The Klein-Gordon oscillator and the proper time formalism in a

Rigged Hilbert space

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Abstract

The implications of manifestly covariant formulation of relativistic quantum mechanics depending on a scalar evolution parameter, canonically conjugated to the variable mass, is still an unsettled issue. In this work we find a complete set of generalized eigenfunctions of the Klein-Gordon Oscillator in the above mentioned formulation, in the Rigged Hilbert Space with Tempered Ultradistributions. We briefly comment on some models where this solution could be applied.

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I. INTRODUCTION

One of the main features of the relativistic mechanics is that time and space coordinates are on the same footing. In contrast, in standard relativistic quantum mechanics the treatment of those quantities is not symmetric. This fact poses a puzzle to physics since long time ago. In a fine set of papers, Dirac introduced the idea of promoting the time coordinate $x^0$ to the rank of operator $[1]$. This proper time formalism introduces an absolute evolution parameter (related to the proper time in the classical limit), which parametrizes the dynamics of the quantum system. The basic idea of these works have probably fallen into oblivion because Dirac himself did not insist on it in his celebrated paper of 1928 [2]. This formalism shortly returned to view with Feynman and Stückelberg antiparticles interpretation as particles moving backward in time [3,4]. A lot of progress have been done in oder to enlight this ideas [5–9]. We think that this formalism is the natural generalization of the relativistic quantum mechanics which gives a consistent way to study quantum systems. In this scenario, we work out the harmonic oscillator (HO) solution of the Klein–Gordon (KG) equation in the Rigged Hilbert Space (RHS) with Tempered Ultradistributions (for an explanation about Rigged Hilbert Spaces and physical applications see ref. [10]) adopting the Collins and Fanchi (CF) formalism [6]. In ref. [11] it is shown how to work out the quantum mechanics in an abstract RHS. The results obtained in our paper are derived from those obtained in ref. [11] and [12]. As Feynman pointed out [3], the proper time formalism introduces exponentially increasing solutions in the motion equations. The introduction of the RHS $(h, \mathcal{H}, \Lambda_\infty)$ and in $(H, \mathcal{H}, \mathcal{U})$ has allowed us to solve this problem. With this tools, our work contributes enrich the study of relativistic quantum systems in terms of the proper
time theory.

We have also a more practical motivation. Our solutions could be also relevant because the HO is present in most problems of physics describing different interactions and we use a consistent mechanism to obtain the eigenvalues and eigenfunctions. In the last decades many works have been done in order to describe, under a quantum relativistic frame, the harmonic oscillator. Kim and Noz [13] proposed a covariant HO in a hyperplane formalism to introduce it in the quark model. They had also studied excited meson decays with this technique. Moshinsky et al have introduced a new kind of interaction in Dirac equation which in the non-relativistic limit becomes an harmonic oscillator with a very strong spin-orbit coupling term. Later on, this procedure was followed by Moreno and Zentella [14] to take into account the quark-antiquark interaction giving the mass spectra for quarkonium systems.

In our work, we present the eigenvalues and eigenfunctions for the harmonic oscillator in the proper time formalism. The solutions we have obtained contains, as a particular case, those for the usual KGHO. Furthermore, we have solutions for particles with negative mass.

The paper is organized as follows: in Sect. 2 we present the proper time formalism. Sect. 3 is devoted to the study of the harmonic oscillator in the above mentioned frame. In Sect. 4 we discuss the results.

II. THE PROPER-TIME FORMALISM

Let us briefly sum up the characteristic of the formalism. The square of the amplitude of the wave function $\Psi$ must be interpreted as the probability of an event corresponding to the position of a particle at a particular time. It is worthwhile to point out that the
latter interpretation differs from the usual one due to that there is a probability distribution associated with time. Therefore, for each $t$ there is a probability that a particle may or may not be found somewhere in space (as in particle decays). Then, the mean value of the position operator $\hat{x}$, can be written as

$$< x^\mu > = \int_{ST} x^\mu \rho \, d^4x \quad (1)$$

where $\rho$ is a distribution on the spacetime manifold satisfying the two following conditions:

$$\rho > 0 \quad \text{and} \quad \int_{ST} \rho \, d^4x = 1$$

with integration understood in the Lebesgue sense, over the entire spacetime manifold.

Taking this into account, the on-shell condition must be understood as the expectation value of the $p^\mu p_\mu$ operator. As shown by CF, the aforementioned operator is proportional to $\partial/\partial \tau$, and the light-cone constraint identifies $\tau$ with the classical proper time $\tilde{\tau}$. Thereby, the free particle equation, in natural units, is given by

$$i \frac{\partial \Psi}{\partial \tau} = H \Psi \quad (2)$$

here $H = p^\mu p_\mu$ and $\tilde{m}$ is defined as the classical limit of the $p^\mu p_\mu$ expectation value.

Our purpose is to work out the relativistic spinless harmonic oscillator in the frame of the formalism we have commented.

**III. KLEIN-GORDON OSCILLATOR**

The hamiltonian for the relativistic harmonic oscillator was studied by Moshinsky and Szczepaniak [15]. They proposed a new type of interaction in the Dirac equation, linear in coordinates and momentum. The corresponding equation has been named “Dirac Oscillator”
because in the non-relativistic limit the harmonic oscillator has been obtained. This kind of interaction was introduced in the Klein-Gordon equation \[16,17\]. To get a closed form for the extended relativistic harmonic oscillator hamiltonian, we present first the Klein-Gordon oscillator developed in the Bruce \[16\] formalism together with the Sakata-Taketani approach \[18\]. The latter selection determines the following KG equation

\[
\left(\Box + m^2 \omega^2 r^2 - 3m \omega + m^2 \right) \Psi = 0
\]  

\[3\]

IV. SCALAR-TIME PARAMETRIZATION OF KLEIN-GORDON OSCILLATOR

Now, we want to work out this equation in the frame of the scalar time parametrization taking into account the above definition of the \(p^\mu p_\mu\) operator and considering a probability distribution associated with time. To solve this equation we shall consider the wave function \(\Psi(x_\mu, \tau)\) as an element of the space of exponentially increasing distributions (in the variable \(\tau\)) \(\Lambda_\infty \[19\]. The space \(\Lambda_\infty\) is formed by distributions \(T\) of the exponential type satisfying:

\[
T = \left(\partial^k/\partial x^k\right) \left[ e^{k|x|} f(x) \right]
\]

\[4\]

where \(k\) is an integer greater than or equal to zero and \(f\) is bounded continuous.

\(\Lambda_\infty\) is the dual of the space \(H\) of all functions \(\phi \in C^\infty\) in \(R\) (real numbers) such that \(e^{k|x|} D^p \phi(x)\) is bounded in \(R\) for all \(k\) and \(p\).

If \(\mathcal{H}\) is the Hilbert space of square integrable functions, the triplet \((H, \mathcal{H}, \Lambda_\infty)\) is a Rigged Hilbert Space (RHS) \[21\]. The Fourier transformed triplet of \((H, \mathcal{H}, \Lambda_\infty)\) is the RHS \((h, \mathcal{H}, \mathcal{U})\). In this triplet \(h\) is the space of analytical test functions, rapidly decreasing in any horizontal band. We denote by \(a_\omega\) the space of all functions \(F(z)\) such that:
i) $F(z)$ is analytic in $\{ z \in C : |Im(z)| > k \}$

ii) $F(z)/z^k$ is bounded continuous in $\{ z \in C : |Im(z)| \geq k \}$ where $k$ is an integer depending on $F(z)$. Let $\Pi$ be the set of all polynomials in the variable $z$. It has been demonstrated in ref. [19] that $\mathcal{U} = a_\omega/\Pi$ where $\mathcal{U}$ is by definition the space of Tempered Ultradistributions [19–22]. In the RHS $(H, \mathcal{H}, \Lambda_\infty)$ (and in $(h, \mathcal{H}, \mathcal{U})$) a linear and symmetric operator $A$ acting on $H(h)$ which admits a self-adjoint prolongation $\bar{A}$ acting on $\mathcal{H}$, has a complete set of eigen-functionals on $\Lambda_\infty(\mathcal{U})$ with real generalized eigenvalues [23,24].

According to ref. [22], if $f \in \Lambda_\infty$, $\phi \in H$, $\hat{f} \in \mathcal{U}$ and $\hat{\phi} \in h$ we have:

$$< f, \phi > = \int_{-\infty}^{+\infty} \hat{f}(\tau) \phi(\tau) \, d\tau = \int_\Gamma \hat{f}(\alpha) \hat{\phi}(\alpha) \, d\alpha = < \hat{f}, \hat{\phi} >$$

(5)

where $\Gamma$ is the path which surrounds all the singularities of $\hat{f}(\alpha)$ placed on a $2k$ wide band that encircles the real axis. The path $\Gamma$ runs from $-\infty$ to $+\infty$ along $Im(\alpha) > k$ and from $+\infty$ to $-\infty$ along $Im(\alpha) < -k$ (ref. [19]) and

$$\overline{f(\tau)} = \int_\Gamma \hat{f}(\alpha)e^{-i\tau\alpha} \, d\alpha = \mathcal{F}^{-1}\{\hat{f}(\alpha)\}$$

(6)

$$\phi(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\phi}(\alpha)e^{i\tau\alpha} \, d\alpha = \mathcal{F}^{-1}\{\hat{\phi}(\alpha)\}$$

(7)

We can define $(2 \hat{m} i \frac{\partial}{\partial\tau})^{1/2}$ operating over a $\Lambda_\infty$ distribution as follows [12]:

$$\left(2 \hat{m} i \frac{\partial}{\partial\tau}\right)^{1/2} f(\tau) = \mathcal{F}^{-1}\{\alpha^{1/2} (\hat{f}(\alpha) + a(\alpha))\}$$

(8)

where
\[ \hat{f}(\alpha) = \mathcal{F}\{f(\tau)\} \quad (9) \]

and \(a(\alpha)\) is an entire analytical function. In view of this definition and (5), (6) and (7) it follows that the operator \((i\partial/\partial\tau)^{1/2}\) is linear and self-adjoint.

With all these requirements eq. (3), takes the form:

\[
\begin{bmatrix}
\Box + 2m i \omega^2 r^2 \frac{\partial}{\partial\tau} - 3 \omega \left(2m i \frac{\partial}{\partial\tau}\right)^{1/2} + 2m i \frac{\partial}{\partial\tau}
\end{bmatrix} \Psi = 0 \quad (10)
\]

Thus, taking into account the Fourier transform, equation (10) could be written as follows:

\[
\int_{\Gamma} \left(\Box + \omega^2 r^2 \alpha - 3\omega \alpha^{1/2} + \alpha \right) \left[\hat{\Psi}_c(x_\mu, \alpha) + a(x_\mu, \alpha)\right] e^{-i\alpha 2m \tau} d\alpha = 0 \quad (11)
\]

where \(\Gamma\) is the usual path which surrounds all the singularities of \(\hat{\Psi}_c\) and \(a(x_\mu, \alpha)\) is an entire analytical function of the variable \(\alpha\). Defining the operator

\[
\Box + \omega^2 r^2 \alpha - 3\omega \alpha^{1/2} + \alpha = L \quad (12)
\]

then, equation (11) is equivalent to

\[
L \left[\hat{\Psi}_c(x_\mu, \alpha) + a(x_\mu, \alpha)\right] = 0 \quad (13)
\]

Thus, we can write:

\[
\hat{\Psi}_c(x_\mu, \alpha) = \hat{f}_c(x_\mu, \alpha) - a(x_\mu, \alpha) \quad (14)
\]

with \(\hat{f}_c\) a general solution of the homogeneous system,

\[
L \hat{f}_c(x_\mu, \alpha) = 0 \quad (15)
\]

The boundary condition we shall impose is to set the entire analytical function \((a(x_\mu, \alpha))\) equal to zero; with this condition we ensure that for a fixed \(\alpha > 0\) we obtain the same energy spectrum as in the usual KG formalism. Therefore we obtain,
\[
\int_{\Gamma} \left( \Box + \omega^2 r^2 \alpha - 3\omega \alpha^{1/2} + \alpha \right) \hat{f}_c(x_\mu, \alpha) e^{-\frac{i\alpha}{2m}r} d\alpha = 0 \quad (16)
\]

On the real axis this expression takes the form,

\[
\int_{-\infty}^{+\infty} \left[ \left( \Box + \omega^2 r^2 \alpha - 3\omega(\alpha+i0)^{1/2} + \alpha \right) \hat{f}_c(x_\mu, \alpha + i0) \right. \\
\left. \left( \Box + \omega^2 r^2 \alpha - 3\omega(\alpha-i0)^{1/2} + \alpha \right) \hat{f}_c(x_\mu, \alpha - i0) \right] e^{-\frac{i\alpha}{2m}r} d\alpha = 0 \quad (17)
\]

If we impose to \( \hat{f}_c \) to satisfy the equality

\[
\int_{-\infty}^{0} \left[ (\alpha + i0)^{1/2} \hat{f}_c(x_\mu, \alpha + i0) - (\alpha - i0)^{1/2} \hat{f}_c(x_\mu, \alpha - i0) \right] e^{-\frac{i\alpha}{2m}r} d\alpha = 0 \quad (18)
\]

equation (17) reads,

\[
\int_{0}^{+\infty} \left( \Box + \omega^2 r^2 \alpha - 3\omega \alpha^{1/2} + \alpha \right) \left[ \hat{f}_c(x_\mu, \alpha + i0) - \hat{f}_c(x_\mu, \alpha - i0) \right] e^{-\frac{i\alpha}{2m}r} d\alpha + \\
\int_{-\infty}^{0} \left( \Box + \omega^2 r^2 \alpha + \alpha \right) \left[ \hat{f}_c(x_\mu, \alpha + i0) - \hat{f}_c(x_\mu, \alpha - i0) \right] e^{-\frac{i\alpha}{2m}r} d\alpha = 0 \quad (19)
\]

Next, we define as in ref. \[20, 19\]

\[
\hat{f}(x_\mu, \alpha) = \hat{f}_c(x_\mu, \alpha + i0) - \hat{f}_c(x_\mu, \alpha - i0) \quad (20)
\]

Bearing this in mind, our system can be reduced to

\[
\left( \Box + \omega^2 r^2 \alpha - 3\omega \alpha^{1/2} + \alpha \right) \hat{f}(x_\mu, \alpha) = 0 \quad \alpha > 0 \quad (21)
\]

\[
\left( \Box + \omega^2 r^2 \alpha + \alpha \right) \hat{f}(x_\mu, \alpha) = 0 \quad \alpha < 0 \quad (22)
\]

which is solved using the standard procedure of quantum mechanics. Thus, introducing

\[
\hat{f} = e^{-i\frac{E}{2m}t} \varphi \quad \text{eq. (21)} \]

becomes

\[
(-\triangle + \alpha \omega^2 r^2 - 3\alpha^{1/2} \omega + \alpha) \varphi = E^2 \varphi \quad (23)
\]
The eingenfunctions that satisfy the last equation are:

\[ \varphi_{\alpha N_1 N_2 N_3} = A_{\alpha N_1 N_2 N_3} e^{-\alpha^{1/2}(x^2 + y^2 + z^2)} \frac{\omega^2}{2} \]

\[ \times H_{N_1} (\sqrt{\alpha^{1/2} \omega} x) H_{N_2} (\sqrt{\alpha^{1/2} \omega} y) H_{N_3} (\sqrt{\alpha^{1/2} \omega} z) \]  

(24)

where \( H_{N_i} \ i = 1, 2, 3 \) are Hermite polynomials and \( E \) satisfies,

\[ E^2 - 2 (N_1 + N_2 + N_3) \frac{\alpha^{1/2} \omega}{2} + \alpha = 0 \]  

(25)

with \( N_i \in \mathbb{N} \) (natural numbers) \( (i = 1, 2, 3) \).

Finally we obtain for \( \hat{\Psi} \)

\[ \hat{\Psi}_{\alpha N_1 N_2 N_3} = A_{\alpha N_1 N_2 N_3} e^{-i\alpha \tau / 2m} e^{-iE(\alpha,N_1,N_2,N_3)t} e^{-(x^2 + y^2 + z^2) \frac{\alpha^{1/2} \omega^2}{2}} \]

\[ \times H_{N_1} (\sqrt{\alpha^{1/2} \omega} x) H_{N_2} (\sqrt{\alpha^{1/2} \omega} y) H_{N_3} (\sqrt{\alpha^{1/2} \omega} z) \]  

(26)

with \( A_{\alpha N_1 N_2 N_3} \) the normalization constant given by

\[ A_{\alpha N_1 N_2 N_3} = \frac{\alpha^{1/2} \omega}{\pi} \frac{1}{\sqrt{2 \pi m 2(N_1 + N_2 + N_3)/2} (N_1! N_2! N_3!)^{1/2}} \]  

(27)

We now turn to the calculation of \( \Psi \) in the case \( \alpha < 0 \). The equation (22) is now

\[ (- \Delta + \alpha \omega^2 r^2 + \alpha) \varphi = E^2 \varphi \]  

(28)

This equation has the following eight independent solutions:
\[ \hat{\Phi}_{\alpha E_1E_2E_3}(x, y, z) = \begin{cases} 
  x y z & \hat{\Phi}_{E_1}(x) \hat{\Phi}_{E_2}(y) \hat{\Phi}_{E_3}(z) \\
  x y & \hat{\Phi}_{E_1}(x) \hat{\Phi}_{E_2}(y) \hat{\Phi}'_{E_3}(z) \\
  x z & \hat{\Phi}_{E_1}(x) \hat{\Phi}'_{E_2}(y) \hat{\Phi}_{E_3}(z) \\
  x & \hat{\Phi}_{E_1}(x) \hat{\Phi}'_{E_2}(y) \hat{\Phi}'_{E_3}(z) \\
  y z & \hat{\Phi}'_{E_1}(x) \hat{\Phi}_{E_2}(y) \hat{\Phi}_{E_3}(z) \\
  y & \hat{\Phi}'_{E_1}(x) \hat{\Phi}_{E_2}(y) \hat{\Phi}'_{E_3}(z) \\
  z & \hat{\Phi}'_{E_1}(x) \hat{\Phi}'_{E_2}(y) \hat{\Phi}_{E_3}(z) \\
  & \hat{\Phi}'_{E_1}(x) \hat{\Phi}'_{E_2}(y) \hat{\Phi}'_{E_3}(z) \end{cases} \] (29)

with

\[ \hat{\Phi}_{E_1\alpha}(x) = \Phi \left( \frac{3}{4} + \frac{i \left( E^2_1 - \frac{\alpha}{2} \right)}{4 (-\alpha)^{1/2} \omega}, \frac{3}{2}, i \sqrt{-\alpha \omega x^2} \right) \] (30)

\[ \hat{\Phi}_{E_2\alpha}(y) = \Phi \left( \frac{3}{4} + \frac{i \left( E^2_2 - \frac{\alpha}{2} \right)}{4 (-\alpha)^{1/2} \omega}, \frac{3}{2}, i \sqrt{-\alpha \omega y^2} \right) \] (31)

\[ \hat{\Phi}_{E_3\alpha}(z) = \Phi \left( \frac{3}{4} + \frac{i \left( E^2_3 - \frac{\alpha}{2} \right)}{4 (-\alpha)^{1/2} \omega}, \frac{3}{2}, i \sqrt{-\alpha \omega z^2} \right) \] (32)

\[ \hat{\Phi}'_{E_1\alpha}(x) = \Phi \left( \frac{1}{4} + \frac{i \left( E^2_2 - \frac{\alpha}{2} \right)}{4 (-\alpha)^{1/2} \omega}, \frac{1}{2}, i \sqrt{-\alpha \omega x^2} \right) \] (33)

\[ \hat{\Phi}'_{E_2\alpha}(y) = \Phi \left( \frac{1}{4} + \frac{i \left( E^2_3 - \frac{\alpha}{2} \right)}{4 (-\alpha)^{1/2} \omega}, \frac{1}{2}, i \sqrt{-\alpha \omega y^2} \right) \] (34)

\[ \hat{\Phi}'_{E_3\alpha}(z) = \Phi \left( \frac{1}{4} + \frac{i \left( E^2_2 - \frac{\alpha}{2} \right)}{4 (-\alpha)^{1/2} \omega}, \frac{1}{2}, i \sqrt{-\alpha \omega z^2} \right) \] (35)

where \( \Phi(\alpha, \beta, s) \) are the degenerate hypergeometric functions \(^{26}\) and \( E^2_1 + E^2_2 + E^2_3 = E^2 \) are real numbers. Thus we obtain for \( \Psi \) the expression:
\[ \hat{\Psi}_{N_1 N_2 N_3} = A_{N_1 N_2 N_3} e^{-i \alpha \tau / 2\hbar} e^{-i E t} \]
\[ \times e^{-i (x^2 + y^2 + z^2) \sqrt{-\alpha \omega / 2}} \hat{\Phi}_{\alpha E_1 E_2 E_3} (x, y, z) \]

(36)

with

\[ A_{\alpha E_1 E_2 E_3} = (|E_1||E_2||E_3|) \times \]
\[ \frac{\left| \Gamma \left( \frac{1}{4} + \frac{i(E_1^2 - \alpha/3)}{4(-\alpha)^{1/2} \omega} \right) \right| \left| \Gamma \left( \frac{1}{4} + \frac{i(E_2^2 - \alpha/3)}{4(-\alpha)^{1/2} \omega} \right) \right| \left| \Gamma \left( \frac{1}{4} + \frac{i(E_3^2 - \alpha/3)}{4(-\alpha)^{1/2} \omega} \right) \right|}{2^{5\pi/2} \omega^{3/2} (-\alpha)^{3/4} \hbar^{1/2}} \]

(37)

The function \( \hat{\Psi} \) has been normalized to

\[ \delta(\alpha - \alpha') \delta(E_1 - E'_1) \delta(E_2 - E'_2) \delta(E_3 - E'_3) \]

(38)

Finally the general solution is given by

\[ \Psi(x, \tau) = \int_0^\infty d\alpha \sum_{N_1 N_2 N_3} C_{\alpha N_1 N_2 N_3} e^{-i E(\alpha N_1 N_2 N_3) t} e^{-i \alpha \tau / 2\hbar} e^{-i (x^2 + y^2 + z^2) \sqrt{-\alpha \omega / 2}} \]
\[ \times H_{N_1} (\sqrt{\alpha^{1/2} \omega} x) H_{N_2} (\sqrt{\alpha^{1/2} \omega} y) H_{N_3} (\sqrt{\alpha^{1/2} \omega} z) + \]
\[ \int_{-\infty}^0 d\alpha \int_{-\infty}^{+\infty} dE_1 dE_2 dE_3 C_{\alpha N_1 N_2 N_3} e^{-i \sqrt{E_1^2 + E_2^2 + E_3^2} t} \]
\[ \times e^{-i \alpha \tau / 2\hbar} e^{-i (x^2 + y^2 + z^2) \sqrt{-\alpha \omega / 2}} \phi_{\alpha E_1 E_2 E_3} (x, y, z) \]

(39)

V. DISCUSSION

In this paper we have shown how to solve a model with a simple interaction in the proper time formalism. We have presented the complete set of eigenfunctions for the harmonic oscillator KG equation. In particular, we have worked out this problem in the frame of the
Rigged Hilbert Space with Tempered Ultradistributions and their inverse Fourier transform space (exponentially increasing distributions). We presented an example of finding pseudodifferential motion equation solutions by the use of ultradistributions.

From the analysis of our equations, we can say that it is possible to select \( a(x_\mu, \alpha) = 0 \). In this case, we obtain for fixed \( \alpha > 0 \), as a particular case, the energy spectrum which coincides with the usual KG harmonic oscillator solution, identifying \( m = \alpha^{1/2} \) (being \( m \) the mass of the oscillator). This can be deduced from the dispersion relation (see eq. (25)). The eigenfunctions given by eq. (24) are the same of those for the usual KG harmonic oscillator. It is clear from the results of this paper that contrary to the usual KG formalism, in the proper time model, solutions for negative values of \( \alpha \) do exist. These solutions represent tachyonic particles. On the contrary to the usual quantum field theory, solutions with \( \alpha < 0 \) are well-behaved in the sense that they are oscillating (as for a normal bradyonic particle), instead of exponentially increasing. We want to stress that the solutions we have found are valid for arbitrary values of the parameter associated with the mass.

In summary, we have found the solutions for the eigenvalues and eigenfunctions for the HO in the proper time formalism. Our contribution is twofold. On one hand, this results contributes to enrich the study of relativistic quantum systems in terms of the proper time theory. On the other hand, our treatment for the harmonic oscillator could be relevant to current problems in contemporary physics. The generalization of our problem to the case of two interacting particles via an harmonic oscillator potential, will be very important to describe bound states. In order to do so, further considerations must be taken into account as de reduction of motion of the mass center plus a relative movement. We hope to report about this issue in a forthcoming paper.
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