On the maximal autocorrelation of Rudin-Shapiro sequences

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Abstract
In this paper, we prove that the maximal aperiodic autocorrelation of the $m$-th Rudin-Shapiro sequence is of the same order as $\lambda^m$, where $\lambda$ is the real root of $x^3 + x^2 - 2x - 4$. This proof was developed independently of the recently published proof given by Katz and van der Linden (2021). A proof of this result for the related periodic autocorrelation is given by Allouche, Choi, Denise, Erdélyi, and Saffari (2019) and Choi (2020) using a translation of the problem into linear algebra. Our approach modifies this linear algebraic translation to deal with aperiodic autocorrelation and provides an alternative method of dealing with the computations given by Choi.

Keywords: Shapiro polynomials, Golay-Rudin-Shapiro sequence, Autocorrelation, Trigonometric polynomials

1. Introduction
Consider a length-$\ell$ sequence $a = (a_0, a_1, \ldots, a_\ell) \in \mathbb{R}_\ell$. We define the aperiodic (sometimes called “acyclic”) autocorrelation of $a$ at shift $k$ to be

$$\sum_{i=0}^{\ell} a_i a_{i+k}$$

where $a_i = 0$ for $i \notin [1, \ell]$. Likewise, we can define the periodic (sometimes called “cyclic”) autocorrelation of $a$ at shift $k$ to be the same sum but taking $a_i = a_{i \mod \ell}$ for all $i \in \mathbb{Z}$. For instance, if we take $a = (1, 2, 3, 4)$, we have

aperiodic autocorrelation of $a$ at shift 2:

\[
\begin{array}{cccc}
  1 & 2 & 3 & 4 \\
  \times & 1 & 2 & 3 & 4 \\
  3 & 8 \\
\end{array}
\]

\[3 + 8 = 11\]

periodic autocorrelation of $a$ at shift 2:

\[
\begin{array}{cccc}
  1 & 2 & 3 & 4 \\
  \times & 3 & 4 & 1 & 2 \\
  3 & 8 & 3 & 8 \\
\end{array}
\]

\[3 + 8 + 3 + 8 = 22\]

where the multiplication is understood to be component-wise. Both notions of autocorrelation are measures of the similarity of a sequence to its translates. In some contexts (e.g. signal processing), it is desirable to find sequences with small autocorrelation at every shift. We will concentrate on studying the maximal aperiodic autocorrelation of Rudin-Shapiro sequences as they are known to have small mean square aperiodic autocorrelation (see [I]).

We define the $m$-th Rudin-Shapiro sequence $(a_0, a_1, \ldots, a_{2^m-1})$ by

$$a_i = (-1)^{\# \text{ of pairs of consecutive ones in the binary expansion of } i}.\]
Associated with the $m$-th Rudin-Shapiro sequence is the $m$-th Shapiro polynomial: $\sum_{i=0}^{2^m-1} a_i x^i$. Both objects are widely studied. It is worth noting that there are several popular definitions for the Rudin-Shapiro sequences which use recursion (see [1], [2], and [3]), some of which are formulated by first building the Shapiro polynomials and defining the Rudin-Shapiro sequences as the sequences of their coefficients (see [4] and [5]). We define

\[
C(m,k) = \text{aperiodic autocorrelation of the } m\text{-th Rudin-Shapiro sequence at shift } k
\]

\[
P(m,k) = \text{periodic autocorrelation of the } m\text{-th Rudin-Shapiro sequence at shift } k.
\]

It is well known that $C(m,k) = P(m,k) = 0$ for even $k$. We will prove the following:

**Theorem 1.** For all $m \in \mathbb{N}$, we have

\[
C_1 \lambda^m \leq \max_k |C(m,k)| \leq C_2 \lambda^m
\]

where $C_1, C_2$ are absolute constants and $\lambda \approx 1.65$ is the largest root of $x^3 + x^2 - 2x - 4$.

In [5], Katz and van der Linden prove this with the best possible $C_2$ (being approximately 0.66) using the language of algebraic number theory. Theorem 1 holds true for $P(m,k)$ in place of $C(m,k)$ and this is proven in [4] and [6]. The idea of this proof begins in [7], in which Taghavi translates the problem of showing $\max_k |P(m,k)| \ll \lambda^m$ into the problem of showing

\[
\max_{(a_1, a_2, \ldots, a_m)} \left| \prod_{i=1}^{m} T_{a_i} \right| \ll \lambda^m
\]

for $(a_1, a_2, \ldots, a_m) \in \{1, 2, 3, 4\}^m$ and some $T_{a_i} \in \mathbb{Z}^{3 \times 3}$. Allouche et al. extend this idea in [4] by reducing the number of matrices considered in the product, which led to showing

\[
\max_k |P(m,k)| \ll (1.00000100000025 \lambda)^m
\]

which is very close to the desired result. Additionally, the lower bound of Theorem 1 adapted to $P(m,k)$ is proven in [6]. Finally, in [6], Choi uses these advances to conclude that $\max_k |P(m,k)| \ll \lambda^m$ where the implied constant is approximately 3.78, proving the upper bound of Theorem 1 for $P(m,k)$. This result on $P(m,k)$ is used to establish results on the oscillation of the modulus of Shapiro polynomials on the unit circle (see [8]). Our proof for $C(m,k)$ was developed independently of the proof of Katz and van der Linden. We use roughly the same ideas as in [4], [6], and [7], although constants are left implicit and the crux of our computations given in Lemma 4 is simpler than those of the computations given in [6] and [5], so we have a more easily verifiable proof of Theorem 1 at the expense of precision – for explicit bounds, see [5]. We follow this with a couple of conjectures on which $k$ gives maximal $C(m,k)$ for fixed $m$. We begin with some preliminary work for the proof of Theorem 1.

**2. Proof of the Theorem**

The proof of Theorem 1 is as follows. First, we construct a vector

\[
v = \left[ C(m,k) \right.
\]

where all components depend on $m$ and $k$. Next, we derive a decomposition of $v$ as a matrix-vector product:

\[
v = \prod_{i=1}^{m} U_i^{a_i} V_i^{b_i} \cdot \left[ \begin{array}{c} 1 \\ -1 \\ 1 \end{array} \right]
\]
for some $U, V \in \mathbb{Z}^{3 \times 3}$ and $(u_i, v_i) \in \{(1, 0), (0, 1)\}$. The lower bound in Theorem 1 is proven quickly using the diagonalization of $U$. Finally, we prove the upper bound by showing

$$\max_{u_i, v_i} \left\| \prod_{i=1}^{m} U^{u_i} V^{v_i} \right\|_2 \ll \lambda^m$$

and using the fact that

$$|C(m, k)| \leq \|v\|_2 \ll \max_{u_i, v_i} \left\| \prod_{i=1}^{m} U^{u_i} V^{v_i} \right\|_2.$$

We begin by defining our vector “$v$”. For $m \in \mathbb{N}$ we fix $k_m$ so that $0 \leq k_m \leq 2^m$ and $k_m$ is odd, and define

$$k'_m = |2^m - k_m|$$

$$k_{m-1} = \begin{cases} k_m & \text{if } k_m \leq 2^{m-1} \\
 k'_m & \text{else} \end{cases}.$$

We turn our attention to

$$v_m := \begin{bmatrix} C(m, k_m) \\
 C(m, k'_m) \\
 C(m-1, k_{m-1}) \end{bmatrix}.$$

**Lemma 2.** For $m \geq 3$ and

$$M = \begin{bmatrix} 0 & 1 & 2 \\
 0 & -1 & 2 \\
 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\
 1 & 0 & 0 \\
 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 0 \end{bmatrix},$$

we have that

$$v_m = \pm \left( \prod_{i=3}^{m} A^{a_i} MB^{b_i} \right) \begin{bmatrix} 1 \\
 -1 \\
 1 \end{bmatrix}$$

where $a_i, b_i \in \{0, 1\}$ for all $i$.

**Proof.** Let $m \geq 3$ and $0 \leq k_m \leq 2^m$. Høholdt, Jensen, and Justesen in Theorem 2.2 of [1] showed that

$$C(m, k_m) = C(m-1, 2^{m-1} - k_m) \quad \text{if } k_m \in (0, 2^{m-2}) \quad \text{(1)}$$

$$C(m, k_m) = C(m-1, 2^{m-1} - k_m) + 2C(m-2, 2^{m-1} - k_m) \quad \text{if } k_m \in (2^{m-2}, 2^{m-1}) \quad \text{(2)}$$

$$C(m, k_m) = -C(m-1, k_m - 2^{m-1}) + 2C(m-2, k_m - 2^{m-1}) \quad \text{if } k_m \in (2^{m-1}, 3 \cdot 2^{m-2}) \quad \text{(3)}$$

$$C(m, k_m) = -C(m-1, k_m - 2^{m-1}) \quad \text{if } k_m \in (3 \cdot 2^{m-2}, 2^m). \quad \text{(4)}$$

Define $S^m_n = (n2^{m-2}, (n+1)2^{m-2})$ for $0 \leq n \leq 3$. We will suppress notation and denote $S^m_n$ as $S_n$. 


Let $k_m \in S_0$. We see that $k_{m-1} = k_m$, which implies $k_{m-1}' = 2^{m-1} - k_{m-1} = 2^{m-1} - k_m$. Using this along with the relations above, we get
\[ C(m, k_m) = C(m - 1, k_{m-1}') \quad \text{using (1)} 
\]
\[ C(m, k_m') = -C(m - 1, k_{m-1}') \quad \text{using (1)} \]
Thus, we get
\[
v_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C(m, k_m) \\ C(m, k_m') \\ C(m - 2, k_{m-2}) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} v_{m-1}.
\]
Now, let $k_m \in S_1$. We see that $k_{m-1} = k_m$, so using (2) and (3) respectively, we get
\[ C(m, k_m) = C(m - 1, k_{m-1}') + 2C(m - 2, k_{m-1}') = C(m - 1, k_{m-1}') + 2C(m - 2, k_{m-2}) 
\]
\[ C(m, k_m') = -C(m - 1, k_{m-1}') + 2C(m - 2, k_{m-1}') = -C(m - 1, k_{m-1}') + 2C(m - 2, k_{m-2}). \]
Thus, we get
\[
v_m = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C(m, k_m) \\ C(m, k_m') \\ C(m - 2, k_{m-2}) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} v_{m-1}.
\]
Now, let $k_m \in S_2$. We see that $k_{m-1} = k_m'$, so using (3) and (2) respectively, we get
\[ C(m, k_m) = -C(m - 1, k_{m-1}') + 2C(m - 2, k_{m-1}') = -C(m - 1, k_{m-1}') + 2C(m - 2, k_{m-2}) 
\]
\[ C(m, k_m') = C(m - 1, k_{m-1}') + 2C(m - 2, k_{m-1}') = C(m - 1, k_{m-1}') + 2C(m - 2, k_{m-2}). \]
Thus, we get
\[
v_m = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C(m, k_m) \\ C(m, k_m') \\ C(m - 2, k_{m-2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} v_{m-1}.
\]
Now, let $k_m \in S_3$. We see that $k_{m-1} = k_m'$, so using (4) and (1) respectively, we get
\[ C(m, k_m) = -C(m - 1, k_{m-1}') 
\]
\[ C(m, k_m') = C(m - 1, k_{m-1}'). \]
Thus, we get
\[
v_m = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C(m, k_m) \\ C(m, k_m') \\ C(m - 2, k_{m-2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} v_{m-1}.
\]
We have shown that
\[ v_m = Tv_{m-1} \]
where
\[
T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]
for $k_m \in S_0, S_1, S_2, S_3$ respectively. Now, noting that each of these matrices is of the form $A^a MB^b$ for $a, b \in \{0, 1\}$ and applying this result inductively, we are done.

\[ \square \]

We may express $v_m$ more simply. In the notation of \textbf{Lemma 2} if $k_m \in S_0^m$, then $k_{m-1} \in S_0^{m-1}$ or $k_{m-1} \in S_1^{m-1}$. Thus, in the matrix-vector product given in \textbf{Lemma 2} $MB$ precedes $MB$ or $M$. Similarly, we derive the rest of the rules for multiplying the 4 matrices given in the proof of \textbf{Lemma 2}.

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• \( MB \) precedes \( MB \) or \( M \)
• \( M \) precedes \( AM \) or \( AMB \)
• \( AM \) precedes \( AM \) or \( AMB \)
• \( AMB \) precedes \( MB \) or \( M \).

Using these relations, we obtain the following lemma.

**Lemma 3.** For \( m \geq 3 \) and

\[
M = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

we have that

\[
v_m = \pm A^\delta_1 \left( \prod_{i=3}^{m} MA^{a_i} B^{b_i} \right) A^\delta_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\]

where \( \delta_1, \delta_2 \in \mathbb{N} \) and \((a_i, b_i) \in \{(0, 1), (1, 0)\} \). In other words, we may express \( v_m \) as an initial vector multiplied by a product of \( MA \) and \( MB \).

**Proof.** We proceed by induction on \( m \). The case \( m = 3 \) is trivial. Assume the appropriate inductive hypothesis. Then, for \( m \geq 4 \), we use Lemma 2 and our inductive hypothesis to get

\[
v_m = \pm A^a MB^b A^\delta_2 \left( \prod_{i=3}^{m-1} MA^{a_i} B^{b_i} \right) A^\delta_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}
\]

with \( a, b \in \{0, 1\} \). Note that \( B \) cannot precede \( A \) and \( AM \) must follow \( M \) given our multiplication rules, so \((b, \delta_i) \in \{(0, 1), (1, 0)\}\). Thus, the leftmost factor of our product is of the form \( A^a MB \) or \( A^a MA \) and we are done.

\[ \Box \]

**Remark 1.** Let

\[
S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
\]

In Lemma 3 of [4], the matrix \( M_1 \) we can see to be our \( SMAS \) and the matrix \( B \) we can see to be our \( SMBS \). Since \( S \) is an isometry and \( S^2 = I \), the matrix products in [4] and [6] are exactly the same as ours in norm. Due to this matrix similarity, we can pass to the calculations given in [6]. However, we will take a different approach to the computations through Lemma 4 and Lemma 5. Note that this may yield another proof to the main theorem in [6] (which we will not exhibit here) due to the similarity of matrices noted above. The nature of the proof of Lemma 4 is asymptotic and will require a check of several base cases. Throughout the rest of this paper, we denote \( \| \cdot \|_2 \) by \( \| \cdot \| \).

**Lemma 4.** We have

\[
\left\| \prod_{n=1}^{2} (MA)^{j_n} (MB)^{k_n} \right\| \leq \prod_{n=1}^{2} \lambda^{j_n+k_n}
\]

where \( j_n, k_n \geq 1 \).
PROOF. Note that for \( m \geq 2 \), we have \((MB)^m = \pm (MB)^2\), so we may assume \( k_n \leq 2 \) without loss of generality. We find the diagonalization of \((MA)^i\):

\[
(MA)^i = \begin{bmatrix}
2 - \lambda^2 & 2 - \alpha^2 & 2 - \alpha^2 \\
-\lambda & -\alpha & -\alpha \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\lambda^i & 0 & 0 \\
0 & \alpha^i & 0 \\
0 & 0 & \alpha^i
\end{bmatrix}
\begin{bmatrix}
1 & \alpha & \alpha \\
\alpha & \alpha & \alpha \\
\alpha & \alpha & \alpha
\end{bmatrix}^{-1}
\]

where \( \lambda = 1.65 \cdots \) and \( \alpha = -1.33 \cdots -0.80 \cdots i \) are roots of \( x^3 + x^2 - 2x - 4 \), and

\[
\Delta = \lambda - \alpha)(\alpha - \lambda) = \sqrt{-236}.
\]

Let

\[
\lambda_{3i} = \lambda, \ \lambda_{3i+1} = \alpha, \ \lambda_{3i+2} = \alpha
\]

for all \( i \in \mathbb{Z} \). Let \( T_{ij} \) denote the \( ij \)-th entry of a matrix \( T \) where \( i, j \geq 1 \). We see then that

\[
p_{i1} := \Delta(P^{-1})_{i1} = \lambda_{i+1} - \lambda_i
\]
\[
p_{i2} := \Delta(P^{-1})_{i2} = (\lambda_i - \lambda_{i+1})(\lambda_i + \lambda_{i+1})
\]
\[
p_{i3} := \Delta(P^{-1})_{i3} = (\lambda_i - \lambda_{i+1})(2 + \lambda_i \lambda_{i+1})
\]

where \( 1 \leq i \leq 3 \) so that \( \Delta P^{-1} = [p_{ij}] \). Note also that

\[
(MB)^{k_n} = \begin{bmatrix}
0 & -(2 - k_n) \alpha & 0 \\
0 & 0 & 0 \\
2 - k_n & k_n - 1 & 0
\end{bmatrix}
\]

as \( 1 \leq k_n \leq 2 \). We use this with the computations above to see that

\[
(MA)^i(MB)^k = \frac{(-1)^j}{\Delta} \begin{bmatrix}
(2 - k) \sum (2\lambda^i - \lambda^i - 1)p_{i3} - \frac{2(\lambda^i - 1)(\alpha^i + 1)}{\alpha^i}((1)^k(2 - k) + 1)(k - 1)p_{i3} & 0 \\
(2 - k) \sum -\frac{2(\lambda^i - 1)(\alpha^i + 1)}{\alpha^i}((1)^k(2 - k) + 1)(k - 1)p_{i3} & 0 \\
(2 - k) \sum \frac{2(\lambda^i - 1)(\alpha^i + 1)}{\alpha^i}((1)^k(2 - k) + 1)(k - 1)p_{i3} & 0
\end{bmatrix}
\]

where all summations are taken over \( 1 \leq i \leq 3 \). We now consider a matrix \( T \) so that

\[
(MA)^j(MB)^k = (-\lambda)^jT.
\]

Our goal is to show that \( \|(1/\lambda)T\| \leq 1 \) for sufficiently large \( j \) so that

\[
\|(MA)^j(MB)^k\| = \|(-\lambda)^jT\| \leq \lambda^{i+1} \leq \lambda^{i+k}
\]

for sufficiently large \( j \). We will rely on the Frobenius norm \( \|\cdot\|_F \) and make use of the well-known fact that

\[
\|T\|_F = \left( \sum_{i,j} |T_{ij}|^2 \right)^{1/2}.
\]

We find that

\[
|\Delta T_{32}| = \left| \sum_{i=1}^{3} \left( \frac{\lambda_{i-1}}{\lambda} \right)^j ((-1)^k(p_{i2} - p_{i1}) + (k - 1)p_{i3}) \right|
\]
\[
= \left| 2\text{Im}(\pi)((-1)^k(2\text{Re}(\alpha) + 1) + (k - 1)(2 + |\alpha|^2)) + 2\text{Im}\left( \left( \frac{\pi}{\lambda} \right)^j (\alpha - \lambda)((-1)^k(\alpha + \lambda + 1) + (k - 1)(2 + \alpha \lambda)) \right) \right|
\]
\[
\leq \left| 2\text{Im}(\alpha)(2\text{Re}(\alpha) + |\alpha|^2 + 3) + \left( \left( \frac{\pi}{\lambda} \right)^j (\alpha - \lambda)((-1)^k(\alpha + \lambda + 1) + (k - 1)(2 + \alpha \lambda)) \right) \right|.
\]
We go through the same calculations for the rest of the entries of $T$ and find that

$$|\Delta T_{11}| \leq |2(2 - \lambda^2)\text{Im}(\alpha)(2 + |\alpha|^2)| + \left| 2 \left( \frac{\pi}{\lambda} \right)^j (2 - \pi^2)(\alpha - \lambda)(2 + \alpha\lambda) \right|$$

$$|\Delta T_{21}| \leq |2\lambda\text{Im}(\alpha)(2 + |\alpha|^2)| + \left| 2 \left( \frac{\pi}{\lambda} \right)^j (2 - \pi^2)(\alpha - \lambda)(2 + \alpha\lambda) \right|$$

$$|\Delta T_{31}| \leq |2\text{Im}(\alpha)(2 + |\alpha|^2)| + \left| 2 \left( \frac{\pi}{\lambda} \right)^j (\alpha - \lambda)(2 + \alpha\lambda) \right|$$

$$|\Delta T_{12}| \leq |2(2 - \lambda^2)\text{Im}(\alpha)(2\text{Re}(\alpha) + |\alpha|^2 + 3)| + \left| 2 \left( \frac{\pi}{\lambda} \right)^j (2 - \pi^2)(\alpha - \lambda)((-1)^k(\alpha + \lambda + 1) + (k - 1)(2 + \alpha\lambda)) \right|$$

$$|\Delta T_{22}| \leq |2\lambda\text{Im}(\alpha)(2\text{Re}(\alpha) + |\alpha|^2 + 3)| + \left| 2 \left( \frac{\pi}{\lambda} \right)^j (\alpha - \lambda)((-1)^k(\alpha + \lambda + 1) + (k - 1)(2 + \alpha\lambda)) \right|$$

$$|\Delta T_{32}| \leq |2\text{Im}(\alpha)(2\text{Re}(\alpha) + |\alpha|^2 + 3)| + \left| 2 \left( \frac{\pi}{\lambda} \right)^j (\alpha - \lambda)((-1)^k(\alpha + \lambda + 1) + (k - 1)(2 + \alpha\lambda)) \right|. $$

We will focus again on bounding $|\Delta T_{32}|$. Note that when $k = 2$, we have

$$|(\alpha - \lambda)(\alpha + \lambda + 1 + (2 + \alpha\lambda))| = 7.460 \cdots$$

and when $k = 1$, we have

$$|(\alpha - \lambda)(\alpha + \lambda + 1)| = 4.804 \cdots$$

With this, we use rational approximations to get

$$\left| 2 \left( \frac{\pi}{\lambda} \right)^j (\alpha - \lambda)(\alpha + \lambda - 1 + (k - 1)(\alpha\lambda)) \right| \leq 2 \left| \frac{\alpha^j}{\lambda^j} \right| \cdot (7.461) \leq (0.936)^j(14.922).$$

Thus, we achieve the bound

$$|\Delta T_{32}| \leq |2\text{Im}(\alpha)(2\text{Re}(\alpha) + |\alpha|^2 + 3)| + (0.936)^j(14.922) \leq 4.416 + (0.936)^j(14.922).$$

Similarly, we get bounds for the rest of the $|\Delta T_{ij}|$ above and divide by $|\Delta|$ and $\lambda$ to get

$$|T_{11}/\lambda| \leq 0.210 + (0.936)^j(0.755)$$

$$|T_{21}/\lambda| \leq 0.462 + (0.936)^j(0.509)$$

$$|T_{31}/\lambda| \leq 0.278 + (0.936)^j(0.328)$$

$$|T_{12}/\lambda| \leq 0.131 + (0.936)^j(1.352)$$

$$|T_{22}/\lambda| \leq 0.288 + (0.936)^j(0.91)$$

$$|T_{32}/\lambda| \leq 0.174 + (0.936)^j(0.586).$$

We find that $\|(1/\lambda)T\|_F = \left( \sum_{i,j} (T_{ij}/\lambda)^2 \right)^{1/2} \leq 1$ if $j \geq 26$. This gives us that

$$\|(MA)^j\cdot(MB)^k\| = \|(-\lambda)^j\cdot(-\lambda)^k\| \leq \|(\lambda^j\cdot\lambda^k)\| \leq \lambda^{j+k} \leq \lambda^{j+k}$$

for $j \geq 26$ and $1 \leq k \leq 2$. A computation using Sage improves this and shows that $\|(MA)^j\cdot(MB)^k\| \leq \lambda^{j+k}$ holds for all $2 \leq j \leq 26$ and $1 \leq k \leq 2$. The only exception is when $k = 1$ and $j = 1$. Using this along with the submultiplicativity of $\|\cdot\|$, we have shown that

$$\left\| \prod_{n=1}^{2} (MA)^{j_n}\cdot(MB)^{k_n} \right\| \leq \prod_{n=1}^{2} \lambda^{j_n+k_n}$$
for \( j_n \geq 2 \) and \( 1 \leq k_n \leq 2 \). Another computation reveals that

\[
\left\| \prod_{n=1}^{2} (MA)^{j_n} (MB)^{k_n} \right\| \leq \prod_{n=1}^{2} \lambda^{j_n+k_n}
\]

for \( j_n = 1 \) and \( 1 \leq k_n \leq 2 \), so we are done.

**Remark 2.** Very similar to the diagonalization of \( MA \) are the diagonalizations of \( M \) and \( AM \):

\[
M = \begin{bmatrix}
\lambda & \alpha & \alpha \\
\lambda^2 - 2 & \alpha^2 - 2 & \alpha^2 - 2 \\
1 & 1 & 1
\end{bmatrix},
AM = \begin{bmatrix}
-\lambda & -\alpha & -\alpha \\
2 - \lambda^2 & 2 - \alpha^2 & 2 - \alpha^2 \\
1 & 1 & 1
\end{bmatrix}.
\]

**Lemma 5.** We have

\[
\| (MA)^j (MB)^k \| \ll \lambda^{j+k}
\]

where \( j, k \geq 0 \) and the implicit constant does not depend on \( j \).

**Proof.** Again, note that for \( n \geq 2 \), we have \((MB)^n = \pm (MB)^2\), so we may assume \( k \leq 2 \) without loss of generality. Using the diagonalization found for \( MA \) in [Lemma 4], we have that \( \| (MA)^j \| \ll \lambda^j \) and this result follows immediately.

The following lemma is used solely for proving the lower bound in [Theorem 1].

**Lemma 6.** Fix \( m \in \mathbb{N} \). If \( m \) is odd, then

\[
\left\lfloor \frac{2m+1}{3} \right\rfloor = 2^{m+1} - \left\lfloor \frac{2m+2}{3} \right\rfloor.
\]

Similarly, if \( m \) is even, then

\[
\left\lfloor \frac{2m+1}{3} \right\rfloor = 2^{m+1} - \left\lfloor \frac{2m+2}{3} \right\rfloor.
\]

**Remark 3.** In the notation of [Lemma 2], if we pick \( k_m = \left\lfloor \frac{2m+1}{3} \right\rfloor \), where \( \lfloor x \rfloor \) denotes the nearest integer to \( x \), then we have that \( k_m \in (3 \cdot 2^{m-2}, 2^m) = S_3^m \). [Lemma 6] tells us that

\[
k_{m-1} = k'_m = \left\lfloor \frac{2^m}{3} \right\rfloor \in S_3^{m-1}
\]

\[
k_{m-2} = k'_{m-1} = \left\lfloor \frac{2^{m-1}}{3} \right\rfloor \in S_3^{m-2}
\]

\[
\vdots
\]

\[
k_3 = k'_4 = 5 \in S_3^3
\]

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so that by Lemma 2 we have

\[ v_m = \begin{bmatrix} C(m, k_m) \\ C(m, k'_m) \\ C(m-1, k_{m-1}) \end{bmatrix} = (AM)^{m-3} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}. \]

We are now ready to prove the main theorem, which we recite below.

**Theorem 1.** For all \( m \in \mathbb{N} \), we have

\[ C_1 \lambda^m \leq \max_k |C(m, k)| \leq C_2 \lambda^m \]

where \( C_1, C_2 \) are absolute constants and \( \lambda \approx 1.65 \) is the largest root of \( x^3 + x^2 - 2x - 4 \).

**Proof.** First, we focus on the upper bound. Fix \( k_m \) so that \( 0 \leq k_m \leq 2^m \) and \( k_m \) is odd, and let

\[ v_m = \begin{bmatrix} C(m, k_m) \\ C(m, k'_m) \\ C(m-1, k_{m-1}) \end{bmatrix}. \]

The idea is to use the fact that

\[ |C(m, k_m)| \leq \|v_m\|. \]

By Lemma 3 we have that

\[ v_m = \pm A^d_1 \left( \prod_{i=3}^{m} MA^a_i B^b_i \right) A^d_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

where \( d_1, d_2 \in \mathbb{N} \) and \( (a_i, b_i) \in \{(0, 1), (1, 0)\} \). We see that

\[ \| \pm A^d_1 \left( \prod_{i=3}^{m} MA^a_i B^b_i \right) A^d_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \| \leq \| A^d_1 \left( \prod_{i=3}^{m} MA^a_i B^b_i \right) A^d_2 \| = \| \prod_{i=3}^{m} MA^a_i B^b_i \| \]

where in the last step we used the facts that \( A \) is an isometry and \( AA^* = I \). Note that \( \|MBv\| \leq \|MAv\| \) for all \( v \in \mathbb{R}^3 \), so we assume that \( (a_m, b_m) = (1, 0) \) without loss of generality. With this assumption, we have that

\[ \prod_{i=3}^{m} MA^a_i B^b_i = \left( \prod_{i=1}^{n} (MA^{j_i}_i MB^{k_i}_i) \right) (MA)^{\ell} \]

where \( j_i, k_i \geq 0 \) and \( \ell + \sum_{i=1}^{n} j_i + k_i = m - 3 \). We use Lemma 4 and Lemma 5 to conclude that

\[ |C(m, k_m)| \leq \|v_m\| \leq \| \prod_{i=3}^{m} MA^a_i B^b_i \| \leq \prod_{i=1}^{n} (MA^{j_i}_i MB^{k_i}_i) \cdot \lambda^\ell \lesssim \lambda^\ell \prod_{i=1}^{n} \lambda^{j_i+k_i} \lesssim \lambda^m \]

where the implicit constants are independent of \( n \).

Now, we concentrate on the lower bound. For this, we use the same idea as the authors in [4]; namely, we exhibit \( C(m, \ell_m) \) for a specific \( \ell_m \) such that \( |C(m, \ell_m)| \geq \lambda^m \). Let

\[ \ell_m = \left\lfloor \frac{2^{m+1}}{3} \right\rfloor \]

where \( \lfloor x \rfloor \) denotes the nearest integer to \( x \). By Lemma 2 and Lemma 6 (see Remark 3), we have that

\[ C(m, \ell_m) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (AM)^{m-3} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}. \]
Using this equation and the diagonalization of \( AM \) given in Remark 2, we find that there exist constants \( a \), \( b \), and \( c \) so that

\[
\max_k |C(m, k)| \geq |C(m, \ell_m)| = |a\lambda^m + b\alpha^m + c\beta^m| \gg \lambda^m
\]

where \( \alpha = -1.33 \cdots - 0.80 \cdots i \) is a root of \( x^3 + x^2 - 2x - 4 \). This concludes the proof.

\[\Box\]

**Remark 4.** In Theorem 2.2 of [1], the authors provide relations (1-4) used in our Lemma 2 for a class of sequences that is more general than the class of Rudin-Shapiro sequences. In particular, they consider the class of sequences \((a_0, a_1, \ldots, a_{2^m-1})\) with

\[
a_0 = 1, \\
a_{2^i+j} = (-1)^{j+f(i)}a^{2^i-j-1}, \quad 0 \leq j \leq 2^i - 1, \\
0 \leq i \leq m - 1,
\]

for all \( m \in \mathbb{N} \) and \( f : \mathbb{N} \to \{0, 1\} \) being an arbitrary function. The Rudin-Shapiro sequences are recovered by choosing \( f \) so that \( f(0) = f(2k - 1) = 0 \) and \( f(2k) = 1 \) for all \( k \in \mathbb{N} \). They show the aforementioned relations but the right-hand side of each is multiplied by \( (-1)^{f(m-1)+f(m-2)} \). Since these relations are the same as the relations in Lemma 2 up to a factor of -1, we may obtain an analogue of Lemma 2 with the matrices \( \pm M, A, \) and \( B \). In norm, products of these matrices are the same as products of \( M, A, \) and \( B \). Thus, we may obtain Theorem 1 for autocorrelations of this more general family of sequences in exactly the same manner as we have done above for Rudin-Shapiro sequences.

### 3. Directions for Further Study

We now concern ourselves with which \( k \) gives maximal \( C(m, k) \). Fix \( m \) and suppose that \( k = k^*_m \) gives maximal \( C(m, k) \).

**Conjecture.** We have that \( k^*_m \) is unique for each \( m \) and \( \lim_{m \to \infty} \frac{3k^*_m}{2^{m+1}} = 1 \).

In other words, the \( k \) that gives maximal autocorrelation of the \( m \)-th Rudin-Shapiro sequence is asymptotically 2/3 the length of the \( m \)-th Rudin-Shapiro sequence.

Let \( \ell_m = \left\lfloor \frac{2^{m+1}}{3} \right\rfloor \) where \( \lfloor x \rfloor \) denotes the nearest integer to \( x \). We find that \( k^*_m \) is unique for each \( 3 \leq m \leq 16 \) and that:

| \( m \) | \( k_m^* - \ell_m \) | \( k_m^*/\ell_m \) |
|---|---|---|
| 3 | 2 | 0.6 |
| 4 | 0 | 1 |
| 5 | 8 | 0.619048... |
| 6 | 0 | 1 |
| 7 | 34 | 0.6 |
| 8 | 2 | 1.011696... |
| 9 | 22 | 1.0645161... |
| 10 | 8 | 1.011713... |
| 11 | 0 | 1 |
| 12 | 34 | 1.012450... |
| 13 | 86 | 1.015748... |
| 14 | 136 | 1.012451... |
| 15 | 18 | 0.999634... |
| 16 | 0 | 1 |
Note that for $m = 4, 6, 11, 16, \ldots$ we have $k_m^* = \ell_m$. That leads us to pose the following question.

**Question.** Assuming that $k_m^*$ is unique for each $m$, do there exist infinitely many integers $m$ such that $k_m^* = \ell_m$?

It is interesting that $|C(m,k)|$ generally increases as $k$ increases from $k = 0$ to a local maximum at $k \approx \frac{2^{m+1}}{5}$ and $|C(m,k)|$ generally decreases as $k$ increases from a local maximum at $k \approx \frac{2^{m+1}}{3}$ to $k = 2^m$ as we can see in the following graphs:

![Graphs showing increase and decrease of |C(m,k)|](image)

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