Short proof of the sharpness of the phase transition for the random-cluster model with $q = 2$

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Abstract: The purpose of this modest note is to provide a short proof of the sharpness of the phase transition for the Random-cluster model with $q = 2$ by extending the approach developed by Duminil-Copin and Tassion [3] for $q = 1$. This in particular implies the exponential decay of the two point-correlation function in the subcritical Ising model.

1 Introduction

Let us start by defining the nearest neighbor random-cluster measure on $\mathbb{Z}^d$. For a finite subgraph $\Lambda = (V, E)$ of $\mathbb{Z}^d$, a percolation configuration $\omega = (\omega)_{e \in E}$ is an element of $\{0, 1\}^E$. A configuration $\omega$ can be seen as a subgraph of $\Lambda$ with vertex-set $V$ and edge-set given by $\{\{x, y\} \in E : \omega_{x,y} = 1\}$. If $\omega_{x,y} = 1$, we say that $\{x, y\}$ is open. Let $k(\omega)$ be the number of connected components in $\omega$ and $o(\omega)$ (respectively $f(\omega)$) the number of open (respectively closed) edges in $\omega$.

Fix $p \in [0, 1], q > 0$. Let $\mu_{\Lambda, p, q}$ be a measure defined for any $\omega \in \{0, 1\}^E$ by

$$\mu_{\Lambda, p, q}(\omega) = \frac{q^{k(\omega)} p^{o(\omega)} (1 - p)^f(\omega)}{Z},$$

where $Z$ is a normalizing constant introduced in such a way that $\mu_{\Lambda, p, q}$ is a probability measure. The measure $\mu_{\Lambda, p, q}$ is called the random-cluster measure on $\Lambda$ with free boundary conditions. For $q \geq 1$, the model undergoes a phase transition: there exists $p_c \in [0, 1]$ satisfying

$$\mu_{\mathbb{Z}^d, p_c}(0 \leftrightarrow \infty) = \begin{cases} 0 & \text{if } p < p_c, \\ > 0 & \text{if } p > p_c. \end{cases}$$

A very nice idea introduced in [3] is to define a new critical parameter $\tilde{p}_c$ for which it is easier to prove sharpness, which will in turn imply that $p_c = \tilde{p}_c$ (see Theorem 1 below). For a finite subset $S$ of $\mathbb{Z}^d$, let $\Delta S$ be the set of edges with exactly one endpoint in $S$ and define

$$\phi_p(S) := p \sum_{\{x,y\} \in \Delta S} \mu_{S, p, q}(0 \leftrightarrow x).$$

Then define the following critical parameter
$\tilde{p}_c := \sup\{p \in [0,1] : \varphi_p(S) < 1 \text{ for some finite } S \subset \mathbb{Z}^d \text{ containing 0}\}.$

The main theorem of this note is the following one.

**Theorem 1.**

1. For $p > \tilde{p}_c$, $\mu_{\mathbb{Z}^d, p, 2}(0 \leftrightarrow \infty) \geq \frac{p - \tilde{p}_c}{p}$.

2. For $p < \tilde{p}_c$, there exists $c = c(p) > 0$ such that for every $x \in \mathbb{Z}^d$

$$\mu_{\mathbb{Z}^d, p, 2}(0 \leftrightarrow x) \leq \exp(-c|x|).$$

**Corollary 1.** We have $p_c = \tilde{p}_c$. In particular, the phase transition for the random-cluster model with $q = 2$ is sharp.

**Corollary 2.** In the Ising model, the two-point correlation function decays exponentially fast with distance.

Corollary 1 follows directly from Theorem 1. Corollary 2 follows from the Edward-Sokal coupling (see \cite{5}).

### 2 Proof of Theorem 1

We will write $\mu_{\Lambda, p}$ instead of $\mu_{\mathbb{Z}^d, p, 2}$. Let us start by proving the second item of Theorem 1. Firstly, we will need the following lemma.

**Lemma 1 (Modified Simon’s inequality).** Let $S$ be a finite set of $\mathbb{Z}^d$ containing 0. For every $z \notin S$,

$$\mu_{\mathbb{Z}^d, p}(0 \leftrightarrow z) \leq p \sum_{\{x,y\} \in \Delta S} \mu_{S, p}(0 \leftrightarrow x) \mu_{\mathbb{Z}^d, p}(y \leftrightarrow z).$$

A similar inequality was proved for the Ising model in \cite{3}. Lemma 1 follows from the latter by the Edward-Sokal coupling by remarking that $\tanh(-\frac{1}{2} \log(1 - p)) \leq p$.

Fix $p < \tilde{p}_c$ and $S$ a finite set containing 0 such that $\varphi_p(S) < 1$. Let $\Lambda_n$ be the box of size $n$ around 0 for the norm $| \cdot |$. Fix $\Lambda_L$ such that $S \subset \Lambda_L$. Then, using Lemma 1, we can write

$$\mu_{\mathbb{Z}^d, p}(0 \leftrightarrow z) \leq p \sum_{\{x,y\} \in \Delta S} \mu_{S, p}(0 \leftrightarrow x) \mu_{\mathbb{Z}^d, p}(y \leftrightarrow z) \leq \varphi_p(S) \max_{y \in \Lambda_L} \mu_{\mathbb{Z}^d, p}(y \leftrightarrow z).$$

Note that $|y - z| \geq |z| - L$. If $|y - z| \leq L$, we bound $\mu_{\mathbb{Z}^d, p}(y \leftrightarrow z)$ by 1, otherwise we apply (4) to $y$ and $z$ instead of 0 and $z$. Iterating $[|z|/L]$ this strategy yields

$$\mu_{\mathbb{Z}^d, p}(0 \leftrightarrow z) \leq \varphi_p(S)^{|z|/L},$$

which proves the second item of Theorem 1.

We now turn to the proof of the first item of Theorem 1. Let $p > \tilde{p}_c$ and $\partial \Lambda_n$ be the boundary of $\Lambda_n$. We will prove the following differential inequality.

**Lemma 2.** Fix $p > \tilde{p}_c$. Then

$$\frac{d}{dp} \mu_{\Lambda_n, p}(0 \leftrightarrow \partial \Lambda_n) \geq \frac{1}{p} (1 - \mu_{\Lambda_n, p}(0 \leftrightarrow \partial \Lambda_n)).$$

Integrating this inequality between $\tilde{p}_c$ and $p$ and taking $n$ to infinity yields the first item of Theorem 1. We will therefore focus on proving Lemma 2. Let $E(\Lambda_n)$ be the set of edges whose endpoints are in $\Lambda_n$. We will need the following result.
Lemma 3. Let $A$ be an increasing event depending on edges of $\Lambda_n$ only. Then
\[
\frac{d}{dp}\mu_{\Lambda_n,p}(A) = \sum_{e \in E(\Lambda_n)} \mu_{\Lambda_n,p}(A|\omega_e = 1) - \mu_{\Lambda_n,p}(A|\omega_e = 0).
\] (6)

The proof is a straightforward computation. Recall that an edge $e$ is pivotal for a configuration $\omega$ and an event $A$ if $\omega(c) \notin A$ and $\omega^{(e)} \in A$, where $\omega(e)$ (respectively $\omega^{(e)}$) is the same configuration as $\omega$ except maybe for $e$ where we close the edge $e$ in $\omega(c)$ (respectively open the edge $e$ in $\omega^{(e)}$). We can use Lemma 3 and the FKG inequality to see that
\[
\frac{d}{dp}\mu_{\Lambda_n,p}(A) = \sum_{e \in E(\Lambda_n)} \mu_{\Lambda_n,p}(A|\omega_e = 1) - \mu_{\Lambda_n,p}(A|\omega_e = 0)
\geq \sum_{e \in E(\Lambda_n)} \mu_{\Lambda_n,p}(\omega^{(e)} \in A) - \mu_{\Lambda_n,p}(\omega_e \in A)
= \sum_{e \in E(\Lambda_n)} \mu_{\Lambda_n,p}(e \text{ pivotal for } A).
\]

Set $A := \{0 \leftrightarrow \partial\Lambda_n\}$. Define the following random set
\[
\gamma := \{z \in \Lambda_n : z \text{ not connected to } \Lambda_n^c\}.
\]

By inclusion of events, we get
\[
\sum_{e \in E(\Lambda_n)} \mu_{\Lambda_n,p}(e \text{ pivotal for } 0 \leftrightarrow \partial\Lambda_n) \geq \sum_{e \in E(\Lambda_n)} \mu_{\Lambda_n,p}(e \text{ pivotal for } 0 \leftrightarrow \partial\Lambda_n, 0 \leftrightarrow \partial\Lambda_n)
= \sum_{0 \in S} \sum_{e \in E(\Lambda_n)} \mu_{\Lambda_n,p}(e \text{ pivotal for } 0 \leftrightarrow \partial\Lambda_n, \gamma = S),
\]
where we decomposed with respect to all possibilities for $\gamma$ in the last line. Remark that $\gamma = S$ and $e = xy$ is pivotal for $0 \leftrightarrow \partial\Lambda_n$ if and only if $\gamma = S$, $0 \leftrightarrow x$ and $y \notin S$. Moreover, the event $\{0 \leftrightarrow x\}$ is measurable with respect to the edges in $S$ and the event $\{\gamma = S\}$ is measurable with respect to the edges that have at least one endpoint outside of $S$. Finally, all the edges in $\Delta S$ are closed. Thus
\[
\mu_{\Lambda_n,p}(e \text{ pivotal for } 0 \leftrightarrow \partial\Lambda_n, \gamma = S) = \mu_{\Lambda_n,p}(0 \leftrightarrow x, \gamma = S) = \mu_{S,p}(0 \leftrightarrow x)\mu_{\Lambda_n,p}(\gamma = S),
\]
where the last equality follows from the Markov property. Plugging this into the inequality above gives
\[
\sum_{0 \in S} \sum_{e \in E(\Lambda_n)} \mu_{\Lambda_n,p}(e \text{ pivotal for } 0 \leftrightarrow \partial\Lambda_n, \gamma = S) = \frac{1}{p} \sum_{0 \in S} \sum_{xy \in \Delta S} p \varphi_p(S) \mu_{\Lambda_n,p}(\gamma = S)
= \frac{1}{p} \varphi_p(S) \mu_{\Lambda_n,p}(\gamma = S)
\geq \frac{1}{p} \mu_{\Lambda_n,p}(0 \leftrightarrow \partial\Lambda_n),
\]
where we used that $\varphi_p(S) \geq 1$ since $p > \bar{p}_c$. Therefore, by combining all the inequalities, we get (5), which finishes the proof of Lemma 2.
3 Concluding remarks

1. It is natural to ask whether this approach can be further generalized for bigger values of $q$. The proof of (5) does not use the fact that $q = 2$ and is valid for all $q \geq 1$, which implies $\hat{\rho}_c(q) \geq \rho_c(q)$. However, it is easy to see that the susceptibility with free boundary conditions is always infinite at $\hat{\rho}_c$, i.e.

$$\sum_{x \in \mathbb{Z}^d} \mu_{\mathbb{Z}^d, \hat{\rho}_c, q}(0 \leftrightarrow x) = \infty.$$ 

But the susceptibility is known to be finite at $\rho_c$ for $q > 4$ on $\mathbb{Z}^2$ (see [4, 2]) and is conjectured to be finite for $q > 2$ on $\mathbb{Z}^d$ with $d \geq 3$ (for $q$ large enough, this result is proved in [6]). This in turn implies that $\rho_c < \hat{\rho}_c$ and therefore Corollary 1 is not longer true in these cases.

2. The argument presented here can be extended to any finite-range coupling constants $(J_{x,y})_{x,y \in \mathbb{Z}^d}$, see [3].

3. For infinite-range coupling constants decaying sub-exponentially fast, the second item of Theorem 1 doesn’t hold. However, as in [3], one can still prove that Lemma 2 holds and that the susceptibility with free boundary conditions is finite for every $p < \hat{\rho}_c$, which implies $\hat{\rho}_c = \rho_c$. One can then use the same reasoning as in [1] to deduce that $\mu_{\mathbb{Z}^d, \rho_c, 2}(0 \leftrightarrow x) \leq cJ_0, x$ for every $p < \rho_c$ and for some positive constant $c$ depending on $p$. Note that the use of the exponential decay of the volume of the connected component of 0 in [1] can be replaced by the existence of $S$ such that $\varphi_p(S) < 1$.

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