ON NICA-PIMSNER ALGEBRAS OF C*-DYNAMICAL SYSTEMS OVER $\mathbb{Z}_n^+$

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ABSTRACT. We examine Nica-Pimsner algebras associated with semigroup actions of $\mathbb{Z}_n^+$ on a C*-algebra $A$ by *-endomorphisms. We give necessary and sufficient conditions on the dynamics for exactness and nuclearity of the Nica-Pimsner algebras. Furthermore we parameterize the KMS states at finite temperature by the tracial states on $A$. Surjective affine homomorphisms are also given for ground states and KMS states at zero temperature, and a formula for tracial states is given.

Our computations can be carried out at the level of multivariable temperatures. However more data are required for the existence of appropriate functions on tubes. We introduce a KMS condition on multivariable temperature associated with a prescribing set and prove the equivalent reformulation. In conclusion this approach is more restrictive: even for trivial systems there are no states on the Nica-Pimsner algebras that satisfy the multivariable KMS condition with respect to any prescribing set.

INTRODUCTION

A C*-dynamical system over $\mathbb{Z}_n^+$ consists of a semigroup action $\alpha: \mathbb{Z}_n^+ \to \text{End}(A)$ on a C*-algebra $A$ by *-endomorphisms. One can associate two natural C*-algebras related to the lattice order of $\mathbb{Z}_n^+$: the Toeplitz-Nica-Pimsner algebra $\mathcal{NT}(A, \alpha)$ and the Cuntz-Nica-Pimsner algebra $\mathcal{NO}(A, \alpha)$. These algebras arise naturally in the theory of product systems, e.g. see Fowler [11], Solel [21], Deaconu, Kumjian, Pask and Sims [8], Sims and Yeend [22], Carlsen, Larsen, Sims and Vittadello [5], and the joint work of Davidson and Fuller with the author [7].

The existence of $\mathcal{NT}(A, \alpha)$ is readily verified by the existence of a Fock representation. For $\mathcal{NO}(A, \alpha)$ the situation is more complicated. Carlsen, Larsen, Sims, and Vittadello [5] show that the Cuntz-Nica-Pimsner algebras over $\mathbb{Z}_n^+$ exist as co-universal objects that satisfy a gauge invariant uniqueness theorem. This remarkable result leaves open the exploration of the algebraic structure of the Cuntz-Nica-Pimsner algebras. With Davidson and Fuller [7], the author gave a concrete picture of $\mathcal{NO}(A, \alpha)$ by using the theory of nonselfadjoint operator algebras: $\mathcal{NO}(A, \alpha)$ is the C*-envelope of

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the “lower triangular” part of $\mathcal{N} T(A, \alpha)$. In particular $\mathcal{NO}(A, \alpha)$ is universal with respect to the Cuntz-Nica covariant representations in the sense of [7]. To accomplish this, a tail adding technique was produced that gives an automorphic dilation $\tilde{\beta}: \mathbb{Z}^n \to \text{Aut}(\tilde{\mathcal{B}})$ of $\alpha: \mathbb{Z}^n_+ \to \text{End}(A)$ such that $\mathcal{NO}(A, \alpha)$ is a full corner of the usual crossed product $\tilde{\mathcal{B}} \rtimes_{\tilde{\beta}} \mathbb{Z}^n$. The circle is completed by showing that $\mathcal{NO}(A, \alpha)$ satisfies a gauge invariant uniqueness theorem and thus it is the Cuntz-Nica-Pimsner algebra in the sense of [5].

In the current paper we use the algebraic characterizations and the gauge invariant uniqueness theorems from [7] to further examine the $\text{C}^*$-structure of $\mathcal{N} T(A, \alpha)$ and $\mathcal{NO}(A, \alpha)$. We strongly feel that our methods extend to the case of Cuntz-Nica-Pimsner algebras for product systems, once an algebraic characterization is achieved in such generality.

In the first part of the paper we give necessary and sufficient conditions for exactness and nuclearity of $\mathcal{N} T(A, \alpha)$ and $\mathcal{NO}(A, \alpha)$. In Theorem 2.1 (resp. Theorem 2.2) we show that $\mathcal{N} T(A, \alpha)$ is exact (resp. nuclear) if and only if $A$ is exact (resp. nuclear), and in Theorem 2.3 we show that $\mathcal{NO}(A, \alpha)$ is exact if and only if $A$ is exact. Nuclearity of $A$ suffices to imply nuclearity of $\mathcal{NO}(A, \alpha)$, however it is not necessary even for injective systems (see [13, Example 7.7] and Remark 2.7). In Theorem 2.6 we give a necessary and sufficient condition for nuclearity of $\mathcal{NO}(A, \alpha)$ in terms of the automorphic dilation $\tilde{\beta}: \mathbb{Z}^n \to \text{Aut}(\tilde{\mathcal{B}})$. There is an analogy with the results of Katsura on $\text{C}^*$-correspondences [13] from which we were inspired. Nevertheless our analysis gets more perplexed because of the lattice structure of $\mathbb{Z}^n_+$.

For the second part we assume that the system is unital, and we examine the $(\sigma, \beta)$-KMS states on the Nica-Pimsner algebras at inverse temperature $\beta = 1/T$. There is a growing interest in the structure of KMS states on $\text{C}^*$-algebras and deducing symmetry and/or phase transition breaking following the seminal work of Bost and Connes [1] (see [6, 10, 12, 14, 15, 16, 17, 18, 19] to mention but a few inspiring works). The gauge action $\{\gamma_z\}_{z \in \mathbb{T}^n}$ induces a non-trivial action $\sigma$ of $\mathbb{R}$ by the automorphisms $\sigma_t = \gamma(e^{\lambda_1 t}, ..., e^{\lambda_n t})$ for $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n \setminus \{0\}$. We note here that the gauge invariant uniqueness theorems of [7] will be proven rather helpful when considering some $\lambda_k = 0$ (Propositions 3.9 and 3.15). As expected $\mathcal{N} T(A, \alpha)$ has a rich structure of KMS states when $\beta \neq 0$. In particular we show that there is an affine homeomorphism from the simplex of the tracial states on $A$ onto the simplex of the $(\sigma, \beta)$-KMS states on $\mathcal{N} T(A, \alpha)$ (Theorem 3.4). Moreover there is an affine homomorphism from the simplex of the tracial states (resp. states) of $A$ onto the simplex of the $\text{KMS}_{\infty}$ states (resp. ground states) of $\mathcal{N} T(A, \alpha)$ (Propositions 3.5 and 3.6). The same characterizations pass to $\mathcal{NO}(A, \alpha)$ (Theorem 3.12 and Corollary 3.13). In particular for the case of $(\sigma, \beta)$-KMS states with $\beta \neq 0$ the theorem reads the same for tracial states on $A$ that vanish on the ideal $I_{(1,...,1)}$ of $A$. Moreover when the system is injective then $\mathcal{NO}(A, \alpha)$ admits only $(\sigma, 0)$-KMS states, i.e. tracial states. In our analysis we follow a similar pattern with
that of Laca and Neshveyev [15, 16], Laca and Raeburn [17], and Laca, Raeburn, Ramagge and Whittaker [19]. Nevertheless, we make an essential use of the dilation method of [7] to show the existence of tracial states on \( \mathcal{NO}(A, \alpha) \), and consequently for \( \mathcal{NT}(A, \alpha) \).

The gauge actions may as well introduce a multivariable action of \( \mathbb{R}^n \) by automorphisms on the \( \mathbb{C}^* \)-algebras. In fact all of our computations hold when considering \( \sigma(t) = \gamma(e^{i t_1}, \ldots, e^{i t_n}) \) for \( (t_1, \ldots, t_n) \in \mathbb{R}^n \setminus \{0\} \). Hence one would suggest to carry out the analysis at the multivariable context. Defining a multivariable KMS condition may seem to be straightforward, one would suggest to carry out the analysis at the multivariable context. However what is important is to be able to associate a KMS condition with an analytic function on an infinite domain. In the appendix we show how this can be achieved for tubes and prescribing sets on the directions of \( \mathbb{R}^n \) (Theorem 4.4). The prescribing sets are required for estimating a uniform bound when applying the Phragmén-Lindelöf principle coordinate-wise. This stronger context affects entirely the multivariable analysis: we show that such multivariable KMS states do not exist for the Toeplitz-Nica-Pimsner algebra even in the simplest case of the trivial system of \( Z_+ \) over \( \mathbb{C} \), unless of course the multivariable actions follow as reductions from the one-variable case (Subsection 4.5).

1. Preliminaries

We write \( 1, \ldots, i, \ldots, n \) for the standard generators of \( Z_+^n \) and \( \emptyset := (0, \ldots, 0) \) and \( \underline{1} := (1, \ldots, 1) \). We denote the support of \( x = (x_1, \ldots, x_n) \in Z_+^n \) by \( \text{supp } x := \{i : x_i \neq 0\} \) and we write

\[
\mathcal{Z}^\perp := \{y \in Z_+^n : \text{supp } y \cap \text{supp } x = \emptyset\} \quad \text{and} \quad \mathcal{Z}^\geq := \{y \in Z_+^n : y \leq x\}.
\]

We use the same notation for \( t \in \mathbb{R}^n \). A grid is a subset of \( \mathbb{R}^n \) or \( Z_+^n \) that is \( \vee \)-closed where \( x \vee y = (\max\{x_i, y_i\})_i \). We will be using the grids

\[
F_m := \{x \in Z_+^n : x_i \leq m \cdot \underline{1}\}.
\]

A \( \mathbb{C}^* \)-dynamical system \( \alpha : Z_+^n \to \text{End}(A) \) will be called unital/injective/automorphic if \( \alpha_x \) is unital/injective/automorphic for all \( x \in Z_+^n \).

1.1. The Toeplitz-Nica-Pimsner algebra. An isometric Nica covariant representation \( V \) of \( Z_+^n \) is a semigroup homomorphism into the isometries of some \( \mathcal{B}(H) \) such that the \( V_i \) doubly commute, i.e. \( V_i V_j^* = V_j^* V_i \) for all \( i \neq j \) [20] (see also [7]). Following the theory of product systems we can define the following universal object.

**Definition 1.1.** A pair \( (\pi, V) \) is called isometric Nica covariant if:

(i) \( \pi : A \to \mathcal{B}(H) \) is a *-representation;

(ii) \( V : Z_+^n \to \mathcal{B}(H) \) is an isometric Nica covariant representation;

(iii) \( \pi(a) V_i = V_i \pi(a) \) for all \( a \in A \) and \( i = 1, \ldots, n \).

The universal \( \mathbb{C}^* \)-algebra \( \mathcal{NT}(A, \alpha) \) generated by \( V_x \pi(a) \) for \( x \in Z_+^n \) and \( a \in A \) under the isometric Nica covariant pairs \( (\pi, V) \) is called the Toeplitz-Nica-Pimsner \( \mathbb{C}^* \)-algebra.
It is immediate that if \((\pi, V)\) is an isometric Nica covariant pair for \(\alpha : \mathbb{Z}^n_{\mathbb{N}} \to \text{End}(A)\) then
\[
C^*(\pi, V) = \overline{\text{span}}\{ V_y \pi(a) V_y^* \mid a \in A, y \in \mathbb{Z}^n_{\mathbb{N}} \}.
\]
Furthermore if \(\alpha\) is unital then \(1 \in A\) is the unit for \(\mathcal{N} T(A, \alpha)\).

A particular isometric Nica covariant pair is given by the Fock representations. Let \(\pi : A \to \mathcal{B}(H)\) be a *-representation of \(A\). On \(K = H \otimes \ell^2(\mathbb{Z}^n_{\mathbb{N}})\), define the orbit representation \(\tilde{\pi} : A \to \mathcal{B}(K)\) and \(V : \mathbb{Z}^n_{\mathbb{N}} \to \mathcal{B}(K)\) by
\[
\tilde{\pi}(a) \xi \otimes e_y = (\pi \alpha_x(a) \xi) \otimes e_x \quad \text{and} \quad V_y(\xi \otimes e_y) = \xi \otimes e_{x+y}
\]
for all \(a \in A, \xi \in H\) and \(x, y \in \mathbb{Z}^n_{\mathbb{N}}\). Due to the Fock representation the ambient C*-algebra \(A\) embeds isometrically inside \(\mathcal{N} T(A, \alpha)\) therefore we omit the symbol \(\tilde{\pi}\). In [7] we showed that when \(\pi\) is faithful then the Fock representation induced by the pair \((\tilde{\pi}, V)\) defines a faithful representation of \(\mathcal{N} T(A, \alpha)\). More generally we obtained the following gauge invariant uniqueness theorem for \(\mathcal{N} T(A, \alpha)\). Recall that \(\mathcal{N} T(A, \alpha)\) admits a gauge action \(\{\gamma_x\}_{x \in \mathbb{T}}\) whose cores will be denoted by
\[
B_{[x, y]} := \overline{\text{span}}\{ V_w a V_w^* \mid a \in A, x \leq w \leq y \}.
\]
We denote the core \(B_{[x, x]}\) simply by \(B_x\). It is not immediate but every \(B_{[x, y]}\) is a C*-subalgebra of \(\mathcal{N} T(A, \alpha)\) [7, Proposition 4.2.6].

**Theorem 1.2.** [7] Let \(\alpha : \mathbb{Z}^n_{\mathbb{N}} \to \text{End}(A)\) be a C*-dynamical system. Then an isometric Nica covariant pair \((\pi, V)\) defines a faithful representation of \(\mathcal{N} T(A, \alpha)\) if and only if it admits a gauge action and
\[
I_{(\pi, V)} := \{ a \in A \mid \pi(a) \in \mathcal{B}_{[0, \infty]} \} = (0),
\]
where \(\mathcal{B}_{[0, \infty]} = \overline{\text{span}}\{ V_x \pi(a) V_x^* \mid a \in A, x \neq 0 \} \).

1.2. The Cuntz-Nica-Pimsner algebra. For each \(x \in \mathbb{Z}^n_{\mathbb{N}} \setminus \{0\}\), consider the ideal \((\bigcap_{x \in \text{supp } x} \ker \alpha_1)^\perp\) and let
\[
I_x = \bigcap_{y \in x^\perp} \alpha_y^{-1} \left( \left( \bigcap_{x \in \text{supp } x} \ker \alpha_1 \right)^\perp \right),
\]
which is an \(\alpha_y\)-invariant ideal for all \(y \in x^\perp\). In particular, \(I_0 = \{0\}\), and \(I_x = I_1 = (\bigcap_{i=1}^n \ker \alpha_1)^\perp\) for all \(x \geq \underline{1}\). The definition of \(I_x\) is inspired by the following observation.

**Proposition 1.3.** [7] Let \((\pi, V)\) be an isometric Nica covariant pair for a C*-dynamical system \(\alpha : \mathbb{Z}^n_{\mathbb{N}} \to \text{End}(A)\) such that \(\pi\) is a faithful representation of \(A\). If \(\sum_{0 < w < x} V_w \pi(a_w) V_w^* = 0\) then \(a_0 \in I_x\).

One has the freedom to define a family of universal objects determined by a system of generators and relations. The interesting part in any case is to prove that the object exists and it is not trivial. For \(\alpha : \mathbb{Z}^n_{\mathbb{N}} \to \text{End}(A)\) we give the following definition.
Definition 1.4. An isometric Nica covariant pair \((\pi, U)\) is called Cuntz-Nica covariant if
\[
\pi(a) \cdot \prod_{i \in \supp Z} (I - U_i U_i^*) = 0, \text{ for all } a \in I_Z.
\]
The universal C*-algebra \(\mathcal{N}O(A, \alpha)\) generated by \(U_p \pi(a)\) for \(p \in \mathbb{Z}_+\) and \(a \in A\) under the Cuntz-Nica covariant pairs \((\pi, U)\) is called the Cuntz-Nica-Pimsner C*-algebra.

Evidently if \(\alpha\) is unital then \(1 \in A\) is the unit for \(\mathcal{N}O(A, \alpha)\). A key ingredient that we used in [7] for proving the existence of \(\mathcal{N}O(A, \alpha)\) is a tail adding technique. With this process we pass from a possibly non-injective system \(\alpha: \mathbb{Z}_+ \to \text{End}(A)\) to an automorphic system \(\beta: \mathbb{Z} \to \text{Aut}(\bar{B})\). We review this construction.

If \(\alpha: \mathbb{Z}_+ \to \text{End}(A)\) is injective then it admits a minimal automorphic extension \(\tilde{\alpha}: \mathbb{Z} \to \text{Aut}(\bar{A})\). Here \(\bar{A}\) is the direct limit C*-algebra associated with the connecting *-homomorphisms \(\alpha_x: A_x \to A_{x+y}\) for \(A_y := A\) and \(x, y \in \mathbb{Z}_+\). The *-automorphism \(\tilde{\alpha}_w\) is defined with respect to the diagram

\[
\begin{array}{ccc}
A_y & \xrightarrow{\alpha_x} & A_{x+y} \\
\downarrow{\alpha_w} & & \downarrow{\tilde{\alpha}_w} \\
A_z & \xrightarrow{\alpha_z} & A_{z+y}
\end{array}
\]

for every \(w \in \mathbb{Z}_+\).

Now suppose that \(\alpha: \mathbb{Z}_+ \to \text{End}(A)\) is not injective. First suppose that \(\alpha\) is unital. Define \(B_x := A/I_x\) for \(x \in \mathbb{Z}_+\) and let \(q_x\) be the quotient map of \(A\) onto \(B_x\). Set \(B = \bigoplus_{x \in \mathbb{Z}_+} B_x\). A typical element of \(B\) is denoted by

\[
b = \sum_{x \in \mathbb{Z}_+} q_x(a_x) \otimes e_x,
\]
where \(a_x \in A\). Observe that \(I_x\) is invariant under \(\alpha_1\) when \(x_i = 0\), thus we can define the action of \(\mathbb{Z}_+\) on \(B\) by

\[
\beta_i(q_x(a) \otimes e_x) = \begin{cases} 
q_x \alpha_i(a) \otimes e_x + q_{x+1}(a) \otimes e_{x+1} & \text{if } x_i = 0, \\
q_x(a) \otimes e_{x+1} & \text{for } x_i \geq 1.
\end{cases}
\]

It is clear that the compression of \(\beta: \mathbb{Z}_+ \to \text{End}(B)\) to the \(i\)-th co-ordinate \(A\) is the system \(\alpha: \mathbb{Z}_+ \to \text{End}(A)\). So \(\beta: \mathbb{Z}_+ \to \text{End}(B)\) is a dilation of \(\alpha: \mathbb{Z}_+ \to \text{End}(A)\). Furthermore \(\beta: \mathbb{Z}_+ \to \text{End}(B)\) is injective and it admits the minimal automorphic extension \(\tilde{\beta}: \mathbb{Z} \to \text{Aut}(\bar{B})\).

If there is at least one \(\alpha_i\) that is not unital then form the unitalization \(\alpha^{(1)}: \mathbb{Z}_+ \to \text{End}(A^{(1)})\) of the system where \(A^{(1)} = A + C\). We consider \(A^{(1)}\) even when \(A\) has a unit, but \(\alpha^{(1)} = \alpha\) when the system is unital. Let \(\beta^{(1)}: \mathbb{Z}_+ \to \text{End}(B^{(1)})\) be the dilation of \(\alpha^{(1)}: \mathbb{Z}_+ \to \text{End}(A^{(1)})\). This is not the unitization of \(\beta: \mathbb{Z}_+ \to \text{End}(B)\) but the ideals \(I_x\) turn out to be the same,
i.e. \( B^{(1)} = \sum_{z \in \mathbb{Z}^n_+} B^1_{z} \) where \( B^1_z = A^{(1)}/I_z \). Further, let \( \beta^{(1)} : \mathbb{Z}^n_+ \to \text{End}(B^{(1)}) \) be the automorphic extension. Since \( \beta^{(1)} : \mathbb{Z}^n_+ \to \text{End}(B^{(1)}) \) extends \( \beta : \mathbb{Z}^n_+ \to \text{End}(B) \) it follows by the gauge invariant uniqueness theorem for crossed products that \( \tilde{B} \times_{\beta} \mathbb{Z}^n \) embeds canonically and isometrically inside \( B^{(1)} \times_{\beta^{(1)}} \mathbb{Z}^n \). Let \( \pi : B^{(1)} \to B(H) \) be a faithful representation and let \( (\tilde{\pi}, U) \) be the left regular pair that defines a faithful representation of \( \tilde{B} \times_{\beta} \mathbb{Z}^n \) on \( H \otimes l^2(\mathbb{Z}^n) \). Let \( 1 \) be the identity of the unitization \( A^{(1)} \) and let \( p = \tilde{\pi}(1 \otimes e_0) \). Then the pair \( (\tilde{\pi}\big|_A, Up) \) defines a faithful representation for \( \mathcal{N}\mathcal{O}(A, \alpha) \). In particular we have the following.

**Theorem 1.5.** [7] Let \( \alpha : \mathbb{Z}^n_+ \to \text{End}(A) \) be a C*-dynamical system, and let \( \beta : \mathbb{Z}^n \to \text{Aut}(B) \) be the automorphic dilation. Then \( \mathcal{N}\mathcal{O}(A, \alpha) \) is a full corner of \( \tilde{B} \times_{\beta} \mathbb{Z}^n \) (by the projection \( p \) above). In particular, if \( \alpha \) is injective then \( \mathcal{N}\mathcal{O}(A, \alpha) \simeq \hat{A} \rtimes_{\hat{\alpha}} \mathbb{Z}^n \).

By [7] the ambient C*-algebra \( A \) embeds isometrically inside \( \mathcal{N}\mathcal{O}(A, \alpha) \) therefore we omit writing the symbol for that representation. Recall that \( \mathcal{N}\mathcal{O}(A, \alpha) \) admits a gauge action \( \{ \gamma_z \}_{z \in \mathbb{T}^s} \) whose cores will be denoted by

\[ B_{[x,y]} := \text{span}\{ U_x a U_x^* \mid a \in A, x \leq \underline{w} \leq y \}. \]

We will also use the notation

\[ B_{[x,\infty]_1,\ldots,\infty_d} = \text{span}\{ U_x a U_x^* \mid a \in A, \text{supp } x \subseteq \{1, \ldots, d\} \}, \]

for some \( d \leq n \). The gauge invariant uniqueness theorem follows.

**Theorem 1.6.** [7] Let \( \alpha : \mathbb{Z}^n_+ \to \text{End}(A) \) be a C*-dynamical system. Then an (isometric) Cuntz-Nica covariant pair \( (\pi, U) \) of \( \mathcal{N}\mathcal{O}(A, \alpha) \) defines a faithful representation if and only if it admits a gauge action and \( \pi \) is faithful.

**Proposition 1.7.** Let \( \alpha : \mathbb{Z}^n_+ \to \text{End}(A) \) be a C*-dynamical system. Then the following are equivalent:

(i) the system is injective;

(ii) \( I_i = A \) for all \( i = 1, \ldots, n \);

(iii) \( U_i \) is a unitary for all \( i = 1, \ldots, n \).

**Proof.** If item (i) holds then \( \ker \alpha_i^\perp = A \) for all \( i = 1, \ldots, n \), hence \( I_i = A \) for all \( i = 1, \ldots, n \). Consequently item (ii) holds. Moreover by Theorem 1.5 we get that \( \mathcal{N}\mathcal{O}(A, \alpha) \simeq \hat{A} \rtimes_{\hat{\alpha}} \mathbb{Z}^n \) so that the \( U_i \) are unitaries.

If item (ii) holds then \( \alpha_i \) is injective, since \( I_i \subseteq \ker \alpha_i^\perp \) for all \( i = 1, \ldots, n \). Consequently item (i) holds.

Now assume that item (iii) holds and let \( a \in \ker \alpha_i \). Then covariance implies \( a U_i = U_i \alpha_i(a) = 0 \). However \( U_i \) is assumed unitary, thus \( a = 0 \) and item (i) follows. \( \blacksquare \)
1.3. **Exactness and Nuclearity.** The reader is addressed to [4] for full details on the subject. For the convenience of the reader, let us state here the results that we will use.

**Lemma 1.8.** (i) [13, Proposition A.6] Suppose that the following diagram with exact rows is commutative

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I & \rightarrow & A & \rightarrow & B & \rightarrow & 0 \\
\varphi_0 & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & I' & \rightarrow & A' & \rightarrow & B & \rightarrow & 0
\end{array}
\]

and \(\varphi\) is injective. Then \(\varphi\) is nuclear if and only if both \(B\) and \(\varphi_0\) are nuclear.

(ii) [9, Proposition 2], [13, Proposition A.13] Let \(\gamma: G \curvearrowright A\) be an action of a compact group \(G\) on a \(C^*\)-algebra \(A\). Then \(A\) is exact (resp. nuclear) if and only if the fixed point algebra \(A^\gamma\) is exact (resp. nuclear).

**Lemma 1.9.** Let the \(C^*\)-algebras \(B_{[n,n]}\) for \(n \in \mathbb{Z}_+\) with \(B_{[0,0]} = A\). Define \(B_{[n,n]} = \sum_{k=0}^n B_{[k,k]}\) and suppose that \(B_{[1,n]} \triangleleft B_{[0,n]}\) (hence the \(B_{[n,n]}\) are \(C^*\)-algebras). Moreover let the inductive limit \(C^*\)-algebra \(B_{[0,\infty]} = \bigcup_{n \geq 0} B_{[n,n]}\) and its ideal \(B_{[0,\infty]} = \bigcup_{n \geq 1} B_{[1,n]}\). If the embeddings \(A \hookrightarrow B_{[0,\infty]}\) and \(B_{[1,n]} \hookrightarrow B_{[0,\infty]}\) are nuclear for all \(n\), and there is an ideal \(I \subset A\) such that

\[
A/I \simeq B_{[0,n]}/B_{[1,n]} \simeq B_{[0,\infty]}/B_{[0,\infty]} \quad \text{for all } n,
\]

then \(B_{[0,\infty]}\) is nuclear.

**Proof.** For \(n = 0\) we obtain the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & I & \rightarrow & A & \rightarrow & A/I & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B_{[0,\infty]} & \rightarrow & B_{[0,\infty]} & \rightarrow & B_{[0,\infty]}/B_{[0,\infty]} & \rightarrow & 0
\end{array}
\]

which by [13, Proposition A.6] asserts that \(A/I \simeq B_{[0,\infty]}/B_{[0,\infty]}\) is nuclear. Applying [13, Proposition A.6] again to the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & B_{[1,n]} & \rightarrow & B_{[0,n]} & \rightarrow & B_{[0,n]}/B_{[1,n]} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & B_{[0,\infty]} & \rightarrow & B_{[0,\infty]} & \rightarrow & B_{[0,\infty]}/B_{[0,\infty]} & \rightarrow & 0
\end{array}
\]

shows that the embedding of \(B_{[0,n]} \hookrightarrow B_{[0,\infty]}\) is nuclear. \(\blacksquare\)

1.4. **Kubo-Martin-Schwinger states.** We review elements of the theory of KMS states. The reader is addressed to [2, 3] for full details. Let \(\sigma: \mathbb{R} \rightarrow \text{Aut}(A)\) be an action on a \(C^*\)-algebra \(A\). For \(x \in A\) define the sequence

\[
x_m = \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \sigma_t(x) e^{-mt^2} dt.
\]
This sequence converges to \( x \) and the function \( t \mapsto \sigma_t(x_m) \) extends to the entire analytic function

\[
z \mapsto f_{x_m}(z) = \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \sigma_t(x)e^{-m(t-z)^2} \, dt.
\]

The elements \( x_m \) for \( x \in \mathcal{A} \) then form the dense \( * \)-subalgebra \( \mathcal{A}_{an} \) of analytic elements in \( \mathcal{A} \) [2, Proposition 2.5.22]. Moreover \( \mathcal{A}_{an} \) is \( \sigma \)-invariant. A state \( \psi \) of \( \mathcal{A} \) is called a \((\sigma, \beta)\)-KMS state if it satisfies

\[
\psi(ab) = \psi(b\sigma_{i\beta}(a)),
\]

for all \( a, b \) in a norm-dense \( \sigma \)-invariant \( * \)-subalgebra of \( \mathcal{A}_{an} \). For \( \beta = 0 \) this amounts to tracial states. (When \( \sigma \) is trivial, then \( \mathcal{A}_{an} = \mathcal{A} \) and the KMS property is again equivalent to \( \tau \) being a tracial state.)

When \( \beta \neq 0 \), the KMS states give rise to particular continuous functions. More precisely for \( \beta > 0 \) let \( D = \{ z \in \mathbb{Z} \mid 0 < \text{Im}(z) < \beta \} \). Then a state \( \psi \) is a \((\sigma, \beta)\)-KMS state if and only if for any pair \( a, b \in \mathcal{A} \) there exists a complex function \( F_{a,b} \) that is analytic on \( D \) and continuous (hence bounded) on \( D \) such that

\[
F_{a,b}(t) = \psi(a\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \psi(\sigma_t(b)a),
\]

for all \( t \in \mathbb{R} \) [3, Proposition 5.3.7]. A similar result is obtained when \( \beta < 0 \) for \( D = \{ z \in \mathbb{Z} \mid \beta < \text{Im}(z) < 0 \} \).

Following [17] we adopt the distinction between ground states and \( \text{KMS}_\infty \) states. A state \( \psi \) of a C*-algebra \( \mathcal{A} \) is called a ground state if the function \( z \mapsto \psi(a\sigma_z(b)) \) is bounded on \( \{ z \in \mathbb{C} \mid \text{Im}z > 0 \} \) for all \( a, b \) inside a dense analytic subset \( \mathcal{A}_{an} \) of \( \mathcal{A} \). A state \( \psi \) of \( \mathcal{N}\mathcal{T}(\mathcal{A}, \alpha) \) is called a KMS\(_\infty\) state if it is the w*-limit of \((\sigma, \beta)\)-KMS states as \( \beta \to \infty \).

When there is a multivariable action \( \sigma : \mathbb{R}^n \to \text{Aut}(\mathcal{A}) \) one may wish to examine KMS states along an induced action \( \sigma F : \mathbb{R} \to \text{Aut}(\mathcal{A}) \) for a continuous function \( F : \mathbb{R} \to \mathbb{R}^n \). Since the action must be by automorphisms then necessarily \( F(t) = (\lambda_1 t, \ldots, \lambda_n t) \) for some \( \lambda_k \in \mathbb{R} \).

2. Exactness and Nuclearity

2.1. The Toeplitz-Nica-Pimsner algebra. We begin by giving necessary and sufficient conditions for \( \mathcal{N}\mathcal{T}(\mathcal{A}, \alpha) \) to be exact.

**Theorem 2.1.** Let \( \alpha : \mathbb{Z}^n_+ \to \text{End}(\mathcal{A}) \) be a C*-dynamical system. Then the following are equivalent:

(i) \( \mathcal{A} \) is exact;

(ii) the fixed point algebra \( \mathcal{N}\mathcal{T}(\mathcal{A}, \alpha)^\gamma \) is exact;

(iii) \( \mathcal{N}\mathcal{T}(\mathcal{A}, \alpha) \) is exact.

**Proof.** Obviously item (iii) implies item (i), and by [13, Proposition A.13] we have the equivalence of items (ii) and (iii). Therefore it suffices to show that exactness of \( \mathcal{A} \) implies exactness of \( \mathcal{N}\mathcal{T}(\mathcal{A}, \alpha)^\gamma \). To accomplish this we use the faithful Fock representation \((\tilde{\pi}, V)\) of \( \mathcal{N}\mathcal{T}(\mathcal{A}, \alpha) \) given by a faithful
representation $\pi: A \to B(H)$. Recall that the fixed point algebra $\mathcal{N}\mathcal{T}(A, \alpha)^\gamma$ can be described as the inductive limit of the C*-subalgebras

$$B_{[0,k]} = \text{span}\{V_m \pi(a)V_m^* | a \in A, 0 \leq m \leq k \cdot 1\}.$$ 

The proof will be completed once we show that every $B_{[0,k]}$ is exact.

**Claim.** The C*-algebra $B_{[0,k]} = \text{span}\{V_l \pi(a)V_l^* | a \in A, 0 \leq l \leq k\}$ is exact.

**Proof of the Claim.** By assumption $A$ is exact, and note that $B_{[0,1]} = A + \mathcal{B}_1$, where $\mathcal{B}_1$ is an ideal of $B_{[0,1]}$. Moreover

$$B_1 = \text{span}\{V_1 \pi(a)V_1^* | a \in A\} = V_1 \pi(A)V_1^*,$$

thus $B_1$ is exact. If $A \cap B_1 = (0)$ then $B_{[0,1]}$ will be exact. This is immediate since $B_1 \subseteq B_{[0,\infty]}$ and $A \cap B_{[0,\infty]} = (0)$.

Now assume that $B_{[0,l]}$ is exact and recall that $B_{[0,(l+1)-1]} = B_{[0,l]} + B_{(l+1)-1}$, where $B_{(l+1)-1}$ is an ideal of $B_{[0,(l+1)-1]}$. Once more

$$B_{(l+1)-1} = \text{span}\{V_{(l+1)-1} \pi(a)V_{(l+1)-1}^* | a \in A\} = V_{(l+1)-1} \pi(A)V_{(l+1)-1}^*,$$

is exact. It suffices to show that $B_{[0,l]} \cap B_{(l+1)-1} = (0)$. To this end let

$$X = V_{(l+1)-1} \pi(a)V_{(l+1)-1}^* \in B_{[0,l]} + B_{(l+1)-1},$$

thus there are $a_m \in A$ such that

$$V_{(l+1)-1} \pi(a)V_{(l+1)-1}^* = \sum_{m=0}^l V_{m-1} \pi(a_m)V_{m-1}^*.$$

Then for every $\xi \in H$ we get that

$$X\xi \otimes e_0 = \sum_{m=0}^l V_{m-1} \pi(a_m)V_{m-1}^* \xi \otimes e_0 = \pi(a_0)\xi \otimes e_0,$$

and

$$X\xi \otimes e_0 = V_{(l+1)-1} \pi(a)V_{(l+1)-1}^* \xi \otimes e_0 = 0.$$ 

Therefore $a_0 = 0$. By induction on the vectors $\xi \otimes e_{m-1}$ we get that $X = 0$, which completes the proof of the claim.

Assume that $B_{[0,x]} = \text{span}\{V_y \pi(a)V_y^* | a \in A, 0 \leq y \leq x\}$ is exact for $x = k \cdot 1 + \cdots + k \cdot (i-1)$, where the support of $i\cdot 1$ is $\{i-1\}$. We will show that

$$B_{[0,x+k-1]} = \text{span}\{V_y \pi(a)V_y^* | a \in A, 0 \leq y \leq x + k \cdot i\}$$

is exact. Let us describe here the method that we will use. By assumption the support of $B_{[0,x]}$ is a square on $(i-1)$-dimensions of size $k$. Seen inside the $i$-th dimensional world it becomes a line segment (a hyperplane) of “length” $k$. In order to enlarge it to a square on $i$-dimensions of size $k$ we add parallel “rows” of length $k$ one after the other to the direction of $i$. We use induction on the rows that we add on the $i$-th direction. Thus, suppose that

$$B_{[0,x+l-1]} = \text{span}\{V_y \pi(a)V_y^* | a \in A, 0 \leq y \leq x + l \cdot i\}$$
is exact. We will show that
\[ B_{[0, \underline{x} + (l+1) \cdot i]} = \text{span}\{V_y \tilde{\varpi}(a) V_y^* \mid a \in A, 0 \leq y \leq \underline{x} + (l+1) \cdot i\} \]
is exact. If \( \underline{z} \in [0, \underline{x} + l \cdot i] \) and \( \underline{w} \in [(l+1) \cdot i, \underline{x} + (l+1) \cdot i] \) then
\[ \underline{z} \vee \underline{w} \in [(l+1) \cdot i, \underline{x} + (l+1) \cdot i]. \]
Therefore \( B_{[(l+1) \cdot i, \underline{x} + (l+1) \cdot i]} \) is an ideal of \( B_{[0, \underline{x} + (l+1) \cdot i]} \). In particular we may write
\[ B_{[0, \underline{x} + (l+1) \cdot i]} = B_{[0, \underline{x} + i]} + B_{[(l+1) \cdot i, \underline{x} + (l+1) \cdot i]}, \]
since the set
\[ \{ y \in \mathbb{Z}_+^n \mid 0 \leq y_1, \ldots, y_{i-1} \leq k, 0 \leq y_i \leq l + 1, y_{i+1} = \cdots = y_n = 0 \} \]
decomposes into the disjoint sets
\[ \{ y \in \mathbb{Z}_+^n \mid 0 \leq y_1, \ldots, y_{i-1} \leq k, 0 \leq y_i \leq l, y_{i+1} = \cdots = y_n = 0 \} \]
and
\[ \{ y \in \mathbb{Z}_+^n \mid 0 \leq y_1, \ldots, y_{i-1} \leq k, y_i = l + 1, y_{i+1} = \cdots = y_n = 0 \}. \]
It remains to show that \( B_{[0, \underline{x} + l \cdot i]} \cap B_{[(l+1) \cdot i, \underline{x} + (l+1) \cdot i]} = (0) \). However this follows in exactly the same way as in the claim since \( B_{[(l+1) \cdot i, \underline{x} + (l+1) \cdot i]} |_{H \otimes e_{\underline{z}}} = 0 \), for all \( \underline{z} \in [0, \underline{x} + l \cdot i] \). Indeed every \( \underline{z} \in [0, \underline{x} + l \cdot i] \) cannot be larger than any element inside \([(l+1) \cdot i, \underline{x} + (l+1) \cdot i] \).

Secondly we give giving necessary and sufficient conditions for \( \mathcal{NT}(A, \alpha) \)
to be nuclear.

**Theorem 2.2.** Let \( \alpha : \mathbb{Z}_+^n \to \text{End}(A) \) be a C*-dynamical system. Then the following are equivalent:

(i) \( A \) is nuclear;
(ii) the fixed point algebra \( \mathcal{NT}(A, \alpha) \) is nuclear;
(iii) \( \mathcal{NT}(A, \alpha) \) is nuclear.

**Proof.** For the implications \([(i) \Rightarrow (ii) \Leftrightarrow (iii)]\) proceed as in the proof of Theorem 2.1 by replacing exactness with nuclearity. It suffices to show that item (ii) implies item (i). To this end note that for \( X = \sum_{z \in \mathbb{Z}^n} \mathbb{Z} \tilde{\varpi}(a_z) V_z^* \) and \( \xi \in H \) we obtain \( X \xi \otimes e_0 = \pi(a_z) \xi \otimes e_0 \in H \otimes e_0 \). However the subspace \( H \otimes e_0 \) is reducing for \( \mathcal{NT}(A, \alpha)^\gamma \), hence the projection on \( H \otimes e_0 \) defines a \( * \)-homomorphism onto \( \tilde{\varpi}(A) \). Thus \( A \) is nuclear, since nuclearity passes to quotients.

2.2. The Cuntz-Nica-Pimsner algebra. Now we pass to the examination of exactness and nuclearity of \( \mathcal{NO}(A, \alpha) \).

**Theorem 2.3.** Let \( \alpha : \mathbb{Z}_+^n \to \text{End}(A) \) be a C*-dynamical system. Then the following are equivalent:

(i) \( A \) is exact;
(ii) the fixed point algebra \( \mathcal{NO}(A, \alpha) \) is exact;
(iii) \( \mathcal{NO}(A, \alpha) \) is exact.
Proof. If item (i) holds then $\mathcal{NT}(A,\alpha)$ and $\mathcal{NT}(A,\alpha)\gamma$ are exact by Theorem 2.1. Since $\mathcal{NO}(A,\alpha)$ and $\mathcal{NO}(A,\alpha)^\gamma$ are quotients of $\mathcal{NT}(A,\alpha)$ and $\mathcal{NT}(A,\alpha)^\gamma$ respectively, we obtain that items (ii) and (iii) hold. Moreover item (ii) implies item (iii) by [13, Proposition A.13], and item (iii) implies item (i) since $A$ is represented faithfully inside $\mathcal{NO}(A,\alpha)$. □

Next we focus on nuclearity of $\mathcal{NO}(A,\alpha)$. We isolate the injective case. Recall that if $\alpha: \mathbb{Z}_{n^+} \to \text{End}(A)$ is injective then it can be extended to the automorphic system $\tilde{\alpha}: \mathbb{Z}^n \to \text{Aut}(\tilde{A})$. In this case we have that $\mathcal{NO}(A,\alpha)^\gamma = \tilde{A}$ and $\mathcal{NO}(A,\alpha) \simeq \tilde{A} \rtimes_{\tilde{\alpha}} \mathbb{Z}^n$.

Proposition 2.4. Let $\alpha: \mathbb{Z}_{n^+} \to \text{End}(A)$ be an injective C*-dynamical system. Then the following are equivalent:

(i) the embedding $A \hookrightarrow \tilde{A}$ is nuclear;
(ii) the fixed point algebra $\tilde{A} = \mathcal{NO}(A,\alpha)^\gamma$ is nuclear;
(iii) $\mathcal{NO}(A,\alpha)$ is nuclear.

Proof. The equivalence of items (ii) and (iii) follows by [13, Proposition A.13]. Also it is immediate that item (ii) implies item (i). In order to show that item (i) implies item (ii) it suffices to show that every embedding $A_\underline{x} = w_\underline{x}(A) \hookrightarrow \tilde{A}$ is nuclear. To this end suppose that the embedding $A \hookrightarrow \tilde{A}$ is nuclear by an approximation $(\psi_n, M_{k(n)}, \varphi_n)$ and fix $\underline{x} \in \mathbb{Z}_{n^+}$. By construction of the system $\tilde{\alpha}: \mathbb{Z}^n \to \text{Aut}(\tilde{A})$ we get that $\tilde{\alpha}_\underline{x}(A_\underline{x}) = A_0 = A$. Therefore the approximation $(\alpha_\underline{x} \psi_n, M_{k(n)}, \varphi_n \alpha_\underline{x})$ implies that the *-homomorphism $\tilde{\alpha}_\underline{x} \alpha_\underline{x}, A_\underline{x} \equiv \text{id}_{A_\underline{x}}: A_\underline{x} \hookrightarrow \tilde{A}$ is nuclear. □

Remark 2.5. The previous result holds also for injective C*-dynamical systems over spanning cones $P$ (see [7] for the pertinent definitions).

Theorem 2.6. Let $\alpha: \mathbb{Z}_{n^+} \to \text{End}(A)$ be a C*-dynamical system. Let the C*-dynamical systems $\beta: \mathbb{Z}_{n^+} \to \text{End}(B)$ and $\tilde{\beta}: \tilde{B} \to \text{Aut}(\tilde{B})$ be as in subsection 1.2. Then the following are equivalent:

(i) the embeddings $B_\underline{x} \hookrightarrow \tilde{B}$ are nuclear for all $\underline{x} \in \mathbb{Z}_{n^+}$;
(ii) the embedding $B \hookrightarrow \tilde{B}$ is nuclear;
(iii) $\tilde{B}$ is nuclear;
(iv) $\tilde{B} \rtimes_{\tilde{\beta}} \mathbb{Z}^n$ is nuclear;
(v) $\mathcal{NO}(A,\alpha)$ is nuclear.

Proof. Since the $B_\underline{x}$ are orthogonal ideals of $B$, and $B$ is an inductive limit of $\bigcup_{\underline{x} \leq k} 1 B_\underline{x}$ then the embedding $B \hookrightarrow \tilde{B}$ is nuclear. The equivalences of items (ii), (iii), and (iv) follow from Proposition 2.4. Moreover since all the $B_\underline{x}$ are C*-subalgebras of $\tilde{B}$, item (iii) implies item (i). The equivalence of items (iv) and (v) is immediate since $\mathcal{NO}(A,\alpha)$ is strong Morita equivalent to $\tilde{B} \rtimes_{\tilde{\beta}} \mathbb{Z}^n \simeq \mathcal{NO}(\tilde{B}, \tilde{\beta})$. □
inductively that the C*-algebras \( N^0 \) that the fixed point algebra \( (\alpha_1 = \cdots = \alpha_n = \id_A) \) has shown that the embedding \( B \hookrightarrow B_{0,\infty} \) implies then that \( B \) are nuclear. We show this inductively. By assumption \( A_n = B \) is nuclear for \( n > 0 \) and \( A_n = D \) is a non-nuclear C*-algebra of \( B \) for all \( n \leq 0 \). Let \( \alpha_1 \) be the forward shift on \( A \) and \( \alpha_2 = \cdots = \alpha_n = \id_A \). Then the fixed point algebra \( N^0(A,\alpha) \) coincides with \( B_{0,\infty} \) which is the direct limit of \( A \) by \( \alpha_1 \), thus coincides with \( \oplus_{n \in \mathbb{Z}} B \). The latter is nuclear hence \( N^0(A,\alpha) \) is nuclear. However by construction \( A \) is not nuclear.

We give a second condition that implies the nuclearity of \( N^0(A,\alpha) \).

**Theorem 2.8.** Let \( \alpha: \mathbb{Z}^n_+ \to \text{End}(A) \) be a C*-dynamical system. If the embedding \( A \hookrightarrow B_{\[0,\infty\]} \) is nuclear for some \( i \in \{1, \ldots, n\} \) then \( N^0(A,\alpha) \) is nuclear.

**Proof.** Without loss of generality we assume that \( i = 1 \). We will show that the fixed point algebra \( N^0(A,\alpha)^\gamma \) is nuclear. To this end we will show inductively that the C*-algebras

\[
B_{\[0,\infty+1\]} = \text{span}\{U_\alpha a U_\alpha^* \mid \text{supp } \alpha = \{1, \ldots, m\}, a \in A\}
\]

are nuclear.

We begin with \( m = 1 \). First we show that

\[
A/I_1 \simeq B_{\[0,1\]}/B_{\[0,1\]} \simeq B_{\[0,\infty\]}\vert_{B_{\[0,\infty\]}},
\]

for all \( l > 0 \). Indeed note that \( B_{\[0,1\]}/B_{\[0,1\]} \simeq A/A \cap B_{\[0,1\]} \) and recall that \( I_1 \subseteq B_1 \subseteq B_{\[0,1\]} \subseteq B_{\[0,\infty\]} \). Now let \( a \in A \cap B_{\[0,1\]} \) (resp. in \( A \cap B_{\[0,\infty\]} \)) and recall that \( U_1 U_1^* \) is an identity for \( B_{\[0,1\]} \) (resp. \( B_{\[0,\infty\]} \)). Thus \( a = a U_1 U_1^* \) which implies that \( a (I - U_1 U_1^*) = 0 \). Hence \( a \in I_1 \). Therefore \( A \cap B_{\[0,1\]} = I_1 \) (resp. \( A \cap B_{\[0,\infty\]} = I_1 \)).

We aim to apply Lemma 1.9 and so we need to verify that the embeddings \( B_{\[0,1\]} \hookrightarrow B_{\[0,\infty\]} \) are nuclear. We show this inductively. By assumption suppose that \( (\psi_n, M_{k(n)}, \varphi_n) \) is an approximation of \( A \hookrightarrow B_{\[0,\infty\]} \). Then \( (\text{ad}_{U_1} \psi_n, M_{k(n)}, \varphi_n \text{ad}_{U_1}^*) \) is an approximation of \( B_{\[0,1\]} = U_1 A U_1^* \hookrightarrow B_{\[0,\infty\]} \). Therefore applying [13, Proposition A.6] to the following diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & B_{\[1\]} & \rightarrow & B_{\[0,1\]} & \rightarrow & B_{\[0,1\]}/B_{\[1\]} & \rightarrow & 0 \\
| & | & | & | & | & | & | & |
0 & \rightarrow & B_{\[0,\infty\]} & \rightarrow & B_{\[0,\infty\]} & \rightarrow & B_{\[0,\infty\]}/B_{\[0,\infty\]} & \rightarrow & 0
\end{array}
\]

yields that the embedding \( B_{\[0,1\]} \hookrightarrow B_{\[0,\infty\]} \) is nuclear. Now assume that we have shown that the embedding \( B_{\[0,1\]} \hookrightarrow B_{\[0,\infty\]} \) is nuclear by an approximation \( (\psi_n, M_{k(n)}, \varphi_n) \). Then the approximation \( (\text{ad}_{U_1} \psi_n, M_{k(n)}, \varphi_n \text{ad}_{U_1}^*) \) shows that the embedding \( B_{\[0,1\]} \hookrightarrow B_{\[0,\infty\]} \) is nuclear. Lemma 1.9 implies then that \( B_{\[0,\infty\]} \) is nuclear.

**Remark 2.7.** It is evident that when \( A \) is nuclear then \( N^0(A,\alpha) \) is nuclear. However the converse is not true. Katsura constructs such a counterexample for the one-variable case in [13, Example 7.7]. The same construction can be extended to give a multivariable counterexample. That is, let \( A = \bigoplus_{n \in \mathbb{Z}} A_n \) where \( A_n = B \) is nuclear for \( n > 0 \) and \( A_n = D \) is a non-nuclear C*-subalgebra of \( B \) for all \( n \leq 0 \). Let \( \alpha_1 \) be the forward shift on \( A \) and \( \alpha_2 = \cdots = \alpha_n = \id_A \). Then the fixed point algebra \( N^0(A,\alpha)^\gamma \) coincides with \( B_{\[0,\infty\]} \) which is the direct limit of \( A \) by \( \alpha_1 \), thus coincides with \( \bigoplus_{n \in \mathbb{Z}} B \). The latter is nuclear hence \( N^0(A,\alpha) \) is nuclear. However by construction \( A \) is not nuclear.
For \( m = 2 \) we repeat the same arguments by substituting \( A \) with the C*-subalgebra \( B_{[0, \infty]} \) of \( B_{[0, \infty]1+\infty} \). Note that \( B_{[0, \infty]} \) is nuclear now, hence the embedding \( B_{[0, \infty]} \hookrightarrow B_{[0, \infty]1+\infty} \) is nuclear. A moment’s thought suggests that \( I_1 \) will have to be substituted by the ideal
\[ \{ X \in B_{[0, \infty]} \mid X \cdot (I - U_2 U_2^*) = 0 \}, \]

since \( U_2 U_2^* \) is an identity for the C*-algebras
\[ \text{span}\{ U_{l, 2} X U_{l, 2}^* \mid X \in B_{[0, \infty]}, 0 \neq l \leq m \}, \]

for all \( m \in \mathbb{Z}_+ \). Induction completes the proof.

3. KMS states on Nica-Pimsner algebras

In this section we work under the assumption that the system \( \alpha: \mathbb{Z}_+ \to \text{End}(A) \) is unital. Consequently the unit of \( A \) is also a unit for \( \mathcal{N} \mathcal{T}(A, \alpha) \) and \( \mathcal{N} \mathcal{O}(A, \alpha) \).

3.1. The Toeplitz-Nica-Pimsner algebra. Let \( \{ \gamma_\lambda \}_{\lambda \in \mathbb{R}^n} \) be the gauge action on \( \mathcal{N} \mathcal{T}(A, \alpha) \). For a fixed \( \lambda \in \mathbb{R}^n \setminus \{ \underline{0} \} \) let the action
\[ \sigma: \mathbb{R} \to \text{Aut}(\mathcal{N} \mathcal{T}(A, \alpha)) : t \mapsto \gamma(e^{i\lambda_1 t}, \ldots, e^{i\lambda_n t}) \].

The monomials of the form \( V_x a V_y^* \) span a dense \(*\)-subalgebra of analytic elements of \( \mathcal{N} \mathcal{T}(A, \alpha) \) since the function
\[ \mathbb{R} \to \mathcal{N} \mathcal{T}(A, \alpha) : t \mapsto \sigma_t(V_x a V_y^*) = e^{i(x-y)\lambda} V_x a V_y^* \]
is analytically continued to the entire function
\[ \mathbb{C} \to \mathcal{N} \mathcal{T}(A, \alpha) : z \mapsto e^{i(x-y)\lambda} V_x a V_y^* \].

For \( \beta \in \mathbb{R} \) the \((\sigma, \beta)\)-KMS condition is then translated into
\[ \psi(V_x a V_y^* V_z b V_w^*) = e^{-\langle x-y, \beta \rangle} \psi(V_z b V_w^* V_x a V_y^*) \].

As we will soon see there are no KMS states when \( \lambda_k \beta < 0 \) for some \( k = 1, \ldots, n \). On the other hand there is a characterization of the KMS states when \( \lambda_k \beta > 0 \) for all \( k = 1, \ldots, n \). We leave the case where some \( \lambda_k \) are zeroes for later. To simplify notation in our computations we will frequently write \( \underline{\beta} = \lambda \beta = (\lambda_1 \beta, \ldots, \lambda_n \beta) \), i.e. \( \beta_k = \lambda_k \beta \) for all \( k = 1, \ldots, n \).

Proposition 3.1. Let \( \alpha: \mathbb{Z}_+ \to \text{End}(A) \) be a unital \( C^*\)-dynamical system, \( \sigma: \mathbb{R} \to \text{Aut}(\mathcal{N} \mathcal{T}(A, \alpha)) \) be the action related to \( \lambda \in \mathbb{R}^n \) as above, and \( \beta \in \mathbb{R} \).

(i) If \( \lambda_k \beta < 0 \) for some \( k = 1, \ldots, n \), then there are no \((\sigma, \beta)\)-KMS states.

(ii) If \( \lambda_k \beta > 0 \) for all \( k = 1, \ldots, n \), then a state \( \psi \) of \( \mathcal{N} \mathcal{T}(A, \alpha) \) is a \((\sigma, \beta)\)-KMS state if and only if
\[ \psi(ab) = \psi(ba) \quad \text{and} \quad \psi(V_x a V_y^*) = \delta_{x,y} e^{-\langle x-y, \beta \rangle} \psi(a), \]

for all \( a, b \in A \) and \( x, y \in \mathbb{Z}_+^n \).
Proof. Since the system is unital then the unit 1 ∈ A is also the unit for \( \mathcal{N}T(A, \alpha) \). For item (i) we compute

\[
1 = \psi(1) \geq \psi(V_k 1 V_k^*) = \psi(1 \sigma_{i,\beta}(V_k^*) V_k) = e^{-\lambda_k \beta},
\]

for all \( k = 1, \ldots, n \).

For item (ii) fix \( \lambda \in \mathbb{R}^n \) and \( \beta \) such that \( \lambda_k \beta > 0 \) for all \( k = 1, \ldots, n \).

Suppose first that \( \psi \) is a \((\sigma, \beta)\)-KMS state. Then we obtain

\[
\psi(ab) = \psi(b \sigma_{i,\beta}(a)) = \psi(ba),
\]

and

\[
\psi(V_{\underline{a}} a V_{\underline{b}}^*) = \psi(a V_{\underline{b}}^* \sigma_{i,\beta}(V_{\underline{a}})) = e^{-(\underline{x} \cdot \lambda)\beta} \psi(a V_{\underline{b}}^* V_{\underline{a}}).
\]

It suffices to show that \( \psi(a V_{\underline{b}}^* V_{\underline{a}}) = 0 \) for all \( a \in A \) when \( \underline{x} \neq \underline{y} \). By using the Nica covariance we may assume that the monomial \( a V_{\underline{b}}^* V_{\underline{a}} \) is written in a reduced form, i.e. \( \text{supp} \underline{x} \cap \text{supp} \underline{y} = \emptyset \) so that \( V_{\underline{b}}^* V_{\underline{a}} = V_{\underline{a}} V_{\underline{b}}^* \). First suppose that \( \underline{x} \neq \underline{1} \) and let an \( x_k > 0 \). Since the isometries \( V_{\underline{i}} \) doubly commute we get

\[
\psi(a V_{\underline{b}}^* V_{\underline{a}}) = \psi(a V_{\underline{b}}^* V_{\underline{a}} - k) = \psi(V_k \alpha_k(a) V_{\underline{b}}^* V_{\underline{a}} - k)
\]

\[
= \psi(\alpha_k(a) V_{\underline{b}}^* V_{\underline{a}} - k \sigma_{i,\beta}(V_k)) = e^{-x_k \lambda_k \beta} \psi(\alpha_k(a) V_{\underline{b}}^* V_{\underline{a}}).
\]

Inductively we get that \( \psi(a V_{\underline{b}}^* V_{\underline{a}}) = (e^{-x_k \lambda_k \beta})^l \psi(\alpha_{i,\beta}(a) V_{\underline{b}}^* V_{\underline{a}}) \) for every \( l \in \mathbb{Z}_+ \). Hence

\[
|\psi(a V_{\underline{b}}^* V_{\underline{a}})| \leq (e^{-x_k \lambda_k \beta})^l \left\| \alpha_{i,\beta}(a) V_{\underline{b}}^* V_{\underline{a}} \right\| \leq (e^{-x_k \lambda_k \beta})^l \|a\|
\]

Taking the limit over \( l \) yields \( \psi(a V_{\underline{b}}^* V_{\underline{a}}) = 0 \) for all \( a \in A \). Now if \( \underline{x} = \underline{1} \) then \( \underline{y} \neq \underline{1} \). By taking adjoints and applying a similar computation we get that

\[
\psi(a V_{\underline{b}}^*) = \overline{\psi(V_{\underline{b}} a V_{\underline{b}}^*)} = (e^{-x_k \lambda_k \beta})^l \psi(\alpha_{i,\beta}(a) V_{\underline{b}}^*).
\]

Taking \( l \longrightarrow +\infty \) yields \( \psi(a V_{\underline{b}}^*) = 0 \).

To prove the converse of item (ii) it suffices to show that if \( \psi \) satisfies the two conditions then

\[
\psi(V_{\underline{a}} a V_{\underline{b}}^* \cdot V_{\underline{z}} b V_{\underline{w}}^*) = e^{-(\underline{z} \cdot \underline{y} \lambda)\beta} \psi(V_{\underline{z}} b V_{\underline{w}}^* \cdot V_{\underline{a}} a V_{\underline{b}}^*),
\]

for all \( \underline{a}, \underline{b}, \underline{z}, \underline{w} \in \mathbb{Z}_+^n \) and \( a, b \in A \). For simplicity set \( f = V_{\underline{a}} a V_{\underline{b}}^* \) and \( g = V_{\underline{a}} b V_{\underline{w}}^* \). By the covariant condition we have that

\[
\psi(f g) = \psi(V_{\underline{z} + \underline{w} - \underline{y}} \cdot \alpha_{\underline{z} - \underline{y} \lambda}(a) \alpha_{\underline{w} - \underline{y} \lambda}(b) V_{\underline{y} + \underline{w} - \underline{y} \lambda})
\]

\[
= \delta_{\underline{z} + \underline{w} - \underline{y} \lambda} e^{-(\underline{z} \cdot \underline{y} \lambda)\beta} \psi(\alpha_{\underline{z} - \underline{y} \lambda}(a) \alpha_{\underline{w} - \underline{y} \lambda}(b))
\]

\[
= \delta_{\underline{z} + \underline{w} - \underline{y} \lambda} e^{-(\underline{z} \cdot \underline{y} \lambda)\beta} \psi(\alpha_{\underline{z} - \underline{y} \lambda}(a) \alpha_{\underline{w} - \underline{y} \lambda}(b)).
\]
On the other hand we have that
\[
\psi(gf) = \psi(V_{z+w}V^*\alpha_{z+w}(b)\alpha_{w-z}(a)V^*_{y+w})
\]
\[
= \delta_{z+x-y-w} \cdot e^{-\langle z+x-y-w,\lambda \rangle} \cdot \psi(\alpha_{z+w}(b)\alpha_{w-z}(a))
\]
\[
= \delta_{z+x-y-w} \cdot e^{-\langle z+x-y-w,\lambda \rangle} \cdot \psi(\alpha_{w-z}(b))
\].

Since \( \beta \neq 0 \), we have to show that if \( z+x = y+w \) then
\[
\sum_{k=1}^{n} (x_k + z_k - \min\{y_k, z_k\}) \lambda_k = \sum_{k=1}^{n} (z_k + x_k - \min\{x_k, w_k\}) \lambda_k + \sum_{k=1}^{n} (x_k - y_k) \lambda_k.
\]

It suffices to show that
\[ x_k + \min\{y_k, z_k\} = y_k + \min\{x_k, w_k\}, \]
for all \( k = 1, \ldots, n \). This is immediate since the equation \( z_k + x_k = y_k + w_k \) implies that \( \min\{y_k, z_k\} = y_k \) if and only if \( \min\{x_k, w_k\} = x_k \).

The next step is to establish the existence of the KMS states when \( \lambda_k\beta > 0 \) for all \( k = 1, \ldots, n \).

**Proposition 3.2.** Let \( \alpha: \mathbb{Z}_+^n \to \text{End}(A) \) be a unital \( C^* \)-dynamical system and \( \sigma: \mathbb{R} \to \text{Aut}(\mathcal{N}T(A, \alpha)) \) be the action related to \( \lambda \in \mathbb{R}^n \). Fix \( \beta \in \mathbb{R} \) such that \( \lambda_k\beta > 0 \) for all \( k = 1, \ldots, n \). Then for every tracial state \( \tau \) of \( A \) there exists a \((\sigma, \beta)\)-KMS state \( \psi \) of \( \mathcal{N}T(A, \alpha) \) such that
\[
\psi(V_x V^*_{y}) = \delta_{x,y} \cdot e^{-\langle x, y, \lambda \rangle} \cdot \prod_{i=1}^{n} (1 - e^{-\lambda_i}) \cdot \sum_{w \in \mathbb{Z}_+^n} e^{-\langle w, \lambda \rangle} \cdot \tau \alpha_w(a),
\]
for all \( a \in A, x, y \in \mathbb{Z}_+^n \).

**Proof.** Write \( \beta = \lambda \beta = (\lambda_1\beta, \ldots, \lambda_n\beta) \) so that \( \beta_i = \lambda_i\beta \) for all \( i = 1, \ldots, n \). Suppose first that \( \psi \) as defined above is a state on \( \mathcal{N}T(A, \alpha) \). We will show that it is a KMS state. For \( a, b \in A \) we obtain that
\[
\psi(ab) = \prod_{i=1}^{n} (1 - e^{-\beta_i}) \cdot \sum_{w \in \mathbb{Z}_+^n} e^{-\langle w, \beta \rangle} \cdot \tau \alpha_w(ab)
\]
\[
= \prod_{i=1}^{n} (1 - e^{-\beta_i}) \cdot \sum_{w \in \mathbb{Z}_+^n} e^{-\langle w, \beta \rangle} \cdot \tau \alpha_w(ba) = \psi(ba),
\]
where we have used that \( \tau \) is a tracial state on \( A \). Furthermore for \( x \in \mathbb{Z}_+^n \) and \( a \in A \) we readily verify that \( e^{-\langle x, \beta \rangle} \psi(a) = \psi(V_x a V^*_x) \). Hence \( \psi \) satisfies the conditions of Proposition 3.1 and is a \((\sigma, \beta)\)-KMS state.

The tricky part is to show that \( \psi \) is indeed a state. To this end consider the Fock representation on \( H_\tau \otimes \mathcal{L}(\mathbb{Z}_+^n) \) associated with the GNS representation \((H_\tau, \pi_\tau, \xi_\tau)\) of a state \( \tau \) of \( A \). Recall the notation
\[
F_m = \{ w \in \mathbb{Z}_+^n \mid w \leq m \cdot 1 = (m, \ldots, m) \}.
\]
For every $\underline{w} \in \mathbb{Z}^n_+$ let

$$\psi_{\underline{w}}(f) := \langle (V_\tau \times \pi_\tau)(f)(\xi_\tau \otimes e_\underline{w}), \xi_\tau \otimes e_\underline{w} \rangle$$

for $f \in \mathcal{NT}(A, \alpha)$, and define

$$\psi(f) := \prod_{i=1}^{n} (1 - e^{-\beta_i}) \cdot \sum_{\underline{w} \in \mathbb{Z}^n_+} e^{-\langle \underline{w}, \beta \rangle} \psi_{\underline{w}}(f).$$

To see that $\psi$ is well defined first note that

$$\sum_{\underline{w} \in F_m} e^{-\langle \underline{w}, \beta \rangle} = \sum_{w_1=0}^{m} e^{-w_1 \beta_1} \cdots \sum_{w_n=0}^{m} e^{-w_n \beta_n} = \prod_{i=1}^{n} \frac{1 - (e^{-\beta_i})^{m+1}}{1 - e^{-\beta_i}}$$

after a proper re-indexing. Taking the limit over $m$ yields

$$\sum_{\underline{w} \in \mathbb{Z}^n_+} e^{-\langle \underline{w}, \beta \rangle} = \prod_{i=1}^{n} (1 - e^{-\beta_i})^{-1}.$$

Let a positive element $f \in \mathcal{NT}(A, \alpha)$. Then the sequence of positives $\left( \sum_{\underline{w} \in F_m} e^{-\langle \underline{w}, \beta \rangle} \psi_{\underline{w}}(f) \right)_m$ is increasing and bounded above by the sequence $\left( \sum_{\underline{w} \in F_m} e^{-\langle \underline{w}, \beta \rangle} \|f\| \cdot 1 \right)_m$, which converges to $\|f\| \cdot \prod_{i=1}^{n} (1 - e^{-\beta_i})^{-1} \cdot 1$. Furthermore a direct computation shows that $\psi_{\underline{w}}(1) = 1$ for the unit $1 \in A$ of $\mathcal{NT}(A, \alpha)$. Therefore $\psi(1) = 1$, hence $\psi$ is a state.

It remains to show that $\psi$ is of the form of the statement. Suppose that $x \neq y$. Then $\psi_{\underline{w}}(V^*_x a V^*_y) = 0$ for all $\underline{w} \in \mathbb{Z}^n_+$, hence $\psi(V^*_x a V^*_y) = 0$. Moreover, since $V^*_x \xi_\tau \otimes e_\underline{w} = 0$ when $x \not\leq \underline{w}$, we obtain

$$\psi(V^*_x a V^*_y) = \prod_{i=1}^{n} (1 - e^{-\beta_i}) \cdot \sum_{\underline{w} \in \mathbb{Z}^n_+} e^{-\langle \underline{w}, \beta \rangle} \langle (V_\tau \times \pi_\tau)(a) V^*_x \pi_\tau(a) \xi_\tau \otimes e_\underline{w}, \xi_\tau \otimes e_\underline{w} \rangle$$

$$= \prod_{i=1}^{n} (1 - e^{-\beta_i}) \cdot \sum_{\underline{w} \in \mathbb{Z}^n_+} e^{-\langle \underline{w}, \beta \rangle} \langle V^*_x \pi_\tau(a) \xi_\tau \otimes e_\underline{w}, V^*_x \xi_\tau \otimes e_\underline{w} \rangle$$

$$= \prod_{i=1}^{n} (1 - e^{-\beta_i}) \cdot \sum_{\underline{w} \in \mathbb{Z}^n_+} e^{-\langle \underline{w}, \beta \rangle} \langle V^*_x \pi_\tau(a) \xi_\tau, \xi_\tau \rangle$$

$$= \prod_{i=1}^{n} (1 - e^{-\beta_i}) \cdot \sum_{\underline{w} \in \mathbb{Z}^n_+} e^{-\langle \underline{w}, \beta \rangle} \tau \alpha_{\underline{w} - x}(a),$$

which shows that $\psi$ is as in the statement.
3.1.1. **KMS states at** \( \beta = 0 \) (tracial states). A tracial state on \( \mathcal{NO}(A,\alpha) \) defines automatically a tracial state on \( \mathcal{NT}(A,\alpha) \). By Proposition 3.10 (that will follow) this works also in the converse direction.

**Proposition 3.3.** Let \( \alpha: \mathbb{Z}_+^n \to \text{End}(A) \) be a unital \( C^* \)-dynamical system and let \( \tilde{\beta}: \mathbb{Z}^n \to \text{Aut}(\tilde{B}) \) be its automorphic dilation as defined in subsection 1.2. For any tracial state \( \tau \) of \( \tilde{B} \) there exists a tracial state \( \psi \) of \( \mathcal{NT}(A,\alpha) \) such that

\[
\psi(V_xaV_y^*) = \delta_{x,y} \tau(\tilde{\beta}_{-2}(a)) \quad \text{for all } a \in A, x, y \in \mathbb{Z}_+^n.
\]

**Proof.** It suffices to find a tracial state \( \varphi \) of \( \mathcal{NO}(A,\alpha) \) so that \( \varphi(U_xaU_y^*) = \delta_{x,y} \tau(\tilde{\beta}_{-2}(a)) \). Then we may set \( \psi = \varphi q \) for the canonical *-epimorphism \( q: \mathcal{NT}(A,\alpha) \to \mathcal{NO}(A,\alpha) \). Recall that \( \mathcal{NO}(A,\alpha) \) is a corner of the crossed product \( \tilde{B} \rtimes \tilde{\beta} \mathbb{Z}^n \) by the element \( p = 1 \in A \subseteq \tilde{B} \). Consequently the tracial states on \( \tilde{B} \rtimes \tilde{\beta} \mathbb{Z}^n \) define tracial states on \( \mathcal{NO}(A,\alpha) \) by restriction. If \( \tau \) is a tracial state on \( \tilde{B} \) then \( \psi := \tau E: \tilde{B} \rtimes \tilde{\beta} \mathbb{Z}^n \to \mathbb{C} \) is a tracial state, where \( E \) is the conditional expectation on the crossed product. It is then readily verified that \( \psi(U_xaU_y^*) = \delta_{x,y} \tau(\tilde{\beta}_{-2}(a)) \) for all \( a \in A \) and \( x, y \in \mathbb{Z}_+^n \).

3.1.2. **Parametrization of** KMS **states when** \( \lambda_k \beta > 0 \). We will show that the correspondence of Proposition 3.2 is in fact a parametrization, when \( \lambda_k \beta > 0 \) for all \( k = 1, \ldots, n \).

**Theorem 3.4.** Let \( \alpha: \mathbb{Z}_+^n \to \text{End}(A) \) be a unital \( C^* \)-dynamical system, \( \sigma: \mathbb{R} \to \text{Aut}(\mathcal{NT}(A,\alpha)) \) be the action related to \( \bar{\lambda} \in \mathbb{R}^n \), and \( \beta \in \mathbb{R} \) such that \( \lambda_k \beta > 0 \). Then there is an affine homeomorphism from the simplex of the tracial states on \( A \) onto the simplex of the \((\sigma,\beta)\)-KMS states on \( \mathcal{NT}(A,\alpha) \).

**Proof.** Let the mapping \( \tau \mapsto \psi_\tau \) where \( \psi_\tau \) is of the form as in Proposition 3.2, i.e.

\[
\psi_\tau(V_xaV_y^*) = \delta_{x,y} \cdot e^{-\langle \bar{\lambda} \rangle \beta} \cdot \prod_{i=1}^n (1 - e^{-\lambda_i \beta}) \cdot \sum_{w \in \mathbb{Z}_+^n} e^{-(w \cdot \bar{\lambda}) \beta} \tau_{\alpha_w}(a),
\]

for all \( a \in A, x, y \in \mathbb{Z}_+^n \). The fact that this is an affine weak* continuous mapping follows by the standard arguments of [19, Proof of Theorem 6.1].

First we show that the mapping is onto. That is, given a \((\sigma,\beta)\)-KMS state \( \varphi \) of \( \mathcal{NT}(A,\alpha) \) we will construct a tracial state \( \tau \) of \( A \) such that \( \varphi = \psi_\tau \).

By Proposition 3.1 it suffices to show that \( \varphi(a) = \psi_\tau(a) \) for all \( a \in A \). For simplicity we will use the notation \( \bar{\beta} = \bar{\lambda} \beta = (\lambda_1 \beta, \ldots, \lambda_n \beta) \) so that \( \beta_i = \lambda_i \beta \) for all \( i = 1, \ldots, n \).

A key step is to isolate a projection \( P \in \mathcal{NO}(A,\alpha) \) such that \( Pa = aP \). To this end let

\[
P = \prod_{i=1}^n (I - V_iV_i^*),
\]
and recall that $aV_i^*V_i = V_i^*aV_i$ for all $a \in A$. We remind that we use the notation

$$F_m = \{ x \in \mathbb{Z}_+^n \mid x \leq m \cdot 1 \} \quad \text{and} \quad x_\geq := \{ y \in \mathbb{Z}_+^n \mid y \leq x \}.$$ 

Then a direct computation shows that

$$P = \sum_{x \in F_1} (-1)^{|x_\geq|} V_x^*V_x,$$

due to the $\varphi(P) = \prod_{i=1}^n (1 - e^{-\beta_i})$. However after a proper re-indexing we get

$$\lim_m \sum_{x \in F_m} e^{-\langle x, \beta \rangle} = \lim_m \prod_{i=1}^n \frac{1 - (e^{-\beta_i})^m}{1 - e^{-\beta_i}} = \left( \prod_{i=1}^n (1 - e^{-\beta_i}) \right)^{-1} = \varphi(P)^{-1},$$

hence $\lim_m \varphi(p_m) = 1$. 

We claim that the function

$$\varphi_P : A \to \mathbb{C} : a \mapsto \frac{\varphi(PaP)}{\varphi(P)},$$

is a tracial state on $A$. Indeed, for $a = 1 \in A$ (which is also the unit of $\mathcal{N}T(A,\alpha)$) we get that $\varphi_P(1) = 1$. Moreover by using that $\varphi$ is a $(\sigma,\beta)$-KMS state and that $\sigma_i\beta(a) = a$ for all $a \in A$ and $\sigma_i\beta(P) = P$ we get that

$$\varphi(PabP) = \varphi(bPPa) = \varphi(bPa) = \varphi(Pba) = \varphi(PbaP).$$

Thus $\varphi_P(ab) = \varphi_P(ba)$ for all $a, b \in A$.

Secondly we construct a sequence of projections $p_m \in \mathcal{N}T(A,\alpha)$ such that $\lim_m \varphi(p_m) = 1$. To this end let

$$p_m = \sum_{w \in F_m} V_w^*P V_w^*.$$

Note that $PV_i = 0$ and consequently $PV_w = 0$ for all $w \neq 0$. Therefore each $p_m$ is a projection. We compute

$$\varphi(p_m) = \sum_{w \in F_m} \varphi(V_w^*PV_w^*) = \sum_{w \in F_m} e^{-\langle w, \beta \rangle} \varphi(P) \sum_{w \in F_m} e^{-\langle w, \beta \rangle},$$

where $\varphi(P) = \prod_{i=1}^n (1 - e^{-\beta_i})$. However after a proper re-indexing we get that
Therefore \( \lim_{m} \varphi(p_m f p_m) = \varphi(f) \) for all \( f \in \mathcal{N}(A, \alpha) \) by [18, Lemma 7.3]. In particular for \( a \in A \) we get that
\[
\varphi(a) = \lim_{m} \varphi(p_m a p_m)
\]
\[
= \lim_{m} \sum_{w \in F_m} \sum_{z \in F_m} \varphi(V_w PV_w^* a \cdot V_z PV_z^*)
\]
\[
= \lim_{m} \sum_{w \in F_m} \sum_{z \in F_m} e^{-\langle w, \beta \rangle} \varphi(PV_w^* a V_z PV_z^* \cdot V_w P)
\]
\[
= \lim_{m} \sum_{w \in F_m} \sum_{z \in F_m} e^{-\langle w, \beta \rangle} \delta_{w,z} \varphi(PV_w^* a V_z P)
\]
\[
= \lim_{m} \sum_{w \in F_m} e^{-\langle w, \beta \rangle} \varphi(PV_w^* a V_w P)
\]
\[
= \lim_{m} \sum_{w \in F_m} e^{-\langle w, \beta \rangle} \varphi(P\alpha_w(a) P)
\]
\[
= \prod_{i=1}^{n} (1 - e^{-\beta_i}) \cdot \sum_{w \in \mathbb{Z}_n^+} \varphi \alpha_w(a),
\]
thus \( \varphi \) coincides with \( \psi_{\varphi_P} \).

Finally a direct computation shows that if \( \psi_{\tau} \) is as in Proposition 3.2 then
\[
(\psi_{\tau})_P(a) = \prod_{i=1}^{n} (1 - e^{-\beta_i})^{-1} \cdot \psi_{\tau}(PaP) = \psi(P) \psi_{\tau}(aP) = \tau(a),
\]
where a computation as before gives that \( \psi(P) = \prod_{i=1}^{n} (1 - e^{-\beta_i})^{-1} \). Indeed for the last equation first write
\[
(*) \quad \psi(a) = \psi(P) \cdot \sum_{w \in \mathbb{Z}_n^+} e^{-\langle w, \beta \rangle} \tau \alpha_w(a)
\]
\[
= \psi(P) \tau(a) + \psi(P) \cdot \sum_{0 < w} e^{-\langle w, \beta \rangle} \tau \alpha_w(a).
\]

Note that we can write
\[
\sum_{0 < w} e^{-\langle w, \beta \rangle} \tau \alpha_w(a) = \lim_{m} \sum_{0 < w \in F_m} e^{-\langle w, \beta \rangle} \tau \alpha_w(a)
\]
\[
= \lim_{m} - \sum_{0 \neq x \in F_1} (-1)^{|x|} \sum_{x \leq w \in F_m} e^{-\langle w, \beta \rangle} \tau \alpha_w(a)
\]
\[
= - \sum_{0 \neq x \in F_1} (-1)^{|x|} \sum_{x \leq w \in \mathbb{Z}_n^+} e^{-\langle w, \beta \rangle} \tau \alpha_w(a)
\]
\[
= - \sum_{0 \neq x \in F_1} (-1)^{|x|} \sum_{w \in \mathbb{Z}_n^+} e^{-\langle w + x, \beta \rangle} \tau \alpha_{w+x}(a).
\]
By using that
\[ \sum_{w \in \mathbb{Z}_n^+} e^{-\langle w, \beta \rangle} \tau a_w = e^{-\langle x, \beta \rangle} \sum_{w \in \mathbb{Z}_n^+} e^{-\langle w, \beta \rangle} \tau a_w, \]
we obtain that
\[ \sum_{0 < w} e^{-\langle w, \beta \rangle} \tau a_w = -\psi(P)^{-1} \sum_{0 \neq x \in F_1} (-1)^{|x|} \psi(a V^*_x). \]
Therefore equation (\*) becomes
\[ \psi(P) \tau(a) = \psi(a) + \sum_{0 \neq x \in F_1} (-1)^{|x|} \psi(a V^*_x) \]
\[ = \psi\left( \sum_{x \in F_1} (-1)^{|x|} a V^*_x \right) \]
\[ = \psi\left( a \prod_{i=1}^{n} (1 - V_i) \right) = \psi(a P). \]
Thus if \( \psi_\tau = \psi_\rho \) for two states in \( \tau, \rho \) of \( A \) then \( \tau = \rho \), which shows that the mapping \( \tau \mapsto \psi_\tau \) is injective.

3.1.3. Ground states. As in [17, 19] we first give a characterization of the ground states and then provide their existence and association with the states on the C*-algebra \( A \).

**Proposition 3.5.** Let \( \alpha: \mathbb{Z}_n^+ \to \text{End}(A) \) be a unital C*-dynamical system. A state \( \psi \) of \( \mathcal{N}_T(A, \alpha) \) is a ground state if and only if
\[ \psi(V_x a V^*_y) = \begin{cases} \psi(a) & \text{for } x = 0 = y, \\ 0 & \text{otherwise}, \end{cases} \]
for a state \( \tau \) of \( A \).

Consequently, there is an affine homomorphism from the state space \( S(A) \) onto the ground states on \( \mathcal{N}_T(A, \alpha) \).

**Proof.** Recall that \( \sigma_z(V_x a V^*_y) = e^{i(\langle z, \Delta \rangle z) V_x a V^*_y} \) for all \( z \in \mathbb{C} \). Suppose that \( \psi \) is a ground state. Suppose that \( y \neq 0 \) and note that the map
\[ r + it \mapsto \psi(V_x a V^*_y) = e^{-i(\langle z, \Delta \rangle z) \tau + it} \psi(V_x a V^*_y), \]
must be bounded when \( t > 0 \), for all \( a \in A \). Therefore \( \psi(V_x a V^*_y) = 0 \). When \( y = 0 \) but \( x \neq 0 \), we have that the function
\[ r + it \mapsto \psi(a^* V^*_x) = e^{-i(\langle z, \Delta \rangle z) \tau + it} \psi(a^* V^*_x) \]
must be bounded for \( t > 0 \); thus \( \psi(a^* V^*_x) = 0 \). Taking adjoints yields \( \psi(V_x a) = 0 \).
Conversely let \( \psi \) be a state on \( \mathcal{N}\mathcal{T}(A, \alpha) \) that satisfies the condition of the statement. Then for \( f = V_\underline{x}aV_\underline{y}^* \) and \( g = V_\underline{w}bV_\underline{u}^* \) we compute
\[
|\psi(f\sigma_{r+it}(g))|^2 = |e^{i(\underline{x} - \underline{w})}(r+it)\psi(fg)|^2 \\
= e^{-i(\underline{x} - \underline{w})t}\psi(fg)^2 \\
\leq e^{-i(\underline{x} - \underline{w})t}\psi(f^*f)\psi(g^*g) \\
\leq e^{-i(\underline{x} - \underline{w})t}\psi(V_\underline{x}a^*aV_\underline{w})\psi(V_\underline{w}b^*bV_\underline{u}).
\]
When \( \underline{y} \neq 0 \) or \( \underline{w} \neq 0 \), then the above expression is 0. When \( \underline{y} = \underline{w} = \underline{0} \) then
\[
|\psi(f\sigma_{r+it}(g))|^2 = e^{-i(\underline{x} - \underline{w})t}\psi(V_\underline{x}aV_\underline{y})^2
\]
which is zero when \( \underline{x} \neq 0 \) or \( \underline{w} \neq 0 \). Finally when \( \underline{x} = \underline{y} = \underline{w} = \underline{0} \), then \( \psi(f\sigma_{r+it}(g)) = \psi(ab) \), which is bounded on \( \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \) for all \( a, b \in A \).

Because of this characterization, every state on \( A \) gives rise to a ground state on \( \mathcal{N}\mathcal{T}(A, \alpha) \). Indeed, let \( \tau \) be a state on \( A \) and let \( (H_\tau, \pi_\tau, \xi_\tau) \) be the associated GNS representation. Then for the Fock representation \((\bar{\pi}_\tau, V_\tau)\) we define the state
\[
\psi_\tau(f) = \langle f\xi_\tau \otimes e_0, \xi_\tau \otimes e_0 \rangle, \quad \text{for all } f \in \mathcal{N}\mathcal{T}(A, \alpha).
\]
It is readily verified that \( \psi_\tau(a) = \tau(a) \) for all \( a \in A \). For \( f = V_\underline{x}aV_\underline{y}^* \) we compute
\[
\psi_\tau(V_\underline{x}aV_\underline{y}^*) = \langle V_{\bar{\tau}y_\underline{x}}\bar{\pi}_\tau(a)V_{\bar{\tau}y_\underline{x}}\xi_\tau \otimes e_0, \xi_\tau \otimes e_0 \rangle \\
= \langle \bar{\pi}_\tau(a)V_{\bar{\tau}y_\underline{x}}\xi_\tau \otimes e_0, V_{\bar{\tau}y_\underline{x}}\xi_\tau \otimes e_0 \rangle \\
= \delta_{\underline{x}}\delta_{\underline{y}}\delta_{\underline{y}}\delta_{\underline{y}}\langle \pi_\tau(a)\xi_\tau, \xi_\tau \rangle \\
= \begin{cases} 
\tau(a) & \text{for } \underline{x} = \underline{0} = \underline{y}, \\
0 & \text{otherwise,}
\end{cases} \\
= \begin{cases} 
\psi_\tau(a) & \text{for } \underline{x} = \underline{0} = \underline{y}, \\
0 & \text{otherwise.}
\end{cases}
\]
Finally note that \( \psi = \psi_\tau \) for \( \tau = \psi|_A \), since the restriction of a ground state to \( A \) defines a state on \( A \).

3.1.4. KMS\(_\infty\) states. We continue with the characterization of the KMS\(_\infty\) states on \( \mathcal{N}\mathcal{T}(A, \alpha) \).

**Proposition 3.6.** Let \( \alpha: \mathbb{Z}_+^n \to \text{End}(A) \) be a unital C\(^\ast\)-dynamical system. A state \( \psi \) is a KMS\(_\infty\) state on \( \mathcal{N}\mathcal{T}(A, \alpha) \) if and only if
\[
\psi(V_\underline{x}aV_\underline{y}^*) = \begin{cases} 
\tau(a) & \text{for } \underline{x} = \underline{0} = \underline{y}, \\
0 & \text{otherwise,}
\end{cases}
\]
for a tracial state $\tau$ of $A$.

Consequently, there is an affine homomorphism from the tracial state space $T(A)$ onto the KMS$_\infty$ states on $\mathcal{NT}(A,\alpha)$.

**Proof.** Let $\psi_\beta$ be $(\sigma,\beta)$-KMS states on $\mathcal{NT}(A,\alpha)$ that converge in the $w^*$-topology to $\psi$. By Proposition 3.1 we obtain that $\psi|_A$ is a tracial state on $A$ and that

$$\psi_\beta(V_xaV_y^*) = \delta_{x,y}e^{-\langle x,\lambda \rangle \beta}\psi(a),$$

which tends to zero when $\beta \to \infty$, if $x \neq 0$ or $y \neq 0$.

Conversely, let $\psi_\tau$ be as in the statement with respect to a tracial state $\tau$ of $A$. Let $\psi_{\tau,\beta}$ be as defined in Proposition 3.2, i.e.

$$\psi_{\tau,\beta}(V_xaV_y^*) = \delta_{x,y} \cdot e^{-\langle x,\lambda \rangle \beta} \sum_{w \in \mathbb{Z}_+^n} e^{-\langle w,\lambda \rangle \beta} \tau \alpha_w(a).$$

By the $w^*$-compactness we may choose a sequence of such states that converges to a state, say $\psi$. By definition $\psi$ is then a KMS$_\infty$ state, and we aim to show that $\psi_\tau = \psi$. When $x,y \neq 0$ then we get that $\lim_{\beta \to \infty} \psi_{\tau,\beta}(V_xaV_y^*) = 0$, as in the preceding paragraph. When $x = y = 0$ then

$$\psi_{\tau,\beta}(V_xaV_y^*) = \prod_{i=1}^n (1 - e^{-\lambda_i \beta}) \cdot \sum_{w \in \mathbb{Z}_+^n} e^{-\langle w,\lambda \rangle \beta} \tau \alpha_w(a).$$

However

$$\left| \sum_{w > 0} e^{-\langle w,\lambda \rangle \beta} \tau \alpha_w(a) \right| \leq \|a\| \cdot \sum_{w > 0} e^{-\langle w,\lambda \rangle \beta}$$

$$= \|a\| (1 - \prod_{i=1}^n (1 - e^{-\lambda_i \beta})^{-1}).$$

Taking $\beta \to \infty$ in the last expression yields that $\sum_{w > 0} e^{-\langle w,\lambda \rangle \beta} \tau \alpha_w(a)$ tends to zero. Trivially $\lim_{\beta \to \infty} \prod_{i=1}^n (1 - e^{-\lambda_i \beta}) = 1$, which implies that $\lim_{\beta \to \infty} \psi_{\tau,\beta}(a) = \tau(a)$, hence $\psi = \psi_\tau$. The proof is completed as in Proposition 3.5.

3.1.5. Applications. Our analysis can be used to treat the particular cases when $\lambda_k = 1$ for all $k = 1,\ldots,n$, or when $\lambda_k = 0$ for some $k = 1,\ldots,n$.

**Example 3.7.** Our original motivation was to examine the action $\sigma : \mathbb{R} \to \text{Aut}(\mathcal{NT}(A,\alpha))$ given by $\sigma_t = \gamma(e^{it}\ldots,e^{it})$. Then the KMS condition translates into

$$\psi(V_xaV_y^* \cdot V_xbV_y^*) = e^{-\langle x,\lambda \rangle \beta} \psi(V_xbV_y^* \cdot V_xaV_y^*),$$

$$\psi(V_xaV_y^* \cdot V_xbV_y^*) = e^{-\langle x,\lambda \rangle \beta} \psi(V_xbV_y^* \cdot V_xaV_y^*),$$
for all $a, b \in A$ and $x, y, z, w \in Z_+^n$, where $|x| \equiv |(x_1, \ldots, x_n)| = x_1 + \cdots + x_n$.

The previous analysis then gives the appropriate characterization of the $(\sigma, \beta)$-KMS states by setting $\lambda_k = 1$:

(i) for $\beta < 0$ there are no $(\sigma, \beta)$-KMS states;

(ii) for $\beta > 0$ a state $\psi$ is $(\sigma, \beta)$-KMS state if and only if $\psi(ab) = \psi(ba)$, and $\psi(V_x^*aV_y^*) = \delta_{x,y}e^{-|x||y|\beta}\psi(a)$, for all $a, b \in A$ and $n, m \in Z_+$;

(iii) for every tracial state $\tau$ of $A$ and $\beta > 0$ there is a $(\sigma, \beta)$-KMS state $\psi_\tau$ of $\mathcal{N}\mathcal{T}(A, \alpha)$ such that

$$
\psi_\tau(V_x^*aV_y^*) = \delta_{x,y}e^{-|x||y|\beta} \prod_{i=1}^n (1 - e^{-\beta_i}) \cdot \sum_{w \in Z_+^n} e^{-|w|\beta} \tau\omega_i(a),
$$

for all $a \in A$ and $x, y \in Z_+^n$. This representation is a parametrization of the $(\sigma, \beta)$-KMS states.

Now let us examine the case where some $\lambda_k$ are zeroes. Without loss of generality we may assume that $\lambda_{d+1} = \cdots = \lambda_n = 0$ and $\lambda_1, \ldots, \lambda_d \neq 0$. Then $\sigma: \mathbb{R} \rightarrow \text{Aut}(\mathcal{N}\mathcal{T}(A, \alpha))$ is given by $\sigma_t = \gamma(e^{\lambda_1}, \ldots, e^{\lambda_d}, 1, \ldots, 1)$; hence

$$
\sigma_t(V_x^*aV_y^*) = e^{-\sum_{k=1}^d (x_k - y_k)t} V_x^*aV_y^*,
$$

and the KMS condition is translated into

$$
\psi(V_x^*aV_y^* V_z^*bV_w^*) = e^{-\sum_{k=1}^d (x_k - y_k - z_k + w_k)t} \psi(V_x^*aV_y^* V_z^*bV_w^*),
$$

for all $a, b \in A$ and $x, y, z, w \in Z_+^n$. We aim to show that the $(\sigma, \beta)$-KMS states on $\mathcal{N}\mathcal{T}(A, \alpha)$ are determined by the $(\sigma', \beta)$-KMS states on $\mathcal{N}\mathcal{T}(Z, \zeta)$ for a suitably chosen C*-dynamical system $\zeta: Z \rightarrow \text{End}(Z)$ and an appropriate action $\sigma': \mathbb{R} \rightarrow \text{Aut}(\mathcal{N}\mathcal{T}(Z, \zeta))$.

To this end recall that $aV_i = V_0\alpha_i(a)$ for all $a \in A$ and $i$. Then $aV_iV_i^* = V_iV_i^*a$ and since $V_i$ is an isometry we obtain that

$$
\alpha_i(a) = V_i^*aV_i, \text{ for all } a \in A.
$$

We define the C*-algebra

$$
Z = \text{span}\{V_x^*aV_y^* | a \in A, \text{supp } x, \text{supp } y \subseteq \{d + 1, \ldots, n\}\}.
$$

Observe that $Z$ is a C*-subalgebra of $\mathcal{N}\mathcal{T}(A, \alpha)$. For $i = 1, \ldots, d$ we can extend the $*$-endomorphism $\alpha_i$ to a $*$-endomorphism $\zeta_i$ of $Z$ defined by

$$
\zeta_i(f) = V_i^*fV_i, \text{ for all } f \in Z.
$$

To see that the $\zeta_i$ is an algebraic endomorphism we remark that

$$
\zeta_i(V_x^*aV_y^*) = V_i^*V_x^*aV_y^*V_i = V_x^*\alpha_1(a)V_x^*V_i^*V_i = V_x^*\alpha_1(a)V_x^*,
$$

where we have used that the $V_i$ are doubly commuting isometries and that $x, y \in i^+$. This is a concrete system and $\mathcal{N}\mathcal{T}(Z, \zeta)$ has a representation in $\mathcal{N}\mathcal{T}(A, \alpha)$ given by the covariant pair $(\text{id}_Z, V)$. We will show that the induced $*$-representation is faithful.
Proposition 3.8. Let the Fock representation of $\alpha: \mathbb{Z}_+^n \to \text{End}(A)$ and let the covariant pair $(\text{id}_Z, V)$ of $\zeta: \mathbb{Z}_+^n \to \text{End}(Z)$. Then the induced $*$-representation on $\mathcal{N}T(Z, \zeta)$ is faithful.

Proof. Note that the representation $(\text{id}_Z, V)$ admits a gauge action given by

$$\gamma: \mathbb{T}^d \to \text{Aut}(\mathcal{N}T(Z, \zeta)),$$

such that $\gamma_{\underline{u}} = \text{ad}_{u_{\underline{u}}}$ where $u_{\underline{u}}(\xi \otimes e_{w}) = (\underline{w} \otimes w) \xi \otimes e_{w}$. Therefore by the gauge invariant uniqueness theorem it suffices to show that $Z \cap B_{(0, \infty]} = (0)$, where

$$B_{(0, \infty]} = \overline{\text{span}}\{V_{\underline{z}} f V_{\underline{w}}^* \mid f \in Z, \text{supp} \underline{z} \subseteq \{1, \ldots, d\}\}.$$ 

For a typical monomial $V_{\underline{z}} a V_{\underline{w}}^* \in Z$ we note that $V_{\underline{z}} a V_{\underline{w}}^*(\xi \otimes e_{w}) = 0$, when $\text{supp} \underline{w} \subseteq \{1, \ldots, d\}$. On the other hand $B_{(0, \infty]}$ is the inductive limit of the C*-subalgebras

$$B_{(0, m \cdot d]} = \text{span}\{V_{\underline{z}} f V_{\underline{w}}^* \mid f \in Z, 0 < \underline{z} \leq m \cdot \underline{d}\}.$$ 

For $f_d \in B_{(0, m \cdot d]}$ it is immediate to deduce that if $f_d(\xi \otimes e_{w}) = 0$ for all $0 < \underline{z} \leq m \cdot \underline{d}$ then $f_d = 0$. Inductively for $f_d \in B_{(0, \infty]}$ we obtain that if $f_d(\xi \otimes e_{w}) = 0$ then $f_d = 0$. This shows that $Z \cap B_{(0, \infty]} = (0)$.

An immediate corollary is that $\mathcal{N}T(A, \alpha) = \mathcal{N}T(Z, \zeta)$. Then the action $\sigma$ induces an action on $\mathcal{N}T(Z, \zeta)$. The gain is that $\sigma|_Z = \text{id}_Z$ and we fall into the previous analysis. In particular note that there is a bijection between the actions $\sigma$ of $\mathcal{N}T(A, \alpha)$ related to $\underline{\lambda}$ with $\lambda_{d+1} = \cdots = \lambda_n = 0$ and the actions $\sigma'$ of $\mathcal{N}T(Z, \zeta)$ defined by $\sigma'_i = \gamma'_i(\epsilon^{i \lambda_t} y_1, \ldots, \epsilon^{i \lambda_t} y_d)$, where $\{\gamma'_i\}_{\underline{y} \in \mathbb{T}^d}$ here is the gauge action of $\mathcal{N}T(Z, \zeta)$.

Proposition 3.9. With the aforementioned notation, a state $\psi$ is a $(\sigma, \beta)$-KMS state on $\mathcal{N}T(A, \alpha)$ if and only if $\psi$ is a $(\sigma', \beta)$-KMS state on $\mathcal{N}T(Z, \zeta)$.

Proof. We will use that every element $\underline{x} \in \mathbb{Z}_+^n$ is decomposed as $\underline{x} = \underline{x}_d + \underline{x}_{n-d}$ where

$$\underline{x}_d = (x_1, \ldots, x_d, 0, \ldots, 0) \quad \text{and} \quad \underline{x}_{n-d} = (0, \ldots, 0, x_{d+1}, \ldots, x_n).$$

Therefore a typical element $V_{\underline{z}} a V_{\underline{w}}^*$ can be written as $V_{\underline{z}_d} f V_{\underline{w}_d}^*$ with $f = V_{\underline{z}_{n-d}} a V_{\underline{w}_{n-d}}^* \in Z$.

First suppose that $\psi$ is a $(\sigma, \beta)$-KMS state on $\mathcal{N}T(A, \alpha)$. We have to establish the equality

$$\psi(V_{\underline{z}_d} f_1 V_{\underline{w}_d}^*, V_{\underline{z}_d} f_2 V_{\underline{w}_d}^*) = \psi(V_{\underline{z}_d} f_2 V_{\underline{w}_d}^* \sigma'_{\beta}(V_{\underline{z}_d} f_1 V_{\underline{w}_d}^*))$$

$$= e^{-\sum_{k=1}^d (x_k - y_k) \lambda_k \beta} \psi(V_{\underline{z}_d} f_2 V_{\underline{w}_d}^* \cdot V_{\underline{z}_d} f_1 V_{\underline{w}_d}^*),$$

for $f_1, f_2 \in \mathcal{N}T(Z, \zeta)$ and $\underline{x}_d, \underline{y}_d, \underline{z}_d, \underline{w}_d \in \mathbb{Z}_+^n$. It suffices to check it for $f_1 = V_{\underline{z}_{n-d}} a V_{\underline{w}_{n-d}}^*$ and $f_2 = V_{\underline{y}_{n-d}} b V_{\underline{w}_{n-d}}^*$, since such elements span a dense subset.
of $Z$. This follows directly by the computation
\begin{align*}
\psi(V_{ξ_d}V_{ξ_{d-1}}aV_{ξ_{d-1}}^*V_{ξ_d}^*V_{ξ_{d-1}}bV_{ξ_{d-1}}^*V_{ξ_d}^*) = \\
= \psi(V_{ξ_d}aV_{ξ_{d-1}}^*V_{ξ_d}bV_{ξ_{d-1}}^*) \\
= \psi(V_{ξ_d}bV_{ξ_{d-1}}^*\sigma_{iβ}(V_{ξ_d}aV_{ξ_{d-1}}^*)) \\
= e^{-\sum_{k=1}^{d}(x_k-y_k)\lambda_k}\psi(V_{ξ_d}bV_{ξ_{d-1}}^*V_{ξ_d}aV_{ξ_{d-1}}^*) \\
= e^{-\sum_{k=1}^{d}(x_k-y_k)\lambda_k}\psi(V_{ξ_d}f_2V_{ξ_{d-1}}^*V_{ξ_d}f_1V_{ξ_{d-1}}^*). \\
\end{align*}

Conversely we have to establish the equality
\begin{align*}
\psi(V_{ξ_d}aV_{ξ_{d-1}}^*V_{ξ_d}bV_{ξ_{d-1}}^*) = e^{-\sum_{k=1}^{d}(x_k-y_k)\lambda_k}\psi(V_{ξ_d}bV_{ξ_{d-1}}^*V_{ξ_d}aV_{ξ_{d-1}}^*),
\end{align*}
for $a, b \in A$ and $ξ_1, ξ_2, ξ_d \in Z^+_\mathbb{N}$. This follows directly by the computation
\begin{align*}
\psi(V_{ξ_d}aV_{ξ_{d-1}}^*V_{ξ_d}bV_{ξ_{d-1}}^*) = \psi(V_{ξ_d}V_{ξ_{d-1}}aV_{ξ_{d-1}}^*V_{ξ_d}^*V_{ξ_{d-1}}bV_{ξ_{d-1}}^*V_{ξ_d}^*) \\
= \psi(V_{ξ_d}f_1V_{ξ_{d-1}}^*V_{ξ_d}f_2V_{ξ_{d-1}}^*) \\
= \psi(V_{ξ_d}f_2V_{ξ_{d-1}}^*\sigma_{iβ}(V_{ξ_d}f_1V_{ξ_{d-1}}^*)) \\
= e^{-\sum_{k=1}^{d}(x_k-y_k)\lambda_k}\psi(V_{ξ_d}f_2V_{ξ_{d-1}}^*V_{ξ_d}f_1V_{ξ_{d-1}}^*) \\
= e^{-\sum_{k=1}^{d}(x_k-y_k)\lambda_k}\psi(V_{ξ_d}bV_{ξ_{d-1}}^*V_{ξ_d}aV_{ξ_{d-1}}^*),
\end{align*}
where we have used that $\psi$ is a $(σ', β)$-KMS state on $\mathcal{NT}(Z, ζ)$.

3.2. The Cuntz-Nica-Pimsner algebra. Recall that $\mathcal{NO}(A, α)$ is the quotient of $\mathcal{NT}(A, α)$ by the ideal generated by
\[ a \cdot \prod_{i \in \text{supp } ξ_d} (I - V_iV_i^*), \text{ for all } a \in I_{ξ_d}. \]
We denote by $q : \mathcal{NT}(A, α) \to \mathcal{NO}(A, α)$ the quotient map. Since $A$ embeds isometrically inside $\mathcal{NO}(A, α)$ we write $a \equiv q(a)$ for all $a \in A$. Moreover we write $\tilde{U}_ξ = q(V_ξ)$ for all $ξ \in Z^+_\mathbb{N}$. The action $σ$ of $\mathcal{NT}(A, α)$ passes naturally to an action of $\mathcal{NO}(A, α)$ which we denote by the same symbol. It is readily verified that the $(σ, β)$-KMS states on $\mathcal{NO}(A, α)$ define $(σ, β)$-KMS states on $\mathcal{NT}(A, α)$. The converse is true when the state vanishes on $\ker q$.

3.2.1. KMS states at $β = 0$. The algebra $\mathcal{NO}(A, α)$ is a C*-subalgebra of $\widehat{\mathcal{B}^{(1)}(\mathbb{Z}^n)}$. Therefore $\mathcal{NO}(A, α)$ admits restrictions of tracial states on this crossed product. On the other hand it shares the “same” tracial states with $\mathcal{NT}(A, α)$.

Proposition 3.10. Let $α : \mathbb{Z}^n_+ \to \text{End}(A)$ be a unital C*-dynamical system. Then a state is tracial on $\mathcal{NT}(A, α)$ if and only if it factors through a tracial state on $\mathcal{NO}(A, α)$.
Proposition 3.11. Let \( \sigma : \mathbb{R} \to \text{Aut}(\mathcal{N} \mathcal{T}(A, \alpha)) \) be the action related to \( \lambda \in \mathbb{R}^n \setminus \{0\} \). If \( \alpha \) is injective then there are no KMS states on \( \mathcal{N} \mathcal{O}(A, \alpha) \) for \( \beta \neq 0 \).

Proof. Recall that injectivity of \( \alpha \) is equivalent to the \( U_i \) being unitaries by Proposition 1.7. Let \( \psi \) be a \((\sigma, \beta)\)-KMS state on \( \mathcal{N} \mathcal{O}(A, \alpha) \). As in the first part of the proof of Proposition 3.1 we can have an estimation for the \( \lambda_k \beta \) by using the \( U_k \) in the place of \( V_k \). However now \( U_k \) is a unitary and we obtain equality, i.e. \( 1 = e^{-\lambda_k \beta} \), thus \( \lambda_k \beta = 0 \) for all \( k = 1, \ldots, n \). Since \( \sigma \) is not trivial there is a \( \lambda_k \neq 0 \) hence \( \beta = 0 \).

Therefore if \( \alpha : \mathbb{Z}_+^n \to \text{End}(A) \) is an injective C*-dynamical system then the only possible KMS states are the tracial states on \( \mathcal{N} \mathcal{O}(A, \alpha) \). Recall that in this case \( \mathcal{N} \mathcal{O}(A, \alpha) \simeq \tilde{A} \rtimes_\alpha \mathbb{Z}_+^n \) for the automorphic extension \( \tilde{\alpha} : \mathbb{Z}_+^n \to \text{Aut}(\tilde{A}) \). When \( \tilde{A} \) admits tracial states then \( \mathcal{N} \mathcal{O}(A, \alpha) \simeq \tilde{A} \rtimes_\alpha \mathbb{Z}_+^n \) admits tracial states that appear as evaluations on the \( (\underline{0}, \underline{0}) \)-entry.

3.2.3. Non-injective systems. For the non-injective case we will use the analysis of the KMS states on \( \mathcal{N} \mathcal{T}(A, \alpha) \).

Theorem 3.12. Let \( \alpha : \mathbb{Z}_+^n \to \text{End}(A) \) be a unital C*-dynamical system, \( \sigma : \mathbb{R} \to \text{Aut}(\mathcal{N} \mathcal{T}(A, \alpha)) \) be the action related to \( \lambda \in \mathbb{R}^n \), and \( \beta \in \mathbb{R} \) such that \( \lambda_k \beta > 0 \) for all \( k = 1, \ldots, n \). Then there is an affine homeomorphism \( \tau \mapsto \varphi_\tau \) from the simplex of the tracial states on \( A \) that vanish on \( I_1 \) onto the simplex of the \((\sigma, \beta)\)-KMS states on \( \mathcal{N} \mathcal{O}(A, \alpha) \) such that

\[
\varphi_\tau(U_\underline{x} a U_\underline{y}^* ) = \delta_{\underline{x}, \underline{y}} \cdot e^{-\langle \underline{\lambda}, \underline{\beta} \rangle} \cdot \prod_{i=1}^n (1 - e^{-\lambda_i \beta}) \cdot \sum_{\underline{w} \in \mathbb{Z}_+^n} e^{-\langle \underline{w}, \underline{\lambda} \rangle \beta} \tau^{\alpha_\underline{w}}(a),
\]

for all \( a \in A, \underline{x}, \underline{y} \in \mathbb{Z}_+^n \).
Proof. Since \( \mathcal{NO}(A, \alpha) \) is a quotient of \( \mathcal{NT}(A, \alpha) \) and because we already have a parametrization of the KMS states on \( \mathcal{NT}(A, \alpha) \) it suffices to show that \( \tau \) is a tracial state on \( A \) that vanishes on \( I_1 \) if and only if \( \psi_\tau \) of Proposition 3.2 defines a KMS state on \( \mathcal{NO}(A, \alpha) \). Since the action \( \sigma \) respects the quotient mapping, this is equivalent to showing that \( \psi_\tau \) vanishes on the elements \( a \cdot \prod_{i \in \text{supp } y} (I - V_i V_i^*) \) for all \( a \in I_y \).

For a fixed \( y \leq 1 \) with support \( \text{supp } y = \{1, \ldots, d\} \) we compute

\[
\psi_\tau(a \cdot \prod_{i=1}^d (I - V_i V_i^*)) = \psi_\tau\left( \sum_{0 \leq z \leq y} (-1)^{|z|} a V_z V_z^* \right)
= \sum_{0 \leq z \leq y} (-1)^{|z|} \psi_\tau(V_z \alpha_\tau(a) V_z^*)
= \sum_{0 \leq z \leq y} (-1)^{|z|} e^{-\langle z, \beta \rangle} \psi_\tau(\alpha_\tau(a)) = (*) .
\]

For convenience let us write \( p = \prod_{i=1}^n (1 - e^{-\beta_i})^{-1} \). Then

\[
(*) = p \sum_{0 \leq z \leq y} (-1)^{|z|} e^{-\langle z, \beta \rangle} \sum_{w \in \mathbb{Z}_+^n} e^{-\langle w, \beta \rangle} \tau \alpha_\tau(\alpha_\tau(a))
= p \sum_{0 \leq z \leq y} (-1)^{|z|} \sum_{w \in \mathbb{Z}_+^n} e^{-\langle x + w, \beta \rangle} \tau \alpha_\tau(\alpha_\tau(a))
= p \sum_{0 \leq z \leq y} (-1)^{|z|} \sum_{w \in \mathbb{Z}_+^n} e^{-\langle w, \beta \rangle} \tau \alpha_\tau(\alpha_\tau(a)) = p \cdot \tau(a).
\]

Recall that \( I_y \subseteq I_1 \) for all \( y \leq 1 \), hence if \( \tau \) vanishes on \( I_1 \) then it vanishes on \( I_y \) and the above computation shows that \( \psi_\tau \) defines a \((\sigma, \beta)\)-KMS state on \( \mathcal{NO}(A, \alpha) \).

Conversely if \( \varphi \) is a \((\sigma, \beta)\)-KMS state on \( \mathcal{NO}(A, \alpha) \) then \( \varphi q \) is a \((\sigma, \beta)\)-KMS state on \( \mathcal{NT}(A, \alpha) \) where \( q : \mathcal{NT}(A, \alpha) \to \mathcal{NO}(A, \alpha) \) is the quotient \(*\)-epimorphism. Hence we obtain that \( \varphi q = \psi_\tau \) for some tracial state \( \tau \) of \( A \), by Theorem 3.4. Then the above computation shows that \( \tau \) vanishes on \( I_1 \) since \( \psi_\tau \) vanishes on \( a \cdot \prod_{i=1}^n (I - V_i V_i^*) \) for all \( a \in I_1 \), and the proof is complete. \( \square \)

3.2.4. Ground states and KMS\(_\infty\) states. Similar computations give the following analogues for the ground states and the KMS\(_\infty\) states on \( \mathcal{NO}(A, \alpha) \). In particular for the KMS\(_\infty\) states we make use of Theorem 3.12.

Corollary 3.13. Let \( \alpha : \mathbb{Z}_+^n \to \text{End}(A) \) be a unital \( C^*\)-dynamical system. The map \( \tau \mapsto \varphi_\tau \) with

\[
\varphi_\tau(U_\alpha a U_\alpha^*) = \begin{cases} 
\tau(a) & \text{for } \alpha = 0 = y, \\
0 & \text{otherwise,}
\end{cases}
\]

...
is an affine homomorphism from the state space $S(A)$ (resp. tracial state space $T(A)$) onto the ground states (resp. KMS$_\infty$ states) of $\mathcal{N}\mathcal{O}(A, \alpha)$.

3.2.5. Applications. As in the case of $\mathcal{N}\mathcal{T}(A, \alpha)$, our analysis can be used to treat the particular cases when $\lambda_k = 1$ for all $k = 1, \ldots, n$, or when $\lambda_k = 0$ for some $k = 1, \ldots, n$.

**Example 3.14.** Let $\sigma: \mathbb{R} \to \text{Aut}(\mathcal{N}\mathcal{O}(A, \alpha))$ given by $\sigma_t = \gamma(e^{it}, \ldots, e^{it})$. Then for every tracial state $\tau$ of $A$ that vanishes on $I_\perp$ and $\beta > 0$ there is a $(\sigma, \beta)$-KMS state $\psi_\tau$ of $\mathcal{N}\mathcal{O}(A, \alpha)$ such that

$$
\psi_\tau(U_x a U_y^*) = \delta_{x,y} \cdot e^{-|x|^2\beta} \prod_{i=1}^{n}(1 - e^{-\beta_i}) \cdot \sum_{w \in \mathbb{Z}^n} e^{-|w|^2\beta} \tau_{\alpha_w}(a),
$$

for all $a \in A$ and $x, y \in \mathbb{Z}_+^n$. This representation is a parametrization of the $(\sigma, \beta)$-KMS states.

Now let us examine the case where some $\lambda_k$ are zeroes. Let us assume that $\lambda_{d+1} = \cdots = \lambda_n = 0$ and $\lambda_1, \ldots, \lambda_d \neq 0$. We aim to show that the $(\sigma, \beta)$-KMS states on $\mathcal{N}\mathcal{O}(A, \alpha)$ are determined by the $(\sigma', \beta)$-KMS states on $\mathcal{N}\mathcal{O}(\Omega, \omega)$ for a suitably chosen C*-dynamical system $\omega: \mathbb{Z}^d \to \text{End}(\Omega)$. To this end let the C*-algebra

$$
\Omega = \text{span}\{ U_x a U_y^* \mid a \in A, \text{supp } x, \text{supp } y \subseteq \{d+1, \ldots, n\} \}.
$$

Observe that $\Omega$ is a C*-subalgebra of $\mathcal{N}\mathcal{O}(A, \alpha)$. For $i = 1, \ldots, d$ we can extend the $\ast$-endomorphism $\alpha_i$ to a $\ast$-endomorphism $\omega_i$ of $\Omega$ defined by

$$
\omega_i(f) = U_i^* f U_i, \quad \text{for all } f \in \Omega.
$$

Then $\sigma: \mathbb{R}^n \to \text{Aut}(\mathcal{N}\mathcal{O}(A, \alpha))$ defines an action $\sigma': \mathbb{R}^d \to \text{Aut}(\mathcal{N}\mathcal{O}(\Omega, \omega))$.

**Proposition 3.15.** With the aforementioned notation, a state $\varphi$ is a $(\sigma, \beta)$-KMS state on $\mathcal{N}\mathcal{O}(A, \alpha)$ if and only if $\varphi$ is a $(\sigma', \beta)$-KMS state on $\mathcal{N}\mathcal{O}(\Omega, \omega)$.

**Proof.** The proof follows in the same way as in Proposition 3.9 once we show that $\mathcal{N}\mathcal{O}(\Omega, \omega) = \mathcal{N}\mathcal{O}(A, \alpha)$. To this end we will show that the representation $(\text{id}_\Omega, U)$ defines a faithful Cuntz-Nica-Pimsner covariant pair of $\mathcal{N}\mathcal{O}(\Omega, \omega)$. It is evident that $\text{id}_\Omega$ is injective on $\Omega$ and that $(\text{id}_\Omega, U)$ admits a gauge action inherited from the gauge action of $\mathcal{N}\mathcal{O}(A, \alpha)$. It remains to show that $(\text{id}_\Omega, U)$ is also Cuntz-Nica-Pimsner covariant.

**Claim.** Fix $w \in \mathbb{Z}^d$; then $f \in (\bigcap_{i \in \text{supp } w} \ker \omega_i)^\perp \subseteq \Omega$ if and only if $f \cdot \prod_{i \in \text{supp } w} (I - U_i U_i^*) = 0$.

**Proof of the Claim.** For convenience let $x = \prod_{i \in \text{supp } w} (I - U_i U_i^*)$. Without loss of generality assume that $\text{supp } w = \{1, \ldots, m\}$ with $m \leq d$. Let $g \in $
\( \cap_{\hat{w} \in \text{supp}\w} \ker \omega_{\hat{w}} \) and \( f \in \Omega \) such that \( fx = 0 \). Then

\[
0 = fx \cdot g = f \left( \prod_{i=1}^{m-1} (I - U_i U_{i+1}^*) \right) \cdot (g - U_m \omega_m(g) U_m^*) = f \prod_{i=1}^{m-1} (I - U_i U_{i+1}^*) (g - 0) = \cdots = fg
\]

which shows that \( f \) is in \( (\cap_{\hat{w} \in \text{supp}\w} \ker \omega_{\hat{w}})^\perp \).

Conversely suppose that \( f \in (\cap_{\hat{w} \in \text{supp}\w} \ker \omega_{\hat{w}})^\perp \). Note that \( x \) commutes with \( \Omega \) and let us form the C*-algebra \( \mathfrak{A} = C^*(\Omega, x) = \Omega x + \Omega(1 - x) \) inside \( \mathcal{NO}(A, \alpha) \). We will denote by \( \tilde{\omega}_1 \) the extension of \( \omega_1 \) to \( \mathfrak{A} \). Then \( x \in \cap_{\hat{w} \in \text{supp}\w} \ker \tilde{\omega}_1 \subseteq \mathfrak{A} \). As a consequence \( f x \in \cap_{\hat{w} \in \text{supp}\w} \ker \tilde{\omega}_1 \subseteq \mathfrak{A} \).

On the other hand every \( \tilde{g} \in \mathfrak{A} \) attains an orthogonal decomposition \( \tilde{g} = gx + \tilde{g}(1 - x) \) for some \( g \in \mathfrak{A} \). We can then write

\[
g = gx + g(1 - x) = \tilde{g} - \tilde{g}(1 - x) + g(1 - x).
\]

Let \( \tilde{g} \in \cap_{\hat{w} \in \text{supp}\w} \ker \tilde{\omega}_1 \); then

\[
\omega_1(g) = U_1^* g U_1 = U_1^* \tilde{g} U_1 - U_1^* \tilde{g}(1 - x) U_1 + U_1^* g(1 - x) U_1 = \tilde{\omega}_1(g) + 0 = 0
\]

because \( (1 - x) U_1 = 0 \). Therefore we have that \( g \in \cap_{\hat{w} \in \text{supp}\w} \ker \omega_{\hat{w}} \subseteq \Omega \). As a consequence we obtain that

\[
fx \tilde{g} = f gx + fx \tilde{g}(1 - x) = fxg = 0,
\]

since \( f \) is perpendicular to \( \cap_{\hat{w} \in \text{supp}\w} \ker \omega_{\hat{w}} \). Thus \( f x \in (\cap_{\hat{w} \in \text{supp}\w} \ker \omega_{\hat{w}})^\perp \) as well. This shows that \( fx = 0 \) and the proof of the claim is complete.

For distinction let us denote by \( J_w \) the ideals of \( \omega : Z^d_+ \to \text{End}(\Omega) \) with \( \text{supp}\w = \{1, \ldots, d\} \). Now let \( f \in J_w \) for \( w \in Z^d_+ \) with \( \text{supp}\w = \{1, \ldots, m\} \). Then in particular \( f \in (\cap_{\hat{w} \in \text{supp}\w} \ker \omega_{\hat{w}})^\perp \). By the claim we obtain that

\[
id_{\Omega}(f) \cdot \prod_{\hat{w} \in \text{supp}\w} (I - U_i U_{i+1}^*) = f \cdot \prod_{\hat{w} \in \text{supp}\w} (I - U_i U_{i+1}^*) = 0
\]

which completes the proof. \( \square \)

4. Appendix

Multivariable systems over \( Z^d_+ \) adopt multivariable gauge actions \( \sigma \) of \( \mathbb{R}^n \). Surprisingly all the computations of Section 3 still hold for the action \( \sigma^L = \gamma_{\text{ext}} \) by substituting \( \lambda_i \beta \) with any \( \beta_i \in \mathbb{R} \). So one may ask why not apply the same analysis in the multivariable context. That is, for an action \( \sigma : \mathbb{R}^n \to \text{Aut}(A) \) of a C*-algebra \( A \) and for \( 0 < \beta \in \mathbb{R}^n \), one may consider the condition

\[
\psi(ab) = \psi(b \sigma_{i\beta}(a)),
\]

for all \( a, b \in A \). However the purpose of the KMS condition is to build a specific analytic function \( F_{a,b} \) on an (unbounded) domain. In the one-variable case this is achieved by using the Phragmén-Lindelöf principle for
strips. The absence of an exact multivariable analogue for tubes in $C^n$ requires the insertion of new data. Here we present a multivariable KMS condition subject to a prescribing set.

**Definition 4.1.** Let $0 < \beta \in \mathbb{R}^n$. A $\beta$-prescribing set $\Lambda_{\beta}$ is a subset of $C_{\beta} := \{ \gamma = \sum \varepsilon_k \beta_k \mid \varepsilon_k = 0, 1 \}$ such that $0 \notin \Lambda_{\beta}$.

For the definition of a multivariable KMS state we will require all 3 data $(\sigma, \beta, \Lambda_{\beta})$.

**Definition 4.2.** Let $\sigma : \mathbb{R}^n \to \text{Aut}(A)$ be a C*-dynamical system and $\Lambda_{\beta}$ be a prescribing set for $0 < \beta \in \mathbb{R}^n$. A state $\tau$ of $A$ is called a $(\sigma, \Lambda_{\beta})$-KMS state on $A$ if for every pair of elements $a, b \in A$ there is a complex-valued function $F_{a,b}$ that is analytic on the interior of $D = \{ z \in \mathbb{C}^n \mid 0 \leq \text{Im}(z_i) \leq \beta_i \}$ and continuous and bounded on $D$ such that

$$F_{a,b}(t + i\gamma) = \begin{cases} \tau(\sigma_\gamma(b)a) & \text{when } \gamma \in \Lambda_{\beta}, \\ \tau(a\sigma_\gamma(b)) & \text{when } \gamma \notin \Lambda_{\beta}, \end{cases}$$

for all $\gamma \in C_{\beta}$.

The first objective is to show that there are enough analytic elements inside $A$ for an action $\sigma$ of $\mathbb{R}^n$.

**Proposition 4.3.** Let $\sigma : \mathbb{R}^n \to \text{Aut}(A)$ be a C*-dynamical system and $\Lambda_{\beta}$ be a prescribing set for $0 < \beta \in \mathbb{R}^n$. A state $\tau$ of $A$ is called a $(\sigma, \Lambda_{\beta})$-KMS state on $A$ if and only if for every pair of elements $a, b \in A$ there is a complex-valued function $F_{a,b}$ that is analytic on the interior of $D = \{ z \in \mathbb{C}^n \mid 0 \leq \text{Im}(z_i) \leq \beta_i \}$ and continuous and bounded on $D$ such that

$$F_{a,b}(t + i\gamma) = \begin{cases} \tau(\sigma_\gamma(b)a) & \text{when } \gamma \in \Lambda_{\beta}, \\ \tau(a\sigma_\gamma(b)) & \text{when } \gamma \notin \Lambda_{\beta}, \end{cases}$$

for all $\gamma \in C_{\beta}$.

Even though this condition is necessary and sufficient for building analytic functions on tubes, it is rather strong. At the end of the appendix we show that the Toeplitz-Nica-Pimsner algebras do not attain such states.

In what follows we use the simplified multivariable notation. We write $t \equiv t \in \mathbb{C}^n$ and we use the symbol $t_i$ to denote either the $i$-th coordinate of $t$ or the vector $t_i \cdot i \in \mathbb{C}^n$. The difference will be made clear by the context.
4.1. **Proof of Proposition 4.3.** For this subsection fix an \( a \in \mathcal{A} \) and define

\[
a_m = \sqrt{\frac{m}{\pi}} \cdot \int_{\mathbb{R}^n} \sigma_t(a) e^{-m \sum_{k=1}^{n} t_k^2} dt
\]

\[
= \sqrt{\frac{m}{\pi}} \cdot \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sigma_t(a) e^{-mt_1^2} \cdots e^{-mt_n^2} dt_1 \cdots dt_n.
\]

By [2, Proposition 2.5.18] the one-variable integral \( \int_{\mathbb{R}} \sigma_{t_1}(a) e^{-mt_1^2} dt_1 \) defines an element in \( \mathcal{A} \) where \( \sigma_{t_1} \equiv \sigma_{(t_1,0,\ldots,0)}. \) By using induction and the formula

\[
\int_{\mathbb{R}} \sigma_t(a) e^{-mt_1^2} \cdots e^{-mt_n^2} dt_1 = \sigma_{t-t_1} \left( \int_{\mathbb{R}} \sigma_{t_1}(a) e^{-mt_1^2} dt_1 \right) e^{-mt_2^2} \cdots e^{-mt_n^2},
\]

we deduce that the \( a_m \) are well defined. Let \( f_m : \mathbb{C}^n \to A \) be the function

\[
f_m(z) = \sqrt{\frac{m}{\pi}} \cdot \int_{\mathbb{R}^n} \sigma_t(a) e^{-m \sum_{k=1}^{n} (t_k-z_k)^2} dt
\]

\[
= \sqrt{\frac{m}{\pi}} \cdot \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sigma_t(a) e^{-m(t_1-z_1)^2} \cdots e^{-m(t_n-z_n)^2} dt_1 \cdots dt_n.
\]

The function \( t_k \mapsto e^{m(t_k-z_k)^2} \) is in \( L^1(\mathbb{R}) \), hence by induction on [2, Proposition 2.5.18] we get that \( f_m(z) \) is a well defined element in \( \mathcal{A} \). For \( z = s \in \mathbb{R}^n \) we compute

\[
f_m(s) = \sqrt{\frac{m}{\pi}} \cdot \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sigma_t(a) e^{-m(t_1-s_1)^2} \cdots e^{-m(t_n-s_n)^2} dt_1 \cdots dt_n
\]

\[
= \sqrt{\frac{m}{\pi}} \cdot \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sigma_{t+s}(a) e^{-mt_1^2} \cdots e^{-mt_n^2} dt_1 \cdots dt_n
\]

\[
= \sigma_s \left( \sqrt{\frac{m}{\pi}} \cdot \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sigma_t(a) e^{-mt_1^2} \cdots e^{-mt_n^2} dt_1 \cdots dt_n \right) = \sigma_s(a_m).
\]

Therefore the mapping \( s \mapsto \sigma_s(a_m) \) extends to a well defined function \( f_m : \mathbb{C}^n \to \mathcal{A} \). We will show that \( f_m \) is entire analytic, i.e. the limit

\[
\lim_{h_k \to 0} h_k^{-1} \left[ f_m(z + h_k) - f_m(z) \right]
\]

exists for all \( z \in \mathbb{C}^n \) and \( k = 1, \ldots, n \). For \( k = 1 \) we calculate

\[
f_m(z + h_1) - f_m(z) = \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} x_a(t)(e^{-m(t_1-z_1-h_1)^2} - e^{-mt_1^2}) dt_1 dt_{n},
\]

where \( \widehat{dt_1} \) is \( dt_2 \cdots dt_n \) and

\[
x_a(t) = \sigma_t(a) e^{-mt_2^2} \cdots e^{-mt_n^2} = \sigma_{t_1} \left( \sigma_{t-t_1}(a) e^{-mt_2^2} \cdots e^{-mt_n^2} \right) = \sigma_{t_1}(a').
\]

Therefore

\[
f_m(z + h_1) - f_m(z) = \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \sigma_{t_1}(a'')(e^{-m(t_1-z_1-h_1)^2} - e^{-mt_1^2}) dt_1,
\]
where
\[ a'' = \sqrt{\frac{m^{n-1}}{\pi}} \int_{\mathbb{R}^{n-1}} \sigma_{t-t_1}(a)e^{-mt_2^2} \ldots e^{-mt_n^2} dt_1, \]
thus \( \|a''\| \leq \|a\| \). Consequently it suffices to show that the limit
\[ \lim_{h_1 \to 0} h_1^{-1} \int_{\mathbb{R}} \sigma_{t_1}(a'')(e^{-m(t_1-z_1-h_1)^2} - e^{-mz_1^2}) dt_1 \]
exists in \( \mathcal{A} \). This follows from [2, Proposition 2.5.22].

It remains to show that the sequence \((a_m)_m\) converges to \(a\) in norm. To this end we compute
\[ a_m - a = \sqrt{\frac{m^n}{\pi}} \int_{\mathbb{R}^n} (\sigma_t(a) - a)e^{-m \sum_{k=1}^n t_k^2} dt. \]
For \( \varepsilon > 0 \) let \( \delta > 0 \) such that \( \|\sigma_t(a) - a\| < \varepsilon/2 \) for all \( |t| < \delta \). Moreover for the same \( \varepsilon > 0 \) let \( m_0 \in \mathbb{Z}^+ \) such that
\[ \sqrt{\frac{m^n}{\pi}} \int_{|t| \geq \delta} \sigma_t(a)e^{-m \sum_{k=1}^n t_k^2} dt \leq \frac{\varepsilon}{4\|a\|}, \]
for all \( m \geq m_0 \). Then for \( m \geq m_0 \) we get that
\[ \|a_m - a\| = \sqrt{\frac{m^n}{\pi}} \int_{|t| \leq \delta} \|\sigma_t(a) - a\| e^{-m \sum_{k=1}^n t_k^2} dt + \]
\[ + \sqrt{\frac{m^n}{\pi}} \int_{|t| \geq \delta} \|\sigma_t(a) - a\| e^{-m \sum_{k=1}^n t_k^2} dt \leq \varepsilon/2 \sqrt{\frac{m^n}{\pi}} \int_{|t| \leq \delta} e^{-m \sum_{k=1}^n t_k^2} dt + 2\|a\| \sqrt{\frac{m^n}{\pi}} \int_{|t| \geq \delta} e^{-m \sum_{k=1}^n t_k^2} dt \leq \varepsilon/2 \sqrt{\frac{m^n}{\pi}} \frac{\pi}{m} + 2\|a\| \frac{\varepsilon}{4\|a\|} = \varepsilon, \]
which finishes the proof.

4.2. Extending the action. The action \( \sigma \) extends pointwise on \( \mathcal{A}_{an} \) to an action of \( \mathbb{C}^n \) in the sense that
\[ \sigma_z \sigma_w(a) = \sigma_{z+w}(a) \]
for all \( a \in \mathcal{A}_{an}, z, w \in \mathbb{C}^n \).

Indeed suppose that \( a = \sqrt{\frac{m^n}{\pi}} \int_{\mathbb{R}^n} \sigma_t(b)e^{-m \sum_{k=1}^n t_k^2} dt, \) for some \( b \in \mathcal{A} \). The function \( \mathbb{R}^n \ni r \mapsto \sigma_r \sigma_w(a) \) is the restriction of the entire analytic function \( z \mapsto \sigma_z \sigma_w(a) \). For the latter first compute
\[ \sigma_r \sigma_w(a) = \sigma_{r+w} \left( \sqrt{\frac{m^n}{\pi}} \int_{\mathbb{R}^n} \sigma_t(b)e^{-m \sum_{k=1}^n (t_k - w_k)^2} dt \right) = \sqrt{\frac{m^n}{\pi}} \int_{\mathbb{R}^n} \sigma_{r+w}(b)e^{-m \sum_{k=1}^n (t_k - w_k)^2} dt = \sqrt{\frac{m^n}{\pi}} \int_{\mathbb{R}^s} \sigma_t(b)e^{-m \sum_{k=1}^n (t_k - r_k - w_k)^2} dt. \]
Hence the function \( r \mapsto \sigma_r \sigma_w(a) \) extends to the entire function
\[
  z \mapsto \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}^n} \sigma_t(b) e^{-m \sum_{k=1}^n (t_k - z_k - w_k)^2} dt,
\]
which implies that \( \sigma_w(a) \) is an analytic element. Thus it has an analytic extension given by \( z \mapsto \sigma_z \sigma_w(a) \). However a second analytic extension is given by
\[
  z \mapsto \sigma_{z+w}(a) := \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}^n} \sigma_t(b) e^{-m \sum_{k=1}^n (t_k - z_k - w_k)^2} dt.
\]
Uniqueness of the analytic extension then shows that \( \sigma_z \sigma_w(a) = \sigma_{z+w}(a) \).

### 4.3. Invariance of KMS states

For an analytic element \( a \) in \( \mathcal{A} \) we write \( \sigma_z(a) := f_a(z) \). The \((\sigma, \Lambda_{\beta})\)-KMS condition for a state \( \tau \) of \( \mathcal{A} \) is then
\[
  \tau(\sigma_z(a)) = \tau(\sigma_z(b)) = \begin{cases} 
\tau(ba) & \text{when } \gamma \in \Lambda_{\beta}, \\
\tau(ab) & \text{when } \gamma \notin \Lambda_{\beta}, 
\end{cases}
\]
for all \( a, b \) in a norm-dense \(*\)-subalgebra of analytic elements of \( \mathcal{A} \).

**Proposition 4.5.** Let \( \tau \) be a state on \( \mathcal{A} \) that satisfies the \((\sigma, \Lambda_{\beta})\)-KMS condition. Then \( \tau \) is \( \sigma \)-invariant.

**Proof.** Let \( \beta = (\beta_1, \ldots, \beta_n) \) in \( \mathbb{R}^n \). Without loss of generality we may assume that \( \beta_k \neq 0 \) for all \( k = 1, \ldots, n \). Otherwise we pass to the subsystem on \( \mathbb{R}^{n-d} \) where \( d \) is the number of the \( \beta_k \) that are zero.

For an element \( a \) inside a norm-dense \(*\)-subalgebra of \( \mathcal{A}_{\text{an}} \) given by the definition of the KMS condition let the function \( F(t) = \tau \sigma_t(a) \). Then \( F(t) \) extends to an entire analytic function \( F(z) = \tau \sigma_z(a) \). Fix attention to the first co-ordinate and let \( a_1 = \sigma_{z-z_1}(a) \) and \( F_1(t) = F(z, z_2, \ldots, z_d) \). Since
\[
|F_1(z_1)| \leq \|\sigma_{z_1}(a_1)\| = \|\sigma_{\text{Re}(z_1)} \sigma_{\text{Im}(z_1)}(a_1)\| \leq \|\sigma_{\text{Im}(z_1)}(a_1)\|
\]
we get that \( F_1 \) is bounded on the strip \( D_1 = \{ z_1 \in \mathbb{C} \mid 0 \leq \text{Im} z_1 \leq \beta_1 \} \) by
\[
M = \sup\{\|\sigma_{it_1}(a_1)\| \mid t_1 \in [0, \beta_1]\}.
\]
Then \( M \) is finite since the function \( [0, \beta_1] \ni t_1 \mapsto \|\sigma_{it_1}(b_1)\| \) is continuous. Let \( (e_j) \) be an approximate identity of \( \mathcal{A} \) and compute
\[
F_1(z_1 + i \beta_1) = \lim_j \tau(e_j \sigma_{i \beta_1} \sigma_{z_1}(a_1)) = \lim_j \tau(\sigma_{z_1}(a_1) e_j) = F(z_1).
\]
Thus \( F_1 \) is periodic, hence bounded by \( M \). By Liouville’s Theorem then \( F_1 \) is constant. Hence we have that \( \partial F_1/\partial z_1 = 0 \), therefore \( F(z_1, \ldots, z_d) = F(z_2, \ldots, z_d) \). Inductively we get that \( F \) is constant in \( \mathbb{C}^n \), i.e.
\[
\tau(a) = F(0) = F_t = \tau \sigma_t(a)
\]
for all \( a \) in a norm-dense \(*\)-subalgebra of \( \mathcal{A}_{\text{an}} \). Since \( \mathcal{A}_{\text{an}} \) is dense in \( \mathcal{A} \) we get that \( \tau(a) = \tau \sigma_t(a) \) for all \( a \in \mathcal{A} \).
4.4. **Proof of Theorem 4.4.** First suppose that \( \tau \) is a state on \( \mathcal{A} \) such that

\[
\tau(a\sigma_i\gamma(b)) = \begin{cases} 
\tau(ba) & \text{when } \gamma \in \Lambda_{\beta}, \\
\tau(ab) & \text{when } \gamma \notin \Lambda_{\beta},
\end{cases}
\]

for all \( a, b \) in a norm-dense \(*\)-subalgebra of \( \mathcal{A}_{\text{an}} \). For such fixed \( a, b \in \mathcal{A}_{\text{an}} \) let the function

\[
\mathbb{C}^n \ni z \mapsto F_{a,b}(z) = \tau(a\sigma_z(b)).
\]

Then \( F_{a,b} \) is entire analytic and

\[
F_{a,b}(t + i\gamma) = \begin{cases} 
\tau(\sigma_t(b)a) & \text{when } \gamma \in \Lambda_{\beta}, \\
\tau(a\sigma_t(b)) & \text{when } \gamma \notin \Lambda_{\beta},
\end{cases}
\]

for all \( \gamma \in \mathbb{C}_{\beta} \). The function \( \mathbb{C}^n \ni z \mapsto \sigma_z(b) \) is analytic hence the function

\[
\{ s \in \mathbb{R}^n \mid 0 \leq s_k \leq \beta_k \} \to \mathbb{R} : s \mapsto \|\sigma_{is}(b)\|
\]

is continuous, thus bounded. For

\[
M = \sup\{\|\sigma_{is}(b)\| \mid 0 \leq s_k \leq \beta_k\}
\]

we get that

\[
|F_{a,b}(t + is)| = |\tau(a\sigma_{is}\sigma_t(b))| \leq M \|a\|
\]

for all \( t + is \in D \).

Now we pass to the general case where \( a, b \in \mathcal{A} \). To this end let \( a_m \) and \( b_m \) in the dense subalgebra of \( \mathcal{A}_{\text{an}} \) with \( \|a_m\| \leq \|a\| \) and \( \|b_m\| \leq \|b\| \) such that

\[
a = \lim_m a_m \quad \text{and} \quad b = \lim_m b_m,
\]

and define \( F_m(z) = F_{a_m,b_m}(z) \). Our aim is to show that the sequence \( (F_m)_m \) is uniformly Cauchy so that defining \( F_{a,b} \) as the limit of \( F_m \) gives rise to a continuous and bounded function on \( D \). Moreover we will eventually have that

\[
F_{a,b}(t + i\gamma) = \lim_m (a_m\sigma_t(b_m)) = \tau(a\sigma_t(b)),
\]

when \( \gamma \notin \Lambda_{\beta} \) and

\[
F_{a,b}(t + i\gamma) = \lim_m F_{a_m,b_m}(t + i\gamma) = \lim_m \tau(a_m\sigma_{t+i\gamma}(b_m)) = \lim_m \tau(\sigma_t(b_m)a_m) = \tau(\sigma_t(b)a),
\]

when \( \gamma \in \Lambda_{\beta} \).

Let us begin with the following remark. Given \( z = (z_1, \ldots, z_n) \) consider the function \( f_m(\zeta_1) = F_m(\zeta_1, z_2, \ldots, z_n) \) as a function on \( \mathbb{C} \). Then \( f_m \) is analytic on \( z_1 \) since \( F_m \) is analytic on \( z \). Therefore \( f_m \) is analytic on \( D_{1} := \{ x + iy \in \mathbb{C} \mid 0 \leq y \leq \beta_1 \} \). Recall that by extending the action we get that

\[
\sigma_{(x + iy, z_2, \ldots, z_n)}(b_m) = \sigma_{(x + iy, 0, \ldots, 0)}(\sigma_{(0, z_2, \ldots, z_n)}(b_m)).
\]

Moreover \( f_m \) is continuous on \( D_1 \) and

\[
|f_m(x + iy)| = |\tau(a_m\sigma'_{x+iy}(b'_m))| \leq \|a_m\| \|\sigma'_{iy}(b'_m)\|,
\]
where $\sigma'_{x+iy} = \sigma_{(x+iy,0,...,0)}$ and $\sigma'_{m} = \sigma_{(0,z_{1},...,z_{n})}(b_{m})$. However the function $[0, \beta] \ni y \mapsto \left\| \sigma'_{iy}(b_{m}) \right\|$ is continuous, hence bounded. Consequently $f_{m}(x + iy)$ is bounded. By the Phragmén-Lindelöf principle, $f_{m}$ admits its supremum at the boundary of $D_{1}$, thus

$$\left| F_{m}(z_{1}, \ldots, z_{d}) \right| = \left| f_{m}(z_{1}) \right| \leq \max \{ \sup_{x \in \mathbb{R}} \left| f(x) \right|, \sup_{x \in \mathbb{R}} \left| f(x + i\beta) \right| \} = \max \{ \sup_{x \in \mathbb{R}} \left| F(x, z_{2}, \ldots, z_{n}) \right|, \sup_{x \in \mathbb{R}} \left| F(x + i\beta_{1}, z_{2}, \ldots, z_{n}) \right| \}.$$ 

The same holds for any co-ordinate.

For $\varepsilon > 0$ let $m_{0} \in \mathbb{Z}_{+}$ such that $\left\| a_{l} - a_{m} \right\| < \varepsilon/2 \left\| a \right\|$ and $\left\| b_{l} - b_{m} \right\| < \varepsilon/2 \left\| b \right\|$ for $m, l \geq m_{0}$. Following inductively the same arguments as above we derive that the function $F_{l}(z) - F_{m}(z)$ is bounded by the maximum of the values

$$\sup_{x_{1}} \ldots \sup_{x_{n}} \left| F_{m}(x_{1}, \ldots, x_{n}) + i\gamma \right| - F_{m}(x_{1}, \ldots, x_{n}) + i\gamma),$$

with respect to $\gamma = \sum \varepsilon_{k}\beta_{k}$ for all choices of $\varepsilon_{k} = 0, 1$. For $\gamma \notin \Lambda_{\beta}$ we obtain

$$\left| F_{m}(x_{1}, \ldots, x_{n}) + i\gamma \right| - F_{m}(x_{1}, \ldots, x_{n}) + i\gamma) =$$

$$= \left| \tau(a_{l}\sigma_{x}(b_{l})) - \tau(a_{m}\sigma_{x}(b_{m})) \right|$$

$$= \left| \tau(a_{l}\sigma_{x}(b_{l})) - \tau(a_{m}\sigma_{x}(b_{m})) \right|$$

$$= \left| \tau(a_{l}\sigma_{x}(b_{l})) + \left\| a_{l} - a_{m} \right\| \left\| b_{l} \right\| + \left\| a_{m} \right\| \left\| b_{l} - b_{m} \right\| < \varepsilon.$$ 

On the other hand for $\gamma \in \Lambda_{\beta}$ we obtain

$$\left| F_{m}(x_{1}, \ldots, x_{n}) + i\gamma \right| - F_{m}(x_{1}, \ldots, x_{n}) + i\gamma) =$$

$$= \left| \tau(a_{l}\sigma_{x}(b_{l})) - \tau(a_{m}\sigma_{x}(b_{m})) \right|$$

$$= \left| \tau(a_{l}\sigma_{x}(b_{l})) - \tau(a_{m}\sigma_{x}(b_{m})) \right|$$

$$= \left| \tau(a_{l}\sigma_{x}(b_{l})) - \tau (a_{m}\sigma_{x}(b_{m})) \right|$$

$$= \left| \tau(a_{l}\sigma_{x}(b_{l})) + \left\| a_{l} - a_{m} \right\| \left\| b_{l} \right\| + \left\| a_{m} \right\| \left\| b_{l} - b_{m} \right\| < \varepsilon.$$ 

Therefore $(F_{m})_{m}$ is indeed a Cauchy sequence uniformly on $D$.

For the converse suppose that for every $a, b \in \mathcal{A}$ there exists a complex function $F_{a,b}$ which is analytic on the interior of $D := \{z \in \mathbb{C}^{n} \mid 0 \leq \Im(z_{k}) \leq \beta_{k} \}$ and continuous on $D$ (hence bounded), such that

$$F(t + i\gamma) = \begin{cases} 
\tau(\sigma_{t}(b)a) & \text{when } \gamma \in \Lambda_{\beta}, \\
\tau(\sigma_{t}(b)) & \text{when } \gamma \notin \Lambda_{\beta}.
\end{cases}$$

For $a, b \in \mathcal{A}_{an}$ define $G_{a,b}(z) = \tau(\sigma_{z}(b))$ with $z \in \mathbb{C}^{n}$. Then $G_{a,b}$ is entire analytic and $G_{a,b}(t) = \tau(\sigma_{t}(b)) = F_{a,b}(t)$ for all $t \in \mathbb{R}^{n}$. By the edge-of-the-wedge theorem for the positive cone $\mathbb{R}^{n}_{+}$ we get that $F_{a,b}(z) = G_{a,b}(z)$ for all $z \in D$ hence $F_{a,b}(z) = \tau(\sigma_{z}(b))$. The second property of $F_{a,b}$ implies

$$\tau(\sigma_{v}(b)) = F_{a,b}(i\gamma) = \tau(ab)$$
for $\gamma \notin \Lambda_\beta$, and
\[
\tau(a\sigma_\gamma(b)) = F_{a,b}(i\gamma) = \tau(ba)
\]
for $\gamma \in \Lambda_\beta$. Hence $\tau$ satisfies the $(\sigma,\Lambda_\beta)$-KMS condition.

As in [3, Proposition 5.3.7], en passant we proved that such an $F_{a,b}$ satisfies
\[
\sup\{|F_{a,b}(z)| : z \in D\} \leq \|a\| \|b\|,
\]
and that when $a, b \in A_{an}$ then $F_{a,b}$ is the restriction of the function $z \mapsto \tau(a\sigma_z(b))$ to $D$.

4.5. A counterexample. Let $\mathcal{N}(Z_+^2)$ be the Toeplitz-Nica-Pimsner of the trivial system $id: Z_+^2 \to \text{End}(\mathbb{C})$. This is the universal C*-algebra generated by two doubly commuting isometries, say $V_{(1,0)}$ and $V_{(0,1)}$. We can extend $V$ to a semigroup action $V: Z_+^2 \to B(H)$ by isometries. Then the monomials $V_{a,b}V_{\tilde{a},\tilde{b}}^*$ span a dense subset of $\mathcal{N}(Z_+^2)$. Moreover $\mathcal{N}(Z_+^2)$ admits a gauge action $\{\gamma_z\} \subset \mathbb{T}^2$. Let the group action $\sigma: \mathbb{R}^2 \to \text{Aut}(\mathcal{N}(Z_+^2))$ defined by $\sigma_t = \sigma((1,1), \gamma(e^{it}, e^{it})).$ We will show that $\mathcal{N}(Z_+^2)$ does not admit $(\sigma,\Lambda_\beta)$-KMS states on $\beta \neq (0,0)$ and any prescribing set $\Lambda_\beta$ (with the trivial exception when $\beta_1 = 0$ or $\beta_2 = 0$; in this case there is reduction to the one-variable case).

First suppose that the prescribing set contains $\{(\beta_1,0), (0,\beta_2)\}$. Then a $(\sigma,\Lambda_\beta)$-KMS state $\psi$ would satisfy
\[
\psi(V_{(1,0)}V_{(1,0)}^*) = \psi(V_{(1,0)}\sigma_{(i\beta_1,0)}(V_{(1,0)})) = e^{-\beta_1}.
\]
On the other hand we get that
\[
\psi(V_{(1,0)}V_{(1,0)}^*) = \psi(V_{(1,0)}\sigma_{(0,i\beta_2)}(V_{(1,0)})) = \psi(V_{(1,0)}V_{(1,0)}) = 1.
\]
Therefore $\beta_1 = 0$; similarly $\beta_2 = 0$. Now suppose that $\Lambda_\beta = \{(\beta_1,0), (0,\beta_2)\}$. Then we obtain
\[
1 = \psi(V_{(1,0)}V_{(1,0)}) = \psi(V_{(1,0)}\sigma_{(i\beta_1,0)}(V_{(1,0)})) = e^{-\beta_1},
\]
hence $\beta_1 = 0$ and similarly $\beta_2 = 0$. Finally assume that $\Lambda_\beta$ contains $(\beta_1,0)$ and does not contain $(0,\beta_2)$. Then $\Lambda_\beta = \{(\beta_1,0), (\beta_1,\beta_2)\}$ because
\[
\psi(V_{\tilde{a},\tilde{b}}V_{\tilde{a},\tilde{b}}^*) = \psi(V_{\tilde{a},\tilde{b}}\sigma_{(i\beta_1,0)}(V_{\tilde{a},\tilde{b}}^*))
\]
\[
= \psi(V_{\tilde{a},\tilde{b}}\sigma_{(i\beta_1,0)}(V_{\tilde{a},\tilde{b}}^*))
\]
\[
= \psi(V_{\tilde{a},\tilde{b}}\sigma_{(\beta_1,\beta_2)}(V_{\tilde{a},\tilde{b}}^*)).
\]
It is evident that the $(\sigma,\Lambda_\beta)$-KMS states in this case are in bijection with the $(\sigma', \beta_1)$-KMS states with respect to the one-variable action $\sigma'_t = \sigma((t,0))$. Such states exist, but they are trivial in the sense that they do not give non-trivial multivariable states. The case when $\Lambda_\beta$ contains $(0,\beta_2)$ is settled in the dual way.
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