Torsional Vibration and Bifurcation Characteristics of Automotive Main Reducer Based on Nonlinear Hybrid Model

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Abstract. In the view of integrating dynamics of systems with dynamics of structures, the pinion shaft of automobile main reducer is divided into several torsion rod elements to establish its dynamic model, while the torsional vibration dynamic model of hypoid gears is still built with lumped mass method, thus the hybrid dynamic model of automotive main reducer is obtained by the combination of two kinds of dynamic models. Considering the nonlinear factors such as time-varying mesh stiffness, backlash and static transmission error of gear pairs, Runge-Kutta method is applied to obtain the time domain spectrum, spectrum graph, phase-plane diagram, Poincaré maps, and bifurcation diagrams of the system. The dynamic response characteristics of system with different parameters such as input rotational speed and mesh stiffness are systematically investigated by numerical simulations.

Keywords: Hybrid model; Dynamics of structures; Torsional vibration; Automotive main reducer.

1. Introduction

As a typical system of automotive transmission, automotive main reducer consists of pinion shaft, hypoid gears, tapered roller bearings(TRB) and differential shell, as shown in Figure 1. Its dynamic performance can make a considerable contribution to the NVH performance of the driveline.

During the last decades, there are many studies on the vibrations of all kinds of gears. Early on these studies, the hot research spot is the vibrations of spur gears [1-3], those studies found out the nonlinear performance of gears is the result of backlash and time-varying mesh stiffness, which settled a theoretical basis of research on the nonlinearity of gears. With the development of nonlinear dynamics, more and more nonlinear factors such as static transmission errors and meshing damping are taken into consideration while establishing the dynamic model [4]. The studies on hypoid gears to obtain the nonlinear parameters such as time-varying mesh stiffness and equivalent mesh point begin to launch based on tooth contact analysis [5]. With the tooth contact analysis, Wang [6] formulated a two-degree-of-freedom(DOF) dynamic model of hypoid gear pair to analysis its nonlinear characteristics. Thereafter, more DOF dynamic model of hypoid gears are modeled to investigate coupled vibrations such as coupled bend-torsion-axes vibration [7]. However, all those studies have been done without considering...
the vibration of the pinion shaft, the pinion shaft is equivalent to an additional moment of inertia adding to the gear.

In this study, the torsion vibration dynamic model of hypoid gears are formulated based on the combination of dynamics of systems and dynamics of structures, not only the nonlinear factors such as time-varying mesh stiffness, backlash, static transmission error and torque fluctuation, but the vibration of pinion shaft are taken into consideration. Based on the more realistic model, the bifurcation characteristics about key parameters are analyzed by numerical solutions. It is anticipated that the thought of model-build could be helpful to the NVH performance research of main reducer, even the whole transmission system.

Figure 1. Schematic diagram of automobile main reducer.

2. Establishment of nonlinear hybrid model

2.1. Torsional vibration model of pinion shaft based on Dynamics of Structures

Based on dynamics of structures, the pinion shaft can be divided into several torsion rod elements, and the mass matrix, stiffness matrix and load matrix of the ith elements can be given by:

\[
M_{ei}^e = \frac{\rho l_i J_{pi}}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K_{ei}^e = \frac{G J_{pi}}{l_i} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad F_{ei}^e = \begin{bmatrix} M_{xi}^e \\ M_{xi+1}^e \end{bmatrix}
\]

(1)

Where \( J_{pi} \) is the area moment of inertia, \( \rho \) is the volume density, \( l_i \) the axial length of the element and \( G \) is Young's modulus. Assuming the pinion shaft is discretized into \( n \) torsion rod elements, the discrete torsion rod elements need to be transformed to eliminate the repeated degrees of freedom(DOF). The transformation \( T_{tr} \) matrix satisfies the requirement:

\[
\begin{bmatrix} \theta_1 \\ \vdots \\ \theta_{n-1} \\ \theta_n \\ \theta_{n+1} \end{bmatrix} = T_{tr} \begin{bmatrix} \theta_{x_1} \\ \vdots \\ \theta_{x_{n-1}} \\ \theta_{x_n} \\ \theta_{x_{n+1}} \end{bmatrix}
\]

(2)
Then the mass matrix, stiffness matrix and load matrix of the pinion shaft can be obtained by:

\[
M_{e}^{T} = T_{e}^{T} \begin{pmatrix}
M_{n}^{e} & M_{s}^{e} & \cdots & M_{t}^{e} \\
M_{s}^{e} & M_{t}^{e} & \cdots & M_{m}^{e} \\
\vdots & \vdots & \ddots & \vdots \\
M_{m}^{e} & M_{m-1}^{e} & \cdots & M_{1}^{e}
\end{pmatrix} T_{e} \quad K_{e}^{T} = T_{e}^{T} \begin{pmatrix}
K_{n}^{e} & K_{s}^{e} & \cdots & K_{t}^{e} \\
K_{s}^{e} & K_{t}^{e} & \cdots & K_{m}^{e} \\
\vdots & \vdots & \ddots & \vdots \\
K_{m}^{e} & K_{m-1}^{e} & \cdots & K_{1}^{e}
\end{pmatrix} T_{e} \quad F_{e}^{T} = T_{e}^{T} \begin{pmatrix}
F_{n}^{e} & F_{s}^{e} & \cdots & F_{t}^{e} \\
F_{s}^{e} & F_{t}^{e} & \cdots & F_{m}^{e} \\
\vdots & \vdots & \ddots & \vdots \\
F_{m}^{e} & F_{m-1}^{e} & \cdots & F_{1}^{e}
\end{pmatrix}
\]

(3)

The damping matrix can be calculated by:

\[
C_{e}^{T} = \alpha M_{e}^{T} + \beta K_{e}^{T}
\]

(4)

2.2. Determination of element density scheme

As the tentative element division scheme is shown in Figure 2, the shaft segments where the tapered roller bearings are installed can be divided into two elements at the equivalent installation point, and the remaining shaft segments are evenly divided into two elements.

The natural frequencies of the pinion shaft can be solved based on the continuous mass model and assuming the natural frequency values are accurate. As shown in Table 1, to compare the natural frequencies obtained by solving the finite element model with the accurate values and guarantee the errors of the first three orders natural frequencies within 5%. Therefore, the tentative element division scheme meets the accuracy requirements.

Table 1. Natural frequencies of torsional vibration based on two models.

| Order | Natural frequencies based on continuous mass model (Hz) | Natural frequencies based on finite element model (Hz) | Error (%) |
|-------|----------------------------------------------------------|--------------------------------------------------------|-----------|
| 1     | 71533                                                    | 71380                                                  | 0.24      |
| 2     | 122946                                                   | 121178                                                 | 1.44      |
| 3     | 188585                                                   | 182940                                                 | 2.99      |

2.3. Meshing model of hypoid gears based on Dynamics of Systems

A dynamic model of hypoid gears based on the lumped mass method is shown in Figure 3. There is no consideration of others vibrations or fiction except torsional vibration. The 2DOF differential equations of hypoid gears are as follows:

\[
I_{px} \ddot{\theta}_{px} + \lambda_{px} F_{n} = M_{rpx}
\]

\[
I_{gy} \ddot{\theta}_{gy} - \lambda_{gy} F_{n} = -T_{g}
\]

(5)
Where $\theta_{px}$ and $\theta_{gy}$ stand for the rotation angle pinion and gear respectively, $I_{px}$ and $I_{gy}$ express the inertia moment of pinion and gear respectively, $\lambda_{px}$ and $\lambda_{gy}$ are the equivalent mesh radius of pinion and gear respectively, the terms $Mr_{px}$ is the torque transmitted from the pinion shaft, $T_g$ is the torque loaded by gear. The nonlinear meshing $F_n$ express the overall damping and elastic forces developed during impact, thus:

$$
F_n = \begin{cases}
  k_n(t)(x_n - b) + c_n \dot{x}_n & x_n > b \\
  0 & |x_n| \leq b \\
  k_n(t)(x_n + b) + c_n \dot{x}_n & x_n < -b
\end{cases}
$$

(6)

Where $b$ is the half of the backlash, $c_m$ is the mesh damping, the total transmission error $x_n$ is a function of angles $\theta_{px}$, $\theta_{gy}$ and the static transmission error $e(t)$, thus:

$$
x_n = \lambda_{px} \theta_{px} - \lambda_{gy} \theta_{gy} - e(t)
$$

(7)

The time-varying mesh stiffness $k_m(t)$ and static transmission error $e(t)$ can be written in the form of Fourier series and consider the effect of first-order fluctuation amplitude merely [8]:

$$
k_n(t) = k_{m0} + k_{m1} \cos(\omega_m t + \phi_{k1})
$$

$$
e(t) = e_0 + e_1 \cos(\omega_m t + \phi_{e1})
$$

(8)

Where $\omega_m$ is the mesh frequency of hypoid gears, can be calculated by the rotational speed of pinion shaft $n$ as:

$$
\omega_m = \frac{n z_p}{60}
$$

(9)

Rewrite the differential equations in the form of matrix as follows:

$$
\begin{bmatrix}
M_{pg} \{ \ddot{\theta}_{pg} \} + \{ C_{pg} \} \{ \dot{\theta}_{pg} \} + \{ K_{pg} \} \{ \theta_{pg} \} = \{ F_{pg} \}
\end{bmatrix}
$$

(10)

Define the nonlinear load coefficients $f_b$ and $f_e$ as follows:

$$
f_b = \begin{cases}
  1 & x_n > b \\
  0 & |x_n| \leq b \\
  -1 & x_n < -b
\end{cases}
$$

$$
f_e = \begin{cases}
  1 & x_n > b \\
  0 & |x_n| \leq b \\
  1 & x_n < -b
\end{cases}
$$

(11)

The mesh stiffness matrix, damping matrix and non-linear load matrix of hypoid gears can be expressed as:

$$
K_{pg} = f_s k_m(t)
\begin{bmatrix}
\lambda_{px}^2 & -\lambda_{px} \lambda_{gy} \\
-\lambda_{px} \lambda_{gy} & \lambda_{gy}^2
\end{bmatrix}
$$

$$
C_{pg} = f_s c_m
\begin{bmatrix}
\lambda_{px}^2 & -\lambda_{px} \lambda_{gy} & -\lambda_{px}^2 \lambda_{gy} \\
-\lambda_{px} \lambda_{gy} & \lambda_{gy}^2
\end{bmatrix}
$$

(12)
$$\{ F_{pg} \} = \begin{bmatrix} M_{rpx} + f_e \lambda_{px} (k_m (t) e(t) + c_m \dot{e}(t)) + f_e \lambda_{bx} b k_m (t) \\ -T_g - f_e \lambda_{gy} (k_m (t) e(t) + c_m \dot{e}(t)) - f_e \lambda_{gy} b k_m (t) \end{bmatrix}$$  \hspace{1cm} (13)

The pinion and gear can be regarded as another torsion rod element, so the pinion shaft and hypoid gears can be assembled in the same way as rod elements assembled into pinion shaft.

2.4. Additional inertia of moment

The torsional stiffness of tapered roller bearings can be neglected. Therefore, tapered roller bearings should be regarded as an additional inertia moment and added to the nodes of the corresponding installation place, the differential shell can be treated in the same way. The value and direction of each additional inertia moment are shown in Table 2.

Table 2. Value and direction of each additional moment of inertia.

| Part                      | Moment of inertia (kg·m²) | Direction |
|---------------------------|---------------------------|-----------|
| Tapered roller bearing 1  | 1.73E-4                   | θ_x       |
| Tapered roller bearing 2  | 2.70E-4                   | θ_x       |
| Tapered roller bearing 3&4| 2.93E-3                   | θ_y       |
| Differential shell        | 1.69E-2                   | θ_y       |

2.5. Numerical Solutions

The material and structural parameters of hypoid gears are listed in Table 3. The total transmission error $x_n$ is defined as the dependent variable solved by Runge-Kutta method to explore the influence of key parameters on the system dynamic characteristics.

Table 3. Material and structural parameters.

| Parameter                      | Pinion    | Gear    |
|--------------------------------|-----------|---------|
| Material                       | 20CrMnTi  |         |
| Teeth numbers                  | 10        | 41      |
| Mass (kg)                      | 1.13      | 4.02    |
| Moment of inertia (kg·m²)      | 7.07E-4   | 3.15E-2 |
| Mesh stiffness amplitude (N/m) | $k_{m0}$=6E8, $k_{m1}$=2E7 |       |
| Half of the backlash (m)      | 5E-5      |         |
| Mesh damping (N·s/m)           | 1878      |         |
| Static transmission error amplitude (m) | $e_0=0, e_1=1.8E-5$ | |
| Phase angle (rad)              | $\Phi_{k1}=0, \Phi_{x1}=0$ | |

2.6. Influence of input rotational speed

Change the input rotational speed of the pinion shaft to obtain the time domain spectrum, spectrum graph, phase-plane diagram, Poincaré map, and bifurcation diagrams of the system. The bifurcation diagram about input rotational speed is shown in Figure 4. With the increase of rotational speed in the range of 1000 to 10000 rpm, the total transmission errors $x_n$ tend to enlarge and fluctuate, the increasing instability of gear transmission as well. The system shows the alterations of 1T-periodic motion, 2T-periodic motion, 3T-periodic motion, and 4T-periodic motion. Where the 1T-periodic motion occurs during the 1000 rpm ~ 4720 rpm, the 2T-periodic motion occurs during the 4730 rpm ~ 5530 rpm and 5910 rpm ~ 6630 rpm, the 3T-periodic motion appears during the 5540 rpm ~ 5530 rpm and 6750 rpm ~ 10000 rpm, the 4T-periodic motion only emerges during the 6640 rpm ~ 6740 rpm. The 1T-periodic motion usually occurs under low speed. In order to observe and verify the result the bifurcation map,
the time domain spectrums, spectrum graphs, phase-plane diagrams, Poincaré maps of several distinguishing speeds are mapped as Figure 5 ~ Figure 7 show.

Figure 4. Bifurcation diagram about input rotational speed.

Figure 5. n=2000rpm, time domain spectrum, spectrum graph, phase-plane diagram, Poincaré map.

Figure 6. n=5000rpm, time domain spectrum, spectrum graph, phase-plane diagram, Poincaré map.
2.7. Influence of mesh stiffness

Introduce the mesh stiffness coefficient $k$ to investigate the bifurcation characteristics with different mesh stiffness. Set the $k$ as 0.5 and 2 respectively, and the bifurcation diagrams about input rotation speed with the different mesh stiffness coefficient are shown as Figure 8 and Figure 9. Compare the results of Figure 4 (the situation $k=1$), Figure and Figure 9, when the mesh stiffness is smaller, the multi-periodic motions account a larger proportion and occur at lower speed, so the stability of the system is better under higher mesh stiffness.

Figure 7. $n=6700$rpm, time domain spectrum, spectrum graph, phase-plane diagram, Poincaré map.

Figure 8. $k=0.5$ Bifurcation diagram about input rotational speed.

Figure 9. $k=2$ Bifurcation diagram about input rotational speed.
3. Conclusion

The torsional vibration model of an automotive main reducer is formulated by integrating dynamics of systems with dynamics of structures, and the non-linear factors such as time-varying mesh stiffness, backlash and static transmission error of gear pairs are considered. The influence of input rotational speed and mesh stiffness on the bifurcation characteristics of the system are investigated based on numerical solutions.

According to the bifurcation diagrams about input rotational speed with different mesh stiffness, multi-periodic motions are observed with the input rotational speed changes, the total transmission errors $x_n$ tends to enlarge and fluctuate with the increase of input rotational speed, the increasing instability of gear transmission as well. Multi-periodic motions are easier to occur with smaller stiffness, the stability of the system is better under higher mesh stiffness and lower speed.

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