STABILITY INEQUALITY FOR A SEMILINEAR ELLIPTIC INVERSE PROBLEM

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ABSTRACT. We establish a logarithmic stability inequality for the inverse problem of determining the non linear term, appearing in a semilinear BVP, from the corresponding Dirichlet-to-Neumann map (abbreviated to DtN map in the rest of this text). Our result can be seen as a stability inequality for an earlier uniqueness result by Isakov and Sylvester [Commun. Pure Appl. Math. 47 (1994), 1403-1410].

1. Introduction

Let \( \Omega \) be a \( C^{1,1} \) bounded domain of \( \mathbb{R}^n \) (\( n \geq 2 \)) with boundary \( \Gamma \). Fix \( c_0 > 0 \), \( c_1 > 0 \) and \( 0 \leq c < \lambda_1(\Omega) \), where \( \lambda_1(\Omega) \) denotes the first eigenvalue of the Laplace operator on \( \Omega \) with Dirichlet boundary condition. We denote by \( \mathcal{A}(\alpha) \), \( \alpha \geq 0 \), the set of continuously differentiable functions \( a : \mathbb{R} \to \mathbb{R} \) satisfying the following two assumptions

\[
|a(u)| \leq c_0 + c_1 |u|^\alpha, \quad u \in \mathbb{R},
\]

and

\[
a'(u) \geq -c, \quad u \in \mathbb{R}.
\]

Consider the non homogenous BVP

\[
\begin{cases}
-\Delta u + a(u(\cdot)) = 0 & \text{in } \Omega, \\
u = f & \text{on } \Gamma.
\end{cases}
\]

For this BVP we have the following existence and uniqueness theorem.

**Theorem 1.1.** Assume that \( \alpha \) is arbitrary if \( n = 2 \) and \( \alpha \leq n/(n-2) \) if \( n \geq 3 \). Let \( a \in \mathcal{A}(\alpha) \) and \( f \in H^{3/2}(\Gamma) \) so that \( ||f||_{H^{3/2}(\Gamma)} \leq M \), for some \( M > 0 \). Then the BVP (1.3) has a unique solution \( u_a(f) \in H^2(\Omega) \) and

\[
\|u_a(f)\|_{H^2(\Omega)} \leq C,
\]

where the constant \( C \) only depends on \( \Omega \), \( M \), \( \alpha \), \( c_0 \), \( c_1 \) and \( c \).

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An example of the function $a$ fulfilling the assumptions in the above theorem is the linear case $a(u) = -ku$ with $k < c$, which models the time-harmonic acoustic wave propagation at the wavenumber $k > 0$. The semilinear equation also covers the Schrödinger equation. We prove this theorem in Section 2 using the variational argument and Leray-Schauder fixed point theorem.

The DtN map associated to $a$ is well defined according to the following theorem that we prove also in the next section. Hereafter, the derivative in the direction of the unit exterior normal vector field $\nu$ on $\Gamma$ of a function $u$ is denoted by $\partial_{\nu}u$.

**Theorem 1.2.** (i) Assume that $\alpha$ is arbitrary if $n = 2$ and $\alpha \leq 3$ if $n = 3$. If $a \in \mathcal{A}(\alpha)$ then the DtN map

$$\Lambda_a : f \in H^{3/2}(\Gamma) \to \partial_{\nu}u_a(f) \in H^{1/2}(\Gamma)$$

is well defined and, under the assumption $\|f\|_{H^{3/2}(\Gamma)} \leq M$, for some $M > 0$, we have

$$\|\Lambda_a(f)\|_{H^{1/2}(\Gamma)} \leq C,$$

where the constant $C$ only depends on $\Omega$, $M$, $\alpha$, $c_0$, $c_1$ and $c$. (ii) Assume that $n > 4$. Let $n/2 < p < n$ and $\alpha \leq q/p$ with $q = 2n/(n-4)$. If $a \in \mathcal{A}(\alpha)$ then

$$\Lambda_a : f \in W^{2-1/p,p}(\Gamma) \to \partial_{\nu}u_a(f) \in W^{1-1/p,p}(\Gamma)$$

is well defined and, under the assumption $\|f\|_{W^{2-1/p,p}(\Gamma)} \leq M$, for some $M > 0$, we have

$$\|\Lambda_a(f)\|_{W^{1-1/p,p}(\Gamma)} \leq C.$$  

Here the constant $C$ only depends on $\Omega$, $M$, $p$, $\alpha$, $c_0$, $c_1$ and $c$. (iii) Assume that $n = 4$. Let $2 < p < 4$, $1 \leq r < 2$, $q = 2r/(2-r)$ and $\alpha \leq q/p$. If $a \in \mathcal{A}(\alpha)$ then

$$\Lambda_a : f \in W^{2-1/p,p}(\Gamma) \to \partial_{\nu}u_a(f) \in W^{1-1/p,p}(\Gamma)$$

is well defined and, under the assumption $\|f\|_{W^{2-1/p,p}(\Gamma)} \leq M$, for some $M > 0$, we have

$$\|\Lambda_a(f)\|_{W^{1-1/p,p}(\Gamma)} \leq C,$$

where the constant $C$ only depends on $\Omega$, $M$, $p$, $r$, $\alpha$, $c_0$, $c_1$ and $c$.

We are concerned with the inverse problem of determining the nonlinear term $a$ from the corresponding DtN map $\Lambda_a$. The main purpose is the stability issue.

We need to restrict the class of the unknown function $a$. To this end we define $\mathcal{A}(\alpha)$ as the set of functions $a \in \mathcal{A}(\alpha)$ satisfying the additional condition: for any $R > 0$, there exists a constant $C_R$ so that

$$|a'(u) - a'(v)| \leq C_R|u - v|, \quad |u|, \ |v| \leq R.$$

Note that the condition (1.8) means that the first derivative of $a$ is Lipschitz continuous on bounded sets of $\mathbb{R}$.

Within this class, we can linearize the inverse problem under consideration. Precisely, we have the following proposition in which, for $j = 0$ or $j = 1$,

$$\mathcal{X}_j = H^{3/2-j}(\Gamma) \text{ if } n = 2, 3 \quad \text{and} \quad \mathcal{X}_j = W^{2-j-1/p,p}(\Gamma) \text{ if } n \geq 4.$$

and the space

$$\mathcal{U} = \mathcal{B}(\mathcal{X}_0, \mathcal{X}_1)$$
denotes the set of bounded linear operators mapping $\mathcal{X}_0$ into $\mathcal{X}_1$.

The Proposition below states that the linearization of the DtN map $\Lambda_a$ is the DtN map of the linearized problem.

**Proposition 1.1.** Under the assumptions and the notations of Theorem 1.2 and the additional condition that $a \in \mathcal{A}(\alpha)$, $\Lambda_a$ is Fréchet differentiable at any $f \in \mathcal{X}_0$ with $\Lambda_a'(f)(h) = \partial_h v_{a,f}(h)$, where $h \in \mathcal{X}_0$ and $v_{a,f}(h)$ is the unique solution of the BVP

$$\begin{aligned}
-\Delta v + a'(u_a(f)(\cdot))v &= 0 \quad \text{in } \Omega, \\
v &= h \quad \text{on } \Gamma.
\end{aligned}$$

Moreover, we have

$$\|\Lambda_a'(f)\|_{\mathcal{Y}} \leq C.$$ 

Here, the constant $C$ only depends on $\Omega$, $M$, $\alpha$, $c_0$, $c_1$ and $c$ if $n = 2$ or $n = 3$; it only depends on $\Omega$, $M$, $p$, $r$, $\alpha$, $c_0$, $c_1$ and $c$ if $n = 4$, and only depends on $\Omega$, $M$, $p$, $\alpha$, $c_0$, $c_1$ and $c$ if $n > 4$.

We are now in position to state the main result of this paper.

**Theorem 1.3.** Assume that $n \geq 3$ and the assumptions of Theorem 1.2 hold for $a, \tilde{a} \in \mathcal{A}(\alpha)$ and let $\beta = 1/2$ if $n = 3$ and $\beta = 2 - n/p$ if $n \geq 4$. If $0 < s < \min(1/2, \beta)$ is fixed, then

$$\max_{|\lambda| \leq M} |a'(\lambda) - \tilde{a}'(\lambda)| \leq C_M \Psi(\mathcal{N}_M),$$

with the constant $C_M = C$ given as in Theorem 1.2. Here

$$\mathcal{N}_M = \sup_{\|f\|_{\mathcal{X}_0} \leq \sqrt{|\Gamma|} M} \|\Lambda_a'(f) - \Lambda_{\tilde{a}}'(f)\|_{\mathcal{Y}}$$

and

$$\Psi(\rho) = |\ln \rho|^{-4s\beta/[(n+2s)(n+2\beta)]} + \rho, \quad \rho > 0,$$

that we extend by continuity at $\rho = 0$ by posing $\Psi(0) = 0$.

It is worth mentioning that the proof of Theorem 1.3 can be adapted to a partial DtN map. A double logarithmic stability inequality for the linearized problem, with a partial DtN map, was recently established by Caro, Dos Santos Ferreira and Ruiz [1]. The result in [1] can serve to obtain a version of Theorem 1.3 in the case of a partial DtN map. We refer to [7] for the first uniqueness result in determining semilinear terms by partial Cauchy data on arbitrary subboundary.

The uniqueness results for recovering semilinear terms from full Cauchy data were obtained by Isakov and Sylvester [6] in three dimensions and by Isakov and Nachman [5] in two dimensions. These results apply to nonlinearities of the form $a = a(x, u)$. For sake of simplicity we only consider here the case $a = a(u)$. However, we believe that Theorem 1.3 can be extended to cover completely the uniqueness result in [6], possibly under some additional conditions.

We point out that the uniqueness results for smooth semilinear terms using partial data in $\mathbb{R}^n$ ($n \geq 2$) were contained in the recent papers by Krupchyk and Uhlmann [10], and Lassas, Liimatainen, Lin and Salo [11]. These two references make use of higher order linearization procedure and contain a detailed overview of semilinear elliptic inverse problems together with a rich list of references.

A similar problem for a semilinear IBVP for a parabolic equation was studied by the first author and Kian [3]. A stability inequality of the determination of a
nonlinear term in a parabolic IBVP from a single measurement was proved by the
first and third authors and Ouhabaz in [4].

We close this introduction by two immediate consequences of Theorem 1.3.

**Corollary 1.1.** Under the assumptions and the notations of Theorem 1.3, if \( a, \hat{a} \in \mathcal{A}(\alpha) \) satisfy \( a(0) = \hat{a}(0) \) then

\[
\max_{|\lambda| \leq M} |a(\lambda) - \hat{a}(\lambda)| \leq C_M \Psi(M),
\]

where the constant \( C_M = C \) is as Theorem 1.2, and \( \Psi \) is as in Theorem 1.3.

**Corollary 1.2.** If \( a, \hat{a} \in \mathcal{A}(\alpha) \) satisfy \( a(0) = \hat{a}(0) \) and \( \Lambda_a = \Lambda_{\hat{a}} \) then \( a = \hat{a} \).

## 2. The Analysis of the Semilinear BVP

Prior to introducing the definition of variational solution of the BVP (1.3), we prove the following lemma.

**Lemma 2.1.** Assume that \( \alpha \) is arbitrary if \( n = 2 \) and \( \alpha \leq (n+2)/(n-2) \) if \( n \geq 3 \).
Let \( a \in \mathcal{A}(\alpha) \) and fix \( \varphi \in H^1(\Omega) \) satisfying \( \|\varphi\|_{H^1(\Omega)} \leq M \) for some \( M > 0 \). Then

\[
\ell(\phi) = \int_{\Omega} a(\varphi(x))\phi(x)dx, \quad \phi \in H^1_0(\Omega),
\]

is bounded on \( H^1_0(\Omega) \) with

\[
\|\ell\|_{H^{-1}(\Omega)} \leq C,
\]

where the constant \( C \) only depends on \( c_0, c_1, \alpha \) and \( M \).

**Proof.** Consider first the case \( n \geq 3 \). In that case \( H^1_0(\Omega) \) is continuously embedded in \( L^q(\Omega) \) with \( q = 2n/(n-2) \). If \( q' = 2n/(n+2) \) is the conjugate component of \( q \), Hölder’s inequality then yields

\[
\left| \int_{\Omega} a(\varphi(x))\phi(x)dx \right| \leq c_0\|\varphi\|_{L^1(\Omega)} + c_1\|\varphi\|_{L^q(\Omega)}^2 \|\phi\|_{L^{q'}(\Omega)},
\]

where we take into account that \( \alpha q' \leq q \).

To complete the proof we see that (2.2) gives in a straightforward manner (2.1).

The case \( n = 2 \) can be carried out similarly by using that \( H^1_0(\Omega) \) is continuously embedded in \( L^q(\Omega) \) for any \( q \geq 1 \). \( \square \)

Let \( f \in H^{1/2}(\Gamma) \). We say that \( u \in H^1(\Omega) \) is a variational solution of the BVP (1.3) if \( u = f \) on \( \Gamma \) in the trace sense and

\[
\int_{\Omega} \nabla u(x) \cdot \nabla \phi(x)dx + \int_{\Omega} a(u(x))\phi(x)dx = 0, \quad \phi \in H^1_0(\Omega).
\]

For \( f \in H^{1/2}(\Omega) \), let \( \mathcal{E} f \in H^1(\Omega) \) denote its harmonic extension. That is \( v = \mathcal{E} f \) is the unique solution of the BVP

\[
\begin{cases}
-\Delta v = 0 & \text{in } \Omega, \\
v = f & \text{on } \Gamma.
\end{cases}
\]

If we can find \( w \in H^1_0(\Omega) \) satisfying

\[
\int_{\Omega} \nabla w(x) \cdot \nabla \phi(x) = -\int_{\Omega} a(w(x) + v(x))\phi(x)dx, \quad \phi \in H^1_0(\Omega),
\]

then \( \phi \) is a variational solution of the BVP (1.3).
then clearly \( u = w + v \) is a variational solution of \((1.3)\).

**Theorem 2.1.** Assume that \( \alpha \) is arbitrary if \( n = 2 \) and \( \alpha \leq (n + 2)/(n - 2) \) if \( n \geq 3 \). Let \( a \in C^2(\Omega) \) and \( f \in H^{1/2}(\Gamma) \) so that \( \|f\|_{H^{1/2}(\Gamma)} \leq M \), for some \( M > 0 \). Then the BVP \((1.3)\) has a unique variational solution \( u_\alpha(f) \in H^1(\Omega) \) satisfying

\[
(2.4) \quad \|u_\alpha(f)\|_{H^1(\Omega)} \leq C.
\]

Here the constant \( C \) only depends on \( \Omega \), \( \alpha \), \( M \), \( c_0 \), \( c_1 \) and \( c \).

**Proof.** In light of the previous discussion, it is enough to prove that \((2.3)\) has a solution \( w \in H_0^1(\Omega) \) and \((2.4)\) holds with \( u_\alpha(f) \) substituted by \( w \).

Let \( T : H_0^1(\Omega) \to H_0^1(\Omega) \) given by: \( \psi = Tw \), \( w \in H_0^1(\Omega) \) is the unique solution of the variational problem

\[
\int_\Omega \nabla \psi(x) \cdot \nabla \phi(x) = - \int_\Omega a(w(x) + v(x))\phi(x)dx, \quad \phi \in H_0^1(\Omega).
\]

It follows readily from Lax-Milgram lemma and Lemma 2.1 that the mapping \( T \) is well defined.

Pick \( w \in H_0^1(\Omega) \) satisfying \( w = \mu Tw \), for some \( \mu \in [0, 1] \). According to the definition of \( T \), \( w \) satisfies

\[
(2.5) \quad \int_\Omega |\nabla w(x)|^2dx = -\mu \int_\Omega a(w(x) + v(x))w(x)dx.
\]

On the other hand, we have

\[
a(w(x) + v(x)) = a(v(x)) + \int_0^1 a'(sw(x) + v(x))w(x)ds, \quad \text{a.e. } x \in \Omega.
\]

This in \((2.5)\) yields

\[
\int_\Omega |\nabla w(x)|^2dx = -\mu \int_\Omega a(v(x))w(x)dx - \mu \int_\Omega \left( \int_0^1 a'(sw(x) + v(x))ds \right) w(x)^2dx.
\]

In light of assumption \((1.2)\) we get

\[
\int_\Omega |\nabla w(x)|^2dx \leq -\mu \int_\Omega a(v(x))w(x)dx + c \int_\Omega w(x)^2dx
\]

which combined with Poincaré’s inequality gives

\[
\int_\Omega |\nabla w(x)|^2dx \leq -\mu \int_\Omega a(v(x))w(x)dx + c\lambda_1(\Omega)^{-1} \int_\Omega |\nabla w(x)|^2dx.
\]

Or equivalently

\[
(1 - c\lambda_1(\Omega)^{-1}) \int_\Omega |\nabla w(x)|^2dx \leq -\mu \int_\Omega a(v(x))w(x)dx.
\]

We can then apply Lemma 2.1 to obtain

\[
\|w\|_{H_0^1(\Omega)} \leq C,
\]

the constant \( C \) only depends on \( \Omega \), \( \alpha \), \( f \), \( c_0 \), \( c_1 \) and \( c \).

In light of this last inequality we can apply Leray-Schauder fixed point theorem (see for instance [8, Theorem 11.3, page 280]) to deduce that there exists \( w \in H_0^1(\Omega) \) so that \( w = Tw \). That is \( w \) is the solution of the variational problem \((1.3)\).
We complete the proof by showing that \((1.3)\) has at most one solution. To this end, let \(u, \tilde{u} \in H^{1}_0(\Omega)\) be two solutions of \((1.3)\) and set \(v = u - \tilde{u}\). Taking into account that
\[
a(u(x)) - a(\tilde{u}(x)) = b(x)v(x), \quad \text{a.e. } x \in \Omega,
\]
with
\[
b(x) = \int_0^1 a'(x, \tilde{u} + s(u(x) - \tilde{u}(x)))ds,
\]
we find that \(v\) is the solution of the BVP
\[
\begin{aligned}
-\Delta v + bv &= 0 \quad \text{in } \Omega, \\
\quad v &= 0 \quad \text{on } \Gamma.
\end{aligned}
\]
Green’s formula then yields
\[
\int_{\Omega} |\nabla v(x)|^2dx + \int_{\Omega} b(x)v(x)^2dx = 0.
\]
Whence
\[
\int_{\Omega} |\nabla v(x)|^2dx = - \int_{\Omega} b(x)v(x)^2dx \leq c \int_{\Omega} v(x)^2dx \leq c\lambda_1(\Omega)^{-1} \int_{\Omega} |\nabla v(x)|^2dx.
\]
As \(c\lambda_1(\Omega)^{-1} < 1\), we deduce that \(v = 0\).

Theorem 1.1 will then follow from the following lemma.

**Lemma 2.2.** Assume that \(\alpha\) is arbitrary if \(n = 2\) and \(\alpha \leq n/(n-2)\) if \(n \geq 3\). Let \(a \in A(\alpha)\) and \(f \in H^{3/2}(\Gamma)\) so that \(\|f\|_{H^{3/2}(\Gamma)} \leq M\), for some \(M > 0\). Then \(u_a(f) \in H^2(\Omega)\) and
\[
\|u_a(f)\|_{H^2(\Omega)} \leq C,
\]
where the constant \(C\) only depends on \(\Omega, M, \alpha, c_0, c_1\) and \(c\).

**Proof.** In this proof \(C\) is a generic constant that can only depend on \(\Omega, \alpha, M, c_0, c_1\) and \(c\).

Consider the case \(n \geq 3\). By \((1.1)\) we have
\[
[a(u_a(f)(x))]^2 \leq 2c_0^2 + 2c_1^2|u_a(f)(x)|^{2\alpha} \quad \text{a.e. } x \in \Omega.
\]
As \(2\alpha \leq 2n/(n-2)\) and \(H^1(\Omega)\) is continuously embedded in \(L^{2\alpha}(\Omega)\), we deduce that \(a(\cdot, u_a(f)(\cdot)) \in L^{2\alpha}(\Omega)\) and using \((2.4)\) we get
\[
\|a(u_a(f)(\cdot))\|_{L^{2\alpha}(\Omega)} \leq C.
\]
By the classical \(H^2\) regularity results, we deduce that \(u_a(f) \in H^2(\Omega)\) and
\[
\|u_a(f)\|_{H^2(\Omega)} \leq \kappa \left( \|f\|_{H^{3/2}(\Gamma)} + \|a(u_a(f)(\cdot))\|_{L^{2\alpha}(\Omega)} \right),
\]
where the constant \(\kappa\) only depends on \(n\) and \(\Omega\).

Finally a combination of \((2.7)\) and \((2.8)\) yields \((2.6)\).

The case \(n = 2\) can be treated similarly using that \(H^1(\Omega)\) is continuously embedded in \(L^r(\Omega)\) for any \(r \geq 1\).
3. The Dirichlet-to-Neumann map

We first observe that as a consequence of Theorem 2.1 and Lemma 2.2 we have the following corollary.

**Corollary 3.1.** Assume that \( \alpha \) is arbitrary if \( n = 2 \) and \( \alpha \leq n/(n-2) \) if \( n \geq 3 \). Let \( a \in \mathcal{A}(\alpha) \) and \( f \in H^{j+1/2}(\Gamma) \) so that \( \|f\|_{H^{j+1/2}(\Gamma)} \leq M \), \( j = 0 \) or \( j = 1 \), for some \( M > 0 \). Then the DN map

\[
\Lambda_a : f \in H^{j+1/2}(\Gamma) \rightarrow \partial_n u_a(f) \in H^{j-1/2}(\Gamma)
\]

is well defined for \( j = 0 \) or \( j = 1 \). Moreover

\[
\|\Lambda_a(f)\|_{H^{j-1/2}(\Gamma)} \leq C,
\]

where the constant \( C \) only depends on \( \Omega \), \( M \), \( \alpha \), \( c_0 \), \( c_1 \) and \( c \).

We recall that \( C^{0,\gamma}(\overline{\Omega}), 0 < \gamma \leq 1 \), is the usual vector space of functions that are Hölder continuous on \( \Omega \) with exponent \( \gamma \). This space is usually endowed with its natural norm

\[
\|w\|_{C^{0,\gamma}(\overline{\Omega})} = \|w\|_{C(\overline{\Omega})} + \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|w(x) - w(y)|}{|x - y|}.
\]

Taking into account that \( H^2(\Omega) \) is continuously embedded in \( C^{0,1/2}(\overline{\Omega}) \), for \( n = 2 \) or \( n = 3 \), we derive as a consequence of the Lemma 2.2 the following corollary.

**Corollary 3.2.** Assume that \( \alpha \) is arbitrary if \( n = 2 \) and \( \alpha \leq 3 \) if \( n = 3 \). Let \( a \in \mathcal{A}(\alpha) \) and \( f \in H^{3/2}(\Gamma) \) so that \( \|f\|_{H^{3/2}(\Gamma)} \leq M \), for some \( M > 0 \). Then \( u_a(f) \in C^{0,1/2}(\overline{\Omega}) \) and

\[
\|u_a(f)\|_{C^{0,1/2}(\overline{\Omega})} \leq C,
\]

where the constant \( C \) only depends on \( \Omega \), \( M \), \( \alpha \), \( c_0 \), \( c_1 \) and \( c \).

**Lemma 3.1.** (i) Assume that \( n > 4 \), let \( n/2 < p < n \) and \( \alpha \leq q/p \) with \( q = 2n/(n-4) \). Let \( a \in \mathcal{A}(\alpha) \) and \( f \in W^{2-1/p,p}(\Gamma) \) so that \( \|f\|_{W^{2-1/p,p}(\Gamma)} \leq M \), for some \( M > 0 \). Then \( u_a(f) \in W^{2,p}(\Omega) \cap C^{0,\beta}(\overline{\Omega}), \beta = 2 - n/p \), and

\[
\|u_a(f)\|_{W^{2,p}(\Omega)} + \|u_a(f)\|_{C^{0,\beta}(\overline{\Omega})} \leq C,
\]

where the constant \( C \) only depends on \( \Omega \), \( M \), \( p \), \( \alpha \), \( c_0 \), \( c_1 \) and \( c \).

(ii) Assume that \( n = 4 \), let \( 2 < p < 4 \), \( 1 \leq r < 2 \), \( q = 2r/(2-r) \) and \( \alpha \leq q/p \). Let \( a \in \mathcal{A}(\alpha) \) and \( f \in W^{2-1/p,p}(\Gamma) \) so that \( \|f\|_{W^{2-1/p,p}(\Gamma)} \leq M \), for some \( M > 0 \). Then \( u_a(f) \in W^{2,p}(\Omega) \cap C^{0,\beta}(\overline{\Omega}), \beta = 2 - 4/p \), and

\[
\|u_a(f)\|_{W^{2,p}(\Omega)} + \|u_a(f)\|_{C^{0,\beta}(\overline{\Omega})} \leq C,
\]

where the constant \( C \) only depends on \( \Omega \), \( M \), \( p \), \( r \), \( \alpha \), \( c_0 \), \( c_1 \) and \( c \).

**Proof.** (i) In this part \( C \) is a generic constant that can only depend on \( \Omega \), \( M \), \( p \), \( \alpha \), \( c_0 \), \( c_1 \) and \( c \).

Noting that \( q/p < n/(n-2) \), we get from Lemma 2.2 that \( u_a(f) \in H^2(\Omega) \) and, since \( H^2(\Omega) \) is continuously embedded in \( L^q(\Omega) \) with \( q = 2n/(n-4) \), \( u_a(f) \in L^q(\Omega) \). Consequently, using (1.1), (2.6) and the assumption on \( \alpha \), we obtain \( a(u_a(f)(\cdot)) \in L^p(\Omega) \) and

\[
\|a(u_a(f)(\cdot))\|_{L^p(\Omega)} \leq C.
\]
By the usual $W^{2,p}$ regularity results, we then have $u_a(f) \in W^{2,p}(\Omega)$ and, since $W^{2,p}(\Omega)$ is continuously embedded in $C^{0,\beta}(\Omega)$, we conclude that $u_a(f) \in C^{0,\beta}(\Omega)$.

Combined with estimate [8, (2.46), page 242], (3.5) yields in straightforward manner

$$
\|u_a(f)\|_{W^{2,p}(\Omega)} \leq C.
$$

Whence (3.3) follows.

(ii) Let $n = 4$ and $1 \leq r < 2$. As $q/p < 2$, we get from Lemma 2.2 that $u_a(f) \in H^2(\Omega)$. Since $H^2(\Omega)$ is continuously embedded in $W^{2,r}(\Omega)$ and $W^{2,r}(\Omega)$ is continuously embedded in $L^q(\Omega)$ with $q = 2r/(2 - r)$, we conclude that $H^2(\Omega)$ is continuously embedded in $L^q(\Omega)$. Hence, if $\alpha p \leq q$, for some $2 < p < 4$, then $u(\cdot, u_a(f)(\cdot)) \in L^p(\Omega)$. The rest of the proof is quite similar to that of (i). □

We end this section by noting that Theorem 1.2 follows readily from Corollary 3.2 and Lemma 3.1.

4. Proof of the stability inequality

We start with the proof of Proposition 1.1.

Proof of Proposition 1.1. We give the proof in case (i). Cases (ii) and (iii) can be proved similarly. On the other hand, since the trace operator $\ell \in H^2(\Omega) \rightarrow (\ell, \partial_\nu \ell)|_{\Gamma} \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is linear and bounded, it is sufficient to prove that $f \in H^{3/2}(\Gamma) \rightarrow u_a(f) \in H^2(\Omega)$ is Fréchet differentiable.

Let $f \in H^{3/2}(\Gamma)$ satisfying $\|f\|_{H^{3/2}(\Gamma)} \leq N$, for some $N > 0$. For $h \in H^{3/2}(\Gamma)$ satisfying $\|h\|_{H^{3/2}(\Gamma)} \leq 1$, we have $\|f + h\|_{H^{3/2}(\Gamma)} \leq M := N + 1$.

Set $w = u_a(f + h) - u_a(f) - v_{a.f}(h)$. Then it is straightforward to check that $w$ is the solution of the BVP

$$
\begin{cases}
-\Delta w = F & \text{in } \Omega, \\
w = 0 & \text{on } \Gamma,
\end{cases}
$$

with

$$
F(x) = -a(u_a(f + h))(x) + a(u_a(f))(x) + a'(u_a(f)(x))v(x)
= -\int_0^1 \{a'(u_a(f))(x) + s[u_a(f + h)(x) - u_a(f)(x)]
\times (u_a(f + h)(x) - u_a(f)(x)) - a'(u_a(f)(x))v(x)\}ds
= -\int_0^1 a'(u_a(f)(x)) + s[u_a(f + h)(x) - u_a(f)(x)]w(x)ds
+ \int_0^1 \{a'(u_a(f)(x)) + s[u_a(f + h)(x) - u_a(f)(x)]
- a'(u_a(f)(x))\}v(x)ds.
$$

We split $F$ into two terms $F = -qw + G$, with

$$
q(x) = \int_0^1 a'(u_a(f)(x)) + s[u_a(f + h)(x) - u_a(f)(x)]ds,
$$

$$
G(x) = \int_0^1 \{a'(u_a(f)(x)) + s[u_a(f + h)(x) - u_a(f)(x)]
- a'(u_a(f)(x))\}v(x)ds.
$$
In other words $w$ is the solution of the BVP
\[
\begin{aligned}
-\Delta w + qw &= G & \quad \text{in } \Omega, \\
w &= 0 & \quad \text{on } \Gamma.
\end{aligned}
\]
According to Corollary 3.2, we have
\[
\|u_a(f + g)\|_{L^\infty(\Omega)} \leq C,
\]
where the constant $C$ only depends on $\Omega, M, \alpha, c_0, c_1$ and $c$. Therefore, by triangle inequality and using (1.8),
\[
\|q\|_{L^\infty(\Omega)} \leq \|a'(0)\|_{L^\infty(\Omega)} + \kappa_C C := \hat{C}.
\]
We obtain from the usual $H^2$ a priori estimates that
\[
\|w\|_{H^2(\Omega)} \leq \hat{C}\|G\|_{L^2(\Omega)},
\]
where the constant $\hat{C}$ only depends on $\Omega$ and $\hat{C}$. But
\[
\|G\|_{L^2(\Omega)} \leq \kappa_C \|u_a(f + h) - u_a(f)\|_{L^2(\Omega)} \|v\|_{L^\infty(\Omega)} \leq \kappa_C C a \|u_a(f + h) - u_a(f)\|_{L^2(\Omega)} \|v\|_{H^2(\Omega)}.
\]
Here the constant $c_2$, only depends on $\Omega$.

Therefore, again from $H^2$ a priori estimates, we have
\[
\|w\|_{H^2(\Omega)} \leq \hat{C}\kappa_C c_2 \|u_a(f + h) - u_a(f)\|_{L^2(\Omega)} \|h\|_{H^{3/2}(\Gamma)}.
\]
We complete the proof of the differentiability of $u_a$ by showing that $f \in H^{3/2}(\Gamma) \to u_a(f) \in H^2(\Omega)$ is continuous. Let then $d = u_a(f + h) - u_a(f)$. Simple computations give that $d$ is the solution of the BVP
\[
\begin{aligned}
-\Delta d + rd &= 0 & \quad \text{in } \Omega, \\
d &= h & \quad \text{on } \Gamma,
\end{aligned}
\]
with
\[
r(x) = \int_0^1 a'(u_a(f))(x) + s[u_a(f + h) - u_a(f)](x) ds.
\]
Once again the $H^2$ a priori estimate yields
\[
\|d\|_{H^2(\Omega)} \leq \hat{C}\|h\|_{H^{3/2}(\Gamma)}.
\]
The continuity of $u_a$ then follows. \hfill \Box

Define
\[
q_{a, f}(x) := a'(u_a(f))(x), \quad x \in \Omega.
\]

Proceeding as in the proof of Proposition 1.1, we prove the following result.

**Lemma 4.1.** Let $\beta$ be as in Theorem 1.3. Under the assumptions and the notations of Proposition 1.1, we have $q_{a, f} \in C^{0, \beta}(\overline{\Omega})$ and
\[
\|q_{a, f}\|_{C^{0, \beta}(\overline{\Omega})} \leq C.
\]
Here the constant $C$ only depends on $\Omega, M, \alpha, c_0, c_1$ and $c$ if $n = 2$ or $n = 3$; the constant $C$ only depends on $\Omega, M, p, r, \alpha, c_0, c_1$ and $c$ if $n = 4$; the constant $C$ only depends on $\Omega, M, p, \alpha, c_0, c_1$ and $c$ if $n > 4$.

According to the definition of $W^{s,p}$ spaces in [9], we can easily check that $C^{0, \gamma}(\overline{\Omega})$ is continuously embedded in $H^s(\Omega)$ for any $0 < s < \gamma$. Whence an immediate consequence of the previous lemma is the following corollary.
Corollary 4.1. Let $\beta$ be as in Theorem 1.3. Under the assumptions and the notations of Proposition 1.1, we have $q_{a,f} \in C^{0,\beta}(\Omega) \cap H^s(\Omega)$ for $0 < s < \min(1/2,\beta)$ and

\[
\|q_{a,f}\|_{C^{0,\beta}(\Omega)} + \|q_{a,f}\|_{H^s(\Omega)} \leq C.
\]

Here the constant $C$ only depends on $\Omega$, $M$, $\alpha$, $s$, $c_0$, $c_1$ and $c$ if $n = 2$ or $n = 3$; the constant $C$ only depends on $\Omega$, $M$, $p$, $r$, $\alpha$, $s$, $c_0$, $c_1$ and $c$ if $n = 4$; the constant $C$ only depends on $\Omega$, $M$, $p$, $\alpha$, $s$, $c_0$, $c_1$ and $c$ if $n > 4$.

The following observation will be useful in the sequel: if $w \in H^s(\Omega)$, $0 < s < 1/2$, then $w\chi_\Omega \in H^s(\mathbb{R}^n)$ (see [9, page 31]).

Proof of Theorem 1.3. We give the proof when $n = 3$. That for the case $n \geq 4$ is quite similar.

Let $a$ and $\bar{a}$ be as in Theorem 1.3. Mimicking the proof of [2, Theorem 2.14] we find four constants $C > 0$, $c > 0$, and $r_0 > 0$, only depending on $\Omega$, $M$, $\alpha$, $s$, $c_0$, $c_1$, and $c$, so that

\[
|\hat{q}(\xi)| \leq C \left( \frac{1}{|\xi| + r} + e^{c(|\xi| + r)} |\mathcal{N}(f)| \right), \quad r \geq r_0,
\]

with

\[
q = (q_{a,f} - q_{a,f})\chi_\Omega \quad \text{and} \quad \mathcal{N}(f) = \|A_a(f) - A_{\bar{a}}(f)\|_{L^2(\Omega \cap \{(|\xi|)^2 \geq \rho^2\})}.
\]

Inequality (4.3) yields, for $\rho > 0$ and changing $c$ if necessary,

\[
|\hat{q}|^2_{L^2(\Omega \cap \{(|\xi|)^2 \geq \rho^2\})} \leq C \left( \frac{\rho^2}{\rho^2 + e^{c(\rho + r)}} \|\mathcal{N}(f)\| \right), \quad r \geq r_0.
\]

Here $\hat{q}$ denotes the Fourier transform of $q$.

On the other hand, with $0 < s < 1/2$, we have

\[
\|\hat{q}\|^2_{L^2(\{(|\xi|)^2 \geq \rho^2\})} \leq \frac{1}{\rho^{2s}} \|\xi|^s |\hat{q}|^2_{L^2(\{(|\xi|)^2 \geq \rho^2\})} \leq \frac{1}{\rho^{2s}} \|q\|^2_{H^s(\mathbb{R}^n)} \leq C^2 \frac{1}{\rho^{2s}}.
\]

Inequalities (4.4) and (4.5) together with Plancherel-Parseval’s identity give

\[
\|q\|^2_{L^2(\Omega)} \leq C \left( \frac{\rho^2}{\rho^2 + e^{c(\rho + r)}} \|\mathcal{N}(f)\| \right), \quad r \geq r_0, \quad \rho > 0.
\]

In this inequality we take $\rho = r^{2/(n+2s)}$. We find

\[
\|q\|^2_{L^2(\Omega)} \leq C \left( \frac{1}{r^{4s/(n+2s)}} + e^{c\rho} \|\mathcal{N}(f)\| \right), \quad r \geq r_0.
\]

In other words, we proved

\[
\|q_{a,f} - q_{a,f}\|^2_{L^2(\Omega)} \leq C \left( \frac{1}{r^{4s/(n+2s)}} + e^{c\rho} \|\mathcal{N}(f)\| \right), \quad r \geq r_0.
\]

In light of the interpolation inequality in [3, Lemma B.1] and Lemma 4.1, inequality (4.6) yields

\[
\|q_{a,f} - q_{a,f}\|_{C^s(\Gamma)} \leq C \left( \frac{1}{r^{4s/(n+2s)}} + e^{c\rho} \|\mathcal{N}(f)\| \right), \quad r \geq r_0.
\]

In particular (4.7) implies the following inequality

\[
\|q_{a,f} - q_{a,f}\|_{C^s(\Gamma)} \leq C \left( \frac{1}{r^{4s/(n+2s)}} + e^{c\rho} \|\mathcal{N}(f)\| \right), \quad r \geq r_0.
\]
We take in this inequality $f = \lambda$ with $|\lambda| \leq M$. We then get
\[
\max_{|\lambda| \leq M} |a'(\lambda) - \tilde{a}'(\lambda)| \leq C \left( \frac{1}{r^{4s/(n+2s)}} + e^{C \mathcal{N} M} \right), \quad r \geq r_0,
\]
with
\[
\mathcal{N} M = \sup_{\|f\|_{H^{3/2}(\Omega)} \leq \sqrt{\|f\|_{M}}} \mathcal{N}(f).
\]
A classical minimization argument with respect to $r$ yields
\[
\max_{|\lambda| \leq M} |a'(\lambda) - \tilde{a}'(\lambda)| \leq \Psi(\mathcal{N} M).
\]
Here,
\[
\Psi(\rho) = \left| \ln \rho \right|^{-4s\beta/(n+2s)(n+2\beta)} + \rho, \quad \rho > 0,
\]
that we extend by continuity at $\rho = 0$ by posing $\Psi(0) = 0$.

The proof is then complete. $\square$

References

[1] P. Caro, D. Dos Santos Ferreira and A. Ruiz, Stability estimates for the Calderón problem with partial data, J. Different. Equat. 260 (3) (2016) 2457-2489.
[2] M. Choulli, Une introduction aux problèmes inverses elliptiques et paraboliques, Mathématiques et Applications, Vol. 65, Springer-Verlag, Berlin, 2009.
[3] M. Choulli and Y. Kian, Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map. Application to the determination of a nonlinear term, J. Math. Pures Appl. 114 (2018) 235-261.
[4] M. Choulli, E. M. Ouhabaz and M. Yamamoto, Stable determination of a semilinear term in a parabolic equation, Commun. Pure Appl. Anal. 5 (3) (2006) 447-462.
[5] V. Isakov and A. Nachman, Global Uniqueness for a two-dimensional elliptic inverse problem, Trans. Amer. Math. Soc. 347 (1995), 3375-3391.
[6] V. Isakov and J. Sylvester, Global uniqueness for a semilinear elliptic inverse problem, Commun. Pure Appl. Math. 47 (1994), 1403-1410.
[7] O. Yu Imanuvilov and M. Yamamoto, Unique determination of potentials and semilinear terms of semilinear elliptic equations from partial Cauchy data, J. Inverse Ill-Posed Probl. 21 (2013), 85-108.
[8] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer, Berlin, 1998.
[9] P. Grisvard, Elliptic problems in nonsmooth domains, Pitman Publishing Inc., 1985.
[10] K. Krupchyk, G. Uhlmann, A remark on partial data inverse problems for semilinear elliptic equations, arXiv:1905.01561.
[11] M. Lassas, T. Liimatainen, Y.-H. Lin, M. Salo, Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations, arXiv:1905.02764.

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