ON THE CAUCHY TRANSFORM OF WEIGHTED BERGMAN SPACES

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Abstract. The problem of describing the range of a Bergman space $B_2^2(G)$ under the Cauchy transform $K$ for a Jordan domain $G$ was solved by Napalkov (Jr) and Yulmukhametov [NYu1]. It turned out that $K(B_2^2(G)) = B_1^2(C_G)$ if and only if $G$ is a quasidisk; here $B_1^2(C_G)$ is a Dirichlet space of the complement of $G$. The description of $K(B_2^2(G))$ for an integrable Jordan domain is given in [M]. In the present paper we give a description of $K(B_2^2(G, \omega))$ analogous to the one given in [M] for a weighted Bergman space $B_2^2(G, \omega)$ with a weight $\omega$ which is constant on level lines of the Green function of $G$. In case $G = D$, the unit disk, and under some additional conditions on the weight $\omega$, $K(B_2^2(D, \omega)) = B_1^2(C_D, \omega^{-1})$, a weighted analogy of a Dirichlet space.

1. Introduction

Let $G$ be a bounded domain in the complex plane $\mathbb{C}$ whose boundary is a rectifiable Jordan curve and let $\varphi$ be a conformal map of the unit disk $\mathbb{D}$ onto $G$. Let $\omega(t)$ be a positive measurable function on $(0, 1]$ that is called a weight. We consider weights $\omega$ for which the integral

$$\int_G \omega(1 - |\psi(z)|) \, dm_2(z)$$

converges, where $\psi$ is the inverse function of $\varphi$ and $dm_2$ is the Lebesgue area measure.

Introduce a weighted Bergman space:

$$B_2^2(G, \omega) = \left\{ g(z) \in \text{Hol}(G), \|g\|_{B_2^2} = \left( \int_G |g(z)|^2 \omega(1 - |\psi(z)|) \, dm_2(z) \right)^{\frac{1}{2}} < \infty \right\}.$$

Consider the weighted Cauchy transform $K$ of functions from the space $B_2^2(G, \omega)$:

$$(Kg)(\zeta) = \frac{1}{\pi} \int_G \frac{\overline{g(z)} \omega(1 - |\psi(z)|)}{z - \zeta} \, dm_2(z), \quad \zeta \in C\overline{G}, \quad g \in B_2^2(G, \omega),$$

where $C\overline{G}$ means the complement of $\overline{G}$. The integrability of $\overline{g(z)} \omega(1 - |\psi(z)|)$ follows from the convergence of (1.1).

The problem of describing the range of $B_2^2(G, \omega)$ under the Cauchy transform for $\omega = 1$ and different domains was studied in [NYu], [NYu1], [M]. In the present paper we are concerned with studying the same question for weighted spaces. The main result of the paper is a theorem describing the range in terms of spaces $W_{\omega}(0, 2\pi)$ which are introduced in §2. The method of proof of the main theorem is similar to the one in [M], but here we use the approximation of functions instead of the domain $G$ when describing $K(B_2^2(G, \omega))$. 

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2. More Spaces

To state the theorem of this paper we need to introduce some more spaces, but first we give an example.

**Example.** Let \( g \in B_2(\mathbb{D}, \omega) \), \( g(z) = \sum a_k z^k \). Taking as a conformal map of \( \mathbb{D} \) onto itself \( \varphi(z) = z \) we can easily calculate the norm \( \| g \|_{B_2} = (\pi \sum |a_k|^2 \omega_k)^{1/2} \), where \( \omega_k = 2 \int_0^1 r^{2k+1} \omega(1 - r) \, dr \) which is a positive decreasing sequence. Also we can compute the Cauchy transform \( Kg \):

\[
(Kg)(\zeta) = \frac{1}{\pi} \int_B \frac{g(z)\omega(1 - |z|)}{z - \zeta} \, dm(z) = \sum b_k \frac{1}{\zeta^k}, \quad \text{where} \quad b_k = -\overline{a}_{k-1} \omega_{k-1}.
\]

So \( K \) is an isometry between \( B_2(\mathbb{D}, \omega) \) and the space

\[
B_2(C\overline{\mathbb{D}}, \omega) = \left\{ \gamma(\zeta) \in \text{Hol}(C\overline{\mathbb{D}}), \, \gamma(\zeta) = \sum b_k \frac{1}{\zeta^k}, \, \| \gamma \|_{B_2}^2 = \left( \pi \sum \frac{|b_k|^2}{\omega_{k-1}} \right)^{1/2} < \infty \right\}.
\]

Now we introduce the spaces \( W_\omega(0, 2\pi) \) and \( W_\omega(0, 2\pi) \):

\[
W_\omega(0, 2\pi) = \left\{ f(e^{it}) \in L^1(0, 2\pi), \, f(e^{it}) \sim \sum_{k=-\infty}^{\infty} f_k e^{ikt}, \right. \quad \rho(f) = \left( \pi \sum_{k=1}^{\infty} \frac{|f_k|^2}{\omega_{k-1}} \right)^{1/2} < \infty \right\}.
\]

Let \( \{ G_n \}_1^\infty \) be a sequence of domains approximating the domain \( G \) from the outside, that is a) \( \overline{G_n} \subset G_{n+1} \subset G \), \( n \in \mathbb{N} \); b) \( \cup_{n=1}^{\infty} G_n = G \). We say that a function \( \gamma(\zeta) \) analytic in \( C\overline{\mathbb{D}} \) belongs to \( W_\omega(0, 2\pi) \) if for some sequence of domains \( \{ G_n \}_1^\infty \) approximating the domain \( G \) from the inside there exists a sequence of functions \( \{ \gamma_n \}_1^\infty \) satisfying the following conditions:

1. \( \gamma_n \) is analytic in \( C\overline{G_n} \);
2. \( \gamma_n \) converges to \( \gamma \) uniformly on every compact \( K \subset C\overline{G} \);
3. \( \sup_n \rho(\gamma_n \circ \varphi) < \infty \), where \( \circ \) means composition.

The space \( W_\omega(0, 2\pi) \) is a kind of closure of the system of functions \( \gamma \) analytic in \( CG \) with \( \gamma \circ \varphi \in W_\omega(0, 2\pi) \), but the closure is in the topology of uniform convergence on compact sets of \( CG \).

In order to prove the theorem of this paper we need to impose an additional condition on the weight \( \omega \). Let \( P_2(G, \omega) \) be the closure of all analytic polynomials in \( B_2(G, \omega) \). Throughout the paper we assume that \( \omega \) is chosen in such a way that

\[
P_2(G, \omega) = B_2(G, \omega).
\]

In connection with (2.2) see, for example, [V], [B], [B1]. One condition that guarantees that \( \omega \) satisfies (2.2) is (see [V])

\[
\{ t \log 1/\omega(t) \uparrow +\infty, \text{ ast } 0, \} \int_0^t \log 1/\omega(t) \, dt = \infty.
\]
3. THE MAIN THEOREM

**Theorem.** Let $G$ be a bounded domain in $\mathbb{C}$ whose boundary is a rectifiable Jordan curve, and let $\omega(t)$ be a positive measurable function on $(0,1]$ such that integral (1.1) converges and condition (2.2) is satisfied. Then a function $\gamma$ analytic in $\overline{G}$ belongs to $K(B_2(G,\omega))$ if and only if $\gamma \in \overline{W}_\omega(0,2\pi)$, i.e.

$$K(B_2(G,\omega)) = \overline{W}_\omega(0,2\pi).$$

**Proof.** Suppose that $\gamma \in \overline{W}_\omega(0,2\pi)$. This means that there exists a sequence of domains $\{G_n\}_{n=1}^\infty$ approximating the domain $G$ from the inside and a sequence of functions $\{\gamma_n\}_{n=1}^\infty$ with

1. $\gamma_n$ analytic in $\overline{G}$;
2. $\gamma_n \to \gamma$ uniformly on every compact set $K \subset \overline{G}$;
3. $\sup_n \rho(\gamma_n \circ \varphi) < \infty$, where $\rho$ is from (2.1).

For every $n \in \mathbb{N}$ consider the linear functional $F_n$ on functions $h \in \text{Hol}(\overline{G})$ given by

$$F_n(h) = -\frac{1}{2\pi i} \int_{\partial G} \gamma_n(\xi) h(\xi) d\xi.$$  \hspace{1cm} (3.1)

We are going to prove that $F_n$ is a bounded linear functional in the norm of the space $B_2(G,\omega)$. If we make the change of variables $\xi = \varphi(e^{i\theta})$ in the integral in (3.1) we get

$$-\frac{1}{2\pi i} \int_0^{2\pi} \gamma_n(\varphi(e^{i\theta})) h(\varphi(e^{i\theta})) \varphi'(e^{i\theta}) d\theta.$$  \hspace{1cm} (3.2)

Since the boundary $\partial G$ is a rectifiable curve, the function $h(\varphi(e^{i\theta})) \varphi'(e^{i\theta})$ is the restriction of the function $h_1(z)zi$ on the unit circumference, where $h_1(z) = h(\varphi(z))\varphi'(z)$ [G, p. 405]. Thus, the last integral is equal to

$$-\frac{1}{2\pi} \int_0^{2\pi} \gamma_n(\varphi(e^{i\theta})) h_1(e^{i\theta}) e^{i\theta} d\theta.$$  \hspace{1cm} (3.3)

The operator $\Phi : h \to h_1$ is an isometry between the spaces $B_2(G,\omega)$ and $B_2(\mathbb{D},\omega)$, that is

$$\|h_1\|_{B_2} = \|h\|_{B_2}.$$  \hspace{1cm} (3.4)

Let $\gamma_n(\varphi(e^{i\theta})) \sim \sum_{k=-\infty}^{\infty} c_k^n e^{ik\theta}$, $h_1(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$-\frac{1}{2\pi i} \int_{\partial G} \gamma_n(\xi) h(\xi) d\xi = -\sum_{k=0}^{\infty} c_k^n (k+1) a_k.$$  \hspace{1cm} (3.5)

Applying the Cauchy-Schwarz inequality we get

$$|F_n(h)| \leq \left( \sum_{k=1}^{\infty} \frac{|c_{-k}^n|^2}{\omega_{k-1}} \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} \omega_k |a_k|^2 \right)^{\frac{1}{2}} = \frac{1}{\pi} \rho(\gamma_n \circ \varphi) \|h\|_{B_2},$$  \hspace{1cm} (3.6)
where we used (3.2). Since polynomials are dense in \( B_2(G, \omega) \) the functional \( F_n \) is uniquely extended to a bounded linear functional, which we also call \( F_n \), on \( B_2(G, \omega) \) with \( \|F_n\| \leq \frac{1}{\pi} \rho(\gamma_n \circ \varphi). \)

The space \( B_2(G, \omega) \) is a Hilbert space, hence for every \( n \in \mathbb{N} \) there exists a function \( g_n \in B_2(G, \omega) \), such that

\[
F_n(h) = \frac{1}{\pi} \int_G h(z)g_n(z)\omega(1 - |\psi(z)|) \, dm_2(z).
\]

Moreover,

\[
||g_n||_{B_2} = ||F_n||. \tag{3.4}
\]

From (3.3), (3.4), condition (3) for the sequence \( \{\gamma_n\}_1^\infty \) and applying the Banach-Alaoglu theorem we conclude that there exists a bounded linear functional \( F \) on \( B_2(G, \omega) \) such that some subsequence \( \{F_{n(k)}\}_1^\infty \) of \( \{F_n\}_1^\infty \) converges to \( F \) in the weak-star topology. Let \( g \in B_2(G, \omega) \) be a function such that

\[
F(h) = \frac{1}{\pi} \int_G h(z)g(z)\omega(1 - |\psi(z)|) \, dm_2(z), \quad h \in B_2(G, \omega)
\]

and \( ||g||_{B_2} = ||F|| \). Computing the value of \( F_n \) at \( h(z) = 1/(z - \zeta), \zeta \in \overline{G} \) we get

\[
\gamma_n(\zeta) = F_n \left( \frac{1}{z - \zeta} \right) = \frac{1}{\pi} \int_G \frac{g_n(z)}{z - \zeta} \omega(1 - |\psi(z)|) \, dm_2(z).
\]

Letting \( n = n(k) \) tend to infinity and using condition (2) for the sequence \( \{\gamma_n\}_1^\infty \) reveals that

\[
\gamma(\zeta) = \frac{1}{\pi} \int_G \frac{g(z)\omega(1 - |\psi(z)|)}{z - \zeta} \, dm_2(z), \quad \zeta \in \overline{G},
\]

i.e. \( \gamma(\zeta) = (Kg)(\zeta) \).

To prove the converse, take any function \( \gamma \in K(B_2(G, \omega)) \), i.e., by (1.2),

\[
\gamma(\zeta) = \frac{1}{\pi} \int_G \frac{g(z)\omega(1 - |\psi(z)|)}{z - \zeta} \, dm_2(z), \quad \zeta \in \overline{G},
\]

\( g \in B_2(G, \omega) \). We shall now prove that \( \gamma \) belongs to \( \mathcal{W} \omega(0, 2\pi) \).

For any \( n \in \mathbb{N} \) consider a function \( \alpha_n \) such that

1. \( \alpha_n \) is continuous on \([0, 1], \; 0 \leq \alpha_n(t) \leq 1; \)
2. \( \alpha_n(t) = 0, \; t \in [0, 1/n]; \alpha_n(t) = 1, \; t \in [2/n, 1]. \)

Set

\[
\gamma_n(\zeta) = \frac{1}{\pi} \int_G \frac{g(z)\omega(1 - |\psi(z)|)}{z - \zeta} \alpha_n(1 - |\psi(z)|) \, dm_2(z). \tag{3.5}
\]

First, it is evident that if we take \( G_n = \{z \in G : \; |\psi(z)| < 1 - 1/n\} \) then \( \gamma_n \) is analytic in \( \overline{C_{G_n}} \). Next we prove that \( \gamma_n \to \gamma \) uniformly on a compact \( K \subset \overline{G} \).

\[
|\gamma(\zeta) - \gamma_n(\zeta)| = \frac{1}{\pi} \left| \int_G \frac{g(z)\omega(1 - |\psi(z)|)}{z - \zeta} \alpha_n(1 - |\psi(z)|) \, dm_2(z) \right|
\leq \frac{C_K}{\pi} ||g||_{B_2} \left( \int_G (1 - \alpha_n(1 - |\psi(z)|))^2 \omega(1 - |\psi(z)|) \, dm_2(z) \right)^{\frac{1}{2}},
\]
where the constant $C_K$ depends on the compact set $K$. The last integral, after changing variables, becomes

$$\int_{1-2/n \leq |z| < 1} |\varphi'(z)|^2 \omega(1 - |z|) \, dm_2(z).$$

Hence $\sup_K |\gamma(\zeta) - \gamma_n(\zeta)| \to 0$ as $n \to \infty$.

Finally, it remains to prove that $\sup_n \rho(\gamma_n \circ \varphi) < +\infty$, where $\gamma_n$ is defined by (3.5). For $f \in L^1(\partial \mathbb{D})$, $f(e^{i\theta}) \sim \sum_{k=-\infty}^{\infty} f_k e^{ik\theta}$ the corresponding Cauchy type integral

$$F(\zeta) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(t)}{t - \zeta} \, dt, \quad \zeta \in \overline{\mathbb{D}}$$

has the Taylor expansion $F(\zeta) = \sum_{k=1}^{\infty} f_{-k}/\zeta^k$, $\zeta \in \overline{\mathbb{D}}$. With this in mind, consider

$$\text{(3.6)} \quad F_n(\zeta) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\gamma_n \circ \varphi(t)}{t - \zeta} \, dt, \quad \zeta \in \overline{\mathbb{D}}.$$ 

If we substitute (3.5) in (3.6), using $\zeta = \varphi(t)$, $t \in \mathbb{D}$, change the order of integration, and compute the inner integral, we get

$$F_n(\zeta) = -\frac{1}{\pi} \int_{\partial \mathbb{D}} \frac{\overline{g(z)} \omega(1 - |\varphi(z)|)}{\varphi(z) - \zeta} \varphi'(\varphi(z)) \alpha_n(1 - |\varphi(z)|) \, dm_2(z).$$

Make the change of variable $w = \varphi(z)$ to obtain

$$F_n(\zeta) = -\frac{1}{\pi} \int_{\partial \mathbb{D}} \frac{g(\varphi(w)) \varphi'(w) \omega(1 - |w|)}{w - \zeta} \alpha_n(1 - |w|) \, dm_2(w)$$

$$= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{1}{c_{k+1}} \int_{\partial \mathbb{D}} g(\varphi(w)) \varphi'(w) w^k \omega(1 - |w|) \alpha_n(1 - |w|) \, dm_2(w).$$

As was noted above, $g(\varphi(z)) \varphi'(z) \in B_2(\mathbb{D}, \omega)$. Let $g(\varphi(z)) \varphi'(z) = \sum_{k=0}^{\infty} a_j z^j$. Then $F_n(\zeta) = \sum_{k=1}^{\infty} b_k^n/\zeta^k$, where $b_k^n = \overline{a_{k-1}} 2 f_0^1 r^{2k-1} (\alpha_n \omega)(1 - r) \, dr$. We need to check that $\sup_n \left( \pi \sum_{k=1}^{\infty} |b_k^n|^2 / \omega_{k-1} \right)^{1/2} < \infty$. Since $0 \leq \alpha_n \leq 1$,

$$\frac{|b_k^n|^2}{\omega_{k-1}} \leq |a_{k-1}|^2 \omega_{k-1}.$$ 

Hence

$$\sup_n \left( \pi \sum_{k=1}^{\infty} \frac{|b_k^n|^2}{\omega_{k-1}} \right)^{1/2} \leq \left( \pi \sum_{k=0}^{\infty} |a_k|^2 \omega_k \right)^{1/2} = \|g\|_{B_2}.$$ 

Thus the theorem is proved.
4. Description of $W_\omega(0, 2\pi)$.

Let $f \in L^1(\partial \mathbb{D})$, that is $f(e^{i\theta}) \in L^1(0, 2\pi)$, $f(e^{i\theta}) \sim \sum_{k=-\infty}^{\infty} f_k e^{ik\theta}$. The Cauchy-type integral $F$ corresponding to $f$ is $F(\zeta) = -\sum_{k=1}^{\infty} f_{-k}/\zeta^k$, $\zeta \in \mathbb{C} \setminus \partial \mathbb{D}$.

We prove that under some condition on the weight $\omega$, $f \in W_\omega(0, 2\pi)$ if and only if $F \in B_1^2(C \mathbb{D}, \omega^{-1})$, where

$$B_1^2(C \mathbb{D}, \omega^{-1}) = \left\{ f(\zeta) \in \text{Hol}(C \mathbb{D}), F(\infty) = 0, \right.$$}

$$\|F\|_{B_1^2} = \left( \iint_{C \mathbb{D}} |F'(\zeta)|^2 \frac{dm_2(\zeta)}{\omega(1 - 1/|\zeta|)} \right)^{1/2}. \right\}$$

**Proposition.** Assume that $\omega$ satisfies the following condition:

$$\sup_k \left\{ (k+1)^2 \int_0^1 r^{2k+1} \omega(1-r) dr \int_0^1 \frac{dr}{r^{2k+1} \omega(1-r)} \right\} \leq C < \infty. \tag{4.1}$$

Then $f \in W_\omega(0, 2\pi)$ if and only if $F \in B_1^2(C \mathbb{D}, \omega^{-1})$.

**Proof.**

First we show that

$$\inf_k \left\{ (k+1)^2 \int_0^1 r^{2k+1} \omega(1-r) dr \int_0^1 \frac{dr}{r^{2k+1} \omega(1-r)} \right\} \geq c, \tag{4.2}$$

where $c$ is some positive constant:

$$\frac{1}{4(k+1)^2} = \left( \int_0^1 r^{2k+1} dr \right)^2 \leq \int_0^1 r^{2k+1} \omega(1-r) dr \int_0^1 \frac{dr}{r^{2k+1} \omega(1-r)},$$

which is (4.2) with $c = 1/4$. Now we are ready to prove the proposition.

$$\iint_{C \mathbb{D}} |F'(\zeta)|^2 \frac{dm_2(\zeta)}{\omega(1 - 1/|\zeta|)} = 2\pi \sum_{k=1}^{\infty} k^2 |f_{-k}|^2 \int_1^\infty \frac{dr}{r^{2k+1} \omega(1-1/r)}$$

$$= 2\pi \sum_{k=1}^{\infty} k^2 |f_{-k}|^2 \int_0^1 \frac{r^{2k-1} dr}{\omega(1-r)} \asymp \pi \sum_{k=1}^{\infty} \frac{|f_{-k}|^2}{\omega_k^{-1}},$$

where $a \asymp b$ means that there exist two positive constants $m, M$ such that $ma \leq b \leq Ma$; here we used (4.1), (4.2). This proves the proposition.

**Remark.** The condition (4.1) is similar to the Mackenhaupt condition [Ga, p. 254].

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