Effective Chern-Simons Theories of Pfaffian and Parafermionic Quantum Hall States, and Orbifold Conformal Field Theories

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Abstract

We present a pure Chern-Simons formulation of families of interesting Conformal Field Theories describing edge states of non-Abelian Quantum Hall states. These theories contain two Abelian Chern-Simons fields describing the electromagnetically charged and neutral sectors of these models, respectively. The charged sector is the usual Abelian Chern-Simons theory that successfully describes Laughlin-type incompressible fluids. The neutral sector is a 2+1-dimensional theory analogous to the 1+1-dimensional orbifold conformal field theories. It is based on the gauge group $O(2)$ which contains a $\mathbb{Z}_2$ disconnected group manifold, which is the salient feature of this theory. At level $q$, the Abelian theory of the neutral sector gives rise to a $\mathbb{Z}_{2q}$ symmetry, which is further reduced by imposing the $\mathbb{Z}_2$ symmetry of charge-conjugation invariance. The remaining $\mathbb{Z}_q$ symmetry of the neutral sector is the origin of the non-Abelian statistics of the (fermionic) $q$-Pfaffian states.

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I. INTRODUCTION

Non-Abelian statistics is an intriguing concept that perhaps may be realized experimentally in two-dimensional electron gases (2DEG) in large magnetic fields. The leading candidates are the fractional quantum Hall (FQH) states at filling factor $\nu = 5/2$ and a number of other unusual cases [1]. A series of quantum Hall states displaying non-Abelian statistics have been proposed for these states, the best known and simplest of them being the Pfaffian FQH state [2,3]. However, still much about the nature of these states remains to be understood in greater detail. All of the wave functions of the proposed states show some sort of pairing correlation, as well as multiparticle generalizations as in the case of the parafermionic states [4]. In all of these cases the spectrum of low-lying states contain excitations with non-Abelian statistics. The notion of non-Abelian statistics [2] is exhibited in these FQH states by exploiting an analogy between the wave functions of interacting electrons in the lowest Landau level and correlators in conformal field theory (CFT) [5].

Although much work has been done on this problem, a general understanding of what particular correlations are responsible for non-Abelian statistics is still lacking. In particular, it is not known as to what extent pairing-type mechanisms and its generalizations are needed for non-Abelian statistics to exist. Specifically, more than one physical realization of non-Abelian statistics may be possible and several of them have been proposed [3,6–9]. It is, therefore, an important issue to understand how these constructions are related to each other and as to what extent they are truly different.

On the one hand, much of our present understanding of non-Abelian FQH states comes from the CFT structure of their edge states, which has been worked out in a number of explicit examples [2,4,6,10]. It has been known from extensive work in the Abelian FQH states [11] that CFT describes successfully the edge excitations, the low-energy degrees of freedom of FQH states.

On the other hand, it has been proven very useful to consider a description of quantum Hall states in terms of an effective low energy field theory describing the basic bulk quasi-
particle excitations \cite{11–14}. These effective field theories generically have the form of a (suitably generalized) Chern-Simons (CS) theory in 2+1-dimensions \cite{15–17}. This powerful approach has been very successful in the description of general Abelian FQH states. However, for the case of non-Abelian FQH states, such as the Pfaffian state and its generalizations, writing down such a CS theory has proven to be rather more subtle \cite{8,9,18–21}. One of the outstanding questions is the connection between the well-defined CFT notion of non-Abelian statistics and its 2+1-dimensional counterpart.

In this paper we construct a class of CS theories that is appropriate for describing the bulk excitations of the Pfaffian states and their parafermionic generalization. The theories that we will discuss here contain two Abelian fields. One field describes the charge quantum numbers of the quasiparticles, while the other field is associated with a set of additional quantum numbers describing the “neutral sector”. The charged sector is given by the familiar construction that yields, for example, the usual Laughlin states \cite{11}. The novelty of our approach relies in the choice of theory describing the neutral sector. It is given by an Abelian CS theory with a gauge group with disconnected components, specifically $O(2)$. These theories are appropriate for describing the 2+1-dimensional analogs of the relevant Abelian orbifold CFTs \cite{16}. Since the Pfaffian models can be also conveniently realized as orbifold CFTs \cite{22}, our approach yields a simple way of dealing with the bulk theories corresponding to the Pfaffian states. Moreover, we could also include in our approach some generalizations of the Pfaffian states, most notably some of the Parafermionic states.

One advantage of this formulation is that, although it makes no explicit assumption on the nature of the microscopic correlations responsible for the non-Abelian statistics, it gives new insights on the meaning of the non-Abelian statistics in these systems. Specifically, for a $O(2) = SO(2) \times \mathbb{Z}_2$ CS theory at level $k$, a symmetry $\mathbb{Z}_{2k}$ characterizes the different disconnected components. One imposes the condition of charge neutrality in this sector by moding out this symmetry by the $\mathbb{Z}_2$ symmetry of charge conjugation, resulting in a $\mathbb{Z}_k$ symmetry that characterizes the neutral sector of the complete theory. This construction defines an *Abelian orbifold* in the context of CS gauge theories \cite{16}. In this context, the
level \( k \) is interpreted as the number of intermediate channels in the fusion rules of the CFT description \([3]\). We will discuss below that the level of this theory may in fact be related naturally with the physics of pairing (and its generalizations). In the CS version of the theory, we find that it is necessary to introduce a set of Wilson loop operators with fractional magnetic flux in order to insure the completeness of the theory. For these operators to make sense, one is led to consider a multiple \( k \)-covering of the \( 2+1 \)-dimensional space-time manifold \([23]\).

Equivalently, one could also define a special class of Wilson loops with a specific regularization that involves the twisting of the framing field \([15]\) by \( k \) units. It is this multivaluedness of some of the Wilson loop operators that is the root of the notion of non-Abelian statistics, that follows from the CFT definition: more than one intermediate channel is involved in the fusion rules. In the CS theory, the fusion rules follow simply from charge conservation, modulo the multivalued character of some of the operators. The physical mechanism of non-Abelian statistics in this context arises, therefore, from considering the identification of several copies of identical (Abelian) theories, by some projection mechanism \([9]\).

In section 2 we review the Abelian CS theory and its connection to CFT, stressing the points that are crucial for generalizing the theory to \( O(2) \) gauge groups. In section 3 we discuss at some length the CS theory that corresponds to orbifold CFTs. More specifically, we discuss the \( 2 + 1 \) dimensional definition of the \( N \) toroidal models \([24,25]\), and their \( \mathbb{Z}_2 \) orbifolds. These models turn out to be relevant for the Pfaffian systems. In section 4 we consider specific applications to the quantum Hall effect, such as the Pfaffian and Parafermionic states, and their CS formulation. Our conclusions are summarized in section 5.

II. ABELIAN CHERN-SIMONS THEORIES

To describe the charged sector we consider an Abelian Chern-Simons field theory, defined by the action

\[
\]
\[ S = \frac{k}{4\pi} \int_{\mathcal{M}} d^3x \, \varepsilon^{\mu\nu\rho} a_\mu(x) \partial_\nu a_\rho(x) , \]  

where \( a_\mu(x) \) is the Chern-Simons field, \( \mathcal{M} \) is the space-time manifold on which coordinates \( x \) are defined, and the coupling constant \( k \) is a non-negative integer, which is referred to as the level. Having in mind physical applications, we consider \( \mathcal{M} = \Sigma \times S^1 \), where \( \Sigma \) is a two-dimensional spatial manifold (a Riemann surface) and \( S^1 \) is the compact time coordinate.

For our applications, \( \Sigma \) is taken to be a manifold without curvature, which could have a boundary (e. g. , a disk or an annulus) or not (e. g. , a torus). The gauge field \( a_\mu(x) \) represents a set of matter currents, as is the case in all effective theories of the FQHE \[11,13\]

Consequently, the conserved hydrodynamic current

\[ J^\mu(x) = \frac{k}{2\pi} \varepsilon^{\mu\nu\rho} \partial_\nu a_\rho(x) . \]  

The conserved hydrodynamic current couples minimally to an externally applied electromagnetic field.

The topological theory defined by (1) has global observables, given by the Wilson loops

\[ W_n[C] = \exp(i \, n \, \int_{\Sigma} dx^\mu a_\mu(x) , \quad \int_{\Sigma} dx^\mu a_\mu(x) \]  

where \( C \) is a closed path in \( \mathcal{M} \), \( \Phi[C] \) the magnetic flux subtended by any surface based on \( C \) and \( n \) labels the \( U(1) \) representation. We consider that \( n \) takes all possible integer values, which means that we can measure and distinguish the value of different charges by coupling the system to an external source. Therefore, the Hilbert space of the theory on a closed spatial manifold \( \Sigma \) is constructed out from a collection of ‘bulk’ states \(| n \rangle \), which are labelled by the holonomy \( n \). The vacuum expectation values of the Wilson operators furnish a representation of the one-dimensional lattice of fluxes (or magnetic charges):

\[ \langle W_n[C] \rangle \cdot \langle W_m[C] \rangle = \langle W_{n+m}[C] \rangle . \]  

The spectrum of the theory is obtained \[24\] by minimally coupling the current (2) to an external electromagnetic field, and integrating out the field \( a_\mu(x) \), yielding

\[ Q_n = \frac{n}{k} , \quad \frac{\theta_n}{\pi} = \frac{n^2}{k} , \]  

\[ 5 \]
where \( n \) is an integer and \( Q_n \) and \( \theta - n/\pi \) are the electric charges and quantum statistics of the quasi-particle excitation corresponding to the state \( | n \rangle \).

The connection of the above theory to the corresponding 1 + 1-dimensional CFT is established along the lines of the classic work of Witten [15]. Namely, we consider a (compact) spacetime manifold \( \mathcal{M} = D \times S^1 \), where \( D \) is the spatial disk and \( S^1 \) represents the (compact) time coordinate \( t \). The presence of the boundary breaks general covariance in 2 + 1 dimensions. Therefore, the topological degrees of freedom of the Chern-Simons theory become dynamical variables of the corresponding \( \tilde{U}(1) \) CFT describing the 1 + 1-dimensional chiral bosonic field \( \varphi(\theta + ivt) \), with \( R, \theta \) the radius and angular coordinate of the disk and \( v \) the velocity of the edge excitations. This free theory has a central charge \( c = 1 \) and Lagrangian density given by

\[
\mathcal{L} = \frac{1}{4\pi} \left( \partial_t \varphi - v \partial_x \varphi \right) \partial_x \varphi ,
\]

where \( x = R\theta \) and we have arbitrarily chosen one chirality. In the following, it will be useful to define a holomorphic coordinate \( z = \exp(-i\theta - vt/R) \) on the entire complex plane. Arbitrary correlators are evaluated using the basic two-point function normalized as

\[
\langle \varphi(z)\varphi(w) \rangle = -\ln(z - w) . \tag{7}
\]

The observables are given by vertex operators

\[
V_{\ell}(z) = : \exp \left( i \frac{\ell}{\sqrt{k}} \varphi(z) \right) : , \quad \ell \in \mathbb{Z} . \tag{8}
\]

The \( \tilde{U}(1) \) symmetry of the theory is manifest through the presence of the current

\[
J(z) = -\frac{1}{2\pi} \partial_z \varphi(z) , \tag{9}
\]

which is the ‘boundary’ analog of the ‘bulk’ expression (4). The spectrum of charges and conformal dimensions spanned by these operators is given by \( Q_{\ell} = \ell/k \) and \( h_{\ell} = \ell^2/2k \), respectively.

Note that it is important to consider the topology of the Abelian gauge group manifold to discuss the class of observables: if the gauge group is \( \mathbb{R} \), the theory is not a rational
CFT (RCFT) because the number of primary fields, which in this case are in one-to-one correspondence with the vertex operators (8), is infinite (and parametrized by \( \ell \)). One can equivalently define a set of conformal highest weight states \( | \ell \rangle = V_\ell(0) | 0 \rangle \), where \( | 0 \rangle \) is the standard \( SL(2, \mathbb{C}) \) vacuum of CFT. Physically, \( | \ell \rangle \) can be put in correspondence with the CS state associated to the Wilson operator \( W_\ell[C] \) that carries magnetic flux and therefore can be thought of as adding \( \ell \) ‘vortices’ to the vacuum. On the other hand, if the gauge group manifold is compact, such as \( \mathbb{R}/\mathbb{Z} \), then the field \( \varphi(z) \) is compactified on a circle of radius \( r \), that is \( \varphi(z) \equiv \varphi(z) + 2\pi r \); the value of \( r \) is obtained by demanding that all vertex operators \( V_\ell(z) \) are well-defined under this shift, which implies \( r = \sqrt{k} \). If \( k \) is an integer, the spectrum of the theory is finite: \( Q_{n+k} = Q_n + 1, h_{n+k} = h_n + k/2 + n \).

This means that there are only \( k \) independent primary fields of the form (8) and the theory possesses a symmetry \( \mathbb{Z}_k \). It is, therefore, a RCFT. Note that if one imposes the condition that the quantum statistics \( \theta/\pi \) of the corresponding quasi-particles in \( 2 + 1 \) dimensions (which is defined up to an even integer) is invariant under this identification, then \( k \) must be even. In physical terms, this means that there are \( k \) different independent quasi-particles, that add up to a \( k \) composite one that can be created or removed from the system without changing the vacuum.

Hence, the connection between CFT and Abelian Chern-Simons field theory can be summarized as follows:

- The coefficient \( k \) in the CS action and the square of the compactification satisfy \( k = r^2 \);
- The Wilson loops correspond to vertex operators;
- Charges are proportional (equal with a convenient definition), i.e. \( \ell = n \);
- Conformal dimensions and quantum statistics of the quasi-particle excitations are related by \( 2h = \theta/\pi \);

Conformal dimensions can be read off from CS data by considering a thickened closed Wilson loop, defined by a framing of the original curve \( C \). A framing is a vector field normal to the
curve, which is needed in order to properly define the holonomies in the quantum theory \[15\]. The writhing number of this field along the curve is known as the self-linking number, and could be equivalently defined as the linking number between the original loop $C$ and the one obtained from it by the displacement action of the framing. Note that the framing should be regraded as a regulator field, in order to make sense of otherwise ill-defined operators in the theory. It could also be regraded as a generalization of the familiar ‘point-splitting’ regularization in quantum field theory.

According to \[15\], the conformal dimension of a primary field is obtained by shifting the framing in $t$ units, so that

$$\langle W_n[C] \rangle \rightarrow \exp{(2\pi i t h)} \cdot \langle W_n[C] \rangle.$$  \hspace{1cm} (10)

Therefore, one defines the conformal dimension of the Wilson loop as $h = \alpha/2\pi$, where $\alpha$ is the $2 + 1$-dimensional holonomy in \[11\], with $t = 1$. For the Abelian theory \[1\], it is given by $h = n^2/2k$. Note that underlying this definition is the important fact that after going around the loop $C$ once, the framing vector field returns to its original value (as implied by $t = 1$). Later on, we will modify this later condition in order to allow for more general Wilson loop operators.

### III. NON-ABELIAN GENERALIZATIONS

Another class of interesting theories arises when one establishes an equivalence relation among the Wilson loops charges $n$. One may encounter a situation in which not all the values of $n$ are actually physically different. The simplest case is to consider that $n$ and $m$ describe equivalent Wilson loops provided that $m = n \text{ mod } 2N$ where $N$ is a given natural number. This equivalence relation introduces a $\mathbb{Z}_{2N}$ symmetry among the observables \[3\] and gives rise to a finite number $(2N)$ of observable Wilson loop charges. The values of the holonomies of Wilson unknotted loops $C$ (assuming a canonical framing in $\mathcal{M}$) have to be given, therefore, by the $2N - th$ roots of unity:
\[ \langle W_n[C] \rangle = \exp \left( 2\pi i \frac{n}{2N} \right), \quad n = 1, \ldots, 2N. \quad (11) \]

The action for this CS theory is given by (1) with \( k = 2N \). In principle, this theory possesses an infinite number of observables, spanned by Wilson loops with charges \( Q = n/2N \) and statistics \( \theta/\pi = n^2/2N \). However, the equivalence relation is not simply a matter of definition: only a finite number of bound states are assumed to be observable, in the sense that not all possible values of the charge \( n \) are observable. In particular, the vacuum should be identified with the state \( |2N\rangle \). So, even though we can couple the system to external sources by means of the current (2), one could only distinguish their \( \mathbb{Z}_{2N} \) structure. Furthermore, one can consider a additional \( \mathbb{Z}_2 \) symmetry among the Wilson loop operators, namely charge conjugation. It amounts to sending \( W_n[C] \rightarrow W^*_n[C] \), or, equivalently, \( \Phi[C] \rightarrow -\Phi[C] \). This is also equivalent to reversing the orientation of Wilson loops. Note also that the action of this symmetry on the Wilson loops is equivalent to the map \( n \rightarrow (2N - n) \).

Finally, we remark that if the CS theory is meant to describe neutrally charged quasi-particles, electric charges should not be observable, but quasi-particles could still carry magnetic flux, which implies a breakdown of Gauss’ Law.

The corresponding 1 + 1-dimensional theories to the proposed CS theories with \( \mathbb{Z}_{2N} \) symmetry are the \( c = 1 \) gaussian \( \mathbb{Z}_{2N} \) models, the so-called \( N \) toroidal models [24] [25]. These are rational CFTs compactified on a circle of radius \( r = \sqrt{2p/p'} \) with \( p \) and \( p' \) coprime integers such that \( N = pp' \). Under the duality \( r \rightarrow 2/r \), which makes sense only after a corresponding isomorphic antichiral theory is glued to the holomorphic one we are discussing, the roles of \( p \) and \( p' \) are exchanged. Without loss of generality, we therefore make here the choice \( p' = 1 \) and \( p = N \), which implies that \( r \geq \sqrt{2} \). The \( N \)-toroidal model possesses only \( 2N \) primary fields, given by the vertex operators (8), with \( n = \ell = 1, \ldots, 2N \). Note that since \( 2N \) is even, there is a further \( \mathbb{Z}_2 \) symmetry which could be eliminated from the spectrum. When this is done, it gives rise to an \( Abelian \mathbb{Z}_2 \) orbifold CFT. The \( \mathbb{Z}_2 \) symmetry action on the chiral field is \( \varphi \rightarrow -\varphi \). Note that enforcing this symmetry automatically describes theories that are electrically neutral, as can be seen recalling the expression of the
current (9). This mechanism is the origin of the non-Abelian statistics, as will be discussed below.

Let us discuss in greater detail these CFTs. The family of vertex operators consistent with the $\mathbb{Z}_{2N}$ symmetry of the $N$-toroidal model is given by

$$V_\ell(z) = : \exp \left(i \frac{\ell}{\sqrt{2N}} \varphi(z) \right) :, \ell = 1, 2, \ldots, 2N ,$$

with charges $Q = \ell/2N$ and conformal dimensions $h = \ell^2/4N$. The corresponding $\mathbb{Z}_{2N}$ symmetry is given by the equivalence relation $| \ell \rangle \equiv | \ell + 2N \rangle$, and the compactification radius is $r = \sqrt{2N}$. Under the operation $\varphi \rightarrow -\varphi$, $\ell \rightarrow 2N - \ell$, some vertex operators remain fixed and the rest exchange among themselves. The remaining theory still possess a $\mathbb{Z}_N \simeq \mathbb{Z}_{2N}/\mathbb{Z}_2$ symmetry. The CFT obtained by moding out the $\mathbb{Z}_2$ symmetry in the toroidal models is a chiral orbifold [25]. These are further CFTs with $c = 1$ and the same compactification radius as the starting gaussian models, but that only retain the sectors of the toroidal Hilbert space which are consistent with the $\mathbb{Z}_2$ discrete symmetry. Moreover, the introduction of additional primary fields is required for consistency of the theory. In CFT these consistency conditions are the fusion rules and the modular invariance of the partition function when $\Sigma$ is an annulus [15] [16]. Let us summarize here the results of [25] for a holomorphic model: the additional primary fields generate the ‘twisted sector’, which contains fields with anti-periodic boundary conditions [1] (for either spatial or - through modular invariance - time-like closed curves), and is independent of the value of $r$. The additional operators add sectors with half-integer charges, and create a new ground state out of the vacuum for fields allowed by the enlargement of acceptable boundary conditions. This new sector is known as the Ramond sector in the string theory literature [1], as opposed to the Neveu-Schwarz sector, which pertains fields with periodic boundary conditions. These

\[1\] This statement applies to a spatial manifold with the topology of a cylinder. On the complex plane, into which it is mapped by a standard conformal transformation, the fields have periodic boundary conditions. This is a well-know phenomenon [1].
operators are generically known as ‘disorder operators’ because of the role they play in the Ising model, and are given by

\[
\sigma(z) = \exp \left( i \frac{1}{2\sqrt{2}} \varphi(z) \right) ; \quad \tau(z) = \exp \left( i \frac{3}{2\sqrt{2}} \varphi(z) \right) ;
\]

of conformal dimensions \( h = 1/16 \) and \( h = 9/16 \), respectively. These operators are non-local in the original toroidal model, but become local in the orbifold. One way of viewing them is as the only two allowed operators with half-integer charges in the smallest \((N = 1)\) toroidal model, with \( r = \sqrt{2} \). This is because conformal dimensions of (holomorphic) primary fields should be less than 1 in the \( N = 1 \) toroidal model, which can in turn be rephrased as the statement that \((13)\) are the only two relevant operators (in the sense of the renormalization group) that could be added with half-integer charges \([25]\). As stated before, these operators generate the twisted sector of the \( \mathbb{Z}_2 \) Abelian orbifolds for all \( N \).

What are the corresponding \( 2 + 1 \)-dimensional disorder operators? To answer this question one should be able to make sense of the analogs of the disorder operators \((13)\) that describe excitations with half-integer flux values. Consider a Wilson loop \((3)\) for \( n = 1/2 \):

\[
W_{1/2}[C] = \exp \left( i \frac{1}{2} \Phi[C] \right)
\]

in a CS theory \((1)\) with \( k = 2N \), corresponding to the \( N \)-toroidal CFT. Consider now the action of twisting \( t \) units the framing field when traversing the closed loop \( C \) in \((14)\) once. The holonomy of the vacuum expectation value of \( W_{1/2} \) is shifted according to \((10)\), with

\[
\alpha = 2\pi \cdot t \cdot \frac{1}{16N} .
\]

We notice that the desired conformal dimension \( h = 1/16 \) is obtained from the previously stated general expression \( h = \alpha/2\pi \), provided that \( t = N \), for all \( N \). A similar reasoning yields a definition of the operator \( W_{3/2} \), with \( h = 9/16 \). The meaning of the newly defined Wilson loops is as follows: the framing field provides a way of giving sense to the \( 2 + 1 \)-dimensional analog of a multi-covering of the complex plane \([27]\). Fields with no branch-cuts in the complex plane are lifted to Wilson loops with \( t = 1 \), whereas fields with branch cuts
of order $N$ are lifted to Wilson loops that are defined with $N$ twists when traversing the loop once. This is the first time in this work we see the appearance of $N$ copies of a given underlying structure, but this idea will arise again in discussing the physical applications of this class of CS theories. One may also ask what is the vacuum holonomy of the operator $W_{1/2}$. We argue that is is given by (11) with $n = 1$, because it is the minimum amount of charge (flux) that exists in the CS theory. However, multivaluedness of the Wilson operator implies that this charge should be considered as evenly distributed over the $N$ different coverings of the loop, as represented by the framing field.

One can gain further support for the correctness of the above definition of the CS theory by considering the consequences of modular covariance of the orbifold CFT, as viewed in the CS formulation. Consider the CS theory defined on the spacetime manifold $M = S^1 \times S^1 \times I$, where the space manifold is given by an annulus $\Sigma = S^1 \times I$, with inner (outer) radius $R_R$ ($R_L$), respectively and $I = [R_L, R_R]$. Completeness of the theory requires that the partition function, which includes both chiral (say, for the outer edge excitations) and anti-chiral sectors (inner edge), should be taken to be modular invariant under the congruence subgroup $\Gamma(2N)$ of the modular group $PSL(2, \mathbb{C})$ [24] [28], instead of the full modular group, in order to accommodate for the disorder operators. Given the general covariance of the CS theory, one may exchange space and time coordinates, and obtain a torus $S^1 \times S^1$, which has been considered in [15]. On this torus with given modulus $\tau$ the generators of the modular group $T$ and $S$ act as $T : \tau \rightarrow \tau + 1$ and $S : \tau \rightarrow -1/\tau$. $T$ can be thought of as the generator of closed paths in $\Sigma$ for loops at fixed $2 + 1$-dimensional time; as such, it probes the boundary conditions of the conformal fields. An explicit realization of the $T$ and $S$ operators in terms of the CS data is given by the Verlinde operators [29]. Note, however, that the relevant subgroup in our discussion is $\Gamma(2N)$ which is generated by $T^{2N}$ and $ST^{2N}S$. This requirement is also the same to impose on the canonical quantization of the CS theory [15] [30], in which a constant time surface is considered. In this setting, the action of $T$ can be associated to the framing field. Therefore, formulating a theory with disorder operators in this framework also amounts to take again $T^{2N}$ as a generator, rather than $T^2$ (which
would be the standard requirement for a theory with fermionic excitations).

What are the resultant CS theories corresponding to the $\mathbb{Z}_2$ orbifolds of the $\mathbb{Z}_{2N}$ gaussian theories? The answer to this question has been discussed in Refs. [16] and [23]: these correspond to theories with a gauge group $G = O(2)$ and level $k = 2N$. This can be understood in view of the previous discussion: denoting by $G_k$ the level $k$ CS theory (1) with $G = U(1) \cong SO(2)$, one has that $O(2)_k \cong (SO(2) \times \mathbb{Z}_2)_k \cong U(1)_k \times \mathbb{Z}_{2k}$. After imposing charge conjugation invariance (i.e., the orbifolding procedure), one is left with a discrete $\mathbb{Z}_k$ symmetry. From now on, we will refer to the $2+1$-dimensional theory corresponding to the $\mathbb{Z}_2$ orbifold simply as an $O(2)_N$ CS theory. Note, however, that in doing so one implicitly assumes the previous definition of the CS disorder operators.

Finally, we would like to remark that one could also view these models as Topological Spin Theories [23]. These are theories in which the space-time manifold $\mathcal{M}$ admits an spin structure. The framing fields allow one to think of these manifolds as multi-covered standard ones, in the sense implied by the definition of fractionally charged Wilson loops. These manifolds were introduced as a natural framework in which $2+1$-dimensional topological field theories can be defined when one considers a gauge group $G$ which is not connected.

We now would like to discuss some examples in more detail:

**A. $N = 1$**

The first example we would like to discuss is the case $N = 1$. Consider first the CFT $N = 1$ toroidal theory: it is a gaussian theory ($c = 1$) with compactification radius $r = \sqrt{2}$. Before the orbifolding procedure is applied, consider the three primary fields: $J^3(z) = i\partial \varphi(z), J^\pm(z) = : \exp(\pm i\sqrt{2}\varphi(z)):$ of conformal weight 1, which satisfy an $SU(2)_1$ current algebra. The theory possesses two vertex operators of the form [12], namely $V_1(z) = \exp(\frac{i}{\sqrt{2}} \varphi(z))$ and $V_2(z) = J^+(z)$ of charges 1/2 and 1, and conformal dimensions 1/4 and 1, respectively. They correspond to the two representations of the affine $SU(2)_1$ algebra, namely the identity $[1] = [V_2]$ and the spinor $[\Psi] = [V_1]$, with $\mathbb{Z}_2$ fusion algebra $\Psi \times \Psi = 1$. The $\mathbb{Z}_2$ orbifold
of this theory is known to be equivalent to the $N = 4$ toroidal model \cite{25}. A counting of representations in the theory is consistent with the $\mathbb{Z}_8$ symmetry of the resulting theory: there are four representations arising from the twisted sectors (two for each of the operators (13)), and four arising from the untwisted sector ($k = 2$ times for each $V_i$, see \cite{25}).

The CS theory of the toroidal model consists of two classes of Wilson loops, namely $W_1[C]$ and $W_2[C]$ with holonomies ($-1$) and $1$, respectively. These operators correspond to the CFT vertex operators $V_1$ and $V_2$, respectively. The holonomies are real, and charge conjugation does not change their values, so both states $|0\rangle$ and $|1\rangle$ are fixed points of the $\mathbb{Z}_2$ transformation. Therefore, both states are retained in the ‘orbifold’ theory. This result is consistent with the expectation that there is no discrete symmetry surviving the $\mathbb{Z}_2$ projection. The Abelian fusion rule of the toroidal CFT is reproduced by charge (flux) addition of the Wilson loops holonomies Eq. (4), i.e., $\langle W_1 \rangle \cdot \langle W_1 \rangle = \langle W_2 \rangle = 1$. In addition, both disorder operators $W_{1/2}$ and $W_{3/2}$ should be added, but in this case one should require $t = 1$, as for the rest of the Wilson loops. Therefore, no special condition distinguishes these operators in this case. On the other hand, consider the CS theory with $k = 8$. The spectrum of Wilson loop charges and conformal dimensions is given by (5). We see that the corresponding Wilson loop operators $W_n$, $n = 1, \ldots, 4$ correspond one-to-one to the previously discussed set. This indicates that the postulated correspondence between $2 + 1$ and $1 + 1$ dimensional theories is consistent with the corresponding orbifolding procedures in this example.

B. $N = 2$

The second example, for $N = 2$, is interesting because it appears in the neutral sector in Pfaffian states of the quantum Hall effect. We first consider the $N = 2$ toroidal model CFT, with compactification radius $r = 2$. The $\mathbb{Z}_4$ structure of the theory is given by the four vertex operators $V_\ell(z) = : \exp \left( i \frac{\ell}{2} \varphi(z) \right) :$, with $\ell = 1, 2, 3, 4$, of charges $Q = 1/2, 1, 3/2, 2$ and conformal dimensions $h = 1/8, 1/2, 9/8, 2$, respectively. The $\mathbb{Z}_2$ orbifold theory yields
two identical copies of the Majorana fermion CFT (the Ising model at the critical point), such that $c = 1 = 1/2 + 1/2$. A description of this theory in terms of a chiral boson has been discussed in \cite{24} \cite{31}.

The CS theory of the $N = 2$ toroidal model has, correspondingly, four classes of Wilson loops, $W_\ell[C] = \exp(2\pi i \ell \Phi[C])$, with holonomies $\exp(i\pi \ell/2)$, $\ell = 1, 2, 3, 4$. Under charge conjugation, $W_1 \to W_3$ with $W_2$ and $W_4$ fixed. Orbifolding amounts to rendering the theory electrically neutral and introducing the ‘disorder’ operators $W_{1/2}$ and $W_{3/2}$ defined in (14), of conformal dimension $1/16$ and $9/16$, respectively. Now, for these operators to make sense, we need to define them by requiring that the framing field undergoes a twist of $t = 2$ units when traversing the loop $C$ once, as discussed above. This is the first CS example in which we encounter this new feature. The other independent Wilson loop operators are $W_2 = \psi$ and $W_4 \equiv W_0 = 1$, of conformal dimensions $1/2$ and $2 \equiv 0$, which is the operator content of the Ising model ($\psi$ is the Majorana field). A minimal set of operators under the Abelian composition law of Wilson loops (4) is given by $W_{1/2}$, $W_2$ and $W_4$. This is because $W_2$ and $W_4$ are invariant under charge conjugation, and $W_{1/2}$ is mapped onto $W_{3/2}$, after taking into account that the actual charges of both operators are $1$ and $3$, respectively, as argued before. Note that $W_2$ and $W_4$ furnish a representation of the surviving $\mathbb{Z}_2$ symmetry.

Furthermore, one realizes that there are actually two copies of the same theory: although the holonomies (11) of $W_2$ and $W_4$ are real, the holonomy of $W_{1/2}$ is $\pm i$, after considering the double covering of $\mathcal{M}$. This sign ambiguity means that there actually two equivalent ways of realizing this theory. More precisely, in the CFT formulation of the Ising model \cite{3}, one decomposes the Weyl fermion of a parent $c = 1$ CFT into two identical but distinct Majorana fermions, each described by a $c = 1/2$ CFT. Next, one introduces a projection, i.e., a reality condition that retains only one of the two Majorana fields as a legitimate degree of freedom. Both possible choices yield equivalent results due to the original symmetrical decomposition. Therefore, the $c = 1 \to c = 1/2$ projection in CFT is a mechanism of halving the theory. Although performing a general projection in CFT calls for a coset construction \cite{3}, in the particular case we are considering this procedure is not necessary.
Correspondingly, we apply an analog approach at the level of the CS theory: the doubling of degrees of freedom analog to the one occurring in the $c = 1 = 1/2 + 1/2$ CFT is paralleled by the existence of two equivalent Wilson loop operators $W_{1/2}$, which could be distinguished by their canonical holonomies $\pm i$. In view of the previous discussion, we adopt as a natural projection condition the picking of a sign determination for the operator $W_{1/2}$, and claim that this realizes in $2+1$ dimensions the corresponding projection $c = 1 \to c = 1/2$ in $1+1$ dimensions.

Moreover, one can also verify the crucial property of the reproducibility of the CFT fusion rules. These are given by charge addition of the corresponding Wilson loops, modulo the issues concerning the multivaluedness of the $W_{1/2}$ operator. Therefore, the product $W_2 \cdot W_2 = W_4$ verifies the corresponding CFT rule $\Psi \times \Psi = 1$. The product $W_{1/2} \cdot W_2 = W_3 \simeq W_{3/2} = W_{1/2}$ verifies the fusion rule $\sigma \cdot \psi = \sigma$, after taking into account the correct charge assignments, and the identification $W_3 \simeq W_{3/2}$ actually means that the operator in the rhs is multivalued, whereas the one with the same charge in the lhs is not; in the same sense, one has $W_1 \simeq W_{1/2}$. The product $W_{1/2} \cdot W_{1/2}$ is the most interesting of all. Due to the double covering associated to each operator, one should consider actually all possible choices of charge addition. Charges should also be provided with a second label indicating in which covering they are considered. This label could be a sign, since the group involved is $\mathbb{Z}_2$. This makes the charges actually behave like the third component of an $SU(2)$ multiplet. Is in now clear that the possible outcome now for the addition of charges is 0 or 2, given that each individual charge is 1. We indicate this by writing $W_{1/2} \cdot W_{1/2} = W_0 + W_2$, which reproduces the CFT fusion rule $\sigma \cdot \sigma = 1 + \Psi$.

\textbf{C. }$N \geq 3$

The $\mathbb{Z}_2$ orbifolds of the $N$ toroidal models for other values of $N$ can be analyzed along the same lines. We remark here that for $N = 3$, the $\mathbb{Z}_4$ parafermionic model is obtained \cite{23}. This model is of relevance for further Quantum Hall universality classes known generically
as *Parafermionic States* [4]. However, more general parafermionic states are not obtained by considering further $\mathbb{Z}_2$ orbifolds of the $N$ toroidal models. This can be understood from the fact that the $K$ parafermion models are described by CFTs with central charge $c = 2(K-1)/(K+2)$, which is larger than 1 for $K > 4$, and therefore beyond the scope of the systems considered here. Further examples of the models considered here are also known: for $N = 4$ one obtains the four-state Potts model, and for $N = 6$ a discrete superconformal model at $c = 1$ [25].

**IV. APPLICATIONS TO THE QUANTUM HALL EFFECT**

For applications to the quantum Hall effect, we consider a direct product of two Abelian CS theories, corresponding to the charged and neutral excitations. In principle, one has several choices among the different possibilities we have discussed above. A consistency principle is therefore required. A usual constraint is to require the presence of an excitation with the quantum numbers of the electron in the spectrum of charges and quantum statistics of the combined theory. However it is not clear what additional conditions (if any) are required in general to determine uniquely the physically acceptable theories.

**A. The Pfaffian states**

Let us now consider the $q$-Pfaffian states [2]. For $q$ even (odd), these models are known as fermionic (bosonic) Pfaffian theories. These theories can be constructed from two Abelian CS theories: one for the charged (+) and another for the neutral (0) sectors, respectively. The charged Abelian CS field is defined with (positive) integer level $k = q$. The physical requirement on the spectra of these theories is that there exists a particle with unit electric charge and odd quantum statistics (electron or hole) for $q$ even. A similar statement, with even quantum statistics, is needed for $q$ odd. In other words, the ‘electron’ excitation has unit charge and fermionic ($q$ even) or bosonic ($q$ odd) statistics. Clearly these conditions cannot be fulfilled by the charged sector alone, with quasiparticle spectrum given by [5],
namely $Q = n/q$ and $\theta^+/\pi = n^2/q$. One therefore considers a suitable choice for the neutral sector. Note, however, that when coupling the combined system to an external electromagnetic probe, only the charged sector contributes to the Hall conduction, yielding the value $\nu = 1/q$ for the filling fraction.

A natural CFT description of the neutral sector of these models is to consider the $\mathbb{Z}_2$ orbifold of the toroidal model with $N = q$ \cite{22}. Given that we now know how to construct the $2 + 1$ dimensional analog of these orbifolds, we consider an Abelian CS theory at level $k = 2q$ for the fermionic Pfaffian and level $k = 2(q + 1)$ for the bosonic case. Equivalently, the compactification radii of charged and neutral sectors, $r_+$ and $r_0$ respectively, satisfy either the condition $r_0^2 = 2r_+^2$ (fermionic Pfaffian) or $r_0^2 = 2(r_+^2 + 1)$ (bosonic Pfaffian).

From previous discussions, the resulting neutral theory has a $\mathbb{Z}_q$ symmetry in the fermionic case, and $\mathbb{Z}_{q+1}$ in the bosonic case. The spectrum of the theory of the neutral sector is $\theta^0/\pi = s^2/2q$, where $s$ is integer (half-integer) in the untwisted (twisted) sector, respectively, for the fermionic Pfaffian. For the bosonic case, one has $\theta^0/\pi = s^2/2(q + 1)$ instead. More explicitly, we consider the total CS theory describing the Pfaffian systems to have symmetry $U(1)_q \times O(2)_{2q}$ for the fermionic case and $U(1)_q \times O(2)_{2(q+1)}$ for the bosonic one, where the first (second) factor refers to the charged (neutral) sector.

The total theory combining both the charged and neutral sectors considered above now admits electrons in its spectrum. Consider first the case of the fermionic Pfaffian, and assume that $q/2$ odd. Then, by choosing $s = q$ in the neutral sector, one has that the condition in the charged sector $Q = 1$ enforces $\theta^+/\pi = q$, and $\theta^0/\pi = q/2$. Adding the contribution from both sectors, a fermionic excitation can be made. If $q = 4r$, again one can choose $s = q/2$ to obtain overall odd statistics provided $r/2$ is odd. Clearly, this analysis can be extended to cover all possible cases along the same lines. The analysis of the bosonic Pfaffian also follows from a similar reasoning. Consider first the case $(q + 1)/2$ odd. Then, it is enough to take the same quantum numbers in the charged sector as in the fermionic case, and $s = q + 1$ in the neutral sector to obtain an excitation with $Q = 1$ and $\theta/\pi = q + (q + 1)/2$. Further cases could be analyzed along the same lines.
For the specific case of the $q = 2$ fermionic Pfaffian, with filling fraction $\nu = 1/2$, we have the following results: the electron operator can be constructed out of two Wilson loops: $W_1^+$, with $Q = 1$ and $\theta^+ / \pi = 1$ in the charged sector and $W_2^0$, with no charge and $\theta^0 / \pi = 1/2$ (Majorana field) in the neutral sector. We consider the Wilson operators for the charged and neutral sectors as defined along the same Wilson loop $C$, since the theory is a direct product of both sectors.

Another known property of these systems is their topological order [11]. For our purposes, it is given by the number of independent states in the CS theory. For simplicity, we focus the discussion on the fermionic case. We have here $q$ independent states for the different charge sectors, and three states for the neutral sector. If all possible pairing of states between the charged and neutral theories lead to a consistent theory, one would in fact have a total of $3q$ states [10]. Actually, $2q$ of these arise from the untwisted sector of the neutral theory, whereas $q$ arise from the twisted sector. In CFT the corresponding consistency condition to establish this pairing is given by modular invariance [28]. In $2 + 1$ dimensions, modular invariance is a consequence of general covariance and gauge invariance [15]. This implies that all possibilities should be taken into account.

B. Non-Abelian Statistics in the Pfaffian states

In the following, consider the fermionic Pfaffian for the sake of simplicity. In CFT, the quasi-hole operator in the $q$-Pfaffian states is

$$\Psi_{qh}(z) = \sigma(z) \cdot : \exp \left( i \frac{1}{\sqrt{q}} \varphi^+(z) \right) : , \quad \sigma(z) = : \exp \left( i \frac{1}{2\sqrt{2}} \varphi^0(z) \right) : , \quad (16)$$

of charge $Q = 1/q$ and conformal dimension $h = 1/16 + 1/2q$. For the case $q = 2$ these expressions yield $Q = 1/2$ and $h = 5/16$. In eq. (16) $\varphi^+(z)$ is the chiral bosonic field of the charged sector and $\sigma(z)$ is the chiral disorder operator, of total charge 0 and conformal dimension 1/16, with $\varphi^+(z)$ being the chiral bosonic field in the neutral sector.

The corresponding CS quasi-hole operator is, therefore, given by
\[ W_{qh}[C] = W_1^+[C] \cdot W_{1/2}^0[C] , \]  

with \( W_1^+[C] \) defined by (3) in terms of the charged CS field \( a_\mu^+(x) \) and \( W_{1/2}^0[C] \) given by (14) in terms of the neutral CS field \( a_\mu^0(x) \). It is important to have in mind that the definition of both operators depend on \( q \) through the CS action (1). There is a further, more important and specific dependence of the operator \( W_{1/2}^0 \) on \( q \), given that it is assumed to be defined after \( q \) twists of the framing field when traversing \( C \) once. The statement that the \( q \)-Pfaffian theories possesses non-Abelian statistics means in our context that the correlation function of four quasi-hole operators yields more than one resulting Wilson loop. This statement is the analog to the CFT result of several resulting channels of conformal blocks in the evaluation of the corresponding CFT correlators. This latter property is a consequence of the fusion rules of the primary fields, which we have already verified. Since, by construction, the Wilson loops corresponding to those primary fields satisfy the CFT fusion rules, the property of non-Abelian statistics is also present in the \( 2 + 1 \)-dimensional theory. Note that in establishing the fusion rules among Wilson operators, the property of multivaluedness of the operator \( W_{1/2}^0 \) is crucial. The origin of this multivaluedness can be traced back to the remaining \( \mathbb{Z}_q \) symmetry in the neutral sector, and could be ultimately held responsible for the appearance of non-Abelian statistics in the Pfaffian models. A similar analysis could be rephrased for the bosonic case.

We conclude by stating that the \( 2 + 1 \) dimensional theory describing the fermionic \( q \)-Pfaffian is given by a topological theory of two Abelian CS fields, with gauge group \( U(1)_q \times O(2)_{2q} \) (in the charged and neutral sectors, respectively). The symmetry of the theory in the neutral sector is \( \mathbb{Z}_q \) and therefore we consider it as the responsible for the appearance of non-Abelian statistics. Similarly, the bosonic Pfaffian has a \( U(1)_q \times O(2)_{2(q+1)} \) gauge group, with symmetry \( \mathbb{Z}_{q+1} \) in the neutral sector.
C. The $K = 4$ Parafermionic states

As we discussed above, within the class of states we consider in this paper, one is able to describe the cases $K = 2$ (Pfaffian) and $K = 4$ Parafermion states. Here we give a brief discussion of these latter cases.

In general, the parafermionic models are the simplest CFTs with symmetry within the class of chiral algebras obtained by adjoining to the Virasoro algebra higher spin primary fields that correspond to Casimir operators of a simply laced Lie group \[32\]. For the $\mathbb{Z}_K$ parafermionic models, the Lie group is $SU(K)$. In the CFT description of the theory, two vertex operators of conformal spin 3 and 4 should be considered as primary fields, in the $N = 3$ toroidal model. This yields a set of vertex operators with conformal dimensions $1/12, 1/3, 3/4$ in the untwisted sector and $1/16, 9/16$ in the twisted sector. The corresponding Wilson loop operators in the corresponding CS theory at level $k = K = 3$ are given by $W_n, n = 1, 2, 3$ in \[3\], and $W_{1/2}$ and $W_{3/2}$ as defined in section 3, respectively. Again, a symmetry $\mathbb{Z}_3$ survives in the neutral sector, which yields the notion of non-Abelian statistics generalizing the case of the Pfaffian.

V. CONCLUSIONS

In this paper we have considered Chern-Simons theories with two Abelian fields, one for the charged and another for the neutral sector. We studied the consequences of considering a group $O(2)$, with two disconnected components, in the neutral sector, verifying in some detail that this theory is the 2 + 1-dimensional analog of the $\mathbb{Z}_2$ orbifolds in Conformal Field Theory.

An interesting physical consequence of our study is that it allows one to think of the non-Abelian statistics from a seemingly unconventional point of view. Indeed, here we found that the non-Abelian statistics follows from the $\mathbb{Z}_q$ structure in the neutral sector for the $q$-Pfaffian models. This result is interesting because it brings together seemingly different types
of theories displaying non-Abelian statistics, such as the Pfaffian and Parafermionic models, the minimal incompressible models of the hierarchical Hall states [7], the Landau-Ginzburg approaches [9,18], and the coset construction [19,20]. In several of these theories, it has been observed that the non-Abelian statistics that characterizes these models also arises form an underlying $\mathbb{Z}_m$ symmetry in the neutral sector, as for the case of the minimal incompressible models [33]. The same structure shows up in the Landau-Ginzburg and coset constructions. However, these theories do not exhibit in an obvious manner an orbifold structure. Likewise, the discrete symmetry associated with the non-Abelian states in some cases emerges from the pairing (or clustering) mechanism behind the microscopic physics of these FQH states. We expect that further connections along these lines could be established among these these and other related theories.

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