INDEPENDENCE TIMES FOR IID SEQUENCES, RANDOM WALKS AND LÉVY PROCESSES

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Abstract. For a sequence in discrete time having stationary independent values (respectively, random walk) \( X \), those random times \( R \) of \( X \) are characterized set-theoretically, for which the strict post-\( R \) sequence (respectively, the process of the increments of \( X \) after \( R \)) is independent of the history up to \( R \). For a Lévy process \( X \) and a random time \( R \) of \( X \), reasonably useful sufficient conditions and a partial necessary condition on \( R \) are given, for the process of the increments of \( X \) after \( R \) to be independent of the history up to \( R \).

1. Introduction

1.1. Motivation and overview of results. The motivation for this investigation is the following. Let \( X \) be an \( \mathbb{R}^d \)-valued process with stationary independent increments (PSII) vanishing a.s. at zero: a random walk in discrete time or a càdlàg Lévy process in continuous time. Let us define the processes \( \Delta X \) and, for a random time \( R \), on \( \{ R < \infty \} \), \( \Delta R X \) and \( \theta R X \), by setting \((\Delta R X)_t := X_{R+t} - X_R\), \((\theta R X)_t := X_{R+t} \) and \((\Delta X)_t := \Delta_t X \) (with \( t \) in \( \mathbb{N}_0 \) or in \([0, \infty)\), as the case may be); and let us call \( R \) an independence (respectively, a regenerative; a Markov) time for \( X \) if, on \( \{ R < \infty \} \), the process \( \Delta R X \) of the increments of \( X \) after \( R \) is independent of the (suitably defined; see the first paragraphs of Sections 2 and 3) history \( F'_R \) up to \( R \) (respectively, and \( \Delta R X \) is equally distributed as \( X \); given \( X_R \), the post \( R \)-process \( \theta R X \) is distributed as \( X \) started at \( X_R \) and independent of \( F'_R \)). It is then well-known that all stopping (but in general not all independence) times are regenerative/Markov [10, Theorem 3.1.2.1] [18, Theorem 40.10]. It is less well-known, and somewhat remarkable, that the regenerative/Markov property already characterizes stopping times in the class of random times that are measurable with respect to \( X \): under certain conditions see [15, Corollary 4.3] [1, Remark 4.14] in continuous and [9, Lemma 3.12] in discrete time; we show below that it holds true in fact in full generality.

The main contribution of this paper, however, is the investigation of independence times: we provide a set-theoretic characterization of (respectively, some sufficient and necessary conditions for) independence times of random walks (respectively, Lévy processes).

More precisely, in the discrete time case, note that the sequence of the consecutive increments of \( X \), \( \Delta X := (X_n - X_{n-1})_{n \in \mathbb{N}} \), has in fact stationary independent values and generates up to a completion the same filtration as \( X \). In this way the investigation of independence times for \( X \) is reduced to the study of the analogous times for the sequence \( \Delta X \). This reduction, besides helping to discern what is really relevant to the argument, from what is just unnecessary ‘background noise’, is also advantageous in that it allows the state space to become an arbitrary measurable space, enlarging the scope of the

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results. Thus, in Section 2 for a sequence with stationary independent values \( Y = (Y_i)_{i \in \mathbb{N}} \) taking values in an arbitrary measurable space, we characterize those random times \( R \), measurable with respect to \( Y \), which render the strict post-\( R \) sequence \( \square_R Y := (Y_{R+i})_{i \in \mathbb{N}} \) independent of the past up to \( R \) (Theorem 2). Noting that a.s. for each \( n \in \mathbb{N}_0 \), \( X_n = \sum_{i=1}^n (\Delta X)_i \), \( (\Delta R X)_n = \sum_{i=1}^n (\square_R \Delta X)_i \) and for each \( n \in \mathbb{N} \), \( (\Delta X)_n = X_n - X_{n-1} \), \( (\square_R \Delta X)_n = (\Delta R X)_n - (\Delta R X)_{n-1} \), this becomes then at the same time a characterization of independence times for the random walk \( X \): loosely speaking, we show that a random time \( R \) of \( X \) is an independence time for \( X \) (with \( F'_R \) generated by \( Z_R \), \( Z \) adapted), precisely when

\[
\{ R = n \} = F_n \cap \{ \Delta_n X \in \Gamma \} \text{ a.s. for all } n \in \mathbb{N}_0 \tag{1.1}
\]

for some measurable \( \Gamma \) in the path space and for some \( F_n \) from the past up to time \( n \in \mathbb{N}_0 \).

There is no analogue of such a reduction in continuous time and the investigation of the independence times of Lévy processes proves much more involved. We provide in Section 3 sufficiency (Proposition 10(II)) dealing with thin random times, i.e. random times whose graphs are included, up to evanescence, in the union of a denumerable family of stopping times; Proposition 19 dealing with strict random times, i.e. random times, whose graphs intersect the graph of any stopping time in an evanescent set only) and partial necessity (Proposition 10(I)) results. They consist in adapting (1.1) to the continuous-time setting. To give an informal flavor of this, up to technicalities, the sufficient condition for \( R \) to be an independence time, in the more intricate case when \( R \) is a strict random time, becomes:

\[
\{ R \leq t \} = \int_{[0,t]} O_s dA_s \text{ for all } t \in [0, \infty) \text{ a.s.,} \tag{1.2}
\]

with \( O \) an optional process and with the increments of the nondecreasing right-continuous process \( A \) after a given deterministic time “depending” only on the increments of \( X \) after that time; whilst thin independence times are characterized in a manner analogous to (1.1), with the deterministic times \( n \) appearing in (1.1) being replaced by a suitable sequence of stopping times. Illustrative examples are given. We leave open the problem of finding a meaningful characterization and/or other/better sufficient/necessary conditions.

In a minor contribution, for PSIIs, we establish (as we have already remarked) the characterization of regenerative/Markov times as stopping times under no assumptions on the underlying space (Corollary 4 and Theorem 26).

1.2. Literature overview. This study falls into the more general context of birth, death, splitting and conditional independence times of Markov processes, together with their associated path decompositions. The literature in this area is considerable: [2] Chapter 12 [20] Section III.9 [14] [16] [12] [13] [9] [15] [17] [7] [8] [4] [5] is an incomplete list. In particular, and as it relates more specifically to our investigation of independence times, random times \( \tau \) of a (temporally homogeneous) Markov process \( Z \) have been studied, for which — speaking somewhat loosely, but giving the correct flavor — the \( \tau \)-future of \( Z \) is independent of the history up to \( \tau \), conditionally on the \( \tau \)-present. It is natural to call such \( \tau \) conditional independence times for \( Z \). Then in discrete time, with \( Z \) being assumed valued in a countable state space \( J \), and defined on the canonical space of \( J \)-valued sequences, [9] gives a set-theoretic characterization of conditional independence times [9] Lemma 3.12] (with the \( \tau \)-present being \( Z_\tau \), the \( \tau \)-future being \( Z_{\tau+} \).
and the $\tau$-past being the $\sigma$-field generated by $W_\tau$ with $W$ adapted). In continuous time (once the notions of the ‘$\tau$-present’, ‘$\tau$-past’ and ‘independence given $\tau$-present’ have been given a precise meaning) provide several sufficient conditions for a random time to be a conditional independence time.

Now, for a PSII $X$ and a random time $R$, the independence, on $\{R < \infty\}$, of the incremental future $\Delta_R X$ from the $R$-past $F'_R$ is not synonymous with the independence, again on $\{R < \infty\}$, of the post-$R$ process $\theta_R X$ from $F'_R$ given the $R$-present $X_R$. Routine conditional independence calculations demonstrate that the first implies the latter (as long as $X_R$ is $F'_R$-measurable), but the converse fails in general (Example 6). Indeed in the latter statement $X_R$ may be replaced by any $\sigma$-field contained in $F'_R$ and with respect to which $X_R$ is measurable. Hence, unless $X_R$ is trivial, the above described ‘conditional independence’ results of the Markov theory do not apply directly to the investigation of whether or not $R$ is an independence time for $X$, and our results are a meaningful complement. (In special cases, indirectly, the conditional independence results may still be brought to bear, even when $X_R$ is not trivial – Remark 24.) Furthermore, while we consider only PSII/i.i.d. sequences and not more general Markov processes, the upshot is that we are able to provide the results in what is otherwise complete generality, in particular without assuming the presence of the usual Markov setup or even canonical realizations. Indeed, for this reason, we briefly include below the characterization, for PSII, of stopping times as regenerative/Markov times: there is no particular claim to originality here, but it is still reassuring to know, viz. the settings of [13] [11] [10], that no special assumptions on the space are needed in order for this to hold true.

We may also mention the investigation of [11]: there, for a two-sided Brownian motion $(B_t)_{t \in \mathbb{R}}$, those random times $T$, measurable with respect to $B$, are characterized, for which $(B_{T+t} - B_T)_{t \in \mathbb{R}}$ is again a two-sided Brownian motion independent of $B_T$.

1.3. Setting. We fix, once and for all, a (not necessarily complete) probability space $(\Omega, \mathcal{G}, P)$. The qualifiers ‘a.s.’, ‘negligible’, ‘evanescent’, ‘with a positive probability’, ‘independent’, ‘law’ and ‘completion’ without reference to a probability measure, refer to the measure $P$. In particular, the completion of a sub-$\sigma$-field $\mathcal{H}$ of $\mathcal{G}$, denoted $\overline{\mathcal{H}}$, we shall mean the $\sigma$-field generated on $\Omega$ by $\mathcal{H}$ and the negligible sets of $\mathcal{G}$. The completion of a filtration is got by completing in this manner each member thereof.

1.4. Miscellaneous notation. For a topological space $U$, $\mathcal{B}_U$ is its Borel $\sigma$-field (under the standard topology for the given $U$ that we will encounter). $2^U$ denotes the power set of a set $U$. The intersection of two sets $a$ and $b$ will be denoted multiplicatively: $ab := a \cap b$. For measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$, $A/\mathcal{B}$ denotes the set of $A/\mathcal{B}$-measurable maps from $A$ into $B$. $\sigma_A(\ldots)$ stands for the smallest $\sigma$-field on $A$ with respect to which whichever maps that stand in between the parentheses, being all defined at least on $A$, are measurable on restriction to $A$ (the $\sigma$-field(s) on the codomain(s) being understood from context); $\sigma(\ldots) := \sigma_{\Omega}(\ldots)$. For a measure $\mu$ on a $\sigma$-field $\mathcal{H}$, $H \in \mathcal{H}$ and $F \in \mathcal{H}/\mathcal{B}_{[-\infty, \infty]}$, $\mu(F) := \int F d\mu$, $\mu(F;H) := \int_H F d\mu$, and $\mu(F|H) = \frac{\mu(F;H)}{\mu(H)}$, whenever well-defined; the parentheses in $\mu(F)$ are omitted if no ambiguity can arise. If $\mathcal{H}$ is a $\sigma$-field on $X$ and $A \subset X$, $\mathcal{H}|_A := \{HA : H \in \mathcal{H}\}$ is the trace $\sigma$-field. Given a probability space $(\Theta, \mathcal{H}, Q)$: (i) for a sub-$\sigma$-field $\mathcal{A}$ of $\mathcal{H}$ and $F \in \mathcal{H}/\mathcal{B}_{[-\infty, \infty]}$ with $QF^+ \land QF^- < \infty$, we shall denote the conditional expectation of $F$ with respect to $\mathcal{A}$ under $Q$ by $Q(F|\mathcal{A})$; (ii) for a random element $X$, indiscriminately, $X_Q = Q_X = Q \circ X^{-1}$ will signify the law of $X$ under $Q$; (iii) for measurable spaces $(E, \mathcal{E})$ and $(F, \mathcal{F})$, random elements $X \in \mathcal{H}/\mathcal{E}$ and $Y \in \mathcal{H}/\mathcal{F},$
and a probability kernel $Q = (Q^x)_{x \in E}$ from $(E, \mathcal{E})$ into $(F, \mathcal{F})$ (meaning: $Q^x$ is a probability measure on $(F, \mathcal{F})$ for each $x \in E$; the map $E \ni x \mapsto Q^x(G)$ belongs to $\mathcal{E}/\mathcal{B}_{[0,1]}$ for each $G \in \mathcal{F}$), we shall say that Y has kernel law $Q$ conditionally on $X$ under $Q$ if for all $G \in \mathcal{F}/\mathcal{B}_{[0,\infty]}$ and $H \in \mathcal{E}/\mathcal{B}_{[0,\infty]}$, $Q(G(Y)H(X)) = Q(Q^X(G)H(X))$. Finally, for a random time $R \in \mathcal{G}/\mathcal{B}_{[0,\infty]}$, $[R]$ will denote the graph of $R$, and for $A \subset \Omega$, $R_A$ will be the random time equal to $R$ on $A$ and $\infty$ otherwise.

2. Discrete time

In this section, let $Y = (Y_n)_{n \in \mathbb{N}}$ be a sequence of random elements on $(\Omega, \mathcal{G}, P)$, adapted to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}_0}$ on $\Omega$ satisfying $\mathcal{F}_\infty \subseteq \mathcal{G}$, taking values in a measurable space $(E, \mathcal{E})$, with the property that $Y_{i+1}$ is independent of $\mathcal{F}_i$ and equally distributed as $Y_1$ for all $i \in \mathbb{N}_0$. That is to say that $Y$ is an adapted process with stationary independent values relative to $\mathcal{F}$. We fix an $R \in \mathcal{G}/\mathcal{B}_{[0,\infty]} \cap \mathcal{G}$, with $P(R < \infty) > 0$. Recall from Subsection 1.1 that, on $(\Omega < \infty), \square_\infty Y = (Y_{R+\tau})_{\tau \in \mathbb{N}}$. We further set (i) $\mathcal{F}'_R := \sigma_{\{R < \infty\}}(Z_R : Z \in \mathcal{F}'_R)$ an $\mathcal{F}$-adapted $(\mathbb{R}, \mathcal{B}_{\mathbb{N}})$-valued process; and (ii) $\mathcal{P}' := (\mathcal{G})_{\{R < \infty\}} \ni A \mapsto P(A \cap \mathcal{G})$. Note that when $R < \infty$.

The proof of the following lemma is elementary – we omit making it explicit.

**Theorem 2.** Of the following two statements, [b] implies [a]. If in addition $\{R = n\} \in \mathcal{F}_n \lor \sigma(\square_\infty Y)$ for each $n \in \mathbb{N}_0$, then [a] implies [b].

(a) $\mathcal{F}'_R$ is independent of $\square R Y$ under $P'$.

(b) There are a $G \in \mathcal{E}^{\square_\infty} Y$ and for all $n \in \mathbb{N}_0$ with $P(R = n) > 0$ an $F_n \in \mathcal{F}_n$, satisfying a.s. $\{R = n\} = F_n \square_\infty Y \in G$.

When [b] prevails, $G$ is $L$-a.s. unique, the $F_n$ are a.s. unique, and $(\square R Y), P' = P(Y \in \cdot | Y \in G) = L(\cdot | G)$.

Furthermore, of the following two statements, [ii] implies [i]. If in addition $\{R = n\} \in \mathcal{F}_n \lor \sigma(\square_\infty Y)$ for each $n \in \mathbb{N}_0$, then [i] implies [ii].

(i) $\mathcal{F}'_R$ is independent of $\square R Y$ under $P'$ and the law of $\square R Y$ under $P'$ is $L$.

(ii) $R$ is a stopping time relative to the completion of $\mathcal{F}$.

**Remark 3.**

1. For sure $\{R = n\} \in \mathcal{F}_n \lor \sigma(\square_\infty Y)$ for each $n \in \mathbb{N}_0$, if $R \in \sigma(\bar{Y})/\mathcal{B}_{[0,\infty]}$.

2. Given the discussion in Subsection 1.1 the theorem has an obvious corollary for random walks.

In particular, for an Euclidean space-valued random walk $X$ on $(\Omega, \mathcal{G}, P)$, adapted and having independent increments relative to $\mathcal{F}$, vanishing a.s. at zero, and for which $R \in \sigma(X)/\mathcal{B}_{[0,\infty]}$, the condition for the independence of $\Delta R X$ and $\mathcal{F}'_R$ under $P'$ writes as

$$\{R = n\} = F_n \{\Delta_n X \in \Gamma\} \text{ a.s. for all } n \in \mathbb{N}_0$$  (2.1)
for a $\Gamma \in (\mathcal{B}_\mathbb{R}^n)^{\otimes \mathbb{N}_0}$ and some $F_n \in \mathcal{F}_n$, $n \in \mathbb{N}_0$.

(3) For the necessity of the conditions, the assumption that $\{R = n\} \in \mathcal{F}_n \vee \sigma(\square_n Y)$ for each $n \in \mathbb{N}_0$ does not in general follow from the rest of the assumptions: for instance when $\mathcal{F}$ is the (completed) natural filtration of $Y$ and $R$ is non-trivial and independent of $Y$.

Proof of Theorem 2. By a monotone class argument, $\square_n Y$ is independent of $\mathcal{F}_n$ for every $n \in \mathbb{N}_0$ (and has clearly the same law as $Y$). Let $M := \{n \in \mathbb{N}_0 : P(R = n) > 0\}$.

Assume (b). We see that for any $H \in \mathcal{E}^{\otimes \mathbb{N}}/\mathcal{B}_{[0,\infty]}$, any $\mathcal{F}$-adapted process $Z$ with values in $([0,\infty], \mathcal{B}_{[0,\infty]})$, and then each $n \in M$,

$$P(Z_R H(\square_R Y); R = n) = P(Z_n H(\square_n Y); R = n) = P(Z_n H(\square_n Y); F_n \{\square_n Y \in G\})$$

$$= P(Z_n; F_n) P(H(\square_n Y); \square_n Y \in G) = P(Z_n; F_n) P(H(Y); Y \in G).$$

Summing over $n \in M$ implies $P(Z_R H(\square_R Y); R < \infty) = P\left(\sum_{n \in M} Z_n 1(F_n)\right) P(H(Y); Y \in G)$. From this we obtain, taking $Z \equiv 1$: $P(H(\square_R Y); R < \infty) = P(\cup_{n \in M} F_n) P(H(Y); Y \in G)$; taking $H \equiv 1$:

$$P(Z_R; R < \infty) = P(\sum_{n \in M} Z_n 1(F_n)) P(Y \in G);$$

taking $Z \equiv 1 \equiv H$: $P(R < \infty) = P(\cup_{n \in M} F_n) P(Y \in G)$.

Then $P(H(\square_R Y); R < \infty) P(Z_R; R < \infty) = P(Z_R H(\square_Q Y); R < \infty)$ whence the desired independence in (a) follows. We also obtain $P(H(\square_R Y)|R < \infty) = P(H(Y)|Y \in G)$ for all $H \in \mathcal{E}^{\otimes \mathbb{N}}/\mathcal{B}_{[0,\infty]}$. If moreover $R$ is a stopping time relative to the completion of $\mathcal{F}$, i.e. $G = E^N \mathcal{L}$-a.s., then this entails $(\square_R Y)_* \mathcal{P}' = \mathcal{L}$.

Now assume (a) and that $\{R = n\} \in \mathcal{F}_n \vee \sigma(\square_n Y)$ for each $n \in \mathbb{N}_0$. We first show that

(I) For each $n \in M$, there exist $A_n \in \mathcal{F}_n/\mathcal{B}_{[0,\infty]}$ and $B_n \in \sigma(\square_n Y)/\mathcal{B}_{[0,\infty]}$, such that a.s. $1(R = n) = A_n B_n$.

Suppose per absurdum that (I) fails for some $n \in M$. Then it must be the case that with a positive probability, the equality $P(R = n) 1(R = n) = P(R = n|\mathcal{F}_n) P(R = n|\square_n Y)$ fails. By the independence of $\mathcal{F}_n$ and $\square_n Y$, the assumption $\{R = n\} \in \mathcal{F}_n \vee \sigma(\square_n Y)$, and a monotone class argument, this implies that there are $A \in \mathcal{F}_n$ and $B \in \sigma(\square_n Y)$ such that

$$P(R = n) P(R = n, A, B) \neq P(R = n, A) P(R = n, B).$$

The assumption of (a) implies that $P(R < \infty) P(\tilde{A}, \tilde{B}, R = n) = P(\tilde{A}, R = n) P(\tilde{B}, R < \infty)$ for any $\tilde{A} \in \mathcal{F}_n$ and $\tilde{B} \in \sigma_{\{R < \infty\}}(\square_R Y)$ (for, $1_{[n]} 1_{\tilde{A}}$ is $\mathcal{F}_n$-adapted, so $\tilde{A}\{R = n\} \in \mathcal{F}_n'$). Taking $\tilde{A} = \Omega$ allows to conclude that

$$P(R < \infty) P(\tilde{B}, R = n) = P(\tilde{B}, R < \infty) P(R = n),$$

hence

$$P(R = n) P(\tilde{A}, \tilde{B}, R = n) = P(\tilde{A}, R = n) P(\tilde{B}, R = n).$$

Now, $\sigma(\square_n Y)|_{\{R = n\}} = \sigma_{\{R < \infty\}}(\square_R Y)|_{\{R = n\}}$. Therefore $B\{R = n\} = \tilde{B}\{R = n\}$ for some $\tilde{B} \in \sigma_{\{R < \infty\}}(\square_R Y)$. Taking finally $\tilde{A} = A$ yields a contradiction.

By Lemma 1 and (I) for each $n \in M$, there are an a.s. uniquely determined $F_n \in \mathcal{F}_n$ and an $\mathcal{L}$-a.s. uniquely determined $G_n \in \mathcal{E}^{\otimes \mathbb{N}}$, such that a.s. $\{R = n\} = F_n\{\square_n Y \in G_n\}$. We now show that

(II) $G_n = G_m \mathcal{L}$-a.s. whenever $\{n, m\} \subset M$. 


Let \( \{m, n\} \subset M \). We compute, for any \( H \in \mathcal{E}^{\otimes \mathbb{N}}/\mathcal{B}_{[0, \infty]} \) (using (2.2) in the last equality):

\[
\mathbb{P}(R = m)\mathbb{P}(H(Y); Y \in G_m) = \mathbb{P}(R = m)\mathbb{P}(H(\square_m Y); \square_m Y \in G_m) = \mathbb{P}(F_m)\mathbb{P}(\square_m Y \in G_m)\mathbb{P}(H(\square_m Y); \square_m Y \in G_m)
\]

\[
= \mathbb{P}(H(\square_m Y); F_m(\square_m Y \in G_m))\mathbb{P}(\{Y \in G_m\}) = \mathbb{P}(H(\square_m Y); R = m)\mathbb{P}(Y \in G_m)
\]

\[
= \mathbb{P}(H(\square_R Y); R = m)\mathbb{P}(Y \in G_m) - \mathbb{P}(R = m)\mathbb{P}(H(\square_R Y); R < \infty)\mathbb{P}(R < \infty),
\]

i.e. \( \mathbb{P}(R < \infty)\mathbb{P}(H(Y); Y \in G_m) = \mathbb{P}(Y \in G_m)\mathbb{P}(H(\square_R Y); R < \infty) \). Then \( \mathbb{P}(H(Y)|Y \in G_m) = \mathbb{P}(H(Y)|Y \in G_n) \) for all \( H \in \mathcal{E}^{\otimes \mathbb{N}}/\mathcal{B}_{[0, \infty]} \), which implies that \( \mathcal{L}(G_n \Delta G_m) = 1 \)

[ take \( H = 1_{G_n} \) and \( H = 1_{G_m} \).]

If we are moreover given that \( (\square_R Y), P' = \mathcal{L} \), then with \( G \) as in [b] by what we have already proven, \( \mathcal{L}(G) = \mathbb{P}(\square_R Y \in G|R < \infty) = \mathbb{P}(Y \in G|Y \in G) = 1 \). \( \square \)

Recall the notation \( \Delta_R X \) and \( \theta_R X \) from Subsection 1.1.

**Corollary 4.** Let \( d \in \mathbb{N} \) and let \( X \) be an \((\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})\)-valued random walk on \((\Omega, \mathcal{G}, \mathbb{P})\), adapted and having independent increments relative to the filtration \( \mathcal{F} \), vanishing a.s. at zero. For \( x \in \mathbb{R}^d \), let \( P^x \) be the law of \( x + X \) under \( P \): \( P := (P^x)_{x \in \mathbb{R}^d} \). Of the following statements, (iii) implies (ii), (ii) and (i) are equivalent. If in addition \( \{R = n\} \in F_n \vee \sigma(\Delta_R X) \) for each \( n \in \mathbb{N}_0 \), then (ii) implies (iii).

(i) Under \( P' \), conditionally on \( X_R, \theta_R X \) is independent of \( \mathcal{F}'_R \) and has kernel law \( P \).

(ii) \( \mathcal{F}'_R \) is independent of \( \Delta_R X \) under \( P' \) and the law of \( \Delta_R X \) under \( P' \) is \( P^0 \).

(iii) \( R \) is a stopping time relative to the completion of \( \mathcal{F} \).

**Proof.** The equivalence of (i) and (ii) is by standard manipulation of conditional independence, using the fact that \( X_R \in \mathcal{F}'_R/\mathcal{B}_{\mathbb{R}^d} \). The rest follows by Theorem 2 applied to the sequence \( \Delta X \) of the consecutive increments of \( X \) (viz. the reduction of \( X \) to \( \Delta X \) of Subsection 1.1). \( \square \)

**Remark 5.** Assume the random walk \( X \) of Remark 3(2) takes values in a denumerable set \( J \) and is the coordinate process on the canonical filtered space \((J^{\otimes \mathbb{N}_0}, \sigma(X), \mathcal{F}^X)\) of \( J \)-valued sequences (with \( \mathcal{F}^X \) the natural filtration of \( X \)). Compare (2.1) with the condition of 9 [Lemma 3.12] for \( \mathcal{F}'_R \) to be independent of \( \theta_R X \) conditionally on \( X_R \) under \( P' \) (i.e. with the condition for \( R \) to be a ‘conditional independence time’ in the terminology of 9), namely that there should be a \( G \in \mathcal{E}^{\otimes \mathbb{N}_0} \) and \( F_n \in \mathcal{F}_n \) for \( n \in \mathbb{N}_0 \), satisfying

\[
\{R = n\} = F_n \{\theta_n X \in G\} \text{ a.s. for all } n \in \mathbb{N}_0.
\]

(2.3)

(2.1) implies the latter, but the converse fails in general, as the next example demonstrates. Note also that the proof method of 9 for establishing condition (2.3) — working on atoms by exploiting the canonical setting — is quite different from our method for establishing condition (2.1).

**Example 6.** Retain the provisions of Remark 5 with further \( J = \mathbb{Z} \), \( X \) drifting to \(-\infty \), and \( P(X_1 = -1), P(X_1 > 0) \) all positive. Let \( R \) be equal to (i) the last time \( X \) is at its running supremum \( \overline{X} \) and \( X \) is not equal to 0, on the event that there is such a time; (ii) the last time \( X \) is at \( \overline{X} \) and \( X \) is equal to 0 and \( X \) jumps down by 1 on its next step, on the event that there is such a time; (iii) \( \infty \) otherwise. Then for \( n \in \mathbb{N}_0 \),

\[
\{R = n\} = \{\overline{X}_n = X_n\} \{X_n = 0, X_{n+1} = -1, X_{n+m} < 0 \text{ for } m \in \mathbb{N}_{\geq 2}\} \cup \{X_n \neq 0, X_{n+m} < X_n \text{ for } m \in \mathbb{N}\}.
\]

Notice that \( P(R < \infty) > 0 \) thanks to the assumptions on \( X \). It is intuitively clear that \( R \) is a conditional independence time, but not an independence time for \( X \). Formally, it follows from (2.3) that \( \Delta_R X \) is
independent of $\mathcal{F}_R'$ given $X_R$ under $P'$, and from (2.1), that this fails if the proviso ‘given $X_R$’ is dropped. Indeed, from the properties of conditional independence it is seen easily, that not only in this example is $\theta_R X$ independent of $\mathcal{F}_R'$ given $X_R$ under $P'$, but that in the latter statement any $\sigma$-field containing $\sigma_{\{R<\infty\}}(X_R)$ and contained in $\mathcal{F}_R'$ may replace $X_R$.

3. Continuous time

In this section, let $d \in \mathbb{N}$ and let $X = (X_t)_{t \in [0, \infty)}$ be a Lévy process on $(\Omega, \mathcal{G}, P)$, relative to a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, \infty)}$ on $\Omega$ satisfying $\mathcal{F}_\infty \subset \mathcal{G}$, taking values in $(\mathbb{R}^d, \mathcal{B}_\mathbb{R})$: so $X$ is adapted and has stationary independent increments relative to $\mathcal{F}$, vanishes at zero a.s., and has càdlàg paths. Denote by $\mathcal{O}$ (respectively, $\text{Prog}$) the optional (respectively, progressive) $\sigma$-field, i.e. the $\sigma$-field on $\Omega \times [0, \infty)$ generated by the $\mathcal{F}$-adapted càdlàg (respectively, $\mathcal{F}$-progressively measurable) $(\mathbb{R}, \mathcal{B}_\mathbb{R})$-valued processes.

We fix an $R \in \mathcal{G}/\mathcal{B}_{[0, \infty]}$ with $P(R < \infty) > 0$. Recall from Subsection 1.1 that, on $\{R < \infty\}$, $\Delta_R X = (X_{R+t} - X_t)_{t \in [0, \infty)}$ (this notation is to be retained for processes and random times other than the given $X$ and $R$) and $\theta_R X = (X_{R+t})_{t \in [0, \infty)}$. We further set (i) $\mathcal{F}_R := \sigma_{\{R<\infty\}}(Z_R : Z \in \mathcal{O}/\mathcal{B}_R)$ (a notation to be retained for random times other than the given $R$) and $\mathcal{F}_R := \sigma_{\{R<\infty\}}(Z_R : Z \in \text{Prog}/\mathcal{B}_R)$; and (ii) $P' := (\mathcal{G}|_{\{R<\infty\}} \ni A \mapsto P(A|R < \infty))$. When $R$ is an $\mathcal{F}$-stopping time, then $\mathcal{F}_{R+} = \mathcal{F}_R = \mathcal{F}_R|_{\{R<\infty\}}$ with $\mathcal{F}_R = \{A \in \mathcal{G} : A\{R \leq t\} \in \mathcal{F}_t$ for all $t \in [0, \infty)\}$ (and equally $\mathcal{F}_\infty$ may replace $\mathcal{G}$ in the latter). Besides, $\mathcal{F}_{R+}/\mathcal{B}_{[0, \infty]} = \{Z_R : Z \in \text{Prog}/\mathcal{B}_{[0, \infty]}\}$ and $\mathcal{F}_R'/\mathcal{B}_{[0, \infty]} = \{Z_R : Z \in \mathcal{O}/\mathcal{B}_{[0, \infty]}\}$.

Remark 7. When $R$ is not a stopping time, various (other) interpretations of the notion of the $\sigma$-field $\mathcal{F}_R$ of the past up to $R$ appear in the literature, for instance:

1. $\mathcal{F}_R$ might be taken to consist of those $A \in \mathcal{F}_\infty$, for which, given any $t \in [0, \infty)$, there is an $A_t \in \mathcal{F}_t$ with $A\{R < t\} = A_t\{R < t\}$. When $R$ is an honest time $\mathcal{F}_R|_{\{R<\infty\}} = \mathcal{F}_{R+}$ [12] Proposition 3.1.
2. One can let $\mathcal{F}_R$ be generated by $\mathcal{F}_S|_{\{S \leq R\}}$ as $S$ ranges over the $\mathcal{F}$-stopping times. According to [15] (that quotes [2]) $\mathcal{F}_R|_{\{R<\infty\}} = \mathcal{F}_R'$.
3. One can take for $\mathcal{F}_R$, at least when $\mathcal{F}$ is the (completed) natural filtration of $X$, the (completed) initial structure $\sigma(X^R, R)$. See [11] Section 4 for some connections between the latter and the natural filtration of $X$.

We set $\mathcal{L} := X_*P$, the law of $X$ on the space $(\mathbb{D}, \mathcal{D})$, where $\mathbb{D}$ are càdlàg paths mapping $[0, \infty) \to \mathbb{R}^d$ and $\mathcal{D}$ is the $\sigma$-field generated by the canonical projections.

We begin with a lemma that describes how the property of being an independence time ‘separates’ over optional sets and also over sets that ‘depend only on the incremental future’.

Lemma 8.

1. Suppose $\mathcal{F}_{R+}$ is independent of $\Delta_R X$ under $P'$, and that $P \in \mathcal{G}/\mathcal{B}_{[0, \infty]}$ satisfies $P(P < \infty) > 0$ and $[P] = A[R]$ with $A \in \mathcal{O} \cup \sigma_{\Omega \times [0, \infty)}(\Delta X)$. Set $\mathcal{L}' := (\Delta_R X)_*P'$ and $P' := (\mathcal{G}|_{\{P<\infty\}} \ni F \mapsto P(F|P < \infty))$.
   (a) If $A \in \mathcal{O}$, then $\Delta_P X$ is independent of $\mathcal{F}_R'$ under $P'$ and $(\Delta_P X)_*P' = \mathcal{L}'$.
   (b) If $A = (\Delta X)^{-1}(\Gamma)$ for a $\Gamma \in \mathcal{D}$, then again $\Delta_P X$ is independent of $\mathcal{F}_R'$ under $P'$ and $(\Delta_P X)_*P' = \mathcal{L}'(\cdot|\Gamma)$. 

(II) Conversely, let \( \{R_1, R_2\} \subset \mathcal{G}/\mathcal{B}_{[0,\infty]} \), \( P(R_i < \infty) > 0 \) for \( i \in \{1,2\} \). Set \( P^i := (\mathcal{G}_{\{R_i<\infty\}} \ni F \mapsto P(F|R_i < \infty)) \) and suppose \( \Delta_{R_i}X \) is independent of \( \mathcal{F}'_{R_i} \) under \( P^i \) for \( i \in \{1,2\} \).

(i) If there is an \( A \in \mathcal{O} \) with, up to evanescence, \([R_1] = [R]A\), \([R_2] = [R]A^c\), and if \( M := (\Delta_{R_1}X)_s P^1 = (\Delta_{R_2}X)_s P^2 \), then \( \mathcal{F}' \) is independent of \( \Delta_{R_i}K \) under \( P' \) and \( (\Delta_{R_1}X)_s P' = M \).

(ii) If there is a \( \Gamma \in \mathcal{D} \) with \([R_1] = [R] \cap \{A X \in \Gamma\} \) and \([R_2] = [R] \cap \{A X \notin \Gamma\} \), is \( \mathcal{F}' \) is independent of \( \Delta_{R_1}X \) under \( P' \) and if there is a law \( M \) on \((\mathbb{D}, \mathcal{D})\) satisfying \( P'(\Delta_{R_1}X \in \Gamma) = M(\Gamma) \), \( (\Delta_{R_1}X)_s P^1 = M(\cdot|\Gamma) \) and \( (\Delta_{R_2}X)_s P^2 = M(\cdot|\mathbb{D} \setminus \Gamma) \), then \( \mathcal{F}' \) is independent of \( \Delta_{R_1}X \) under \( P' \) and \( (\Delta_{R_2}X)_s P' = M \).

Remark 9.

1. Recall \((A X)_t = \Delta_tX\) for \( t \in [0,\infty) \), and note that \( A X \in \mathcal{G} \otimes \mathcal{B}_{[0,\infty]}/\mathcal{D} \).

2. In terms of \( \mathcal{O} \) and \( \sigma_{\Omega \times [0,\infty]}(A X) \), the property of \( R \) being an independence time for \( X \) can be rephrased as follows. Let \( P^R \) be the unique probability measure \( \mu \) on \( \mathcal{G} \otimes \mathcal{B}_{[0,\infty]} \) satisfying \( \mu(Z) = P'(Z_R) \) for \( Z \in \mathcal{G} \otimes \mathcal{B}_{[0,\infty]}/\mathcal{B}_{[0,\infty]} \). Then \( \Delta_{R}X \) is independent of \( \mathcal{F}'_{R} \) under \( P' \) iff \( \mathcal{O} \) is independent of \( \sigma_{\Omega \times [0,\infty]}(A X) \) under \( P^R \).

Proof of Lemma 8. Take \( Z \in \mathcal{O}/\mathcal{B}_{[0,\infty]} \) and \( H \in \mathcal{D}/\mathcal{B}_{[0,\infty]} \). The fact that \( R \) is an independence time for \( X \), together with the possibility of the process \( Z\mathbb{1}_A \), yields \( P(R < \infty)P(Z_R H(\Delta_{X}R); R < \infty) = P(Z_R; P < \infty)P(H(\Delta_{R}X); R < \infty) \). Setting \( Z \equiv 1 \) and plugging back in, everything follows. Now \( H\mathbb{1}_\mathcal{D} \in \mathcal{D}/\mathcal{B}_{[0,\infty]} \) and one proceeds in a similar fashion. We have \( P((Z\mathbb{1}_A)\mathcal{R} H(\Delta_{R}X) < \infty) \mathcal{R} M(H) \) and \( P((Z\mathbb{1}_A^c)\mathcal{R} H(\Delta_{R}X) < \infty) \mathcal{R} M(H) \). Summing the two, the desired conclusion follows upon taking \( Z \equiv 1 \) and plugging it back in. The proof of (II)(i) is similar.

Consider now the family \( \mathfrak{F} \) of collections \( \mathcal{T} \) of \( \mathcal{F} \)-stopping times, identified up to a.s. equality, satisfying

\[
\{T_1, T_2\} \subset \mathcal{T} \land P(T_1 = T_2) \neq 1 \Rightarrow P(T_1 = T_2 < \infty) = 0 \land (T \in \mathcal{T} \Rightarrow P(R = T < \infty) > 0).
\]

\( \mathfrak{F} \) is non-empty, is partially ordered by inclusion, and every linearly ordered subset of \( \mathfrak{F} \) admits an upper bound. By Zorn’s lemma there is a maximal element \( \mathcal{T} \in \mathfrak{F} \). Since \( P(R < \infty) \leq 1 < \infty \), \( \mathcal{T} \) is denumerable. By the maximality of \( \mathcal{T} \), it follows that \( A := \cup_{T \in \mathcal{T}} [T] \) is an optional set for which \([R]A^c[S] = \emptyset \) up to evanescence for each \( \mathcal{F} \)-stopping time \( S \). Let the random times \( R_1 \) and \( R_2 \) be defined by \([R_1] = [R]A\) and \([R_2] = [R]A^c\). Then the graph of \( R_1 \) is included up to evanescence in the union of some denumerable family of \( \mathcal{F} \)-stopping times \( (R_1) \) is a ‘thin random time’), whilst \( R_2 \) satisfies \([R_2][S] = \emptyset \) up to evanescence for each \( \mathcal{F} \)-stopping time \( S \) \( (R_2) \) is a ‘strict random time’). Moreover, when both \( P(R_1 < \infty) \) and \( P(R_2 < \infty) \) are positive, then according to Lemma 8, \( R \) is an independence time for \( X \) if and only if \( R_1 \) and \( R_2 \) are both independence times for \( X \) with \( \Delta_{R_1}X \) having the same law on \( \{R_1 < \infty\} \) as does \( \Delta_{R_2}X \) on \( \{R_2 < \infty\} \).

We (may thus) deal with thin and strict random times separately. The former are described completely in:

Proposition 10.

(I) Suppose \( \mathcal{F}'_{R} \) is independent of \( \Delta_{R}X \) under \( P' \). Then, for any \( \mathcal{F} \)-stopping time \( S \), satisfying \( P(R = S < \infty) > 0 \) and \( \{R = S < \infty\} \in \mathcal{F}'_{S} \), there are an a.s. unique \( F_S \in \mathcal{F}'_{S} \)
and an $\mathcal{L}$-a.s. unique $G_S \in \mathcal{D}$, such that a.s. $\{R = S < \infty\} = F_S\{\Delta S X \in G_S\}$, in which case furthermore $(\Delta R)_{X}P' = P(X \in \cdot | X \in G_S) = \mathcal{L}(\cdot | G_S)$. In particular, if $S_1$ and $S_2$ are two such $\mathcal{F}$-stopping times, then $\mathcal{L}$-a.s. $G_{S_1} = G_{S_2}$.

(II) If a.s. $\{R < \infty\} = \bigcup_{i \in \mathbb{N}} \{R = S_i < \infty\}$ for a sequence $(S_i)_{i \in \mathbb{N}}$ of $\mathcal{F}$-stopping times, and if there are $G \in \mathcal{D}$, and for each $i \in \mathbb{N}$ an $F_i \in \mathcal{F}'_{S_i}$, satisfying $\{R = S_i < \infty\} = F_i\{\Delta S_i X \in G\}$ a.s., then $\mathcal{F}'_{R+}$ is independent of $\Delta R X$ under $P'$ and $(\Delta R)_{X}P' = P(X \in \cdot | X \in G) = \mathcal{L}(\cdot | G)$.

In particular, if $R \in \sigma(X)/B_{[0,\infty]}$ and $R$ is a thin random time, then $\mathcal{F}'_{R}$ is independent of $\Delta R X$ under $P'$ iff there exist $O \in \mathcal{O}$ and $G \in \mathcal{D}$ such that $[R] = O\{\Delta X \in G\}$ up to evanescence, and when so then even $\mathcal{F}'_{R+}$ is independent of $\Delta R X$ under $P'$ with $(\Delta R)_{X}P' = \mathcal{L}(\cdot | G)$.

Remark 11.

(1) For sure $\{R = S < \infty\} \in \mathcal{F}'_{S} \cup \sigma_{\{S < \infty\}}(\Delta S X)$ for all $\mathcal{F}$-stopping times $S$, if $R \in \sigma(X)/B_{[0,\infty]}$.
(2) $X$ is also a Lévy process with respect to the usual augmentation of $\mathcal{F}$.
(3) It is conjectured that there is no independence time for $X$ belonging to $\overline{\sigma(X)}/B_{[0,\infty]}$, that is finite and equal to a stopping time of $X$ with a positive probability, yet whose graph fails to be included, up to evanescence, in the union of the graphs of a denumerable family of stopping times of the right-continuous augmentation of $\mathcal{F}$. However the author was not able to prove this.

Example 12. Let $X$ be a linear Brownian motion with strictly negative drift and continuous sample paths, vanishing at zero. Denote by $\overline{X}$ the running supremum of $X$ and by $L$ the last time that $X$ is at its running supremum ($L = \infty$ on the event that there is no such last time). Let $R$ be equal to 0 on the event that $X$ never reaches the level 1; or else let it be equal to the first hitting time of $-1$ by the process $\Delta L X$ on the event $\{L < \infty\}$; $R = \infty$ otherwise. For $q \in \mathbb{Q}_{>0}$ let $T_q$ be the first entrance time into the set $\{1\}$ of the process $\overline{X} - X$ strictly after time $q$, and let $S_q := (T_q\{X_s \neq \overline{X}_s \text{ for } s \leq q, T_q\})$. Then $\{R < \infty\} = \bigcup_{q \in \mathbb{Q}_{>0}} \{R = S_q < \infty\} \cup \{R = 0\} = \{X \text{ never reaches } 1\}$ and for $q \in \mathbb{Q}_{>0}$, letting $L_q$ be the last time $X = \overline{X}$ on $[0,q]$, $\{R = S_q < \infty\} = \{S_q < \infty\} \{1 \leq \overline{X}_q > X_s > \overline{X}_q - 1 \text{ for } s \in (L_q, S_q)\}\{\Delta S_q X \text{ never reaches } 1\}$. By Proposition (II)[II] $R$ is an independence time for $X$ (even with $\mathcal{F}'_{R}$ replaced by $\mathcal{F}'_{R+}$).

Proof of Proposition (II) (I) Uniqueness of $G_S$ and $F_S$ is a consequence of Lemma 1 and of the strong Markov property of $X$. Set $P' = (\mathcal{G}|_{\{S < \infty\} \ni A \mapsto P(A|S < \infty)}$. We verify that $P'$-a.s.
\[ P'(R = S) = P'(R = S|\mathcal{F}'_{S})P'(R = S|\sigma_{\{S < \infty\}}(\Delta S X)) \]
Since $\{R = S < \infty\} \in \mathcal{F}'_{S} \cup \sigma_{\{S < \infty\}}(\Delta S X)$ and since $\mathcal{F}'_{S}/\mathcal{B}_{\{0,1\}} = \{Z_S : Z \in O/\mathcal{B}_{\{0,1\}}\}$, by monotone class and the strong Markov property of $X$, this will follow, if we have shown that for all $Z \in O/\mathcal{B}_{\{0,\infty\}}$ and $H \in \mathcal{D}/\mathcal{B}_{\{0,\infty\}}$, $P'(R = S)P'(Z_S H(\Delta S X); R = S) = P'(Z_S; R = S)P'(H(\Delta S X); R = S)$. But this follows from Lemma 8 [II][I(a)] applied to the optional set [S]. By Lemma 1, this establishes the existence of $G_S$ and $F_S$. Next we compute, for any $H \in \mathcal{D}/\mathcal{B}_{\{0,\infty\}}$,
\[ P'(R = S)P'(H(X); X \in G_S) = P'(F_S)P'(\Delta S X \in G_S)P'(H(\Delta S X); \Delta S X \in G_S) = P'(H(\Delta S X); F_S \{\Delta S X \in G_S\})P(X \in G_S) \]
\[ = P'(H(\Delta S X); R = S)P(X \in G_S) = P'(H(\Delta R X); R = S)P(X \in G_S) = P(X \in G_S)P'(R = S)P'(H(\Delta R X); R < \infty)/P(R < \infty), \]
where the final equality is a consequence of the independence of $\Delta R X$ and $\mathcal{F}'_{R}$ under $P'$ (as applied to the optional process $1_{[q]}$). Similarly as in the proof of Theorem 2 the rest of the claims of (II) follow.

(II) Let $S$ be an $\mathcal{F}$-stopping time. Then a.s. $\{R = S < \infty\} = \bigcup_{i \in \mathbb{N}} \{R = S < \infty\} \{S = S_i\} = \bigcup_{i \in \mathbb{N}} F_i\{S = S_i\}\{\Delta S X \in G\}$ where $\cup_{i \in \mathbb{N}} F_i\{S = S_i\} \in \mathcal{F}'_{S}$. Hence Theorem 3.31 we may assume
without loss of generality that the graphs of the $S_i$, $i \in \mathbb{N}$, are pairwise disjoint. Then we may compute for $Z \in \text{Prog}/\mathcal{B}_{0,\infty}$ and $H \in \mathcal{D}/\mathcal{B}_{0,\infty}$, using the strong Markov property of $X$: 
\[ P(Z_RH(\Delta_RX); R < \infty) = \sum_{i \in \mathbb{N}} P(Z_{S_i}H(\Delta_{S_i}X); F_i(\Delta_{S_i}X \in G)) = \sum_{i \in \mathbb{N}} P(Z_{S_i}; F_{S_i}(S_i < \infty))P(H(X); X \in G) \]
and we conclude by taking $Z \equiv 1$, $H \equiv 1$ and $Z \equiv 1 \equiv H$ in turn.

For the final observation of the proposition, we note as follows. By definition, $R$ being a thin random time means that there exists a denumerable family of $\mathcal{F}$-stopping times $(S_i)_{i \in \mathcal{I}}$, the graphs of the members of which may be assumed pairwise disjoint, and such that $[R] \subset \bigcup_{i \in \mathbb{N}}[S_i]$ up to evanescence and $P(R = S_i) > 0$ for all $i \in \mathcal{I}$. Then if $R$ is an independence time for $X$, we may use [I] setting $O = \bigcup_{i \in \mathcal{I}}[S_i](F_{S_i} \times [0, \infty))$ and $G$ equal to any of the $G_{S_i}$. This establishes the necessity of the condition. Sufficiency follows from [II] upon taking $F_i \in \mathcal{F}'_{S_i}$, so that $1_{F_i} = (1_O)_{S_i}1(S_i < \infty)$ for $i \in \mathcal{I}$.

Strict independence times appear to be more subtle. We give below a sufficient condition for a strict random time to be an independence time for $X$ (Proposition [19]). Some preliminary notions and results are needed to this end.

**Definition 13.** A $T \in \mathcal{G}/\mathcal{B}_{0,\infty}$ is an incremental terminal time (ITT) if, for all $t \in [0, \infty)$ with $P(T > t) > 0$, conditionally on $\{T > t\}$, $T - t$ is independent of $\mathcal{F}_t|\{T > t\}$ and has the same law as $T$.

**Remark 14.**

1. If $T$ is an ITT, then: by the characterization of the solutions to Cauchy’s functional equation, $P(T = 0) = 1$, or $P(T = \infty) = 1$, or $T$ has an exponential distribution; furthermore, for all $\mathcal{F}$-stopping times $S$, satisfying $P(T > S) > 0$, via the usual extension from deterministic to stopping times, conditionally on $\{T > S\}$, $T - S$ is independent of $\mathcal{F}_S|\{T > S\}$ and has the same law as $T$.

2. ITTs are the analogue of terminal times (e.g. [5] Definition 4.1) from the Markov context: If $T$ is an $\mathcal{F}$-stopping time for which there is a $K \in \mathcal{D}/\mathcal{B}_{[0,\infty]}$ with $T = t + K(\Delta_tX)$ a.s. on $\{T > t\}$ for each $t \in [0, \infty)$, then by the simple Markov property of $X$, $T$ is an ITT.

3. If $T$ is an ITT and $e \in \mathcal{G}/\mathcal{B}_{[0,\infty]}$ is an exponentially distributed random time independent of $\mathcal{F}_\infty \vee \sigma(T)$ then $e \wedge T$ is an ITT.

**Examples 15.** ITTs that fall under Remark [14][2] include the deterministic times $\infty$ and $0$ and, assuming $\mathcal{F}$ is right-continuous — this is not a major assumption, since $X$ is a Lévy process also relative to the right-continuous augmentation of $\mathcal{F}$ — for an open $O \subset \mathbb{R}^d$, the time of the first entrance into (or hitting of) the set $O$ by the jump process $\Delta X$ of $X$ ([19] proves that such hitting times are stopping times when $d = 1$ – a perusal of the proof given, convinces one that the dimension $d = 1$ has nothing special to it in this regard; when $\mathcal{F}$ is not right-continuous, then still these hitting times are ITTs, though they may fail to be stopping times). Indeed, by the Début theorem [6] Theorem 4.2], when $\mathcal{G}$ is universally complete, any first entrance time of the jump process of $X$ into a set from $\mathcal{B}_{\mathbb{R}^d}$, is an ITT (and an $\mathcal{F}$-stopping time if further $(\Omega, \mathcal{G}, \mathcal{F}, P)$ satisfies “the usual hypotheses” [6 Theorem 4.30]).

**Definition 16.** Let $T$ be an ITT. A process $A \in \mathcal{G} \otimes \mathcal{B}_{[0,\infty]}/\mathcal{B}_{[0,\infty]}$, nondecreasing and right-continuous, is an incremental functional (IF) up to $T$ if (i) $A_0 = 0$ a.s., (ii) $A = A^T$ a.s. and (iii) for all $t \in [0, \infty)$, if $P(T > t) > 0$, then, conditionally on $\{T > t\}$, $(\Delta_tX, \Delta_tA)$ is independent of $\mathcal{F}_t|\{T > t\}$ and has the same law as $(X, A)$. 
Remark 14(1) there is then a probability space on which there lives a mean one exponentially distributed random time \( e \).

We may assume that \( f \) is an \( F \)-stopping time that is an ITT rendering \( A^T = A \) a.s., and if there is a \( J \in \mathcal{D}/\mathcal{D} \) such that \( \Delta_t A = J(\Delta_t X) \) a.s. on \( \{ T > t \} \) for all \( t \in [0, \infty) \), then by the simple Markov property of \( X, A \) is an IF up to \( T \). IFs are the analogue of raw additive functionals (e.g. \([5, \text{Definition 4.2}]\)) from the Markov context.

(3) If \( A \) is an IF up to \( T \), then in fact for all \( \mathcal{F} \)-stopping times \( S \), satisfying \( P(T > S) > 0 \), conditionally on \( \{ T > S \} \), \((\Delta_S X, \Delta_S A)\) is independent of \( \mathcal{F}_S \vert_{\{ T > S \}} \) and has the same law as \((X, A)\). This is proved in a manner that is entirely analogous to the proof of the strong Markov property of \( X \) from the simple Markov property of \( X \) (see e.g. \([18, \text{proof of Theorem 40.10}]\)).

(4) If \( A \) is an \( F \)-stopping time and \( e \in \mathcal{G}/\mathcal{B}_{[0, \infty]} \) is an exponentially distributed random time independent of \( \mathcal{F}_\infty \vee \sigma(A, T) \), then the stopped process \( A^e \) is an IF up to \( T \land e \) (cf. Remark 14(3)).

Lemma 18. Let \( A \) be an IF up to \( T \), \( H \in \mathcal{D}/\mathcal{B}_{[0, \infty]} \) and \( M \in \mathcal{O}/\mathcal{B}_{[0, \infty]} \). Set \( \lambda := (PT)^{-1} \in [0, \infty) \). Assume that (i) for all \( N \in (0, \infty) \), \( P \int_{[0, \epsilon]} H(\Delta_u X) \wedge N \, dA_u < \infty \) for some \( \epsilon \in (0, \infty) \) and (ii) \( P(\Delta A_S; S < \infty) = 0 \) for all \( \mathcal{F} \)-stopping times \( S \) (\( \Delta A \) being the jump process of \( A \)). Then

\[
P \int_{[0, \infty)} M_u H(\Delta_u X) \, dA_u = P \int_0^1 H(\Delta_u X) \, dA_u \int_0^T M_u \, du \lambda \frac{\lambda}{1 - e^{-\lambda}}
\]

with the quotient understood in the limiting sense when \( \lambda \in [0, \infty) \). (It is part of the statement that the inclusion or the exclusion of the upper delimiter 1 in the integral \( P \int_0^1 \ldots \, dA_u \) is immaterial.)

Proof. The case \( P(T = 0) = 1 \) is trivial and may be ignored. By Remark 17(4) and monotone convergence, we may assume without loss of generality (possibly by exploiting an extension of the underlying filtered probability space on which there lives a mean one exponentially distributed random time \( e \) independent of \( \mathcal{F}_\infty \vee \sigma(A, T) \)) that \( P(T < \infty) = 1 \) (one minimizes \( T \) by \( ne \), stops \( A \) at \( T \land (ne) \), and sends \( n \to \infty \)). By Remark 14(3) there is then a \( \lambda \in (0, \infty) \) with \( P(T > t) = e^{-\lambda t} \) for all \( t \in [0, \infty) \). By monotone convergence we may assume that \( M \) are \( H \) bounded and hence (by assumption (i)) that \( P \int_{[0, \epsilon]} H(\Delta_u X) \, dA_u < \infty \) for some \( \epsilon \in (0, \infty) \).

Define \( f_H := ([0, \infty) \ni t \mapsto P \int_{[t, s]} H(\Delta_u X) \, dA_u) \). For \( t \in [0, \infty) \), \( s \in [t, \infty) \), we have

\[
P \left( \int_{[t, s]} H(\Delta_u X) \, dA_u \right) = P \left( \int_{[t, s-t]} H(\Delta_u X) \, d(\Delta_t A)_u; t < T \right) = P(t < T) P \int_{[0, s-t]} H(\Delta_u X) \, dA_u.
\]

(3.1)

We find that \( f_H \) is finite-valued, nondecreasing, and satisfies the functional equation \( f_H(s) - f_H(t) = e^{-\lambda t} f_H(s-t) \). Set \( s = t + 1 \), to see that the limit \( f_H(\infty) := \lim_{s \to \infty} f_H(\infty) \) is finite, then send \( s \to \infty \), to obtain \( f_H(\infty) = e^{-\lambda t} f_H(\infty) \). It follows that \( f_H(\infty) = \frac{f_H(1)}{1 - e^{-\lambda t}} PT \).

It now follows from Remarks 14(1) and 17(3), and from assumption (ii), that when \( M = 1_{[S, \infty]} \) for an \( \mathcal{F} \)-stopping time \( S \), then

\[
P \int_{[0, \infty)} M_u H(\Delta_u X) \, dA_u = P \left( \int_{[S, \infty)} H(\Delta_u X) \, dA_u; S < T \right) = P \left( \int_{[0, \infty)} H(\Delta_u \Delta_S X) \, d(\Delta_S A)_u; S < T \right)
\]
\[ \Pr(S < T) \Pr \left( \frac{f_H(1)}{1 - e^{-\lambda}} \right) = \Pr(T - S; S < T) \frac{f_H(1)}{1 - e^{-\lambda}} = \frac{\lambda}{1 - e^{-\lambda}} \Pr \int_0^1 H(\Delta_u X) dA_u \Pr \int_0^T M_u du. \]

The class of processes \( M \) of the form considered is closed under multiplication and generates the optional \( \sigma \)-field [6 Theorem 3.17]. By monotone class we conclude. \( \square \)

**Proposition 19.** Suppose there exists an IF \( A \) up to some \( T \), with \( PA_e < \infty \) for some (then all) \( e \in (0, \infty) \) and with \( \Pr(\Delta A_S; S < \infty) = 0 \) for all \( F \)-stopping times \( S \), and that there exists an \( O \in \mathcal{O}/\mathcal{B}_{[0, \infty)} \), such that there is the following equality of random measures on \( \mathcal{B}_{[0, \infty)} \):

\[ \delta_R \mathbb{1}(R < \infty) = O \cdot dA \text{ a.s.,} \quad (3.2) \]

where \( \delta_R \) is the Dirac mass at \( R \) and \( (O \cdot dA)(C) := \int_C O_s dA_s \) for \( C \in \mathcal{B}_{[0, \infty)} \). Then \( \mathcal{F}'_R \) is independent of \( \Delta R X \) under \( \mathcal{P}' \), and the law of \( \Delta R X \) under \( \mathcal{P}' \) is given as follows: for \( H \in \mathcal{D}/\mathcal{B}_{[0, \infty)} \), \( \mathcal{P}'(H(\Delta R X)) = \Pr \int_0^1 H(\Delta_u X) dA_u / PA_1 \).

**Remark 20.**

1. The assumptions of Proposition 19 imply \( \mathbb{E}[R] \mathbf{1} = \emptyset \) up to evanescence for any \( F \)-stopping time \( S \). In particular, they are mutually exclusive with those of Proposition 10(11).
2. The presence of the ITT \( T \) in the statement of Proposition 19 allows some non-trivial added generality. For instance, independent exponential killing: if for the random time \( R \), an IF \( A \) up to some \( T \) and an \( O \) have been found satisfying the assumptions of Proposition 19 and if \( e \in \mathcal{G}/\mathcal{B}_{[0, \infty]} \) is an exponentially distributed time independent of \( F \) \( \supseteq \sigma(A, T) \), then \( A^e \) and \( O \) also satisfy the assumptions of Proposition 19 but for \( R_{(R < e)} \) replacing \( R \) (see Remark 17(4)).
3. (3.2) is a continuous-time analogue of (2.1); indeed the latter can be written as \( \delta_R \mathbf{1}(R < \infty) = Z \cdot \mu \) with \( \mu = \sum_{n \in \mathbb{N}_0} \delta_n \mathbf{1} \mathcal{F}(\Delta_n X) \) and \( Z = \sum_{n \in \mathbb{N}_0} \mathbf{1}_{[n]} F_n \).
4. Another way of writing (2.1) is as \( \mathbb{E}[R] = \mathbb{E}[\mathbf{1}(X \in \Gamma)] \) up to evanescence, with \( O = \bigcup_{n \in \mathbb{N}_0} F_n \times \{ n \} \).

This, and the final statement of Proposition 10, suggest the following as being the most natural analogue of (2.1) in continuous time: \( \mathbb{E}[R] = \mathbb{E}[\mathbf{1}(X \in \Gamma)] \) up to evanescence for an \( O \in \mathcal{O} \) and a \( \Gamma \in \mathcal{D} \). Establishing the precise relationship (conjectured equivalence, when \( R \in \sigma(X)/\mathcal{B}_{[0, \infty)} \)) between the property of \( R \) being an independence time for \( X \) and the latter condition, escapes the author in general: Proposition 10 establishes this equivalence for thin random times.

**Example 21.** Let \( X \) be the difference between a strictly positive unit drift and a non-zero compound Poisson subordinator whose paths are all piecewise constant (to avoid measurability issues). Let \( R \) be the unique time at which \( X \) is one temporal unit away from jumping for the second time and \( X \) is above zero, if there is such a time, \( R = \infty \) otherwise. Note that \( \Pr(R < \infty) > 0 \). Now let \( A \) be the right-continuous nondecreasing process vanishing a.s. at zero which increases by one at precisely those times \( t \in [0, \infty) \) when \( \Delta_t X \) performs the following: drifts for one spatial unit, and then jumps. By Remark 17(2), \( A \) is an IF up to \( \infty \). Clearly \( PA_1 < \infty \) and by the strong Markov property of \( X \) it follows that \( \Pr(\Delta A_S; S < \infty) = 0 \) for all \( F \)-stopping times \( S \). Let next \( N \) be the right-continuous adapted process that counts the number of jumps of \( X \) by, inclusive of, a given time; finally let \( O := \mathbf{1}(N = 1, X \geq 0) \) record the times when \( X \) has had one jump to-date and is not below zero. Then \( O \in \mathcal{O}/2^{[0,1)} \) and (3.2) obtains. So \( R \) is an independence time for \( X \).
Lemma 18. Conclude that $R$ since a.s. all $n$ we obtain that $R$.

Remark 23. One concludes as in the proof of Proposition 10(II). □

Example 24. Assume $X$ is a sample path continuous linear Brownian motion with strictly negative drift and suppose $\mathcal{F}$ is the completed natural filtration of $X$. Let $R$ be the last time $X$ is at its running supremum ($R = \infty$ on the event that there is no such time). For $\epsilon \in (0, \infty)$, let $A^\epsilon$ be the right-continuous nondecreasing process vanishing a.s. at zero that increases by 1 at precisely those $t \in [0, \infty)$ for which $\Delta_t X$ hits the level $\epsilon$ before hitting zero, never to return to $\epsilon$ again. Since a.s. between any two increases of $A^\epsilon$, $X$ must ascend by $\epsilon$ and then decrease by at least $\epsilon$, it follows that the set of such times is locally finite in $[0, \infty)$ and that moreover $PA^\epsilon_1 < \infty$. By the strong Markov property of $X$ it follows that $P(\Delta_{A^\epsilon_1}; S < \infty) = 0$ for all $\mathcal{F}$-stopping times $S$. By Remark 17(2), $A^\epsilon$ is an IF up to $\infty$. Set $O^\epsilon := \mathbb{1}(X + \epsilon \geq \bar{X})$. Clearly $O^\epsilon \in \mathcal{O}/2^{(0,1)}$. Let $R^\epsilon$ be equal to the last time that we are, in spatial terms, $\epsilon$ away from attaining the global supremum ($R^\epsilon = \infty$, when there is no such time). By Proposition 19, we obtain that $R^\epsilon$ is an independence time for $X$. In particular, for all continuous bounded $Z \in \mathcal{O}/B_\mathbb{R}$, all $n \in \mathbb{N}$, all real $0 < t_1 < \cdots < t_n$, and all bounded continuous $h : \mathbb{R}^n \to \mathbb{R}$:

$$P(R^\epsilon < \infty)P(Z_{R^\epsilon} h((\Delta_{R^\epsilon} X)_{t_1}, \ldots, (\Delta_{R^\epsilon} X)_{t_n}); R^\epsilon < \infty) = P(Z_{R^\epsilon}; R^\epsilon < \infty)P(h((\Delta_{R^\epsilon} X)_{t_1}, \ldots, (\Delta_{R^\epsilon} X)_{t_n}); R^\epsilon < \infty).$$

Since a.s. $R_\epsilon \to R$ as $\epsilon \downarrow 0$, we may pass to the limit. Finally a monotone class argument allows to conclude that $R$ is an independence time for $X$ (in a Brownian filtration optionality is synonymous with predictability).

Proof of Proposition 19. Set $\lambda := (PT)^{-1} \in [0, \infty)$. Let $M \in \mathcal{O}/B_{[0,\infty]}$ and $H \in \mathcal{D}/B_{[0,\infty]}$. Then by Lemma 18

$$P(M_R H(\Delta_R X); R < \infty) = P\left( \int M_u H(\Delta_u X) d\delta_R(du); R < \infty \right) =$$

$$P\left( \int_{[0,\infty)} M_u H(\Delta_u X) O_u dA_u \right) = P \int_0^1 H(\Delta_u X) dA_u P \int_0^T M_u O_u du \frac{\lambda}{1 - e^{-\lambda}}.$$  

One concludes as in the proof of Proposition 10(II). □

Remark 23. The preceding proposition was inspired by, and should be compared with [5] Theorem 5.7. There is an extra possibility in the latter, which in our case would correspond to $\delta_{R_\epsilon}(R < \infty)$ having a mixture of the two forms, (4.2) and the one from Proposition 10(II) by separating them according to a subset $M \subset \Omega \times [0, \infty)$ that is both optional and also of the form $\{ X \in \Gamma \}$ for a $\Gamma \in \mathcal{D}$. But if $M$ is such a set, then, as $S$ ranges over the class of finite $\mathcal{F}$-stopping times, $1_M(S) = 1_{\Gamma}(\Delta_S X)$ is at the same time $\mathcal{F}_S$ measurable and, by the strong Markov property, independent of $\mathcal{F}_S$ and of constant expectation. It is to say that the process $1_M$ is a.s. equal to zero at every finite stopping time $S$ or a.s. equal to one at every finite stopping time $S$. Hence any such $M$ equals $\emptyset$ or $\Omega \times [0, \infty)$ up to evanescence [6 Corollary 4.11], and the separation would be trivial (see, however, Lemma 13(III)). Indeed, the separation in [5] Theorem 5.7] happens according to an optional homogeneous (in the sense of Markov processes, e.g. [5 p. 315]) set and, by contrast, there are many non-trivial sets of such a form (e.g. $\{ X \in \Gamma \}$ for $\Gamma \in \mathcal{B}_{\mathbb{R}^d}$).

Remark 24. In special cases, the ‘conditional independence’ results from the Markov theory (see Subsection 1.2) may also be applied successfully in the present context – modulo said results’ provisions. Either directly, when $X_R$ is trivial, or indirectly when there is some other strong Markov process $Z$ for which $Z_R$ is trivial and for which $R$ is a conditional independence time if and only if it is an independence time for $X$. For instance, $X$ reflected in its running supremum is known to be strong Markov and a sample-path continuous process $Z$ is, for $a$ from its state space, equal to $a$ on its last visit to $a$ (viz. Example 22).
We conclude by characterizing regenerative/Markov times as stopping times.

**Lemma 25.** Assume $\mathcal{G} = \sigma(X)$ (respectively, up to negligible sets). Then (respectively, up to evanescent sets) $\mathcal{G} \otimes \mathcal{B}_{\{0,\infty\}} = O \vee \sigma_{\Omega \times [0,\infty)}(\Delta X)$.

**Proof.** Since every optional process is measurable and by Remark 7, $\mathcal{G} \otimes \mathcal{B}_{\{0,\infty\}} \supset O \vee \sigma_{\Omega \times [0,\infty)}(\Delta X)$. For the reverse inclusion, by monotone class, it suffices to show that for $n \in \mathbb{N}_0$, real numbers $0 = r_0 < \cdots < r_n$, $\{\alpha_0, \cdots, \alpha_n\} \subseteq \mathbb{R}^d$ and $r \in [0,\infty)$, the process $Z := \prod_{k=0}^n e^{i\alpha_k \cdot X_{r_k}} 1_{[r,\infty)}$ belongs to $O \vee \sigma_{\Omega \times [0,\infty)}(\Delta X)$. Set $r_{n+1} := \infty$ and let $j \in \{0,\ldots,n\}$ be the unique index for which $r \in [r_j, r_{j+1})$. Conclude by identifying $Z_t = 1_{[r,\infty)}(t) \left( e^{i(\alpha_j + \cdots + \alpha_n) \cdot X_{r_k}} \prod_{k=0}^j e^{i\alpha_k \cdot X_{r_k}} \left(\prod_{k=j+1}^n e^{i\alpha_k \cdot (\Delta X)_{t \wedge (r_k-r)}}\right) \right)$ for $t \in [0,\infty)$ and noting that deterministic stopping is $D$/$D$-measurable. □

**Theorem 26.** For $x \in \mathbb{R}^d$, denote by $P^x$ be the law of $x + X$ under $P$ (so $\mathcal{L} = \mathcal{P}^0$), $P := (P^x)_{x \in \mathbb{R}^d}$. Of the following statements, (iii) implies (ii), (i) and (ii) are equivalent. If in addition $R \in \sigma(X)/\mathcal{B}_{[0,\infty]}$, then (iii) implies (ii)

(i) Under $P'$, conditionally on $X_R$, $\theta R X$ is independent of $\mathcal{F}'_R$ and has kernel law $P$.

(ii) $\Delta R X$ is independent of $\mathcal{F}'_R$ under $P'$ and the law of $\Delta R X$ under $P'$ is $\mathcal{L}$.

(iii) $R$ is a stopping time relative to the completion of $\mathcal{F}$.

**Proof.** The equivalence of (i) and (ii) follows by routine manipulation of conditional independence, using the fact that $X_R \in \mathcal{F}'_R/\mathcal{B}_{\mathbb{R}^d}$. Condition (iii) is known to be sufficient for (ii). Necessity. Since the completed natural filtration of $X$ is right-continuous (every Lévy process is a Feller process), we may assume without loss of generality that the filtered probability space $(\Omega, \mathcal{G}, P, \mathcal{F})$ satisfies the “usual hypotheses” of the general theory of stochastic processes with $\mathcal{F}$ the completed natural filtration of $X$ and $\mathcal{G} = \overline{\sigma(X)}$. Consider the measure $\mu_R$ on $\mathcal{G} \otimes \mathcal{B}_{[0,\infty]}$ associated to the raw increasing process $1_{[R,\infty)}$: $\mu_R M = P(M_R; R < \infty)$ for $M \in \mathcal{G} \otimes \mathcal{B}_{[0,\infty]}/\mathcal{B}_{[0,\infty]}$. It suffices [6, Theorem 5.13] to establish $\mu_R$ is an optional measure, i.e. that for $M \in \mathcal{G} \otimes \mathcal{B}_{[0,\infty]}/\mathcal{B}_{[0,\infty]}$, $\mu_R M = \mu_R(\sigma M)$ with $\sigma M$ the optional projection of $M$. By monotone class and Lemma 25, it suffices to check it when $M = Z H(\Delta X)$ with $Z \in \mathcal{L}/\mathcal{B}_{[0,\infty]}$ and $H \in D/\mathcal{B}_{[0,\infty]}$. In such case, by the strong Markov property of $X$, for all $\mathcal{F}$-stopping times $S$, a.s. on $\{S < \infty\}$, one has $P(M_S|\mathcal{F}_S) = P(Z_S H(\Delta S X)|\mathcal{F}_S) = Z_S \mathcal{L}(H)$, i.e. $\sigma M = Z \mathcal{L}(H)$. Now, since $R$ is an independence time for $X$, indeed $\mu_R M = P(Z_R H(\Delta R X); R < \infty) = P(Z_R; R < \infty) \mathcal{L}(H) = \mu_R(\sigma M)$. □

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