Determination of constraint stabilization parameters with multiple roots of characteristic equation

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Abstract. During a numerical integration of motion equations of a constrained mechanical system effect of deviations from initial data can be neglected with the help of Baumgarte stabilization method. According to this method first time derivatives of constraint equations are not considered as first integrals and are equated to an arbitrarily linear form of constraints themselves. The coefficients of these forms are called perturbation parameters. So, the problem of constraint stabilization is reduced to a problem of finding an appropriate range of perturbation parameters’ values. This problem can be solved by obtaining these values with the help of stability conditions. The procedure of evaluation is getting even more easier if the perturbation coefficients correspond to the multiple root case of its characteristic equation.

1. Introduction
Application of the simplest methods of numerical integration to systems of differential equations with constraints does not always lead to a satisfactory result. The accumulation of constraint deviations at each integration step causes an instability of the numerical solution with respect to the constraint equations. To limit this kind of accumulation, the method of constraint stabilization suggested by Baumgarte is applied [1]. According to this method, when determining arbitrary Lagrange multipliers, the total time derivative of the constraint equations is equated to the linear form according to them. The resulting expression, in fact, is an equation of perturbed constraints. Then the determination of the optimal range of values of the coefficients of the linear form, which will be called in the future perturbation parameters, is the goal of constraint stabilization. The paper [2] presents a mechanism for estimating the perturbation parameters, calculated with the condition of not increasing the deviation from the constraints. When implementing the numerical integration of the equations of motion with constraint stabilization, it is convenient to use the method of self-determination of the perturbation parameters presented in [3]. However, when considering the equation of perturbation due to multiple roots, the process of self-determination becomes much easier, due to the reduction in the number of required quantities. In this paper, we demonstrate an algorithm for self-determination of the root of the characteristic equation of perturbed constraints, which sets the values of all perturbation parameters.

2. Problem statement
Let’s consider that a position of a mechanical system is determined by a set of generalized coordinates \( x = (x_1, \ldots, x_n) \) and velocities \( v = (v_1, \ldots, v_n) \). Then a system of system’s motion equations can be written as follows:
\[
\begin{align*}
\dot{x}_i &= \frac{dx_i}{dt} = v_i, \\
\dot{v}_i &= \frac{dv_i}{dt} = a_i(x, v, t), & i = 1, \ldots, n.
\end{align*}
\]
Let the system be restricted with a set of holonomic constraints

$$f_\mu(x) = 0, \ \mu = 1, ..., m \leq n. \quad (2)$$

The method of arbitrary multipliers ($\lambda_1, ..., \lambda_m$) is used to solve the system of equations with constraints. Then the system of equations (1) will be rewritten as:

$$\dot{x}_i = v_i, \ \dot{v}_i = a_i(x, v, t) + \lambda_\mu \frac{\partial f_\mu}{\partial x_i}, \ i = 1, ..., n, \quad (3)$$

here and later, repeated indexes mean summation.

Taking into account the method of Baumgarte’s stabilization, the equations of perturbed constraints are chosen so that the roots of the characteristic equation are multiple. So, we have

$$\ddot{f}_\mu + 2k_\mu \dot{f}_\mu + k_\mu^2 f_\mu = 0, \ \mu = 1, ..., m \leq n, \quad (4)$$

where $k_\mu$ is a perturbation parameter.

The equations of motion (3) together with the equations of perturbed constraints (4) are a system of differential algebraic equations that allow to uniquely determine Lagrange multipliers as functions of generalized coordinates and velocities.

Let the numerical integration be carried out using a first order difference scheme:

$$x_{(n+1)} = x_{(n)} + \tau v_{(n)}, \ v_{(n+1)} = v_{(n)} + \tau V_{(n)}, \ t_{(n+1)} = t_{(n)} + \tau, \quad (5)$$

where the index in parentheses indicates the integration step and $V_{(n)} = a(x_{(n)}, v_{(n)}, t) + \lambda_\mu(x_{(n)}, v_{(n)}, t) \frac{\partial f_\mu}{\partial x} \Big|_{x=x_{(n)}}$.

To control the accumulation of errors, we introduce estimations of the deviation from the constraints at each step. If $||f(x_{(n)})|| \leq \varepsilon$, then, it is necessary that $||f(x_{(n+1)})|| \leq \varepsilon, \ \forall n = 1, ..., N$ for the deviation values not to increase. Here $f = (f_1, ..., f_\mu, \ ..., f_m)$ is a vector of values of the deviations from the constraints and their derivatives. Then the equations of perturbed constraints (4) can be rewritten in vector form:

$$\dot{f} = Kf, \quad (6)$$

where $K$ is a matrix of perturbation parameters corresponding to the linear form (4).

Then we expand the value of the deviation at the $n + 1$ step in the Taylor series by the integration step $\tau$. With regard for stabilization, the expansion will take form

$$f(x_{(n+1)}) = f(x_{(n)}) + \tau Kf(x_{(n)}) + \frac{\tau^2}{2} \frac{d^2f(x_{(n+1)})}{dt^2} \big|_t=\xi, \ \xi \in [t_{(n)}, t_{(n+1)}]. \quad (7)$$

Then, based on the condition of nonincreasing of the deviation value at each step, the following relation takes place:

$$||I_{2\mu} + \tau K|| \leq 1 - \frac{\tau^2}{2\varepsilon} \bigg| \frac{d^2f(x_{(n+1)})}{dt^2} \bigg|_{t=\xi}, \ \xi \in [t_{(n)}, t_{(n+1)}]. \quad (8)$$

where $I_{2\mu}$ is a unit matrix $[2\mu \times 2\mu]$.

This condition allows us to determine the range of values of the perturbation parameters $k_\mu, \ \mu = 1, ..., m$. If there is only one holonomic constraint, the expression (8) allows us to uniquely determine the values of the perturbation parameter. Then for the Euclidean norm the expression (8) with one connection will be rewritten as:

$$\sqrt{1 + (1 - 2\tau k)^2 + \tau^2 k^4} \leq 1 - \frac{\tau^2}{2\varepsilon} \bigg| \frac{d^2f(x_{(n+1)})}{dt^2} \bigg|_{t=\xi}. \quad (9)$$
3. **Example and conclusion**

As an example (3) the equation of a mathematical pendulum with a constraint is considered:

\[
\begin{align*}
\ddot{x} &= g + 2\lambda x, \\
\ddot{y} &= 2\lambda y, \\
x^2 + y^2 - R^2 &= 0
\end{align*}
\]

(10)

Numerical integration (10) was carried out using a difference scheme (5) taking into account constraint stabilization (4). The perturbation parameter was automatically determined during integration by the formula (9). Its numerical values are shown in figure 1.

![The graphic perturbation parameter k on time t](image)

**Figure 1.** Dependence of the perturbation parameter on time

Unambiguous determination of the perturbation parameter during numerical integration is possible with only one holonomic constraint. However, this method allows us to automatically solve the problem of constraint stabilization.

**References**

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