AN ASYMPTOTICALLY STABLE CUSP-FOLD SINGULARITY IN 3D PIECEWISE SMOOTH VECTOR FIELDS.

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1. Introduction

In this paper we study piecewise smooth vector fields (PSVF for short) $Z$ on $\mathbb{R}^3$. Our goal is to describe the local dynamics around typical singularities of $Z$ consisting of two smooth vector fields $X, Y$ in $\mathbb{R}^3$ such that on one side of a smooth surface $\Sigma = \{z = 0\}$ we take $Z = X$ and on the other side $Z = Y$.

PSVF are widely used in Electrical and Electronic Engineering, Physics, Economics, among other areas. In our approach Filippov’s convention (see [9]) is considered. So, the vector field is discontinuous across the switching manifold $\Sigma$ and it is possible for its trajectories to be confined onto the switching manifold itself. The occurrence of such behavior, known as sliding motion, has been reported in a wide range of applications (see for instance [2] and references therein).

The main tool treated here concerns the non transversal contact between a general smooth vector field and the boundary $\Sigma$ of a manifold. Such points are distinguished singularities — important objects to be analyzed when one studies Filippov systems (see [5] for a planar analysis on this subject). In the 3-dimensional case, there are two important distinguished generic singularities: the points where this contact is either quadratic or cubic, which are called fold points and cusp points respectively. As it is fairly known, from a generic cusp point emanate two branches of fold points.
points (see Figure 1), one of such branches formed by visible fold points, where the trajectories tangent are visible and one of such branches formed by invisible fold points, where the trajectories tangent are not visible. Moreover, it is possible for a point $p \in \Sigma$ be a tangency point for both $X$ and $Y$. When $p$ is a fold point of both $X$ and $Y$ we say that $p$ is a two-fold singularity and when $p$ is a cusp point for $X$ and a fold point for $Y$ we say that $p$ is a cusp-fold singularity (see Figure 1 below). In [7, 8, 11, 12] two-fold singularities are studied and their normal forms and phase portraits are exhibited and in [3, 4] are exhibited applications of such theory in electrical and control systems, respectively. This singularity is particularly relevant because in its neighborhood some of the key features of a piecewise smooth system are present: orbits that cross $\Sigma$, those that slide along it according to Filippov’s convention, among others.

In this paper we analyze the bifurcation diagram and the asymptotical stability of the following family presenting a fold-cusp singularity:

\[
Z_{\lambda}(x, y, z) = \begin{cases} 
X_{a,b}^\lambda = \begin{pmatrix} a \\ \lambda \\ b(y + x^2) \end{pmatrix} & \text{if } z \geq 0, \\
Y_{c,d} = \begin{pmatrix} c \\ d \\ x \end{pmatrix} & \text{if } z \leq 0,
\end{cases}
\]

with $a, b, c, d, \lambda \in \mathbb{R}$, $b \cdot c \neq 0$ and $\lambda$ is arbitrarily small. Moreover, we observe that the topological dynamic of this still poor studied object is even more sophisticated than that one exhibited by the two-fold singularity. In fact, by means of the variation of the parameter $\lambda$ occurs the birth of two-fold singularities approaching the cusp-fold singularity.

The paper is organized as follows: In Section 2 we formalize some basic concepts on PSVF's and the concept of first return map in this scenario is formalized. In Section 3 the problem is described, the main results are stated and we pave the way in order to prove the main results in Section 4.
2. Basic Theory about PSVF's

Let $K = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < \delta\}$, where $\delta > 0$ is arbitrarily small. Consider $\Sigma = \{(x, y, z) \in K \mid z = 0\}$. Clearly the switching manifold $\Sigma$ is the separating boundary of the regions $\Sigma_+ = \{(x, y, z) \in K \mid z \geq 0\}$ and $\Sigma_- = \{(x, y, z) \in K \mid z \leq 0\}$.

Designate by $\chi$ the space of $C^r$-vector fields on $K$ endowed with the $C^r$-topology with $r = \infty$ or $r \geq 1$ large enough for our purposes. Call $\Omega^r$ the space of vector fields $Z : K \to \mathbb{R}^3$ such that

\begin{equation}
Z(x, y, z) = \begin{cases} 
X(x, y, z), & \text{for } (x, y, z) \in \Sigma_+, \\
Y(x, y, z), & \text{for } (x, y, z) \in \Sigma_-, 
\end{cases}
\end{equation}

where $X = (X_1, X_2, X_3)$ and $Y = (Y_1, Y_2, Y_3)$ are in $\chi$. We may consider $\Omega^r = \chi^r \times \chi^r$ endowed with the product topology and denote any element in $\Omega^r$ by $Z = (X, Y)$, which we will accept to be multivalued in points of $\Sigma$. The basic results of differential equations, in this context, were stated by Filippov in [9]. Related theories can be found in [2, 20] and references therein. On $\Sigma$ we generically distinguish the following regions:

- **Crossing Region:** $\Sigma^c = \{p \in \Sigma \mid X_3(p).Y_3(p) > 0\}$. Moreover, we denote $\Sigma^{c+} = \{p \in \Sigma \mid X_3(p) > 0, Y_3(p) > 0\}$ and $\Sigma^{c-} = \{p \in \Sigma \mid X_3(p) < 0, Y_3(p) < 0\}$.
- **Sliding Region:** $\Sigma^s = \{p \in \Sigma \mid X_3(p) < 0, Y_3(p) > 0\}$.
- **Escaping Region:** $\Sigma^e = \{p \in \Sigma \mid X_3(p) > 0, Y_3(p) < 0\}$.

When $q \in \Sigma^s$, following the Filippov’s convention, the **sliding vector field** associated to $Z \in \Omega^r$ is the vector field $\hat{Z}^s$ tangent to $\Sigma^s$ expressed in coordinates as

\begin{equation}
\hat{Z}^s(q) = \frac{1}{(Y_3 - X_3)(q)}((X_1Y_3 - Y_1X_3)(q), (X_2Y_3 - Y_2X_3)(q), 0).
\end{equation}

Associated to [3] there exists the planar **normalized sliding vector field**

\begin{equation}
Z^s(q) = ((X_1Y_3 - Y_1X_3)(q), (X_2Y_3 - Y_2X_3)(q)).
\end{equation}

Observe that $\hat{Z}^s$ and $Z^s$ are topologically equivalent in $\Sigma^s$ and $Z^s$ can be $C^r$-extended to the closure $\overline{\Sigma^s}$ of $\Sigma^s$.

**Lemma 1.** Given $Z = (X, Y) \in \Omega^r$ if $q \in \Sigma$ is a two-fold or a cusp-fold singularity of $Z$, then $q$ is an equilibrium point of the normalized sliding vector field given in [4].

**Proof.** It is straightforward since both $X$ and $Y$ are tangent to $\Sigma$ at $q$. So, $X_3(q) = Y_3(q) = 0$ and $Z^s(q) = (0, 0)$. \qed

In fact, the previous lemma remains true when the trajectories of both $X$ and $Y$ have a non transversal contact point at $q$ regardless the order of such contact.

The points $q \in \Sigma$ such that $Z^s(q) = 0$ are called **pseudo equilibria** of $Z$ and the points $p \in \Sigma$ such that $X_3(p).Y_3(p) = 0$ are called **tangential**
singularities of \( Z \) (i.e., the trajectory through \( p \) is tangent to \( \Sigma \)). Note that two-fold and cusp-fold singularities are both pseudo equilibria and tangential singularities.

Notations:

- We denote the flow of a vector field \( W \in \chi^r \) by \( \phi_W(t,p) \) where \( t \in I \) with \( I = I(p,W) \subset \mathbb{R} \) being an interval depending on \( p \in K \) and \( W \).
- Given a vector field \( W \) defined in \( A \subset K \), we denote the \textbf{backward trajectory} \( \phi_W^{-1}(A) \) (respectively, \textbf{forward trajectory} \( \phi_W(A) \)) the set of all negative (respectively, positive) orbits of \( W \) through points of \( A \).
- We denote the boundary of an arbitrary set \( A \subset K \) by \( \partial A \).

Following [10], page 1971, we consider the definition:

**Definition 2.** The forward local trajectory \( \phi_Z^+(t,p) \) of a PSVF given by (2) through \( p \in \Sigma \) is defined as follows:

(i) \( \phi_Z^+(t,p) = \phi_X(t,p) \) (respectively, \( \phi_Z^+(t,p) = \phi_Y(t,p) \)) provided that \( p \in \Sigma^c^+ \) (respectively, \( p \in \Sigma^c^- \)).

(ii) \( \phi_Z^+(t,p) = \phi_Z^-(t,p) \) provided that \( p \in \Sigma^s \).

(iii) \( \text{splits in two orbits} \) \( \phi_Z^+(t,p) = \phi_X(t,p) \) and \( \phi_Z^+(t,p) = \phi_Y(t,p) \) provided that \( p \in \Sigma^c \).

(iv) For \( p \in \partial \Sigma^s \cup \partial \Sigma^c \cup \partial \Sigma^e \) such that the definitions of forward trajectories for points in a full neighborhood of \( p \) in \( \Sigma \) can be extended to \( p \) and coincide, the trajectory through \( p \) is this trajectory.

(v) For any other point \( \phi_Z^+(t,p) = p \) for all \( t \in \mathbb{R} \).

2.1. \textbf{The fist return map.} The following construction is presented in [20]. Let \( Z = (X,Y) \in \Omega^r \) such that \( q \) is an invisible fold point of \( X \). From Implicit Function Theorem, for each \( p \in \Sigma \) in a neighborhood \( \mathcal{V}_q \) of \( q \) there exists a unique \( t(p) \in (-\delta,\delta) \), a small interval, such that \( \phi_X(t,p) \) meets \( \Sigma \) in \( p = \phi_X(t(p),p) \). Define the map \( \gamma_X : \mathcal{V}_q \cap \Sigma \rightarrow \mathcal{V}_q \cap \Sigma \) by \( \gamma_X(p) = \bar{p} \). This map is a \( C^r \)-diffeomorphism and satisfies: \( \gamma_X^2 = Id \). Analogously, when \( \bar{q} \) is an invisible fold point of \( Y \) we define the map \( \gamma_Y : \mathcal{V}_{\bar{q}} \cap \Sigma \rightarrow \mathcal{V}_{\bar{q}} \cap \Sigma \) associated to \( Y \) which satisfies \( \gamma_Y^2 = Id \). Now we can give the following definition:

**Definition 3.** Let \( \mathcal{T} \subset \Sigma^c \) be an open region of \( \Sigma \). The \textbf{First Return Map} \( \varphi_Z : \mathcal{T} \rightarrow \mathcal{T} \) is defined by the composition \( \varphi_Z = \gamma_Y \circ \gamma_X \) when \( \gamma_X(\mathcal{T}) \subset \Sigma^c \) and \( \gamma_X, \gamma_Y \) are well defined in \( \mathcal{T}, \gamma_X(\mathcal{T}) \) respectively.

Considering the PSVF given in (1), we get the expression of the first return map

(5) \[ \varphi_Z(x,y) = \left( \frac{2ax + \Delta_1}{4a}, y + \frac{d(2ax + \Delta_1)}{2ac} + \frac{\lambda(-6ax - \Delta_1)}{4a^2} \right) \]

where \( \Delta_1 = 3\lambda - \sqrt{9\lambda^2 + 36a\lambda x - 12a^2(x^2 + 4y)} \).
Note that we can extend $\varphi_{Z_\lambda}$ to the boundary of $SwR$. In this way, the unique fixed point of $\varphi_{Z_\lambda}$, in a neighborhood of origin, is the origin. Let

$$\Delta_2 = (ad)^2 - aed\lambda.$$ 

When $\lambda \neq 0$, the eigenvalues of $D\varphi_{Z_\lambda}$ at origin are

$$\xi_\pm = \frac{2ad - c\lambda \pm 2\sqrt{\Delta_2}}{c\lambda},$$ 

the eigenvectors associated to $\xi_\pm$ and $\xi_-\pm$ respectively, are

$$v_\pm = (\omega_\pm, 1),$$ 

where $\omega_\pm = \frac{ac}{ad \pm \sqrt{\Delta_2}}$ and the eigenspaces associated to $\xi_\pm$, respectively, are tangent to the straight lines

$$S_\pm = \left\{ (x, y, 0) \in \Sigma | x = \frac{ac}{ad \pm \sqrt{\Delta_2}}y \right\}.$$ 

3. Main Results

The main results of the paper are now stated.

**Theorem A.** Let $Z_\lambda$ given by (1) presenting a fold-cusp singularity. If $a < 0$, $b < 0$, $c > 0$, $d > 0$ and $a + bd > 0$ then:

- $Z_\lambda$ is asymptotically stable when $\lambda \geq 0$
- $Z_\lambda$ is not asymptotically stable when $\lambda < 0$.

3.1. **Proof for the case $\lambda = 0$.** Let (1) with $\lambda = 0$, i.e., the following normal form presenting a cusp-fold singularity at the origin:

$$Z_0(x, y, z) = \begin{cases} 
X_{a,b} = \begin{pmatrix} a \\ 0 \\ b(y + x^2) \end{pmatrix} & \text{if } z \geq 0, \\
Y_{c,d} = \begin{pmatrix} c \\ d \\ x \end{pmatrix} & \text{if } z \leq 0.
\end{cases}$$ 

Note that $S_X = \{(x, y, z) \in \Sigma | y = -x^2\}$ and $S_Y = \{(x, y, z) \in \Sigma | x = 0\}$ are the sets of tangential singularities of $X$ and $Y$ respectively.

3.1.1. **Local dynamics of the normalized sliding vector fields.** Using (1), the normalized sliding vector field is given by

$$Z_0^\Sigma = (ax - bc(y + x^2), -db(y + x^2)).$$ 

So, the eigenvalues of $Z_0^\Sigma$ are

$$\lambda_1^0 = a \text{ and } \lambda_2^0 = -db,$$
the eigenvectors associated to $\lambda_1^0$ and $\lambda_2^0$ respectively, are

$$v_1^0 = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \quad \text{and} \quad v_2^0 = \left( \frac{bc}{a+bd} \right) \frac{1}{1}$$

and the eigenspaces associated to $\lambda_1^0$ and $\lambda_2^0$ respectively, are

$$E_1^0 = \{(x, y, 0) \in \Sigma \mid y = 0\} \quad \text{and} \quad E_2^0 = \{(x, y, 0) \in \Sigma \mid y = \frac{(a + bd)x}{bc}\}.$$  

In order to obtain a cusp-fold singularity asymptotically stable some hypotheses must be imposed in the parameters.

**Hypothesis 1 ($H_1$):** The fold point generated by the vector field $Y$ must be invisible. So, $c > 0$.

**Hypothesis 2 ($H_2$):** The origin must be asymptotically stable for $Z^\Sigma$. So $\lambda_1^0 = a < 0$ and $-\lambda_2^0 = bd > 0$.

Note that, since $a < 0$, the vector field $X^0$ goes from the right to the left. This implies that the vector field $Y$ goes from the left to the right in order to permit recurrences. In fact this happens because $c > 0$, according to $H_1$.

Following $H_1$ and $H_2$, the phase portrait of $Z$, in $\Sigma^s$, is given by one of the following illustrations, in Figure 2:

![Figure 2](image_url)

**Figure 2.** The two possible local dynamics of $Z_0$ with hypothesis $H_1$ and $H_2$.

However, as can be easily checked, just at the case (a) of Figure 2 we hope some asymptotical stability. So we consider the next hypothesis:

**Hypothesis 3 ($H_3$):** The cusp singularity generated by the vector field $X$ must be of the topological type described in Figure 2. So, $b < 0$.

By consequence of $H_2$ and $H_3$ we conclude that $d < 0$. Moreover, in $\Sigma^s$ we get $x > 0$ and $y + x^2 > 0$, so $Y_3 - X_3 = x - b(y + x^2) > 0$ and the sliding vector field in 4 has the same orientation of 3.

**Lemma 4.** The eigenspace $E_2^0$ associated to $\lambda_2$ is tangent to the curve $S_X$, in $\Sigma$. 

Proof. Straightforward according to (9). □

Faced to $H_2$, in order to obtain that the sliding region $\Sigma^s$ is invariant for $Z_0^\Sigma$ and the origin is asymptotically stable, we impose the following hypothesis:

**Hypothesis 4 ($H_4$):** $E_2^0$ is stronger than $E_1^0$, i.e., $|\lambda_1| < |\lambda_2|$. So,

$$-a < bd \Rightarrow 0 < a + bd.$$  

As an immediate consequence of $H_4$, we get $(bc)/(a + bd) < 0$ and $E_2^0 \cap \Sigma^s = \emptyset$ (so, in fact, $\Sigma^s$ is an invariant for $Z_0^\Sigma$). See Figure 3.

![Figure 3. Local dynamics of $Z^\Sigma$.](image)

3.1.2. **Local dynamics of the first return map.** Now, in order to determine the dynamics of the positive trajectories of $Z_0$ we consider the First Return Map, given in (5), with $\lambda = 0$. We get

$$\varphi_{Z_0}(x,y) = \left(\frac{ax - \sqrt{-3a^2(x^2 + 4y)}}{2a}, y + \frac{d(ax - \sqrt{-3a^2(x^2 + 4y)}}{ac}\right).$$

Given a point $p \in \mathbb{R}^3$, it is easy to see that the positive trajectory $\phi^+_{Z_0}(p)$ of $Z$ passing through $p$ intersects $\Sigma^s \cup \Sigma^{c+}$. In what follows we prove that $\phi^+_{Z_0}(p) \cap \Sigma^s \neq \emptyset$ and, after an appropriated choice on the parameters $a, b, d$, we obtain that $\phi^+_{Z_0}(p)$ converges to the origin when $t \to +\infty$.

**Lemma 5.** The image of the curve $y = -x^2$, with $x > 0$, by the First Return Map $\varphi_{Z_0}$ is the curve $y = -\frac{x^2}{4} + 2\frac{d}{c}x$ with $x > 0$, i.e.,

$$\varphi_{Z_0}(\{y = -x^2 \text{ with } x > 0\}) = \left\{y = -\frac{x^2}{4} + 2\frac{d}{c}x \text{ with } x > 0\right\}.$$  

**Proof.** Consider the point $P_0 = (x_0, -x_0^2, 0)$, with $x_0 > 0$. The trajectory of $X_0$ by $P_0$ intersects $\Sigma$ at $P_1 = (-2x_0, -x_0^2, 0)$ after a time $t_1 = -3x_0/a$. The trajectory of $Y_0$ by $P_1$ intersects $\Sigma$ at $P_2 = (2x_0, 4dx_0/c - x_0^2, 0)$ after a time $t_2 = 4x_0/c$. Considering the chance of variables $x = 2x_0$, after a time $\tilde{t} = t_1 + t_2 = \frac{(4a-3c)x_0}{ac}$, the curve $y = -x^2$ return to $\Sigma$ at the curve $y = -\frac{x^2}{4} + 2\frac{d}{c}x$. □
Lemma 6. The image of the curve \( x = 0 \), with \( y < 0 \), by the First Return Map \( \varphi_{Z_0} \) is the curve \( y = -\frac{x^2}{3} + 2\frac{d}{c}x \) with \( x > 0 \), i.e.,
\[
\varphi_{Z_0}(\{x = 0 \text{ with } y < 0\}) = \left\{ y = -\frac{x^2}{3} + 2\frac{d}{c}x \text{ with } x > 0 \right\}.
\]

Proof. Consider the point \( P_0 = (0, y_0, 0) \), with \( y_0 < 0 \). The trajectory of \( X_0 \) by \( P_0 \) intersects \( \Sigma \) at \( P_1 = (-\sqrt{-3 y_0}, -y_0, 0) \) after a time \( t_1 = -\frac{\sqrt{-3 y_0}}{a} \). The trajectory of \( Y_0 \) by \( P_1 \) intersects \( \Sigma \) at \( P_2 = (\sqrt{-3 y_0}, \frac{\sqrt{-3 y_0}}{c}, 0) \) after a time \( t_2 = \frac{2\sqrt{-3 y_0}}{c} \). Considering the change of variables \( x = \sqrt{-3 y_0} \), after a time \( t = t_1 + t_2 = (\frac{2a-c}{\sqrt{-3 y_0}}) \), the curve \( x = 0 \) return to \( \Sigma \) at the curve \( y = y = -\frac{x^2}{3} + 2\frac{d}{c}x \).

Lemma 7. The image of the set \( \Sigma^{c^+} \), by the First Return Map \( \varphi_{Z_0} \) remains between the curves \( y = -\frac{x^2}{3} + 2\frac{d}{c}x \) and \( y = -\frac{x^2}{4} + 2\frac{d}{c}x \) with \( x > 0 \), i.e.,
\[
\varphi_{Z_0}(\Sigma^{c^+}) \subset \left\{ (x, y, 0) \in \Sigma \mid -\frac{x^2}{3} + 2\frac{d}{c}x < y < -\frac{x^2}{4} + 2\frac{d}{c}x \text{ with } x > 0 \right\}.
\]

Proof. Given a point \( P_0 = (x_0, y_0, 0) \in \Sigma^{c^+} \) (where \( x_0 > 0 \) and \( y_0 < 0 \)), it is easy to see that the trajectory of \( X_0 \) by \( P_0 \) intersects \( \Sigma \) at \( P_1 \in \Sigma^{c^-} \) and the trajectory of \( Y_0 \) by \( P_1 \) intersects \( \Sigma \) at \( P_2 \), where \( P_2 \) is situated between the curves \( y = -\frac{x^2}{3} + 2\frac{d}{c}x \) and \( y = -\frac{x^2}{4} + 2\frac{d}{c}x \) which correspond to the images of the curves \( x = 0 \), with \( y < 0 \) and \( y = -x^2 \), with \( x > 0 \), respectively.

Lemma 8. Given \( p_0 = (x_0, y_0, 0) \in \Sigma^{c^+} \), call \( p_1 = (x_1, y_1, 0) = \varphi_{Z_0}(p_0) \) and \( p_n = (x_n, y_n, 0) = \varphi_{Z_0}(p_n) \), when it is well defined. Then \( x_1 > x_0 \) and \( x_n \to \infty \) when \( n \to \infty \).

Proof. Given \( p_0 = (x_0, y_0, 0) \in \Sigma^{c^+} \), a straightforward calculus show that \( x_1 = \frac{x_0}{2} + \frac{\sqrt{3} \sqrt{-(x_0^2 + 4y_0)}}{2} \) where \( p_1 = (x_1, y_1, 0) = \varphi_{Z_0}(p_0) \). Since \( p_0 \in \Sigma^{c^+} \) we conclude that \( y_0 \leq -\frac{x_0^2}{3} \). So,
\[
y_0 < -\frac{x_0^2}{3} \Rightarrow -4x_0^2 - 12y_0 > 0 \Rightarrow 3(-(x_0^2 + 4y_0)) > x_0^2 \Rightarrow \frac{\sqrt{3} \sqrt{-(x_0^2 + 4y_0)}}{2} > \frac{x_0}{2} \Rightarrow x_1 > x_0.
\]

If \( p_1 \in \Sigma^s \) then a First Return Map is not defined. Otherwise we repeat the previous argument. A recursive analysis prove that \( x_{n+1} > x_n \). In fact, repeating the previous argument
\[
x_{n+1} = \frac{x_n + \sqrt{3} \sqrt{-(x_n^2 + 4y_n)}}{2} > 2x_n \Rightarrow x_{n+1} > 2.
\]
Moreover, \( \frac{x_{n+1}}{x_n} > 1 \) implies, by a test of convergence of sequences, that \( x_n \to \infty \).

Proposition 9. For all \( p \in \mathbb{R}^3 \) it happens \( \phi_0^+(p) \cap \Sigma^s \neq \emptyset \).
Proof. As we observed above, given a point \( p \in \mathbb{R}^3 \), it is easy to see that \( \phi_{Z_0}^+(p) \cap [\Sigma^s \cup \Sigma^{c+}] \neq \emptyset \). So, it is enough to prove that \( \varphi_{Z_0}^{n_0}(\Sigma^{c+}) \subset \Sigma^s \) for some \( n_0 > 0 \).

By Lemmas 5, 6 and 7 we obtain that
\[
\varphi_{Z_0}(\Sigma^{c+}) \subset \{ (x, y, 0) \in \Sigma \mid \frac{x^2}{3} + 2\frac{d}{c}x \leq y \leq -\frac{x^2}{4} + 2\frac{d}{c}x \text{ with } x > 0 \}.
\]

By Lemma 8, there exists \( n_0 > 0 \) such that \( p_{n_0} = (x_{n_0}, y_{n_0}, 0) = \varphi_{Z_0}^{n_0}(p) \) satisfies \( y_{n_0} + x_{n_0}^2 \geq 0 \). Therefore \( p_{n_0} \in \Sigma^s \). □

3.2. Proof of the case \( \lambda \neq 0 \). When \( \lambda \neq 0 \), we consider the normal form (1), presenting a fold-fold singularity at the origin, since \( b \neq 0 \). The local dynamics for \( Z_\lambda \) is given in Figure 4. The tangential sets \( S_X \) and \( S_Y \) remains the same as the ones established in Subsection 3.1.

![Figure 4. The local dynamics of \( Z_\lambda \) with hypothesis \( H_1 \) and \( H_3 \).](image)

3.3. Local dynamics of the normalized sliding vector fields. According to (1), the normalized sliding vector field is given by
\[
Z_\lambda^\Sigma = (ax - bc(y + x^2), \lambda x - db(y + x^2)).
\]

Let
\[
\Delta_3 = (a + bd)^2 - 4bc\lambda.
\]
So, the eigenvalues of \( DZ_\lambda^\Sigma(0, 0) \) are
\[
\lambda_1 = \frac{a - bd - \sqrt{\Delta_3}}{2} \quad \text{and} \quad \lambda_2 = \frac{a - bd + \sqrt{\Delta_3}}{2},
\]
the eigenvectors associated to \( \lambda_1 \) and \( \lambda_2 \) respectively, are
\[
v_1^\lambda = \left( \frac{a + bd - \sqrt{\Delta_3}}{2\lambda} \right) \quad \text{and} \quad v_2^\lambda = \left( \frac{a + bd + \sqrt{\Delta_3}}{2\lambda} \right)
\]
and the eigenspaces associated to \( \lambda_1 \) and \( \lambda_2 \) respectively, are
\[
E_1^\lambda = \left\{ (x, y, 0) \in \Sigma \mid y = \frac{2\lambda}{a + bd - \sqrt{\Delta_3}} x \right\}
\]
\[
E_2^\lambda = \left\{ (x, y, 0) \in \Sigma \mid y = \frac{2\lambda}{a + bd + \sqrt{\Delta_3}} x \right\}.
\]
Under the hypotheses $H_1, \ldots, H_4$, we get the following results:

**Lemma 10.** The eigenspace $E_1^\lambda \subset \Sigma^c$ and

- (a) $E_2^\lambda \subset [\Sigma^s \cup \Sigma^u]$ since $\lambda > 0$
- (b) $E_2^\lambda \subset \Sigma^c$ since $\lambda < 0$, see Figure 5.

**Proof.** Straightforward according to (11). □

![Figure 5](image.png)

**Figure 5.** The local dynamics of $Z^\Sigma_\lambda$ with hypothesis $H_1, \ldots, H_4$.

Note that in case $\lambda < 0$, the sliding vector fields has a transient behavior in $\Sigma^s$, i.e., all the obits in $\Sigma^s$ will be iterated by the first return map, whereas in case $\lambda > 0$, $Z^\Sigma_\lambda$ is asymptotic stable at origin.

3.3.1. **Local dynamics of the first return map.** Now, in order to determine the dynamics of the positive trajectories of $Z_\lambda$ we consider the First Return Map $\varphi_{Z_\lambda}$ of $Z_\lambda$ whose expression is given in (5).

**Lemma 11.** Under the hypothesis $H_1, \ldots, H_4$ the origin is a hyperbolic saddle fixed point for $\varphi_{Z_\lambda}$ and

- (a) $S^+_{\lambda} \subset \Sigma^c$ since $\lambda > 0$
- (b) $S^+_{\lambda} \subset \Sigma^c, S^-_{\lambda} \subset [\Sigma^e \cup \Sigma^s]$ since $\lambda < 0$.

Besides, $S^+_{\lambda}, S^-_{\lambda}$ is a expansive, contraction direction, respectively.

**Proof.** Follows by the expressions (6) and (7), of the eigenvalues and the eigenspaces of $D\varphi_{Z_\lambda}(0)$, respectively. □

By Lemma 11 in case $\lambda > 0$, we get that given $p \in \Sigma^{c+}$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^{|n_0|}_{Z_\lambda}(p) \in \Sigma^s$. And the Lemma 11 under the hypothesis $H_1, \ldots, H_4$, provides that $Z^\Sigma_\lambda$ is asymptotic stable at origin. See Figure 6 when the orbits in red represent the iterated of $\varphi_{Z_\lambda}$ and the orbits in blue the dynamic of $Z^\Sigma_\lambda$.

In this case, we get that $Z_\lambda$ is asymptotic stable at origin, under the hypothesis $H_1, \ldots, H_4$.

In case $\lambda < 0$, the Lemma 11 provides that the trajectories of the sliding vector field $Z^\Sigma_\lambda$ have a transient behavior in $\Sigma^s$. In fact, in the present case, we shall prove that $Z_\lambda$ is not Lyapunov stable at the origin (a fold-fold singularity).
Lemma 12. Given \( p_0 = (x_0, -x_0^2, 0) \) (under the curve \( y = -x^2 \)), with \( x_0 > 0 \), we get
\[
\varphi_{Z_\lambda}(x_0, -x_0^2, 0) = \left( 2x_0 + \frac{3\lambda}{2a}, -x_0^2 - \frac{3\lambda(\lambda + 2ax_0)}{2a^2} + \frac{d(3\lambda + 4ax_0)}{ac}, 0 \right).
\]

Proof. Straightforward. \( \square \)

We denote \( \varphi_{Z_\lambda}(p_0) = p_1^\lambda = (x_1^\lambda, y_1^\lambda, 0) \), that can be situated at \( \Sigma^{c+} \) and in this case, by Lemma 11 its distance to the origin increase when compared with \( p_0 \). Otherwise, \( p_1^\lambda \) can be situated at \( \Sigma^s \) and in this case the trajectory by this point slides to the parabola \( y = -x^2 \). The intersection point will be called \( p_2^\lambda = (x_2^\lambda, y_2^\lambda, 0) = (x_2^\lambda, -[x_2^\lambda]^2, 0) \). As the origin is an attractor for \( Z_\lambda^\Sigma \) we have to answer if the attractiveness \( Z_\lambda^\Sigma \) is greater or less than the repulsiveness of the first return map \( \varphi_{Z_\lambda} \).

Denote by \( d(p, 0) \) the euclidian distance between the point \( p \) to the origin \( 0 \).

Lemma 13. Under the hypothesis \( H_1, \ldots, H_4, \lambda < 0 \) and with the previous notation,
\[
d(p_2^\lambda, 0) > d(p_0, 0).
\]

Proof. By \( \text{[10]} \) we obtain a vectorial equation of the straight line
\[
r = \{(x(\alpha), y(\alpha), 0)|x(\alpha) = x_0 + \alpha ax_0, y(\alpha) = -x_0^2 + \alpha \lambda x_0, \text{ with } \alpha \in \mathbb{R}\},
\]
tangent to the trajectory of \( Z_\lambda^\Sigma \) by \( p_0 = (x_0, -x_0^2, 0) \).

Note that \( r \) splits \( \Sigma^s \) in two regions, denoted by \( V^+ \) and \( V^- \), see Figure 7. Consider the vertical straight line \( s : p = p_1^\lambda + \beta(0, 1, 0) \), with \( \beta \in \mathbb{R} \), see Figure 7. Some calculus show that \( r \cap s = p_2^\beta \), where \( p_2^\beta = (x_2^\beta, y_2^\beta, 0) = \left( x_2^\lambda, -x_2^2 + \frac{\lambda}{a} \left( \frac{3\lambda}{2a} + x_0 \right), 0 \right) \). Comparing \( y_1^\lambda \) with \( y_1^\beta \) we get \( y_1^\lambda < y_1^\beta \). Therefore \( p_1^\beta \), and consequently \( p_2^\beta \), are situated at the region \( V^- \) described in Figure 7. So \( d(p_2^\beta, 0) > d(p_0, 0) \). \( \square \)

Lemma 14. \( Z_\lambda \) is not Lyapunov stable at origin for \( \lambda < 0 \).

Proof. By Lemma \( \text{[11]} \) we get that \( Z_\lambda^\Sigma \) has a transient behavior, i.e., \( \Sigma^{c+} \) is a attractor set for \( Z_\lambda^\Sigma \) and by Lemma \( \text{[11]} \) we conclude that all points in \( \Sigma^{c+} \) converge to \( \Sigma^s \). So, to analyze the stability of \( Z_\lambda \) at origin it is sufficient study the iterates of \( Z_\lambda \) at boundary of \( \Sigma^s \). By Lemma \( \text{[13]} \) we obtain that
the distance between the origin and a point in $\Sigma$ increases along the time. Therefore, we conclude that $Z_\lambda$ is not Lyapunov stable at origin for this case.

**Remark 1.** As consequence of Lemmas 10 and Lemma 11 we get that $Z_0$ has codimension at least two because the eigenspaces of the normalized sliding vector field and the first return map are tangent to $S_X$.

4. Proof of main results

4.1. **Case $\lambda = 0$.** When $\lambda = 0$, by Proposition 9, the trajectories of all points in $\mathbb{R}^3$ intersect $\Sigma$. By hypotheses $H_2$ and $H_4$, the omega limit set of all trajectories in $\Sigma$ is the origin. So, $Z_0$ is asymptotically stable.

4.2. **Case $\lambda > 0$.** When $\lambda > 0$, by item (a) of Lemma 11 the trajectories of all points in $\mathbb{R}^3$ intersect $\Sigma$. Since the origin is a hyperbolic attractor for $\lambda = 0$, the same holds when $\lambda \neq 0$, sufficiently small. By Lemma 10 we get $E_1^\lambda \subset \Sigma^e$ and $E_2^\lambda \subset \Sigma^s$, for $x > 0$. So, $Z_\lambda$ is asymptotically stable.

4.3. **Case $\lambda < 0$.** When $\lambda < 0$, the result is an immediate consequence of Lemma 14

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