GALVIN’S QUESTION ON NON-$\sigma$-WELL ORDERED TOTAL ORDERS

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Abstract. Assume $\mathcal{C}$ is the class of all linear orders $L$ such that $L$ is not a countable union of well ordered sets, and every uncountable subset of $L$ contains a copy of $\omega_1$. We show it is consistent that $\mathcal{C}$ has minimal elements. This answers an old question due to Galvin in [3].

1. Introduction

A linear order $L$ is said to be $\sigma$-well ordered if it is a countable union of well ordered subsets. Galvin asked whether or not every non-$\sigma$-well ordered linear order has to contain a real type, Aronszajn type, or $\omega_1^*$. Baumgartner answered Galvin’s question negatively by proving the following theorem.

Theorem 1.1 ([3]). There are non-$\sigma$-well ordered linear orders $L$ such that every uncountable suborder of $L$ contains a copy of $\omega_1$.

Recall that a linear order $L$ is said to be a real type, if it is isomorphic to an uncountable subset of the real numbers. An uncountable linear order $L$ is said to be an Aronszajn type, if it does not contain any real type or copies of $\omega_1, \omega_1^*$. Here $\omega_1^*$ is $\omega_1$ with the reverse ordering.

Let $\mathcal{C}$ be the class of non-$\sigma$-well ordered linear orders such that every uncountable suborder of $L$ contains a copy of $\omega_1$. Note that the elements in $\mathcal{C}$ together with real types, Aronszajn types, and $\omega_1^*$ form a basis for the class of non-$\sigma$-well ordered linear orders. Baumgartner’s theorem asserts it is essential to include $\mathcal{C}$ in this basis.

In the final section of [3], Baumgartner mentions the following question which is due to Galvin.

Question 1.2 ([3], Problem 4). $L \in \mathcal{C}$ is said to be minimal provided that whenever $L' \subset L$, $|L'| = |L|$ and $L' \in \mathcal{C}$ then $L$ embeds into $L'$. Does $\mathcal{C}$ have minimal elements?

Key words and phrases. trees, linear orders, $\sigma$-well ordered, $\sigma$-scattered.
Before we answer Question 1.2 we discuss the motivation behind this question. The following two deep theorems are about the minimality of non-$\sigma$-well ordered order types.

**Theorem 1.3** ([1]). Assume MA$_{\omega_1}$. Then it is consistent that there is a minimal Aronszajn line.

**Theorem 1.4** ([2]). Assume PFA. Then every two $\aleph_1$-dense subsets of the reals are isomorphic.

In particular, these theorems show it is consistent that real types and Aronszajn types have minimal elements. It is trivial that $\omega_1^*$ is a minimal non-$\sigma$-well ordered linear order as well. So it is natural to ask whether or not $\mathcal{C}$ can have minimal elements. A consistent negative answer to Question 1.2 is provided in [4].

**Theorem 1.5.** Assume PFA$^+$. Then every minimal non-$\sigma$-scattered linear order is either a real type or an Aronszajn type.

Recall that a linear order $L$ is said to be scattered if it does not contain a copy of $(\mathbb{Q}, \leq)$. $L$ is called $\sigma$-scattered if it is a countable union of scattered suborders. In order to see that the theorem above provides consistent negative answer to Question 1.2 recall that a linear order is $\sigma$-well ordered if and only if it is $\sigma$-scattered and it does not contain a copy of $\omega_1^*$. In this paper we provide a consistent positive answer to Question 1.2 by proving the following theorem.

**Theorem 1.6.** Assume $\mathcal{C}$ is the class of all non-$\sigma$-well ordered linear orders $L$ such that every uncountable suborder of $L$ contains a copy of $\omega_1$. Then it is consistent with ZFC that $\mathcal{C}$ has a minimal element.

This theorem should be compared to the following theorem from [6].

**Theorem 1.7.** It is consistent with ZFC that there is a minimal non-$\sigma$-scattered linear order $L$, which does not contain any real type or Aronszajn type.

Theorem 1.7 does not answer Question 1.2. The reason is that the linear orders which witness Theorem 1.7 in [6], are dense suborders of the set of all branches of a Kurepa tree $K$. Note that such linear orders have to contain copies of $\omega_1^*$. Moreover, the only way to show that a suborder $L$ of the set of branches of $K$ is not $\sigma$-scattered was to show that $L$ is dense in a Kurepa subtree. In particular, it was

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1A linear order is said to be $\aleph_1$-dense if every non-empty interval has size $\aleph_1$.

2This fact probably exists in classical texts. Since we do not have a reference for it, we provide a proof in the next section.
unclear how to keep the tree $K$ non-$\sigma$-scattered, if $K$ had only $\aleph_1$ many branches. In this paper, aside from eliminating copies of $\omega^*_1$, we provide a different way of keeping $\omega_1$-trees like $K$ non-$\sigma$-scattered, in certain forcing extensions.

2. Preliminaries

In this section we review some facts and terminology regarding $\omega_1$-trees, linear order and countable support iteration of some type of forcings. The material in this section can also be found in [4] and [5].

Recall that an $\omega_1$-tree is a tree which has height $\omega_1$ and countable levels. If $T$ is a tree we assume that it does not branch at limit heights. More precisely, if $s, t$ are distinct elements in the same level of limit height then they have different sets of predecessors. Assume $T$ is a tree and $U \subset T$. We say that $U$ is nowhere dense if for all $t \in T$ there is $s > t$ such that $U$ has no element above $s$. Assume $T, U$ are trees. The function $f : T \to U$ is said to be a tree embedding if $f$ is one-to-one, it is level preserving and $t < s$ if and only if $f(t) < f(s)$. Assume $T$ is a tree, then $T_t$ is the collection of all $s \in T$ which are comparable with $t$. We call a chain $b \subset T$ a cofinal branch, if it intersects all levels of $T$. If $b \subset T$ is a branch then $b(\alpha)$ refers to the element $t \in b$ which is of height $\alpha$. If $b, b'$ are two different maximal chains then $\Delta(b, b')$ is the smallest ordinal $\alpha$ such that $b(\alpha) \neq b'(\alpha)$. The collection of all cofinal branches of $T$ is denoted by $\mathcal{B}(T)$. We use the following fact which is easy to check.

**Fact 2.1.** Assume $T$ is a lexicographically ordered $\omega_1$-tree such that $(T, <_{\text{lex}})$ has a copy of $\omega^*_1$. Then there is a branch $b$ and a sequence of branches $\langle b_\xi : \xi \in \omega_1 \rangle$ such that:

- for all $\xi \in \omega_1$, $b <_{\text{lex}} b_\xi$
- $\sup\{\Delta(b, b_\xi) : \xi \in \omega_1\} = \omega_1$.

**Definition 2.2.** [4] Assume $L$ is a linear order. We use $\hat{L}$ in order to refer to the completion of $L$. In other words, we add all the Dedekind cuts to $L$ in order to obtain $\hat{L}$. For any set $Z$ and $x \in L$ we say $Z$ captures $x$ if there is $z \in Z \cap \hat{L}$ such that $Z \cap L$ has no element which is strictly in between $z$ and $x$.

**Fact 2.3.** [4] Assume $L$ is a linear order, $M \prec H_\theta$ where $\theta$ is a regular large enough cardinal, $x \in L$ and $M$ captures $x$. Then there is a unique $z \in \hat{L}$ such that $M \cap L$ has no element strictly in between $z, x$. In this case we say that $M$ captures $x$ via $z$. 
Definition 2.4. \cite{4} The invariant $\Omega(L)$ is defined to be the set of all countable $Z \subseteq \hat{L}$ such that $Z$ captures all elements of $L$. We let $\Gamma(L) = [\hat{L}]^\omega \setminus \Omega(L)$.

If $T$ is an $\omega_1$-tree which is lexicographically ordered, then both $T, B(T)$ can be considered as linear orders. Then the following fact is a routine definition chasing.

Fact 2.5. Assume $T$ is a lexicographically ordered $\omega_1$-tree such that for every $t \in T$, there is a cofinal branch $b \subseteq T$ with $t \in b$. Let $\theta$ be a regular cardinal such that $\mathcal{P}(T) \in H_\theta$, $M \prec H_\theta$ be countable. Then $M \in \Omega(B(T))$ iff $M \in \Omega(T)$.

We will use the following lemma in order to characterize $\sigma$-scattered linear orders.

Theorem 2.6. \cite{4} $L$ is $\sigma$-scattered iff $\Gamma(L)$ is not stationary in $[\hat{L}]^\omega$.

The following lemma will be used in order to determine which linear orders are $\sigma$-well ordered. Most likely an equivalent of this lemma exists in classical texts, but for more clarity we include the proof. Our proof uses the ideas in the proof of the previous theorem from \cite{4}.

Lemma 2.7. Assume $L$ is a linear order which does not have a copy of $\omega_1^*$. Then $L$ is $\sigma$-well ordered iff it is $\sigma$-scattered.

Proof. Assume $L$ is a linear order of size $\kappa$ which does not have a copy of $\omega_1^*$ and which is $\sigma$-scattered. We will show that it is $\sigma$-well ordered. Let $\theta$ be a regular cardinal such that $\mathcal{P}(L) \in H_\theta$. Let $\langle M_\xi : \xi \in \kappa \rangle$ be a continuous $\in$-chain of elementary submodels of $H_\theta$ such that $L, \Omega(L)$ are in $M_0$, $\xi \in M_\xi$, and $|M_\xi| = |\xi| + \aleph_0$. Observe that for all $x \in L$ and $\xi \in \omega_1$ there is a unique $z \in L \cap M_\xi$ such that $M_\xi$ captures $x$ via $z$. Moreover, if $M_\xi$ captures $x$ via $z$ then $x \leq z$. This is because $\omega_1^*$ does not embed into $L$.

For each $x \in L$ let $g_x : \kappa \to \hat{L}$ such that for all $\xi \in \omega_1$, $g_x(\xi) \in M_\xi \cap \hat{L}$ and $M_\xi$ captures $x$ via $g_x(\xi)$. We note that the map $x \mapsto g_x$ is order preserving when we consider the lexicographic order on all functions from $\omega_1$ to $\hat{L}$. It is easy to see that the function $g_x$ is decreasing and $\text{range}(g_x)$ is finite.

For each $x \in L$ with $|\text{range}(g_x)| = n + 1$ let $\sigma(x)$ be the decreasing $(n + 1)$-tuple $\langle z_0, z_1, ..., z_n \rangle$ such that for each $i \leq n$, $z_i \in \text{range}(g_x)$. Note that if $i < j$ then $z_i$ appears before $z_j$ in $g_x$. Let $U = \{\sigma(x) : x \in L\}$. We define an order on $U$ as follows. For $\sigma, \tau$ in $U$ let $\sigma < \tau$ if either $\tau$ is an initial segment of $\sigma$ or $\sigma$ is below $\tau$ in the lexicographic order. It is easy to see that $x \mapsto \sigma(x)$ is order preserving.
For each $k \in \omega$, let $U_k = \{ \sigma \in U : |\sigma| = k \}$. We will show by induction on $k$ that $U_k$ is $\sigma$-well ordered for all $k \in \omega$. This is obvious if $k \leq 1$. Assume $U_k$ is $\sigma$-well ordered. For each $\xi \in \kappa$ fix an $\omega$ enumeration of all $z \in \hat{L} \cap (M_{\xi+1} \setminus M_\xi)$ such that for some $x \in L$, $M_{\xi+1}$ captures $x$ via $z$.

For each $\sigma \in U_k$ and $i \in \omega$ let $U_{k,\sigma,i} \subset U_{k+1}$ be the set of all $\sigma z$ such that for some $\xi \in \omega_1$, $z$ is the $i$'th element of $\hat{L} \cap (M_{\xi+1} \setminus M_\xi)$. Note that $U_{k,\sigma,i}$ is well ordered for all $\sigma \in U_k$ and $i \in \omega$. Also $U_{k+1} = \sum_{\sigma \in U_k} (\bigcup_{i \in \omega} U_{k,\sigma,i})$. This means that $U_{k+1}$ is $\sigma$-well ordered as desired. □

Now we review some definitions and facts about the forcings which we are going to use.

**Definition 2.8.** [5] Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. A poset $P$ is said to be $S$-complete if every descending $(M, P)$-generic sequence $\langle p_n : n \in \omega \rangle$ has a lower bound, for all $M$ with $M \cap X \in S$ and $M$ suitable for $X, P$.

We note that $S$-complete posets preserve the stationary subsets of $S$. It is also easy to see that if $X, S$ are as above and $P$ is an $S$-complete forcing then it preserves $\omega_1$ and adds no new countable sequences of ordinals.

**Lemma 2.9.** [5] Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. Then $S$-completeness is preserved under countable support iterations.

**Lemma 2.10.** [5] Assume $T$ is an $\omega_1$-tree which has no Aronszajn subtree in the ground model $V$. Also assume $\Omega(T) \subset [\mathcal{B}(T)]^\omega$ is stationary and $P$ is an $\Omega(T)$-complete forcing. Then $T$ has no Aronszajn subtree in $V^P$. Moreover, $P$ adds no new branches to $T$.

### 3. A Generic Element of $\mathcal{C}$

We will introduce forcing which adds a generic lexicographically ordered $\omega_1$-tree $T$. The tree $T$ has no Aronszajn subtrees. Moreover, the set of all cofinal branches of $T$, which is denoted by $B$, has no copy of $\omega^*_1$. We will also show that $B$ is not $\sigma$-scattered. In the next section by iterating two types of posets we make $B$ a minimal element of $\mathcal{C}$.

**Definition 3.1.** Fix a set $\Lambda$ of size $\aleph_1$. The forcing $Q$ is the poset consisting of all conditions $(T_q, b_q, d_q)$ such that the following hold.
Lemma 3.2. Assume \(<q_n : n \in \omega>\) is a decreasing sequence of conditions in \(Q\), \(m \leq \omega\) and for each \(i \in m\) let \(c_i \subset \bigcup_{n \in \omega} T_{q_n}\) be a cofinal branch. Then there is a lower bound \(q\) for the sequence \(<q_n : n \in \omega>\) in which every \(c_i\) has a maximum with respect to the tree order in \(T_q\). Moreover, for every \(t \in (T_q)_{\alpha_q}\) either there is \(i \in m\) such that \(t\) is above all elements of \(c_i\) or there is \(\xi \in D = \bigcup_{n \in \omega} \text{dom}(b_{q_n})\) such that \(t\) is above all elements of \(\{b_{q_n}(\xi) : n \in \omega \land \xi \in \text{dom}(b_{q_n})\}\). In particular, \(Q\) is \(\sigma\)-closed.

Proof. For each \(n \in \omega\), let \(\alpha_n = \alpha_{q_n}\) and \(T_n = T_{q_n}\). If the set of all \(\alpha_n\)'s has a maximum, it means that after some \(n\), the sequence \(q_n\) is constant. So without loss of generality assume \(\alpha = \sup \{\alpha_n : n \in \omega\}\) is a limit ordinal above all \(\alpha_n\)'s. Let \(T = \bigcup_{n \in \omega} T_n\). For each \(\xi \in D = \bigcup_{n \in \omega} \text{dom}(b_{q_n})\), let \(b_{\xi}\) be the set of all \(t \in T\) such that for some \(n \in \omega\), \(t \leq b_{q_n}(\xi)\). Observe that \(b_{\xi}\) is a cofinal branch in \(T\). Since we are going to put an element on top of every \(b_{\xi}\), from now on, assume that \(s_i\)'s are different from \(b_{\xi}\).

Now we are ready to define the lower bound \(q\). We let \(\alpha_q = \alpha\) and obviously \((T_q)_{\alpha_q} = T\). We put distinct element \(t_\xi\) on top of \(b_{\xi}\) for each \(\xi \in D\). We also put distinct element \(s_i\) on top of \(c_i\) for each \(i \in m\). Therefore, \((T_q)_{\alpha_q} = \{t_\xi : \xi \in D\} \cup \{s_i : i \in m\}\). Let \(E \subset \omega_1 \setminus D\) such that \(|E| = m\). Let \(b_q : D \cup E \to (T_q)_{\alpha_q}\) be any bijective function such that \(b_q(\xi) = t_\xi\) for each \(\xi \in D\). For each \(\xi \in D\), let \(d_q(\xi) = d_{q_n}(\xi)\) where \(n \in \omega\) such that \(\xi \in \text{dom}(d_{q_n})\). For each \(\eta \in E\) let \(d_q(\eta) = \alpha + 1\).
We need to show that $q$ is a lower bound in $Q$. We only show Condition 3 of Definition 3.1 for $q \in Q$. The rest of the conditions and the fact that $q$ is an extension of all $q_n$’s are obvious. Let $A = \{t_\xi : \xi \in D\}$, $S = \{s_i : i \in m\}$, and $u < \text{lex} v$ be two distinct elements in $(T_\eta)_\alpha$. If $u \in S$, then $\Delta(u, v) < \alpha < d_q(\eta)$, where $\eta \in E$ such that $b_q(\eta) = u$. If $u, v$ are both in $A$, and $\xi, \xi'$ are in $D$ such that $b_q(\xi) = u, b_q(\xi') = v$, let $n \in \omega$ such that $\xi, \xi'$ are in $\text{dom}(b_{q_n})$. Then $\Delta(u, v) = \Delta(b_{q_n}(\xi), b_{q_n}(\xi')) < d_{q_n}(\xi) = d_q(\xi)$. If $u \in A, v \in S$ and $\xi \in D$ such that $b_q(\xi) = u$, fix $n \in \omega$ such that $\xi \in \text{dom}(b_{q_n})$ and $\alpha_{q_n} > \Delta(u, v)$. Let $u', v'$ be the elements in $T_\alpha$ which are below $u, v$ respectively. It is obvious that $b_{q_n}(\xi) = u'$. Then $\Delta(u, v) = \Delta(u', v') < d_{q_n}(\xi) = d_q(\xi)$. Therefore $q$ is a condition in $Q$. \qed

We will use the following terminology and notation regarding the forcing $Q$. Assume $G$ is a generic filter for $Q$. We let $T = \bigcup_{q \in G} T_q$. We also let $B = (\mathcal{B}(T), <_{\text{lex}})$. By $b_\xi$ we mean the set of $t \in T$ such that for some $q \in G$, $b_q(\xi) = t$.

**Definition 3.3.** For every $\xi \in \omega_1$, $d(\xi) = \sup \{\Delta(b_\xi, b_q) : b_\xi < \text{lex} b_q\}$, and if $b = b_\xi$ we sometimes use $d(b)$ instead of $d(\xi)$.

It is worth pointing out that, by Fact 2.1, the role of $d$ in this forcing is to control $<_{\text{lex}}$ so that $(B, <_{\text{lex}})$ has no copy of $\omega_1^*$. The behavior of $d$ plays an essential role from the technical point of view, mostly in the density lemmas for the forcing which adds embedding from $B$ to its subsets.

**Lemma 3.4.** The function $d$ is countable to one function, i.e. for all $\alpha \in \omega_1$ there are countably many $\xi \in \omega_1$ with $d(\xi) = \alpha$.

**Proof.** Assume that the set $A = \{\xi : d(\xi) = \alpha\}$ is uncountable. Then for each pair of distinct ordinals $\xi, \eta$ in $A$, $b_\xi(\alpha + 1) \neq b_\eta(\alpha + 1)$. But this means that $T$ has an uncountable level which is a contradiction. \qed

**Lemma 3.5.** For every $t_0 \in T$ and $\beta > \text{ht}(t)$, there is an $\alpha > \beta$ such that $(T_\alpha \cap T_{t_0}, <_{\text{lex}})$ contains a copy of the rationals.

**Proof.** We will show that for all $q \in Q$ and $t_0 \in T_q$, the set $\{p \leq q : (Q, \langle) \rightarrow (\{s \in (T_p)_\alpha : t \leq T_p, \text{lex}\})\}$ is dense blow $q$. Fix $r \leq q$ with $\alpha_r > \beta$ and $t \in (T_r)_\alpha \cap T_{t_0}$. Let $\xi \in \omega_1$ such that $b_r(\xi) = t$. Without loss of generality we can assume that $d_r(\xi) < \alpha_r$. Fix $X \subseteq \Lambda \setminus T_r$ an infinite countable set and $u \in X$. Let $p < r$ be the condition such that the following hold.

- $\alpha_p = \alpha_r + 1$, and $\text{dom}(b_p) = \text{dom}(r) \cup E$ where $E$ consists of the first $\omega$ ordinals after $\sup(\text{dom}(r))$. 

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\( T_r \subset T_p, (T_r)_\alpha = (T_p)_\alpha \) and for all \( s \in (T_r)_\alpha \setminus \{ t \} \) there is a unique \( s' \in (T_p)_\alpha \) with \( s' > s \).

- \( (T_p)_\alpha \) consists of the set of all \( s' \) as above union with \( X \). Moreover, for every \( x \in X \), \( t \) is below \( x \), in the tree order.
- Define \( \langle \rangle \text{lex} \) on \( X \) so that \( X \) becomes a countable dense linear order without smallest element and with \( \max(X) = u \).

It is easy to see that \( p \in Q \) is an extension of \( r \) and the set \( X \setminus \{ u \} \) is a copy of the rationals whose elements are above \( t_0 \).

**Lemma 3.6.** Every uncountable downward closed subset of \( T \) contains \( b_\xi \) for some \( \xi \in \omega_1 \). In particular, \( \{ b_\xi : \xi \in \omega_1 \} \) is the set of all branches of \( T \).

**Proof.** Let \( \dot{A} \) be a \( Q \)-name for an uncountable downward closed subset of \( T \) and \( p \in Q \) forces that \( \dot{A} \) contains non of the \( b_\xi \)'s. Let \( M \prec H_\theta \) be a countable where \( \theta \) is a regular large enough cardinal such that \( A, p \) are in \( M \). By Lemma 3.2 for \( m = 0 \), there is an \( (M, Q) \)-generic condition \( q \leq p \) such that \( \alpha_q = \delta \) where \( \delta = M \cap \omega_1 \) and for each \( t \in (T_q)_\delta \) there is \( \xi \in M \) such that \( b_q(\xi) = t \). But then \( q \) forces that \( \dot{A} \) has no element of height \( \delta \) which is a contradiction.

The proof of the following lemma is very similar to the one above.

**Lemma 3.7.** \( \Omega(T) \) is stationary.

We note that if CH holds then the forcing \( Q \) satisfies the \( \aleph_2 \) chain condition. On the other hand, if \( \kappa > \omega_1 \) and we consider \( \kappa \) many branches for \( T \) in the definition of \( Q \), then \( Q \) collapses \( \kappa \) to \( \omega_1 \). This is because \( Q \) adds a countable to one function from \( \kappa \) to \( \omega_1 \).

4. Making \( B \) Minimal in \( C \)

In this forcing we will introduce the forcings which make \( B \) a minimal element of \( C \). The idea is as follows. If \( L \subset B \) is too small, in the sense that there is not forcing which preserves \( B \in C \) and which adds an embedding from \( B \) to \( L \), then we make \( L \) \( \sigma \)-well ordered. In other words \( L \not\in C \) anymore. For other suborders of \( B \) we introduce a forcing which adds embedding from \( B \) to them and which keeps \( B \) inside \( C \).
Definition 4.1. Assume $L \subset B$ is nowhere dense. Define $S_L$ to be the poset consisting of all increasing continuous countable sequences $\langle \alpha_i : i \in \beta + 1 \rangle$ such that $\beta \in \omega_1$ and for all $i$ and $t \in T_{\alpha_i} \cap (\bigcup L)$ there is $\xi < \alpha_i$ with $t \in b_\xi$. If $p, q \in S_L$, $q$ is an extension of $p$ if $p$ is an initial segment of $q$.

It is easy to see that for every nowhere dense $L \subset B$, $S_L$ is $\Omega(T)$-complete. Therefore, as long as $\Omega(T)$ is stationary, $S_L$ preserves $\omega_1$. Moreover, after forcing with $S_L$, $\Gamma(L)$ is non-stationary. So in any forcing extension of with $S_L$, $L$ is $\sigma$-well ordered. This uses Lemmas 2.7, 2.6, and the fact that $L$ has no copy of $\omega^*_1$. Note that there are many nowhere dense subsets $L \subset B$ which are not $\sigma$-scattered and which do not contain a copy of $B$ in any forcing extension which has the same $\omega_1$. This means that the forcing $S_L$ are necessary in order to make $B$ minimal in $\mathcal{C}$.

Definition 4.2. Assume $U = T_t$ for some $t \in T$ and $L \subset B(U)$ is dense in $B(U)$. Define $E_L$ to be the poset consisting of all conditions $q = (f_q, \phi_q)$ such that:

1. $f_q : T \upharpoonright A_q \rightarrow U \upharpoonright A_q$ is a $\leq_{\text{lex}}$-preserving tree embedding where $A_q$ is a countable and closed subset of $\omega_1$ with $\text{max}(A_q) = \alpha_q$,
2. $\phi_q$ is a countable partial injection from $\omega_1$ into $\{ \xi \in \omega_1 : b_\xi \in L \}$ such that the map $b_\xi \mapsto b_{\phi_q}(\xi)$ is $\leq_{\text{lex}}$-preserving,
3. for all $t \in T_{\alpha_q}$ there are at most finitely many $\xi \in \text{dom}(\phi_q) \cup \text{range}(\phi_q)$ with $t \in b_\xi$,
4. $f_q, \phi_q$ are consistent, i.e. for all $\xi \in \text{dom}(\phi_q), f_q(b_\xi(\alpha_q)) \in b_{\phi_q}(\xi)$,
5. for all $\xi \in \text{dom}(\phi_p), d(\xi) \leq d(\phi_p(\xi))$

We let $q \leq p$ if $A_p$ is an initial segment of $A_q$, $f_p \subset f_q$, and $\phi_p \subset \phi_q$.

Lemma 4.3. For all $\beta \in \omega_1$ the set of all conditions $q \in E_L$ with $\alpha_q > \beta$ is dense in $E_L$.

Proof. Fix $p \in E_L$ and let $D_p = \text{dom}(\phi_p)$ and $R_p = \text{range}(\phi_p)$. We consider the following partition of $U = T_{\alpha_p} \cap \text{range}(f_p)$. Let $U_0$ be the set of all $u \in U$ such that if $u \in b \in R_p$ then there is a $c \in B$ with $u \in c$ and $b \leq_{\text{lex}} c$. Note that if $u \in U$ and there is no $b \in R_p$ with $u \in b$ then $u \in U_0$. We let $U_1 = U \setminus U_0$.

First we will show that if $u \in U_0$ then there is $\alpha_u \in \omega_1$ and $X_u \subset T_{\alpha_u} \cap T_u$ such that:

a. $\alpha_u > \max(\{ \Delta(b, c) : b, c \text{ are in } A \} \cup \{ \beta \})$, where $A$ is the set of all $b \in R_p$ such that $u \in b$,

b. $(X_u, \leq_{\text{lex}})$ is isomorphic to the rationals, and
In order to see this, let $b_m$ be the maximum of $A$ with respect to $\lex$. Let $c \in B$ such that $u \in c$ and $b_m \prec \lex c$. Let $t_m$ be the element in $c \setminus b$ which has the lowest height. By Lemma 3.3, there is a copy of the rationals $X_{t_m}$ in some level $\alpha_{t_m}$ which is above $t_m$ in the tree order. In other words, $u \prec t_m \prec x$ for all $x \in X_{t_m}$ and $X_{t_m}$ is isomorphic to the rationals when it is considered with $\lex$. Moreover, $b_m(\alpha_{t_m}) \prec \lex x$ for all $x \in X_{t_m}$.

We can find $\alpha_{b/b} > \beta$ such that if $b' \prec \lex b$ are in $A$ then there is a copy of the rationals $X_{b/b} \subset T_{\alpha_{b/b}} \cap T_u$ and for all $x \in X_{b/b}$, $b'(\alpha) \prec \lex x \prec \lex b(\alpha)$. This is because there is no restriction for branching to the left in the tree $T$. More precisely, for all $\gamma \in \omega_1$ there is a $c \prec \lex b$ in $B$ such that $\Delta(b, c) \succ \gamma$. Similarly, if $a$ is the minimum of $A$ with respect to $\lex$, there is $\alpha_a \succ \beta$ and $X_a \subset T_{\alpha_a} \cap T_u$ which is isomorphic to the rationals and if $x \in X_a$ then $u \prec x$ and for all $b \in A$, $x \prec \lex b(\alpha)$. Now let $\alpha$ be above $\alpha_{t_m}$, $\alpha_a$ and all of $\alpha_{b/b}$. Then $T_{\alpha_a} \cap T_u$ contains $X_u$ which is a copy of the rationals and $\{b(\alpha_u) : b \in A\} \subset X_u$.

Note that $U_1$ is the set of all $u \in U$ such that for some $b_u \in R_p$, $u \in b_u$ and if $u \in c \in B$ then $c \prec \lex b_u$. By the same argument as above we can show for all $u \in U_0$ there is $\alpha_u \in \omega_1$ and $X_u \subset T_{\alpha_u} \cap T_u$ such that:

d. $\alpha_u \succ \max\{\{\Delta(b, c) : b, c \text{ are in } A\} \cup \{\beta\}\}$, where $A$ is the set of all $b \in R_p$ such that $u \in b$,

e. $(X_u \setminus \{b_m(\alpha)\}, \prec \lex)$ is isomorphic to the rationals, where $b_m$ is the maximum of $A$ with respect to $\lex$,

f. $\{b(\alpha_u) : b \in A\} \subset X_u$, and

g. $\max(X_u, \prec \lex) = b_m(\alpha)$.

Now we are ready to introduce the extension $q \leq p$. Let $\alpha \in \omega_1$ and $\alpha > \alpha_u$ for all $u \in U$. Let $A_q = A_p \cup \{\alpha\}$, $\phi_q = \phi_p$. If $u \in U_0$ then $T_{\alpha} \cap T_u$ contains $X_u$ such that conditions b,c hold. If $u \in U_0$ and $f_p(t) = u$, let $f_q \upharpoonright (T_{\alpha} \cap T_t)$ be any $\prec \lex$ preserving function which is consistent with $\phi_q$. If $u \in U_1$ then $T_{\alpha} \cap T_u$ contains $X_u$ such that conditions e,f,g hold. In addition, if $u \in U_1$, $f_p(t) = u$, $b_m = \max(A)$ and $b$ is mapped to $b_m$ by $\phi_q$, then $d(b) < d(b_m)$. So $b(\alpha) = \max((T_{\alpha} \cap T_t), \prec \lex)$. This means that if $u \in U_1$ and $f_p(t) = u$, we can find $f_q \upharpoonright (T_{\alpha} \cap T_t)$ which is $\prec \lex$ preserving and which is consistent with $\phi_q$. 

The proof of the following lemma uses Lemma 3.4 and the same argument as above.

**Lemma 4.4.** For all $\xi \in \omega_1$, the set of all $q \in E_L$ with $\xi \in \text{dom}(\phi_q)$ is dense in $E_L$. 
The following lemma is trivial.

**Lemma 4.5.** The forcing $E_L$ is $\Omega(T)$-complete.

Now we are ready to introduce the forcing extension in which $C$ has a minimal element. Let’s fix some notation first. $P = P_{\omega_2}$ is a countable support iteration $\langle P_i, \dot{Q}_j : i \leq \omega_2, j < \omega_2 \rangle$ over a model of CH such that $Q_0 = Q$, and for all $0 < j < \omega_2$, $\dot{Q}_j$ is a $P_j$-name for either $E_L$ or $S_L$ depending on whether or not $L$ is somewhere dense. As usual, the bookkeeping is such that if $L \subset B$ is in $V^P$, either $E_L$ or $S_L$ has appeared in some step of the iteration. $T, B$ are the generic objects that are introduced by $Q$, as in the previous section.

Since $E_L, S_L$ are $\Omega(T)$-complete forcings, by Lemma 2.10 the countable support iteration consisting of the posets $E_L, S_L$ do not add new branches to $T$. It is worth pointing out that although $P$ is a $\sigma$-closed forcing, the posets $E_L, S_L$ that are involved in the iteration are not even proper.

Regarding $\Gamma(T)$ in the final model, the following fact plays an important role. Assume $M$ is a suitable model for $P$ in $V$ with $M \cap \omega_1 = \delta$ and $\langle p_n : n \in \omega \rangle$ is a descending $(M, P)$-generic sequence. Let $p$ be a lower bound for $\langle p_n : n \in \omega \rangle$, $q_n = p_n \uparrow 1$, $q = p \uparrow 1$. Then $p$ forces that $M \cap B(T) \in \Gamma(B)$ iff $\text{dom}(b_p) \neq \delta$.

**Lemma 4.6.** Assume $G \subset P$ is $V$-generic. Then $\Gamma(B)$ is stationary in $V[G]$.

**Proof.** Assume $M$ is a suitable model for $P$ in $V$ with $M \cap \omega_1 = \delta$ and $\langle p_n : n \in \omega \rangle$ is a descending $(M, P)$-generic sequence. Let $q_n = p_n \uparrow 1$ and $R = \bigcup_{n \in \omega} T_{q_n}$. Note that every $P$-generic filter over $V$ which contains $\langle p_n : n \in \omega \rangle$ extends $V$ to $V^P$ such that $T_{\leq \delta} = R$. We will find an $(M, P)$-generic condition $p$ below $\langle p_n : n \in \omega \rangle$ which forces that $M \cap B(T) \in \Gamma(T)$.

Before we deal with the details we explain the idea how to find such a condition $p$. Let $q = p \uparrow 1$. Since $\langle q_n : n \in \omega \rangle$ is $(M, P)$-generic, we have to have that $\text{dom}(b_q) \supset \delta$. If we allow $\text{dom}(b_q) = \delta$, the advantage is that it is easy to find lower bounds for the rest of the sequences $\langle p_n(\beta) : n \in \omega \rangle$, by induction on $\beta$. But then, the resulting lower bound is going to force that $M \cap B(T) \notin \Gamma(T)$. We use a diagonalization argument in $T$ to add branches other than the ones that are listed by some $\xi \in \delta$ in the generic sequence. Then we will show that we can add enough of such branches and make sure that for the rest of the iteration we can find a lower bound.

Note that for all $\beta \in M \cap \omega_2$, and $\dot{L}$ a $P_\beta$-name for a nowhere dense subset of $B$ in $M$, $\langle p_n : n \in \omega \rangle$ decides $(\bigcup \dot{L}) \cap T_{< \delta}$. More precisely,
there exists \( U \subseteq R \) in \( \mathbf{V} \) such that if \( G \) is a a \( P \)-generic filter over \( \mathbf{V} \) with \( \langle p_n : n \in \omega \rangle \subseteq G \) then \( [(\bigcup \hat{L})]_G \cap [T_{<\delta}]_G = U \). This is because \( [(\bigcup \hat{L})]_G \cap [T_{<\delta}]_G \) is a countable subset of \( R \) and \( R \in \mathbf{V} \). Here we use the fact that countable support iteration of \( \Omega(T) \)-complete forcings do not add new reals. Let \( \mathcal{U} \) be the set of all countable \( U \subseteq R \) such that for some \( \beta \in M \cap \omega_2 \) and \( \hat{L} \in M \) which is a \( P_\beta \)-name for a nowhere dense subset of \( B \) \( \langle p_n : n \in \omega \rangle \) decides \( (\bigcup \hat{L}) \cap T_{<\delta} \) to be \( U \).

Assume \( \beta \in M \cap \omega_2 \) and \( \hat{L} \) is a \( P_\beta \)-name for a somewhere dense subset of \( B \) and \( \hat{Q}_\beta \) is a \( P_\beta \)-name for the forcing \( E_{\hat{L}} \). Let’s denote the canonical name for the generic filter of \( E_{\hat{L}} \) by \( (\hat{f}, \hat{\phi}) \). Similar to the case of nowhere dense subsets of \( B \), \( \langle p_n : n \in \omega \rangle \) decides \( (\bigcup \hat{L}) \cap T_{<\delta} \), \( f \restriction R \) and \( \phi \restriction \delta \). Moreover, \( \hat{f}_G \restriction R \) is in \( \mathbf{V} \), for any \( V \)-generic filter \( G \).

Now, let \( \mathcal{F} \) be the set of all finite compositions \( g_0 \circ g_1 \circ \ldots \circ g_n \), such that for all \( i \leq n \), \( g_i \) or \( g_i^{-1} \) is a partial function on \( R \) which is of the form \( \hat{f}_G \restriction R \) where for some \( \phi, (f, \hat{\phi}) \) is the canonical name for the generic filter added by \( E_{\hat{L}} \), \( \hat{L} \in M \) is a name for a somewhere dense subset of \( B \), and \( G \) is a \( P \)-generic filter over \( \mathbf{V} \) which contains \( \langle p_n : n \in \omega \rangle \). Also let \( \mathcal{W} \) be the collection of all \( g[U] \) such that \( U \in \mathcal{U} \) and \( g \in \mathcal{F} \). Note that every \( W \in \mathcal{W} \) is nowhere dense in \( R \).

Fix \( \langle W_n : n \in \omega \rangle \) an enumeration of \( \mathcal{W} \) and \( \langle \xi_n : n \in \omega \rangle \) an enumeration of \( \delta \). Let \( \langle t_m : m \in \omega \rangle \) be a chain in \( R \) such that

- if \( m = 2k \) then \( W_k \) has no element above \( t_m \), and
- if \( m = 2k+1 \) then \( t_m \) is not in the downward closure of \( \{b_{q_n}(\xi_k) : n \in \omega \} \).

By Lemma 3.2 \( \langle q_n : n \in \omega \rangle \) has a lower bound \( q \) such that whenever \( c \) is a cofinal branch of \( R \), then there is an element on top of \( c \) if and only if one of the following holds.

- For some \( \xi \in \delta \), \( \{b_{q_n}(\xi) : n \in \omega \} \) is cofinal in \( c \).
- For some \( g \in \mathcal{F} \), the image of the downward closure of \( \langle t_m : m \in \omega \rangle \) under \( g \) is cofinal in \( c \).

Moreover, we can choose \( q \) such that if \( t \) is on top of \( c \), \( c \) satisfies the second condition, and \( \eta \in \omega_1 \) with \( b_q(\eta) = t \) then \( d_q(\eta) = \delta + 1 \). For the rest of the proof, assuming that \( p \restriction \beta \) is given, we find \( p(\beta) \). If \( \beta \notin M \cap \omega_2 \) we define \( p(\beta) \) to be the trivial condition of the corresponding forcing \( Q_\beta \). For each \( \beta \in M \), since \( p \restriction \delta \) is \( (M, P_\beta) \)-generic, it decides \( p_n(\beta) \) for all \( n \in \omega \).

Assume \( \hat{Q}_\beta \) is a \( P_\beta \)-name for some \( S_{\hat{L}} \in M \). We define \( p(\beta) = \{(\delta, \delta)\} \cup \bigcup_{n \in \omega} p_n(\beta) \). Observe that if \( \hat{L} \) is a \( P_\beta \)-name for a nowhere dense subset of \( B \), \( t \in (T_q)_\delta \), \( c_t \) is the set of all elements of \( T_\eta \) that are
less than \( t \), and \((b_\eta)^{-1}(t) \geq \delta \) then there is \( s \in c_t \) such that \( p \upharpoonright \beta \) forces that \( \bigcup \hat{L} \) has no element above \( s \). This makes \( p(\beta) \) a lower bound for \( \langle p_n(\beta) : n \in \omega \rangle \).

Assume \( \dot{Q}_{\beta} = E_\dot{L} \) is a \( P_3 \)-name where \( \dot{L} \in M \). By Lemmas 4.3 sup( \( \bigcup_{n \in \omega} A_{p_n(\beta)} \)) = \( \delta \). Moreover, Lemma 4.4 implies that for all \( \xi \in \delta \) there is \( n \in \omega \) such that \( \xi \in \text{dom}(\phi_{p_n(\beta)}) \). We define \( p(\beta) = (f, \phi) \) as follows. Let \( \phi = \bigcup_{n \in \omega} \phi_{p_n(\beta)} \), \( A_{p(\beta)} = \{ \delta \} \cup \bigcup_{n \in \omega} A_{p_n(\beta)} \), \( \text{dom}(f) = T_\eta \upharpoonright A_{p(\beta)} \). If \( \text{ht}(s) \in A_{p_n(\alpha)} \) for some \( n \in \omega \), let \( f(s) = f_{p_n(\alpha)}(s) \). If \( \text{ht}(s) = \delta \) and \( c \) is a cofinal branch in \( R \) whose elements are below \( s \), let \( f(s) \) be the element on top of the chain \( \{ f(v) : v \in c \cap \text{dom}(f) \} \). This makes sense, because there is an element on top of \( \{ f(v) : v \in c \cap \text{dom}(f) \} \) in \( T_\eta \).

In order to see this, first assume that for some \( \xi \in \delta \), \( \{ b_\eta(\xi) : n \in \omega \} \) is cofinal in \( c \). Let \( n \in \omega \) such that \( \xi \in \text{dom}(\phi_{p_n(\beta)}) \), and \( \eta = \phi_{p_n(\beta)}(\xi) \). Then \( b_\eta(\eta) \) is the top element of \( \{ f(v) : v \in c \cap \text{dom}(f) \} \). If for some \( g \in F \), the image of the downward closure of \( \langle t_m : m \in \omega \rangle \) under \( g \) is cofinal in \( c \), then the image of the downward closure of \( \langle t_m : m \in \omega \rangle \) under \( f \circ g \) is cofinal in \( \{ f(v) : v \in c \cap \text{dom}(f) \} \). This finishes defining \( f, \phi \). It is obvious that \( p(\beta) \) is a lower bound for \( \langle p_n(\beta) : n \in \omega \rangle \).

Now we are ready to show that \( B \) is a minimal element of \( C \). Lemma 4.6 implies that \( B \in C \). If \( L \subset B \) and \( L \in C \), then \( L \) has to be somewhere dense. But then the forcing \( E_L \) adds an embedding from \( B \) to \( L \). Hence \( B \) is minimal.

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