The structure of mode-locking regions of piecewise-linear continuous maps: II. Skew sawtooth maps

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Abstract

In two-parameter bifurcation diagrams of piecewise-linear continuous maps on \( \mathbb{R}^N \), mode-locking regions typically have points of zero width known as shrinking points. Near any shrinking point, but outside the associated mode-locking region, a significant proportion of parameter space can be usefully partitioned into a two-dimensional array of annular sectors. The purpose of this paper is to show that in these sectors the dynamics is well-approximated by a three-parameter family of skew sawtooth circle maps, where the relationship between the skew sawtooth maps and the \( N \)-dimensional map is fixed within each sector. The skew sawtooth maps are continuous, degree-one, and piecewise-linear, with two different slopes. They approximate the stable dynamics of the \( N \)-dimensional map with an error that goes to zero with the distance from the shrinking point. The results explain the complicated radial pattern of periodic, quasi-periodic, and chaotic dynamics that occurs near shrinking points.

Keywords: border-collision bifurcation, nonsmooth, piecewise-smooth, circle map

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(Some figures may appear in colour only in the online journal)

1. Introduction

This paper extends the study of [1] for piecewise-linear maps on \( \mathbb{R}^N \) (\( N \geq 2 \)) that are continuous and involve two pieces. For such maps, coordinates \( x \in \mathbb{R}^N \) can be chosen so that the
are the left and right Farey roots of \( M \) be the rotation number associated with \( 00 \), \( \sigma \geq 0 \) for some \( \kappa \) and \( \sigma \), -mode-locking regions, form two sequences that approach the element, such as trade cycle models with non-negativity constraints [11].

Maps of the form (1.2) also arise as models of discrete-time phenomena involving a switching dominant nearby mode-locking regions have rotation numbers \( s \) implies \( \tau (\kappa, \mu) \) dependency on a parameter \( \kappa \) \( (K \geq 2) \) dependency on a parameter \( \xi \in \mathbb{R}^M \). The assumption that \( f \) is continuous on \( s = 0 \) implies \( A_R = A_L + C \xi \) for some \( C \in \mathbb{R}^N \).

Maps of the form (1.2) describe the dynamics near border-collision bifurcations of piecewise-smooth continuous maps [2]. Analogous bifurcations of discontinuous maps have a different character, see [3, 4]. Piecewise-smooth maps arise as return maps for nonsmooth differential equations which serve as useful mathematical models of phenomena with events that are discontinuous or at least fast relative to the usual motion of the system. Examples of such systems exhibiting border-collision bifurcations include neurons with square wave forcing [5, 6], mechanical oscillators with friction [7, 8], and DC/DC power converters [9, 10]. Maps of the form (1.2) also arise as models of discrete-time phenomena involving a switching element, such as trade cycle models with non-negativity constraints [11].

To illustrate the ideas of this paper, consider

\[
A_L = \begin{bmatrix}
\tau_L & 0 \\
-\sigma_L & 0
\end{bmatrix}, \quad A_R = \begin{bmatrix}
\tau_R & 0 \\
-\sigma_R & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
\mu \\
0
\end{bmatrix},
\]

(1.3)

where \( \xi = (\tau_L, \sigma_L, \delta_L, \tau_R, \sigma_R, \delta_R, \mu) \in \mathbb{R}^7 \). The map (1.2) with (1.3) is commonly known as the border-collision normal form in three dimensions [12]. Figure 1 shows mode-locking regions of (1.2) with (1.3) and

\[
\tau_L = 0, \quad \sigma_L = -1, \quad \sigma_R = 0, \quad \delta_R = 2, \quad \mu = 1.
\]

(1.4)

Each coloured region is a mode-locking region where (1.2) has an attracting periodic solution of a fixed rotation number (defined in section 2.1). The regions are roughly ordered by rotation number and are more narrow for higher periods.

Unlike mode-locking regions of smooth maps, the mode-locking regions in figure 1 have points of zero width. These are termed shrinking points and are where the map has a periodic solution with two points on the switching manifold. They have been described in a wide variety of mathematical models that can be put in the form (1.2), or are at least well-approximated by a map of this form [6, 13–16]. In a mode-locking region, as we cross a shrinking point the number of points that the corresponding periodic solution has on each side of the switching manifold changes by one. In a neighbourhood of a shrinking point, the mode-locking region is bounded by four curves along which a certain \( S \)-cycle (periodic solution with symbolic itinerary \( S \in \{L, R\}^\mathbb{Z} \)) has one point on the switching manifold, and so the shrinking point is referred to as an \( S \)-shrinking point.

Now consider the dynamics near an \( S \)-shrinking point but outside its corresponding mode-locking region. Let \( m/n \) be the rotation number associated with \( S \). As explained in [1], the most dominant nearby mode-locking regions have rotation numbers \( m/n \), \( m/n \), and \( m/n \), where \( k \in \mathbb{Z}^+ \) and \( m/n \) and \( m/n \) are the left and right Farey roots of \( m/n \). These regions, termed \( G_k \)-mode-locking regions, form two sequences that approach the \( S \)-shrinking point from.
opposite sides as \( k \to \infty \). In figure 2 this is indicated for the above example with \( m = \frac{3}{8} \). Here \( G_{k}^{\pm} \)-mode-locking regions are shaded dark grey and all other nearby mode-locking regions are shaded light grey.

As shown in [1], the existence and location of shrinking points on \( G_{k}^{\pm} \)-mode-locking regions is determined by various scalar quantities associated with the \( S \)-shrinking point. This was established by performing calculations on one-dimensional centre manifolds. In the present paper the approximate return to a fundamental domain of such a manifold is described using circle maps. The circle maps are continuous, piecewise-linear, and degree-one, and written as

\[
g(z; a_L, a_R, w) := \begin{cases} (w + a_L (z - z_{sw}) + z_{sw}) \mod 1, & 0 \leq z \leq z_{sw}, \\ (w + a_R (z - z_{sw}) + z_{sw}) \mod 1, & z_{sw} \leq z < 1, \end{cases} \tag{1.5}
\]

where \( z \in [0, 1) \) is the state variable,

\[
a_L < 1, \quad a_R > 1, \quad w \in \mathbb{R}, \tag{1.6}
\]

are parameters, and

\[
z_{sw} := \frac{a_R - 1}{a_R - a_L}. \tag{1.7}
\]

As shown in figure 3, \( a_L \) and \( a_R \) are the slopes of (1.5) and \( w \) is the vertical displacement (modulo 1) of (1.5) at the kink \( z = z_{sw} \). Given (1.6), the map is a homeomorphism if and only if \( a_L > 0 \). It is termed a skew sawtooth map because with \( a_L + a_R = 2 \) (giving \( z_{sw} = \frac{1}{2} \)) it is commonly known as a sawtooth map [17]. As a three-parameter family, (1.5) is equivalent to the ‘tip maps’ studied in [18].

Here we explain how (1.5) can be used to approximate the dynamics of (1.2) near shrinking points, with details to be provided in later sections. In a neighbourhood of an \( S \)-shrinking point, let \( \gamma_{k}^{\pm} \) denote the piecewise-smooth boundary of the \( G_{k}^{\pm} \)-mode-locking region on the side of the \( S \)-shrinking point (coloured green in figure 2). We then partition each strip between \( \gamma_{k}^{\pm} \) and \( \gamma_{k+1}^{\pm} \) into sectors \( \Sigma_{k,\ell}^{\pm} \), where \( \Delta \ell \) is used to specify angular bounds. Figure 2 shows
two representative sectors, $\Sigma_{+}^{2,0}$ and $\Sigma_{+}^{3,1}$, and the mode-locking regions that they contain. Figure 4 shows the approximate location of the sectors as given by truncating an asymptotic expansion in powers of $\frac{1}{k}$ to leading order.

Within each $\Sigma_{\pm}^{\Delta \ell}$, we use $\delta \geq 0$ to measure the distance from $\gamma_{\ell}^{\pm}$ towards the $S$-shrinking point, and $\theta \in [0, 2\pi)$ to measure the angle about the $S$-shrinking point. In $(\delta, \theta)$-coordinates the sectors are rectangular in the approximation of figure 4:

$$
\Sigma_{\pm}^{\Delta \ell} \approx \{(\delta, \theta) \mid 0 \leq \delta \leq \frac{1}{2} \tau, \theta_{\min} \leq \theta \leq \theta_{\max}\},
$$

where $\theta_{\min}, \theta_{\max} \in [0, 2\pi)$ depend only on the value of $\Delta \ell$.

For $\Sigma_{+}^{2,0}$ and $\Sigma_{+}^{3,1}$, figure 5 shows mode-locking regions of (1.5) using

$$
a_L = \frac{\tan(\theta)}{\tan(\theta_{\min})}, \quad a_R = \frac{\tan(\theta)}{\tan(\theta_{\max})}, \quad w = k^2 \delta.
$$
Equation (1.9) represents the appropriate transformation from \((\delta, \theta)\)-coordinates to the parameter space of the skew sawtooth family for \(\Sigma^+_{2,0}\) and \(\Sigma^+_{1,1}\). By comparing figures 2 and 5, it is evident that the dynamics of (1.2) in \(\Sigma^+_{2,0}\) and \(\Sigma^+_{1,1}\) matches well to that of (1.5) using (1.9). Indeed the dynamics of (1.2) in each \(\Sigma^+_{k,0}\) and \(\Sigma^+_{k,1}\) matches that of (1.5) using (1.9) with \(O\left(\frac{1}{k}\right)\) error and where one iterate of (1.5) corresponds to \(k \left(n + O\left(\frac{1}{k}\right)\right)\) iterates of (1.2) (see section 2.5 and theorem 5.2).
The remainder of this paper is organised as follows. In section 2, the sectors $\Sigma^\pm k$ and $\Delta \ell$ and $((\delta, \theta))$-coordinates are defined more precisely. The correspondence between (1.2) and (1.5) is clarified and used to explain the bifurcation structure of (1.2) near a typical shrinking point. In section 3 some important identities for the symbolic itineraries of periodic solutions in $G^\pm$-mode-locking regions are derived and an attracting one-dimensional centre manifold $W_c$ and a fundamental domain $\Omega^\Delta \ell \subset W_c$ are introduced. In section 4, $\Omega^\Delta \ell$ is enlarged into an $N$-dimensional set $\Phi$ that forward orbits of (1.2) regularly visit. The set $\Phi$ is interpreted as a cylinder and topological arguments are used to prove that the first return map $F: \Phi \to \Phi$ has an attracting invariant set that is homotopic to $\Omega^\Delta \ell$. This shows that (1.2) has an attracting invariant set homotopic to a circle on which the dynamics is well approximated by (1.5). In section 5 the necessary calculations to achieve this approximation are performed, with the main result given by theorem 5.2. Finally section 6 provides concluding remarks.

Throughout this paper, but mostly in proofs, particular theorems and equations etc. in [1] are referred to. Some proofs are deferred to appendix A and some formulas of [1] are given in appendix B. For brevity, sections 3–5 only provide calculations for sectors $\Sigma^\pm k, \Delta \ell$ with $\Delta \ell \geq 0$.

2. Main results

In this section we first describe periodic solutions of (1.2), section 2.1. We then discuss shrinking points and briefly review the notation of [1] in section 2.2. In section 2.3 we describe $G^\pm$-mode-locking regions and in section 2.4 define the sectors $\Sigma^\pm k, \Delta \ell$. We then detail the correspondence between (1.2) and (1.5) in section 2.5 and finally use this to explain the dynamics of (1.2) near a typical shrinking point in section 2.6.
2.1. Periodic solutions

We denote the two components of (1.2) by
\[ f^L(x; \xi) := A_L(\xi)x + B(\xi), \quad f^R(x; \xi) := A_R(\xi)x + B(\xi). \]  
(2.1)

Given a periodic symbol sequence \( S \in \{L, R\}^\mathbb{Z} \) of period \( n \), we let
\[ f^S := f^{S_{n-1}} \circ \ldots \circ f^{S_0}, \]  
(2.2)
denote the composition of \( f^L \) and \( f^R \) in the order specified by \( S \). The map \( f^S \) is given by
\[ f^S(x) = M_Sx + P_SB, \]  
(2.3)
where
\[ M_S := A_{S_{n-1}} \cdots A_{S_0}, \]  
(2.4)
\[ P_S := I + A_{S_{n-1}} + A_{S_{n-2}}A_{S_{n-1}} + \cdots + A_{S_1}A_{S_2}\cdots A_{S_0}. \]  
(2.5)

An \( S \)-cycle is defined as an \( n \)-tuple, \( \{x^S_i\}_{i=0}^{n-1} \), for which
\[ f^{S_n} (x^S_0) = x^S_1, \quad f^{S_{n-1}} (x^S_1) = x^S_2, \ldots, \quad f^{S_1} (x^S_{n-1}) = x^S_0. \]  
(2.6)

If \( x^S_i \leq 0 \) for every \( i \) for which \( S_i = L \) and \( x^S_i \geq 0 \) for every \( i \) for which \( S_i = R \), then each point \( x^S_0 \) lies on the side of the switching manifold corresponding to \( S_0 \). In this case the \( S \)-cycle is a periodic solution of \( f \) and we say it is \emph{admissible}. If it is not admissible we say it is \emph{virtual}.

Each point \( x^S_0 \) is a fixed point of \( f^{S_0} \), where \( S_0 \) denotes the \( \ell \)th left shift permutation of \( S \) (i.e. \( S_0^{(j)} = S_{j+\ell} \) for all \( j \in \mathbb{Z} \)). If \( \det (I - M_S) \neq 0 \), then \( x^S_0 \) is unique and given by
\[ x^S_0 = (I - M_S)^{-1} P_{S_0}B. \]  
(2.7)

Given \( \ell, m, n \in \mathbb{Z}^+ \), with \( \ell < n \), \( m < n \) and \( \gcd(m,n) = 1 \), we define a symbol sequence \( \mathcal{F}[^\ell, m, n] \in \{L, R\}^\mathbb{Z} \) by
\[ \mathcal{F}[^\ell, m, n] := \begin{cases} L, & \text{im mod } n < \ell, \\ R, & \text{im mod } n \geq \ell. \end{cases} \]  
(2.8)

Geometrically, \( \mathcal{F}[^\ell, m, n] \) describes the action of stepping through \( n \) points on a circle, \( \ell \) of which lie to the left of the switching manifold, with rotation number \( \frac{\ell m}{n} \) [19, 20]. Indeed, we call \( \frac{\ell m}{n} \) the \emph{rotation number} of \( \mathcal{F}[^\ell, m, n] \), and refer to such symbol sequences and their shift permutations as \emph{rotational}. In this way we can assign a rotation number to certain periodic solutions of (1.2); if a periodic solution is an \( \mathcal{F}[^\ell, m, n] \)-cycle then we say it has rotation number \( \frac{\ell m}{n} \). For a detailed comparison of this combinatorial notion of a rotation number to more standard geometric definitions, refer to section 4.1 of [1].

We use the notation \( S^\ell \) to denote the symbol sequence created from \( S \) by flipping the symbols \( S_{i+jn} \) (from \( L \) to \( R \), or vice-versa) for all \( j \in \mathbb{Z} \). Rotational symbol sequences (2.8) satisfy the important identity
\[ \mathcal{F}[^\ell, m, n]^\ell \circ \mathcal{F}[^\ell, m, n]^{\ell m/n} = \mathcal{F}[^\ell, m, n]^{-\ell m/n}, \]  
(2.9)
where \( d \) denotes the multiplicative inverse of \( m \) modulo \( n \). That is, flipping indices 0 and \( \ell d \mod n \) is equivalent to a shift permutation of \( -d \) places to the left.
For example consider $S = F[3, 3, 8]$ (as in figure 2) for which $d = 3$. By the definition (2.8), $S$ is the infinite repetition of the word $S_0 \cdots S_{l} = LRRLRRLR$. Flipping $S_0$ and $S_{ld \mod n} = S_1$ produces $RLRLRRLR$. By inspection this also results from a cyclic permutation of $-d \mod n = 5$ places to the left, verifying (2.9) for this example.

2.2. Shrinking points

We use rotational symbol sequences to define shrinking points of (1.2). Let $S = F[\ell, m, n]$ be a rotational symbol sequence with $2 \leq \ell \leq n - 2$. Suppose that at some point in parameter space $\xi$, the following three genericity conditions are satisfied:

\[ e_T^{\ell} \ adj(I - A_L) B \neq 0, \quad \det(I - M_{S}) \neq 0, \quad \det(I - M_{S'\ell}) \neq 0. \]  

(2.10)

In view of the second condition, the $S_0$-cycle is unique. If the $S_0$-cycle is also admissible, with $s_{S_0}^{\ell} = 0$ if and only if $i = 0$ or $i = \ell d \mod n$, then we say that $\xi$ is an $S$-shrinking point.

At this point a periodic solution (the $S_0$-cycle) has two points on the switching manifold. As in [1], at an $S$-shrinking point we let

\[ a := \det(I - M_{S}), \quad b := \det(I - M_{S'\ell}), \]  

(2.11)

and

\[ y_i := x_i^{S'}, \quad t_i := s_i^{S'}, \]  

(2.12)

for each $i$. Many identities involving quantities associated with an $S$-shrinking point, such as (2.11) and (2.12), are given in [1]. Of these, perhaps the most fundamental is

\[ \frac{a}{b} = \frac{t_d(t - 1)d}{t - d(t + 1)d}, \]  

(2.13)

which was first given in [21].

By lemma 5.8 of [1], at an $S$-shrinking point 1 is an eigenvalue of $M_S$ with multiplicity one. We let

\[ \rho_{max} := \max_{j=2, \ldots, N} |\rho_j|, \]  

(2.14)

where $\rho_j$ are the eigenvalues of $M_S$, counting multiplicity, with $\rho_1 = 1$, and

\[ c := \prod_{j=2}^{N} (1 - \rho_j). \]  

(2.15)

If $\rho_{max} < 1$ then $S$-cycles are stable for some parameter values near the $S$-shrinking point ensuring that the $S$-shrinking point is connected to part of a mode-locking region involving $S$-cycles (instead of merely a periodicity region involving only unstable $S$-cycles). Note that $\rho_{max} < 1$ implies $c > 0$ (lemma 7.4 of [1]).

2.3. $g_{\pm}^{\ell}$-mode-locking regions

Let $F[\ell, m, n]$ be a rotational symbol sequence. Let $\frac{m^-}{n^-}$ and $\frac{m^+}{n^+}$ denote the left and right Farey roots of $\frac{m}{n}$, and let $\ell^- := \left\lfloor \frac{tn^-}{n} \right\rfloor$ and $\ell^+ := \left\lceil \frac{tn^+}{n} \right\rceil$, where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and ceiling functions. Let $k \in \mathbb{Z}^+$ and $\Delta \ell \in \mathbb{Z}$ with $|\Delta \ell| < k$. We then define
\[
G^\pm [k, \Delta \ell] := F \left[ \ell_k^\pm + \Delta \ell, m_k^\pm, n_k^\pm \right],
\]
where
\[
\ell_k^\pm := k\ell + \ell^\pm, \quad m_k^\pm := km + m^\pm, \quad n_k^\pm := kn + n^\pm.
\]
We also let \(d_k^\pm\) denote the multiplicative inverse of \(m_k^\pm\) modulo \(n_k^\pm\). The rotation number of \(G^\pm [k, \Delta \ell]\) is \(m_k^\pm n_k^\pm\). These rotation numbers represent the first level of complexity relative to \(m_k n_k\) [22]. As mentioned in section 1, we refer to a mode-locking region with rotation number \(m_k^\pm n_k^\pm\) as a \(G^\pm_k\)-mode-locking region.

For simplicity we assume \(\xi \in \mathbb{R}^2\) and write \(\xi = (\xi_1, \xi_2)\). In a neighbourhood of an \(S\)-shrinking point, where \(S = F[\ell, m, n]\), let
\[
\eta := s_0^S (\xi_1, \xi_2), \quad \nu := s_{1d}^S (\xi_1, \xi_2).
\]
Throughout this paper we assume that \((\xi_1, \xi_2) \rightarrow (\eta, \nu)\) is a locally invertible coordinate change. That is, \(\det(J) \neq 0\), where \(J := \begin{vmatrix} \frac{\partial \eta}{\partial \xi_1} & \frac{\partial \eta}{\partial \xi_2} \\ \frac{\partial \nu}{\partial \xi_1} & \frac{\partial \nu}{\partial \xi_2} \end{vmatrix}\) evaluated at the shrinking point. The curves \(s_0^S = 0\) and \(s_{1d}^S = 0\) are boundaries of one component of the mode-locking region connected to the \(S\)-shrinking point. Thus in \((\eta, \nu)\)-coordinates the \(S\)-shrinking point is located at the origin, and, locally, the boundaries of one component of the mode-locking region coincide with the coordinate axes, figure 6.

As in [1], we define polar coordinates \((r, \theta)\) by
\[
\eta = \frac{ct \ell d}{a} r \cos(\theta), \quad \nu = \frac{ct (\ell - 1) d}{a} r \sin(\theta),
\]
where \(a, c, \ell, d, \) and \(t(\ell - 1) d\) are defined by (2.11) and (2.12) and (2.15). If \(a < 0\), then \(G^+_k\)-mode-locking regions exist for \(\theta \in \left(\frac{\pi}{2}, 2\pi\right)\) and \(G^-_k\)-mode-locking regions exist for \(\theta \in \left(\frac{\pi}{2}, \pi\right)\), see figure 6. If \(a > 0\), then the opposite is true.
For large values of \( k \), the \( G^\pm_k \)-mode-locking regions are narrow regions that lie within \( \mathcal{O}(\frac{1}{k}) \) of the curve

\[
r = \frac{1}{k} \Gamma(\theta),
\]

where \( \Gamma \) is defined by

\[
\Gamma(\theta) := \left\{ \frac{\ln(\cos(\theta)) - \ln(\sin(\theta))}{\sqrt{2}}, \quad \theta \in (0, \frac{\pi}{2}) \setminus \{ \frac{\pi}{4} \}, \right.
\]

\[
\theta = \frac{\pi}{4}, \quad \right\}
\]

for \( \theta \in (0, \frac{\pi}{2}) \), and defined by \( \Gamma(\theta) = \Gamma(\theta \mod \frac{\pi}{2}) \) for all other non-integer multiples of \( \frac{\pi}{2} \). For \( \theta \in (0, \frac{\pi}{2}) \), the function \( r = \Gamma(\theta) \) is a curve of solutions to \( Xe^{-X} = Ye^{-Y} \), where \( X = r\cos(\theta) \) and \( Y = r\sin(\theta) \) (this arises in the asymptotic expansion of the \( G^\pm_k \)-mode-locking region boundaries, see equation (A.14) of [1]).

The existence of shrinking points on the \( G^\pm_k \)-mode-locking regions for large values of \( k \) is determined by scalar quantities \( \kappa^\pm_{\Delta \ell} \) associated with the \( \Sigma \)-shrinking point. These are defined in appendix B as dot products involving left and right eigenvectors of \( M_{Z^j} \), for certain indices \( j \). The discovery of these crucial quantities is one of the main results of [1]. If \( \rho_{\max} < 1 \) and certain admissibility conditions are satisfied, then given \( \Delta \ell \in \mathbb{Z} \) and a choice of \( + \) or \( - \), \( G^\pm_k \)-mode-locking regions have \( G^\pm[k, \Delta \ell] \)-shrinking points for all sufficiently large values of \( k \) if \( \kappa^\pm_{\Delta \ell} > 0 \). If instead \( \kappa^\pm_{\Delta \ell} < 0 \), then the mode-locking regions have short boundaries where \( M_{G^\pm[k, \Delta \ell]} \) has an eigenvalue of \(-1\). Here the \( G^\pm[k, \Delta \ell] \)-cycle loses stability by attaining a stability multiplier of \(-1\).

To leading order, the \( \theta \)-values of the \( G^\pm[k, \Delta \ell] \)-shrinking points and stability-loss boundaries are independent of \( k \). We denote the leading order components by \( \theta^\pm_{\Delta \ell} \) and provide formulas for them in appendix B.

### 2.4. Annular sectors

Let \( \Delta \ell \in \mathbb{Z} \) and a choice of \(+\) or \(-\) be given. Suppose \( \kappa^\pm_{\Delta \ell} \) and \( \kappa^\pm_{\Delta \ell-1} \) are non-zero (as is generically the case) with which \( \theta^\pm_{\Delta \ell} \) and \( \theta^\pm_{\Delta \ell-1} \) are well-defined. Assuming admissibility, for any sufficiently large value of \( k \in \mathbb{Z} \) the inner boundary of the \( G^\pm_k \)-mode-locking region, called \( \gamma^k \), is \( C^k \) between endpoints, call them \( p^k_1 \) and \( q^k_1 \), at \( \theta = \theta^\pm_{\Delta \ell} + \mathcal{O}(\frac{1}{k}) \) and \( \theta = \theta^\pm_{\Delta \ell-1} + \mathcal{O}(\frac{1}{k}) \), see figure 7. If \( \kappa^\pm_{\Delta \ell} > 0 \), then \( p^k_1 \) is a \( G^\pm[k, \Delta \ell] \)-shrinking point, while if \( \kappa^\pm_{\Delta \ell} < 0 \), then at \( p^k_1 \) there is a \( G^\pm[k, \Delta \ell] \)-cycle with a stability multiplier of \(-1\). Similarly if \( \kappa^\pm_{\Delta \ell-1} > 0 \), then \( q^k_1 \) is a \( G^\pm[k, \Delta \ell - 1] \)-shrinking point, while if \( \kappa^\pm_{\Delta \ell-1} < 0 \), then at \( q^k_1 \) there is a \( G^\pm[k, \Delta \ell - 1] \)-cycle with a stability multiplier of \(-1\).

We define \( \Sigma_{\Delta \ell} \) as the region bounded by the curves \( \gamma^k \) and \( \gamma^k_{k+1} \), the line segment connecting \( p^k_1 \) and \( p^k_{k+1} \), and the line segment connecting \( q^k_1 \) and \( q^k_{k+1} \). The boundaries \( \gamma^k \) and \( \gamma^k_{k+1} \) are given by (2.20), while \( \theta \) is approximately constant along the two line segments. Thus \( \Sigma_{\Delta \ell} \) is an annular sector formed by arcs that resemble hyperbolas.

Equations (2.20) and (B.3) and (B.4) provide expressions for the boundaries of \( \Sigma_{\Delta \ell} \) with an \( \mathcal{O}(\frac{1}{k^2}) \) error. Figure 4 illustrates the leading order approximation for the example of figure 2, and figure 7 shows both \( \Sigma_{\Delta \ell} \) and its approximation.
For a given value of $\Delta \ell$ and a choice of $+$ or $-$, let $r_k(\theta)$ denote the radial coordinate of $\gamma_{k}^{\pm}$ (i.e. the inner boundary of the $G_{k}^{\pm}$-mode-locking region) for each $k$. Extending (2.20), we have

$$r_k(\theta) = \frac{1}{k} \Gamma(\theta) + \frac{1}{k^2} \Gamma_2(\theta) + O\left(\frac{1}{k^3}\right),$$

for some $C^k$ function $\Gamma_2$. The difference in $r$-values between the inner and outer boundaries of $\Sigma_{k,\Delta\ell}^{\pm}$ is then

$$r_k(\theta) - r_{k+1}(\theta) = \frac{1}{k^2} \Gamma(\theta) + O\left(\frac{1}{k^3}\right),$$

(2.22)

because $O\left(\frac{1}{k^2}\right)$-terms involving $\Gamma_2$ vanish.

We then define

$$\delta := \frac{1}{k} \left(1 - \frac{r}{r_k(\theta)}\right),$$

(2.23)

with which each point in $\Sigma_{k,\Delta\ell}^{\pm}$ is uniquely represented by the pair $(\delta, \theta)$. We have $\delta = 0$ at any point on the outer boundary of $\Sigma_{k,\Delta\ell}^{\pm}$ and by (2.22) we have $\delta = \frac{1}{k^2} + O\left(\frac{1}{k^3}\right)$ at any point on the inner boundary of $\Sigma_{k,\Delta\ell}^{\pm}$. Let

$$\theta_{\text{min}} := \min \left(\theta_{k,\Delta\ell}^{\pm}, \theta_{k,\Delta\ell-1}^{\pm}\right), \quad \theta_{\text{max}} := \max \left(\theta_{k,\Delta\ell}^{\pm}, \theta_{k,\Delta\ell-1}^{\pm}\right).$$

(2.24)

On the two linear boundaries of $\Sigma_{k,\Delta\ell}^{\pm}$ we have $\theta = \theta_{\text{min}} + O\left(\frac{1}{k}\right)$ and $\theta = \theta_{\text{max}} + O\left(\frac{1}{k}\right)$.

The particular definition (2.23) is useful because, to leading order, $\Sigma_{k,\Delta\ell}^{\pm}$ is a rectangle in $(\delta, \theta)$-coordinates as given by (1.8). We include the factor $\frac{1}{k}$ in (2.23) so that an $O(1)$ change in the value of $\delta$ corresponds to an $O(1)$ change in the value of $r$.

2.5. The approximation by skew sawtooth maps

Throughout each sector $\Sigma_{k,\Delta\ell}^{\pm}$ we can approximate the dynamics of (1.2) by using the skew sawtooth map (1.5) with appropriate formulas for $a_L, a_R$ and $w$ in terms of $\delta$ and $\theta$. The value of $w$ is independent of $\theta$ and, as shown in section 5.2, is given by
\[
\begin{array}{c|c|c}
\text{Table 1. Formulas for } a_L \text{ and } a_R \text{ inside a sector } \Sigma_{k,\Delta t}^+ \\
\hline
\text{For } \Sigma_{k,\Delta t}^+ \text{ with } \kappa_{\Delta t}^+ > 0, \kappa_{\Delta t}^+ > 0 & \tan(\theta) & \tan(\theta) \\
\text{For } \Sigma_{k,\Delta t}^+ \text{ with } \kappa_{\Delta t}^+ \kappa_{\Delta t}^- < 0 & \tan(\theta) & \tan(\theta) \\
\text{For } \Sigma_{k,\Delta t}^+ \text{ with } \kappa_{\Delta t}^- > 0, \kappa_{\Delta t}^- > 0 & \tan(\theta) & \tan(\theta) \\
\text{For } \Sigma_{k,\Delta t}^+ \text{ with } \kappa_{\Delta t}^- \kappa_{\Delta t}^- < 0 & \tan(\theta) & \tan(\theta) \\
\end{array}
\]

\[
w = k^2 \delta,
\]

as it varies linearly from 0 to 1 across \( \Sigma_{k,\Delta t}^\pm \) at any constant angle. The values of \( a_L \) and \( a_R \) are independent of \( \delta \) and given in table 1. Note that we do not consider sectors with \( \kappa_{\Delta t}^+ < 0 \) and \( \kappa_{\Delta t}^- < 0 \) as such sectors do not seem to involve admissible \( G^\pm[k,\Delta t] \)-cycles.

As indicated in table 1, the formulas for \( a_L \) and \( a_R \) take slightly different forms in different cases, but in each case \( \frac{a_R}{a_L} \) is constant throughout \( \Sigma_{k,\Delta t}^\pm \). Therefore the ratio \( \frac{a_R}{a_L} \) characterises the two-dimensional slice of parameter space of (1.5) to which \( \Sigma_{k,\Delta t}^\pm \) corresponds. In terms of \( a_L \), \( a_R \), and \( w \), in the approximation of (1.8) the four parts of the boundary of \( \Sigma_{k,\Delta t}^\pm \) are given by \( w = 0 \), \( w = 1 \), \( a_R = 1 \), and either \( a_L = 1 \) or \( a_L = -1 \).

One iteration of the skew sawtooth map (1.5) corresponds to many iterations of (1.2). To be more specific, let

\[
g_{\text{lift}}(z) := \begin{cases} 
      w + a_L (z_i - \zsw) + \zsw, & 0 \leq z_i \leq \zsw, \\
      w + a_R (z_i - \zsw) + \zsw, & \zsw \leq z_i < 1,
   \end{cases}
\]

be a lift of \( g \). At any point in \( \Sigma_{k,\Delta t}^\pm \), for all but an \( O \left( \frac{1}{k} \right) \) set of \( z \)-values for which (1.2) and (1.5) are misaligned, if \( 0 \leq z \leq \zsw \) then \( g(z) \) corresponds to \( f^\pm[k+\Delta t] \Delta t \), while if \( \zsw \leq z < 1 \) then \( g(z) \) corresponds to \( f^\pm[k+\Delta t,\Delta t-1] \), where

\[
\Delta k = g_{\text{lift}}(z) - g(z).
\]

A formal statement of this approximation is given below by theorem 5.2.

2.6. Dynamics of skew sawtooth maps

If \( \frac{a_R}{a_L} > 0 \), as in both plots shown in figure 5, then (1.5) is a homeomorphism and has a unique rotation number as defined in the classical sense. The dynamics of (1.5) approximates that of (1.2) in \( \Sigma_{k,\Delta t}^+ \) for all \( k \geq 1 \), see figure 2. For instance, in the large grey region of the left plot of figure 5 the map (1.5) has a stable fixed point with \( 0 \leq z \leq \zsw \) and \( g_{\text{lift}}(z) - g(z) = 0 \). Therefore, in the corresponding region of each \( \Sigma_{k,\Delta t}^+ \), the map (1.2) has a stable \( G^+[k+1,0] \)-cycle (this belongs to the \( G_{\zsw}^{-1} \)-mode-locking-region).

As a further illustration, consider the mode-locking region of the left plot of figure 5 with rotation number \( \frac{1}{2} \). This region has two components connected by a shrinking point. In the upper component the map (1.5) has a stable period-two solution \( \{z_0, z_1\} \) with \( g_{\text{lift}}(z_0) - g(z_0) = 0 \), \( g_{\text{lift}}(z_1) - g(z_1) = 1 \), \( 0 \leq z_0 \leq \zsw \), and \( 0 \leq z_1 \leq \zsw \), while in the lower component it has a stable period-two solution \( \{z_0, z_1\} \) with the same properties except \( zsw \leq z_1 < 1 \). Therefore, in the upper (lower) component of each \( \Sigma_{k,\Delta t}^+ \), the map (1.2) has a stable \( G^+[k,0]G^+[k+1,0] \)-cycle.
\[ \theta \approx \theta_{\text{max}} \approx 6.0014, \quad \theta_{\min} \approx 3.0811 \]

For \( \Sigma_{-5,0} \), \( \theta_{\min} \approx 2.2254, \quad \theta_{\max} \approx 3.0811 \)

The dot inside \( \Sigma_{+5,0} \) indicates the parameter values of figure 12(B).

**Figure 9.** Mode-locking regions of (1.5) with (1.9) corresponding to \( \Sigma_{k,0}^+ \) and \( \Sigma_{k,0}^- \) of figure 8, where \( k \in \mathbb{Z}^+ \). For \( \Sigma_{k,0}^+ \), \( \theta_{\min} = \theta_0^+ \approx 4.9786 \) and \( \theta_{\max} = \theta_{-1}^+ \approx 6.0014 \). For \( \Sigma_{-5,0}^+ \), \( \theta_{\min} = \theta_0^- \approx 2.2254 \) and \( \theta_{\max} = \theta_1^- \approx 3.0811 \). The dot inside \( \Sigma_{+5,0}^+ \) indicates the parameter values of figure 12(B).

\[ [G^+ [k, 0] G^+ [k + 1, -1] \text{-cycle}] \]

These belong to a mode-locking region with rotation number \( \frac{m^+_{k+1} + m^+_{k+2}}{n_{k+1} + n_{k+2}} \). In this way (1.5) can be used to describe all mode-locking regions of (1.2) near a shrinking point.
Figure 8 shows a second example using the $S$-shrinking point of figure 1 with $S = LRRLR$. The sectors $\Sigma_{3,0}^+$ and $\Sigma_{5,0}^-$ are highlighted and figure 9 shows the mode-locking regions of (1.5) corresponding to these sectors.

As expected, the dynamics of (1.5) shown in figure 9 matches well to the dynamics of (1.2) in $\Sigma_{3,0}^+$ and $\Sigma_{5,0}^-$ shown in figure 8. For each $\Sigma_{3,0}^+$, we have $\frac{\theta_k}{m} = 21\frac{1}{4}$, and so the right plot of figure 9 is similar to the left plot of figure 5 (for which $\frac{\theta_k}{m} \approx 14.7884$). For each $\Sigma_{5,0}^-$, we have $\frac{\theta_k}{m} = -12\frac{4}{5}$. Here $\frac{d_k}{m} < 0$ because $\kappa_0^+ < 0$ (thus $G^+[k,0]$-shrinking points do not exist) and $\kappa_{-1}^+ > 0$, see table 1. With $\frac{d_k}{m} < 0$, the map (1.5) is not invertible and exhibits a fundamentally different bifurcation structure. There are no shrinking points. Mode-locking regions emanate from $\theta = \theta_{\max}$ and terminate at critical values of $\theta$ where the corresponding periodic solution attains a stability multiplier of $-1$. Beyond these critical values the dynamics of (1.5) can be chaotic and multiple attractors coexist for some parameter values [18].

3. A centre manifold and approximately recurrent dynamics

In this section we first discuss symmetries of the centre manifolds associated with an $S$-shrinking point to explain why the majority of the calculations involve the map $f^{S_{(-d)}}$, section 3.1. We then derive new properties of $G^+[k,\Delta\ell]$ in the case $\Delta\ell \geq 0$, section 3.2. In section 3.3 we describe the dynamics of $f^{S_{(-d)}}$ in relation to its centre manifold and in section 3.4 identify a fundamental domain of this manifold.

3.1. Centre manifolds

At an $S$-shrinking point (1.2) has a unique $S^j$-cycle, denoted $\{y_j\}$. There are also uncountably many $S$-cycles and the union of these forms a non-planar polygon with vertices $y_j$, see figure 10.

Each point on the line passing through $y_0$ and $y_d$ is a fixed point of $f^S$. Of these, only points between $y_0$ and $y_d$ are admissible (i.e. are fixed points of $f^S$). More generally, for each $j = 0, 1, \ldots, n - 1$, every point on the line passing through $y_j$ and $y_{(j+d) \mod n}$ is a fixed point of $f^{S_{(-d)}}$. Each line is a centre manifold persisting as an extended centre manifold for parameter values near the $S$-shrinking point.

As shown in [1], the dynamics near an $S$-shrinking point can be analysed by working with these (extended) centre manifolds for the four values $j \in \{0, (\ell - 1)d, \ell d, -d\}$, where from now on we omit ‘$\mod n$’ in indices for brevity. These four values of $j$ correspond to the four line segments of the polygon that intersect the switching manifold. Also, the four curves that bound the mode-locking region associated with the $S$-shrinking point are where the $j^{th}$ point of an $S$-cycle lies on the switching manifold for these four values of $j$.

For the remainder of this paper we work only with the $j = -d$ centre manifold. As explained in the next section, we can use this manifold to extract information about $G^+[k,\Delta\ell]$-cycles with $\Delta\ell \geq 0$. Information about $G^+[k,\Delta\ell]$-cycles with $\Delta\ell < 0$ and $G^-[k,\Delta\ell]$-cycles can be deduced by utilising two forms of symmetry associated with shrinking points. First, we can simply swap the symbols $L$ and $R$ throughout the analysis. More precisely, if we flip all the symbols of $F[\ell, m, n]$ (from $L$ to $R$ and vice-versa) we produce $F[\ell - n, m, \ell n]$. Second, rather than viewing $F[\ell, m, n]$ as clockwise rotation with rotation number $\frac{\ell m}{n}$, we can view it as anticlockwise rotation with rotation number $\frac{m - \ell n}{n}$. This corresponds to the identity $F[\ell, n - m, n] = F[\ell, m, n]^{(\ell - 1)\ell}$. It follows that by replacing $\ell$ and $m$ with $n - \ell$ and $n - m$, respectively.
respectively, in the results below we can generate analogous results for $G^+[k, \Delta \ell]$-cycles with $\Delta \ell < 0$ and $G^- [k, \Delta \ell]$-cycles with any $\Delta \ell \in \mathbb{Z}$.

3.2. Properties of $G^+[k, \Delta \ell]$ with $\Delta \ell \geq 0$

Here we provide three symbolic results for $G^+[k, \Delta \ell]$ with $\Delta \ell \geq 0$ obtained in a straightforward manner from formulas given in [1]. Here it is helpful to work with words rather than sequences. As explained in [1], any periodic symbol sequence $S$ of minimal period $n$ is given by the infinite repetition of the primitive word $S_0 \cdots S_{n-1}$. In this way there is a one-to-one correspondence between periodic symbol sequences and primitive words, and so we also denote the word $S_0 \cdots S_{n-1}$ by $S$.

For any $S = F[\ell, m, n]$, we define the words

$$X := S_0 \cdots S_{(\ell d - 1) \mod n},$$

$$Y := S_{\ell d \mod n} \cdots S_{n-1},$$

$$\hat{X} := S_0 \cdots S_{(-d-1) \mod n}.$$  \hspace{1cm} (3.1)

We first show that the word $G^+[k, \Delta \ell]$ ends in powers of $S^{(-d)}$.

**Lemma 3.1.** For all $k \in \mathbb{Z}^+$ and $\Delta \ell = 0, \ldots, k - 1$,

$$G^+[k, \Delta \ell] = (X Y^k)^{\Delta \ell} \hat{X} \left(S^{(-d)}\right)^{k-\Delta \ell}. \hspace{1cm} (3.4)$$

**Proof.** By proposition 4.8 of [1], $G^+[k, \Delta \ell] = (X Y^k)^{\Delta \ell} S^{\ell-\Delta \ell} \hat{X}$. This can rewritten as (3.4) because $S \hat{X} = \hat{X} S^{(-d)}$. \hfill $\Box$

The next result shows that if we flip the first symbol of $G^+[k, \Delta \ell]$ the result is a cyclic permutation of $G^+[k, \Delta \ell - 1]$.

**Lemma 3.2.** For all $k \in \mathbb{Z}^+$ and $\Delta \ell = 0, \ldots, k - 1$,

$$G^+[k, \Delta \ell]^{-1} = G^+[k, \Delta \ell - 1]^{(-d^+)}.$$  \hspace{1cm} (3.5)
Proof. The result is obtained by simply applying the general identity
\[ F[ℓ, m, n] = F[ℓ - 1, m, n]^{(-d)} \]
(equation (4.3) of [1]) to \( G^+[k, \Delta ℓ] = F[ℓ_k^+ + \Delta ℓ, m_k^+, n_k^+] \).

The \( j = -d \) centre manifold is useful for analysing \( G^+[k, \Delta ℓ] \)-cycles with \( \Delta ℓ > 0 \) because, by lemma 3.1, the word \( G^+[k, \Delta ℓ] \) ends in a large power of \( S^{(-d)} \). Thus, under certain assumptions, the fixed point of \( f^{S^{(-d)}}[k, \Delta ℓ] \) lies close to the \( j = -d \) centre manifold. Also, by lemma 3.2, \( G^+[k, \Delta ℓ - 1] \)-cycles can be analysed by studying the fixed point of \( f^{S^{(-d)}}[k, \Delta ℓ] \) which lies close to the \( j = -d \) centre manifold for the same reason.

We also provide an additional result used in later proofs.

Lemma 3.3. For all \( k \in \mathbb{Z}^+ \) and \( \Delta ℓ = 0, \ldots, k - 1 \),
\[ S^{(-d)} G^+[k, \Delta ℓ] = G^+[k, \Delta ℓ]^{d} (I + \Delta ℓ)_{d} S^{(-d)}. \]

Proof. Since \( d_k^+ = n \) (see lemma 4.5 of [1]), by (3.4),
\[ G^+[k, \Delta ℓ]^{(-d_k^+)} = S^{(-d)} \left( \frac{\partial}{\partial x} \right)^{\Delta ℓ} S^{(-d)} \left( \frac{\partial}{\partial x} \right)^{k-\Delta ℓ-1}. \]

Also, by (2.9),
\[ G^+[k, \Delta ℓ]^{(d_k^+ + \Delta ℓ)} = G^+[k, \Delta ℓ]^{(-d_k^+)}. \]

Then (3.6) is obtained by combining (3.4), (3.7) and (3.8).

3.3. Dynamics near the centre manifold

The map \( f^{S^{(-d)}} \) is affine with matrix part \( M_{S^{(-d)}} \). At the \( S \)-shrinking point, \((η, ν) = (0, 0)\), \( M_{S^{(-d)}} \) has a unit eigenvalue with multiplicity one. Thus in a neighbourhood of \((η, ν) = (0, 0)\), \( M_{S^{(-d)}} \) has an eigenvalue \( λ(η, ν) \) with \( λ(0, 0) = 1 \) and a \( C^\infty \) dependency on \( η \) and \( ν \). Throughout this neighbourhood let \( ω_{-d}^T \) and \( ζ_{-d} \) be the corresponding left and right eigenvectors of \( M_{S^{(-d)}} \) normalised by
\[ e_1^T ζ_{-d} = 1, \quad ω_{-d}^T ζ_{-d} = 1. \]

At points \((η, ν) \) where \( λ ≠ 1 \), the \( S \)-cycle, denoted \( \{ x_s^d \} \), is unique. The point \( x_{-d}^d \) is a fixed point of \( f^{S^{(-d)}} \). As shown in [1] (lemma 7.2), the quantities
\[ x_{-d}^{int} := \left( I - ζ_{-d} e_1^T \right) x_{-d}^S, \quad x_{-d}^{step} := (1 - λ) e_1^T x_{-d}^S, \]
whose purpose is explained below, can be extended in a \( C^\infty \) fashion to a neighbourhood of \((η, ν) = (0, 0)\) (i.e. including points where \( λ = 1 \)).

The line
\[ W^c := \left\{ x_{-d}^{int} + h ζ_{-d} : h ∈ \mathbb{R} \right\}, \]
has direction \( ζ_{-d} \) and passes through \( x_{-d}^S \) whenever \( λ ≠ 1 \) and so is a centre manifold of \( f^{S^{(-d)}} \). Notice \( e_1^T x_{-d}^{int} = 0 \), by (3.9) and (3.10), thus \( x_{-d}^{int} \) is the intersection of \( W^c \) with the switching manifold, figure 11.
Any $x \in \mathbb{R}^N$ can be uniquely written as

$$x = x_{\text{int}}^0 + h\zeta_{\omega} - d$$

(3.12)

where $h \in \mathbb{R}$ and $q \in \mathbb{R}^N$ satisfies

$$\omega^T q = 0$$

(equation (3.12) represents a partial decomposition by eigenspaces). Define two functions $u : \mathbb{R}^N \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}^N$ by

$$u(x) := \omega^T \left( x - x_{\text{int}}^0 \right), \quad v(h) := x_{\text{int}}^0 + h\zeta_{\omega} - d.$$  

(3.13)

The function $u$ returns the value of $h$ in (3.12); the vector $q$ is given by

$$q = (I - \zeta_{\omega} \omega^T)^{-1} \left( x - x_{\text{int}}^0 \right).$$

(3.14)

The function $v$ returns the point $x \in W^c$ with $q = 0$ in (3.12). The composition $v \circ u$ thus represents the projection onto $W^c$ in directions orthogonal to $\omega_{\omega} - d$.

Figure 11. A sketch illustrating dynamics near the centre manifold $W^c$ (3.11). We show an arbitrary point $x$ and its first and second images under $f_{f_{\omega}}^{(-\rho)}$ and indicate the decomposition of these points as given by (3.15). We also show the fundamental domain $\Omega_{\omega}$ (3.19).

Given $x \in \mathbb{R}^N$ in the form (3.12), we can write an arbitrary power of $f_{f_{\omega}}^{(-\rho)}$ as

$$f_{f_{\omega}}^{(-\rho)^k}(x) = x_{\text{int}} + \left( s_{-\rho}^{\text{step}} \sum_{j=0}^{k-1} \lambda^j + h \lambda^k \right) \zeta_{\omega} - d + M_{k}^{\rho} q,$$  

(3.15)

for any $k \in \mathbb{Z}^+$. Equation (3.15) is illustrated in figure 11. It has the form (3.12) because the middle term is a scalar multiple of $\zeta_{\omega}$ and the last term is orthogonal to $\omega_{\omega}$. The scalar quantity $s_{-\rho}^{\text{step}}$, defined by (3.10), is used in (3.15) to describe the action of iteration under $f_{f_{\omega}}^{(-\rho)}$. By (3.15) with $k = 1$, we have

$$f_{f_{\omega}}^{(-\rho)}(x_{\text{int}}) = x_{\text{int}} + s_{-\rho}^{\text{step}} \zeta_{\omega} - d.$$  

Thus $s_{-\rho}^{\text{step}}$ can be interpreted as the first coordinate of the image of $x_{\text{int}}$ under $f_{f_{\omega}}^{(-\rho)}$.

If $\rho < 1$, where $\rho_{\text{max}}$ is defined by (2.14), then forward orbits of $f_{f_{\omega}}^{(-\rho)}$ approach $W^c$. Indeed, by (3.15) the vector $M_{k}^{\rho} q$ is the displacement of $f_{f_{\omega}}^{(-\rho)^k}(x)$ from $W^c$ in directions orthogonal to $\omega_{\omega}$. If $\rho_{\text{max}} < 1$, then $M_{k}^{\rho} q = O(\rho_{\text{max}}^k)$ for any fixed $q$ orthogonal to $\omega_{\omega}$. In view of (3.14) we can therefore write
\[ M_{S(-\ell)}^k (I - \zeta_d \omega^{-\ell}_d) = \mathcal{O} \left( \mu_{\text{max}}^k \right). \] 

(3.16)

### 3.4. An approximately recurrent line segment, \( \Omega_{\Delta\ell} \)

Here we identify a useful fundamental domain \( \Omega_{\Delta\ell} \subset W^c \), then provide three lemmas. Lemma 3.4 gives an algebraic identity, lemma 3.5 indicates the approximate range of \( \eta \) and \( \nu \) values relevant for \( \Omega_{\Delta\ell} \), and lemma 3.6 relates \( \Omega_{\Delta\ell} \) to one boundary of the \( G^+_k \)-mode-locking region.

For any \( \hat{x} \in W^* \), consider a half-open line segment with endpoints \( \hat{x} \) and \( f^{S(-\ell)}(\hat{x}) \). Assuming \( \hat{x} \neq x^S_{-\ell} \), by (2.9) all orbits of \( f^{S(-\ell)}(\hat{x}) \) have exactly one point in this line segment, and in this sense the line segment is a fundamental domain.

We choose \( \hat{x} \in W^* \) such that its image under \( f^\pi(y^\pi x)^{\Delta\ell} \) lies on the switching manifold. We do this because by (3.4) the first \( (\ell \Delta t + \Delta s) \) symbols of \( G^+[k, \Delta\ell] \) are \( \left( x^\pi \right)^{\Delta\ell} \), which, upon flipping the first symbol, can be rewritten as \( \left( y^\pi x \right)^{\Delta\ell} \). Thus, if there exists a \( G^+[k, \Delta\ell] \)-cycle for which \( f^\pi(y^\pi x)^{\Delta\ell} \) lies on the switching manifold, then the \( G^+[k, \Delta\ell] \)-cycle is also a \( G^+[k, \Delta\ell] \)-cycle, and by (2.9) it is also a permutation of a \( G^+[k, \Delta\ell] \)-cycle.

That is, we let \( \hat{x} = v(h^R_{\Delta\ell}) \), where \( h^R_{\Delta\ell} \in \mathbb{R} \) is defined implicitly by

\[ e_1^T f^\pi(y^\pi x)^{\Delta\ell} (v(h^R_{\Delta\ell})) = 0. \] 

(3.17)

The image of \( v(h^R_{\Delta\ell}) \) under \( f^{S(-\ell)} \) is \( v(h^L_{\Delta\ell}) \) where

\[ h^L_{\Delta\ell} := e_1^T f^{S(-\ell)} (v(h^R_{\Delta\ell})). \] 

(3.18)

Then our fundamental domain is

\[ \Omega_{\Delta\ell} := \left\{ v(h) \mid h^L_{\Delta\ell} \leq h < h^R_{\Delta\ell} \right\}, \]

(3.19)

assuming \( h^L_{\Delta\ell} < h^R_{\Delta\ell} \).

Before we continue we must verify that \( h^R_{\Delta\ell} \) is unique and well-defined by (3.17). To do this we solve for \( h^R_{\Delta\ell} \) in (3.17) giving

\[ h^R_{\Delta\ell} = -\frac{e_1^T f^\pi(y^\pi x)^{\Delta\ell}(x^\text{int}_{-\ell,d})}{e_1^T M^\pi(y^\pi x)^{\Delta\ell} \zeta_{-\ell,d}}, \] 

(3.20)

assuming that the denominator of (3.20) is non-zero. Next we derive an explicit expression for the leading order component of this denominator. Then lemma 3.5 gives formulas for \( h^R_{\Delta\ell} \) and \( h^L_{\Delta\ell} \) and is proved in appendix A.

**Lemma 3.4.** Suppose \( \text{det}(J) \neq 0 \), where \( J \) is defined in section 2.3. Then for any \( \Delta\ell \geq 0 \),

\[ e_1^T M^\pi(y^\pi x)^{\Delta\ell} \zeta_{-\ell,d} = \begin{cases} \frac{\eta}{\nu} \pm \mathcal{O} \left( \eta, \nu \right), & \Delta\ell = 0, \\ \frac{\eta^\prime}{\nu^\prime} \pm \mathcal{O} \left( \eta^\prime, \nu^\prime \right), & \Delta\ell > 1. \end{cases} \]

(3.21)

**Proof.** If \( \Delta\ell = 0 \), then the denominator of (3.20) is \( e_1^T M^\pi \zeta_{-\ell,d} \) and the result follows im-
immediately from equation (6.9) of [1]. If $\Delta \ell \geq 1$, then at $(\eta, \nu) = (0, 0)$ we have

$$u_0^T (I - M_{S \Sigma}) = \frac{b \ell}{c_{(\ell+1)d}} e_1^T M_X,$$

(3.22)

where $u_0^T$ is the left eigenvector of $M_S$ corresponding to the unit eigenvalue 1 (this is equation (A.34) of [1]). Also

$$X^\Pi \left( Y^\Pi X \right)^{\Delta \ell} = X^\Pi Y^\Pi \left( X Y^\Pi \right)^{\Delta \ell - 1} X = S^{(-d)} \left( S^{\Sigma} \right)^{\Delta \ell - 1} X,$$

and so by (3.22) at $(\eta, \nu) = (0, 0)$ we have

$$e_1^T M_X X^\Pi (Y^\Pi X)^{\Delta \ell} \zeta_{-d} = \frac{c_{(\ell+1)d}}{b \ell d} u_0^T (I - M_{S \Sigma}) M^{\Delta \ell - 1} \zeta_{-d}.$$

This yields (3.21) for $\Delta \ell \geq 1$ by using (2.13) and the definition of $\kappa_{\Delta \ell}^+$ (B.1).

\[ \square \]

**Lemma 3.5.** Suppose $\det(J) \neq 0$. Then for any $\Delta \ell \geq 0$,

$$h_{\Delta \ell}^L = \begin{cases} \eta - \frac{l_{\Delta \ell}^+}{\ell (\ell+1)d} \nu + \mathcal{O} \left( (\eta, \nu)^2 \right), & \Delta \ell = 0, \\ \frac{a}{\ell^2 \Delta \ell} \left( \eta - \frac{l_{\Delta \ell}^+}{\ell (\ell+1)d} \nu \right) + \mathcal{O} \left( (\eta, \nu)^2 \right), & \Delta \ell \geq 1, \end{cases}$$

(3.23)

$$h_{\Delta \ell}^R = \begin{cases} \eta - \frac{l_{\Delta \ell}^+}{\ell (\ell+1)d} \nu + \mathcal{O} \left( (\eta, \nu)^2 \right), & \Delta \ell = 0, \\ \frac{a}{\ell^2 \Delta \ell} \left( \eta - \frac{l_{\Delta \ell}^+}{\ell (\ell+1)d} \nu \right) + \mathcal{O} \left( (\eta, \nu)^2 \right), & \Delta \ell \geq 1, \end{cases}$$

(3.24)

assuming $\kappa_{\Delta \ell}^+ \neq \kappa_{\Delta \ell - 1}^+$ in the case $\Delta \ell \geq 1$.

Inside the $G^+_k$-mode-locking region and between the $G^+ [k, \Delta \ell]$ and $G^+ [k, \Delta \ell - 1]$-shrinking points (if they exist), there are $G^+ [k, \Delta \ell]$ and $G^+ [k, \Delta \ell - 1]$-cycles, one of which is attracting (as determined by the sign of $a$). The mode-locking region is bounded by a curve where $x_0^{G^+ [k, \Delta \ell]}$ lies on the switching manifold, and a curve where $x_0^{G^+ [k, \Delta \ell]} \left( \epsilon_{i+1}^{\ell} + \Delta \ell - 1 \right) d^i$ lies on the switching manifold. These boundaries are where $P_{G^+ [k, \Delta \ell]}(0)$ is singular for $i = 0$ and $i = (\ell^2 + \Delta \ell - 1) d^i$.

**Lemma 3.6.** Suppose $\det(J) \neq 0$ and $\rho_{\text{max}} < 1$. Then for any $\Delta \ell \geq 0$ there exists a neighbourhood of the $S$-shrinking point within which $\det \left( P_{G^+ [k, \Delta \ell]}(0) \right) = 0$ implies

$$f^{G^+ [k, \Delta \ell]} (x_{-d}^{\text{int}}) = x_{-d}^{\text{int}} + \mathcal{O} \left( \rho_{\text{max}}^k \right),$$

(3.25)

where $k \in \mathbb{Z}^+$.  

**Proof.** Suppose $\det \left( P_{G^+ [k, \Delta \ell]}(0) \right) = 0$. Also suppose, for the moment, that the $G^+ [k, \Delta \ell]$-cycle exists and is unique (as is generically the case, although it may not be admissible). Then $x_0^{G^+ [k, \Delta \ell]}$ lies on the switching manifold. Since $\rho_{\text{max}} < 1$ and $G^+ [k, \Delta \ell]$ ends in a power of $S^{(-d)}$ proportional to $k$, $x_0^{G^+ [k, \Delta \ell]}$ is an $O \left( \rho_{\text{max}}^k \right)$ distance from $W$ (this is demonstrated formally in the proof of lemma 7.7 of [1]). Thus $x_0^{G^+ [k, \Delta \ell]} = x_{-d}^{\text{int}} + \mathcal{O} \left( \rho_{\text{max}}^k \right)$, and then (3.25) follows from the fact that $x_0^{G^+ [k, \Delta \ell]}$ is a fixed point of $f^{G^+ [k, \Delta \ell]}$.  

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Equation (3.25) is also true if the \( G^+[k, \Delta \ell] \)-cycle does not exist or is non-unique (i.e. \( \det (I - M G^+[k, \Delta \ell]) = 0 \)) because the components of (3.25) are smooth functions of the parameters and \( \det (I - M G^+[k, \Delta \ell]) \neq 0 \) on a dense subset of parameter space.

### 4. An attracting invariant set

In this section we enlarge the one-dimensional fundamental domain \( \Omega_{\Delta \ell} \subset W^c \) into an \( N \)-dimensional set \( \Phi \). The set \( \Phi \) will have the property that iterations of any \( x \in \Phi \) under \( f \) will return to \( \Phi \) after following \( G^+[k, \Delta \ell] \), or a similar sequence, with the assumption \( \rho_{\max} < 1 \). For this reason \( \Phi \) is referred to as a \textit{recurrent set}.

We first define \( \Phi \) in section 4.1. We then study the return dynamics on \( \Phi \) in sections 4.2 and 4.3. Lastly we use \( \Phi \) to identify an attracting invariant set in section 4.4.

#### 4.1. The construction of a recurrent set

By the definition of \( \Omega_{\Delta \ell} \), see (3.19), the surface

\[
\left\{ x \in \mathbb{R}^N \mid e_1^T f^H (y^T x)^{\Delta \ell} (x) = 0 \right\}
\]

intersects \( \Omega_{\Delta \ell} \) at its right endpoint \( v \left( \kappa^R_{\Delta \ell} \right) \).

**Lemma 4.1.** Suppose \( \det(J) \neq 0 \). Let \( \Delta \ell \geq 0 \) and suppose \( k^+_{\Delta \ell} \neq k^+_{\Delta \ell-1} \) if \( \Delta \ell \geq 1 \). Then (4.1) is a hyperplane not parallel to \( \Omega_{\Delta \ell} \).

**Proof.** By (2.3),

\[
e_1^T f^H (y^T x)^{\Delta \ell} (x) = e_1^T M x \sigma (\gamma^T x)^{\Delta \ell} x + e_1^T P \sigma (\gamma^T x)^{\Delta \ell} B.
\]

This is the equation for a hyperplane with normal vector \( e_1^T M x \sigma (\gamma^T x)^{\Delta \ell} \). By lemma 3.4, \( e_1^T M x \sigma (\gamma^T x)^{\Delta \ell} \neq 0 \). Thus (4.1) is a hyperplane (because the normal vector is nonzero) and is not parallel to \( \Omega_{\Delta \ell} \) (because \( \Omega_{\Delta \ell} \) has direction \( \zeta_{-d} \)).

Given \( k \in \mathbb{Z}^+, |\Delta \ell| < k \) and \( Q > 0 \), let

\[
H_{k, \Delta \ell, Q} := \left\{ x \in \text{range} \left( f^{S(-\ell)} \right) \mid e_1^T f^H (y^T x)^{\Delta \ell} (x) = 0, \right. \\
\left. \| (I - \zeta_{-d} \omega_{-d}^T) (x - x_{\text{int}}) \| \leq Q \rho_{\max}^k \right\},
\]

be a subset of (4.1) that contains \( v \left( \kappa^R_{\Delta \ell} \right) \). If we write any \( x \in H_{k, \Delta \ell, Q} \) in the form

\[
x = x_{\text{int}} + h \zeta_{-d} + q (3.12),
\]

then \( \| q \| \leq Q \rho_{\max}^k \).

If 0 is an eigenvalue of \( M_{S(-\ell)} \), then the range of \( f^{S(-\ell)} \) has dimension less than \( N \) and the condition \( H_{k, \Delta \ell, Q} \subset \text{range} \left( f^{S(-\ell)} \right) \) is helpful below. It is tempting to ignore this complication, as it is a special case within the space of maps of the form (1.2), however it is in fact typical for 0 to be an eigenvalue of \( M_{S(-\ell)} \) in the application to grazing-sliding bifurcations [23].

Let

\[
\Phi_{k, \Delta \ell, Q} := \left\{ \alpha x_1 + (1 - \alpha) x_2 \mid x_1 \in H_{k, \Delta \ell, Q}, x_2 \in f^{S(-\ell)} (H_{k, \Delta \ell, Q}) , 0 \leq \alpha < 1 \right\},
\]

(3.3)
For brevity we just write $\Phi$ when it is clear what values of $k$, $\Delta \ell$ and $Q$ are being used. The set $\Phi$ is the convex hull of $H_{k,\Delta \ell,Q}$ and its image under $f^{S(-\ell)}$, but not including $H_{k,\Delta \ell,Q}$. Figure 12 shows the set $\Phi$ for two examples.

4.2. Return dynamics

Here we show that, under certain assumptions, $\Phi$ is a recurrent set for $f$ for parameter values throughout $\Sigma^+_{k,\Delta \ell}$. Let $x \in \mathbb{R}^N$ and suppose $\rho_{\max} < 1$. Then $f^{\Delta \ell + k, \Delta \ell}(x)$ is an $O(\rho^k)$ distance from $W^e$ (assuming $k \gg \Delta \ell$), see section 3.3. Also

$$f^{\Delta \ell + k, \Delta \ell}(x) = f^{S(-\ell)}(f^{\Delta \ell + k, \Delta \ell}(x)), \quad (4.4)$$

for all $\Delta k \geq 0$ by (3.4). Moreover, (4.4) holds for all $\Delta k \geq -k - \Delta \ell$ where we take $f^{S(-\ell)}(x)^{-1}$ to be the unique inverse belonging to range $f^{S(-\ell)}$. Therefore $\{f^{\Delta \ell + k, \Delta \ell}(x)\}_{\Delta k \geq -k}$ is a sequence of points near $W^e$ mapping from one to the next under $f^{S(-\ell)}$.

For any $y \in \text{range } f^{S(-\ell)}$, let $\beta(y)$ denote the smallest value of $\Delta k$ for which

$$e^{\frac{\Delta k}{\lambda^*}(y^x,v)} \Delta k^\beta(y) > 0, \quad (4.5)$$

and assume $\beta(y) \gg -k$. Also let

$$T(y) := \left(f^{S(-\ell)}\right)^{\beta(y)}(y). \quad (4.6)$$

**Lemma 4.2.** Suppose $\det(J) \neq 0$ and $\rho_{\max} < 1$. Let $\Delta \ell \geq 0$ and suppose $\Sigma^+_{k,\Delta \ell}$ is well-defined for arbitrarily large values of $k$ with $(\kappa^+_{\Delta \ell} - \kappa^+_{\Delta \ell - 1}) a > 0$. If $k \in \mathbb{Z}^+$ and $Q \in \mathbb{R}$ are sufficiently large, then at any point in $\Sigma^+_{k,\Delta \ell}$ and any $x \in \Phi_{k,\Delta \ell,Q}$:

(i) $\beta\left(f^{\Delta \ell + k, \Delta \ell}(x)\right)$ is well-defined and $f^{\Delta \ell + k, \Delta \ell}(x) \in \Phi$ if and only if
\( \Delta k = \beta \left( f^{G^+[k, \Delta \ell]}(x) \right) \) and similarly 
(ii) \( \beta \left( f^{G^+[k, \Delta \ell]}(x) \right) \) is well-defined and \( f^{G^+[k+\Delta k, \Delta \ell]}(x) \in \Phi \) if and only if 
\( \Delta k = \beta \left( f^{G^+[k, \Delta \ell]}(x) \right) \).

A proof of lemma 4.2 is given in appendix A. In lemma 4.2 the assumption on the sign 
\( \left( \kappa_{\Delta \ell}^+ - \kappa_{\Delta \ell^-}^+ \right) a \) is provided as an alternative to a stronger condition of admissibility at 
\( G^+[k, \Delta \ell] \)-shrinking points and is used to show that the left hand-side of (4.5) is an increasing 
function of \( \Delta k \).

4.3. Continuity of the return map in a cylindrical topology

Let 
\[
F(x) := \begin{cases} 
T \left( f^{G^+[k, \Delta \ell]}(x) \right), & s \leq 0, \\
T \left( f^{G^+[k+\Delta k, \Delta \ell]}(x) \right), & s \geq 0.
\end{cases}
\]  

(4.7)

Equivalently
\[
F(x) = \begin{cases} 
T \left( f^{G^+[k+\Delta k, \Delta \ell]}(x) \right), & s \leq 0, \\
T \left( f^{G^+[k+\Delta k, \Delta \ell]}(x) \right), & s \geq 0.
\end{cases}
\]

(4.8)

where
\[
\Delta k = \begin{cases} 
\beta \left( f^{G^+[k, \Delta \ell]}(x) \right), & s \leq 0, \\
\beta \left( f^{G^+[k+\Delta k, \Delta \ell]}(x) \right), & s \geq 0.
\end{cases}
\]

(4.9)

By lemma 4.2, \( F : \Phi \to \Phi \) throughout \( \Sigma_{k, \Delta \ell}^+ \).

Here we identify the left and right faces of \( \Phi \) to create a topology in which \( \Phi \) is cylindrical. More specifically, we identify each \( x^+ \in H_{k, \Delta \ell, Q} \) with \( x^- = f^{S^{(-\ell)}}(x^+) \).

Lemma 4.3. The map \( F : \Phi \to \Phi \) is continuous in the cylindrical topology of \( \Phi \).

Proof. \( F \) is continuous on the switching manifold because if \( s = 0 \) then 
\( f^{G^+[k, \Delta \ell]}(x) = f^{G^+[k, \Delta \ell]}(x) \) by the continuity of \( f \). Also \( F \) is continuous at any \( x \in \Phi \) with 
\( F(x) \in f^{S^{(-\ell)}}(H_{k, \Delta \ell, Q}) \), because for any small \( \Delta x \) the point \( F(x + \Delta x) \) will lie near either \( F(x) \) (if the value of \( \beta(y) \) in \( T \) is unchanged) or its preimage in \( H_{k, \Delta \ell, Q} \) under \( f^{S^{(-\ell)}} \) (if the value of \( \beta(y) \) in \( T \) decreases by 1) and these two points are identical in the cylindrical topology of \( \Phi \).

Choose any \( x^- \in f^{S^{(-\ell)}}(H_{k, \Delta \ell, Q}) \), and let \( x^+ \) be the preimage of \( x^- \) in \( H_{k, \Delta \ell, Q} \) under \( f^{S^{(-\ell)}} \). It remains to show that the points 
\[
F(x^-) = T \left( f^{G^+[k, \Delta \ell]}(x^-) \right),
\]

(4.10)

\[
F(x^+) = T \left( f^{G^+[k+\Delta k, \Delta \ell]}(x^+) \right),
\]

(4.11)

are identical.

Since \( x^- = f^{S^{(-\ell)}}(x^+) \), by (3.6) we have
Since \( x^+ \in H_{k, \Delta, Q} \), by the continuity of \( f \) this is the same as

\[
f^{G^+[k, \Delta]_Q}(x^-) = f^{G^+[k, \Delta]_Q}(x^+). \]

Therefore (4.10) can be rewritten as

\[
F(x^-) = T \left( f^{S^{(-\alpha)}} \left( f^{G^+[k, \Delta]_Q}(x^+) \right) \right),
\]

which is identical to (4.11) because \( T \circ f^{S^{(-\alpha)}} = T \) by the definition of \( T \).

\[
\square
\]

4.4. An attracting invariant set and its topology

Figure 12 shows the set \( F(\Phi) \) for two examples. If we continue to iterate \( \Phi \) under \( F \) we obtain the attracting invariant set

\[
\Lambda_{\Delta \ell} := \bigcap_{i=0}^{\infty} F^i(\Phi). \tag{4.12}
\]

**Lemma 4.4.** The sets \( \Lambda_{\Delta \ell} \) and \( \Omega_{\Delta \ell} \) are homotopic in the cylindrical topology of \( \Phi \).

Roughly speaking this means that \( \Lambda_{\Delta \ell} \) can be transformed to \( \Omega_{\Delta \ell} \) by distorting it in a continuous fashion [24, 25]. Below we prove lemma 4.4 by using the following result.

**Lemma 4.5.** Let \( \Psi \subset \Phi \) be connected. If \( F(\Psi) \subset \Psi \), then \( \Psi \) and \( F(\Psi) \) are homotopic in the cylindrical topology of \( \Phi \).

To say that \( \Psi \) and \( F(\Psi) \) are homotopic means that there exists a continuous function \( K : \Phi \times [0, 1] \to \Phi \) with \( K(x, 0) = x \) and \( K(x, 1) = F(x) \) for all \( x \in \Psi \). The function \( K \) is said to be a homotopy between \( \iota_{\Phi} \) (the inclusion of \( \Psi \)) and \( F|_{\Phi} \) (the restriction of \( F \) to \( \Psi \)).

**Proof of lemma 4.5.** Define \( K : \Phi \times [0, 1] \to \Phi \) by

\[
K(x, \alpha) := \begin{cases} 
T \left( (1 - \alpha)x + \alpha f^{G^+[k, \Delta]_Q}(x) \right), & s \leq 0, \\
T \left( (1 - \alpha)x + \alpha f^{G^+[k, \Delta]_Q}(x) \right), & s \geq 0.
\end{cases} \tag{4.13}
\]

By (4.6), \( K(x, 0) = x \) for all \( x \in \Psi \). Also, by (4.7), \( K(x, 1) = F(x) \) for all \( x \in \Psi \).

Choose any \( x^- \in f^{S^{(-\alpha)}}(H_{k, \Delta, Q}) \), and let \( x^+ \) be the preimage of \( x^- \) in \( H_{k, \Delta, Q} \) under \( f^{S^{(-\alpha)}} \). We wish to show that

\[
K(x^-, \alpha) = T \left( (1 - \alpha)x^- + \alpha f^{G^+[k, \Delta]_Q}(x^-) \right), \tag{4.14}
\]

\[
K(x^+, \alpha) = T \left( (1 - \alpha)x^+ + \alpha f^{G^+[k, \Delta]_Q}(x^+) \right), \tag{4.15}
\]

are identical for all \( \alpha \in [0, 1] \). Following the steps of the proof of lemma 4.3, we obtain
\[ K(x^-, \alpha) = T \left( (1 - \alpha) f^{\mathcal{S}(-\alpha)}(x^+) + \alpha f^{\mathcal{S}(\alpha)} \left( f^{\mathcal{G}^+[k, \Delta \ell]}(x^+) \right) \right). \]

Since \( f^{\mathcal{S}(-\alpha)} \) is affine, this is the same as
\[ K(x^-, \alpha) = T \left( f^{\mathcal{S}(-\alpha)} \left( (1 - \alpha)x^+ + \alpha f^{\mathcal{G}^+[k, \Delta \ell]}(x^+) \right) \right), \]
which is identical to (4.15) because \( T \circ f^{\mathcal{S}(-\alpha)} = T \).

This shows that \( K \) is continuous at any \( x^- \in f^{\mathcal{S}(-\alpha)}(H_{k, \Delta \ell, Q}) \). As with \( F \), the function \( K \) is continuous on the switching manifold and at any \( x \in \Phi \) with \( K(x, \alpha) \in f^{\mathcal{S}(-\alpha)}(H_{k, \Delta \ell, Q}) \). Thus \( K \) is continuous throughout \( \Phi \) for any \( \alpha \in [0, 1] \). Thus \( K \) is a homotopy and \( \Psi \) and \( F(\Psi) \) are homotopic.

**Proof of lemma 4.4.** By applying lemma 4.5 to \( \Psi = \Phi \), \( \Psi = F(\Phi) \), \( \Psi = F^2(\Phi) \), and so on, we deduce that \( \Phi \) and \( \Lambda_{\Delta \ell} \) are homotopic. It is clear that \( \Omega_{\Delta \ell} \) and \( \Phi \) are homotopic in view of the definition of \( \Phi \), thus \( \Lambda_{\Delta \ell} \) and \( \Omega_{\Delta \ell} \) are homotopic. \( \square \)

5. The skew sawtooth map as an approximate return map

5.1. One-dimensional approximation

Recall, for any \( x \in \mathbb{R}^N \) written as \( x = x^{\text{int}} + h \zeta - d + q \) (3.12), the function \( u(x) \) extracts the value of \( h \), and for any \( h \in \mathbb{R} \) the function \( v(h) \) returns the point \( x^{\text{int}} + h \zeta - d \in \mathbb{R}^N \). Thus for any \( x \in \mathbb{R}^N \) we have \( v(u(x)) = x \) and for any \( x \) near \( \mathbb{R}^N \) we have \( v(u(x)) \approx x \).

The set \( \Phi = \Phi_{k, \Delta \ell, Q} \) is an \( O(p_{\text{max}}) \) fattening of the fundamental domain \( \Omega_{\Delta \ell} \subset \mathbb{R}^N \). Thus for any \( x \in \Phi \) we have \( v(u(x)) \approx x \) and \( v(u(F(x))) \approx F(x) \) because \( F(x) \in \Phi \). In this way the \( N \)-dimensional map \( F \) is well-approximated by the one-dimensional map taking \( u(x) \) to \( u(F(x)) \). This is the map
\[ G := u \circ F \circ v. \]

The map \( G \) is piecewise-linear but over the interval \([\tilde{h}^L_{\Delta \ell}, \tilde{h}^R_{\Delta \ell}]\) can involve many different linear pieces corresponding to different values of \( \Delta k \). Next we show that, with fixed values of \( \Delta \ell \) and \( \theta \), the leading order components of the slopes of these pieces only take two different values.

**Lemma 5.1.** Suppose \( \det(J) \neq 0 \) and \( p_{\text{max}} < 1 \). Let \( \Delta \ell \geq 0 \) and suppose \( \Sigma_{k, \Delta \ell}^{+} \) is well-defined for arbitrarily large values of \( k \) with \( (\kappa_{\Delta \ell}^{+} - \kappa_{\Delta \ell-1}^{+}) a > 0 \). Then at any point in \( \Sigma_{k, \Delta \ell}^{+} \) and any \( h \in \mathbb{R} \),
\[ \frac{d}{dh} u \left( f^{\mathcal{G}^+[k, \Delta \ell]}(v(h)) \right) = \frac{t_d \kappa_{\Delta \ell}^{+} \tan(\theta)}{t_d} + O \left( \frac{1}{k} \right), \quad (5.2) \]
\[ \frac{d}{dh} u \left( f^{\mathcal{G}^+[k, \Delta \ell]}(v(h)) \right) = \begin{cases} \frac{t_d \kappa_{\Delta \ell}^{+} \tan(\theta)}{t_d} + O \left( \frac{1}{k} \right), & \Delta \ell = 0, \\ \frac{t_d \kappa_{\Delta \ell}^{+} \tan(\theta)}{t_d} + O \left( \frac{1}{k} \right), & \Delta \ell \geq 1, \end{cases} \quad (5.3) \]
where \( k \in \mathbb{Z}^{+} \).
Lemma 5.1 is proved in appendix A. Since $G$ is given by (5.1), where $F$ is equal to $f^{\theta}|_{k+\Delta k,\Delta \ell}$ or $f^{\theta^*}|_{k+\Delta k,\Delta \ell'}$ for some $O(1)$ value of $\Delta k$, the slopes of $G$ are given by (5.2) and (5.3) to leading order.

5.2. Circle map formulation

The function

$$\varphi(h) := \begin{cases} \frac{h-h^*_L}{\theta^*_L-h^*_L}, & a < 0, \\ \frac{h-h^*_R}{\theta^*_R-h^*_R} \mod 1, & a > 0. \end{cases}$$ (5.4)

transforms the interval $[h^*_L, h^*_R]$ into $[0, 1)$ (the domain of the skew sawtooth map (1.5)). We use this particular $a$-dependent transformation because it leads to convenient $a$-independent formulas for $w$ and $\Delta k$.

**Theorem 5.2.** Suppose $\det(J) \neq 0$ and $\rho_{\text{max}} < 1$. Let $\Delta \ell \geq 0$ and suppose $\Sigma^+_{k,\Delta \ell}$ is well-defined for arbitrarily large values of $k$ with $(\kappa^+_\Delta \ell - \kappa^+_\Delta \ell-1) a > 0$. Let $\theta_{\text{min}} = \min \{ \theta^+_\Delta \ell, \theta^+_\Delta \ell-1 \}$ and $\theta_{\text{max}} = \max \{ \theta^+_\Delta \ell, \theta^+_\Delta \ell-1 \}$. Then at any point in $\Sigma^+_{k,\Delta \ell}$ there exists $\Xi \subset \Phi$ with $\max(\Xi) = O\left( \frac{1}{k^2} \right)$ such that throughout $\Phi \setminus \Xi$.

$$F = v \circ \varphi^{-1} \circ g \circ \varphi \circ u + O\left( \frac{1}{k^2} \right),$$ (5.5)

where $g$ is given by (1.5) with

$$a_L = \begin{cases} \frac{\tan(\theta)}{\tan(\theta_{\text{min}}^+)}, & \kappa^+_\Delta \ell > 0, \kappa^+_\Delta \ell-1 > 0, \\ -\frac{\tan(\theta)}{\tan(\theta_{\text{min}}^+)}, & \kappa^+_\Delta \ell \kappa^+_\Delta \ell-1 < 0, \end{cases}, \quad a_R = \frac{\tan(\theta)}{\tan(\theta_{\text{max}}^+)} = \frac{\tan(\theta_{\text{max}}^+)}{\tan(\theta_{\text{min}}^+)}, \quad w = k^2 \delta.$$ (5.6)

Moreover, for any $x \in \Phi \setminus \Xi$,

$$F(x) = \begin{cases} f^{\theta^*}|_{k+\Delta k,\Delta \ell'}(x), & 0 \leq z \leq z_w, \\ f^{\theta}|_{k+\Delta k,\Delta \ell}(x), & z_w < z < 1, \end{cases}$$ (5.7)

where $\Delta k = g_\text{in}(z) - g(z)$ and $z = \varphi(u(x))$.

Theorem 5.2 indicates explicitly how $g$ (1.5) approximates the dynamics of $f$ (1.2). Specifically, $g$ can be used to approximate the return map $F$ via the formula (5.5), where the parameters of $g$ are given by (5.6) and the relationship of $F$ to $f$ is given by (5.7). For $x \in \Xi$, either $u(x) \notin \Omega_{\Delta \ell}$, in which case (5.5) is not well-defined, or $F(x)$ corresponds to a different sequence of iterates of $f$ than that indicated by (5.7). These discrepancies have not been seen to cause qualitatively different dynamics for $f$ and $g$ because these are continuous maps and $\Xi$ constitutes a vanishingly small fraction of $\Phi$ as $k \to \infty$.

6. Discussion

This paper continues the work of [1] to rigorously characterise the dynamics of (1.2) near an $\mathcal{S}$-shrinking point (where $\mathcal{S} = F[\ell, m, n]$). Relative to the distance from an $\mathcal{S}$-shrinking point, the $G_k^\pm$-mode-locking regions are the largest. This is because, symbolically, they belong to
the first level of complexity relative to $S$. The $G^k_i^\pm$-mode-locking regions, and their shrinking points, were analysed in [1]. The results of the present paper help explain the dynamics that occurs between the $G^k_i^\pm$-mode-locking regions: higher period solutions, quasiperiodic dynamics, and chaos.

We have shown that such dynamics is captured by a skew sawtooth map (1.5) with three parameters: the slopes $a_L$ and $a_R$, and the vertical displacement $w$. To relate (1.2) to (1.5) quantitatively, we defined an array of sectors $\Sigma_{\pm}^{k,\Delta}$ near the $S$-shrinking point. Within each $\Sigma_{\pm}^{k,\Delta}$ the dynamics of (1.2) is well-approximated by that of (1.5) with fixed formulas for $a_L$, $a_R$ and $w$ in terms of $(\delta, \theta)$-coordinates. We identified a region $\Phi$ to which forward orbits regularly return, and considered the first return map $F : \Phi \to \Phi$ that equals the composition of (1.2) with itself many times (proportional to $kn$). Theorem 5.2 indicates precisely how (1.5) can be used to approximate $F$.

The results of section 4.4 describe the nature of the attracting invariant set $\Lambda_{\Delta}$ on which the dynamics is well approximated by (1.5). By Lemma 4.4, $\Lambda_{\Delta}$ is homotopic to a loop with a cylindrical topology on $\Phi$. This tells us that the invariant set $\cup_{i \geq 0} f^i(\Lambda_{\Delta})$ of (1.2) is either an invariant loop or a complicated geometric object that is homotopic to an invariant loop in $\mathbb{R}^2$.

For the map (1.5), the value of $v$ varies linearly from zero to one as we move from the outer boundary of $\Sigma_{\pm}^{k,\Delta}$ to the inner boundary of $\Sigma_{\pm}^{k,\Delta}$, and so is given by $v = k^2 \theta$. Lemma 5.1 essentially tells us that the slopes $a_L$ and $a_R$ are proportional to $\lambda$, where $\lambda$ is the critical eigenvalue of $M_S$ (that is, $\lambda = 1$ at the $S$-shrinking point). By equations (7.43)-(7.44) of [1], in $\Sigma_{\pm}^{k,\Delta}$ we have $\lambda = \tan(\theta) + O\left(\frac{1}{2}\right)$, whereas in $\Sigma_{\pm}^{k,\Delta}$ we have $\lambda = \frac{1}{\tan(\theta)} + O\left(\frac{1}{2}\right)$. The slopes $a_L$ and $a_R$ are also such that $a_R = 1$ on one of the two linear boundaries of $\Sigma_{\pm}^{k,\Delta}$, and either $a_L = 1$ or $a_L = -1$ on the other linear boundary of $\Sigma_{\pm}^{k,\Delta}$, see table 1.

The dynamics in $\Sigma_{\pm}^{k,\Delta}$ corresponds to a two-dimensional cross-section of the three-dimensional parameter space of (1.5) as characterised by the ratio $\frac{a_L}{a_R}$ (which is constant throughout $\Sigma_{\pm}^{k,\Delta}$). Qualitatively, there are two distinct scenarios as determined by the sign of $\frac{a_L}{a_R}$. If $\frac{a_L}{a_R} > 0$, then (1.5) is a homeomorphism and has a unique rotation number. Periodic dynamics occurs within mode-locking regions that exhibit shrinking points and non-periodic dynamics is quasiperiodic. The value of $\frac{a_L}{a_R}$ dictates the overall fatness of the mode-locking regions, see figure 5. This explains the presence of the roughly horizontal strip in the left half of figure 2 where mode-locking regions are relatively sparse. Here $G^{+}[k,0]$ and $G^{-}[k,1]$-shrinking points are relatively close together, hence $\theta^-_1 - \theta^-_0$ is relatively small. For sectors between these two sequences of shrinking points, we have $\theta^-_{\min} = \theta^-_0$ and $\theta^-_{\max} = \theta^-_1$, thus the value of $\frac{a_L}{a_R} = \frac{\tan(\theta^-_{\min})}{\tan(\theta^-_{\max})}$ is relatively small and so the mode-locking regions are relatively narrow. It remains to quantify this observation by calculating, say, the fraction of the parameter space of (1.5), with $\frac{a_L}{a_R} > 0$ fixed, for which (1.5) has a rational rotation number.

If $\frac{a_L}{a_R} < 0$, then (1.5) is non-invertible. There may be coexisting attractors and chaotic dynamics [18]. Mode-locking regions do not have shrinking points but rather terminate due to a loss of stability by attaining a stability multiplier of $-1$. It remains to further understand these dynamics to explain how curves connecting shrinking points can form a boundary for chaotic dynamics, as identified numerically in [26].

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Appendix A. Additional proofs

**Proof of lemma 3.5.** For brevity we derive (3.23) and (3.24) only for $\Delta \ell \geq 1$. The result for $\Delta \ell = 0$ can be obtained in a similar fashion (and is a little simpler).

By (3.15) and (3.18),
\[ h_{\Delta \ell}^1 = s_{\Delta \ell}^{\text{step}} + h_{\Delta \ell}^R. \] (A.1)

Then (3.23) is obtained by combining (3.24), $\lambda = 1 + O(\eta, \nu)$, and equation (A.9) of [1]:
\[ s_{\Delta \ell}^{\text{step}} = \frac{a t_{\ell - d} - d}{c t_{(\ell - 1)d}} \nu + O((\eta, \nu)^2). \] (A.2)

It therefore just remains for us to derive (3.24).

We begin by manipulating the numerator of (3.20), where $\{\psi^{S^\infty}_{\ell}\}$ denotes the unique $S^\infty$ cycle:
\[ e^T f^\chi(y^\chi, x)^{\Delta \ell} \left( x_{\text{int}}^{\ell - d} \right) = e^T f(x) \left( x_{\text{int}}^{\ell - d} \right) \]
\[ = e^T f(x^{S^\infty})\chi \left( x_{\text{int}}^{\ell - d} \right) \]
\[ = e^T f(x^{S^\infty})\chi \left( x^{\text{int}}_0 \right) + e^T M_x \left( x^{\text{int}}_0 - x^{S^\infty}_0 \right) \]
\[ = e^T f(x^{S^\infty}) \left( x^{\text{int}}_0 \right) + e^T M_x \left( x^{\text{int}}_0 - x^{S^\infty}_0 \right) \]
\[ = s^{S^\infty}_d + e^T M_x M^{\Delta \ell}_{S^\infty} \left( s^{\text{int}}_{\ell - d} - x^{S^\infty}_0 \right). \] (A.3)

In a neighbourhood of the $S$-shrinking point, one component of its corresponding mode-locking region is where $\eta \geq 0$ and $\nu \geq 0$, while the other component is where $\eta \leq \psi_1(\nu)$ and $\nu \leq \psi_2(\eta)$, for some $C^\infty$ functions $\psi_1$ and $\psi_2$ with $\psi_1(0) = \psi_2(0) = \psi_2'(0) = 0$, see theorem 6.9 of [1].

First suppose $\eta = \psi_1(\nu)$. Then $s^{S^\infty}_d = 0$ and so by (A.3) the numerator of (3.20) is
\[ e^T f(x^{\chi, y^{\chi}})^{\Delta \ell} \left( x^{\text{int}}_{\ell - d} \right) = e^T M_x \left( x^{\text{int}}_{\ell - d} - x^{S^\infty}_0 \right). \] (A.4)

By (3.22),
\[ e^T M_x M^{\Delta \ell}_{S^\infty} \left|_{(\eta, \nu) = (0, 0)} \right. = \frac{c t_{(\ell + 1)d}}{b t_d} u_0^T M^{\Delta \ell}_{S^\infty} \left( I - M_{S^\infty} \right) \left|_{(\eta, \nu) = (0, 0)} \right.. \] (A.5)

Since $x^{S^\infty}_0$ is a fixed point of $f^{S^\infty}$, we have $x^{S^\infty}_0 = (I - M_{S^\infty})^{-1} P_{S^\infty} B \mu$, from which we obtain
\[ (I - M_{S^\infty}) \left( x^{\text{int}}_{\ell - d} - x^{S^\infty}_0 \right) = (I - M_{S^\infty}) x^{\text{int}}_{\ell - d} - P_{S^\infty} B \mu \]
\[ = x^{\text{int}}_{\ell - d} - \left( M_{S^\infty} x^{\text{int}}_{\ell - d} + P_{S^\infty} B \mu \right) \]
\[ = x^{\text{int}}_{\ell - d} - f^{S^\infty} \left( x^{\text{int}}_{\ell - d} \right) \]
\[ = -s_{\ell - d}^{\text{step}}. \] (A.6)
using also (3.15) in the last line. By combining (2.13), (A.2), (A.4)–(A.6) and the formula for \( \kappa + \Delta \ell \) given in appendix B, we obtain

\[
e^T f y^X (x^\text{int}) = \kappa + \Delta \ell \nu + O(\nu^2),
\]

under the assumption \( \eta = \psi_1(\nu) \).

Now suppose \( \nu = \psi_2(\eta) \). Then \( S_0 = 0 \) and the \( S \)-cycle is unique with \( x^i = x^S_i \) for each \( i \). Thus by (3.10),

\[
x^\text{int} = (I - \zeta_d e^T) x^0 = x^S.
\]

The numerator of (3.20) is then

\[
e^T f (y^X (x^\text{int}) = e^T f y^X (x^0) \Delta \ell (x^\text{int} - d) = e^T f y^X (x^0) \Delta \ell (x^S - d) = e^T f y^X (x^S) \Delta \ell (x^S - d) = \frac{t(\ell - 1)d}{\ell - d} \eta + O(\eta^2).\]

(3.21)

Then (3.24) is given by (3.21) divided by (3.21).

\( \square \)

**Proof of lemma 4.2.** For brevity we prove the result only for \( G^+[k, \Delta \ell] \). The result for \( G^+[k, \Delta \ell] \) can be proved in the same fashion.

Step 1. We first use lemma 3.4 to show that

\[
e^T M (y^X (x^\text{int}) \zeta - d) < 0,
\]

throughout \( \Sigma^+_{k, \Delta \ell} \).

At the \( S \)-shrinking point the \( S \)-cycle, denoted \( \{y_i\} \), is assumed to be admissible with only \( y_0 \) and \( y_{\ell d} \) on the switching manifold. Therefore \( t(\ell - 1)d < 0 \) and \( t - d > 0 \). Since \( r_{\text{max}} < 1 \), we have \( c > 0 \). Also \( (k^+ - k^+_{\Delta \ell - 1}) a > 0 \) by assumption. By applying these inequalities to (3.21) we obtain (3.21).

Step 2. Next we show that \( h_d^{k, \Delta \ell} h_d^{k, \Delta \ell} < h_d^{k, \Delta \ell} \).

From now on we only consider parameter values in \( \Sigma^+_{k, \Delta \ell} \). Then \( \text{sgn}(\nu) = \text{sgn}(a) \), see section 2.3. Also \( \eta \) and \( \nu \) are \( O(\frac{1}{k}) \). Thus, by (A.2), \( s^\text{step}_d < 0 \) for sufficiently large values of \( k \).

By (3.13), (3.18), and (3.15),
\[ h^R_{\Delta t} = s^{\text{step}}_{-d} + h^R_{\Delta t} \lambda. \] (A.12)

Since \( s^{\text{step}}_{-d} = O \left( \frac{1}{\lambda} \right) \), \( \lambda = 1 + O \left( \frac{1}{\lambda} \right) \), and \( h^R_{\Delta t} \) and \( h^R_{\Delta t} \) are \( O \left( \frac{1}{\lambda} \right) \) by (3.23)–(3.24), we have

\[ h^R_{\Delta t} - h^R_{\Delta t} = -s^{\text{step}}_{-d} + O \left( \frac{1}{\lambda} \right). \] (A.13)

Therefore \( h^R_{\Delta t} < h^R_{\Delta t} \) (throughout \( \Sigma_{\Delta t}^{+} \)) for sufficiently large values of \( k \).

**Step 3.** We have thus shown that \( \Omega_{\Delta t} \) is well-defined by (3.19). Moreover, \( \Omega_{\Delta t} \) is an \( O \left( \frac{1}{\lambda} \right) \) distance from the switching manifold, and so is \( \Phi \). In this step we show that \( f^{G^{+,\Delta t}} (\Phi) \) is also an \( O \left( \frac{1}{\lambda} \right) \) distance from the switching manifold.

On the outer boundary of \( \Sigma^{+}_{\Delta t} \), the matrix \( P_{G^{+,\Delta t} (0)} \) is singular for either \( i = 0 \) or \( i = (\ell_{k} + \Delta t - 1) h_{k} \). For brevity we just treat the case \( i = 0 \). Then by (3.25), on the outer boundary of \( \Sigma^{+}_{\Delta t} \), the point \( f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) \) is an \( O \left( \rho_{\text{max}} \right) \) distance from the switching manifold.

In order to describe \( f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) \) for parameter values on the inner boundary of \( \Sigma^{+}_{\Delta t} \), we first note that on this boundary \( P_{G^{+,\Delta t} (\chi)} \) is singular. Thus by (3.25)

\[ f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) = x^{\text{int}}_{-d} + O \left( \rho_{\text{max}}^{k} \right). \] (A.14)

By (4.4), \( f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) \) is the unique inverse of \( f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) \) under \( f^{S_{(-i)}} \) that belongs to the range of \( f^{S_{(-i)}} \). Thus by (3.15)

\[ f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) = x^{\text{int}}_{-d} - \frac{s^{\text{step}}_{-d}}{\lambda} \zeta_{-d} + O \left( \rho_{\text{max}}^{k} \right), \] (A.15)

on the inner boundary of \( \Sigma^{+}_{\Delta t} \). Therefore \( f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) \) is an \( O \left( \frac{1}{\lambda} \right) \) distance from the switching manifold on the inner boundary of \( \Sigma^{+}_{\Delta t} \). It follows that \( f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) \) is an \( O \left( \frac{1}{\lambda} \right) \) distance from the switching manifold throughout \( \Sigma^{+}_{\Delta t} \).

Now we show that for any \( x \in \Phi \) we have

\[ f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) (x) - f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) = O \left( \frac{1}{\lambda} \right), \] (A.16)

which will complete our demonstration that \( f^{G^{+,\Delta t}} (\chi^{\text{int}}_{-d}) (\Phi) \) is an \( O \left( \frac{1}{\lambda} \right) \) distance from the switching manifold. In view of (3.4), since \( x - x^{\text{int}}_{-d} = O \left( \frac{1}{\lambda} \right) \) we have

\[ f^{S_{(-i)}} (x^{\text{int}}_{-d}) - f^{S_{(-i)}} (x^{\text{int}}_{-d}) = O \left( \frac{1}{\lambda} \right) \] because \( f^{S_{(-i)}} (x^{\text{int}}_{-d}) \) is independent of \( k \). Since \( \rho_{\text{max}} < 1 \), all eigenvalues of the matrix part of \( f^{S_{(-i)}} \) have modulus less than 1 except possibly \( \lambda \). But \( \lambda = O (1) \) because \( \lambda = 1 + O(\eta, \nu) \) and \( \eta \) and \( \nu \) are \( O \left( \frac{1}{\lambda} \right) \). Therefore \( f^{S_{(-i)}} \) is not expanding by more than an \( O(1) \) factor, which verifies (A.16) by (3.4).

**Step 4.** Next we show that \( \beta(y) \) is well-defined with \( y = f^{G^{+,\Delta t}} (\chi) \) and any \( x \in \Phi \).

For any \( y = v(h) \in W^{c} \), by (3.15) we have

\[ e^{T} f^{S_{(-i)}} (y) = e^{T} f^{S_{(-i)}} (y) \]

\[ = e^{T} M f^{S_{(-i)}} (y) \]

\[ = (s^{\text{step}}_{-d} + (\lambda - 1) h_{k}) e^{T} M f^{S_{(-i)}} (y) \zeta_{-d}. \] (A.17)
Since $s_{\text{step}} < 0$, by (A.11) we have that (A.17) is negative for sufficiently small values of $h$. Moreover, since $s_{\text{step}}$ and $\lambda - 1$ are $O(\frac{1}{\lambda})$, (A.17) is negative if $h = O(\frac{1}{\lambda})$, and this is also true for points $y$ sufficiently close to $W^c$.

For any $x \in \Phi$, $f^{\gamma^*}[k, \Delta \ell](x)$ is an $O\left(\frac{1}{\lambda}\right)$ distance from the switching manifold and an $O\left(\rho_{\text{max}}^k\right)$ distance from $W^c$, and the same is true for $f^{\gamma^*}[k + \Delta k, \Delta \ell](x)$ with $\Delta k = O(1)$. This shows that (A.17) is negative using $y = f^{\gamma^*}[k + \Delta k, \Delta \ell](x)$. Therefore the left hand side of (4.5) is an increasing function of $\Delta k$. Thus $\beta(y)$ is well-defined.

**Step 5.** Now we show that if $\Delta k = \beta\left(f^{\gamma^*}[k, \Delta \ell](x)\right)$ for some $x \in \Phi$, then $f^{\gamma^*}[k + \Delta k, \Delta \ell](x) \in \Phi$.

Let $\omega_{\Delta \ell}^d$ denote the orthogonal complement of $\omega_{-d}$. We define a function $\psi : \omega_{-d}^d \to \mathbb{R}$ by

$$\psi(q) := h_{\Delta \ell}^R - \frac{e_1^T M_{\mathcal{X}}(\gamma^* X)^{\Delta \ell} q}{e_1^T M_{\mathcal{X}}(\gamma^* X)^{\Delta \ell} \zeta_{-d}}. \tag{A.18}$$

Given any $q \in \omega_{-d}^d$, let $x = x^{\text{int}_{-d}} + \psi(q) \zeta_{-d} + q$. It is straightforward to show that $e_1^T f^{\gamma^*}(\gamma^* X)^{\Delta \ell}(x) = 0$. Thus if $q$ is sufficiently small and $x \in \text{range}(f^{\gamma^*(-d)})$, then $x \in H_{k, \Delta \ell, \ell, Q}$.

Choose any $x \in \Phi$ and let $\Delta k = \beta\left(f^{\gamma^*}[k, \Delta \ell](x)\right)$. By the definition of $\beta$ we have

$$e_1^T f^{\gamma^*}(\gamma^* X)^{\Delta \ell}\left(f^{\gamma^*}[k + \Delta k, \Delta \ell](x)\right) > 0, \tag{A.19}$$

and

$$e_1^T f^{\gamma^*}(\gamma^* X)^{\Delta \ell}\left(f^{\gamma^*}[k + \Delta k - 1, \Delta \ell](x)\right) \leq 0. \tag{A.20}$$

Write $f^{\gamma^*}[k + \Delta k, \Delta \ell](x) = x^{\text{int}}_{-d} + h \zeta_{-d} + q$. Let $h_1 = \psi(q)$ and $x_1 = x^{\text{int}}_{-d} + h_1 \zeta_{-d} + q$. Then $x_1 \in H_{k, \Delta \ell, \ell, Q}$, assuming $Q$ is sufficiently large. Moreover,

$$e_1^T f^{\gamma^*}(\gamma^* X)^{\Delta \ell}\left(f^{\gamma^*}[k + \Delta k, \Delta \ell](x)\right) = (h - h_1)e_1^T M_{\mathcal{X}}(\gamma^* X)^{\Delta \ell} \zeta_{-d},$$

thus $h < h_1$ by (A.11) and (A.19).

Similarly write $f^{\gamma^*}[k + \Delta k - 1, \Delta \ell](x) = x^{\text{int}}_{-d} + h_1 \zeta_{-d} + \tilde{q}$. Let $\tilde{h}_1 = \psi(\tilde{q})$ and $\tilde{x}_1 = x^{\text{int}}_{-d} + h_1 \zeta_{-d} + \tilde{q}$. Then $\tilde{x}_1 \in H_{k, \Delta \ell, \ell, Q}$, assuming $Q$ is sufficiently large. Let $h_2 = s_{\text{step}} + h_1 \lambda$ and $x_2 = x^{\text{int}}_{-d} + h_2 \zeta_{-d} + q$. Then $x_2 \in f^{\gamma^*(-d)}(H_{k, \Delta \ell, \ell, Q})$ because $x_2 = f^{\gamma^*(-d)}(\tilde{x}_1)$. Moreover,

$$e_1^T f^{\gamma^*}(\gamma^* X)^{\Delta \ell}\left(f^{\gamma^*}[k + \Delta k, \Delta \ell](x)\right) = (h - h_2)e_1^T M_{\mathcal{X}}(\gamma^* X)^{\Delta \ell} \zeta_{-d},$$

thus $h > h_2$ by (A.11) and (A.20).

Let $\alpha = \frac{h_{\Delta \ell}}{h_2}$. Then $f^{\gamma^*}[k + \Delta k, \Delta \ell](x) = \alpha x_1 + (1 - \alpha)x_2$ and $0 \leq \alpha < 1$, hence $f^{\gamma^*}[k + \Delta k, \Delta \ell](x) \in \Phi$.

**Step 6.** Choose any $x \in \Phi$ and suppose $f^{\gamma^*}[k + \Delta k, \Delta \ell](x) \in \Phi$ for some $\Delta k = O(1)$. To complete the proof it remains to show that $\Delta k = \beta\left(f^{\gamma^*}[k, \Delta \ell](x)\right)$.

Write $f^{\gamma^*}[k + \Delta k, \Delta \ell](x) = \alpha x_1 + (1 - \alpha)x_2$ where $x_1 \in H_{k, \Delta \ell, \ell, Q}$, $x_2 \in f^{\gamma^*(-d)}(H_{k, \Delta \ell, \ell, Q})$ and $0 \leq \alpha < 1$. Since $x_1 = x^{\text{int}}_{-d} + h^R_{\Delta \ell} \zeta_{-d} + O\left(\rho_{\text{max}}^k\right)$ and $x_2 = x^{\text{int}}_{-d} + h^L_{\Delta \ell} \zeta_{-d} + O\left(\rho_{\text{max}}^k\right)$, we have $x_1 - x_2 = (h^R_{\Delta \ell} - h^L_{\Delta \ell}) \zeta_{-d} + O\left(\rho_{\text{max}}^k\right)$. Thus by (A.11),

$$\text{dist}(x, \Phi) = \Delta k.$$

Thus $\Delta k = \beta\left(f^{\gamma^*}[k, \Delta \ell](x)\right)$. \qed
\(e_1^T M_{X} \sigma(\beta \alpha) \Delta (x_1 - x_2) < 0.\) (A.21)

By writing \(f^{\Delta + \Delta}(x) = x_1 - (1 - \alpha)x_1 - x_2,\) we obtain
\[e_1^T M_{X} \sigma(\beta \alpha) \Delta (f^{\Delta + \Delta}(x_1) = -(1 - \alpha) e_1^T M_{X} \sigma(\beta \alpha) \Delta (x_1 - x_2),\]
which is positive by (A.21). This verifies (A.19). Equation (A.20) can be verified in the same fashion which shows that \(\Delta = \beta \left(f^{\Delta + \Delta}(x)\right)\) by the definition of \(\beta.\)

**Proof of lemma 5.1.** By (2.3) and (3.13),
\[u \left(f^{\Delta + \Delta}(\varphi(h)) = \omega T \left(M_{G}^{\Delta + \Delta} \left(x_{\alpha}^1 + h \zeta_d - \zeta_d\right) + P_{\Delta + \Delta}(B - x_{\alpha}^1)\right),\]
thus
\[
\frac{d}{dh} u \left(f^{\Delta + \Delta}(\varphi(h)) = \omega T M_{G}^{\Delta + \Delta} \zeta_d.\right.\]
(A.23)

By (3.4) and (3.16),
\[M_{G}^{\Delta + \Delta} = M_{G}^{\Delta + \Delta} + \omega T M_{X} M_{S}^{\Delta + \Delta} + O \left(\rho_{\beta}^k\right).\]
(A.24)

Thus using \(M_{G}^{\Delta + \Delta} = \lambda \zeta_d - \lambda \zeta_d\) and \(\omega T \zeta_d - \zeta_d = 1\) we obtain
\[
\frac{d}{dh} u \left(f^{\Delta + \Delta}(\varphi(h)) = \lambda^{\Delta + \Delta} \omega T M_{X} M_{S}^{\Delta + \Delta} \zeta_d + O \left(\rho_{\beta}^k\right).\right.\]
(A.25)

By lemma 7.1 of [1],
\[
\frac{d}{dh} u \left(f^{\Delta + \Delta}(\varphi(h)) = \lambda^{\Delta + \Delta} u_{0}^T M_{S}^{\Delta + \Delta} M_{X} v_0 \right|_{(\eta, \alpha) = (0, 0)} + O \left(\frac{1}{\beta}\right),\]
where \(u_{0}^T\) and \(v_0\) are the left and right eigenvectors of \(M_S\) with \(u_{0}^T v_0 = 1\) and \(e_1^T v_0 = 1.\)

In \(\Sigma_{\Delta + \Delta}\) we have \(\lambda^{\Delta + \Delta} = \lambda^{\Delta} + O \left(\frac{1}{\beta}\right) = -\tan(\theta) + O \left(\frac{1}{\beta}\right)\) (see equation (A.24) of [1]). Then by the identity \(\Delta = \frac{-u_{0}^T M_{X} v_0}{e_1^T v_0}\) (lemma 6.6 of [1]) and the formula for \(\kappa_{\Delta + \Delta}\) (B.1), equation (A.26) reduces to (5.2). Equation (5.3) can be derived in a similar fashion. □

**Proof of theorem 5.2.** For brevity we just prove the result in the case \(\alpha < 0.\)

**Step 1.** Define \(g_{\text{lift.approx}} : [0, 1] \to \mathbb{R}\) by
\[
g_{\text{lift.approx}} := \left\{\begin{array}{cl}
\varphi \circ u \circ f^{\Delta + \Delta} \circ \varphi^{-1}, & \text{on } [0, 1] \\
\varphi \circ u \circ f^{\Delta + \Delta} \circ \varphi^{-1}, & \text{on } [0, 1].
\end{array}\right.
\]
(A.27)

We first show that
\[
g_{\text{lift.approx}} = g_{\text{lift}} + O \left(\frac{1}{\beta}\right).
\]
(A.28)
Both \( g_{\text{lift,approx}} \) and \( g_{\text{lift}} \) are real-valued, piecewise-linear continuous functions on \([0, 1]\) that are comprised of two pieces. The slopes of \( g_{\text{lift,approx}} \) are given by (5.2) and (5.3). The slopes of \( g_{\text{lift}} \) are \( a_L \) and \( a_R \) (5.6). To show that the leading order components of \( g_{\text{lift,approx}} \) and \( g_{\text{lift}} \) are the same, observe that \( \kappa^+_{\Delta t} < \theta^+_{\Delta t-1} \) because \( a < 0 \) and \( (\kappa^+_{\Delta t} - \kappa^+_{\Delta t-1}) a > 0 \) by assumption. Thus \( \theta_{\text{min}} = \theta^+_{\Delta t} \) and \( \theta_{\text{max}} = \theta^+_{\Delta t-1} \) Then by (B.1), \( a_L \) and \( a_R \) match (5.2) and (5.3) to leading order. The kink of \( g_{\text{lift,approx}} \) is located at \( z = \frac{-h_{\Delta t}}{\kappa^+_{\Delta t} - \kappa^+_{\Delta t-1}} \). Using (3.23), (3.24), and \( \frac{\kappa^+_{\Delta t}}{\kappa^+_{\Delta t-1}} \tan(\theta) \) (2.19), we see that this matches \( z_{\text{sw}} = \frac{\delta_a - 1}{\delta_a - \delta_W} \) (the kink of \( h_{\text{lift}} \)) to leading order. To complete our demonstration of (A.28), we show that the values of \( g_{\text{lift,approx}} \) and \( g_{\text{lift}} \) differ by \( O \left( \frac{1}{k^2} \right) \) at their kink points.

The value of \( g_{\text{lift}} \) at its kink is

\[
g_{\text{lift}}(z_{\text{sw}}) = k^2 \delta + z_{\text{sw}}. \tag{A.29}
\]

To evaluate \( g_{\text{lift,approx}} \) at \( z = \frac{-h_{\Delta t}}{\kappa^+_{\Delta t} - \kappa^+_{\Delta t-1}} \), notice that

\[
V \left( \varphi^{-1} \left( \frac{-h_{\Delta t}}{h_R^+ - h_L^+} \right) \right) = V(0) = x_{-d}^\text{int}. \tag{A.30}
\]

At any point on the outer boundary of \( \Sigma^+_{\Delta t} \), where \( \delta = 0 \), we have \( \det \left( P_{G^+[k, \Delta t]} \right) = 0 \). This is because \( a < 0 \) and is a straightforward consequence of theorem 2.4 of [1]. Thus by lemma 3.6, \( f^{G^+[k, \Delta t]}(x_{\text{int}}^\text{d}) = x_{\text{int}}^\text{d} + O \left( \rho_{\text{max}}^k \right) \) when \( \delta = 0 \).

At any point on the inner boundary of \( \Sigma^+_{\Delta t} \), where \( \delta = \frac{1}{k} + O \left( \frac{1}{k^2} \right) \) (see section 2.4), we similarly have \( \det \left( P_{G^+[k+1, \Delta t]} \right) = 0 \). Thus by lemma 3.6, \( f^{G^+[k+1, \Delta t]}(x_{\text{int}}^\text{d}) = x_{\text{int}}^\text{d} + O \left( \rho_{\text{max}}^k \right) \). Since \( f^{G^+[k, \Delta t]} = f^{S^{(-d)}} \circ f^{G^+[k, \Delta t]} \), by (3.15) we have \( f^{G^+[k, \Delta t]}(x_{\text{int}}^\text{d}) = x_{\text{int}}^\text{d} - s_{\text{step}} \zeta_{-d} + O \left( \rho_{\text{max}}^k \right) \), where here \( \delta = \frac{1}{k} + O \left( \frac{1}{k^2} \right) \).

Linear interpolation of these two values of \( f^{G^+[k, \Delta t]}(x_{\text{int}}^\text{d}) \) gives

\[
f^{G^+[k, \Delta t]}(x_{\text{int}}^\text{d}) = x_{\text{int}}^\text{d} - k^2 \delta s_{\text{step}} \zeta_{-d} + O \left( \frac{1}{k^2} \right), \tag{A.31}
\]

where the error term can be justified from the smoothness of \( f^{G^+[k, \Delta t]} \). Then

\[
\varphi \left( u \left( f^{G^+[k, \Delta t]}(x_{\text{int}}^\text{d}) \right) \right) = \frac{-k^2 \delta s_{\text{step}} - h_R^+}{h_R^+ - h_L^+} + O \left( \frac{1}{k^2} \right), \tag{A.32}
\]

and so by (A.13) and (A.30) we obtain

\[
g_{\text{lift,approx}} \left( \frac{-h_{\Delta t}^+}{h_R^+ - h_L^+} \right) = k^2 \delta - \frac{h_R^+}{h_R^+ - h_L^+} + O \left( \frac{1}{k^2} \right),
\]

matching the value of \( g_{\text{lift}}(z_{\text{sw}}) \) (A.29) to leading order.

**Step 2.** Next we derive a formula that is used below to show that \( \Delta k = g_{\text{lift}}(z) - g(z) \).

By (3.15), \( u \left( f^{S^{(-d)}}(v(h)) \right) = s_{\text{step}} + h \lambda \). Using (A.13), since \( \lambda = 1 + O \left( \frac{1}{k} \right) \), for any \( O \left( \frac{1}{k} \right) \) value of \( h \) we can write

\[
u \left( f^{S^{(-d)}}(v(h)) \right) = h_R^+ - h_L^+ + h \lambda + O \left( \frac{1}{k^2} \right).
\]
Then
\[ \varphi \left( u \left( f^{S_{-1}}(v(h)) \right) \right) = \varphi(h) - 1 + O \left( \frac{1}{k} \right). \]

Repeating this result yields the desired formula
\[ \varphi(u(T(v(h)))) = \varphi(h) - \Delta k + O \left( \frac{1}{k} \right), \tag{A.33} \]
where \( T = \left( f^{S_{-n}} \right) \Delta k. \)

**Step 3.** Define
\[ g_{\text{approx}} := \varphi \circ G \circ \varphi^{-1}, \tag{A.34} \]
where \( G \) given by (5.1). Here we show that
\[ g_{\text{approx}}(z) = g(z) + O \left( \frac{1}{k} \right), \tag{A.35} \]
for all \( z \in [0, 1) \setminus \Xi_1 \), for some set \( \Xi_1 \subset [0, 1) \) with \( \text{meas}(\Xi_1) = O \left( \rho^k_{\max} \right). \)

By (4.7) and (5.1),
\[ (\Xi_1 \setminus \Xi) \in \mathbb{R}^d. \]

Then by (A.33),
\[ g_{\text{approx}}(z) = g_{\text{lift,approx}} - \Delta k + O \left( \frac{1}{k} \right), \tag{A.36} \]
and by (A.35),
\[ g_{\text{approx}} = g_{\text{lift}} - \Delta k + O \left( \frac{1}{k} \right). \tag{A.37} \]

But \( F \left( \varphi^{-1}(z) \right) \in \Phi \) for all \( z \in [0, 1) \). Thus \( u \left( F \left( \varphi^{-1}(z) \right) \right) \in [h^R_{\Delta t}, h^R_{\Delta t}] \) for all \( z \in [0, 1) \setminus \Xi_1 \), where \( \Xi_1 \subset [0, 1) \) contains points near the left and right faces of \( \Phi \) and \( \text{meas}(\Xi_1) = O \left( \rho^k_{\max} \right). \) Thus \( g_{\text{approx}}(z) \in [0, 1) \) for all \( z \in [0, 1) \setminus \Xi_1 \). Therefore \( g_{\text{approx}}(z) = g_{\text{lift}}(z) \mod 1 + O \left( \frac{1}{k} \right) \) for all \( z \in [0, 1) \setminus \Xi_1 \), which verifies (A.35).

**Step 4.** For any \( x \in \Phi \), we have \( x - v(u(x)) = O \left( \rho^k_{\max} \right). \) It is straightforward to show that similarly \( F(x) - F(v(u(x))) = O \left( \rho^k_{\max} \right) \) because iterates approach \( W^c \) under \( f^{S_{-n}} \) and \( \lambda^k = O(1). \) Since \( F(v(u(x))) \in \Phi \), we also have \( F(v(u(x))) = O \left( \rho^k_{\max} \right) \).

Thus \( F(x) - v(G(u(x))) = O \left( \rho^k_{\max} \right). \) Thus by (A.34) and (A.35),
\[ F = v \circ \varphi^{-1} \circ g \circ \varphi \circ u + O \left( \frac{1}{k} \right), \tag{A.39} \]
for all \( x \in \Phi \) with \( z \in [0, 1) \setminus \Xi_1 \) where \( z = \varphi(u(x)) \).
By (A.35) and (A.38), \( \Delta k = g_{\text{lift}}(z) - g(z) \) for all but a measure \( O \left( \frac{1}{k} \right) \) subset of \([0, 1)\) due to the error terms in these expressions. In view of (4.8), equation (5.7) holds unless \( z = \varphi(u(x)) \) belongs to an \( O \left( \frac{1}{k} \right) \) subset of \([0, 1)\) where (5.7) is not satisfied due to the \( O \left( \frac{1}{k} \right) \) difference in the kink points \( \frac{h^+}{\Delta k} \) and \( z_{\text{ow}} \).

Appendix B. Additional formulas

Suppose (1.2) is at an \( S \)-shrinking point where \( S = \mathcal{F}[^{\ell}m, n] \). For each \( j = 0, (\ell - 1)d, \ell d \) and \(-d \) (taken modulo \( n \)), let \( u^j_1 \) and \( v_1 \) be the left and right eigenvectors of \( M_{S^{\ell}m} \) corresponding to the eigenvalue 1 and normalised by \( u^j_1 v_1 = 1 \) and \( e^j_1 v_1 = 1 \). Then for all \( \Delta \ell \in \mathbb{Z} \), let

\[
\begin{align*}
\kappa^+_{\Delta \ell} &:= \begin{cases} u^j_1 M_{S^{\ell}m}^{-1} v_{(\ell - 1)d}, & \Delta \ell \leq -1, \\ u^j_0 M_{S^{\ell}m}^{-1} v_{-d}, & \Delta \ell \geq 0, \end{cases} \\
\kappa^-_{\Delta \ell} &:= \begin{cases} u^j_1 M_{S^{\ell}m}^{-1} v_{0}, & \Delta \ell \leq 0, \\ u^j_0 M_{S^{\ell}m}^{-1} v_{(\ell - 1)d}, & \Delta \ell \geq 1, \end{cases}
\end{align*}
\]  

and assuming \( \kappa^\pm_{\Delta \ell} \neq 0 \), let

\[
\begin{align*}
\theta^+_{\Delta \ell} &:= \begin{cases} \tan^{-1} \left( \frac{\ell_{(\ell - 1)d}}{\ell_{d}} \right), & \Delta \ell \leq -1, \\ \tan^{-1} \left( \frac{\ell_{d}}{\ell_{(\ell - 1)d}} \right), & \Delta \ell \geq 0, \end{cases} \\
\theta^-_{\Delta \ell} &:= \begin{cases} \tan^{-1} \left( \frac{\ell_{(\ell - 1)d} \kappa^-_{\Delta \ell}}{\ell_{d}} \right), & \Delta \ell \leq 0, \\ \tan^{-1} \left( \frac{\ell_{d} \kappa^+_{\Delta \ell}}{\ell_{(\ell - 1)d}} \right), & \Delta \ell \geq 1, \end{cases}
\end{align*}
\]  

where \( \theta^+_{\Delta \ell} \in \left( \frac{\pi}{2}, 2\pi \right) \) and \( \theta^-_{\Delta \ell} \in \left( \frac{\pi}{2}, -\pi \right) \) if \( a < 0 \), and \( \theta^+_{\Delta \ell} \in \left( \frac{\pi}{2}, \pi \right) \) and \( \theta^-_{\Delta \ell} \in \left( \frac{3\pi}{2}, 2\pi \right) \) if \( a > 0 \).

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