Properties of the symplectic structure of General Relativity
for spatially bounded spacetime regions.

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Abstract

We continue a previous analysis of the covariant Hamiltonian symplectic
structure of General Relativity for spatially bounded regions of spacetime.
To allow for wide generality, the Hamiltonian is formulated using any fixed
hypersurface, with a boundary given by a closed spacelike 2-surface. A main
result is that we obtain Hamiltonians associated to Dirichlet and Neumann
boundary conditions on the gravitational field coupled to matter sources, in
particular a Klein-Gordon field, an electromagnetic field, and a set of Yang-
Mills-Higgs fields. The Hamiltonians are given by a covariant form of the
Arnowitt-Deser-Misner Hamiltonian modified by a surface integral term that
depends on the particular boundary conditions. The general form of this sur-
face integral involves an underlying “energy-momentum” vector in the space-
time tangent space at the spatial boundary 2-surface. We give examples of the

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resulting Dirichlet and Neumann vectors for topologically spherical 2-surfaces in Minkowski spacetime, spherically symmetric spacetimes, and stationary axisymmetric spacetimes. Moreover, we establish the relation between these vectors and the ADM energy-momentum vector for a 2-surface taken in a limit to be spatial infinity in asymptotically flat spacetimes. We also discuss the geometrical properties of the Dirichlet and Neumann vectors and obtain several striking results relating these vectors to the mean curvature and normal curvature connection of the 2-surface. Most significantly, the part of the Dirichlet vector normal to the 2-surface depends only on the spacetime metric at this surface and thereby defines a geometrical normal vector field on the 2-surface. We show that this normal vector is orthogonal to the mean curvature vector, and its norm is the mean null extrinsic curvature, while its direction is such that there is zero expansion of the 2-surface i.e. the Lie derivative of the surface volume form in this direction vanishes. This leads to a direct relation between the Dirichlet vector and the condition for a spacelike 2-surface to be (marginally) trapped.
I. INTRODUCTION

In a previous paper [1] we began an investigation of the covariant symplectic structure associated to the Einstein equations for the gravitational field in any fixed spatially compact region \( \Sigma \times \mathbb{R} \) of spacetime whose spacelike slices \( \Sigma \) possess a closed 2-surface boundary \( \partial \Sigma \), with a fixed time-flow vector field tangent to the timelike boundary hypersurface \( \partial \Sigma \times \mathbb{R} \). Through an analysis of boundary conditions required for existence of a Hamiltonian variational principle, we derived Dirichlet, Neumann, and mixed type boundary conditions for the spacetime metric at the spatial boundary 2-surface \( \partial \Sigma \). The corresponding Hamiltonians we obtained are given by a covariant form of the ADM Hamiltonian plus a surface integral term whose form depends on the boundary conditions. We also showed that these Hamiltonians naturally yield covariant field equations which are equivalent to a 3+1 split of the Einstein equations into the well-known constraint equations and geometrical time-evolution equations for the spacetime metric.

The present paper continues the previous analysis in two significant ways. First, in Sec. II, we investigate the covariant symplectic structure of the Einstein equations coupled to matter sources in any fixed spatially bounded region of spacetime. Specifically, we consider Dirichlet and Neumann boundary conditions for a scalar field, an electromagnetic field, and a set of Yang-Mills/Higgs fields. Furthermore, we allow the fixed time-flow vector field on spacetime to have an arbitrary direction (i.e. not necessarily timelike) in the spacetime region. Such freedom in the choice of the time-flow vector field is useful for relating the Hamiltonian boundary terms to expressions for total energy, momentum, angular momentum associated to the gravitational and matter fields on given hypersurfaces in spacetime.

Next, in Sec. III, we discuss in detail the geometrical structure of the gravitational part of the Dirichlet and Neumann Hamiltonian boundary terms. In particular, as noted in Ref. [1], these each involve an underlying locally constructed “energy-momentum” vector at each point in the tangent space at the 2-surface. We show that the form of the boundary term vectors is closely related to the mean curvature vector and normal curvature connection 1-form which describe the extrinsic geometry of the spatial boundary 2-surface in spacetime. Most striking, we further show that the part of the Dirichlet boundary term vector orthogonal to the 2-surface yields a direction in which the 2-surface has zero expansion in spacetime.

Finally, through several examples, we illustrate the properties of the Dirichlet and Neumann boundary term vectors for topologically spherical 2-surfaces in various physically interesting spacetimes in Sec. IV. As a main result, we show that in asymptotically flat spacetimes the Dirichlet vector at spatial infinity can be identified in a natural way with the ADM energy-momentum vector.

We make some concluding remarks in Sec. V. (The notation and conventions of Ref. [1] are used throughout.)

II. MATTER FIELDS

It is convenient here to employ the tetrad formulation of the Einstein equations, since this simplifies the analysis of boundary conditions and Hamiltonian boundary terms as shown in Ref. [1]. We focus on Dirichlet and Neumann boundary conditions and make some remarks on more general boundary conditions at the end.
A. Preliminaries

On a given smooth orientable spacetime \((M, g_{ab})\), let \(\xi^a\) be a complete, smooth time-flow vector field, allowed to be timelike, spacelike, or null. Let \(\Sigma\) be a region contained in a fixed hypersurface in \(M\) such that the boundary of the region is a closed orientable spacelike 2-surface \(\partial \Sigma\) (with the hypersurface allowed to be otherwise arbitrary).

For treatment of boundary conditions when the time-flow \(\xi^a\) is not necessarily timelike, it is helpful to introduce the following structure associated to the boundary 2-surface \(\partial \Sigma\).

Let \(P_{\partial \Sigma}\) and \(P_{\perp \partial \Sigma}\) denote projection operators onto the tangent subspaces \(T(\partial \Sigma)\) and \(T(\partial \Sigma)\perp\) with respect to the surface \(\partial \Sigma\) in local coordinates in \(M\). Note \(P_{\partial \Sigma} + P_{\perp \partial \Sigma}\) is the identity map on the tangent space \(T(M)\) at \(\partial \Sigma\). Define the metric on \(\partial \Sigma\) by

\[
\sigma^{ab} = P_{\partial \Sigma}(g^{ab}), \quad \sigma_{ab} = g_{ac}g_{bd}\sigma^{bc}
\]  

(2.1)

Let \(\epsilon_{ab}\) be the metric volume form on \(\partial \Sigma\), and define

\[
\epsilon^{*ab} = \epsilon^{cd}\epsilon_{abcd}(g)
\]

(2.2)

in terms of the spacetime volume form \(\epsilon_{abcd}(g)\). Note that \(P_{\partial \Sigma}(\epsilon_{ab}) = P_{\perp \partial \Sigma}(\epsilon_{ab}) = 0\). A useful identity is given by

\[
2(P_{\partial \Sigma})^c_{[a}(P_{\partial \Sigma})^d_{b]} = \epsilon_{ab}^{cd}.
\]

(2.3)

Let

\[
\zeta^a = P_{\perp \partial \Sigma}(\xi^a), \quad N^a = P_{\partial \Sigma}(\xi^a),
\]

(2.4)

so \(\xi^a = \zeta^a + N^a\) decomposes into a sum of normal and tangential vectors with respect to \(\partial \Sigma\). We now suppose \(\zeta^a\) is not tangential to \(\partial \Sigma\), i.e. \(\zeta^a \neq 0\) everywhere on the surface \(\partial \Sigma\). In this situation, much of the formalism and results given in Sec. 3 of Ref. [1] can be paralleled.

Let \(B\) denote the hypersurface given by the image of \(\partial \Sigma\) under a one-parameter diffeomorphism generated by \(\xi^a\) on \(M\). Note that the dual vector field \(\epsilon^{ab}\zeta_b\) is hypersurface orthogonal since it is annihilated by all tangent vectors (in particular \(\xi^a\)) in \(B\). Define a basis \(\{s^a, t^a\}\) for \(T^*(\partial \Sigma)\perp\) by diagonalization of the identity map

\[
\delta^b_a = \sigma^b_a + s^a s^b + t^a t^b
\]

(2.5)

such that \(s_a \propto *\epsilon_{ab}\zeta^b\) is hypersurface orthogonal to \(B\), with \(\{s^*a, t^*a\}\) denoting a basis for \(T(\partial \Sigma)\perp\) that is dual to \(\{s^a, t^a\}\). In particular, \(s^*a s_a = t^*a t_a = 1\), and \(s^*a t_a = t^*a s_a = 0\). This leads to a corresponding decomposition of the spacetime metric

\[
g_{ab} = \sigma_{ab} + s_a s^* + t_a t^*
\]

(2.6)

with \(s^*a = g_{ab}s^b\) and \(t^*a = g_{ab}t^b\). Now, define a projection operator \(P_B\) with respect to \(B\) by
\[ h_{a}^{b} = \delta_{a}^{b} - s_{a}^{b} s^{a} \]  

satisfying
\[ h_{a}^{b} s_{b} = 0, h_{a}^{b} s^{a} = 0. \]  

Then
\[ h_{ab} = g_{ab} - s_{a} s_{b} = \sigma_{ab} + t_{a} t_{a}^{*} \]  
defines the induced metric on \( B \). Also, define the volume form on \( B \) by
\[ \epsilon_{abc}(h) = \epsilon_{abcd}(g) s_{d} = 3 t_{[ab]} c_{[bc]}^{e}. \]  

Finally, note that
\[ *\epsilon_{ab} = 4 t_{[ab]} s_{b}, \quad \epsilon_{abcd} = 3 \epsilon_{[ab]cde} = 4 \epsilon_{(abc) s_{d}}, \]  
\[ \zeta^{a} = - N t^{*a}, \quad \zeta^{a} t_{a} = - N, \quad \zeta^{a} s_{a} = \zeta^{a} \epsilon_{ab} = 0, \]  
for some scalar function \( N \). This yields the identities
\[ \xi^{a} = - N t^{*a} + N^{a}, \]  
\[ \xi^{a} \epsilon^{bc} \epsilon_{abcd}(g) = \xi^{a} \epsilon_{ad} = - 2 N s_{d}, \]  
\[ \xi^{a} \epsilon^{bc} \epsilon_{abc}(h) = - 2 N, \quad \mathcal{P}_{\mathcal{O}}(\xi^{a} \epsilon_{abc}(h)) = - N \epsilon_{bc}. \]  

These will be important in the analysis of boundary conditions for both the gravitational field and matter fields.

Now we introduce an orthonormal frame for \( g_{ab} \) given by
\[ \theta^{\mu}_{a} = \sigma_{a}^{\mu} + s_{a} s^{\mu} + t_{a} t^{*\mu} \]  
where \( \sigma_{a}^{\mu} = \sigma_{a}^{b} \theta^{\mu}_{b} \) is an orthonormal frame for \( \sigma_{ab} \), with the coefficients
\[ s^{*\mu} = s^{*a} \theta^{\mu}_{a}, \quad t^{*\mu} = t^{*a} \theta^{\mu}_{a} \]  
defined to satisfy \( \mp t^{*\mu} t^{*\nu} \pm s^{*\mu} s^{*\nu} = diag(-1, 1, 0, 0) \) if \( \xi^{a} \) is timelike or spacelike, or \( 2 s^{*\mu} t^{*\nu} = diag(-1, 1, 0, 0) \) if \( \xi^{a} \) is null. Consequently, the frame components of \( s^{a}, t^{a}, \sigma^{ab}, g^{ab} \) are given by
\[ s^{\mu} = s^{a} \theta^{\mu}_{a}, \quad t^{\mu} = t^{a} \theta^{\mu}_{a}, \]  
\[ \sigma^{a} = \sigma_{ab} \theta^{\mu}_{a} \theta^{\nu}_{b} = diag(0,0,1,1), \quad \eta^{\mu \nu} = g^{ab} \theta^{\mu}_{a} \theta^{\nu}_{b} = diag(\mp 1, \pm 1, 1, 1), \]  
where \( \{s^{\mu}, t^{\mu}\} \) are dual to \( \{s^{a} = \eta_{\mu \nu} s^{*\nu}, t^{*_{a} = \eta_{\mu \nu} t^{*\nu}}\} \). Hereafter we fix the frame coefficients (2.17) and (2.18) to be independent of the spacetime metric \( g_{ab} \), so therefore under a variation \( \delta g_{ab} \), the induced variations \( \delta s_{a}, \delta t_{a}, \delta \sigma_{a}^{\mu}, \delta \theta^{\mu}_{a} \) satisfy
\[ \delta \theta^\mu_a = \delta \sigma^\mu_a + s^\mu \delta s_a + t^\mu \delta t_a \]  
(2.20)

\[ \delta s_a = s_\mu \delta \theta^\mu_a, \quad \delta t_a = t_\mu \delta \theta^\mu_a, \quad \delta \sigma^\mu_a = \sigma_\nu \delta \sigma^\nu_a, \]  
(2.21)

\[ \delta \sigma_{ab} = 2\sigma_{\mu \nu} \theta^\mu_{(a} \delta \theta^\nu_{b)} = \sigma_{(a} \delta \sigma_{b)}, \quad \delta g_{ab} = 2\eta_{\mu \nu} \theta^\mu_{(a} \delta \theta^\nu_{b)}. \]  
(2.22)

Note, by hypersurface orthogonality of \( s_a \), it also follows that

\[ \mathcal{P}_B(\delta s_a) = 0, \quad \mathcal{P}_{\partial \Sigma}(\delta t_a) = 0, \quad \mathcal{P}_{\partial \Sigma}(\delta \sigma_{ab}) = 0. \]  
(2.23)

Let

\[ h^\mu_a = h^b_a \theta^\mu_b = \theta^\mu_a - s_a s^\mu \]  
(2.24)

which yields a decomposition of the frame with respect to \( B \), satisfying

\[ \mathcal{P}_{\partial \Sigma}(h^\mu_a) = \sigma^\mu_a, \quad \mathcal{P}_{\partial \Sigma}(h^\mu_a) = t_a t^\mu. \]  
(2.25)

It is convenient for later to also introduce a fixed frame adapted to \( \partial \Sigma \) and \( \xi^a \). Let

\[ \vartheta^0_a = t_a, \quad \vartheta^1_a = s_a, \quad \vartheta^2_a, \vartheta^3_a = \epsilon_{ab}, \quad \vartheta^2_a = \epsilon^b_a \vartheta^3_a, \]  
(2.26)

which defines the frame \( \vartheta^\mu_a \) uniquely up to rotations of \( \vartheta^2_a, \vartheta^3_a \). Thus, in this formalism, \( \vartheta^\mu_a \) is an orthonormal frame when \( \xi^a \) is timelike or spacelike, and a null frame when \( \xi^a \) is null.

In the case when \( \zeta^a \) is timelike, the previous formalism reduces to that in Ref. [1]. Most important, the formalism here applies equally well to the cases when \( \zeta^a \) is spacelike or null.

Finally, in the case that \( \zeta^a = 0 \), i.e. \( \xi^a \) is tangential to \( \partial \Sigma \), we simply fix any basis \( \{s_a, t_a\} \) of \( T^*(\partial \Sigma)^\perp \) and define a frame \( \theta^\mu_a \) to satisfy the previous equations (2.16) to (2.19). This yields the same formalism as in the case that \( \xi^a \) is not tangential to \( \partial \Sigma \), except that there does not exist a hypersurface \( B \) generated by \( \zeta^a = 0 \).

Now, with the frame \( \theta^\mu_a \) used as the gravitational field variable, the Lagrangian for the vacuum Einstein equations is given by

\[ L_{abcd}(\theta) = \epsilon_{abcd}(\theta)R(\theta). \]  
(2.27)

Here \( R(\theta) = \theta^b_\nu R^\nu_b(\theta) \) and \( R^\mu_a(\theta) = \theta^b_\nu R^\mu_{ab}(\theta) \) are the scalar curvature and Ricci curvature obtained from the curvature 2-form

\[ R^\mu_{ab}(\theta) = 2\partial^a \Gamma^\mu_{b\nu}(\theta) + 2\Gamma^\mu_{a\sigma}(\theta) \Gamma^\nu_{b\sigma}(\theta) \]  
(2.28)

in terms of the frame-connection given by

\[ \Gamma^\mu_{ab}(\theta) = \theta^b_\mu \nabla_a \theta^\nu_b - \theta^b_\mu \theta^\nu_{(a} \partial \theta^\alpha_{b)}. \]  
(2.29)

A variation of \( L_{abcd}(\theta) \) yields
\[ \delta L_{abcd}(\theta) = \mathcal{E}_{bcd}(\theta) \delta \theta^\mu_a + \partial_a \Theta_{bcd}(\theta, \delta \theta) \]  
(2.30)

where

\[ \mathcal{E}_{bcd}(\theta) = 8 \epsilon_{abcd}(\theta)(R^\mu_{\alpha}(\theta) - \frac{1}{2} \theta^\mu_{\alpha} R(\theta)) = 0 \]  
(2.31)

are the vacuum Einstein field equations for \( \theta^\mu_a \), and where

\[ \Theta_{bcd}(\theta, \delta \theta) = 8 \epsilon_{abcd}(\theta) \theta^\nu_{\beta} \theta^\alpha_{\gamma} \delta \theta^\mu \]  
(2.32)

is the symplectic potential 3-form. It follows that the Noether current associated to \( \xi^a \) is given by the 3-form

\[ J_{abc}(\xi, \theta) = \Theta_{abc}(\theta, \mathcal{L}_\xi \theta) + 4 \xi^d L_{abcd}(\theta) = \epsilon_{abcd}(g)(8 \theta^\mu_{\alpha} \theta^\nu_{\beta} \mathcal{L}_\xi \Gamma_{e\mu\nu}(\theta) + 4 \xi^d R(\theta)) \]  
(2.33)

which simplifies to

\[ J_{abc}(\xi; \theta) = 3 \partial_a Q_{bc}(\xi; \theta) - \xi^c \theta^e \mathcal{E}_{abc}(\theta) \]  
(2.34)

where

\[ Q_{bc}(\xi; \theta) = 4 \xi^c \epsilon_{abcd}(g) \theta^d_{\mu} \theta^e_{\nu} \Gamma_{e\mu\nu}(\theta) \]  
(2.35)

is the Noether charge potential.

The gravitational Noether charge associated to \( \partial \Sigma \) is determined by the pullback of \( Q_{bc}(\xi; \theta) \). A simple expression for the pullback is obtained through identities (2.2) and (2.11), yielding

\[ \epsilon^{bc} \xi^a \epsilon_{abcd}(g) \theta^d_{\mu} \theta^e_{\nu} \Gamma_{e\mu\nu}(\theta) = 8 \epsilon_{abc}(g) \theta^d_{\mu} \theta^e_{\nu} \Gamma_{e\mu\nu}(\theta) \]  
(2.36)

where, recall, \( \epsilon_{bc} \) is the volume form on \( \partial \Sigma \). Hence, the surface integral

\[ Q_{\Sigma}(\xi; \theta) = \int_{\partial \Sigma} Q_{bc}(\xi; \theta) = 8 \int_{\partial \Sigma} \epsilon_{bc} \xi^a \theta^d_{\mu} \theta^e_{\nu} \Gamma_{e\mu\nu}(\theta) \]  
(2.37)

gives the gravitational Noether charge.

When \( \xi^a \) is not tangential to \( \partial \Sigma \), the pullback of \( \Theta_{bcd}(\theta, \delta \theta) \) to the hypersurface \( \mathcal{B} \) can be simplified similarly by the identities (2.14) and (2.11) and the frame decomposition (2.24),

\[ \frac{1}{8} \epsilon^{ab} \xi^c \Theta_{abc}(\theta, \delta \theta) = \epsilon^{ab} \xi^c \epsilon_{abcd}(g) \theta^d_{\mu} \theta^e_{\nu} \delta \Gamma_{e\mu\nu}(\theta) \]
\[ = -2 N \theta^e_{\mu} \delta(\theta^e_{\nu} \nabla_{\nu} \theta^\mu) \]
\[ = -2 N h^e_{\nu} \delta(s^a_{\mu} h^d_{\nu} c^g_{\nu} \nabla_{\nu} \theta^\mu) \]
\[ = \epsilon^{ab} \xi^c \epsilon_{abc}(h) h^e_{\nu} \delta K_{e\nu} \]  
(2.38)

where we define
\[ K_a^\mu = s_\nu h_a^d h^e_\mu g \nabla_d \theta^\nu_c. \] (2.39)

(Note these expressions have the same form as those obtained in Ref. [1] when \( \xi^a \) is timelike.)

Hence, one obtains

\[ P_B \Theta_{abc}(\theta, \delta \theta) = 8 \epsilon_{abc} (h) h^e_\nu \delta K^\nu_e. \] (2.40)

This expression leads to a simple form for the gravitational symplectic flux associated to \( B \),

\[
\int_B \omega_{abc}(\theta, \delta_1 \theta, \delta_2 \theta) = \int_B \delta_1 \Theta_{abc}(\theta, \delta_2 \theta) - \delta_2 \Theta_{abc}(\theta, \delta_1 \theta) \\
= 8 \int_B \epsilon_{abc}(h) \left( (\delta_1 h^d_\mu - h^e_\mu h^e_\nu \delta_1 h^\nu_e) \delta_2 K^\mu_d - (\delta_2 h^d_\mu - h^e_\mu h^e_\nu \delta_2 h^\nu_e) \delta_1 K^\mu_d \right). \tag{2.41}
\]

The vanishing of this flux determines the allowed boundary conditions on the frame \( \theta^\mu_c \) at the boundary hypersurface \( B \).

We remark that in a frame (2.26) adapted to \( \partial \Sigma \), one sees that \( K^\mu_a = h^e_\mu h^a dg \nabla_d s_c \) represents the frame components of the extrinsic curvature tensor \( K_{ab} = h^e_\mu h^a dg \nabla_d s_c \) of the boundary hypersurface \( B \). Moreover, the Noether charge (2.37) is simply \( Q_\Sigma (\xi; \theta) = 8 \int_{\partial \Sigma} \epsilon_{bc} a^t_\xi t^\mu_a \theta^\mu_c = 8 \int_{\partial \Sigma} \epsilon_{bc} a^t_\xi \theta^\mu_c K^\mu_a \).

For the sequel, we now introduce Dirichlet and Neumann symplectic vectors

\[
P^D_a(\theta) = s_\nu \epsilon^e_\sigma \sigma^d_\sigma \nabla_d \theta^\nu_c - s_\nu \epsilon^e_\sigma \sigma^d_\sigma \nabla_d \theta^\mu_d + t_\nu s_a \sigma_{bdg} \nabla_b \theta^\nu_d = \frac{1}{2} \epsilon_{bcde} \theta^d_b \Gamma^\mu_a(\theta), \tag{2.42}
\]

\[
P^N_a(\theta) = s_\nu \epsilon^e_\sigma \nabla_a \theta^\nu_c = \frac{1}{4} \epsilon_{bcde} \theta^d_b \Gamma^\mu_a(\theta), \tag{2.43}
\]

associated to the boundary 2-surface \( \partial \Sigma \) and the frame \( \theta^\mu_a \). In a frame (2.26) adapted to the hypersurface \( B \), these vectors take the more geometrical form

\[
P^D_a(\vartheta) = \epsilon^e_\sigma \sigma^d_\sigma \nabla_d s_c - t_\sigma \sigma^d_\sigma \nabla_d s_d + s_a \sigma_{bdg} \nabla_b s_d, \tag{2.44}
\]

\[
P^N_a(\vartheta) = \epsilon^e_\sigma \nabla_a s_c. \tag{2.45}
\]

Similarly to the derivation in Ref. [1] holding for the situation when \( B \) is timelike, here the projection of the vectors (2.42) and (2.43) along \( \xi^a \) yields the respective boundary terms required to define a covariant Hamiltonian for the vacuum Einstein equations with \( \xi^a \) as the time-flow vector field in a spacetime region with spatial boundary 2-surface \( \partial \Sigma \), subject to Dirichlet or Neumann boundary conditions on the frame \( \theta^\mu_a \), for a timelike, spacelike, or null boundary hypersurface \( B \). The significance and properties of the full vectors (2.42) and (2.43) will be discussed in Sec. III.

Lastly, we make some remarks on the gauge invariance of the preceding results, which follow from the detailed gauge transformation analysis given in Sec. 3 of Ref. [1]. Under a local \( SO(3, 1) \) transformation on the frame \( \theta^\mu_a \), the Noether charge \( Q_{bc}(\xi; \theta) \) transforms inhomogeneously due to its explicit dependence on the frame connection. However, the curvature \( R_{ab}^{\mu\nu}(\theta) \) is invariant, and consequently so is the Lagrangian \( L_{abcd}(\theta) \). Therefore the symplectic current \( \omega_{abc}(\theta, \delta_1 \theta, \delta_2 \theta) \) is necessarily gauge invariant. As a result, up to
addition of a locally constructed exact 2-form, the symplectic current obtained here for
the frame formulation of the vacuum Einstein equations must agree with the analogous
current derived from the standard metric formulation. This means that the presymplectic
forms \( \Omega_\Sigma(\theta, \delta \theta, L_\xi \theta) \) and \( \Omega_\Sigma(g, \delta g, L_\xi g) \) in the two formulations differ by only a boundary
term (i.e. a locally constructed 2-form integrated over the 2-surface \( \partial \Sigma \)). Correspondingly,
the Dirichlet and Neumann symplectic vectors associated to \( \Omega_\Sigma(g, \delta g, L_\xi g) \) in the metric
formulation are found to be the same as the ones given here for the frame formulation, up
to certain gradient terms. Furthermore, if \( \xi^a \) is timelike and orthogonal to \( \partial \Sigma \), then these
gradient terms can be shown to vanish. In this situation, \( \xi^a P^D_\alpha(\vartheta) \) and \( \xi^a P^N_\alpha(\vartheta) \) are precisely
the Dirichlet and Neumann boundary terms in the covariant Hamiltonian determined by the
metric formulation of the vacuum Einstein equations (see Ref. [15] for a discussion of this
Hamiltonian). Consequently, as noted in Ref. [1], we find that the expression \( \xi^a P^D_\alpha(\vartheta) \) reduces to the boundary term derived by Brown and York [16,17] in the standard canonical
formalism, with Dirichlet boundary conditions on the canonical variables in the case of a
hypersurface boundary \( \mathcal{B} \) where \( \xi^a \) is timelike. In comparison, the covariant formulation we
have presented here applies equally well when \( \xi^a \) is null or spacelike.

B. Electromagnetic field

We start by considering a free electromagnetic field \( A^a \) on \((M, g_{ab})\), coupled to the gravi-
tational field, generalizing the Minkowski background spacetime considered in Sec. 2 in Ref.
[1]. The Lagrangian 4-form for \( A^a \) is given by

\[
L_{abcd}(A; \theta) = \frac{1}{2} \epsilon_{abcd}(g) F_{mn} F^{mn} = 3 F_{[ab} * F_{cd]} (2.46)
\]

where \( F_{ab} = g^{\alpha \nu} [A_\alpha]_b = \partial_\alpha A^\alpha_a \) is the electromagnetic field strength and \( * F_{ab} = \epsilon_{abcd}(g) F^{cd} \) is the dual field strength in terms of \( F^{cd} = g_a^c g_b^d F_{bd} \), with \( g_{ab} = \theta^\mu_a \theta^\nu_b \eta_{\mu \nu} \). A useful fact here
is that \( g^\alpha \nabla_\alpha \) reduces to \( \partial_\alpha \) in any skew derivative expression on \( M \). By variation of \( A^a \) and \( \theta^\mu \) in this Lagrangian, one obtains

\[
\delta L_{abcd}(A; \theta) = \epsilon_{abcd}(g) \left( g^\nu \nabla_\nu (\delta A^\mu_\nu F^{mn}) - \delta A^\mu_\nu g^\nu \nabla_\mu F^{mn} - T^{\mu}_{\nu}(A; \theta) \delta \theta^\nu \right) (2.47)
\]

where \( T^{\mu}_{\nu}(A; \theta) \theta^\nu = T^\nu_{\mu}(A; g) \) is the electromagnetic stress-energy tensor given by

\[
T^b_a(A; g) = 2 F_{ac} F^{bc} - \frac{1}{2} \delta^b_a F_{mn} F^{mn}. (2.48)
\]

From the coefficient of the variation \( \delta A^a \) in \( \delta L_{abcd}(A; \theta) \), the field equation for \( A^a \) is given
by the Maxwell equations

\[
* \mathcal{E}_a(A; g) = g^b \nabla^b F_{ba} = g^b \nabla^b \partial_\nu A^\nu_a = 0. (2.49)
\]

The symplectic potential 3-form obtained from \( L_{abcd}(A; \theta) \) is given by the total derivative
term in Eq. (2.47), which yields
\[ \Theta_{bcd}(A, \delta A; \theta) = 4 \epsilon_{abcd}(g) \delta A_e F^{ae}. \]  

(2.50)

Hence, the Noether current associated to \( \xi^a \) for \( A_a \) is given by the 3-form

\[ J_{abc}(\xi, A; g) = \Theta_{abc}(A, \mathcal{L}_\xi A; \theta) + 4 \xi^d L_{abcd}(A; \theta) \]

\[ = \epsilon_{abcd}(g)(-4 F^{de} \mathcal{L}_\xi A_e + 2 \xi^d F_{mn} F^{mn}) \]

\[ = 4 \epsilon_{abcd}(g)(\xi^e T^d_e(A; g)) + \epsilon^e A_e \mathcal{E}^d(A; g)) + 6 \partial_l (\ast F_{bc}) \xi^e A_e \]

(2.51)

by a similar derivation as in Minkowski spacetime, with

\[ \mathcal{L}_\xi A_a = \xi^e \nabla_e A_a + A_e \nabla^e \xi_a = 2 \xi^e \nabla_{[e} A_{a]} + \nabla^e (\xi^e A_e). \]

(2.52)

This yields the electromagnetic Noether charge

\[ Q_{\Sigma}(\xi; A) = \int_{\Sigma} J_{abc}(\xi, A; g) = 4 \int_{\Sigma} \xi^e \epsilon_{dabc}(g) T^d_e(A; g) + 2 \int_{\partial \Sigma} \epsilon_{bcda} F^{da} \xi^e A_e \]

(2.53)

for solutions \( A_a \) of the Maxwell equations (2.49).

The total Lagrangian for the Maxwell equations coupled to the Einstein equations

\[ R^a_{\mu}(\theta) - \frac{1}{2} \theta^a_{\mu} R(\theta) = T^a_{\mu}(A; \theta) \]

(2.54)

using the field variables \( A_a \) and \( \theta^a_{\mu} \) is given by \( L_{abcd}(\theta, A) = L_{abcd}(\theta) - L_{abcd}(A; \theta) \) from Eqs. (2.27) and (2.46). One then obtains the total Noether current

\[ J_{abc}(\xi, \theta, A) = J_{abc}(\xi, \theta) - J_{abc}(\xi, A; \theta) \]

\[ = 8 \epsilon_{dabc}(g) \xi^e \theta^\mu_e (R^d_{\mu}(\theta)) - \frac{1}{2} \theta^d \theta^\mu R(\theta) - T^d_{\mu}(A; \theta)) + 3 \partial_l \epsilon_{bcda} \xi^e A_e \]

(2.55)

where

\[ Q_{bc}(\xi, \theta, A) = \xi^a \epsilon_{bcda}(g)(4 \theta^\mu_e \Gamma^\mu_{\nu a} - 2 F_{de} A_a) \]

(2.56)

is the Noether charge potential. Hence, on solutions of the coupled Einstein-Maxwell equations, the total Noether charge is given by the surface integral

\[ Q_{\Sigma}(\xi; \theta, A) = \int_{\partial \Sigma} Q_{bc}(\xi, \theta, A). \]

(2.57)

The electromagnetic part of this expression simplifies through identities (2.2) and (2.11), yielding

\[ \epsilon^a_{bc} \ast F_{bc} \xi^d A_d = \epsilon^a_{bc} F^{bc} \xi^d A_d = 4 t^a_{bc} \xi^d A_d. \]

(2.58)

Then, substituting Eq. (2.36) for the gravitational part, one obtains

\[ Q_{\Sigma}(\xi; \theta, A) = \int_{\partial \Sigma} \epsilon_{bc} \xi^a (8 s^b t^c \nabla_e \theta^\mu e - 4 t^d s^e F^{de} A_a). \]

(2.59)
The Noether current gives a Hamiltonian conjugate to $\xi^a$ on $\Sigma$ under compact support variations $\delta\theta^\mu_a$ and $\delta A_a$,

$$H(\xi; \theta, A) = 8 \int_\Sigma \epsilon_{dabc}(g) \xi^e \theta^\mu_e (R_\mu^d(\theta) - \frac{1}{2} \theta^d_\mu R(\theta) - T_\mu^d(A; \theta))$$  \quad (2.60)

up to a boundary term (2.59). For variations $\delta\theta^\mu_a$ and $\delta A_a$ with support on $\partial\Sigma$, after taking into account boundary terms, one has

$$\delta H(\xi; \theta, A) = \int_{\partial\Sigma} \delta Q_{ab}(\xi, \theta, A) - \xi^c \Theta_{abc}(\theta, A, \delta\theta, \delta A)$$  \quad (2.61)

for Einstein-Maxwell solutions, where

$$\Theta_{abc}(\theta, A, \delta\theta, \delta A) = \Theta_{abc}(\theta, \delta\theta) - \Theta_{abc}(A, \delta A; \theta)$$  \quad (2.62)

is the total symplectic potential 3-form from Eqs. (2.32) and (2.50). The electromagnetic part of the symplectic potential terms in the Hamiltonian variation (2.61) can be simplified similarly to expression (2.38) for the gravitational part, yielding

$$\frac{1}{4} \epsilon^{ab} \xi^c \Theta_{abc}(A, \delta A; \theta) = \epsilon^{ab} \xi^c \epsilon_{dabc}(g) F^{de} \delta A_e = 2Ns_d F^{de} \delta A_e = -\epsilon^{ab} \epsilon_{abc} (h)s_d F^{de} \delta A_e$$  \quad (2.63)

through identities (2.14) and (2.15). Thus, one obtains

$$\mathcal{P}_{\partial\Sigma}(\xi^c \Theta_{abc}(\theta, A, \delta\theta, \delta A)) = \mathcal{P}_{\partial\Sigma}(\xi^c \epsilon_{abc}(h)(8h^\nu_\nu \delta K_\nu^\nu + 4s_d F^{de} \delta A_e))$$  \quad (2.64)

$$= -N\epsilon_{ab} (8h^\nu_\nu \delta K_\nu^\nu + 4s_d F^{de} \delta A_e).$$

Hence, for existence of a Hamiltonian conjugate to $\xi^a$ on $\Sigma$, there must exist a locally constructed 3-form $\tilde{B}_{abc}(\theta, A)$ such that

$$\mathcal{P}_{\partial\Sigma}(\xi^c \Theta_{abc}(\theta, A, \delta\theta, \delta A)) = \mathcal{P}_{\partial\Sigma}(\xi^c \tilde{B}_{abc}(\theta, A) - \partial_{[a} \alpha_{b]}(\xi, \theta, A, \delta\theta, \delta A))$$  \quad (2.65)

for some locally constructed 1-form $\alpha_b(\xi, \theta, A, \delta\theta, \delta A)$ in $T^* (\partial\Sigma)$. Then the total Hamiltonian is given by $H(\xi; \theta, A)$ plus a boundary term

$$H_B(\xi; \theta, A) = \int_{\partial\Sigma} Q_{ab}(\xi, \theta, A) - \xi^c \tilde{B}_{abc}(\theta, A).$$  \quad (2.66)

We now consider Dirichlet and Neumann type boundary conditions on the fields $\theta^\mu_a$ and $A_a$ at $\partial\Sigma$.

First, consider the case when $\xi^a$ is tangential to $\partial\Sigma$. Then one finds

$$\mathcal{P}_{\partial\Sigma}(\xi^c \Theta_{abc}(\theta, A, \delta\theta, \delta A)) = 0,$nach which leads to the following result.
Proposition 2.1. Suppose $\xi^a$ is tangential to $\partial \Sigma$. Then no boundary conditions are necessary for existence of a Hamiltonian conjugate to $\xi^a$ on $\Sigma$. Consequently, a Hamiltonian is given by $H_\Sigma(\xi; \theta, A) = H(\xi; \theta, A) + Q_\Sigma(\theta, A)$.

Next, assume $\xi^a$ is not tangential to $\partial \Sigma$, and consider Dirichlet and Neumann boundary conditions on the electromagnetic and gravitational field variables.

Theorem 2.2. Suppose $\xi^a$ is nowhere tangential to $\partial \Sigma$. Let

\begin{align}
(D) & \quad \delta(h^b_a A_b)|_{\partial \Sigma} = 0, \quad \delta(h^\mu_a)|_{\partial \Sigma} = 0 \\
(N) & \quad \delta(|h|s^a h^a F_{eb})|_{\partial \Sigma} = 0, \quad \delta(K^\mu_a)|_{\partial \Sigma} = 0
\end{align}

where $|h| = \text{det}(h^\mu_a)$ is the determinant of the components of the frame $h^\mu_a$ associated to $\mathcal{B}$.

Under Dirichlet (D) or Neumann (N) boundary conditions for both $A_a$ and $\theta^\mu_a$, there exists a Hamiltonian $H(\xi; \theta, A) + H_B(\xi; \theta, A)$ conjugate to $\xi^a$ on $\Sigma$, with the boundary term (2.66) given by

\begin{align}
H^D(\xi; \theta, A) &= 8 \int_{\partial \Sigma} \xi^a (P^D_a(\theta) - \frac{1}{2} t_d s^e F^{de} A_a) dS, \\
H^N(\xi; \theta, A) &= 8 \int_{\partial \Sigma} \xi^a (P^N_a(\theta) - \sigma_{[a} c_{d]} s^e F^{de} A_c) dS,
\end{align}

in terms of the Dirichlet and Neumann symplectic vectors (2.42) and (2.43).

Proof:

For case (D), first note from Eq. (2.63) that $\epsilon_{bc} \xi^a \Theta_{abc}(A, \delta A; \theta) = 8Ns^d F^{de} h^e_c \delta A_c$. Now, using the boundary condition (D) on $\delta A_a$, one has

$$h^e_c \delta A_c = h^e_c \delta(s^b s^h A_b) = s^b A_b h^e_c \delta s^c = 0$$

by the hypersurface orthogonality relations (2.7) and (2.23). Thus,

$$\mathcal{P}_{\partial \Sigma}(\xi^a \Theta_{abc}(A, \delta A; \theta)) = 0.$$ \hspace{1cm} (2.72)

Then, in Eq. (2.38), since $\delta \epsilon_{abc}(h) = \delta \epsilon_{bc} = 0$ by the boundary condition (D) on $\delta \theta^\mu_a$, one has

$$\epsilon_{bc} \xi^a \Theta_{abc}(\theta, \delta \theta) = \epsilon_{bc} \delta(8 \xi^a \epsilon_{abc}(h)K)$$

and thus,

$$\mathcal{P}_{\partial \Sigma}(\xi^a \Theta_{abc}(\theta, \delta \theta) - \delta(8 \xi^a \epsilon_{abc}(h)K)) = 0$$ \hspace{1cm} (2.74)

where $K = h_{\nu}^e K_{\nu}^e$. Hence, substitution of Eqs. (2.72) and (2.74) into Eq. (2.65) yields

$$\xi^a \tilde{B}_{abc}(\theta, A) = 8 \xi^a \epsilon_{abc}(h)K$$ \hspace{1cm} (2.75)
and $\alpha_b = 0$. This leads to the boundary term (2.69) through Eq. (2.66) as follows. The pullback of $\xi^a \tilde{B}_{abc}(\theta, A)$ to $\partial \Sigma$ is given by

$$8\xi^a \epsilon^{bc} \epsilon_{abc}(h) K = 8\xi^a t_a s_d e^g \nabla_e \theta^{d\nu} = 16\xi^a t_a s_d e^g \nabla_e \theta^{d\nu}$$

(2.76)

which, when combined with expression (2.36) for the pullback of $Q_{bc}(\xi, \theta)$, yields

$$\epsilon^{bc}(Q_{bc}(\xi, \theta) - \xi^a \tilde{B}_{abc}(\theta, A)) = 16\xi^a (s_t e^g \nabla_a \theta^{\nu} - t_a s_d e^g \nabla_e \theta^{d\nu})$$

$$= 16\xi^a s_t (t_d \sigma_a - t_a \sigma^{de}) g \nabla_e \theta^{\nu}$$

$$= 16\xi^a P^D_a(\theta)$$

(2.77)

by the metric decompositions (2.6) and (2.9) and the orthogonality $\xi^a s_a = 0$. Finally, combining this expression with the pullback of $Q_{bc}(\xi, A)$ given by Eq. (2.58), we obtain Eq. (2.69).

For case (N), one has from Eq. (2.63) that

$$\epsilon^{bc} \xi^a \Theta_{abc}(A, \delta A; \theta) = \epsilon^{bc} \delta(4\xi^a \epsilon_{abc}(h)s_e A_d F^{de}) + 4\epsilon^{bc} A_e \xi^a \delta(\epsilon_{abc}(h)s_d F^{de}).$$

(2.78)

Now, since

$$\delta(\epsilon_{abc}(h)s_d F^{de}) = \epsilon_{abc}(h) \delta(h) \delta(s_d h_m e F^{dm}) + \delta \ln |h| s_d h_m e F^{dm},$$

(2.79)

this term vanishes by boundary condition (N) for $\delta F^{ab}$, and thus

$$\mathcal{P}_{\partial \Sigma}(\xi^a \Theta_{abc}(A, \delta A; \theta) - \delta(4\xi^a \epsilon_{abc}(h)s_e A_d F^{de})) = 0.$$

(2.80)

Next, $\mathcal{P}_{\partial \Sigma}(\xi^a \Theta_{abc}(\theta, \delta \theta)) = 0$ holds immediately by boundary condition (N) for $\delta K^\alpha_a$. Hence, from Eqs. (2.65) and (2.80), one has $\alpha_d = 0$ and

$$\xi^a \tilde{B}_{abc}(\theta, A) = 4\xi^a \epsilon_{abc}(h)s_e A_d F^{de}.$$  

(2.81)

Then this leads to the boundary term (2.70) through Eq. (2.66) similarly to the derivation of the boundary term (2.69) above.

C. Klein-Gordon scalar field

We next consider a free Klein-Gordon scalar field $\varphi$ coupled to the gravitational field on $(M, g_{ab})$, with the standard Lagrangian 4-form given by

$$L_{abcd}(\varphi; \theta) = \frac{1}{2} \epsilon_{abcd}(g)(g^{\nabla_{a}\varphi} g^{\nabla_{b}\varphi} + m^2 \varphi^2)$$

(2.82)

where $m = \text{const}$ is the mass. Note, here $g^{\nabla_{a}\varphi} = \partial_{a} \varphi$, $g^{\nabla^{a}\varphi} = g^{ab} \partial_{b} \varphi$, and $g_{ab} = \theta^\mu_a \theta^\nu_b \eta_{\mu \nu}$. A variation of this Lagrangian with respect to $\varphi$ and $\theta^\mu_a$ yields
\[ \delta L_{abcd}(\varphi; \theta) = \epsilon_{abcd}(g)(g^{e}g^{ed}\varphi - m^{2}\varphi) - T_{\mu}^{\varphi}(\varphi; \theta)\delta \theta_{\mu}^{a} \]  

(2.83)

where \( T_{\mu}^{\varphi}(\varphi; \theta)\delta \theta_{\mu}^{a} = T_{d}^{\varphi}(\varphi; g) \) is the Klein-Gordon stress-energy tensor given by

\[ T_{d}^{\varphi}(\varphi; g) = g^{d\varphi}g_{a\varphi} - \frac{1}{2}\delta_{a}^{d}(g^{e}g_{e\varphi} + m^{2}\varphi^{2}). \]  

(2.84)

Hence, from the coefficient of the variation \( \delta \varphi \) in Eq. (2.83), the Klein-Gordon field equation for \( \varphi \) is given by

\[ \ast \mathcal{E}(\varphi; g) = g^{a}\partial_{a}\varphi - m^{2}\varphi = 0. \]  

(2.85)

The symplectic potential 3-form obtained from \( L_{abcd}(\varphi; \theta) \) is given by

\[ \Theta_{bcd}(\varphi, \delta \varphi; \theta) = 4\epsilon_{abcd}(g)g^{a}\partial_{a}\varphi. \]  

(2.86)

This yields the Noether current associated to \( \xi^{a} \)

\[ J_{abc}(\xi, \varphi; g) = \Theta_{abc}(\varphi, \xi \varphi; \theta) + 4\epsilon_{abcd}(g)g^{a}\partial_{a}\varphi \]  

(2.87)

where \( \xi \varphi = \xi^{a}g_{a\varphi} = \xi^{a}\partial_{a}\varphi \). Hence, one obtains the Noether charge

\[ Q_{\Sigma}(\xi; \varphi) = \int_{\Sigma} J_{abc}(\xi, \varphi; g) = 4\int_{\Sigma} \epsilon_{dabc}(g)\xi^{a}T_{d}(\varphi; g). \]  

(2.88)

In contrast to the situation for the electromagnetic field, here, due to the scalar nature of the Klein-Gordon field, the Noether charge does not have a surface integral term.

The total Lagrangian for the Klein-Gordon equation coupled to the Einstein equations

\[ R_{\mu}^{\varphi}(\theta) - \frac{1}{2}\theta_{\mu}^{\theta}R(\theta) = T_{\mu}^{\varphi}(\varphi; \theta) \]  

(2.89)

using the field variables \( \varphi \) and \( \theta_{\mu}^{a} \) is obtained through Eqs. (2.27) and (2.82) by \( L_{abcd}(\theta, \varphi) = L_{abcd}(\theta) - L_{abcd}(\varphi; \theta) \). The resulting total Noether current is given by

\[ J_{abc}(\xi, \theta, \varphi) = J_{abc}(\xi, \theta) - J_{abc}(\xi, \varphi; \theta) \]  

(2.90)

where \( Q_{bc}(\xi, \theta) \) is the gravitational Noether charge potential (2.35). Thus, there is no contribution from \( \varphi \) to the total Noether charge.

The Noether current gives a Hamiltonian conjugate to \( \xi^{a} \) on \( \Sigma \) under compact support variations \( \delta \theta_{\mu}^{a} \) and \( \delta \varphi \).
\[ H(\xi; \theta, \varphi) = 8 \int_{\Sigma} \epsilon_{dabc}(g) \xi^e \theta^\mu (R^d_{\mu}(\theta) - \frac{1}{2} \theta^d_{\mu} R(\theta) - T^d_{\mu}(\varphi; \theta)) \] (2.91)

up to a boundary term \( \int_{\partial \Sigma} Q_{ab}(\xi, \theta) \). For variations \( \delta \theta^\mu_a \) and \( \delta \varphi \) with support on \( \partial \Sigma \), one has for Einstein-Klein-Gordon solutions,

\[ \delta H(\xi; \theta, \varphi) = \int_{\partial \Sigma} \delta Q_{ab}(\xi, \theta) - \xi^c \Theta_{abc}(\theta, \delta \theta) \] (2.92)

where \( \Theta_{abc}(\theta, \delta \theta) \) is the expression (2.38) for the gravitational symplectic potential. Thus, there exists a Hamiltonian conjugate to \( \xi^a \) on \( \Sigma \) if

\[ \mathcal{P}_{\partial \Sigma}(\xi^c \Theta_{abc}(\theta, \varphi, \delta \theta, \delta \varphi)) = \mathcal{P}_{\partial \Sigma}(\xi^c \tilde{B}_{abc}(\theta, \varphi) - \partial_{[a} \alpha_{b]}(\xi, \theta, \varphi, \delta \theta, \delta \varphi)) \] (2.93)

holds for a locally constructed 3-form \( \tilde{B}_{abc}(\theta, \varphi) \) and 1-form \( \alpha_b(\xi, \theta, \varphi, \delta \theta, \delta \varphi) \). Then the total Hamiltonian is given by \( H(\xi; \theta, \varphi) \) plus a boundary term

\[ H_B(\xi; \theta, \varphi) = \int_{\partial \Sigma} Q_{ab}(\xi, \theta) - \xi^c \tilde{B}_{abc}(\theta, \varphi). \] (2.94)

Now, by an analysis similar to that for the Einstein-Maxwell equations, we obtain the following results.

**Proposition 2.3.** Suppose \( \xi^a \) is tangential to \( \partial \Sigma \). Then no boundary conditions are necessary for existence of a Hamiltonian conjugate to \( \xi^a \) on \( \Sigma \). Consequently, a Hamiltonian is given by \( H(\xi; \theta, \varphi) = H(\xi; \theta, \varphi) + Q_{\Sigma}(\theta) \).

**Theorem 2.4.** Suppose \( \xi^a \) is nowhere tangential to \( \partial \Sigma \). Let

\[
\begin{align*}
(D) & \quad \delta(\varphi)|_{\partial \Sigma} = 0, \quad \delta(h^\mu_a)|_{\partial \Sigma} = 0 \\
(N) & \quad \delta(|h|^a s^a \partial_a \varphi)|_{\partial \Sigma} = 0, \quad \delta(K^\mu_a)|_{\partial \Sigma} = 0
\end{align*}
\] (2.95, 2.96)

where \( |h| = \det(h^\mu_a) \) is the determinant of the components of the frame \( h^\mu_a \) associated to \( \mathcal{B} \). Under Dirichlet (D) or Neumann (N) boundary conditions for both \( \varphi \) and \( \theta^\mu_a \), there exists a Hamiltonian \( H(\xi; \theta, \varphi) + H_B(\xi; \theta, \varphi) \) conjugate to \( \xi^a \) on \( \Sigma \), with the boundary term (2.94) given by

\[
\begin{align*}
H^D(\xi; \theta, \varphi) &= 8 \int_{\partial \Sigma} \xi^a \mathcal{P}^D_a(\theta) dS, \tag{2.97} \\
H^N(\xi; \theta, \varphi) &= 8 \int_{\partial \Sigma} \xi^a (\mathcal{P}^N_a(\theta) - \frac{1}{2} t^d_a \varphi s^d \partial_d \varphi) dS. \tag{2.98}
\end{align*}
\]

in terms of the Dirichlet and Neumann symplectic vectors (2.42) and (2.43).
D. Yang-Mills and Higgs fields

Last, we generalize the previous two examples by considering on \((M, g_{ab})\) a set of Yang-Mills fields \(A^\tau_a\) and Higgs fields \(\varphi^\tau\), \(\tau = 1, \ldots, n\), with a gauge group given by any \(n\)-dimensional semi-simple Lie group \(\mathcal{G}\), \(n \geq 3\). Let \(C^\tau_{\Delta A}\) be the commutator structure constants of the Lie algebra \(A\) of \(\mathcal{G}\) (in a fixed basis). The structure constants are skew \(C^\tau_{\Delta A} = -C^\tau_{A\Delta}\) and satisfy the Jacobi relation \(C^\tau_{[\Delta A} C^\tau_{B \Delta]} \varphi = 0\). Let \(\Gamma^\tau_a(A) = C^\tau_{A\Delta} A^\Delta_a\) be the Yang-Mills connection, and define \(k^\tau_{\Delta A} = -\frac{1}{2} C^\tau_{\Delta A} H C^\tau_{H H} A^\Delta_a\), which denotes the positive definite Cartan-Killing metric on \(A\).

The Yang-Mills Lagrangian for \(A^\tau_a\) is given by the 4-form

\[
L_{abcd}(A; \theta) = \frac{1}{2} \epsilon_{abcd}(g) k^A_{\Delta \tau} F^A_{mn} F^\tau_{mn} = 3 k^A_{\Delta \tau} F^A_{[ab} F^\tau_{cd]} (2.99)
\]

where

\[
F^A_{ab} = g^{[a} A^A_{b]} + \frac{1}{2} C^A_{\Delta \tau} A^\Delta_a A_b^\tau (2.100)
\]

is the Yang-Mills field strength, \(F^A_{ac} = \epsilon_{abcd}(g) F^A_{bcd}\) is the dual field strength in terms of \(F_{[ac} = g^{ca} g^{db} F^{A}_{bd}\), with \(g_{ab} = \theta^\mu_{a} \theta^\nu_{b} \eta_{\mu \nu}\). For \(\varphi^\tau\), it is convenient to introduce the gauge-covariant Higgs field strength

\[
W^A_{a} = g^{[a} \varphi^A + e \Gamma^A_{\mu \tau}(A) \varphi^\tau. (2.101)
\]

In terms of this field strength the Higgs Lagrangian is given by the 4-form

\[
L_{abcd}(\varphi; \theta) = \epsilon_{abcd}(g) \left(\frac{1}{2} k^A_{\Delta \tau} W^A_{[m} W_{n]}^\tau + V(|\varphi|)\right) (2.102)
\]

where \(V(|\varphi|)\) is a Higgs potential with \(|\varphi|^2 = k^A_{\Delta \tau} \varphi^A \varphi^\tau\), and \(e = \text{const}\) is a coupling constant. These Lagrangians are gauge invariant under Yang-Mills gauge symmetries on the fields \(A^\tau_a\) and \(\varphi^\tau\). (In particular, if \(U^\tau_A\) denotes a homomorphism of \(A\) given by a function of the spacetime coordinates \(x^\eta\), it is straightforward to show that the Yang-Mills gauge symmetry is then given by \(A^\tau_a \rightarrow U^{-1 \tau}_{A} A^\Delta_a U^\Delta_{\tau A} - C^\tau_{\Delta A} U^{-1 \tau}_{\eta A} \theta_{\eta} U^\Delta_{\tau A}\) and \(\varphi^\tau \rightarrow U^{-1 \tau}_{A} \varphi^A\), where \(C^\tau_{\Delta A} = k^\tau_{\eta A} C^\tau_{\eta A}\) denotes the structure constants of the dual Lie algebra \(A^\tau\) and \(k^\tau_{\eta A}\) is the inverse of the Cartan-Killing metric. Under these transformations, the field strengths are gauge covariant, \(F^\tau_{ab} \rightarrow U^{-1 \tau}_{A} F^A_{ab}\) and \(W^\tau_{a} \rightarrow U^{-1 \tau}_{A} W^A_{a}\).

To proceed, we consider the combined Yang-Mills-Higgs Lagrangian,

\[
L_{abcd}(A, \varphi; \theta) = L_{abcd}(A; \theta) + L_{abcd}(\varphi; \theta). (2.103)
\]

First, the coefficient of the variation \(\delta A^\tau_a\) yields the Yang-Mills field equations

\[
k^\tau_{\Delta A} \star E^\tau_{a \tau}(A; g) = g^{ab} F^A_{ba} + \Gamma^A_{\eta \tau}(A) F^\tau_{ba} - e C^\tau_{\eta H} \varphi^H W^\tau_{a} = 0 (2.104)
\]
where $eC_{\mu
u}^A \varphi^\mu W_a^\nu$ has the role of a current source. The coefficient of the variation $\delta \varphi^T$ similarly yields the Higgs field equations

$$k^T A \ast \mathcal{E}_T (A; g) = g \nabla a W_a^A + \Gamma^a_{\mu\nu} (A) W_a^\nu - \frac{1}{|\varphi|} V'(|\varphi|) \varphi^A = 0$$

(2.105)

Next, by variation of $\varphi^\mu$, one obtains the Yang-Mills-Higgs stress-energy tensor $T_\mu^\nu (A, \varphi; \theta)^\mu = T_d^\nu (A, \varphi; g)$ where

$$T_a^b (A, \varphi; g) = k_{AT} (2 F_a^c F^T_{bc} + W_a^c W^T_b) - \frac{1}{2} \delta^b_a k_{AT} (F^A_{mn} F^T_{mn} + W_m^A W^m_T) - \delta^b_a V(|\varphi|).$$

(2.106)

The symplectic potential 3-form arising from the Lagrangian (2.103) is given by

$$\Theta_{bcd} (A, \varphi, \delta A, \varphi; \theta) = 4 \epsilon_{abcd} (g) k_{AT} (\delta A^e_a F^{Tae} + \delta \varphi^A W^T_a).$$

(2.107)

This yields the Noether current

$$J_{abc} (\xi, A, \varphi; g) = \Theta_{abc} (A, \varphi, L_{\xi} A, L_{\xi} \varphi; \theta) + 4 \epsilon_{abc} (g) k_{AT} \left( \delta A^e_a F^{Tae} + \delta \varphi^A W^T_a \right).$$

(2.108)

for Yang-Mills-Higgs solutions. Hence, one obtains the Noether charge

$$Q_{\Sigma} (\xi; A, \varphi) = \int_{\Sigma} J_{abc} (\xi, A, \varphi; g) = 4 \int_{\Sigma} \xi^e \epsilon_{abc} (g) T_e^d (A, \varphi; g) + 2 \int_{\partial \Sigma} k_{AT} \epsilon_{bcda} F^A_{bd} \xi^e A^T_e.$$

(2.109)

For the Yang-Mills-Higgs equations coupled to the Einstein equations

$$R_{\mu}^\alpha (\theta) - \frac{1}{2} \theta^\mu R (\theta) = T_{\mu}^\alpha (A, \varphi; \theta)$$

(2.110)

using the field variables $A^T_a$, $\varphi^T$, $\theta^\mu$, the total Lagrangian is given by $L_{abcd} (\theta, A, \varphi) = L_{abcd} (\theta) - L_{abcd}(A, \varphi; \theta)$ from Eqs. (2.27) and (2.103).

Through same analysis as used in the Maxwell and Klein-Gordon examples, we obtain the following results.

**Proposition 2.5.** Suppose $\xi^a$ is tangential to $\partial \Sigma$. Then no boundary conditions are necessary for existence of a Hamiltonian conjugate to $\xi^a$ on $\Sigma$. Consequently, a Hamiltonian is given by

$$H (\xi; \theta, A, \varphi) = 8 \int_{\Sigma} \epsilon_{abc} (g) \xi^e \theta^\mu (R_{\mu}^d (\theta) - \frac{1}{2} \theta^d R (\theta) - T_{\mu}^d (A, \varphi; \theta))$$

(2.111)

up to an inessential boundary term.
**Theorem 2.6.** Suppose $\xi^a$ is nowhere tangential to $\partial \Sigma$. Let

\begin{align}
(D) \quad \delta(A^T_a)|\partial \Sigma &= 0, \quad \delta(\varphi^T)|\partial \Sigma = 0, \quad \delta(h^\mu_a)|\partial \Sigma = 0 \quad (2.112) \\
(N) \quad \delta(|h|s_bh^a F^{Tcb})|\partial \Sigma &= 0, \quad \delta(|h|s^aW_a^T)|\partial \Sigma = 0, \quad \delta(K^\mu_a)|\partial \Sigma = 0 \quad (2.113)
\end{align}

where $|h| = \det(h^\mu_a)$ is the determinant of the components of the frame $h^\mu_a$ associated to $\mathcal{B}$. Under Dirichlet (D) or Neumann (N) boundary conditions for both $\varphi$ and $\theta^\mu$, there exists a Hamiltonian $H(\xi; \theta, A, \varphi) + H_B(\xi, \theta, A, \varphi)$ conjugate to $\xi^a$ on $\Sigma$, with the boundary term given by

\begin{align}
H^D(\xi; \theta, A, \varphi) &= 8 \int_{\partial \Sigma} \xi^a(P^D_a(\theta) - P^D_a(A)) dS, \quad (2.114) \\
H^N(\xi; \theta, A, \varphi) &= 8 \int_{\partial \Sigma} \xi^a(P^N_a(\theta) - P^N_a(A, \varphi)) dS, \quad (2.115)
\end{align}

where

\begin{align}
P^D_a(A) &= \frac{1}{2} k_{AT} t_a s^e F^{Tde} A^A_e, \\
P^N_a(A, \varphi) &= k_{AT} (\sigma_{[a} c_{d]} s^e F^{Tde} A^A_e + \frac{1}{2} t_a s^d W^T_d \varphi^A), \quad (2.116) \quad (2.117)
\end{align}

and $P^D_a(\theta)$, $P^N_a(\theta)$ are the symplectic vectors given by (2.42) and (2.43).

\section*{E. Remarks}

Clearly, the previous results when $\xi^a$ is not tangential to $\partial \Sigma$ are easily generalized to mixed Dirichlet-Neumann boundary conditions on the tetrad and matter fields similar to Theorem 2.4 and Theorems 3.5 and 3.6 in Ref. [1]. In particular, for allowed boundary conditions, note that one can have the tetrad satisfying (D) while the matter fields satisfy (N), and vice versa.

\section*{III. PROPERTIES OF THE SYMPLECTIC VECTORS}

We first review some geometry of spatial 2-surfaces in spacetime (most of this material is standard, e.g., [2,5,6]). Then we describe the properties of the Dirichlet and Neumann symplectic vectors regarded as locally constructed geometrical vector fields associated to a fixed spatial 2-surface in spacetime, independently of any Hamiltonian structure.

\subsection*{A. 2-surface geometry}

Let $(S, \sigma_{ab})$ be a closed, orientable smooth spacelike 2-surface in a spacetime $(M, g_{ab})$, where $\sigma_{ab}$ is the pullback of $g_{ab}$ to $S$. Let $T(S)$ and $T(S)\perp$ denote, respectively, the tangent space of $S$ and the normal space to $S$ (defined by the orthogonal complement of $T(S)$ in
\(T(M)\). Since \(T(S) \oplus T(S)^\perp = T(M)\), every vector in \(T(M)\) has a unique decomposition into vectors tangent and normal to \(T(S)\), given by projection operators \(P_S : T(M) \rightarrow T(S)\), \(P_S^\perp : T(M) \rightarrow T(S)^\perp\).

Fix an oriented orthonormal frame \(\{t^a, s^a\}\) for \(T(S)^\perp\),

\[
t^a s_a = 0, -t^a t_a = s^a s_a = 1, \tag{3.1}
\]

with \(t^a\) being a future timelike unit vector and \(s^a\) being an outward spacelike unit vector. (If \(M\) is spatially non-compact, we define the “outward” direction by the exterior of the set \(M - S\). If \(M\) is spatially compact, there is no preferred way in general to distinguish the sets \(S\) and \(M - S\), so we then make an arbitrary consistent choice for an “outward” direction.)

The metric on \(S\) is given by

\[
\sigma_{ab} = g_{ab} + t_a t_b - s_a s_b. \tag{3.2}
\]

The compatible volume form on \(S\) is given by

\[
\epsilon_{ab} = \epsilon_{abcd}(g)s^b t^c, \tag{3.3}
\]

satisfying \(\sigma_{c[a}\sigma_{b]d} = \epsilon_{ab} \epsilon_{cd}\). The projection operators for \(T(S)\) and \(T(S)^\perp\) are given by

\[
(P_S)^b_a = \sigma^b_a = \epsilon_{ac} \epsilon^c_b, (P_S^\perp)^b_a = s_a s_b - t_a t_b = \sigma^\perp_{ab} = \epsilon^\perp_{ac} \epsilon^\perp_{bc} \tag{3.4}
\]

where \(\epsilon_{ab}^\perp = 2s_{[a} t_{b]}\) and

\[
\sigma^\perp_{ab} = s_a s_b - t_a t_b. \tag{3.5}
\]

Both \(\sigma_{ab}\) and \(\epsilon_{ab}\) are independent of choice of the orthonormal frame. Since \(\sigma_{ab}^\perp\) is a two-dimensional Lorentz metric, any two oriented orthonormal frames \(\{t^a, s^a\}\) and \(\{t'^a, s'^a\}\) differ by a local boost

\[
t'^a = (\cosh \chi)t^a + (\sinh \chi)s^a, s'^a = (\cosh \chi)s^a + (\sinh \chi)t^a, \tag{3.6}
\]

where \(\chi\) is a function on \(S\). Under an arbitrary boost (3.6), \(\sigma_{ab}\) and \(\epsilon_{ab}\) are invariant.

The intrinsic geometry of the 2-surface \(S\) is completely determined by the metric \(\sigma_{ab}\). In particular, the intrinsic curvature of \(S\) is given by

\[
[D_a , D_b]v_c = \mathcal{R}_{abc}^d v_d \tag{3.7}
\]

where \(v_c\) is any dual tangent vector field on \(S\), and \(D_a\) denotes the metric compatible (torsion-free) derivative operator on \(S\) defined by \(D_a \sigma_{bc} = 0\). Since \(S\) is two-dimensional, it follows that the intrinsic curvature tensor has only one linearly independent component

\[
\mathcal{R}_{abc}^d = \frac{1}{2} \sigma_{c[b} \sigma_{a]}^d \mathcal{R} = \frac{1}{2} \epsilon_{ab} \epsilon_c^d \mathcal{R} \tag{3.8}
\]
where $\mathcal{R}$ denotes the scalar curvature of $S$.

The 2-surface $S$ also has an extrinsic geometry with respect to $(M, g_{ab})$, which is characterized by the following curvatures [2,3]. Let $\nabla^S_a = \sigma_a^b \nabla_b$ where $\nabla_b$ is the metric compatible (torsion-free) derivative operator on $(M, g_{ab})$. Then $\nabla^S_a$ can be decomposed into the tangential derivative operator $\mathcal{D}_a$ and a normal derivative operator $\mathcal{D}_a^\perp$, defined by $\mathcal{D}_a^\perp v^b = \sigma_c^b \nabla^S_a v^c$ for any vector field $v^a$ in $T(M)$ at $S$. Now consider $\nabla^S_a t_b$ and $\nabla^S_a s_b$. The tangential parts yield the extrinsic curvature tensors of $S$ with respect to the orthonormal frame $\kappa_{ab}(t) = \mathcal{D}_a t_b$, $\kappa_{ab}(s) = \mathcal{D}_a s_b$, \hspace{1cm} (3.9)

which are symmetric tensors on $S$. These measure the spatial rotation of the orthonormal frame in $T(S)^\perp$ under displacement on $S$. The normal parts of $\nabla^S_a t_b$ and $\nabla^S_a s_b$ give

$$\mathcal{D}_a^\perp t_b = s_b \mathcal{J}_a^\perp, \quad \mathcal{D}_a^\perp s_b = -t_b \mathcal{J}_a^\perp,$$

where

$$\mathcal{J}_a^\perp = s^c \nabla^S_a t^c, \hspace{1cm} (3.10)$$

which measures the boost of the orthonormal frame in $T(S)^\perp$ under displacement on $S$. The commutator of $\mathcal{D}_a^\perp$ defines the normal curvature of $S$

$$[\mathcal{D}_a^\perp, \mathcal{D}_b^\perp] v^c = \mathcal{R}_a^\perp_{\quad d} v^d,$$

with

$$\mathcal{R}_a^\perp_{\quad cd} = 2 \mathcal{D}_a^\perp \mathcal{J}_b^\perp \mathcal{J}_c^\perp, \hspace{1cm} (3.12)$$

where $v_c$ is any dual normal vector field on $S$. Hence, $\mathcal{J}_a^\perp$ is geometrically a connection 1-form on $S$ associated to the normal curvature of $S$. Since $S$ is two-dimensional, note $\mathcal{R}_a^\perp_{\quad cd}$ has only one linearly independent component, which is proportional to $\epsilon^{ab} \nabla^S_a \mathcal{J}_b^\perp$.

The trace of the extrinsic curvatures (3.9) of $S$

$$\kappa(t) = \sigma^{ab} \kappa_{ab}(t), \quad \kappa(s) = \sigma^{ab} \kappa_{ab}(s),$$

measure how the 2-surface area changes under infinitesimal dragging of $S$ along each direction of the orthonormal frame. In particular, for any vector field $v^a$ in $T(S)^\perp$,

$$\mathcal{P}_S(\mathcal{L}_v \epsilon_{ab}) = \kappa(v) \epsilon_{ab}, \hspace{1cm} (3.15)$$

with

$$\kappa(v) = \mathcal{D}_a v^a = \frac{1}{2} \sigma^{ab} \mathcal{L}_v \sigma_{ab},$$

(3.16)
Thus, $S$ is “expanding” or “contracting” in the direction $v^a$ according to whether its trace extrinsic curvature $\kappa(v)$ is positive or negative. (More precisely, $\kappa(v)$ equals the rate of change of the area of the image of $S$ under any diffeomorphism of $M$ whose generator agrees with $v^a$ at $S$.) We say that the expansion of $S$ defined by (3.15) for a direction $v^a$ is spacelike, timelike, or null, if $v^a v_a$ is, respectively, positive, negative, or zero. If $v^a$ is non-null, we refer to $\frac{1}{|v|} |\kappa(v)|$ as the absolute expansion of $S$ in the direction $v^a$ (with $|v| = \sqrt{|v^a v_a|}$).

A preferred direction in $T(S)^\perp$ is given by the mean curvature vector $H^a = \kappa(s)s^a - \kappa(t)t^a$. (3.17)

If $H^a$ is spacelike or timelike, then this is the direction of, respectively, minimum absolute spacelike or minimum absolute timelike expansion of $S$. Furthermore, the minimum value of the absolute expansion is given by the mean extrinsic curvature of $S$,

$$\frac{1}{|H|} |\kappa(H)| = \sqrt{\kappa(s)^2 - \kappa(t)^2}|.$$ Note, here, the norm of $H^a$ is $H^a H_a = \kappa(s)^2 - \kappa(t)^2 \equiv H^2$.

The mean curvature vector and normal curvature tensor of $S$ are each independent of choice of the orthonormal frame, namely, $H^a$ and $R^\perp_{abcd}$ are invariant under boosts (3.6) of $\{t^a, s^a\}$. In contrast, the extrinsic curvatures of $S$ are not invariant but instead transform like the orthonormal frame, while the normal connection transforms like a SO(1,1) connection

$$J^\perp_a = J^\perp_a + \nabla^S_a \chi$$ (3.18)

with respect to the SO(1,1) group generated by the boosts (3.6).

**B. Dirichlet Symplectic vector**

It is convenient to work with a null frame for $T(S)^\perp$. Let

$$\sqrt{2}\theta^+_a = t_a + s_a, \sqrt{2}\theta^-_a = t_a - s_a,$$ (3.19)

which respectively define outgoing and ingoing future pointing null dual vectors satisfying

$$\theta^+_a \theta^-_a = -1.$$ (3.20)

Note that any two such oriented null frames $\{\theta^+_a, \theta^-_a\}$ and $\{\theta^+_a, \theta^-_a\}$ are related by a local boost

$$\theta^+_a = e^\chi \theta^+_a, \theta^-_a = e^{-\chi} \theta^-_a,$$ (3.21)

where $\chi$ is a boost parameter given by a function on $S$.

The extrinsic curvatures of $S$ in the $\theta^+_a$ directions are given by

$$\kappa^\pm_{ab} = D^a \theta^\pm_b.$$ (3.22)

Then
\[ \kappa^\pm = D_a \theta^\pm_a = \frac{1}{2} \sigma^{ab} \mathcal{L}_{\theta^\pm} \sigma_{ab} \]  

are the trace extrinsic curvatures which measure the expansion of \( S \) in the \( \theta^\pm_a \) directions. Specifically, \( \kappa^\pm \) is the rate of change of 2-surface area of \( S \)

\[ \mathcal{P}_S(\mathcal{L}_{\theta^\pm} \epsilon_{ab}) = \kappa^\pm \epsilon_{ab} \]  

under any diffeomorphism of \( M \) whose generator is given by \( \theta^\pm_a \) at \( S \). Equivalently, \( \kappa^\pm \) measures the focusing of a congruence of null geodesics normal to \( S \).

The mean curvature vector of \( S \) is given by

\[ H^a = - (\kappa^- \theta^+ + \kappa^+ \theta^-) \]  

and the connection for the normal curvature of \( S \) is given by

\[ J^\perp_a = \theta^b \nabla^S_a \theta_b^- \]  

We now consider the Dirichlet symplectic vector (2.44) associated to \( S \) in the frame \( \{ t^a, s^a \} \), and separate it into vectors that are normal and tangential to \( S \)

\[ \mathcal{P}_P^\perp (P^a) = \kappa^\perp \theta^b \nabla^S_a \theta_b^- \]  

We call \( \mathcal{P}_P^\perp \) the Dirichlet normal vector associated to \( S \). It has the important property that it is independent of choice of the null frame at \( S \) [4–6].

**Proposition 3.1.** \( \mathcal{P}_P^\perp \) is invariant under arbitrary boosts (3.21) of the oriented null frame.

Consequently, \( \mathcal{P}_P^\perp \) depends only on the 2-surface \( S \) and the spacetime metric \( g_{ab} \).

(In particular, its components in any coordinate system can be locally constructed out of the components of \( g \) and their partial derivatives, but not in a coordinate invariant form.) Moreover, \( \mathcal{P}_P^\perp \) has three significant geometrical properties.

First of all, we consider the extrinsic curvature of \( S \) in the direction \( \mathcal{P}_P^\perp \)

\[ \kappa^\perp_{ab} = D^\perp_a (P^\perp)_b \]  

Remarkably, the trace of this extrinsic curvature vanishes

\[ \kappa^\perp = D^\perp_a (P^\perp)_a = \kappa^+ D^\perp_a \theta^- + \kappa^- D^\perp_a \theta^+ = 0 \]  

by Eq. (3.23) and \( \theta^\pm_a \nabla^S_a = 0 \). Then we have

\[ \mathcal{L}_{\mathcal{P}_P^\perp} \epsilon_{ab} = \kappa^+ \mathcal{L}_{\theta^-} \epsilon_{ab} - \kappa^- \mathcal{L}_{\theta^+} \epsilon_{ab} = \kappa^\perp \epsilon_{ab} = 0. \]  

This result yields the following key geometrical property of \( \mathcal{P}_P^\perp \).
Theorem 3.2. The normal direction \((P_\perp)^a\) to \(S\) in the spacetime \((M, g_{ab})\) is area preserving, i.e. \(S\) has zero expansion in the direction \((P_\perp)^a\).

Moreover, this property essentially characterizes the directional part of \((P_\perp)^a\) since there is a unique area-preserving normal direction at all points of \(S\), except, if any, where \(\kappa^+ = \kappa^- = 0\) (in which case all normal directions to \(S\) are area-preserving).

Second, we find that the norm of \((P_\perp)^a\) is given by
\[
(P_\perp)^2 = 2\kappa^+\kappa^-.
\] (3.32)

Hence, the direction of \((P_\perp)^a\) is timelike, spacelike, or null if the expansions \(\kappa^\pm\) of \(S\) are, respectively, opposite sign, same sign, or at least one is zero. (These are boost invariant properties.) In general, the signs of \(\kappa^\pm\) can vary on \(S\) even if the spacetime curvature satisfies positive energy conditions. Therefore, the sign of \((P_\perp)^2\) need not be the same everywhere on \(S\). We note that the situation \((P_\perp)^2 > 0\) characterizes \(S\) as a trapped surface, related to the formation of black-holes [7]. Further remarks on this aspect of \((P_\perp)^a\) will be made in Sec. V.

Third, we see that \((P_\perp)^a\) is closely related to the mean curvature vector of \(S\).

Proposition 3.3.
\[
(P_\perp)^a H_a = 0, \quad (P_\perp)^2 = -H^2.
\] (3.33)

Thus, \((P_\perp)^a\) is respectively timelike, spacelike, or null as \(H^a\) is spacelike, timelike, or null. Let \(|H| = \sqrt{|H^2|}, |P_\perp| = \sqrt{|(P_\perp)^2|}\) denote the absolute norms of \(H^a\) and \((P_\perp)^a\). Then, in the non-null case, the relations (3.33) give a unique characterization of \((P_\perp)^a\) (up to a sign) as a vector in \(T(S)^\perp\) orthogonal to \(H^a\) and with the same absolute norm as \(H^a\). Consequently, we will write \((P_\perp)^a = H^a_\perp\) and refer to
\[
H^a_\perp = \kappa^+\theta^a - \kappa^-\theta^a
\] (3.34)
as the normal mean curvature vector of \(S\), with \(H^a_\perp H_a = 0, H^2_\perp = -H^2\).

Lemma 3.4. Suppose \(|H| \neq 0\) or equivalently \(|P_\perp| \neq 0\), i.e. \(H^a\) and \(H^a_\perp = (P_\perp)^a\) are non-null. Then \(\left\{\frac{1}{|H|}H^a, \frac{1}{|H|}H^a_\perp\right\}\) is an orthonormal frame for \(T(S)^\perp\). Correspondingly, the pair of vectors
\[
\hat{\theta}^+ = \frac{1}{\sqrt{2|H|}}(H^a + H^a_\perp) = \frac{-\kappa^-}{\sqrt{\kappa^+\kappa^-}}\theta^+,
\] (3.35)
\[
\hat{\theta}^- = \frac{1}{\sqrt{2|H|}}(H^a - H^a_\perp) = \frac{-\kappa^+}{\sqrt{\kappa^+\kappa^-}}\theta^-,
\] (3.36)
is a null frame for \(T(S)^\perp\).
Thus, in the non-null case, \( H^a \) and \( H^a_\perp = (P_\perp)^a \) determine a preferred orthonormal frame (or null frame) of \( T(S)^\perp \) which depends just on the 2-surface \( S \) and spacetime metric \( g_{ab} \). Then we can summarize the geometrical properties of these vectors in terms of the following orthonormal vectors in \( T(S)^\perp \),

\[
\hat{H}^a = \frac{1}{\sqrt{2|\kappa^+ \kappa^-|}}(\kappa^- \theta^a + \kappa^+ \theta^a - \kappa^- \theta^a) \quad \text{(3.37)}
\]

\[
\hat{H}_\perp^a = \frac{1}{\sqrt{2|\kappa^+ \kappa^-|}}(\kappa^- \theta^a - \kappa^+ \theta^a) \quad \text{(3.38)}
\]

**Theorem 3.5.** Suppose \( \kappa^+ \kappa^- \neq 0 \) on \( S \), i.e. \( H^a \) and \( H^a_\perp \) are non-null. Then:

1. The expansion of \( S \) is zero in the unique normal direction \( \hat{H}_\perp^a \), which is spacelike or timelike as \( \kappa^+ \kappa^- \) is positive or negative on \( S \).

2. The absolute expansion of \( S \) in the orthogonal normal direction \( \hat{H}^a \) is \( \sqrt{2|\kappa^+ \kappa^-|} \). This is the minimum absolute spacelike expansion or minimum absolute timelike expansion where \( \kappa^+ \kappa^- \) is, respectively, positive or negative on \( S \).

We now turn to consider the geometrical properties of \( (P_\parallel)^a \). To begin, \( (P_\parallel)^a \) can be identified [5,6] with the connection \( J^\perp_a \) for the normal curvature of \( S \), in the null frame \( \{\theta^+_a, \theta^-_a\} \).

**Proposition 3.6.**

\[
(P_\parallel)^a = -\sigma^{ab} J^\perp_b \quad \text{(3.39)}
\]

where

\[
J^\perp_a = \theta^+_a \nabla^S \theta^-_a \quad \text{(3.40)}
\]

Hence, in contrast to the invariance of \( (P_\perp)^a \) under boosts (3.21) of the null frame, \( (P_\parallel)^a \) is not invariant but instead transforms as a SO(1,1) connection

\[
(P_\parallel)'_a = (P_\parallel)_a - \nabla^S_a \chi \quad \text{(3.41)}
\]

where \( \chi \) is a boost parameter given by a function on \( S \). This describes a gauge transformation of \( (P_\parallel)^a \) associated to the boosts (3.21) acting on \( T(S)^\perp \) as an SO(1,1) gauge group. Consequently, the curl of \( (P_\parallel)^a \) has the role of the gauge invariant curvature.

**Proposition 3.7.**

\[
-D_{[a}(P_\parallel)_{b]} = \frac{1}{4} \mathcal{R}_{abcd}^\perp \epsilon^{bcd} \quad \text{(3.42)}
\]

is invariant under arbitrary boosts (3.21) of the null frame, where \( \mathcal{R}_{abcd}^\perp \) is the normal curvature of \( S \).
Thus, the curvature \( D_{(P_\parallel)_b} \) depends only on the 2-surface \( S \) and the spacetime metric \( g_{ab} \).

Interestingly, in the case when \( H^a \) is non-null, we can use the preferred orthonormal frame or null frame given by Lemma 3.4 to gauge-fix \((P_\parallel)^a\). We introduce

\[
(\hat{P}_\parallel)_a = \frac{1}{H^2} H^b \mathcal{D}_a H_b = \frac{1}{H^2} H^b \mathcal{D}_a H_{\perp b} 
\]

(3.43)

related to \((P_\parallel)_a\) by a gauge transformation (3.41) with boost parameter \( \chi = \frac{1}{2} \ln(\kappa^+ / \kappa^-) \) on \( S \).

**Proposition 3.8.** \((\hat{P}_\parallel)_a\) is invariant under arbitrary boosts (3.21) of the oriented null frame.

Consequently, we call \((\hat{P}_\parallel)_a\) the invariant Dirichlet tangent vector associated to \( S \). In particular, \((\hat{P}_\parallel)_a\) depends only on the 2-surface \( S \) and the spacetime metric \( g_{ab} \). We now state the main geometrical property of \((\hat{P}_\parallel)_a\), which follows from Eq. (3.43).

**Theorem 3.9.** Suppose \( \kappa^+ \kappa^- \neq 0 \) on \( S \), i.e. \( H^a \) and \( H^a_{\perp} \) are non-null. Then the boost (with respect to \( T(S)^\perp \)) of the area-preserving unit normal vector \( \hat{H}^a_{\perp} \) to \( S \) under displacement on \( S \) is a maximum in the direction \((\hat{P}_\parallel)_a\). By orthogonality of \( H^a \) and \( H^a_{\perp} \), this is equivalent to the tangent direction on \( S \) in which the boost of the unit mean curvature vector \( \hat{H}^a \) under displacement on \( S \) is a maximum.

Finally, from Propositions 3.1 and 3.8, when the mean curvature vector is non-null we can define an invariant locally constructed Dirichlet 4-vector associated to \( S \) by

\[
\hat{P}^a = (P_\perp)^a + (\hat{P}_\parallel)^a = \kappa^+ \theta^{-a} - \kappa^- \theta^{+a} + \sigma^{ac} \theta^{-b} \nabla^c \theta^{+} + \frac{1}{2} \nabla^a \ln(\kappa^+ / \kappa^-).
\]

(3.44)

Note that this vector depends only on \( S \) and \( g_{ab} \) and is independent of the choice of null frame \( \{\theta^+_a, \theta^-_a\} \). Indeed, in terms of purely geometrical structure associated to \( S \),

\[
\hat{P}^a = H^a_{\perp} + \frac{1}{H^2} H^b_{\perp} \mathcal{D}_a H_b
\]

(3.45)

where \( H^a \) is the mean curvature vector (3.25) and \( H^a_{\perp} \) is the normal mean curvature vector (3.34) of \( S \).

**C. Neumann symplectic vector**

Finally, we consider the Neumann symplectic vector (2.45) associated to \( S \),

\[
P^a = g^{ac} \theta^{-b} \nabla^c \theta^{+}_b.
\]

(3.46)
Notice, first of all, the tangential part of $P^a$ with respect to $S$

$$(P_{\parallel})^a = \mathcal{P}_S(P^a) = \sigma^{ac} \theta^{-b} \nabla_c \theta^+_b$$

(3.47)

is identically equal to the tangential part of the Dirichlet symplectic vector (3.28). Hence, similarly to Theorem 3.9, $(P_{\parallel})^a$ gives the direction in which the boost of the null frame $\theta^+_a$ under displacement on $S$ is a maximum. In contrast, the normal part of $P^a$ with respect to $S$,

$$(P_{\perp})^a = \mathcal{P}_S^\perp(P^a) = \sigma^{+ac} \theta^{-b} \nabla_c \theta^-_b = -\sigma^{+ac} \theta^{+b} \nabla_c \theta^-_b$$

(3.48)

involves derivatives of the null frame $\theta^\pm_a$ in normal directions to $S$. In particular, note through substitution of

$$\sigma^{+b}_a = - (\theta^+_a \theta^{-b} + \theta^-_a \theta^{+b})$$

(3.49)

that

$$(P_{\perp})^a = -\theta^{+a} \theta^{-b} \theta^{-c} \nabla_c \theta^+_b + \theta^{-a} \theta^{+b} \theta^{+c} \nabla_c \theta^-_b = \sigma^{+a}_c [\theta^-, \theta^+]^c$$

(3.50)

since $\theta^{+b} \nabla_c \theta^+_b = \theta^{-b} \nabla_c \theta^-_b = 0$.

**Proposition 3.10.** $(P_{\perp})^a$ is the normal part of the commutator $[\theta^-, \theta^+]^a$ of the null frame, $(P_{\perp})^a = \mathcal{P}_S^\perp[\theta^-, \theta^+]^a$.

Consequently, unlike $(P_{\parallel})^a$ which is well-defined just given the 2-surface $S$ and a null frame $\theta^\pm_a$ of $T(S)^\perp$, it is necessary to consider “nearby” 2-surfaces $S'$, diffeomorphic to $S$, to extend the null frame $\theta^\pm_a$ of $T(S)^\perp$ off $S$ so that $(P_{\perp})^a$ is well-defined.

Let $S_{(\lambda_+, \lambda_-)}$ denote a two-parameter $(\lambda_+, \lambda_-)$ local null congruence of 2-surfaces $S'$ diffeomorphic to $S$ in $(M, g_{ab})$ with $S_{(0,0)} = S$. (The congruence is defined to be ingoing as a function of $\lambda_-$ and outgoing as a function of $\lambda_+$. ) Extend the null frame $\{\theta^+^a, \theta^-_a\}$ of $T(S)^\perp$ off $S$ to $S'$ using boosts (3.21). Then $(P_{\perp})^a = \sigma^{\perp ac} \theta^{-b} \nabla_c \theta^-_b$ is a well-defined normal vector at each point on $S$. We call $(P_{\perp})^a$ the *Neumann normal vector* associated to $S$ in a null congruence $S_{(\lambda_+, \lambda_-)} \simeq S \times (\lambda_+, \lambda_-)$. It depends, of course, on the congruence but also on the choice of null frame for $T(S_{(\lambda_+, \lambda_-)})^\perp$.

**Proposition 3.11.** Under boosts (3.21) of the null frame on the 2-surfaces $S_{(\lambda_+, \lambda_-)} \simeq S \times (\lambda_+, \lambda_-)$, $(P_{\perp})^a$ transforms as

$$(P_{\perp})'_a = (P_{\perp})_a - \sigma^{+b}_a \nabla_b \chi$$

(3.51)

where $\chi$ is a boost parameter given by a function of $(\lambda_+, \lambda_-)$.
By Proposition 3.6, \((P_{\parallel})^a\) has a similar boost transformation property, which has the geometrical meaning of a SO(1, 1) connection for the normal curvature of \(S\). This suggests that, geometrically, \(P^a = (P_{\perp})^a + (P_{\parallel})^a\) is also related to a SO(1, 1) connection associated to an extrinsic curvature of \(S\).

Consider the derivative operator \(\nabla^\perp_a\) defined by \(\nabla^\perp_a v^b = \sigma^b_c \nabla_a v^c\) for any normal vector field \(v^a\) on the 2-surfaces \(S_{(\lambda^+, \lambda^-)}\). The commutator of \(\nabla^\perp_a\) gives the curvature

\[
[\nabla^\perp_a, \nabla^\perp_b] v_c = R^\perp_{abc} d v_d. 
\]

(Note that, on functions, \([\nabla^\perp_a, \nabla^\perp_b] f = 2\nabla_a [\nabla_b] f = 0\). Clearly, \(P_{\parallel}^b ([\nabla^\perp_a, \nabla^\perp_b]) v_c = [\mathcal{D}_a, \mathcal{D}_b] v_c = R^\perp_{abc} d v_d\) yields the normal curvature of \(S\). Hence, \(R^\perp_{abcd}\) is a generalization of \(R^\perp_{abcd}\), which we call the sectional curvature normal to \(S\) in the null congruence \(S_{(\lambda^+, \lambda^-)}\).

**Proposition 3.12.**

\[
R^\perp_{abcd} = 2\nabla^\perp_{[a} J^\perp_{b]} c_d 
\]

where \(J^\perp_a = \theta^+ b \nabla_a \theta^-\).

Here \(J^\perp_a\) is geometrically a connection 1-form for \(R^\perp_{abcd}\). In particular, boosts of the null frame act as an SO(1, 1) gauge group on \(T(S_{(\lambda^+, \lambda^-)})^\perp\) under which \(J^\perp_a\) transforms as \(J^\perp_{a'} = J^\perp_a + \nabla_a \chi\) where \(\chi\) is a function on \(S_{(\lambda^+, \lambda^-)} \simeq S \times (\lambda^+, \lambda^-)\). Note that the curvature \(R^\perp_{abcd}\) is invariant under these boosts. This leads to the main geometrical result concerning \(P^a\).

**Theorem 3.13.** In any null congruence of 2-surfaces \(S_{(\lambda^+, \lambda^-)}\), \(P_a = 2J^\perp_a\) is a connection 1-form for the sectional curvature normal to \(S\),

\[
-\nabla^\perp_{[a} P_{b]} = \frac{1}{4} R^\perp_{abcd} c_{cd}. 
\]

Thus the curl \(\nabla^\perp_{[a} P_{b]}\) is invariant under arbitrary boosts of the null frame on \(S_{(\lambda^+, \lambda^-)}\). It depends, still, on the choice of null congruence \(S_{(\lambda^+, \lambda^-)} \simeq S \times (\lambda^+, \lambda^-)\).

In general, there is no unique null congruence determined just by \(S\) and \(\theta_{ab}\). However, a natural choice is given by ingoing and outgoing null geodesics congruences \(S_{\lambda^\pm}\) through \(S\), with \(S_{(\lambda^+, \lambda^-)}\) defined as \((S_{\lambda^+})_{\lambda^-}\) or \((S_{\lambda^-})_{\lambda^+}\) corresponding to constructing successive one-parameter ingoing and outgoing congruences [8].

If \(S_{(\lambda^+, \lambda^-)} \simeq S \times (\lambda^+, \lambda^-)\) is chosen to be a null geodesic congruence, then the geodesic equation implies that \(\theta^\pm a\) satisfies \(\theta^{+b} \nabla_b \theta^{+a} = 0\) where \(\theta^{+a}\) is given by a boost (3.21) for some function \(\chi\) of \((\lambda^+, \lambda^-)\). Thus,

\[
\theta^{+b} \nabla_b \theta^{+a} = -\theta^{+a} \theta^{+b} \nabla_b \chi, \quad \theta^{+b} \nabla_b \theta^{-a} = \theta^{-a} \theta^{+b} \nabla_b \chi. 
\]

This leads to a simplification of \((P_{\perp})^a\) from Eq. (3.50),

\[
(P_{\perp})^a = \theta^{+a} \theta^{+b} \nabla_c \theta^c_{+b} - \theta^{-a} \theta^{-b} \theta^{+c} \nabla_c \theta^c_{-b} 
\]

\[
= -\theta^{+a} \theta^{-c} \nabla_c \chi - \theta^{-a} \theta^{+c} \nabla_c \chi. 
\]
Proposition 3.14. In a null geodesic congruence, \((P_\perp)^a = \sigma \nabla_b \chi\), and consequently, 
\[ \mathcal{P}_S (R_{abcd}) = 0. \]

The converse of this result also holds, since if the normal part of \(R_{abcd}\) vanishes, then 
\((P_\perp)_a\) is a gradient and hence \(\theta^\pm a\) satisfies the geodesic equation (3.55) so that the congruence 
\(S(\lambda_+, \lambda_-)\) arises from null geodesics through \(S\).

Geometrically, the boost function \(\chi\) in the geodesic equation (3.55) is related to the
choice of parameterization of the null congruence. Indeed, we can fix the parameterization in
a natural way by \(\chi = 0\), which implies a corresponding gauge-fixing of \((P_\perp)^a\).

\[ (\hat{P}_\perp)^a = 0. \quad (3.57) \]

To conclude, we remark that the use of ingoing and outgoing null congruences in defining
\((P_\perp)^a\) can be replaced by using timelike and spacelike congruences, denoted \(S_{\lambda_+}\) and \(S_{\lambda_-}\),
through \(S\). Moreover, if \(g_{ab}\) has isometries then it may be possible to fix a unique local
two-parameter congruence \(S(\lambda_+, \lambda_-)\) constructed in a natural way from the Killing vectors and
invariant surfaces of the isometries. In general, then \((P_\perp)^a\) is no longer just a gradient. This
will be illustrated in the examples in the next section.

Finally, it is important to note that there is no ambiguity in \((P_\perp)^a\) appearing in the
Neumann Hamiltonian boundary term (2.43), since this involves only the component of
\((P_\perp)^a\) in the direction of the time-flow vector, which is well-defined using the unique timelike
congruence through \(S\) generated by the time-flow vector on \(M\).

IV. EXAMPLES

We now consider examples for the Dirichlet and Neumann symplectic vectors described
in Sec. III. In particular, we calculate these vectors and their properties for spacelike,
topologically spherical 2-surfaces in (A) Minkowski Spacetime, (B) Spherically Symmetric Spacetimes, (C) Axisymmetric Spacetimes, (D) Homogeneous Isotropic Spacetimes, (E) Asymptotically Flat Spacetimes.

A. Minkowski Spacetime

Consider a closed orientable spacelike 2-surface \(S\) embedded in a spacelike hyperplane
in Minkowski spacetime \((\mathbb{R}^4, \eta)\), using spherical coordinates
\[ \eta_{ab} = -(dt)_a (dt)_b + (dr)_a (dr)_b + r^2 ((d\theta)_a (d\theta)_b + \sin^2 \theta (d\phi)_a (d\phi)_b), \quad (4.1) \]
where \(S\) is given by \(t = t_0, r = R(\theta, \phi)\) for some function \(R(\theta, \phi)\) and constant \(t_0\). Fix an
orthonormal frame \(\vartheta^\mu_a\) adapted to \(S\) in \((\mathbb{R}^4, \eta)\) by
\[ \vartheta^0_a = (dt)_a, \quad \vartheta^1_a = \frac{1}{\psi} ((dr)_a - \partial_a R), \quad (4.2) \]
\[ \vartheta^2_a = \frac{1}{\mu} \left( r (d\theta)_a + \frac{\partial_a R}{r} (dr)_a \right), \quad \vartheta^3_a = \frac{\mu}{\psi} \left( r \sin \theta (d\phi)_a + \frac{\partial_a R}{\mu^2 r \sin \theta} ((dr)_a - \partial_a R (d\theta)_a) \right) \quad (4.3) \]
where
\[
\mu = \sqrt{1 + r^{-2}(\partial_{\theta}R)^2}, \quad \psi = \sqrt{1 + r^{-2}((\partial_{\phi}R)^2 + (\partial_{\phi}R/\sin\theta)^2)}.
\] (4.4)

Note the metric associated to \(S\) is given in spherical coordinates by
\[
\sigma_{ab} = (1 - \psi^{-2})(dr)_a(dr)_b + 2\psi^{-2}(dr)_a(\partial_b)R - \psi^{-2}\partial_a R \partial_b R \\
+ r^2((d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b).
\] (4.5)

The pullback of \(\sigma_{ab}\) to \(S\) yields the induced metric on \(S\)
\[
\sigma_{ab}|_S = (R^2 + (\partial_{\theta}R)^2)(d\theta)_a(d\theta)_b + (R^2 \sin^2\theta + (\partial_{\phi}R)^2)(d\phi)_a(d\phi)_b + 2\partial_y R\partial_y R(d\theta)(d\phi)_a(d\phi)_b.
\] (4.6)

Correspondingly, let \(\psi_S = \psi|_S = \sqrt{1 + R^{-2}((\partial_{\theta}R)^2 + (\partial_{\phi}R/\sin\theta)^2)}\).

The trace of the extrinsic curvatures of \((S, \sigma_{ab})\) with respect to the frame on \(T(S)^{\perp}\)
\[
t_a = \tilde{\theta}_a^0|_S = (dt)_a, \quad s_a = \tilde{\theta}_a^1|_S = \frac{1}{\psi_S}((dr)_a - \partial_a R)
\] (4.7)
are respectively
\[
\kappa(t) = \sigma^{ab}\nabla_a t_b = - (\Gamma_2 \tilde{\theta}^2 + \Gamma_3 \tilde{\theta}^3)|_S = 0
\] (4.8)
and
\[
\kappa(s) = \sigma^{ab}\nabla_a s_b = -(\Gamma_2 \tilde{\theta}^1 + \Gamma_3 \tilde{\theta}^3)|_S = 2(\tilde{\theta}_1^0\tilde{\theta}_2^b\partial_a\tilde{\theta}_b^2 + \tilde{\theta}_1^0\tilde{\theta}_3^b\partial_a\tilde{\theta}_b^3)|_{r=R(\theta,\phi)}.
\] (4.9)

Calculated in terms of the Ricci rotation coefficients
\[
\Gamma_\lambda^{\mu\nu}(\vartheta) = \tilde{\theta}_\lambda^\mu\tilde{\theta}^{\nu\vartheta}\nabla_a\tilde{\theta}_a^\vartheta = 2\tilde{\theta}_\lambda^\mu\tilde{\theta}^{b\nu}\partial_a\tilde{\theta}_b^\nu - \tilde{\theta}^{b\mu}\tilde{\theta}^{c\nu}\partial_b\tilde{\theta}_c^\mu.
\] (4.10)

Here \(\kappa(s)\) is the standard Euclidean extrinsic curvature of \(S\) in \(\mathbb{R}^3\) [2]. (The explicit expression for \(\kappa(s)\) as a function of the spherical coordinates is lengthy and will be omitted.)

A preferred direction in \(T(S)^{\perp}\) is given by the mean curvature vector
\[
H^a = s^a \kappa(s) - t^a \kappa(t) = \frac{\kappa(s)}{\psi_S} \left((\partial_r)^a - \frac{\partial_{\theta}R}{R^2}(\partial_{\theta})^a - \frac{\partial_{\phi}R}{R^2\sin^2\theta}(\partial_{\phi})^a\right),
\] (4.11)
which is spacelike. This vector gives the direction of the minimum absolute spacelike expansion of \(S\) in \((\mathbb{R}^4, \eta)\). Furthermore, the value of the expansion is the mean extrinsic curvature of \(S\) given by
\[
\frac{1}{\sqrt{H^2}}\kappa(H) = |\kappa(s)| = \sqrt{H^2}.
\] (4.12)

Note \(S\) is convex or concave according to where the sign of \(\kappa(s)\) is negative or positive.
The normal part of the Dirichlet symplectic vector is given by the normal mean curvature vector

\[(P_D^\perp)^a = H^a = t^a \kappa(s) - s^a \kappa(t) = \kappa(s)(\partial_t)^a.\]  

Note that \((P_D^\perp)^a\) is timelike, orthogonal to \(H^a\), with the same absolute norm as \(H^a\). Most significant, \((P_D^\perp)^a\) gives the direction of zero expansion of \(S\).

A preferred orthonormal frame for \(T(S)^\perp\) is

\[
\hat{t}^a = \frac{1}{\sqrt{H^2}} H^a = (\partial_t)^a|_S, \quad \hat{s}^a = \frac{1}{\sqrt{H^2}} H^a = \frac{1}{\psi_S} ((\partial_r)^a - \frac{\partial \theta}{r^2} (\partial_\theta)^a - \frac{\partial \phi}{r^2 \sin^2 \theta} (\partial_\phi)^a)|_S, 
\]

which depend only on \(S\) and \(\eta_{ab}\) but not on the Minkowski frame \(\vartheta^\mu_a\). In the preferred frame (4.14), the tangential part of the Dirichlet symplectic vector is

\[
(P_D^\parallel)^a = \sigma^{ac} [\hat{t}, \hat{s}]^a = -\frac{1}{\psi_S} \sigma^{ac} \left( (\partial_r)^b - \frac{\partial \theta}{R^2} (\partial_\theta)^b - \frac{\partial \phi}{R^2 \sin^2 \theta} (\partial_\phi)^b \right) \nabla_c (dt)^b = 0, \tag{4.15}
\]

and thus the normal curvature of \(S\) is zero.

Hence the complete Dirichlet symplectic vector is

\[
P^a = (P_D^\perp)^a + (P_D^\parallel)^a = \kappa(s)(\partial_t)^a, \tag{4.16}
\]

which depends only on \(S\) and \(\eta_{ab}\). In particular, it is independent of choice of the original orthonormal frame (4.2) and (4.3) on Minkowski space and of the normals (4.7) in \(T(S)^\perp\).

To define the Neumann symplectic vector, it is natural to extend the preferred orthonormal frame (4.14) off \(S\) by using the obvious isometries of \(\eta_{ab}\). With respect to this extension, the normal part of the Neumann symplectic vector is given by the commutator

\[
(P_N^\perp)^a = P_S^\perp [\hat{t}, \hat{s}]^a = \frac{1}{\psi_S} P_S^\perp \left( (\partial_t)^a - \frac{1}{R^2 \sin^2 \theta} [(\partial_t)^a, \sin^2 \theta \partial_\theta R (\partial_\theta) + \partial_\phi R (\partial_\phi)]^a \right) = 0. \tag{4.17}
\]

(Alternatively, the same result for \((P_N^\perp)^a\) is obtained by extending (4.14) off \(S\) to the congruence of 2-surfaces \(t = \text{const}, \ r = R(\theta, \phi) = \text{const}\), which lie in parallel hyperplanes to \(S\) and are isometric to \(S\).) Then, since \((P_N^\parallel)^a = (P_D^\parallel)^a = 0\), the complete Neumann symplectic vector in the congruence of 2-surfaces associated to \(S\) under isometries of \(\eta_{ab}\) is given by

\[
P^a = (P_N^\perp)^a + (P_N^\parallel)^a = 0. \tag{4.18}
\]

Thus the sectional curvature normal to \(S\) vanishes, reflecting the fact that \(S\) lies in a flat hyperplane.
Light cone 2-sphere

Next, consider a closed orientable spacelike 2-surface \( S \) embedded in a light cone in \((\mathbb{R}^4, \eta)\). Let \( u = (t - r)/\sqrt{2}, \ v = (t + r)/\sqrt{2}, \ \theta, \ \phi \) be light cone coordinates (i.e. \( r, \theta, \phi \) are spherical coordinates with respect to the origin point for the cone), with

\[
\eta_{ab} = -2(du)_a(dv)_b + \frac{1}{2}(v - u)^2((d\theta)_a(d\theta)_b + \sin^2 \theta(d\phi)_a(d\phi)_b).
\] (4.19)

Then \( S \) is given by \( u = u_0, \ v = V(\theta, \phi) \) for some function \( V(\theta, \phi) \) and constant \( u_0 \). Note that \((du)_a\) and \((dv)_a - \partial_a V\) are, respectively, a null normal and spacelike normal for \( S \), while \((\partial_\theta)^a + \partial_\theta V(\partial_\phi)^a\) and \((\partial_\phi)^a + \partial_\phi V(\partial_\theta)^a\) are orthogonal tangent vectors on \( S \).

Fix a null frame \( \vartheta^\mu_a \) adapted to \( S \) in \((\mathbb{R}^4, \eta)\) by

\[
\vartheta^0_a = (du)_a, \quad \vartheta^1_a = \frac{1}{2}\psi^2(du)_a + (dv)_a - \partial_a V,
\] (4.20)

\[
\vartheta^2_a = r(d\theta)_a - \frac{\partial_\theta V}{r}(du)_a, \quad \vartheta^3_a = r \sin \theta(d\phi)_a - \frac{\partial_\phi V}{r \sin \theta}(du)_a
\] (4.21)

satisfying \( \vartheta^\mu_a \vartheta^\nu_b \eta^{ab} = -2\delta_0^\mu \delta_1^\nu + \delta_2^\mu \delta_2^\nu + \delta_3^\mu \delta_3^\nu \), where

\[
\psi = |dV| = r^{-1}\sqrt{(\partial_\theta V)^2 + (\partial_\phi V/\sin \theta)^2}.
\] (4.22)

Note the metric associated to \( S \) is given by

\[
\sigma_{ab} = \psi^2(du)_a(du)_b - 2(du)_a \left( \partial_\theta V(d\theta)_b + \partial_\phi V(d\phi)_b \right) + r^2((d\theta)_a(d\theta)_b + \sin^2 \theta(d\phi)_a(d\phi)_b),
\] (4.23)

where

\[
\sigma_{ab}|S = \frac{1}{2}(V - u_0)^2((d\theta)_a(d\theta)_b + \sin^2 \theta(d\phi)_a(d\phi)_b)
\] (4.24)

yields the induced metric on \( S \).

The trace of the extrinsic curvatures of \((S, \sigma_{ab})\) with respect to the null frame on \( T(S)^\perp \)

\[
u_a = \vartheta^0_a|S = (du)_a,
\] (4.25)

\[
u_a = \vartheta^1_a|S = \psi_2(du)_a + (dv)_a - \partial_\theta V(d\theta)_a - \partial_\phi V(d\phi)_a
\] (4.26)

are respectively

\[
\kappa(u) = \sigma^{ab}\nabla_a u_b = -(\Gamma_2^{02}(\vartheta) + \Gamma_3^{03}(\vartheta))|S = -\frac{2}{R},
\] (4.27)

and
\[ \kappa(v) = \sigma^{ab} \nabla_a v_b = -(\Gamma^2_{12}(\vartheta) + \Gamma^3_{13}(\vartheta))|_{S} \]
\[ = \frac{2}{R} (1 + \psi^2_{S}) - \frac{2\partial_\theta R \cos \theta}{R^2 \sin \theta} - \frac{2\partial_\phi R}{R^2} - \frac{2\partial^2_\phi R}{R^2 \sin^2 \theta} \]  
(4.28)

where
\[ \psi_S = \frac{1}{\sqrt{2}} \psi|_S = R^{-1} \sqrt{(\partial_\theta R)^2 + (\partial_\phi R/ \sin \theta)^2}, \quad R = \sqrt{2r}|_S = V - u_0. \]  
(4.29)

A preferred direction in \( T(S)^\perp \) is given by the mean curvature vector
\[ H^a = -u^a \kappa(v) - v^a \kappa(u) \]  
(4.30)
in terms of the null vectors
\[ u^a = -(\partial_\vartheta)^a|_S, \quad v^a = -\left( (\partial_\vartheta)^a + \frac{\psi^2}{2}(\partial_\vartheta)^a + \frac{\partial_\theta V}{r^2}(\partial_\theta)^a + \frac{\partial_\phi V}{r^2 \sin^2 \theta}(\partial_\phi)^a \right)|_S. \]  
(4.31)

The norm of \( H^a \) gives the mean extrinsic curvature of \( S \)
\[ \frac{1}{|H|} |\kappa(H)| = \sqrt{2|\kappa(u)\kappa(v)|} = |H|. \]  
(4.32)

Now, the normal part of the Dirichlet symplectic vector is given by the normal mean curvature vector \( (P^D)_\perp^a = H^a_\perp = v^a \kappa(u) - u^a \kappa(v) \), which simplifies to
\[ (P^D)_\perp^a = -\kappa(u)(\partial_\vartheta)^a - \left( \kappa(u)\psi^2_S - \kappa(v) \right)(\partial_\vartheta)^a - \kappa(u) \frac{2\partial_\theta R}{R^2}(\partial_\theta)^a - \kappa(u) \frac{2\partial_\phi R}{R^2 \sin^2 \theta}(\partial_\phi)^a. \]  
(4.33)

This vector gives the direction of zero expansion of \( S \) in \( (\mathbb{R}^4, \eta) \).

A preferred null frame for \( T(S)^\perp \) consists of
\[ \hat{u}^a = \frac{1}{\sqrt{2|H|}} (H^a_\perp + H^a), \]  
(4.34)
\[ \hat{v}^a = \frac{1}{\sqrt{2|H|}} (H^a_\perp - H^a). \]  
(4.35)

which depend only on \( S \) and \( \eta_{ab} \) but not on the Minkowski frame \( \partial_\mu^a \). In the preferred frame (4.34) and (4.35), the tangential part of the Dirichlet symplectic vector is given by
\[ (P^D)_{\parallel}^a = \sigma^{ac} \hat{v}^b \nabla_c \hat{u}_b = \sigma^{ac} v^b \nabla_c (du)_b + \frac{1}{2} \sigma^{ac} \partial_c \ln(\kappa(u)/\kappa(v)). \]  
(4.36)

This simplifies to
\[
(P_D^D)^a = \frac{1}{R^2} \left( \left( \partial \theta \right)^a + \partial_{\theta} R \left( \partial_{\phi} \right)^a \right) \partial_{\phi} \ln(\kappa(u)/\kappa(v)) \\
+ \frac{1}{R^2 \sin^2 \theta} \left( \left( \partial_{\phi} \right)^a + \partial_{\phi} R \left( \partial_{\phi} \right)^a \right) \partial_{\phi} \ln(R^2 \kappa(u)/\kappa(v)) \right) .
\] (4.37)

Therefore, since the dual vector \((P_D^D)^a = \frac{1}{2} \nabla_a^S \ln(R^2 \kappa(u)/\kappa(v))\) is a gradient on \(S\), the normal curvature of \(S\) is zero.

The complete Dirichlet symplectic vector is
\[
(P^a = (P_D^D)^a + (P_D^D)^a
\] (4.38)

which depends only on \(S\) and \(\eta_{ab}\). In particular, it is independent of choice of the original null frame (4.20) and (4.21) on Minkowski space and of the corresponding frame (4.25) and (4.26) on \(T(S)^\perp\). Geometrically, the dual vector \((P_D^D)^a\) provides a preferred normal direction for a family of hypersurfaces defined to cut the light cone at \(S\), with vanishing normal curvature.

Finally, the commutator of the null frame (4.31) yields the normal part of the Neumann symplectic vector
\[
(P_D^N)^a = \mathcal{P}_S^\perp[v, u]^a = \mathcal{P}_S^\perp[\partial_u + \frac{\psi^2}{2} \partial_v + \frac{\partial_{\theta} V}{r^2} \partial_{\theta} + \frac{\partial_{\phi} V}{r^2 \sin^2 \theta} \partial_{\phi}, \partial_v]^a |_S
\]
\[
= \frac{\sqrt{2}}{r} \mathcal{P}_S^\perp \left( \frac{\psi^2}{2} \left( \partial_{\theta} \right)^a + \frac{\partial_{\theta} V}{r^2} \left( \partial_{\theta} \right)^a + \frac{\partial_{\phi} V}{r^2 \sin^2 \theta} \left( \partial_{\phi} \right)^a \right) |_S
\]
\[
= -2 \frac{\psi^2}{R} \mathcal{P}_S^\perp (\partial_v)^a
\] (4.39)

through \(\partial_v \psi^2 = -\sqrt{2} \psi^2 / r\). The tangential part of the Neumann symplectic vector is simply \((P_D^N)^a = (P_D^D)^a\). Hence, this yields the complete Neumann symplectic vector
\[
P^a = (P_\perp^N)^a + (P_D^N)^a,
\] (4.40)

which depends only on the congruence of spacelike 2-surfaces \(u = \text{const}, v - V(\theta, \phi) = \text{const},\) lying on the light cones in Minkowski space.

**Constant curvature 2-sphere**

In the special case of a constant curvature 2-sphere \(S\), viewed as embedded either in a hyperplane \(t = t_0 = \text{const}, r = R = r_0 = \text{const}\), or in a light cone, \(u = u_0 = (t_0 - r_0)/\sqrt{2},\) \(v = V = v_0 = (t_0 + r_0)/\sqrt{2}\), the mean curvature vector of \(S\) is simply \(H^a = \frac{2}{r_0} (\partial_t)^a\), and the complete Dirichlet symplectic vector reduces to
\[
P^a = H^a = \frac{2}{r_0} (\partial_t)^a,
\] (4.41)

while the complete Neumann symplectic vector vanishes.
B. Spherically Symmetric Spacetimes

In a spherically symmetric spacetime \((\mathbb{R} \times \Sigma, g_{ab})\),

\[
g_{ab} = -e^{2\psi}(dt)_a(dt)_b + e^{-2\nu}(dr)_a(dr)_b + r^2((d\theta)_a(d\theta)_b + \sin^2 \theta(d\phi)_a(d\phi)_b),
\]

(4.42)

where \(\psi = \psi(t, r)\) and \(\nu = \nu(t, r)\), consider a spacelike 2-surface \(S\) given by an isometry sphere \(r = r_0 = \text{const}\) and \(t = t_0 = \text{const}\). The metric on \(S\) is

\[
\sigma_{ab} = r_0^2(d\theta)_a(d\theta)_b + r_0^2 \sin^2 \theta(d\phi)_a(d\phi)_b,
\]

(4.43)

and the area of \(S\) is \(A(S) = 4\pi r_0^2\). Fix an orthonormal frame adapted to \(S\) by

\[
\vartheta^0_a = e^\psi(dt)_a, \quad \vartheta^1_a = e^{-\nu}(dr)_a, \quad \vartheta^2_a = r(d\theta)_a, \quad \vartheta^3_a = r \sin \theta(d\phi)_a.
\]

(4.44)

The Ricci rotation coefficients of the frame \(\Gamma^\lambda_{\mu\nu}(\vartheta) = \vartheta^a_{\lambda} \vartheta^{b\nu} \nabla_a \vartheta^\mu_{\lambda} = 2\vartheta^a_{\lambda} \vartheta^{b\mu} \partial_\nu \vartheta^\lambda_{[a} - \vartheta^b_{[\nu} \vartheta^c_{\mu]} \partial_\lambda \partial_{a]} \vartheta^\mu_{c]}\)

have the following non-vanishing components:

\[
\Gamma^0_{10} = -(\partial_r e^\psi)e^\nu e^{-\psi},
\]

(4.46)

\[
\Gamma^0_{01} = -(\partial_t e^\nu)e^{-\psi},
\]

(4.47)

\[
\Gamma^1_{22} = -\frac{e^\nu}{r} = \Gamma^3_{33},
\]

(4.48)

\[
\Gamma^3_{23} = -\frac{\cos \theta}{r \sin \theta}.
\]

(4.49)

The trace of the extrinsic curvatures of \((S, \sigma_{ab})\) with respect to the frame on \(T(S)^\perp\)

\[
t_a = \vartheta^0_a|_S, \quad s_a = \vartheta^1_a|_S,
\]

(4.50)

are respectively

\[
\kappa(t) = \sigma^{ab} \nabla_a t_b = - (\Gamma^2_{22} + \Gamma^3_{33})|_S = 0,
\]

(4.51)

and

\[
\kappa(s) = \sigma^{ab} \nabla_a s_b = - (\Gamma^2_{22} + \Gamma^3_{33})|_S = \frac{2 e^{2\nu(t_0, r_0)}}{r_0}.
\]

(4.52)

A preferred direction in \(T(S)^\perp\) is given by the mean curvature vector

\[
H^a = s^a \kappa(s) - t^a \kappa(t) = \frac{2 e^{2\nu(t_0, r_0)}}{r_0} (\partial_r)^a,
\]

(4.53)

which is spacelike (outside of any horizon). This vector gives the direction of the minimum absolute spacelike expansion of \(S\). Furthermore, the value of the expansion is given by the trace extrinsic curvature of \(S\) with respect to the unit vector in the direction \(H^a\).
\[
\frac{1}{\sqrt{H^2}} \kappa(H) = |\kappa(s)| = \frac{2 \epsilon^\nu(t_0, r_0)}{r_0},
\]  

which is equal to the norm of \( H^a \).

The normal part of the Dirichlet symplectic vector is given by the normal mean curvature vector

\[
(P_{\perp}^D)^a = H_{\perp}^a = t^a \kappa(s) - s^a \kappa(t) = \frac{2 \epsilon^\nu(t_0, r_0) e^{-\psi(t_0, r_0)}}{r_0} (\partial_t)^a. \tag{4.55}
\]

Here \((P_{\perp}^D)^a \) is timelike (outside of any horizon), orthogonal to \( H^a \), with the same absolute norm as \( H^a \). Most significant, \((P_{\perp}^D)^a \) gives the direction of zero expansion of \( S \).

A preferred orthonormal frame for \( T(S)^\perp \) is

\[
\tilde{t}^a = \frac{1}{\sqrt{H^2}} H_{\perp}^a = e^{-\psi}(\partial_t)^a, \quad \tilde{s}^a = \frac{1}{\sqrt{H^2}} H^a = e^\nu(\partial_r)^a, \tag{4.56}
\]

which depend only on \( S \) and \( \eta_{ab} \) but not on the chosen frame \( \tilde{\vartheta}_a^\mu \). In the preferred frame (4.56) the tangential part of the Dirichlet symplectic vector is

\[
(P_{\parallel}^D)^a = \sigma^{ac} \tilde{t}^b \nabla_c \tilde{s}_b = (\psi^{2a} \Gamma_2^1 + \psi^{3a} \Gamma_3^1)|_S = 0, \tag{4.57}
\]

and thus the normal curvature of \( S \) is zero.

Hence the complete Dirichlet symplectic vector is

\[
P^a = (P_{\perp}^D)^a + (P_{\parallel}^D)^a = \frac{2 \epsilon^\nu(t_0, r_0) - \psi(t_0, r_0)}{r_0} (\partial_t)^a, \tag{4.58}
\]

which depends only on \( S \) and \( \eta_{ab} \). In particular, it is independent of choice of the original orthonormal frame (4.44) for \( g_{ab} \) and (4.50) for \( \sigma_{ab}^\parallel \).

To define the Neumann symplectic vector, it is natural to use the orthonormal frame (4.56) extended off \( S \) to the congruence of isometry spheres \( t = \text{const}, r = \text{const} \). Then, for this extension, the normal part of the Neumann symplectic vector is given by the commutator

\[
(P_{\perp}^N)^a = \mathcal{P}_{\perp}^S \{ \tilde{t}, \tilde{s} \}^a = \left. (t^a \Gamma_0^1 + s^a \Gamma_1^0) \right|_S = \left. \left( - (\partial_t \epsilon^\nu) e^\nu e^{-2\psi}(\partial_t)^a + (\partial_r \epsilon^\nu) e^{-\psi}(\partial_r)^a \right) \right|_S. \tag{4.59}
\]

Since \((P_{\parallel}^N)^a = (P_{\parallel}^D)^a = 0\), the complete Neumann symplectic vector with respect to the congruence of isometry spheres associated to \( S \) is given by

\[
P^a = (P_{\perp}^N)^a + (P_{\parallel}^N)^a = e^\nu(t_0, r_0) - \psi(t_0, r_0) \left( - \partial_r \psi(t_0, r_0)(\partial_t)^a + \partial_t \nu(t_0, r_0)(\partial_r)^a \right). \tag{4.60}
\]

Finally, as a special case, consider the Reissner-Nordström black hole spacetime obtained for

\[
e^\psi = e^\nu = \sqrt{1 - 2m/r + q^2/r^2} \tag{4.61}
\]

where \( m = \text{const} \) and \( q = \text{const} \) are the black hole mass and charge; \( q = 0 \) yields the Schwarzschild black hole. The mean curvature vector of an isometry sphere \( S, t = \text{const}, r = \text{const} \), outside of the horizon is given by
\[ H^a = \frac{2}{r} \left( 1 - \frac{2m}{r} + \frac{q^2}{r^2} \right) (\partial_r)^a \] (4.62)

which gives the direction of the minimum absolute spacelike expansion of \( S \). Furthermore, the value of the expansion is given by the norm of \( H^a \),

\[ |\kappa(s)| = \frac{2}{r} \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}}. \] (4.63)

The complete Dirichlet symplectic vector is given by

\[ P^a = \frac{2}{r} (\partial_t)^a \] (4.64)

which depends only on \( S \) and \( \eta_{ab} \). Note that \( P^a \) is timelike, orthogonal to \( H^a \), with the same absolute norm as \( H^a \), and it gives the direction of zero expansion of \( S \).

With respect to the congruence of isometry spheres associated to \( S \), the complete Neumann symplectic vector is given by

\[ P^a = - \left( \frac{m}{r^2} - \frac{q^2}{r^3} \right) \left( 1 - \frac{2m}{r} + \frac{q^2}{r^2} \right)^{-1} (\partial_t)^a. \] (4.65)

The curl of this vector yields the sectional curvature normal to \( S \).

C. Axisymmetric Spacetimes

Now consider a stationary axisymmetric spacetime \((\mathbb{R} \times \Sigma, g_{ab})\),

\[ g_{ab} = -e^{2\psi}(dt)_a(dt)_b + e^{-2\nu}(dr)_a(dr)_b + e^{-2\mu_1}(d\theta)_a(d\theta)_b + e^{-2\mu_2}((d\phi)_a - w(dt)_a)((d\phi)_b - w(dt)_b), \] (4.66)

where \[ w = w(r, \theta), \psi = \psi(r, \theta), \nu = \nu(r, \theta), \mu_1 = \mu_1(r, \theta) \text{ and } \mu_2 = \mu_2(r, \theta). \] Let \( S \) be a spacelike 2-surface given by \( r = r_0 = \text{const} \) and \( t = t_0 = \text{const} \), which has the metric

\[ \sigma_{ab} = e^{-2\mu_1(r_0, \theta)}(d\theta)_a(d\theta)_b + e^{-2\mu_2(r_0, \theta)}(d\phi)_a(d\phi)_b. \] (4.67)

The area of \( S \) is \( A(S) = 2\pi \int_0^{r_0} e^{-\mu_1(r_0, \theta) - \mu_2(r_0, \theta)}d\theta \). A natural orthonormal frame adapted to \( S \) is given by

\[ \vartheta^0_a = e^{\psi}(dt)_a, \quad \vartheta^1_a = e^{-\nu}(dr)_a, \quad \vartheta^2_a = e^{-\mu_1}(d\theta)_a, \quad \vartheta^3_a = e^{-\mu_2}((d\phi)_a - w(dt)_a). \] (4.68)

The Ricci rotation coefficients of the frame

\[ \Gamma^\mu_{\lambda\nu}(\vartheta) = \vartheta^a_{(\lambda} \nabla_b \vartheta^b_{\nu)} = 2 \vartheta^a_{(\lambda} \vartheta^{b|\nu} \partial_a \vartheta_{b)} - \vartheta^b_{\mu} \vartheta^{c\nu} \partial_b \vartheta_{c)}_{\lambda} \] (4.69)

have the following non-vanishing components:
\( \Gamma_0^{01} = e^{-\psi}e^{\nu}\partial_{\nu}e^{\psi}, \Gamma_3^{01} = \frac{1}{2}e^{-\psi}e^{\nu}e^{-\mu_2}\partial_{\nu}w, \) (4.70)

\( \Gamma_0^{02} = e^{-\psi}e^{\mu_1}h_{\mu_1}e^{\psi}, \Gamma_3^{02} = \frac{1}{2}e^{-\psi}h_{\mu_1}e^{-\mu_2}\partial_{\nu}w, \) (4.71)

\( \Gamma_1^{03} = \frac{1}{2}e^{-\psi}e^{\nu}e^{-\mu_2}\partial_{\nu}w, \Gamma_2^{03} = \frac{1}{2}e^{-\psi}h_{\mu_1}e^{-\mu_2}\partial_{\nu}w, \) (4.72)

\( \Gamma_1^{12} = e^{\nu}h_{\mu_1}e^{\psi}, \Gamma_2^{12} = -e^{\nu}h_{\mu_1}\partial_{\nu}e^{-\mu_1}, \) (4.73)

\( \Gamma_0^{13} = \frac{1}{2}e^{-\psi}e^{\nu}e^{-\mu_2}\partial_{\nu}w, \Gamma_3^{13} = -e^{\nu}h_{\mu_2}\partial_{\nu}e^{-\mu_2}, \) (4.74)

\( \Gamma_0^{23} = \frac{1}{2}e^{-\psi}h_{\mu_1}e^{-\mu_2}\partial_{\nu}w, \Gamma_3^{23} = -e^{\mu_1}h_{\mu_2}\partial_{\nu}e^{-\mu_2}. \) (4.75)

The trace of the extrinsic curvatures of \((S, \sigma_{ab})\) with respect to the frame on \(T(S)^\perp\)

\[ t_a = \partial_a^0 |_{S}, \quad s_a = \partial_a^1 |_{S} \] (4.76)

are respectively

\[ \kappa(t) = \sigma^{ab}\nabla_a t_b = - (\Gamma_2^{02} + \Gamma_3^{03}) |_{S} = 0, \] (4.77)

and

\[ \kappa(s) = \sigma^{ab}\nabla_a s_b = - (\Gamma_2^{12} + \Gamma_3^{13}) |_{S} = -e^{\nu(r_0, \theta)}\partial_{\nu}(r_0, \theta), \] (4.78)

where \(\mu(r, \theta) = \mu_1 + \mu_2\). A preferred direction in \(T(S)^\perp\) is given by the mean curvature vector

\[ H^a = s^a \kappa(s) - t^a \kappa(t) = -e^{2\nu(r_0, \theta)}\partial_{\nu}(r_0, \theta)(\partial_r)^a, \] (4.79)

which is spacelike (outside of any horizon). This vector gives the direction of the minimum absolute spacelike expansion of \(S\). Furthermore, the value of the expansion is given by the trace extrinsic curvature of \(S\) with respect to the unit vector in the direction \(H^a\),

\[ \frac{1}{\sqrt{H^2}} \kappa(H) = |\kappa(s)| = e^{\nu(r_0, \theta)}|\partial_r\mu(r_0, \theta)|, \] (4.80)

which is equal to the norm of \(H^a\).

The normal part of the Dirichlet symplectic vector is given by the normal mean curvature vector

\[ (P^D_\perp)^a = H^a_\perp = t^a \kappa(s) - s^a \kappa(t) = -e^{-\psi(r_0, \theta)}e^{\nu(r_0, \theta)}\partial_{\nu}(r_0, \theta) \left( (\partial_t)^a + w(\partial_\phi)^a \right). \] (4.81)

Here \((P^D_\perp)^a\) is timelike (outside of any horizon), orthogonal to \(H^a\), with the same absolute norm as \(H^a\). Most significant, \((P^D_\perp)^a\) gives the direction of zero expansion of \(S\).

A preferred orthonormal frame for \(T(S)^\perp\) is

\[ t^a = \frac{1}{\sqrt{H^2}} H^a_\perp = e^{-\psi} \left( (\partial_t)^a + w(\partial_\phi)^a \right), \quad s^a = \frac{1}{\sqrt{H^2}} H^a = e^{\nu}(\partial_r)^a, \] (4.82)
which depend only on $S$ and $\eta_{ab}$ but not on the chosen frame $\vartheta^\mu_a$. In the preferred frame (4.82) the tangential part of the Dirichlet symplectic vector is

$$(P^D)^a = \sigma^{ac} b^b \nabla e^c_b = (\vartheta^2 \Gamma_2^{10} + \vartheta^3 \Gamma_3^{10})|_S = \frac{1}{2} e^{-\psi(r_0, \theta)} e^{\nu(r_0, \theta)} \partial_r w(r_0, \theta)(\partial_\phi)^a. \quad (4.83)$$

The curl of this vector yields the normal curvature of $S$.

Hence the complete Dirichlet symplectic vector is

$$P^a = (P^D^\|)^a + (P^D^\perp)^a$$

$$= e^{-\psi(r_0, \theta) + \nu(r_0, \theta)} \left(-\partial_r \partial_\mu (r_0, \theta) \left((\partial_t)^a + w(\partial_\phi)^a\right) + \frac{1}{2} \partial_\mu w(r_0, \theta)(\partial_\phi)^a\right), \quad (4.84)$$

which depends only on $S$ and $\eta_{ab}$. In particular, it is independent of choice of the original orthonormal frame (4.68) for $g_{ab}$ and (4.76) for $\sigma^a_{ab}$.

To define the Neumann symplectic vector, it is natural to use the orthonormal frame (4.82) extended off $S$ to the congruence of 2-surfaces $t = \text{const}$, $r = \text{const}$. With respect to this extension, the normal part of the Neumann symplectic vector is given by the commutator

$$(P^N)^a = (P^D)^a \parallel \parallel S = (\hat{t}, \hat{s})^a = (\hat{a} \Gamma_0^{10} + \hat{s} \Gamma_1^{10})|_S = -e^{-\psi(r_0, \theta)} e^{\nu(r_0, \theta)} \partial_r \psi(r_0, \theta) \left((\partial_t)^a + w(\partial_\phi)^a\right). \quad (4.85)$$

Since $(P^N)^a = (P^D)^a$, the complete Neumann symplectic vector in this congruence of 2-surfaces associated to $S$ is given by

$$P^a = (P^N)^a + (P^D)^a$$

$$= e^{-\psi(r_0, \theta) + \nu(r_0, \theta)} \left(-\partial_\phi \psi(r_0, \theta) \left((\partial_t)^a + w(\partial_\phi)^a\right) + \frac{1}{2} \partial_\mu w(r_0, \theta)(\partial_\phi)^a\right). \quad (4.86)$$

The curl of this vector yields the sectional curvature normal to $S$.

As a special case, consider the Kerr black hole spacetime obtained for

$$e^\psi = \sqrt{\Delta} \rho \gamma, e^{-\nu} = \frac{\rho}{\sqrt{\Delta}}, e^{-\mu_1} = \rho, e^{-\mu_2} = \frac{\gamma \sin \theta}{\rho}, w = \frac{2amr}{\gamma^2} \quad (4.87)$$

where

$$\gamma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \Delta = r^2 - 2mr + a^2, \rho^2 = r^2 + a^2 \cos^2 \theta. \quad (4.88)$$

Here $m = \text{const}$ and $a = \text{const}$ are the black hole mass and angular momentum; $a = 0$ yields the Schwarzschild black hole. The mean curvature vector of a 2-surface $S$, $t = \text{const}, r = \text{const}$, outside the horizon is given by

$$H^a = \frac{\Delta}{\rho^2} \frac{\partial_r \gamma}{\gamma} (\partial_r)^a = \frac{\Delta}{\rho^2 \gamma^2} (2r(r^2 + a^2) - a^2(r - m) \sin^2 \theta)(\partial_r)^a \quad (4.89)$$

which gives the direction of the minimum absolute spacelike expansion of $S$. Furthermore, the value of the expansion is given by the norm of $H^a$. 

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$|\kappa(s)| = \frac{\sqrt{\Delta}}{\rho Y^2} (2r(r^2 + a^2) - a^2(r - m) \sin^2 \theta)$.  \hfill (4.90)

The complete Dirichlet symplectic vector is given by

$$P^a = \frac{Y}{\rho^2} \left( \frac{\partial_r Y}{Y} \left( (\partial_t)^a + w(\partial_\phi)^a \right) + \frac{\rho \partial_r w}{2} (\partial_\phi)^a \right)$$

$$= \frac{2r(r^2 + a^2) - a^2(r - m) \sin^2 \theta}{\rho^2 Y} (\partial_t)^a + \frac{am}{\rho^2 Y} (\partial_\phi)^a$$ \hfill (4.91)

which depends only on $S$ and $\eta_{ab}$. Note that the normal part of $P^a$ is timelike, orthogonal to $H^a$, with the same absolute norm as $H^a$, and it gives the direction of zero expansion of $S$.

With respect to this congruence of 2-surfaces $S$, the complete Neumann symplectic vector is given by

$$P^a = \frac{Y}{\rho^2} \left( \frac{\partial_r Y}{Y} - \frac{\partial_r \Delta}{2 \Delta} - \frac{\partial_r \rho}{2 \rho} \right) \left( (\partial_t)^a + w(\partial_\phi)^a \right) + \frac{\rho \partial_r w}{2} (\partial_\phi)^a$$

$$= -\frac{2m}{\rho^4 \Gamma \Delta} \left( (r^2 + a^2)^2 (r^2 - a^2) + a^2((r^2 + a^2)^2 - 4mr^3) \sin^2 \theta \right) (\partial_t)^a$$

$$+ \frac{am}{\rho^4 \Gamma \Delta} \left( (r^2 - a^2)^2 - 4r^3(r - m) - a^2(a^2 - r^2) \sin^2 \theta \right) (\partial_\phi)^a.$$ \hfill (4.92)

### D. Homogeneous Isotropic Spacetimes

Next, consider a Friedmann-Robertson-Walker spacetime ($\mathbf{R} \times \Sigma, g_{ab}$),

$$g_{ab} = -(dt)_a (dt)_b + a^2(t) \left( \frac{1}{1 - kr^2} (dr)_a (dr)_b + r^2((d\theta)_a (d\theta)_b + \sin^2 \theta (d\phi)_a (d\phi)_b) \right),$$ \hfill (4.93)

where $k = 0, 1, -1$ ($\Sigma$ is $\mathbf{R}^3$ if $k = 0, 1$ or $S^3$ if $k = 1$) corresponding to a spatially flat, spherical, or hyperbolic geometry on $\Sigma$. Let $S$ be a spacelike 2-surface given by an isometry sphere $r = r_0 = \text{const}$ and $t = t_0 = \text{const}$. The metric on $S$ is

$$\sigma_{ab} = a^2(t_0) r_0^2 \left( (d\theta)_a (d\theta)_b + \sin^2 \theta (d\phi)_a (d\phi)_b \right)$$ \hfill (4.94)

and the area of $S$ is $A(S) = 4\pi a(t_0) r_0^2$. Fix an orthonormal frame adapted to $S$ by

$$\vartheta^0 = (dt)_a, \quad \vartheta^1 = \frac{a(t)}{\sqrt{1 - kr^2}} (dr)_a, \quad \vartheta^2 = a(t) r (d\theta)_a, \quad \vartheta^3 = a(t) r \sin \theta (d\phi)_a.$$ \hfill (4.95)

The Ricci rotation coefficients of the frame

$$\Gamma^\mu_{\lambda \nu} (\vartheta) = \vartheta^\alpha_{(\lambda} \nabla_{\nu} \vartheta^\beta_{\mu)} - 2 \vartheta^\alpha_{(\lambda} \vartheta^{\beta \mu}_{\nu)} \partial_{[\alpha} \vartheta^\nu_{\beta]} - \vartheta^\beta_{\mu} \vartheta^{\alpha \nu} \partial_{[\beta} \vartheta^\nu_{\alpha]}.$$ \hfill (4.96)
have the following non-vanishing components:

\[ \Gamma_0^{01} = \frac{\dot{a}(t)}{a(t)} = \Gamma_2^{02} = \Gamma_3^{03}, \]
\[ \Gamma_1^{12} = \frac{\sqrt{1 - 2kr}}{a(t)r} = \Gamma_3^{13}, \]
\[ \Gamma_3^{23} = -\frac{\cos \theta}{a(t)r \sin \theta}. \]  

(Here, an over-dot “\(\cdot\)” denotes a derivative with respect to \(t\).)

The trace of the extrinsic curvatures of \((S, \sigma_{ab})\) with respect to the frame on \(T(S)\) is

\[ t_a = \dot{v}_a^0|_S, \quad s_a = \dot{v}_a^1|_S, \]  

are respectively

\[ \kappa(t) = \sigma^{ab} \nabla_a t_b = - (\Gamma_2^{02} + \Gamma_3^{03})|_S = \frac{2\ddot{a}(t_0)}{a(t_0)} \]  

and

\[ \kappa(s) = \sigma^{ab} \nabla_a s_b = - (\Gamma_2^{12} + \Gamma_3^{13})|_S = \frac{2\sqrt{1 - kr_0^2}}{r_0 a(t_0)}. \]

A preferred direction in \(T(S)^\perp\) is given by the mean curvature vector

\[ H^a = s^a \kappa(s) - t^a \kappa(t) = 2 \left( \frac{1 - kr_0^2}{a(t_0)^2 r_0^2} (\partial_0)^a - \frac{\dot{a}(t_0)}{a(t_0)} (\partial_r)^a \right). \]

This vector gives the direction of the minimum absolute spacelike expansion of \(S\). Furthermore, the value of the expansion is given by the trace extrinsic curvature of \(S\) with respect to the unit vector in the direction \(H^a\),

\[ \frac{1}{\sqrt{H^2}} \kappa(H) = |\kappa(s)| = \frac{2}{a(t_0)} \sqrt{\frac{1 - kr_0^2}{r_0^2} - \ddot{a}^2(t_0)}, \]

which is equal to the norm of \(H^a\).

The normal part of the Dirichlet symplectic vector is given by the normal mean curvature vector

\[ (P_\perp)^a = H_\perp^a = t^a \kappa(s) - s^a \kappa(t) = \frac{2\sqrt{1 - kr_0^2}}{a(t_0)} \left( \frac{1}{r_0} (\partial_0)^a - \frac{\dot{a}(t_0)}{a(t_0)} (\partial_r)^a \right). \]

Here \((P_\perp)^a\) is orthogonal to \(H^a\), with the same absolute norm as \(H^a\). Most significant, \((P_\perp)^a\) gives the direction of zero expansion of \(S\). Note that \((P_\perp)^a\) is timelike (and \(H^a\) is spacelike) if and only if the acceleration of \(\Sigma\) satisfies \(|\ddot{a}(t_0)| \leq \sqrt{1 - kr_0^2}/r_0\), depending on the radius of \(S\).
A preferred orthonormal frame for $T(S)^\perp$ is
\[ \hat{t}^a = \frac{1}{\sqrt{H}} H^a_{\perp} = (\partial_t)^a, \quad \hat{s}^a = \frac{1}{\sqrt{H}} H^a = \frac{\sqrt{1 - kr^2}}{a(t)} (\partial_r)^a, \]
which depend only on $S$ and $\eta_{ab}$ but not on the chosen frame $\vartheta^\mu$. In the preferred frame (4.56) the tangential part of the Dirichlet symplectic vector is
\[ (P^D_D)^a = \sigma^{ac}_{\perp} \Gamma_2^{10} + \sigma^{3a}_{\perp} \Gamma_3^{10} |_{S} = 0, \]
and thus the normal curvature of $S$ is zero.
Hence the complete Dirichlet symplectic vector is
\[ P^a = (P^D_D)^a + (P^D_D)^a = \frac{2\sqrt{1 - kr^2}}{a(t)} \left( \frac{1}{r_0} (\partial_t)^a - \frac{\dot{a}(t_0)}{a(t_0)} (\partial_r)^a \right), \]
which depends only on $S$ and $\eta_{ab}$. In particular, it is independent of choice of the original orthonormal frame (4.95) for $g_{ab}$ and (4.100) for $\sigma_{ab}^\perp$.
To define the Neumann symplectic vector, it is natural to use the orthonormal frame (4.106) extended off $S$ to the congruence of isometry spheres $t = \text{const}, r = \text{const}$. Then, for this extension, the normal part of the Neumann symplectic vector is given by the commutator
\[ (P_N^N)^a = \mathcal{P}_S^N [\hat{t}, \hat{s}]^a = (\hat{t}^a \Gamma_0^{10} + \hat{s}^a \Gamma_1^{10}) |_{S} = - \left( \frac{\sqrt{1 - kr^2}}{a(t_0)^2} \right) \dot{a}(t_0) (\partial_r)^a. \]
Since $(P_N^N)^a = (P_D^D)^a = 0$, the complete Neumann symplectic vector with respect to the congruence of isometry spheres associated to $S$ is given by
\[ P^a = (P_N^N)^a + (P_N^N)^a = - \sqrt{1 - kr^2} \frac{\dot{a}(t_0)}{a(t_0)^2} (\partial_r)^a. \]
For an isometry sphere $S$, $r = \text{const}, t = \text{const}$, in a time-symmetric hypersurface $\Sigma$, since $\dot{a}(t) = 0$ it follows that $H^a$ is spacelike, $(P_D^D)^a$ is timelike. Then the complete Dirichlet symplectic vector is
\[ P^a = 2\sqrt{1 - kr^2} (\partial_t)^a, \]
while the complete Neumann symplectic vector vanishes.

E. Asymptotically flat spacetimes
Consider an asymptotically flat spacetime $(M, g_{ab})$ with $g_{ab} = \eta_{ab} + O(1/r)$ and $\partial_{c}g_{ab} = O(1/r^2)$ as $r \to \infty$ at fixed $t$, where $\eta_{ab}$ is a flat metric (4.1) in Minkowski spherical coordinates $t, r, \theta, \phi$. Suppose the total ADM mass, $m$, of the spacetime $(M, g_{ab})$ is finite and positive. Then the metric has the asymptotic form \[10,11\]
\[ g_{ab} = -(1 - 2m/r + O(1/r^2))(dt)_a(dt)_b + (1 + 2m/r + O(1/r^2))(dr)_a(dr)_b + r^2((d\theta)_a(d\theta)_b + \sin^2 \theta (d\phi)_a(d\phi)_b + O(1/r^3)) \quad \text{as } r \to \infty \text{ at fixed } t. \] 

(4.112)

(Note that any gravitational radiation terms vanish in this limit.) We first discuss the ADM energy-momentum vector \([8]\). In the spacetime \((M, g_{ab})\), spatial infinity, \(t^0\), can be represented as the set of asymptotic 2-spheres \(S_\infty\) given by \(t = \text{const}, r = \to \infty\), which are regarded as being identified under asymptotic time translations generated by the Killing vector \((\partial_t)^a\) of \(\eta_{ab}\). Now, for a spacelike hypersurface \(\Sigma_t, t = \text{const}\), the ADM energy and momentum in standard asymptotic Minkowski coordinates \(x^\mu\) on \(M\) are given by \([12]\)

\[
P_\mu = \begin{cases} 
\sum_{\nu, \rho \neq 0} \frac{1}{16\pi} \int_{S_\infty} s_\nu(\partial_\rho g_{\mu\nu} - \partial_\nu g_{\rho\mu})dS = m, & \mu = 0 \\
\sum_{\nu \neq 0} \frac{1}{16\pi} \int_{S_\infty} (s_\nu \partial_t g_{\mu\nu} - s_\mu \partial_t g_{\nu\nu})dS = 0, & \mu = 1, 2, 3.
\end{cases}
\]

(4.113)

Hence, the ADM energy-momentum vector at spatial infinity is represented by \((P^{\text{ADM}})_a = m(dt)_a|_{S_\infty}\). Let \(n^a = (\partial_t)^a|_{S_\infty}\). Then, geometrically, the vector

\[
\frac{1}{m} (P^{\text{ADM}})^a = -n^a
\]

(4.114)

corresponds to an asymptotic stationary unit-norm Killing vector of the asymptotically flat metric (4.112).

To proceed, let \(S\) be any family of spacelike 2-surfaces that approaches the 2-sphere \(S_\infty\) as \(r \to \infty\) at fixed \(t\). Since the spacetime metric is asymptotic to the Schwarzschild metric, we may use the results obtained in Example (B) to calculate the Dirichlet and Neumann symplectic vectors in this limit.

The Dirichlet symplectic vector is given by

\[
(P^D)^a = \frac{2}{r}((1 - m/r)t^a + O(1/r^3))
\]

(4.115)

where \(t^a = (1 + m/r + O(1/r^2))(\partial_t)^a\) is a unit timelike vector of \((M, g_{ab})\). For a comparison with \((P^{\text{ADM}})^a\), we scale \((P^D)^a\) by the area of \(S\), \(A(S) = 4\pi r^2 + O(1/r)\), which yields

\[
A(S)(P^D)^a = 8\pi r^2 - m + O(1/r))t^a.
\]

(4.116)

Note that the first term in this expression is singular as \(r \to \infty\). It corresponds to the Dirichlet symplectic vector for \(S_\infty\) with respect to the flat metric \(\eta_{ab}\) on \(M\). We extend this vector in a natural geometrical manner from \(S_\infty\) to \(S\) by

\[
(P^D)^a = \frac{2}{r}t^a
\]

(4.117)

which depends only on the radius of \(S\) and the timelike unit vector \(t^a\) with respect to \(g_{ab}\). We now subtract \((P^D)^a\) from \((P^D)^a\) to obtain the normalized Dirichlet symplectic vector

\[
(P^D)^a = A(S)((P^D)^a - (P^D)^a) = (-8\pi m + O(1/r)t^a.
\]

(4.118)

Then the limit \(r \to \infty\) yields a well-defined (finite) vector associated to \(S_\infty\) in terms of \(t^a \to n^a\). This establishes our main result.
Theorem 4.1. For an asymptotically flat spacetime \((M, g_{ab})\), at spatial infinity the normalized Dirichlet symplectic vector (4.118) is equal to \(8\pi\) times the ADM energy-momentum vector (4.114),

\[
\frac{1}{8\pi}(P^D)^a|_{S_\infty} = (P^{ADM})^a = -mn^a
\]  

(The \(8\pi\) factor reflects the normalization chosen for the Hamiltonian variational principle for the Einstein equations in Sec. II.)

We remark that the ADM vector \((P^{ADM})^a\) can be derived \([13]\) directly from the symplectic structure of the Einstein equations similarly to the analysis given in Sec. 3 in Ref. [1] by using asymptotically flat boundary conditions at \(S_\infty\) in place of the Dirichlet boundary condition at \(S\) on the spacetime metric.

Finally, we discuss the Neumann symplectic vector. Note that the normalized symplectic vector (4.118) is obtained from the locally constructed Dirichlet symplectic vector \((P^D)^a\) associated to a spacelike 2-surface \(S\), where \((P^D)^a\) depends only on \(S\) and \(g_{ab}\). In contrast, the Neumann symplectic vector \((P^N)^a\) associated to \(S\) also depends on a choice of congruence of 2-surfaces \(S'\) diffeomorphic to \(S\). If a suitably parameterized null geodesic congruence through \(S\) is used to define \((P^N)^a\), it follows from Proposition 3.14 that the normal part of \((P^N)^a\) vanishes. Moreover, the tangential part of \((P^N)^a\) is equal to the tangential part of \((P^D)^a\). Thus for any topological 2-sphere \(S\) that approaches \(S_\infty\) as \(r \to \infty\), by Eq. (4.115) the resulting vector \((P^N)^a\) is at most \(O(1/r^3)\) and is tangential to \(S\). Consequently, if we consider the normalized symplectic vector

\[
(\tilde{P}^N)^a = A(S)((P^N)^a - (P^N\text{ flat})^a)
\]  

\[
(\tilde{P}^N)^a|_{S_\infty} = 0.
\]  

Note that for an asymptotically flat metric (4.112), as \(S\) approaches \(S_\infty\), all null geodesic congruences through \(S\) approach future and past null infinity, \(I^\pm\), and thereby provide a natural congruence of spacelike 2-spheres \(S_{I^\pm}\) associated to the 2-sphere \(S_\infty\) representing spatial infinity, \(i^0\). In particular, \(S_{I^\pm}\) are related to \(S_\infty\) by null geodesic asymptotic isometries of \(g_{ab}\). Hence, the normalized symplectic vector (4.121) effectively depends only on \(S\) and \(g_{ab}\) (including its asymptotic structure), similarly to the vector (4.118).

V. CONCLUDING REMARKS

In this paper we have considered the covariant symplectic structure associated to the Einstein equations with matter sources. One main result is that we derive a covariant Hamiltonian under Dirichlet and Neumann type boundary conditions for both the gravitational field and matter fields in any fixed spatially bounded region of spacetime \((M, g_{ab})\), allowing the time-flow vector \(\xi^a\) to be timelike, spacelike, or null.
The Dirichlet and Neumann Hamiltonians evaluated on solutions of the coupled gravitational and matter field equations reduce to a surface integral over the spatial boundary 2-surface, $S$. (In fact, this result is known to hold for any diffeomorphism covariant spacetime field theory [14].) For each of the boundary conditions this surface integral has the form of $\int_S \xi^a P_a dS$ where $P_a$ is a locally constructed dual vector field associated to the 2-surface $S$ and boundary conditions, which we call the Dirichlet and Neumann symplectic vectors. Similar results are discussed in Ref. [18,19].

Our principle result is to show that the purely gravitational part of the Dirichlet symplectic vector ($P^D_a$) has very interesting geometrical properties when decomposed into its normal and tangential parts, ($P^D_\perp$) and ($P^D_\parallel$), with respect to $S$. First, ($P^D_\perp$) depends only on the 2-surface $S$ and spacetime metric $g_{ab}$ and thus yields a geometrical vector field normal to $S$ in spacetime. This vector ($P^D_\perp$) is shown to be orthogonal to the mean curvature vector of $S$ and, most importantly, it gives the direction of zero expansion of $S$ in spacetime, i.e. $\mathcal{L}_{P^\perp_a} \epsilon_{ab}(S) = 0$ where $\epsilon_{ab}(S)$ is the area volume form of $S$. Furthermore, the norm of the vector ($P^D_\perp$) is equal to the product of the expansions of $S$ with respect to ingoing and outgoing null geodesics, $\theta^- a$ and $\theta^+ a$ (and is independent of parameterization of the geodesics). This expression is obviously related to the condition for a spatial 2-surface $S$ to be trapped (or marginally trapped), namely, $\kappa^+ \kappa^-$ is positive (or zero) on $S$, where $\mathcal{L}_{\theta^\pm} \epsilon_{ab}(S) = \kappa^\pm \epsilon_{ab}(S)$. Consequently, $S$ is trapped (or marginally trapped) precisely when ($P^D_\perp$) is spacelike (or null) on $S$. If this notion is applied to the ingoing and outgoing null geodesics at each point $p$ on $S$ (i.e. the pair of null geodesics through $p$ is “trapped” (or “marginally trapped”) if $\kappa^+ \kappa^-$ is positive (or zero) at $p$), then, in this sense, ($P^D_\perp$) measures point-wise how close $S$ is to being a trapped surface.

In contrast, ($P^D_\parallel$) depends not only on the 2-surface $S$ and spacetime metric $g_{ab}$ but also on a choice of orthonormal frame or null frame for the normal tangent space $T(S)^\perp$ of $S$. Geometrically, ($P^D_\parallel$) is shown to be a connection for the normal curvature of $S$ in spacetime and consequently changes by a gradient under a boost of the frame. However, if the normal vector ($P^D_\parallel$) is non-null, then ($P^D_\perp$) and the mean curvature vector of $S$ comprise a preferred frame for $T(S)^\perp$ and hence there exists a corresponding preferred tangential vector ($P^D_\parallel$) (evaluated in this frame). Thus, in this situation, the complete Dirichlet symplectic vector is a well-defined geometrical vector field depending only on $S$ and $g_{ab}$. We refer to this as the invariant Dirichlet symplectic vector associated to $S$.

Apart from its geometrical interest, the Dirichlet symplectic vector is also related to definitions of canonical energy, momentum, and angular momentum given by the value of the Dirichlet Hamiltonian for solutions of the Einstein (and matter) equations. In particular, we have shown that in an asymptotically flat spacetime in the limit of $S$ approaching spatial infinity $S_\infty$, the Dirichlet symplectic vector reduces in a suitable sense to the ADM energy-momentum vector. Hence, the integral $\int_{S_\infty} \xi^a (P^D_a) dS$ yields total energy, momentum, angular momentum of the spacetime when $\xi^a$ is chosen to be an asymptotic Killing vector associated to time-translations, space-translations, or rotations of the asymptotic flat background metric.

In addition, for a compact spatial 2-surface $S$ in $(M, g_{ab})$, it follows from results in Refs.
\[ \int_{S} \xi^{a}(P^{D})_{a}dS \] for \( \xi^{a} \) chosen to be normal and tangential to \( S \) reproduce Brown and York’s [16,17] quasilocal energy, momentum, and angular momentum quantities. (See also Refs. [1,5].) Furthermore, we have obtained matter contributions to these quantities, for an electromagnetic field and a set of Yang-Mills-Higgs fields. In a forthcoming paper we will explore geometrical quasi-local quantities defined purely in terms of \((P^{\perp})^{a}\) and \((P^{\parallel})^{a}\). We will also explore the use of \((P^{D})^{a}\) as a time flow vector for a boundary-initial value formulation of the Einstein equations.

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