On the additive bases problem

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Abstract

We prove that if $G$ is an Abelian group and $A_1, \ldots, A_k \subseteq G$ satisfy $mA_i = G$ (the $m$-fold sumset), then $A_1 + \ldots + A_k = G$ provided that $k \geq c_m \log n$. This generalizes a result of Alon, Linial, and Meshulam [Additive bases of vector spaces over prime fields. J. Combin. Theory Ser. A, 57(2):203–210, 1991] regarding the so called additive bases.

1 Introduction

Let $p$ be a fixed prime, and let $\mathbb{Z}_p^n$ denote the $n$-dimensional vector space over the field $\mathbb{Z}_p$. Given a multiset $B$ with elements from $\mathbb{Z}_p^n$, let $S(B) = \{ \sum_{b \in B} b \mid S \subseteq B \}$. The set $B$ is called an additive basis if $S(B) = \mathbb{Z}_p^n$.

Jaeger, Linial, Payan, and Tarzi [JLPT92] made the following conjecture and showed that if true, it would provide a beautiful generalization of many important results regarding nowhere zero flows. In particular the case $p = 3$ would imply the weak 3-flow conjecture, which has been proven only recently by Thomassen [Tho12].

Conjecture 1. [JLPT92] For every prime $p$, there exists a constant $k_p$ such that the union (with repetitions) of any $k_p$ bases for $\mathbb{Z}_p^n$ forms an additive basis.

Let us denote by $k_p(n)$ the smallest $k \in \mathbb{N}$ such that the union of any $k$ bases for $\mathbb{Z}_p^n$ forms an additive basis. In [ALM91] two different proofs are given to show that $k_p(n) \leq c_p \log n$, where here and throughout the paper the logarithms are in base 2. The first proof is based on exponential sums and yields the bound $k_p(n) \leq 1 + (p^2/2)\log 2pn$, and the second proof is based on an algebraic method and yields $k_p(n) \leq (p-1)\log n + p - 2$. As it is observed in [ALM91], it is easy to construct examples showing that $k_p(n) \geq p$, and in fact, to the best of our knowledge, it is quite possible that $k_p(n) = p$.

Let $G$ be an Abelian group, and for $A, B \subseteq G$, define the sumset $A + B = \{ a + b \mid a \in A, b \in B \}$. For $A \subseteq G$ and $m \in \mathbb{N}$, let $mA = A + \ldots + A$ denote the $m$-fold sumset of $A$. Note that for a basis $B$ of $\mathbb{Z}_p^n$, we have $(p-1)S(B) = \mathbb{Z}_p^n$. On the other hand if $B = B_1 \cup \ldots \cup B_k$ is a union with repetitions of $k$ bases for $\mathbb{Z}_p^n$, then $S(B) = S(B_1) + \ldots + S(B_k)$. Hence Theorem 2 below is a generalization of the above mentioned theorem of Alon et al [ALM91].

Theorem 2 (Main theorem). Let $G$ be a finite Abelian group and suppose that $A_1, \ldots, A_{2K} \subseteq G$ satisfy $mA_i = G$ for all $1 \leq i \leq 2K$ where $K \geq m \ln \log |G|$. Then $A_1 + \ldots + A_{2K} = G$. Moreover for $m = 2$, it suffices to have $K \geq \log \log |G|$.

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We present the proof of Theorem 2 in Section 2. While it is quite possible that Conjecture 1 is true, the following example shows that its generalization, Theorem 2, cannot be improved beyond $\Theta(\log \log |G|)$ even when $m = 2$.

**Example 3.** Let $n = 2^k$ and for $i = 1, \ldots, k$, let $C_i \subseteq \mathbb{Z}^2_p$ be the set of vectors in $\mathbb{Z}^2_p \setminus \{0\}$ in which the first half or the second half (but not both) of the coordinates are all 0’s. Note that $C_i + C_i = \mathbb{Z}^2_p$. Define $A_0 = (\mathbb{Z}_p \setminus \{0\})^{2^k}$ and for $i = 1, \ldots, k$, let

$$A_i = C_i \times \ldots \times C_i \subseteq \mathbb{Z}^n_p.$$  

It follows from $C_i + C_i = \mathbb{Z}^2_p$ that $A_i + A_i = \mathbb{Z}^n_p$. On the other hand a simple induction shows that for $j \leq k$,

$$A_0 + \ldots + A_j = (\mathbb{Z}_p^{2^j} \setminus \{0\})^{2^k-j} \neq \mathbb{Z}^n_p.$$

**Remark 4.** Theorem 2 in particular implies that $k_p(n) \leq 2(p - 1) \ln n + 2(p - 1) \ln \ln p$, and $k_3(n) \leq 2 \log n + 2$. Note that for $p > 3$, the algebraic proof of [ALM91] provides a slightly better constant, however unlike the theorem of [ALM91], Theorem 2 can be applied to the case where $p$ is not necessarily a prime. □

## 2 Proof of Theorem 2

The proof is based on the Plünnecke-Ruzsa inequality.

**Lemma 5** (Plünnecke-Ruzsa). If $A, B$ are finite sets in an Abelian group satisfying $|A + B| \leq \alpha |B|$, then

$$|kA| \leq \alpha^k |B|,$$

provided that $k > 1$.

Next we present the proof of Theorem 2. For $2 \leq i \leq K$, substituting $k = m$, $A = A_i$ and $B = A_1 + \ldots + A_{i-1}$ in Lemma 5 we obtain

$$|G| = |mA_i| \leq \left( \frac{|A_1 + \ldots + A_{i-1} + A_i|}{|A_1 + \ldots + A_{i-1}|} \right)^m |A_1 + \ldots + A_{i-1}|,$$

which simplifies to

$$|G|^{1/m} |A_1 + \ldots + A_{i-1}|^{m^{-1}} \leq |A_1 + \ldots + A_{i-1} + A_i|.$$

Consequently

$$|G|^{1-\lambda} |A_1|^\lambda \leq |A_1 + \ldots + A_K|,$$

where $\lambda = \left( \frac{m-1}{m} \right)^K$. Since $K \geq m \ln \log |G|$, we have $\lambda = \left( \frac{m-1}{m} \right)^K < e^{-K/m} \leq 1/\log |G|$, and thus $|G|\lambda < 2$ and $|G|/2 < |A_1 + \ldots + A_K|$. Similarly we obtain

$$|G|/2 < |A_{K+1} + \ldots + A_{2K}|.$$

Since $A + B = G$ if $|A|, |B| > |G|/2$, we conclude

$$A_1 + \ldots + A_{2K} = G.$$

Finally note that for $m = 2$, we have $\lambda = 2^{-K}$, and thus to obtain $|G|/2 < |G|^{1-\lambda} |A_1|^\lambda$, it suffices to have $K \geq \log \log |G|$.
3 Quasi-random Groups

While Example 3 shows that the bound of $\Theta(\log \log |G|)$ is essential in Theorem 2 for certain non-Abelian groups, it is possible to achieve the constant bound similar to what is conjectured in Conjecture 1. A finite group $G$ is called $D$-quasirandom if all non-trivial unitary representations of $G$ have dimension at least $D$. The terminology “quasirandom group” was introduced explicitly by Gowers in the fundamental paper [Gow08] where he showed that the dense Cayley graphs in quasirandom groups are quasirandom graphs in the sense of Chung, Graham, and Wilson [CGW89]. The group $\text{SL}_2(\mathbb{Z}_p)$ is an example of a highly quasirandom group. The so called Frobenius lemma says that $\text{SL}_2(\mathbb{Z}_p)$ is $(p-1)/2$-quasirandom. This has to be compared to the cardinality of this group, $|\text{SL}_2(\mathbb{Z}_p)| = p^3 - p$. The basic fact that we will use about the quasirandom groups is the following theorem of Gowers (See also [Tao15, Exercise 3.1.1]).

**Theorem 6 ([Gow08]).** Let $G$ be a $D$-quasirandom finite group. Then every $A, B, C \subseteq G$ with $|A||B||C| > |G|^3/D$ satisfy $ABC = G$.

We will also need the noncommutative version of Ruzsa’s inequality.

**Lemma 7 (Ruzsa inequality).** Let $A, B, C \subseteq G$ be finite subsets of a group $G$. Then

$$|AC^{-1}| \leq \frac{|AB^{-1}||BC^{-1}|}{|B|}.$$

**Proof.** The claims follows immediately from fact that by the identity $ac^{-1} = ab^{-1}bc^{-1}$, every element $ac^{-1}$ in $AC^{-1}$ has at least $|B|$ distinct representations of the from $xy$ with $(x, y) \in (AB^{-1}) \times (BC^{-1})$.

Finally we can state the analogue of Theorem 2 for quasi-random groups.

**Theorem 8.** Let $G$ be a $|G|^{\delta}$-quasirandom finite group for some $\delta > 0$. If the sets $A_1, \ldots, A_K \subseteq G$ satisfy $A_iA_i^{-1} = G$ for all $1 \leq i \leq K$ where $K > \log(3/\delta)$. Then $A_1 \cdots A_K = G$.

**Proof.** Obviously $|A_1| \geq |G|^{1/2}$. For $2 \leq i \leq K$, substituting $A = C = A_i^{-1}$ and $B = A_1 \cdots A_{i-1}$ in Lemma 7 we obtain

$$\sqrt{|G||A_1 \cdots A_{i-1}|} \leq |A_1 \cdots A_i|,$$

which in turn shows

$$|G|^{1-2^{-K}} \leq |A_1 \cdots A_K|.$$

Since $K > \log(3/\delta)$, we have

$$|G||G|^{-\delta/3} < |A_1 \cdots A_K|.$$

We obtain a similar bound for $|A_{K+1} \cdots A_{2K}|$ and $|A_{2K+1} \cdots A_{3K}|$, and the result follows from Theorem 6.

**Remark 9.** Note that in particular for $G = \text{SL}_2(\mathbb{Z}_p)$, if $p \geq 7$, and $A_1, \ldots, A_{12} \subseteq G$ satisfy $A_iA_i^{-1} = G$, then $A_1 \cdots A_{12} = G$.

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