Almost Convex Groups and the Eight Geometries

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Abstract. If $M$ is a closed Nil geometry 3-manifold then $\pi_1(M)$ is almost convex with respect to a fairly simple “geometric” generating set. If $G$ is a central extension or a $\mathbb{Z}$ extension of a word hyperbolic group, then $G$ is also almost convex with respect to some generating set. Combining these with previously known results shows that if $M$ is a closed 3-manifold with one of Thurston’s eight geometries, $\pi_1(M)$ is almost convex with respect to some generating set if and only if the geometry in question is not Sol.

Introduction.

In [C], Cannon introduced the notion of a finitely generated group being almost convex with respect to a given generating set. This property is formulated in terms of the geometry of the Cayley graph, and gives a simple and efficient algorithm for constructing the Cayley graph. One class of groups one would like to study in terms of this property is the class of fundamental groups of closed 3-manifolds carrying one of Thurston’s eight geometries. (For an account of these, see [Sc].) In most cases, the answer is already known. If $G = \pi_1(M)$ where $M$ is a Riemannian 3-manifold whose universal cover is $S^3$, $\mathbb{H}^3$, or $S^2 \times \mathbb{R}$, then $G$ is word hyperbolic and thus almost convex with respect to any finite generating set. If $M$ is covered by Euclidean 3-space, $E^3$, then Cannon [C] shows that $G$ is almost convex. Now Cannon et al. [CFGT] show that if $M$ is a compact quotient of Sol, then $G$ is not almost convex with respect to any generating set.

For the remaining three geometries, it is known that $G$ is almost convex with respect to some generating set for special cases. Specifically, if the universal cover is Nil and $M$ fibers over a torus, then $G$ is almost convex with respect to a standard generating set [Sh1]. If the universal cover is $\widetilde{PSL}_2 \mathbb{R}$ and $M$ fibers over a closed orientable surface, then $G$ is almost convex with respect to a standard generating set [Sh2]. And if the universal cover is $\mathbb{H}^2 \times \mathbb{R}$ and $G = H \times \mathbb{Z}$ where $H$ is a hyperbolic surface group, then $G$ is almost convex with respect to any split generating set. Note that in all the remaining cases, $G$ is a finite index supergroup of the Nil, $\widetilde{PSL}_2 \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ cases we have just discussed. Now almost convexity is not a commensurability invariant. Indeed, as Thiel has shown [T], it is not a group invariant, but does in fact depend on generating set. Still, it has long seemed likely that the remaining Nil, $\widetilde{PSL}_2 \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ groups would turn out to be almost convex. We will show this to be true, thus proving

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**Theorem 1.** Suppose $G$ is the fundamental group of a closed 3-manifold $M$ carrying one of Thurston’s eight geometries. Then $G$ is almost convex with respect to some generating set if and only if $M$ is not a Sol geometry manifold.

This paper is organized as follows. Section 1 contains background and definitions. In Section 2 we establish almost convexity for the remaining Nil groups. In Section 3 we establish almost convexity for the remaining $\overline{PSL_2}\mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ cases. In the course of this, we prove the following general result.

**Theorem 2.** Suppose that $H$ is word hyperbolic, that $A$ is finitely generated abelian, and that

\[ 1 \to A \to G \xrightarrow{\pi} H \to 1 \]

is a central extension. Then there is a finite generating set $G$ for $G$ so that $G$ is almost convex with respect to $G$.

1. Background and definitions.

Let $G$ be a finitely generated group and $\mathcal{G}$ a finite set, and $a \mapsto \overline{a}$ a map of $\mathcal{G}$ to a monoid generating set $\overline{\mathcal{G}} \subset G$. We use $\mathcal{G}^\ast$ to denote the free monoid on $\mathcal{G}$. We refer to the elements of $\mathcal{G}^\ast$ as *words*. A word $w = a_1 \ldots a_n$ is said to have *length* $n$. This is denoted $\ell(w) = n$. The map of $\mathcal{G}$ into $G$ extends to a unique monoid homomorphism which we denote by $w \mapsto \overline{w}$. We will assume that $\mathcal{G}$ is supplied with an involution $a \mapsto a^{-1}$ and that this involution respects inverses in $G$, that is, $\overline{a^{-1}} = (\overline{a})^{-1}$. In all cases that we will consider, the map of $\mathcal{G}$ into $G$ is an injection. Thus we can make the following convention: any list of generators will be taken to include the inverses of those listed. Thus a definition such as $\mathcal{X} = \{x, y\}$ will mean $\mathcal{X} = \{x^{\pm 1}, y^{\pm 1}\}$. If a generating set contains some element $\rho$ so that $\overline{\rho}$ has order two, we will take $\rho = \rho^{-1}$.

Given such a group $G$ and $\mathcal{G}$, we can form the Cayley graph $\Gamma$ of $G$ with respect to $\mathcal{G}$. This is a directed labelled graph $\Gamma = \Gamma_\mathcal{G}(G)$. The vertices of $\Gamma$ are the elements of $G$. There is a directed edge $(g, a, g')$ from $g$ to $g'$ with label $a$ exactly when $g' = ga$ with $a \in \mathcal{G}$. Since $\overline{\mathcal{G}}$ generates $G$, $\Gamma$ is connected. One turns $\Gamma$ into a metric space by declaring each edge isomorphic with the unit interval and taking the induced path metric. We denote this metric by $d(\cdot, \cdot) = d_\mathcal{G}(\cdot, \cdot)$. This, in turn, gives each element $x \in \Gamma$ a *length*, $\ell(x) = \ell_\mathcal{G} = d_\mathcal{G}(1, x)$. As usual, we take the *ball of radius* $r$, $B(r)$ to be $\{x \in \Gamma \mid \ell(x) \leq r\}$.

Each edge path in $\Gamma$ is labelled by a unique element of $\mathcal{G}^\ast$. We identify each word with the edge path it labels starting at $1 \in G$. We take this path to be parameterized with unit speed, and extend each word $w$ to a map of $[0, \infty)$ by setting $w(t) = \overline{w}$ for $t \geq \ell(w)$. The translate of the path $w$ by the group element $g$ is denoted $gw$. This is the path based at $g$ bearing label $w$. We say that $w$ is a *geodesic* if $w\big|_{[0,\ell(w)]}$ is an isometry. Equivalently, $w$ is a geodesic if $\ell(w) = \ell(\overline{w})$. We say that $w$ is a $(\lambda, \epsilon)$ *quasigeodesic* if for every subword $u$ of $w$, $\ell(u) \leq \lambda \ell(\overline{u}) + \epsilon$.

Following [C], we say that $G$ is *almost convex* (m) with respect to $\mathcal{G}$ if there is a constant $K(m)$ with the following property: if $\ell_\mathcal{G}(g) = \ell_\mathcal{G}(g') = n$ and $d_\mathcal{G}(g, g') \leq m$ then there is an edgepath $p$ in $\Gamma$ which runs from $g$ to $g'$, lies inside $B(n)$ and has length bounded by $K(n)$. We say that $G$ is *almost convex* with respect to $\mathcal{G}$ if it is almost convex.
(m) with respect to G for all m. It is a result of [C] that if G is almost convex (2) with respect to G, then G is almost convex with respect to G. We will say that G is almost convex if there is some G so that G is almost convex with respect to G.

In Section 3, we will need some standard results concerning word hyperbolic groups. For an account of these, see, for example, [Sho]. We say that G is word hyperbolic if there is a generating set G and a constant δ so that if α, β and γ are geodesic edge paths forming a triangle in ΓG(G), then if p is any point on α, d(p, β ∪ γ) ≤ δ. In fact, the existence of such a δ is independent of choice of G. Given G and G, we say D = \{r_1, ..., r_k\} ⊂ G* is a Dehn’s algorithm if for each r_i ∈ D, we have πi = 1 and if for any w ∈ G*, if πw = 1 then w contains more than half of some r_i ∈ D as a subword. In fact, the existence of a Dehn’s algorithm can be taken as a definition of a word hyperbolic group. That is, G is word hyperbolic if and only if for any generating set G, there is a Dehn’s algorithm D ⊂ G*.

Given a Dehn’s algorithm D, we say a word w is D-reduced if w does not contain more than half of any word in D. By standard methods, one can check that given a Dehn’s algorithm D, there are λ and ε so that D reduced words are (λ, ε) quasigeodesics. It is a standard hyperbolic result that for any distance m, there is a constant k(m) so that if u, v are (λ, ε) quasigeodesics with d(πu, πv) ≤ m then each point of u lies within distance k(m) of v and vice versa.

2. Nil manifold groups.

It is shown in [Sh1] that if M in a closed 3-manifold with Nil geometry, and M fibers over a torus then π1(M) is almost convex. More specifically,

Theorem 3. Let

\[ N = \langle x, y \mid [[x, y], x] = [[x, y], y] = 1 \rangle, \]

\[ N^e = \langle x, y, z \mid [x, z] = [y, z] = 1, \ [x, y] = z^e \rangle. \]

We take

\[ \mathcal{N} = \{x, y\} \subset N, \]

\[ \mathcal{N}^e = \{x, y, z\} \subset N^e. \]

Then N is almost convex with respect to \( \mathcal{N} \). For e ≥ 1, \( N^e \) is almost convex with respect to \( \mathcal{N}^e \).

Proof. The assertions about N and \( N^e \) for e > 1 are proven in [Sh1]. The assertion about \( N^1 \) uses the following lemma.

Lemma 4. Suppose \( G \) and \( H \) are generating sets for G such that G is almost convex with respect to \( G \), and there exists k such that for all g ∈ G, |\( \ell_G(g) - \ell_H(g) \)| ≤ k. Then G is almost convex with respect to H.

Proof. Let λ = max\{\( \ell_G(h) \mid h ∈ H \}\} \cup \{\( \ell_H(g) \mid g ∈ G \}\). Suppose \( \ell_H(h) = \ell_H(h') = n \) and \( d_H(h, h') ≤ 2 \). We must find a path of bounded length from h to h' lying inside the H ball of radius n. We can assume that n ≥ 2k + λ + 1, for otherwise, h and h' are connected by a path of length at most 4k + 2λ + 2. Let \( a_1 \ldots a_n \) and \( b_1 \ldots b_n \) be H geodesics for h and h'. Then \( g = a_1 \ldots a_{n-2k-\lambda-1} \) and \( g' = b_1 \ldots b_{n-2k-\lambda-1} \) both lie inside the G ball of...
radius \( n - k - \lambda - 1 \), and lie within \( \mathcal{G} \) distance \( \lambda(4k + 2\lambda + 4) \) of each other. Since \( G \) is almost convex with respect to \( \mathcal{G} \), they are connected by a \( \mathcal{G} \) path \( p \) of length at most \( K \) which lies entirely within the \( \mathcal{G} \) ball of radius \( n - k - \lambda - 1 \). The path \( p \) is easily turned into a \( \mathcal{H} \) path \( p' \) of \( \mathcal{H} \) length at most \( \lambda K \) whose vertices all lie in the \( \mathcal{G} \) ball of radius \( n - k - 1 \). In particular, \( p' \) lies entirely within the \( \mathcal{H} \) ball of radius \( n \), and so does the path from \( h \) to \( h' \) labelled \( a_{n-2k-\lambda}^{-1} \ldots a_{n-2k-\lambda}^{-1} b_{n-2k-\lambda} \ldots b_n \).

Continuing with the proof of Theorem 3, we first note that \( N \) is isomorphic to \( N^1 \). We now observe that if \( g \in N < N^e \) then \( \ell_N(g) \leq \ell_N^e(g) \leq \ell_N^e(g) + 20 \).

The first inequality is obvious. To see the second, notice that a geodesic cannot contain both \( z \) and \( z^{-1} \), for these could be commuted together and deleted. Nor can any geodesic contain more than 25 \( z \)'s or \( z^{-1} \)'s. For these could be commuted to the end of the word and replaced with \([x^5, y^5]^{\pm 1}\), thus shortening the word. Finally, check that for \(-25 \leq i \leq 25\), \( \ell_N(z^i) \leq 20 \). Now applying Lemma 4 completes the proof that \( N \) is almost convex with respect to \( N^1 \).

We have seen that if \( M \) is a Nil manifold that fibers over the torus, then \( \pi_1(M) \) is almost convex with respect to the above generating sets. We would like to extend this result to the general case. Now, every Nil manifold fibers* over a 2-dimensional Euclidean orbifold. In fact, the list of orbifolds \( E \) which occur in this role is quite restrictive. To see this, first recall from [Sc] that there are no orientation reversing isometries of Nil. This implies that \( E \) can not have any reflector curves, for each reflector curve must lift to an orientation reversing element of \( \pi_1(M) \). Thus the underlying surface of \( E \) must be closed and its singularities (if any) are all cone points. Now \( E \) must have 0 as its orbifold Euler characteristic. Thus, if \( E \) is orientable, then \( E \) is either the torus or one of \( S(2,2,2,2) \), \( S(2,4,4) \), \( S(3,3,3) \), \( S(2,3,6) \). (These denote the sphere with cone points of the orders listed.) If \( E \) is not orientable, then \( E \) is either the Klein bottle or \( P(2,2) \), the projective plane with two cone points of order 2.

In each of these cases, if \( G = \pi_1(M) \), we have the diagram

\[
\begin{array}{ccccccccc}
1 & \rightarrow & \mathbb{Z} & \rightarrow & N^e & \rightarrow & \mathbb{Z}^2 & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathbb{Z} & \rightarrow & G & \rightarrow & \pi_{1}^{\text{orb}}(E) & \rightarrow & 1 \\
\downarrow & & \downarrow & & Q & \cong & Q & & \\
\downarrow & & \downarrow & & 1 & & 1 & & \\
\end{array}
\]

If \( E \) is orientable, each of the \( \mathbb{Z} \) kernels is central. In the case where \( E \) is not orientable, each element of \( \pi_{1}^{\text{orb}}(E) \) acts trivially or nontrivially on the \( \mathbb{Z} \) kernel depending on whether it is orientation preserving or reversing.

* More properly, we should say “Seifert fibers,” but we will not keep up this distinction.
Our strategy for all of these groups is to use the extension

\[ 1 \to N^e \to G \to Q \to 1 \]

to lift the almost convexity of \( N^e \) up to \( G \). We first establish the lemmas we will need.

**Lemma 5.** Suppose

\[ 1 \to H \overset{i}{\to} G \overset{p}{\to} Q \to 1 \]

with \( Q \) finite. Suppose \( H \) and \( G \) are generating sets for \( H \) and \( G \) respectively so that \( H \subset \mathcal{G} \), 
\( i \) is an isometry, and the elements of \( \mathcal{G} \setminus H \) permute the elements of \( H \) when acting by conjugation. Then there is a finite set \( T \subset (\mathcal{G} \setminus H)^* \) so that every element of \( G \) has \( \mathcal{G} \) geodesic lying in \( H^*T \).

**Proof.** We will take \( T = \{ t \in (\mathcal{G} \setminus H)^*: \ell(t) \leq \#Q \} \).

Let \( g \in G \) and suppose \( w \) is a geodesic with \( \overline{w} = g \). Since the elements of \( \mathcal{G} \setminus H \) permute the elements of \( H \), it is easy to see that we can find \( w' \) so that \( \overline{w} = \overline{w'} \), \( \ell(w) = \ell(w') \) and \( w' = h_1 \ldots h_m g_1 \ldots g_n \) where \( h_i \in H \) for \( 1 \leq i \leq m \) and \( g_i \in \mathcal{G} \setminus H \) for \( 1 \leq i \leq n \). If \( n \leq \#Q \), we are done. If not, then by the pigeon hole principle, there are \( j \) and \( k \), \( 1 \leq j < k \leq n \) so that \( p(g_1 \ldots g_j) = p(g_1 \ldots g_k) \). Then \( g_{j+1} \ldots g_k \in H \). Since \( i \) is an isometry, we can replace \( g_{j+1} \ldots g_k \) by an expression of equal length in \( H^* \). Continuing with these two processes produces a geodesic of the desired form. \( \blacksquare \)

**Corollary 6.** Suppose

\[ 1 \to H \overset{i}{\to} G \overset{p}{\to} Q \to 1 \]

with \( Q \) finite. Suppose \( H \) and \( G \) are generating sets for \( H \) and \( G \) so that \( H \subset \mathcal{G} \), the elements of \( \mathcal{G} \setminus H \) permute the elements of \( H \) when acting by conjugation, and \( i \) is an isometry. If \( H \) is almost convex with respect to \( H \), then \( G \) is almost convex with respect to \( G \).

**Proof.** Suppose that \( g, g' \in G, \ell(g) = \ell(g') \) and \( d(g, g') \leq 2 \). We invoke the previous lemma and perhaps interchange \( g \) and \( g' \) to find geodesics \( g = \overline{u}, g' = \overline{u'v't'} \), where \( u, u'v' \in \mathcal{H}^* \), \( t, t' \in T \), and \( \ell(v't') = \ell(t) \leq \#Q \). Consequently, \( \overline{u}, \overline{w} \in H \), with \( d(\overline{u}, \overline{w}) \leq 2\#Q + 2 \). Using the fact that \( H \) is almost convex, there is a path of length at most \( K = K(2\#Q + 2) \) connecting \( \overline{u} \) to \( \overline{w} \) inside the ball of radius \( \ell(u) \) in \( H \). This path lies inside the ball of radius \( \ell(u) \) in \( G \). But this gives us a path of length at most \( K + 2\#Q \) connecting \( g \) to \( g' \) inside the ball of radius \( \ell(g) \).

Now we will use Corollary 6 to show that when \( E \) is the Klein bottle, \( S(2,2,2,4) \), \( S(2,2,2) \) or \( P(2,2) \), \( G \) is almost convex. First we note that in these cases, the action of \( Q \) on \( \mathbb{Z}^2 \) preserves the standard generating set. Now in each of these cases, we can find a generating set for \( G \) by lifting the orbifold generating set for \( \pi^1_{\text{orb}}(E) \) and appending \( z \). Conjugation by these generators of \( G \) preserves a generating set of \( N^e \) of the form

\[ N^e_{\mathfrak{s}} = \{ x, xz^{i_1}, \ldots, xz^{i_s}, y, yz^{j_1}, \ldots, yz^{j_t}, z^{k_1}, \ldots, z^{k_v} \}. \]

We shall call such a generating set *saturation* of \( N^e \). In order to use Corollary 6, we need to establish
Theorem 7. $N^e$ is almost convex with respect to any saturation $N^e_s$ of $N$.

This follows from Lemma 4 together with the following

Lemma 8. Let $N^e_s$ be a saturation of $N^e$. Then there is a constant $K$ so that for all $g \in N^e$, $|\ell_{N^e_s}(g) - \ell_{N^e}(g)| \leq K$.

In order to do this, we need to recall some facts about geodesics in $N$ in the generating set $N$, established in [Sh1]. We review this since we will need to use these methods later for the $(3,3,3)$ and $(3,6,6)$ groups. The viewpoint followed in [Sh1] is to use the short exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow N \rightarrow \mathbb{Z}^2 \rightarrow 1$$

to view words in $N^*$ as lifts of paths in the Cayley graph of $\mathbb{Z}^2$ based at the identity. Two such paths $\alpha$ and $\beta$ represent the same element of $N$ if and only if they end at the same point in $\mathbb{Z}^2$, and the concatenation $\alpha \beta^{-1}$ encloses zero signed area. For a given $g = (a, b) \in \mathbb{Z}^2$, we let $B_g(n) = B_{g^{-\ell(g)}} = B(n) \cap p^{-1}(g)$. (Recall that $B(n)$ is the ball of radius $n$ in $\Gamma$. Since $p^{-1}(g)$ consists only of group elements, so does $B_g(n) = B_{g^{-\ell(g)}}$.)

For each $g = (a, b) \in \mathbb{Z}^2$, we identify $p^{-1}(g)$ with $\mathbb{Z}$ by carrying $x^a y^b [x, y]^t$ to $t$. We observe

1. Geodesics for elements in $p^{-1}(0,0)$ project to closed paths; in fact, to simple closed curves.

2. If you can enclose $n$ squares with a loop of length $l$, you can enclose $n - 1$ squares with a loop of length at most $l$. This shows that $B^n_{(0,0)}$ is an interval. In fact,

3. For any $g \in \mathbb{Z}^2$, $B^n_g$ is an interval.

We will call the elements of $B^n_g$ at the extremes of the interval extremals. A geodesic or a projection of a geodesic for an extremal will also be called extremal. We also observe

4. The number of squares enclosed by a closed extremal is strictly monotone in the length of the extremal.

From this we deduce that

- Closed extremals are rectangles. Indeed, their sides differ in length by at most one.

We call such a rectangle an almost square.

For suppose $r$ is any simple closed curve. Let $R$ be the unique minimal rectangle containing $r$. $R$ encloses at least as many squares as little $r$, but $R$ is no longer than $r$. This is easy to see: first notice that minimality of $R$ ensures that $r$ meets all sides of $R$. But now given two points on successive sides of $R$, then the subpath of $R$ that connects them is geodesic, so it is no longer than the corresponding subpath of $r$. Now, since the largest rectangles for a fixed perimeter are almost squares, (4) follows.

We are now prepared to describe a sublanguage of the geodesics in $N$ which contains at least one geodesic for each element of $N$ and suffices for our purposes.

First notice that for each $g \in \mathbb{Z}^2$, all words representing elements of $p^{-1}(g)$ differ in length from $g$ by an even amount. This follows from the fact that all relators are of even length.

(G1) For each $g \in \mathbb{Z}^2$, the elements of $B^n_g$ are all represented by geodesics which project to geodesics in $\mathbb{Z}^2$. (Indeed, every geodesic in $\mathbb{Z}^2$ occurs in this role.)

(G2) Any geodesic for an extremal element projects to a subpath of an almost square.
(G3) For every element of $B_{g}^{2k}$, $k \geq 1$, there is a geodesic whose projection first follows the almost square given by an extremal of $B_{g}^{2k-2}$, then crosses via a single edge to follow the almost square given by an extremal of $B_{g}^{2k}$.

We are now ready to prove Lemma 8.

Proof of Lemma 8. Recall our generating sets $N^e = \{x, y, z\}$, and

$$N^e_s = \{x, xz^{j}, \ldots, xz^{i}, y, yz^{j}, \ldots, yz^{i}, z^{k}, \ldots, z^{k_e}\}.$$ Choose $g \in N^e$. Let $w$ be a geodesic word in $N^e_s$ with $\overline{w} = g$. Now let $w'$ be the word in $N^e_s$ obtained by deleting all generators of type $z^a$ in $w$ and replacing each $x^{\pm 1}z^b$ and $y^{\pm 1}z^c$ generator by $x^{\pm 1}$ and $y^{\pm 1}$ respectively. Let $g' = w' \in N < N^e$. Now, $g = g'z^{\pm t}$ where $0 \leq t \leq k\ell(w)$ with

$$k = \max\{|i_1|, \ldots, |i_a|, |j_1|, \ldots, |j_b|, |k_1|, \ldots, |k_e|\}.$$ Notice that $[x^m, y^m] = [x, y]^{m^2} = z^{em^2}$. Hence the $N^e$-length of $z^{\pm t}$ is at most $4\sqrt{t/e} + e \leq 4\sqrt{t/e} + e + 4$. So now if $v \in \{x, y\}^*$ is an $N$ geodesic for $g' \in N$, then

$$\ell(w) \leq \ell(v) + 4\sqrt{t/e} + e + 4 \leq \ell(v) + 4\sqrt{\frac{k\ell(w)}{e}} + e + 4.$$ So if $\ell(w) = \ell_G(g)$ is sufficiently large, we will have

$$1/2\ell(w) \leq \ell(v) \leq \ell(w).$$ Now $g = vz^{\pm t}$ where $t \leq k\ell(w) \leq 2k\ell(v)$. But we can assume that the projection of $v$ looks like one of the projections described in (G1) — (G3), and hence has a straight piece of length $\ell(v)/4$. So by inserting either $x$ and $x^{-1}$ or $y$ and $y^{-1}$, around the piece of $v$ corresponding to this side, we increase the word length by 2, but increase enclosed area by $\ell(v)/4$ squares. Continuing in this way, and introducing at most $\lceil 8k/e \rceil$ such pairs of generators and at most $e$ $z$’s, we produce a word for $g$. This has increased length by at most $\lceil 16k/e \rceil + e$. Thus we have

$$\ell_{N^e_s}(g) \leq \ell_{N^e}(v) \leq [16k/e] + e \leq \ell_{N^e_s}(g) + [16k/e] + e.$$ This proves Lemma 8 with $K = [16k/e] + e$.

We are now ready to establish the almost convexity of the Nil groups $G = \pi_1(M)$ such that $M$ fibers over $E$ where $E$ is the Klein bottle, $S(2, 2, 2, 2)$, $S(2, 4, 4)$ or $P(2, 2)$. We will use a generating set of the form $G = \overline{\mathcal{N}^e_s} \cup S$ where $S$ is a lift to $G$ of a standard generating set for $\pi_1^{orb}(E)$, and $\mathcal{N}^e_s$ is a saturated generating set preserved by $S$. We observed that there are saturated generating sets preserved by $S$, and that $N^e$ is almost convex with respect to any saturated generating set. So all that remains is to choose a saturated generating set $\mathcal{N}^e_s$ preserved by $S$ so that the inclusion $N^e \hookrightarrow G$ is geodesic with respect to the generating sets $\mathcal{N}^e_s$ and $\overline{\mathcal{N}^e_s} \cup S$. 7
We begin by noting that the inclusion $\mathbb{Z}^2 \hookrightarrow \pi_1^{orb}(E)$ is geodesic with respect to the generating sets $X = \{x, y\}$ and $E = \{\rho, x, y\}, \{a, b, c, d, x, y\}, \{p, q, r, x, y\}, \{a, b, \rho, x, y\}$ depending on whether $E$ is the Klein bottle, $S(2, 2, 2, 2)$ or $S(2, 4, 4)$ or $P(2, 2)$ (in each case, $\rho$ is orientation reversing, $a, b, c, d, p$ are rotations of order 2 and $q$ and $r$ are rotations of order 4.) We spell out the argument for $E = S(2, 2, 2, 2)$. We need to show that no geodesic in $\{x, y\}^*$ may be shortened by rewriting it in the $\{a, b, c, d, x, y\}$ generating set. So suppose $w'$ is an $\{x, y\}$ geodesic which can be shortened to a $\{a, b, c, d, x, y\}$ geodesic $w$. We can push the $\{a, b, c, d\}$ letters all to the right without changing the number of $\{x, y\}$ letters. Thus we assume can assume $w = uv$, where $u$ is composed of $\{x, y\}$ letters and $v$ is composed of $\{a, b, c, d\}$ letters. But now, since $Q = \mathbb{Z}_2$, any substring $v$ of length 3 or more includes a substring which evaluates into $\mathbb{Z}^2$. One checks that words in $\{a, b, c, d\}$ of length less than 3 which evaluate into $\mathbb{Z}^2$ are not shorter than the corresponding $\{x, y\}$ words for the elements they represent. By this process we can eliminate all $\{a, b, c, d\}$ letters from $w$ without increasing length. Since $w'$ was assumed to be an $X$ geodesic, this contradicts the assumption that $w$ is shorter than $w'$.

The argument for the other cases proceeds similarly. When $E$ is the Klein bottle, we check $\{\rho\}$ words of length 2. Since $\rho^2 = x$ the result is immediate. For $E = S(2, 4, 4)$, one checks all $\{p, q, r\}$ words of length at most 4. Similarly for the case $E = P(2, 2)$, we must check $\{a, b, \rho\}$ words of lengths at most 4.

We now wish to lift this to $N^e < G$. Choose a saturated generating $N'_s$ for $N^e$ which is preserved by conjugation by $S$. Suppose the inclusion $N^e \hookrightarrow G$ is not geodesic with respect to $N'_s$ and $G' = N'_s \cup S$. As above any failure of this inclusion to be geodesic can be observed in some short word $v \in S^*$ with $\ell(v) \leq \#Q$. Consider the projection of $v$ into $\pi_1^{orb}(E)$. Since the inclusion down below is geodesic, there is an $\{x, y\}$ word $v'$ which evaluates to this projection and $\ell(v') \leq \ell(v)$. We now consider $v'$ as a word in the generators for $N^e$. Note that $\overline{v} = v'z^k$ for some $k$. Further, only finitely many such $k$ occur since there are only finitely many short $v'$s. Let $K$ be the maximum of the absolute values of those $k$ which arise. We take

$$N^e_s = \{az^t \mid a \in N'_s, |t| \leq K\}.$$  

Clearly the inclusion of $N^e$ into $G$ is geodesic with respect to $N^e_s$ and $G = N^e_s \cup S$, and $N^e_s$ is preserved by the action of $S$.

Thus we have shown that if $G = \pi_1(M)$ is a Nil group so that $M$ fibers over the Klein bottle, $S(2, 2, 2, 2), S(2, 4, 4)$ or $P(2, 2)$, there is a generating set $G$ so that $G$ is almost convex with respect to $G$.

It is now easy to see that $G$ is almost convex with respect to the generating set $N^e \cup S$ or $N \cup S$ if this latter generates $G$. This is because every element of $G$ has a $G$ geodesic in which there are at most $\#Q S$ letters and these occur at the end. But now Lemma 8 tells us that replacing the $N^e_s$ part by the corresponding $N^e$ or $N$ geodesic increases length by at most a bounded amount. Thus by Lemma 4, $G$ is almost convex with respect to this reduced generating set.

We are now prepared to pursue the case where $M$ fibers over $S(3, 3, 3)$ or $S(2, 3, 6)$. We wish to pursue the same program. However, the action of our finite quotient no longer preserves the $x$ and $y$ directions in the plane, and hence does not preserve the generating
set \( \{x, y\} \). Rather, it preserves hexagonal symmetry, and hence preserves generating sets of the form \( \{x, y, xy\} \). (We may think of our \( x \) and \( y \) directions as labelling nonadjacent rays in this hexagonal symmetry.) Thus we will need to work with generating sets of the form

\[
X = \{x, y, t\} \subset \mathbb{Z}^2, \\
N = \{x, y, t\} \subset N, \\
N^e = \{x, y, t, z\} \subset N^e.
\]

(In each case, we take \( t = xy \).) To carry out our program, we must first prove

**Theorem 9.** \( N \) is almost convex with respect to \( N \).

**Theorem 10.** \( N^e \) is almost convex with respect to \( N^e \).

Any generating set of the form

\[
N^e_s = \{x, xz^{i_1}, \ldots xz^{i_a}, y, yz^{j_1}, \ldots yz^{j_b}, t, tz^{k_1}, \ldots tz^{k_c}, z, z^{l_1}, \ldots, z^{l_d}\}
\]

is called a *saturation* of \( N^e \).

**Corollary 11.** \( N^e \) is almost convex with respect to any saturation \( N^e_s \).

Once again, we study \( N \) by studying the projection of paths in \( N \) into \( \mathbb{Z}^2 \). This time, however, we have taken the generating set \( N = \{x, y, t\} \) and its projection \( X \) in \( \mathbb{Z}^2 \). The Cayley graph of \( \mathbb{Z}^2 \) with respect to \( X \) is the 1-skeleton of the tessellation of the plane by equilateral triangles. Now \( \pi^1_{orb} S(3, 3, 3) \) acts as the orientation preserving subgroup of the symmetries of this tessellation. There are two orbits of triangles under this action, those with boundary label \( xyt^{-1} \) (which we color white) and those with boundary label \( tx^{-1}y^{-1} \) (which we color black). A circuit around a white triangle lifts to the identity in \( N \). A circuit around a black triangle lifts to \([x, y]\) in \( N \). Consequently, a closed curve in the plane lifts to \([x, y]^n\), where \( n \) is the signed number of black triangles enclosed. As before, we can use this technology to treat \( N \) as equivalence classes of based edgepaths in the Cayley graph of \( \mathbb{Z}^2 \). Then after replacing the words “almost square” with “almost regular hexagon” (1) — (4) and (G1) — (G3) hold. (A hexagon is an almost regular hexagon if no two of its sides differ in length by more than 2.)

¿From this description of geodesics in \( N \) we can deduce

**Lemma 12.** Suppose that \( g, g' \in N \) and that \( \ell(g) = \ell(g') = n \) and that \( g' = g\overline{r} \) with \( \ell(r) \leq 2 \). Then one of the following occurs

1) Both \( g \) and \( g' \) are small, i.e., each has length less than 50.
2) There are standard geodesics for \( g \) and \( g' \) whose projections in \( \mathbb{Z}^2 \) lie in a 25 neighborhood of each other and 50-fellow travel.
3) The projection of \( gr \) crosses an axis. If it crosses exactly one axis, then there is \( g'' \) so that \( \ell(g'') = n \), \( d(g, g'') \leq 2 \), the projection of \( g'' \) lies on the axis, \( d(g', g'') \leq 2 \), and 2) above holds for each of the pairs \( (g, g'') \) and \( (g'', g') \).

The above estimates are not sharp.
Sketch of the proof of Lemma 12. We suppose that \( g, g', \) and \( r \) are given as above, and suppose that \( \alpha' \) and \( \beta' \) are geodesics for \( g \) and \( g' \). If \( g \in B^0 \), then the projection of \( \alpha' \) lies entirely in the parallelogram determined by the tessellation of the plane with corners at \((0,0)\) and \( p(g) \). In this case we can further demand that the projection of \( \alpha' \) have a specific form: we demand that it stay along one side of the parallelogram as long as possible. In this case we take \( \alpha \) to be the path in the plane given by the projection of \( \alpha' \). If \( g \notin B^0 \), then \( \alpha' \) can be taken to lie (in projection) close to the boundary of an almost regular hexagon. We take \( \alpha \) to be the path along the boundary of this hexagon determined by \( \alpha' \). We choose \( \beta \) similarly.

One can then perform a systematic enumeration of the possibilities using the following facts:

(E1) Each of \( \alpha \) and \( \beta \) is either a geodesic of the special form we have given or consists of between one and six sides of an almost regular hexagon.

(E2) Each of \( \alpha \) and \( \beta \) starts along one of the 6 axial directions.

(E3) If \( \alpha \) or \( \beta \) lies along an almost regular hexagon, it turns either clockwise or counterclockwise.

Without loss of generality, one picks an axial direction for \( \alpha \) and quickly eliminates the cases in which \( \alpha \) and \( \beta \) turn in opposite directions.

One has the following data:

(D1) The lengths of \( \alpha \) and \( \beta \) differ by at most 4.

(D2) The end points of \( \alpha \) and \( \beta \) are separated by a path \( s \) of length at most 4.

(D3) The path given by \( \alpha' r \beta'^{-1} \) in projection encloses 0 signed area, where area is measured by the number of black triangles enclosed.

(D4) The projection of \( \alpha' \) and \( \alpha \) are either identical or lie separated by a straight strip of width 1. In the case where \( \alpha \) is more than 2 sides of an almost regular hexagon, we can take \( \alpha \) to have the same endpoint as the projection of \( \alpha' \), and similarly for \( \beta \). Thus, when \( \alpha \) and \( \beta \) are more than 2 sides of an almost regular hexagon, \( s \) is simply the projection of \( gr \).

One then proceeds to examine each of these cases using elementary Euclidean geometry. In each of the cases where at least one of \( \alpha \) or \( \beta \) does not consist of 6 sides of an almost regular hexagon, we find that we are in case 1) or 2) of the Lemma. That is, either \( g \) and \( g' \) are both short, or there are geodesic paths for \( g \) and \( g' \) which fellow travel in projection.

The case where both \( \alpha \) and \( \beta \) are 6 sides of an almost regular hexagon is more interesting. If \( \alpha \) and \( \beta \) both end in the interior of a common sextant (i.e., in the region between two axial rays), then \( \alpha \) and \( \beta \) must start in the same axial direction and have approximately the same size. Thus, they fellow travel. On the other hand, if, say, \( \alpha \) ends in the interior of a sextant and \( \beta \) ends on an axis which defines that sextant, \( \alpha \) and \( \beta \) need not fellow travel in projection. For here, \( \alpha \) may start out in (say) the \( t \) direction, while \( \beta \) can start out in the \( y \) direction, so that we have, say, \( \beta = y^i t^j x^j y^{-j} t^{-j} x^{-j} \), with \( i < j \). However, the fact that \( \beta \) starts in the \( y \) direction and ends on the \( y \) axis allows us to “slide” \( \beta \) in the following manner: for any \( \delta \leq i \), the path \( y^{i-\delta} t^{j} x^j y^{-j} t^{-j} x^{-j} y^\delta \) evaluates to the same element of \( N \) as \( \beta \). By taking \( \delta = j \) we replace \( \beta \) with an equivalent geodesic which starts in the same direction as \( \alpha \). Their almost regular hexagons are approximately
Finally, if the path $g\ell s$ from the end of $\alpha$ to the end of $\beta$ crosses exactly one axis, we need only note that there is an element $g''$ which is within distance 2 of each of $g$ and $g'$, has the same length as these and projects to a point lying on this axis.

**Proof of Theorem 9 from Lemma 12.**

Let $\ell(g) = \ell(g') = n$ and $d(g, g') \leq 2$, say $g' = g\ell$, with $\ell(r) \leq 2$. Thus, we are in the situation of Lemma 12. We must exhibit a path of bounded length which connects $g$ to $g'$ and lies in $B(n)$.

If situation 1) of the Lemma holds, then we can connect $g$ to $g'$ by going through the identity, and this path has length at most 100.

Suppose situation 2) of the Lemma holds. We shall also assume that $g$ and $g'$ have length at least 900 for otherwise, we proceed as above. (We remind the reader that our estimates are in fact very crude!). Let $w = uvw$ and $w' = u'v'w'$ be geodesics for $g$ and $g'$ with $\ell(v) = \ell(v') = 900$. Let $q$ be the lift to $\hat{N}$ of a path connecting $p(\gamma)$ to $p(\gamma')$. We can assume $\ell(q) \leq 50$. Then the path given by $v^{-1}q\gamma v^{-1}$ in projection encloses an area $A$ with $|A| \leq (50)(900)$. We let $\gamma$ be a geodesic for $z^{-A}$. This has length at most $4\sqrt{A} + 4 < 900$. We then let $P$ be the path based at $g$ bearing the label $v^{-1}q\gamma v'$. It is easy to see that this connects $g$ to $g'$ and lies inside $B(n)$.

We now suppose that we are in situation 3) of the Lemma. If the projection of $gr$ crosses only one axis, then we can perform the previous process twice, once connecting $g$ to $g''$, and once connecting $g''$ to $g'$. But in fact, $r$ has length at most 2, so the projection of $gr$ can cross at most 2 axes, and this happens only when $g$ and $g'$ project to points within distance 1 of $(0,0)$. In this case we repeat the previous method twice and are done.

**Proof of Theorem 10.** This follows from Theorem 9 by observing that as usual an $N^e$ geodesic in $N^e$ can have very few $z$'s. Then if $g, g' \in N^e$ with $\ell(g) = \ell(g')$ and $d(g, g') \leq 2$, we write geodesics $w = uz^m$ and $w' = uz'^{m'}$ for $g$ and $g'$, and $|m|$ and $|m'|$ are bounded, and $u$ and $u'$ are free of $z$'s. Without loss of generality we can assume $|m| \leq |m'|$. We let $r$ be the terminal segment of $u$ of length $|m'| - |m|$, so that $u = u'r$ with $\ell(u'^{m'}) = \ell(u')$. Now the almost convexity of $N$ gives us a path $Q$ of bounded length connecting $u''$ to $u'$ inside the ball of radius $\ell(g) - |m'|$. The path we seek is the one which starts at $g$ and is labelled $z^{-m}r^{-1}Qz^{m'}$.

The Corollary now follows by the methods of Lemma 8.

We now have all the tools in place that we used for the proof in the square case. The proof in the present case follows along exactly the same lines. One checks that the embedding of $\mathbb{Z}^2$ into the appropriate orbifold groups is geodesic with respect to the generating set $X = \{x, y, t\}$ for $\mathbb{Z}^2$ and the appropriate orbifold generating sets $S \cup X$. The action of $S$ preserves $X$. Once again, this lifts to the inclusion $N^e \hookrightarrow G$ giving almost convexity with respect to a generating set of the form $N^e_s \cup S$, where $N^e_s$ is a saturated generating set. As before, this gives almost convexity with respect to the generating set $N^e \cup S$ or $N \cup S$ if this latter generates.

3. Central extensions of word hyperbolic groups.
We will now prove Theorem 2.

Proof of Theorem 2. We take \( \rho : H \times H \to A \) to be the cocycle defining \( G \). Thus we can identify \( G \) with the set \( A \times H \) endowed with the multiplication \((a, h)(a', h') = (a + a' + \rho(h, h'), hh')\). Let \( G' \) be a generating set for \( G \). Then \( \mathcal{H} = \pi G' \) is a generating set for \( H \). We will extend \( G' \) to a generating set \( G \) so that \( G \) is almost convex with respect to \( G \).

Lemma 13.

Suppose \( G' \) is a generating set for \( G \) with \( \pi G' = \mathcal{H} \). Suppose that \( D \) is a Dehn’s algorithm for \( H \) with respect to \( \mathcal{H} \). Then there are sets \( G'' \subset \pi^{-1} \mathcal{H} \) and \( \mathcal{A}' \subset A \) so that if \( \mathcal{A}' \subset \mathcal{A} \) and \( G = G' \cup G'' \cup \mathcal{A} \), then for any \( g \in G \) there is a geodesic \( w \in G^* \) so that \( w = g \) and \( \pi w \) is \( D \)-reduced.

Proof. We start by taking

\[
G'' = \{(1, h) \mid h \in \mathcal{H}\}.
\]

We enlarge \( D \) if necessary to make sure it is closed under inversion and cyclic permutation. We wish to lift the words in \( D \) to \( A \). For each \( d = h_1 \ldots h_k \in D \), we take \( \tilde{d} = (1, h_1) \ldots (1, h_k) \in A \). For each \( d = h_1 \ldots h_k \in D \), let

\[
R_d = \{(a_1, h_1) \ldots (a_k, h_k) : (a_i, h_i) \in G' \cup G'' \text{ for } 1 \leq i \leq k\}.
\]

For each \( r = (a_1, h_1) \ldots (a_k, h_k) \in R_d \) and each \( i \leq k \), let \( a_i(r) = a_1 + \ldots + a_i \). We take

\[
\mathcal{A}' = \{a_i(r) + \tilde{d} : r \in R_d, \ d \in D\}.
\]

We take an arbitrary geodesic \( w'' = (a_1, h_1) \ldots (a_p, h_p) \) for \( g \in G \). Since \( A \) is central in \( G \), we may replace \( w'' \) with \( w' \) of the form

\[
w' = (a_1, 1) \ldots (a_i, 1)(a_{i+1}, h_{i+1}) \ldots (a_p, h_p)
\]

where \( h_j \neq 1 \) for \( j > i \). We then have \( \pi(w') = h_{i+1} \ldots h_p \). If this is \( D \)-reduced we are done. If not, there are \( m \) and \( n \) with \( i < m < n \leq p \) so that \( h_m \ldots h_n \) is more than half a relator in \( D \). Hence \( h_m \ldots h_n = h'_m \ldots h'_{n'} \) with \( n' < n \) and with \( d = h_m \ldots h_n(h'_m \ldots h'_{n'})^{-1} \in D \). We then have

\[
(a_m, h_m) \ldots (a_n, h_n) = (a_m + \ldots + a_n, 1)(1, h_m) \ldots (1, h_n)
\]

\[
= (a_m + \ldots + a_n + \tilde{d}, 1)(1, h'_m) \ldots (1, h'_{n'}).
\]

By construction, \( (a_m + \ldots + a_n + \tilde{d}, 1) \in G \), so this last expression lies in \( G^* \), and since \( n' < n \), it is no longer than the first expression. Thus, we may use it to replace the first expression in \( w' \). This reduces the length of \( \pi w' \), so continuing in this way produces a geodesic \( w \) whose projection is \( D \)-reduced.
Notice that this means that $A'$ is a generating set for $A$. For suppose $w$ is as guaranteed by Lemma 13. If $w \in A$, then $\pi w = 1 \in H$, and a $D$-reduced path for the identity is the empty path. In particular $A$ is geodesic in $G$, that is, given a generating set $G$ of the form $G' \cup G'' \cup A$ as above, the inclusion of $A$ into $G$ is an isometry.

We continue with our proof that $G$ is almost convex. We must choose our generating set. Recall that $D$-reduced words are $(\lambda, \epsilon)$ quasigeodesics in $H$, and that there is a $k$ so that if two $(\lambda, \epsilon)$ quasigeodesics end at most distance 2 apart, then they lie in $k$ neighborhoods of each other. We will take

$$A = A' \cup \{a \in A : a = \pi x \text{ with } r \in (G' \cup G'')^* \text{ and } \ell(r) \leq \lambda(2k+3) + \epsilon + 2k + 3\}.$$  

We take $G = G' \cup G'' \cup A$.

We now suppose that $g, g' \in G$ with $\ell(g) = \ell(g') = n$ and $g' = gg$ with $\ell(q) \leq 2$. We write $g = \pi x$ where $w = uv$ so that each letter of $u$ projects to $1 \in H$ and no letter of $v$ does. Similarly, we write $g' = \pi y$ with $w' = u'v'$. Notice that $d(\pi x, \pi y') \leq 2$. Since $\pi v$ and $\pi v'$ are both $D$-reduced, they are $(\lambda, \epsilon)$ quasigeodesics lying in $k$ neighborhoods of each other.

Suppose first that $\ell(v) \geq k+1$, say $v = xy$ with $\ell(y) = k+1$. Then $\ell(\pi x) = n-k-1$ and $d(\pi x, \pi w') \leq k$. Choose $z$ so that $\ell(z) \leq k$ and $\pi z$ labels a path from $\pi x$ to a point on $\pi v'$. We will suppose this point to be $\pi y$ where $v' = x'y'$. Notice that the path $uxz$ stays inside the ball of radius $n-1$. Since $\pi v'$ is a $(\lambda, \epsilon)$-quasigeodesic, it follows that $\ell(y') \leq \lambda(2k+3)+\epsilon$. Hence the path $y'gy^{-1}z$ has length at most $\ell(y') \leq \lambda(2k+3)+\epsilon + 2k + 3$. It projects to a closed path, so, in particular $y'gy^{-1}z = (a, 1) \in A$. We now have $gy^{-1}z(a, 1) = u'x'$. But the path labelled $y^{-1}x(a, 1)$ based at $g$ stays inside the ball of radius $n$. Clearly the path labelled $y'$ based at $u'x'$ also stays inside this ball. Thus, the path labelled $y^{-1}z(a, 1)^{-1}y'$ runs from $g$ to $g'$ staying inside the ball of radius $n$. Its length is clearly bounded.

We must now check that we can produce such a path when $\ell(v) \leq k$. Suppose $\ell(v) \geq 1$. Then “backing up along $\pi v$” takes us to $1 \in H$, and we can perform the same argument as above taking $y = v$ and $z$ to be trivial.

We are now left with the case where $\ell(v) = 0$ and by symmetry, we may assume $\ell(v') = 0$. In that case $g$ and $g'$ lie in the abelian group $A$. We have seen that $A$ is geodesic in $G$, so we are done.

This gives the following

**Scholium 14.** Suppose that $H$ is word hyperbolic and that

$$1 \rightarrow \mathbb{Z} \rightarrow G \rightarrow H \rightarrow 1.$$  

Then there is a generating set $G$ so that $G$ is almost convex with respect to $G$.

**Proof.** There are only two actions on $\mathbb{Z}$, namely the trivial action and the action which inverts elements of $\mathbb{Z}$. Thus at the possible cost of inverting elements of $\mathbb{Z}$, we can move each of these to the beginning of a word. Now this process cannot increase length. Consequently, each element of $G$ has a geodesic in which every $\mathbb{Z}$ generator appears the beginning. Now one can proceed as above.
Corollary 15. Let $M$ be a closed 3-manifold with $\widetilde{\text{PSL}_2 \mathbb{R}}$ or $\mathbb{H}^2 \times \mathbb{R}$ geometry. Then there is a generating set $G$ so that $\pi_1(M)$ is almost convex with respect to $G$.

Proof. In this case

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(M) \rightarrow H \rightarrow 1,$$

where $H$ is the orbifold fundamental group of a hyperbolic surface orbifold. Since this is necessarily word hyperbolic, the result follows.

This completes the proof of Theorem 1.

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