Quasi-local energy from a Minkowski reference

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Abstract  The specification of energy for gravitating systems has been an unsettled issue since Einstein proposed his pseudotensor. It is now understood that energy-momentum is quasi-local (associated with a closed 2-surface). Here we consider quasi-local proposals (including pseudotensors) in the Lagrangian-Noether-Hamiltonian formulations. There are two ambiguities: (i) there are many possible expressions, (ii) they depend on some non-dynamical structure, e.g., a reference frame. The Hamiltonian approach gives a handle on both problems. The Hamiltonian perspective helped us to make a remarkable discovery: with an isometric Minkowski reference a large class of expressions—namely all those that agree with the Einstein pseudotensor’s Freud superpotential to linear order—give a common quasi-local energy value. Moreover, with a best-matched reference on the boundary this is the Wang-Yau mass value.

Keywords  Hamiltonian · quasi-local energy · Minkowski reference

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1 Introduction

Along with his celebrated general relativity (GR) field equations, in November 1915 Einstein also proposed his gravitational energy-momentum density, which is a pseudotensor, not a proper tensor with coordinate reference frame independent meaning [1]. Many objected, including Lorentz, Levi-Civita, Felix Klein and Schrödinger (he later called it “sham” [2]). Einstein understood their concerns, but believed that his pseudotensor had physical meaning. Emmy Noether’s paper with her two famous theorems concerning symmetry in dynamical systems was written to clarify issues regarding energy raised by the investigations of Einstein, Hilbert and Klein [3,4]. She conclusively showed that there is no proper conserved energy-momentum density for any geometric gravity theory.

The topic of gravitational energy has remained an outstanding puzzle for over a century. Various pseudotensor and quasi-local expressions obtained from different perspectives have been proposed. Those that fit into the Lagrangian-Noether-Hamiltonian framework all depend on some non-dynamic structure, e.g., a reference frame. Here we present in detail our recent discovery, which was briefly described in the essay [5], regarding how choosing an isometric Minkowski reference sheds new light on the gravitational energy issue. In order to place our discovery in a suitable framework where it can be easily verified and its import can be better appreciated, we believe it appropriate to first review some of the history, survey many of the proposed energy expressions, explain how the Hamiltonian perspective clarifies the issues, and briefly describe our covariant Hamiltonian approach.

2 The pseudotensors

The Einstein Lagrangian density is quadratic in the connection, it differs from Hilbert’s scalar curvature Lagrangian by a total divergence [1]

\[
2\kappa L_E := -\sqrt{|g|} \Gamma^\alpha_{\beta\gamma} \Gamma^\gamma_{\beta\nu} \delta_{\alpha\sigma} \equiv \sqrt{|g|} R - \text{div}
\]

(here \(\kappa := 8\pi G/c^3\)). It depends on the metric and its first derivatives. The \textit{Einstein pseudotensor} is the associated canonical energy-momentum density:

\[
t^\mu_\nu := \frac{\partial L_E}{\partial g_{\alpha\beta}} g^{\alpha\beta} \partial_\mu g_{\alpha\beta}, \quad \partial_\mu t^\mu_\nu \equiv \frac{\delta L_E}{\delta g_{\alpha\beta}} \partial_\mu g_{\alpha\beta},
\]

where \(\delta L_E/\delta g_{\alpha\beta} = 0\) is the Einstein equation. With a source it reads

\[
\sqrt{|g|} G^\mu_\nu = \kappa \Sigma^\mu_\nu := \kappa \sqrt{|g|} T^\mu_\nu.
\]

Applying the contracted Bianchi identity yields \(\nabla_\nu \Sigma^\mu_\nu = 0\). With \(\Sigma\) this gives a (noncovariantly) conserved total energy-momentum complex:

\[
\partial_\mu (\Sigma^\mu_\nu + t^\mu_\nu) = 0.
\]

\footnote{Our notation follows in general MTW [6].}
From this relation one can infer the existence of some superpotential $\Pi^{\mu \lambda \nu} \equiv \Pi^{[\mu \lambda \nu]}$ such that

$$\kappa^{-1} \sqrt{|g|} G^\mu_{\nu} + t^\mu_{\nu} = \partial_\nu \Pi^{\mu \lambda \nu}.$$ \hspace{1cm} (5)

Such a superpotential was presented only in 1939 by Freud [7,8]:

$$2c_2 \Pi^{\mu \lambda \nu} := - \sqrt{|g|} g^{\beta \sigma} \Gamma^\alpha_{\beta \gamma} \delta^{\lambda \alpha \gamma \nu} \equiv - \sqrt{|g|} g^{\beta \sigma} g^{\alpha \delta} \delta^{\lambda \alpha \gamma \nu} \partial_\gamma g_{\sigma \delta},$$ \hspace{1cm} (6)

Other pseudotensors were proposed from various perspectives by Landau-Lifshitz [9], Papapetrou [10], Bergmann-Thomson [11], Goldberg [12], Möller [13], and Weinberg [14] (also used in [6]). They likewise follow from their respective superpotentials. Here, with the aid of the Minkowski reference metric $\bar{g}_{\mu \nu}$ and its associated covariant derivative $\bar{\nabla}_\mu$, we list the well-known superpotentials; all are written as weight one tensor densities$^2$ in forms that reveal their interrelationships:

$$2c_2 \Pi^{\mu \lambda \nu} := - \sqrt{|g|} g^{\beta \sigma} (\Gamma^\alpha_{\beta \gamma} - \bar{\Gamma}^\alpha_{\beta \gamma}) \delta^{\lambda \alpha \gamma \nu} \equiv - \sqrt{|g|} g^{\beta \sigma} g^{\alpha \delta} \delta^{\lambda \alpha \gamma \nu} \bar{\nabla}_\gamma g_{\sigma \delta},$$ \hspace{1cm} (7)

$$2c_2 \Pi^{\mu \lambda \nu} := \bar{\delta}^{\lambda \alpha \gamma \nu} \bar{\delta}^{\mu \alpha \beta \sigma} \bar{\nabla}_\pi |g| g^{\alpha \beta \gamma \tau} \equiv \bar{\delta}^{\lambda \alpha \gamma \nu} \bar{\delta}^{\mu \alpha \beta \sigma} g^{\alpha \beta \gamma \tau} (|g| \bar{g}^{\gamma \tau}),$$ \hspace{1cm} (8)

$$\Pi^{[\mu \lambda \nu]} := \bar{g}^{\beta \sigma} \Pi^{\mu \lambda \nu} \equiv \bar{g}^{\beta \sigma} 2c_2 \Pi^{\mu \lambda \nu},$$ \hspace{1cm} (9)

$$\Pi^{[\mu \lambda \nu]} := \bar{g}^{[\beta \sigma} \Pi^{\mu \lambda \nu]} \equiv \bar{g}^{[\beta \sigma} 2c_2 \Pi^{\mu \lambda \nu]},$$ \hspace{1cm} (10)

$$\Pi^{[\mu \lambda \nu]} := \bar{g}^{[\beta \sigma} \Pi^{\mu \lambda \nu]} \equiv \bar{g}^{[\beta \sigma} 2c_2 \Pi^{\mu \lambda \nu]},$$ \hspace{1cm} (11)

$$\Pi^{[\mu \lambda \nu]} := \bar{g}^{[\beta \sigma} \Pi^{\mu \lambda \nu]} \equiv \bar{g}^{[\beta \sigma} 2c_2 \Pi^{\mu \lambda \nu]},$$ \hspace{1cm} (12)

$$2c_2 \Pi^{\mu \lambda \nu} := 2|g| \bar{\delta}^{\lambda \alpha \gamma \nu} \bar{\delta}^{\mu \alpha \beta \sigma} \bar{\nabla}_\pi |g| g^{\beta \sigma},$$ \hspace{1cm} (13)

$$2c_2 \Pi^{\mu \lambda \nu} := 2|g| \bar{\delta}^{\lambda \alpha \gamma \nu} \bar{\delta}^{\mu \alpha \beta \sigma} \bar{\nabla}_\pi |g| g^{\beta \sigma}.$$ \hspace{1cm} (14)

This list includes Goldberg’s Freud (11) and Landau-Lifshitz (12) density-weighted superpotentials. In a Minkowski coordinate frame, $y^\mu$, where $\bar{g}_{\mu \nu} = \text{diag}(-1, +1, +1, +1)$, $\bar{\nabla}_\mu = \partial_\mu$, $\bar{\Gamma}^\alpha_{\beta \gamma} = 0$ and $|g| = 1$, the above superpotential expressions revert to their traditional form. All of these superpotentials define energy-momentum values which depend on the Minkowski reference geometry, or equivalently a Minkowski coordinate system.

There are two unsatisfactory issues: (i) which expression? (ii) and which Minkowski reference geometry-coordinate system? On the other hand (a) these expressions do provide a description of energy-momentum conservation, (b) they (like connections) have well defined values for each Minkowski reference geometry/Minkowski coordinate reference frame, (c) all (except for Möller) give the expected total energy-momentum values at spatial infinity$^3$. But none

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$^2$ These superpotentials can all be put into the standard form of (7) type expressions by using the Minkowski metric $\bar{g}_{\alpha \beta}$ to lower the $\nu$ index.

$^3$ Ref. [6], §20.2 uses the linearized theory to argue that the total energy-momentum for an asymptotically flat spacetime is given by the integral over the 2-surface boundary at infinity of effectively the expression (13) or any other superpotential that agrees with it to linear order.
of them gives positive energy for small vacuum regions [15]. An 11-parameter set of new pseudotensor superpotentials with this desirable property was constructed by So [16].

A widely held opinion was expressed in an influential textbook (Ref. [6], p 467): Anyone who looks for a magic formula for “local gravitational energy-momentum” is looking for the right answer to the wrong question. Unhappily, enormous time and effort were devoted in the past to trying to “answer this question” before investigators realized the futility of the enterprise. Here we will present a somewhat different view.

We note that some time ago it was, surprisingly, found that the fundamentally different pseudotensors of Einstein, Landau-Lifshitz, Papapetrou and Weinberg could sometimes—in particular for all Kerr-Schild metrics—give the same energy value [17]. Our discovery is a far more general result—both with respect to the class of metrics and the class of expressions.

3 The Hamiltonian approach

How can one understand the physical significance of all these expressions? The Hamiltonian approach offers a way. Note that pseudotensors are related to the Hamiltonian. This can be seen as follows. For any region covered by a single coordinate system one can choose a vector field $Z^\nu$ with constant components in that frame. The associated total energy-momentum, $P^\nu(V)$, in the region can then be determined, using (5), to be

$$-Z^\nu P^\nu(V) := -\int_V Z^\nu (\mathcal{T}^\mu_{\nu} + t^\mu_{\nu}) d\Sigma_\mu$$

$$\equiv \int_V Z^\nu |g|^{\frac{1}{2}} \left( \frac{1}{k} G^\mu_{\nu} - T^\mu_{\nu} \right) d\Sigma_\mu - \oint_{V} Z^\nu \mathcal{H}_{GR}^\mu_{\nu} \frac{1}{2} dS_{\mu\lambda}$$

$$\equiv \int_V Z^\nu H^\nu_{GR} + \oint_{S=\partial V} B(Z) \equiv H(Z, V). \quad (15)$$

Note that $H^\nu_{GR}$ is just the covariant expression for the ADM Hamiltonian density (see, e.g. [6] Ch. 21), and the boundary term 2-surface integrand, $B(Z)$, is determined by the superpotential. The value of the pseudotensor/Hamiltonian is thus quasi-local, determined just by this boundary term, since the spatial volume integral vanishes “on shell” (initial value constraints).

From the Hamiltonian variation one gets information that tames the freedom in the boundary term—namely boundary conditions. (The boundary term in the variation of the Hamiltonian indicates what must be held fixed on the boundary.) The pseudotensor values thus are the values of the Hamiltonian with the associated boundary conditions [18]. Hence the first problem is under

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4 Following the appearance of this seminal work, numerous papers have been continually appearing comparing the energy-momentum values given by various pseudotensors in various metrics.
control: the generally different energy-momentum values from different pseudotensors correspond to the value of the associated Hamiltonians which evolve the system with their respective boundary conditions.

4 Quasi-local energy-momentum

The modern concept is quasi-local energy-momentum: i.e., associated with a closed 2-surface (pseudotensors always had this property, but this was not easily recognized before Freud’s paper \cite{7,12}, and its importance only became appreciated after Penrose \cite{19}, where the term quasi-local was introduced). A comprehensive review \cite{20} states that we “have no ultimate, generally accepted expression for the energy-momentum and especially for the angular momentum, ...” Some of the frequently proposed criteria for conserved quasi-local energy-momentum expressions will play a role in our presentation: (i) they should vanish for Minkowski, (ii) give the standard linearized theory values at spatial infinity, and (iii) positive energy.

Notable quasi-local energy expressions include Komar \cite{21}, Møller’s tetrad-teleparallel expression \cite{22}, the spinor Hamiltonian boundary term 2-form associated with the Witten positive energy proof \cite{23}, the teleparallel gauge current \cite{24,25,26}, Brown and York \cite{27}, Bičák, Katz and Lynden-Bell \cite{28} (equivalent to our favored expression discussed below), our 2-parameter set \cite{29,30,31,32}, Tung’s expression from his quadratic spinor Lagrangian \cite{33}, Kijowski’s “free energy” \cite{34}, Epp’s energy \cite{35}, the “new superpotential” of Petrov-Katz \cite{36}, the (positive) energy of Liu-Yau \cite{37} (the same as Kijowski’s free energy), and the Wang-Yau mass \cite{38}.

5 The covariant Hamiltonian formulation

To provide a framework for our discovery we briefly review some parts of our covariant Hamiltonian formulation \cite{18,29,30,31,32}. From a first order Lagrangian $\mathcal{L} = d\varphi \wedge p - A(\varphi, p)$ for a $k$-form field $\varphi$, by variation one obtains a pair of first order dynamical equations:

$$\delta \mathcal{L} = d(\delta \varphi \wedge p) + \delta \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \delta p. \tag{16}$$

From an infinitesimal diffeomorphism, a “local translation” along a vector field $Z$, with the aid of the formula for the Lie derivative acting on forms, $\mathcal{L}_Z \equiv i_Z d + di_Z$, one obtains an identity by replacing $\delta$ in (16) with $\mathcal{L}_Z$:

$$di_Z \mathcal{L} \equiv \mathcal{L}_Z \mathcal{L} \equiv d(\mathcal{L}_Z \varphi \wedge p) + \mathcal{L}_Z \varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \mathcal{L}_Z p. \tag{17}$$

This leads to the identification of the Noether current 3-form, which, moreover, is the Hamiltonian density:

$$\mathcal{H}[Z] := \mathcal{L}_Z \varphi \wedge p - i_Z \mathcal{L}. \tag{18}$$
It satisfies the differential identity

$$-dH[Z] \equiv \mathcal{L}Z\varphi \wedge \frac{\delta \mathcal{L}}{\delta \varphi} + \frac{\delta \mathcal{L}}{\delta p} \wedge \mathcal{L}Zp,$$

(19)

so it is a conserved “current” on shell (i.e., when the field equations are satisfied). This 3-form is linear in the displacement vector field up to a total differential:

$$H[Z] \equiv \zeta iZ\varphi \wedge dp + \zeta d\varphi \wedge iZp + iZA + d(iZ\varphi \wedge p) =: Z^\mu H_\mu + dB[Z].$$

(20)

Comparing the differential of the latter form, $dH[Z] \equiv dZ^\mu \wedge H_\mu + Z^\mu dH_\mu,$ with (19) shows that from local diffeomorphism invariance $H_\mu$ is proportional to field equations and thus vanishes on shell; hence the translational Noether current conservation reduces to a differential identity between Euler-Lagrange expressions (an instance of Noether’s second theorem). The value of the Hamiltonian $H(Z, V)$, which determines the energy-momentum, is thus quasi-local (associated with a closed 2-surface):

$$-P(Z, V) = H(Z, V) := \int_V H[Z] = \oint_{\partial V} B[Z].$$

(21)

As with other Noether conserved currents, without loss of the conservation property one can add a “curl” to $H(Z)$, i.e., the boundary term inherited from the Lagrangian, $iZ\varphi \wedge p$, can be adjusted—this changes the boundary conditions that are obtained from the boundary term in the variation of the Hamiltonian.

That $H(Z, V)$, the integral of $H[Z]$, is indeed the Hamiltonian follows from the easily established identity $31,32$:

$$\delta H[Z] = -\delta \varphi \wedge \mathcal{L}Zp + \mathcal{L}Z\varphi \wedge \delta p + dZ(\delta \varphi \wedge p) - iZ(\delta \varphi \wedge \mathcal{L} \delta \varphi + \mathcal{L} \delta p \wedge \delta p).$$

(22)

On shell, the total differential term gives rise to a boundary term that vanishes for the boundary condition $\delta \varphi |_{S} = 0$. With a modified boundary term, $B'[Z]$, the Hamiltonian will yield a modified boundary condition.

We were led to a set of general boundary terms which are linear in $\Delta \varphi := \varphi - \bar{\varphi}$, $\Delta p := p - \bar{p}$, where $\bar{\varphi}, \bar{p}$ are non-dynamic reference values. For Einstein’s GR our “covariant-symplectic” Hamiltonian boundary terms are

$$2\kappa B[Z](a, b) = \Delta \Gamma^\alpha \beta \wedge iZ[(1 - a)\eta_\alpha \beta + a\bar{\eta}_\alpha \beta] + [(1 - b)\nabla_\beta Z^\alpha + b\nabla_\beta Z^\alpha] \Delta \eta_\alpha \beta,$$

(23)

where $\eta^{\alpha \beta \ldots} := *(\vartheta^\alpha \wedge \vartheta^\beta \ldots)$, $\vartheta^\alpha$ is the coframe and $\Gamma^\alpha \beta$ is the connection one-form. Here one may freely choose $a, b$. The choices $(0, 0), (0, 1), (1, 0), (1, 1)$ select essentially Dirichlet (fixed field) or Neumann (fixed momentum) boundary conditions for the space and time parts of the fields $39$. For asymptotically flat spaces the Hamiltonian with these boundary term expressions is well defined, i.e., the boundary term in its variation vanishes and the quasi-local

\textsuperscript{5} Where $\zeta := (-1)^k$. 
quantities are well defined—at least on the phase space of fields satisfying Regge-Teitelboim [40] like asymptotic parity and fall-off conditions.

Our preferred GR Hamiltonian boundary term is

\[ 2\kappa \mathcal{B}[Z] = \Delta \Gamma^\alpha_\beta \wedge i_Z \eta^\beta_\alpha + \nabla_\beta Z^\alpha \Delta \eta^\beta_\alpha. \] (24)

It corresponds to holding the metric fixed on the boundary. Like many other boundary term choices, at spatial infinity it gives the ADM, MTW [6] §20.2, Regge-Teitelboim [40], Beig-Ó Murchadha [41], Szabados [42] energy, momentum, angular-momentum, center-of-mass. One of its special virtues is that at null infinity it gives the Bondi-Trautman energy and energy flux [31].

6 The reference

Regarding the second ambiguity inherent in the discussed quasi-local energy-momentum expressions, the choice of reference: one could use any physically appropriate reference, preferably a very symmetrical one. If the chosen reference is a space of constant curvature (de Sitter, anti-de Sitter or Minkowski) one has 10 reference Killing vector fields that can be used for the vector Z to define all 10 quasi-local quantities: energy-momentum, angular momentum/center-of-mass. One of its special virtues is that at null infinity it gives the Bondi-Trautman energy and energy flux [31].

In a neighborhood of the desired spacelike boundary 2-surface \( S \), any 4 smooth functions \( y^\mu = y^\mu(x^\nu) \), \( \mu = 0, 1, 2, 3 \) with \( dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \neq 0 \) define a Minkowski reference \( \tilde{g} = -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2 \).

Locally this defines an embedding of a neighborhood of \( S \) into a Minkowski space. A Killing field of \( \tilde{g} \) has the infinitesimal Poincaré transformation form: \( \tilde{Z}^\mu = \alpha^\mu + \lambda^\mu_\nu y^\nu \), where \( \alpha^\mu \) and \( \lambda^\mu_\nu = \lambda_\mu^\nu \) are constants. Then the Hamiltonian boundary term quasi-local quantity integral has the form

\[ E(\tilde{Z}, S) = \oint_S \mathcal{B}[\tilde{Z}] = -\alpha^\mu p_\mu(S) + \frac{1}{2} \lambda^\mu_\nu J^\mu_\nu(S), \] (25)

giving the quasi-local energy-momentum and angular momentum values associated with this reference. With suitable fall-offs, the integrals \( p_\mu(S) \), \( J^\mu_\nu(S) \) for our expressions agree with the standard expressions asymptotically [6,40,41,42].

The reference metric on the dynamical space has the components \( \tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu} = \delta_{\alpha\beta} y^\alpha_\mu y^\beta_\nu \), and the reference connection one-form (with \( dx^\mu = x^\alpha_\mu dx^\alpha \), \( dy^\alpha = y^\alpha_\nu dx^\nu \)) is

\[ \tilde{\Gamma}^\alpha_\beta = x^\alpha_\mu (\tilde{\Gamma}^\mu_\nu y^\nu_\beta + dy^\mu_\beta) = x^\alpha_\mu dy^\mu_\beta, \] (26)
since the Minkowski connection coefficients \( \bar{\Gamma}^{\mu}_{\nu \rho} \) vanish in the \( y^\rho \) frame. With \( \bar{Z}^\mu \) being a translational Killing field of the Minkowski reference, the second term in (24) vanishes, then our quasi-local expression takes the form

\[
2\kappa B(\bar{Z}) = \bar{Z}^\nu x^\mu (\bar{\Gamma}^{\alpha}_{\beta \rho} - x^{\alpha \rho} \mathrm{d}y^\beta_{\beta}) \wedge \eta_{\nu \alpha} \beta 
\]

\[
\equiv \bar{Z}^\nu \Gamma^{\alpha}_{\beta \rho} \wedge \eta_{\nu \alpha} \beta \equiv \bar{Z}^\nu |g|^{1/2} g_{\rho \sigma} \Gamma^{\alpha}_{\rho \beta \sigma} \frac{1}{2} dS_{\mu \nu} .
\]

Thus, when expressed in the Minkowski reference coordinate frame, it reduces to the Freud superpotential (6).

Our first criterion for fixing the reference is 4D isometric matching to a Minkowski reference on the boundary of the region. The hard part is the isometric embedding of the 2D surface \( S \) into Minkowski space (Wang and Yau have made deep investigations into this \[38\]). For isometric matching of the 2-surface, in terms of quasi-spherical foliation adapted coordinates \( t, r, \theta, \phi \) with \( i, j = 1, 2, 3 \) and \( A, B = \theta, \phi \):

\[
g_{AB} = \bar{g}_{AB} = \bar{g}_{\rho \sigma} y^\rho A y^\sigma B = -y^0 A y^0 B + \delta_{ij} y^i A y^j B
\]

on \( S \). From a classic closed 2-surface into \( \mathbb{R}^3 \) embedding theorem, there is a unique embedding (but no explicit formula) — as long as the choice of \( S \) and \( y^0 \) are such that \( \bar{g}^\rho_{AB} := g_{AB} + y^0 A y^0 B \) is convex on \( S \). If this is not satisfied the isometric embedding is not guaranteed.

Complete 4D isometric matching on \( S \) was proposed in 2000 by Epp \[35\] and by Szabados\[6\]. There are 10 constraints: \( g_{\alpha \beta}|S = \bar{g}_{\alpha \beta}|S = \bar{g}_{\rho \sigma} y^\rho \alpha y^\sigma \beta|S \) and 12 embedding functions on a constant \( t, r \) 2-surface: \( y^\rho (\Rightarrow y^\rho \theta, y^\rho \phi), y^\mu t, y^\mu r \). The 10 constraints split into 3 for the already discussed 2D isometric matching of \( g_{AB} \) whereby \( y^0 \) determines \( y^1, y^2, y^3 \) on \( S \), and 7 algebraic equations that determine the other embedding variables. One can take \( y^0, y^0 r \) as the embedding control variables; \( y^0 r \) controls a boost in the normal plane \[43,44\].

An alternative regards 4D isometric matching in terms of orthonormal frames. The reference geometry has a frame of the form \( \bar{\hat{\vartheta}}^{\hat{\mu}} = d\bar{y}^\mu \). With 4D isometric matching one can choose a dynamical \( \vartheta^\alpha \) frame that can be Lorentz transformed to match such a reference frame on \( S \):

\[
L^\rho_{\hat{\alpha}}(p) \vartheta^\hat{\hat{\alpha}}(p) = \bar{\hat{\vartheta}}^\rho(p) = dy^\rho(p), \quad \forall p \in S .
\]

Then restricted to \( S \) one has four integrability conditions:

\[
d(L^\rho_{\hat{\alpha}} \vartheta^\hat{\hat{\alpha}})|S = 0 ,
\]

each a one-component 2-form, thus 4 restrictions on 6 Lorentz parameters, hence again 2 degrees of freedom.

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\[6\] Szabados, L.B.: in a talk at Tsinghua Univ., Hsinchu, Taiwan, July 2000.; Szabados, L.B.: “Quasi-local energy-momentum and angular momentum in GR: the covariant Lagrangian approach,” unpublished draft, 2005.
A common value

We have reviewed some of the history and surveyed many of the proposed energy expressions and have laid out the Hamiltonian perspective as a framework for clarifying the issues. With this foundation we can now easily explain our remarkable discovery.

With a 4D isometric matching reference the values of many distinct expressions coincide. This is obvious for the covariant symplectic Hamiltonian boundary terms (23), as one then has \( \eta_{\alpha\beta}|_S = \bar{\eta}_{\alpha\beta}|_S \). If the reference is Minkowski the expression (24) becomes (28), which is holonomically the Freud expression (6, 7), and orthonormally it is the teleparallel gauge current. Furthermore, with \( g_{\mu\nu}|_S = \bar{g}_{\mu\nu}|_S \) the superpotentials (8), (9), (10), (11), (14) will then coincide on \( S \). Checking other expressions we discovered a surprising concord.

For any closed 2-surface in a dynamical Riemannian spacetime, with 4D isometric matching to a Minkowski reference, there is a common quasi-local energy value for all the expressions that linearly agree with the Freud superpotential, (6) or (7), in the Minkowski limit.

Linearly Freud-like expressions having a concordant value include: Landau-Lifshitz (8), Papapetrou (9), Bergmann-Thomson (10), Goldberg’s weighted Freud densities (11), Møller’s 1961 tetrad-teleparallel expression,

\[
B_{M61}(\partial_\nu) = \Gamma^{\dot{\alpha}}_{\dot{\beta}} \wedge \epsilon^\mu_{\nu\dot{\alpha}\dot{\beta}},
\]

the teleparallel gauge current [24,25,26],

\[
B_\parallel(e_\mu) = \Gamma^{\dot{\alpha}}_{\dot{\beta}} \wedge \bar{\eta}_{\mu\dot{\alpha}\dot{\beta}},
\]

Weinberg (14), the spinor Hamiltonian boundary term 2-form associated with the Witten positive energy proof [23],

\[
B_\psi = 2[\bar{\psi}\gamma_{0123}\gamma_\mu \gamma^\mu \wedge D\bar{\psi} + D\bar{\psi} \wedge \gamma_{0123}\gamma_\mu \gamma^\mu \bar{\psi}]
\equiv 2[\bar{\psi}\gamma_{0123}\gamma_\nu \gamma^\nu \wedge d\bar{\psi} + d\bar{\psi} \wedge \gamma_{0123}\gamma_\nu \gamma^\nu \bar{\psi}] + \bar{\psi}\gamma^\mu \psi \Gamma^{\dot{\alpha}}_{\dot{\beta}} \wedge \eta_{\alpha\beta\mu},
\]

for a spinor having constant components in the Minkowski reference frame on the boundary, Tung’s spinor expression [33], our 2-parameter covariant-symplectic boundary expressions [24], Bičák-Katz-Lynden-Bell [28], Petrov-Katz [36] (essentially it replaces the first term in (24) by (9), the Papapetrou superpotential), So’s 11-parameter superpotentials [16], and, as we shall explain, Wang-Yau [38].

In view of the quasi-local desiderata of (i) vanishing for Minkowski and (ii) agreeing with the standard spatial infinity linear results, this concord could have been expected, even though these various expressions differ beyond the linear order and may not agree for other than an isometric Minkowski reference. Different energy expressions can have a common value far more generally

\footnote{Here \( 70123 := 70717273 \).}
than Ref. [17] had imagined, in terms of both the class of metrics and the class of expressions.

Some pseudotensor/Hamiltonian-boundary term expressions give other values for the quasi-local energy. This includes those that do not have the aforementioned desirable spatial asymptotic linearized theory limit or do not choose the reference by embedding into Minkowski space, e.g., Møller [13], Komar, $B(Z) = \ast d(Z_\mu \partial^\mu)$, [21], the expression [31] (with the spinor boundary values needed for the Witten positivity proof), Brown-York [27] with $S$ embedded into $\mathbb{R}^3$ reference, Kijowski’s “free energy” [34], Epp [35], and Liu-Yau [37]. Also, we should mention that some well-known quasi-local proposals are not formulated in the Lagrangian/Hamiltonian framework, e.g. Penrose’s twistor expression [19, Hawking’s mass [45], and Hayward’s expression [46].

8 A best matched reference

What has been described in the previous section is not a specific reference, but a whole class of references. Within the set of isometric Minkowski references can one find a “best matched” Minkowski reference geometry? For any of the common value expressions, e.g. (27), there are 12 embedding variables subject to 10 isometric conditions (or 6 orthonormal frame parameters subject to 4 conditions). To determine the two embedding control variables, one can use the boundary term value. Its critical points are distinguished [43, 44] and can be used to select a specific reference.

There are 2 quantities which could be considered: $m^2 c^2 = -\bar{g}_{\mu \nu} \bar{p}^\mu \bar{p}^\nu = p_0^2 - p_1^2 - p_2^2 - p_3^2$ and $p_0$. Technically $m^2 c^2$ is more complicated: one is extremizing a linear combination of quadratic quantities, each an integral over $S$; this would determine the reference up to Poincaré transformations. The Lorentz “gauge” freedom could then be used to specialize to the frame with vanishing momentum: $p = 0$. In this “center-of-momentum” frame $m^2 c^2$ reduces to $p_0^2$. But the critical points of $p_0$ are also critical points of $p_0^2$. So one can get the same reference from the much more simple expression $-c p_0 = E(\partial \phi, S)$, which is a smooth function of the 2 embedding control variables.

From our 4D approach we have not found a general analytic formula for the critical points, however for the special case of axisymmetric metrics (including Kerr) we can explicitly find the critical point analytically [47]. Do suitable critical points generally exist? Can one find them? Here is a practical computational argument. Consider being given a set of data from a numerical relativity calculation. Compute the energy given by a large number of reference choices; the critical value(s) will stand out.

8.1 Wang-Yau

From another perspective, in a milestone work Wang and Yau [38] used a quasi-local expression in terms of surface geometric quantities (metric, normals, extrinsic curvatures) based on the Hamilton-Jacobi approach of Brown
and York [27] as developed in [48]. They found a way to determine an opti-
mally embedded isometric Minkowski reference analytically and thereby ob-
tained their quasi-local mass; moreover, they were able to show that their
quasi-local mass is non-negative and, furthermore, vanishes for Minkowski.
An outstanding achievement.

8.2 The link

In a recent work Liu and Yu [49] found that the expression (24) with a 4D
isometric matching Minkowski reference is closely related to the expression
used by Wang and Yau; consequently a saddle critical value of the associated
energy agrees with the Wang-Yau mass. This is not very surprising, as both
approaches start with the Einstein-Hilbert Lagrangian of GR and follow (albeit
different) paths that lead to a GR Hamiltonian boundary term. Nevertheless
this is an important link, especially since, as we indicated, all the linearly-
like-Freud expressions with a 4D isometric Minkowski reference give the same
energy value as the expression (24). We now realize that the results was not a
special property of the expression (24); one could have used any other Freud
linear expression and established the link had they thought to do so.
Hence there is, thanks to [38] (with the link established by [49]) for all
the Freud-linear expressions a specific quasi-local energy, which satisfies the
main criteria: it is non-negative and vanishes if the dynamical spacetime is
Minkowski. This simple observation is an additional result on top of our earlier
mentioned discovery.

9 Concluding discussion

We have surveyed much of the work that had been done regarding the unsettled
issue of gravitational energy. While the energy of gravitating systems cannot
be localized, with the aid of the Hamiltonian framework we explained that with
an appropriate Minkowski reference one can find a good quasi-local energy.
The quasi-local energy with nice properties referred to at the end of the
previous section could have been found even as early as 1939. It was found
only 70 years later by Wang and Yau [38] via a Hamilton-Jacobi approach. We
came to this value a few years later by a 4D covariant Hamiltonian route [43].
Einstein with his pseudotensor was close (as were many others), merely lacking
a good way to choose the coordinate reference frame on the boundary.
As noted, much effort by many people was expended in trying to find the
“best” energy-momentum expression. Our group investigated the roles of the
Hamiltonian boundary term. We found a preferred 4D covariant (reference
dependent) expression. Only then did we turn to finding a good reference.
But just taking (almost) any of the proposed expressions and looking for the
“best” reference could have led anyone directly to this energy.
The complaint was that gravitational energy is ill defined: (i) no unique
expression, (ii) reference frame dependent expressions with no unique reference
frame. But we find that: (a) one can generally have a 4D isometric Minkowski reference, (b) with such a reference the quasi-local expressions in a large class give the same energy-momentum (and via (25) angular momentum), (c) and one can find a “best matched” reference.

As the textbook said: there is no proper local energy-momentum density—not now it is understood that energy-momentum is not local but rather quasi-local—and, although much effort was expended on finding the best expression, indeed there still is no single accepted quasi-local expression. However, we discovered that a consensus on the expression is not needed—associated with any region for a large class of reasonable proposed expressions (those which have the desired asymptotics) there is a well defined energy value (fixed by the 4D best-matched isometric Minkowski reference), which does have the desired properties.

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References

1. Einstein, A.: Der Feldgleichungen der Gravitation Königlich Preussische Akademie der Wissenschaften (Berlin) Sitzungsberichte 844–847 (1915); The Field Equations of gravitation Doc. 25 in The Collected Papers of Albert Einstein Vol. 6. Princeton Univ. Press, Princeton (1997), online at http://einsteinpapers.press.princeton.edu
2. Schrödinger, E.: Die Energiekomponenten des Gravitationsfeldes. Phys. Zeit. 19, 4 (1918); Spacetime structure. Cambridge University Press, Cambridge (1950)
3. Rowe, D.E.: The Göttingen Response to General Relativity and Emmy Noether’s Theorems. In: Gray, J. (ed.) The Symbolic Universe: Geometry and Physics 1890–1930, pp 189–233. Oxford University Press, Oxford (1999)
4. Kosmann-Schwarzbach, Y.: The Noether Theorems: Invariance and Conservation Laws in the Twentieth Century. Springer, New York (2011)
5. Chen, C.M., Liu, J.L., Nester, J.M.: Gravitational energy is well defined. Int. J. Mod. Phys. D 27, 1847017 (2018) Honorable Mention Gravity Research Foundation Essay (2018) doi:10.1142/S021827181847017X [arXiv:1805.07692 [gr-qc]]
6. Misner, C.W., Thorne, K., Wheeler, J.A.: Gravitation. Freeman, San Francisco (1973)
7. Freud, Ph.: Uber Die Ausdrücke Der Gesamtenergie Und Des Gesamtimpulses Eines Materiellen Systems in Der Allgemeinen Relativitätstheorie. Ann. Math. 40, 417 (1939)
8. Böhmer, C.G., Hehl, F.W.: Freud superpotential in general relativity and in Einstein-Cartan theory. Phys. Rev. D 97, 044028 (2018) doi:10.1103/PhysRevD.97.044028 [arXiv:1712.05208 [gr-qc]]
9. Landau, L.D., Lifshitz, E.M.: The Classical Theory of Fields. 2nd ed. Addison-Wesley, Reading, MA (1962)
10. Papapetrou, A.: Einstein’s theory of gravitation and flat space. Proc. Roy. Irish Acad. (Sect. A) 52, 11–23 (1948); Gupta, S.N.: Gravitation and Electromagnetism. Phys. Rev. 96, 1683 (1954) doi:10.1103/PhysRev.96.1683; Bak, D., Cangemi, D., Jackiw, R.;

* Much larger than had found previously in [17].
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Energy-momentum conservation in gravity theories. Phys. Rev. D 49, 5173–5181 (1994); Erratum: 52 3753 (1995) doi:10.1103/PhysRevD.52.3753, 10.1103/PhysRevD.49.5173 [hep-th/9310025].

11. Bergmann, P.G.; Thomson, R.: Spin and angular momentum in general relativity. Phys. Rev. 89, 400–407 (1953) doi:10.1103/PhysRev.89.400

12. Goldberg, J.N.: Conservation Laws in General Relativity. Phys. Rev. 111, 315 (1958) doi:10.1103/PhysRev.111.315

13. Møller, C.: On the localization of the energy of a physical system in general theory of relativity. Ann. Phys. (N.Y.) 4, 347–71 (1958) doi:10.1016/0003-4916(58)90053-8

14. Weinberg, S.: Gravitation and Cosmology. Wiley, New York (1972)

15. So, L.L., Nester, J.M., Chen, H.: Energy-momentum density in small regions: The Classical pseudotensors. Class. Quantum Gravity 26, 085004 (2009) doi:10.1088/0264-9381/26/8/085004 [arXiv:0901.3884 [gr-qc]].

16. So, L.L., Nester, J.M.: Positive small-vacuum-region gravitational-energy expressions. Phys. Rev. D 79, 084028 (2009) doi:10.1103/PhysRevD.79.084028 [arXiv:0901.2400 [gr-qc]].

17. Aguirregabiria, J.M., Chamorro, A., Virbhadra, K.S.: Energy and angular momentum of charged rotating black holes. Gen. Relativ. Gravit. 28, 1393 (1996) doi:10.1007/BF02109529 [gr-qc/9501002]; Virbhadra, K.S.: Naked singularities and Seifert's conjecture. Phys. Rev. D 60, 104041 (1999) doi:10.1103/PhysRevD.60.104041 [gr-qc/9809077].

18. Chang, C.C., Nester, J.M., Chen, C.M.: Pseudotensors and Quasilocal Gravitational Energy Momentum. Phys. Rev. Lett. 83, 1897–1901 (1999) doi:10.1103/PhysRevLett.83.1897 [gr-qc/9809040].

19. Penrose, R.: Quasi-local mass and angular momentum in general relativity. Proc. R. Soc. London Ser. A 381, 53 (1982) doi:10.1098/rspa.1982.0058

20. Szabados, L.B.: Quasi-Local Energy-Momentum and Angular Momentum in General Relativity. Living Rev. Relativ. 12, 4 (2009) doi:10.12942/lrr-2009-4

21. Komar, A.: Covariant Conservation Laws in General Relativity. Phys. Rev. 113, 934–36 (1959) doi:10.1103/PhysRev.113.934

22. Møller. C.: Further Remarks on the Localization of the Energy in the General Theory of Relativity. Ann. Phys. 12, 118–33 (1961) doi:10.1016/0003-4916(61)90148-8; Conservation laws and absolute parallelism in general relativity. Mat. Fys. Dan. Vid. Selsk. 1, No. 10 1–50 (1961)

23. Nester, J.M.: A New gravitational energy expression with a simple positivity proof. Phys. Lett. A 83, 241 (1981) doi:10.1016/0375-9601(81)90972-5; The Gravitational Hamiltonian. In: Flaherty, F. (Ed.) Asymptotic Behavior of Mass and Space-time geometry. Lecture Notes in Physics 202, pp 155–163, Springer (1984)

24. Duan, Y.S., Zhang, J.Y.: Acta Physica Sinica 19, 689 (1963); Duan, Y.S.: Gauge Theories of Gravitation. Commun. Theor. Phys. 4, 661–674 (1985)

25. Wallner, R.P.: Superpotential-Forms and Total Energy/Momentum in General Relativity. Acta Physica Austriaca 52, 121–124 (1980)

26. de Andrade, V.C., Guillen, L.C.T., Pereira, J.G.: Gravitational energy momentum density in teleparallel gravity. Phys. Rev. Lett. 84, 4533 (2000) doi:10.1103/PhysRevLett.84.4533 [gr-qc/0003100].

27. Brown, J.D., York, J.W. Jr.: Quasilocal energy and conserved charges derived from the gravitational action. Phys. Rev. D 47, 1407–1419 (1993) doi:10.1103/PhysRevD.47.1407 [gr-qc/9209012].

28. Lynden-Bell, D., Katz, J., Bičák, J.: Mach's Principle from the Relativistic Constraint Equations. Mon. Not. R. Astron. Soc. 272, 150 (1995) doi:10.1093/mnras/272.1.150; Katz, J., Bičák, J., Lynden-Bell, D.: Relativistic conservation laws and integral constraints for large cosmological perturbations. Phys. Rev. D 55, 5957 (1997) doi:10.1103/PhysRevD.55.5957 [gr-qc/9504041].

29. Chen, C.M., Nester, J.M., Tung, R.S.: Quasilocal energy momentum for gravity theories. Phys. Lett. A 203, 5 (1995) doi:10.1016/0375-9601(95)92844-T [gr-qc/9411048].

30. Chen, C.M., Nester, J.M.: Quasilocal quantities for GR and other gravity theories. Class. Quantum Gravity 16, 1279 (1999) doi:10.1088/0264-9381/16/4/018 [gr-qc/9809020]; A Symplectic Hamiltonian derivation of quasilocal energy momentum for GR. Gravit. Cosmol. 6, 257 (2000) [gr-qc/0001088].
31. Chen, C.M., Nester, J.M., Tung, R.S.: The Hamiltonian boundary term and quasi-local energy flux. Phys. Rev. D 72, 104020 (2005) doi:10.1103/PhysRevD.72.104020 [gr-qc/0508026].
32. Chen, C.M., Nester, J.M., Tung, R.S.: Gravitational energy for GR and Poincaré gauge theories: a covariant Hamiltonian approach. Int. J. Mod. Phys. 24, 1530026 (2015) doi:10.1142/S0218271815300268 [arXiv:1507.07300 [gr-qc]]; In: Ni, W.T. (Ed.) One Hundred Years of General Relativity: From Genesis and Empirical Foundations to Gravitational Waves, Cosmology and Quantum Gravity Vol 1. World Scientific, Singapore p I-187 (2017)
33. Nester, J.M., Tung, R.S.: A Quadratic spinor Lagrangian for general relativity. Gen. Relativ. Gravit. 27, 115 (1994) doi:10.1007/BF02107951 [gr-qc/9407004].
34. Kijowski, J.: A simple derivation of canonical structure and quasi-local Hamiltonians in general relativity. Gen. Relativ. Gravit. 29, 307–343 (1997) doi:10.1023/A:1010268818255
35. Epp, R.J.: Angular momentum and an invariant quasilocal energy in general relativity. Phys. Rev. D 62, 124018 (2000) doi:10.1103/PhysRevD.62.124018 [gr-qc/0003035].
36. Petrov, A.N., Katz, J.: Conserved currents, superpotentials and cosmological perturbations. Proc. Roy. Soc. Lond. A 458, 319–337 (2002) doi:10.1098/rspa.2001.0865 [gr-qc/9911025].
37. Liu, C.C.M., Yau, S.T.: Positivity of quasilocal mass. Phys. Rev. Lett. 90, 231102 (2003) doi:10.1103/PhysRevLett.90.231102 [gr-qc/0303019].
38. Wang, M.T., Yau, S.T.: Quasilocal mass in general relativity. Phys. Rev. Lett. 102, 021101 (2009) doi:10.1103/PhysRevLett.102.021101 [arXiv:0801.1174 [gr-qc]]; Isometric embeddings into the Minkowski space and new quasi-local mass. Commun. Math. Phys. 288, 919 (2009) doi:10.1007/s00220-009-0745-0 [arXiv:0805.1370 [math.DG]].
39. So, L.L.: A Modification of the Chen-Nester quasilocal expressions. Int. J. Mod. Phys. D 16, 875 (2007) doi:10.1142/S0218271807010444 [gr-qc/0605149].
40. Regge, T., Teitelboim, C.: Role of Surface Integrals in the Hamiltonian Formulation of General Relativity. Ann. Phys. (N.Y.) 88, 286 (1974) doi:10.1016/0003-4916(74)90404-7
41. Beig, R., Ó Murchadha, N.: The Poincaré group as the symmetry group of canonical general relativity. Ann. Phys. (N.Y.) 174, 463 (1987) doi:10.1016/0003-4916(87)90037-6
42. Szabados, L.B.: On the roots of the Poincaré structure of asymptotically flat spacetimes. Class. Quantum Gravity 20, 2627 (2003) doi:10.1088/0264-9381/20/13/312 [gr-qc/0302033].
43. Nester, J.M., Chen, C.M., Liu, J.L., Sun, G.: A reference for the covariant Hamiltonian boundary term. Springer Proc. Phys. 157, 177 (2014) doi:10.1007/978-3-319-06761-2_22 [arXiv:1210.6148 [gr-qc]].
44. Sun, G., Chen, C.M., Liu, J.L., Nester, J.M.: A Reference for the Quasi-Local Gravitational Energy and Angular Momentum. Chin. J. Phys. 53, 110107 (2015) doi:10.6122/CJIP.20150819 [arXiv:1301.1510 [gr-qc]].
45. Sun, G., Chen, C.M., Liu, J.L., Nester, J.M.: An Optimal Choice of Reference for the Quasi-Local Gravitational Energy and Angular Momentum. Chin. J. Phys. 52, 111 (2014) doi:10.6122/CJIP.20141032 [arXiv:1309.4109 [gr-qc]].
46. Hawking, S.W., Horowitz, G.T.: The Gravitational Hamiltonian, action, entropy and surface terms. Class. Quantum Gravity 13, 1487 (1996) doi:10.1088/0264-9381/13/6/017 [gr-qc/9501014].