Derivation of the Burgers equation with finite particle correction from the derivative nonlinear Schrödinger equation

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We investigate the dynamics of the asymmetric simple exclusion process (ASEP) in the continuous limit. We derive the Burgers equation with finite particle correction from the quantum derivative nonlinear Schrödinger equation which is equivalent to the ASEP in the continuous limit. We numerically confirm that the obtained Burgers equation describes the dynamics of the ASEP with few particles better than the conventional one.

\textit{Introduction}.— Although almost all phenomena in the world are out of equilibrium, the theory of nonequilibrium physics is not still completed. Therefore, to investigate universal law among nonequilibrium phenomena is an important work in the modern theoretical physics. The asymmetric simple exclusion process (ASEP), which is a continuous time Markov model describing asymmetric diffusion of hard-core particles in one dimension, is one of the most hopeful models to study nonequilibrium transport phenomena \cite{1,5}. The schematic drawing of the ASEP is shown in Fig.1. The update rules of this model are simple. Each particle hop to the right (left) nearest site with probability $p dt$ ($q dt$) in the time interval $dt$ if that site is vacant. If another particle already exist in that site, hoppings do not occur. Despite of its simplicity, the ASEP describes various interesting nonequilibrium phenomena, such as boundary-induced phase transition \cite{2,3}. Moreover, the current fluctuation is known to belong to the Kardar-Parisi-Zhang universality class to which the surface growth phenomena belongs \cite{4,5}. The ASEP is originally introduced as a model of biopolymerization \cite{6}. Then it was applied to a wide range of nonequilibrium phenomena, such as traffic flow, pedestrian dynamics \cite{7,8} and biophysical transport \cite{9,10}. In addition to these rich properties about nonequilibrium physics, the mathematical aspect of the ASEP is also fascinating many scientists. The ASEP is a quantum integrable system and its Hamiltonian is exactly diagonalized by the Bethe ansatz \cite{11,12}. If we can express initial state by Bethe vector, the dynamics could be calculated exactly. Moreover, the exact stationary state is constructed from the matrix product ansatz \cite{13}.

In this letter, we investigate the mathematical structure of the ASEP. Previous study shows that the ASEP is related to the two type of integrable systems: the Burgers equation which is the classical integrable system \cite{14,15} and the quantum derivative nonlinear Schrödinger equation (QDNLS) which is the quantum integrable system \cite{16}. It was shown that Burgers equation is derived from the ASEP by taking mean field approximation and continuous limit. On the other hand, the QDNLS is obtained via the bosonization of the ASEP and considering continuous limit \cite{17}. We study the relation between these integrable systems and derive the Burgers equation from the QDNLS. Moreover, we find the Burgers equation obtained from this method contains finite particles correction.

\textit{Formulation of the ASEP}.— Let us consider the ASEP on a ring. We introduce the variable $n_j$ as the number of particle at site $j$. In the case of ASEP, $n_j$ is 0 or 1 due to the exclusive volume effect. Number of lattice is denoted by $L$ and the state of the system by $(n) = (n_1, n_2, \cdots, n_L)$. We introduce the orthogonal basis $|n\rangle = |n_1, n_2, \cdots, n_L\rangle$ which corresponds to the state $(n)$. Then the state vector of the system at time $t$ as

$$|\psi(t)\rangle = \sum_n \psi(n,t)|n\rangle$$

(1)

where $\psi(n,t)$ means the probability that the system is in a state $(n)$ at time $t$. The ladder operators $\hat{s}_j^\pm$ and the particle number operator $\hat{n}_j$ are defined by the Pauli matrices $\sigma$ as $\hat{s}_j^\pm = \frac{1}{2}(\hat{\sigma}_x^j \pm \hat{\sigma}_y^j)$, $\hat{n}_j = \frac{1}{2}(1-\hat{\sigma}_z^j)$. The time evolution of this state obeys the master equation

$$\frac{d}{dt}|\psi(t)\rangle = -\hat{H}_{\text{ASEP}}|\psi(t)\rangle$$

(2)
where the Markov matrix $\hat{H}_{\text{ASEP}}$ is given by

$$
\hat{H}_{\text{ASEP}} = \sum_{j=1}^{L} \left[ -p\hat{s}_j^+ \hat{s}_{j+1} - q\hat{s}_j^- \hat{s}_{j+1} + p\hat{n}_j (1 - \hat{n}_{j+1}) + q (1 - \hat{n}_j) \hat{n}_{j+1} \right].
$$

This equation is the same format as the imaginary time Schrödinger equation, so we call $\hat{H}_{\text{ASEP}}$ as Hamiltonian hereafter. We introduce the projection state $\langle s \rangle$ defined as

$$
\langle s \rangle = \langle 0 \rangle \exp \left( \sum_{j=1}^{L} \hat{s}_j^+ \right) = \sum_n \langle n | n \rangle
$$

where $|0\rangle$ means the vacuum state $|0\rangle = |0,0,\cdots,0\rangle$. Then, the normalization condition of the state vector is given by

$$
\langle s | \hat{A} | \psi(t) \rangle = 1.
$$

The time evolution of the particle density is obtained from eq. (2), eq. (3) and eq. (6) and given as

$$
\frac{d}{dt} \langle \hat{n}_i \rangle = p(\hat{n}_{i-1} - \hat{n}_{i+1}) + q(\hat{n}_{i+1} + \hat{n}_{i-1} - 2\hat{n}_i)
$$

$$
- (p + q)(\hat{n}_i) + (p - q)(\hat{n}_{i+1} + \hat{n}_{i-1} - 2\hat{n}_i).
$$

**The Burgers equation derived from the continuous limit**— Before explain our results, we review the conventional derivation of the Burgers equation from the ASEP. First, we consider the mean field approximation, i.e. we ignore the correlation between particles. We replace the two-point correlation function $\langle \hat{n}_i \hat{n}_{i+1} \rangle$ with $\langle \hat{n}_i \rangle \langle \hat{n}_{i+1} - \hat{n}_{i-1} \rangle$. Then the time evolution equation of the particle density is written as

$$
\frac{d}{dt} \langle \hat{n}_i \rangle = p(\hat{n}_{i-1} - \hat{n}_{i+1}) + q(\hat{n}_{i+1} + \hat{n}_{i-1} - 2\hat{n}_i)
$$

$$
- (p + q)(\hat{n}_i) + (p - q)(\hat{n}_{i+1} + \hat{n}_{i-1} - 2\hat{n}_i).
$$

Second, we take the continuous limit. We denote the total length of the system as $l$ and the lattice distance as $a = \frac{L}{l}$. Then we take the infinite limit of the lattice number $L$. We write $\langle \hat{n}_j \rangle = \rho(x_j, t)$ and rescale time $at \rightarrow t$, then we obtain the Burgers equation

$$
\partial_t \rho = aD\partial_x^2 \rho - 2\alpha \rho + 4\alpha \rho \partial_x \rho.
$$

Therefore, the Burgers equation describes the time evolution of the particle density of the ASEP in the continuous limit.

In order to know how well this equation describes the dynamics of the ASEP, we simulate this by Monte Carlo simulation and compare with the numerical solution of the Burgers equation. We show the results in Fig. 2 and 3. The particle density of the ASEP against position is plotted. Red dots correspond to the ASEP and green line to the Burgers equation. Configuration of the simulation of the ASEP is as below. The number of lattice $L$ is 50 and the asymmetric diffusion rate $(p, q)$ is $(1, 0)$. We simulate for various number of particles $N$ and show three pattern ($N = 1, 2, 25$) in this paper. The parameter $a$ of the Burgers equation is determined by the configuration of the ASEP. We choose the total length of the system as $l = 1$ and the lattice distance $a$ as $a = l/L = 1/50$. According to the initial configuration of the ASEP, we set the initial condition of the Burgers equation as the rectangular function: if site $j$ is occupied in the ASEP, then $\rho(x_j, t) = 1$ for $a(j - 1) \leq x \leq aj$, otherwise 0. For example, if $N = 1$ and located in site $i$ in the initial state, we denote the initial condition of the Burgers equation as $\rho_{ini}(x, 0) = \theta(x - i) - \theta(i + 1 - x)$, where $\theta(x)$ is the step function.

From these figures, we see the solution of the Burgers equation (9) roughly describes the dynamics of particle density of the ASEP. However, when we look at Fig. 2 and Fig. 3 we notice the solution of the Burgers equation is receding against the particle density wave of the ASEP. This receding phenomena is observed in various initial conditions especially in the particle number of the ASEP is small. On the other hand, in the large particle number case (Fig. 4), receding phenomenon is not observed. Thus, it seems that the Burgers equation derived from the conventional way overestimates the exclusive volume effect of the ASEP when the number of particles in the ASEP is small. This reason of this overestimate is that we consider the expectation value of the arbitrary state in time $t$ in eq. (1) which contains states with large number of particles. However, the number of particles of the ASEP is conserved in the periodic boundary condition. Therefore, when the initial condition with the small number of particles is considered, the Burgers equation overestimates the exclusive volume effect.

**The relation between the ASEP and the DNLS.**— Next we consider the non-exclusive boson model in one dimensional lattice in which each site contains more than one particle. It is defined by the Hamiltonian

$$
-\hat{H}_{\text{boslomb}} = \sum_j \hat{a}_j^\dagger \left[ p\hat{a}_{j-1} + q\hat{a}_{j+1} - (p + q)\hat{a}_j\right]
$$

$$
+ (p - q) \sum_j \hat{a}_j^\dagger \left[ \hat{a}_j^\dagger - \hat{a}_{j+1}^\dagger \right] \hat{a}_j \hat{a}_{j+1}
$$

where $\hat{a}_j^\dagger$ and $\hat{a}_j$ are the bosonic creation and annihilation operators whose commutation relations are provided by $[\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0, [\hat{a}_j, \hat{a}_k^\dagger] = \delta_{j,k},$. The time evolution equation of particle density is the same as that of the ASEP [17]. The first term denotes the asymmetric diffusion of particles and the second one mimics
the exclusive volume effect of the ASEP. As well as in the case of the ASEP, the projection state \( \langle s \rangle \) is given by \( \langle s \rangle = \langle 0 \rangle \text{exp}(\sum \hat{a}_j) \) and the expectation value of the physical quantity \( \hat{A} \) in the state \( \hat{\psi}(t) \) is provided by \( \langle s \rangle \hat{A} \hat{\psi}(t) \rangle \) in this case. When the time evolution obeys the imaginary time Schrödinger equation which is the same form as eq. (2), the time evolution equation of the particle density of this boson model is identical with that of the ASEP.

We take the continuous limit in this model, namely we take the infinitesimal limit of the lattice distance denoted as \( a = \frac{t}{\lambda} \) as we did in the previous section. We replace the boson operators \( \hat{a}, \hat{a}^\dagger \) with the field operators \( \hat{\psi}, \hat{\psi}^\dagger \) as \( \hat{a}_j \rightarrow \sqrt{a} \hat{\psi}(x_j), \hat{a}^\dagger_j \rightarrow \sqrt{a} \hat{\psi}^\dagger(x_j) \) where \( x_j := a_j \). Then the Hamiltonian is expressed as

\[
\hat{H}_{\text{DNLS}} = a^2 D \int dx \hat{\psi}^{\dagger} \partial_x^2 \hat{\psi} - 2a \alpha \int dx \hat{\psi}^{\dagger} \partial_x \hat{\psi} - 2a^2 \alpha \int dx \hat{\psi}^{\dagger} \partial_x \hat{\psi} \hat{\psi},
\]

(11)

where we introduce two parameters \( D = \frac{\hbar^2}{2m} \) and \( \alpha = \frac{\hbar c}{2m} \). The commutation relations of the field operators \( \hat{\psi}, \hat{\psi}^{\dagger} \) are given by \( [\hat{\psi}(x), \hat{\psi}(y)] = [\hat{\psi}^{\dagger}(x), \hat{\psi}^{\dagger}(y)] = 0, [\hat{\psi}(x), \hat{\psi}^{\dagger}(y)] = \delta(x - y) \). This Hamiltonian \( \hat{H}_{\text{DNLS}} \) is the derivative nonlinear Schrödinger type Hamiltonian.

**Derivation of the Burgers equation from the DNLS.**

Here we explain the derivation of Burgers equation from the DNLS type Hamiltonian \( \hat{H}_{\text{DNLS}} \). We introduce the orthogonal basis of the states with \( N \) particles as \( |x_1, \cdots, x_N \rangle := \frac{1}{N!} \hat{\psi}^{\dagger}(x_1) \cdots \hat{\psi}^{\dagger}(x_N) |0 \rangle \) and denote the state vector as \( |f \rangle = a^N \int dx_1 \cdots dx_N f(x_1, \cdots, x_N) |x_1 \cdots x_N \rangle \)

(12)

where \( f(x_1, \cdots, x_N) \) is the distribution function of \( N \) particles. Then the state vector at time \( t \) is given by \( |f(t) \rangle = e^{-\hat{H}_{\text{DNLS}}t} |f \rangle \).

The projection state in the continuous limit is expressed as \( |s \rangle = e^{\hat{A}^\dagger} |0 \rangle \) where \( \hat{A} = \int dx a^2 \hat{\psi} \hat{\psi} \). The commutation relation between \( \hat{\psi} \) and \( e^{\hat{A}} \) is \( [\hat{\psi}(x), e^{\hat{A}}] = a^{-2} e^{\hat{A}} \hat{\psi} \). Acting both size of this equation on the vacuum \( |0 \rangle \), we obtain

\[
\hat{\psi}(x) |s \rangle = a^{-\frac{1}{2}} |s \rangle.
\]

(13)

This relation means that the projection state \( |s \rangle \) is the eigenvector of the annihilation operator \( \hat{\psi} \) with the eigenvalue \( a^{-\frac{1}{2}} \). In addition to this property, the projection state satisfies below:

\[
\langle s | \hat{H}_{\text{DNLS}} | s \rangle = 0.
\]

(14)

Thus, the projection state is the stationary state of this model. The expectation value of the physical quantity \( \hat{A} \) in the \( N \)-particles state \( |f \rangle \) is provided by \( \langle s | \hat{A} | f \rangle \). The time evolution of the physical quantity \( \hat{A}(t) \) in the Heisenberg picture is given by

\[
\partial_t \hat{A} = -[\hat{A}, \hat{H}_{\text{DNLS}}].
\]

(15)
Therefore, the time evolution equation of the annihilation operator \( \hat{\psi} \) is described by the quantum DNLS equation:

\[
\partial_t \hat{\psi}(x, t) = a^2 D \partial_x^2 \hat{\psi}(x, t) - 2a \alpha \partial_x \hat{\psi}(x, t) + 4a^2 \alpha \psi(x, t) \partial_x \hat{\psi}(x, t). 
\] (16)

We denote the density operator as \( \hat{\rho}(x) = \hat{\psi}^\dagger(x) \hat{\psi}(x) \). Therefore the particle distribution function at time \( t \) is given by

\[
\rho(x, t) = \langle s \vert \hat{\rho}(x) \vert f, t \rangle = a^{-\frac{3}{2}} \langle s \vert \hat{\psi}(x) \vert f, t \rangle. 
\] (17)

Here we use the property of the projection state eq. (13).

Now we are ready to derive the Burgers equation from the quantum DNLS equation. Acting the projection state \( \langle s \rangle \) from the left and the state vector \( |f\rangle \) from the right on the quantum DNLS equation (16) and use eq. (13) and (17), then we obtain

\[
a^2 \partial_t \rho(x, t) = a^2 D \partial_x^2 \rho(x, t) - 2a \alpha \partial_x \rho(x, t) + 4a^2 \alpha \rho(x, t) \partial_x \rho(x, t). 
\] (18)

In order to calculate the third term in the right hand side, we consider the mean field approximation, where the distribution function of the \( N \) particles state is given by the product of that for one particle state:

\[
f(x_1, \ldots, x_N, t) \sim \prod_{j=1}^{N} f(x_j, t) 
\] (19)

where \( f(x, t) \) is the one particle distribution function. Under this approximation, the particle distribution (17) is expressed as

\[
\rho(x, t) = N f(x, t). 
\] (20)

Then, the third term in the right hand side of the equation (18) is calculated as

\[
\langle s \vert \hat{\psi}(x) \partial_x \hat{\psi}(x) \vert f, t \rangle \sim \partial_x \left( \frac{N}{2} f(x, t)^2 \right) = \frac{N-1}{N} \rho(x, t) \partial_x \rho(x, t). 
\] (21)

Substituting eq. (21) to the equation (18), dividing both sides of this equation by \( a^2 \) and rescaling time \( at \rightarrow t \), obtains the Burgers equation:

\[
\partial_t \rho = a D \partial_x^2 \rho - 2a \alpha \partial_x \rho + 4a \frac{N-1}{N} \rho \partial_x \rho. 
\] (22)

In order to distinguish the Burgers equation derived from the quantum DNLS equation from the conventional one, we call this equation “Boson-Burgers equation”. The difference between the Boson-Burgers equation and the conventional one is the coefficient of the third term in the right hand side. The coefficient \( \frac{N-1}{N} \) in the Boson-Burgers equation means the finite particle effect. In the limit of \( N \rightarrow \infty \), the Boson-Burgers equation corresponds to the conventional equation since \( \frac{N-1}{N} \rightarrow 1 \). As we mentioned before, the conventional Burgers equation overestimates the exclusive volume effect of the ASEP when particles number \( N \) is small. The reason for this deviation is that we consider the expectation value among all possible states in the conventional derivation. However, in the case of the Boson-Burgers equation, we are able to calculate the expectation value of \( N \) states in the frame work of the field theory. Therefore we obtain the finite particle correction.

In order to investigate the validity of this finite particle correction, we compare the numerical solution of the Boson-Burgers equation with the Monte Carlo simulation of the ASEP. The results are shown in Fig. 2 and Fig. 3. The particle density of the ASEP is plotted against position. The red dots represent the Monte Carlo simulation of the ASEP and the blue lines represent the numerical solution of the Boson-Burgers equation. The settings and parameters of the simulations are the same as in the previous section.

From these figures, one sees that the solution of Boson-Burgers equation is almost match the particle density of the ASEP. In Fig. 2 and Fig. 3 where the particle number \( N \) is small, it is observed that the receding phenomena which is observed in the conventional Burgers equation does not happen in this equation.

**Conclusion.** — In this letter, we have investigated the relationship between the quantum and classical integrable systems which is related to the asymmetric simple
exclusion process (ASEP). First, we derived the Burgers equation by the conventional method and found that its numerical solution shifts backwards against the particle density of the ASEP. Next, we introduce the non-exclusive boson model in which the time evolution of the one point correlation function is equivalent to that of the ASEP. The derivative nonlinear Schrödinger (DNLS) type Hamiltonian, which is the quantum integrable system, emerges in the continuous limit of this model. Then we got the time evolution equation of the annihilation operator which is the quantum DNLS equation. Considering the mean field approximation in the $N$-particle state, we obtain the Boson-Burgers equation. The Boson-Burgers equation contains finite particle correlation and we found that this equation describes the dynamics of the ASEP with few particles better than the conventional one.

* A footnote to the article title
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