A Remark on Integrability of Stochastic Systems
Solvable by Matrix Product Ansatz

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Abstract
Within the Matrix Product Formalism we have already introduced a multi-species exclusion process [1],[2], in which different particles hop with different rates and fast particles stochastically overtake slow ones. In this letter we show that on an open chain, the master equation of this process can be exactly solved via the coordinate Bethe ansatz. It is shown that the N-body S-matrix of this process is factorized into a product of two-body S-matrices, which in turn satisfy the quantum Yang-Baxter equation (QYBE). This solution is to our knowledge, a new solution of QYBE.

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1 Introduction

The static Matrix Product Ansatz (MPA) as formulated in [3], relates the problem of finding the steady state of a one dimensional homogenous stochastic system, to a quadratic algebra and its representations.

Looking back at the history of the subject, one can look at static MPA and its final formulation by Krebs and Sandow [3], as an abstractization of some recursion relations between certain quantities for systems of different sizes. In this way the technique of using recursion relations [4] [5] acquires a purely algebraic character, and the results obtained by the former method [4] [5] are rederived [6] by computing matrix elements of product of operators. Conceptually MPA is the generalization of the ordinary Bernoulli measure where matrix elements of operators replace numbers to produce correlations.

Many of the questions regarding the steady state and the correlation functions can then be answered at least in principle, by purely algebraic manipulations. We say ”in principle”, since in general, finding the representations of the algebra may be as difficult as the original problem. The special cases where the static MPA has been used for calculating concrete physical quantities include the one-species asymmetric simple exclusion process (ASEP) on a ring and on an open chain with and without an impurity [1], [2], [3], [4], [5], [6] (see also [1], [2] and references therein). The works on the multi-species asep include [3], [4], [5], and [6].

When one comes to the dynamical MPA [4], [5], the corresponding algebra becomes time-dependent and much more involved. Even when all the particles are identical, only in certain limited cases the dynamical algebra has been used for calculating physical quantities, namely the symmetric exclusion process [3] and a non-equilibrium model of spin relaxation [6]. In these two cases the algebra is simple enough to be used for finding the spectrum of the corresponding quantum Hamiltonian and r-point density correlators. These two cases happen to be integrable. In fact they belong to a 10-parameter family of diffusion-reaction systems which are equivalent through a similarity transformation to a generalized Heisenberg chain [6], which is known to be integrable.

For a general stochastic process, dynamical MPA, like the static one is just a rewriting of the master equation and its real power is revealed only when one can solve the resulting algebraic relations or can find a representation of the algebra.

It is useful to agree upon a notion, however vague, of MPA solvability. Let us call a process static-MPA (dynamic-MPA) solvable, if a nontrivial representation of the static (dynamic) MPA algebra can be found, or the ensuing recursion relations can be solved.

At present there is no definite connection between static or dynamic MPA-solvability and integrability, although from the above examples it appears that the dynamic-MPA solvable systems are integrable, since for these models the entire spectrum has been found. However it is not known what kind of integrable systems are MPA solvable.
To see if there is any connection between MPA-solvability and integrability, it is desirable to study a number of examples which hint to such a connection and to investigate the common properties of these examples.

In this letter we provide an example of a multi-species stochastic process which on the one hand is related to a very simple algebra in the MPA formalism and on the other hand is integrable in the sense that its N-particle S-matrix is factorized into a product of two-body S-matrices, which in turn satisfy the quantum Yang-Baxter equation (QYBE). The algebra is generated by \( p + 1 \) generators \( E \) and \( D_i \), \( i = 1 \ldots p \) with relations

\[
D_i E = \frac{1}{v_i} D_i + E \quad i = 1 \ldots p
\]

where \( v_1 \leq v_2 \leq \ldots \leq v_p \) are finite non-zero real numbers. In this process a particle of type \( j \) hops with rate \( v_j \) and when it encounters a particle of type \( i \), with \( v_i < v_j \) the two particles interchange their sites, with a rate \( v_j - v_i \) as if the fast particle stochastically overtakes the slow one. This model seems to be natural as a simple model of one-way traffic flow. We have shown elsewhere \[1\], \[2\] that on a finite chain this process is MPA-solvable in the above mentioned sense. We now consider the same process on an open chain and show that the master equation of this process can be solved via the coordinate Bethe ansatz. That is, by a suitable transformation we turn the master equation into an eigenvalue equation, and find the eigenvalues and eigenfunctions of this equation, the latter being in the form of Bethe wavefunctions. In this case, since we have \( p \) species of particles, the two particle S-matrix is not a c-number as in \[17\], \[18\] but is a \( p^2 \times p^2 \) matrix. Thus the factorizability of the N-particle S-matrix puts a stringent condition on this two-body S-matrix, in the form of the Yang-Baxter equation. We will show that this is indeed the case, and as a byproduct provide a new solution of QYBE with spectral parameter.

## 2 The Master Equation

Suppose there are \( N \) particles of different types on an infinite one dimensional chain. The basic objects that we are interested in, are denoted by \( P_{\alpha_1, \alpha_2, \ldots, \alpha_N}(x_1, x_2, \ldots, x_N) \) which are the probabilities of finding at time \( t \), a particle of type \( \alpha_1 \) at site \( x_1 \), a particle of type \( \alpha_2 \) at site \( x_2 \), etc. Following \[18\], these functions to define probabilities only in the physical region \( x_1 < x_2 < \cdots < x_N \). The hypersurfaces where any of the two adjacent coordinates are equal are the boundaries of the physical region. In the subset of the physical region where \( x_{i+1} - x_i > 1, \forall i \) we have free hopping of particles and the master equation is written as

\[
\frac{\partial}{\partial t} P_{\alpha_1, \alpha_2, \ldots, \alpha_N}(x_1, x_2, \ldots, x_N; t) = v_{\alpha_1} P_{\alpha_1, \alpha_2, \ldots, \alpha_N}(x_1 - 1, x_2, \ldots, x_N; t) + \cdots
\]
\[ + v_{\alpha_1} P_{\alpha_1, \alpha_2, \cdots, \alpha_N}(x_1, x_2, \cdots, x_N - 1; t) \]
\[- (v_{\alpha_1} + \cdots v_{\alpha_N}) P_{\alpha_1, \alpha_2, \cdots, \alpha_N}(x_1, x_2, \cdots, x_N; t). \] (3)

We assume hereafter for convenience that the time variable has been rescaled so that all the hopping rates are dimensionless quantities, i.e. they are ratios of actual hopping rates to some standard hopping rate.

Extending the above equation to the whole physical region, where \( x_{i+1} - x_i \geq 1 \), causes terms on the boundary surfaces to appear on the right hand side of the master equation, which should be fixed by a choice for the boundary condition. The choice of boundary condition in fact determines the type of interaction between particles (see [19]). On any such surface where for example \( x_{i+1} - x_i = 1 \) for some \( i \), we supplement the above equation with the following boundary condition, where we suppress for simplicity the time variable and all the other coordinates:

\[ v_{\beta} P_{\alpha, \beta}(x, x) = v_\alpha P_{\alpha, \beta}(x, x + 1) + (v_{\beta} - v_\alpha) P_{\beta, \alpha}(x, x + 1) \quad \beta \geq \alpha \] (4)
\[ P_{\beta, \alpha}(x, x) = P_{\beta, \alpha}(x, x + 1) \quad \beta \geq \alpha \] (5)

The master equation (3) and the boundary conditions (4,5), replace the very large number of equations which one should write by considering the multitude of cases which arise according to which group of particles are adjacent to each other. Instead of giving a general proof we provide a few examples in the two and three particle sectors.

Consider the two particle sector. From the definition of the process we have:

\[ \frac{\partial}{\partial t} P_{12}(x, x + 1) = v_1 P_{12}(x - 1, x + 1) + (v_2 - v_1) P_{21}(x, x + 1) - v_2 P_{12}(x, x + 1) \] (6)
\[ \frac{\partial}{\partial t} P_{21}(x, x + 1) = v_2 P_{21}(x - 1, x + 1) - (v_1 + (v_2 - v_1)) P_{21}(x, x + 1) \] (7)

These are exactly the equations which are obtained from combination of (3) and (4,5). As a couple of examples in the three particle sector, consider \( P_{123}(x, x + 1, x + 2) \) and \( P_{213}(x, x + 1, x + 2) \). From the definition of the process we have

\[ \frac{\partial}{\partial t} P_{123}(x, x + 1, x + 2) = v_1 P_{123}(x - 1, x + 1, x + 2) + (v_2 - v_1) P_{213}(x, x + 1, x + 2) + (v_3 - v_2) P_{132}(x, x + 1, x + 2) - v_3 P_{123}(x, x + 1, x + 2) \] (8)
\[ \frac{\partial}{\partial t} P_{213}(x, x + 1, x + 2) = v_2 P_{213}(x - 1, x + 1, x + 2) + (v_3 - v_1) P_{231}(x, x + 1, x + 2) - (v_3 + (v_2 - v_1)) P_{213}(x, x + 1, x + 2) \] (9)

These are precisely obtained from combination of (3) and (4,5). This pattern repeats in all sectors.
3 The Bethe Ansatz Solution

We rewrite \( P_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1, x_2, \ldots, x_N; t) \) as
\[
P_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1, x_2, \ldots, x_N; t) = e^{-(E+v_{\alpha_1}+\cdots+v_{\alpha_N})t} (v_{\alpha_1})^{x_1} \cdots (v_{\alpha_N})^{x_N} \Psi_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1, \ldots, x_N; t)
\]
and insert it into (3) to obtain
\[
\Psi_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1-1, x_2, \ldots, x_N; t) + \Psi_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1, x_2-1, \ldots, x_N; t) + \cdots + \Psi_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1, x_2, \ldots, x_N-1; t) = -E \Psi_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1, x_2, \ldots, x_N; t)
\]
In terms of \( \Psi \) the boundary conditions (4, 5) are rewritten as
\[
\Psi_{\alpha,\beta}(x, x) = v_\alpha \Psi_{\alpha,\beta}(x, x+1) + \frac{v_\alpha}{v_\beta} (v_\beta - v_\alpha) \Psi_{\beta,\alpha}(x, x+1) \quad \beta \geq \alpha
\]
\[
\Psi_{\beta,\alpha}(x, x) = v_\alpha \Psi_{\beta,\alpha}(x, x+1) \quad \beta \geq \alpha
\]
Hereafter we use a compact notation as follows. \( \Psi \) is a tensor of rank \( N \), whose components are \( \Psi_{\alpha_1,\alpha_2,\ldots,\alpha_N}(x_1, x_2, \ldots, x_N) \). The boundary conditions are written in the form
\[
\Psi(\cdots, \xi, \xi, \cdots) = b_{k,k+1} \Psi(\cdots, \xi, \xi+1, \cdots)
\]
where \( b_{k,k+1} \) is the embedding of \( b \) (the matrix derived from (12, 13)) in the locations \( k \) and \( k+1 \).
\[
b_{k,k+1} = 1 \otimes \cdots \otimes b_{k,k+1} \otimes \cdots \otimes 1
\]
In the two species case \( b \) is equal to
\[
b = \begin{pmatrix}
v_1 & . & . & . \\
. & v_1 & \frac{v_1}{v_2} (v_2 - v_1) & . \\
. & . & v_1 & . \\
. & . & . & v_2
\end{pmatrix}
\]
In the \( p \)-species case it is :
\[
b = \sum_{i,j} \tilde{v}_{ij} E_{ii} \otimes E_{jj} + \sum_{i<j} \frac{v_i}{v_j} (v_i - v_j) E_{ij} \otimes E_{ji}
\]
where \( (E_{ij})_{k,l} = \delta_{ik} \delta_{jl} \) and \( \tilde{v}_{ij} = v_{\min(i,j)} \).
To solve the eigenvalue equation (14), we write \( \Psi \) as a Bethe wave function
\[
\Psi(x_1, \cdots, x_N) = \sum_\sigma A_\sigma e^{i\sigma(p) \cdot x}
\]
Here \( x \) and \( p \) denote \( n \)-tuples of coordinates and momenta respectively, and the summation runs over all the elements \( \sigma \) of the permutation group. For each element \( \sigma \) of the permutation
group, the corresponding coefficient (a tensor of rank N) is denoted by $A_\sigma$. Inserting (18) in (11), yields
\[ \sum_{\sigma} A_\sigma e^{i\sigma(p) \cdot x} (\sum_j e^{-i\sigma(p_j)} + E) = 0 \] (19)
From this, one obtains the eigenvalues (as one can remove $\sigma$ from the summations in the parenthesis)
\[ E = -(e^{-ip_1} + e^{-ip_2} + \cdots + e^{-ip_N}). \] (20)
The Bethe wavefunction should also satisfy the boundary condition (14). Inserting (18) in (14) we obtain
\[ \sum_{\sigma} e^i \sum_{j \neq k,k+1} \sigma(p_j)x_j + i(\sigma(p_k)+\sigma(p_{k+1})) \xi \left(1 - e^{i\sigma(p_{k+1})b_{k,k+1}}\right) A_\sigma = 0 \] (21)
We now note that $x_j$ and $\xi$ are arbitrary and the coefficient of $\xi$ is symmetric with respect to the interchange of $p_k$ and $p_{k+1}$. This interchange is effected by the element $\sigma_k$ of the permutation group. Thus symmetrizing with respect to this interchange, we obtain
\[ \left(1 - e^{i\sigma(p_{k+1})b_{k,k+1}}\right) A_\sigma + \left(1 - e^{i\sigma(p_k)b_{k,k+1}}\right) A_{\sigma\sigma_k} = 0 \] (22)
where $\sigma\sigma_k$ is the product of the elements $\sigma$ and $\sigma_k$ in the permutation group and $A_{\sigma\sigma_k}$ is the corresponding coefficient. Thus we obtain
\[ A_{\sigma\sigma_k} = S_{k,k+1}(\sigma(p_k),\sigma(p_{k+1})) A_\sigma \] (23)
where
\[ S_{k,k+1}(z_1,z_2) = 1 \otimes \cdots \otimes S(z_1,z_2) \otimes \cdots \otimes 1 \] (24)
and
\[ S(z_1,z_2) = -(1 - z_1 b)^{-1}(1 - z_2 b) \] (25)
Here we have denoted $e^{ip_k}$ by $z_k$. $S(z_1,z_2)$ is the two particle S-matrix. The above equation allows all the $A_\sigma$’s to be calculated recursively in terms of $A_1$. The first few members of the permutation group are 1, $\sigma_1$, $\sigma_2$, $\sigma_1\sigma_2$, $\sigma_2\sigma_1$ and $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Accordingly we find from (23)
\[ A_{\sigma_1} = S_{12}(p_1,p_2)A_1 \quad A_{\sigma_2} = S_{23}(p_2,p_3)A_1 \quad A_{\sigma_1\sigma_2} = S_{23}(p_1,p_3)A_{\sigma_1} \] (26)
\[ A_{\sigma_2\sigma_1} = S_{12}(p_1,p_3)A_{\sigma_2} \quad A_{\sigma_1\sigma_2\sigma_1} = S_{12}(p_2,p_3)A_{\sigma_1\sigma_2}. \] (27)
Moreover we have
\[ A_{\sigma_2\sigma_1\sigma_2} = S_{23}(p_1,p_2)A_{\sigma_2\sigma_1} \] (28)
At this point one may think that the master equation (3) can be solved by Bethe ansatz for any type of matrix $b$ and hence for any type of nearest neighbour interaction. The answer is
that we are not yet done with the Bethe ansatz solution, since for consistency of the solution we should check that different ways of obtaining a coefficient yield the same result. It is sufficient to check this consistency only for two cubic elements of the permutation group \( [20] \), namely \( \sigma_1 \sigma_2 \sigma_1 \) and \( \sigma_2 \sigma_1 \sigma_2 \) which are equal as elements of this group. Thus we should have

\[
A_{\sigma_1 \sigma_2 \sigma_1} = A_{\sigma_2 \sigma_1 \sigma_2}
\]  

This yields via (26) - (28) the following Yang-Baxter equation:

\[
S_{12}(p_2, p_3)S_{23}(p_1, p_3)S_{12}(p_1, p_2) = S_{23}(p_1, p_2)S_{12}(p_1, p_3)S_{23}(p_2, p_3)
\]  

which is crucial for consistency of the method.

**Remark:** With the transformation \( R := PS \), where \( P \) is the permutation operator, we obtain the more familiar form:

\[
R_{23}(p_2, p_3)R_{13}(p_1, p_3)R_{12}(p_1, p_2) = R_{12}(p_1, p_2)R_{13}(p_1, p_3)R_{23}(p_2, p_3)
\]  

We can calculate the S-matrix from (18) and (25). For example in the two-species case we find

\[
S(z, w) = \begin{pmatrix}
\frac{1-v_1 w}{v_1 z-1} & \cdots & \frac{1-v_1 w}{v_1 z-1} & \frac{w-z}{v_2 (v_2 - v_1)} \\
\frac{w-z}{v_2 (v_2 - v_1)} & \ddots & \frac{1-v_1 w}{v_1 z-1} & \cdots \\
\frac{1-v_1 w}{v_1 z-1} & \cdots & \frac{1-v_1 w}{v_1 z-1} & \frac{w-z}{v_2 (v_2 - v_1)} \\
\frac{w-z}{v_2 (v_2 - v_1)} & \cdots & \frac{1-v_1 w}{v_1 z-1} & \ddots
\end{pmatrix}
\]  

Note that the last diagonal element is different from the others. We have checked that this matrix satisfies exactly equation (30), although its proof requires lengthy calculations which we will not reproduce here. It has moreover the following properties

1- In the special case where \( v_1 = v_2 \) we have \( S(z, w) = \frac{1-w}{z-1}1 \) which is nothing but the S-matrix found in \([17]\) for the single species case. Note that in the latter case the S-matrix is a c-number and (30) is satisfied trivially.

2- \( S^{-1}(z, w) = S(w, z) \)

3- \( S(z, z) = 1 \quad \forall \quad v_1, v_2 \)

One can easily obtain from (18) and (25), the S-matrix for higher values of \( p \). The final result is

\[
S(z, w) = \sum_{i,j}^{p} \frac{1-\tilde{v}_{ij} w}{\tilde{v}_{ij} z-1} E_{ii} \otimes E_{jj} + \sum_{i<j}^{p} \frac{v_i}{v_j} (v_j - v_i) \frac{w-z}{(1-v_i z)^2} E_{ij} \otimes E_{ji}
\]

where \( \tilde{v}_{ij} = v_{\min(i,j)} \).

In conclusion we have provided an example of a multi-species stochastic process (the number of species being arbitrary), which is both static-MPA solvable (see the introduction) and integrable on an infinite chain. Integrability of the model is ensured by testing a very
stringent condition on the S-matrix, namely the 2-particle S-matrix satisfies the Quantum Yang-Baxter equation. This solution is to our knowledge a new solution of QYBE. Given these facts, it would be quite interesting to see if this process is also dynamic-MPA solvable. At a concrete level and for sectors of low number of particles, one can use the Bethe wave functions to calculate the same quantities that were calculated in [18], namely the diffusion and drift constant, and see what is the effect of overtaking on these characteristics. Questions specific to the multispecies case can also be answered. For example, given an initial situation of particle 2, \( l \) steps behind particle 1, what is the probability that at a given time particle 2 overtakes particle 1, or what is the average passing time.

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