $\partial_1 \tilde{U}$. Moreover, assume that $\rho$ is even under the action of $T$.

Let $p : \tilde{U} \to \partial C_2(M)$ be the natural projection. Then consider the form $\hat{\eta} = \rho p^* \eta$. Since $\eta$ is a global angular form, $d\hat{\eta} = d\rho p^* \eta$ is a representative of the Thom class of the normal bundle of $\Delta$. Therefore,
if we extend \( \tilde{\eta} \) by zero on the whole of \( C_2(M) \), we have that \( d\tilde{\eta} \) is the pullback of a representative of the Poincaré dual of the diagonal.

This might not be our choice \( \chi_\Delta \). Anyhow, \( d\tilde{\eta} = \pi^*(\chi_\Delta + d\alpha) \), and it is not difficult to check that one can choose \( \alpha \in \Omega^2(M \times M) \). So we set \( \hat{\eta} = \tilde{\eta} - \pi^*\alpha \), and it is an immediate check that properties (4) hold.

Notice that, as is clear from the proof, the definition of \( \hat{\eta} \) is not unique, even for fixed \( \eta \) and \( \chi_\Delta \).

With such a form \( \hat{\eta} \) we can finally define a parametrix. In fact, let us denote by \( \rho_1 \) and \( \rho_2 \) the projections from \( M \times M \) to each factor, and by \( \pi_1 \) and \( \pi_2 \) the corresponding projections from \( C_2(M) \). By defining the push-forward as acting from the left, we have the following

**Proposition 3.2.** If \( \hat{\eta} \in \Omega^2(M) \) satisfies (4a) and (4b), then

\[
P\alpha = -\pi_2^*(\hat{\eta} \pi_1^*\alpha), \quad \alpha \in \Omega^*(M),
\]

is a parametrix with \( S = -\rho_2^*(\chi_\Delta \rho_1^*\alpha) \).

The proof is a simple exercise of fiber integration in the case when the fiber has a boundary (s. [4]).

**Remark 3.3.** Property (4c) is not needed in the definition of the parametrix, but is natural and simplifies the writing of the invariants. Notice incidentally that the propagator in perturbative Witten–Chern–Simons theory is odd under the action of \( T \), being related to the expectation value of a connection 1-form placed at two different points.

### 3.1. The global angular form.

To end with the construction, we have to specify a choice for the global angular form.

Following [3] and [4], we pick up a Riemannian metric \( g \). Then \( \partial C_2(M) \simeq OM \times_{SO(3)} S^2 \). Let

\[
p : OM \times S^2 \to OM \times_{SO(3)} S^2
\]

be the natural projection, and let \( \theta \) be a connection form on \( OM \) (i.e., a metric connection). By abuse of notation, we will write \( \theta \) also for its pullback to \( OM \times S^2 \).

**Definition 3.4.** We call a global angular form \( \bar{\eta} \in \Omega^2(OM \times S^2) \) equivariant if it is a polynomial in \( \theta \) and \( d\theta \)—with coefficients in \( \Omega^*(S^2) \)—and it is basic (i.e., \( \bar{\eta} = p^*\eta \)).

For a given \( \theta \), the equivariant global angular form is unique and its \( \theta \)-independent part is the \( SO(3) \)-invariant unit volume form on \( S^2 \), which we will denote by \( \omega \) throughout. See [4] for an explicit construction.
We assume that the restriction of \( \hat{\eta} \) to the boundary is such that its pullback to \( OM \times S^2 \) is the equivariant global angular form.

We denote by \( \omega_{ij} \) the pullbacks of \( \omega \) by the projections \( \hat{C}_n(\mathbb{R}^3) \rightarrow \hat{C}_2(\mathbb{R}^3) = S^2 \). Then a useful property proved in [8] (s. also [9]) is expressed by the following

**Lemma 3.6 (Kontsevich).** If \( x_i, 1 \leq i \leq n \), is any coordinate in \( \hat{C}_n(\mathbb{R}^3) \), then, for any indices \( j \) and \( k \) (\( j \neq i, k \neq i \))

\[
\int_{x_i} \omega_{ij} \omega_{ik} = 0.
\]

### 3.2. Parametrizing the choices.

We have constructed an integral kernel \( \hat{\eta} \) for a parametrix depending on the following choices: a metric \( g \), a compatible connection form \( \theta \), a representative \( \chi_\Delta \) of the Poincaré dual of \( \Delta \) (and a function \( \rho \) and a 2-form \( \alpha \) as in the proof of Prop. 3.1).

To take care of these choices, we introduce the parameter space \( I = [0, 1] \). Then, denoting by \( \sigma \) the inclusion \( M \times M \hookrightarrow M \times M \times I \), we take \( \chi_\Delta \in \Omega^3(M \times M \times I) \) such that \( d\chi_\Delta = 0 \) and \( \sigma^*\chi_\Delta \) is a representative of the Poincaré dual of \( \Delta \). (We treat similarly \( \rho \) and \( \alpha \).)

As for \( g \) and \( \theta \), we operate as follows. We take a block-diagonal metric \( g \) on \( M \times I \) (i.e., a metric such that \( g(m,t)(v,w) = 0 \) for all \( v \in T_m M \) and \( w \in T_t I \)), and consider the orthonormal frame bundle \( \tilde{OM} \) of \( TM \rightarrow M \times I \). Then we choose a connection form \( \theta \) on \( \tilde{OM} \) and define the equivariant global angular form on \( \tilde{OM} \times S^2 \).

Using the projections \( C_n(M) \times I \rightarrow C_2(M) \times I \), we can pull back the form \( \hat{\eta} \) in \( n(n-1)/2 \) different ways which we denote by \( \hat{\eta}_{ij} \).

**Definition 3.7.** We call a form on \( \Omega^*(C_n(M)) \) special if it is a product of pullbacks of \( \hat{\eta} \).

Each special form can be graphically associated to a diagram, each edge representing a pullback of \( \hat{\eta} \).

Let \( \mathcal{S}_{n,k} \) denote, as in the previous section, the face in \( \partial C_n(M) \) corresponding to the collapse of the first \( k \) points (we consider only this case since all other codimension-one faces can be reduced to this one by simply applying a permutation of the factors). Let \( \pi^S \) denote the induced projection \( \mathcal{S}_{n,k} \times I \rightarrow C_{n-k+1}(M) \times I \), and let \( \pi_1 : C_{n-k+1}(M) \times I \rightarrow M \times I \) denote the projection on the first point (i.e., where the first \( k \) points have collapsed). Then we have the following
Lemma 3.8 (Axelrod and Singer). If \( \alpha \in \Omega^*(S_n,k) \) is the restriction of a special form, then

\[
\pi^S \alpha = \beta \pi^1 \gamma,
\]

where \( \beta \) is special and \( \gamma \) is either a constant or a multiple of the first Pontrjagin form \( p_1 \) associated to \( \theta \).

For the proof, s. [3] or [4]. Using the same notations, we also have the following

Corollary 3.9. \( \gamma \) (and hence \( \pi^1 \gamma \)) is a constant in the case when no parameter space is introduced.

4. An invariant for rational homology spheres

In this section we assume \( M \) to be a 3-dimensional r.h.s. We then choose a representative \( v \) of the unit generator of \( H^3(M) \), so we can take, as the Poincaré dual of the diagonal in \( M \times M \),

\[
\chi_{\Delta} = v_2 - v_1,
\]

(5)

where \( v_i = \rho_i^* v \), and \( \rho_i \), \( i = 1,2 \), is the projection to the \( i \)-th factor.

We now define our form \( \hat{\eta} \) as in the preceding section—i.e., satisfying (4) and Condition 3.5—with \( \chi_{\Delta} \) as in (5).

Next we consider the three projections \( \pi_{ij} : C_3(M) \to C_2(M) \), and write \( \hat{\eta}_{ij} = \pi_{ij} \hat{\eta} \). Then (4) implies \( d \hat{\eta}_{ij} = v_{ij} - v_{i} \). Thus, we can define the following non-trivial closed form in \( \Omega^2(C_3(M)) \):

\[
\hat{\eta}_{123} = \hat{\eta}_{12} + \hat{\eta}_{23} + \hat{\eta}_{31}.
\]

(6)

Notice that on any configuration space \( C_n(M) \), \( n > 2 \), we can analogously define closed forms \( \hat{\eta}_{ijk} \) for any triple of distinct indices \( i, j, k \).

Now consider graphs with numbered vertices, and set equivalent to zero all graphs with an edge connecting a vertex to itself. We have then an induced orientation of the edges (viz., each edge is oriented from the lower to the higher end-point).

To each trivalent graph \( \Gamma \) of the above type we can associate the following number:

\[
A_{\Gamma}(M) = \int_{C_{n+1}(M)} v_0 \prod_{(ij) \in E(\Gamma)} \hat{\eta}_{ij0},
\]

(7)

where \( n \) is the number of vertices and \( E(\Gamma) \) is the set of ordered edges. The point labeled by 0 is an extra point and not a vertex of \( \Gamma \). We extend \( A_{\Gamma} \) to combinations of graphs by linearity.

We are interested in the dependence of \( A_{\Gamma} \) on the choices in the construction of \( \hat{\eta} \). So we introduce a parameter space \( I \) as in the
preceding section and consider \( A_{\Gamma} \) as a function on \( I \). (As for \( v \), we take it in \( \Omega^3(M \times I) \) and such that it is closed and its restriction to \( M \) is a representative of the unit generator of \( H^3(M) \).) Then we consider the differential of this function. Since the integrand form is closed, this differential is given only by boundary terms. These are dealt with by using Lemmata 3.6 and 3.8.

Remark 4.1. \( A_{\Gamma} \) can be defined also if \( M \) is not a r.h.s. However, in this case the integrand form is not closed. So in differentiating \( A_{\Gamma} \) we also have a bulk contribution which we do not know how to deal with.

We now define a coboundary operator \( \delta \) that acts on graphs by contracting each edge one at a time, with a sign given by the parity of the higher end-point. In \([4]\), it is shown that \( \delta \) is a coboundary operator.

We call a cocycle a \( \delta \)-closed combination of graphs. We say that it is connected (trivalent) if all its terms are connected (trivalent) graphs.

Finally, we consider the Chern–Simons integral,

\[
\text{CS}(M, f) = -\frac{1}{8\pi^2} \int_M f^* \text{Tr} \left( \theta d\theta + \frac{2}{3} \theta^3 \right),
\]

where \( f \) is a section of \( OM \) and \( \theta \) is the same connection form as in the construction of \( \hat{\eta} \). In \([4]\), the following was proved:

Theorem 4.2. If \( \Gamma \) is a connected, trivalent cocycle, then there exists a constant \( \phi(\Gamma) \) such that

\[
I_{\Gamma}(M, f) = A_{\Gamma}(M) + \phi(\Gamma) \text{CS}(M, f)
\]

is an invariant for the framed rational homology 3-sphere \( M \).

Remark 4.3. Instead of defining the equivariant global angular form, one could repeat the previous construction by choosing a trivialization of \( S(TM) \) and by defining the global angular form to be the (pullback of the) \( SO(3) \)-invariant unit volume form \( \omega \) on \( S^2 \) (as suggested in \([8]\)).

The equivariant treatment shows, cf. Thm. 4.2, that under a change of framing the invariants \( A_{\Gamma} \) behave as multiples of the Chern–Simons integral. In particular, they are invariant under a homotopic change of framing.

All graphs in a cocycle have the same number \( n \) of vertices. If the cocycle is trivalent, this number is even. In this case, one can define

\[
\text{ord} \Gamma = \frac{n}{2}.
\]
The constant \( \phi(\Gamma) \) depends only on \( \Gamma \) and not on \( M \). One can show \([3, 4]\) that \( \phi(\Gamma) = 0 \) if \( \text{ord } \Gamma \) is even. Moreover, in \([4]\) it is shown that \( \phi(\Theta) = 1/4 \), with

\[
A_\Theta = \int_{C_3(M)} v_0 \hat{\eta}_{012}.
\]

This allows for the definition of the following unframed invariants (for \( \Gamma \neq \Theta \)):

\[
J_\Gamma(M) = A_\Gamma(M) - 4 \phi(\Gamma) A_\Theta(M).
\]

A computation in \([4]\) shows that \( J_\Gamma(S^3) = J_\Gamma(SO(3)) = 0 \) if \( \text{ord } \Gamma \) is odd.

### 4.1. Knot invariants

In \([4]\), invariants for knots in a r.h.s. are studied.

If \( K \) is an imbedding \( S^1 \hookrightarrow M \), one has induced imbeddings \( \tilde{C}_n(S^1) \hookrightarrow C_n(M) \), where \( \tilde{C}_n(S^1) \) is the connected component of \( C_n(S^1) \) defined by an ordering of the points on \( S^1 \). The configuration space \( \tilde{C}_n^k(M) \) of \( n \) points on the knot and \( t \) points in \( M \) is then defined by pulling back the bundle \( C_{n+t}(M) \rightarrow C_n(M) \).

All the forms introduced before can be pulled back to \( \tilde{C}_n^k(M) \), and by abuse of notation we will keep calling them with the same names. One should keep in mind, however, that the pulled-back forms depend on the imbedding \( K \). This understood, one defines the self-linking number

\[
\text{sln}(K, M) \doteq \int_{\tilde{C}_n^k(M)} \hat{\eta}_{12}.
\]

Now consider graphs with a distinguished loop, which represents the knot. We call \emph{external} the vertices and the edges on this loop, and \emph{internal} all the others. To a trivalent graph we can then associate the number

\[
A_\Gamma(K, M) \doteq \int_{\tilde{C}_n^k(M)} v_0 \prod_{(ij) \in I(\Gamma)} \hat{\eta}_{ij0},
\]

where \( n \) and \( t \) are the numbers of external and internal vertices in \( \Gamma \), and \( I(\Gamma) \) is the set of internal edges. Again we extend \((11)\) to combinations of graphs by linearity.

Next we define a coboundary operator \( \delta \) as before with the only difference that now \( \delta \) does not contract internal edges connecting two external vertices. Graph combinations killed by \( \delta \) are called cocycles. (An explicit computation of these cocycles is presented in \([4]\).)

We name \emph{prime} a graph which is connected after removing any pair of external edges (in \([4]\) a graph of this kind was called connected).
A cocycle will be called prime (trivalent) if all its terms are prime (trivalent). In [4], the following was proved:

**Theorem 4.4.** If $K$ is a knot in the rational homology 3-sphere $M$, and $\Gamma$ is a prime, trivalent cocycle, then there exists a constant $\mu(\Gamma)$ such that

$$I_{\Gamma}(K, M) = A_{\Gamma}(K, M) + \mu(\Gamma) \text{ sln}(K, M)$$

is a knot invariant. Moreover, $\mu(\Gamma) = 0$ if $\text{ord} \, \Gamma$ is even.

5. Relationship with Kontsevich’s proposal

As we recalled in the Introduction, in [8] Kontsevich proposed a different way of constructing invariants for r.h.s.’s. His proposal differs from our construction since i) one point is removed from $M$ in order to make its rational homology trivial, and ii) the global angular form on the boundary of the configuration space is defined via a trivialization of $TM$. Let us consider part i) of the proposal first.

We introduce the compactified configuration space $C_n(M, x_{\infty})$ of $n$ points on $M \setminus x_{\infty}$ (where $x_{\infty}$ is an arbitrary base point) as the fiber of $C_{n+1}(M) \to M$ over the point $x_{\infty}$ in the last copy of $M$; viz.:

$$
\begin{array}{ccc}
C_n(M, x_{\infty}) & \longrightarrow & C_{n+1}(M) \\
\downarrow & & \downarrow \pi_{n+1} \\
x_{\infty} & \longrightarrow & M
\end{array}
$$

Consider now the projections $\pi_{ij} : C_{n+1}(M) \to C_2(M)$, $i < j \leq n + 1$, and set $p_{ij} = \pi_{ij} \circ p$ for $i < j \leq n$ and $p_{i\infty} = \pi_{i,n+1} \circ p$ for $i \leq n$. Then we will denote by $\hat{\eta}_{ij}$ and $\hat{\eta}_{i\infty}$ the pullbacks of $\eta$ to $C_n(M, x_{\infty})$. Accordingly, we will define $\hat{\eta}_{ijk}$ and $\hat{\eta}_{i\infty \infty}$. On the boundary faces we will have the pullbacks of global angular forms $\eta_{ij}$ and $\eta_{i\infty}$. Observe that $\eta_{i\infty} = \omega_{i\infty}$, with the notations of Lemma 3.6.

We also have projections $C_n(M, x_{\infty}) \to C_n(M)$. The forms $\hat{\eta}_{ij}$ we have written above can also be seen as the pullbacks of the forms with the same name on $C_n(M)$.

In particular, $C_1(M, x_{\infty})$ is just $M$ blown up at $x_{\infty}$, and $\partial C_1(M, x_{\infty}) = S^2$. If we pull $v$ back by the projection $\tau : C_1(M, x_{\infty}) \to M$, we get an exact form $\tau^* v = dw$, where the two-form $w$ restricted to the boundary must be a representative of the unit generator of $H^2(S^2)$. We may choose this representative to be $\omega$ since, by Thm. 4.2, the explicit choice of $v$ does not affect the invariant. We have then the following
THEOREM 5.1. If $\Gamma$ is a connected, trivalent cocycle and $M$ is a rational homology 3-sphere, then

$$A_\Gamma(M) = A'_\Gamma(M) + B_\Gamma,$$

with

$$A'_\Gamma(M) = \int_{C_0(M, x_\infty)} \prod_{(ij) \in E(\Gamma)} \hat{\eta}_{ij\infty},$$

$$B_\Gamma = \int_{\hat{C}_{n+2}(\mathbb{R}^3)} \omega_{0\infty} \prod_{(ij) \in E(\Gamma)} \omega_{ij0}.$$

Moreover, if $\text{ord } \Gamma$ is odd, $B_\Gamma$ vanishes.

Notice that $B_\Gamma$ does not depend on $M$ or on any arbitrary choice. Thus, even if it should not vanish, it would just represent a constant shift in the invariant. As a consequence, Thm. 4.2 holds with $A_\Gamma(M)$ replaced by $A'_\Gamma(M)$.

**Proof.** We can pull back the integrand form in (7) to $C_{n+1}(M, x_\infty)$ and integrate it over there. Using the fact that the pullback of $v_0$ is exact, by Stokes’s theorem we can rewrite (7) as

$$A_\Gamma(M) = \int_{\partial C_{n+1}(M, x_\infty)} w_0 \prod_{(ij) \in E(\Gamma)} \hat{\eta}_{ij0}.$$

The codimension-one faces in $\partial C_{n+1}(M, x_\infty)$ are labeled by subsets of $\{0, 1, \ldots, n, \infty\}$. Denote by $S$ any of these subsets.

Assume now that the cardinality of $S' = S \cap \{1, \ldots, n\}$ is $k$. Since points in $S'$ label vertices in the graph and the graph is trivalent, we have the relation

$$3k = 2e + e_0,$$

where $e$ denotes the number of edges connecting points in $S'$, and $e_0$ denotes the number of edges with exactly one end-point in $S'$. Now we have four cases, according to $S\setminus S' = \{0\}$ (a), $\emptyset$ (b), $\{0, \infty\}$ (c), $\{\infty\}$ (d).

The cardinality $r$ of $S$ is then $k + 1, k, k + 2$ and $k + 1$ respectively. The boundary face labeled by $S$ is a bundle over $C_{n+2-r}(M, x_\infty)$ with projection $\pi^S$ and fiber $\hat{C}_r(\mathbb{R}^3)$. So the fiber dimension is $3k-1, 3k-4, 3k+2$ and $3k-1$ respectively.
We now write the integrand form $\alpha$ restricted to this boundary as $(\pi^*S^*\beta)\alpha'$, where $\beta$ is the product of the pullbacks of $\hat{\eta}$ corresponding to edges with at least one end-point not in $S$, times $w_0$ in cases (a), (b) and (d).

In cases (b) and (d), the term $\hat{\eta}_{ij0}$ contributes to $\alpha'$ only if both $i$ and $j$ are in $S'$. In case (a) and (c), also terms with either $i$ or $j$ in $S'$ contribute. Moreover, $w_0$ contributes to $\alpha'$ only in case (c). As a consequence, the degree of $\alpha'$ will be: (a) $2e + 2e_0$, (b) $2e$, (c) $2e + 2e_0 + 2$, (d) $2e$.

By using all the above results, we see that the degree of $\gamma$ in Corollary 3.9 is $e_0 + 1, 4 - e_0, e_0$ and $1 - e_0$ respectively. Since $\gamma$ must be a constant zero-form, we see that the contribution of the face $S$ vanishes unless we are in case (b) with $e_0 = 4$, in case (c) with $e_0 = 0$, or in case (d) with $e_0 = 1$. Notice, moreover, that we can replace $\hat{\eta}$ by $\omega$ in $\alpha'$. Thus, in the last case above we conclude that the contribution vanishes by Lemma 3.6. The first case is taken care of by the fact that $\Gamma$ is a cocycle.

We are then left with case (c) and $e_0 = 0$. Since $\Gamma$ is connected, there are only two possibilities: 1) only point 0 has collapsed at $x_\infty$. 2) all points have collapsed at $x_\infty$. In case 1), $\alpha' = \omega_{0\infty}$ and the fiber is $S^2$. After this trivial integration we get $A'_\Gamma(M)$. Case 2) yields $B_\Gamma$.

To prove that $B_\Gamma$ vanishes if ord $\Gamma = n/2$ is odd, consider the involution $x_i \rightarrow -x_i, i = 0, 1, \ldots, n, \infty$. All the pullbacks of $\omega$ change signs. Since the number of edges is $3n/2$, the integrand form gets the sign $(-1)^{3n/2+1}$. On the other hand, since $\hat{C}_{n+2}(\mathbb{R}^3)$ is $S^{3n+2}$ with some submanifolds blown up, under the involution the orientation gets the sign $(-1)^{n+1}$.

In the particular case of the $\Theta$-invariant (9), we have

$$A'_\Theta(M) = \int_{C_2(M, x_\infty)} \hat{\eta}_{12\infty}^3.$$

By our construction, $\hat{\eta}_{12\infty}$ is a closed form on $C_2(M, x_\infty)$ which reduces to the global angular form when restricted to the faces $(1\infty)$, $(2\infty)$ and $(12)$.

As observed in remark 4.3, one can also modify the construction by choosing a trivialization of $S(TM)$ at the very beginning, and this corresponds to part ii) of Kontsevich's proposal. Invariance under homotopic changes of framing is then guaranteed (while under non-homotopic changes, the invariant behaves as $-1/4\ CS$). In this case, we have the additional property that $\hat{\eta}_{12\infty}^2$ vanishes close to the faces.
(1∞), (2∞) and (12). However, close to (12∞) neither ˆη_{12∞} nor ˆη_{12∞} vanish.

This is to be compared with Taubes’s invariant
\[ \tilde{A}_\Theta(M) = \int_{C_2(M, x_{\infty})} \omega^3, \]
where \( \omega \) is a 2-form on \( C_2(M, x_{\infty}) \) with the following properties:

1. \( \omega \) restricted to the faces (1∞), (2∞) and (12) is a global angular form;
2. \( \omega^2 \) vanishes not only close to (1∞), (2∞) and (12) but also close to (12∞).

The latter property is achieved only by choosing what Taubes names a singular framing for \( T(M \setminus x_{\infty}) \). As a consequence, \( \omega^2 \) (and hence \( \omega^3 \)) is a form with compact support, and Taubes’s \( \Theta \)-invariant can actually be defined as an integral over the uncompactified configuration space \( C_2^0(M, x_{\infty}) \). Moreover, property 2. is crucial in Taubes’s proof that his invariant is trivial on integral homology spheres.

Now the main question is if there is any relationship between the two different ways, \( A'_\Theta \) and \( \tilde{A}_\Theta \), of realizing Kontsevich’s proposal for the \( \Theta \)-invariant.

5.1. The case of knots. Let us consider an imbedding \( K \) of \( S^1 \) in the interior of \( M \setminus x_{\infty} \). This induces imbeddings \( \tilde{C}_n(S^1) \hookrightarrow C_n(M, x_{\infty}) \). By pulling back the bundles \( C_{n+t}(M, x_{\infty}) \to C_n(M, x_{\infty}) \), we then obtain the configuration spaces \( C_{n,t}^K(M, x_{\infty}) \). We have the following

**Theorem 5.2.** If \( \Gamma \) is a prime, trivalent cocycle and \( K \) is a knot in the rational homology 3-sphere \( M \), then
\[
A_\Gamma(K, M) = \int_{\tilde{C}_{n,t}^K(M, x_{\infty})} \prod_{(ij) \in I(\Gamma)} \hat{\eta}_{ij\infty},
\]
\[
\text{sln}(K, M) = \int_{\tilde{C}_{2,0}^K(M, x_{\infty})} \hat{\eta}_{12}.
\]

In particular, if \( M = S^3 \) and we choose the Euclidean metric on \( \mathbb{R}^3 = S^3 \setminus x_{\infty} \), we recover Bott and Taubes’s result [6]. As a consequence, the anomaly coefficients \( \mu(\Gamma) \) are the same in the two cases.

**Proof.** We work as in the proof of Thm. 5.1. The only difference is that we must distinguish between the cases when the collapse is at a point on \( K \) or otherwise.

Notice that, since \( x_{\infty} \) does not belong to the image of \( K \), there is no such term as \( B_\Gamma \). By the same reason, when we consider a collapse at
a point on $K$, we only have points in $\{0, 1, \ldots, n + t\}$. If 0 is involved, the term vanishes since $w_0$ is basic and is a 2-form. If 0 is not involved, reasoning as in the proof of Thm. 5.1 and applying Corollary 3.9 shows that the term vanishes unless $\epsilon_0 = 2$. But this is taken care of by the fact that $\Gamma$ is a cocycle.

\[\square\]

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