Notes on observational and radar coordinates for localized observers

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(Dated: November 12, 2019)

The worldline of a uniformly accelerated localized observer in Minkowski space is restricted in the Rindler wedge, where the observer can in principle arrange experiments repeatedly, and the Cauchy problem for quantum fields in that Rindler wedge can be well defined. However, the observer can still receive the signals sourced by the events behind the past horizon, and coordinatize those events in terms of some kind of observational coordinates. We construct such observational coordinates in some simple cases with the localized observers in Minkowski, de Sitter, and Schwarzschild-like spacetimes, and compare them with radar coordinates for the same observers.

I. INTRODUCTION

In special relativity, an observer is considered to be localized in space with a clock. Such a localized observer moving in a constant velocity in Minkowski space can operationally define a reference frame, called “radar coordinates”, for the events in spacetime by sending a radar pulse at her/his proper time to some event and then recording the receiving time of the echo from the event. Accordingly each event can be coordinatized in terms of radar time , radar distance , and the direction of the radar signals indicated by the sign of . There are a few advantages in applying Rindler radar coordinates to field theory, e.g. a Lorentz boost about the origin in the Minkowski coordinates is simply a time-translation in Rindler radar coordinates, and the Cauchy problem for quantum fields can be well defined in Rindler radar coordinates. Nevertheless, the uniformly accelerated observer in wedge R should be able to receive the classical signal emitted from an event behind the past horizon of radar coordinates, e.g. the outcome of a local measurements in wedge P on a field. How could the uniformly accelerated observer coordinatize that event, which is never reachable by her/his radar signals?

When we look into the sky, we are receiving the information along the past light cones extended from our eyes. Astronomers can see the classical signals emitted by an object billions lightyears away from the Earth, while radar was invented for less than one hundred years. Thus the visible universe is much larger than the region that can be practically covered by any radar coordinates. To coordinatize an observed event beyond the reach of radar coordinates, one may follow astronomers and adopt “optical coordinates” or “geodesic light-cone coordinates”, in terms of the signal-receiving time in the observer’s clock together with the distance and the direction of the event seen by the observer.

In , the astronomical distances which may be useful for observational coordinates include the binocular (parallax) distance, luminosity distance, angular diameter distance, and so on. In the ideal cases the affine distance and advanced/retarded distance by mathematical constructions would also be convenient for theorists. To determine the astronomical distance of an object at some moment, either the observer or the observed object has to be extended in the directions orthogonal to the null geodesic connecting the object and the observer at that moment, while the sizes of the observer and the observed object/event are considered to be infinitesimal in this paper to ensure causality and reduce ambiguity. Indeed, the angular diameter distance of an object cannot be determined

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1 In this paper, we call them “localized” observers to distinguish them from the “local” observers in general relativity referring to the observers confined to a finite region where the variation of the gravitational field is unobservably small [2]. Our localized observer is extended in space while her/his volume is infinitesimal.
if its angular diameter is zero, and one needs a baseline between two eyes or telescopes to determine the binocular
distance, and an antenna of finite area for measuring the apparent luminosity to determine the luminosity distance
from an event. In (1+1)D, however, while radar distance can still be defined, those astronomical distances cannot
be determined physically and we have to rely on the distances by mathematical constructions to depart from radar
coordinates.

The idea of the observational coordinates is not new. Similar ideas have been applied to curved spacetimes to
construct the advanced coordinates, which are the time reversal of the retarded coordinates [17, 18]. And yet,
some details of observational coordinates and even radar coordinates are not fully explored. What would the events
behind the past horizon of Rindler radar coordinates look like in the viewpoint of a uniformly accelerated localized
observer? Does an accelerated localized observer always see a past horizon of radar coordinates and the events
behind it? There is no nontrivial coordinate singularity at finite values of Rindler radar coordinates [1]. How
about observational coordinates? Is the acceleration of the localized observer necessary for the presence of nontrivial
coordinate singularities?

To answer these questions, below we study a few simple cases of the localized observers in various motions and
background geometries. In section II we look at the localized observer in uniform acceleration, non-uniform accelera-
tion, and spinning without center-of-mass motion in (3+1)D Minkowski space. The cases with a comoving localized
observer in de Sitter space and in a non-eternal inflation background are discussed in section III. Then the case with
the localized observer fixed at a constant radius from the center of a spherical shell in (1+1)D is considered in section
IV. Finally we summarize our findings with discussion in section V.

II. LOCALIZED OBSERVERS IN MINKOWSKI SPACE

The observational coordinates with the observer’s clock \( \tilde{\tau} \), the distance \( \tilde{r} \) and the direction \((\tilde{\theta}, \tilde{\phi})\) of the event seen
by a non-spinning localized observer in inertial motion in Minkowski space will coincide with the radar coordinates for
the same observer and the conventional Minkowski coordinates up to a Lorentz transformation if we define the time
coordinate as \( t = \tilde{\tau} - \tilde{r} \) (Figure 1 (left)). Similar observational coordinate systems can also be determined for localized
observers in general motions and background spacetimes, but they usually deviate from radar or other conventional
coordinates for the same observers.

A. uniformly accelerated observer

Consider the simplest case of non-inertial motions, with the localized observer uniformly accelerated along the
worldline \( z^\mu(\tau) = (a^{-1} \sinh a \tau, 0, 0, a^{-1} \cosh a \tau) \) in Minkowski coordinates \( x^\mu \) in (3+1)D Minkowski space. Here \( \tau \) is
the proper time of the observer and \( a \) is the proper acceleration. For this uniformly accelerated localized observer, a
natural coordinate system would be the Rindler coordinates given by

\[
ds^2 = e^{2a \zeta} (-d\eta^2 + d\zeta^2) + (dx^1)^2 + (dx^2)^2,
\]

FIG. 1: Rindler coordinates in Minkowski space (left) and its maximal analytic extension (right).
with \(-\infty < \eta < \infty\), \(-\infty < \zeta < \infty\), and \(x^1\) and \(x^2\) are identical to those of Minkowski coordinates. The metric \(d\tilde{s}^2\) is transformed from Minkowski coordinates \(d\tilde{s}^2 = \eta_{\mu\nu} dx^\mu dx^\nu\), \(\eta_{\mu\nu} = \text{diag}(-1,1,1,1)\) by \(x^0 = a^{-1} e^{\alpha \eta} \sinh \eta\) and \(x^3 = a^{-1} e^{\alpha \eta} \cosh \eta\). The uniformly accelerated observer appears to be at rest along the worldline \((\tau,0,0,0)\) in the Rindler coordinates \((\eta, x^1, x^2, \zeta)\).

Unlike the case in \((1+1)D\), the Rindler coordinates \((2)\) in \((3+1)D\) is not a radar coordinate system since \(\zeta\) is not the radar distance of any event off the plane of the observer’s motion, \(x^1 = x^2 = 0\). The radar coordinates for the uniformly accelerated observer going along \(z^\mu(\tau)\) read

\[
ds^2 = \frac{1}{(\cosh ar - \cos \theta \sinh ar)^2} \left[ -d\eta^2 + dr^2 + \left( \frac{\sinh ar}{a} \right)^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]
\]

with \(\eta \in (-\infty, \infty), r \in [0, \infty), \theta \in [0, \pi]\) and \(\varphi \in [0, 2\pi]\). The line element \((3)\) is obtained using the same operations as in the inertial cases, as follows. Suppose the observer emitted a radar pulse at \(\tau = \tau_i\) to an event \(E\) at \(x^\mu = x^\mu_E = (t,x,y,z)\) in Minkowski coordinates and received the echo at \(\tau = \tau_f\). Then the radar time and radar distance of the event will be \(\eta = (\tau_f + \tau_i)/2\) and \(r = (\tau_f - \tau_i)/2\), respectively, and the event will be somewhere in the \(\eta\)-slice of \(x^0 = \zeta \sinh \eta, x^2 = \zeta \cosh \eta\) and \(x^1, x^2, \zeta \in \mathbb{R}\) in terms of Minkowski coordinates. Transforming the \(\eta\)-slice to the \(\eta = 0\) (and so \(t = 0\)) hypersurface by a Lorentz boost in the \(x^3\)-direction about the origin, then in the new coordinates the event \(E\) is at \(x^\mu_E = x^\mu_E = (0,x,y,Z)\), which satisfies the null condition \(\sigma(x^\mu_E, z^\mu(\pm r)) = 0\), or

\[
(Z - a^{-1} \cosh ar)^2 + \rho^2 - (a^{-1} \sinh ar)^2 = 0
\]

since \(x^\mu_E\) is on the future and past light cones of the observer at \(\tau = -r\) and \(+r\), respectively. Here \(\rho^2 \equiv x^2 + y^2\), and \(\sigma(A^\mu, B^\nu) \equiv -(A^\mu - B^\mu)(A^\nu - B^\nu)/2\) is the Synge’s world function. Now in the instantaneous Lorentz frame of the observer at \(\tau = \pm r\) with the Lorentz factor \(\gamma = \cosh ar\), the 3-speed \(v = \tanh a(\pm r)\), and the origin set to be the observer’s spacetime point of emitting or receiving the radar signal, the event \(E\) is located at \((Z \sinh a(\pm r), x, y, Z \cosh ar - a^{-1})\), implying that \(\tan \phi|_{\tau = r} = \tan \phi|_{\tau = -r} = y/x \equiv \tan \varphi\) and

\[
\tan \theta|_{\tau = r} = \tan \theta|_{\tau = -r} = \frac{\rho}{Z \cosh ar - a^{-1}} \equiv \tan \theta
\]

and so the observer would perceive that both the directions of emitting and receiving the radar signal are \((\theta, \varphi)\). Combining Eqs. \((4)\) and \((5)\) yields \(\sin \theta = \rho/(Z \sinh ar)\) and \(\cos \theta = (Z \cosh ar - a^{-1})/(Z \sinh ar)\), which gives \(Z = |a(\cosh ar - \cos \theta \sinh ar)|^{-1} > 0\) and the transformation

\[
t = Z \sinh a\eta, \quad z = Z \cosh a\eta, \quad \rho = Z \sinh ar \sin \theta
\]
between Minkowski coordinates and radar coordinates. With $\varphi$ invariant and other coordinates transformed as

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + \rho^2 d\varphi^2$$

of Minkowski cylindrical coordinates can be transformed to (8), which coincides the Rindler coordinates (2) in the plane of $\theta = 0$ and $\pi$, i.e. $x^1 = x^2 = 0$ (Figure 3).

Both Rindler coordinates (2) and radar coordinates (3) cover wedge $R$ (Figure 1 (left)) with $U \equiv x^0 - x^3 (= -a^{-1}e^{\alpha(\zeta - \eta)} = -e^{-\alpha\eta}Z) < 0$ and $V \equiv x^0 + x^3 (= a^{-1}e^{\alpha(\zeta + \eta)} = e^{\alpha\eta}Z) > 0$ in Minkowski coordinates. Each spacetime point in wedge $R$ is in principle accessible by the localized observer using those radar or light pulses, and so causally connected with the observer both in the past and future directions. The hypersurface $U = 0$ is the event horizon and $V = 0$ is the past horizon for the uniformly accelerated localized observer.

The events behind the past horizon cannot be reached by any radar pulse from the uniformly accelerated observer and so radar coordinates are not defined around those events. Nevertheless, the localized observer can passively receive the signal emitted by an event in wedge $P$ and then determine the distance, which can be finite, from the emission event to the observer. Using the receiving time in the observer’s clock, the distance, and the direction of the event perceived by the observer, the localized observer can coordinatize that event along the observer’s past light cone.

Among the distances determined in different ways, the advanced distance may be the most convenient one in Minkowski space for theorists. It can be read off from the field amplitude of a massless scalar field emitted by a point source as a standard candle [19–21]. Mathematically, the advanced distance of an event seen by the localized observer is determined by $\sigma (z^\mu(\tilde{\tau}), x^\mu_E) = 0$, which gives

$$\tilde{\tau} = \frac{1}{a} \ln \frac{a}{2|U|} \left( X - UV + \rho^2 + \frac{1}{a^2} \right),$$

(7)

with $U \equiv t - z$, $V \equiv t + z$, $\rho^2 \equiv x^2 + y^2$, and $X \equiv \sqrt{(t^2 - z^2 - \rho^2 + a^{-2})^2 + 4a^{-2}\rho^2}$ [19–21]. At the moment $\tilde{\tau}$, the 4-velocity of the observer is $u^\mu(\tilde{\tau}) = \frac{d\tilde{\tau}}{d\tau} = (\cos a\tilde{\tau}, 0, 0, \sinh a\tilde{\tau}) = (\gamma_\tau, 0, 0, \gamma_\tau v_\tau)$, thus $\gamma_\tau = \cosh a\tilde{\tau}$ and $v_\tau = \sinh a\tilde{\tau}$.

FIG. 3: The $\eta = 0$ slice in radar coordinates (3) for the uniformly accelerated observer. The contours of $ar$ and $\theta$ are represented in thick and thin curves, respectively. The gray lines represent the contours of $a\zeta$ in the Rindler coordinates (2) for the same observer.

Hereafter our “past horizon for an observer” refers to the past horizon of the spacetime region covered by the radar coordinates for that observer. No null geodesic starting at the observer can go beyond her/his past horizon, to which the observer would infer a divergent radar distance.
\[ v_\tilde{r} = \tanh a \tilde{r}. \]

Choosing this spacetime point of the observer when receiving the signal, \( z^\mu(\tilde{r}) \), as the origin, an extrapolated Lorentz frame with the instantaneous 4-velocity \( u^\mu(\tilde{r}) \) of the observer in Minkowski coordinates can be obtained by the Lorentz transformation

\[ \tilde{\mathbf{x}} = \gamma_{\tilde{r}} \left[ x^0 - z^0(\tilde{r}) - v_\tilde{r}(x^3 - z^3(\tilde{r})) \right], \tag{8} \]
\[ \tilde{\mathbf{x}}^3 = \gamma_{\tilde{r}} \left[ x^3 - z^3(\tilde{r}) - v_\tilde{r}(x^0 - z^0(\tilde{r})) \right], \tag{9} \]
\[ \tilde{x}^1 = x^1 - z^1(\tilde{r}) = x^1, \quad \tilde{x}^2 = x^2 - z^2(\tilde{r}) = x^2. \tag{10} \]

Then one has \( \tilde{x}^0 = \tilde{r} \equiv \sqrt{(\tilde{x}^1)^2 + (\tilde{x}^2)^2 + (\tilde{x}^3)^2} = aX/2 = |\partial_\tau \sigma(z(\tau), x)|_{\tau = \tilde{r}} \), which is the advanced distance from the emitting event \((\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)\) to the origin of this new coordinates, that is, the observer at \( \tilde{r} \) \[21\]. Using the four parameters \((\tilde{r}, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)\) as observational coordinates, the observer can uniquely identify the event at \( x^\mu_{\tilde{r}} \). To find the metric for these observational coordinates, one can re-arrange the above relations in an inverse Lorentz transformation,

\[ x^0 - z^0(\tilde{r}) = \gamma_{\tilde{r}} \left( \tilde{x}^0 + v_\tilde{r}\tilde{x}^3 \right) = \gamma_{\tilde{r}} \left( -\tilde{r} + v_\tilde{r}\tilde{x}^3 \right), \]
\[ x^3 - z^3(\tilde{r}) = \gamma_{\tilde{r}} \left( \tilde{x}^3 + v_\tilde{r}(\tilde{r}) \right) \tag{11} \]

and express \( dx^\mu \) as the linear combinations of \( d\tilde{r}, d\tilde{x}^1, d\tilde{x}^2, \) and \( d\tilde{x}^3 \), or more conveniently, with the spatial part in the spherical coordinates \((\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = (\tilde{r} \sin \theta \cos \phi, \tilde{r} \sin \theta \sin \phi, \tilde{r} \cos \theta)\). Then one obtains the line element

\[ ds^2 = - \left[ (1 + a\tilde{r} \cos \tilde{\theta})^2 - (a\tilde{r})^2 \right] d\tilde{r}^2 + 2 \left( d\tilde{r} + a\tilde{r}^2 \sin \tilde{\theta} d\tilde{\theta} \right) d\tilde{\theta} + \tilde{r}^2 \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \right), \tag{12} \]

which is almost a special case of those in Refs. \[17\] \[22\] except that here \( g_{\tilde{r}\tilde{r}} = g_{\tilde{r}\tilde{r}} = 1 \) instead of \(-1\) because we are looking at the past light cones (advanced coordinates) for the observer instead of the future light cones (retarded coordinates \[18\]) considered in \[17\] \[22\]  \[3\].

In the plane of the observer’s motion \( x^1 = x^2 = 0 \), the line element reduces to

\[ ds^2 = -(1 \pm 2a\tilde{r})d\tilde{r}^2 + 2d\tilde{r}d\tilde{\theta} \tag{13} \]

with \( + \) and \( - \) for \( \tilde{\theta} = 0 \) and \( \pi \), respectively. We show an example in Figure 2 (middle), where one can see that the contours of \( \tilde{r} \) are timelike in wedge R \((2a\tilde{r} < 1)\) but spacelike in wedge P \((2a\tilde{r} > 1)\). Such a pattern is similar to those contours of \( \zeta \) in the maximally extended Rindler radar coordinates in wedges R and P in Figure 1 (right), though they are not exactly the same.

\[3\] The line element \[12\] is almost identical to Eq.(4) in Ref. \[23\] except the sign of \( g_{\tilde{r}\tilde{r}}(g_{\varphi\varphi} \text{ there}) \).
FIG. 5: Contour surfaces of constant advanced distance \( \tilde{r} \) in view of the uniformly accelerated observer moving along \( z''(\tau) = (a^{-1} \sinh a\tau, 0, 0, a^{-1} \cosh a\tau) \) with \( a = 1 \). The blue and orange surfaces represent the region with \( 0 \leq \tilde{\theta} \leq \pi/2 \) and \( \pi/2 < \tilde{\theta} \leq \pi \) in the localized observer’s point of view (\( \tilde{\theta} = 0 \) and \( \pi \) in the \( +z \)- and \( -z \)-directions, respectively). The hypersurfaces \( t = z \) and \( t = -z \) (gray) are the event and past horizons, respectively, for the uniformly accelerated observer.

A coordinate singularity in the observational coordinates \( \tilde{\theta} \) occurs at \( (1 + a \tilde{r} \cos \tilde{\theta})^2 - (a \tilde{r})^2 = 0 \), which implies \( (x^0)^2 - (x^3)^2 = 0 \) in Minkowski coordinates from (11). In particular, the hypersurface \( V = x^0 + x^3 = 0 \) is the past horizon of radar coordinates (3) for the uniformly accelerated observer, on which

\[
\tilde{r} = \frac{a^{-1}}{1 - \cos \tilde{\theta}}
\]

is independent of the azimuthal angle \( \tilde{\phi} \). Eq. (14) indicates that in view of the uniformly accelerated observer the past horizon is a paraboloid of revolution with the focus at \( \tilde{r} = 0 \) (where the observer is located), the semi-latus rectum \( a^{-1} \), and the open end in the direction of the acceleration (Figure 4).

A person standing on the surface of the earth experiences a roughly uniform gravitational acceleration \( a = g = 9.8 \text{ m/s}^2 \). If this gravitational acceleration is strictly uniform in the Universe, that person will see a paraboloidal past horizon with the semi-latus rectum \( c^2/a \approx 10^{16} \text{ m} \approx 1 \text{ lightyear} \), which is much larger than the scale of the Earth.

Extended to the spacetime with \( \rho^2 = (x^1)^2 + (x^2)^2 > 0 \), the constant-\( \tilde{r} \) hypersurfaces in Minkowski coordinates are given by

\[
(x^0)^2 - (x^3)^2 = \rho^2 \pm 2a^{-1} \sqrt{\tilde{r}^2 - \rho^2} - a^{-2}
\]

from (11), where “−” and “+” correspond to the cases with \( \tilde{\theta} \in [0, \pi/2) \) and \( [\pi/2, \pi] \), respectively. In Figure 5 we show some constant-\( \tilde{r} \) hypersurfaces for example. While every constant-\( \tilde{r} \) hypersurface with \( \tilde{r} < 1/(2a) \) is timelike, each constant-\( \tilde{r} \) hypersurface with \( \tilde{r} \geq 1/(2a) \) is not timelike everywhere. For every timelike worldline of a point-like light source behind the past horizon (i.e., in wedge P) for the observer, the source’s \( \tilde{r} \) and \( \tilde{\phi} \) can never both be constants of time in the observation of the localized observer, a fact associated with the signature change of \( g_{\tilde{r}\tilde{r}} \) in (12) (or \( g_{tt} \) in (17)). Thus the past horizon is also a static limit surface for the uniformly accelerated observer. Moreover, the accelerated observer will see that all the point-like sources moving along timelike worldlines not going to future null infinity will eventually approach the past horizon and then stop there. For example, in Figure 5 two point-like emitters moving along blue timelike worldlines AB and CDE emit light rays continuously to a uniformly accelerated observer going along the red worldline. The observer would see in the direction of \( \tilde{\theta} = \pi \) that the emitter started
with point A at distance $\tilde{r} \approx 5/(2a)$ would go towards the observer and then stop right behind the past horizon (B) at $\tilde{r} = 1/(2a)$ and never cross it. The other emitter would be seen also in the direction of $\tilde{\theta} = \pi$ that one emitter started with A, went towards the observer and stopped right behind the past horizon (B). The other emitter would be seen as started with point C, went towards the observer and crossed the past horizon (D), reached the minimum distance $\tilde{r} \approx 0.5/(2a)$ of this trip, then dropped back and eventually stopped around the past horizon and never crossed it again (E).

If a collection of the point-like sources are not exactly in the directions $\theta = 0$ or $\pi$ of the localized observer, as they are observed to approach the past horizon, they would also be concentrating toward the direction of the observer’s velocity as special relativity implies [10].

A constant-$\tilde{r}$ hypersurface in our observational coordinates [12] is the past light cone of $x''(\tilde{r})$ and so $\tilde{r}$ is a null coordinate. One may further define a time coordinate $\tilde{t}$ by letting

$$d\tilde{t} \equiv d\tilde{r} - \frac{d\tilde{r} + a\tilde{r}^2 \sin \tilde{\theta} d\tilde{\theta}}{(1 + a\tilde{r} \cos \tilde{\theta})^2 - (a\tilde{r})^2},$$

(16)

then

$$ds^2 = - \left[(1 + a\tilde{r} \cos \tilde{\theta})^2 - (a\tilde{r})^2\right] d\tilde{t}^2 + \left(\frac{d\tilde{r} + a\tilde{r}^2 \sin \tilde{\theta} d\tilde{\theta}}{(1 + a\tilde{r} \cos \tilde{\theta})^2 - (a\tilde{r})^2}\right)^2 + \tilde{r}^2 \left(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2\right),$$

(17)

such that the constant-$\tilde{r}$ slices are timelike in wedge R and spacelike in wedge P, and all of them will intersect at the origin of Minkowski coordinates as the observer goes to future infinity, similar to the Rindler-time slices in Figure 1 (right). From (16), one has

$$\tilde{t} = \tilde{r} + \frac{1}{2a} \ln \left|\frac{1 + a\tilde{r} \cos \tilde{\theta} - a\tilde{r}}{1 + a\tilde{r} \cos \tilde{\theta} + a\tilde{r}}\right|,$$

(18)

which goes to $-\infty$ both in wedges R and P as the observed events goes to the past horizon $x^0 + x^3 = 0$ (cf. Eq. (14)), and so the past horizon is part of the past infinity with respect to $\tilde{t}$ for the observer. In the plane of the observer’s motion, the line element (17) reduces to $ds^2 = -(1 + 2a\tilde{r}) d\tilde{t}^2 + (1 + 2a\tilde{r})^{-1} d\tilde{r}^2$ for $\tilde{\theta} = 0$ and $\pi$ (cf. (13)).

Interesting enough, when we fix $\tilde{\theta} = \pi/2$, $\tilde{\varphi} =$-constant, and $d\tilde{\theta} = d\tilde{\varphi} = 0$, the line element (17) on this slice looks very much like the static de Sitter coordinates [30] with the angular dimensions suppressed and $a$ here being identified as the Hubble constant $H$ there. Indeed, the static de Sitter coordinates have some properties similar to (17), as we will discuss in next section.
B. Non-uniform linear acceleration

As the motion of the observer is turned from non-inertial to inertial, the spacelike part of the constant-$\tilde{r}$ hypersurfaces behind the past horizon for the accelerated localized observer will evolve to timelike surfaces. These constant-$\tilde{r}$ hypersurfaces in our observational coordinates will not be smooth if the acceleration of the localized observer suddenly changes while the 4-velocity changes continuously. For example, in Figure 2 (right), when the acceleration is suddenly switched to zero, while the tangent vector of the worldline of the observer evolves continuously, the constant-$\tilde{r}$ hypersurfaces is not differentiable around the past light cone of the moment that the observer changes acceleration. To make it differentiable the observer’s acceleration has to be changed smoothly. The radar coordinates for the same observer behave better: The hypersurfaces of constant radar distance evolve in the same way as the observer’s motion, namely, the first derivatives are continuous [7].

C. Spinning observer without center-of-mass motion

Suppose an observer is situated at the origin and spinning about the $z$-axis at a constant angular frequency $\omega$. Radar coordinates of event $E$ at $(t, \rho, z, \varphi)$ in Minkowski cylindrical coordinates can be constructed by the spinning localized observer in a way similar to those for a non-spinning rest observer. If a radar signal is emitted by the observer at $\tau_1$ and the echo from the event is received at $\tau_f$ in the observer’s clock, then the radar time and radar distance of the event are again $\tilde{t} = (\tau_f + \tau_1)/2$ and $r = (\tau_f - \tau_1)/2$ for the observer. The polar angle of the event $\theta = \tan^{-1}(z/\rho)$ is a constant of time for the observer and so the radar polar angle is still $\theta$. The only modification is that the radar azimuthal angle $\varphi'$ of the event should be determined by the average of the emitting and receiving azimuthal angles in the instantaneous frames of the observer at $\tau_f$ and $\tau_1$, namely, $\varphi' = (\varphi(\tau_f) + \varphi(\tau_1))/2 = \varphi - \omega(\tau_f + \tau_1)/2 = \varphi - \omega \tilde{t}$. Thus the line element in radar coordinates can be obtained from Minkowski cylindrical coordinates by the transformation $(t', \rho', \varphi') = (t, \rho, \varphi - \omega t)$:

$$ds^2 = -(1 - \omega^2 \rho^2)dt'^2 + 2\omega \rho^2 dt' d\varphi' + dz'^2 + d\rho'^2 + \rho'^2 d\varphi'^2,$$

(19)

which is the rotating cylindrical coordinates [1, 21, 23]. It is well known that clocks along a closed curve in this coordinate system cannot be synchronized uniquely since $g_{t'\varphi'} \neq 0$ [23]. One may define a new time coordinate as

$$dT' = dt' - \frac{\omega \rho^2 d\varphi}{1 - \omega^2 \rho^2},$$

(20)

to diagonalize (19) into

$$ds^2 = -(1 - \omega^2 \rho^2) dT'^2 + \frac{\rho'^2 d\varphi'^2}{1 - \omega^2 \rho'^2} + dz'^2 + d\rho'^2,$$

(21)

where the timelike and spacelike properties of $T'$ and $\varphi'$ coordinates will be switched when the observed events are crossing the cylinder $\rho' = 1/\omega$.

The same event $E$, now represented as $(t, r, \theta, \varphi)$ in Minkowski spherical coordinates, will be observed by the spinning observer at her/his proper time $\tilde{\tau} = t + r$. At that moment the observer will see the event in the direction $(\tilde{\theta}, \tilde{\varphi}) = (\theta, \varphi - \omega \tilde{\tau}) = (\theta, \varphi - \omega(t + r))$ at the distance $\tilde{r} = r$ away from the observer. Thus the observational coordinates for the spinning observer read

$$ds^2 = -d\tilde{r}^2 + 2d\tilde{r}d\tilde{\varphi} + \tilde{r}^2 \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} (d\tilde{\varphi} + \omega d\tilde{\tau})^2 \right)$$

$$= - \left(1 - \omega^2 \tilde{r}^2 \sin^2 \tilde{\theta} \right) d\tilde{r}^2 + 2d\tilde{r} \left( d\tilde{\varphi} + \omega \tilde{r}^2 \sin^2 \tilde{\theta} d\tilde{\varphi} \right) + \tilde{r}^2 \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2 \right).$$

(22)

Define a new time-parameter $\tilde{t}$ by

$$d\tilde{t} = d\tilde{r} - \frac{d\tilde{\varphi} + \omega \tilde{r}^2 \sin^2 \tilde{\theta} d\tilde{\varphi}}{1 - \omega^2 \tilde{r}^2 \sin^2 \tilde{\theta}},$$

(23)

then (22) becomes

$$ds^2 = - \left(1 - \omega^2 \tilde{r}^2 \sin^2 \tilde{\theta} \right) d\tilde{t}^2 + \frac{\left( d\tilde{r} + \omega \tilde{r}^2 \sin^2 \tilde{\theta} d\tilde{\varphi} \right)^2}{1 - \omega^2 \tilde{r}^2 \sin^2 \tilde{\theta}} + \tilde{r}^2 \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2 \right).$$

(24)
Transform to the cylindrical coordinates, $\tilde{\rho} = \tilde{r} \sin \tilde{\theta}$ and $\tilde{z} = \tilde{r} \cos \tilde{\theta}$, the above line element becomes

$$ds^2 = -(1 - \omega^2 \tilde{r}^2) d\tilde{t}^2 + \frac{\tilde{\rho}^2}{1 - \omega^2 \tilde{r}^2} \left[ d\tilde{\varphi} + \frac{\omega (\tilde{\rho} d\tilde{\varphi} + \tilde{z} d\tilde{z})}{\sqrt{\tilde{\rho}^2 + \tilde{z}^2}} \right]^2 + d\tilde{z}^2 + d\tilde{\rho}^2,$$

(25)

which is not the same as (21) because here $\varphi = \varphi - \omega (t + r)$ in observational coordinates but there $\varphi' = \varphi - \omega t$ in radar coordinates.

The coordinate-singularity cylinder of perceived radius $\tilde{\rho} = 1/\omega$ is a static limit surface, beyond which nothing along a timelike worldline in Minkowski coordinates can be at rest in view of the spinning observer. Indeed, for $\tilde{\rho} > 1/\omega$, $d\tilde{t}$ becomes spacelike and $d\tilde{r} + \omega^2 \tilde{z} \sin \tilde{\theta} d\tilde{\varphi}$ becomes timelike. For our Earth, $2\pi/\omega = 1$ day $\approx 86400$ seconds, so the static limit surface would be positioned at $\tilde{\rho} \approx 13751$ light seconds $\approx 4 \times 10^{12}$ m away from the Earth.

Two features of the observational and radar coordinates here are different from those for a non-spinning, uniformly accelerated observer. First, radar coordinates (21) have a nontrivial coordinate singularity at $\tilde{\rho}' = 1/\omega$, which is the same static limit surface in the observational coordinates (25) for the same observer, while the radar coordinates for a uniformly accelerated observer (3) are regular for all finite values of the coordinates. Second, assuming the emitting and receiving operations of the localized observer have been started early around past timelike infinity, then radar coordinates (19) and observational coordinates (24) will cover almost the same region in the Penrose diagram of Minkowski space except the neighborhood of past null infinity. In contrast, for a non-spinning uniformly accelerated observer started the operations around past null infinity, the spacetime region covered by her/his observational coordinates are clearly larger than the region covered by radar coordinates.

### III. COMOVING OBSERVER IN DE SITTER SPACE

Similar to Rindler coordinates in Minkowski space, the static and flat de Sitter coordinates do not cover the whole de Sitter space. One may wonder if the observational and radar coordinates of a localized observer in de Sitter space can be related to the static, flat, or other conventional de Sitter coordinates. To answer this, below we are constructing the observational and radar coordinates for a comoving observer localized in de Sitter space. Then we will see if they have the features similar to those in Minkowski space.

#### A. Observer localized at the origin

Consider the global coordinates in de Sitter space [26, 27],

$$ds^2 = -dt^2 + H^{-2} \cosh^2 Ht \left[ d\chi^2 + \sin^2 \chi d\Omega_{11} \right],$$

(26)

$$= (H \sin T)^{-2} \left[ -dT^2 + d\chi^2 + \sin^2 \chi d\Omega_{11} \right],$$

(27)

where $d\Omega_{11} = d\theta^2 + \sin^2 \theta d\varphi^2$, $H$ is the Hubble constant, and $T = 2 \tan^{-1}[\tanh(HT/2)]$ so that $T = 0$ when $t = 0$. The Penrose diagram of the de Sitter space with $\theta$ and $\varphi$ suppressed can thus be represented as a square with $T \in (-\pi/2, \pi/2)$ and $\chi \in (0, \pi)$ in the $T\chi$-plane, as shown in Figure 7 (left).

Let

$$H\tilde{t} = \cosh Ht \sin \chi = \frac{\sin \chi}{\sin T},$$

(28)

$$H\tilde{\varphi} = \tanh^{-1}(\tanh HT \sec \chi).$$

(29)

Then we get the static de Sitter coordinates

$$ds^2 = -(1 - H^2 \tilde{r}^2) d\tilde{t}^2 + \frac{d\tilde{r}^2}{1 - H^2 \tilde{r}^2} + \tilde{r}^2 d\Omega_{11},$$

(30)

where a coordinate singularity occurs at $\tilde{r} = 1/H$. For an observer situated at $\tilde{r} = \chi = 0$, the coordinate time $\tilde{t}$ in [30] is identical in value to the coordinate time $t$ in [26] as well as the localized observer’s proper time. Interesting enough, $\tilde{r}$ here is actually the angular diameter distance and the affine distance of the null geodesic from an event $(\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\varphi})$ to the observer at the spatial origin $\chi = \tilde{r} = 0$, as will be shown in section IIIB. For $\tilde{r} < 1/H$, the static coordinates (30) has the metric component $g_{\tilde{t}\tilde{t}} > 0$ and covers the region R in Figure 7 (left), which is the counterpart of wedge R in Minkowski space. Outside region R, one can keep using $\tilde{r}$ in (28), which is well defined and ranges from $1/H$ (the past horizon $T = \chi$) to $\infty$ (past null infinity $T = 0$) in region F. Note that from (28), one has the contours
of \( \tilde{r} \) in the \( T \chi \)-plane as \( \chi = \sin^{-1} (H \tilde{r} \sin T) \) for \( H \tilde{r} < 1 \), which are timelike in region R, and \( T = \sin^{-1} [(H \tilde{r})^{-1} \sin \chi] \) for \( H \tilde{r} > 1 \), which are spacelike in region P. The boundaries of the regions, \( T = \chi \) and \( T = \pi - \chi \) for \( H \tilde{r} = 1 \), are lightlike.

One may define the null coordinate

\[
d\tilde{\tau} = d\tilde{t} + \frac{d\tilde{r}}{1 - H^2 \tilde{r}^2},
\]

then [30] becomes

\[
ds^2 = -(1 - H^2 \tilde{r}^2) d\tilde{\tau}^2 + 2 d\tilde{r} d\tilde{\tau} + \tilde{r}^2 d\Omega_{II}.
\]

Here \( \tilde{\tau} \) is the clock reading of the observer localized at \( \tilde{r} = 0 \) (where \( \tilde{\tau} = \tilde{t} \)). The coordinates consist of the observer’s clock \( \tilde{\tau} \), the distance \( \tilde{r} \) and angles \( \theta \) and \( \varphi \) in the line element [32] would be a good observational coordinate system to specify the observed events in regions P and R for the observer localized at the origin.

A radar coordinate system for the same localized observer can be obtained from [30] after identifying radar distance

\[
r = H^{-1} \tanh^{-1} H \tilde{r},
\]

such that \( r \to \infty \) as \( \tilde{r} \to 1/H \), and

\[
ds^2 = \frac{1}{\cosh^2 H r} \left[-d\tilde{\tau}^2 + dr^2 + \left(\frac{\sinh H r}{H} \right)^2 d\Omega_{II}\right],
\]

which turns out to be conformally equivalent to [3] for a uniformly accelerated observer at proper acceleration \( a = H \) in Minkowski space. Similar to the Rindler coordinates in Minkowski space, the above radar coordinate system has no nontrivial coordinate singularity for all finite values of the coordinates, and it only covers region R of the de Sitter space [26] in the Penrose diagram. Again, the visible universe of the localized observer situated at \( \tilde{r} = r = 0 \) is not restricted in region R where radar coordinates can be defined. The observer can see the events in region P behind the past horizon of radar coordinates [34] \( (r \to \infty, \text{ or } \tilde{r} = 1/H) \) and coordinatize those events in observational coordinates [32]. Moreover, the observer will feel that all the timelike worldlines not going to future infinity of \( \chi = 0 \) will be going towards and eventually stop around the past horizon, which is a sphere of radius \( 1/H \) centered at the observer.
B. Flat coordinates

Suppose the same comoving observer happens to use the flat de Sitter coordinates,
\[ ds^2 = -dt^2 + a^2(t) \left[ d\rho^2 + \rho^2 d\Omega_{II} \right] , \] (35)
where \( a(t) = e^{Ht}/H \), and the observer is localized at \( \rho = 0 \). Here \( \hat{t} \) and \( \rho \) are transformed from the global coordinates (26) by
\[ e^{H\hat{t}} = \sinh Ht + \cosh Ht \cos \chi, \] (36)
\[ \rho = \sin \chi \tanh Ht + \cos \chi, \] (37)
and the flat de Sitter coordinates (35) covers the region \( F \cup R \) bounded by \( \chi = 0 \), \( T = \pi/2 \), and \( \chi = T \) in the Penrose diagram (Figure 7). For every null geodesic from past infinity to the observer at \( \hat{t} = t_o \), (35) implies
\[ \frac{d\hat{t}}{d\rho} = -a(\hat{t}(\rho)) \] (38)
where the minus sign corresponds to the past light cone. Requiring \( \rho = 0 \) at \( t = t_o \), the solution for (38) is
\[ \rho = e^{-H\hat{t}} - e^{-Hi_o} \] (39)
along the null geodesic. Suppose the observer uses \( \lambda \equiv a(\hat{t}_o)\rho \) to parametrize the null geodesic so that the value of \( \lambda \) matches radar distance in the neighborhood of the localized observer. By virtue of the spherical symmetry, the null geodesic can be described by the equation
\[ \frac{d^2z^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dz^\alpha}{d\lambda} \frac{dz^\beta}{d\lambda} = \kappa(\lambda) \frac{dz^\mu}{d\lambda}, \] (40)
where we find \( \kappa(\lambda) = -2He^{H(\hat{t}(\lambda))-\hat{t}_o} = -2H/(H\lambda + 1) \) after taking \( z^\mu = (\hat{t}(\lambda), \rho(\lambda), \theta, \varphi) \) and introducing (38) and (39). While \( \kappa \) is not zero and so \( \lambda \) is not an affine parameter, one can generate an affine parameter \( \lambda^* \) from \( \lambda \) by solving (28)
\[ \frac{d\lambda^*}{d\lambda} = \exp \int^\lambda \kappa(\lambda') d\lambda' = \frac{1}{(H\lambda + 1)^2}, \] (41)
which gives
\[ \lambda^* = \frac{1}{H} \left( 1 - \frac{1}{H\lambda + 1} \right) = \frac{1}{H} \left( 1 - e^{H(\hat{t}(\rho))} \right). \] (42)
with the condition \( \lambda^* = 0 \) at \( \lambda = 0 \). It turns out that \( \lambda^* = a(\hat{t})\rho = H^{-1} \cosh Ht \sin \chi = \hat{r} \) from (42), (39), (36), (37), and (28), namely, \( \lambda^* \) coincides with the radial coordinate \( \hat{r} \), which is nothing but the angular diameter distance \( a(\hat{t})\rho \), in the static coordinates (36) or (32) along the null geodesics on the past light cones of the localized observer. Thus the coordinates in (32) would be a natural observational coordinates after identifying \( \tau = \hat{t}_o \) and \( \lambda^* = \hat{r} \). The observer can see through the past horizon \( \hat{t} \to -\infty \) of the flat coordinates into region \( P \) of de Sitter space, though the flat coordinates do not cover that region.

C. Non-Eternal Inflation

The above result is valid in de Sitter space, corresponding to an eternally inflationary universe. In the usual non-eternal inflation model, a comoving localized observer cannot see beyond the null surface \( T = \chi \) and the observational coordinates for that observer would cover only wedge \( R \) in the \( T\chi \)-plane, as shown in Figure 7 (right).

For example, suppose an inflation era of a flat spacetime is started with a Minkowski space at \( \hat{t} = \hat{t}_s \), and the metric of the spacetime is given by
\[ ds^2 = -dt^2 + a^2(\hat{t}) \left( d\rho^2 + \rho^2 d\Omega_{II} \right) \] (43)
where $a(t) = e^{Hi}/H$ is exponentially growing in time for $\hat{t} \geq \hat{t}_s$ and $a(t) = e^{Hi_s}/H$ is a constant for $\hat{t} < \hat{t}_s$. Using the inverse transformations of (36) and (37) from $(\hat{t}, \rho)$ to $(t, \chi)$, one can see that the flat coordinates in (43) still covers region $F \cup R$ in the Penrose diagram for de Sitter space in the $T\chi$-plane.

For a localized observer at the spatial origin ($\rho = \chi = 0$) with the metric in (35), the light pulse from the event at $(\hat{t}, \rho, \theta, \varphi)$ is received by the observer at her/his proper time $\hat{t}_o$ determined by (39) for $\hat{t} \geq \hat{t}_s$ and by $ho = e^{-Hi_s} - e^{-Hi_s} + e^{-Hi_s} - \hat{t}$ for $\hat{t} < \hat{t}_s$. Thus the angular diameter distance of the event for the localized observer is

$$a(t)\rho = \begin{cases} H^{-1} \left( 1 - e^{Hi(t - \hat{t}_o)} \right) & \text{for } \hat{t} \geq \hat{t}_s, \\ H^{-1} \left( 1 - e^{Hi(t - \hat{t}_o)} + \hat{t}_s - \hat{t} \right) & \text{for } \hat{t} < \hat{t}_s, \end{cases}$$

which diverges as $\hat{t} \to -\infty$. In other words, the distance from the localized observer at the origin to the hypersurface $\bar{t} \to -\infty$ is infinity for the observer. The region $F \cup R$ is geodesically complete, and in Figure 7 (right) there would be nothing behind past infinity at $T = \chi$ to be visible for the observer.

In Figure 7 (right) one can also see that the physical objects with timelike worldlines going through the event horizon (the null surface labeled $\tilde{r} \to \infty$) after the onset of inflation ($\hat{t} > \hat{t}_s$) would appear to go away from the observer and approach the illusory horizon at $\tilde{r} = 1/H$ at late times for the observer. Other physical objects going through the event horizon before $\hat{t}_s$ would be observed at late times as frozen at some distances greater than $1/H$ and their clock readings would never reach $\hat{t}_s$, if all the clocks have been synchronized initially at $\hat{t} \to -\infty$.

### IV. OBSERVER OUTSIDE A SPHERICAL SHELL IN (1+1)D

The null geodesics around a black hole in (3+1)D can be complicated even in the simplest case of Schwarzschild spacetime [10, 29, 30], and so observational and radar coordinates in terms of perceived and radar distances for a localized observer may not be convenient for analysis. Nevertheless, the observational and radar coordinates for a localized observer in (1+1)D Schwarzschild geometry can be simple enough to gain insights.

Kruskal coordinates for an (1+1)D Schwarzschild black hole look very similar to Rindler coordinates in (1+1)D Minkowski space. One may be tempted to think that an observer localized outside an eternal black hole or a collapsing star would be able to coordinate the events behind the past horizon at the Schwarzschild radius in Kruskal coordinates, and would observe that most of the timelike worldlines would eventually approach the past horizon with increasing redshift. Similar to the cases in de Sitter space, such a speculation would be true only in the maximally extended Schwarzschild coordinates for an eternal black hole (with the white hole singularity visible by a localized observer outside), but not in the case of a spherical collapsing star, which is a non-eternal black hole as in the example below.

Consider a (1+1)D spacetime in the presence of a spherical thin shell of mass $M$ and radius $r = r_s > 2M$ [28, 31],

$$ds^2 = -A(r)dt^2 + B(r)dr^2,$$  \hspace{1cm} (45)

where

$$A(r) = 1/B(r) = 1 - \frac{2M}{r} \quad \text{for } r > r_s,$$  \hspace{1cm} (46)

$$A(r) = A_s \equiv 1 - \frac{2M}{r_s}, \quad B(r) = 1 \quad \text{for } r \leq r_s,$$  \hspace{1cm} (47)

and a localized observer outside of the shell is fixed at a constant radial distance $r = r_o > r_s$ from the center of the spherical shell in the above bookkeeper coordinates [32]. The event at $(t, r)$ can be specified by radar time $t' = (\tau_f + \tau_i)/2 = \sqrt{A_o} t$ with $A_o \equiv A(r_o) = 1 - (2M/r_o)$, and radar distance

$$r' = \sqrt{A_o} \frac{\Delta t}{2} = \sqrt{A_o} \left| \int_{r_o}^r \sqrt{\frac{B(\bar{r})}{A(\bar{r})}} d\bar{r} \right|$$

$$= \begin{cases} \sqrt{A_o} (r - r_o) + 2M \ln \frac{r - 2M}{r_o - 2M} & \text{for } r > r_s, \\ \sqrt{A_o} \left( \frac{1}{\sqrt{A_s}} (r_s - r) - r_s + r_o - 2M \ln \frac{r_o - 2M}{r_s - 2M} \right) & \text{for } r \leq r_s. \end{cases}$$  \hspace{1cm} (48)

Then (45) can be transformed to radar coordinates

$$ds^2 = \frac{A(r)}{A_o} (-dt'^2 + dr'^2)$$  \hspace{1cm} (49)
In a static spacetime (45), while in (1+1)D the astronomical distances for a localized observer cannot be determined, one can still formally define the affine distance of an event as the difference of the normalized affine parameter along a null geodesic connecting the event at \( r = r_o \) and some point of the worldline of the localized observer at \( r_o \) in the future of the event. From the geodesic equations, the affine distance reads \( \tilde{r} = \alpha \left| \int_{r_o}^{r} \sqrt{AB}dr \right| \) up to a constant factor \( \alpha \). We choose \( \alpha = 1/\sqrt{A_o} \) to match the radar distance in the neighborhood of the localized observer. Rewrite \( r_e \) as \( r \), we find the affine distance

\[
\tilde{r} = \left| \frac{r-r_o}{\sqrt{A_o}} \right|
\]
for the events outside the spherical shell \((r > r_s)\), and
\[
\hat{r} = \frac{1}{\sqrt{A_o}} \left[ r_o - r_s + \sqrt{A_s(r - r)} \right]
\] (54)
for the events inside \((r < r_s)\). One can see that \(\hat{r}\) is finite for \(r = 0\) and \(r = 2M\). The observational coordinates for the observer localized at \(r = r_o\) then read
\[
ds^2 = \begin{cases} 
\frac{1}{A_o} \left[ - \left( 1 - \frac{2M}{r_o + r/\sqrt{A_o}} \right) d\tilde{t}^2 + 2d\tilde{r}d\tilde{r} \right] & \text{for } r > r_s, \\
-\frac{A_o}{d_s^2} d\tilde{r}^2 + 2d\tilde{r}d\tilde{r} & \text{for } r \leq r_s,
\end{cases}
\] (55)
where \(d\tilde{r} = dt' + (d\tilde{r}/A(r))\) for \(r > r_s\), \(d\tilde{r} = dt' + (A_o/A_s)d\tilde{r}\) for \(r \leq r_s\), and + and − correspond to the cases of \(r > r_o\) and \(r_s < r < r_o\), respectively. Observational coordinates \((55)\) cover almost the same region that radar coordinates \((49)\) do at late times in the Penrose diagram except the neighborhood of past null infinity. Since the affine distance \(\tilde{r}\) in \((53)\) and \((54)\) is proportional to \(r\), the contours of \(\tilde{r}\) in the \(\eta\rho\)-plane and the Penrose diagram for \((55)\) have the same pattern as those of \(r\) for \((45)\).

If the worldline of the observer is started at some point in past null infinity rather than past timelike infinity, then the situation will be similar to the one with the uniformly accelerated observer in Minkowski space: the spacetime region \(R_o\) covered by observational coordinates will be significantly larger than the region \(R_r\) covered by radar coordinates in the Penrose diagram. The border of \(R_r\) and \(R_o - R_r\) is the past horizon where the radar distance is infinity but the affine distance is finite for the localized observer.

In Figure 8 (right) we sketch a scenario of a collapsing thin shell \((33)\) (rigorous calculations can be found in literature, e.g. \([18]\)). The union of wedges R and F is maximal and there is no need of attaching wedges L and P or a white hole. Suppose the thin shell contains a few point-like light emitters inside and the shell is semi-transparent. As the shell radius is approaching the Schwarzschild radius \((r \rightarrow 2M)\), an observer localized at \(r_o\) outside the shell would perceive that all the emitters inside the shell squeeze with the shell surface around the horizon since the depth information of the emitters would be suppressed as \(\sqrt{A_o} \rightarrow 0\) in this limit (note that the \((r_s - r)\) term in \((54)\) is proportional to \(\sqrt{A_o}\)). While the depth information in terms of radar distance would be well resolvable whenever \(r \neq 2M\) (the \((r_s - r)\) term in \((48)\) is proportional to \(1/\sqrt{A_o}\)). Measuring radar distance could be much harder than measuring the affine distance in the limit of \(r \rightarrow 2M\) because the former needs additional knowledge about the received signals (echoes) for the observer. Similar observation in \((3+1)D\) in terms of more measures of distance may be relevant to the area law of black hole entropy \([30]\). The observed horizon at the affine distance \(\tilde{r}|_{r \rightarrow 2M} = \sqrt{r_o(r_o - 2M)}\) from the observer is the illusory horizon \([16]\) rather than the past horizon of the observer’s radar coordinates or the event horizon, the latter is defined globally and will never be seen by any localized observer outside of it since no past light cone from the observer will intersect the event horizon.

V. SUMMARY AND DISCUSSION

We have considered the observational coordinates and radar coordinates for the localized observers in inertial motion (Mi), in uniform acceleration (Ma), and spinning without center-of-mass motion (Ms) in Minkowski space, and for those observers comoving in de Sitter space (dS), fixed at constant radius in Schwarzschild geometry outside of static (Ss) and collapsing (Sc) spherical shells, as well as the cases of non-uniform acceleration in Minkowski space (MnU) and non-eternal inflation (nEI) where the universe was similar to Minkowski space before the onset of inflation.

A. Regions covered by radar and observational coordinates

Observational coordinates are determined by a localized observer according to the light or radar signals received. Radar coordinates are determined with a stronger condition that those received signals are echoes of the radar signals emitted earlier by the same observer. Thus the region \(R_r\) covered by radar coordinates must be contained by the region \(R_o\) covered by the observational coordinates for the same observer. For an observer localized at the origin in Minkowski space, either non-spinning (Mi) or spinning (Ms), and for a localized observer (Ss) fixed in \((1+1)D\) Schwarzschild geometry at a constant radius from the center of a spherical shell, radar coordinates and observational coordinates for the localized observers at late times appear to cover the same spacetime regions in the Penrose diagrams, where \(R_o\) is the closure of \(R_r\). The situations are similar in the cases of the non-eternal inflation (nEI) and the collapsing star (Sc), where neither observational nor radar coordinates can cover the whole universe due to the presence of the event horizon. The region \(R_o - R_r\) become significant in the Penrose diagrams for the uniformly accelerated observer.
in Minkowski space (Ma) and the comoving observer situated at the origin in de Sitter space (dS). In these cases, the observers can see through the past horizons of radar coordinates and coordinatize the events beyond the reach of radar coordinates with some physical assumptions.

B. Static limit surface and past horizon

Coordinate singularities at finite distances in observational coordinates arise in the cases (Ma) and (Ms) for non-inertial observers in Minkowski space and (dS) for a comoving observer in de Sitter space. These coordinate singularities are associated with signature change of the metric component $g_{\tau\tau}$ or $g_{tt}$ in each case and correspond to the static limit surfaces, beyond which no point-like physical object is possible to be seen at rest in the viewpoint of the localized observer. In (Ma) and (dS) the static limit surfaces of observational coordinates coincide with the past horizons of the radar coordinates for the same observer. However, in (Ms) observational and radar coordinates for the spinning observer share the same static limit surface which is not the past horizon for the observer.

C. Coordinate singularity and acceleration

While the observers in the cases (Ma) and (Ms) are accelerated, and the comoving objects in the case (dS) look accelerated in the viewpoint of the observer, the accelerations of the observer and/or the comoving objects are not always associated with coordinate singularities in observational coordinates. In the case (Ss), while a localized observer fixed at a constant radius from the center of a static massive spherical shell is in a constant acceleration, there is no coordinate singularity or past horizon in observational or radar coordinates. Neither does the case of the non-eternal inflation (nEI).

D. Event horizon and illusory horizon

There exist event horizons in the cases (Ma), (dS), (Sc), and (nEI). The localized observer at late times in (Ma), (dS), or (Sc) would see an illusory horizon at some finite distance in her/his observational coordinates. All the visible physical objects with timelike worldlines passing through the event horizon would appear to approach the illusory horizon and eventually frozen there. In the Penrose diagrams of these cases an illusory horizon does not have to coincide with the past horizon (in (Sc) there is even no past horizon.) In (nEI) only the timelike worldlines going through the event horizon after the onset of inflation would be observed like that: they would appear to go away from the observer and approach the illusory horizon at late times. Other physical objects would simply be seen at late times as frozen behind the illusory horizon.

The localized observers in the cases without event horizon could not see any illusory horizon that physical objects appear to approach, even if there exist coordinate singularities in those cases (e.g. the spinning observer in (Ms).)
[8] W. Pauli, *Die Relativitätstheorie*, Encyclopädie der mathematischen Wissenschaften 5 (1921) 539; In English: *Theory of Relativity* (Dover, New York, 1981).

[9] W. Rindler, *Kruskal space and the uniformly accelerated frame*, Am. J Phys. 34 (1966) 1174.

[10] H. Lass, *Accelerating frames of reference and the clock paradox*, Am. J. Phys. 31 (1963) 274.

[11] G. Temple, *New system of normal coordinates for relativistic optics*, Proc. R. Soc. London. A 168 (1938) 122-148.

[12] G. F. R. Ellis, S. D. Nel, R. Maartens, W. R. Stoeger, and A. P. Whitman, *Ideal observational cosmology*, Phys. Rep. 124 (1985) 315-417.

[13] M. Gasperini, G. Marozzi, F. Nugier, and G. Veneziano, *Light-cone averaging in cosmology: formalism and applications*, JCAP 07(2011), 008.

[14] F. Nugier, *From GLC to double-null coordinates and illustration with static black holes*, JCAP 09(2016), 019.

[15] S. Weinberg, *Gravitation and cosmology: Principles and applications of the general theory of relativity*, (Wiley, New York, 1972).

[16] A. J. S. Hamilton and G. Polhemus, *Stereoscopic visualization in curved spacetime: seeing deep inside a black hole*, New J. Phys. 12 (2010) 123027.

[17] E. T. Newman and T. W. J. Unti, *A Class of Null Flat-Space Coordinate Systems*, J. Math. Phys. 4 (1963) 1467.

[18] E. Poisson, *The Motion of Point Particles in Curved Spacetime*, Living Rev. Rel. 7 (2004) 6 [Online Article]: cited [07/25/2015].

[19] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Redwood, 1965).

[20] S.-Y. Lin and B. L. Hu, *Accelerated detector - quantum field correlations: From vacuum fluctuations to radiation flux*, Phys. Rev. D 73 (2006) 124018.

[21] D. C. M. Ostapchuk, S.-Y. Lin, R. B. Mann, and B. L. Hu, *Entanglement dynamics between inertial and non-uniformly accelerated detectors*, JHEP 07(2012) 072.

[22] W. Kinnersley, *Field of an Arbitrarily Accelerating Point Mass*, Phys. Rev. 186 (1969) 1335.

[23] W. B. Bonner, *The photon rocket*, Class. Quantum Grav. 11 (1994) 2007.

[24] M. Born, *Die Theorie des starren Elektrons in der Kinematik des Relativitäts-Prinzipes*, Ann. Phys. 30, (1909) 1.

[25] L. D. Landau and L. M. Lifshitz, *The Classical Theory of Field*, 3rd English Ed (Pergamon, New York, 1971).

[26] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime*, (Cambridge University Press, Cambridge, 1973).

[27] E. Mottola, *Particle creation in de Sitter space*, Phys. Rev. D 31, 754 (1985).

[28] E. Poisson, *A Relativist’s Toolkit: The Mathematics of Black Hole Mechanics* (Cambridge University Press, 2004).

[29] V. Perlick, *Gravitational Lensing from a Spacetime Perspective*, Living Rev. Rel. 7, 9 (2004) [Online Article]: cited [07/25/2015].

[30] S.-Y. Lin, *Seeing into a nearly black star*, preprint [arXiv:1910.13198].

[31] W. Israel, *Singular Hypersurfaces and Thin shells in General Relativity*, Nuovo Cimento 44B (1966) 1; erratum 48B (1967) 463.

[32] E. F. Taylor and J. A. Wheeler, *Exploring Black Holes – Introduction to General Relativity*, (Addison Wesley Longman, San Francisco, 2000).

[33] A. J. S. Hamilton, *Collapse to a black hole* (1998), http://casa.colorado.edu/~ajsh/collapse.html, cited [04/15/2019].