Demand-Flow of Agents with Gross-Substitute Valuations

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Abstract

We consider the class of valuations on indivisible items called \textit{gross-substitute} (GS). This class was introduced by Kelso and Crawford (1982) and is widely used in studies of markets with indivisibilities. GS is a condition on the demand-flow in a specific scenario: some items become more expensive while other items retain their price. We prove that GS implies a much stronger condition, describing the demand-flow in the general scenario in which all prices may change. We prove that the demand of GS agents always flows (weakly) downwards, i.e, from items with higher price-increase to items with lower price-increase. We show that this property is equivalent to GS and is not true when there are complementarities.

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1. Introduction

Many markets involve a set of distinct indivisible goods that can be bought and sold for money. The analysis of such markets crucially depends on the agents’ valuation functions - the functions that assign monetary values to bundles. It is common to assume that agents’ valuations are weakly increasing (more goods mean weakly more value) and quasi-linear in money. Even so, without further restrictions on the valuations, the market may fail to have desirable properties such as the existence of a price-equilibrium. Kelso and Crawford [1] introduced a property of valuations which they called gross-substitutes (GS). An agent’s valuation has the GS property if, when the price of an item increases, the agent’s demand for other items weakly increases. In other words, a GS agent will not stop wanting an item, only because another item became more expensive. Kelso and Crawford [1] proved that a market in which all agents are GS always has a price-equilibrium. Gul and Stacchetti [2, 3] complemented this result by proving that the GS condition is, in some sense, necessary to ensure existence of a price-equilibrium. The GS condition has been widely used in the study of matching markets [4], auctions [5] and algorithmic mechanism design [6].

The GS condition specifies the behavior of an agent in a very specific situation: some items become more expensive, while other items retain their original price. During our research on double auctions [7], we needed to understand how GS agents behave in the more general situation, in which the prices of all items change in different ways and in different directions. Consider two price-vectors: old and new. For every item $x$, define $\Delta_x$ as the price-increase of $x$ (the new price minus the old price). Add a ”null item” $\emptyset$ and set its price-increase to 0. Arrange the items vertically according to their price-increase. Then, our main result is that:

The demand of a GS agent always flows weakly downwards.

I.e, an agent may switch from wanting an item whose price increased more to an item whose price decreased less, but not vice-versa. This property is trivially true for a unit-demand agent, but it is not true when the agent regards some items as complementaries.

**Example 1.1.** There are three items: $x,y,z$. Initially their prices are $10,10,10$. Then, the prices increase by $\Delta_x = 20$, $\Delta_y = 30$, $\Delta_z = 40$, so that the new prices are $30,40,50$. Consider two agents with the following valuations:

|       | $x$  | $y$  | $z$  | $x,y$ | $x,z$ | $y,z$ | $x,y,z$ |
|-------|------|------|------|-------|-------|-------|---------|
| Alice | $65$ | $70$ | $75$ | $70$  | $75$  | $75$  | $75$    |
| Bob   | $0$  | $0$  | $75$ | $90$  | $75$  | $75$  | $90$    |

Alice has unit-demand: she needs only one item and values each bundle as the maximum item in that bundle. Bob regards $x$ and $y$ as complementaries: they are worthless in separation, but together they are worth more than $z$.

In the initial prices Alice’s preferred bundle is $z$, and after the price-change her preferred bundle is $x$, so her demand flows downwards - towards the smaller price-increase.

In contrast, Bob’s demand is initially $x+y$, and after the price-change his demand is $z$, so his demand flows upwards - towards the item with the larger price-increase.

Our main result is that GS agents behave like unit-demand agents in this regard: their demand flows only downwards.
2. Model and Notation

There is a finite set of indivisible items, \( M = \{1, \ldots, m\} \). There is an \( m \)-sized price-vector \( p \): a price per item. The price of a bundle is the sum of the prices of the items in it: \( p(X) := \sum_{x \in X} p_x \).

The present paper focuses on a single agent with a single valuation-function \( u : 2^M \rightarrow \mathbb{R} \). \( u \) is assumed to be weakly-increasing: if a bundle \( X \subseteq Y \) then \( u(X) \leq u(Y) \). It is normalized such that \( u(\emptyset) = 0 \).

The agent’s utility is quasi-linear in money. Given a utility function \( u \) and a price-vector \( p \), the agent’s net-utility function \( u_p \) is: \( u_p(X) := u(X) - p(X) \).

Definition 2.1. Given a valuation function \( u \) and a price-vector \( p \), we say that a bundle \( P \) is a \( p \)-demand if it is optimal for the agent to buy this bundle when the prices are \( p \), i.e, the set \( P \) maximizes the net-utility function \( u_p(\cdot) \) over all bundles of items: \( \forall X : u_p(P) \geq u_p(X) \).

Definition 2.2. Given a valuation function \( u \) and a price-vector \( p \), we say that an item \( x \) is \( p \)-demanded if there exists a \( p \)-demand \( P \) such that \( P \ni x \).

Definition 2.3. Given an agent, an item \( x \) and a pair of price-vectors \((p, q)\), we say that:

(a) The agent abandoned item \( x \) if \( x \) is \( p \)-demanded but not \( q \)-demanded.
(b) The agent discovered item \( x \) if \( x \) is \( q \)-demanded but not \( p \)-demanded.

Definition 2.4. An agent’s valuation function has the downward-demand-flow (DDF) property if the following are true for every pair of price-vectors \((p, q)\) (where \( \Delta_x := q_x - p_x \)):

(a) If the agent abandoned an item \( x \) with \( \Delta_x \leq 0 \), then the agent must have discovered some item \( y \) with \( \Delta_y < \Delta_x \).
(b) If the agent discovered an item \( x \) with \( \Delta_x \geq 0 \), then the agent must have abandoned some item \( y \) such that \( \Delta_y > \Delta_x \).

DDF clearly implies GS.

Proof. The DDF property implies that, if an agent abandoned an item \( x \) whose price remained constant (\( \Delta_x = 0 \)), then the agent must have discovered some item \( y \) whose price decreased (\( \Delta_y < 0 \)). Hence, if there is no item whose price decreased, then the agent have not abandoned any item whose price remained constant. But this is exactly the definition of the GS property. \( \square \)

Our main result is the converse implication: GS implies DDF.

3. M#-concavity

Our main technical tool is the following characterization of GS valuations [8]:

Definition 3.1. A valuation function \( u \) is \( M# \)-concave if-and-only-if, for every two bundles \( X, Y \) and for every \( X' \subseteq X \setminus Y \) with \(|X'| = 1\) (i.e., \( X' \) is a singleton), there exists a subset \( Y' \subseteq Y \setminus X \) with \(|Y'| \leq 1\) (i.e., \( Y' \) is either empty or a singleton) such that:

\[
 u(X \setminus X' \cup Y') + u(Y \setminus Y' \cup X') \geq u(X) + u(Y)
\]

Lemma 3.1 ([8]). A valuation function \( u \) is \( M# \)-concave if-and-only-if it is gross-substitute.

Below we prove that \( M# \)-concavity is preserved in net-utility functions and marginal valuation functions.
Lemma 3.2. Let \( p \) be an arbitrary price vector. A valuation function \( u \) is \( M\# \)-concave if-and-only-if the net-utility function \( u_p \) is \( M\# \)-concave.

Proof. The price function \( p(\cdot) \) is additive. Hence, for all \( X' \subseteq X \setminus Y \) and \( Y' \subseteq Y \setminus X \):

\[
p(X \setminus X' \cup Y') + p(Y \setminus Y' \cup X') = p(X) + p(Y)
\]

Hence, the \( M\# \)-concave condition is independent of price:

\[
u(X \setminus X' \cup Y') + u(Y \setminus Y' \cup X') \geq u(X) + u(Y) \\
\iff u_p(X \setminus X' \cup Y') + u_p(Y \setminus Y' \cup X') \geq u_p(X) + u_p(Y)
\]

Definition 3.2. Given a valuation \( u \) and a constant bundle \( Z \), the marginal valuation \( u_{Z^+} \) is a function that returns, for every bundle \( X \) that does not intersect \( Z \), the additional value that an agent holding \( Z \) gains from having \( X \):

\[
u_{Z^+}(X) := u(Z \cup X) - u(Z) \quad \text{for all } X \text{ with } X \cap Z = \emptyset
\]

Lemma 3.3. A valuation function \( u \) is \( M\# \)-concave if-and-only-if, for every bundle \( Z \), the marginal-valuation function \( u_{Z^+} \) is \( M\# \)-concave.

Proof. The "if" direction is obvious since \( \emptyset \) is also a bundle and \( u_{\emptyset^+} \equiv u \).

For the "only if" direction, suppose \( u \) is \( M\# \)-concave and let \( Z \) be an arbitrary bundle. We have to prove that \( u_{Z^+} \) is \( M\# \)-concave, i.e., for all bundles \( X, Y \) with \( X \cap Z = Y \cap Z = \emptyset \), and for every \( X' \subseteq X \setminus Y \) with \( |X'| = 1 \), there exists a \( Y' \subseteq Y \setminus X \) with \( |Y'| \leq 1 \) such that:

\[
u(Z \cup (X \setminus X' \cup Y')) + u(Z \cup (Y \setminus Y' \cup X')) \geq u(Z \cup X) + u(Z \cup Y)
\]  

(1)

Since \( u \) is \( M\# \)-concave, we can apply the definition of \( M\# \)-concave to the bundles \( Z \cup X \) and \( Z \cup Y \). For every \( X'' \subseteq (Z \cup X) \setminus (Z \cup Y) \) with \( |X''| = 1 \), there exists \( Y'' \subseteq (Z \cup Y) \setminus (Z \cup X) \) with \( |Y''| \leq 1 \) such that:

\[
u((Z \cup X) \setminus X'' \cup Y'') + u((Z \cup Y) \setminus Y'' \cup X'') \geq u(Z \cup X) + u(Z \cup Y)
\]  

(2)

This is particularly true when \( X'' = X' \) from above, since \( (Z \cup X) \setminus (Z \cup Y) \equiv X \setminus Y \). We can take \( Y' := Y'' \), since \( (Z \cup Y) \setminus (Z \cup X) \equiv Y \setminus X \). It remains to prove that (2) implies (1).

Indeed, since \( X \cap Z = X' \cap Z = Y' \cap Z = \emptyset \):

\[
Z \cup (X \setminus X' \cup Y') = (Z \cup X) \setminus X' \cup Y',
\]

since it does not matter whether we first add \( X \) to \( Z \) and then remove some items from the union, or first remove these items from \( X \) and then add the remaining items to \( Z \). Similarly, since also \( Y \cap Z = \emptyset \):

\[
Z \cup (Y \setminus Y' \cup X') = (Z \cup Y) \setminus Y' \cup X'
\]

so (2) and (1) are equivalent. \( \square \)
4. Telescopic Arrangement of Maximizing Bundles

By definition, an agent’s demanded bundles are maximizing-bundles - bundles that maximize his net-utility over all $2^m$ possible bundles. In addition to the global maximizing-bundles, we can consider the maximizing-bundles in each size-group, i.e., the maximizing-bundles among the bundles with 1 item, with 2 items, etc. In this section we prove that, when the agents’ valuation is $M^\#$-concave, the maximizing-bundles in the different size-groups have a telescopic arrangement: each maximizing-bundle contains smaller maximizing-bundles and is contained in larger maximizing-bundles.

**Definition 4.1.** Given valuation $u$ on $m$ items and a number $i \in \{0, \ldots, m\}$, a bundle $Z_i$ is called $i$-maximizer of $u$ if it maximizes $u$ among all bundles with $i$ items. I.e., $|Z_i| = i$ and for every other bundle $X_i$ with $i$ items, $u(Z_i) \geq u(X_i)$.

**Lemma 4.1.** For every $M^\#$-concave valuation $u$ on $m$ items and two integers $i, j$ such that $0 \leq i < j \leq m$:

(a) For every $i$-maximizer $Z_i$ there is a $j$-maximizer $Z_j'$ such that $Z_j' \supset Z_i$.

(b) For every $j$-maximizer $Z_j$ there is an $i$-maximizer $Z_i'$ such that $Z_i' \supset Z_j$.

**Proof.** The lemma is obviously true when $i = 0$ since there is a unique 0-maximizer (the empty set). It is also true when $j = m$ since there is a unique $m$-maximizer (the set containing all items). We have to prove it for $1 \leq i < j \leq m - 1$, which is possible only when $m \geq 3$. The proof is by induction on $m$.

**Base:** $m = 3, j = 2, i = 1$. Let $Z_1$ be a 1-maximizer and $Z_2$ a 2-maximizer. If $Z_1 \subseteq Z_2$ then we are done. Otherwise, $Z_1$ contains a single item, e.g. $\{x\}$, and $Z_2$ contains the other two items, $\{y, z\}$. Apply the $M^\#$-concavity definition with $X = Z_2$ and $Y = Z_1$ and $X' = \{y\}$. Then, $Y'$ can be either $\emptyset$ or $\{x\}$:

- If $Y' = \emptyset$, then by the $M^\#$-concavity condition: $u(\{z\}) + u(\{x, y\}) \geq u(\{x\}) + u(\{y, z\})$. Then $\{z\}$ must be a 1-maximizer and $\{x, y\}$ must be a 2-maximizer; the former is contained in $Z_2$ and the latter contains $Z_1$ so we are done.

- If $Y' = \{x\}$, then by the $M^\#$-concavity condition: $u(\{x, z\}) + u(\{y\}) \geq u(\{x\}) + u(\{y, z\})$. Then $\{y\}$ must be a 1-maximizer and $\{x, z\}$ must be a 2-maximizer; the former is contained in $Z_2$ and the latter contains $Z_1$ so we are done.

**Step:** we assume that the lemma is true when there are less than $m$ items and prove that it is true for $m$ items, where $m \geq 4$. Let $Z_i$ be an $i$-maximizer and $Z_j$ a $j$-maximizer. We consider several cases.

**Case 1:** There is an item which is not in $Z_i$ nor in $Z_j$. W.l.o.g. call it item 1. Let $u'$ be the restriction of $u$ to the items $\{2, \ldots, m\}$. Then $Z_i$ is an $i$-maximizer of $u'$ and $Z_j$ is a $j$-maximizer of $u'$. By the induction assumption, the lemma is true for $u'$. Hence, there is a $j$-maximizer of $u'$, say $Z_j'$, which contains $Z_i$. Since both $Z_j$ and $Z_j'$ are $j$-maximizers of $u'$, $u'(Z_j) = u'(Z_j')$. Hence $u(Z_i) = u(Z_j')$. Hence, $Z_j'$ is also a $j$-maximizer of $u$, so part (a) is done. Similarly, there is an $i$-maximizer of $u'$, say $Z_i'$, which is contained in $Z_j$. Since both $Z_i$ and $Z_i'$ are $i$-maximizers of $u'$, $u'(Z_i) = u'(Z_i')$. Hence $u(Z_i) = u(Z_i')$. Hence, $Z_i'$ is also an $i$-maximizer of $u$, so part (b) is done.

**Case 2:** There is an item which is in both $Z_i$ and $Z_j$. W.l.o.g. call it item 1. Let $u'$ be the marginal valuation function $u_{\{1\}}$. By Lemma 3.3, $u'$ also is $M^\#$-concave. It is a valuation function on $m - 1$ items, $\{2, \ldots, m\}$. The bundle $Z_{i-1} = Z_i \setminus \{1\}$ is an $(i - 1)$-maximizer of $u'$ and the bundle $Z_{j-1} = Z_j \setminus \{1\}$ is a $(j - 1)$-maximizer of $u'$. By the induction assumption the lemma is true for
Lemma 5.1. Let $p$ be a price-vector, $d$ a real constant, and $p'$ another price-vector such that for every item $x$: $p'_x = p_x + d$. If the valuation is $M^\#$-concave, then the following is true for every bundle $P$:

(a) If $d \leq 0$ and $P$ is a $p$-demand, then there exists a $p'$-demand $P' \supseteq P$.

(b) If $d \geq 0$ and $P$ is a $p'$-demand, then there exists a $p$-demand $P' \supseteq P$.

Proof. It is obviously sufficient to prove part (a), since part (b) is its mirror-image.

Let $P$ be a $p$-demand and $Q$ a $p'$-demand. We consider two cases.

Case 1: $|Q| \leq |P|$. Note that $u_{p'}(Q) = u_p(Q) - d \cdot |Q|$ and $u_{p'}(P) = u_p(P) - d \cdot |P|$, so $u_{p'}(Q) - u_{p'}(P) = u_p(Q) - u_p(P)$. This means that, in the move from $p$ to $p'$, $Q$ gained weakly less net-utility than $P$. Hence, if $Q$ is a $p'$-demand, $P$ is necessarily a $p'$-demand too. $P \supseteq P$ so we are done.

Case 2: $|Q| > |P|$. Let $i = |P|$ and $j = |Q|$. Then, $P$ is an $i$-maximizer of the net-utility function $u_p$ and $Q$ is a $j$-maximizer of the net-utility function $u_{p'}$. But, the change in price between
$p$ and $p'$ does not affect the preference relation between bundles of the same size. Hence, $P$ is also an $i$-maximizer of $u_{p'}$. Since $j > i$, by Lemma 4.1/a there exists a $j$-maximizer of $u_{p'}$ that contains $P$. Call it $P'$. By definition of a $j$-maximizer, $u_{p'}(P') \geq u_{p'}(Q)$. Hence, $P'$ is also a $p'$-demand. $P' \supseteq P$ so we are done.

**Remark 5.1.** Lemma 5.1 is not true when there are complementary items. E.g, in Example 1.1, when the prices change from $10, 10, 10$ to $50, 50, 50$, Bob discovers item $z$.

### 6. Downward Demand-Flow Property (Main Result)

**Theorem 1.** If a valuation function is $M^\#$-concave, then it has the DDF property.

**Proof.** Let $p, q$ be two price-vectors and $\Delta_x = q_x - p_x$. We now prove part (a) in the DDF definition: if the agent abandoned an item $x$ with $\Delta_x \leq 0$, then the agent must have discovered some item $y$ with $\Delta_y < \Delta_x$. The proof of part (b) is analogous.

Consider an item $x$ with $\Delta_x \leq 0$ that is $p$-demanded but not $q$-demanded. Define the price-vector $p'$ as (see Figure 1):

$$\forall y : p'_y = p_y + \Delta_x$$

By Lemma 5.1(a), all items that are $p$-demanded, including item $x$, are also $p'$-demanded.

Define the price-vector $q'$ as (see Figure 1):

$$\begin{align*}
\Delta_y \leq \Delta_x & : q'_y = p'_y = p_y + \Delta_x \\
\Delta_y \geq \Delta_x & : q'_y = p'_y + (\Delta_y - \Delta_x) = p_y + \Delta_y = q_y
\end{align*}$$

Between $p'$ and $q'$, the prices of items above $x$ weakly increased while the prices of item $x$ and the items below $x$ remained the same. By the GS property, item $x$ is $q'$-demanded, and all items below $x$ that were $p$-demanded are $q'$-demanded.

The last step of the proof - the move from $q'$ to $q$ - is true for arbitrary valuations (not only GS). Since $x$ was $q'$-demanded, there was a $q'$-demand $Q'$ that contained $x$. Since $x$ is not $q$-demanded, $Q'$ is not a $q$-demand. This means that there must be a different $q$-demand, say $Q$, that became more attractive than $Q'$, i.e, $u_q(Q) > u_q(Q')$. But $u_{q'}(Q) \leq u_{q'}(Q')$, so necessarily, in the move from $q'$ to $q$, the bundle $Q$ became cheaper more than $Q'$. Since the only items that became cheaper from $q'$ to $q$ are items with $\Delta_y < \Delta_x$, the bundle $Q$ must contain at least one of these items $y$ which was not previously demanded. This implies that our agent, who abandoned $x$, has discovered $y$. 

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7. Remark

Gul and Stacchetti [2] prove that GS is equivalent to two other properties: Single Improvement (SI) and No Complementaries (NC). They also present a property which they call Strong No Complementaries (SNC) and prove that it implies NC. They do not prove that NC implies SNC, but they also do not prove otherwise, i.e., they do not give an example of a valuation that is NC and not SNC. Based on our failure to find such an example ourselves, and based on the similarity between the SNC condition and the MC condition, we conjecture that SNC is actually equivalent to MC (and hence, to NC and SI and GS).

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