Bohr Radius for Some Classes of Harmonic Mappings

K. Gangania\(^1\) \& S. Sivaprasad Kumar\(^1\)

Received: 16 October 2021 / Accepted: 18 February 2022 / Published online: 13 May 2022
© The Author(s), under exclusive licence to Shiraz University 2022

Abstract
In this note, we introduce a general class of sense-preserving harmonic mappings defined as follows:

\[ B^0_H(M) := \{ f = h + \bar{g} : \sum_{m=2}^{\infty} \gamma_m |a_m| + \delta_m |b_m| \leq M, \ M > 0 \}, \]

where \( h(z) = z + \sum_{m=2}^{\infty} a_m z^m \), \( g(z) = \sum_{m=2}^{\infty} b_m z^m \) are analytic functions in \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and 
\[ \gamma_m, \delta_m \geq \alpha_2 := \min\{\gamma_2, \delta_2\} > 0, \]

for all \( m \geq 2 \). We obtain Growth Theorem, Covering Theorem, and derive the Bohr radius for the class \( B^0_H(M) \). As an application of our results, we obtain the Bohr radius for many classes of harmonic univalent functions and some classes of univalent functions.

Keywords Bohr radius \cdot Harmonic mappings \cdot Univalent analytic functions \cdot Growth theorem

Mathematics Subject Classification 30C45 \cdot 30C50 \cdot 30C80

1 Introduction

Let \( \mathcal{H} \) denotes the class of complex valued harmonic functions \( f \) (which satisfy the Laplacian equation \( \Delta f = 4f_z z = 0 \)) defined on the unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \); then we can write \( f = h + \bar{g} \), where \( h \) and \( g \) are analytic and satisfy \( f(0) = g(0) \). If the Jacobian \( J_f := |h|^2 - |g'|^2 > 0 \), we say \( f \) is sense-preserving in \( \mathbb{D} \). Let \( \mathcal{H}_0 \) be the class of functions \( f \) with \( f_z(0) = 0 \) and \( f = h + \bar{g} \), where \( h(z) = z + \sum_{m=2}^{\infty} a_m z^m \) and \( g(z) = \sum_{m=2}^{\infty} b_m z^m \) are analytic functions in \( \mathbb{D} \). For \( g \equiv 0 \), \( \mathcal{H}_0 \) reduces to the class \( A \) of analytic functions \( f \) with normalization \( f(0) = 0 = f_z(0) - 1 \). Let \( S^0_H \) denotes the class of harmonic and univalent functions, which clearly includes the well-known class of normalized univalent functions \( S \). Recall that if \( f(z) = \phi(w(z)) \) and \( w \) is a Schwarz function, we say \( f \) is subordinate to \( \phi \), which is written as \( f \prec \phi \). Further, if \( \phi \) is univalent, then \( f \prec \phi \) if and only if \( f(0) = \phi(0) \) and \( f(\mathbb{D}) \subseteq \phi(\mathbb{D}) \). Motivated by Sakaguchi class of starlike functions with respect to the symmetric points using subordination (Sakaguchi 1959), and (Cho and Dziok 2020) considered a subclass of \( S^0_H \), which is given by

\[ S^0_H(C, D) := \left\{ f \in \mathcal{H}_0 : \frac{2D_Hf(z)}{f(z) - f(-z)} < \frac{1 + Cz}{1 + Dz}, -D \leq C < D \leq 1 \right\}, \]

where \( D_Hf(z) := zH'(z) - \bar{g}'(z) \). Further, in light of Silverman’s work (Silverman 1998), they defined the class \( S^+\bigcap \mathcal{H}_0(C, D) \), where \( t^a(a \in (0, 1]) \) is the
class of functions in $H_0$ with the coefficients $a_m$ and $b_m$ are replaced respectively by $-a_m$ and $(-1)^{m}b_m$ for all $m$. By involving Janowski functions, (Dziok 2015) studied the following classes:

$$S_{H_0}(C, D) := \left\{ f \in H_0 : \frac{2Dh(f)}{f(z)} < 1 + Cz, \quad -D \leq C < D \leq 1 \right\},$$

and $S_{H_0}(C, D) := \left\{ f \in S_{H_0} : Dh(f) \in S_{H_0}(C, D) \right\}$ in addition to the classes $S_{H_0}(C, D) := \tau^0 \cap S_{H_0}(C, D)$ and $S_{H_0}(C, D) := \tau^1 \cap S_{H_0}(C, D)$. Singh (1977) studied the subclass $F_{\lambda}(\lambda) := \left\{ f \in A : |f(z) - 1| < \lambda, \lambda \in (0, 1) \right\}$ of close-to-convex functions. Later, Nagpal and Ravichandran (2014a) examined its harmonic extension defined as:

$$F_{\lambda}^{H_0}(\lambda) := \left\{ f = h + \bar{g} : |f'(z) - 1| < \lambda - |f'(z)|, \lambda \in (0, 1) \right\}.$$

Gao and Zou (2007) investigated a subclass of close-to-convex functions given by $W(\mu, \rho) := \left\{ f \in A : R(h'(z) + \mu h''(z)) > \rho, \mu \geq 0, 0 \leq \rho < 1 \right\}$. Rajbala and Prajapati (2020) also explored the subclass of close-to-convex harmonic mappings defined as:

$$W_{H_0}^{\mu}(\mu, \rho) := \left\{ f = h + \bar{g} : R(h'(z) + \mu h''(z)) > |g'(z)|, \mu \geq 0, 0 \leq \rho < 1 \right\},$$

where $\mu \geq 0$ and $0 \leq \rho < 1$, which is the harmonic extension of $W(\mu, \rho)$. This generalizes the classes studied in Ghosh and Vasudevarao (2019); Nagpal and Ravichandran (2014b). In a similar way, (Dixit and Porwal 2010) considered the class $R_{H_0}(\beta) := \left\{ f = h + \bar{g} : R(h'(z) + g'(z)) \leq \beta, \beta \geq 1 \right\}$, where $h(z) = z + \sum_{m=2}^{\infty} |a_m|z^m$, $g(z) = \sum_{m=1}^{\infty} |b_m|z^m$ with $|b_1| < 1$. Now if we take $b_1 = 0$, then we get the class

$$R_{H_0}^{\beta}(\beta) := \left\{ f = h + \bar{g} : R(h'(z) + g'(z)) \leq \beta, \beta > 1 \right\},$$

comprising of functions with positive coefficients and reduces to the class $R(\beta)$ explored by Uralegaddi for the case $\beta \equiv 0$. Altinkaya et al. (2018) studied the class $k - \tilde{S}_q^T(z)$ of functions in $A$ with negative coefficients associated with the conic domains defined by $R(p(z)) > k[p(z) - 1] + \alpha$, where $0 \leq k < \infty$, $0 \leq \alpha < 1$, $0 < q < 1$, $p(z) = z(D_qf)(z)/f(z)$ and

$$(D_qf)(z) = \left\{ \begin{array}{ll} f(qz) - f(q^{-1}z) & , \quad z \neq 0 \\ (q - q^{-1})z & , \quad z = 0 \end{array} \right. \quad \text{and} \quad [\bar{m}]_q = q^m - q^{-m}/q - q^{-1}.$$
2 Bohr-Radius for the Class $W^0_H(\mu, \rho)$

Let us recall the subclass of close-to-convex harmonic mappings introduced by Rajbala and Prajapat (2020):

$$W^0_H(\mu, \rho) := \{ f = h + g \in H_0 : (h(\bar{z}) + \mu h''(\bar{z}) - \rho) > |g'(z) + \mu g''(z)| \},$$

where $\mu \geq 0$ and $0 \leq \rho < 1$.

Lemma 1 (Rajbala and Prajapat 2020) let $f = h + g \in W^0_H(\mu, \rho)$. Then for $m \geq 2$ the following sharp inequality holds:

$$|a_m| + |b_m| \leq \frac{2(1 - \rho)}{m(1 + \mu(m - 1))}.$$

Lemma 2 (Rajbala and Prajapat 2020) let $f = h + g \in W^0_H(\mu, \rho)$. Then for $|z| = r$, we have the sharp inequality

$$|f(z)| \leq |z| - 2 \sum_{m=2}^{\infty} \frac{(-1)^{m-1}(1 - \rho)}{m(1 + \mu(m - 1))} |z|^m.$$

Theorem 1 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \in W^0_H(\mu, \rho)$. Then

$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|) |z|^m \leq d(f(0), \partial f(\mathbb{D}))$$

for $|z| \leq r^*$, where $r^*$ is the unique positive root in $(0, 1)$ of

$$r + \sum_{m=2}^{\infty} \frac{2(1 - \rho)}{m(1 + \mu(m - 1))} r^m = 1 - 2 \sum_{m=2}^{\infty} \frac{(-1)^{m-1}(1 - \rho)}{m(1 + \mu(m - 1))}.$$

The radius $r^*$ is the Bohr radius for the class $W^0_H(\mu, \rho)$.

Proof From Lemma 2, it follows that the distance between origin and the boundary of $f(\mathbb{D})$ satisfies

$$d(f(0), \partial f(\mathbb{D})) \geq \left(1 - 2 \sum_{m=2}^{\infty} \frac{(-1)^{m-1}(1 - \rho)}{m(1 + \mu(m - 1))}\right). \quad (2)$$

Let us consider the continuous function

$$H(r) := r + \sum_{m=2}^{\infty} \frac{2(1 - \rho)}{m(1 + \mu(m - 1))} r^m - \left(1 - 2 \sum_{m=2}^{\infty} \frac{(-1)^{m-1}(1 - \rho)}{m(1 + \mu(m - 1))}\right).$$

Now

$$H'(r) = 1 + \sum_{m=2}^{\infty} \frac{2m(1 - \rho)}{m(1 + \mu(m - 1))} r^{m-1} > 0$$

for all $r \in (0, 1)$, which implies that $H$ is a strictly increasing continuous function. Note that $H(0) < 0$ and

$$H(1) = \sum_{m=2}^{\infty} \frac{2(1 - \rho)}{m(1 + \mu(m - 1))} + \sum_{m=2}^{\infty} \frac{2(-1)^{m-1}(1 - \rho)}{m(1 + \mu(m - 1))} > 0.$$ 

Thus by Intermediate Value Theorem for continuous function, we let $r^*$ be the unique root of $H(r) = 0$ in $(0, 1)$. Now using Lemma 1 and the inequality (2), we have

$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|) |z|^m \leq r$$

$$+ \sum_{m=2}^{\infty} \frac{2(1 - \rho)}{m(1 + \mu(m - 1))} r^m \leq r^*$$

$$+ \sum_{m=2}^{\infty} \frac{2(-1)^{m-1}(1 - \rho)}{m(1 + \mu(m - 1))}(r^*)^m$$

$$\leq \left(1 - 2 \sum_{m=2}^{\infty} \frac{(-1)^{m-1}(1 - \rho)}{m(1 + \mu(m - 1))}\right) \leq d(f(0), \partial f(\mathbb{D})),$$

which hold for $r \leq r^*$. Now consider the analytic function

$$f(z) = z + \sum_{m=2}^{\infty} \frac{2(-1)^{m-1}(1 - \rho)}{m(1 + \mu(m - 1))} z^m.$$

Then clearly $f \in W^0_H(\mu, \rho)$ and at $|z| = r^*$, we get

$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|) |z|^m = d(f(0), \partial f(\mathbb{D}))$$. Hence the radius $r^*$ is the Bohr radius for the class $W^0_H(\mu, \rho)$. \hfill \Box

Now using Theorem 1, we can obtain the Bohr radius for the classes $W^0_H(\mu, 0) = W^0(\mu), W^0_H(0, \rho) = P^0_H(\rho)$, $W^0_H(1, 0) = W^0_H$ and $W^0_H(0, 0) = P^0_H$. Here we mention the following:

Corollary 1 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \in P^0_H$. Then

$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|) |z|^m \leq d(f(0), \partial f(\mathbb{D}))$$

for $|z| \leq r^*$, where the Bohr radius $r^*$ is the unique positive root in $(0, 1)$ of

$$r + \sum_{m=2}^{\infty} \frac{2}{m} r^m = 1 - 2 \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m}.$$ \quad (3)

For the case $g = 0$, the class $W^0_H(\mu, \rho)$ reduces to $W(\mu, \rho)$, the class introduced by Gao and Zohu (2007) and we have:

Corollary 2 The Bohr radius of the classes $W(\mu, \rho)$ and $W^0_H(\mu, \rho)$ is same.
3 The Class $B_0^2(\mathcal{M})$ and its Applications

**Theorem 2** (Growth Theorem) Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \in B_0^2(\mathcal{M})$. Then
$$r - \frac{M}{a_2} r^2 \leq |f(z)| \leq r + \frac{M}{a_2} r^2 \quad (|z| = r).$$

The inequalities are sharp for the functions $f(z) = z + \frac{M}{a_2} z^2$ and $f(z) = z + \frac{M}{a_2} z^2$ with the suitable choice of $a_2$.

**Proof** From the condition (1), we see that
$$\sum_{m=2}^{\infty} (\gamma_m |a_m| + \delta_m |b_m|) \leq M$$
implies
$$\sum_{m=2}^{\infty} (|a_m| + |b_m|) \leq \frac{M}{a_2},$$
(5)
where $\gamma_m$, $\delta_m$, and $a_2$ are as defined in (1) and the equality in (5) holds for the function $f(z) = z + (M/a_2) z^2$. Now using the inequality $|a| - |b| \leq |a \pm b| \leq |a| + |b|$ and then (5), we have for $|z| = r$
$$|z| - \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \leq |f(z)| \leq |z|$$
$$+ \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m,$$
which immediately yields the required inequality. \hfill \square

From the above result, we also see that functions in the class $B_0^2(\mathcal{M})$ are bounded. Now taking $r$ tending to $1^{-}$, we get the covering theorem:

**Corollary 3** (Covering Theorem) Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \in B_0^2(\mathcal{M})$. Then
$$\left\{ w \in \mathbb{C} : |w| \leq 1 - \frac{M}{a_2} \right\} \subset f(\mathbb{D}).$$

Now, we are ready to obtain the Bohr-radius for the class $B_0^2(\mathcal{M})$.

**Theorem 3** Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \in B_0^2(\mathcal{M})$. Then
$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|) |z|^m \leq d(f(0), \partial f(\mathbb{D}))$$
for $|z| \leq r^*$, where
$$r^* = \frac{-1 + \sqrt{1 + 4 \left( \frac{M}{a_2} \right) \left( 1 - \frac{M}{a_2} \right)}}{2 \left( \frac{M}{a_2} \right)}.$$
Bohr radius $r^*$ is achieved by the function $f(z) = z - \frac{M}{a_2} z^2$.

**Proof** From the Growth Theorem 2 and Corollary 3(Covering Theorem), we see that the distance between origin and the boundary of $f(\mathbb{D})$ satisfies
$$d(f(0), \partial f(\mathbb{D})) \geq 1 - \frac{M}{a_2}.$$ (6)

Let us consider the continuous function defined in $(0, 1)$ as
$$H(r) := r + \frac{M}{a_2} r^2 - \left( 1 - \frac{M}{a_2} \right),$$
such that $H'(r) > 0$ for $r \in (0, 1)$ with $H(0)<0$ and $H(1) > 0$. Therefore, by Intermediate Value Theorem for continuous functions, we let $r^*$ be the unique positive root in $(0, 1)$ as mentioned in the Theorem statement. Thus for $r = r^*$ we have
$$r^* + \frac{M}{a_2} (r^*)^2 = 1 - \frac{M}{a_2}.$$ Now from (5), (6) and the above equality, it follows that
$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|) |z|^m \leq r + \frac{M}{a_2} r^2 \leq r^* + \frac{M}{a_2} (r^*)^2$$
$$d(f(0), \partial f(\mathbb{D})) \leq d(f(0), \partial f(\mathbb{D}))$$
for $|z| = r \leq r^*$. Let us consider the analytic function $f : \mathbb{D} \rightarrow \mathbb{C}$
$$f(z) = z - \frac{M}{a_2} z^2,$$
which by suitably choosing $a_2$ and using (4) belongs to $B_0^2(\mathcal{M})$. Further, for $|z| = r^*$ we have
$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|) |z|^m = d(f(0), \partial f(\mathbb{D})).$$
Hence the result is sharp. \hfill \square

**Remark 1** Note that we can extend our results by considering the analytic functions of the following form:
$$h(z) = z + \sum_{m=k}^{\infty} a_m z^m \quad \text{and} \quad g(z) = \sum_{m=k}^{\infty} b_m z^m, \quad (k \geq 2)$$
and the class

$$B_k(\mathcal{M}) = \left\{ f(z) = z + \sum_{m=k}^{\infty} a_m z^m + \sum_{m=k}^{\infty} b_m z^m : f(z) \in B_0(\mathcal{M}) \right\}$$
Corollary 4 Let \( f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \in S_+^\ast(C, D) \). Then
\[
r - \frac{D - C}{2(1 + D)} r^2 \leq |f(z)| \leq r + \frac{D - C}{2(1 + D)} r^2 \quad (|z| = r)
\]
and
\[
|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|)|z|^m \leq d(f(0), \partial f(\mathbb{D}))
\]
for \(|z| \leq r^*\), where
\[
r^* = -1 + \sqrt{1 + 4 \left( \frac{D - C}{2} \right) \left( \frac{D - C}{2} \right)}^2 \frac{2 (D - C)}{1 + D - C^2},
\]
where \(x_2\) is as defined in (7). Bohr radius \(r^*\) is achieved by the function \(f(z) = z - \frac{D - C}{2(1 + D)} z^2\).

Proof Cho and Dziok (2020) showed that \(f \in S_+^{**}(C, D)\) if and only if
\[
\sum_{m=2}^{\infty} (|a_m| + |b_m|) \leq D - C,
\]
where
\[
z_m = m(1 + D) - \frac{(1 + C)(1 - (-1)^m)}{2}
\]
and
\[
\beta_m = m(1 + D) + \frac{(1 + C)(1 - (-1)^m)}{2}.
\]
Note that for all \(m \geq 2\), we have \(z_m < \beta_m\) which shows that \(0 < x_2 \leq x_m < \beta_m\). Therefore from (7) we obtain that
\[
\sum_{m=2}^{\infty} (|a_m| + |b_m|) \leq D - C
\]
and also the condition in (1) holds by choosing \(\gamma_m = x_m\), \(\delta_m = \beta_m\) and \(M = D - C\). Thus using Theorem 2 and Theorem 3, we get the result. \(\square\)

Corollary 5 Let \(f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \in S_+^{**}(C, D)\). Then
\[
r - \frac{D - C}{1 + 2D - C} r^2 \leq |f(z)| \leq r + \frac{D - C}{1 + 2D - C} r^2, \quad (|z| = r)
\]
and
\[
|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|)|z|^m \leq d(f(0), \partial f(\mathbb{D}))
\]
for \(|z| \leq r^*\), where
\[
r^* = -1 + \sqrt{1 + 4 \left( \frac{D - C}{2(1 + D)} \right) \left( 1 - \frac{D - C}{2(1 + D)} \right)}^2 \frac{2 (D - C)}{1 + D - C^2}.
\]
Bohr radius \(r^*\) is achieved by the function \(f(z) = z - \frac{D - C}{2(1 + D)} z^2\).

Proof Dziok (2015) proved that \(f \in S_+^{**}(C, D)\) if and only if
\[
\sum_{m=2}^{\infty} (|a_m| + |b_m|) \leq D - C
\]
and \(f \in S_+^{**}(C, D)\) if and only if
\[
\sum_{m=2}^{\infty} (m |a_m| + m |b_m|) \leq D - C,
\]
where \(z_m = m(1 + D) - (1 + C)\) and \(\beta_m = m(1 + D) + (1 + C)\). We note that \(\beta_m > z_m \geq x_2 > 0\) for all \(m \geq 2\). Therefore from (8), we obtain that
\[
\sum_{m=2}^{\infty} (|a_m| + |b_m|) \leq D - C
\]
and
\[
\sum_{m=2}^{\infty} (m |a_m| + m |b_m|) \leq D - C.
\]
Now choosing \(\gamma_m = x_m\), \(\delta_m = \beta_m\) and \(M = D - C\), the condition (1) holds and thus using Theorem 2 and Theorem 3, we get the desired radius. \(\square\)

Again using \(\beta_m > x_m \geq x_2\) for all \(m \geq 2\) in (9), we obtain that if \(f \in S_+^{**}(C, D)\) then the following inequality holds:
\[
\sum_{m=2}^{\infty} (|a_m| + |b_m|) \leq D - C.
\]
Now choosing \( \gamma_m = m \xi_m \), \( \delta_m = m \eta_m \) and \( M = D - C \), the condition (1) holds and thus using Theorem 2 and Theorem 3, we obtain the following result:

**Corollary 6** Let \( f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \in S_1^*(C, D) \). Then

\[
r - \frac{D - C}{2(1 + 2D - C)} r^2 \leq |f(z)| \leq r + \frac{D - C}{2(1 + 2D - C)} r^2, \quad (|z| = r)
\]

and

\[
|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|) r^m \leq d(f(0), \partial f(D))
\]

if \( \gamma = \frac{\lambda}{2} \). Therefore, applying Theorem 2 and Theorem 3, we get the following result:

If \( g = 0 \) then \( \mathcal{F}_0^{\lambda}(\gamma) \) reduces to the (Singh 1977) class \( \mathcal{F}(\lambda) \), which is contained in the MacGregor subclass \( \mathcal{F} = \{ f \in A : |f'(z) - 1| < 1 \} \) of close-to-convex functions.

**Corollary 8** Bohr radius for the classes \( \mathcal{F}(\lambda) \) and \( \mathcal{F}_0^{\lambda}(\gamma) \) is same, whenever the condition (10) holds.

The following two corollaries are for the classes \( k - \mathcal{S}_q^{-}\left(\alpha\right) \) and \( R_0^\lambda(\beta) \) respectively:

**Lemma 3** (Altinkaya et al. 2018) Let \( 0 \leq k < \infty, 0 < q < 1 \) and \( 0 \leq \alpha < 1 \). Then \( f \in k - \mathcal{S}_q^{-}\left(\alpha\right) \) if and only if

\[
\sum_{m=2}^{\infty} (|\overline{a}_m| k + 1 - (k + \alpha)) a_m \leq 1 - \alpha.
\]

From Lemma 3, we see that choosing \( \gamma_m = (\overline{a}_m)(k + 1) - (k + \alpha) \), \( \delta_m = 0 \) and \( M = 1 - \alpha \), condition in (1) holds. Therefore, applying Theorem 2 and Theorem 3, we get the following result:

**Corollary 9** Let \( f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in k - \mathcal{S}_q^{-}\left(\alpha\right) \). Then

\[
r - \frac{q(1 - \alpha)}{(q^2 + 1)(k + 1) - q(k + \alpha)} r^2 \leq |f(z)| \leq r + \frac{q(1 - \alpha)}{(q^2 + 1)(k + 1) - q(k + \alpha)} r^2,
\]

where \( |z| = r \) and

\[
|z| + \sum_{m=2}^{\infty} |a_m| r^m \leq d(f(0), \partial f(D))
\]

for \( |z| \leq r^* \), where

\[
r^* = -1 + \sqrt{1 + 2 \lambda \left(1 - \frac{q}{q^2 + 1} \right)}
\]

and the Bohr radius \( r^* \) for \( \mathcal{F}_0^{\lambda}(\gamma) \) is obtained when \( f(z) = z - \frac{q}{q^2 + 1} z^2 \).

From Lemma 4, we see that choosing \( \gamma_m = \delta_m = m \) and \( M = \beta - 1 \), condition in (1) holds. Therefore applying Theorem 2 and Theorem 3, we get the following result:
Corollary 10 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=2}^{\infty} b_m z^m \in R^0_{\beta}(\beta)$. Then

$$r - \frac{\beta - 1}{2} r^2 \leq |f(z)| \leq r + \frac{\beta - 1}{2} r^2, \quad (|z| = r)$$

and

$$|z| + \sum_{m=2}^{\infty} (|a_m| + |b_m|) |z|^m \leq d(f(0), \partial f(D))$$

for $|z| \leq r^*$, where

$$r^* = \frac{2(\gamma_2 + \alpha - 1)}{\gamma_2 + \sqrt{\gamma_2^2 + 4\gamma_2(1 - \alpha) - 4(1 - \alpha)^2}}$$

and the Bohr radius $r^*$ is obtained when $f(z) = z + \frac{\beta - 1}{2} z^2$.

Corollary 11 Bohr radius for the classes $R(\beta)$ and $R^0_{\beta}(\beta)$ is same.

Silverman considered the class with negative coefficients as follows:

$$T := \left\{ f \in S : f(z) = z - \sum_{m=2}^{\infty} a_m z^m, a_m \geq 0 \right\}.$$ 

Using this recently, (Ali et al. 2019) considered the following general class defined as:

$$T(\alpha) := \left\{ f \in T : \sum_{m=2}^{\infty} g_m a_m \leq 1 - \alpha \right\},$$

where $g_m \geq g_2 > 0$ and $\alpha < 1$. Note that if we choose in $T(\alpha)$, $\gamma_m = g_m$, $\delta_m = 0$ and $M = 1 - \alpha$, then the class $B^0_{M}(M)$ contains $T(\alpha)$, which satisfies the required conditions. Thus using Theorem 2 and Theorem 3, we have the following result obtained in Ali et al. (2019):

Theorem 4 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in T(\alpha)$. Then

$$r - \frac{1 - \alpha}{\gamma_2} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{\gamma_2} r^2, \quad (|z| = r)$$

and

$$|z| + \sum_{m=2}^{\infty} |a_m||z|^m \leq d(f(0), \partial f(D))$$

for $|z| \leq r^*$, where

$$r^* = \frac{2(\gamma_2 + \alpha - 1)}{\gamma_2 + \sqrt{\gamma_2^2 + 4\gamma_2(1 - \alpha) - 4(1 - \alpha)^2}}$$

Choosing $\gamma_m = m - \alpha$ and $\gamma_m = m(m - \alpha)$ in (1), the class $B^0_{\gamma}(\gamma)$ contains all the classes $B_{\gamma}(\gamma)$ and $B^0_{\gamma}(\gamma)$.

Theorem 4 and Theorem 5.

Ali et al. (2019) considered the class $T_{\beta} := T \cap T_{\beta}$, $0 \leq \alpha \leq 1$, where $T_{\beta}$ is the class of close to convex functions and showed that if $f \in T_{\beta}$, then

$$\sum_{m=2}^{\infty} m(2n + \alpha)a_m \leq 2 + \alpha.$$ 

Thus choosing $\gamma_m = m(2n + \alpha)$ and $\delta_m = 0$, it follows from (1), the class $T_{\beta}$ satisfies all conditions of $B^0_{\gamma}(\gamma)$, and Theorem 3 (or Theorem 4) reduces to (Ali et al. 2019, Corollary 2.6).

For $\alpha > 1$, (Owa and Nishiwaki 2002) considered the classes of analytic functions $M(\alpha) := \{ f \in A : R(f^0(z)/f(z)) < \alpha \}$ and $N(\alpha) := \{ f \in A : f(f^0(z)) < \alpha \}$. They showed that the conditions:

$$\sum_{m=2}^{\infty} \left( (m - \mu) + |m + \mu - 2n| \right)|a_m| \leq 2(\alpha - 1)$$

and

$$\sum_{m=2}^{\infty} m(m - \mu - 1 + |m + \mu - 2n|)|a_m| \leq 2(\alpha - 1),$$

where $0 \leq \mu \leq 1$ are sufficient for the function $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ to be in $M(\alpha)$ and $N(\alpha)$ respectively. It is easy to see that the above two conditions also become necessary for the classes $T_M(\alpha) := T \cap M(\alpha)$ and $T_N(\alpha) := T \cap N(\alpha)$ respectively. Thus choosing $\gamma_m = (m - \mu) + |m + \mu - 2n|$, $\gamma_m = m(m - \mu + 1 + |m + \mu - 2n|)$ with $\delta_m = 0$ and $M = 2(\alpha - 1)$, from Theorem 2 and Theorem 3 we obtain the following result respectively:

Corollary 12 Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in T_M(\alpha)$ (or $T_N(\alpha)$). Then

$$r - \frac{2(\alpha - 1 - \gamma_2 r^2 \leq |f(z)| \leq r + \frac{2(\alpha - 1 - \gamma_2)}{\gamma_2} r^2, \quad (|z| = r),$$

where $\gamma_2 := (2 - \mu) + |(m - \mu)(1 - \alpha)|$ (or $2(3 - \mu + |\mu - 2(1 - \alpha)|)$) and

$$|z| + \sum_{m=2}^{\infty} |a_m||z|^m \leq d(f(0), \partial f(D))$$

for $|z| \leq r^*$, where

$$r^* = \sqrt{1 + \frac{4(2(\alpha - 1 - \gamma_2))}{\gamma_2}(1 - \frac{2(\alpha - 1 - \gamma_2)}{\gamma_2})}.$$
4 Conclusion and Future Scope

We studied the Bohr phenomenon for the class $B^p_0(M)$ and pointed out its several applications in context of various known classes. Further, the Bohr radius for the classes of $q$-starlike and $q$-convex functions studied in Altintas and Mustafa (2019) can be obtained by mere an application of our result. Similarly many other classes can also be dealt for Bohr radius. Since $S^+_H(C, D) \subseteq S^+_H(C, D)$, $S^+(C, D) \subseteq S^+(C, D)$ and $S^+(C, D) \subseteq S^+(C, D)$. If we let $r_*$ be the Bohr radius for a well defined class $F$, then we conclude that $r_* \leq r^*$ whenever $F = S^+_H(C, D)$ or $S^+_H(C, D)$ or $S^+_H(C, D)$. However, finding $r_*$ is still open.

Acknowledgement We are thankful to the Editor and the Reviewers for their valuable suggestions to improve the previous version of this manuscript. All authors contributed Equally.

Funding The work of Kamajeet Gangania is supported by University Grant Commission, New-Delhi, India under UGC-Ref. No.:1051/ (CSIR-UGC NET JUNE 2017).

Data availability None

Declarations

Conflict of interest The authors declare that they have no conflict of interest

References

Ahuja OP, Anand S, Jain NK (2020) Bohr radius problems for some classes of analytic functions using quantum calculus approach. Mathematics 8:623
Ali RM, Barnard RW, Solybin AYu (2017) A note on the Bohr’s phenomenon for power series. J Math Anal Appl 449:154–167
Ali RM, Jain NK, Ravichandran V (2019) Bohr radius for classes of analytic functions. Results Math 74(179):13
Altinkaya S, Kanas S, Yalcin S (2018) Subclass of $-\alpha$-uniformly starlike functions. J Math Anal Appl 420:124–136
Altinkaya S, Kilisalan M, Yalcin S (2018) Subclass of $-\alpha$-uniformly starlike functions defined by symmetric -derivative operator. Ukrains’kyi Matematychnyi Zhurnal 70(11):1499–1510
Altintas O, Mustafa N (2019) Coefficient bounds and distortion theorems for the certain analytic functions. Turkish J Math 43:985–997
Boas HP, Khavinson D (1997) Bohr’s power series theorem in several variables. Proc Amer Math Soc 125:2975–2979
Bohr H (1914) A theorem concerning power series. Proc London Math Soc 13:1–5
Cho NE, Dziok J (2020) Harmonic starlike functions with respect to symmetric points. Axioms 9:3
Dixit KK, Porwal S (2010) A subclass of harmonic univalent functions with positive coefficients. Tamkang J Math 41(3):261–269
Dziok J (2015) On Janowski harmonic functions. J Appl Anal 21(2):99–107
Dziok J (2019) Classes of harmonic functions associated with Ruscheweyh derivatives. RACSAM 113:1315–1329
Gangania K, Kumar SS, Bohr-Rogosinski phenomenon for $S^*(\psi)$ and $C(\psi)$. Mediterr J Math (To Appear). arXiv:2105.08684v1
Gao CY, Zoub SQ (2007) Certain subclass of starlike functions. Appl Math Comput 187:176–182
Ghosh N, Vasudevarao A (2019) On a subclass of harmonic close-to-convex mappings. Monatsh Math 188(2):247–267
Kayumov IR, Ponnusamy S (2017) Bohr inequality for odd analytic functions. Comput Methods Funct Theory 17:679–688
Kayumov IR, Ponnusamy S (2018) Improved version of Bohr’s inequality. C R Math Acad Sci Paris 356:272–277
Kumar SS, Gangania K (2020) On certain Generalizations of $S^*(\psi)$. arXiv:2007.06069v1
Kumar SS, Gangania K (2021) On geometrical properties of certain analytic functions. Iran J Sci Technol Trans Sci 45:1437–1445. https://doi.org/10.1007/s40995-021-01116-1
Liu M, Ponnusamy S, Wang J (2020) Bohr’s phenomenon for the classes of quasi-subordination and $K$-quasiregular harmonic mappings. RACSAM 114(115):15
Ma WC, Minda D (1992) A unified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis, Tianjin, Conf Proc Lecture Notes Anal., I Int Press, Cambridge, MA. 157–169
Muhanna YA (2010) Bohr’s phenomenon in subordination and bounded harmonic classes. Complex Var Elliptic Equ 55:1071–1078
Muhanna YA, Ali RM, Ng ZC, Hansi SFM (2014) Bohr radius for subordinating families of analytic functions and bounded harmonic mappings. J Math Anal Appl 420:124–136
Muhanna YA, Ali RM, Ponnusamy S (2016) On the Bohr inequality. In: Progress in Approximation Theory and Applicable Complex Analysis (Edited by N.K. Govil et al.), Springer Optimization and Its Applications. 117:265–295
Nagpal S, Ravichandran V (2014) A subclass of close-to-convex harmonic mappings. Complex Var Elliptic Equ 59(2):204–216
Nagpal S, Ravichandran V (2014) Construction of subclasses of univalent harmonic mappings. J Korean Math Soc 53:567–592.
Owa S, Nishiwaki J (2002) Coefficient estimates for certain classes of analytic functions. J Inequal Pure Appl Math, 3(5), Article 72, 5
Rajbala, Prajapat JK (2020) Certain geometric properties of close-to-convex harmonic mappings. https://doi.org/10.1142/ S1793557121501023
Sakaguchi K (1959) On a certain univalent mapping. J Math Soc Jpn 5:72–75
Silverman H (1998) Harmonic univalent functions with negative coefficients. J Math Anal Appl 220:283–289
Singh V (1977) Univalent functions with bounded derivative in the unit disc. Ind J Pure Appl Math 8:1370–1377