Singular instantons in higher derivative theories

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Abstract
We study the Hawking-Turok (HT) instanton solutions which have been employed to describe the creation of an open inflationary universe, in the context of higher derivative theories. We consider the effects of adding quadratic and cubic terms of the forms $\alpha R^2$ and $\beta R^3$ to the gravitational action. Using a conformal transformation to convert the higher derivative theories into theories of self-interacting scalar fields minimally coupled to Einstein gravity, we argue that the cubic term represents a generic perturbation of the polynomial type to the action and obtain the conditions on the parameters of these theories for the existence of singular and non-singular instanton solutions. We find that, relative to the quadratic case, there are significant changes in the nature of the constraints on the parameters for the existence of these instantons, once cubic (and higher order perturbations) are added to the action.

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1 Introduction

It is well known that the creation of an open homogeneous universe requires either very special potentials [1] or fine tuned initial conditions [10]. Thus, for example, the early mechanism suggested by Coleman and De Luccia [1], requires a special class of inflaton effective potentials which have a metastable minimum followed by a small slope region which permits a further slow-roll inflation. The inflaton field is supposed to be initially trapped in the false vacuum leading to a period of inflation, which gives an almost de Sitter space with small quantum fluctuations. The field eventually undergoes a quantum tunnelling, nucleating bubbles within which it slowly rolls down to the true vacuum. The interior of such a bubble is actually an open universe. To keep the quantum fluctuations small, a very flat potential is required while one also requires a metastable minimum. Thus this scenario can only be realised at the cost of having very special potentials and fine tunings.

An interesting alternative mechanism was suggested recently by Hawking and Turok (HT) [2, 3], which aims to remove some of these shortcomings. The HT mechanism makes use of the no–boundary proposal of Hartle and Hawking [4] and provides a novel technique for creating an open inflationary universe described by a singular instanton obtained in a minisuperspace model with an inflaton field. The HT instanton provides a method of calculating the probability of the creation of a homogeneous open universe while removing the requirement of a false vacuum and at the same time giving rise to a universe with small quantum fluctuations. Despite its novelty, several aspects of this mechanism have been criticised. For example Vilenkin [5] has criticised the use of a singular instanton and has produced a counter-example, whereby a model with a minimally coupled massless scalar field which permits a singular instanton has been shown to lead to catastrophic results. The proposal and this controversy have led to considerable work in this field. Gratton and Turok [6] have argued that singular instantons should not be ignored, as suggested by Vilenkin, as the field configurations contributing to the path integral are in any case non-differentiable. Wu [7] has pointed out that the singular instantons should be treated as constrained solutions which are not the stationary points of the gravitational action. Turok [8] has further suggested that a careful treatment of these instantons may remove the instability pointed out by Vilenkin. Unruh [9] has shown that the instanton creates a finite bounded universe of which the homogeneous hyperbolic geometry is a part. Linde [10] has suggested the introduction of a change of sign in the Euclidean action for an instanton in quantum cosmology, giving the probability $P \sim e^S$, different from what one obtains with the proposal of Hartle and Hawking. Bousso and Hawking [11] have shown that with Linde’s prescription the creation of a universe with large numbers of black holes is favoured resulting in a universe without a radiation dominated era. Further work on open inflation has also been done in the context of scalar-tensor gravity theories [12] and in higher dimensions. Garriga [13] has given a method for obtaining HT instantons in four dimensions by dimensional reduction of higher dimensional non-
singular instantons. By including a cosmological constant in the theory, he obtains a non-singular instanton, although eleven dimensional supergravity theory does not have a cosmological constant. Such a theory has, however, been shown [14] to permit also a singular instanton.

Given the potential importance of the HT scenario, it is of interest to study its robustness with respect to additional ingredients that are expected to be present. Here as a step in this direction, we shall consider the effects of including higher derivative corrections to the gravitational action, on the existence of both non-singular de Sitter and singular HT-type solutions. Quadratic and higher-order terms in the Riemann curvature tensor and its traces appear in the low-energy limit of superstrings [24], as well as when the usual perturbation expansion is applied to General Relativity [25, 26]. For example, it is known that with suitable counter terms viz $C^{\mu\nu\rho\delta}C_{\mu\nu\rho\delta}, R^2, \Lambda$ added to the Einstein action one may obtain a perturbation theory which is well behaved, formally renormalizable and asymptotically free. Some singular as well as non-singular instanton solutions have recently been presented in the case of quadratic Lagrangians [15]. The renormalisation of higher loop contributions introduces terms into the effective action that are higher than quadratic order. Consequently it is important to also study the effects of these additional terms. In this paper we shall study the effects of including higher order terms, by looking at the effects of including $R^3$-contributions to the action. By employing the conformal equivalence of higher-order gravity theories with Einstein gravity coupled to matter fields, we argue that this term is prototypical of the higher order terms than quadratic, and in this sense represents a more general perturbation to the Einstein–Hilbert action than the $R^2$-correction, at least within the context of four-dimensional FLRW space-times. This allows us to find the range of parameters in each theory for which the singular HT as well as non-singular instantons exist in these more general settings.

The outline of the paper is as follows. In section 2, we obtain the conditions for the existence of de Sitter instanton solutions in higher derivative theories, in particular in quadratic and cubic theories. Section 3 contains an analogous study in the presence of a scalar field, using the equivalence picture, obtained by a conformal transformation of the higher order gravitational Lagrangians. Included are also the conditions for the existence of both singular and the non-singular instantons permitted in the two theories, as well a comparative study of the two. Finally section 4 contains our conclusions.

2 Instantons in higher derivative theories

We consider a theory described by the following Euclidean action

$$I_E = \frac{1}{16\pi} \int f(R) \sqrt{g} \, d^4x - \frac{1}{8\pi} \int_{\partial M} K f'(R) \sqrt{h} \, d^3x$$

(2.1)
where \( g \) is the determinant of the 4-dimensional metric, \( f \) is a differentiable function of the scalar curvature \( R \), \( h_{ij} \) is the metric induced on \( \partial M \), \( K = h^{ij} K_{ij} \) is the trace of the second fundamental form, and prime denotes differentiation with respect to the argument of the function. Here we shall mainly be concerned with the case where \( f(R) \) is given by the general polynomial function

\[
f(R) = \sum_{n=0}^{N} \lambda_n R^n. \tag{2.2}
\]

Choosing \( N = 2 \), one obtains the quadratic Lagrangian which, as was mentioned above, is known to have a number interesting features. Such quadratic terms in the Riemann curvature tensor and its trace also appear in the Gauss-Bonnet combination, as well as in the low energy limit of the superstring theory. If one only considers the conformally flat space-times in 4 dimensions, then in the quadratic theory the Gauss-Bonnet combination essentially reduces to a \( R^2 \) term. We shall, however, also consider the general cases, the prototype of which is the cubic case.

As in the work of Hawking and Turok [3], we are interested in \( O(4) \) symmetric Euclidean instanton solutions described by the metric

\[
ds^2 = d\sigma^2 + b^2(\sigma) \left( d\psi^2 + \sin^2 \psi \ d\Omega^2 \right) \tag{2.3}
\]

where \( d\Omega^2 \) is the metric of 2-sphere. The scalar curvature for this metric is given by

\[
R = -6 \left( \frac{\dddot{b}}{b} + \frac{\ddot{b}^2}{b^2} - \frac{1}{b^2} \right), \tag{2.4}
\]

where a dot denotes a derivative with respect to \( \sigma \). Considering \( b \) and \( R \) as independent variables we rewrite the action (2.1), including the constraint given by (2.4), through a Lagrange multiplier \( \beta \), in the form

\[
I_E = -\frac{\pi}{4} \int d\sigma \left[ f(R)b^3 - \beta \left( R + 6 \frac{\ddot{b}}{b} + 6 \frac{\dot{b}^2}{b^2} - \frac{6}{b^2} \right) \right]

- \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{h} K f'(R). \tag{2.5}
\]

The variation of the action (2.5) with respect to \( R \) gives

\[
\beta = b^3 f'(R). \tag{2.6}
\]

Substituting from (2.6) in Eq.(2.3) we obtain

\[
I_E = -\frac{\pi}{4} \int_{\sigma=0}^{\sigma_{\partial M}} \left[ (f(R) - R f'(R))b^3 + 6 \left( f'(R)b\dot{b}^2 + f''(R)\dot{b}^2\dot{b} + b f'(R) \right) \right] d\sigma

+ 3\pi \left[ \dot{b}^2 f'(R) \right]_{\sigma=0}. \tag{2.7}
\]
The variation of the action (2.7) with respect to $b$ yields
\[
\frac{\ddot{b}}{b} + \frac{\dot{b}^2 - 1}{b^2} + \frac{f''(R)}{f'(R)} \dot{R}^2 + \frac{f''(R)}{f'(R)} \ddot{R} + 2 \frac{f''(R)}{f'(R)} \frac{\dot{b}}{b} \dot{R} - \frac{1}{2} \frac{f(R)}{f'(R)} + \frac{1}{2} R = 0. \tag{2.8}
\]

The de Sitter instanton solution has the form
\[
b = H_o^{-1} \sin H_o \sigma \tag{2.9}
\]
where $H_o$ is a constant which upon using (2.4) is found to be $R_o = 12H_o^2$. For a de Sitter instanton solution, equation (2.8) therefore reduces to an equation for $R_o$ in the form,
\[
\frac{f'(R_o)}{f(R_o)} = \frac{2}{R_o}, \tag{2.10}
\]
which in turn shows that for a polynomial action of the form (2.2), a de Sitter instanton will exist for any real solution $x = R_o$ of the equation
\[
\sum_{n=0}^{N} (n - 2) \lambda_n x^n = 0. \tag{2.11}
\]

The two cases of special interest for us are the quadratic and cubic Lagrangians:

I. The quadratic case

Considering the quadratic ($N = 2$) truncation of the Lagrangian (2.2), in the form
\[
f_2(R) = -2\Lambda + R + \alpha R^2 \tag{2.12}
\]
the condition (2.10) for the existence of a de Sitter instanton solution becomes $R = 4\Lambda = 12H_o^2$, which implies that in this case no such solutions exist for $\Lambda = 0$.

II. The cubic case

Considering the cubic truncation ($N = 3$) of the action (2.2), in the form
\[
f_3(R) = -2\Lambda + R + \alpha R^2 + \beta R^3, \tag{2.13}
\]
Eq. (2.10) implies that in this case a de Sitter instanton will exist for any real positive root of the equation
\[
\beta x^3 - x + 4\Lambda = 0.
\]
Note that as opposed to the quadratic case, in this case de Sitter instanton solutions can also exist for $\Lambda = 0$, and are given by $R = 12H_o^2 = \frac{1}{\sqrt{\beta}}$. Note also that in both the quadratic and cubic cases, the existence conditions for the instanton solutions are
independent of the parameter $\alpha$, the reason being that the quadratic term in each action satisfies the equation (2.11) identically. The corresponding action, however, does depend on it.

III. The general polynomial case

The existence of de Sitter instanton solutions in the case of a higher order theory based on the general Lagrangian (2.2) depends on the algebraic equation (2.11) having a real solution. This raises two important questions. Firstly, given the existence of de Sitter instanton solutions in a lower order Lagrangian theory, what are the effects of switching on higher order terms? Secondly, given that as $N$ increases the number of roots of the equation (2.11) also increase, what is the likelihood that such an equation has a real root and hence a de Sitter instanton solution.

The former question relates to the question of stability or fragility of the de Sitter instanton solution with respect to addition of higher order terms to the Lagrangians [19]. In this connection it is clear that as $N \to \infty$, there is a real solution for every odd $N$ but not necessarily for an even $N$. More generally it is known that if the series expansions are convergent (representing some analytic function say on $\{|z| < r\}$) and if $T : \{|z| < s\}$, where $s < r$, then for large $N$, the number of real zeros of the infinite series, representing the function in $T$ is the same as the number of real zeros of any partial sum of size greater than $N$, counting zeros by multiplicity [21]. Regarding the latter, the results by Kac [22] concerning the probability of n-th order polynomials to have real roots demonstrates that this is in fact rare, thus demonstrating that such theories are likely to have only few de Sitter instanton solutions (see also [19], [20]). The above two points are important to bear in mind when considering such solutions in more general settings where higher order Lagrangian terms are present.

Now the primary criterion for deciding whether an instanton solution is physically favoured is to compute the Euclidean action $S_E$ [4]. The wave function for the system to the leading approximation is then given by $e^{-S_E}$. This allows the calculation of the probability of creation of a de Sitter universe, described by these instantons. The Euclidean action, which is obtained by integrating over half of the $S^4$ is, for the general action (2.11) considered here, given by

$$I_{ds} = -\frac{\pi}{6H_o^4} f(H_o^2). \quad (2.14)$$

For the quadratic and cubic cases considered here, these actions take the explicit forms

$$^2I_{ds} = -\left[\frac{3\pi}{2\Lambda} + 12\pi\alpha\right], \quad (2.15)$$

$$^3I_{ds} = -\left[\frac{2\pi}{H_o^2} - \frac{\pi\Lambda}{2H_o^2} + 12\pi\alpha\right] \quad (2.16)$$
respectively, where $H_o$ is given by $12H_o^2 = R_o$ and $R_o$ is a real positive root of the equation (2.11). Note that an increase in the values of $\alpha$ and $\beta$ enhances the probability of the creation of the de Sitter instanton. Although the solution (2.9) does not depend on $\alpha$, the action cannot remain negative unless $\alpha > -\frac{1}{8\Lambda}$. Thus we obtain a lower bound on $\alpha$ for the quadratic case. In the cubic case, if $\Lambda = 0$, $R = 12H_o^2 = 1/\sqrt{\beta}$, and hence a negative $\beta$ is not allowed in this case. Since $H_o$ is independent of $\alpha$, it is obvious that a larger $\alpha > 0$, will give greater probability of creation.

To obtain an open inflationary universe, one considers the $O(4)$ symmetric metric (2.3) and substitutes $\psi = \frac{\pi}{2} + i\tau$ to obtain

$$ds^2 = d\sigma^2 + b^2(\sigma) \left( -d\tau^2 + \cosh^2 \tau \ d\Omega_2^2 \right)$$

(2.17)

which is a spatially inhomogeneous de Sitter like metric. Now setting $\sigma = i t$ and $\tau = i \frac{\pi}{2} + \chi$, the metric (2.17) becomes

$$ds^2 = -dt^2 + a^2(t) \left( d\chi^2 + \sinh^2 \chi \ d\Omega_2^2 \right)$$

(2.18)

with $a(t) = -i \ b(i \ t)$, which is a singular expanding open model.

Our aim here is to find out the precise conditions under which the HT singular instantons [2, 3] are permitted in each case considered above. We shall do this in the next section by employing a conformal transformation which converts both the quadratic and cubic Lagrangian theories into Einstein gravity with a minimally coupled, self interacting scalar field $\phi$.

3 Instantons in the conformally equivalence picture

It is well known that the theories with higher order Lagrangians are conformally equivalent to Einstein gravity with a matter sector containing a minimally coupled, self interacting scalar field $\phi$, with an effective potential $V(\phi)$ [18, 23]. The precise form of the self interaction of the scalar field is determined by the higher derivative terms in the action. For a general polynomial Lagrangian of order $n$ of the form (2.2), the form of the potential is extremely complicated. However, one can obtain the qualitative behavior of the potentials for all values of $n$ at small and large values of the field $\phi$. The asymptotic behavior of potentials at infinity depends on the combination of the highest degree of the Lagrangian polynomial and the dimensionality $D$ of space-time. In particular, for $D > 2N$ the potential is unbounded from above, for $D = 2N$ it flattens into a plateau and for $D < 2N$ it has an exponentially decaying tail [18]. Importantly, for $D < 2N$, $V(\phi)$ remains qualitatively the same as $N$ increases. As a result, in the 4 dimensional case, the qualitative behavior of $V(\phi)$ does not change.
relative to the $R^3$ case even when higher order terms with $N > 3$ are considered. This implies that the $n = 2$ contribution is rather special in four dimensions, and that the $R^3$-term is in this sense more generic, being prototypical of higher order perturbations to the action. Thus, it is important to check if the results of the quadratic theory are robust with respect to the more general $R^3$ perturbations. It is also worth mentioning in this connection the results of [13] which show that in the case of the cubic theories, there exists a region of parameter space in which neither the tunnelling nor the Hartle-Hawking boundary conditions predict a suitable inflationary scenario, good enough to solve the horizon and flatness problems, contrary to what is obtained in the case of quadratic theories. We note that another cubic term, $R \Box R$, which may also be reduced to a theory of Einstein gravity, minimally coupled to two scalar fields, after a suitable conformal transformation, is not considered here.

As was mentioned above, the theory represented by the Lagrangian $f(R)$ can be written as a scalar field theory minimally coupled to Einstein gravity. More precisely, considering a conformal transformation of the form

$$\tilde{g}_{\mu\nu} = \Omega^2 \, g_{\mu\nu}$$

(3.19)

where $\ln \Omega = \frac{\phi}{\sqrt{6}} = \frac{1}{2} \ln |f'(R)|$, allows such a general action to be transformed into

$$I = - \int \sqrt{\tilde{g}} \left[ R(\tilde{g}) - \frac{1}{2} (\tilde{\partial} \phi)^2 - V(\phi) \right],$$

(3.20)

with $V(\phi) = \frac{1}{2} e^{-2\sqrt{2}/3} \left[ R \frac{\partial f}{\partial R} - f(R) \right]$. Thus knowing the precise form of $f(R)$, allows the corresponding potential to be determined, in principle.

The Einstein field equations corresponding to the action (3.20) is derived using the equation $\frac{\partial I}{\partial \tilde{g}^{\mu\nu}} = 0$ and the scalar field equation is obtained from $\frac{\partial I}{\partial \phi} = 0$. Choosing $\tilde{g}$ to be given by the metric

$$ds^2 = d\tilde{\sigma}^2 + a^2(\tilde{\sigma}) \left( d\psi^2 + \sin^2 \psi \, d\Omega_2^2 \right),$$

(3.21)

the corresponding field equations become

$$3 \left[ \frac{a'^2 - 1}{a^2} \right] = \left[ \frac{1}{2} \phi'^2 - V(\phi) \right],$$

(3.22)

$$\phi'' + 3 \frac{a'}{a} \phi' = \frac{\partial V}{\partial \phi},$$

(3.23)

where primes denote derivatives with respect to $\tilde{\sigma}$. We now look for non-singular de Sitter as well as singular HT instanton solutions of these equations. The two cases of interest to us here are the theories with quadratic and cubic Lagrangians.
I. The quadratic case

In this case the Lagrangian takes the form \( f(R) = R + \alpha R^2 - 2\Lambda \), with the corresponding scalar field potential given by

\[
V(\phi) = \frac{1}{8\alpha} e^{-2\sqrt{2}\phi} \left[ e^{\sqrt{2}\phi} - 1 \right]^2 + \Lambda e^{-2\sqrt{2}\phi}.
\]  

(3.24)

With a non-zero cosmological constant, the de Sitter instanton solution exists for \( \alpha > 0, \Lambda > 0 \) as well as for \( \alpha < 0, 0 < \Lambda < \frac{1}{8|\alpha|} \), and is given by

\[
\phi = \phi_o = \sqrt{\frac{3}{2}} \ln(1 + 8\alpha\Lambda),
\]

\[
a(\tilde{\sigma}) = \sqrt{\frac{3(1 + 8\alpha\Lambda)}{\Lambda}} \sin \sqrt{\frac{\Lambda}{3(1 + 8\alpha\Lambda)}} \tilde{\sigma}.
\]

(3.25)

The corresponding Euclidean action evaluated in this case becomes

\[
I_E = -\left( \frac{3\pi}{2\Lambda} + 12\pi\alpha \right)
\]

(3.26)

which is the same as in the original quadratic theory.

Considering the conditions for the singular HT instanton solutions, we note that such solutions cannot be obtained for all values of the parameters \( \alpha \) and \( \Lambda \) in the quadratic theory. The existence conditions can be obtained by recalling the general features of the potential \( V(\phi) \) in this case (with \( \alpha' = -\alpha \)), namely:

- \( V(\phi) \) has two zeros, \( \phi_+ \) and \( \phi_- \), given by

\[
\phi_{\pm} = \sqrt{\frac{3}{2}} \ln(1 \pm \sqrt{8\alpha'\Lambda}).
\]

(3.27)

- \( V(\phi) \) has a maximum at \( \phi = \phi_m = \sqrt{\frac{3}{2}} \ln(1 - 8\alpha'\Lambda) \), with \( V(\phi_m) = \frac{\Lambda}{1 - 8\alpha'\Lambda} \).
- \( V(\phi) \rightarrow -\frac{1}{8\alpha'} \) as \( \phi \rightarrow \infty \).

The choice of initial conditions play a dominant role in the evolution of these solutions. Let us assume that at \( \tilde{\sigma} = 0, \phi(0) = \phi_+ \) and \( V(\phi_+) = 0 \). It then follows that \( \phi'(0) = 0 \) and \( \frac{dV}{d\phi}|_{\phi_+} = -\sqrt{\frac{\Lambda}{3\alpha'\Lambda}} \). Since the point \( \tilde{\sigma} = 0 \) is a non-singular point, the manifold looks locally like \( R^4 \) in spherical polar coordinates and we may assume \( a(\tilde{\sigma}) = v_o \tilde{\sigma} + 0(\tilde{\sigma}^2) \) where \( v_o \) is the initial velocity, i.e., \( a'(0) = v_o \). The potential has a negative gradient at \( \tilde{\sigma} = 0 \). The initial conditions along with the field Eqs. (3.22) and (3.23) determine the evolutions of \( b \) and \( \phi \).
To find conditions for the existence of singular HT in this case, we assume that close to the singularity \( \tilde{\sigma}_f \), ( \( \tilde{\sigma}_f - \tilde{\sigma} < 1 \) ) behave as

\[
\phi = q \ln(\tilde{\sigma}_f - \tilde{\sigma}) \quad (3.28)
\]

\[
a \sim (\tilde{\sigma}_f - \tilde{\sigma})^n \quad (3.29)
\]

Eq. (3.22) then determines \( q = \sqrt{\frac{3}{2}} \) for \( n < 1 \) and Eq.(3.23) gives

\[
n = \frac{3}{4} = \frac{1}{3} \left[ 1 + \frac{1 - 8\alpha'\Lambda}{6\alpha'} \right] . \quad (3.30)
\]

This in turn determines \( \Lambda \) in terms of \( \alpha' \)

\[
\Lambda = f(\alpha') = \frac{2 - 15\alpha'}{16\alpha'}
\]

and hence results in the bound \( \alpha' < \frac{2}{15} \) on the coupling constant of the quadratic term. Thus the conditions for the existence of a singular HT instanton (a singularity at which \( a \) vanishes as \( \tilde{\sigma} \to \tilde{\sigma}_f \) and \( \phi \) diverges logarithmically) are in this case given by: \( \alpha < 0, \Lambda = f(\alpha) \) and \( 8|\alpha|\Lambda < 1 \). There are therefore the following possibilities in this case:

- The scalar field may move uphill and become stabilised at \( \phi_m \), giving the non-singular instanton solution.
- The universe may end up in a singularity at \( \tilde{\sigma} = \tilde{\sigma}_f \), giving the singular HT instanton.

The important point here is that the existence of singular HT instantons in this case requires \( \alpha < 0 \), which makes them rather special. We also note that in this model, \( V(\phi) \) cannot be neglected even close to the singularity, in contradiction to the case considered by [2].

II. The cubic case

In this case the Lagrangian takes the form \( f(R) = R + \alpha R^2 + \beta R^3 - 2\Lambda \), with the corresponding scalar field potential given by

\[
V(\phi) = \frac{\alpha^3}{27\beta^2} e^{-2\sqrt{\frac{2}{3}} \phi} \left[ -1 + \frac{9\beta}{2\alpha^2} \left( 1 - e^{\frac{2}{\sqrt{3}} \phi} \right) + \left( 1 - \frac{3\beta}{\alpha^2} \left( 1 - e^{\frac{2}{\sqrt{3}} \phi} \right) \right)^\frac{3}{2} \right]
\]

\[
+ \Lambda e^{-2\sqrt{\frac{2}{3}} \phi} . \quad (3.31)
\]

With a non-zero cosmological constant, there exist de Sitter instanton solutions in this case of the form

\[
\phi = \phi_o .
\]
Figure 1: Plots of the potential $V(\phi)$ (3.31), with $\Lambda = 0$, $\alpha = 1$ and various values of $\beta$. The dark, dashed, broken and large-dashed lines correspond to $\beta = 0.05, 0.002, -0.001$ and $-0.05$ respectively.

$$a(\tilde{\sigma}) = \frac{1}{H_o} \sin H_o \tilde{\sigma}$$

(3.32)

where $\phi_o$ is the solution of $dV(\phi)/d\phi = 0$ for the potential (3.31) and $H_o$ is given by $3H_o^2 = V(\phi_o)$. In the case of $\Lambda = 0$, the Euclidean action can be evaluated to be

$$I_E = -12\pi(\alpha + 2\sqrt{\beta}),$$

(3.33)

which is real and negative for $\alpha > 0, \beta > 0$.

We now look for conditions for the existence of singular HT instanton in the cubic theory. We recall that the scalar field potential in this case has a maximum for $\alpha > 0$ and $\beta \geq 0$, but becomes flat at large $\phi$ for $\beta \to 0$, as in the quadratic theory. Fig. 1 shows the potential $V(\phi)$ (3.31) for a number of values of $\beta$ with $\alpha = 1$ and $\Lambda = 0$. Given that in this case $\alpha < 0$ makes $V(\phi) < 0$, we shall restrict ourselves to $\alpha > 0$.

Now in the case of the cubic theory with $\Lambda = 0$, one obtains HT instantons corresponding to the solution of the equations (3.28) and (3.29) with

$$q = \sqrt{\frac{3}{2}} \quad \text{and} \quad n = \frac{3}{4}.$$

(3.34)

On the other hand, when $\Lambda \neq 0$, HT instantons are allowed if $\Lambda$ satisfies

$$\Lambda = -\frac{15}{16} - \frac{\alpha^3}{27\beta^2} \left[ -1 + \frac{9\beta}{2\alpha^2} + \left(1 - \frac{3\beta}{\alpha^2}\right)^{3/2}\right].$$

(3.35)
Thus $\Lambda$ in this case is determined in terms of the dimensional constants $\alpha$ and $\beta$ satisfying the constraint $\beta < \frac{\alpha^2}{3}$. Note that $\alpha$ can assume some small negative values if $\Lambda \neq 0$. The conditions for the existence of HT instantons in this case are:

- $V(\phi)$ has a maximum.
- $\alpha^2 > 3\beta$.
- $\Lambda$ satisfies the constraint given by (3.35) and could be positive, zero or negative, depending upon the values of $\alpha$ and $\beta$.

4 Discussion

We have studied the existence of singular and non-singular instanton solutions in theories of gravity with quadratic and cubic Lagrangians. We find that in the quadratic case, the existence of both types of instantons requires a non–zero cosmological constant; whereas in the cubic case instanton solutions can also exist with a vanishing cosmological constant.

To study the existence of HT instantons and clarify the special features of the two types of solutions, expressed the higher order Lagrangian theories as Einstein gravity minimally coupled to a scalar field, using a conformal transformation of the metric. In the case of the quadratic theory, the non-singular instanton corresponds to the scalar field sitting at the maximum of the potential. This gives an indication that the Lorentzian continuation of this solution ( with $\alpha < 0$ ) is stable, as is the case with the de Sitter solution in the quadratic theory for $\alpha < 0$ [19]. The fact that HT instantons cannot be obtained with $\alpha > 0$ shows that they are rather special in the quadratic theory. There is also no HT instantons for $\Lambda < 0$ in this case. In the case of the cubic theory, HT instantons can exist for both $\alpha < 0$ and $\alpha > 0$ subject to a constraint which relates the cosmological constant to the dimensional constants $\alpha$ and $\beta$ in the gravitational action.

Finally, in view of our discussion of higher order polynomial Lagrangians in section 2, we would expect our general results regarding the behaviour of the HT instantons, obtained using the cubic theory, to hold in higher order Lagrangian settings, although the algebraic relations among the dimensional constants, $\alpha$, $\beta$ etc. and $\Lambda$ would be different at different orders.

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