Shortest Non-trivial Cycles in Directed and Undirected Surface Graphs

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Abstract
Let \( G \) be a graph embedded on a surface of genus \( g \) with \( b \) boundary cycles. We describe algorithms to compute multiple types of non-trivial cycles in \( G \), using different techniques depending on whether or not \( G \) is an undirected graph. If \( G \) is undirected, then we give an algorithm to compute a shortest non-separating cycle in \( 2^{O(g)} n \log \log n \) time. Similar algorithms are given to compute a shortest non-contractible or non-null-homologous cycle in \( 2^{O(g+b)} n \log \log n \) time. Our algorithms for undirected \( G \) combine an algorithm of Kutz with known techniques for efficiently enumerating homotopy classes of curves that may be shortest non-trivial cycles.

Our main technical contributions in this work arise from assuming \( G \) is a directed graph with possibly asymmetric edge weights. For this case, we give an algorithm to compute a shortest non-contractible cycle in \( G \) in \( O((g^2 + gb) n \log n) \) time. In order to achieve this time bound, we use a restriction of the infinite cyclic cover that may be useful in other contexts. We also describe an algorithm to compute a shortest non-null-homologous cycle in \( G \) in \( O((g^2 + gb) n \log n) \) time, extending a known algorithm of Erickson to compute a shortest non-contractible cycle. In both the undirected and directed cases, our algorithms improve the best time bounds known for many values of \( g \) and \( b \).

1 Introduction
There is a long line of work on computing shortest non-trivial cycles in surface embedded graphs. Cabello and Mohar [11] claim that finding short non-trivial cycles is arguably one of the most natural problems for graphs embedded on a surface. Additionally, finding these cycles has many benefits both for theoretical combinatorial problems [1, 18, 38, 42] and more practical applications in areas such as graphics and graph drawing [3, 23, 31, 33, 39, 49].

Researchers have focused primarily on finding short non-contractible and non-separating cycles. Consider a graph \( G \) embedded on a surface \( \Sigma \). Informally, a cycle in \( G \) is contractible if its image on \( \Sigma \) can be continuously deformed to a single point. The cycle is separating if removing its image from \( \Sigma \) disconnects \( \Sigma \). Every non-separating cycle is non-contractible, but there may be non-contractible cycles that are separating. See Figure 1 for examples.

The history of non-trivial cycles in undirected graphs goes back several years to a result of Itai and Shiloach [34]. They give an \( O(n^2 \log n) \) time algorithm to find a shortest non-trivial cycle in an annulus as a subroutine for computing minimum \( s,t \)-cuts in planar graphs. Their result has seen several improvements, most recently by Italiano et al. [28, 35, 44].

Thomassen [47] gave the first efficient algorithm for computing non-trivial cycles on surfaces with arbitrary genus. His algorithm runs in \( O(n^3) \) time and relies on a property of certain families of cycles known as the 3-path condition; see also Mohar and Thomassen [42, Chapter 4]. Erickson and HarPeled [23] gave an \( O(n^3 \log n) \) time algorithm, which remains the fastest known for graphs of arbitrary genus. Cabello and Mohar [11] gave the first results parameterized by genus, and Kutz [41] showed it is possible to find short non-trivial cycles in time near-linear in the number of vertices if we allow an exponential dependence on the genus. Kutz’s algorithm requires searching \( g^{O(g)} \) subsets of the universal cover. Cabello, Chambers, and Erickson [5, 6] later showed the near-linear time dependence is possible with only a polynomial dependence on the genus by avoiding use of the universal cover. The current best running time in terms of the number of vertices is \( g^{O(g)} n \log \log n \) due to a modification to Kutz’s algorithm by Italiano et al. [35]. For other results related to finding interesting cycles on surfaces, see [4, 8–10, 12, 14, 22, 27].

Unfortunately, all of the above results rely on properties that exist only in undirected graphs; shortest paths intersect at most once (assuming uniqueness), and the reversal of any shortest path is a shortest path. Due to the difficulty in avoiding these assumptions, there are few results for finding shortest non-trivial cycles in directed surface graphs, and all of these results are relatively recent. Befittingly, the short history of these results appears to coincide nicely with the history given above for undirected graphs.

Janiga and Kouček [36] gave the first near-linear time algorithm for computing a shortest non-
The first algorithm computes shortest non-separating 3-path condition and balanced separators, respectively. \[\text{complexity but only a polynomial dependence on}\]

algorithms with a near-linear dependency on graph mentioned in this paragraph, we ideally would like \[\text{graphs}\]

flows and minimum cuts in surface embedded \[\text{Similar trends appear in the computation of maximum}\]

dependence in the complexity of the embedded graph. \[\text{exponential dependence in the genus, but near-linear}\]

others supplemented them with algorithms with \[\text{non-separating and non-contractible cycles, and}\]
directed graph settings, researchers presented \[\text{1.1 Our results}\]

In both the undirected and \[\text{large)}\] subset of the universal cover.  \[\text{algorithm in}\]

g \[\text{non-contractible cycles (which may be separating)}\]
in \[g^{O(g)} n \log n\] time in a manner similar to Kutz’s algorithm \[41\], by lifting the graph to a finite (but large) subset of the universal cover.

\[1\]Unfortunately, their minimum cut algorithm has a subtle error \[37\] which may lead to an incorrect result when the minimum \[t, s\]-cut is smaller than the minimum \[s, t\]-cut.

genus. Of course, we are also interested in pushing down the dependence on graph complexity even if it means sacrificing a bit in the genus dependency when \(g\) is sufficiently small.

Our first result is improved algorithms for computing non-trivial cycles in undirected surface graphs. Our algorithms run in \(2^{O(g)} n \log n\) time and can be used to find shortest non-separating, non-contractible, or non-null-homologous cycles. Informally, a non-null-homologous cycle is one that is either non-separating or separates a pair of boundary cycles. These algorithms improve the running times achieved by Italiano et al. \[35\] for finding shortest non-separating and non-contractible cycles and show that it is possible to take advantage of the universal cover as in Kutz’s algorithm in order to minimize the dependency on \(n\), without searching a super-exponential in \(g\) number of subsets of the covering space. For surfaces with \(b\) boundary cycles, the shortest non-contractible and non-null-homologous cycle algorithms run in time \(2^{O(g)} n \log \log n\), while the shortest non-separating cycle algorithm continues to run in \(2^{O(g)} n \log \log n\) time. The main idea behind these algorithms is to construct fewer subsets of the universal cover by only constructing subsets corresponding to certain weighted triangulations of a dualized polygonal schema as in \[12, 14\]. These algorithms are described in Section 3.

Next, we sketch an algorithm to compute a shortest non-null-homologous cycle in a directed surface graph in \(O((g^2 + g b) n \log n)\) time. This algorithm is actually a straightforward extension to Erickson’s algorithm for computing shortest non-separating cycles \[21\], but we must work out some non-trivial details for the sake of completeness. The key change to Erickson’s algorithm is that we compute shortest paths in an additional \(O(b)\) copies of a covering space defined using shortest paths between boundary cycles. These additional computations will find a shortest non-null-homologous cycle if all shortest non-null-homologous cycles are separating. This algorithm is given in

Figure 1. Left: A contractible cycle on \(\Sigma\). Center: A non-contractible but separating cycle on \(\Sigma\). Right: A non-contractible and non-separating cycle on \(\Sigma\).
Section 4. Along with being an interesting result in its own right, we use this algorithm as a subroutine for our primary result described below.

Our final, primary, and most technically interesting result is an $O(g^3 n \log n)$ time algorithm for computing shortest non-contractible cycles in directed surface graphs, improving the result of Erickson [21] for all positive $g$ and showing it is possible to have near-linear dependency in graph complexity without suffering an exponential dependency on genus. On a surface with $b$ boundary cycles, our algorithm runs in $O((g^3 + gb)n \log n)$ time. In order to achieve this running time, we choose to forgo using a subset of the universal cover in favor of subsets of a different covering space known as the infinite cyclic cover. If any shortest non-contractible cycle is non-separating, then the algorithm of Erickson [21] will find such a cycle in $O(g^2 n \log n)$ time. On the other hand, if any shortest non-contractible cycle is separating, then it will lift to a non-null-homologous cycle in the subset of the infinite cyclic cover if we lift the graph to the covering space in the correct way.

2 Preliminaries

We begin by recalling several useful definitions related to surface-embedded graphs. For further background, we refer the reader to Gross and Tucker [30] or Mohar and Thomassen [42] for topological graph theory, and to Hatcher [32] or Stillwell [46] for surface topology and homology. We adopt the presentation of our terminology and notation directly from previous works [14, 21, 22, 24, 25].

2.1 Surfaces and Curves A surface (more formally, a 2-manifold with boundary) is a compact Hausdorff space in which every point has an open neighborhood homeomorphic to either the plane $\mathbb{R}^2$ or a closed halfplane $\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0 \}$. The points with halfplane neighborhoods make up the boundary of the surface; every component of the boundary is homeomorphic to a circle. A surface is non-orientable if it contains a subset homeomorphic to the Möbius band, and orientable otherwise. For this paper, we consider only compact, connected, orientable surfaces.

A path in a surface $\Sigma$ is a continuous function $p : [0, 1] \rightarrow \Sigma$. A loop is a path whose endpoints $p(0)$ and $p(1)$ coincide; we refer to this common endpoint as the basepoint of the loop. An arc is a path internally disjoint from the boundary of $\Sigma$ whose endpoints lie on the boundary of $\Sigma$. A cycle is a continuous function $\gamma : S^1 \rightarrow \Sigma$; the only difference between a cycle and a loop is that a loop has a distinguished basepoint. We say a loop $\ell$ and a cycle $\gamma$ are equivalent if, for some real number $\delta$, we have $\ell(t) = \gamma(t + \delta)$ for all $t \in [0, 1]$. We collectively refer to paths, loops, arcs, and cycles as curves. A curve is simple if it is injective; we usually do not distinguish between simple curves and their images in $\Sigma$. A simple curve $p$ is separating if $\Sigma \setminus p$ is disconnected.

The reversal $rev(p)$ of a path $p$ is defined by setting $rev(p)(t) = p(1 - t)$. The concatenation $p \cdot q$ of two paths $p$ and $q$ with $p(1) = q(0)$ is the path created by setting $(p \cdot q)(t) = p(2t)$ for all $t \leq 1/2$ and $(p \cdot q)(t) = q(2t - 1)$ for all $t \geq 1/2$. Finally, let $p[x, y]$ denote the subpath of a path $p$ from point $x$ to point $y$.

The genus of a surface $\Sigma$ is the maximum number of disjoint simple cycles in $\Sigma$ whose complement is connected. Up to homeomorphism, there is exactly one such surface with any genus $g \geq 0$ and any number of boundary cycles $b \geq 0$; the Euler characteristic $\chi$ of this surface is $\chi := 2 - 2g - b$.

2.2 Graph Embeddings An embedding of an undirected graph $G$ on a surface $\Sigma$ maps vertices to distinct points and edges to simple, interior-disjoint paths. The faces of the embedding are maximal connected subsets of $\Sigma$ that are disjoint from the image of the graph. An embedding is cellular if each of its faces is homeomorphic to the plane; in particular, in any cellular embedding, each component of the boundary of $\Sigma$ must be covered by a cycle of edges in $G$. Euler's formula implies that any cellularly embedded graph with $n$ vertices, $m$ edges, and $f$ faces lies on a surface with Euler characteristic $\chi = n - m + f$, which implies that $m = O(n + g)$ and $f = O(n + g)$ if the graph is simple. We consider only such cellular embeddings of genus $g = O(\sqrt{n})$, so that the overall complexity of the embedding is $O(n)$.

Any cellular embedding in an orientable surface can be encoded combinatorially by a rotation system, which records the counterclockwise order of edges incident to each vertex. Two paths or cycles in a combinatorial surface cross if no continuous infinitesimal perturbation makes them disjoint; if such
a perturbation exists, then the paths are non-crossing.

We redundantly use the term arc to refer to a walk in the graph whose endpoints are boundary vertices. Likewise, we use the term cycle to refer to a closed walk in the graph. Cutting a combinatorial surface along a cycle or arc modifies both the surface and the embedded graph. For any combinatorial surface \( S = (\Sigma, G) \) and any simple cycle or arc \( \gamma \) in \( G \), we define a new combinatorial surface \( S \setminus \gamma \) by taking the topological closure of \( \Sigma \setminus \gamma \) as the new underlying surface; the new embedded graph contains two copies of each vertex and edge of \( \gamma \), each bordering a new boundary.

Any undirected graph \( G \) embedded on a surface \( \Sigma \) without boundary has a dual graph \( G^* \), which has a vertex \( f^* \) for each face \( f \) of \( G \), and an edge \( e^* \) for each edge \( e \) in \( G \) joining the vertices dual to the faces of \( G \) that \( e \) separates. The dual graph \( G^* \) has a natural cellular embedding in \( \Sigma \), whose faces corresponds to the vertices of \( G \). For any subgraph \( F = (U, D) \) of \( G = (V, E) \), we write \( G \setminus F \) to denote the edge-complement \( (V, E \setminus D) \). We also abuse notation by writing \( F^* \) to denote the subgraph of \( G^* \) corresponding to any subgraph \( F \) of \( G \).

A tree-cotree decomposition \((T, L, C)\) of an undirected graph \( G \) embedded on a surface without boundary is a partition of the edges into three disjoint subsets; a spanning tree \( T \) of \( G \), a spanning cotree \( C \) (the dual of a spanning tree \( C^* \) of \( G^* \)), and leftover edges \( L = G \setminus (T \cup C) \). Euler’s formula implies that in any tree-cotree decomposition, the set \( L \) contains exactly \( 2g \) edges [19]. The definitions for dual graphs and tree-cotree decompositions given above extend to surfaces with boundary, but we do not require these extensions in this paper.

For some of the problems we consider, the input is actually a directed edge-weighted graph \( G \) with a cellular embedding on some surface. We use the notation \( u \rightarrow v \) to denote the directed edge from vertex \( u \) to vertex \( v \). Without loss of generality, we consider only symmetric directed graphs, in which the reversal \( v \rightarrow u \) of any edge \( u \rightarrow v \) is another edge, possibly with infinite weight. We also assume that in the cellular embedding, the images of any edge in \( G \) and its reversal coincide (but with opposite orientations). Thus, like Cabello et al. [7] and Erickson [21], we implicitly model directed graphs as undirected graphs with asymmetric edge weights. Duality can be extended to directed graphs [13], but the results in this paper do not require this extension.

Let \( p = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k \) be a simple directed cycle or arc in an embedded graph \( G \). We say an edge \( u \rightarrow v \) enters \( p \) from the left (resp. right) if the vertices \( v_{i-1}, u, \) and \( v_{i+1} \) (module \( k \) in the case of a cycle) are ordered clockwise (resp. counterclockwise) around \( v_i \), according to the embedding’s rotation system. An edge \( v_i \rightarrow u \) leaves \( p \) from the left (resp. right) if its reversal \( u \rightarrow v_i \) enters \( p \) from the left (resp. right). If \( p \) is an arc, the above definitions require that \( 0 < i < k \) and that \( u \) is not a vertex in \( p \). Recall an arc’s endpoints lie on boundary cycles. Let \( t_0v_0 \) and \( v_0w_0 \) be the boundary edges incident to \( v_0 \) with vertices \( t_0, v_1, \) and \( w_0 \) appearing in clockwise order around \( v_0 \). We say \( t_0 \rightarrow v_0 \) enters \( p \) from the left. We say \( w_0 \rightarrow v_0 \) enters \( p \) from the right. Similarly, if \( t_kv_k \) and \( v_kw_k \) are boundary edges incident to \( v_k \) with vertices \( t_k, w_{k+1}, \) and \( v_{k-1} \) appearing in clockwise order around \( v_k \), we say \( t_k \rightarrow v_k \) enters \( p \) from the left and \( w_k \rightarrow v_k \) enters \( p \) from the right. Finally, we treat \( t_0 \) as \( v_{-1} \) and \( t_k \) as \( v_{k+1} \) to define entering from the left (resp. right) for any other edges \( u \rightarrow v_0 \) or \( u \rightarrow v_k \) where \( u \) does not appear in \( p \).

To simplify our presentation and analysis, we assume that any two vertices \( x \) and \( y \) in \( G \) are connected by a unique shortest directed path, denoted \( \sigma(x, y) \). The Isolation Lemma [43] implies that this assumption can be enforced (with high probability) by perturbing the edge weights with random infinitesimal values [23].

Our algorithms rely on a result by Cabello et al. [5, 6] which generalizes a result of Klein [40] for planar graphs.

Lemma 2.1 (Cabello et al. [5, 6]). Let \( G \) be a directed graph with non-negative edge weights, cellularly embedded on a surface \( \Sigma \) of genus \( g \), and let \( f \) be an arbitrary face of \( G \). We can preprocess \( G \) in \( O(gn \log n) \) time\(^2\) and \( O(n) \) space, so that the length of any shortest path from any vertex incident to \( f \) to any other vertex can be retrieved in \( O(\log n) \) time.

2.3 Homotopy and Homology. Two paths \( p \) and \( q \) in \( \Sigma \) are homotopic if one can be continuously deformed into the other without changing their endpoints. More formally, a homotopy between \( p \) and \( q \) is a continuous map \( h : [0, 1] \times [0, 1] \rightarrow \Sigma \) such that \( h(0, \cdot) = p \), \( h(1, \cdot) = q \), \( h(\cdot, 0) = p(0) = q(0) \), and \( h(\cdot, 1) = p(1) = q(1) \). Homotopy defines an equivalence relation over the set of paths with any fixed pair of endpoints. The set of homotopy classes of loops in \( \Sigma \) with basepoint \( x_0 \) defines a group \( \pi_1(\Sigma, x_0) \) under concatenation, called the fundamental group of \( \Sigma \). (For all basepoints \( x_0 \) and \( x_1 \), the groups \( \pi_1(\Sigma, x_0) \) and \( \pi_1(\Sigma, x_1) \) are isomorphic.) A cycle is contractible if it is homotopic to a constant map.

\(^{2}\)The published version of this algorithm [5] has a weaker time bound of \( O(2^n n \log n) \). Using the published version increases the running time of our algorithms for directed graphs by a factor of \( g \).
Homology is a coarser equivalence relation than homotopy, with nicer algebraic properties. Like several earlier papers [14,15,21,22,24], we consider only one-dimensional cellular homology with coefficients in the finite field \( \mathbb{Z}_2 \).

Fix a cellular embedding of an undirected graph \( G \) on a surface \( \Sigma \) with genus \( g \) and \( b \) boundary cycles. An even subgraph is a subgraph of \( G \) in which every node has even degree, or equivalently, the union of edge-disjoint cycles. An even subgraph is null-homologous if it is the boundary of the closure of a subset of faces of \( G \). Two even subgraphs \( \eta \) and \( \eta' \) are homologous, or in the same homology class, if their symmetric difference \( \eta \oplus \eta' \) is null-homologous. The set of all homology classes of even subgraphs defines the first homology group of \( \Sigma \), which is isomorphic to the finite vector space \( (\mathbb{Z}_2)^{2g+b} \). If \( b \leq 1 \), then a simple cycle \( \gamma \) is separating if and only if it is null-homologous; however, when \( b > 1 \), some separating cycles are not null-homologous.

2.4 Covering spaces A continuous map \( \pi : \Sigma' \rightarrow \Sigma \) between two surfaces is called a covering map if each point \( x \in \Sigma \) lies in an open neighborhood \( U \) such that (1) \( \pi^{-1}(U) \) is a countable union of disjoint open sets \( U_1 \cup U_2 \cup \cdots \) and (2) for each \( U \), the restriction \( \pi|_U : U \rightarrow \pi(U) \) is a homeomorphism. If there is a covering map \( \pi \) from \( \Sigma' \) to \( \Sigma \), we call \( \Sigma' \) a covering space of \( \Sigma \). The universal cover \( \hat{\Sigma} \) is the unique simply-connected covering space of \( \Sigma \) (up to homeomorphism). The universal cover is so named because it covers every path-connected covering space of \( \Sigma \).

For any path \( p : [0,1] \rightarrow \Sigma \) such that \( \pi(x') = p(0) \) for some point \( x' \in \Sigma' \), there is a unique path \( p' \) in \( \Sigma' \), called a lift of \( p \), such that \( \pi(p'(0)) = x' \) and \( p \circ p' = p \). We also say that \( p \) lifts to \( p' \). Conversely, for any path \( p' \) in \( \Sigma' \), the path \( \pi \circ p' \) is called a projection of \( p' \).

We define a lift of a cycle \( \gamma : S^1 \rightarrow \Sigma \) to be the infinite path \( \gamma' : \mathbb{R} \rightarrow \Sigma' \) such that \( \pi(\gamma'(t)) = \gamma(t \mod 1) \) for all real \( t \). We call the path obtained by restricting \( \gamma' \) to any unit interval a single-period lift of \( \gamma \); equivalently, a single-period lift of \( \gamma \) is a lift of any loop equivalent to \( \gamma \). We informally say that a cycle is the projection of any of its single-period lifts.

3 Non-trivial Cycles in Undirected Graphs

Let \( G \) be an undirected graph with non-negative edge weights,cellularly embedded on an orientable surface \( \Sigma \) of genus \( g \). We sketch an algorithm to compute a shortest non-separating, non-contractible, or non-null-homologous cycle in \( G \). We assume the surface has no boundary, and consider the case with boundary at the end of this section. Recall any shortest non-null-homologous cycle is a shortest non-separating cycle in a surface without boundary.

We begin by reviewing Kutz’s [41] algorithm for computing shortest non-trivial cycles. He begins by computing a greedy system of loops \( \Lambda = \{ \lambda_1, \lambda_2, \ldots, \lambda_{2g} \} \) using a construction of Erickson and Whittlesey [26]. The construction can be performed in \( O(gn) \) time using our assumption that \( g = O(\sqrt{n}) \) [41]. The surface \( D = \Sigma \setminus \Lambda \) is a topological disk with each loop \( \lambda_i \in \Lambda \) appearing twice upon its boundary. See Figure 2. Kutz argues that there exists some shortest non-trivial cycle \( \gamma \) that meets three criteria: (1) \( \gamma \) crosses each loop \( \lambda_i \) at most twice [41, Lemma 1]; (2) the crossing sequence of \( \gamma \) with regards to the loops contains no curls; there is never any instance where \( \gamma \) crosses a loop \( \lambda_i \) from left-to-right (right-to-left) only to immediately cross again right-to-left (left-to-right) [41, Lemma 3]; and (3) \( \gamma \) is simple. Given a cycle \( \gamma \), there exists a sequence of crossings between \( \gamma \) and the loops of \( \Lambda \). Kutz uses the above observations to find shortest cycles corresponding to \( g^{O(\varepsilon)} \) crossing sequences of length \( O(g) \) where at least one of the crossing sequences corresponds to a shortest non-trivial cycle. For each crossing sequence \( X \), he describes how to determine if a cycle corresponding to \( X \) meets the criteria above and, if so, how to find a shortest cycle corresponding to \( X \) in \( O(gn \log n) \) time using an algorithm of Colin de Verdière and Erickson [17]. Italiano et al. [35] later improved the running time of Colin de Verdière and Erickson’s algorithm to \( O(n \log \log n) \). The final running time for Kutz’s algorithm with the modification by Italiano et al. is therefore \( g^{O(\varepsilon)} n \log \log n \).

In order to improve the running time, we show how to reduce the number of crossing sequences that need to be considered by Kutz’s algorithm using a similar strategy to that seen in [12,14]. As mentioned, the greedy system of loops \( \Lambda \) used by Kutz cuts the surface into a topological disk \( D \). By replacing each loop in \( \Lambda \) with a single edge in \( D \), we transform \( D \) into an abstract polygonal schema. Each loop of \( \Lambda \) corresponds to two edges of the polygon. Any non-self-crossing cycle \( \gamma \) in \( \Sigma \) is cut into arcs by the polygon where an arc exists between two edges if \( \gamma \) consecutively crosses the corresponding loops of \( \Lambda \). We dualize the polygonal schema by replacing each edge with a vertex and each vertex with an edge. Cycle \( \gamma \) now corresponds to a weighted triangulation of the dualized polygonal schema where each pair of consecutive crossings by \( \gamma \) between loops of \( \Lambda \) is represented by an edge between the corresponding vertices. Each edge of the triangulation receives a weight equal to the number of times \( \gamma \) performs the corresponding consecutive crossings. Some shortest
non-trivial cycle crosses each member of \( \Lambda \) at most twice, so the edge weights on its triangulation are all between 0 and 2.

Our algorithm for computing a shortest non-trivial cycle in \( G \) enumerates all weighted triangulations of the dualized polygonal schema with weights between 0 and 2 by brute force. There are \( 2^{O(g)} \) weighted triangulations considered. For each triangulation, the algorithm then checks if it corresponds to a single cycle in \( O(g) \) time by brute force. If the triangulation does correspond to a single cycle, then its crossing sequence is calculated. The algorithm uses Italiano et al.’s modification to Kutz’s algorithm to determine if the crossing sequence meets the aforementioned criteria and, if so, to calculate a shortest cycle corresponding to that crossing sequence. Our algorithm will eventually return a shortest cycle corresponding to the correct crossing sequence for some shortest non-trivial cycle. The overall running time is \( 2^{O(g)} n \log \log n \).

### 3.1 Surfaces with Boundary

We now extend the above algorithm to work on surfaces with boundary. For computing a shortest non-separating cycle, we reduce to the case without boundary by pasting disks into each of the boundary components. This transformation does not change the set of non-separating cycles. Our algorithm still runs in time \( 2^{O(g)} n \log \log n \).

In order to compute a shortest non-contractible cycle, we use a greedy system of \( O(g + b) \) arcs instead of a greedy system of loops. The necessary properties of the greedy system (the shortest non-contractible cycle being simple, crossing each arc at most twice, and being curl-free) are easily shown using the same proofs given in [41]. We still use a dualized polygonal schema, except it now has \( O(g + b) \) vertices, and our algorithm must enumerate \( 2^{O(g+b)} \) weighted triangulations. The rest of the details are essentially the same. The overall running time is \( 2^{O(g+b)} n \log \log n \).

Finally, we can compute a shortest non-null-homologous cycle by slightly modifying the algorithm for finding a shortest non-contractible cycle. The only difference is we ignore weighted triangulations where the corresponding crossing sequence does not correspond to a non-null-homologous cycle. Testing if a crossing sequence corresponds to a non-null-homologous cycle can be done using techniques shown in [14].

With these extensions to surfaces with boundary, we get the following theorem.

**Theorem 3.1.** A shortest non-separating cycle in an undirected graph embedded on an orientable surface of genus \( g \) with \( b \) boundary cycles can be computed in \( 2^{O(g)} n \log \log n \) time. Further, a shortest non-contractible or non-null-homologous cycle can be computed in \( 2^{O(g+b)} n \log \log n \) time.

### 4 Shortest Non-null-homologous Cycles in Directed Graphs

Now let \( G \) be a symmetric directed graph with non-negative edge weights, cellerly embodied on an orientable surface \( \Sigma \) of genus \( g \) with \( b \) boundary cycles. We continue by giving an overview of an algorithm to compute a shortest cycle in \( G \) that is not null-homologous.

In [21], Erickson describes a system \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{2g}\} \) of \( 2g \) non-separating cycles where each cycle \( \lambda_i \) is composed of two shortest paths in \( G \) along with an extra edge. We actually describe and use this construction explicitly in Section 6. For each cycle \( \lambda_i \in \Lambda \), Erickson gives an \( O(\sqrt{n} \log n) \) time algorithm to find a shortest cycle that crosses \( \lambda_i \) an odd number of times. Any non-separating cycle must cross at least one member of \( \Lambda \) an odd number of times, so an \( O(\sqrt{n} \log n) \) time algorithm for finding a shortest non-separating cycle follows immediately.

In a similar vain, we claim it is possible to compute in \( O(\sqrt{n} \log n) \) time a shortest cycle crossing any non-separating arc \( \lambda \) an odd number of times assuming \( \lambda \) is a shortest path. Our algorithm for finding a shortest
non-null-homologous cycle begins by calling Erickson’s algorithm as a subroutine in case any shortest non-null-homologous cycles are non-separating. We then perform the following steps in case all the shortest non-null-homologous cycles are separating. Arbitrarily label the boundary cycles of $G$ as $B_0, B_1, \ldots, B_{b-1}$. Let $s$ be an arbitrary vertex on $B_0$. We compute the shortest path tree $T$ from $s$ using Dijkstra’s algorithm in $O(n \log n)$ time. For each index $i \geq 1$, let $\lambda_i$ be a shortest directed path in $T$ from $B_0$ to $B_i$ that contains exactly one vertex from each boundary cycle $B_0$ and $B_i$. Let $\Lambda = \{ \lambda_1, \lambda_2, \ldots, \lambda_{b-1} \}$ be the set of shortest paths computed above. Each path must be non-separating as it connects two distinct boundary cycles. We can easily compute $\Lambda$ in $O(bn)$ time once we have the shortest path tree $T$. If a shortest non-null-homologous cycle is separating, then it must separate $B_0$ from some other boundary cycle $B_i$ with $i \geq 1$.

**Lemma 4.1.** If a simple cycle $\gamma$ separates boundary cycle $B_0$ from a different boundary cycle $B_i$, then $\lambda_i$ crosses $\gamma$ an odd number of times.

**Proof:** Cycle $\gamma$ separates $\Sigma$ into two components $A$ and $B$ containing boundary cycles $B_0$ and $B_i$, respectively. Arc $\lambda$ must cross $\gamma$ from $A$ to $B$ one more time than it crosses from $B$ to $A$. Therefore, $\lambda$ crosses an odd number of times. \hfill $\square$

Lemma 4.1 implies that any shortest non-null-homologous cycle $\gamma$ crosses some arc $\lambda_i$ an odd number of times if $\gamma$ is separating.

All that remains is to present a slightly modified lemma of Erickson [21, Lemma 3.4].

**Lemma 4.2.** Let $\lambda$ be any arc in $\Lambda$. The shortest cycle $\gamma$ that crosses $\lambda$ an odd number of times can be computed in $O(gn \log n)$ time.

The proof remains essentially unchanged for our version of the lemma. In short, we compute the cyclic double cover $\Sigma^2_\lambda$ as described in Appendix A. The lift of $G = (V, E)$ to $\Sigma^2_\lambda$ contains the vertex set $V \times \{0, 1\}$. Lemma A.3 implies that $\gamma$ lifts to a shortest path from $(s, 0)$ to $(s, 1)$, for some vertex $s$ of $\lambda$. We can compute this path using a single multiprocessor shortest path computation in $O(gn \log n)$ time (Lemma 2.1).

Applying Lemma 4.2 to each arc $\lambda \in \Lambda$ and comparing the results to the shortest non-separating cycle found by Erickson’s algorithm, we immediately get Theorem 4.3.

**Theorem 4.3.** A shortest non-null-homologous cycle in a directed graph embedded on an orientable surface of genus $g$ with $b$ boundary cycles can be computed in $O((g^2 + gb)n \log n)$ time.

5 The Infinite Cyclic Cover

As in the previous section, let $G$ be a symmetric directed graph with non-negative edge weights, cellurally embedded on an orientable surface $\Sigma$ of genus $g$ with $b$ boundary cycles. We begin to describe our algorithm for computing a shortest non-contractible cycle in $G$. Our job is easy if any shortest non-contractible cycle is non-null-homologous; we can just run the algorithm given in Section 4 in $O((g^2 + gb)n \log n)$ time. We must work harder, though, to find a shortest non-contractible cycle $\gamma$ if every shortest non-contractible cycle is null-homologous. Our high-level strategy is to construct $O(g)$ subsets of a covering space we call the infinite cyclic cover. In Lemma 7.1, we show at least one of the subsets contains a non-null-homologous cycle that projects to $\gamma$.

Let $\lambda$ be an arbitrary simple non-separating cycle in $\Sigma$. We define the covering space $\Sigma^\infty_\lambda$, which we call the infinite cyclic cover, as follows. Cutting the surface $\Sigma$ along $\lambda$ gives us a new surface $\Sigma'$ with $b + 2$ boundary cycles where two of the boundary cycles are copies of $\lambda$ denoted $\lambda^+$ and $\lambda^-$. The infinite cyclic cover is obtained by pasting together an infinite number of copies of $\Sigma'$ along corresponding boundary cycles $\lambda^\pm$. Specifically, we have a copy $(\Sigma', i)$ of $\Sigma'$ for each integer $i$. Let $(\lambda^+, i)$ and $(\lambda^-, i)$ denote copies of $\lambda^+$ and $\lambda^-$ in $(\Sigma', i)$. The infinite cyclic cover is defined by identifying $(\lambda^+, i)$ and $(\lambda^-, i + 1)$ for every $i$. Any graph $G$ cellurally embedded on $\Sigma$ lifts to an infinite graph $G^\infty_\lambda$ embedded in $\Sigma^\infty_\lambda$. Note that for any pair of simple non-separating cycles $\lambda$ and $\mu$, the infinite cyclic covers $\Sigma_\lambda$ and $\Sigma_\mu$ are homeomorphic, but the lifted graphs $G^\infty_\lambda$ and $G^\infty_\mu$ may not be isomorphic.

We would like to use the infinite cyclic cover to aid us in finding a shortest non-contractible cycle. As explained in Section 6, it is possible to consider only a finite portion of $\Sigma^\infty_\lambda$ if we choose $\lambda$ carefully. We call this subset the restricted infinite cyclic cover. Again, let $\lambda$ be an arbitrary simple non-contractible cycle in $\Sigma$ and define $\Sigma'$ as above with boundaries $\lambda^+$ and $\lambda^-$. Instead of pasting together an infinite number of copies of $\Sigma'$, we only paste together five copies. Specifically, we have a copy $(\Sigma', i)$ of $\Sigma'$ for each integer $i \in \{1, \ldots, 5\}$. Again, let $(\lambda^+, i)$ and $(\lambda^-, i)$ denote copies of $\lambda^+$ and $\lambda^-$ in $(\Sigma', i)$. The restricted infinite cyclic cover is defined by identifying $(\lambda^+, i)$ and $(\lambda^-, i + 1)$ for every $i \in \{1, \ldots, 4\}$. See Figure 3. Now any graph $G$ cellurally embedded on $\Sigma$ lifts to a finite graph $G^\infty_\lambda$ embedded in $\Sigma^\infty_\lambda$ with at most six times as many vertices and edges. Note that $\Sigma^\infty_\lambda$ still has two lifts of $\lambda$ acting as boundary cycles. We continue

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3Named for the infinite cyclic group.
We declare that a cycle in $G$ is infinite cyclic cover $G$ if and only if its projection to $u$ for all edges $\lambda$ in pairs $(G, \Sigma)$. Let $\gamma$ define the restricted infinite cyclic cover using a construction from the left, and 0 otherwise. We can further restrict $\lambda$ to be a simple non-separating cycle in $G$. For any path or cycle $p$, we define the crossing count $c_\lambda(p)$ to be the number of times $p$ crosses $\lambda$ from left to right minus the number of times $p$ crosses $\lambda$ from right to left. Equivalently, we have

$$c_\lambda(p) = \sum_{u \rightarrow v \in p} c_\lambda(u \rightarrow v)$$

where for any directed edge $u \rightarrow v$, we define $c_\lambda(u \rightarrow v)$ to be 1 if $u \rightarrow v$ enters $\lambda$ from the left, $-1$ if $u \rightarrow v$ leaves $\lambda$ from the left, and 0 otherwise. We can define the restricted infinite cyclic cover using a voltage construction [30, Chapters 2,4] for combinatorial surfaces. Let $G'_\lambda$ be the graph whose vertices are the pairs $(v, i)$, where $v$ is a vertex of $G$ and $i$ is an integer in $\{1, \ldots, 6\}$ if $v$ lies along $\lambda$, or $\{1, \ldots, 5\}$ if $v$ does not lie along $\lambda$. The edges of $G'_\lambda$ are the ordered pairs

$$(u \rightarrow v, i) := (u, i) \rightarrow (v, i + c_\lambda(u \rightarrow v))$$

for all edges $u \rightarrow v$ of $G$ and all $i \in \{1, \ldots, 6\}$. Let $\pi : G'_\lambda \rightarrow G$ denote the obvious covering map $\pi(v, i) = v$.

We declare that a cycle in $G'_\lambda$ bounds a face of $G'_\lambda$ if and only if its projection to $G$ bounds a face of $G$. The resulting embedding of $G'_\lambda$ defines the restricted infinite cyclic cover $\Sigma'_\lambda$.

6 Lifting Shortest Non-contractible Cycles

Consider the following procedure also used in [21]. We construct a greedy tree-cotree decomposition $(T, L, C)$ of $G$, where $T$ is a shortest path tree rooted at some arbitrary vertex of $G$. Euler’s formula implies that $L$ contains exactly $2g$ edges; label these edges arbitrarily as $u_1 v_1, u_2 v_2, \ldots, u_{2g} v_{2g}$. For each index $i$, let $\lambda_i$ denote the unique cycle in the undirected graph $T \cup u_i v_i$ oriented so that it contains the directed edge $u_i \rightarrow v_i$. If there are no boundary cycles in $\Sigma$, then the set of cycles $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{2g}\}$ is a basis for the first homology group of $\Sigma$ [19]. We refer to the construction as a partial homology basis. Every non-separating cycle in $\Sigma$ crosses at least one cycle in $\Lambda$ an odd number of times [11, Lemma 3]. The greedy tree-cotree decomposition $(T, L, C)$ can be constructed in $O(n \log n)$ time using Dijkstra’s algorithm. Afterward, we can easily compute the partial homology basis in $O(gn)$ time.

Recall that a single period lift of a cycle $\gamma$ to a covering space refers to any lift of a loop equivalent to $\gamma$. Let $\sigma$ be an arbitrary shortest path in $G$. Erickson [21] argues that the lift of any shortest non-contractible cycle to the universal cover does not intersect many lifts of $\sigma$. This observation applies to the infinite cyclic cover as well. The following lemma and its corollary are essentially equivalent to Lemma 4.6 and Corollary 4.7 of [21], but modified for our setting.
Lemma 6.1. Let \( \gamma \) be a shortest non-contractible cycle in \( \Sigma \); let \( \lambda \) be any simple non-separating cycle in \( \Sigma \); and let \( \sigma \) be any shortest path in \( \Sigma \). Any single-period lift of \( \gamma \) to the infinite cyclic cover \( \Sigma_\lambda \) intersects at most two lifts of \( \sigma \).

**Proof:** The covering space \( \Sigma_\lambda \) is path connected, so it is itself covered by the universal cover \( \hat{\Sigma} \). Any single period lift of \( \gamma \) to \( \Sigma_\lambda \) in turn has one or more lifts in \( \hat{\Sigma} \). Any one of these single period lifts of \( \gamma \) to \( \hat{\Sigma} \) intersects at most two lifts of \( \sigma \) [21, Lemma 4.6]. Covering maps are functions, so lifting from \( \Sigma_\lambda \) to \( \hat{\Sigma} \) cannot decrease the number of intersecting lifts of \( \sigma \). □

Corollary 6.2. Let \( \Lambda \) be a partial homology basis in \( \Sigma \); let \( \lambda \) be any cycle in \( \Lambda \); and let \( \gamma \) be a shortest non-contractible cycle in \( \Sigma \). Any single-period lift of \( \gamma \) to \( \Sigma_\lambda \) intersects at most four lifts of \( \lambda \).

**Proof:** Every vertex of \( \lambda \) belongs to one of two directed shortest paths. By Lemma 6.1, any single-period lift of \( \gamma \) intersects at most two lifts of either shortest path. □

Recall the restricted infinite cyclic cover defined in Section 5 is constructed by pasting together five copies of the surface cut along the simple non-separating cycle \( \lambda \). We immediately get the following lemma stating the restricted infinite cyclic cover is large enough to contain a lift of any shortest non-contractible cycle.

Lemma 6.3. Let \( \Lambda \) be a partial homology basis in \( \Sigma \); let \( \lambda \) be any cycle in \( \Lambda \); and let \( \gamma \) be a shortest non-contractible cycle in \( \Sigma \). There exists a single period lift of \( \gamma \) to \( \Sigma_\lambda' \).

In fact, we show below that \( \gamma \) lifts to a shortest non-contractible cycle in \( \Sigma_\lambda' \) if \( \gamma \) is separating. This statement actually holds for any non-separating cycle \( \lambda \) made of two shortest paths optionally connected by an edge. In Lemma 7.1, we explain that the correct choice of \( \lambda \) guarantees the lift of \( \gamma \) to be non-null-homologous.

We continue by noting that every shortest non-contractible cycle is simple [7, Lemma 3]. We show that if any shortest non-contractible cycle \( \gamma \) is separating, then it lifts to a cycle in \( \Sigma_\lambda' \) for any \( \lambda \) in the partial homology basis. Recall the definition of the crossing count \( c_\lambda(\gamma) \).

Lemma 6.4. Let \( \lambda \) be any simple non-separating cycle in \( \Sigma \), and let \( \gamma \) be a loop in \( \Sigma \) with a lift in \( \Sigma_\lambda' \). Then, \( \gamma \) lifts to a loop in \( \Sigma_\lambda' \) if and only if \( c_\lambda(\gamma) = 0 \).

Proof: Let \( \Sigma' \) be the surface \( \Sigma \) cut along \( \lambda \). By construction, \( \Sigma_\lambda' \) is composed of five copies of \( \Sigma' \), denoted \( (\Sigma',i) \) for each integer \( i \in \{1, \ldots, 5\} \). Each copy is separated by a lift of \( \lambda \). Consider a lift of \( \gamma \) contained in \( \Sigma_\lambda' \) which we denote \( \gamma_\lambda \). For every instance of \( \gamma \) crossing \( \lambda \) from left to right, there is an instance of \( \gamma_\lambda \) crossing a lift of \( \lambda \) from \( (\Sigma',i) \) to \( (\Sigma',i+1) \) for some \( i \). Likewise, every time \( \gamma \) crosses \( \lambda \) from right to left, \( \gamma_\lambda \) crosses \( \lambda \) from \( (\Sigma',i+1) \) to \( (\Sigma',i-1) \). If \( \gamma_\lambda \) begins in \( (\Sigma',i) \), then it ends at a copy of the same point in \( (\Sigma',i+c_\lambda(\gamma)) \).

□

Lemma 6.5. Let \( \lambda \) be any simple non-separating cycle in \( \Sigma \) and let \( \gamma \) be any simple separating cycle. We have \( c_\lambda(\gamma) = 0 \).

Proof: Cycle \( \gamma \) separates \( \Sigma \) into two components denoted \( A \) and \( B \) so that a path crossing \( \gamma \) exactly once starts in \( A \) and ends in \( B \) if it crosses from left to right. Let \( x \) be an arbitrary point on \( \lambda \) and consider the loop \( \ell \) equivalent to \( \lambda \) based at \( x \). Every time \( \ell \) crosses \( \gamma \) from left to right, we see \( \ell \) goes from \( A \) to \( B \). Further \( \gamma \) crosses \( \ell \) once from right to left. Similarly, every time \( \ell \) crosses \( \gamma \) from right to left, we see \( \ell \) goes from \( B \) to \( A \) and \( \gamma \) crosses \( \ell \) once from left to right. Loop \( \ell \) must cross from \( A \) to \( B \) the same number of times it crosses from \( B \) to \( A \). Therefore, \( \gamma \) crosses \( \ell \) and \( \lambda \) from right to left the same number of times it crosses left to right. By definition, \( c_\lambda(\gamma) = 0 \). □

Corollary 6.6. Let \( \Lambda \) be a partial homology basis in \( \Sigma \); let \( \lambda \) be any cycle in \( \Lambda \); and let \( \gamma \) be a shortest non-contractible cycle in \( \Sigma \). If \( \gamma \) is separating, then \( \gamma \) lifts to a loop in \( \Sigma_\lambda' \).

We can finally show that if any shortest non-contractible cycle \( \gamma \) is separating, then it actually lifts to a shortest non-contractible cycle in \( \Sigma_\lambda \) for any \( \lambda \) in a partial homology basis.

Lemma 6.7. Let \( \gamma_\lambda \) be a loop in \( \Sigma_\lambda' \) that projects to a simple loop \( \gamma \) in \( \Sigma \). Loop \( \gamma_\lambda \) is contractible if and only if \( \gamma \) is contractible.

Proof: Suppose \( \gamma_\lambda \) is contractible. There exists a homotopy \( h \) from \( \gamma_\lambda \) to a constant map. The paths in \( h \) can be projected to \( \Sigma \), yielding a homotopy from \( \gamma \) to a constant map. Therefore, \( \gamma \) is contractible.

Now, suppose \( \gamma \) is contractible. There exists a homotopy \( h \) from \( \gamma \) to a constant map. There exists a unique homotopy \( h_\lambda \) of \( \gamma_\lambda \) that lifts the paths in \( h \) to the infinite cyclic cover \( \Sigma_\lambda \) [32, Proposition 1.30]. Homotopy \( h_\lambda \) finishes with a constant map, so \( \gamma_\lambda \) is contractible in \( \Sigma_\lambda \). Loop \( \gamma_\lambda \) must be simple to project to a simple loop \( \gamma \), so it bounds a disk \( D \) in \( \Sigma_\lambda \). Disk \( D \)
contains no faces outside of $\Sigma^r_\lambda$, because $\gamma_\lambda$ contains no edges outside of $\Sigma^r_\lambda$ to bound those outside faces. Therefore, $\gamma_\lambda$ bounds a disk ($D$) in $\Sigma^r_\lambda$ implying $\gamma_\lambda$ is contractible in $\Sigma^r_\lambda$. □

Lemma 6.8. Let $\Lambda$ be a partial homology basis in $\Sigma$; let $\lambda$ be any cycle in $\Lambda$; and let $\gamma$ be a shortest non-contractible cycle in $\Sigma$. If $\gamma$ is separating, then $\gamma$ lifts to a shortest non-contractible cycle in $\Sigma^r_\lambda$.

7 Computing Shortest Non-contractible Cycles in Directed Graphs

We now describe our algorithm for computing a shortest non-contractible cycle. We assume the surface has genus $g \geq 1$. Otherwise, every non-contractible cycle is null-homologous, and we can simply use the algorithm given in Section 4. Further, we begin by assuming the surface has exactly one boundary cycle. Instances where $\Sigma$ has more than one boundary cycle or no boundary cycles are handled as simple reductions to the one boundary cycle case given at the end of this section.

Let $\partial \Sigma$ denote the one boundary cycle on $\Sigma$. We compute a partial homology basis $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{2g}\}$ in $O(n \log n + gn)$ time as described in Section 6. The following lemma states that one of the cycles in the homology basis can be used to build a restricted infinite cyclic cover that is useful for our computation. Surprisingly, the boundary introduced by restricting the infinite cyclic cover plays a key role in the proof of the lemma.

Lemma 7.1. Let $\gamma$ be a shortest non-contractible cycle in $\Sigma$. If $\gamma$ is separating, then there exists a non-separating cycle $\lambda \in \Lambda$ such that $\gamma$ lifts to a shortest non-null-homologous cycle in the restricted infinite cyclic cover $\Sigma^r_\lambda$.

Proof: Every shortest non-contractible cycle is simple [7, Lemma 3]. So by assumption, $\gamma$ is a simple separating cycle. There is exactly one boundary $\partial \Sigma$, so $\gamma$ bounds the closure $A$ of a set of faces. The component $A$ must have genus, or $\gamma$ would bound a disk and be contractible. There exists a simple non-separating cycle $\omega$ on $\Sigma$ contained entirely within $A$. Cycle $\omega$ must cross some other cycle $\lambda \in \Lambda$ an odd number of times [11, Lemma 3]. See Figure 3. Consider the infinite cyclic cover $\Sigma_1$ and its restriction $\Sigma^r_\lambda$.

Let $p$ be a path in $\Sigma$ from $\partial \Sigma$ to $\omega$ such that $p$ does not cross $\lambda$. Path $p$ must exist, because $\lambda$ is non-separating. Further, $p$ crosses $\gamma$ an odd number of times. Let $\partial \Sigma_1$ be a lift of $\partial \Sigma$ to $\Sigma_1$, and let $p_1$ be the lift of $p$ to $\Sigma_1$ that begins on $\partial \Sigma_1$. Let $\gamma_\lambda$ be a lift of $\gamma$ to $\Sigma_\lambda$ such that $p_\lambda$ crosses $\gamma_\lambda$ an odd number of times.

By symmetry and Lemma 6.8, we may assume $\gamma_\lambda$ is a cycle in $\Sigma^r_\lambda$. We note $\gamma_\lambda$ is simple as it projects to simple cycle $\gamma$.

Suppose that $\gamma_\lambda$ is separating. Let $\omega_\lambda$ denote a lift of cycle $\omega$ to $\Sigma_\lambda$ such that $p_\lambda$ ends on $\omega_\lambda$. Curve $\omega_\lambda$ is not a cycle in $\Sigma_\lambda$, because $\omega$ crosses $\lambda$ an odd number of times in $\Sigma$ (see Lemma 6.4). Therefore, $\omega_\lambda$ is a simple infinite path that does not cross any lift of $\gamma$. Let $\omega'_\lambda = \omega_\lambda \cap \Sigma^r_\lambda$. Path $\omega'_\lambda$ is a simple arc from $\lambda^-$ to $\lambda^+$ in $\Sigma^r_\lambda$ which does not cross $\gamma_\lambda$. Path $p$ does not cross $\lambda$, implying that $p_\lambda$ is a path in $\Sigma^r_\lambda$ with endpoints on $\partial \Sigma_\lambda$ and $\omega'_\lambda$. Further, $p_\lambda$ crosses $\gamma_\lambda$ an odd number of times, implying that $\gamma_\lambda$ separates $\partial \Sigma_\lambda$ from $\omega'_\lambda$ and $\lambda^-$.

We see either $\gamma_\lambda$ is non-separating or it separates a pair of boundary cycles. Therefore, $\gamma_\lambda$ is non-null-homologous in $\Sigma^r_\lambda$. Lemma 6.8 implies $\gamma_\lambda$ is actually a shortest non-null-homologous cycle in $\Sigma^r_\lambda$. □

In the above proof, it would actually be preferable if $\gamma_\lambda$ was separating. In this case, we could find $\gamma_\lambda$ in $O(gn \log n)$ time by applying Lemma 4.2 along shortest paths between $\lambda^-$ and each lift of $\partial \Sigma$. As written, the lemma requires us to apply the full algorithm of Section 4 in $O(g^2n \log n)$ time if we wish to find $\gamma_\lambda$.

We now finish considering the case where $\Sigma$ has one boundary cycle. Applying lemmas 6.8 and 7.1, we construct the restricted infinite cyclic cover $\Sigma^r_\lambda$ and find a shortest non-null-homologous cycle in $\Sigma^r_\lambda$ once for each cycle $\lambda \in \Lambda$ using the algorithm of Section 4. This procedure gives us a shortest non-contractible cycle in $O(g^3n \log n)$ time if any are separating. We apply the algorithm of Section 4 (or Erickson’s [21] algorithm) once to $G$ directly to account for the case where every shortest non-contractible cycle is non-separating. All that remains is to consider the cases where $\Sigma$ has several boundary cycles or no boundary cycles.

7.1 Surfaces with Several Boundary We now consider the case where $\Sigma$ has $b > 1$ boundary cycles. We apply the algorithm of Section 4 to find any shortest non-contractible cycles that are non-null-homologous. Next, we paste disks into all but one of the boundary cycles. This transformation does not introduce any new non-contractible cycles, because it does not remove any paths from any homotopies. Further, it does not restrict the set of non-contractible null-homologous cycles. Every such cycle $\gamma$ still separates a subset of faces (with genus) from the one remaining boundary cycle. We now apply the algorithm as given for one boundary cycle to find any shortest non-contractible cycles that happen to be null-homologous.
7.2 Surfaces without Boundary  Finally, we extend our algorithm to consider the case where $\Sigma$ has no boundary. We apply the algorithm of Section 4 to find any shortest non-contractible cycles that are non-null-homologous (we can also apply Erickson’s [21] algorithm as every non-null-homologous cycle is also non-separating on a surface without boundary). We then perform the following reduction in case every shortest non-contractible cycle is null-homologous. We compute one cycle $\lambda$ of a greedy homology basis using a greedy tree-cotree decomposition in $O(n \log n)$ time and reduce the problem of finding the shortest non-contractible cycle for the surface $\Sigma$ with genus $g$ and no boundary to the same problem on the larger surface $\Sigma'_b$, which has two boundary cycles and genus $5g - 5$. Note that the shortest non-contractible cycle in $\Sigma'_b$ may be non-separating. The reduction is correct according to Lemma 6.8. We then apply the algorithm for several boundary on the new surface $\Sigma'_b$. Using both extensions and the algorithm as given above, we get our desired theorem.

**Theorem 7.2.** A shortest non-contractible cycle in a directed graph embedded on an orientable surface of genus $g$ with $b$ boundary cycles can be computed in $O((g^3 + gb)n \log n)$ time.

8 Conclusions and Future Work
We gave algorithms to compute shortest non-trivial cycles in both directed and undirected surface embedded graphs. In undirected graphs, our algorithms find shortest non-contractible and non-null-homologous cycles in $2O(gb)n \log log n$ time and shortest non-separating cycles in $2O(gb)n \log n$ time. For directed graphs, our algorithms find shortest non-null-homologous cycles in $O((g^2 + gb)n \log n)$ time and shortest non-contractible cycles in $O((g^3 + gb)n \log n)$ time.

The most obvious question remaining is whether we can reduce these times further. In particular, it is natural to ask if we can compute a shortest non-contractible cycle in a directed surface graph in $O((g^2 + gb)n \log n)$ time, matching the algorithm of Cabello *et al.* [5, 6] for undirected surface graphs. The main bottleneck appears to be the need to compute shortest non-null-homologous cycles in the restricted infinite cyclic cover. If the proof of Lemma 7.1 can be improved to show an appropriate arc or cycle of $\Sigma'_b$ is crossed an odd number of times by the lift of a shortest non-contractible cycle, then we can easily reduce the cost of searching each cover to $O(gn \log n)$. Another question is whether or not the $O(n \log \log n)$ running time achieved by Italiano *et al.* [35] can be achieved in directed graphs and if its use requires lifting to subsets of the universal cover.

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Appendix

A Extending the Cyclic Double Cover

Let $G$ be a symmetric directed graph with non-negative edge weights, cellularly embedded on an orientable surface $\Sigma$ of genus $g$ with $b$ boundary cycles. We describe an extension to the cyclic double cover of Erickson [21] that works with simple arcs instead of cycles. Let $\lambda$ be an arbitrary simple non-separating arc in $\Sigma$.

Define the covering space $\Sigma^2_\lambda$, which we call the cyclic double cover\footnote{Named for the cyclic group of order 2.} as follows. Cutting the surface $\Sigma$ along $\lambda$ gives us a new surface $\Sigma'$ with at least one boundary cycle. One boundary cycle of $\Sigma'$ contains two copies of $\lambda$ denoted $\lambda^+$ and $\lambda^-$. Let $(\Sigma', 0)$ and $(\Sigma', 1)$ denote two distinct copies of $\Sigma'$. For any point $p \in \Sigma'$, let $(p, 0)$ and $(p, 1)$ denote the corresponding points in $(\Sigma', 0)$ and $(\Sigma', 1)$, respectively. In particular, let $(\lambda^+, 0)$ and $(\lambda^-, 0)$ denote the copies of $\lambda^+$ and $\lambda^-$ in $(\Sigma', 0)$. Finally, let $\Sigma^2_\lambda$ be the surface obtained by identifying $(\lambda^+, 0)$ and $(\lambda^-, 1)$ to a single arc, denoted $(\lambda, 0)$, and identifying $(\lambda^+, 1)$ and $(\lambda^-, 0)$ to a single arc, denoted $(\lambda, 1)$. Any graph $G$ that is cellularly embedded in $\Sigma$ lifts to a graph $G^2_\lambda$ with twice as many vertices and edges that is cellularly embedded in $\Sigma^2_\lambda$. There are also twice as many faces in the embedding of $G^2_\lambda$ on $\Sigma^2_\lambda$ and at least $2b - 2$ boundary cycles, so Euler’s formula implies the genus of $\Sigma^2_\lambda$ is at most $2g$. See Figure 4.

For combinatorial surfaces, we can equivalently define the cyclic double cover using a standard voltage construction [30, Chapters 2,4]. Here, we assume $\lambda$ is an arc in $G$. For any directed edge $u \to v$, we define $\epsilon_{\lambda}(u \to v)$ to be 1 if $u \to v$ enters $\lambda$ from the left or leaves $\lambda$ from the left, and 0 otherwise. Let $G^2_\lambda$ be the graph whose vertices are the pairs $(v, z)$, where $v$ is a vertex of $G$ and $z$ is a bit, and whose edges are the ordered pairs

$$(u \to v, z) := (u, z) \to (v, z \oplus \epsilon_{\lambda}(u \to v))$$

for all edges $u \to v$ of $G$ and both bits $z$. Here, $\oplus$ denotes addition modulo 2. Let $\pi : G^2_\lambda \to G$ denote the obvious covering map $\pi(v, z) = v$. We declare that a cycle in $G^2_\lambda$ bounds a face of $G^2_\lambda$ if and only if its projection to $G$ bounds a face of $G$. The resulting embedding of $G^2_\lambda$ defines the cyclic double cover $\Sigma^2_\lambda$. For any directed cycle $\gamma$, we define the crossing parity $\epsilon_{\lambda}(\gamma)$ to be 1 if $\gamma$ crosses $\lambda$ an odd number of times and 0 otherwise. Equivalently, we have

$$\epsilon_{\lambda}(\gamma) = \bigoplus_{u \to v \in \gamma} \epsilon_{\lambda}(u \to v).$$

As in [21], the following lemmas are immediate.

**Lemma A.1.** Let $\lambda$ be any simple non-separating arc in $\Sigma$; let $\gamma$ be any cycle in $\Sigma$; and let $s$ be any vertex of $\gamma$. Then $\gamma$ is the projection of a unique path in $\Sigma^2_\lambda$ from $(s, 0)$ to $(s, \epsilon_{\lambda}(\gamma))$.

**Lemma A.2.** Let $\lambda$ be any simple non-separating arc in $\Sigma$. Every lift of a shortest directed path in $G$ is a shortest directed path in $G^2_\lambda$.

**Lemma A.3.** Let $\lambda$ be any simple non-separating arc in $\Sigma$; let $\gamma$ be the shortest cycle in $\Sigma$ that crosses $\lambda$ an odd number of times; and let $s$ be any vertex of $\gamma$. Then $\gamma$ is the projection of a shortest path in $\Sigma^2_\lambda$ from $(s, 0)$ to $(s, 1)$.