Directions in Causal Set Quantum Gravity*

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Abstract

In the causal set approach to quantum gravity the spacetime continuum arises as an approximation to a fundamentally discrete substructure, the causal set, which is a locally finite partially ordered set. The causal set paradigm was elucidated in a classic paper by Bombelli, Lee, Meyer and Sorkin in 1987 [1]. While early kinematical results already showed promise, the program received a substantial impetus about a decade ago with the work of Rideout and Sorkin on a classical stochastic growth dynamics for causal sets [2]. Considerable progress has been made ever since in our understanding of causal set theory while leaving undisturbed the basic paradigm set out in [1]. Recent highlights include a causal set expression for the Einstein-Hilbert action [3, 4] and the construction of a scalar field Feynman propagator on a fixed causal set [5, 6]. The aim of the present article is to give a broad overview of the results in causal set theory while pointing out directions for future investigations.

1 Introduction

The origins of the causal set paradigm lie in the rich soil of Lorentzian geometry. Unlike a Riemannian space, a Lorentzian spacetime $(M, g)$ possesses an additional structure, the causal structure $(M, \prec)$ where for a pair of events $x, y \in M$, $x \prec y$ if $x$ is to the causal past of $y$ [7, 8]. For a causal spacetime, i.e., one which has no closed causal curves, $(M, \prec)$ is a partially ordered set (or poset), namely $\prec$ is

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1. Reflexive: $x \prec x$.

2. Acyclic: $x \prec y$ and $y \prec x$ implies that $y = x$.

3. Transitive: $x \prec y$ and $y \prec z$ implies $x \prec z$.

In addition to the causal relation $\prec$ one also has the chronological relation $\ll$ which satisfies both acyclicity and transitivity but not reflexivity. Viewed in terms of the partial order $(M, \prec)$, $M$ is merely the set of spacetime events, stripped of its standard manifold-like topological character.

Spacetime causal structure was studied extensively in the 60’s and 70’s and led to a deeper understanding of the fundamentally Lorentzian features of spacetime like black holes and their associated singularities [7, 8, 10, 11, 12, 13]. These investigations suggested that causal structure plays a primitive, rather than a derivative role in Lorentzian geometry leading naturally to the question – can spacetime topology and geometry be derived from the poset $(M, \prec)$?

The answer to the former question has been known for several decades now, when restricted to the class of strongly causal spacetimes [7, 8]. First, notice that the relational connectivity of a generic poset ensures that it contains non-trivial topological information – indeed, posets admit a host of different non-trivial topologies [14, 15]. Of these, the Alexandroff topology $\mathcal{A}$ generated from a basis of Alexandroff intervals $I(p, q) \equiv I^+(p) \cap I^-(q)$, where $I^+(p) \equiv \{s | p \ll s\}$ and $I^-(q) \equiv \{s | s \ll q\}$, is of particular interest. It is not difficult to prove that in strongly causal spacetimes $\mathcal{A}$ is homeomorphic to the manifold topology $\mathcal{M}$, or equivalently, $\mathcal{A}$ is homeomorphic to $\mathcal{M}$ iff $\mathcal{A}$ is Hausdorff [10]. It was recently demonstrated that $\mathcal{M}$ can also be obtained from a more general causal topology in spacetimes satisfying weaker causality restrictions [9]. These results are quite startling since they tell us that the manifold topology, constructed from purely Riemannian considerations, can be derived entirely from the poset $(M, \prec)$. They provide the first indication of the underlying primacy of the causal structure in Lorentzian geometry.

That there can, additionally, be a relation between geometry and causal structure is not altogether surprising: it is a simple fact that the causal structure is unchanged under conformal transformations of the spacetime geometry. However, the relationship is far more profound. Using results due to Hawking, King and McCarthy [13] Malament [16, 17] showed that causal bijections (i.e., maps that preserve the causal structure) between 4-dimensional future and past distinguishing spacetimes are also conformal isometries. In other words,

\[\text{This relation can be derived from } \prec \text{ when the spacetime is either past or future distinguishing [9].}\]
if a pair of 4-dimensional spacetimes admit a bijective map that preserves
the causal structure then these spacetimes are in fact conformally related. It is
possible to show that causal bijections cannot exist between spacetimes with
different dimensions \(d > 2\) and hence causal structure also determines the
spacetime dimension \([9]\). These results imply that the conformal geometry is
completely determined by the causal space \((M, \prec)\) provided the latter satisfies
certain causality conditions. In a 4-dimensional spacetime the causal structure
is therefore “9/10th” of the spacetime geometry and the remaining “1/10” is
determined by the local volume element so that \([18, 19]\)

\[
\text{GEOMETRY} = \text{CAUSAL STRUCTURE} + \text{VOLUME} \tag{1}
\]

This is one of the starting points for the causal set paradigm.

The second ingredient that goes into causal set theory comes from the quan-
tum world, and its conjunction with gravity. This suggests that there is a funda-
mental Plank scale cut-off \(l_p\) in spacetime – a suggestion that is reinforced by the
apparent “quantisation” of the black hole area law\(^2\). How this fundamental cut-
off is implemented as a “discretisation” of spacetime is of course dependent on
what other continuum baggage one wishes to retain. Rather than employing a
Lorentz violating cut-off like the Plank energy (or length or time), it is desirable
to use the covariantly well-defined Plank spacetime volume \(V_p = l_p^4\) as a cut-off.
Introducing such a cut-off in the poset \((M, \prec)\) produces a causal set \(C\), which
apart from being a poset is also “locally finite”. Namely, if \(J^+(x) \equiv \{y | x \prec y\}\)
and \(J^-(x) \equiv \{y | y \prec x\}\), then the interval \(J(x, y) = J^+(x) \cap J^-(y)\) is required to
be of finite cardinality. The prescription of local finiteness formalises the idea
of a fundamental spacetime discreteness – spacetime regions that have finite
volume should contain only a finite number of fundamental spacetime atoms.

Indeed, it is precisely this added ingredient of discreteness which makes it
possible to reconstruct, in an appropriate sense, the full spacetime geometry
from the causal set in the spirit of Statement \([11]\). Normalising the volume with
respect to a volume cut-off, this suggests the natural correspondence

\[
\begin{align*}
\text{Order} & \rightarrow \text{Causality} \\
\text{Number} & \rightarrow \text{Volume}. \tag{2}
\end{align*}
\]

The particulars of course depend on the desired nature of the discrete-continuum
correspondence. Importantly, if the discrete substratum is to be thought of as
fundamental, then the continuum can arise only as an approximation. The
\[^2\] An adage attributed to Mark Kac “Be Wise – Discretise” while applicable to various
physical and mathematical situations, seems particularly appropriate to quantum gravity!
analogy to keep in mind is that of a fluid whose apparently continuum nature has an underlying, discrete, molecular structure. In lattice gauge theory, or Causal Dynamical Triangulations [20], in contrast, the continuum is regarded as fundamental and is approximated by the discrete lattice. In this case, the continuum limit is essential, since it represents the physical limit of a discrete approximating scheme. However, it is precisely because the continuum is not a fundamental construction in the causal set approach that the continuum limit plays no fundamental role in the theory. As in statistical physics, one is therefore interested in seeing how coarse grained continuum structures emerge from the underlying causal set.

However, to decide on the rules for when the approximation is a good one, it is useful to start with the reverse question: what is the causal set that underlies a continuum spacetime? The naivest such discretisation of Minkowski spacetime is via a regular lattice of the sort depicted in Fig 1. In the figure on the left the spacetime interval depicted has a volume $V = 4V_c$ and contains precisely 4 elements of the lattice. However, as shown in the figure on the right, under a boost this same region contains no lattice elements. Thus, the second correspondence in Statement [2] is not consistent with a regular lattice, no matter how clever the construction. The only discretisation that can implement the spirit of this correspondence is a “random lattice” of the sort described in

\[\text{Figure 1: Diamond lattice under a boost. A fixed volume region containing 4 elements before the boost becomes empty after the boost.}\]

\[\text{An equivalent, passive viewpoint, is the observation that most intervals of spacetime volume } V = 4V_c \text{ in the figure on the left contain no lattice elements.}\]
Thus, a causal set $C$ can be generated from a continuum spacetime $(M, g)$ via a random sprinkling of points ordered by the induced causal order. The most appropriate (and possibly unique) random distribution for our purpose is the Poisson distribution $P_V(n) = (1/n!) \exp^{-V/V_c}(V/V_c)^n$, where $P_V(n)$ is the probability of finding $n$ elements in a spacetime region of volume $V$ for a given cut-off $V_c$. For this distribution, $\langle n \rangle = V/V_c$ which means that the number to volume correspondence holds on average, thus satisfying the spirit of the Malament theorem. Thus, one has the causal set maxim due to Rafael Sorkin as a discrete analogue of (1). The use of the random lattice in turn has the added bonus of preserving local Lorentz invariance. This has a profound significance for causal set phenomenology, and one that is definitely falsifiable. As an example, Fig 2 depicts a causal set obtained by a random discretisation of a region of 2d Minkowski spacetime.

We have described how to obtain a causal set which underlies a continuum spacetime but the more fundamental question we seek to answer is how the continuum emerges as an approximation of the theory. Given a causal set $C$, we will call an embedding $\Phi : C \to (M, g)$ order preserving if the order relation in $C$ is mapped bijectively to the causal order induced on $\Phi(C)$. It is an important fact that not all causal sets can be embedded via an order preserving map into a given spacetime. In addition, the embedding $\Phi$ is faithful if it is order
preserving and $\Phi(C) \subset M$ is a high probability Poisson distribution in $(M, g)$ \cite{24}. A causal set $C$ is then said to be approximated by a continuum spacetime $(M, g)$ if $C$ can be faithfully embedded into $(M, g)$. Note that the continuum spacetime that one obtains via the approximation cannot be too detailed in its description — structure (both topological and geometrical) on scales at or smaller than the Planck scale cannot be encoded in the causal set. It is thus important for a faithful embedding to be approximately unique – a given causal set should not approximate to spacetimes that differ on scales much larger than the discreteness scale. If this were the case, then it means that spacetimes cannot be distinguished solely by the underlying causal set. A Hauptvermutung or fundamental conjecture of causal set theory states

**Hauptvermutung:** If a causal set $C$ faithfully embeds at the same density into two distinct spacetimes $(M_1, g_1)$ and $(M_2, g_2)$ then these spacetimes are related by an approximate isometry.

Here, an approximate isometry captures the notion that the two spacetimes differ only at scales of order $V_c$. The above conjecture is true when the locally finite condition is relaxed to locally countable and dense \cite{24}. However, the locally finite case is still unproven. Much of the difficulty lies in defining the closeness of two Lorentzian geometries, although progress has been made in this direction \cite{26,27}. On the other hand, a considerable body of circumstantial evidence has accumulated over the years in support of the conjecture from the construction of continuum geometrical and topological information from the causal set and we will describe this “kinematics” in Section 2. In Section 3 we discuss phenomenological predictions that arise from causal sets. The most important of these is the prediction of the value of the cosmological constant \cite{28}. In Section 4 we discuss both the classical and quantum dynamics of causal sets and end with some concluding remarks in Section 5.

The aim of this review is to give a broad outline of causal set theory and point to directions that are currently being pursued. The assumption of a fundamental random discreteness means that many of the standard tools of continuum physics cannot be used. This has been a big challenge, but one that has been met with a fair amount of success in recent years. After acquainting the reader with some of these new tools, the hope of this review is to open a new window on the quantum nature of spacetime.
2 Kinematics

In this section, we will focus our attention on the question, “Where is the geometry and topology of a continuum spacetime \((M, g)\) encoded in a causal set \(C\)?” Finding answers to this query requires a sometimes laborious and sometimes inspired search for continuum-like structures within the causal set. Every successful identification of such a continuum property is a confirmation of the Hauptvermutung since the same causal set \(C\) cannot then embed into a spacetime \((M', g')\) with a significantly different continuum property.

Importantly, standard poset theory does not always provide ready answers to this question—while a poset itself admits several natural geometric and topological structures, these are not necessarily appropriate in the continuum approximation. The main difficulty arises from the randomness of the discretisation. A useful contrast is a Regge-discretisation [29] which uses a simplicial decomposition—not only are topological structures retained under the discretisation, but there is a simple procedure for calculating the local curvature using nearest neighbours. In a causal set the nearest neighbour of an element \(p\) is one that is linked to it: \( p \prec q \) is said to be a link if for every \( r \) with \( p \prec r \prec q \), either \( r = p \) or \( r = q \). A link is thus an irreducible relation in a causal set since it cannot be deduced from transitivity. For a causal set that is approximated by Minkowski spacetime, every element therefore has an infinite number of nearest neighbours extending all along its future and past light-cones (see Fig.3). This points to an inherent non-locality in causal sets, a legacy of discreteness combined with local Lorentz invariance [30, 31]. This non-locality makes it non-trivial to construct familiar continuum quantities and it is a happy fact that, despite this, much progress has been made.

![Figure 3: The links from an element in a causal set that is approximated by flat spacetime lies all along the null cones. The figure shows a few (of the infinite) future directed links.](image)

Figure 3: The links from an element in a causal set that is approximated by flat spacetime lies all along the null cones. The figure shows a few (of the infinite) future directed links.
The most rudimentary topological property of a spacetime is its dimension. Let \( C \) be a causal set which faithfully embeds into a flat spacetime interval \( J(p, q) \) of dimension \( d \) and volume \( V \). The Myrheim-Meyer dimension estimator for this causal set uses the distribution of “chains”. A chain \( c \) is a totally ordered subset of \( C \), i.e., for every pair of elements \( x, y \in c \), either \( x \prec y \) or \( y \prec x \) (Fig 4).

It was shown in [23] that for \( V_c \ll V \), the average number of \( k \)-element chains \( \langle C_k \rangle = f(d,k)\langle N \rangle^k \), where \( \langle N \rangle = (V/V_c) \) and \( f(d,k) \) is an explicit function of \( d \) and \( k \). This function can then be inverted to obtain the dimension. For \( k = 2 \), this estimator is called the Myrheim-Meyer dimension [32], where \( \langle C_2 \rangle \) is simply the average number of relations in \( C \). \( f(d,2) \) is then half of the so-called “ordering fraction” of \( C \), i.e., the ratio of the number of relations in \( C \) to the number of possible relations between \( n \) elements.

Another dimension estimator is the mid-point scaling dimension, which uses the relationship between the volume \( V \) of a spacetime interval \( J(p, q) \) in \( M^d \) and the length \( \tau \) of the time-like geodesic from \( p \) to \( q \), i.e., \( 2^d = V/V' \), where the midpoint \( m \in J(p, q) \) is the one that maximises the volume \( V' \) of the smaller of the two intervals \( J(p,m) \) and \( J(m,q) \) [19, 33]. Numerical tests show that these dimension estimators reproduce the continuum dimension fairly accurately for causal sets that approximate to flat spacetimes for small \( d \) [23, 34].

In [23] an analysis of \( \langle C_k \rangle \) for causal sets that approximate to both anti-deSitter and deSitter spacetimes suggested that not only dimension, but also
the Gaussian curvature and the height of the conformally related flat interval could be obtained by solving a system of three equations. The numerical results though promising, were not entirely conclusive since relatively small causal sets were used. In [34] a procedure was proposed for finding Minkowski-like intervals in a class of conformally flat spacetimes from which to calculate the spacetime dimension, and numerical simulations were carried out to verify the procedure. For large enough causal sets these simulations work very well for the Myrheim-Meyer dimensions [34].

While these two dimensional estimators do seem to yield the correct values for Minkowski spacetime, it is an important open question whether and how they can be extended to arbitrary curved spacetimes. It is tempting to think that these results should still be valid since dimension is locally defined in the continuum. Indeed, in the continuum every event lies in a neighbourhood which is diffeomorphic to flat spacetime. Since there is no intrinsically causal definition of such a neighbourhood, how does one construct it in the causal set?

The continuum result that the Alexandroff topology is the manifold topology in strongly causal spacetimes suggest that this is the most natural topology for a causal set. While this will give rise to the manifold topology in the continuum limit, what is of relevance to causal sets is not the fine manifold topology but some coarser sub-topology. In [35] it was suggested on general grounds that it may suffice to consider a finitary topology generated by locally finite covering of a space. In Riemannian spaces such locally finite sub-topologies are easy to construct from a discretisation. For example, the circumspheres of a Delaunay triangulation of a space provide a locally finite sub-topology. However, Alexandroff intervals do not provide locally finite coverings – every element of the causal set is contained in an infinite number of such intervals in Minkowski spacetime.

In order to construct a locally finite covering, one can implement a localisation via the causal set analog of a spacelike hypersurface. This is an inextendible antichain $A$ or a maximal collection of unrelated elements in $C$ (see Fig 4). Since a choice of $A$ is a choice of frame, one must also sample over a sufficiently large ensemble of randomly chosen inextendible antichains. This was the strategy adopted in [36, 37] for constructing the homology of a causal set which is approximated by a globally hyperbolic spacetime. Since $A$ itself carries only the discrete topology, it needs to borrow information from the larger causal set. This is done via a “volume” thickening $T_n(A) \equiv \{x | x \succ A, |J(A, x)| \leq n\}$, where $|.|$ denotes cardinality and $J(A, x) \equiv \{r | \exists a \in A, a \prec r \prec x\}$ (Fig 5). The past of a maximal element $m$ of $T_n(A)$ then casts a “shadow” on $A$, $J^-(m) \cap A$, and
the set of shadows from each maximal element provides a covering $\mathcal{O}_v(A)$ of $A$. In [36] it was shown that the nerve simplicial complex $\mathcal{N}_v(A)$ constructed from $\mathcal{O}_v(A)$ gives the correct continuum homology for a large range of contiguous values of $v$, provided $A$ satisfies certain conditions. Subsequent numerical work using ensembles of randomly selected inextendible antichains [37] suggests that these conditions are satisfied quite generically. This work lends support to a homological version of the fundamental conjecture, namely, that if a causal set faithfully embeds into two distinct globally hyperbolic spacetimes, then these are “approximately” homological.

Figure 5: A diagram of a causal set that is approximated by a region in 2d Minkowski spacetime. The antichain in marked as a jagged red line and the blue region denotes an associated thickened antichain.

Although these results are robust, they nevertheless rely on a splitting of spacetime into space and time which is unnatural in a causal set context. It is still an open question whether interesting topological invariants can be obtained without recourse to inextendible antichains. Since Alexandroff intervals do not give rise to locally finite coverings of the causal set, is it possible to get around this, and work with non-locally finite coverings? Or are there other poset structures that one might use to define a new, more pertinent topology? Moreover, while homology is an important characterisation of topology, there are other topological invariants, some of which could be of importance to causal sets. Recent explorations on a Gauss-Bonnet-like relation in 2-d causal sets are suggestive and it would be interesting to gain a deeper understanding of these results [38].
The simplest geometric quantity that can be constructed in a causal set is the analog of timelike distance. For a generic pair \( x \prec y \) in \( C \), there are several chains that can be constructed with starting point \( x \) and end point \( y \). Each chain can be thought of as a possible time-like curve from \( x \) to \( y \). Since the continuum timelike distance is obtained by maximising over the length of timelike geodesics, a natural discrete analog of timelike distance is the length of the longest chain between \( x \) and \( y \) (Fig 2) [32]. For a causal set that is approximated by an interval in flat spacetime it was shown by Brightwell and Gregory [39] that this length converges rapidly to a multiple of the proper time \( \tau \), for large \( \tau \). This multiple is known for \( d = 1, 2 \) and has a fairly stringent bound for higher dimensions. Numerical simulations [40] suggest that the timelike distance estimator holds in 2,3 and 4 conformally flat spacetimes. More recently, progress has been made on recovering spacelike distances in flat spacetime [41]. Extending these results to generic curved spacetime are important but one will, again, have to address the question of localisation.

Using the antichain as a localisation device, we can also obtain the induced geometry of a spatial hypersurface \( \Sigma \), with the volume parameter defined in the homology estimation being used to define a spatial metric. If \( A \) is an antichain which is approximated by \( \Sigma \) and \( a_i, a_j \in A \), one can define the predistance function \( \tilde{d}(a_i, a_j) \) as the smallest \( n \) such that \( a_i, a_j \in J^-(x) \) and \( |J(A, x)| = n \) if it exists, and infinite otherwise. This gives a labelled graph on \( A \) with vertices \( a_i \) and edges labelled by \( \tilde{d}(a_i, a_j) \). The distance function \( d(a_i, a_j) \) is then the shortest path on \( A \). A similarly constructed distance function in the continuum then allows an analytical comparison with the continuum spatial geometry [42], but numerical simulations have not yet been carried out.

Apart from direct topological and geometric constructions, a causal set is also characterised by the way fields propagate on it. The causal set analog of a Green’s function for a given field is thus an intrinsic property of that causal set. Every causal set \( C \) has a representation as a causal matrix \( M \) obtained from an order preserving labelling of \( C \): \( M_{ij} \) is 1 if \( i \prec j \) and 0 otherwise, where \( i, j \) denote labelled elements of \( C \). Similarly, one can construct a link matrix \( L \), for which \( L_{ij} \) is 1 or 0 depending on whether \( i \) is linked to \( j \) or not. In [43] it was shown that for a causal set which is approximated by 2d Minkowski spacetime, the Green’s function for a scalar field satisfies \( G_{ret} + G_{adv} = \frac{1}{2}(M + MT) \) [31] [43] and a similar relation holds using the link matrix for causal sets that approximate to 4-d spacetimes. The D’Alembertian for this propagator is obtained by symmetrising \( G_{ret} \) and then inverting it. Simulations show that when restricted to causal sets that approximate to spacetime intervals,
the continuum D’Alembertian can be recovered for suitable test functions.

A different approach was adopted in [31, 44] for constructing the D’Alembertian operator, which has been very successful both in resolving the problem of non-locality as well as admitting a generalisation to curved spacetimes [3, 4]. The basic insight used for taming non-locality comes from the following observation in [31]. Consider a field $\phi(u, v)$ in Minkowski spacetime that is slowly varying in a particular frame. If $(u, v)$ are the lightcone coordinates in this frame, the D’Alembertian $\Box \phi(u, v) = \partial_u \partial_v \phi(u, v)$ on an interval $I(p, q)$ with $q = (u, v)$ and $p = (u - a, v - a)$ of area $a^2$ admits a discretisation $\Box_d = \phi(u, v) \frac{1}{a^2} (\phi(u, v) - \phi(u - a, v) - \phi(u, v - a) + \phi(u - a, v - a))$. Even though the system is Lorentz invariant, the contribution from other intervals of volume $a^2$ (corresponding to boosts of $I$) need not be the same since a given field configuration is not itself required to be Lorentz invariant. In particular, consider the contribution from a “narrow” interval $I'(p', q)$ with $p' = (u - a_1, v - a_2)$, where $a_1 a_2 = a^2$. As the interval becomes narrower, i.e., $a_1 \to 0$, the first two terms and the last two terms of the D’Alembertian each sum to small values because of the slow variation of the field. Thus, narrow intervals or those highly boosted with respect to the slowly varying frame do not contribute significantly to the D’Alembertian. Non-locality is embodied in a causal set by the existence of a large number of such narrow intervals of a fixed volume and this simple observation motivated the more complicated causal set construction which we now describe [31, 44].

In a causal set the nearest neighbours are links and hence one would expect an expression for the discrete D’Alembertian to include the sum $\sum_{y \in N^-_1(x)} \phi(y)$ where $N^-_1(x)$ denotes the set of elements that are in the past of $x$ and linked to it. Since all these terms come with equal weight, one has to counter this with terms from the next nearest neighbours, and so on. In other words, the discrete D’Alembertian should be a sum of terms of alternating sign, with each term of the form $\sum_{y \in N^-_i(x)} \phi(y)$, where $N^-_i(x)$ denotes the set of elements to the past of $x$ whose order interval $J(y, x)$ with $x$, contains precisely $i - 1$ elements besides $x$ and $y$. As with any approximation, it seems reasonable to truncate the operator by including only a finite number of layers and in order to obtain the correct continuum approximation, the exact coefficient with which each such “layer” appears in the sum has to be determined.

In 2d, for a volume cut-off $l^2$, the discrete operator takes on a particularly simple form [31]

$$B \phi(x) = \frac{4}{l^2} (-\frac{1}{2} \phi(x) + \sum_{y \in N^-_1(x)} \phi(y) - 2 \sum_{y \in N^-_2(x)} \phi(y) + \sum_{y \in N^-_3(x)} \phi(y)),$$  (4)
namely, there is a truncation at the third layer. The mean of this expression over Poisson sprinklings of $C$ converges to the continuum expression for the D’Alembertian as $l \to 0$. This averaging process also allows an expression in terms of an integral kernel

$$B(x - y) = \frac{4}{l^4} p(\zeta) e^{-\zeta} - \frac{2}{l^2} \delta^{(2)}(x - y)$$

where $p(\zeta) = 1 - 2\zeta + \frac{1}{2}\zeta^2$ and $\zeta l^2$ is the volume of the order interval $J(y, x)$ in units of $l^2$. However, although the expression (4) averages to the right continuum value, the fluctuations grow like $N$. In [31] a mesoscale $l_k >> l$ was introduced to damp these fluctuations and to rewrite the integral kernel (5) in terms of $l_k$. Discretising this kernel one gets the discrete operator

$$B_k \phi(x) = \frac{4\epsilon}{l^2} \left(-\frac{1}{2} \phi(x) + \epsilon \sum_{y \prec x} f((n(x, y), \epsilon)) \phi(x)\right)$$

where $f(n, \epsilon) = (1 - \epsilon)^n \left(1 - \frac{2\epsilon n}{1-\epsilon} + \frac{\epsilon^2 n(n-1)}{(1-\epsilon)^2}\right)$ and $\epsilon = l^2/l_k^2$. In writing this expression we have reincorporated terms from more than the first three layers but in the $l_k \to l$ limit one recovers the expression (4). The introduction of $l_k$ means that rather than summing over order intervals of volumes in units of $l^2$ one is doing so over thickened layers with volumes in units of $l_k^2$. This effectively damps out fluctuations although it does lead to the appearance of a new non-locality scale. This form of the D’Alembertian has now been generalised to all spacetime dimensions [3, 4, 45] and a prescription for finding the exact coefficients is given in [4].

What is the generalisation of this construction to curved spacetime? It turns out surprisingly enough that one can indeed use the very same flat spacetime expression for the operator $B_k$, except that additional curvature dependent terms also arise [3]. Using the Gibbons-Solodukin expression for the Ricci scalar in terms of the volume and height (or proper distance) of a small causal intervals $J(x, Y)$ [33] the authors of [3] showed that

$$\lim_{l_k \to 0} B_k \phi(x) = (\Box - \frac{1}{2} R(x)) \phi(x)$$

in both 2 and 4 spacetime dimensions, where $\phi$ is assumed to vary slowly with respect to $l_k$ and the radius of curvature is much larger than $l_k$. Using the constant field, this was then used to obtain a causal set version of the Einstein-
Hilbert action, the Benincasa-Dowker action in 2 and 4 dimensions

\[
\frac{1}{\hbar} S^{(2)}[C] = N - 2N_1 + 4N_2 - 2N_3 \quad (8)
\]

\[
\frac{1}{\hbar} S^{(4)}[C] = N - N_1 + 9N_2 - 16N_3 + 8N_4 \quad (9)
\]

where \( N \) is the number of elements in the causal set \( C \) and \( N_i \) the number of elements in the order interval which contains precisely \( i - 1 \) elements besides the end points. Thus, \( N_1 \) counts the number of links or 2 element chains, \( N_2 \) the number of 3-element chains, \( N_3 \) the number of 4-element chains plus the number of diamond causal sets. The Benincasa-Dowker action is a clear proof that causal sets can give rise to local physics, and is a big milestone for the theory. The action was generalised to arbitrary dimensions in [4].

Earlier work on finding a causal set version of the Einstein-Hilbert and matter actions for causal sets adopted a somewhat different approach [46]. The Gibbons-Solodukhin results were used to find certain components of the Ricci tensor in terms of the volume and heights of small spacetime intervals, and in order to find a way of including all possible intervals, a minimisation procedure was adopted. However, this procedure does not lead to a closed form for the action. It is possible that a procedure similar to the layer construction of [31] can be used and this could be a useful direction to pursue.

One of the promises one tends to associate with discretisation is that of curing the UV divergences of quantum field theory. Typical discretisations break Lorentz invariance since a UV cut-off is put in by hand. As we have seen, causal set discretisation is very different in that it preserves Lorentz invariance and it is therefore an open question whether it can tame these divergences. A reasonable hope might be that causal set discretisation can provide a Lorentz invariant renormalisation procedure but one is currently a long way from showing this [31]. To begin with, one has to first construct a quantum field theory on a causal set which is approximated by Minkowski spacetime. In [5] a Feynman-diagram inspired prescription was given to obtain the retarded Green’s function \( G(x - y) \) of a quantum scalar field in 2 and 3 spacetime dimensions. In 2-d, one sums over all chains from \( x \) to \( y \), weighting each vertex or element with an amplitude \( b \) and each leg or relation that appears in the chain by an amplitude \( a \). In 4-d one sums only over chains in which all the relations are strictly links. Comparison of the continuum propagator with the discrete version allows the amplitudes \( a \) and \( b \) to be evaluated. The 2d causal set propagator is well approximated by the continuum and in 4d, this is true in the continuum limit.
In [6] the next big step was taken by constructing the Feynman propagator using a rather novel procedure. To begin with, the Pauli-Jordan function is a matrix on the causal set $\Delta = G_R - G_A$, where $G_R$ and $G_A$ are the 2d or 4d causal set versions of the retarded and advanced Green’s functions that were obtained previously. $i\Delta$ is a Hermetian skew-symmetric matrix, and hence of even rank $2s$. This means that its non-zero eigenvalues appear in real positive and negative pairs $(\lambda_i, -\lambda_i)$ with $i = 1, \ldots, s$. The associated positive and negative eigenvectors $\{u_i\}$ and $\{v_i\}$ are used to define a matrix $Q = \sum_{i=1}^{s} \lambda_i u_i u_i^\dagger$, so that $i\Delta = Q - Q^*$, and bosonic quantum field operators $\hat{\phi}_x$ are associated with each causal set element. Apart from satisfying the standard Hermiticity and commutation conditions these operators satisfy a new condition which replaces the standard requirement that $\hat{\phi}$ satisfy the Klein Gordon equation. Namely, all operators linear in $\hat{\phi}$ which commute with $\hat{\phi}$ are required to themselves be zero. Thus

$$i\Delta w = 0 \Rightarrow \sum_{x=1}^{N} w_x \hat{\phi}_x = 0,$$

where $N$ is the number of causal set elements. The positive and negative eigenfunctions $\{u_i\}$ and $\{v_i\}$ of $i\Delta$ allows one to define creation and annihilation operators and using the conditions on $\hat{\phi}_x$, a consistent mode decomposition for the field operator is obtained. The vacuum associated with the creation and annihilation operators is then used to construct the two point function $Q_{xy} = \langle 0 | \hat{\phi}_x \hat{\phi}_y | 0 \rangle$ and the natural order-preserving labelling on the causal set gives the requisite time-ordering. The Feynman operator is then given by $G_F = G_R + iQ$. Numerical tests in 2-d and 4-d demonstrate that the discrete operator has the recognisable continuum form as the discreteness scale goes to zero.

What is makes this construction particularly interesting is that a vacuum state is obtained purely algebraically without explicit reference to a coordinate system. The positive and negative mode decomposition arise from properties of the Pauli-Jordan matrix which itself is defined independent of any frame of reference. Whether this construction has consequences for quantum field theory on curved spacetime is clearly an important direction to pursue. What special role does the new ingredient, Eqn (11) play, for example?

The existing calculations for the blackhole area and entropy bounds can also be categorised as kinematics since the causal set itself remains non-dynamical [17, 35, 19]. In [17] the area of a black hole like region is calculated by counting the links that cross the horizon. This and similar calculations require a past volume cut-off, but yield the desired geometric result, namely that the area of the
blackhole horizon is proportional to the number of links that cross it. Whether assigning these links the term “horizon degrees of freedom” offers a glimpse into a more complete description of this relationship can only be determined by a better understanding of the quantum dynamics of causal sets.

3 Phenomenology

An important criticism levied against quantum gravity by most other physicists is that it offers us no experimental signatures. Almost as a response to this criticism, recent years have witnessed a plethora of models of quantum gravity — from large extra dimensions which bring down the energy scale of quantum gravity to those in which there are violations of Lorentz invariance, one of the most stringently verified symmetries of nature. It is therefore of interest to see what it is that causal set theory can offer by way of an experimental signature. Without a complete theory of causal set quantum gravity — kinematics and dynamics — one may worry that any model building is futile. However, there is a big desert of energy scales between the Plank energy scale and that of current colliders and it is a legitimate query to ask whether the causal set discretisation of a spacetime has observable consequences. The small effects of discreteness can in principle be amplified by cosmological distances leading to clear cosmological signatures.

The most significant phenomenological signature from causal set theory was the prediction of a cosmological constant in the late ’80s [28]. Indeed, this preceded the observation of a small but non-zero value for Λ, and is of the same order of magnitude as the causal set prediction. The basic argument is very straightforward consequence of the random discreteness for spacetime and the number to volume correspondence (2). The cosmological constant appears in the Einstein-Hilbert action as the volume term \( \Lambda \int \sqrt{-g} d^4x = \Lambda V \) where \( V \) can be taken to be the total past volume of the universe. Fixing this volume in a unimodular modification of gravity gives rise to an uncertainty relation between the fluctuations in \( \Lambda \) and \( \Delta \): \( \Delta \Lambda \Delta V \sim 1 \) in natural units. In a causal set discretisation of the FRW universe, the fluctuation in volume can be determined from the Poisson distribution, i.e., \( \Delta N = \sqrt{N} \) so that \( \Delta V = \sqrt{V} \). One finds that \( \rho_\Lambda \sim \sqrt{\rho_{\text{critical}}} \sim H^2 \) at all times and that given the age of our universe and hence \( V \), \( \Delta \Lambda \) in our present epoch is the small value \( 10^{-120} \) in natural units. Such a small value for \( \Lambda \) has been notoriously difficult to calculate from standard quantum field theory arguments and hence it is impressive that it emerges so naturally from causal set theory. However, it is important to point out that the
argument above only predicts the value for the fluctuation in $\Lambda$. The hope is that a complete answer to this question lies within the full causal set quantum gravity.

In [50] a stochastic model for the evolution of a fluctuating $\Lambda$ was incorporated into a modified FRW model which retains only the Hamiltonian constraint. The stochastic models are characterised by a single free parameter $\alpha$ and for suitable choices of $\alpha$ numerical simulations show a $\Lambda$ which fluctuates between negative and positive values with a significant fraction of the trials giving rise to a small but positive cosmological constant in the present epoch. In these same trials, the Big Bang Nucleosynthesis phase is characterised by a large negative $\Lambda$ which the authors suggest may help explain the reduced Helium abundance of observations. Expanding this stochastic model to one in which $\Lambda$ varies spatially is of great importance in getting observational constraints on the free parameters [51]. Moreover, an answer to why the mean value of $\Lambda$ is zero may only be possible in a more developed cosmological model which incorporates quantum fields.

As we have noted, the basic argument for a fluctuating $\Lambda$ is very simple and seemingly transcends the details of the theory. Indeed, it has been suggested that since spacetime discreteness is a standard feature of several approaches to quantum gravity, they must also admit a similar solution to the cosmological constant problem. This would in principle be reasonable if we could view this problem purely as a macroscopic manifestation of a fundamental theory. After all, several disparate approaches to quantum gravity yield the same black hole area law. However, it is the precise manifestation of causal set discreteness via a number to spacetime volume correspondence that distinguishes it from the other approaches and is key to the arguments of [28]. If this correspondence is taken seriously, then one is forced to consider a random discretisation of spacetime and no other approach to quantum gravity currently uses this as a starting point.

Are there other observational consequences of causal set discreteness? The Poisson sprinkling of causal set elements allow for the possibility of very large voids or continuum volumes which are entirely unpopulated by causal set elements. Such voids, if ubiquitous, would surely contradict experimental observation since in such regions the continuum description would entirely breakdown. In spacetimes of infinite extent it is indeed the case that there will almost surely be arbitrarily large voids, but it is more pertinent to ask this question of our observable universe. In [52] it was argued that the probability of a single nuclear sized void in the observable universe is less than $\sim 10^{252} \times e^{-10^{72}}$. Thus, one needs to find more significant observational signatures for causal set discre-
ness.

In [52] a model of a massless classical particle propagating on a causal set was constructed. A causal set $C$ which faithfully embeds into flat spacetime offers no straight line path for a particle which thus must “swerve”, i.e., the direction of the momentum must change while hopping from one element $e_n$ to the next $e_{n+1}$. The momentum $p_{n+1}$ associated with such a hop can be taken to be proportional to the spacetime vector in the spacetime embedding, with starting point $e_n$ and end point $e_{n+1}$. Since this is a classical motion, the particle is assumed to choose a path in $C$ which is as close to a straight-line as possible. This means that, given a particular pair $(e_n, p_n)$, the next element $e_{n+1}$ is chosen as close to the direction $p_n$ as possible so that $|p_{n+1} - p_n|$ is minimised. Additionally, $e_{n+1}$ is chosen to lie within a certain proper time $\tau_f$ which corresponds to a “forgetting time”. In the continuum this process manifests itself as a proper time diffusion in phase space $\mathbb{H}^3 \times M^4$, where $\mathbb{H}^3$ is the momentum mass shell and $M^4$ is Minkowski spacetime. The probability distribution $\rho(p^n, x^\mu, \tau)$ then satisfies the Lorentz invariant diffusion equation

\[
\frac{\partial}{\partial \tau} \rho = k \Delta^2 p \rho - \frac{1}{mc^2} p^\mu \frac{\partial}{\partial x^\mu} \rho,
\]

where $k$ is a parameter which will depend on the forgetting time $\tau_f$. Assuming the probability distribution to be spatially uniform, the Gaussian evolution of the probability in momentum space means high energy events become significant over sufficiently long periods of time. This affords a Fermi-acceleration type mechanism for generating very high energy cosmic rays. For protons, laboratory and astrophysical data can be used to put too stringent a limit on $k$ for protons [54]. Nevertheless, the possibility of neutrino sources giving rise to similar signatures hasn’t yet been ruled out. Since the mechanism does not depend on a specific matter distribution, if there are causal set origins for ultra high gamma rays then these should be distributed isotropically. Moreover, as in a standard diffusion process the swerves do not conserve energy, but as discussed in [55] this does not contradict the current astrophysical bounds on Lorentz violation. Subsequent work in this direction includes the massive particle models of [56, 57].

More recently, a model has been proposed to study the effects of Lorentz invariant discretisations on the CMB polarisation [58]. Here, in addition to the phase space one has the set of polarisation states $B$ given by the Bloch sphere. Neglecting effects of phase spacetime diffusion, one gets a Lorentz invariant diffusion on $B$ which reduces for linearly polarised states to a diffusion cum a drift equation. In the cosmic frame these are frequency dependent. The authors
of [58] propose that this can give rise to signatures in CMB polarisation and the hope is that this exciting proposal will find useful constraints when new data becomes available from Planck.

Before going on to examining dynamics, it is perhaps important to emphasise that while it may seem tempting to some to construct a 3+1 Cauchy-type formulation of causal sets based on thickened antichains, such an attempt is doomed to failure for a reason that can, once again, be traced to non-locality. Instead of being able to mimic a Cauchy surface whose intrinsic and extrinsic geometry captures all the required information for future evolution, an inextendible antichain $\mathcal{A}$ is like a “sieve”—a large amount of geometric information in fact by-passes it, because of what I will call “missed links”. Consider a causal set $\mathcal{C}$ that faithfully embeds into Minkowski spacetime and let $\mathcal{A}$ be an inextendible antichain in $\mathcal{C}$. While $J^+(\mathcal{A}) \cap J^-(\mathcal{A}) = \mathcal{A}$, where $J^+(\mathcal{A}) \equiv \{ r | \exists a \in \mathcal{A}, a \prec r \}$ and similarly $J^-(\mathcal{A}) \equiv \{ r | \exists a \in \mathcal{A}, r \prec a \}$, so that $\mathcal{A}$ neatly divides the elements of $\mathcal{C}$ into the disjoint subsets $\mathcal{C} = \mathcal{A} \sqcup (J^+(\mathcal{A}) \setminus \mathcal{A}) \sqcup (J^-(\mathcal{A}) \setminus \mathcal{A})$, this is not true of the relations. In particular there exists pairs $p, q$ with $p \in J^-(\mathcal{A}) \setminus \mathcal{A}$ and $q \in J^+(\mathcal{A}) \setminus \mathcal{A}$ such that $p \prec q$ is a link, which $\mathcal{A}$ however “misses” (see Fig 6). In fact $\mathcal{A}$ misses an infinite number of links in Minkowski spacetime. Are these causal links relevant to signal propagation [30, 52, 56, 59]? If so, then far from capturing all the information relevant to its future, $\mathcal{A}$ in fact lets most of it slip through! A finite volume thickening of $\mathcal{A}$ clearly cannot eliminate this problem, and one has to contend with the question of how to recover the approximate global hyperbolicity of our physical laws. It is important that the missed links tend to be those that are highly boosted with respect to the rest frame associated with the antichain and hence it seems physically reasonable that such a signals will tend to pass through a macroscopic body unnoticed. However, given

Figure 6: A missing link $p \prec q$ through an inextendible antichain $\mathcal{A}$ is marked with a bold red line.
that there are an infinity of such links in Minkowski spacetime it is important to know whether the effect is cumulative. Does it imply, for example that in order for causal sets to be compatible with known physics, that the universe be past finite, at least until the last cosmological bounce?

4 Dynamics

Given that causal sets do not admit a natural space and time split, it is clear that a Hamiltonian framework is not viable for describing causal set dynamics. Indeed, it is far more natural to talk of dynamics in a path integral or sum-over histories language. In a path integral representation of 4-dimensional quantum gravity, the histories space is the space of all Lorentzian 4-geometries. In causal set theory this is replaced by the set of all (unlabelled) locally finite posets \( \Omega \). Importantly, \( \Omega \) includes causal sets that do not have a continuum approximation, and the hope is that a suitable dynamics would, in the classical limit, pick out causal sets that are approximated by continuum spacetimes. But does a typical causal set look anything like a spacetime and if not does it only differ from it “in the small”? Dynamical triangulations have already taught us that such an expectation is naive, however reasonable the discretisation may seem – it is a well known problem that most simplicial manifolds have a branched polymer structure, which dominate the path integral [60]. Causal sets also potentially face such an entropy problem. As the size of the causal set \( N \) gets larger, most causal sets are of the Kleitman-Rothschild or (KR) form shown in Fig 7 [61]. These have only three layers, with a large number of relations between each layer and are therefore most unlike spacetime. However, their numbers grow like \( 2^{N^2/4} \) where \( N \) is the size of the causal set and hence seem like a threatening presence in the path integral. Moreover, it was shown in [62]

![Figure 7: A three layer \( n \) element Kleitman-Rothschild poset](image)

that apart from the KR posets there is a hierarchy of classes of sub-dominant
causal sets, with the number of layers increasing as one goes down the hierarchy. The entropic landscape is therefore complex and characterised by first order transitions between these sub-dominant classes \[62\]. Any causal set dynamics must therefore work hard to counter this dense entropy landscape.

It is thus an important and surprising result that a dynamics can be constructed which counters this entropy. These are the Rideout-Sorkin classical sequential growth models alluded to in the abstract \[2, 63\]. These models use a bottom-up approach to dynamics – rather than deduce a rule from the continuum one begins with a set of fundamental principles. Starting from the empty set, a causal set is grown element by element in a Markovian fashion by assigning probabilities to each transition from an \(n\) element causal set to an \(n + 1\) element causal set. The \(n + 1\)th element can be added either to the future of or be unrelated to an existing element, but it cannot be added to its past. Apart from being Markovian, the dynamics is required to obey label invariance (a discrete avatar of covariance) as well as a causality condition dubbed Bell causality. The latter encodes the requirement that the transition probabilities should be independent of the “spectator” elements, i.e., those that are not involved explicitly in the transition. These simple rules can be manipulated to extract a set of coupling constants, \(t_n\), one for each stage of the growth, which then fully determine the dynamics. In the cosmological context, starting from generic initial conditions, these coupling constants are seen to get renormalised after every cosmological epoch separated by a “bounce”, with a line of fixed points corresponding to a simple, transitive percolation dynamics, with \(t_n = t^n\) for \(t \geq 0\) \[64\]. This last era of transitive percolation gives rise typically to causal sets which have more than a fleeting resemblance to continuum spacetime \[65\] although they do not seem to admit an exact spacetime approximation. Nevertheless they teach us that it is possible to overcome the threat of KR posets with an appropriate choice of dynamics.

Since the growth process cannot add a new element to the past of an existing one, the stochastic growth models give rise to causal sets that are past finite. Each causal set is moreover labelled by the specific growth process and hence, even though the probabilities are label invariant, the causal sets themselves are labelled. Because of the nature of the growth the labelling is order preserving, i.e., \(x < y \Rightarrow l(x) < l(y)\) where \(l(x) \in \mathbb{N}\) denotes the labelling of the causal set element \(x\). Thus, the space of histories or sample space \(\Omega\) is the space of past finite labelled causal sets. An obvious and important question is – what are the covariant or label invariant questions that one can ask of such a system? This question was addressed in \[66\] by re-expressing the dynamics as
a probability measure space. A finite $n$-element labelled causal set $c_n$ can be used to construct a cylinder subset $\text{cyl}(c_n)$ of $\Omega$ which is the set of all labelled causal sets in $\Omega$ whose first $n$ elements are the causal set $c_n$. The event sigma algebra $\mathcal{S}$ is then constructed by taking countable unions and intersections of all such cylinder sets constructed from the set of finite labelled causal sets. In the growth process, any $c_n$ is assigned a probability $P(c_n)$ and hence we can take $\mathcal{P}(\text{cyl}(c_n)) \equiv P(c_n)$. This measure is known to extend to all of $\mathcal{S}$ by the Kolmogorov-Caratheodory-Hahn extension theorem, so that the triple $(\Omega, \mathcal{S}, P)$ is a well defined probability measure space. A label invariant measure space can then be constructed formally [66] by taking its quotient under relabellings: if $\tilde{\Omega}$ is the set of unlabelled past finite causal sets, then $\tilde{\mathcal{S}}$ is the sub-sigma algebra of $\mathcal{S}$ constructed as follows: a subset $\tilde{\mathcal{S}} \subset \tilde{\Omega}$ is an element of $\tilde{\mathcal{S}}$ if for every $c \in \tilde{\mathcal{S}}$, its relabelling $c'$ is also in $\tilde{\mathcal{S}}$. The restriction of $P$ to $\tilde{\mathcal{S}}$ then provides a label invariant measure space $(\tilde{\Omega}, \tilde{\mathcal{S}}, P_{\tilde{\mathcal{S}}})$ [66].

But what are these label invariant or covariant events and can they be assigned a physically useful meaning? Of interest to quantum cosmology are covariant questions of the type: how many bounces has the universe gone through? In [66] it was shown that the formal event algebra $\tilde{\mathcal{S}}$ can be replaced, up to set of measure zero, with a sub-sigma algebra $\mathcal{S}'$, which is generated from physically well defined events. A finite unlabelled sub-causal set $\tilde{c}_n$ of $\tilde{c} \in \tilde{\Omega}$ is said to be a partial stem if it contains its own past, and is the causal set analogue of a “past-set” $J^-(X)$ in a spacetime. A stem set $\text{stem}(\tilde{c}_n)$ is a subset of $\tilde{\Omega}$ such that every $\tilde{c} \in \text{stem}(\tilde{c}_n)$ contains the partial stem $\tilde{c}_n$. Thus, for these cosmologies it is possible to ask, for example, what the probability is for the universe to have a single element which lies to the past of every other element (originary growth). For transitive percolation, this is given by the Euler function $\phi(q) = \prod_{i=1}^{\infty}(1 - q^i)$, where $q = 1 - p$ is the probability for a new element to be unrelated to all other elements [67]. Thus, the importance of this sub-sigma algebra is clear – it makes it possible to pose covariant cosmological questions and to therefore construct the covariant “observables” or events relevant to cosmology. It is the hope that at least a part of this analysis survives into a quantum dynamics for causal sets.

These stochastic growth models in fact admit a simple quantum generalisation with the transition probabilities replaced by transition amplitudes. Instead of a probability measure space the growth process generates a quantum pre-measure space $(\Omega, \mathcal{A}, \mu)$, where $\mathcal{A}$ is the algebra generated from cylinder sets, closed only under finite unions and intersections, and $\mu$ is a quantum pre-measure $\mu : \mathcal{A} \rightarrow \mathbb{R}^+$ which satisfies the more general quantum sum rule (QSR)
\[ \mu(A \cup B \cup C) = \mu(A \cup B) + \mu(A \cup C) + \mu(B \cup C) - \mu(A) - \mu(B) - \mu(C), \quad (13) \]

for the disjoint sets \( A, B, C \in \mathcal{A} \). \( \mu \) is not in general additive since there can be interference between different sets in \( \mathcal{A} \). This means that the extension of \( \mu \) to the full event sigma algebra \( \mathcal{S} \) is not guaranteed. But such an extension is required if we want to construct covariant observables in analogy with the classical growth models. Using the Histories Hilbert space construction of [71] the quantum pre-measure can be recast as a vector pre-measure [67] for which a Kolmogorov-Caratheodary-Hahn type extension theorem exists provided \( \mu \) satisfies certain condition. This analysis was used to explore certain simple “complex percolation” models in [67]. These models are useful quantum generalisations of the classical stochastic theories since they incorporate both the Markovian evolution as well as label invariance.\(^4\)

Underlying the above construction of a quantum measure space is another deep current of enquiry into the measurement question of quantum theory. The interpretational problems of quantum theory become of tantamount importance in quantum cosmology where one needs to give meaning to cosmological events in the absence of measuring devices. A quiet revolution in quantum interpretation was initiated by Sorkin in a series of papers starting from a quantum measure formulation [68, 69] and culminating more recently in the anhomomorphic logic proposal [74]. This is a realistic interpretation of quantum theory, independent of external observers and measuring devices, which nevertheless passes the stringent Kochen-Specker test [75]. While an exciting development for quantum theory, a more detailed description is outside the scope of this review. From a functional point of view however, it suffices to think of causal set dynamics simply as a quantum measure space. Questions related to observation reduce to events that are measurable – an event of quantum measure zero, for example, can be taken to “not occur”.

The construction of the causal set action described in Section 2 on the other hand allows a “top-down” approach to quantisation along more conventional lines (keeping the interpretational issues at bay). Using a parameter Wick rotation of the type described in [76] the Lorentzian action can be Euclideanised to obtain a partition function for causal sets. Importantly, unlike the standard Euclideanisation which changes the set of histories from Lorentzian to Euclidean

\(^4\)However, is not quite clear what the proper quantum analog of the Bell causality condition is [72, 73]. While its easy to implement a classical analog of this condition in these simple models, a more general implementation seems riddled with complications.
geometries, this procedure does not change the space of causal sets. Causal set dynamics can then be simulated using Monte Carlo methods and efforts in this direction are currently underway. Early simulations show promise, though much work remains to be done in understanding the complex phase structure of the entropy landscape. While an important direction to pursue, however, from a fundamental point of view it is unclear that the causal set action should provide us with the correct quantum dynamics. It is entirely possible that dynamics is dictated by yet undiscovered fundamental rules and that the Einstein-Hilbert action makes its appearance only as an effective dynamics much like the Navier-Stokes equation.

An interesting, though limited, example of a top-down approach to constructing causal set quantum is a 2d model of causal sets which exhibits unexpected results. It is important to reiterate that unlike most other theories of quantum gravity in which dimension is inbuilt, Ω includes causal sets that approximate to spacetimes of all dimensions. Hence much more is expected from causal set dynamics than from other approaches, since spacetime dimension is also required to be a prediction of the theory. On the other hand it is a useful exercise to examine such a dimensional restriction of causal sets. All finite element causal sets that are approximated by conformally flat 2-d spacetimes with trivial topology belong to the class of 2d-orders, obtained by taking the intersection of two linear or total orders. One can think of these two total orders as the discrete lightcone coordinates \((u, v)\) of a causal set that faithfully embeds into such a spacetime – an element \(e_1 = (u_1, v_1)\) precedes an element \(e_2 = (u_2, v_2)\) iff both \(u_1 < u_2\) and \(v_1 < v_2\). Note that in a sprinkling, the probability of an element to lie exactly on the light cone of another element (i.e., either their \(u\) or \(v\) coordinates are the same) is precisely zero. However, not all 2d orders admit a continuum approximation, and thus, though there is an intrinsic causal set dimension given by the number of total orders, this does not itself imply manifoldlikeness. In particular it means that the entropy problem could also potentially rear its head in such a model. What is surprising is that even for a uniform measure on 2d orders, the dominant contribution comes from those which faithfully embed into flat 2d spacetime. A uniform measure can be motivated from a continuum inspired model of quantum gravity which soups up this order theoretic result into a genuine 2d model of quantum gravity. Generalisations to higher dimensions though much harder to do, would be a natural direction to pursue.

\footnote{Indeed, causal sets are so inherently Lorentzian it is difficult or impossible to imagine a Euclideanised version!}
5 Conclusions

What is the current state of causal set theory? For years, criticisms have ranged from saying, on the one hand, that the approach is too minimalist to give us any recognisable physics, and on the other that it is too ad hoc and not well motivated enough. The latter complaint in my opinion is entirely unjustified. All approaches to quantum gravity necessarily use new ingredients, even when guised in the modest robes of “naturalness”. Questions of choice and emphasis are motivated either by principles of simplicity (however that is interpreted) and by phenomenological restrictions. Causal set theory is well motivated in this sense – a causal set is the simplest possible manifestation of a quantum Lorentzian geometry. It takes the Lorentzian character of spacetime, in particular the classical results of [13, 16], most seriously adding only the simplest ingredient of discreteness in a manner consistent with the classical results. They are also, at a basic level, phenomenologically justified since they satisfy Lorentz invariance, a well verified symmetry of nature. The fact that it is too minimalist is a criticism that perhaps understandably kept interested researchers at bay. It was unclear for a long time how to recover locality from such a fundamentally non-locality theory, without going to the continuum limit. In retrospect, these fears are unfounded given the recent developments in the field, in particular the construction of a well defined local D’Alembertian operator and Einstein-Hilbert action for causal sets. It gives hope that with continued work other seemingly insurmountable hurdles could also be one day be cleared.

I now present a short list of questions of immediate interest to causal set workers. It is far from exhaustive, but hopefully it is representative. Arguably, one of the most interesting questions is whether the formalism developed in [5, 6] can be applied to black-holes to study Hawking radiation. Discreteness is often taken to be synonymous with a ultraviolet cut-off, but this is not true in causal set theory, since the discretisation is Lorentz invariant. It is therefore not clear whether causal sets cure the transplankian problem. Is there instead a manifestation of non-locality which modifies the thermal spectrum as in other non-local theories [81]? Constructing a decoherence functional (or quantum measure) in terms of the Feynman propagator is also an important question to address, as is the question of how interactions manifest themselves in this framework. In [82] an interesting proposal has been made for constructing an S matrix on the causal set for interacting fields and it is important to know how far this analysis can be taken. In [3, 4, 31] in the process of constructing discrete operators, non-local continuum actions were constructed and it would be an
interesting exercise to study such non-local field theories and obtain constraints on the non-locality scale.

The numerical work on the Monte Carlo simulations for quantum dynamics of causal sets, while still in its infancy, should be able to tell us whether a Euclideanised dynamics of causal sets can give rise (after a reverse Wick rotation) to causal sets that are manifoldlike in the classical limit. In the Causal Dynamical Triangulation (CDT) approach, a Euclideanised dynamics has been employed successfully to cure the entropy problem encountered when the causality constraint is not included \[20\] and it is a somewhat analogous success that we seek with causal sets. On the other hand, while interesting and of value, the success or failure of this approach may not be related, in a simple way, to the construction of a truly bottom-up quantum dynamics.

As described in Section 4 one would like to obtain a fundamental class of quantum dynamics based on first principles much like the classical stochastic growth dynamics, but the implementation of causality remains an important open question \[72, 73\]. In addition, any examination of quantum dynamics ties in intimately with the measurement problem (which we have touched upon perhaps too lightly in this review.) The quantum measure approach provides a framework to do so and while one now has an understanding of certain finite dimensional situations, developments in the infinite dimensional case will be crucial to quantum gravity \[67\].

Another very important question which, when answered, should make the approach more amenable to the wider community, is how matter should be incorporated into causal set theory. The combination of discreteness with Lorentz invariance means that there is no ready prescription for putting vector fields or higher spin fields on the causal set. Should these instead emerge from the theory or does one simply need an (as yet undiscovered) well defined way of putting them in “by hand”? Fermions are perhaps easier to incorporate – one can assign spins to the links, but relating these to spacetime fermions brings us back to the problem encountered with putting higher spins on the causal set. Progress on such questions would be extremely valuable in constructing more realistic phenomenological models. In \[82\] a suggestion for a Feynman chequerboard model was made for introducing spin on the causal set. These discussions suggest that rather than getting classical fields to sit on a causal set, one should construct the more fundamental quantum field operators which in the continuum approximation will give rise to the classical fields.

Another very important direction that needs to be actively pursued is the search for phenomenological signatures of the sort described in Section 3.
begin with, more realistic models of a varying $\Lambda$ need to be constructed to match observations more precisely. Can the neutrino origins of cosmic rays give hope to the swerve model of particles and are there other ways in which the Lorentz invariant discreteness of causal sets can manifest itself in observations? One can at this stage ask bolder questions – can causal set theory explain the other big puzzles of cosmology like dark matter or structure formation, and can it provide an alternative explanations to the scale invariance of the CMB power spectrum? These may seem too broad a class of questions, but it seems possible at this point in the development of the theory, to begin building more realistic models to constrain the effects of spacetime discreteness on observations.

In this article I have outlined causal set theory in broad strokes and pointed to the directions which are currently being pursued. A modest hope is that the reader will become familiar with the basic tenets of this approach, while a more ambitious hope is that she or he will then begin actively ruminating on causal sets!

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