Homotopy fibre sequences induced by 2-functors

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Abstract

This paper contains some contributions to the study of the relationship between 2-categories and the homotopy types of their classifying spaces. Mainly, generalizations are given of both Quillen’s Theorem B and Thomason’s Homotopy Colimit Theorem to 2-functors.

Keywords: 2-category, monoidal category, classifying space, homotopy fibre, homotopy colimit, Grothendieck construction, loop space

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1. Introduction and summary

The construction of nerves and classifying spaces of categorical structures has become an essential part of the machinery in algebraic topology and algebraic K-theory. This paper provides a contribution to the study of classifying spaces of 2-categories and, in particular, of monoidal categories.

To help motivate the reader, and to establish the setting for our discussions, let us briefly recall that it was Grothendieck [19, 20] who first associated a simplicial set $\mathcal{N}_C$ to a small category $\mathcal{C}$, calling it its nerve. The set of $n$-simplices

$$N_n\mathcal{C} = \bigcup_{(x_0, \ldots, x_n) \in \text{Ob} \mathcal{C}^{n+1}} \mathcal{C}(x_1, x_0) \times \mathcal{C}(x_2, x_1) \times \cdots \times \mathcal{C}(x_n, x_{n-1})$$

consists of length $n$ sequences of composable morphisms in $\mathcal{C}$. Milnor’s realization [30] of its nerve is the classifying space of the category, $B\mathcal{C} = |\mathcal{N}\mathcal{C}|$. We can stress the historical relevance of this construction by noting that, in Quillen’s development of higher algebraic K-theory [25], K-groups are defined by taking homotopy groups of classifying spaces of certain categories. When a monoid (or group) $\mathcal{M}$ is regarded as a category with only one object, then $B\mathcal{M}$ is its classifying space in the traditional sense. Therefore, many weak homotopy types thus occur, since every path-connected space has the weak homotopy type of the classifying space of a monoid, as it was proved by McDuff [26, Theorem 1] (see

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also [13, Theorem 3.5]). Moreover, any CW-complex is homotopy equivalent to the classifying space of a small category, as Illusie showed in [22, Theorem 3.3]: The category of simplices $\int_{\Delta} S$, of a simplicial set $S$, has as objects pairs $(p, x)$ where $p \geq 0$ and $x$ is a $p$-simplex of $S$; and arrow $\xi : (p, x) \rightarrow (q, y)$ is an arrow $\xi : [p] \rightarrow [q]$ in $\Delta$ with the property $x = \xi^* y$. Then there exists a homotopy equivalence $|S| \simeq B \int_{\Delta} S$ between the geometric realization of $S$ and the classifying space of $\int_{\Delta} S$ (this result is, in fact, a very particular case of the homotopy colimit theorem of Thomason [37]). Then, by [30, Theorem 4], if $X$ is any CW-complex and we take $S = SX$, the total singular complex of $X$, it follows that $X \simeq |SX| \simeq B \int_{\Delta} SX$.

In [33], Segal extended Milnor’s realization process to simplicial topological spaces. He observed that, if $\mathcal{C}$ is a topological category, then $\mathbf{NC}$ is, in a natural way, a simplicial space and he defines the classifying space $B\mathcal{C}$ of a topological category $\mathcal{C}$ as the realization of the simplicial space $\mathbf{NC}$. This general construction given by Segal provides, for instance, the definition of classifying spaces of 2-categories. A 2-category $\mathcal{C}$ is a category endowed with categorical hom-sets $\mathcal{C}(x, y)$, for any pair of objects, in such a way that composition is a functor $\mathcal{C}(y, z) \times \mathcal{C}(x, y) \rightarrow \mathcal{C}(x, z)$ satisfying the usual identities. By replacing the hom-categories $\mathcal{C}(x, y)$ by their classifying spaces $B\mathcal{C}(x, y)$, one obtains a topological category with a discrete space of objects, of which Segal’s realization is the classifying space $B\mathcal{C}$ of the 2-category. Thus,

$$B\mathcal{C} := |\mathbf{NC}| \cong |\text{diag} \mathbf{NC}|,$$

where $\text{diag} \mathbf{NC}$ is the diagonal of the bisimplicial set obtained from the simplicial category $\mathbf{NC}$ by replacing each category $\mathcal{C}(x, y)$ by its nerve (i.e., the double nerve of the 2-category). For instance, the classifying space of a (strict) monoidal category $(\mathcal{M}, \otimes)$ is the classifying space of the one object 2-category, with category of endomorphisms $\mathcal{M}$, that it defines.

The category $\mathbf{2Cat}$ of small 2-categories and 2-functors has a Quillen model structure [41], such that the functor $\mathcal{C} \mapsto B\mathcal{C}$ induces an equivalence between the corresponding homotopy category of 2-categories and the ordinary homotopy category of CW-complexes. By this correspondence, 2-groupoids correspond to spaces whose homotopy groups $\pi_n$ are trivial for $n > 2$ [32]. From this point of view the use of 2-categories and their classifying spaces in homotopy theory goes back to Whitehead [40] and Mac Lane-Whitehead [28] since 2-groupoids with only one object (= strict 2-groups, in the terminology of Baez [2]) are the same as crossed modules (this observation is attributed to Verdier in [6]). But, beyond homotopy theory, the use of classifying spaces of 2-categories has also shown its relevance in several other mathematical contexts such as in algebraic K-theory [21], conformal field theory [39], or in the study of geometric structures on low-dimensional manifolds [38].

This paper contributes to clarifying several relationships between 2-categories and the homotopy types of their classifying spaces. Here, we deal with questions such as, when do 2-functors induce homotopy equivalences or homotopy cartesian squares of classifying spaces? In fact, research work aiming to answer that
question was started by the author with Bullejos in [7], a paper to which this one is a sequel, and where a generalization to 2-functors of Quillen’s Theorem A [25] was shown as a primary outcome. That paper relied heavily on the fact that realizations of Duskin-Street’s geometric nerves [9, 34] yield classifying spaces for 2-categories [7, Theorem 1]. In this paper we mainly state and prove an extension of the relevant and well-known Quillen’s Theorem B [25] to 2-categories by showing, under reasonable necessary conditions, a category-theoretical interpretation of the homotopy fibres of the realization map $BF : B\mathcal{B} \to B\mathcal{C}$, of a 2-functor $F : \mathcal{B} \to \mathcal{C}$. More precisely, in Theorem 3.2 we replace the concept of homotopy fibre category of a functor by Gray’s notion [18, §3.1] of homotopy fibre 2-category $\mathcal{B}(z//F)$ of a 2-functor $F : \mathcal{B} \to \mathcal{C}$ at an object $z$ of $\mathcal{C}$ (the double bar notation // avoids confusion with the homotopy fibre category of the underlying functor, see §3 for details). Then, we prove the existence of induced homotopy fibre sequences $B(z//F) \to B\mathcal{B} \to B\mathcal{C}$, whenever the induced maps $B(z_0//F) \to B(z_1//F)$, for the different morphisms (1-cells) $z_1 \to z_0$ of $\mathcal{C}$, are homotopy equivalences. This says that the name “homotopy fibre 2-category” was well-chosen, since the classifying spaces of these homotopy fibres are the homotopy fibres of the map of classifying spaces. Clearly if the homotopy fibre 2-categories are contractible then $BF$ is a homotopy equivalence, and Theorem A for 2-functors in [7] becomes an immediate corollary.

When both $\mathcal{B}$ and $\mathcal{C}$ are small categories, regarded as discrete 2-categories, one obtains the ordinary Theorem B, and the methods used in the proof of Theorem 3.2 we give follow along similar lines to those used by Goerss and Jardine in [16, §IV, 5.2] for proving Quillen’s theorem, though the generalization to 2-categories is highly nontrivial.

Our result is applied to the homotopy theory of lax functors $\mathcal{F} : \mathcal{C}^\circ \to 2\text{Cat}$, where $\mathcal{C}$ is any 2-category, and hence to acting monoidal categories. The application is carried out through an enriched Grothendieck construction $\int_{\mathcal{C}} \mathcal{F}$, which is actually a special case of the one considered by Baković in [3, §4] and, for the case where $\mathcal{C}$ is a category, it is a special case of the ones given by Tamaki in [37, §3] and by Carrasco, Cegarra and Garzón in [10, §3] (see also the construction $\int_{\Gamma}$ in [11, Theorem 3.3], where $\Gamma$ is a group). For any lax functor $\mathcal{F} : \mathcal{C}^\circ \to 2\text{Cat}$, $\int_{\mathcal{C}} \mathcal{F}$ is a 2-category that assembles all 2-categories $\mathcal{F}_z$, $z \in \text{Ob}\mathcal{C}$, and, in Theorem 4.3 when every map $BF_{z_0} \to BF_{z_1}$ induced by a morphism $z_1 \to z_0$ of $\mathcal{C}$ is a homotopy equivalence, we prove the existence of homotopy fibre sequences $BF_{z} \to B\mathcal{F}_z \to B\mathcal{C}$, for the different objects $z$ of $\mathcal{C}$.

The usual Grothendieck construction [20] on a lax functor $\mathcal{F} : \mathcal{C}^\circ \to \text{Cat}$, for $\mathcal{C}$ a category, underlies our 2-categorical construction $\int_{\mathcal{C}} \mathcal{F}$. Recall also that the well-known Homotopy Colimit Theorem by Thomason [37] establishes that
the Grothendieck construction on a diagram of categories is actually a categorical model for the homotopy type of the homotopy colimit of the diagram of categories. The notion of homotopy colimit has been well generalized in the literature to 2-functors \( F : C^o \to \text{Cat} \) (see [21, 2.2], for example), where \( C \) is any 2-category. Our Theorem 4.5 states two generalizations of Thomason’s theorem both to 2-diagrams of categories, that is to 2-functors \( F : C^o \to \text{Cat} \), with \( C \) a 2-category, and to diagrams of 2-categories, that is to functors \( F : C^o \to \text{2Cat} \), with \( C \) a category, by showing the existence of respective homotopy equivalences

\[
\text{Bhocolim}_C F \simeq \text{B}_F \int_C F.
\]

The plan of this paper is, briefly, as follows. After this historical and motivating introductory Section 1, the paper is organized in three sections. Section 2 contains a minimal amount of notation as well as various auxiliary statements on classifying spaces of 2-categories. In Section 3 we state and prove the main result of the paper, Theorem 3.2, with the generalization to 2-functors of Quillen’s Theorem B which, in Section 4, is applied to the study of 2-diagrams of 2-categories by means of the higher Grothendieck construction.

2. Preliminaries and notations

For the general background on 2-categories used in this paper, we refer to [4], [27] and [35], and on simplicial sets to [29], [25] and, mainly, to [16].

The simplicial category is denoted by \( \Delta \). It has as objects the ordered sets \( [n] = \{0, \ldots, n\}, n \geq 0 \), and as arrows the (weakly) monotone maps \( \alpha : [n] \to [m] \). This category is generated by the directed graph with edges all maps

\[
[n + 1] \xrightarrow{\sigma_i} [n] \xleftarrow{\delta_i} [n - 1], \quad 0 \leq i \leq n,
\]

where

\[
\sigma_i(j) = \begin{cases} j & \text{if } j \leq i \\ j - 1 & \text{if } j > i \end{cases} \quad \text{and} \quad \delta_i(j) = \begin{cases} j & \text{if } j < i \\ j + 1 & \text{if } j \geq i \end{cases}.
\]

Segal’s geometric realization functor \( [33] \), for simplicial spaces \( S : \Delta^o \to \text{Top} \), is denoted by \( S \mapsto |S| \). For instance, by regarding a set as a discrete space, the (Milnor’s, [30]) geometric realization of a simplicial set \( S : \Delta^o \to \text{Set} \) is \( |S| \).

The category of small categories is \( \text{Cat} \), and \( N : \text{Cat} \to \text{Simpl.Set} \) denotes the nerve functor to the category of simplicial sets. Any ordered set \( [n] \) is considered as a category with only one morphism \( j \to i \) when \( 0 \leq i \leq j \leq n \), and the nerve

\[
NC : \Delta^o \to \text{Set},
\]

of a category \( C \), is the simplicial set with \( N_n C = \text{Func}([n], C) \), the set of all functors \( x : [n] \to C \), which usually we write by

\[
x = (x_i \xrightarrow{x_{i,j}} x_j)_{0 \leq i \leq j \leq n}.
\]
In other words, for \( n > 0 \), an \( n \)-simplex is a string of composable arrows in \( C \):

\[
\begin{array}{cccccccc}
x_0 & \\ \xrightarrow{x_{0,1}} & x_1 & \\ \xrightarrow{x_{1,2}} & x_2 & \cdots & \xrightarrow{x_{n-1,n}} & x_n
\end{array}
\]

and \( N_0C = \text{Ob}C \), the set of objects of the category. By applying realization to \( NC \), we obtain the classifying space \( BC \) of the category \( C \), that is

\[ BC = |NC| \]

We also use the notion of classifying space \( BC \) for a simplicial category \( C : \Delta^o \to \text{Cat} \). That is, the realization of the bisimplicial set \( NC : ([p], [q]) \mapsto N_pC_q \), obtained by composing \( C \) with the above functor nerve. Recall that, when a bisimplicial set \( S : \Delta^o \times \Delta^o \to \text{Set} \) is regarded as a simplicial object in the simplicial set category and one takes geometric realizations, then one obtains a simplicial space \( \Delta^o \to \text{Top}, [p] \mapsto |S_{p,*}| \), whose Segal realization is taken to be \( |S| \), the geometric realization of \( S \). As there are natural homeomorphisms \([25, \text{Lemma in p. 86}]\),

\[ |[p] \mapsto |S_{p,*}|| \cong |\text{diag}S| \cong |[q] \mapsto |S_{*,q}|| , \]

where \( \text{diag}S \) is the simplicial set obtained by composing \( S \) with the diagonal functor \( \Delta \to \Delta \times \Delta, [n] \mapsto ([n], [n]) \), one usually takes

\[ |S| = |\text{diag}S|. \]

The following relevant fact is used several times along the paper (see [5, Chapter XII, 4.2 and 4.3] or [16, IV, Proposition 1.7], for example):

**Fact 2.1.** If \( f : S \to S' \) is a bisimplicial map such that the simplicial maps \( f_{p,*} : S_{p,*} \to S'_{p,*} \) (respect. \( f_{*,q} : S_{*,q} \to S'_{*,q} \)) are weak homotopy equivalences for all \( p \) (respect. \( q \)), then so is the map \( \text{diag}f : \text{diag}S \to \text{diag}S' \).

A 2-category \( C \) is just a category enriched in the category of small categories. Then, \( C \) is a category in which the hom-set between any two objects \( x, y \in C \) is the set of objects of a category \( C(x, y) \), whose arrows are called deformations (or 2-cells) and are denoted by \( \alpha : u \Rightarrow v \) and depicted as

\[
\begin{array}{ccc}
x & \downarrow^\alpha & y \\
v \end{array}
\]

Composition in each category \( C(x, y) \), usually referred to as the vertical composition of deformations, is denoted by juxtaposition. Moreover, the horizontal composition is a functor \( C(x, y) \times C(y, z) \to C(x, z) \) that is associative and has identities \( 1_x \in C(x, x) \).

Several times throughout the paper, categories \( C \) are considered as 2-categories in which all deformations are identities, that is, in which each category \( C(x, y) \)
is discrete. For any 2-category \(\mathcal{C}\), \(\mathcal{C}^{o}\) is the 2-category with the same objects as \(\mathcal{C}\) but whose hom-categories are \(\mathcal{C}^{o}(x, y) = \mathcal{C}(y, x)\).

The \textit{nerve} of a 2-category \(\mathcal{C}\) is the simplicial category

\[
\mathcal{N}\mathcal{C} : \Delta^o \to \mathbf{Cat}
\]

whose category of \(n\)-simplices is

\[
\mathcal{N}_n\mathcal{C} = \bigsqcup_{(x_0, \ldots, x_n) \in \text{Ob}\mathcal{C}^{n+1}} \mathcal{C}(x_1, x_0) \times \mathcal{C}(x_2, x_1) \times \cdots \times \mathcal{C}(x_n, x_{n-1}),
\]

where a typical arrow \(\chi\) is a string of deformations in \(\mathcal{C}\)

\[
\chi = x_0 \xrightarrow{\alpha_{0,1}} x_1 \xrightarrow{\alpha_{1,2}} x_2 \cdot \cdot \cdot x_{n-1} \xrightarrow{\alpha_{n-1,n}} x_n,
\]

and \(\mathcal{N}_0\mathcal{C} = \text{Ob}\mathcal{C}\), as a discrete category. The face and degeneracy functors are defined in a standard way by using the horizontal composition in \(\mathcal{C}\), that is, \(d_0(\chi) = (\alpha_{1,2}, \ldots, \alpha_{n-1,n})\), \(d_1(\chi) = (\alpha_{0,1} \circ \alpha_{1,2}, \ldots, \alpha_{n-1,n})\), and so on.

The \textit{classifying space} \(\mathcal{B}\mathcal{C}\) of the 2-category \(\mathcal{C}\) is, by definition, the realization of the simplicial category \(\mathcal{N}\mathcal{C}\), that is,

\[
\mathcal{B}\mathcal{C} = |\text{diag}\mathcal{N}\mathcal{N}\mathcal{C}|,
\]

where \(\mathcal{N}\mathcal{N}\mathcal{C} : ([p], [q]) \mapsto \mathcal{N}_q\mathcal{N}_p\mathcal{C}\) is the bisimplicial set \textit{double nerve} of the 2-category.

\textbf{Example 2.2.} A strict monoidal category \((\mathcal{M}, \otimes)\) [27], that is, an internal monoid in \(\mathbf{Cat}\), can be viewed as a 2-category with only one object, say 1, the objects \(x\) of \(\mathcal{M}\) as morphisms \(x : 1 \to 1\) and the morphisms of \(\mathcal{M}\) as deformations. It is the horizontal composition of morphisms and deformations given by the tensor \(\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}\) and the vertical composition of deformations given by the composition of arrows in \(\mathcal{M}\). The identity at the object is the unit object of the monoidal category.

Then, \(\mathcal{N}(\mathcal{M}, \otimes)\), the nerve of the monoidal category as in [1], is exactly the simplicial category that \((\mathcal{M}, \otimes)\) defines by the reduced bar construction; that is,

\[
\mathcal{N}(\mathcal{M}, \otimes) = \mathcal{W}(\mathcal{M}, \otimes) : \Delta^o \to \mathbf{Cat}, \quad [n] \mapsto \mathcal{M}^n.
\]

Therefore, the ordinary classifying space \(\mathcal{B}(\mathcal{M}, \otimes)\), of the monoidal category, is just the classifying space of the one object 2-category it defines. \(\square\)

A \textit{2-functor} \(F : \mathcal{B} \to \mathcal{C}\) between 2-categories is an enriched functor and so it takes objects, morphisms and deformations in \(\mathcal{B}\) to objects, morphisms and deformations in \(\mathcal{C}\) respectively, in such a way that all the 2-category structure is preserved. It is clear how any 2-functor between 2-categories \(F : \mathcal{B} \to \mathcal{C}\) induces...
a simplicial functor between the corresponding nerves $NF : NB \rightarrow NC$, whence a cellular map on classifying spaces 

$$BF : BB \rightarrow BC.$$ 

Furthermore, as we shall explain below, any normal lax functor, written 

$$F : B \rightsquigarrow C,$$

also induces a map $BF : BB \rightarrow BC$, well defined up to homotopy equivalence. Let us recall that a normal lax functor between 2-categories, $F : B \rightsquigarrow C$, consists of three maps that assign:

- to each object $x \in B$, an object $Fx \in C$;
- to each pair of objects $x, y$ of $B$, a functor $F : B(x, y) \rightarrow C(Fx, Fy)$;
- to each pair of composable arrows $x \xrightarrow{u} y \xrightarrow{v} z$ in $B$, a deformation in $C$

$$F_{u,v} : Fu \circ Fv \Rightarrow F(u \circ v).$$

These data are required to satisfy the normalization condition: $F1_x = 1_{Fx}$, $F_{u,1} = 1_{Fu} = F_{1,u}$; the naturality condition: deformations $F_{u,v}$ are natural in $(u, v) \in B(y, z) \times B(x, y)$; and the coherence (or cocycle) condition that, for any triple of composable arrows $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} t$ in $B$, the square of deformations below commutes.

$$
\begin{array}{ccc}
F_{u,v} \circ 1 & \xrightarrow{1 \circ F_{v,w}} & F_{u} \circ F(v \circ w) \\
F_{u,v} \circ 1 & \xrightarrow{F_{u,v} \circ w} & F_{u,v} \circ Fw \\
F(u \circ v) \circ Fw & \xrightarrow{F_{u,v} \circ Fw} & F(u \circ v \circ w).
\end{array}
$$

A normal lax functor in which all constraints $F_{u,v}$ are identities is precisely a 2-functor.

The geometric nerve of a 2-category $C$, \cite{9, 34}, is defined to be the simplicial set

$$\Delta C = \text{laxFunc}(\cdot, C) : \Delta^o \rightarrow \text{Set},$$

with $\Delta_n C = \text{laxFunc}([n], C)$ the set of normal lax functors $x : [n] \rightsquigarrow C$.

Thus, for a 2-category $C$, the vertices of its geometric nerve $\Delta C$ are the objects $x_0$ of $C$, the 1-simplices are the morphisms $x_0 \xleftarrow{x_{0,1}} x_1$ and the 2-simplices are triangles

$$
\begin{array}{ccc}
x_0 & \xrightarrow{x_{0,1}} & x_1 \\
\downarrow x_{0,2} & \swarrow & \downarrow x_{1,2} \\
x_0 & \xleftarrow{x_{0,1,2}} & x_2
\end{array}
$$
with $x_{0,1,2} : x_{0,1} \circ x_{1,2} \Rightarrow x_{0,2}$ a deformation in $\mathcal{C}$. For $n \geq 3$, an $n$-simplex of $\Delta \mathcal{C}$ consists of a family
\[
x = \{x_i, x_{i,j}, x_{i,j,k}\}_{0 \leq i \leq j \leq k \leq n}
\]
with $x_i$ objects, $x_{i,j} : x_j \to x_i$ morphisms and $x_{i,j,k} : x_{i,j} \circ x_{j,k} \Rightarrow x_{i,k}$ deformations in $\mathcal{C}$, which is geometrically represented by a diagram in $\mathcal{C}$ with the shape of the 2-skeleton of an orientated standard $n$-simplex whose faces are triangles
\[
\begin{array}{c}
\xymatrix{ & x_j \\
& x_{i,j} \\
x_i & x_{i,j} & x_{i,k} \\
& x_{i,j,k} & x_k & \}
\end{array}
\]

These data are required to satisfy the condition that each tetrahedron
\[
\begin{array}{c}
\xymatrix{ & x_l \\
& x_{i,l} \\
x_i & x_{i,j} & x_{i,k} & x_{i,k,l} & x_{i,l} \\
& x_{i,j,k} & x_{i,k,l} & x_{i,l} & \}
\end{array}
\]

for $0 \leq i \leq j \leq k \leq l \leq n$ is commutative in the sense that the following square of deformations
\[
\begin{array}{c}
\xymatrix{ & x_{i,j} \circ x_{j,k} \circ x_{k,l} \\
& x_{i,j} \circ x_{j,l} \\
x_{i,j} \circ 1 & x_{i,j} \circ x_{j,l} & x_{i,j,l} & x_{i,l} & \}
\end{array}
\]

commutes in the category $\mathcal{C}(x_l, x_i)$, and, moreover, the following normalization equations hold:
\[
x_{i,i} = 1_{x_i}, \quad x_{i,j,j} = 1_{x_{i,j}}, \quad x_{i,i,j} = x_{i,j,i}.
\]

Note that, if $\mathcal{C}$ is a category, regarded as a 2-category with all deformations identities, then $\Delta \mathcal{C} = N \mathcal{C}$. As a main result in [7], the following is proved:

**Theorem 2.3.** For any 2-category $\mathcal{C}$ there is a natural homotopy equivalence
\[
B \mathcal{C} \simeq |\Delta \mathcal{C}|.
\]

**Example 2.4.** Let $(\mathcal{M}, \otimes)$ be a strict monoidal category, regarded as a 2-category with only one object as in Example [2.2]. Then, by Theorem 2.3, the geometric nerve $\Delta(\mathcal{M}, \otimes)$ realizes the classifying space of the monoidal category, that is, $B(\mathcal{M}, \otimes) \simeq |\Delta(\mathcal{M}, \otimes)|$. This geometric nerve $\Delta(\mathcal{M}, \otimes)$ is a 3-coskeletal
reduced simplicial set whose simplices have the following simplified interpretation: the 1-simplices are the objects of $M$, the 2-simplices are morphisms of $M$ of the form

$$x_{0,1} \otimes x_{1,2} \rightarrow x_{0,2}$$

and the 3-simplices are commutative squares in $M$ of the form

$$x_{0,1} \otimes x_{1,2} \otimes x_{2,3} \rightarrow x_{0,2} \otimes x_{2,3}$$

$$1 \otimes x_{1,2,3} \rightarrow x_{0,1,3} \rightarrow x_{0,3}$$

The geometric nerve construction on 2-categories $C \mapsto \Delta C$ is clearly functorial on normal lax functors between 2-categories. Therefore, Theorem 2.3 gives the following:

**Corollary 2.5.** Any normal lax functor between 2-categories $F : B \rightarrow C$ induces a continuous map $BF : BB \rightarrow BC$, well defined up to homotopy equivalence.

The following fact in Lemma 2.6 will be needed later. Recall that a lax transformation $\alpha : F \Rightarrow G$, where $F, G : B \rightarrow C$ are normal lax functors between 2-categories, consists of a family of morphisms $\alpha_x : Fx \rightarrow Gx$ in $C$, one for each object $x$ of $B$, and deformations $\alpha_u : \alpha_y \circ Fu \Rightarrow Gu \circ \alpha_x$, which are natural in $u \in B(x, y)$, subject to the usual two axioms: for each object $x$ of $B$, $\alpha_1_x = 1_{\alpha_x}$, and for each pair of composable morphisms $x \rightarrow y \rightarrow z$ in $B$, the diagram below commutes.

Replacing the structure deformations above by $\alpha_u : Gu \circ \alpha_x \Rightarrow \alpha_y \circ Fu$ we have the notion of oplax transformation $\alpha : F \Rightarrow F'$. If $F, F' : B \rightarrow C$ are 2-functors, then a lax (or oplax) transformation $\alpha : F \Rightarrow F'$ whose components $\alpha_u$ at the different morphisms $u$ of $B$ are all identities is called a 2-natural transformation. Thus, a 2-natural transformation is a $\text{Cat}$-enriched natural transformation, that is, a natural transformation between the underlying ordinary functors that also respects 2-cells.
Lemma 2.6. If two normal lax functors between 2-categories, $F, G : B \rightarrow C$, are related by a lax or an oplax transformation, $F \Rightarrow G$, then the induced maps on classifying spaces, $BF, BG : B \rightarrow BC$, are homotopic.

Proof. Suppose $\alpha : F \Rightarrow G : B \rightarrow C$ is a lax transformation. There is a normal lax functor $H : B \times [1] \rightarrow C$ making the diagram commutative

\[
\begin{array}{ccc}
B \times [0] \cong B & \xleftarrow{1 \times \delta_0} & B \times [1] \\
\downarrow \quad & & \downarrow \quad \\
B \times [0] \cong B & \xleftarrow{1 \times \delta_1} & B \times [1]
\end{array}
\]

that carries a morphism in $B \times [1]$ of the form $(x \xrightarrow{v} y, 1 \rightarrow 0) : (x, 1) \rightarrow (y, 0)$ to the composite morphism in $C$

$F_x \xrightarrow{\alpha_x} G_x \xrightarrow{G \alpha_y} G_y,$

and a deformation $(\phi, 1_{1 \rightarrow 0}) : (x \xrightarrow{u} y, 1 \rightarrow 0) \Rightarrow (x \xrightarrow{v} y, 1 \rightarrow 0)$ to

$G \phi \circ 1_{\alpha_x} : G u \circ \alpha_x \Rightarrow G v \circ \alpha_x.$

For $x \xrightarrow{v} y \xrightarrow{u} z$ two composable morphisms in $B$, the structure constraints

$H(y \xrightarrow{u} z, 1 \rightarrow 0) \circ H(x \xrightarrow{v} y, 1 \rightarrow 1) \Rightarrow H(x \xrightarrow{uv} z, 1 \rightarrow 0)$

and

$H(y \xrightarrow{u} z, 0 \rightarrow 0) \circ H(x \xrightarrow{v} y, 1 \rightarrow 0) \Rightarrow H(x \xrightarrow{uv} z, 1 \rightarrow 0)$

are, respectively, given by the composite deformations

$G u \circ \alpha_y \circ F v \xrightarrow{1 \circ \alpha_x} G u \circ G v \circ \alpha_x \xrightarrow{G u \circ \alpha_x} G(u \circ v) \circ \alpha_x$

and

$G u \circ G v \circ \alpha_x \xrightarrow{G u \circ \alpha_x} G(u \circ v) \circ \alpha_x.$

Applying geometric nerve construction to diagram (3), we obtain a diagram of simplicial set maps

\[
\begin{array}{ccc}
\Delta B \times \Delta [0] \cong \Delta B & \xleftarrow{\Delta 1 \times \delta_0} & \Delta B \times \Delta [1] \\
\downarrow \quad & & \downarrow \quad \\
\Delta B \times \Delta [0] \cong \Delta B & \xleftarrow{\Delta 1 \times \delta_1} & \Delta B \times \Delta [1]
\end{array}
\]
showing that the simplicial maps \( \Delta F, \Delta G : \Delta B \to \Delta C \) are made homotopic by \( \Delta H \), whence the lemma follows by Theorem 2.3.

The proof is similar for the case in which \( \alpha : F \Rightarrow G : B \Rightarrow C \) is an oplax transformation, but with a change in the construction of the lax functor \( H : B \times [1] \Rightarrow C \) that makes diagram (3) commutative: now define \( H \) such that
\[
H(x \to y, 1 \to 0) = \alpha_y \circ Fu : Fx \to Gy.
\]

\[\square\]

3. Homotopy cartesian squares induced by 2-functors

Suppose \( F : B \to C \) any given 2-functor between 2-categories \( B \) and \( C \). For each object \( z \) of \( C \), by the homotopy fibre 2-category \( z//F \) we mean Gray’s lax comma category \( \lbrack z \rbrack//F \) where \( \lbrack z \rbrack : 1 \Rightarrow C \) is the “name of an object” 2-functor \([18, \S 3.1]\). Its objects are the pairs \( (x, v) \), where \( x \) is an object of \( B \) and \( v : z \Rightarrow Fx \) is a morphism in \( C \). A morphism \( (u, \beta) : (x, v) \Rightarrow (x', v') \) consists of a morphism \( u : x \Rightarrow x' \) in \( B \) together with a deformation \( \beta : Fu \circ v \Rightarrow v' \) in \( C \)

\[
\begin{array}{ccc}
Fx & \xrightarrow{v} & z \\
\downarrow{\beta} & & \\
Fx' & \xleftarrow{v'} & \end{array}
\]

and a deformation

\[
\begin{array}{ccc}
(u, \beta) & \xleftarrow{(x, v)} & (x', v') \\
\downarrow{\alpha} & & \\
(u', \beta') & \end{array}
\]

is a deformation \( \alpha : u \Rightarrow u' \) in \( B \) such that the diagram in \( C(z, Fx') \)

\[
\begin{array}{ccc}
Fu \circ v & \xrightarrow{\beta} & v' \\
\downarrow{\alpha} & & \\
F\alpha \circ 1_v & \xrightarrow{\beta'} & \\
\end{array}
\]

commutes. Compositions in \( z//F \) derive naturally from those in \( B \) and \( C \).

Similarly, one has the homotopy fibre 2-categories \( F//z \), with objects the pairs \( (x, Fx \Rightarrow z) \). The arrows \( (u, \beta) : (x, v) \Rightarrow (x', v') \), are pairs where \( u : x \Rightarrow x' \) is a morphism in \( B \) and \( \beta : v' \Rightarrow Fu \Rightarrow v \) is a deformation in \( C \); and a deformation \( \alpha : (u, \beta) \Rightarrow (u', \beta') : (x, v) \Rightarrow (x', v') \) is a deformation \( \alpha : u \Rightarrow u' \) in \( B \) such that \( \beta'(1_v \circ F\alpha) = \beta \).

In particular, when \( F = 1_C \) is the identity 2-functor on \( C \) we have the comma 2-categories \( z//C \), of objects under an object \( z \), and \( C//z \), of objects over \( z \). The following fact is proved in \([11, \text{Proposition 4.1}]\):
Lemma 3.1. For any object \( z \) of a 2-category \( C \), the classifying spaces of the comma 2-categories \( B(z//C) \) and \( B(C//z) \) are contractible.

PROOF. We shall discuss the case of \( B(z//C) \). Let \( C_t : z//C \rightarrow z//C \) be the constant 2-functor sending every object of \( z//C \) to \( (z, 1_z) \), every morphism to the identity morphism \( 1_{(z, 1_z)} = (1_z, 1_{1_z}) \) and every deformation to the identity deformation \( 1_{1_z} : 1_{(z, 1_z)} \Rightarrow 1_{(z, 1_z)} \). There is a canonical oplax transformation \( C_t \Rightarrow 1_{z//C} \) whose component at an object \( (x, v) \rightarrow (x, v) \) is the morphism \( (v, 1_v) : (1_z, 1_z) \Rightarrow (1_z, 1_z) \). From Lemma 2.6 it follows that \( B(1_{z//C}) = 1_{B(z//C)} \) and \( B C_t = C t B \) are homotopic, whence the lemma. \( \square \)

Returning to an arbitrary 2-functor \( F : B \rightarrow C \), observe that, for any object \( z \) of \( C \), there is a 2-functor \( \Phi : z//F \rightarrow B \), defined by forgetting the second components

\[
(x, v) \xrightarrow{\alpha} (x', v') \xrightarrow{\Phi} x \xrightarrow{\alpha} x',
\]

and there is a commutative square of 2-functors

\[
\begin{array}{ccc}
z//F & \xrightarrow{\Phi} & B \\
F' \downarrow & & \downarrow F \\
z//C & \xrightarrow{\Phi} & C,
\end{array}
\]

where \( F' : z//F \rightarrow z//C \) is given by

\[
(x, v) \xrightarrow{\alpha} (x', v') \xrightarrow{F'} (Fx, v) \xrightarrow{\alpha} (Fx', v').
\]

Moreover, for any morphism \( w : z_1 \rightarrow z_0 \) in \( C \) the assignment

\[
(x, v) \xrightarrow{\alpha} (x', v') \xrightarrow{w^*} (x, v \circ w) \xrightarrow{\alpha} (x', v' \circ w),
\]

defines a 2-functor \( w^* : z_0//F \rightarrow z_1//F \), and the analogous to Quillen’s Theorem B for 2-functors is stated as follows:
Theorem 3.2. Let $F : B \to C$ be a 2-functor such that, for every morphism $z_1 \to z_0$ in $C$, the induced map $B(z_0//F) \to B(z_1//F)$ is a homotopy equivalence. Then, for every object $z$ of $C$, the induced square by (4) on classifying spaces

$$
\begin{array}{ccc}
B(z//F) & \to & B\mathcal{B} \\
\downarrow & & \downarrow \\
B(z//C) & \to & BC
\end{array}
$$

is homotopy cartesian. Therefore, for each object $x \in F^{-1}z$, there is a homotopy fibre sequence

$$B(z//F) \to B\mathcal{B} \to BC,$$

relative to the base points $z$ of $BC$, $x$ of $B\mathcal{B}$ and $(x,1_z)$ of $B(z//F)$, that induces a long exact sequence on homotopy groups

$$\cdots \to \pi_{n+1}BC \to \pi_nB(z//F) \to \pi_nB\mathcal{B} \to \pi_nBC \to \cdots.$$ 

The remainder of this section is mainly devoted to the proof of this theorem. To do so, we shall start with the following construction of 2-categories $[q]//F$, one for each integer $q \geq 0$, which, in the case of $q = 0$ and up to almost obvious identification, yields the coproduct $\bigsqcup_z z//F$, of the different homotopy fibre 2-categories:

For each $q \geq 0$, let $[q]//F$ be the 2-category whose objects are pairs $(x,v)$, where $x$ is an object of $B$ and $v : [q+1] \to C$ is a normal lax functor such that $v_0 =Fx$. A morphism $(u,y)$ consists of a morphism $u : x \to x'$ in $B$ together with a normal lax functor $y : [q+2] \to C$ such that $y_0 = Fu$; the source of $(u,y)$ is $(x,y\delta_0)$ and its target is $(x',y\delta_1)$. Note that, since the square

$$
\begin{array}{ccc}
[q] & \xleftarrow{\delta_0} & [q+1] \\
\delta_0 & \downarrow & \delta_1 \\
[q+1] & \xrightarrow{\delta_0} & [q+2]
\end{array}
$$

is commutative and cartesian in $\mathbf{Cat}$, given two objects $(x,v)$ and $(x',v')$ of $[q]//F$, the existence of a morphism between them requires that $v\delta_0 = v'\delta_0$, and then such a morphism $(u,y) : (x,v) \to (x',v')$ is completely specified by the morphism $u : x \to x'$ and the deformations

$$
\begin{array}{ccc}
Fx & \xleftarrow{v_0,i+1} & Fz \\
\downarrow \downarrow & & \downarrow \downarrow \\
v_0,i+1 & \leftarrow & v_{i+1}
\end{array}
$$
for $0 \leq i \leq q$. A deformation in $[q]//F$

$$
\begin{array}{c}
(u, y) \\
\downarrow \alpha \\
(u', y')
\end{array}
\xrightarrow{(x, v)}
\begin{array}{c}
(u, y) \\
\downarrow \alpha \\
(u', y')
\end{array}
$$

is a deformation $\alpha : u \Rightarrow u'$ in $\mathcal{B}$ such that the triangles

$$
\begin{array}{ccc}
Fu' \circ v_{0,i+1} & \xrightarrow{y_{0,1,i+2}} & v'_{0,i+1} \\
F\alpha \circ 1 & \Downarrow & y_{0,1,i+2} \\
Fu' \circ v_{0,i+1} & \xrightarrow{y_{0,1,i+2}} & v'_{0,i+1}
\end{array}
$$

commute, for $0 \leq i \leq q$.

Compositions in $[q]//F$ come from those in $\mathcal{B}$ and $\mathcal{C}$ in the natural way. Thus, a morphism $(u, y) : (x, v) \rightarrow (x', v')$ composes horizontally with a morphism $(u', y') : (x', v') \rightarrow (x'', v'')$ yielding the morphism $(u' \circ u, y'') : (x, v) \rightarrow (x'', v'')$, where, for $0 \leq i \leq q$, the deformation $y''_{0,1,i+2} : F(u' \circ u)_{0,i+1} \Rightarrow v''_{0,i+1}$ is the composition

$$
Fu' \circ Fu \circ v_{0,i+1} \xrightarrow{1 \circ y_{0,1,i+2}} F_{u'} \circ v'_{0,i+1} \xrightarrow{y'_{0,1,i+2}} v''_{0,i+1}.
$$

The identity morphism for each object in $[q]//F$ is provided by the surjection $\sigma_0 : [q+1] \rightarrow [q]$ which repeats the 0 element, by $1_{(x, v)} = (x, v\sigma_0)$. And deformations in $[q]//F$ compose, both horizontally and vertically, as in $\mathcal{B}$.

Similarly, one has the 2-categories

$$
F//[q], \quad q \geq 0,
$$

with objects the pairs $(x, v)$, where $x$ is an object of $\mathcal{B}$ and $v : [q+1] \rightarrow \mathcal{C}$ is a normal lax functor such that $v_{q+1} = Fx$. The morphisms $(u, y) : (x, v) \rightarrow (x', v')$, are pairs where $u : x \rightarrow x'$ is a morphism in $\mathcal{B}$ and $y : [q+2] \rightarrow \mathcal{C}$ is a normal lax functor such that $y_{q+1,q+2} = Fu$, $y\delta_{q+1} = v$ and $y\delta_{q+2} = v'$; and a deformation $\alpha : (u, y) \Rightarrow (u', y') : (x, v) \rightarrow (x', v')$ is a deformation $\alpha : u \Rightarrow u'$ in $\mathcal{B}$ such that $y'_{i,q+1,q+2}(1_{v'_{i,q+2}} \circ F\alpha) = y_{i,q+1,q+2},$ for $0 \leq i \leq q$.

Regarding the set $\Delta_q \mathcal{C}$, of geometric $q$-simplices of the 2-category $\mathcal{C}$, as a discrete 2-category whose morphisms and deformations are all identities, the 2-functors

$$
\Psi_q : [q]//F \longrightarrow \Delta_q \mathcal{C}, \quad \Psi'_q : F//[q] \longrightarrow \Delta_q \mathcal{C}
$$

are respectively defined by

$$
(x, v) \mapsto v\delta_0, \quad (x, v) \mapsto v\delta_{q+1}.
$$
And these 2-functors give rise to a decomposition of the 2-categories \([q]/F\) and \(F/[q]\) as

\[
[q]/F = \bigsqcup_{z: [q] \to C} z//F, \quad F/[q] = \bigsqcup_{z: [q] \to C} F//z,
\]

where, for each geometric \(q\)-simplex \(z: [q] \hookrightarrow C\), the homotopy fibre 2-categories of \(F\) at \(z\)

\[
z//F := \Psi_q^{-1}(z), \quad F//z := \Psi_q^{-1}(z),
\]

are respectively defined to be the full 2-subcategories of \([q]/F\) and \(F/[q]\) fibre of \(\Psi_q\) and \(\Psi'_q\) at \(z\).

In particular, when \(F = 1_C\) is the identity 2-functor on \(C\) we have the comma 2-categories

\[
z//C, \quad C//z,
\]

of objects under and over a normal lax functor \(z: [q] \to C\).

The following required lemma does not use any hypothesis on the 2-functor \(F\) in Theorem (3.2).

**Lemma 3.3.** Let \(z: [q] \to C\) be any given normal lax functor. Then,

(i) There are 2-functors

\[
\Gamma: z_0//F \to z//F, \quad \Gamma': F//z_q \to F//z,
\]

both inducing homotopy equivalences on classifying spaces

\[
B(z_0//F) \simeq B(z//F), \quad B(F//z_q) \simeq B(F//z).
\]

(ii) The classifying spaces of the comma 2-categories \(B(z//C)\) and \(B(C//z)\) are contractible.

**Proof.** (i) We shall discuss the case of \(\Gamma\), which is defined as follows: It carries an object \((x, v)\) of \(z_0//F\) to the object \((x, v^z)\) of \(z//F\), where \(v^z: [q+1] \to C\) is the normal lax functor defined by the equalities

\[
v^z_i \delta_0 = z, \quad v^z_0 = Fx,
\]

\[
v^z_{0,i+1} = v \circ z_0,i : z_i \to z_0 \Rightarrow v^z_i \Rightarrow Fx,
\]

\[
v^z_{0,i+1,j+1} = 1_{v} \circ z_{0,i,j} : v \circ z_{0,i} \circ z_{i,j} \Rightarrow v \circ z_{0,j}.
\]

A morphism \((u, y): (x, v) \to (x', v')\) in \(z_0//F\) is carried by \(\Gamma\) to the morphism \((u, y^z): (x, v^z) \to (x', v'^z)\) of \(z//F\) specified by the deformations

\[
y^z_{0,1,i+2} = y \circ z_{0,i} : Fv \circ z_{0,i} \Rightarrow v' \circ z_{0,i},
\]

for \(0 \leq i \leq q\), and on deformations, the 2-functor acts by the simple rule \(\Gamma(\alpha) = \alpha\).
Actually, this 2-functor \( \Gamma \) embeds \( z_0//F \) into \( z//F \) as a deformation retract, with retraction given by the 2-functor

\[
\Theta : z//F \rightarrow z_0//F,
\]

\[
\begin{array}{c}
(\alpha, y) \\
\downarrow \alpha
\end{array}
\xrightarrow{\Theta}
\begin{array}{c}
(x, v) \\
\downarrow \alpha
\end{array}
\xrightarrow{\Theta}
\begin{array}{c}
(x_0, v) \\
\downarrow \alpha
\end{array}
\xrightarrow{\Theta}
\begin{array}{c}
(x', v') \\
\downarrow \alpha
\end{array}
\xrightarrow{\Theta}
\begin{array}{c}
(x', v) \\
\downarrow \alpha
\end{array}
\xrightarrow{\Theta}
\begin{array}{c}
(x', v') \\
\downarrow \alpha
\end{array}.
\]

One observes that \( \Theta \Gamma = 1_{z_0//F} \). Furthermore, there is a 2-natural transformation \( \Gamma \Theta \Rightarrow 1_{z//F} \), whose component at an object \((x, v)\) of \( z//F \) is the morphism \( (x, v) : (x, v_0, v_1) \rightarrow (x, v) \) with

\[
\tilde{v}_{0,1,i+2} = v_{0,1,i+1} \circ z_0 = v_{0,1,i+1}.
\]

for \( 0 \leq i \leq q \). Then, by Lemma 2.6 it follows that the induced map \( B\Gamma \) embeds the space \( B(z_0//F) \) into \( B(z//F) \) as a deformation retract, with \( B\Theta \) a retraction.

The proof for \( \Gamma' \) is similar. It carries an object \((x, v)\) of \( F//z \) to the object \((x, zv)\) of \( F//z \), where \( zv_{i,q+1} = zv_{i,q} \circ v \), etc.

(ii) After what has already been proven above, the result follows from Lemma 3.1.

\[\square\]

Actually, we have a simplicial 2-category

\[
[-][]//F = \bigsqcup_{q \geq 0} [q][]//F,
\]

in which the induced 2-functor \( \xi^* : [n][]//F \rightarrow [q][]//F \), by a map \( \xi : [q] \rightarrow [n] \) in the simplicial category, is given by

\[
\begin{array}{c}
(\alpha, y) \\
\downarrow \alpha
\end{array}
\xrightarrow{\xi^*}
\begin{array}{c}
(x, v) \\
\downarrow \alpha
\end{array}
\xrightarrow{\xi^*}
\begin{array}{c}
(x, v) \\
\downarrow \alpha
\end{array}
\xrightarrow{\xi^*}
\begin{array}{c}
(x', v') \\
\downarrow \alpha
\end{array}
\xrightarrow{\xi^*}
\begin{array}{c}
(x', v') \\
\downarrow \alpha
\end{array}
\xrightarrow{\xi^*}
\begin{array}{c}
(x', v') \\
\downarrow \alpha
\end{array}.
\]

where, for any integer \( p \geq 0 \), the map \( \xi + p : [q + p] \rightarrow [n + p] \) is

\[
(\xi + p)(i) = \begin{cases} 
i & \text{if } 0 \leq i < p, \\ \xi(i + p) & \text{if } p \leq i \leq q + p. \end{cases}
\]

Note that, for any \( \xi : [q] \rightarrow [n] \), the square below in the simplicial category commutes.

\[
\begin{array}{ccc}
[q] & \xrightarrow{\xi} & [n] \\
\delta_0 \downarrow & & \downarrow \delta_0 \\
[q+1] & \xrightarrow{\xi+1} & [n+1]
\end{array}
\]

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Hence, for any \( z : [n] \to C \), the 2-functor \( \xi^* : [n]//F \to [q]//F \) maps the homotopy fibre 2-category \( z//F \) into \( z\xi//F \). As a crucial result in our discussion for proving Theorem 3.2, we have the following:

**Lemma 3.4.** Under the hypothesis in Theorem 3.2, for any given normal lax functor \( z : [n] \to C \) and map in the simplicial category \( \xi : [q] \to [n] \), the 2-functor

\[
\xi^* : z//F \to z\xi//F
\]

induces a homotopy equivalence on classifying spaces \( B(z//F) \simeq B(z\xi//F) \).

**Proof.** We have the square of 2-functors

\[
\begin{array}{ccc}
z_0//F & \xrightarrow{\Gamma_z} & z//F \\
\downarrow \xi^* & & \downarrow \xi^* \\
z_0,\xi(0)//F & \xrightarrow{\Gamma_{z,\xi}} & z\xi//F,
\end{array}
\]

where the \( \Gamma' s \) are those 2-functors in Lemma 3.3(i) corresponding to \( z \) and \( z\xi \) respectively. The two composite 2-functors in the square are related by a 2-natural transformation \( \Gamma_{z,\xi} \xi^* z_0,\xi(0) \Rightarrow \xi^* \Gamma_z \), whose component at an object \((x, v)\) of \( z_0//F \) is the morphism

\[
(1_x, y) : (x, (v \circ z_0,\xi(0))z\xi) \to (x, v^* (\xi + 1))
\]

in \( z\xi//F \) with

\[
y_{0, i + 2} = 1_v \circ z_0,\xi(0),\xi(i) : v \circ z_0,\xi(0) \circ z_\xi(0),\xi(i) \Rightarrow v \circ z_0,\xi(i),
\]

for \( 0 \leq i \leq q \). Hence, by Lemma 2.6 the induced square on classifying spaces

\[
\begin{array}{ccc}
B(z_0//F) & \xrightarrow{B\Gamma_z} & B(z//F) \\
\downarrow B\xi^* & & \downarrow B\xi^* \\
B(z_0,\xi(0)//F) & \xrightarrow{B\Gamma_{z,\xi}} & B(z\xi//F)
\end{array}
\]

is homotopy commutative; that is, there is a homotopy \( B\Gamma_{z,\xi} B\xi^* z_0,\xi(0) \simeq B\xi^* B\Gamma_z \). Since both maps \( B\Gamma_z \) and \( B\Gamma_{z,\xi} \) are homotopy equivalences by Lemma 3.3 above, and, by hypothesis, \( B\xi^* z_0,\xi(0) \) is also a homotopy equivalence, the result follows.

\( \square \)

The next lemma plays, in our discussion, the role of the relevant Lemma 5.3, p. 90 in Quillen’s proof of his Theorem B for functors. The simplicial 2-category \( [\square] \) has a classifying space \( B([-]//F) \); namely, the geometric realization of the simplicial space \([q] \to B([q]//F)\) obtained by replacing each 2-category \([q]//F\) by its classifying space, or

\[
B([-]//F) = |\text{diag}\Delta([-]//F)|,
\]

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where \( \Delta([-]//F) : (p, q) \mapsto \Delta_p([q]//F) \) is the bisimplicial set obtained by applying the geometric nerve functor to the simplicial 2-category \([-]//F\). Furthermore, regarding the simplicial set \( \Delta C \), geometric nerve of the 2-category \( C \), as a simplicial discrete 2-category whose morphisms and deformations are all identities, a simplicial 2-functor

\[
\Psi : [-]//F \rightarrow \Delta C, \quad (x, v) \mapsto v\delta_0,
\]

is defined by the 2-functors (9). This 2-functor yields an induced map on classifying spaces \( B([-]//F) \rightarrow BC \), and we have the following:

**Lemma 3.5.** For any object \( z_0 \) of \( C \), the induced square by the inclusion of \( z_0//F \) into \([-]//F\) and the simplicial 2-functor (9)

\[
\begin{array}{ccc}
B(z_0//F) & \longrightarrow & B([-]//F) \\
\downarrow & & \downarrow B\Psi \\
pt & \longrightarrow & BC,
\end{array}
\]

where \( pt \) is the one-point space, is homotopy cartesian. That is, \( B(z_0//F) \) is homotopy equivalent to the homotopy fibre of \( B\Psi \), relative to the base point \( z_0 \) of \( BC \).

**Proof.** By Theorem 2.3, it suffices to prove that the diagram of bisimplicial sets

\[
\begin{array}{ccc}
\Delta(z_0//F) & \longrightarrow & \Delta([-]//F) \\
\downarrow & & \downarrow \Delta\Psi \\
\Delta[0] & \longrightarrow & \Delta C
\end{array}
\]

induces on diagonals a homotopy cartesian square of simplicial maps. And, to do that, we are going to prove the following two facts:

(i) The pullback square of bisimplicial sets

\[
\begin{array}{ccc}
\Delta[0] \times_{\Delta C} \Delta([-]//F) & \longrightarrow & \Delta([-]//F) \\
\downarrow & & \downarrow \Delta\Psi \\
\Delta[0] & \longrightarrow & \Delta C
\end{array}
\]

induces on diagonals a homotopy cartesian square of simplicial maps.

(ii) The induced map on diagonals by the bisimplicial map

\[
\Delta(z_0//F) \rightarrow \Delta[0] \times_{\Delta C} \Delta([-]//F)
\]

is a weak homotopy equivalence.
Recall that the homotopy fibre of $\text{diag}\Delta\Psi$ at $z_0$ is the pullback

$$Y \times_{\Delta C} \text{diag}\Delta([\cdot]/F),$$

where $\Delta[0] \to Y \to \Delta C$ is a (any) factorization of $\Delta[0] \xrightarrow{z_0} \Delta C$ into a trivial cofibration (= injective weak equivalence) followed by a Kan fibration. By using the “small object argument” \cite{24, 16} to find such a factorization, the trivial cofibration $\Delta[0] \to Y$ is a colimit of a sequence of pushouts of coproducts of simplicial inclusions $\Lambda^k[n] \hookrightarrow \Delta[n]$, of $k^\text{th}$-horn subcomplexes of standard $n$-simplex simplicial complexes. Since pullbacks along $\text{diag}\Delta\Psi$ commute with colimits in the comma category of simplicial sets over $\Delta C$, to prove (i) it is sufficient to show that for each composite

$$\Lambda^k[n] \hookrightarrow \Delta[n] \xrightarrow{z} \Delta C,$$

with $z_0$ the given, the bisimplicial map obtained by pulling back along $\Delta\Psi$

$$\Lambda^k[n] \times_{\Delta C} \Delta([-]/F) \longrightarrow \Delta[n] \times_{\Delta C} \Delta([-]/F)$$  \hspace{1cm} (12)

induces a weak homotopy equivalence on diagonals.

Fix a simplicial map $z : \Delta[n] \to \Delta C$, that is, a normal lax functor $z : [n] \Rightarrow C$.

The pullback square of bisimplicial sets

$$
\begin{array}{ccc}
\Delta[n] \times_{\Delta C} \Delta([-]/F) & \longrightarrow & \Delta([-]/F) \\
\downarrow & & \downarrow \\
\Delta[n] & \xrightarrow{z} & \Delta C,
\end{array}
$$

is that induced on geometric nerves by the pullback of simplicial 2-categories

$$
\begin{array}{ccc}
\Delta[n] \times_{\Delta C} [-]/F & \longrightarrow & [-]/F \\
\downarrow & & \downarrow \\
\Delta[n] & \xrightarrow{z} & \Delta C,
\end{array}
$$

where

$$\Delta[n] \times_{\Delta C} [-]/F = \bigsqcup_{q} \bigsqcup_{[q] \to [n]} z\xi/F,$$

is the simplicial 2-subcategory of $[-]/F$ generated by the 2-subcategory $z/F$ of $[n]/F$.

On considering the product simplicial 2-category

$$\Delta[n] \times z_0/F = \bigsqcup_{q} \bigsqcup_{[q] \to [n]} z_0/F,$$
we have a simplicial 2-functor
\[
\gamma : \Delta[n] \times z_0//F \longrightarrow \Delta[n] \times \Delta C [-]//F,
\]
where \(\Gamma\) is the 2-functor (7). Since, for any \(q \geq 0\), the map induced by \(\gamma_q\) on geometric nerves
\[
\bigsqcup_{[q] \hookrightarrow [n]} \left( \Delta(z_0//F) \xrightarrow{\Delta \Gamma} \Delta(z//F) \xrightarrow{\Delta \xi^*} \Delta(z\xi//F) \right)
\]
is a coproduct of weak homotopy equivalences, by Lemmas 3.3 and 3.4, it follows that the simplicial map induced on diagonals
\[
\Delta \gamma : \Delta[n] \times \Delta(z_0//F) \longrightarrow \mathrm{diag} \left( \Delta[n] \times \Delta([-]//F) \right)
\]
is a weak homotopy equivalence as well, by Fact 2.1.

Similarly, the bisimplicial set \(\Lambda^k[n] \times \Delta C [-]//F\) is the geometric nerve of the simplicial 2-category
\[
\Lambda^k[n] \times \Delta C [-]//F = \bigsqcup_{q \geq 0} \bigsqcup_{[q] \hookrightarrow [n]} z\xi//F,
\]
and, by taking into account the product simplicial 2-category
\[
\Lambda^k[n] \times z_0//F = \bigsqcup_{q \geq 0} \bigsqcup_{[q] \hookrightarrow [n]} z_0//F,
\]
the simplicial 2-functor \(\gamma\) above restricts to a simplicial 2-functor
\[
\gamma' : \Lambda^k[n] \times z_0//F \longrightarrow \Lambda^k[n] \times \Delta C [-]//F,
\]
such that the map induced on diagonals
\[
\Delta \gamma' : \Lambda^k[n] \times \Delta(z_0//F) \longrightarrow \mathrm{diag} \left( \Lambda^k[n] \times \Delta([-]//F) \right)
\]
is also a weak homotopy equivalence of simplicial sets.

Since the square of simplicial maps
\[
\begin{array}{ccc}
\Lambda^k[n] \times \Delta(z_0//F) & \xrightarrow{\Delta \gamma'} & \Delta[n] \times \Delta(z_0//F) \\
\Delta \gamma | & & | \Delta \gamma \\
\mathrm{diag}(\Lambda^k[n] \times \Delta C [-]//F) & \longrightarrow & \mathrm{diag}(\Delta[n] \times \Delta C [-]//F)
\end{array}
\]
is commutative, the vertical maps are weak homotopy equivalences, and the top map is the trivial cofibration product of the inclusion \( \Lambda^k[n] \to \Delta[n] \) with the identity map on \( \Delta(z_0/F) \), it follows that the bottom map, that is, that induced on diagonals by (12), is a weak homotopy equivalence of simplicial sets, as claimed. This proves (i), but actually we have also shown (ii) since the induced on diagonals by the map (11) is precisely the weak equivalence (14) for the case \( n = 0 \). □

With the following lemma we will be ready to complete the proof of Theorem 3.2. For any \( q \geq 0 \), forgetting in the second component gives a 2-functor

\[
\Phi : [\neg]/F \to B,
\]

and we have an augmented simplicial 2-category

\[
[-]/F \xrightarrow{\Phi} B. \tag{15}
\]

Lemma 3.6. The simplicial 2-functor (15) induces a homotopy equivalence on classifying spaces, \( B([-]/F) \simeq BB \).

Proof. By Theorem 3.3 it suffices to prove that the bisimplicial map

\[
\Delta \Phi : \Delta([-]/F) \to \Delta B
\]

induces a weak equivalence \( \text{diag} \Delta \Phi : \text{diag} \Delta([-]/F) \to \Delta B \). Now, for every \( p \geq 0 \), the augmentation

\[
\Delta_p \Phi : \Delta_p([-]/F) = \bigsqcup_{q \geq 0} \Delta_p([q]/F) \to \Delta_p B
\]

is actually a weak equivalence since, for any geometric simplex \( x : [p] \to B \) of \( B \), the fibre \( (\Delta_p \Phi)^{-1}(x) \) is precisely \( \Delta(C/Fx) \), the geometric nerve of the comma category \( C/Fx \), which, by Lemma 3.3 (ii), has a contractible classifying space; that is,

\[
|\Delta_p([-]/F)| = \bigsqcup_{x : [p] \to B} B(C/Fx) \simeq \bigsqcup_{x : [p] \to B} pt \simeq \Delta_p B.
\]

Therefore, by Fact 2.1 the map induced on diagonals \( \text{diag} \Delta \Phi \) is also a weak homotopy equivalence, as claimed. □
Consider the diagram of spaces

\[
\begin{array}{ccc}
B(z_0//F) & \longrightarrow & B([-]//F) \\
\downarrow^{BF'} & & \downarrow^{B\Phi} \\
B([-]//F)(a) & \longrightarrow & BF
\end{array}
\]

\[
\begin{array}{ccc}
B(z_0//C) & \longrightarrow & B([-]//1_C) \\
\downarrow^{B\Phi} & & \downarrow^{BF}
\end{array}
\]

\[
\begin{array}{ccc}
pt & \longrightarrow & B\Psi \\
\downarrow^{z_0} & & \downarrow^{BF}
\end{array}
\]

in which the simplicial 2-functor \( F' : [-]/F \to [-]/1_C \) is given by

\[
\begin{array}{ccc}
(x, v) & \downarrow^\alpha & (x', v') \\
(u, y) & \downarrow^F & (Fx, v) \\
(u', y') & \downarrow^{F\alpha} & (Fx', v')
\end{array}
\]

By Lemma 3.5, both squares \((a) + (b)\) and \((b)\) are homotopy cartesian. It follows that \((a)\) is homotopy cartesian. Hence, \((a) + (c) = (5)\) is also since both maps \(B\Phi\) are homotopy equivalences byLemma 3.6, whence the theorem. \(\square\)

The following corollary is a direct generalization of [25, Theorem A], and it was proved in [7]:

**Corollary 3.7.** Let \( F : B \to C \) be a 2-functor between 2-categories. If the classifying space \( B(z//F) \) is contractible for every object \( z \) of \( C \), then the induced map \( BF : BB \to BC \) is a homotopy equivalence.

**Example 3.8.** Recall that the classifying space of a strict monoidal category is the classifying space of the one-object 2-category it defines (see examples 2.2 and 2.4). Then, Theorem 3.2 is applicable to strict monoidal functors (= 2-functors) between strict monoidal categories.

However, we should stress that the homotopy fibre 2-category of a strict monoidal functor \( F : (M, \otimes) \to (M', \otimes) \), at the unique object of the 2-category defined by \((M', \otimes)\), is not a monoidal category but a genuine 2-category: Its objects are the objects \( x' \in M' \), its morphisms are pairs \((x, u') : x' \to y'\) with \( x \) an object in \( M \) and \( u' : F(x) \otimes x' \to y'\) a morphism in \( M' \), and its deformations

\[
\begin{array}{ccc}
(x, u') & \downarrow^u & (y, v') \\
x' & \downarrow^F & y'
\end{array}
\]

The following corollary is a direct generalization of [25, Theorem A], and it was proved in [7]:

**Corollary 3.7.** Let \( F : B \to C \) be a 2-functor between 2-categories. If the classifying space \( B(z//F) \) is contractible for every object \( z \) of \( C \), then the induced map \( BF : BB \to BC \) is a homotopy equivalence.
are those morphisms \( u : x \to y \) in \( \mathcal{M} \) such that the following triangle commutes

\[
\begin{array}{ccc}
F_x \otimes x' & \xrightarrow{u'} & y' \\
Fu \otimes 1_{x'} & \downarrow & \downarrow \\
Fy \otimes x' & \xrightarrow{v'} & \\
\end{array}
\]

In [7], this 2-category was called the homotopy fibre 2-category of the monoidal functor \( F : (\mathcal{M}, \otimes) \to (\mathcal{M}', \otimes) \), and it was denoted by \( \mathcal{K}_F \). Every object \( z' \) of \( \mathcal{M}' \) determines a 2-endofunctor \(- \otimes z' : \mathcal{K}_F \to \mathcal{K}_F\), defined by

\[
(\langle x, u' \rangle, x') \quad \xrightarrow{\quad \otimes z' \quad} \quad (\langle x, u' \otimes 1_{z'} \rangle, x')
\]

and, by Theorem 3.2 whenever the induced maps \( B(\otimes z') : \mathcal{K}_F \to \mathcal{K}_F \), for all \( z' \in \text{Ob} \mathcal{M}' \), are homotopy auto-equivalences, there is a homotopy fibre sequence

\[
\mathcal{K}_F \to B(\mathcal{M}, \otimes) \to B(\mathcal{M}', \otimes).
\]

The reader interested in the study of classifying spaces of monoidal categories can find in the above fact a good reason to also be interested in the study of classifying spaces of 2-categories.

4. Homotopy fibre sequences induced by lax 2-diagrams

Theorem 3.2 can be applied to homotopy theory of lax functors \( F : \mathcal{C} \to \mathbf{2Cat} \), where \( \mathcal{C} \) is any 2-category, hence to acting monoidal categories, through an enriched Grothendieck construction

\[
\int_{\mathcal{C}} F,
\]

that we explain in Definition 4.1 below.

Hereafter, \( \mathbf{2Cat} \) means the 2-category whose objects are 2-categories \( \mathcal{A} \), morphisms the 2-functors \( F : \mathcal{A} \to \mathcal{B} \), and deformations \( \xrightarrow{\alpha} \mathcal{B} \) the 2-natural transformations \( \alpha : F \Rightarrow F' \), as we recalled in the preliminary Section 2. To set some additional notations, let us briefly say that, in the 2-categorical structure of \( \mathbf{2Cat} \), a 2-natural transformation \( \alpha : F \Rightarrow F' \) composes vertically with a 2-natural transformation \( \alpha' : F' \Rightarrow F'' \), that is, in the category \( \mathbf{2Cat}(\mathcal{A}, \mathcal{B}) \), yielding the 2-natural transformation \( \alpha \circ \alpha' : F \Rightarrow F'' \), whose component at an object \( x \) of \( \mathcal{A} \) is \( (\alpha' \circ \alpha)_x = \alpha'_x \circ \alpha_x \), the horizontal composition in \( \mathcal{B} \) of the morphisms

\[
F_x \xrightarrow{\alpha_x} F'_x \xrightarrow{\alpha'_x} F''_x.
\]

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Composition of 2-functors $\mathcal{A} \overset{F}{\to} \mathcal{B} \overset{G}{\to} \mathcal{C}$, which we are denoting by juxtaposition, that is, $\mathcal{A} \overset{GF}{\to} \mathcal{C}$, is the function on objects of the horizontal composition functors $\mathbf{2Cat}(\mathcal{B}, \mathcal{C}) \times \mathbf{2Cat}(\mathcal{A}, \mathcal{B}) \to \mathbf{2Cat}(\mathcal{A}, \mathcal{C})$, which on morphisms of the hom-categories works as follows: for 2-natural transformations $\mathcal{A} \overset{F}{\to} \mathcal{B}$ and $\mathcal{B} \overset{G}{\to} \mathcal{C}$, then $\beta \alpha : GF \Rightarrow G'F'$ is the 2-natural transformation with $(\beta \alpha)_x = \beta_{Fx} \circ G \alpha_x = G' \alpha_x \circ \beta_{Fx}$, the horizontal composite in $\mathcal{B}$ of the morphisms $GFx \overset{G\alpha_x}{\to} GFx \overset{\beta_{Fx}}{\to} G'Fx$.

A lax 2-diagram of 2-categories with the shape of a 2-category $\mathcal{C}$, or normal lax 2-functor $\mathcal{F} : \mathcal{C}^o \to \mathbf{2Cat}$, provides us with the following data:

- a 2-category $\mathcal{F}_x$, for each object $x$ of $\mathcal{C}$,
- a 2-functor $u^* : \mathcal{F}_y \to \mathcal{F}_x$, for each morphism $u : x \to y$ in $\mathcal{C}$,
- a 2-natural transformation $\mathcal{F}_y \overset{u^*}{\Rightarrow} \mathcal{F}_x$, for each deformation $\mathcal{F}_y \overset{v^*}{\Rightarrow} \mathcal{F}_x$ of $\mathcal{C}$,
- a 2-natural transformation $\mathcal{F}_z \overset{\zeta_{u,v}^*}{\Rightarrow} \mathcal{F}_x$, for each pair of composable morphisms $x \overset{u}{\Rightarrow} y \overset{v}{\Rightarrow} z$ in $\mathcal{C}$,

and these data must satisfy the following conditions:

- for any object $x$ and any morphism $u : x \to y$ in $\mathcal{C}$, $1^*_x = 1_{\mathcal{F}_x}$, $1^*_u = 1_{u^*}$ and $\zeta_{u,1_x} = 1_{u^*} = \zeta_{1_y,u}$,
- for any two vertically composable deformations in $\mathcal{C}$, $\mathcal{F}_x \overset{\alpha}{\Rightarrow} y$, $(\alpha' \alpha)^* = \alpha'^* \circ \alpha^*$,
• for any two horizontally composable deformations in \( C \), the following diagram of 2-natural transformations commutes:

\[
\begin{array}{ccc}
\alpha^* \beta^* & \xrightarrow{\zeta_{v,w}} & (\beta \circ \alpha)^* \\
\downarrow & & \downarrow \\
u^*v^* & \xrightarrow{\zeta_{v,u}} & (v \circ u)^*
\end{array}
\]

• for any three composable morphisms in \( C \), the following diagram of 2-natural transformations commutes:

\[
\begin{array}{ccc}
u^*w^* & \xrightarrow{\zeta_{v,w}} & (v^*w^*)^* \\
\downarrow & & \downarrow \\
u^* & \xrightarrow{\zeta_{w,v}} & (\omega \circ v)^* \\
\end{array}
\]

The so-called Grothendieck construction on a lax functor \( F : C \rightarrow \mathbf{Cat} \), for \( C \) a category, underlies the following 2-categorical construction, which is actually a special case of the one considered by Baković in [3, §4] and, for the case where \( C \) is a category, it is an special case of the ones given by Tamaki in [36, §3] and by Carrasco, Cegarra and Garzón in [10, §3]:

**Definition 4.1.** Let \( F : C \rightarrow \mathbf{2Cat} \) be a normal lax functor, where \( C \) is a 2-category. The Grothendieck construction on \( F \) is the 2-category, denoted by

\[ \int_C F, \]

whose objects are pairs \((a, x)\) where \( x \) is an object of \( C \) and \( a \) an object of the 2-category \( F_x \). A morphism \((f, u) : (b, y) \rightarrow (a, x)\), is a pair of morphisms where \( u : y \rightarrow x \) is in \( C \) and \( f : b \rightarrow u^*a \) in \( F_y \); and a deformation

\[
\begin{array}{ccc}
(f, u) & \xrightarrow{(\phi, \alpha)} & (a, x) \\
(b, y) & \xrightarrow{\phi} & (f', u')
\end{array}
\]

consists of a deformation \( y \xrightarrow{u} x \) in \( C \), together with a deformation

\[
\begin{array}{ccc}
u^*a & \xrightarrow{\phi} & f \\
\downarrow & & \downarrow \\
u^*a & \xrightarrow{\phi} & f'.
\end{array}
\]
that is, $\phi : \alpha^*_a \circ f \Rightarrow f'$, in $\mathcal{F}_y$.

The vertical composition in $\int_c \mathcal{F}$ of deformations

\[
\begin{array}{c}
\begin{array}{c}
(f, u) \\
\downarrow (\phi, \alpha) \\
(b, y)
\end{array} \\
\begin{array}{c}
(a, x) \\
\downarrow (\phi', \alpha') \\
(f'', u'')
\end{array}
\end{array}
\]

is given by the formula $(\phi', \alpha')(\phi, \alpha) = (\phi'(1 \circ \phi), \alpha' \alpha)$, where $\alpha' \alpha$ is the vertical composition in $\mathcal{C}$, and $\phi'(1 \circ \phi) : (\alpha' \alpha)^*_a \circ f \Rightarrow f''$ is the deformation in $\mathcal{F}_y$ obtained by pasting the diagram

The horizontal composition in $\int_c \mathcal{F}$ of two arrows

\[
\begin{array}{c}
\begin{array}{c}
(c, z) \\
\downarrow (g, v)
\end{array} \\
\begin{array}{c}
(b, y) \\
\downarrow (f, u)
\end{array} \\
\begin{array}{c}
(a, x)
\end{array}
\end{array}
\]

is given by

\[
(f, u) \circ (g, v) = (\zeta_a \circ v^* f \circ g, u \circ v) : (c, z) \to (a, x),
\]

where $u \circ v$ is the horizontal composition of $u$ and $v$ in $\mathcal{C}$ and $\zeta_a \circ v^* f \circ g$ is the horizontal composite

\[
\begin{array}{c}
\begin{array}{c}
(c) \\
\downarrow g
\end{array} \\
\begin{array}{c}
v^* b \\
\downarrow v^* f
\end{array} \\
\begin{array}{c}
v^* u^* a \\
\downarrow \zeta_a
\end{array} \\
\begin{array}{c}
(u \circ v)^* a
\end{array}
\end{array}
\]

in the 2-category $\mathcal{F}_z$; and the horizontal composition of two deformations

\[
\begin{array}{c}
\begin{array}{c}
(c, z) \\
\downarrow (g, v)
\end{array} \\
\begin{array}{c}
(b, y) \\
\downarrow (f, u)
\end{array} \\
\begin{array}{c}
(a, x)
\end{array}
\end{array}
\]

is given by the formula

\[
(\phi, \alpha) \circ (\psi, \beta) = ((1 \circ \psi)(1 \circ v^* f \circ 1), \alpha \circ \beta),
\]

where $\alpha \circ \beta$ is the horizontal composition in $\mathcal{C}$ and $(1 \circ \psi)(1 \circ v^* f \circ 1)$ is the
deformation in $\mathcal{F}_z$ obtained by pasting the diagram

\[
\begin{array}{c}
(u \circ v)^*a \xrightarrow{\zeta} v^*u^*a \xrightarrow{vf} v^*b \\
(\alpha \circ \beta)^*a \xrightarrow{\phi} v^*\alpha \xrightarrow{\psi} v^*f \xrightarrow{g} c \\
(u' \circ v')^*a \xrightarrow{\zeta} v^*u'^*a \xrightarrow{v^*f'} \xrightarrow{\phi} v^*b.
\end{array}
\]

Remark 4.2. We should note that, with the necessary natural changes, the above Grothendieck construction makes sense on lax morphisms of tricategories $\mathcal{F} : \mathcal{C}^{o} \rightsquigarrow \mathbf{2-Cat}$ with all of its coherence 3-cells invertible, from any 2-category $\mathcal{C}$ (regarded as a strict tricategory in which the 3-cells are all identities) to the larger tricategory of small 2-categories $\mathbf{2-Cat}$, that is, the full subtricategory given by the 2-categories of the tricategory $\mathbf{Bicat}$ [17 §5] of bicategories, pseudo-functors, pseudo-natural transformations, and modifications. However, the resulting $\int_{\mathcal{C}} \mathcal{F}$ is not a 2-category, but rather a bicategory (see [3 §3] or [11 Definition 3.1] for details.) □

For any given normal lax functor $\mathcal{F} : \mathcal{C}^{o} \rightsquigarrow \mathbf{2Cat}$, the 2-category $\int_{\mathcal{C}} \mathcal{F}$, whose construction is natural both in $\mathcal{C}$ and $\mathcal{F}$, assembles all 2-categories $\mathcal{F}_x$ in the following precise sense: There is a projection 2-functor

\[
\pi : \int_{\mathcal{C}} \mathcal{F} \to \mathcal{C},
\]

given by

\[
(f, u) \xrightarrow{(h, y) \Downarrow (\phi, \alpha) (a, x) \xrightarrow{\pi} y \Downarrow \alpha, x}
\]

and, for each object $z$ of $\mathcal{C}$, there is a pullback square of 2-categories

\[
\begin{array}{ccc}
\mathcal{F}_z & \xrightarrow{j} & \int_{\mathcal{C}} \mathcal{F} \\
\downarrow \pi & & \downarrow \pi \\
[z] & \to & \mathcal{C}
\end{array}
\]

where $j : \mathcal{F}_z \to \int_{\mathcal{C}} \mathcal{F}$ is the embedding 2-functor defined by

\[
a \xrightarrow{\Downarrow \phi, g} b \xrightarrow{j} (a, z) \xrightarrow{(f, 1_z)} (b, 1_z) \, .
\]
Thus, $\mathcal{F}_z \cong \pi^{-1}(z)$, which is the fibre 2-category of $\pi$ at $z$.

The following main result in this section is consequence of Theorem 3.2:

**Theorem 4.3.** Suppose that $\mathcal{F} : C \rightarrow \mathbf{2Cat}$ is a normal lax functor, where $C$ is a 2-category, such that the induced map $Bw^* : BF_{z_0} \rightarrow BF_{z_1}$, for each morphism $w : z_1 \rightarrow z_0$ in $C$, is a homotopy equivalence. Then, for every object $z$ of $C$, the square induced by (10) on classifying spaces

$$
\begin{array}{ccc}
BF_z & \rightarrow & BF_C \\
\downarrow & & \downarrow \\
pt & \rightarrow & BC
\end{array}
$$

is homotopy cartesian.

**Proof.** Observe that, for each object $z \in C$, the homotopy fibre 2-category $z//\pi$ has objects the triples $(a,x,v)$ with $v : z \rightarrow x$ a morphism in $C$ and $a$ an object of $\mathcal{F}_x$. A morphism $(f,u,\beta) : (a,x,v) \rightarrow (a',x',v')$ consists of a morphism $u : x \rightarrow x'$ together with a deformation $\beta : u \circ v \Rightarrow v'$ in $C$ and a morphism $f : a \rightarrow u^*a'$ in $\mathcal{F}_z$; and a deformation

$$
(f,u,\beta) \\
(a,x,v) \Downarrow (\phi,\alpha) \quad (a',x',v') \Downarrow (f',u',\beta')
$$

is a pair with $\alpha : u \Rightarrow u'$ a deformation in $C$ such that $\beta'(\alpha \circ 1_v) = \beta$ and

$$
\begin{array}{c}
\begin{array}{c}
\alpha_w^a \\
\downarrow \phi \\
u^*a' \begin{array}{c}
\leftarrow \\
f'
\end{array}
f
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\leftarrow a,
\end{array}
\end{array}
$$

a deformation in $\mathcal{F}_z$.

We have an embedding 2-functor $i = i_z : \mathcal{F}_z \rightarrow z//\pi$ given by

$$
\begin{array}{c}
\begin{array}{c}
a \quad \begin{array}{c}
f \\
\downarrow \phi \\
f'
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\leftarrow \begin{array}{c}
(a,z,1_z) \\
(f',1_z,1_z)
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\leftarrow \begin{array}{c}
(f,1_z,1_z) \\
(\phi,1_z)
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\leftarrow \begin{array}{c}
(a',z,1_z) \\
(\phi',1_z)
\end{array}
\end{array}
\begin{array}{c}
\leftarrow \begin{array}{c}
(a',x',v') \\
(v^*a',v^*u^*a',v^*a')
\end{array}
\end{array}
\begin{array}{c}
\leftarrow (u \circ v)^*a' \beta_{a'}^* v^*a'.
\end{array}
$$

and there is also a 2-functor $p = p_z : z//\pi \rightarrow \mathcal{F}_z$ that is defined as follows: it carries an object $(a,x,v)$ of $z//\pi$ to the object $v^*a$ of $\mathcal{F}_z$, a morphism $(f,u,\beta) : (a,x,v) \rightarrow (a',x',v')$ is mapped by $p$ to the composite morphism

$$
\begin{array}{c}
\begin{array}{c}
\leftarrow v^*a \\
\begin{array}{c}
\leftarrow v^*u^*a' \leftarrow (u \circ v)^*a' \beta_{a'}^* v^*a'.
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\leftarrow (u \circ v)^*a' \beta_{a'}^* v^*a'.
\end{array}
$$

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and a deformation \((\phi, \alpha) : (f, u, \beta) \Rightarrow (f', u', \beta')\), as in (18), to the deformation

\[1 \circ v^\ast \phi : p(f, u, \beta) \Rightarrow p(f', u', \beta')\]

obtained by pasting the diagram

\[
\begin{array}{cccc}
v^\ast f & \rightarrow & (u \circ v)^\ast a' & \rightarrow \\
v^\ast a & \downarrow v^\ast \phi & (\alpha \circ 1_v)^\ast a' & \downarrow \\
v^\ast f' & \rightarrow & (u' \circ v)^\ast a' & \rightarrow
\end{array}
\]

\[
\begin{array}{cccc}
\zeta_{a'} & = & (\beta \circ a')^\ast & = \\
(\alpha \circ v)^\ast & v^\ast a & \downarrow & v^\ast a'.
\end{array}
\]

By Lemma 2.6 both 2-functors \(i\) and \(p\) induce homotopy equivalences on classifying spaces

\[
Bi : B\mathcal{F}_z \simeq B(\mathcal{C} / \pi), \quad Bp : B(\mathcal{C} / \pi) \simeq B\mathcal{F}_z,
\]

since \(pi = 1\) and there is an oplax transformation \(\theta : i p \Rightarrow 1\), whose component at an object \((a, x, v)\) of \(\mathcal{C} / \pi\) is the morphism

\[
\theta_{(a,x,v)} = (v^\ast a, v, 1_v) : (v^\ast a, z, 1_z) \rightarrow (a, x, v),
\]

and whose component at a morphism \((f, u, \beta) : (a, x, v) \rightarrow (a', x', v')\),

\[
\theta_{(f,u,\beta)} : (f, u, \beta) \circ \theta_{(a,x,v)} \Rightarrow \theta_{(a',x',v')} \circ ip(f, u, \beta),
\]

is the deformation \((1, \beta)\),

\[
(\zeta_{a'} \circ v^\ast f, u \circ v, \beta)
\]

\[
(\beta^\ast_a \circ \zeta_{a'} \circ v^\ast f, v', 1_{v'})
\]

The square (16) is the composite of the squares

\[
\begin{array}{cccc}
\mathcal{F}_z & \rightarrow & \mathcal{C} / \pi & \rightarrow \\
\downarrow i_z & \downarrow \Phi & \downarrow \int_\mathcal{C} \mathcal{F} & \\
(0, z, 1_z) & \rightarrow & \pi & \rightarrow
\end{array}
\]

where, in (a) both horizontal 2-functors induce homotopy equivalences on classifying spaces (recall Lemma 3.1). Therefore, the induced square (17) is homotopy cartesian if and only if the one induced by (b) is as well. But Theorem 3.2 actually implies that the square induced by (b)

\[
\begin{array}{cccc}
B(\mathcal{C} / \pi) & \rightarrow & B\int_\mathcal{C} \mathcal{F} & \\
\downarrow & \downarrow & \downarrow & \\
B(\mathcal{C}) & \rightarrow & BC
\end{array}
\]
is homotopy cartesian: to verify the hypothesis of Theorem 3.2 let \( w : z_1 \to z_0 \)
any given morphism in \( C \). We have the square of 2-functors

\[
\begin{array}{ccc}
z_0 // \pi & \xrightarrow{p} & F_{z_0} \\
w^* & & w^* \\
z_1 // \pi & \xrightarrow{p} & F_{z_1}.
\end{array}
\]

The two composite 2-functors in the square are related by a 2-natural transformation \( w^* p \Rightarrow p w^* \), whose component at an object \((a,x,v)\) of \( z_0 // \pi \) is the morphism \( \zeta_a : w^* v^* a \to (v \circ w)^* a \) in \( z_1 // F \). Hence, by Lemma 2.6 the induced square on classifying spaces

\[
\begin{array}{ccc}
B(z_0 // \pi) & \xrightarrow{Bp} & F_{z_0} \\
Bw^* & & Bw^* \\
B(z_1 // \pi) & \xrightarrow{Bp} & F_{z_1}.
\end{array}
\]

is homotopy commutative; that is, there is a homotopy \( Bw^* Bp \simeq Bp Bw^* \).
Since the induced maps \( Bp : Bz_i // \pi \to BF_{z_i}, i = 0,1 \), and \( Bw^* : BF_{z_0} \to BF_{z_1} \)
are all homotopy equivalences, the map \( Bw^* : B(z_0 // \pi) \to B(z_1 // \pi) \) is also. \( \square \)

**Example 4.4.** The 2-category \( \text{Cat} \), of small categories, functors and natural transformations, is a full 2-subcategory of \( 2\text{Cat} \), regarding any category as a 2-category whose deformations are all identities. Hence, the Grothendieck construction in Definition 4.1 works on normal lax functors \( C \, \Rightarrow \, \text{Cat} \), with \( C \) any 2-category. For any object \( x \) in a 2-category \( C \), we have the 2-functor \( \int_x C(-, x) : C \Rightarrow \text{Cat} \) on which the Grothendieck construction gives

\[ \int_x C(-, x) = C // x, \]

the comma 2-category of objects over \( x \), which, by Lemma 3.1 has a contractible classifying space. Therefore, if the object \( x \) is such that the induced maps \( Bu^* : B\mathcal{C}(z, x) \to B\mathcal{C}(y, x) \) are homotopy equivalences for the different morphisms \( u : y \to z \) in \( C \), then Theorem 3.3 implies the existence of a homotopy equivalence

\[ \Omega(B\mathcal{C}, x) \simeq B(\mathcal{C}(x, x)), \]

between the loop space of the classifying space of the 2-category \( C \) with base point \( x \) and the classifying space of the category of endomorphisms of \( x \) in \( C \). \( \square \)

The well-known Homotopy Colimit Theorem by Thomason [37] establishes that the Grothendieck construction on a diagram of categories is actually a categorical model for the homotopy type of the homotopy colimit of the diagram.
of categories. The notion of homotopy colimit has been well generalized in the literature to 2-functors $\mathcal{F} : \mathcal{C}^o \to \mathbf{Cat}$ (see [21, 2.2], for example), where $\mathcal{C}$ is any 2-category and $\mathbf{Cat} \subseteq 2\mathbf{Cat}$, the 2-category of small categories, functors and natural transformations. Next, Theorem 4.5 generalizes Thomason’s theorem both to 2-diagrams of categories $\mathcal{F} : \mathcal{C}^o \to \mathbf{Cat}$, with $\mathcal{C}$ a 2-category, and to diagrams of 2-categories $\mathcal{F} : \mathcal{C}^o \to 2\mathbf{Cat}$, with $\mathcal{C}$ a category.

Recall that the homotopy colimit of a 2-functor $\mathcal{F} : \mathcal{C}^o \to \mathbf{Cat}$, where $\mathcal{C}$ is a 2-category, is defined [21, Definition (2.2.2)] to be the simplicial category

$$\text{hocolim}_\mathcal{C} \mathcal{F} : \Delta^o \to \mathbf{Cat},$$

whose category of $n$-simplices is

$$\bigsqcup_{(x_0, \ldots, x_n) \in \text{Ob} \mathcal{C}^{n+1}} \mathcal{F}_{x_0} \times \mathcal{C}(x_1, x_0) \times \mathcal{C}(x_2, x_1) \times \cdots \times \mathcal{C}(x_n, x_{n-1});$$

faces and degeneracies are defined as follows: the face functor $d_0$ maps the component category $\mathcal{F}_{x_0} \times \mathcal{C}(x_1, x_0) \times \mathcal{C}(x_2, x_1) \times \cdots \times \mathcal{C}(x_n, x_{n-1})$ into $\mathcal{F}_{x_1} \times \mathcal{C}(x_2, x_1) \times \cdots \times \mathcal{C}(x_n, x_{n-1})$, and it is induced by

$$\begin{array}{ccc}
F_{x_0} \times C(x_1, x_0) & d_0 & F_{x_1}, \\
(a \to b, x_1 & \overset{u}{\underset{\alpha}{\to}} x_0) & \mapsto \quad u^*a \overset{\alpha_b \circ u^*f}{\underset{\circ v^*}{\to}} v^*b.
\end{array}$$

The other face and degeneracy functors are induced by the operators $d_i$ and $s_i$ in $\mathcal{N} \mathcal{C}$ as $1_{\mathcal{F}_{x_0}} \times d_i$ and $1_{\mathcal{F}_{x_0}} \times s_i$, respectively. Note that by composing $\text{hocolim}_\mathcal{C} \mathcal{F}$ with the nerve of categories functor one obtains the bisimplicial Borel construction $E_\mathcal{C} \mathcal{F}$ in the sense of Tillmann [38].

**Theorem 4.5.** (i) For any 2-functor $\mathcal{F} : \mathcal{C}^o \to \mathbf{Cat}$, where $\mathcal{C}$ is a 2-category, there exists a natural homotopy equivalence

$$\text{B} \text{hocolim}_\mathcal{C} \mathcal{F} \simeq \text{B} \int_\mathcal{C} \mathcal{F},$$

where $\text{hocolim}_\mathcal{C} \mathcal{F}$ is given by (19).

(ii) For any functor $\mathcal{F} : \mathcal{C}^o \to 2\mathbf{Cat}$, where $\mathcal{C}$ is a category, there exists a natural homotopy equivalence

$$\text{B} \text{hocolim}_\mathcal{C} \mathcal{F} \simeq \text{B} \int_\mathcal{C} \mathcal{F},$$

where $\text{hocolim}_\mathcal{C} \mathcal{F} := \text{hocolim}_\mathcal{C} \Delta \mathcal{F} : \Delta^o \to \mathbf{Set}$ is the homotopy colimit, [3, Ch. XII], of the $\mathcal{C}^o$-diagram of simplicial sets obtained by composing $\mathcal{F}$ with the geometric nerve functor $\Delta : 2\mathbf{Cat} \to \mathbf{Simpl.Set}$ given by [3].

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Proof. We shall use the bar construction on a bisimplicial set $\overline{W}S$, also called its “codiagonal” or “total complex” \[8\]. Let us recall that the functor $W : \text{Bisimpl.Set} \to \text{Simpl.Set}$ is the right Kan extension along the ordinal sum functor $\Delta \times \Delta \to \Delta, ([p], [q]) \mapsto [p + 1 + q]$. For an explicit description of the $W$ construction, it is often convenient to view a bisimplicial set $S : \Delta^o \times \Delta^o \to \text{Set}$ as a (horizontal) simplicial object in the category of (vertical) simplicial sets. For this case, we write $d_h^i = S(\delta_i, 1) : S_{p,q} \to S_{p-1,q}$ and $s_h^i = S(\sigma_i, 1) : S_{p,q} \to S_{p+1,q}$ for the horizontal face and degeneracy maps and, similarly, $d_v^j = S(1, \delta_j)$ and $s_v^j = S(1, \sigma_j)$ for the vertical ones. Then, for any given bisimplicial set $S$, $WS$ can be described as follows (cf. \[1, \S III\]): the set of $p$-simplices of $WS$ is
\[
\left\{ (t_0, \ldots, t_p) \in \prod_{m=0}^p S_{m,p-m} \mid d_h^m t_m = d_h^{m+1} t_{m+1}, 0 \leq m < p \right\}
\]
and, for $0 \leq i \leq p$, the faces and degeneracies of a $p$-simplex are given by
\[
\begin{align*}
    d_i(t_0, \ldots, t_p) &= (d_h^0 t_0, \ldots, d_h^i t_i, d_h^i t_{i+1}, \ldots, d_h^p t_p), \\
    s_i(t_0, \ldots, t_p) &= (s_h^0 t_0, \ldots, s_h^i t_i, s_h^{i+1} t_{i+1}, \ldots, s_h^p t_p).
\end{align*}
\]
For any bisimplicial set $S$, there is a natural Alexander-Whitney type diagonal approximation, the so-called Zisman comparison map (see \[8\])
\[
\eta : \text{diag} S \to \overline{W}S,
\]
which carries a $p$-simplex $t \in S_{p,p}$ to
\[
\eta t = ((d_h^0 t_0), (d_h^1 t_1), \ldots, (d_h^p t_p)).
\]
And the following is a useful result (see \[12, 13, \text{Theorem 1.1, Theorem 9}\]):

**Fact 4.6.** For any bisimplicial set $S$, the simplicial map $\eta : \text{diag} S \to \overline{W}S$ is a weak homotopy equivalence.

*Proof of (i).* The strategy of the proof is to apply the weak homotopy equivalences \[20\] above on the following two bisimplicial sets $S_1$ and $S_2$. We let
\[
S_1 = \text{N hocollim}_C F : \Delta^o \times \Delta^o \to \text{Set},
\]
the bisimplicial set obtained by composing the simplicial category $\text{hocolim}_C F$ \[19\] with the nerve functor of categories, and
\[
S_2 = \text{NN } \int_C F : \Delta^o \times \Delta^o \to \text{Set},
\]
the bisimplicial set double nerve of the 2-category $\int_C F$. Since
\[
B \text{hocolim}_C F = |\text{diag} S_1|, \quad B \int_C F = |\text{diag} S_2|,
\]
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we have homotopy equivalences induced by the maps \((20)\)

\[ B \hocolim_C \mathcal{F} \simeq |\mathcal{W}S_1|, \quad B \int_C \mathcal{F} \simeq |\mathcal{W}S_2| \]

We shall now observe the nature of the simplicial sets \(\mathcal{W}S_1\) and \(\mathcal{W}S_2\). A \(p\)-simplex of \(\mathcal{W}S_1\), say \(\chi\), is a list of data

\[
\chi = \left( u_{m}^{m}, a_{m-1} \xrightarrow{f_{m}} a_{m}, x_{m-1} \xrightarrow{\psi \alpha_{m}} x_{m} \right)_{0 \leq m \leq p, 1 \leq j \leq m}. \tag{21}
\]

in which \(x_0, \ldots, x_p\) are objects, the \(u_j^m\) morphisms and the \(\alpha_j^m\) are deformations in \(\mathcal{C}\), each \(a_m\) is an object in the category \(\mathcal{F}_{x_m}\) and each \(f_m\) is a morphism of the category \(\mathcal{F}_{x_m}\); while a \(p\)-simplex \(\chi' \in \mathcal{W}S_2\) is a list of objects, morphisms and deformations in \(\int_C \mathcal{F}\) of the form

\[
\chi' = \left( \left( a_{m-1}, a_{m-1} \right) \xrightarrow{\psi \alpha_{m}} (a_{m}, x_{m}) \right)_{0 \leq m \leq p, 1 \leq j \leq m}. \tag{22}
\]

Since in \(\chi'\) all triangles

\[
\begin{array}{ccc}
  a_m & \xrightarrow{f_j} & u_j^m \\
  \downarrow \alpha_j^m & & \downarrow \alpha_j^m \\
  f_j^{j-1} & \xrightarrow{a_m} & u_j^{j-1} a_{m-1}
\end{array}
\]

must be commutative, we have equalities

\[
f_j^{j-1} = (\alpha_j^m)^* \circ f_j^m = (\alpha_j^m)^* \circ (\alpha_j^{j+1})^* \circ f_j^{j+1} = \cdots \]

\[
\cdots = (\alpha_m^m \cdots \alpha_j^m)^* \circ f_m^m.
\]

Hence, it is straightforward to verify that there is a simplicial bijection

\[
\mathcal{W}S_1 \cong \mathcal{W}S_2,
\]

that carries a \(p\)-simplex \(\chi\) of \(\mathcal{W}S_1\), as in \((21)\), to the \(p\)-simplex \(\chi'\) of \(\mathcal{W}S_2\), as in \((22)\), in which all data \(x's\), \(u's\) \(\alpha's\) and \(\alpha's\) are the same as those in \(\chi\), and the morphisms \(f_j^m : a_m \to u_j^m a_{m-1}\) are respectively given by:

\[
\begin{cases}
  f_m^m = f_m, \\
  f_j^{j-1} = (\alpha_m^m \cdots \alpha_j^m)^* \circ f_m \text{ for } 1 < j < m.
\end{cases}
\]
This makes the proof for (i) complete.

**Proof of (ii).** In this case, we show a weak homotopy equivalence of simplicial sets

\[ \text{hocolim}_C \Delta F \to \Delta \int_C F, \]

and, to do so, we first give a description of both simplicial sets.

On the one hand, \( \text{hocolim}_C \Delta F \) is the simplicial set diagonal of the bisimplicial set

\[ S = \bigsqcup_{x \in NC} \Delta F_{x_0} = \bigsqcup_{x : [q] \to C} \text{laxFunc}([p], F_{x_0}), \tag{23} \]

whose \((p,q)\)-simplices are pairs

\( (y, x) \)

consisting of a functor \( x : [q] \to C \) and a normal lax functor \( y : [p] \to F_{x_0} \). The horizontal face and degeneracy maps are given by

\[ d^h_i(y, x) = (y\delta_i, x), \quad s^h_i(y, x) = (y\sigma_i, x), \]

for \( 0 \leq i \leq p \), and the vertical ones by

\[ d^v_j(y, x) = (y, x\delta_j), \quad s^v_j(y, x) = (y, x\sigma_j), \]

for \( 0 \leq j \leq q \), except the vertical 0th face which is defined by

\[ d^v_0(y, x) = (x_{0,1}^*y, x\delta_0), \]

where \( x_{0,1}^*y : [p] \to F_{x_1} \) is the lax functor obtained by the composition of \( y \) with the 2-functor \( x_{0,1}^* : F_{x_0} \to F_{x_1} \) attached in diagram \( F : C \to 2\text{Cat} \) at the morphism \( x_{0,1} : x_1 \to x_0 \) of \( C \).

On the other hand, a \( p \)-simplex of \( \Delta \int_C F \) is a normal lax functor \([p] \to \int_C F\), which can be described as a pair

\( (y', x) \),

where \( x : [p] \to C \) is a functor, that is, a \( p \)-simplex of \( NC \), and \( y' : [p] \to F \) is a normal \( x \)-crossed lax functor [14 §4.1], that is, a family

\[ y' = \{ y'_i, y'_{i,j}, y'_{i,j,k} \}_{0 \leq i \leq j \leq k \leq p} \tag{24} \]

in which each \( y'_i \) is an object of the 2-category \( F_{x_i} \), each \( y'_{i,j} : y'_j \to x_{i,j}^* y'_i \) is a morphism in \( F_{x_j} \), and the \( y'_{i,j,k} : x_{j,k}^* y'_{i,j} \circ y'_{j,k} \Rightarrow y'_{i,k} \) are deformations in \( F_{x_k} \).
satisfying the condition that, for $0 \leq i \leq j \leq k \leq l \leq p$, the following diagram of deformations

\[
x_k^*(x_{j,k}^*y_{i,j,k} \circ y_{j,k}^l) \circ y_{k,l}^i = x_j^*y_{i,j} \circ x_k^*y_{j,k}^l \circ y_{k,l}^i
\]

commutes in the category $F_\mathcal{Q}$; and, moreover, the following normalization equations hold:

\[
y_{i,i} = 1, \quad y_{i,j,j} = 1, \quad y_{i,j} = y_{i,i,j}.
\]

The face and degeneracy maps are given by

\[
d_i(y', x) = (y' \delta_i, x \delta_i), \quad s_i(y', x) = (y \sigma_i, x \sigma_i), \quad \text{for } 0 \leq i \leq p.
\]

Our strategy now is to apply the weak homotopy equivalences (20) on the bisimplicial set $S$, defined in (23). Since $\text{diag} S = \operatorname{hocolim}_C \Delta F$, we have a weak homotopy equivalence $\eta : \operatorname{hocolim}_C \Delta F \to \overline{\mathcal{W}} S$ and the proof will be complete once we show a simplicial isomorphism

\[
\overline{\mathcal{W}} S \cong \Delta \int_C \mathcal{F}.
\]

For, note that a $p$-simplex of $\overline{\mathcal{W}} S$, say $\chi$, can be described as a list of pairs

\[
\chi = ((y_o^0, x_o^0), \ldots, (y_m^m, x_m^m), \ldots, (y_p^p, x_p^p)),
\]

in which each $x^m : [p - m] \to \mathcal{C}$ is a functor and each $y^m : [m] \to \mathcal{F}_{x^m}$ is a normal lax functor, such that $x^m \delta_0 = x^{m+1}$ and $y^{m+1} \delta_{m+1} = x_{0,m}^m y^m$, for all $0 \leq m < p$. Denoting $x^0 : [p] \to \mathcal{C}$ simply by $x : [p] \to \mathcal{C}$, an iterated use of the above equalities proves that

\[
x^m = x(\delta_0)^m : [p - m] \to [p] \xrightarrow{\delta_0^m} \mathcal{C},
\]

for $0 \leq m \leq p$, and

\[
y^m \delta_m \cdots \delta_{k+1} = x_{k,m}^* y_{k}^k : [k] \to \mathcal{F}_{x^m},
\]

for $0 \leq k < m \leq p$. These latter equations mean that

\[
\begin{cases}
y_{i}^i = x_{i,i}^* y_{i}^i & \text{for } i \leq j, \\
y_{i,j}^k = x_{j,k}^* y_{i,j}^k & \text{for } i \leq j \leq k, \\
y_{i,j,k}^k = x_{k,l}^* y_{i,j,k}^k & \text{for } i \leq j \leq k \leq l,
\end{cases}
\]

whence we see how the $p$-simplex $\chi$ of $\overline{\mathcal{W}} S$ is uniquely determined by $x : [p] \to \mathcal{C}$, the objects $y_{i}^i$ of $\mathcal{F}_{x^i}$, the morphisms $y_{i,j}^j : y_{i}^j \to y_{i}^i = x_{i,j}^* y_{i}^i$ of $\mathcal{F}_{x^j}$ and the

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deformations $y^k_{i,j,k} : y^j_{i,j} \circ y^k_{j,k} = x^*_{j,k} y^j_{i,j} \circ y^k_{j,k} \Rightarrow y^k_{i,k}$ in $\mathcal{F}_{x^k}$, all for $0 \leq i \leq j \leq k \leq p$. At this point, we observe that there is a normal $\mathbf{x}$-crossed lax functor $\mathbf{y}' = \{y'_i, y'_{i,j}, y'_{i,j,k} : [p] \to \mathcal{F}\}$, as in (21), defined just by stating $y'_i = y_i^k$, $y'_{i,j} = y_{i,j}^k$ and $y'_{i,j,k} = y_{i,j,k}^k$ (the commutativity of diagrams (25) follows from $\mathbf{y}'$ being a lax functor). Thus, the $p$-simplex $\chi \in \mathcal{W}S$ defines the $p$-simplex $(\mathbf{y}', \mathbf{x})$ of $\Delta \int_{\mathcal{F}} \mathcal{F}$, which itself uniquely determines $\chi$. In this way, we obtain an injective simplicial map

$$j : \mathcal{W}S \to \Delta \int_{\mathcal{F}} \mathcal{F}$$

$$(y_0^0, x_0^0, \ldots, (y_0^p, x_0^p)) \overset{j}{\mapsto} (\mathbf{y}', \mathbf{x}) = (\{y'_i, y'_{i,j}, y'_{i,j,k}\}, x^0),$$

which is also surjective, that is, actually an isomorphism, as we can see by retracing our steps: To any pair $(\mathbf{y}', \mathbf{x})$ describing a $p$-simplex of $\Delta \int_{\mathcal{F}} \mathcal{F}$, that is, with $\mathbf{x} = \{x_i, x_{i,j} : [p] \to \mathcal{C}\}$ a functor and $\mathbf{y}' = \{y'_i, y'_{i,j}, y'_{i,j,k} : [p] \to \mathcal{F}\}$ a normal $\mathbf{x}$-crossed lax functor, we associate the $p$-simplex $\chi = ((y^m, x^m))$ of $\mathcal{W}S$, where, for each $0 \leq m \leq p$, $x^m : [p-m] \to \mathcal{C}$ is the composite $[p-m] \xrightarrow{(h_{p-m})^m} [p] \xrightarrow{x} \mathcal{C}$, and the normal lax functor $y^m : [m] \to \mathcal{F}_{x^m} = \mathcal{F}_{x^0} \simeq \mathcal{F}$ is defined by the objects $y^m_i = x^m_i y_i^k$, the morphisms $y^m_{i,j} = x^m_{i,m} y^k_{i,j} = y^k_{j} \circ y^m_i$ and the deformations $y^m_{i,j,k} = x^m_{i,j,k} y^k_{i,j,k} : y^m_i \circ y^m_j = y^m_{i,j}$, which itself uniquely determines $\chi$. Since one easily checks that $j(\chi) = (\mathbf{y}', \mathbf{x})$, the proof is complete. \hfill \Box

**Remark 4.7.** If $\mathcal{F} : \mathcal{C}^\circ \to \mathbf{Cat}$ is any 2-functor from a 2-category $\mathcal{C}$ such that any morphism $z_1 \to z_0$ in $\mathcal{C}$ induces a homotopy equivalence $\mathcal{B}\mathcal{F}_{z_0} \to \mathcal{B}\mathcal{F}_{z_1}$, then it is a consequence of Theorems 4.3 and 4.5 that, for each object $z$ of $\mathcal{C}$, there is a homotopy cartesian induced square

$$\begin{array}{ccc}
\mathcal{B}\mathcal{F}_z & \xrightarrow{\text{Bhocolim}} & \text{Bhocolim}\mathcal{C}\mathcal{F}
\downarrow & & \downarrow \text{BC}.
\end{array}$$

This is a fact that, alternatively, can be obtained from a general result on simplicial categories acting on simplicial sets by Moerdijk [31, Theorem 2.1]. The analogous result for a functor $\mathcal{F} : \mathcal{C}^\circ \to \mathbf{2Cat}$, where $\mathcal{C}$ is a category, can also be obtained directly from Quillen’s Lemma [25, p. 90], [16, §IV, Lemma 5.7]. \hfill \Box

**Example 4.8.** Let $(\mathcal{M}, \otimes)$ be a strict monoidal category. If we regard the monoidal category as a 2-category with only one object, say 1, then we can identify a 2-functor

$$\mathcal{N} : (\mathcal{M}, \otimes)^\circ \to \mathbf{Cat}$$

with a category $\mathcal{N} (= \mathcal{N}1$, the one associated to the unique object of the 2-category) endowed with an associative and unitary right action of $\mathcal{M}$ by a functor $\otimes : \mathcal{N} \times \mathcal{M} \to \mathcal{N}$; namely, that given by

$$(a \xrightarrow{f} b) \otimes (u \xrightarrow{\alpha} v) = (u^*a \xrightarrow{\alpha_b \circ u^*f} v^*b).$$
Since there is an identification of simplicial categories

\[ \text{hocolim}_{\mathcal{M}, \otimes} \mathcal{N} = E_{\mathcal{M}, \otimes} \mathcal{N}, \]

where \( E_{\mathcal{M}, \otimes} \mathcal{N} : \Delta^o \to \text{Cat} \), \([n] \mapsto \mathcal{N} \times \mathcal{M}^n\), is the simplicial category obtained by the so-called Borel construction (or bar construction) for the action, it follows from Theorem 4.5 that \( \int_{\mathcal{M}, \otimes} \mathcal{N} \) is a 2-category modelling the homotopy type of the Borel simplicial category \( E_{\mathcal{M}, \otimes} \mathcal{N} \), that is, there is a homotopy equivalence

\[ B \int_{\mathcal{M}, \otimes} \mathcal{N} \simeq BE_{\mathcal{M}, \otimes} \mathcal{N}. \]

This 2-category \( \int_{\mathcal{M}, \otimes} \mathcal{N} \) has the following easy description: its objects are the same as \( \mathcal{N} \). A morphism \((f, u) : a \to b \) in \( \int_{\mathcal{M}, \otimes} \mathcal{N} \) is a pair \((f, u) \) with \( u \) an object of \( \mathcal{M} \) and \( f : a \to b \otimes u \) a morphism in \( \mathcal{N} \), and a deformation

\[ a \overset{(g, v)}{\Rightarrow} \begin{array}{c} \downarrow \alpha \end{array} \]

is a morphism \( \alpha : u \to v \) in \( \mathcal{M} \) such that the following triangle commutes

\[ \begin{array}{c} f \\ \downarrow \alpha \end{array} \begin{array}{c} g \\ 1 \otimes \alpha \\ \downarrow \end{array} \begin{array}{c} b \\ \otimes u \end{array} \]

The compositions in \( \int_{\mathcal{M}, \otimes} \mathcal{N} \) are given in an obvious manner.

Many of the homotopy-theoretical properties of the classifying space of a monoidal category, \( B(\mathcal{M}, \otimes) \), can now be more easily reviewed by using Grothendieck 2-categories \( \int_{\mathcal{M}, \otimes} \mathcal{N} \) instead of the Borel simplicial categories \( E_{\mathcal{M}, \otimes} \mathcal{N} \).

Thus, one sees, for example, that if the action is such that multiplication by each object \( u \) of \( \mathcal{M} \), \( a \mapsto a \otimes u \), induces a homotopy equivalence \( BN \simeq BN \), then, by Theorem 4.3, \( BN \) is homotopy equivalent to the homotopy fibre of the map \( B \int_{\mathcal{M}, \otimes} \mathcal{N} \to B(\mathcal{M}, \otimes) \) (cf. [23, Proposition 3.5]); that is, one has a homotopy fibre sequence

\[ BN \to B \int_{\mathcal{M}, \otimes} \mathcal{N} \to B(\mathcal{M}, \otimes). \]

In particular, the right action of \((\mathcal{M}, \otimes)\) on the underlying category \( \mathcal{M} \) leads to the 2-category \( \int_{\mathcal{M}, \otimes} \mathcal{M} = 1/(\mathcal{M}^\prime, \otimes) \), the comma 2-category whose classifying space is contractible by Lemma 3.1 (cf. [23, Proposition 3.8]). Then, it follows the well-known fact that there is a homotopy equivalence

\[ \text{BM} \simeq \Omega B(\mathcal{M}, \otimes), \]

between the classifying space of the underlying category and the loop space of the monoidal category, whenever multiplication for each object \( u \in \mathcal{M} \), \( v \mapsto v \otimes u \), induces a homotopy autoequivalence on \( \text{BM} \).
Example 4.9. Let us recall that the category of simplices of a simplicial set $S : \Delta^o \to \text{Set}$, $\int_{\Delta^o} S$, has as objects pairs $(x, m)$ where $m \geq 0$ and $x$ is a $m$-simplex of $S$; and arrow $\xi : (x, m) \to (y, n)$ is an arrow $\xi : [m] \to [n]$ in $\Delta$ with the property $x = \xi^* y$. It is a well-known result, due to Illusie [22, Theorem 3.3], that there exists a homotopy equivalence $|S| \simeq B\int_{\Delta^o} S$ between the geometric realization of $S$ and the classifying space of $\int_{\Delta^o} S$. This result is, in fact, a very particular case of the homotopy colimit theorem of Thomason [37]: If $C : \Delta^o \to \text{Cat}$ is any simplicial category, then there is a homotopy equivalence $B C \simeq B \int_{\Delta} C$, where $\int_{\Delta} C$ is the category Grothendieck construction on $C$.

Now, from Theorem 4.5 (ii), the homotopy type of any given simplicial 2-category $C : \Delta^o \to \text{2Cat}$, $[n] \mapsto C_n$, is the same as the homotopy type of the 2-category $\int_{\Delta} C$, that is, $BC \simeq B \int_{\Delta} C$. To describe this 2-category, note that its set of objects is

$$\text{Ob} \int_{\Delta} C = \bigsqcup_{n \geq 0} \text{Ob} C_n,$$

and its hom-categories are

$$\int_{\Delta} C ((x, m), (y, n)) = \bigsqcup_{[m] \leftrightarrow [n]} C_m(x, \xi^* y),$$

where the disjoint union is taken over all maps $[m] \to [n]$ in $\Delta$.

For instance, if $C$ is any 3-category, that is, a $\text{2Cat}$-enriched category, then $N C$ is a simplicial 2-category whose classifying space is the classifying space of the 3-category. Therefore, we have homotopy equivalences

$$BC \simeq B \int_{\Delta} N C \simeq B \int_{\Delta} N (\int_{\Delta} N C).$$

□

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