Stability and Hopf Bifurcation Analysis of the Delay Logistic Equation

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Abstract—Logistic functions are good models of biological population growth. They are also popular in marketing in modelling demand-supply curves and in a different context, to chart the sales of new products over time.

Delays being inherent in any biological system, we seek to analyse the effect of delays on the growth of populations governed by the logistic equation. In this paper, the local stability analysis, rate of convergence and local bifurcation analysis of the logistic equation with one and two delays is carried out and it can be extended to a system with multiple delays.

Since fluctuating populations are susceptible to extinction due to sudden and unforeseen environmental disturbances, a knowledge of the conditions in which the population density is fluctuating or stable is of great interest in planning and designing control as well as management strategies.

Index Terms—delay logistic equation, stability, rate of convergence, Hopf bifurcation analysis.

I. INTRODUCTION

The logistic equation is a simple model of population growth in conditions where there are limited resources. It was proposed by Verhulst in 1838 to describe the self limiting growth of a biological population [1]. The equation has a variety of applications. It is used in neural networks to clamp signals to within a specified range [2], in economics to illustrate the progress of the diffusion of an innovation through its life cycle [3], in medicine to model the growth of tumours [4] and in linguistics to model language change [5].

The growth of a population is conceptualised in Fig. 1. A biological population with plenty of food, space to grow and no threat from predators tends to grow at a rate that is proportional to the population. However, most populations are constrained by environmental limitations. Growth is eventually limited by a factor, usually one from amongst many essential resources. When a population is far from its limits of growth or the carrying capacity of the ecosystem, it can grow exponentially. The feedback about the availability of resources reaches with a delay due to various factors such as generation and maturation periods, differential resource consumption with respect to age structure, hunger threshold levels, migration and diffusion of populations, markedly differing birth rates in interaction species and delays in behavioural responses to a changing environment (including changes in density of prey or predators or competing species) [6-8]. When nearing its limits, the population can fluctuate, even chaotically.

According to the logistic equation, the growth rate of a population is directly proportional to the current population and the availability of resources in the ecosystem. The logistic equation model is as follows,

\[
\frac{dy}{dt} = ay \left(1 - \frac{y}{k}\right),
\]

where \(y\) is the population at that instant, \(a\) is called the Malthusian parameter which represents the growth rate and \(k\) is the carrying capacity of the ecosystem.

Hutchinson incorporated the effect of delays into the logistic equation [9]. Delays bring about interesting topological changes in the population size like damped oscillations, limit cycles and even chaos [10]. The bifurcation analysis of a system with a single delay has been performed in [11]. The importance of two delays in the logistic equation can be seen in [12-15]. In this paper, the effect of such delays will be analysed. Analysis will primarily focus on local stability, local rate of convergence and local bifurcation phenomena. The procedure adopted here to characterise a system with two delays can be easily extended to a case of multiple delays.

The rest of the paper is organized as follows. In section 2, the model is described. In section 3, It is linearised, local stability analysis of the model is carried out and the conditions for stability are presented. In section 4, rate of convergence is analysed. Bifurcation analysis of the system is carried out in section 5. Finally in section 6, the graphical results of the analysis are presented.

II. MODEL DESCRIPTION

In this section, we present the model employed with the accompanying assumptions for a single delay system and a two delay system.

A. Logistic equation with single delay

The model we have employed to demonstrate population dynamics is as follows:

- The initial normalised population is chosen to be small (typically 0.01) as it cannot be zero. A zero initial population signifies a non-existent species.
- The equation is normalized.
- The growth rate is \(a\) which is finite, positive and time independent.
- The delay \(\tau_1\) is also finite and positive.

With these assumptions, the single delay logistic equation simplifies to
Figure 1. Schematic of the model. The population growth depends both on the current population (reproductive growth) as well as the availability of resources. The logistic equation with delay model the abstraction well and is widely used.

\[
\frac{dx}{dt} = ax(t) [1 - bx(t - \tau_1)]. \tag{2}
\]

**B. Logistic equation with multiple delays**

The same assumptions as made is the single delay logistic equation may be applied here as well. In the case of a system with \( n \) delays, the equation takes the form

\[
\frac{dx}{dt} = ax(t) \left[ 1 - \sum_{i=1}^{n} b_i x(t - \tau_i) \right]. \tag{3}
\]

The case of two delays is analysed in this paper.

### III. LOCAL STABILITY ANALYSIS

The delay logistic equation is non-linear. In this section, we first linearise it and proceed to extract conditions for the stability of both the single delay as well as the multiple delay logistic equation. We end the section by presenting stability charts of the system.

**A. Stability analysis of logistic equation with single delay**

Let the equilibrium point (where the growth rate is zero) be denoted by \( x^* \). Then \( x(t) = x(t - \tau_1) = x^* \). Also,

\[
\frac{dx}{dt} \equiv 0 = ax^* (1 - bx^*). \tag{4}
\]

Solving for this, we get \( x^* = 0 \) (signifying the initial stages) and \( x^* = \frac{1}{b} \) (signifying the saturating stages).

Now, to linearise the equation, substitute \( x(t) \equiv x^* + p(t) \), where \( p(t) \) is a small variation in the population. Higher powers of \( p \) may be neglected. The equation therefore is

\[
\frac{dx}{dt} = a [x^* + p(t)] [1 - bx^* - bp(t - \tau_1)].
\]

1) *Stability analysis around initial value:* Substituting \( x^* = 0 \) or linearising around the initial value, we have

\[
\frac{dp}{dt} = ap(t), \tag{5}
\]

the solution of which is easily obtained as

\[
p(t) = b_1 e^{at},
\]

which is to say

\[
x(t) = be^{at}, \tag{6}
\]

around \( x = 0 \). This equilibrium point is not stable as the characteristic equation of (5) has as its root \( a \) which is in the right half plane.

2) *Stability analysis around saturation value:* This value is of greater interest to us. Substituting \( x^* = \frac{1}{b} \) or linearising around the saturation value, we have

\[
\frac{dp}{dt} = -ap(t - \tau_1).
\]

To solve this, we take

\[
p(t) = b_2 e^{\lambda t}.
\]

Then, the characteristic equation reduces to

\[
\lambda + ae^{-\lambda \tau_1} = 0. \tag{7}
\]

For the system to be stable, the roots of the above equation must lie on the left-half of the \( \lambda \) plane. Bifurcation point is the value of the parameters for which the roots lie on the imaginary axis.

Substituting \( \lambda = j\omega \) in (7), we get

\[
j\omega + ae^{-j\omega \tau_1} = 0
\]

\[
\implies j\omega + a [\cos(\omega \tau_1) - j \sin(\omega \tau_1)] = 0. \tag{8}
\]

Equating real part to zero, we get

\[
a \cos(\omega \tau_1) = 0
\]

\[
\implies \omega \tau_1 = (2n + 1) \frac{\pi}{2}, \quad n = 0, 1, 2... \tag{9}
\]

Equating imaginary part of (8) to zero, we get

\[
\omega - a \sin(\omega \tau_1) = 0.
\]

To get minimum order solution, we use \( n = 0 \) in (9).

\[
\frac{\pi}{2} - a \tau_1 = 0.
\]

For \( \tau = 0 \), the characteristic equation is

\[
\lambda = -a,
\]

which implies that \( \lambda < 0 \) for \( a > 0 \) and the system is stable. i.e., for \( a\tau_1 = 0 \), the roots of the characteristic equation...
lie of the left-half plane. Therefore, necessary and sufficient condition for stability is,
\[ a\tau_1 < \frac{\pi}{2}. \]  
(10)

It is apparent that if the system has no delays or \( \tau_1 = 0 \), the system is always stable.

B. Stability analysis of logistic equation with two delays

At the fixed points, \( x(t - \tau_1) = x(t - \tau_2) = x(t) = x^* \). Here \( x^* \)is the equilibrium point. The delay logistic equation then becomes:
\[ \frac{dx}{dt} \equiv 0 = ax^*[1 - (b_1 + b_2)x^*]. \]  
(11)
Equating \( \frac{dx}{dt} \) to zero, we get \( x^* \) to be 0 or \( \frac{1}{b_1 + b_2} \). We have seen in the previous subsection that the first equilibrium is unstable and is not of much interest. We explore the latter value.

The delay logistic equation is then linearised by expanding it by a Taylor series and neglecting all higher order terms. We obtain the following linearised equation about the fixed point \( x^* = \frac{1}{1 + b_2} \), such that \( x(t) = x^* + y(t) \). We obtain:
\[ \frac{dy}{dt} = -ax^*[b_1 y(t - \tau_1) + b_2 y(t - \tau_2)]. \]  
(12)
To solve this, we take
\[ y(t) = b_2 e^{\lambda t}. \]

If \( b_1 \gg b_2 \), then the second delay term in (12) can be neglected and the analysis reduces to the case of a single delay system. Consider the case when \( b_1 = b_2 \),
\[ \lambda e^{\lambda t} = - a \left( e^{\lambda(t - \tau_1)} + e^{\lambda(t - \tau_2)} \right). \]
The characteristic equation therefore is
\[ \lambda + \frac{a}{2} e^{-\lambda \tau_1} + \frac{a}{2} e^{-\lambda \tau_2} = 0. \]  
(13)
For the system to be stable, the roots of the above equation must lie on the left-half of the \( \lambda \) plane. Bifurcation point is the value of the parameters for which the roots lie on the imaginary axis. Substituting \( \lambda = j\omega \) in (13), we get
\[ j\omega + \frac{a}{2} e^{-j\omega \tau_1} + \frac{a}{2} e^{-j\omega \tau_2} = 0 \]
\[ \implies j\omega + \frac{a}{2} \left[ \cos(\omega \tau_1) - j \sin(\omega \tau_1) \right] + \frac{a}{2} \left[ \cos(\omega \tau_2) - j \sin(\omega \tau_2) \right] = 0. \]
Equating imaginary part of (13) to zero, we get
\[ \omega = \frac{a}{2} \sin(\omega \tau_1) - \frac{a}{2} \sin(\omega \tau_2) = 0. \]
Equating real part to zero, we get
\[ \frac{a}{2} \cos(\omega \tau_1) + \frac{a}{2} \cos(\omega \tau_2) = 0 \]
\[ \implies \cos \left( \frac{\omega \tau_1 + \tau_2}{2} \right) \cos \left( \frac{\omega \tau_1 - \tau_2}{2} \right) = 0. \]
The second term cannot be 0 as the condition on the imaginary component will imply that \( \omega = 0 \) which is not true for all \( \omega \).
\[ \omega = \frac{2(\pm \pi + 2\pi C_1)}{\tau_1 + \tau_2}, \]  
(14)
where \( C_1 \in \mathbb{Z} \).

Using the value of \( \omega \) obtained in (14), we get
\[ a = \frac{2\pi}{(\tau_1 + \tau_2) \left( \sin \left( \frac{\pi \tau_1}{\tau_1 + \tau_2} \right) - \sin \left( \frac{\pi \tau_2}{\tau_1 + \tau_2} \right) \right)}, \]  
(15)
where \( (\tau_1 + \tau_2) \left[ \sin \left( \frac{\pi \tau_1}{\tau_1 + \tau_2} \right) - \sin \left( \frac{\pi \tau_2}{\tau_1 + \tau_2} \right) \right] \neq 0. \)
On simplifying this, we get the bifurcation point at
\[ a = \frac{\pi}{(\tau_1 + \tau_2) \cos \left( \frac{\pi (\tau_1 - \tau_2)}{2(\tau_1 + \tau_2)} \right)}. \]  
(16)
The sufficient and necessary condition is:
\[ a(\tau_1 + \tau_2) \cos \left( \frac{\pi (\tau_1 - \tau_2)}{2(\tau_1 + \tau_2)} \right) < \pi. \]  
(17)
Substituting \( \tau_1 = \tau_2 = \tau \), we get the familiar single delay case \( a\tau < \frac{\pi}{2} \). These conditions are valid for any positive value of \( a, \tau_1 \) and \( \tau_2 \). We also get the following sufficient condition for the specific case of \( b_1 = b_2 \) from (17),
\[ a(\tau_1 + \tau_2) < \pi. \]

C. Sufficient conditions for stability

In this sub-section, we explore the sufficient conditions for stability in both the single as well as multiple delay models. Consider the characteristic equation of a system with two delays:
\[ \lambda + ax^* b_1 e^{-\lambda \tau_1} + ax^* b_2 e^{-\lambda \tau_2} = 0. \]
This can be re-written as follows [16], [17]:
\[ \lambda + ax^* b_1 + ax^* b_2 + \frac{\lambda \tau_1 ax^* b_1}{\lambda \tau_1} (e^{-\lambda \tau_1} - 1) + \frac{\lambda \tau_2 ax^* b_2}{\lambda \tau_2} (e^{-\lambda \tau_2} - 1) = 0. \]
We now define
\[ H_1(\lambda) = \lambda + ax^* b_1 + ax^* b_2, \]
and
\[ H_2(\lambda) = \lambda \tau_1 ax^* b_1 \left( \frac{e^{-\lambda \tau_1} - 1}{\lambda \tau_1} \right) + \lambda \tau_2 ax^* b_2 \left( \frac{e^{-\lambda \tau_2} - 1}{\lambda \tau_2} \right). \]
Clearly, \( H_1(\lambda) \) has no zeros on the right half plane. Consider the imaginary axis,
For any real $\theta$, $|1 - e^{-j\theta}| < 1$.

Hence, on the imaginary axis we have,

$$|H_2(\lambda)| < |\lambda| \tau_1 a x^* b_1 + \tau_2 a x^* b_2|.$$  \hspace{1cm} (18)

If

$$\tau_1 a x^* b_1 + \tau_2 a x^* b_2 < 1,$$  \hspace{1cm} (18)

then:

$$|H_2(\lambda)| < |\lambda|.$$  \hspace{1cm} (18)

Hence, by Rouche’s Theorem, $H_1(\lambda) + H_2(\lambda) \neq 0$ on the imaginary axis. There cannot be any zeros in the right half plane and the system is stable. Hence a sufficient condition for stability is:

$$a(b_1\tau_1 + b_2\tau_2) < b_1 + b_2.$$  \hspace{1cm} (19)

For a system with only one delay, i.e. $b_2 = 0$, the sufficient condition is

$$a\tau_1 < 1.$$  \hspace{1cm} (20)

It is noted that the condition (19) is conservative.

D. Stability charts

In Fig. 2, the stability chart of a system with single delay is seen. The plot simply corresponds to the bifurcation point condition $a\tau = \pi/2$. Fig. 3 has the stability chart of the two delay system. Fig. 4, the stability of the two delay logistic equation is shown with respect to the growth rate and the two delays.

E. Nyquist Plots

In this subsection, we see a few representative Nyquist plots of the system when stable or unstable.

1) Single delay system: In Fig. 5, we observe the Nyquist plot of the characteristic equation (7). If there are any encirclements about the origin, the system is unstable as the characteristic equation does not have poles.

2) Two delay system: In Fig. 6, the Nyquist plot for a system with two delays is shown. There are no encirclements about the origin, hence the system is stable. This matches with our calculation as the parameters satisfy the conditions in (17).

IV. RATE OF CONVERGENCE

In this section, the analysis of rate of convergence is carried out for a system with a single delay. We define the rate of convergence ($R$) as the inverse of the settling time based on a tolerance band of $\pm36.8\%$.

Consider the characteristic equation (7). If there are no oscillations in the system, the imaginary part of $\lambda$ is zero. Let $\Re\{\lambda\} = \sigma$, where $\Re(x)$ denotes the real part of $x$. The equation now becomes,
Figure 4. Stability chart of a two delay system. The system is stable below the surface and enters a Hopf bifurcation at the contour shown. It is not locally stable for points which lie above the contour.

Figure 5. Nyquist plot for an unstable system with a single delay. $a = 2, \tau = 1$ in this diagram and it does not satisfy the sufficient and necessary conditions highlighted in (10). Encirclements are seen about the origin.

\[ \sigma + ae^{-\sigma \tau} = 0. \tag{21} \]

On differentiating (21) with respect to \( \sigma \), we obtain,

\[ 1 - a\tau e^{-\sigma \tau} = 0. \tag{22} \]

Eliminating \( \sigma \) from (21) and (22), we find,

\[ a\tau = \frac{1}{e}. \tag{23} \]

This point can be considered as the critical damping point. For values of \( a\tau < \frac{1}{e} \), the system behaves in an overdamped fashion and converges without any overshoot. When \( \frac{1}{e} < a\tau < \frac{1}{2} \), underdamped behaviour is observed. i.e. convergent oscillations are present.

Figure 6. Nyquist plot of a system with two delays. $a = b_1 = b_2 = 1, \tau_1 = 0.5$ and $\tau_2 = 1$ is chosen such that conditions in (17) are met. There are no encirclements about the origin and the system is stable.

The characteristic equation (7) can be rewritten as in [18],

\[ \lambda \tau_1 e^{\lambda \tau_1} = -a\tau_1 \]

\[ \Rightarrow \lambda = \frac{W(-a\tau_1)}{\tau_1}, \]

where $W(x)$ is the Lambert W function. The rate of convergence $R$ is given by,

\[ R = \left| \Re \left( \frac{W(-a\tau_1)}{\tau_1} \right) \right|. \tag{24} \]

**Rate of convergence charts**

In Fig. 7, the rate of convergence of the logistic equation with single delay is seen with respect to the time delay. It can be seen that the system converges fastest for a slightly underdamped system. Fig. 8 shows the variation of rate of convergence of solutions to (7) with respect to the growth rate. It can be observed that rate of convergence tends to zero as growth rate tends to zero. In other words, a population which multiplies very slowly requires a very large time to reach its carrying capacity. Fig. 9 shows the variation of the rate of convergence with both parameters of the single delay system.

**V. Bifurcation Analysis**

In this section, it is shown that the delay logistic equation undergoes a Hopf bifurcation at a critical value of the parameters growth rate ($a$) or delay ($\tau_1$). The nature of the bifurcation is characterized in the subsequent sub-sections as are the properties of the resulting periodic oscillations. Bifurcation diagrams are presented at the end of the section.
Figure 7. The rate of convergence is plotted with respect to the time delay \( \tau_1 \) in a system with single delay where the growth rate \( a = 1 \). It is seen to converge fastest for a slightly underdamped system when \( \tau_1 \) is slightly greater than the critical damping point \( \tau_1 = \frac{\pi}{2} \).

Figure 8. The rate of convergence is plotted with respect to the time delay in a system with single delay where \( \tau_1 = 1 \).

Figure 9. Rate of convergence with respect to growth rate \( a \) and delay \( \tau_1 \) in the single delay system. The red portions represent the fastest rate of convergence and blue the slowest. The white portions are for the regions where the system is not stable. As can be seen from the chart, fastest rates of convergence are obtained for systems with large values of growth rate and a small value of the delay.

**A. Existence of Hopf bifurcation**

From (10), at the critical point for the single delay case, \( a\tau_1 = \frac{\pi}{2} \). Differentiating characteristic equation (7) with respect to the growth rate, we get:

\[
\frac{d\lambda}{da} = \frac{e^{-\lambda \tau_1}}{a \tau_1 e^{-\lambda \tau_1} - 1}.
\]

Evaluating at critical point \( a^* = \frac{\pi}{2\tau_1} \),

\[
\Re \left\{ \frac{d\lambda}{da} \right\}_{a=a^*} = \frac{\pi}{4} + 1 > 0.
\]

(25)

Similarly differentiating (7) with respect to the delay, we obtain:

\[
\frac{d\lambda}{d\tau_1} = \frac{\lambda a e^{-\lambda \tau_1}}{1 - a \tau_1 e^{-\lambda \tau_1}}.
\]

Evaluating at critical point \( \tau_1^* = \frac{\pi}{2a} \),

\[
\Re \left\{ \frac{d\lambda}{d\tau_1} \right\}_{\tau_1 = \tau_1^*} = \frac{a^2}{\frac{\pi^2}{4} + 1} > 0.
\]

(26)

The transversality condition of the Hopf spectrum with respect to the growth rate and time delay is satisfied in (25) and (26) respectively. Thus, logistic equation with single delay undergoes a Hopf bifurcation at the critical point given by \( a\tau_1 = \frac{\pi}{2} \).

Consider the logistic equation with two delays given by (13). Differentiating (13) with respect to the growth rate, we get,

\[
\frac{d\lambda}{da} = \frac{e^{-\lambda \tau_1} + e^{-\lambda \tau_2}}{a \tau_1 e^{-\lambda \tau_1} + a \tau_2 e^{-\lambda \tau_2} - 2}.
\]

Evaluating at the critical point given by

\[
a^* = \frac{\pi}{(\tau_1 + \tau_2) \cos \left( \frac{\pi(\tau_1 - \tau_2)}{2(\tau_1 + \tau_2)} \right)},
\]

(27)

we get,

\[
\Re \left\{ \frac{d\lambda}{da} \right\}_{a=a^*} = \frac{\pi \tau_1 + \tau_2}{\tau_1 + \tau_2} \frac{2\tau_1 \tau_2}{\tau_1 + \tau_2} > 0.
\]

(28)

Hence the transversality condition of the Hopf bifurcation is satisfied with respect to the growth rate. Thus, the two delay logistic equation undergoes Hopf bifurcation with respect to the growth rate at the critical point given by (27).
B. Direction and stability of the Hopf bifurcation in the logistic equation with growth rate as the parameter

The logistic equation with a single delay or with two delays undergoes a Hopf at a critical value of the growth rate as shown in the previous sub-section. In this section the direction, stability and period of the bifurcating solutions is analysed. The procedure adopted is based on the centre manifold theory [19] (See also [20]). The logistic equation with two delays as described in (3) can be written as,

\[ \dot{u}(t) = ax(t) \times f(x(t - \tau_1), x(t - \tau_2)), \]

where,

\[ f(x(t - \tau_1), x(t - \tau_2)) = 1 - \left( b_1 x(t - \tau_1) + b_2 x(t - \tau_2) \right), \]  
(29)

Without loss of generality assume that \( \tau_2 \geq \tau_1 \). (29) can also be written as,

\[ \dot{u}(t) = L_\mu u(t) + F(u(t), \mu), \quad u(t) = x(t), \ t > 0, \ \mu \in R, \ \text{where for} \ \tau_2 > 0, \]

\[ u(t) = u(t + \theta), \ u : [-\tau_2, 0] \rightarrow R, \ \theta \in [-\tau_2, 0]. \]

Also, \( L_\mu : C[-\tau_2, 0] \rightarrow R \) is

\[ L_\mu \phi = -\left( a_0 + \mu \right)x^* \left( b_1 \phi(-\tau_1) + b_2 \phi(-\tau_2) \right) \]

and \( F(u(t), \mu) : C[-\tau_2, 0] \rightarrow R \) is

\[ F(u(t), \mu) = \left( a_0 + \mu \right)x^* \left( b_1 u(t)u(-\tau_1) + b_2 u(t)u(-\tau_2) \right). \]

Here \( a = a_0 + \mu \) and \( a_0 \) refers to the critical value of the growth rate at the bifurcation.

By the Riesz representation theorem, there exists a matrix-valued function with bounded variation components \( \eta(\theta, \mu), \theta \in [-\tau_2, 0] \), such that

\[ L_\mu \phi = \int_{-\tau_2}^{0} d\eta(\theta, \mu) \phi(\theta), \]
(31)

where \( \phi \in C([-\tau_2, 0], \mathbb{R}) \) and,

\[ d\eta(\theta, \mu) = -(a_0 + \mu) x^* \left[ b_1 \delta(\theta + \tau_1) + b_2 \delta(\theta + \tau_2) \right]. \]

Here \( \delta(\theta) \) is the Dirac-Delta function.

For \( \phi \in C^1([-\tau_2, 0], \mathbb{R}) \), define

\[ A_\mu \phi(\theta) = \left \{ \begin{array}{ll} \frac{d\phi}{d\theta} , & \theta \in [-\tau_2, 0) \\ \int_{-\tau_2}^{0} d\eta(\xi, \mu) \phi(\xi) = L_\mu \phi, & \theta = 0, \end{array} \right. \]
(32)

and

\[ R = \left \{ \begin{array}{ll} 0, & \theta \in [-\tau_2, 0), \\ F, & \theta = 0. \end{array} \right. \]
(33)

Now, (30) can be written as

\[ \dot{u}_1(t) = A_\mu u_1 + R_\mu u_1. \]
(34)

The bifurcating solutions of \( u(t, \mu(\epsilon)) \) of (29) have amplitude \( O(\epsilon) \), period \( P(\epsilon) \) and non-zero Floquet exponent \( \beta(\epsilon) \), where \( \mu, P \) and \( \beta \) have the following expansions:

\[ \mu = \mu_2 \epsilon^2 + \mu_4 \epsilon^4 + O(\epsilon^6), \]

\[ P = 4\tau_1 \left( 1 + T_2 \epsilon^2 + T_4 \epsilon^4 + O(\epsilon^6) \right), \]

\[ \beta = \beta_2 \epsilon^2 + \beta_4 \epsilon^4 + O(\epsilon^6). \]

If \( \mu_2 > 0 \), the bifurcation is supercritical and if \( \mu_2 < 0 \), it is subcritical. If \( \beta_2 < 0 \), \( u(t, \mu(\epsilon)) \) shows asymptotic orbital stability and instability if \( \beta_2 > 0 \). These coefficients will now be calculated.

Define the adjoint operator \( A^* \) as,

\[ A_0^* \psi(s) = \left \{ \begin{array}{ll} \frac{-d\psi}{ds}, & s \in (0, \tau_1] \\ \int_{-\tau_2}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{array} \right. \]
(35)

Here, \( \psi \in C([0, \tau_2], \mathbb{R}) \) and \( \eta^T \) denotes the transpose of \( \eta \).

For \( \phi \in C^1([-\tau_2, 0], \mathbb{R}) \) and \( \psi \in C([0, \tau_2], \mathbb{R}) \) define an inner product

\[ \langle \psi, \phi \rangle = \psi(0)\phi(0) - \int^{0}_{\theta=-\tau_2} \int^{\theta}_{\xi=0} \psi^T(\xi-\theta) d\eta(\theta,0)\phi(\xi)d\xi. \]
(36)

Let \( q(\theta) \) be the eigenfunction for \( A_0 \) corresponding to \( \lambda(0) \), namely

\[ A_0 q(\theta) = j \omega_0 q(\theta), \]

\[ q(\theta) = e^{j\omega_0 \theta}. \]

Let \( q^*(s) \) be an eigen vector of \( A_0^* \) such that

\[ q^*(s) = De^{j\omega_0 s}, \]

and

\[ \langle q^*, q \rangle = 1, \ \langle q^*, \bar{q} \rangle = 0. \]

From the above equation \( D \) can be determined as shown below.

\[ \langle q^*, q \rangle = \bar{D} - \bar{D} \int^{0}_{\theta=-\tau_2} \int^{\theta}_{\xi=0} e^{-j\omega_0 (\xi-\theta)} d\eta(\theta)e^{j\omega_0 \xi}d\xi \]

\[ \Rightarrow 1 = \bar{D} - \bar{D} \int^{0}_{\theta=-\tau_2} \int^{\theta}_{\xi=0} e^{j\omega_0 \theta}d\eta(\theta) \]

\[ \Rightarrow 1 = D - D \int_{-\tau_2}^{0} [\tau_1 e^{-j\omega_0 \tau_1}(a_0 + \mu)b_1 + \tau_2 e^{-j\omega_0 \tau_2}(a_0 + \mu)b_2] \]

\[ \Rightarrow D = \frac{1}{1 - (a_0 + \mu) x^* [b_1 \tau_1 e^{j\omega_0 \tau_1} + b_2 \tau_2 e^{j\omega_0 \tau_2}]} \]
It can be easily verified that \( \langle q^*,\bar{q}\rangle = 0 \).

Define,

\[
\dot{z}(t) = \langle q^*, u_t \rangle,
\]

\[
w(t, \theta) = u_t(\theta) - 2\Re\{ z(t)q(\theta) \}.
\]

Then on the centre manifold \( C_0 \),

\[
w(t, \theta) = w(z(t), \bar{z}(t), \theta),
\]

where

\[
w(z, \bar{z}, \theta) = w_{20}(\theta)z^2 + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\bar{z}^2 + \ldots
\]

and \( z \) and \( \bar{z} \) are local coordinates for centre manifold \( C_0 \) in the direction of \( q^* \) and \( \bar{q}^* \). Note that \( w \) is also real if \( u_t \) is real, we consider only real solutions. For solutions \( u_t \in C_0 \) of (11) at \( \mu = 0 \),

\[
\dot{z}(t) = \langle q^*, Au_t + Ru_t \rangle = j\omega_0(\dot{z}(t) + \bar{q}^*(0)F_0(z, \bar{z})) = j\omega_0z(t) + g(z, \bar{z}),
\]

(37)

here,

\[
g(z, \bar{z}) = \bar{q}^*(0)F_0(z, \bar{z}) = g_{20}z^2 + g_{11}z\bar{z} + g_{02}\bar{z}^2 + g_{21}z^2\bar{z} + \ldots
\]

Now using (34) and (35) we get,

\[
w = \dot{u}_t - \dot{z}q - \dot{\bar{q}} \bar{q}
\]

(38)

or,

\[
w = \begin{cases}
Aw - 2\Re\{ \bar{q}^*(0)F_0q(\theta) \}, & \theta \in [-\tau_1, 0) \\
Aw - 2\Re\{ q^*(0)F_0q(\theta) \} + F_0, & \theta = 0,
\end{cases}
\]

which can be written as

\[
w = Aw + H(z, \bar{z}, \theta),
\]

(39)

where

\[
H(z, \bar{z}, \theta) = H_{20}(\theta)z^2 + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\bar{z}^2 + \ldots
\]

(40)

Expanding the above series and comparing the co-efficients we get,

\[
(2j\omega_0 - A)w_{20}(\theta) = H_{20}(\theta),
\]

(41)

\[-Aw_{11} = H_{11}(\theta),
\]

(42)

\[-(2j\omega_0 + A)w_{02}(\theta) = H_{02}(\theta).
\]

(43)

From (38), we have

\[
u_t(\theta) = w(z, \bar{z}, \theta) + zq(\theta) + \bar{q}\bar{q}(\theta)
\]

\[
w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\bar{z} + w_{02}(\theta)\frac{\bar{z}^2}{2} + ze^{j\omega_0\theta + \bar{z}e^{-j\omega_0\theta} + \ldots
\]

from which \( u_t(0) \) and \( u_t(-\tau_1) \) can be obtained. As we only need the coefficients of \( z^2, z\bar{z}, z^2 \) and \( z^2\bar{z} \), we get

\[
u_t(0) = w(z, \bar{z}, 0) + z + \bar{z},
\]

\[
u_t(-\tau_1) = w(z, \bar{z}, -\tau_1) + ze^{-j\omega_0\tau_1} + z\bar{e}^{j\omega_0\tau_1},
\]

\[
u_t(0)u_t(-\tau_1) = w(0)(w(-\tau_1) + w(-\tau_1)(z + \bar{z}) + w(0)(\bar{e}^{-j\omega_0\tau_1} + z\bar{e}^{j\omega_0\tau_1}) + z^2e^{-j\omega_0\tau_1} + z\bar{e}^{j\omega_0\tau_1} + e^{-j\omega_0\tau_1} + z^2\bar{e}^{j\omega_0\tau_1},
\]

\[
u_t(0)u_t(-\tau) = z^2e^{-j\omega_0\tau_1} + z\bar{e}^{j\omega_0\tau_1} + z\bar{e}^{j\omega_0\tau_1} + e^{-j\omega_0\tau_1}) + z^2\bar{e}^{j\omega_0\tau_1} + (2w_{11}(0)e^{-j\omega_0\tau_1} + \frac{w_{20}(0)}{2})e^{-j\omega_0\tau_1}
\]

\[
+w_{11}(0) + \frac{w_{20}(0)}{2} + \ldots
\]

Now we can write,

\[
g(z, \bar{z}) = \bar{q}^*(0)F_0(z, \bar{z}) = g_{20}z^2 + g_{11}z\bar{z} + g_{02}\bar{z}^2 + g_{21}z^2\bar{z} + \ldots
\]

\[
g_{20} = -\bar{q}^*(0)2ax^*\left[b_1e^{j\omega_0\tau_1} + b_2e^{j\omega_0\tau_2}\right]
\]

\[
g_{11} = -\bar{D}ax^*\left[b_1(e^{j\omega_0\tau_1} + e^{-j\omega_0\tau_1}) + b_2(e^{j\omega_0\tau_2} + e^{-j\omega_0\tau_2})\right],
\]

(44)

\[
g_{02} = -\bar{D}ax^*[b_1e^{j\omega_0\tau_1} + b_2e^{j\omega_0\tau_2}]
\]

(45)

\[
g_{21} = -\bar{D}2ax^*\left[b_1w_{11}(0)e^{-j\omega_0\tau_1} + \frac{w_{20}(0)}{2}e^{j\omega_0\tau_1}
\]

\[+w_{11}(-\tau_1) + \frac{w_{20}(0)}{2})e^{-j\omega_0\tau_2}
\]

\[+w_{11}(-\tau_2) + \frac{w_{20}(0)}{2}e^{j\omega_0\tau_2}
\]

\[+w_{20}(0)\right)\}

We have, for \( \theta \in [-\tau_1, 0) \),

\[
H(z, \bar{z}, \theta) = 2\Re\bar{q}^*(0)F_0q(\theta)
\]

\[= -g\bar{q}(\theta) - \bar{g}\bar{q}(\theta)
\]

\[= -\left(\frac{g_{20}z^2}{2} + g_{11}z\bar{z} + g_{02}\bar{z}^2 + \ldots\right)q(\theta)
\]

\[= -\left(\frac{g_{20}z^2}{2} + g_{11}z\bar{z} + g_{02}\bar{z}^2 + \ldots\right)\bar{q}(\theta)
\]
which yields
\[ H_{20} = -g_{20}q(\theta) - \bar{g}_{02}q(\theta) , \]
\[ H_{11} = -g_{11}q(\theta) - \bar{g}_{11}q(\theta) . \]

Using the above in (41) to (43), we get
\[ \dot{w}_{20}(\theta) = 2j\omega_0 w_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}q(\bar{\theta}) \]
\[ \dot{w}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}q(\bar{\theta}). \]

On solving the above differential equations,
\[ w_{20}(\theta) = \frac{-g_{20}}{j\omega_0} q(0)e^{i\omega_0\theta} - \frac{\bar{g}_{02}}{3j\omega_0} q(0)e^{-i\omega_0\theta} + E_1 e^{2j\omega_0\theta} \]
\[ w_{11}(\theta) = \frac{g_{11}}{j\omega_0} q(0)e^{i\omega_0\theta} - \frac{\bar{g}_{11}}{j\omega_0} q(0)e^{-i\omega_0\theta} + E_2. \]

for some \( E_1, E_2 \) which will soon be determined. Similarly, for \( \theta = 0 \),
\[ H(z, \bar{z}, 0) = -3\Re^* F_0 q(0) + F_0 \]
\[ H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}q(0) - 2a\left( b_1 e^{-j\omega_0\tau_1} + b_2 e^{-j\omega_0\tau_2} \right) \]
\[ = ax^* \left( b_1 w_{20}(-\tau_1) + b_2 w_{20}(-\tau_2) \right) + 2j\omega_0 w_{20}(0) \]
\[ H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}q(0) - 2a\left( b_1 (e^{j\omega_0\tau_1} + e^{-j\omega_0\tau_1}) \right) + b_2 (e^{j\omega_0\tau_2} + e^{-j\omega_0\tau_2}) \]
\[ = ax^* \left( b_1 w_{11}(-\tau_1) + b_2 w_{11}(-\tau_2) \right). \]

The expression for \( w_{20}(-\tau_1), w_{20}(-\tau_2), w_{20}(0), w_{11}(-\tau_1), w_{11}(-\tau_2), \) and \( w_{11}(0) \) can be found from (49) and (50). On substituting them in (51) and (52), \( E_1, E_2 \) can be found. From the above analysis the following can be calculated:
\[ c_1(0) = \frac{j}{2\omega_0} \left( g_{20}g_{11} - 2 |g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right) + \frac{g_{21}}{2}, \]
\[ \mu_2 = -\frac{\Re(c_1(0))}{\Re(\lambda'(a_0))} , \]
\[ P = 4\tau(1 + \epsilon^2 T_2 + O(\epsilon^4)), \]
\[ T_2 = -\left( \frac{3(c_1(0) + \mu_2 \lambda'(a_0))}{\omega_0} \right), \]
\[ \beta = \epsilon^2 \beta_2 + O(\epsilon^4), \beta_2 = 2\Re(c_1(0)), \epsilon = \sqrt{\frac{\mu}{\mu_2}}, \]
where \( c_1(0) \) is the lyapunov coefficient. The asymptotic form of the bifurcating periodic solutions is
\[ u(t, \mu(\epsilon)) = 2\Re \left( q(0)e^{j\omega_0t} + \epsilon^2 \Re \left( E_1 e^{2j\omega_0t} + E_2 \right) \right) + O(\epsilon^3) \]

for \( 0 \leq t \leq P(\epsilon) \).

On substituting \( \tau_1 = \tau_2 \) in (29) the analysis reduces to that of the single delay logistic equation.

C. Direction and stability of the Hopf bifurcation in the logistic equation with delay as the parameter

In the previous sub-section the direction and stability of the bifurcation was considered with the growth rate as the bifurcation parameter. It has been shown in Section 5.1 that the single delay equation undergoes Hopf bifurcation with an increase in the delay also. A similar analysis for this case is carried out here.

Let \( \tau_0 \) be the critical value of the delay at bifurcation. The logistic equation can also be written as,
\[ \dot{u}(t) = L_\mu u_t + F(u_t, \mu), \]
where, \( u(t) = x(\tau_t), t > 0, \mu \in R, \tau_1 = \tau_0 + \mu, \) and
\[ u_t(\theta) = u(t + \theta)u : [-1, 0] \to R, \theta \in [-1, 0]. \]

Also, \( L_\mu : C([-1, 0]) \to R \) is
\[ L_\mu \phi = -ab\phi(-1). \]
\[ F(u_t, \mu) : C([-1, 0]) \to R \] is
\[ F(\phi, \mu) = -ab\phi(-1). \]

By the Riesz representation theorem, there exists a matrix function with bounded variation components \( \eta(\theta, \mu), \theta \in [-1, 0], \) such that
\[ L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta). \]
where,
\[ d\eta(\theta, \mu) = -ab\delta(\theta + 1)d\theta, \]
where \( \delta(\theta) \) is the Dirac delta function.

Proceeding further as shown in the previous sub-section we get,
\[ \mu_2 = \left( \frac{3\pi - 2}{10} \right) > 0. \]
Hence the Hopf bifurcation is supercritical.
Numerical Example: If we take the case of a system with a two delays with \( b_1 = b_2 = 0.5 \), \( \tau_1 = 1 \) and \( \tau_2 = 2 \), the system undergoes a Hopf bifurcation at growth parameter \( a_0 = 1.2092 \). The value of the lyapunov coefficient \( c_1(0) = -0.1691 - 0.2290i \), the real part of which is less than zero which makes the resulting periodic solutions asymptotically orbitally stable. Parameter \( \mu_2 = 0.5175 > 0 \), implying that the Hopf bifurcation is supercritical. The period of these oscillations is \( P = 6 \). Fig. 10 has the resulting state space diagram where the limit cycles can be seen and Fig. 11 compares the bifurcation diagram obtained through simulation to the analytical solution.

D. Bifurcation diagrams

In Fig. 12 and 13, the bifurcation diagrams of the single delay system are drawn with respect to both parameters namely, the growth rate \( a \) and delay \( \tau_1 \). Both curves look similar. In Fig. 14 and 15, the bifurcation diagrams of the two delay system are shown with respect to various parameters. The only parameter that affects the equilibrium value is the scaling factor of the delay terms, \( b_1 \) and \( b_2 \). The amplitude of the periodic oscillations beyond the critical value of the parameter varies more with changes in the growth rate \( a \) than it does with delays \( \tau_1 \) and \( \tau_2 \).
VI. RESULTS

In this section, we show the behaviour of the delay logistic system with time under various conditions. In Fig. 16, we see the sigmoid curve of the logistic function. As growth rate is increased, the function reaches saturation value faster. In Fig. 17, the time domain response of a converging system is observed. In Fig. 18, the logistic equation with two delays is observed in the time domain. Varying the parameters changes the behaviour from asymptotically stable to forming limit cycles. In the latter case, the sufficient and necessary conditions for stability are not met.

VII. CONCLUSIONS

We have performed local stability analysis of the logistic equation with and without multiple time delays. Sufficient conditions to aid design were also extracted. The rate of convergence for the single delay logistic equation was analysed. While the ideal system (without delay) is always perfectly stable, the actual system that we have considered using delays
undergoes a Hopf bifurcation which is supercritical. The nature of the resulting periodic oscillations was analytically characterized for the single delay system and a methodology to ascertain the same in the two delay case was presented.

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