Two Remarks on Marcinkiewicz decompositions by Holomorphic Martingales

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October 11, 1993
1 Introduction

The real part of $H^{\infty}(\mathbb{T})$ is not dense in $L^\infty_{\mathbb{R}}(\mathbb{T})$. The John-Nirenberg theorem in combination with the Helson-Szegö theorem and the Hunt Muckenaupt Wheeden theorem has been used to determine whether $f \in L^\infty_{\mathbb{R}}(\mathbb{T})$ can be approximated by $\text{Re} \, H^{\infty}(\mathbb{T})$ or not: $\text{dist}(f, \text{Re} \, H^{\infty}) = 0$ if and only if for every $\epsilon > 0$ there exists $\lambda_0 > 0$ so that for $\lambda > \lambda_0$ and any interval $I \subseteq \mathbb{T}$.

$$\{x \in I : |\tilde{f} - (\tilde{f})_I| > \lambda\} \leq |I| e^{-\lambda / \epsilon},$$

where $\tilde{f}$ denotes the Hilbert transform of $f$. See [G] p. 259. This result is contrasted by the following

**Theorem 1** Let $f \in L^\infty_{\mathbb{R}}$ and $\epsilon > 0$. Then there is a function $g \in H^{\infty}(\mathbb{T})$ and a set $E \subset \mathbb{T}$ so that $|\mathbb{T} \setminus E| < \epsilon$ and

$$f = \text{Re} \, g \quad \text{on } E.$$ 

This theorem is best regarded as a corollary to Men’shov’s correction theorem. For the classical proof of Men’shov’s theorem see [Ba, Ch VI §1-§4].

Simple proofs of Men’shov’s theorem – together with significant extensions – have been obtained by S.V. Khruschev in [Kh] and S.V. Kislyakov in [K1], [K2] and [K3].

In [S] C. Sundberg used $\partial$-techniques (in particular [G, Theorem VIII.1] gave a proof of Theorem 1 that does not mention Men’shov’s theorem.

The purpose of this paper is to use a Marcinkiewicz decomposition on Holomorphic Martingales to give another proof of Theorem 1. In this way we avoid uniformly convergent Fourier series as well as $\partial$-techniques.

Holomorphic Martingales enter in the proof of the following lemma.

**Lemma 2** There exist $c_1, c_2 > 0$ so that for every $f \in \text{BMOA}$, where $||f|| \leq 1$, $\epsilon > 0$ and $\lambda \in \mathbb{R}^+$ there exists $g \in H^{\infty}(\mathbb{T})$ and $E \subset \mathbb{T}$

$$||g||_{\infty} \leq \lambda$$

$$|f(\theta) - g(\theta)| \leq \epsilon \quad \text{on } E$$

$$|\mathbb{T} \setminus E| \leq \frac{1}{\epsilon} e^{-\lambda c_1 c_2}.$$
Consider complex Brownian Motion \((z_t)_{t \geq 0}\) on the Wiener space \((\Omega, (\mathcal{F}_t), \mathcal{F}, P)\). A complex valued random variable \(X\) on \(\Omega\) is called holomorphic if the conditional expectation

\[ X_t = E(X | \mathcal{F}_t) \]

admit a stochastic integral representation of the form \(X_t = X_0 + \int_0^t F_s dz_s\), where \(F_s\) is adapted to \(\mathcal{F}_s\).

\(H^p(\Omega)\) denotes the closure in \(L^p(\Omega)\) of holomorphic random variables. \(\text{BMO}(\Omega)\) denotes the closure of holomorphic random variables under the norm

\[ \sup_t ||E(|X - X_t| |\mathcal{F}_t)||_\infty. \]

The connection to analytic functions is provided by operators \(M, N\) so that

\[ H^p(\mathbb{T}) \xrightarrow{M} H^p(\Omega) \xrightarrow{\text{Id}} \xrightarrow{N} H^p(\mathbb{T}) \]

where \(||M||_p = ||N||_p = 1\) and

\[ \text{BMOA}(\mathbb{T}) \xrightarrow{M} \text{BMO}(\Omega) \xrightarrow{\text{Id}} \xrightarrow{N} \text{BMOA}(\mathbb{T}) \]

where \(||M||_{\text{BMO}} \leq C_0, ||N||_{\text{BMO}} \leq C_0\).

These probabilistic ideas have a quite long history and were useful in several problems of Analysis. See [F], [G-S], [Ma] and [V].

2 Proofs of the results

Proof of Lemma 2. Fix \(\lambda > 0\) and let

\[ \sigma = \inf \{t : |z_t| > 1\} \]
\[ \tau = \inf \{t \leq \sigma : |f(z_t)| > \lambda\} \]
\[ G_t = f(z_{t\wedge \tau}) \]
\[ F_t = f(z_t) \]
\[ g(\theta) = N(G)(\theta). \]
Then
\[ ||g||_\infty \leq \lambda \]
\[ g \in H^\infty \]
and
\[ |\{ \theta : |f - g| > \epsilon \}| \leq \frac{1}{\epsilon} ||f - g||_1 \]
\[ \leq \frac{1}{\epsilon} ||N(F - G)||_1. \]

By [M, Lemma 1] we get
\[ ||N(F - G)||_1 \leq ||F - G||_1 \leq 2 \int_{\{F^* > \lambda\}} |F|dP \]
where \( F^* = \sup |F_t| \). By Cauchy-Schwartz we obtain
\[ \int_{\{F^* > \lambda\}} |F|dP \leq \{ F^* \geq \lambda \}^{1/2} c_2 \leq e^{-\lambda c_1} c_2, \]
because \( F \in \text{BMO}(\Omega) \) implies \( Ee^{F^*c} < \infty \). This implies the estimate
\[ |\{ \theta : |g - f| > \epsilon \}| \leq \frac{1}{\epsilon} e^{-\lambda c_1} c_2. \]

**Proof of Theorem 1.** Given \( \epsilon > 0 \) we select \( \lambda_n \in \mathbb{R}^+ \) so that
\[ \sum_{n=0}^{\infty} e^{-\lambda_n c_1} c_2 2^n < \epsilon \]
and
\[ \sum_{n=0}^{\infty} \lambda_n 2^{-n} < \infty. \]
Then given a function \( h : \mathbb{T} \to \mathbb{C} \) and \( \delta > 0 \) we define
\[ T_\delta(h)(\theta) = \begin{cases} h(\theta) & \text{if } |h(\theta)| \leq \delta \\ \delta & \text{if } |h(\theta)| \geq \delta \end{cases} \]
Now consider \( u_0 \in L^\infty_R(\mathbb{T}) \) with \( \|u_0\|_\infty = 1 \) and let \( \tilde{u}_0 \) be the Hilbert transform of \( u_0 \) then \( u_0 + i\tilde{u}_0 \in \text{BMOA} \) and

\[
\|u_0 + i\tilde{u}_0\|_{\text{BMOA}} \leq C\|u_0\|_\infty.
\]

We next apply an interaction procedure from [S].

**Step 1.** Use Lemma 2 to obtain \( E_1 \subset \mathbb{T}, g_1 \in H^\infty \) with \( \|g_1\|_\infty \leq \lambda_1 \) so that

\[
|u_0 + i\tilde{u}_0 - g_1| < 1/2 \text{ on } E_1
\]

and

\[
|\mathbb{T} \setminus E_1| \leq 2e^{-\lambda_1c_1c_2}.
\]

**Induction Step.** We have already constructed \( u_0, \ldots, u_{n-1} \in L^\infty_R, g_1, \ldots, g_n \in H^\infty(\mathbb{T}) \) and \( E_1, \ldots, E_n \leq \mathbb{T} \) so that for \( j \leq n \)

\[
\|g_j\|_\infty \leq \lambda_j 2^{-j}
\]

\[
|u_{j-1} + i\tilde{u}_{j-1} - g_j| \leq 2^{-j} \text{ on } E_j
\]

\[
|\mathbb{T} \setminus E_j| \leq e^{-c_1\lambda_jc_2}2^j.
\]

Now we let

\[
u_n := T_{2^{-n}}(u_{n-1} - \text{ Re } g_n)
\]

and we have

\[
u_n = u_{n-1} - \text{ Re } g_n \text{ on } E_n
\]

\[
\|u_n\|_\infty \leq 2^{-n}.
\]

By Lemma 2 we find \( g_{n+1} \in H^\infty(D), E_{n+1} \leq \mathbb{T} \) so that

\[
\|g_{n+1}\|_\infty \leq \lambda_{n+1}2^{-n-1},
\]

\[
|u_n + i\tilde{u}_n - g_{n+1}| \leq 2^{-n-1} \text{ on } E_{n+1},
\]

\[
|\mathbb{T} \setminus E_{n+1}| \leq e^{-c_1\lambda_{n+1}c_2}2^{n+1}.
\]

Having completed the construction we set

\[
g := \sum_{j=1}^{\infty} g_j
\]
which defines an element in \( H^\infty(\mathbb{T}) \). Tracing back we see that

\[
\sum_{n=1}^{\infty} u_n = \sum_{n=0}^{\infty} u_n - \sum_{n=1}^{\infty} \text{Re} g_n \quad \text{on} \quad \bigcap_{n=1}^{\infty} E_n
\]

or

\[
u_0 = \text{Re} g \quad \text{on} \quad \bigcap_{n=1}^{\infty} E_n.
\]

It remains to estimate \( |\bigcap_{n=1}^{\infty} E_n| \) from below:

\[
|\bigcap_{n=1}^{\infty} E_n| \geq |\mathbb{T}| - \sum_{n=1}^{\infty} |\mathbb{T} \setminus E_n|
\]

\[
\geq |\mathbb{T}| - \sum_{n=1}^{\infty} e^{-\lambda_n c_1 2^n c_2}
\]

\[
\geq |\mathbb{T}| - \epsilon.
\]

3 A Refinement of Lemma 2

In the above argument we gave just an estimate for the size of the set

\[
\{ \theta : |f(\theta) - g(\theta)| < \epsilon \}
\]

but did not give any indication where to find this set. A more detailed analysis of the “conditional expectation” operator \( \hat{N} \) gives estimates which relate the probabilistic Marcinkiewicz decomposition to classical maximal functions.

Let \( h : \mathbb{T} \to \mathbb{C} \) be a function, then let \( h^\# \) be the non tangential maximal function and define

\[
M_{HL}(h)(\theta) := \sup_I \int_I |h| \frac{dt}{|I|}
\]

where the sup is taken over intervals in \( \mathbb{T} \) which contain \( \theta \). Let \( g \) be defined as in the proof of Lemma 2 then we have the pointwise estima
**Theorem 3**

1. \(|f(\theta) - g(\theta)| \leq C(|f(\theta)| + \lambda)M_{HL}(\chi_{H_\lambda})(\theta)\) \(\text{whe}\ H_\lambda = \{f^# > \lambda\}\).

2. Let \(f \in BMO\), with \(|f| \leq 1\), then for every \(N > 0\) there exists \(\lambda > 0\) and \(B \subset \{\theta \in \mathbb{T} : |f(\theta)| \leq N\}\) so that
   \[|\mathbb{T} \setminus B| \leq e^{-Nc_1c_2} ,\]
   \(M_{HL}(\chi_{H_\lambda})(\theta) \leq e^{-c_3\lambda}\) for \(\theta \in B\).

**Proof.** ad 1. For \(\theta \in \mathbb{T}\) and \(z \in D\) let
   \[P_\theta(z) := \frac{1 - |z|^2}{|e^{i\theta} - z|^2}.\]

Fix \(0 < r < 1\) consider the stopping times
   \[\sigma_r := \inf\{t : |z_t| > r\}\]

and let
   \[F_\lambda := \{\omega \in \Omega : \tau(\omega) < \sigma(\omega)\}\]
   \[E_\lambda := \{z \in D : |f(z)| > \lambda\}\]

where the stopping time \(\tau\) has been defined during the proof of Lemma 2. Then for \(\theta \in \mathbb{T}\) we have (using formula (1) in [Du, Section 3.2])
   \[g(\theta) = N(F - G)(\theta) = \lim_{r \to 1} E((f(z_{\sigma_r}) - f(z_{\sigma_r}\wedge \tau))P_\theta(z_r)) \leq (|f(\theta)| + \lambda)\lim_{r \to 1} E(\chi_{F_\lambda}P_\theta(z_{\sigma_r})).\]

For \(A \subset D\) let
   \[\omega(A) := \mathbb{P}\{z_t \in A, \text{ for some } t < \sigma\}.\]

Then \(\omega(E_\lambda) = \mathbb{P}(F_\lambda)\) and by the strong Markov Property we have: (see [D], p. 923 or [V], p. 112)
   \[\lim_{r \to 1} E(\chi_{F_\lambda}P_\theta(z_{\sigma_r})) = \int_{\partial E_\lambda} P_\theta(z)\omega(dz).\]
The integral on the RHS is called balyage or sweep of $\omega|_{\partial E_\lambda}$ and has been much studied because of its relation to Carleson-measures and BMO. See [G], pp. 229, 239 and 240. The argument in [G], p. 239 gives the estimate

$$\int_{\partial E_\lambda} P_\theta(z) \omega(dz) \leq C_3 \sup_{0 \leq h < 1} \frac{\omega(\partial E_\lambda \cap S_h)}{h}$$

where

$$S_h := \{ re^{i\psi} : 1 - h \leq r < 1, |\psi - \theta| \leq h \}.$$ 

The result of Burkholder Gundy Siverstein gives for every harmonic function $u : D \to \mathbb{R}$

$$\omega\{u > \lambda\} \leq C|\{u^# > \lambda\}|.$$ 

See [P], p. 36. Therefore by [G, Lemma I.5.5] $\omega$ is a Carleson Measure. Hence a simple stopping time argument gives for every $0 \leq h \leq 1$

$$\omega(\partial E_\lambda \cap S_h) \leq C|\{f^# > \lambda\} \cap 3I_h|$$

where $I_h = S_h \cap \mathbb{T}$. We therefore have the estimate.

$$\int_{\partial E_\lambda} P_\theta(z) \omega(dz) \leq C \sup_{0 \leq h \leq 1} \frac{|\{f^# > \lambda\} \cap 2I_h|}{h}$$

And by choice of $I_h$ the LHS is dominated by

$$CM_{H\Lambda}(\chi_{H\Lambda})(\theta)$$

whe $H_{\Lambda} = \{\psi \in \mathbb{T} : f^#(\psi) > \lambda\}$. 

ad 2. As $f \in \text{BMO}$ there exists $\delta_0 > 0$ and $C_0 > 0$ so that for each $\lambda > 0$

$$|\{f^# > \lambda\}| \leq e^{-\lambda \delta_0} C_0.$$ 

Now choose $\delta_1 = \delta_0/2$ and let

$$H = \{f^# > \lambda\},$$

$$G = \{|f| < N\},$$

$$J = \{I \subset \mathbb{T} : |H \cap I| > e^{-\lambda \delta_1} |I|, I \text{ Intervall}\},$$

$$J = \bigcup_{I \in J} I,$$

$$B = G \setminus J.$$
The weak type $1 : 1$ bound for the Hardy Littlewood maximal function gives

$$|J| \leq |H| e^{\lambda \delta_1} C \leq e^{-\lambda \delta_1} C.$$  

Hence

$$|T \setminus B| \leq |T \setminus G| + |J| \leq e^{-N\delta_0} C_0 + e^{-\lambda \delta_1} C.$$  

Moreover, by definition, we have

$$M_{HL}(\chi_H)(\theta) \leq e^{-\lambda \delta_1} \text{ for } \theta \in B$$

and this completes the proof.  

\[\blacksquare\]
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