MICROSCOPIC DERIVATION OF TIME-DEPENDENT POINT INTERACTIONS
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Abstract. We study the dynamics of the three-dimensional polaron – a quantum particle coupled to bosonic fields – in the quasi-classical regime. In this case the fields are very intense and the corresponding degrees of freedom can be treated semiclassically. We prove that in such a regime the effective dynamics for the quantum particles is approximated by the one generated by a time-dependent point interaction, i.e., a singular time-dependent perturbation of the Laplacian supported in a point. As a by-product, we also show that the unitary dynamics of a time-dependent point interaction can be approximated in strong operator topology by the one generated by time-dependent Schrödinger operators with suitably rescaled regular potentials.

1. Introduction and Main Results
Point interactions, also called zero-range interactions or Fermi pseudo-potentials have been widely studied in mathematical physics as solvable models for realistic quantum systems (see [AGH-KH] for an extensive review of the topic). By formally replacing a typically complicated interaction potential with a sum of singular distributions (Dirac deltas) supported at isolated points or on curves or surfaces, one aims at summing up all the features of the interaction in a minimal number of physical parameters (e.g., scattering length, effective range, . . . ). The models obtained by this approximation are almost exactly solvable, while the salient physical features of the original systems should be retained in the procedure. Often, however, there is no conclusive evidence of the latter assertion: despite being known that point interactions can be approximated by suitable rescaled potentials, it is not clear whether they can be derived from realistic models, in suitable physical regimes.

In this paper we would like to derive a class of zero-range models as the effective description of physical systems, originating from a well-defined approximation that we call quasi-classical limit (see [CF, CFO1, CFO2] and references therein). This should clarify the importance of zero-range models in mathematical and theoretical investigations. As a by-product, we also prove that there exist lattice field quantum states in which a polaron is completely ionized (this is discussed in more detail below).

The class of zero-range models considered is the so-called time-dependent point interactions: solvable models with singular potentials whose “strength” may change in time. They are typically useful to investigate the ionization of a bound state by the action of a time-dependent localized interaction. In the zero-range approximation, one studies the formal Hamiltonian

$$-\Delta + \mu(t)\delta(x)$$ (1.1)

in $L^2(\mathbb{R}^d)$, and the system is assumed to be in a bound state at initial time, e.g., in the ground state, and the asymptotic probability of ionization is computed [CDFM, CD2] (see also [CCL] and [CD1] for the one-dimensional version of the same model and other time-dependent singular perturbations, respectively). As already discussed, the physical relevance of such a minimal model is still unclear and it is one of the main goals of the present investigation.

We prove that time-dependent point interactions can indeed be derived from the microscopic dynamics of a quantum particle interacting with bosonic scalar quantum fields, in suitable configurations in which the fields are very intense, and the average number of carriers is much larger than one. In particular, we consider the coupling of the particle with two distinct species of force-carrying fields. The presence of several force fields is not uncommon in physical systems, both in condensed matter and at high energies. The model we are considering, see (1.4) below,
is one describing two species of phonons interacting with a quantum particle (e.g., an electron or a molecule) in a lattice. Although the original Fröhlich polaron contains only one species of phonons, the optic ones (see [Fr]), there are physical systems for which the presence of two species is meaningful. The typical example of a model in which quantum particles are coupled with both acoustic and optic phonons is that of a compound ionic crystal (see, e.g., [Ki, Chpt. 4]), i.e., with at least two species of atoms per unit cell. Let us remark that it is the presence of two different species of phonons, with different scaling properties and dispersion relations, that produces the ionization of the particle in our model. In mathematical terms, it is the interplay between the two species that yields a time-dependent point interaction, instead of a time-independent one (see also [Remark 1.2]).

As mentioned above, we consider configurations in which the fields are very intense, due to the presence of a very large number of force carriers. Here, the reference scale is the number of non-relativistic particles in the system, i.e., just 1 in our setting, which is also of the same order as the commutator between creation and annihilation operators of the two fields, set equal to one in the considered units. Equivalently, it is more convenient to set
\[
[a_{\varepsilon}(x), a^\dagger_{\varepsilon}(y)] = [b_{\varepsilon}(x), b^\dagger_{\varepsilon}(y)] = \varepsilon \delta(x - y) ,
\]
and to take the limit
\[
\varepsilon \to 0 ,
\]
where the parameter $\varepsilon$ has the physical meaning of the inverse of the average number of field excitations. Here $a(x), b(x)$ and $a^\dagger(x), b^\dagger(x), x \in \mathbb{R}^3$, are the usual operator-valued distributions associated to the annihilation and creation of field carriers, for the two species of phonons. The limit $\varepsilon \to 0$ is precisely the aforementioned quasi-classical limit. In the case of the Fröhlich’s polaron, in which the particle is coupled with optic phonons only, such a limit is known to be equivalent, at least in the stationary picture, to the so-called strong coupling regime, i.e., of a very intense coupling between the particles and the Bose field. The strong coupling dynamics is however slightly different from the one considered here because the field is frozen to leading order.

Polaron models were originally introduced in [Fr] to describe the interaction between one or more electrons with a crystal of nuclei, vibrating around the rest positions on the lattice crystal, but, more recently, have been widely used in solid and condensed matter physics, notably also as effective models to describe the behavior of an impurity in a Bose-Einstein condensate (see, e.g., [GD], for a review of the topic). These latter models can be concretely realized in the lab [Ho et al.], by immersing a Cs impurity in a Rb condensate. Typically, the polaron is studied in three dimensions, but lower dimensional models may also be of a certain interest from the physical point of view [GAD]. For instance, one-dimensional tight-binding models are known to well-approximate the motion of electrons in organic semiconductors [DeF et al]. Furthermore, in these investigations, the phonon degrees of freedom are typically treated classically, as in the ideal quasi-classical limit described above.

When the limit $\varepsilon \to 0$ is taken, the bosonic fields become classical fields, whose dynamics a priori would depend on both its initial configuration and the coupling with the particle. We are going to scale the coupling in such a way that the classical field either evolves freely or is frozen in the limit, while the effective particle dynamics is generated by an effective time-dependent interaction, which is time-periodic and point-like at the origin, i.e., it is the rigorous counterpart of [Li]. Physically speaking, the quasi-classical limit we are considering is such that there is no back-reaction of the particle on the classical fields.

Mathematically, the dimensions lower than three are easier to deal with, therefore we prefer to focus on a three-dimensional model. Our techniques could however be adapted to study one- and two-dimensional systems. Notice that, since we derive a time-dependent point interaction from the polaron Hamiltonian and the former model shows complete asymptotic ionization [CDFM, Thm. 4.4], we also prove that there exist quantum field configurations that yield complete ionization of the polaron (compare with [Ho]). Let us stress that our model is at zero-temperature, and the ionization is due to the fields’ configuration only; a discussion of temperature-induced ionization for polaron systems can also be found in the physical literature (see, e.g., [HP]).
In the next [§ 1.1] and [1.2], we describe our setting in more detail, introducing the microscopic system and the effective point interaction model, respectively. The convergence of the dynamics is discussed in [§ 1.3], where we also state a technical result about the approximation of the dynamics generated by a time-dependent point interaction by means of the dynamics generated by the corresponding approximating regular potentials.

Let us conclude this section by fixing some basic notations. We use the convention of denoting by calligraphic letters all the quantities referring to the effective model, e.g., $\mathcal{U}_{\text{eff}}$ and $\mathcal{H}_{\text{eff}}$ denote the effective dynamics and generator, respectively, while regular letters (e.g., $U_{\varepsilon}$ and $H_{\varepsilon}$) are attached to the microscopic counterparts, i.e., to the operators acting on the full particle-field Hilbert space. Bold letters denotes vectors ($\mathbf{x}$), while italic roman letters are used for scalar quantities. When there is no risk of confusion, we use the corresponding italic letter ($x$) to denote the modulus of a vector ($\mathbf{x}$). Concerning operators and quadratic forms, we denote by $\mathcal{D}(A)$ and $\mathcal{D}[A]$ the operator and form domains, respectively.

1.1. **Microscopic model.** As anticipated, we want to investigate the quasi-classical limit of a system composed of a quantum spinless particle interacting with two bosonic fields. Other models could however be considered as well.

The Hilbert space of microscopic states is thus $\mathcal{H} := L^2(\mathbb{R}^3) \otimes \Gamma_{\text{sym}}(\mathcal{S}) \otimes \Gamma_{\text{sym}}(\mathcal{S})$, where the single excitation (phonon) space is $\mathcal{S} = L^2(\mathbb{R}^3)$, and $\Gamma_{\text{sym}}$ stands for the symmetric Fock space. The Hamiltonian of the full system reads

$$H_{\varepsilon} = H_0 + H_I = -\Delta \otimes 1 \otimes 1 + 1 \otimes d \Gamma_x^{(a)}(\omega) \otimes 1 + \frac{\kappa}{\varepsilon} \otimes 1 \otimes d \Gamma_x^{(b)}(1) + a_{\varepsilon}^{\dagger} \left( \lambda_x^{(a)} \right) + a_{\varepsilon} \left( \lambda_x^{(a)} \right) + b_{\varepsilon} \left( \lambda_x^{(b)} \right) + b_{\varepsilon}^{\dagger} \left( \lambda_x^{(b)} \right),$$

(1.4)

where $1$ stands for the identity operator on either $L^2(\mathbb{R}^3)$ or $\Gamma_{\text{sym}}(\mathcal{S})$, and

$$H_0 = -\Delta \otimes 1 \otimes 1 + 1 \otimes d \Gamma_x^{(a)}(\omega) \otimes 1 + \frac{\kappa}{\varepsilon} (1 \otimes 1 \otimes d \Gamma_x^{(b)}(1)).$$

(1.5)

The creation and annihilation operators $a_\varepsilon$, $a_\varepsilon^{\dagger}$ and $b_\varepsilon$, $b_\varepsilon^{\dagger}$ refer to the acoustic and optic phonons respectively, and $\kappa > 0$ is a frequency parameter for the optic phonons. Although more general choices are possible, we assume that the dispersion relation $\omega$ of the acoustic phonons is simply

$$\omega(\mathbf{k}) = k.$$  

(1.6)

In the Hamiltonian above, we have used the $\varepsilon$-dependent representation of the canonical commutation relations for the creation and annihilation operators $a_\varepsilon$, $b_\varepsilon$ and $a_\varepsilon^{\dagger}$, $b_\varepsilon^{\dagger}$, i.e.,

$$\left[ a_\varepsilon(\xi), a_\varepsilon^{\dagger}(\eta) \right] = \left[ b_\varepsilon(\xi), b_\varepsilon^{\dagger}(\eta) \right] = \varepsilon \langle \xi | \eta \rangle_\mathcal{S};$$

(1.7)

however the two fields have kinetic terms that scale differently in the quasi-classical limit. The acoustic phonons yield a *time-independent* effective potential, while the optic phonons yield a *time-dependent* one. The form factor for the optic phonons $\lambda^{(a)}_x(\mathbf{k}) = e^{i \mathbf{k} \cdot \mathbf{x}} \lambda_0(\mathbf{k})$ is such that $\lambda_0$ is invertible for almost all $\mathbf{k} \in \mathbb{R}^3$, and

$$\lambda_0(\mathbf{k}) , k^{-1/2}\lambda_0(\mathbf{k}) \in \mathcal{S}.$$  

(1.8)

Furthermore, we assume that $\lambda_0$ is polynomially decaying at large $k$, or, more precisely,

$$\frac{1}{|\lambda_0(\mathbf{k})|} \leq C(k^2 + 1)^M, \quad \text{for some } M \geq 0.$$  

(1.9)

On the other hand, the form factor for the optic phonons $\lambda^{(b)}_x$ has the explicit form

$$\lambda^{(b)}_x(\mathbf{k}) = \frac{e^{i \mathbf{k} \cdot \mathbf{x}}}{k}.$$  

(1.10)

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1The spin can be easily added to the picture, however since it would make the notations more cumbersome, we avoid it. Similarly, we can include more quantum particles in the model and possibly some interaction or a trapping potential, but we consider the simplest model for the sake of clarity.
The commutation relations \( (1.7) \) can be thought of as a quasi-classical rescaling of the usual relations; such rescaling is convenient to investigate the regime \( \langle a^\dagger a \rangle \sim \langle b^\dagger b \rangle \gg [a, a^\dagger] = [b, b^\dagger] = 1 \), where \( a^\dagger, b^\dagger \) are the usual creation and annihilation operators. In fact, one can set \( a^\dagger := \sqrt{\epsilon} a^\dagger, b^\dagger := \sqrt{\epsilon} b^\dagger \), from which \( (1.7) \) follows. Moreover, given any self-adjoint one-particle operator \( h \) on \( \mathcal{H} \),

\[
\text{d} \Gamma^{(a)}(h) = \varepsilon \text{d} \Gamma^{(a)}(h), \quad \text{d} \Gamma^{(b)}(h) = \varepsilon \text{d} \Gamma^{(b)}(h),
\]

where the left hand side is written w.r.t. \( a^\dagger, b^\dagger \) and the latter w.r.t. \( a^\dagger, b^\dagger \). The parameter \( \varepsilon \) is the quasi-classical parameter that describes the energy scale of the macroscopic phonon field, which is of order \( \mathcal{O}(\varepsilon^{-1}) \), and is assumed to be small, i.e., \( \varepsilon \ll 1 \). This corresponds to high field energies, due to the presence of a large number of excitations.

It is worth noting that if we get rid of the optic phonons, our quasi-classical limit is equivalent, up to a suitable rescaling of time and lengths, to the strong-coupling regime. We refer to [GrWe] for further details on the strongly coupled polaron.

Since for any \( x \in \mathbb{R}^3, \lambda_x(b) \not\in \mathcal{H}_x \), \( H_x \) can be written explicitly as the above sum only as a quadratic form, acting on the form domain of the non-interacting part \( \mathcal{D}[H_0] \) (see, e.g., [LT, FS, Fa3, GrWa]). The form \( (\cdot | H_\varepsilon | \cdot)_\mathcal{H} \) is closed and bounded from below on \( \mathcal{D}[H_0] \), and therefore \( H_\varepsilon \) can be defined as a self-adjoint operator on a suitable dense domain \( \mathcal{D}(H_\varepsilon) \subset \mathcal{D}[H_0] \). Moreover, \( H^2(\mathbb{R}^3) \otimes \mathcal{D}(\text{d} \Gamma^{(a)}(\omega)^{1/2}) \otimes \mathcal{D}(\text{d} \Gamma^{(b)}(1)^{1/2}) \) is dense in \( \mathcal{H} \) and contained in the form domain \( \mathcal{D}[H_0] \).

1.2. Effective dynamics. As anticipated above, we want to derive an effective dynamics generated by the formal operator \( (1.1) \), i.e., \(-\Delta + \gamma \mu(t) \delta(x)\). In two or three dimensions the rigorous definition of the self-adjoint counterpart of this formal expression is not straightforward: one can not simply consider the corresponding energy form and investigate its closedness, as it is done in one dimension. The typical way to address this question (see, e.g., [AGH-KH]) is to consider the operator \(-\Delta\) restricted to functions vanishing at the origin and classify its self-adjoint extensions, which form a one-parameter family of operators \( \{ \mathcal{H}_\beta \}_{\beta \in \mathbb{R}} \) given by

\[
\mathcal{H}_\beta \psi = -\Delta \phi, \quad \mathcal{D}(\mathcal{H}_\beta) = \left\{ \psi \in L^2(\mathbb{R}^3) \left| \psi = \phi + \frac{q}{4\pi \varepsilon}, \phi \in H^2_{\text{loc}}(\mathbb{R}^3), \Delta \phi \in L^2(\mathbb{R}^3), q \in \mathbb{C}, \phi(0) = \beta q \right. \right\}.
\]

The interaction is thus encoded into the boundary condition \( \phi(0) = \beta q \), which has to be satisfied by any function in \( \mathcal{D}(\mathcal{H}_\beta) \). The action of \( \mathcal{H}_\beta \) on the other hand coincides with the one of \(-\Delta\), although on the regular part of the wave function \( \phi \). In particular, if \( \phi(0) = 0 \), it is immediate to verify that \( \mathcal{H}_\beta \phi = -\Delta \phi \) and, in this respect, \( \mathcal{H}_\beta \) defines a self-adjoint realization of the formal expression \(-\Delta + \mu(t) \delta(x)\). However, it is important to remark that the meaning of the parameter \( \beta \) in \( (1.12) \) is not the strength of the interaction but it is rather proportional to the inverse of the scattering length, so that for instance the free operator \(-\Delta\) corresponds to \( \beta = +\infty \).

As already discussed, our main goal is thus to prove a rigorous derivation of the effective particle dynamics generated by the time-dependent operator \( \mathcal{H}_{\beta(t)} \), with \( \beta(t) \) a periodic function. The existence of such a dynamics has already been studied in the literature [SY2] (see also [CFT, DFT, Po, SY2, Ya] for further details and similar results) and it is known that, under minimal regularity assumptions on \( \beta(t) \), i.e., \( \beta(t) \in L^\infty_{\text{loc}}(\mathbb{R}) \), there exists a two-parameter unitary group \( \mathcal{U}_{\text{eff}}(t, s) \), \( t, s \in \mathbb{R} \), which is generated by \( \mathcal{H}_{\beta(t)} \) for any \( s \in \mathbb{R} \) and any \( \psi \in \mathcal{D}(\mathcal{H}_{\beta(s)}) \),

\[
\lim_{h \to 0} \frac{i}{h} \left( (\mathcal{U}_{\text{eff}}(s + h, s) - 1) \psi \right) = \mathcal{H}_{\beta(s)} \psi \quad . \tag{1.13}
\]

In fact, the time-evolution generated by \( \mathcal{H}_{\beta(t)} \) can be explicitly characterized: for any \( \psi_s \in H^2_{0}(\mathbb{R}^3 \setminus \{0\}) \),

\[
\psi(t) := (\mathcal{U}_{\text{eff}}(t, s) \psi_s)(x) = (U_0(t - s) \psi_s)(x) + i \int_s^t d\tau U_0(t - \tau; x) q(\tau), \quad \tag{1.14}
\]
where we have denoted by $U_0(t)$ the unitary time-evolution generated by $\mathcal{H}_0 = -\Delta$, and by $U_0(t; x)$ the corresponding integral kernel, which reads

$$U_0(t; x) := \frac{1}{(4\pi it)^{3/2}} \exp \left\{ \frac{ix^2}{4t} \right\}.$$  \hfill (1.15)

The charge $q(t) \in \mathbb{C}$ solves the Volterra-type integral equation

$$q(t) + 4\sqrt{\pi} \int_s^t d\tau \frac{\beta(\tau) q(\tau)}{\sqrt{t - \tau}} = 4\sqrt{\pi} \int_s^t d\tau \frac{1}{\sqrt{t - \tau}} (U_0(\tau - s) \psi_s)(0),$$  \hfill (1.16)

and, in fact, is the unique solution of such an equation in the space of continuous functions. Notice that it is easy to verify heuristically that (1.14) solves the time-dependent Schrödinger equation, since

$$i\partial_t \psi_t = -\Delta \left( \psi_t - \frac{q(t)}{4\pi x} \right),$$

while it is much harder to derive the charge equation (1.16), which is related to the boundary condition in (1.12). The evolution $\mathcal{U}_{\text{eff}}$ can be expressed as in (1.14) for any $\psi$ vanishing around the origin, but it can be extended to any state in $L^2(\mathbb{R}^3)$ by density.

1.3. Main results. We now describe in more detail the quasi-classical regime. Let $\mathcal{B} \in \mathcal{B}(L^2(\mathbb{R}^3))$ be a bounded observable associated to the particle. The Heisenberg evolution of $\mathcal{B}$ in the microscopic dynamics is given by

$$B_\varepsilon(t, s) = e^{iH_\varepsilon(t-s)} \mathcal{B} \otimes 1 e^{-iH_\varepsilon(t-s)}.$$  \hfill (1.17)

Since the microscopic system describes an interaction between the particle and the phonon fields, for almost all $t \neq s \in \mathbb{R}$, $B_\varepsilon(t, s)|_{\Gamma_{\text{sym}}(\mathcal{B}) \otimes \Gamma_{\text{sym}}(\mathcal{B})} \neq 1$, thus acting non-trivially on the field. However, since we are interested on a description of the subsystem consisting of the particle alone, let us take the partial expectation of $B_\varepsilon(t, s)$ with respect to some initial state of the fields $\Psi_\varepsilon \in \Gamma_{\text{sym}}(\mathcal{B}) \otimes \Gamma_{\text{sym}}(\mathcal{B})$ at time $s$ (to be fixed later):

$$\mathcal{B}_\varepsilon(t, s) := \langle \Psi_\varepsilon | B_\varepsilon(t, s) | \Psi_\varepsilon \rangle_{\Gamma_{\text{sym}}(\mathcal{B}) \otimes \Gamma_{\text{sym}}(\mathcal{B})} = \langle e^{-iH_\varepsilon(t-s)} \Psi_\varepsilon | \mathcal{B} \otimes 1 | e^{-iH_\varepsilon(t-s)} \Psi_\varepsilon \rangle_{\Gamma_{\text{sym}}(\mathcal{B}) \otimes \Gamma_{\text{sym}}(\mathcal{B})} \in \mathcal{B}(L^2(\mathbb{R}^3)).$$  \hfill (1.18)

Our main goal is thus to prove that in the quasi-classical limit $\varepsilon \to 0$, $\mathcal{B}_\varepsilon(t, s)$ converges to $\mathcal{B}(t, s)$ in some topology (that turns out to be at most the strong operator topology), the latter being defined as

$$\mathcal{B}(t, s) := \mathcal{U}_{\text{eff}}(t, s) \mathcal{B} \mathcal{U}_{\text{eff}}(t, s) \frac{1}{\varepsilon^1},$$  \hfill (1.19)

where the two-parameter unitary group $\mathcal{U}_{\text{eff}}(t, s)$ is defined in § 1.2. In other words, the effective quasi-classical evolution of the particle is, whenever the field is in a suitable microscopic state described by $\Psi_\varepsilon$, generated by the time-dependent point interaction Hamiltonian $H_{\beta_\varepsilon(t)}$. The expression (1.19) does not hold true for any choice of the field state $\Psi_\varepsilon$, but only for a restricted class of vectors; the relevant example on which we focus our attention is a time-dependent coherent state of the form

$$\Xi_{\varepsilon,s} := W_\varepsilon^{(a)} \left( \frac{a_\varepsilon}{\varepsilon} \right) W_\varepsilon^{(b)} \left( \frac{b_\varepsilon}{\varepsilon} \right) \Omega,$$  \hfill (1.20)

where

$$W_\varepsilon^{(a)}(\xi) = e^{i(a_\varepsilon(\xi) + a_\varepsilon^*(\xi))}, \quad W_\varepsilon^{(b)}(\xi) = e^{i(b_\varepsilon(\xi) + b_\varepsilon^*(\xi))}$$

are the Weyl operators for the two fields and $\Omega$ is the $\varepsilon$-independent Fock vacuum vector of $\Gamma_{\text{sym}}(\mathcal{B}) \otimes \Gamma_{\text{sym}}(\mathcal{B})$. Our proof takes advantage of the convenient coherent structure of $\Xi_{\varepsilon,s}$. The functions $a_\varepsilon, b_\varepsilon \in \mathcal{F}$ are chosen of the following form:\footnote{The analysis can be easily extended to suitable unbounded observables, but we restrict the discussion to bounded operators for the sake of simplicity.}

$$a_\varepsilon(k) = c_\varepsilon k \sigma_0^{-1}(k), \quad b_\varepsilon(k) = c'_\varepsilon k \sigma_0^\dagger(k),$$

and

$$\sigma_\varepsilon(k) = c_\varepsilon \sigma_\varepsilon k \sigma_\varepsilon^\dagger(k).$$

\footnote{When $\kappa' = 0$, the optic phonons’ subsystem decouples from the other subsystems, and therefore it does not play a role in the analysis. The choice of $\beta_\varepsilon$ given in (1.22) is thus to be intended only for $\kappa' > 0$.}
where \( \mathcal{W} \in C^\infty_0(\mathbb{R}^3) \) is a resonant potential in the sense of Definition 1.6, and \( \hat{\mathcal{W}} \) stands for its Fourier transform, \( c_\varepsilon, c'_\varepsilon \) are \( \varepsilon \)-uniformly bounded constants
\[
c_\varepsilon = \frac{1}{2} \gamma_\alpha \sigma_\varepsilon + o(\sigma_\varepsilon), \quad c'_\varepsilon = \frac{1}{2} \gamma_\beta \sigma_\varepsilon + o(\sigma_\varepsilon),
\]
with \( \gamma_\alpha, \gamma_\beta \in \mathbb{R} \) and \( \sigma_\varepsilon > 0 \) is such that
\[
\varepsilon^{1/j_*} \ll \sigma_\varepsilon \ll 1,
\]
where \( j_* := 6 + 8M \) is inherited from (1.9).

We can now state our main result:

**Theorem 1.1** (Effective dynamics).

Let \( \mathcal{B} \) be a bounded operator on \( L^2(\mathbb{R}^3) \). Let also \( \mathcal{B}_\varepsilon(t,s) \) and \( \mathcal{B}(t,s) \) be defined as in (1.18) and (1.19), respectively. If the field is in the coherent state (1.20) at time \( s \in \mathbb{R} \), then, for any \( t \in \mathbb{R} \),
\[
\mathcal{B}_\varepsilon(t,s) \xrightarrow[\varepsilon \to 0]{} \mathcal{B}(t,s) = \mathcal{U}_{\text{eff}}(t,s) \mathcal{U}(t,s) \mathcal{U}_{\text{eff}}(t,s),
\]
where \( \mathcal{U}_{\text{eff}} \) is the dynamics generated by \( \mathcal{H}_{\beta(t)} \), with
\[
\beta(t) = \gamma_\alpha + \gamma_\beta \cos \kappa(t - s).
\]

**Remark 1.2** (Time-dependence).

It is also possible to obtain time-independent point interactions in the limit, i.e., to prove the analogous of Theorem 1.1 with \( \beta(t) \) replaced by \( \gamma_\beta \). This is achieved whenever the optic phonons do not play any role in the limit. Concretely, this can be done by choosing suitable initial configurations for the optical field \( \hat{\beta}_i^2 \); possible choices are the vacuum, or suitable coherent, whose average number of excitations is subleading w.r.t. 1.

**Remark 1.3** (Initial state).

The choice of the initial state \( \Xi_{\varepsilon,s} \) is crucial for the derivation of the effective dynamics. Let us stress that \( \Xi_{\varepsilon,s} \) depends on \( \varepsilon \) through the phonon fields (of order \( \sqrt{\varepsilon} \)), through the factor \( \varepsilon^{-1} \) in the argument, and additionally through \( \alpha_\varepsilon, \beta_\varepsilon \). This latter \( \varepsilon \)-dependence, that is new compared to other situations in which coherent states are used to investigate quasi- and semi-classical limits (see, e.g., [He, GV2, GV1, GNV, RS, Fa2]), is chosen so that the effective potential becomes singular in the limit, due to the \( \sigma \)-scaling; in the language of semiclassical analysis, \( \Xi_{\varepsilon,s} \) converges to an additive but not \( \sigma \)-additive cylindrical Wigner measure on the classical space of fields (see [Fa1, Fa5] for additional details on cylindrical Wigner measures). In particular, the results proven in [CF02] do not apply to such a class of states, which call for an alternative approach.

**Remark 1.4** (Field dynamics).

The effective dynamics \( \mathcal{U}_{\text{eff}} \) of the particle is time-dependent in the quasi-classical regime, although the original microscopic dynamics \( \mathcal{U}_\varepsilon \) generated by \( \mathcal{H}_\varepsilon \) is time-independent. The reason is clearly the interaction of the particle with the optic phonons, which produces an entanglement of any initial product state, and gives rise, together with the particle-acoustic phonons interaction, to the non-trivial dynamics \( \mathcal{U}_{\text{eff}} \) for the particle in the limit \( \varepsilon \to 0 \). One may wonder however what is the effect on the fields themselves of such an interaction and, as we are going to show, such an effect actually disappears as \( \varepsilon \to 0 \), since in the chosen scaling the classical acoustic field is constant in time, while the optic field undergoes an evolution which is asymptotically free: if we denote by \( \alpha(k;t), \beta(k;t) \) the classical counterparts of the field operators, they satisfy \( i\dot{\alpha}(t) = 0, i\dot{\beta}(t) = \kappa \beta(t) \), or, equivalently,
\[
\alpha(k;t) = \alpha(k), \quad \beta(k;t) = e^{-ik(t-s)} \beta(k),
\]
where \( \alpha(k) \) and \( \beta(k) \) stands for the classical fields at initial time \( s \), i.e., the classical counterparts of (1.22). In other words, there is no back-reaction of the particle on the fields in the chosen quasi-classical scaling.

**Remark 1.5** (Heisenberg evolution).

The convergence proven in (1.25) implies by duality that the effective Schrödinger dynamics on states in \( L^2(\mathbb{R}^3) \) is also given by the two-parameter unitary group \( \mathcal{U}_{\text{eff}}(t,s) \): in the limit \( \varepsilon \to 0 \), a state \( \psi \in L^2(\mathbb{R}^3) \) at time \( s \) is mapped to \( \mathcal{U}_{\text{eff}}(t,s) \psi \) at time \( t \).
An important step in the proof of the main Theorem above is the approximation of the effective dynamics $U_{\text{eff}}$ by the one generated by Schrödinger operators with suitable regular potentials. It is indeed very well known [AGH-KH Sect. I.1.2] that a point interaction Hamiltonian can be obtained as the strong resolvent limit of a sequence of operators with rescaled smooth potentials: let $W \in C_0^\infty(\mathbb{R}^3)$ and let $0 < \sigma \ll 1$; set
\[ W_{\beta,\sigma}(x) := \nu(\sigma)W_{\sigma}(x), \quad W_{\sigma}(x) := \frac{1}{\sigma^2}W(x/\sigma), \quad (1.28) \]
for some $\nu \in C([0,1])$ given in (1.30) below and
\[ K_{\beta,\sigma} := -\Delta + W_{\beta,\sigma}(x), \quad (1.29) \]
which is obviously self-adjoint on $H^2(\mathbb{R}^3)$. In order to generate a point interaction, it is well-known (see again [AGH-KH Sect. I.1.2]) that a zero-energy resonance must be present. Therefore, we formulate the following

**Definition 1.6** (Resonant potential).

Let $W \in C_0^\infty(\mathbb{R}^3)$. We say that $W$ is resonant, if
\[ -1 \text{ is a simple eigenvalue of } \text{sgn}(W)|W|^{1/2}(-\Delta)^{-1}|W|^{1/2} \text{ with eigenvector } \phi \in L^2(\mathbb{R}^3); \]
\[ \langle |W|^{1/2} | \phi \rangle \neq 0. \]

Hence, if $W$ is resonant, then $-\Delta + W$ has a zero-energy resonance. If in addition
\[ \nu(\sigma) = 1 + \beta \sigma + o(\sigma), \quad (1.30) \]
for some $\beta \in \mathbb{R}$, then [AGH-KH Thm. 1.2.5]
\[ K_{\beta,\sigma} \xrightarrow{\| \cdot \| _{\text{res}} \to 0} \mathcal{H}_\beta, \quad (1.31) \]
where $\mathcal{H}_\beta$ is defined in (1.12) and $\| \cdot \|_{\text{res}}$ is short for norm resolvent convergence.

Obviously, by standard results (see, e.g., [Ka Thm. 2.16]), the one-parameter unitary group $e^{-itK_{\beta,\sigma}}$ generated by $K_{\beta,\sigma}$ strongly converges to $e^{i\beta t}$, when $\sigma \to 0$. Whether the same strong convergence holds true when $\beta$ depends on time, and therefore the propagator becomes a two-parameter unitary group, is not a consequence of some general result of operator theory: there are indeed adaptations of the aforementioned results to time-dependent operators [Sl, Yo], but they typically work only for bounded operators or in presence of a common core independent of time (which is not the case for $H_{\beta(t)}$). In the Theorem below we fill this gap for time-dependent point interactions. We believe this result might be of interest on its own.

**Theorem 1.7** (Time-dependent point interaction dynamics).

Let $\beta \in C^1(\mathbb{R})$ and $H_{\beta(t)}$ and $\mathcal{K}_{\beta(t),\sigma}$ be defined as in (1.12) and (1.29), respectively, with $\nu(\sigma)$ given by (1.30) and with $\beta(t)$ in place of $\beta$. Let also $U_{\text{eff}}(t,s)$ and $U_\sigma(t,s)$, $t,s \in \mathbb{R}$, be the two-parameter unitary groups generated by $H_{\beta(t)}$ and $\mathcal{K}_{\beta(t),\sigma}$, then
\[ U_\sigma(t,s) \xrightarrow{\sigma \to 0} U_{\text{eff}}(t,s). \quad (1.32) \]

**Remark 1.8** (Many-center point interactions).

The result in Theorem 1.7 is proven for a single time-dependent point interaction at the origin, but we expect that it is possible to extended it to point interactions with finitely many centers. The proof can indeed be adapted to take into account also the off-diagonal terms appearing in the time-evolution generated by a Schrödinger operator with many point interactions.

**Acknowledgments.** R.C., M.F. and M.O. are thankful to INdAM group GNFM for the financial support, through the grant Progetto Giovani GNFM 2017 “Dinamica quasi-classica del polaron”. M.C. and M.O. are especially grateful to the Institut Mittag-Leffler, where part of this work was completed. M.F. has been supported by the Swiss National Science Foundation via the grant “Mathematical Aspects of Many-Body Quantum Systems”, and by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (ERC CoG UniCoSM, grant agreement n.724939).
2. Approximation of the Effective Dynamics

The main goal of this Section is the proof of Theorem 1.7. We thus focus on the limiting dynamics described in §1.2 as anticipated, the existence of such a dynamics was originally proven in [SY1], although later other alternative approaches have been developed to deal with the same problem (see, e.g., [Po]). The key idea is to show that the ansatz (1.14) preserves the domain of the quadratic form associated to $\mathcal{K}_t\beta(t)$, which is independent of $t$. Next, one has to prove that the evolved state still satisfies the boundary condition in (1.12) and therefore belongs to $\mathcal{D}(\mathcal{K}_t\beta(t))$. This second step is easy to obtain if $\beta(t)$ is regular enough: by direct inspection of the Volterra integral equation (1.10), one can indeed prove that, if $\beta \in C^2_{\text{loc}}(\mathbb{R})$, then $q \in C^2_{\text{loc}}(\mathbb{R})$ as well [SY1 Thm. 1], and it is also easy to verify that the boundary condition is satisfied. In the general case, the argument is more involved but still the existence of the dynamics, as a two-parameter unitary group, can be proven for any $\beta \in L^\infty_{\text{loc}}(\mathbb{R})$ [SY1 Thm. 2].

An important intermediate step towards Theorem 1.7 is the approximation of the time-dependent dynamics generated by $\mathcal{K}_t\beta(t)$ in terms of a product of time-independent ones, in the spirit of [Yo, Thm. 1, p. 432]. The idea is to split the interval $[s,t]$ into smaller intervals and replace in each of those the propagator of $\mathcal{K}_t\beta(t)$ with the one associated to $\mathcal{K}_t\beta_{\cdot}(\cdot)$. $\beta_{\cdot}(t)$ being the step function approximation of $\beta(t)$. The proof is then divided into three parts:

- first, we show that the dynamics generated by $\mathcal{K}_t\beta(t)$ strongly converges to $U_{\text{eff}}$ (Proposition 2.1);
- next, we discuss the approximation of $\mathcal{K}_t\beta(t)$ and its unitary group in terms of the Schrödinger operators (1.29) with rescaled potentials $W_{\beta(t),\sigma}$ (Proposition 2.5);
- finally, we show that we can undo the step approximation of $\beta(t)$ at the level of the dynamics generated by the approximant Schrödinger operators $K_{\beta(t),\sigma}$ (Proposition 2.7).

All the results are then collected at the end of the Section to complete the proof of Theorem 1.7. We remark that, as it is customary in the investigations of such questions, all the steps above are verified on a dense subset of the Hilbert space, which is given in our case by functions belonging to

$$\mathcal{D} := \{ \psi \in C^\infty_0(\mathbb{R}^3) \mid \psi(x) = O(|x|^a), \text{ as } |x| \to 0 \}, \quad \text{for some } a > \frac{5}{2},$$

which is obviously a dense set for both $K_{\beta,\sigma}$ and $\mathcal{K}_t\beta$, for any $\beta \in \mathbb{R}$ and $\sigma > 0$. Each result is then extended to the whole Hilbert space by density and unitarity of the propagators. In the following we do not track the dependence of the constants appearing in the statements on, e.g., $\beta$ or time $t$, whenever the constant is finite for parameters varying on bounded intervals, as it is always the case in our setting.

Let us then fix the initial time $s \in \mathbb{R}$. Then, for any $t \in \mathbb{R}$, with $t > s$, we divide the interval $[s,t]$ into $n \in \mathbb{N}$ smaller intervals

$$I_j := [t_j, t_{j+1}), \quad t_j := s + \frac{j(t-s)}{n}, \quad j = 0, \ldots, n - 1.$$  

Next, we define $\beta_{\cdot}(t) := \beta(t_j)$, for $t \in I_j$.

Thanks to [SY1 Thm. 2] already mentioned above, the dynamics $\mathcal{V}_n(t,s)$ generated by $\mathcal{K}_t\beta_{\cdot}(t)$ exists as a two-parameter unitary group and, if the initial state $\psi_s$ at time $s$ belongs to $H^2(\mathbb{R}^3 \setminus \{0\})$, it can be represented as in (1.14), i.e.,

$$(\mathcal{V}_n(t,s)\psi_s)(x) = (U_0(t-s)\psi_s)(x) + i \int_s^t d\tau U_0(t-\tau;x)q_n(\tau),$$

where the charge $q_n$ now solves the Volterra equation

$$q_n(t) + 4\sqrt{\pi i} \int_s^t d\tau \frac{\beta_{\cdot}(\tau)q_n(\tau)}{\sqrt{t-\tau}} = 4\sqrt{\pi i} \int_s^t d\tau \frac{1}{\sqrt{t-\tau}} (U_0(\tau-s)\psi_s)(0).$$

Note that $\mathcal{V}_n(t,s)$ can be equivalently represented as

$$\mathcal{V}_n(t,s) = e^{-i\mathcal{K}_s(t-s)}e^{-i\mathcal{K}_{s-1}(t_{n-1}-t_{n-2})} \cdots e^{-i\mathcal{K}_{s_0}(t_1-s)}.$$
Proposition 2.1 (Step function approximation of $U_{\text{eff}}(t,s)$). Let $\beta(t) \in C^1(\mathbb{R})$. Then, for any $s,t \in \mathbb{R}$,

$$\mathcal{V}_n(t,s) \xrightarrow{n \to \infty} U_{\text{eff}}(t,s).$$

(2.7)

Proof. The idea is to prove the result for a dense subset of initial states $\psi_s$ and then extend it to the rest of the Hilbert space by density. Let us thus assume that $\psi_s \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, in which case the representation (2.4) holds true with $q_n \in L^\infty_{\text{loc}}(\mathbb{R})$ solving (2.5), as proven in [SY1]. Furthermore, the independence on $n$ of the forcing term (term on the r.h.s. of (2.5)) and the uniform boundedness of $\beta_s$ also imply that the $L^\infty$ norm of $q_n(t)$ on any finite interval is bounded uniformly in $n \in \mathbb{N}$ (see also the general theory of Volterra integral equations in [Mi]).

By bootstrap, it is easy to see that, if $q_n \in L^\infty_{\text{loc}}(\mathbb{R})$, then also $q_n \in W^{1,1}_{\text{loc}}(\mathbb{R})$. Indeed, the forcing term can be easily seen to be differentiable, under the assumptions made: thanks to the properties of the Abel 1/2-operator [GV], one has

$$\frac{d}{dt} \int_s^t \frac{1}{\sqrt{t - \tau}} (U_0(\tau - s)\psi_s)(0) = \int_s^t \frac{1}{\sqrt{t - \tau}} \frac{d}{d\tau} (U_0(\tau - s)\psi_s)(0),$$

and

$$\left| \frac{d}{d\tau} (U_0(\tau - s)\psi_s)(0) \right| = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} dk k^2 e^{-ik^2(\tau-s)} \hat{\psi}_s(k) \leq C \int_{\mathbb{R}^3} dk k^2 \left| \hat{\psi}_s(k) \right| \leq C,$$

so that the derivative of the forcing term is bounded on finite intervals. Note that, as a direct consequence, the $W^{1,1}$ norm of $q_n$ on finite intervals is bounded uniformly in $n \in \mathbb{N}$.

Next, we prove that the sequence $q_n$ pointwise converges to $q$, as $n \to +\infty$: taking the difference between the equations (1.16) and (2.5) and setting $\chi_n := q - q_n$ for short, we obtain

$$\chi_n(t) = 4\sqrt{\pi} \int_s^t \frac{\beta_s(\tau) q_n(\tau) - \beta(\tau) q(\tau)}{\sqrt{t - \tau}} d\tau = 4\sqrt{\pi} \int_s^t \frac{\partial \beta(\tau) \chi_n(\tau)}{\sqrt{t - \tau}} =: F_n(t).$$

It follows that $\chi_n \in C(\mathbb{R})$ (recall that both $q$ and $q_n$ belong to $W^{1,1}_{\text{loc}}(\mathbb{R})$) is a solution of the Volterra equation

$$\chi_n(t) + 4\sqrt{\pi} \int_s^t \frac{\beta(\tau) \chi_n(\tau)}{\sqrt{t - \tau}} = F_n(t).$$

(2.8)

Furthermore,

$$F_n(t) \xrightarrow{n \to +\infty} 0,$$

(2.9)

pointwise, since

$$|F_n(t)| \leq C \sup_{\tau \in [s,t]} |q_n(\tau)| \int_s^t \frac{|\partial \beta_s(\tau) - \partial \beta(\tau)|}{\sqrt{t - \tau}} \xrightarrow{n \to +\infty} 0,$$

because

$$\beta_s \xrightarrow{n \to +\infty} \beta,$$

for any $1 \leq p < +\infty$, (2.10)

and $q_n$ is uniformly bounded. Hence, it just remains to observe that the l.h.s. of (2.8) is invertible for $t$ small enough, since

$$\int_s^t \frac{\partial \beta(\tau) \chi_n(\tau)}{\sqrt{t - \tau}} \leq C \sup_{\tau \in [s,t]} |\chi_n(\tau)| \sqrt{t - s} \xrightarrow{s \to t} 0,$$

(2.11)

which implies that the l.h.s. of (2.8) can be rewritten as $(1 + J) \chi_n$, with $J : C_{\text{loc}}(\mathbb{R}) \to C_{\text{loc}}(\mathbb{R})$ and such that $\|J\|_{C \to C} < 1$, if $t-s$ is small enough. Hence, we get $\chi_n = (1 + J)^{-1} F_n$, with $F_n \to 0$ and $(1 + J)^{-1}$ bounded from $C_{\text{loc}}(\mathbb{R})$ to $C_{\text{loc}}(\mathbb{R})$. We conclude that $\chi_n \to 0$ in $C_{\text{loc}}(\mathbb{R})$, if $t-s$ is small enough.

The argument above guarantees that $\chi_n \to 0$ in an interval $[s,t_1]$, where $t_1$ is independent of $n$ and determined only by the condition that the r.h.s. of (2.11) is strictly smaller than $\|\chi_n\|_{\infty}$. As such, $t_1$ depends only on $s$ and $\|\beta\|_{\infty}$. Hence it is not difficult to see that, picking $n$ large enough
and exploiting the convergence to 0 of $\chi_n$ in $[s,t_1]$, it is possible to reproduce the argument in the interval $[s,2t_1]$. A bootstrap then gives pointwise convergence to 0 of $\chi_n$ in any finite interval $[s,t]$. In the last bootstrap, it is crucial that the considered interval is relatively compact.

Next, exploiting that the $W^{1,1}$ and $L^\infty$ norms of $q_n$ are bounded uniformly in $n$, we use the charge equation (2.3) once more, to show that the convergence of $q_n$ to $q$ is actually uniform on bounded intervals. Indeed, we claim that the sequence of functions $\{q_n\}_{n\in\mathbb{N}}$ is uniformly equicontinuous: for any $0 < t' < t$, we get

$$q_n(t) - q_n(t') = -4\sqrt{\pi}t \int_{t'}^{t} d\tau \frac{\beta_n(\tau) q_n(\tau)}{\sqrt{t - \tau}} + 4\sqrt{\pi}t \int_{t'}^{t} d\tau \frac{1}{\sqrt{t - \tau}} (U_0(\tau - s)\psi_s)(0).$$  \hspace{1cm} (2.12)

Now, the second term on the r.h.s. is obviously uniformly continuous, since we have shown that the function is actually $C^1$. Hence, we have only to consider the first term on the r.h.s. of (2.12). However, by the uniform boundedness of both $\beta_n$ and $q_n$, we can bound

$$\left| \int_{t'}^{t} d\tau \frac{\beta_n(\tau) q_n(\tau)}{\sqrt{t - \tau}} \right| \leq C\sqrt{t - t'},$$

which is independent of $n \in \mathbb{N}$, so implying uniform equicontinuity of the sequence. A direct application of the Ascoli-Arzelà theorem then yields uniform convergence of $q_n$ to $q$ on finite intervals.

Moreover, using (1.14) and (2.4), we get

$$\|(U_{\text{eff}}(t,s) - \mathcal{V}_n(t,s))\psi_s\|_2 \leq C \sup_{\tau \in [s,t]} |q(\tau) - q_n(\tau)|,$$  \hspace{1cm} (2.13)

and therefore the strong convergence of the propagators follows from the uniform convergence of the charges, for $\psi_s \in C^\infty_0(\mathbb{R}^3 \setminus \{0\})$. \hfill \Box

We recall that for any time-independent $\beta \in \mathbb{R}$, the family of operators $\mathcal{K}_{\beta,\sigma}$ defined in (1.29) and given by $\mathcal{K}_{\beta,\sigma} = -\Delta + W_{\beta,\sigma}(x)$, where $-\Delta + W$ is assumed to have a zero-energy resonance, $W_{\beta,\sigma}(x) = \nu(\sigma)\sigma^2 W(x^/\sigma)$ and $\nu(\sigma) = 1 + \beta \sigma + o(\sigma)$, converges in norm resolvent sense to $\mathcal{H}_\beta$, as $\sigma \to 0$ [AGH-KH Thm. 1.2.5]. A by-product of this result is the strong convergence of the corresponding unitary operators obtained by the piecewise approximations of $\beta(t)$: in the following we denote by $\mathcal{V}_{n,\sigma}(t,s)$ the two-parameter unitary group associated with the time evolution generated by $\mathcal{K}_{\beta_n(t),\sigma}$, i.e., such that

$$i\partial_t \mathcal{V}_{n,\sigma}(t,s) = \mathcal{K}_{\beta_n(t),\sigma} \mathcal{V}_{n,\sigma}(t,s),$$  \hspace{1cm} (2.14)

and study the strong limit $\sigma \to 0$ of such operators. Note that, by (2.3), one obviously gets that

$$\mathcal{V}_{n,\sigma}(t,s) := e^{-i\mathcal{K}_{\beta_n(t),\sigma}(t-s)} e^{-i\mathcal{K}_{\beta_n(t_0),\sigma}(t_0-s)} \cdots e^{-i\mathcal{K}_{\beta_n(t_{j-1}),\sigma}(t_{j-1}-s)}.$$  \hspace{1cm} (2.15)

Before doing that, we have however to formulate a technical lemma concerning the approximation of functions evolved with $e^{-i\mathcal{K}_{\beta(t),\sigma}}$ or, more in general, with $\mathcal{V}_n(t,s)$, which is simply a consequence of the density of $D$ in $L^2(\mathbb{R}^3)$. We state a quantitative bound however for concreteness.

**Lemma 2.2.**

Let $\chi \in L^2(\mathbb{R}^3)$, such that $\|\chi\|_2 = 1$. Let also $s < t \in \mathbb{R}$ varying in a compact set. Then, there exists a sequence of functions $\{\chi_m\}_{m \in \mathbb{N}} \subset D$, and a finite constant $C > 0$ independent of $n$, such that $\|\chi_m\|_2 \leq \|\chi\|_2$ and

$$\|\mathcal{V}_n(t,s)\chi - \chi_m\|_2 \leq Cm^{-2/3}.$$  \hspace{1cm} (2.16)

**Proof.** The first simple observation is that, by direct inspection of the charge equation (2.4), which reduces to (1.10) in each interval $[t_j,t_{j+1}]$ with $\beta = \beta(t_j)$ and initial datum $\psi(t_j)$, one gets that the charge $q(t)$ is uniformly bounded on compact sets irrespective of $n \in \mathbb{N}$.

Then, it is sufficient to set for $m \in \mathbb{N}_0$

$$\chi_m(x) := e^{-\frac{i\pi}{2}}(\mathcal{V}_n(t,s)\chi)(x).$$  \hspace{1cm} (2.17)
We claim that such a sequence satisfies (2.18). By the characterization of the operator domain, we know that \( \mathcal{V}_n(t, s) \chi \in \mathcal{D}(\mathcal{H}_{\beta, (\tau)}) \), i.e., we can decompose for any \( \tau \in [s, t] \)

\[
(\mathcal{V}_n(\tau, s) \chi)(x) = \phi_\tau(x) + \frac{q(\tau)e^{-x}}{4\pi x},
\]
where \( q(\tau) \) is a solution of (2.4) and \( \phi_\tau \in H^2(\mathbb{R}^3) \). Note that we have used a slightly different domain decomposition than the one given in (1.12): starting from the domain given there, one simply recovers the expression above adding and subtracting \( \frac{q(\tau)e^{-x}}{4\pi x} \) and observing that \( \frac{e^{-x}}{x} \in L^2(\mathbb{R}^3) \). Hence,

\[
\| \mathcal{V}_n(t, s) \chi - \chi_m \|_2^2 = \int_{\mathcal{B}_R} dx \left( 1 - e^{-\frac{1}{4\pi x}} \right)^2 \left| \phi_t(x) + \frac{q(t)e^{-x}}{4\pi x} \right|^2 + \int_{\mathcal{B}_R} dx \left( 1 - e^{-\frac{1}{4\pi x}} \right)^2 \left| \phi_t(x) + \frac{q(t)e^{-x}}{4\pi x} \right|^2
\]

where \( R > 0 \) is a parameter to be chosen later. Next, we estimate

\[
\int_{\mathcal{B}_R} dx \left( 1 - e^{-\frac{1}{4\pi x}} \right)^2 \left| \phi_t(x) + \frac{q(t)e^{-x}}{4\pi x} \right|^2 \leq C m^{-3} \int_{\mathcal{B}_{mR}} dx \left[ \left| \phi_t(x) \right|^2 + \frac{m^2 |q(t)|^2}{x^2} \right] \leq C \left( R^3 + R \right),
\]

by (2.1) and the fact that \( \phi_t \in H^2(\mathbb{R}^3) \). On the other hand,

\[
\int_{\mathcal{B}_R} dx \left( 1 - e^{-\frac{1}{4\pi x}} \right)^2 \left| \phi_t(x) + \frac{q(t)e^{-x}}{4\pi x} \right|^2 \leq \frac{C}{m^2} \int_{\mathcal{B}_R} dx \frac{1}{x^2} \left[ \left| \phi_t(x) \right|^2 + \frac{|q(t)|^2}{x^2} \right] \leq C m^{-2} \left( R^2 + R^{-1} \right).
\]

Altogether the above estimates yield

\[
\| \mathcal{V}_n(t, s) \chi - \chi_m \|_2^2 \leq C \left[ R^3 + m^{-2} R^2 + (R + m^2 R^{-1}) \right] \leq C m^{-2/3}
\]

after an optimization over \( R \), i.e., taking \( R = m^{-2/3} \).

Before discussing the convergence of \( \mathcal{V}_{n, \sigma} \) to \( \mathcal{V}_n \), we need one more technical result: we are going to show that a strong estimate over a dense set of the difference between the unitary evolutions generated by \( \mathcal{H}_\beta \) and \( \mathcal{X}_{\beta, \sigma} \) is sufficient to control the difference between the unitaries \( \mathcal{V}_n(t, s) \) and \( \mathcal{V}_{n, \sigma}(t, s) \) in strong sense. The result is going to be used in next Proposition 2.5, where we are going to show that the bound (2.23) holds true on a dense set with some explicit error \( \delta \). The result below then allows to translate the bound (2.23) into an estimate of the difference between the unitary groups on the same dense set.

**Lemma 2.3.**

Let \( t_0 = s < t_1 < \cdots < t_n = t \) be a partition of the bounded interval \([s, t]\) as in (2.2). Let \( \psi \in \mathcal{D} \), with \( \| \psi \|_2 \leq 1 \). Assume that there exists \( \delta > 0 \) such that, for all \( \phi \in \mathcal{D} \) and for all finite \( \tau \),

\[
\left\| \left( e^{-i\mathcal{X}_{\beta, \sigma} \tau} - e^{-i\mathcal{X}_{\beta, \sigma} \tau} \right) \phi \right\|_2 \leq \delta \| \phi \|_2.
\]

Then,

\[
\| (\mathcal{V}_n(t, s) - \mathcal{V}_{n, \sigma}(t, s)) \psi \|_2 \leq n \delta + O(n^{-1}).
\]

**Proof.** The result is proved iteratively on the quantity

\[
\| (\mathcal{V}_n(t, s) - \mathcal{V}_{n, \sigma}(t, s)) \psi \|_2.
\]

We have that

\[
\| (\mathcal{V}_n(t, s) - \mathcal{V}_{n, \sigma}(t, s)) \psi \|_2 \leq \left\| e^{i\mathcal{X}_{\beta, \sigma} (t-t_n-1)} - e^{i\mathcal{X}_{\beta, \sigma} (t-t_{n-1})} \right\|_2 \| \mathcal{V}_n(t_n-1, s) \psi \|_2 + \left\| e^{i\mathcal{X}_{\beta, \sigma} (t-t_{n-1})} (\mathcal{V}_n(t_n-1, s) - \mathcal{V}_{n, \sigma}(t_n-1, s)) \psi \|_2.
\]
Now, we can apply Lemma 2.2 with $\chi = \mathcal{V}_n(t_n, s)\psi$, and we get a sequence of functions $\chi_m \in \mathcal{D}$, such that $\|\chi_m\|_2 \leq \|\mathcal{V}_n(t_n, s)\psi\|_2 \leq 1$ and

$$\mathcal{V}_n(t_n, s)\psi - \chi_m = O(m^{-2/3}),$$

for any $m \in \mathbb{N}_0$. In particular, if we take $m = n^2$, we obtain

$$\mathcal{V}_n(t_n, s)\psi - \chi_m = O(n^{-2}).$$

(2.25)

Therefore,

$$\|\mathcal{V}_n(t, s) - \mathcal{V}_{n,\sigma}(t, s)\|_2 \leq \left\| \left( e^{i\mathcal{K}_{\beta, n-1}(t-t_{n-1})} - e^{i\mathcal{K}_{\beta, n-1,\sigma}(t-t_{n-1})} \right) \chi_m \right\|_2$$

$$+ \left\| \left( \mathcal{V}_n(t_n, s) - \mathcal{V}_{n,\sigma}(t_n, s) \right)\psi \right\|_2 + O(n^{-2}).$$

Now, using the assumption (2.23), we get

$$\|\mathcal{V}_n(t, s) - \mathcal{V}_{n,\sigma}(t, s)\|_2 \leq \|\mathcal{V}_n(t_n, s) - \mathcal{V}_{n,\sigma}(t_n, s)\|_2 + \delta + O(n^{-2}).$$

Iterating such reasoning for the remaining $n - 1$ intervals yields

$$\|\mathcal{V}_n(t, s) - \mathcal{V}_{n,\sigma}(t, s)\|_2 \leq n\delta + O(n^{-1}),$$

thus concluding the proof.

The technical results proven in Lemmas 2.2 and 2.3 are used to prove the convergence of $\mathcal{V}_{n,\sigma}(t, s)$ to $\mathcal{V}_n(t, s)$, which in turn is going to play a key role in the proof of Theorem 1.7. We are going to use a known representation formula for the propagator, provided in next Lemma.

Lemma 2.4.

For any $t \in \mathbb{R}$ finite and for any $\psi \in \mathcal{D}(\mathcal{K}_\beta) \cap \mathcal{D}(\mathcal{K}_{\sigma,\beta})$, with $\|\mathcal{K}_{\sigma,\beta}\psi\| \leq C$,

$$\left\| e^{-it\mathcal{K}_{\sigma,\beta}^t} - e^{-it\mathcal{K}_{\sigma,\beta}} \psi \right\|_2 = \left( \frac{k}{|t|} \right)^k \sup_{\chi \in L^2(\mathbb{R}^3), \|\chi\|_2 \leq 1} \left| \left\langle \chi, \left( \mathcal{K}_{\sigma,\beta} - \frac{ik}{t} \right)^{-k} \psi \right\rangle \right|_2$$

$$+ |t| o_k(1),$$

(2.26)

uniformly in $\sigma \in [0, 1]$.

Proof. The starting point is the representation formula [P2, Thm. 8.3, Ch. 1] for a unitary group $e^{-itA}$ generated by a self-adjoint operator $A$: for any $\psi \in L^2(\mathbb{R}^3)$, one has the following convergence in norm, uniformly w.r.t. $t$ on bounded intervals:

$$e^{-itA} \psi = \lim_{k \to +\infty} \left( 1 + \frac{itA}{k} \right)^{-k} \psi = \lim_{k \to +\infty} \left( \frac{k}{it} \right)^k \left( A - \frac{ik}{t} \right)^{-k} \psi.$$  

(2.27)

Since the vector $\psi$ belongs to the intersection of the domains of the two operators, it is possible to write a more explicit bound on the error:

$$\left( e^{-it\mathcal{K}_{\beta}} - \left( \frac{k}{it} \right)^k (\mathcal{K}_{\beta} - \frac{ik}{t})^{-k} \right) \psi = -i \int_0^t \left( e^{-it\mathcal{K}_{\beta}} - \left( 1 + \frac{it}{k} \mathcal{K}_{\beta} \right)^{-k} \right) \mathcal{K}_{\beta} \psi \, d\tau
$$

$$= |t| o_k(1),$$

where $o_k(1) \in L^2(\mathbb{R}^3)$ converges to zero as $k \to \infty$ uniformly with respect to $t$ and $\beta$ on bounded intervals.

Let us prove that a similar estimates holds true for $\mathcal{K}_{\sigma,\beta}$: following [P2], we write

$$\left\| e^{-it\mathcal{K}_{\sigma,\beta}} - \left( \frac{k}{it} \right)^k (\mathcal{K}_{\sigma,\beta} - \frac{ik}{t})^{-k} \right\|_2^2 = \frac{k^{k+1}}{k!} \int_0^{+\infty} d\xi \, \xi^k e^{-k\xi} \left( e^{-it\mathcal{K}_{\sigma,\beta}} - e^{-it\mathcal{K}_{\sigma,\beta}} \right) \psi \right\|_2$$

$$\leq C \frac{k^{k+1}}{k!} \int_0^{+\infty} d\xi \, \xi^k e^{-k\xi} = \frac{C}{k!} \int_0^{+\infty} d\xi \, \xi^k e^{-\xi} = C,$$

so that

$$\left\| e^{-it\mathcal{K}_{\sigma,\beta}} - \left( \frac{k}{it} \right)^k (\mathcal{K}_{\sigma,\beta} - \frac{ik}{t})^{-k} \psi \right\|_2 \to 0, \quad k \to +\infty.$$  

(2.29)
by dominated convergence and pointwise convergence to 0 of the integrand:

\[
\frac{\xi^k e^{-\xi}}{k!} \leq \frac{k^k e^{-k}}{k!} \leq \frac{C}{\sqrt{k}} \xrightarrow{k \to +\infty} 0.
\]

Note that the convergence is uniform in \( \sigma \), since the estimates above are independent of \( \sigma \).

Furthermore, for any \( \psi \in \mathcal{D}(\mathcal{K}_{\sigma,\beta}) \),

\[
\left\| \left( \mathcal{K}_{\sigma,\beta} - \frac{i k}{\tau} \right)^{-1} \psi \right\|_2 = \left\| \left( \mathcal{K}_{\sigma,\beta} - \frac{i k}{\tau} \right)^{-1} \mathcal{K}_{\sigma,\beta} \psi \right\|_2 \leq \frac{C t/k}{\sqrt{k}} \xrightarrow{k \to +\infty} 0,
\]

(2.30)

uniformly in \( \sigma \). This implies the analogue of (2.28) for \( \mathcal{K}_{\sigma,\beta} \), via triangular inequality.

The core of the proof of Theorem 1.7 is next Proposition 2.5, where we show that the norm resolvent convergence of \( \mathcal{K}_{\sigma,\beta} \) to \( \mathcal{H}_\beta \) is in fact sufficient, via Lemmas 2.2 to 2.4 to deduce a quantitative estimate of the convergence on a suitable dense subset of the unitary groups \( \mathcal{V}_{n,\sigma} \) and \( \mathcal{V}_n \) as \( \sigma \to 0 \). It is precisely at this point that we need to restrict the argument to the set \( \mathcal{D} \) introduced in (2.1).

**Proposition 2.5** (Rescaled potential approximation of \( \mathcal{V}_n(t,s) \)).

For any finite \( t, s \in \mathbb{R} \) and for any \( \psi, \phi \in \mathcal{D} \), with \( \| \phi \|_2 = \| \psi \|_2 = 1 \), as \( \sigma \to 0 \), with \( \sigma n \ll 1 \),

\[
|\langle \phi | (\mathcal{V}_{n,\sigma}(t,s) - \mathcal{V}_n(t,s)) \psi \rangle | = O(n^5 \sigma^2) + n^{-1} o_{\sigma}(1) + o_n(1).
\]

(2.31)

**Proof.** The idea is to use Lemma 2.3 and thus focus on estimating the quantity

\[
\left\| e^{-i \xi_{\beta} \tau} - e^{-i \xi_{\sigma,\beta} \tau} \psi \right\|_2
\]

(2.32)

for any \( \psi \in H^2(\mathbb{R}^3) \) such that \( \psi(0) = 0 \), \( \beta \in \mathbb{R} \) and

\[
\tau \sim \frac{1}{n}.
\]

(2.33)

By assumption \( \psi \in \mathcal{D}(\mathcal{H}_\beta) \cap \mathcal{D}(\mathcal{K}_{\sigma,\beta}) \) and we are thus in position to apply Lemma 2.4, yielding

\[
\left\| e^{-i \xi_{\beta} \tau} - e^{-i \xi_{\sigma,\beta} \tau} \psi \right\|_2
\]

\[
= \left( \frac{k}{|\tau|} \right)^k \sup_{\chi \in L^2(\mathbb{R}^3), \| \chi \|_2 \leq 1} \left\| \chi \left| \left( \mathcal{H}_\beta - \frac{i k}{\tau} \right)^{-k} - \left( \mathcal{K}_{\sigma,\beta} - \frac{i k}{\tau} \right)^{-k} \right| \psi \right\|_2 + |\tau| o_k(1),
\]

(2.34)

uniformly in \( \sigma \).

Now we claim that, given two self-adjoint operators \( A \) and \( B \), \( z \in \rho(A) \cap \rho(B) \) and \( k \in \mathbb{N} \), if there exists some \( \delta_z < +\infty \) such that, for any \( \phi, \psi \in L^2(\mathbb{R}^3) \), such that \( \| \psi \|_2 \leq 1 \), \( \| \phi \|_2 \leq 1 \),

\[
|\langle \phi | \left( (A - z)^{-1} - (B - z)^{-1} \right) \psi \rangle | \leq \delta_z \| \phi \|_2 \| \psi \|_2,
\]

(2.35)

then

\[
|\langle \phi | \left( (A - z)^{-k} - (B - z)^{-k} \right) \psi \rangle | \leq \frac{k \delta_z}{|\text{Im}(z)|^{k-1}}.
\]

(2.36)

The result can be proven by induction writing

\[
(A - z)^{-k} - (B - z)^{-k} = ((A - z)^{-1} - (B - z)^{-1}) (A - z)^{-k+1}
\]

\[
+ (B - z)^{-1} \left( (A - z)^{-k+1} - (B - z)^{-k+1} \right),
\]

and using the inequalities

\[
\| (A - z)^{-1} \| \leq \frac{1}{|\text{Im}(z)|}, \quad \| (B - z)^{-1} \| \leq \frac{1}{|\text{Im}(z)|}.
\]
in the consequent estimate

\[
\left| \langle \phi \left( (A - z)^{-k} - (B - z)^{-k} \right) \psi \rangle \right|_2 \leq \left| \langle \phi \left( (A - z)^{-1} - (B - z)^{-1} \right) (A - z)^{-k+1} \psi \rangle \right|_2 + \left| \langle (B - z)^{-1} \phi \left( (A - z)^{-k+1} - (B - z)^{-k+1} \right) \psi \rangle \right|_2
\] 

\[
\leq \frac{k \delta_{n,k,\sigma}}{|\tau|_2} + \left| \langle (B - z)^{-1} \phi \left( (A - z)^{-k+1} - (B - z)^{-k+1} \right) \psi \rangle \right|_2,
\]

which leads to (2.36) by the induction hypothesis.

Therefore, using (2.36) in (2.34), we get (keeping in mind that \( \psi \) has norm one)

\[
\left\| e^{i\tau \sigma} - e^{-i\sigma \tau} \right\|_2 \leq \frac{k^2 \delta_{n,k,\sigma}}{|\tau|} + |\tau| \rho_k(1),
\] (2.37)

where \( \delta_{n,k,\sigma} \) is such that, for any normalized \( \phi, \psi \in L^2(\mathbb{R}^3) \),

\[
\left\| \phi \left( (\mathcal{H} - i \frac{k}{\tau})^{-1} - (\mathcal{K}_{\sigma,\beta} - i \frac{k}{\tau})^{-1} \right) \psi \right\|_2 \leq \delta_{n,k,\sigma}.
\] (2.38)

Notice that we already know that such a \( \delta_{n,k,\sigma} \) does exist thanks to the norm resolvent convergence of \( \mathcal{K}_{\sigma,\beta} \) to \( \mathcal{H}_\beta \) stated in [AGH-KH] Thm. 1.2.5] and, more precisely, \( \delta_{n,k,\sigma} \to 0 \), as \( \sigma \to 0 \), for fixed \( n, k \in \mathbb{N} \). However, we are going to estimate the dependence of \( \delta_{n,k,\sigma} \) on the parameters \( n, k, \) and \( \sigma \), showing that

\[
\delta_{n,k,\sigma} = O \left( \sigma^2 \sqrt{n k} \right) + n^{-5/2} k^{-5/2} \rho_{\sigma}(1).
\] (2.39)

In fact, the result is proven by simply tracking down in [AGH-KH] Proof of Thm. 1.2.5] the dependence of the remainders on the spectral parameter. With the same notation as in [AGH-KH], we have (recall the definition (1.30) of \( \nu(\sigma) \))

\[
(\mathcal{H}_\beta + \lambda_n)^{-1} - (-\Delta + \lambda_n)^{-1} =: -\frac{1}{\beta + \sqrt{|\lambda_n|}} |G_{\lambda_n}| \langle G_{\lambda_n} \rangle,
\] (2.40)

\[
(\mathcal{K}_{\sigma,\beta} + \lambda_n)^{-1} - (-\Delta + \lambda_n)^{-1} =: -\sigma \nu(\sigma) A_{\sigma,n} (1 + B_{\sigma,n})^{-1} C_{\sigma,n},
\] (2.41)

where the operators \( A_{\sigma,n}, B_{\sigma,n} \) and \( C_{\sigma,n} \) are defined in (2.34) below, we have set for short

\[
\lambda_n := \frac{i k}{\tau},
\] (2.42)

\( G_{\lambda}(x) := (-\Delta + \lambda)^{-1}(x) \) is the Green function of the Laplacian, \( v = \sqrt{|W|} \), \( u = \text{sgn}(W) \sqrt{|W|} \) and \( \phi_0 \) is the \( L^2 \) solution of the zero-energy equation

\[
u(-\Delta)^{-1} \phi_0 = -\phi_0,
\] (2.43)

which is known to exist and being non-trivial thanks to the resonance condition, which also ensures that \( \langle v \rangle \neq 0 \). Note that, since \( \lambda_n \) is purely imaginary ad its modulus diverges as \( n, k \to +\infty \), the Green function \( G_{\lambda_n} \) belongs to \( L^2(\mathbb{R}^3) \) uniformly in \( n \) and \( k \). The operators \( A_{\sigma,n}, B_{\sigma,n} \) and \( C_{\sigma,n} \) are integral operators whose kernels are given by (see also [AGH-KH] Dafs. (1.2.12) – (1.2.14)])

\[
A_{\sigma,n}(x, x') := G_{\lambda_n}(x - \sigma x') v(x'),
\] (2.44)

\[
B_{\sigma,n}(x, x') := \nu(\sigma) u(x) G_{\sigma^2 \lambda_n}(x - x') v(x'),
\] (2.45)

\[
C_{\sigma,n}(x, x') := u(x) G_{\lambda_n}(\varepsilon x - x').
\] (2.46)

At fixed \( n \) and \( k \) it is not difficult to see [AGH-KH] Lemma 1.2.2 that

\[
A_{\sigma,n} \to A_n := \langle G_{\lambda_n} \rangle \langle v \rangle, \tag{2.47}
\]

\[
B_{\sigma,n} \to B := u(-\Delta)^{-1} v, \tag{2.48}
\]

\[
C_{\sigma,n} \to C_n := \langle u \rangle \langle G_{\lambda_n} \rangle, \tag{2.49}
\]

where \( u \) and \( v \) are meant as the multiplication operators by \( u \) and \( v \), respectively. The convergences above can be proven in Hilbert-Schmidt norm. Note that the operator \( B \) is independent of \( n \) and \( k \).
In order to estimate the difference between the resolvent we use (2.40) and (2.41) to write
\[
\left\langle \phi \left| \left( (\mathcal{H}_0 + \lambda_n)^{-1} - (\mathcal{H}_0 + \lambda_n)^{-1} \right) \psi \right\rangle \right|_2 = -\sigma \nu(\sigma) \left\langle \phi \left| (A_{\sigma,n} - A_n) \left( 1 + B_{\sigma,n} \right)^{-1} C_{\sigma,n} \psi \right\rangle \right|_2
\]
\[
\quad - \sigma \nu(\sigma) \left\langle \psi \left| A_n \left( 1 + B_{\sigma,n} \right)^{-1} (C_{\sigma,n} - C_n) \psi \right\rangle \right|_2
\]
\[
\quad - \left\langle \phi \left| \sigma \nu(\sigma) A_n \left( 1 + B_{\sigma,n} \right)^{-1} C_n - \frac{1}{\beta + \frac{2\lambda_n}{\pi}} \left| G_{\lambda_n} \right| \left| G_{\lambda_n} \right| \psi \right\rangle \right|_2 .
\]

Note that the quantity \( \beta + \frac{\sqrt{\lambda_n}}{\pi} \) is invertible because
\[
\text{Im} \left( \beta + \frac{\sqrt{\lambda_n}}{\pi} \right) = \frac{1}{\pi} \sqrt{\frac{k}{\tau_n}} \text{Im} \left( \sqrt{t} \right) \neq 0,
\]
for any \( k, n \). Terms (1) and (2) above are the easiest to bound and can in fact bounded in the very same way: since however the bound requires an estimate of the norm of \( B_{\sigma,n} \), which is also involved in term (3), we start from this last one. By (2.47) and (2.49),
\[
\left\langle \phi \left| \left[ \sigma \nu(\sigma) A_n \left( 1 + B_{\sigma,n} \right)^{-1} C_n - \frac{1}{\beta + \frac{2\lambda_n}{\pi}} \left| G_{\lambda_n} \right| \left| G_{\lambda_n} \right| \right] \psi \right\rangle \right|_2
\]
\[
= \left\langle \phi \left| G_{\lambda_n} \right|_2 \left( G_{\lambda_n} \right| \psi \right|_2 \left\langle \psi \left| \sigma \nu(\sigma) \left( 1 + B_{\sigma,n} \right)^{-1} \right| u \right\rangle \right|_2 - \frac{1}{\beta + \frac{2\lambda_n}{\pi}} \right|_2 \leq \frac{C}{|\lambda_n|} \| \phi \|_2 \| \psi \|_2 \left\langle \psi \left| \sigma \nu(\sigma) \left( 1 + B_{\sigma,n} \right)^{-1} \right| u \right\rangle \right|_2 - \frac{1}{\beta + \frac{2\lambda_n}{\pi}} .
\]

where we have used that \( \| G_{\lambda_n} \|_2 \leq C |\lambda_n|^{-1/4} \). The estimate of the last factor can essentially be done as in [AGH-KH] Lemma 1.2.4: expanding around \( \sigma = 0 \), we get
\[
\nu(\sigma) G_{\sigma^2 \lambda_n} (x - x') = \left( 1 + \sigma \nu'(\sigma \theta_1(\sigma)) \right) G_0(x - x') - \frac{\sigma \sqrt{\lambda_n}}{4\pi} e^{-\sqrt{\lambda_n} \sigma \theta_2(\sigma)} |x - x'| \]
\[
= \left( 1 + \beta \sigma + o(\sigma) \right) G_0(x - x') - \frac{\sigma \sqrt{\lambda_n}}{4\pi} e^{-\sqrt{\lambda_n} \sigma \theta_2(\sigma)} |x - x'| ,
\]
for some \( 0 \leq \theta_1, \theta_2 \leq 1 \) and the remainder \( o(\sigma) \) in the first term is uniform in \( n \). If we plug the above expansion in the expression of \( B_{\sigma,n} \), we deduce
\[
B_{\sigma,n} = (1 + \beta \sigma) B - \frac{\sigma \sqrt{\lambda_n}}{4\pi} |u| \langle u \rangle + O (\sigma^2 |\lambda_n|) + o(\sigma),
\]
where we have estimated the Hilbert-Schmidt norm (recall that \( W \) is smooth and compactly supported).
\[
\left\| B_{\sigma,n} - (1 + \sigma \beta) B + \frac{\sigma \sqrt{\lambda_n}}{4\pi} |u| \langle u \rangle \right\|_2^2 \leq C \sigma^2 |\lambda_n| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dxdx' \left| 1 - (1 + \beta \sigma) e^{-\sqrt{\lambda_n} \sigma \theta_2(\sigma)} |x - x'| \right|^2 \left| W(x) \right| \left| W(x') \right| \leq C \sigma^4 \left( |\lambda_n|^2 + |\lambda_n| \right) .
\]

We now reproduce the same estimates as in [AGH-KH] eqs. (1.245) – (1.247): setting \( B_1 := \beta B_0 - (4\pi)^{-1} \sqrt{\lambda_n} |u| \langle u \rangle \) and denoting by \( \epsilon_\sigma \) the error \( O(\sigma^2 \lambda_n) + o(\sigma) \) in (2.52), we get
\[
\sigma (1 + B_{\sigma,n})^{-1} = \sigma (1 + B_0 + \sigma B_1 + \epsilon_\sigma)^{-1} = (1 + \sigma (1 + \sigma + B_0)^{-1} (B_1 - 1))^{-1} (1 + \sigma + B_0)^{-1} + \epsilon_\sigma .
\]

Let \( \phi \) be the resonant function appearing in [Definition 1.6] and set \( P := -|\phi\rangle \langle \phi| \), where \( \phi := \text{sgn}(W)\phi \), we have the expansion
\[
\sigma (1 + \sigma + B_0)^{-1} = P + O(\sigma),
\]
which plugged in the previous expression yields
\[ \sigma (1 + B_{\sigma,n})^{-1} - \epsilon_\sigma = (1 + (P + \tilde{O}(\sigma))(B_1 - 1))^{-1} (P + \tilde{O}(\sigma)) \]
\[ = (1 + P(B_1 - 1) + \tilde{O}(\sigma))^{-1} P + \tilde{O}(\sigma) \]
\[ = \left( 1 + \left( \beta + \frac{\sqrt{n}}{4\pi} |\langle v | \phi \rangle|^2 \right)^{-1} (1 - \beta) |\phi \rangle \langle \phi | + \frac{\sqrt{n}}{4\pi} \left( \beta + \frac{\sqrt{n}}{4\pi} |\langle v | \phi \rangle|^2 \right)^{-1} \langle \phi | v \rangle |\phi \rangle \langle v | \right) P + \tilde{O}(\sigma) \]
\[ = - \left( \beta + \frac{\sqrt{n}}{4\pi} |\langle v | \phi \rangle|^2 \right)^{-1} P + \tilde{O}(\sigma), \]
or, equivalently,
\[ \sigma (1 + B_{\sigma,n})^{-1} = - \left( \beta + \frac{\sqrt{n}}{4\pi} |\langle v | \phi \rangle|^2 \right)^{-1} P + \tilde{O}(\sigma) + O(\sigma^2 \lambda_n) + o(\sigma). \quad (2.53) \]

Therefore, we have
\[ |(3)| = O(\sigma^2 |\lambda_n|^{1/2}) + O(\sigma |\lambda_n|^{-1/2}) + o(\sigma) |\lambda_n|^{-1/2} \quad (2.54) \]
which, imposing the condition (since we are going to choose \( k = n \) later, this would result in the condition \( \sigma n \ll 1 \) in the statement)
\[ \sigma \sqrt{|\lambda_n|} = C \sigma \sqrt{kn} \ll 1, \quad (2.55) \]
becomes
\[ |(3)| = O(\sigma^2 |\lambda_n|^{1/2}) + o(\sigma). \quad (2.56) \]

Note that this implies that
\[ \|\sigma v(\sigma)(1 + B_{\sigma,n})^{-1}\| \leq C |\lambda_n|^{-1/2}. \quad (2.57) \]

Let us now consider the terms (1) and (2) and, since the argument is the same, let us focus on (1). A direct inspection of [AGH-KH, Lemma 1.2.2] reveals that the Hilbert-Schmidt convergence of \( A_{\sigma,n} \) to \( A_n \) is in fact uniform in \( n \) and \( k \). More precisely, the dependence on \( \lambda_n \) can be easily scaled out: by setting \( y = \sqrt{|\lambda_n|} x \) and \( y' = \sqrt{|\lambda_n|} x' \), one has
\[ |\lambda_n|^{-3/4} \|(A_{\sigma,n} - A)\tilde{\psi} \| (|\lambda_n|^{-1/2} y) = |\lambda_n|^{-1} \int_{\mathbb{R}^3} dy' \left[ G_i(y) - G_i(y - \sigma y') \right] v(|\lambda_n|^{-1/2} y') \tilde{\psi}(y') \]
\[ =: |\lambda_n|^{-1} \left( D_{\sigma,n} \tilde{\psi} \right)(y), \quad (2.58) \]
where we have set
\[ \tilde{\psi}(y) = |\lambda_n|^{-3/4} \psi(|\lambda_n|^{-1/2} y), \]
so that the \( L^2 \) norms of the function is preserved. Exploiting then the smoothness and boundedness of \( v \), one can apply the argument of [AGH-KH, Proof of Lemma 1.2.2] to show that
\[ \lim_{\sigma \to 0} \|D_{\sigma,n}\| = 0, \quad \text{uniformly in } n \in \mathbb{N}. \quad (2.59) \]
Moreover, by the very same scaling argument, one can also easily show that
\[ \|A_{\sigma,n}\| \leq \frac{C}{|\lambda_n|}, \quad \|C_{\sigma,n}\| \leq \frac{C}{|\lambda_n|}. \quad (2.60) \]

Therefore, using (2.57) and the above estimates, we obtain
\[ |(1)| = k^{-5/2} n^{-5/2} o_n(1), \quad (2.61) \]
which combined with (2.56) yields (2.39).

Hence, if \( \tau \sim \frac{1}{n}, \quad (2.37) \) becomes
\[ \left\| e^{-i2\lambda \tau} - e^{-i2\lambda_n \tau} \tilde{\psi} \right\|_2 \leq C \left( k^{5/2} n^{3/2} \sigma^2 + k^{-1/2} n^{-3/2} o_n(1) \right) + |n^{-1}| o_k(1). \]

If we finally choose \( k = n \) for simplicity, we deduce
\[ \left\| e^{-i2\lambda \tau} - e^{-i2\lambda_n \tau} \tilde{\psi} \right\|_2 \leq C n^4 \sigma^2 + n^{-2} o_n(1) + o_n(n^{-1}), \]
and plugging this in (2.24), we get for any $\phi, \psi \in \mathcal{D}$ normalized,

$$\langle \phi \mid (V_n(t,s) - V_{n,\sigma}(t,s)) \psi \rangle \leq Cn^{5} \sigma^{2} + n^{-1} o_\sigma(1) + o_\sigma(1).$$

\[\square\]

Finally, we address the approximation à la Yoshida for the two-parameter unitary group $\mathcal{U}_\sigma(t,s)$ generated by the approximant operators $\mathcal{K}_{\beta(t),\sigma}$ (recall the definition (1.29) of the operator $\mathcal{K}_{\beta,\sigma}$). Before addressing the main question, it is useful to state one more technical Lemma.

**Lemma 2.6.**
Let $\beta \in C(\mathbb{R})$ and $\beta' \in \mathbb{R}$ finite, with $\beta(s) = \beta'$. Let also $\psi \in \mathcal{D}$ with $\|\psi\|_2 \leq 1$. Then, for any finite $s < t \in \mathbb{R}$, with $t - s \gg \sigma$, and for any $0 < \epsilon < 1$,

$$\left\| \left( \mathcal{U}_\sigma(t,s) - e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \right) \psi \right\|_2^2 = \mathcal{O}(\sigma^{-2}(t-s)^{8-\epsilon}).$$

(2.62)

**Proof.** Since at time $\tau = 0$ the l.h.s. of (2.62) vanishes, it is sufficient to estimate its time derivative:

$$\partial_t \left\| \left( \mathcal{U}_\sigma(t,s) - e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \right) \psi \right\|_2^2 = -2\text{Re} \left[ \partial_t \left( \mathcal{U}_\sigma(t,s) \psi \right) \left| e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \psi \right| \right]$$

$$= 2\sigma (\beta' - \beta(t)) \text{Im} \left( \left| e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \psi \right| \left( \mathcal{W}_\sigma \left( e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} - \mathcal{U}_\sigma(t,s) \right) \psi \right) \right),$$

(2.63)

where we used that the expectation of $\mathcal{W}_\sigma$ is real. Hence, exploiting the differentiability of $\beta(t)$ to bound $|\beta(t) - \beta'| \leq C(t-s)$, we get

$$\partial_t \left\| \left( \mathcal{U}_\sigma(t,s) - e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \right) \psi \right\|_2^2 \leq C(\sigma(t-s)) \left\| \mathcal{W}_\sigma e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \psi \right\|_2 \left\| \left( \mathcal{U}_\sigma(t,s) - e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \right) \psi \right\|_2.$$ 

Now, we claim that, for any $\epsilon > 0$ for any $s, t \in \mathbb{R}$ such that $t - s \gg \sigma$,

$$\left\| \mathcal{W}_\sigma e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \psi \right\|_2 = \mathcal{O}(\sigma^{-2}(t-s)^{8-\epsilon}),$$

(2.64)

so that we obtain the result, i.e.,

$$\left\| \left( \mathcal{U}_\sigma(t,s) - e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \right) \psi \right\|_2 \leq \mathcal{O}(\sigma(t-s)) \int_s^t d\tau \left\| \mathcal{W}_\sigma e^{-i\mathcal{K}_{\beta',\sigma}(\tau-s)} \psi \right\|_2 \leq C\sigma^{-1}(t-s)^{4-\epsilon}. \quad (2.65)$$

Let us now prove (2.64). We first observe that, for any $\psi \in \mathcal{D}$ and for some $a > 5/2$,

$$\left\| \mathcal{W}_\sigma \psi \right\|_2^2 = \frac{1}{\sigma} \int_{\mathbb{R}^3} dx \mathcal{W}^2(x) |\psi(\sigma x)|^2 = \mathcal{O}(\sigma^{2a-1}), \quad (2.66)$$

$$\left\| \mathcal{W}^2 \psi \right\|_2^2 = \frac{1}{\sigma^5} \int_{\mathbb{R}^3} dx \mathcal{W}^4(x) |\psi(\sigma x)|^2 = \mathcal{O}(\sigma^{2a-5}),$$

(2.67)

since $W$ is compactly supported and by (2.1). Furthermore,

$$\partial_t \left\| \mathcal{W}_\sigma e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \psi \right\|_2^2 = 2\text{Im} \left( \langle \mathcal{K}_{\beta',\sigma} \psi(t) \mid \mathcal{W}_\sigma^2 \psi(t) \rangle \right), \quad (2.68)$$

where we abbreviated $\psi(t) := e^{-i\mathcal{K}_{\beta',\sigma}(t-s)} \psi$. Hence,

$$\partial_t \left\| \mathcal{W}_\sigma \psi(t) \right\|_2^2 \leq C \left\| \mathcal{K}_{\beta',\sigma} \psi \right\|_2 \left\| \mathcal{W}^2 \psi(t) \right\|_2 \leq C \left( 1 + \sigma^{a-\frac{1}{2}} \right) \sigma^{-\frac{5}{2}} = \mathcal{O}(\sigma^{\alpha-\frac{5}{2}}).$$

(2.69)

Finally, we compute now and estimate the second derivative of the quantity in (2.64):

$$\partial_t^2 \left\| \mathcal{W}_\sigma \psi(t) \right\|_2^2 = 2\text{Re} \left\{ \langle \mathcal{K}_{\beta',\sigma}^2 \psi(t) \mid \mathcal{W}_\sigma^2 \psi(t) \rangle - \langle \mathcal{K}_{\beta',\sigma} \psi(t) \mid \mathcal{W}_\sigma^2 \mathcal{K}_{\beta',\sigma} \psi(t) \rangle \right\}$$

$$\leq 2 \left\| \mathcal{K}_{\beta',\sigma}^2 \psi(t) \right\|_2 \left\| \mathcal{W}^2 \psi(t) \right\|_2 \leq \frac{C}{\sigma^2} \left\| \mathcal{K}_{\beta',\sigma}^2 \psi(t) \right\|_2 \left\| \mathcal{W}_\sigma \psi(t) \right\|_2 \leq \frac{C}{\sigma^2} \left\| \mathcal{W}_\sigma \psi(t) \right\|_2,$$

(2.70)

where we have bounded the $L^\infty$ norm of $\mathcal{W}_\sigma$ by $C/\sigma^2$ and estimated

$$\left\| \mathcal{K}_{\beta',\sigma}^2 \psi(t) \right\|_2 = \left\| \mathcal{K}_{\beta',\sigma}^2 \psi(s) \right\|_2 \leq C \left[ \left\| (\Delta)^2 \psi \right\|_2 + \left\| \mathcal{W}_\sigma^2 \psi \right\|_2 \right] = \mathcal{O}(1) + \mathcal{O}(\sigma^{a-\frac{5}{2}}).$$

(2.71)
The idea is to start with a preliminary bound on \( \| W_\sigma \psi(t) \|_2 \) and then refine recursively through (2.66), (2.69) and (2.70). The starting point is the bound
\[
\left\| W_\sigma e^{-iK_{\beta',\sigma}(t-s)} \psi \right\|_2 = O(\sigma^{-2}(t-s)),
\]  
which can be proven as follows: using the \( L^\infty \) bound on \( W_\sigma \) in (2.68), we get
\[
\partial_t \left\| W_\sigma \psi(t) \right\|_2^2 \leq \frac{C}{\sigma^2} \left\| K_{\beta',\sigma}(t-s) \right\|_2 \left\| W_\sigma \psi(t) \right\|_2,
\]  
yielding \( \partial_t \left\| W_\sigma \psi(t) \right\|_2 \leq C\sigma^{-2} \), which implies (2.72) via (2.66) and the condition \( t-s \gg \sigma \). Plugging (2.72) into (2.70) and combining it with (2.66) and (2.69), we find
\[
\left\| W_\sigma \psi(t) \right\|_2^2 = O(\sigma^{2a-1}) + O(\sigma^{a-\frac{5}{2}(t-s)}) + O(\sigma^{-4}(t-s)^3) = O(\sigma^{-4}(t-s)^3).
\]  
If now we plug the above in place of (2.72) into (2.70), we end up with the bound \( \left\| W_\sigma \psi(t) \right\|_2 = O(\sigma^{-4}(t-s)^7/2), \) i.e., at each step we improve the dependence on \( t-s \). After a direct check, one realizes that after \( N \in \mathbb{N} \) steps of the bootstrap, the exponent of \( (t-s) \) in the estimate above reads \( 2 + \sum_{k=0}^{N-1} \frac{1}{2^k} \), which yields (2.64), after suitably large number of repetitions of the argument. 

We are now in position to prove the last estimate on the Yoshida approximation for the dynamics generated by \( K_{\beta(t),\sigma} \).

**Proposition 2.7** (Step function approximation of \( \mathcal{U}_\sigma(t,s) \)).

Let \( \beta \in C(\mathbb{R}) \) and let \( \psi \in \mathcal{D} \) with \( \| \psi \|_2 = 1 \). Then, for any finite \( s < t \in \mathbb{R} \) and for any \( 0 < \epsilon < 1 \), there exists a constant \( C > 0 \) independent of \( t,s \)
\[
\| (V_{\sigma,n}(t,s) - \mathcal{U}_\sigma(t,s)) \psi \|_2^2 = O(n^{-6+\epsilon} \sigma^{-2}).
\]  

**Proof.** By a direct application of [Lemma 2.6](#) with \( t = t_1 \), we get
\[
\| (V_{\sigma,n}(t_1,s) - \mathcal{U}_\sigma(t_1,s)) \psi \|_2^2 = O(n^{-8+\epsilon} \sigma^{-2})
\]  
by differentiability of \( \beta(t) \). Note the condition \( n\sigma \ll 1 \) inherited from \( t_1 - s \gg \sigma \) in the statement of [Lemma 2.6](#). Furthermore, we claim that, for any \( j \in \{1, \ldots, n-1\} \),
\[
\| (V_{\sigma,n}(t_j+1,s) - \mathcal{U}_\sigma(t_j+1,s)) \psi \|_2^2 \leq \| (V_{\sigma,n}(t_j,s) - \mathcal{U}_\sigma(t_j,s)) \psi \|_2 + Cn^{-4+\epsilon/2} \sigma^{-1},
\]  
and the result then follows by a trivial recursive argument. In order to show that the above estimate holds true, we write
\[
\| (\mathcal{U}_\sigma(t_j+1,s) - V_{\sigma,n}(t_j+1,s)) \psi \|_2^2 \\
\leq \| (\mathcal{U}_\sigma(t_j+1,s) - \mathcal{U}_\sigma(t_j+1,t_j) V_{n,\sigma}(t_j,s)) \psi \|_2 + \| (\mathcal{U}_\sigma(t_j+1,t_j) V_{n,\sigma}(t_j,s) - V_{\sigma,n}(t_j+1,s)) \psi \|_2 \\
\leq \| (\mathcal{U}_\sigma(t_j,s) - V_{\sigma,n}(t_j,s)) \psi \|_2 + \| (\mathcal{U}_\sigma(t_j+1,t_j) - V_{\sigma,n}(t_j+1,t_j)) \psi(t_j) \|_2 \\
\leq \| (\mathcal{U}_\sigma(t_j,s) - V_{\sigma,n}(t_j,s)) \psi \|_2 + \| (\mathcal{U}_\sigma(t_j+1,t_j) - V_{\sigma,n}(t_j+1,t_j)) \psi \|_2 + Cn^{-2/3}
\]  
where we have set \( \psi(t_j) := V_{n,\sigma}(t_j,s) \psi \) for short and applied once more [Lemma 2.2](#). Now, we can use [Lemma 2.6](#) to bound the second term in the expression above and choose \( m \gg n^{6-3\epsilon} \sigma^{3/2} \) to get that the sum of last two terms above is bounded by \( Cn^{-4+\epsilon/2} \sigma^{-1} \).

We now complete the proof of the main result in this Section.

**Proof of Theorem 1.7.** The idea is to prove the result in three steps: we first replace \( \mathcal{U}_{\text{eff}}(t,s) \) with its Yoshida approximants; the resulting dynamics is then generated by a time-independent point interaction and, as such, can be approximated by \( V_{n,\sigma}(t,s) \), i.e., the dynamics generated by \( K_{3_j,\sigma} \) in the corresponding interval; finally, we undo the step function approximation of \( \beta(t) \) and obtain \( \mathcal{U}_\sigma(t,s) \).

To prove strong convergence of unitary operators, it is sufficient to prove weak convergence on a dense subset. This can be easily proved as follows. Let \( V_n \xrightarrow{n \to \infty} V \) on a dense subset \( \mathcal{D} \subset \mathcal{H} \).
In addition, given $\psi \in \mathcal{H}$, let us denote by $\{\psi_m\}_{m \in \mathbb{N}}$ its approximation in $\mathcal{D}$. Then
\[
\frac{1}{2}\| (V - V_n) \psi \|^2 \leq \| (V - V_n) \psi_m \|^2 + \| (V - V_n) (\psi - \psi_m) \|^2
\leq \langle \psi_m \mid (2 - V^* V_n - V_n^* V) \psi_m \rangle_2 + 4 \| \psi - \psi_m \|^2
\leq 2 \text{Re} \langle V \psi_m \mid (V - V_n) \psi_m \rangle_2 + 4 \| \psi - \psi_m \|^2
\]
that converges to zero as $n \to \infty$, since $m$ can be chosen arbitrarily large. Hence, it is sufficient to prove the convergence for all $\psi, \phi \in \mathcal{D}$.

Now, for any $\psi, \phi \in \mathcal{D}$ and for all $t, s \in \mathbb{R}$ with $s < t$:
\[
|\langle \phi \mid (\mathcal{U}_\text{eff}(t, s) - \mathcal{U}_\sigma(t, s)) \psi \rangle| \leq |\langle \phi \mid (\mathcal{U}_\text{eff}(t, s) - \mathcal{V}_n(t, s)) \psi \rangle|
+ |\langle \phi \mid (\mathcal{V}_n(t, s) - \mathcal{V}_n,\sigma(t, s)) \psi \rangle| + |\langle \phi \mid (\mathcal{V}_n,\sigma(t, s) - \mathcal{U}_\sigma(t, s)) \psi \rangle|.
\]
Hence, using the results proven in Propositions 2.1, 2.5 and 2.7, respectively, we get
\[
|\langle \phi \mid (\mathcal{U}_\text{eff}(t, s) - \mathcal{U}_\sigma(t, s)) \psi \rangle| = O(n^5 \sigma^2) + n^{-1} o_n(1) + o_n(1) + O(n^{-3+\epsilon/2} \sigma^{-1})
\]
provided that $\sigma n \ll 1$. If we now optimize the first and last terms, i.e., we pick
\[
n = n_\sigma = \sigma^{-3/(8-\epsilon/2)} \xrightarrow{\sigma \to 0} +\infty,
\]
for $\epsilon > 0$ small enough the condition $n\sigma \ll 1$ is satisfied, and
\[
|\langle \phi \mid (\mathcal{U}_\text{eff}(t, s) - \mathcal{U}_\sigma(t, s)) \psi \rangle| = O(\sigma^{1/8-C_\epsilon}) + o_\sigma(1) \xrightarrow{\sigma \to 0} 0.
\]

\[\square\]

3. Convergence of Fluctuations

In this Sect. we prove the quasi-classical convergence, in strong topology, of the unitary operator of microscopic coherent quantum fluctuations, perturbing the quasi-classical solution. The key idea is to use coherent states that, in the quasi-classical limit, are “singular enough” to produce an effective point interaction. The strong convergence of fluctuations is sufficient to prove the strong convergence of evolved particle observables given in [Theorem 1.1]. Let us start with some preliminary definitions and remarks.

In order to have a more compact notation for the two phonon fields, we introduce a single boson field encompassing both. As it is well known, $\Gamma_{\text{sym}}(\delta) \otimes \Gamma_{\text{sym}}(\tilde{\delta}) \cong \Gamma_{\text{sym}}(\delta \oplus \delta)$. With this identification, we can introduce vector creation and annihilation operators
\[
a_{\lambda}^\dagger(\eta) = a_{\lambda}^\dagger(\eta_1) + b_{\lambda}^\dagger(\eta_2), \quad \forall \eta = (\eta_1, \eta_2) \in \delta \oplus \tilde{\delta};
\]
and analogously the second quantization
\[
d\Gamma_{\xi}(h) = d\Gamma_{\xi}^{(a)}(h_1) + d\Gamma_{\xi}^{(b)}(h_2),
\]
where $h_1, h_2$ are self-adjoint operators on $\delta$. Hence it follows that, defining $\lambda_{\chi} = (\chi^{(a)}_\lambda, \chi^{(b)}_\lambda)$, and $\omega_{\chi} = (\omega, \tilde{\omega})$, the Hamiltonian $H_{\chi}$ can be rewritten in the compact form:
\[
H_{\chi} = -\Delta + d\Gamma_{\xi}(\omega_{\chi}) + a_{\lambda}(\lambda_{\chi}) + a_{\lambda}^\dagger(\lambda_{\chi}).
\]
Let us remark that the form factor $\lambda_{\chi}(k) = e^{ik \cdot x}(\lambda_0(k), k^{-1})$ has the following important property, as first remarked in [LT]:
\[
\lambda_{\chi} = \lambda_{\chi} > \chi + \lambda_{\chi} < \chi;
\]
\[
\lambda_{\chi} > \chi \in L^\infty(\mathbb{R}^3; \delta \oplus \tilde{\delta});
\]
\[
\lambda_{\chi} < \chi \in \{-i \nabla, \xi_{\chi}\}, \quad \xi_{\chi} \in L^\infty(\mathbb{R}^3; (\delta \oplus \tilde{\delta})^3).
\]
More precisely, for all $r > 0$, it is possible to make the splitting $\lambda_{\chi} = (\lambda_1, \lambda_2, \chi), \chi = (0, \xi_{\chi}, \xi_{\chi})$ in a way such that $||\xi_{\chi}||_{L^\infty(\mathbb{R}^3; (\delta \oplus \tilde{\delta})^3)} \xrightarrow{r \to +\infty} 0$ and $||\lambda_{\chi} < ||_{L^\infty(\mathbb{R}^3; \delta)} \xrightarrow{r \to +\infty} +\infty$.

Throughout the rest of the paper, whenever a Fock space estimate is used, it will be a direct application of the following basic estimate, whose well-known proof stems combining a direct calculation and the canonical commutation relations: for any couple of positive self-adjoint operators
\( \tau = (\tau_1, \tau_2) \) on \( \mathfrak{F} \oplus \mathfrak{F} \), for any \( f, g \) such that \( f, g, \tau^{-\frac{1}{2}}f, \tau^{-\frac{1}{2}}g \in L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3) \), and for any \( 0 < \varepsilon \leq 1 \),

\[
\left\| \left( a_f^\dagger (f) + a_{g}^\dagger (g) \right) (d\Gamma_{\varepsilon}(\tau) + 1)^{-\frac{1}{2}} \right\| \leq C \left( \left\| \tau^{-\frac{1}{2}}f \right\|_{L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3)} + \left\| \tau^{-\frac{1}{2}}g \right\|_{L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3)} \right. \\
\left. + \varepsilon \left\| g \right\|_{L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3)} \right),
\]

where the norm on the l.h.s. is the operator norm on \( \mathfrak{H} \). As a straightforward application, making use of (3.3) this inequality yields that \( H_{\varepsilon} \) is a densely defined quadratic form on \( \mathcal{D}[H_0] \): there exist \( A > 0 \) such that for all \( \Theta \in \mathcal{D}[H_0] \), and for all \( \delta > 0 \), there exists \( B_\delta > 0 \) such that, uniformly w.r.t. \( \varepsilon \in (0, 1) \),

\[
(\Theta | H_{\varepsilon} | \Theta)_{\mathfrak{H}} \leq A \left( \delta \left\| \mathbf{\lambda}_\varepsilon \right\|_{L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3)} + \left\| \xi_{2, r} \right\|_{L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3)} \right) (\Theta | H_0 | \Theta)_{\mathfrak{H}} + B_\delta \left\| \Theta \right\|^2_{\mathfrak{H}},
\]

where \( B_\delta \) depends additionally on \( \left\| \mathbf{\lambda}_\varepsilon \right\|_{L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3)} \) and \( \left\| \xi_{2, r} \right\|_{L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3)} \). Furthermore, it is possible to choose \( \delta > 0 \) and \( r > 0 \) in the above bound such that

\[
A \left( \delta \left\| \mathbf{\lambda}_\varepsilon \right\|_{L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3)} + \left\| \xi_{2, r} \right\|_{L^\infty(\mathbb{R}^3; (\mathfrak{F} \oplus \mathfrak{F})^3)} \right) < 1,
\]

so that the quadratic form induced by \( H_{\varepsilon} \) is bounded from below and symmetric by KLMN’s theorem. As already remarked, the above argument is completely analogous to the one commonly given to prove self-adjointness of the optical polaron model (see, e.g., [FS]).

The operator of fluctuations for coherent states is now defined as follows. Let \( W_{\varepsilon}(\frac{\alpha_{\varepsilon}(t)}{\varepsilon}) = W^{(a)}_{\varepsilon}(\frac{\alpha_{\varepsilon}(t)}{\varepsilon})W^{(b)}_{\varepsilon}(\frac{\alpha_{\varepsilon}(t)}{\varepsilon}) \) be the Weyl operator appearing in the definition of \( \Xi_{\varepsilon} \) (recall (1.20)). Then, the operator of \textit{microscopic fluctuations} \( Z_{\varepsilon}(t, s) \) is defined by

\[
Z_{\varepsilon}(t, s) := \mathcal{W}_{\varepsilon}\left( \frac{\alpha_{\varepsilon}(t)}{\varepsilon} \right) e^{-iH_{\varepsilon}(t-s)} \mathcal{W}_{\varepsilon}\left( \frac{\alpha_{\varepsilon}}{\varepsilon} \right),
\]

where \( \alpha_{\varepsilon}(t) = (\alpha_{\varepsilon}(t), \beta_{\varepsilon}(t)) \) satisfies the classical dynamics (1.27), i.e., recalling (1.22),

\[
\alpha_{\varepsilon}(k; t) := (\alpha_{\varepsilon}(k), e^{-ik(t-s)}\beta_{\varepsilon}(k)).
\]

The strong limit of \( Z_{\varepsilon}(t, s) \) as \( \varepsilon \to 0 \), of which we prove the existence, is the operator of \textit{quasi-classical fluctuations} \( Z(t, s) \), defined by

\[
Z(t, s) := \mathcal{U}_{\text{eff}}(t, s) \otimes e^{-i(t-s)d\Gamma((0, s))} = \mathcal{U}_{\text{eff}}(t, s) \otimes 1 \otimes e^{-i\xi(t-s)}d\Gamma^{(b)}(1),
\]

where \( d\Gamma, d\Gamma^{(b)} \) stand for the second-quantized operators defined in terms of the unscaled creation and annihilation operators \( a^\dagger, b^\dagger \) (recall (1.11)). Therefore, \( Z \) is a factorized unitary operator on the full space \( \mathfrak{H} \), and its factorization is due to the chosen scaling in \( H_{\varepsilon} \), that guarantees no quasi-classical back-reaction on the field.

The relation between the fluctuation operators \( Z_{\varepsilon}(t, s), Z(t, s) \) and the full Heisenberg evolution of a particle observable \( \mathcal{B} \) can be derived as follows. Let \( \mathcal{B} \) be a bounded particle operator, acting on \( L^2(\mathbb{R}^3) \), and let \( \mathcal{B}_{\varepsilon}(t, s) \) and \( \mathcal{B}(t, s) \) be the associated microscopic and quasi-classical Heisenberg evolved operators as defined in (1.18) and (1.19), respectively, then, for any \( t, s \in \mathbb{R} \) and any \( \psi \in L^2(\mathbb{R}^3) \),

\[
\left\| (\mathcal{B}(t, s) - \mathcal{B}_{\varepsilon}(t, s))\psi \right\|_{L^2(\mathbb{R}^3)} = \left\| (B(t, s) - B_{\varepsilon}(t, s))\psi \otimes \Xi_{\varepsilon, s} \right\|_{\mathfrak{H}}^2
\]

\[
= \left\langle \psi \otimes \Xi_{\varepsilon, s} \left\| B(t, s)^2 \right\| + \left| B_{\varepsilon}(t, s) - B_{\varepsilon}^*(t, s)B_{\varepsilon}(t, s) - B_{\varepsilon}^*(t, s)B(t, s) \right| \psi \otimes \Xi_{\varepsilon, s} \right\rangle_{\mathfrak{H}}.
\]

If we now plug in the definition of \( Z \) and \( Z_{\varepsilon} \) and use (1.20), we can write

\[
\left\| (\mathcal{B}(t, s) - \mathcal{B}_{\varepsilon}(t, s))\psi \right\|_{L^2(\mathbb{R}^3)} = \left\langle \psi \otimes \Omega \left| W_{\varepsilon}^\dagger (B(t, s) - B_{\varepsilon}(t, s))^2 W_{\varepsilon} \right| \psi \otimes \Omega \right\rangle_{\mathfrak{H}}
\]

\[
= \left\langle \psi \otimes \Omega \left| Z_{\varepsilon}^\dagger B^2 \otimes 1Z_{\varepsilon} + Z_{\varepsilon}^\dagger B^2 \otimes 1Z_{\varepsilon} - Z_{\varepsilon}^\dagger B \otimes 1Z_{\varepsilon}Z_{\varepsilon}^\dagger B \otimes 1Z_{\varepsilon} - Z_{\varepsilon}^\dagger B \otimes 1Z_{\varepsilon}Z_{\varepsilon}^\dagger B \otimes 1Z_{\varepsilon} \right| \psi \otimes \Omega \right\rangle_{\mathfrak{H}}
\]
where we used the following property of Weyl operators (recall \( \text{1.27} \))
\[
e^{-i(t-s)\Gamma(h)} W_\epsilon \left( \frac{\alpha_{\epsilon}(t)}{\epsilon} \right) = W_\epsilon \left( \frac{e^{-i(t-s)\Gamma(h)}}{\epsilon} \right) e^{-i(t-s)\Gamma(h)} = W_\epsilon \left( \frac{\alpha_{\epsilon}(t)}{\epsilon} \right) e^{-i(t-s)\Gamma(h)},
\]
which, combined with \( \text{1.11} \), i.e., \( e^{-1}\Gamma(h) = \Gamma(h) \), implies, for any particle operator \( A \),
\[
W_\epsilon^\dagger \left( \frac{\alpha_{\epsilon}(t)}{\epsilon} \right) \left( \mathcal{U}_{\text{eff}} \otimes 1 \right) A \otimes 1 \left( \mathcal{U}_{\text{eff}} \otimes 1 \right) W_\epsilon \left( \frac{\alpha_{\epsilon}}{\epsilon} \right) = Z^\dagger A \otimes 1 Z,
\]
and we omit the dependence on \( t \) and \( s \) of \( Z(t,s) \) and \( Z_\epsilon(t,s) \) for convenience. If we now exploit the identity
\[
Z^\dagger B^2 \otimes 1 Z_e + Z^\dagger B^2 \otimes 1 Z - Z^\dagger B \otimes 1 Z Z^\dagger B \otimes 1 Z_e - Z^\dagger B \otimes 1 Z_e Z^\dagger B \otimes 1 Z
\]
\[
= (Z_e^\dagger - Z^\dagger B^2 \otimes 1 Z_e + Z^\dagger B^2 \otimes 1 (Z - Z_e) - Z^\dagger B \otimes 1 Z (Z_e^\dagger - Z^\dagger) B \otimes 1 Z_e + \text{h.c.}
\]
in the expression above, we deduce that
\[
\| (B(t,s) - B_\epsilon(t,s)) \psi \|^2_{L^2(\mathbb{R}^3)} \leq 4 \| B \|^2 \| \psi \|_2 \| \Omega \|_{\Gamma \otimes \Gamma^*} \| (Z - Z_\epsilon) \psi \otimes \Omega \|_{\mathcal{H}}.
\]

The estimate \( \text{3.14} \) makes apparent the link between the Heisenberg evolution of the observable \( B \) and the fluctuation operators \( Z_e, Z \). More precisely, the convergence stated in \[ \text{Theorem 1.1} \] is equivalent to show strong convergence of the fluctuation operator \( Z_\epsilon \) to \( Z \), which we are going to prove in \[ \text{Proposition 3.1} \]. Note that a similar quasi-classical limit of coherent state fluctuations has been studied (with less singular coherent states that do not carry any \( \epsilon \)-dependence on the classical solution \( \alpha \)) for the renormalized Nelson model in \[ \text{GNV} \].

### 3.1. Strong convergence

In order to prove strong convergence of \( Z_\epsilon \), we make use of an intermediate auxiliary operator
\[
Y_\sigma(t,s) = \mathcal{U}_{\sigma}(t,s) \otimes e^{-i(t-s)\Gamma((0,s))},
\]
with \( \sigma = \sigma_\epsilon \) properly chosen, i.e., such that
\[
\epsilon^{1/j} \ll \sigma \ll 1,
\]
and where \( \mathcal{U}_{\sigma} \) is the two-parameter group defined in \[ \text{8.2} \] and generated by \( \mathcal{K}_{\beta(t),\sigma} \). The precise result is given in the following

**Proposition 3.1** (Convergence of fluctuations).
For any \( \Phi \in \mathcal{H} \),
\[
\lim_{\epsilon \to 0} \| (Z(t,s) - Z_\epsilon(t,s)) \Phi \|_{\mathcal{H}} = 0.
\]

Before proving the above result, let us give some preparatory lemmas. The first one is a well-known result about Weyl operators (for an explicit proof, see, e.g., \[ \text{FaI} \]). For the last property, it is useful to remark that \( \alpha_\epsilon(k;t) \) is by construction a rapidly decaying function in Schwartz class and therefore its scalar product by any polynomial, as, e.g., \( \omega \), is always bounded.

**Lemma 3.2.**
The Weyl operators \( W_\epsilon \left( \frac{\alpha_{\epsilon}(t)}{\epsilon} \right) \) are strongly differentiable with respect to \( t \in \mathbb{R} \) on \( \mathcal{D}(d\Gamma(1)^{1/2}) \), with derivative given by
\[
i \partial_t W_\epsilon \left( \frac{\alpha_{\epsilon}(t)}{\epsilon} \right) = \frac{i}{\epsilon} \left( a_\epsilon^\dagger(\alpha_\epsilon) - a_\epsilon(\alpha_\epsilon) - i \text{Im} \langle \alpha_\epsilon | \alpha_\epsilon \rangle_{B \otimes B} \right) W_\epsilon \left( \frac{\alpha_{\epsilon}(t)}{\epsilon} \right)
\]
\[
= \frac{i}{\epsilon} W_\epsilon \left( \frac{\alpha_{\epsilon}(t)}{\epsilon} \right) \left( a_\epsilon^\dagger(\alpha_\epsilon) - a_\epsilon(\alpha_\epsilon) + i \text{Im} \langle \alpha_\epsilon | \alpha_\epsilon \rangle_{B \otimes B} \right).
\]

In addition, \( W_\epsilon(z) \) maps \( \mathcal{D} |H_0| \) and \( \mathcal{D}(d\Gamma((h,h'))^{1/2}) \) into themselves for any self-adjoint and positive \( h, h' \), provided that \( z \in \mathcal{D}(h) \oplus \mathcal{D}(h') \).

Another useful result is the weak differentiability of \( Z_\epsilon \) in a suitable dense domain.
Lemma 3.3.

The operator $Z_\varepsilon(t,s)$ is weakly differentiable, with respect to both $t \in \mathbb{R}$ and $s \in \mathbb{R}$, on $\mathcal{D}[H_0] \cap \mathcal{D}(d\Gamma_\varepsilon(1))$. The weak derivatives have the following form:

$$i\partial_t Z_\varepsilon(t,s) = L_\varepsilon(t)Z_\varepsilon(t,s),$$
$$i\partial_s Z_\varepsilon(t,s) = -Z_\varepsilon(t,s)L_\varepsilon(s),$$

where $(L_\varepsilon(t))_{t \in \mathbb{R}}$ is the family of operators

$$L_\varepsilon(t) = -\Delta \otimes 1 + 2\Re \langle \lambda_\varepsilon | \alpha_\varepsilon(t) \rangle_{\mathcal{D}(\omega,\Gamma)} + a_\varepsilon(\lambda_\varepsilon) + a_\varepsilon^\dagger(\lambda_\varepsilon) + d\Gamma_\varepsilon(\omega_\varepsilon).$$

Proof. Using Lemma 3.2 and the definition of $Z_\varepsilon(t,s)$, it is easy to see that it is possible to differentiate with respect to both $t$ and $s$ the quantity

$$\langle \Theta | Z_\varepsilon(t,s) \Phi \rangle_{\mathcal{H}},$$

for any $\Theta, \Phi \in \mathcal{D}[H_0] \subset \mathcal{D}(d\Gamma_\varepsilon((\omega,1))^{1/2})$. Indeed, $e^{-i(t-s)H_\varepsilon}$ is weakly differentiable on $\mathcal{D}[H_\varepsilon] = \mathcal{D}[H_0]$, and $W_\varepsilon(\cdot)$ is strongly differentiable on $\mathcal{D}(d\Gamma_\varepsilon(1)^{1/2})$, and maps $\mathcal{D}[H_0]$ into itself. The explicit form of the derivative is given using again Lemma 3.2, the action as translations of Weyl operators when acting on creation and annihilation operators, and the equation for the time derivative of $\alpha_\varepsilon$, i.e., $i\partial_t \alpha_\varepsilon(t) = \pi \{ 0, \kappa \} \alpha_\varepsilon(t)$. □

The two final preparatory results are essentially Gronwall-type estimates for the time-evolved expectation of the Laplace operator.

Lemma 3.4.

For any $t \in \mathbb{R}$, there exists a finite constant $C_t > 0$ such that for every $\Phi \in \mathcal{D}[H_+] = \mathcal{D}[H_0]$,

$$\langle \Phi | e^{itH_+}(-\Delta)e^{-itH_+} | \Phi \rangle_{\mathcal{H}} \leq C_t \left( \langle \Phi | H_+ + H_I | \Phi \rangle_{\mathcal{H}} + \| \Phi \|^2_{\mathcal{H}} \right),$$

where $H_+ := -\Delta + d\Gamma_\varepsilon((\omega,\kappa)) \geq 0$.

Before proving the above result, let us point out the more regular behavior as $\varepsilon \to 0$ of the operator $H_+$ compared with $H_0$, since the former has no prefactor $\varepsilon^{-1}$ in front of the optic phonon’s energy.

Proof. Since $-\Delta \leq H_+$, adding and subtracting $H_I$, it is possible to write for any $\Phi \in \mathcal{D}[H_0]$

$$\langle \Phi | e^{itH_+(-\Delta)e^{-itH_+}} | \Phi \rangle_{\mathcal{H}} \leq \langle \Phi | e^{itH_+H_Ie^{-itH_+}} | \Phi \rangle_{\mathcal{H}} = \langle \Phi | e^{itH_+H_+}e^{-itH_+} | \Phi \rangle_{\mathcal{H}} - \langle \Phi | e^{itH_+H_Ie^{-itH_+}} | \Phi \rangle_{\mathcal{H}}. \quad (3.20)$$

Let us control the two terms of the last sum separately. The form associated to $H_+I$ as it has been proved in other papers dealing with the polaron Hamiltonian (see, e.g., [LLF]). Indeed, there exist $A \in (0,1)$ and $B > 0$, such that

$$\langle \Phi | e^{itH_+H_Ie^{-itH_+}} | \Phi \rangle_{\mathcal{H}} \leq A \langle \Phi | e^{itH_+H_Ie^{-itH_+}} | \Phi \rangle_{\mathcal{H}} + B \| \Phi \|^2_{\mathcal{H}}. \quad (3.21)$$

It remains to bound the first term of the sum. We will use a Gronwall and density argument. Without loss of generality, we can assume that $t > 0$. Let $-M, M \geq 0$, be a lower bound for $H_+$, and suppose that $\Phi \in \mathcal{D}((H_+ + M)^{3/2})$, the latter being a dense domain. Since $e^{-itH_\varepsilon}$ is weakly differentiable on $\mathcal{D}[H_0] \subset \mathcal{D}((H_+ + M)^{3/2})$, and maps $\mathcal{D}((H_+ + M)^{3/2})$ into itself for any $t \in \mathbb{R}$, it is possible to write

$$\langle \Phi | e^{itH_+H_+}H_Ie^{-itH_+} | \Phi \rangle_{\mathcal{H}} \leq \langle \Phi | H_+ + H_I | \Phi \rangle_{\mathcal{H}} + \int_0^t ds \partial_s \langle \Phi | e^{isH_+H_+}H_Ie^{-isH_+} | \Phi \rangle_{\mathcal{H}}$$

$$\leq \langle \Phi | H_+ + H_I | \Phi \rangle_{\mathcal{H}} + \int_0^t ds \langle \Phi | e^{isH_+H_+H_IH_0 + H_I}e^{-isH_+} | \Phi \rangle_{\mathcal{H}}. \quad (3.22)$$

Now, the commutator, as a quadratic form, yields

$$[H_+ + H_I, H_0 + H_I] = [H_+, H_I] + [H_I, H_0] = (\varepsilon - 1) \left( \beta_0^\dagger(\lambda^b_\varnothing) - \beta_\varepsilon(\lambda^b_\varnothing) \right).$$
Lemma 3.5. First of all, let us notice that Gronwall lemma then yields the analogous case where we can rewrite the previous inequality as (recall that it is well-known \[CF,\ Prop. A.2\] that)

\[\left| \langle \Theta \left( b_e^\dagger (\lambda^b_\infty) - b_e (\lambda^b_\infty) \right) \rangle \right| \leq 2 \left| \langle \Theta \left( b_e^\dagger (\lambda^b_\infty) \right) \rangle \right| \leq 2 A' \langle \Theta | H_+ | \Theta \rangle \| \Theta \|^2_{\mathcal{K}}. \] (3.22)

By KLMN theorem, the above bound yields an estimate from below for \( H_+ + H_1 \): for any \( \Theta \in \mathcal{D}[H_+] \),

\[\langle \Theta | H_+ + H_1 | \Theta \rangle \geq (1 - A') \langle \Theta | H_+ | \Theta \rangle - B' \| \Theta \|^2_{\mathcal{K}}. \] (3.23)

or, equivalently,

\[A' \langle \Theta | H_+ | \Theta \rangle + \langle \Theta | H_1 | \Theta \rangle + B' \| \Theta \|^2_{\mathcal{K}} \geq 0. \] (3.24)

Now, we use (3.23) in (3.22) to obtain (remembering that \( 3A' < 1 \))

\[\left| \langle \Theta \left( b_e^\dagger (\lambda^b_\infty) - b_e (\lambda^b_\infty) \right) \rangle \right| \leq 3A' \langle \Theta | H_+ | \Theta \rangle + \langle \Theta | H_1 | \Theta \rangle + 3B' \| \Theta \|^2_{\mathcal{K}} \]

\[\leq \langle \Theta | H_+ + H_1 | \Theta \rangle + 3B' \| \Theta \|^2_{\mathcal{K}}. \]

Therefore, using the fact that \(|1 - \varepsilon| < 1\), we get

\[\langle \Phi | e^{itH_+(H_++H_1)}e^{-itH_1} | \Phi \rangle \leq \int_0^t ds \langle \Phi | e^{isH_+(H_++H_1)}e^{-isH_1} | \Phi \rangle \]

\[+ \langle \Phi | H_++H_1 | \Phi \rangle + 3B't \| \Phi \|^2_{\mathcal{K}}. \]

If we now set

\[P(t) := \langle \Phi | e^{itH_+(H_++H_1)}e^{-itH_1} | \Phi \rangle, \quad B(t) := 3B't \| \Phi \|^2_{\mathcal{K}}, \]

we can rewrite the previous inequality as (recall that \( t \geq 0 \))

\[P(t) \leq P(0) + \int_0^t d\tau P(\tau) + B(t). \] (3.25)

Gronwall lemma then yields

\[P(t) \leq P(0) + B(t) + \int_0^t d\tau (P(0) + B(\tau)) e^{\tau} \leq e^t P(0) + 3B'te^t \| \Phi \|^2_{\mathcal{K}}. \]

Therefore inserting this bound in (3.20), we get for the expectation of \( H_+ \) (considering now also the analogous case \( t < 0 \)):

\[(1 - A) \langle \Phi | e^{itH_+(H_++H_1)}e^{-itH_1} | \Phi \rangle \leq e^{|t|} \langle \Phi | H_++H_1 | \Phi \rangle + \left( 3B'|t|e^{|t|} + B \right) \| \Phi \|^2_{\mathcal{K}}. \] (3.26)

This concludes the proof for \( \Phi \in \mathcal{D}((H_+ + M)^{3/2}) \), since \( A < 1 \). The proof is then extended to any \( \Phi \in \mathcal{D}[H_0] \) by a density argument. \( \square \)

**Lemma 3.5.** Let \( W \in C^\infty_0(\mathbb{R}^3) \). Then, for any \( t, s \in \mathbb{R} \), there exists a constant \( C_{t,s} > 0 \) depending only on \( \| \nabla W \|_\infty \) and \( \| \Delta W \|_\infty \), such that, for every \( \Phi \in \mathcal{D}[H_0] \),

\[\langle \Phi | Y^\nu_\sigma(t,s)(-\Delta \otimes 1)Y_\sigma(t,s) | \Phi \rangle \leq C_{t,s} \nu^{-6} \left( \| \nabla \phi \|_\infty^2 + \| \phi \|^2_{\mathcal{K}} \right). \] (3.27)

**Proof.** First of all, let us notice that \( Y_\sigma(t,s)^{(1)}(\Delta \otimes 1)Y_\sigma(t,s) = \mathbb{U}_\sigma^3(t,s)\Delta \mathbb{U}_\sigma(t,s) \otimes 1 \). Omitting the multiplication by the identity when it is obvious from the context, we can write

\[\langle \Phi | Y^\nu_\sigma(t,s)(-\Delta \otimes 1)Y_\sigma(t,s) | \Phi \rangle = \langle \mathbb{U}_\sigma(t,s)\Phi | -\Delta | \mathbb{U}_\sigma(t,s)\Phi \rangle =: R(t,s). \]

Suppose now that \( \Phi \in \mathcal{D}(H_0) \). Without loss of generality, we can also assume that \( t \geq s \geq 0 \). Hence, we get the integral inequality

\[R(t,s) \leq R(s,s) + \int_s^t d\tau \left| \langle \mathbb{U}_\sigma(t,s)\Phi | -\Delta, W_{\beta(\tau),\sigma}(x) \mathbb{U}_\sigma(t,s)\Phi \rangle \right|, \]

where \( \nu(\sigma) \) and \( W_\sigma \) are given by (1.30) and (1.28), respectively. The commutator has the following explicit form:

\[-\Delta, W_{\beta(\tau),\sigma} = (-\Delta W_{\beta(\tau),\sigma})(x) + 2(\nabla W_{\beta(\tau),\sigma})(x) \cdot \nabla. \]
Therefore, keeping in mind that $|\beta(t)| \leq C$ for all $t \in \mathbb{R}$, we obtain

$$
R(t, s) \leq R(s, s) + (t - s)\sigma^{-2}||\Delta W||_{\infty} \Phi \| \Phi \|_{\mathcal{H}}^2 + 2\sigma^{-3}||\nabla W||_{\infty} \Phi \| \Phi \|_{\mathcal{H}} \int_s^t d\tau \| \nabla U_{\sigma}(\tau, s) \Phi \|_{\mathcal{H}} \leq R(s, s) + \left[ (t - s)\sigma^{-2}||\Delta W||_{\infty} + \sigma^{-6}||\nabla W||_{\infty}^2 \right] \Phi \| \Phi \|_{\mathcal{H}}^2 + \int_s^t d\tau R(\tau, s)
$$

The result for $\Phi \in \mathcal{D}(H_0)$ then follows by Gronwall lemma, and it is extended by density to $\Phi \in \mathcal{D}[H_0]$. $\square$

We can now conclude the proof of Proposition 3.1.

**Proof of Proposition 3.1.** Since the operators $Z_\varepsilon$ and $Z$ are bounded, in order to prove strong convergence it suffices to show that $Z_\varepsilon$ weakly converges to $Z$. Therefore, we want to prove that for all $\Phi, \Theta \in \mathcal{H}$,

$$
\lim_{\varepsilon \to 0} \langle \Theta | (Z(t, s) - Z_\varepsilon(t, s)) \Phi \rangle_{\mathcal{H}} = 0;
$$

however, by triangular inequality, we can show separately the convergences of $Z_\varepsilon$ to $Y_\sigma$ and of $Y_\sigma$ to $Z$. By Theorem 1.7, we already know that $U_\sigma$ converges strongly to $U_{\text{eff}}$ on $L^2(\mathbb{R}^3)$, which implies that $Y_\sigma$ converges to $Z$, provided that $\sigma \to 0$, as $\varepsilon \to 0$, which we are going to assume. Therefore, it suffices to prove convergence of $Z_\varepsilon$ to $Y_\sigma$.

We restrict to the dense set

$$
\mathcal{D} := \mathcal{D}(H_0) \cap (\mathcal{D}(d\Gamma(((k^2 + 1)^{2M}, (k^2 + 1)^{2N}))^{1/2}) \subset \mathcal{D}[H_0],
$$

where $M$ is defined by (1.7), and $N > 1$, and prove weak convergence to zero on $\mathcal{D}$ for the quadratic form associated to $Y_\sigma(t, s)Z_\varepsilon(t, s) - 1$, whenever $\sigma = \mathcal{O}(\varepsilon^\gamma)$, with suitable $\gamma > 0$. Note that, since $k^2 + 1 \geq 1$, $d\Gamma(1) \leq d\Gamma(((k^2 + 1)^{2M}, (k^2 + 1)^{2N}))$ and consequently $\mathcal{D}(d\Gamma(((k^2 + 1)^{2M}, (k^2 + 1)^{2N}))^{1/2}) \subset \mathcal{D}(d\Gamma(1)^{1/2})$. By the polarization identity

$$
\langle \Theta | (Y_\sigma - Z_\varepsilon) \Phi \rangle_{\mathcal{H}} = \langle Y_\sigma^2 \Theta | (Y_\sigma Z_\varepsilon - 1) \Phi \rangle_{\mathcal{H}} = \sum_{j=1}^4 c_j \langle \eta_j (Y_\sigma^2 \Theta, \Phi) | (Y_\sigma Z_\varepsilon - 1) \eta_j (Y_\sigma^2 \Theta, \Phi) \rangle_{\mathcal{H}},
$$

where the coefficients $c_j \in \mathbb{C}$ and the linear combinations $\eta(Y_\sigma^2 \Theta, \Phi) \eta_j$ of $Z_\varepsilon^2 \Theta$ and $\Phi$ are suitably given. Hence, on one hand, weak convergence of $Y_\sigma(t, s) - Z_\varepsilon(t, s)$ implies convergence of the quadratic form, via the identification $\Theta = Y_\sigma \Psi$ and Theorem 1.7 and, on the other hand, if

$$
\langle \Theta | (Y_\sigma Z_\varepsilon - 1) \Phi \rangle_{\mathcal{H}} \to 0, \quad \forall \Phi \in \mathcal{H},
$$

then, by the same polarization identity, it follows that $Y_\sigma(t, s) - Z_\varepsilon(t, s)$ converges weakly to zero.

Furthermore, by the uniform boundedness in $\varepsilon$ of both $Y_\sigma$ and $Z_\varepsilon$, it is sufficient to prove the convergence of the quadratic form on the dense set $\mathcal{D}$, which we are going to do now. The operator $Z_\varepsilon^2$ is strongly differentiable on $\mathcal{D}(H_0)$ and maps $\mathcal{D}[H_0]$ into itself, while $Z_\varepsilon$ is weakly differentiable on $\mathcal{D}[H_0]$. Without loss of generality, we can suppose that $t \geq s \geq 0$. Therefore, it is possible to write

$$
\left| \left\langle \Phi \left| \left( Y_\sigma Z_\varepsilon - 1 \right) \Phi \right\rangle_{\mathcal{H}} \right| \leq \int_s^t d\tau \left| \partial_\tau \left\langle \Phi \left| \left( Y_\sigma Z_\varepsilon - 1 \right) \Phi \right\rangle_{\mathcal{H}} \right| = \int_s^t d\tau \left| \left\langle \Phi \left| Y_\sigma (L_\varepsilon(\tau) - K_{\beta(\tau), \sigma} \otimes 1 \otimes 1 - 1 \otimes d\Gamma((0, \kappa)) \right) Z_\varepsilon(\tau, s) \Phi \right\rangle_{\mathcal{H}} \right|.
$$

Now, since by hypothesis $2\Re \langle \lambda_\varepsilon \mid \alpha_\varepsilon \rangle_{\mathcal{D}(\mathbb{H})} = W_{\beta(t), \sigma}(\mathbf{x})$, and $\frac{1}{\varepsilon} d\Gamma^{(b)}_\varepsilon(1) = d\Gamma^{(b)}(1)$ by (1.11), we have that

$$
L_\varepsilon(t) - K_{\beta(\tau), \sigma} \otimes 1 \otimes 1 - 1 \otimes d\Gamma((0, \kappa)) = a_\varepsilon(\lambda_\varepsilon) + a_\varepsilon^\dagger(\lambda_\varepsilon) = \sqrt{\varepsilon} \left( a(\lambda_\varepsilon) + a^\dagger(\lambda_\varepsilon) \right).
$$
Therefore, we obtain the following bound
\[
\left| \left\langle \Phi \left( \frac{Y^d}{\sqrt{\varepsilon}} Z_{\varepsilon} - 1 \right) \Phi \right\rangle \right| \leq \varepsilon \int_t^s d\tau \left| \left\langle \left( a^d(\lambda_{\varepsilon}) + a^d(\lambda_{\varepsilon}) \right) Y_\sigma(\tau, s) \Phi \mid Z_\varepsilon(\tau, s) \Phi \right\rangle \right|. \tag{3.30}
\]

Let us define
\[
S := \int_t^s d\tau \left| \left\langle a(\lambda_{\varepsilon}) Y_\sigma(\tau, s) \Phi \mid Z_\varepsilon(\tau, s) \Phi \right\rangle \right|,
\]
\[
T := \int_t^s d\tau \left| \left\langle a^d(\lambda_{\varepsilon}) Y_\sigma(\tau, s) \Phi \mid Z_\varepsilon(\tau, s) \Phi \right\rangle \right|.
\]
The term \( S \) is easy to bound, exploiting the fact that \( d\Gamma((k^2 + 1)^{1/2}, (k^2 + 1)^N) \) commutes with \( d\Gamma((0, \kappa)) \), and therefore with \( Z_\varepsilon \): \[
S \leq \int_t^s d\tau \left\| a(\lambda_{\varepsilon}) Y_\sigma(\tau, s) \Phi \right\|_{\mathcal{H}} \left\| \Phi \right\|_{\mathcal{H}} \leq (t - s) \sup_{\tau \in (t, t)} \left\| a(\lambda_{\varepsilon}) Y_\sigma(\tau, s) \Phi \right\|_{\mathcal{H}} \left\| \Phi \right\|_{\mathcal{H}} \leq (t - s) \left( (k^2 + 1)^{-M}, (k^2 + 1)^{-N} \right) \left\| \lambda_{\varepsilon} \right\|_{L^\infty(\mathbb{R}^3; \mathfrak{H})} \left\| d\Gamma \left( (k^2 + 1)^{2M}, (k^2 + 1)^N \right) \right\|_{\mathcal{H}} \left\| \Phi \right\|_{\mathcal{H}} \leq C \left\| d\Gamma \left( (k^2 + 1)^{2M}, (k^2 + 1)^N \right) \right\|_{\mathcal{H}} \left\| \Phi \right\|_{\mathcal{H}}, \tag{3.31}
\]
which is uniformly bounded on \( (3.28) \).

The \( T \) term requires some additional care: recalling \( (3.4) \), we observe that one can split \( T \) into a regular and a singular part, i.e., \( T \leq T_{\text{reg}} + T_{\text{sing}} \), where
\[
T_{\text{reg}} := \int_t^s d\tau \left| \left\langle a^d(\lambda_{\varepsilon}) Y_\sigma(\tau, s) \Phi \mid Z_\varepsilon(\tau, s) \Phi \right\rangle \right|,
\]
\[
T_{\text{sing}} := \int_t^s d\tau \left| \left\langle a^d(\lambda_{\varepsilon}) Y_\sigma(\tau, s) \Phi \mid Z_\varepsilon(\tau, s) \Phi \right\rangle \right|.
\]
The regular part can be treated analogously to \( S \): let \( \omega \) be the operator on \( \mathfrak{H} \) acting as \( \omega \eta = (\omega \eta_1, \kappa \eta_2) \), then
\[
T_{\text{reg}} \leq (t - s) \left( \left\| \omega^{-1/2} \lambda_{\varepsilon} \right\|_{L^\infty(\mathbb{R}^3; \mathfrak{H})} + \left\| \lambda_{\varepsilon} \right\|_{L^\infty(\mathbb{R}^3; \mathfrak{H})} \right) \left\| d\Gamma(\omega) + 1 \right\|^{1/2}_{\mathcal{H}} \left\| \Phi \right\|_{\mathcal{H}}, \tag{3.32}
\]
which is uniformly bounded w.r.t. \( \varepsilon \).

It remains to estimate the singular part. Let us split again such term in two parts: \( T_{\text{sing}} = \bar{T}_1 + \bar{T}_2 \), where
\[
\bar{T}_1 := \int_t^s d\tau \left| \left\langle a^d(\xi_{\varepsilon}) \cdot \nabla Y_\sigma(\tau, s) \Phi \mid Z_\varepsilon(\tau, s) \Phi \right\rangle \right|,
\]
\[
\bar{T}_2 := \int_t^s d\tau \left| \left\langle \nabla \cdot a^d(\xi_{\varepsilon}) Y_\sigma(\tau, s) \Phi \mid Z_\varepsilon(\tau, s) \Phi \right\rangle \right|.
\]
The first one is bounded as follows:
\[
\bar{T}_1 \leq (t - s) \sup_{\tau \in (s, t)} \left\| a^d(\xi_{\varepsilon}) \cdot \nabla Y_\sigma(\tau, s) \Phi \right\|_{\mathcal{H}} \left\| \Phi \right\|_{\mathcal{H}} \leq C \sup_{\tau \in (s, t)} \sum_{j=1}^3 \left\| a^d(\xi_{\varepsilon}) \partial_j Y_\sigma(\tau, s) \Phi \right\|_{\mathcal{H}} \leq C \left( \frac{\Phi^{-1/2}}{\Phi} \right) \left\| \nabla Y_\sigma(\tau, s) \mid T_{\text{reg}}(\varepsilon) \Phi \right\|_{\mathcal{H}} \leq C \left( \left\| Y_\sigma(\tau, s) \right\|_{\mathfrak{H}} \left\| \Theta \right\|_{\mathfrak{H}} \right) \left| \left( \nabla Y_\sigma(\tau, s) \mid T_{\text{reg}}(\varepsilon) \Phi \right\|_{\mathcal{H}} \right| \leq C \left( \left\| Y_\sigma(\tau, s) \right\|_{\mathfrak{H}} \left\| \Theta \right\|_{\mathfrak{H}} \right)^{1/2} \left( \left\| Y_\sigma(\tau, s) \right\|_{\mathfrak{H}} \left\| \Theta \right\|_{\mathfrak{H}} \right)^{1/2},
\]
with \( \Theta := d\Gamma(\omega)^{1/2} \). Therefore, by Lemma \( 3.5 \) it follows that there exists a constant \( C_{t, s} > 0 \), depending on \( \Phi \), such that
\[
\bar{T}_1 \leq C_{t, s} \sigma^{-3}. \tag{3.33}
\]
The second term is bounded using Lemma 3.4

\[
\tilde{T}_2 \leq \sum_{j=1}^{3} \left\| \alpha^1 \left( (\xi_x)_j \right) Y_{\alpha}(\tau, s) \Phi \right\|_{\mathcal{K}} \left\| \partial_j Z_{\epsilon}(\tau, s) \Phi \right\|_{\mathcal{K}} \leq C \left( \left\| |\omega|^{-1/2} \xi_{\epsilon} \right\|_{L^{\infty}(\mathbb{R}^3; (\mathbb{S}\oplus \mathbb{S}))} \right) \left\| d\Gamma(\omega)^{1/2} \Phi \right\|_{\mathcal{K}} + \left\| \xi_{\epsilon} \right\|_{L^{\infty}(\mathbb{R}^3; (\mathbb{S}\oplus \mathbb{S}))} \left( \left\| (-\Delta)^{1/2} Z_{\epsilon}(\tau, s) \Phi \right\|_{\mathcal{K}} \right).
\]

Now, setting \( W_t := W_\epsilon (\alpha(t)) \) and using the fact that \(-\Delta\) commutes with \( W_t \) for any \( t \in \mathbb{R} \), we get by Lemma 3.3

\[
\left\| (-\Delta)^{1/2} Z_{\epsilon}(\tau, s) \Phi \right\|^2_{\mathcal{K}} \leq C \left( \left\| (W_t \Phi | H_+ + H_I | W_t \Phi)_{\mathcal{K}} + \left\| \phi \right\|^2_{\mathcal{K}} \right).
\]

However, by the translation properties of Weyl operators, one has

\[
W_1^t (H_+ + H_I) W_s = -\Delta + d\Gamma_\epsilon(\omega) + H_I + |\omega\alpha_\epsilon(s)|^2 + a^1(\omega\alpha_\epsilon(s)) + a^1(\omega\alpha_\epsilon(s)) + 2\Re \langle \alpha_\epsilon(s) | \lambda_\alpha \rangle,
\]

so that

\[
\left\| (-\Delta)^{1/2} Z_{\epsilon}(\tau, s) \Phi \right\|^2_{\mathcal{K}} \leq C \left( \left\| \Phi \right\|_{\mathcal{K}} \left\| (-\Delta + d\Gamma_\epsilon(\omega) + H_I + a^1(\omega\alpha_\epsilon(s)) + a^1(\omega\alpha_\epsilon(s)) \right) \right.
\]

\[
\left. + \left( 2\Re \langle \alpha_\epsilon(s) | \lambda_\alpha \rangle + |\omega\alpha_\epsilon(s)|^2 + 1 \right) \right\| \phi \|^2_{\mathcal{K}} \right).
\]

Now, since by (3.31) with (3.32), (3.33) and (3.36), we finally get, for any \( \Phi \in \mathcal{D} \),

\[
\left\| \langle \alpha_\epsilon(s) | \lambda_\alpha \rangle \right\|_{\mathcal{D} \oplus \mathcal{D}} \leq 2 \left\| W_{\beta(t), \sigma}(x) \right\|_{\mathcal{D}} = O \left( \sigma_{\epsilon}^{-2} \right),
\]

and the other terms are either uniformly bounded or subdominant with respect to \( \sigma \). It follows that

\[
\tilde{T}_2 \leq C \left( \sigma_{\epsilon}^{-3} - 4M + \sigma_{\epsilon}^{-1} \right).
\]

Hence, putting together (3.31) with (3.32), (3.33) and (3.36), we finally get, for any \( \Phi \in \mathcal{D} \),

\[
\left\| \left( \Phi \left( Y_{\sigma}^1 Z_{\epsilon} - 1 \right) \Phi \right) \right\|_{\mathcal{K}} \leq O \left( \epsilon^{1/2} \sigma^{-3} \right) + O \left( \epsilon^{1/2} \sigma^{-3} - 4M \right) = o(1),
\]

as long as \( \sigma \gg \epsilon^{1/j_x} \), with \( j_x = 6 + 8M \).

\[ \square \]

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