Projective Representations

I. Projective lines over rings

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Dedicated to Armin Herzer on the occasion of his 70th birthday.

Abstract

We discuss representations of the projective line over a ring $R$ with 1 in a projective space over some (not necessarily commutative) field $K$. Such a representation is based upon a $(K,R)$-bimodule $U$. The points of the projective line over $R$ are represented by certain subspaces of the projective space $\mathbb{P}(K,U \times U)$ that are isomorphic to one of their complements. In particular, distant points go over to complementary subspaces, but in certain cases, also non-distant points may have complementary images.

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1 Introduction

Each ring $R$ with 1, containing in its centre a (necessarily commutative) field $F$ with $1 \in F$, gives rise to a chain geometry $\Sigma(F,R)$. For a survey, see [11]. In [4] we introduced the concept of a generalized chain geometry $\Sigma(F,R)$; now $R$ is a ring with 1 containing a (not necessarily commutative) field $F$ subject to $1 \in F$. In both cases the point set of $\Sigma(F,R)$ is $\mathbb{P}(R)$, i.e., the projective line over $R$, and the chains are the $F$-sublines.

In the present paper we introduce representations of the projective line over an arbitrary ring $R$ in a projective space over some field $K$. In a second publication our results will be applied to obtain representations of generalized chain geometries.

The starting point of our investigation is A. Herzer’s approach [11] to obtain a model of a chain geometry $\Sigma(F,R)$ for a finite-dimensional $F$-algebra $R$ by means of a faithful right $R$-module $U$ with finite $F$-dimension, say $r$. Here the points of the projective line $\mathbb{P}(R)$ are in one-one correspondence with certain $(r-1)$-dimensional subspaces of the $(2r-1)$-dimensional projective space $\mathbb{P}(F,U \times U)$. In our more general setting we use a $(K,R)$-bimodule $U$; so $U$ is a left $K$-vector space and at the same time a right $R$-module. We neither do assume that $K$ is a subset of $R$, nor that $U$ is a faithful $R$-module, nor that the $K$-dimension of $U$ is finite. A projective representation obtained in this way maps the points of $\mathbb{P}(R)$ into the set of those subspaces of the projective space $\mathbb{P}(K,U \times U)$ that are isomorphic to one

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of their complements. This mapping is injective if, and only if, \( U \) is a faithful \( R \)-module. In this case we speak of a \textit{projective model} of \( \mathbb{P}(R) \).

If \( U' \) is a sub-bimodule of \( U \) then there are representations of \( \mathbb{P}(R) \) that stem from the action of \( R \) on \( U' \) and \( U/U' \). In general, these \textit{induced representations} are not injective, even if \( U \) is faithful. This is one of the reasons why we also discuss non-injective representations. The examples at the end of the paper illustrate how these induced representations can sometimes be used in order to describe models of \( \mathbb{P}(R) \) in terms of \( \mathbb{P}(K, U \times U) \).

2 \hspace{1em} \textbf{The projective line over a ring}

Let \( R \) be a ring. Throughout this paper we shall only consider rings with 1 (where the trivial case \( 1 = 0 \) is not excluded). The group of invertible elements of the ring \( R \) will be denoted by \( R^* \). Consider the free left \( R \)-module \( R^2 \). Its automorphism group is the group \( \text{GL}_2(R) \) of invertible \( 2 \times 2 \)-matrices with entries in \( R \). According to [4], [11], the \textit{projective line over} \( R \) is the orbit

\[
\mathbb{P}(R) := R(1, 0)^{\text{GL}_2(R)}
\]

of the free cyclic submodule \( R(1, 0) \) under the action of \( \text{GL}_2(R) \). Since \( R^2 = R(1, 0) \oplus R(0, 1) \), the elements (the \textit{points}) of \( \mathbb{P}(R) \) are exactly those free cyclic submodules of \( R^2 \) that have a free cyclic complement.

A pair \( (a, b) \in R^2 \) is called \textit{admissible}, if there exist \( c, d \in R \) such that \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(R) \). So \( \mathbb{P}(R) = \{ R(a, b) \subset R^2 \mid (a, b) \text{ admissible} \} \). However, in certain cases the points of \( \mathbb{P}(R) \) may also be represented by non-admissible pairs, as we will see below.

We recall that a pair \( (a, b) \in R^2 \) is \textit{unimodular}, if there exist \( x, y \in R \) such that \( ax + by = 1 \), i.e., if there is an \( R \)-linear form \( R^2 \to R \) mapping \( (a, b) \) to 1. This is equivalent to saying that the right ideal generated by \( a \) and \( b \) is the whole ring \( R \).

Obviously, each admissible pair \( (a, b) \) is unimodular. If \( R \) is commutative, then admissibility and unimodularity are equivalent. W. Benz in [1] considers only commutative rings and defines the projective line using unimodular pairs.

\textbf{Proposition 2.1} \hspace{1em} \textit{Let} \( (a, b) \in R^2 \) \textit{be admissible}, and let \( s \in R \). Put \( (a', b') := s(a, b) \). \textit{Then}

\begin{enumerate}
    \item \( s \) \textit{is left invertible} \iff \( R(a, b) = R(a', b') \).
    \item \( s \) \textit{is right invertible} \iff \( (a', b') \) \textit{is admissible}.
\end{enumerate}

\textbf{Proof}: (1): If there is an \( l \in R \) with \( ls = 1 \), then \( (a, b) = l(a', b') \). So \( R(a, b) = R(a', b') \).

If \( R(a, b) = R(a', b') \), then there is an \( l \in R \) such that \( (a, b) = l(a', b') \). Since \( (a, b) \) is admissible, it is also unimodular, and so there are \( x, y \in R \) with \( 1 = ax + by = lsax + lsby = ls \). Hence \( s \) is left invertible.

(2): If \( s \) is right invertible, then \( sr = 1 \) for some \( r \in R \). An easy calculation shows that

\[
\gamma = \begin{pmatrix} s & 0 \\ 1 - rs & r \end{pmatrix} \in \text{GL}_2(R), \quad \text{with} \quad \gamma^{-1} = \begin{pmatrix} r & 1 - rs \\ 0 & s \end{pmatrix}.
\]
There is a matrix \((a,b)\) \(\in\) GL\(_2\)(\(R\)), whence \((a',b') = \gamma (a,b) \in\) GL\(_2\)(\(R\)), as required. If \((a',b')\) is admissible, then there are \(x', y' \in R\) with \(a'x' + b'y' = 1\). So \(s(ax' + by') = 1\), i.e., \(s\) has a right inverse. □

Note that the statement of Proposition 2.1 remains true if one substitutes “admissible” by “unimodular”, however, the proof of (2)\(\Rightarrow\)” then has to be modified.

Rings with the property that \(ab = 1\) implies \(ba = 1\) are called Dedekind-finite (see e.g. [13]). From Proposition 2.1 we obtain

**Proposition 2.2** Let \(R\) be a ring. Then the following are equivalent:

1. \(R\) is Dedekind-finite.
2. If \((a,b) \in \mathbb{P}(R)\), then \((a,b)\) is admissible.
3. No point of \(\mathbb{P}(R)\) is properly contained in another point of \(\mathbb{P}(R)\).

**Remark 2.3** If \(R\) is not Dedekind-finite, then each point \(p \in \mathbb{P}(R)\) belongs to an infinite sequence of points

\[
\cdots \subset p_{-2} \subset p_{-1} \subset p_0 = p \subset p_1 \subset p_2 \subset \cdots
\]

Namely, let \(\gamma\) be the matrix of formula (1), where \(sr = 1 \neq rs\). Then Proposition 2.1 shows that the points \(p_i := p^{\gamma^i}\) are as desired.

Recall that according to F.D. Veldkamp [15], [16] the ring \(R\) has stable rank 2, if for each unimodular pair \((a,b) \in R^2\) there is a \(c \in R\) such that \(a + bc\) is right invertible. The following results on rings of stable rank 2 can be found in [15] (results 2.10 and 2.11):

**Remark 2.4** Let \(R\) be of stable rank 2. Then \(R\) is Dedekind-finite and each unimodular \((a,b) \in R^2\) is admissible.

Note that Herzer’s definition of stable rank 2 in [11] seems to be stronger but actually coincides with Veldkamp’s because of 2.4. Moreover, it is not necessary to distinguish between left and right stable rank 2 because by [15], 2.2, the opposite ring (with reversed multiplication) of a ring of stable rank 2 also has stable rank 2.

Using results of [13], § 20, one obtains that each left (or right) artinian ring has stable rank 2 (called “left stable range 1” in [13]). We shall need the following special case:

**Remark 2.5** Assume that \(R\) contains a subfield \(K\) such that \(R\) is a finite-dimensional left (or right) vector space over \(K\). Then \(R\) is of stable rank 2. In particular, \(R\) is Dedekind-finite.

Here by a subfield we mean a not necessarily commutative field \(K \subset R\) with \(1 \in K\).

We turn back to the projective line over an arbitrary ring. The point set \(\mathbb{P}(R)\) is endowed with the symmetric relation \(\triangle\) ("distant") defined by

\[
\triangle := \{R(1,0), R(0,1)\}^{\text{GL}_2(R)}
\]

i.e., two points \(p, q \in \mathbb{P}(R)\) are distant exactly if there is a \(\gamma \in \text{GL}_2(R)\) mapping \(R(1,0)\) to \(p\) and \(R(0,1)\) to \(q\). Distance can also be expressed in terms of coordinates:
Remark 2.6  Let \( p = R(a, b), q = R(c, d) \in \mathbb{P}(R) \) with admissible \((a, b), (c, d)\). Then
\[
p \triangle q \iff \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(R).
\]

Note that this is independent of the choice of the admissible representatives \((a, b), (c, d)\). In addition, \( \triangle \) is anti-reflexive exactly if \( 1 \neq 0 \); compare [11].

By definition, two points of \( \mathbb{P}(R) \) are distant if, and only if, they are complementary sub-modules of \( R^2 \). There are several possibilities for points being non-distant, which all can occur as the following examples show:

Examples 2.7  \( (1) \) Let \( R \) be a ring that is not Dedekind-finite. Let \( \gamma \in \text{GL}_2(R) \) be defined as in Remark 2.3. Then \( p = R(1, 0) \gamma = R(s, 0) \) and \( q = R(0, 1) \) are non-distant: They have a trivial intersection but they do not span \( R^2 \).

Now consider \( p' = R(1, 0) \gamma^{-1} = R(r, 1 - rs) \) (see formula (1)). Then \( p' \) and \( q \) are non-distant: They span \( R^2 \), but \( (1 - rs)(r, 1 - rs) = (0, 1 - rs) \neq (0, 0) \) lies in their intersection.

\( (2) \) Let \( R \) contain a subfield \( K \) such that \( R \), considered as left vector space over \( K \), has finite dimension \( n \). Then all points of \( \mathbb{P}(R) \) are \( n \)-dimensional subspaces of the left vector space \( R^2 \). In particular, two points have a trivial intersection exactly if they span \( R^2 \).

In Example 4.7 below we will see an example of a commutative (and hence Dedekind-finite) ring where non-distant points intersect trivially.

### 3 Homomorphisms

Now we want to study mappings between projective lines over rings that are induced by ring homomorphisms.

From now on, we will follow the convention that whenever a point of \( \mathbb{P}(R) \) is given in the form \( R(a, b) \), we always assume that the pair \((a, b) \in R^2\) is admissible.

Let \( R, S \) be rings. The distance relations on \( \mathbb{P}(R) \) and \( \mathbb{P}(S) \) are denoted by \( \triangle_R \) and \( \triangle_S \), respectively. Consider a ring homomorphism \( \varphi : R \to S \), where we always suppose that \( 1_R \) is mapped to \( 1_S \). Associated to \( \varphi \) is a homomorphism \( M(2 \times 2, R) \to M(2 \times 2, S) \), mapping \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) to \( \left( \begin{array}{cc} a^\varphi & b^\varphi \\ c^\varphi & d^\varphi \end{array} \right) \), which will also be denoted by \( \varphi \). Its restriction to \( \text{GL}_2(R) \) is a group homomorphism into \( \text{GL}_2(S) \). This implies that if \((a, b) \in R^2\) is admissible, so is \((a^\varphi, b^\varphi) \in S^2\), and we can introduce the mapping
\[
\varphi : \mathbb{P}(R) \to \mathbb{P}(S) : R(a, b) \mapsto S(a^\varphi, b^\varphi).
\]

**Proposition 3.1**  Let \( \varphi : R \to S \) be a ring homomorphism. Then for \( \bar{\varphi} : \mathbb{P}(R) \to \mathbb{P}(S) \) the following statements hold:

1. \( \bar{\varphi} \) preserves distance, i.e., \( \forall p, q \in \mathbb{P}(R) : p \triangle_R q \Rightarrow p^\varphi \triangle_S q^\varphi \).
According to J.R. Silvester \cite{14} we introduce the following notions for a ring $\mathbb{R}$.

(2) $\phi$ is compatible with the action of $\text{GL}_2(\mathbb{R})$, i.e. $\forall p \in \mathbb{P}(\mathbb{R}) \forall \gamma \in \text{GL}_2(\mathbb{R}) : p^\gamma \phi = p^{\phi} \gamma^\phi$.

(3) $\phi$ is injective if, and only if, $\phi$ is.

Proof: Only (3) deserves our attention. Let $\phi$ be injective. Assume that $R(a,b)^\phi = R(c,d)^\phi$ holds for $R(a,b)$, $R(c,d) \in \mathbb{P}(\mathbb{R})$. Then there is an $s \in S^*$ with $(a^\phi,b^\phi) = s(c^\phi,d^\phi)$. Since $(c,d) \in \mathbb{R}^2$ is unimodular, there are $x, y \in \mathbb{R}$ with $s = s1 = s1^\phi = s(cx+dy)^\phi = a^\phi x^\phi + b^\phi y^\phi \in \mathbb{R}^\phi$. Analogously, one sees that $s^{-1} \in \mathbb{R}^\phi$. Hence $s \in (\mathbb{R}^\phi)^*$, which equals $(\mathbb{R}^\phi)^\phi$, since $\phi$ is injective. So $R(a,b) = R(c,d)$.

Now let $\phi$ be injective, and assume $a^\phi = b^\phi$ for $a, b \in \mathbb{R}$. Then $R(1,a)^\phi = S(1,a^\phi) = S(1,b^\phi) = R(1,b)^\phi$, whence $R(1,a) = R(1,b)$ and so $a = b$. \qed

We call the mapping $\phi : \mathbb{P}(\mathbb{R}) \rightarrow \mathbb{P}(\mathbb{S})$ the homomorphism of projective lines induced by $\varphi : \mathbb{R} \rightarrow \mathbb{S}$. Such homomorphisms map distant points to distant points. However, they may also map non-distant points to distant points: Consider e.g. the homomorphism $\mathbb{P}(\mathbb{Z}) \rightarrow \mathbb{P}(\mathbb{Q})$ induced by the natural inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$. This injective homomorphism actually is a bijection, since each element of $\mathbb{P}(\mathbb{Q})$ can be represented by a pair of relatively prime integers. The points $\mathbb{Z}(1,0)$ and $\mathbb{Z}(1,2)$ are non-distant because $(\begin{smallmatrix}1 & 0 \\ 1 & 2 \end{smallmatrix})$ is not invertible over $\mathbb{Z}$.

However, their image points $\mathbb{Q}(1,0)$ and $\mathbb{Q}(1,2)$ are different and hence distant in $\mathbb{P}(\mathbb{Q})$.

The following gives a characterization of the homomorphisms $\phi$ that preserve also non-distance. By $\text{rad}(\mathbb{R})$ we denote the (Jacobson) radical of the ring $\mathbb{R}$ (cf. \cite{13}).

**Proposition 3.2** Let $\phi : \mathbb{P}(\mathbb{R}) \rightarrow \mathbb{P}(\mathbb{S})$ be induced by the ring homomorphism $\varphi : \mathbb{R} \rightarrow \mathbb{S}$. Then the following statements are equivalent:

1. $\forall p, q \in \mathbb{P}(\mathbb{R}) : p^\phi \triangle q^\phi \Rightarrow p \triangle \text{rad}(\mathbb{R}) q$.
2. $\forall y \in \mathbb{R} : y^\phi \in S^* \Rightarrow y \in R^*$.
3. $\ker(\phi) \subset \text{rad}(\mathbb{R})$ and $(R^\phi)^* = S^* \cap R^\phi$.

Proof: (1) $\Rightarrow$ (2): For $r \in \mathbb{R}$ with $r^\phi \in S^*$ we have $S(1,0)\triangle S(1,r^\phi)$. Hence condition (1) implies $R(1,0)\triangle R(1,r)$ and thus $r \in R^*$. (2) $\Rightarrow$ (1): Let $p^\phi \triangle q^\phi$ hold for $p, q \in \mathbb{P}(\mathbb{R})$. Choose $\gamma \in \text{GL}_2(\mathbb{R})$ with $\gamma^\phi = R(1,0)$. Then $q^\gamma = R(x,y)$ for a certain admissible pair $(x,y) \in \mathbb{R}^2$. By 3.1(2), we have $S(1,0) = p^{\phi\gamma} = p^{\phi \gamma^\phi} \triangle q^{\phi^\gamma}$, and hence $y^\phi \in S^*$. So, by (2), $y \in R^*$. This implies $p^{\phi \gamma} \triangle R q^\gamma$ and thus also $p \triangle \text{rad}(\mathbb{R}) q$.

(2) $\Leftrightarrow$ (3): See \cite{8}, Lemma 1.5. \qed

As the example $\mathbb{P}(\mathbb{Z}) \rightarrow \mathbb{P}(\mathbb{Q})$ above shows, the ring homomorphism $\phi$ need not be surjective if $\phi$ is.

We now consider the case where $\varphi : \mathbb{R} \rightarrow \mathbb{S}$ is a surjective homomorphism of rings. It is not clear whether in general $\phi$ also is surjective. We study special cases.

According to J.R. Silvester \cite{14} we introduce the following notions for a ring $\mathbb{R}$:

The elementary linear group $E_2(\mathbb{R})$ is the subgroup of $\text{GL}_2(\mathbb{R})$ generated by the elementary transvections, i.e. by all matrices of the form $\begin{pmatrix}1 & 0 \\ x & 1 \end{pmatrix}$ or $\begin{pmatrix}1 & x \\ 0 & 1 \end{pmatrix}$ ($x \in \mathbb{R}$). The group $\text{GE}_2(\mathbb{R})$
is the subgroup of $\text{GL}_2(R)$ generated by $E_2(R)$ and all diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \text{GL}_2(R)$. Note that $E_2(R)$ is normal in $\text{GE}_2(R)$. If $\text{GE}_2(R) = \text{GL}_2(R)$, then $R$ is called a $\text{GE}_2$-ring. Examples of $\text{GE}_2$-rings and also of rings that are not $\text{GE}_2$-rings can be found in [14], p.114 and p.121, respectively. Important for us is the following:

**Remark 4.1** (see [2]). Let $U$ be a left vector space over a field $K$, and let $S = \text{End}_K(U)$ be its endomorphism ring. Moreover, let $\mathcal{G}$ be the set of all subspaces of the projective space $\mathbb{P}(K, U \times U)$ that are isomorphic to one of their complements. Then

$$\Psi : \mathbb{P}(S) \to \mathcal{G} : S(\alpha, \beta) \mapsto U^{(\alpha, \beta)} := \{ (u^\alpha, u^\beta) \mid u \in U \}$$

**4 Projective representations**

The projective representations we are now aiming at are based upon the following.

**Remark 3.3** (See [9], 4.2.5.) Let $R$ be a ring of stable rank 2. Then $R$ is a $\text{GE}_2$-ring.

**Lemma 3.4** Let $R$ be a $\text{GE}_2$-ring. Then $\mathbb{P}(R) = R(1, 0)^{E_2(R)}$.

**Proof:** Let $p = R(1, 0)^{\gamma} \in \mathbb{P}(R)$, with $\gamma \in \text{GL}_2(R)$. Since $\text{GL}_2(R) = \text{GE}_2(R)$ and $E_2(R)$ is normal in $\text{GE}_2(R)$, we have $\gamma = \delta \eta$, where $\delta = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\eta \in E_2(R)$. So $p = R(1, 0)^{\gamma} = R(a^{-1}, 0)^{\gamma} = R(1, 0)^{\eta} \in R(1, 0)^{E_2(R)}$. □

Now we can state conditions that imply that with $\varphi : R \to S$ also $\bar{\varphi}$ is surjective.

**Proposition 3.5** Let $\varphi : R \to S$ be a surjective homomorphism of rings. Then also $\bar{\varphi} : \mathbb{P}(R) \to \mathbb{P}(S)$ is surjective, if one of the following conditions is satisfied:

1. $S$ is a $\text{GE}_2$-ring.
2. $\ker(\varphi) \subseteq \text{rad}(R)$.
3. $R$ is the internal direct product of $\ker(\varphi)$ and some ideal $R' \subset R$.

**Proof:** (1): Consider a point $q \in \mathbb{P}(S)$. By Lemma 3.4 we have $q = S(1, 0)^{\eta}$, where $\eta \in E_2(S)$, i.e., $\eta$ is a product of elementary transvections. Since $\varphi : R \to S$ is surjective, each elementary transvection has a preimage under $\varphi : \text{M}(2 \times 2, R) \to \text{M}(2 \times 2, S)$ that is an elementary transvection over $R$. Hence $\eta = \gamma \bar{\varphi}$, where $\gamma \in E_2(R)$, and so by 3.1(2) we obtain $q = R(1, 0)^{\gamma \bar{\varphi}} = R(1, 0)^{\eta \bar{\varphi}} \in \mathbb{P}(R)^\bar{\varphi}$.

(2): Follows from [5], Lemma 1.14.

(3): In this case, $\text{GL}_2(R)$ consists exactly of the sums $\gamma + \gamma'$, where $\gamma \in \text{GL}_2(\ker(\varphi))$ and $\gamma' \in \text{GL}_2(R')$. Moreover, $\varphi|_{\text{GL}_2(R')} : \text{GL}_2(R') \to \text{GL}_2(S)$ is an isomorphism of groups. This yields the assertion. □

Note that one could also use Proposition 3.2 in order to prove assertion (2) above, since the radical of $\text{M}(2 \times 2, R)$ consists exactly of all matrices with entries in $\text{rad}(R)$.
is a well-defined bijection mapping distant points of \( \mathbb{P}(S) \) to complementary subspaces in \( \mathcal{G} \) and non-distant points to non-complementary subspaces. Moreover, the groups \( \text{GL}_2(S) \) and \( \text{Aut}_K(U \times U) \) are isomorphic and their actions on \( \mathbb{P}(S) \) and on \( \mathcal{G} \), respectively, are equivalent via \( \Psi \). In particular, the mappings induced on \( \mathcal{G} \) by \( \text{GL}_2(S) \) arise from projective collineations of the projective space \( \mathbb{P}(K, U \times U) \).

Let now \( K \) be a field and let \( R \) be a ring. A left vector space \( U \) over \( K \) is called a \((K,R)\)-bimodule, if \( U \) is a (unitary) right \( R \)-module such that for all \( k \in K \), \( u \in U \), \( a \in R \) the equality \( k(u \cdot a) = (ku) \cdot a \) holds. If \( U \) is a \((K,R)\)-bimodule, then \( \varphi : R \to \text{End}_K(U) \) with \( a\varphi = \rho_a : u \mapsto u \cdot a \) is a ring homomorphism.

If, on the other hand, there is a homomorphism \( \varphi : R \to \text{End}_K(U) \), then \( U \) becomes a \((K,R)\)-bimodule by setting \( u \cdot a := u^{\rho_a} \), where \( \rho_a = a\varphi \). A homomorphism \( \varphi : R \to \text{End}_K(U) \) is also called a \( K \)-linear representation of \( R \).

So, the concepts of a \( K \)-linear representation of \( R \) and a \((K,R)\)-bimodule are equivalent. Whenever we consider a \((K,R)\)-bimodule \( U \), we denote by \( \varphi \) the associated linear representation, and for \( a \in R \) we write \( \rho_a \) for the endomorphism \( a\varphi : u \mapsto u \cdot a \).

A \((K,R)\)-bimodule \( U \) and the associated linear representation \( \varphi \) are called faithful, exactly if \( \varphi \) is an injection.

Combining 3.1 and 4.1, we obtain our main result:

**Theorem 4.2** Let \( U \) be a \((K,R)\)-bimodule. Then the mapping

\[
\Phi := \varphi \Psi : \mathbb{P}(R) \to \mathcal{G} : R(a,b) \mapsto U^{(\rho_a,\rho_b)}
\]

maps distant points of \( \mathbb{P}(R) \) to complementary subspaces in \( \mathbb{P}(K, U \times U) \). The bimodule \( U \) is faithful if, and only if, \( \Phi \) is injective.

Thus, to each homomorphism \( \varphi : R \to \text{End}_K(U) \) corresponds a mapping \( \Phi \) (see above). We call \( \Phi \) a projective representation of \( \mathbb{P}(R) \), and a faithful projective representation if \( U \) is faithful. We are interested in the image of \( \mathbb{P}(R) \) under a projective representation. If the representation is faithful, then \( \Phi : \mathbb{P}(R) \to \mathbb{P}(R)^{\Phi} \) is a bijection, and the image \( \mathbb{P}(R)^{\Phi} \) can be seen as a model of \( \mathbb{P}(R) \) in the projective space; we then call \( \mathbb{P}(R)^{\Phi} \) a projective model of \( \mathbb{P}(R) \). Otherwise, one obtains a model of the projective line over another ring:

**Proposition 4.3** Let \( J = \text{ann}(U) \) be the annihilator of \( U \), i.e., the kernel of the representation \( \varphi : R \to \text{End}_K(U) \). Then the following statements hold:

1. **The mapping** \( \varphi_f : R/J \to \text{End}_K(U) \) with \( \rho_{a+J} : u \mapsto u^{\rho_a} \) is a faithful \( K \)-linear representation of \( R/J \). Hence \( \Phi_f = \varphi_f \Psi \) is a faithful projective representation of \( \mathbb{P}(R/J) \).

2. **The projective model** \( \mathbb{P}(R/J)^{\Phi_f} \) contains \( \mathbb{P}(R)^{\Phi} \).

3. **The mapping** \( \tilde{\pi} : \mathbb{P}(R) \to \mathbb{P}(R/J) \) induced by the canonical epimorphism \( \pi : R \to R/J \) is surjective if, and only if, \( \mathbb{P}(R/J)^{\Phi_f} = \mathbb{P}(R)^{\Phi} \).

Recall that Proposition 3.5 gives conditions under which the assumptions of statement (3) are met.
The representation $\varphi : R \to S = \text{End}_K(U)$ gives rise to a group homomorphism $\varphi : \text{GL}_2(R) \to \text{GL}_2(S) \cong \text{Aut}_K(U \times U)$. Using 3.1(2) and 4.1 we obtain the following:

**Proposition 4.4** Let $U$ be a $(K, R)$-bimodule, and let $\gamma \in \text{GL}_2(R)$. Then the induced mapping

$$\mathbb{P}(R)^\Phi \to \mathbb{P}(R)^\Phi : R(a, b)^\Phi \mapsto R(a, b)^\gamma^\Phi$$

is induced by a projective collineation of $\mathbb{P}(K, U \times U)$.

Finally, Proposition 3.2 yields the following:

**Proposition 4.5** Let $U$ be a $(K, R)$-bimodule. Then the corresponding projective representation $\Phi$ maps non-distant points to non-complementary subspaces exactly if for each $a \in R$ the condition $\rho_a \in \text{Aut}_K(U)$ implies $a \in R^*$.

Note that from $\rho_a \in \text{Aut}_K(U)$ and $a \in R^*$ one obtains that $(\rho_a)^{-1} = \rho_{a^{-1}}$.

We mention two classes of examples where the condition of Proposition 4.5 is satisfied.

**Examples 4.6** (1) Let $R$ contain $K$ as a subfield. Then $U = R$ is a left vector space over $K$, and $\varphi : R \to \text{End}_K(U)$ with $\rho_a : x \mapsto xa$ is a faithful linear representation of $R$, called the regular representation. In this case $\Phi$ is the identity, where the submodule $R(a, b) \in \mathbb{P}(R)$ is considered as a projective subspace of $\mathbb{P}(K, U \times U)$. So points of $\mathbb{P}(R)$ are distant exactly if their $\Phi$-images are complementary. This reflects the algebraic fact that the endomorphism $\rho_a : R \to R : x \mapsto xa$ is a bijection exactly if $a \in R^*$.

(2) Let $U$ be a faithful $(K, R)$-bimodule. Assume moreover that $R$ contains a subfield $L$ such that $R$ is a finite-dimensional left vector space over $L$. Then the projective representation $\Phi$ maps non-distant points to non-complementary subspaces:

In view of (1), it suffices to show that for each $a \in R$ with $\rho_a \in \text{Aut}_K(U)$ the $L$-linear mapping $R \to R : x \mapsto xa$ is injective. Suppose $xa = 0$ for $x \in R$. Then for all $u \in U$ we have $0 = u \cdot 0 = u \cdot (xa) = (u \cdot x)^\rho_a$. Since $\rho_a$ is an automorphism, this implies $u \cdot x = 0$ for all $u \in U$, and hence $x = 0$ because $U$ is a faithful $R$-module.

We proceed by giving an example of a faithful projective representation where non-distant points appear as complementary subspaces:

**Example 4.7** Let $K$ be any commutative field, let $R$ be the polynomial ring $R = K[X]$, and let $U = K(X)$ be its field of fractions. Then $U$ contains $K$ and $R$, and thus is a faithful $(K, R)$-bimodule in a natural way. Obviously, $\rho_X : u \mapsto uX$ is a bijection on $U$, but $X \not\in R^*$.

This means that e.g. $R(1, 0)$ and $R(1, X)$ are non-distant points of $\mathbb{P}(R)$, but their images $U^{(1, 0)} = U \times \{0\}$ and $U^{(1, \rho_X)} = \{(u, uX) \mid u \in U\}$ are complementary subspaces of $\mathbb{P}(K, U \times U)$. Note that $R(1, 0)$ and $R(1, X)$, considered as submodules of $R^2$, also intersect trivially, but they do not span $R^2$ (compare 2.7).

Note, moreover, that we could also interpret the elements $R(a, b)^\Phi = U(a, b)$ as points of the projective line over the field $U$. Hence any two such elements must be complementary.

In a similar way one can also construct examples where $R$ is not contained in any field: Let $R$ and $U$ be as above. Let $R[\varepsilon]$ be the ring of dual numbers over $R$, with $\varepsilon$ central, $\varepsilon \not\in K$,
and \( \varepsilon^2 = 0 \). Then \( \varepsilon \) is a zero-divisor and hence \( R[\varepsilon] \) is not embeddable into any field. Now take \( U[\varepsilon] \) and proceed as above.

Let \( U \) be a \((K, R)\)-bimodule. A subset \( U' \subset U \) is called a sub-bimodule of \( U \), if \( U' \) is a subspace of the left vector space \( U \) over \( K \) and at the same time a submodule of the right \( R \)-module \( U \). The linear representation of \( R \) given by the bimodule \( U' \) is \( \varphi' : a \mapsto \rho_a|_{U'} \).

The faithful representation \((\varphi')_f : R/\text{ann}(U') \to \text{End}_K(U') \) will be called the induced faithful representation.

Let again \( \rho \in \text{End}_K(U) \) and consider an arbitrary \( \rho \in \text{End}_K(U) \) such that \( \rho|_{U'} \) is contained in \( \rho(U') \). The kernel of this representation is the ideal consisting of all \( a \in R \) such that the image of \( \rho_a \) is contained in \( U' \). As above, we obtain an induced faithful representation \((\varphi')_f : R/\text{ann}(U') \to \text{End}_K(U') \).

Let \( \varphi' \) be a sub-bimodule of the \((K, R)\)-bimodule \( U \), and let \( \varphi' \) and \( \varphi \) be the associated projective representations of \( \mathbb{P}(R)\). Then for each \( p \in \mathbb{P}(R) \) we have

\[
p^{\varphi'} = p^\varphi \cap (U' \times U').
\]

In particular, each \( p^\varphi \) meets the projective subspace \( \mathbb{P}(K, U' \times U') \) in an element of \( \mathcal{G}' \).

**Proof:** First consider \( p = R(1, 0) \). Then \( p^{\varphi'} = U' \times \{0\} = (U \times \{0\}) \cap (U' \times U') = p^\varphi \cap (U' \times U') \).

Now consider an arbitrary \( p \in \mathbb{P}(R) \). Then \( p = R(1, 0)^\gamma \) for some \( \gamma \in \text{GL}_2(R) \). The induced automorphism \( \gamma \varphi' \) of \( U \times U \) leaves \( U' \times U' \) invariant, it coincides on \( U' \times U' \) with \( \gamma \varphi' \in \text{Aut}_K(U' \times U') \). This yields the assertion. \( \square \)

Note that the \( \Phi'^\gamma \)-image of \( \mathbb{P}(R) \) is contained in the image of \( \mathbb{P}(R/\text{ann}(U')) \) under the induced faithful representation \((\Phi')_f \). According to 4.3(3), the two sets coincide exactly if the mapping \( \tilde{\pi} : \mathbb{P}(R) \to \mathbb{P}(R/\text{ann}(U')) \), associated to the canonical epimorphism \( \pi : R \to R/\text{ann}(U') \), is surjective.

**Proposition 4.9** Let \( U = U' \oplus U'' \) be a \((K, R)\)-bimodule. Let \( \varphi, \varphi', \varphi'' \) be the associated representations of \( R \). Then for each \( p \in \mathbb{P}(R) \) we have \( p^\varphi = p^{\varphi'} \oplus p^{\varphi''} \).

**Proof:** As in the proof of Proposition 4.8 we first verify the assertion for \( p = R(1, 0) \) (with the help of 4.8) and then use the action of \( \text{GL}_2(R) \). \( \square \)

Let again \( U' \) be a sub-bimodule of the \((K, R)\)-bimodule \( U \). Then also \( \widetilde{U} = U/U' \) is a \((K, R)\)-bimodule, corresponding to the representation \( \widetilde{\varphi} : R \to \text{End}_K(\widetilde{U}) \), where \( \widetilde{\rho}_a : u + U' \mapsto u^{\rho_a} + U' \). The kernel of this representation is the ideal consisting of all \( a \in R \) such that the image of \( \rho_a \) is contained in \( U' \). As above, we obtain an induced faithful representation \((\widetilde{\varphi})_f : R/\ker(\widetilde{\varphi}) \to \text{End}_K(\widetilde{U}) \).

The projective representation \( \widetilde{\Phi} \) maps \( \mathbb{P}(R) \) into the set \( \mathcal{G} \) of all subspaces of \( \mathbb{P}(K, \widetilde{U} \times \widetilde{U}) \) that are isomorphic to one of their complements. Now the projective space \( \mathbb{P}(K, \widetilde{U} \times \widetilde{U}) \) is...
canonically isomorphic to the projective space of all subspaces of $\mathbb{P}(K, U \times U)$ containing $U' \times U'$, because $(U \times U)/(U' \times U') \cong \bar{U} \times \bar{U}$. We shall identify the elements of $\bar{G}$ with their images under this isomorphism. So we can compare $\bar{\Phi}$ and $\Phi$, and the same procedure as before yields

**Proposition 4.10** Let $\bar{U} = U/U'$, and let $\bar{\Phi}$ be the associated projective representation of $\mathbb{P}(R)$. Then for each $p \in \mathbb{P}(R)$ we have

$$p^\bar{\Phi} = p^\Phi + (U' \times U').$$

In particular, each $p^\Phi + (U' \times U')$ is an element of $\bar{G}$.

As before, one may also consider the induced faithful representation $(\bar{\Phi})_f$ of $\mathbb{P}(R/ \ker(\bar{\phi}))$.

### 5 Examples

In this section we study some examples. Note that we consider only rings $R$ that are finite-dimensional left vector spaces over a subfield $K$. Then for each ideal $I$ of $R$ also the ring $R/I$ is finite dimensional over $K$, whence $R/I$ is of stable rank 2 and hence a $GE_2$-ring (compare 2.5 and 3.3). So Proposition 3.5 implies that in all our examples the mapping $\tilde{\pi} : \mathbb{P}(R) \to \mathbb{P}(R/I)$ induced by the canonical epimorphism $\pi : R \to R/I$ is surjective.

**Example 5.1** Let $K = R$ be any (not necessarily commutative) field and let $U = K^2$ with componentwise action $(x_1, x_2) \cdot k = (x_1 k, x_2 k)$. Then $U$ is the direct sum of the sub-bimodules $U_1 = K(1, 0)$ and $U_2 = K(0, 1)$, on which $R = K$ acts faithfully in the natural way. The representations induced in the skew lines $U_i \times U_i$ are faithful and map $\mathbb{P}(K)$ onto the set of all points of $U_i \times U_i$. Moreover, $\beta := \Phi_1^{-1} \Phi_2$ is a bijection between these two projective lines, which is linearly induced and hence a projectivity. The elements of the projective model $\mathbb{P}(K)^\Phi$ in $\mathbb{P}(K, U \times U)$ are exactly the lines joining a point of $U_1 \times U_1$ and its $\beta$-image in $U_2 \times U_2$. So $\mathbb{P}(K)^\Phi$ is a regulus in 3-space (compare [6]).

The same applies if $U = K^n$. Then one obtains a regulus in a $(2n - 1)$-dimensional projective space (see [3]), i.e., a generalization to the not necessarily pappian case of a family of $(n-1)$-dimensional subspaces on a Segre manifold $S_{n-1, 1}$ (compare [7]).

**Example 5.2** Example 5.1 above can be modified in the following way: Let $\alpha_1, \alpha_2 : K \to K$ be field monomorphisms. Then $K$ acts faithfully on $U = K^2$ via $(x_1, x_2) \cdot k = (x_1 k^{\alpha_1}, x_2 k^{\alpha_2})$. The induced projective models of $\mathbb{P}(K)$ in the projective lines $U_i \times U_i$ are projective sublines over the subfields $K^{\alpha_i}$. In general, the bijection $\beta$ between the two models is not $K$-semilinearly induced.

We mention one special case: If $K = \mathbb{C}$, $\alpha_1 = \text{id}$, and $\alpha_2$ is the complex conjugation, then the projective model of $\mathbb{P}(\mathbb{C})$ is a set of lines in the 3-space $\mathbb{P}(\mathbb{C}, U \times U)$. It can be interpreted as follows: The $\alpha_2$-semilinear bijection $\beta$ extends to a collineation of order two which fixes a Baer subspace (with $\mathbb{R}$ as underlying field). The lines of the projective model of $\mathbb{P}(\mathbb{C})$ meet this Baer subspace in a regular spread (elliptic linear congruence). See [10] for a generalization of this well-known classical result that the regular spreads of a real 3-space can...
be characterized (in the complexified space) as those sets of lines that join complex-conjugate points of two skew complex-conjugate lines.

**Example 5.3** Let $K$ be any field. Let $U = R = K^n$, with componentwise addition and multiplication. For $i \in \{1, \ldots, n\}$, let $U_i = Kb_i$, where $b_i$ runs in the standard basis. Then $U_i$ is a sub-bimodule of $U$, the induced faithful action is the ordinary action of $K$. Hence the projective model $\mathbb{P}(R) = \mathbb{P}(R)$ meets the line $U_i \times U_i$ in all points. Moreover, each $(n - 1)$-dimensional projective subspace of $\mathbb{P}(K, U \times U)$ that meets all the lines $U_i \times U_i$ belongs to $\mathbb{P}(R)$, because $\text{GL}_2(R) \cong \text{GL}_2(K) \times \ldots \times \text{GL}_2(K)$.

If $n = 2$, the set $\mathbb{P}(R)$ is a generalization to the not necessarily pappian case of a hyperbolic linear congruence.

**Example 5.4** Let $K$ be any field. Let $U = R = K[\varepsilon]$, where $\varepsilon \notin K$, $\varepsilon^2 = 0$ and $\varepsilon k = k\alpha \varepsilon$ for some fixed $\alpha \in \text{Aut}(K)$. This is a ring of twisted dual numbers over $K$. It is a local ring with $I = K\varepsilon$ the maximal ideal of all non-invertible elements. So $U' = I$ is a sub-bimodule of $U = R$, with $\text{ann}(U') = I$, and on $U'$ we have the induced faithful representation $(\varphi')_f$ of $R/I \cong K$ with $k\varepsilon \cdot \alpha = k\alpha \varepsilon$. So each point of $U' \times U'$ is given by $K(k^\alpha \varepsilon, l^\varepsilon)$. This bijection $\beta$ is given by $K(k^\alpha \varepsilon, l^\varepsilon) \mapsto K(k, l) \oplus (U' \times U')$.

Moreover, one can compute that the projective model $\mathbb{P}(R)$ consists of all lines in $\mathbb{P}(K, U \times U)$ that meet $U' \times U'$ in a unique point, say $q$, and then lie in the plane $q^\beta$.

In case $\alpha = \text{id}$ the bijection $\beta$ is a projectivity. So then the set $\mathbb{P}(R)$ is a generalization of a parabolic linear congruence. The ring $R$ is then the ordinary ring of dual numbers over $K$. In the general case $\beta$ is only semilinearly induced. If $K = \mathbb{C}$ and $\alpha$ is the complex conjugation, then $R$ is the ring of Study’s quaternions (see [12], p.445).

**Example 5.5** Let $R$ be the ring of upper triangular $2 \times 2$-matrices with entries in $K$. Then $U = K^2$ is in a natural way a faithful $(K, R)$-bimodule. Moreover, $U' = K(0, 1)$ is a sub-bimodule with $\text{ann}(U') = \{(a, b) \mid a, b \in K\}$. So $R/\text{ann}(U') \cong K$, and the induced faithful representation is the ordinary action of $K$ on $U'$. This means that each point of $U' \times U'$ is on a line of the projective model $\mathbb{P}(R)$.

Now consider $\tilde{U} = U/U'$. The kernel of the induced action is $J = \{(0, b) \mid b, c \in K\}$. So $R/J \cong K$, and also here we have the ordinary action of $K$ on $\tilde{U} \cong K(1, 0)$. Hence each plane through $U' \times U'$ contains a line of $\mathbb{P}(R)$.

Up to now, we are in the same situation as in Example 5.4. An easy calculation shows that the projective model $\mathbb{P}(R)$ consists of all lines that meet $U' \times U'$ in a point. This is the generalization of a special linear complex to the not necessarily pappian case.
Example 5.6 Let \( U = R = K[\varepsilon, \delta] \) with \( \varepsilon \notin K, \delta \notin K[\varepsilon], \varepsilon, \delta \) central, and \( \varepsilon^2 = \delta^2 = \varepsilon\delta = 0 \). The projective model \( \mathbb{P}(R)^\Phi = \mathbb{P}(R) \) is a set of planes in 5-space.

The ring \( R \) is a local ring with maximal ideal \( I = K\varepsilon + K\delta = U' \). Moreover, \( \text{ann}(U') = I \), and \( R/I \cong K \) acts on \( U' \) componentwise. So according to 5.1 the induced model of \( \mathbb{P}(K) \) in the 3-space \( U' \times U' \) is a regulus \( \mathcal{R} \).

Now consider \( \overline{U} = R/U' \). Then \( \text{ker}(\overline{\phi}) = I \), and we have the ordinary faithful action of \( K \) on \( \overline{U} \cong K \). So all hyperplanes (4-spaces) through \( U' \times U' \) contain an element of \( \mathbb{P}(R) \).

As in Example 5.4 the elements of \( \mathbb{P}(R) \) fall into equivalence classes with respect to \( \not\sim \), such that equivalent elements have the same intersection and the same join with \( U' \times U' \). This yields a bijection \( \beta \) between the regulus \( \mathcal{R} \) and the set of all hyperplanes through \( U' \times U' \). As in 5.4, case \( \alpha = \text{id} \), this bijection is a projectivity. A calculation shows that \( \mathbb{P}(R) \) consists of all planes that meet the 3-space \( U' \times U' \) in an element of \( \mathcal{R} \), say \( X \), and then lie in the hyperplane \( X^3 \).

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