UNIVERSAL DEFORMATION OF A CURVE AND A DIFFERENTIAL

EMMA CARBERRY AND MARTIN ULRICH SCHMIDT

Abstract. We construct a universal local deformation (Kuranishi family) for pairs consisting of a compact complex curve and a meromorphic 1-form. Each pair is assumed to be locally planar, a condition which in particular forces the periods of the meromorphic differential to be preserved by local deformations. The hyperelliptic case yields a universal local deformation for the spectral data of integrable systems such as simply-periodic solutions of the KdV equation or of the sinh-Gordon equation (cylinders of constant mean curvature). This is the first of two papers in which we shall develop a deformation theory of the spectral curve data of an integrable system.

In this paper we construct a universal local deformation of certain pairs \((X, \Theta)\) where \(X\) is a compact one-dimensional complex analytic space and \(\Theta\) is a meromorphic differential on \(X\) with prescribed poles. The pairs are such that away from the poles of \(\Theta\), any local primitive \(x\) of \(\Theta\) can be complemented by another local function \(y\) to give a local embedding \((x, y)\) of \(X\) into the affine plane \(\mathbb{C}^2\). In a neighbourhood of a pole, instead an appropriate power of any primitive \(x\) provides a local coordinate for the curve. These locally planar pairs \((X, \Theta)\) are defined in Definition 1.5.

When the curve \(X\) is hyperelliptic then a complementary function \(y\) may be provided by a local parameter for the quotient of \(X\) by its hyperelliptic involution. Such hyperelliptic \((X, \Theta)\) arise for example as the spectral data for simply periodic solutions of KdV or the Sinh-Gordon equation.

The “locally planar” condition has the important consequence that the periods of \(\Theta\) are preserved by local deformations of the data \((X, \Theta)\), as shown in Remark 1.7. This is particularly pertinent to the spectral curve applications, as then the periods of the differential \(\Theta\) frequently satisfy restrictive conditions, such as being integer-valued.

These period-preserving deformations can be studied purely locally. There are only finitely many points \(q\) of \(X\) where the data \((X, \Theta)\) may deform non-trivially, namely the plane curve singularities or ramification points of the local Weierstraß covering \(x_q\). We show in Theorem 2.2 that every deformation of locally planar \((X, \Theta)\) is obtained by patching deformations of finitely many such germs and is uniquely determined by these germ deformations. Hence for deformations of locally planar \((X, \Theta)\) it is not necessary to create a global theory along the lines of [ACG11]. Instead we take a local
approach more closely related to the deformation theory of plane curve singularities [dJP00, GLS07]. This approach is furthermore well adapted to the spectral curve examples, as in the finite-type situation the Bloch curve has infinite arithmetic genus but finite geometric genus, and is such that the deformations permitted for it to remain a spectral curve are non-trivial only in the neighbourhood of finitely many points. Patching arguments similar to Theorem 2.2 treat such examples also.

There are two key ways in which the deformation theory of locally planar \((X, \Theta)\) is simpler than that of the curve alone. The first is the reduction of the study of deformations of the global object \((X, \Theta)\) to deformations of finitely many germs. The analogous result of course does not hold for deformations of the curve alone, as otherwise the complex structure on a Riemann surface could not be deformed. Secondly, deformations of germs of \((X, \Theta)\) have a rigidity not enjoyed by deformations of curve singularities. It is this rigidity that enables the construction of a universal local deformation, as opposed to merely a semi-universal one.

For plane curve singularities, using the infinitesimal deformations one can construct a semi-universal local deformation [dJP00, GLS07]. There one cannot hope for a universal local deformation as the dimension of the space of infinitesimal deformations is not locally constant. Smoothing out a singularity of \(f(x, y) = 0\) decreases the dimension of the Tjurina algebra \(\mathbb{C}\{x, y\}/\langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle\).

By contrast, we find that the dimension of infinitesimal deformations \(\mathbb{C}\{x, y\}/\langle f, \frac{\partial f}{\partial y} \rangle\) of \((X, \Theta)\) is locally constant, in fact the infinitesimal deformations form a vector bundle over the base (Theorem 3.5). This helps us to show that the local deformation \((19)\) is universal (Theorem 5.1). Our arguments are furthermore simpler than the construction of semi-universal deformations of plane curve singularities referenced above in that we do not rely upon Grauert’s powerful Approximation Theorem [dJP00, Chapter 8.2].

One may ask whether the base space \(T\) of this universal deformation locally parameterises isomorphism classes of pairs \((X, \Theta)\). It is a general property of universal deformations that this cannot be the case whenever \((X, \Theta)\) possesses a non-trivial automorphism (Example 5.3), as this extends to an isomorphism of fibres over distinct elements of the base \(T\). We provide also a heuristic argument suggesting that such isomorphic fibres exist only when the base fibre has a non-trivial automorphism.

Our interest in this topic arose from wanting to develop a deformation theory for the spectral data of an integrable system. Spectral curves are typically endowed with two meromorphic differentials and local primitives of these provide a locally planar embedding of the curve. Thus the locally planar condition is automatically satisfied in the integrable systems applications. In this manuscript we develop the theory of deformations of a curve with a single differential. In a subsequent paper we shall build upon these results to deal with deformations of two differentials.
The objects, deformations and morphisms we shall consider are defined in Section 1. Throughout the paper we incorporate the option of the data satisfying a reality condition. Section 2 explains that deformations of a curve $X$ and differential $\Theta$ as above are both generated and uniquely determined by deformations of finitely many germs of such data. In Section 3 we use the infinitesimal deformations of $(X, \Theta)$ to construct a specific local deformation (18) of it. Importantly, the infinitesimal deformations of the fibres (18) all have the same dimension, in fact they form a vector bundle over the base. Our main result, Theorem 5.1, is that this local deformation (18) is universal. To prove this we utilise a decomposition of certain holomorphic functions which is the subject of Section 4. Section 5 completes the proof of Theorem 5.1. It also contains a discussion of the role of automorphisms of the base fibre $(X, \Theta)$. We conclude in Section 6 by extending Theorem 5.1 to the case when $X$ and the curves through which it may be deformed are hyperelliptic. This immediately yields for example a universal local deformation for simply periodic solutions of the KdV equation or for constant mean curvature cylinders in $S^3$, $\mathbb{R}^3$, $H^3$ or $S^2 \times \mathbb{R}$.

1. Locally planar deformations

In this section we describe the data and the deformations we shall consider. Our primary objects of study are local deformations of pairs $(X, \Theta)$, where $X$ is a compact one-dimensional complex space and $\Theta$ is a meromorphic differential on $X$ with prescribed poles. We make the assumption that the pair $(X, \Theta)$ is locally planar (see Definition 1.5), which will have the pleasant consequence that such pairs are locally parameterised by a smooth manifold. The deformations of locally planar $(X, \Theta)$ (see Definition 1.6) furthermore preserve the periods of $\Theta$ (Remark 1.7). Throughout we shall use $x$ to denote a local primitive of $\Theta$.

We begin by recalling the basic notions of deformations of a complex space, which we will extend to define appropriate deformations of the data we will discuss. A deformation $X \hookrightarrow Y \rightarrow S$ of a complex space $X$ is a pair of complex spaces $Y$ and $S$ together with a flat map $Y \rightarrow S$, a marked point $s_0 \in S$ and an isomorphism from $X$ to the preimage of the point $s_0$ in $Y$. The space $X$ is called the special fibre, $Y$ the total space and $S$ the base space of the deformation. Flatness ensures that the fibres of the map $Y \rightarrow S$ depend in a regular way on the points in $S$. A morphism from a deformation $X \hookrightarrow Y \rightarrow S \ni s_0$ to a deformation $X \hookrightarrow Z \rightarrow T \ni t_0$ is a commutative diagram of holomorphic maps

$$
\begin{array}{ccc}
X & \hookrightarrow & Y \\
\| & & \downarrow \\
X & \hookrightarrow & Z \\
 & & \downarrow \\
 & & T \ni t_0
\end{array}
$$

such that the surjective horizontal maps are flat. For any deformation $X \hookrightarrow Y \rightarrow S \ni s_0$ and any open neighbourhood $O \subset S$ of $s_0$ let $U \subset Y \rightarrow S$ be the preimage of $O$. The resulting deformation $X \hookrightarrow U \rightarrow O \ni s_0$ is called
the restriction of $X \hookrightarrow Y \to S \ni s_0$ to $O$. To avoid double subscripts we shall denote the base point as $0$; frequently $S \subset \mathbb{C}^r$ and then without loss of generality we take the base point to be the origin.

**Definition 1.1.** A local deformation $X \hookrightarrow Y \to S$ is an equivalence class of deformations $X \hookrightarrow Y \to S$ defined on a neighbourhood of $0 \in S$, where two such deformations are equivalent if their restrictions to some neighbourhood of $0 \in S$ are isomorphic.

One often wishes to deform a singularity or other non-generic structure and so it is also convenient to work locally on the special fibre $X$ and consider (local) deformations of a space germ $X_p$. The above definition of a deformation is modified in the obvious way with the total space $Y$ and base space $S$ being replaced by space germs $Y_p$ and $S_0$ respectively.

Our primary concern is with the case where the complex space $X$ is a curve and away from finitely many smooth points is locally planar with respect to a primitive $x$ of $\Theta$ and some other local function $y$, meaning that $(x, y)$ locally embed $X$ into $\mathbb{C}^2$. Within the context of our main application, namely spectral curves of integrable systems, the locally planar condition appears naturally.

**Definition 1.2.** A one-dimensional complex space germ $X_q$ is locally planar with respect to $(x_q, y_q) \in \mathcal{O}_{X,q} \times \mathcal{O}_{X,q}$ if these vanish at $q$ and map $X_q$ biregularly onto the zero set of some $f \in \mathbb{C}\{x, y\}$. Let us write $V(f)$ for the space germ given by the vanishing set of a germ $f \in \mathbb{C}\{x, y\}$, and similarly define $V(F)$ for $F \in \mathbb{C}\{x, y\} \hat{\otimes} \mathcal{O}_{S_0}$ as above. A curve $X$ with two regular coordinate functions $(x, y) : X \to \mathbb{C}^2$ is called locally planar with respect to $x, y$ at $p \in X$ if the germ $X_p$ is locally planar with respect to $((x - x(p))_p, (y - y(p))_p)$ and $X$ is locally planar with respect to $x, y$ if this holds at every $p \in X$.

An example where this condition fails is that the normalisation of the cusp $y^2 = x^3$ is not locally planar with respect to the pullbacks of $(x, y)$ by the normalisation map $t \mapsto (t^2, t^3)$ at $t = 0$, since the local ring of regular functions of the cusp does not contain any function which pulls back to the function $t$.

For a germ $S_0$ of a complex space $S$ at $0 \in S$ we denote by $\mathcal{C}\{x, y\} \hat{\otimes} \mathcal{O}_{S_0}$ the stalk of the holomorphic functions on $\mathbb{C}^2 \times S$ at $(x, y, s) = (0, 0, 0)$ (c.f. [dJP00, Definition 7.3.6]).

**Definition 1.3.** A deformation

$$
(X_q, x_q, y_q) \leftrightarrow (Y_q, x_{Y,q}, y_{Y,q}) \to S
$$

of a locally planar space germ $(X_q, x_q, y_q)$ consists of a deformation $X_q \hookrightarrow Y_q \to S_0$ together with holomorphic extensions $x_{Y,q}, y_{Y,q} \in \mathcal{O}_{Y,q}$ of $x_q, y_q$ such that $(x_{Y,q}, y_{Y,q}, s)$ maps $Y_q$ biregularly onto $V(F)$ with $F \in \mathbb{C}\{x, y\} \hat{\otimes} \mathcal{O}_{S_0}$. 

Explicitly, such a deformation is given by \( F \in \mathbb{C}\{x, y\} \hat{\otimes} \mathcal{O}_{S_0} \) and a unit \( h \in \mathbb{C}\{x, y\} \) such that

\[
F(x, y, 0) = h(x, y)f(x, y).
\]

Let us write \( V(f) \) for the space germ given by the vanishing set of a germ \( f \in \mathbb{C}\{x, y\} \) as above. It is proven in [GLS07, Corollary II.1.6] that all local deformations of \( V(f) \) are indeed of the form \( V(f) \hookrightarrow V(F) \to S_0 \) with \( F \in \mathbb{C}\{x, y\} \hat{\otimes} \mathcal{O}_{S_0} \); that is that all such deformations are embeddable. Hence this definition places no restriction on the corresponding deformations of the space germs \( X_q \).

With an eye to our goal of considering deformations of data \((X, \Theta)\) with \( x \) a local primitive of \( \Theta \), the morphisms which we consider between deformations of \((X_q, x_q, y_q)\) respect the maps \( x_q \) in the following sense.

**Definition 1.4.** An \( x \)-morphism from a deformation

\[
(X_q, x_q, y_q) \hookrightarrow (Y_q, x_{Y, q}, y_{Y, q}) \to S_0
\]

to a deformation

\[
(X_q, x_q, y_q) \hookrightarrow (\tilde{Y}_q, \tilde{x}_{Y, q}, \tilde{y}_{Y, q}) \to \tilde{S}_0
\]

is a morphism of deformations

\[
\begin{align*}
X_q & \hookrightarrow Y_q \to S_0 \\
\| & \quad \downarrow \psi \quad \downarrow \phi \\
X_q & \hookrightarrow \tilde{Y}_q \to \tilde{S}_0
\end{align*}
\]

so that \( \tilde{x}_{Y, q} \circ \psi = x_{Y, q} \).

We now define the global data whose deformations we shall consider.

**Definition 1.5.** Let \( X \) be a compact one-dimensional complex space with finitely many smooth marked points \( p_1, \ldots, p_K \) and a meromorphic 1-form \( \Theta \). Write \( X^\circ := X \setminus \{p_1, \ldots, p_K\} \). We say that \((X, p_1, \ldots, p_K, \Theta)\) or more briefly \((X, \Theta)\) is locally planar with prescribed poles if
(a) **Prescribed poles:** For each \( p \in \{ p_1, \ldots, p_K \} \) there exists \( u_p \in \mathcal{O}_{X,p} \) which vanishes at \( p \) and maps \( X_p \) biregularly onto \( \mathbb{C}_0 \) such that
\[
\Theta_p = d(u_p^{-M_p}).
\]

(b) **Locally planar on \( X^\circ \), compatible with \( \Theta \):** For each \( q \in X^\circ \) there exist \( (x_q, y_q) \in \mathcal{O}_{X,q} \times \mathcal{O}_{X,q} \) which vanish at \( q \) and map \( X_q \) biregularly onto the zero set of some \( f_q \in \mathbb{C}[x, y] \). Further we require that
\[
\Theta_q = d(x_q).
\]

If in addition the following condition holds then we say that \((X, p_1, \ldots, p_K, \Theta)\) is real.

(c) **Real:** On \( X \) there exists an anti-holomorphic involution \( \eta \) which permutes \( p_1, \ldots, p_K \) and acts as:
\[
\begin{align*}
x_{\eta(q)} &= \eta^* x_q, \quad y_{\eta(q)} = \eta^* y_q, \quad u_{\eta(q)} = \eta^* u_q, \quad f_{\eta(q)}(x, y) = \bar{f}_q(\bar{x}, \bar{y}).
\end{align*}
\]

We briefly mention some consequences of these conditions. The condition (a) guarantees that \( p_1, \ldots, p_K \) are smooth points of the 1-dimensional complex space \( X \). Further, the germ \( u_p \) of the local coordinate of \( X \) at \( p \in \{ p_1, \ldots, p_K \} \) is determined by the 1-form \( \Theta \) and the condition that the germ of \( \Theta \) at \( p \) is equal to \( d(u_p^{-M_p}) \) at \( p \) up to multiplication with a \( M_p \)-th root of unity. Finally, the condition (c) implies that \( \eta \) transforms \( \Theta \) as \( \eta^* \Theta = \bar{\Theta} \). We also note that there is a slight abuse of terminology here in that whilst \( X^\circ \) is locally planar with respect to a primitive of \( \Theta \) and another coordinate, in fact \( X \) does not satisfy this condition at the points \( p_1, \ldots, p_K \). There it is instead locally biregular to an open set in \( \mathbb{C} \).

We now expand our notion of deformations to include the differential \( \Theta \) and also to include the points \( p_1, \ldots, p_K \) where \( X \) instead of being locally planar is there locally biregular to an open set in \( \mathbb{C} \).

**Definition 1.6.** A deformation
\[
(X, \Theta) \hookrightarrow (Y, \Theta_Y) \rightarrow \mathcal{S}
\]
of locally planar \((X, \Theta)\) with prescribed poles is given by a deformation
\[
X \hookrightarrow Y \rightarrow \mathcal{S}
\]
of the complex space \( X \) together with a meromorphic 1-form \( \Theta_Y \) on \( Y \), such that the following conditions hold:

(a) **Prescribed poles:** For each \( p \in \{ p_1, \ldots, p_K \} \), the germ \( u_p \) of Definition 1.5 (a) extends to \( u_{Y,p} \in \mathcal{O}_{Y,p} \) such that \( (u_{Y,p}, s) \) maps \( Y_p \) biregularly onto \( \mathbb{C} \times S_{(0,0)} \) and \( \Theta_{Y,p} = d(u_{Y,p}^{-M_p}) \).

(b) **Locally planar on \( X^\circ \), compatible with \( \Theta \):** For each \( q \in X^\circ = X \setminus \{ p_1, \ldots, p_K \} \), the induced deformation
\[
(X_q, x_q, y_q) \hookrightarrow (Y_q, x_{Y,q}, y_{Y,q}) \rightarrow \mathcal{S}
\]
of the locally planar space germ \((X_q, x_q, y_q)\) satisfies \( \Theta_{Y,q} = d(x_{Y,q}) \).

(c) **Real:** If \((X, \Theta)\) is real, i.e. satisfies Definition 1.5 (c), then \( \eta \) extends to involutions of \( Y \) and \( \mathcal{S} \) which commute with the maps.
\[ X \mapsto Y \rightarrow S \text{ and act as} \]

\[
x_{Y,\eta(q)} = \eta^*x_{Y,q} \quad y_{Y,\eta(q)} = \eta^*y_{Y,q} \quad u_{Y,\eta(q)} = \eta^*u_{Y,q}
\]

Here \( Y_p \) denotes for all \( p \in X \) the space germ of \( Y \) at \( p \) and \( \Theta_{Y,p} \) the germ of the 1-form \( \Theta_Y \) at \( p \). Analogously \( \mathbb{C} \times S_{(0,0)} \) denotes the space germ of \( \mathbb{C} \times S \) at \( (0,0) \in \mathbb{C} \times S \).

Again (a) guarantees that \( p_1, \ldots, p_K \) are deformed into smooth points of the fibres of \( Y \rightarrow S \) and for each \( p \in \{p_1, \ldots, p_K\} \), \( u_{Y,p} \) is determined by \( \Theta_Y \) up to multiplication by a \( M_p \)-th root of unity. The reality condition (c) implies that \( \Theta_Y \) transforms as \( \eta^*\Theta_Y = \Theta_Y \).

The locally planar condition (b) of Definition 1.6 forces the periods of \( \Theta \) to be locally preserved along fibres of the deformation, as we now explain.

**Remark 1.7.** Let \( (X, \Theta) \mapsto (Y, \Theta_Y) \mapsto S \) be a deformation of locally planar \((X, \Theta)\) with prescribed poles. Suppose \( s : (-\epsilon, \epsilon) \rightarrow S \) and \( \gamma : S^1 \times (-\epsilon, \epsilon) \rightarrow Y \) are continuous maps such that the diagram

\[
\begin{array}{ccc}
S^1 \times (-\epsilon, \epsilon) & \xrightarrow{\gamma} & Y \\
\downarrow & & \downarrow \\
(-\epsilon, \epsilon) & \xrightarrow{s} & S
\end{array}
\]

commutes, where the left vertical map is the projection onto the second factor and \( s(0) = 0 \). Then for \( t \in (-\epsilon, \epsilon) \) the element of \( \pi_1(Y) \) defined by \( \gamma(\cdot, t) \) and the period \( \int_{\gamma(\cdot, t)} \Theta_Y \) are independent of \( t \).

**Proof.** The continuous map \( \gamma \) provides a homotopy between \( \gamma(\cdot, a) \) and \( \gamma(\cdot, b) \) and hence the resulting element of \( \pi_1(Y) \) is independent of \( t \). For the second statement, we note that if \( Y \) and \( \gamma \) are smooth and \( \gamma \) avoids the poles of \( \Theta_Y \) then the remark is a consequence of Stokes' Theorem and the fact that \( \Theta_Y \) is locally exact and hence closed. We adapt the proof of Stokes' Theorem and give an argument which does not require reformulation when these conditions fail. Fix \( a < b \in (-\epsilon, \epsilon) \). We consider the map \( \beta : [0, 2\pi] \times [a, b] \rightarrow Y \) with \( \beta(\varphi, t) = \gamma(e^{i\varphi}, t) \). For each \( (\varphi, t) \in [0, 2\pi] \times [a, b] \) we may take an open rectangular neighbourhood \( Q_{(\varphi, t)} \subset [0, 2\pi] \times [a, b] \) such that the differential \( \Theta_Y \) has a local integral \( x_{Y,\beta(\varphi, t)} \) defined on \( \beta(Q_{(\varphi, t)}) \). If \( \beta(\varphi, t) \) is not a pole of \( \Theta_Y \) then this local function is that guaranteed by condition (b) of Definition 1.6. If \( \beta(\varphi, t) \) is a pole then we take \( x_{Y,\beta(\varphi, t)} = u_{Y,p}^{-M_p} \) as in condition (a) of Definition 1.6. Taking a finite subcover \( Q_1, \ldots, Q_N \), on each \( Q_n \) we have a function \( \tilde{x}_n \) obtained by pulling back the corresponding local integral of \( \Theta_Y \). Writing \( \bar{Q}_n(t) \) for the slice where the second coordinate takes the value \( t \), for each \( t \in [a, b] \) then there are finitely many \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\} \) so that

\[
\int_{\gamma(\cdot, t)} \Theta_Y = \sum_{j=1}^k \int_{Q_{ij}(t)} d\tilde{x}_{ij}
\]

and moreover by construction, this is locally constant in \( t \). \( \square \)
Now we shall describe the morphisms of these deformations.

**Definition 1.8.** A morphism from one deformation \((X, \Theta) \hookrightarrow (Y, \Theta_Y) \rightarrow S\) to another \((X, \Theta) \hookrightarrow (\tilde{Y}, \tilde{\Theta}_Y) \rightarrow \tilde{S}\) is a morphism

\[
\begin{align*}
X & \hookrightarrow Y \rightarrow S \\
\| & \downarrow \psi \downarrow \phi \\
X & \hookrightarrow \tilde{Y} \rightarrow \tilde{S}.
\end{align*}
\]

of the underlying deformation of \(X\) which satisfies \(u_{\tilde{Y}} \circ \psi_p = u_Y \circ p\) for each \(p \in \{p_1, \ldots, p_K\}\) and such that for each \(q \in X^0\), the induced morphism of deformations of space germs \((X_q, x_q, y_q)\) is an \(x\)-morphism.

It immediately follows that \(\Theta_Y\) is equal to \(\psi^* \Theta_{\tilde{Y}}\), and so no special provision needs to be made for the differential in the above definition.

As for deformations of complex spaces, we have the following definition.

**Definition 1.9.** A local deformation \((X, \Theta) \hookrightarrow (Y, \Theta_Y) \rightarrow S_0\) is an equivalence class of deformations \((X, \Theta) \hookrightarrow (Y, \Theta_Y) \rightarrow S\) defined on a neighbourhood of \(0\), where two such deformations are equivalent if their restrictions to some neighbourhood of \(0\) are isomorphic.

The notion of morphism extends to local deformations in the obvious way. Our primary interest lies in the category of local deformations of locally planar \((X, \Theta)\). The main result of this paper is the construction of a universal object in this category (Theorem 5.1).

### 2. Patching deformations

Recall that we denote by \(q_l, l = 1, \ldots, L\) the finitely many points where either \(X\) has a singularity or the local primitive \(x\) of \(\Theta\) has a ramification point. The main result of this section is that a deformation of the corresponding germs \((X_l, x_l, y_l)\) can be uniquely extended to a local deformation of \((X, \Theta)\). This reduces our study of the compact curve \(X\) and differential \(\Theta\) to that of finitely many germs and we shall make use of this simplification throughout the rest of the paper. To avoid double indices we have written \(l\) rather than \(q_l\) as the index for the germs.

Let \((X, \Theta)\) be locally planar with prescribed poles at marked points \(p_1, \ldots, p_K\) and \((X, \Theta) \hookrightarrow (Y, \Theta_Y) \rightarrow S\) a deformation. The restriction of a deformation to the complement of sufficiently small open neighbourhoods of \(q_1, \ldots, q_L\) is locally \(x\)-isomorphic to the trivial deformation. To see this, note first that condition (a) of Definition 1.6 ensures that the space germs of \(Y\) at the marked points are isomorphic to the space germs of \(C \times S\) at \((0, 0)\). Furthermore, at smooth points \(q \in X^0\) where \(\Theta_q = d(x_q)\) does not vanish, \(x_q\) maps \(X_q\) biregularly onto \(C_0\) and \((x_{Y,q}, s)\) maps \(Y_q\) biregularly onto \(C \times S_{(0,0)}\). This observation, which we formalise below, proves the uniqueness part of Theorem 2.2. Note that although this lemma is stated
Lemma 2.1. Let \((X, \Theta)\) be locally planar with prescribed poles and denote by \(q_1, \ldots, q_L \in X\) the points at which either \(X\) is not smooth or \(\Theta\) has a root. Let

\[(10) \quad (X, \Theta) \mapsto (Y, \Theta_Y) \mapsto \mathcal{S}, \quad (X, \Theta) \mapsto (Z, \Theta_Z) \mapsto \mathcal{T}\]

be deformations such that for each \(q_l, l = 1, \ldots, L\) the corresponding deformations of space germs at \(q_l\) are \(x\)-isomorphic with respect to an isomorphism \(\phi_0 : \mathcal{S}_0 \to \mathcal{T}_0\). Then the deformations \((10)\) are isomorphic.

Proof. For each of the marked points \(p \in \{p_1, \ldots, p_K\}\), recall from Definition 1.6 that the maps \((u_{Y,p}, s) : Y_p \to \mathcal{S} \times \mathbb{C}\) and \((u_{Z,p}, t) : Z_p \to \mathcal{T} \times \mathbb{C}\) are isomorphisms. The germ \(\psi_p : Y_p \to Z_p\) defined by \(\psi_p = (u_{Z,p}, t)^{-1} \circ (\phi_0, \text{id}_\mathbb{C}) \circ (u_{Y,p}, s)\) satisfies \(u_{Z,p} \circ \psi_p = u_{Y,p}\), so \(\psi_p\) is an \(x\)-morphism. Analogously for each \(q \in X^0 \setminus \{q_1, \ldots, q_L\}\) since \((x_{Y,q}, s)\) and \((x_{Z,q}, t)\) are isomorphisms, replacing \(u\) by \(x\) yields \(\psi_q : Y_q \to Z_q\). Since for each \(q \in \{q_1, \ldots, q_L\}\) the germs \(\psi_q : Y_q \to Z_q\) are \(x\)-isomorphisms, all these germs \(\psi_q : Y_q \to Z_q\) with \(q \in X\) fit together to form a holomorphic map \(\psi\) from an open neighbourhood of the compact subset \(X \subset Y\) into \(Z\). The complement of this open neighbourhood of \(X\) is mapped by \(Y \to \mathcal{S}\) onto the complement of an open neighbourhood \(\mathcal{S}'\) of \(0 \in \mathcal{S}\), since a proper continuous map to a metrisable space is closed. We just constructed a morphism from the restriction of the deformation on the left hand side of \((10)\) to \(\mathcal{S}'\) to the deformation on the right hand side. Reversing the roles of these two deformations yields the inverse morphism. This shows uniqueness of \((6)\) up to isomorphism and restriction to an open neighbourhood of \(0\) in \(\mathcal{S}\).

We now proceed to the more technically difficult existence part of Theorem 2.2. That is, we now show that given deformations of each of the space germs \(X_{q_l}\) one can “patch” these together with this trivial deformation to yield a deformation of \((X, \Theta)\). To avoid double indices we abbreviate the index \(q_l\) for each \(l = 1, \ldots, L\) by the index \(l\) so that for example we will hitherto write \(X_l\) for the space germ \(X_{q_l}\). In particular, \(x_l = x_{q_l}, y_l = y_{q_l}, f_l = f_{q_l}\) and \(F_l = F_{q_l}\). Furthermore, we abbreviate the space germs \(X_l = X_{q_l}\) and \(Y_l = Y_{q_l}\).

Theorem 2.2. Let \((X, \Theta)\) be locally planar with prescribed poles and denote by \(q_1, \ldots, q_L \in X\) the points at which either \(X\) is not smooth or \(\Theta\) has a root. For each \(l = 1, \ldots, L\) suppose we are given a deformation \((X_l, x_l, y_l) \mapsto (\tilde{Y}_l, x_{Y_l}, y_{Y_l}) \mapsto \mathcal{S}_0\) of the locally planar space germ \((X_l, x_l, y_l)\). Then there exists a unique local deformation \((X, \Theta) \mapsto (Y, \Theta_Y) \mapsto \mathcal{S}_0\) such that for each \(l = 1, \ldots, L\) the induced deformation \((X_l, x_l, y_l) \mapsto (Y_l, x_{Y_l}, y_{Y_l}) \mapsto \mathcal{S}_0\) is \(x\)-isomorphic to the given one.
Proof. We shall prove this theorem by constructing a deformation \((X, \Theta) \hookrightarrow (Y, \Theta_Y) \rightarrow S\) as in (6) with the above properties. The uniqueness of (9) is an immediate consequence of Lemma 2.1. From Definition 1.5 each \(X_l\) is biregular via \((x_l, y_l)\) to the vanishing set germ \(V(f_l)\) of some \(f_l \in \mathbb{C}\{x, y\}\) and \(\tilde{Y}_l\) is biregular to \(V(F_l)\) for some \(F_l \in \mathbb{C}\{x, y\} \otimes O_S\) where \(F_l(x, y, 0) = H_l(x, y) f_l(x, y)\) for a unit \(H_l \in \mathbb{C}\{x, y\}\). By replacing \(f_l\) by \(H_l f_l\) if necessary, we assume without loss of generality that \(F_l(x, y, 0) = f_l(x, y)\). The space germ \(S_0\) is embedded in some \(\mathbb{C}_k^k\). The holomorphic functions \(F_1, \ldots, F_L\) clearly extend to holomorphic functions in \(\mathbb{C}\{x, y\} \otimes O_{\mathbb{C}_k^k}\). This extension is not unique, but the restriction to the subspace \(S_0\) is unique. We prove the Lemma for \(S_0 = \mathbb{C}_k^k\) and construct a deformation with base equal to an open set in \(\mathbb{C}_k^k\). Since the restriction of this deformation to a subspace \(S\) of the base is a deformation with base \(S\), the general case follows.

We apply the Weierstraß Preparation Theorem [PR94, Chapter I Theorem 1.4] and replace \(F_1, \ldots, F_L\) by Weierstraß polynomials \(f_1, \ldots, f_L \in \mathbb{C}\{x\}[y] \otimes O_S\). Furthermore, we replace \(f_1, \ldots, f_L\) by the corresponding evaluations \(f_1, \ldots, f_L \in \mathbb{C}\{x\}[y]\) of these Weierstraß polynomials at \(s = 0\). For \(l = 1, \ldots, L\) the polynomials \(F_l\) and \(f_l\) have highest coefficient 1 and all other coefficients vanish at \((x, s) = (0, 0)\) and \(x = 0\), respectively.

We choose an open ball \(S\) around \(0 \in \mathbb{C}_k^k\) and \(\delta_1 > 0\) such that the polynomials \(F_l\) are holomorphic on \((x, y, s) \in B(0, \delta_1) \times \mathbb{C} \times S\). For sufficiently small \(\delta_1\) there exist disjoint open neighbourhoods \(O_l\) of \(q_l\) in \(X\) such that the maps \((x_l, y_l)\) in Definition 1.5 (b) extend to biregular maps
\[
(11) \quad (x_l, y_l) : O_l \rightarrow U_l, \quad U_l = \{(x, y) \in B(0, \delta_1) \times \mathbb{C} \mid f_l(x, y) = 0\}.
\]
The set \(\{q_1, \ldots, q_L\}\) is comprised of the singular points of \(X\) and the smooth points which are roots of \(\Theta\). Hence
\[
(12) \quad U_l \rightarrow B(0, \delta_1), \quad (x, y) \mapsto x
\]
is a Weierstraß covering with a single branch point at \((0, 0)\). We can take \(\delta_1\) sufficiently small and shrink \(S\) so that for all \(l = 1, \ldots, L\),
\[
V_l = \{(x, y, s) \in B(0, \delta_1) \times \mathbb{C} \times S \mid F_l(x, y, s) = 0\}
\]
is exhibited as a Weierstraß covering
\[
(13) \quad V_l \rightarrow B(0, \delta_1) \times S, \quad (x, y, s) \mapsto (x, s)
\]
and is unbranched over \((x, s) \in A \times S\) for the annulus
\[
A = B(0, \delta_1) \setminus \overline{B(0, \delta_1/2)}.
\]
We denote the restrictions of the two coverings (12) and (13) to \(A\) and \(A \times S\) as
\[
\tilde{U}_l \rightarrow A, \quad \tilde{V}_l \rightarrow A \times S,
\]
respectively. In order to extend the given deformation “trivially” to a deformation of \(X\) we now demonstrate the existence of a unique biholomorphic map \(\psi : \tilde{U}_l \times S \rightarrow \tilde{V}_l\) of the form \((x, y(x), s) \mapsto (x, y(x), s), s\) which is equal
to the identity for \( s = 0 \). The coverings \( \hat{U}_i \) and \( \hat{V}_i \) are unbranched Weierstraß coverings with an equal number of sheets. Therefore on \( \hat{U}_l \), we have that \( y \) is locally a holomorphic function \( y = y(x) \) of \( x \in A \). Similarly on \( \hat{V}_i \), locally \( y \) is a holomorphic function \( y = y(x, s) \) with \( (x, s) \in A \times S \). Furthermore \( y(x, 0) = y(x) \). Given a local “branch” of \( y(x) \), it extends uniquely to any simply connected subset of \( A \) containing the original point and similarly any specific branch of \( y(x, s) \) extends uniquely on simply connected subsets of \( A \times S \). Let \( W_{l_i} \) be an open subset of \( \hat{U}_i \) which is mapped biholomorphically by \( x \) onto an open simply connected subset \( B \) of \( A \). The cartesian product \( W_{l_i} \times S \) is also simply connected since \( S \) is an open ball in \( \mathbb{C}^k \). Hence there exists a unique biholomorphic map from \( W_{l_i} \times S \) onto a simply connected open subset of \( \hat{V}_i \) which preserves \( (x, s) \) and is equal to the identity for \( s = 0 \).

\[
\begin{array}{ccc}
W_{l_i} \times S & \longrightarrow & \hat{V}_i \\
\downarrow & & \downarrow \\
B \times S & \longrightarrow & B \times S
\end{array}
\]

\[
(x, y(x), s) \quad \longrightarrow \quad (x, y(x), s) \quad \longrightarrow \quad (x, y(x), s) \quad \longrightarrow \quad (x, s).
\]

We first cover \( A \) by finitely many simply connected open subsets \( B \) such that the intersection of any two of them is either empty or connected. Denoting by \( n_l \) the number of sheets of the Weierstrass coverings \((12)\) and \((13)\), there are \( n_l \) simply connected open subsets \( W_{l_i} \) which each map biholomorphically onto \( B \) via the projection \( x \), and taken altogether these cover \( \hat{U}_l \). Since \( \hat{V}_l \) is unbranched over \( (x, s) \in A \times S \), the maps in the above diagram patch together to form a unique biholomorphic map

\[
\psi_l : \hat{U}_l \times S \rightarrow \hat{V}_l, \quad (x, y(x), s) \mapsto (x, y(x), s).
\]

For each \( l = 1, \ldots, L \) let \( C_l \) denote the preimage in \( O_l \subset X \) of \( U \setminus \hat{U}_l \) with respect to the map \((11)\). Writing \( \hat{O}_l = O_l \setminus C_l \), we have that the open subsets \( \hat{O}_l \times S \) of \((X \setminus (C_1 \cup \ldots \cup C_L)) \times S\) are mapped by \( \psi_l \circ (x_l, y_l, s) \) biregularly onto \( \hat{V}_l \). This identification of the two “annuli” allows us to glue \((X \setminus (C_1 \cup \ldots \cup C_L)) \times S\) with \( V_1 \cup \ldots \cup V_L \) along the maps \( \psi_l \circ (x_l, y_l, s) \) for \( l = 1, \ldots, L \). This defines \( Y \) together with the map \( Y \rightarrow S \). The differential \( \Theta \) on \( X \) extends to a differential on \((X \setminus (C_1 \cup \ldots \cup C_L)) \times S\). Its restriction to \( \hat{O}_l \times S \) agrees with the pullback under \( \psi_l \circ (x_l, y_l, s) : \hat{O}_l \times S \rightarrow \hat{V}_l \) of the differential \( \Theta \) on \( V \). Hence we obtain a global differential \( \Theta_Y \) on \( Y \) which for \( q \notin \{p_1, \ldots, p_K\} \) satisfies \( \Theta_{Y,q} = d(x_{Y,q}) \). The sets \( \{p_1, \ldots, p_K\} \) and \( \{q_1, \ldots, q_L\} \) are disjoint and so each \( p \in \{p_1, \ldots, p_K\} \) has an open neighbourhood \( O_p \times S \) in \( X \) such that \( O_p \times S \) is a neighbourhood of \( p \) in \( Y \). Hence the germ \( u_p \) of Definition \((1.5)\) (a) extends to \( u_{Y,p} \in O_{Y,p} \) such that \( (u_{Y,p}, s) \) maps \( Y_p \) biregularly onto \( \mathbb{C} \times S_{(0,0)} \) and \( \Theta_{Y,p} = d(u_{Y,p}^{-M_p}) \). We claim that the map \( Y \rightarrow S \) is proper and flat. It is the composition of a Weierstraß map \( Y_l \rightarrow B(0, \delta_1) \times S \) and the projection \( B(0, \delta_1) \times S \rightarrow S \). By \([PR94\text{ Chapter II Proposition 2.10}]\) and the Weierstraß Isomorphism \([GR84\text{ Chapter 2 §4.2}]\),
Weierstraß maps are flat. Furthermore, by [PR94, Chapter II Corollary 2.7] projections of complex spaces are flat. This shows flatness of $Y \rightarrow S$, since the composition of flat maps is flat [PR94, Chapter II Proposition 2.6]. It is also proper since $X$ and $C_1, \ldots, C_L$ are compact. Hence we have constructed a deformation (6) with the properties in the theorem. □

3. Infinitesimal deformations

In this section we will construct a local deformation (19) of a locally planar $(X, \Theta)$ from the space of isomorphism classes of infinitesimal deformations of $(X, \Theta)$. Furthermore, we prove that the isomorphism classes of infinitesimal deformations of the fibres of (18) form a vector bundle over its base. We will later use this vector bundle structure to show that the local deformation is universal. The universality will then imply that the dimension of the isomorphism classes of infinitesimal deformations of the fibres of any deformation of $(X, \Theta)$ are locally constant.

Definition 3.1. An infinitesimal deformation is a deformation whose base space is the complex space germ $\mathcal{I}_\epsilon$ consisting of a single point with local ring (i.e. holomorphic functions) $\Pi = \mathbb{C}[\epsilon]/(\epsilon^2) \simeq \mathbb{C}[\epsilon]/(\epsilon^2)$.

Informally, $\mathcal{I}_\epsilon$ may be considered as a point with a single tangent direction. For an analytic curve germ $V(f)$ in the plane, the Tjurina algebra $\mathbb{C}\{x, y\}/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ parameterises isomorphism classes of infinitesimal deformations of $V(f)$ [dJP00, Theorem 10.1.5]. The natural analog holds for $x$-isomorphism classes:

Lemma 3.2. The $x$-isomorphism classes of the infinitesimal deformations of a space germ $V(f)$ with $f \in \mathbb{C}\{x, y\}$, $f(0, 0) = 0$ are in one-to-one correspondence with

$$\mathbb{C}\{x, y\}/(f, \frac{\partial f}{\partial y}).$$

Proof. An infinitesimal deformation of $V(f)$ is given by $V(F)$ for $F(\epsilon) = f + \epsilon g$ and $g \in \mathbb{C}\{x, y\}$. Suppose we are given two infinitesimal deformations of $V(f)$, defined by $F_1(\epsilon) = f + \epsilon g_1$ and $F_2(\epsilon) = f + \epsilon g_2$ respectively. An $x$-morphism between them has the form

$$\begin{array}{ccc}
V(f) & \leftrightarrow & V(F_1) \\
\| & \downarrow \psi & \downarrow \phi \\
V(f) & \leftrightarrow & V(F_2)
\end{array}$$

Note however that any automorphism $\phi$ of $\mathcal{I}_\epsilon$ can be lifted to an automorphism of the infinitesimal deformation. Hence for the purpose of classifying $x$-isomorphism classes of the infinitesimal deformations, we may assume that $\phi$ is the identity. An $x$-morphism for which $\phi$ is the identity is explicitly described, as in (11), by $u, H \in \mathbb{C}\{x, y\}$ such that

$$(1 + \epsilon H(x, y))(f(x, y + \epsilon u(x, y)) + \epsilon g_2(x, y + \epsilon u(x, y)) = f(x, y) + \epsilon g_1(x, y)$$
holds in $\mathbb{C}\{x,y\} \otimes \Pi$. The $\epsilon^0$-term and the $\epsilon^1$-term yield

$$f(x, y) = f(x, y), \quad H(x, y)f(x, y) + u(x, y)\frac{\partial f}{\partial y}(x, y) + g_2(x, y) = g_1(x, y).$$

We conclude that the two infinitesimal deformations $F_1(\epsilon) = f + \epsilon g_1$ and $F_2(\epsilon) = f + \epsilon g_2$ are isomorphic if and only if $g_1 = g_2$ in $\mathbb{C}\{x, y\}/\langle f, \frac{\partial f}{\partial y} \rangle$. \hfill $\Box$

We now describe how the space of isomorphism classes of infinitesimal deformations of a locally planar $(X, \Theta)$ induces a local deformation. Let $(X, \Theta)$ be locally planar and as above let $q_1, \ldots, q_L \in X$ be the points at which either $X$ is not smooth or $\Theta$ has a root. Take $f_l \in \mathbb{C}\{x, y\}$ such that $X_l = X_{q_l}$ is biregular to $V(f_l)$ via $(x_l, y_l)$. By Lemma 3.3 and Theorem 2.2, taking a basis $g_l$ of each $\mathbb{C}\{x, y\}/\langle f_l \rangle$ can be thought of as a choice of basis for the space of infinitesimal deformations of $(X, \Theta)$. To see that the spaces $\mathbb{C}\{x, y\}/\langle f_l \rangle$ are finite dimensional and to choose an appropriate basis of each, we begin with some observations.

The Weierstraß Preparation Theorem [PR94, Chapter I Theorem 1.4] tells us that each space $\mathbb{C}\{x, y\}/\langle f_l \rangle$ is unchanged when we replace $f_l$ by the unique Weierstraß polynomial $f_l \in \mathbb{C}\{x\}[y]$ whose degree is the order $d_l$ of $f_l$ in $y$ at 0 and which has highest coefficient 1 and lower coefficients vanishing at $x = 0$. For sufficiently small $\delta_1 > 0$ these Weierstraß polynomials $f_1, \ldots, f_L$ are holomorphic on $x \in B(0, \delta_1)$ and belong to $\mathbb{C}[y] \otimes \mathcal{O}_{B(0, \delta_1)}$. The corresponding complex space $U_l$ is a Weierstraß covering over $x_l \in B(0, \delta_1)$ with $d_l$ sheets. Condition (b) of Definition 1.5 guarantees that for sufficiently small $\delta_1$ the maps $(x_l, y_l)$ extend to give a biregular morphism of an open neighbourhood $O_l \subset X$ of each $q_l$ to $U_l$, as in (11).

Applying the Weierstraß Division Theorem [PR94, Chapter I Theorem 1.1], we have that the quotient $\mathbb{C}\{x, y\}/\langle f_l \rangle$ is the same as $\mathbb{C}\{x\}[y]/\langle f_l, \frac{\partial f_l}{\partial y} \rangle$. This implies $\mathbb{C}\{x, y\}/\langle f_l, \frac{\partial f_l}{\partial y} \rangle = \mathbb{C}\{x\}[y]/\langle f_l, \frac{\partial f_l}{\partial y} \rangle$. Moreover, $\frac{\partial f_l}{\partial y}$ does not vanish identically on the zero locus of $f_l$ in $(x, y) \in \mathbb{C}^2$ and $\mathbb{C}\{x\}[y]/\langle f_l, \frac{\partial f_l}{\partial y} \rangle$ is finite dimensional.

For each $l = 1, \ldots, L$, choose tuples $g_l = (g_{l,1}, \ldots, g_{l,r_l})$ of $g_{l,i}(x, y) \in \mathbb{C}[y] \otimes \mathcal{O}_{B(0, \delta_1)}$, polynomial with respect to $y$ of degree less than $d_l$, which induce a basis of

$$\mathbb{C}\{x\}[y]/\langle f_l, \frac{\partial f_l}{\partial y} \rangle \simeq \mathbb{C}[y] \otimes \mathcal{O}_{B(0, \delta_1)}/\langle f_l, \frac{\partial f_l}{\partial y} \rangle.$$

We may choose small open balls $T_l \subset \mathbb{C}^{r_l}$, such that for all $t_l \in T_l$, the roots of the discriminant of the polynomial $f_l + t_l \cdot g_l$ with respect to $y_l$ belong to $x \in B(0, \delta_1/2)$. Each basis element $g_l$ defines a deformation

$$U_l \to V_l \to T_l$$

where

$$(16) \quad G_l(x, y, t_l) = f_l(x, y) + t_l \cdot g_{l,1}(x, y) + \ldots + t_l \cdot g_{l,r_l}(x, y)$$

$$=: f_l(x, y) + t_l \cdot g_l(x, y)$$
and

\[ V_l = \{(x, y, t_l) \in B(0, \delta_1) \times \mathbb{C} \times T_l \mid G_l(x, y, t_l) = 0\}. \]

As in the proof of Theorem 2.2, each \( V_l \) is a Weierstraß covering with \( d_l \) sheets over \( x \in B(0, \delta_1) \times T_l \), unbranched over \( A \times T_l \) with \( A \) the annulus in (14). Write \( T = T_1 \times \ldots \times T_L \). We may glue \((X \setminus (C_1 \cup \ldots \cup C_L)) \times T_l\) with \( V_1 \cup \ldots \cup V_L \), as was done in the proof of Theorem 2.2 in such a way that \( d(x_1), \ldots, d(x_L) \) extend to a global meromorphic 1-form obeying the conditions (a)-(b) of Definition 1.6.

**Definition 3.3.** We say that the deformation

\[ (X, \Theta) \hookrightarrow (Z, \Theta_Z) \to \mathcal{T} \]

described above is the deformation of \((X, \Theta)\) defined by the choice of basis \( g_1, l = 1, \ldots, L \) for the isomorphism classes of infinitesimal deformations of \((X, \Theta)\). The corresponding local deformation of \((X, \Theta)\) is denoted

\[ (X, \Theta) \hookrightarrow (Z, \Theta_Z) \to \mathcal{T}_0. \]

We shall show in Theorem 5.1 that this local deformation is universal. In particular, the isomorphism class of (18) or equivalently the local deformation (19) do not depend on the choice of the basis \( g_l \).

If we additionally impose the reality condition (c) of Definition 1.5, then the maps \((x_l, y_l) : X_l \to \mathbb{C}^2\) obey (5). We define an action \( l \mapsto \eta_l \) of \( \eta \) on \( \{1, \ldots, L\} \) such that \( \eta(q_l) = q_{\eta l} \). Due to (5) the unique Weierstraß polynomials \( f_1, \ldots, f_L \) obey

\[ f_{\eta l}(x, y) = \bar{f}_l(\bar{x}, \bar{y}). \]

We choose the bases \( g_1, \ldots, g_L \) in such a way that they obey:

\[ g_{\eta l}(x, y) = \bar{g}_l(\bar{x}, \bar{y}). \]

In particular, for \( \eta_l = l \) the coefficients of \( g_l \) take real values for real \( x_l \in B(0, \delta_1) \). Consequently the deformation (18) obeys condition (c) in Definition 1.6 with \( \eta(t_l) = t_{\eta l} \) also denoted by \( \eta \).

Our next task is to show in Theorem 3.5 that the spaces of isomorphism classes of infinitesimal deformations of the fibres of (18) form a vector bundle over \( \mathcal{T} \). In the proof we shall use the following description of the regular forms (Rosenlicht differentials).

**Lemma 3.4.** Let \( f \in \mathbb{C}[y] \otimes \mathcal{O}_W \) be a Weierstraß polynomial of degree \( d \) with respect to \( y \), whose coefficients are holomorphic functions on the open subset \( x \in W \subset \mathbb{C} \). If the roots of the discriminant of \( f \) are isolated in \( W \), then the regular forms of the complex space \( U = \{(x, y) \in W \times \mathbb{C} \mid f(x, y) = 0\} \) have the following form:

\[ \omega = g(x, y)\Theta/\partial f/\partial y(x, y), \text{ with } g \in \mathcal{O}_O. \]
Proof. By definition, a regular form is a meromorphic form whose product with any local holomorphic function has no residues [Ser88, Chapter IV §3] and [Ros52]. The statement of the lemma is a local statement. Therefore it suffices to show that for each \((x_0, y_0) \in O\) there exists a ball \(B(x_0, \epsilon) \subset W\) such that the lemma holds on the restriction of the Weierstraß covering \(O \to W\) to the preimage \(\hat{O}\) of \(B(x_0, \epsilon)\). In particular we may choose \(B(x_0, \epsilon)\) small enough such that the all connected components of \(\hat{O}\) contain only one point \((x_0, y_0)\) over \(x_0\). Since the holomorphic functions on different connected components are independent, it suffices to prove the statement of the lemma for each connected component of \(\hat{O}\) separately. Consequently we may assume that \(\hat{O} \to B(x_0, \epsilon)\) is a Weierstraß covering with only one point \((x_0, y_0)\) over \(x_0\). In this case the residue of \(h\omega\) at \((x_0, y_0)\) is the residue at \(x_0\) of the meromorphic 1-form on \(B(x_0, \epsilon)\), which is the sum of \(h\omega\) over the sheets of \(\hat{O} \to B(x_0, \epsilon)\). For \(\omega\) which are regular at \((x_0, y_0)\) and for all \(h \in \mathcal{O}_0\) the sum of \(h\omega\) over the \(d\) sheets of \(\hat{O} \to B(x_0, \epsilon)\) has no residue at \(x_0\). This implies that this sum of \(h\omega\) is holomorphic at \(x_0\) as a 1-form on \(x \in W\). Hence it suffices to prove that a meromorphic function \(g = \frac{\partial f}{\partial y} \omega\) on \(\hat{O}\) is holomorphic at \((x_0, y_0)\), if and only if for all \(h \in \mathcal{O}_0\) the sum of \(hg/\partial y\) over the \(d\) sheets of \(\hat{O} \to B(x_0, \epsilon)\) is holomorphic at \(x_0\).

Let \(f(y)\) be a complex polynomial of degree \(d\) with highest coefficient 1 and pairwise different roots \(y_1, \ldots, y_d\). For any polynomial \(g(y)\) of degree less than \(d\) we have the identity

\[ g(y) = \sum_{i=1}^{d} g(y_i) \prod_{j \neq i} (y - y_j) / \frac{\partial f}{\partial y}(y_i). \]

(20)

to see this, observe that the values of both sides at \(y = y_1, \ldots, y_d\) coincide. Since both sides are polynomials with respect to \(y\) of degree less than \(d\) they are equal. In particular, the sum \(\sum_{i=1}^{d} g(y_i) / \frac{\partial f}{\partial y}(y_i)\) is equal to the coefficient of the monomial \(y^{d-1}\) in \(g(y)\). For the monomials \(g(y) = 1, y, \ldots, y^{d-1}\) this implies

\[ \sum_{i=1}^{d} y_i^n / \frac{\partial f}{\partial y}(y_i) = \begin{cases} 0 & \text{if } 0 \leq n < d - 1 \\ 1 & \text{if } n = d - 1. \end{cases} \]

(21)

We conclude that the same is true if the coefficients of \(f\) are holomorphic functions depending on \(x \in W\), provided the roots of the discriminant of \(f\) are isolated in \(W\). Due to the Weierstraß Divison Theorem [PR94, Chapter I Theorem 1.1] each holomorphic function on \(\hat{O}\) may be written as a polynomial with respect to \(y\) of degree less than \(d\) whose coefficients are holomorphic functions depending on \(x \in B(x_0, \epsilon)\). Furthermore, the meromorphic functions on \(\hat{O}\) can be written as polynomials with respect to \(y\) of degree less than \(d\), whose coefficients are meromorphic functions on \(x \in B(x_0, \epsilon)\). Hence the sum of such a function \(g\) over the sheets of the Weierstraß covering \(\hat{O} \to B(x_0, \epsilon)\) is holomorphic at \((x_0, y_0)\), if and only if
the coefficient corresponding to \( y^{d-1} \) is holomorphic at \( x_0 \). We multiply \( g \) with the monomials \( y^{0}, y^{1}, \ldots, y^{d-1} \) and subtract from the product multiples of \( f \) such that the difference again has degree less than \( d \) with respect to \( y \). Therefore \( \sum_{i=1}^{d} g(y_i) y_i^n / \partial y_i \) is holomorphic at \( x_0 \) for holomorphic \( g \) and \( n = 1, \ldots, d - 1 \). Conversely we may conclude inductively that all coefficients of \( g \) are holomorphic at \( x_0 \), if \( \sum_{i=1}^{d} g(y_i) y_i^n / \partial y_i \) are holomorphic at \( x_0 \) for \( n = 0, \ldots, d - 1 \).

Theorem 3.5. Let \( g_l, l = 1, \ldots, L \) be a basis for the isomorphism classes of infinitesimal deformations of locally planar \( (X, \Theta) \) and

\[
(X, \Theta) \rightarrow (Z, \Theta_Z) \rightarrow T
\]

be the induced deformation of \( (X, \Theta) \), as in \([\ref{15}]\). The isomorphism classes of infinitesimal deformations of the fibres of \([\ref{13}]\) comprise a vector bundle over \( T \). On a neighbourhood of \( 0 \in T \), the \( g_l \) are linearly independent sections and hence provide a trivialisation of this bundle.

Proof. From Theorem \([\ref{22}]\) it is equivalent to show that the spaces of \( x \)-isomorphism classes of infinitesimal deformations of the \( Z(t)_l \) form a vector bundle over the base \( T \), where \( X(t) \) denotes the fibre of the projection over \( t \) and \( X(t)_l \) its germ at the point \( q_l(t) \) resulting from deforming the point \( q_l \in X \). Lemma \([\ref{22}]\) tells us that for each \( t_l \), these are given by \( \mathbb{C}\{x, y\}/(G_l(x, y, t_l), \partial G_l(x, y, t_l)) \), where \( G_l \) was defined in \([\ref{16}]\). Here \( G_l(x, y, t_l) \) is for each fixed \( t_l \) considered as a function of \( x \) and \( y \). Writing

\[
U_l(t_l) = \{(x, y) \in B(0, \delta_l) \times \mathbb{C} | G_l(x, y, t_l) = 0\},
\]

since \( U_l(t_l) \) is an open subset of the vanishing set of \( G_l(x, y, t_l) \), these \( x \)-isomorphism classes are given by the elements of

\[
\mathcal{O}_{U_l(t_l)} \frac{\partial G_l(x, y, t_l)}{\partial y}(x, y, t_l) \mathcal{O}_{U_l(t_l)}).
\]

We shall demonstrate the theorem then by showing that for \( l = 1, \ldots, L \) and small \( t_l \in T_l \), the \( g_l \) form a basis of the above quotient space.

Let \( \tilde{V}_l \) be the subspace of \( V_l \) \([\ref{17}]\) defined by the vanishing of \( \partial G_l(x, y, t_l) \) and note that \( \tilde{V}_l \) is the support of the quotient sheaf \([\ref{23}]\). By shrinking \( \tilde{V}_l \) if necessary we may assume that each \( U_l(t_l) \) is contained in a compact set on which \( \partial G_l(x, y, t_l) \) is holomorphic and hence this derivative has only finitely many roots on \( U_l(t_l) \). By the construction of the deformation \([\ref{13}]\) each \( V_l \) is isomorphic to an open subspace of \( Z \) and under this isomorphism the union \( \tilde{V}_1 \cup \ldots \cup \tilde{V}_L \) defines a closed subspace \( \tilde{Z} \) of \( Z \). The map \( \tilde{Z} \rightarrow T \) is finite and we shall show that it is again flat. As proven in \([\ref{PR94}, \text{Chapter II Proposition 2.10}]\), a finite holomorphic map \( \tilde{Z} \rightarrow T \) to a reduced complex analytic space is flat precisely when for \( l = 1, \ldots, L \) and \( t = (t_1, \ldots, t_L) \) the sum of the dimensions of \([\ref{23}]\) over the finitely many points \( (x, y, t_l) \in \tilde{V}_l \) is locally constant on \( t \in T \).
Let \( S(t) \) be the locally free subsheaf of the sheaf of meromorphic functions on the fibres \( X(t) \) of (18) which is generated by \( \left( \frac{\partial G_l}{\partial y}(x, y, t_l) \right)^{-1} \) near \( q_l \) for all \( l = 1, \ldots, L \) and generated by 1 away from \( q_1, \ldots, q_L \). The aforementioned sum of dimensions of (23) is equal to

\[
\dim H^0(X(t), S(t)/\mathcal{O}_{X(t)}) =: \deg(S(t)).
\]

We remark that a finitely generated subsheaf of the sheaf of meromorphic functions is termed a generalised divisor and is a natural extension of the notion of a divisor to one-dimensional complex analytic spaces with singularities; the above is the usual definition of the degree of a generalised divisor [KLSS20]. The isomorphism of Lemma 3.4 extends to give an isomorphism

\[
S(t) \to \Omega^1_{X(t)}(D), \quad h \mapsto h\Theta
\]

where \( D \) is the divisor \( \sum_{k=1}^K (M_k + 1)p_k \) and \( \Omega^1_{X(t)}(D) \) denotes the sheaf of meromorphic 1-forms \( \omega \) on the fibre \( X(t) \), which are regular away from \( p_1, \ldots, p_K \) and have for \( k = 1, \ldots, K \) poles of order at most \( M_k + 1 \) at the smooth point \( p_k \). Consider the long exact cohomology sequence of

\[
\cdots \to \mathcal{O}_{X(t)} \to S(t) \to S(t)/\mathcal{O}_{X(t)} \to 0.
\]

The alternating sum of the dimensions vanishes and using that the first cohomology of the quotient sheaf vanishes since it has finite support, we obtain that

\[
\deg(S(t)) = 2g - 2 + K + M, \quad M = \sum_{i=1}^K M_i.
\]

The Semi-Continuity Theorem also states [PR94, Chapter III Theorem 4.7d] that the direct image of the structure sheaf under a proper flat map is a vector bundle. Applying this result here yields that the direct image of the structure sheaf of \( \tilde{Z} \) with respect to \( \tilde{Z} \to \mathcal{T} \) is a holomorphic vector bundle on \( \mathcal{T} \) of rank \( 2g - 2 + M + K \). Since \( \tilde{Z} \) is the support of the quotient sheaves (23), this is the statement that the isomorphism classes of infinitesimal deformations form a vector bundle. Then any choice of sections, such as the \( g_l \), which form a basis in the fibre over \( 0 \in \mathcal{T} \) form a basis of all fibres in a neighbourhood of \( 0 \in \mathcal{T} \). \( \square \)

4. Decomposition of holomorphic functions

In this section we prove a decomposition result which we shall use in our proof that (19) is a universal local deformation. Universality means that given a deformation \( (X, \Theta) \hookrightarrow (Y, \Theta_Y) \to \mathcal{S} \), then after shrinking \( \mathcal{S} \) if necessary, there exists a unique holomorphic map \( \phi : \mathcal{S} \to \mathcal{T} \) such that the pullback of (19) under \( \phi \) is isomorphic to the given deformation.
As explained in Theorem 2.2, such an isomorphism is determined by \( x \)-isomorphisms

\[
(\mathbf{X}_l,\mathbf{x}_l,\mathbf{y}_l) \mapsto (\mathbf{Y}_l,\mathbf{y}_l,\mathbf{y}_l) \mapsto S_0
\]

\[
(\mathbf{X}_l,\mathbf{x}_l,\mathbf{y}_l) \mapsto (\mathbf{Z}_l,\mathbf{z}_l,\mathbf{y}_l) \mapsto T_0
\]

of deformations of the corresponding germs \( \mathbf{X}_l = X_{\mathbf{q}_l} \) for \( l = 1, \ldots, L \).

Explicitly, these \( x \)-morphisms are comprised for each \( l = 1, \ldots, L \) of a triple \((\phi_l, u_l, H_l)\), such that

\[
H_l(x,y,s)G_l(x,u_l(x,y,s),\phi_l(s)) = F_l(x,y,s),
\]

where \( G_l \) was defined in (10) and \((x_{\mathbf{q}_l},y_{\mathbf{q}_l})\) defines a biregular map from \( Y_l \) to \( V(F_l) \). We shall differentiate (4) with respect to \( s \in S \) and derive a vector field for such triples and then apply the Picard-Lindelöf Theorem. The differential of the left hand side in (4) consists of a linear combination of \( G_l, \frac{\partial G_l}{\partial u_l} \) and \( g_l \). In order to derive the vector field, we prove in the next lemma that the derivative of the right hand side in (4) has a unique such decomposition.

We supplement the choice of \( \delta_1 > 0 \) in the proof of Theorem 2.2 by a \( \delta_2 > 0 \) chosen so that for each \( l = 1, \ldots, L \) the complex space \( U_l \) in (11) is a subset of the multidisc \((x, y) \in B(0, \delta_1) \times B(0, \delta_2)\). Furthermore, we choose the balls \( \mathcal{T}_l \subset \mathbb{C}^{r_l} \) sufficiently small such that \( V_l \) is a subset of the multidisc \((x, y, t_l) \in B(0, \delta_1) \times B(0, \delta_2) \times \mathcal{T}_l \). Then let \( H \) denote the Banach space of bounded holomorphic functions on \((x, y) \in B(0, \delta_1) \times B(0, \delta_2)\) with the supremum norm \( \| \cdot \|_{\infty} \). Recalling that \( d_l \) is the degree with which \( x \) exhibits \( V_l \) as a Weierstraß covering over \( B(0, \delta_1) \times \mathcal{T}_l \), define \( H_l \) to be the subset of \( H \) consisting of those elements which are polynomials in \( y \) of degree less than \( d_l \).

**Lemma 4.1.** There exist \( \epsilon_1, \epsilon_2 > 0 \) such that for each fixed \( t_l \in \mathcal{T}_l \) and \( u \in H_l \) with \( |t_l| < \epsilon_1 \) and \( \|u - y\|_{\infty} < \epsilon_2 \), every \( h \in H \) has a unique decomposition on \((x, y) \in B(0, \delta_1) \times B(0, \delta_2)\) as

\[
h(x, y) = a(x, y)G_l(x, u(x, y), t_l) + b(x, y)\frac{\partial G_l}{\partial u}(x, u(x, y), t_l) + c \cdot g_l(x, u(x, y)),
\]

where \((a, b, c) \in H \times H_l \times \mathbb{C}^{r_l}\). Furthermore, \((a, b, c)\) depends holomorphically on such \((h, u, t_l) \in H \times H_l \times \mathcal{T}_l\).

**Proof.** Since \( u \) is a polynomial with respect to \( y \), for small \( \|u - y\|_{\infty} \) the derivative \( \frac{\partial u}{\partial y} - 1 \) is small. Furthermore, by Banach’s Fixed Point Theorem, for all \((x, y) \in B(0, \delta_1) \times B(0, \delta_2)\) the map \( v \mapsto y - (u(x, v) - v) \) has on \( B(y, \delta_2) \) a unique fixed point \( v(x, y) \) obtained by iterating this map with the initial input \( v = 0 \). Therefore the map \((x, y) \mapsto (x, u(x, y))\) is a biholomorphism from \( B(0, \delta_1) \times B(0, \delta_2) \) onto an open subset \( U \subset \mathbb{C}^2 \), with inverse map \( \varphi : (x, y) \mapsto (x, v(x, y)) \). The function \( h(x, y) = h(x, v(x, y)) \) is holomorphic on \( U \) and bounded. In the proof of Theorem 3.5 it was shown that for
sufficiently small \( t_1 \in \mathcal{T}_1 \), the \( g_l \) form a basis of \( \mathcal{O}_{U_l(t_1)}/\frac{\partial G_l}{\partial y}(x, y, t_1)\mathcal{O}_{U_l(t_1)}. \) Hence there exists a unique \( c \in \mathbb{C}^{nu} \) such that
\[
\left( \hat{h}(x, y) - c \cdot g_l(x, y) \right) / \frac{\partial G_l}{\partial y}(x, y, t_1)
\]
is holomorphic on the complex space \( U_l(t_1) \). Here we assume that \( \delta_1 > 0 \) is for our choice of \( \delta_2 > 0 \) sufficiently small that \( U_l(t_1) \) is contained in \( U \). The space \( U_l(t_1) \) is by definition a \( d_l \)-fold Weierstraß covering over \( x \in B(0, \delta_1) \). Hence \( \varphi(U_l(t_1)) \subset B(0, \delta_1) \times B(0, \delta_2) \) is also a \( d_l \)-fold covering over \( x \in B(0, \delta_1) \). In particular there exists a unique \( F \in \mathbb{H}_t \) such that
\[
\varphi(U_l(t_1)) = \{(x, y) \in B(0, \delta_1) \times B(0, \delta_2) \mid F(x, y) = 0\}.
\]
We conclude that the function
\[
(26) \quad (h(x, y) - c \cdot g_l(x, u(x, y))) / \frac{\partial G_l}{\partial u}(x, u(x, y), t_1)
\]
is holomorphic on \( \varphi(U_l(t_1)) \). Then by the Weierstraß Division Theorem [PR94 Chapter I Theorem 1.1], on the zero set of \( F \) this function is equal to a unique \( b \in \mathbb{H}_t \). Hence
\[
h(x, y) - c \cdot g_l(x, u(x, y)) - b(x, y)\frac{\partial G_l}{\partial u}(x, u(x, y), t_1)
\]
vanishes on
\[
\varphi(U_l(t_1)) = \{(x, y) \in B(0, \delta_1) \times B(0, \delta_2) \mid G_l(x, u(x, y), t_1) = 0\}.
\]
Since \( \varphi^{-1} \) is a biholomorphism from this space to the reduced space \( U_l(t_1) \), \( \varphi(U_l(t_1)) \) is reduced. Therefore there exists a unique \( a \in \mathbb{H} \) so that \( (a, b, c) \) solve (25).

It remains to prove that such triples \( (a, b, c) \) depend holomorphically on \( (u, h, t) \). For sufficiently small \( \|u - y\|_\infty \) and \( (x, y) \in B(0, \delta_1) \times B(0, \delta_2) \), Banach’s Fixed Point Theorem guarantees that the unique fixed point of \( v \mapsto y - (u(x, v) - v) \) is obtained by iterating this map starting with \( v = 0 \). These iterations depend holomorphically on \( (x, y) \) and converge uniformly in \( H \) to a map \( v \) such that \( (x, y) \mapsto (x, v(x, y)) \) is the inverse of \( (x, y) \mapsto (u(x, y)). \) Therefore \( v \in H \) depends holomorphically on \( u \) and \( \hat{h}(x, y) = h(x, v(x, y)) \) depends holomorphically on \( u \) and \( \hat{h} \).

For each \( l = 1, \ldots, L \) the function \( \frac{\partial f_l}{\partial y}(x, y) \) has only one root at \( (0, 0) \) on \( U_l \) ([11]). We know from Lemma 3.3 that a holomorphic function \( g \) belongs to the ideal \( \langle f_l, \frac{\partial f_l}{\partial y} \rangle \) of \( \mathbb{C}[y] \odot \mathcal{O}_{B(0, \delta_1)} \) if for all \( f \in \mathcal{O}_{U_l} \) the residue of \( g f(\frac{\partial f_l}{\partial y})^{-2} \Theta \) on \( X_l \) vanishes. Moreover, for \( f \in \frac{\partial f_l}{\partial y} \mathcal{O}_{U_l} \) this pairing vanishes. Therefore we obtain the following non-degenerate pairing
\[
(27) \quad \mathcal{O}_{U_l}/\frac{\partial f_l}{\partial y} \mathcal{O}_{U_l} \times \mathcal{O}_{U_l}\Theta/\frac{\partial f_l}{\partial y} \mathcal{O}_{U_l}\Theta \rightarrow \mathbb{C} \quad , (f, g) \mapsto \operatorname{Res}_{U_l} g f(\frac{\partial f_l}{\partial y})^{-2} \Theta.
\]
Using this pairing, the basis \( g_{l1}, \ldots, g_{l,r_l} \) of \( \mathcal{O}_{U_l}/\partial f/\mathcal{O}_{U_l} \) induces a dual basis \( h_{l1}, \ldots, h_{l,r_l} \in \mathbb{C}[y] \otimes \mathcal{O}_{B(0,\delta_1)} \) of \( \mathcal{O}_{U_l}(\Theta)/\partial f/\mathcal{O}_{U_l}(\Theta) \). The \( r_l \times r_l \) matrix

\[
B_ij(t) = \text{Res}_{U_l(t_i)} \left( g_{l,i} h_{l,j}(\Theta)/\left( \partial G_l/\partial y \right)^2 \right).
\]

depends holomorphically on small \( t_l \in T_l \) and stays nearby \( 1 \). In particular \( B \) is invertible. For small \( t_l \in T_l \), the proof of Theorem 3.5 shows that the basis \( h_{l1}, \ldots, h_{l,r_l} \) of \( \mathcal{O}_{U_l}(\Theta)/\partial f/\mathcal{O}_{U_l}(\Theta) \) defines also a basis of \( \mathcal{O}_{U_l(t_l)}(\Theta)/\partial f/\mathcal{O}_{U_l(t_l)} \).

Taking the residue on \( U_l(t_l) \), the pairing (27) remains non-degenerate and with respect to the bases \( g_l, h_l \) is given by

\[
c_i = \sum_{j=1}^r B_{ij}^{-1} \text{Res}_{U_l(t_l)} \left( \tilde{h}_{l,j}(\Theta)/\left( \partial G_l/\partial y \right)^2 \right)
\]

and depends holomorphically on \( h, u \) and \( t_l \).

Given \( x \in B(0, \delta_1) \), for sufficiently small \( \delta_1, |t_l| \) and \( \|u-y\|\infty \), the polynomial \( y \mapsto G_l(x, u(x,y), t_l) \) has exactly \( d_l \) roots in \( y \in B(0, \delta_2) \). These are also the roots of the Weierstraß polynomial \( y \mapsto F(x, y) \). For \( n = 0, \ldots, d_l - 1 \) the sum of the \( n \)-th powers of these roots are equal to

\[
\text{Res}_{y \in B(0, \delta_2)} \left( \partial G_l/\partial u \right)(x, u(x,y), t_l) \partial u(x,y) y^n dy/G_l(x, u(x,y), t_l).
\]

For the same \( x \in B(0, \delta_1) \) the coefficients of the Weierstraß polynomials \( F \) are polynomials with respect these residues [Mac79 (2.14')]. In particular, these coefficients depend holomorphically on \( (x, u, t_l) \in B(0, \delta_1) \times H \times T_l \).

It remains to show that the coefficients of \( b \) depend holomorphically on \( (x, h, u, t_l) \in B(0, \delta_1) \times H \times H \times T_l \). For any function \( g : \mathbb{C} \to \mathbb{C} \) the right hand side of (20) defines a polynomial of degree \( d - 1 \), which takes at the roots \( y_1, \ldots, y_d \) of \( f(y) \) the same values as \( g \). Hence for such \( g \) the difference of the left hand side minus the right hand side of (20) is always a multiple of \( f \). We claim that the coefficients of the right hand side are polynomials in the coefficients of the polynomials \( f \) and

(28) \[
\sum_{i=1}^d g(y_i) y_i^n/\partial y (y_i) \quad 0 \leq n \leq d - 1.
\]

In order to prove the claim we may assume that \( g(y) \) is a polynomial of degree \( d - 1 \). Due to (21) the highest coefficient, in front of \( y^{d-1} \), is equal to (28) for \( n = 0 \). Inductively the difference of \( y^n g(y) \) minus a multiple of \( f(y) \) has degree less than \( d \). Consequently the coefficient of \( g \) in front of \( y^{d-1-n} \) is equal to (28) minus a polynomial with respect to the coefficients of \( f \) and higher coefficients of \( g \). This proves the claim.
Given \( x \in B(0, \delta_1) \) we apply this claim to the function \( g(y) \) given by (26). For given \( x \in B(0, \delta_1) \) the corresponding expression (28) is equal to

\[
\text{Res}_{y \in B(0, \delta_2)} \left( h(x, y) - c \cdot g_l(x, u(x, y)) \right) \frac{\partial u}{\partial y}(x, y)y^n dy / G_l(x, u(x, y), t_l).
\]

Hence the coefficients of \( F \) and of \( b \) depend holomorphically on \( (x, h, u, t_l) \in B(0, \delta_1) \times H \times H \times T_l \).

Finally, given \( (u, h, t_l) \), \( c \) and \( b \), \( a(x, y) \) is equal to

\[
\left( h(x, y) - b(x, y) \frac{\partial G_l}{\partial u}(x, u(x, y), t_l) + c \cdot g_l(x, u(x, y)) \right) / G_l(x, u(x, y), t_l).
\]

By choice of \( b \) and since \( U_l(t_l) \) (22) is reduced, this expression belongs to \( H \) and depends holomorphically on \( (h, u, t_l) \in H \times H \times T_l \). \( \square \)

5. Existence of a universal deformation

In this section we shall prove our main result, which is the following theorem.

**Theorem 5.1.** Let \( g_1, \ldots, g_L \) be a basis for the isomorphism classes of infinitesimal deformations of a locally planar pair \((X, dx)\) and

\[
(X, \Theta) \hookrightarrow (Z, \Theta_Z) \to \mathcal{T}
\]

the corresponding deformation, constructed in (18). Then given any deformation \((X, \Theta) \hookrightarrow (Y, d\Theta_Y) \to \mathcal{S}\), after reducing the base \( \mathcal{S} \) if necessary, there exists a unique holomorphic map \( \phi : \mathcal{S} \to \mathcal{T} \) such that the pullback of (18) with respect to \( \phi \) is isomorphic to the given deformation.

Thus the local deformation

\[
(X, \Theta) \hookrightarrow (Z, \Theta_Z) \to \mathcal{T}_0
\]

of (19) is a universal object in the category of local deformations under morphism.

In particular then, (19) is independent of the choice of basis \( g_l \).

**Proof.** Let \((X, dx)\) be locally planar with prescribed poles, and adopt the notation of Definition (1.5). Let \((X, \Theta) \hookrightarrow (Y, d\Theta_Y) \to \mathcal{S}\) be a deformation of \((X, dx)\). It was shown in Theorem (2.2) that the corresponding local deformation \((X, \Theta) \hookrightarrow (Y, d\Theta_Y) \to \mathcal{S}_0\) is uniquely determined by deformations \((X_q, x_q, y_q) \hookrightarrow (Y_q, x_{Y,q}, y_{Y,q}) \to \mathcal{S}_0\) where \( q_1, \ldots, q_L \) denote the points where \( x_i \) is not a local coordinate. These points contain the singularities of \( X \) together with the smooth points of \( X \) which are roots of \( \Theta \). It was argued at the beginning of the proof of Theorem (2.2) that we may without loss of generality take a representative deformation whose base \( \mathcal{S} \) is an open ball around 0 \( \in \mathbb{C}^k \). The deformations of the space germs may be represented as

\[
Y_l = \{ (x, y, s) \in B(0, \delta_1) \times B(0, \delta_2) \times \mathcal{S} \mid F_l(x, y, s) = 0 \}
\]
with \( F_l \in \mathbb{C}\{x,y\} \otimes \mathcal{O}_S \). Furthermore, by the Weierstraß Preparation Theorem [PR94, Chapter I Theorem 1.4] we may assume that each \( F_l(x,y,s) \) is a Weierstraß polynomial of degree \( d_l \) in \( \mathbb{C}\{x,y\} \otimes \mathcal{O}_S \).

The theorem calls for a holomorphic map \( \phi : S \to T \) such that the pullback of \( (18) \) is isomorphic to the given deformation. By Theorem 2.2 this is equivalent to the construction of holomorphic maps \( \phi = (\phi_1, \ldots, \phi_L) : S \to T = T_1 \times \ldots \times T_L \) such that for all \( l = 1, \ldots, L \) the pullback of \( X_l \hookrightarrow Z_l \to T_l \) with respect to \( \phi_l \) is \( x \)-isomorphic to \( X_l \hookrightarrow Y_l \to S \). As in \([3]\), alongside the maps \( \phi_1, \ldots, \phi_L \) we shall construct units \( H_1, \ldots, H_L \) and holomorphic functions \( u_1, \ldots, u_L \) such that for all \( l = 1, \ldots, L \) we have

\[
H_l(x,y,s)G_l(x,u_l(x,y,s),\phi_l(s)) = F_l(x,y,s)
\]

for all \( s \in S \). These maps define an \( x \)-morphism from the deformation \( X_l \hookrightarrow Y_l \to S \) to the pullback of \( X_l \hookrightarrow Z_l \to T_l \) with respect to \( \phi \). The derivative of \((30)\) with respect to \( s \) gives

\[
\frac{\partial H_l}{\partial s}(x,y,s)G_l(x,u_l(x,y,s),\phi_l(s)) = F_l(x,y,s)
\]

(31)

\[
+ H_l(x,y,s) \frac{\partial G_l}{\partial u_l}(x,u_l(x,y,s),\phi_l(s)) \frac{\partial u_l}{\partial s}(x,y,s)
\]

\[
+ H_l(x,y,s) \frac{\partial \phi_l}{\partial s}(s) \cdot g_l(x,u_l(x,y,s)) = \frac{\partial F_l}{\partial s}(x,y,s).
\]

For fixed \( s \in S \) the \( x \)-morphism \( (x,y) \mapsto (x,u_l(x,y,s)) \) should be an isomorphism. Therefore there should exist \( \eta_l \in \mathbb{C}\{x,y,s_1,\ldots,s_K\} \) such that \( \eta_l(x,u_l(x,y,s),s) = y \) and such that \( (x,y) \mapsto (x,\eta_l(x,y,s)) \) is the inverse of \( (x,y) \mapsto (x,u_l(x,y,s)) \). The Weierstraß Division Theorem [PR94, Chapter I Theorem 1.1] proves that the map \( (x,y) \mapsto (x,u_l(x,y,s)) \) can be represented uniquely as a polynomial with respect to \( y \) of degree less than \( d_l \) . If the distance \( \|u_l - y\|_\infty \) of this polynomial \( u_l \) to \( y \) is sufficiently small, then \( |\frac{\partial u_l}{\partial y} - 1| \) is bounded on \( (x,y) \in B(0,\delta_1) \times B(0,\delta_2) \) by \( \frac{1}{2} \) and for all fixed \( x \in B(0,\delta_1) \) the map \( y \mapsto u_l(x,y,s) \) has bounded inverse \( y \mapsto v_l(x,y) \) from an open subset of \( \mathbb{C} \) onto \( B(0,\delta_2) \).

We apply Lemma 1.1 for \( l = 1, \ldots, L \) and small \( s \in S \) to

\[
h(x,y) = H_l^{-1}(x,y,s) \frac{\partial F_l}{\partial s}(x,y,s), \quad u(x,y) = u_l(x,y,s), \quad t_l = \phi_l(s)
\]

and obtain from (31) and (25) that the equations we wish to solve are

\[
(32) \quad \frac{\partial H_l}{\partial s}(x,y,s) = a(x,y)H_l(x,y,s), \quad \frac{\partial u_l}{\partial s}(x,y,s) = b(x,y), \quad \frac{\partial \phi_l}{\partial s}(s) = c.
\]

These equations define for small \( s \in S \) a smooth vector field along the straight line connecting \( 0 \) with \( s \). For all sufficiently small \( s \in S \) we apply the Picard-Lindelöf Theorem and integrate these vector fields along these straight lines up to \( s \), hence obtaining at \( s \) a solution of (30). For sufficiently small \( s \) the corresponding \( u_1, \ldots, u_L \) have small distance \( \|u_l - y_l\|_\infty \) from \( y_l \) so that \( (x_l,y_l) \mapsto (x_l,u_l(x_l,y_l,s)) \) are \( x \)-isomorphisms. Consequently, on a
neighbourhood of 0 the given deformation is \( x \)-isomorphic to the pullback of (18) with respect to \( \phi \).

It remains only to prove the uniqueness of the map \( \phi \). Any map \( \phi : S \to T \) with respect to which the pullback of (18) is isomorphic to the given deformation is described by solutions \((H_l, u_l, \phi_l)_{l \in \{1, \ldots, L\}}\) of (30) obeying (31). Hence uniqueness follows from Lemma 4.1 and the Picard-Lindelöf Theorem.

**Corollary 5.2.** Let \((X, dx)\) be a locally planar pair, \(g_1, \ldots, g_L\) a basis for the infinitesimal deformations of \((X, dx)\) and \((X, \Theta) \hookrightarrow (Z, \Theta_Z) \to T\) the deformation (18). Then after shrinking the base \(T\) if necessary, (18) represents the local universal deformation of all fibres of \(Z \to T\).

**Proof.** By Theorem 3.5, there is an open neighbourhood of the base point \(0 \in T\) such that the local sections \(g_1, \ldots, g_L\) are at each point a basis for the isomorphism classes of infinitesimal deformations of the corresponding fibre of \(Z \to T\). Clearly then by Theorem 5.1 the restriction of the deformation \((X, \Theta) \hookrightarrow (Z, \Theta_Z) \to T\) to this open subset represents the local universal deformation of all its fibres. \(\square\)

We shall henceforth take \(T\) to be this reduced neighbourhood, so that it enjoys the property of Corollary 5.2. If in Theorem 5.1 we impose the reality condition (c) on the fibres of (6), then the involution \(\eta\) extends to an involution of (6) and (18). In this case Theorem 5.1 implies that the morphism \(\phi\) commutes with \(\eta\). Furthermore \(\phi\) maps the fixed point set of \(\eta\) in \(S\) into the fixed point set of \(\eta\) in \(T\). In particular, the restriction of (18) to the fixed point set of \(\eta\) in \(T\) defines a universal local deformation of locally planar \((X, \Theta)\) obeying the reality condition.

We find it convenient below to extend Definitions 1.6 and 1.8 in the obvious way to encompass germs of locally planar pairs \((V(f), \Theta)\); the isomorphism classes of these pairs are clearly the same as the \(x\)-isomorphism classes of germs \(V(f)\).

We now turn to the question of whether \(T\) locally parameterises the isomorphism classes of locally planar \((X, \Theta)\) with prescribed poles (Definition 1.5). In fact it does not, as is easily revealed by considering the case when the special fibre has a non-trivial automorphism. Since the deformation \(X \hookrightarrow Y \to T\) is universal, the automorphism of \(X\) induces automorphisms of \(Y\) and \(T\), with respect to which the above maps are equivariant. The fibres over distinct points of the base space are mapped to one another under the automorphism of the total space and are hence isomorphic. Using Theorem 2.2 the universal deformation is described by the corresponding universal deformations of space germs \(V(f_l) \hookrightarrow V(G_l) \to T_l\) for \(l = 1, \ldots, L\).

We present now an example which demonstrates how a symmetry in \(f_l\) induces an isomorphism between the fibres over different elements of \(T\).

**Example 5.3.** Taking \(f(x, y) = y^3 - x^2\), as before we write \(V(f)\) for the space germ defined by \(f\) at \((0,0)\). Using Lemma 3.2 the isomorphism
classes of infinitesimal deformations \([15]\) of \((V(f), dx)\) are parameterised by \(\mathbb{C}\{x, y\}/\langle f, f_y \rangle\), for which \(1, x, y, xy\) provides a basis. The corresponding deformation \((V(f), \Theta) \hookrightarrow (Z, \Theta) \to T\) is given by
\[
G(x, y, t_1, t_2, t_3, t_4) = y^3 - x^2 + t_1 + t_2x + t_3y + t_4xy
\]
and as shown above this represents a universal local deformation. Writing \(q = e^{\frac{\pi i}{2}}\) and using the notation of \([4]\), the pairs \((H, u) = (1, qy)\) and \((H, u) = (1, q^2y)\) each define isomorphisms of \((V(f), \Theta)\). They lift to isomorphisms between the fibres of \((Z, \Theta) \to T\) over \((t_1, t_2, t_3, t_4), (t_1, t_2, qt_3, qt_4)\) and \((t_1, t_2, q^2t_3, q^2t_4)\). Note that the presence of these isomorphisms does not contradict the uniqueness property of the universal local deformation; uniqueness demands that there is no non-trivial \(x\)-isomorphism of \(Z \to T\) which acts trivially on the fibre \(V(f)\) over \(0 \in T\) but the above are allowable since they do not restrict to the identity automorphism of \(V(f)\).

In this example, the isomorphism classes of locally planar pairs \((X, dx)\) in a neighbourhood of \((V(f), \Theta)\) are locally parameterised not by \(T\) but by \(T/\mathbb{Z}_3\), giving locally the structure of an orbifold whose finite group action is the induced action of the automorphism group of the central fibre. This will be made explicit later in this section by the construction of a 3-fold cover \(\text{Dis} : T \to \mathbb{C}^4 \cong \mathbb{C}\{x, y\}/\langle f, f_y \rangle\), where this map is the symmetrisation of the branch values of \(x\).

It is straightforward to generalise the above example and see that non-trivial automorphisms of the pair \((X_t, dx)\) force distinct fibres of \((Z_t, \Theta) \to T_l\) to be isomorphic. We expect furthermore that whenever \((Z_t, \Theta) \to T_l\) has such fibres, the base space \((X_t, dx)\) possesses a non-trivial automorphism. We give below a heuristic argument which we anticipate could be expanded into a proof of this statement.

We shall describe the fibres of the universal deformation \([18]\) as local coverings with respect to the regular functions \(x_q\), and show that this family is locally parameterised by the branch values of the local functions \(x\). Simplest are those points of \(T\) for which the corresponding fibre of \([18]\) is smooth and contains only simple ramification points of \(x\). We begin by showing that such points of \(T\) are generic.

**Lemma 5.4.** Let \(X(t) \subset Z\) denote for all \(t \in T\) the fibres of \([18]\). Then the following set is open and dense in \(T\):
\[
T^* := \{ t \in T \mid X(t) \text{ is smooth and contains only simple roots of } \Theta_Z \}
\]

**Proof.** By Theorem 2.2 it suffices to prove the statement for the analogous subsets \(T_l^* \subset T_l\). Then have \(T^* = T_1^* \times \cdots \times T_L^*\). We shall show that for each \(l = 1, \ldots, L\) the set
\[
T_l^* := \{ t_l \in T_l \mid U_l(t_l) \text{ is smooth} \}
\]
is the complement of an analytic proper subset of \(T_l\) and that
\[
T_l^* := \{ t_l \in T_l^0 \mid U_l(t_l) \text{ contains only simple roots of } \Theta_l \}
\]
is the complement of an analytic proper subset of $T_0$. We note first that
\[ \frac{\partial G_l}{\partial y} dy + \frac{\partial G_l}{\partial x} \Theta = 0, \]
and on $T_0$ neither the partial derivatives nor the differentials have a common root. Hence we may equivalently characterise $T_1^*$ as
\[ T_1^* := \{ t_l \in T_l^0 \mid U_l(t_l) \text{ contains only simple roots of } \frac{\partial G_l}{\partial y} \}. \]

The complement of $T_0^*$ is the set of all $t_l \in T_l$ such that $\frac{\partial G_l}{\partial x}$ and $\frac{\partial G_l}{\partial y}$ have a common root on $U_l(T)$. The complement of $T_1^*$ in $T_0^*$ is the set of all $t_l \in T_l$ such that $\frac{\partial G_l}{\partial y}$ and $\frac{\partial^2 G_l}{\partial y^2}$ have a common root on $U_l(T)$. Thus these complements are each given by the vanishing of a resultant and hence the Weierstraß Preparation Theorem [PR94, Chapter I Theorem 1.4] implies they are analytic subsets of $T_l$ and $T_0^*$, respectively. It remains to prove that they are proper subsets. For this, consider the deformation
\[ V(f_l) \hookrightarrow V(f_l - s) \rightarrow \mathcal{S} := \{ s \in \mathbb{C} \}. \]

We have hitherto used the notation $V(f)$ for the germ of the vanishing set of $f$ at $(0,0)$; here we also allow it to indicate an open neighbourhood of $(0,0)$ representing this germ. The fibre over $s$ contains a singularity if and only if $s$ is a critical value of $f$. By Sard’s Theorem these critical values have measure zero, and the deformation contains in every neighbourhood of $s = 0$ a fibre without singularities. Since this deformation is the pullback of $X_l \hookrightarrow Z_l \rightarrow T_l$ with respect to a unique map from $\mathcal{S}$ to $T_l$, every neighbourhood of $0 \in T_l$ contains an element of $T_0^*$. This shows that $T_1^*$ is open and dense in $T_l$.

Now we give a similar argument for the analytic subset $T_1^*$ of $T_0^*$. Suppose, for $f_l \in \mathbb{C}\{x, y\}$, that $x : V(f_l) \rightarrow \mathbb{C}$ has a smooth ramification point of order $m \geq 2$ at $(0,0)$. We may assume that $f_l$ is a polynomial of degree $m+1$ with respect to $y$ with highest coefficient equal to 1 and vanishing second highest coefficient:
\[ f_l(x,y) = y^{m+1} + a_{m-1}(x)y^{m-1} + \ldots + a_0(x). \]

All coefficients $a_0, \ldots, a_{m-1} \in \mathbb{C}\{x\}$ vanish at $x = 0$ and since $V(f_l)$ is smooth, furthermore $a_0'(0) \neq 0$. We consider the deformation
\[ V(f_l) \hookrightarrow V(f_l - s_0 - s_1 y) \rightarrow \mathcal{S} := \{(s_0, s_1) \in \mathbb{C}^2 \}. \]

For sufficiently small $s_0$, the inverse function theorem guarantees that the coefficient $a_0(x) - s_0$ has a unique small root $x(s_1) \in \mathbb{C}$. Then for $s_1 \neq a_1(x(s_0))$, $V(f_l - s_0 - s_1 y)$ contains no ramification point of order $m$, since all coefficients $a_0 - s_0, a_1 - s_1, a_2, \ldots, a_{m-1}$ must vanish at such ramification points. Again the given deformation is a pullback of $X_l \hookrightarrow Z_l \rightarrow T_l$ and all $t_l \in T_l$ whose fibres contain a ramification point of order $m$ have nearby fibres with ramification points of order $m - 1$. Due to Theorem 3.5 the deformation $(X_l, \Theta_l) \hookrightarrow (Z_l, \Theta_l) \rightarrow T_l$ represents a universal deformation of
all its fibres. Hence we can iterate the lowering of the ramification order. This completes the proof that $\mathcal{T}_l^*$ is open and dense in $\mathcal{T}_l^*$ and thus in $\mathcal{T}_l$. □

Let be a $X_l = V(f_l)$ be a complex space germ and $(X_l, dx_l) \hookrightarrow (Z_l, dx) \rightarrow \mathcal{T}_l$ denote the universal deformation defined in (16). It deforms higher order ramification points of the map $x_l$ (i.e. higher order roots of $\frac{\partial f_l}{\partial y}$ at $(0,0)$) into lower order ramification points. On the fibres of $t_l \in \mathcal{T}_l^*$ the map $x_l$ has only first order ramification points, and as proven in the above lemma, $\mathcal{T}_l^*$ is open and dense in $\mathcal{T}$.

We will show that the fibres $U_l(t_l)$ of each $Z_l \rightarrow \mathcal{T}_l$ are locally parameterised by the (symmetrisation of the) branch points of $x_l : U_l(t_l) \rightarrow \mathbb{C}$. We begin by considering only smooth fibres with simple ramification points.

**Lemma 5.5.** We may locally define a map on $\mathcal{T}_l^*$ by sending $t_l \in \mathcal{T}_l^*$ to the branch points of $x_l : U_l(t_l) \rightarrow \mathbb{C}$. There are $r_l = \dim \left( \mathcal{O}_{U_l(t_l)}/\frac{\partial G_l}{\partial y}(x,y,t_l)\mathcal{O}_{U_l(t_l)} \right)$ such branch points and this defines a local diffeomorphism

$$\mathcal{T}_l^* \rightarrow \mathbb{C}^{r_l}.$$  

**Proof.** There is no global ordering of the branch points, but locally on $\mathcal{T}_l^*$ we may choose any ordering and take the corresponding map. Any curve in $\mathcal{T}_l^*$ can be lifted to a curve $s \mapsto (x(s),y(s),t_l(s))$ with $x(0)$ a branch point. Then differentiating $G(x,y,t) = 0$ with respect to $t$ gives

$$\dot{x}(s) = -t_l(s) \cdot g_l(x(s),y(s)) / \frac{\partial G_l(x(s),y(s),t(s))}{\partial x}$$

and for $t_l \in \mathcal{T}_l^*$ the denominator is non-vanishing. Hence it suffices to show that the $r$ vectors obtained by evaluating $g_l = (g_{l,1}, \ldots, g_{l,r_l})$ at the $r$ ramification points of $x_l$ are linearly independent and that $r = r_l$. As stated in (24), the $x$-isomorphism classes of infinitesimal deformations of $U_l(t_l)$ are given by the elements of $\mathcal{O}_{U_l(t_l)}/\frac{\partial G_l}{\partial y}(x,y,t_l)\mathcal{O}_{U_l(t_l)}$. For $t_l \in \mathcal{T}_l^*$, we define a local map

$$\mathcal{O}_{U_l(t_l)}/\frac{\partial G_l}{\partial y}(x,y,t_l)\mathcal{O}_{U_l(t_l)} \rightarrow \mathbb{C}^r$$

by sending a function to its values at the (simple) ramification points of $x_l$, where the ordering of these may be fixed locally. We shall argue that this is an isomorphism and hence in particular conclude that $r = r_l$. Note that these branch points are precisely the (simple) roots of $\frac{\partial G_l}{\partial y}$. The map is injective since the ideal in $\mathcal{O}_{U_l(t_l)}$ generated by $\frac{\partial G_l}{\partial y}$ consists precisely of the functions vanishing at these ramification points. It is surjective by Theorem 3.26.7 of [For81]. Then from Theorem 3.3 we see that for $t_l \in \mathcal{T}_l^*$ the $r_l$ vectors obtained by evaluating $g_l = (g_{l,1}, \ldots, g_{l,r_l})$ at the ramification points are linearly independent. □

In order to define a global analogue of the local map in the above lemma, we symmetrise with respect to permutations of the branch points. For $t_l \in \mathcal{T}_l^*$, the branch points of $x_l$ are precisely the roots of $\frac{\partial G_l}{\partial y}$ on $U_l(t_l)$. 

and so they coincide with the roots of the discriminant of the polynomial \( G_l \) with respect to \( y_l \) (holding \( t_l \) fixed). This discriminant \( \text{Dis}_l(x_l, t_l) \) is a Weierstraß polynomial of degree \( r_l \) with respect to \( x_l \), with coefficients depending holomorphically on \( t_l \in T_l \). The elementary symmetric functions of the roots of \( \text{Dis}_l(x_l, t_l) \) are given by its coefficients. Hence these coefficients define a global holomorphic map on \( T_l^* \) which by Lemma 5.4 we may extend holomorphically to \( T_l \). We write this map also as \( \text{Dis}_l \).

**Remark 5.6.**

1. The map 
   \[ \text{Dis}_l : T_l \to \mathbb{C}^{r_l} \]
   is a covering map whose ramification points all lie in \( T_l \setminus T_l^* \).

2. If the fibres of \( \text{Dis}_l \) over \( t_l \neq t'_l \in T_l \) are isomorphic then \( \text{Dis}_l(x_l, t_l) = \text{Dis}_l(x_l, t'_l) \). Hence if every sufficiently small neighbourhood of \( t \in T_l \) contains exactly \( n \) distinct \( t_l \) with isomorphic fibres, then the map \( \text{Dis}_l \) has a ramification point of order \( n \) at \( t \).

3. If there is an isomorphism between the fibres of \( \text{Dis}_l \) over \( t_l \neq t'_l \in T_l^* \) then it extends to an isomorphism between the fibres of all nearby pairs \( t_l \neq t'_l \in T_l^* \) with \( \text{Dis}_l(x_l, t_l) = \text{Dis}_l(x_l, t'_l) \).

**Proof.**

1. Lemma 5.5 guarantees that the differential of \( \text{Dis}_l \) is invertible on \( T_l^* \).

2. If the fibres over \( t_l \neq t'_l \in T_l \) are isomorphic then the values of \( x_l \) at the branch points of \( x_l \) must agree and so \( \text{Dis}_l(x_l, t_l) = \text{Dis}_l(x_l, t'_l) \).

3. This is an immediate consequence of Theorem 3.5.

The above arguments provide some evidence for our conjecture that if every neighbourhood of \( t \in T \) has \( n \) distinct points over which the fibres of \( \text{Dis}_l \) are isomorphic then the fibre over \( t \) has an automorphism of order \( n \). In order to prove this conjecture it would suffice to show that

1. the \( n \) points can be assumed to lie in \( T^* \)
2. the local automorphism of \( T^* \) induced by permuting the fibres of \( \text{Dis}_l \) extends to an automorphism of \( T \), fixing the corresponding branch point of \( \text{Dis}_l \).

6. **Deformations of hyperelliptic curves with one differential**

Hitherto we have considered deformations of locally planar pairs \((X, \Theta)\) with prescribed poles, comprised of a curve and a single differential satisfying Definition 1.5. In a forthcoming work we shall extend our results to triples \((X, \Theta, dy)\) which satisfy a natural analogue of Definition 1.5, namely that poles of \( dy \) are also prescribed and that the locally planar structure is compatible with both \( \Theta \) and \( dy \). There is a particular case, arising in a number of integrable systems, for which the existence of a local universal deformation of such triples \((X, \Theta, dy)\) follows quite easily from our results above. Namely, we consider in this section the case of locally planar pairs
(X, Θ) where X possesses a global meromorphic function y of degree two, and hence is hyperelliptic. Instead of directly requiring the existence of the global function y of degree two, we impose an involution σ upon (X, Θ) and the local functions y_q of Definition [13] such that the quotient X/σ has arithmetic genus zero.

More precisely, we consider locally planar (X, Θ) with local functions x_q, y_q for each q ∈ X^0 such that X possesses a holomorphic involution σ which interchanges p_1, . . . , p_K, satisfies

\begin{equation}
\sigma^* \Theta = -\Theta \quad \sigma^* x_q = -x_{\sigma(q)} \quad \sigma^* y_q = y_{\sigma(q)}
\end{equation}

and is such that each y_q is a local parameter of the quotient X/σ.

In particular then X/σ is smooth, since it has a local parameter about every point in X^0/σ whilst the remaining points p_1, . . . , p_K ∈ X are smooth and remain so in the quotient as σ has only isolated fixed points. We shall only consider the case when X is hyperelliptic with hyperelliptic involution σ (i.e. X/σ ∼= P^1). As arithmetic genus is preserved in families and the quotient is smooth, hyperellipticity will be preserved under our deformations.

For the the initial (X, Θ) we choose particular local parameters y_q. Let again q_1, . . . , q_L ∈ X^0 denote the points where x_q is not a local coordinate. These points are interchanged by σ. The corresponding action of σ on the index set {1, . . . , L} is denoted by l ↦ σl. Again we abbreviate the indices q by indices l and the indices q_{σ(l)} by indices σl. In particular x_{σl} = −x_l and y_{σl} = y_l. Nearby q_l ≠ q_{σl} we choose parameters such that

\begin{align*}
x_l &= y_l^{r+2} \quad x_{σl} = −y_{σl}^{r+2} \quad f_l(x_l, y_l) = y_l^{r+2} − x_l \quad f_{σl}(x_{σl}, y_{σl}) = y_{σl}^{r+2} + x_{σl}
\end{align*}

with some r ∈ N (depending on l). Nearby q_l = q_{σl} we may instead assume

\begin{align*}
x_l^2 &= y_l^{r+2} \quad f_l(x_l, y_l) = y_l^{r+2} − x_l^2.
\end{align*}

Definition 6.1. A deformation of a hyperelliptic locally planar (X, Θ) with local functions y_q as above is a deformation (X, Θ) ↦ (Y, Θ_Y) → S such that

1. the involution σ extends to an involution of Y which acts trivially on S, commutes with the maps X ↦ Y → S and satisfies

\begin{equation}
\sigma^* \Theta_Y = -\Theta_Y \quad \sigma^* x_{Y,q} = -x_{Y,\sigma(q)} \quad \sigma^* y_{Y,q} = y_{Y,\sigma(q)};
\end{equation}

2. at q ∈ X^0, (y_{Y,q}, s) are local parameters for the quotient Y/σ.

We could consider more general deformations with non-trivial actions of σ on S, but only the fibres of Y → S over the fixed points of σ in S obey (33).

The morphisms of these deformations are the morphisms ψ of deformations of (X, Θ) (see Definition [13]) such that ψ commutes with σ. Finally, a local deformation (X, Θ, y_k) ↦ (Y, Θ_Y, y_{Y,q}) → S_0 is an equivalence class of deformations (X, Θ, y_k) ↦ (Y, Θ_Y, y_{Y,q}) → S, under this notion of isomorphism.

Theorem 2.2 carries over and shows that a local deformation of hyperelliptic (X, Θ, y_k) is equivalent to a deformation of each X_l = X_{q_l} preserving
the involution $\sigma$. In order to describe these deformations, we decompose the algebra $\mathbb{C}\{x,y\}$ into a direct sum of symmetric and anti-symmetric elements with respect to the involution $\sigma^* x = -x$ and $\sigma^* y = y$:

$$g = g^+ + g^- \quad g^+ = \frac{1}{2}(g + \sigma^* g) \quad g^- = \frac{1}{2}(g - \sigma^* g).$$

Since $\sigma$ acts trivially on the base, we choose for $\sigma l = l$ a basis of the symmetric part of $\mathbb{C}\{x,y\}/\langle f_l, \frac{\partial f_l}{\partial y} \rangle$. In all cases we choose the basis in (22) to be of the form

$$(g_l,1,\ldots,g_l,r) = (g_{\sigma l},1,\ldots,g_{\sigma l},r) = (1,y_l,\ldots,y_l^{r-1}).$$

This basis again defines a particular deformation denoted by

$$(X,\Theta,q) \hookrightarrow (Z,\Theta_Z,y_Y,q) \rightarrow \mathcal{T},
$$

and the corresponding local deformation is independent of the choice of $\sigma$-symmetric basis $g_l$. Our arguments carry over to the hyperelliptic case, yielding the following analog to Theorem 5.1:

**Corollary 6.2.** Let $(X,\Theta,y,q)$ be hyperelliptic and locally planar as described above. Then (35) defines a universal local deformation $(X,\Theta,q) \hookrightarrow (Z,\Theta_Z,y_Y,q) \rightarrow \mathcal{T}_0$ of $(X,\Theta,y,q)$.

If we additionally require $(X,\Theta,y,q)$ to be real with respect to an anti-holomorphic involution $\eta$, as in Definition 1.5(c), then $\eta$ lifts to an involution of (35). By restricting to the fixed point set $\mathcal{T}^R$ of $\eta$ acting on $\mathcal{T}$ describes a universal local deformation of real hyperelliptic locally planar data $(X,\Theta,y,q)$. In particular this includes the periodic solutions of the KdV equation and of the sinh-Gordon equation. In [GS95] the space of spectral curves is investigated as a covering with respect to the local parameters $t \in \mathcal{T}$. For real periodic solutions of the KdV equation they are globally one-to-one [MOk75, Kor08]. In [HKS16] these deformations are applied to the periodic solutions of the sinh-Gordon equation. We shall discuss these and other examples in more detail in forthcoming work.

**References**

[ACG11] E. Arbarello, M. Cornalba, and P. A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[dJP00] T. de Jong and G. Pfister. *Local analytic geometry*. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 2000. Basic theory and applications.

[For81] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981. Translated from the German by Bruce Gilligan.

[GLS07] G.-M. Greuel, C. Lossen, and E. Shustin. *Introduction to singularities and deformations*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[GR84] H. Grauert and R. Remmert. *Coherent analytic sheaves*, volume 265 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984.
[GS95] P. G. Grinevich and M. U. Schmidt. Period preserving nonisospectral flows and the moduli space of periodic solutions of soliton equations. Physica D, 87:73, 1995.

[HKS16] L. Hauswirth, M. Kilian, and M. U. Schmidt. Mean-convex Alexandrov embedded constant mean curvature tori in the 3-sphere. Proc. Lond. Math. Soc. (3), 112(3):588–622, 2016.

[KLSS20] S. Klein, E. Lübcke, M. U. Schmidt, and T. Simon. Singular curves and Baker-Akhiezer functions, 2020.

[Kor08] E. Korotyaev. A priori estimates for the Hill and Dirac operators. Russ. J. Math. Phys., 15(3):320–331, 2008.

[Mac79] I. G. Macdonald. Symmetric functions and Hall polynomials. The Clarendon Press, Oxford University Press, New York, 1979. Oxford Mathematical Monographs.

[MOk75] V. A. Marčenko and I. V. Ostrovs’kiĭ. A characterization of the spectrum of the Hill operator. Mat. Sb. (N.S.), 97(139)(4(8)):540–606, 633–634, 1975.

[PR94] Th. Peternell and R. Remmert. Differential calculus, holomorphic maps and linear structures on complex spaces. In Several complex variables, VII, volume 74 of Encyclopaedia Math. Sci., pages 97–144. Springer, Berlin, 1994.

[Ros52] Maxwell Rosenlicht. Equivalence relations on algebraic curves. Ann. of Math. (2), 56:160–191, 1952.

[Ser88] Jean-Pierre Serre. Algebraic groups and class fields, volume 117 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1988. Translated from the French.

Email address: emma.carberry@sydney.edu.au

School of Mathematics and Statistics, University of Sydney, Australia

Email address: schmidt@math.uni-mannheim.de

Mathematics Chair III, Universität Mannheim, D-68131 Mannheim, Germany