SOJOURN FUNCTIONALS FOR SPATIOTEMPORAL RANDOM FIELDS WITH LONG–MEMORY

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Abstract
This paper considers the asymptotic behaviour of sojourn functionals of spatiotemporal Gaussian random fields with long-range dependence (LRD) in time also known as long memory. Specifically, reduction theorems are derived for local functionals of nonlinear transformation of such fields, under general covariance structures, for Hermite rank $m \geq 1$. Then for $m = 1$, the properly normalized Minkowski functional of a Gaussian long–memory spatiotemporal random field has an asymptotically normal distribution. These results are proven to hold, in particular, for a family of non–separable covariance structures belonging to Gneiting class. For $m = 2$, under separability of the spatiotemporal covariance function in space and time, the properly normalized Minkowski functional, involving the modulus of a Gaussian random field, is proven to converge in distribution to the Rosenblatt type limiting distribution, represented in terms of a double Wiener-Itô stochastic integral, for a suitable range of the long memory parameter.

Key Words. Asymptotic normality, excursion sets, Gaussian fields with long memory, Rosenblatt-type distribution, spatiotemporal random fields.

1 Introduction
Geometric characteristics of random surfaces play a critical role in areas such as geostatistics, environmetrics, astrophysics, and medical imaging. There exists an extensive literature on data analysis based on Gaussian random field modeling. Minkowski functionals have played an important role in the geometrical analysis of their sample paths. In [31], Minkowski functionals are applied to the characterization of hot regions (i.e., the excursion sets), where the normalized temperature fluctuation field exceeds a given threshold. The normalized temperature fluctuation field, associated with CMB temperature on the sky, is represented in terms of a spherical random field (see also [23]; [31]). Furthermore, Minkowski functionals are attractive due to their geometrical interpretation in two dimensions, in relation to the total area of all hot regions, the total length of the boundary between hot and cold regions, and the Euler characteristic, which counts the number of isolated hot regions minus the number...
of isolated cold regions. Minkowski functionals have also been applied to brain mapping analysis, and, in general, to the description of texture models in medical imaging analysis, in relation to anatomy segmentation, and pathology detection and diagnosis (see, e.g., [35]). Truncated Gaussian processes or sequential indicator simulation play a crucial role in geosciences to model the spatial distribution of the materials. Here, Minkowski functionals are used as morphological measures (see, e.g., [30]; [33]). In that sense; a wide research area has been developed in the multiscale analysis of media with complex internal structures (see [3]), including soils, sedimentary rocks, foams, ceramics and composite materials (see, e.g., [12]; [15]; [32], and [38]). Also, a good overview and introduction to some of these applications can be found in [1] and [28].

Since the nighties sojourn measures for Gaussian processes with long–range dependence have been extensively analyzed (see, e.g., Berman [4]). Sojour functional were also analyzed in the context of weak–dependent random fields (see, e.g., Bulinski \textit{et al.} [5]; Ivanov and Leonenko [14]; Leonenko and Olenko [18]; among others). A parallel literature has also been developed in the long–range dependence random field context (see Ivanov and Leonenko [14]; Leonenko [17]; Leonenko and Olenko [18]; Makogin and Spodarev [26]; Marinucci, Rossi and Vidotto [29], just to mention a few). The approach adopted in this paper continues this research line.

There has been a growing interest on covariance function modeling for spatiotemporal random fields. Marinucci, Rossi and Vidotto [29] consider isotropic in space and stationary in time spatiotemporal Gaussian random fields on the two–dimensional unit sphere, and investigate the asymptotic behaviour of the empirical measure (excursion area), as time goes to infinity, covering both cases when the underlying field exhibits short and long memory in time. It turns out that the limiting distribution is not universal, depending both of the memory parameter and the threshold or level of sojourn functional.

This paper presents an alternative approach to the one adopted in Marinucci, Rossi and Vidotto [29]. Specifically, we start from a spatially homogeneous and isotropic Gaussian random field family, displaying stationarity and LRD in time, defined on $\mathbb{R}^d \times \mathbb{R}$. We consider the restriction of this family to a convex compact set in space, and an increasing sequence of intervals in time, that define the temporal increasing domain asymptotics adopted. Note that our methodology is applicable, in particular, to considering the restriction to a compact two–points homogeneous space, like the sphere, of our original family of spatiotemporal Gaussian random fields on $\mathbb{R}^d \times \mathbb{R}$ (see, e.g., Leonenko and Ruiz–Medina [20]).

We present a general reduction principle (Theorem 3), discovered first by Taqqu [36] (see also [8]; [21]; [22]; [37]), which allows to reduce the problem of the limiting distributions of properly normalised non-linear transformation of spatiotemporal Gaussian random fields to the integral functionals of
Hermite polynomials of such Gaussian random fields. The method of proof is standard. Indeed, we use the expansion of the local functional of Gaussian field into series of Hermite polynomials of such a field. But the novelty of the paper is that we consider spatiotemporal random fields beyond the regularly varying condition on the spatiotemporal covariance function. Hence, we can analyze a large class of spatiotemporal covariance functions, including Gneiting class (see [11]). This class of covariance functions is popular in many applications, including Meteorology or Earth sciences, among others.

In Theorems 1 and 2, we particularly address the derivation of the limiting probability distribution of the first Minkowski functional, above referred as a well–known sojourn functional in spatiotemporal Gaussian random field applications. For very general conditions on the decaying of covariance function to zero in time, the limiting distribution of normalized first Minkowski functional is asymptotically normal for large classes of covariances, including Gneiting class. For the modulus of a Gaussian random field, the limiting distribution is given in the form of a multiple Wiener–Itô stochastic integral, assuming the separability condition of the covariance structure, with respect to space an time. We also assume that the covariance function is a regular–varying function in time. The derived limiting distribution is of Rosenblatt type. We plane further investigation on these limiting distributions, for a wider family of spatiotemporal covariance functions including Gneiting class.

The outline of the paper is as follows. We first review some results on geometric probabilities in Section 2. In Section 3 we present the asymptotic behaviour of sojourn functionals of spatiotemporal Gaussian random fields with LRD in time, under general conditions on their spatiotemporal covariance structure. The general reduction theorem for subordinated Gaussian spatiotemporal random fields with LRD in time is presented in Section 4. These results are applied for sojourn functionals introduced in Section 3 providing the asymptotic normality of the first Minkowski functional of Gaussian random field, and limiting distribution of Rosenblatt type, for the sojourn functional of modulus of Gaussian random field. In Section 5 we provide the examples in terms of separable covariance structures, while in Section 6 we present examples of covariance structures for which main results hold for non-separable covariance structures. We restrict out exposition by the covariances known as Gneting class covariance structures.

2 Geometric probability

Some fundamental elements and basic results on geometric probability are now introduced (see Ahronyan and Khlatayan [2]; Ivanov and Leonenko [14]; Lellouche and Souries [16]; Lord [24], and the references therein).

Let $\nu_d$ be the Lebesgue measure on $\mathbb{R}^d$, $d \geq 1$, and $\mathcal{K}$ be a convex body
in $\mathbb{R}^d$, i.e., a compact convex set with not empty interior. We will denote by $D(\mathcal{K}) = \{\max \|x - y\|, x, y \in \mathcal{K}\}$ the diameter of $\mathcal{K}$. Let $\nu_d(\mathcal{K}) = |\mathcal{K}|$ be the volume of $\mathcal{K}$, and for $d \geq 2$, $\nu_{d-1}(\delta \mathcal{K}) = \mathcal{U}_{d-1}(\mathcal{K})$ be the surface area of $\mathcal{K}$, where $\delta \mathcal{K}$ denotes the boundary of $\mathcal{K}$. For $d = 1$, we put $\mathcal{U}_0(\mathcal{K}) = 0$.

For example, let $\mathcal{K} = B(1) = \{x \in \mathbb{R}^d; \|x\| \leq 1\}$ be the unit ball. Hence, $\delta \mathcal{K} = \delta B(1) = S_{d-1} = \{x \in \mathbb{R}^d; \|x\| = 1\}$ is the unit sphere. Thus,

$$D(B(1)) = 2, \quad |B(1)| = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}, \quad \mathcal{U}_{d-1}(B(1)) = \frac{2\pi^{d/2}}{\Gamma(d/2 + 1)}.$$ (1)

Let $Q$ the space of straight in $\mathbb{R}^d$, and $d\Gamma$ is an element of a locally finite measure in the space $Q$, which is invariant with respect to the group $\mathcal{M}$ of all Euclidean motions in the space $\mathbb{R}^d$. Let now consider a chord length distribution function of body $\mathcal{K}$, given by

$$F_{\mathcal{K}}(v) = \frac{2(d-1)}{|S_{d-2}|} \int_{\chi(\Gamma) \leq v} d\Gamma,$$ (2)

where $\chi(\Gamma) = \Gamma \cap \mathcal{K}$ is a chord in $\mathcal{K}$. For example, if $\mathcal{K} = B(1)$, then

$$F_{B(1)}(v) = \begin{cases} 
0, & v \leq 0 \\
1 - \left(1 - \left(\frac{v}{2}\right)^2\right)^{\frac{d-1}{2}}, & 0 \leq v \leq 2 \\
1, & v \geq 2
\end{cases}$$ (3)

(see Ahoronyan and Khalatyan [2] for details).

Let now consider two points $P_1, P_2 \in \mathcal{K}$ randomly and independently selected, with uniform distribution in $\mathcal{K}$. We consider the probability density $\psi_{\rho_{\mathcal{K}}}$ of the random variable $\rho_{\mathcal{K}} = \|P_1 - P_2\|$, given by

$$\psi_{\rho_{\mathcal{K}}}(z) = \frac{d}{dz} \mathbb{P} (\rho_{\mathcal{K}} \leq z).$$

In the particular case $\mathcal{K} = [-1, 1]$, $d = 1$, we have

$$\psi_{\rho_{\mathcal{K}}}(u) = 1 - \frac{u}{2}, \quad 0 \leq u \leq 2,$$

while for $d \geq 2$ (see Ivanov and Leonenko, 1989; Lord, 1954)

$$\psi_{\rho_{B(1)}}(z) = \mathcal{I}_{1-(\frac{d}{2})^2} \left(\frac{d+1}{2}, \frac{1}{2}\right), \quad 0 \leq z \leq 2,$$ (4)

where $\mathcal{I}_\mu(p, q)$ denotes the incomplete Beta function, given by

$$\mathcal{I}_\mu(p, q) = \frac{\Gamma(p + q)}{\Gamma(p)\Gamma(q)} \int_0^\mu t^{p-1}(1 - t)^{q-1}dt, \quad \mu \in [0, 1].$$ (5)
It is also known (see, e.g., equation (2.6) in Ahoronyan and Khalatyan [2]) that

\[
\psi_{\rho K}(z) = \frac{1}{|K|^2} \left[ z^{d-1} |S_{d-1}| |K| \left( z d - 1 \right) \right]^{d-1} \left[ 1 - F_K(v) \right] dv, \quad 0 \leq z \leq \mathcal{D}(K). \tag{6}
\]

In particular, for the ball \( K = B(1) \), we obtain an alternative to equation (4), given by, for \( 0 \leq z \leq 2 \),

\[
\psi_{\rho B(1)}(z) = z^{d-1} \left[ \frac{2\Gamma \left( \frac{d+1}{2} \right)}{\pi^{\frac{d-1}{2}}} - \frac{4\Gamma \left( \frac{d+1}{2} \right)}{\pi^{\frac{d+1}{2}} \Gamma \left( \frac{d+1}{2} \right) (d-1)} \int_0^z \left( 1 - \left( \frac{u}{2} \right)^2 \right)^{\frac{d-1}{2}} du \right]. \tag{7}
\]

## 3 Sojourn functionals

In the following we denote by \((\Omega, \mathcal{A}, P)\) the basic probability space, and \( Z : (\Omega, \mathcal{A}, P) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) defines a spatiotemporal real–valued random field \( \{ Z(x, t), \ x \in \mathbb{R}^d, t \in \mathbb{R} \} \).

**Condition 1.** Let \( Z \) be a measurable mean–square continuous homogeneous and isotropic in space and stationary in time Gaussian random field with \( \mathbb{E}[Z(x, t)] = 0, \mathbb{E}[Z^2(x, t)] = 1 \), and covariance function \( \tilde{C}(\|x - y\|, |t - s|) = E[Z(x, t)Z(y, s)] \geq 0 \). In spherical coordinates we denote

\[
C(z, \tau) = \tilde{C}(\|x - y\|, |t - s|), \quad z = \|x - y\| \geq 0, \quad \tau = |t - s| \geq 0. \tag{8}
\]

For simplicity, we will use \( dx \) instead of \( \nu_d(dx) \), and \( dt \) instead of \( \nu(dt) \).

We now introduce the following sojourn functional motivated by the first Minkowski functional:

\[
M_T^{(1)}(u) = \left| \left\{ 0 \leq t \leq T; \ Z(x, t) \geq u, \ x \in K \right\} \right| = \int_0^T \int_K 1_{\{Z(x, t) \geq u\}}(x, t) dx dt,
\]

where \( 1_{\{Z(x, t) \geq u\}}(\cdot, \cdot) \) denotes the indicator function of the set \( \{ (x, t) \in K \times [0, T]; Z(x, t) \geq u \} \). In the spatiotemporal isotropic spherical random field case the sojourn function [9] has been analyzed in [29].

Similarly, we can define

\[
M_T^{(2)}(u) = \left| \left\{ 0 \leq t \leq T; \ |Z(x, t)| \geq u, \ x \in K \right\} \right| = \int_0^T \int_K 1_{\{|Z(x, t)| \geq u\}}(x, t) dx dt.
\]

(10)
In other words, for each time $t$ fixed, the random area

$$A_u(t) = |Z^{-1}(\cdot, t) ([u, \infty))| = |\{x \in \mathcal{K}; Z(x, t) \geq u\}|$$

provides the empirical measure (i.e., the excursion area) of $Z(\cdot, t)$ corresponding to the level $u$. The integrated area over the temporal interval $[0, T]$ is then computed as $M^{(1)}_T(u) = \int_0^T A_u(t) dt$. Similar geometric interpretation has the functional $M^{(2)}_T(u)$ involving the random area

$$\tilde{A}_u(t) = |Z^{-1}(\cdot, t) ((-\infty, u] \cup [u, \infty))| = |\{x \in \mathcal{K}; |Z(x, t)| \geq u\}|$$

temporally integrated over $[0, T]$. Let $Z \sim \mathcal{N}(0, 1)$ be a standard Gaussian random variable with probability density $\phi$, and distribution $\Phi$ functions

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right), \quad \Phi(u) = \int_{-\infty}^u \phi_Z(z) dz, \quad z, u \in \mathbb{R}.$$

Let now $G$ be a Borel measurable function such that

$$\int_{\mathbb{R}} [G(z)]^2 \phi(z) dz < \infty.$$

Then, $G$ has an expansion with respect to the normalized Hermite polynomials that converges in $L_2(\mathbb{R}, \phi(z) dz)$:

$$G(z) = \sum_{n=0}^{\infty} \frac{G_n}{n!} H_n(z), \quad G_n = \int_{\mathbb{R}} H_n(\xi) G(\xi) \phi(\xi) d\xi, \quad z \in \mathbb{R}, \ q \geq 1, \quad (11)$$

where the Hermite polynomial of order $q \geq 1$, denoted as $H_q$ satisfies the equation:

$$\frac{d^n}{dz^n} \phi(z) = (-1)^n H_n(z) \phi(z). \quad (12)$$

Note that

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1 \quad H_3(x) = x^3 - 3x, \quad H_4(x) = x^4 - 6x^2 + 3, \ldots \quad (13)$$

Particularly, if $G_u(z) = 1_{\{z \geq u\}}$, then we obtain

$$G_u(Z(x, t)) = E[1_{\{Z(x, t) \geq u\}} (x, t)]$$

$$+ \sum_{q=1}^{\infty} \frac{G_q(u)}{q!} H_q(Z(x, t)), \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}. \quad (14)$$
Here,
\[
\mathcal{G}_0(u) = E[1_{\{|Z(x,t)| \geq u\}}(x,t)] = |1 - \Phi(u)| = \int_u^\infty \phi(\xi) d\xi
\]
\[
\mathcal{G}_q(u) = \phi(u) H_{q-1}(u), \quad q \geq 1.
\]

For the second functional corresponding to \(\tilde{\mathcal{G}}_u(z) = 1_{\{|z| \geq u\}}\), we have, for any even \(n \geq 0\),
\[
\tilde{\mathcal{G}}_0(u) = E[1_{\{|Z(x,t)| \geq u\}}(x,t)] = 2[1 - \Phi(u)] = 2 \int_u^\infty \phi(\xi) d\xi
\]
\[
\tilde{\mathcal{G}}_q(u) = 2\phi(u) H_{q-1}(u), \quad q \geq 1,
\]
and \(\tilde{\mathcal{G}}_q(u) = 0\), for odd \(n \geq 1\).

In what follows, from (14)–(15), we will consider the induced expansions of the functionals \(M^{(i)}_T(u), i = 1, 2\), given by
\[
M^{(1)}_T(u) = (1 - \Phi(u)) T|K| + \phi(u) \sum_{n=1}^\infty \frac{H_{n-1}(u)}{n!} \eta_n
\]
\[
M^{(2)}_T(u) = 2(1 - \Phi(u)) T|K| + 2 \phi(u) \sum_{n=1}^\infty \frac{H_{2n-1}(u)}{(2n)!} \eta_n,
\]
where
\[
\eta_n = \int_0^T \int_K H_n(Z(x,t)) dxd\tau,
\]
and
\[
\mathbb{E}[\eta_n] = 0, \quad \mathbb{E}[\eta_n \eta_l] = 0, \quad n \neq l,
\]
\[
\sigma_{n,K}(T)^2 = \mathbb{E}[\eta_n^2] = 2n! \int_0^T \left(1 - \frac{\tau}{T}\right) \int_{K \times K} \tilde{C}^n(\|x - y\|, \tau) d\tau dy
\]
\[
= 2n! |K| \int_0^T \left(1 - \frac{\tau}{T}\right) \mathbb{E} \left[\tilde{C}^n(\|P_1 - P_2\|, \tau)\right] d\tau
\]
\[
= 2n! |K|^2 \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^{D(K)} \psi_{p,K}(z) C^n(z, \tau) dz d\tau.
\]

In particular for \(K = B(1)\), from (14) and (16),
\[
\sigma_{n,B(1)}^2(T) = \frac{8n! T^d}{d \left[\Gamma \left(\frac{d}{2}\right)\right]^2} \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^2 z^{d-1} C(z, \tau)
\]
\[
\times I_{1-\frac{d}{2}} \left(\frac{d + 1 + \frac{1}{2}}{2}\right) dz d\tau.
\]

**Condition 2.** Assume that
\( \sup_{z \in [0, D(K)]} |C(z, \tau)| \to 0, \tau \to \infty \)

(ii) For certain fixed \( m \in \{1, 2, \ldots\} \), there exists \( \delta \in (0, 1) \) such that

\[
\lim_{T \to \infty} \frac{1}{T^2} \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^{D(K)} C^m(z, \tau) \psi_{\rho_K}(z) dz d\tau = \infty.
\]

(20)

Under the above conditions we formulate the following results:

**Theorem 1** Under Conditions 1 and 2, for \( m = 1 \), the random variables

\[
X_{1,T} = \frac{M_T^{(1)}(u) - T|\mathcal{K}|(1 - \Phi(u))}{\phi(u) \left[ 2T \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^{D(K)} C(z, \tau) \psi_{\rho_K}(z) dz d\tau \right]^{1/2}},
\]

and

\[
\int_0^T \int_\mathcal{K} Z(x,t) dx dt \left[ 2T \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^{D(K)} C(z, \tau) \psi_{\rho_K}(z) dz d\tau \right]^{1/2}
\]

have the same limit as \( T \to \infty \). Namely, (22) has a standard normal distribution.

The analogous result to Theorem 1 for functional \( M_T^{(2)} \) is now formulated.

**Theorem 2** Under Conditions 1 and 2, with \( m = 2 \), the random variables

\[
X_{2,T} = \frac{M_T^{(2)}(u) - 2T|\mathcal{K}|(1 - \Phi(u))}{2\phi(u) \left[ 2T \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^{D(K)} C^2(z, \tau) \psi_{\rho_K}(z) dz d\tau \right]^{1/2}},
\]

and

\[
\int_0^T \int_\mathcal{K} (Z^2(x,t) - 1) dx dt \left[ 2T \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^{D(K)} C^2(z, \tau) \psi_{\rho_K}(z) dz d\tau \right]^{1/2}
\]

have the same limit distribution in the sense that if one exists then so does the other and the two are equal.

The proofs of Theorems 1 and 2 are obtained from Theorem 3 below in Section 4. In Sections 5 and 6, we will present some examples of covariance functions displaying long-range dependence in time, for which Conditions 2(i)–(ii) hold true.
4 Reduction theorems for spatiotemporal random fields with LRD in time

In this section, we extend the results by [36] and [37] to the case of spatiotemporal random fields with LRD in time. For a function $G \in L_2(\mathbb{R}, \phi(u)du)$, under Condition 1, we consider the following local functional

$$A_T = \int_0^T \int_{K} G(Z(x,t))dxdt$$

$$= T|K|G_0 + \sum_{n=1}^{\infty} \frac{G_n}{n!} \int_0^T \int_{K} H_n(Z(x,t))dxdt,$$  \hspace{0.5cm} (25)

where $G_n$ has been introduced in (14), and the series (25) converges in $L_2(\Omega, \mathcal{A}, P)$.

Applying (18), we obtain

$$\sigma_T^2 = \text{Var}(A_T) = \mathbb{E}[A_T - \mathbb{E}[A_T]]^2 = \sum_{n=0}^{\infty} \sigma_n^2(T),$$

where $\sigma_n^2(T)$ has also been introduced in (18).

The following condition was first introduced by [36].

**Condition 3.** We say that an integer $m \geq 1$ is the Hermite rank of function $G$, if for $m = 1$, $G_1 \neq 0$, or for $m \geq 2$, $G_1 = \cdots = G_{m-1} = 0$, $G_m \neq 0$.

**Theorem 3** Under Conditions 1, 2, and 3, the random variables

$$Y_T = \frac{A_T - \mathbb{E}[A_T]}{|G_m|\sigma_{m,K}(T)(1/m!)}$$  \hspace{0.5cm} (26)

and

$$Y_{m,T} = \frac{\text{sgn}\{G_m\} \int_0^T \int_{K} H_m(Z(x,t))dxdt}{\sigma_{m,K}(T)}$$  \hspace{0.5cm} (27)

have the same limiting distributions (if one of it exists).

**Proof.** We split

$$A_T - \mathbb{E}[A_T] = S_{1,T} + S_{2,T},$$

where using notation (18), and applying Parseval identity,

$$S_{1,T} = \frac{G_m}{m!}\xi_m, \quad S_{2,T} = \sum_{n=m+1}^{\infty} \frac{G_n}{n!}\xi_n, \quad \sum_{n=m}^{\infty} \frac{G_n^2}{n!} < \infty.$$  \hspace{0.5cm} (28)

From (18) and (28) we get

$$\text{Var}(A_T) = \text{Var}(S_{1,T}) + \text{Var}(S_{2,T}),$$  \hspace{0.5cm} (29)

9
and we have to show that
\[
\frac{\text{Var}(S_2, T)}{\sigma_{m,K}^2(T)} \to 0, \quad T \to \infty.
\]

Under **Condition 2(i),**
\[
\sup_{z \in [0, D(K)], \tau \geq T^\delta} |C(z, \tau)| \to 0, \quad T \to \infty, \quad (30)
\]
where \(\delta\) satisfies **Condition 2(ii).** Note that, for \(0 \leq \tau \leq T^\delta\), the unit variance of \(Z\) allows to work with the uniform estimate \(|C(z, \tau)|^{m+1} \leq 1, \quad z \in \mathbb{R}_+\). From (28), we then have
\[
\text{Var}(S_2, T) \leq \sum_{n=m+1}^{\infty} \frac{G_n^2}{(n!)^2} \sigma_{m,K}^2(T)
\]
\[
\leq M_1 \left\{ 2T \left[ \int_0^{T^\delta} + \int_{T^\delta}^T \right] \right\} \int_0^{D(K)} C^{m+1}(z, \tau) \psi_{\rho_K}(z) dzd\tau,
\]
\[
(31)
\]
for \(M_1 > 0\), whose value follows from (18). In addition, from (6),
\[
\psi_{\rho_K}(z) \leq \frac{z^{d-1}}{|K|} |S_{d-1}|, \quad 0 \leq z \leq D(K),
\]
leading to
\[
\text{Var}(S_2, T) \leq M_1 \left\{ M_2 T^{\delta+1} + 2T \int_{T^\delta}^T \left( 1 - \frac{T}{\tau} \right) \int_K C^{m+1}(z, \tau) \psi_{\rho_K}(z) dzd\tau \right\}
\]
\[
\leq M_3 \left\{ T^{\delta+1} + 2T \sup_{z \in [0, D(K)], \tau \geq T^\delta} (C(z, \tau)) \right\}
\times \left( 1 - \frac{T}{\tau} \right) \int_K C^m(z, \tau) \psi_{\rho_K}(z) dzd\tau \right\},
\]
\[
(33)
\]
Hence,
\[
\frac{\text{Var}(S_2, T)}{\sigma_{m,K}^2(T)} \leq M_1 \left\{ \frac{1}{T^{-(\delta+1)} \sigma_{m,K}^2(T)} \right\}
\]
\[
+ M_5 \sup_{z \in [0, D(K)], \tau \geq T^\delta} |C(z, \tau)| \frac{\int_T^{\infty} (1 - \frac{T}{\tau}) \int_K C^m(z, \tau) \psi_{\rho_K}(z) dzd\tau}{\int_0^{T} (1 - \frac{T}{\tau}) \int_K C^m(z, \tau) \psi_{\rho_K}(z) dzd\tau}\right\}.
\]
\[
(34)
\]
From (18), under **Condition 2(ii),**
\[
\frac{\sigma_{m,K}^2(T)}{T^{\delta+1}} \to \infty, \quad T \to \infty,
\]
\[
(35)
\]
and under **Condition 2(i)**,
\[
\sup_{z \in [0, D(K)], \tau \geq T^d} |C(z, \tau)| \to 0, \quad T \to \infty.
\]

Note that,
\[
\frac{\int_T^T (1 - \frac{\tau}{T}) \int_K C^m(z, \tau) \psi_{\rho c}(z) \,dz \,d\tau}{\int_0^T (1 - \frac{\tau}{T}) \int_K C^m(z, \tau) \psi_{\rho c}(z) \,dz \,d\tau} \leq 1. \tag{36}
\]

The convergence to zero of \(\frac{\text{Var}(S_{2,T})}{\sigma^2_{m,K}(T)}\) then follows from equation (34) under **Condition 2**. The result then follows, since \(\mathbb{E}[Y_T - Y_{m,T}]^2 = \frac{\text{Var}(S_{2,T})}{\sigma^2_{m,K}(T)}\).

**Remark 1** We have applied in (36) that, under **Condition 1**, the correlation function \(C(z, \tau) \geq 0\), for every \(\tau, z \in \mathbb{R}_+\).

Theorems 1 and 2 are respectively obtained from Theorem 3 for \(m = 1\) and \(m = 2\).

## 5 Separable covariance structures

Under **Condition 1**, the covariance function \(\tilde{C}(\|x - y\|, |t - s|)\) is said to be separable if it can be factorized as the product of a spatial \(C_S\) and temporal \(C_T\) covariance functions (see Cressie and Huang [7], and Christakos [6]). That is,
\[
\tilde{C}(\|x - y\|, |t - s|) = \tilde{C}_S(\|x - y\|)\tilde{C}_T(|t - s|), \tag{37}
\]
where, as before, \(\|x - y\| = z \geq 0\), and \(\tau = |t - s| \geq 0\).

**Condition 4**. Consider the covariance function
\[
C_T(\tau) = \frac{L(\tau)}{\tau^\alpha}, \quad \tau \geq 0, \quad \alpha \in (0, 1), \tag{38}
\]
where \(L\) is a slowly varying function locally bounded, i.e., bounded at each bounded interval.

Under **Conditions 1** and 4, for \(\alpha \in (0, \frac{1}{m})\), for separable covariance functions as given in (37), we obtain
\[
\sigma^2_{n}(T) = 2n!T \int_0^T \left(1 - \frac{\tau}{T}\right) C_T^\alpha(\tau) \,d\tau
\]
\[
= T^{2-n\alpha}\mathcal{L}^n(T) \left[2n! \int_0^1 (1 - \tau)^{-n\alpha} d\tau\right] (1 + o(1)). \tag{39}
\]
From (18) and (39), as \(T \to \infty\),
\[
\sigma^2_{n,K}(T) = c_K(n, \alpha)T^{2-n\alpha}\mathcal{L}^n(T) (1 + o(1)),
\]
where
\[ c_K(n, \alpha) = 2n! \left( \int_0^1 (1 - \tau) \frac{d\tau}{\tau^{\alpha + 1}} \right) |K|^2 \int_0^{D(K)} C_S(z) \psi_K(z) dz. \]

**Proposition 1** Under Conditions 1 and 3, for separable covariance functions (37), Condition 2(ii) holds for \( \alpha \in (0,1) \), if \( m = 1 \), and for \( \alpha \in (0,1/2) \) if \( m = 2 \). Moreover, for \( \alpha \in (0,1/2) \), the random variables (23) and (24) have, as \( T \to \infty \), the limiting distribution \( R \) of Rosenblatt type, given by the following Wiener-Itô integral representation, with respect to spatiotemporal complex Gaussian white noise random measure \( W \) on \( \mathbb{R}^2 \times \mathbb{R}^d \) (integration over hyperdiagonals are excluded, see, e.g., [8])

\[ R = c_T(\alpha) \oint_{\mathbb{R}^2} \int_{\mathcal{K}} \exp \left\{ i \langle \mu, \omega \rangle \right\} W(d\mu, d\omega), \tag{40} \]

where the Tauberian constant
\[ c_T(\alpha) = \frac{\Gamma \left( \frac{1}{2} - \alpha \right)}{2^{\alpha} \Gamma \left( \frac{3}{2} \right) \sqrt{\pi}}, \tag{41} \]

Note that \( E[R^2] < \infty \).

**Proof.** The proof of Proposition 1 is standard (see, e.g., Leonenko and Olenko [18]). A sketch of the proof is now given. Note that the spectral density of a spatiotemporal random field with separable covariance function (37) is also separable, i.e.,

\[ f(\omega, \mu) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R} \times \mathbb{R}^d} \exp (-i\mu \tau) \exp (-i \langle \omega, x \rangle) \tilde{C}_S(\|x\|) \tilde{C}_T(\|\tau\|) dx d\tau \]
\[ = \left[ \frac{1}{2\pi} \int_{\mathbb{R}^d} \exp (-i \langle \omega, x \rangle) \tilde{C}_S(\|x\|) dx \right] \left[ \frac{1}{2\pi} \int_{\mathbb{R}} \exp (-i\mu \tau) \tilde{C}_T(\|\tau\|) d\tau \right] \]
\[ = f_S(\omega) f_T(\mu), \quad \omega \in \mathbb{R}^d, \quad \mu \in \mathbb{R}, \tag{42} \]

Using Tauberian Theorems (see Leonenko and Olenko [19]), under Condition 4, we get convergence

\[ f_T(\mu) \sim c_T(\alpha) \frac{L \left( \frac{1}{\mu} \right)}{|\mu|^{1-\alpha}}, \quad \mu \to 0, \tag{43} \]

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for $0 < \alpha < \frac{1}{2}$, where the Tauberian constant $c_T(\alpha)$ has been introduced in (11). Applying the Wiener–Itô stochastic integral (see, e.g., [25], and Section 4.4.2 in [28]), we obtain isonormal representation:

$$Z(x, t) = \int_{\mathbb{R}^d} \exp(i\mu t) \exp(i\langle\omega, x\rangle) \sqrt{f_T(\mu) f_S(\omega)} W(d\mu, d\omega)$$  \hspace{1cm} (44)$$

with $W$ denoting complex–valued white noise measure.

For $d = d$ denoting the identity in probability distribution, applying now the self–similarity of Gaussian white noise random measure

$$W(d\mu, d\omega) = d\sqrt{ab^d/2} W(d\mu, d\omega), \quad \forall \mu \in \mathbb{R}, \ \omega \in \mathbb{R}^d,$$  \hspace{1cm} (45)$$

and the Itô formula (see, e.g., [8]; [27]), from equation (44), we obtain

$$Y_{2,T} = \int_0^T \int_K (Z^2(x, t) - 1) \, dx \, dt$$

\hspace{1cm} (46)$$

We denote

$$I_K = \frac{1}{c_K(2, \alpha)} \int_{\mathbb{R}^d} \left| \int_K \exp(i\langle x, \omega + \omega_2 \rangle) \, dx \right|^2 \left[ \prod_{j=1}^2 f_S(\omega_j) \right] \, d\omega_1 d\omega_2,$$

\hspace{1cm} (47)$$

where in the last identity, we have applied similar steps to (18).

From (46) and (47), we then obtain

$$E[Y_{2,T} - \mathcal{R}]^2 = [c_T(\alpha)]^2 I_K \int_{\mathbb{R}^d} \left| \int_0^1 \exp(i(\mu_1 + \mu_2)t) \, dt \right|^2$$

\hspace{1cm} (48)$$

\times \frac{1}{|\mu_1 \mu_2|^{1-\alpha}} Q_T(\mu_1, \mu_2) d\mu_1 d\mu_2,$
where

\[ Q_T(\mu_1, \mu_2) = \left( \frac{1}{c_T(\alpha)} |\mu_1 \mu_2| \prod_{j=1}^{2} f_{\frac{j}{T}} \left( \frac{\mu_j}{T} \right) \frac{1}{T^{1-\alpha} L(T)} \right)^2. \quad (49) \]

Applying Tauberian Theorems (see (43)), and Dominated Convergence Theorem, as \( T \to \infty \), \((48)\) converges to zero for \( \alpha \in (0, 1/2) \). Hence, the convergence in probability distribution of the random variable \( Y_{2,T} \) to \( \mathcal{R} \) holds (see Leonenko and Olenko [18], for more details).

### 6 Non–separable covariance functions

Let \( \varphi(v) \geq 0 \) be a completely monotone function. That is, an infinite differentiable function satisfying

\[ (-1)^n \frac{d^n \varphi}{dv^n}(v) \geq 0, \quad v > 0, \quad n \geq 0. \]

By Bernstein’s Theorem

\[ \varphi(v) = \int_0^v \exp(-v \xi) \mu(d\xi), \]

where \( \mu \) is a positive measure over \([0, \infty)\) (see Gneiting [11]).

Suppose further that \( \psi : [0, \infty) \to [0, \infty) \) has completely monotone derivatives, i.e., it is a Bernstein function. The Gneiting class of spatiotemporal covariance function is defined as follows (see Gneiting [11])

\[ \tilde{C}(|x - y|, |t - s|) = \frac{1}{\psi(|t - s|)^{d/2}} \varphi \left( \frac{|x - y|^2}{\psi(|t - s|)^2} \right) \]

\[ = C(z, \tau) = \frac{1}{[\psi(\tau^2)]^{d/2}} \varphi \left( \frac{z^2}{\psi(\tau^2)} \right) \]

\[ x, y \in \mathbb{R}^d, \; t, s \in \mathbb{R}, \; \tau, z \geq 0. \quad (50) \]

It is known that the one–parameter Mittag–Leffler function \( E_\nu \), for \( 0 < \nu \leq 1 \), is a completely monotone function (see Feller [10], p. 147), given by

\[ E_\nu(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + 1)}, \quad z \in \mathbb{C}, \; 0 < \beta < 1 \]

(see Erdélyi et al. [9]; Haubold, Mathai and Saxena [13]).

For every \( \nu \in (0, 1) \), uniformly in \( x \in \mathbb{R}_+ \), the following two–sided estimates are obtained with optimal constants (see [34], Theorem 4):

\[ \frac{1}{1 + \Gamma(1-\nu)x} \leq E_\nu(-x) \leq \frac{1}{1 + [\Gamma(1+\nu)]^{-1}x}. \quad (51) \]
Note that the function
\[ \psi(u) = (1 + au^\alpha)^\beta, \quad a > 0, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 1, \quad u \geq 0 \]
has completely monotone derivatives (as well as the functions, for \( b > 1, \psi_2(u) = \frac{\log(b + au^\alpha)}{\log(b)}, \) and \( \psi_3(u) = \frac{(b + au^\alpha)}{b^{(1+au^\alpha)}}, \) for \( 0 < b \leq 1). \) Thus, we consider the Gneiting class of covariance functions
\[ C_Z(z, \tau) = \frac{1}{(a\tau^{2a} + 1)^{3d/2}} E_\nu \left( \frac{z^{2\gamma}}{(a\tau^{2a} + 1)^{\beta \gamma}} \right). \]
\[ z, \tau \geq 0, \quad \nu, \alpha, \beta, \gamma \in (0, 1), \quad a > 0. \] (52)

From (51), the following proposition is derived.

**Proposition 2** Under Condition 1, and for the Gneiting class of covariance functions introduced in (52), Condition 2(ii) holds if \( m = 1, \) for \( 0 < 2\alpha \beta (d/2 - \gamma) < 1, \) and for \( 0 < 2\alpha \beta (d/2 - \gamma) < 1/2 \) if \( m = 2. \)

**Proof.** The proof follows straightforward from equation (52), applying (51). Specifically, for \( m = 1, \)
\[
\sigma_{1,K}(T) = 2T^2|K| \int_{[0,1]} (1 - \tau) \int_0^{D(K)} C_Z(z, T\tau)\psi_{\rho_K}(z) dzd\tau
\leq 2T^2|K|^2 \int_{[0,1]} (1 - \tau) \int_0^{D(K)} \frac{1}{(a[T\tau]^{2\alpha} + 1)^{3d/2}}
\times E_\nu \left( \frac{z^{2\gamma}}{(a[T\tau]^{2\alpha} + 1)^{\beta \gamma}} \right) \psi_{\rho_K}(z) dzd\tau
\geq 2T^2|K|^2 \int_{[0,1]} (1 - \tau) \int_0^{D(K)} \frac{1}{(a[T\tau]^{2\alpha} + 1)^{3d/2}}
\times \frac{1}{1 + \Gamma(1 - \nu)\frac{z^{2\gamma}}{(1 + a[T\tau]^{2\alpha})^{\beta \gamma}}} \psi_{\rho_K}(z) dzd\tau
\geq 2T^2|K|^2 \int_{[0,1]} (1 - \tau) \int_0^{D(K)} \frac{1}{a^{\beta \gamma}T^{2\alpha \beta \gamma} \tau^{2\alpha \beta \gamma}}
\times \frac{1}{[1 + aT^{2\alpha \beta \gamma} \tau^{2\alpha \beta \gamma} + \Gamma(1 - \nu)z^{2\gamma}]} \psi_{\rho_K}(z) dzd\tau
= 2T^2(1-\alpha(\frac{d}{2}-\gamma))|K|^2 \int_{[0,1]} (1 - \tau) \int_0^{D(K)} \frac{1}{(aT^{2\alpha} + 1)^{(\beta \gamma) \frac{3d}{2}}}
\times \frac{1}{[1 + aT^{2\alpha} \tau^{2\alpha} + \Gamma(1 - \nu)z^{2\gamma}]} \psi_{\rho_K}(z) dzd\tau.
\] (53)
From \((53)\), **Condition (ii)** holds for 
\[ 2 \alpha \beta (\frac{d}{2} - \gamma) < 1. \]
In a similar way to (53), it can be proved that for \(m = 2\), **Condition (ii)** also holds for 
\[ 2 \alpha \beta (\frac{d}{2} - \gamma) < \frac{1}{2}. \]

As a direct consequence of Proposition 2, we obtain that Theorems 1 and 2 hold for the family of spatiotemporal Gaussian random fields with covariance function (52).

Similar assertions hold for the family of spatiotemporal covariance functions
\[
C_{Z}(z, \tau) = \frac{\sigma^2}{\psi(\tau^2)\sqrt[3]{2}} \varphi \left( \frac{\|z\|^2}{\psi(\tau^2)} \right), \quad \sigma^2 \geq 0, \ (z, \tau) \in \mathbb{R}^d \times \mathbb{R}
\]
\[
\varphi(u) = \frac{1}{(1 + cu)^\nu}, \quad u > 0, \ c > 0, \ 0 < \gamma \leq 1, \nu > 0
\]
\[
\psi(u) = (1 + au^\alpha)^\beta, \quad a > 0, \ 0 < \alpha \leq 1, \ 0 < \beta \leq 1, \ u \geq 0,
\]
for \(2 \alpha \beta (\frac{d}{2} - \gamma \nu) < 1\) if \(m = 1\), and for \(2 \alpha \beta (\frac{d}{2} - \gamma \nu) < 1/2\) if \(m = 2\).

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