LOCAL AND GLOBAL WELL-POSEDNESS FOR THE 2D GENERALIZED ZAKHAROV-KUZNETSOV EQUATION

FELIPE LINARES AND ADEMIR PASTOR

Instituto Nacional de Matemática Pura e Aplicada - IMPA,
Estrada Dona Castorina 110, CEP 22460-320, Rio de Janeiro, RJ, Brazil.

Abstract. This paper addresses well-posedness issues for the initial value problem (IVP) associated with the generalized Zakharov-Kuznetsov equation, namely,
\[
\begin{align*}
  u_t + \partial_x \Delta u + u^k u_x &= 0, \\
  u(x, y, 0) &= u_0(x, y).
\end{align*}
\]

For 2 \leq k \leq 7, the IVP above is shown to be locally well-posed for data in $H^s(\mathbb{R}^2)$, $s > 3/4$. For $k \geq 8$, local well-posedness is shown to hold for data in $H^s(\mathbb{R}^2)$, $s > s_k$, where $s_k = 1 - 3/(2k - 4)$. Furthermore, for $k \geq 3$, if $u_0 \in H^s(\mathbb{R}^2)$ and satisfies $\|u_0\|_{H^s} \ll 1$, then the solution is shown to be global in $H^s(\mathbb{R}^2)$. For $k = 2$, if $u_0 \in H^s(\mathbb{R}^2)$, $s > 53/63$, and satisfies $\|u_0\|_{L^2} < \sqrt{3} \|\varphi\|_{L^2}$, where $\varphi$ is the corresponding ground state solution, then the solution is shown to be global in $H^s(\mathbb{R}^2)$.

1. Introduction

This paper is concerned with the initial value problem (IVP) associated with the generalized Zakharov-Kuznetsov (gZK) equation in two space dimensions, namely,
\[
\begin{align*}
  u_t + \partial_x \Delta u + u^k u_x &= 0, \\
  u(x, y, 0) &= u_0(x, y),
\end{align*}
\]

where $u$ is a real-valued function, and $k \geq 2$ is an integer number.

When $k = 1$, the equation in (1.1) (termed simply as ZK equation) was formally deduced by Zakharov and Kuznetsov \cite{ZK} (see also \cite{KKT} and references therein) as an asymptotic model to describe the propagation of nonlinear ion-acoustic waves in a magnetized plasma. The equation in (1.1) may also be seen as a natural, two-dimensional extension of the one-dimensional generalized Korteweg-de Vries (KdV) equation
\[
  u_t + u_{xxx} + u^k u_x = 0.
\]

The aim of this paper is to establish local and global well-posedness to the IVP (1.1). The notion of well-posedness will be the usual one in the context of nonlinear dispersive equations, that is, it includes existence, uniqueness, persistence property, and continuous dependence upon the data.

2000 Mathematics Subject Classification. Primary 35Q53, 35B65; Secondary 35Q60.

Key words and phrases. Zakharov-Kuznetsov equation, local well-posedness, global well-posedness.
Before describing our results, let us recall what has been done so far regarding the gZK equation. In [8], Faminskii considered the IVP associated with the ZK equation. He showed local and global well-posedness for initial data in $H^m(\mathbb{R}^2)$, $m \geq 1$ integer. In [2], Biagioni and Linares dealt with the IVP associated with the modified ZK equation (i.e. that one in (1.1) with $k = 2$). They proved local and global well-posedness for data in $H^1(\mathbb{R}^2)$. Linares and Pastor ([14]) studied the IVP associated with both the ZK and modified ZK equations. They improved the results in [2], [8], [14] by showing that both IVP’s are locally well-posed for initial data in $H^s(\mathbb{R}^2)$, $s > 3/4$. Moreover, by using the techniques introduced in Birnir at al. [3], [4], they proved that the IVP associated with the modified ZK equation is ill-posed, in the sense that the flow-map data-solution is not uniformly continuous, for data in $H^s(\mathbb{R}^2)$, $s \leq 0$. It should be noted that the method employed in [2], [8], [14] to show local well-posedness, was the one developed by Kenig, Ponce, and Vega [11] (when dealing with the generalized KdV equation), which combines smoothing effects, Strichartz-type estimates, and a maximal function estimate together with the Banach contraction principle.

It is worth mentioning that in [14], the authors proved that if $u_0 \in H^1(\mathbb{R}^2)$ and satisfies $\|u_0\|_{L^2} < \sqrt{3}\|\varphi\|_{L^2}$, where $\varphi$ is the unique positive radial solution (hereafter refereed to as the ground state solution) of the elliptic equation
\[
-\Delta \varphi + \varphi - \varphi^3 = 0, \tag{1.2}
\]
then the solution $u(t)$ of (1.1), with $k = 2$, may be globally extended in $H^1(\mathbb{R}^2)$. It should be pointed out that if $\|u_0\|_{L^2} \geq \sqrt{3}\|\varphi\|_{L^2}$, the question of showing whether or not the $H^1$-solution, with $u(0) = u_0$, blows up in finite time is currently open.

It should also be observed that questions of existence and orbital stability of solitary-wave solutions, and unique continuation were addressed, respectively, by de Bouard [7], and Panthee [16]. In [7], the author proved that the positive, radially symmetric solitary waves are orbitally stable if $k = 1$, and orbitally unstable otherwise. In [16], the author established that if the solution of the ZK equation is sufficiently regular and is compactly supported in a nontrivial time interval, then it vanishes identically.

Now, let us describe our results. We first recall that the quantities
\[
I_1(u(t)) = \int_{\mathbb{R}^2} u^2(t) \, dx \, dy \tag{1.3}
\]
and
\[
I_2(u(t)) = \int_{\mathbb{R}^2} \left\{ u_x^2(t) + u_y^2(t) - \frac{2}{(k+1)(k+2)} u^{k+2}(t) \right\} \, dx \, dy \tag{1.4}
\]
are conserved by the flow of the gZK equation, that is, $I_1(u(t)) = I_1(u(0))$ and $I_2(u(t)) = I_2(u(0))$, as long as the solution exists. Thus, these quantities could lead local rough solutions to global ones. So, it is natural to ask what would be the largest Sobolev space where local well-posedness holds. To answer this question, we perform a scaling argument, by noting that if $u$ solves (1.1), with initial data $u_0$, then
\[
u_{\lambda}(x, y, t) = \lambda^{2/k} u(\lambda x, \lambda y, \lambda^3 t)
\]
also solves (1.1), with initial data $u_{\lambda}(x, y, 0) = \lambda^{2/k} u_0(\lambda x, \lambda y)$, for any $\lambda > 0$. Hence,
\[
\|u(\cdot, 0)\|_{\dot{H}^s} = \lambda^{2/k + s - 1} \|u_0\|_{\dot{H}^s}, \tag{1.5}
\]
where \( \dot{H}^s = \dot{H}^s(\mathbb{R}^2) \) denotes the homogeneous Sobolev space of order \( s \). As a consequence of (1.3), the scale-invariant Sobolev space for the gZK equation is \( H^{s_c(k)}(\mathbb{R}^2) \), where \( s_c(k) = 1 - 2/k \). Therefore, one expects that the Sobolev spaces \( H^s(\mathbb{R}^2) \) for studying the well-posedness of (1.1) are those with indices \( s > s_c(k) \).

We divide the paper into two parts. The first one deals with local and global well-posedness in the case \( k \geq 3 \), whereas the second part is devoted to establishing the global well-posedness in the case \( k = 2 \) (the critical case).

Our first result regards local well-posedness of (1.1) for \( 3 \leq k \leq 7 \). More precisely, we prove the following.

**Theorem 1.1.** Assume \( 3 \leq k \leq 7 \). For any \( u_0 \in H^s(\mathbb{R}^2) \), \( s > 3/4 \), there exist \( T = T(\|u_0\|_{H^s}) > 0 \) and a unique solution of the IVP (1.1), defined in the interval \([0, T]\), such that

\[
    u \in C([0, T]; H^s(\mathbb{R}^2)),
\]

\[
    \|D_x^s u_x\|_{L_x^\infty L_y^2} + \|D_y^s u_x\|_{L_x^\infty L_y^2} < \infty,
\]

\[
    \|u\|_{L_x^{p_k} L_y^\infty} + \|u_x\|_{L_x^{12/5} L_y^\infty} < \infty,
\]

and

\[
    \|u\|_{L_x^{4} L_y^\infty} < \infty,
\]

where \( p_k = \frac{12(k-1)}{7-12\gamma} \) and \( \gamma \in (0, 1/12) \). Moreover, for any \( T' \in (0, T) \) there exists a neighborhood \( W \) of \( u_0 \) in \( H^s(\mathbb{R}^2) \) such that the map \( \tilde{u}_0 \mapsto \tilde{u}(t) \) from \( W \) into the class defined by (1.6)–(1.9) is smooth.

To prove Theorem 1.1 we use the technique introduced by Kenig, Ponce, and Vega [11] to study the IVP associated to the KdV equation. We point out that the proof of Theorem 1.1 in [14] does not apply to the case \( k \geq 3 \). Here, instead of using an \( L_x^4 \) maximal function estimate, we use an \( L_x^2 \) one (see Proposition 2.6 below). However, the main new ingredient is the embedding given in Lemma 2.3. Observe that for \( 3 \leq k \leq 7 \), we obtain \( s_c(k) < 3/4 \), so that, our result does not reach the indices conjectured by the scaling argument.

Next, we deal with the case \( k \geq 8 \). Our main result in this case reads as follows.

**Theorem 1.2.** Let \( k \geq 8 \) and \( s_k = 1 - \frac{3}{2(k-2)} \). For any \( u_0 \in H^s(\mathbb{R}^2) \), \( s > s_k \), there exist \( T = T(\|u_0\|_{H^s}) > 0 \) and a unique solution of the IVP (1.1), defined in the interval \([0, T]\), such that

\[
    u \in C([0, T]; H^s(\mathbb{R}^2)),
\]

\[
    \|D_x^s u_x\|_{L_x^\infty L_y^2} + \|D_y^s u_x\|_{L_x^\infty L_y^2} < \infty,
\]

\[
    \|u\|_{L_x^{\tilde{p}_k} L_y^\infty} + \|u_x\|_{L_x^{12/5} L_y^\infty} < \infty,
\]

and

\[
    \|u\|_{L_x^{4} L_y^\infty} < \infty,
\]

where \( \tilde{p}_k = \frac{2(k-2)}{1-2\gamma} \) and \( \gamma > 0 \) is sufficiently small. Moreover, for any \( T' \in (0, T) \) there exists a neighborhood \( \tilde{U} \) of \( u_0 \) in \( H^s(\mathbb{R}^2) \) such that the map \( \tilde{u}_0 \mapsto \tilde{u}(t) \) from \( \tilde{U} \) into the class defined by (1.10)–(1.13) is smooth.
The proof of Theorem 1.2 is very close to that of Theorem 1.1. In this case, because of the scaling argument, we do not expect to prove local well-posedness for all \( s > 3/4 \). Indeed, note that, for \( k \geq 8 \), we always have, \( s_k \geq s_c(k) \geq 3/4 \). Moreover, \( s_k = s_c(k) \) if and only if \( k = 8 \). Also observe that in the case \( k = 8 \), we get \( s_8 = s_c(8) = 3/4 \). This implies that our result, for \( k = 8 \), is “almost” sharp, but for \( k > 8 \) there is still a gap between the scaling and our results, which is evidenced by the theorem below.

**Theorem 1.3.** Let \( k \geq 3 \). Then, the IVP (1.1) is ill-posed for data in \( H^{s_c(k)}(\mathbb{R}^2) \), \( s_c(k) = 1 - 2/k \), in the sense that the map data-solution is not uniformly continuous.

Note that the well-posedness sense in Theorem 1.3 requires additional smoothness of the map data-solution, and not only that of merely continuity. However, this is not too strong because as affirmed in Theorems 1.1 and 1.2, the map data-solution, in those cases, is sufficiently smooth. The argument to establish Theorem 1.3 is similar to that of Theorem 1.2 in [14], and goes back to the techniques introduced in [3] and [4].

One of the main difficulties to obtain possible sharp results is the lack of some needed estimates in mixed spaces. There is not an available Leibniz rule for fractional derivatives in mixed spaces \( L^p_xL^q_yL^r_T \) for instance. This makes a difference with the analysis for the generalized KdV for \( k \geq 4 \). Another point we should remark is the gain of derivatives we have for the Strichartz estimate for the linear group. We only get \( 1/4 - \epsilon \) derivatives, \( 0 < \epsilon \ll 1 \), (see Lemma 2.2 below) in contrast to the gain of exactly \( 1/4 \) derivatives of the KdV linear group. Because of this we also loose some regularity.

We now turn our attention to the global well-posedness issue. Our main result is proved under a smallness condition on the initial data.

**Theorem 1.4.** Let \( k \geq 3 \). Let \( u_0 \in H^1(\mathbb{R}^2) \) and assume \( \|u_0\|_{H^1} \ll 1 \), then the local solutions given in Theorems 1.1 and 1.2 can be extended to any time interval \([0,T]\).

Theorem 1.4 is proved in a standard fashion, and relies on a combination of the conserved quantities (1.3) and (1.4) with the Gagliardo-Nirenberg interpolation inequality.

Next, we will focus on the second part of the paper. As we already mentioned, the local well-posedness of (1.1) with \( k = 2 \) for initial data in \( H^s(\mathbb{R}^2) \), \( s > 3/4 \), was obtained in [14]. Furthermore, we announced that (1.1) was globally well-posed for the initial data \( u_0 \) in \( H^s(\mathbb{R}^2) \), \( s > 19/21 \) satisfying \( \|u_0\|_{L^2} < \sqrt{3}\|\varphi\|_{L^2} \), where \( \varphi \) is the ground state solution of equation (1.2).

In the present paper, we reaffirm that this result holds, however, we slightly modify the proof of the local well-posedness in [14], to improve that announced result. More precisely, we prove the following.

**Theorem 1.5.** Let \( k = 2 \). Let \( u_0 \in H^s(\mathbb{R}^2) \), \( s > 53/63 \), and assume that \( \|u_0\|_{L^2} < \sqrt{3}\|\varphi\|_{L^2} \), where \( \varphi \) is the ground state solution of equation (1.2), then (1.1) is globally well-posed.

The method we use to prove Theorem 1.5 is that one developed in [9] and [10], which combines the smoothing effects for the solution of the linear problem with the iteration process introduced by Bourgain [5]. Since we are in the critical case, as in [10], controlling the \( L^2 \)-norm of the initial data could bring some difficult. Nevertheless, with a suitable decomposition of the initial data into low and high frequencies, we are able to handle this.
Let us highlight what enables us to improve the global result announced in [14]. The reason is quite simple. In [14], to apply the contraction principle, we get a factor of $T^{2/3}$ in front of the estimates for the nonlinear terms. Here, modifying a little bit the functional spaces, we get a factor of $T^{5/12}$ (see proof of Theorem 4.1), this in turn, is relevant in the method described in [9], [10].

As we have pointed out in [14], the Fourier restriction method does not seem to work to proving a local well-posedness result for the generalized ZK equation. So, it is not clear that the I-method, introduced by Colliander et al. [6], work either to establish a better global well-posedness result.

The paper is organized as follows. In Section 2, we state the results concerned with the linear problem associated with (1.1). In Section 3, we deal with the case $k \geq 3$. We show our local (and global) well-posedness result as well as the ill-posedness one. Finally, in Section 4 we establish the global well-posedness for $k = 2$ announced in Theorem 1.5.

**Notation.** The symbol $a \pm$ means that there exists an $\varepsilon > 0$, small enough, such that $a \pm = a \pm \varepsilon$. For $\alpha \in \mathbb{C}$, the operators $D_x^\alpha$ and $D_y^\alpha$ are defined via Fourier transform by $\hat{D_x^\alpha f}(\xi, \eta) = |\xi|^{\alpha} \hat{f}(\xi, \eta)$ and $\hat{D_y^\alpha f}(\xi, \eta) = |\eta|^\alpha \hat{f}(\xi, \eta)$. The mixed space-time norm is defined by (for $1 \leq p, q, r < \infty$)

$$\|f\|_{L^p_x L^q_y L^r_T} = \left( \int_{-\infty}^{+\infty} \left( \int_{0}^{T} \left| f(x, y, t)^r \right| dt \right)^{q/r} dy \right)^{1/p} \quad \text{dx},$$

with obvious modifications if either $p = \infty$, $q = \infty$ or $r = \infty$.

2. Preliminary results

In this section, we recall some results concerning the linear IVP associated to the gZK equation, which will be useful throughout the paper.

Consider the linear IVP

$$\begin{cases}
  u_t + \partial_x \Delta u = 0, & (x, t) \in \mathbb{R}^2, \ t \in \mathbb{R}, \\
  u(x, y, 0) = u_0(x, y),
\end{cases}$$

(2.1)

The solution of (2.1) is given by the unitary group $\{U(t)\}_{t=-\infty}^{\infty}$ such that

$$u(t) = U(t)u_0(x, y) = \int_{\mathbb{R}^2} e^{i(t(\xi^2 + \xi^2) + x\xi + y\eta)} \hat{u}_0(\xi, \eta) d\xi d\eta.$$  (2.2)

We begin by remembering the smoothing effect of Kato type, and the Strichartz-type estimates.

**Lemma 2.1. (Smoothing effect)** Let $u_0 \in L^2(\mathbb{R}^2)$. Then,

$$\|\partial_x U(t)u_0\|_{L^\infty_x L^2_y L^T_T} \leq c\|u_0\|_{L^2_y}$$  (2.3)

and

$$\|\partial_x \int_0^t U(-t')f(\cdot, \cdot, t')dt'\|_{L^2_y} \leq c\|f\|_{L^1_y L^2_T}.$$  (2.4)

Moreover, the same still hold if we replace $\partial_x$ with $\partial_y$. 
Proof. See Faminskii [8, Theorem 2.2] for the proof of (2.3). The inequality (2.4) is just the dual version of (2.3).

□

Proposition 2.2. (Strichartz-type estimates) Let \(0 \leq \varepsilon < 1/2\) and \(0 \leq \theta \leq 1\). Then, the group \(\{U(t)\}_{t=-\infty}^{\infty}\) satisfies

\[
\|D_{\theta\varepsilon/2}^x U(t)f\|_{L_t^p L_x^q} \leq c\|f\|_{L_x^2},
\]

(2.5)

where \(p = \frac{2}{1-\theta}\) and \(\frac{2}{q} = \frac{\theta(2+\varepsilon)}{3}\).

Proof. See Linares and Pastor [14, Proposition 2.4].

□

The next lemmas are useful to recover the “loss of derivative” present in the nonlinear term of the gZK equation.

Lemma 2.3. Let \(0 \leq \varepsilon < 1/2\). Then, the group \(\{U(t)\}_{t=-\infty}^{\infty}\) satisfies

\[
\|U(t)f\|_{L_t^p L_x^q} \leq cT^{\gamma_1} \|D_x^{-\varepsilon/2} f\|_{L_x^2},
\]

(2.6)

where \(1 \leq p \leq \frac{6}{2+\varepsilon}\) and \(\gamma_1 = \frac{1}{p} - \frac{2+\varepsilon}{6}\). In particular, if \(0 < T \leq 1\), then

\[
\|U(t)f\|_{L_t^{12/5} L_x^\infty} \leq c\|D_x^{-\varepsilon/2} f\|_{L_x^2}.
\]

(2.7)

Proof. By using Holder’s inequality (in \(t\)), we get

\[
\|U(t)f\|_{L_t^p L_x^q} \leq cT^{\gamma_1} \|U(t)f\|_{L_t^p L_x^\infty},
\]

where \(\frac{1}{p} = \gamma_1 + \frac{1}{q}\). Thus, taking \(\theta = 1\) and \(q = 6/(2+\varepsilon)\) in Proposition 2.2, the estimate (2.6) then follows.

□

Lemma 2.4. Let \(\delta > 0\). Then,

\[
\|f\|_{L_x^\infty} \leq c\left\{\|f\|_{L_x^{p_\delta}} + \|D_x^\delta f\|_{L_x^{p_\delta}} + \|D_y^\delta f\|_{L_y^{p_\delta}}\right\},
\]

where \(p_\delta > 2/\delta\). In particular, \(p_\delta \to \infty\) as \(\delta \to 0\).

Proof. See Kenig and Ziesler [13, Lemma 3.4].

□

Lemma 2.5. Let \(0 < \delta < 1\). Assume \(1 - \delta < \theta < 1\) and \(2 \leq r \leq 3/\theta\). Then,

\[
\|U(t)f\|_{L_t^r L_x^\infty} \leq cT^{\gamma_2} \left\{\|f\|_{L_x^2} + \|D_x^\delta f\|_{L_x^2} + \|D_y^\delta f\|_{L_y^2}\right\},
\]

for some \(\gamma_2 = \frac{1}{r} - \frac{\theta}{3} \geq 0\).

Proof. We first note that taking \(\varepsilon = 0\) in Proposition 2.2, we obtain

\[
\|U(t)f\|_{L_t^{3/\theta} L_x^{2/(1-\theta)}} \leq c\|f\|_{L_x^2},
\]

(2.8)
Now, applying Holder’s inequality followed by Lemma 2.4 we deduce
\[
\|U(t)f\|_{L^r_t L^\infty_x} \leq c T^{\gamma_2} \|U(t)f\|_{L^r_t L^\infty_x} \\
\quad \leq c T^{\gamma_2} \left\{ \|U(t)f\|_{L^r_t L^\infty_y}^p + \|D_y^\delta U(t)f\|_{L^r_t L^\infty_x}^p + \|D_y^\delta U(t)f\|_{L^r_t L^\infty_y}^p \right\}.
\]
If we now choose \( r' = 3/\theta \) and \( p_\delta = 2/(1 - \theta) \), then an application of \((2.8)\) yields the affirmation. Note that \( p_\delta > 2/\delta \) implies \( 1 - \delta < \theta \), and \( \gamma_2 \geq 0 \) implies \( r \leq 3/\theta \). This completes the proof of the lemma.

As we commented before, Kenig, Ponce, and Vega’s technique combines the smoothing effect and Strichartz estimate with a maximal function estimate. Here, we present the \( L^2_x \) and \( L^4_x \) maximal function estimates we will use in our arguments.

**Proposition 2.6. (Maximal function)**

(i) For any \( s_1 > 1/4, r_1 > 1/2 \) and \( 0 < T \leq 1 \), we have
\[
\|U(t)f\|_{L^3_t L^\infty_y} \leq c \|(1 + D_x)^{s_1}(1 + D_y)^{r_1}f\|_{L^2_y}.
\]

(ii) For any \( s > 3/4 \), we have
\[
\|U(t)f\|_{L^4_t L^\infty_y} \leq c(s,T)\|f\|_{H^s_y},
\]
where \( c(s,T) \) is a positive constant depending only on \( T \) and \( s \).

**Proof.** See Linares and Pastor [14, Proposition 1.5] for part (i), and Faminskii [8, Theorem 2.4] for part (ii).

**Corollary 2.7.** For any \( s > 3/4 \) and \( 0 < T \leq 1 \), we have
\[
\|U(t)f\|_{L^4_t L^\infty_y} \leq c \|f\|_{H^s_y}.
\]

**Proof.** Let \( s_1 \) and \( r_1 \) be as in Proposition 2.6(i). In view of Plancherel’s theorem, we obtain
\[
\|(1 + D_x)^{s_1}(1 + D_y)^{r_1}f\|_{L^2_y}^2 = \int_{\mathbb{R}^2} (1 + |\xi|^2)^{s_1} (1 + |\eta|^2)^{r_1} |\hat{f}(\xi,\eta)|^2 d\xi d\eta
\]
\[
\quad \leq c \int_{\mathbb{R}^2} (1 + |\xi|^{2s_1} + |\eta|^{2r_1} + |\xi|^{2s_1}|\eta|^{2r_1}) |\hat{f}(\xi,\eta)|^2 d\xi d\eta
\]
\[
\quad \leq c \int_{\mathbb{R}^2} (1 + |\xi|^{2s_1} + |\eta|^{2r_1} + |\xi|^{6s_1} + |\eta|^{3r_1}) |\hat{f}(\xi,\eta)|^2 d\xi d\eta,
\]
where in the last inequality we applied the Young inequality. Now, splitting the integral into \( B_1(0) \) and \( \mathbb{R}^2 \setminus B_1(0) \), where \( B_1(0) \) denotes the ball of radius 1 centered at the origin, it is easy to see that
\[
\int_{\mathbb{R}^2} (1 + |\xi|^{2s_1} + |\eta|^{2r_1} + |\xi|^{6s_1} + |\eta|^{3r_1}) |\hat{f}(\xi,\eta)|^2 d\xi d\eta \leq c \int_{\mathbb{R}^2} (1 + |\xi|^{6s_1} + |\eta|^{3r_1}) |\hat{f}(\xi,\eta)|^2 d\xi d\eta.
\]

(2.9)
Write \( s_1 = \frac{1}{4} + \rho / 3 \) and \( r_1 = \frac{1}{2} + 2\rho / 3 \), where \( \rho > 0 \). Thus,
\[
(1 + |\xi|^{6s_1} + |\eta|^{3r_1}) \leq c(1 + |\xi|^2 + |\eta|^2)^{3/4 + \rho}.
\]
(2.10)
Using (2.10) in (2.9) one easily shows the desired conclusion.

Finally, we also recall the Leibniz rule for fractional derivatives.

**Lemma 2.8.** Let \( 0 < \alpha < 1 \) and \( 1 < p < \infty \). Then
\[
\| D^\alpha (fg) - f D^\alpha g - g D^\alpha f \|_{L^p(\mathbb{R})} \leq c \|g\|_{L^\infty(\mathbb{R})} \| D^\alpha f \|_{L^p(\mathbb{R})},
\]
where \( D^\alpha \) denotes either \( D_\xi^\alpha \) or \( D_y^\alpha \).

**Proof.** See Kenig, Ponce, and Vega [11, Theorem A.12].

3. Proofs of Theorems 1.1–1.4

We begin this section by showing Theorem 1.1. Since the proof of Theorem 1.2 is similar, we only sketch it. We finish the section by proving Theorem 1.4.

**Proof of Theorem 1.1.** As usual, we consider the integral operator
\[
\Psi(u)(t) = \Psi_{u_0}(u)(t) := U(t)u_0 + \int_0^t U(t - t')(u^k u_x)(t')dt',
\]
and define the metric spaces
\[
\mathcal{Y}_T = \{ u \in C([0, T]; H^s(\mathbb{R}^2)); \| u \| < \infty \}
\]
and
\[
\mathcal{Y}_T^a = \{ u \in \mathcal{X}_T; \| u \| \leq a \},
\]
with
\[
\| u \| = \| u \|_{L^\infty_t H^s_{xy}} + \| u \|_{L^p T L^\infty_{xy}} + \| u_x \|_{L^{12/5}_T L^2_{xy}} + \| D^k_x u_x \|_{L^\infty_t L^2_{xy}} + \| D^k_y u_x \|_{L^\infty_t L^2_{xy}} + \| u \|_{L^1 T L^{\infty}_{xy}},
\]
where \( a, T > 0 \) will be chosen later. We assume that \( 3/4 < s < 1 \) and \( T \leq 1 \).

First we estimate the \( H^s \)-norm of \( \Psi(u) \). Let \( u \in \mathcal{Y}_T \). By using Minkowski’s inequality, group properties and then Hölder’s inequality, we have
\[
\| \Psi(u)(t) \|_{L^2_{xy}} \leq c \| u_0 \|_{H^s} + c \int_0^T \| u \|_{L^2_{xy}} \| u^{k-1} u_x \|_{L^\infty_{xy}} dt'
\]
\[
\leq c \| u_0 \|_{H^s} + c \| u \|_{L^\infty_t L^2_{xy}} \int_0^T \| u \|_{L^2_{xy}}^{k-1} \| u_x \|_{L^\infty_{xy}} dt'
\]
\[
\leq c \| u_0 \|_{H^s} + c^T \| u \|_{L^\infty_t L^2_{xy}} \| u_x \|_{L^{12/5}_T L^2_{xy}} \| u \|_{L^1 T L^{\infty}_{xy}}^{k-1}.
\]
(3.2)
Using group properties, Minkowski and Hölder's inequalities and twice Lemma 2.8 we have

\[ \|D^s_x \Psi(u(t))\|_{L^2_{xy}} \leq \|D^s_x u_0\|_{L^2_{xy}} + \int_0^T \|D^s_x (u^k u_x)\|_{L^2_{xy}} dt' \]

\[ \leq c\|u_0\|_{H^s} + c\int_0^T \|u_x\|_{L^\infty_{xy}} \|D^s_x (u^k)\|_{L^2_{xy}} dt' + c\int_0^T \|u^k D^s_x u_x\|_{L^2_{xy}} dt' \]

\[ \leq c\|u_0\|_{H^s} + c\int_0^T \|u_x\|_{L^\infty_{xy}} \|u||^{k-1} \|D^s_x u\|_{L^2_{xy}} dt' + c\int_0^T \|u^k D^s_x u_x\|_{L^2_{xy}} dt' \]

\[ \leq c\|u_0\|_{H^s} + c\|u\|_{L^\infty_T L^\infty_{xy}} \int_0^T \|u_x\|_{L^\infty_{xy}} \|u|^{k-1} \|D^s_x u_x\|_{L^2_{xy}} dt' + c\int_0^T \|u^k D^s_x u_x\|_{L^2_{xy}} dt'. \]  \hspace{1cm} (3.3)

As in (3.2), from Hölder’s inequality, we get

\[ \int_0^T \|u_x\|_{L^\infty_{xy}} \|u||^{k-1} \|D^s_x u_x\|_{L^2_{xy}} dt' \leq c T^\gamma \|u_x\|_{L^{12/5}_{xy}} \|u|^{k-1} \|D^s_x u_x\|_{L^2_{xy}} \]

\[ \hspace{1cm} \text{(3.4)} \]

Moreover,

\[ \int_0^T \|u^k D^s_x u_x\|_{L^2_{xy}} dt' \leq \int_0^T \|u\|_{L^\infty_{xy}} \|u^2 D^s_x u_x\|_{L^2_{xy}} dt' \]

\[ \leq T^\gamma \|u||^{k-2} \|D^s_x u_x\|_{L^\infty_{xy}} \|D^s_x u_x\|_{L^2_{xy}} \]

\[ \text{(3.5)} \]

where \( \tilde{p}_k = \frac{2(k-2)}{1-\gamma} \). Note that for \( 3 \leq k \leq 7 \) we have \( \tilde{p}_k < p_k \). Thus, combining (3.4)-(3.5) with (3.3), we then deduce

\[ \|D^s_x \Psi(u(t))\|_{L^2_{xy}} \leq c\|u_0\|_{H^s} + c T^\gamma \|u\|_{L^\infty_T H^s_{xy}} \|u_x\|_{L^{12/5}_{xy}} \|u|^{k-1} \|D^s_x u_x\|_{L^2_{xy}} \]

\[ + c T^\gamma \|u||^{k-2} \|D^s_x u_x\|_{L^\infty_{xy}} \|D^s_x u_x\|_{L^2_{xy}} \]  \hspace{1cm} (3.6)

A similar analysis can be carried out to see that

\[ \|D^s_y \Psi(u(t))\|_{L^2_{xy}} \leq c\|u_0\|_{H^s} + c T^\gamma \|u\|_{L^\infty_T H^s_{xy}} \|u_x\|_{L^{12/5}_{xy}} \|u|^{k-1} \|D^s_x u_x\|_{L^2_{xy}} \]

\[ + c T^\gamma \|u||^{k-2} \|D^s_x u_x\|_{L^\infty_{xy}} \|D^s_x u_x\|_{L^2_{xy}} \]  \hspace{1cm} (3.7)

Therefore, from (3.2), (3.6) and (3.7), we deduce

\[ \|\Psi(u)\|_{L^\infty_T H^s} \leq c\|u_0\|_{H^s} + c T^\gamma \|u||^{k+1}. \]  \hspace{1cm} (3.8)

Next, we estimate the remaining norms. By taking \( \delta = 3/4 \) and \( \theta = 1/4 + \sigma, 0 < \sigma \leq \frac{1-12\gamma}{24} \), in Lemma 2.5 we see that \( p_k \leq 3/\theta \). Thus, Lemma 2.5 group properties and the arguments
used to obtain (3.8) yield
\[
\|\Psi(u)\|_{L^p_\nu L^\infty_x y} \leq \|U(t)u_0\|_{L^p_\nu L^\infty_x y} + \left\| U(t) \left( \int_0^t U(-t')(u^k u_x)(t')dt' \right) \right\|_{L^p_\nu L^\infty_x y} \\
\leq c\|u_0\|_{H^{3/4}} + c \int_0^T \|u^k u_x\|_{H^{3/4}} dt' \\
\leq c\|u_0\|_{H^s} + c \int_0^T \|u^k u_x\|_{H^s} dt' \\
\leq c\|u_0\|_{H^s} + cT^\gamma\|u\|^{k+1}. 
\]
(3.9)

By choosing \(\varepsilon \sim 1/2\) such that \(1 - \varepsilon/2 \leq s\), an application of Lemma 2.3 together with arguments similar to those ones used to derive (3.8) imply

\[
\|\partial_x \Psi(u)\|_{L^{12/5}_\nu L^\infty_x y} \leq \|U(t)\partial_x u_0\|_{L^{12/5}_\nu L^\infty_x y} + \left\| U(t) \left( \int_0^t U(-t')\partial_x (u^k u_x)(t')dt' \right) \right\|_{L^{12/5}_\nu L^\infty_x y} \\
\leq c\|D_x^{-\varepsilon/2}\partial_x u_0\|_{L^2_y} + c \int_0^T \|D_x^{-\varepsilon/2}\partial_x (u^k u_x)\|_{L^2_y} dt' \\
\leq c\|u_0\|_{H^s} + c \int_0^T \|u^k u_x\|_{L^2_y} dt' + c \int_0^T \|D_x^s (u^k u_x)\|_{L^2_y} dt' \\
\leq c\|u_0\|_{H^s} + cT^\gamma\|u\|^{k+1}. 
\]
(3.10)

Applying Lemma 2.1 group properties, Minkowski and Hölder inequalities, we obtain

\[
\|D_x^s \partial_x \Psi(u)\|_{L^\infty_\nu L^2_x y} \leq \|\partial_x U(t)D_x^s u_0\|_{L^\infty_\nu L^2_x y} \\
+ \left\| \partial_x U(t) \left( \int_0^t U(-t')D_x^s (u^k u_x)(t')dt' \right) \right\|_{L^\infty_\nu L^2_x y} \\
\leq c\|D_x^s u_0\|_{L^2_y} + c \int_0^T \|D_x^s (u^k u_x)\|_{L^2_y} dt' \\
\leq c\|u_0\|_{H^s} + cT^\gamma\|u\|^{k+1} 
\]
(3.11)

and

\[
\|D_y^s \partial_x \Psi(u)\|_{L^\infty_\nu L^2_x y} \leq \|\partial_x U(t)D_y^s u_0\|_{L^\infty_\nu L^2_x y} \\
+ \left\| \partial_x U(t) \left( \int_0^t U(-t')D_y^s (u^k u_x)(t')dt' \right) \right\|_{L^\infty_\nu L^2_x y} \\
\leq c\|D_y^s u_0\|_{L^2_y} + c \int_0^T \|D_y^s (u^k u_x)\|_{L^2_y} dt' \\
\leq c\|u_0\|_{H^s} + cT^\gamma\|u\|^{k+1}. 
\]
(3.12)
Finally, an application of Corollary 2.7, Minkowski’s inequality, group properties, and arguments previously used yield

\[
\| \Psi(u) \|_{L^4_x L^\infty_y} \leq \| U(t)u_0 \|_{L^4_x L^\infty_y} + \left\| U(t) \left( \int_0^t U(-t')(u^k u_x)(t') dt' \right) \right\|_{L^4_x L^\infty_y}
\]

\[
\leq c \| u_0 \|_{H^s} + c \int_0^T \| u^k u_x \|_{H^s} dt'
\]

\[
\leq c \| u_0 \|_{H^s} + cT^\gamma \| u \|^{k+1}.
\]

Therefore, from (3.8)–(3.13), we infer

\[
\| \Psi(u) \| \leq c \| u_0 \|_{H^s} + cT^\gamma \| u \|^{k+1}.
\]

Choose \( a = 2c \| u_0 \|_{H^s} \), and \( T > 0 \) such that

\[
ca^{kT^\gamma} \leq \frac{1}{4}.
\]

Then, it is easy to see that \( \Psi : \mathcal{Y}_T^s \rightarrow \mathcal{Y}_T^{s, k} \) is well defined. Moreover, similar arguments show that \( \Psi \) is a contraction. To finish the proof we use standard arguments, thus, we omit the details. This completes the proof of Theorem 1.1. \( \square \)

**Proof of Theorem 1.2.** The proof is very similar to that of Theorem 1.1. So, we give only the main steps. Assume \( s_k < s < 1 \) and \( 0 < T \leq 1 \). Define the metric space

\[
\mathcal{X}_T = \{ u \in C([0, T]; H^s(\mathbb{R}^2)); \| u \|_{s,k} < \infty \}
\]

with

\[
\| u \|_{s,k} := \| u \|_{L^\infty_x H^s_y} + \| u \|_{L^\infty_T L^\infty_y} + \| u_x \|_{L^{12/5}_T L^\infty_y} + \| D_x^s u_x \|_{L^\infty_x L^2_y} + \| D_y^s u_x \|_{L^\infty_x L^2_y} + \| u \|_{L^4_x L^\infty_y}.
\]

We first note that since \( k \geq 8 \) we have \( \tilde{p}_k > p_k \), where \( p_k = \frac{12(k-1)}{k-2} \) is given in Theorem 1.1. Hence, similarly to estimates (3.2)–(3.7), we establish that

\[
\| \Psi(u) \|_{L^\infty_x H^s} \leq c \| u_0 \|_{H^s} + cT^\gamma \| u \|^{k+1}_{s,k},
\]

where \( \Psi \) is the integral operator given in (3.1). The estimates (3.10)–(3.13) also hold here without any change. What is left, is to show a similar estimate as (3.9). Here, to use Lemma 2.5 we take \( \delta = s \) and \( \theta = 1 - s + \sigma \), where \( \sigma \) and \( \gamma \) are chosen such that

\[
s > s_k + \frac{6\gamma}{2(k-2)} + \sigma.
\]

The inequality (3.15) promptly implies that \( \tilde{p}_k \leq 3/\theta \). Thus, in view of Lemma 2.5 we obtain

\[
\| \Psi(u) \|_{L^\infty_T L^\infty_y} \leq c \| u_0 \|_{H^s} + c \int_0^T \| u^k u_x \|_{H^s} dt'
\]

\[
\leq c \| u_0 \|_{H^s} + cT^\gamma \| u \|^{k+1}_{s,k}.
\]

Collecting all of our estimates, we then deduce

\[
\| \Psi(u) \|_{s,k} \leq c \| u_0 \|_{H^s} + cT^\gamma \| u \|^{k+1}_{s,k}.
\]
integrating once, we see that

\[-c \varphi_c + \Delta \varphi_c + \frac{1}{k+1} \varphi_c^{k+1} = 0.\]  

(3.16)

The following lemma is well known and will be sufficient to establish our result.

**Lemma 3.1.** Let \(c > 0\). Then equation \((3.16)\) admits a positive, radially symmetric solution \(\varphi_c \in H^1(\mathbb{R}^2)\). Moreover, \(\varphi_c \in C^\infty(\mathbb{R}^2)\), and there exists \(\rho > 0\) such that for all multi-index \(\alpha \in \mathbb{N}^2\) with \(|\alpha| \leq 2\), one has \(|D^\alpha \varphi_c(x)| \leq C_\alpha e^{-\rho|x|}\), where \(C_\alpha\) depends only on \(\alpha\).

**Proof.** See Berestycki and Lions [1].

It is easy to see that

\[\varphi_c(x, y) = c^{1/k} \varphi_1 \left( \sqrt{cx}, \sqrt{cy} \right), \quad \text{for all} \quad c > 0,\]

where \(\varphi_1\) is the solution of \((3.16)\) with \(c = 1\). Thus, since

\[\tilde{\varphi}_c(\xi, \eta) = c^{1/k-1} \tilde{\varphi}_1 \left( \frac{\xi}{\sqrt{c}}, \frac{\eta}{\sqrt{c}} \right),\]

one easily checks that

\[\|\varphi_c\|_{\dot{H}^{s_c(k)}} = c^{1/k-1/2+s_c(k)/2} \|\varphi_1\|_{\dot{H}^{s_c(k)}} = \|\varphi_1\|_{\dot{H}^{s_c(k)}} =: a_0\]

(3.18)

Note that the constant \(a_0\) does not depend on \(c\).

Next, for any \(c > 0\) fixed, we consider

\[u_c(x, y, t) = \varphi_c(x - ct, y).\]

Hence, at \(t = 0\), we have \(u_c(0) = \varphi_c\). Moreover, for any \(c_1, c_2 > 0\), we obtain

\[\|\varphi_{c_1} - \varphi_{c_2}\|_{\dot{H}^{s_c(k)}}^2 = \|\varphi_{c_1}\|_{\dot{H}^{s_c(k)}}^2 + \|\varphi_{c_2}\|_{\dot{H}^{s_c(k)}}^2 - 2\langle \varphi_{c_1}, \varphi_{c_2} \rangle_{\dot{H}^{s_c(k)}}.\]

(3.19)

But, using \((3.17)\) again, we obtain

\[\langle \varphi_{c_1}, \varphi_{c_2} \rangle_{\dot{H}^{s_c(k)}} = \int_{\mathbb{R}^2} D^{s_c(k)} \varphi_{c_1}(x, y) \overline{D^{s_c(k)} \varphi_{c_2}(x, y)} \, dx \, dy\]

\[= \int_{\mathbb{R}^2} \langle (\xi, \eta) \rangle^{2s_c(k)} \tilde{\varphi}_{c_1}(\xi, \eta) \overline{\tilde{\varphi}_{c_2}(\xi, \eta)} \, d\xi \, d\eta\]

\[= (c_1 c_2)^{1/k-1} \int_{\mathbb{R}^2} \langle (\xi, \eta) \rangle^{2s_c(k)} \tilde{\varphi}_1 \left( \frac{\xi}{\sqrt{c_1}}, \frac{\eta}{\sqrt{c_1}} \right) \overline{\tilde{\varphi}_1} \left( \sqrt{\frac{c_1}{c_2}}, \sqrt{\frac{c_1}{c_2}} \right) \, d\xi \, d\eta\]

\[= \left( \frac{c_2}{c_1} \right)^{1/k-1} \int_{\mathbb{R}^2} \langle (\xi, \eta) \rangle^{2s_c(k)} \tilde{\varphi}_1(\xi, \eta) \overline{\tilde{\varphi}_1} \left( \sqrt{\frac{c_1}{c_2}}, \sqrt{\frac{c_1}{c_2}} \right) \, d\xi \, d\eta.\]

Therefore, as \(\theta := c_1/c_2 \to 1\), we get

\[\lim_{\theta \to 1} \langle \varphi_{c_1}, \varphi_{c_2} \rangle_{s_c(k)} = a_0^2.\]

(3.20)
As a consequence of (3.18)–(3.20), we then get
\[
\lim_{\theta \to 1} \| \varphi_{c_1} - \varphi_{c_2} \|_{H^{s_c(k)}} = 0.
\]

On the other hand, for any \( t > 0 \),
\[
\| u_{c_1}(t) - u_{c_2}(t) \|_{H^{s_c(k)}}^2 = \| u_{c_1}(t) \|_{H^{s_c(k)}}^2 + \| u_{c_2}(t) \|_{H^{s_c(k)}}^2 - 2 \langle u_{c_1}(t), u_{c_2}(t) \rangle_{H^{s_c(k)}}.
\]
But, since
\[
\langle \varphi_1, \varphi_2 \rangle = e^{1/k-1}e^{-it\xi t} \varphi_1 \left( \frac{\xi}{\sqrt{c}}, \frac{\eta}{\sqrt{c}} \right),
\]
we deduce
\[
\langle u_{c_1}(t), u_{c_2}(t) \rangle_{H^{s_c(k)}} = \left( e^{1/k-1} \int_{\mathbb{R}^2} e^{-it\xi t} |\xi c t|^{s_c(k)} |\xi c t|^{2s_c(k)} \varphi_1 \left( \frac{\xi}{\sqrt{c}}, \frac{\eta}{\sqrt{c}} \right) d\xi d\eta \right)^{1/2}.
\]
By choosing \( c_1 = m + 1 \) and \( c_2 = m \in \mathbb{N} \) and letting \( m \to \infty \), an application of the Riemann-Lebesgue lemma, yields
\[
\lim_{m \to \infty} \langle u_{c_1}(t), u_{c_2}(t) \rangle_{H^{s_c(k)}} = 0.
\]
Therefore, for any \( t > 0 \),
\[
\lim_{\theta \to 1} \| u_{c_1}(t) - u_{c_2}(t) \|_{H^{s_c(k)}} = \sqrt{2} a_0.
\]
This completes the proof of the theorem.

**Proof of Theorem 1.4** By using the Gagliardo-Nirenberg interpolation theorem it follows that
\[
\| u(t) \|_{L^{k+2}}^{k+2} \leq c \| u(t) \|_{L^2}^2 \| \partial_x u(t) \|_{L^2}^k.
\]
Combining (1.3), (1.4) and (3.21), we obtain that
\[
\| u(t) \|_{H^1}^2 = I_1(u(t)) + I_2(u(t)) + c \| u(t) \|_{L^{k+2}}^{k+2} \\
\leq I_1(u_0) + I_2(u_0) + c \| u_0 \|_{L^2}^2 \| \partial_x u(t) \|_{L^2}^k.
\]
Denote \( X(t) = \| u(t) \|_{H^1} \). Since \( k \geq 3 \), we then have
\[
X(t) \leq C(\| u_0 \|_{H^1}^2) + c \| u_0 \|_{L^2}^2 X(t)^{1 - \frac{k+2}{k+3}}.
\]
Thus, if \( \| u_0 \|_{H^1} \) is small enough, a standard argument leads to \( \| u(t) \|_{H^1} \leq C(\| u_0 \|_{H^1}) \) for \( t \in [0, T] \). Therefore, we can apply the local theory to extend the solution.

4. Global well-posedness for the modified ZK

In this section, we consider the Cauchy problem associated with the modified ZK. The main goal is to prove the global well-posedness result stated in Theorem 1.5.
4.1. Auxiliary results. We start with the following local well-posedness result. The proof is slightly different from that of Theorem 1.1 in [14].

**Theorem 4.1.** Let \( k = 2 \). For any \( u_0 \in H^s(\mathbb{R}^2) \), \( s > 3/4 \), there exist \( T = T(\|u_0\|_{H^s}) > 0 \) and a unique solution of the IVP (1.1), defined in the interval \([0, T]\), such that

\[
u \in C([0, T]; H^s(\mathbb{R}^2)),
\]

\[
\|D_x^s u_x\|_{L^\infty_T L^2_y} + \|D_y^s u_y\|_{L^\infty_T L^2_x} < \infty,
\]

\[
\|u\|_{L^p_T L^\infty_y} + \|u_x\|_{L^{12/5}_T L^\infty_x} < \infty,
\]

and

\[
\|u\|_{L^2_T L^\infty_y} < \infty,
\]

where \( p = \frac{2}{1-2\gamma} \) and \( \gamma \in (0, 5/12) \). In addition, the following statements hold:

(i) For any \( T' \in (0, T) \) there exists a neighborhood \( V \) of \( u_0 \) in \( H^s(\mathbb{R}^2) \) such that the map \( \tilde{u}_0 \mapsto \tilde{u}(t) \) from \( V \) into the class defined by (4.1)–(4.4) is smooth.

(ii) The existence time \( T \) is given by

\[
T \sim \|u_0\|_{H^s}^{2/\gamma}.
\]

To simplify the exposition and for further references, we prove first the following lemma.

**Lemma 4.2.** Assume \( u, v, w \) are sufficiently smooth. Let \( p \) be as in Theorem 4.1.

(i) For any \( T > 0 \),

\[
\int_0^T \|vwu_x\|_{L^2_y} dt' \leq cT^\gamma \|u_x\|_{L^{12/5}_T L^\infty_x} \|v\|_{L^\infty_T L^2_y} \|w\|_{L^p_T L^\infty_y}.
\]

(ii) For any \( T > 0 \) and \( s \in (0, 1) \),

\[
\int_0^T \|D_x^s(vwu_x)\|_{L^2_y} dt' \leq cT^\gamma \|u_x\|_{L^{12/5}_T L^\infty_x} \left\{ \|v\|_{L^\infty_T H^s_y} \|w\|_{L^p_T L^\infty_y} + \|w\|_{L^p_T H^s_y} \|v\|_{L^p_T L^\infty_y} \right\}
\]

\[
+ cT^\gamma \|w\|_{L^p_T L^\infty_y} \|v\|_{L^2_T L^\infty_y} \|D_x^s u_x\|_{L^\infty_T L^2_y}.
\]

The same still holds if we replace \( D_x^s \) by \( D_y^s \).

**Proof.** The estimate (i) follows after applying Hölder inequality. The proof of (ii) is roughly an application of Lemma 2.8 combined with the Hölder inequality. Indeed, applying twice Lemma 2.8 we deduce

\[
\|D_x^s(vwu_x)\|_{L^2_y} \leq c \|D_x^s w\|_{L^2_y} \|v\|_{L^\infty_y} \|u_x\|_{L^\infty_y} + c \|D_x^s v\|_{L^2_y} \|w\|_{L^\infty_y} \|u_x\|_{L^\infty_y} + c \|vwD_x^s u_x\|_{L^2_y}.
\]

For the first two terms, from Hölder’s inequality, we obtain

\[
\int_0^T \left\{ \|D_x^s w\|_{L^2_y} \|v\|_{L^\infty_y} \|u_x\|_{L^\infty_y} + \|D_x^s v\|_{L^2_y} \|w\|_{L^\infty_y} \|u_x\|_{L^\infty_y} \right\} dt'
\]

\[
\leq cT^\gamma \|u_x\|_{L^{12/5}_T L^\infty_x} \left\{ \|v\|_{L^\infty_T H^s_y} \|w\|_{L^p_T L^\infty_y} + \|w\|_{L^p_T H^s_y} \|v\|_{L^p_T L^\infty_y} \right\}.
\]
Finally, gathering together all estimates we see that

\[ \int_0^T \|wvD_x^s u_x\|_{L^2_y} dt' \leq \int_0^T \|w\|_{L^\infty_x} \|vD_x^s u_x\|_{L^2_y} dt' \]

\[ \leq \left( \int_0^T \|w\|_{L^\infty_x}^2 dt' \right)^{1/2} \|vD_x^s u_x\|_{L^2_y} \]

\[ \leq c T^{\gamma} \|w\|_{L^p_T L^\infty_y} \|v\|_{L^p_T L^\infty_y} \|D_x^s u_x\|_{L^\infty_T L^2_y}. \]

This completes the proof of the lemma.

\[ \square \]

**Sketch of proof of Theorem 4.1.** The proof is similarly carried out as the proof of Theorem 1.1 (see also [14]). The main difference is that instead of using the maximal function in Proposition 2.6(i), we use the one in (ii).

Thus, we consider the integral operator

\[ \Phi(u)(t) = \Phi_{u_0}(u)(t) := U(t)u_0 + \int_0^t U(t - t')(u^2 u_x)(t')dt', \]

and define the metric spaces

\[ Z_T = \{ u \in C([0, T]; H^s(\mathbb{R}^2)); \|u\|_{s, 2} < \infty \} \]

and

\[ Z_T^a = \{ u \in X_T; \|u\|_{s, 2} \leq a \}, \]

with

\[ \|u\|_{s, 2} := \|u\|_{L^\infty_T H^s_x} + \|u\|_{L^p_T L^\infty_y} + \|u_x\|_{L^{12/5}_T L^\infty_y} + \|D_x^s u_x\|_{L^\infty_T L^2_y} + \|D_y^s u_x\|_{L^\infty_T L^2_y} + \|u\|_{L^2_T L^\infty_y}, \]

where \( a, T > 0 \) will be chosen later. We assume that \( 3/4 < s < 1 \) and \( T \leq 1 \).

Here, we only estimate the \( L^\infty_T H^s_y \)-norm, because the others ones are obtained as in Theorem 1.1. From group properties, Minkowski’s inequality, and Lemma 4.2 it follows that

\[ \|\Phi(u)(t)\|_{H^s_y} \leq \|u_0\|_{H^s_y} + \int_0^T \|u^2 u_x\|_{H^s_y} dt' \]

\[ \leq \|u_0\|_{H^s_y} + c T^{\gamma} \|u_x\|_{L^{12/5}_T L^\infty_y} \left\{ \|u\|_{L^\infty_T H^s_x} \|u\|_{L^p_T L^\infty_y} + \|u\|_{L^\infty_T H^s_x} \|u\|_{L^p_T L^\infty_y} \right\} \]

\[ + c T^{\gamma} \|u\|_{L^p_T L^\infty_y} \|u\|_{L^2_T L^\infty_y} \|D_x^s u_x\|_{L^\infty_T L^2_y} + c T^{\gamma} \|u\|_{L^p_T L^\infty_y} \|u\|_{L^2_T L^\infty_y} \|D_y^s u_x\|_{L^\infty_T L^2_y}, \]

(4.7)

Thus,

\[ \|\Phi(u)\|_{L^\infty_T H^s_y} \leq \|u_0\|_{H^s_y} + c T^{\gamma} \|u\|_{s, 2}^3. \]

Finally, gathering together all estimates we see that

\[ \|\Phi(u)\|_{s, 2} \leq c \|u_0\|_{H^s_y} + c T^{\gamma} \|u\|_{s, 2}^3. \]

Choosing \( a = 2c \|u_0\|_{H^s} \), and then \( T \) such that

\[ c T^{\gamma} a^2 < \frac{1}{20}, \]

(4.8)
we deduce that $\Phi : Z_T^a \rightarrow Z_T^a$ is well defined and is a contraction. The rest of the proof follows standard arguments. So we will omit it. □

**Proposition 4.3.** Consider the IVP

$$\begin{cases} v_t + \partial_x \Delta v + v^2 v_x = 0, \quad (x, y) \in \mathbb{R}^2, \quad t > 0, \\ v(x, y, 0) = v_0(x, y) \in H^1(\mathbb{R}^2), \end{cases}$$ (4.9)

Let $T \sim \|v_0\|_{H_1}^{-2/\gamma}$ be the existence time given by Theorem 4.1. If $v_0$ satisfies

$$\|v_0\|_{L^2} < \sqrt{3}\|\varphi\|_{L^2} \quad \text{and} \quad \|v_0\|_{H^1} \sim N^{1-s},$$ (4.10)

where $\varphi$ is the ground state solution of (1.2), then

(i) the solution $v$ of (4.9) satisfies

$$\sup_{[0,T]} \|v(t)\|_{H^1} \leq cN^{1-s}. \quad (4.11)$$

(ii) For any $\rho \in (3/4, 1)$, the solution $v$ of (4.9) satisfies

$$\|v\|_{\rho,2} \sim N^{\rho(1-s)}. \quad (4.12)$$

**Proof.** The proof of (4.11) is similar to the proof of Theorem 1.4, but instead of using (3.21), we use the following (sharp) Gagliardo-Nirenberg inequality (see [17]):

$$\frac{1}{6} \|u(t)\|_{L^4}^4 \leq \frac{1}{3} \left( \frac{\|u(t)\|_{L^2}}{\|\varphi\|_{L^2}} \right)^2 \|\nabla u(t)\|_{L^2}^2.$$ 

The proof of (4.12) follows immediately from the proof of Theorem 4.1 and the inequality $\|v_0\|_{H^\rho} \leq cN^{\rho(1-s)}$. □

**Proposition 4.4.** Let $v_0 \in H^1(\mathbb{R}^2)$ and $w_0 \in H^\rho(\mathbb{R}^2)$, $\rho > 3/4$, and let $v$ be the solution given in Proposition 4.3. Then there exists a unique solution $w$ of the IVP

$$\begin{cases} w_t + \partial_x \Delta w + w^2 w_x + 2wv v_x + 2w w v_x + v^2 w_x + w^2 v_x = 0, \quad (x, y) \in \mathbb{R}^2, \quad t > 0, \\ w(x, y, 0) = w_0(x, y), \end{cases}$$ (4.13)

defined in the same interval of existence of $v$, $[0, T]$, such that

$$w \in C([0, T]; H^\rho(\mathbb{R}^2)), \quad (4.14)$$

$$\|D_x^\rho w_x\|_{L^\infty_x L^2_y} + \|D_y^\rho w_x\|_{L^\infty_x L^2_y} < \infty, \quad (4.15)$$

$$\|w\|_{L^\rho_x L^\infty_y} + \|w_x\|_{L^\rho_x L^{12/5} y} < \infty, \quad (4.16)$$

and

$$\|w\|_{L^2_x L^\infty_y} < \infty, \quad (4.17)$$

where $p = \frac{2}{1-2\gamma}$ and $\gamma \in (0, 5/12)$. 
Sketch of the proof. Define the integral operator

\[
\bar{\Phi}(w)(t) = \bar{\Phi}_{w_0}(w)(t) := U(t)w_0 + \int_0^t U(t-t')F(t')dt'
\]

where

\[
F = w^2 w_x + 2 wvw_x + 2 wvw + v^2 w_x + w^2 v_x.
\]

and the function metric spaces

\[
\mathcal{W}_T = \{ w \in C([0, T]; H^p(\mathbb{R}^2)); \| w \|_{\rho, 2} < \infty \}
\]

and

\[
\mathcal{W}_T^a = \{ w \in \mathcal{W}_T; \| w \|_{\rho, 2} \leq a \},
\]

with

\[
\| w \|_{\rho, 2} := \| w \|_{L^\infty_T H^p_y} + \| w \|_{L^p_T L^\infty_y} + \| w_x \|_{L^{12/5}_T L^{6}_y} + \| D^p_x w_x \|_{L^\infty T L^2_y} + \| D^p_w \|_{L^\infty T L^2_y} + \| w \|_{L^2_T L^\infty_y},
\]

As before, we only estimate the \( L^\infty_T H^p_y \)-norm, because from our linear estimates, all the others estimates reduce to this one.

First, we note that

\[
\| D^p_x \bar{\Phi}(w) \|_{L^2} \leq \| w_0 \|_{H^p} + \int_0^T \| D^p_x F \|_{L^2} dt'.
\]

But, successive applications of Lemma 4.2(ii) lead to

\[
\int_0^T \| D^p_x (w^2 w_x) \|_{L^2_y} dt' \leq c T^\gamma \| w_x \|_{L^{12/5}_T L^{6}_y} \| w \|_{L^\infty T H^p_y} \| w \|_{L^p_T L^\infty_y} + c T^\gamma \| w \|_{L^p_T L^\infty_y} \| w \|_{L^2_T L^\infty_y} \| D^p_x w_x \|_{L^2 T L^2_y}.
\]

\[
\int_0^T \| D^p_x (wvw) \|_{L^2_y} dt' \leq c T^\gamma \| w_x \|_{L^{12/5}_T L^{6}_y} \| w \|_{L^\infty T H^p_y} \| w \|_{L^p_T L^\infty_y} + c T^\gamma \| w \|_{L^p_T L^\infty_y} \| w \|_{L^2_T L^\infty_y} \| D^p_x w_x \|_{L^2 T L^2_y}.
\]

\[
\int_0^T \| D^p_x (wvw_x) \|_{L^2_y} dt' \leq c T^\gamma \| w_x \|_{L^{12/5}_T L^{6}_y} \| w \|_{L^\infty T H^p_y} \| w \|_{L^p_T L^\infty_y} + c T^\gamma \| w \|_{L^p_T L^\infty_y} \| w \|_{L^2_T L^\infty_y} \| D^p_x w_x \|_{L^2 T L^2_y}.
\]

\[
\int_0^T \| D^p_x (v^2 w_x) \|_{L^2_y} dt' \leq c T^\gamma \| w_x \|_{L^{12/5}_T L^{6}_y} \| w \|_{L^\infty T H^p_y} \| w \|_{L^p_T L^\infty_y} + c T^\gamma \| w \|_{L^p_T L^\infty_y} \| w \|_{L^2_T L^\infty_y} \| D^p_x w_x \|_{L^2 T L^2_y}.
\]

\[
\int_0^T \| D^p_x (w^2 v_x) \|_{L^2_y} dt' \leq c T^\gamma \| v_x \|_{L^{12/5}_T L^{6}_y} \| w \|_{L^\infty T H^p_y} \| w \|_{L^p_T L^\infty_y} + c T^\gamma \| w \|_{L^p_T L^\infty_y} \| w \|_{L^2_T L^\infty_y} \| D^p_x v_x \|_{L^2 T L^2_y}.
\]
Thus, we see that
\[ \|D_y^\nu \Phi(w)\|_{L^2} \leq \|w_0\|_{H^\rho} + cT^\gamma \left\{ \|w\|_{\rho,2}^2 + \|v\|_{1,2}^2 + \|w\|_{\rho,2} \|v\|_{1,2} \right\} \|w\|_{\rho,2}. \]

Analogously, we deduce
\[ \|D_y^\nu \Phi(w)\|_{L^2} \leq \|w_0\|_{H^\rho} + cT^\gamma \left\{ \|w\|_{\rho,2}^2 + \|v\|_{1,2}^2 + \|w\|_{\rho,2} \|v\|_{1,2} \right\} \|w\|_{\rho,2} \]
and
\[ \|\Phi(w)\|_{L^2} \leq \|w_0\|_{L^2} + cT^\gamma \left\{ \|w\|_{\rho,2}^2 + \|v\|_{1,2}^2 + \|w\|_{\rho,2} \|v\|_{1,2} \right\} \|w\|_{\rho,2}. \]

Therefore, we have established that
\[ \|\Phi(w)\|_{\rho,2} \leq \|w_0\|_{H^\rho} + cT^\gamma \left\{ \|w\|_{\rho,2}^2 + \|v\|_{1,2}^2 + \|w\|_{\rho,2} \|v\|_{1,2} \right\} \|w\|_{\rho,2}. \quad (4.19) \]

Now, by choosing \( a = 2c \max\{\|v_0\|_{H^1}, \|w_0\|_{H^\rho}\} \), we see that
\[ cT^\gamma a^2 < \frac{1}{20}. \quad (4.20) \]

As a consequence, \( \tilde{\Phi} : \mathcal{W}^\rho_T \to \mathcal{W}^\rho_T \) is well defined. To finish the proof, one proceeds as usual. This proves the theorem.

\[ \square \]

**Corollary 4.5.** Let \( s \in (3/4, 1) \) be fixed. Let \( v_0 \in H^1(\mathbb{R}^2) \) and \( w_0 \in H^\rho(\mathbb{R}^2) \) with \( 3/4 < \rho \leq s \). Assume that the initial data \( w_0 \) satisfies \( \|w_0\|_{H^\rho} \sim N^{\rho-s} \) and let \( v \) and \( w \) be the corresponding solutions given in Propositions 4.3 and 4.4 respectively. Then
\[ \|w\|_{\rho,2} \leq cN^{\rho-s}. \]

**Proof.** From the proof of Proposition 4.4, we have
\[ w(t) = U(t)w_0 + \int_0^t U(t-t')F(t')dt', \quad t \in [0, T]. \quad (4.21) \]

Moreover, since \( \tilde{\Phi} : \mathcal{W}^\rho_T \to \mathcal{W}^\rho_T \), we obtain \( \|w\|_{\rho,2} \leq a \), where \( a = 2c \max\{\|v_0\|_{H^1}, \|w_0\|_{H^\rho}\} \). Analogously, \( \|v\|_{1,2} \leq a \). Hence, from (4.19) and (4.8), we get
\[ \|w\|_{\rho,2} \leq \|w_0\|_{H^\rho} + \frac{3}{20}\|w\|_{\rho,2}. \]

This completes the proof of the lemma.

\[ \square \]

**Lemma 4.6.** Define
\[ \|w\|_0 := \|w\|_{L^\infty_T L^2_{x,y}} + \|w\|_{L^\infty_T L^2_{y,x}}. \]

Let \( v_0 \in H^1(\mathbb{R}^2) \) and \( w_0 \in H^\rho(\mathbb{R}^2), \) \( 3/4 < s < 1 \), such that \( \|v_0\|_{H^1} \sim N^{1-s} \) and \( \|w_0\|_{L^2} \sim N^{-s} \). Let \( v \) and \( w \) be the solutions given in Propositions 4.3 and 4.4 respectively. Then,
\[ \|w\|_0 \leq cN^{-s}. \]
Proof. It follows from (4.21) and Lemma 2.1 that
\[ \|w_x\|_{L^\infty_T L^2_y} \leq c\|w_0\|_{L^2} + c \int_0^T \|F(t')\|_{L^2} dt'. \]

Now, applying Lemma 4.2(i), we deduce
\[ \int_0^T \|F(t')\|_{L^2} dt' \leq cT^{\gamma} \left\{ \|w\|_{L^0_T L^\infty_x L^2_y} \|w_x\|_{L^{12/5}_T L^\infty_x L^2_y} + \|v\|_{L^0_T L^\infty_x L^2_y} \|v_x\|_{L^{12/5}_T L^\infty_x L^2_y} \right\} \|w\|_0. \]

Hence, as in Corollary 4.5 we obtain
\[ \|w_x\|_{L^\infty_T L^2_y} \leq c\|w_0\|_{L^2} + \frac{1}{4} \|w\|_0. \]

Similarly, we have
\[ \|w\|_{L^\infty_T L^2_y} \leq c\|w_0\|_{L^2} + \frac{1}{4} \|w\|_0. \]

This proves the lemma.

Proposition 4.7. Assume that \( w_0 \) and \( v_0 \) satisfy the hypotheses of Corollary 4.5 and Lemma 4.6. Let \( v \) and \( w \) be the solutions given in Propositions 4.3 and 4.4, respectively. Define
\[ z(t) = \int_0^t U(t - t') F(t') dt', \tag{4.22} \]
where \( F \) is given in (4.18). Then,
\[ \|z\|_{L^\infty_T H^1} \sim N^{\frac{-2\alpha}{12}}. \]

Proof. We begin by estimating \( \|\partial_x z\|_{L^2_{xy}} \). The main tool here is the estimate (2.4). Indeed,
\[ \|\partial_x z\|_{L^2_{xy}} \leq \|\partial_x \int_0^T U(t - t') F(t') dt'\|_{L^2_{xy}} \]
\[ \leq c \left\{ \|w^2 w_x\|_{L^2_{xy}} + \|w v w_x\|_{L^2_{xy}} + \|w w v_x\|_{L^2_{xy}} + \|v^2 w_x\|_{L^2_{xy}} + \|v w^2 v_x\|_{L^2_{xy}} \right\} \]
\[ = A_1 + A_2 + A_3 + A_4 + A_5. \]

Now, from Holder’s inequality, we obtain
\[ A_1 \leq c \|w\|_{L^2_{xy}}^2 \|w_x\|_{L^\infty_T L^2_y} \leq c \|w\|_{\rho,2}^2 \|w\|_0. \]

Applying Corollary 4.5 and Lemma 4.6, we then get
\[ A_1 \leq cN^{-s} \leq cN^{\frac{-3s}{2}}. \]

From this point on we apply several times Hölder’s inequality without mentioning it.
\[ A_2 \leq cT^{1/12} \|v\|_{L^0_T L^\infty_y} \|v_x\|_{L^{12/5}_T L^\infty_y} \|w\|_{L^\infty_T L^2_y} \leq cT^{1/12} \|v\|_{\rho,2}^2 \|w\|_0. \]
Thus, Proposition 4.3(ii) and Lemma 4.6 yield $A_2 \sim N^{3-5s}$. To estimate $A_3$, we note that

$$A_3 \leq cT^{1/12}\|v\|_{L^2_T L^\infty_y} \|w_x\|_{L^{12/5}_T L^2_y} \|w\|_{L^\infty_T L^2_y} \leq cT^{1/12}\|v\|_{\rho,2}\|w\|_{\rho,2}\|w\|_0.$$ 

Since $T \sim N^{-2(1-s)/\gamma}$, $\gamma \in (0, 5/12)$, we deduce that $T^{1/12} \sim N^{-2(1-s)/5}$. Hence, from Proposition 4.3 Corollary 4.5 and Lemma 4.6 we infer $A_3 \sim N^{3-5s}$. Similarly, since

$$A_4 \leq cT^{1/12}\|w\|_{L^2_T L^\infty_y} \|v_x\|_{L^{12/5}_T L^2_y} \|w\|_{L^\infty_T L^2_y} \leq cT^{1/12}\|v\|_{\rho,2}\|w\|_{\rho,2}\|w\|_0,$$

we get $A_4 \sim N^{3-5s}$.

Finally,

$$A_5 \leq c\|v\|_{L^2_T L^\infty_y} \|w_x\|_{L^\infty_T L^2_y} \leq c\|v\|_{\rho,2}\|w\|_0.$$ 

Therefore, Proposition 4.3 and Lemma 4.6 yield $A_5 \sim N^{3-5s}$. The same analysis can be performed to estimate $\|\partial_{y}z\|_{L^2_y}$ and $\|z\|_{L^2_y}$. This completes the proof of the proposition. 

**Proof of Theorem 1.5** Let us consider the IVP

$$\begin{cases} 
  u_t + \partial_x \Delta u + u^2 u_x = 0, \\
  u(x, y, 0) = u_0(x, y).
\end{cases} \quad (4.23)$$

Assume that $u_0 \in H^s(\mathbb{R}^2)$, $3/4 < s < 1$ (a priori) and satisfies $\|u_0\|_{L^2} < \sqrt{3}\|\varphi\|_{L^2}$. We split the datum $u_0$ as

$$u_0(x) = (\chi_{|x|<\gamma N}) \tilde{u}_0(x) + (\chi_{|x|\geq\gamma N}) \tilde{u}_0(x) = v_0(x) + w_0(x),$$

where $N \gg 1$ will be chosen later. First, we note that

$$\|v_0\|_{L^2} < \sqrt{3}\|\varphi\|_{L^2}, \quad \|v_0\|_{H^1} \sim N^{1-s}, \quad (4.24)$$

and

$$\|w_0\|_{H^\rho} \sim N^{\rho-s}, \quad 3/4 < \rho \leq s < 1.$$ 

In view of Propositions 4.3 and 4.4 we can solve the IVPs (4.9) and (4.13), with initial data $v_0$ and $w_0$, respectively, obtaining solutions $v(t) \in H^1(\mathbb{R}^2)$ and $w(t) \in H^\rho(\mathbb{R}^2)$, for $t \in [0, T]$, where $T \sim N^{-2(1-s)/\gamma}$, $\gamma \in (0, 5/12)$. Moreover, the solution $u$ of (4.23) can be rewritten as

$$u(t) = v(t) + U(t)w_0 + z(t), \quad t \in [0, T],$$

where $z(t)$ is given by (4.22).

Given any $\tilde{T} > 0$, our goal now is to extend the solution $u$ on the whole interval $[0, \tilde{T}]$ by an iteration process.

At the point $t = T$, we have

$$u(T) = v(T) + U(T)w_0 + z(T). \quad (4.25)$$

Since $U$ is an unitary group, the function $U(T)w_0$ remains in $H^s(\mathbb{R}^2)$. We shall show that $v(T) + z(T)$ still satisfies the condition in (4.24).
Note that (4.25) and (1.3) imply
\[ \| v(T) + z(T) \|_{L^2} \leq \| u(T) - U(T)w_0 \|_{L^2} \]
\[ \leq \| u_0 \|_{L^2} + \| w_0 \|_{L^2} \]
\[ \leq \| u_0 \|_{L^2} + N^{-s}. \]  
(4.26)

Thus, for \( N \) large enough, we get
\[ \| v(T) + z(T) \|_{L^2} < \sqrt{3} \| \varphi \|_{L^2}. \]

Now, Proposition 4.3 leads to
\[ \| v \|_{L^\infty T H^1} \leq c N^{1-s}, \]
and Proposition 4.7 tells us that
\[ \| z \|_{L^\infty T H^1} \leq c N^{3-5s/2}. \]
Hence, at each step, there is a contribution of \( N^{3-5s/2} \) from \( \| z \|_{L^\infty T H^1} \). To reach the time \( \tilde{T} \), we need to iterate \( \tilde{T}/T \) times, and to guarantee that the \( H^1 \)-norm grows on the interval \([0, \tilde{T}]\) as \( N^{1-s} \), that is,
\[ \frac{\tilde{T}}{T} N^{3-5s/2} \sim \tilde{T} N^{2(1-s)/\gamma} N^{3-5s/2} \leq c N^{1-s}, \]
we need to choose \( N = N(\tilde{T}) \sim N^{63s-53}/63 \), for \( 53/63 < s < 1 \).
Finally, from (4.26), we have a contribution from the \( L^2 \)-norm of \( N^{-s} \). But, since \( 53/63 < s < 1 \), for that chosen \( N(\tilde{T}) \), we deduce
\[ \tilde{T} N^{2(1-s)/\gamma} N^{-s} \leq c N^{63s-53/63} N^{2(1-s)/\gamma} N^{-s} \leq c. \]
This proves the theorem. \( \square \)

Acknowledgement. F. L. was partially supported by CNPq-Brazil and A. P. was supported by CNPq/Brazil under grant 152234/2007-1.

References
[1] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations, Arch. Rational Mech. Anal. 82, (1983), 313–375.
[2] H. A. Biagioni and F. Linares, Well-posedness results for the modified Zakharov-Kuznetsov equation, in Nonlinear Equations: Methods, Models and Applications, Progr. Nonlinear Differential Equations Appl., 54, Birkhuser, Basel, 2003, 181–189.
[3] B. Birnir, G. Ponce, and N. Svanstedt, The local ill-posedness of the modified KdV equation, Ann. Inst. H. Poincaré Anal. Non Linéaire, 13 (1996), 529–535.
[4] B. Birnir, C. E. Kenig, G. Ponce, N. Svanstedt, and L. Vega, On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations, J. London Math. Soc., 53 (1996), 551–559.
[5] J. Bourgain, Refinements of Strichartz’s inequality and applications to 2D-NLS with critical nonlinearity, Internat. Math. Res. Notices 5 (1998), 253–283.
[6] J. Colliander, J. Keel, G. Staffilani, H. Takaoka, and T. Tao, Sharp global well-posedness for periodic and nonperiodic KdV and mKdV, J. Amer. Math. Soc. 16 (2003) 705–749.
[7] A. de Bouard, Stability and instability of some nonlinear dispersive solitary waves in higher dimension, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996), 89–112.
[8] A. V. Faminskii, The Cauchy problem for the Zakharov-Kuznetsov equation, Differ. Equ. 31 (1995), 1002–1012.
[9] G. Fonseca, F. Linares, and G. Ponce, Global well-posedness for the modified Korteweg-de Vries equation, *Comm. PDE*, 24 (1999), 683–705.

[10] G. Fonseca, F. Linares, and G. Ponce, Global existence for the critical generalized KdV equation, *Proc. Amer. Math. Soc.*, 131 (2003), 1847–1855.

[11] C. E. Kenig, G. Ponce, and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the Contraction principle, *Commun. Pure Appl. Math.*, 46 (1993), 527–620.

[12] C. E. Kenig and S. N. Ziesler, Local well posedness for modified Kadomtev-Petviashvili equation, *Differential Integral Equations* 18 (2005), 1111-1146.

[13] C. E. Kenig and S. N. Ziesler, Maximal function estimates with applications to a Kadomtev-Petviashvili equation, *Commun. Pure Appl. Anal.* 4 (2005), 45–91.

[14] F. Linares and A. Pastor, Well-posedness for the two-dimensional modified Zakharov–Kuznetsov equation, *SIAM J. Math Anal.* 41 (2009), 1323–1339.

[15] F. Linares and J.-C Saut, The Cauchy problem for the 3D Zakharov–Kuznetsov equation, *Discrete Contin. Dyn. Syst.*, 24 (2009), 547–565.

[16] M. Panthee, A note on the unique continuation property for Zakharov-Kuznetsov equation, *Nonlinear Anal.* 29 (2004), 425–438.

[17] M. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.*, 87 (1983), 567–576.

[18] V.E. Zakharov, and E.A. Kuznetsov, On three-dimensional solitons, *Sov. Phys. JETP* 39 (1974), 285–286.

*E-mail address: linares@impa.br, apastor@impa.br*