Galactic dynamics in general relativity: the role of gravitomagnetism

Matteo Luca Ruggiero\textsuperscript{1,2,*,}\footnote{Author to whom any correspondence should be addressed.}, Antonello Ortolan\textsuperscript{2}\textsuperscript{,} and Clive C Speake\textsuperscript{3}

\textsuperscript{1} Dipartimento di Matematica ‘G.Peano’, Università degli studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
\textsuperscript{2} INFN—LNL, Viale dell’Università 2, 35020 Legnaro (PD), Italy
\textsuperscript{3} School of Physics and Astronomy, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom

E-mail: matteoluca.ruggiero@unito.it

Received 17 March 2022; revised 13 September 2022
Accepted for publication 11 October 2022
Published 26 October 2022

Abstract

It is a well-known fact that, in the absence of dark matter, the observation of the rotation curves of galaxies cannot be explained in terms of Newtonian gravity. Rotation curves become flat in the outer regions, in contrast to what is expected according to Keplerian motion. Far from the galactic center, the gravitational field is supposed to be weak enough so we expect to be able to use Newtonian gravity; however, even in the weak-field approximation, there are general relativistic effects without a Newtonian counterpart, such as the gravitomagnetic effects originating from mass currents. Using the gravitoelectromagnetic approach to the solution of Einstein equations in the weak-field and slow-motion approximation, we discuss some simple arguments that suggest the surprising result that gravitomagnetic effects may have a relevant role in better understanding the impact of dark matter on galactic dynamics. In addition, treating matter as a fluid of dust, we study the influence of post-Newtonian effects on the fluid vorticity.

Keywords: gravitomagnetism, galactic rotation curves, post-Newtonian effects

1. Introduction

Thanks to the pioneering work of Rubin et al [1] and taking into account subsequent analyses [2], we know that the rotation curves of galaxies become flat far from the center: the observed...
non-Keplerian features contradict what is expected on the basis of a Newtonian gravity analysis. The observation of flatness of rotation curves is one element in the case supporting the existence of dark matter [3, 4]. It is also worth pointing out that an ad hoc modification of Newtonian dynamics, the so-called MOND has also been proposed by Milgrom [5] to explain the flatness problem.

At first sight, more than 100 years after the publication of general relativity (GR), it could appear surprising that Newtonian rather than Einsteinian physics is applied to study the gravitational dynamics of galaxies. The motivations are apparently sound, since the stars' velocities are small compared to the speed of light and, moreover, in the outer regions of the galaxies far from the center, the gravitational field is weak. However, a subtler analysis suggests a different perspective. In fact, while we know that Newtonian gravity is a good approximation to GR, we should remember that there are general relativistic effects without Newtonian counterparts: for instance, gravitational waves and the effects produced by mass currents, the so-called gravitomagnetic field which gives rise to the Lense–Thirring effect [6, 7]. Recently, gravitomagnetic effects due to passage of a plane gravitational wave have been investigated [8–10]. In both cases, GR introduces new phenomena that go beyond corrections to known Newtonian effects.

Based on these motivations, there have been some attempts to use suitable solutions of GR field equations to model galaxies and, in this framework, the flatness problem is analyzed without requiring extra unknown sources. We refer, for instance, to the works published by Cooperstock and Tieu [11] and Balasin and Grumiller [12]. In both cases, a solution is obtained with the assumptions that (a) the system is axially symmetric and stationary, (b) matter can be treated as dust (pressureless and perfect—hence inviscid—fluid interacting only gravitationally). The model proposed by Cooperstock and Tieu [11] has been questioned (see e.g. Cross [13], Menzies and Mathews [14]), while the model developed by Balasin and Grumiller [12] even though it develops singularities, has been recently reconsidered by Crosta et al [15] and generalized by Astesiano et al [16]. The corrections due to the galactic rotation on the velocity of circular orbits were calculated by [17], using an approximate form of the Kerr solution. The role of post-Newtonian corrections in galactic dynamics was emphasized by Ramos-Caro et al [18] while, in a recently published paper, Ludwig [19] developed a model of galactic dynamics based on the equations that govern the motion of a weakly relativistic perfect fluid. All these research efforts share the idea that the gravitomagnetic effects, originating from mass currents, have a relevant impact on the observed galactic dynamics. Consequently, if these purely non-Newtonian effects are properly taken into account, there is perhaps no need to invoke dark matter to explain observations or it is likely that the impact of dark matter should be reconsidered.

The purpose of this paper is to discuss, from a quite general perspective, the role of gravitomagnetic effects in the study of galactic dynamics. This might help to shed new light on dark matter, which plays a central role not only in this context, but in many domains of modern cosmology. To this end, we will not resort to specific GR solutions or solve the fluid equation coupled to the gravitational field using specific mass density models. We will just assume that, in the outer regions of the galaxies, a weak-field and slow-motion approximation can be used to describe the gravitational field acting upon stars, which can themselves be treated as test masses and whose motion can be studied using the geodesic equation: in particular, we will consider the stars as dust. In addition, we will analyze the limits of the Maxwellian analogy, since new terms that were neglected in previous analysis have an effective impact on the fluid vorticity and can be relevant for astrophysical systems.
2. Gravitoelectromagnetism in a nutshell

The solution of Einstein field equations in weak-field and slow-motion approximation, leads to the gravitoelectromagnetic analogy; in particular, it is possible to write a set of Maxwell-like equations, where the mass density and current play the role of the charge density and current, respectively \(^{20-22}\). Accordingly, if we neglect in the metric tensor terms that are \(O(c^{-4})\), the line element can be written\(^4\) as \(^{23, 24}\)

\[
ds^2 = -c^2 \left( 1 - 2 \frac{\Phi}{c^2} \right) dt^2 - 4 \frac{A \cdot d\mathbf{x}}{c} dt + \left( 1 + 2 \frac{\Phi}{c^2} \right) \delta_{ij} dx^i dx^j
\]

where, in stationary conditions, the gravitoelectric (\(\Phi\)) and gravitomagnetic (\(A\)) potentials are solutions of Poisson equations

\[
\nabla^2 \Phi = -4\pi G \rho,
\]

\[
\nabla^2 A = -8\pi G j,
\]

in terms of the mass density \(\rho\) and current \(j\) of the sources. The analogy with electromagnetism is strengthened by the geodesic equation. In fact, in stationary conditions, if we define the gravitoelectric (\(E\)) and gravitomagnetic (\(B\)) fields

\[
E = -\nabla \Phi, \quad B = \nabla \wedge A,
\]

the spatial components of the geodesic equation, up to linear order in \(|v|c\), can be expressed in analogy with the Lorentz-like force equation acting upon a test mass \(m\)

\[
m \frac{dv}{dt} = -mE - 2m \frac{v}{c} \wedge B.
\]

Notice that the hypothesis of stationarity is used not only in the definition of the gravitoelectromagnetic fields (4), but also in the expression of the geodesic equation; if we relax the stationary conditions and define the gravitoelectromagnetic fields

\[
B = \nabla \wedge A, \quad E = -\nabla \Phi - 2 \frac{\partial A}{\partial t},
\]

the above equation (5) becomes \(^{22, 25, 26}\)

\[
m \frac{dv}{dt} = -mE - 2m \frac{v}{c} \wedge B - 3m \frac{v}{c} \frac{\partial \Phi}{\partial t}.
\]

In particular, we notice that the last term in equation (7) is non-Maxwellian and, hence, breaks the gravitoelectromagnetic analogy: we will discuss its impact in what follows.

The above approach is aimed at emphasizing the similarity between electromagnetism and GR in the weak-field and slow-motion approximation. In analogy with the electric field of a point charge, \(\Phi\) differs by a minus sign from the actual Newtonian potential of point mass \(M\), \(\Psi = -\frac{GM}{|\mathbf{x}|}\). In what follows, it will be useful to make a comparison with Newtonian quantities: so, to this end, we introduce the gravitational field \(g = -\nabla \Psi = -E\) and the Newtonian potential \(\Psi = -\Phi\).

\(^4\) Bold symbols like \(v\) refer to space vectors, in particular \(\mathbf{x} = (x, y, z)\) is the position vector at location \(P(x, y, z)\); the spacetime signature is \((-1, 1, 1, 1)\); Latin indices run from 1 to 3; \(G\) is the gravitational constant and \(c\) is the speed of light in vacuum.
On the other hand we compute the gravitomagnetic field in the same way as in Maxwell’s equations. However the contribution to the Lorentz force from the gravitomagnetic field is multiplied by a factor \((-2)\) (see e.g. equation (5)).

3. The generalized vorticity

Using the gravitoelectromagnetic analogy, we can define some properties of a weakly relativistic dust. To this end, we remember that when we have fluids and electromagnetic fields it is possible to introduce the generalized vorticity \(\Omega_G = \nabla \wedge P\) starting from the canonical momentum \(P = mv + \frac{q}{c}A\), where \(q\) is the electromagnetic charge and \(A\) the electromagnetic vector potential. Similarly, for a weakly relativistic dust we may define \(P = v - \frac{2}{c}A\), that is the canonical momentum of the fluid element per unit mass [24]; for the sake of simplicity, here we focus only on gravitational effects and do not consider electromagnetic ones. Using these definitions, we may write the momentum equation (5) in the form:

\[
\frac{\partial P}{\partial t} + (\nabla \wedge P) \wedge v = \nabla \left( \Phi - \frac{1}{2}v^2 \right).
\]

In addition, we may define the gravitational generalized vorticity \(\Omega_G = \nabla \wedge P = \Omega + 2\Omega_L\) in terms of the fluid vorticity \(\Omega = \nabla \wedge v\), and the gravitomagnetic Larmor frequency \(\Omega_L = -\frac{1}{c}B\) [27]. Due to the formal analogy with the electromagnetic case (see e.g. Mahajan and Yoshida [28]), the rate of change of circulation

\[
\Gamma = \oint_L P \cdot dA = \int_S \Omega_G \cdot dS = \int_S \Omega \cdot dS + \int_S 2\Omega_L \cdot dS = \Gamma_N + \Gamma_{GM}
\]

along a closed loop \(L\), which is the contour of the surface \(S\), is zero. Hence, we see that a gravitomagnetic field is indistinguishable from a vorticity field, and thus it contributes to establish a rotational fluid. The generalized gravitational vorticity is conserved, and cannot emerge from a zero initial value; accordingly, the Newtonian circulation \(\Gamma_N\) is not conserved due to the presence of the gravitomagnetic term \(\Gamma_{GM}\): the latter, is simply related to the gravitomagnetic flux and its presence can be relevant in coalescence of compact binaries or accretion phenomena around black holes [29, 30]. The presence of the gravitomagnetic field can explain rotational motion in an inviscid fluid as a consequence of the definition of generalized vorticity. Moreover, the generalized gravitational helicity

\[
H = \int_V P \cdot \Omega_G dV
\]

is conserved (see e.g. Mahajan [31], Alves et al [32] and references therein; see also Bini et al [33] for the gravitomagnetic helicity). Again, the Newtonian helicity is not conserved, which means that the gravitomagnetic field modifies the topology of the line of force of the Newtonian fluid.

If we want to analyze our fluid of stars using this approach, we must remember the gravitomagnetic field experienced by a test mass at a given location depends both on matter moving inside and outside its orbit [34, 35]. Consequently, it is necessary to treat the system as a whole, and to look for a self-consistent solution of the gravitational and fluid equations. To this end, in stationary conditions the momentum equation (8) can be written in the form

\[
\nabla \left( \frac{v^2}{2} + \Psi \right) = v \wedge \Omega_G.
\]
This equation, together with the source equations (2) and (3) can be used to describe the equilibrium of the dust: so, starting from the knowledge of the mass density profile, it is possible to use the above equations, to obtain \( v, \mathbf{A}, \Psi \) (remember that the mass current is defined by \( j = \rho v \), in terms of the mass density and fluid velocity).

Actually, this is what is done in the paper by Ludwig [19].

3.1. Non-stationary conditions

The gravitoelectromagnetic analogy has however some limitations, which are important to emphasize, in particular if we consider the non-stationary state, that is in the evolution of the fluid of dust. In fact, we noticed that in these conditions the momentum equation (7) has a non-Maxwellian term which is missing in previous analysis (see e.g. Ludwig [19]) but it is reasonable that it cannot be neglected in an evolving scenario. In fact, at a distance \( r \) from a point-like source of mass \( M \), moving with speed \( v_s \), its magnitude is

\[
\frac{v_s}{c} \frac{\partial (\Phi)}{\partial t} \approx \frac{M v_s}{c^2 r^2}.
\]

Accordingly, it is of the same order as the translational gravitomagnetic field of the source. The non-Maxwellian term can be neglected if we assume that the source is at rest, but in evolving scenarios it cannot be neglected. The presence of the non Maxwellian term, then, leads to the non conservation of the generalized gravitational vorticity and the corresponding helicity. Moreover, starting from equation (7) and using the continuity equation, it is possible to show that \( \Omega_G \) satisfies the following vorticity equation

\[
\frac{d}{dt} \left( \frac{\Omega_G}{\rho} \right) = \left( \frac{\Omega_G}{\rho} \cdot \nabla \right) v + \frac{1}{\rho} \mathbf{S}
\]

where \( \mathbf{S} = -\frac{3}{c^2} \nabla \left( \frac{\partial \Phi}{\partial t} \right) \wedge v - \frac{3}{c^2} \frac{\partial \Phi}{\partial t} \nabla \wedge v \). This is a new result that generalizes what was obtained in previous works [19], and emphasizes the importance of all post-Newtonian effects (and not only of the gravitomagnetic ones) in determining the evolution of a fluid element. Indeed, the presence of the gravitomagnetic effect and of the non-Maxwellian term may produce vorticity in an initially irrotational motion. It is expected that this may have implications in galaxies dynamics and, more in general, in the evolution of astrophysical systems.

4. The Newtonian case

Initially we examine the possibility of describing an isolated, axially symmetric and rotating dust solution using only Newtonian physics. The limitations of this approach in comparison with GR were pointed out by Bonnor [36] in 1977 and we briefly discuss his argument. We consider dust particles steadily rotating around a symmetry axis: we use a reference frame, with origin \( O \), and choose cylindrical coordinates \( \{r, \varphi, z\} \) within it, such that \( z \) is the rotation axis; \( \mathbf{u}_r, \mathbf{u}_\varphi, \mathbf{u}_z \) are the unit vectors. If \( \omega = \omega \mathbf{u}_z \) is the rotation rate and \( \mathbf{x} \) is the position vector of a dust particle, we may write its velocity in the form

\[
v = \omega \wedge \mathbf{x},
\]

where we suppose that \( \omega \) can be a function of \( r \) and \( z \). The momentum equation for a dust cloud acted upon by the gravitational field is:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \Psi,
\]
where v is the velocity and again $\Psi$ the Newtonian potential, and we have used the convective derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla$. The continuity equation may be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$  \hspace{1cm} (15)

In stationary conditions ($\frac{\partial}{\partial t} = 0$), the above equations become

$$(v \cdot \nabla) v = -\nabla \Psi,$$  \hspace{1cm} (16)

$$\nabla \cdot (\rho v) = 0.$$  \hspace{1cm} (17)

The gravitational potential and the mass density are related by the Poisson equation

$$\nabla^2 \Psi = 4\pi G \rho.$$  \hspace{1cm} (18)

It is useful for what follows to write the following mathematical identities

$$\nabla^2 \Psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Psi}{\partial r} \right) + \frac{\partial^2 \Psi}{\partial z^2},$$  \hspace{1cm} (19)

and

$$(v \cdot \nabla) v = (\nabla \wedge v) \wedge v + \frac{1}{2} \nabla (v \cdot v).$$  \hspace{1cm} (20)

Since, according to equation (13) $v_z = 0$ everywhere, from the momentum equation (14) we obtain

$$\frac{\partial \Psi}{\partial z} = 0.$$  \hspace{1cm} (21)

In other words there is no gravitational force along the rotation axis acting on the dust particles. As a consequence, by differentiating the Poisson equation (18), we eventually deduce that there is no density gradient parallel to the rotation axis:

$$\frac{\partial \rho}{\partial z} = 0.$$  \hspace{1cm} (22)

Eventually, on taking into account axial symmetry and using equations (14) and (20) turns out to be

$$\omega^2 r = \frac{\partial \Psi}{\partial r},$$  \hspace{1cm} (23)

which means that also $\omega$ is independent of $z$, and the motion of the dust particles is the same in every plane $z = \text{constant}$. In summary: in Newtonian gravity, stationary, axially symmetric motion of dust is necessarily cylindrically symmetric and, moreover, the motion is the same in every plane orthogonal to the symmetry axis. Hence, no compact or finite dust object can exist in Newtonian gravity in the given symmetry conditions. As Bonnor pointed out, things are quite different in GR where ‘a non-Newtonian force, arising from the spin of the central body, permits non-equatorial circular orbits’. Actually, as we are going to see below, this can be explained in terms of a gravitomagnetic force acting on moving masses.

In several papers and textbooks pertaining to galactic dynamics, it is maintained that for those that are rotationally supported, such as spirals or irregular, relation (23) provides a link between rotation curves and gravitational potential [37]. The analysis of the rotation curves is based on the following hypothesis: since the luminous mass density is rapidly decreasing from the center, the galaxy is modeled as a point mass, and it is then expected that stars far away from the center move as test masses around a point-like mass, just like the planets around the
Sun. Accordingly, Newtonian dynamics suggests that velocities should decrease, which is not observed, since rotation curves remain flat. Hence, it is expected that extra, non-visible matter is present and acts upon stars, thus increasing their velocity, the so-called dark matter. This is different from the Newtonian analysis of a self-consistent dust solution that we have considered, since in the standard approach it is supposed that there is a compact mass distribution, which constitutes the great part of the galaxy mass content, and distant stars move like test particles around it.

Our discussion of Newtonian solutions has been limited to models with smoothly varying functions of density. However we note that the solutions proposed by Kuzmin [38] and Toomre [39] are based on infinitely thin discs of finite radial extent but therefore feature unphysical discontinuities in the axial gravity field. Other models having mass distributions that have a finite axial extent have been found by Miyamoto and Nagai [40], but these cannot form time-independent structures as suggested by the above analysis. The stability of these structures is generally attributed to the dispersive random component of velocity and the deepening of the potential well of the galaxy itself by the dark matter halo [41]. Axial forces are required to maintain equilibrium against the axial gravity field as we will see below.

Starting from the Poisson equation (18), and substituting in it equation (23), we obtain

\[ 2\omega^2 + 2\omega \frac{\partial \omega}{\partial r} = 4\pi G \rho. \]  

(24)

This equation locally relates the matter density to the rotation rate and its derivative: in the regime \( v = \omega r \approx \) constant, that is in the flat region of the rotation curves, we have \( \frac{\partial \omega}{\partial r} r + \omega \approx 0 \), hence we get \( \rho = 0 \) from (24). As a consequence, in the framework of Newtonian gravity the flat velocity profile for the infinite dust cylinder is not allowed [42, 43]. Stated in a different way, using a purely Newtonian approach, it is not clear how to relate the matter density with the rotation rate in the flat zone. The situation is quite different if we resort to GR.

5. The Einsteinian case

Let us now extend the analysis of an axially symmetric and rotating dust solution using the general relativistic approach; in particular, dust will be coupled to the gravitational field described in terms of the gravitoelectromagnetic analogy. Before entering into details, we describe the underlying hypotheses. A galaxy as a whole is a complex object in which very different gravitational fields are present: think of the Milky Way, with a supermassive black hole [44] at its center, and the disk which extends up to 50 kpc. Then, it is reasonable to say that the dust approximation cannot be used to describe the galactic center, but in the outer regions it is expected that stars can be thought of as test particles, moving in a background spacetime, that can be considered a solution of Einstein equations in weak-field and slow-motion approximation, with the additional requirements of axial symmetry and stationarity. Accordingly, this solution can be expressed in the form (1). We point out that for our purposes it is not necessary to know the explicit form of the spacetime metric: we just want to focus on the role of the gravitomagnetic effects. Stars move along geodesics defined by the Lorentz-like equation

\[ \mathbf{a} = \mathbf{g} - 2 \frac{\mathbf{v}}{c} \wedge \mathbf{B}, \]  

(25)

where their acceleration can be written as a convective derivative of velocity \( \mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{v}}{\partial r} + \mathbf{v} \cdot \nabla \mathbf{v} \). This is the momentum equation for the dust; the difference with respect to the corresponding Newtonian equation (14) is the presence of the gravitomagnetic term. In order to focus on the consequences provoked by this additional term, we remember that according to
our hypothesis the gravitational field is axially symmetric, hence the gravitomagnetic field can be written in the cylindrical coordinate system introduced in the previous section in the form \( \mathbf{B} = B_r \mathbf{u}_r + B_z \mathbf{u}_z \). We are interested in circular orbits of test masses, hence we write the velocity in the form \( \mathbf{v} = \omega r \mathbf{u}_\phi \). The component of equation (25) parallel to symmetry and rotation axis \( z \) reads

\[
a_z = \frac{2}{c} v_\phi B_z = g_z. \tag{26}
\]

We see that, thanks to the presence of the gravitomagnetic term, non equatorial orbits \((a_z = 0)\) are possible: this is a first remarkable difference from the Newtonian case, as pointed out by Bonnor [36]. In addition, equation (26) suggests that the effects of the gravitomagnetic field are comparable to the Newtonian field, contrary to the common belief, deriving from the study of other physical systems, where post-Newtonian effects are always smaller than Newtonian ones: for instance, this is the case of the Solar System. However, this is not always true: for instance, if we consider a uniformly rotating hollow homogeneous sphere, the gravitomagnetic field is constant within the sphere (see e.g. Ciufolini et al [34]), while the corresponding gravitational field is null: this shows that it is not generally true that gravitomagnetic fields are always smaller than the Newtonian ones. If we assume the acceleration of stars along the rotation axis are negligible, equation (26) can be used to measure gravitomagnetic effects. Notice that this statement is a natural consequence of the model the we considered, i.e. an axisymmetric fluid of dust: this conclusion is not necessarily true if more complex models are considered, including for instance pressure or baryonic effects.

Following the discussion above on generalized vorticity, we might expect a natural magnitude of the gravitomagnetic field to be that of the fluid vorticity.

Let us evaluate the impact of the gravitomagnetic field on the rotation velocity; to this end, we may consider the \( r \) component in the momentum equation (25), which turns out to be

\[
a_r + \frac{2}{c} v_\phi B_z = g_r. \tag{27}
\]

We suppose that test particles are moving on circles in planes orthogonal to the symmetry axis, we have \( a_z = 0, a_r = -\omega^2 r \) and \( v_\phi = \omega r \), hence

\[
-\omega^2 r + \frac{2}{c} \omega r B_z = g_r. \tag{28}
\]

Now, we remember that \( \mathbf{g} = -\nabla \Psi \), so \( g_r = -\frac{\partial \Psi}{\partial r}, g_z = -\frac{\partial \Psi}{\partial z} \). Hence, from equations (26) and (28), we obtain

\[
\frac{\partial \Psi}{\partial r} = \omega^2 r - \frac{2}{c} \omega r B_z, \tag{29}
\]

\[
\frac{\partial \Psi}{\partial z} = \frac{2}{c} \omega r B_r. \tag{30}
\]

A comparison with the corresponding Newtonian equations (21) and (23) emphasizes the role of the gravitomagnetic field. As we said before, equation (23) relates rotation velocities and gravitational potential: we see that the non-Newtonian effect changes this equation, and it is expected that the link between mass distribution and rotation velocities is equally modified, as we are going to show. Substituting from equations (29) and (30) into the Poisson equation (18) and using equation (19), we get the following expression
\[
2\omega^2 + 2\omega \frac{\partial \omega}{\partial r} r - \frac{4}{c} \omega B_z - 2 \frac{\partial \omega}{\partial r} r B_z - 2 \frac{\partial B_z}{\partial r} r \\
+ 2 \frac{\partial \omega}{\partial z} r B_r + \frac{2}{c} \omega B_r = 4\pi G \rho. \quad (31)
\]

This equation can be directly derived in vector notation from the momentum equation in stationary conditions \((11)\), and taking into account the Ampère law for the gravitomagnetic field (see e.g. Ruggiero and Tartaglia \([23]\), Mashhoon \([24]\)):

\[
\nabla \wedge \mathbf{B} = \frac{8\pi G}{c} \mathbf{j}, \quad (32)
\]

and we obtain:

\[
4\pi G \rho \left(1 - 4\frac{v^2}{c^2}\right) + \frac{2}{c} \mathbf{B} \cdot \mathbf{\Omega} = -\nabla \cdot \left[(\mathbf{v} \cdot \nabla) \mathbf{v}\right]. \quad (33)
\]

Since we are working at linear order in \(v/c\), we can neglect the quadratic term and write

\[
4\pi G \rho + \frac{2}{c} \mathbf{B} \cdot \mathbf{\Omega} = -\nabla \cdot \left[(\mathbf{v} \cdot \nabla) \mathbf{v}\right]. \quad (34)
\]

For comparison, the same equation without gravitomagnetic field reads

\[
4\pi G \rho = -\nabla \cdot \left[(\mathbf{v} \cdot \nabla) \mathbf{v}\right], \quad \text{(35)}
\]

which corresponds to equation \((24)\).

The meaning of equation \((35)\) is easy to understand if we take into account the momentum equation \((16)\): in fact, it corresponds to Gauss’ law for the gravitational field.

Then, from equation \((34)\) we see that the coupling between the gravitomagnetic field and the fluid vorticity influences the local relation between density and the fluid velocity. We can evaluate the impact of the gravitomagnetic field on the density profile setting

\[
\delta \rho = -\frac{1}{2\pi G c} \mathbf{B} \cdot \mathbf{\Omega}. \quad (36)
\]

For instance, in the region where the rotation curves are flat, we may set \(v = \omega r \approx \text{constant}\). Accordingly, in the equatorial plane \((B_r = 0, \Omega_z = \omega)\), we obtain

\[
\delta \rho = -\frac{1}{2\pi G c} B_z \omega. \quad (37)
\]

Remember that, according to equation \((24)\), in this regime the Newtonian density is null: accordingly, we see that a gravitomagnetic field antiparallel to the \(z\) contributes to define the matter density.

Similar conclusions about the impact of the additional degree of freedom due to the gravitomagnetic field were obtained in previous works \([11, 12, 15, 16]\) studying suitable solutions of Einstein equations.

Our simple argument, which rests upon few reasonable hypotheses and does not require a model for the description of the galaxy, suggests that the gravitomagnetic field can be relevant to better estimate the impact of dark matter on the galactic rotation curves: in fact, equation \((36)\) shows that part of the missing mass density needed to fit the rotation curves could derive from general relativistic effects.
6. Conclusions

In this paper, we have taken a quite general approach to show the relevance of the gravitomagnetic effects in galactic dynamics. In our simplified model we consider the stars to behave as dust particles in a fluid; we expect that, far from the galactic center, the dynamics of the fluid is determined by the background spacetime that is an axially symmetric and stationary solution of Einstein’s equations in the weak-field and slow-motion limit. We do not need to know the explicit form of this spacetime metric because, in any case, the gravitomagnetic formalism can be applied.

Gravitomagnetic fields could be relevant to the evolution of rotating structures like galaxies: in fact, dissipative processes in the early phase of galaxy formation are expected and these can generate fluid vorticity in protogalactic clouds (see e.g. Silk [45], Wang and Scheuerle [46]). In later evolution of galaxies, the conservation of the sum of fluid vorticity and the gravitomagnetic Larmor frequency can give rise to non-Keplerian rotation velocity curves. In this regard, it is useful to remember that the role and the generation of angular momentum, which is source of gravitomagnetic effects, is poorly understood: recently, it has been showed that angular momentum can be generated on unexpectedly large scales [47].

In addition, we stressed that the analogy between electromagnetism and linear gravity is limited: in non-stationary conditions, non-Maxwellian terms appear that cannot be neglected. As a consequence in this regime the fluid vorticity is related not only to gravitomagnetic effects but, more in general, to post-Newtonian effects.

By rephrasing an old argument by Bonnor, we showed that diverse circular orbits in planes orthogonal to the rotation axis are allowed thanks to the presence of the gravitomagnetic force that balances the Newtonian force in the direction of the rotation axis. But in Newtonian gravity the motion is the same in every plane orthogonal to the symmetry axis and there can be no variation of density in the axial direction. This raises the possibility that rotational velocities of galaxies can have a dependency on axial distance from the equatorial plane, which is contrary to what is currently assumed. It is important to emphasize that if the galaxy can be modeled as an axisymmetric fluid of dust, equation (26) suggests that non equatorial circular orbits require gravitomagnetic effects of the same order as Newtonian ones: we suggest that this fact can be considered as new test of GR. We showed that velocity, gravitomagnetic field and mass density are connected to the fluid vorticity: namely, there is a coupling between the gravitomagnetic field and the fluid vorticity which influences the local relation between density and the fluid velocity. In particular, in the Newtonian case, where this coupling is not present (see e.g. equation (24)), the local definition of mass density seems to be meaningless in the region where the flatness of the rotation curves is observed: this fact suggests that, in this region, the contribution of the gravitomagnetic field becomes important.

In summary, our heuristic approach suggests that the gravitomagnetic field may play a relevant role in understanding galactic dynamics as it allows for an extra rotational degree of freedom associated with the fluid vorticity. It is important to emphasize that we do not claim that this approach can eliminate the need for dark matter. Since dark matter plays a central role in modern cosmology (and the rotation curves of galaxies are only one of many motivations for its existence) we believe that the physics presented in this paper will help us to better understand the actual impact of dark matter: our simple argument suggests, in fact, that an extra contribution to the matter density can derive from the coupling between the gravitomagnetic field and the fluid vorticity. A more sophisticated analysis based on a numerical approach (some numerical codes that can be used to simulate gravitomagnetism are discussed by Adamek et al [48]) is required to understand the details of the role that gravitomagnetism
has in the formation of galaxies, however we hope that our discussion will motivate further attention to this topic.

Data availability statement

No new data were created or analysed in this study.

Note added. After the submission of this paper to Classical and Quantum Gravity, two papers were published which focused on the role of GR and, in particular, of the gravitomagnetic effects on the rotation curves. Ciotti [49] concludes, after a detailed analysis based on analytical models, that GR effects cannot compensate by any detectable amount the Keplerian fall of the rotational velocity. We note also that a simple application of Gauss’ law and Ampere’s circuital law to an annular element of a disk galaxy shows that the ratio of the axial components of the Newtonian, $g$, to Lorentz (i.e. gravitomagnetic) $L$, accelerations is $L/g \simeq v^2/c^2$ which supports Ciotti’s analysis. However Astesiano and Ruggiero [50] obtain a different result, starting from exact solutions of Einstein equations for stationary axisymmetric systems, thus showing that GR may have a relevant impact in understanding galactic dynamics, since it introduces an additional degree of freedom with respect to the Newtonian case. In particular, in the latter paper it is shown that the strong gravitomagnetic limit can provide contributions of the same order of the Newtonian ones. Accordingly, the debate is still open and our paper can contribute to add new elements to the discussion.

ORCID iDs

Matteo Luca Ruggiero  https://orcid.org/0000-0002-1844-5863
Antonello Ortolan  https://orcid.org/0000-0002-7280-6498

References

[1] Rubin V C, Ford W K Jr and Thonnard N 1978 Astrophys. J. 225 L107
[2] Sofue Y and Rubin V 2001 Annu. Rev. Astron. Astrophys. 39 137
[3] Strigari L E 2013 Phys. Rep. 531 1
[4] Amendola L et al 2018 Living Rev. Relativ. 21 1
[5] Milgrom M 1983 Astrophys. J. 270 365
[6] Iorio L, Lichtenegger H I M, Ruggiero M L and Corda C 2011 Astrophys. Space Sci. 331 351
[7] Pfister H 2014 Relativity and Gravitation (Berlin: Springer) pp 191–7
[8] Ruggiero M L and Ortolan A 2020 J. Phys. Commun. 4 055013
[9] Ruggiero M L and Ortolan A 2020 Phys. Rev. D 102 101501
[10] Ruggiero M L 2021 Am. J. Phys. 89 639
[11] Cooperstock F I and Tieu S 2007 Int. J. Mod. Phys. A 22 2293
[12] Balasin H and Grumiller D 2008 Int. J. Mod. Phys. D 17 475
[13] Cross D J 2006 (arXiv:astro-ph/0611191)
[14] Menzies D and Mathews G J 2007 (arXiv:astro-ph/0701019)
[15] Crosta M, Giammaria M, Lattanzi M G and Poggio E 2020 Mon. Not. R. Astron. Soc. 496 2107
[16] Astesiano D, Cacciatori S L and Re F 2021 (arXiv:2106.12818)
[17] Vogt D and Letelier P S 2005 Mon. Not. R. Astron. Soc. 363 268
[18] Ramos-Caro J, Agon C A and Pedraza J F 2012 Phys. Rev. D 86 043008
[19] Ludwig G 2021 Eur. Phys. J. C 81 1
[20] Mashhoon B, Gronwald F and Lichtenegger H I M 2001 Gyros, Clocks, Interferometers…: Testing Relativistic Gravity in Space (Berlin: Springer) pp 83–108
[21] Ciufolini I and Wheeler J A 1995 Gravitation and Inertia vol 101 (Princeton, NJ: Princeton University Press)
