Global attractors for the damped nonlinear wave equation in unbounded domains

Djiby Fall
Mathematics Program, Division of Science
NewYork University Abu Dhabi, UAE
Email: dfall@nyu.edu, falldjiby@gmail.com

Yuncheng You
Department of Mathematics and Statistics
University of South Florida
Email: you@mail.usf.edu

Abstract. The existence of a global attractor for wave equations in unbounded domains is a challenging problem due to the non-compactness of the Sobolev embeddings. To overcome this difficulty, some authors have worked with weighted Sobolev spaces which restrict the choice of the initial data. Using the "tail estimation method" introduced by B. Wang for reaction diffusion equations, we establish in this paper the existence of a global attractor for two wave equations in the traditional Hilbert spaces $H^1(\Omega) \times L^2(\Omega)$ where $\Omega$ is an unbounded domain of $\mathbb{R}^N$. The first equation, with a mass term is studied in the whole space $\mathbb{R}^N$ and the second one without mass term is considered in a domain bounded in only one direction so that Poincaré inequality will hold.

Keywords. wave equation, global attractor, unbounded domain, asymptotic compactness, absorbing set, “tail estimates”, Poincaré Inequality.

1 Introduction

We study in this paper the existence of a global attractor for the following two damped nonlinear wave equations in an unbounded domain of $\mathbb{R}^N$:

\[ u_{tt} + \lambda u_t - \Delta u + u + f(u) = g(x), \quad t > 0 \] (1)

and

\[ u_{tt} + \lambda u_t - \Delta u + f(u) = g(x), \quad t > 0 \] (2)

where $\lambda$ is a positive constant, $g$ is a given function and $f$ is a nonlinear term satisfying some growth conditions to be specified later.

The long-time behavior of solutions of such equations in a bounded domain was studied by many authors, for instance in [48], [43] and the references therein.

In the unbounded domain case, there also exists an extensive literature. In 1994, E. Feireisl [17] showed that the more challenging equation (2) admits a global attractor in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ when $N = 3$. For arbitrary $N$, he obtains in [18] the same result in the phase space $H^{1}_{loc}(\mathbb{R}^N) \times L^{2}_{loc}(\mathbb{R}^N)$.

In 2001, S.V. Zelik [54] considered the nonautonomous case for equation (1), in which the forcing term $g$ depends on time. He obtained the existence of locally compact global attractor and the upper and lower bounds for their Kolomogorov’s $\varepsilon$-entropy. Some other authors have also considered different types of wave equations in unbounded domains ([25], [26]) in weighted spaces.

In this paper, we establish the existence of global attractors in the usual Hilbert spaces $H^1 \times L^2$, for equations (1) and (2) in unbounded domains of $\mathbb{R}^N$. To this end we cannot apply the same procedure as for bounded domains, since the Sobolev embeddings are no longer compact. We will apply the "tail estimation" method, introduced for the first order lattice systems and the reaction diffusion equations ([4, 50, 51]). It features an "approximation" of $\mathbb{R}^N$ by sufficiently large bounded domains $\Omega_k$, then using the compactness of the embeddings in $\Omega_k$ and showing the uniform null convergence of the solutions on $\mathbb{R}^N - \Omega_k$, we finally arrive to get the asymptotical compactness of the semiflow.

Before we get into the details of our work, let us first introduce the basic definitions and results relevant to the general theory of dynamical systems. We present the notions of semiflows and attractors along with a brief presentation of the theory of semigroups and its relation to solving abstract nonlinear equations in a Banach space.

1.1 Semiflows and Attractors

We will use in this work the definition of semiflows as in Temam [48]. A stronger version can be found in Sell & You [43], where the only difference is the continuity property.

Definition 1.1 Let $(H, d)$ be a complete metric space. A family of operators $\{S(t)\}_{t \geq 0}$ is called a semiflow on $H$, if it satisfies the following properties:
1.2 Semigroups of Linear Operators

1. \( S(0) = I \) (identity in \( H \)), i.e. \( S(0)u = u \ \forall u \in H \),
2. \( S(t)S(s) = S(t+s), \ \forall s, t \in \mathbb{R}^+ \),
3. The mapping \( S(t) : H \rightarrow H \) is continuous for every \( t \geq 0 \).

We now introduce the concepts of invariant sets and attractors of a semiflow.

**Definition 1.2** Let \( S(t) \) be a semiflow on \( H \) and \( K \subset H \). We say that \( K \) is positively invariant if \( S(t)K \subset K \), for all \( t \geq 0 \). \( K \) is invariant if \( S(t)K = K \), for all \( t \geq 0 \).

To define attractors we will need the following asymmetric Hausdorff pseudodistance:

\[
\sup_{a \in A} \inf_{b \in B} d(A, b) \tag{3}
\]

where \( A, B \) are bounded sets in \( H \).

We say that \( A \) attracts \( B \) if

\[
h(S(t)B, A) \rightarrow 0, \quad \text{as } t \to \infty, \tag{4}
\]

that is: for every \( \epsilon > 0 \), there exists \( T \geq 0 \) such that \( d(S(t)u, A) \leq \epsilon \), for all \( t \geq T \) and \( u \in B \).

**Definition 1.3** A subset \( A \) of \( H \) is called an attractor for the semiflow \( S(t) \) provided that

1. \( A \) is a compact, invariant set in \( H \), and
2. there is a neighborhood \( U \) of \( A \) in \( H \) such that \( A \) attracts every bounded set in \( U \).

An attractor \( A \) that attracts every bounded set in \( H \) is called a global attractor.

The existence of a global attractor is in general related to what some authors call the “dissipativity” of the dynamical system. This is equivalent to the existence of absorbing sets.

**Definition 1.4** Let \( B \) be a subset of \( H \) and \( U \) an open set containing \( B \). We say that \( B \) is an absorbing set in \( U \) if the orbit of any bounded set in \( U \) enters into \( B \) after a finite time (which may depend on the set):

\[
\left\{ \begin{array}{l}
\forall B_0 \subset U, \ B_0 \text{ bounded} \\
\exists t_1(B_0) \text{ such that } S(t)B_0 \subset B, \ \forall t \geq t_1(B_0).
\end{array} \right.
\]

We also say that \( B \) attracts the bounded sets of \( U \).

We have also the related concept of asymptotical compactness.

**Definition 1.5** A semiflow \( \{S(t)\}_{t \geq 0} \) is said to be asymptotically compact on \( U \) if for every bounded sequence \( \{u_n\} \) in \( U \) and \( t_n \to \infty \), \( \{S(t_n)u_n\}_{t \geq 0} \) is precompact in \( H \).

We are now ready to present a standard result on the existence of global attractors which can be found in [23] [48].

**Theorem 1.6** Let \( \{S(t)\}_{t \geq 0} \) be a semiflow in \( X \). If \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set and is asymptotically compact in \( H \), then \( \{S(t)\}_{t \geq 0} \) possesses a global attractor which is a compact invariant set that attracts every bounded set in \( H \).

### 1.2 Semigroups of Linear Operators

In practice, semiflows are generated by the solutions of differential equations. We will consider abstract nonlinear ODEs of the form

\[
\frac{du}{dt} + Au = F(u, t) \tag{5}
\]

in a Banach space \( X \), where \( A \) is an unbounded linear operator in \( X \) and \( F : X \times \mathbb{R} \rightarrow X \) is a nonlinear functional. In this section we will present the general existence theory for equations such as (5). This will apply directly to a wide range of evolutionary partial differential equations. We will first give some basic notions on semigroup theory which is related to solving the corresponding linear problem

\[
\frac{du}{dt} + Au = 0. \tag{6}
\]

In the remainder of this section, \( X \) denotes a Banach space with norm \( \| \cdot \|_X \) and \( \mathcal{L}(X) \) is the space of bounded linear operators on \( X \).

**Definition 1.7** We will say that a family of operators \( \{T(t)\}_{t \geq 0} \) is a \( C_0 \)-semigroup of linear operators on \( X \), if \( T(t) \in \mathcal{L}(X) \) for all \( t \in [0, +\infty) \) and the following hold:

(i) \( T(0) = I \) (identity in \( X \))
(ii) \( T(t)T(s) = T(t+s), \quad s, t \in [0, +\infty) \)
(iii) \( \lim_{t \to +\infty} T(t)x = x, \quad \text{for all } x \in X \).

We see that a \( C_0 \)-semigroup is a typical example of a semiflow on \( X \).

**Definition 1.8** Let \( T(t) \) be a \( C_0 \)-semigroup on \( X \), its infinitesimal generator is the linear operator \( A \) on \( X \) defined as follows

- The domain of \( A \) is:
  \[
  D(A) = \{ x \in X : \lim_{h \to 0^+} \frac{T(h)x - I}{h} \text{ exists in } X \}
  \]
- for \( x \in D(A) \) we set:
  \[
  Ax = \lim_{h \to 0^+} \frac{T(h)x - I}{h} = \frac{d^+(T(t)x)}{dt}_{t=0}.
  \]
Next we will give a necessary and sufficient condition for an operator to be the infinitesimal generator of a \(C_0\)-semigroup in a Hilbert space \(H\). We need to introduce first some concepts. Let \(H\) be a Hilbert space with inner product \(.\,.\). A linear operator \(A : D(A) \subseteq H \to H\) is said to be \textit{accretive} if

\[
\text{Re} \langle Ax, x \rangle \geq 0, \quad \forall x \in D(A).
\]

If in addition we have \(R(I + A) = H\) (range of \(I + A\) is equal to \(H\)) then we say that \(A\) is \textit{maximal accretive}.

A \(C_0\)-semigroup is said to be \textit{nonexpansive} if \(\|T(t)\| \leq 1\) for every \(t \geq 0\).

**Theorem 1.9 (Lumer-Phillips)** Let \(H\) be a Hilbert space. Then a linear operator \(-A : D(A) \subseteq H \to H\) is the infinitesimal generator of a nonexpansive \(C_0\)-semigroup \(e^{-At}\) on \(H\) if and only if both the following conditions are satisfied:

1. \(A\) is a closed linear operator and \(D(A)\) is dense in \(H\), and
2. \(A\) is a maximal accretive operator.

This is a classical result on semigroups and their generators. The proof can be found in [39][43]. However this result applies only to Hilbert spaces; there is a more general one on Banach spaces, namely the \textit{Hille-Yosida} theorem.

Now we will briefly present the existence theory for abstract nonlinear evolutionary equations in a Banach space \(X\). There exists a vast literature on this issue, but we will just give some basic results; see [39][43][48].

We consider the following initial value problem in the Banach space \(X\):

\[
\begin{cases}
  \frac{du}{dt} + Au = F(u) \\
  u(t_0) = u_0 \in X,
\end{cases}
\tag{7}
\]

Assume that the nonlinearity \(F\) belongs to \(F \in CLip = CLip(X, X)\), the collection of all continuous functions \(G : X \to X\) that are Lipschitz continuous on every bounded set \(B\) in \(X\). We suppose also that \(-A\) generates a \(C_0\)-semigroup \(T(t)\) on \(X\).

At first, we give different notions of solution for problem \(7\) and then present some existence results for such types of solutions.

**Definition 1.10** Let \(I = [t_0, t_0 + \tau)\) be an interval in \(\mathbb{R}^+\) where \(\tau > 0\). A strongly continuous mapping \(u : I \to X\) is said to be a \textit{mild solution} of \(7\) in \(X\) if it solves the following integral equation

\[
u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s)F(u(s))\,ds, \quad t \in I.
\tag{8}
\]

If \(u\) is differentiable almost everywhere in \(I\) with \(u_t, Au \in L^1_{loc}(I, X)\), and satisfies the differential equation

\[
\frac{du}{dt} + Au = F(u), \quad \text{on } (t_0, t_0 + \tau), \quad \text{and } u(t_0) = u_0,
\tag{9}
\]

then \(u\) is called a \textit{strong solution} of \(7\). If in addition, one has \(u_t \in C(I, X)\) and the differential equation in \(9\) is satisfied for \(t_0 < t < t_0 + \tau\), then \(u\) is called a \textit{classical solution} of \(7\) on \(I\).

We have the following result which is a particular case of Theorems 46.1 & 46.2 in G. Sell & Y. You [43].

**Theorem 1.11** Let \(-A\) generate a \(C_0\)-semigroup \(T(t)\) on \(X\) and \(F \in CLip(X, X)\). Then for every \(u_0 \in X\) and \(t_0 > 0\), the Initial Value Problem \(7\) has a unique mild solution \(u\) in \(X\) on some interval \(I = [t_0, t_0 + \tau)\), for some \(\tau > 0\).

Assume that \(X = H\) is a Hilbert space or a reflexive Banach space. If \(u_0 \in D(A)\) or \(T(t)\) is a differentiable semigroup, then the mild solution is a strong one.

**Remark 1.12** The solution \(u\) in Theorem 1.11 can be extended to a maximum possible interval \(I\). Indeed \(u\) is maximally defined if either \(\tau = +\infty\) or \(\\lim_{t \to -\infty} \|u(t)\|_X = +\infty\).

### 1.3 Some Useful Inequalities

We present in this section some inequalities that will be used in the remaining of this paper. The most used inequality throughout our work is the Gronwall inequality which comes in different forms. We present here some variants of it.

**Lemma 1.13** (The Gronwall inequality) Suppose that \(a\) and \(b\) are nonnegative constants and \(u(t)\) a nonnegative integrable function. Suppose that the following inequality holds for \(0 \leq t \leq T\):

\[
u(t) \leq a + b \int_{0}^{t} u(s)\,ds.
\tag{10}
\]

Then for \(0 \leq t \leq T\), we have

\[
u(t) \leq ae^{bt}.
\tag{11}
\]

**Lemma 1.14** (The uniform Gronwall inequality) Let \(g, h, y\) be nonnegative functions in \(L^{1}_{loc}[0, T; \mathbb{R}]\), where \(0 < T \leq \infty\). Assume that \(y\) is absolutely continuous on \((0, T)\) and that

\[
\frac{dy}{dt} \leq gy + h \quad \text{almost everywhere on } (0, T).
\tag{12}
\]

Then \(y \in L^{1}_{loc}(0, T; \mathbb{R})\) and one has

\[
y(t) \leq y(t_0) \exp\left(\int_{t_0}^{t} g(s)\,ds\right) + \int_{t_0}^{t} \exp\left(\int_{s}^{t} g(r)\,dr\right) h(s)\,ds,
\tag{13}
\]

for \(0 < t_0 < t < T\). If in addition one has \(y \in C([0, T; \mathbb{R}]\), then inequality \text{13} is valid at \(t_0 = 0\).

The following theorem is concerned with the Poincaré inequality.

**Theorem 1.15** (Poincaré inequality) Let \(\Omega\) be a domain of \(\mathbb{R}^N\) bounded only in one direction and let \(u \in H^1_0(\Omega)\). Then there is a positive constant \(C\) depending only on \(\Omega\) and \(n\) such that

\[
\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{(L^2(\Omega))^N}, \quad \forall u \in H^1_0(\Omega).
\tag{14}
\]

**Remark 1.16** The Poincaré inequality is usually presented for bounded domains but the proof requires only the boundedness in one direction \(x_i\).
2 The Damped Wave Equation in \( \mathbb{R}^N \)

We consider in this section, the nonlinear wave equation with mass term,

\[
    u_{tt} + \lambda u_t - \Delta u + u + f(u) = \sigma(x), \quad t > 0
\]

in \( \mathbb{R}^N \). We shall establish first the existence and boundedness of solutions, then we shall prove the asymptotic compactness of the corresponding semiflow to obtain the global attractor.

2.1 Existence of Solutions and Absorbing Set

We start by transforming our problem into an abstract ODE in the space \( L^2 \times H^1 \) and prove that the new operator is maximal accretive. This will allow us to show the existence of solutions and the uniform boundedness of such solutions.

We consider the system

\[
    u_{tt} + \lambda u_t - \Delta u + u + f(u) = g(x), \quad x \in \mathbb{R}^N, \quad t > 0
\]

with initial conditions

\[
    u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \quad x \in \mathbb{R}^N
\]

where \( \lambda > 0, \quad g \in L^2(\mathbb{R}^N), \) and \( f \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies the following condition:

\[
    f(0) = 0, \quad f(s)s \geq \nu F(s) \geq 0, \quad \forall s \in \mathbb{R}
\]

where \( \nu \) is a positive constant and \( F(s) = \int_0^s f(t) \, dt \). In addition we assume that

\[
    0 \leq \limsup_{\delta \to \infty} \frac{f(s)}{\delta} < \infty
\]

Now set \( H = L^2(\mathbb{R}^N), \quad V = H^1(\mathbb{R}^N), \) and \( X = V \times H \) with the usual norms and scalar products. We define the operator \( G \) in \( X \) by:

\[
    D(G) = H^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)
\]

\[
    Gw = \begin{pmatrix} \delta u - v \\ -\Delta u + (\lambda - \delta)v + (\delta^2 - \delta \lambda + 1)u \end{pmatrix}
\]

for \( w = (u, v) \in D(G) \).

Then (15), (16) are equivalent to the initial value problem in \( X \):

\[
    \begin{cases}
        w_t + Gw = R(w), & t > 0, \quad w \in X \\
        w(0) = w_0 = (u_0, v_0 + \delta u)
    \end{cases}
\]

where

\[
    R(w) = \begin{pmatrix} 0 \\ -f(u) + g \end{pmatrix}
\]

The next result establishes the maximal accretivity of the the operator \( G \) in \( X \).

Lemma 2.1 For a suitable \( \delta \) chosen to be \( \delta = \frac{\lambda}{\lambda^2 + 4} \), the operator \( G \) defined previously is maximal accretive in \( X \), and there exists a constant \( C(\delta) > 0 \) depending on \( \delta \) such that

\[
    (Gw, w)_X \geq C(\delta)\|w\|_X^2, \quad \forall w \in D(G)
\]

Proof: We first prove the positivity. Let \( w = (u, v) \in D(G) \), then

\[
    (Gw, w)_X = (\delta u - v, v)_V + (-\Delta u + (\lambda - \delta)v + (\delta^2 - \delta \lambda + 1)u, v)_H
\]

\[
    = \delta\|u\|_V^2 - \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx - (u, v)_H + \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx
\]

\[
    + (\lambda - \delta)\|v\|_H^2 + (\delta^2 - \delta \lambda + 1)\langle u, v \rangle_H
\]

\[
    = \delta\|u\|_V^2 + (\lambda - \delta)\|v\|_H^2 + (\delta^2 - \delta \lambda)\langle u, v \rangle_H
\]

Then setting

\[
    \sigma = \frac{\lambda}{\sqrt{\lambda^2 + 4(\lambda + \sqrt{\lambda^2 + 4})}}
\]

we have

\[
    (Gw, w)_X - \sigma \left( \|u\|_V^2 + \|v\|_H^2 \right) - \frac{\lambda}{2}\|v\|_H^2 \geq (\delta - \sigma)\|u\|_V^2 + (\frac{\lambda}{2} - \delta - \sigma)\|v\|_H^2
\]

\[
    - \delta\|u\|_V\|v\|_H
\]

\[
    \geq 2\sqrt{(\delta - \sigma)(\frac{\lambda}{2} - \delta - \sigma)}\|u\|_V\|v\|_H
\]

We can check that \( 4(\delta - \sigma)(\frac{\lambda}{2} - \delta - \sigma) = \lambda^2\delta^2 \) so that

\[
    (Gw, w)_X - \sigma \|w\|_X^2 - \frac{\lambda}{2}\|v\|^2 \geq 0.
\]
It suffices to take $C(\delta) = \sigma$

Now we prove that the range of $G + I$ equals $X$. Let $f = (h, g) \in X$; the question is whether there exists a $w = (u, v) \in D(G)$ such that:

$$Gw + w = f$$

i.e. \[
\begin{aligned}
\delta u - v + u &= h \\
-\Delta u + (\lambda - \delta)u + (\delta^2 - \delta\lambda + 1)u &= g
\end{aligned}
\]

which implies that

$$\int$$

and then by Uniform Gronwall inequality we get

$$w$$

by (17) we have

$$\text{Lemma 2.3}$$

$$2$$

$$\text{Lemma 2.1}$$

$$\text{Lemma 2.2}$$

$$\text{Lemma 2.1}$$

$$\text{Lemma 2.1}$$

Note that the operator $Au = -\Delta u$ in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$ is a sectorial operator and there exists $\omega \in \mathbb{R}$ such that $g(\lambda) \in \mathbb{C} : \text{Re} \lambda \geq \omega$. So the equation

$$-\Delta u + (\lambda - \delta + 1)u = g + (\lambda - \delta)h$$

has a unique solution $u \in H^2(\mathbb{R}^N)$, thus letting $v = (\delta + 1)u - h$ and $w = (u, v)$, we get a unique $w \in D(G)$ such that $Gw + w = f$. So the range of $G + I$ equals $X$. This, with (21), shows that $G$ is maximal accretive and finishes the proof of lemma 2.1.

**Lemma 2.2** If $g \in L^2(\mathbb{R}^N)$ and $f$ satisfies (15), then for any initial data $w_0 = (u_0, v_0) \in X$, there exists a unique solution $w(t) = (u(t), v(t))$ of (20) such that $w \in C^4([-T_0, T_0])$ for some $T_0 = T_0(w_0) > 0$.

In fact we will show that the local solution $w(t)$ of (20) is bounded and exists globally.

**Lemma 2.3** Assume that (17) and (18) are satisfied and that $g \in H$. Then any solution $w(t)$ of problem (20) satisfies

$$\|w(t)\|_X \leq M, \quad t \geq T_1$$

(23)

where $M$ is a constant depending only on $(\lambda, g)$ and $T_1$ depending on the data $(\lambda, g, R)$ when $\|w_0\|_X \leq R$.

**Proof:** Let $w_0 \in D(G)$ be the initial condition in (20). Taking the inner-product of (20) with $w$ in $X$ we find that

$$\frac{1}{2} \frac{d}{dt} \|w\|^2_X = -(Gw, w)_X + (f, w)_E$$

$$= -(Gw, w)_X + (g, v)_H - (f, v)_H$$

$$\leq -C(\alpha)\|w\|^2_X + \|g\|_H \|v\|_H - \delta(f(u), u)_H - (f(u), u_t)_H,$$

by (17) we have

$$-\delta(f(u), u)_H \leq -\nu \int_{\mathbb{R}^N} F(u) \, dx$$

and

$$-(f(u), u_t)_H = -\frac{d}{dt} \int_{\mathbb{R}^N} F(u) \, dx.$$

Then using the Young inequality, it follows for any $\alpha > 0$ that

$$\frac{1}{2} \frac{d}{dt} \|w\|^2_X \leq -C(\alpha)\|w\|^2_X + \frac{\alpha}{2} \|v\|^2_H + \frac{1}{2\alpha} \|g\|^2_H - \delta \int_{\mathbb{R}^N} F(u) \, dx - \frac{d}{dt} \int_{\mathbb{R}^N} F(u) \, dx$$

which implies that

$$\frac{d}{dt} \left( \|w\|^2_X + 2 \int_{\mathbb{R}^N} F(u) \, dx \right) \leq 2(\alpha - C(\delta)) \|w\|^2_X - 2\nu \int_{\mathbb{R}^N} F(u) \, dx + \frac{1}{\alpha} \|g\|^2_H$$

Now we can choose $\alpha$ small enough so that $\alpha - C(\delta) < 0$ and taking $\mu = \min\{-2(\alpha - C(\delta)): \nu \}$ we have

$$\frac{d}{dt} \left( \|w\|^2_X + 2 \int_{\mathbb{R}^N} F(u) \, dx \right) \leq -\mu \left( \|w\|^2_X + 2 \int_{\mathbb{R}^N} F(u) \, dx \right) + \frac{1}{\alpha} \|g\|^2_H$$

(24)

and then by Uniform Gronwall inequality we get

$$\|w\|^2_X + 2 \int_{\mathbb{R}^N} F(u) \, dx \leq e^{-\mu t} \left( \|w_0\|^2_X + 2 \int_{\mathbb{R}^N} F(u_0) \, dx \right) + (1 - e^{-\mu t}) \frac{1}{\mu \alpha} \|g\|^2_H$$

which yields

$$\|w\|^2_X \leq e^{-\mu t} \left( \|w_0\|^2_X + 2 \int_{\mathbb{R}^N} F(u_0) \, dx \right) + \frac{1}{\mu \alpha} \|g\|^2_H$$

(25)

Now by (17) we have

$$\int_{\mathbb{R}^N} F(u_0) \, dx \leq \frac{1}{\nu} \int_{\mathbb{R}^N} f(u_0) \, dx \leq \frac{C}{\nu} \int_{\mathbb{R}^N} u_0^2(x) \, dx.$$

Then we deduce from (20) that for every $w_0 \in D(G)$,

$$\|w\|^2_X \leq e^{-\mu t} \left( \|w_0\|^2_X + \frac{C}{\nu} \|u_0\|^2_H \right) + \frac{1}{\mu \alpha} \|g\|^2_H$$

(26)
And by density of $D(G)$ in $X$ and the continuity of the solution of (20) in $X \times (0, T(w_0))$ we see that (20) holds for every $w_0 \in X$.

Now let $R > 0$ and $\|w_0\|_X \leq R$, then $\|w_0\|_H \leq R$ and
\[
\|w\|_X^2 \leq e^{-\mu t} \left( R^2 + \frac{CR^2}{\nu} \right) + \frac{1}{\mu \alpha} \|g\|_H^2
\]
which yields
\[
\|w\|_X^2 \leq \frac{2}{\mu \alpha} \|g\|_H^2, \quad \text{for } t \geq T_1 = \frac{1}{\mu} \ln \left( \frac{\mu \lambda R^2 + CR^2}{\|g\|_H^2} \right)
\]
and (23) follows with $M = \frac{2}{\mu \alpha} \|g\|_H^2$ and the proof is complete.

By (27), we have also the following result.

**Lemma 2.4** Let $g \in H$. Then for any given $T > 0$, every solution $w$ of (20) satisfies
\[
\|w\|_X \leq L, \quad 0 \leq t \leq T
\]
where $L$ depends on $(\lambda, \delta, \|g\|_H), T$ and $\|w_0\|_X$.

Lemma 2.3 implies that the solution $w(t)$ exists globally, that is $T(w_0) = +\infty$, which implies that the system (20) generates a continuous semiflow $\{S(t)\}_{t \geq 0}$ on $X$. Denote by $O$ the ball
\[
O = \{w \in X : \|w\|_X \leq M\}
\]
where $M$ is the constant in (23). Then it follows from (23) that $O$ is an absorbing set for $S(t)$ in $X$ and that for every bounded set $B$ in $X$ there exists a constant $T(B)$ depending only on $(\lambda, g)$ and $B$ such that
\[
S(t)B \subseteq O, \quad t \geq T(B).
\]
In particular there exists a constant $T_0$ depending only on $(\lambda, g)$ and $O$ such that
\[
S(t)O \subseteq O, \quad t \geq T_0.
\]

### 2.2 Global Attractor

The existence of an absorbing set is the first step toward the existence of a global attractor. We need now to prove the asymptotic compactness of $S(t)$. The key idea lies in establishing uniform estimates on “Tail Ends” of solutions, that is, the norm of the solutions $w(t)$ is uniformly small with respect to $t$ outside a sufficiently large ball.

**Lemma 2.5** If (17) and (18) hold, $g \in H$ and $w_0 = (u_0, v_0) \in O$, then for every $\varepsilon > 0$, there exists positive constants $T(\varepsilon)$ and $K(\varepsilon)$ such that the solution $w(t) = (u(t), v(t))$ of problem (20) satisfies
\[
\int_{|x| \geq k} \left( |w|^2 + |\nabla w|^2 + |v|^2 \right) dx \leq \varepsilon, \quad t \geq T(\varepsilon), \quad k \geq K(\varepsilon).
\]

**Proof:** Choose a smooth function $\theta$ such that $0 \leq \theta(s) \leq 1$ for $s \in \mathbb{R}^+$, and
\[
\theta(s) = 0 \quad \text{for } 0 \leq s \leq 1; \quad \theta(s) = 1 \quad \text{for } s \geq 2.
\]
Then there exists a constant $C > 0$ such that $|\theta'(s)| \leq C$ for $s \in \mathbb{R}^+$.

Let $w(t) = (u(t), v(t))$ be the solution of problem (20) with initial condition $w_0 = (u_0, v_0) \in O$ then $v(t) = \delta u + u_t$ satisfies the equation
\[
v_t - \Delta u + (\lambda - \delta)v + (\delta^2 - \lambda\delta + 1)u = -f(u) + g
\]
Taking inner product of (20) with $\theta(|x|^2 k^2)w$ in $H$ we get
\[
\int_{\mathbb{R}^N} \theta(|x|^2 k^2)wv dx - \int_{\mathbb{R}^N} \Delta u \theta(|x|^2 k^2)wv dx + (\lambda - \delta) \int_{\mathbb{R}^N} \theta(|x|^2 k^2)|v|^2 dx
\]
\[\quad + (\delta^2 - \lambda\delta + 1) \int_{\mathbb{R}^N} \theta(|x|^2 k^2)uv dx = - \int_{\mathbb{R}^N} f(u) \theta(|x|^2 k^2)v dx + \int_{\mathbb{R}^N} \theta(|x|^2 k^2)gv dx
\]
But
\[
- \int_{\mathbb{R}^N} \Delta u \theta(|x|^2 k^2)wv dx = \int_{\mathbb{R}^N} \theta(|x|^2 k^2)\nabla u \cdot \nabla v + \frac{2}{k^2} \int_{\mathbb{R}^N} \theta'(|x|^2 k^2)ux \cdot \nabla u
\]
\[\quad = \int_{\mathbb{R}^N} \theta(|x|^2 k^2) [\delta |\nabla u|^2 + \nabla u \cdot \nabla u] + \frac{2}{k^2} \int_{\mathbb{R}^N} \theta'(|x|^2 k^2)ux \cdot \nabla u
\]
\[\quad = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \theta(|x|^2 k^2)|\nabla u|^2 + \delta \int_{\mathbb{R}^N} \theta(|x|^2 k^2)|\nabla u|^2
\]
\[\quad + \frac{2}{k^2} \int_{\mathbb{R}^N} \theta'(|x|^2 k^2)ux,\]
and
\[
(\delta^2 - \lambda\delta + 1) \int_{\mathbb{R}^N} \theta(|x|^2 k^2)uv dx = (\delta^2 - \lambda\delta + 1) \int_{\mathbb{R}^N} \theta(|x|^2 k^2)\delta |u|^2 + u_t
\]
\[\quad = \frac{1}{2} (\delta^2 - \lambda\delta + 1) \int_{\mathbb{R}^N} \frac{d}{dt} \theta(|x|^2 k^2)|u|^2 + \delta |\nabla u|^2 + (\delta^2 - \lambda\delta + 1) \int_{\mathbb{R}^N} \theta(|x|^2 k^2)|u|^2.
\]
Then \( \Theta \) becomes

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) \left[ (\delta^2 - \lambda \delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 \right]
+ \delta \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) \left[ (\delta^2 - \lambda \delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 \right] + (\lambda - 2\delta) \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right)|v|^2
= - \int_{\mathbb{R}^N} \left( \frac{|x|^2}{k^2} \right) f(u)(v^u + u_t) + \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) g v \, dx - \frac{2}{k^2} \int_{\mathbb{R}^N} \theta' \left( \frac{|x|^2}{k^2} \right) u_x \cdot \nabla u,
\]

and since

\[
\int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) f(u)(v^u + u_t) \geq \frac{d}{dt} \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) F(u) + \delta \nu \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) F(u),
\]

we deduce that

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) \left[ (\delta^2 - \lambda \delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 + 2F(u) \right]
+ \delta \alpha \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) \left[ (\delta^2 - \lambda \delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 + 2F(u) \right]
\leq - (\lambda - 2\delta) \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) |v|^2 + \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) g v \, dx - \frac{2}{k^2} \int_{\mathbb{R}^N} \theta' \left( \frac{|x|^2}{k^2} \right) u_x \cdot \nabla u,
\]

where \( \alpha = \min \{ 1, \nu \} \). Now, there exists a constant \( K(\varepsilon) > 0 \) such that for \( k \geq K \), we have

\[
-(\lambda - 2\delta) \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) |v|^2 + \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) g v \, dx - \frac{2}{k^2} \int_{\mathbb{R}^N} \theta' \left( \frac{|x|^2}{k^2} \right) u_x \cdot \nabla u \leq \frac{\varepsilon}{2},
\]

which implies by Uniform Gronwall inequality that

\[
\int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) \left[ (\delta^2 - \lambda \delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 + 2F(u) \right]
\leq e^{-\delta \alpha t} \int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) \left[ (\delta^2 - \lambda \delta + 1)|u_0|^2 + |\nabla u_0|^2 + |v_0|^2 + 2F(u_0) \right] + \frac{1 - e^{-\delta \alpha}}{2\delta \alpha}.
\]

Now since \( u_0 \in \Omega \), there exist a constant \( M > 0 \), uniformly chosen for \( u_0 \in \Omega \), such that

\[
\int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) \left[ (\delta^2 - \lambda \delta + 1)|u_0|^2 + |\nabla u_0|^2 + |v_0|^2 + 2F(u_0) \right] \leq M.
\]

Then we get for \( k \geq K(\varepsilon) \),

\[
\int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) \left[ (\delta^2 - \lambda \delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 + 2F(u) \right] \leq Me^{-\delta \alpha t} + \frac{1 - e^{-\delta \alpha}}{2\delta \alpha}.
\]

Choosing \( T(\varepsilon) = \frac{1}{\delta} \ln \left( \frac{2M\delta \alpha}{2e\delta \alpha - \varepsilon} \right) \), we deduce that

\[
\int_{\mathbb{R}^N} \theta \left( \frac{|x|^2}{k^2} \right) \left[ (\delta^2 - \lambda \delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 \right] \leq \varepsilon \quad \text{for} \ t \geq T(\varepsilon), \ k \geq K(\varepsilon)
\]

which yields \( E \) since \( 0 < \delta^2 - \lambda \delta + 1 < 1 \) for the particular choice of \( \delta \), and the proof is complete.

By multiplying equation \( 34 \) with \( v \) and integrating we deduce the following energy equation

\[
\frac{d}{dt} E(w(t)) + 2\delta E(w(t)) = G(w(t)) \quad \forall t > 0,
\]

where \( E(w) \) is the quasi-energy functional,

\[
E(w) = (\delta^2 - \lambda \delta + 1)\|u\|^2_H + \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \|v\|^2_H,
\]

and

\[
G(w) = -2(\lambda - 2\delta)\|v\|^2_H + 2 \int_{\mathbb{R}^N} \frac{g}{2} \, dx - 2 \int_{\mathbb{R}^N} f(u) v \, dx.
\]

This energy functional \( E \) will be used later as an equivalent norm, more suitable in proving the asymptotical compactness.

The following lemma will be also useful in proving the asymptotical compactness.

**Lemma 2.6** Let \( w_n = (u_n, v_n) \rightharpoonup w_0 = (u_0, v_0) \) weakly in \( X \), then for every \( T > 0 \) we have

\[
S(t)w_n \rightharpoonup S(t)w_0 \quad \text{weakly in} \ L^2(0, T; X)
\]

and

\[
S(t)w_n \rightharpoonup S(t)w_0 \quad \text{weakly in} \ X, \quad \text{for} \ 0 \leq t \leq T.
\]
2.2 Global Attractor

PROOF: Since \( \{w_n\}_n \) converges weakly in \( X \), then it is bounded in \( X \) so that, by lemma 2.4, \( \{S(t)w_n\}_n \) is bounded in \( L^\infty(0,T;H^{-1}(\mathbb{R}^N)) \). This, with (20), implies that
\[
\frac{\partial}{\partial t} S(t)w_n \text{ is bounded in } L^\infty(0,T;H^{-1}(\mathbb{R}^N))
\]
and
\[
S(t)w_n \text{ is bounded in } L^\infty(0,T;L^2(\mathbb{R}^N)).
\]
We infer that there exists a subsequence \( \{w_{n_j}\}_j \) and \( w_\infty = (u_\infty,v_\infty) \in L^\infty(0,T;X) \) such that
\[
S(t)w_{n_j} \rightarrow w_\infty \text{ weakly in } L^2(0,T;X),
\]
\[
\frac{\partial}{\partial t} S(t)w_{n_j} \rightarrow \frac{\partial}{\partial t} u_\infty \text{ weakly in } L^\infty(0,T;H^{-1}(\mathbb{R}^N))
\]
and
\[
\frac{\partial}{\partial t} S(t)u_{n_j} \rightarrow \frac{\partial}{\partial t} u_\infty \text{ weakly in } L^2(0,T;H).
\]
We can show that \( w_\infty \) is a solution of (20) with \( w_\infty(0) = w_0 \). Indeed, we have by the mild solution formula,
\[
S(t)w_{n_j} = e^{-Gt}w_{n_j} + \int_0^t e^{-G(t-s)} R(S(s)w_{n_j}) \, ds.
\]
And, since \( w_{n_j} \rightarrow w_0 \) weakly in \( X \), we deduce from (13) that
\[
e^{-Gt}w_{n_j} + \int_0^t e^{-G(t-s)} R(S(s)w_{n_j}) \, ds \rightarrow e^{-Gt}w_0 + \int_0^t e^{-G(t-s)} R(w_\infty(s)) \, ds,
\]
weakly in \( X \). which implies, by the uniqueness of weak limit that
\[
w_\infty(t) = e^{-Gt}w_0 + \int_0^t e^{-G(t-s)} R(w_\infty(s)) \, ds.
\]
That is \( w_\infty \) is a solution of (20) and by the uniqueness of solutions we have \( w_\infty(t) = S(t)w_0 \). This shows that any subsequence of \( S(t)w_n \) has a weakly convergent subsequence in \( L^2(0,T;X) \), therefore we conclude (11). A similar argument yields (22).

Similar to (11) we also have that if \( w_n \rightarrow w \) weakly in \( X \), then for \( 0 \leq s \leq T \),
\[
S(t)w_n \rightarrow S(t)w \text{ weakly in } L^2(s,T;X)
\]
We state here another useful lemma.

**Lemma 2.7** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). Suppose \( u_n \rightarrow u \) in \( L^2(\Omega) \) and \( v_n \rightarrow v \) weakly in \( L^2(\Omega) \), then \( \int_\Omega f(u_n)v_n \, dx \rightarrow \int_\Omega f(u)v \, dx \) in \( \mathbb{R} \) (up to a subsequence extraction).

**Proof:** By (13) we can show, up to a subsequence, that \( f(u_n) \rightarrow f(u) \) in \( L^2(\Omega) \). Now define the linear functionals \( I_n \) and \( I \) on \( L^2(\Omega) \) by
\[
I_n(v) = \int_\Omega f(u_n)v \, dx, \quad I(v) = \int_\Omega f(u)v \, dx.
\]
Then \( I_n \rightarrow I \) in \( L^2(\Omega)^* \) (the dual space of \( L^2(\Omega) \). Indeed
\[
|I_n(v) - I(v)| \leq \int_\Omega |f(u_n) - f(u)||v| \, dx \leq \|f(u_n) - f(u)\|_{L^2} \|v\|_{L^2},
\]
which implies that
\[
\|I_n - I\|_{L^2} \leq \|f(u_n) - f(u)\|_{L^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]
So \( I_n \rightarrow I \) in \( L^2(\Omega)^* \) and \( v_n \rightarrow v \) weakly in \( L^2(\Omega) \), then it follows, by a classical result in functional analysis that \( I_n(v_n) \rightarrow I(v) \), which proves the lemma.

We are now ready to prove the asymptotic compactness of the semiflow \( S(t) \).

**Theorem 2.8** The semiflow \( S(t) \) generated by the system (20) is asymptotically compact in \( X \), that is if \( \{w_n\}_n \) is a bounded sequence in \( X \) and \( t_n \rightarrow +\infty \), then \( \{S(t_n)w_n\}_{n \geq 1} \) is precompact in \( X \).

**Proof:** Let \( w_n \) be a bounded sequence in \( X \) with \( \|w_n\|_X \leq R \) and \( t_n \rightarrow +\infty \) then by (31) there exists a constant \( T(R) > 0 \) depending only on \( R > 0 \) such that
\[
S(t_n)w_n \in O, \quad \forall t_n \geq 1, \quad \forall t \geq T(R).
\]
Since \( t_n \rightarrow +\infty \), there exists \( N_1(R) \) such that \( n \geq N_1 \) implies \( t_n \geq T(R) \) so that
\[
S(t_n)w_n \in O, \quad \forall n \geq N_1(R).
\]
Then there exists \( w \in X \) such that, up to a subsequence
\[
S(t_n)w_n \rightarrow w \quad \text{weakly in } X.
\]
Now for every \( T > 0 \) there exists \( N_2(R,T) \) such that for \( n \geq N_2(R,T) \) we have \( t_n - T \geq T(R) \) so that
\[
S(t_n - T)w_n \in O \quad \forall n \geq N_2(R,T).
\]
Thus there is a \( w_T \in O \) such that
\[
S(t_n - T)w_n \rightarrow w_T \quad \text{weakly in } X,
\]
and by the weak continuity, we must have \( w = S(T)w_T \) which implies that
\[
\liminf_{n \to \infty} \|S(t_n)w_n\|_X \geq \|w\|_X.
\] (57)

So we only need to prove that
\[
\limsup_{n \to \infty} \|S(t_n)w_n\|_X \leq \|w\|_X.
\] (58)

By the energy equation, it follows that any solution \( u(t) = S(t)w \) of (20) satisfies
\[
E(\hat{S}(t)w) = e^{-25(t-s)}E(S(s)w) + \int_s^t e^{-25(t-r)}G(S(r)w) \, dr, \quad t \geq s \geq 0.
\] (59)

where \( E \) and \( G \) are given by (59) and (10), respectively.

In the following, \( T_0 \) is the constant in (24), and for \( \varepsilon > 0, T(\varepsilon) \) is the constant in (24). Let \( T_0(\varepsilon) \) be a fixed constant such that \( T_0(\varepsilon) \geq \max\{T(\varepsilon), T_0\} \). Taking \( T \geq T_0(\varepsilon) \), and applying (59) to the solution \( S(t)(S(t_0 - T)w_n) \) with \( s = T_0 \) and \( t = T \), then we get, for \( n \geq N_2(R, T) \),
\[
E(S(t_n)w_n) = E(S(T)(S(t_n - T)w_n)) \leq e^{-25(T - T_0)}E(S(T_0)(S(t_n - T)w_n)) + \int_{T_0}^T e^{-25(T-r)}G(S(r)(S(t_n - T)w_n)) \, dr.
\] (60)

Since \( T_0 \geq T_0 \) we have \( S(T_0)(S(t_n - T)w_n) \in O \) for \( n \geq N_2(R, T) \), therefore by the definition of \( E \) we find that
\[
e^{-25(T - T_0)}E(S(T_0)(S(t_n - T)w_n)) \leq Ce^{-25(T - T_0)}, \quad \forall n \geq N_2(R, T).
\] (61)

On the other hand, we have
\[
\int_{T_0}^T e^{-25(T-r)}G(S(r)(S(t_n - T)w_n)) \, dr = -2(\lambda - \delta) \int_{T_0}^T e^{-25(T-r)}\|S(r)S(t_n - T)v_n\|^2 \, dr + 2 \int_{T_0}^T e^{-25(T-r)} \int_{\mathbb{R}^N} gS(r)S(t_n - T)v_n \, dx \, dr
\]
\[
-2 \int_{T_0}^T e^{-25(T-r)} \int_{\mathbb{R}^N} f(S(r)S(t_n - T)v_n)S(r)S(t_n - T)v_n \, dx \, dr
\] (62)

Let’s handle the first and last term of (62). Since we have,
\[
e^{-25(T-r)}S(r)S(t_n - T)v_n \longrightarrow e^{-25(T-r)}S(r)v \text{ weakly in } L^2(T_0, T; H),
\]

it follows that:
\[
\liminf_{n \to \infty} \|e^{-25(T-r)}S(r)S(t_n - T)v_n\|_{L^2(T_0, T; H)} \geq \|e^{-25(T-r)}S(r)v\|_{L^2(T_0, T; H)}
\]
which implies that
\[
\limsup_{n \to \infty} -2(\lambda - \delta)\|e^{-25(T-r)}S(r)S(t_n - T)v_n\|_{L^2(T_0, T; H)} \leq -2(\lambda - \delta)\|e^{-25(T-r)}S(r)v\|_{L^2(T_0, T; H)}.
\] (63)

Also by (60) and (61) we have
\[
\int_{T_0}^T e^{-25(T-r)} \int_{\mathbb{R}^N} gS(r)S(t_n - T)v_n \, dx \, dr \longrightarrow \int_{T_0}^T e^{-25(T-r)} \int_{\mathbb{R}^N} gS(r)v_T \, dx \, dr
\] (64)

Now let’s handle the nonlinear term of (62). We have
\[
-2 \int_{T_0}^T e^{-25(T-r)} \int_{\mathbb{R}^N} f(S(r)S(t_n - T)v_n)S(r)S(t_n - T)v_n \, dx \, dr
\]
\[
= -2 \int_{T_0}^T e^{-25(T-r)} \int_{|x| \geq k} f(S(r)S(t_n - T)v_n)S(r)S(t_n - T)v_n \, dx \, dr
\]
\[
-2 \int_{T_0}^T e^{-25(T-r)} \int_{|x| \leq k} f(S(r)S(t_n - T)v_n)S(r)S(t_n - T)v_n \, dx \, dr.
\] (65)

Handling the first term on the right-hand side of (62) gives
\[
\left| 2 \int_{T_0}^T e^{-25(T-r)} \int_{|x| \geq k} f(S(r)S(t_n - T)v_n)S(r)S(t_n - T)v_n \, dx \, dr \right|
\]
\[
\leq C \int_{T_0}^T e^{-25(T-r)} \int_{|x| \geq k} |S(r)S(t_n - T)v_n||S(r)S(t_n - T)v_n| \, dx \, dr
\]
\[
\leq C \int_{T_0}^T e^{-25(T-r)} \left( \int_{|x| \geq k} |S(r)S(t_n - T)v_n|^2 \right)^{\frac{1}{2}} \left( \int_{|x| \geq k} |S(r)S(t_n - T)v_n|^2 \right)^{\frac{1}{2}}
\]
\[
\leq \varepsilon^2 C \int_{T_0}^T e^{-25(T-r)} \, dr \leq \frac{\varepsilon^2 C}{25}, \quad n \geq N_2(R, T).
\] (66)
We treat now the second term on the right-hand side of (63). We want to prove that as \( n \to +\infty, \)
\[
\int_{T_0}^{T} e^{-2\delta(T-r)} \int_{|x| \leq k} f(S(r)S(t_n - T)u_n)S(r)S(t_n - T)v_n \, dx \, dr \to \int_{T_0}^{T} e^{-2\delta(T-r)} \int_{|x| \leq k} f(S(r)w_T)S(r)v_T \, dx \, dr
\]  
(67)
Set \( \Omega_k = \{ x \in \mathbb{R}^N : |x| \leq k \} \) and let \( r \in [T_0,T]. \) Then we have
\[ S(r)S(t_n - T)w_n \to S(r)v_T, \quad \text{weakly in } X. \]
By the compactness of the Sobolev embedding \( H^1(\Omega_k) \subset L^2(\Omega_k), \) we infer that
\[ S(r)S(t_n - T)u_n \to S(r)v_T, \quad \text{strongly in } L^2(\Omega_k) \]
and
\[ S(r)S(t_n - T)v_n \to S(r)v_T, \quad \text{weakly in } L^2(\Omega_k) \]  
(69)
then (67) follows from lemma 3.4.
By (65), (66) and (67) we find that for \( k \geq K(\varepsilon), \)
\[
\limsup_{n \to \infty} -2 \int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} f(S(r)S(t_n - T)u_n)S(r)S(t_n - T)v_n \, dx \, dr \leq \varepsilon C - 2 \int_{T_0}^{T} e^{-2\delta(T-r)} \int_{|x| \leq k} f(S(r)v_T)S(r)v_T \, dx \, dr.
\]
Letting \( k \to \infty \) we obtain
\[
\limsup_{n \to \infty} -2 \int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} f(S(r)S(t_n - T)u_n)S(r)S(t_n - T)v_n \, dx \, dr \leq \varepsilon C - 2 \int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} f(S(r)v_T)S(r)v_T \, dx \, dr.
\]  
(70)
By (62), (63), (64) and (66), we finally obtain
\[
\limsup_{n \to \infty} \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)(S(t_n - T)w_n)) \, dr \leq -2(\lambda - \delta) \int_{T_0}^{T} e^{-2\delta(T-r)} \| S(r)v_T \|^2 \, dr + 2 \int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} gS(r)v_T \, dx \, dr
\]
\[-2 \int_{T_0}^{T} e^{-2\delta(T-r)} \int_{\mathbb{R}^N} f(S(r)v_T)S(r)v_T \, dx \, dr + \varepsilon C, \]
that is
\[
\limsup_{n \to \infty} \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)(S(t_n - T)w_n)) \, dr \leq \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)v_T) \, dr + \varepsilon C.
\]  
(71)
Taking limit of (60), (61) and (71) we get, as \( n \to \infty, \)
\[
\limsup_{n \to \infty} E(S(t_n)w_n) \leq Ce^{-2\delta(T-T_0)} + \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)v_T) \, dr + \varepsilon C.
\]  
(72)
On the other hand, since \( w = S(T)w_T, \) by (29) we also have that
\[
E(u) = E(S(T)v_T) = e^{-2\delta(T-T_0)} E(S(T_0)w_T) + \int_{T_0}^{T} e^{-2\delta(T-r)} G(S(r)v_T) \, dr.
\]  
(73)
Hence it follows from (72) - (73) that
\[
\limsup_{n \to \infty} E(S(t_n)w_n) \leq E(u) + Ce^{-2\delta(T-T_0)} + \varepsilon C - e^{-2\delta(T-T_0)} E(S(T_0)v_T).
\]  
(74)
Now since \( w_T \in O \) and \( T_0 \geq T(O) \) we find that
\[
|e^{-2\delta(T-T_0)} E(S(T_0)v_T)| \leq Ce^{-2\delta(T-T_0)}.
\]
Then from (74) we have
\[
\limsup_{n \to \infty} E(S(t_n)w_n) \leq E(u) + Ce^{-2\delta(T-T_0)} + \varepsilon C.
\]  
(75)
Now taking limit of (65) as \( T \to \infty \) and then letting \( \varepsilon \to 0, \) we obtain
\[
\limsup_{n \to \infty} E(S(t_n)w_n) \leq E(u).
\]
Lemma 3.1

For \( w \) and \( \delta \), choose equivalent norms in \( V \) as desired in (58). Therefore we get the strong convergence of

Theorem 2.9

Assume that \( \lambda \) and \( \delta \) are positive constants. Then, problem (77) is equivalent to the norm of \( X \), and the norm of \( X \) is defined by it. Then we have

\[
\limsup_{n \to \infty} \| S(t_n)w_n \|_X \leq \| w \|_X
\]
as desired in (58). Therefore we get the strong convergence of \( S(t_n)w_n \) to \( w \) in \( X \). The proof is complete.

Now we state our main result obtained in this section.

Theorem 2.9 Assume that \( f \) satisfies (70), (71) and \( g \in L^2(\mathbb{R}^N) \). Then, problem (70) possesses a global attractor in \( X = H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \) which is a compact invariant subset that attracts every bounded set of \( X \) with respect to the norm topology.

Proof: Since we have established the existence of an absorbing set in (30) and the asymptotic compactness of the semiflow \( S(t) \) in \( X \) in Theorem 2.8, the conclusion follows from Theorem 2.9.

3 The Wave Equation Without Mass Term

In this section we will study the existence of global attractor for the wave equation without mass term,

\[
\begin{cases}
u_{tt} + \lambda \nu - \Delta \nu + f(u) = 0, & x \in \Omega, \quad t > 0, \\
u|_{\partial \Omega} = 0, \\
u(0, x) = u_0(x), \quad \nu_t(0, x) = u_1(x)
\end{cases}
\]

where \( \Omega \) is a domain of \( \mathbb{R}^N \) bounded only in one direction, with smooth boundary. The case \( \Omega = \mathbb{R}^N \), for this equation is still an open problem due to some difficulties in getting an inequality such as (21) for the operator \( G \). In our case we will use an equivalent norm (provided by the Poincaré inequality) for which the desired estimate works. We assume the same conditions (17) and (18) for the nonlinear function \( f \).

We will work in the phase space \( X = V \times H \) where \( V = H^1_0(\Omega) \), \( H = L^2(\Omega) \). \( H \) is endowed with the norm and inner product for \( L^2 \) and \( V \) is endowed with the inner product and norm defined as follows,

\[
(u, v)_V = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in V \quad \text{and} \quad \| u \|_V = \| \nabla u \|_{H^1}, \quad u \in V.
\]

Now define the following bilinear operator in \( V \):

\[
(u, v)_1 = \int_{\Omega} u v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad u, v \in V,
\]

which is also an inner product in \( V \) with induced norm \( \| u \|_1 = \left[ \| u \|_V^2 + \| \nabla u \|_{H^1}^2 \right]^{\frac{1}{2}} \). By the Poincaré inequality \( \| \cdot \|_V \) and \( \| \cdot \|_1 \) are equivalent norms in \( V \). That is, there are positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \| u \|_V \leq \| u \|_1 \leq C_2 \| u \|_V, \quad \forall u \in V.
\]

Let’s make a transformation to write the equation (77) as a first order abstract ODE.

Choose \( \delta = \frac{\lambda}{\lambda^2 + 4} \) and set \( v = \delta u + u_t \), \( w = \left( \begin{array}{c} u \\ v \end{array} \right) \). Then, problem (77) is equivalent to

\[
\begin{cases}
w_{t} + Gw = R(w), & t > 0, \quad w \in X \\
w(0) = w_0 = (u_0, u_1 + \delta u_0)
\end{cases}
\]

where

\[
R(w) = \left( \begin{array}{c} 0 \\ -f(u) \end{array} \right)
\]

and

\[
Gw = \left( \begin{array}{c} 0 \\ -\Delta u + (\lambda - \delta)v + (\delta^2 - \delta^\lambda)u \end{array} \right)
\]

for \( w = \left( \begin{array}{c} u \\ v \end{array} \right) \in D(G) = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1(\Omega) \).

As in Lemma 2.1 we show the positivity of the operator \( G \) with a similar estimate.

Lemma 3.1 For \( \delta = \frac{\lambda}{\lambda^2 + 4} \), the operator \( G \) is maximal accretive in \( X \) and verifies the following

\[
(G(w), w)_X \geq \sigma \| w \|_X^2 + \frac{\lambda}{2} \| v \|_{H^1}^2, \quad \forall w = \left( \begin{array}{c} u \\ v \end{array} \right) \in X,
\]

where

\[
\sigma = \frac{\lambda}{\sqrt{\lambda^2 + 4(\lambda + \sqrt{\lambda^2 + 4})}}.
\]
Proof: Let \( w = \left( \begin{array}{c} u \\ v \end{array} \right) \in X \) then we have:

\[
(G(w), w)_X = (\delta u - v, u)_V + (\Delta u + (\lambda - \delta)(v - \delta u), v)_V \\
= \delta \|u\|^2_V - (\nabla u, \nabla v)_V + (\Delta u, v)_V + (\lambda - \delta)\|v\|^2_V \\
= \delta \|u\|^2_V + (\lambda - \delta)\|v\|^2_V - \delta(\lambda - \delta)(u, v)_V \\
\geq \delta \|u\|^2_V + (\lambda - \delta)\|v\|^2_V - \delta \lambda \|u\|_H \|v\|_H.
\]

Then setting \( \sigma = \frac{\lambda}{\sqrt{\lambda^2 + 4(\lambda + \sqrt{\lambda^2 + 4})}} \) as in (22), we have:

\[
(G(w), w)_X - \sigma (\|u\|^2_V + \|v\|^2_V) - \frac{\lambda}{2} \|v\|^2_V \geq (\delta - \sigma)\|u\|^2_V + (\lambda^2 - \delta - \sigma)\|v\|^2_V \\
- \delta \lambda \|u\|_V \|v\|_V \\
\geq 2\sqrt{(\delta - \sigma)(\lambda^2 - \delta - \sigma)} \|u\|_V \|v\|_V \\
- \delta \lambda \|u\|_V \|v\|_V.
\]

we can check that \( 4(\delta - \sigma)(\lambda^2 - \delta - \sigma) = \lambda^2 \delta^2 \) so that

\[
(G(w), w)_X - \sigma \|u\|^2_X - \frac{\lambda}{2} \|v\|^2_H \geq 0.
\]

The proof is complete.

The existence of solution for (84) follows in the same approach as for equation (20). Similarly, we can prove an analogous result as in theorem (25) and we have shown that there also exists a bounded absorbing set \( O \) in \( X \).

Now let's establish the tail ends estimates for equation (85).

**Lemma 3.2** If (17), (83) hold, \( g \in H \) and \( w_0 = (u_0, v_0) \in O \), then for every \( \varepsilon > 0 \), there exists \( T(\varepsilon) \) and \( K(\varepsilon) \) such that the solution \( u(t) = (u(t), v(t)) \) of problem (85) satisfies

\[
\int_{\Omega \cap \{|x| \geq k\}} \left[ |u(t)|^2 + |
abla u(t)|^2 + |v(t)|^2 \right] dx \leq \varepsilon, \quad t \geq T(\varepsilon), \quad k \geq K(\varepsilon).
\]  

**(Proof)**

The proof works basically like that for equation (20). Any solution \( u(t) = \left( \begin{array}{c} u(t) \\ v(t) \end{array} \right) \) satisfies:

\[
v_t - \Delta u + (\lambda - \delta)v + (\delta^2 - \lambda \delta)u = f(u) + g
\]

and

\[
u_t + \delta u = v.
\]

We choose the same cut-off function \( \theta \).

Now take inner product in \( H \) of \( \theta \left( \frac{|x|^2}{k^2} \right)v(x) \) with (83) to get

\[
\int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right)vv dx - \int_{\Omega} \Delta \theta \left( \frac{|x|^2}{k^2} \right)v dx + (\lambda - \delta) \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right)|v|^2 dx \\
+ (\delta^2 - \lambda \delta) \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right)uv dx = - \int_{\Omega} f(u) \theta \left( \frac{|x|^2}{k^2} \right)v dx + \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right)gv dx.
\]  

But

\[
- \int_{\Omega} \Delta \theta \left( \frac{|x|^2}{k^2} \right)v dx = \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right)\nabla u \cdot \nabla v + \frac{2}{k^2} \int_{\Omega} \theta' \left( \frac{|x|^2}{k^2} \right)vx \cdot \nabla u \\
= \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right) \left[ |\nabla u|^2 + \nabla u \cdot \nabla u \right] + \frac{2}{k^2} \int_{\Omega} \theta' \left( \frac{|x|^2}{k^2} \right)vx \cdot \nabla u \\
= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right) |\nabla u|^2 + \delta \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right) |u|^2 dx \\
+ \frac{2}{k^2} \int_{\Omega} \theta' \left( \frac{|x|^2}{k^2} \right)vx \cdot \nabla u,
\]

and

\[
(\delta^2 - \lambda \delta) \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right)uv dx = (\delta^2 - \lambda \delta + 1) \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right)(\delta |u|^2 + uu_t) \\
= \frac{1}{2} (\delta^2 - \lambda \delta + 1) \frac{d}{dt} \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right) |u|^2 dx + \delta (\delta^2 - \lambda \delta + 1) \int_{\Omega} \theta \left( \frac{|x|^2}{k^2} \right) |u|^2.
\]
Then \( \delta^2 - \lambda \delta \) could be negative for certain values of \( \lambda \). Since \( \delta^2 - \lambda \delta + 1 > 0 \), let’s introduce another equation to get a more desirable identity.

Taking inner product of \( \theta (|x|^2 u(x)) \) with \( \delta \theta (\delta u + u_t) \), we get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \theta \frac{|x|^2}{k^2} |u|^2 \, dx + \int_\Omega \theta \frac{|x|^2}{k^2} (|u|^2 + |v|^2) \, dx = \int_\Omega \theta \frac{|x|^2}{k^2} uv \, dx.
\]

And adding the above and (88) yields
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \theta \frac{|x|^2}{k^2} \left[ (\delta^2 - \lambda \delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 \right] + \delta \int_\Omega \theta \frac{|x|^2}{k^2} \left[ (\delta^2 - \lambda \delta + 1)|u|^2 + |\nabla u|^2 + |v|^2 \right] + (\lambda - 3\delta) \int_\Omega \theta \frac{|x|^2}{k^2} |v|^2 \, dx - \frac{2}{k^2} \int_\Omega \theta' \frac{|x|^2}{k^2} v_x \cdot \nabla u.
\]

Then the conclusion follows the same way as in the proof of lemma 2.8.

Similarly, we have the following energy equation for the solution of (89),
\[
\frac{d}{dt} E(w(t)) + 2\delta E(w(t)) = G(w(t)), \quad \forall t > 0,
\]
where
\[
E(w) = (\delta^2 - \lambda \delta + 1)||u||^2_H + ||\nabla u||^2_{H^1} + ||v||^2_{H^1},
\]
and
\[
G(w) = -2(\lambda - 3\delta)||v||^2_{H^1} + 2 \int_\Omega gv \, dx - 2 \int_\Omega f(u)v \, dx.
\]

The rest of the proof of existence of a global attractor is again similar to the case with mass term. We get the main result in this section.

**Theorem 3.3** Let \( \Omega \) be a domain of \( \mathbb{R}^N \) bounded in only one direction. Assume that \( f \) satisfies (17), (19) and \( g \in L^2(\Omega) \). Then, problem (87) possesses a global attractor in \( X = H_0^1(\Omega) \times L^2(\Omega) \) which is a compact invariant subset that attracts every bounded set of \( X \) with respect to the norm topology.

**References**

[1] R. A. Adams, *Sobolev spaces*, 2d Ed., Academic Press, 2003.
[2] A.V. Babin, M.I. Vishik, *Attractors of partial differential evolution equations in an unbounded domain*, Proc. R. Soc. Edinburgh 116A (1990), 221-243.
[3] A. Babin, B. Nicolaenko, *Exponential attractors of reaction-diffusion systems in an unbounded domain*, J. of dyn. and diff. Eq., Vol. 7, No. 4, (1995), 567-590.
[4] P. Bates, K. Lu & B. Wang, *Attractors for lattice dynamical systems*, Int. J. Bifurc. Chaos, 11, No. 1, (2001), 143-153.
[5] S.-N. Chow, J. Mallet-Paret, W. Shen, *Traveling wave in lattice dynamical systems*, J. Diff. Eq. 149, (1998), 248-291.
[6] S.-N. Chow, R. Conti, R. Johnson, J. Mallet-Paret, R. Nussbaum, *Dynamical systems*, Lecture Notes in Mathematics 1822, Springer Verlag, Berlin Haidelberg 2003.
[7] A. Eden, C. Foias, B. Nicolaenko, R. Temam, *Exponential attractors for dissipative evolution equations*, Research in Applied Mathematics, vol. 37, John Wiley-Masson, New York, 1994.
[8] A. Eden, V.K. Kalantarov, *On the discrete squeezing property for semilinear wave equations*, Tr. J. Of Math. 22 (1998), 335-341.
[9] M. Efendiev, A. Miranville, S. Zelik, *Exponential attractors for a nonlinear reaction-diffusion system in \( \mathbb{R}^3 \), C.R. Acad. Paris, t. 330, Série I, 713-718, 2000.
[10] M. Efendiev, A. Miranville, S. Zelik, *Finite-dimensional exponential attractors for nonlinear reaction-diffusion systems in unbounded domains and their approximation*, Proceedings A of the Royal Society 460 (2004), 1107-1129, 2004.
[11] M. Efendiev, A. Miranville, S. Zelik, *Global and exponential attractors for nonlinear reaction-diffusion systems in unbounded domains*, To appear in Proc. R. Soc. Edinburgh: Section A.
[52] Y. You, *Global dynamics of nonlinear wave equations with cubic non-monotone damping*, Dynamics of PDE, Vol.1, No.1 (2004), 65-86.

[53] Y. You, *Spectral barriers and inertial manifolds for time-discretized dissipative equations*, Comp. & Math. with Appl., 48, (2004), 1351-1368.

[54] S.V. Zelik, *The attractor for a nonlinear hyperbolic equation in the unbounded domain*, Disc. Cont. Dyn. Sys., 7, No. 3, (2001), 593-611.

[55] S. Zelik, *The Attractor for a Nonlinear Reaction-Diffusion System in an Unbounded Domain and Kolmogorov’s Epsilon-Entropy*, Math. Nachr. 232 (2001), No. 1, 129-179.

[56] S. Zheng, *Nonlinear evolution equations*, Monographs and surveys in Pure and Applied Mathematics 133, Chapman & Hall/CRC, 2004.

[57] S. Zhou, *Attractors for second order lattice dynamical systems*, J. Differential Equations 179 (2002), 605-624.

[58] E. Zuazua, *Exponential decay for the semilinear wave equation with localized damping in unbounded domains*, J. Math. Pures et appl., 70, (1991), 513-529.