ON THE TOPOLOGICAL COMPLEXITY OF TORAL
RELATIVELY HYPERBOLIC GROUPS

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Abstract. We prove that the topological complexity $TC(\pi) = cd(\pi \times \pi)$
for certain toral relatively hyperbolic groups $\pi$.

1. Introduction

The (reduced) topological complexity $TC(X)$ of a space $X$ is defined as the
minimal integer $n$ for which there exists a cover of $X \times X$ by $n + 1$ open subsets
$U_0, \ldots, U_n$ such that the path fibration $X^{[0,1]} \to X \times X$ admits a local section
over each $U_i$. This quantity, which is similar in spirit to the classical Lusternik–
Schnirelmann category, was introduced by Farber [Far03] in the context of robot
motion planning. In fact, $TC(\pi)$ is a homotopy invariant and hence one can define
the topological complexity $TC(\pi)$ of a group $\pi$ to be $TC(B\pi)$, where $B\pi$ is the
classifying space for $\pi$. There are bounds $cd(\pi) \leq TC(\pi) \leq cd(\pi \times \pi)$, where
$cd(\pi)$ denotes the cohomological dimension. However, the precise value of $TC(\pi)$
is known only for a small class of groups, which contains for instance the abelian
groups, hyperbolic groups, free products of the form $H \ast H$ for $H$ geometrically
finite, right-angled Artin groups, and certain subgroups of braid groups. We refer
to [FM20] and [Dra20] for a more thorough account on this topic.

It is the decisive insight of [FGLO19] that the topological complexity of groups
can be expressed in terms of classifying spaces for families of subgroups, which
are well-studied objects in equivariant topology. For a family $F$ of subgroups of
a group $G$, the classifying space $E_FG$ is a terminal object, up to $G$-homotopy,
among $G$-CW-complexes with stabilizers in $F$. Farber, Grant, Lupton, and Oprea
showed that $TC(\pi)$ equals the minimal integer $n$ for which the canonical $(\pi \times \pi)$-
map $E(\pi \times \pi) \to E_D(\pi \times \pi)$ is equivariantly homotopic to a map with values in the
$n$-skeleton $E_D(\pi \times \pi)^{(n)}$. Here $D$ is the family of subgroups of $\pi \times \pi$ consisting of all
conjugates of the diagonal subgroup $\Delta(\pi)$ and their subgroups. Using this character-
ization of $TC(\pi)$, in a recent breakthrough Dranishnikov [Dra20] has computed
the topological complexity of torsionfree hyperbolic groups and more generally, of
geometrically finite groups with cyclic centralizers.

Theorem 1.1 (Dranishnikov). Let $\pi$ be a geometrically finite group with $cd(\pi) \geq 2$
such that the centralizer $Z_\pi(b)$ is cyclic for any $b \in \pi \setminus \{e\}$. Then $TC(\pi) = cd(\pi \times \pi)$.

Recall that a group $\pi$ is called geometrically finite if it admits a finite model
for $B\pi$. Note that for geometrically finite groups $\pi$ we have $cd(\pi \times \pi) = 2 cd(\pi)$,

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see [Dra19]. Previously, Farber and Mescher [FM20] had shown for groups \( \pi \) as in Theorem 1.1 that \( \text{TC}(\pi) \) equals either \( \text{cd}(\pi \times \pi) \) or \( \text{cd}(\pi \times \pi) - 1 \). The main contribution of the present note is the following generalization of Theorem 1.1.

**Theorem 1.2.** Let \( \pi \) be a torsionfree group with \( \text{cd}(\pi) \geq 2 \). Suppose that \( \pi \) admits a malnormal collection of abelian subgroups \( \mathcal{P} = \{P_i \mid i \in I\} \) satisfying \( \text{cd}(P_i \times P_i) < \text{cd}(\pi \times \pi) \) such that the centralizer \( Z_{\pi}(b) \) is cyclic for any \( b \in \pi \) that is not conjugate into any of the \( P_i \). Then \( \text{TC}(\pi) = \text{cd}(\pi \times \pi) \).

Recall that a set \( \mathcal{P} = \{P_i \mid i \in I\} \) of subgroups of \( \pi \) is called a malnormal collection if for any \( P_i, P_j \in \mathcal{P} \) and \( g \in \pi \), we have \( gP_ig^{-1} \cap P_j = \{e\} \) or \( i = j \) and \( g \in P_i \). Our main examples of groups satisfying the assumptions of Theorem 1.2 are torsionfree relatively hyperbolic groups \( \pi \) with \( \text{cd}(\pi) \geq 2 \) and finitely generated abelian peripheral subgroups \( P_1, \ldots, P_k \) satisfying \( \text{cd}(P_i) < \text{cd}(\pi) \). Note that Theorem 1.2 recovers Theorem 1.1 as a special case when \( \mathcal{P} \) consists only of the trivial subgroup and that the assumption of geometric finiteness has been dropped.

In light of the upper bound \( \text{TC}(\pi) \leq \text{cd}(\pi \times \pi) \), Theorem 1.1 and Theorem 1.2 are statements about the maximality of topological complexity. They share a common strategy of proof based on the characterization of \( \text{TC}(\pi) \) in terms of classifying spaces from [TGLO19]. Namely, we construct a “small” model for \( E_\mathcal{P}(\pi \times \pi) \) from \( E(\pi \times \pi) \) allowing us to show that the map \( E(\pi \times \pi) \to E_\mathcal{D}(\pi \times \pi) \) induces a non-trivial map on cohomology in degree \( \text{cd}(\pi \times \pi) \). Hence one has equality \( \text{TC}(\pi) = \text{cd}(\pi \times \pi) \). Nevertheless, even for the case when \( \mathcal{P} \) consists only of the trivial subgroup, our proof is different from Dranishnikov’s. He constructed a specific model for \( E_D(\pi \times \pi) \) and used cohomology with compact support, while we employ a general construction due to Lück and Weiermann and use equivariant Bredon cohomology. Lück and Weiermann’s construction (Theorem 2.1) is a general recipe to efficiently construct \( E_\mathcal{F}G \) from \( E_\mathcal{E}G \) for two families of subgroups \( \mathcal{E} \subset \mathcal{F} \) of a group \( G \) satisfying a certain maximality condition. While for the group \( \pi \times \pi \) this condition is not satisfied for the families \( \{\{e\}\} \subset \mathcal{D} \), we define an intermediate family \( \{\{e\}\} \subset \mathcal{F}_1 \subset \mathcal{D} \) such that we can apply two iterations of the construction.

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2. Preliminaries on classifying spaces for families

We briefly review the notion of classifying spaces for families of subgroups due to tom Dieck and their equivariant Bredon cohomology. For a survey on classifying spaces for families we refer to [Liu18] and for an introduction to Bredon cohomology to [Flu]. Let \( G \) be a group, which shall always mean a discrete group.

A family of subgroups \( \mathcal{F} \) is a non-empty set of subgroups of \( G \) that is closed under conjugation by elements of \( G \) and under taking subgroups. Typical examples are \( \mathcal{TR} = \{\{e\}\} \), \( \mathcal{FLN} = \{\text{finite subgroups}\} \), \( \mathcal{VCY} = \{\text{virtually cyclic subgroups}\} \), and \( \mathcal{ALL} = \{\text{all subgroups}\} \). For a set \( \mathcal{H} \) of subgroups of \( G \), one can consider \( \mathcal{F}(\mathcal{H}) = \{\text{conjugates of subgroups in } \mathcal{H} \text{ and their subgroups}\} \) which is the smallest family containing \( \mathcal{H} \) and called the family generated by \( \mathcal{H} \). When \( \mathcal{H} = \{H\} \) consists
of a single subgroup, we denote $\mathcal{F}(\{H\})$ instead by $\mathcal{F}(H)$ and call it the family generated by $H$. For a family $\mathcal{F}$ of subgroups of $G$ and any subgroup $H \subset G$, we denote by $\mathcal{F}|_H$ the family $\{K \cap H \mid K \in \mathcal{F}\}$ of subgroups of $H$. (In the literature this family is sometimes denoted by $\mathcal{F} \cap H$ instead.)

A classifying space $E_{\mathcal{F}}G$ for the family $\mathcal{F}$ is a terminal object in the $G$-homotopy category of $G$-CW-complexes with stabilizers in $\mathcal{F}$. It can be shown that $E_{\mathcal{F}}G$ always exists and that a $G$-CW-complex $X$ is a model for $E_{\mathcal{F}}G$ if and only if the fixed-point set $X^H$ is contractible for $H \in \mathcal{F}$ and empty otherwise. In particular, there exists a $G$-map $EG \to E_{\mathcal{F}}G$ which is unique up to $G$-homotopy.

The orbit category $\mathcal{O}_{\mathcal{F}}G$ has as objects $G/H$ for $H \in \mathcal{F}$ and as morphisms $G$-maps. Let $\mathcal{O}_{\mathcal{F}}G$,Mod denote the category of contravariant functors $M: \mathcal{O}_{\mathcal{F}}G \to \text{Z-Mod}$ with values in the category of $\mathbb{Z}$-modules, which are called $\mathcal{O}_{\mathcal{F}}G$-modules. For a $G$-CW-complex $X$ with stabilizers in $\mathcal{F}$, the $G$-equivariant Bredon cohomology $H^*_G(X;M)$ with coefficients in an $\mathcal{O}_{\mathcal{F}}G$-module $M$ is the cohomology of the cochain complex $\text{Hom}_{\mathcal{O}_{\mathcal{F}}G\text{-Mod}}(C^*(X^i), M)$, where $C^*(X^i)(G/H) = C^*(X^H)$ is the cellular chain complex.

**Passage to larger families.** Let $G$ be a group and $\mathcal{E} \subset \mathcal{F}$ be two families of subgroups.

We say that $G$ satisfies condition $(M_{\mathcal{E} \subset \mathcal{F}})$ if every element $H \in \mathcal{F} \setminus \mathcal{E}$ is contained in a unique element $M \in \mathcal{F} \setminus \mathcal{E}$ which is maximal in $\mathcal{F} \setminus \mathcal{E}$ (with respect to inclusion). We say that $G$ satisfies condition $(NM_{\mathcal{E} \subset \mathcal{F}})$ if every maximal element $M \in \mathcal{F} \setminus \mathcal{E}$ is self-normalizing, i.e. $M$ equals its normalizer $N_GM$ in $G$. Let $\mathcal{M} = \{M_i \mid i \in I\}$ be a complete set of representatives for the conjugacy classes of maximal elements in $\mathcal{F} \setminus \mathcal{E}$, i.e. each $M_i$ is maximal in $\mathcal{F} \setminus \mathcal{E}$ and any maximal element in $\mathcal{F} \setminus \mathcal{E}$ is conjugate to precisely one of the $M_i$. The following [LW12, Corollary 2.8] is a special case of a more general construction due to Lück and Weiermann.

**Theorem 2.1** (Lück–Weiermann). Let $G$ be a group satisfying condition $(M_{\mathcal{E} \subset \mathcal{F}})$ for two families of subgroups $\mathcal{E} \subset \mathcal{F}$. Consider a cellular $G$-pushout of the form

$$
\begin{array}{ccc}
\bigsqcup_{i \in I} G \times_{N_GM_i} E_{\mathcal{N}_G M_i}(N_GM_i) & \xrightarrow{\varphi} & E_{\mathcal{E}}G \\
\bigsqcup_{i \in I} \text{id}_G \times_{N_GM_i} f_i & \downarrow & \\
\bigsqcup_{i \in I} G \times_{N_GM_i} E_{\mathcal{A} \mathcal{L} \mathcal{C}|_{M_i \cup \mathcal{E}}M_i}(N_GM_i) & \xrightarrow{} & X
\end{array}
$$

such that each $f_i$ is a cellular $N_GM_i$-map and $\varphi$ is an inclusion of $G$-CW-complexes, or such that each $f_i$ is an inclusion of $N_GM_i$-CW-complexes and $\varphi$ is a cellular $G$-map. Then $X$ is a model for $E_{\mathcal{E}}G$.

Note that a $G$-pushout as in Theorem 2.1 with maps $f_i$ and $\varphi$ as required always exists by using equivariant cellular approximation and mapping cylinders.

**Corollary 2.2.** Let $G$ be a group and $\mathcal{E} \subset \mathcal{F}$ be two families of subgroups.

(i) If $G$ satisfies condition $(M_{\mathcal{TR} \subset \mathcal{F}})$, then a model for $E_{\mathcal{F}}G$ can be constructed as a $G$-pushout of the form

$$
\begin{array}{ccc}
\bigsqcup_{i \in I} G \times_{N_GM_i} E(N_GM_i) & \xrightarrow{} & EG \\
\bigsqcup_{i \in I} G \times_{N_GM_i} E(N_GM_i/M_i) & \xrightarrow{} & E_{\mathcal{F}}G
\end{array}
$$

(ii) If $G$ satisfies condition $(NM_{\mathcal{TR} \subset \mathcal{F}})$, then a model for $E_{\mathcal{F}}G$ can be constructed as a $G$-pushout of the form

$$
\begin{array}{ccc}
\bigsqcup_{i \in I} G \times_{N_GM_i} E(N_GM_i) & \xrightarrow{} & EG \\
\bigsqcup_{i \in I} G \times_{N_GM_i} E(N_GM_i/M_i) & \xrightarrow{} & E_{\mathcal{F}}G
\end{array}
$$
Lemma 3.1. Let $\pi$ be a group. Then $TC(\pi) = hdim_{\pi \times \pi}$.

Proof. This follows from Theorem 2.1 by observing that if $E|_{N_GM_i} \subset \mathcal{AC}|_{M_i}$, then a model for $E_{\mathcal{AC}|_{M_i},\mathcal{AC}|_{N_GM_i}}(N_GM_i)$ is given by $E(N_GM_i/M_i)$ regarded as a $N_GM_i$-CW-complex. $\square$

Homotopy dimension and cohomological dimension of maps. Let $G$ be a group and $\mathcal{E} \subset \mathcal{F}$ be two families of subgroups. The following notation is not standard.

We denote by $cd_{G}(\mathcal{E})$ the minimal integer $n$ for which the canonical $G$-map $E_G \rightarrow E_{\mathcal{F}}$ is $G$-homotopic to a $G$-map with values in the $n$-skeleton $(E_{\mathcal{F}})^{(n)}$. We denote by $cd_{G}(\mathcal{E})$ the maximal integer $k$ for which the induced map on Bredon cohomology $H^{G}_{E}(E_{\mathcal{F}};M) \rightarrow H^{G}_{E}(E_G;M)$ is non-trivial for some $O_{G}G$-module $M$. One clearly has the inequality

$$cd_{G}(\mathcal{E}) \leq hdim_{G}(\mathcal{F}).$$

Topological complexity as homotopy dimension. Let $\pi$ be a group and $\Delta(\pi) \subset \pi \times \pi$ be the diagonal subgroup. Consider the family $\mathcal{D} := \mathcal{F}(\Delta(\pi))$ of subgroups of $\pi \times \pi$ that is generated by $\Delta(\pi)$. The following is the main result of [FGLO19, Theorem 3.3].

Theorem 2.3 (Farber–Grant–Lupton–Oprea). Let $\pi$ be a group. Then $TC(\pi) = hdim_{\pi \times \pi}(\mathcal{D})$.

Theorem 2.3 was recently generalized to families generated by a single subgroup in [BCE, Theorem 1.1] and to arbitrary families in [CLM, Proposition 7.5].

3. Structure of the diagonal family of $\pi \times \pi$

Let $\pi$ be a group and $\Delta : \pi \rightarrow \pi \times \pi$ be the diagonal map. For a subset $S \subset \pi$, denote by $Z(\pi)(S)$ the centralizer of $S$ in $\pi$. The following notation is adopted from [FGLO19] and [Dra20].

For $\gamma \in \pi$ and a subset $S \subset \pi$, define the subgroup $H_{\gamma,S}$ of $\pi \times \pi$ to be

$$H_{\gamma,S} := (\gamma,e) \cdot \Delta(Z_{\pi}(S)) \cdot (\gamma^{-1},e).$$

When $S$ is a singleton set $\{b\}$, we write $H_{\gamma,b}$ instead of $H_{\gamma,(b)}$. Note that $H_{e,e} = \Delta(\pi)$. The proof of the following identities is elementary and left to the reader.

Lemma 3.1. Let $\gamma, \delta \in \pi$ and $S,T \subset \pi$ be subsets. Then the following hold:

(i) $(g,h) \cdot H_{\gamma,S} \cdot (g^{-1},h^{-1}) = H_{\gamma(\delta^{-1})h^{-1},\delta^{-1}}$ for any $(g,h) \in \pi \times \pi$;

(ii) $H_{\gamma,S} \cap H_{\delta,T} = H_{\gamma,S \cup \delta^{-1}}$;

(iii) $N_{\pi}H_{\gamma,S} = \{k \in \pi \times \pi \mid h \in Z_{\pi}(S), k \in Z_{\pi}(Z_{\pi}(S))\}.

We define the families $\mathcal{G}_1 \subset \mathcal{D}$ of subgroups of $\pi \times \pi$ to be

$$\mathcal{G} := \mathcal{F}(\Delta(\pi));$$

$$\mathcal{G}_1 := \mathcal{F}(\{H_{\gamma,b} \mid \gamma \in \pi, b \in \pi \setminus \{e\}\}).$$


In view of Lemma 3.1(i) and (ii) the family $F_1$ is generated by the intersections of conjugates of the diagonal subgroup $\Delta(\pi)$.

**Lemma 3.2.** Let $\pi$ be a group. Then condition $(M_{F_1 \subset D})$ holds for the group $\pi \times \pi$. Moreover, if the center $Z_{\pi}(\pi)$ of $\pi$ is trivial, then condition $(NM_{F_1 \subset D})$ holds.

**Proof.** If $F_1$ equals $D$, then the statement is vacuous, so we may assume that $F_1$ is strictly contained in $D$. For $\gamma \in \pi$, conjugates of $H_{\gamma,e}$ are of the form $H_{\delta,e}$ for some $\delta \in \pi$ by Lemma 3.1(i). If $\gamma \neq \delta$, then $H_{\gamma,e} \cap H_{\delta,e} \in F_1$ by Lemma 3.1(ii). Hence the $\{H_{\gamma,e} \mid \gamma \in \pi\}$ are precisely the maximal elements in $D \setminus F_1$ and condition $(M_{F_1 \subset D})$ holds. Moreover, given that $Z_{\pi}(\pi)$ is trivial, we have $N_{\pi \times \pi}(H_{\gamma,e}) = H_{\gamma,e}$ by Lemma 3.1(iii). □

From now on and for the remainder of this note, we specialize to the following situation.

**Setup 3.3.** Let $\pi$ be a torsionfree group admitting a malnormal collection of abelian subgroups $\mathcal{P} = \{P_i \mid i \in I\}$ such that the centralizer $Z_{\pi}(b)$ is cyclic for any $b \in \pi$ that is not conjugate into any of the $P_i$.

Note that in the situation of Setup 3.3, we have $N_{\pi}(Z_{\pi}(P_i)) = Z_{\pi}(P_i) = P_i$ for every $P_i \in \mathcal{P}$. Our main examples of groups as in Setup 3.3 are torsionfree relatively hyperbolic groups with finitely generated abelian peripheral subgroups, so-called toral relatively hyperbolic groups.

The following lemma for the case when $\mathcal{P} = \{\{e\}\}$ can be found in [FGLO19] Lemma 8.0.4 from where the first part of the proof is recalled.

**Lemma 3.4.** Let $\pi$ be a group as in Setup 3.3. Then for $b, c \in \pi \setminus \{e\}$, we have either $Z_{\pi}(b) = Z_{\pi}(c)$ or $Z_{\pi}(b) \cap Z_{\pi}(c) = \{e\}$.

**Proof.** Let $b, c \in \pi \setminus \{e\}$ be two elements. Suppose neither $b$ nor $c$ are conjugate into any of the $P_i$ and that $Z_{\pi}(b) \cap Z_{\pi}(c)$ is non-trivial. Let $Z_{\pi}(b)$, $Z_{\pi}(c)$ and $Z_{\pi}(b) \cap Z_{\pi}(c)$ be generated by $x$, $y$ and $z$, respectively. Then $x^n = z = y^m$ for some $n, m \in \mathbb{Z}$. Observe that $z$ is not conjugate into any of the $P_i$. Thus its centralizer $Z_{\pi}(z)$ is infinite cyclic and contains both $x$ and $y$. Therefore, $x$ and $y$ commute and it follows that $Z_{\pi}(b) = Z_{\pi}(c)$.

Suppose $b \in \pi \setminus \{e\}$ and $c \in gP_ig^{-1}$ for some $g \in \pi$, $P_i \in \mathcal{P}$. Note that $Z_{\pi}(c) = gP_ig^{-1}$. If $Z_{\pi}(b) \cap gP_ig^{-1}$ is non-trivial, then $b \in gP_ig^{-1}$ by malnormality of $\mathcal{P}$ and hence $Z_{\pi}(b) = Z_{\pi}(c)$. □

**Lemma 3.5.** Let $\pi$ be a group as in Setup 3.3. Then we have the following:

1. Condition $(M_{TR \subset F_1})$ holds for the group $\pi \times \pi$. Moreover, for $\gamma \in \pi$ and $b \in \pi \setminus \{e\}$ there is an isomorphism $N_{\pi \times \pi}H_{\gamma,b} \cong Z_{\pi}(b) \times Z_{\pi}(b)$;
2. Conditions $(M_{TR \subset F_1} | H_{c,e})$ and $(NM_{TR \subset F_1} | H_{c,e})$ hold for the group $H_{c,e}$.

**Proof.** (i) For $\gamma \in \pi$ and $b \in \pi \setminus \{e\}$, conjugates of $H_{\gamma,b}$ are of the form $H_{\delta,c}$ for some $\delta \in \pi$, $c \in \pi \setminus \{e\}$ by Lemma 3.1(i). We have either $H_{\gamma,b} = H_{\delta,c}$ or $H_{\gamma,b} \cap H_{\delta,c} = \{e\}$ by Lemma 3.1(ii) and Lemma 3.4. Hence the $\{H_{\gamma,b} \mid \gamma \in \pi, b \in \pi \setminus \{e\}\}$ are precisely the maximal elements in $F_1 \setminus TR$ and condition $(M_{TR \subset F_1})$ holds. Moreover, for $b \in \pi$ that is not conjugate into any of the $P_i$, observe that $N_{\pi}(Z_{\pi}(b))$ is torsionfree virtually cyclic and hence infinite cyclic. It follows that $N_{\pi}(Z_{\pi}(b)) = \{e\}$. □
Theorem 4.1. Let $\pi$ be a torsionfree group with $\text{cd}(\pi) \geq 2$. Suppose that $\pi$ admits a malnormal collection of abelian subgroups $\mathcal{P} = \{P_i \mid i \in I\}$ satisfying $\text{cd}(P_i \times P_i) < \text{cd}(\pi \times \pi)$ such that the centralizer $Z_\pi(b)$ is cyclic for any $b \in \pi$ that is not conjugate into any of the $P_i$. Then $\text{cd}_{\mathcal{TR} \subseteq \mathcal{D}}(\pi \times \pi) = \text{cd}(\pi \times \pi)$.

Proof. We denote $\text{cd}(\pi \times \pi)$ by $n$ and may assume that it is finite. Consider the families $\mathcal{TR} \subseteq \mathcal{F}_1 \subseteq \mathcal{D}$ of subgroups of $\pi \times \pi$ as defined in (1).

First, condition $(M_{\mathcal{TR} \subseteq \mathcal{F}_1})$ holds by Lemma 3.5 (1) and hence Corollary 2.2 (1) yields a $(\pi \times \pi)$-pushout

$$
\begin{array}{ccc}
\prod_{\gamma,b \in \mathcal{M}} (\pi \times \pi) \times \text{N}_{n \times n} \text{H}_{\gamma,b} E(N_{\pi \times \pi} \text{H}_{\gamma,b}) & \longrightarrow & E(\pi \times \pi) \\
\downarrow & & \downarrow \\
\prod_{\gamma,b \in \mathcal{M}} (\pi \times \pi) \times \text{N}_{n \times n} \text{H}_{\gamma,b} E(N_{\pi \times \pi} \text{H}_{\gamma,b}/\text{H}_{\gamma,b}) & \longrightarrow & E_{\mathcal{F}_1}(\pi \times \pi),
\end{array}
$$

(2)

where $\mathcal{M}$ is a complete set of representatives of conjugacy classes of maximal elements in $\mathcal{F}_1 \setminus \mathcal{TR}$. Moreover, in Lemma 3.5 (i) we identified $N_{\pi \times \pi} \text{H}_{\gamma,b} \cong Z_\pi(b) \times Z_\pi(b)$ which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ or $P_i \times \mathbb{Z}$ for some $P_i \in \mathcal{P}$ and hence has cohomological dimension strictly less than $n$. Thus, for any $\mathcal{O}_D(\pi \times \pi)$-module $M$, we have

$$H^n_{\pi \times \pi}(E_{\mathcal{F}_1}(\pi \times \pi); M) = 0.$$ 

Applying the Mayer–Vietoris sequence for $H^n_{\pi \times \pi}(\pi \times \pi; M)$ to the pushout (2) yields that the map

$$H^n_{\pi \times \pi}(E_{\mathcal{F}_1}(\pi \times \pi); M) \to H^n_{\pi \times \pi}(E(\pi \times \pi); M)$$

is surjective.

Second, conditions $(M_{\mathcal{F}_1 \subseteq \mathcal{D}})$ and $(NM_{\mathcal{F}_1 \subseteq \mathcal{D}})$ hold by Lemma 3.2 and hence Corollary 2.2 (ii) yields a $(\pi \times \pi)$-pushout

$$
\begin{array}{ccc}
(\pi \times \pi) \times H_{c,e} E_{\mathcal{F}_1/H_{c,e}}(H_{c,e}) & \longrightarrow & E_{\mathcal{F}_1}(\pi \times \pi) \\
\downarrow & & \downarrow \\
(\pi \times \pi)/H_{c,e} & \longrightarrow & E_{\mathcal{D}}(\pi \times \pi).
\end{array}
$$

(3)

Applying the Mayer–Vietoris sequence for $H^n_{\pi \times \pi}(\pi \times \pi; M)$ to the pushout (3) yields that the map

$$H^n_{\pi \times \pi}(E_{\mathcal{D}}(\pi \times \pi); M) \to H^n_{\pi \times \pi}(E_{\mathcal{F}_1}(\pi \times \pi); M)$$
is surjective provided that

\[ H_n^\pi \times \pi \times \pi_{H_{e,e}}(E_{T_1}; M) = 0. \]

The latter is true by another application of Corollary 2.2 (ii) using that conditions \((M \subset \mathcal{F}_1 \setminus H_{e,e})\) and \((NM \subset \mathcal{F}_1 \setminus H_{e,e})\) hold for the group \(H_{e,e}\) by Lemma 3.5 (ii). It yields an \(H_{e,e}\)-pushout

\[ \bigoplus_{H_{e,b} \in M'} \bigoplus_{H_{e,e}} E(H_{e,b}) \to E(H_{e,e}) \]

\[ \bigoplus_{H_{e,b} \in M'} \bigoplus_{H_{e,e}} E(H_{e,b}) \to E_{T_1}(H_{e,e}), \]

where \(M'\) is a complete set of representatives of conjugacy classes of maximal elements in \(\mathcal{F}_1 \setminus H_{e,e} \setminus \mathcal{T}\). The Mayer–Vietoris sequence for \(H_{H_{e,e}}(\pi; M)\) applied to the pushout \(\bigoplus\) shows that (4) indeed holds, using that \(\text{cd}(H_{e,e}) < n\) and \(\text{cd}(H_{e,b}) < n - 1\) for \(b \in \pi \setminus \{e\}\).

Together, the map

\[ H_n^\pi \times \pi(\pi; E_D(\pi \times \pi); M) \to H_n^\pi(\pi; \pi; M) \]

is surjective for any \(O_D(\pi \times \pi)-\)module \(M\). Finally, the coefficients \(M\) can be chosen such that \(H_n^\pi(\pi; \pi; M)\) is non-trivial. This concludes the proof. □

Proof of Theorem 1.2. It follows from Theorem 4.1 that the inequalities

\[ \text{cd}_{T_1}(\pi \times \pi) \leq \text{TC}(\pi) \leq \text{cd}(\pi \times \pi) \]

are in fact equalities. □

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