THE GAUSSIAN PROCESS FOR PARTICLE MASSES IN THE
NEAR-CRITICAL ISING MODEL

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Abstract. We review the construction of a stationary Gaussian process \( X(t) \) starting from the near-critical continuum scaling limit \( \Phi^h \) of the Ising magnetization and its relation to the mass spectrum of the relativistic quantum field theory associated to \( \Phi^h \). Then for the near-critical Ising model on \( \mathbb{Z}^2 \) with external field \( a \frac{15}{8} h \geq 0 \), we study the renormalized magnetization along a vertical line (with horizontal coordinate approximately \( t \)) and prove that the limit as \( a \downarrow 0 \) is the same Gaussian process \( X(t) \). We also explore the possible extension of this approach to dimensions \( d > 2 \).

1. Introduction

1.1. Overview. Consider the Ising model at inverse critical temperature \( \beta_c \) on \( \mathbb{Z}^2 \) with \( a > 0 \) and external field \( a \frac{15}{8} h \geq 0 \). Let \( \{\sigma_x : x \in \mathbb{Z}^2\} \) denote the basic spin random variables of the Ising model, and \( \Phi^{a,h} \) denote the magnetization field

\[ \Phi^{a,h} := a^{\frac{15}{8}} \sum_{x \in \mathbb{Z}^2} \sigma_x \delta_x, \]  

where \( \delta_x \) is a unit Dirac point measure at \( x \). In Theorem 1.2 of [2] and Theorem 1.4 of [3], it was proved that

\[ \Phi^{a,h} \Rightarrow \Phi^h \]  

where \( \Rightarrow \) denotes convergence in distribution, and \( \Phi^h \) with \( h \geq 0 \) is a generalized random field on \( \mathbb{R}^2 \); see [7] for a review. Euclidean random fields such as \( \Phi^h \) on Euclidean “space-time” \( \mathbb{R}^d := \{x = (x_0, y_1, \ldots, y_{d-1})\} \) are related to quantum fields on relativistic space-time, \( \{(t, y_1, \ldots, y_{d-1})\} \), essentially by replacing \( x_0 \) with a complex variable and analytically continuing from the purely real \( x_0 \) to pure imaginary \((-it)\)—see [31, 32], Chapter 3 of [17] and [29] for background. It was predicted in [37, 36] that the relativistic quantum field theory associated to the Euclidean field \( \Phi^h \) with \( h > 0 \) should have remarkable properties including the existence of eight distinct types of particles, with relations between the masses of those particles and the Lie algebra \( E_8 \) [13, 1, 26]—see also [12] (respectively, [9]) for experimental (respectively, numerical) studies.

In [4, 5], exponential decay of truncated correlations in \( \Phi^h \) with \( h > 0 \) was proved; this roughly says that in the relativistic quantum field theory associated to \( \Phi^h \) with \( h > 0 \), there is at least one particle with strictly positive mass and no smaller mass particles. In [6], the authors took the limit of \( \Phi^h(t, y) \) as the spatial coordinate \( y \) scales to infinity with \( t \) fixed and proved that it is a stationary Gaussian process \( X(t) \) whose covariance function \( K(t) \) should provide a useful tool for analyzing particle masses in the associated quantum field theory.

1.2. Why are \( X \) and \( K \) of interest? \( K \) should capture some important information about particle masses of the quantum field theory associated with \( \Phi^h \). In [6], based on [13, 37, 36] and [18] below, it was conjectured that there exist \( m_1, m_2, m_3 \in (0, \infty) \) and \( B_1, B_2, B_3 \in (0, \infty) \) such that, for large \( t \),

\[ K(t) = B_1 \exp(-m_1 |t|) + B_2 \exp(-m_2 |t|) + B_3 \exp(-m_3 |t|) + \mathcal{O}(\exp(-2m_1 |t|)), \]  

where \( \mathcal{O}(\cdot) \) denotes the vanishing of the remainder term as \( t \) goes to infinity.
with $m_1 < m_2 < m_3 < 2m_1$; the mass $m_1$ should be the same as in [15], and $m_2/m_1$, $m_3/m_1$ should take the predicted values, as in (1.8) of [17].

The exponential decay result in [3, 5] only shows that in the relativistic quantum field theory associated to $\Phi^h$, there is a mass gap—i.e., no particles with masses in $[0, m_1)$. A natural question is then whether the mass spectral measure $\rho$ (defined in (17) below) has an atom with strictly positive weight at $m_1$. As proved in Appendix B of [6], this would follow from Ornstein-Zernike behavior of the covariance function $H(t, y)$ of $\Phi^h$. We state that result as a proposition here.

**Proposition 1** (Propositions A and B of [6]). Suppose that there exist a constant $C_1 \in (0, \infty)$ such that

\[
\lim_{t \to \infty} \frac{H(t, 0)}{t^{-1/2} \exp(-m_1 t)} = C_1 = \rho(\{m_1\}) \sqrt{m_1/(2\pi)}.
\]

Then we have

\[
\lim_{t \to \infty} \frac{K(t)}{\exp(-m_1 t)} = C_1 \sqrt{2\pi/m_1} = \rho(\{m_1\}).
\]

Recently, a new proof of exponential decay (for truncated correlations in $\Phi^h$) based on the random current representation of the Ising model was given in [23]. We believe that it is possible to combine the methods in [33] and [23] to give a rigorous proof of (4). Then the next step would be to show that the mass spectrum has an upper gap $(m_1, m_1 + \epsilon)$ for some $\epsilon > 0$.

In this paper, we first review in Section 2 the main results of [6]. Then in Section 3 we state some new results which are closely related to those of Section 2. In Section 3.1 the main result is that the same Gaussian process $X_t(y)$ can be obtained directly from the near-critical lattice Ising model on $a\mathbb{Z}^2$ by appropriate scalings of the vertical and horizontal coordinates without use of the continuum field $\Phi^h$. Then in Section 3.2 we explore the possible extension of this approach to dimensions $d > 2$. In Section 4 we prove the results stated in Section 3.

2. The Gaussian Process from $\Phi^h$: A Review

2.1. Construction of the Gaussian Process. Let $H(t, y)$ be the covariance function of $\Phi^h$. Roughly speaking,

\[
H(t, y) := \text{Cov} \left( \Phi^h(t_0, y_0), \Phi^h(t_0 + t, y_0 + y) \right) \quad \text{for any } (t_0, y_0) \in \mathbb{R}^2.
\]

The existence of $H$ follows from Proposition 6.1.4 of [17]; this is because $\mathbb{E} \left( \exp(i z \Phi^h(f)) \right)$ is analytic in $z$, which can be proved, for example, by arguments based on the Lee-Yang theorem [25] and the GHS inequality [20]. For each fixed $L > 0$, we define a collection of random variables $\{X_L(s) : s \in \mathbb{R} \}$ by

\[
X_L(s) := \frac{\Phi^h(1_{[-L,L]}(y)\delta_s(t)) - \mathbb{E}\Phi^h(1_{[-L,L]}(y)\delta_s(t))}{\sqrt{2L}},
\]

where $1_{[-L,L]}(y)\delta_s(t)$ is the product of an interval indicator function in $y$ and a delta function in $t$, and $\mathbb{E}$ is the expectation with respect to the random field $\Phi^h$. Formally,

\[
\Phi^h(f) := \int_{\mathbb{R}^2} \Phi^h(x)f(x)dx.
\]

In [6], it was shown that [8] is well-defined for any $f$ in the dual of the Sobolev space $\mathcal{H}^{-1/8}(\mathbb{R}^2)$. This was later generalized in [16] to any $f$ in the dual of the Besov space $\mathcal{B}_{p,q}^{-1/8-\epsilon,\text{loc}}(\mathbb{R}^2)$ where $\epsilon > 0$ and $p,q \in [1,\infty]$. Since the test function $1_{[-L,L]}(y)\delta_s(t)$
is in neither of those two spaces, we instead refer to Lemma A in Appendix A of [6] for a justification of such a paring; the idea is to approximate the delta function with smooth functions. Let \( \{X(s) : s \in \mathbb{R}\} \) be a mean zero stationary Gaussian process with covariance function

\[
\text{Cov}(X(s), X(t)) = K(t - s) := \int_{-\infty}^{\infty} H(t - s, y)dy \text{ for any } s, t \in \mathbb{R}.
\]  

(9)

The following was proved in [6].

**Theorem 1** (Theorem 1 of [6]). Fix \( h > 0 \). For any \( n \in \mathbb{N} \) and distinct \( s_1, \ldots, s_n \in \mathbb{R} \),

\[
(X_L(s_1), \ldots, X_L(s_n)) \Rightarrow (X(s_1), \ldots, X(s_n)) \text{ as } L \to \infty,
\]

(10)

where \( \Rightarrow \) denotes convergence in distribution.

\( H \) is really a function of the radial variable \( \sqrt{t^2 + y^2} \). Note that \( H \) actually depends on \( h \); we only distinguish whether \( h = 0 \) or not since all results in this paper are insensitive to the exact value \( h > 0 \). When \( h = 0 \), we always write \( H^0 \) for the covariance function of \( \Phi^0 \). By Proposition 4 below and Wu’s result [35, 27] (see also Remark 1.4 of [11] and Theorem 3.1 of [10]), we have

\[
H^0(t, y) = C_2(t^2 + y^2)^{-1/8}, \text{ for any } (t, y) \in \mathbb{R}^2 \text{ with } (t, y) \neq (0, 0),
\]

(11)

where \( C_2 \in (0, \infty) \) is a universal constant. The following small distance/time behavior of \( H \) and \( K \) was also proved in [6].

**Theorem 2** (Theorem 2 of [6]). Fix \( h > 0 \).

\[
\lim_{\lambda \downarrow 0} \lambda^{1/4} H(0, \lambda) = H^0(0, 1) = C_2.
\]

(12)

\[
\lim_{\epsilon \downarrow 0} \frac{K(0) - K(\epsilon)}{\epsilon^{3/4}} = 2 \int_{0}^{\infty} [H^0(0, y) - H^0(1, y)] dy \in (0, \infty).
\]

(13)

2.2. **Relation to quantum field theory.** Since \( \Phi^h \) is a Euclidean field satisfying the Osterwalder-Schrader axioms, by the Källén-Lehmann spectral formula (see Theorem 6.2.4 of [17]), we have

\[
H(s, y) = \int_{0}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(ipy) \exp(-E|s|) \delta(m^2 + p^2 - E^2) dEdp \right) d\tilde{\rho}(m),
\]

(14)

where \( \tilde{\rho} \) is a mass spectral measure of the relativistic quantum field theory obtained from \( \Phi^h \) via the Osterwalder-Schrader reconstruction theorem [31, 32]. Here for fixed \( p \) and \( m \),

\[
\delta(m^2 + p^2 - E^2) = \frac{\delta(\sqrt{m^2 + p^2} + E) + \delta(\sqrt{m^2 + p^2} - E)}{2 \sqrt{m^2 + p^2}}.
\]

(15)

Then an easy computation (see [6] for the details) gives

\[
K(s) = \pi \int_{0}^{\infty} \frac{\exp(-|s|m)}{m} d\tilde{\rho}(m), \text{ for any } s \in \mathbb{R}.
\]

(16)

By Theorem 1.4 and Remark 1.7 of [4], the support of \( \tilde{\rho} \) is in \([m_1, \infty)\) for some \( m_1 > 0 \). If we define a new measure \( \rho \) by the Radon-Nikodym derivative

\[
\frac{d\rho}{d\tilde{\rho}}(m) = \frac{\pi}{m},
\]

(17)

then we have the following.
Proposition 2 (Theorem 1 and Corollary 1 of [9]). There exists \( m_1 \in (0, \infty) \) such that
\[
K(s) = \int_{m_1}^{\infty} \exp(-m|s|)d\rho(m).
\]
Moreover, \( \rho \) is a finite measure, but its first moment is infinite.

3. A Gaussian process from the discrete Ising model

3.1. Two-dimensional results. Theorem 1 says that the limit of the centered \( \Phi^h \) as the spatial coordinate \( y \) scales to infinity with the Euclidean time coordinate \( t \) fixed is a mean zero stationary Gaussian process \( X(t) \) with covariance function \( K(t) \). We prove in Section 4 below that this Gaussian process can also be obtained directly from the near-critical Ising model on \( a\mathbb{Z}^2 \).

Denote by \( P^a_h \) the infinite volume Ising measure at the inverse critical temperature \( \beta_c \) on \( a\mathbb{Z}^2 \) with external field \( a^{15/8}h > 0 \). Let \( \langle \cdot \rangle_{a,h} \) denote expectation with respect to \( P^a_h \).

For \( s \in \mathbb{R} \), let \( s_a \) denote a point in \( a\mathbb{Z} \) that is closest to \( s \). For \( L > 0 \) and \( s \in \mathbb{R} \), we define
\[
X_L(s) := \frac{a^{7/8} \sum_{k \in a\mathbb{Z} \cap [-L,L]} [\sigma(s_a,k) - \langle \sigma(s_a,k) \rangle_{a,h}]}{\sqrt{2L}}.
\]

Our main result for \( d = 2 \) is the following.

Theorem 3. Suppose \( L(a) \geq 0 \) is a function of \( a \) satisfying \( L(a) \to \infty \) as \( a \downarrow 0 \). Then for any \( n \in \mathbb{N} \) and distinct \( s_1, \ldots, s_n \in \mathbb{R} \), we have
\[
(X_{L(a)}(s_{1,a}), \ldots, X_{L(a)}(s_{n,a})) \Rightarrow (X(s_1), \ldots, X(s_n)) \quad \text{as} \quad a \downarrow 0,
\]
where \( s_{j,a} \) for each \( 1 \leq j \leq n \) is a point in \( a\mathbb{Z} \) that is closest to \( s_j \).

The relation between Theorems 1 and 3 and the results in 3 can be summarized in the following diagram:

\[
\begin{array}{c}
\{\sigma_x : x \in a\mathbb{Z}^2\} \\
\xrightarrow{\text{n.i.d.}}
\Phi^h(t, y) \\
\downarrow_{L \uparrow \infty}
X(t),
\end{array}
\]

where the right and down arrows represent results in 3 and Theorem 1 respectively, and the diagonal arrow represents Theorem 3.

3.2. Results and discussions for general dimension. For general \( d \geq 2 \), we try to derive the desired Gaussian process (denoted by \( \hat{X} \)) in two steps. In the first step, we construct a Gaussian process \( \hat{X}^a(t) \) from the critical Ising model on \( a\mathbb{Z}^d \) with fixed magnetic field \( \tilde{h} \); in the second step, we choose \( \tilde{h} \) as an appropriate function of \( a \) and obtain \( \hat{X} \) from \( \hat{X}^a(t) \) by sending \( a \downarrow 0 \).

On \( a\mathbb{Z}^d \), we write \( x \in a\mathbb{Z}^d \) as \( x = (t, \tilde{y}) \) where \( t \in a\mathbb{Z} \) and \( \tilde{y} \in a\mathbb{Z}^{d-1} \). For \( L > 0 \) and \( s \in \mathbb{R} \), we define
\[
\hat{X}^a_L(s) := \frac{\sum_{\tilde{y} \in a\mathbb{Z}^{d-1} \cap [-L,L]^{d-1}} [\sigma(s_a,\tilde{y}) - \langle \sigma(s_a,\tilde{y}) \rangle]}{\sqrt{\text{Var} \left( \sum_{\tilde{y} \in a\mathbb{Z}^{d-1} \cap [-L,L]^{d-1}} \sigma(s_a,\tilde{y}) \right)}}.
\]

where \( \langle \cdot \rangle \) denotes the expectation for the critical Ising model on \( a\mathbb{Z}^d \) with fixed magnetic field \( \tilde{h} \); for general \( d \geq 2 \) we do not add any subscripts to brackets since the corresponding probability measure should be clear from the context. Note that \( \hat{X}^a_L(s) \) is standardized.
with mean 0 and variance 1. For \( z, w \in a\mathbb{Z}^d \), let \( \langle \sigma_z; \sigma_w \rangle \) denote the truncated two-point function, i.e.,
\[
\langle \sigma_z; \sigma_w \rangle = \langle \sigma_z \sigma_w \rangle - \langle \sigma_z \rangle \langle \sigma_w \rangle. \tag{22}
\]
In the first step, we prove

**Theorem 4.** Consider the critical Ising model on \( a\mathbb{Z}^d \) with fixed magnetic field \( \hat{h} > 0 \). Then for any \( n \in \mathbb{N} \) and distinct \( s_1, \ldots, s_n \in a\mathbb{Z} \), we have
\[
\left( \hat{X}^a_0(s_1), \ldots, \hat{X}^a_L(s_n) \right) \Rightarrow \left( \hat{X}^a(s_1), \ldots, \hat{X}^a(s_n) \right) \text{ as } L \uparrow \infty,
\tag{23}
\]
where \( \{ \hat{X}^a(s), s \in a\mathbb{Z} \} \) is a mean zero stationary Gaussian process with covariance function
\[
\text{Cov}(\hat{X}^a(s), \hat{X}^a(t)) = \hat{K}^a(t - s) := \sum_{\tilde{y} \in a\mathbb{Z}^{d-1}} \frac{\langle \sigma(0, \tilde{y}); \sigma_i(t - s, \tilde{y}) \rangle}{\sum_{\tilde{y} \in a\mathbb{Z}^{d-1}} \langle \sigma(0, \tilde{y}); \sigma(0, \tilde{y}) \rangle} \text{ for } s, t \in a\mathbb{Z}. \tag{24}
\]

**Remark 1.** \( \hat{K}^a(s) \) is non-increasing as a function of \( |s| \). This follows from the monotonicity of \( \langle \sigma(0, \tilde{y}); \sigma(t, \tilde{y}) \rangle \) (and hence also \( \langle \sigma(0, \tilde{y}); \sigma_i(t, \tilde{y}) \rangle \)) in \( |t| \) (see \([34]\) and \([28]\)).

We recall that the correlation function for the critical Ising model on \( \mathbb{Z}^d \) with \( \hat{h} = 0 \) is expected to scale in the following way
\[
\langle \sigma_0 \sigma_x \rangle \approx |x|^{-d+2-\eta} \text{ for } 0, x \in \mathbb{Z}^d \text{ and large } |x|,
\tag{25}
\]
where \( |x| := \|x\|_2 \) denotes the Euclidean distance and \( \eta \geq 0 \). It is known that \( \eta = 1/4 \) when \( d = 2 \) \([33, 27]\) and the conjecture for \( d \geq 4 \) is \( \eta = 0 \). If we take \( \hat{h} = a^{(d+2-\eta)/2}h \) for some \( h > 0 \), we conjecture that the following limit exists
\[
\hat{K}(s) := \lim_{a \downarrow 0} \hat{K}^a(s_a) = \lim_{a \downarrow 0} \frac{\sum_{\tilde{y} \in a\mathbb{Z}^{d-1}} \langle \sigma(0, \tilde{y}); \sigma_{i(s, \tilde{y})} \rangle}{\sum_{\tilde{y} \in a\mathbb{Z}^{d-1}} \langle \sigma(0, \tilde{y}); \sigma(0, \tilde{y}) \rangle}, \text{ for } s \in \mathbb{R}. \tag{26}
\]

Now it is easy to prove (by showing the convergence of the corresponding characteristic functions)

**Proposition 3.** Under the assumption that the limit in \((26)\) exists, we have
\[
(\hat{X}^a(s_1), \ldots, \hat{X}^a(s_n)) \Rightarrow (\hat{X}(s_1), \ldots, \hat{X}(s_n)) \text{ as } a \downarrow 0, \tag{27}
\]
where \( \{ \hat{X}(s) : s \in \mathbb{R} \} \) is a mean zero stationary Gaussian process with covariance function
\[
\text{Cov}(\hat{X}(s), \hat{X}(t)) = \hat{K}(t - s). \tag{28}
\]

**Remark 2.** From \((26)\) and Remark \[\] below, one can see that \( \hat{K}(s) = K(s)/K(0) \) when \( d = 2 \). Therefore, \( \hat{X}(s) \overset{d}{=} X(s)/\sqrt{K(0)} \) when \( d = 2 \).

As mentioned in \([4]\), for \( d = 3 \), the covariance function \( \hat{K}(t) \) for small \( t \) would be nondifferentiable at \( t = 0 \), like in Theorem \([2]\) but with \( \hat{K}(0) - \hat{K}(\epsilon) \) behaving like \( \epsilon^{1-\eta} \) as \( \epsilon \downarrow 0 \), rather than \( \epsilon^{3/4} \), with \( \eta \) the corresponding critical exponent for \( d = 3 \). For \( d > 4 \), \( \hat{K} \) would be differentiable while for \( d = 4 \), there is the possibility of logarithmic behavior.

The diagram right after Theorem \([8]\) and Remark \[\] imply that, when \( d = 2 \), the two limits “\( L \uparrow \infty \)” and “\( a \downarrow 0 \)” commute. This should be contrasted with a result and a conjecture for the high dimensional Ising model in \([8]\). In Remark 3 of \([8]\), it was proved that for large \( d \), the limit of the near-critical magnetization field on \( \Lambda^a \) := \([-L, L]^d \cap a\mathbb{Z}^d \), \( \Phi^h_{\Lambda^a} \), converges (after subtracting its mean) as \( L \uparrow \infty \) in distribution to a massless Gaussian free field on \( \mathbb{R}^d \). But Conjecture 1 of \([8]\) says that the near-critical magnetization field on \( a\mathbb{Z}^d \) converges (after subtracting its mean) as \( a \downarrow 0 \) in distribution to a massive...
Gaussian free field on \( \mathbb{R}^d \). So in that case, the two limits “\( L \uparrow \infty \)” and “\( a \downarrow 0 \)” should not commute.

4. Proof of results in Section 3

The main ingredients for the proof of Theorem 3 are the convergence of the covariance \( \text{Cov}(X_{(a)}(s), X_{(a)}(t)) \) and an inequality for FKG systems from [30]. Since \( H \) is a function only of the radial variable, we define

\[
\hat{H}(\sqrt{t^2 + y^2}) := H(t, y), \quad \text{for any } (t, y) \in \mathbb{R}^2.
\]

(29)

Recall that \( s_a \) is a point in \( a\mathbb{Z} \) that is closest to \( s \). For \( z \in \mathbb{R}^2 \), let \( z_a \) be a point in \( a\mathbb{Z} \) that is closest to \( z \).

**Proposition 4.** Fix \( h \geq 0 \). For any \( L \in (0, \infty) \) and any \( s, t \in \mathbb{R} \), we have

\[
\lim_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z}[\mathbb{Z}[L, L]]} \langle \sigma(s_a, 0); \sigma(t_a, k) \rangle_{a,h} = \int_{-L}^{L} H(t - s, y)dy,
\]

(30)

\[
\lim_{a \downarrow 0} a^{-1/4} \langle \sigma_{z_a}; \sigma_{w_a} \rangle_{a,h} = \hat{H}(|z - w|), \quad \text{for all } z \neq w \in \mathbb{R}^2.
\]

(31)

**Remark 3.** The limit (31) generalizes a classical result by Wu, which corresponds to \( h = 0 \) in (31). See [35, 27] (also [11] and Theorem 3.1 of [11]) for Wu’s result.

We recall two inequalities which will be important for the proof of Proposition 4. The first one is the SMM (for Schrader, Messager and Miracle-Sole) inequality [34, 28]: in a region \( \Lambda \) with reflection symmetry, the correlation \( \langle \prod_{x \in A} \sigma_x \prod_{x \in B} \sigma_x \rangle_{\Lambda, a, h} \) with \( A \) and \( B \) on the same side of a reflection plane can only decrease when \( B \) is replaced by its reflected image \( \bar{B} \), i.e.,

\[
\langle \prod_{x \in A} \sigma_x \prod_{x \in \bar{B}} \sigma_x \rangle_{\Lambda, a, h} \geq \langle \prod_{x \in A} \sigma_x \prod_{x \in B} \sigma_x \rangle_{\Lambda, a, h}.
\]

(32)

In the infinite volume limit on \( a\mathbb{Z}^2 \), this inequality holds for reflections with respect to (a) lines parallel to the principal axes and passing through points in \( (\frac{1}{2}a\mathbb{Z}) \times (\frac{1}{2}a\mathbb{Z}) \) — in particular for any \( z = (z_1, z_2), w = (w_1, w_2) \in a\mathbb{Z}^2 \),

\[
\langle \sigma_0 \sigma_z \rangle_{a, h} \leq \langle \sigma_0 \sigma_w \rangle_{a, h} \quad \text{if } z_1 = w_1 \text{ and } |z_2| \geq |w_2|;
\]

(33)

(b) “diagonal” lines, i.e., lines with slope \( \pm 1 \) and passing through points in \( a\mathbb{Z}^2 \).

The SMM inequality also holds for infinite-volume truncated two-point functions since the one-point function is constant. The second inequality is the GKS inequality (see Corollary 1 of [19] and also [22]), which says that

\[
\langle \sigma_z; \sigma_w \rangle_{a,h} \geq 0 \text{ for any } z, w \in a\mathbb{Z}^2.
\]

(34)

We will also use the following lemma about convergence of moments.

**Lemma 1.** Fix \( h \geq 0 \). Suppose \( f \in C_c^\infty(\mathbb{R}^2) \), the space of \( C^\infty \) functions on \( \mathbb{R}^2 \) whose support is compact. Then we have

\[
\lim_{a \downarrow 0} \langle \Phi^{a,h}(f) \rangle_{a,h} = \mathbb{E} \Phi^h(f).
\]

(35)

Moreover, for any \( f, g \in C_c^\infty(\mathbb{R}^2) \), we have

\[
\lim_{a \downarrow 0} \langle \Phi^{a,h}(f) \Phi^{a,h}(g) \rangle_{a,h} = \mathbb{E} \langle \Phi^h(f) \Phi^h(g) \rangle.
\]

(36)
Remark 4. Lemma 1 actually holds for any bounded \( f, g \) in the dual of the Sobolev space \( \mathcal{H}^{-3}(\mathbb{R}^2) \). In our applications, \( f \) and \( g \) will be indicator functions of some bounded regions in \( \mathbb{R}^2 \).

Proof of Lemma 1. \( \Phi^{a,h}(f) \Rightarrow \Phi^h(f) \).

So (35) follows if we can show that

\[
\langle \exp \left(t \Phi^{a,h}(f)\right) \rangle_{a,h} \text{ is bounded for any } t \in \mathbb{R}
\]

(uniformly as \( a \downarrow 0 \)). We may choose a bounded \( \Lambda \) such that the support of \( f \) is contained in \( \Lambda \). Then the GKS inequalities [18, 22] imply that

\[
\langle \exp \left(t \Phi^{a,h}(f)\right) \rangle_{a,h} \leq \langle \exp \left(\Phi^{a,0} ((|t||f||_{\infty} + h)1_\Lambda)\right) \rangle_{a,h}^+,
\]

where \( \langle \cdot \rangle^+_{a,h} \) is the expectation of the critical Ising model on \( \Lambda \cap a\mathbb{Z}^2 \) with plus boundary conditions. The GHS inequality [20] gives that, for any \( M \in \mathbb{R} \),

\[
\langle \exp \left(\Phi^{a,0}(M1_\Lambda)\right) \rangle^+_{a,h} \leq \exp \left[ \langle \Phi^{a,0}(M1_\Lambda) \rangle^+_{a,h} + \frac{1}{2} \text{Var}^+_{a,0}(\Phi^{a,0}(M1_\Lambda)) \right].
\]

By Proposition B.1 in Appendix B.1 of [2], which bounds the mean and variance in (40), we see that (39) and (40) imply (38). For the proof of (36), we note that

\[
\Phi^{a,h}(f + g) \Rightarrow \Phi^h(f + g) \text{ as } a \downarrow 0,
\]

and (37) and (38) imply that

\[
\lim_{a \downarrow 0} \left\langle \left( \Phi^{a,h}(f + g) \right)^2 \right\rangle_{a,h} = \mathbb{E} \left[ \Phi^h(f + g) \right]^2, \quad \lim_{a \downarrow 0} \left\langle \left( \Phi^{a,h}(f) \right)^2 \right\rangle_{a,h} = \mathbb{E} \left[ \Phi^h(f) \right]^2,
\]

which completes the proof. \( \square \)

Proof of Proposition 4. We first consider the easy case: \( s \neq t \). Without loss of generality, we may assume that \( s < t \). An application of (33) and (34) gives that, for each fixed \( \epsilon \) satisfying \( 2a \leq \epsilon < (t - s)/2 \),

\[
\sum_{k \in a\mathbb{Z} \cap [-L,L]} \left\langle \sigma_{(s,u,0)}; \sigma_{(u,k)} \right\rangle_{a,h} \leq \frac{1}{([\epsilon/a] - 1)^3} \sum_{z,w \in a\mathbb{Z}^2} 1_{\{z \in [s,s+\epsilon] \times [-\epsilon/2,\epsilon/2], w \in [t-\epsilon,t] \times [-L-\epsilon,L+\epsilon]\}} \left\langle \sigma_z; \sigma_w \right\rangle_{a,h},
\]

where \([ \cdot ]\) denotes the greatest integer function. It is clear that

\[
\lim_{a \downarrow 0} \frac{(\epsilon/a)^3}{([\epsilon/a] - 1)^3} = 1.
\]

Lemma 1, (33) and (44) imply that

\[
\limsup_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [-L,L]} \left\langle \sigma_{(s,u,0)}; \sigma_{(u,k)} \right\rangle_{a,h} \leq \frac{1}{e^3} \int_{[s,s+\epsilon] \times [-\epsilon/2,\epsilon/2]} \int_{[t-\epsilon,t] \times [-L-\epsilon,L+\epsilon]} H(\|z - w\|) \, dz \, dw.
\]

(45)
Note that $\hat{H}$ is real analytic on $(0, \infty)$ (see, e.g., Corollary 19.5.6 of [17]). Therefore, by letting $\epsilon \downarrow 0$ in (45), we get
\[
\limsup_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [-L, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \leq \int_{-L}^{L} \hat{H}(\sqrt{(t-s)^2 + y^2})dy. \tag{46}
\]

Another application of (33) and (34) gives that, for each fixed $\epsilon$ with $2a \leq \epsilon < L$,
\[
\sum_{k \in a\mathbb{Z} \cap [-L, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \geq \frac{1}{(\lceil \epsilon/a \rceil + 1)^3} \sum_{z,w \in a\mathbb{Z}^2} 1\{z \in [s-\epsilon,s) \times [-\epsilon/2, \epsilon/2], w \in [t-\epsilon, t+\epsilon] \} \langle \sigma_z; \sigma_w \rangle_{a,h}. \tag{47}
\]

The same arguments leading to (46) imply that
\[
\liminf_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [-L, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \geq \int_{-L}^{L} \hat{H}(\sqrt{(t-s)^2 + y^2})dy. \tag{48}
\]

The limit (30) with $s < t$ follows from (46) and (48).

The proof of (31) is similar but simpler than that of (30) with $s \neq t$. We next consider the more involved case: (31) with $s = t$. By the SMM inequality (33), we have
\[
\sum_{k \in a\mathbb{Z} \cap [-L, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \geq \sum_{k \in a\mathbb{Z} \cap [-L, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \text{ for any } s < t. \tag{49}
\]

Using (49) and (30) with $s \neq t$, we obtain
\[
\liminf_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [-L, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \geq \liminf_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [-L, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} = \int_{-L}^{L} H(t-s, y)dy \text{ for any } s < t. \tag{50}
\]

Taking $t \downarrow s$, we get
\[
\liminf_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [-L, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \geq \int_{-L}^{L} H(0, y)dy. \tag{52}
\]

So (30) with $s = t$ would follow if we can show
\[
\limsup_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [0, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \leq \int_{0}^{L} H(0, y)dy. \tag{53}
\]

To do this, we first observe that, for any $\eta \in (0, L)$,
\[
\limsup_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [0, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \leq \limsup_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [0, \eta]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h} \tag{54}
\]
\[
+ \limsup_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [\eta, L]} \langle \sigma(s_{a,0}); \sigma(t_{a,k}) \rangle_{a,h}. \tag{55}
\]
It is not hard to see that for any \( \epsilon \) satisfying \( 2a \leq \epsilon < \eta/2 \)
\[
\sum_{k \in a\mathbb{Z} \cap [\eta, L]} \langle \sigma(0,0); \sigma(0,k) \rangle_{a,h} \leq \frac{1}{(\lfloor \epsilon/a \rfloor - 1)^3} \sum_{z,w \in a\mathbb{Z}} 1_{\{z \in [-\epsilon/2, \epsilon/2] \times [0, \epsilon], w \in [-\epsilon/2, \epsilon/2] \times [\eta, \eta + \epsilon]\}} \langle \sigma_z; \sigma_w \rangle_{a,h}; \tag{56}
\]
this is because for any \( z \in a\mathbb{Z}^2 \cap \{[-\epsilon/2, \epsilon/2] \times [0, \epsilon]\} \) and any \( w_1 \in a\mathbb{Z} \cap [-\epsilon/2, \epsilon/2], \) by the SMM inequality, we have
\[
\sum_{k \in a\mathbb{Z} \cap [\eta - \epsilon, L + \epsilon]} \langle \sigma_z; \sigma(\eta, L) \rangle_{a,h} \geq \sum_{k \in a\mathbb{Z} \cap [\eta, L]} \langle \sigma(0,0); \sigma(0,k) \rangle_{a,h}. \tag{57}
\]
Let us check \refeq{57} for the special case \( z = (-\epsilon/2, 0) \) and \( w_1 = \epsilon/2. \) (We assume \( \epsilon/2 \) is a multiple of \( a; \) otherwise we may take \( \lfloor \epsilon/(2a) \rfloor a \) instead.) Part (b) of the SMM inequality (reflection with respect to the line with slope 1 and passing through \( (\epsilon/2, 0) \)) implies that
\[
\sum_{k \in a\mathbb{Z} \cap [\eta - \epsilon, L + \epsilon]} \langle \sigma(-\epsilon/2,0); \sigma(\epsilon/2,k) \rangle_{a,h} \geq \sum_{k \in a\mathbb{Z} \cap [\eta - \epsilon, L + \epsilon]} \langle \sigma(\epsilon/2,-\epsilon); \sigma(\epsilon/2,k) \rangle_{a,h}. \tag{58}
\]
Then translation invariance implies
\[
\sum_{k \in a\mathbb{Z} \cap [\eta - \epsilon, L + \epsilon]} \langle \sigma(-\epsilon/2,0); \sigma(\epsilon/2,k) \rangle_{a,h} \geq \sum_{k \in a\mathbb{Z} \cap [\eta - \epsilon, L + \epsilon]} \langle \sigma(\epsilon/2,-\epsilon); \sigma(\epsilon/2,k) \rangle_{a,h} \tag{59}
\]
\[
\geq \sum_{k \in a\mathbb{Z} \cap [\eta, L]} \langle \sigma(0,0); \sigma(0,k) \rangle_{a,h}. \tag{60}
\]
Letting \( a \downarrow 0 \) in \refeq{56} and applying Lemma \ref{lemma} we get
\[
\limsup_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [\eta, L]} \langle \sigma(0,0); \sigma(0,k) \rangle_{a,h} \leq \frac{1}{\epsilon^3} \int_{[-\epsilon, \epsilon] \times [0, \epsilon]} \int_{[-\epsilon, \epsilon] \times [\eta, \eta + \epsilon]} \tilde{H}(\|z - w\|) dz dw. \tag{61}
\]
Letting \( \epsilon \downarrow 0 \) in the last displayed inequality, we have
\[
\limsup_{a \downarrow 0} a^{3/4} \sum_{k \in a\mathbb{Z} \cap [\eta, L]} \langle \sigma(0,0); \sigma(0,k) \rangle_{a,h} \leq \int_{\eta}^{L} H(0,y) dy. \tag{62}
\]
Next, we deal with the sum in the RHS of \refeq{54}. By the GHS inequality \ref{E19},
\[
\sum_{k \in a\mathbb{Z} \cap [0, \eta]} \langle \sigma(0,0); \sigma(0,k) \rangle_{a,h} \leq \sum_{k \in a\mathbb{Z} \cap [0, \eta]} \langle \sigma(0,0); \sigma(0,k) \rangle_{a,h=0}; \tag{63}
\]
where \( \langle \cdot \rangle_{a,h=0} \) is expectation with respect to the critical Ising model on \( a\mathbb{Z}^2 \) without external field (i.e., \( h = 0 \)). Wu’s result \ref{E1}, \ref{E27} says that there exists \( C_3 \in (0, \infty) \) such that
\[
\lim_{N \to \infty} \frac{\langle \sigma(0,0) \sigma(0,N) \rangle_{a=1, h=0}}{N^{-1/4}} = C_3. \tag{64}
\]
This implies that, for all small enough \( a, \)
\[
\langle \sigma(0,0) \sigma(0,k) \rangle_{a,h=0} \leq 2C_3 \left( \frac{k}{a} \right)^{-1/4} \text{ for any } k \in a\mathbb{Z} \cap [a^{1/2}, \infty). \tag{65}
\]
Combining this with (63), we get
\[
\sum_{k \in aZ \cap [0,a]} \langle \sigma(0,0) ; \sigma(0,k) \rangle_{a,h} \leq \sum_{k \in aZ \cap [0,1/2]} 1 + \sum_{k \in aZ \cap [1/2,a]} \langle \sigma(0,0) ; \sigma(0,k) \rangle_{a,h=0} \leq \frac{a^{1/2}}{a} + 1 + 2C_3 \sum_{k \in aZ \cap [1/2,a]} \left( \frac{k}{a} \right)^{-1/4}.
\] (67)

Multiplying each side by \(a^{3/4}\) and taking limits, we obtain
\[
\limsup_{a \downarrow 0} a^{3/4} \sum_{k \in aZ \cap [0,a]} \langle \sigma(0,0) ; \sigma(0,k) \rangle_{a,h} \leq 2C_3 \int_0^{\eta} y^{-1/4} dy = \frac{8C_3}{3} \eta^{3/4}.
\] (68)

Substituting (68) and (62) into (54) and (55) respectively, we get that for all small \(\eta\)
\[
\limsup_{a \downarrow 0} a^{3/4} \sum_{k \in aZ \cap [0,L]} \langle \sigma(0,0) ; \sigma(0,k) \rangle_{a,h} \leq \frac{8C_3}{3} \eta^{3/4} + \int_{\eta}^{L} H(0,y) dy.
\] (69)

This completes the proof of (53) by letting \(\eta \downarrow 0\). \(\square\)

**Remark 5.** For any \(h > 0\), from (51) and Theorem 1.1 of [1], we know that \(\hat{H}(t)\) decays exponentially for large \(t\). Using this fact, it is easy to see that Proposition 4 also extends to \(L = \infty\).

The following corollary, based on Proposition 4 and Remark 5, will be used to prove convergence of \(\text{Cov}(X_{L(a)}(s), X_{L(a)}(t))\).

**Corollary 1.** Fix \(h > 0\). Suppose \(L(a) > 0\) is a function of \(a\) satisfying \(L(a) \to \infty\) as \(a \downarrow 0\). Then for any \(s, t \in \mathbb{R}\), we have
\[
\lim_{a \downarrow 0} a^{3/4} \sum_{k \in aZ \cap [-L(a),L(a)]} \langle \sigma(s,a) ; \sigma(t,a) \rangle_{a,h} = \int_{-\infty}^{\infty} H(t-s,y) dy = K(t-s).
\] (70)

**Proof of Corollary 4.** The GKS inequality (34) and Remark 5 imply that
\[
\limsup_{a \downarrow 0} a^{3/4} \sum_{k \in aZ \cap [-L(a),L(a)]} \langle \sigma(s,a) ; \sigma(t,a) \rangle_{a,h} \leq \limsup_{a \downarrow 0} a^{3/4} \sum_{k \in aZ} \langle \sigma(s,a) ; \sigma(t,a) \rangle_{a,h} = \int_{-\infty}^{\infty} H(t-s,y) dy.
\] (71)

On the other hand, for any fixed \(M > 0\), one may choose \(\epsilon > 0\) such that \(L(a) \geq M\) for each \(a \in (0, \epsilon)\). Then the GKS inequality (34) and Proposition 4 imply that
\[
\liminf_{a \downarrow 0} a^{3/4} \sum_{k \in aZ \cap [-L(a),L(a)]} \langle \sigma(s,a) ; \sigma(t,a) \rangle_{a,h} \geq \liminf_{a \downarrow 0} a^{3/4} \sum_{k \in aZ \cap [-M,M]} \langle \sigma(s,a) ; \sigma(t,a) \rangle_{a,h} = \int_{-M}^{M} H(t-s,y) dy.
\] (72)

Letting \(M \to \infty\), we obtain
\[
\liminf_{a \downarrow 0} a^{3/4} \sum_{k \in aZ \cap [-L(a),L(a)]} \langle \sigma(s,a) ; \sigma(t,a) \rangle_{a,h} \geq \int_{-\infty}^{\infty} H(t-s,y) dy.
\] (73)

The corollary follows from (71) and (73). \(\square\)

We are ready to prove convergence of \(\text{Cov}(X_{L(a)}(s), X_{L(a)}(t))\) as \(a \downarrow 0\).
Proposition 5. Fix $h > 0$. Suppose $L(a) > 0$ is a function of a satisfying $L(a) \to \infty$ as $a \downarrow 0$. Then for any $s, t \in \mathbb{R}$, we have
\[
\lim_{a \downarrow 0} \text{Cov}(X_{L(a)}(s), X_{L(a)}(t)) = K(t - s).
\] (74)

Proof of Proposition 5. By the definition of $X_{L(a)}(s)$ in [19], the SMM inequality (33), and Corollary 1, we have
\[
\limsup_{a \downarrow 0} \text{Cov}(X_{L(a)}(s), X_{L(a)}(t)) = \limsup_{a \downarrow 0} \frac{a^{7/4}}{2L(a)} \sum_{k,j \in a\mathbb{Z} \cap [-L(a),L(a)]} \langle \sigma(s,a,k); \sigma(t,a,j) \rangle_{a,h}
\leq \limsup_{a \downarrow 0} \frac{a^{7/4}}{2L(a)} \left( \frac{2L(a)}{a} + 1 \right) \sum_{j \in a\mathbb{Z} \cap [-L(a),L(a)]} \langle \sigma(s,a,0); \sigma(t,a,j) \rangle_{a,h}
= K(t - s).
\] (75)

For any fixed $\epsilon \in (0, 1)$, the GKS inequality (34), translation invariance of the Ising model on $a\mathbb{Z}^2$ and Corollary 1 imply that
\[
\liminf_{a \downarrow 0} \text{Cov}(X_{L(a)}(s), X_{L(a)}(t)) \geq (1 - \epsilon)K(t - s).
\] (76)

Letting $\epsilon \downarrow 0$, we get
\[
\liminf_{a \downarrow 0} \text{Cov}(X_{L(a)}(s), X_{L(a)}(t)) \geq K(t - s).
\] (77)

The proposition now follows from (75) and (77). \qed

Another important ingredient in the proof of Theorem 3 is the following inequality from [30].

Theorem 5 (Theorem 1 in [30]). Suppose $U_1, \ldots, U_m$ have finite variance and satisfy the FKG inequality; then for any $r_1, \ldots, r_m \in \mathbb{R}$,
\[
\left| \exp \left( i \sum_{l=1}^m r_l U_l \right) - \prod_{l=1}^m \exp (ir_l U_l) \right| \leq \frac{1}{2} \sum_{l \neq n} |r_l r_n| \text{Cov}(U_l, U_n),
\] (80)

where $\langle \cdot \rangle$ denotes expectation.

Proof of Theorem 3. For fixed $\vec{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ and $\vec{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n$, we define
\[
Y_L = Y_L(\vec{z}, \vec{s}) = z_1X_L(s_1) + \cdots + z_nX_L(s_n).
\] (81)

Note that (20) is equivalent to
\[
\lim_{a \downarrow 0} \langle \exp(iY_L(a)) \rangle_{a,h} = \exp \left( -\frac{1}{2} \sum_{j=1}^n \sum_{l=1}^n z_j z_l K(s_j - s_l) \right),
\] (80)

for each $(z_1, \ldots, z_n) \in \mathbb{R}^n$. We will prove (80) for $(z_1, \ldots, z_n) \in [0, \infty)^n$. We claim that this yields a proof of Theorem 3 by applying the same argument used in the proof of Theorem 1 of [6]; for completeness, we reproduce below. The basic idea is to define
\[
Y^+_L(a) := \sum_{j:z_j \geq 0} z_j X_L(a)(s_j),
Y^-_L(a) := \sum_{j:z_j < 0} |z_j| X_L(a)(s_j),
\] (81)
and note that \( cY_{L(a)}^+ + dY_{L(a)}^- \) for any \( c \geq 0, d \geq 0 \) converges (as \( a \downarrow 0 \)) in distribution to a Gaussian distribution with mean zero and variance

\[
\sum_{j=1}^{n} \sum_{l=1}^{n} (c1_{\{z_j \geq 0\}} + d1_{\{z_l < 0\}}) (c1_{\{z_j \geq 0\}} + d1_{\{z_l < 0\}}) |z_j z_l| K(s_j - s_l). \tag{82}
\]

So we may assume \( (Y_{L(a)}^+, Y_{L(a)}^-) \) has a subsequential limit in distribution \( (Y^+, Y^-) \). Then \( cY^+ + dY^- \) for \( c, d \geq 0 \) is a mean zero Gaussian random variable with variance given by \( (82) \). Theorem 3 of \cite{21} then implies that \( (Y^+, Y^-) \) is a bivariate normal vector whose distribution is already determined by \( cY^+ + dY^- \) for \( c, d \geq 0 \). Therefore,

\[
(Y_{L(a)}^+, Y_{L(a)}^-) \Rightarrow (Y^+, Y^-) \text{ as } a \downarrow 0. \tag{83}
\]

In particular, we have

\[
Y_{L(a)} = Y_{L(a)}^+ - Y_{L(a)}^- \Rightarrow Y^+ - Y^- \text{ as } a \downarrow 0, \tag{84}
\]

which is the claim.

For \( j \in \mathbb{N} \) and \( j \in [-|L^{1/2}|, |L^{1/2}|] \), we define

\[
Y_{L(j)} := \begin{cases} 
\sum_{l=1}^{n} z_l L^{-1/4} a_l^{7/8} \sum_{k \in a \mathbb{Z} \cap [j |L^{1/2}|, (j+1) |L^{1/2}|]} [\sigma_{(s_l,a,k)} - \langle \sigma_{(s_l,a,k)} \rangle_{a,h}], & j \leq |L^{1/2}| - 1 \\
2^{1/2} L^{-1/4} Y_{L} - \sum_{k=-|L^{1/2}|}^{0} Y_{L(k)}, & j = |L^{1/2}|.
\end{cases} \tag{85}
\]

where \( s_l,a \) is a point in \( a \mathbb{Z} \) which is closest to \( s_l \). Note that by translation invariance, \( Y_{L(j)} \) for all \( j \)'s with \(-|L^{1/2}| \leq j \leq |L^{1/2}| - 1 \) are identically distributed random variables. Then Theorem \cite{3} implies that (this is where we use the assumption that all \( z_l \)'s are nonnegative)

\[
\left| \left\langle \exp(iY_{L(a)}) \right\rangle_{a,h} - \prod_{j=-|L(a)|^{1/2}}^{0} \left\langle \exp(i2^{-1/2} L(a)^{-1/4} Y_{L(j)}) \right\rangle_{a,h} \right| \leq \frac{1}{2} \sum_{j \neq l} \sum_{j=-|L(a)|^{1/2}}^{0} 2^{-1} L(a)^{-1/2} \operatorname{Cov}\left( Y_{L(a)}^{(j)}, Y_{L(a)}^{(l)} \right)
\]

\[
= \frac{1}{2} \left[ \operatorname{Var}(Y_{L(a)}) - 2^{-1} L(a)^{-1/2} \sum_{j=-|L(a)|^{1/2}}^{0} \operatorname{Cov}(Y_{L(a)}^{(j)}, Y_{L(a)}^{(j)}) \right]
\]

\[
= \frac{1}{2} \left[ \operatorname{Var}(Y_{L(a)}) - \frac{|L(a)|^{1/2}}{L(a)^{1/2}} \operatorname{Var}(Y_{L(a)}^{(0)}) - \frac{1}{2 \sqrt{L(a)}} \operatorname{Var}(Y_{L(a)}^{(|L(a)|^{1/2}/2)}) \right], \tag{86}
\]

where we have used that \( Y_{L(a)}^{(j)} \) has the same distribution as \( Y_{L(a)}^{(0)} \) if \( j \leq |L(a)|^{1/2} - 1 \). By Proposition \cite{5} we have

\[
\lim_{a \downarrow 0} \operatorname{Var}(Y_{L(a)}) = \sum_{j=1}^{n} \sum_{l=1}^{n} z_j z_l K(s_j - s_l), \tag{87}
\]

\[
\lim_{a \downarrow 0} \operatorname{Var}(Y_{L(a)}^{(0)}) = \sum_{j=1}^{n} \sum_{l=1}^{n} z_j z_l K(s_j - s_l). \tag{88}
\]
By \(^{(33)}\), translation invariance and the Cauchy-Schwarz inequality, we have
\[
\text{Var}(Y_{L(a)}^{([L(a)^{1/2}])}) \leq 4\text{Var}(Y_{L(a)}^{(0)}).
\] (89)
Therefore,
\[
\lim_{a \to 0} \left| \left\langle \exp(iY_{L(a)}) \right\rangle_{a, h} - \prod_{j = -[L(a)^{1/2}]}^{[L(a)^{1/2}]} \left\langle \exp(i2^{-1/2}L(a)^{-1/4}Y_{j}) \right\rangle_{a, h} \right| = 0. \tag{90}
\]
Since \(^{(33)}\) implies that \(\text{Var}(Y_{L(a)}^{(0)}) < \infty\) for all small \(a\), a standard Taylor expansion result (see, e.g., Theorem 3.3.20 of [14]) implies that
\[
\text{Var}(Y_{L(a)}^{(0)}) \rightarrow 0 \text{ as } a \rightarrow 0. \tag{91}
\]
We conclude with a proof of Theorem 4.

**Proof of Theorem 4** Without loss of generality, we may assume \(a = 1\) in the proof. Theorem 4 is equivalent to
\[
\lim_{L \to \infty} \left\langle \exp(i(z_1 \tilde{X}_L^a(s_1) + \cdots + z_n \tilde{X}_L^a(s_n))) \right\rangle = \exp \left( -\frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} z_j z_l \tilde{K}^a(s_j - s_l) \right), \tag{92}
\]
for each \((z_1, \ldots, z_n) \in \mathbb{R}^n\). By the same argument as in the proof of Theorem 3, it is enough to show \(^{(92)}\) for \((z_1, \ldots, z_n) \in [0, \infty)^n\). Note that \(\langle \sigma_x; \sigma_y \rangle\) decays exponentially as \(|z - w| \to \infty\) (see [24, 15]). This and (26) in Lemma 4 of [30] imply that
\[
\lim_{L \to \infty} \frac{\text{Var}(\sum_{j \in \mathbb{Z}^{d-1} \cap [-L, L]^{d-1}} \sigma(s_j) \tilde{g})}{(2L)^{d-1}} = \sum_{\tilde{g} \in \mathbb{Z}^{d-1}} \left\langle \sigma(0, 0); \sigma(0, \tilde{g}) \right\rangle, \forall s \in \mathbb{Z}. \tag{93}
\]
A direct application of Theorem 2 in [30] implies that
\[
\sum_{j=1}^{n} z_j \sum_{\tilde{g} \in \mathbb{Z}^{d-1} \cap [-L, L]^{d-1}} \sigma(s_j) \tilde{g} - \sum_{j=1}^{n} z_j \sum_{\tilde{g} \in \mathbb{Z}^{d-1} \cap [-L, L]^{d-1}} \left\langle \sigma(s_j) \tilde{g} \right\rangle \tag{94}
\]
converges as \(L \uparrow \infty\) in distribution to a normal distribution with mean 0 and variance
\[
\sum_{j=1}^{n} \sum_{l=1}^{n} z_j z_l \sum_{\tilde{g} \in \mathbb{Z}^{d-1}} \left\langle \sigma(s_j, 0); \sigma(s_l, \tilde{g}) \right\rangle. \tag{95}
\]
This combined with \(^{(93)}\) completes the proof of \(^{(92)}\). \(\square\)

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