Hedging with transient price impact for non-covered and covered options

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We solve the superhedging problem for European options in a market with finite liquidity where trading has transient impact on prices, and possibly a permanent one in addition. Impact is multiplicative to ensure positive asset prices. Hedges and option prices depend on the physical and cash delivery specifications of the option settlement. For non-covered options, where impact at the inception and maturity dates matters, we characterize the superhedging price as a viscosity solution of a degenerate semilinear pde that can have gradient constraints. The non-linearity of the pde is governed by the transient nature of impact through a resilience function. For covered options, the pricing pde involves gamma constraints but is not affected by transience of impact. We use stochastic target techniques and geometric dynamic programming in reduced coordinates.

Keywords: Superreplication, multiplicative price impact, resilience, transient impact, permanent impact, option settlement, non-covered options, covered options, hedging, viscosity solutions, stochastic target problem, geometric dynamical programming.

1 Introduction

The key insight from the Black-Scholes formula is, that it shows how to delta-hedge the derivative in order to eliminate the risk by replicating the options’s payout by dynamic trading in the underlying, while the minimal capital required to do so yields the unique arbitrage free pricing. In the present paper, we study the superhedging problem in a market model with finite liquidity where dynamic hedging has a multiplicative and transient impact on the price of the underlying risky asset for the derivative. This relaxes the assumption of perfect liquidity (or traders being small) from the frictionless Black-Scholes model, where it is assumed that arbitrary quantities of the underlying could be bought or sold without affecting its price. In spirit of the additive impact model in [PSS11], multiplicative transient impact can be interpreted in terms of a volume effect process that reflects transient volume imbalances in a limit order book, where asset prices are ensured to remain positive and the inter-temporal price impact is persistent but tends to cease over time, see [BBF18, Remark 2.3]. The superhedging problem in illiquid market models leads to non-linear feedback effects, in the sense that a hedging strategy directly influences the price of the underlying risky asset with respect to which the option’s payoff at maturity is defined, cf. [SW00]. We study the hedging

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and pricing for non-covered (and covered) options, whose hedging strategies are (respectively are not, for covered options) subject to price impact also at inception and maturity. The analysis shows how transience of impact can affect the pricing and hedging equation for non-covered options through a resilience function.

For non-covered options, one needs to distinguish between physical and cash delivery specifications for the option settlement. It is intuitively clear that the price of such options, at least at maturity, can depend on the current level of market impact since delivering the asset physically could require a terminal block trade whose cost can depend on the current volume imbalance in the limit order book. We formulate the superhedging problem for non-covered options as a stochastic target problem and prove a Dynamic Programming Principle (DPP) along reduced coordinates for the effective price and impact processes, which represent the price and impact levels that would prevail if the large trader were to unwind her (long or short) position in the underlying risky asset immediately. Along these reduced coordinates, the DPP provides a way to compare at stopping times the instantaneous liquidation wealth and the (minimal) superhedging price. This enables us to characterize the superhedging price as the viscosity solution to a non-linear pricing pde, which is a semi-linear Black-Scholes equation. Its nonlinearity involves the (non-parametric) price impact and the resilience functions as well. Moreover, if it has a sufficiently regular solution, it yields an optimal strategy which is replicating the option payoff as the solution to the superhedging problem. This strategy incorporates the transient nature of impact in that it depends on the effective level of impact. Our analysis can be generalized to combined transient and permanent price impact, where the latter impact component is one that is persistent over time without resilience effects, see Section 6.

It turns out that the current deviation of the asset price from the unaffected fundamental price is a relevant state variable on which the price of the option and the hedging strategies depend. Moreover, having physical or cash delivery at maturity leads to different boundary conditions for the pricing pde and hence typically different prices. For instance, the superhedging price of a European call option with cash delivery is smaller than that with physical delivery. The aforementioned analysis is derived for general (non-parametric) impact functions, being assumed to be bounded away from zero and infinity, meaning that the possible relative price impact on the risky asset by large block trades is limited to certain fractions. Extending the analysis beyond such a condition, Section 5.2 moreover studies the hedging problem for a setting where the price impact function is of exponential form. There, delta constraints on the admissible hedging strategies occur to ensure wellposedness for the pricing pde. In this setup, the pricing pde for a typical European option, whose payoff is given by a function of the price of the underlying only, reduces to the Black-Scholes pde with gradient constraints. In the case of covered options, we show that the resilience of price impact is immaterial for the price, irrespectively of a particular form for the resilience function, what has been observed likewise in [BLZ17, Section 4]. We explain in Section 8 how a similar analysis carries over to our setup and derive a singular pricing pde, the analysis of which induces gamma constraints. It turns out that the current deviation of the asset price from the unaffected price becomes a relevant state variable for describing the solution.

The most closely related articles are [BLZ16, BLZ17]; they study the superhedging problem in an additive model with permanent impact, cf. Remark 2.2, and inspired our analysis. In comparison, for non-covered options the transient impact setting requires to consider an extended state space of reduced coordinates which in addition includes the level of market impact. It appears that pricing and hedging of derivatives in models with price impact has been studied so far mainly for models where price impact occurs in an instantaneous or permanent way, or a combination of both, see e.g. [Fre98, BB04, ÇJP04, CST10, GP17, BSV16] and more references therein. Instantaneous impact means that only the price for current trades are affected but not those for future trades, and hence it can be viewed as non-proportional transaction costs, whereas a permanent impact component has an inter-temporal effect on future prices which is persistent without any relaxation over time. In
contrast, the focus for this paper is on price impact of a transient nature and how, if at all, it affects the minimal superhedging price and the hedging strategy for European options.

The paper is organized as follows. The multiplicative price impact model is explained in Section 2, while in Section 3 we discuss the notion of non-covered options. The superhedging problem for those is reformulated in Section 4 as a stochastic target problem for which a DPP is derived along a suitable reduced coordinates, see Theorem 4.1. This leads to the derivation of the pricing PDEs in Section 5, see Theorems 5.5 and 5.9. In Section 6 we explain how the analysis extends when additional permanent impact is considered. Section 7 presents a numerical example, while Section 8 sketches how the superhedging problem for covered options can be solved. Technical proofs on viscosity solution properties and comparison are delegated to Section 9.

2 Transient price impact model

This section describes the multiplicative market impact model for this paper. An extension with additional permanent impact is postponed to Section 6. We fix a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\), with a filtration \(\mathbb{F}\) satisfying the usual conditions, that supports a Brownian motion \(W\). In the absence of the large trader, the unaffected price process \(\tilde{S}\) of the single risky asset evolves according to the stochastic differential equation

\[
d\tilde{S}_t = \tilde{S}_t(\mu_t \, dt + \sigma \, dW_t), \quad \tilde{S}_0 \in \mathbb{R}_+,
\]

with constant \(\sigma > 0\) and bounded progressive process \(\mu\). Let the càdlàg adapted process \(\Theta\) denote the evolution of her holdings in the risky asset, let us say a stock. As in [BBF17b], we define the market impact process \(Y = Y^{\Theta}\) pathwise, on the Skorohod space of càdlàg paths, via

\[
dY_t^{\Theta} = -h(Y_t^{\Theta}) \, dt + d\Theta_t, \quad Y_0 = y \in \mathbb{R},
\]

for \(h : \mathbb{R} \to \mathbb{R}\) being a Lipschitz continuous function with \(\text{sgn}(x) h(x) \geq 0\). When the large trader follows a strategy \(\Theta\), the risky asset price observed on the market, being the marginal price at which additional infinitesimal quantities could be traded, is

\[
S_t^{\Theta} = S_t = f(Y_t^{\Theta}) \tilde{S}_t, \quad t \geq 0,
\]

where the price impact function \(f : \mathbb{R} \to \mathbb{R}_+\) is increasing and in \(C^1\) with \(f(0) = 1\). In particular, \(\lambda := f'/f\) is a non-negative and locally integrable \(C^0\) function, satisfying

\[
f(x) = \exp \left( \int_0^x \lambda(u) \, du \right), \quad x \in \mathbb{R}.
\]

By the monotonicity of \(f\), the price impact from her trades is adverse to the large trader. During periods where the large trader is inactive, the impact process \(Y\) recovers towards its neutral state 0, so that the relative price impact \(S/\tilde{S} = f(Y)\) w.r.t. the unaffected (fundamental) price \(\tilde{S}\) is persistent but lessens over time, rendering the impact as transient.

Next, we specify the large trader’s proceeds (negative expenses) \(L\), which are the variations of her cash account to finance the dynamic holdings \(\Theta\) in the risky asset. For simplicity, we assume zero interest and a riskless asset with constant price 1 as cash, i.e. prices are discounted in units of this numeraire asset. For continuous strategies \(\Theta\) of finite variation,

\[
L(\Theta) = -\int_0^\infty S^{\Theta} \, d\Theta
\]

are the proceeds. And there is a unique continuous extension of the functional \(\Theta \mapsto L(\Theta)\) in
(2.5) to general (bounded)\(^1\) semimartingale strategies \(\Theta\), that is given by

\[
L(\Theta) := \int_0^T F(Y_t^{\Theta}) \, dS_t - \int S_t (f h)(Y_t^{\Theta}) \, dt - (SF(Y_T^{\Theta}) - S_0 F(Y_0^{\Theta})),
\]

as shown in [BBF17b, Theorem 3.8], with antiderivative

\[
F(x) := \int_0^x f(u) \, du, \quad x \in \mathbb{R}.
\]

More precisely, every (cadlag) semimartingale can be approximated (in probability) in the Skorokhod \(M_1\) topology by a sequence of continuous processes of finite variation, and that if semimartingales \(\Theta^n \xrightarrow{P} \Theta\) in \((D([0,T]), M_1)\) for a semimartingale \(\Theta\), then \(L(\Theta^n) \xrightarrow{P} L(\Theta)\) in \((D([0,T]), M_1)\). Thus, it is natural to define \(L\) by (2.6) as the continuous extension of \(L\) from (2.5) to all semimartingales. The proceeds from a block trade of size \(\Delta \Theta\) at time \(t\) are given by

\[
- \bar{S}_t \int_0^{\Delta \Theta_t} f(Y_t^{\Theta} + x) \, dx,
\]

showing that the price per share that the large trader pays (resp. obtains) for a block buy (resp. sell) order is between the price before the trade \(f(Y_t^{\Theta}) \bar{S}_t\) and the price after the trade \(f(Y_t^{\Theta}) \bar{S}_t\). The form of proceeds and price impact from block trades can be interpreted from the perspective of a latent limit order book, where a block trade is executed against available orders in the order book for prices between \(f(Y_t^{\Theta}) \bar{S}_t\) and \(f(Y_t^{\Theta} + \Delta \Theta_t) \bar{S}_t\), see [BBF17a, Section 2.1]. In this sense, \(Y\) is a volume effect process in spirit of [PSS11].

For a self-financing portfolio \((\beta, \Theta)\), in which the dynamic holdings in cash (the riskless asset) and in stock (the risky asset) evolve as \(\beta\) and \(\Theta\), the self-financing condition is

\[
\beta = \beta_0 - L(\Theta).
\]

In order to define the wealth dynamics induced by the large trader’s strategy, one needs to specify the dynamics of the value of the risky asset position in the portfolio. If the large trader were forced to liquidate her stock position immediately by a single block trade, the instantaneous liquidation wealth \(V^{\text{liq}}_t\) is

\[
V^{\text{liq}}_t = V^{\text{liq}}_t(\Theta) := \beta_t + \bar{S}_t \int_0^{\Theta_t} f(Y_t^{\Theta} - x) \, dx.
\]

The dynamics for this notion of wealth is mathematically tractable and continuous, satisfying

\[
dV^{\text{liq}}_t = (F(Y_t -) - F(Y_t - - \Theta_t -)) \, d\bar{S}_t - \bar{S}_t (f(Y_t -) - f(Y_t - - \Theta_t -)) \, h(Y_t) \, dt.
\]

One obtains from (2.10) absence of arbitrage within the following set of admissible strategies

\[
\mathcal{A}^\text{NA} := \{ (\Theta_t)_{t \geq 0} | \text{ bounded semimartingale, with } \Theta_t = 0 \text{ and } \Theta_t = 0 \text{ on } t \in [T, \infty) \text{ for some } T < \infty \}.
\]

**Proposition 2.1.** The market is free of arbitrage up to any finite time horizon \(T \in [0, \infty)\) in the sense that there exists no \(\Theta \in A^\text{NA}\) with \(\Theta_t = 0\) on \(t \in [T, \infty)\) such that for the self-financing strategy \((\beta, \Theta)\) with \(\beta_0 = 0\) we have \(P[V^{\text{liq}}_T \geq 0] = 1\) and \(P[V^{\text{liq}}_T > 0] > 0\).

**Proof.** The claim is proven as in [BBF17b, Section 4]. We note that there it was additionally required for admissible strategies that \(V^{\text{liq}}\) is bounded from below. The latter condition however can be omitted in the current setup of bounded controls. To see this, observe that for any \(\Theta \in \mathcal{A}^\text{NA}\) there exists an equivalent martingale measure \(Q^{\Theta} \approx P\) (on \(F_T\)), constructed as in [BBF17b, pf. of Thm. 4.3], under which the process \(V^{\text{liq}}\) is a true martingale.

\(^1\)Results in [BBF17b] are stated in a more general setup where \(\bar{S}\) could have jumps and trading strategies do not need to be semimartingales and bounded. Yet, for the analysis here we restrict to bounded semimartingale strategies.
Unlike to the frictionless situation, there is more than one sensible way to define wealth in an illiquid market with price impact. For the analysis in Section 8, we shall also make use of another notion of book wealth. For a strategy with dynamic holdings $\Theta$ and $\beta$ in the risky and the riskless asset, the book wealth process is given by

$$V^{\text{book}} := \beta + \Theta S,$$  \hspace{1cm} (2.11)

with the risky asset being evaluated at the current (marginal) market prices $S$. In illiquid markets, the liquidation wealth (2.9) which is achievable by the large trader if she were to unwind her risky asset holdings immediately is usually different from book wealth (2.11).

**Remark 2.2.** For comparison, note that [BLZ16, cf. equation (2.1)] study a model where price impact is permanent and additive, in the sense that (using notation similar to ours) resilience $h = 0$ is zero, thus $Y = \Theta$ for $Y_{0-} := 0$, and the stock price after a small (infinitesimal) trade of size $\delta$ becomes $s(\theta + \delta) \approx s(\theta) + \delta f(s(\theta))$ where $f: \mathbb{R} \rightarrow (0, \infty)$ is a smooth function of the current stock price $s(\theta)$ which prevails if the large trader holds $\theta$ stocks just before the trade. That means, more precisely, $\frac{\delta}{\delta s} s(\theta) = f(s(\theta))$. For comparison, it is instructive to pretend, just formally, that one could choose a ‘multiplicative’ form $f(x) := \lambda x$. With $\tilde{s} := s(0)$ one then would get $s(\theta) = \left( \exp \left( \int_0^\theta \lambda(x) dx \right) \right) \tilde{s}$, being reminiscent to (2.3)–(2.4), and taking $\lambda$ to be constant would give $s(\theta) = \exp(\lambda \theta) \tilde{s}$, what is the permanent impact variant of the basic case for multiplicative (transient) impact that is studied in Section 5.2. However, a choice like $f(x) = \lambda x$ in (multiplicative) form with $\lambda > 0$ does not fit with assumptions (H1) and (H2) in [BLZ16]: Neither is $x \mapsto \lambda x$ (strictly) positive on $\mathbb{R}$, nor is $x \mapsto \exp(\lambda x)$ a surjective function from $\mathbb{R} \rightarrow \mathbb{R}$. Observe that asset prices in [BLZ16] take values $x$ in $\mathbb{R}$ (instead of $(0, \infty)$) and the instructive basic example for their setting is the case of fixed (constant) impact with $f(x) := \lambda > 0$ where $s(\theta) = \tilde{s} + \lambda \theta$, and with the unaffected asset price $\tilde{s}$ evolving as in the Bachelier model (say), see [BLZ16, Section 3.4].

### 3 Hedging of non-covered options in illiquid markets

We solve in Sections 3-7 the problem of dynamic hedging for non-covered options, where the issuer who wants to hedge the option is to receive the option premium in cash, whereas Section 8 discusses the same problem for the, less familiar, case of covered options, like it was posed in [BLZ17]. For the latter, a part of the premium (the initial ‘delta’) is to be paid in cash and with the unaffected asset price $\bar{s}$ evolving as in the Bachelier model (say), see [BLZ16, Section 3.4].
price impact, the hedging problem becomes more complex since the large trader by hedging
the option influences the price of the underlying, which on the other hand defines the option’s
payout at maturity. As a consequence, this may give the large trader an incentive to influence
in her favor the price, and thus the payout. We restrict the possibility of manipulations by
distinguishing between physical and cash delivery part in the option’s payoff and requiring
that the physical part shall be delivered exactly. Thus, doing trades shortly before maturity
that shall be unwound right after delivery, hence influencing the option’s payout to favor the
large trader, will not be allowed.

Among her admissible trading strategies \( \Gamma \) (to be specified precisely in Section 4.1), she
is going to look for the cheapest strategies to super-replicate the option’s payout in the
following sense.

**Definition 3.2 (Hedging of a non-covered option).** A superhedging strategy is a self-financing
strategy \((\beta, \Theta)\) with \( \Theta \in \Gamma, \Theta_{0−} = 0, \) and

\[
\beta_T \geq g_0(S_T, Y_T) \quad \text{and} \quad \Theta_T = g_1(S_T, Y_T).
\]

We stress again that a hedging strategy has to deliver exactly the physical component
\( g_1(S_T, Y_T) \) at maturity, and that any further (long or short) position in the underlying has to
be unwound before the options are settled at the resulting price \( S_T \) and impact level \( Y_T \). In
particular, a hedging strategy for a payoff with pure cash delivery part is a round trip, i.e. it
begins and ends with zero shares in the underlying, while the hedging strategy for a payoff
with non-trivial physical delivery part should be such that the amount of risky assets held
at maturity will meet exactly the physical delivery requirement. Thus, hedging strategies
for European contingent claims with physical delivery can be different from those with pure
cash delivery part, and as it will turn out, their respective prices can also differ.

The (minimal) superhedging price of a non-covered option with payoff \((g_0, g_1)\), which
we will denote by \( p_{(g_0,g_1)} \), is the minimal (inffimum of) initial capital \( \beta_{0−} \) for which such a
superhedging strategy \((\beta, \Theta)\) exists.

Options with pure cash settlement are described by \( g_1 = 0 \). In fact, every (reasonable)
option can be represented by a payoff with pure cash settlement. Indeed, if \( \Gamma \) is stable under
adding an additional jump at terminal time, meaning that \( \Theta \in \Gamma \) implies that \( \Theta + \Delta 1_{\{T\}} \in \Gamma \)
for every \( F_T\)-measurable \( \Delta \), then any European option can be represented by an option with
pure cash settlement. To see this for an option with payoff \((g_0, g_1)\), let for \((s, y) \in \mathbb{R}_+ \times \mathbb{R}

\[
H(s,y) := \inf \left\{ g_0\left( f_y, y + \theta \right) + s \frac{F(y + \theta) - F(y)}{f(y)} \mid \theta = g_1\left( f_y, y + \theta \right) \right\}. \quad (3.1)
\]

The value \( H(s, y) \) is the minimal amount of cash (riskless assets) needed to hedge the payoff
\((g_0, g_1)\) with a single (instant) block trade at maturity, when just before that trade (at time
\( T− \)) the level of impact is \( y \) and there are no holdings in the risky asset whose price is \( s \).
Indeed, a block trade of size \( \theta \) will result in the new price \( \hat{s} = sf(y + \theta)/f(y) \) and impact
\( \hat{y} = y + \theta \), it will incur the cost \( s(F(y + \theta) − F(y))/f(y) \). Thus, it will hedge the claim
\((g_0, g_1)\) if \( \theta = g_1(\hat{s}, \hat{y}) \) and we have enough capital to pay for the block trade and to cover the
cash-delivery part that after the block trade equals \( g_0(\hat{s}, \hat{y}) \), see Definition 3.2.

We have the following result.

**Lemma 3.3.** For a European option with payoff \((g_0, g_1)\), let \( H \) from \((3.1)\) be finite and
measurable. Then we have \( p_{(g_0,g_1)} = p_{(H,0)} \). In the case of \( \lambda \) being constant, the function \( H \)
does not depend on \( y \), if \( g_0 \) and \( g_1 \) do not depend on \( y \).

**Proof.** Suppose that \((\beta, \Theta)\) is a superhedging strategy for \((g_0, g_1)\). This means that \( \Theta_T =
= g_1(S_T, Y_T) \) and \( \beta_{0−} + L_T(\Theta) \geq g_0(S_T, Y_T) \). Consider the strategy \( \tilde{\Theta} := \Theta − \Theta_T 1_{\{T\}} \). The
price and impact at \( T \) resulting from \( \tilde{\Theta} \) are \( \tilde{S}_T = S_T f(Y_T - \Theta_T)/f(Y) \) and \( \tilde{Y}_T = Y_T - \Theta_T \)
respectively, and the generated proceeds, which here are equal to \( V_T^{\text{lin}}(\Theta) \) since \( \tilde{\Theta}_T = 0 \), are

\[
V_T^{\text{lin}}(\tilde{\Theta}) = \beta_{0−} + L_T(\tilde{\Theta}) + \tilde{S}_T(F(Y_T) − F(Y_T − \Theta_T))).
\]
Hence,
\[
V_T^{\text{liq}}(\hat{\Theta}) \geq g_0(S_T, Y_T) + \bar{S}_T(F(Y_T) - F(Y_T - \Theta_T)) \\
= g_0(\bar{S}_T, \frac{f(Y_T + \Theta_T)}{f(Y_T)}, \tilde{Y}_T + \Theta_T) + \bar{S}_T \frac{F(Y_T + \Theta_T) - F(Y_T)}{f(Y_T)} \\
\geq H(\bar{S}_T, \tilde{Y}_T).
\]

Therefore, the self-financing trading strategy \( \hat{\Theta} \) with initial capital \( \beta_{0-} \) is a superhedging strategy for the European claim with payoff \((H, 0)\), implying \( p_{(g_0, g_1)} \geq p_{(H, 0)} \).

To show the reverse inequality, let \((\beta, \Theta)\) be a superhedging strategy for \((H, 0)\), meaning that \( \Theta_T = 0 \) and \( V_T^{\text{liq}}(\Theta) \geq H(S_T, Y_T) \). A measurable selection argument yields that for any \( \varepsilon > 0 \), there exists an \( \mathcal{F}_T\)-measurable random variable \( \Theta^\varepsilon_T \) such that \( \Theta^\varepsilon_T = g_1(\frac{f(Y_T + \Theta^\varepsilon_T)}{f(Y_T)}, Y_T + \Theta^\varepsilon_T) \) and
\[
H(S_T, Y_T) + \varepsilon \geq g_0(S_T, \frac{f(Y_T + \Theta^\varepsilon_T)}{f(Y_T)}, Y_T + \Theta^\varepsilon_T) + S_T \frac{F(Y_T + \Theta^\varepsilon_T) - F(Y_T)}{f(Y_T)}.
\]
Thus, the strategy \( \tilde{\Theta}^\varepsilon := \Theta + \Theta^\varepsilon_T 1_{\{T\}} \) with initial capital \( \beta_{0-} + \varepsilon \) is superhedging for the claim with payoff \((g_0, g_1)\); indeed, the proceeds generated from \( \tilde{\Theta}^\varepsilon \) are
\[
V_T^{\text{liq}}(\Theta) + \varepsilon - ST \frac{1}{f(Y_T)} (F(Y_T + \Theta^\varepsilon_T) - F(Y_T))
\]
where the last term is the cost of acquiring \( \Theta^\varepsilon_T \) assets. Hence, with reference to the preceding inequality, the generated proceeds from \( \tilde{\Theta}^\varepsilon \) are sufficient to deliver the cash part of the payoff, and also physical part by choice of \( \Theta^\varepsilon \). Since \( \varepsilon > 0 \) was arbitrary, we conclude \( p_{(H, 0)} \geq p_{(g_0, g_1)} \), and thus the claim.

If \( \lambda \) is constant, then we have \( f(x) = \exp(\lambda x) \) and thus \( (F(y + \theta) - F(y))/f(y) = F(\theta) \) and \( f(y + \theta)/f(y) = f(\theta) \). In particular, \( H \) does not depend on \( y \), if \( g_0 \) and \( g_1 \) do not.

**Example 3.4.** 1. A cash-settled European call option with strike \( K \) is specified by the payoff \((g_0(s, y), g_1(s, y)) = ((s - K)^+, 0)\).

2. In comparison, a European call option with strike \( K \) and physical settlement has the payoff \((-K 1_{\{s \geq K\}}, 1_{\{s \geq K\}})\). Note that although the payoff profile \((g_0, g_1)\) does not depend on the level of impact \( y \), the equivalent pure cash settlement profile \( H \) from Lemma 3.3 could still depend on it, if the function \( \lambda \) is not constant. Indeed, the effect on the relative price change \( f(y + \theta)/f(y) \) from a block trade \( \theta \) can depend on the level \( y \) of impact before the trade in general, unless \( f(x) = \exp(\lambda x) \) for constant \( \lambda \).

**Remark 3.5.** We discuss an example to show how the hedging problem for the large trader could be related to hedging in a market with perfect liquidity but with portfolio constraints, if \( F \) from (2.7) is not surjective onto \( \mathbb{R} \). In particular, in this case our market model will not be complete in the sense that not every contingent claim can be perfectly replicated. A prototypical example is the special case of purely permanent impact, i.e. \( h \equiv 0 \), constant \( \lambda \), i.e. \( f(x) = \exp(\lambda x) \), and a claim with payoff \((H, 0)\), i.e. only cash settlement. Hence, we are in the setup of [BB04] with the smooth family of semimartingales \( P(x, t) := \exp(\lambda x)\bar{S}_t \). If \( Y_{0-} = 0 \) and \( \lambda = 1 \), (2.10) takes the form
\[
dV^{\text{liq}}_t = (\exp(\Theta_t) - 1) d\bar{S}_t,
\]
By the conditions from Definition 3.2, any hedging strategy \( \Theta \) satisfies \( \Theta_T = 0 \), and hence at maturity \( S_T = \bar{S}_T \) and \( Y_T = Y_{0-} = 0 \). Thus, the superreplication condition becomes \( V_T^{\text{liq}}(\Theta) \geq H(S_T, 0) \). This means that, after a reparametrization \( \Theta \mapsto \exp(\Theta) - 1 \) of strategies, the superreplication problem in this large investor model becomes equivalent to the problem in the respective frictionless model with price process \( \bar{S} \) for a small investor and with constraints on the delta \( (\text{greater than } -1) \), i.e. the number of risky assets that a hedging strategy might have. In particular, one should expect that in such situations (where \( F \) is not invertible) the
pricing equation should contain gradient constraints. Note that this is different from [BB04] because for this particular \( f \) the crucial Assumption 5 there is violated, and also different from [BLZ16] because their assumption (H2) would not hold in this case.

In the presence of resilience in the market impact \( (h \neq 0) \), the situation becomes more complex since the evolution of the price and impact processes depend on the full history of the trading strategy, and a simplification as above is not applicable. But we will see later in Section 5.2 that in the case \( f = \exp(\lambda \cdot) \) a lower bound on the delta will also emerge naturally in order to make sense of the pricing equation.

4 Superhedging by geometric dynamic programming

In this section, we formulate the superhedging problem for non-covered options as a stochastic target problem. We prove a geometric Dynamic Programming Principle (DPP) for the control problem whose value function will be characterized subsequently. Notably, we are going to show that a DPP holds with respect to suitably chosen coordinates, which correspond to modified state processes describing the evolution of effective price and impact levels that would result from an immediate unwinding of the risky asset holdings by the large trader.

With respect to these new effective coordinates, we will characterize the value function of the control problem as a viscosity solution to a partial differential equation, cf. (PDE) and (PDE\(^{\delta} \)) in Section 5, that is the pricing pde generalizing the (frictionless) Black-Scholes equation.

4.1 Stochastic target formulation

We consider strategies that take values in the constraint set \( \mathcal{K} \subseteq \mathbb{R} \), for one of the two cases

\[
\mathcal{K} = [-K, +\infty) \text{ for some } K > 0, \text{ or} \\
\mathcal{K} = \mathbb{R}. 
\]  

(4.1) \hspace{1cm} (4.2)

The short-selling constraints (4.1) will be needed when \( F \) is not surjective onto \( \mathbb{R} \), see Remark 3.5, in which case we will consider in Section 5.2 \( f(x) = \exp(\lambda x) \) for some \( \lambda > 0 \), while \( \mathcal{K} = \mathbb{R} \) will be in force when \( f \) is bounded away from 0 and \( +\infty \), meaning that the (relative) change of the price from a block trade cannot be arbitrarily big.

For our analysis we need to allow for jumps in the admissible trading strategies in order to obtain a DPP. For \( k \in \mathbb{N} \), let \( \mathcal{U}_k \) denote the set of random \( \{0, \ldots, k\} \)-valued measures \( \nu \) supported on \( (\mathcal{K} \cap [-k, k]) \times [0,T] \) that are adapted in the following sense: for every \( A \in B(\mathcal{K}) \), the process \( t \mapsto \nu(A, [0,t]) \) is adapted to the underlying filtration. Note that the elements of \( \mathcal{U}_k \) have the representation

\[
\nu(A, [0,t]) = \sum_{i=0}^{k} \mathbb{1}_{\{(\delta_i, \tau_i) \in A \times [0,t]\}},
\]

where \( 0 \leq \tau_1 < \cdots < \tau_k \leq T \) are stopping times and \( \delta_i \) is a real-valued \( \mathcal{F}_{\tau_i} \)-random variable (might take values 0 as well). Consider also \( \mathcal{U} := \bigcup_{k \geq 1} \mathcal{U}_k \).

The admissible trading strategies \( \Theta \) that we will consider are bounded, take values in \( \mathcal{K} \) and have the representation

\[
\Theta_t = \Theta_0 - \int_0^t a_s \, ds + \int_0^t b_s \, dW_s + \int_0^t \int_{\mathbb{R}} \delta \nu(d\delta, ds), 
\]  

(4.3)

in which \( \Theta_0 \in \mathcal{K} \), \( \nu \in \mathcal{U} \) and \( (a, b) \in \mathcal{A} \), where

\[
\mathcal{A} := \{(a, b) \mid a \text{ and } b \text{ are predict. with } a \in L^1(dt \otimes d\mathbb{P}) \text{ and } b \in L^2(dt \otimes d\mathbb{P})\}. 
\]
In this sense, we identify the trading strategies by triplets \((a, b, \nu)\) ∈ \(A \times U\). For \(k \in \mathbb{N}\) set
\[
\Gamma_k := \{(a, b, \nu) \in A \times U_k : \Theta \text{ from (4.3) takes values in } K \cap [-k, k]\}
\]
and let \(\Gamma := \bigcup_{k \geq 1} \Gamma_k\).

To reformulate the superhedging problem in our market impact model as a stochastic target problem, consider for \((t, z) = (t, s, y, \theta, v) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times K \times \mathbb{R}\) and \(\gamma \in \Gamma\) the (dynamic version of) the state process
\[
(Z^{t,z;\gamma})_{u \in [t,T]} = (S_u^{t,z;\gamma}, Y_u^{t,z;\gamma}, \Theta_u^{t,z;\gamma}, V^{\text{liq},t,z;\gamma})_{u \in [t,T]},
\]
where the processes \(S^{t,z;\gamma}, Y^{t,z;\gamma}, \Theta^{t,z;\gamma}\) and \(V^{\text{liq},t,z;\gamma}\) correspond to the price, impact, risky asset position and instantaneous liquidation wealth processes on \([t, T]\) for the control \(\Theta^{t,z;\gamma}\) associated with \(\gamma\) (from the decomposition (4.3) on \([t, T]\) instead), when started at time \(t\) and plays a crucial role in deriving a pde that characterizes the value function (in a viscosity sense). The aim of this section is to provide a suitable DPP.

Following the discussion in Section 3, for a non-covered European option with payoff function given by a measurable map \((s, y) \in \mathbb{R}_+ \times \mathbb{R} \mapsto (g_0(s, y), g_1(s, y))\), a strategy \(\gamma \in \Gamma\) is superhedging if the state process at time \(T\) is (a.s.) in the set
\[
\Theta := \{(s, y, \theta, v) \in \mathbb{R}_+ \times \mathbb{R} \times K \times \mathbb{R} : \theta = g_1(s, y), v - s(F(y) - F(y - \theta))/f(y) \geq g_0(s, y)\}
\]
that we call the target set. The superhedging strategies for initial position \(\theta\) in the risky asset are
\[
G(t, s, y, \theta, v) := \bigcup_{k \geq 1} G_k(t, s, y, \theta, v)
\]
with
\[
G_k(t, s, y, \theta, v) := \{\gamma \in \Gamma_k : Z_T^{t,s,y,\theta,v;\gamma} \in \Theta\}.
\]
We are interested in the superhedging price, when the hedger starts with no risky assets initially, i.e.
\[
w(t, s, y) := \inf_{k \geq 1} w_k(t, s, y), \quad \text{where } w_k(t, s, y) := \inf\{v : G_k(t, s, y, 0, v) \neq \emptyset\}. \quad (4.4)
\]
Let us point out that the value function depends on the constraint set \(K\) (via the target set \(\Theta\)). Note also that the set of admissible superhedging strategies (identified with \(G(t, s, y, 0, v)\)) is a subset of \(A^{NA}\), meaning that the superhedging price of a positive payoff \(H\), the pure cash delivery equivalent of \((g_0, g_1)\) as in Lemma 3.3, is strictly positive.

### 4.2 Effective coordinates and dynamic programming principle

For stochastic target problems usually a form of the dynamic programming principle holds and plays a crucial role in deriving a pde that characterizes the value function (in a viscosity sense). The aim of this section is to provide a suitable DPP.

Let us first note that the formulation for the superhedging problem above looks not time-consistent, because in the definition (4.4) of the superhedging price \(w\) it is assumed that the initial position in risky assets is zero, whereas at later times it typically will not be. To obtain a time-consistent formulation, the first naive idea could be to make the risky asset position a new variable, that means to work with the function \(\bar{w}\) defined on \([0, T] \times \mathbb{R}_+ \times \mathbb{R} \times K\) by
\[
\bar{w}(t, s, y, \theta) := \inf_{k \geq 1} \bar{w}_k(t, s, y, \theta) \quad \text{with } \bar{w}_k(t, s, y, \theta) := \inf\{v : G_k(t, s, y, \theta, v) \neq \emptyset\}. \quad (4.5)
\]
But the function \(\bar{w}(t, \cdot, \cdot, \cdot)\) would have to respect a functional relation along suitable orbits of the coordinates \((s, y, \theta)\) at any time \(t\), because of the equations (2.3) and (2.8), namely
\[
\bar{w}(t, s, y, \theta) = \bar{w}(t, s, y - \Delta)/f(y), y - \Delta, \theta - \Delta) + (s/f(y)) \int_0^\Delta f(y - x)dx, \quad \Delta \in \mathbb{R}.
\]
This suggests that one coordinate dimension is redundant and a \textquote{pde on curves} may be required to describe \( \dot{w} \). Indeed, it turns out that the state space can be reduced by considering the problem in new reduced coordinates with respect to which a DPP and a viscosity characterization can be shown for the function \( w \), following ideas from [BLZ16].

To derive dynamic programming principle for \( w \), we want to compare it at different points in time with the wealth process. Since \( w \) assumes zero initial risky assets, it is natural to consider the (fictitious) state process that would prevail if the trader would be forced to liquidate her position in the risky asset immediately (with a block trade). To this end, let

\[
S(S_t, Y^\Theta_t, \Theta_t) := S_t f(Y^\Theta_t - \Theta_t) = S_t f(Y^\Theta_t - \Theta_t) / f(Y^\Theta_t),
\]

\[
y(Y^\Theta_t, \Theta_t) := Y^\Theta_t - \Theta_t.
\]

The process \( S(y, \theta, \gamma) \) for \( y \) can be interpreted as the price of the asset that would prevail after \( \theta \) assets were liquidated, when \( s \) and \( y \) are the price of the risky asset and the market impact just before the trade, while \( y \) would be the level of the market impact after this trade. In this sense, we refer to the processes \( S(S_t, Y^\Theta_t, \Theta_t) \) and \( y(Y^\Theta_t, \Theta_t) \) as the effective price and impact processes, respectively, for a self-financing trading strategy \( \Theta \). Observe that both processes are continuous, even though the trading strategy \( \Theta \) may have jumps.

For the dynamic programming principle in Theorem 4.1, we are going to compare the liquidation wealth \( V^{\text{liq}} \) with the value function \( w \) along evolutions of \( (S(S_t, Y^\Theta_t, \Theta_t), y(Y^\Theta_t, \Theta_t)) \). While for the proof of the DPP we can follow [BLZ16, Proposition 3.3], we like to mention that the arguments simplify in technical terms and become more transparent when expressed in terms of our choice of \( V^{\text{liq}} \), instead of \( V^{\text{book}} \).

**Theorem 4.1** (Geometric DPP). \( \text{Fix } (t, s, y, v) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \).

(i) If \( v > w(t, s, y) \), then there exists \( \gamma \in \Gamma \) and \( \theta \in \mathcal{K} \) such that

\[
V^{\text{liq}, t, z, \gamma}_\tau \geq w(\tau, \mathcal{S}(S^{t, z, \gamma}_\tau, Y^{t, z, \gamma}_\tau, \Theta^{t, z, \gamma}_\tau), Y^{t, z, \gamma}_\tau - \Theta^{t, z, \gamma}_\tau)
\]

for all stopping times \( \tau \geq t \), where \( z = (S(s, y, -\theta), y + \theta, \tau, v) \).

(ii) Let \( k \geq 1 \). If \( v < w_{2k+2}(t, s, y) \), then for every \( \gamma \in \Gamma_k \), \( \theta \in \mathcal{K} \cap [-k, k] \) and a stopping time \( \tau \geq t \) we have

\[
P \left[ V^{\text{liq}, t, z, \gamma}_\tau > w_k(\tau, \mathcal{S}(S^{t, z, \gamma}_\tau, Y^{t, z, \gamma}_\tau, \Theta^{t, z, \gamma}_\tau), Y^{t, z, \gamma}_\tau - \Theta^{t, z, \gamma}_\tau) \right] < 1
\]

where \( z = (S(s, y, -\theta), y + \theta, \theta, v) \).

**Proof.** We follow the ideas of [BLZ16, proof of Prop.3.3], but present details for completeness. It is easy to see that for all \( k \geq 2 \) and \( (t, s, y, \theta) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R} \times (\mathcal{K} \cap [-k, k]) \)

\[
w_k(t, s, y, \theta) \geq w_{k+1}(t, S(s, y, \theta), y(\theta),) \text{ for all } k \geq 2 \quad (4.6)
\]

\[
w_{k-1}(t, S(s, y, \theta), y(\theta)) \geq \bar{w}_k(t, s, y, \theta) \quad (4.7)
\]

Now suppose that \( v > w(t, s, y) \). Then by definition of \( w \) there exists \( \theta \in \mathcal{K} \) and some \( \gamma \in \mathcal{G}(t, z) \) for \( z = (S(s, y, -\theta), y + \theta, \tau, v) \). As in [ST02, proof of Thm.3.1, Step 1], we have for all stopping times \( \tau \geq t \) (the first part of) the DPP for \( \bar{w} \): \( V^{\text{liq}, t, z, \gamma}_\tau \geq \bar{w}(\tau, S^{t, z, \gamma}_\tau, Y^{t, z, \gamma}_\tau, \Theta^{t, z, \gamma}_\tau) \). Then (i) follows from (4.6) by taking \( k \to \infty \).

To prove (ii), let \( v < w_{2k+2}(t, s, y) \) and suppose that there exists \( \gamma \in \Gamma_k \), \( \Theta \in \mathcal{K} \cap [-k, k] \) and a stopping time \( \tau \geq t \) such that \( V^{\text{liq}, t, z, \gamma}_\tau > w_k(\tau, \mathcal{S}(S^{t, z, \gamma}_\tau, Y^{t, z, \gamma}_\tau, \Theta^{t, z, \gamma}_\tau), Y^{t, z, \gamma}_\tau - \Theta^{t, z, \gamma}_\tau) \) for \( z = (S(s, y, -\theta), y + \theta, \tau, v) \). Then by (4.7), \( V^{\text{liq}, t, z, \gamma}_\tau > \bar{w}_{k+1}(S^{t, z, \gamma}_\tau, Y^{t, z, \gamma}_\tau, \Theta^{t, z, \gamma}_\tau) \) and thus, by [ST02, proof of Thm.3.1, Step 2], we get that \( v \geq \bar{w}_{2k+1}(t, S(s, y, -\theta), y + \theta, \theta) \). In particular, by (4.6) we conclude that \( v \geq w_{2k+2}(t, s, y) \), hence a contradiction.

**Remark 4.2.** Part (ii) of the theorem is stated in terms of \( w_k \) instead of \( w \) because of a measurable-selection argument employed in the proof, cf. [BLZ16, Remark 3.2].
To derive the pricing pde from the dynamic programming principle in Theorem 4.1, we need the dynamics of the continuous processes
\[ t \mapsto V^\text{liq}_t = \varphi(t, S_t, Y^\Theta_t, \Theta_t), \quad \psi(Y^\Theta_t, \Theta_t) \]  
for sufficiently smooth functions \( \varphi : [0, T] \times \mathbb{R} \times \mathbb{R} \) that will later serve as test functions when characterizing the value function as a viscosity solution.

**Lemma 4.3.** For every \( \Theta = (a, b, \gamma) \in \Gamma \) and every \( \varphi \in C^{1,2,1}([0, T] \times \mathbb{R} \times \mathbb{R}) \), we have
\[
d(V^\text{liq}_t - \varphi(t, S_t, Y_t)) = 
S_t \left( \frac{F(Y_t) - F(Y_t)}{f(Y_t)} \right) \{(\mu_t - \lambda_t h(y_t + \Theta_t)) dt + \sigma_t dW_t \} 
+ \left\{ -\varphi_t - 1/2\sigma_t^2 S_t \varphi_{S} + h(y_t + \Theta_t) \varphi_Y + \tilde{\gamma}(S_t, y_t, \Theta_t) \right\} dt, 
\]
with
\[ \tilde{\gamma}(s, y, \theta) = s h(y + \theta) \left( \lambda(s) F(y + \theta - y) - \frac{f(y + \theta) - f(y)}{f(y)} \right), \]
where \( S_t = S(S_t, Y^\Theta_t, \Theta_t), \quad y_t = y(Y^\Theta_t, \Theta_t) \) and the derivatives of \( \varphi \) are evaluated at \((t, S_t, Y_t)\).

**Proof.** Since \( S_t = S(S_t, Y^\Theta_t, \Theta_t) \) equals \( S_t f(Y^\Theta_t - \Theta_t) \), the product rule and \( f' = \lambda_f \) imply
\[
dS_t = S_t \left\{ (\mu_t - \lambda_t Y^\Theta_t - \Theta_t) h(Y^\Theta_t) \right\} dt + \sigma_t dW_t. \tag{4.9} \]

By Itô’s formula, we obtain
\[
d\varphi_t(S_t, Y^\Theta_t - \Theta_t) = \varphi_t dt + \varphi_S dS_t + \varphi_Y d(Y^\Theta_t - \Theta_t) + 1/2 \varphi_S^2 d[S]_t 
= \left\{ -\varphi_t + \lambda(Y^\Theta_t - \Theta_t) h(Y^\Theta_t) S_t \varphi_S - h(Y^\Theta_t) \varphi_Y + 1/2 \sigma_t^2 S_t^2 \varphi_{SS} \right\} dt 
+ \mu_t S_t \varphi_S dt + \sigma_t S_t \varphi dW_t. \tag{4.10} \]

With reference to (2.10), we have
\[
dV^\text{liq}_t = -h(Y^\Theta_t) S_t \frac{f(Y^\Theta_t) - f(Y^\Theta_t - \Theta_t)}{f(Y^\Theta_t - \Theta_t)} dt 
+ \mu_t S_t \frac{F(Y^\Theta_t) - F(Y^\Theta_t - \Theta_t)}{f(Y^\Theta_t - \Theta_t)} dt + \sigma_t S_t \frac{F(Y^\Theta_t) - F(Y^\Theta_t - \Theta_t)}{f(Y^\Theta_t - \Theta_t)} dW_t. \tag{4.11} \]

Combining (4.10) and (4.11) and rearranging the terms completes the proof. \( \square \)

**Remark 4.4.** Consider the case when \( \lambda \) is constant, i.e. \( f = \exp(\lambda) \). Then we simply have \( \tilde{\gamma} \equiv 0 \) and the dynamics of \( V^\text{liq} \) can be stated in a surprisingly simplified form, namely
\[
dV^\text{liq}_t = F(\Theta_t) dS_t, \]  
where \( S_t = S(S_t, Y^\Theta_t, \Theta_t) \) has the dynamics (4.9). As a consequence, the superhedging price (for the large investor) of an option with maturity \( T \) and pure cash settlement \( H(S_T) \) is at least the small investor’s price of \( H, \) in absence of the large trader, when the price process is \( S \) instead. Indeed, for each (bounded) superhedging strategy \( \Theta \) (by the large investor) with initial capital \( v \) there exists \( P^\Theta \approx P \) (on \( \mathcal{F}_T \)) such that \( S = S_0 \cdot \mathcal{E}(\sigma W) \) under \( P^\Theta \) for a \( P^\Theta \)-Brownian motion \( W \). Hence, \( V^\text{liq}(\Theta) \) is a \( P^\Theta \)-martingale and thus \( v \leq E^{P^\Theta}[H(S_T)] = E^{P^\Theta}[H(S_T)] \) [recall that \( \Theta_T = 0 \), implying \( S_T = S_T \)]. On the other hand, a Feynman-Kac argument shows that \( E^{P^\Theta}[H(S_T)] \) is just the classical Black-Scholes price for a small investor in a frictionless market with risky asset process \( S \). As \( \Theta \) was an arbitrary superhedging strategy with initial capital \( v \), taking the infimum yields the claim.
The above observation shows a notable difference to the model in [BB04, Thm. 5.3], where the price for the large investor would be typically smaller. This is mainly due to their specification of superhedging strategies, according to which a large trader can try to reduce at maturity the payoff of the option to a larger extend, by exploiting her price impact on the underlying at maturity. That means, she can vary at maturity her risky asset position in order to minimize the payoff with less constraints, and immediately afterwards could unwind any residual at no additional cost (by absence bid-ask spread). In contrast, our setup is more restrictive on such manipulative behavior by imposing as a constraint on the strategies that they have to replicate the physical delivery part exactly, i.e. after settlement the large trader has to hold a non-negative cash position without residual holdings in the risky asset.

At this point, we like to note that an argument as above does not apply in the general case with non-constant $\lambda$ for our price impact model. In fact, examples in Section 7 also reveal situations where superhedging can be cheaper for the large trader, cf. Example 7.1.

5 The pricing PDEs and main results

Next, we determine the terminal value for the function $w$ at maturity date $T$, that will serve as a boundary condition for the pricing pde. Recall that $\mathcal{K}$ is the (constraint) set in which trading strategies take values and set $\mathcal{K}_n = \mathcal{K} \cap [-n, n]$ for $n \in \mathbb{N}$.

**Lemma 5.1** (Boundary condition). For $n \in \mathbb{N}$, let

$$H_n(s, y) := \inf \left\{ f_0 \left( s \frac{f(s + \theta)}{f(y)}, y + \theta \right) + s \frac{F(y + \theta) - F(y)}{f(y)} \mid \theta \in \mathcal{K}_n, \theta = g_1 \left( s \frac{f(s + \theta)}{f(y)}, y + \theta \right) \right\}.$$  

Then we have $w_n(T, \cdot) = H_n(\cdot)$ and $w(T, \cdot) = H(\cdot)$, where the function $H$ is given by

$$H := \inf_{n \geq 0} H_n.$$

*Proof.* At maturity time $T$, the hedger of the option has to do a block trade of size $\theta$ in order to meet the physical delivery part specified by $g_1$, thereby moving the price of the underlying from $s$ to $s \frac{f(s + \theta)}{f(y)}$ and the impact level from $y$ to $y + \theta$. Such a block trade incurs costs of size $s \frac{F(y + \theta) - F(y)}{f(y)}$ and hence it superreplicates the payoff $(g_0, g_1)$ if the hedger can cover these costs and the required cash delivery part, which after the block trade is $g_0 \left( s \frac{f(s + \theta)}{f(y)}, y + \theta \right)$. 

*Remark 5.2.* Note that $H(s, y) = +\infty$ holds if the equation $\theta = g_1 \left( s \frac{f(s + \theta)}{f(y)}, y + \theta \right)$ does not have a solution $\theta$ in $\mathcal{K}$. 

As we do not know at this point whether the value function $w$ is continuous, we need to work with discontinuous viscosity solutions and hence to consider the relaxed semi-limits

$$w_\ast(t, s, y) := \lim_{(t', s', y', k) \to (t, s, y, \infty)} w_k(t', s', y'),$$

$$w^\ast(t, s, y) := \lim_{(t', s', y', k) \to (t, s, y, \infty)} w_k(t', s', y'),$$

where the limits are taken over $t' < T$. Recall that $w$ is a (discontinuous) viscosity solution (of our pricing equations, see Sections 5.1 and 5.2) if $w_\ast$ (resp. $w^\ast$) is a supersolution (resp. subsolution). For proving the viscosity property we make the following assumption.

**Assumption 5.3.**

**(Bounded value function):** The functions $w_\ast$ and $w^\ast$ are bounded on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$;

**(Regular payoff):** $H$ from (BC) is continuous, bounded, and $H_n \downarrow H$ uniformly on compacts.

In particular, Assumption 5.3 implies that $w(T, \cdot)$ is finite. This means that the payoff is well-behaved in terms of the physical delivery part, i.e. if the trader was supposed to fulfill his obligation from selling the option immediately, he would be able to do so in any situation (in any state of $(s, y)$) with an admissible trade, provided that he has enough capital.
5.1 Case study for a general bounded price impact function \( f \)

In this section, the following assumption is supposed to hold.

**Assumption 5.4.** The resilience function \( h \) is Lipschitz and bounded, the price impact function \( f \) is bounded away from 0 and \( \infty \), i.e. \( \inf_{R} f > 0 \) and \( \sup_{R} f < +\infty \), \( \lambda \) is bounded and continuously differentiable with bounded derivative, and \( K = R \) (no delta constraints).

Under Assumption 5.4, the antiderivative \( F \) from (2.7) and its inverse \( F^{-1} \) are bijections \( R \to R \) and Lipschitz continuous with Lipschitz constants \( \sup_{R} f < +\infty \) and \( 1/\inf_{R} f \), respectively.

To derive the pricing pde just formally (at first) in this case, let \( (t, s, y) \in [0, T) \times R+ \times R \) and apply formally part (i) of DPP in Theorem 4.1 to \( v = w(t, s, y) \) (assuming that the infimum in the definition of \( w \) is attained) and \( \tau = t+ \), together with Lemma 4.3 for \( \varphi = w \), assuming that \( w \) is smooth enough. Thus we get the existence of \( \theta^* \) such that

\[
0 \leq s(F(y + \theta^*) - F(y)) - w_{S}(t, s, y)\{\langle \mu_t - \lambda(y)h(y + \theta^*) \rangle dt + \sigma dW_t \}
\]

\[
+ \{ - w_t(t, s, y) - \frac{1}{2}\sigma^2 s^2 w_{SS}(t, s, y) + h(y + \theta^*)w_{Y}(t, s, y) + \tilde{\mathbb{G}}(s, y, \theta^*) \} dt.
\]

Still arguing just at a formal level, this cannot hold unless

\[
F(y + \theta^*) = F(y)w_{S}(t, s, y) + F(y) \quad \text{and} \quad -w_{t}(t, s, y) - 1/2\sigma^2 s^2 w_{SS}(t, s, y) + h(y + \theta^*)w_{Y}(t, s, y) + \tilde{\mathbb{G}}(s, y, \theta^*) \geq 0. \tag{5.3}
\]

In particular, \( \theta^* = \theta^*(y, t, s) = F^{-1}(f(y)w_{S}(t, s, y) + F(y)) - y \). The second part of DPP in Theorem 4.1 will actually give that the drift term must be 0, i.e. we should have equality in (5.3). This formally motivates that the form of the pricing pde for \( w \) should be

\[
-w_{t} - \frac{1}{2}\sigma^2 s^2 w_{SS} + \tilde{h}(t, s, y)(w_{Y} + s\lambda(y)w_{S}) + s\tilde{h}(t, s, y)(1 - \frac{f(t, s, y)}{F(y)}) = 0, \tag{PDE}
\]

where for \( (t, s, y) \in [0, T) \times R+ \times R \)

\[
\tilde{h}(t, s, y) := h \circ F^{-1}(f(y)w_{S}(t, s, y) + F(y)),
\]

\[
\tilde{f}(t, s, y) := f \circ F^{-1}(f(y)w_{S}(t, s, y) + F(y)).
\]

Observe that the pde is semilinear and degenerate (since not containing second order derivatives involving the \( y \)-variable). Our main result is as follows.

**Theorem 5.5.** Under Assumption 5.3 and Assumption 5.4, the value function \( w \) of the superhedging problem is continuous and is the unique bounded viscosity solution to (PDE) with the boundary condition \( w(T, \cdot) = H(\cdot) \), where \( H \) is defined in (BC).

**Proof.** The viscosity property, i.e. that \( w \) (respectively \( w^* \)) is a viscosity supersolution (respectively subsolution), follows by the dynamic programming principle in Theorem 4.1 together with Lemma 4.3. The key arguments are presented in Section 9 in detail for the case where \( \lambda \) is constant, which actually leads to a slightly more involved pricing pde (PDE\( ^* \)) (including gradient constraints) requiring additional justifications.

The comparison result of Theorem 9.5 proves uniqueness and continuity, cf. Remark 9.7. \( \square \)

Let us conclude this section by commenting on some consequences from Theorem 5.5 for the superhedging price and the existence of a respective hedging strategy. A numerical example is presented in Section 7.

**Remark 5.6** (Dependence on displacement from unaffected price). Like in the classical case of liquid markets (without market impact), the superhedging price does not depend on the drift in the unperturbed price process. This may be seen more directly by working
under the equivalent martingale measure for \( \hat{S} \) from the beginning. On the other hand, the
superhedging price depends non-trivially on the initial level of impact \( y \) and the resilience
function \( h \), and can do so even for option payoffs of the form \((g_0(s),0)\), i.e. not depending
on the level of impact. So it turns out that for the pricing and hedging (cf. Remark 5.8) the
perturbation of the market price from the ‘unaffected’ value is a relevant state variable.

**Remark 5.7** (Permanent impact). Observe that for only permanent impact, that means for
\( h \equiv 0 \), (PDE) simplifies to the classical (frictionless) Black-Scholes pricing equation and
hence the superhedging price for the large trader equals the Black-Scholes price for the option
with payoff \( H \).

**Remark 5.8** (Replicating the option payout). Under sufficient regularity, it turns out that a
strategy can be constructed that is perfectly replicating the option payout from the (minimal)
superhedging price. This means, we have dynamic hedging in the sense of replication, like in the
frictionless complete Black-Scholes model.

To this end, suppose that a function \( w \in C^{1,1,1}_b([0,T] \times \mathbb{R}_+ \times \mathbb{R}) \) solves the pricing pde
(PDE) with the boundary condition \( w(T,\cdot) = H(\cdot) \). Then for any \( \varepsilon > 0 \) a superhedging
strategy with an initial cost of \( w(0,s,y) + \varepsilon \) can be constructed as follows. Consider the
self-financing strategy \((\beta, \Theta)\) with \( \beta_{\text{yo}} = w(0,s,y) + \varepsilon, \Theta_0 = F^{-1}(f(y)w_S(0,s,y) + F(y)) - y \),
meaning that a block trade of size \( \Delta \Theta_0 = \Theta_0 \) is performed at time 0, and
\[
\Theta_t = F^{-1}(f(y^\Theta)w_S(t,S(t),T,\Theta_t),Y(t)) - y^\Theta = 0 \quad \text{for } t \in [0,T),
\]
(5.4)
\[
\Theta_T = 0, \quad \text{i.e. } \Delta \Theta_T = \Theta_{T-} = 0,
\]
(5.5)
where \( y^\Theta = Y^\Theta - \Theta \). Then by Lemma 4.3, together with (5.4) and (PDE) we conclude that
\[
\varepsilon = V^{\text{liq}}(0,\Theta) - w(0,s,y) = V^{\text{liq}}(T,\Theta) - w(T,S(T),T,\Theta),Y(T))
\]
\[
= V^{\text{liq}}(0,\Theta) - H(S(T),T,\Theta, Y(T)),
\]
\[
= V^{\text{liq}}(T,\Theta) - H(S(T),T,\Theta_T), \Theta_T = 0,
\]
where the last line follows from (5.5). By definition of \( H \), having \( H + \varepsilon \) in cash at time \( T \)
will be enough to superreplicate the European claim with payoff \( (g_0,g_1) \) by doing a possible
additional final block trade of size \( \Delta^T \). Note that such a block trade would not affect \( V^{\text{liq}}_T \).
Hence, the strategy \( \Theta + \mathbb{1}_{(T)} \Delta^T \) will be superreplicating for the European claim. Note that
one could take \( \varepsilon = 0 \) if the constructed strategy is bounded and the infimum in the definition
of \( H_n \) is attained (cf. Lemma 5.1), i.e. we have a replicating strategy in this case.

An application of Itô’s formula gives that a strategy \( \Theta \) satisfying the fixed-point problem
(5.4) can be obtained, under suitable regularity, by solving the following system of SDEs
\[
dS_t = S_t(\mu_t - \lambda(y^\Theta_t)h(y^\Theta_t + \Theta_t)) dt + \sigma dW_t,
\]
\[
d\Theta_t = a(t,S_t,y^\Theta_t,\Theta_t) dt + b(t,S_t,y^\Theta_t) dW_t,
\]
(5.6)
\[
dy^\Theta_t = -h(y^\Theta_t + \Theta_t) dt,
\]
with initial conditions \( S_0 = s, y^\Theta_0 = y \) and \( \Theta_0 = F^{-1}(f(y)w_S(0,s,y) + F(y)) - y \), where
\[
a(t,s,y,\theta) := h(y + \theta) \left( 1 - \frac{\lambda f w_S - f - \lambda s w_S}{f(F^{-1}(f w_S + F))} \right) + \frac{w_S + s \mu w_S + 1/2 \sigma^2 s^2 w_{SSS}}{f(F^{-1}(f w_S + F))},
\]
\[
b(t,s,y) := \frac{\sigma s w_{SSS}}{f(F^{-1}(f w_S + F))},
\]
and where we write \( f = f(y), \lambda = \lambda(y) \), etc., when arguments of functions have not been
specified, to ease the notation. Thus, an optimal superhedging strategy accounts for the
transient nature of price impact.
Let us comment here on Assumption 5.4 that implies bijectivity of \( F \) on \( \mathbb{R} \). Observe that its inverse \( F^{-1} \) is used to describe the optimal control \( \theta^* \). Similar conditions are also crucial for the results in [BB04] and [BLZ16]: See the surjectivity assumption (A5) in [BB04] and the invertibility assumption (H2) in [BLZ16]. The next section shows how departing from this assumption leads naturally to singularity in the pricing pde with respect to the gradient. Indeed, the lack of invertibility of \( F \) requires conditions on \( w_S \) so that \( \theta^* \) could be derived. Therefore, the analysis there will involve constraints on the ‘delta’, i.e. on the holdings in the risky asset, what in pde terms translates to constraints on the spacial gradient \( w_S \).

5.2 Case study for price impact of exponential form

In this section, we extend the analysis to a natural case where the antiderivative of the price impact function is not assumed to be surjective. To this end, we take the price impact function to be of exponential form \( f(x) = \exp(\lambda x) \) with \( \lambda \) being a constant, meaning that the relative marginal price impact function \( \lambda = f'/f > 0 \) is constant. A peculiarity of this case is that at any time \( t \), knowing the (marginal) price \( S_t \) for the stock is sufficient to know the impact from an instant block trade, since after a block trade of size \( \Delta \) the price would be \( S_t f(Y_t + \Delta) = S_t \exp(\lambda \Delta) \). Hence, the relative displacement \( f(Y^\theta) \) of \( S \) from the fundamental price \( S \) is immaterial to determine the price impact from a block trade, in difference to the situation of Section 5.1. Motivated by Remark 3.5, we consider short-selling constraints, i.e. trading strategies are required to take values in \( \mathcal{K} = [-K, \infty) \) for some \( K > 0 \).

To derive (heuristically, at first) the pricing pde, let us apply formally Theorem 4.1 for \( v = w(t, s, y) \) at \( t, s, y, \tau = t^+ \), provided that \( w \) is smooth enough, to get the existence of \( \theta^* \in \mathcal{K} \) such that, using Lemma 4.3, we have

\[
\mathcal{L}^{\theta^*} w(t, s, y) dt - s(w_s(t, s, y) - \exp(\lambda y)/\lambda + 1/\lambda)(\sigma dW_t + \eta_t dt) \geq 0, \tag{5.7}
\]

where \( \eta_t = \mu_t - \lambda h(y + \theta^*) \) and

\[
\mathcal{L}^{\theta^*} w(t, s, y) := -w_t(t, s, y) + h(y + \theta^*)w_y(t, s, y) - \frac{1}{2} \sigma^2 s^2 w_{SS}(t, s, y).
\]

As in Section 5.1, the diffusion part in (5.7) should vanish, giving the optimal control

\[
\theta^* = \frac{1}{\lambda} \log \left( \lambda w_S(t, s, y) + 1 \right),
\]

and from the drift part we identify the pricing pde \( \mathcal{L}^{\theta^*} w(t, s, y) = 0 \). The constraint \( \theta^* \in \mathcal{K} \) is now equivalent to \( \mathcal{H}_K w(t, s, y) \geq 0 \), where for a smooth function \( \varphi \) we set

\[
\mathcal{H}_K \varphi(t, s, y) := \lambda \varphi_S(t, s, y) + 1 - e^{-\lambda K}.
\]

Thus we conclude, just formally, that \( w \) should be a solution to the variational inequality

\[
\mathcal{F}_K[w] := \min \{ \mathcal{L}^{\theta[w]} w, \mathcal{H}_K w \} = 0 \quad \text{on } [0, T) \times \mathbb{R}_+ \times \mathbb{R}, \tag{PDE^\mathcal{F}}
\]

where

\[
\theta[w](t, s, y) := 1/\lambda \cdot \log \left( \lambda w_S(t, s, y) + 1 \right). \tag{5.8}
\]

As usual, the gradient constraints propagate to the boundary, meaning that the boundary condition for (PDE^\mathcal{F}) should be

\[
\min \{ w(T, \cdot) - H, \mathcal{H}_K w \} = 0. \tag{BC^\mathcal{F}}
\]

After this motivation, we state the main result for exponential price impact \( f = \exp(\lambda \cdot) \).
Theorem 5.9. Suppose that the resilience function $h$ is Lipschitz continuous and Assumption 5.3 holds. Then the value function $w$ of the superhedging problem is continuous and is the unique bounded viscosity solution to the variational inequality (PDE\textsuperscript{P}) with boundary condition (BC\textsuperscript{P}).

**Proof.** The proofs are postponed to Section 9. The viscosity super-/sub-solution property are proven in Theorem 9.2 and Theorem 9.3 respectively, while uniqueness and continuity follow from the comparison result of Theorem 9.6, cf. Remark 9.7. 

Corollary 5.10. In the setup from Theorem 5.9, suppose moreover that the payoff function $(g_0, g_1)$ is not depending on the level of impact $y$ but just on the price $s$ of the underlying. Then the superhedging price is a function in $(t, s)$ only and the pricing pde (PDE\textsuperscript{P}) simplifies to the Black-Scholes pde with gradient constraints. In this case, if the face-lifted payoff $F_{\mathcal{K}}[H]$ is continuously differentiable with bounded derivative, where

$$F_{\mathcal{K}}[H](s) := \sup_{x \geq 0} \left\{ H(s + x) + \frac{1 - e^{-\lambda K}}{\lambda} x \right\}, \quad s \in \mathbb{R}_+,$$

with the convention that $H = H(0)$ on $(-\infty, 0]$, then the superhedging price (for the large trader) coincides with the friction-less Black-Scholes price for the face-lifted payoff $F_{\mathcal{K}}[H]$.

**Proof.** If $(g_0, g_1)$ is a function of the price $s$ of the underlying only (but not of $y$), then it is easy to see that $H$ is such as well and that the dimension of the state process can be reduced by omitting the impact process $Y$. In this case, the stochastic target problem in Section 4 could be formulated for the new state process and thus the value function would be a function on $(t, s)$ only. The same analysis can be carried over to derive the pricing pde and to prove viscosity solution property of the value function. The pricing pde in this case would be the Black-Scholes pde with gradient constraints since the term $h(Y)\varphi_Y$ in Lemma 4.3 would not be present. Hence, the superhedging price in our large investor model would coincide with the superhedging price under delta constraints in the small investor model for the payoff $H$ (because it solves the same pde). In this one-dimensional setup, this price coincides with the Black-Scholes price for the face-lifted payoff $F_{\mathcal{K}}[H]$, cf. [CEK15, Proposition 3.1].

6 Combined transient and permanent price impact

Here we show how our analysis can be extended to a more general form of persistent inter-temporal impact, which has a transient and as well an additional permanent component. In the absence of future trading by the large investor, the former component can decay completely as time goes by, whereas the latter is given by an additional term $\eta \Theta, \delta \Theta$ constituting a permanent (non-decaying) impact from cumulative trading until $u$ onto prices at all future times $t \geq u$. To this end, we generalize the model as follows.

For $\eta \geq 0$, the marginal price of the risky asset (for trading an infinitesimal quantity) is

$$S_t := f(\eta \Theta_t + Y_t^{\Theta}) S_t,$$

in modification of equation (2.3), with $Y^{\Theta}$ being given by (2.2). Following the arguments in [BBF17b, Section 5.4], the (stable) proceeds from a general semimartingale strategy $\Theta$ are

$$\tilde{L}(\Theta) := \frac{1}{1 + \eta} \left( \int_0^t F(\eta \Theta_t + Y_t^{\Theta}) \, d\tilde{S}_t - \int_0^t \tilde{S}_t f(\eta \Theta_t + Y_t^{\Theta}) h(Y_t^{\Theta}) \, dt - SF(\eta \Theta + Y^{\Theta})|_{t=0} \right).$$

In particular, a block trade $\Delta \Theta_t$ yields the proceeds $-\tilde{S}_t \frac{1}{1 + \eta} \int_0^1 \int_0^1 f(\eta \Theta_t + Y_t^{\Theta}) \, d\tilde{S}_t$. Thus, following the discussion in Section 2, the volume effect process (in the spirit of [PSS11]) in this case is $\eta \Theta + Y^{\Theta}$ and thereby has a permanent and a transient component. The dynamics of the instantaneous liquidation value process $V^{\Theta}$ now satisfies

$$(1 + \eta) dV^{\Theta}_t = (F(\eta \Theta_t + Y_t^{\Theta}) - F(Y_t^{\Theta} - \Theta_t)) \, d\tilde{S}_t - h(Y_t^{\Theta})(f(\eta \Theta_t + Y_t^{\Theta}) - f(Y_t^{\Theta} - \Theta_t)) \, dt.$$
It is worth noting that the generalization by an additional permanent impact is not changing the effective price and impact processes $S(S, Y^{θ*}, Θ^*)$ and $y(Y^{θ*}, Θ)$ because the permanent component vanishes for asset holdings with zero shares in the risky asset. Therefore, the previous analysis carries over to additional permanent impact, with adjustments as follows:

- The boundary condition in Lemma 5.1 needs to be modified by adding the prefactor $1 + η$ to $θ$, when $θ$ appears as an argument of a function;
- In Lemma 4.3, $F(Y^{θ*})$ should be substituted by $F(ηθ + Y^{θ*})$, all the fractions should be divided by $1 + η$ and $ś$ will now become $ś''$ with
  
  \[
  ś''(s, y, θ) := s'h(y + θ) \left( λ(y) \frac{F(y + (1 + η)θ) - F(y)}{(1 + η)f(y)} - \frac{f(y + (1 + η)θ) - f(y)}{(1 + η)f(y)} \right).
  \]

Let us first discuss the setup of Section 5.1 which essentially required $F$ to be invertible. In this case, the pricing pde will have the same structure as (PDE) with the following modifications $\tilde{h} = \tilde{h}''$ and $\tilde{f} = \tilde{f}''$ replacing the former $\tilde{h}$ and $\tilde{f}$, namely

\[
\tilde{h}''(t, s, y) = \tilde{h} \left( \frac{1}{1 + η} F^{-1}(1 + η) f(y) φS(t, s, y) + F(y) + \frac{y}{1 + η} y \right),
\]

\[
\tilde{f}''(t, s, y) = f \circ F^{-1}(1 + η) f(y) φS(t, s, y) + F(y)).
\]

An optimal hedging strategy $Θ^*$, if it exists, satisfies (as in Remark 5.8 for $η = 0$)

\[
(1 + η)θ^*_1 = F^{-1}(1 + η) f(y_1^*) φS(t, s, y_1^*) + F(y_1^*) - y_1^*,
\]

where $S^* = S(S, Y^{θ^*}, Θ^*)$ and $y^* = y(Y^{θ^*}, Θ^*)$. Hence, the large trader’s optimal strategy also reflects the permanent component in addition to the displacement from the fundamental price process tracked by $Y^{θ*}$.

In the setup from Section 5.2, we again consider portfolio constraints $θ ∈ K = [−K, +∞)$ in order to derive the pricing pde. Thanks to $ś'' = 0$, the pricing pde here simplifies to

\[
\min \{-w - \frac{1}{2} σ^2 Y S^2 W + \tilde{h}(y + θ^*) W V, \lambda(1 + η) φS + 1 - e^{-λ(1 + η)K} \} = 0, \quad ∀(t, s, y) ∈ [0, T) \times \mathbb{R}^+ \times \mathbb{R},
\]

where $θ^* = \frac{1}{λ(1 + η)} \log \left( λ(1 + η) W S + 1 \right)$, with boundary condition

\[
\min \{w(T, \cdot) - H, \lambda(1 + η) φS + 1 - e^{-λ(1 + η)K} \} = 0,
\]

where $H$ is the modified boundary condition from Lemma 5.1, as explained above. In particular, the pricing pde with permanent impact coincides with the pricing pde with purely transient impact but with suitably modified $λ$, that in this case becomes $λ(1 + η)$.

### 7 Numerical example

This section discusses numerical results on the superhedging price $w$ characterized by (PDE), cf. Theorem 5.5. For the computations we consider an impact function

\[
f(x) = 1 + \arctan(x)/10, \quad x ∈ \mathbb{R},
\]

satisfying Assumption 5.4. Note that $λ(x) = 1/(10(1 + x^2)f(x))$ varies most within the range of about $(-4, 4)$; Here, the change in impact is significant, see Figure 1a. Apart from satisfying our assumptions and having $F(x) = x + (x \arctan(x) - 1/2 \log(1 + x^2))/10$ in explicit form, being useful for the implementation, it turns out that similar shape of impact has been observed in the calibration of a related propagator model to real data, see [BL12, Appendix].

For $h(y) = βy$ with $β = 1$, we compare the large trader’s price of a European call option with physical delivery at maturity $T = 0.5$ and strike $K = 50$, and the option’s frictionless
price, i.e. the classical Black-Scholes price of a European call option for the same model parameters. Let us recall that the case \( f = 1 \) in our market impact model coincides with the Black-Scholes model. The volatility \( \sigma \) is set to 0.3. The payoff for the large trader is
\[
H(s, y) = \left( s \frac{F(y+1) - F(y)}{f(y)} - K \right) I_{\{s \geq K\}}
\] that we “smooth out” by approximating the indicator function by linearly interpolating 0 and 1 between \( K - 0.5 \) and \( K \).

To approximate both prices, we solve the corresponding pdes using (semi-implicit) finite difference scheme in the bounded region \((y, s) \in [-20, 20] \times [0, 200] \). For our simulation we set the following boundary condition for \( t < T \):
\[
\frac{\partial w}{\partial s} = \frac{(F(y+1) - F(y))}{f(y)} \quad \text{on} \quad [-20, 20] \times \{200\}, \quad \frac{\partial w}{\partial y} = 0 \quad \text{on} \quad [-20, 20] \times [0, 200] \cup [-20, 20] \times \{0\}.
\]
Indeed, for initial impact \( y \) close to -20 or +20 the impact function is approximately constant and until maturity \( T \) resilience would be unlikely to bring back the level of impact to the region where the changes in \( f \) are significant, see Figure 1a; Thus we might expect that the price would not depend that much on the level of impact. On the other hand, for larger values of \( s \) one may expect the price to depend approximately linearly on \( s \) (like the payoff profile). The difference between the Black-Scholes price and the large trader’s price (as a function of the risky asset price \( s \) and the level of impact \( y \)) is shown in Figure 1b. Let us point out that the Black-Scholes price does not depend on level of impact \( y \).

![Figure 1](image-url)

(a) Impact function \( f \) (in blue) and its logarithmic derivative \( \lambda \) (in purple)
(b) The difference \( p_{BS} - p_{large} \) of prices of a European call option with physical delivery, resilience rate is \( \beta = 1 \)
(c) The frictionless Black-Scholes price and the large trader’s price for call option with physical delivery, resilience rate \( \beta = 1 \) and initial impact level \( y = 0 \)
(d) Difference between large trader’s prices \( p_{\beta=1}^{large} \) and \( p_{\beta=0}^{large} \) for a call option with physical delivery with \( \beta = 1 \) and without \( \beta = 0 \) resilience, for initial impact level \( y = 0 \)

Figure 1: Numerical computations with impact function \( f \) from (7.1), \( \sigma = 0.3 \), \( T = 0.5 \), strike \( K = 50 \), resilience function \( h(y) = \beta y \).

Numerical examples indicate that the superrepllication price for the large trader dominates the frictionless Black-Scholes price for the call option with physical delivery, see Figure 1c.
But let us note here that such property does not hold in general. For instance, it does not appear to be the case for the European call with pure cash delivery, where numerical examples can show that the price could also be smaller for the large investor, typically if the impact level at inception is away from zero, see also Example 7.1. On the other hand, superhedging becomes more expensive for the large trader when she has to deliver physically the asset at maturity, since she has to have bought at maturity the physical asset, if the option settles in-the-money, meaning that she needs to do a final block trade to buy what is lacking for one physical unit. But this trade has price impact on the underlying in a direction that is unfavorable to her, as it increases the call option payout.

In addition, observe that the presence of resilience renders the level of impact (or the displacement from the fundamental price) to be a relevant state variable for the problem. For the setup of our numerical example for instance, the price of a European call option with physical delivery, when hedging is initiated at neutral impact level \( y = 0 \), is cheaper in the presence of resilience than in the case of no resilience, i.e. only permanent impact, see Figure 1d. This is however not always the case, for example if impact at initiation is negative \( y < 0 \). To conclude, the dependence in \( y \) of the option’s price is complex: apart from the drift on the prices that the level of impact induces, it also determines the price impact from intermediate tradings and the final trade (enforced by settlement rules). And we have mentioned examples where superhedging could be less or more expensive for the large investor in the presence or absence of resilience.

**Example 7.1.** Here we show that the price of an European option in the Black-Scholes model (for a small investor) could indeed be greater than the superhedging price for the large trader of this option with pure cash delivery. More specifically, for maturity \( T > 0 \) consider the solution \( v^{BS} \) of the Black-Scholes pde with bounded and smooth terminal condition \( H \) that has bounded derivatives, where we moreover assume that \( \partial_S H \geq 0 \), for instance a smooth approximation of a bull call spread option. Note that in particular \( \partial_S v^{BS} \geq 0 \) and the derivatives of \( v^{BS} \) are bounded. We compare now \( v^{BS}(0, \cdot) \) with \( v(0, \cdot, y) \) for large values of \( y \), where \( v = w \) with \( w \) from Theorem 5.5 with terminal condition \( H \). Note that when \( y = Y_{0-} > 0 \) the affected price process includes additional drift in favorable for the large trader direction.

Let \( \Theta \) with \( \Theta_{0-} = 0 \) be such that \( \Theta_T = 0 \) (corresponding to only cash delivery at maturity) and for \( t \in [0, T] \)

\[
\Theta_t = F^{-1}(\partial_S v^{BS}(t, S_t) f(Y^\Theta_t) + F(Y^\Theta_t)) - y^\Theta_t, \tag{7.2}
\]

where \( y^\Theta = Y^\Theta - \Theta \) and \( S = f(Y^\Theta) S \). Since \( v^{BS} \) is smooth, the arguments in Remark 5.8 ensure existence of such \( \Theta \), while positivitity of \( \partial_S v^{BS} \) implies \( \Theta \geq 0 \) on \([0, T] \). Now for the self-financing portfolio \((\beta, \Theta)\) with initial cash holdings \( \beta_{0-} = v^{BS}(0, S_{0-}) \) we have by (4.9), (4.11) and (7.2) (recall that \( S_{0-} = S_0 \))

\[
V_{T-}^{eq} = v^{BS}(0, S_0) + \int_0^T \partial_S v^{BS}(t, S_t) dS_t
- \int_0^T S_t h(Y^\Theta_t) \left( \frac{f(Y_t^\Theta) - f(Y_t^\Theta - \Theta_t)}{F(Y_t^\Theta) - F(Y_t^\Theta - \Theta_t)} - \lambda(Y_t^\Theta - \Theta_t) \right) dt
= H(S_T) - \int_0^T S_t h(Y^\Theta_t) \left( \frac{f(Y_t^\Theta) - f(Y_t^\Theta - \Theta_t)}{F(Y_t^\Theta) - F(Y_t^\Theta - \Theta_t)} - \lambda(Y_t^\Theta - \Theta_t) \right) dt. \tag{7.3}
\]

In particular, if the integrand in (7.3) is negative on \([0, T] \), then \((\beta, \Theta)\) would be a superhedging strategy for the large trader with initial capital \( \beta_{0-} = v^{BS}(0, S_{0-}) \) and hence

\[
v(0, S_{0-}, Y_{0-}) \leq v^{BS}(0, S_{0-}). \tag{7.4}
\]

One can show that the integrand will be negative for instance when \( Y^\Theta \geq 0 \) on \([0, T] \) and \( \lambda \) is strictly decreasing (at least on a compact set containing the range of \( Y^\Theta \) and \( Y^\Theta - \Theta \)).

Such a situation could arise if for example \( Y_{0-} \) is large enough. Alternatively, a negative
8 Pricing and hedging of covered options

A key conclusion from [BLZ16, BLZ17] is that the way in which the hedger forms the hedging strategy and delivers the payoff is crucial for the pricing equation. In our setup so far, the trading actions of the hedger at inception and maturity have been assumed to have price impact on the underlying and the respective superhedging price for the option was characterized by a degenerate semilinear pde. In contrast, the present section now studies the problem for covered options, for which the buyer of the option can be required (at discretion of the hedger) to provide the required initial (delta) hedging position and to accept a mix of cash and stocks (at current market prices) as a final settlement. In this sense, the hedger of the option is not exposed to initial and terminal impact when forming and unwinding the hedging position for covered options. The corresponding pricing equation turns out to be fully non-linear and singular in the second-order term. This induces gamma constraints, whereas for non-covered options singularity arises in the first order derivative and induces delta constraints, see Section 5.2. We restrict ourselves to a sketch of the derivation of the pricing pde and explain, how it can be obtained by adapting the analysis from [BLZ16].

Let us consider continuous hedging strategies that are Itô processes

\[
d\Theta_t = a(t)\,dt + b(t)\,dW_t, \quad \Theta_0 = \theta_0 \in \mathbb{R}, \tag{8.1}
\]

where \(a\) and \(b\) are continuous processes with some integrability conditions. For such controls \(\Theta\), the market impact process and the perturbed price process take the form:

\[
dY_t = (-b(Y_t) + a(t))\,dt + b(t)\,dW_t, \quad Y_0 = y,
\]

\[
dS_t = d\left(f(Y_t)S_t\right) = S_t\left[\xi(t)\,dt + (\sigma + \lambda(Y_t)b(t))\,dW_t\right]
\]

\[
= S_t\left[\mu - \lambda(Y_t)h(Y_t) + \lambda(Y_t)a(t) + 0.5(\lambda(Y_t)^2 + \lambda'(Y_t))\,dt + \lambda(Y_t)\sigma b(t)\right]
\]

\[
+ (\sigma + \lambda(Y_t)b(t))\,dW_t, \tag{8.2}
\]

with \(S_0 = f(y)\bar{s}\) (initial impact from acquiring \(\theta_0\) shares at \(t = 0\) is omitted) and with

\[
\xi(t) := \mu - \lambda(Y_t)h(Y_t) + \lambda(Y_t)a(t) + 0.5(\lambda(Y_t)^2 + \lambda'(Y_t))\,dt + \lambda(Y_t)\sigma b(t) \tag{8.3}
\]

as an abbreviation for the long term for the drift coefficient process in the last equation. Note that \(\theta_0, a\) and \(b\) have to be to be determined for a (super-)replicating strategy. Using integration by parts in (2.6), the proceeds from a continuous strategy \(\Theta\) rewrite as

\[
L_T(\Theta) = -\int_0^T S_t\,d\Theta_t - \frac{1}{2}[S, \Theta]_T - \frac{1}{2}\int_0^T \sigma S_t\,d[\Theta, W]_t. \tag{8.4}
\]

For a self-financing strategy \((\beta, \Theta)\), the book wealth (recall (2.11)) at time \(T\) is

\[
V_T^{\text{book}}(\Theta) = \beta_0 + L_T(\Theta) + \Theta_T S_T.
\]

Consider a contingent claim of the form \(H = g(S_T)\) written on the risky asset price. For a superhedging strategy \(\Theta\) with initial capital \(p\), the hedger needs to set up the initial position in the risky asset \(\Theta_0\) that incurs the cost \(\Theta_0 S_0\). Hence, at maturity we have

\[
p + L_T(\Theta) + \Theta_T S_T - \Theta_0 S_0 \geq g(S_T). \tag{8.5}
\]
Therefore, a replicating strategy $\Theta$ with initial capital $p$ should satisfy
\[ p + \int_0^T \Theta_t \, dS_t + \int_0^T \frac{1}{2} S_t \lambda(Y_t) b^2(t) \, dt = g(S_T). \]

To construct a replicating strategy we look for a pair of processes $(a, b)$ (or equivalently a strategy $\Theta$) such that the process $G_t := G_0 + \int_0^t \Theta_u \, dS_u + \int_0^t \frac{1}{2} S_u \lambda(Y_u) b^2(u) \, du$ satisfies $G_T = g(S_T)$. To find such a process, we try the following Ansatz:
\[ a(t) \, dt + b(t) \, dW_t = \Theta_t \, dS_t. \]

Using (8.4), the change in the book wealth satisfies
\[ dL_t + d(S_t \Theta_t) = \Theta_t \, dS_t + \frac{1}{2} \, d[S_t, \Theta_t] + \frac{1}{2} \sigma S_t \, d[\Theta_t, W_t]. \]
\[ = \Theta_t \, dS_t + \frac{1}{2} S_t \lambda(Y_t) b^2(t) \, dt. \quad (8.6) \]

Comparing the drift and the diffusion terms, we need
\[ v_t + \frac{1}{2} S_t^2 (\sigma + \lambda(Y_t)b(t))^2 v_{ss} + v_s S_t \xi(t) = \frac{1}{2} S_t \lambda(Y_t) b^2(t) + \Theta_t S_t \xi(t), \quad (8.7) \]
\[ (\sigma + \lambda(Y_t)b(t)) S_t v_s(t, S_t) = \Theta_t S_t (\sigma + \lambda(Y_t)b(t)). \quad (8.8) \]

Now (8.8) is satisfied for $\Theta_t = v_s(t, S_t)$ and for this choice of $\Theta$, (8.7) reduces to
\[ v_t(t, S_t) + \frac{1}{2} S_t^2 (\sigma + \lambda(Y_t)b(t))^2 v_{ss}(t, S_t) - \frac{1}{2} S_t \lambda(Y_t) b^2(t) = 0. \quad (8.9) \]

To get the form of $b$, we have by Itô’s formula
\[ a(t) \, dt + b(t) \, dW_t = d\Theta_t = dv_s(t, S_t) = v_s dt + v_s dS_t + \frac{1}{2} v_{ss} d[S_t, S_t]. \]

and comparing the diffusion coefficients we get that $b(t) = v_{ss} S_t (\sigma + \lambda(Y_t)b(t))$, i.e.
\[ b(t) = \frac{\sigma S_t v_{ss}(t, S_t)}{1 - \lambda(Y_t) S_t v_{ss}(t, S_t)}. \quad (8.10) \]

Similarly, we get
\[ a(t) = v_s dt + v_s S_t \xi(t) + \frac{1}{2} v_{ssss} S_t^2 (\sigma + \lambda(Y_t)b(t))^2. \]

Using the definition of $\xi$ in (8.2), we get (with $\lambda$ and $\lambda'$ evaluated at $Y_t$)
\[ a(t) = v_s dt + v_s S_t (\mu - \lambda b(Y_t)) + 0.5(\lambda^2 + \lambda') b^2(t) + \lambda \sigma b(t)] + \frac{1}{2} v_{ssss} S_t^2 (\sigma + \lambda b(t))^2 \]
\[ - \lambda S_t v_{ss} \]

Note that $\sigma + \lambda(Y_t)b(t) = \sigma/(1 - \lambda(Y_t) S_t v_{ss}(t, S_t))$. Thus, (8.9) yields the PDE
\[ v_t(t, s) + \frac{1}{2} \frac{\sigma^2 s^2 v_{ss}(t, s)}{1 - \lambda(y) s v_{ss}(t, s)} = 0. \quad (8.11) \]

Note that this pricing pde is structurally similar to the equations derived in [LY05, FP11, BLZ17]. Due to the singularity at $\lambda(y)s v_{ss} = 1$ in (8.11), constraints on $s v_{ss}$ (upper bound $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}$) need to be imposed in order to have a well-posed pde.
Following the analysis in [BLZ17], it turns out that (8.11) characterizes the superhedging price under appropriate gamma constraints. Indeed, let us write (8.1) as
\[ d\Theta_t = \sigma_{a,b}^\Theta(S_t) \, dS_t + \mu_{a,b}^\Theta(S_t) \, dt, \]
with \( S_t\sigma_{a,b}^\Theta(S_t) = \frac{b_t}{\sigma + \lambda(Y_t)b_t}, \)
\( \mu_{a,b}^\Theta(S_t) = a_t - \xi_t S_t\sigma_{a,b}^\Theta(S_t). \)
Like in [BLZ17], we restrict the admissible trading strategies \( \Theta = (a, b) \) to these with Lipschitz continuous and bounded \( a, b \), for which \( b \) is an Itô diffusion with Lipschitz continuous and bounded drift and diffusion processes, and such that \( S_t\sigma_{a,b}^\Theta(S_t) \) is bounded from above by some \( \tau(S_t) \) and also bounded from below. Under suitable conditions on \( \tau \) and \( \lambda \) (cf. Remark 8.1 below), then arguments in [BLZ17] would carry over to our setup and would give that the superhedging price
\[ v_\gamma(t, y, s) = \inf_\{p \mid \exists \text{admissible } \Theta \text{ so that (8.5) holds}\} \]
satisfies \( v_\gamma(t, y, s) = v^\gamma(t, s) \), where \( v^\gamma(t, s) \) is the unique viscosity solution of the pricing pde
\[ \mathbb{F}^y[\varphi](t, s) := \min\left\{ -\varphi(t, s) - \frac{1}{2} \sigma^2 \varphi_{ss}(t, s), \tau(s) - s\varphi_s \right\} = 0 \quad \text{on } [0, T) \times \mathbb{R}_+, \]
with terminal condition given by the face-lifted payoff \( \hat{g} \), where \( \hat{g} \) is the smallest function dominating \( g \) that is a viscosity supersolution of the equation \( \tau - s\varphi_{ss} \geq 0 \).
We conclude the section by highlighting some features of the superhedging price for covered options and pointing out differences to the case of non-covered options.

**Remark 8.1.** The arguments from [BLZ17] can be adapted to the present setup for bounded continuous \( \tau \) satisfying
\[ \sup_{y \in \mathbb{R}_+} \frac{\gamma(s)}{1 - \lambda(y)s} \in \mathbb{R}_+, \]
and continuous bounded payoffs \( g \). The main reason is that for every \( y \in \mathbb{R} \) and \( s \in \mathbb{R}_+ \), the map \( M \in (-\infty, \gamma(s)) \mapsto \sigma^2 s^{-1} M \) is non-decreasing and convex, like in [BLZ17, Remark 3.1], ensuring that smoothing techniques from [BLZ17, Section 3.1] can be applied here as well.

**Remark 8.2.** 1) The resilience function \( h \) does not appear in the pricing pde (8.12). Note that this is different from the results in Section 5.1, where the resilience function enters the pricing equation in a non-trivial way. However, the derived superhedging price for a covered option will depend on the initial level of impact \( y \) through \( \lambda \).
2) The superhedging price is decreasing in the impact \( \lambda \) in the sense that if \( \lambda \geq \lambda \), then \( v^\gamma_\lambda \geq v^\gamma_{\lambda} \) and \( v^\gamma_{\lambda} \geq v^{\text{BS}} \), where \( v^{\text{BS}} \) is the Black-Scholes price for the option with payoff \( g \), i.e. \( v^{\text{BS}} \) solves \(-\partial_tv^{\text{BS}} - \frac{1}{2}\sigma^2 v^{\text{BS}}_{ss} = 0 \) on \([0, T) \times \mathbb{R}_+ \) with terminal condition \( v^{\text{BS}}(T, \cdot) = g(\cdot) \), see [BLZ17, Remark 2.9].

### 9 Proofs
This section provides the proofs delegated from Section 4, in particular the proof of Theorem 5.9. Recall that in this case \( f(x) = \exp(\lambda x) \) for \( \lambda > 0 \) and thus the effective price simplifies to \( S(s, y, \theta) = xe^{-\lambda M} \equiv S(s, \theta) \), i.e. the level of impact is not needed in order to determine the price change of a block trade, given the price before the trade. We consider strategies taking values in \( K = [-K, +\infty) \) for \( K > 0 \). This yields a gradient constraint for the pde that is needed because of a singularity in the pde, for the expression (5.8) for the form of the optimal strategy to be finitely defined.
First, we verify in Section 9.1 that if the pricing pde (PDE) admits a sufficiently smooth classical solution, then a replicating strategy in feedback form can be constructed. Such a
construction will be needed also for the contradiction argument in the proof of the subsolution property in Section 9.2 where, using smooth test functions, one constructs locally strategies which, roughly speaking, behave like replicating strategies. The viscosity property proofs are collected in Section 9.2 and in Section 9.3 we prove comparison results that imply uniqueness of the viscosity solutions of the pricing pdes and continuity of the value function for the superhedging problem.

9.1 Verification argument for exponential impact function

Suppose that \(w \in C^{1,2,1}([0,T] \times \mathbb{R}_+ \times \mathbb{R})\) is such that for every \((t, s, y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}\)

1. \(\theta|w|(t, s, y) \in \mathcal{K}\), recalling the definition in (5.8), and
2. \(L^\theta|w|(t, s, y)w(t, s, y) = 0\) when \(t < T\), and
3. \(w(T, s, y) = H(s, y)\).

Suppose further that \(w\) is sufficiently regular (see the subsequent remark) so that there exists an admissible strategy \(\Theta \in \Gamma\) of the form

\[
\Theta_t = \frac{1}{\lambda} \log(a(t, S_t, Y_t^\Theta, \Theta_t) + 1) \quad \text{for } t \in [0, T),
\]

\(\Theta_T = 0\), \text{ i.e. } \Delta \Theta_T = \Theta_{T-}.

In particular, \(\Theta_0 = 1/\lambda \log(\lambda w_S(0, s, y) + 1)\) and \(\Delta \Theta_T \in \mathcal{K}\). Consider the self-financing portfolio \((\beta, \Theta)\) with \(\beta_0 = w(0, s, y)\). Then as in Remark 5.8 we get

\[
V^\text{liq}_T(\Theta) = H(S_T, Y^\Theta_T), \quad \Theta_T = 0.
\]

By definition of \(H\), this shows that \(V^\text{liq}_T(\Theta)\) at maturity \(T\) is enough capital to (super-)replicate the European claim with payoff \((g_0, g_1)\) with a possible additional block trade (provided that the infima in the definition of \(H\), cf. Lemma 5.1, are attained). Hence, \((\beta, \Theta)\) will be a (super-)replicating strategy for the European claim \((g_0, g_1)\) with initial capital \(w(0, s, y)\), meaning that its price is exactly \(w(0, s, y)\).

Remark 9.1 (On the form of a replicating strategy) To construct a replicating strategy (9.1), suppose moreover that \(w \in C^{1,3,1}([0,T] \times \mathbb{R}_+ \times \mathbb{R})\) and apply Itô’s formula, similarly as in Remark 5.8, to get for \(t < T\)

\[
d\Theta_t = \frac{1}{\lambda} \left( \frac{1}{\lambda w_S + 1} d(\lambda w_S + 1) - \frac{1}{2(\lambda w_S + 1)^2} d[\lambda w_S + 1], \right)
\]

where for \(S_t := S(S_t, \Theta_t)\) and \(Y^\Theta_t = Y_t^\Theta - \Theta_t\) we set

\[
a(t, S_t, Y^\Theta_t, \Theta_t) := \frac{1}{\lambda w_S + 1} \left( w_t + w_s S_t (\mu_t - \lambda h(Y^\Theta_t)) - w_S Y_t^\Theta \right) + \frac{1}{2} w_{ss} \sigma^2 \frac{S_t^2}{(\lambda w_S + 1)},
\]

\[
b(t, S_t, Y^\Theta_t) := \frac{\sigma S_t w_{ss} \lambda w_S}{\lambda w_S + 1}.
\]

all the derivatives of \(w\) above are evaluated at \((t, S(S_t, \Theta_t), Y_t - \Theta_t)\). Thus, a replicating strategy, which is superhedging the payout at a minimal cost, can be constructed as the \((\Theta_t)_{t \in [0,T]}\)-part (plus a terminal block trade) from a solution, if it exists, to the SDE system

\[
dS_t = S_t [(\mu_t - \lambda h(Y^\Theta_t + \Theta_t))] dt + \sigma dW_t,
\]

\[
d\Theta_t = a(t, S_t, Y^\Theta_t, \Theta_t) dt + b(t, S_t, Y^\Theta_t) dW_t,
\]

\[
dY^\Theta_t = -h(Y^\Theta_t + \Theta_t) dt
\]

for \(t \in [0, T]\), with initial condition \(S_0 = s, Y^\Theta_0 = y\) and \(\Theta_0 = 1/\lambda \log(\lambda w_S(0, s, y) + 1)\).
9.2 Viscosity solution property of $w$ for exponential impact function

Now we prove the viscosity property for the result from Section 5.2.

**Theorem 9.2.** The function $w_*$ from (5.1) is a viscosity supersolution of \((\text{PDE}^\delta)\) on \([0,T) \times \mathbb{R}_+ \times \mathbb{R}\) with the boundary condition \((\text{BC}^\delta)\) on \([T] \times \mathbb{R}_+ \times \mathbb{R}\).

**Proof.** First, let \((t_0,s_0, y_0) \in [0,T) \times \mathbb{R}_+ \times \mathbb{R}\) and \(\varphi \in C^\infty_b([0,T] \times \mathbb{R}_+ \times \mathbb{R})\) be a smooth function such that

\[
(\text{strict}) \min_{[0,T] \times \mathbb{R}_+ \times \mathbb{R}} (w_* - \varphi) = (w_* - \varphi)(t_0,s_0,y_0) = 0.
\]

**Case 1:** Suppose that \(\mathcal{H}_K \langle \varphi \rangle(t_0,s_0,y_0) < 0\). By continuity of the operator \(\mathcal{H}_K\) there exists an open neighborhood \(\mathcal{O}\) of \((t_0,s_0,y_0)\) whose closure is contained in \([0,T) \times \mathbb{R}_+ \times \mathbb{R}\), such that \(\mathcal{H}_K \langle \varphi \rangle(t,s,y) < -\varepsilon\) in \(\mathcal{O}\) for some \(\varepsilon > 0\). Therefore, after possibly decreasing the neighbourhood \(\mathcal{O}\), there exists a constant \(k_\varepsilon > 0\) such that

\[
|s| \mathcal{H}_K \langle \varphi \rangle(t, s, y) + 1/\lambda - e^{\lambda \theta}/\lambda \geq k_\varepsilon \quad \text{for all } \theta \in K, (t,s,y) \in \mathcal{O}. \tag{9.3}
\]

Let \((t_n, s_n, y_n)_n \subset \mathcal{O}\) be a sequence converging to \((t_0, s_0, y_0)\) with \(w(t_n, s_n, y_n) \rightarrow w_*(t_0, s_0, y_0)\) (note that \(w_*\) is the lower-semicontinuous envelope of \(w\)). Set \(v_n := w(t_n, s_n, y_n) + 1/n\). Since \(v_n > w(t_n, s_n, y_n)\), Theorem 4.1 implies the existence of \(\tau_n \in K\) and strategies \(\gamma_n \in \Gamma\) such that for stopping times \(\tau_n \geq t_n\) (to be suitably chosen later) we have \(\mathbb{P}\)-a.s.

\[
V_{t_n \wedge \tau_n}^{\text{lin}, t_n; z_n; \gamma_n} \geq w(\cdot, S(t_n; z_n; \gamma_n), Y(t_n; z_n; \gamma_n) - \Theta(t_n; z_n; \gamma_n))_{t_n \wedge \tau_n}, \quad t \in [t_n, T], \tag{9.4}
\]

where \(z_n = (s_n e^{\lambda \theta_n} + y_n, \theta_n, y_n, v_n)\). To abbreviate notation, in the sequel we write \(n\) as superscript instead of \((t_n, s_n, \gamma_n)\), with \(S^n := S(S^n_{t_n}; z_n; \gamma_n), Y^n := Y(t_n; z_n; \gamma_n) - \Theta(t_n; z_n; \gamma_n)\).

Take \(\tau_n = \inf \{t \geq t_n \mid (t, S^n_t, Y^n_t) \notin \mathcal{O}\}\), which is the first entrance time of the parabolic boundary of the open region \(\mathcal{O}\). In particular, \(\tau_n < T\). Since \(w \geq w_* \geq \varphi\) and \(w_* - \varphi\) has a strict local minimum at \((t_0, s_0, y_0)\), there exists \(\iota > 0\) such that

\[
(w - \varphi)(\tau_n, S^n_{\tau_n}, Y^n_{\tau_n}) \geq \iota.
\]

Hence, \(V_{\iota \wedge \tau_n}^{\text{lin}, n} - \varphi(\tau_n, S^n_{\tau_n}, Y^n_{\tau_n}, \gamma_n) \geq \iota\). Now, Lemma 4.3 together with the fact that \(S^n_{\tau_n} = s_n, Y^n_{\tau_n} = y_n\), gives that \(\mathbb{P}\)-a.s.

\[
\iota \leq v_n - \varphi(t_n, s_n, y_n) - \int_{t_n}^{\tau_n} S^n_u \left(\varphi_S(u, S^n_u, Y^n_u) + 1/\lambda - e^{\lambda \theta_u}/\lambda\right) (\sigma dW_u + \zeta^n_u du) \tag{9.5}
\]

where

\[
\zeta^n_t := \eta^n_t - \frac{L_{\Theta t}^{\gamma^n_t} \varphi}{S^n_t(\varphi_S(u, S^n_u, Y^n_u) + 1/\lambda - e^{\lambda \theta_t}/\lambda)} \quad \text{for } t \in [t_n, \tau_n]
\]

with \(\eta^n_t := \mu_t - \lambda h(Y^n_t)\). Note that \(\zeta^n_t\) is well-defined on \([t_n, \tau_n]\) and uniformly bounded, noting (9.3) and the fact that \(Y^n\) is bounded since \(\Theta^n\) is so. Hence, by Girsanov’s theorem, there exists a measure \(\mathbb{P}^n\) which is equivalent to \(\mathbb{P}\) such that

\[
\int_{t_n}^{\tau_n} S_u (\varphi_S(u, S^n_u, Y^n_u) + 1/\lambda - e^{\lambda \theta_u}/\lambda) (\sigma dW_u + \zeta^n_u du), \quad t \geq t_n,
\]

is a square-integrable martingale under \(\mathbb{P}^n\) as the integrand of the stochastic integral is uniformly bounded, because of the definition of \(\tau_n\), the continuity of \(\varphi_S\) and the boundedness of the range of \(\Theta\), noting \(\tau_n \leq T\). Taking expectation under \(\mathbb{P}^n\) of the right-hand side of (9.5) leads to \(v_n - \varphi(t_n, s_n, y_n) \geq \iota > 0\), what yields a contradiction as by our choice of \(v_n\) and the sequence \((t_n, s_n, y_n)_n\)

\[
v_n - \varphi(t_n, s_n, y_n) \longrightarrow w_*(t_0, s_0, y_0) - \varphi(t_0, s_0, y_0) = 0.
\]
Case 2: From Case 1 we know that $\mathcal{H}_{K^0} \varphi(t_0, s_0, y_0) \geq 0$. Hence

$$\theta[\varphi](t_0, s_0, y_0) = 1/\lambda \log(\lambda \varphi_S(t_0, s_0, y_0) + 1)$$

is well-defined (also in a neighborhood of $(t_0, s_0, y_0)$). Let us suppose that $L^\theta[\varphi](t_0, s_0, y_0) < 0$. By continuity of the operator $\mathcal{L}$, there exists an open neighborhood $\mathcal{O} \subset [0, T] \times \mathbb{R}_+ \times \mathbb{R}$ of $(t_0, s_0, y_0)$ and some $r, \varepsilon > 0$ such that

$$L^\theta \varphi(t, s, y) < -\varepsilon \quad \text{for all } (t, s, y) \in \mathcal{O}, \quad \theta \in (\theta[\varphi](t, s, y) - r, \theta[\varphi](t, s, y) + r).$$

In particular, by continuity of the functions involved we have (after possibly decreasing the open set $\mathcal{O}$) that for every $(t, s, y) \in \mathcal{O}$ and for some $r' > 0$

$$L^\theta \varphi(t, s, y) < -\varepsilon \quad \text{whenever } |\varphi_S(t, s, y) + 1/\lambda - e^{\lambda \varepsilon}/\lambda| \leq r'.$$

As in Case 1, consider a sequence $(t_n, s_n, y_n) \in \mathcal{O}$ which converges to $(t_0, s_0, y_0)$ and such that $w(t_n, s_n, y_n) \to w^*(t_0, s_0, y_0)$. Set $v_n := w(t_n, s_n, y_n) + 1/n$ and let $\theta_n \in K$ and strategies $\gamma_n \in \Gamma$ be such that the dynamic programming principle (9.4) holds for the stopping times $\tau_n$ that are the first exit times of $\langle \cdot, S^n, Y^n \rangle$ from the set $\mathcal{O}$. Now, a contradiction follows similarly as in Case 1 with the following adjustment: we have

$$V^{3n}_{t\wedge \tau_n} - \varphi(t_n, S^n, Y^n)_{t\wedge \tau_n} = v_n - \varphi(t_n, s_n, y_n)$$

$$- \int_{t_n}^{t\wedge \tau_n} S^n_u(\varphi_S + 1/\lambda - e^{\lambda \varepsilon]/\lambda}) (\sigma dW_u + \zeta^n_u du$$

$$+ \int_{t_n}^{t\wedge \tau_n} L^\theta \varphi(u, S^n_u, Y^n_u) I_{\{\varphi_S + 1/\lambda - e^{\lambda \varepsilon]/\lambda} \leq r\}} du$$

$$\leq v_n - \varphi(t_n, s_n, y_n) - \int_{t_n}^{t\wedge \tau_n} S^n_u(\varphi_S + 1/\lambda - e^{\lambda \varepsilon]/\lambda}) (\sigma dW_u + \zeta^n_u du),$$

where we set

$$\zeta^n_u := \eta^n_u - \frac{L^\theta \varphi}{\varphi_S + 1/\lambda - e^{\lambda \varepsilon]/\lambda}} \I_{\{|\varphi_S + 1/\lambda - e^{\lambda \varepsilon]/\lambda}\geq r\}} \quad \text{for } t \in [t_n, \tau_n],$$

with the functions $\varphi$ and $\varphi_S$ in the expressions above being evaluated at $(t_n, s_n, y_n)$. The contradiction now follows by taking expectation under $\mathbb{P}^n \approx \mathbb{P}$ and letting $n \to \infty$.

**Boundary condition.** Let $(s_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}$ and $\varphi$ be a smooth function such that

$$(\text{strict}) \quad \min_{[0,T] \times \mathbb{R}_+ \times \mathbb{R}} (w_s - \varphi) = (w_s - \varphi)(T, s_0, y_0) = 0.$$

Suppose that

$$\min\{w_s(T, s_0, y_0) - H(s_0, y_0), \mathcal{H}_{K^0} \varphi(T, s_0, y_0)\} < 0.$$

The case $\mathcal{H}_{K^0} \varphi(T, s_0, y_0) < 0$ leads to a contradiction by the same arguments as in Case 1 above, using that $\mathcal{H}_{K^0} \varphi < 0$ in a small neighborhood of $(T, s_0, y_0)$. Hence we have $\mathcal{H}_{K^0} \varphi(T, s_0, y_0) \geq 0$.

Now, if $w_s(T, s_0, y_0) < H(s_0, y_0)$ then also $\varphi(T, s_0, y_0) - H(s_0, y_0) < 0$. After possibly modifying the test function $\varphi$ by $(t, s, y) \mapsto \varphi(t, s, y) - \sqrt{T-t}$, we can assume that $\partial_t \varphi(t, s, y) \to +\infty$ when $t \to T$, uniformly on compacts. Hence, in an $\varepsilon$-neighborhood $[T - \varepsilon, T] \times B_z(s_0, y_0)$ around $(T, s_0, y_0)$ we have $L^\theta \varphi < 0$. Moreover, after possibly decreasing $\varepsilon$ we have $\varphi(T, \cdot) \leq H(\cdot) - t_1$ on $B_z(s_0, y_0)$ for some $t_1 > 0$. We can argue as in Cases 1-2 above (by starting from $(t_n, s_n, y_n)$ in $[T - \varepsilon, T] \times B_z(s_0, y_0)$, with $(t_n, s_n, y_n) \to (T, s_0, y_0)$ and $w(t_n, s_n, y_n) \to w^*(T, s_0, y_0)$, stopping at the (parabolic) boundary at time $\tau_n$, and using $w(T, \cdot) = H(\cdot)$ to get

$$V^{3n}_{t\wedge \tau_n} - \varphi(\cdot, S^n(\Theta^n), Y^n - \Theta^n)_{t\wedge \tau_n} \geq t_1 \wedge t_2,$$

where $t_2 := \inf_{[T - \varepsilon, T] \times \partial B_z(s_0, y_0)} (w_s - \varphi) > 0$. A contradiction follows as in Case 2 above.
Now we prove the subsolution property.

**Theorem 9.3.** The function \( w^* \) from (5.2) is a viscosity subsolution of \((\text{PDE}^\delta)\) on \([0, T) \times \mathbb{R}_+ \times \mathbb{R}\) with the boundary condition \((\text{BC}^\delta)\) on \(\{T\} \times \mathbb{R}_+ \times \mathbb{R}\).

**Proof.** The proof is similar to and inspired by the one for the subsolution property in [BLZ16, Theorem 3.7]. The reason is that in this case, the gradient constraints will ensure that a test function \( \varphi \), that would possibly contradict the subsolution property, should satisfy \( \mathcal{H}_K \varphi > 0 \) locally and hence would be sufficiently “nice” to define (locally) control processes (employing the verification argument in Remark 9.1) that would lead to a contradiction like in [BLZ16]. For completeness, we outline differences in the line of proof and sketch the main steps.

Let \( \varphi \) be a \( C^\infty([0, T], \mathbb{R}_+ \times \mathbb{R}) \) test function such that \((t_0, s_0, y_0) \in [0, T) \times \mathbb{R}_+ \times \mathbb{R}\) is a strict (local) maximum of \( w^* - \varphi \), i.e.

\[
\left( \text{strict} \right) \max_{[0, T] \times \mathbb{R}_+ \times \mathbb{R}} (w^* - \varphi) = (w^* - \varphi)(t_0, s_0, y_0) = 0.
\]

First assume that \( t_0 < T \). To ease the notations, we will use the variable \( x \) to denote the pair \((s, y)\). Because of the special form of the second part of DPP, cf. Theorem 4.1 (ii), we need to employ \( w_k \) (instead of \( w \) as we did in the proof of the supersolution property). By [Bar13, Lemma 6.1] we can take a sequence \((k_n, t_n, x_n)_{n \geq 1}\) such that \( k_n \to \infty \), any \((t_n, x_n)\) is a local maximum of \( w^*_{k_n} - \varphi \), and \((t_n, x_n, w_k(t_n, x_n)) \to (t_0, x_0, w^*(t_0, x_0))\).

Assume that \( \mathcal{F}_K[\varphi](t_0, x_0) > 0 \) and let \( \varphi_n(t, x) = \varphi(t, x) + |t - t_n|^2 + |y - y_n|^2 + |s - s_n|^4 \). Then \( \mathcal{F}_K[\varphi_n] > 0 \) holds in a neighborhood \( B \) of \((t_0, x_0)\) that contains \((t_n, x_n)\), for all \( n \) large enough. Since we will be working on the local neighborhood \( B \) where also \( \mathcal{H}_K \varphi_n > 0 \), we can modify (in a smooth way) the functions \( h \) and \( \varphi_n \) outside of \( B \) to be supported on a slightly bigger compact set where \( \mathcal{H}_K \varphi_n > 0 \) holds. Thus, after possibly passing to a suitable subsequence, there exists \( \gamma_n \in \Gamma_{k_n} \) such that

\[
\Theta_t^{t_n, z_n, \gamma_n} = 1 / \lambda \log \left( \lambda \frac{D\varphi_n}{D\gamma}(t, \Theta_t^{t_n, z_n, \gamma_n}, y_t^{t_n, z_n, \gamma_n}) + 1 \right), \quad t \geq t_n,
\]

where for \( z_n = (s_n, y_n, 0), w_k(t_n, x_n) - n^{-1} \) we set \( \Theta_t^{t_n, z_n, \gamma_n} = \Theta(t_n + z_n, \gamma_n) \) and \( y_t^{t_n, z_n, \gamma_n} = (Y - \Theta)^{t_n, z_n, \gamma_n} \), see Remark 9.1. Let \( \tau_n \) be the first time after \( t_n \) at which the process \( (\Theta_t^{t_n, z_n, \gamma_n}, y_t^{t_n, z_n, \gamma_n})_{t \geq t_n} \) leaves \( B \). Like in [BLZ16, proof of Thm. 3.7] we conclude, by applying Itô’s formula, using Lemma 4.3 and \( F[\varphi_n] > 0 \) on \( B \), that \( \mathbb{P}\) a.s.

\[
V_{\text{liq}}^{t_n, z_n, \gamma_n} \geq \varphi_n(t_n, \Theta_t^{t_n, z_n, \gamma_n}, (Y - \Theta)^{t_n, z_n, \gamma_n}) + v_n - \varphi_n(t_n, x_n).
\]

Now, a contradiction follows as in [BLZ16, proof of Thm. 3.7, subsolution property, (a)].

For the boundary condition, i.e. the case \( t_0 = T \), the arguments are exactly the same as in [BLZ16, proof of Thm. 3.7, subsolution property, (b)].

\[\square\]

### 9.3 Comparison results for viscosity solutions

First we provide a comparison result for the pricing pde (PDE), needed for the proof of Theorem 5.5. Note that (PDE) has the structure

\[
0 = -\partial_t \varphi - \frac{\sigma^2 s^2}{2} \partial_{ss} \varphi - B_1(y, f(y) \partial_x \varphi) \partial_y \varphi - s B_2(y, f(y) \partial_x \varphi) \partial_x \varphi - s B_3(y, f(y) \partial_x \varphi), \quad (9.6)
\]

where \( B_i : \mathbb{R}^2 \to \mathbb{R}, i = 1, 2, 3, \) are bounded and Lipschitz continuous functions. By a change of coordinates, one can transform the pde as follows.

**Lemma 9.4.** Let \( u \) be viscosity subsolution (resp. supersolution) of the pde (9.6). Fix \( \kappa > 0 \). Then the function \( \tilde{u} \), which is defined by

\[
\tilde{u}(t, s, y) = e^{\kappa t} u(t, s f(y), y) \quad \text{for all } (t, s, y) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R},
\]

Electronic copy available at: https://ssrn.com/abstract=3212870
is a viscosity subsolution (resp. supersolution) of the pde

\[
0 = \kappa \varphi - \partial_t \varphi - \frac{x^2}{2} \partial_{ss} \varphi - B_1(y, e^{-xt} \partial_s \varphi) \partial_y \varphi + \lambda(y) B_1(y, e^{-xt} \partial_s \varphi) \partial_y \varphi
- s B_2(y, e^{-xt} \partial_s \varphi) \partial_y \varphi - e^{-xt} s f(y) B_3(y, e^{-xt} \partial_s \varphi). \tag{9.7}
\]

**Proof.** We have formally (if derivatives exist)

\[
\begin{align*}
\bar{u}_n(t, s, y) &= e^{xt} f(y) u_n(t, s f(y), y) \\
\bar{u}_n(t, s, y) &= e^{xt} f^2(y) u_n(t, s f(y), y) \\
\bar{u}_n(t, s, y) &= e^{xt} (\lambda(y) f(y) u_n(t, s f(y), y) + e^{xt} u_n(t, s f(y), y) \\
\bar{u}_n(t, s, y) &= e^{xt} u_n(t, s f(y), y) + \kappa e^{xt} u(t, s f(y), y).
\end{align*}
\]

Writing now the equation (9.6) for \( u \) at \( (t, s f(y), y) \), we can read off the equation (9.7) for \( \bar{u} \).

Now clearly the viscosity property of \( u \) implies the viscosity property of \( \bar{u} \) by definition of viscosity solutions. \( \Box \)

By Lemma 9.4 it now suffices to prove comparison for equation (9.7) since this would imply a comparison result for (9.6). This is done in the following result.

**Theorem 9.5.** Let \( u \) (respectively \( v \)) be a bounded upper-semicontinuous subsolution (resp. lower-semicontinuous supersolution) on \( [0, T) \times \mathbb{R}_+ \times \mathbb{R} \) of (9.7). Suppose that \( u \leq v \) on \( [T] \times \mathbb{R}_+ \times \mathbb{R} \). Then \( u \leq v \) on \( [0, T] \times \mathbb{R}_+ \times \mathbb{R} \).

**Proof.** To prove the claim by contradiction, let us suppose that

\[
\sup_{(t, s, y) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}} (u - v)(t, s, y) > 0.
\]

Then we can find \( R > 1 \) such that

\[
\sup_{(t, s, y) \in [0, T] \times \mathcal{O}_R \times [-R, R]^2} (u - v)(t, s, y) > 0,
\]

where \( \mathcal{O}_R := (1/R, R) \). In particular, there exists \( \delta > 0 \) and \( (t_0, s_0, y_0) \in \overline{\mathcal{O}_R} \times [-R, R] \) such that \( (u - v)(t_0, s_0, y_0) = \delta > 0 \).

Now consider, for \( n \in \mathbb{N} \), the bounded upper-semicontinuous function

\[
\Phi_n(t, s_1, s_2, y_1, y_2) := u(t, s_1, y_1) - v(t, s_2, y_2) - \frac{n}{2} (s_1 - s_2)^2 - \frac{n}{2} (y_1 - y_2)^2.
\]

It attains its maximum at some \( (t^n, s^n_1, s^n_2, y^n_1, y^n_2) \in [0, T] \times \overline{\mathcal{O}_R} \times [-R, R]^2 \) by compactness of the set, and we clearly have

\[
\Phi_n(t^n, s^n_1, s^n_2, y^n_1, y^n_2) \geq \delta \quad \text{for all } n \in \mathbb{N}. \tag{9.8}
\]

By the arguments in [BLZ16, proof of Lemma 3.11] one obtains (after possibly passing to a subsequence) that

\[
n(s^n_1 - s^n_2)^2 + n(y^n_1 - y^n_2)^2 \to 0 \quad \text{as } n \to \infty. \tag{9.9}
\]

Note that (9.9) also implies \( n(s^n_1 - s^n_2)(y^n_1 - y^n_2) \to 0 \) as \( n \to \infty \).

Now, by Ishii’s lemma as stated in [CIL92, Theorem 8.3], there exist \( (b^n, X^n, Y^n) \in \mathbb{R} \times S_2 \times S_2 \) such that with \( p^n = n(s^n_1 - s^n_2) \) and \( q^n = n(y^n_1 - y^n_2) \) we have

\[
\begin{align*}
(b^n, (p^n, q^n), X^n) &\in \mathcal{P}_{\mathcal{O}_n}^2 + u(t^n, s^n_1, y^n_1), \\
(b^n, (p^n, q^n), Y^n) &\in \mathcal{P}_{\mathcal{O}_n}^2 - v(t^n, s^n_2, y^n_2).
\end{align*}
\]

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where \( X^n \) and \( Y^n \) satisfy
\[
\begin{pmatrix} X^n & 0 \\ 0 & -Y^n \end{pmatrix} \leq 3n \begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix},
\]
and where \( S_2 \) denotes the set of \( 2 \times 2 \) symmetric non-negative matrices and \( I_2 \in S_2 \) is the identity matrix. Using the viscosity property of \( u \) and \( v \) at \((t^n, s^n_1, y^n_1)\) and \((t^n, s^n_2, y^n_2)\) respectively, we have
\[
\kappa u(t^n, s^n_1, y^n_1) - b_n - \frac{1}{2}\sigma^2(s^n_1)^2X^n_{11} + L(s^n_1, y^n_1, p^n, q^n) \leq 0
\]
\[
\kappa v(t^n, s^n_2, y^n_2) - b_n - \frac{1}{2}\sigma^2(s^n_2)^2Y^n_{11} + L(s^n_2, y^n_2, p^n, q^n) \geq 0,
\]
where
\[
L(t, s, y, p, q) := -B_1(y, e^{-\kappa t}p)q + \lambda(y)B_1(y, e^{-\kappa t}p)p - sB_2(y, e^{-\kappa t}p)p - e^{\sigma t}s f(y)B_3(y, e^{-\kappa t}p).
\]
As a consequence,
\[
0 < \kappa \delta < \kappa(u(t^n, s^n_1, y^n_1) - v(t^n, s^n_2, y^n_2)) \leq
\]
\[
\leq -\frac{1}{2}\sigma^2(s^n_1)^2X^n_{11} + \frac{1}{2}\sigma^2(s^n_2)^2Y^n_{11} + L(t^n, s^n_1, y^n_1, p^n, q^n) - L(t^n, s^n_2, y^n_2, p^n, q^n).
\]
On the other hand, by (9.10) we get that
\[
\frac{1}{2}\sigma^2(s^n_1)^2X^n_{11} - \frac{1}{2}\sigma^2(s^n_2)^2Y^n_{11} \leq \frac{\sigma^2}{2}n(s^n_1 - s^n_2)^2,
\]
what converges to 0 for \( n \to \infty \) due to (9.9). Let us now analyze the difference \( L(t^n, s^n_1, y^n_1, p^n, q^n) - L(t^n, s^n_1, y^n_1, p^n, q^n) \). With \( C \) (resp. \( C_R \)) denoting a Lipschitz constant (depending on \( R \)), that may change from line to line, we get estimates for the corresponding terms as follows:
\[
|B_1(y^n_1, e^{-\kappa t}p^n)q^n - B_1(y^n_2, e^{-\kappa t}p^n)q^n| \leq C|y^n_1 - y^n_2||q^n|,
\]
\[
|\lambda(y^n_1)B_1(y^n_1, e^{-\kappa t}p^n)p^n - \lambda(y^n_2)B_1(y^n_2, e^{-\kappa t}p^n)p^n| \leq C|y^n_1 - y^n_2||p^n|,
\]
\[
|s^n_1 B_2(y^n_1, e^{-\kappa t}p^n)p^n - s^n_2 B_2(y^n_2, e^{-\kappa t}p^n)p^n| \leq C|s^n_1 - s^n_2||p^n| + C_R|y^n_1 - y^n_2||p^n|,
\]
\[
|e^{\sigma t\delta} s^n_1 f(y^n_1)B_3(y^n_1, e^{-\kappa t}p^n) - e^{\sigma t\delta} s^n_2 f(y^n_2)B_3(y^n_2, e^{-\kappa t}p^n)| \leq C_R|s^n_1 - s^n_2| + |y^n_1 - y^n_2|.
\]
As all estimates from above vanish for \( n \to \infty \), the right-hand side in (9.11) is bounded by something that converges to 0 as \( n \to \infty \). But this yields a contradiction for large \( n \).

Because of lack of precise reference, we provide a comparison result also in the case of delta constraints leading to the variational inequality (PDE\(^D\)).

**Theorem 9.6.** Suppose that the resilience function \( h \) is Lipschitz continuous and Assumption 5.3 is in force. Let \( u \) (resp. \( v \)) be bounded upper (resp. lower-) semicontinuous viscosity subsolution (resp. supersolution) of the variational inequality (PDE\(^D\)) with the terminal condition (BC\(^D\)). Then \( u \leq v \) on \([0, T] \times R_+ \times R\).

**Proof.** We argue by contradiction. For any \( a > 0 \), set \( C_a := [a, \infty) \times [-1/a, 1/a] \). Suppose that
\[
\sup_{(t,s,y) \in [0,T] \times R_+ \times R} (u-v) > 0.
\]
Then there exists some \( a > 0 \) such that \( \sup_{(t,s,y) \in [0,T] \times C_a} (u-v) > 0 \). For \( \kappa > 0 \), consider \( \bar{u} := e^{\kappa t}u \) and \( \bar{v} := e^{\kappa t}v \). Then \( \bar{u} \) (resp. \( \bar{v} \)) is a viscosity sub- (resp. super-)solution of
\[
\min\{ \kappa \varphi + \bar{L}[\varphi], \mathcal{H}_{\kappa,t}[\varphi] \} = 0
\]
\]
with the boundary condition \( \min\{v(T,\cdot) - H(\cdot), \mathcal{H}_{\mathcal{K},T}v\} = 0 \), where
\[
\tilde{L}[v](t, s, y) = -\partial_t v + h(y + 1/\lambda \log(\lambda e^{-\kappa t} \partial_x v + 1))\partial_y v - 1/2 \sigma^2 x^2 \partial_{xx} v
\]
and \( \mathcal{H}_{\mathcal{K},T}v = \lambda e^{-\kappa t} \partial_x v + 1 - e^{-\lambda K} \) for \( t \in [0,T] \).

Consider
\[
\Theta_n := \sup_{(t,x_1,x_2) \in [0,T] \times \mathcal{O}_n^2} \tilde{u}(t,x_1) - \tilde{v}(t,x_2) - \frac{n}{2}(x_1 - x_2)^2.
\]

We have \( \Theta_n > \iota \) for some \( \iota > 0 \). Since \( \tilde{u} - \tilde{v} \) is upper-semicontinuous, for any \( n \) the supremum is attained as a maximum at some \((t_n,x_1^n,x_2^n)\) in the compact set \([0,T] \times \mathcal{O}_n^2\). By arguments as in [BLZ16, proof of Lemma 3.11], after possibly passing to a subsequence, we obtain
\[
\lim_{n \to \infty} \Theta_n = \sup_{(t,s,y) \in [0,T] \times \mathcal{O}_n} (\tilde{v} - \tilde{u}) \geq \iota > 0, \quad \text{and} \quad n|x_1^n - x_2^n|^2 \to 0 \quad \text{as} \quad n \to \infty.
\]

Note also that
\[
\lim_{n \to \infty} \tilde{u}(t_n,x_1^n) - \tilde{v}(t_n,x_2^n) \geq \iota. \quad (9.14)
\]

**Case 1:** Suppose, after passing to a subsequence, that \( t_n = T \) for all \( n \). Then Ishii’s lemma together with the viscosity property of \( \tilde{u} \) and \( \tilde{v} \) give
\[
\min \left\{ \tilde{u}(T,x_1^n) - H(x_1^n), \lambda e^{-\kappa T} p_n + 1 - e^{-\lambda K} \right\} \leq 0,
\]
\[
\min \left\{ \tilde{v}(T,x_2^n) - H(x_2^n), \lambda e^{-\kappa T} p_n + 1 - e^{-\lambda K} \right\} \geq 0,
\]
where \( p_n = n(s_1^n - s_2^n) \). Hence we conclude that \( \tilde{u}(T,x_1^n) \leq H(x_1^n) \) for all \( n \). However, in this case since \( \tilde{v}(T,x_2^n) \geq H(x_2^n) \) for all \( n \) we have
\[
\tilde{v}(T,x_2^n) \geq H(x_2^n) \geq H(x_1^n) + \tilde{u}(T,x_1^n),
\]
which contradicts (9.14) for large \( n \) by continuity of \( H \).

**Case 2:** We can now assume (after passing to a subsequence) that \( t_n < T \) for all \( n \). Set
\[
p_n := n(s_1^n - s_2^n), \quad q_n := n(y_1^n - y_2^n).
\]

By Ishii’s lemma, see [CIL92, Theorem 8.3], using the viscosity property of \( \tilde{u} \) and \( \tilde{v} \), there exist \( a_n \in \mathbb{R} \) and symmetric \( 2 \times 2 \) matrices \( A_n, B_n \) (that satisfy a bound like in (9.10)) with
\[
(a_n, (p_n,q_n), A_n) \in \mathcal{P}_e^2(\Phi_n^0)(t_n,x_1^n), \quad (a_n, (p_n,q_n), B_n) \in \mathcal{P}_e^2(-\Phi_n^0)(t_n,x_2^n)
\]
such that
\[
\min \left\{ -a_n + L(t_n,x_1^n,\tilde{u}(t_n,x_1^n),p_n,q_n,A_n), \lambda e^{-\kappa t_n} p_n + 1 - e^{-\lambda K} \right\} \leq 0,
\]
\[
\min \left\{ -a_n + L(t_n,x_2^n,\tilde{v}(t_n,x_2^n),p_n,q_n,B_n), \lambda e^{-\kappa t_n} p_n + 1 - e^{-\lambda K} \right\} \geq 0,
\]
where for \( t \in [0,T], x = (x_1,y_1) \in \mathbb{R}^2, \ell, p, q, A \in \mathbb{R} \) and a \( 2 \times 2 \) matrix \( A \)
\[
L(t,x = (x_1,y_1),\ell, p, q, A) := \kappa \ell + h(y_1 + 1/\lambda \log(\lambda e^{-\kappa t} p + 1))q - 1/2 \sigma^2 x_1^2 A_{11}.
\]

Therefore, we have
\[
-a_n + L(t_n,x_1^n,\tilde{u}(t_n,x_1^n),p_n,q_n, A_n) \leq 0.
\]

Note also that on the set \( \{(t,y,p) \in [0,T] \times \mathbb{R} \times \mathbb{R} \mid \lambda e^{-\kappa t} p + 1 - e^{-\lambda K} \geq 0\} \), the function
\[
(t,y,p) \mapsto h(y + 1/\lambda \log(\lambda e^{-\kappa t} p + 1))
\]
is Lipschitz continuous. Thus, we are exactly in the setup the proof of Theorem 9.5 and a contradiction argument follows like in the proof there: one gets the estimate
\[
\kappa(\tilde{u}(t_n,x_1^n) - \tilde{v}(t_n,x_2^n)) \leq C \left( n|x_1^n - x_2^n|^2 + 1/n \right)
\]
for some constant \( C > 0 \) that does not depend on \( n \). For large \( n \) this contradicts \( \Theta_n > \iota \) due to 9.12 from above.

\[\square\]
Remark 9.7. By Theorem 9.2 and Theorem 9.3 we know that $w_* \ (\text{resp. } w^*)$ is a supersolution (subsolution) of (PDE) with boundary condition (BC) and hence Theorem 9.6 gives that $w_* \geq w^*$ on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$. However, by definition it is clear that $w_* \leq w^*$ and hence we have the $w_* = w^*$ on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$. On the other hand, $w_* \leq w \leq w^*$ on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$. To obtain equality also for $t = T$, note that the super-/sub-solution property of $w_*/w^*$ respectively implies also $w_*(T, \cdot) \geq H(\cdot)$ and $w_*(T, \cdot) \leq H(\cdot)$, hence the $T$-value of $w_*$ is exactly $H$. Since also $H(\cdot) = w(T, \cdot)$ by definition, we conclude the equality $w_* = w^* = w$ also on $\{T\} \times \mathbb{R}_+ \times \mathbb{R}$. Hence, $w_* = w^* = w$ on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}$, what implies continuity.

The same conclusion holds for (PDE) with the boundary condition (BC).

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