One-parameter families containing three-dimensional toric Gorenstein singularities

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1 Introduction

(1.1) Let \( \sigma \) be a rational, polyhedral cone. It induces a (normal) affine toric variety \( Y_\sigma \) which may have singularities. We would like to investigate its deformation theory. The vector space \( T^1_{Y_\sigma} \) of infinitesimal deformations is multigraded, and its homogeneous pieces can be determined by combinatorial formulas developed in \cite{Al1}.
If \( Y_\sigma \) only has an isolated Gorenstein singularity, then we can say even more (cf. \cite{Al2}, \cite{Al3}): \( T^1_{Y_\sigma} \) is concentrated in one single multidegree, the corresponding homogeneous piece allows an elementary geometric description in terms of Minkowski summands of a certain lattice polytope, and it is even possible (cf. \cite{Al4}) to obtain the entire versal deformation of \( Y_\sigma \).

(1.2) The first aim of the present paper is to provide a geometric interpretation of the \( T^1_{Y_\sigma} \)-formula for arbitrary toric singularities in every multidegree. This can be done again in terms of Minkowski summands of certain polyhedra. However, they neither need to be compact, nor do their vertices have to be contained in the lattice anymore (cf. \cite{Al2}).
In \cite{Al2} we have studied so-called toric deformations only existing in negative (i.e. \( \in -\sigma^\vee \)) multidegrees. They are genuine deformations with smooth parameter space, and they are characterized by the fact that their total space is still toric. Now, having a new description of \( T^1_{Y_\sigma} \), we will describe in Theorem \ref{thm:main} the Kodaira-Spencer map in these terms.
Moreover, using partial modifications of our singularity \( Y_\sigma \), we extend in \ref{thm:partial_mod} the construction of genuine deformations to non-negative degrees. Despite the fact that the total spaces are no longer toric, we can still describe them and their Kodaira-Spencer map combinatorially.

(1.3) Afterwards, we focus on three-dimensional, toric Gorenstein singularities. As already mentioned, everything is known in the isolated case. However, as soon as \( Y_\sigma \) contains one-dimensional singularities (which then have to be of transversal type \( A_k \)), the situation changes dramatically. In general, \( T^1_{Y_\sigma} \) is spread into infinitely many multidegrees. Using our geometric description of the \( T^1 \)-pieces, we detect in \ref{thm:one_dim} all non-trivial ones and determine their dimension (which will be one in most cases). The easiest example of that kind is the cone over the weighted projective plane \( \mathbb{P}(1,2,3) \) (cf. \ref{ex:cone}).
At least at the moment, it seems to be impossible to describe the entire versal deformation; it is an infinite-dimensional space. However, the infinitesimal deformations corresponding to the one-dimensional homogeneous pieces of \( T^1_{Y_\sigma} \) are unobstructed, and we lift them in \ref{thm:unobstructed} to genuine one-parameter families. Since the corresponding multidegrees are in general non-negative, this can be done using the construction introduced in \ref{thm:construction}. See section \ref{sec:example} for a corresponding sequel of example \( \mathbb{P}(1,2,3) \).
Those one-parameter families form a kind of skeleton of the entire versal deformation. The most
important open questions are the following: Which of them belong to a common irreducible component of the base space? And, how could those families be combined to find a general fiber (a smoothing of $Y_\sigma$) of this component? The answers to these questions would provide important information about three-dimensional flips.

2 Visualizing $T^1$

(2.1) **Notation:** As usual when dealing with toric varieties, denote by $N, M$ two mutually dual lattices (i.e. finitely generated, free abelian groups), by $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$ their perfect pairing, and by $N_R, M_R$ the corresponding $\mathbb{R}$-vector spaces obtained by extension of scalars.

Let $\sigma \subseteq N_R$ be the polyhedral cone with apex in 0 given by the fundamental generators $a^1, \ldots, a^M \in N$. They are assumed to be primitive, i.e. they are not proper multiples of other elements from $N$. We will write $\sigma = \langle a^1, \ldots, a^M \rangle$.

The dual cone $\sigma^\vee := \{ r \in M_R \mid \langle \sigma, r \rangle \geq 0 \}$ is given by the inequalities assigned to $a^1, \ldots, a^M$. Intersecting $\sigma^\vee$ with the lattice $M$ yields a finitely generated semigroup. Denote by $E \subseteq \sigma^\vee \cap M$ its minimal generating set, the so-called Hilbert basis. Then, the affine toric variety $Y_\sigma := \text{Spec}\mathcal{O}[\sigma^\vee \cap M] \subseteq \mathcal{O}^E$ is given by equations assigned to the linear dependencies among elements of $E$. See [Od] for a detailed introduction into the subject of toric varieties.

(2.2) Most of the relevant rings and modules for $Y_\sigma$ are $M$-(multi)graded. So are the modules $T^1_Y$, which are important for describing infinitesimal deformations and their obstructions. Let $R \in M$, then in [Al 1] and [Al 3] we have defined the sets

$$E^R_j := \{ r \in E \mid \langle a^j, r \rangle < \langle a^j, R \rangle \} \quad (j = 1, \ldots, M).$$

They provide the main tool for building a complex $\text{span}(E^R)_\bullet$, of free Abelian groups with the usual differentials via

$$\text{span}(E^R)_{-k} := \bigoplus_{\tau < \sigma, \dim \tau = k} \text{span}(E^R_\tau) \quad \text{with} \quad E^R_0 := \bigcup_{j=1}^N E^R_j, \quad \text{and} \quad E^R_\tau := \bigcap_{a^j \in \tau} E^R_j \text{ for faces } \tau < \sigma.$$

**Theorem:** (cf. [Al 1], [Al 3]) For $i = 1$ and, if $Y_\sigma$ is additionally smooth in codimension two, also for $i = 2$, the homogeneous pieces of $T^1_Y$ in degree $-R$ are

$$T^1_Y(-R) = H^i \left( \text{span}(E^R)_\bullet \otimes_{\mathbb{Z}} \mathcal{O} \right).$$

In particular, to obtain $T^1_Y(-R)$, we need to determine the vector spaces $\text{span}_\sigma E^R_j$ and $\text{span}_\sigma E^R_{jk}$, where $a^j, a^k$ span a two-dimensional face of $\sigma$. The first one is easy to get:

$$\text{span}_\sigma E^R_j = \begin{cases} 0 & \text{if } \langle a^j, R \rangle \leq 0 \\ \langle a^j \rangle & \text{if } \langle a^j, R \rangle = 1 \\ M_{\sigma} & \text{if } \langle a^j, R \rangle \geq 2. \end{cases}$$

The latter is always contained in $(\text{span}_\sigma E^R_j) \cap (\text{span}_\sigma E^R_k)$ with codimension between 0 and 2. As we will see in the upcoming example, its actual size reflects the infinitesimal deformations of the two-dimensional cyclic quotient singularity assigned to the plane cone spanned by $a^j, a^k$. (These singularities are exactly the transversal types of the two-codimensional ones of $Y_\sigma$.)
(2.3) **Example:** If \( Y(n, q) \) denotes the two-dimensional quotient of \( \mathbb{R}^2 \) by the \( \mathbb{Z}_n/\mathbb{Z} \)-action via \( \left( \frac{\xi}{n} \right) \) (\( \xi \) is a primitive \( n \)-th root of unity), then \( Y(n, q) \) is a toric variety and may be given by the cone \( \sigma = \langle (1, 0); (-q, n) \rangle \subseteq \mathbb{R}^2 \). The set \( E \subseteq \sigma^\vee \cap \mathbb{Z}^2 \) consists of the lattice points \( r^1, \ldots, r^w \) along the compact faces of the boundary of \( \text{conv}(\sigma^\vee \setminus \{0\}) \cap \mathbb{Z}^2 \). There are integers \( a_v \geq 2 \) such that \( r^{v-1} + r^{v+1} = a_v r^v \) for \( v = 1, \ldots, w - 1 \). They may be obtained by expanding \( n/\langle n, q \rangle \) into a negative continued fraction (cf. [Od], §1(6)).

Assume \( w \geq 2 \), let \( a^1 = (1, 0) \) and \( a^2 = (-q, n) \). Then, there are only two sets \( E_1^R \) and \( E_2^R \) involved, and the previous theorem states

\[
T^1_Y(-R) = \left( \frac{\text{span}_E E_1^R \cap \text{span}_E E_2^R}{\text{span}_E (E_1^R \cap E_2^R)} \right)^*.
\]

Only three different types of \( R \in \mathbb{Z}^2 \) provide a non-trivial contribution to \( T^1_Y \):

(i) \( R = r^1 \) (or analogously \( R = r^{w-1} \)) : \( \text{span}_E E_1^R = (a^1)^\perp \), \( \text{span}_E E_2^R = \mathcal{Q}^2 \) (or \( (a^2)^\perp \), if \( w = 2 \), and \( \text{span}_E E_2^R \perp = 0 \). Hence, \( \dim T^1_Y(-R) = 1 \) (or \( = 0 \), if \( w = 2 \)).

(ii) \( R = r^v \) (\( 2 \leq v \leq w - 2 \)) : \( \text{span}_E E_1^R = \text{span}_E E_2^R = \mathcal{Q}^2 \), and \( \text{span}_E E_{12}^R = 0 \). Hence, we obtain \( \dim T^1_Y(-R) = 2 \).

(iii) \( R = p \cdot r^v \) (\( 1 \leq v \leq w - 1 \), \( 2 \leq p < a_v \) for \( v \geq 3 \); or \( v = 1 = w - 1 \), \( 2 \leq p \leq a_1 \) for \( w = 2 \)) : \( \text{span}_E E_1^R = \text{span}_E E_2^R = \mathcal{Q}^2 \), and \( \text{span}_E E_{12}^R = \mathcal{Q} \cdot R \). In particular, \( \dim T^1_Y(-R) = 1 \).

(2.4) **Definition:** For two polyhedra \( Q', Q'' \subseteq \mathbb{R}^n \) we define their Minkowski sum as the polyhedron \( Q' + Q'' := \{ p' + p'' \mid p' \in Q', p'' \in Q'' \} \). Obviously, this notion also makes sense for translation classes of polyhedra in arbitrary affine spaces.

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Every polyhedron \( Q \) is decomposable into the Minkowski sum \( Q = Q^c + Q^\infty \) of a (compact) polytope \( Q^c \) and the so-called cone of unbounded directions \( Q^\infty \). The latter one is uniquely determined by \( Q \), whereas the compact summand is not. However, we can take for \( Q^c \) the minimal one - given as the convex hull of the vertices of \( Q \) itself. If \( Q \) was already compact, then \( Q^c = Q \) and \( Q^\infty = 0 \).

A polyhedron \( Q' \) is called a Minkowski summand of \( Q \) if there is a \( Q'' \) such that \( Q = Q' + Q'' \) and if, additionally, \( (Q')^\infty = Q^\infty \).

In particular, Minkowski summands always have the same cone of unbounded directions and, up to dilatation (the factor 0 is allowed), the same compact edges as the original polyhedron.

(2.5) **The setup for the upcoming sections** is the following: Consider the cone \( \sigma \subseteq N_\mathbb{R} \) and fix some element \( R \in M \). Then \( A(R) := \{ R = 1 \} := \{ a \in N_\mathbb{R} \mid \langle a, R \rangle = 1 \} \subseteq N_\mathbb{R} \) is an affine space; if \( R \) is primitive, then it comes with a lattice \( L(R) := \{ R = 1 \} \cap N \). The assigned vector space is \( A_0(R) := \{ R = 0 \} \); it is always equipped with the lattice \( L_0(R) := \{ R = 0 \} \cap N \). We define the cross cut of \( \sigma \) in degree \( R \) as the polyhedron

\[
Q(R) := \sigma \cap \{ R = 1 \} \subseteq A(R).
\]

It has the cone of unbounded directions \( Q(R)^\infty = \sigma \cap A_0(R) \subseteq N_\mathbb{R} \). The compact part \( Q(R)^c \) is given by its vertices \( \tilde{a}^j := a^j/\langle a^j, R \rangle \), with \( j \) meeting \( \langle a^j, R \rangle \geq 1 \). A trivial but nevertheless
important observation is the following: The vertex $\bar{a}^j$ is a lattice point (i.e. $\bar{a}^j \in L(R)$), if and only if $\langle \bar{a}^j, R \rangle = 1$.

Fundamental generators of $\sigma$ contained in $R^\perp$ can still be “seen” as edges in $Q(R)^\infty$, but those with $\langle \bullet, R \rangle < 0$ are "invisible" in $Q(R)$. In particular, we can recover the cone $\sigma$ from $Q(R)$ if and only if $R \in \sigma^\perp$.

\[(2.6)\] Denote by $d^1, \ldots, d^N \in R^\perp \subseteq N\mathbb{R}$ the compact edges of $Q(R)$. Similar to [Al 4], §2, we assign to each compact 2-face $\varepsilon < Q(R)$ its sign vector $\varepsilon \in \{0, \pm 1\}^N$ by

$$\varepsilon_i := \begin{cases} 
\pm 1 & \text{if } d^i \text{ is an edge of } \varepsilon \\
0 & \text{otherwise}
\end{cases}$$

such that the oriented edges $\varepsilon_i \cdot d^i$ fit into a cycle along the boundary of $\varepsilon$. This determines $\varepsilon$ up to sign, and any choice will do. In particular, $\sum_i \varepsilon_i d^i = 0$.

**Definition:** For each $R \in M$ we define the vector spaces

\[
V(R) := \{ (t_1, \ldots, t_N) | \sum_i t_i \varepsilon_i d^i = 0 \text{ for every compact 2-face } \varepsilon < Q(R) \}
\]

\[
W(R) := \mathbb{I}_R^{\#(Q(R))} \cdot \mathbb{R}^N_{\geq 0} \text{vertices not belonging to } N.
\]

Measuring the dilatation of each compact edge, the cone $C(R) := V(R) \cap \mathbb{I}_R^{\#(Q(R))} \cdot \mathbb{R}^N_{\geq 0}$ parametrizes exactly the Minkowski summands of positive multiples of $Q(R)$. Hence, we will call elements of $V(R)$ “generalized Minkowski summands”; they may have edges of negative length. (See [Al 4], Lemma (2.2) for a discussion of the compact case.) The vector space $W(R)$ provides coordinates $s_j$ for each vertex $\bar{a}^j \in Q(R) \setminus N$, i.e. $\langle a^j, R \rangle \geq 2$.

\[(2.7)\] To each compact edge $d^{jk} = \bar{a}^j \bar{a}^k$ we assign a set of equations $G_{jk}$ which act on elements of $V(R) \oplus W(R)$. These sets are of one of the following three types:

1. $G_{jk} = \{ s_j - s_k = 0 \}$ provided both coordinates exist in $W(R)$, set $G_{jk} = \emptyset$ otherwise, or
2. $G_{jk} = \{ t_{jk} - s_j = 0, t_{jk} - s_k = 0 \}$, dropping equations that do not make sense.

Restricting $V(R) \oplus W(R)$ to the (at most) three coordinates $t_{jk}, s_j, s_k$, the actual choice of $G_{jk}$ is made such that these equations yield a subspace of dimension $1 + \dim T^1_{\langle a^j, a^k \rangle}(-R)$. Notice that the dimension of $T^1(-R)$ for the two-dimensional quotient singularity assigned to the plane cone $\langle a^j, a^k \rangle$ can be obtained from Example [3].

**Theorem:** The infinitesimal deformations of $Y_\sigma$ in degree $-R$ equal

$$T^1_Y(-R) = \{ (\underline{L}, \underline{s}) \in V_\sigma(R) \oplus W_\sigma(R) | \langle \underline{L}, \underline{s} \rangle \text{ fulfills the equations } G_{jk} \} / \mathfrak{G} \cdot (1, 1).$$

In some sense, the vector space $V(R)$ (encoding Minkowski summands) may be considered the main tool to describe infinitesimal deformations. The elements of $W(R)$ can (depending on the type of the $G_{jk}$’s) be either additional parameters, or they provide conditions excluding Minkowski summands not having some prescribed type.

If $Y$ is smooth in codimension two, then $G_{jk}$ is always of type (2). In particular, the variables $\underline{s}$ are completely determined by the $\underline{L}$’s, and we obtain the

**Corollary:** If $Y$ is smooth in codimension two, then $T^1_Y(-R)$ is contained in $V_\sigma(R) / \mathfrak{G} \cdot (1)$. It is built from those $\underline{s}$ such that $t_{jk} = t_{kl}$ whenever $d^{jk}, d^{kl}$ are compact edges with a common non-lattice vertex $\bar{a}^k$ of $Q(R)$.
Thus, $T^1_Y(-R)$ equals the set of equivalence classes of those Minkowski summands of $R_{\geq 0} \cdot Q(R)$ that preserve up to homothety the stars of non-lattice vertices of $Q(R)$.

(2.8) **Proof:** (of previous theorem)

**Step 1:** From Theorem (2.2) we know that $T^1_Y(-R)$ equals the complexification of the cohomology of the complex

$$N_{RI} \to \bigoplus \left( \text{span}_{RI} E^R_j \right)^* \to \bigoplus_{\langle a^j, a^k \rangle} \left( \text{span}_{RI} E^R_j \right)^* .$$

According to (2.2), elements of $\bigoplus \left( \text{span}_{RI} E^R_j \right)^*$ can be represented by a family of

$$b^j \in N_{RI} \quad \text{if} \quad \langle a^j, R \rangle \geq 2 \quad \text{and} \quad b^j \in N_{RI} / R \cdot a^j \quad \text{if} \quad \langle a^j, R \rangle = 1 .$$

Dividing by the image of $N_{RI}$ means to shift this family by common vectors $b \in N_{RI}$. On the other hand, the family $\{ b^j \}$ has to map onto 0 in the complex, i.e. for each compact edge $\bar{\omega}^j, \bar{\omega}^k < Q$ the functions $b^j$ and $b^k$ are equal on $\text{span}_{RI} E^R_j$. Since

$$\langle a^j, a^k \rangle \subseteq \text{span}_{RI} E^R_j \subseteq \left( \text{span}_{RI} E^R_j \right) \cap \left( \text{span}_{RI} E^R_j \right),$$

we immediately obtain the necessary condition $b^j - b^k \in I_R a^j + I_R a^k$. However, the actual behavior of $\text{span}_{RI} E^R_j$ will require a closer look (in the third step).

**Step 2:** We introduce new “coordinates”:

- $\bar{b}^j := b^j - \langle b^j, R \rangle \bar{a}^j \in R^\perp$, being well defined even in the case $\langle a^j, R \rangle = 1$;
- $s_j := -\langle b^j, R \rangle$ for $j$ meeting $\langle a^j, R \rangle \geq 2$ (inducing an element of $W(R)$).

The shift of the $b^j$ by an element $b \in N_{RI}$ (i.e. $(b^j)' = b^j + b$) appears in these new coordinates as

$$\begin{align*}
(b^j)' & = (b^j)' - \langle (b^j)', R \rangle \bar{a}^j = b^j + b - \langle b^j, R \rangle \bar{a}^j - \langle b, R \rangle \bar{a}^j \\
 & = \bar{b}^j + b - \langle b, R \rangle \bar{a}^j , \\
s_j' & = -(b^j)' , R = s_j - \langle b, R \rangle .
\end{align*}$$

In particular, an element $b \in R^\perp$ does not change the $s_j$, but shifts the points $\bar{b}^j$ inside the hyperplane $R^\perp$. Hence, the set of $\bar{b}^j$ should be considered modulo translation inside $R^\perp$ only. On the other hand, the condition $b^j - b^k \in I_R a^j + I_R a^k$ changes into $\bar{b}^j - \bar{b}^k \in I_R \bar{a}^j + I_R \bar{a}^k$ or even $\bar{b}^j - \bar{b}^k \in I_R (\bar{a}^j - \bar{a}^k)$ (consider the values of $R$). Hence, the $\bar{b}^j$’s form the vertices of an at least generalized Minkowski summand of $Q(R)$. Modulo translation, this summand is completely described by the dilatation factors $t_{jk}$ obtained from

$$\bar{b}^j - \bar{b}^k = t_{jk} \cdot (\bar{a}^j - \bar{a}^k) .$$

Now, the remaining part of the action of $b \in N_{RI}$ comes down to an action of $\langle b, R \rangle \in R$ only:

$$\begin{align*}
t_{jk}' & = t_{jk} - \langle b, R \rangle \quad \text{and} \\
s_j' & = s_j - \langle b, R \rangle , \text{ as we already know}.
\end{align*}$$

Up to now, we have found that $T^1_Y(-R) \subseteq V_\sigma(R) \oplus W_\sigma(R) / (1, 1)$.

**Step 3:** Actually, the elements $b^j$ and $b^k$ have to be equal on $\text{span}_{RI} E^R_{jk}$, which may be a larger space than just $(a^j, a^k)^\perp$. To measure the difference we consider the factor $\text{span}_{RI} E^R_{jk} / (a^j, a^k)^\perp$ contained in the two-dimensional vector space $M_{RI} / (a^j, a^k)^\perp = \text{span}_{RI} (a^j, a^k)^*$. Since this factor coincides with the set $\text{span}_{RI} E^R_{jk}$ assigned to the two-dimensional cone $(a^j, a^k) \subseteq \text{span}_{RI} (a^j, a^k)$,
where \( \bar{R} \) denotes the image of \( R \) in \( \text{span}_{R}(a^j, a^k)^* \), we may assume that \( \sigma = \langle a^1, a^2 \rangle \) (i.e. \( j = 1, k = 2 \)) represents a two-dimensional cyclic quotient singularity. In particular, we only need to discuss the three cases (i)-(iii) from Example (3.3).

In (i) and (ii) we have \( \text{span}_{R^1}E^{\bar{R}}_{12} = 0 \), i.e. no additional equation is needed. This means \( G_{12} = \emptyset \) is of type (0). On the other hand, if \( T^0_1 = \emptyset \), then the vector space \( R^1_{(12)} \)(\( \emptyset, \emptyset, \emptyset \)) has to be killed by identifying the three variables \( t_{12}, s_1, \) and \( s_2 \); we obtain type (2).

Case (iii) provides \( \text{span}_{R^1}E^{\bar{R}}_{12} = R \cdot R \). Hence, as an additional condition we obtain that \( b^1 \) and \( b^2 \) have to be equal on \( R \). By the definition of \( s_j \) this means \( s_1 = s_2 \), and \( G_{12} \) has to be of type (1). \( \square \)

### 3 Genuine deformations

(3.1) In [Al 2], we have studied so-called toric deformations in a given multidegree \( -R \in M \). They are genuine deformations in the sense that they are defined over smooth parameter spaces; they are characterized by the fact that the total spaces together with the embedding of the special fiber still belong to the toric category. Despite the fact they look so special, it seems that toric deformations cover a big part of the versal deformation of \( Y_\sigma \). They do only exist in negative degrees (i.e. \( R \in \sigma^\vee \cap M \)), but here they form a kind of skeleton. If \( Y_\sigma \) is an isolated toric Gorenstein singularity, then toric deformations even provide all irreducible components of the versal deformation (cf. [Al 2]).

After a quick reminder of the idea of this construction, we describe the Kodaira-Spencer map of toric deformations in terms of the new \( T^1_1 \)-formula presented in (Al 3). It is followed by the investigation of non-negative degrees: If \( R \notin \sigma^\vee \cap M \), then we are still able to construct genuine deformations of \( Y_\sigma \); but they are no longer toric.

(3.2) Let \( R \in \sigma^\vee \cap M \). Then, following [Al 2] §3, toric \( m \)-parameter deformations of \( Y_\sigma \) in degree \( -R \) correspond to splittings of \( Q(R) \) into a Minkowski sum

\[
Q(R) = Q_0 + Q_1 + \ldots + Q_m
\]

meeting the following conditions:

(i) \( Q_0 \subseteq A(R) \) and \( Q_1, \ldots, Q_m \subseteq A_0(R) \) are polyhedra with \( Q(R) \in \infty \) as their common cone of unbounded directions.

(ii) Each supporting hyperplane \( t \) of \( Q(R) \) defines faces \( F(Q_0, t), \ldots, F(Q_m, t) \) of the indicated polyhedra; their Minkowski sum equals \( F(Q(R), t) \). With at most one exception (depending on \( t \)), these faces should contain lattice vertices, i.e. vertices belonging to \( N \).

**Remark:** In [Al 2], we have distinguished between the case of primitive and non-primitive elements \( R \in M \); if \( R \) is a multiple of some element of \( M \), then \( A(R) \) does not contain lattice points at all. In particular, condition (ii) just means that \( Q_1, \ldots, Q_m \) have to be lattice polyhedra. On the other hand, for primitive \( R \), the \( (m+1) \) summmands \( Q_i \) have equal rights and may be put into the same space \( A(R) \). Then, their Minkowski sum has to be interpreted inside this affine space.

If a Minkowski decomposition is given, how do we obtain the assigned toric deformation? Defining \( \hat{N} := N \oplus \mathbb{Z}^m \) (and \( \hat{M} := M \oplus \mathbb{Z}^m \)), we have to embed the summmands as \( (Q_0, 0), (Q_1, e^1), \ldots, (Q_m, e^m) \) into the vector space \( \hat{N}_R \); \( \{e^1, \ldots, e^m\} \) denotes the canonical basis of \( \mathbb{Z}^m \). Together with \( (Q(R))^{\infty} \), these polyhedra generate a cone \( \hat{\sigma} \subseteq \hat{N} \) containing \( \sigma \) via \( N \hookrightarrow \hat{N}, a \mapsto (a; (a, 1, R), \ldots, a, R) \). Actually, \( \sigma \) equals \( \hat{\sigma} \cap N_R \), and we obtain an inclusion \( Y_\sigma \hookrightarrow X_{\hat{\sigma}} \) between the associated toric varieties.

On the other hand, \( [R, 0] : \hat{N} \to \mathbb{Z}^m \) and \( \text{pr}_m : \hat{N} \to \mathbb{Z}^m \) induce regular functions \( f : X_{\hat{\sigma}} \to \mathbb{C}^m \) and \( (f^1, \ldots, f^m) : X_{\hat{\sigma}} \to \mathbb{C}^m \), respectively. The resulting map \( (f^1-f, \ldots, f^m-f) : X_{\hat{\sigma}} \to \mathbb{C}^m \) is
flat and has $Y_\sigma \to X_\bar{\sigma}$ as special fiber.

(3.3) Let $R \in \sigma^\vee \cap M$ and $Q(R) = Q_{0} + \ldots + Q_{m}$ be a decomposition satisfying (i) and (ii) mentioned above. Denote by $(\bar{\alpha})_i$ the vertex of $Q_i$ induced from $\bar{\alpha}^j \in Q(R)$, i.e. $\bar{\alpha}^j = (\bar{\alpha})_0 + \ldots + (\bar{\alpha})_m$.

**Theorem:** The Kodaira-Spencer map of the corresponding toric deformation $X_{\bar{\sigma}} \to \mathfrak{A}^m$ is

$$\varrho: \mathfrak{A}^m \to T_{\mathfrak{A}^m} = T_{\mathfrak{A}^m}^{1}(-R) \subseteq V_{\mathfrak{A}}(R) \oplus W_{\mathfrak{A}}(R)/\mathfrak{A} \cdot \{1, 1\}$$

sending $e^i$ onto the pair $[Q_i, \bar{s}^i] \in V(R) \oplus W(R)$ $(i = 1, \ldots, m)$ with

$$s^i_j := \begin{cases} 0 & \text{if the vertex } (\bar{\alpha})_i \text{ of } Q_i \text{ belongs to the lattice } N \\ 1 & \text{if } (\bar{\alpha})_i \text{ is not a lattice point.} \end{cases}$$

**Remark:** Setting $e^0 := -(e^1 + \ldots + e^m)$, we obtain $\varrho(e^0) = [Q_0, \bar{s}^0]$ with $\bar{s}^0$ defined similar to $s^i$ in the previous theorem.

(3.4) **Proof** (of previous theorem): We would like to derive the above formula for the Kodaira-Spencer map from the more technical one presented in \cite{Al2}, Theorem (5.3). Under additional use of \cite{Al2} (6.1), the latter one describes $\varrho(e^i) \in T^1_{\mathfrak{A}}(-R) = H^1(\text{span}_\mathbb{Q}(E^R))$ in the following way:

Let $E = \{r^1, \ldots, r^w\} \subseteq \sigma^\vee \cap M$. Its elements may be lifted via $\tilde{M} \to M$ to $\tilde{r}^v \in \tilde{\sigma}^\vee \cap \tilde{M}$ $(v = 1, \ldots, w)$; denote their $i$-th entry of the $\mathbb{Z}^m$-part by $\eta^i_v$, respectively. Then, given elements $v^j \in \text{span}E^R$, we may represent them as $v^j = \sum q^j_v r^v$ $(q^j_v \in \mathbb{Z}^{E^R})$, and $\varrho(e^i)$ assigns to $v^j$ the integer $-\sum q^j_v \eta^i_v$. Using our notation from \cite{Al3} for $\varrho(e^i)$, this means that $b^j$ sends elements $r^v \in E^R$ onto $-\eta^i_v \in \mathbb{Z}$.

By construction of $\bar{\sigma}$, we have inequalities

$$\langle (\bar{\alpha})_0, \tilde{r}^v \rangle \geq 0 \quad \text{and} \quad \langle (\bar{\alpha})_i, e^i \rangle, \tilde{r}^v \rangle \geq 0 \quad (i = 1, \ldots, m)$$

summing up to $\langle \bar{\alpha}^j, r^v \rangle = \langle (\bar{\alpha}^j), 1 \rangle, \tilde{r}^v \rangle \geq 0$. On the other hand, the fact $r^v \in E^R_j$ is equivalent to $\langle \bar{\alpha}^j, r^v \rangle < 1$. Hence, whenever $(\bar{\alpha})_i \in Q_i$ belongs to the lattice, the corresponding inequality $(i = 0, \ldots, m)$ becomes an equality. With at most one exception, this always has to be the case. Hence,

$$\langle (\bar{\alpha})_i, r^v \rangle + \eta^v_i = \begin{cases} 0 & \text{if } (\bar{\alpha})_i \in N \\ \langle \bar{\alpha}^j, r^v \rangle & \text{if } (\bar{\alpha})_i \notin N \end{cases} \quad (i = 1, \ldots, m)$$

meaning that $b^j = (\bar{\alpha})_i$ or $b^j = (\bar{\alpha})_i - \bar{\alpha}^j$, respectively. By the definitions of $\tilde{b}^j$ and $s^j$ given in \cite{Al3}, we are done.

(3.5) Now we treat the case of non-negative degrees; let $R \in M \setminus \sigma^\vee$. The easiest way to solve a problem is to change the question until there is no problem left. We can do so by changing our cone $\sigma$ into some $\tau^R$ such that the degree $-R$ becomes negative. We define

$$\tau := \tau^R := \sigma \cap [R \geq 0] \quad \text{that is} \quad \tau^\vee = \sigma^\vee + R \geq 0 \cdot R.$$

The cone $\tau$ defines an affine toric variety $Y_\tau$. Since $\tau \subseteq \sigma$, it comes with a map $g: Y_\tau \to Y_\sigma$, i.e. $Y_\tau$ is an open part of a modification of $Y_\sigma$. The important observation is

$$\tau \cap [R = 0] = \sigma \cap [R = 0] = Q(R)$$

and

$$\tau \cap [R = 1] = \sigma \cap [R = 1] = Q(R)$$
implying $T^1_{\chi}(-R) = T^1_{\chi}(-R)$ by Theorem (3.4). Moreover, even the genuine toric deformations $X_\tau \to A^m$ of $Y_\tau$ carry over to $m$-parameter (non-toric) deformations $X \to A^m$ of $Y_\sigma$:

**Theorem:** Each Minkowski decomposition $Q(R) = Q_0 + Q_1 + \ldots + Q_m$ satisfying (i) and (ii) of (3.3) provides an $m$-parameter deformation $X \to A^m$ of $Y_\sigma$. Via some birational map $\tilde{g} : X_\tau \to X$ it is compatible with the toric deformation $X_\tau \to A^m$ of $Y_\tau$ presented in (3.3).

\[
\begin{array}{ccc}
X_\tau & \xrightarrow{\tilde{g}} & X \\
\downarrow g & & \downarrow \pi \\
Y_\tau & \xrightarrow{\pi} & Z_{\tilde{\sigma}}
\end{array}
\]

The total space $X$ is not toric anymore, but it sits via birational maps between $X_\tau$ and some affine toric variety $Z_{\tilde{\sigma}}$ still containing $Y_\sigma$ as a closed subset.

**Proof:** First, we construct $N, \tilde{\tau} \subseteq \tilde{N}_R$ by the recipe of (3.3). In particular, $N$ is contained in $\tilde{N}$, and the projection $\pi : \tilde{M} \to M$ sends $[r; g_1, \ldots, g_m]$ onto $r + (\sum_i g_i) R$. Defining $\tilde{\sigma} := \tilde{\tau} + \sigma$ (hence $\tilde{\sigma} = \tilde{\tau} \cap \pi^{-1}(\sigma)$), we obtain the commutative diagram

\[
\begin{array}{ccc}
A[\tilde{\tau} \cap \tilde{M}] & \xrightarrow{\tilde{\pi}} & A[\tilde{\sigma} \cap \tilde{M}] \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
A[\tau \cap M] & \xrightarrow{\pi} & A[\sigma \cap M]
\end{array}
\]

with surjective vertical maps. The canonical elements $e_1, \ldots, e_m \in Z^m \subseteq \tilde{M}$ together with $[R; 0] \in \tilde{M}$ are preimages of $R \in M$. Hence, the corresponding monomials $x^{e_1}, \ldots, x^{e_m}, x^{[R, 0]}$ in the semigroup algebra $A[\tilde{\tau} \cap M]$ (called $f_1, \ldots, f_m, f$ in (3.3)) map onto $x^m \in A[\tau \cap M]$ which is not regular on $Y_\sigma$. We define $Z_{\tilde{\sigma}}$ as the affine toric variety assigned to $\tilde{\sigma}$ and $X$ as

\[X := \text{Spec } B\text{ with } B := A[\tilde{\sigma} \cap \tilde{M}][f_1 - f, \ldots, f_m - f] \subseteq A[\tilde{\tau} \cap \tilde{M}].\]

That means, $X$ arises from $X_\tau$ by eliminating all variables except those lifted from $Y_\sigma$ or the deformation parameters themselves. By construction of $B$, the vertical algebra homomorphisms $\pi$ induce a surjection $B \twoheadrightarrow A[\sigma \cap M]$.

**Lemma:** Elements of $A[\tilde{\tau} \cap \tilde{M}]$ may uniquely be written as sums

\[
\sum_{(v_1, \ldots, v_m) \in \Gamma^m} c_{v_1, \ldots, v_m} \cdot (f_1 - f)^{v_1} \cdot \ldots \cdot (f_m - f)^{v_m}
\]

with $c_{v_1, \ldots, v_m} \in A[\tilde{\tau} \cap \tilde{M}]$ such that $s - e_i \notin \tilde{\tau}$ for any of its monomial terms $x^s$. Moreover, those sums belong to the subalgebra $B$, if and only if their coefficients $c_{v_1, \ldots, v_m}$ do.

**Proof:** (a) Existence. Let $s - e_i \notin \tilde{\tau}$ for some $s, i$. Then, with $s' := s - e_i + [R, 0]$ we obtain

\[x^s = x^{s'} + x^{s-e_i} (x^{e_i} - x^[R,0]) = x^{s'} + x^{s-e_i} (f^i - f).
\]

Since $e_i = 1$ and $[R, 0] = 0$ if evaluated on $(Q_i, e^i) \subseteq \hat{\tau}$, this process eventually stops.

(b) $B$-Membership. For the previous reduction step we have to show that if $s \in A[\tilde{\tau} \cap \tilde{M}]$, then the same is true for $s'$ and $s - e_i$. Since $\pi(s') = \pi(s) \in \sigma'$, this is clear for $s'$. It remains to check that $\pi(s - e_i) \in \sigma'$. Let $a \in \sigma$ be an arbitrary test element; we distinguish two cases:

Case 1: $\langle a, R \rangle \geq 0$. Then $a$ belongs to the subcone $\tau$, and $\pi(s - e_i) \in \tau'$ yields $\langle a, \pi(s - e_i) \rangle \geq 0$. 

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Case 2: \( \langle a, R \rangle \leq 0 \). This fact implies \( \langle a, \pi(s-e_i) \rangle = \langle a, s \rangle - \langle a, R \rangle \geq \langle a, s \rangle \geq 0 \).

(c) Uniqueness. Let \( p := \sum c_{v_1,\ldots,v_m} (f^1 - f) v^1 \cdots (f^m - f) v^m \) (meeting the above conditions) be equal to 0 in \( \mathcal{O}[\tilde{\mathcal{V}} \cap \tilde{M}] \). Using the projection \( \pi : \tilde{M} \rightarrow M \), everything becomes \( M \)-graded. Since the factors \((f^i - f)\) are homogeneous (of degree \( R \)), we may assume this fact also for \( p \), hence for its coefficients \( c_{v_1,\ldots,v_m} \).

Claim: These coefficients are just monomials. Indeed, if \( s, s' \in \tilde{\mathcal{V}} \) had the same image via \( \pi \), then we could assume that some \( e_i \)-coordinate of \( s' \) would be smaller than that of \( s \). Hence, \( s - e_i \) would still be equal to \( s \) on \( (Q_0, 0) \) and on any \( (Q_j, e^j) \) \((j \neq i)\), but even greater or equal than \( s' \) on \( (Q_i, e^i) \). This would imply \( s - e_i \in \tilde{\mathcal{V}} \), contradicting our assumption for \( p \).

Say \( c_{v_1,\ldots,v_m} = \lambda_{v_1,\ldots,v_m} x^s \); we use the projection \( \tilde{M} \rightarrow \mathbb{Z}^m \) for carrying \( p \) into the ring \( \mathcal{O}[\mathbb{Z}^m] = \mathcal{O}[y_1^{\pm1}, \ldots, y_m^{\pm1}] \). The elements \( x^s, f^1, f \) map onto \( y^s, y_i, 1 \), respectively. Hence, \( p \) turns into

\[
\tilde{p} = \sum_{(v_1,\ldots,v_m) \in \mathbb{N}^m} \lambda_{v_1,\ldots,v_m} \cdot (y_1-1)^{v_1} \cdots (y_m-1)^{v_m} .
\]

By induction through \( \mathbb{N}^m \), we obtain that vanishing of \( \tilde{p} \) implies the vanishing of its coefficients: Replace \( y_i - 1 \) by \( z_i \), and take partial derivatives. \( \Box \)

Now, we can easily see that \( X \rightarrow \mathcal{O}^m \) is flat and has \( Y_\sigma \) as special fiber: The previous lemma means that for \( k = 0,\ldots, m \) we have inclusions

\[
B/(f^1 - f, \ldots, f^k - f) \hookrightarrow \mathcal{O}[\tilde{\mathcal{V}} \cap \tilde{M}]/(f^1 - f, \ldots, f^k - f) .
\]

The values \( k < m \) yield that \((f^1 - f, \ldots, f^m - f)\) forms a regular sequence even in the subring \( B \), meaning that \( X \rightarrow \mathcal{O}^m \) is flat. With \( k = m \) we obtain that the surjective map \( B/(f^1 - f, \ldots, f^m - f) \rightarrow \mathcal{O}[\sigma^\vee \cap M] \) is also injective. \( \Box \)

4 Three-dimensional toric Gorenstein singularities

(4.1) By [Ish], Theorem (7.7), toric Gorenstein singularities always arise from the following construction: Assume we are given a lattice polytope \( P \subseteq \mathbb{R}^n \). We embed the whole space (including \( P \)) into height one of \( N_{\mathbb{R}} := \mathbb{R}^n \oplus \mathbb{R} \) and take for \( \sigma \) the cone generated by \( P \); denote by \( M_{\mathbb{R}} := (\mathbb{R}^n)^* \oplus \mathbb{R} \) the dual space and by \( N, M \) the natural lattices. Our polytope \( P \) may be recovered from \( \sigma \) as

\[
P = Q(R^*) \subseteq A(R^*) \quad \text{with} \quad R^* := [0, 1] \in M .
\]

The fundamental generators \( a^1, \ldots, a^M \in \mathbb{L}(R^*) \) of \( \sigma \) coincide with the vertices of \( P \). (This involves a slight abuse of notation; we use the same symbol \( a^i \) for both \( a^i \in \mathbb{Z}^n \) and \((a^1, 1) \in M \).) If \( a^1a^2 \) forms an edge of the polytope, we denote by \( \ell(j, k) \in \mathbb{Z} \) its “length” induced from the lattice structure \( \mathbb{Z}^n \subseteq \mathbb{R}^n \). Every edge provides a two-codimensional singularity of \( Y_\sigma \) with transversal type \( A_{\ell(j, k)} - 1 \). In particular, \( Y_\sigma \) is smooth in codimension two if and only if all edges of \( P \) are
primitive, i.e. have length \( \ell = 1 \).

(4.2) As usual, we fix some element \( R \in M \). From (2.10) we know what the vector spaces \( V(R) \) and \( W(R) \) are; we introduce the subspace

\[
V'(R) := \{ \underline{a} \in V(R) \mid t_{jk} \neq 0 \implies 1 \leq \langle a^j, R \rangle = \langle a^k, R \rangle \leq \ell(j,k) \}
\]

representing Minkowski summands of \( Q(R) \) that have killed any compact edge not meeting the condition \( \langle a^j, R \rangle = \langle a^k, R \rangle \leq \ell(j,k) \).

**Theorem:** For \( T^1_\gamma(-R) \), there are two different types of \( R \in M \) to distinguish:

(i) If \( R \leq 1 \) on \( P \) (or equivalently \( \langle a^j, R \rangle \leq 1 \) for \( j = 1, \ldots, M \)), then \( T^1_\gamma(-R) = V_\partial(R)/(1) \).

Moreover, concerning Minkowski summands, we may replace the polyhedron \( Q(R) \) by its compact part \( P \cap [R = 1] \) (being a face of \( P \)).

(ii) If \( R \) does not satisfy the previous condition, then \( T^1_\gamma(-R) = V'(R) \).

**Proof:** The first case follows from Theorem (2.10) just because \( W(R) = 0 \). For (ii), let us assume there are vertices \( a^j \) contained in the affine half space \( [R \geq 2] \). They are mutually connected inside this half space via paths along edges of \( P \).

The two-dimensional cyclic quotient singularities corresponding to edges \( a^j a^k \) of \( P \) are Gorenstein themselves. In the language of Example (4.11) this means \( w = 2 \), and we obtain

\[
\dim T^1_{(a^j,a^k)}(-R) = \begin{cases} 1 & \text{if } \langle a^j, R \rangle = \langle a^k, R \rangle = 2, \ldots, \ell(j,k) \text{ (case (iii) in (4.11))} \\ 0 & \text{otherwise.} \end{cases}
\]

In particular, \( T^1_{(a^j,a^k)}(-R) \) cannot be two-dimensional, and (using the notation of (2.10)) the equations \( s_j - s_k = 0 \) belong to \( G_{jk} \) whenever \( \langle a^j, R \rangle, \langle a^k, R \rangle \geq 2 \). This means for elements of

\[
T^1_\gamma \subseteq \left( V_\partial(R) \oplus W_\partial(R) \right)/\mathcal{A} \cdot (1,1)
\]

that all entries of the \( W_\partial(R) \)-part have to be mutually equal, or even zero after dividing by \( \mathcal{A} \cdot (1,1) \). Moreover, if not both \( \langle a^j, R \rangle \) and \( \langle a^k, R \rangle \) equal one, vanishing of \( T^1_{(a^j,a^k)}(-R) \) implies that \( G_{jk} \) also contains the equation \( t_{jk} - s_* = 0 \).

**Corollary:** Condition (4.11)/(ii) to build genuine deformations becomes easier for toric Gorenstein singularities: \( Q_1, \ldots, Q_m \) just have to be lattice polyhedra.

**Proof:** If \( R \leq 1 \) on \( P \), then \( Q(R) \) itself is a lattice polyhedron. Hence, condition (ii) automatically comes down to this simpler form.

In the second case, there is some \( W(R) \)-part involved in \( T^1_\gamma(-R) \). On the one hand, it indicates via the Kodaira-Spencer map which vertices of which polyhedron \( Q_\gamma \) belong to the lattice. On the other, we have observed in the previous proof that the entries of \( W(R) \) are mutually equal. This implies exactly our claim. \( \square \)

(4.3) In accordance with the title of the section, we focus now on *plane lattice polygons* \( P \subseteq \mathbb{R}^2 \). The vertices \( a^1, \ldots, a^M \) are arranged in a cycle. We denote by \( d^j := a^{j+1} - a^j \in \mathbb{L}_0(R^*) \) the edge going from \( a^j \) to \( a^{j+1} \), and by \( \ell(j) := \ell(j,j+1) \) its length (\( j \in \mathbb{Z}/M \mathbb{Z} \)).

Let \( s^1, \ldots, s^M \) be the fundamental generators of the dual cone \( \sigma^* \) such that \( \sigma \cap (s^j)^\perp \) equals the face spanned by \( a^j, a^{j+1} \in \sigma \). In particular, skipping the last coordinate of \( s^j \) yields the (primitive) inner normal vector at the edge \( d^j \) of \( P \).
Remark: Just for convenience of those who prefer living in $M$ instead of $N$, we show how to see
the integers $\ell(j)$ in the dual world: Choose a fundamental generator $s^j$ and denote by $r, r' \in M$
the closest (to $s^j$) elements from the Hilbert bases of the two adjacent faces of $\sigma^\vee$, respectively.
Then, $\{R^*, s^j\}$ together with either $r$ or $r'$ form a basis of the lattice $M$, and $(r + r') - \ell(j) R^*$
is a positive multiple of $s^j$. See the figure in (4.3).

In the very special case of plane lattice polygons (or three-dimensional toric Gorenstein singularities), we can describe $T_Y^1$ and the genuine deformations (for fixed $R \in M$) explicitly. First, we can
easily spot the degrees carrying infinitesimal deformations:

**Theorem:** In general (see the upcoming exceptions), $T_Y^1(-R)$ is non-trivial only for
\begin{enumerate}
\item $(1)$ $R = R^*$ with $\dim T_Y^1(-R) = M - 3$;
\item $(2)$ $R = q R^*$ ($q \geq 2$) with $\dim T_Y^1(-R) = \max \{0, \# \{q \mid \ell(q) \leq 0\}\} - 2$, and
\item $(3)$ $R = q R^* + p s^j$ with $2 \leq q \leq \ell(j)$ and $p \in \mathbb{Z}$ sufficiently large such that $R \notin \text{int}(\sigma^\vee)$. In this
case, $T_Y^1(-R)$ is one-dimensional.
\end{enumerate}

Additional degrees exist only in the following two (overlapping) exceptional cases:
\begin{enumerate}
\item $(4)$ Assume $P$ contains a pair of parallel edges $d^j, d^k$, both longer than every other edge. Then
$\dim T_Y^1(-q R^*) = 1$ for $\max \{|\ell(l)| l \neq j, k\} < q \leq \min \{\ell(j), \ell(k)\}$.
\item $(5)$ Assume $P$ contains a pair of parallel edges $d^j, d^k$ with distance $d$ ($d := \langle a^j, s^k \rangle = \langle a^k, s^j \rangle$). If
$\ell(k) > d$ $(\geq \max \{|\ell(l)| l \neq j, k\})$, then $\dim T_Y^1(-R) = 1$ for $R = q R^* + p s^j$ with $1 \leq q \leq \ell(j)$
and $1 \leq p \leq (\ell(k) - q)/d$.
\end{enumerate}

The cases (1), (2), (4), and (5) yield at most finitely many (negative) $T_Y^1$-degrees. Type (3) consists
of $\ell(j) - 1$ infinite series to any vertex $a^j \in P$, respectively; up to maybe the leading elements ($R$
might sit on $\partial \sigma^\vee$), they contain only non-negative degrees.

**Proof:** The previous claims are straight consequences of Theorem (4.2). Hence, the following
short remark should be sufficient: The condition $\langle a^j, R \rangle = \langle a^{j+1}, R \rangle$ means $d^j \in R^\perp$. Moreover, if
$R \notin \mathbb{Z} \cdot R^*$, then there is at most one edge (or a pair of parallel ones) having this property. □

(4.4) **Example:** A typical example of a non-isolated, three-dimensional toric Gorenstein
singularity is the cone over the weighted projective space $\mathbb{P}(1, 2, 3)$. We will use it to demonstrate
our calculations of $T^1$ as well as the upcoming construction of genuine one-parameter families. $P$
has the vertices $(-1, -1), (2, -1), (-1, 1)$, i.e. $\sigma$ is generated from

$$a^1 = (-1, -1; 1), \quad a^2 = (2, -1; 1), \quad a^3 = (-1, 1; 1).$$

Since our singularity is a cone over a projective variety, $\sigma^\vee$ appears as a cone over some lattice
polygon, too. Actually, in our example, $\sigma$ and $\sigma^\vee$ are even isomorphic. We obtain

$$\sigma^\vee = \langle s^1, s^2, s^3 \rangle \quad \text{with} \quad s^1 = [0, 1; 1], \quad s^2 = [-2, -3; 1], \quad s^3 = [1, 0; 1].$$

The Hilbert basis $E \subseteq \sigma^\vee \cap \mathbb{Z}^3$ consists of these three fundamental generators together with

$$R^* = [0, 0; 1], \quad v^1 = [-1, -2; 1], \quad v^2 = [0, -1; 1], \quad w = [-1, -1; 1].$$
respectively.

Proposition: The cone one-parameter family in degree $-g$ constitutes genuine one-parameter deformations should be no problem: Just split the polygon and infinitesimal deformation. To show that they are unobstructed by describing how they lift to their degrees come in three series:

(i) $2R^* - p^s s^3$ with $p^s \geq 1$. Even the leading element $R^s = [-1, 0, 1]$ is not contained in $\sigma^\vee$.

(ii) $2R^* - p^\beta s^1$ with $p^\beta \geq 1$. The leading element equals $R^\beta = v^2 = [0, -1, 1]$ and sits on the boundary of $\sigma^\vee$.

(iii) $3R^* - p^\gamma s^1$ with $p^\gamma \geq 2$. The leading element is $R^\gamma = [0, -2, 1] \notin \sigma^\vee$.

Each degree belonging to type (3) (i.e. $R = qR^* - p s^j$ with $2 \leq q \leq \ell(j)$) provides an infinitesimal deformation. To show that they are unobstructed by describing how they lift to genuine one-parameter deformations should be no problem: Just split the polygon $Q(R)$ into a Minkowski sum meeting conditions (i) and (ii) of (3.2), then construct $\tilde{\sigma}, \tilde{\tau}$, and $(f^1 - f)$ as in (3.5) and (3.2). However, we prefer to present the result for our special case all at once by using new coordinates.

Let $P \subseteq A(R^*) = \mathbb{R}^2 \times \{1\} \subseteq \mathbb{R}^3 = N_{R^f}$ be a lattice polygon as in (4.3), let $R = qR^* - p s^1$ be as just mentioned. Then $\sigma, \tau \subseteq N_{R^f}$ are the cones over $P$ and $P \cap \{R \geq 0\}$, respectively, and the one-parameter family in degree $-R$ is obtained as follows:

**Proposition:** The cone $\tilde{\tau} \subseteq N_{R^f} \oplus R = \mathbb{R}^4$ is generated by the elements

(i) $(a, 0) - \langle a, R \rangle (0, 1)$, if $a \in P \cap \{R \geq 0\}$ runs through the vertices from the $R^2$-line until $a^j$,

(ii) $(a, 0) - \langle a, R \rangle (d^j/\ell(j), 1)$, if $a \in P \cap \{R \geq 0\}$ runs from $a^{j+1}$ until the $R^1$-line again, and

(iii) $(0, 1)$ and $(d^j/\ell(j), 1)$.

The vector space $N_{R^f}$ containing $\sigma$ sits in $N_{R^f} \oplus R$ as $N_{R^f} \times \{0\}$. Via this embedding, one obtains $\tilde{\sigma} = \tilde{\tau} + \sigma$ as usual. The monomials $f$ and $f^1$ are given by their exponents $[R, 0], [R, 1] \in M \oplus \mathbb{Z}$, respectively.
Since $\tau$ is a lift of $\sigma$ if

$$q/\ell(j) \cdot a^j a^{j+1}$$

and some top on the $R^\perp$-line; take $-\langle \cdot, R \rangle$ as an additional, fourth coordinate. Then, $[R^*, 0]$ is still 1 on $P'$ and equals 0 on $I$. Moreover, $[R, 0]$ vanishes on $I$ and on the $R^\perp$-edge of $P'$; $[R, 1]$ vanishes on the whole $P'$.

**Proof:** We change coordinates. If $g := \gcd(p, q)$ denotes the “length” of $R$, then we can find an $s \in M$ such that $\{s, R/g\}$ forms a basis of $M \cap (d^j)_\perp$. Adding some $r \in M$ with $\langle d^j / \ell(j), r \rangle = 1$ ($r$ from Remark (4.3) will do) yields a $\mathbb{Z}$-basis for the whole lattice $M$. We consider the following commutative diagram:

$$
\begin{array}{c}
\stepcounter{equation}
\tag{4.6}
N \xrightarrow{(s,r,R/g)} \mathbb{Z}^3 \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
N \oplus \mathbb{Z} \xrightarrow{(\id, 0) \oplus [r, 0] \cdot [R/g, 0], [R, 1]} \mathbb{Z}^3 \oplus \mathbb{Z}
\end{array}
$$

The left hand side contains the data being relevant for our proposition. Carrying them to the right yields:

- $[0, 0, g] \in (\mathbb{Z}^3)^*$ as the image of $R$;
- $[0, 0, g, 0], [0, 0, 1] \in (\mathbb{Z}^4)^*$ as the images of $[R, 0]$ and $[R, 1]$, respectively;
- $\tau$ becomes a cone with affine cross cut
  $$Q(\langle 0, 0, g \rangle) = \conv \left( \langle a, s \rangle/\langle a, R \rangle; \langle a, r \rangle/\langle a, R \rangle; 1/g \right) \big| a \in P \cap [R \geq 0] ;$$
- $I$ changes into the unit interval $(Q_1, 1)$ reaching from $(0, 0, 0, 1)$ to $(0, 1, 0, 1)$;
- finally, cone($P'$) maps onto the cone spanned by the convex hull $(Q_0, 0)$ of the points $(\langle a, s \rangle/\langle a, R \rangle; \langle a, r \rangle/\langle a, R \rangle; 1/g; 0)$ for $a \in P \cap [R \geq 0]$ on the $a^j$-side and $(\langle a, s \rangle/\langle a, R \rangle; \langle a, r \rangle/\langle a, R \rangle - 1; 1/g; 0)$ for $a$ on the $a^{j+1}$-side, respectively.

Since $Q(\langle 0, 0, g \rangle)$ equals the Minkowski sum of the interval $Q_1 \subseteq A_0([0, 0, g])$ and the polygon $Q_0 \subseteq A([0, 0, g]),$ we are done by (3.3). \hfill \Box

To see how the original equations of the singularity $Y_s$ will be perturbed, it is useful to study first the dual cones $\tau^\vee$ or $\tilde{\tau}^\vee = \tau^\vee \cap \pi^{-1}(\sigma^\vee)$:

**Proposition:** If $s \in \sigma^\vee \cap M$, then the $(M \oplus \mathbb{Z})$-element

$$S := \begin{cases} 
 [s, 0] & \text{if } \langle d^j, s \rangle \geq 0 \\
 [s, -\langle d^j / \ell(j), s \rangle] & \text{if } \langle d^j, s \rangle \leq 0
\end{cases}
$$

is a lift of $s$ into $\tilde{\sigma}^\vee \cap (M \oplus \mathbb{Z})$. (Notice that it does not depend on $p, q$, but only on $j$.) Moreover, if $s^\vee$ runs through the edges of $P \cap [R \geq 0]$, the elements $S^\vee$ together with $[R, 0]$ and $[R, 1]$ form
versions of both equations yields the result.

Proof: Since we know \( \tilde{\tau} \) from the previous proposition, the calculations are straightforward and will be omitted.

\[ (4.7) \quad \text{Recall from (4.6) that } E \text{ denotes the minimal set generating the semigroup } \sigma^\vee \cap M. \]

To any \( s \in E \) there is a assigned variable \( z_s \), and \( Y_\sigma \subseteq \mathcal{A}^E \) is given by binomial equations arising from linear relations among elements of \( E \). Everything will be clear by considering an example: A linear relation such as \( s^1 + 2s^3 = s^2 + s^4 \) transforms into \( z_1 z_2^2 = z_2 z_4 \).

The fact that \( \sigma \) defines a Gorenstein variety (i.e. \( \sigma \) is a cone over a lattice polytope) implies that \( E \) consists only of \( R^* \) and elements of \( \partial \sigma^\vee \) including the fundamental generators \( s^\vee \). If \( E \cap \partial \sigma^\vee \) is ordered clockwise, then any two adjacent elements form together with \( R^* \) a \( \mathbb{Z} \)-basis of the three-dimensional lattice \( M \).

In particular, any three sequenced elements of \( E \cap \partial \sigma^\vee \) provide a unique linear relation among them and \( R^* \). (We met this fact already in Remark (4.3); there \( r, s^i \), and \( r' \) were those elements.) The resulting “boundary” equations do not generate the ideal of \( Y_\sigma \subseteq \mathcal{A}^E \). Nevertheless, for describing a deformation of \( Y_\sigma \), it is sufficient to know about perturbations of this subset only. Moreover, if one has to avoid boundary equations “overlapping” a certain spot on \( \partial \sigma^\vee \), then it will even be possible to drop up to two of them from the list.

![Diagram](image)

**Theorem:** The one-parameter deformation of \( Y_\sigma \) in degree \( -(q R^* - p s^j) \) is completely determined by the following perturbations:

(i) **(Boundary) equations involving only variables that are induced from \([d^j \geq 0] \subseteq \sigma^\vee \) remain unchanged. The same statement holds for \([d^j \leq 0] \).**

(ii) The boundary equation \( z_r z_{r'} - z^{(j)}_{R^*} z^{k}_{s} = 0 \) assigned to the triple \( \{r, s^j, r'\} \) is perturbed into \( (z_r z_{r'} - z^{(j)}_{R^*} z^{k}_{s}) - t z^{(j)-q}_{R^*} z^{k+p}_{s} = 0 \). Divide everything by \( z^{k}_{s} \) if \( k < 0 \).

Proof: Restricted to either \([d^j \geq 0] \) or \([d^j \leq 0] \), the map \( s \mapsto S \) lifting \( E \)-elements into \( \tilde{\sigma} \cap (M \oplus \mathbb{Z}) \) is linear. Hence, any linear relation remains true, and part (i) is proven.

For the second part, we consider the boundary relation \( r + r' = \ell(j) R^* + k s^j \) with a suitable \( k \in \mathbb{Z} \). By Lemma (4.10), the summands involved lift to the elements \([r, 0], [r', 1], [R^*, 0], \) and \([s^j, 0], \) respectively. In particular, the relation breaks down and has to be replaced by

\[
[r, 0] + [r', 1] = [R, 1] + (\ell(j) - q) [R^*, 0] + (k + p) [s^j, 0]
\]

and

\[
\ell(j) [R^*, 0] + k [s^j, 0] = [R, 0] + (\ell(j) - q) [R^*, 0] + (k + p) [s^j, 0].
\]

The monomials corresponding to \([R, 1] \) and \([R, 0] \) are \( f^1 \) and \( f \), respectively. They are not regular on the total space \( X \), but their difference \( t := f^1 - f \) is. Hence, the difference of the monomial versions of both equations yields the result.
Finally, we should remark that (i) and (ii) cover all boundary equations except those overlapping the intersection of $\partial\sigma'$ with $\mathbb{R} - \mathbb{R}$.

\begin{equation}
(4.8) \quad \text{We return to Example } \text{(4.4)} \text{ and discuss the one-parameter deformations occurring in degree } -R^a, -R^3, \text{ and } -R^1, \text{ respectively:}
\end{equation}

Case $\alpha$: $R^\alpha = [-1, 0, 1] = 2R^* - s^3$ means $j = 3$, $q = \ell(3) = 2$, and $p = 1$. Hence, the line $R^\perp$ has distance $q/p = 2$ from its parallel through $a^3$ and $a^1$. In particular, $\tau = \langle a^3, c^1, c^3, a^3 \rangle$ with $c^1 = (1, -1, 1)$ and $c^3 = (3, -1, 3)$.

We construct the generators of $\tilde{\tau}$ by the recipe of Proposition (4.5): $a^3$ treated via (i) and $a^1$ treated via (ii) yield the same element $A := (-1, 1, 1, -2)$; from the $R^2$-line we obtain $C^1 := (1, -1, 1, 0)$ and $C^3 := (3, -1, 3, 0)$; finally (iii) provides $X := (0, 0, 0, 1)$ and $Y := (0, -1, 0, 1)$. Hence, $\tilde{\tau}$ is the cone over the pyramid with plane base $XYC^1C^3$ and $A$ as top. (The relation between the vertices of the quadrangle equals $3C^1 + 2X = C^3 + 2Y$.) Moreover, $\tilde{\sigma}$ equals $\sigma = \tilde{\sigma} + \mathbb{R}_{>0}a^2$ with $a^2 := (a^2, 0)$. Since $A + 2X + 2a^2 = C^3$ and $A + 2Y + 2a^2 = 3C^1$, $\tilde{\sigma}$ is a simplex generated by $A$, $X$, $Y$, and $a^2$.

Denoting the variables assigned to $s^1, s^2, s^3, R^*, v^1, v^2, w \in E \subseteq \sigma' \cap M$ by $Z_1, Z_2, Z_3, U, V_1, V_2$, and $W$, respectively, there are six boundary equations:

\[ Z_3WZ_1 - U^3 = Z_1Z_2 - W^2 = WV_1 - UZ_2 = Z_2V_2 - V_1^2 = V_1Z_3 - V_2^2 = V_2Z_1 - U^2 = 0. \]

Only the four latter ones are covered by Theorem (4.7). They will be perturbed into

\[ WV_1 - UZ_2 = Z_2V_2 - V_1^2 = V_1Z_3 - V_2^2 = V_2Z_1 - U^2 - t_\alpha Z_3 = 0. \]

Case $\beta$: $R^\beta = [0, -1, 1] = 2R^* - s^1$ means $j = 1$, $q = \ell(1) = 3$, $q = 2$, and $p = 1$. Hence, $R^\perp$ still has distance 2, but now from the line $a^1a^2$.

We obtain $\tilde{\tau} = \langle (-1, -1, 1, -2); (0, -1, 1, -2); (-1, 1, 1, 0); (0, 0, 0, 1); (1, 0, 0, 1) \rangle$.

The boundary equation corresponding to Theorem (4.4)(ii) is $Z_3WZ_1 - U^3 = 0$; it perturbs into $Z_3WZ_1 - U^3 - t_\beta UZ_1 = 0$.

Case $\gamma$: $R^\gamma = [0, -2, 1] = 3R^* - 2s^1$ means $j = 1$, $q = \ell(1) = 3$, and $p = 2$. 

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Here, we have \( \tilde{\tau} = \langle (-1, -1, 1, -3); (-2, 1, 2, 0); (-1, 2, 4, 0); (0, 0, 0, 1); (1, 0, 0, 1) \rangle \), and the previous boundary equation provides \( Z_3 W Z_1 - U^3 - t_\gamma Z_1^2 = 0 \).

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