Smoking epidemic model with density-dependent death rate and numerical sensitivity analysis

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Abstract. Tobacco smoking is one of the biggest health problem in Indonesia. The government has established policies to control the number of smokers in this country. In this paper, an epidemic model of smoking behaviours will be developed. The model divides the population into four classes, namely non-smokers, normal smokers, heavy smokers and ex-smokers. In this model, the death rates of each class are distinguished and density-dependent death rate is also considered. The discussion of equilibrium points and their stability is used to obtain qualitative information about the behavior of each class. The smoking-free equilibrium point and its stability are solved analytically while the endemic equilibrium point and its stability are solved numerically. The final section will discuss the effects of the rate of interaction that causes a non-smoker to become a smoker. The results of this discussion can be used as a consideration in determining the control measures for the number of smokers in this country.

1. Introduction
Tobacco smoking is known to be the cause of twenty-five serious health problems, such as lung cancer, bronchitis and emphysema. According to WHO [1], tobacco has killed more than sixty million people in the period 1950-2000. This number is more than the deaths caused by World War II. If the situation remains, it is estimated that cigarettes can kill more than one hundred million people at the beginning of the 21st decade.

Indonesia is one of the countries with the highest number of smokers in the world. According to the \textit{Buku Fakta Tembakau dan Permasalahannya di Indonesia} published by the Ministry of Health of the Republic of Indonesia[2], the prevalence of smoking in Indonesia tends to increase from year to year. Based on the Figure 1, the death rate of people due to smoking in Indonesia has entered the red zone. Cigarettes have killed 225,720 people in Indonesia every year. This represents 14.7 percent of the total deaths in Indonesia in one year. In addition to health, cigarettes also cause socio-economic losses in Indonesia. This is confirmed in the [2], that the total cost of treating diseases due to tobacco use in 2013 is estimated at 2.25 trillion rupiah.

There have been many researches that study the evolution of smoking behavior ([3],[4],[5],[6]). For example, [3] analyzed the impact of smoke-free legislation in Spain (2006) using an epidemic model where the smoking habits spread through social interaction. In addition, [4] analyzed the impact smoke-free law on air nicotine concentration using linear regression models on log-transformed nicotine and [5] analyzed the behaviour of smokers from different ethnic groups, using a longitudinal and qualitative panel study. Meanwhile, [6] discusses the spread of smoking
behavior in Indonesia by distinguishing the mortality rates of each class and considering density-dependent deaths.

In this paper, we will develop an epidemic model of smoking behaviour based on the model of [3]. We add new parameter, migration and density-dependent death rate as in [6]. We assumed the mortality rate of each class is different as in [6]. We also assumed that an ex-smokers re-adapt the smoking habit because they have certain interaction with the smokers. The epidemic model of smoking behavior is described in a system of non-linear differential equations. The system is used on the proportion scale. We will analyze the proportion scale system just as in [7]. In general, it is not possible to solve the nonlinear system; however great deal of qualitative information about the local behaviour of the solution is determined by its equilibrium points. We will show the existence of its equilibrium points: the smoke-free equilibrium point and the endemic equilibrium point, and analyze the stability of those points. For this, we refer to the basic reproductive number with the next generation matrix method, as in Driessche P and Watmough J [8] and the Liapunov Stability-Criterion, as in Wigg. S [9].

The model development and the existence and the stability of the smoking-free equilibrium are discussed in Section 2. In Section 3, we discussed the endemic equilibrium point and the effect of the parameters on smokers.

2. Mathematical Model Formulation

In this section we will discuss the formation of an epidemic model of the spread of smoking behavior and its analysis includes the smoking-free equilibrium point and its stability.

Suppose the total population at time t is denoted by $P(t)$. In this model, we divide the total population $P(t)$ into four different subclasses. The classes are described as follows: potential smokers (non-smokers) $N(t)$, that is, part of the population that does not habitually smoke at all, normal smoker $S(t)$, which is part of the population who smoke with the number of cigarettes consumed less than or equal to 12 cigarettes a day, heavy smokers $S_e(t)$, which is part of the population who smoke with the number of cigarettes consumed more than 12 cigarettes in a day and the last class is the class of ex-smoker $E(t)$, which is part of the population that has stopped smoking. Thus, we can write $P(t) = N(t) + S(t) + S_e(t) + E(t)$.

2.1. Flow of People Among the Compartments

People join the non-smoker class with a constant recruitment rate $\mu P$, which is the birth rate in the population. The number of migrants who are non-smokers $\Lambda_{N}$ also increases the number.
of individuals in the non-smoker class. Non-smokers decreased due to non-smoker deaths, both natural deaths $d_nN$ and density-dependent deaths $rPN$. Some non-smokers decide to become smokers due to their social interaction with smokers at a rate of $\beta \frac{S + S_e}{P}$, consequently they move to the smoker class.

Normal smokers, $S(t)$ increases when non-smokers start smoking, with rates $\beta \frac{S + S_e}{P}$ and the presence of migrants who are normal smokers $\Lambda \frac{S}{P}$. Heavy smokers who reduce the number of cigarettes in a day will move to the normal smoker class at a rate of $\alpha S_e$. Conversely, normal smokers move to heavy smoker classes because it increases the number of cigarettes he consumes in a day, with a rate $\gamma S$. A normal smoker who stops smoking will go to the ex-smoker class, with a rate $\lambda S$, whereas the ex-smoker who returns to smoking habit will move to the normal smoker class, with a rate $\rho E \frac{S + S_e}{P}$. The mortality rate for normal smokers, both deaths from smoking habits $d_sS$ and death-dependent density $rPS$ decreases the number of normal smokers.

People join the heavy smoker class $S_e(t)$ at the rate, $\gamma S_e$ and $\Lambda \frac{S_e}{P}$. Some will leave heavy smoker classes at a rate of $\alpha S_e$ and $\delta S_e$, where $\alpha$ and $\delta$ is the rate of a heavy smoker being a normal smoker and the rate of a heavy smoker quitting, respectively. Heavy smoking deaths, both smoking deaths and density-dependent deaths, of which $(d_{se} + rP)S_e$, reduce the number of heavy smokers.

People will enter the ex-smoker class at the rate of $\delta S_e$ and $\Lambda \frac{E}{P}$. Some left the ex-smoker class at the rate of $\rho E \frac{S + S_e}{P}$ and $(d_e + rP)E$.

2.2. Flow Diagram of Smoking Epidemic Model

Here, we have given the flow diagram of the model. The compartments of the model are represented by rectangular boxes. The flow direction of the people among the compartments are represented by directed arrows.

![Flow Diagram of Smoking Epidemic Model](image)

2.3. Model Assumptions

In this section, for a diagram of the model of the spread of smoking behavior, the following assumptions are made.

- For this model, the transition parameters are constant over time because we want to observe the model in a short time.
• The size of the sub-populations (classes) and their changes over time cause dynamic changes to smoking habits.
• Transitions between non-smokers, normal smokers, heavy smokers and ex-smokers occur, proportional to the size of the class.
• Transitions from non-smokers to normal smokers occur because of the influence of smokers on non-smokers.
• Death rates that do not depend on the total population of each class vary with $d_n < d_e < d_s < d_{se}$.
• We assume, if an ex-smoker returns to smoking habit then he will enter the normal smoker class. Therefore, the transition between sub-populations $Se$ and $E$ is not taken into account.
• In addition, a smoker is said to be a heavy smoker if the number of cigarettes he consumes in a day is more than the average, which is 12 cigarettes a day.

2.4. Model Equations

By considering the assumptions and transitions that occur in the smoking behavior model, we get a system of differential equations for the model of the spread of smoking behavior as follows.

$$\frac{dN}{dt} = \Lambda \frac{N}{P} + \mu P - \beta N \frac{S + S_e}{P} - f_n N$$
$$\frac{dS}{dt} = \Lambda \frac{S}{P} + \beta N \frac{S + S_e}{P} + \alpha S_e + \rho E \frac{S + S_e}{P} - \gamma S - \lambda S - f_s S$$
$$\frac{dS_e}{dt} = \Lambda \frac{S_e}{P} + \gamma S - \alpha S_e - \delta S_e - f_{se} S_e$$
$$\frac{dE}{dt} = \Lambda \frac{E}{P} + \lambda S + \delta S_e - \rho E \frac{S + S_e}{P} - f_e E.$$ (1)

Because in this study we use data in the form of percentages, so we change the scale of the system equation (1) to the same unit with the data obtained. After scaling, we get the system of equations as follows.

$$\frac{ds}{dt} = (d_s - d_n - \beta) s^2 + (\beta - \mu - \gamma - \lambda - d_s + d_n) s + (d_{se} - d_n - 2\beta) ss_e$$
$$+ (d_e - d_n - \beta) se - \beta s_e^2 + (\beta + \alpha) s_e - \beta s_e e + \rho c(s + s_e)$$
$$\frac{ds_e}{dt} = \gamma s + (d_s - d_n) ss_e + (d_{se} - d_n) s_e^2 + (d_n - \alpha - \delta - \mu - d_{se}) s_e + (d_e - d_n) e s_e$$
$$\frac{de}{dt} = \lambda s + (d_s - d_n) se + \delta s_e + (d_{se} - d_n) s_e e + (d_e - d_n) e^2 + (d_n - d_e - \mu) e$$
$$- \rho c(s + s_e)$$
$$\frac{dP}{dt} = \Lambda - \tau P^2 - ((d_s - d_n) s + (d_{se} - d_n) s_e + (d_e - d_n) e + d_n - \mu) P.$$ (2)

where the new variable scale is,

$$s = \frac{S}{P}, \quad s_e = \frac{S_e}{P}, \quad e = \frac{E}{P},$$

and

$$n = (1 - s - s_e - e).$$
The solution of the system (2) above is contained in the domain:

\[ D = \left\{ \begin{bmatrix} s \\ s_E \\ e \\ P \end{bmatrix} \in \mathbb{R}^4 \ \middle| \ \begin{array}{l} s \geq 0, \\ s_E \geq 0, \\ e \geq 0, \\ s + s_E + e \leq 1, \\ P > 0 \end{array} \right\}. \]  

(3)

2.5. Existence of Smoking-free Equilibrium Points

The smoke-free equilibrium point is a steady-state solution with no smoker in the population. Therefore, the system (2) is evaluated for \( s = 0 \) and \( s_E = 0 \).

By setting \( \frac{ds}{dt} = 0 \), \( \frac{de}{dt} = 0 \), \( \frac{dp}{dt} = 0 \) we get, \( e = s_E = e = 0 \) and \( P^* = \frac{(\mu - d_n(1-\epsilon)-d_e) + \sqrt{(\mu - d_n(1-\epsilon)-d_e)^2 + 4\Lambda r}}{2r} \). Thus, the smoke-free equilibrium point obtained for the system of equations (2) is

\[ x_{sfe} = (0,0,0,P^*) \] or \[ x_{sfe} = (0,0,0,\frac{\mu - d_n + \sqrt{(\mu - d_n)^2 + 4\Lambda r}}{2r}). \]  

(4)

2.6. The Basic Reproduction Number \((R_0)\)

In this paper, basic reproductive numbers are used to analyze the stability of the smoke-free equilibrium point of the model of the spread of smoking behavior.

Basic reproductive numbers, denoted \( R_0 \), is defined as the number of new infections that occur in vulnerable classes. If \( R_0 > 1 \), the infected individual transmits more than one individual so that the disease can invade the population. Conversely, if \( R_0 < 1 \) then the infected individual on average transmits less than one individual so that the disease cannot grow in the population.

The method used to determine \( R_0 \) is the Watmough and Driessche method, which is the next generation matrix method with a value of \( R_0 \geq 0 \). By this method, we get the reproductive number of the system (2) is given below,

\[ R_0 = \frac{\beta(\mu + \delta + \lambda + d_{se} - d_n + \gamma)}{(\gamma + \lambda + \mu + d_a - d_n)(\mu + \delta + \alpha + d_{se} - d_n) - \alpha \gamma}. \]

Here, reproductive numbers are used to analyze the sensitivity of the smoker-free equilibrium point, which is explained in the following theorem.

**Theorem 1.** Given the smoking-free equilibrium point of the system (2) is,

\[ x_{sfe} = (0,0,0,P^*) \]

\[ = \left(0,0,0,\frac{\mu - d_n + \sqrt{(\mu - d_n)^2 + 4\Lambda r}}{2r}\right) \]

and the reproductive number,

\[ R_0 = \frac{\beta(\mu + \delta + \lambda + d_{se} - d_n + \gamma)}{(\gamma + \lambda + \mu + d_a - d_n)(\mu + \delta + \alpha + d_{se} - d_n) - \alpha \gamma}. \]

If \( R_0 < 1 \) then \( x_{sfe} \) is locally asymptotically stable, but unstable if the \( R_0 > 1 \).
3. Numerical Analysis

In this section, we hold the numerical analysis using the Python programming language which includes the existence of the endemic equilibrium point and the effect of selected parameters on the class of smokers.

The value of parameters of the birth rate and the pure death rate of the population are obtained from [10] and the population mortality rate due to smoking is obtained from [11]. Meanwhile, the level of population migration is obtained from [12]. The value of the rate of social interaction rate that causes a non-smoker to be a \( \beta \) smoker, the rate of a normal smoker quitting \( \lambda \), the rate of a heavy smoker quitting \( \delta \), the rate of a normal smoker being a heavy smoker \( \gamma \) and the rate of a heavy smoker being a normal smoker \( \alpha \) is obtained from [3]. The other parameter value is the author’s assumption while the initial value is obtained from [2]. These values are presented as follows:

| Parameter | Value (year\(^{-1}\)) |
|-----------|---------------------|
| \( \mu \) | 0.00162             |
| \( \Lambda \) | 180                |
| \( d_n \) | 0.00065            |
| \( d_s \) | 0.0013             |
| \( d_{se} \) | 0.002          |
| \( d_e \) | 0.000975          |
| \( r \) | 0.000000000065    |
| \( \beta \) | 0.0381          |
| \( \gamma \) | 0.1175         |
| \( \alpha \) | 0.1244        |
| \( \lambda \) | 0.0498        |
| \( \delta \) | 0.0398        |
| \( \rho \) | 0.0825         |

and

\[
 z_0 = [0.3176, 0.0762, 0.045, 2000000] \tag{5}
\]

3.1. The Existence and Stability of The Endemic Equilibrium

The existence of endemic equilibrium points will be discussed numerically by considering the stability of each proportion over a long period of time.

Based on Table 1, we obtain the basic reproductive number \( R_0 = 0.5694 < 1 \). Then, we get the result of the numerical analysis as shown in Figure 3 below.
In the Figure (3), it shown that each proportion will be constant at the smoke-free equilibrium point after 400 years. In other words, there is no endemic equilibrium point. It also shown that smoker classes disappear from the population after 200 years. Therefore, it can be said that it takes a long time to destroy the smoking habit of the population if the parameter value in table 1 maintained. However, if the parameter $\beta$, $\lambda$ and $\delta$ are each converted to 0.0031, 0.0798 and 0.0698 so that the basic reproductive number is $R_0 = 0.033$, then we get the numerical analysis results as shown in picture below.

In the figure 4, it is shown that smokers classes disappear faster than the results of numerical analysis shown in Figure 3. Smoker classes disappear from the population after 50 years, whereas in Figure 3 it takes 200 years to eliminate the class of smokers from the population. Therefore, we can say to eliminate smoker classes from a population of 150 years faster, the $\beta$ parameter value needs to be reduced to 0.0031 while the $\lambda$ and $\delta$ parameter values need to be increased to 0.0798 and 0.0698.
We will discuss about the existence and stability of the equilibrium point with the existence of the class of smokers if the value of basic reproductive numbers is $R_0 > 1$. Thus, the value of $\beta = 0.0881$, $\lambda = 0.0398$, and $\delta = 0.0298$, while the other parameter values are in accordance with Table 1. Based on this value the basic reproductive number is obtained $R_0 = 1.66 > 1$. Based on parameter values and initial values, the results of numerical analysis are shown in the figure below.

Figure 5. The Proportion of $s$, $s_e$ and $e$ vs Time (year) for $R_0 > 1$.

The figure 3.1 shows that the three proportions of $s$, $s_e$ and $t$ are constant with time after 125 years. The equilibrium point with the existence of the smoker class when $R_0 > 1$ exists. Next, we will discuss the stability of the equilibrium point. The discussion was carried out using the definition presented as follows:

**Definition 1. (Liapunov)** Given $\bar{x}(t)$ is the solution of the system (2). Point $\bar{x}(t)$ is said to be stable if, given $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ so that for each solution $y(t)$ from the system (2) with $|\bar{x}(t_0) - y(t_0)| < \delta$ applies,

$$|\bar{x}(t) - y(t)| < \epsilon$$

for $t > t_0, t_0 \in \mathbb{R}$.

Furthermore, the definition of asymptotic stability is given below.

**Definition 2. (Asymptotic stability)** The point $\bar{x}(t)$ is said to be asymptotically stable if $\bar{x}(t)$ is stable and for each solution $y(t)$ of the system (2), there is $b > 0$ so, for $|\bar{x}(t_0) - y(t_0)| < b$ applies

$$\lim_{t \to \infty} |\bar{x}(t) - y(t)| = 0.$$

According to the above definition of stability, it can be said that if a equilibrium point is stable then a small change in the initial value only causes a very small effect on the solution around the equilibrium point.

Based on the Definition 2, we choose $c = 0.08$ so that the initial value becomes

$$z_0 = [s_0, s_{e0}, e_0, P_0]$$
where

\[
0.2376 \leq s_0 \leq 0.3976, \\
0.0038 \leq s_e_0 \leq 0.1562, \\
0.035 \leq e_0 \leq 0.125 \\
\text{and } P_0 = 2000000.
\] (6)

Based on the parameter values and the initial values obtained results as shown in figure below.

Figure 6. The Proportion of \( s \) vs time while \( R_0 > 1 \).

Figure 7. The Proportion of \( s_e \) vs time while \( R_0 > 1 \).
In the Figure 3.1, Figure 3.1, and Figure 3.1, it shown that if the initial value is changed slightly, then the solution of each proportion will lead to the same point, namely the endemic equilibrium point. In other words, the equilibrium point satisfies the Definition 2. Thus, the equilibrium point with the existence of a smoker class is local asymptotically stable for the basic reproductive number of $R_0 > 1$.

### 3.2. Sensitivity Analysis

In this section we will discuss how the influence of the rate of social interaction between non-smokers and smokers ($\beta$) on the proportion of smokers. We will see how sensitive the proportion of normal-smokers to $\beta$. We set the value of $\beta$ is in the estimation $0.0181 \leq \beta \leq 0.0681$ and the other parameters is in Table 1. By initial value (5) and the parameters value, we obtain the result of numerical analysis as described on Figure 9.

![Figure 9](image.png)

**Figure 9.** The parameter $\beta$ vs the Proportion of Normal Smokers and Heavy Smokers.

It shows that the increase of $\beta$ value results in the sizes of proportion of the normal smoker $s$ and the heavy smoker $s_e$ also increase. The black dot in the Figure 9 shows the value of the $\beta$ parameter set for the current time of 0.0381. According to the results of numerical analysis, if the value increases changes by 0.0005 then the size of the proportion of the normal smoker class $s$ and heavy smoker $s_e$ also changes by 0.0007 and 0.0004. Based on this, the $s$ normal smoker class is more sensitive to the $\beta$ parameter than the $s_e$ heavy smoker class. Thus, according to
the results of numerical analysis it was found that for the next 25 years when the rate of social interaction that caused a non-smoker to become a smoker was reduced by 0.0005, the proportion of the proportion of normal smokers and heavy smokers decreased by 0.0007 and 0.0004.

4. Conclusion
In this paper, we have proposed an epidemic model of smoking behaviour. This model consists of a system of non-linear differential equations describing the rate of changes in each proportion. This model has an equilibrium point, both a smoke-free equilibrium point and an endemic equilibrium point. Using basic reproductive numbers, we get an asymptotically stable smoking-free equilibrium point. While the reproductive number is greater than zero, the endemic equilibrium point is exist and asymptotically stable. Based on the sensitivity analysis of the rate parameters of social interactions that cause a non-smoker to become a smoker, an increase in the value of these parameters causes the size of the proportion of smoker classes to increase. However, the class of normal smokers is more sensitive to these parameters compared to heavy smokers.

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