SLANT NULL CURVES ON NORMAL ALMOST CONTACT B-METRIC 3-MANIFOLDS WITH PARALLEL REEB VECTOR FIELD

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Abstract. In this paper we study slant null curves with respect to the original parameter on 3-dimensional normal almost contact B-metric manifolds with parallel Reeb vector field. We prove that for non-geodesic such curves there exists a unique Frenet frame for which the original parameter is distinguished. Moreover, we obtain a necessary condition this Frenet frame to be a Cartan Frenet frame with respect to the original parameter. Examples of the considered curves are constructed.

1. Introduction

Many of the results in the classical differential geometry of curves have analogues in the Lorentzian geometry. Since null curves have very different properties compared to space-like and time-like curves, we have special interest in studying the geometry of null curves. The general theory of null curves is developed in [1, 2], where there are established important applications of these curves in general relativity.

Let $\mathbf{F}$ be a Frenet frame along a null curve $C$ on a Lorentzian manifold. According to [1], $\mathbf{F}$ and the Frenet equations with respect to $\mathbf{F}$ depend on both the parametrization of $C$ and the choice of a screen vector bundle. However, if a non-geodesic null curve $C$ is properly parameterized, then there exists only one Frenet frame, called a Cartan Frenet frame, for which the corresponding Frenet equations of $C$, called Cartan Frenet equations, have minimum number of curvature functions (2).

In this paper we consider 3-dimensional almost contact B-metric manifolds $(M, \varphi, \xi, \eta, g)$, which are Lorentzian manifolds equipped with an almost contact B-metric structure. We study slant null curves on considered manifolds belonging to the class $\mathcal{F}_1$ of the Ganchev-Mihova-Gribachev classification given in [3]. Also, the special class $\mathcal{F}_0$, which is subclass of $\mathcal{F}_1$, is object of considerations.

A slant curve $C(t)$ on $(M, \varphi, \xi, \eta, g)$, defined by the condition $g(\dot{C}(t), \xi) = \text{const}$ for the tangent vector $\dot{C}(t)$, is a natural generalization of a cylindrical helix in an Euclidean space. Let us remark that if we change the parameter $t$ of a slant curve $C(t)$ with another parameter $p$, then we have $\dot{C}(p) = \dot{C}(t) \frac{dt}{dp}$. Hence $g(\dot{C}(p), \xi)$ is a 

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constant if and only if \( t = ap + b \), where \( a, b \) are real numbers. This means that a slant null curve \( C(t) \) is not slant with respect to a special parameter \( p \) in general. Motivated by this fact, our aim in the present work is to study slant null curves with respect to the original parameter. In [8] there are considered non-geodesic slant null curves with respect to a special parameter \( p \) on 3-dimensional normal almost paracontact metric manifolds.

The paper is organized as follows. Section 2 is a brief review of almost contact B-metric manifolds and geometry of null curves on a 3-dimensional Lorentzian manifold. The main results are presented in Section 3. First, in Proposition 3.2 we express a general Frenet frame \( F \) and the functions \( h \) and \( k_1 \) along a slant null curve \( C(t) \) on \( (M, \varphi, \xi, \eta, g) \) in terms of the almost contact B-metric structure. In Theorem 3.6 we prove that for a non-geodesic slant null curve \( C(t) \) on a 3-dimensional \( F_1 \)-manifold there exists a unique Frenet frame \( F_1 \) for which the original parameter \( t \) is distinguished. Let us remark that in [4] it is shown that for any non-geodesic null curve on a 3-dimensional Minkowski space there exists a unique Frenet frame for which the original parameter is distinguished. In Theorem 3.7 we give a necessary condition for a slant null curve \( (C(t), F_1) \) to be a non-geodesic Cartan framed null curve with respect to the original parameter \( t \). The last Section 4 is devoted to some examples of the investigated curves.

2. Preliminaries

Let \( (M, \varphi, \xi, \eta, g) \) be an almost contact manifold with B-metric or an almost contact B-metric manifold. This means that \( M \) is a \((2n+1)\)-dimensional differentiable manifold, \( (\varphi, \xi, \eta) \) is an almost contact structure consisting of an endomorphism \( \varphi \) of the tangent bundle, a Reeb vector field \( \xi \) and its dual contact 1-form \( \eta \), i.e. the following relations are satisfied ([3]):

\[
\varphi^2 X = -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1,
\]

where \( \text{id} \) denotes the identity; as well as \( g \) is a pseudo-Riemannian metric, called a B-metric, such that ([3])

\[
g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y).
\]

Here and further \( X, Y \) and \( Z \) are tangent vector fields on \( M \), i.e. \( X, Y \in \Gamma(TM) \).

Immediate consequences of the above conditions are:

\[
(2.1) \quad \eta \circ \varphi = 0, \quad \varphi \xi = 0, \quad \text{rank}(\varphi) = 2n, \quad \eta(X) = g(X, \xi), \quad g(\xi, \xi) = 1.
\]

The tensor \( \tilde{g} \) given by \( \tilde{g}(X, Y) = g(X, \varphi Y) + \eta(X)\eta(Y) \) is a B-metric, too. Both metrics \( g \) and \( \tilde{g} \) are of signature \((n+1, n)\).

Let \( \nabla \) be the Levi-Civita connection of \( g \). The tensor field \( F \) of type \((0, 3)\) on \( M \) is defined by \( F(X, Y, Z) = g(\nabla_X \varphi Y, Z) \) and it has the following properties:

\[
F(X, Y, Z) = F(X, Z, Y) = F(X, \varphi Y, \varphi Z + \eta(Y)F(X, \xi, Z) + \eta(Z)F(X, Y, \xi).
\]

Moreover, we have

\[
(2.2) \quad F(X, \varphi Y, \xi) = (\nabla_X \eta)Y = g(\nabla_X \xi, Y).
\]
The following 1-forms, called Lee forms, are associated with $F$:

$$
\theta(X) = g^{ij}F(e_i, e_j, X), \quad \theta^*(X) = g^{ij}F(e_i, \varphi e_j, X), \quad \omega(X) = F(\xi, \xi, X),
$$

where $\{e_i, \xi\}, i = \{1, \ldots, 2n\}$ is a basis of $T_xM, x \in M$, and $(g^{ij})$ is the inverse matrix of $(g_{ij})$.

A classification of the almost contact B-metric manifolds with respect to $F$ is given in [3] and eleven basic classes $F_i (i = 1, 2, \ldots, 11)$ are obtained. If $(M, \varphi, \xi, \eta, g)$ belongs to $F_i$ then it is called an $F_i$-manifold.

The special class $F_0$ is the intersection of all basic classes. It is known as the class of the cosymplectic B-metric manifolds, i.e. the class of the considered manifolds with parallel structure tensors with respect to $\nabla$, namely $\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = \nabla \tilde{g} = 0$ and consequently $F = 0$.

The lowest dimension of the considered manifolds is dimension three. In [3] it is established that the class of 3-dimensional almost contact B-metric manifolds is $F_1 \oplus F_4 \oplus F_5 \oplus F_8 \oplus F_9 \oplus F_{10} \oplus F_{11}$. According to [7], the class of the normal almost contact B-metric manifolds is $F_1 \oplus F_2 \oplus F_4 \oplus F_8 \oplus F_6$, since the Nijenhuis tensor of almost contact structure vanishes there. Therefore, the class of the 3-dimensional normal almost contact B-metric manifolds is $F_1 \oplus F_4 \oplus F_5$.

In this paper we consider 3-dimensional almost contact B-metric manifolds belonging to $F_1$, determined by (3)

$$
F_1 : F(X, Y, Z) = \frac{1}{2} \{g(X, \varphi Y)\theta(\varphi Z) + g(\varphi X, \varphi Y)\theta(\varphi^2 Z) + g(X, \varphi Z)\theta(\varphi Y) + g(\varphi X, \varphi Z)\theta(\varphi^2 Y)\}.
$$

Taking into account (2.2) and (2.3), for $F_1$-manifolds we have that

$$
\nabla \xi = 0.
$$

Let us remark that $\nabla \xi \neq 0$ for the rest 3-dimensional normal almost contact B-metric manifolds. Therefore, we can determine $F_1$ as the class of 3-dimensional normal almost contact B-metric manifolds with parallel Reeb vector field. Obviously, $F_0$-manifolds are $F_1$-manifolds with vanishing Lee forms, i.e. $F_0$ is the subclass of $F_1$ of the so-called balanced manifolds of the considered type.

Let us remark that on a 3-dimensional $(M, \varphi, \xi, \eta, g)$ the metric $g$ has signature $(2, 1)$, i.e. $(M, g)$ is a 3-dimensional Lorentzian manifold.

Let $C : I \longrightarrow M$ be a smooth curve on $M$ given locally by

$$
x_i = x_i(t), \quad t \in I \subseteq \mathbb{R}, \quad i \in \{1, 2, 3\}
$$

for a coordinate neighborhood $U$ of $C$. The tangent vector field is given by

$$
\frac{d}{dt} = (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \dot{C},
$$

where we denote $\frac{d x_i}{d t}$ by $\dot{x}_i$ for $i \in \{1, 2, 3\}$. The curve $C$ is called a regular curve if $\dot{C} \neq 0$ holds everywhere.

Let a regular curve $C$ be a null (lightlike) curve on $(M, g)$, i.e. at each point $x$ of $C$ we have

$$
g(\dot{C}, \dot{C}) = 0, \quad \dot{C} \neq 0.
$$
A general Frenet frame on $M$ along $C$ is denoted by $F = \{\dot{C}, N, W\}$ and determined by

$$g(\dot{C}, N) = g(W, W) = 1, \quad g(N, N) = g(N, W) = g(\dot{C}, W) = 0.$$ \hfill (2.5)

The following general Frenet equations with respect to $F$ and $\nabla$ of $(M, g)$ are known from [2]

$$\begin{align*}
\nabla_{\dot{C}}\dot{C} &= h\dot{C} + k_1 W, \\
\nabla_{\dot{C}}N &= -hN + k_2 W, \\
\nabla_{\dot{C}}W &= -k_2 \dot{C} - k_1 N,
\end{align*}$$

where $h, k_1$ and $k_2$ are smooth functions on $U$. The functions $k_1$ and $k_2$ are called curvature functions of $C$.

The general Frenet frame $F$ and its general Frenet equations (2.6) are not unique as they depend on the parameter and the choice of the screen vector bundle of $C$ (for details see [1, pp. 56-58], [2, pp. 25-29]). It is known [1, p. 58] that there exists a parameter $p$ called a distinguished parameter, for which the function $h$ vanishes in (2.6). The pair $(C(p), F)$, where $F$ is a Frenet frame along $C$ with respect to a distinguished parameter $p$, is called a framed null curve (see [2]). In general, $(C(p), F)$ is not unique since it depends on both $p$ and the screen distribution. Therefore we look for a Frenet frame with the minimum number of curvature functions which are invariant under Lorentzian transformations. Such frame is called Cartan Frenet frame of a null curve $C$. In [2] it is proved that if the null curve $C(p)$ is non-geodesic such that the following condition for $\ddot{C}$ holds

$$g(\ddot{C}, \ddot{C}) = k_1 = 1,$$

then there exists only one Cartan Frenet frame $F$ with the following Frenet equations

$$\begin{align*}
\nabla_{\dot{C}}\dot{C} &= W, \\
\nabla_{\dot{C}}N &= \tau W, \\
\nabla_{\dot{C}}W &= -\tau \dot{C} - N.
\end{align*}$$

The latter equations are called the Cartan Frenet equations of $C(p)$ whereas $\tau$ is called a torsion function and it is invariant upto a sign under Lorentzian transformations. A null curve together with its Cartan Frenet frame is called a Cartan framed null curve.

Let us consider a smooth curve $C$ on an almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$. We say that $C$ is a slant curve on $M$ if

$$g(\dot{C}, \xi) = \eta(\dot{C}) = a$$

and $a$ is a real constant. The curve $C$ is called a Legendre curve if $a = 0$.

For the sake of brevity, let us use the following denotation

$$g(\ddot{C}, \varphi\dot{C}) = b,$$

where $b$ is a smooth function on $C$. 
3. Cartan framed slant null curves with respect to the original parameter on 3-dimensional $\mathcal{F}_1$-manifolds

**Lemma 3.1.** Let $C$ be a null curve on a 3-dimensional almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$. Then the triad of vector fields $\{\dot{C}, \xi, \varphi \dot{C}\}$ is a basis of $T_x M$ at $x \in C$.

**Proof.** At each point $x$ of $C$ we have (2.1) and (2.4), i.e. $\xi$ is space-like and $\dot{C}$ is lightlike. Hence $\xi$ and $\dot{C}$ are linearly independent vector fields along $C$. If we assume that $\varphi \dot{C}$ belongs to the plane $\alpha = \text{span}\{\xi, \dot{C}\}$, we have $\varphi \dot{C} = u \xi + v \dot{C}$ for some functions $u$ and $v$. Applying $\varphi$ to the both sides of this equality, we obtain $-\dot{C} + \eta(\dot{C}) \xi = uv \xi + v^2 \dot{C}$ which implies $v^2 = -1$. Thus, $\{\dot{C}, \xi, \varphi \dot{C}\}$ are linearly independent vector fields along $C$ which confirms our assertion. □

**Proposition 3.2.** Let $C$ be a slant null curve with conditions (2.8) and (2.9) on $(M, \varphi, \xi, \eta, g)$, $\dim M = 3$. If $F = \{\dot{C}, N, W\}$ is a general Frenet frame on $M$ along $C$ which has the same positive orientation as a basis $\{\dot{C}, \xi, \varphi \dot{C}\}$ at each $x \in C$, then

$$W = \alpha \xi + \beta \dot{C} + \gamma \varphi \dot{C}, \quad N = \lambda \xi + \mu \dot{C} + \nu \varphi \dot{C},$$

where $\beta$ is an arbitrary function and $\alpha, \gamma, \lambda, \mu, \nu$ are the following functions

$$\alpha = -\frac{b}{\sqrt{a^4 + b^2}}, \quad \gamma = \frac{a}{\sqrt{a^4 + b^2}}$$

$$\lambda = \frac{a^3 + \beta b \sqrt{a^4 + b^2}}{a^4 + b^2},$$

$$\mu = -\frac{a^2 + \beta^2 (a^4 + b^2)}{2 (a^4 + b^2)},$$

$$\nu = \frac{b - \beta a \sqrt{a^4 + b^2}}{a^4 + b^2}.$$

Moreover, the functions $h$ and $k_1$ with respect to $F$ are given by

$$h = -\lambda g(\dot{C}, \nabla_\dot{C} \xi) + \frac{\nu}{2} \left[\dot{C}(b) - F(\dot{C}, \dot{C}, \dot{C})\right],$$

$$k_1 = \alpha g(\dot{C}, \nabla_\dot{C} \xi) + \frac{\gamma}{2} \left[\dot{C}(b) - F(\dot{C}, \dot{C}, \dot{C})\right].$$

**Proof.** According to Lemma 3.1, we have $W = \alpha \xi + \beta \dot{C} + \gamma \varphi \dot{C}$ for some functions $\alpha, \beta, \gamma$. Using (2.5), we obtain the following system of equations for $\alpha, \beta, \gamma$

$$a \alpha + \gamma b = 0,$$

$$\alpha^2 + a^2 \gamma^2 + 2 \beta (a \alpha + \gamma b) = 1.$$
Let us express \( N = \lambda \xi + \mu \dot{C} + \nu \varphi \dot{C} \) for some functions \( \lambda, \mu, \nu \). From (2.5) we obtain the following system of equations for \( \lambda, \mu, \nu \)

\[
a\lambda + \nu b = 1, \\
\lambda \alpha + \beta + a^2 \nu \gamma = 0, \\
\lambda^2 + a^2 \nu^2 + 2\mu = 0,
\]

which yields the following solution

\[
\lambda = \frac{1 - \nu b}{a}, \quad \nu = -\frac{a + a\beta}{a^3\gamma - \alpha b}, \quad \mu = -\frac{1}{2} (\lambda^2 + a^2 \nu^2).
\]

The frame \( \mathbf{F} \) is positive oriented if \( \gamma = \frac{a}{\sqrt{a^4 + b^2}} \) holds. This implies

\[
\alpha = -\frac{b}{\sqrt{a^4 + b^2}}.
\]

Then we obtain (3.3) by direct substitutions. Now, from (2.6) and (3.1) we have

\[
h = g \left( \nabla_\dot{C} \dot{\varphi}, N \right) \\
= \lambda g \left( \nabla_\dot{C} \dot{\varphi}, \xi \right) + \mu g \left( \nabla_\dot{C} \dot{\varphi}, \dot{C} \right) + \nu g \left( \nabla_\dot{C} \dot{\varphi}, \varphi \dot{C} \right),
\]

\[
k_1 = g \left( \nabla_\dot{C} \dot{W}, C \right) \\
= \alpha g \left( \nabla_\dot{C} \dot{\varphi}, \xi \right) + \beta g \left( \nabla_\dot{C} \dot{\varphi}, \dot{C} \right) + \gamma g \left( \nabla_\dot{C} \dot{\varphi}, \varphi \dot{C} \right).
\]

Since (2.4) and (2.8) are valid, it follows that

\[
g \left( \nabla_\dot{C} \dot{\varphi}, \dot{C} \right) = 0, \quad g \left( \nabla_\dot{C} \dot{\varphi}, \xi \right) = -g \left( \dot{\varphi}, \nabla_\dot{C} \dot{\varphi} \right).
\]

Using the following expressions

\[
F(\dot{C}, \dot{C}, \dot{\varphi}) = g \left( \nabla_\dot{C} \varphi \dot{C}, \dot{C} \right) - g \left( \varphi \nabla_\dot{C} \dot{C}, \dot{C} \right),
\]

\[
\dot{C}(b) = g \left( \nabla_\dot{C} \dot{\varphi}, \dot{C} \right) + g \left( \dot{C}, \nabla_\dot{C} \varphi \dot{C} \right),
\]

we obtain

\[
g \left( \nabla_\dot{C} \dot{\varphi}, \varphi \dot{C} \right) = \frac{1}{2} \left[ \dot{C}(b) - F(\dot{C}, \dot{C}, \dot{\varphi}) \right].
\]

Substituting (3.7) and (3.8) in (3.6), we get (3.4).

\[\square\]

**Corollary 3.3.** Let the assumptions of Proposition 3.2 are satisfied and \((M, \varphi, \xi, \eta, g)\) be in \( \mathcal{F}_1 \). Then we have

\[
h = \frac{\nu}{2} \left[ \dot{C}(b) + a^2 \theta(\dot{C}) - b \theta(\varphi \dot{C}) \right],
\]

\[
k_1 = \frac{\gamma}{2} \left[ \dot{C}(b) + a^2 \theta(\dot{C}) - b \theta(\varphi \dot{C}) \right].
\]
Proof. By virtue of \((2.3)\) we get \(\theta(\xi) = 0\) and hence \(\theta(\varphi^2 \dot{C}) = -\theta(\dot{C})\). Then
\[
(3.10) \quad F(\dot{C}, \dot{C}, \dot{C}) = b\theta(\varphi \dot{C}) - a^2 \theta(\dot{C}).
\]
Substituting \((3.10)\) in \((3.4)\), we obtain \((3.9)\). \hfill \Box

**Proposition 3.4.** A slant null curve \(C\) on a 3-dimensional \(F_1\)-manifold is geodesic if and only if the following equality holds
\[
\dot{C}(b) = b\theta(\varphi \dot{C}) - a^2 \theta(\dot{C}).
\]
Proof. As it is known \([2]\), a null curve is geodesic if and only \(k_1\) vanishes. Taking into account \((3.9)\) and \(\gamma \neq 0\), we establish the truthfulness of the statement. \hfill \Box

Immediately we obtain the following

**Corollary 3.5.** A slant null curve \(C\) on a 3-dimensional \(F_0\)-manifold is geodesic if and only if \(b\) is a constant.

Bearing in mind the statement for the general Frenet frames in Proposition 3.2, we have the following

**Theorem 3.6.** Let \(C(t)\) be a non-geodesic slant null curve on a 3-dimensional \(F_1\)-manifold \((M, \varphi, \xi, \eta, g)\). Then there exists a unique Frenet frame \(F_1 = \{\dot{C}, N_1, W_1\}\) for which the original parameter \(t\) is distinguished and
\[
(3.11) \quad W_1 = \alpha \xi - \frac{\alpha}{a} \dot{C} + \gamma \varphi \dot{C}, \quad N_1 = \frac{1}{a} \xi - \frac{1}{2a^2} \dot{C},
\]
where \(\alpha\) and \(\gamma\) are given by \((3.2)\).

Proof. The original parameter \(t\) of a null curve \(C(t)\) is distinguished if \(h(t)\) vanishes. Since \(C(t)\) is non-geodesic, by using Proposition 3.4 and Corollary 3.5, we obtain for an \(F_1\)-manifold that \(h = 0\) if and only if \(\nu = 0\). The second equality of \((3.5)\) implies that \(\nu = 0\) if and only if \(\beta = -\frac{a}{\alpha}\). Substituting \(\beta = -\frac{a}{\alpha}\) in \((3.3)\), we get \(\lambda = \frac{1}{a}\) and \(\mu = -\frac{1}{2a^2}\). The vector fields \(W_1\) and \(N_1\) in \((3.11)\) are obtained from \((3.1)\) by \(\beta = -\frac{a}{\alpha}, \lambda = \frac{1}{a}, \mu = -\frac{1}{2a^2}\) and \(\nu = 0\). Hence the Frenet frame \(F_1 = \{\dot{C}, N_1, W_1\}\) is the unique frame for which the original parameter \(t\) is distinguished. \hfill \Box

**Theorem 3.7.** Let \(C(t)\) be a slant null curve on a 3-dimensional \(F_1\)-manifold \((M, \varphi, \xi, \eta, g)\) and \(b\) satisfies the following ordinary differential equation
\[
(3.12) \quad \dot{C}(b) = b\theta(\varphi \dot{C}) - a^2 \theta(\dot{C}) + \frac{2}{a} \sqrt{a^4 + b^2}.
\]
Then \((C(t), F_1)\) is a non-geodesic Cartan framed slant null curve, where \(F_1 = \{\dot{C}, N_1, W_1\}\) is the unique Frenet frame of \(C(t)\) from Theorem 3.6. Moreover, the torsion function is \(\tau = -\frac{1}{2a^2}\). In particular, if \((M, \varphi, \xi, \eta, g)\) is an \(F_0\)-manifold then \(b\) is given by
\[
(3.13) \quad b = \frac{1}{2} \left[ \exp\left(\frac{2(t + u)}{a}\right) - a^4 \exp\left(-\frac{2(t + u)}{a}\right) \right],
\]
where \(u\) is an arbitrary real constant.
Proof. Taking into account Proposition 3.4 and (3.12), we conclude that $C(t)$ is non-
geodesic. Then it follows from Theorem 3.6 that there exists a unique Frenet frame $F_1 = \{\dot{C}, N_1, W_1\}$ for which the original parameter $t$ is distinguished. Substituting the expression for $\gamma$ from (3.2) and (3.12) in (3.9), we obtain $k_1(t) = 1$. Hence $F_1$ is a Cartan Frenet frame with respect to $t$. The Cartan Frenet equations (2.7) with respect to $F_1$ imply $\tau = g(\nabla_\dot{C}N_1, W_1)$. Using (3.11) and (2.7), we obtain the equality $\nabla_\dot{C}N_1 = -\frac{1}{2}a^2W_1$. Hence $\tau = -\frac{1}{2a^2}$ holds.

In the particular case, when $(M, \varphi, \xi, \eta, g)$ is an $F_0$-manifold, $b$ satisfies the following ordinary differential equation
\[
\frac{a \, db}{2\sqrt{a^4 + b^2}} = dt,
\]
which is obtained from (3.12) by $\theta = 0$. Integrating the latter equation, we get
\[
a \left(\frac{b}{\sqrt{b^2 + a^4}}\right) = t + u, \quad u \in \mathbb{R}.
\]
The last equality implies (3.13).

4. Examples of non-geodesic Cartan framed slant null curves with respect to the original parameter on 3-dimensional $F_0$- and $F_1$-manifolds

4.1. A Minkowski space equipped with a cosymplectic B-metric structure. An almost contact structure $(\varphi, \xi, \eta)$ and a B-metric $g$ on $\mathbb{R}^{2n+1}$ are defined in [3, Example 1., p. 270] and it is shown that $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ is an $F_0$-manifold. Now, we consider such an $F_0$-manifold in dimension 3, where the structure $(\varphi, \xi, \eta, g)$ is defined by
\[
\xi = \frac{\partial}{\partial x_3}, \quad \eta = dx_3, \quad \varphi \left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial x_2}, \quad \varphi \left(\frac{\partial}{\partial x_2}\right) = -\frac{\partial}{\partial x_1}, \quad \varphi \left(\frac{\partial}{\partial x_3}\right) = 0,
\]
for $x = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}$. It is clear that $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ is a Minkowski space, which is equipped with an almost contact B-metric structure.

Let $C_1(t) = (x_1(t), x_2(t), x_3(t)), \ t \in I$, be a slant null curve on $(\mathbb{R}^3, \varphi, \xi, \eta, g)$. Then the conditions (2.4) and (2.8) imply
\[
-x_1^2 + x_2^2 + x_3^2 = 0
\]
and
\[
\dot{x}_3 = a,
\]
respectively. Taking into account that $\varphi \dot{C}_1 = (-\dot{x}_2, \dot{x}_1, 0)$, we get
\[
b = g(\dot{C}_1, \varphi \dot{C}_1) = 2\dot{x}_1\dot{x}_2.
\]
According to Theorem 3.7, $(C_1(t), F_1)$ is a non-geodesic Cartan framed slant null curve on $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ if the condition (3.13) holds, i.e.
\[
2\ddot{x}_1\ddot{x}_2 = \frac{1}{2} \left[ \exp \left( \frac{2(t + u)}{a} \right) - a^4 \exp \left( -\frac{2(t + u)}{a} \right) \right].
\]
From (4.1) we obtain
\[ \dot{x}_1 = \pm \sqrt{\dot{x}_2^2 + a^2} \]
which implies \( \dot{x}_1 \neq 0 \) for all \( t \in I \). If we take
\( \dot{x}_1 = \sqrt{\dot{x}_2^2 + a^2} \), then (4.3) becomes
\[ 2\dot{x}_2 \sqrt{\dot{x}_2^2 + a^2} = \frac{1}{2} \left[ \exp \left( \frac{2(t + u)}{a} \right) - a^4 \exp \left( -\frac{2(t + u)}{a} \right) \right]. \]
We solve the latter equation and obtain
\[ (4.4) \quad \dot{x}_2 = \frac{1}{2} \left[ \exp \left( \frac{t + u}{a} \right) - a^2 \exp \left( -\frac{t + u}{a} \right) \right]. \]
Substituting (4.4) in (4.3), we find
\[ (4.5) \quad \dot{x}_1 = \frac{1}{2} \left[ \exp \left( \frac{t + u}{a} \right) + a^2 \exp \left( -\frac{t + u}{a} \right) \right]. \]
Integrating (4.2), (4.4) and (4.5), we get
\[ C_i(t) = \left( \frac{a}{2} \left[ \exp \left( \frac{t + u}{a} \right) - a^2 \exp \left( -\frac{t + u}{a} \right) \right] + c_1, \right. \]
\[ \left. \frac{a}{2} \left[ \exp \left( \frac{t + u}{a} \right) + a^2 \exp \left( -\frac{t + u}{a} \right) \right] + c_2, \quad at + c_3 \right), \]
where \( c_1, c_2, c_3 \in \mathbb{R} \).

Analogously, in the case \( \dot{x}_1 = -\sqrt{\dot{x}_2^2 + a^2} \) we have
\[ C_2(t) = \left( \frac{a}{2} \left[ \exp \left( \frac{t + u}{a} \right) - a^2 \exp \left( -\frac{t + u}{a} \right) \right] + c_4, \right. \]
\[ \left. \frac{a}{2} \left[ \exp \left( \frac{t + u}{a} \right) + a^2 \exp \left( -\frac{t + u}{a} \right) \right] + c_5, \quad at + c_6 \right), \]
where \( c_4, c_5, c_6 \in \mathbb{R} \).
Thus, we state

**Theorem 4.1.** The unique non-geodesic Cartan framed slant null curves with respect to the original parameter \( t \) on \( (\mathbb{R}^3, \varphi, \xi, \eta, g) \) are \( (C_i(t), F_i) \), where \( F_i = \{ \dot{C}_i, N_i, W_i \} \) \( (i = 1, 2) \) and
\[ N_i = \frac{1}{a} \xi - \frac{1}{2a^2} \dot{C}_i, \]
\[ W_i = -\rho \left( \xi - \frac{1}{a} \dot{C}_i - \frac{2a}{\exp \left( \frac{2(t+u)}{a} \right) - a^4 \exp \left( -\frac{2(t+u)}{a} \right)} \varphi \dot{C}_i \right), \]
where
\[ \rho = \frac{\exp \left( \frac{2(t+u)}{a} \right) - a^4 \exp \left( -\frac{2(t+u)}{a} \right)}{\exp \left( \frac{2(t+u)}{a} \right) + a^4 \exp \left( -\frac{2(t+u)}{a} \right)}. \]
4.2. A non-geodesic Cartan framed slant null curve on an $\mathcal{F}_1$-manifold constructed on a Lie group. Let $L$ be a 3-dimensional real connected Lie group and $I$ its corresponding Lie algebra. If $\{E_0, E_1, E_2\}$ is a basis of left invariant vector fields on $I$ then $L$ is equipped with an almost contact structure $(\varphi, \xi, \eta)$ and a left invariant B-metric $g$ in [5] as follows:

\begin{align*}
\varphi E_0 &= 0, \quad \varphi E_1 = E_2, \quad \varphi E_2 = -E_1, \quad \xi = E_0, \\
\eta(E_0) &= 1, \quad \eta(E_1) = \eta(E_2) = 0, \\
g(E_0, E_0) &= g(E_1, E_1) = -g(E_2, E_2) = 1, \\
g(E_0, E_1) &= g(E_0, E_2) = g(E_1, E_2) = 0.
\end{align*}

(4.6)

Let $(L, \varphi, \xi, \eta, g)$ be a 3-dimensional almost contact B-metric manifold belonging to the class $\mathcal{F}_1$. It is proved in [5, Theorem 1.1] that the corresponding Lie algebra $\mathfrak{g}_1$ of $L$ is determined by the following commutators:

$$[E_0, E_1] = [E_0, E_2] = 0, \quad [E_1, E_2] = \alpha E_1 + \beta E_2,$$

where $\alpha, \beta$ are arbitrary real parameters such that $(\alpha, \beta) \neq (0, 0)$ and

$$\alpha = \frac{1}{2} \theta_1, \quad \beta = \frac{1}{2} \theta_2.$$

In the latter equalities, by $\theta_1$ and $\theta_2$ are denoted $\theta(E_1)$ and $\theta(E_2)$, respectively. We note that $\theta_0 = \theta(\xi) = 0$.

Let $G$ be the compact simply connected Lie group with the same Lie algebra as $L$ as well as $G$ is isomorphic to $L$. Consider the curve $C(t) = \exp(tX)$ on $G$, where $t \in \mathbb{R}$ and $X \in \mathfrak{g}_1$. Hence the tangent vector to $C(t)$ at the identity element $e$ of $G$ is $\dot{C}(0) = X$. Let $X$ satisfies the following conditions

$$g(X, X) = 0, \quad \eta(X) = a, \quad a \in \mathbb{R} \setminus \{0\}.$$

(4.7)

Since $g$ is left invariant, from (4.7) it follows that $g(\dot{C}(t), \dot{C}(t)) = 0$ and $\eta(\dot{C}(t)) = a$ for all $t \in \mathbb{R}$. This means that $C(t)$ is a slant null curve on $G$. Also we have $b = g(\dot{C}(t), \varphi \dot{C}(t)) = g(X, \varphi X)$ for all $t \in \mathbb{R}$ and therefore $b$ is a constant. Taking into account (4.7), the coordinates $(p, q, r)$ of $X$ with respect to the basis $\{E_0, E_1, E_2\}$ satisfy the following equalities

$$p^2 + q^2 - r^2 = 0, \quad p = a.$$  

(4.8)

From (4.8) we obtain $r = \pm \sqrt{a^2 + q^2}$. Let us suppose that $r = \sqrt{a^2 + q^2}$. Having in mind (4.6), we get $\varphi X = (0, -\sqrt{a^2 + q^2}, q)$. Then we have

$$b = g(X, \varphi X) = -2q\sqrt{a^2 + q^2}.$$  

(4.9)

Using (4.2), we obtain

$$\theta(\varphi X) = 2\beta q - 2\alpha \sqrt{a^2 + q^2}, \quad \theta(X) = 2\alpha q + 2\beta \sqrt{a^2 + q^2}.$$  

(4.10)

Now, according to Theorem 3.7, $(C(t), F_1)$ is a non-geodesic Cartan framed slant null curve on $G$ if (3.12) holds. By using (4.9) and (4.10), the equation (3.12) becomes

$$(a^2 + 2q^2) \left(a\beta \sqrt{a^2 + q^2} - \alpha aq - 1\right) = 0.$$
which is equivalent to
\[(4.11) \quad a\beta \sqrt{a^2 + q^2} - \alpha aq - 1 = 0\]
since \(a^2 + 2q^2 \neq 0\).

Consider an \(F_1\)-manifold \((L_1, \varphi, \xi, \eta, g)\) such that \(\alpha = \beta \neq 0\) and \(b \neq 0\). For this manifold \((4.11)\) becomes
\[(4.12) \quad \alpha \sqrt{a^2 + q^2} = 1 + \alpha aq.\]
If \(\alpha a > 0\) then \((4.12)\) has a unique solution \(q = \frac{\alpha^2 a^4 - 1}{2\alpha a}\). Then we get \(r = \frac{\alpha^2 a^4 + 1}{2\alpha a}\).
The condition \(b \neq 0\) implies \(q \neq 0\) which means that \(\alpha \neq \pm \frac{1}{\alpha^2}\).

Thus, for \(F_1 = \{X, N_1, W_1\} \in T_eG\) and \(\varphi X\) we obtain
\[(4.13) \quad X = \left( a, \frac{\alpha^2 a^4 - 1}{2\alpha a}, \frac{\alpha^2 a^4 + 1}{2\alpha a} \right),\]
\[(4.14) \quad N_1 = \frac{1}{a} \xi - \frac{1}{2a^2} X,\]
\[(4.15) \quad W_1 = \frac{\alpha^4 a^8 - 1}{\alpha^4 a^8 + 1} \left( \xi - \frac{1}{a} X + \frac{2\alpha^2 a^3}{\alpha^4 a^8 - 1} \varphi X \right),\]
\[(4.16) \quad \varphi X = \left( 0, -\frac{\alpha^2 a^4 - 1}{2\alpha a}, \frac{\alpha^2 a^4 + 1}{2\alpha a} \right),\]
where \(\alpha a > 0\) and \(\alpha \neq \pm \frac{1}{\alpha^2}\).

Further, we find the matrix representation of \(C(t)\) and \(F_1\). In [5, Theorem 2.1] it is found explicitly the 3-dimensional matrix representation \(\pi\) of the Lie algebra \(g_1\). It is well known that \(\pi\) is the following Lie algebra homomorphism \(\pi : g_1 \rightarrow \text{Hom}(V)\) such that \(Y \rightarrow \pi(Y)\), where \(V\) is a 3-dimensional real vector space. Notice that the linear operators \(\pi(Y) \in \text{Hom}(V)\) do not need to be invertible.

Their matrices \(A\) are called briefly matrix representation of \(g_1\). It is proved that
\[
\pi(E_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi(E_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\alpha & -\beta \end{pmatrix},
\]
\[
\pi(E_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & 0 \end{pmatrix}.
\]

Now, taking into account that \(\alpha = \beta\) and using \((4.13), (4.15)\), we have
\[(4.16) \quad \pi(X) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\alpha^2 a^4 + 1}{2a} & \frac{\alpha^2 a^4 - 1}{2a} \\ 0 & \frac{1-\alpha^2 a^4}{2a} & \frac{1-\alpha^2 a^4}{2a} \end{pmatrix},\]
(4.17) \[
\pi(\varphi X) = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{\alpha^2a^4 - 1}{2a} & \frac{\alpha^2a^4 - 1}{2a} \\
0 & \frac{\alpha^2a^4 + 1}{2a} & \frac{\alpha^2a^4 + 1}{2a}
\end{pmatrix}.
\]

Let \( \Pi : G \rightarrow \text{End}(V) \) be the matrix representation of the simply connected Lie group \( G \). By \( \text{End}(V) \) are denoted the invertible linear operators of \( V \). The representations \( \Pi \) and \( \pi \) are related as follows:

\[
\Pi(\exp(Y)) = \exp(\pi(Y))
\]

for all \( Y \in g \). Since \( tX \in g, t \in \mathbb{R} \), we get

(4.18) \[
\Pi(C(t)) = \Pi(\exp(tX)) = \exp(\pi(tX)).
\]

In [5], the group of the matrices of the endomorphisms \( \Pi(b), b \in G \), is denoted by \( G_1 \) and it is called the matrix Lie group representation. According to [5, Theorem 2.1], we have

(4.19) \[
G_1 = \left\{ \exp(A) = E + \left( \frac{\exp(\text{tr}A) - 1}{\text{tr}A} \right) A \right\},
\]

where \( \text{tr}A \neq 0 \). Using (4.16), we find \( \text{tr}(\pi(tX)) = \frac{t}{a} \) and taking into account (4.19), we get

(4.20) \[
\exp(\pi(tX)) = E + a \left( \frac{t}{a} - 1 \right) \pi(X).
\]

Substituting (4.20) in (4.18) and taking into account (4.16), we obtain the following matrix representation of \( C(t) \)

\[
\Pi(C(t)) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + \exp\left(\frac{t}{a} - 1\right) \frac{\alpha^2a^4 + 1}{2a} & \exp\left(\frac{t}{a} - 1\right) \frac{\alpha^2a^4 + 1}{2a} \\
0 & \exp\left(\frac{t}{a} - 1\right) \frac{1 - \alpha^2a^4}{2a} & 1 + \exp\left(\frac{t}{a} - 1\right) \frac{1 - \alpha^2a^4}{2a}
\end{pmatrix}.
\]

Finally, using (4.14), (4.16) and (4.17), we obtain the matrix representations of \( N_1 \) and \( W_1 \) as follows:

\[
\pi(N_1) = \begin{pmatrix}
0 & 0 & 0 \\
0 & -\frac{\alpha^2a^4 + 1}{4a^3} & -\frac{\alpha^2a^4 + 1}{4a^3} \\
0 & \frac{\alpha^2a^4 - 1}{4a^3} & \frac{\alpha^2a^4 - 1}{4a^3}
\end{pmatrix},
\]

\[
\pi(W_1) = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{1 - \alpha^2a^4}{2a^2} & \frac{1 - \alpha^2a^4}{2a^2} \\
0 & \frac{1 + \alpha^2a^4}{2a^2} & \frac{1 + \alpha^2a^4}{2a^2}
\end{pmatrix}.
\]
REFERENCES

[1] Duggal, K. L., Bejancu, A.: *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*. Kluwer Academic, 364 (1996)

[2] Duggal, K. L., Jin, D. H.: *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*. World Scientific Publishing, Singapore, (2007)

[3] Ganchev, G., Mihova, V., Gribachev, K.: *Almost contact manifolds with B-metric*. Math. Balkanica 7, 262–276 (1993)

[4] Honda, K., Inoguchi, J.: *Deformation of Cartan framed null curves preserving the torsion*. Differ. Geom. Dyn. Syst. 5, 31–37 (2003)

[5] Manev, H.: *Matrix Lie groups as 3-dimensional almost contact B-metric manifolds*. Facta Universitatis (Niš), Ser. Math. Inform. 30 (3), 341–351 (2015)

[6] Manev, H.: *On the structure tensors of almost contact B-metric manifolds*. Filomat 29 (3), 427–436 (2015)

[7] Manev, M., Ivanova, M.: *Canonical type connections on almost contact manifold with B-metric*. Ann. Global Anal. Geom. 43 (4), 397–408 (2013)

[8] Welyczko, J.: *Slant curves in 3-dimensional normal almost paracontact metric manifolds*. Mediterr. J. Math. 11, 965–978 (2014)

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