Stability analysis of anisotropic Bianchi type-I cosmological model in teleparallel gravity

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Abstract: In this work, we study a cosmological model of Bianchi type-I Universe in teleparallel gravity for a perfect fluid. To obtain the cosmological solution of the model, we assume that the deceleration parameter is a linear function of the Hubble parameter $H$ i.e. $q = -1 + \beta H$ (where $\beta$ as a positive constant). Consequently, we get a model of our Universe, where it goes from the initial phase of deceleration to the current phase of acceleration. We have discussed some physical and geometric properties such as Hubble parameter, deceleration parameter, energy density, pressure, and equation of state (EoS) parameter and study their behavior graphically in terms of redshift and compare it with observational data such as Type Ia supernovae (SNIa). We also discussed the behavior of other parameters such as the Jerk parameter, Statefinder parameters and we tested the validity of the model by studying the stability analysis and energy conditions.

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I. INTRODUCTION

Since Einstein [1] published his theory of General Relativity (GR) in 1916 until the end of the previous century, cosmologists believed that the Universe was in a decelerated phase of expansion, due to Friedmann’s equations in the standard model of cosmology [2]. But recently, a group of theoretical and observational studies appeared in cosmology which showed the opposite, that is to say, that our current Universe is in a phase of accelerated expansion [3–7]. This contradiction between theories and observational data has led many researchers to suggest other alternatives that agree between the two [8]. The most famous is the idea that there is a new form of energy called dark energy (DE) which is causing the accelerating expansion of the Universe. According to the observational data, DE is characterized by negative pressure or in some other way a negative EoS parameter $\omega$, where $\omega = -\frac{p}{\rho}$ with $\rho$ represents the energy density of the Universe and $p$ represents the pressure. Many researchers recommend another idea, namely that the accelerating expansion of the Universe could be the result of modifications of gravity, and they called this class the Modified Gravity Theories (MGT), see [9–14]. Another class combines the holographic principle with DE, such as [15]. Many researchers around the world are striving to uncover the causes of cosmic acceleration, but despite this, the question of cosmic acceleration or DE remains a mystery in the scientific arena. Another simple idea about the nature of DE in the context of GR was to add the cosmological constant $\Lambda$ that Einstein had introduced into his equations in another context, but soon many problems arose, for example, it was a difficulty on the theoretically predicted order of magnitude compared to that of the observed vacuum energy [16] [17].

Recently, a new type of study has appeared, which attempts to explain the accelerated expansion of the Universe by assuming cosmological models in various gravitational theories that contain the deceleration parameter or in another way the scale factor which varies over time [18–20]. Indeed, this hypothesis is supported by observational data which shows that the Universe has passed from the stage of early deceleration ($q > 0$) to the stage of current acceleration ($q < 0$).

Among all these modified gravity theories found in the literature, in this work, we will focus on another approach to examine alternatives to GR which is Teleparallel Gravity (TG) which uses the Weitzenbock connection in place of the Levi–Civita connection and therefore does not have curvature but has a torsion which is responsible for the acceleration of the Universe. Some studies on this subject have gone so far as to replace scalar torsion $T$ with a generalized function $f(T)$ for the latter [21]. For example, Bhoyar et al. study Bianchi type-I space-time for the linear and quadratic form of $f(T)$ gravity with a hybrid expansion law for scale factor [22]. Holographic DE has also been discussed in this context by Shaikh et al. using the power and exponential laws [23]. In another work, Shaikh studies DE in a teleparallel gravity framework using the hybrid expansion law for scale factor with the thermodynamic aspects of the model [24]. See also other works in this context [25–29]. Several studies support the idea that the geometry of the Universe at the end of the inflationary era is homogeneous and isotropic [30], so that FLRW models play an important role in this period. However, defects in the cosmic microwave background (CMB) due to quantum fluctuations of the inflation period confirm the existence of an anisotropic phase which then transformed into an isotropic phase. Recently, with the advent of Planck observational data [31], Bianchi cosmological models describing the anisotropic Universe have attracted the attention of many authors. In
the literature, there are several types of anisotropic and inhomogeneous Bianchi space-times. The Bianchi type-I space-time is the mathematically simplest case that describes an anisotropic and homogeneous Universe. Also defined as a direct generalization of the FLRW Universe with a different scale factor in each spatial direction. Several Bianchi type-I cosmological models have been studied in various contexts by many researchers. Recently, Hossienkhani et al. studied the spatially homogeneous and anisotropic Bianchi type-I Universe with the interacting holographic and new agegraphic scalar fields models of DE [32]. The dynamical evolution of an $f(R)$ model of gravity in a viscous and anisotropic background given by Bianchi type-I space-time is discussed by Saaidi et al. [33]. Also, the anisotropy effects on Baryogenesis in $f(R)$ gravity are examined in Ref. [34].

In this paper, motivated by the above works we study a Bianchi type-I cosmological model under teleparallel gravity $f(T) = T$ with the perfect fluid-like matter. We assume that the deceleration parameter varies with time as a linear function of the Hubble parameter, and we discuss the behavior of each of the last two parameters and compare them to the current values of the observational data in Sect. 4. Sect. 5 is devoted to the physical and geometric properties of the model. In the last section, the main results of the model are discussed.

II. $f(T)$ GRAVITY FORMALISM

As usual, in this section, we will give a brief description of the $f(T)$ gravity and its field equations. The action of $f(T)$ gravity as a generalization of teleparallel gravity is given by the following relation

$$ S = \int [T + f(T) + L_{\text{matter}}] \, dt, $$

where $T$ is the torsion scalar, $f(T)$ is a general differentiable function of torsion, $L_{\text{matter}}$ is the matter Lagrangian density and $e = \sqrt{-g} = \det [e_{ij}]$. The torsion scalar is defined as

$$ T = T^\alpha_{\mu \nu} S^\mu_{\alpha}, $$

where $S^\mu_{\alpha}$ and torsion tensor $T^\alpha_{\mu \nu}$ are given as follows

$$ S^\mu_{\alpha} = \frac{1}{2} \left( K^\mu_{\alpha} + \delta^\mu_{\alpha} T^\beta_{\beta} - \delta^\alpha_{\beta} T^\beta_{\mu} \right), $$

$$ T^\alpha_{\mu \nu} = \Gamma^\alpha_{\mu \nu} - \Gamma^\alpha_{\nu \mu} = e^\alpha_{i} \left( \partial_{\mu} e^i_{\nu} - \partial_{\nu} e^i_{\mu} \right). $$

In the previous equation, $e^i_{\mu}$ represents the components of the non-trivial tetrad field $e_i$ in the coordinate base. One chooses at random the tetrad field related to the metric tensor $g_{\mu \nu}$ by the following relation

$$ g_{\mu \nu} = \eta_{ij} e_{i}^{\mu} e^{j}_{\nu}, $$

where $\eta_{ij} = \text{diag}(1, -1, -1, -1)$ is the Minkowski space-time metric and $e^i_{\mu} e^j_{\nu} = \delta^i_j$ or $e^i_{\mu} e^i_{\nu} = \delta^\nu_{\mu}$. The process for evaluating the tetrad field has been provided in [33][37]. Note that the Latin alphabets ($i, j, \ldots = 1, 2, 3$) will be used to denote the indices of the tetrad field and the Greek alphabets ($\mu, \nu, \ldots = 0, 1, 2, 3$) to denote the space-time indices.

The contortion tensor $K^\mu_{\alpha \nu}$ is defined as the difference between the Levi-Civita and Weitzenböck connections, it is defined as

$$ K^\mu_{\alpha \nu} = \frac{1}{2} \left( T^\mu_{\alpha \nu} - T^\nu_{\alpha \mu} - T^\alpha_{\mu \nu} \right). $$

The functional variation of the action in Eq. (1) with respect to tetrads leads to the following field equations

$$ S^\mu_{\alpha} \partial_{\mu} T + \left[ e^{-1} e^i_{\mu} \partial_{\mu} (e^i_{\nu} S^\nu_{\alpha}) + T^\alpha_{\mu \nu} S^\mu_{\nu} \right] (1 + f_T) + \frac{1}{4} \delta^\nu_{\alpha} (T + f) = k^2 T^\nu_{\mu}. $$

Here, $f_T = \frac{df(T)}{dT}$, $f_{TT} = \frac{d^2 f(T)}{dT^2}$, $k^2 = 8\pi G = 1$ and $T^\nu_{\mu}$ is the energy-momentum tensor of matter. The field equation [7] for $f(T)$ gravity is written in terms of tetrad and partial derivatives and appears very different from Einstein’s equations in GR. If we consider the function $f(T) = \text{constant}$ this leads to Einstein’s ordinary field equations.
III. METRIC AND FIELD EQUATIONS

There are eleven types of Bianchi metrics (I-IX). In this article, we will study the Bianchi type-I (B-I) metric, which is one of the simplest models of the spatially homogeneous and anisotropic Universe. This metric is a direct generalization of the spatially homogeneous and isotropic FRW metric with a scale factor in each direction and is given as follows

\[ ds^2 = dt^2 - A^2(t) \, dx^2 - B(t)^2 \left( dy^2 + dz^2 \right), \]  

where \( A \) and \( B \) are functions of cosmic time \( t \) only. The corresponding torsion scalar is given by

\[ T = -2 \left( \frac{\dot{A}B}{AB} + \frac{B^2}{AB^2} \right). \]  

The energy-momentum tensor \( T^\nu_\mu \) for the perfect fluid distribution can be represented as

\[ T^\nu_\mu = (p + \rho) \, u^\nu u_\mu - pg^\nu_\mu, \]  

where \( p \) and \( \rho \) are the pressure and energy density of the cosmic fluid respectively and \( u^\nu = (0, 0, 0, 1) \) is the four-velocity vector satisfying \( u^\nu u_\nu = 1 \).

Now, using the field equation (7) for the Bianchi type-I metric (8) and perfect fluid distribution in Eq. (10), the modified Friedmann field equations are given by

\[ (T + f) + 4 (1 + f_T) \left\{ \frac{\dot{B}}{B} + \frac{B^2}{AB^2} + \frac{\dot{A}B}{AB} \right\} + 4 \frac{B}{B} Tf_{TT} = -p(t), \]  

\[ (T + f) + 2 (1 + f_T) \left\{ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{B^2}{AB^2} + 3 \frac{\dot{A}B}{AB} \right\} + 2 \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) Tf_{TT} = -p(t), \]  

\[ (T + f) + 4 (1 + f_T) \left\{ \frac{\dot{B}^2}{B^2} + 2 \frac{\dot{A}B}{AB} \right\} = \rho(t). \]

where the dot (.) denotes the derivative with respect to time \( t \).

The field equations (11)-(13) are a set of three differential equations that contain five unknowns \( A, B, f(T), p(t), \rho(t) \). In order to solve the field equations explicitly, we need two additional constraints which we will assume in the next section. Now we will know some of the physical and geometric quantities that we will need later.

The mean scale factor \( a \) of the Bianchi type-I Universe is given by

\[ a = (AB^2)^{\frac{1}{3}}. \]  

The spatial volume \( V \) of the Universe is defined as

\[ V = a^3 = AB^2. \]

Now, the directional Hubble parameters \( H_i \) \( (i = 1, 2, 3) \) are respectively

\[ H_1 = \frac{\dot{A}}{A}, \quad H_2 = \frac{\dot{B}}{B} \quad \text{and} \quad H_2 = H_3. \]

The mean Hubble’s parameter is defined as

\[ H = \frac{1}{3} (H_1 + 2H_2). \]
Using Eqs. (14)-(17), we find

\[ H = \frac{1}{3} \dot{V} = \frac{1}{3} \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right). \]  

(18)

Other physical parameters, the expansion scalar \( \theta \), the mean anisotropic parameter \( A_m \) and the shear scalar \( \sigma^2 \), are defined for the Bianchi type-I Universe, as

\[ \theta = \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} = 3H, \]  

(19)

\[ A_m = \frac{2}{3} \frac{\sigma^2}{H^2} = \frac{1}{3} \sum \left( \frac{\Delta H_i}{H} \right)^2, \]  

(20)

\[ \sigma^2 = \frac{1}{2} \left[ \left( \frac{\dot{A}}{A} \right)^2 + 2 \left( \frac{\dot{B}}{B} \right)^2 \right] - \frac{\theta^2}{6}, \]  

(21)

where \( \Delta H_i = H_i - H \) and \( H_i (i = 1, 2, 3) \) represent the directional Hubble parameters.

**IV. SOLUTIONS OF THE FIELD EQUATIONS**

In this section, we find exact solutions of field equations using the linear form of \( f(T) \) gravity, i.e.

\[ f(T) = T. \]  

(22)

Using Eqs. (11) and (12) yields

\[ \frac{d}{dt} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) + \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) \frac{\dot{V}}{V} = 0. \]  

(23)

By integrating the previous equation, we get

\[ \frac{A}{B} = k_2 \exp \left[ k_1 \int \frac{dt}{V} \right], \]  

(24)

where \( k_1 \) and \( k_2 \) are constants of integration.

Using Eq. (15), we get the metric potentials as follows

\[ A = D_1 V^{\chi_1} \exp \left( \chi_1 \int \frac{1}{V} dt \right), \]  

(25)

\[ B = D_2 V^{\chi_2} \exp \left( \chi_2 \int \frac{1}{V} dt \right), \]  

(26)

where \( D_i (i = 1, 2) \) and \( \chi_i (i = 1, 2) \) satisfy the relation \( D_1 D_2 = 1 \) and \( \chi_1 + 2\chi_2 = 0 \). To complete the additional constraints to find the exact solutions to the field equations, we reduce the second constraint on the scale factor or in some other way the deceleration parameter (DP).

The DP is given by

\[ q = -\frac{\ddot{a}}{\dot{a}} = \frac{d}{dt} \left( \frac{1}{H} \right) - 1. \]  

(27)
The DP is an important tool to describe the evolution of the Universe. If $q > 0$ indicates the cosmic deceleration while $q < 0$ shows the cosmic acceleration. According to recent observational data of the SNIa, our Universe goes from the initial deceleration phase to the current acceleration phase, that is, it goes from a positive value of the DP ($q > 0$) to a negative value ($q < 0$), indicating that the DP is a function that varies with cosmic time $t$. Therefore, in this article, we assume that the DP varies with cosmic time as a linear function of the Hubble parameter $H$

$$q = \alpha + \beta H. \quad (28)$$

Here $\alpha$ and $\beta$ are arbitrary constants. We solve Eq. (28) for $\alpha = -1$, we find the scale factor as follows

$$a (t) = \exp \left( \frac{1}{\beta} \sqrt{2\beta t + c} \right), \quad (29)$$

where $c$ is an integrating constant, and for the rest of the article, we’ll use that $F = \frac{1}{\beta} \sqrt{2\beta t + c}$.

Similarly, the Hubble parameter $H$ and DP $q$ in terms of cosmic time $t$ are obtained as

$$H = \frac{1}{\beta F}, \quad (30)$$

$$q = -1 + \frac{1}{F}. \quad (31)$$

The choice $\alpha = -1$ is appropriate to obtain a Hubble parameter which depends on cosmic time instead of being constant [39-43]. Also, we use $\alpha = -1$ to get the time-dependent DP, but if we choose $\alpha \neq -1$, we will find that the DP takes a constant value $q = -1$. By Eqs. (30) and (31), we show that $H \to 0$ and $q \to -1$ as $t \to \infty$. Moreover, $q \geq 0$ for $t \leq \frac{\beta}{2} - \frac{c}{2\beta}$ and $q < 0$ for $t > \frac{\beta}{2} - \frac{c}{2\beta}$. To study the behavior of certain cosmological parameters in terms of redshift $z$, we must first give the relation between the redshift $z$ and the scale factor $a (t)$, which is written as follows

$$z = \frac{a (t_0)}{a (t)} - 1, \quad (32)$$

where $a (t_0)$ is the current value of scale factor.

Using Eqs. (29) and (32), we find the cosmic time $t$ in terms of redshift $z$ as

$$t (z) = \frac{\beta}{2} \left[ \{F_0 - \log (1 + z)\}^2 - \frac{c}{\beta^2} \right], \quad (33)$$

where, $F_0 = \frac{1}{\beta} \sqrt{2\beta t_0 + c}$ and $t_0$ denotes the present time.

With a simple calculation, we find the Hubble parameter $H$ and the DP $q$ in terms of redshift $z$ as follows

$$H (z) = \frac{1}{\beta (F_0 - \log (1 + z))}, \quad (34)$$

$$q (z) = -1 + \frac{1}{F_0 - \log (1 + z)}. \quad (35)$$

From Eqs. (34) and (35), it is clear that $H \to 0$ and $q \to -1$ as $z \to -1$. In Tab. 1, we summarize the dynamics of the Universe for $a (t) = \exp (F)$. Now, in Fig. 1, we plot the behavior of the Hubble parameter $H$ and the deceleration parameter $q$ in terms of redshift $z$ for the value of the pair $(\beta, c)$ as $(3, 2.85)$, respectively. Our model is transforming from $q > 0$ (deceleration) to $q < 0$ (acceleration) phases. According to observational data, the DP $q$ value lies between the range $-1 < q < 0$ and the expansion of the current Universe is accelerating. Therefore, the current value of the deceleration parameter is consistent with recent observations i.e. $q_0 = -0.68$. In the following, we have given the values of the deceleration parameter and the Hubble parameter for the current time according to some observational data:

- **Case I Based on SNIa union data [44]**: Based on these observational data, $q_0 = -0.73$ and $H_0 = 73.8$.
- **Case II Based on SNIa data in combination with BAO and CMB observations [45]**: In that case, $q_0 = -0.54$ and $H_0 = 73.8$. 

\[ \begin{array}{cccc}
\text{Time (t)} & \text{Redshift (z)} & a & q \\
\hline
\rightarrow 0 & \rightarrow \exp \left[ \frac{1}{\beta} \left( \sqrt{2\beta \tilde{H}_0 + \tilde{c}} - \sqrt{\tilde{c}} \right) \right] & \exp \left( \frac{\chi_1}{\beta} \right) & -1 + \frac{2}{\beta} \frac{1}{\sqrt{\beta}} \\
\rightarrow \infty & \rightarrow -1 & \infty & -1 \\
\end{array} \]

TABLE I: Dynamics of the Universe for \( a(t) = \exp \left( \frac{1}{\beta} \sqrt{2\beta + \tilde{c}} \right) \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{plot.png}
\caption{(left) The plot of \( H \) versus \( z \), (right) The plot of \( q \) versus \( z \).}
\end{figure}

- **Case III** Based on current data in combination with OHD and JLA observations [46]. In that case, \( q_0 = -0.52 \) and \( H_0 = 69.2 \).

Using Eqs. (29) in (25) and (26), we obtain the metric potentials as follows

\[ A(t) = D_1 \exp (F) \exp \left[ -\frac{\chi_1}{9} \beta (3F + 1) \exp (-3F) \right], \quad (36) \]

and

\[ B(t) = D_2 \exp (F) \exp \left[ -\frac{\chi_2}{9} \beta (3F + 1) \exp (-3F) \right]. \quad (37) \]

Using Eqs. (32) and (33), the metric in Eq. (8) becomes as follows

\[ ds^2 = dt^2 - D_1^2 \exp (2F) \exp \left[ -\frac{2\chi_1}{9} \beta (3F + 1) \exp (-3F) \right] dx^2 \]

\[ - D_2^2 \exp (2F) \exp \left[ -\frac{2\chi_2}{9} \beta (3F + 1) \exp (-3F) \right] [dy^2 + dz^2]. \quad (38) \]

The torsion scalar \( T \) for the model becomes

\[ T = \frac{-6}{\beta^2 F^2} - 2\chi_2 (2\chi_1 + \chi_2) \exp (-6F). \quad (39) \]
V. PHYSICAL AND GEOMETRICAL PROPERTIES OF THE MODEL

The directional Hubble parameters, which determine the expansion rate of the Universe, are given by

\[ H_1 = \chi_1 \exp(-3F) + \frac{1}{\beta F}, \quad (40) \]

\[ H_2 = H_3 = \chi_2 \exp(-3F) + \frac{1}{\beta F}. \quad (41) \]

The expansion scalar \( \theta \) and the shear \( \sigma^2 \) are obtained as

\[ \theta = 3H = \frac{3}{\beta F}, \quad (42) \]

\[ \sigma^2 = \frac{1}{2} \left( \frac{\chi_1^2 + 2\chi_2^2}{a^2} \right) = \frac{1}{2} \left( \chi_1^2 + 2\chi_2^2 \right) \exp(-6F). \quad (43) \]

Using Eq. (29) into Eq. (15) we get the spatial volume as

\[ V = \exp(3F). \quad (44) \]

The average anisotropy parameter \( A_m \) is given as

\[ A_m = \frac{1}{3} \left( \chi_1^2 + 2\chi_2^2 \right) \frac{\beta^2 F^2}{a^2} \exp(-6F). \quad (45) \]

From Eq. (44), it is clear that the spatial volume of the model is finite at the initial singularity (i.e. at \( t = 0 \)) and approaches infinity as \( t \to \infty \). Moreover, the average scale factor \( a(t) \) in Eq. (29) is also finite at the early epoch of the Universe. It shows that the obtained model of the Universe is expanding continuously with cosmic time \( t \). Eqs. (40)-(43) show the directional Hubble parameters \( H_i \), the scalar expansion \( \theta \) and the scalar shear \( \sigma^2 \to 0 \) as \( t \to \infty \) and they approach finite value as \( t \to 0 \). Finally, from Eq. (45), we observe that the average anisotropy parameter \( A_m \to 0 \) as \( t \to \infty \). This indicates that our model contains a transition from the early anisotropic Universe to the current isotropic Universe as shown by observational data.

Using Eqs. (36) and (37) in the field equations (11)-(13), with simple math, the physical parameters such as energy density \( \rho(t) \), cosmic pressure \( p(t) \) are obtained as

\[ \rho(t) = \frac{12}{\beta^2 F^2} + 4\chi_2 (2\chi_1 + \chi_2) \exp(-6F) \quad (46) \]

\[ p(t) = \frac{8}{\beta^2 F^3} + \frac{24\chi_2}{\beta F} \exp(-3F) - 12 \left( \chi_2 \exp(-3F) + \frac{1}{\beta F} \right)^2 \quad (47) \]

Using the relationship between cosmic time \( t(z) \) and redshift \( z \) in Eq. (33) and Eqs. (46) and (47), we plot the behavior of the energy density \( \rho(z) \) and pressure \( p(z) \) of the Universe versus redshift \( z \) in Fig. 2, respectively. First of all, note that \( \rho(z) \) and \( p(z) \to 0 \) as \( z \to -1 \) (or \( t \to \infty \)), which is similar behavior to the big-bang model. From Fig. 2 (left), we can observe that the energy density remains positive throughout the evolution of the Universe and is a decreasing function of redshift \( z \). The pressure in Fig. 2 (right), evolves from early positive values to present negative ones. As per the observation, the negative pressure is due to DE in the context of accelerated expansion of the Universe. Hence, the behavior of pressure in our model is consistent with this observation.

Using the equation of state for a perfect fluid \( \left( \omega = \frac{2}{3} \right) \), and using Eqs. (46) and (47), we find the EoS parameter as follows

\[ \omega(t) = \frac{2}{3} + \frac{6\chi_2}{\beta F} \exp(-3F) - 3 \left( \chi_2 \exp(-3F) + \frac{1}{\beta F} \right)^2 \]

\[ \frac{3}{\beta F^2} + \chi_2 (2\chi_1 + \chi_2) \exp(-6F) \quad (48) \]
The EoS parameter is among the basic tools for studying the different phases of the Universe as well as the history of the Universe. If $\omega = -1$, it represents $\Lambda CDM$ model, $-1 < \omega < -\frac{1}{3}$, represents quintessence model and $\omega < -1$, indicates phantom behavior of the model. We have plotted the EoS parameter $\omega(z)$ for redshift $z$ in Fig. 3 (left) for a fixed value of the pair $(\beta, c)$, the EoS parameter $\omega \in$ quintessence region for high redshift $z$ and over time, $\omega \to -1$ in infinite future (i.e. $z \to -1$). The present value of the EoS parameter of our model is consistent with the observational data on $\omega$ from Planck data [47]:

- $\omega = -1.56^{+0.60}_{-0.84}$ (Planck + TT + lowE),
- $\omega = -1.58^{+0.52}_{-0.41}$ (Planck + TT, EE + lowE),
- $\omega = -1.57^{+0.50}_{-0.46}$ (Planck + TT, TE, EE + lowE + lensing),
- $\omega = -1.04^{+0.10}_{-0.10}$ (Planck + TT, TE, EE + lowE + lensing + BAO).

From Fig. 3 (left), it is clear that the EoS parameter of our model is within the range of the above observational data. Accordingly, our results are in agreement with previous observational data.

### A. Energy Conditions

Energy conditions are a set of conditions that describe matter in the Universe and are used in many approaches to understanding the evolution of the Universe. The role of energy conditions is to verify the acceleration of the expansion of the Universe. There are many forms of energy conditions such as null energy condition (NEC), weak energy condition (WEC), dominant energy condition (DEC), and strong energy condition (SEC). Here [48–52], a group of authors who have done work on energy conditions. In $f(T)$ gravity with known energy density and pressure, these energy conditions are given as follows:

- **WEC:** $\rho \geq 0$
- **NEC:** $\rho + p \geq 0$
- **DEC:** $\rho - p \geq 0$
- **SEC:** $\rho + 3p \geq 0$

Fig. 3 (right) represents the energy conditions as a function of time for our model under study, i.e. the Bianchi type-I Universe with the DP varies with cosmic time as a linear function of the Hubble parameter $H$. From the figure below, we notice that the WEC, NEC, and DEC are well satisfied throughout the cosmic evolution, while there is a clear violation of the SEC. Thus, the violation of the SEC gives us the acceleration of the Universe.
FIG. 3: (left) The plot of $\omega$ versus $z$, (right) The plot of energy conditions versus $z$.

B. Perturbation and stability of the obtained solution

To study the stability of our solutions, we will follow the same approach found in this work [53, 54]. We will use the perturbation approach to check the obtained expanding background solution stability against perturbation of scale factors or the metric field. Now, we will consider the existence of a perturbation for the three scale factors $a_i (i = 1, 2, 3)$ as

$$a_i \rightarrow a_{Bi} + \delta a_i = a_{Bi} \left(1 + \frac{\delta a_i}{a_{Bi}}\right) = a_{Bi} (1 + \delta b_i),$$  \hspace{1cm} (49)

where $\delta b_i = \frac{\delta a_i}{a_{Bi}}$.

In the same way, we write the perturbation in the spatial volume $V = \prod_{i=1}^{3}a_i$, directional Hubble parameters $H_i = \frac{a_i}{a_{i}}$ and mean Hubble parameter $H = \frac{1}{3} \sum_{i=1}^{3} H_i$ as follows

$$V \rightarrow V_B + V_B \sum_i \delta b_i,$$  \hspace{1cm} (50)

$$H_i \rightarrow H_{Bi} + \sum_i \delta b_i,$$  \hspace{1cm} (51)

$$H \rightarrow H_B + \frac{1}{3} \sum_i \delta b_i.$$  \hspace{1cm} (52)

Here, $V_B$, $H_{Bi}$, and $H_B$ are the background spatial volume, directional Hubble parameters, and mean Hubble parameter respectively. Now, it can be shown that the metric linear order perturbations $\delta b_i$ satisfy the following differential equations

$$\sum_i \ddot{\delta b_i} + 2 \sum_i H_{Bi} \delta b_i = 0,$$  \hspace{1cm} (53)

$$\ddot{\delta b_i} + \frac{\dot{V}_B}{V_B} \delta b_i + \sum_j \delta b_j H_{Bi} = 0,$$  \hspace{1cm} (54)
\[ \sum_i \delta \dot{b}_i = 0. \]

With a little math, we can easily find through Eqs. (53)-(55) the following relation

\[ \delta \ddot{b}_i + \frac{V_B}{V_B} \delta \dot{b}_i = 0. \]

(56)

For our model, \( V_B \) is given by

\[ V_B = \exp \left( \frac{3}{\beta} \sqrt{2 \beta t + c} \right). \]

(57)

Using the above condition in Eq. (56) and after integration, we find

\[ \delta b_i = -c_i \left[ \frac{(\beta + 3\sqrt{2 \beta t + c})}{9 \exp \left( \frac{3\sqrt{2 \beta t + c}}{\beta} \right)} \right], \]

(58)

where \( c_i \) is an integrating constant. Thus, for each scale factor \( a_i \), the actual fluctuations are given by

\[ \delta a_i = -c_i \left[ \frac{(\beta + 3\sqrt{2 \beta t + c})}{9 \exp \left( \frac{3\sqrt{2 \beta t + c}}{\beta} \right)} \right]. \]

(59)

From the above equation, it is clear that \( \delta a_i \) approaches zero as \( t \to \infty \). The same behavior is illustrated by Fig. 4 (left) which represents the variation of \( \delta a_i \) in terms of the redshift \( z \), i.e. \( \delta a_i \to 0 \) as \( z \to -1 \). Thus, the background solution is stable against the perturbation of the metric.

C. Jerk parameter

As it is known in the literature, the jerk parameter is one of the fundamental physical quantities to describe the dynamics of the Universe. The Jerk parameter is a dimensionless third derivative of the scale factor \( a (t) \) for cosmic time \( t \) and is defined as

\[ j = \frac{\dddot{a}}{a H^3}. \]

(60)

Eq. (60) can be written in terms of a DP as

\[ j = q + 2q^2 - \frac{\dot{q}}{H}. \]

(61)

Using Eqs. (30) and (31), the jerk parameter for our model is

\[ j = \frac{3\beta^2}{2\beta t + c} - \frac{3\beta}{\sqrt{2\beta t + c}} + 1. \]

(62)

To study the behavior of the jerk parameter \( j (z) \), it is better to express it in terms of redshift \( z \)

\[ j (z) = \frac{3}{(F_0 - \log (1 + z))^2} - \frac{3}{F_0 - \log (1 + z)} + 1. \]

(63)

For the \( \Lambda CDM \) model, the value of the jerk parameter is \( j = 1 \). The Universe shifts from the early deceleration phase to the current acceleration phase with a positive jerk parameter \( j_0 > 0 \) and a negative DP \( q_0 < 0 \) according to the \( \Lambda CDM \) model. Fig. 4 (right) represents the variation of jerk parameter \( j \) versus redshift \( z \), i.e. \( \delta a_i \to 0 \) as \( z \to -1 \). Thus, the background solution is stable against the perturbation of the metric.
D. Statefinder diagnostic

The statefinder pair is a very important geometrical diagnostic tool used to distinguish different DE models such as $\Lambda$CDM, HDE, CG, SCDM, and Quintessence. The state-finder pair $\{r, s\}$ is defined as

$$ r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r - 1}{3(q - \frac{1}{2})} \quad (64) $$

We can find different models of DE according to the values of the couple $r$ and $s$. In particular,

- $\Lambda$CDM corresponds to $(r = 1, s = 0)$,
- $SCDM$ corresponds to $(r = 1, s = 1)$,
- $HDE$ corresponds to $(r = 1, s = \frac{2}{3})$,
- $CG$ corresponds to $(r > 1, s < 0)$,
- $Quintessence$ corresponds to $(r < 1, s > 0)$

Using Eqs. (29), (30) and (31), the values of the state-finder parameters in terms of redshift $z$ for our model are

$$ r(z) = \frac{3}{(F_0 - \log (1 + z))^2} - \frac{3}{F_0 - \log (1 + z)} + 1 \quad (65) $$

$$ s(z) = \frac{2(F_0 - \log (1 + z)) - 2}{[3(F_0 - \log (1 + z)) - 2](F_0 - \log (1 + z))} \quad (66) $$

From Fig. 5, we notice that the statefinder parameters $\{r, s\}$ evolve from the CG (Chaplygin Gas) region $(r > 1, s < 0)$ to the quintessence region $(r < 1, s > 0)$ at present, and later time to $\Lambda$CDM point $(r = 1, s = 0)$. As a result, our model current behaves like a quintessence model for DE.

VI. DISCUSSIONS AND CONCLUSIONS

In this paper, we have studied a cosmological model with a variable deceleration parameter in a Bianchi type-I Universe in $f(T)$ gravity by assuming a particular form of the deceleration parameter as a linear function of the Hubble parameter i.e. $q = -1 + \beta H$, $\beta > 0$. We consider $f(T) = T$ and find the field equations for our model and graphically represent the different physical and geometric parameters as a function of the redshift. The important results of our model are:
The DP of our model gives us two phases of the Universe, the early deceleration phase, and the current acceleration phase, as indicated by the observational data. The Hubble parameter of our model is a decreasing function of redshift $z$. Also, $H \to 0$ as $z \to -1$ (i.e. $t \to \infty$).

Thus, our model contains a transition from the early anisotropic Universe ($A_m \neq 0$) to the current isotropic Universe ($A_m = 0$).

The energy density of the Universe decreases over time and remains positive throughout cosmic evolution, while the pressure starts with positive values then changes to negative values for the current time, the negative pressure is caused by cosmic acceleration.

The EoS parameter $\omega$ of our model evolves from the quintessence region to the $\Lambda CDM$ model region in the future $\omega \to -1$.

All the energy conditions are satisfied throughout cosmic evolution, except the SEC condition is violated, and the reason is due to cosmic acceleration.

For the stability analysis, the background solution is stable against the perturbation of the metric.

The jerk parameter is positive throughout the evolution of the Universe. Thus, our model can be expected to adopt the behavior of another DE model instead of the $\Lambda CDM$ model at present $z = 0$, but our model is similar to the $\Lambda CDM$ model $j = 1$ in the future $z \to -1$.

The statefinder parameters $\{r, s\}$ evolve from the CG region to the quintessence region at present, and later time to $\Lambda CDM$ point. As a result, our model behaves like the $\Lambda CDM$ model in the future.

The obtained results are similar to several works that discuss the issue of dark energy and cosmic acceleration in different contexts: $f(R, T)$ gravity, $f(G)$ gravity, $f(T)$ gravity, etc. The only difference is the choice of a different background for the study. In this reference [58] Sharma et al. obtained similar results for our model by studying the simplest non minimal matter-geometry coupling in the framework of the $f(R, T)$ gravity with power law expansion of the scale factor. It is noticeable that such forms of the scale factor produce a constant deceleration parameter [59-61], while in the present work we chose the deceleration parameter as a linear function of the Hubble parameter which leads to the production of the deceleration parameter varies with cosmic time, such as [41, 62].

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