Abstract. Given a Calabi-Yau manifold $X$ acted by a group $G$ and considering the $B$-branes on $X$ as objects in the derived category of coherent sheaves, we give a definition of equivariant branes, which generalizes the concept of equivariant sheaves. We also propose a definition of equivariant charge of an equivariant brane. The spaces of strings joining the branes $\mathcal{F}$ and $\mathcal{G}$, are the groups $\text{Ext}^i(\mathcal{F}, \mathcal{G})$. We prove that the spaces of strings between two $G$-equivariant branes support representations of $G$. Thus, these spaces can be decomposed in direct sum of invariant spaces for the $G$-action. We show some particular decompositions, when $X$ is a toric variety and when $X$ is a flag manifold of a semisimple Lie group.

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1. Introduction

As it is known, a $D$-brane of type $B$ in a Calabi-Yau manifold $X$ is an object of the derived category of coherent sheaves on $X$ [1, 2, 3, 11, 21, 28]. In this note we will consider such objects in manifolds acted by a Lie group $G$.

Given a $G$-manifold $X$, some objects related with $X$ admit an “equivariant” version. For example, the equivariant vector bundles on $X$ are vector bundles equipped with a structure compatible with the $G$-action on the base. In the same way, it is natural to consider “equivariant” $B$-branes on $X$. In this article, we deal with equivariant branes on a $G$-manifold, with the spaces of open strings connecting them and we will relate these spaces with representations of the group $G$.

Henceforth, the space $X$ will be a Kähler $G$-manifold, and we put $\mathcal{O}_X$ for the corresponding structure sheaf. By $\textbf{D}(\mathcal{O}_X)$ we denote the bounded derived category of coherent $\mathcal{O}_X$-modules [20]. In the context mentioned above, given the $B$-branes $\mathcal{F}$ and $\mathcal{G}$, that are objects of $\textbf{D}(\mathcal{O}_X)$, an open string between $\mathcal{F}$ and $\mathcal{G}$ is an element of the Ext group $\text{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ [11, 30], where $i + \text{ghost number of } \mathcal{G} –$ 

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ghost number of $\mathcal{F}$ can be considered as the ghost number of the corresponding strings.

We denote by $\mu : G \times X \to X$ an analytic action of a reductive Lie group $G$ on $X$. Essentially, a $G$-equivariant structure on the $\mathcal{O}_X$-module $\mathcal{H}$ is given by a family $\{\alpha_{g,x}\}$ of isomorphisms between the stalks \[
\alpha_{g,x} : \mathcal{H}_x \to \mathcal{H}_{\mu(g,x)}, \quad \text{for all } g \in G, \; x \in X
\] compatible with the multiplication in $G$.

For a precise definition, we introduce the map $b : (g, x) \in G \times X \mapsto x \in X$. A $G$-equivariant $\mathcal{O}_X$-module is a pair $(\mathcal{H}, \alpha)$, where $\mathcal{H}$ is an $\mathcal{O}_X$-module and $\alpha$ is an isomorphism
\[
\alpha : b^*\mathcal{H} \to \mu^*\mathcal{H}
\]
of $\mathcal{O}_{G \times X}$-modules, where $b^*$ and $\mu^*$ are the functors inverse image defined by the respective maps [13, page 136]. Furthermore, $\alpha$ must satisfy the cocycle condition (see [6, page 2] and equation (2.2) below).

The same definition of $G$-equivariance is applicable to an object $A$ of the derived category $D(\mathcal{O}_X)$, now $b^*$ and $\mu^*$ are functors from $D(\mathcal{O}_X)$ to the derived category of $\mathcal{O}_{G \times X}$-modules (see Subsection 2.1).

We will put $j : S \hookrightarrow X$ for the inclusion of an open subset $S$ of $X$, and $j!$ will denote the corresponding functor direct image with compact support [20, page 103].

In Section 2, we will prove the following theorems.

**Theorem 1.** If $(\mathcal{H}, \alpha)$ is a $G$-equivariant coherent sheaf on $X$ and $S$ is a $G$-invariant open subset of $X$, then the isomorphism $\alpha$ determines a representation of $G$ on $\text{St}^i(j!(\mathcal{O}_S), \mathcal{H})$, all $i$. In particular, each space $\text{St}^i(\mathcal{O}_X, \mathcal{H})$ carries a representation of $G$ induced by $\alpha$.

**Theorem 2.** If $(\mathcal{G}, \beta)$ and $(\mathcal{F}, \gamma)$ are $G$-equivariant objects of the category $D(\mathcal{O}_X)$, then the isomorphisms $\beta$ and $\gamma$ determine a representation of $G$ on $\text{St}^i(\mathcal{F}, \mathcal{G})$, in a natural way.

In the particular case that $\mathcal{F}$ and $\mathcal{G}$ are the sheaves of sections of $G$-equivariant vector bundles, the action $g \in G$ on a morphism \[
\Phi \in \text{St}^0(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})
\]is given by the natural representation $(g \cdot \Phi)(-)=g\Phi(g^{-1}(-))$.

The decomposition of the representations of compact groups in direct sum of irreducible representations permits to classify the elements of a given space $\text{St}^i(\mathcal{F}, \mathcal{G})$ in subspaces, which can be labelled by the characters of the corresponding irreducible representations. Thus, we have the following theorem.
Theorem 3. Let $G$ be a compact group $G$. If $\mathcal{F}$ and $\mathcal{G}$ are $G$-equivariant objects of $D(X)$ then for each $i$,

\[(1.2) \quad St^i(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_A n_A A,\]

where the sum runs over a complete set of pairwise nonisomorphic representations of $G$, $n_A$ is a natural number and $n_A A$ is the direct sum of $n_A$ summands of the irreducible representation $A$.

In Section 2, we give an equivariant version of the charge of a $G$-equivariant brane, that coincides with the usual one, when the group $G$ is trivial (see Subsection 2.4). This equivariant charge can be evaluated using the localization formulas in equivariant cohomology [4].

Examples manifolds in which a group action is an important ingredient of its structure are the coadjoint orbits of a Lie group and the toric manifolds. In Section 3, we will show the form which Theorem 3 adopts in some examples of pairs of $O_X$-modules, when $X$ is a toric manifold and when $X$ is a flag manifold of a semisimple group.

Notations. Besides the already introduced notations, we also use the following:

The category of complex vector spaces will be denoted by $\mathfrak{Vect}$, and we let $D(\mathfrak{Vect})$ for the corresponding bounded derived category.

Given a locally compact space $Z$, if $\mathcal{R}$ a sheaf of $\mathbb{C}$-algebras on $Z$, the bounded derived category of sheaves on $Z$ which are $\mathcal{R}$-modules is denoted by $D(\mathcal{R})$. As usual, $\Gamma(Z, .)$ will be the functor global sections and we put

\[R\Gamma(Z, .) : D(\mathcal{R}) \to D(\mathfrak{Vect})\]

for its derived [20, 30]. The composition of this functor with the cohomology functor $H^i$ is denoted by $R^i\Gamma(Z, .)$

\[R^i\Gamma(Z, .) : D(\mathcal{R}) \to \mathfrak{Vect}.\]

If $f : Y \to Z$ is a continuous map,

\[Rf_* : D(f^*\mathcal{R}) \to D(\mathcal{R})\]

will denote the derived functor of the direct image functor $f_*$. In general, if $Z$ is a ringed space the structure sheaf will be denoted by $\mathcal{O}_Z$.

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2. Proofs of the results

2.1. The cocycle condition. To formulate the cocycle condition, above mentioned, we introduce the following notations

\[ m : G \times G \rightarrow G, \quad m(g_1, g_2) = g_1g_2. \]

\[ p : G \times G \times X \rightarrow G \times X, \quad p(g_1, g_2, x) = (g_2, x). \]

Thus, one has the maps \( p, m \times 1_X \) and \( 1_G \times \mu \) from \( G \times G \times X \) to \( G \times X \) and the corresponding functors

\[
\begin{align*}
D(O_X) \xrightarrow{b^*} & D(O_{G \times X}) \xrightarrow{p^*} D(O_{G \times G \times X}) \\
& \xrightarrow{(m \times 1_X)^*} \xrightarrow{(1_G \times \mu)^*} \xrightarrow{(1_G \times \mu)^*}
\end{align*}
\]

where an asterisk as superscript is used for denoting the inverse image functor between the corresponding derived categories. The equalities

\[ b \circ (m \times 1_X) = b \circ p, \quad b \circ (1_G \times \mu) = \mu \circ p, \quad \mu \circ (1_G \times \mu) = \mu \circ (m \times 1_X) \]

give rise to equalities between the respective compositions of the functors in (2.1).

Given an object \( A \) of the category \( D(O_X) \), an isomorphism \( \alpha : b^*A \rightarrow \mu^*A \) satisfies the cocycle condition if

\[
(m \times 1_X)^*(\alpha) = (1_G \times \mu)^*(\alpha) \circ p^*(\alpha).
\]

In this case, we say that the pair \((A, \alpha)\) is an \( G \)-equivariant object.

Both sides of equation (2.2) are isomorphisms between two objects on the derived category of \( O_{G \times G \times X} \)-modules; more precisely, between the objects \( d_0^*A \) and \( d^*A \), where \( d, d_0 : G \times G \times X \rightarrow X \) are defined by \( d_0(g_1, g_2, x) = x \) and \( d(g_1, g_2, x) = (g_1g_2)x \). In other words, the cocycle condition means the commutativity of the following triangle

\[
\begin{align*}
& Z_1 \xrightarrow{p^*(\alpha)} Z_2 \\
& \downarrow^{(m \times 1_X)^*(\alpha)} \quad \downarrow^{(1_G \times \mu)^*(\alpha)} \quad \downarrow^{Z_3,}
\end{align*}
\]

where

\[
\begin{align*}
Z_1 := p^*b^*(A) = (m \times 1_X)^*b^*(A), \quad Z_2 := p^*\mu^*(A) = (1_G \times \mu)^*b^*(A) \\
Z_3 := (m \times 1_X)^*\mu^*(A) = (1_G \times \mu)^*\mu^*(A).
\end{align*}
\]
2.2. **Equivariant sheaves.** Now we suppose that the $G$-equivariant object $A$ is a coherent sheaf $\mathcal{H}$ on $X$. One can define the category $\text{Coh}^G(X)$, whose objects are the $G$-equivariant coherent sheaves on $X$. If $(\mathcal{H}', \alpha')$ and $(\mathcal{H}, \alpha)$ are objects in this category, a morphism in $\text{Coh}^G(X)$ from $(\mathcal{H}', \alpha')$ to $(\mathcal{H}, \alpha)$ is a sheaf morphism $f : \mathcal{H}' \to \mathcal{H}$ such that $ab^*(f) = \mu^*(f)\alpha'$.

Given an open subset $U \subset X$ and $g \in G$, we put $U_g := \{g\} \times U \subset G \times X$. If $(\mathcal{H}, \alpha)$ is an object of $\text{Coh}^G(X)$, the restriction to $U_g$ of the morphism of sheaves $\alpha$ is denoted $\alpha|_{U_g}$.

\[(\alpha|_{U_g} : b^*\mathcal{H}(U_g) = \mathcal{H}(U) \longrightarrow \mu^*\mathcal{H}(U_g) = \mathcal{H}(gU))\]

Hence, one has the following proposition.

**Proposition 4.** $\alpha|_{U_g}$ determines an isomorphism of complex vector spaces

\[(\mathcal{H}(U) \sim \mathcal{H}(gU))\]

The image of $\sigma_U \in \mathcal{H}(U)$ will be denoted $g \cdot \sigma_U$. Thus, the element $g \in G$ determines an isomorphism

\[(\alpha_{g,x} : \mathcal{H}_x \to \mathcal{H}_{g,x})\]

for any $x \in X$.

**Lemma 5.** Let $g, h$ be elements of $G$ and $U$ an open set of $X$, then

\[\alpha|_{(gU)} \circ \alpha|_{U_g} = \alpha|_{U_{hg}}.\]

**Proof.** We consider the commutative triangle (2.3) when $A$ is the sheaf $\mathcal{H}$ and we restrict this triangle to $\{h\} \times U_g \subset G \times G \times X$. The restriction of $(m \times 1_X)^*(\alpha)$ is the morphism

\[Z_1(\{h\} \times U_g) = \mathcal{H}(U) \longrightarrow Z_2(\{h\} \times U_g) = \mathcal{H}((hg)U)\]

induced by $\alpha$. Thus, by (2.4), the mentioned restriction is $\alpha|_{U_{hg}}$.

The restriction of $p^*(\alpha)$ to $\{h\} \times U_g$

\[Z_1(\{h\} \times U_g) = \mathcal{H}(U) \longrightarrow Z_2(\{h\} \times U_g) = \mathcal{H}(gU)\]

is (2.3).

Finally, we consider the restriction of $(1_G \times \mu)^*(\alpha)$. It is the morphism

\[Z_2(\{h\} \times U_g) = \mathcal{H}(gU) \longrightarrow Z_2(\{h\} \times U_g) = \mathcal{H}(h(gU))\]

induced by $\alpha$, and according to (2.4) it is $\alpha|_{(gU)}$. Then the lemma follows from the commutativity of (2.3). \qed

As a direct consequence of Lemma

\[(\alpha_{h,gx} \circ \alpha_{g,x} = \alpha_{hg,x})\]
Since \( gX = X \), \( \alpha \big|_{X_g} \) is an automorphism of the vector space \( \mathcal{H}(X) \), from the Lemma, one deduces

\[
(2.8) \quad \alpha \big|_{X_{kh}} = \alpha \big|_{X_{hg}}.
\]

The following proposition is a consequence from (2.8).

**Proposition 6.** The automorphisms \( \{ \rho_g := \alpha \big|_{X_g} \}_g \) form a representation of \( G \) in the vector space \( \mathcal{H}(X) \).

**Lemma 7.** Given \((\mathcal{H}, \alpha)\) an object of \( \text{Coh}^G(X) \), there is a resolution of \((\mathcal{H}, \alpha)\) in \( \text{Coh}^G(X) \) consisting of injective \( \mathcal{O}_X \)-modules.

**Proof.** Given a point \( x \in X \), \( \mathcal{H}_x \) is a \( \mathbb{C} \)-vector space, so is an injective \( \mathbb{Z} \)-module (i.e. a divisible abelian group). We set \( I_x := \text{Hom}_{\mathbb{Z}}(\mathcal{O}_x, \mathcal{H}_x) \), where \( \mathcal{O}_x \) is the stalk of the sheaf \( \mathcal{O}_X \) at the point \( x \). Then the inclusion \( \mathcal{H}_x \hookrightarrow I_x \) is an embedding of the \( \mathcal{O}_x \)-module \( \mathcal{H}_x \) in an injective \( \mathcal{O}_x \)-module (see [24, III.7], [27, page 123]).

By means of the \( I_x \) one can construct an injective \( \mathcal{O}_X \)-module \( \mathcal{J} \) in which \( \mathcal{H} \) can be embedded,

\[
\mathcal{J} = \prod_{x \in X} (j_x)_*(I_x),
\]

where \( j_x : \{x\} \hookrightarrow X \) (see [17, page 207]).

As it is well-known, \( \mathcal{H} \hookrightarrow \mathcal{J} \) is the 0th-term of an injective resolution of the \( \mathcal{O}_X \)-module \( \mathcal{H} \). As \( \mathcal{H} \) is \( G \)-equivariant, by (2.6), for each \( g \in G \), there is an isomorphism of \( \mathbb{Z} \)-modules \( \mathcal{H}_x \rightarrow \mathcal{H}_{gx} \). Since the \( G \)-action on \( X \) is analytic, one has an isomorphism \( \mathcal{O}_{gx} \rightarrow \mathcal{O}_x \). So, we have an isomorphism \( I_x \rightarrow I_{gx} \). Thus, for any open subset \( U \subset X \), there is an isomorphism of \( \mathbb{C} \)-vector spaces \( \mathcal{J}(U) \rightarrow \mathcal{J}(gU) \), making commutative the following diagram, where the horizontal sequences are exact

\[
\begin{array}{ccc}
0 & \rightarrow & \mathcal{H}(U) \\
\alpha_{|U} \downarrow & & \downarrow i_U \\
0 & \rightarrow & \mathcal{H}(gU) \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{J}(U) & \rightarrow & \mathcal{J}(U) \\
\downarrow & & \downarrow \\
\mathcal{J}(gU) & \rightarrow & \mathcal{J}(gU)
\end{array}
\]

The isomorphisms \( \mathcal{J}(U) \rightarrow \mathcal{J}(gU) \) give rise to an isomorphism of \( \mathcal{O}_X \)-modules, \( \alpha^0 : b^* \mathcal{J} \rightarrow \mu^* \mathcal{J} \).

From (2.7), it follows the equality of \( \mathcal{J}(U) \rightarrow \mathcal{J}(gU) \rightarrow \mathcal{J}(h(gU)) \) and \( \mathcal{J}(U) \rightarrow \mathcal{J}((hg)U) \). That is, \( \alpha^0 \) satisfies the cocycle condition. Hence, \( \mathcal{J} \) is an object of \( \text{Coh}^G(X) \). By the commutativity of (2.9), it follows that \( i : \mathcal{H} \rightarrow \mathcal{J} \) a morphism in that category.
Diagram (2.9) can be continued with the cokernels of \(i_U\) and \(i_{gU}\). We denote \(C(V) = J(V)/H(V)\) and put \(C^+\) for denoting the sheaf associated to the presheaf \(C\). There are isomorphisms of \(\mathbb{C}\)-vector spaces induced canonically between the vector spaces of the following commutative diagram.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H(U) & \xrightarrow{i_U} & J(U) & \xrightarrow{} & C(U) & \xrightarrow{} & C^+(U) \\
\alpha_{|U}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H(gU) & \xrightarrow{i_{gU}} & J(gU) & \xrightarrow{} & C(gU) & \xrightarrow{} & C^+(gU)
\end{array}
\]

The term \(J^1\) of an injective resolution of \(H\) can be obtained from \(C^+\), by embedding \(C^+\) in an injective object, as we have made with \(H\). Hence, there exists an isomorphism \(\alpha^1\) of \(\mathcal{O}_{G\times X}\)-modules, making commutative the following diagram, where we put \(J^0\) for the preceding \(J\).

\[
\begin{array}{cccccccc}
0 & \rightarrow & b^*H & \xrightarrow{b^*i} & b^*J^0 & \xrightarrow{b^*\partial^0} & b^*J^1 \\
\alpha\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mu^*H & \xrightarrow{\mu^*i} & \mu^*J^0 & \xrightarrow{\mu^*\partial^0} & \mu^*J^1
\end{array}
\]

Continuing the process, we obtain a complex \(J^*\) which satisfies the Lemma.

**Proposition 8.** If \((H, \alpha)\) is a \(G\)-equivariant \(\mathcal{O}_X\)-module, then for each \(i\) the cohomology group \(H^i(X; H)\) supports a representation of \(G\) induced by the isomorphism \(\alpha\).

**Proof.** Let \(J^*\) the injective resolution of \(H\) constructed in Lemma 7. As \(J^i\) is \(G\)-equivariant, by Proposition the space the \(J^i(X)\) carries the representation \(\rho^i\) of \(G\) defined by \(\rho^i_g = \alpha^i|X_g\). Since the diagrams

\[
\begin{array}{ccc}
J^i(X) & \xrightarrow{d^i} & J^{i+1}(X) \\
\rho^i_g\downarrow & & \downarrow\rho^{i+1}_g \\
J^i(X) & \xrightarrow{d^i} & J^{i+1}(X)
\end{array}
\]

are commutative, one has a representation of \(G\) on each cohomology group \(h^i(J^*(X))\) of the complex \(J^*(X)\). As \(J^*\) is an injective resolution of the \(\mathcal{O}_X\)-module \(H\) and \(H^i(X, H)\) is by definition the cohomology group \(h^i(\Gamma(X, J^*))\), the proposition follows. \(\square\)
The arguments given in the proof of Proposition 8 are also valid when $X$ is substituted by a $G$-invariant open subset $S \xrightarrow{j} X$, thus $H^i(S, \mathcal{H})$ also carries a representation of $G$.

**Proof of Theorem 1.** As we said, let $j_!$ for the corresponding functor direct image with compact support. The theorem follows from the above observation together with the following identities

$$H^i(S, \mathcal{H}) = R^i\Gamma(S, \mathcal{H}) = \text{Ext}^i_{\mathcal{O}_X}(j_!(\mathcal{O}_S), \mathcal{H}).$$

□

Let $(\mathcal{F}, \gamma)$, $(\mathcal{G}, \beta)$ be $G$-equivariant $\mathcal{O}_X$-modules. By Proposition 8, $\mathcal{F}(X)$ and $\mathcal{G}(X)$ support representations of $G$. We put $\mathcal{K} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the sheaf of homomorphisms from $\mathcal{F}$ to $\mathcal{G}$. Given an open subset $U \subset X$, for $\Phi_U \in \mathcal{K}(U)$ and $g \in G$, we define $g \cdot \Phi_U \in \mathcal{K}(gU)$ as follows:

Given $\sigma \in \mathcal{F}(gU)$, then $g^{-1} \cdot \sigma \in \mathcal{F}(U)$, with the notation introduced in Proposition 4. Then $\Phi_U(g^{-1} \cdot \sigma) \in \mathcal{G}(U)$ and we put

$$\text{(2.13)} \quad (g \cdot \Phi_U)(\sigma) := g \cdot (\Phi(g^{-1} \cdot \sigma)).$$

So, we have constructed an isomorphism

$$\eta|_{U_g} : \mathcal{K}(U) \rightarrow \mathcal{K}(gU), \quad \Phi_U \mapsto g \cdot \Phi_U.$$

Moreover,

$$\eta|_{(gU)_h} \circ \eta|_{U_g} = \eta|_{U_{hg}}.$$

Therefore, the isomorphisms $\{\eta|_{X_g}\}_{g}$ define a representation of $G$ on the space $\mathcal{K}(X)$. That is,

**Proposition 9.** Let $(\mathcal{F}, \gamma)$, $(\mathcal{G}, \beta)$ be $G$-equivariant $\mathcal{O}_X$-modules, then $\mathcal{K}(X)$, with $\mathcal{K} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, supports a representation of $G$ induced by the isomorphisms $\gamma$ and $\beta$.

2.3. **Equivariant complexes.** The results of Subsection 2.2 can be generalized to case when $\mathcal{F}$ and $\mathcal{G}$ are $G$-equivariant objects of the category $\mathcal{D}(\mathcal{O}_X)$, the derived category of coherent sheaves on $X$. As we said, a $G$-equivariant object of $\mathcal{D}(\mathcal{O}_X)$ is a pair $(\mathcal{A}, \alpha)$ consisting of an object $\mathcal{A}$ of $\mathcal{D}(X)$ and an isomorphism in the derived category of $\mathcal{O}_{G \times X}$-modules, which satisfies (2.2).

Given $g \in G$, and open subset $U$ of $X$, we denote by $L_g$ the diffeomorphism

$$L_g : (g, x) \in U_g \mapsto (g, gx) \in (gU)_g.$$

Thus, $b \circ L_g = \mu : U_g \rightarrow gU$. As $L_g$ is a diffeomorphism the functors $L_g^*$ and $(L_g^{-1})^*$ are the same, one has the following relations among objects
of the derived category $D(\mathcal{V}ect)$, when $(\mathcal{A}, \alpha)$ is an equivariant object. 

$$R\Gamma(U, \mathcal{A}) = R\Gamma(U_g, b^*\mathcal{A}) \simeq R\Gamma(U_g, \mu^*\mathcal{A}) = R\Gamma(U_g, L_g^*b^*\mathcal{A})$$

$$= R\Gamma(U_g, R(L_g^{-1}), b^*\mathcal{A}) = R\Gamma((gU)_g, b^*\mathcal{A}) = R\Gamma(gU, \mathcal{A}).$$

That is, we have an isomorphism $R\Gamma(U, \mathcal{A}) \xrightarrow{\sim} R\Gamma(gU, \mathcal{A})$. In particular, when $U = X$, for any $i$ there is an isomorphism

$$R\Gamma(X, \mathcal{A}) \xrightarrow{\hat{r}_g} R\Gamma(X, \mathcal{A}).$$

By the cocycle condition $\hat{r}_{hg} = \hat{r}_h \circ \hat{r}_g$.

On the other hand, as $\alpha$ is an isomorphism between two complexes, the $\alpha^i$'s intertwine with the boundary operators; so, the representation $\hat{r}$ induces a representation $r$ on each space $R^i\Gamma(X, \mathcal{A})$. Since

$$R^i\Gamma(X, \mathcal{A}) = Ext^i(O_X, \mathcal{A}) = H^i(X, \mathcal{A}),$$

$H^i(X, \mathcal{A})$ carries a representation of $G$ induced by $\alpha$. This result is a generalization of Proposition 6.

**$G$-equivariant Horseshoe lemma.** The well-known Horseshoe lemma (see [27, page 349], [31, page 37]) admits a $G$-equivariant version, which can be proved following the steps of the proof for the no-equivariant case. This equivariant version can be formulated as follows:

Let $(\mathcal{H}', \alpha')$, $(\mathcal{H}, \alpha)$ and $(\mathcal{H}'', \alpha'')$ be $G$-equivariant $O_X$-modules and

$$0 \to \mathcal{H}' \to \mathcal{H} \to \mathcal{H}'' \to 0$$

be a short exact sequence in the category $\text{Coh}^G(X)$. If $\mathcal{H}' \to \mathcal{J}'$ and $\mathcal{H}'' \to \mathcal{J}''$ are resolutions of $\mathcal{H}'$ and $\mathcal{H}''$ (resp.) in $\text{Coh}^G(X)$ consisting of injective $O_X$-modules. Then there are a resolution $\mathcal{J}'$ of $(\mathcal{H}, \alpha)$ in $\text{Coh}^G(X)$, formed by injective $O_X$-modules, and morphisms between the resolutions such that

$$0 \to \mathcal{J}' \to \mathcal{J} \to \mathcal{J}'' \to 0$$

is an exact sequence of complexes in $\text{Coh}^G(X)$.

**Proof of Theorem 2.** Since $(\mathcal{G}, \beta) = (\{\mathcal{G}^\bullet\}, \{\beta^\bullet\})$ is a $G$-equivariant object of $D(X)$, the $O_X$-module $\mathcal{G}^j$ together with $\beta^j$ is a $G$-equivariant $O_X$-module. We denote by $\partial^j : \mathcal{G}^j \to \mathcal{G}^{j+1}$ the corresponding boundary operator, as $b^*$ and $\mu^*$ are exact functors, $\ker \partial^j$, $\text{im} \partial^j$ and the cohomology $h^j(\mathcal{G}^\bullet)$ are $G$-equivariant $O_X$-modules.

According to Lemma 7 there are resolutions in $\text{Coh}^G(X)$ consisting of injective $O_X$-modules for $h^j(\mathcal{G}^\bullet)$ and $\text{im} \partial^{j-1}$. By the $G$-equivariant Horseshoe lemma applied to the exact sequence

$$0 \to \text{im} \partial^{j-1} \to \ker \partial^j \to h^j(\mathcal{G}^\bullet) \to 0,$$
there is a resolution for \( \ker \partial^j \) satisfying the properties above stated. A new application of the equivariant Horseshoe lemma to the exact sequence

\[
0 \to \ker \partial^j \to \mathcal{G}^j \to \text{im} \partial^j \to 0,
\]

permits the construction of a \( G \)-equivariant Cartan-Eilenberg resolution \( \mathcal{J}^{\bullet \bullet} \) of the complex \( \mathcal{G}^\bullet \), in which each term is an injective \( G \)-equivariant \( \mathcal{O}_X \)-module (see [27, Theorem 10.45]).

The isomorphism between the complexes \( b^* \mathcal{F} \) and \( \mu^* \mathcal{F} \) implies that the representations on \( \mathcal{F}^a(X) \) and on \( \mathcal{F}^{a+1}(X) \) satisfies \( \partial^a(g \cdot \sigma) = g \cdot (\partial^a \sigma) \), for all \( \sigma \in \mathcal{F}^a(X) \). The total complex \( \mathcal{I} = \text{Tot}(\mathcal{J}^{\bullet \bullet}) \) is a complex in \( \text{Coh}^G(X) \) quasi-isomorphic to \( \mathcal{G}^\bullet \) and it is formed by injective \( \mathcal{O}_X \)-modules. Thus, by Proposition 6, each vector space \( \mathcal{I}^i(X) \) carries a representation of \( G \), which also intertwine with the boundary operators of \( \mathcal{I}^\bullet(X) \).

Since \( (\mathcal{F}, \gamma) \) is a \( G \)-equivariant object of \( \mathcal{D}(X) \), from the argument before Proposition 9, it follows that \( \text{Hom}_{\mathcal{O}_X(X)}(\mathcal{F}^a(X), \mathcal{I}^b(X)) \) supports a representation of \( G \). So,

\[
C^n := \prod_a \text{Hom}_{\mathcal{O}_X(X)}(\mathcal{F}^a(X), \mathcal{I}^{a+n}(X)).
\]

(2.14)

This carries also a representation \( \rho \) of \( G \). For \( C^\bullet \) one defines the following boundary operator [19, page 17]

\[
\delta^n(f_a) = (\partial^{a+n} \circ f_a + (-1)^{a+1} f_{a+1} \circ \partial^a),
\]

(2.15)

where,

\[
f_a \in \text{Hom}_{\mathcal{O}_X(X)}(\mathcal{F}^a(X), \mathcal{I}^{a+n}(X)).
\]

On the other hand, by (2.13), the action of \( g \in G \) on \( f_a \) is given by \( g \cdot (f_a)(\sigma) = g \cdot f_a(g^{-1} \cdot \sigma) \), for all \( \sigma \in \mathcal{F}^a(X) \). Since \( \partial^{a+n} \circ (g \cdot f_a(g^{-1} \cdot \sigma)) = g \cdot (\partial^{a+n}(f_a(g^{-1} \cdot \sigma))) \), \( \partial^n(g^{-1} \cdot \sigma) = g^{-1} \cdot \partial^n(\sigma) \), then \( \delta^n((g \cdot (f_a))) = g \cdot \delta^n(f_a) \). Hence, \( \rho \) induces a representation \( r \) of \( G \) on the cohomology of \( C^\bullet \).

On the other hand, the functor

\[
\text{RHom}(\_ , \_ ) : \mathcal{D}(X) \times \mathcal{D}(X) \to \mathcal{D}(\mathfrak{Vect})
\]

assigns to the pair \( (\mathcal{F}, \mathcal{G}) \) the object represented by the complex \( C^\bullet \). As \( \text{Ext}^i(\mathcal{F}, \mathcal{G}) = h^i(\text{RHom}(\mathcal{F}, \mathcal{G})) \), the representation \( r \) is the one claimed in the statement of the theorem. \( \square \)
2.4. **Equivariant charges.** The charge of a brane $F$ [1] [18] [25] is an element of the cohomology of $X$ defined from certain characteristic classes of $X$ and $F$. For a $G$-equivariant brane, it is natural to define a $G$-equivariant charge through the respective $G$-equivariant characteristic classes. The resulting charge will be an element of the equivariant cohomology $H^*_G(X)$.

Given a $G$-equivariant brane $F \in D(X)$, the complex $F^\bullet$ is quasi-isomorphic to a complex $E^\bullet$ consisting of locally free sheaves; that is, $E^i$ is the sheaf of sections of a $G$-equivariant vector bundle $V^i$.

Each $V^i$ has the corresponding $G$-equivariant Chern character $\text{ch}^G(V^i)$ (see [4, page 212]). We put $\text{ch}^G(E^\bullet)$ for denoting the $\sum_i (-1)^i \text{ch}^G(V^i)$.

On the other hand, one can consider the $G$-equivariant $\hat{A}$-class of $X$, which will be denoted by $\hat{A}^G(X)$. The equivariant charge of the equivariant brane $F$ can be defined by the formula

$$Q^G(F) := \text{ch}^G(E^\bullet) \hat{A}^G(X).$$

Taking into account the relation between the $\hat{A}$-roof class and the Todd class [23, page 231] the above definition coincides, when $G$ is the trivial group, with the one given in [1].

In some particular cases, the preceding definition has a natural interpretation in terms of the index of an elliptic operator. The exterior bundle $\Lambda^*T^*X$ of $X$ with the connection induced by the Levi-Civita connection and the standard Clifford multiplication is a Dirac bundle (see [23, page 114]). This Dirac bundle has associated the corresponding Dirac operator $D^X$. If $G$ acts as a group of isometries of $X$, then $D^X$ is a $G$-operator [23, page 211]; i.e. $D^X$ is $G$-equivariant.

Let us assume that the complex $E^\bullet$ consists of only one nonzero element, $E^0$. The compactness of $G$ allows us to average over the group for obtaining $G$-invariant metrics and $G$-invariant connections on $V^0$. On the other hand, the tensor product of $(\Lambda^*T^*X) \otimes V^0$ is a Dirac bundle (see [23, page 122]) and the corresponding Dirac operator $D$ is also $G$-equivariant, by the $G$-invariance of the metric and the connection. Since $D$ is elliptic, if $X$ is compact, $\ker D$ and $\coker D$ are representations of $G$ of finite dimension. Denoting by $R(G)$ the character ring of $G$, the $G$-index of $D$, $\text{ind}_G(D)$, is an element of $R(G)$. For $g \in G$ the virtual character $\chi(D)(g)$ of $D$ at $g$ is defined by

$$\chi(D)(g) = \text{trace}(g|_{\ker D}) - \text{trace}(g|_{\coker D}).$$
The equivariant index theorem \cite[Chapter 8]{4}, asserts that in a neighborhood of \(0 \in \mathfrak{g} := \text{Lie}(G)\)

\begin{equation}
\chi(D) \circ \exp = \int_X Q^G(\mathcal{E}^0).
\end{equation}

(2.17)

The value of \(\chi(D)(\exp(\xi))\), for \(\xi \in \mathfrak{g}\), can be calculated by the localization formula in equivariant cohomology. The result is the Atiyah-Segal-Singer fixed point formula \cite[23]{4}.

3. PARTICULAR CASES.

In this section, we show the form that the results of Section 2 adopt in some simple cases.

A consequence of the Grothendieck spectral sequence \cite[page 207]{13} is the known Local-to-Global Ext spectral sequence, which allows to determine the \(\text{Ext}\) groups from the sheaves \(\mathcal{E}xt\). Given the \(\mathcal{O}_X\)-modules \(\mathcal{F}\) and \(\mathcal{G}\), the first quadrant spectral sequence

\[ E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \]

abuts to \(\text{Ext}^n(\mathcal{F}, \mathcal{G})\).

Let \(\mathcal{F}\) be a locally free \(\mathcal{O}_X\)-module of finite rank, then \(0 \to \mathcal{F} \to \mathcal{F} \to 0\) is a resolution of \(\mathcal{F}\), which can be used to determine the sheaves \(\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})\) \cite[Proposition 6.5, page 234]{17}. So, the sheaves \(\mathcal{E}xt\) are the cohomology of the trivial complex consisting of the sheaf \(\mathcal{Hom}(\mathcal{F}, \mathcal{G})\) at the position 0 and zeros in the other positions. Thus,

\[ \mathcal{E}xt^0(\mathcal{F}, \mathcal{G}) = \mathcal{Hom}(\mathcal{F}, \mathcal{G}). \]

and \(\mathcal{E}xt^q(\mathcal{F}, \mathcal{G}) = 0\), for \(q \neq 0\). Therefore, the spectral sequence degenerates at the second page and

\[ \text{Ext}^p(\mathcal{F}, \mathcal{G}) = H^p(X, \mathcal{Hom}(\mathcal{F}, \mathcal{G})). \]

Thus, we have the following proposition.

**Proposition 10.** Let \(\mathcal{F}\) be a locally free sheaf of finite rank on \(X\), then for any coherent sheaf \(\mathcal{G}\),

\[ S\ell^q(\mathcal{F}, \mathcal{G}) = H^q(X, \mathcal{F}^\vee \otimes \mathcal{G}), \]

where \(\mathcal{F}^\vee\) is the dual sheaf of \(\mathcal{F}\).

Given an invertible sheaf \(\mathcal{G}\), i.e. a rank 1 locally free \(\mathcal{O}_X\)-module, we put \(\mathcal{G}^n\) for denoting the tensor product \(\mathcal{G}^\otimes m\). The following corollary is a consequence from the proposition together with the fact that \(\mathcal{G}\) is ample (see \cite[Proposition 5.3, page 229]{17}).
Corollary 11. Let $\mathcal{F}$ be a locally free sheaf of finite rank and $\mathcal{G}$ an ample invertible sheaf, then there is an integer $n_0$ such that

$$St^q(\mathcal{F}, \mathcal{G}^n) = 0,$$

for all $q > 0$ and all $n > n_0$.

Let $\mathcal{V}$ and $\mathcal{W}$ be holomorphic vector bundles over $X$ with finite rank. We put $\mathcal{F} := \mathcal{O}(\mathcal{V})$ and $\mathcal{G} := \mathcal{O}(\mathcal{W})$ for denoting the respective sheaves of holomorphic sections. Then $\mathcal{F}$ is a locally free sheaf and by the Proposition 10,

$$St^0(\mathcal{O}(\mathcal{V}), \mathcal{O}(\mathcal{W})) = H^p(X, \mathcal{O}(\mathcal{V}^\vee \otimes \mathcal{W})),$$

where $\mathcal{V}^\vee$ is the dual vector bundle of $\mathcal{V}$.

Let us assume that $\mathcal{V}$ and $\mathcal{W}$ are $G$-equivariant vector bundles on $X$, with $G$ a compact Lie group. We denote by $\chi_\mathcal{V}$ and $\chi_\mathcal{W}$ the characters of the corresponding representations on the spaces of sections. Then $St^0(\mathcal{O}(\mathcal{V}), \mathcal{O}(\mathcal{W}))$ supports the representation with character $\chi := \bar{\chi}_\mathcal{V} \chi_\mathcal{W}$. If we write $\chi = \sum_k n_k \chi_k$, where the $\chi_k$ are characters of a complete family of irreducible representations of $G$, then the open string states with ghost number 0, between two branes wrapped on the whole $X$ with gauge bundles $\mathcal{V}$ and $\mathcal{W}$, can be expressed as the following sum direct of $G$-invariant subspaces

$$St^0(\mathcal{O}(\mathcal{V}), \mathcal{O}(\mathcal{W})) \simeq \bigoplus_k n_k B_k,$$

where $B_k$ is a subspace on which the representation of $G$ has character $\chi_k$. The natural number $n_k$ before the subspace $B_k$ means the direct sum of $n_k$ summands equal to $B_k$.

3.1. Flag manifolds. When $X$ is a flag manifold of semisimple group, the result stated in Theorem 3 admits, for certain spaces of strings, a more precise formulation derived from the Borel-Bott-Weil theorem (see [7, 32], for a brief exposition [22, pages 13-22]).

We remind some basic facts about flag manifolds; for details see [8].

Let $\mathfrak{g}$ be the Lie algebra of a linear connected semisimple complex Lie group $G_C$. We assume that a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ has been fixed. If $\mathfrak{v}$, an $\text{ad}(\mathfrak{h})$-invariant subspace of $\mathfrak{g}$, then the set roots of $\mathfrak{h}$ in $\mathfrak{v}$ will be denoted by $\Delta(\mathfrak{v})$. Given a system $\Delta^+ \subset \Delta(\mathfrak{g})$ of positive roots, the parabolic subalgebras of $\mathfrak{g}$ can be constructed as follows: If $\Gamma$ a set of simple positive roots, we put $\Delta(\Gamma) = \text{Span}_\mathbb{Z}(\Gamma) \cap \Delta(\mathfrak{g})$. The set $\Gamma$ determines the parabolic subalgebra $\mathfrak{p} := \mathfrak{l} \oplus \mathfrak{u}$, where

$$\mathfrak{l} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\Gamma)} \mathfrak{g}_\alpha, \quad \text{and} \quad \mathfrak{u} := \bigoplus_{\alpha \in \Delta^+ \setminus \Delta(\Gamma)} \mathfrak{g}_\alpha.$$
When $\Gamma = \emptyset$, the corresponding parabolic algebra is the Borel subalgebra determined by $\Delta^+$. The parabolic subgroup $P$ associated with the algebra $\mathfrak{p}$ can be expressed $P = L_C U_C$ (Levi decomposition), where $L_C \cap U_C = \{1\}$, $L_C$ is a reductive group and $U_C$ is nilpotent.

Henceforth in this Subsection, we assume that a parabolic subgroup $P$ of $G_C$ has been fixed. As $G_C$ is connected, the normalizer $N_{G_C}(P)$ coincide with $P$. Hence, the flag variety $X = G_C/P$ can be identified with the set of all the parabolic subalgebras which are $G_C$-conjugated to $\mathfrak{p}$. $X$ is a compact simply connected Kähler manifold (see for example [8, 32]). In [15], Grantcharov showed several examples flag manifolds which are Calabi-Yau.

By $G \subset G_C$ we denote a real form of $G_C$; i.e. a Lie subgroup of $G_C$, such that $\mathfrak{g} = \text{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$. As subgroup of $G_C$, $G$ acts on $X$ and there is only finitely many $G$-orbits on $X$. In the case $G$ is a compact real form of $G_C$, the $G$-action on $X$ is transitive. In [31], there is a detailed exposition of the properties $G$-action on $X$, a shorter one can be looked up in [33].

We assume that $G$ is a compact real form of $G_C$. We put $L := L_C \cap G$, then transitive action of $G$ on $X$ permits identify $X$ and $G/L$. Using the above Levi decomposition of $P$, an irreducible representation $r$ of $L$ on the finite dimensional complex vector space $V$ can be extended to a holomorphic representation of $P$, in which the action of the factor $U_C$ is trivial. With the $P$-action on $V$ one can define the following $G$-homogeneous vector bundle over $X = G_C/P$

\[ V = G \times_P V. \]  

(3.2)

Let $\lambda \in \mathfrak{h}^*$ be the highest weight of the above representation $r$ of $L$. Define

\[ i(\lambda) := \# \{ \alpha \in \Delta^+ | \langle \lambda + \rho, \alpha \rangle < 0 \}, \]

$\rho$ being the half sum of the positive roots.

As $\text{St}^i(\mathcal{O}_X, \mathcal{O}(V)) = H^i(X, \mathcal{O}(V))$, by a direct application of Borel-Bott-Weil theorem, we deduce the following proposition about the spaces of string on the flag manifold $X = G_C/P$.

**Proposition 12.**

1. If $\lambda + \rho$ is singular, that is $\langle \lambda + \rho, \alpha \rangle = 0$ for some root $\alpha \in \Delta(\mathfrak{g})$, then $\text{St}^i(\mathcal{O}_X, \mathcal{O}(V)) = 0$ for all $i$.

2. If $\lambda + \rho$ is regular, that is $\langle \lambda + \rho, \alpha \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{g})$, then there exists $w$ in the Weyl group so that $w(\lambda + \rho)$ is dominant with respect to $\Delta^+ \cap \Delta(\mathfrak{l})$. In this case, $\text{St}^i(\mathcal{O}_X, \mathcal{O}(V)) = 0$ for
i \neq i(\lambda) \text{ and } St^i(O_X, O(V)) \text{ is the irreducible representation of } G \text{ of highest weight } w(\lambda + \rho) - \rho.

As a direct consequence of a vanishing theorem proved in [29], we can state the following result, which yields a upper bound for the ghost number of the strings between two particular types of branes in \( X \); in other words, a upper bound in the number of nonzero summands of (1.2).

**Proposition 13.** Let \( S \) be an open \( G \)-orbit (\( G \) not necessarily compact) in the flag manifold \( X = G_C/P \). We denote by \( j : S \hookrightarrow X \) the inclusion and by \( s \) de complex dimension of a maximal compact subvariety of \( S \). If \( \mathcal{H} \) is a coherent sheaf on \( X \), then \( St^i(j_!(O_S), \mathcal{H}) = 0 \), for \( i > s \). Where \( j_!(O_S) \) is the direct image of \( O_S \) by the inclusion.

### 3.2. Toric varieties

Known properties of the cohomology of toric varieties will allow us to express the decomposition (1.2) in a more precise terms, when \( \mathcal{F} \) and \( \mathcal{G} \) are particular branes on a toric manifold. We will also apply the localization formula in cohomology equivariant to (2.17), when \( X \) is a toric manifold.

Let \( \Sigma \) be a fan in \( N = \mathbb{Z}^r \), we will denote by \( X \) the toric variety defined by \( \Sigma \) [9, 10, 12, 26]. We put \( M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \), \( N_R := N \otimes_{\mathbb{Z}} \mathbb{R} \) and \( T \) for the torus

\[
T = N \otimes \mathbb{C}^\times = \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^\times).
\]

Given \( m \in M \) we denote by \( \chi^m \) the homorphism

\[
\chi^m : t \in T \mapsto t(m) \in \mathbb{C}^\times.
\]

That is, the \( \chi^m \)'s are the characters of the irreducible representations of \( T \).

We put \( \Sigma(1) \) for denoting the set of 1-dimensional cones in \( \Sigma \), and given \( \rho \in \Sigma(1) \), there is a unique primitive element \( v_\rho \in N \cap \rho \), such that the cone \( \rho \) can be expressed as \( \mathbb{R}_{\geq 0}v_\rho \), and any cone \( \sigma \) of \( \Sigma \) can be written

\[
(3.4) \quad \sigma = \sum_{\rho \in \Sigma(1) \cap \sigma} \mathbb{R}_{\geq 0}v_\rho.
\]

We assume that \( X \) is nonsingular. Given a family \((a_\rho) \in \mathbb{Z}^{\Sigma(1)}\), we put \( \psi(v_\rho) = a_\rho \), for all \( \rho \in \Sigma(1) \). By (3.4), \( \psi \) can be extended to a function \( \psi \) defined on the support of \( \Sigma \), which is linear on each cone of \( \Sigma \). On the other hand, the family \((a_\rho)\) determines the following divisor on \( X \)

\[
(3.5) \quad A = - \sum_\rho a_\rho V(\rho),
\]
where \(V(\rho)\) is the closure of the orbit of \(\rho\) under the \(T\)-action. \(A\) is a \(T\)-invariant divisor of \(X\), which determines a \(T\)-equivariant line bundle \(L\), in the usual way.

By Proposition \(\ref{prop:representation}\) \(H^i(X, O(L))\) supports a representation of \(T\). As \(\text{Ext}^i(O_X, O(L)) = H^i(X, O(L))\), the decomposition of \(\text{St}^i(O_X, O(L))\) stated in Theorem \(\ref{thm:decomposition}\) can be expressed, for this particular case, in terms of the local cohomology of \(N_R\). In fact, from Theorem 2.6 of \(\cite{26}\) (see also \(\cite{12}\,\text{page } 74\)) we deduce the following proposition.

**Proposition 14.** If \(\Sigma\) is a smooth fan and \(L\) is the line \(T\)-equivariant bundle associated to the divisor (3.5), then
\[
\text{St}^i(O_X, O(L)) = \sum_{m \in M} H^i_{Z(m)}(N_R, \mathbb{C}) \chi^m,
\]

where \(Z(m) = \{v \in N_R, | \langle m, v \rangle \geq \psi(v)\}\).

Let \(A\) and \(A'\) be \(T\)-invariant divisors of \(X\). We denote by \(\psi\) and \(\psi'\) the corresponding linear support functions associated to \(A\) and \(A'\), respectively. We put \(L\) and \(L'\) for the respective line bundles. One says that \(\psi\) is strictly convex if \(\lambda \psi(u) + (1 - \lambda) \psi(v) < \psi(\lambda u + (1 - \lambda v))\), for all \(\lambda \in [0, 1]\) and \(u, v \in N_R\). A known fact is that \(O(L)\) is ample if \(\psi\) is strictly convex \(\cite{12}\,\text{page } 70\). The following proposition shows the form adopted by Theorem 3, when \(\mathcal{F} = O(L)\) and \(\mathcal{G} = O(L')\).

**Proposition 15.** With the above notations, if \(\psi' - \psi\) is strictly convex, then
\[
\text{St}^q(O(L), O(L')) = 0, \quad \text{for } q \neq 0,
\]
and
\[
\text{St}^q(O(L), O(L')) = \bigoplus_{m \in P} \mathbb{C} \cdot \chi^m,
\]

where \(P = \{m \in M | \langle m, n \rangle \geq \psi'(n) - \psi(n), \text{ for all } n \in N_R\}\).

**Proof.** By Proposition 10, \(\text{St}^q(O(L), O(L')) = H^q(X, O(L' \otimes L'))\). As the support function associated to \(O(L' \otimes L')\) is \(\psi' - \psi\), the proposition follows from Theorem 2.7 and Corollary 2.9 in \(\cite{26}\). \(\square\)

In order to apply the fixed point formula to (2.17) when \(X\) is a toric manifold, we make two Remarks.

**Remark 1.** Let us assume that the torus \(T\) acts trivially on a connected manifold \(S\) and that \(\mathcal{W}\) is a \(T\)-equivariant vector bundle over \(S\) with rank \(m\). By the equivariant splitting principle, we can assume that \(\mathcal{W}\) is a direct sum of \(T\)-equivariant line bundles
\[
\mathcal{W} = \bigoplus_{j=1}^m L_j.
\]
The action of $T$ on $W$ is defined by $m$ weights $\varphi_j$ and the $T$-equivariant Chern class of $L_j$ is given by (see [14, page 317])

$$c^T_1(L_j) = c_1(L_j) + \frac{1}{2\pi} \varphi_j.$$  

The $T$-equivariant Chern character of $W$ is (see [23, page 234])

$$\text{ch}^T(W) = \sum_{j=1}^m \exp(c^T_1(L_j)).$$

**Remark 2.** Let $p$ be a fixed point for the $T$-action on the toric manifold associated with the fan $\Sigma \subset \mathbb{R}^n$. We denote by $\nu_{i,p} \in 2\pi(\mathbb{Z})^n$, $i = 1, \ldots, n$, the weights of the isotropy representation of $T$ on $T_pX$. The fixed points of the $T$-action are in bijective correspondence with the $n$-dimensional cones in $\Sigma$ [10, §3.2]. If the point $p$ is associated with the cone $\sigma$, then

$$\omega_{i,p} = \frac{\nu_{i,p}}{2\pi}$$

are the generators of $\sigma^\vee \cap M$, where $\sigma^\vee$ is the dual cone of $\sigma$.

In the statement of the following proposition, $\mathcal{V}^0$ is the vector bundle introduced in Subsection 2.4.

**Proposition 16.** Let $X$ be the toric manifold associated to the fan $\Sigma$, denoting by $\{\varphi_{j,p}\}_{j=1,\ldots,m}$ the weights of the representation of $T$ on the fibre of $\mathcal{V}^0$ at $p$, then (2.17) is equal to

$$\chi(D) \circ \exp = (2\pi)^n \sum_{p \in X^T} \left( \sum_{j=1}^m e^{\frac{1}{2\pi} \varphi_{j,p}} \right)^n \prod_{i=1}^n \left( \sinh \left( \frac{\omega_{i,p}}{2} \right) \right)^{-1},$$

where $X^T$ is the set of fixed points of $X$ for the $T$-action.

**Proof.** The localization theorem in equivariant cohomology [16, §10.9] allows us to calculate the value (2.17) as a sum of contributions of the connected components of $X^T$. As $X^T$ is discret, the localization formula adopts the following form

$$\chi(D) \circ \exp = (2\pi)^n \sum_{p \in X^T} \frac{Q^G(\mathcal{E}^0)(p)}{\prod_{i=1}^n \nu_{i,p}},$$

where the $\nu_{i,p}$ are the weights of the isotropy representation of $T$ at the fixed point $p$.

From (3.6) and (3.7), it follows

$$\text{ch}^T(\mathcal{V}^0)|_p = \sum_{j=1}^m e^{\frac{1}{2\pi} \varphi_{j,p}}.$$
Similarly (see \[23\] page 231),
\[
\hat{A}^T(TX)|_p = \prod_{i=1}^{n} \frac{\omega_{i,p}}{2} \left( \sinh \left( \frac{\omega_{i,p}}{2} \right) \right)^{-1}.
\]

The proposition follows from (2.16) together with (3.8). □

Note that the contribution of the manifold \(X\) to (3.9) is encoded in the \(n\)-dimensional cones of the fan \(\Sigma\).

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