Maximum-width Axis-Parallel Empty Rectangular Annulus

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Abstract. Given a set $P$ of $n$ points on $\mathbb{R}^2$, we address the problem of computing an axis-parallel empty rectangular annulus $A$ of maximum-width such that no point of $P$ lies inside $A$ but all points of $P$ must lie inside, outside and on the boundaries of two parallel rectangles forming the annulus $A$. We propose an $O(n^3)$ time and $O(n)$ space algorithm to solve the problem. In a particular case when the inner rectangle of an axis-parallel empty rectangular annulus reduces to an input point we can solve the problem in $O(n \log n)$ time and $O(n)$ space.

1 Introduction

A set of $n$ points $P$ on $\mathbb{R}^2$ is said to be enclosed by a geometric (or enclosing) object $C$ if all points of $P$ must lie inside $C$ and on the boundary of $C$. The problem of enclosing the input point set $P$ using a minimum sized geometric object $C$ such as a circle \cite{16}, rectangle \cite{18}, triangle \cite{15}, circular annulus \cite{19,17,11,3,2}, rectilinear annulus \cite{12}, rectangular annulus \cite{14} etc has been extensively studied in computational geometry over the last few decades. Here we start the discussion with the enclosing problem that uses various annulus as an enclosing object. Among various types of annulus, the enclosing problem using circular annulus has been studied extensively \cite{19,17,11,3,2}. The objective of this problem is to find a circular annulus of minimum-width that encloses $P$. Here circular annulus region is formed by two concentric circles. Gluchshenkoa et al. \cite{12} considered the problem of finding a rectilinear annulus of minimum-width which encloses $P$. For this problem, the annulus region is formed by two concentric axis-parallel squares. Recently, Bae \cite{4} studied this square annulus problem in arbitrary orientation where annulus is the open region between two concentric squares. Mukherjee et al. \cite{14} considered the problem of identifying a rectangular annulus of minimum-width which encloses $P$. In this problem, it is interesting to note that two mutually parallel rectangles forming the annulus region, are not necessary to be concentric. Moreover the orientation of such rectangles is not restricted to be axis-parallel. Further details on various annulus problem can be found in \cite{4,6,7,5,1,10} and the references therein.

As per we are aware, there has been little work on finding empty annulus of maximum-width for an input point set $P$. Díaz-Báñez et al. \cite{9} first studied
the problem of finding an empty circular annulus of maximum-width and proposed \(O(n^3 \log n)\) time and \(O(n)\) space algorithm to solve it. Mahapatra [13] considered the problem of identifying an axis-parallel empty rectangular annulus of maximum-width for the point set \(P\) and proposed an incorrect \(O(n^2)\) time algorithm to solve it. Given a point set \(P\), note that, for an axis-parallel minimum-width rectangular annulus which encloses \(P\), the outer or larger rectangle is always the minimum enclosing rectangle of \(P\) [14]. This observation leads to develop an \(O(n)\) time algorithm to find an axis-parallel rectangular annulus of minimum-width which encloses \(P\) [14]. However for the empty axis-parallel rectangular annulus problem of maximum-width, the number of potential outer rectangles forming an empty rectangular annulus is \(O(n^4)\). This implies that \(O(n^3)\) algorithm can be developed to solve this empty annulus problem using the result in [14]. Here we propose an \(O(n^3)\) time and \(O(n)\) space algorithm for finding an axis-parallel empty rectangular annulus of maximum-width for a given point set \(P\). Note that the problem of axis-parallel empty rectangular annulus of maximum-width is equivalent to the problem when the empty annulus region is generated by two concentric rectangles.

The paper is organized as follows: In Section 2 we discuss the problem of identifying an axis-parallel empty rectangular annulus of maximum-width after introducing some notations. In Section 3 we describe our new algorithm and prove its correctness. Section 4 concludes the paper.

2 Problem definition and terminologies

We begin by introducing some notations. Let \(P = \{p_1, \ldots, p_n\}\) be a set of \(n\) points on \(\mathbb{R}^2\). Let the \(x\) and \(y\)-coordinate of a point \(p_i\) be denoted as \(x(i)\) and \(y(i)\) respectively. Two axis-parallel rectangles \(R\) and \(R'\) are said to be parallel to each other when one of the sides of rectangle \(R\) is parallel to a side of \(R'\). Let \(R_{\text{in}}\) and \(R_{\text{out}}\) be two axis parallel rectangles such that \(R_{\text{in}} \subset R_{\text{out}}\). The rectangular annulus \(A\) formed by two such axis-parallel rectangles \(R_{\text{in}}\) and \(R_{\text{out}}\) is the open region between \(R_{\text{in}}\) and \(R_{\text{out}}\) where \(R_{\text{in}}\) has non-zero area. See Fig. 1 for a demonstration. We use the term inner (resp. outer) for the smaller (resp. larger) rectangle of rectangular annulus \(A\). In this paper the rectangle will always imply an axis parallel rectangle. The top-width of the rectangular annulus \(A\) is the perpendicular distance between top sides of its inner and outer rectangles. Similarly we define the bottom-width, right-width and left-width of \(A\). The minimum width among the top-, right-, bottom- and left-widths of a rectangular annulus \(A\) is defined as the width of \(A\) and is denoted by \(W(A)\). A rectangular annulus \(A\) formed by rectangles \(R_{\text{in}}\) and \(R_{\text{out}}\) (\(R_{\text{in}} \subset R_{\text{out}}\)) is said to be empty if the following two conditions are satisfied.

(i) No points of \(P\) lie inside \(A\).

(ii) All points of \(P\) lie inside the rectangle \(R_{\text{in}}\) and outside the rectangle \(R_{\text{out}}\).

The input points may lie on the boundaries of both \(R_{\text{in}}\) and \(R_{\text{out}}\).

The objective of our problem is to compute an axis-parallel empty rectangular annulus of maximum-width from the given point set \(P\). Note that the solution
of this problem is not unique. From now onwards the term annulus is used to mean an axis-parallel empty rectangular annulus.

3 Proposed Algorithm

An annulus is defined by its eight edges (Four edges of outer rectangle and four edges of inner rectangle). Each edge of an annulus passes through a point \( p \in P \). See Fig. 1 as an illustration.

\[
\text{Fig. 1. Different configurations of empty annulus.}
\]

Initially sort \( n \) points of \( P \) in ascending order on the basis of their \( x \)-coordinates and in descending order on the basis of their \( y \)-coordinates. Throughout the paper we have assumed that all points are in general position i.e. no horizontal or vertical line pass through two points. Two horizontal lines \( \text{Top}_{\text{out}} \) and \( \text{Bot}_{\text{out}} \) sweeps vertically from top to bottom over the plane and these two lines denote the current positions of the top and bottom sides of the outer rectangle defining an empty annulus. Depending on the position of \( \text{Top}_{\text{out}} \) and \( \text{Bot}_{\text{out}} \) a horizontal strip is defined as follows.

**Definition 1** A horizontal strip \( S(a, b) \) is defined as the open region bounded by two parallel lines \( \text{Top}_{\text{out}} \) and \( \text{Bot}_{\text{out}} \) where the lines \( \text{Top}_{\text{out}}, \text{Bot}_{\text{out}} \) pass through the points \( a \) and \( b \) respectively, having \( y(a) > y(b) \) and \( a, b \in P \).

**Definition 2** \( E(a, b) \) is the set of all empty annuli in \( S(a, b) \) such that the top (\( \text{Top}_{\text{out}} \)) and bottom (\( \text{Bot}_{\text{out}} \)) edges of outer rectangle of any annulus \( A \in E(a, b) \) pass through the points \( a \) and \( b \) respectively.

We now state the following simple observation.

**Observation 1** \([14]\) Given an outer rectangle \( R_{\text{out}} \) generated from the point set \( P \) on \( \mathbb{R}^2 \), the empty annulus \( A \) having \( R_{\text{out}} \) as the outer rectangle can be computed in \( O(m) \) time, where \( m \) is the number of points inside \( R_{\text{out}} \).
Our proposed algorithm computes an empty annulus $A_{(a,b)}^{\max}$ of maximum-width within each strip $S(a,b)$, for all such possible pairs $(a,b)$, where $a, b \in P$. Finally an annulus of maximum-width among all those annuli ($A_{(a,b)}^{\max}$) is reported.

3.1 Finding an empty annulus of maximum-width in a horizontal strip

Consider a strip $S(a, b)$ where we are looking for an empty annulus of maximum-width from $E(a, b)$. The following approach presents the way to achieve our goal.

Let $Q$ be the set of ordered points in increasing order (w.r.t. value of x-coordinates) including points $a$ and $b$ in the strip $S(a, b)$. Note that $Q$ can be determined from ordered set of points $P$ in linear time. Also let the leftmost and rightmost points of $Q$ be $l$ and $r$ respectively. Without loss of generality we have assumed that $x(a) > x(b)$ in $S(a, b)$. To compute the elements in $E(a, b)$, we use two vertical segments Left out and Right out. These two lines define the left and right edges of outer rectangle $R_{out}$ of an annulus $A \in E(a, b)$. It can be observed that Left out is required to sweep over the points of $Q$ on the left of $b$ and Right out is required to sweep over the points of $Q$ on the right of $a$ to generate elements of $E(a, b)$. If any one of these segments moves to a point of $Q$ an annulus defined by the current positions of Left out and Right out is required to update and therefore the points of $Q$ are the event points of the proposed sweep line algorithm. We now partition $Q$ into 3 groups - (i) Points starting from $x(l)$ to $x(b)$ are in set $L$, (ii) Points inside the rectangle formed by corner points $x(a)$ and $x(b)$ in set $M$ and (iii) Points starting from $x(a)$ to $x(r)$ in set $R$ (See Fig. 2). Left out starts sweeping from $x(b)$ and moves towards $x(l)$ where the event points of Left out are the input points in $L$. Similarly Right out starts sweeping from $x(a)$ and moves towards $x(r)$ and its event points are input points in $R$.

Let $p$ and $q$ denote the immediate right point of $a$ and immediate left point of $b$ respectively. Also let $p'$ and $q'$ denote the immediate right and immediate
left points of \( p \) and \( q \) respectively. See Fig. 2

Depending on the cardinality of \( M \) we have the following three cases - (i) \(|M| \geq 2\), (ii) \(|M| = 0\), and (iii) \(|M| = 1\).

**Case I** (\(|M| \geq 2\)): We first take an initial annulus \( A \) in \( S(a, b) \) from which we generate other annuli in the strip. This annulus \( A \) has outer rectangle say \( R_{out} \). The top-right corner and bottom-left corner of \( R_{out} \) are at points \( a \) and \( b \) respectively. Construct an inner rectangle \( R_{in} \) within this \( R_{out} \) using Observation 1. Let \( \text{Top}_{in}, \text{Right}_{in}, \text{Bot}_{in} \) and \( \text{Left}_{in} \) denote the top, right, bottom and left edges of \( R_{in} \) respectively. \( W(A) \) is the width of annulus \( A \).

**Lemma 1.** Consider any annulus \( K \in E(a, b) \) and assume that \( W(K) \) is determined by top-, bottom or left-width of annulus \( K \). If the right edges of outer and inner rectangles of annulus \( K \) is shifted towards right to obtain another annulus \( K' \), then \( W(K') \leq W(K) \).

**Proof.** Let the top- (\( \text{Top}_{in} \)), right- (\( \text{Right}_{in} \)), bottom- (\( \text{Bot}_{in} \)), left- (\( \text{Left}_{in} \)) edges of inner rectangle \( R_{in} \) of annulus \( K \) pass through the points \( p_t, p_r, p_b, \) and \( p_l \) respectively. The \( \text{Left}_{out} \) and \( \text{Right}_{out} \) edges of \( K \) pass through the points \( s \) and \( t \) where \( s \in L \), \( t \in R \) and \( W(K) \) is determined by the left-width of \( K \). See Fig 3(a) for an illustration. We now shift \( \text{Right}_{out} \) of annulus \( K \) from \( t \) to a point \( t' \) in the right where \( x(t) < x(t') \) and \( t' \in R \) keeping \( \text{Left}_{out} \) fixed at \( s \). Annulus \( K' \) is constructed where \( K' \in E(a,b) \) and let us assume that \( W(K') > W(K) \). Since the left edges of outer and inner rectangles of both \( K \) and \( K' \) pass through same points \( s \) and \( p_l \), their left-widths are equal. If the
Bot$_{in}$ edge of $K'$ is determined by a point say $p'_b$ such that $x(p'_b) \geq x(t)$ and $p_b \in R$ then bottom-width of $K'$ is less than bottom-width of $K$. Similarly we can say this for top-width of $K'$. Right-width of $K'$ can be equal, greater or smaller than right-width of $K$. If any one of the top-, right- or bottom-widths of $K'$ is smaller than its left-width then $W(K') < W(K)$. So it contradicts our assumption. If left-width of $K'$ determines $W(K')$ then we have $W(K') = W(K)$.

Now consider annulus $K$ where its top-width determines $W(K)$. Now we shift Right$_{out}$ of annulus $K$ from $t$ to any point $t'$ in the right where $x(t) < x(t')$ and $t, t' \in R$ (See Fig.3(b)). Annulus $K'$ is formed. Right$_{in}$ of $K'$ pass through $p'_r$ where $p'_r \in R$. To achieve better solution i.e. $W(K') > W(K)$ we have to increase the top-width of $K'$. However this is not possible because the point $p_t$ will lie in the open region between Right$_{out}$ and Right$_{in}$ of annulus $K'$. This means that no further sweeping of Right$_{out}$ and Right$_{in}$ of annulus $K$ is required. Using symmetry the assertion that $W(K') \leq W(K)$ holds when $W(K)$ is determined by the bottom-width of $K$ and $K'$ is any annulus whose left edge of outer rectangle lies at the same position where Left$_{out}$ of $K$ lies, and right edge of outer rectangle lies to the right of Right$_{out}$ of $K$. $\Box$

Similarly we can prove the following result.

**Lemma 2.** Assume that $K$ is an annulus in $E(a, b)$ where $W(K)$ is determined by top-, bottom- or right-width of annulus $K$. If the left edges of outer and inner rectangles of annulus $K$ is shifted towards left to obtain another annulus $K'$, then $W(K') \leq W(K)$.

![Fig. 4. Demonstration shows construction of $A_1$ from $A'$. Left-width of $A'$ becomes $W(A')$. Top$_{in}$, Right$_{in}$, Bot$_{in}$ and Left$_{in}$ edges of $A'$ pass through the points $p_r$, $p_t$, $p_b$, $p_l$ respectively. Right$_{out}$ of both $A'$ and $A_1$ pass through $t$. Left$_{out}$ of $A'$ and $A_1$ pass through $p'_l$ and $s$.](image)

Algorithm 1 is based on the computation of a new annulus from an initial configuration. Let $A'$ be a given annulus in $E(a, b)$. See Fig.4 as illustration. Depending on $W(A')$ we shift Left$_{out}$ of $A'$ from $p'_l$ to the next event point in left $s$, where $p'_l, s \in L$. Therefore a new outer rectangle is formed. Since $p'_l$ lies in the open region between this new outer rectangle and $R_{in}$ of $A'$, $p'_l$ is compared
with the points $p_1$, $p_2$, and $p_3$. We thus create new annulus $A_1$. Note that this operation requires constant time. Now we describe Algorithm 1 to compute the set $E(a, b)$ for Case I. In each step our algorithm keeps information about the best solution computed so far. Let the best solution in $S(a, b)$ is stored in $W(A_{(a,b)}^{\max})$ where $A_{(a,b)}^{\max}$ is an maximum-width annulus in the strip.

Algorithm 1: Algorithm for computing an empty annulus of maximum-width in $S(a, b)$.

**Input:** Annulus $A$ whose outer rectangle is defined by two opposite corner points $a$ and $b$ and its $Left_{out}$ and $Right_{out}$ passes through $b$ and $a$. $L, M, R$ are set of ordered points in $S(a, b)$ in increasing order (w.r.t. the value of x-coordinates) where $L, M, R$ are obtained from $Q$ in $S(a, b)$.

**Output:** The width $W(A_{(a,b)}^{\max})$ of an empty annulus $A_{(a,b)}^{\max}$ of maximum-width in $S(a, b)$.

1. $W(A_{(a,b)}^{\max}) \leftarrow W(A)$.
2. while $Left_{out}$ and $Right_{out}$ do not pass through $l$ and $r$ respectively do
   3. if top-width (or bottom-width) of $A$ determines $W(A)$ then
      4. if $W(A) > W(A_{(a,b)}^{\max})$ then
         5. $W(A_{(a,b)}^{\max}) \leftarrow W(A)$
   6. Exit
   7. if left-width of $A$ determines $W(A)$ & $Left_{out}$ passes through $l$ then
      8. if $W(A) > W(A_{(a,b)}^{\max})$ then
         9. $W(A_{(a,b)}^{\max}) \leftarrow W(A)$
    10. Exit
   11. if right-width of $A$ determines $W(A)$ & $Right_{out}$ passes through $r$ then
      12. if $W(A) > W(A_{(a,b)}^{\max})$ then
         13. $W(A_{(a,b)}^{\max}) \leftarrow W(A)$
    14. Exit
   15. if left-width of $A$ determines $W(A)$ then shift $Left_{out}$ of $A$ to the next event point in left. Let $A'$ be the new annulus formed then
      16. if $W(A') > W(A_{(a,b)}^{\max})$ then
         17. $W(A_{(a,b)}^{\max}) \leftarrow W(A')$
     18. $A \leftarrow A'$ (Ref. Lemma 1)
   19. if right-width of $A$ determines $W(A)$ then shift $Right_{out}$ of $A$ to the next event point in right. Let $A'$ be the new annulus formed then
      20. if $W(A') > W(A_{(a,b)}^{\max})$ then
         21. $W(A_{(a,b)}^{\max}) \leftarrow W(A')$
     22. $A \leftarrow A'$ (Ref. Lemma 2)
23. Return $W(A_{(a,b)}^{\max})$.

Note that if top- (or bottom) width becomes width of an annulus and is equal to its left- (or right) width we consider its top- (or bottom) width as its width (Followed from Lemma 1). Also if any annulus have width from its left-width width.
and right-width simultaneously then we consider any one of them as its width and proceed accordingly.

![Diagram](image)

**Fig. 5.** (a) $W(A')$ and $W(A_1)$ are determined by left-widths of $A'$ and $A_1$. (b) $W(A')$ and $W(A_1)$ are determined by right-widths of $A'$ and $A_1$.

In Algorithm 1, all elements of $E(a, b)$ are not computed. It starts with the initial configuration of annulus $A$. Depending on $W(A)$ we shift either its left edge or right edge of outer rectangle. Assume that left-width of $A$ determines $W(A)$. Now consider $A'$ and $A_1$ are two annuli computed in Case I where left-widths of $A'$ and $A_1$ determines $W(A')$ and $W(A_1)$ respectively. See Fig 5(a) as an illustration. $Left_{out}$ of $A'$ and $A_1$ pass through $s'$ and $s$, and $s', s \in L$. $Right_{out}$ of $A'$ and $A_1$ pass through point $t$, $t \in R$. Let $A_L$ be the set of all annuli whose left edges ($Left_{out}$) of outer rectangle pass through any point between $s'$ and $s$. The right edge of outer rectangle of any annulus in $A_L$ is fixed at $t$. If we shift the $Right_{out}$ of $A'$ to any point $t'$ such that $x(t') > x(t)$ and $t' \in R$ then the annulus that will be created have width either less or equal to $W(A')$ (Ref. Lemma 1). This fact is true for all annuli in $A_L$. This means that there is no requirement to generate all those annuli whose $Left_{out}$ pass through any point from $s'$ to $s$ and $Right_{out}$ passes through any point in the right of $t$. It may happen that $Left_{out}$ of $A_1$ reaches $l$ and left-width of $A_1$ determines $W(A_1)$ then our algorithm terminates and reports the best solution in $S(a, b)$. If right-width of $A_1$ determines $W(A_1)$ then we shift the $Right_{out}$ of $A_1$ and compute annuli further. Now assume that $W(A')$ and $W(A_1)$ are determined by right-widths of $A'$ and $A_1$. $Right_{out}$ of $A'$ and $A_1$ pass through $t'$ and $t$ and $t', t \in R$ and their $Left_{out}$ is fixed at $s'$, $s' \in L$. Let $A_R$ be the set of all annuli whose right edges of outer rectangle pass through any point between $t'$ and $t$ and left edges of outer rectangle fixed at $s'$ (See Fig 5(b)). In a similar way we can say that there is no requirement to shift the left edge of outer rectangle of any annulus in $A_R$.

We now consider the case when $|M| = 0$.

**Case II** ($|M| = 0$): We use Algorithm 1 to generate annuli of $E(a, b)$. It requires an initial configuration. We need at least two points to create an inner rectangle. These two points to form inner rectangle can lie in the left side of both $a$ and $b$, in the right side of both $a$ and $b$, or one in the right side of $a$ and other in the
Thus we need three initial configurations of the annuli from which we can generate other annuli in $E(a, b)$. They are as follows.



(i) Outer rectangle formed by point $a$ on the top right corner, $b$ in the bottom and $Left_{out}$ passing through the point $q_1$ where $q_1$ is immediate left point of $q'$ (See Fig. 6(a)). Two points $q$ and $q'$ lie on the two opposite corner of the inner rectangle. We name this annulus as $A_1$.

(ii) Outer rectangle formed by point $b$ on the lower left corner, $a$ on the above and $Right_{out}$ passing through the point $p_1$ where $p_1$ is immediate right point of $p'$ (See Fig. 6(b)). Two points $p$ and $p'$ lie on the two opposite corner of the inner rectangle. We name this annulus as $A_2$.

(iii) Outer rectangle formed by point $q'$ on the left, $a$ on the above, $p'$ on the right and $b$ lying at bottom. The inner rectangle is formed by two opposite corner points $p$ and $q$. Say this annulus $A_3$ (See Fig. 6(c)).

For each initial configuration we invoke Algorithm 1. We compare the solutions obtained from them and finally report an empty annulus of maximum-width in $S(a, b)$.

**Case III**($|M| = 1$): A single point, say $z$ is present inside the outer rectangle formed by two opposite corner points $a$ and $b$ in $S(a, b)$. Algorithm 1 requires an initial configuration to start with. We need at least two points to create an
inner rectangle. One of them is \( z \) and the other point can lie either in the left side or right side of \( z \). Therefore we form two initial configurations of annuli to compute other annuli in \( E(a,b) \).

(i) Outer rectangle formed by point \( a \) on the top right corner, \( b \) in the bottom and \( \text{Left}_{\text{out}} \) passing through the point \( q' \) (See Fig. 7(a)). Opposite corner points \( q \) and \( z \) form inner rectangle. This annulus is \( A_1 \).

(ii) Outer rectangle formed by point \( b \) on the lower left corner, \( a \) on the above and \( \text{Right}_{\text{out}} \) passing through the point \( p' \). Here \( p \) and \( z \) are used to form inner rectangle. See Figure 7(b). Let this annulus be \( A_2 \).

![Fig. 7. Demonstration shows \( A_1, A_2 \).](image)

We invoke Algorithm 1 separately on \( A_1, A_2 \) and report an empty annulus of maximum-width in \( S(a,b) \). As stated in Case I, we do not compute all elements of \( E(a,b) \) for Case II and Case III. We report an empty annulus of maximum-width in \( S(a,b) \) from those annuli which are computed in Case II (resp. Case III).

Now we have the following result.

**Theorem 1.** For a given set of points \( P = \{p_1, \ldots, p_n\} \) in \( \mathbb{R}^2 \), an empty annulus of maximum-width can be computed in \( O(n^3) \) time using \( O(n) \) space.

**Proof.** Note that the number of horizontal strips formed by any two points of \( P \) is \( O(n^2) \). Algorithm 1 requires \( O(m) \) time where \( m (\leq n) \) is the number of input points in any such strip. Thus the result follows. \( \square \)

In the above axis-parallel empty rectangular annulus problem, the inner rectangle forming such an empty annulus always have non-zero area. However if \( R_{m} \) reduces to a single point \( p \in P \) then we have following result.

**Corollary 1.** Given a set \( P \) of \( n \) points on \( \mathbb{R}^2 \), an axis-parallel empty rectangular annulus \( A \) of maximum-width can be computed in \( O(n \log n) \) time using
When the inner rectangle of the annulus $A$ reduces to a single point $p \in P$.

**Proof.** One can construct the Voronoi diagram for point set $P$ in $O(n \log n)$ time using $O(n)$ size data structure [8]. For any query point $q \in P$, a nearest point of $q$ among the points from $P$ can be computed in $O(\log n)$ time. Therefore the computation of nearest input points for all points of $P$ requires $O(n \log n)$ time. Hence the result follows. $\square$

## 4 Conclusion and discussion

In the annulus problem studied by Mukherjee et al. [14], the outer rectangle of an annulus of minimum-width which encloses $P$ must be the minimum enclosing rectangle enclosing $P$ where $P$ is the set of $n$ input points in $\mathbb{R}^2$. However in the empty axis-parallel rectangular annulus problem of maximum-width, the number of potential outer rectangles is $O(n^4)$. This observation implies that an $O(n^2)$ algorithm is trivial to find an empty rectangular axis-parallel annulus of maximum width. Therefore the proposed $O(n^5)$ time algorithm to solve the maximum-width empty annulus problem is a non-trivial one. Note that we didn’t give any lower bound for this problem but proposed $O(n \log n)$ time algorithm in Corollary 1 to solve the problem for a particular case. In this context, it would be interesting to give a sub-quadratic algorithm or to prove the problem $O(n^2)$-hard. Note that for each empty rectangular annulus problem discussed so far, the orientation is fixed. In future it remains as a challenge to solve this problem where the annuli are of arbitrary orientation.

## 5 Acknowledgements

This work is supported by Project (Ref. No. 248(19) 2014 R & D II 1045) from The National Board for Higher Mathematics (NBHM), Government of India awarded to P. Mahapatra where Arpita Baral is a research scholar under this Project.

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