OSCILATION STABILITY FOR CONTINUOUS MONOTONE SURJECTIONS

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Abstract. We prove that for every integer \( b \geq 2 \) and positive real \( \varepsilon \) there exists a finite number \( t \) such that for every finite coloring of the nondecreasing surjections from \( b^\omega \) onto \( b^\omega \), there exist \( t \) many colors such that their \( \varepsilon \)-fattening contains a cube.

1. Introduction

Recall the statement of the dual Ramsey theorem for infinite partitions of \( \omega \) (see [CS] or [To]): For every finite Borel coloring of the space \( C_{\text{sur}}^r(\omega) \) of all rigid surjections from \( \omega \) onto \( \omega \) there is a rigid surjection \( h : \omega \to \omega \) such that the set

\[
C_{\text{sur}}^r(\omega) \mid h = \{ f \circ h : f \in C_{\text{sur}}^r(\omega) \}
\]

is monochromatic. In this note we examine this kind of dual Ramsey statement with the index-set \( \omega \) replaced by the Cantor set \( 2^\omega \), or more generally, powers of the form \( b^\omega \) for \( b \) any positive integer. More precisely, we focus on the space \( C_{\text{sur}}^r(b^\omega) \) of all nondecreasing surjections and we examine to which extent a similar result holds. Unlike the dual Ramsey theorem, in our case the structure under consideration admits a Ramsey degree and this degree can be realized only in an approximate sense. In Section 5 we establish the necessity of the approximations, while in Section 6 we prove that the Ramsey degree provided by the statement of the main result (Theorem 1 below) is the best possible. However, to state our result precisely we need some notation.

By \( \omega \), we denote the set of the natural numbers starting from 0. For every \( k \) in \( \omega \), \( k \) also stands for the set of the natural numbers strictly less than \( k \). For \( b, k \in \omega \), by \( b^k \) (resp. \( b <^k \)) we denote the set of all sequences in \( b \) of length \( k \) (resp. strictly less than \( k \)) and by \( b^\omega \) (resp. \( b <^\omega \)) we denote the set of all sequences in \( b \) of infinite (resp. finite) length. For \( 2 \leq b < \omega \), it is well known that the space \( b^\omega \) is a metrizable compact space. Throughout this note we will consider the following metric witnessing this fact. For every distinct \( x, y \) in \( b^\omega \), we set \( \rho_b(x, y) = 2^{-n_0} \) where \( n_0 = \min\{ n < \omega : x(n) \neq y(n) \} \). Moreover we endow the set \( b^\omega \) with the lexicographical order \( \leq_{\text{lex}} \), i.e. for \( x, y \in b^\omega \), we write \( x \leq_{\text{lex}} y \) if either \( x = y \) or \( x(n_0) < y(n_0) \) where \( n_0 = \min\{ n < \omega : x(n) \neq y(n) \} \). Then \( (b^\omega, \leq_{\text{lex}}) \) is a linearly
ordered set. Similarly the lexicographical order $\leq_{\text{lex}}$ is defined on $\sqsubseteq$-incomparable pairs of $b^{<\omega}$ inducing a linear order on every subset of $b^{<\omega}$ consisting of pairwise $\sqsubseteq$-incomparable elements.

We are interested in the following subspace of the continuous maps from $b^\omega$ into itself

$$C^\uparrow_{\text{sur}}(b^\omega) = \{ f : b^\omega \to b^\omega : f \text{ is continuous, onto and nondecreasing} \},$$

where by nondecreasing we mean $f(x) \leq_{\text{lex}} f(y)$ whenever $x \leq_{\text{lex}} y$. We endow $C^\uparrow_{\text{sur}}(b^\omega)$ with the following metric

$$\rho_\infty(f, g) = \sup \{ \rho_b(f(x), g(x)) : x \in b^\omega \}$$

for all $f, g \in C^\uparrow_{\text{sur}}(b^\omega)$. Finally, let us recall the sequence of the odd tangent numbers $(t_k)_{k=1}^\infty$ defined by $t_k = \tan(2^{k-1})$ for every positive integer $k$. The main result of this note is the following.

\textbf{Theorem 1.} Let $b \in \omega$ with $b \geq 2$. Then for every positive real $\varepsilon$ there exists a positive integer $t = t(\varepsilon)$ such that for every positive integer $K$ and every coloring $c : C^\uparrow_{\text{sur}}(b^\omega) \to K$ there exist $h \in C^\uparrow_{\text{sur}}(b^\omega)$ and $B \subseteq K$ with at most $t$ elements such that for every $f \in C^\uparrow_{\text{sur}}(b^\omega)$ there exists $g \in C^\uparrow_{\text{sur}}(b^\omega)$ satisfying $\rho_\infty(g, f \circ h) < \varepsilon$ and $c(g) \in B$. In particular, $t = t(\varepsilon) = t_b(1) + 1$.

To state a corollary of this result we need the following notion whose relationship to Ramsey theory was already pointed out before (see \cite{KPT}). Let $(X, d)$ be a metric space, $t$ a positive integer and $\delta$ a positive real. We will say that a subset $Y$ of $X$ is of $\delta$-covering number at most $t$, if there exists a finite subset $A$ of $X$ of cardinality at most $t$ such that $Y \subseteq \bigcup_{x \in A} B_d(x, \delta)$. Under this terminology, the above result has the following immediate consequence.

\textbf{Corollary 2.} Let $b \in \omega$ with $b \geq 2$. Then for every positive real $\varepsilon$ there exists a positive integer $t$ such that for every compact metric space $(K, d)$, every map $c : C^\uparrow_{\text{sur}}(b^\omega) \to K$ and every positive real $\delta$ there exist $h \in C^\uparrow_{\text{sur}}(b^\omega)$ and $B \subseteq K$ of $\delta$-covering number $t$ such that for every $f \in C^\uparrow_{\text{sur}}(b^\omega)$ there exists $g \in C^\uparrow_{\text{sur}}(b^\omega)$ satisfying $\rho_\infty(g, f \circ h) < \varepsilon$ and $c(g) \in B$.

Moreover Theorem 1 has the following corollary.

\textbf{Corollary 3.} Let $b \in \omega$ with $b \geq 2$. Then for every positive reals $\varepsilon, M$ there exists a positive integer $t = t(\varepsilon, M)$ such that for every bounded metric space $(K, d)$, every $M$-Lipschitz map $c : C^\uparrow_{\text{sur}}(b^\omega) \to K$ there exist $h \in C^\uparrow_{\text{sur}}(b^\omega)$ and $B \subseteq K$ of $\varepsilon$-covering number at most $t$ such that for every $f \in C^\uparrow_{\text{sur}}(b^\omega)$ we have $c(f \circ h) \in B$. In particular, $t(\varepsilon, M) = t(\frac{\varepsilon}{M}).$

\textbf{Proof.} Let $\varepsilon, M$ be positive reals and $t = t(\varepsilon, M)$. Also let $(K, d)$ be a bounded metric space and $c : C^\uparrow_{\text{sur}}(b^\omega) \to K$ be an $M$-Lipschitz map. By the boundness of $(K, d)$ there exist a positive integer $N$ and $x_0, \ldots, x_{N-1} \in K$ such that $\bigcup_{i=0}^{N-1} B_d(x_i, \varepsilon/2) =$
Let us start with the following fact concerning the minimum and maximum of nonempty closed or clopen subsets of $b^\omega$ with respect to $\leq_{\text{lex}}$.

**Fact 4.** Let $b < \omega$ with $b \geq 2$. Every nonempty closed subset of $b^\omega$ admits a maximum and a minimum. Moreover if the subset is clopen then its maximum is eventually equal to $b - 1$ and its minimum is eventually equal to 0.

**Proof.** Let $F$ be a nonempty closed subset of $b^\omega$. Clearly $F$ is a compact set. For every $n < \omega$, we pick a finite subset $G_n$ of $F$ such that $F \subseteq \cup_{x \in G_n} B_{\rho_n}(x, 1/n)$ and we set $x_n$ to be the maximum of $G_n$ with respect to $\leq_{\text{lex}}$. It is easy to check that $x_m \in B_{\rho_n}(x_n, 1/n)$, for all $n \leq m < \omega$. Hence $(x_n)_n$ is a Cauchy sequence and therefore converges to some element $x$ of $F$. Moreover we have that

\[
\rho_n(x, x_n) \leq 1/n
\]

for all $n < \omega$. We claim that $x$ is the maximum of $F$. Indeed, assume on the contrary that there exists $y \in F$ such that $x <_{\text{lex}} y$. Let $n_0$ positive integer such that $1/n_0 < \rho_0(x, y)/2$. Thus by (ii) we have that $y \notin B_{\rho_n}(x_{n_0}, 1/n_0)$ and since $x <_{\text{lex}} y$ we have that $z <_{\text{lex}} y$ for all $z \in B_{\rho_n}(x_{n_0}, 1/n_0)$. This in particular, by the choice of $x_{n_0}$, yields that $y \notin \cup_{x \in G_{n_0}} B_{\rho_n}(x, 1/n_0)$, which contradicts that the latter union covers $F$. Similar arguments yield that $F$ admits a minimum.

Let us now assume that $F$ is a nonempty clopen subset of $b^\omega$. Assume that the maximum $x$ of $F$ is not eventually equal to $b - 1$. Then we can pick a sequence $(x_n)_n$ in $b^\omega$ convergent to $x$ such that $x <_{\text{lex}} x_n$ for all $n < \omega$. Thus, since $x$ is the maximum of $F$ we have that $x_n$ belongs to the complement of $F$ for all $n < \omega$, which contradicts that $F$ is also open. Similar arguments yield that the minimum of $F$ is eventually equal to 0. \qed

Let $b < \omega$ with $b \geq 2$. A subset $U$ of $b^\omega$ is called interval if for every $x, y \in U$ and $z \in b^\omega$ satisfying $x \leq_{\text{lex}} z \leq_{\text{lex}} y$ we have that $z \in U$. Central role in our analysis possesses the following notion.
Definition 5. Let \( b < \omega \) with \( b \geq 2 \). A family \((U_s)_{s \in b^\omega}\) of nonempty clopen intervals of \( b^\omega \) is called a filtering on \( b^\omega \), if the following are satisfied

(i) \( U_0 = b^\omega \),
(ii) \( (U_s \setminus \{ b \})_{s \in b^\omega} \) is a (disjoint) partition of \( U_s \) for every \( s \in b^\omega \) and
(iii) \( \{ \max U_s \setminus \{ b \} \}_{s \in b^\omega} \) is \( \leq \text{lex} \)-increasing for every \( s \in b^\omega \).

Let \( \mathcal{F}_b \) be the set of all filterings on \( b^\omega \). For \((V_s)_{s \in b^\omega}\) and \((U_s)_{s \in b^\omega}\) in \( \mathcal{F}_b \) we will write \((V_s)_{s \in b^\omega} \preceq (U_s)_{s \in b^\omega}\) if for every \( s \in b^\omega \), the set \( V_s \) is element of the algebra generated by the members of the family \((U_s)_{s \in b^\omega}\). For \((U_s)_{s \in b^\omega}\) in \( \mathcal{F}_b \) we set

\[
\mathcal{F}_b((U_s)_{s \in b^\omega}) = \{(V_s)_{s \in b^\omega} \in \mathcal{F}_b : (V_s)_{s \in b^\omega} \preceq (U_s)_{s \in b^\omega}\}.
\]

Moreover, for every \( f \in C^\uparrow_{\text{sur}}(b^\omega) \), we set \((U_s^f)_{s \in b^\omega} = (f^{-1}(W_s))_{s \in b^\omega}\), where \( W_s = \{ x \in b^\omega : s \sqsubset x \} \) for all \( s \in b^\omega \). It is easy to check that \((U_s^f)_{s \in b^\omega}\) is a filtering on \( b^\omega \). The following lemma describes the relation between the elements of \( C^\uparrow_{\text{sur}}(b^\omega) \) and the filterings on \( b^\omega \). Finally, for \( y \in b^\omega \) and \( n < \omega \) by \( x|n \) we denote the initial segment of \( x \) of length \( n \).

Lemma 6. Let \( b < \omega \) with \( b \geq 2 \). Then the map \( D : C^\uparrow_{\text{sur}}(b^\omega) \to \mathcal{F}_b \) sending each \( f \) to \((U_s^f)_{s \in b^\omega}\) is 1-1 and onto. Moreover, for every \( h \in C^\uparrow_{\text{sur}}(b^\omega) \) we have

\[
\mathcal{F}_b((U_s^h)_{s \in b^\omega}) = \{ (U_s^{f \circ h})_{s \in b^\omega} : f \in C^\uparrow_{\text{sur}}(b^\omega) \}.
\]

Proof. To prove that \( D \) is 1-1, we fix \( f \neq g \) in \( C^\uparrow_{\text{sur}}(b^\omega) \). Then there exists \( x \in b^\omega \) such that \( f(x) \neq g(x) \). Pick \( s \in b^\omega \) such that \( s \) is initial segment of \( f(x) \) but not of \( g(x) \). Thus \( f(x) \in W_s \) and \( g(x) \notin W_s \). That is \( x \in f^{-1}(W_s) = U_s^f \) and \( x \notin g^{-1}(W_s) = U_s^g \). Hence \( U_s^f \neq U_s^g \) and therefore \( D(f) \neq D(g) \).

In order to prove that \( D \) is onto, let us fix some \((U_s)_{s \in b^\omega}\) in \( \mathcal{F}_b \) and define \( f : b^\omega \to b^\omega \) as follows. Fix an \( x \in b^\omega \). Since \((U_s)_{s \in b^\omega}\) is a filtering, by (i) and (ii) of definition \( \mathcal{F}_b \) there exists a sequence \((s_n)_{n < \omega}\) in \( b^\omega \) such that \( x \in U_{s_n} \), \( s_n \) is of length \( n \) and \( s_n \sqsubseteq s_{n+1} \) for all \( n < \omega \). Actually there exists unique such a sequence. Let \( y \) be the unique element of \( b^\omega \) satisfying \( s_n \sqsubseteq y \) for all \( n < \omega \). Set \( f(x) = y \).

Let us check that \( f \) belongs to \( C^\uparrow_{\text{sur}}(b^\omega) \) satisfying \((U_s^f)_{s \in b^\omega} = (U_s)_{s \in b^\omega}\). By (iii) of Definition \( \mathcal{F}_b \) it follows that \( f \) is increasing. To check that \( f \) is onto let us fix some \( y \in b^\omega \). Observe that \( \cap_{n < \omega} U_{y|n} \) is non-empty. Picking any \( x \) from this intersection we have that \( f(x) = y \). To justify the continuity of \( f \) let us fix a convergent sequence \((x_n)_{n < \omega}\) to some \( x \) in \( b^\omega \). Let \( y = f(x) \) and \( y_n = f(x_n) \) for all \( n < \omega \). Moreover we set \( s_n \) to be the initial segment of \( y \) of length \( n \) for all \( n < \omega \). By the definition of \( f \) we have that \( x \in U_{s_n} \). We pass to a subsequence \((x_{k_n})_{n < \omega}\) of \((x_n)_{n < \omega}\) such that \( x_{k_n} \in U_{s_n} \) for every \( n < \omega \). By the definition of \( f \) we have that \( s_n \) is initial segment of \( y_{k_n} \) and by the definition of \((s_n)_{n < \omega}\) we get that \( y_{k_n} \) converges to \( y \).

Hence \( f \) is continuous. Up to now we have proven that \( f \) belongs to \( C^\uparrow_{\text{sur}}(b^\omega) \). To
see that \((U^f_s)_{s\leq \omega} = (U_s)_{s\leq \omega}\) let us fix an arbitrary \(s \in b^{<\omega}\). Then 

\[
x \in U_s \iff s \text{ is initial segment of } f(x) \\
\text{iff } f(x) \in W_s \\
\text{iff } x \in U^f_s.
\]

To prove the second part of the lemma we fix some \(h \in C^*_\text{sur}(b^\omega)\). Since every clopen set can be written as a finite union of the basic clopen sets \((W_s)_{s \leq b^{<\omega}}\), we easily get that \(F^h \cap C^*_\text{sur}(b^\omega) \supseteq \{(U^f_s)_{s \leq b^{<\omega}} : f \in C^*_\text{sur}(b^\omega)\}\). In order to prove the inverse inclusion, let us pick \((V_s)_{s \leq b^{<\omega}}\) from \(F^h \cap C^*_\text{sur}(b^\omega)\). Since \(D\) is onto, there exists \(g \in C^*_\text{sur}(b^\omega)\) such that \((U^g_s)_{s \leq b^{<\omega}} = (V_s)_{s \leq b^{<\omega}}\). It suffices to find \(f \in C^*_\text{sur}(b^\omega)\) such that \(f = g \circ h\). We define \(f\) as follows. Let \(y \in b^\omega\) and \(A_y = \cap_{n < \omega} U^h_{y|n}\). Since \((U^g_s)_{s \leq b^{<\omega}} \in F^h \cap C^*_\text{sur}(b^\omega)\), we have that there exists a sequence \((s_n)_{n < \omega}\) such that \(A_y \subseteq U^g_{s_n}\), \(s_n\) is initial segment of \(s_{n+1}\) and \(s_n\) is of length \(n\) for all \(n < \omega\). Finally, let \(z\) be the unique element in \(b^\omega\) such that \(s_n\) is initial segment of \(z\) for all \(n < \omega\) and set \(f(y) = z\). Arguing as in the proof of the first part of the lemma, one can show that \(f\) belongs to \(C^*_\text{sur}(b^\omega)\). To prove that \(f \circ h = g\) it suffices to show that \((U^f_s)_{s \leq b^{<\omega}} = (U^g_s)_{s \leq b^{<\omega}}\). By the definition of the map \(g\), it suffices to show that \((U^f_s)_{s \leq b^{<\omega}} = (V_s)_{s \leq b^{<\omega}}\). First, let us observe that since \((V_s)_{s \leq b^{<\omega}} \leq (U^h_s)_{s \leq b^{<\omega}}\), we have that \(x \in V_s\) if and only if \(h^{-1}(|h(x)|) \subseteq V_s\), for all \(x \in b^\omega\) and \(s \in b^{<\omega}\). Moreover, for every \(x \in b^\omega\) we have that \(A_{h(x)} = \cap_{n < \omega} U^h_{h(x)|n} = h^{-1}(|h(x)|) \subseteq V_x\), for all \(x \in b^\omega\) and \(s \in b^{<\omega}\). Hence for every \(x \in b^\omega\) and \(s \in b^{<\omega}\) we have 

\[
x \in V_s \\
\text{iff } h^{-1}(|h(x)|) \subseteq V_s \\
\text{iff } A_{h(x)} \subseteq V_s \\
\text{iff } s \text{ is initial segment of } f(h(x)) \\
\text{iff } f(h(x)) \in W_s \\
\text{iff } x \in U^f_s.
\]

\[\square\]

3. Order isomorphic copies of \(Q\) in \(b^\omega\)

Let \(b < \omega\) with \(b \geq 2\). We set \(\mathcal{A}_b\) to be the set of the elements of \(b^\omega\) being eventually equal to \(b - 1\) excluding \(\max b^\omega\). It is easy to check that \((\mathcal{A}_b, \leq_{\text{lex}})\) is a countable unbounded dense linearly ordered set and therefore order isomorphic to \(Q\). Moreover, for every filtering \((U_s)_{s \leq b^{<\omega}}\) on \(b^\omega\), the subset 

\[\{ \max U_s : s \in b^{<\omega} \} \setminus \{ \max b^\omega \}\]

of \(\mathcal{A}_b\) (see Fact \[\bullet\]) is order isomorphic to \(Q\).
We shall need a result due to D. Devlin (see [D] and [To]). In order to state it we need some additional notation. For a linear ordered set \((P, \leq)\) and a positive integer \(k\) by \([P]^k\) we denote the set of all \(\leq\)-increasing \(k\)-tuples in \(P\). Moreover, let us recall the sequence of the odd tangent numbers \((t_k)_{k=1}^\infty\) defined by \(t_k = \tan^{2k-1}(0)\) for every positive integer \(k\).

**Theorem 7** (D. Devlin). For every positive integer \(l\) and every finite coloring of \([Q]^l\), there exists a subset \(Y\) of \(Q\) order isomorphic to \(Q\) such that \(|Y|^l\) uses at most \(t_l\) colors.

The above has the following immediate consequence.

**Corollary 8.** Let \(b < \omega\) with \(b \geq 2\). For every positive integer \(k\) and every finite coloring of \([A_b]^k\), there exists a subset \(Y\) of \(A_b\) order isomorphic to \(A_b\) such that \(|Y|^k\) uses at most \(t_k\) colors.

For every positive integer \(k\), let \(l_k = b^k - 1\) and \((s_k^i)_{i=0}^{l_k}\) be the \(\leq_{lex}\)-increasing enumeration of \(b^k\). Clearly for every positive integer \(k\) and every \(f \in C_{\text{sur}}^+(b^\omega)\) we have that \((\max U_{s_k^i})_{i < l_k} \in [A_b]^l_k\).

**Lemma 9.** Let \(b < \omega\) with \(b \geq 2\) and \(k\) be a positive integer. For every \(x \in [A_b]^l_k\) we have that there exists \(f \in C_{\text{sur}}^+(b^\omega)\) such that \(x = (\max U_{s_k^i})_{i < l_k}\).

**Proof.** Let \(x = (x_i)_{i < l_k} \in [A_b]^l_k\) and \((U_s)_{s \in b^\omega}\) be the unique partition of \(b^\omega\) into nonempty clopen intervals satisfying \(\max U_{s_k^i} = x_i\) for all \(i < l_k\) and \(\max U_{s_k^i} = \max b^\omega\). Pick any filtering \((V_s)_{s \in b^\omega}\) such that \(V_{s_k^i} = U_{s_k^i}\) for all \(i \leq l_k\). Finally, by Lemma 8 there exists \(f \in C_{\text{sur}}^+(b^\omega)\) such that \((U_{s_k^i}^f)_{s \in b^\omega} = (V_s)_{s \in b^\omega}\). Clearly \(f\) is the desirable one and the proof is complete. \(\square\)

We will also need some notation concerning the order isomorphic copies of \(Q\) in \(A_b\). Let \(b < \omega\) with \(b \geq 2\). Also let \(Y\) be a subset of \(A_b\) order isomorphic to \(Q\). We set

\[|Y|^\eta = \{Z \subseteq Y : Z\text{ is order isomorphic to }Q\}.\]

Moreover, for every \(f \in C_{\text{sur}}^+(b^\omega)\) let us set

\[Y_f = \{\max U_s^f : s \in b^{<\omega}\} \setminus \{\min b^\omega\}.\]

It is easy to check that \(Y_f \in [A_b]^\eta\), for all \(f \in C_{\text{sur}}^+(b^\omega)\). The relation between the elements of \(C_{\text{sur}}^+(b^\omega)\) and \(A_b\) is even stronger and it is described by the following lemma.

**Lemma 10.** Let \(b < \omega\) with \(b \geq 2\). Then for every \(Y \in [A_b]^\eta\) there exists \(f \in C_{\text{sur}}^+(b^\omega)\) such that \(Y_f = Y\). More generally, for every \(h \in C_{\text{sur}}^+(b^\omega)\) we have that \([Y_h]^\eta = \{Y_{foh} : f \in C_{\text{sur}}^+(b^\omega)\}\).
Proof. Let $Y \in [\mathcal{A}_b]^\eta$. In order to determine $f \in C^\uparrow_{\text{sur}}(b^\omega)$ such that $Y_f = Y$, by Lemma 8 it suffices to construct a filtering $(U_s)_{s \in b^{<\omega}}$ on $b^\omega$ such that $\{\max U_s : s \in b^{<\omega}\} \subseteq \{\max b^\omega\}$. Since $Y$ is countable, let $\{y_n : n < \omega\}$ be an enumeration of $Y$. We set $U_0 = b^\omega$. Suppose that for some $k < \omega$ with $k > 0$ the elements $(U_s)_{s \in b^{<k}}$ have been constructed. We are going to construct $(U_s)_{s \in b^k}$. Let $s \in b^{k-1}$. We set $i_0 = \min\{i < \omega : y_i \in U_s \setminus \{\max U_s\}\}$ and for every $p < b - 1$ with $p > 0$ we inductively define $i_p = \min\{i < \omega : y_{i_{p-1}} < \text{lex} y_i \text{lex} \max U_s\}$. We set $U_s^{i_0} = \{x \in U_s : x \leq_{\text{lex}} y_{i_0}\}$, for every $p < b - 1$ with $p > 0$ we set $U_s^{i_p} = \{x \in U_s : y_{i_{p-1}} < \text{lex} x \leq_{\text{lex}} y_i\}$ and $U_s^{i_p} = \{x \in U_s : y_{i_{p-2}} < \text{lex} x\}$.

It is clear that $(U_s)_{s \in b^{<\omega}}$ is a coloring of $b^\omega$ and $\{\max U_s : s \in b^{<\omega}\} \subseteq \{\max b^\omega\}$. The proof of the first part of the lemma is complete.

Let $h \in C^\uparrow_{\text{sur}}(h^\omega)$. Also let $Y \in [Y_h]^\eta$. By the first part of the lemma we may pick $g \in C^\uparrow_{\text{sur}}(b^\omega)$ satisfying $Y_g = Y$. Since $Y_g \in [Y_h]^\eta$, we have that $(U^g_s)_{s \in b^{<\omega}} \ll (U^h_s)_{s \in b^{<\omega}}$. Indeed, let $k < \omega$. Then $\{\max U^g_{s_i} : i \leq k\} \subseteq Y_g \cup \{\max b^\omega\} \subseteq Y_h \cup \{\max b^\omega\}$ (see the notation introduced before Lemma 9) and therefore we may pick $k' < \omega$ such that $\{\max U^g_{s_i} : i \leq k\} \subseteq \{\max U^h_{s_i} : j \leq k'\}$. Thus, there exist $0 \leq j_0 < j_1 < \ldots < j_k = k'$ such that $\max U^g_{s_k} = \max U^h_{s_k}$ for all $i \leq k$. Let $I_0 = \{s_{j_i} : j < j_0\}$, for every $i \leq k$ with $i > 0$ let $I_i = \{s_{j_i} : j < j_i\}$ and observe that $U^g_{s_i} = \bigcup_{s \in I_i} U^g_s$, for all $i \leq k$. Hence $(U^g_s)_{s \in b^{<\omega}} \ll (U^h_s)_{s \in b^{<\omega}}$. By Lemma 8 there exists $f \in C^\uparrow_{\text{sur}}(b^\omega)$ such that $f = g \circ h$. Hence $Y = Y_g = Y_{f \circ h}$ and therefore $[Y_h]^\eta \subseteq \{Y_{f \circ h} : f \in C^\uparrow_{\text{sur}}(b^\omega)\}$. The inverse inclusion is straightforward since for every $f \in C^\uparrow_{\text{sur}}(b^\omega)$ we have that $(U^f_{s_{1\leq\omega}})_{s \in b^{<\omega}} \ll (U^h_s)_{s \in b^{<\omega}}$ and therefore $Y_{f \circ h} \in [Y_h]^\eta$.

4. Proof of Theorem 11

By the definition of the metric $\rho_{\infty}$ on $C^\uparrow_{\text{sur}}(b^\omega)$ the following is immediate.

**Lemma 11.** Let $b < \omega$ with $b \geq 2$ and $\varepsilon$ be a positive real. For every $f, g \in C^\uparrow_{\text{sur}}(b^\omega)$ we have that $\rho_{\infty}(f, g) < \varepsilon$ if and only if $(\max U^f_s)_{s \in b^k} = (\max U^g_s)_{s \in b^k}$, where $k = \lfloor \log_2(1/\varepsilon) \rfloor + 1$.

We are ready to give the proof of the main result of this note.

**Proof of Theorem 11.** Let $\varepsilon$ be a positive real and $k = \lfloor \log_2(1/\varepsilon) \rfloor + 1$. We set $t = t_{b-1}$. Let $K$ be a positive integer and $c : C^\uparrow_{\text{sur}}(b^\omega) \to K$ be a coloring of $C^\uparrow_{\text{sur}}(b^\omega)$. Let $\ell = b^k - 1$ and $(s_i)_{i=0}^\ell$ the increasing enumeration of the set $b^k$ with respect to the lexicographical order on it. As we have already mentioned $([\mathcal{A}_b], \leq_{\text{lex}})$ is a countable dense unbounded linear order and therefore order isomorphic to $\mathbb{Q}$. For every $x$ in $[\mathcal{A}_b]^\ell$, using Lemma 8 we fix $f_x$ in $C^\uparrow_{\text{sur}}(b^\omega)$ satisfying $\{\max U^f_x : s \leq t_{b-1}\} = x$. We define a coloring $\tilde{c} : [\mathcal{A}_b]^\ell \to K$, by setting $\tilde{c}(x) = c(f_x)$. By Corollary 8...
there exists a subset $Y$ of $\mathcal{A}_0$ order isomorphic to $\mathbb{Q}$ and $B \subseteq K$ of cardinality at most $t$ such that the image of $[Y]^{\omega}$ through $c$ is equal to $B$. By Lemma 11 we may pick $h$ in $C_{\text{sur}}^t(b^\omega)$ such that $Y_h = Y$. Then for every $f \in C_{\text{sur}}^t(b^\omega)$, setting $x = (\max U_{\omega}^h)_{i < t}$, by Lemma 11 we have that $\rho_{\infty}(f \circ h, f_x) < \varepsilon$ and by the definition of the coloring $\bar{c}$ and the choice of $B$ we get that $c(f_x) = \bar{c}(x) \in B$. □

5. Necessity of the approximations

We recall some notation from [KT] adapted in our setting. A subset $T$ of $2^{<\omega}$ is called subtree if for every $s, t$ in $b^{<\omega}$ with $t \in T$ and $s$ initial segment of $t$ we have that $s \in T$. A node $s$ of a subtree $T$ is called a splitting node of $T$ if there exist $t, t'$ in $T$ such that $s \setminus (0)$ is initial segment of $t$ and $s \setminus (1)$ is initial segment of $t'$, while by $\text{Sp}(T)$ we denote the set of all splitting nodes of $T$. A subtree $T$ is called perfect if for every $s \in T$ there exists $t \in \text{Sp}(T)$ such that $s$ is proper initial segment of $t$. For every perfect subtree $T$ we denote by

$$Bd(T) = \{x \in 2^\omega : x|n \in T \text{ for all } n < \omega\}$$

the body of $T$. For every subset $A$ of $2^\omega$ we set

$$A^\uparrow = \{t \in 2^{<\omega} : \text{there exists } y \in A \text{ such that } t \text{ is initial segment of } y\}.$$ 

Finally, a subset $A$ of $2^\omega$ is called non scattered if it contains a subset order isomorphic to $\mathbb{Q}$. We recall the following result form [KT].

Lemma 12. Let $A \subseteq 2^\omega$. If $A$ is non scattered then the set $T = \{s \in 2^{<\omega} : W_s \cap A \text{ is non scattered}\}$ is a perfect subtree, where $W_s = \{x \in b^\omega : s \text{ is an initial segment of } x\}$.

Although the following result is well known, we could not find a reference and we include its proof for the convenience of the reader.

Theorem 13. There exists a coloring of $|[\mathbb{Q}]^\omega$ into $\omega$ colors such that for every $Y \in [\mathbb{Q}]^\omega$, the set $[Y]^\omega$ witnesses all the colors.

Proof. Since $\mathcal{A}_2$ and $\mathbb{Q}$ are order isomorphic, it suffices to construct $c : [\mathcal{A}_2]^\omega \rightarrow \omega$ such that for every $Y \in [\mathcal{A}_2]^\omega$ we have that $[Y]^\omega$ witnesses all the colors. We define $c : [\mathcal{A}_2]^\omega \rightarrow \omega$ as follows. Let $Y \in [\mathcal{A}_2]^\omega$. We set

$$T_Y = \{s \in 2^{<\omega} : W_s \cap Y \text{ is non scattered}\}.$$ 

By Lemma 12 we have that $T_Y$ is a perfect subtree. Since $T_Y$ is perfect we have that $Bd(T_Y)$ is a non-empty closed subset of $2^\omega$ (see [K]) and by Fact 4 it admits a maximum $m_Y$ and a minimum $t_Y$. Let $(t^n_Y)_{n<\omega}$ be the $\sqsubseteq$-increasing enumeration of the set $\{t_Y\}^\uparrow \cap \text{Sp}(T)$ and $(s^n_Y)_{n<\omega}$ be the $\sqsubseteq$-increasing enumeration of the set $\{m_Y\}^\uparrow \cap \text{Sp}(T)$. We set $c(Y) = \max\{i < \omega : |s^n_Y| < |t^n_Y|\}$. 

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Let \( Y \in [A_2]^\eta \) and a color \( r < \omega \). We will construct \( Z \in [Y]^\eta \) such that \( c(Z) = r \). We pick \( n < \omega \) such that \( m = \max\{i < \omega : |s_i^Y| < |t_i^Y|\} \geq r \). For notational simplicity we set \( t_0 = b_n^Y \) and \( s_0 = s_{m-r+1}^Y \). We define \( T = \{t \in T_Y : t \text{ is } \subseteq \text{-comparable with either } t_0 \text{ or } s_0\} \).

We set \( I = \{y \in 2^{\omega} : \max_{\text{lex}} W_{t_0} \leq_{\text{lex}} y \leq_{\text{lex}} \min_{\text{lex}} W_{s_0}\} \), \( Z' = Y \setminus I \) and \( Z \) a \( \subseteq \)-maximal subset of \( Z' \) order isomorphic to \( \mathbb{Q} \). Clearly \( Z \in [Y]^\eta \). It suffices to show that \( T_Z = T \). It is easy to check that \( |Z' \setminus Z| \leq 1 \). Thus setting \( T_{Z'} = \{s \in 2^{\omega} : W_s \cap Z' \text{ is non scattered}\} \), we have that \( T_Z = T_{Z'} \). By the definition of \( T \) and \( I \), for every \( t \in T_Y \setminus T \) we have that \( W_t \subseteq I \), while for every \( t \in T \) we have that there exists \( s_t \in T \) such that \( t \) is initial segment of \( s_t \) and \( W_{s_t} \cap I = \emptyset \). Hence, for every \( t \in T_Y \setminus T \) we have that \( W_t \cap Z' = \emptyset \) and therefore \( t \notin T_{Z'} \), while for every \( t \in T \) we have that \( W_t \cap Z' \supseteq W_{s_t} \cap Z' = W_{s_t} \cap Y \) and therefore \( t \in T_{Z'} \). That is, \( T_{Z'} \cap T_Y = T \). Since \( Z' \subseteq Y \), we have that \( T_{Z'} \subseteq T_Y \). Thus \( T_{Z'} = T_Y \) and the proof is complete. \( \Box \)

**Corollary 14.** Let \( b < \omega \) with \( b \geq 2 \). There exists a coloring \( c : C_{\text{sur}}^+(b^\omega) \to \omega \) such that for every \( h \in C_{\text{sur}}^+(b^\omega) \) the set \( \{f \circ h : f \in C_{\text{sur}}^+(b^\omega)\} \) witnesses all the colors.

**Proof.** Since \( A_b \) and \( \mathbb{Q} \) are order isomorphic, by Theorem [13] we have that there exists a coloring \( c : [A_b]^\eta \to \omega \) such that for every \( Y \in [A_b]^\eta \) the set \( [Y]^\eta \) witnesses all the colors. We define \( c : C_{\text{sur}}^+(b^\omega) \to \omega \) as follows. For every \( f \in C_{\text{sur}}^+(b^\omega) \) we set \( c(f) = \overline{c}(Y_f) \). Then for every \( h \in C_{\text{sur}}^+(b^\omega) \) and \( r < \omega \) there exists \( Z \in [Y_h]^\eta \) such that \( \overline{c}(Z) = r \) and by Lemma [10] there exists \( f \in C_{\text{sur}}^+(b^\omega) \) such that \( Z = Y_{f \circ h} \) and therefore \( c(f \circ h) = \overline{c}(Y_{f \circ h}) = \overline{c}(Z) = r \). \( \Box \)

6. **Accuracy of the Ramsey degree**

In this section we show that the Ramsey degree estimated in the Theorem [1] is the best possible. In particular, we have the following result.

**Proposition 15.** Let \( b \in \omega \) with \( b \geq 2 \). Also let \( \varepsilon > 0 \). Then there exists a coloring \( c : C_{\text{sur}}^+(b^\omega) \to t(\varepsilon) \) such that for every \( B \subseteq t(\varepsilon) \) such that there exists \( h \in C_{\text{sur}}^+(b^\omega) \) satisfying that for every \( f \in C_{\text{sur}}^+(b^\omega) \) there exists \( g \in C_{\text{sur}}^+(b^\omega) \) with \( \rho_\infty(f \circ h, g) < \varepsilon \) and \( c(g) \in B \), we have that \( B = t(\varepsilon) \).

The above result is essential an application of the following result.

**Theorem 16** (Devlin). Let \( \ell \in \omega \). Then there exists a coloring of \([Q]^\ell\) into \( t_\ell \) colors such that for every \( Y \in [Q]^\eta \), we have that \( [Y]^\ell \) witnesses all the colors.

**Lemma 17.** Let \( b \in \omega \) with \( b \geq 2 \). Also let \( \ell \in \omega \), \( h \in C_{\text{sur}}^+(b^\omega) \) and \((x_i)_{i < \ell} \in [Y_h]^\ell\). We set \( U_0 = \{x \in b^\omega : x \leq_{\text{lex}} x_0\} \) and \( V_0 = \{y \in b^\omega : y \leq_{\text{lex}} h(x_0)\} \),
while for every $i = 1, \ldots, \ell - 1$ we set

$$U_i = \{x \in b^\omega : x_{i-1} <_{\text{lex}} x \leq_{\text{lex}} x_i\} \quad \text{and} \quad V_i = \{y \in b^\omega : h(x_{i-1}) <_{\text{lex}} y \leq_{\text{lex}} h(x_i)\}.$$  

Then $(h(x_i))_{i < \ell}$ and $(U_i)_{i < \ell} = (h^{-1}(V_i))_{i < \ell}$.

**Proof.** For every $i < \ell$ we pick $s_i \in b^{<\omega}$ such that $x_i = \max U_{s_i} = h^{-1}(W_{s_i})$. Moreover, we may assume that $s_0, \ldots, s_{\ell-1}$ are of the same length, by extending the shorter ones by $b - 1$. Let $m$ be the common length of $s_0, \ldots, s_{\ell-1}$. Since $x_0, \ldots, x_{\ell-1}$ are distinct, we have that $s_0, \ldots, s_{\ell-1}$ are distinct too. Since $h$ is nondecreasing, we have that $s_0 <_{\text{lex}} \ldots <_{\text{lex}} s_{\ell-1}$. We set $I_0 = \{s \in b^m : s \leq_{\text{lex}} s_0\}$ and for every $1 \leq i < \ell$ we set $I_i = \{s \in b^m : s_{i-1} <_{\text{lex}} s \leq_{\text{lex}} s_i\}$. Then we have that $U_i = \bigcup_{s \in I_i} U_{s} = \bigcup_{s \in I_i} h^{-1}(W_s) = h^{-1}(\bigcup_{s \in I_i} W_s)$ for all $i < \ell$. Moreover, since $h$ is onto and increasing, we have that $h(x_i) = \max W_{s_i}$ for all $i < \ell$ and therefore $V_i = \bigcup_{s \in I_i} W_s$ for all $i < \ell$. Hence $(U_i)_{i < \ell} = (h^{-1}(V_i))_{i < \ell}$ and $(h(x_i))_{i < \ell} = (\max V_i)_{i < \ell}$ belongs to $[A_b]^\ell$.

We will also need the following strengthening of Lemma 9.

**Lemma 18.** Let $b < \omega$ with $b \geq 2$ and $k$ be a positive integer. Also let $h \in C_{\text{sur}}^+(b^\omega)$. Then $[Y_h]_k = \{(\max U_{s_{i}}^{f_{\text{oh}}})_{i < k} : f \in C_{\text{sur}}^+(b^\omega)\}$.

**Proof.** Clearly $\{(\max U_{s_{i}}^{f_{\text{oh}}})_{i < k} : f \in C_{\text{sur}}^+(b^\omega)\} \subseteq [Y_h]_k$. In order to prove the inverse inclusion, let $x = (x_i)_{i < l_k} \in [Y_h]_k$. By Lemma 16 we have that $(h(x_i))_{i < l_k}$ belongs to $[A_h]^{l_k}$. By Lemma 9 there exists a map $f$ in $C_{\text{sur}}^+(b^\omega)$ such that $(\max U_{s_{i}}^{f_{\text{oh}}})_{i < l_k} = (h(x_i))_{i < l_k}$. We set $U_0 = \{x \in b^\omega : x \leq_{\text{lex}} x_0\}$ and $U_i = \{x \in b^\omega : x_{i-1} <_{\text{lex}} x \leq_{\text{lex}} x_i\}$ for all $1 \leq i < l_k$. We also set $V_0 = \{y \in b^\omega : y \leq_{\text{lex}} h(x_0)\}$ and $V_i = \{y \in b^\omega : h(x_{i-1}) <_{\text{lex}} y \leq_{\text{lex}} h(x_i)\}$ for all $1 \leq i < l_k$. Then $(U_f)_{i < l_k} = (V_i)_{i < l_k}$. By Lemma 16 we have that $(U_f)_{i < l_k} = (h^{-1}(V_i))_{i < l_k}$. Thus $(f \circ h)^{-1}(W_{s_{i}}) = h^{-1}(f^{-1}(W_{s_{i}})) = h^{-1}(V_i) = U_i$ for all $i < l_k$. Hence $(\max U_{s_{i}}^{f_{\text{oh}}})_{i < l_k} = x$ and the proof is complete.

**Proof of Proposition 4.** Let $k = [\log_2(1/\varepsilon)] + 1$ and $\ell = b^k - 1$. Then $t_\ell = t(\varepsilon)$. Since $A_b$ and $Q$ are order isomorphic, by Theorem, we have that 16 there exists a coloring $\bar{c} : [A_b]^\ell \rightarrow t_\ell$ such that for every $Y \in [A_h]^\ell$, the set $[Y]^\ell$ witnesses all the colors. Let $(s_i)_{i=0}^{t_\ell}$ the increasing enumeration of the set $b^k$ with respect to the lexicographical order on it. We define a coloring $c : C_{\text{sur}}^+(b^\omega) \to t_\ell$ by setting $c(f) = \bar{c}(\max U_{s_i}^{f_{\text{oh}}})$, for all $f$ in $C_{\text{sur}}^+(b^\omega)$.

Let $B \subseteq t_\ell$ such that there exists a map $h \in C_{\text{sur}}^+(b^\omega)$ satisfying that for every $f \in C_{\text{sur}}^+(b^\omega)$ there exists $g \in C_{\text{sur}}^+(b^\omega)$ with $\rho(\bar{c} h, g) < \varepsilon$ and $c(g) \in B$. We need to show that $B = t_\ell$. Indeed, let $r < t_\ell$. By the choice of $\bar{c}$ there exists $x \in [Y_h]^\ell$ such that $\bar{c}(x) = r$. By Lemma 18 there exists $f \in C_{\text{sur}}^+(b^\omega)$ such that $(\max U_{s_i}^{f_{\text{oh}}})_{i < \ell} = x$ and therefore $c(f \circ h) = \bar{c}(x) = r$. By Lemma 11 we have that for every $g \in C_{\text{sur}}^+(b^\omega)$ satisfying $\rho(\bar{c} h, g) < \varepsilon$ we have that $(\max U_{s_i}^{f_{\text{oh}}})_{i < \ell} = (\max U_{s_i}^{g_{\text{oh}}})_{i < \ell}$ and therefore $c(f \circ h) = c(g) = r$. Hence $r \in B$ and the proof is complete.
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