CYLINDRIC $P$-TABLEAUX FOR 3+1-FREE POSETS

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Abstract. For a $(3+1)$-free poset $P$, we define a hybrid of $P$-tableaux and cylindric $P$-tableaux called cylindric $P$-tableaux. We introduce $P$-analogs of cylindric Schur functions, defined by a determinantal formula, and prove that they are the weight generating functions of cylindric $P$-tableaux. We deduce that certain sums of the $e$-expansion coefficients of the chromatic symmetric function $X_{\text{inc}(P)}$ are positive. This improves on Gasharov’s theorem on the Schur positivity of $X_{\text{inc}(P)}$ and gives further evidence for the Stanley-Stembridge conjecture.

1. Introduction

The chromatic symmetric function $X_G(x)$ of a graph $G$ is the sum $\sum_\kappa x_\kappa$ over all proper colorings $\kappa : V(G) \to \mathbb{Z}_{>0}$ where $x_\kappa = \prod_{v \in V(G)} x_{\kappa(v)}$. It is of particular interest when $G$ is the incomparability graph $\text{inc}(P)$ of a $(3+1)$-free poset $P$. In this case, a theorem of Haiman showed that $X_{\text{inc}(P)}(x)$ is Schur positive [11], and a combinatorial formula for the Schur expansion was given by Gasharov [8]. Stanley and Stembridge [18] conjectured that $X_{\text{inc}(P)}(x)$ is a positive sum of elementary symmetric functions; the coefficient of $e_\lambda$ is known to be positive when $\lambda$ is a rectangular [21], two-column [22,13], or hook shape [22,13]. Additionally, $X_G$ is known to be $e$-positive for several classes of graphs [9,6,12,3,5,2].

In the recent paper [13] and in a companion paper to this one [1], the authors adapted the Fomin-Greene theory of noncommutative Schur functions [7] to study chromatic symmetric functions. This is the starting point for our work here, so we review the basic setup. For a finite poset $P$, let $\mathbb{Z}[u]$ be the polynomial ring in the commuting variables $u = \{u_p : p \in P\}$. We define the $P$-elementary function $e_k^P(u) \in \mathbb{Z}[u]$ by

$$ e_k^P(u) = \sum_{i_1 < i_2 < \ldots < i_k} u_{i_1}u_{i_2}\ldots u_{i_k}. $$

([13] and [1] are largely concerned with the setting where the $u_p$ are noncommuting variables but we only need the commuting setup here.)

Let $\Lambda(x)$ denote the ring of symmetric functions in variables $x = x_1, x_2, \ldots$. Now, for $f(x) \in \Lambda(x)$, we define the $P$-analog $f^P(u)$ of $f(x)$ to be the image of $f(x)$ under the homomorphism

$$ \psi : \Lambda(x) \to \mathbb{Z}[u], \quad e_k(x) \mapsto e_k^P(u). $$

The following result is a consequence of [1, Theorem 2.6 and Eq. (2.21)].
Theorem 1.1. The e-expansion of $X_{\text{inc}(P)}(x)$ can be expressed in terms of the $P$-analogues $m_{\lambda}^P(u)$ of the monomial symmetric functions $m_{\lambda}(x)$ as

$$X_{\text{inc}(P)}(x) = \sum_{\lambda} \langle u_P \rangle m_{\lambda}^P(u) e_\lambda(x),$$

where $\langle u_P \rangle m_{\lambda}^P(u)$ is the coefficient of $u_P = \prod_{p \in P} u_p$ in the $u$-monomial expansion of $m_{\lambda}^P(u)$.

Hence the $u$-monomial positivity of $m_{\lambda}^P(u)$ implies the Stanley-Stembridge conjecture (in fact, they’re equivalent by [1] Remark 3.15).

On the other hand, Gasharov’s result [8] can be rephrased as giving a $u$-monomial positive formula for $s_{\lambda}^P(u)$ in terms of $P$-tableaux (see Theorem 2.1). Hence a natural intermediate goal to go from the known $u$-monomial positivity of $s_{\lambda}^P(u)$ to the conjectured $u$-monomial positivity of $m_{\lambda}^P(u)$ is to establish $u$-monomial positivity for various $P$-analogues $\psi(f(x))$ for symmetric $f(x)$ which “lie in between” monomial symmetric functions and Schur functions.

One such class are the cylindric Schur functions $s_{\lambda/\mu/d}(x)$, where “lie in between” has the precise meaning that they are positive sums of some subset of monomials which sum to a Schur function. The cylindric Schur functions are based on cylindric partitions of Gessel and Krattenthaler [10], and were used by Postnikov to study the quantum cohomology of the Grassmannian [17]. Lam showed that cylindric Schur functions are special cases of skew affine Schur functions [14]. McNamara [16] conjectured and Lee [15] proved that cylindric skew Schur functions expand positively in terms of cylindric Schur functions and that the coefficients of this expansion are the same as 3-point Gromov-Witten invariants.

We define $P$-analogues $s_{\lambda/\mu/d}^P(u) := \psi(s_{\lambda/\mu/d}(x))$ of the cylindric Schur functions and prove that these also have a formula in terms of cylindric $P$-tableaux, a hybrid of cylindric tableaux and $P$-tableaux (Theorem 2.6). In particular, this establishes that the $s_{\lambda/\mu/d}^P(u)$ are $u$-monomial positive and that certain sums of the coefficients $c_{\lambda}^P$ in $X = \sum_{\lambda} c_{\lambda}^P e_\lambda(x)$ are positive (Corollary 2.7); these sums are smaller than those obtained from Gasharov’s result in the same fashion. As a special case, we recover a positive formula of Stembridge [20] and Clearman-Hyatt-Shelton-Skandera [1] for the coefficient $c_{\lambda}^P$ when $\lambda$ is a rectangle.

2. $P$-cylindric Schur functions are $u$-monomial positive

2.1. Preliminaries. For an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l) \vdash n$, the (English style) Young diagram of shape $\lambda$ is the set $\{(i, j) : 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}$, drawn as boxes labeled with matrix-style coordinates where the box $(i, j)$ is in row $i$ and column $j$. We often identify partitions with their corresponding Young diagrams so that, for partitions $\lambda, \mu$, we write $\mu \subset \lambda$ to mean that the Young diagram of $\mu$ is contained in the Young diagram of $\lambda$. For $\mu \subset \lambda$, the skew shape $\lambda/\mu$ is the difference of Young diagrams $\lambda - \mu$. We write $\lambda'$ for the transpose partition of $\lambda$ and $\ell(\lambda)$ for the number of nonzero parts of $\lambda$.

A semistandard Young tableau of shape $\lambda/\mu$ is a function $T : \lambda/\mu \to \mathbb{Z}_{\geq 0}$ so that $T(i, j) < T(i + 1, j)$ and $T(i, j) \leq T(i, j + 1)$, i.e. a Young tableau is an assignment of positive integers to the boxes of $\lambda/\mu$ so that the entries strictly increase down columns and weakly increase across rows. We write $\text{SSYT}(\lambda/\mu)$ for the set of semistandard Young tableaux of shape $\lambda/\mu$. 
2.2. Reformulation of Gasharov’s theorem. To set the stage for our main theorem on $P$-cylindric Schur functions, we discuss several precursors of this result.

It is well known from the theory of symmetric functions that the following two quantities are equal; either can be taken as the definition of the skew Schur function $s_{\lambda/\mu}(x)$.

\begin{equation}
\sum_{T \in SSYT(\lambda/\mu)} x^T = \det[e_{\lambda'_j-\mu'_i-j}(x)]_{i,j=1}^{(\lambda')},
\end{equation}

where $x^T = \prod_{b \in \lambda/\mu} x_{T(b)}$. Thus, for a poset $P$, the $P$-analog of $s_{\lambda/\mu}(x)$ is given by

\begin{equation}
s_{\lambda/\mu}^P(u) := \psi(s_{\lambda/\mu}(x)) = \det[e_{\lambda'_j-\mu'_i-j}(u)]_{i,j=1}^{(\lambda')},
\end{equation}

We say $T : \lambda/\mu \rightarrow P$ is a $P$-tableau of shape $\lambda/\mu$ if $T(i,j) <_P T(i+1,j)$ and $T(i,j) \not<_P T(i,j+1)$, i.e. an assignment of elements of $P$ to the boxes of $\lambda/\mu$ is a $P$-tableau if it is increasing in $P$ down columns and non-decreasing in $P$ across rows.

**Theorem 2.1.** For a $(3+1)$-free poset $P$,

\begin{equation}
s_{\lambda/\mu}^P(u) = \sum_{T \in SSYT_P(\lambda/\mu)} u^T,
\end{equation}

where $SSYT_P(\lambda/\mu)$ denotes the set of $P$-tableaux of shape $\lambda/\mu$ and $u^T = \prod_{b \in \lambda/\mu} u_{T(b)}$.

For straight shapes ($\mu = \emptyset$), this is a reformulation of Gasharov’s theorem \cite{Gasharov} (it also follows directly from [\cite{Lam} Theorem 3.9]); for skew shapes, it will follow from the more general Theorem 2.7 below.

2.3. $P$-cylindric Schur functions.

**Definition 2.2** (Cylindric $P$-tableaux). For a poset $P$, a $P$-tableau $T$ of shape $\lambda/\mu$, and a nonnegative integer $d$, define $T^d$ to be the diagram obtained by gluing a copy of the first column of $T$ to the last column of $T$ so that the copied column is shifted up by $d$ cells. i.e. $T^d$ has shape $\nu/\theta$ where $\nu' = (\lambda_1'+d, \lambda_2'+d, \ldots, \lambda_l'+d)$ and $\theta' = (\mu_1'+d, \mu_2'+d, \ldots, \mu_r'+d)$.

A cylindric $P$-tableau of shape $\lambda/\mu/d$ is a function $T : \lambda/\mu \rightarrow P$ such that $T^d$ is a $P$-tableau. Let

\begin{equation}
CT_P(\lambda/\mu/d) = \text{the set of cylindric $P$-tableau of shape $\lambda/\mu/d$.}
\end{equation}

Note that $T^d$ will never be a $P$-tableau when $d < \lambda_1' - \lambda_{\lambda_1}'$ or $d < \mu_1' - \mu_{\lambda_1}'$, and $CT_P(\lambda/\mu/d) = CT_P(\lambda/\mu/d')$ when $d, d' \geq \lambda_1'$. Therefore we take $\max(\lambda_1' - \lambda_{\lambda_1}', \mu_1' - \mu_{\lambda_1}') \leq d \leq \lambda_1'$, and in this case we say $\lambda/\mu/d$ is a cylindric shape.

**Example 2.3.** Let $P$ be the total order on $\mathbb{Z}_{>0}$ and $T$ be the following tableau

\begin{equation}
\begin{array}{cccc}
1 & 2 & 3 & \\
1 & 3 & 5 & 5 \\
2 & 2 & 4 & 6 & 6 \\
3 & 4 & 5 & 7 & \\
\end{array}
\end{equation}
Then

\[ T^3 = \begin{array}{ccc}
1 & 2 & 3 \\
1 & 3 & 5 \\
2 & 4 & 6 \\
3 & 4 & 5 \\
\end{array} \begin{array}{c}
3 \\
5 \\
6 \\
7 \\
\end{array} \quad \text{and} \quad T^2 = \begin{array}{ccc}
1 & 2 & 3 \\
1 & 3 & 5 \\
2 & 4 & 6 \\
3 & 4 & 5 \\
\end{array} \begin{array}{c}
2 \\
5 \\
6 \\
7 \\
\end{array} \]

When \( P \) is the total order on \( \mathbb{Z}_{>0} \), we call cylindric \( P \)-tableaux \textit{cylindric semistandard Young tableaux} and write \( \text{CSSYT}(\lambda/\mu/d) = CT_P(\lambda/\mu/d) \).

For a cylindric shape \( \lambda/\mu/d \) define the cylindric Schur function \( s_{\lambda/\mu/d}(x) \) by

\[ s_{\lambda/\mu/d}(x) = \sum_{T \in \text{CSSYT}(\lambda/\mu/d)} x^T. \]

**Remark 2.4.** Our definition of cylindric semistandard Young tableaux and cylindric Schur functions are related to other definitions in the literature as follows.

(i) Our definition of the shape of a cylindric tableau differs from the definition used in [10]. Our definition is slightly more general. For a cylindric shape \( \lambda/\mu/d \) in our notation with \( d > 0 \), we convert to a cylindric shape \( \nu/\eta/m \) in the notation of Gessel and Krattenthaler by taking

\[ \nu = (\lambda_1 + \lambda_1 + d + \lambda_1 + 2d + \cdots, \lambda_2 + \lambda_2 + d + \lambda_2 + 2d + \cdots, \cdots, \lambda_d + \lambda_d + d + \cdots), \]

\[ \eta = \mu, \quad \text{and} \quad m = \lambda_1. \]

(ii) For \( d > 0 \), our definition [10] of \( s_{\lambda/\mu/d}(x) \) is equivalent to the definition used by Postnikov [17] and McNamara [16], but our conventions for cylindric shapes are different. Following [16, §4], we convert a cylindric shape \( \lambda/\mu/d \) in our notation to a shape \( \nu/m/\theta \) in the notation of Postnikov and McNamara in the following way: let \( k = d \) and \( n = \lambda_1 + d \). Then \( \theta = \mu \), and \( \nu \) is obtained from \( \lambda \) by removing \( m \) \( n \)-ribbons from the border of \( \lambda \), where \( m \) is the smallest number so that \( \nu \) has at most \( k \) rows.

For example, for \( \lambda = (4, 4, 4, 4, 2, 1, 1), \mu = (2, 1), \) and \( d = 3, \) to convert to the notation of Postnikov and McNamara, we repeatedly remove ribbons with 7 cells from the bottom edge of \( \lambda \) until the resulting shape has at most 3 rows. In the following figure, we label the cells removed in the first ribbon with 1’s and label the cells in the second ribbon with 2’s.

Therefore we have that the equivalent shape \( \nu/m/\theta \) in the notation of Postnikov and McNamara is \( \nu = (3, 3), m = 2, \) and \( \theta = (2, 1). \)
Gessel-Krattenthaler [10] describe a method for expressing cylindric Schur functions as a sum of determinants and give an explicit formula in the case when \( \lambda \) is a rectangular shape and \( d = 0 \). Postnikov [17, Eq. (11)] (see also [16, §6]) then gave the following Jacobi-Trudi-like identity for cylindric Schur functions, making the result of Gessel-Krattenthaler explicit for arbitrary \( \lambda \) and \( d > 0 \):

**Theorem 2.5.**

\[
(13) \quad s_{\lambda/\mu/d}(x) = \sum_{k_1+k_2+\cdots+k_{\lambda_1} = 0} \det \left[ e_{k_i(\lambda_1+d)+\lambda'_j-i+j}(x) \right].
\]

Now define the \( P \)-cylindric Schur functions by

\[
(14) \quad s_{\lambda/\mu/d}^P(u) := \psi(s_{\lambda/\mu/d}^P(x)) = \sum_{k_1+k_2+\cdots+k_r = 0} \det \left[ e_{k_i(\lambda_1+d)+\lambda'_j-i+j}(u) \right].
\]

Our main result is the following \( P \)-version of Theorem 2.5.

**Theorem 2.6.** For any \((3+1)\)-free poset \( P \) and cylindric shape \( \lambda/\mu/d \),

\[
(15) \quad s_{\lambda/\mu/d}^P(u) = \sum_{T \in \text{CT}_P(\lambda/\mu/d)} u^T.
\]

By (3), this \( u \)-monomial positive formula for the \( P \)-cylindric Schur functions has the following consequence for the expansion of \( X_{\text{inc}(P)} \) into elementary symmetric functions.

**Corollary 2.7.** Letting \( a_\nu \) denote the coefficients in the monomial expansion \( s_{\lambda/\mu/d}(x) = \sum_\nu a_\nu m_\nu(x) \) and \( c_\lambda^P \) the coefficients in \( X_{\text{inc}(P)} = \sum_\lambda c_\lambda^P e_\lambda(x) \), we have

\[
(16) \quad \sum_\nu a_\nu c_\lambda^P = \langle u_P \rangle s_{\lambda/\mu/d}^P(u) = \# \{ T \in \text{CT}_P(\lambda/\mu/d) : T \text{ is standard} \},
\]

where a cylindric \( P \)-tableau is standard if it contains each element of \( P \) exactly once.

Note that in the special case \( s_{\lambda/\mu/d}(x) = s_\lambda(x) \) (when \( \mu = \emptyset \), \( d = \lambda'_1 \)), \( a_\nu \) is the Kostka coefficient \( K_{\lambda\nu} \) and (16) follows directly from Gasharov’s result [8]. Thus (16) can be regarded as a strengthening of this result as the \( a_\nu \) are typically smaller than Kostka coefficients.

See §4 for two other interesting cases of (16).

3. **Proof of Theorem 2.6**

Our proof of Theorem 2.6 combines ideas of Gessel-Krattenthaler [10, Proposition 1] and Gasharov [8, Theorem 3].

We begin by generalizing Young diagrams to non-partition shapes. For a composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \) the Young diagram of shape \( \text{col}(\alpha) \) is the set \( \{(i, j) : 1 \leq j \leq l, 1 \leq i \leq \alpha_j \} \) using matrix style coordinates. For a partition \( \beta \) with \( \text{col}(\beta) \subset \text{col}(\alpha) \), write \( \text{col}(\alpha, \beta) \) for \( \text{col}(\alpha) - \text{col}(\beta) \). We define a \( P \)-array of shape \( \text{col}(\alpha, \beta) \) to be a function \( A : \text{col}(\alpha, \beta) \to P \) that is increasing in \( P \) down columns. We write \( P\text{-Array}(\alpha, \beta) \) for the set of \( P \)-arrays of shape \( \text{col}(\alpha, \beta) \).
As with $P$-tableaux, for a $P$-array $A$ let $A^d$ be the $P$-array obtained by gluing the first column of $A$ to the right of the last column of $A$ so that the copied column is shifted up by $d$ cells, i.e. $A^d$ has shape $\text{col}(\bar{\alpha}, \bar{\beta})$ where $\bar{\alpha} = (\alpha_1 + d, \alpha_2 + d, \ldots, \alpha_l + d, \alpha_1)$ and $\bar{\beta} = (\beta_1 + d, \beta_2 + d, \ldots, \beta_l + d, \beta_1)$.

**Example 3.1.** For $P$ the total order on $\mathbb{Z}_{>0}$,

(17) $A = \begin{array}{ccc}
2 & 1 & 2 \\
3 & 3 & 3 \\
5 & 4 & \\
6 & & 7
\end{array}$

is a $P$-array of shape $\text{col}((4, 6, 2, 3), (2, 1, 1))$, and

(18) $A^2 = \begin{array}{ccc}
2 & 1 & 3 \\
3 & 3 & 5 \\
5 & 4 & \\
6 & & 7
\end{array}$ and $A^3 = \begin{array}{ccc}
3 & & 1 \\
2 & 1 & 2 \\
3 & 3 & 3 \\
5 & 4 & 6 \\
7 & & 7
\end{array}$

Now fix a cylindric shape $\lambda/\mu/d$. Write $\tilde{\mathbb{Z}}^{\lambda_1}$ for the set of $k = (k_1, \ldots, k_{\lambda_1}) \in \mathbb{Z}^{\lambda_1}$ such that $k_1 + k_2 + \cdots + k_{\lambda_1} = 0$. For a permutation $\pi \in S_{\lambda_1}$ and $k \in \tilde{\mathbb{Z}}^{\lambda_1}$, let $\pi^k_d(\lambda)$ be the sequence of $\lambda_1$ integers given by

(19) $\pi^k_d(\lambda)_i = k_i(\lambda_1 + d) + \lambda'_\pi(i) + i - \pi(i)$

for $i \in [\lambda_1]$. Note that when $\pi$ is the identity permutation and $k = 0$, $\pi^k_d(\lambda) = \lambda'$.

Let

(20) $\tilde{B} = \{ (\pi, k, A) : \pi \in S_{\lambda_1}, k \in \tilde{\mathbb{Z}}^{\lambda_1}, A \in P\text{-Array}(\text{col}(\pi^k_d(\lambda), \mu')) \}$.

(21) $B = \{ (\pi, k, A) \in \tilde{B} : A^d \text{ is not a } P\text{-tableau} \}$.

Note that if $\pi^k_d(\lambda)_i < \mu'_i$ for some $i$ and $k$, there are no $P$-arrays of shape $\pi^k_d(\lambda)$. Say that the sign of $(\pi, k, A) \in \tilde{B}$ is the sign of $\pi$. We will give a sign-reversing involution $\Phi$ on $B$, therefore showing that

(22) $\sum_{(\pi, k, A) \in B} \text{sgn}(\pi) u^A = 0$,

where $u^A$ is the product of $u_a$’s over the entries $a$ of $A$.
Definition 3.2. For a $P$-array $A = [a_{s,t}]_{(s,t) \in \text{col}(\alpha, \beta)}$ of shape $\text{col}(\alpha, \beta)$, we say $a_{s,t}$ is empty if $(s, t) \notin \text{col}(\alpha, \beta)$ and $s > \beta_t$. For columns $i, j, i < j$, we say column $i$ and column $j$ intersect if there is some index $m$ so that $a_{m,j}$ is nonempty, $m - j + i + 1 > \beta_i$, and

1. $a_{m-j+i+1,i}$ is empty, or
2. $a_{m-j+i+1,i} > P a_{m,j}$.

We say $a_{m,j}$ is an intersection point of columns $i$ and $j$, and that $a_{m-j+i+1,i}$ is a witness to the intersection point.

The following lemma applied to $A^d$ shows that $B$ consists of all triples $(\pi, k, A) \in \tilde{B}$ such that $A^d$ has an intersection point.

Lemma 3.3. Let $P$ be a $(3+1)$-free poset and $A$ be a $P$-array. The following are equivalent:

(i) $A$ is a $P$-tableau
(ii) $A$ has no adjacent intersecting columns
(iii) $A$ has no intersecting columns

Proof. We first show that the definition of a $P$-tableau is the same as a $P$-array with no adjacent intersecting columns. A $P$-tableau is defined to be a $P$-array of skew partition shape that is nondecreasing in $P$ across rows. Now a $P$-array $A$ is of skew partition shape $\lambda/\mu$ only if there is no $c > 1$ and $r > \mu_{i-1}$ so that $a_{r,c}$ is nonempty and $a_{r,c-1}$ is empty. But that is exactly condition (1) of the definition of intersecting columns applied to $i = c-1$ and $j = c$. Likewise, the nondecreasing row condition of $P$-tableau is equivalent to saying that no adjacent columns satisfy condition (2) of the definition of intersecting columns. Therefore (i) and (ii) are equivalent.

Now if $A$ has adjacent intersecting columns, then $A$ has intersecting columns. So (iii) $\implies$ (ii).

It remains to show that if some columns $i$ and $j$ intersect, then there is some adjacent column that intersect. To do this, choose a pair $i, j, i < j$, of intersecting columns so that $j - i$ is as small as possible. We then show that $j - i = 1$. Suppose $j - i > 1$. Let $a_{m,j}$ be the intersection point between columns $i$ and $j$. Now $\beta_{j-1} \geq \beta_j$, so if $a_{m-1,j-1}$ is empty, columns $j - 1$ and $j$ intersect and we are done. Therefore we assume that $a_{m-1,j-1}$ is nonempty. Now if $a_{m-j+i+1,i}$ is empty, then, as $a_{m-1,j-1}$ is nonempty, columns $i$ and $j - 1$ intersect, which is a contradiction. We can then assume that $a_{m-j+i+1,i}$ is nonempty. As $m - j + i + 1 \geq \beta_i \geq \beta_j$, we have that $a_{m-1,j}$ is nonempty, and $a_{m,j}$ is an intersection point between columns $i$ and $j$, we have $a_{m-1,j} < P a_{m,j} < P a_{m-j+i+1,i}$. If $a_{m-1,j} < P a_{m-1,j-1}$ then columns $j - 1$ and $j$ intersect, and if $a_{m-1,j-1} < P a_{m-j+i+1,i}$ then column $i$ and $j - 1$ intersect, which is a contradiction. However, if $a_{m-1,j} \not< P a_{m-1,j-1}$ and $a_{m-1,j-1} \not< P a_{m-j+i+1,i}$, then $a_{m-1,j}, a_{m,j}, a_{m-j+i+1,i}$, and $a_{m-1,j-1}$ form an induced $(3+1)$, which contradicts the fact that $P$ has no induced $(3+1)$ subposet.

Therefore, if $A$ is a $P$-array with a pair of intersecting columns, then $A$ must have an adjacent pair of intersecting columns, as desired.$\square$

The next lemma shows that $B \setminus \tilde{B}$ consists of exactly the triples $(\pi, 0, A)$ with $A$ a cylindric $P$-tableau of shape $\lambda/\mu/d$. 
Lemma 3.4. If $A$ is a $P$-array of shape $\text{col}(\pi_d^k(\lambda), \mu')$ such that $A^d$ is a $P$-tableau, then $\pi$ is the identity permutation and $k = 0$.

Proof. Let $A$ be a $P$-array of shape $\text{col}(\pi_d^k(\lambda), \mu')$ such that $A^d$ is a $P$-tableau. By Lemma 3.3 for any $1 < i \leq \lambda_1$, column $i$ does not intersect with column 1 or column $\lambda_1 + 1$. Therefore, from the first condition in the definition of intersecting columns, we have the following inequalities

\begin{equation}
\pi_d^k(\lambda)_1 \geq \pi_d^k(\lambda)_i - i + 2
\end{equation}

and

\begin{equation}
\pi_d^k(\lambda)_i + d \geq \pi_d^k(\lambda)_1 - (\lambda_1 + 1) + i + 1
\end{equation}

which are equivalent to

\begin{equation}
\lambda_{\pi(1)}' - \lambda_{\pi'(1)}' + \pi(i) - (\pi(1) + 1) \geq (k_i - k_1)(\lambda_1 + d)
\end{equation}

and

\begin{equation}
(k_i - k_1 + 1)(\lambda_1 + d) \geq \lambda_{\pi(1)}' - \lambda_{\pi'(1)}' + \pi(i) - (\pi(1) + 1).
\end{equation}

Now consider the case when $\pi(i) > \pi(1)$. Then we have that $0 \leq \pi(i) - (\pi(1) + 1) < \lambda_1$ and $0 \leq \lambda_{\pi(1)}' - \lambda_{\pi'(i)}' \leq \lambda_1 - \lambda_1' \leq d$. Therefore we have

\begin{equation}
\lambda_1 + d \geq \lambda_{\pi(1)}' - \lambda_{\pi'(i)}' + \pi(i) - (\pi(1) + 1) \geq (k_i - k_1)(\lambda_1 + d),
\end{equation}

so $k_i - k_1 < 1$. Furthermore, we have

\begin{equation}
(k_i - k_1 + 1)(\lambda_1 + d) \geq \lambda_{\pi(1)}' - \lambda_{\pi'(i)}' + \pi(i) - (\pi(1) + 1) + 1 \geq 2,
\end{equation}

so $k_i - k_1 + 1 > 0$. Therefore we have that $k_i = k_1$.

Now consider the case when $\pi(i) < \pi(1)$. Then $0 \geq (\pi(i) - (\pi(1) + 1) \geq -\lambda_1$ and

\begin{equation}
0 \geq \lambda_{\pi(1)}' - \lambda_{\pi'(i)}' \geq \lambda_1' - \lambda_1 \geq -d.
\end{equation}

Then

\begin{equation}
0 \geq \lambda_{\pi(1)}' - \lambda_{\pi'(i)}' + \pi(i) - (\pi(1) + 1) \geq (k_i - k_1)(\lambda_1 + d),
\end{equation}

so $k_i < k_1$, and

\begin{equation}
(k_i - k_1 + 1)(\lambda_1 + d) \geq \lambda_{\pi(1)}' - \lambda_{\pi'(i)}' + \pi(i) - (\pi(1) + 1) + 1 \geq -\lambda_1 - d + 1,
\end{equation}

so $k_i - k_1 + 1 > -1$. Therefore we have that $k_1 = k_i + 1$.

Then

\begin{equation}
k_1 + k_2 + \cdots + k_{\lambda_1} = \lambda_1 k_1 - \#\{i : \pi(i) < \pi(1)\} = 0.
\end{equation}

But $0 \leq \#\{i : \pi(i) < \pi(1)\} < \lambda_1$, so $k_1 = k_2 = \cdots = k_{\lambda_1} = 0$. Now we show that $\pi$ must be the identity permutation. With $k = 0$, we have that if $i < j$ and $\pi(i) > \pi(j)$ then $\lambda_{\pi(i)}' \leq \lambda_{\pi(j)}'$. Then

\begin{equation}
\pi_d^k(\lambda)_i = \lambda_{\pi(i)}' + i - \pi(i) < \lambda_{\pi(j)}' + j - \pi(j) - j + i + 1 = \pi_d^k(\lambda)_j - j + i + 1,
\end{equation}

so columns $i$ and $j$ intersect. Therefore we have that $\pi(i) < \pi(j)$ whenever $i < j$, so $\pi$ must be the identity permutation. \qed
Definition 3.5. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ be a composition and $\beta \subset \alpha$ be a partition. For a $P$-array $A$ of shape $\text{col}(\alpha, \beta)$ with columns $i$ and $j$ intersecting, we define a swap at $i$ and $j$ by letting $\text{swap}(A, i, j)$ be the $P$-array obtained in the following way. Let $a_{m,j}$ be the $P$-minimal intersection point between columns $i$ and $j$, and let $a_{m-i+j+1, i}$ be the witness to this intersection. Let

\begin{equation}
R_{i,j} = \{ a_{r,j} : r > m \} \quad \text{and} \quad L_{i,j} = \{ a_{r,i} : r \geq m - j + i + 1 \}.
\end{equation}

Then $\text{swap}(A, i, j)$ is the $P$-array obtained by moving $R_{i,j}$ to column $i$, $L_{i,j}$ to column $j$, and fixing the rest of the cells.

Given an integer $d$, $\beta_1 - \beta_l \leq d$, and columns $i$ and $j$ so that $i$ and $j$ intersect in $A^d$, we define the cylindric swap at $i$ and $j$ by letting $\text{swap}(A, i, j, d)$ be the $P$-array obtained in the following way. If $j \neq l + 1$, $\text{swap}(A, i, j, d) = \text{swap}(A, i, j)$. Now if $j = l + 1$, we again let $a_{m,j}$ be the $P$-minimal intersection point between columns $i$ and $j$, and take

\begin{equation}
R_{i,j} = \{ a_{r,j} : r > m \} = \{ a_{r,1} : r > m + d \} \quad \text{and} \quad L_{i,j} = \{ a_{r,i} : r \geq m - j + i + 1 \}.
\end{equation}

Then $\text{swap}(A, i, j, d)$ is the $P$-array obtained from $A$ by moving $R_{i,j}$ to column $i$, $L_{i,j}$ to column 1, and fixing the rest of the cells of $A$.

Note that the cylindric swap is an operation on $A$ rather than on $A^d$, even though we use an intersection point of $A^d$ to define it.

Example 3.6. Let $P$ be the poset on $\mathbb{Z}_{>0}$ so that $i <_P j$ if $i < j$ and $j - i > 1$, and consider the following $P$-arrays $A$ and $A^1$

\begin{equation}
A = \begin{bmatrix}
2 & 3 & 1 \\
1 & 4 & 5 \\
4 & 9 & 7 \\
6 &
\end{bmatrix}, \quad A^1 = \begin{bmatrix}
2 & 3 & 1 \\
1 & 4 & 5 & 4 \\
4 & 9 & 7 & 6 \\
6 &
\end{bmatrix}.
\end{equation}

The underlined entries indicate the intersection points of $A^1$.

We then have

\begin{equation}
\text{swap}(A, 2, 3, 1) = \begin{bmatrix}
1 & 4 & 5 \\
4 & 7 &
\end{bmatrix} \quad \text{and} \quad \text{swap}(A, 3, 4, 1) = \begin{bmatrix}
1 & 4 & 6 \\
3 & 9 &
\end{bmatrix}.
\end{equation}

Lemma 3.7. If $A$ is a $P$-array of shape $\text{col}(\pi^k_d(\lambda), \mu')$ so that columns $i$ and $j$ intersect in $A^d$, then $\text{swap}(A, i, j, d)$ is a $P$-array of shape $\text{col}(\sigma^k_d(\lambda), \mu')$ where

\begin{equation}
\sigma = \begin{cases}
\pi \circ (i, j) & \text{if } j \neq \lambda_1 + 1 \\
\pi \circ (1, i) & \text{if } j = \lambda_1 + 1
\end{cases}
\end{equation}
and

\[ \begin{align*}
\mathbf{k}' &= \begin{cases} 
(k_1, \ldots, k_j, \ldots, k_i, \ldots, k_{\lambda_1}) & \text{if } j \neq \lambda_1 + 1 \\
(k_1 + 1, k_2, \ldots, k_{i-1}, k_1 - 1, k_{i+1}, \ldots, k_{\lambda_1}) & \text{if } j = \lambda_1 + 1.
\end{cases}
\end{align*} \tag{38}\]

**Proof.** Let \( A \) be a \( P \)-array of shape \( \col(\pi^k_d(\lambda), \mu') \) such that columns \( i \) and \( j \) of \( A^d \) intersect. We first consider the case when \( 1 \leq i < j \leq \lambda_1 \). We then have that \( \swap(A, i, j) \) has shape \( \col(\alpha, \mu') \) where \( \alpha_i = \pi^k_d(\lambda) + i - j \), \( \alpha_j = \pi^k_d(\alpha_i) + j - i \), and the rest of the columns are fixed. So we have

\[ \alpha_i = \lambda'_{\pi(i)} + j - \pi(j) + k_j(\lambda_1 + d) + i - j = \lambda'_{\pi(i)} + j - \pi(j) + k_j(\lambda_1 + d), \]

and

\[ \alpha_j = \lambda'_{\pi(i)} + i - \pi(i) + k_i(\lambda_1 + d) + j - i = \lambda'_{\pi(i)} + j - \pi(i) + k_i(\lambda_1 + d). \]

So we have \( \alpha = \sigma^k_d(\lambda) \) where \( \sigma = \pi \circ (i \ j) \) and \( \mathbf{k}' = (k_1, \ldots, k_j, \ldots, k_i, \ldots, k_{\lambda_1}) \) as desired.

Now consider the case when \( j = \lambda_1 + 1 \). Then \( \swap(A, i, j, d) \) has shape \( \col(\alpha, \mu') \) where

\[ \begin{align*}
\alpha_i &= \pi^k_d(\lambda) - (\lambda_1 + 1) + i - d = \lambda'_{\pi(1)} + 1 - \pi(1) + k_1(\lambda_1 + d) - (\lambda_1 + 1) + i - d \\
\alpha_j &= \lambda'_{\pi(i)} + i - \pi(i) + k_i(\lambda_1 + d) + j - i = \lambda'_{\pi(i)} + j - \pi(i) + k_i(\lambda_1 + d),
\end{align*} \tag{41} \tag{42} \]

\[ \begin{align*}
\alpha_1 &= \pi^k_d(\lambda) - i + (\lambda_1 + 1) + d = \lambda'_{\pi(i)} + i - \pi(i) + k_i(\lambda_1 + d) - i + (\lambda_1 + 1) + d \\
\alpha_j &= \lambda'_{\pi(i)} + 1 - \pi(i) + (k_i + 1)(\lambda_1 + d), \tag{43} \tag{44}
\end{align*} \]

and the rest of the columns are fixed. Therefore we have that \( \alpha = \sigma^k_d(\lambda) \) where \( \sigma = \pi \circ (1 \ i) \) and \( \mathbf{k}' = (k_1 + 1, k_2, \ldots, k_{i-1}, k_1 - 1, k_{i+1}, \ldots, k_{\lambda_1}) \) as desired. \( \square \)

We define the involution \( \Phi \) on \( B \) in the following way: for a \( P \)-array \( A \), let \( a_{m,j} \) be the rightmost \( P \)-minimal intersection point in \( A^d \). Let \( i \) be the rightmost column index so that columns \( i \) and \( j \) intersect with intersection point \( a_{m,j} \). We let \( \phi(A) = \swap(A, i, j, d) \) and \( \Phi((\pi, \mathbf{k}, A)) = (\sigma, \mathbf{k}', \phi(A)) \), where \( \sigma \) and \( \mathbf{k}' \) are as in Lemma 3.7.

Lemma 3.7 shows that \( \Phi \) is sign-reversing. To show that \( \Phi \) is an involution, we need to show that the rightmost \( P \)-minimal intersection of \( A^d \) is also the rightmost \( P \)-minimal intersection point of \( \phi(A)^d \). This is established in the next two lemmas.

**Lemma 3.8.** If \( a_{m,j} \) is the rightmost \( P \)-minimal intersection point of \( A^d \), then \( a_{m,j} \) is a \( P \)-minimal intersection point of \( \phi(A)^d \). Furthermore, if \( i \) is the rightmost column so that \( a_{m,j} \) is an intersection point between columns \( i \) and \( j \), \( i \) is the rightmost column in \( \phi(A)^d \) so that \( a_{m,j} \) is an intersection point between columns \( i \) and \( j \).

**Proof.** First we show that \( a_{m,j} \) is an intersection point in \( \phi(A)^d \). As \( \phi \) swaps the positions of \( \{a_{m-j+i+1,i}, a_{m-j+i+2,i}, \ldots, a_{m,i}\} \) and \( \{a_{m+1,j}, a_{m+2,j}, \ldots, a_{m,j}\} \), and either \( a_{m,j} <_P a_{m+1,j} \) or \( a_{m+1,j} \) is not defined, \( a_{m,j} \) remains an intersection point of columns \( i \) and \( j \). As \( \phi \) fixes the entries above row \( m \) of column \( j \) and the entries above \( m - j + i \) in column \( i \), \( a_{m,j} \) remains the \( P \)-minimal intersection point between columns \( i \) and \( j \). As \( \phi \) fixes columns \( \{i + 1, i + 2, \ldots, j - 1\} \), column \( i \) is the rightmost column so that \( a_{m,j} \) is an intersection point.
Now we will show that $a_{m,j}$ is $P$-minimal among intersection points of $\phi(A)^d$. Consider some intersection point $x$ of $\phi(A)^d$ such that $x \prec_P a_{m,j}$. As each entry that is different in $\phi(A)^d$ from the corresponding entry of $A$ is greater than $a_{m,j}$ in $P$, each such entry is also greater than $x$ in $P$. Therefore, $x$ relates to each entry of $A^d$ in the same way as it relates to the corresponding entry of $\phi(A)^d$. So, if $x$ is an intersection point of $\phi(A)^d$ it must have also been an intersection point of $A^d$, which contradicts the fact that $a_{m,j}$ is a $P$-minimal intersection point. Therefore $a_{m,j}$ is a $P$-minimal intersection point in both $A^d$ and $\phi(A)^d$ as desired.

**Lemma 3.9.** If $a_{m,j}$ is the rightmost $P$-minimal intersection point of $A^d$, $a_{m,j}$ is the rightmost $P$-minimal intersection point of $\phi(A)^d$.

**Proof.** Suppose $x \not= a_{m,j}$ is the rightmost $P$-minimal intersection point of $\phi(A)^d$. Then $x$ is in some column $k$ with $j < k \leq \lambda_1 + 1$. As every entry that is moved by $\phi$ is greater than $a_{m,j}$ in $P$, $x$ must be in the same location in $A^d$ and $\phi(A)^d$. Next observe that if $y$ is a witness to $x$ being an intersection point, $y$ must be an entry of the southeast diagonal directly below the diagonal containing $x$. e.g. the bullets in the following array indicate the possible witnesses of $x$ being an intersection point.

\[
\begin{array}{|c|c|c|c|}
\hline
\bullet & \bullet & \bullet & x \\
\hline
\end{array}
\]

Now as $j \not= \lambda_1 + 1$, $\phi$ fixes southeast diagonals in columns $1, 2, \ldots, \lambda_1$. Therefore if $x$ is an intersection point to the right of column $j$ and incomparable to $a_{m,j}$ in $\phi(A)^d$, $x$ must have been an intersection point in $A^d$. But this is a contradiction. \qed

We now combine the previous lemmas to prove Theorem 2.6

**Proof.** We wish to show that

\[
(46) \quad s^p_{\lambda/\mu/d}(u) = \sum_{k \in \mathbb{Z}^{\lambda_1}} \det[e^P_{k,1} + \lambda_j - \mu_j - i + j](u) = \sum_{T \in CT_P(\lambda/\mu/d)} u^T.
\]

By Lemmas 3.7, 3.8, and 3.9 $\Phi$ is a sign-reversing involution, and by Lemma 3.4 $\tilde{B} \setminus B$ consists of exactly the triples $(id, 0, A)$ with $A$ a cylindric $P$-tableau of shape $\lambda/\mu/d$. Hence

(47) \[
\sum_{T \in CT_P(\lambda/\mu/d)} u^T = \sum_{k \in \mathbb{Z}^{\lambda_1}} \sum_{\pi \in S_{\lambda_1}} \text{sgn}(\pi) \sum_{T \in P-Array(\sigma^k)} u^T.
\]

Now for a composition $\alpha$ and partition $\beta \subset \alpha$, we have

(48) \[
\sum_{T \in P-Array(\alpha, \beta)} u^T = \prod_{i=1}^{\ell(\alpha)} e^P_{\alpha_i - \beta_i}(u),
\]

so

(49) \[
\sum_{T \in CT_P(\lambda/\mu/d)} u^T = \sum_{k \in \mathbb{Z}^{\lambda_1}} \sum_{\pi \in S_{\lambda_1}} \text{sgn}(\pi) e^P_{k,1}(\lambda_j + \lambda_j - \mu_j - i + j - \pi(i) - \mu_i}(u).
\]
Permutation of the indices fixes $\tilde{Z}^{\lambda_1}$, so we have

$$\sum_{k \in \tilde{Z}^{\lambda_1}} \sum_{\pi \in S_{\lambda_1}} \text{sgn}(\pi) e_{k_{\pi(i)}}^{P} (\lambda_{\pi(i)} + i - \pi(i) - \mu'_i (u)) = \sum_{\pi \in S_{\lambda_1}} \sum_{k \in \tilde{Z}^{\lambda_1}} \text{sgn}(\pi) e_{k_{\pi(i)}}^{P} (\lambda_{\pi(i)} + i - \pi(i) - \mu'_i (u)).$$

As the determinant of an $n \times n$ matrix $X$ is given by $\det X = \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{i=1}^{n} X_{\pi(i),i}$, we have

$$\sum_{T \in CT_P(\lambda/\mu/d)} u^T = \sum_{k \in \tilde{Z}^{\lambda_1}} \det[e_{k_{\pi(i)}}^{P} (\lambda_{\pi(i)} + i - \mu'_i + j)]$$

as desired. \qed

**Remark 3.10.** In the proof of Theorem 2.6 we used the fact that $S_{\lambda_1}$ acts on $\tilde{Z}^{\lambda_1}$ to show that

$$\sum_{k \in \tilde{Z}^{\lambda_1}} \sum_{\pi \in S_{\lambda_1}} \text{sgn}(\pi) e_{k_{\pi(i)}}^{P} (\lambda_{\pi(i)} + i - \pi(i) - \mu'_i (u)) = \sum_{k \in \tilde{Z}^{\lambda_1}} \sum_{\pi \in S_{\lambda_1}} \text{sgn}(\pi) e_{k_{\pi(i)}}^{P} (\lambda_{\pi(i)} + i - \pi(i) - \mu'_i (u)).$$

Therefore we can obtain two equivalent determinantal formulas for $s_{\lambda/\mu/d}^{P}(u)$ by replacing $k_i$ with $k_j$:

$$s_{\lambda/\mu/d}^{P}(u) = \sum_{k \in \tilde{Z}^{\lambda_1}} \det[e_{k_{\pi(i)}}^{P} (\lambda_{\pi(i)} + i - \mu'_i - j)] = \sum_{k \in \tilde{Z}^{\lambda_1}} \det[e_{k_{\pi(i)}}^{P} (\lambda_{\pi(i)} + i - \mu'_i - j)].$$

4. **Special Cases**

We now examine two special cases of cylindric Schur functions with particularly nice monomial expansions.

Take the partition $\lambda = (r^c)$ whose Young diagram is a rectangle with width $r$ and height $c$. Then for $T \in \text{CSSYT}(\lambda/\emptyset/0)$, if $a_1, a_2, \ldots, a_r$ are the entries of a row in $T$, then

$$a_1 \leq a_2 \leq \cdots \leq a_r \leq a_1,$$

so $a_1 = a_2 = \cdots = a_r$. Therefore we have

$$s_{(r^c)/\emptyset/0}(x) = m_{(r^c)}(x).$$

If $\lambda = (r^c) \vdash n$ is the partition with $c$ rows of length $r$ and another row of length $t \leq r$, then any cylindric tableau $T$ of shape $\lambda/\emptyset/1$ has the following form: the word $a_1 \cdots a_n$ formed by concatenating the rows of $T$ from top to bottom must satisfy $a_1 \leq a_2 \leq \cdots \leq a_n$ and $a_i < a_{i+r}$ for all $1 \leq i \leq n - r$. Therefore, given any multiset of positive integers so that no number is repeated more than $r$ times, we can fill the diagram of shape $\lambda/\emptyset/1$ in exactly one way. Hence

$$s_{(r^c)/\emptyset/1}(x) = \sum_{\lambda_1 \leq r} m_{\lambda_1}(x).$$

The monomial expansions (53) and (54) combined with Corollary 2.7 yield the following results.
Corollary 4.1. For a $(3+1)$-free poset $P$, the coefficient of $e_{(r^c)}(x)$ in the $e$-expansion of $X_{\text{inc}(P)}(x)$ is the number of standard cylindric $P$-tableaux of shape $(r^c)/\emptyset/0$.

Corollary 4.2. For a $(3+1)$-free poset $P$ and positive integer $r$, letting $c^P_{\lambda}$ denote the coefficients in $X_{\text{inc}(P)}(x) = \sum_{\lambda} c^P_{\lambda} e_{\lambda}(x)$, we have

\begin{equation}
\sum_{\lambda \vdash |P|, \lambda_1 \leq r} c^P_{\lambda} = \# \{ T \in \text{CT}_P((r^c)t)/\emptyset/1 : T \text{ is standard} \},
\end{equation}

where $(r^c)t$ is the partition of $|P|$ with the maximal number of rows of length $r$.

Corollary 4.1 recovers a result of Clearman-Hyatt-Shelton-Skandera [4, Theorem 4.7 (v-b)] which gave a combinatorial interpretation to a theorem of Stembridge [20, Theorem 2.8].

Corollary 4.2 is reminiscent of the following theorem of Stanley [18]:

Theorem 4.3. Let $c^G_{\lambda}$ be the coefficient of $e_{\lambda}$ in a chromatic symmetric function $X_G$. Then

\begin{equation}
\sum_{\lambda : \ell(\lambda) = j} c^G_{\lambda} = \text{sink}(G, j)
\end{equation}

where $\text{sink}(G, j)$ is the number of acyclic orientations of $G$ with $j$ sinks.

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