NON-WEAKLY AMENABLE BEURLING ALGEBRAS

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Abstract. Weak amenability of a weighted group algebra, or a Beurling algebra, is a long-standing open problem. The commutative case has been extensively investigated and fully characterized. We study the non-commutative case. Given a weight function \( \omega \) on a locally compact group \( G \), we characterize derivations from \( L^1(G, \omega) \) into its dual in terms of certain functions. Then we show that for a locally compact IN group \( G \), if there is a non-zero continuous group homomorphism \( \varphi: G \to \mathbb{C} \) such that \( \varphi(x)/\omega(x)\omega(x^{-1}) \) is bounded on \( G \), then \( L^1(G, \omega) \) is not weakly amenable. Some useful criteria that rule out weak amenability of \( L^1(G, \omega) \) are established. Using them we show that for many polynomial type weights the weighted Heisenberg group algebra is not weakly amenable, neither is the weighted \( ax+b \) group algebra. We further study weighted quotient group algebra \( L^1(G/H, \hat{\omega}) \), where \( \hat{\omega} \) is the canonical weight on \( G/H \) induced by \( \omega \). We reveal that the kernel of the canonical homomorphism from \( L^1(G, \omega) \) to \( L^1(G/H, \hat{\omega}) \) is complemented. This allows us to obtain some sufficient conditions under which \( L^1(G/H, \hat{\omega}) \) inherits weak amenability of \( L^1(G, \omega) \). We study further weak amenability of Beurling algebras of subgroups. In general, weak amenability of a Beurling algebra does not pass to the Beurling algebra of a subgroup. However, in some circumstances this inheritance can happen. We also give an example to show that weak amenability of both \( L^1(H, \omega|_H) \) and \( L^1(G/H, \hat{\omega}) \) does not ensure weak amenability of \( L^1(G, \omega) \).

1. Introduction

Let \( G \) be a locally compact group. As usual, we denote the integral of a function \( f \) against a fixed left Haar measure by

\[
\int f(x)dx.
\]

The group algebra \( L^1(G) \) is the Banach algebra consisting of all Haar integrable functions on \( G \) with the convolution product and the \( L^1 \)-norm

\[
\|f\|_1 = \int |f(x)|dx.
\]

Two functions in \( L^1(G) \) are regarded as the same if they are equal almost everywhere on \( G \) with respect to the Haar measure.

A weight function on \( G \) is a locally bounded positive measurable function \( \omega: G \to \mathbb{R}^+ \) that satisfies the submultiplicative inequality

\[
\omega(xy) \leq \omega(x)\omega(y) \quad (x, y \in G).
\]

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Date: November 23, 2014.

2010 Mathematics Subject Classification. Primary 22D15, 43A20. Secondary 46H10, 43A10.

Key words and phrases. weight, locally compact group, Beurling algebra, weak amenability, subgroup, quotient group, IN group, Heisenberg group, \( ax+b \) group.

† Supported by NSERC Grant 238949.
Given a weight $\omega$ on $G$, consider
$$L^1(G, \omega) = \{ f : f \omega \in L^1(G)\}.$$ Equipped with the norm
$$\|f\|_\omega := \int_G |f(x)|\omega(x) \, dx$$
and the convolution product, $L^1(G, \omega)$ becomes a Banach algebra, called a weighted group algebra or a Beurling algebra. The dual space of $L^1(G, \omega)$ may be identified with
$$L^\infty(G, \frac{1}{\omega}) := \{ f : f/\omega \in L^\infty(G) \}$$
whose norm is given by
$$\|f\|_{L^\infty,1/\omega} = \text{ess sup}_{x \in G} \frac{|f(x)|}{\omega(x)} \quad (f \in L^\infty(G, \frac{1}{\omega})).$$
Obviously, as a Banach space $L^1(G, \omega)$ is isometrically isomorphic to $L^1(G)$. However, as Banach algebras these two are very different. For example, it is well-known that $L^1(G)$ is a typical quantum group algebra [10], while $L^1(G, \omega)$ is usually not, although $L^\infty(G, \frac{1}{\omega})$ is a von Neumann algebra with the product $f \cdot g = \frac{1}{\omega}fg$. In fact, $L^1(G, \omega)$ is not even an F-algebra, unless the weight is trivial (meaning that the weight is multiplicative). We refer to [18] for the relation between quantum groups and F-algebras.

The investigation of $L^1(G, \omega)$ goes back to A. Beurling [3], where $G = \mathbb{R}$ was considered. One may find a good account of elementary theory concerning the general weighted group algebra in [24].

Two weight functions $\omega_1$ and $\omega_2$ on $G$ are called equivalent if there are constants $c_1, c_2 > 0$ such that
$$c_1\omega_1(x) \leq \omega_2(x) \leq c_2\omega_1(x)$$
locally almost everywhere on $G$. It is readily seen that if $\omega_1$ and $\omega_2$ are equivalent weights, then $L^1(G, \omega_1)$ and $L^1(G, \omega_2)$ are isomorphic as Banach algebras. It is well-known that a weight on $G$ is always equivalent to a continuous weight on $G$ (see [28], or [24, Theorem 3.7.5] for a proof; note that in [24] the condition $\omega \geq 1$ is not necessary if we do not require the weighted algebra to be a subalgebra of $L^1(G)$). For this reason, unless otherwise is specified, in this paper we always assume that a weight is continuous.

We are concerned with weak amenability of the Beurling algebra $L^1(G, \omega)$. We refer to [7, 8, 22] for research of other aspects regarding Beurling algebras. Special types of groups have been studied in [12, 20, 29]. Related research concerning weighted Fourier algebras may be found in [19, 21].

Recall that a derivation from a Banach algebra $A$ to a Banach $A$-bimodule $X$ is a linear mapping $D_A \rightarrow X$ satisfying $D(ab) = a \cdot D(b) + D(a) \cdot b$ $(a, b \in A)$. For every $x \in X$ the map $a \mapsto a \cdot x - x \cdot a$ is a continuous derivation, called an inner derivation. Given a Banach $A$-bimodule $X$, its dual space $X^\ast$ is naturally a Banach $A$-bimodule (called the dual module of $X$) with the module actions defined by
$$\langle x, a \cdot f \rangle = \langle x \cdot a, f \rangle, \quad \langle x, f \cdot a \rangle = \langle a \cdot x, f \rangle \quad (a \in A, f \in X^\ast, x \in X).$$
Following B. E. Johnson [14], we call $A$ amenable if every continuous derivation from $A$ into any dual Banach $A$-bimodule $X^\ast$ is inner. Johnson showed in [14] that the group algebra $L^1(G)$ is amenable if and only if $G$ is an amenable group. Later
N. Gronbaek showed in \[10\] that the weighted group algebra $L^1(G, \omega)$ is amenable if and only if $G$ is an amenable group and $\omega$ is a diagonally bounded weight, i.e., the function $\omega(x)\omega(x^{-1})$ is bounded on $G$. The latter conditions actually imply that the weight $\omega$ is bounded up to a multiplicative factor. Hence, a nontrivial weighted group algebra is intrinsically not an amenable Banach algebra.

Weak amenability for commutative Banach algebras was introduced by Bade, Curtis, and Dales in \[2\]. Based on a characterization result of \[2\], Johnson later called a general Banach algebra $A$ weakly amenable if every continuous derivation from $A$ into $A^*$ is inner. He showed in \[15\] that $L^1(G)$ is weakly amenable for all locally compact groups $G$.

Weak amenability of Beurling algebras has been studied by many authors. In \[2\] it was shown that $L^1(\mathbb{Z}, \omega_\alpha)$ for the additive group $\mathbb{Z}$ and the polynomial weight $\omega_\alpha(x) = (1 + |x|)\alpha$ is weakly amenable if and only if $0 \leq \alpha < 1/2$. The same conclusion holds if $\mathbb{Z}$ is replaced with $\mathbb{R}$ (\[5, 25, 30\]). In \[11\] N. Gronbaek showed that the Beurling algebra of a commutative discrete group $G$ is weakly amenable if and only if every non-trivial group homomorphism $\Phi : G \rightarrow \mathbb{C}$ satisfies

\[
sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} = \infty.
\]  

It turns out that this characterization is still valid for a general commutative locally compact group.

**Theorem 1.1.** \[30\] Theorem 3.1\] Let $G$ be an Abelian locally compact group, and $\omega$ be a weight on $G$. The Beurling algebra $L^1(G, \omega)$ is weakly amenable if and only if \[1\] holds for every continuous non-zero group homomorphism $\Phi : G \rightarrow \mathbb{C}$.

However, condition \[1\] is far from being sufficient for $L^1(G, \omega)$ to be weakly amenable if the group $G$ is not commutative. A counterexample associated to discrete $SL_2(\mathbb{R})$ was obtained in \[4\]. In \[26\] the first author showed that with a non-trivial polynomial weight $\omega_\alpha$ the algebra $\ell^1(\mathbb{F}_2, \omega_\alpha)$ is never weakly amenable. This contrasts with the results on commutative groups $\mathbb{Z}$ and $\mathbb{R}$ mentioned above. Similar investigations concerning the discrete $ax + b$ group were also conducted there. Overall, weak amenability of a non-commutative Beurling algebra is still very unclear. So far we have not even seen a non-trivial example of weakly amenable Beurling algebra which is not commutative. The related problem of weak amenability of the center algebra of a Beurling algebra has been studied in \[1, 27, 30\].

In this paper, in Section 2 we first characterize continuous derivations from $L^1(G, \omega)$ into its dual in terms of certain functions from $L^\infty(G \times G, \omega \times \omega)$. We then show that the necessity part of Theorem 1.1 remains true if $G$ is an IN group, improving a result of \[30\]. We further establish a criterion that rules out weak amenability of a Beurling algebra. As an application, we show that the weighted group algebra of the topological Heisenberg group with certain type of “polynomial weights” is not weakly amenable.

In Section 3 we continue the investigation of \[26\] on weighted $ax + b$ group algebras. For the topological $ax + b$ group, we show that the Beurling algebra on $ax + b$ with a polynomial weight is never weakly amenable. For the discrete case we show that if the weight is independent of $b$, then the corresponding Beurling algebra is weakly amenable only when the weight is diagonally bounded. This provides us with an example of a locally compact group $G$ with a closed normal subgroup $H$ and a weight $\omega$ such that both Beurling algebras $L^1(H, \omega|_H)$ and $L^1(G/H, \hat{\omega})$ are
weakly amenable, but $L^1(G, \omega)$ is not weakly amenable, where $\hat{\omega}$ is a weight on $G/H$ naturally induced from $\omega$.

In Section 4 we study Beurling algebras associated to quotient groups. If $H$ is a closed normal subgroup of $G$ then

$$L^1(G/H, \hat{\omega}) \cong L^1(G, \omega)/J_\omega(G, H),$$

where $J_\omega(G, H)$ is a closed ideal of $L^1(G, \omega)$. We show that $J_\omega(G, H)$ is always complemented in $L^1(G, \omega)$. This allows us to establish a sufficient condition under which weak amenability of $L^1(G, \omega)$ is inherited by $L^1(G/H, \hat{\omega})$. Using this result, we prove that weak amenability of the tensor product $L^1(G_1, \omega_1)\hat{\otimes}L^1(G_2, \omega_2)$ implies weak amenability of both $L^1(G_1, \omega_1)$ and $L^1(G_2, \omega_2)$, provided the weights $\omega_1$, $\omega_2$ are bounded away from zero. The question whether the converse is true remains open except for the case when $G$ is Abelian [30, Corollary 3.10]. We also improve a result of [17] concerning weak amenability of a complemented subalgebra.

In Section 5 we investigate Beurling algebras of subgroups. Example 5.1 shows that, even in the Abelian case, weak amenability of a Beurling algebra does not imply weak amenability of the induced Beurling algebra of a subgroup. However, the implication is true under some circumstances. We also investigate the problem of extending a group homomorphism from a subgroup to the whole group in Section 5.

2. Criteria Ruling Out Weak Amenability of $L^1(G, \omega)$

We start from a characterization of a bounded derivation from $L^1(G, \omega)$ into its dual $L^1(G, \omega)^* \cong L^\infty(G, \frac{1}{\omega})$. It generalizes a result of B. E. Johnson [15] which deals with the case $\omega \equiv 1$.

Let $G_1$ and $G_2$ be two locally compact groups and $\omega_i$ be a weight on $G_i$ ($i = 1, 2$). We denote by $\omega_1 \times \omega_2$ the weight on $G_1 \times G_2$ defined by

$$(\omega_1 \times \omega_2)(x_1, x_2) = \omega_1(x_1)\omega_2(x_2) \quad (x_1 \in G_1, \ x_2 \in G_2).$$

**Lemma 2.1.** Let $G$ be a locally compact group and $\omega$ be a weight on $G$. Then for every bounded derivation $D : L^1(G, \omega) \to L^\infty(G, \frac{1}{\omega})$ there exists a function $\alpha \in L^\infty(G \times G, \frac{1}{\omega \times \omega})$ such that

(2) $\alpha(xy, z) = \alpha(x, yz) + \alpha(y, z) \ (\text{locally a.e.} \ (x, y, z) \in G \times G \times G)$ and

(3) $\langle g, D(f) \rangle = \iint_{G \times G} \alpha(x, y)f(x)g(y) \ dx \ dy \quad (f, g \in L^1(G, \omega)).$

Conversely, every function $\alpha \in L^\infty(G \times G, \frac{1}{\omega \times \omega})$ satisfying (2) defines a bounded derivation $D : L^1(G, \omega) \to L^\infty(G, 1/\omega)$ by the formula (3).

**Proof.** Given a bounded derivation $D : L^1(G, \omega) \to L^\infty(G, \frac{1}{\omega})$, the map $(f, g) \mapsto \langle g, D(f) \rangle$ is a bilinear functional on $L^1(G, \omega) \times L^1(G, \omega)$ and we have

$$|\langle g, D(f) \rangle| \leq ||D|| \ ||g||_\omega ||f||_\omega.$$

Hence this map defines a bounded linear functional

$$\alpha \in (L^1(G, \omega) \hat{\otimes}L^1(G, \omega))^* \cong L^\infty(G \times G, \frac{1}{\omega \times \omega})$$

by

$$\langle f \otimes g, \alpha \rangle = \langle g, D(f) \rangle \quad (f, g \in L^1(G, \omega)).$$
It follows that relation (3) holds.

Let \( \pi : L^\infty(G, \frac{1}{\omega}) \to L^\infty(G \times G, \frac{1}{\omega \times \omega}) \) be the operator defined by

\[
\pi(f)(x, y) = f(xy) \quad (f \in L^\infty(G, \frac{1}{\omega})).
\]

From [24 Corollary 3.3.32] it is readily seen that \( \pi(f) \in L^\infty(G \times G, \frac{1}{\omega \times \omega}) \) if \( f \in L^\infty(G, \frac{1}{\omega}) \), and \( \|\pi(f)\|_{\infty, 1/(\omega \times \omega)} = \|f\|_{\infty, 1/\omega} \).

Applying \( \pi \otimes \text{id} \) to \( \alpha \), where \( \text{id} \) stands for the identity operator on \( L^\infty(G, \frac{1}{\omega}) \), we see that the function \( \alpha_1(x, y, z) = \alpha(xy, z) \) belongs to \( L^\infty(G \times G \times G, \frac{1}{\omega \times \omega \times \omega}) \). Similarly, the functions \( \alpha_2(x, y, z) = \alpha(x, yz) \) and \( \alpha_3(x, y, z) = \alpha(y, zx) \) also belong to \( L^\infty(G \times G \times G, \frac{1}{\omega \times \omega \times \omega}) \). In order to show that identity (2) holds, it suffices to verify

\[
\{f \otimes g \otimes h, \alpha_1\} = \{f \otimes g \otimes h, \alpha_2\} + \{f \otimes g \otimes h, \alpha_3\}.
\]

In fact,

\[
\{f \otimes g \otimes h, \alpha_1\} = \int_{G^3} \alpha(xy, z)f(x)g(y)h(z)dxdydz
\]

\[
= \int_{G^2} \alpha(y, z)(f \ast g)(y)h(z)dydz = \{h, D(f \ast g)\}
\]

\[
= \{h, f \ast D(g) + D(f) \ast g\} = \{h \ast f, D(g)\} + \{g \ast h, D(f)\}
\]

\[
= \{f \otimes g \otimes h, \alpha_2\} + \{f \otimes g \otimes h, \alpha_3\}
\]

for all \( f, g, h \in L^1(G, \omega) \). Therefore (2) holds.

The converse can be easily verified by computation. The proof is complete.

Recall that a locally compact group \( G \) is an IN group if it has a compact neighborhood of the unit element \( e \) which is invariant under all inner automorphisms, i.e., if there is a compact neighborhood \( U \) of \( e \) such that \( gUg^{-1} = U \) for all \( g \in G \). It was shown in [30, Remark 3.2] by the second author that for an IN group \( G \) the necessity part of Theorem 1.1 remains true under some extra condition. We now can remove this condition. Precisely, we have the following.

**Theorem 2.2.** Let \( G \) be an IN group and \( \omega \) be a weight on \( G \). Suppose that there exists a non-trivial continuous group homomorphism \( \Phi : G \to \mathbb{C} \) such that

\[
\sup_{t \in G} \frac{|\Phi(t)|}{\omega(t)\omega(t^{-1})} < \infty.
\]

Then \( L^1(G, \omega) \) is not weakly amenable.

**Proof.** We use \( \Phi \) to construct a continuous non-inner derivation \( D : L^1(G, \omega) \to L^\infty(G, \frac{1}{\omega}) \). Let \( B \) be an invariant compact neighborhood of \( e \). Define \( D \) as in [30 Theorem 3.1] by

\[
D(h)(t) = \int_B \Phi(t^{-1}\xi)h(t^{-1}\xi)\, d\xi \quad (h \in L^1(G, \omega), \ t \in G).
\]
As indicated in [30] Remark 3.2, $D$ is indeed a continuous derivation. Here we show this by using Lemma 2.1. For all $g, h \in L^1(G, \omega)$ we have

$$
\langle g, D(h) \rangle = \int \int \Phi(\xi) h(\xi) \frac{d\xi}{G} g(t) dt = \int \int \chi_{t^{-1}B}(\xi) \Phi(\xi) h(\xi) g(t) d\xi dt.
$$

Let $\alpha(t, \xi) = \chi_B(t\xi) \Phi(\xi)$. Then $\alpha$ is clearly a measurable function on $G \times G$. Also,

$$
\sup_{(\xi, t) \in G \times G} \frac{|\alpha(t, \xi)|}{\omega(t)} = \sup_{\xi, t \in G} \frac{|\chi_B(t\xi)\Phi(\xi)|}{\omega(\xi)\omega(t)} = \sup_{\xi, t \in G, \xi \in B} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(t)} \leq \sup_{\xi \in G} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(\xi^{-1})} \cdot \sup_{\xi, t \in G, \xi \in B} \frac{\omega(\xi^{-1})}{\omega(t)} \leq \sup_{\xi \in G} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(\xi^{-1})} \cdot \sup_{\xi, t \in G, \xi \in B} \omega((t\xi)^{-1}) < \infty,
$$

since $\sup_{\xi \in G} \frac{|\Phi(\xi)|}{\omega(\xi)\omega(\xi^{-1})} < \infty$ and $\omega$ is bounded on the compact set $B^{-1}$ as a continuous function. So we have shown that $\alpha \in L^\infty(G \times G, \frac{1}{\omega(t)})$. Next we prove that

$$
\alpha(xy, z) = \alpha(x, yz) + \alpha(y, zx) \quad (x, y, z \in G).
$$

Fix $x, y, z \in G$. Since $yzz = y(yzx)y^{-1} = y$ and $B$ is invariant under inner automorphisms, we have that $\chi_B(yyz) = \chi_B(y)$. Then we can use the fact that $\Phi$ is a group homomorphism to obtain

$$
\alpha(xy, z) = \chi_B(yyz)\Phi(xy) = \chi_B(yyz)(\Phi(xy) + \Phi(y)) = \chi_B(yyz)\Phi(x) + \chi_B(yyz)\Phi(y) = \alpha(xyz) + \alpha(y, zx).
$$

So identity (5) is verified. By Lemma 2.1 $D$ is a bounded derivation from $L^1(G, \omega)$ to $L^\infty(G, \frac{1}{\omega(t)})$.

We now show that for every $h \in L^1(G, \omega)$ the function $D(h) \in L^\infty(G, 1/\omega)$ is continuous. Fix any $t_0 \in G$ and let $C$ be a compact neighborhood of $t_0$. Let

$$
\beta(x) = \begin{cases} 
\Phi(x)h(x), & x \in C^{-1}B, \\
0, & x \notin C^{-1}B.
\end{cases}
$$

Then,

$$
D(h)(t) = \int_B \beta(t^{-1}\xi) d\xi = \int_B L_t(\beta)(\xi) d\xi \quad (t \in C),
$$

where $L_t$ is the left translation operator. Since $\Phi$ is continuous, $C^{-1}B$ is compact, $h \in L^1(G, \omega)$, and $\omega$ is bounded away from zero on compact sets, we have that $\beta \in L^1(G)$. Therefore, for $t \in C$ we have:

$$
|D(h)(t) - D(h)(t_0)| = \int_B (L_t\beta(\xi) - L_{t_0}\beta(\xi)) d\xi \leq \int_G |L_t\beta(\xi) - L_{t_0}\beta(\xi)| d\xi.
$$

Hence, $D(h)$ is continuous at $t_0$. Since $t_0$ was taken arbitrarily, we conclude that $D(h)$ is a continuous function on $G$ for each $h \in L^1(G, \omega)$. 

We are now ready to show that $D$ is not an inner derivation. Suppose, to the contrary, that there exists $f \in L^\infty(G, \frac{1}{2})$ such that

$$D(h) = f \cdot h - h \cdot f \quad (h \in L^1(G, \omega)).$$

Fix any $t_0 \in G$ and consider $h_0 = \chi_{t_0^{-1}B} \in L^1(G, \omega)$. Then

$$D(h_0)(t_0) = (f \cdot h_0)(t_0) - (h_0 \cdot f)(t_0) = \int_G f(yt_0)h_0(y) \, dy - \int_G f(t_0y)h_0(y) \, dy$$

$$= \int_{t_0^{-1}B} f(yt_0) \, dy - \int_{t_0^{-1}B} f(t_0y) \, dy = \int_B f(y) \, dy - \int_B f(y) \, dy = 0.$$  

As we have already shown, $D(h_0)$ is a continuous function. It is also standard that $f \cdot h_0 - h_0 \cdot f$ is a continuous function when $f \in L^\infty(G, \frac{1}{2})$ (see, for example, Proposition 7.17]). Therefore,

$$0 = D(h_0)(t_0) = \int_B \Phi(t_0^{-1}x)h_0(t_0^{-1}x) \, dx = \int_B \Phi(t_0^{-1}x) \, dx.$$ 

Since $\Phi$ is a homomorphism, we obtain

$$0 = \int_B \Phi(t_0^{-1}x) \, dx = \int_B (\Phi(x) - \Phi(t_0)) \, dx = \int_B \Phi(x) \, dx - \Phi(t_0)\mu(B),$$

which implies that

$$\Phi(t_0) = \frac{\int_B \Phi(x) \, dx}{\mu(B)},$$

where $\mu$ denotes the Haar measure on $G$ ($\mu(B) > 0$ since $B$ is a neighborhood of identity and thus contains an open subset). Because $t_0 \in G$ was chosen arbitrarily, it follows that $\Phi$ is constant on $G$, which can happen for a homomorphism $\Phi$ only if $\Phi \equiv 0$. This contradiction shows that $D$ is not an inner derivation. The proof is complete.

Our next result provides another criterion to rule out weak amenability for a Beurling algebra. For the discrete case it was first obtained by Borwick in his Ph.D. thesis [4] (see also [26]), and has been used in [4] and [26] to study weak amenability of Beurling algebras on discrete $SL_2(\mathbb{R})$, $\mathbb{F}_2$, and discrete $ax + b$ group.

Let $G$ be a group. Recall that the conjugacy class of $x \in G$ is the set $C_x = \{gxg^{-1} : g \in G\}$. Given a subset $B$ of $G$, we denote

$$C_B = \{gxg^{-1} : g \in G, x \in B\} = \bigcup_{x \in B} C_x$$

and call it the conjugacy class of $B$.

**Theorem 2.3.** Let $G$ be a locally compact group, $B \neq \emptyset$ be an open set in $G$ with compact closure, and $\omega$ be a weight on $G$ that is bounded away from zero on $C_B$, i.e., there is a constant $\delta > 0$ such $\omega(x) \geq \delta$ for $x \in C_B$. Suppose that there exists a measurable function $\psi : G \to \mathbb{C}$ bounded on $B$ and such that

$$\frac{\text{ess sup}_{x,y \in G} |\psi(xy) - \psi(yx)|}{\omega(x)\omega(y)} < \infty \quad \text{and}$$

$$\frac{\text{ess sup}_{x \in G} |\psi(x)|}{\omega(x)} < \infty.$$ 

Then $D$ is not an inner derivation.
\[(8) \quad \text{ess sup}_{z \in C_B} \frac{\vert \psi(z) \vert}{\omega(z)} = \infty.\]

Then \(L^1(G, \omega)\) is not weakly amenable.

**Proof.** Suppose that \(\psi\) is a function satisfying all aforementioned conditions. Then \(\Psi(x, y) = \psi(xy) - \psi(yx)\) is measurable on \(G \times G\), and condition (7) ensures that \(\Psi \in L^\infty(G \times G, \frac{1}{\omega \times \omega})\). Moreover,

\[
\Psi(xyz, z) = \psi(xyz) - \psi(zxy) = (\psi(xyz) - \psi(yzx)) + (\psi(yzx) - \psi(zxy))
\]

\[
= \Psi(x, yz) + \Psi(y, zx) \quad (x, y, z \in G).
\]

Then by Lemma 2.1 \(\Psi\) defines a continuous derivation \(D : L^1(G, \omega) \rightarrow L^\infty(G, \frac{1}{\omega})\) that satisfies

\[
\langle g, D(f) \rangle = \int_{G^2} (\psi(xy) - \psi(yx))f(x)g(y) \, dx \, dy \quad (f, g \in L^1(G, \omega)).
\]

We show that this derivation \(D\) is not inner, which will imply that \(L^1(G, \omega)\) is not weakly amenable.

Suppose, to the contrary, that \(D\) is inner. Then there exists a function \(\varphi \in L^\infty(G, \frac{1}{\omega})\) such that

\[
D(f) = \varphi \cdot f - f \cdot \varphi \quad (f \in L^1(G, \omega)).
\]

It follows that

\[
\langle g, D(f) \rangle = \int_{G^2} (\varphi(xy) - \varphi(yx))f(x)g(y) \, dx \, dy \quad (f, g \in L^1(G, \omega)).
\]

Denote \(\Phi(x, y) = \varphi(xy) - \varphi(yx)\). Then \(\Phi \in L^\infty(G \times G, \frac{1}{\omega \times \omega})\) and

\[
\langle f \otimes g, \Psi - \Phi \rangle = 0 \quad (f, g \in L^1(G, \omega)).
\]

Therefore, \(\Psi = \Phi\) as the elements of \(L^\infty(G \times G, \frac{1}{\omega \times \omega})\). We then have

\[
\int_{G^2} (\Psi(x, y) - \Phi(x, y))U(x, y) \, dx \, dy = 0 \quad (U \in L^1(G \times G, \omega \times \omega)).
\]

On the other hand, if \(U\) is in \(L^1(G \times G, \omega \times \omega)\), then so is the function \(\chi_B(xy)U(x, y)\). Hence,

\[
\int_{G^2} (\Psi(x, y) - \Phi(x, y))\chi_B(xy)U(x, y) \, dx \, dy = 0.
\]

In particular, the last equality holds for all \(U\) in \(C_{00}(G \times G)\), the space of continuous functions with compact support. For any \(U \in C_{00}(G \times G)\), let \(V(x, y) = U(x, xy)\). It is evident that \(V \in C_{00}(G \times G)\). Thus,

\[
0 = \int_{G^2} (\Psi(x, y) - \Phi(x, y))\chi_B(xy)V(x, y) \, dx \, dy
\]

\[
= \int_{G \times B} (\Psi(x, x^{-1}y) - \Phi(x, x^{-1}y))U(x, y) \, dx \, dy
\]
for all \( U \in C_00(G \times G) \). Since \( C_00(G \times G) \) is dense in \( L^1(G \times G, \omega \times \omega) \), we have
\[
\int_{G \times B} (\Psi(x, x^{-1}y) - \Phi(x, x^{-1}y))U(x, y) \, dx \, dy = 0 \quad (U \in L^1(G \times G, \omega \times \omega)).
\]
This implies that \( \Psi(x, x^{-1}y) - \Phi(x, x^{-1}y) = 0 \) locally almost everywhere on \( G \times B \), i.e.,
\[
\psi(x^{-1}yx) = \psi(y) - \varphi(y) + \varphi(x^{-1}yx) \quad (\text{locally a.e. on } G \times B).
\]
Dividing both sides by \( \omega(x^{-1}yx) \) and noting that
\[
\frac{\psi(x^{-1}yx)}{\omega(x^{-1}yx)} = \frac{\varphi(x^{-1}yx)}{\omega(x^{-1}yx)} + \frac{\psi(y)}{\omega(x^{-1}yx)} \quad (\text{locally a.e.})
\]
we obtain
\[
\varphi(x^{-1}yx) = \varphi(y) + \psi(y) - \psi(x^{-1}yx) \quad (\text{locally a.e.})
\]
Since \( \| \varphi \|_{\infty, 1/\omega} < \infty \), \( \psi \) and \( \omega \) are bounded on \( B \), and \( \omega \) is bounded away from zero on \( C_B \), we derive
\[
\text{ess sup}_{(x, y) \in G \times B} \frac{\varphi(x^{-1}yx)}{\omega(x^{-1}yx)} < \infty,
\]
which is a contradiction to condition (8). Therefore, \( D \) is not inner. The proof is complete.

As an application of Theorem 2.3, let us consider the topological Heisenberg group. Recall that the Heisenberg group \( G_H \) is a 3-dimensional Lie group consisting of all \( 3 \times 3 \) matrices of the form
\[
\begin{pmatrix}
1 & u & w \\
0 & 1 & v \\
0 & 0 & 1
\end{pmatrix}
\]
\((u, v, w \in \mathbb{R})\).

It is a unimodular locally compact group with the ordinary Euclidean norm topology and the Lebesgue measure of \( \mathbb{R}^3 \) as a Haar measure (see [23, Section 12.1.18]). To simplify the notation, we represent the elements of \( G_H \) by \((u, v, w)\) so that \( G_H = \mathbb{R}^3 \) with the product and inverse operations given by
\[
(u, v, w)(a, b, c) = (u + a, v + b, w + c + ub), \quad (u, v, w)^{-1} = (-u, -v, uv - w).
\]

Proposition 2.4. Let \( \omega \) be a weight on \( G_H \) of the form
\[
\omega(u, v, w) = W(|u|, |v|) \quad ((u, v, w) \in G_H).
\]
Suppose that
\[
\lim_{(x, y) \to \infty} W(x, y) = \infty.
\]
Then \( L^1(G_H, \omega) \) is not weakly amenable.

Proof. Consider \( B = \{(u, v, w) : |u| < 1, |v| < 1, |w| < 1\} \). Then \( B \) is an open set in \( G_H \) with compact closure. From [23] we have
\[
(u, v, w)(a, b, c)(u, v, w)^{-1} = (a, b, c + ub - va).
\]
Therefore $C_B = \{(u, v, w) : |u| < 1, |v| < 1, w \in \mathbb{R}\}$. Since $\omega > 0$ is continuous and depends only on the first two variables, it is obviously both bounded and bounded away from zero on $C_B$. Consider

$$\tilde{\omega}(t) = \inf\{W(u, v) : u \geq 0, v \geq 0, u + v > |t|\}.$$ 

It is readily seen that $\tilde{\omega}$ is a positive increasing unbounded continuous function on $\mathbb{R}$ and $\tilde{\omega}(-t) = \tilde{\omega}(t)$ ($t \in \mathbb{R}$). Moreover, $\tilde{\omega}$ is a weight on $(\mathbb{R}, +)$. To see this we note that if $u_i, v_i \geq 0, t_i \in \mathbb{R}$ and $u_i + v_i > |t_i|$ ($i = 1, 2$), then

$$\tilde{\omega}(t_1 + t_2) \leq W(u_1 + u_2, v_1 + v_2) \leq W(u_1, v_1)W(u_2, v_2).$$

Taking infimum on the right side over all possible $(u_1, v_1)$ and $(u_2, v_2)$, we derive the desired inequality

$$\tilde{\omega}(t_1 + t_2) \leq \tilde{\omega}(t_1)\tilde{\omega}(t_2) \quad (t_1, t_2 \in \mathbb{R}).$$

Let

$$\psi(u, v, w) = \chi_{C_B}(u, v, w) \ln \tilde{\omega}(w) \quad ((u, v, w) \in G_H),$$

where $\chi_{C_B}$ is the characteristic function of $C_B$. We aim to show that $\psi$ satisfies all the conditions of Theorem 2.3. It is readily seen that $\psi$ is a locally bounded measurable function on $G_H$ which is unbounded on $C_B$ by (10). Since $\omega$ is bounded on $C_B$, it follows that $\psi$ satisfies condition (5). To show that (7) is satisfied, we let $x = (u, v, w) \in G_H$ and $y = (a, b, c) \in G_H$. Then $xy$ and $yx$ belong to the same conjugacy class. If $xy \notin C_B$, then $yx \notin C_B$ and condition (7) is obviously satisfied. Assume now that $xy, yx \in C_B$. Then

$$|\psi(xy) - \psi(yx)| = \left| \ln \frac{\tilde{\omega}(w + c + ub)}{\tilde{\omega}(w + c + av)} \right| \leq |\ln \tilde{\omega}(|ub - av|)| = \ln \tilde{\omega}(|ub - av|).$$

To obtain the last inequality, we used symmetry and submultiplicativity of $\tilde{\omega}$ together with the fact that $\tilde{\omega} \geq 1$ as a symmetric weight function. Since $xy \in C_B$, we have that $|u + a| < 1$ and $|v + b| < 1$. So,

$$|ub - av| = |(u + a)b - a(v + b)| \leq |a| + |b|.$$

Similarly, $|ub - av| \leq |u| + |v|$. Then the monotonicity of $\tilde{\omega}$ implies

$$\ln \tilde{\omega}(|ub - av|) \leq \frac{1}{2} \ln \left( \tilde{\omega}(|a| + |b|) \tilde{\omega}(|u| + |v|) \right) \leq \frac{1}{2} \ln (W(|a|, |b|)W(|u|, |v|)) = \frac{1}{2} \ln (\omega(x)\omega(y)) \leq \frac{1}{2} \omega(x)\omega(y).$$

In the last step we used the fact that $\omega > 1$, which is true since $\omega$ is a symmetric weight by the assumption. Combining the last inequality with (11), we see that $\psi$ satisfies condition (7). By Theorem 2.3, $L^1(G_H, \omega)$ is not weakly amenable, and the proof is complete.

$$\Box$$

It is readily seen that the function $\omega_\alpha(u, v, w) = (1 + |u| + |v|)^\alpha$ is a weight on $G_H$ satisfying the condition of Proposition 2.4. So we have

**Example 2.5.** The Beurling algebra $L^1(G_H, \omega_\alpha)$ is not weakly amenable for any $\alpha > 0$.

It is worth to restate Theorem 2.3 for the discrete group case. We will use this discrete version to study weak amenability of $\ell^1(ax + b, \omega)$ in Section 3.
Corollary 2.6. Let $G$ be a discrete group, $B \neq \emptyset$ be a finite set in $G$, and $\omega$ be a weight on $G$ that is bounded away from zero on the conjugacy class $C_B$. Suppose that there exists a function $\psi : G \to \mathbb{R}$ and a constant $c > 0$ such that

$$|\psi(xy) - \psi(yx)| \leq c \omega(x)\omega(y) \quad (x, y \in G) \quad \text{and}$$

$$\sup_{z \in C_B} \frac{|\psi(z)|}{\omega(z)} = \infty.$$  

Then $\ell^1(G, \omega)$ is not weakly amenable.

For a discrete group $G$, Lemma 2.7 ensures that each bounded derivation $D : \ell^1(G, \omega) \to \ell^\infty(G, \frac{1}{\omega})$ gives rise to a function $\alpha \in \ell^\infty(G \times G, \frac{1}{\omega \times \omega})$ such that

$$\alpha(xy, z) = \alpha(x, yz) + \alpha(xz, y) \quad \text{and} \quad D(\delta_y)(y) = \alpha(x, y) \quad (x, y, z \in G).$$

With an additional assumption we can derive further that $D$ must be in the form

$$D(\delta_x) = f \cdot \delta_x - \delta_x \cdot f, \quad \text{i.e.,} \quad \alpha(x, y) = f(xy) - f(yx) \quad (x, y \in G)$$

for some function $f$ on $G$. We note that although $\alpha \in \ell^\infty(G \times G, \frac{1}{\omega \times \omega})$, in general one cannot expect that $f \in \ell^\infty(G, \frac{1}{\omega})$, which happens only when $D$ is an inner derivation.

Lemma 2.7. Let $G$ be a discrete group, $\omega$ be a weight on $G$, and $D : \ell^1(G, \omega) \to \ell^\infty(G, \frac{1}{\omega})$ be a bounded derivation. If $D(\delta_x)(y) = 0$ for all commuting elements $x, y \in G$, then there exists a function $f$ on $G$ such that

$$D(\delta_x)(y) = f(xy) - f(yx) \quad (x, y \in G).$$

Proof. Since every element commutes with the unit $e$, from our assumption it follows that $D(\delta_x)(e) = D(\delta_x)(x) = 0$ for all $x \in G$. In particular, $D(xy)(e) = 0$, which implies that $D(\delta_x)(y) = -D(\delta_y)(x)$ for all $x, y \in G$.

We note that $G$ is the disjoint union of all conjugacy classes. To construct $f$ we consider each conjugacy class separately. Let $x_0 \in G$ be fixed. Define $f$ on $C_{x_0} = \{yx_0y^{-1} : y \in G\}$ as follows:

$$f(yx_0y^{-1}) = -D(\delta_{x_0y^{-1}})(y) \quad (y \in G).$$

We first clarify that $f$ is well-defined. Suppose that $u \in C_{x_0}$ has two representations $u = yx_0y^{-1} = zx_0z^{-1}$. Then $x_0y^{-1} = y^{-1}zx_0z^{-1}$. Using the derivation identity, we obtain

$$D(\delta_{x_0y^{-1}})(y) = D(\delta_{y^{-1}z}(x_0z^{-1}))(y) = (D(\delta_{y^{-1}z}) \cdot \delta_{x_0z^{-1}} + \delta_{y^{-1}z} \cdot D(\delta_{x_0z^{-1}}))(y)$$

$$= D(\delta_{y^{-1}z})(x_0z^{-1}y) + D(\delta_{z^{-1}})(z).$$

Since $yx_0y^{-1} = zx_0z^{-1}$, it is readily seen that the elements $y^{-1}z$ and $x_0z^{-1}y$ commute. By assumption, we then have $D(\delta_{y^{-1}z})(x_0z^{-1}y) = 0$. Thus,

$$D(\delta_{x_0y^{-1}})(y) = D(\delta_{x_0z^{-1}})(z).$$

This shows that the function $f$ is well-defined on $C_{x_0}$, so it is well-defined on the whole $G$. (Here, of course, the Axiom of Choice is assumed.) We now prove (14).
For any \( x, y \in G \) the elements \( xy \) and \( yx \) belong to the same conjugacy class, say \( C_{x_0} \). Let \( xy = ax_0a^{-1} \). Then
\[
\begin{align*}
  f(xy) &= -D(\delta_{x_0a^{-1}})(a) = D(\delta_a)(x_0a^{-1}), \\
  f(yx) &= f(yax_0(ya)^{-1}) = D(\delta_{y_0a})(x_0a^{-1}y^{-1}) = D(\delta_a)(x_0a^{-1}) + D(\delta_y)(x).
\end{align*}
\]
In the last step we used the relation \( ax_0a^{-1}y^{-1} = x \). Therefore,
\[
  f(xy) - f(yx) = -D(\delta_y)(x) = D(\delta_x)(y).
\]
The proof is complete. \hfill \Box

**Proposition 2.8.** Let \( G \) be a discrete group and \( \omega \) be a weight on \( G \) such that
\[
\sup_{n \in \mathbb{N}} \frac{n}{\omega(x^n)\omega(x^{-n})} = \infty \quad (x \in G).
\]
Then for every bounded derivation \( D : \ell^1(G, \omega) \to \ell^\infty(G, \frac{1}{\omega}) \) there exists a function \( f \) on \( G \) such that
\[
D(\delta_x)(y) = f(xy) - f(yx) \quad (x, y \in G).
\]

**Proof.** Due to Lemma 2.7 it suffices to show that \( D(\delta_x)(y) = 0 \) for all bounded derivations \( D : \ell^1(G, \omega) \to \ell^\infty(G, \frac{1}{\omega}) \) and all commuting elements \( x, y \in G \). Suppose, to the contrary, that \( xy = yx \) and \( D(\delta_x)(y) = c \neq 0 \) for some bounded derivation \( D \). Then, by induction, we have
\[
D(\delta_{x^n})(yx^{1-n}) = cn \quad (n \in \mathbb{N}).
\]
In fact, this is trivial for \( n = 1 \). Now assume that (15) holds for \( n \in \mathbb{N} \). Then
\[
\begin{align*}
  D(\delta_{x^{n+1}})(yx^{-n}) &= (D(\delta_x) \cdot \delta_{x^n} + \delta_x \cdot D(\delta_{x^n}))(yx^{-n}) \\
  &= D(\delta_x)(y) + D(\delta_{x^n})(yx^{1-n}) = c + cn = c(n + 1).
\end{align*}
\]
So (15) holds for all \( n \in \mathbb{N} \). It then follows that
\[
\begin{align*}
  \|D\| &\geq \sup_{n \in \mathbb{N}} \frac{\|D(\delta_{x^n})\|_{\ell^1(G,1/\omega)}}{\|\delta_{x^n}\|_{\ell^2(G,\omega)}} \geq \sup_{n \in \mathbb{N}} \frac{|D(\delta_{x^n})(yx^{1-n})|}{\omega(yx)\omega(x^n)} \\
  &= \sup_{n \in \mathbb{N}} \frac{|c|n}{\omega((yx)x^{-n})\omega(x^n)} \geq \sup_{n \in \mathbb{N}} \frac{|c|n}{\omega(yx)\omega(x^{-n})\omega(x^n)} \\
  &= \frac{|c|}{\omega(yx)} \sup_{n \in \mathbb{N}} \frac{n}{\omega(x^{-n})\omega(x^n)} = \infty
\end{align*}
\]
due to the condition on \( \omega \). This contradicts to the boundedness of \( D \). The proof is complete. \hfill \Box

**Remark 2.9.** Taking into account Lemma 2.4 we see that the function \( f \) ensured in Lemma 2.7 and Proposition 2.8 satisfies
\[
\sup_{x, y \in G} \frac{|f(xy) - f(yx)|}{\omega(x)\omega(y)} < \infty.
\]
3. The affine motion group

In this section we consider the $ax + b$ group of all affine transformations $x \mapsto ax + b$ of $\mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}$. Precisely, $ax + b = \{(a, b) : a \in \mathbb{R}^+, b \in \mathbb{R}\}$ with product and inverse given by

$$(a, b)(c, d) = (ac, ad + b), \quad (a, b)^{-1} = \left( \frac{1}{a}, \frac{-b}{a} \right) \quad (a, c \in \mathbb{R}^+, b, d \in \mathbb{R}).$$

With the Euclidean metric topology inherited from $\mathbb{R}^2$, $ax + b$ is a locally compact group whose left Haar measure is $da \, db/a^2$.

Let $\omega$ where $\Psi(\cdot) = \int \Psi(\omega) \, \omega(\cdot)$ satisfies

$$\alpha \in \mathbb{R}, \omega \in \mathbb{B} \quad \text{with product and inverse given by} \quad (a, b)(c, d) = (ac, ad + b), \quad (a, b)^{-1} = \left( \frac{1}{a}, \frac{-b}{a} \right) \quad (a, c \in \mathbb{R}^+, b, d \in \mathbb{R}).$$

This shows that $\omega \in \mathbb{B}$ is indeed a (continuous) weight on $ax + b$.

Proposition 3.1. Let $\omega_\alpha$ $(\alpha > 0)$ be the weight on $ax + b$ defined as above. Then $L^1(ax + b, \omega_\alpha)$ is not weakly amenable.

Proof. Clearly, $\omega_\alpha \geq 1$ on $ax + b$. Let $B = \{(a, b) : 1 < a < 2, 1 < b < 2\}$. Then $B$ is open and $\overline{B}$ is compact in $ax + b$. Since

$$(c, d)(a, b)(c, d)^{-1} = (ac, bc + d) \left( \frac{1}{c}, \frac{-d}{c} \right) = (a, -ad + bc + d),$$

we have that $C_B = \{(a, b) : 1 < a < 2, b \in \mathbb{R}\}$. Consider the auxiliary function $\Psi : ax + b : \mathbb{R} \to \mathbb{R}^+$ defined by

$$\Psi(a, b) = \begin{cases} \max\{a - 1, |b|\} & \text{if } 1 < a < 2, \\ 1 & \text{otherwise}. \end{cases}$$

Obviously, $\Psi$ is a positive measurable function on $ax + b$. We show that it also satisfies

$$\frac{\Psi(yz)}{\Psi(z)} \leq \omega_1(y)\omega_1(z) \quad (y, z \in ax + b),$$

where $\omega_1(a, b) = (1 + a + |b|)$. Let $y = (a, b)$, $z = (c, d) \in ax + b$. Then $yz = (ac, ad + b)$ and $zy = (ac, bc + d)$. If $0 < ac \leq 1$ or $ac \geq 2$, then $\Psi(yz) = \Psi(z) = 1$ and hence (16) holds trivially. Now assume $1 < ac < 2$. Then by the definition of $\Psi$ we have

$$ac - 1 \leq \Psi(z) \leq \Psi(yz) \leq \Psi(y)\omega_1(y)\omega_1(z) \quad \text{and}$$

$$|ad + b| = |a(bc + d) - b(ac - 1)| \leq a|bc + d| + |b|(ac - 1) \leq \max\{ac - 1, |bc + d|\}(a + |b|) = \Psi(yz)(a + |b|) \leq \Psi(yz) \omega_1(y)\omega_1(z).$$

Thus

$$\Psi(yz) = \max\{ac - 1, |ad + b|\} \leq \Psi(yz)\omega_1(y)\omega_1(z).$$

This shows that (16) still holds if $1 < ac < 2$. Therefore, (16) holds for all $y, z \in ax + b$.

We now let $\psi = \ln \Psi$. Clearly, $\psi$ is a measurable function supported on $C_B$ and bounded on $B$. We show that it also satisfies the conditions

$$\overset{\text{ess sup}}{z \in C_B} \frac{\psi(z)}{\omega_\alpha(z)} = \infty \quad \text{and}$$

$$\overset{\text{ess sup}}{z \in C_B} \frac{\psi(z)}{\omega_\alpha(z)} = \infty \quad \text{and}$$
So it suffices to verify (20) for $zy$. Note that 
\[ |\psi(z) - \psi(yz)| \leq C\omega_{\alpha}(y)\omega_{\alpha}(z) \quad (y, z \in ax + b) \]
for some constant $C > 0$. Indeed, 
\[ \sup_{z \in C_B} \frac{|\psi(z)|}{\omega_{\alpha}(z)} \geq \sup_{1 < a < 2} \frac{|\psi(a, a - 1)|}{\omega_{\alpha}(a, a - 1)} = \sup_{1 < a < 2} \frac{\ln(a - 1)}{(2a)^{\alpha}} = \infty. \]
So (17) is verified. To show (18) we may assume, without loss of generality, that $\Psi(yz) \geq \Psi(zy)$. Then, using (16), we obtain 
\[ \omega_{\alpha}(y)\omega_{\alpha}(z) = (\omega_{\alpha}(z))^{\alpha} \geq \left( \frac{\Psi(yz)}{\Psi(zy)} \right)^{\alpha} = \alpha \left[ \frac{\Psi(yz)}{\Psi(zy)} \right] \]
\[ = \alpha \left[ \ln \frac{\Psi(yz)}{\Psi(zy)} \right] = \alpha |\psi(yz) - \psi(zy)|. \]
It follows that $\psi$ satisfies (18) with $C = 1/\alpha$. Therefore, the function $\psi$ satisfies all the conditions of Theorem 2.3. This shows that $L^1(ax + b, \omega_{\alpha})$ is not weakly amenable. The proof is complete.

We now equip $ax + b$ with the discrete topology. It is readily seen that $H_b = \{(1, b) : b \in \mathbb{R}\}$ is a normal subgroup of $ax + b$, and $(ax + b)/H_b \cong (\mathbb{R}^+, \cdot)$ through the group homomorphism $[(a, b)] \mapsto a$.

**Proposition 3.2.** Let $\omega$ be a weight on $ax + b$ that is bounded away from zero and is bounded on $H_b$. Then $\ell^1(ax + b, \omega)$ is weakly amenable if and only if $\omega$ is diagonally bounded on $ax + b$.

**Proof.** The sufficiency is due to [26, Proposition 4.1].

For the necessity, we assume that $\omega$ is not diagonally bounded. Let $\hat{\omega}$ be the function on $ax + b$ defined by $\hat{\omega}(z) = \inf_{h \in H_b} \omega(hz)$. Clearly, $\hat{\omega}$ is submultiplicative on $ax + b$ and $\hat{\omega}(a, b)$ is independent of $b$. We simply denote $\hat{\omega}(a, b)$ by $\hat{\omega}(a)$. Then $\hat{\omega}$ is a submultiplicative function on $\mathbb{R}^+$. It is easy to verify further that 
\[ \hat{\omega}(a) \leq \omega(a, b) \leq \hat{\omega}(a) \quad ((a, b) \in ax + b), \]
where $\hat{\omega} = \sup_{h \in H_b} \omega(h)$. By our assumption $0 < \hat{\omega} < \infty$.

Consider the singleton set $B = \{(1, 1)\}$. The conjugacy class of $B$ is
\[ C_B = \{ y \cdot (1, 1) \cdot y^{-1} : y \in ax + b \} = \{(1, b) : b > 0\}. \]

Define $\psi : ax + b \to \mathbb{R}$ by
\[ \psi(a, b) = \begin{cases} \ln \left( \hat{\omega}(b)\hat{\omega}(b^{-1}) \right) & \text{if } a = 1, b > 0, \\ 0 & \text{otherwise}. \end{cases} \]

By definition, $\psi$ vanishes outside the conjugacy class $C_B$. We show that 
\[ |\psi(zy) - \psi(yz)| \leq \omega(y)\omega(z) \quad (y, z \in ax + b). \]
Note that $zy$ and $yz$ always belong to the same conjugacy class for $y, z \in ax + b$. So it suffices to verify (20) for $zy, yz \in C_B$. Let $yz = (1, b), \text{ and } z = (k, l), b, k > 0, l \in \mathbb{R}$. Then 
\[ y = (yz)z^{-1} = (k^{-1}, (l + bk)k^{-1}), \quad z = (1, bk). \]
It follows that
\[ |\psi(zy) - \psi(yz)| = |\psi(1, bk) - \psi(1, b)| = \left| \ln \frac{\hat{\omega}(bk)\hat{\omega}((bk)^{-1})}{\omega(b)\hat{\omega}(b^{-1})} \right| \leq |\ln(\hat{\omega}(k)\hat{\omega}(k^{-1}))| \]

since
\[ \frac{1}{\hat{\omega}(k)\omega(k^{-1})} \leq \frac{\hat{\omega}(bk)\hat{\omega}((bk)^{-1})}{\omega(b)\hat{\omega}(b^{-1})} \leq \hat{\omega}(k)\hat{\omega}(k^{-1}). \]

But \( \hat{\omega}(k)\hat{\omega}(k^{-1}) \geq \hat{\omega}(e) \geq 1 \). So \( |\ln(\hat{\omega}(k)\hat{\omega}(k^{-1}))| = |\ln(\hat{\omega}(k)\hat{\omega}(k^{-1}))| \leq \hat{\omega}(k)\hat{\omega}(k^{-1}) \), which implies
\[ |\psi(zy) - \psi(yz)| \leq \hat{\omega}(k)\hat{\omega}(k^{-1}). \]

On the other hand, relation \((\ref{eq:19})\) yields
\[ \omega(y) \geq \hat{\omega}(k^{-1}), \quad \omega(z) \geq \hat{\omega}(k). \]

Thus we obtain \((\ref{eq:20})\) as desired. Moreover, using \((\ref{eq:19})\) again, we have
\[ \sup_{x \in C_B} \frac{|\psi(x)|}{\omega(x)} = \sup_{b > 0} \frac{|\psi(1, b)|}{\omega(1, b)} = \sup_{b > 0} \frac{|\ln(\hat{\omega}(b)\hat{\omega}(b^{-1}))|}{\omega(1, b)} \geq \sup_{z \in ax + b} \frac{|\ln(\omega(z)\omega(z^{-1}))| - |\ln \tilde{c}|}{\tilde{c}} = \infty, \]

since \( \omega \) is not diagonally bounded on \( G \). From Corollary \((\ref{cor:2.6})\), \( \ell^1(ax + b, \omega) \) is not weakly amenable. The proof is complete.

\[ \square \]

4. Beurling algebra of quotient groups

Let \( G \) be a locally compact group, \( \omega \) be a weight on \( G \), and \( H \) be a closed normal subgroup of \( G \). Define \( \hat{\omega} \) on the quotient group \( G/H \) by
\[ \hat{\omega}([x]) = \inf_{z \in [x]} \omega(z) = \inf_{\xi \in H} \omega(x\xi), \]
where \([x]\) stands for the coset of \( x \) in \( G/H \) \( (x \in G) \). From \[ (\text{I}) \] Theorem 11.0 we know that \( \hat{\omega} \) is a nonnegative upper semicontinuous and hence is a locally bounded measurable function on \( G/H \). To avoid \( \hat{\omega} \) being trivial, here and in the rest of this section we assume that \( \omega \) is bounded away from zero. Then \( \hat{\omega} \) is a locally bounded measurable weight function on \( G/H \) \[ (\text{II}) \] Theorem 3.7.13. As indicated in Section \[ (\text{I}) \] \( \hat{\omega} \) is equivalent to a continuous weight. We note that in studying the weighted group algebra \( L^1(G, \omega) \), requiring \( \omega \) to be bounded away from zero is not really a restriction if \( G \) is an amenable group. Indeed, if \( G \) is amenable, then by \[ (\text{II}) \] Lemma 1 there exists a continuous positive character \( \phi : G \to (\mathbb{R}^+, \cdot) \) such that \( \phi \leq \omega \) on \( G \). Then \( \hat{\omega} = \omega/\phi \geq 1 \) is a weight on \( G \) and \( L^1(G, \omega) \) is isometrically isomorphic to \( L^1(G, \hat{\omega}) \) as a Banach algebra.

We are concerned with the relation between weak amenability of \( L^1(G, \omega) \) and that of \( L^1(G/H, \hat{\omega}) \). First, as a simple consequence of Theorems \[ (\text{II}) \] and \[ (\text{I}) \] we obtain the following.

**Proposition 4.1.** Let \( G \) be an IN group and \( H \) be a closed normal subgroup of \( G \) such that \( G/H \) is Abelian. Suppose that \( \omega \) is a weight on \( G \) that is bounded away from zero. If \( L^1(G, \omega) \) is weakly amenable, then so is \( L^1(G/H, \hat{\omega}) \).
Proof. If $L^1(G/H, \hat{\omega})$ were not weakly amenable, according to Theorem 1.1, there would exist a continuous non-trivial group homomorphism $\Phi : G/H \to \mathbb{C}$ such that
\[
\sup_{x \in G/H} \frac{|\Phi([x])|}{\hat{\omega}([x])\hat{\omega}([x]^{-1})} < \infty.
\]
Then the natural extension $\tilde{\Phi}$ of $\Phi$ to $G$ defined by $\tilde{\Phi}(x) = \Phi([x])$ ($x \in G$) is a non-trivial continuous group homomorphism from $G$ to $\mathbb{C}$ and
\[
\sup_{x \in G} \frac{|\tilde{\Phi}(x)|}{\omega(x)\omega(x^{-1})} \leq \sup_{x \in G/H} \frac{|\Phi([x])|}{\hat{\omega}([x])\hat{\omega}([x]^{-1})} < \infty,
\]
since $\hat{\omega}([x]) \leq \omega(x)$ ($x \in G$). By Theorem 2.2 this implies that $L^1(G, \omega)$ is not weakly amenable, contradicting our assumption.

For the general case, according to the theory established in [24], there is a standard Banach algebra homomorphism $T$ from $L^1(G, \omega)$ onto $L^1(G/H, \hat{\omega})$ defined by
\[
(Tf)([x]) = \int_H f(xh) \, dh \quad (f \in L^1(G, \omega), \, x \in G).
\]
The kernel of $T$ is a closed ideal in $L^1(G, \omega)$ and we denote it by $J_\omega(G, H)$. It was proved in [24, Theorem 3.7.13] that $T$ induces an isometric isomorphism between $L^1(G, \omega)/J_\omega(G, H)$ and $L^1(G/H, \hat{\omega})$. So we are in the situation concerned by the following well-known result.

**Proposition 4.2.** [9, Proposition 2.4] Let $A$ be a weakly amenable Banach algebra and $I$ be a closed ideal in $A$. Then $A/I$ is weakly amenable if and only if $I$ has the trace extension property as described in the following.

For every $\lambda \in I^*$ satisfying $a \cdot \lambda = \lambda \cdot a$ ($a \in A$), there is a $\tau \in A^*$ such that $\tau|_I = \lambda$ and $\tau(ab) = \tau(ba)$ ($a, b \in A$).

We now investigate when $J_\omega(G, H)$ has the trace extension property as a closed ideal of $L^1(G, \omega)$. We start from proving that $J_\omega(G, H)$ is always complemented in $L^1(G, \omega)$ as a Banach subspace. For this we need two technical lemmas.

**Lemma 4.3.** [24, Proposition 8.1.16] Let $H$ be a closed subgroup of a locally compact group $G$ and $U$ be a non-empty open set in $G$ with compact closure. Then there is a subset $Y$ of $G$ such that the family $\{UyH\}_{y \in Y}$ covers $G$ and is locally finite, i.e., every point of $G$ has a neighborhood intersecting at most finitely many members of the family.

The second lemma we need generalizes the investigation in [24, Section 8.1] of the Bruhat function associated to a normal subgroup.

**Lemma 4.4.** Let $G$ be a locally compact group, $H$ be a closed normal subgroup of $G$, and $\omega$ be a weight on $G$ bounded away from zero. Then there exists a continuous function $g \geq 0$ on $G$ and a constant $c > 0$ such that the following two conditions are satisfied:
\[
(22) \quad \int_H g(xh) \, dh = 1 \quad (x \in G) \quad \text{and}
\]
Proof. We first construct a continuous function \( g_1 \) on \( G \) that satisfies
\[
0 < \int_{H} g_1(xh) \, dh < \infty \quad (x \in G)
\]
for some constant \( c \). Let
\[
\{ U \} = \{ x \in G : \omega(x) \leq c \hat{\omega}([x]) \}
\]
where \( c > 0 \) is a constant.

Consider a non-trivial non-negative function \( f \in C_{00}(G) \). Let
\[
U = \{ x \in G : f(x) > 0 \}.
\]
Then \( U \neq \emptyset \) is an open set with a compact closure. Let \( \hat{c} > 0 \) be a constant such that \( \omega(u), \omega(u^{-1}) \leq \hat{c} \) for every \( u \in U \). (The existence of such \( \hat{c} \) is justified by the compactness of \( U \) and the continuity of \( \omega \).) We set \( c = 2\hat{c}^2 \).

By Lemma 4.3, there exists a set \( Y \subset G \) such that the family \( \{ UyH \}_{y \in Y} \) covers \( G \) and is locally finite. For every \( y \in Y \), by the definition of \( \hat{\omega} \), there is \( y_0 \in [y] \) such that \( \hat{\omega}(y_0) \leq 2\hat{\omega}([y]) \). We define \( g_{1,y}(x) = f(xy_0^{-1}) \) \( (x \in G) \). Clearly, \( g_{1,y} \geq 0 \) is a continuous function with compact support, and
\[
\{ x : g_{1,y}(x) \neq 0 \} = \{ x : f(x) > 0 \} \cdot y_0 = Uy_0 \subset UyH.
\]

We now show that \( g_{1,y} \) satisfies \( \text{(26)} \), which is equivalent to
\[
Uy_0 \subset \{ x \in G : \omega(x) \leq c \hat{\omega}([x]) \}.
\]
In fact, for each \( u \in U \), by the choice of \( y_0 \) we have
\[
\omega(uy_0) \leq \omega(u) \omega(y_0) \leq 2\hat{c} \hat{\omega}([y]) = 2\hat{c} \inf_{h \in H} \omega(y_0 h) \leq 2\hat{c} \omega(u^{-1}) \inf_{h \in H} \omega(uy_0 h) \leq 2\hat{c}^2 \hat{\omega}([uy_0]).
\]
So \( \text{(26)} \) holds. Next we prove that \( g_{1,y} \) satisfies
\[
0 < \int_{H} g_{1,y}(xh) \, dh < \infty \quad (x \in UyH).
\]
By definition, \( g_{1,y} \) is a non-negative continuous function with a compact support. So the upper inequality holds. Since \( H \) is a normal subgroup of \( G \), when \( x \in UyH \) we have \( xy_0^{-1} \in UH \), and hence there is \( h_0 \in H \) such that \( xy_0^{-1} h_0 \in U \). Because \( U \) is open, there is a non-trivial open subset \( V \) of \( H \) such that \( xy_0^{-1} V \subset U \). Let \( V_0 = y_0^{-1} V y_0 \). Then \( V_0 \neq \emptyset \) is an open subset of \( H \) such that \( xV_0 y_0^{-1} \subset U \). Since \( f > 0 \) on \( U \), \( g_{1,y} > 0 \) on \( xV_0 \). Therefore,
\[
\int_{H} g_{1,y}(xh) \, dh \geq \int_{V_0} g_{1,y}(xh) \, dh > 0.
\]

Now we let
\[
g_1 = \sum_{y \in Y} g_{1,y}.
\]
Note that since \( \{ x : g_{1,y}(x) \neq 0 \} \subset UyH \) \( (y \in Y) \) and the family \( \{ UyH \}_{y \in Y} \) is locally finite, the sum in the definition of \( g_1 \) has only finitely many non-zero terms in a neighborhood of every point. This implies that \( g_1 \) is well-defined, and
because each $g_{1,y}$ is continuous, $g_1$ is also continuous on $G$. From \eqref{eq:21} and the local finiteness of $\{U_yH\}_{y \in Y}$ it follows that \eqref{eq:24} holds. The inclusion \eqref{eq:25} also holds since it holds for each $g_{1,y}$. So the function $g_1$ satisfies all our requirements.

We then define the function $g$ by

$$g(x) = \frac{g_1(x)}{\int_H g_1(xh) \, dh} \quad (x \in G).$$

Clearly, $g$ is a continuous non-negative function on $G$ and it satisfies

$$\int_H g(xh) \, dh = \int_H \frac{g_1(xh)}{g_1(xht)} \, dt \, dh = \int_H \frac{g_1(xh) \, dh}{g_1(xt) \, dt} = 1 \quad (x \in G).$$

So \eqref{eq:22} is satisfied. Moreover, it follows directly from \eqref{eq:25} and \eqref{eq:22} that

$$\int_H g(xh) \omega(xh) \, dh \leq c \hat{\omega}([x]) \int_H g(xh) \, dh = c \hat{\omega}([x]).$$

So \eqref{eq:23} is also satisfied. The proof is complete. \hfill \Box

Let $g$ be a function ensured in Lemma 4.4 and $T$ be the homomorphism given by \eqref{eq:21}. Define

\begin{equation}
(Pf)(x) = (Tf)([x]) g(x) \quad (x \in G, \ f \in L^1(G, \omega)).
\end{equation}

Then for each $f \in L^1(G, \omega)$, the function $P(f)$ is clearly measurable. By Weil’s Formula and inequality \eqref{eq:23} we have

$$\int_G |(Pf)(x)| \omega(x) \, dx = \int_{G/H} \int_H |(Tf)([x])| g(xh) \omega(xh) \, dh \, d[x]
= \int_{G/H} |(Tf)([x])| \int_H g(xh) \omega(xh) \, dh \, d[x]
\leq \int_{G/H} |(Tf)([x])| \cdot c \hat{\omega}([x]) \, d[x] = c \|Tf\|_1 \omega \leq c \|f\|_1 \omega.
$$

So $P : L^1(G, \omega) \to L^1(G, \omega)$ is a bounded operator with $\|P\| \leq c$.

**Theorem 4.5.** Let $G$ be a locally compact group, $H$ be a closed normal subgroup of $G$, and $\omega$ be a weight on $G$ bounded away from zero. Then the mapping $P : L^1(G, \omega) \to L^1(G, \omega)$ defined by \eqref{eq:28} is a continuous projection whose kernel is $J_{\omega}(G, H)$.

**Proof.** Obviously, $\ker(P) = \ker(T) = J_{\omega}(G, H)$. So we only need to verify that $P^2 = P$. In fact,

$$\begin{align*}
(P^2 f)(x) &= (P Pf)(x) = (T Pf)([x]) g(x) = \left( \int_H (Pf)(xh) \, dh \right) g(x) \\
&= g(x) \int_H (Tf)([xh]) g(xh) \, dh = g(x) (Tf)([x]) \int_H g(xh) \, dh \\
&= (Tf)([x]) g(x) = (P f)(x) \quad (x \in G, \ f \in L^1(G, \omega)).
\end{align*}$$
Therefore, \( P \) is a projection. The proof is complete.

We do not know whether \( J_\omega(G,H) \) has the trace extension property in general. The next lemma provides a sufficient condition for a complemented ideal to have the trace extension property.

**Lemma 4.6.** Let \( A \) be a Banach algebra and \( I \) be a closed complemented ideal in \( A \). Denote by \( I_0 \) the closure of
\[
\text{lin}\{at - ta : a \in A, t \in I\}.
\]
Suppose that \( A = I \oplus X \), where \( X \) is a closed subspace of \( A \) such that
\[
xy - yx \in I_0 \oplus X \quad (x,y \in X).
\]
Then \( I \) has the trace extension property.

**Remark 4.7.** There are two important special cases for which conditions of Lemma 4.6 are satisfied:
1. the complement \( X \) of \( I \) is a subalgebra of \( A \);
2. the complement \( X \) is commutative, i.e., \( xy = yx \) for all \( x, y \in X \) (note that \( xy \) may not be in \( X \)). In particular, this is the case if \( A \) is Abelian.

Our Lemma 4.6 generalizes [17] Lemma 2.3, where only the first case was concerned.

**Proof of Lemma 4.6.** Let \( \lambda \in I^* \) satisfy \( \lambda \cdot a = a \cdot \lambda \) (\( a \in A \)). The condition really means \( \lambda(at) = \lambda(at) \) for all \( t \in I \) and \( a \in A \), or, equivalently, \( \lambda|_{I_0} = 0 \). Since \( A = I \oplus X \), we have that \( A^* = I^* \oplus X^* \). We show that \( \tau = \lambda \oplus 0 \) is a trace extension of \( \lambda \). Obviously, \( \tau \) is a continuous linear functional on \( A \), \( \tau|_I = \lambda \), and \( \tau|_{I_0 \oplus X} = 0 \). Now let \( a, b \in A \) such that \( a = t_1 + x_1 \) and \( b = t_2 + x_2 \) with \( t_1, t_2 \in I \) and \( x_1, x_2 \in X \). We have \( \lambda(t_1 b) = \lambda(t_1) \) and \( \lambda(t_2 x_1) = \lambda(x_1 t_2) \). So
\[
\tau(ab) = \lambda(t_1 b + x_1 t_2) + \tau(x_1 x_2) = \lambda(bt_1 + t_2 x_1) + \tau(x_1 x_2) = \tau(bt_a + \tau(x_1 x_2 - x_2 x_1)).
\]
Since \( x_1 x_2 - x_2 x_1 \in I_0 \oplus X \) by the assumption, \( \tau(x_1 x_2 - x_2 x_1) = 0 \). Therefore, \( \tau(ab) = \tau(ba) \). This completes the proof.

Combining Theorem 4.5 with Proposition 4.2 and Lemma 4.6 we obtain the following.

**Proposition 4.8.** Let \( G \) be a locally compact group, \( H \) be a closed normal subgroup of \( G \), and \( \omega \) be a weight on \( G \) bounded away from zero. Suppose that \( X \) is a Banach space complement of \( J_\omega(G,H) \) in \( L^1(G,\omega) \) such that
\[
xy - yx \in J_0 \oplus X \quad (x,y \in X),
\]
where \( J_0 \) is the closure of \( \text{lin}\{j \ast j - j \ast f : f \in L^1(G,\omega), j \in J_\omega(G,H)\} \). Then weak amenability of \( L^1(G,\omega) \) implies weak amenability of \( L^1(G/H,\omega) \).

We now consider the special case when \( G = G_1 \times G_2 \), \( H = G_2 \), and \( \omega = \omega_1 \times \omega_2 \) with \( \omega_i \) bounded away from zero on \( G_i \) (\( i = 1, 2 \)). In this case \( G/H = G_1 \),
\[
\hat{\omega}(x_1) = \omega_1(x_1) \quad \inf_{x_2 \in G_2} \omega_2(x_2) = \text{const} \cdot \omega_1(x_1),
\]
and the operator \( T : L^1(G,\omega) \to L^1(G/H,\omega) \cong L^1(G_1,\omega_1) \) is precisely given by
\[
T(f)(x_1) = \int_{G_2} f(x_1, x_2) dx_2 \quad (x_1 \in G_1).
\]
Consider a non-negative function $h \in C_{00}(G_2)$ such that
\[
\int_{G_2} h(x_2) \, dx_2 = 1.
\]
Then $g(x_1, x_2) = h(x_2)$ satisfies
\[
\int_{G_2} g(x_1, x_2) \, dx_2 = \int_{G_2} h(x_2) \, dx_2 = 1,
\]
\[
\int_{G_2} g(x_1, x_2) \omega(x_1, x_2) \, dx_2 = \omega_1(x_1) \int_{G_2} h(x_2) \omega_2(x_2) \, dy = \text{const} \cdot \hat{\omega}(x_1) \quad (x_1 \in G_1).
\]
Note that $L^1(G, \omega) = L^1(G_1, \omega_1) \hat{\otimes} L^1(G_2, \omega_2)$, and so we have
\[
J_\omega(G, H) = L^1(G_1, \omega_1) \hat{\otimes} I_2, \quad X = L^1(G_1, \omega_1) \hat{\otimes} (Ch),
\]
where
\[
I_2 = \left\{ f \in L^1(G_2, \omega_2) : \int_{G_2} f(x_2) \, dx_2 = 0 \right\}.
\]

**Proposition 4.9.** Let $G_1, G_2$ be locally compact groups and $\omega_i$ be a weight on $G_i$ bounded away from zero ($i = 1, 2$). Suppose that $L^1(G_1 \times G_2, \omega_1 \times \omega_2)$ is weakly amenable. Then both $L^1(G_1, \omega_1)$ and $L^1(G_2, \omega_2)$ are also weakly amenable.

**Proof.** Because of the symmetry, it is enough to show that $L^1(G_1, \omega_1)$ is weakly amenable. For this case, as has been discussed,
\[
L^1(G_1 \times G_2, \omega_1 \times \omega_2) = J_\omega(G, H) \oplus X
\]
with $J_\omega(G, H)$ and $X$ being given by (29).

For $f_1, f_2 \in L^1(G_1, \omega_1)$ we have
\[
(f_1 \otimes h)(f_2 \otimes h) - (f_2 \otimes h)(f_1 \otimes h) = (f_1 \ast f_2 - f_2 \ast f_1) \otimes (h \ast h)
\]
\[
= (f_1 \ast f_2 - f_2 \ast f_1) \otimes (h \ast h - h) + (f_1 \ast f_2 - f_2 \ast f_1) \otimes h.
\]
The second term of the last expression belongs to $X$. We show that the first term belongs to $J_0$. Denote $k = h \ast h - h$. It is easy to see that $k \in I_2$ and so $f_2 \otimes k \in J_\omega(G, H)$. Let $(e_i)$ be a bounded approximate identity of $L^1(G_2, \omega_2)$. Then for each $i$
\[
(f_1 \otimes e_i)(f_2 \otimes k) - (f_2 \otimes k)(f_1 \otimes e_i) \in J_0,
\]
and hence
\[
(f_1 \ast f_2 - f_2 \ast f_1) \otimes k = \lim_i (f_1 \ast f_2 \otimes (e_i \ast k) - (f_2 \ast f_1) \otimes (k \ast e_i))
\]
\[
= \lim_i ((f_1 \otimes e_i)(f_2 \otimes k) - (f_2 \otimes k)(f_1 \otimes e_i)) \in J_0.
\]
So we have shown that $(f_1 \otimes h)(f_2 \otimes h) - (f_2 \otimes h)(f_1 \otimes h) \in J_0 \oplus X$, and the condition of Proposition 4.8 holds. Thus, $L^1(G_1, \omega_1) \cong L^1(G/H, \hat{\omega})$ is weakly amenable if $L^1(G_1 \times G_2, \omega_1 \times \omega_2)$ is weakly amenable. 

\[\Box\]
5. **Beurling algebra of subgroups**

In spite of Proposition 4.9 weak amenability of \( L^1(G_1 \times G_2, \omega) \) does not necessarily imply weak amenability of \( L^1(G_1, \omega_1) \) even if the groups \( G_1, G_2 \) are commutative, where \( \omega_1(x) = \omega(x, e_2) \) and \( e_2 \) is the unit of \( G_2 \). We give a counterexample in the following.

Let \( G_1, G_2 \) be Abelian locally compact groups and \( G = G_1 \times G_2 \). Suppose that there exist continuous non-zero group homomorphisms \( \Phi_i : G_i \to \mathbb{R} \) \((i = 1, 2)\). For any \( \alpha, \beta > 0 \) we define the function \( \omega \) on \( G \) as follows:

\[
\omega(x) = (1 + |\Phi_1(x)|)^\alpha (1 + |\Phi_1(x) + \Phi_2(y)|)^\beta, \quad (x \in G_1, y \in G_2).
\]

It is readily seen that \( \omega \) is a weight on \( G \), and

\[
\omega(x) = \omega(x, e_2) = (1 + |\Phi_1(x)|)^{\alpha + \beta}, \quad (x \in G_1).
\]

**Example 5.1.** Let \( G_1, G_2, \) and \( \omega \) be as above. If \( 0 < \alpha, \beta < 1/2 \) and \( \alpha + \beta \geq 1/2 \), then \( L^1(G_1, \omega) \) is weakly amenable, but \( L^1(G_1, \omega_1) \) is not weakly amenable.

**Proof.** Since \( \Phi_1 : G_1 \to \mathbb{R} \) is a non-trivial continuous group homomorphism and

\[
\sup_{x \in G_1} \frac{|\Phi_1(x)|}{\omega_1(x)^{\alpha} |\Phi_1(x)|^{\alpha + \beta}} < \infty
\]

if \( \alpha + \beta \geq 1/2 \) weak amenability of \( L^1(G_1, \omega_1) \) is not weakly amenable due to Theorem 1.3. To show that \( L^1(G_1, \omega) \) is weakly amenable, we consider any non-trivial continuous group homomorphism \( \Phi : G \to \mathbb{R} \). We have

\[
\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} = \sup_{x \in G_1, y \in G_2} \frac{|\Phi(x, e_2) + \Phi(e_1, y)|}{(1 + |\Phi_1(x)|)^{2\alpha} (1 + |\Phi_1(x) + \Phi_2(y)|)^{2\beta}}.
\]

**Case 1.** If there is \( y \in G_2 \) such that \( \Phi(e_1, y) \neq 0 \), then

\[
\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} = \sup_{n \in \mathbb{N}} \frac{|\Phi(e_1, y^n)|}{\omega(e_1, y^n)\omega(e_1, y^{-n})} = \sup_{n \in \mathbb{N}} \frac{n|\Phi(e_1, y)|}{(1 + n|\Phi_2(y)|)^{2\beta}} = \infty,
\]

since \( \beta < 1/2 \).

**Case 2.** If \( \Phi(e_1, y) = 0 \) for all \( y \in G_2 \), then we can choose \( x_0 \in G_1 \) such that \( \Phi(x, e_2) \neq 0 \). We can also choose \( y \in G_2 \) such that \( \Phi_2(y) \neq 0 \). For each \( x \in G_1 \), we take an \( n = n(x) \in \mathbb{N} \) such that

\[
\left| n + \frac{\Phi_1(x)}{\Phi_2(y)} \right| \leq 1.
\]

It then follows that

\[
|\Phi_1(x) + \Phi_2(y^n)| = |\Phi_1(x) + n\Phi_2(y)| \leq |\Phi_2(y)|.
\]

Hence,

\[
\sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} \geq \sup_{x \in G_1} \frac{|\Phi(x, e_2)|}{(1 + |\Phi_1(x)|)^{2\alpha} (1 + |\Phi_1(x) + \Phi_2(y^n)|)^{2\beta}} \geq \sup_{x \in G_1} \frac{|\Phi(x, e_2)|}{(1 + |\Phi_1(x)|)^{2\alpha} (1 + |\Phi_2(y)|)^{2\beta}} \geq \frac{m|\Phi(x_0, e_2)|}{(1 + m|\Phi_1(x_0)|)^{2\alpha} (1 + |\Phi_2(y)|)^{2\beta}} = \infty,
\]

because \( \alpha < 1/2 \).
So, we have shown that
\[ \sup_{g \in G} \frac{|\Phi(g)|}{\omega(g)\omega(g^{-1})} = \infty \]
for every non-trivial continuous group homomorphism \( \Phi : G \to \mathbb{R} \). Therefore, \( L^1(G, \omega) \) is weakly amenable by Theorem 1.1 (see [12] Theorem 3.5).

Example 5.1 also shows that, unlike the group algebra case, in general weak amenability of a Beurling algebra on an Abelian group \( G \) does not imply weak amenability of the induced Beurling algebra on a subgroup of \( G \). However, the implication is true for certain “large” open subgroups. We first give a technical lemma dealing with extension of a group homomorphism.

**Lemma 5.2.** Let \( G \) be a locally compact Abelian group and \( H \) be an open subgroup of \( G \). Then any continuous group homomorphism \( \Phi : H \to \mathbb{C} \) can be extended to a continuous group homomorphism \( \hat{\Phi} : G \to \mathbb{C} \).

**Proof.** By Zorn’s Lemma, it suffices to show that for every \( g \in G \) we can extend \( \Phi \) to the open subgroup \( H_g = \bigcup_{n \in \mathbb{Z}} g^nH = \{g^n h : h \in H, n \in \mathbb{Z}\} \) of \( G \).

Suppose first that there exists \( m \in \mathbb{N} \) such that \( g^m \in H \). Let \( m_0 \) be the smallest such number. Then we denote \( \alpha = \frac{1}{m_0} \Phi(g^{m_0}) \) and define \( \hat{\Phi}(g^n h) = n\alpha + \Phi(h) \) (\( h \in H, n \in \mathbb{Z} \)). It is easy to see that \( \hat{\Phi} \) is a group homomorphism on \( H_g \). In fact, the only non-trivial assertion one needs to verify is that the extension is well-defined, i.e., if \( g^{n_1} h_1 = g^{n_2} h_2 \) then \( n_1 \alpha + \Phi(h_1) = n_2 \alpha + \Phi(h_2) \). But in this case \( g^{n_1-n_2} = h_2 h_1^{-1} \in H_g \), and so \( n_1 - n_2 = km_0 \) for some \( k \in \mathbb{Z} \). Because \( \Phi \) is a group homomorphism on \( H_g \), we then have
\[
\hat{\Phi}(g^{n_1-n_2} h_1) = \Phi(g^{n_1-n_2} h_1) = k\Phi(g^{m_0}) = km_0\alpha = (n_1 - n_2)\alpha ,
\]
which implies the desired equality \( n_1 \alpha + \Phi(h_1) = n_2 \alpha + \Phi(h_2) \). The extension \( \hat{\Phi} \) is also continuous on \( H_g \). Indeed, let \( \{t_{c_1} = g^{\gamma} h_1\}_{c_1} \subset H_g \) be a net that converges to some \( t = g^n h \in H_g \). We have \( g^{n-\gamma} h \gamma \to h \). Since \( H \) is open, there is \( \gamma_0 \in \Gamma \) such that \( g^{n-\gamma} \in H \) for \( \gamma \geq \gamma_0 \). Then from the continuity of \( \Phi \) on \( H \) it follows that \( \Phi(g^{n-\gamma} h) \to \Phi(h) \).

Using the fact that \( \hat{\Phi} \) is a group homomorphism, we finally obtain
\[
\hat{\Phi}(t_{c_1}) = \hat{\Phi}(g^{\gamma} h_{c_1}) = \hat{\Phi}(g^{\gamma-n} h_{c_1}) + \hat{\Phi}(g^n h_{c_1}) + \hat{\Phi}(g^n) = \hat{\Phi}(g^n h) = \hat{\Phi}(t).
\]

Now assume that \( g^n \not\in H \) for all \( n \in \mathbb{N} \). Then we put \( \hat{\Phi}(g^n h) = \Phi(h) \) (\( h \in H, n \in \mathbb{Z} \)). Obviously, \( \Phi \) is a group homomorphism on \( H_g \). We now show that it is continuous. Let \( g^{n_1} h_{c_1} \to g^n h \) (\( n_1, n \in \mathbb{Z}, h_{c_1}, h \in H \)). Then, as above, \( g^{n_1-n} h_{c_1} \to h \) and, because \( H \) is open, there is \( \gamma_0 \) such that \( g^{n_1-n} \in H \) for \( \gamma \geq \gamma_0 \). But our assumption on \( g \) implies that this is possible only when \( n_1 = n \) for \( \gamma \geq \gamma_0 \), and so \( h_{c_1} \to h \). Therefore, \( \hat{\Phi}(g^{n_1} h_{c_1}) = \Phi(h_{c_1}) \to \Phi(h) = \hat{\Phi}(g^n h) \).

This shows that \( \hat{\Phi} \) is continuous. The proof is complete. \( \square \)
In general, one cannot expect that a group homomorphism $\Phi$ from a normal subgroup $H$ of $G$ has an extension to the whole $G$. In fact, if such extension exists then $\Phi$ must satisfy $\Phi(g_0 h^{-1}) = \Phi(h)$ for all $g \in G$ and $h \in H$. It turns out that the latter condition is also sufficient for semidirect product group $G = L \rtimes H$, where $H$ is a normal subgroup and $L$ is a subgroup of $G$ such that $L \cap H = \{e\}$.

**Proposition 5.3.** Let $G = L \rtimes H$ and $\Phi : H \to \mathbb{R}$ be a group homomorphism. Then $\Phi$ extends to a group homomorphism $\tilde{\Phi} : G \to \mathbb{R}$ if and only if

$$\Phi(l h l^{-1}) = \Phi(h) \quad (l \in L, h \in H).$$

Moreover, if $H$ is open in $G$ then $\tilde{\Phi}$ is continuous whenever $\Phi$ is continuous.

**Proof.** The necessity part is trivial.

For sufficiency, we note that every $g \in G$ may be uniquely expressed in the form $g = l h$. Suppose that (31) holds. We then extend $\Phi$ to $\tilde{\Phi}$ on the whole $G$ simply by letting $\tilde{\Phi}(g) = \Phi(h)$ ($g = l h$, $l \in L$, $h \in H$). It is a group homomorphism because for any $g_1 = l_1 h_1, g_2 = l_2 h_2 \in G$ we have

$$\tilde{\Phi}(g_1 g_2) = \tilde{\Phi}(l_1 h_1 l_2 h_2) = \tilde{\Phi}((l_1 l_2)(l_1^{-1} h_1 l_2 h_2)) = \Phi(l_2^{-1} h_1 l_2) = \Phi(l_2^{-1} h_1) + \Phi(h_2) = \tilde{\Phi}(g_1) + \tilde{\Phi}(g_2).$$

Assume now that $H$ is open in $G$ and that $\tilde{\Phi}$ is continuous on $H$. Let $g_i = l_i h_i \to g = l h$ ($h_i \in H$, $l_i \in L$). Then $l^{-1} l_i h_i \to h \in H$. Since $H$ is open, it follows that $l^{-1} l_i h_i \in H$ ($i \geq i_0$) for some $i_0$. Then $l^{-1} l_i \in H \cap L$ and hence $l_i = l$ for $i \geq i_0$. This implies that $h_i \to h$. Using the continuity of $\tilde{\Phi}$ we finally obtain

$$\tilde{\Phi}(g) = \tilde{\Phi}(h_1) = \tilde{\Phi}(h) = \tilde{\Phi}(g).$$

Therefore, $\tilde{\Phi}$ is also continuous. $\square$

**Proposition 5.4.** Let $G$ be a locally compact IN group and $\omega$ be a weight on it. Suppose that $H$ is a commutative subgroup of $G$, and suppose that every continuous group homomorphism $\Phi : H \to \mathbb{C}$ can be extended to the whole $G$. If there is $c > 0$ such that for each $x \in G$ there is $k = k(x) \in \mathbb{N}$ for which $x^k \in H$ and

$$\frac{\omega(x^k) \omega(x^{-k})}{k} \leq c \omega(x) \omega(x^{-1}),$$

then weak amenability of $L^1(G, \omega)$ implies weak amenability of $L^1(H, \omega|_H)$.

**Remark 5.5.** In particular, the conditions of Proposition 5.4 are satisfied when $G/H$ is a torsion group (see [13, A.1]).

**Proof of Proposition 5.4.** If $L^1(H, \omega|_H)$ is not weakly amenable, by Theorem 1.4 there is a non-trivial continuous group homomorphism $\Phi : H \to \mathbb{C}$ such that

$$\sup_{h \in H} \frac{|\Phi(h)|}{\omega(h) \omega(h^{-1})} = r < \infty.$$

By our assumption, $\Phi$ can be extended to a continuous group homomorphism $\tilde{\Phi} : G \to \mathbb{R}$. We have

$$\frac{|\tilde{\Phi}(x)|}{\omega(x) \omega(x^{-1})} = \frac{|\Phi(x^k)|}{\omega(x^k) \omega(x^{-k})} \frac{\omega(x^k) \omega(x^{-k})}{k} \frac{1}{\omega(x) \omega(x^{-1})} \leq r c$$

since $x^k \in H$, where $k = k(x) \in \mathbb{N}$ is such that (32) is satisfied. Then, by Theorem 2.2 $L^1(G, \omega)$ is not weakly amenable.
Corollary 5.6. Let $G$ be a locally compact $\mathbb{IN}$ group and $H$ be a commutative subgroup of $G$ of finite index. Suppose that each continuous group homomorphism from $H$ to $\mathbb{C}$ can be continuously extended to the whole $G$. Then, for every weight $\omega$ on $G$ such that $L^1(G, \omega)$ is weakly amenable, $L^1(H, \omega|_H)$ is also weakly amenable.

Proof. Suppose, to the contrary, that $L^1(H, \omega|_H)$ is not weakly amenable. Then, since $H$ is commutative, Theorem 1.1 implies the existence of a non-trivial continuous group homomorphism $\Phi : H \rightarrow \mathbb{C}$ and a constant $c > 0$ such that

$$\frac{|\Phi(h)|}{\omega(h)\omega(h^{-1})} \leq c \quad (h \in H).$$

By the assumption $\Phi$ extends to a continuous group homomorphism $\tilde{\Phi} : G \rightarrow \mathbb{C}$. Because $H$ is of finite index, there exist $g_1, g_2, \ldots, g_n \in G$ such that $G = \cup_{i=1}^n g_iH$. Hence, every $g \in G$ can be written in the form $g = g_ih$ for some $1 \leq i \leq n$, $h \in H$, and so

$$\frac{|\tilde{\Phi}(g)|}{\omega(g)\omega(g^{-1})} \leq \frac{|\tilde{\Phi}(h)| + |\tilde{\Phi}(g_i)|}{\omega(g_i)\omega(g_i^{-1})} \leq \frac{|\tilde{\Phi}(h)| + |\tilde{\Phi}(g_i)|}{\omega(h)\omega(h^{-1})} \cdot \omega(g_i)\omega(g_i^{-1})$$

$$\leq \max_{1 \leq i \leq n} \left( c + \max_{1 \leq i \leq n} \left( |\tilde{\Phi}(g_i)| \right) \right) \omega(g_i)\omega(g_i^{-1}) = \text{const.}$$

It follows that $L^1(G, \omega)$ is not weakly amenable by Theorem 1.1, which contradicts our assumption. □

Given a locally compact group $G$ and a closed normal subgroup $H$ of it, we have seen that weak amenability of $L^1(G, \omega)$ does not pass to $L^1(H, \omega|_H)$ in general even $G$ is commutative. One may wonder whether the condition that both $L^1(H, \omega|_H)$ and $L^1(G/H, \omega)$ are weakly amenable forces $L^1(G, \omega)$ to be weakly amenable. It turns out that the answer is also negative. A counterexample is as follows.

Example 5.7. We consider $G = ax + b$ and $H = H_b$. Suppose that $w$ is a weight on $(\mathbb{R}^+, \cdot)$ that is not diagonally bounded, but such that $\ell^1(\mathbb{R}^+, w)$ is weakly amenable. (For example, we can take $w(a) = (1 + |\ln a|)^{\alpha}$, $0 < \alpha < 1/2$.) We then define $\omega$ on $ax + b$ by $\omega(a, b) = w(a) \cdot a > 0$. Clearly, $\omega$ is a weight on $G$, $\omega|_H = \text{const}$, and $\omega = w$. So $\ell^1(H, \omega|_H)$ and $\ell^1((ax + b)/H, \omega)$ are both weakly amenable. But by our assumption $\omega$ is not diagonally bounded, and so $\ell^1(ax + b, \omega)$ is not weakly amenable due to Proposition 3.2.

Even $G$ is finitely generated, this situation could happen.

Example 5.8. Let $\mathbb{Z}[\frac{1}{2}]$ denote the set of all dyadic fractions, i.e., the set of all rational numbers whose binary expansion is finite. Consider the countable subgroup $G_2$ of $ax + b$ defined by

$$G_2 = \left\{(2^n, b) : n \in \mathbb{Z}, b \in \mathbb{Z} \left[ \frac{1}{2} \right] \right\}.$$

In fact, $G_2$ is the subgroup of $ax + b$ generated by the elements $(2, 0)$ and $(1, 1)$, and so it is a finitely generated amenable group. Let

$$H_2 = H_b \cap G_2 = \left\{(1, b) : b \in \mathbb{Z} \left[ \frac{1}{2} \right] \right\}.$$
Then $H_2$ is a normal subgroup of $G_2$ and $G_2/H_2 \cong (\mathbb{Z}, +)$. On $G_2$ we consider the weight $\omega_\alpha$ ($0 < \alpha < 1/2$) defined by

$$\omega_\alpha(n, b) = (1 + |n|)^\alpha \quad (n \in \mathbb{Z}).$$

The same argument as in Example 5.7 shows that $\ell^1(G_2, \omega_\alpha)$ is not weakly amenable while both $\ell^1(H_2, \omega_\alpha)$, which is isomorphic to $\ell^1(\mathbb{Z})$, and $\ell^1(G_2/H_2, \hat{\omega}_\alpha)$, which is isometrically isomorphic to $\ell^1(\mathbb{Z}, \omega_\alpha)$, are weakly amenable. We are grateful to N. Spronk for this observation.

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