A 2D domain boundary estimation

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Abstract. This paper presents a numerical method for solving a problem usually encountered in thermal imaging. The goal is to estimate an interior boundary of a material by applying a known heat flux and measuring the induced temperature response on its external boundary. The interior boundary is assumed to be under a homogeneous Neumann condition. This boundary is first parameterized by a finite-term of Fourier series and the corresponding approximate inverse problem is numerically optimized using an iterative Newton method. The required gradient is established using the domain derivative techniques. The system of heat equations is treated using finite element method for space and implicit scheme for time. Some numerical tests are provided to illustrate the performances of the proposed method.

1. Introduction
The ability to inspect the interior of an object without destroying it, is known as a nondestructive technique. Thermal imaging is one of these techniques being used for detecting damage or corrosion in composite materials. This technique consists in applying a heat flux on the accessible part of the boundary of the object and measuring the temperature response over an interval of time. An inverse procedure is then used to determine some internal characteristics of the material.

In this context we try to identify the internal shape of a material non directly observable by applying the above technique. More precisely, let $D_i$ and $D_e$ be two simply connected bounded domain in $\mathbb{R}^2$ such that $D_i \subset D_e$. The boundaries of $D_i$ and $D_e$ are denoted respectively by $\Gamma_i$ and $\Gamma_e$. Setting $D := D_e \setminus \overline{D_i}$, then the temperature $u$, in a point $x$ at time $t$, is assumed to be governed by the following dimensionless system,

\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) - \nabla u(x,t) &= f(x,t), \quad (x,t) \in D \times ]0, T[, \\
\frac{\partial u}{\partial \nu}(x,t) &= \varphi(x,t), \quad (x,t) \in \Gamma_e \times ]0, T[, \\
\frac{\partial u}{\partial \nu}(x,t) &= 0, \quad (x,t) \in \Gamma_i \times ]0, T[, \\
u(x,0) &= u_0(x), \quad x \in D,
\end{align*}
\]

where $f$ represents the rate of heat exchanged with the exterior environment defined on $(D \cup \Gamma_i \cup \Gamma_e) \times ]0, T[$. The vector $\nu$ is the outward unit normal on $\Gamma = \Gamma_i \cup \Gamma_e$. The term
The input heat flux applied on the exterior boundary $\Gamma_e$ while the internal boundary $\Gamma_i$ is perfectly insulated. Thus we are concerned in this paper with the following inverse problem: for known $\varphi$, $f$ and $u_0$ and given temperature measurements $\phi$ performed on the external boundary $\Gamma_e$ over the time interval $[0, T]$, determine the internal boundary $\Gamma_i$.

The problem, as it is posed, is a generalization of that treated by [5] in the case of homogeneous heat equation and homogeneous initial condition, ($f = 0$ and $u_0 = 0$), has been considered. The term $f$, introduced to take into account of a non negligible energy exchange with exterior, can also be viewed as a heat source specially introduced to improve the quality of the estimation. The authors in [5] proposed a regularized Newton method to deal with the inverse problem and a boundary element method as a solver for the direct problem. In our previous work [3], which also constitutes a generalization of the work [4], we have solved the same problem but when the boundary to estimate is under a Dirichlet condition. More precisely, we have derived an estimation algorithm based on the finite element method easy to implement with the FreeFem software. This software allows the resolution of partial differential equations just by writing a few lines of code. Further, FreeFem provides some well known optimization procedures (Linear and non linear conjugate gradient, Newton-Raphson,...) making it an appropriate environment to easily handle inverse problems for partial differential equations.

In view of the good results obtained in [3], we attempt to use this method for the problem posed here. Let us underline that the Neumann case presents more difficulties, both theoretical and numerical, compared with the Dirichlet case.

Noting that more recently, the authors in [12] and in [7] have consider similar problems of estimating boundaries under Neumann condition or Robin condition. They adopted in their mathematical model the steady heat equation. For problems dealing with the unsteady heat equation, we cite the work [2], however the geometry of the problem and the resolution method differ from that we propose in this paper.

This paper is structured as follows. In section 2, we first begin by establishing some domain derivative results, then we parameterize the unknown boundary by approximating its radial function using trigonometric polynomials. This section is ended by formulating the approximate inverse problem. In section 3, we present the identification algorithm and how it is implemented using FreeFem software. In the last section, we compare the present method with that used by [5] and make a variety of numerical tests.

## 2. Approximate inverse problem

In this section, we first present some results related to the domain derivative. After that we present a parametrization of the boundary and we formulate an approximate inverse problem.

### 2.1. Domain derivative

Let $A$ be the operator which associates to the boundary $\Gamma_i$ the function defined on $\Gamma_e \times [0, T]$ by $A(\Gamma_i) = u|_{\Gamma_e \times [0, T]}$ where $u$ is the solution of (1) corresponding to $\Gamma_i$. Then the concerned inverse problem is equivalent to solving the non linear operator equation $A(\Gamma_i) = \phi$.

First we seek for the derivative of $A$ with respect to $\Gamma_i$ in the direction $a$ is defined by

$$A'(\Gamma_i, a) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (A(\Gamma_i^\varepsilon) - A(\Gamma_i))$$

(2)

By adapting the computations used in [5] to the situation where $f$ and $u_0$ are non homogenous, we can prove the following main result.
Theorem 2.1 Let $a \in C^2(\Gamma_i, \mathbb{R}^2)$ and $u$ the solution of (1), then

$$A'(\Gamma_i, a) = u'|_I,$$

on $I = \Gamma_e \times [0, T]$, where $u'$ solves the system

$$
\begin{align*}
\frac{\partial u'}{\partial t} - \Delta u' &= 0, & \text{in } D \times [0, T], \\
\frac{\partial u'}{\partial \nu} &= 0, & \text{on } I, \\
\frac{\partial u'}{\partial \nu} &= -a.\nu(\frac{\partial u}{\partial t} - f) + \frac{\partial}{\partial s}(a.\nu \frac{\partial u}{\partial s}), & \text{on } \Gamma_i \times [0, T], \\
u'(x, 0) &= 0, & \text{in } D,
\end{align*}
$$

and $\frac{\partial}{\partial s}$ denotes differentiation with respect to arc length on $\Gamma_i$.

If $f = 0$, then

$$\frac{\partial u'}{\partial \nu} = -a.\nu(\frac{\partial u}{\partial t}) + \frac{\partial}{\partial s}(a.\nu \frac{\partial u}{\partial s}).$$

The latter expression is given in [5].

2.2. Boundary parametrization

For numerical purpose the unknown boundary has to be parameterized. We assume that $\Gamma_i$ is starlike with respect to the origin, therefore it can represented as follows

$$\Gamma_i = \{r(s)(\cos s, -\sin s), \ s \in [0, 2\pi]\}$$

where $r$ is the radial function $2\pi$ periodic. The choice of clockwise orientation is kept to guarantee the injectivity of the operator associating to a direction $a = \alpha(s)(\cos s, -\sin s)$ its domain derivative $A'(\Gamma_i, a)$.

We approximate the radial function in the subspace of trigonometric polynomials of degree less than or equal to $K$ by

$$r_b(s) = \sum_{j=0}^{2K} b_j q_j(s), \ s \in [0, 2\pi].$$

The basis functions are given by

$$q_j(s) = \begin{cases} 
\cos js & 0 \leq j \leq K, \\
\sin(j - K)s & K < j \leq 2K.
\end{cases}$$

Now, let $E$ be the operator defined from a subset $X \subset \mathbb{R}^{2K+1}$ to $L^2(I)$ by $E(b) = u|_I$, where $u$ is always the solution of (1) obtained for the internal boundary

$$\Gamma_i^b = \{r_b(s)(\cos s, -\sin s), \ s \in [0, 2\pi]\}.$$ 

The estimation problem of $\Gamma_i$ is approximated by the problem of solving the non linear equation $E(b) = \phi$. Since measurements $\phi$ are always affected by noise, this equation will be solved in a least squares sense, i.e, by minimizing

$$J(b) = \|E(b) - \phi\|^2_{L^2(I)}$$

The linearization of $E(b) - \phi$ requires the knowledge of the gradient of $E$.

It is easy to establish, by application of Theorem (2.1), that the $i$th component $\partial E/\partial b_i$ of the gradient $E'$ is given by

$$\frac{\partial E}{\partial b_i} = 6th International Conference on Inverse Problems in Engineering: Theory and Practice IOP Publishing
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\[
\frac{\partial E}{\partial b_i}(b) = u'\big|_I, \quad \text{on } I = \Gamma_e \times ]0, T[ \tag{5}
\]

where \(u'\) is solution of system (3) with \(a = q_i(s)(\cos s, -\sin s)\).

The derivative so obtained will be used to build an iterative method to obtain numerical estimation of \(\Gamma_i\).

3. Estimation Algorithm

This section describes the essential steps of the algorithm which will be used to estimate the unknown boundary.

Most optimization method used in the domain of shape estimation are based on the Newton method or Landweber method [4], [5], [9], [11], [12]. Furthermore, in the case of the unsteady heat equation, the comparison made in the work [6] shows that the regularized Newton, as proposed in [4], is more accurate that the Landweber method in estimating non convex curves. For this reason, in this paper we employ a regularized Newton method.

For the problem posed in this paper, the iterative Newton method consists in replacing the problem of minimizing

\[
J(b) = \| E(b) - \phi \|^2_{L^2(I)}
\]

by that of minimizing

\[
R(p) = \| E(b) + p \cdot E'(b) - \phi \|^2_{L^2(I)} + \varepsilon \| p \|^2 \tag{6}
\]

with respect to new variable \(p \in X \subset \mathbb{R}^{2K+1}\). This improves a given approximation \(b\) into \(b + p\). The term in \(\varepsilon\) is a regularizing one introduced to deal with the ill-posedness of this inverse problem [1].

Finally the basic steps of the estimation algorithm can be summarized as follows:

(i) start with initial known vector \(b\) and get the corresponding boundary \(\Gamma_i^b\);
(ii) solve the direct system (1). At this stage the quantity \(-a \cdot \nu(\frac{\partial u}{\partial t} - f) + \frac{\partial}{\partial s}(a \cdot \nu \frac{\partial u}{\partial s})\) is available.
(iii) for \(k\) (varying from 0 to \(2K\)), set \(a = q_k(s)(\cos s, -\sin s)\) and solve the system (3). This gives the gradient \(E'(b)\).
(iv) minimize (6) to yield \(p\).
(v) update \(b := b + p\), if the stopping test is satisfied or the allowed number of iterations is reached, terminate; else go to (ii).

As a stopping test, we take \(\frac{|r_p|}{|r_b|} < \delta\), where \(\delta\) is a positive real and \(\| r_p \|^2 = \int_0^{2\pi} (r_p(s))^2 ds\).

This algorithm is implemented with the free finite element software FreeFem [8]. Systems (1) and (3) are solved here using finite element method in space since it is more suitable to deal with the non homogeneous heat equation where all the domain \(D\) must be meshed. In time, we employ an implicit scheme with a step time \(\delta t = \frac{T}{M}\) (M is integer). The mesh of domain \(D\) is generated by dividing the boundaries \(\Gamma_i\) and \(\Gamma_e\) in \(N\) equal segments. For more details on this procedure, we refer to [3].

The optimization of (6) is performed in FreeFem using the linear conjugate gradient subroutine "linearCG". This subroutine requires an initial guess, which we take \(p = 0 \in \mathbb{R}^{2K+1}\), and the gradient of \(R\) is expressed as follows

\[
\frac{\partial R}{\partial p_i}(p) = 2 \int_0^T \int_{\Gamma_e} \left[ \frac{\partial E}{\partial b_i}(b)(E(b) + p \cdot E'(b) - \phi)(x,t) \right] ds(x) \ dt + 2\varepsilon \ p_i. \tag{7}
\]
We conclude this section by mentioning that the term $\frac{\partial}{\partial s}(a.\nu \frac{\partial u}{\partial s})$ can be easily evaluated in FreeFem in the following manner. Let $v \in H^1(D)$ and $\nabla_\tau u$ the tangential gradient of $u$ such that $\nabla u = (\nabla u.\nu)\nu + \nabla_\tau u$. Then the integral appearing in the variational form of system (3)

$$\int_{\Gamma_1} \frac{\partial}{\partial s}(a.\nu \frac{\partial u}{\partial s})v \, ds$$

is replaced by

$$-\int_{\Gamma_1} a.\nu \nabla_\tau u.\nabla_\tau v \, ds.$$

The gradient of $u$ and its normal component are obtained in FreeFem in a simple way by writing $\nabla u = (dx(u), dy(u))$ and $\frac{\partial u}{\partial \nu} = (N.x, N.y)$. Consequently the tangential gradient can be deduced.

4. Numerical tests

In this section, we investigate the performances of the proposed method by means of some tests.

4.1. Homogeneous heat equation

This subsection aims to compare the proposed method with that given in [5]. To this end, we consider the same test example as that adopted in [5]. This test is defined by an input heat flux $\varphi(x, t) = t^2 \exp(-4t + 2)$ applied on the circle centered at the origin and of radius 1.5. This circle represents the external boundary. The initial condition and the heat source are identically zero.

We use synthetic data, i.e., the measurements $\phi$ are generated by numerically solving the direct system (1). This resolution is done using FreeFem with $\delta t = 0.1$ and $N = 64$. The simulation time is $T = 1$.

The curve to be estimated is defined by its radial function $r_*$ given by

$$r_*(s) = \sqrt{\cos^2(s) + 0.26 \sin^2(s + 0.5)}, \quad 0 \leq s \leq 2\pi.$$

The inverse problem is solved by the proposed method with the following specifications: $N = 64$, $\delta t = 0.1$, the stopping test value is $\delta = 0.005$ and without any regularization. The initial guess is the circle of radius 1.

Denoting by $r$ the radial function obtained by the inverse resolution. To check the quality of estimation, the error of estimation defined by $\|r - r_*\|$ is introduced. All computations are performed on a PC with Processor 1.66Ghz and RAM 512Mb.

By considering trigonometric polynomials of degree $K = 3$, the estimation algorithm converges after 4 iterations in a computing time of 61.9s. The error obtained is 0.0305. When the degree becomes $K = 4$, the error obtained after convergence at iteration 4 is 0.0061. This convergence necessitates 85,76s as computing time.

Table 1 presents the values of the objective function $J$, given by (4), for various numbers of iterations.

| Iterations | $J$      |
|------------|----------|
| 0          | 21.448   |
| 1          | 1.720    |
| 2          | 0.0997   |
| 3          | 0.00599  |
| 4          | 7.64 $10^{-6}$ |
While comparing these results with those obtained by [5](0.0058 for \(K = 4\)), it can be concluded that the proposed identification method is more accurate than the method of [5]. When a random noise of level 1% is added to the exact data \(\Phi\), the convergence of the algorithm is possible and the obtained relative error is 0.0069 \((K = 4)\). Figure 1 shows good agreement between exact and estimated boundaries in this noisy case.

4.2. General heat equation

In this subsection, the method under investigation is tested in a general situation where the heat source \(f\) and the initial condition are non homogenous. As expressions for \(f\) and \(u_0\), we take \(f(x, t) = (x^2 + y^2 - 0.25)(t + 1)\) and \(u_0(x) = x^2 + y^2 - 0.25\). On the exterior circle of radius 1.5, the applied heat flux is \(\varphi(x, t) = t^2 \exp(-4t + 2)\).

Here we consider three test curves, denoted by \(C_2, C_3\) and \(C_4\) and given respectively by:

\[
\begin{align*}
    r_{s,2}(s) &= (0.8 + 0.2 \cos(4s)) & 0 \leq s \leq 2\pi \\
    r_{s,3}(s) &= \frac{1 + 0.9 \cos s + 0.1 \sin 2s}{1 + 0.75 \cos s} & 0 \leq s \leq 2\pi \\
    r_{s,4}(s) &= \frac{2}{3}((\cos s)^{10} + (\frac{2}{3} \sin s)^{10})^{-0.1}, & 0 \leq s \leq 2\pi
\end{align*}
\]

The measurements \(\phi\) are generated with the same parameters of the previous subsection. However, in order to avoid the inverse crime phenomena resulting in using the same parameters to generate the measurements and to solve the inverse problem, we take in the inverse problem \(N = 80\). The regularizing parameter is \(\varepsilon = 0.01\), chosen by trial and error.

The first test curve \(C_2\) can be exactly retrieved by trigonometric polynomials of degree 4. After 3 iterations, we obtained the following approximation (neglecting coefficients under \(10^{-4}\)) \(r(s) \approx 0.797 + 0.198 \cos(4s),\quad 0 \leq s \leq 2\pi\). Moreover the obtained exact error is 0.0030. Figure 2.a compare the computed boundary and the exact one in this case.

Figures 2.b and 2.c show the estimated boundaries of the curve \(C_3\) respectively with \(K = 4\) and
$K = 6$. The errors obtained are 0.0176 and 0.0055. These errors are achieved respectively at iterations 3 and 4. The test curve $C_4$ is an example of a convex curve. The convergence occurs at iteration 3. The obtained error, when $K = 4$, is 0.0535 which is relatively higher compared to those obtained for all previous curves. This is related to the shape of $C_4$ which almost presents corners. The figure 2.d depicts the comparison between $C_4$ and its estimation obtained by taking $K = 6$. Figures 2.a, 2.b, 2.c, 2.d indicate good matching between the estimated curves and the exact curves.

We want to end this discussion by mentioning that if the distance between the measurements locations and the boundary to estimate increases the quality of the reconstruction decreases. Namely with the test curve $C_3$, if the radial function is divided by a factor 2, the error of estimation increases from 0.0176 (obtained after 3 iterations) to 0.0357 (after 5 iterations). This loss of precision can be compensated by augmenting the time interval of measurements.

5. Conclusion
In this paper, we have presented an algorithm to estimate an internal boundary from the temperature measurements performed on the accessible part of the boundary of the material. This algorithm is based on the finite element method and is implemented with the help of the FreeFem software. The main advantage of the proposed approach is conceptually simple to implement and as the numerical examples indicate leads to accurate estimations.

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Figure 2. Comparison between exact and computed boundaries.