A sufficient condition for the fiber of the tangent bundle of a scheme and its Zariski tangent space to be isomorphic

Colas Bardavid
IMSc — CIT Campus, Taramani
Chennai 600 113 India

January 15, 2013

Abstract — In this note, we give a simple sufficient condition for the Zariski relative tangent space and the Grothendieck relative tangent space to be isomorphic.

2000 Mathematics Subject Classification: 14A05, 13N05, 13N10

Keywords: Zariski tangent space, tangent bundle
Two notions of tangent space have been proposed in scheme theory: the Zariski tangent space $T_x^\text{Zar}(X)$ and the Grothendieck relative tangent space $T_{X/S}^{(\text{Gro})}(x)$. The relation between these two tangent spaces is known only in a very special case: in [2], Grothendieck shows that for a scheme $X$, when $S = \text{Spec } k$ (where $k$ is a field) and when $x \in X$ is $k$-rational point of $X$, then these two objects coincide. The aim of this note is to compare these two tangent spaces in more general situations. After having introduced a relative tangent space in the Zariski fashion, we show that there always exists a morphism from the Grothendieck relative tangent space to the Zariski one. The main result is that this morphism is an isomorphism whenever the extension $\kappa(x)/\kappa(s)$ is algebraic and separable. We also give a counter-example showing that in general these two tangent spaces do not coincide.

The note is organized as follows: in Section 1, we introduce three different tangent spaces $T_x^\text{Zar}(X)$, $T_{X/S}^{(\text{Gro})}(x)$ and $T_{X/S}^{(\text{Zar})}(x)$. In Section 2, we prove that in general $T_{X/S}^{(\text{Gro})}(x)$ and $T_{X/S}^{(\text{Zar})}(x)$ are not isomorphic. In section 3, we construct a morphism $T_{X/S}^{(\text{Gro})}(x) \rightarrow T_{X/S}^{(\text{Zar})}(x)$. In section 4, we give a condition for this morphism to be an isomorphism. In Section 5, we prove the main result of this note: $T_{X/S}^{(\text{Gro})}(x)$ and $T_{X/S}^{(\text{Zar})}(x)$ are isomorphic whenever $\kappa(x)/\kappa(s)$ is an algebraic and separable extension.

We wish to thank David Madore for a helpful discussion on this matter.

1 Introducing the two tangent spaces

(1.1) Global notations. The Zariski tangent space. In this paragraph, we recall some very classical facts and set various notations. Let $X \rightarrow S$ be schemes and $x \in X$ an element above $s \in S$. The structure sheaf of $X$ is denoted by $\mathcal{O}_X$, its stalk over $x$ by $\mathcal{O}_{X,x}$. Its maximal ideal is denoted by $\mathfrak{m}_x$ and $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$. For any $f \in \mathcal{O}_{X,x}$, the image of $f$ in $\kappa(x)$ under the canonical projection is denoted by $f(x)$. The residue field $\kappa(x)$ is viewed as an $\mathcal{O}_{X,x}$-module via $f \cdot \lambda := f(x)\lambda$. The ideal $\mathfrak{m}_x$ is an $\mathcal{O}_{X,x}$-module and admits as a sub-$\mathcal{O}_{X,x}$-module the ideal $\mathfrak{m}_x^2$. The quotient $\mathcal{O}_{X,x}$-module $\mathfrak{m}_x/\mathfrak{m}_x^2$ is in fact a $\kappa(x)$-vector space. We will also denote by $[f]$ the image of $f \in \mathfrak{m}_x$ in $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Definition 1.1. The Zariski tangent space of $X$ at $x$ is the $\kappa(x)$-vector space $T_x^\text{Zar}(X) := \text{Hom}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x))$.

(1.2) The Grothendieck relative tangent space. It is defined in §(16.5.13) of [2] as follows.

Definition 1.2. The Grothendieck relative tangent space of $X/S$ at $x$ is the $\kappa(x)$-vector space $T_{X/S}^{(\text{Gro})}(x) := \text{Hom}_{\kappa(x)}(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \kappa(x), \kappa(x))$. 
Let us recall that $\Omega^1_{X/S}$ is the $\mathcal{O}_X$-module of 1-differentials of $X/S$. We don’t go further in the description of this object since Fact 1.3 will give a more handy definition of $T^{(\text{Gro})}_{X/S}(x)$.

The main advantage of this construction over the Zariski’s one is that $T^{(\text{Gro})}_{X/S}(x)$ appears as the fiber of a tangent bundle. The tangent bundle of $X$ relatively to $S$ is a vector bundle $T_{X/S}$ above $X$. By definition, the $\kappa(x)$-rational points of its fiber over $x$ is precisely $T^{(\text{Gro})}_{X/S}(x)$. Let us give now the more handy description of $T^{(\text{Gro})}_{X/S}(x)$, which has also been noticed by Kunz in [3]:

**Fact 1.3.** $T^{(\text{Gro})}_{X/S}(x) \simeq \text{Der}_{\mathcal{O}_{S,x}}(\mathcal{O}_{X,x}, \kappa(x))$.

**Proof.** — It follows from two observations. First, as in [2], one can write

$$T^{(\text{Gro})}_{X/S}(x) = \text{Hom}_{\kappa(x)} \left( \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}} \otimes \mathcal{M}_x \cdot \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}}, \kappa(x) \right).$$

Second, if $k$ is a ring and $(A, \mathfrak{m})$ a local $k$-algebra with residual field $K$, then

$$\text{Hom}_{K-\text{ev}}(\Omega_{A/k}/\mathfrak{m} \cdot \Omega_{A/k}, K) \simeq \text{Der}_k(A, K),$$

as one can easily check. Then, apply this formula with $k = \mathcal{O}_{S,s}$, $A = \mathcal{O}_{X,x}$ to conclude. ■

**1.3 The Zariski relative tangent space.** Let us give a definition of the relative tangent space, in the Zariski fashion. When $X$ and $S$ are schemes, with $f : X \rightarrow S$ sending $x$ to $s$, one would like to define the differential of $f$ in $x$, mapping $T_xX$ to $T_sS$ (or better, to $T_sS \otimes_{\kappa(s)} \kappa(x)$). Imagine there is such a $\kappa(x)$-linear map

$$T_xf : T_xX \rightarrow T_sS \otimes_{\kappa(s)} \kappa(x).$$

Then, the relative tangent space would be the kernel of this map. Intuitively, it corresponds to the tangent space of the fiber $X_s$ at $x$. Actually, such a map $T_xf$ does not exist, but we can still define a very similar morphism and, subsequently, the relative tangent space.

The morphism $f$ induces a morphism $f^\#: \mathcal{O}_{S,s} \rightarrow \mathcal{O}_{X,x}$ that maps $\mathfrak{m}_s$ into $\mathfrak{m}_x$. Hence, we get a map $i_x : \kappa(s) \rightarrow \kappa(x)$ which is an injection between fields, as well as

$$j_x : \mathfrak{m}_s/\mathfrak{m}_s^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$$

which is $\kappa(s)$-linear. So, starting with $\bar{v} \in T_xX$, one obtains a $\kappa(s)$-linear map that we will denote

$$\overline{T_xf} \bullet \bar{v} := \bar{v} \circ j_x \in \text{Hom}_{\kappa(s)}(\mathfrak{m}_s/\mathfrak{m}_s^2, \kappa(x)).$$

The application

$$\overline{T_xf} : \begin{array}{ccc}
T_xX & \longrightarrow & \text{Hom}_{\kappa(s)}(\mathfrak{m}_s/\mathfrak{m}_s^2, \kappa(x)) \\
\bar{v} & \mapsto & \overline{T_xf} \bullet \bar{v}
\end{array}$$

3
is \( \kappa(x) \)-linear. The problem to define properly the differential of \( f \) is that, in general, \( \text{Hom}_{\kappa(s)}(\mathcal{M}_s/\mathcal{M}_s^2, \kappa(x)) \) and \( T_xS \otimes_{\kappa(s)} \kappa(x) \) are not isomorphic. But here, the map \( \bar{T}_xf \) will play the role of the differential.

**Definition 1.4.** The Zariski relative tangent space of \( X/S \) at \( x \) is the \( \kappa(x) \)-vector space

\[
T_{X/S}^{(\text{Zar})}(x) := \ker \bar{T}_xf.
\]

(1.4) An alternative description of the Zariski relative tangent space. As intuition suggests, the relative Zariski tangent space can be described as in the following lemma. We will use later this description.

**Lemma 1.5.** \( T_{X/S}^{(\text{Zar})}(x) \) is isomorphic to the tangent space \( T_{X,s,x} \) at \( x \) of the fiber \( X_s \) above \( s \).

**Proof.** — First, let us describe \( T_{X,s,x} \). The local ring \( \mathcal{O}_{X,s,x} \) is isomorphic to \( \mathcal{O}_{X,x}/\mathcal{M}_s\mathcal{O}_{X,x} \). Its maximal ideal is \( \mathcal{M}_x/\mathcal{M}_s\mathcal{O}_{X,x} \) and the tangent space is

\[
T_{X,s,x} \simeq \text{Hom}_{\kappa(x)}\left( \frac{\mathcal{M}_x/\mathcal{M}_s\mathcal{O}_{X,x}}{\mathcal{M}_x^2/\mathcal{M}_s\mathcal{O}_{X,x}}, \kappa(x) \right).
\]

As just seen, the relative Zariski tangent space can be described as

\[
T_{X/S}^{(\text{Zar})}(x) = \text{Hom}_{\kappa(x)}\left( \frac{\mathcal{M}_x/\mathcal{M}_s^2}{\mathcal{M}_s/\mathcal{M}_s^2}, \kappa(x) \right).
\]

So, let us prove that

\[
\frac{\mathcal{M}_x/\mathcal{M}_s\mathcal{O}_{X,x}}{\mathcal{M}_x^2/\mathcal{M}_s\mathcal{O}_{X,x}} \simeq \frac{\mathcal{M}_x/\mathcal{M}_s^2}{\mathcal{M}_s/\mathcal{M}_s^2}.
\]

Clearly, there is a map

\[
\frac{\mathcal{M}_x/\mathcal{M}_s^2}{\mathcal{M}_x/\mathcal{M}_s\mathcal{O}_{X,x}} \rightarrow \frac{\mathcal{M}_x/\mathcal{M}_s\mathcal{O}_{X,x}}{\mathcal{M}_s/\mathcal{M}_s^2},
\]

whose kernel can be described as the image of \( \mathcal{M}_s \rightarrow \mathcal{M}_x/\mathcal{M}_x^2 \). To conclude, remark that this last map factors through \( \mathcal{M}_s/\mathcal{M}_s^2 \). ■

2 Grothendieck and Zariski are not isomorphic in general

At this point, it is very easy to give a counter-example. Indeed, let us consider \( k \) a field and \( X := \text{Spec} \ k(t), S := \text{Spec} \ k \) and \( x \) the unique element of \( X \). Then, \( T_xX = 0 \) and so \( T_{X/S}^{(\text{Zar})}(x) = 0 \). But,

\[
T_{X/S}^{(\text{Gro})}(x) = \text{Der}_k(k(t), k(t))
\]

which is isomorphic to \( k(t) \).
3 From Grothendieck to Zariski

Let $D \in T^{(\text{Gro})}_{X/S}(x)$, in other words, in virtue of Fact 1.3, let $D : \mathcal{O}_{X,x} \rightarrow \kappa(x)$ be a $\mathcal{O}_{S,s}$-derivation. We can associate to $D$ an element of $T^{(\text{Zar})}_{X/S}(x)$, indeed, the restriction of $D$ to $\mathcal{M}_x$ factors through $\mathcal{M}_x \rightarrow \mathcal{M}_x/\mathcal{M}_x^2$, since for any $f, g \in \mathcal{M}_x$ one has

$$D(fg) = f \cdot D(g) + g \cdot D(f) = f(x) \cdot D(g) + g(x) \cdot D(f) = 0.$$ 

If

$$\phi_D : \mathcal{M}_x/\mathcal{M}_x^2 \rightarrow \kappa(x) \quad \begin{array}{c} \phi \\ \downarrow \end{array} \rightarrow D(\phi)$$

denotes the factored map, then let us check that $\phi_D \in T^{(\text{Zar})}_{X/S}(x)$. We have to show that the morphism

$$\mathcal{M}_x/\mathcal{M}_x^2 \cdot j_x \rightarrow \mathcal{M}_x/\mathcal{M}_x^2 \cdot \phi_D \rightarrow \kappa(x)$$

is zero. This is straightforward since $D$ is zero on $\mathcal{O}_{S,s}$. Hence, we have defined a $\kappa(x)$-linear map

$$\Phi^{\text{X/S}} := T^{(\text{Gro})}_{X/S}(x) \rightarrow T^{(\text{Zar})}_{X/S}(x) \quad \begin{array}{c} \phi_D \\ \downarrow \end{array} .$$

4 A condition for Grothendieck and Zariski to be isomorphic

To begin with, let us construct analogs of $\mathcal{M}_x$ and $\mathcal{M}_x/\mathcal{M}_x^2$ with a structure of $\kappa(x)$-algebra. First, let us denote

$$\tilde{\mathcal{O}}_{X/S,x} := \kappa(x) \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$$

and point out some facts:

- Any derivation $D \in \text{Der}_{\mathcal{O}_{S,s}}(\mathcal{O}_{X,x}, \kappa(x))$ gives rise to a derivation

$$\tilde{D} \in \text{Der}_{\kappa(x)}(\tilde{\mathcal{O}}_{X/S,x}, \kappa(x)).$$

It is simply defined by $\tilde{D}(\lambda \otimes \varphi) = \lambda \otimes D(\varphi)$.

- On the ring $\tilde{\mathcal{O}}_{X/S,x}$, we still have an evaluation map. It is

$$\tilde{\mathcal{O}}_{X/S,x} \rightarrow \kappa(x) \quad \begin{array}{c} \lambda \otimes \varphi \\ \downarrow \end{array} \rightarrow \lambda \cdot \varphi(x) .$$
We then define \( \tilde{M}_{x/s} := \ker(\tilde{O}_{X/S,x} \longrightarrow \kappa(x)) \). Remark that as in classical case, we have \( (\tilde{O}_{X/S,x})/(\tilde{M}_{x/s}) \cong \kappa(x) \), so that the \( \tilde{O}_{X/S,x} \)-module \( \tilde{M}_{x/s}/(\tilde{M}_{x/s})^2 \) is actually a \( \kappa(x) \)-vector space.

- If \( \ell_x : \mathcal{O}_{X,x} \longrightarrow \tilde{O}_{X/S,x} \) denotes the ring morphism that sends \( \varphi \) to \( 1 \otimes \varphi \), then \( \ell_x \) maps \( M_x \) into \( \tilde{M}_{x/s} \) and so, one can consider the following

\[
\vartheta_{x/s} : M_x \longrightarrow \tilde{M}_{x/s}/(\tilde{M}_{x/s})^2
\]

which is a morphism of \( \kappa(x) \)-vector spaces. It makes the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{M}_x & \xrightarrow{\ell_x} & \tilde{M}_{x/s} \\
\downarrow & & \downarrow \\
M_x/M_x^2 & \xrightarrow{\vartheta_{x/s}} & \tilde{M}_{x/s}/(\tilde{M}_{x/s})^2
\end{array}
\] (1)

- One has \( \tilde{D}(\ell_x(\varphi)) = D(\varphi) \) and, as it happens for \( D \), one can factor \( \tilde{D} \) through \( \tilde{M}_{x/s}/(\tilde{M}_{x/s})^2 \).

- We still denote by \([\varphi] \) the image of \( \varphi \in \tilde{M}_{x/s} \) in \( \tilde{M}_{x/s}/(\tilde{M}_{x/s})^2 \).

Now, let us state and prove the main lemma of this note.

**Lemma 4.1.** If \( \vartheta_{x/s} \) is an isomorphism, then \( \Phi_{X/S}^x \) is also one.

**Proof.** — Let us assume that \( \vartheta_{x/s} \) is an isomorphism. First, we construct a \( \kappa(x) \)-linear map \( \Upsilon_{X/S} : T^{(\text{Zar})}_{X/S}(x) \longrightarrow T^{(\text{Gro})}_{X/S}(x) \). Let \( \bar{v} \in T^{(\text{Zar})}_{X/S}(x) \). We associate to \( \bar{v} \) the following map

\[
D_{\bar{v}} : \mathcal{O}_{X,x} \longrightarrow \kappa(x)
\]

\[
\varphi \longmapsto \bar{v} \bullet (\vartheta_{x/s})^{-1}[1 \otimes \varphi - \varphi(x) \otimes 1].
\]

It is an \( \mathcal{O}_{S,s} \)-derivation. Indeed,

- First, if \( \varphi \in \mathcal{O}_{S,s} \) then \( 1 \otimes \varphi = \varphi(x) \otimes 1 \) and so

\[
D_{\bar{v}}(f^{\#}(\varphi)) = \bar{v} \bullet (\vartheta_{x/s})^{-1}[1 \otimes \varphi - \varphi(x) \otimes 1] = \bar{v} \bullet (\vartheta_{x/s})^{-1}[\varphi(x) \otimes 1 - \varphi(x) \otimes 1] = 0.
\]

- Second, let us verify the Leibniz rule. Let \( \varphi, \psi \in \mathcal{O}_{X,x} \). In \( \tilde{M}_{x/s}/(\tilde{M}_{x/s})^2 \), one has

\[
[1 \otimes \varphi - \varphi(x) \otimes 1] \cdot [1 \otimes \psi - \psi(x) \otimes 1] = 0.
\]
and so
\[ [\psi(x) \otimes \varphi] + [\varphi(x) \otimes \psi] = [1 \otimes \varphi\psi] + [\varphi(x)\psi(x) \otimes 1] \]  \hspace{1cm} (2)

Then,
\[
\psi(x) \cdot [1 \otimes \varphi - \varphi(x) \otimes 1] + \varphi(x) \cdot [1 \otimes \psi - \psi(x) \otimes 1] \\
= [\psi(x) \otimes \varphi] + [\varphi(x) \otimes \psi] - 2[\varphi(x)\psi(x) \otimes 1] \\
= [1 \otimes \varphi\psi] - [\varphi(x)\psi(x) \otimes 1] \hspace{1cm} \text{by (2)}.
\]

This implies that \( D_\varphi(\varphi\psi) = \varphi(x) \cdot D_\psi(\varphi) + \psi(x) \cdot D_\varphi(\varphi). \)

Now, let us prove that \( \Upsilon_{X/S} \circ \Phi_{X/S} = \Id. \) Let \( D : \mathcal{O}_{X,x} \longrightarrow \kappa(x) \) be an \( \mathcal{O}_{S,s} \)-derivation. Let \( \varphi \in \mathcal{O}_{X,x}. \) Let us compute \( \Upsilon_{X/S} \circ \Phi_{X/S}(D)(\varphi) = \Phi_{X/S}(D) \cdot (\partial_{x/s})^{-1}[1 \otimes \varphi - \varphi(x) \otimes 1]. \)

Let \( \psi \in \mathcal{O}_{X,x} \) such that \( [\psi] = (\partial_{x/s})^{-1}[1 \otimes \varphi - \varphi(x) \otimes 1]. \) \hspace{1cm} (3)

Then, by definition, one has \( \Phi_{X/S}(D) \cdot (\partial_{x/s})^{-1}[1 \otimes \varphi - \varphi(x) \otimes 1] = D(\psi). \) Applying \( \partial_{x/s} \) to (3), one gets, with (1), that
\[ [1 \otimes \varphi - \varphi(x) \otimes 1] = [1 \otimes \psi]. \]

Applying \( \tilde{D} \), and since \( \tilde{D}(\varphi(x) \otimes 1) = 0 \), one obtains that
\[ \tilde{D}(1 \otimes \varphi - \varphi(x) \otimes 1) = D(\varphi) = D(\psi). \]

So, we have got the required identity, \( \Upsilon_{X/S} \circ \Phi_{X/S}(D) = D. \)

Let us prove now that \( \Phi_{X/S} \circ \Upsilon_{X/S} = \Id. \) Let \( \tilde{v} : \mathfrak{M}_x/\mathfrak{M}_x^2 \longrightarrow \kappa(x) \) be a Zariski tangent vector and let \( \varphi \in \mathfrak{M}_x. \) Then,
\[
\Phi_{X/S} \circ \Upsilon_{X/S}(\tilde{v}) \cdot \varphi = \Upsilon_{X/S}(\tilde{v})(\varphi) \\
= \tilde{v} \cdot (\partial_{x/s})^{-1}[1 \otimes \varphi - \varphi(x) \otimes 1] \\
= \tilde{v} \cdot (\partial_{x/s})^{-1}[1 \otimes \varphi] = \tilde{v} \cdot ((\partial_{x/s})^{-1} \circ \partial_{x/s})(\varphi) \\
= \tilde{v} \cdot \varphi.
\]

\[ \square \]

5 The main theorem

Theorem 5.1. When the extension \( i_x : \kappa(s) \longrightarrow \kappa(x) \) is algebraic and separable
\[ \Phi_{X/S} : T_{X/S}^{\text{(Gro)}}(x) \longrightarrow T_{X/S}^{\text{(Zar)}}(x) \]
is an isomorphism of \( \kappa(x) \)-vector spaces.
Proof. — To begin with, remark that we can replace the relative situation $X \to S$ by the “absolute situation” $X_s \to \text{Spec } \kappa(s)$. Indeed, let us consider the following cartesian square

\[
\begin{array}{ccc}
X_s & \to & X \\
\downarrow & & \downarrow \\
\text{Spec } \kappa(s) & \to & S
\end{array}
\]

First, by Lemma 1.5, the relative Zariski tangent spaces are isomorphic one to each other in this case. For the Grothendieck tangent spaces, one can say the following:

— By (16.5.13.2) of [2], when $\Omega^1_{X/S}$ is an $\mathcal{O}_X$-module of finite type, the Grothendieck tangent space is invariant under base extension, and so $T^{(\text{Gro})}_{X/S}(x)$ and $T^{(\text{Gro})}_{X_s/\kappa(s)}(x)$ are isomorphic.

— But, actually, the latter is true without any condition of finiteness, as we prove it in Lemma 5.2.

So, in what follows, we will assume that $\mathcal{O}_{S,s} = \kappa(s)$. In particular, $\mathcal{O}_{X,x}$ is a $\kappa(s)$-algebra.

Now, let us apply the second fundamental exact sequence of Kähler differentials (Theorem 58 of [4]), respectively with

1) first, $k = \kappa(s)$, $A = \mathcal{O}_{X,x}$ and $\mathcal{M} = \mathcal{M}_x$

2) second, $k = \kappa(x)$, $A = \tilde{\mathcal{O}}_{X/S,x}$ and $\mathcal{M} = \tilde{\mathcal{M}}_{x/s}$

to get the following two exact sequences:

\[
\frac{\mathcal{M}_x}{\mathcal{M}_x^2} \to \Omega_{\mathcal{O}_{X,x}/\kappa(s)} \otimes_{\mathcal{O}_{X,x}} \kappa(x) \to \Omega_{\kappa(x)/\kappa(s)} \to 0 \quad (4)
\]

and

\[
\frac{\tilde{\mathcal{M}}_{x/s}/(\tilde{\mathcal{M}}_{x/s})^2}{\tilde{\mathcal{M}}_{x/s} \otimes_{\mathcal{O}_{X/S,x}} \tilde{\mathcal{O}}_{X/S,x}} \to \Omega_{\tilde{\mathcal{O}}_{X/S,x}/\kappa(x)} \otimes_{\tilde{\mathcal{O}}_{X/S,x}} \kappa(x) \to \Omega_{\kappa(x)/\kappa(x)} = 0. \quad (5)
\]

In the second one, the left-hand morphism is injective. Indeed, $\tilde{\mathcal{O}}_{X/S,x} \to \kappa(x)$ has a section and so, the criterion given by Proposition 16.12 of [4] applies. So, (5) can be written

\[
0 \to \tilde{\mathcal{M}}_{x/s}/(\tilde{\mathcal{M}}_{x/s})^2 \to \Omega_{\tilde{\mathcal{O}}_{X/S,x}/\kappa(x)} \otimes_{\tilde{\mathcal{O}}_{X/S,x}} \kappa(x) \to 0
\]

and hence gives an isomorphism. But we also know that

\[
\Omega_{\tilde{\mathcal{O}}_{X/S,x}/\kappa(x)} \simeq \Omega_{\mathcal{O}_{X,x}/\kappa(s)} \otimes_{\mathcal{O}_{X,x}} \tilde{\mathcal{O}}_{X/S,x}
\]

so that, we have

\[
\Omega_{\tilde{\mathcal{O}}_{X/S,x}/\kappa(x)} \otimes_{\tilde{\mathcal{O}}_{X/S,x}} \kappa(x) \simeq \Omega_{\mathcal{O}_{X,x}/\kappa(s)} \otimes_{\mathcal{O}_{X,x}} \kappa(x).
\]
Hence, in the exact sequence (4), we can replace the second vector space by $\widehat{M}_{x/s}/(\widehat{M}_{x/s})^2$. We get

$$M_{x}/M_{x}^2 \longrightarrow \widehat{M}_{x/s}/(\widehat{M}_{x/s})^2 \longrightarrow \Omega_{\kappa(x)/\kappa(s)} \longrightarrow 0$$

and one can check that the first arrow in this sequence is $\vartheta_{x/s}$.

By Corollary 16.13 of [1], a sufficient condition for $\vartheta_{x/s}$ to be injective is that the extension $\kappa(x)/\kappa(s)$ is separable. A sufficient condition for $\vartheta_{x/s}$ to be surjective is that $\Omega_{\kappa(x)/\kappa(s)} = 0$. Hence, if $\kappa(x)/\kappa(s)$ is separable and algebraic, by Lemma 16.15 of [1], $\vartheta_{x/s}$ is an isomorphism and so is $\Phi_{X/S}$.

Lemma 5.2. For any schemes $X/S$ and any $x \in X$ above $s \in S$,

$$T^{(\text{Gro})}_{X/S}(x) \simeq T^{(\text{Gro})}_{X_s/\text{Spec}\kappa(s)}(x).$$

Proof. — We use the same description of the local ring of $X_s$ at $x$ as in Lemma 1.5. So, we want to prove that

$$\text{Der}_{\mathcal{O}_S,x}(\mathcal{O}_{X,x},\kappa(x)) \quad \text{and} \quad \text{Der}_{\kappa(s)}(\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x},\kappa(x))$$

are isomorphic. The two inverse map are the most natural ones to describe, and the check that it works is left to the reader as an exercise.

References

[1] Eisenbud, D. *Commutative algebra*, vol. 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[2] Grothendieck, A. *Éléments de géométrie algébrique*. IV. étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, 32 (1967), 361.

[3] Kunz, E. On the tangent bundle of a scheme. *Univ. Iagel. Acta Math.*, 37 (1999), 9–24. Effective methods in algebraic and analytic geometry (Bielsko-Biała, 1997).

[4] Matsumura, H. *Commutative algebra*. W. A. Benjamin, Inc., New York, 1970.