Four Fermion Operator Matching with NRQCD Heavy and AsqTad Light Quarks

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We present one-loop matching coefficients between continuum and lattice QCD for the heavy-light four-fermion operators relevant for neutral B meson mixing both within and beyond the Standard Model. For the lattice theory we use nonrelativistic QCD (NRQCD) to describe b quarks and improved staggered fermions (AsqTad) for light quarks. The gauge action is the tree-level Symanzik improved gauge action. Matching to full QCD is carried out through order \( \alpha_s, \Lambda_{QCD}/M_b, \) and \( \alpha_s/(aM_b) \).

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I. INTRODUCTION

The difference \( \Delta M_q = B^0_q - \overline{B}^0_q \), mixing, with \( q = d \) or \( q = s \), has been the focus of much attention by experimentalists and theorists in recent years, and it will continue to be probed extensively in the LHC era [1]. On the experimental front the mass differences between the “heavy” and “light” eigenstates in the \( B^0 \) systems, \( \Delta M_d \) and \( \Delta M_s \), are now known very accurately [2, 3]. Measurements of the decay width differences \( \Delta \Gamma_s \) and the phase \( \phi_s \) by D\O and CDF have also appeared [4, 5, 6]. On the theory side neutral \( B \) systems are of particular interest as a possible window into New Physics (NP). In the Standard Model \( B^0_q - \overline{B}^0_q \) mixing does not occur at tree level and must go through box diagrams involving the exchange of two W’s at lowest order. NP could enter through the exchange of new particles in the box diagrams, or through new tree level contributions. Studies of the neutral \( B \) meson parameters can impose important constraints on different NP scenarios [1].

Theoretical estimates of mixing rates employ effective Hamiltonians involving four fermion operators. Matrix elements of these operators between \( B^0_q \) and \( \overline{B}^0_q \) states are needed to complete the calculations and this requires control over non-perturbative QCD. For instance the Standard Model expression for the mass difference \( \Delta M_q \) is given by [2]:

\[
\Delta M_q = \frac{G_F M_W^2}{6\pi^2} |V_{tb} V_{td}|^2 \eta^B \phi S_0(x_t) M_{B_q} f_{B_q} \bar{B}_{B_q},
\]

where \( x_t = m_t^2/M_W^2 \), \( \eta^B \) is a perturbative QCD correction factor, \( S_0(x_t) \) the Inami-Lim function and \( V_{tb} \) and \( V_{td} \) the appropriate Cabibbo-Kobayashi-Maskawa (CKM) matrix elements. The nonperturbative QCD input into this formula is the combination \( f_{B_q} \bar{B}_{B_q} \) where \( f_{B_q} \) is the \( B_q \) meson decay constant and \( \bar{B}_{B_q} \) the renormalization group invariant bag parameter. Lattice QCD provides a first principles approach to obtaining these crucial non-perturbative factors. The first realistic lattice results that include effects from two very light and the strange sea quarks (so-called \( N_f = 2 + 1 \) simulations) have now appeared [7, 8, 9].

An important step in all lattice calculations is the matching between four-fermion operators in continuum QCD that enter into formulas such as eq. (1) and the operators of the lattice theory used in the lattice QCD simulations. In this article we present one loop matching results for a complete basis of \( \Delta B = 2 \) four fermion operators relevant for mixing both within and beyond the Standard Model. In the lattice theory we employ a nonrelativistic QCD (NRQCD) action for the \( b \) quarks, an improved staggered quark action (AsqTad) for the \( s \) and \( d \) quarks and the Symanzik improved glue action.

A subset of these results was used already in reference [1] in the calculation of all the hadronic matrix elements relevant for the determination of \( \Delta M_s \) and \( \Delta \Gamma_s \) in the Standard Model. That was the first lattice determination of these parameters with \( N_f = 2 + 1 \) sea quarks.

II. THE FOUR FERMION OPERATORS AND MATRIX ELEMENTS IN QCD

In order to study neutral \( B \) meson mixing phenomena, we focus on the following five \( \Delta B = 2 \) four-fermion operators (\( i \) and \( j \) are color indices)

\[
\begin{align*}
Q_1 &= \left( \overline{\Psi}_b \gamma^\nu P_L \Psi^j \right) \left( \overline{\Psi}^i \gamma_\nu P_L \Psi_q \right) \\
Q_2 &= \left( \overline{\Psi}_b P_L \Psi^j \right) \left( \overline{\Psi}^i P_L \Psi_q \right) \\
Q_3 &= \left( \overline{\Psi}_b P_L \Psi^j \right) \left( \overline{\Psi}^i P_L \Psi_q \right) \\
Q_4 &= \left( \overline{\Psi}_b P_L \Psi^j \right) \left( \overline{\Psi}^i P_L \Psi_q \right) \\
Q_5 &= \left( \overline{\Psi}_b P_L \Psi^j \right) \left( \overline{\Psi}^i P_L \Psi_q \right).
\end{align*}
\]

The subscript \( q \) stands for either the \( d \) or the \( s \) quark, both of which we take to be massless in our matching calculations, and \( P_{R,L} \equiv (I \pm \gamma_5) \).

Operators \( Q_1, Q_2 \) and \( Q_3 \) appear in the Standard Model and are relevant for the mass and width differ-
ences $\Delta M_q$ and $\Delta \Gamma_q$. Matrix elements of $Q1$, for instance, lead to $f^{2L}_B$. Additional $\Delta B = 2$ operators, such as $Q4$ and $Q5$ are required when going to extensions of the Standard Model. The above five operators go under the name of the “SUSY basis of operators” in the literature [11,12]. At intermediate stages of the matching calculation, we found it useful to introduce matrix elements of two more operators,

$$Q6 = (\overline{\Psi}_c \gamma^\nu P_L \Psi^\dagger_q) (\overline{\Psi}_b \gamma^\nu P_R \Psi^\dagger_q)$$  \hspace{1cm} (7)

$$Q7 = (\overline{\Psi}_c \gamma^\nu P_L \Psi^\dagger_q) (\overline{\Psi}_d \gamma^\nu P_R \Psi^\dagger_q).$$  \hspace{1cm} (8)

Within perturbation theory matching can be carried out by considering scattering from an incoming state with [heavy antiquark + light quark] into an outgoing state with [heavy quark + light antiquark]. Symbolically,

$$|in\rangle = q_i^B \gamma^\nu q_f^c,$$  \hspace{1cm} (9)

$$|out\rangle = \gamma^\alpha Q^\dagger.$$

where “A”, “B”, “C”, and “D” are color indices. We also introduce external Dirac spinors $u_q$ and $v_q$ for the incoming light quark and outgoing light antiquark respectively, and similarly $\overline{\sigma}_Q$ and $\overline{\sigma}_q$ for the outgoing heavy quark and incoming heavy antiquark.

At the tree-level, matrix elements of operators $Q1$, $Q2$, $Q4$, and $Q6$ become,

$$\langle out | (\overline{\Psi}_c \gamma^\nu P_L \Psi^\dagger_q) (\overline{\Psi}_d \gamma^\nu P_R \Psi^\dagger_q) | in\rangle_{tree} =$$  \hspace{1cm} (10)

$$\delta_{AB} \delta_{CD} [(\overline{\sigma}_Q \Gamma_1 u_q)(\overline{\sigma}_Q \Gamma_2 v_q) + (\overline{\sigma}_Q \Gamma_2 u_q)(\overline{\sigma}_Q \Gamma_1 v_q)]$$  \hspace{1cm} (11)

Diagrammatically, we will denote the two Dirac structures , $S1 = (\overline{\sigma}_Q \Gamma_1 u_q)(\overline{\sigma}_Q \Gamma_2 v_q)$ and $S2 = (\overline{\sigma}_Q \Gamma_2 u_q)(\overline{\sigma}_Q \Gamma_1 v_q)$ by the first and second diagrams in Fig.1 respectively.

For operators $Q3$, $Q5$ and $Q7$ one has similarly,

$$\langle out | (\overline{\Psi}_c \gamma^\nu P_L \Psi^\dagger_q) (\overline{\Psi}_b \gamma^\nu P_R \Psi^\dagger_q) | in\rangle_{tree} =$$  \hspace{1cm} (12)

$$\delta_{AB} \delta_{CD} [(\overline{\sigma}_Q \Gamma_1 u_q)(\overline{\sigma}_Q \Gamma_2 v_q) + (\overline{\sigma}_Q \Gamma_2 u_q)(\overline{\sigma}_Q \Gamma_1 v_q)].$$

Continuum one-loop corrections to tree-level matrix elements are obtained by evaluating the diagrams in Fig.2a - Fig.2f. We carry out these calculations in the $\overline{MS}$-NDR scheme with definitions of general dimension four-fermion operators (i.e. of “evanescent” operators) as given in Appendix B [13,14]. We retain only terms of $O(\alpha_s)$ and discard $O(\alpha_s A_{QCD} / M)$ contributions, where $M$ is the mass of the $b$ quark. We thus expand the full QCD calculations to the same order to which our effective theory (lattice NRQCD) perturbative calculations of the next section are carried out. In this limit the heavy quark spinors obey.

$$\overline{\sigma}_Q \gamma_0 = \overline{\sigma}_Q, \quad \overline{\sigma}_Q \gamma_0 = -\overline{\sigma}_Q.$$  \hspace{1cm} (13)

As is well known, there is mixing among the four-fermion operators at one-loop. For instance,

$$\langle Q1 \rangle^{MS} = \langle Q1 \rangle_{tree}$$

$$\hspace{2cm} + \alpha_s \left[ c_{11} \langle Q1 \rangle^{(0)}_{tree} + c_{12} \langle Q2 \rangle^{(0)}_{tree} \right].$$
All matrix elements are taken between \( \langle \text{out} \rangle \) and \( \langle \text{in} \rangle \) and the superscript \((0)\) means that we are working with spinors obeying [13]. Similarly, one finds that \( \langle Q2 \rangle^{MS} \) has contributions from \( Q2 \) and \( Q1 \), \( \langle Q3 \rangle^{MS} \) from \( Q3 \) and \( Q1 \), \( \langle Q4 \rangle^{MS} \) from \( Q4 \) and \( Q6 \), and \( \langle Q5 \rangle^{MS} \) from \( Q5 \) and \( Q7 \). The one-loop coefficients \( c_{xy} \) depend on the \( b \) quark mass \( M \), the gluon mass \( \lambda \) that acts as an IR regulator, and the \( MS \)-NDR renormalization scale \( \mu \). Their values for the operators in the basis (2) are given by,

\[
\begin{align*}
c_{11} &= \frac{1}{4\pi} \left\{ -35 \left[ -2 \ln \frac{\mu^2}{M^2} - 4 \ln \frac{\lambda^2}{M^2} \right] \right\}, \\
c_{12} &= -\frac{8}{4\pi} \\
c_{22} &= \frac{1}{4\pi} \left\{ 10 + 16 \left[ -2 \ln \frac{\mu^2}{M^2} - 4 \ln \frac{\lambda^2}{M^2} \right] \right\}, \\
c_{21} &= \frac{1}{4\pi} \left\{ 3 + \frac{2}{3} \ln \frac{\mu^2}{M^2} + 2 \ln \frac{\lambda^2}{M^2} \right\}, \\
c_{33} &= \frac{1}{4\pi} \left\{ -2 - 8 \ln \frac{\mu^2}{M^2} - 4 \ln \frac{\lambda^2}{M^2} \right\}, \\
c_{31} &= \frac{1}{4\pi} \left\{ 3 + \frac{4}{3} \ln \frac{\mu^2}{M^2} + 2 \ln \frac{\lambda^2}{M^2} \right\}, \\
c_{44} &= \frac{1}{4\pi} \left\{ 143 \left[ -2 \ln \frac{\mu^2}{M^2} - 7 \ln \frac{\lambda^2}{M^2} \right] \right\}, \\
c_{46} &= \frac{1}{4\pi} \left\{ -23 - \frac{3}{8} \left[ 3 \ln \frac{\mu^2}{M^2} + 3 \ln \frac{\lambda^2}{M^2} \right] \right\}, \\
c_{55} &= \frac{1}{4\pi} \left\{ -85 \left[ -2 \ln \frac{\mu^2}{M^2} - 7 \ln \frac{\lambda^2}{M^2} \right] \right\}, \\
c_{57} &= \frac{1}{4\pi} \left\{ 13 - \frac{3}{8} \left[ 3 \ln \frac{\mu^2}{M^2} + 3 \ln \frac{\lambda^2}{M^2} \right] \right\}.
\end{align*}
\]

The one-loop coefficients \( c_{11}, c_{12}, c_{22} \) and \( c_{21} \) agree with the values already published in reference [13]. Using eq.\((12)\) \( c_{46} \) and \( c_{57} \) can be replaced by,

\[
\begin{align*}
c_{45} &= \frac{1}{4\pi} \left\{ -23 - \frac{3}{2} \ln \frac{\lambda^2}{M^2} \right\}, \\
c_{54} &= \frac{1}{4\pi} \left\{ 13 + \frac{3}{4} \left[ 3 \ln \frac{\mu^2}{M^2} - \frac{3}{2} \ln \frac{\lambda^2}{M^2} \right] \right\}.
\end{align*}
\]

### III. MATRIX ELEMENTS IN THE EFFECTIVE THEORY

In effective theories such as HQET or NRQCD one works separately with heavy quark fields that create heavy quarks \( (\bar{Q}Q) \) and with those that annihilate heavy antiquarks \( (\bar{Q}_{\overline{Q}}) \). At lowest order in \( 1/M \), matrix elements such as \( \text{\langle 10 \rangle} \) and \( \text{\langle 11 \rangle} \), are reproduced by working in the effective theory with

\[
\hat{O} = (\bar{Q}_{\overline{Q}} \Gamma_1 \Psi_q) \bigg( \bar{Q}_{\overline{Q}} \Gamma_2 \Psi_q \bigg) + \bigg( \bar{Q}_{\overline{Q}} \Gamma_1 \Psi_q \bigg) \bigg( \bar{Q}_{\overline{Q}} \Gamma_2 \Psi_q \bigg).
\]

If one introduces an effective theory field,

\[
\bar{Q}_{\overline{Q}}^{\text{eff}} = \bar{Q} + \bar{Q}_{\overline{Q}}
\]

then \( \bar{Q}_{\overline{Q}}^{\text{eff}} \) and the QCD field \( \bar{Q}_{\overline{Q}} \) are related by a Foldy-Wouthuysen-Tani transformation. In particular,

\[
\bar{Q}_{\overline{Q}} = \bar{Q}_{\overline{Q}}^{\text{eff}} \left[ I + \frac{1}{2M} \nabla \cdot \nabla + \mathcal{O}(1/M^2) \right],
\]

where the \( \nabla \) acts to the left. By inserting (20) into the expressions for the four-fermion operators \( Q1 - Q7 \), one sees that \( \mathcal{O}(\Lambda_{QCD}/M) \) corrections to (27) can be obtained within the effective theory by adding the following \( 1/M \) operator corrections

\[
\hat{O} j 1 = \frac{1}{2M} \bigg[ \left( \bar{Q}_{\overline{Q}} \cdot \nabla \nabla \Gamma_1 \Psi_q \right) \bigg( \bar{Q}_{\overline{Q}} \Gamma_2 \Psi_q \bigg) \bigg]
\]

and similarly for all the other \( Qk \)’s. Several of the lattice one-loop integrals are IR divergent and we use a gluon mass \( \lambda \) to extract the IR finite contributions as explained in reference [10]. In this effective theory, we have evaluated one-loop corrections to matrix elements of \( \hat{O} \) and \( \hat{O} j 1 \). The one-loop corrections to \( \hat{O} \) involve the same diagrams Fig.2a - 2f as in continuum QCD. One obtains

\[
\text{\langle Q1 \rangle}^{\text{eff}} = \left[ 1 + \alpha_s c_{11}^L \text{\langle Q1 \rangle}^{(0)}_{\text{tree}} + \alpha_s c_{12}^L \text{\langle Q2 \rangle}^{(0)}_{\text{tree}} \right],
\]

and similarly for all the other \( Qk \)’s. Of the lattice one-loop integrals are IR divergent and we use a gluon mass \( \lambda \) to extract the IR finite contributions as explained in reference [10]. The \( c_{xy}^L \)’s have the same IR divergent \( \ln(\lambda^2) \) terms as the corresponding \( c_{xy} \) in continuum QCD, and they also depend on the bare heavy quark mass. The divergent terms will cancel when we do the matching (see below) and things are reduced to finite differences such as \( [c_{xy} - c_{xy}^L] \). We carry out the lattice perturbative calculations in both Feynman and Landau gauges and use gauge invariance as a check on our results.

In order to calculate the one-loop renormalization coefficients for the matrix elements of the operators \( \hat{O} j 1 \) the diagrams of Figs.3 & 4 need to be evaluated. One finds,

\[
\text{\langle Q1 j 1 \rangle}^{\text{eff}} = \text{\langle Q1 j 1 \rangle}^{(0)}_{\text{tree}} + \alpha_s \left[ c_{11}^L \text{\langle Q1 \rangle}^{(0)}_{\text{tree}} + c_{12}^L \text{\langle Q2 \rangle}^{(0)}_{\text{tree}} \right],
\]

(32)
where we ignore $O(\alpha_s)$ corrections to $\langle Q1j1^{(0)}\rangle_{\text{tree}}$. Similar expressions are obtained for the other $\hat{O}j1$.

The coefficients $\zeta^{xy}$ tell us about the “mixing down” of a dimension seven operator $\hat{O}j1$ onto dimension six operators $\hat{O}$. On dimensional grounds these coefficients go as $1/(aM)$, “$a$” being the lattice spacing. They represent “power law” contributions from matrix elements in the effective theory. Power law terms are unavoidable when working with effective theories and they need to be subtracted in order that the effective theory produce the same physics as full QCD (which does not suffer from power law contributions). Our matching procedure, described in the next section, will be such that power law contributions are removed from matrix elements of $\hat{O}j1$ through $O(\alpha_s/(aM))$. Errors from the mismatch between QCD and effective theory due to power law contributions will come in at $O(\alpha_s^2/(aM))$.

### IV. MATCHING

We wish to relate the continuum QCD matrix elements $\langle Qk \rangle_{\text{MS}}$ to the matrix elements $\langle Qk \rangle_{\text{eff}}$ in the effective theory. The latter will ultimately be replaced by output from nonperturbative simulations. We will focus on matching of $Q1$. The other $Qk$’s are handled identically.

If one expands the first term on the RHS of (13) in powers of $1/M$, i.e. $\langle Q1 \rangle_{\text{tree}} \rightarrow \langle Q1^{(0)} \rangle_{\text{tree}} + \langle Q1j1^{(0)} \rangle_{\text{tree}}$, then this equation becomes

$$
\langle Q1 \rangle_{\text{MS}} = [1 + \alpha_s c_{11}] \langle Q1 \rangle_{\text{tree}}^{(0)} + \alpha_s c_{12} \langle Q2 \rangle_{\text{tree}}^{(0)} + \langle Q1j1 \rangle_{\text{tree}}^{(0)}.
$$

(33)

The next step is to rewrite the matrix elements $\langle \cdots \rangle_{\text{tree}}^{(0)}$ appearing on the RHS in terms of matrix elements in the effective theory $\langle \cdots \rangle_{\text{eff}}$. This can be accomplished by inverting (31) and (32). To the order that we are working one has,

$$
\langle Q1 \rangle_{\text{tree}}^{(0)} = \langle Q1 \rangle_{\text{tree}}^{\text{eff}} - \alpha_s \left[ c_{11}^{(0)} \langle Q1 \rangle_{\text{tree}}^{\text{eff}} + c_{12}^{(0)} \langle Q2 \rangle_{\text{tree}}^{\text{eff}} \right],
$$

(34)

and

$$
\langle Q1j1 \rangle_{\text{tree}}^{(0)} = \langle Q1j1 \rangle_{\text{tree}}^{\text{eff}} - \alpha_s \left[ \zeta^{11} \langle Q1 \rangle_{\text{tree}}^{\text{eff}} + \zeta^{12} \langle Q2 \rangle_{\text{tree}}^{\text{eff}} \right].
$$

(36)

Upon inserting the last three equations into (33) one ends up with,

$$
\langle Q1 \rangle_{\text{MS}}^{(0)} = [1 + \alpha_s \rho_{11}] \langle Q1 \rangle_{\text{tree}}^{\text{eff}} + \alpha_s \rho_{12} \langle Q2 \rangle_{\text{tree}}^{\text{eff}} + \langle Q1j1 \rangle_{\text{tree}}^{\text{eff}} - \alpha_s \left[ \zeta^{11} \langle Q1 \rangle_{\text{tree}}^{\text{eff}} + \zeta^{12} \langle Q2 \rangle_{\text{tree}}^{\text{eff}} \right] + O(\alpha_s^2, \alpha_s \Lambda_{\text{QCD}}^2),
$$

(37)

where,

$$
\rho_{xy} = e_{xy} - e_{xy}^L.
$$

(38)

With (37) we have achieved the goal of relating the continuum full QCD matrix element $\langle Q1 \rangle_{\text{MS}}^{(0)}$ to matrix elements in the effective theory. Although this perturbative matching calculation was carried out with external scattering states, one carries over matchings such as (37) to the case of hadronic matrix elements between $B^0_q$ and $\bar{B}^0_q$ states where then $\langle Q1 \rangle_{\text{tree}}^{\text{eff}}$ or $\langle Q1j1 \rangle_{\text{tree}}^{\text{eff}}$ must be evaluated nonperturbatively. With such nonperturbative matrix elements in mind we define

$$
\langle Q1j1 \rangle_{\text{tree}}^{\text{eff}} - \alpha_s \left[ \zeta^{11} \langle Q1 \rangle_{\text{tree}}^{\text{eff}} + \zeta^{12} \langle Q2 \rangle_{\text{tree}}^{\text{eff}} \right] \equiv \langle Q1j1 \rangle_{\text{sub}}^{\text{eff}}.
$$

(39)
The combination $\langle Q\bar{Q}\rangle_{\text{sub}}^{\gamma f}$ represents thus the matrix element of the dimension seven $1/M$ correction in the effective theory with power law contributions subtracted out through $O(\alpha_s/(aM))$. This matrix element gives us the physical $\Lambda_{QCD}/M$ contributions to $\langle Q\bar{Q}\rangle_{\text{sub}}^{\gamma f}$ up to corrections of $O((aM)^{-1})$. Further discussion of power law subtractions for the case of heavy-light currents are given in reference [17]. A more complete derivation of one-loop matching formulas including all contributions at $O(\alpha_s\Lambda_{QCD}/M)$ is provided (again for heavy-light currents) in reference [18].

A final technical detail is that eq.(37) differs from eq.(10) of [3] in that here we assume the same normalization of states in the effective theory as in continuum QCD. For the purposes of evaluating matching coefficients it is convenient to do so. Any differences in normalization of states are taken care of at the stage of doing the non-perturbative calculations and of extracting decay constants and bag parameters.

V. RESULTS

In this section we summarize results for the effective theory coefficients $c_{xy}^L$ and $\zeta^{xy}$ of eqns. (31) and (32), and for the matching coefficients $\rho_{xy}^L$ of eq. (37). We present numbers for three values of the bare heavy quark mass in lattice units, $aM_0$, corresponding to the $b$ quark mass on lattices with spacings $0.09\,fm$, $0.12\,fm$ and $0.17\,fm$, as fixed in previous studies of the $\Upsilon$ system [19]. These values of $a$ correspond to the so-called MILC fine, coarse and super-coarse lattices, which have been used extensively in recent studies of heavy-heavy [19] and heavy-light [20, 21] quantities with the same choice of lattice actions as the one in the present article.

In Table I we list values for the one-loop renormalization coefficients $c_{xy}^L$ after subtracting the IR divergent $\ln(a\lambda)^2$ pieces, together with the one-loop $\zeta^{xy}$, which are IR finite. Table II shows values for $\rho_{xy}^L$ at scale $\mu$ equal to the heavy quark mass $M$. The parameter $n$ in these two tables is the stability parameter in the NRQCD action.

In Table III we illustrate how the different diagrams contribute to $c_{44}^L$ for $aM_0 = 2.8$. Calculations were done in both Feynman ($\xi = 1$) and Landau ($\xi = 0$) gauges to check for gauge invariance. We have evaluated $c_{44}^L$ by collecting all contributions that are proportional to $\delta_{AB}^{\text{QCD}}$ and that can be written (using Fierz relations to convert where necessary) in terms of Dirac structures $(\Gamma_Q P_L u_q)(\Gamma_Q P_R v_q)$ or $(\Gamma_Q P_R u_q)(\Gamma_Q P_L v_q)$. We could just as well have collected terms proportional to $-\delta_{AB}\delta_{aB}$ and of Dirac structure $(\Gamma_Q P_L v_q)(\Gamma_Q P_R u_q)$ or $(\Gamma_Q P_R u_q)(\Gamma_Q P_L v_q)$ to obtain the same final result. Table IV illustrates different contributions to $c_{46}^L$ again for $aM_0 = 2.8$. Here we collect terms proportional to $\delta_{AB}^{\text{QCD}}$ and of Dirac structure $(\Gamma_Q\gamma^\nu P_L u_q)(\Gamma_Q\gamma^\nu P_R v_q)$ or $(\Gamma_Q\gamma^\nu P_R u_q)(\Gamma_Q\gamma^\nu P_L v_q)$. From $c_{46}^L$ one easily obtains $c_{45}^L = -2c_{46}^L$.

| $aM_0$ (n) | 1.95 (n = 2) | 2.8 (n = 2) | 4.0 (n = 2) |
|------------|--------------|--------------|--------------|
| $c_{11}^L$ | -1.196       | -0.735       | -0.403       |
| $c_{12}^L$ | -1.802       | -1.315       | -0.960       |
| $c_{22}^L$ | 0.010        | 0.014        | -0.004       |
| $c_{21}^L$ | 0.020        | -0.018       | -0.050       |
| $c_{53}^L$ | -0.890       | -0.644       | -0.483       |
| $c_{51}^L$ | 0.133        | 0.064        | 0.010        |
| $c_{44}^L$ | 0.692        | 0.599        | 0.529        |
| $c_{46}^L$ | 0.097        | 0.029        | -0.022       |
| $c_{55}^L$ | -0.886       | -0.553       | -0.311       |
| $c_{57}^L$ | -0.467       | -0.383       | -0.322       |
| $c_{45}^L$ | -0.194       | -0.058       | 0.044        |
| $c_{54}^L$ | 0.934        | 0.766        | 0.644        |

TABLE I: The one-loop coefficients $c_{xy}^L$ and $\zeta^{xy}$ for three values of $aM_0$. $n$ is the stability parameter in the NRQCD action. IR divergent $\ln(a\lambda)^2$ terms are omitted. Numerical integration errors are of order one or less in the last digit.

VI. SUMMARY

We have completed the one-loop matching of a complete set of $\Delta B = 2$ heavy-light four fermion operators through $O(\alpha_s\Lambda_{QCD}/M_b, \alpha_s/(aM_b))$. The main results are the coefficients $\rho_{xy}^L$ of Table II and the $\zeta^{xy}$ of Table I.

We find that with the lattice actions employed in this article (and in our simulations) matching coefficients are all well behaved. None of them are particularly large, and in fact many are considerably smaller than one. An interesting feature that can be extracted from the results in Table III and IV, and which holds also for the matrix elements of the other operators, is that the one-loop matching coefficients are dominated by the current-like diagrams, $a$ and $b$, and the wave function renormalizations. The contributions from the pure four-fermion diagrams are at least an order of magnitude smaller. We are currently investigating non-perturbative matching methods for heavy(NRQCD)-light(staggered) currents. The same methodology could be applied here for four-fermion operators to calculate non-perturbatively the main contribution to the renormalization coefficients. In this way
we could considerably reduce the uncertainty associated with the matching process, which is one of the main sources of error at present in our calculation of $f_{B_s} \sqrt{B_{B_s}}$ and $f_{B_d} \sqrt{B_{B_d}}$.

The matching calculation in this article is an important part of the HPQCD collaboration’s studies of $B_s$ and $B_d$ meson mixing via lattice QCD methods both in the Standard Model [8, 9, 22] and beyond. Values for the mass and decay width differences, $\Delta M_q$ and $\Delta \Gamma_q$ with $q = s, d$, as well as for the ratio $\xi = \frac{f_{B_s} \sqrt{B_{B_s}}}{f_{B_d} \sqrt{B_{B_d}}}$, which use the results presented here will be available soon [22]. Extensions of this work to matching with other lattice actions such as the HISQ light quark action are straightforward and are planned for the future.

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APPENDIX A: FIERZ RELATIONS IN 4 D

In this appendix we collect some Fierz relations that were found to be useful in our calculations. They are written in terms of fixed external spinors rather than fermionic fields, which means that interchanging two spinors does not bring in a minus sign. In our calculations, fermionic signs come in at the stage of doing the Wick contractions, for instance to get the RHS’s of eq. (10) and (11).

$$\bar{\psi}_Q \gamma^\nu P_L v_q \] [\bar{\psi}_Q \gamma_\nu P_L u_q] = - [\bar{\psi}_Q \gamma^\nu P_L u_q] [\bar{\psi}_Q \gamma_\nu P_L v_q]$$

(A1)

$$[\bar{\psi}_Q P_L v_q] [\bar{\psi}_Q P_L u_q] = \frac{1}{2} [\bar{\psi}_Q P_L u_q] [\bar{\psi}_Q P_L v_q] + \frac{1}{8} [\bar{\psi}_Q \sigma^{\mu\nu} P_L u_q] [\bar{\psi}_Q \sigma_{\mu\nu} P_L v_q]$$

where $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. 

$$[\bar{\psi}_Q P_L v_q] [\bar{\psi}_Q P_R u_q] = \frac{1}{2} [\bar{\psi}_Q \gamma^\nu P_R u_q] [\bar{\psi}_Q \gamma_\nu P_L v_q]$$

(A3)

$$[\bar{\psi}_Q \gamma^\nu P_L v_q] [\bar{\psi}_Q \gamma_\nu P_R u_q] = 2 [\bar{\psi}_Q P_R u_q] [\bar{\psi}_Q P_L v_q]$$

(A4)

In the large $M$ limit, i.e. with spinors obeying (13), the relation (A2) simplifies to (22),

$$[\bar{\psi}_Q P_L v_q] [\bar{\psi}_Q P_L u_q] = 0$$

(A5)

$$[\bar{\psi}_Q P_L u_q] [\bar{\psi}_Q P_L v_q] + \frac{1}{2} [\bar{\psi}_Q \gamma^\nu P_L u_q] [\bar{\psi}_Q \gamma_\nu P_L v_q]$$

TABLE II: The one-loop matching coefficients $\rho_{xy}$ in [38] with $\mu \equiv M$ for three values of $\alpha M_0$.

| $\alpha M_0$ (n) | $2.8$ (n = 2) | $4.0$ (n = 2) |
|-----------------|----------------|----------------|
| $\rho_{11}$    | 0.693          | 0.462          | 0.357          |
| $\rho_{12}$    | 1.165          | 0.678          | 0.323          |
| $\rho_{22}$    | 0.927          | 1.000          | 1.094          |
| $\rho_{21}$    | 0.029          | 0.028          | 0.022          |
| $\rho_{33}$    | 0.873          | 0.703          | 0.618          |
| $\rho_{31}$    | 0.035          | 0.065          | 0.082          |
| $\rho_{44}$    | 0.628          | 0.923          | 1.192          |
| $\rho_{46}$    | 0.052          | 0.077          | 0.085          |
| $\rho_{55}$    | 0.694          | 0.563          | 0.520          |
| $\rho_{57}$    | 0.258          | 0.131          | 0.027          |
| $\rho_{45}$    | -0.104         | -0.154         | -0.170         |
| $\rho_{54}$    | -0.516         | -0.262         | -0.054         |

TABLE III: Contributions to the coefficient $c_{14}$ from the diagrams of Fig.2 for $\alpha M_0 = 2.8$. $\xi = 1$ and $\xi = 0$ refer to Feynman or Landau gauge respectively. The second and fourth columns give the IR finite contributions. Columns 3 and 5 list IR divergent terms in units of $\frac{1}{\alpha} \ln(\alpha)^2$. $Z_q$ and $Z_Q$ are the light and heavy quark wave function renormalizations respectively and are taken from reference [10]. The last row gives the full $c_{14}$.

| diagram | $\xi = 1$ | $\xi = 0$ |
|---------|-----------|-----------|
| $a + b$ | $0.836 \times 2$ | $-\frac{1}{2} \times 2$ | $0.3274 \times 2$ | $0$ |
| $a' + b'$ | $0.0823 \times 2$ | $0.0823 \times 2$ | $0$ | $0$ |
| $c + d$ | $-0.0495 \times 2$ | $\frac{1}{2} \times 2$ | $0.0139 \times 2$ | $0$ |
| $e$ | $0.0331$ | $\frac{1}{2}$ | $-0.0304$ | $\frac{1}{2}$ |
| $e'$ | $0.0269$ | $0$ | $0.0269$ | $0$ |
| $f$ | $0.0640$ | $\frac{1}{2}$ | $0.0066$ | $0$ |
| $Z_q$ | $-0.924$ | $\frac{1}{2}$ | $-0.416$ | $0$ |
| $Z_Q$ | $-0.338$ | $\frac{1}{4}$ | $0.171$ | $-4$ |
| Total | $0.5996$ | $-\frac{1}{2}$ | $0.5993$ | $-\frac{1}{2}$ |

TABLE IV: Same as Table III for the coefficient $c_{16}$.
Appendix B: Some Formulas in General Dimensions

We carry out the continuum one-loop calculations using dimensional regularization in the \( \overline{MS} \) and NDR scheme. As is by now well known, this information is insufficient to fix one’s renormalization scheme unambiguously. One needs to specify in addition how one handles \( d \)-dimensional Dirac structures appearing at intermediate stages of the calculations and how they are projected onto some 4-dimensional basis. This is because the Dirac algebra is infinite dimensional for non-integer \( d \), and requires in addition to the 4-d basis set an infinite set of “evanescent operators”. The \( \mathcal{O}(\epsilon) \) (we use \( d \equiv 4 - \epsilon \) terms in the projections onto the 4-d basis defines one’s choice of evanescent operators, and is convention dependent \([13, 14, 24, 25]\). In our calculations we have adopted the following conventions taken from the literature \([24, 25]\):

\[
[\not{\bar{u}} \gamma^\alpha \gamma^\beta \gamma^\nu P_L u_q] [\not{\bar{u}} \gamma^\alpha \gamma^\beta \gamma^\nu P_L u_q] \\
\Rightarrow (16 - 2\epsilon)[\not{\bar{u}} \gamma^\nu P_L u_q] [\not{\bar{u}} \gamma^\nu P_L u_q] 
\]

(B1)

\[
[\not{\bar{u}} \gamma^\alpha \gamma^\beta \gamma^\nu P_L u_q] [\not{\bar{u}} \gamma^\nu \gamma^\mu P_L u_q] \\
\Rightarrow (8 - 4\epsilon)[\not{\bar{u}} \gamma^\nu P_L u_q] [\not{\bar{u}} \gamma^\nu P_L u_q] 
\]

(B2)

\[
[\not{\bar{u}} \gamma^\alpha \gamma^\nu P_L u_q] [\not{\bar{u}} \gamma^\alpha \gamma^\nu P_L u_q] \\
\Rightarrow (8 - 2\epsilon)[\not{\bar{u}} \gamma^\nu P_L u_q] [\not{\bar{u}} \gamma^\nu P_L u_q] 
\]

(B3)

\[
[\not{\bar{u}} \gamma^\alpha \gamma^\beta \gamma^\mu P_L u_q] [\not{\bar{u}} \gamma^\nu \gamma^\alpha P_R v_q] \\
\Rightarrow (4 + 2\epsilon)[\not{\bar{u}} \gamma^\mu P_L u_q] [\not{\bar{u}} \gamma^\nu P_R v_q] 
\]

(B4)

\[
[\not{\bar{u}} \gamma^\alpha \gamma^\beta \gamma^\mu P_L u_q] [\not{\bar{u}} \gamma^\nu \gamma^\alpha P_R v_q] \\
\Rightarrow (4 - 4\epsilon)[\not{\bar{u}} \gamma^\mu P_L u_q] [\not{\bar{u}} \gamma^\nu P_R v_q] 
\]

(B5)

In adopting \([13]\) we are following the conventions of reference \([24]\).

We note that \([13]\) has different conventions from \([24]\). Instead of eq. \([B3]\) the \([24]\) conventions translate into the relation

\[
[\not{\bar{u}} \gamma^\alpha \gamma^\beta \gamma^\nu P_L u_q] [\not{\bar{u}} \gamma^\alpha \gamma^\beta P_L v_q] \\
\Rightarrow (8 - \epsilon)[\not{\bar{u}} \gamma^\nu P_L u_q] [\not{\bar{u}} \gamma^\nu P_L v_q] \\
- 8[\not{\bar{u}} \gamma^\nu P_L v_q] [\not{\bar{u}} \gamma^\nu P_L u_q] 
\]

(B6)

Using \([B6]\) leads to different constant terms in results for the one-loop coefficients \( c_{22}, c_{21}, c_{33} \) and \( c_{31} \). The \( \ln m^2 \) and \( \ln \lambda^2 \) terms remain unchanged, however. Denoting by \( \tilde{c}_{ij} \) the coefficients obtained by using the \([24]\) conventions, one finds that eqns. \([17], [18], [19] \) and \([20]\) are modified to,

\[
\tilde{c}_{22} = \frac{1}{4\pi} \left\{ \frac{16}{3} \ln \frac{\mu^2}{M^2} - \frac{4}{3} \ln \frac{\lambda^2}{M^2} \right\} 
\]

(B7)

\[
\tilde{c}_{21} = \frac{1}{4\pi} \left\{ \frac{4}{3} \ln \frac{\mu^2}{M^2} + \frac{2}{3} \ln \frac{\lambda^2}{M^2} \right\} 
\]

(B8)

\[
\tilde{c}_{33} = \frac{1}{4\pi} \left\{ -\frac{8}{3} \ln \frac{\mu^2}{M^2} - \frac{4}{3} \ln \frac{\lambda^2}{M^2} \right\} 
\]

(B9)

\[
\tilde{c}_{31} = \frac{1}{4\pi} \left\{ \frac{67}{12} + \frac{4}{3} \ln \frac{\mu^2}{M^2} + \frac{2}{3} \ln \frac{\lambda^2}{M^2} \right\} . 
\]

(B10)

Care is required to ensure that a consistent set of conventions is applied to different parts of a calculation contributing to a physical quantity such as \( \Delta M_q \).

[1] For a recent review see R.Fleischer; arXiv:0802.2882 [hep-ph].
[2] V. Abazov et al. [DØ Collaboration]; Phys. Rev. Lett. 97, 021802 (2006).
[3] A. Abulencia et al. [CDF Collaboration]; Phys. Rev. Lett. 97, 242003 (2006).
[4] V. Abazov et al. [DØ Collaboration]; arXiv:0802.2255 [hep-ex].
[5] V. Abazov et al. [DØ Collaboration]; Phys. Rev. Lett. 98, 121801 (2007).
[6] T. Aaltonen et al. [CDF Collaboration]; arXiv:0712.2348 [hep-ex]; arXiv:0712.2397 [hep-ex].
[7] A. Buras, M. Jamin and F. Weisz; Nucl. Phys. B 347, 491 (1990).
[8] E. Dalicet et al.; Phys. Rev. D76, 011501(R) (2007).
[9] E. Gámiz et al.; PoS(Lattice 2007) 349.
[10] R. Todd Evans et al.; PoS(Lattice 2007) 354.
[11] F. Gabbiani et al.; Nucl. Phys. B 477, 321 (1996).
[12] D. Becirevic et al.; JHEP 0204:025 (2002).
[13] S. Collins et al.; Nucl. Phys. B 527, 66 (1999).
[14] S. Herrlich and U. Nierste; Nucl. Phys. B 455, 39 (1995).
[15] S. Hashimoto et al.; Phys. Rev. D72, 145020 (2005).
[16] E. Guler, J. Shigemitsu and M. Wingate; Phys. Rev. D69, 074501 (2004).
[17] P. Horgan and L. C. Storoni, J. Comput. Phys. 199, 340 (2005).
[18] R. J. O. tilt et al.; Phys. Rev. D63, 034505 (2001).
[19] C. Morningstar and J. Shigemitsu; Phys. Rev. D57, 6741 (1998).
[20] A. Gray et al.; Phys. Rev. D72, 094507 (2005).
[21] E. Daligic, A. Gray, M. Wingate, C. T. H. Davies, G. P. Lepage and J. Shigemitsu, Phys. Rev. D 73, 074502 (2006) [Erratum-ibid. D 75, 119906 (2007)] [arXiv:hep-lat/0601021].

[22] E.Gámiz et al. (HPQCD Collaboration); work in progress.

[23] J.Flynn, O.Hernandez and B.Hill; Phys. Rev. D43, 3709 (1991).

[24] M.Beneke et al.; Phys.Lett.B 459, 631 (1999).

[25] A.Buras, M.Misiak and J.Urban; Nucl.Phys.B 586, 397 (2000).