Inductive algebras for the affine group of a finite field

Promod Sharma¹ · M. K. Vemuri¹

Received: 1 January 2022 / Accepted: 22 March 2022 / Published online: 6 April 2022
© African Mathematical Union and Springer-Verlag GmbH Deutschland, ein Teil von Springer Nature 2022

Abstract
Each irreducible representation of the affine group of a finite field has a unique maximal inductive algebra, and it is self-adjoint.

Keywords Inductive algebra · Induced representation · Affine group · Finite field

Mathematics Subject Classification 20C15

1 Introduction

Let \( G \) be a separable locally compact group and \( \pi \) an irreducible unitary representation of \( G \) on a separable Hilbert space \( \mathcal{H} \). Let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of bounded operators on \( \mathcal{H} \). An inductive algebra is a weakly closed abelian sub-algebra \( A \) of \( \mathcal{B}(\mathcal{H}) \) that is normalized by \( \pi(G) \), i.e., \( \pi(g)A\pi(g)^{-1} = A \) for each \( g \in G \). If we wish to emphasize the dependence on \( \pi \), we will use the term \( \pi \)-inductive algebra. A maximal inductive algebra is a maximal element of the set of inductive algebras, partially ordered by inclusion.

The identification of inductive algebras can shed light on the possible realizations of \( \mathcal{H} \) as a space of sections of a homogeneous vector bundle (see e.g. [7–10]). For self-adjoint maximal inductive algebras there is a precise result known as Mackey’s Imprimitivity Theorem, as explained in the introduction to [7]. Inductive algebras have also found applications in operator theory (see e.g. [3, 4]).

In [6], it was shown that finite dimensional inductive algebras for a connected group are trivial. However, there are interesting inductive algebras for finite groups (see e.g. [5]). In this note, we classify the maximal inductive algebras for the representations of the affine group (the “\( ax + b \)” group) of a finite field.

In Sect. 2, we recall the structure of the affine group of a finite field, and set up the notation. In Sect. 3, we recall its representation theory, and formulate our main result. The main result is proved in Sect. 4.

¹ Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi 221005, India
2 The affine group of a finite field

Let \( k \) be a finite field of order \( q = p^n \), where \( p \) is prime. Let \( k^\times \) denote the multiplicative group of non-zero elements of \( k \). Recall that the affine group of \( k \) is the group \( G \) of affine automorphisms of \( k \). Thus an element \( g \) of \( G \) is a map \( g : k \rightarrow k \) of the form \( g(x) = ax + b \) where \( a \in k^\times \) and \( b \in k \), and the group law is composition. The group \( G \) may be identified with the group of matrices

\[
\left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a \in k^\times, \ b \in k \right\}.
\]

Let \( \iota : k \rightarrow G \), \( p : G \rightarrow k^\times \) and \( s : k^\times \rightarrow G \) be defined by

\[
\iota(b) = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad p \left( \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right) = a, \quad \text{and} \quad s(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.
\]

Then \( \iota, p \) and \( s \) are homomorphisms, \( \iota(k) \triangleleft G \) and

\[
0 \longrightarrow k \xrightarrow{\iota} G \xrightarrow{p} k^\times \xrightarrow{s} 1
\]

is an exact sequence with splitting \( s \). Thus \( G \) is a semidirect product \( k^\times \rtimes k \). We note for future reference that \( s(a')\iota(b)s(a')^{-1} = \iota(a'b) \).

3 The representations and their inductive algebras

The irreducible unitary representations of \( G \) may be constructed using the Mackey machine (see [1, Sect. 3.9]). There are \( q - 1 \) characters (one-dimensional representations), and one \( (q - 1) \)-dimensional representation (up to unitary equivalence).

Obviously, the characters have only the trivial inductive algebra \( \mathbb{C} \), which is self-adjoint. The \( (q - 1) \)-dimensional representation is

\[
\pi = \text{Ind}_{\iota(k)}^G \chi,
\]

where \( \chi : k \rightarrow \mathbb{C}^\times \) is a non-trivial homomorphism (i.e. \( \chi \neq 1 \)). Let \( \mathcal{H} \) denote the Hilbert space of all complex-valued functions on \( k^\times \) equipped with the inner product

\[
\langle F_1, F_2 \rangle = \sum_{a' \in k^\times} F_1(a')\overline{F_2(a')}.
\]

The representation \( \pi \) may be realized on \( \mathcal{H} \) by

\[
(\pi(g)F)(a') = \chi(a'b)F(a'a), \quad g = \iota(b)s(a).
\]

For each \( \varphi \in \mathcal{H} \), let \( m_\varphi : \mathcal{H} \rightarrow \mathcal{H} \) be defined by \( m_\varphi(F) = \varphi F \). Let

\[
\mathcal{B} = \{ m_\varphi \mid \varphi \in \mathcal{H} \}.
\]

Then \( \mathcal{B} \) is a maximal-abelian subalgebra of \( \mathcal{B}(\mathcal{H}) \), and \( \mathcal{B} \) is \( \pi \)-inductive. Therefore \( \mathcal{B} \) is a maximal \( \pi \)-inductive algebra. Moreover, it is self-adjoint. Our main result is the following theorem.

**Theorem 1** \( \mathcal{B} \) is the only maximal \( \pi \)-inductive algebra.
4 The proof

Let $\mathcal{A}$ be a maximal $\pi$-inductive algebra.

**Lemma 2** There are no non-zero nilpotent elements in $\mathcal{A}$.

**Proof** Let $\mathcal{N}$ denote the set of nilpotent elements in $\mathcal{A}$ (the nilradical of $\mathcal{A}$). Let

$$K = \{ F \in \mathcal{H} \mid T(F) = 0, \forall T \in \mathcal{N} \}.$$ 

By (a trivial case of) Engel’s theorem [2, Sect. 3.3], $K \neq \emptyset$. Observe that $\mathcal{N}$ is normalized by $\pi(G)$, so $K$ is $\pi(G)$-invariant. However, since $\pi$ is irreducible, it follows that $K = \mathcal{H}$, whence $\mathcal{N} = 0$. □

**Corollary 3** $\dim(\mathcal{A}) \leq q - 1$.

**Proof** By Lemma 2, the Jordan–Chevalley decomposition [2, §4.2], and the fact that $\mathcal{A}$ is abelian, it follows that there is a (not necessarily orthonormal) basis for $\mathcal{H}$ in which each element of $\mathcal{A}$ is diagonal. Since $\dim(\mathcal{H}) = q - 1$, the result follows. □

For $b' \in k$, define $\kappa(b') : \mathcal{A} \to \mathcal{A}$ by $\kappa(b')T = \pi((b'))T\pi((b'))^{-1}$. Then $\kappa$ is a representation of the finite abelian group $k$ on the vector space $\mathcal{A}$, which decomposes as

$$\mathcal{A} = \bigoplus_{b \in k} \mathcal{A}_b,$$

where

$$\mathcal{A}_b = \{ T \in \mathcal{A} \mid \kappa(b')T = \chi(bb')T, \forall b' \in k \}.$$ 

Here we are using the fact that every character of $k$ is of the form $\chi_b$ where $\chi_b(b') = \chi(bb')$ for all $b' \in k$.

Observe that

1. if $T \in \mathcal{A}_b$ and $T' \in \mathcal{A}_{b'}$, then $TT' \in \mathcal{A}_{b+b'}$, and
2. for each $a \in k^\times$, the map $T \mapsto \pi(s(a))^{-1}T\pi(s(a))$ is a linear isomorphism $\mathcal{A}_b \to \mathcal{A}_{ab}$.

**Lemma 4** $\mathcal{A}_1 = 0$.

**Proof** Suppose not. Then there exists a non-zero element $T \in \mathcal{A}_1$. By the first observation above, $T^p \in \mathcal{A}_0$. Since $T$ is not nilpotent (see Lemma 2), it follows that $\dim(\mathcal{A}_0) \geq 1$. For $b \neq 0$, we have $\dim(\mathcal{A}_b) = \dim(\mathcal{A}_1) \geq 1$, by the second observation. Therefore

$$\dim(\mathcal{A}) = \sum_{b \in k} \dim(\mathcal{A}_b) \geq |k| = q,$$

contradicting Corollary 3. □

**Lemma 5** $\mathcal{A}_0 \subseteq \mathcal{B}$.

**Proof** Observe that $\mathcal{H}$ is spanned by the set $\{ \chi_{b'} | b' \in k \}$, hence $\mathcal{B}$ is spanned by the set $\{ m_{\chi_{b'}} | b' \in k \}$. If $T \in \mathcal{A}_0$, then $T$ commutes with $m_{\chi_{b'}}$ for each $b' \in k$, whence $T$ commutes with $\mathcal{B}$. Since $\mathcal{B}$ is maximal-abelian, it follows that $T \in \mathcal{B}$. □

By Lemma 4, and the second observation above, it follows that $\mathcal{A}_b = 0$ for all $b \in k^\times$, whence $\mathcal{A} = \mathcal{A}_0 \subseteq \mathcal{B}$. Since $\mathcal{A}$ is maximal, it follows that $\mathcal{A} = \mathcal{B}$.
References

1. George, W.: Mackey. The Theory of Unitary Group Representations. The University of Chicago Press, Chicago, London (1976)
2. Humphreys, James E.: Introduction to Lie algebras and representation theory, Springer-Verlag, New York-Berlin, Graduate Texts in Mathematics, Vol. 9. (1972)
3. Korányi, A.: Homogeneous bilateral block shifts. Proc. Indian Acad. Sci. Math. Sci. 124(2), 225–233 (2014)
4. Prasad, A., Vemuri, M.K.: Inductive algebras and homogeneous shifts. Complex Anal. Oper. Theory 4(4), 1015–1027 (2010)
5. Prasad, A., Vemuri, M.K.: Inductive algebras for finite Heisenberg groups. Comm. Algebra 38(2), 509–514 (2010)
6. Raghavan, K.N.: Finite dimensional inductive algebras are trivial. Comm. Algebra 33(10), 3783–3785 (2005)
7. Stegel, G.: Inductive algebras for trees. Pacific J. Math. 216(1), 177–200 (2004)
8. Stegel, G.: Even automorphisms of trees and inductive algebras. Int. J. Pure Appl. Math. 29(4), 521–552 (2006)
9. Steger, T., Vemuri, M.K.: Inductive algebras for SL(2, R). Illinois J. Math. 49(1), 139–151 (2005)
10. Vemuri, M.K.: Realizations of the canonical representation. Proc. Indian Acad. Sci. Math. Sci. 118(1), 115–131 (2008)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.