On pseudo-stochastic matrices and pseudo-positive maps

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Abstract

Stochastic matrices and positive maps in matrix algebras have proved to be very important tools for analysing classical and quantum systems. In particular they represent a natural set of transformations for classical and quantum states, respectively. Here we introduce the notion of pseudo-stochastic matrices and consider their semigroup property. Unlike stochastic matrices, pseudo-stochastic matrices are permitted to have matrix elements which are negative while respecting the requirement that the sum of the elements of each column is one. They also allow for convex combinations, and carry a Lie group structure which permits the introduction of Lie algebra generators. The quantum analog of a pseudo-stochastic matrix exists and is called a pseudo-positive map. They have the property of transforming a subset of quantum states (characterized by maximal purity or minimal von Neumann entropy requirements) into quantum states. Examples of qubit dynamics connected with ‘diamond’ sets of stochastic matrices and pseudo-positive maps are dealt with.

Keywords: positive maps, stochastic matrices, quantum dynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

Stochastic matrices and linear positive maps are well established tools for dealing with many problems in stochastic processes, stochastic evolution and quantum information theory [1–3]. A stochastic matrix \( T_{ij} \) satisfies two basic properties: \( T_{ij} \geq 0 \) and \( \sum_j T_{ij} = 1 \). These properties guarantee that stochastic matrices map probability vector into a probability vector and hence may be used to describe legitimate operations on the classical states (described by probability vectors).

States of quantum systems are described by density matrices, their evolution is usually described by positive or completely positive maps [4–7]. The evolution equation of Markovian type for density matrices was introduced by Kossakowski [8] and further elaborated by Gorini et al [9] and independently by Lindblad [10]. Nowadays, open quantum systems and their dynamical features are attracting increasing attention [11–14]. They are of paramount importance in the study of the interaction between a quantum system and its environment, causing dissipation, decay, and decoherence [15].

It was observed [16–18] that quantum states may also be described by tomographic probability distributions both for finite (qudit) and infinite (photon quadratures) dimensional Hilbert spaces. According to this picture standard quantum evolution can be related with the evolution of probabilities describing the quantum states, it was first observed on simple examples [19, 20] and then considered in its generality [21] that the evolution of probability vectors is related with the analog of stochastic matrices which can have negative matrix elements. The violation of positivity is associated with the observation that probability vectors describing quantum states occupy only a subset of the simplex. Such a phenomenon
does not seem to be well known in the existing literature. It is
to worthy to mention that recently [22] non-positive maps of
Gaussian states have made their appearance in the discussion
of properties of quantum channels. A non-positive map
obtained by rescalling the argument of the Wigner function

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In this paper we would like to study linear maps on the
space of probability vectors which need not be stochastic but
are only pseudo-stochastic, as we are going to call them.
These maps naturally appear when we only consider the
transformation of subdomains in the simplex. Another aspect
of this paper is the introduction of non-positive maps as maps
acting on the space of density matrices. It is natural to con-
side the relation between non-positive maps of density states
with the notion of pseudo-stochastic matrices. The subset of
density matrices associated with subdomains of the prob-
ability vectors are just the objects which can be transformed
by means of pseudo-positive maps. To this aim, we consider
subsets of density matrices and characterize them by maximal
purity or minimal entropy requirements. To illustrate these
ideas we reconsider in details the dynamics of qubit states
following [24, 25]. Pseudo-positive maps which are positive
only on the convex subset \( K \) may be considered as witnesses
of ‘not being an element of \( K \)’ in the same way as positive but
non-completely positive maps are witnesses of being non-separable state [26].

In this paper we point out the importance of the clear
understanding of the introduction of the notion of pseudo-
stochastic matrices and pseudo-positive maps for quantum
information. It turns out that the use of these matrices and
maps is natural in quantum mechanics and represents another
aspects of classical-to-quantum transition. In the classical
setting it was sufficient to use stochastic matrices for the
description of kinetic phenomena associated with random
variables and probability distributions as well as with their
time evolution. In the quantum setting the evolution of states
considered in the framework of probability distributions
demands the use of pseudo-stochastic matrices. Moreover,
when considering the density matrices and their evolution as
well as quantum channels in all their diverse facets we need
to introduce pseudo-positive maps. It appears that these maps
are new elements to be taken into account for the analysis of
quantum correlation properties and the analysis of quantum
information processes.

The paper is organized as follows. In section 2 we study
the semigroup of stochastic and pseudo-stochastic matrices
and identify convex subsets of these matrices. In section 3 we
provide an instructive example of such subsets which we call
‘diamond’ subsets. In section 4 we discuss an example of
classical evolution of a two-level system in connection to
‘diamond’ subsets in \( \mathbb{R}^2 \). In section 5 the quantum pseudo-
positive maps are dealt with and considered on the example of
qubit state dynamics. In section 6 we draw some conclusions
and advance some perspectives. In the appendix the Lie
algebra structure of the group of pseudo-stochastic matrices
for \( n = 2 \) and \( n = 3 \) is shortly discussed.

2. A semigroup of pseudo-stochastic matrices

A real \( n \times n \) matrix \( T \) is 

\[ \begin{bmatrix} T_{ij} \end{bmatrix} \]

\[ \sum_{i,j=1}^{n} T_{ij} = 1 \]

iff \( T_{ij} \geq 0 \) and \( \sum_{i,j=1}^{n} T_{ij} = 1 \) [27, 28]. It defines a compact convex subset
of \( S_0 \subset \mathbb{R}^{n(n-1)} \). Stochastic matrices define a semigroup: if
\( T_i, T_j \in S_0 \), then \( T_i T_j \in S_0 \). It is not a group because \( T \) does
not need to be invertible and even if \( T^{-1} \) exists it needs not
belong to \( S_0 \). Actually, \( T^{-1} \in S_0 \) iff \( T \in Per_n \), where \( Per_n \)
denotes a set of \( n \times n \) permutation matrices. It is clear that
\( Per_n \) defines a discrete group being a subgroup of the unitary
group \( U(n) \). Stochastic matrices satisfying the additional
condition \( \sum_{j=1}^{n} T_{ij} = 1 \) define a proper convex subset
\( S_n \subset S_0 \) of bistochastic matrices. According to the cele-
brated Birkhoff theorem [28] any bistochastic matrix \( T \) is a
convex combination of permutation matrices.

Now, we relax the condition \( T_{ij} \geq 0 \) and call the matrix
\( T \in M_n(\mathbb{R}) \) pseudo-stochastic iff \( \sum_{i,j=1}^{n} T_{ij} = 1 \). A set \( PS_n \) of
\( n \times n \) pseudo-stochastic matrices is isomorphic to \( \mathbb{R}^{n(n-1)} \)
and \( S_0 \) defines a convex subset of \( PS_n \). It is clear that \( PS_n \)
defines a semigroup: if \( T_i, T_j \in PS_n \), then \( T_i T_j \in PS_n \). Let us observe
that if \( T \in S_0 \) is invertible, then \( T^{-1} \in PS_n \).

If \( T \in PS_n \) and \( T \) is invertible, then \( T^{-1} \in PS_n \). Hence

\[ GPS_n = \{ T \in PS_n \mid \det T = 0 \} \subset PS_n \]

defines a group of pseudo-stochastic matrices. It is a subgroup
of \( GL(n, \mathbb{R}) \) and contains \( Per_n \) as a discrete subgroup. Note, that
\( GPS_n \) containing invertible matrices from \( PS_n \) such that
\( \det T > 0 \) defines a subgroup of \( GPS_n \)—the connected
component of identity.

Stochastic matrices provide mathematical representation
of classical channels

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^n, \]

that is \( T(\Sigma_n) \subset \Sigma_n \), where

\[ \Sigma_n = \left\{ p = (p_1, \ldots, p_n) \in \mathbb{R}^n \mid p_k \geq 0, \sum_{k=1}^{n} p_k = 1 \right\} \]

defines a simplex of probability distributions (classical states).
Consider a convex subset \( K \subset \Sigma_n \) and define

\[ (1) \quad S(K) \subset S_n \quad \text{such that for all } T \in S(K) \text{ one has } T(K) \subset K, \]

\[ (2) \quad PS(K) \subset PS_n \quad \text{such that for all } T \in PS(K) \text{ one has } T(K) \subset \Sigma_n, \]

\[ (3) \quad S_0(K) \subset S(K) \quad \text{such that for all } T \in S_0(K) \text{ one has } T(S_n) \subset K. \]

One immediately finds

\[ S_0(K) \subset S(K) \subset S_n \subset PS(K) \subset PS_n, \]

and if \( K = \Sigma_n \), then

\[ S_0(\Sigma_n) = S(\Sigma_n) \subset S_n \subset PS(\Sigma_n) = PS_n. \]

Interestingly, if \( K = \{ p_k \} \) with \( p_k = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \), then
\( S(\{ p_k \}) \) defines a set of bistochastic matrices and \( S_0(\{ p_k \}) \)
contains only one element $T_n$ defined by $(T_n)_{ij} = \frac{1}{n}$ (a maximally mixing bistochastic matrix).

A set $S(K)$ has a clear interpretation: a subset $K$ is $T$-invariant for all $T \in S(K)$. Note, that if $T_1, T_2 \in S(K)$ in general $T_1T_2$ needs not belong to $S(K)$. However, one proves

**Proposition 1.** For any $T_1, T_2 \in S_0(K)$, $T_1T_2 \in S_0(K)$, that is, $S_0(K)$ defines a semigroup (subsemigroup of $S_n$).

Convex sets $S_0(K)$ and $S(K)$ contain only stochastic matrices. A set $PS(K)$ contains also pseudo-stochastic matrices which are not stochastic. The interpretation of these matrices is provided by the following

**Proposition 2.** An element $p \in \Sigma_n$ belongs to $K$ if and only if $T_p \in \Sigma_n$ for all $T \in PS(K)$.

Hence, element from $PS(K) - \Sigma_n$ may be used to witness that $p$ does not belong to $K$.

**Corollary 1.** An element $p \in \Sigma_n$ does not belong to $K$ if and only if there exists $T \in PS(K) - \Sigma_n$ such that $T_p \not\in \Sigma_n$.

### 3. Example: ‘diamond’ subsets

Consider the following convex subset $K_2$ of $\Sigma_2$

$$\varepsilon \leq p_1, p_2 \leq 1 - \varepsilon,$$

with $0 \leq \varepsilon \leq \frac{1}{2}$. Clearly, $K_0 = \Sigma_2$ and $K_2 = \{ p_\varepsilon \}$, where $p_\varepsilon = (\frac{1}{2}, \frac{1}{2})$ is the maximally mixed state. Moreover $K_2 \subset K \subset K_0 \subset \Sigma_2$ for $\varepsilon' \leq \varepsilon$. Any $2 \times 2$ pseudo-stochastic matrix may be parameterized by two real numbers $(a, b)$ as follows

$$T = \begin{pmatrix} a & 1 - b \\ 1 - a & b \end{pmatrix}.$$

Two convex sets $S(K_2)$ and $PS(K_2)$ are represented by diamond shape bodies displayed in figure 1: $S(K_2)$ corresponds to the inner violet diamond and $PS(K_2)$ corresponds to the outer yellow diamond. One finds for the corresponding vertices

$$A = \frac{1 - \varepsilon}{1 - 2\varepsilon}(1, 1), B = \frac{\varepsilon}{1 - 2\varepsilon}(-1, -1), C = (\varepsilon, 1 - \varepsilon), D = (1 - \varepsilon, \varepsilon).$$

In this case $S_2$ is represented by the red square $[0, 1] \times [0, 1]$ and $PS_2$ is the whole $(a, b)$-plane $\mathbb{R}^2$. Finally, $S_0(K_2)$ corresponds to the inner dark blue square $[\varepsilon, 1 - \varepsilon] \times [\varepsilon, 1 - \varepsilon]$. If $\varepsilon \rightarrow \frac{1}{2}$, then $S(K_2)$ defines the set of bi-stochastic matrices and $PS(K_2)$ the set of pseudo-bistochastic matrices represented by the line $a = b$. Finally $S_0(K_2)$ shrinks to the point $\left( \frac{1}{2}, \frac{1}{2} \right)$. Note, that

$$\det T = \text{tr} T - 1,$$

and hence $T$ is invertible iff $\text{tr} T = 1$. It gives rise to a group of pseudo-stochastic matrices

$$GPS_2 = \{ T \in PS_2 \mid \text{tr} T = 1 \},$$

and the proper subgroup of stochastic matrices contains only two elements

$$GS_2 = \{ T_0, T_1 \} \subset GPS_2,$$

where $T_0 = \mathbb{I}_2$ and $T_1 = \sigma_a$ is a permutation matrix. The group $GPS_2$ has two connected components: $GPS_2^+$ corresponding to $\det T > 0$ and $GPS_2^-$ corresponding to $\det T < 0$. $GPS_2^+$ contains identity matrix whereas $GPS_2^-$ contains permutation matrix. The Lie algebra properties of the ‘diamond’ group are shortly discussed in the appendix.

### 4. Divisible dynamical maps and pseudo-stochastic propagators

Consider now a classical evolution $p_t \in \Sigma_n$ described by the dynamical map $T(t) \in S_n$ satisfying initial condition $T(t = 0) = \mathbb{I}_n$, that is, $p_t = T(t)p_0$. An example of such a map is provided by Markovian semi-group $T(t) = e^{Lt}$, where $L \in \mathbb{M}_n(\mathbb{R})$ is the corresponding generator satisfying well
known conditions \[ L_{ij} \leq 0, \quad (i \neq j); \quad \sum_{i=1}^{n} L_{ij} = 0. \] (9)

Note, that \( T(t) = e^{itL} \) is invertible \( T^{-1}(t) = e^{-itL} \) and clearly \( T^{-1}(t) \in \text{PS}_n \setminus \mathcal{S}_n \) for \( t > 0 \). However, the corresponding propagator

\[
V(t, s) = T(t) \cdot T^{-1}(s) = e^{(t-s)L},
\] (10)

belongs to \( \mathcal{S}_n \) for \( t \geq s \). It is clear that if \( L \) satisfies only \( \sum_{i=1}^{n} L_{ij} = 0 \), then \( T(t) = e^{itL} \) defines a 1-parameter semi-group of pseudo-stochastic maps.

A dynamical map \( T(t) \) is divisible if for any \( t > s \) one has

\[
T(t) = V(t, s)T(s),
\] (11)

and \( V(t, s) \in \mathcal{S}_n \). Note, that if \( T(t) \) is invertible, then \( V(t, s) = T(t) \cdot T^{-1}(s) \). Now, the family \( T(t) \) is divisible if and only if it satisfies the time-local master equation

\[
\frac{d}{dt}T(t) = L(t)T(t),
\] (12)

with time-dependent local generator satisfying \( L_{ij}(t) \leq 0 \) for \( i \neq j \), and \( \sum_{i=1}^{n} L_{ij}(t) = 0 \). The corresponding propagator is given by

\[
V(t, s) = T \exp \left( \int_{s}^{t} L(u)du \right).
\] (13)

General dynamical map needs not be divisible.

**Example 1.** Consider the following time-local generator for \( n = 2 \)

\[ L(t) = \begin{pmatrix} -x(t) & y(t) \\ x(t) & -y(t) \end{pmatrix}. \] (14)

It is clear that it generates divisible dynamical map iff \( x(t), y(t) \geq 0 \) for \( t \geq 0 \). Note that

\[
L(t) = \begin{pmatrix} y(t) & y(t) \\ x(t) & x(t) \end{pmatrix} - \begin{pmatrix} \gamma(t) & 0 \\ 0 & \gamma(t) \end{pmatrix},
\] (15)

with \( \gamma(t) = x(t) + y(t) \), and hence

\[
L(t)q = \gamma(t) [q(t)(p_1 + p_2) - q(t)],
\] (16)

where \( q_1(t) = q_1(t), q_2(t) \), with \( q_1(t) = y(t)/\gamma(t) \) and \( q_2(t) = x(t)/\gamma(t) \). Interestingly, one has

\[
L(t)q(t) = 0,
\] (17)

that is, \( q(t) \) is a time-dependent invariant vector. One easily finds the corresponding solution

\[
p_t := T(t)p_0 = e^{-\Gamma(t)}p_0 + \left[ 1 - e^{-\Gamma(t)} \right] q(t),
\] (18)

with

\[
\Gamma(t) = \int_{0}^{t} \gamma(u)du. \] (19)

and \( \tau(t) = \int_{0}^{t} \gamma(u)du \). Equivalently one has

\[
T(t) = e^{-\Gamma(t)}I_2 + \left[ 1 - e^{-\Gamma(t)} \right] \begin{pmatrix} Q_1(t) & Q_2(t) \\ Q_2(t) & Q_1(t) \end{pmatrix} = \begin{pmatrix} Q_1(t) + e^{-\Gamma(t)}Q_2(t) & Q_1(t) - e^{-\Gamma(t)}Q_1(t) \\ Q_2(t) - e^{-\Gamma(t)}Q_2(t) & Q_2(t) + e^{-\Gamma(t)}Q_1(t) \end{pmatrix}.
\] (20)

where we have used

\[
Q_1(t) + Q_2(t) = 1.
\] (21)

Formula (20) shows that \( T(t) \) is a convex combination of two pseudo-stochastic matrices (actually, one of them \( I_2 \) is stochastic). Now, \( T(t) \) defines a legitimate dynamical map if and only if

\[
Q_1(t) + e^{-\Gamma(t)}Q_2(t) \geq 0, \quad Q_2(t) + e^{-\Gamma(t)}Q_1(t) \geq 0.
\] (22)

for all \( t \geq 0 \). In particular, if

\[
\Gamma(t) \geq 0, \quad Q_1(t) \geq 0, \quad Q_2(t) \geq 0,
\] (23)

then (22) is satisfied and \( T(t) \in \mathcal{S}_2 \). It means that \( Q(t) \) is a legitimate state for all \( t \geq 0 \). Note, that if \( x, y \) are time dependent, then \( q(t) = Q(t) = q \) and

\[
p_t = e^{-\gamma}p_0 + \left[ 1 - e^{-\gamma} \right] q.
\] (24)

which means that the evolution is convex combination of the initial state \( p_0 \) and the asymptotic invariant state \( q \)—Markovian semigroup. Condition (23) provides highly nontrivial constraints for admissible functions \( x(t) \) and \( y(t) \). It is clear that if \( x(t) \not\geq 0 \) or \( y(t) \not\geq 0 \), then \( T(t) \) is not divisible.

Let \( K \subset \Sigma_n \) be a convex set. We say that a dynamical map \( T(t) \) is \( K \)-divisible if and only if \( (11) \) is satisfied with \( V(t, s) \in \text{PS}(K) \) for all \( t \geq s \). Note, that if \( K = \Sigma_n \) then \( \Sigma_n \)-divisibility reduces to divisibility. Moreover, if \( K_1 \subset K_2 \), then \( K_2 \)-divisibility implies \( K_1 \)-divisibility. Hence, if \( T(t) \) is divisible then it is \( K \)-divisible for any \( K \).

**Example 2.** Dynamical map \( T(t) \) from example 2 gives rise to the following family of propagators

\[
V(t, s) = \begin{pmatrix} Q_1(t, s) + e^{-\Gamma(t,s)}Q_2(t, s) \\ Q_2(t, s) - e^{-\Gamma(t,s)}Q_2(t, s) \end{pmatrix},
\] (25)

where

\[
\Gamma(t, s) = \int_{s}^{t} \gamma(u)du,
\]

\[
Q_k(t, s) = \frac{1}{1 - e^{-\Gamma(t,s)}} \int_{s}^{t} \gamma(u)e^{\Gamma(u)}q_k(u)du.
\] (26)

Taking \( K_\varepsilon \) from example 1 one finds that \( K_\varepsilon \)-divisibility provides extra constraints for \( x(t) \) and \( y(t) \) in order to \( V(t, s) \in \text{PS}(K_\varepsilon) \) for any \( t \geq s \).
5. Pseudo-positive maps

A linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is Hermitian if $(\Phi[A])^\dagger = \Phi[A]^\dagger$. It is positive if $\Phi[A] \succeq 0$ for all $A \succeq 0$. Finally, it is trace-preserving if $\text{tr}(\Phi[A]) = \text{tr} A$. It is easy to show that positive maps are necessarily Hermitian. Note that $\Phi$ is positive trace-preserving (PTP) iff any orthonormal basis $\{e_1, \ldots, e_d\}$ in $\mathcal{H}$ the following matrix

$$T_j = \text{tr}(E_j \Phi[E_j]),$$

(27)

is stochastic (we define the standard matrix units $E_{ij} := |e_i\rangle\langle e_j|$). We call $\Phi$ pseudo-PTP if it is Hermitian and trace-preserving but not necessarily positive. Note, that $\Phi$ is Pseudo-PTP iff the matrix $T_{ij}$ defined in (27) is pseudo-stochastic. It is clear that pseudo-PTP maps form a semigroup, that is, if $\Phi_1$ and $\Phi_2$ are pseudo-PTP so is $\Phi_1 \Phi_2$.

Denote by $\mathcal{S}$ a convex set of density operators in $\mathcal{H}$. It is clear that for any PTP map $\Phi$ maps $\mathcal{S}$ into $\mathcal{S}$. Now, let $K$ be a convex subset in $\mathcal{S}(\mathcal{H})$ and let us define

1. $\mathcal{P}(K) \subset \mathcal{P}$ such that for all $\Phi \in \mathcal{P}(K)$ one has $\Phi[K] \subset K$, 
2. $p\mathcal{P}(K) \subset p\mathcal{P}$ such that for all $\Phi \in p\mathcal{P}(K)$ one has $\Phi[K] \subset \mathcal{S}$, 
3. $\mathcal{P}_0(K) \subset \mathcal{P}(K)$ such that for all $\Phi \in \mathcal{P}_0(K)$ one has $\Phi[\mathcal{S}] \subset K$,

where $\mathcal{P} = \text{PTP}$ maps and $p\mathcal{P} = \text{pseudo-PTP}$ maps. Again, one has the following chain of inclusions

$$\mathcal{P}_0(K) \subset \mathcal{P}(K) \subset \mathcal{P} \subset p\mathcal{P}(K) \subset p\mathcal{P},$$

(28)

and if $\mathcal{K} = \mathcal{S}$, then

$$\mathcal{P}_0(K) = \mathcal{P}(K) = \mathcal{P} \subset p\mathcal{P}(K) = p\mathcal{P}.$$

(29)

If $K = \{\rho_\lambda = \frac{1}{d} I\}$ contains only maximally mixed state then $\mathcal{P}(K)$ defines a set of bistoochastic positive maps and $\mathcal{P}_0(K)$ contains only one element $\Phi_\lambda$ defined by

$$\Phi_\lambda[\rho] = \rho_\lambda \text{ tr } \rho.$$

(30)

Convex sets $\mathcal{P}_0(K)$ and $\mathcal{P}(K)$ contain only PTP maps. A set $p\mathcal{P}(K)$ contains also pseudo-PTP maps which are not positive. The interpretation of these maps is provided by the following

Proposition 3. A density operator $\rho \in \mathcal{S}$ belongs to $K$ if and only if $\Phi[\rho] \in \mathcal{S}$ for all $\Phi \in p\mathcal{P}(K)$.

Hence, a map from $p\mathcal{P}(K) - \mathcal{P}$, i.e. pseudo-PTP but not positive, may be used to witness that $\rho$ does not belong to $K$.

Corollary 2. A density operator $\rho \in \mathcal{S}$ does not belong to $K$ if and only if there exists $\Phi \in p\mathcal{P}(K) - \mathcal{P}$ such that $\Phi[\rho] \notin \mathcal{S}$.

Example 3. Consider $\mathcal{H} = \mathbb{C}^2$. In this case $\mathcal{S}$ may be represented by the Bloch ball, that is

$$\rho = \frac{1}{2} \left( I + \sum_{k=1}^{3} x_k a_k^\dagger a_k \right).$$

(31)

and hence

$$\mathcal{S} = \{ x \in \mathbb{R}^3 \mid |x| \leq 1 \}.$$  

(32)

Now, consider a convex subset

$$K_\varepsilon = \{ \rho \in \mathbb{R}^3 \mid |\rho| \leq 1 - \varepsilon \} \subset \mathcal{S},$$

(33)

and let us analyze convex sets of pseudo-PTP bistochastic maps. Note, that density operators $\rho \in K_\varepsilon$ satisfy

$$\text{Purity}[\rho] = \text{tr} \rho^2 \leq \frac{1 + (1 - \varepsilon)^2}{2}.$$  

Equivalently, we may characterize this set via the von Neumann entropy: $\rho \in K_\varepsilon$, if

$$S[\rho] \geq 2 - \frac{1}{2} \left((2 - \varepsilon) \ln(2 - \varepsilon) + \varepsilon \ln \varepsilon\right).$$

(34)

A unital pseudo-PTP map $\Phi : \mathcal{S} \to \mathcal{S}$ may be represented in terms of the Bloch vectors as follows

$$s'_k = \sum_{i=1}^{3} A_{ik} x_i,$$

(35)

with $A_{ik}$ being matrix elements of $3 \times 3$ real matrix $A$. Now, Singular Value Decomposition of $A$ gives rise to

$$A = O_1 D O_2^T,$$

(36)

where $O_1, O_2$ are orthogonal matrices and $D$ is the diagonal matrix of singular values $s_k$ of $A$. It is clear that $s' \in K_\varepsilon$ iff the singular values $s_k$ satisfy

$$s_k \leq \frac{1}{1 - \varepsilon},$$

(37)

for $k = 1, 2, 3$.

Example 4. Let us consider well known reduction map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$\Phi[\rho] = \text{tr} \rho \rho - \rho,$$

(38)

which is evidently positive since it maps any projector $|\psi\rangle\langle \psi|$ to the orthogonal one $|\psi^\perp\rangle\langle \psi^\perp| = 1 - |\psi\rangle\langle \psi|$. Let us define the family of trace-preserving maps

$$\Phi_\mu[\rho] = \frac{1}{2 - \mu} [\text{tr} \rho - \mu\rho],$$

(39)

for $\mu \in [1, 2]$. Clearly these maps are pseudo-positive and only $\Phi_{\mu_1}$ is positive. One easily checks that for

$$1 < \mu \leq \frac{1}{1 + (1 - \varepsilon)^2},$$

(40)

the map is not positive but $\Phi \in p\mathcal{P}(K_\varepsilon)$. Hence, if $\mu$ satisfies (40) and $\Phi_\mu[\rho] \neq 0$, then $\rho \notin K_\varepsilon$, that is, the purity $P[\rho] > \frac{1}{2}[1 + (1 - \varepsilon)^2]$.

Remark 1. In the recent paper [29] authors use the inverse to the reduction map in $M_n(\mathbb{C})$

$$\Phi[X] = \frac{1}{n - 1} [\text{tr} X - X],$$

(41)

given by

$$\Phi^{-1}[X] = \text{tr} X - (n - 1)X,$$

(42)
to construct an entanglement witness in $\mathbb{C}^n \otimes \mathbb{C}^n$. Note, that $\Phi^{-1}$ is not a positive map (unless $n = 2$) but clearly it is pseudo-positive.

6. Non-Markovian $K$-divisible evolution

Evolution of quantum system living in the Hilbert space $\mathcal{H}$ is described by the dynamical map, that is, a family of quantum channels

$$\Lambda_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),$$

satisfying $\Lambda_0 = 1$ (identity map). Consider now the dynamical map $\Lambda_t$ satisfying time-local master equation

$$\dot{\Lambda}_t = L_t \Lambda_t,$$

(44)

with the time-dependent generator $L_t$. The map $\Lambda_t$ represents Markovian evolution if and only if $\Lambda_t$ is CP-divisible [31–34] (see also [35, 36] for recent reviews), that is

$$\Lambda_t = V_{t,s} \Lambda_s,$$

(45)

and $V_{t,s}$ is completely positive for $t \geq s$. If the maps $V_{t,s}$ are only positive then one calls $\Lambda_t$ $P$-divisible. Our approach enables one to generalize this notion: if $K$ is a convex subset of $\mathcal{G}$, then one calls $\Lambda_t$ $K$-divisible iff $V_{t,s} \in \mathcal{P}(K)$. If $K = \mathcal{G}$, then $K$-divisibility reduces to $P$-divisibility. $K$-divisible evolution has the following property: $V_{t,s}$ maps any density operator from $K$ into the legitimate state. However, for $\rho \notin K$ the result of the action $V_{t,s}[\rho]$ needs not be a legitimate state.

Example 5. Consider the evolution of a qubit governed by the following generator

$$L_t[\rho] = \frac{1}{2} \sum_{k=1}^{3} \gamma_k(t) \{\sigma_k \rho \sigma_k - \rho\},$$

(46)

with time dependent decoherence rates $\gamma_k(t)$. The corresponding solution reads [24, 25]

$$\Lambda_t[\rho] = \sum_{\alpha=0}^{3} p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha,$$

(47)

with real $p_\alpha(t)$ and $\sum_{\alpha=0}^{3} p_\alpha(t) = 1$ given by

$$p_\alpha(t) = \frac{1}{4} \sum_{\beta=0}^{3} H_{\alpha,\beta} \lambda_\beta(t),$$

(48)

and $H_{\alpha,\beta}$ is the Hadamard matrix

$$H = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 \\
\end{pmatrix}.$$

Finally, the quantities $\lambda_\alpha(t)$ define eigenvalues of the map $\Lambda_t$

$$\Lambda_t[\sigma_\alpha] = \lambda_\alpha(t) \sigma_\alpha,$$

(49)

and they are given by: $\lambda_0 = 1$ (any trace-preserving Hermitian map satisfy this property) and

$$\lambda_1(t) = \exp\left[\frac{-\Gamma_2(t,0) - \Gamma_3(t,0)}{2}\right] + \text{cyclic perm.}$$

(50)

with

$$\Gamma_k(t, s) = \int_s^t \gamma_k(\tau) d\tau.$$

(51)

One has the following result

1. $\Lambda_t$ is CP-divisible iff $\gamma_k(t) \geq 0$ for $k = 1, 2, 3$,

2. $\Lambda_t$ is $P$-divisible iff

$$\gamma_1(t) + \gamma_2(t) \geq 0, \quad \gamma_2(t) + \gamma_3(t) \geq 0, \quad \gamma_3(t) + \gamma_1(t) \geq 0,$$

3. $\Lambda_t$ is $K$-divisible iff

$$\Gamma_1(t, s) + \Gamma_2(t, s) \geq \ln[1 - \varepsilon], \quad \Gamma_2(t, s) + \Gamma_3(t, s) \geq \ln[1 - \varepsilon], \quad \Gamma_3(t, s) + \Gamma_1(t, s) \geq \ln[1 - \varepsilon],$$

for $t > s$.

It is clear that if $\varepsilon \to 0$, then 3. reduces to 2. Conversely, if $\varepsilon \to 1$, then $\gamma_k(t)$ are completely arbitrary.

7. Conclusions

Stochastic matrices preserve the simplex of probability vectors in $\mathbb{R}^n$. Similarly, trace-preserving positive maps preserve the convex set of density matrices in $\mathcal{B}(\mathcal{H})$. It is therefore clear that both objects proved to be very important for the analysis of various properties of classical and quantum systems. In this paper we introduced the notions of pseudo-stochastic $n \times n$ matrices and pseudo-positive maps acting in $\mathcal{B}(\mathcal{H})$. These objects provide a natural generalization of stochastic matrices and positive maps. They naturally appear when we only consider the transformation of a convex subdomains in the set of states. Actually, one is often interested not in the whole set of states but only in a suitable convex subset satisfying some extra properties (like for example additional symmetries and/or special preparation procedure). In a realistic laboratory scenario one usually has an access only to a subset of states defined by the admissible preparation scheme. Therefore, it is natural to extend the notion of stochastic matrices and positive maps to deal with more general scenarios as well. Interestingly, these more general matrices or maps may be used as witnesses that a given state does not belong to a convex subdomain in perfect analogy to entanglement witnesses.

Moreover, given a dynamical map—classical $T(t)$ or quantum $\Lambda(t)$—the corresponding propagators $T(t,s) = T(t)T^{-1}(s)$ and $\Lambda(t,s) = \Lambda(t)\Lambda^{-1}(s)$ are always pseudo-stochastic and pseudo-positive, respectively. We have shown that these objects are useful for the refinement of the notion of divisible maps and hence may be used to further characterization of non-Markovian classical and/or quantum evolution. Indeed, if $T(t,s)$ is stochastic for any $t > s$, then classical evolution is Markovian. Similarly, if $\Lambda(t,s)$ is positive, then quantum evolution is $P$-divisible which is considered as a natural notion of Markovianity in the quantum case [36]. Possible new applications are currently investigated.
The six generators of the Lie algebra read
\[
L_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\quad
L_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\quad
L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\quad
L_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\quad
L_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\quad
L_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (56)

The six generators of the Lie algebra read
\[
T_1 = \begin{pmatrix} 1 - a & b_1 & c_1 \\ a_1 & 1 - b_1 & c_2 \\ a_2 & b_2 & 1 - c_1 - c_2 \end{pmatrix},
\quad
T_2 = \begin{pmatrix} 1 - a & b \\ a & 1 - b \end{pmatrix}.
\] (52)

One can measure the pseudo-stochasticity by means of the negativity, \(-\max|\langle a|\{1 - a\rangle\). If the matrices are written for the Lie group, the generators of the Lie algebra and their commutator read
\[
[L_\alpha, L_\beta] = L_\alpha - L_\beta.
\] (53)

This is a solvable Lie algebra corresponding to the Lie algebra of the solvable group of matrices
\[
g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.
\] (54)

Another example is the qutrit case. Then, the pseudo-stochastic matrix has the form
\[
T_3 = \begin{pmatrix} 1 - a_1 & a_2 & c_1 \\ a_1 & 1 - a_2 & c_2 \\ a_2 & c_2 & 1 - c_1 - c_2 \end{pmatrix}.
\] (55)

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