We propose a model of gravity in which a General Relativity metric tensor and an effective metric generated from a single scalar formulated in Geometric Scalar Gravity are mixed. We show that the model yields the exact Schwarzschild solution, along with accelerating behavior of scale factors in cosmological solutions.

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I. INTRODUCTION

Bigravity theory is an alternative theory of gravity that describes both massless and massive gravitons [1, 2] (for a review, see [3]). The low-energy modification from General Relativity (GR) is expected to explain mysterious components of the universe known as dark matter and dark energy. Bigravity models generally contain two Einstein–Hilbert terms representing the two metrics, as well as the mixing term of the two metrics, in order to generate mass for a graviton. Recent developments in the study of bigravity or massive gravity theories [4–8] have stemmed from the discovery of the appropriate mass term for ghost-free nonlinear bimetric action.¹

More recently, Nojiri, Odintsov, and their collaborators have considered extensions of the bigravity theory [9–14] in which the pure Einstein–Hilbert terms are replaced by the Lagrangian of $F(R)$ gravity or scalar–tensor theories.² These authors have studied models having scalar degrees of freedom with a view to resolving the cosmological problems in the very early era, as well as in the present universe.

In 2014, Novello and collaborators presented a new theory of gravity, called Geometric Scalar Gravity (GSG) [18–23].³ In this theory, the dynamics of gravity is described by a single scalar field. A normalized derivative of the scalar field expresses part of the dynamical metric, as well as the scalar field itself. Novello and his collaborators found a specific form of the scalar field potential from which the Schwarzschild spacetime is derived as an exact solution. These researchers also discussed the (exotic) cosmology based on GSG [18–23]. The novel behavior of the scale factor in the GSG cosmology provides a very interesting supplementary perspective on the issue of the initial singularity [21]. On the other hand, GSG predicts scalar gravitational waves [18, 26], which may conflict with recent direct observations of gravitational waves from a black hole binary [27]. Further, more scalar degrees of freedom may be needed in order to explain the gravitational field around a spinning source [23, 28]. Therefore, the simplest GSG model has practical difficulty in describing astrophysical processes.

In this paper, we propose a GR–GSG hybrid model of gravity. Our model consists of the dynamics of a fundamental metric tensor in GR and an effective metric in GSG. This

¹ In a certain sense, massive gravity is just bigravity in which one of the metrics is non-dynamic.
² For the various models of modified gravity and their cosmological meanings, see [15–17].
³ See also [24, 25].
theory naturally possesses a massless mode for the symmetric tensor field and yields the Schwarzschild solution exactly. Nevertheless, the cosmological solutions in the model are expected to be interesting, because they may inherit novel characteristics from GSG solutions. In the next section, we define our model. In Sec. III, we investigate a spherically symmetric solution of the model in weak gravity. In Sec. IV, we explore cosmological solutions for our model. We consider two cases: two metrics independently coupled to corresponding matter and the case in which the “composite” metric is considered to be the physical metric coupled to matter. Finally, we summarize our work and remark on the general significance of our study in Sec. V.

II. THE GR–GSG HYBRID MODEL

A. Brief review of GSG

First, we provide a brief review of GSG [18] to render the present paper self-contained. The effective metric $q_{\mu\nu}$ in GSG is described by a scalar field $\Phi$ as

$$q_{\mu\nu} = e^{2\Phi} \left[ \eta_{\mu\nu} - \frac{e^{-4\Phi} V(\Phi) - 1 \partial_\mu \Phi \partial_\nu \Phi}{w} \right],$$

(2.1)

where $w \equiv \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi$ and $\eta_{\mu\nu}$ is a flat Minkowski metric with the signature $(-+++)$. The inverse of the effective metric is then written as

$$q^{\mu\nu} = e^{-2\Phi} \left[ \eta^{\mu\nu} + \frac{e^{-4\Phi} V(\Phi) - 1}{w} \eta^{\mu\rho} \eta^{\nu\sigma} \partial_\rho \Phi \partial_\sigma \Phi \right].$$

(2.2)

Note that

$$\sqrt{-\det q} = \frac{e^{6\Phi}}{\sqrt{V(\Phi)}} \sqrt{-\det \eta}, \quad q^{\mu\nu} \partial_\nu \Phi = e^{-6\Phi} V(\Phi) \eta^{\mu\nu} \partial_\nu \Phi.$$

(2.3)

We consider the following action governing the dynamics of $\Phi$ with a potential $V(\Phi)$:

$$S_{GSG} = -M_q^2 \int d^4x \sqrt{-\det q} q^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi,$$

(2.4)

where $M_q$ is a constant with mass dimensions. The variation of the action with respect to $\Phi$ is calculated as

$$\delta S_{GSG} = 2M_q^2 \int d^4x \sqrt{-\det q} \sqrt{V(\Phi)(\Box_q \Phi)} \delta \Phi,$$

(2.5)
where
\[ \Box_q \Phi \equiv \frac{1}{\sqrt{-\det q}} \partial_\mu (\sqrt{-\det qq^{\mu\nu}} \partial_\nu \Phi), \]
\[ = e^{-6\Phi} V(\Phi) \left[ \frac{1}{\sqrt{-\det \eta}} \partial_\mu (\sqrt{-\det \eta \eta^{\mu\nu}} \partial_\nu \Phi) + \frac{w}{2} \frac{d}{d\Phi} \ln V(\Phi) \right]. \]

\[ (2.6) \]

Novello et al. [18] have stated that the form for \( V(\Phi) \) must be chosen so as to realize the exact Schwarzschild solution. Therefore, they employ
\[ V(\Phi) = \frac{1}{4} e^{2\Phi} (1 - 3e^{2\Phi})^2. \]

\[ (2.7) \]

We also adopt this potential in the present paper.

**B. Construction of hybrid model**

The following Einstein–Hilbert action is exploited in order to provide the dynamics of the metric tensor \( g_{\mu\nu} \):
\[ S_{GR}(g) = \frac{M_g^2}{2} \int d^4 x \sqrt{-\det g} R_g, \]
where \( M_g \) is a constant with mass dimensions and \( R_g \) is the Ricci scalar constructed from the metric tensor \( g \). The variation of the action (2.8) with respect to \( g \) yields
\[ \delta S_{GR}(g) = -\frac{M_g^2}{2} \int d^4 x \sqrt{-\det g} \left[ R_{\mu\nu}^g - \frac{1}{2} R_g g_{\mu\nu} \right] \delta g_{\mu\nu}, \]
where \( R_{\mu\nu}^g \) denotes the Ricci tensor constructed from \( g \).

Next, we consider the mixing term of \( g \) and the effective metric \( q \). For simplicity, we adopt that used in the minimal case of the ghost-free bigravity, which is expressed as follows [1, 5, 9–14, 29, 30]:
\[ S_{mix}(g, q) = m^2 M_0^2 \int d^4 x \sqrt{-\det g} \left[ 3 - \text{tr} \sqrt{g^{-1}q} + \det \sqrt{g^{-1}q} \right], \]
\[ (2.10) \]
where \( m \) and \( M_0 \) are two constants with mass dimensions. Note that \( M_0 \) is implicitly considered to be of the same order as \( M_g \) and \( M_q \). The tensor \( \sqrt{g^{-1}q} \) means \( (\sqrt{g^{-1}q})^\mu_\rho (\sqrt{g^{-1}q})^\rho_\nu = g^{\mu\rho} q_{\rho\nu} \). The variation of (2.10) is given by
\[ \delta S_{mix}(g, q) = \frac{m^2 M_0^2}{2} \int d^4 x \sqrt{-\det g} \left[ g^{\mu\nu} \left( 3 - \text{tr} \sqrt{g^{-1}q} \right) + \frac{1}{2} (\sqrt{g^{-1}q})^\mu_\rho g^{\rho\nu} + \frac{1}{2} (\sqrt{g^{-1}q})^\nu_\rho g^{\mu\rho} \right] \delta g_{\mu\nu}. \]
\[ + \frac{m^2 M^2_0}{2} \int d^4x \sqrt{-\det g} \left[ q^{\mu\nu} \det \sqrt{g^{-1}q} \right. \]
\[ - \frac{1}{2} \left( \sqrt{g^{-1}q} \right)^{-1\rho g^{\rho\nu}} - \frac{1}{2} \left( \sqrt{g^{-1}q} \right)^{-1\nu g^{\mu\nu}} \right] \delta q_{\mu\nu}. \quad (2.11) \]

Now, we define the total action for the graviton sector as the following combination:

\[ S = S_{GR}(g) + S_{GSG} + S_{mix}(g, q). \quad (2.12) \]

Note that a possible additional action for matter fields \( S_{\text{matter}} \) will be considered later, in Sec. IV. The equations of motion derived from \( S \) can be expressed as

\[ M^2_g \left[ R^\mu_\nu - \frac{1}{2} R g^{\mu\nu} \right] - m^2 M^2_0 \left[ g^{\mu\nu} \left( 3 - \text{tr} \sqrt{g^{-1}q} \right) + \frac{1}{2} \left( \sqrt{g^{-1}q} \right)^{\mu\rho g^{\rho\nu}} + \frac{1}{2} \left( \sqrt{g^{-1}q} \right)^{\nu g^{\mu\nu}} \right] = 0, \quad (2.13) \]

and

\[ M^2_q \sqrt{V(\Phi)}(\Box_\Phi) + \frac{m^2 M^2_0}{2} \frac{\sqrt{-\det g}}{\sqrt{-\det q}} \left[ \tau + \left( 2 - \frac{1}{2} \frac{dV}{d\Phi} \right) \varepsilon + \nabla^g_\mu \chi^\mu \right] = 0, \quad (2.14) \]

where

\[ \tau \equiv 4 \det \sqrt{g^{-1}q} - \text{tr} \sqrt{g^{-1}q}, \quad (2.15) \]
\[ \varepsilon \equiv \det \sqrt{g^{-1}q} - \frac{1}{\Omega} \left( \sqrt{g^{-1}q} \right)^{-1\mu} g^{\mu\nu} \partial_\nu \Phi \partial_\nu \Phi, \quad (2.16) \]
\[ \chi^\mu \equiv \frac{e^{-4\Phi} V - 1}{\Omega} \left( -\frac{1}{2} \left( \sqrt{g^{-1}q} \right)^{-1\mu} g^{\mu\nu} - \frac{1}{2} \left( \sqrt{g^{-1}q} \right)^{-1\nu} g^{\mu\nu} \right. \]
\[ + \frac{1}{\Omega} \left( \sqrt{g^{-1}q} \right)^{-1\lambda} g^{\mu\sigma} \partial_\lambda \Phi \partial_\sigma \Phi q^{\mu\nu} \right) \partial_\nu \Phi, \quad (2.17) \]

with

\[ \Omega \equiv q^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi = e^{-6\Phi} V w, \quad \nabla^g_\mu \chi^\mu \equiv \frac{1}{\sqrt{-\det g}} \left( \partial_\mu \sqrt{-\det g} \chi^\mu \right). \quad (2.18) \]

Here, we have used [18]

\[ \delta q_{\mu\nu} = \delta \left[ e^{2\Phi} \left[ \eta_{\mu\nu} - \frac{e^{-4\Phi} V - 1}{e^{-4\Phi} V w} \partial_\mu \Phi \partial_\nu \Phi \right] \right], \]
\[ = \left[ 2q_{\mu\nu} + \left( 4 - \frac{1}{V} \frac{dV}{d\Phi} \right) \frac{\partial_\mu \Phi \partial_\nu \Phi}{\Omega} \right] \delta \Phi \]
\[ + \frac{e^{-4\Phi} V - 1}{\Omega} \left[ 2 \frac{e^{-6\Phi} V}{\Omega} \partial_\mu \Phi \partial_\nu \Phi \partial_\lambda \Phi \partial_\lambda \delta \Phi - \partial_\mu \delta \Phi \partial_\nu \Phi - \partial_\mu \Phi \partial_\nu \delta \Phi \right]. \quad (2.19) \]
III. STATIC SPHERICAL SOLUTIONS

In this section, we consider static vacuum solutions with the spherically symmetric ansatz. First, we consider the flat metric in spherical coordinates

\[ \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dR^2 + R^2 d\Omega^2. \]  

(3.1)

where \( d\Omega^2 \) is the line element on a unit sphere. Because the spherical symmetry enforces the fact that the GSG scalar field \( \Phi \) has only radial-coordinate dependence, i.e., \( \Phi = \Phi(R) \), the effective line element \( ds^2_q \) becomes

\[ ds^2_q = q_{\mu\nu} dx^\mu dx^\nu = -e^{2\Phi} dt^2 + e^{6\Phi} V(\Phi) dR^2 + e^{2\Phi} R^2 d\Omega^2. \]  

(3.2)

Now, converting the radial coordinate to \( r \equiv e^\Phi R \), we find

\[ ds^2_q = q_{\mu\nu} dx^\mu dx^\nu = -B(r) dt^2 + 2H(r) dr^2 + r^2 d\Omega^2. \]  

(3.3)

where

\[ B(r) = e^{2\Phi} \quad \text{and} \quad H(r) = \frac{e^{4\Phi}}{V(\Phi)} \left( 1 - r \frac{d\Phi}{dr} \right)^2. \]  

(3.4)

Next, we impose the bidiagonal spherically symmetric ansatz [31], i.e., \( g \) is also diagonal and assumed to be

\[ ds^2_g = g_{\mu\nu} dx^\mu dx^\nu = -D(r) dt^2 + \frac{dr^2}{\Delta(r)} + r^2 \gamma(r) d\Omega^2. \]  

(3.5)

From these assumptions, the quantities defined in (2.15, 2.16, 2.17) are expressed as

\[ \tau = \frac{4}{\gamma} \sqrt{\frac{HB\Delta}{D}} - \left( \frac{B}{D} + \sqrt{H\Delta} + \frac{2}{\sqrt{\gamma}} \right), \quad \varepsilon = \frac{1}{\gamma} \sqrt{\frac{HB\Delta}{D}} - \sqrt{H\Delta}, \quad \chi'' = 0, \]  

(3.6)

and (2.14) becomes

\[ \frac{|3B - 1|}{2r^2\sqrt{H}} \left( \frac{r^2B'}{\sqrt{HB}} \right)' + \frac{m^2M^2_0}{M_q^2} \gamma \sqrt{\frac{D}{HB\Delta}} \left[ \frac{4}{\gamma} \sqrt{\frac{HB\Delta}{D}} - \left( \frac{B}{D} + \sqrt{H\Delta} + \frac{2}{\sqrt{\gamma}} \right) \right. \]
\[ \left. + \frac{1 + 3B}{1 - 3B} \left( \frac{1}{\gamma} \sqrt{\frac{HB\Delta}{D}} - \sqrt{H\Delta} \right) \right] = 0, \]  

(3.7)

where the prime (’) indicates the derivative with respect to \( r \). On the other hand, (2.13) reads

\[ \left( 1 + \frac{r'\gamma}{2\gamma} \right) \frac{\Delta'}{r} + \frac{\gamma\Delta - 1}{r^2\gamma} + \frac{\Delta}{r} \left( \frac{3\gamma'}{\gamma} - \frac{r'\gamma^2}{4\gamma^2} + \frac{r''\gamma}{\gamma} \right) \]
\[ - \frac{m^2M^2_0}{M_q^2} \left[ 3 - \left( \frac{B}{D} + \sqrt{H\Delta} + \frac{2}{\sqrt{\gamma}} \right) + \sqrt{\frac{B}{D}} \right] = 0, \]  

(3.8)
\[ \frac{\Delta}{r} \left( 1 + \frac{r' \gamma'}{2 \gamma} \right) \frac{D'}{D} + \frac{\gamma \Delta - 1}{r^2 \gamma} + \frac{\Delta}{r} \left( \frac{\gamma'}{\gamma} + \frac{r \gamma'^2}{4 \gamma^2} \right) \]
\[ - \frac{m^2 M_0^2}{M_g^2} \left[ 3 - \left( \sqrt{\frac{B}{D}} + \sqrt{H \Delta} + \frac{1}{\sqrt{\gamma}} \right) \right] = 0, \quad (3.9) \]

\[ \frac{\Delta}{2} \left( \frac{D''}{D} + \frac{D'}{r D} - \frac{D'^2}{2 D^2} + \frac{\gamma'^2}{\gamma^2} - \frac{2 \gamma'}{r \gamma} + \frac{\gamma'^2}{2 \gamma^2} + \frac{D' \gamma'}{2 D \gamma} \right) + \frac{\Delta}{2r} \left( 1 + \frac{r}{2} \left( \frac{D'}{D} + \frac{\gamma'}{\gamma} \right) \right) \]
\[ - \frac{m^2 M_0^2}{M_g^2} \left[ 3 - \left( \sqrt{\frac{B}{D}} + \sqrt{H \Delta} + \frac{1}{\sqrt{\gamma}} \right) \right] = 0. \quad (3.10) \]

Note that, in the above expressions, the function \( H(r) \) is defined as
\[ H \equiv \frac{4B}{(1 - 3B)^2} \left( 1 - \frac{r B'}{2 B} \right)^2. \quad (3.11) \]

From these field equations, one can find that the Schwarzschild metric is obtained as an exact solution, i.e.,
\[ B(r) = D(r) = \Delta(r) = 1 - \frac{2M_1}{r}, \quad \gamma(r) = 1, \quad (3.12) \]
where \( M_1 \) is an arbitrary constant. Unfortunately, it is difficult to obtain general solutions of the field equations explicitly, because of their severe nonlinearity. Therefore, we perturb the metric around the Minkowski space to the first order. Then, the field equations (3.7, 3.8, 3.9, 3.10) give an asymptotic solution in vacuum having
\[ B(r) = 1 - \frac{2M_1}{r} - e^{-\mu r} \frac{M_2}{r}, \quad (3.13) \]
\[ D(r) = 1 - \frac{2M_1}{r} + e^{-\mu r} \left( \frac{\zeta M_2}{r} + O(r^{-2}) \right), \quad (3.14) \]
\[ \Delta(r) = 1 - \frac{2M_1}{r} + e^{-\mu r} \left( \mu l_0 M_2 + \frac{\zeta M_2}{r} + O(r^{-2}) \right), \quad (3.15) \]
\[ \gamma(r) = 1 + e^{-\mu r} \left( \mu g_0 M_2 + \frac{g_1(1 + \zeta) M_2}{r} + O(r^{-2}) \right), \quad (3.16) \]
where \( \mu \) is given by
\[ \mu^2 = \frac{2(1 + \zeta) m^2 M_0^2}{\zeta M_g^2 + M_q^2}, \quad (3.17) \]
and \( \zeta \) is a constant. The other coefficients are determined to be
\[ l_0 = \frac{\zeta(5\zeta + 2) M_g^4 - 2M_g^2 M_q^2 - M_q^4}{2M_g^2(\zeta M_g^2 + M_q^2)}, \quad g_0 = -\frac{\zeta M_g^2 - M_q^2}{2M_g^2}, \quad g_1 = \frac{\zeta M_q^2 - M_q^2}{\zeta M_g^2 + M_q^2}. \quad (3.18) \]
Hence, $H(r)$ is calculated as

$$
H(r)^{-1} = 1 - \frac{2M_1}{r} - e^{-\mu r} \frac{M_2(1 - \mu r)}{r} + \text{higher orders in } M_1, M_2 \text{ and } e^{-\mu r}.
$$  (3.19)

Interestingly, we note that $|q_{00} - q_{11}^{-1}| = B(r) - H(r)^{-1} = \frac{M_2}{r} e^{-\mu r} \to 0$ in the small mass limit $\mu \to 0$, up to this order. In the same manner, we also find that $|g_{00} - g^{-1}_{11}|$ vanishes in the small mass limit $\mu \to 0$, up to this order. These asymptotic behaviors show a different case from the bigravity theory with two tensor fields [3, 31].

In this section, we have found that the static spherical solution of our model is very similar to the GR solution. In the next section, we consider the cosmology based on our model.

### IV. COSMOLOGICAL SOLUTIONS

#### A. Cosmology with two metrics

In this section, we attempt to study the cosmological solution for our hybrid model. We expect new and interesting scale-factor behavior, as GSG is known to give non-standard evolution of the scale factor [20].

We first assume the total action as $S + S_{\text{matter}}$, where

$$
S_{\text{matter}} = \int d^4x \sqrt{-\det g} \mathcal{L}_g(g, \varphi_g) + \int d^4x \sqrt{-\det q} \mathcal{L}_q(q, \varphi_q).
$$  (4.1)

This form is known as the safest and most interesting choice in bigravity theories for cosmology [3, 32]. It is often referred to as a “twin matter” model. Of course, this assumption involves the original Hassan-Rosen theory [1] for $\mathcal{L}_q = 0$. Now, the field equations including matter are

$$
M_g^2 \left[ R_{g}^{\mu\nu} - \frac{1}{2} R_g g^{\mu\nu} \right] - m^2 M_0^2 \left[ g^{\mu\nu} \left( 3 - \text{tr} \sqrt{g^{-1}q} \right) + \frac{1}{2} (\sqrt{g^{-1}q})_{\rho}^{\nu} g^{\rho\mu} + \frac{1}{2} (\sqrt{g^{-1}q})_{\rho}^{\mu} g^{\rho\nu} \right] = T_{g}^{\mu\nu},
$$  (4.2)

where

$$
T_{g}^{\mu\nu} = -\frac{2}{\sqrt{-\det g}} \frac{\partial}{\partial g^{\rho\sigma}} \left( \sqrt{-\det g} \mathcal{L}_g \right) g^{\rho\mu} g^{\sigma\nu},
$$  (4.3)
and

\[ M_q^2 \sqrt{V(\Phi)}(\Box_q \Phi) + \frac{m^2 M_0^2}{2} \sqrt{-\det q} \left[ \tau + \left( 2 - \frac{1}{2V} \frac{dV}{d\Phi} \right) \varepsilon + \nabla^q_{\mu} \chi^\mu \right] = -\frac{1}{2} \left[ T_q + \left( 2 - \frac{1}{2V} \frac{dV}{d\Phi} \right) E_q + \nabla^q_{\mu} X^\mu \right], \tag{4.4} \]

where

\[ T_q \equiv T^{\mu\nu}_q q_{\mu\nu}, \quad E_q \equiv \frac{1}{\Omega} T^{\mu\nu}_q \partial_{\mu} \Phi \partial_{\nu} \Phi, \quad X^\mu \equiv \frac{e^{-4\Phi} V - 1}{\Omega} (T^{\mu\nu}_q - Eq^{\mu\nu}) \partial_{\nu} \Phi, \tag{4.5} \]

with

\[ T^{\mu\nu}_q \equiv -\frac{2}{\sqrt{-\det q}} \frac{\partial(\sqrt{-\det q} L_q)}{\partial q^{\rho\sigma}} q^{\rho\mu} q^{\sigma\nu}, \quad \nabla^q_{\mu} X^\mu \equiv \frac{1}{\sqrt{-\det q}} (\partial_{\mu} \sqrt{-\det q} X^\mu). \tag{4.6} \]

To consider solutions of time-dependent homogeneous space, we take the GSG scalar as a time dependent function, \( \Phi = \Phi(t') \). Then, \( ds_q^2 \) becomes

\[ ds_q^2 = q'_{\mu\nu} dx_{\mu} dx_{\nu} = -\frac{e^{6\Phi}}{V(\Phi)} dt'^2 + e^{2\Phi} d\mathbf{x}^2. \]

If a coordinate transformation is performed such that \( dt' = N(t) \sqrt{e^{-6\Phi} V(\Phi)} dt \), a new expression is attained:

\[ ds_q^2 = q_{\mu\nu} dx_{\mu} dx_{\nu} = q_{\mu\nu} dx_{\mu} dx_{\nu} = -N(t)^2 dt^2 + e^{2\Phi(t)} d\mathbf{x}^2 = -N(t)^2 dt^2 + b(t)^2 d\mathbf{x}^2, \tag{4.8} \]

where the scale factor for \( q \) is defined as \( b(t) \equiv e^{\Phi(t)} \).

We again consider the bidiagonal ansatz and assume that \( g \) takes the form

\[ ds_g^2 = g_{\mu\nu} dx_{\mu} dx_{\nu} = -c(t)^2 dt^2 + a(t)^2 d\mathbf{x}^2, \tag{4.9} \]

where \( a(t) \) is the scale factor for \( g \). Note that the metric is the usual flat Friedmann–Lemaître–Robertson–Walker metric for \( c(t) = 1 \). If we set \( c(t) = a(t) \), we obtain the conformal form of this metric.

We further assume that each energy-momentum tensor is given in the form of a perfect fluid

\[ T^{\mu\nu}_g = (\rho_g + p_g) u^\mu_g u^\nu_g + p_g g^{\mu\nu}, \quad T^{\mu\nu}_q = (\rho_q + p_q) u^\mu_q u^\nu_q + p_q q^{\mu\nu}, \tag{4.10} \]

where \( u^\mu_g \) and \( u^\mu_q \) are the four-velocities that satisfy \( g_{\mu\nu} u^\mu_g u^\nu_g = -1 \) and \( q_{\mu\nu} u^\mu_q u^\nu_q = -1 \), respectively. In the present case, the four-velocities have the time-like component only.
Applying these ansätze, we find that the field equations (4.2, 4.4) can be rewritten as

\[
3M_g^2 \frac{\dot{a}^2}{c^2a^2} + 3m^2M_0^2 \left( 1 - \frac{b}{a} \right) = \rho_g, \quad (4.11)
\]

\[
M_g^2 \frac{1}{c^2} \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{2\dot{a} \dot{c}}{ac} \right) + m^2M_0^2 \left( 3 - \frac{2b}{a} - \frac{N}{c} \right) = -p_g, \quad (4.12)
\]

\[
-M_g^2 \frac{b}{2N^2} |3b^2 - 1| \left( \frac{\ddot{b}}{b} + 2\frac{b^2}{b^2} - \frac{bN}{bN} \right) + \frac{m^2M_0^2}{2} a^3c \left[ \tau - \frac{3b^2 + 1}{3b^2 - 1} \varepsilon \right]
\]

\[
= -\frac{1}{2} \left[ T_q - \frac{3b^2 + 1}{3b^2 - 1} E_q \right], \quad (4.13)
\]

where

\[
\tau = 4 \frac{b^3N}{a^3c} - 3 \frac{b}{a} - \frac{N}{c}, \quad \varepsilon = \frac{b^3N}{a^3c} - \frac{N}{c}, \quad (4.14)
\]

and

\[
T_q = -\rho_q + 3p_q, \quad E_q = -\rho_q. \quad (4.15)
\]

The dot (\(\dot{}\)) in the above equations denotes the derivative with respect to the time coordinate \(t\).

We further assume the conservation of the energy-momentum tensor:

\[
\nabla_{\mu} T_{\mu\nu} = \frac{1}{\sqrt{-\det g}} \partial_{\mu}(\sqrt{-\det g} T_{\mu\nu}) + \Gamma(g)_{\rho\sigma}T_{g}^{\rho\sigma} = 0, \quad (4.16)
\]

where

\[
\Gamma(g)_{\rho\sigma} = \frac{1}{2} g^{\nu\lambda}(\partial_{\rho}g_{\lambda\sigma} + \partial_{\sigma}g_{\lambda\rho} - \partial_{\lambda}g_{\rho\sigma}). \quad (4.17)
\]

The conservation gives rise to a simple equation for the energy density and the pressure

\[
\dot{\rho}_g + 3\frac{\dot{a}}{a}(\rho_g + p_g) = 0. \quad (4.18)
\]

Then, by applying the Bianchi identity to (4.2), or rearranging (4.11), (4.12), and (4.18), one can find a simple relation

\[
\frac{\dot{a}}{c} = \frac{\dot{b}}{N}. \quad (4.19)
\]

We can obtain an equation without the second derivative term using (4.12), (4.13) and (4.19). Further, using (4.11) and (4.19) again to eliminate \(\dot{a}\) and \(\dot{b}\), we obtain an algebraic equation incorporating \(a\), \(b\), \(N/c\), \(\rho_g\), \(p_g\), \(p_q\), and \(p_q\). The equation can be solved for \(N/c\) and yields

\[
N = \frac{\frac{\rho_g + 3p_q}{3M_g^2} - \frac{6m^2M_0^2}{M_g^2} \frac{a}{b^2 - 1} + \frac{m^2M_0^2}{M_g^2} (2 - \frac{b}{a})}{\frac{4a^3p_q}{3bM_g^2} - \frac{2}{a^2(b^2 - 1)} \left[ T_q - \frac{3b^2 + 1}{3b^2 - 1} E_q \right] - \frac{2m^2M_0^2}{a^2(b^2 - 1)M_g^2} \left[ 4 - \frac{a^3}{b^3} - \frac{3b^2 + 1}{3b^2 - 1} (1 - \frac{a^3}{b^3}) \right] + \frac{m^2M_0^2}{M_g^2} (5 - 4\frac{a}{b})}. \quad (4.20)
\]
We further assume
\[ \nabla^q T^\mu_\nu = \frac{1}{\sqrt{-\det q}} \partial_\mu (\sqrt{-\det q} T^\mu_\nu) + \Gamma(q)^\nu_\rho_\sigma T^\rho_\sigma = 0, \quad (4.21) \]
where
\[ \Gamma(q)^\nu_\rho_\sigma = \frac{1}{2} q^{\nu\lambda} (\partial_\rho q_{\lambda\sigma} + \partial_\sigma q_{\lambda\rho} - \partial_\lambda q_{\rho\sigma}). \quad (4.22) \]
This conservation equation implies
\[ \dot{\rho}_q + \frac{\dot{b}}{b} (\rho_q + p_q) = 0. \quad (4.23) \]
Simple ansätze for the equations of state
\[ p_g = \omega_g \rho_g, \quad p_q = \omega_q \rho_q \quad (4.24) \]
where \( \omega_g \) and \( \omega_q \) are constants, give the dependence on scale factors, i.e.,
\[ \rho_g = \rho_{g0} \left( \frac{a_0}{a} \right)^{3(1+\omega_g)} \quad \rho_q = \rho_{q0} \left( \frac{b_0}{b} \right)^{3(1+\omega_q)}, \quad (4.25) \]
where \( \rho_{g0}, \rho_{q0}, a_0, \) and \( b_0 \) are constants.

Using all of the ansätze, the \( N/c \) given by (4.20) can be expressed as a function of \( a \) and \( b \). Then, we can obtain the time-development of \( a \) and \( b \) by solving the differential equations (4.11), (4.19), and (4.20).

Figure 1 shows the results of numerical calculations. Here, we set \( M_g = M_q = M_0 \) for simplicity. We also set \( c = 1 \) so that the parameter \( t \) becomes the standard cosmological time. The “initial” value of \( a(t) \) is set to \( a(1) = a_0 = 1 \). We regard the matter as dust (\( \omega_g = 0 \)). In these calculations, we set \( \rho_q = 0 \) for simplicity.

The solutions are obtained for two cases of different mass parameters \( m \): \( 3M_g^2m^2/\rho_{g0} = 10 \) (Fig. 1) and \( 3M_g^2m^2/\rho_{g0} = 1 \) (Fig. 2).\(^4\) As the “initial” value \( b(1) \), we take \( b(1) = 2, 3, 4, \) and \( 5 \) in each case.

For all cases, \( b(t) \) approaches \( a(t) \) in the later stages. Thus, for the limit \( t \rightarrow \infty \), the scale factors behave as the standard Friedmann universe; this can be understood from (4.11) and (4.20). Along with the increase of \( a \) and \( b \), the last term of both the numerator and

\(^4\) Slightly large values are used to demonstrate accelerated expansion in the numerical results explicitly.

The acceleration can be tuned almost arbitrarily. For details, see the discussion below (4.27).
FIG. 1. Time evolution of scale factors $a$ (solid curves) and $b$ (broken curves) with $3M_g^2m^2/\rho_g0 = 10$. The lines correspond to $b(1) = 2, 3, 4,$ and $5$. For the parameters employed here, please see the text.

FIG. 2. Time evolution of scale factors $a$ (solid curves) and $b$ (broken curves) with $3M_g^2m^2/\rho_g0 = 1$. The lines correspond to $b(1) = 2, 3, 4,$ and $5$. For the parameters employed here, please see the text.

denominator of $N/c$ (4.20) become dominant. Hence, in later stages, we find that $a \approx b$ and $\dot{a} \approx \dot{b}$.

The Friedmann-like equation (4.11) indicates that the matter is also dominant when $a$ is very small, as $b$ approaches $1/\sqrt{3}$ in the early stage. We examined the case with $\rho_q \neq 0$ and found that there is no qualitative difference in the behavior of the scale factors if there is a comparable amount of ordinary matter ($\omega_q \geq 0$) coupled to $q$, i.e., $\rho_q0 \approx \rho_g0$. This is because $b$ soon becomes large, as $N/c$ is large when $a$ is small: then, $\rho_q \propto 1/b^{3(1+\omega_q)}$ becomes small.

The relative evolution of the two scale factors can be clearly seen if the solutions are plotted on an $(a, b)$-plane as shown in Figs. 3 and 4. The arrows in the figures indicate the normalized vector

$$\frac{1}{\sqrt{\dot{a}^2 + b^2}}(\dot{a}, \dot{b}) = \frac{1}{\sqrt{1 + \frac{N^2}{c^2}}}(1, N/c), \quad (4.26)$$
at each point. In these figures, the shaded regions indicate that the right hand side of (4.11) becomes negative.

![FIG. 3. Solutions plotted on \((a, b)\)-plane for \(3M_g^2m^2/\rho_{g0} = 10\).](image)

FIG. 3. Solutions plotted on \((a, b)\)-plane for \(3M_g^2m^2/\rho_{g0} = 10\).

![FIG. 4. Solutions plotted on \((a, b)\)-plane for \(3M_g^2m^2/\rho_{g0} = 1\).](image)

FIG. 4. Solutions plotted on \((a, b)\)-plane for \(3M_g^2m^2/\rho_{g0} = 1\).

Returning to Figs. 1 and 2, we find that accelerated expansion occurs for a relatively large \(b(1)\). From (4.11) and (4.12), one arrives at the equation

\[
2\frac{1}{ca} \left( \frac{\dot{a}}{c} \right) = -\frac{1}{3M_g^2} (\rho_g + 3p_g) - \frac{m^2 M_0^2}{M_g^2} \left( 2 - \frac{b}{a} + \frac{N}{c} \right). \tag{4.27}
\]

If we take \(c = 1\) here, in other words, \(t\) is the standard cosmological time in the system described by \(g\), the left hand side of (4.27) reads \(2\ddot{a}/a\). Thus, we can confirm that accelerated expansion is only possible if \(b/a - N/c > 2\). Because the value of \(N/c\) is negative or almost zero at \(t = 1\), cosmic acceleration is feasible for a large value of \(b(1)/a(1)\) and for a large value of \(3M_g^2m^2/\rho_{g0}\), which is confirmed by the numerical calculations.

A subtle point to note is that \(|q_{00}| = N^2\) vanishes at certain points for sufficiently large \(b(1)\) and \(m^2\): in other words, there are determinant singularities [33]. As we hold that \(q\) is
not a genuine metric, no problem exists especially in the case of $\mathcal{L}_q = 0$. The degenerate metric, however, may induce field theoretical problems if the matter field is coupled to $q$, i.e., $\mathcal{L}_q \neq 0$.

In this subsection, we have found that the $a$ of the physical $g$ can exhibit accelerated expansion if $m$ and the initial value of $b/a$ are sufficiently large, even if there is no exotic matter. It is worth noting that the expansion decelerates in the early stages. The acceleration occurs subsequently and ends in the later stages.

### B. Cosmology with composite metric

Next, we consider a model with a “composite” metric, similar to the bigravity models proposed in \cite{34, 35}. The action for matter is now assumed to be described with the composite metric

$$G_{\mu\nu} = \alpha^2 g_{\mu\nu} + 2\alpha\beta g_{\mu\rho}(\sqrt{g^{-1}q})^{\rho}_{\nu} + \beta^2 q_{\mu\nu}, \quad (4.28)$$

where $\alpha$ and $\beta$ are constants. The composite line element for cosmology is given by

$$ds_G^2 = G_{\mu\nu}dx^\mu dx^\nu = -(\alpha c + \beta N)^2 dt^2 + (\alpha a + \beta b)^2 d\mathbf{x}^2 \equiv -C^2 dt^2 + A^2 d\mathbf{x}^2, \quad (4.29)$$

where we use the same symbols for the components of the two metrics as in the previous subsection. The energy-momentum tensor of the matter field is understood to be conserved with respect to the description when the composite metric is employed, i.e.,

$$\nabla_G^\mu T_G^{\mu\nu} = \frac{1}{\sqrt{-\det G}}\partial_\mu(\sqrt{-\det G} T_G^{\mu\nu}) + \Gamma(G)^\mu_{\rho\sigma} T_G^{\rho\sigma} = 0, \quad (4.30)$$

and we further assume the perfect fluid form for the isotropic and homogeneous universe to be

$$T_G^{\mu\nu} = (\rho_G + p_G)u_G^\mu u_G^\nu + p_G G^{\mu\nu}, \quad (4.31)$$

where $u_G^\mu$ satisfies $G_{\mu\nu}u_G^{\mu}u_G^{\nu} = -1$.

The field equations are found using the treatment in \cite{35} and following a similar calculation to previously, yielding

$$3M_g^2 \frac{\dot{a}^2}{c^2 a^2} + 3m^2 M_0^2 \left(1 - \frac{b}{a}\right) = \alpha \frac{A^3}{a^3} \rho_G, \quad (4.32)$$

$$M_g^2 \frac{\dot{a}^2}{c^2} \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - 2\frac{\dot{a}\dot{c}}{ac}\right) + m^2 M_0^2 \left(3 - 2\frac{b}{a} - \frac{N}{c}\right) = -\alpha \frac{CA^2}{ca^2} \rho_G, \quad (4.33)$$
\[-M_q^2 \frac{b}{2N^2} |3b^2 - 1| \left( \frac{\ddot{b}}{b} + 2 \frac{\dot{b}^2}{b^2} - \frac{\dot{b}N}{bN} \right) + \frac{m^2 M_0^2}{2} \frac{a^3 c}{b^2 N} \left[ \tau - \frac{3b^2 + 1}{3b^2 - 1} \varepsilon \right] \]

\[= - \frac{1}{2} \frac{A^3}{b^2} \beta \left[ 2 \frac{\rho_G + 3 Cb}{NA^2 p_G} \right], \quad (4.34)\]

with the definition given in (4.14). Then, the Bianchi identity yields the same relation as (4.19).\(^5\)

We can solve all the above equations to obtain

\[\frac{N}{c} = \frac{N}{C}, \quad (4.35)\]

where

\[N = \frac{A^3 \alpha}{3a^3 M_g^2} \left( \rho_G + \frac{3 \alpha a}{A} \rho_G \right) + \frac{2A^3 \alpha \beta}{ab^2 |3b^2 - 1| M_q^2 P_G}
- \frac{6m^2 M_0^2}{M_g^2} \frac{a}{b^2 |3b^2 - 1|} + \frac{m^2 M_0^2}{M_g^2} \left( 2 - \frac{b}{a} \right), \quad (4.36)\]

and

\[C = \frac{4A^3 \alpha}{3ba^2 M_g^2} \left( \rho_G - \frac{3b}{4A} \rho_G \right) - \frac{2A^3 \beta}{ab^3 |3b^2 - 1| M_q^2} \left( \frac{2}{3b^2 - 1} \rho_G + \frac{3b}{A} \rho_G \right)
- \frac{2m^2 M_0^2}{a |3b^2 - 1| M_g^2} \left[ 4 - \frac{a^3}{b^3} - \frac{3b^2 + 1}{3b^2 - 1} \left( 1 - \frac{a^3}{b^3} \right) \right] + \frac{m^2 M_0^2}{M_g^2} \left( 5 - 4 \frac{a}{b} \right). \quad (4.37)\]

Because \(N/c = \dot{b}/\dot{a}\), (4.32) and (4.35, 4.36, 4.37) can express the development of the scale factors, if the equation of state for matter is given. Note that \(N/c \rightarrow 1\) if \(\rho_G, p_G \rightarrow 0\) and \(b \rightarrow 0\), as in the case examined in the previous subsection.

Here, we again take a simple assumption for the equation of state

\[p_G = \omega_G \rho_G, \quad (4.38)\]

where \(\omega_G\) is a constant. Then, the dependence on the scale factor \(A\) is

\[\rho_G = \rho_{G0} \left( \frac{A_0}{A} \right)^{3(1 + \omega_G)}, \quad (4.39)\]

where \(\rho_{G0}\) and \(A_0\) are constants.

Note that the following relation holds:

\[\frac{C}{c} = \frac{ac + \beta N}{c} = \alpha + \beta \frac{N}{c} = \alpha + \beta \frac{\dot{b}}{\dot{a}} = \frac{\dot{a}}{\dot{a}} = \frac{\dot{A}}{\dot{a}}. \quad (4.40)\]

\(^5\) To be precise, the identity leads to \(m^2 M_0^2 - \alpha \beta \frac{A^2}{a^2} p_G \left( \frac{\dot{b}}{c} - \frac{N}{c} \dot{a} \right) = 0.\)
If we choose a new cosmological time $T$, which satisfies $dT = C dt$, this relation is no more than $\frac{dA}{dT} = \frac{\dot{a}}{c}$. Thus, if $\dot{a}/c > 0$, $A$ also increases with $T$.

From (4.32) and (4.33), we find the second-order differential equation

$$2 \frac{1}{ca} \left( \frac{\dot{a}}{c} \right)^2 = -\frac{\alpha}{3M_g^2} \left( \frac{A^3}{a^3} \rho_G + 3 \frac{A^2}{a^2} p_G \right) - \frac{m^2 M_0^2}{M_g^2} \left( 2 - \frac{b}{a} + \frac{N}{c} \right).$$

(4.41)

Using $T$, this equation becomes

$$2 \frac{C}{ca} \frac{d^2A}{dT^2} = -\frac{\alpha}{3M_g^2} \left( \frac{A^3}{a^3} \rho_G + 3 \frac{A^2}{a^2} p_G \right) - \frac{m^2 M_0^2}{M_g^2} \left( 2 - \frac{b}{a} + \frac{N}{c} \right).$$

(4.42)

Provided $C/c$ is positive, we expect that accelerated expansion of $A(T)$ is only possible if $b/a - N/c > 2$, as in the case treated in the previous subsection.

Figure 5 shows the results of numerical calculations for the composite metric model. Here, we set $M_g = M_q = M_0$ for simplicity. We also set $c = 1$ so as to make the parameter $t$ the standard cosmological time. The parameters $(\alpha, \beta)$ are taken to be $(0.5, 0.5)$, $m$ is chosen to satisfy $3M_g^2 m^2 / \rho_G = 10$, and the “initial” value of the scale factor $a(t)$ is set to $a(1) = a_0 = 1$. We consider dust matter ($\omega_G = 0$). As the “initial” value $b(1)$, we take $b(1) = 2, 3, 4,$ and $5$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5}
\caption{Time evolution of scale factors $A = 0.5a + 0.5b$ (solid curves), $a$ (gray solid curves), and $b$ (gray broken curves). The lines correspond to $b(1) = 2, 3, 4,$ and $5$. For the parameters employed here, please see the text.}
\end{figure}

The behaviors of $a$ and $b$ are similar to the case treated in the previous subsection. This time, however, the “physical” and unique scale factor is $A$. One can see the novel behavior of $A(t)$ from Fig. 5; however, we must bear in mind that $t$ is not the most suitable cosmological time.
We plot $A(T)$ for $b(1) = 2$ against $T$ in Fig. 6. For small and large $T$, the solution resembles that of the Friedmann universe. $A$ only seems to accelerate in the intermediate era.

![Graph of A(T) for b(1) = 2 and α = β = 0.5 (T = 1 at t = 1).]

FIG. 6. Plot of $A(T)$ for $b(1) = 2$ and $\alpha = \beta = 0.5$ ($T = 1$ at $t = 1$).

Figure 7 shows a plot of the solutions on the $(a, b)$-plane. In this figure, the shaded region indicates where the right hand side of (4.11) or $C$ becomes negative. For a sufficiently large $b(1)$, we find that the solution passes the point where $C = 0$ (and $\frac{4A}{dt} = 0$ at the same time, for (4.40)). The degenerate metric with $G_{00} = 0$ is too curious to assign a physical meaning.

![Graph of solutions on (a, b)-plane for α = β = 0.5.]

FIG. 7. Solutions plotted on $(a, b)$-plane for $\alpha = \beta = 0.5$.

Thus, the initial value $b(1)$ cannot be large, for instance, $b(1)$ must be less than $\approx 2$ in the case of $\alpha = \beta = 0.5$. If the value of $\alpha/\beta$ is larger, larger $b(1)$ is allowed. This fact can be seen from Figs. 8 and 9, in which the solutions for $\alpha = 0.65$ and $\beta = 0.35$ are plotted.

The evolution of $A(T)$ is plotted in Fig. 10 for $\alpha = 0.65$ and $\beta = 0.35$. A larger initial value of $b/a$ yields larger acceleration in the permitted parameter range.
In this subsection, we have found that the “physical” scale factor $A = \alpha a + \beta b$ can exhibit accelerated expansion if $m$ and the initial value of $b/a$ are sufficiently large. The parameters are restricted by the condition that the “physical” metric should be non-degenerate (i.e., $G_{00}$ does not vanish). The expansion shows successive deceleration, acceleration, and deceleration.

V. SUMMARY AND PROSPECTS

In this paper, we have presented a GR–GSG hybrid model of gravity. We have shown that the exact Schwarzschild solution is produced and the accelerating phase of the universe is obtained without the cosmological constant in this model. In this paper, we have shown only qualitative analyses, because there are many tunable parameters of our model, such as $M_q/M_g$, $\rho_{q0}$, and $\omega_g(\omega_q)$, as well as $m^2$, $b(1)$, and $a(1)$. Moreover, we can assume a general
FIG. 10. Plot of $A(T)$ for $b(1) = 2$ (lower curve) and $b(1) = 3$ (upper curve) for $\alpha = 0.65$ and $\beta = 0.35$ ($T = 1$ at $t = 1$).

mixing of $g$ and $q$ in $S_{mix}(g, q)$ other than the minimal choice considered in the present paper. Further research should be followed in future.

Unfortunately, our model does not provide a mechanism for inflation. However, because the early phase of the universe in our model resembles the Friedmann universe, incorporation of the inflation dynamics can be naturally introduced in the very early phase. The aspect of inflation in the GR–GSG hybrid model is an important subject for future study.

The quantum cosmology of our GR–GSG hybrid model is another very interesting subject, as the evolution of scale factors is naively dependent on the initial conditions. In particular, our classical model cannot avoid the singularity problem, unfortunately. Quantum cosmological approaches to the problem of singularities are common topics of study to which bimetric theories are applied [29].

In future, we hope to investigate many aspects of GR–GSG hybrid gravity, such as compact objects, instability problems (including initial fluctuations\(^6\)), and anisotropic solutions in the model, as well as the above-mentioned subjects. Through these investigations, we will find the phenomenological limit of the theory and obtain the ability to construct a more realistic model based on the present model.

\(^6\) If twin matter exists in our model ($L_q \neq 0$), the primordial fluctuations may exhibit novel behavior because of the rapid expansion of $b(t)$. However, the problem of a degenerate effective metric arises for some parameter choices.
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