Rapid exponential stabilization of nonlinear continuous systems via event-triggered impulsive control

Abstract: This article investigates the problem of rapid exponential stabilization for nonlinear continuous systems via event-triggered impulsive control (ETIC). First, we propose a trigger mechanism that, when triggered by a predefined event, causes the closed-loop system exponentially stable. Then, the exponential stabilization is achieved by the designed ETIC with or without data dropout. The case where there are delays in the ETIC signals is also studied, and the exponential stabilization is proved. Finally, a numerical study is presented, along with numerical illustrations of the stability results.

Keywords: impulsive control, event-triggered control, rapid exponential stabilization, time-delays, data dropout

MSC 2020: 34H15, 93D23, 93C10, 93C27

1 Introduction

It has long been recognized that continuous control is incapable of solving many control problems of nonlinear systems [1]. Moreover, even in cases where continuous control can be used, many practical and theoretical obstacles arise when using this type of control, such as the lack of accurate and up-to-date information at all time despite the availability of frequent state measurements, the lack of accurate state information at all times, or even the sensitivity of the closed-loop system behavior to state measurement errors [2]. Furthermore, the frequent use of the latest innovations in computer and communication technologies in control theory has added new difficulties of other kinds, such as the limited storage capacity of the large amount of data measured continuously or the network saturation in front of the large amount of data to be transported. All of these difficulties and challenges motivated scientists to develop new control methods. Initially, periodic control was applied, which was activated after a predetermined period of time, giving rise to time-triggered impulsive control (TTIC) [3–5]. To make the control system more effective against disturbances caused by impulsive control, the duration of the control period was shortened in some cases. However, in the case of system stability and the absence of disturbances, this control period becomes useless and represents a waste of system resources [2]. To further reduce the computational cost and waste of system resources, an alternative control strategy is proposed in which the impulsive control is activated only when certain predefined events occur [6,7]. In terms of this strategy, the event-triggered impulsive control (ETIC) is a control that is updated aperiodically only when certain predefined events occur [6,8,9].

In the last decade, various event-triggered control strategies have been proposed for many practical control systems, such as intelligent transportation systems [10], marine structures [11], active vehicle...
problem statement

Let us consider the following nonlinear system:

\[
\dot{x} = f(t, x(t), d(t)), \quad t \geq t_0, \quad x(t_0) = x_0,
\]

where \( x(t) \in \mathbb{R}^n \) is the system state, \( d(t) \in \mathbb{R}^n \) is an external disturbance, \( t_0 \) is the initial time, and the function \( f(t, x) \) is piecewise continuous in \( t \), locally Lipschitz in \( x \) with constant \( L \), and satisfies \( f(t, 0, 0) = 0 \) for all \( t \geq t_0 \).

For systems (1) and (2), it is assumed that the state \( x(t) \) can be influenced at certain times \( 0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots \), with \( t_k \) tending to \( +\infty \) as \( t \) toward \( +\infty \). This sequence will be chosen to achieve the exponential stability of systems (1) and (2) by an accurate choice of a certain controller \( u(t_k) \). The controller \( u(t_k) \) is chosen among \( q \) available values \( \{g_1(x), \ldots, g_q(x)\} \), where for all \( 1 \leq j \leq q \), \( g_j \) is a function from \( \mathbb{R}^n \) into \( \mathbb{R}^n \). More precisely, we are interested in constructing a sequence \( (t_k, u(t_k)) \) such that the solution of the following system is exponentially convergent:

\[
\dot{x}(t) = f(t, x(t), d(t)), \quad t \geq t_0, \quad t \neq t_k
\]

\[
x(t_{k}) = u_k(x(t_{k})), \quad t = t_k, \quad k = 1, 2, \ldots
\]

\[
x(t_{0}) = x_0.
\]

The following is a precise definition of the notion of rapid exponential stabilization of a solution.

**Definition 1.** The solution of systems (1) and (2) is said to be rapidly exponentially stabilizable if, for any decay rates \( \gamma > 0 \), there exists a sequence of functions \( u_k(\cdot) \), \( k = 1, 2, \ldots \), and a constant \( c > 0 \) such that, for every solution of the closed-loop systems (3)–(5) with \( x(t_0) \in \mathbb{R}^n \), one has

\[
\|x(t)\| \leq c\|x(t_0)\|e^{-\gamma(t-t_0)}, \quad \forall t \geq t_0.
\]
3 Main result

Throughout this section, we will set forth our main findings on how to design a control strategy that rapidly stabilizes our systems (1) and (2) in the sense of the aforementioned definition.

3.1 Event-triggered control design

Our strategy for constructing our impulsive control is dependent on two conditions: The first condition allows us to choose the intervention instants $t_k$ from a set of available choices, whereas the second condition imposes some constraints on the values $u(t_k^i)$ that can be assigned to the control $u$ to the right of the instants $t_k$ during the intervention. More specifically, we consider the following two assumptions.

**Assumption $\mathcal{A}_1$:** The control instants satisfy:
\[
\forall k \in \mathbb{N}, \quad t_{k+1} - t_k \in \{\tau_1, \ldots, \tau_p\},
\]
where $p$ is a positive integer and $\tau_i$ are positive constants for all $i \in \{1, \ldots, p\}$.

**Assumption $\mathcal{A}_2$:** There exists a positive real number $\mu < 1$ such that
\[
\min_{1 \leq j \leq q} \|g(x(t))\| \leq \mu \|x(t)\|.
\]

Let $\beta < 1$ be a positive real number, which we will fix later. Now, before adopting a recurrence construction for our impulsive control $(t_k, x(t_k^i))$, we consider the following notations, $\tau_{\text{max}} = \max_{1 \leq i \leq p} \tau_i$, $\tau = \min_{1 \leq i \leq p} \frac{\tau_i}{\|x(t + \tau_i)\|} \geq \beta \|x(t^+)^j\|$, and $j^*(t) = \arg \min_{1 \leq j \leq q} \|g(x(t))\|$.

Now, we construct our sequence $(t_k, x(t_k^i))$ in the following way: For $t = 0$, we take $t_0^i = 0$ and $u(t_0^i) = x(t_0)$. Suppose we have constructed the pair $(t_k, u(t_k^i))$, then the choice of the pair $(t_{k+1}, u(t_{k+1}^i))$ will be made according to the occurrence of the following events:

**$E_1$:**
\[
\begin{align*}
\text{If} & \quad \max_{1 \leq i \leq p} \|x(t_k + \tau_i)\| < \beta \|x(t_k^i)\|, \quad \text{we take:} \\
& \quad t_{k+1} = t_k + \tau_{\text{max}}, \\
& \quad x(t_{k+1}^i) = x(t_k^i)
\end{align*}
\]

**$E_2$:**
\[
\begin{align*}
\text{If} & \quad \exists \tau_i \in \{\tau_1, \ldots, \tau_p\}, \quad \|x(t_k + \tau_i)\| \geq \beta \|x(t_k^i)\|, \quad \text{we take:} \\
& \quad t_{k+1} = t_k + \tau_i, \\
& \quad x(t_{k+1}^i) = g_j^*(t_k,x(t_k))
\end{align*}
\]

**Remark 1.** Note that according to the definitions of events $E_1$ and $E_2$, control is triggered only when event $E_2$ occurs. If $N_k$ is the number of occurrences of this event during the time interval $[0, t_k]$, it is clear that $N_k \leq k$.

**Remark 2.** Although a similar ETIC scheme in [25] focuses on the exponential stabilization of infinite-dimensional systems, the effectiveness of the presented method in the face of disturbances such as the loss of some information causing control triggering and delay in the transmission of the control signal has not been studied. Therefore, this study complements the aspects discussed in [25] by considering such perturbations that may affect the results of this strategy.

**Remark 3.** A related ETIC strategy is given in [23] for the ISS study of nonlinear systems, with the difference that the convergence conditions are formulated in terms of the number of $E_1$ and $E_2$ events that can occur (Theorems 1 and 2 in [23]), which cannot be checked in practice before running the procedure. In addition,
there is another difference between the results obtained in this study and those obtained in [23], which concerns the nature of the convergence in both studies. While this study demonstrated rapid exponential stabilization, the previous study only focused on ISS stabilization.

**Remark 4.** In light of our ETIC $E_1 - E_2$ strategy, systems (3)–(5) are controlled by the event-triggered control $u$ defined as follows:

$$
u(x(t_k)) = \begin{cases} 
   x(t_k) & \text{if } t_k \text{ results from } E_1, \\
   g_f^*(x(t_k)) & \text{if } t_k \text{ results from } E_2.
\end{cases}$$

The next execution time $t_{k+1}$ of the control $u$ is determined by the event-triggering mechanism $E_1 - E_2$, which continuously checks whether the condition

$$\max_{1 \leq i \leq p} \|x(t_k + \tau_i)\| < \beta \|x(t_k^+\|$$

is satisfied or not. This condition contains information about the state variable $x(t_k)$ at the previous execution time $t_k$, and $t_{k+1}$ can be written as

$$t_{k+1} = \inf \left\{ t_k + \inf_{i \leq p} \{ \tau_i, \tau_i \neq \tau_{\max}, \|x(t_k + \tau_i)\| \geq \|x(t_k^+)\| \}, t_k + \tau_{\max} \right\}.$$  

Note here the difference from the classical ETIC, which can be formally written in our case as follows (see [6]):

$$t_{k+1} = \inf \{ t > t_k : \|x(t)\| \geq \|x(t_k^+)\| \}.$$  

The difference between strategies (11) and (12) is that in (12), condition (10) must be satisfied every time $t > t_k$, whereas in our proposed strategy (11), condition (10) must be satisfied only at points $t_k + \tau_i, 1 \leq i \leq p$. This allows us to consider our ETIC strategy as a hybrid of the classical ETIC strategy and the TTIC strategy, but it complicates the checking process more difficult because the solution can take large values between the two time points $t_k$ and $t_{k+1}$, contributing to a slow convergence of the solution to zero.

### 3.2 Rapidly exponential stabilization under ETIC

At this point, our first main result about the exponential stabilization of systems (1) and (2) can be established.

**Theorem 1.** Assume that assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$ are satisfied, and consider the designed events defined earlier. Then, systems (3)–(5) are exponentially stable.

**Proof.** We will proceed as in the proof of Theorem 1 in [25], noting first that, since we have for all $k \geq 0$, $t_{k+1} - t_k \geq \min_{1 \leq i \leq p} \tau_i$, it is obvious that Zeno behavior is avoided for systems (3)–(5) (see [4]).

Let $\gamma$ be any positive constant, and we seek, under assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$, a positive constant $c$ such that the solutions of systems (3)–(5) verify the inequality in (6).

Let $x(t)$ be the solution systems (3)–(5), and as mentioned earlier, once $(t_k, u(t_k^+))$ is constructed, the construction of the element $(t_{k+1}, u(t_{k+1}^+))$ is done according to the occurrence of two events $E_1$ and $E_2$, resulting in the following two cases:

1. If $t_{k+1}$ results from the occurrence of the event $E_1$, then from the definition of $t_{k+1}$, we have

$$\|x(t_{k+1}^+)\| = \|x(t_{k+1}^+)\| \leq \beta \|x(t_k^+)\|.$$  

$$\|x(t_{k+1}^+)\| = \|x(t_{k+1}^+)\| \leq \beta \|x(t_k^+)\|. \quad (13)$$
(ii) If $t_{k+1}$ results from the occurrence of event $E_2$, then from the definition of $x(t_{k+1})$, and Grönwall’s inequality applied to the solution $x(t)$ of systems (3) and (4) on the interval $(t_k, t_{k+1}]$, we have

$$
\|x(t_{k+1}^-)\| = \|g_f(t_{k+1}, x(t_{k+1}))\|,
\leq \mu \|x(t_k)\| \leq \mu \left(\|x(t_k^-)\| + \int_{t_k}^{t_{k+1}} \|f(s, x(s), w(s))\| \, ds\right)
\leq \mu \|x(t_k^-)\| e^{L(t_{k+1}^- - t_k)}
\leq \mu e^{L_{\max}|x(t_k^-)|}.
$$

Taking into account (13) and (14), we obtain

$$
\|x(t_{k+1}^-)\| \leq (\mu e^{L_{\max}})^N \beta^{k-N} \|x_0\|,
\leq e^{N(\ln(\mu) + L_{\max} + (k-N) \ln(\beta))} \|x_0\|,
\leq e^{N(\ln(\mu) + L_{\max} - \ln(\beta)) + \ln(\beta)k} \|x_0\|.
$$

By choosing $\beta = e^{-\gamma_{\max}}$ and $\mu \in (0, e^{-(L+\tau_{\max})})$, it is clear that $\beta, \mu \in (0, 1)$ and $\ln(\mu) + L_{\max} - \ln(\beta) \leq 0$. Then, it yields that

$$
N_k(\ln(\mu) + L_{\max} - \ln(\beta)) + \ln(\beta)k \leq -\gamma_{\max} k,
$$

which gives

$$
\|x(t_{k+1}^-)\| \leq e^{-\gamma_{\max} k} \|x_0\|.
$$

In addition, by applying Grönwall’s inequality for $t \in (t_k, t_{k+1}]$, we obtain

$$
\|x(t)\| \leq e^{L_{\max}} \|x(t_k^-)\|,
$$

and assumption $\mathcal{A}_1$ states

$$
t - t_0 \leq (k + 1)\tau_{\max}.
$$

Therefore, combining (17), (18), and (19), it follows that

$$
\|x(t)\| \leq ce^{-\gamma(t-t_0)} \|x_0\|,
$$

where $c = e^{(L+\tau)\gamma_{\max}}$. Hence, systems (3)–(5) are exponentially stable with decay rate $\gamma > 0$ and the proof of the theorem is finished.

### 3.3 Rapid exponential stabilization under data dropout in the ETIC

Section 3.2 has shown that exponential stabilization occurs when there are no data dropouts in the control signals. However, the stability of the controlled system, and hence the ETIC strategy, can be seriously affected if impulsive signals from the communication network fail. We say that the failure of the impulsive control signal for systems (1) and (2) occurs at time $t_k$ if systems (3)–(5) do not receive some $u(t_j)$ at $t_j$ for $j \leq k$. Thus, if the data dropout occurs at time $t_k$, then there is no jump for $x(t_k)$, and $x(t_k^-) = x(t_k)$. Note that there is no data dropout for the ETIC event $E_1$ because there is no ETIC entry when $E_1$ occurs. Let $d_j(t_0, t_k)$ be the number of dropouts of event $E_2$ during the period $(t_0, t_k)$, and let $s_j(t_0, t_k)$ represent the number of events $E_2$ that were received successfully by the system during the same period $(t_0, t_k)$. Finally, let $m_2$ denote the maximum allowable dropout rate of the event $E_2$. Obviously, we have

$$
d_j(t_0, t_k) \leq m_2 N_2.
$$
It follows from
\[ N_2 = d_2(t_0, t_k) + s_2(t_0, t_k) \]
that
\[ s_2(t_0, t_k) \geq (1 - m_2)N_2. \] (22)

Now, we can set up the following criteria for rapid exponential stabilization of systems (3)–(5) with existing data dropouts in the ETIC.

**Theorem 2.** Under ETIC strategy \((E_1)\) and \((E_2)\) with data dropouts and assumptions \(A_1\) and \(A_2\), systems (3)–(5) are exponentially stable.

**Proof.** From the proof of Theorem 1, it is easy to see that if there exist data dropouts of ETIC, then (15) should be changed to:
\[
\|x(t_{k+1}^-)\| \leq (\mu e^{L_{\text{max}}})^N N_2 \|x_0\| \\
= (\mu e^{L_{\text{max}}})^N \|x_0\| \\
= \mu e^{L_{\text{max}}^N} N_2 \|x_0\|. 
\]

It follows from (22) that
\[
\|x(t_{k+1}^-)\| \leq \mu(1 - m_2)N_2 e^{N_2 L_{\text{max}}^N} \|x_0\| \\
\leq e^{N_2((1 - m_2) \ln(\mu) + L_{\text{max}} \ln(\beta))} \|x_0\| \\
\leq e^{N_2((1 - m_2) \ln(\mu) + L_{\text{max}} \ln(\beta) + \ln(\beta))k} \|x_0\|.
\]

Choosing \(\beta = e^{-\gamma_{\text{max}}}\) and \(\mu \in \left(0, e^{\frac{L_{\text{max}}}{1 - m_2}}\right)\). Clearly \(\beta\) and \(\mu\) are strictly positive reals and strictly smaller than 1. In addition, we have \((1 - m_2) \ln(\mu) + L_{\text{max}} - \ln(\beta) \leq 0\), which gives
\[
N_2(1 - m_2) \ln(\mu) + L_{\text{max}} - \ln(\beta) \leq (1 - m_2) \ln(\mu) + L_{\text{max}} - \ln(\beta) - \gamma_{\text{max}}k. \] (23)

Using this, we obtain
\[
\|x(t_{k+1}^-)\| \leq e^{-\gamma_{\text{max}} k}\|x_0\|,
\]
and by using (19), we obtain
\[
\|x(t_{k+1}^-)\| \leq e^{\gamma_{\text{max}}^N} e^\gamma\|x_0\|. \] (24)

Therefore, systems (3)–(5) are rapidly exponentially stable with convergence rate \(\gamma\). This completes the proof. \(\square\)

**Remark 5.** In comparison to the result proved in [23], we can observe that condition (11) considered in [23] to prove ISS stabilization with data dropouts in ETIC is expressed in terms of number of incidents \(E_1\) and \(E_2\) that lie in the future and thus have not yet occurred at the beginning of the process. However, in our research, convergence was not measured in terms of the number of these incidents. Moreover, our decay rate was randomly chosen to achieve exponential convergence under ETIC \((E_1) - (E_2)\), which demonstrates our strategy’s resilience in the face of data dropout during the transmission of the trigger signals.

### 3.4 Rapid exponential stabilization under time-delay in the ETIC

The results presented in Sections 3.2 and 3.3 were obtained without taking into account the communication delays that may exist in the communication network and affect the robustness of the ETIC scheme \((E_1) - (E_2)\). In this part, we will take into consideration the possibility of delays in the triggering of control.
Let $h_1, \ldots, h_p$ denote the delays at the time of the measurements of the values of $x$ in $t + \tau_1, \ldots, t + \tau_p$. Without loss of generality, we can assume that they verify $0 \leq \max_{i \leq p} h_i \leq \tau_{\max}$. In the delayed ETIC, at $t_k$, the systems (3)–(5) will receive a previous ETIC signal $u_i(x(t_k - h_i))$, which will be used as the value of $x(t'_k)$ rather than $u_i(x(t_k))$. Thus, systems (3)–(5) take the following form:

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), d(t)), \quad t \geq t_0, t \neq t_k \\
x(t'_k) &= u_i(x(t_k - h_i)), \quad k = 1, 2, \ldots \\
x(t'_0) &= x_0.
\end{align*}
\] (25) (26) (27)

**Assumption $\mathcal{A}_3$:** Assume that there is no data dropout with regard to event $E_2$.

The result concerning the stability of systems (25)–(27) in the presence of delays in the transmission of control signals is given in the following theorem:

**Theorem 3.** Suppose that assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ are satisfied. Furthermore, it is assumed that there exists some $m \in \mathbb{N}$ such that

\[
t_{k-m} \leq t_k + \tau_{(k)} - h_{(k)} \leq t_{k+1-m}, \quad \forall k \in \mathbb{N},
\] (28)

and $\max_{i \leq p} h_i \leq \tau_{\max}$. Then, under the ETIC $(E_1)$–$(E_2)$, systems (25)–(27) are rapidly exponentially stable.

**Proof.** The proof of the avoidance of Zenon phenomena is the same as in Theorem 1. To prove the rapid exponential stability of systems (25)–(27), we distinguish two cases:

**Case 1:** As only the delay occurring at the time of event $E_2$ can influence the control of the system, we can paraphrase this to state that if $t_{k+1}$ results from the event $E_2$, then $h_k = 0$ and

\[
\|x(t'_k)\| = \|x(t_k + \tau_{\max})\| \leq \beta\|x(t'_k)\|.
\] (29)

**Case 2:** If $t_{k+1}$ results from the event $E_2$, it follows that $t_{k+1} = t_k + \tau_{(k)}$, where $\tau_{(k)} = \min\{\tau_1, \ldots, \tau_p\}$ such that

\[
\|x(t_k + \tau_{(k)} - h_{(k)})\| \geq \beta\|x(t'_k)\|.
\]

Let $\mathbb{N}_m = \{-m, \ldots, -1, 0\}$ and let $g_{-m} = \cdots = g_{-1} = g_0 = \text{Id}$, where $\text{Id}$ denotes the identity application of $\mathbb{R}^{2n}$. Now, it yields from (28) that

\[
\|x(t'_k)\| = \|g_{f(t_{k-1})}(x(t_k + \tau_{(k)} - h_{(k)}))\| \\
\leq \mu\|x(t_k + \tau_{(k)} - h_{(k)})\| \\
\leq \mu e^{L(t_0 + \tau_{(k)} - h_{(k)})-t_{k-1}}\|x(t'_{k-m})\|, \\
\leq \mu e^{L(t_{k-m} + t_{k-m})-t_{k-1}}\|x(t'_{k-m})\|, \\
\leq \mu e^{L_{\max}}\|x(t'_{k-m})\|.
\] (30)

Let $z(k) = \|x(t'_k)\|$, $\tilde{z}(k) = \sup_{n \in \mathbb{N}_m}\|x(t'_{k+1})\|$ and $\lambda = \max\{\beta, \mu e^{L_{\max}}\}$. It follows from (29) and (30) that

\[
z(k+1) \leq \lambda z(k), \quad \forall k \in \mathbb{N}.
\] (31)

As $0 < \lambda < 1$, it follows from Theorem 4.2 in [26] that the discrete time-delay system (31) is exponentially stable. Then, according to the comparison principle for discrete systems (e.g., Proposition 1 in [27]), there exist two positive numbers $\alpha$ and $M$ such that for all $k \in \mathbb{N}$,

\[
\|x(t'_k)\| \leq Me^{\alpha k}\|x_0\|.
\] (32)

Combining (18), (19), and (32), we obtain

\[
\|x(t)\| \leq M' e^{-\alpha \min\{t-t_{0}\}}\|x_0\|,
\] (33)

where $M' = Me^{\alpha_{\max}}$. Hence, systems (3)–(5) are exponentially stable with decay rate $\frac{\alpha}{\alpha_{\max}}$ and the proof is complete. $\square$
Remark 6.
(i) We should note that the assumption \( \max_{i \leq k} p_i \leq \tau_{\text{max}} \) does not confer any constraints and can be replaced by the standard assumption, which simply stipulates that the delays are finite from above. This is because if the delay exceeds \( \tau_{\text{max}} \), then our strategy will act as if event \( E_i \) had occurred.
(ii) Similarly, hypothesis (28) is not so restrictive, since we are concerned with studying the asymptotic behavior of the solutions of systems (25)–(27), we know that as soon as \( t_k > t_0 + \tau_{\text{max}} \), there exists \( j, \ 0 \leq j < k \) such that
\[
t_j < t_k + h_{\ell(k)} - h_{\ell(k)}(t) \leq t_{j+1}.
\]

4 Numerical application

To illustrate the effectiveness of the proposed event-triggered control strategy, we will apply this strategy to study the stability of trajectory tracking for a wheeled robot. The coordinates \((x, y)\) of the vehicle center satisfy the following system [28–31]:
\[
\begin{align*}
\dot{x}(t) &= v(t) \cos(\theta(t)), \\
\dot{y}(t) &= v(t) \sin(\theta(t)), \\
\dot{\theta}(t) &= \omega(t),
\end{align*}
\]
where \( \theta \) is the angle between the heading direction and the \( x \)-axis (Figure 1).

The problem of tracking a reference robot was originally introduced in [32] as follows:
\[
\begin{align*}
\dot{x}_r(t) &= v_r(t) \cos(\theta_r(t)), \\
\dot{y}_r(t) &= v_r(t) \sin(\theta_r(t)), \\
\dot{\theta}_r(t) &= \omega_r(t).
\end{align*}
\]

The error coordinates will be defined similarly to that in [29–33]
\[
\begin{bmatrix}
\dot{x}_e(t) \\
\dot{y}_e(t) \\
\dot{\theta}_e(t)
\end{bmatrix} =
\begin{bmatrix}
\cos(\theta(t)) & \sin(\theta(t)) & 0 \\
- \sin(\theta(t)) & \cos(\theta(t)) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_e(t) - x(t) \\
y_e(t) - y(t) \\
\theta_e(t) - \theta(t)
\end{bmatrix}.
\]

Figure 1: Wheeled robot coordinates.
Taking into account the correlations (34)–(36) and (37)–(39), it follows that the error dynamics are governed by the following equations:

\[ \dot{x}_e(t) = \omega(t)x(t) - \nu(t) + \nu(t) \cos(\theta_e(t)), \quad (40) \]
\[ \dot{y}_e(t) = -\omega(t)x(t) + \nu(t) \sin(\theta_e(t)), \quad (41) \]
\[ \dot{\theta}_e(t) = \omega_e(t) - \omega(t). \quad (42) \]

Systems (40)–(42) have the same structure as systems (1) and (2) with \( x(t) = (x_e(t), y_e(t), \theta_e(t))^T \), and \( d(t) = (-\nu(t), 0, \omega_e(t) - \omega(t))^T \). The state trajectories of systems (40)–(42) without control and with initial condition \( \left( -\frac{\pi}{2}, \frac{\pi}{2}, \pi \right)^T \) are depicted in Figure 2, which shows their instability.

To stabilize systems (40)–(42), we will apply Theorem 1, and for the simulation, we consider the following parameters:

\[ u(t) \in \{B_1(x(t_{k+1})), B_2(x(t_{k+1})), B_3(x(t_{k+1}))\} \]

\[ B_1 = \begin{bmatrix} 0.2 & 0.5 & 0.5 \\ 0.2 & 0.1 & 0.1 \\ 0.2 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.5 & 0.6 & 0 \\ 0 & 0.2 & 0.6 \end{bmatrix}, \quad \text{and} \quad B_3 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}. \]

The impulsive control system takes the following form:

\[ \dot{x}_e(t) = \omega(t)x_e(t) - \nu(t) + \nu(t) \cos(\theta_e(t)), \quad t \in [t_k, t_{k+1}] \quad (43) \]
\[ \dot{y}_e(t) = -\omega(t)x_e(t) + \nu(t) \sin(\theta_e(t)), \quad t \in [t_k, t_{k+1}] \quad (44) \]
\[ \dot{\theta}_e(t) = \omega_e(t) - \omega(t), \quad t \in [t_k, t_{k+1}] \quad (45) \]
\[ x(t_k^+) = \begin{cases} x(t_k), & t_k \in E_1 \\ B_f(t_k)(x(t_k)), & t_k \in E_2. \end{cases} \quad (46) \]

The rest of the parameters considered in the numerical simulation are as follows: \( T = 1.2, \beta = 0.9, \tau \in \{0.1, 0.3, 0.6\}, \nu(t) = 1.25e^{-0.5t}, \nu(t) = e^{-0.5t}, \omega(t) = \sin(t), \) and \( \omega_e(t) = \frac{1}{2} \sin(t/5) + \sin(t) \). Figure 3 depicts the results of the numerical simulation showing the exponential convergence of the solutions of systems (40)–(42) under the impulsive control described in ETIC \((E_1)\)–\((E_2)\). Finally, in Figure 4, systems (40)–(42) are under the influence of a periodic control in time with period \( \Delta t = 0.01 \). Figures 3 and 4 clearly show the difference in the number of pulses derived by the TTIC and ETIC methods.

Figure 2: The solution of systems (40)–(42) (without control).
5 Conclusion

In this study, rapid exponential stabilization via event-triggered control has been investigated for nonlinear continuous dynamical systems. We have also studied the robustness of the ETIC for network systems where delays and data dropouts occur. In the presence of disturbances, the proposed ETIC is shown to be robust to dropouts and delays in data transmission. Nevertheless, larger delays may equate to a slower convergence speed in exponential stabilization. In addition, as an illustration of the theoretical results in this article, the stability of the trajectory tracking of a wheeled robot has been investigated. However, the complicated case of data dropout and delay in the transmission of ETIC simultaneously is still an open issue, and it will be on our agenda for future research. Other important future works of the ETIC method could involve the stabilization of some neutral fractional differential equations (see [34]) or some nonlinear fractional differential problems with boundary conditions (see [35]).
Acknowledgements: The author would like to thank the anonymous referees and the handling editor for their careful reading and relevant remarks/suggestions that helped them to improve the article. The author would like to thank the Deanship of Scientific Research, Qassim University for funding the publication of this project.

Conflict of interest: The author declares no conflict of interest.

Data availability statement: No data were used to support this study.

Ethical approval: The conducted research is not related to either human or animal use.

References

[1] C. Prieur and E. Trélat, Robust optimal stabilization of the Brockett integrator via a hybrid feedback, Math. Control Signals Systems 17 (2005), no. 3, 201–216.
[2] M. Donkers and M. Heemels, Output-based event-triggered control with guaranteed $L^\infty$ gain and improved and decentralized event-triggering, IEEE Trans. Automat Control 57 (2012), no. 6, 1362–1376.
[3] W. Elmenreich, Time-triggered fieldbus networks state of the art and future applications, 2008 11th IEEE International Symposium on Object and Component-Oriented Real-Time Distributed Computing (ISORC), 2008, pp. 436–442.
[4] M. Heemels, K. H. Johansson, and P. Tabuada, An introduction to event-triggered and self-triggered control, 2012 IEEE 51st IEEE Conference on Decision and Control (CDC), 2012, pp. 3270–3285.
[5] C. Albea and A. Seuret, Time-triggered and event-triggered control of switched affine systems via a hybrid dynamical approach, Nonlinear Anal. Hybrid Syst. 41 (2021), 101039.
[6] P. Tabuada, Event-triggered real-time scheduling of stabilizing control tasks, IEEE Trans. Autom. Control 52 (2007), no. 9, 1680–1685.
[7] D. Yue, E. Tian, and Q. L. Han, A delay system method for designing event-triggered controllers of networked control systems, IEEE Trans. Autom. Control 58 (2013), no. 2, 475–481.
[8] A. Girard, Dynamic triggering mechanisms for event-triggered controls, IEEE Trans. Autom Control 60 (2015), 1992–1997.
[9] M. Heemels, M. Donkers, and A. R. Teel, Periodic event-triggered control for linear systems, IEEE Trans. Autom. Control 54 (2013), no. 4, 847–861.
[10] X. M. Zhang and Q. L. Han, Network-based $h_{\infty}$ filtering using a logic jumping-like trigger, Automatica J. IFAC 49 (2013), 1428–1435.
[11] B. L. Zhang, Q. L. Han, and X. M. Zhang, Event-triggered $h_{\infty}$ reliable control for offshore structures in network environments, J. Sound Vib. 368 (2016), 1–21.
[12] Y. L. Wang and Q. L. Han, Network-based modelling and dynamic output feedback control for unmanned marine vehicles in network environments, Automatica J. IFAC 91 (2018), 43–53.
[13] H. Zhang, X. Zheng, H. I. Li, Z. Wang, and H. Yan, Active suspension system control with decentralized event-triggered scheme, IEEE Trans. Ind. Electron. 67 (2020), no. 12, 10798–10808.
[14] C. Nowzari, E. Garcia, and J. Cortes, Event-triggered communication and control of networked systems for multi-agent consensus, Automatica J. IFAC 105 (2019), 1–27.
[15] J. Qin, Q. Ma, Y. Shi, and L. Wang, Recent advances in consensus of multi-agent systems: A brief survey, IEEE Trans. Ind. Electron. 64 (2017), no. 6, 4972–4983.
[16] V. Dolk, D. Borgers, and W. Heemels, Output-based and decentralized dynamic event-triggered control with guaranteed $l_2$ gain performance and zero-freeness, IEEE Trans. Autom. Control 62 (2017), no. 1, 34–94.
[17] Y. Guan, Q. L. Han, and X. Ge, On asynchronous event-triggered control of decentralized networked systems, Inform Sci. 425 (2018), 127–139.
[18] M. S. Mahmoud and Y. Xia, Networked Control Systems, Elsevier, New York, 2019.
[19] Y. Q. Xia, Y. L. Gao, L. P. Yan, and M. Y. Fu, Recent progress in networked control systems - A survey, Int. J. Autom. Comput. 12 (2015), 343–367.
[20] W. Zhu, D. Wang, and L. Liu, Event-based impulsive control of continuous-time dynamic systems and its application to synchronization of memristive neural networks, IEEE Trans. Neural Netw. Learn. Syst. 29 (2018), no. 8, 3599–3609.
[21] X. Li, D. Peng, and J. Cao, Lyapunov stability for impulsive systems via event-triggered impulsive control, IEEE Trans. Autom. Control 65 (2020), no. 11, 4908–4913.
[22] B. Liu, D. J. Hill, and Z. Sun, Stabilisation to input-to-state stability for continuous-time dynamical systems via event-triggered impulsive control with three levels of events, IET Control Theory Appl. 12 (2018), no. 9, 1167–1179.
[23] M. Cao, Z. Ai, and L. Peng, *Input-to-state stabilization of nonlinear systems via event-triggered impulsive control*, IEEE Access **48** (2019), no. 5, 826–836.

[24] M. Dlala and S. Obaid Alrashidi, *Rapid exponential stabilization of Lotka-McKendrick’s equation via event-triggered impulsive control*, Math. Biosci. Eng. **18** (2021), 9121.

[25] M. Dlala and A. S. Almutairi, *Rapid exponential stabilization of nonlinear wave equation derived from brain activity via event-triggered impulsive control*, Mathematics **9** (2021), no. 5.

[26] B. Liu and H. J. Marquez, *Razumikhin-type stability theorems for discrete delay systems*, Automatica J. IFAC **43** (2007), no. 7, 1219–1225.

[27] G. Bitsoris and E. Gravalou, *Comparison principle, positive invariance and constrained regulation of nonlinear systems*, Automatica J. IFAC **31** (1995), no. 2, 217–222.

[28] A. Loria and E. Panteley, *Cascaded nonlinear time-varying systems: Analysis and design*, in: Advanced Topics in Control Systems Theory, Vol. 311 of Lecture Notes in Control and Information Sciences, Springer, London, 2005, Chap. 2, pp. 23–64.

[29] E. Panteley, E. Lefeber, A. Loria, and H. Nijmeijer, *Exponential tracking control of a mobile car using a cascaded approach*, vol. 31, 1998 IFAC Workshop on Motion Control (MC’98), Grenoble, France, 1998, 21–23 September, pp. 201–206.

[30] N. O. Sedova, *The global asymptotic stability and stabilization in nonlinear cascade systems with delay*, Russian Math. (Iz. VUZ) **52** (2008), 60–69.

[31] M. Dlala, *Exponential stability of impulsive cascaded systems and its application in robot control*, IEEE Access **10** (2022), 6319–6327.

[32] Y. Kanayama, Y. Kimura, F. Miyazaki, and T. Noguchi, *A stable tracking control method for an autonomous mobile robot*, in: Proceedings of IEEE International Conference on Robotics and Automation vol. 1, 1990, pp. 384–389.

[33] J. Jakubiak, E. Lefeber, K. Tchon, and H. Nijmeijer, *Two observer-based tracking algorithms for a unicycle mobile robot*, Int. J. Appl. Math. Comput. Sci. **12** (2002), no. 4, 513–522.

[34] N. Abdellouahab, B. Tellab, and K. Zennir, *Existence and stability results for the solution of neutral fractional integro-differential equation with nonlocal conditions*, Tamkang J. Math. **53** (2021), no. 5, 3509–3520.

[35] A. Naimi, B. Tellab, Y. Altayeb, and A. Moumen, *Generalized Ulam-Hyers-Rassias stability results of solution for nonlinear fractional differential problem with boundary conditions*, Math. Probl. Eng. **2021** (2021), 7150739.