Exact Solutions for Cosmological Models with a Scalar Field

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Abstract

We consider the existence of a Noether symmetry in the scalar-tensor theory of gravity in flat Friedman Robertson Walker (FRW) cosmology. The forms of coupling function $\omega(\phi)$ and generic potential $V(\phi)$ are obtained by requiring the existence of a Noether symmetry for such theory. We derive exact cosmological solutions of the field equations from a point-like Lagrangian.

1 Introduction

Scalar-tensor theories of gravity have been widely used in recent years. In principle, they are related to many fundamental theories as super-string theories, grand unified theories and quantum gravity.\textsuperscript{1,2} These theories allow the gravitational coupling to vary and act as a dynamical field.\textsuperscript{3,4}

In the Kaluza-Klein models this arises from a variation of the size of the internal dimensions. In a cosmological context such theories allow to look for dynamical answers for equations.

The general form of the extended gravitational actions in scalar-tensor theories can be written in terms of the Brans-Dicke field $\phi$ and the strength of the coupling between the scalar field and gravity which is represented by the dynamical coupling function $\omega(\phi)$. Furthermore, a nontrivial potential $V(\phi)$ may be introduced in terms of a scalar field which clearly affect the dynamics. Higher-order gravitational theories are equivalent to a scalar-tensor theory when $V(\phi)$ is non-zero but $\omega(\phi) = 0$, which introduces Yukawa-type corrections to the Newtonian potential.\textsuperscript{5} The case of $V(\phi) = 0$ ensures a strictly Newtonian weak field limit to lowest order. When the coupling functional $\omega(\phi)$ is considered to be a constant parameter, the scalar-tensor theory reduces to the Brans-Dicke theory.

In general, there is no unique method to determine the functional forms of $\omega(\phi)$ and $V(\phi)$. In this paper, we first prove the existence of the Noether symmetry and then we use this symmetry to find these functionals. The resulting functional forms for $\omega(\phi)$ and $V(\phi)$ are not independent
of each other. One may note that the Lagrangian in the action becomes point-like if we impose a flat Friedman Robertson Walker (FRW) metric.

The paper is organized as follows: the existence of the Noether symmetry is proved in section 2. In section 3 the exact solutions of the field equations are derived from a point-like Lagrangian for a quartic potential. The concluding remarks appear in the section 4.

## 2 Noether Symmetry

We consider the action of scalar-tensor theory in the form

\[
S[\Phi] = \int d^4x \sqrt{-g} \left\{ \Phi R + \frac{\omega(\Phi)}{\Phi} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - V(\Phi) \right\}
\]  

(1)

where \( R \) is the scalar curvature, \( \Phi \) denotes a real scalar field, non-minimally coupled to gravity, and \( \omega(\Phi) \) and \( V(\Phi) \) represent the coupling function and generic potential, respectively. The action (1) can be rewritten, with redefinition \( \Phi = \phi^2 \), as

\[
S[\phi] = \int d^4x \sqrt{-g} \left\{ \phi^2 R + 4\omega(\phi) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right\}.
\]  

(2)

In the cosmological case, when the space-time manifold is described by a flat FRW metric, the scalar curvature has the expression

\[
R = -6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right),
\]

in which \( a \) is the scale factor of the universe and the dot denotes the derivative with respect to time. One can show simply that in this space-time manifold the Lagrangian density related to the action (2) takes the point-like form

\[
L = 12a^2 \dot{a} \dot{\phi} \dot{\phi} + 6a \ddot{a} \phi^2 + a^3 (4\omega(\phi) \dot{\phi}^2 - V(\phi)).
\]  

(3)

The corresponding Euler-Lagrange equations are given by

\[
\left[ \frac{\ddot{a}}{a} + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 \right] \phi^2 + \left[ \dot{\phi} + 2\left( \frac{\dot{a}}{a} \right) \dot{\phi} \right] \phi + \dot{\phi}^2 - (\omega(\phi) \dot{\phi}^2 - \frac{1}{4} V(\phi)) = 0,
\]  

(4)

and

\[
\dot{\phi} \omega(\phi) + 3\dot{\phi} \left( \frac{\dot{a}}{a} \right) \omega(\phi) + \frac{3}{2} \left( \frac{\dot{a}}{a} \right)^2 \phi + \frac{1}{2} \left( \omega(\phi) \dot{\phi}^2 + \frac{1}{4} V'(\phi) \right) = 0.
\]  

(5)

These equations are equivalent to the second order Einstein equation and the Klein-Gordon equation, in the flat FRW space-time, respectively.

We examine Noether symmetry of the Lagrangian (3) in order to determine the two unknown functions \( \omega(\phi) \) and \( V(\phi) \), as well as solving its dynamical field equations.

The infinitesimal generator of the Noether symmetry, namely the lift vector field \( X \), is

\[
X = \alpha(\phi, a) \frac{\partial}{\partial a} + \beta(\phi, a) \frac{\partial}{\partial \phi} + \frac{d\alpha}{dt} \frac{\partial}{\partial a} + \frac{d\beta}{dt} \frac{\partial}{\partial \phi}.
\]

We try to determine \( \alpha(\phi, a) \) and \( \beta(\phi, a) \) by imposing the following condition

\[
L_X \mathcal{L} = 0,
\]  

(6)
where $L_X$ is the Lie derivative with respect to the vector field $X$. Condition (6) leads to the following differential equations

\[
3\alpha V(\phi) + a\beta V'(\phi) = 0 \quad (7)
\]
\[
3\alpha \omega(\phi) + \beta a\omega'(\phi) + 3\phi \frac{\partial \alpha}{\partial \phi} + 2a\omega(\phi) \frac{\partial \beta}{\partial \phi} = 0 \quad (8)
\]
\[
\alpha \phi^2 + 2\beta a\phi + 2a\phi^2 \frac{\partial \alpha}{\partial a} + 2a^2 \phi \frac{\partial \beta}{\partial a} = 0 \quad (9)
\]
\[
2\phi\alpha + a\beta + a\phi \frac{\partial \alpha}{\partial a} + \phi^2 \frac{\partial \alpha}{\partial \phi} + a\phi \frac{\partial \beta}{\partial \phi} + 2a^2 \omega(\phi) \frac{\partial \beta}{\partial a} = 0. \quad (10)
\]

From Eq. (7) we have

\[
\alpha = -a\beta U(\phi), \quad (11)
\]

where we have used the definition

\[
U(\phi) = \frac{V'(\phi)}{3 V(\phi)}. \quad (12)
\]

Substituting (11) into (8) and (9), we find that the variables $\phi$ and $a$ in $\beta(\phi, a)$ must separate as

\[
\beta(\phi, a) = f(\phi)a^n, \quad (13)
\]

where the separation constant $n$ is given by

\[
n = \frac{3\phi U(\phi) - 1}{1 - \phi U(\phi)}, \quad (14)
\]

and functional $f(\phi)$ has the following form

\[
f(\phi) = exp \left\{ \int \frac{\omega(\phi)U(\phi) + \phi U'(\phi) - \frac{1}{3}\omega'(\phi)}{\frac{4}{3}\omega(\phi) - \phi U(\phi)} d\phi \right\}. \quad (15)
\]

From the definition (12) and using (14) we obtain

\[
V(\phi) = \lambda \phi^m, \quad m = \frac{3(n + 1)}{n + 3/2} \quad (16)
\]

and

\[
U(\phi) = \frac{1}{3} m \phi^{-1} \quad (17)
\]

where $\lambda$ is an arbitrary integration constant. Putting these results into Eq. (10) one has

\[
(m - 3)\phi \omega'(\phi) + m\omega^2(\phi) - m^2 + m(1 + 4n) - 6\omega(\phi) + 4nm + 3m - 6 = 0. \quad (18)
\]

This is a first-order differential equation for the dynamical coupling function $\omega(\phi)$ with the solution

\[
\omega(\phi) = k_1 \left[ \frac{(\phi/\phi_0)^{k_2} - 1}{(\phi/\phi_0)^{k_2} + 1} \right] + \omega_0. \quad (19)
\]
where $\phi_0$ is an integration constant and $k_1$, $k_2$ and $\omega_0$ are defined as
\[
    k_1 = \frac{3}{(2n + 3)^2}, \quad k_2 = \frac{-2n}{2n + 3}, \quad \omega_0 = \frac{3(8n^2 + 24n + 17)}{(2n + 3)^2}.
\]
Substituting (17) and (19) into (15) and using (13) and (11), one gets
\[
    \alpha(\phi, a) = -\frac{1}{3} \beta_0 m \phi_0^{-1} a^n (\phi/\phi_0)^{l_1-1} [((\phi/\phi_0)^{k_2} - \xi l_2 [(\phi/\phi_0)^{k_2} + 1])^{l_3} \tag{20}
\]
and
\[
    \beta(\phi, a) = \beta_0 a^n (\phi/\phi_0)^{l_1} [((\phi/\phi_0)^{k_2} - \xi l_2 [(\phi/\phi_0)^{k_2} + 1])^{l_3}, \tag{21}
\]
where $\beta_0$ is an integration constant, and we have used the definitions
\[
    l_1 = \frac{m(\omega_0 - k_1 - 1)}{2\omega_0 - 2k_1 - m},
    l_2 = \frac{2k_1^2 k_2 - k_2 (m - 2\omega_0)^2/2 - 2m (m - 2)}{4k_1^2 - (m - 2\omega_0)^2},
    l_3 = \frac{-2k_1^2 k_2 - k_2 (m - 2\omega_0)^2/2 - 2m (m - 2)(k_1 + 1)}{4k_1^2 - (m - 2\omega_0)^2},
    \xi = \frac{2k_1 - (m - 2\omega_0)}{2k_1 + (m - 2\omega_0)}.
\]
Therefore, the vector field $X$ exists, and the existence of the Noether symmetry leads to the determination of the class of potentials (16) and the dynamical coupling function (19), for arbitrary values of $n$, except $n = -3/2$. In principle, the existence of the Noether symmetry means that there must exist a constant of motion. In this case one may compute it using the Cartan one-form associated with the Lagrangian (3)
\[
    \theta_L \equiv \frac{\partial L}{\partial \dot{a}} da + \frac{\partial L}{\partial \dot{\phi}} d\phi.
\]
Contracting $\theta_L$ with $X$ gives the following required constant of motion
\[
    i_X \theta_L = 12a a\alpha(\phi, a) \{a\dot{\phi} + \dot{a} \phi\} + 4a^2 \beta(\phi, a) \{3\dot{a} \phi + 2a\phi \omega(\phi)\}
\]
where $\omega(\phi)$, $\alpha(\phi, a)$ and $\beta(\phi, a)$ are given by (19), (20) and (21), respectively.

## 3 Dynamical Field Equations and Solutions

Scalar-tensor theory reduces to Brans-Dicke theory when the coupling function $\omega(\phi)$ is taken to be a constant. For mathematical simplicity, we analyze the solutions of the field equations (4) and (5) for $n = -3$ and in the case that $\omega(\phi)$ is a constant parameter such as $\omega$. In this case the Lagrangian (3) takes the form
\[
    L = 12\dot{a} a^2 \dot{\phi} + 6\dot{a}^2 a\phi^2 + a^3 (4\omega \dot{\phi}^2 - \lambda \phi^4). \tag{22}
\]
The corresponding field equations are given by
\[
    \left(\frac{\ddot{a}}{a} + \frac{1}{2} \left(\frac{\dot{a}}{a}\right)^2\right) \phi^2 + \left[\dot{\phi} + 2\left(\frac{\dot{a}}{a}\right)\dot{\phi}\right] \phi + \dot{\phi}^2 + \frac{1}{4} \lambda \phi^4 = 0 \tag{23}
\]
and
\[
\omega \dddot{\phi} + 3\omega \dot{\phi} \left( \frac{\ddot{a}}{a} \right) + \frac{3}{2} \frac{\ddot{a}}{a} + \left( \frac{\ddot{a}}{a} \right)^2 \phi + \frac{1}{2} \lambda \phi^3 = 0. \tag{24}
\]

One may select the initial conditions of the field equations (23) and (24) such that the energy function associated with the Lagrangian (22) vanishes
\[
E_L = \frac{\partial L}{\partial \dot{\phi}} \dot{\phi} - L = 12\dot{a}^2 \phi \dot{\phi} + 6\dot{a}^2 \phi^2 + a^3 (4\omega \dot{\phi}^2 + \lambda \phi^4) = 0. \tag{25}
\]

In fact, by choosing \(E_L = 0\), one imposes a constraint on the integration constants. From (20) and (21) we have
\[
\alpha = -\frac{4}{3} \frac{\beta_0}{a^2 \phi^2}, \quad \beta = \frac{\beta_0}{a^3 \phi} \tag{26}
\]
for \(n = -3\). Now, the existence of the vector field \(X\) can be used to find a non-degenerate point transformation
\[
(a, \phi) \rightarrow (p, q),
\]
such that the transformed Lagrangian is cyclic in one of the new variables. A possible way is to compute the Cartan one-form associated with Lagrangian \(L\), namely
\[
\begin{cases}
i_X(dp) = 1 \\
i_X(dq) = 0,
\end{cases}
\]
or explicitly
\[
\begin{cases}
\alpha \frac{\partial p}{\partial a} + \beta \frac{\partial p}{\partial \phi} = 1 \\
\alpha \frac{\partial q}{\partial a} + \beta \frac{\partial q}{\partial \phi} = 0.
\end{cases} \tag{27}
\]
Substituting (26) into (27), yields the solutions
\[
\begin{cases}
p = -\frac{1}{2\beta_0} a^3 \phi^2 \\
q = \gamma_0 a^3 \phi^{4/\gamma},
\end{cases} \tag{28}
\]
where \(\gamma_0\) and \(\gamma\) are constant. Under these transformations, the Lagrangian (22) takes the form
\[
\mathcal{L} = 2\beta_0 \left\{ 2\gamma (\omega - 5/3) \frac{\dot{p} q}{q} + (\omega - 4/3) \left( \frac{p^2}{p} + \gamma q^2 \frac{pq}{q^2} \right) + \frac{\lambda}{2\beta_0 \gamma_0 \gamma} \right\}. \tag{29}
\]
One can easily check that Eq. (18) has two solutions: \(\omega = 3/2\) and \(\omega = 4/3\) for \(n = -3\). Now, if one computes the Hessian determinant \(\mathcal{H} = |\partial^2 \mathcal{L}/\partial \dot{\phi} \partial \dot{a}|\), one finds that \(\omega = 3/2\) leads to the degeneration of \(\mathcal{L}\). In other words, this value of \(\omega\) does not allow a Hamiltonian formulation for the theory and leads to pathological dynamics. Thus, one can disregard \(\omega = 3/2\). For \(\omega = 4/3\) the second term of (29) disappears and \(\mathcal{L}\) takes the form
\[
\mathcal{L} = -\frac{4}{3} \beta_0 \gamma \frac{\dot{p} q}{q} + \frac{\lambda}{\gamma_0 q^\gamma}. \tag{30}
\]
This Lagrangian clearly does not depend on $p$. Therefore, in new configuration space $(p, q)$, the variable $p$ is cyclic. This implies the existence of the Noether symmetry. The corresponding field equations for the last Lagrangian are given by

$$\ddot{p} + \frac{3\lambda}{4\beta_0\gamma_0}q^\gamma = 0$$

and

$$\dot{q} - r_0q = 0$$

where $r_0$ is constant of motion. The solutions of these equations are

$$p = s_0(e^{r_0\gamma t} + p_0) + \dot{p}_0 t$$

and

$$q = q_0 e^{r_0t},$$

where $p_0, \dot{p}_0$ and $q_0$ are integration constants and $s_0$ is given by

$$s_0 = -\frac{3\lambda q_0^\gamma}{4\beta_0\gamma_0\gamma^2 r_0^2}.$$

In new configuration space $(p, q)$, the condition (25) can be rewritten as

$$E_L = -\frac{4}{3}\beta_0\gamma\frac{\dot{p} \dot{q}}{q} - \frac{\lambda}{\gamma_0} q^\gamma = 0.$$

(33)

Substituting (31) and (32) into (33), leads to $\ddot{p}_0 = 0$, and so the solution (31) reduces to

$$p = s_0(e^{r_0\gamma t} + p_0).$$

(34)

Putting (32) and (34) into (28), one gets the following solutions

$$\phi = \psi_0 e^{r_0\gamma t/2}(e^{r_0\gamma t} + p_0)^{-1/2}, \quad \psi_0 = \left(-\frac{q_0^\gamma}{2s_0\beta_0\gamma_0}\right)^{1/2}$$

and

$$a = a_0 e^{-r_0\gamma t}(e^{r_0\gamma t} + p_0)^{2/3}, \quad a_0 = \left(\frac{4s_0^2\beta_0^2\gamma_0^\gamma}{q_0^0}\right)^{1/3}.$$

In the limit $t \to \infty$ the scalar field $\phi$ approaches a constant value $\psi_0$, and then the Einstein gravity is recovered and the Newtonian gravitational constant is identified as $G_N = \frac{1}{\psi_0^2}$.

### 4 Concluding Remarks

We have studied scalar-tensor theories of gravity in which the coupling function $\omega(\phi)$ and the generic potential $V(\phi)$ are unknown. The form of the coupling function and the potential are determined using Noether symmetry in a flat FRW background. In the special case $n = -3$, we showed that Noether symmetry exists for $\omega = 3/4$, and we derived the corresponding constant of motion, $r_0$. Furthermore, the exact solutions of the field equations were derived from a point-like Lagrangian for a quartic potential. It is also shown that in the case of $\omega = 3/2$, there is no Noether symmetry for the Lagrangian. However, it is interesting to note that the theory is exactly conformal invariant. Further attempts seems to be necessary to understand this relationship more deeply.
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