Abstract

We construct semiclassical solutions of the symplectically covariant Schrödinger phase-space equation rigorously studied in a previous paper; we use for this purpose an adaptation of Littlejohn’s nearby-orbit method. We take the opportunity to discuss in some detail the so fruitful notion of squeezed coherent state and the action of the metaplectic group on these states.

1 Introduction

Let $\psi$ be a square-integrable solution of Schrödinger’s equation

$$i\hbar \frac{\partial \psi}{\partial t} = H(x, -i\hbar \partial_x)\psi$$

and set $\Psi = U_\phi \psi$, where the wave-packet transform $U_\phi$ is defined as follows [8]:

$$U_\phi \psi(x, p) = \left( \frac{1}{2\pi\hbar} \right)^{n/2} e^{\frac{1}{\hbar} p \cdot x} \int e^{-\frac{1}{\hbar} p \cdot x'} \psi(x') \phi(x - x') dx'$$

here $\phi$ is an arbitrary rapidly decreasing function. The function $\Psi$ satisfies the equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \tilde{H}_{ph} \Psi$$
where the operator $\hat{H}_{\text{ph}}$ is defined by the intertwining formula

$$U_\phi \hat{H}_{\text{ph}} = H(x, -i \hbar \partial_x) U_\phi;$$

A straightforward calculation shows that we have

$$U_\phi(x\psi) = \left(\frac{1}{2} x + i \hbar \partial_p\right) U_\phi \psi, \quad U_\phi \left(-i \hbar \partial_x \psi\right) = \left(\frac{1}{2} p - i \hbar \partial_x\right) U_\phi \psi; \quad (3)$$

these relations motivate the notation

$$\hat{H}_{\text{ph}} = H\left(\frac{1}{2} x + i \hbar \partial_p, \frac{1}{2} p_j - i \hbar \partial_x\right)$$

and we may thus rewrite (2) as the phase-space Schrödinger equation

$$i \hbar \frac{\partial}{\partial t} \Psi = H\left(\frac{1}{2} x + i \hbar \partial_p, \frac{1}{2} p_j - i \hbar \partial_x\right) \Psi. \quad (4)$$

In a recent paper [8] (also see [7]) one of us exposed in some detail the properties of the solutions of this equation, originally proposed by Torres-Vega and Frederick [21, 22] in the particular case where $\phi$ is a Gaussian.

In this paper we pursue the study of the properties of the solutions of this equation, and, in particular we show how to construct in an easy way approximate solutions to that equation for a large class of initial wave functions. Our approach is an adaptation to phase space of Littlejohn’s [17] near orbit method (we remark that Heller [13] has considered a more restrictive notion).

**Notation**

The position vector will be denoted by $x = (x_1, \ldots, x_n)$ and the momentum vector by $p = (p_1, \ldots, p_n)$, and we write $z = (x, p)$ for the generic phase space variable. We will use the generalized gradients

$$\partial_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \quad \partial_p = \left(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\right)$$

and $\partial_z = (\partial_x, \partial_p)$.

The symplectic product of $z = (x, p)$, $z' = (x', p')$ is denoted by $\sigma(z, z')$; by definition:

$$\sigma(z, z') = p \cdot x' - p' \cdot x$$

where the dot $\cdot$ is the usual (Euclidean) scalar product. In matrix notation:

$$\sigma(z, z') = (z')^T J z, \quad J = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{pmatrix}.$$

The corresponding symplectic group is denoted by $\text{Sp}(n)$: the relation $S \in \text{Sp}(n)$ means that $S$ is a real $2n \times 2n$ matrix such that $\sigma(Sz, Sz') = \sigma(z, z')$; equivalently $S^T J S = SJS^T = J$.

For $z_0 = (x_0, p_0)$ the Heisenberg–Weyl operator $\hat{T}(z_0)$ acts on functions of $x$ via

$$\hat{T}(z_0) \psi(x) = e^{i \frac{x}{\hbar} p_0 \cdot x - \frac{1}{2} \hbar \partial_x} \psi(x - x_0). \quad (5)$$
The Wigner–Moyal transform of a pair \((\psi, \phi)\) of functions in the Schwartz space \(S(\mathbb{R}^n)\) is defined by
\[
W(\psi, \phi)(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{-\frac{\hbar}{2} p \cdot y} \psi(x + \frac{1}{2} y) \overline{\phi(x - \frac{1}{2} y)} d^n y. \tag{6}
\]

2 Coherent States in Phase Space

In this Section we review known results about coherent states; the action of the metaplectic group on these states will be studied in the next section.

We denote by \(\phi^\hbar\) the standard coherent state:
\[
\phi^\hbar(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4} e^{-\frac{\hbar}{2} x^2}; \tag{7}
\]
for \(z_0 = (x_0, p_0)\) we define more generally the coherent state \(\phi^\hbar_{z_0} = \hat{T}(z_0)\phi^\hbar\) that is
\[
\phi^\hbar_{z_0}(x) = e^{-\frac{\hbar}{2} p_0 \cdot x} \phi^\hbar(x - x_0). \tag{8}
\]
(These states are often denoted respectively by \(|0\rangle\) and \(|z_0\rangle\) in bra-ket notation \[17\]). The following properties are well-known (see for instance. \[15, 20\]; for the sake of self-containedness we however give a proof.

**Proposition 1**

(i) We have
\[
\left(\frac{1}{2\pi\hbar}\right)^n \int \phi^\hbar_{z_0}(x) \overline{\phi^\hbar_{z_0}(y)} d^{2n} z_0 = \delta(x - y). \tag{9}
\]

(ii) For every \(\psi \in L^2(\mathbb{R}^n)\)
\[
\psi(x) = \left(\frac{1}{2\pi\hbar}\right)^n \int (\psi, \phi^\hbar_{z_0})_{L^2} \phi^\hbar_{z_0}(x) d^{2n} z_0 \tag{10}
\]
and
\[
||\psi||_{L^2}^2 = \left(\frac{1}{2\pi\hbar}\right)^n \int |(\psi, \phi^\hbar_{z_0})_{L^2}|^2 d^{2n} z_0. \tag{11}
\]

(iii) Let \(\hat{A}\) be a continuous linear operator \(S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)\) with Schwartz kernel \(K\); we have
\[
K(x, y) = \left(\frac{1}{2\pi\hbar}\right)^n \int \hat{A}\phi^\hbar_{z_0}(x) \overline{\phi^\hbar_{z_0}(y)} d^{2n} z_0. \tag{12}
\]

2.1 Squeezed coherent states

Despite the interest of the standard coherent states it is often advantageous to work with their generalization, called “squeezed coherent states” (they anyway appear automatically when one lets the metaplectic group act on coherent states, see Subsection 3.2). A squeezed coherent state is a Gaussian of the type
\[
\phi^\hbar_{(X,Y)}(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4} (\det X)^{1/4} e^{-\frac{\hbar}{2} (X + iY) x^2} \tag{13}
\]
where $X$ and $Y$ are real symmetric $n \times n$ matrices with $X > 0$; we have $||\phi^h_{(X,Y)}||_{L^2} = 1$. We will find it convenient to set

$$M = i(X + iY), \quad X = X^T > 0, \quad Y = Y^T,$$

and to write $\phi^h_{(X,Y)} = \phi^h_M$; thus:

$$\phi^h_M(x) = \left(\frac{1}{\pi \hbar}\right)^{n/4} (\det X)^{1/4} e^{\frac{M x^2}{2}}, \quad X = \text{Im} M.$$

The Wigner transform of $\phi^h_M$ is given by the formula

$$W\phi^h_M(z) = \left(\frac{1}{\pi \hbar}\right)^n (\det X)^{-1/2} e^{-\frac{1}{2} G z^2} \quad (14)$$

where $G$ is the symmetric matrix

$$G = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix}; \quad (15)$$

an essential remark is that $G$ is in addition symplectic (this fact was apparently first observed by Bastiaans [1]). More precisely, we have $G = S^T S$ with

$$S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix} \in \text{Sp}(n) \quad (16)$$

as results from a direct calculation; notice that $S$ belongs to the isotropy subgroup of the Lagrangian plane $\ell_p = 0 \times \mathbb{R}^n$ in $\text{Sp}(n)$. For $z_0 \in \mathbb{R}^{2n}$ we set

$$\phi^h_{z_0,M} = \hat{T}(z_0)\phi^h_M;$$

the Wigner transform of $\phi^h_{z_0,M}$ is given by

$$W\phi^h_{z_0,M}(z) = W\phi^h_M(z - z_0); \quad (17)$$

it is thus a Gaussian centered at $z_0$.

Let us generalize formula (14) by calculating the Wigner-Moyal transform $W(\phi^h_{z_0,M}, \phi^h_{z_0,M'})$ of a pair of squeezed coherent states; recall for this purpose the “Fresnel formula”

$$\left(\frac{1}{\pi \hbar}\right)^{n/2} \int e^{-\frac{i}{\hbar} \xi \cdot x} e^{-\frac{1}{4 \hbar K x^2}} d^n x' = (\det K)^{-1/2} e^{-\frac{1}{4 \hbar} K^{-1} \xi^2} \quad (18)$$

valid for $\xi \in \mathbb{C}^n$, $K = K^T$, $\text{Im} K > 0$; here $(\det K)^{-1/2} = \lambda_1^{-1/2} \cdots \lambda_n^{-1/2}$ where $\lambda_j^{-1/2}$ is the square root with positive real part of the eigenvalue $\lambda_j^{-1}$ of $K^{-1}$ (see [4] [16]).

**Proposition 2** We have

$$W(\phi^h_M, \phi^h_{M'})(z) = \left(\frac{1}{\pi \hbar}\right)^n (\det XX')^{-1/4} e^{-\frac{1}{4 \hbar} F z^2} \quad (19)$$

where $F$ is the matrix

$$F = \begin{pmatrix} 2i(M-M')^{-1}M & -i(M+M')(M-M')^{-1} \\ -i(M-M')^{-1}(M+M') & 2i(M-M')^{-1} \end{pmatrix}. \quad (20)$$
**Proof.** Setting \( W_{M,M'} = W(\phi^h_M, \phi^h_{M'}) \) we have

\[
W_{M,M'}(z) = \left( \frac{1}{2\pi \hbar} \right)^n \int e^{-\frac{\pi}{\hbar} p y \phi^h_M(x + \frac{1}{2} y)\phi^h_{M'}(x - \frac{1}{2} y)} d^n y
\]

that is, replacing \( \phi^h_M \) and \( \phi^h_{M'} \) by their expressions,

\[
W_{M,M'}(z) = C(X, X')e^{\frac{iz}{\hbar}(M + M') z^2} \int e^{-\frac{\pi}{\hbar} p y e^{\frac{i}{\hbar} \Phi(x,y)} d^n y}
\]

where \( M = i(X + iY), \ M' = i(X' + iY') \), the terms \( C(X, Y) \) and \( \Phi(x, y) \) being given by

\[
C(X, Y) = 2^{-n} \left( \frac{1}{\pi \hbar} \right)^{2n} (\det XX')^{1/4}
\]

\[
\Phi(x, y) = M(x + \frac{1}{2} y)^2 + \overline{M'}(x - \frac{1}{2} y)^2.
\]

Expanding \( \Phi(x, y) \) and using [18], we finally get

\[
W_{M,M'}(z) = \left( \frac{1}{2\pi \hbar} \right)^n (\det XX')^{-1/4} e^{-\frac{i}{\hbar}Fz^2}
\]

where \( F \) is the matrix \([20]\) (the left-upper block is obtained by using the identity

\[
M + M' - (M - M')(M + M')^{-1}(M - M') = 4i\overline{M}(M + M')^{-1}M.
\]

Noting the following formula, which is an easy consequence of the definitions of the Wigner–Moyal transform and the Heisenberg–Weyl operators:

**Lemma 3** For all \( z_0, z'_0 \) in \( \mathbb{R}^{2n} \) and \( f, g \) in \( L^2(\mathbb{R}^{2n}) \) we have

\[
W(\hat{T}(z_0)\psi, \hat{T}(z'_0)\psi')(z) = e^{\frac{i}{\hbar}(\sigma(z_0 - z'_0, z) - \frac{1}{2}\sigma(z_0, z'_0))} \times W(\psi, \psi')(z - \frac{1}{2}(z_0 + z'_0))
\]

we immediately get the following generalization of Proposition [2]

**Corollary 4** We have

\[
W(\phi^h_{z_0,M}, \phi^h_{z_0,M'})(z) = \left( \frac{1}{2\pi \hbar} \right)^n e^{\frac{i}{\hbar}(\sigma(z_0 - z'_0, z) - \frac{1}{2}\sigma(z_0, z'_0))}(\det XX')^{-1/4} e^{-\frac{i}{\hbar}F(z - \frac{1}{2}(z_0 + z'_0))^2}
\]

where \( F \) is given by \([24]\); hence in particular

\[
W(\phi^h_{z_0,M}, \phi^h_{z_0,M'})(z) = \left( \frac{1}{2\pi \hbar} \right)^n (\det XX')^{-1/4} e^{-\frac{i}{\hbar}F(z - z_0)^2}.
\]

### 2.2 Phase-space coherent states

For each \( \phi \in \mathcal{S}(\mathbb{R}^n) \) the operator

\[
U_{\phi}\psi(z) = \left( \frac{\pi \hbar}{2} \right)^{n/2} W(\psi, \phi)(\frac{1}{2}z)
\]

is an isometry of \( L^2(\mathbb{R}^n) \) onto its range \( \mathcal{H}_\phi \); it follows that:
Proposition 5  Let \( \Phi^h_{z_0} = U_\phi(\phi^h_{z_0}) \). For each \( \Psi \in \mathcal{H}_\phi \) we have
\[
\Psi(z) = \left(\frac{1}{2\pi\hbar}\right)^n \int (\Psi, \Phi^h_{z_0})_{L^2} \Phi^h_{z_0}(z) d^2n z_0  \tag{21}
\]
and
\[
||\Psi||_{L^2} = \left(\frac{1}{2\pi\hbar}\right)^n \int |(\Psi, \Phi^h_{z_0})_{L^2}|^2 d^2n z_0. \tag{22}
\]

Proof. Let \( \psi \) be defined by \( \Psi = U_\phi \psi \); In view of part (ii) of Proposition 1 (formula (10)) we have
\[
\psi = \left(\frac{1}{2\pi\hbar}\right)^n \int (\psi, \phi^h_{z_0})_{L^2} \phi^h_{z_0} d^2n z_0
\]
hence, since \( U_\phi \) is continuous,
\[
\Psi = \left(\frac{1}{2\pi\hbar}\right)^n \int (\psi, \phi^h_{z_0})_{L^2} U_\phi(\phi^h_{z_0}) d^2n z_0
\]
\[
= \left(\frac{1}{2\pi\hbar}\right)^n \int (\psi, \phi^h_{z_0})_{L^2} \Phi^h_{z_0} d^2n z_0.
\]
In view of formula (22) we have
\[
(\Psi, \Phi^h_{z_0})_{L^2} = (U_\phi \psi, U_\phi \phi^h_{z_0})_{L^2} = (\psi, \phi^h_{z_0})_{L^2}
\]
hence (21); formula (22) follows by a similar argument from formula (11). \( \blacksquare \)

3  Metaplectic Group and Coherent States

The metaplectic group \( \text{Mp}(n) \) is a faithful unitary representation of \( \text{Sp}_2(n) \), the double cover of the symplectic group \( \text{Sp}(n) \). There are several different ways to describe the elements of \( \text{Mp}(n) \) (see for instance [9, 16, 23]); for our purposes the most adequate definition makes use the notion of generating function for a free symplectic matrix because it is the simplest way to arrive at the Weyl symbol of metaplectic operators (and hence to their extension to phase space).

The interest of the metaplectic representation comes from the fact that it links in a crucial way classical (Hamiltonian) mechanics to quantum mechanics (see for instance [18] or [9] and the references therein). Assume in fact that \( H \) is a Hamiltonian function which is a quadratic polynomial in the position and momentum variables (with possibly time-dependent coefficients): thus
\[
H(z, t) = \frac{1}{2} z^T H''(t) z = \frac{1}{2} H''(t) z^2
\]
where \( H''(t) \) is a symmetric matrix (it is the Hessian matrix of \( H \)). The associated Hamilton equations \( \dot{z} = \partial_z H(z, t) \) determine a (generally time-dependent) flow consisting of symplectic matrices \( S_t \). We thus have a continuous path
$t \mapsto S_t$ in the symplectic group $\text{Sp}(n)$ passing through the identity at time $t = 0$: $S_0 = I$. Following general principles, this path can be lifted (in a unique way) to a path $t \mapsto \hat{S}_t$ in $\text{Mp}(n)$ such that $\hat{S}_0 = \hat{I}$ (the identity in $\text{Mp}(n)$).

Choose now an initial wavefunction $\psi_0$ in, say, $\mathcal{S}(\mathbb{R}^n)$ and set $\psi(x, t) = \hat{S}_t \psi_0(x)$.

The function $\psi$ satisfies Schrödinger’s equation

$$i\hbar \frac{\partial \psi}{\partial t} = H(x, i\hbar \partial_x, t) \psi$$

where the operator $H(x, i\hbar \partial_x)$ is given by

$$H(x, i\hbar \partial_x, t) = \frac{1}{2} (x, i\hbar \partial_x)^T H''(t)(x, i\hbar \partial_x).$$

3.1 Description of $\text{Mp}(n)$ and $\text{IMp}(n)$

Let $W$ be quadratic form of the type

$$W(x, x') = \frac{1}{2} P x^2 - L x \cdot x' + \frac{1}{2} Q x^2$$

with $P = P^T$, $Q = Q^*$, $\det L \neq 0$ (we have set $P x^2 = P x \cdot x$, etc.). To each such quadratic form we associate the generalized Fourier transform

$$\hat{S}_{W,m} \psi(x) = \left( \frac{1}{2\pi \hbar} \right)^{n/2} i^m \sqrt{|\det L|} \int e^{\frac{i}{\hbar} W(x, x')} \psi(x') dx'$$

where $m$ corresponds to a choice of the argument of $\det L$ modulo $2\pi$. One proves (see [5]) that every $\hat{S} \in \text{Mp}(n)$ can be written (of course in a non-unique way) as the product of two operators of the type (23), and that the integer

$$m(\hat{S}) = m + m' - \text{Inert}(P' + Q)$$

is independent modulo 4 of the factorization $\hat{S}_{W,m} \hat{S}_{W',m'}$ of $\hat{S}$; the class modulo 4 of $m(\hat{S})$ is the Maslov index of the metaplectic operator $\hat{S}$. Since $\text{Mp}(n)$ is a realization of the double cover of $\text{Sp}(n)$ there exists a natural projection

$$\pi^{\text{Mp}} : \text{Mp}(n) \rightarrow \text{Sp}(n);$$

that projection is a 2-to-1 group epimorphism defined by the condition: $S_W = \pi^{\text{Mp}}(\hat{S}_{W,m})$ is the free symplectic matrix generated by $W$, that is:

$$(x, p) = S_W(x', p') \iff \begin{cases} p = \partial_x W(x, x'), \\ p' = -\partial_{x'} W(x, x'). \end{cases}$$

Writing $S \in \text{Sp}(n)$ in block form:

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$S$ is free if and only if $\det B \neq 0$, and the generating function $W$ is the quadratic form in $2n$ variables

$$W(x, x') = \frac{1}{2} DB^{-1} x^2 - (B^{-1})^T x \cdot x' + \frac{1}{2} B^{-1} Ax^2$$

We will use the following covariance properties of the metaplectic operators:
• We have
  \[ \hat{S}\hat{T}(z_0) = \hat{T}(S z_0)\hat{S} \]  
(27)
for all \( \hat{S} \in \text{Mp}(n) \) and \( z_0 \in \mathbb{R}^{2n} \);

• If \( \hat{S} \in \text{Mp}(n) \) has projection \( S = \pi\text{Mp}(\hat{S}) \) then
  \[ W(\hat{S}\psi, \hat{S}\phi) = W(\psi, \phi) \circ S^{-1}. \]

The group generated by the Weyl–Heisenberg operators \( \hat{T}(z_0) \) and the operators \( \hat{S} \in \text{Mp}(n) \) is a group of unitary operators in \( L^2(\mathbb{R}^n) \); it is denoted by \( \text{IMp}(n) \) and called the inhomogeneous metaplectic group.

3.2 The action of \( \text{Mp}(n) \) on coherent states

To describe the action of \( \hat{S} \in \text{Mp}(n) \) on the squeezed coherent states \( \phi_{\mathbf{z}_0, M}^h \) we will need the following Lemma. Let us denote by \( \Sigma(n) \) the Siegel half-space, that is
\[
\Sigma(n) = \{ M : M = M^T, \text{Im} M > 0 \}
\]
\( (M \text{ is a complex } n \times n \text{ matrix}) \).

**Lemma 6**  Let \( S \in \text{Sp}(n) \) be given by (25) and \( M \in \Sigma(n) \). Then \( \det(A + BM) \neq 0, \det(C + DM) \neq 0 \) and
\[
\alpha(S)M = (C + DM)(A + BM)^{-1} \in \Sigma(n)
\]
(28)
(in particular \( \alpha(S)M \) is symmetric), and
\[
\alpha(SS')M = \alpha(S)\alpha(S')M.
\]
(29)
The action \( \text{Sp}(n) \times \Sigma(n) \rightarrow \Sigma(n) \) defined by (28) is transitive: if \( M, M' \in \Sigma(n) \) then there exists \( S \in \text{Sp}(n) \) such that \( M' = \alpha(S)M \).

For a proof see [4, 12]; also see the preprint [2] by Combescure and Robert.

Notice that (29) implies that \( S \rightarrow \alpha(S) \) is a true representation of the symplectic group in the Siegel half-space.

Let \( \hat{S} \in \text{Mp}(n) \) have projection
\[
S = \pi\text{Mp}(\hat{S}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
on \( \text{Sp}(n) \); then
\[
\hat{S}\phi^h(x) = \left( \frac{1}{\pi \hbar} \right)^{n/4} \frac{e^{im(\hat{S})}}{\sqrt{\det(A + iB)}} \exp \left[ -\frac{1}{2\hbar} \alpha(S)x^2 \right]
\]
where the branch cut of the square root of \( \det(A + iB) \) is taken to lie just under the positive real axis; \( m(\hat{S}) \) is the Maslov index (24) of \( \hat{S} \). Using this formula
\[ \hat{S}\phi_{z_0}^h(x) \] is easily calculated: since by definition \( \phi_{z_0}^h = \hat{T}(z_0)\phi^h \) the metaplectic covariance formula \( \tag{27} \) immediately yields
\[ \hat{S}\phi_{z_0}^h(x) = \hat{T}(S z_0)\hat{S}\phi^h(x) \]
(see Littlejohn \[17\] for an explicit formula).

The results above can be generalized to arbitrary squeezed coherent states:

**Proposition 7** Let \( \phi^h_{z_0,M}, M \in \Sigma(n) \), be a squeezed coherent state and \( \hat{S} \in \text{Mp}(n), S = \pi^{\text{Mp}}(\hat{S}) \). We have
\[ \hat{S}\phi_{z_0,M}^h = \phi_{\alpha(S)M}^h, \quad \hat{S}\phi_{z_0,M}^h = \hat{T}(S z_0)\phi_{\alpha(S)M}^h. \] \( \tag{30} \)
(see for instance \[2, 4, 17\]).

The Gaussian character of a wavepacket is thus preserved by metaplectic operators; as a consequence the solution of a Schrödinger equation with Gaussian initial value remains Gaussian when the Hamiltonian operator is associated to a quadratic Hamiltonian function.

### 3.3 The Weyl symbol of a metaplectic operator

For \( S \in \text{Sp}(n) \) such that \( \det(S - I) \neq 0 \) we set
\[ M_S = \frac{1}{2} J(S + I)(S - I)^{-1} \]
(it is the symplectic Cayley transform of \( S \)). In \[6\] one of us proved the following result:

**Proposition 8** (i) If \( \hat{S}_{W,m} \) is such that \( S_W = \pi^{\text{Mp}}(\hat{S}_{W,m}) \) has no eigenvalue equal to one, then
\[ \hat{S}_{W,m}\psi(x) = \left( \frac{1}{2\pi\hbar} \right)^n e^{\frac{m}{\hbar} \text{Inert}_{W_{xx}}}(S) \int e^{\frac{\pi}{\hbar} M_S z_0^2 \hat{T}(z_0)\psi(x)} \psi(x) d^{2n} z_0 \] \( \tag{31} \)
for \( \psi \in S(R^n) \); here \( \text{Inert}_{W_{xx}} \) is the number of negative eigenvalues of the Hessian matrix of the function \( x \mapsto W(x,x) \). (ii) Every \( \hat{S} \in \text{Mp}(n) \) can be written as the product of two metaplectic operators of this type. (iii) More generally, every \( \hat{S} \in \text{Mp}(n) \) with projection \( S \) such that \( \det(S - I) \neq 0 \) can be written as
\[ \hat{S}\psi(x) = \left( \frac{1}{2\pi\hbar} \right)^n \nu(\hat{S}) \int e^{\frac{\pi}{\hbar} M_S z_0^2 \hat{T}(z_0)\psi(x)} \psi(x) d^{2n} z_0. \] \( \tag{32} \)
where \( \nu(\hat{S}) \) is the Conley–Zehnder index of \( \hat{S} \).
Formulae (31) and (32) yield the Weyl representations of the metaplectic operators; for a definition and a precise study of the Conley–Zehnder index $\nu(\hat{S})$ appearing in (32) see [9]. In [6] it was also proven that formula (32) can be rewritten alternatively as

$$\hat{S} = \left(\frac{1}{2\pi\hbar}\right)^n i^{\nu(\hat{S})} \sqrt{|\det(S - I)|} \int e^{-\frac{\pi}{\hbar}(Sz, z)} \hat{T}((S - I)z)d^{2n}z \quad (33)$$

or

$$\hat{S} = \left(\frac{1}{2\pi\hbar}\right)^n i^{\nu(\hat{S})} \sqrt{|\det(S - I)|} \int \hat{T}(Sz) \hat{T}(-z)d^{2n}z. \quad (34)$$

4 The Nearby Orbit Method

We now have the background material that is needed to study the nearby-orbit method for the phase-space Schrödinger equation. We begin by describing this method in the usual case; we refer to Littlejohn’s excellent review [17] for a discussion and interpretation.

4.1 The “standard” case

Consider an arbitrary (possibly time-dependent) Hamiltonian function $H$ on $\mathbb{R}^{2n}$; we denote by $(f_t)$ its flow: $t \mapsto f_t(z_0)$ is the solution of Hamilton’s equations $\dot{z} = J\partial_z H(z, t)$ passing through the phase-space point $z_0$ at time $t = 0$. We have, for every $z$,

$$f_t(z) = f_t(z_0) + S_t(z_0)(z - z_0) + O((z - z_0)^2)$$

where $S_t(z_0) = Df_t(z_0)$ is the Jacobian matrix of $f_t$ at $z_0$. Since Hamiltonian flows consist of canonical transformations, $S_t(z_0)$ is a symplectic matrix: $S_t(z_0) \in Sp(n)$. Suppose now the point $z$ is close to $z_0$; then

$$f_t(z) \approx f_t(z_0) + S_t(z_0)(z - z_0);$$

the nearby orbit approximation consists in replacing $f_t(z)$ by $f_t(z_0) + S_t(z_0)(z - z_0)$. Noticing that

$$f_t(z_0) + S_t(z_0)(z - z_0) = T(f_t(z_0))S_t(z_0)T(z_0)^{-1}z$$

where $T(z_0)$ is the translation operator $z \mapsto z + z_0$, an educated guess is that we will obtain the quantum counterpart of this approximation by replacing translations by Weyl–Heisenberg operators, and symplectic matrices by metaplectic operators. In fact this guess is correct up to a phase factor arising from the non-commutativity of the Heisenberg–Weyl operators: the “true” semiclassical approximation to the solution $\psi$ of Schrödinger’s equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi, \quad \psi(t = 0) = \psi_0 \quad (35)$$
The approximation (36) is valid when the initial state \( \psi_0 \) is given by

\[
U(t, z_0)\psi_0 = e^{\frac{i}{\hbar}\gamma(z_0, t)}\widehat{T}(f_t(z_0))\widehat{S}_t(z_0)\widehat{T}(z_0)^{-1}\psi_0
\]

(36)

where the phase \( \gamma(t) \) is given by

\[
\gamma(z_0, t) = \int_0^t \frac{1}{2}\sigma(z(t'), \dot{z}(t')) - H(z(t'), t')dt'.
\]

(37)

One shows \([3, 17]\) that the function \( \psi = U(t, z_0)\psi_0 \) is the solution of the Schrödinger equation associated to the Hamiltonian function

\[
H_{z_0}(z, t) = H(f_t(z_0), t) + H'(f_t(z_0), t)(z - f_t(z_0)) + \frac{1}{2}H''(f_t(z_0), t)(z - f_t(z_0))^2
\]

obtained by truncating the Taylor series for \( H \) at \( f_t(z_0) \) after second order.

The approximation \([30]\) is valid when the initial state \( \psi_0 \) is localized near \( z_0 = (x_0, p_0) \), that is when \( x_0 \) and \( p_0 \) are chosen to be the position and momentum expectation vectors \( (x_0\psi_0, \psi_0)_L \) and \( p_0 = (p_0\psi_0, \psi_0)_L \) at initial time \( t = 0 \); the metaplectic operators \( \widehat{S}_t(z_0) \) are obtained by the usual lifting-procedure: the function \( t \mapsto \widehat{S}_t(z_0) \) is a path in the symplectic group passing through the identity of \( \text{Sp}(n) \) at time \( t = 0 \); the function \( t \mapsto \widehat{S}_t(z_0) \) is then the path in the metaplectic group passing through the identity at time \( t = 0 \), and such that \( \pi^{\text{Mp}}(\widehat{S}_t(z_0)) = S_t(z_0) \) for all \( t \). Formula \([30]\) has the following interpretation (Littlejohn \([17]\)): the operator \( \widehat{T}(z_0)^{-1} \) rigidly translates the initial wavepacket to a wavepacket centered at the origin; the metaplectic operator \( \widehat{S}_t(z_0) \) then “squeezes” it, and \( \widehat{T}(f_t(z_0)) \) finally rigidly translates this squeezed wavepacket so that it becomes centered at \( f_t(z_0) \); one then multiplies the result by \( e^{\frac{i}{\hbar}\gamma(z_0, t)} \).

Let us work out explicitly formula \([30]\) when the initial wavepacket \( \psi_0 \) is a squeezed coherent state:

**Proposition 9** Let \( \phi_{z_0, M}^h \) be an arbitrary squeezed coherent state. We have:

\[
U(t, z_0)\phi_{z_0, M}^h = e^{\frac{i}{\hbar}\gamma(z_0, t)}\phi_{f_t(z_0), \alpha(S_t(z_0))M}^h
\]

where \( \alpha(S_t(z_0)) \) is defined by \([28]\) with \( S = S_t(z_0) \).

**Proof.** By definition \( \phi_{z_0, M}^h = \widehat{T}(z_0)\phi_M^h \) hence

\[
U(t, z_0)\phi_{z_0, M}^h(x) = e^{\frac{i}{\hbar}\gamma(z_0, t)}\widehat{T}(f_t(z_0))\widehat{S}_t(z_0)\phi_M^h
\]

that is, taking the first formula \([30]\) into account,

\[
U(t, z_0)\phi_{z_0, M}^h = e^{\frac{i}{\hbar}\gamma(z_0, t)}\phi_{f_t(z_0), \alpha(S_t(z_0))M}^h
\]

and

\[
= e^{\frac{i}{\hbar}\gamma(z_0, t)}\phi_{f_t(z_0), \alpha(S_t(z_0))M}^h
\]

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which completes the proof. ■

Formula (36) can be extended to arbitrary square-integrable initial wavefunctions using the machinery of coherent states reviewed in Section 2: applying formula (10) to \( \psi_0 \in L^2(\mathbb{R}^n) \) one has

\[
\psi_0(x) = \left( \frac{1}{2\pi \hbar} \right)^n \int (\psi_0, \phi^h_z)_{L^2} \phi^h_z(x) d^{2n} z_0
\]

and one then takes as semiclassical solution

\[
U(t)\psi_0(x) = \left( \frac{1}{2\pi \hbar} \right)^n \int (\psi_0, \phi^h_z)_{L^2} U(t, z_0) \phi^h_z(x) d^{2n} z_0.
\]

Since by definition \( \hat{T}(z_0)^{-1}\phi^h_{z_0} = \phi^h \) is the standard coherent state, this formula can be rewritten

\[
U(t)\psi_0(x) = \left( \frac{1}{2\pi \hbar} \right)^n \int \left( \psi_0, \phi^h_z \right)_{L^2} \hat{T}(f_t(z_0)) e^{i \gamma(t)} \hat{S}(z_0) \phi^h(x) d^{2n} z_0
\]

(cf. equation (7.33) in [17]).

**Remark 10** Observe that neither \( U(t, z_0) \) nor \( U(t) \) are linear operators.

Of course, a natural question is arising at this point, namely “How good are the semiclassical approximations (36), (37)?”. Very precise estimates have been given in [3, 10, 11] (see the discussion in the last section); we will content us here with quoting the following result:

**Proposition 11** Let \( \psi \) be the exact solution to Schrödinger’s equation (35) with initial datum \( \psi_0 = \phi^h_{z_0} \). Then for each \( T > 0 \) there exists \( C_T \geq 0 \) such that

\[
||U(t, z_0)\phi^h_{z_0} - \psi(\cdot, t)||_{L^2(\mathbb{R}^2)} \leq C_T \sqrt{\hbar}
\]

provided that the Hamiltonian function \( H \) satisfies uniform estimates of the type

\[
|\partial^\alpha_z H(z, t)| \leq C'_\alpha,T (1 + |z|)^m
\]

for \( |\alpha| \geq m, |t| < T \) and \( z \in \mathbb{R}^{2n} \).

(It is in fact possible to obtain precise bounds for the constant \( C_T \); see [3]).

### 4.2 Weyl operators on phase space

In [8] (also see [12] for details) we noticed that the wave-packet transform (1) is related to the Wigner–Moyal transform (6) by the simple formula

\[
U_\phi \psi(z) = \left( \frac{\pi \hbar}{2} \right)^{n/2} W(\psi, \phi)(\frac{z}{\hbar}) , \ \psi \in L^2(\mathbb{R}^n)
\]

It follows that we have

\[
(U_\phi \psi, U_\phi \psi')_{L^2(\mathbb{R}^{2n})} = (\psi, \psi')_{L^2(\mathbb{R}^n)}
\]
and hence each of the linear mappings $U_\phi$ is an isometry of $L^2(\mathbb{R}^n_\sigma)$ onto a closed subspace $\mathcal{H}_\phi$ of $L^2(\mathbb{R}^{2n}_\sigma)$ (the square integrable functions on phase space). It follows that $U_\phi^*U_\phi$ is the identity operator on $L^2(\mathbb{R}^n_\sigma)$ and that $P_\phi = U_\phi U_\phi^*$ is the orthogonal projection onto the Hilbert space $\mathcal{H}_\phi$. Defining $\hat{T}_{\text{ph}}(z_0)$ by

$$
\hat{T}_{\text{ph}}(z_0)\Psi(z) = e^{-\frac{i}{\hbar}\sigma(z_0, z)}\Psi(z - z_0)
$$

we have the fundamental relation

$$
\hat{T}_{\text{ph}}(z_0)U_\phi = U_\phi\hat{T}(z_0).
$$

Formula 44 allows us to associate to every Weyl operator

$$
\hat{A}\Psi(x) = \left(\frac{i}{2\pi\hbar}\right)^n \int a_\sigma(z_0)\hat{T}(z_0)\psi(x)d^{2n}z_0
$$

the phase space operator

$$
\hat{A}_{\text{ph}}\Psi(z) = \left(\frac{i}{2\pi\hbar}\right)^n \int a_\sigma(z_0)\hat{T}_{\text{ph}}(z_0)\Psi(z)d^{2n}z_0;
$$

of course

$$
\hat{A}_{\text{ph}}U_\phi = U_\phi\hat{A}.
$$

Formulae 46 and 41 allow us to give an explicit description of the action of the metaplectic representation on phase-space functions: if $\det(S - I) \neq 0$ we define

$$
\hat{S}_{\text{ph}}\Psi(z) = \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{\frac{i}{\hbar}\sigma(S, z)}\int e^{i\frac{M_\sigma z z}{2}}\hat{T}_{\text{ph}}(z_0)\Psi(z)d^{2n}z_0;
$$

the operators $\hat{S}_{\text{ph}}$ are in one-to-one correspondence with the metaplectic operators $\hat{S}$ and thus generate a group which we denote by $Mp_{\text{ph}}(n)$; that group is of course isomorphic to $Mp(n)$. Of course, as an immediate consequence of The following equivalent formulae are immediate consequences of the corresponding expressions 43 and 44 of $\hat{S}$:

$$
\hat{S}_{\text{ph}} = \left(\frac{1}{2\pi\hbar}\right)^n e^{i\nu(S)}\sqrt{\det(S - I)}\int \hat{T}_{\text{ph}}((S - I)z)d^{2n}z
$$

and

$$
\hat{S}_{\text{ph}} = \left(\frac{1}{2\pi\hbar}\right)^n e^{i\nu(S)}\sqrt{\det(S - I)}\int \hat{T}_{\text{ph}}(Sz)\hat{T}_{\text{ph}}(-z)d^{2n}z.
$$

Notice that the well-known “metaplectic covariance” relation $\hat{A} \circ \hat{S} = \hat{S}^{-1}\hat{A}\hat{S}$ valid for any $\hat{S} \in Mp(n)$ with projection $S \in Sp(n)$ extends to the phase-space Weyl operators $\hat{A}_{\text{ph}}$: we have

$$
\hat{S}_{\text{ph}}\hat{T}_{\text{ph}}(z_0)\hat{S}_{\text{ph}}^{-1} = \hat{T}_{\text{ph}}(Sz)\; , \; \hat{A}_{\text{ph}}\hat{S}_{\text{ph}} = \hat{S}_{\text{ph}}\hat{A}_{\text{ph}}\hat{S}_{\text{ph}}.
$$

In Subsection 2.2 we defined coherent states in phase space. The metaplectic action on coherent states described in Proposition 4 carries over to this case without difficulty, yielding the formulae

$$
\hat{S}_{\text{ph}}\Phi^h_{\sigma(S)M} = \Phi^h_{\sigma(S)M}\; , \; \hat{S}_{\text{ph}}\Phi^h_{z_0, M} = \hat{T}_{\text{ph}}(Sz_0)\hat{S}_{\text{ph}}\Phi^h_{\sigma(S)M}.
$$
4.3 Nearby-orbit method in phase space

Let us now state and prove the main result of this paper:

**Proposition 12** (i) Let $U(t, z_0)$ be the semiclassical propagator for Schrödinger’s equation and set $\psi = U(t, z_0)\psi_0$. The wavepacket transform $U_\phi$ takes $\psi$ to the function $\Psi$ defined by

$$\Psi = e^{i\pi t(z_0, t)}\widehat{T}_{ph}(f_t(z_0))(\widehat{S}_t(z_0))_{ph}\widehat{T}(z_0)_{ph}^{-1}\Psi_0$$

with $\Psi_0 = U_\phi\psi_0$. (ii) In particular, if $\psi_0 = \phi^h_{z_0}$

$$U_\phi(U(t, z_0)\phi^h_{z_0}) = e^{i\pi t(z_0, t)}\widehat{T}_{ph}(f_t(z_0))(\widehat{S}_t(z_0))_{ph}\Phi^h_{z_0}.$$  

**Proof.** Set $\psi = U(t, z_0)\psi_0$; by definition of $U(t, z_0)$ we have

$$\psi = e^{i\pi t(z_0, t)}\widehat{T}(f_t(z_0))\widehat{S}_t(z_0)\widehat{T}(z_0)_{ph}^{-1}\psi_0$$

hence, using successively (51) and (52),

$$U_\phi\psi = e^{i\pi t(z_0, t)}U_\phi(\widehat{T}(f_t(z_0))\widehat{S}_t(z_0)\widehat{T}(z_0)_{ph}^{-1}\psi_0)
= e^{i\pi t(z_0, t)}\widehat{T}_{ph}(f_t(z_0))(\widehat{S}_t(z_0))_{ph}\widehat{T}(z_0)_{ph}^{-1}U_\phi\psi_0.$$  

Formula (52) follows since we have

$$\widehat{T}(z_0)_{ph}^{-1}\Phi^h_{z_0} = \widehat{T}(z_0)_{ph}^{-1}U_\phi(\phi^h_{z_0}) = U_\phi(\widehat{T}(z_0)_{ph}^{-1}\phi^h_{z_0}) = U_\phi(\phi^h).$$

The result above therefore suggests that the phase-space version of the semiclassical nearby-orbit propagator $U(t, z_0)$ should be given by the formula

$$U_{ph}(t, z_0) = e^{i\pi t(z_0, t)}\widehat{T}_{ph}(f_t(z_0))(\widehat{S}_t(z_0))_{ph}\widehat{T}(z_0)_{ph}^{-1}$$

and we have:

**Proposition 13** Let $\Psi$ be the solution of the phase-space Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{H}_{ph}\Psi, \quad \Psi(t = 0) = \Phi^h_{z_0}$$

with $\Phi^h_{z_0} = U_\phi(\phi^h_{z_0})$. Suppose that $H$ satisfies the conditions in Proposition 14. Then, for $|t| < T$ there exists a constant $C_T > 0$ such that

$$||U_{ph}(t, z_0)\Phi^h_{z_0} - \Psi(\cdot, t)||_{L^2(\mathbb{R}^n)} \leq C_T \sqrt{\hbar}.$$ (54)

**Proof.** The inequality (54) is an easy consequence of the estimate (59): the solution $\Psi$ is given by $\Psi(\cdot, t) = U_\phi(\psi(\cdot, t))$ where $\psi$ is the solution of the usual Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \widehat{H}\psi, \quad \psi(t = 0) = \phi^h_{z_0};$$

since $U_\phi$ is a linear isometry we have

$$||U_{ph}(t, z_0)\Phi^h_{z_0} - \Psi(\cdot, t)||_{L^2(\mathbb{R}^n)} = ||U(t, z_0)\phi^h_{z_0} - \psi(\cdot, t)||_{L^2(\mathbb{R}^n)} \leq C_T \sqrt{\hbar}.$$
5 Conclusion and Discussion

There are several problems and questions we have not discussed in this paper, and to which we will come back in forthcoming publications. Needless to say, there is one outstanding omission: we haven’t analyzed the domain of validity of the nearby-orbit method much in detail. There are many results in the literature for the standard nearby-orbit method. For instance, Hagedorn [10, 11] obtains precise estimates using the Lie–Trotter formula; similar results were rediscovered and sharpened by Combescure-Robert [3] using the Duhamel principle. Also see [19] (Ch.2, §2.1). It shouldn’t be too difficult to obtain corresponding estimate for the phase-space Schrödinger equation, using the properties of the wavepacket transform. It is on the other hand well-known that there are problems with long times when the associated classical systems exhibits a chaotic behavior; as Littlejohn points out in [17], the nearby orbit methods probably fails for long times near classically unstable points.

Acknowledgement 14 This work has been supported by the FAPESP grant 2005/51766–7 during the author’s stay at the University of São Paulo. I take the opportunity to thank Professor Paolo Piccione for his kind and generous invitation.

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