Quantum dense coding by spatial state entanglement

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We have presented a theoretical extended version of dense coding protocol using entangled position state of two particles shared between two parties. A representation of Bell states and the required unitary operators are shown utilizing symmetric normalized Hadamard matrices. In addition, some explicit and conceivable forms for the unitary operators are presented by using some introduced basic operators. It is shown that, the proposed version is logarithmically efficient than some other multi-qubit dense coding protocols.

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The quantum entanglement property is providing new methods of information transfer, in some cases much more powerful than their classical counterparts. In quantum information theory, entanglement, as a key concept is used for a wide range of applications, such as dense coding, teleportation, secret sharing and key distribution. To see more about the mentioned topics and the efforts done on them, both theoretically and experimentally, one can refer to and references therein.

Quantum dense coding protocol proposed originally by Bennett and Wiesner in 1992. The protocol describes a way to transmit two bits of classical information through manipulation of only one of the entangled pair of spins. While each of the pair individually could carry only one bit of classical information. The first experimental realization of dense coding has been reported by Mattle et al. in 1996.

There have been attempts to generalize dense coding protocol to achieve higher channel capacity. The first proposition in this regard was due to the original reference of dense coding by using a pair of n-state particles prepared in a completely entangled state (instead of an EPR spin pair in a singlet state) to encode n² values. In practice, there might be some limitations for finding n-state particles with high n’s and controlling them. Although very recently, some progress have been made in this direction, but still it is worth, both theoretically and experimentally, examining other alternatives to achieve this goal. For example, Bose et al. studied N pairwise entangled states in which each party gets one particle except Bob with N qubits. Also in a more efficient scheme, they considered N + 1 particles sharing maximally entangled qubits in a way that each party, including Bob, possesses one qubit. In addition, some recent attempts on generalization of quantum dense coding can be found in. In an elegant alternative, Vaidman proposed a method for utilizing canonical continuous variables x (position variable) and p (linear momentum variable) to perform a quantum communication. After that, Braunstein and Kimble presented a typical realization for continuous variable dense coding using the quadrature amplitudes of the electromagnetic fields in which the mean photon number in each channel should be considered very large. Recently, in another way, some effort has been also done on experimental realization of continuous variables dense coding.

In this work, we have theoretically presented another extended version of dense coding which uses discrete spatial variables along with only two entangled particles. In this regard, all necessary Bell states and their corresponding unitary operators are presented to encode and decode information. This version at large N’s can be considered as a conceivable scheme for Vaidman’s idea except that we have considered just the position variable (not both canonical variables x and p) for communication. Finally, the efficiency of our scheme is compared with some other known ones. Furthermore, according to Werner’s general classification on dense coding and teleportation schemes, it is possible to find a teleportation protocol corresponding to our dense coding scheme.

Consider an original EPR source which emits isotropically a pair of identical (fermionic or bosonic) particles with vanishing total linear momentum in a two dimensional space. The similar EPR source is also used elsewhere to clarify some discussions on the foundations of quantum mechanics. The source S is placed exactly in the middle of the two parties, “Alice” and “Bob”, where each one has an array of receivers aligned on a vertical line. The receivers just receive the particles and do not perform any destructive measurement. Total number of the receivers of each party is considered to be 2N. Since Alice and Bob are assumed to be very far from each other and the source is considered isotropic, there is an equal probability for every receiver to obtain one of the emitted particles. We label the receivers placed at the upper (lower) part of the x-axis with positive (negative) integers; 1, 2, , N (-1, -2, , -N). Figure shows an illustration of this scheme.

Now, the position state of the system (the arrays + the
FIG. 1: Schematic of the proposed dense coding scheme. The 2N receivers of each party are connected to its corresponding lab by quantum channels. The quantum channels can be connected together in each lab using side quantum channels. Alice’s lab is equipped with \( O_{k,r,j} \) encoder unitary operators, and Bob’s lab contains a decoder unitary operator like \( H \). The Bell state measurement is completed by just a simple position measurement on Bob’s outgoing channels.

source) can be written in the form

\[
|\psi_{1,2}\rangle = \frac{1}{\sqrt{2^N}} \sum_{n=1}^{N} [n,n] \pm |n-n,n\rangle
\]

(1)

where the subscripts 1 and 2 are related to ± signs, respectively, and \( n \) refers to label of the receivers. In addition, the order in writing the state is according to direct product of Alice’s state and Bob’s. The signs ± in \( |\psi_{1,2}\rangle \) indicate the symmetry and anti-symmetry property of the position state with respect to the particles exchange. Without loss of generality, we assume bosonic property for our system. In general, the entangled state **can** be considered as a member of a larger family (with a bit different notation), that is,

\[
|\psi_{(1,j)}\rangle = \frac{1}{\sqrt{2^N}} \sum_{n=1}^{N} [h_{j,2n-1} |n,n\rangle + h_{j,2n} |n,n\rangle]
\]

\[1 \leq j \leq 2N \]

(2)

in which \( h_{j,j} = \sqrt{2^N} |H|_{j,j} \) and \( H \) is a 2N-dimensional normalized symmetric Hadamard matrix which satisfies the property \( H^2 = I \). In addition, the above entangled states can be generalized to a more complete set of orthonormal and maximally entangled states, which can be defined as

\[
|\psi_{(k,r,j)}\rangle = \frac{1}{\sqrt{2^N}} \sum_{n=1}^{N} [h_{j,2n-1} |n,k_r(n)\rangle + h_{j,2n} |n,-k_r(n)\rangle]
\]

\[1 \leq k \leq N, 1 \leq j \leq 2N \]

(3)

where \( k_r \) and \( j \) are called family and member indices, respectively. Furthermore, we have adopted the conventions \( h_{j,ij} < 0 \) or \( i < 0 \) = 0 and \( f_{k_r}(n) = r(n+k-1)_{\text{mod}(N)} \) where \( r = \pm 1 \). Now, the states **form Bell bases** for our dense coding scheme. In fact, it is straightforward to check that these states have the properties

\[
\langle \psi_{(k,r,j)} | \psi_{(k',r',j')} \rangle = \delta_{j,j'} \delta_{k,k'} \delta_{r,r'} \]

(4)

\[\text{tr}_{1(2)}(|\psi_{(k,r,j)}\rangle \langle \psi_{(k,r,j)}|) = \frac{1}{2^N} I_{1(2)} \]

(5)

Here it should be noted that, the order of a typical real Hadamard matrix can be 1, 2 and 4k where \( k \) is a positive integer [19]. Physically it means that in other cases one cannot make enough necessary entangled orthonormal states to perform an efficient dense coding.

For allowed \( N \)'s, Alice should find unitary encoding operators which transform the state describing the system in Eq. (1), \( |\psi_1\rangle \), as a Bell state with \( j = 1 \) (and \( k = 1, r = -1 \)) into the others given in Eq. (6). The representation of \( 4N^2 \) suitable unitary operators for this task is

\[
O_{k,r,j} = \sum_{n=1}^{N} [h_{j,2n-1} |n, f_{k_r}(n)\rangle + h_{j,2n} |n, -f_{k_r}(n)\rangle]
\]

(6)

which acts as follows

\[
O_{k,r,j} |\psi_{(k',r',j')=1}\rangle = |\psi_{(k'',r'',j'')}\rangle
\]

(7)

in which

\[k''_r := \frac{(k+k'-1)_r_{\text{mod}(N)}}{2}. \]

(8)

It is a relevant question how to implement \( O_{k,r,j} \) operators in practice. So here, we concisely propose one way to perform this task using some basic and conceivable operators. For example, the same as Pauli’s operator in the spin-\( \frac{1}{2} \) space, suitable basic operators in the position space can be considered to be

\[N_n|\pm n\rangle = |\pm |n \rangle \]

(9)

\[P_n|\pm n\rangle = |\mp |n \rangle \]

(10)

where the subscript \( n \) means that the operator acts as a local gate on the \( \pm n \)-th channel and as identity on the other channels. The latter operator, \( P_n \), is defined to relate the two groups of receivers in the upper and lower halves of the \( x \)-axis. We also need to define a ladder operator \( L_+ \) to relate the receivers located on each half of the \( x \)-axis, so that

\[L_+ |\pm n\rangle = |\pm (n+1) \rangle_{\text{mod}(N)}. \]

(11)

It is easy to understand that the main task of the basic operators \( H, T \) is really displacement of a channel or equivalently displacement of a particle from a channel to the other. This task can be performed using, for example, some side quantum channels which connect the main channels together in a suitable way. So, if our introduced channels in the scheme are considered, for instance, superconductor wires containing an entangled current according to [16], then these basic operators can be conceived to be realized by using switching process in the superconductor circuits. In fact, when a basic gate is OFF, its main channel(s) is (are) superconducting and the side channels are not superconducting, and if the gate is ON then the main channel(s) is (are) not superconducting and the suitable and corresponding side channel(s) is (are) superconducting. However, full realization of this proposition is left as an experimental challenge.
Now using the above basic gates, we have shown that one can construct operators such that transform any member of a family to any member of the other ones. Here, our construction mechanism is to find a set of operators, $O_j$, to transform the $j$ label of the Bell states of any given family. Next, we introduce another set of operators, $F_{k,r}$, to transform $k$ label of the Bell states of any given member. Thus, our desired total operators are

$$O_{(k,r,j)} = F_{k,r}O_j. \quad (12)$$

The explicit form of the $O_j$ operators can be constructed using a normalized symmetric Hadamard matrix as follows

$$O_j = \prod_{i=1}^{N} (P_i P_i'|P_i(P_i - 1 - P_i')/2 N_1^{(h_{a,2i-1} - h_{j,2i})}/2 \quad 1 \leq j, a \leq 2N \quad (13)$$

where $a$ is an arbitrary fixed positive integer. One can easily check that, the operator $O_j$ has the property $O_j|\psi(k_r, j')\rangle = |\psi(k_{r'}, j')\rangle$ where $h_{r'} = h_ri h_{j'j}$. Furthermore, the explicit form for $F_{k,r}$ can be considered as

$$F_{k,r} = L_{1}^{(1-k)} \prod_{i=1}^{N} P_i^{(1-r)/2}. \quad (14)$$

It can be seen that they transform families to each other according to $F_{k,r}|\psi(k'_r, j')\rangle = |\psi(k'_r, j')\rangle$ where $k'_r$ satisfies the rule in Eq. $\mathcal{S}$. Now, Bob is ready to perform a Bell state measurement (BSM) on his particle and the processed particle which is sent to him by Alice. To do a BSM, Bob needs apply a grand unitary and Hermitian operator on all the channels in his lab which should be followed by a position state measurement on his outgoing channels. To simplify the representation of the grand operator we first move into another set of Bell states which have the compact form as $|\psi(k_r, j)\rangle = \sum_{n=1}^{2N} h_{j,n} |a, f_{k_r}(j)\rangle$ where $f_{k_r}(j) = r(j + k - 1)_{\text{mod}(2N)}$. These states can be obtained from the firstly defined states, Eq. $\mathcal{S}$, by application of some local unitary operator. Now, the representation of the grand operator can be considered as

$$\mathcal{H} = \sum_{j=1}^{2N} \sum_{k=1}^{N} \sum_{r=\pm 1} |j, f_{k_r}(j)| \langle \psi(k_r, j)|. \quad (15)$$

It is easy to check the properties $\mathcal{H}|\psi(k_r, j)\rangle = |j, f_{k_r}(j)\rangle$ and also $\mathcal{H}^2 = I$.

Again, one can find an explicit form for $\mathcal{H}$ operator based on the introduced basic operators. At first, Bob needs one operator to disentangle the two particles. We have considered this operator as a non-local operator which acts conditionally on the $|l, m\rangle$ state as

$$\mathrm{PCS}|l, m\rangle = \theta(-l)(I \otimes \mathcal{P}_m)|l, m\rangle + \theta(l)|l, m\rangle \quad (16)$$

where $\mathrm{PCS}$ stands for position controlled swap operator and $\theta(l)$ is the conventional unit step function. Elsewhere, we have proposed a method to realize this gate using four usual CNOT gates $[20, 21]$. On the other hand, the same as the spin-$\frac{1}{2}$ case, Bob can use Hadamard operators in the position space with the form

$$H_{x_n} = \frac{1}{\sqrt{2}}(P_n + N_n) \quad (17)$$

where $H_{x_n}$ is a local operator acting on the $\pm n$-th channels. But for $N \geq 2$ cases, applying just $\mathrm{PCS}$ and $H_{x_n}$ operators does not produce pure position states in a disentangled and measurable form. So, we should introduce a non-local unitary operator for this mean which acts like

$$U_{(N)}|rl, r'm\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N}|h_{m,m+n-1}|f_{r}(n), f_{m, r}(n)\rangle_{\text{mod}(N)} \quad (18)$$

where $l$ and $m$ are positive integers. Moreover, here, $h_{m,n}$ is an element of an $N$-dimensional normalized symmetric Hadamard matrix. It is straightforward to check that, $U_{(N)}^2 = I$. Considering the works of Barenco et al. $[22]$ as well as Bremer et al. $[22]$ and the Eq. $\mathcal{S}$, it can be concluded that in construction of $U_{(N)}$ operator, at most $(N - 1)$ successive $L_r$ operators for each particle and $O(N)$ parallel PCS gates are certainly needed. The same as $O_{(k_r,j)}$ operator, it is also possible to find an explicit form for $U_{(N)}$ operator based on the basic gates $[21]$, but its details are omitted here. Now, using the above mentioned operators, Bob can perform the position BSM in this way

$$U_{(N)}(H_{x_1} H_{x_2} \ldots H_{x_n} \otimes I)\mathrm{PCS}|\psi(k_r,j)\rangle = |m, n\rangle \quad (19)$$

which should be followed by a position measurement on the outgoing channels. In Eq. $\mathcal{S}$, the operator $U_{(N)}$ acts on all the channels, and $m$ as well as $n$ are some unique functions of $k_r$ and $j$. In addition, action of the set of operators in $\mathcal{S}$ are equivalent to the action of the grand operator $\mathcal{H}$.

We have seen that, there are $4N^2$ Bell basis and the same number of different unitary operators $O_{(k_r,j)}$ for encoding in our dense coding scheme. This obviously corresponds to encoding $4N^2$ different messages by Alice. Thus, she can send $2 \log_2(2N)$ bits of classical information per particle to Bob. Now, he needs $N$ PCS, $2N$ Hadamard and one $U_{(N)}$ gates to read out the sent classical information during the BSM. Since Bob performs one BSM on just the two particles, it is possible to consider that all PCS and also all Hadamard gates operate in a parallel form, i.e. concurrently. If the operation times for the PCS, Hadamard and $U_{(N)}$ gates are $t_p$, $t_h$ and $t_u$, respectively, the rate of classical information gain $R$, defined as sent classical bits of information per unit time and sent particle, is

$$R_e = 2 \log_2(2N)/(t_p + t_h + t_u). \quad (20)$$

Similarly, one can calculate $R$ for a dense coding protocol which works using $N$ pairwise entangled qubits and/or $N$ maximally entangled qubits shared between two parties $[10]$. In the pairwise entangled case, $2N$ classical bits
of information are transferred from Alice to Bob. Meanwhile, in this case, \(N\) separate CNOT and Hadamard gates are required by Bob to decode the Bell states. Thus, its \(R\) in terms of bits per unit time per sent particle is

\[
R_p = \frac{2N}{N^2 (t_c + t_h)}
\]

(21)

where \(t_c\) is the operation time of a CNOT gate. On the other hand, in the maximally entangled case, the number of sent classical bits is \(N\). In this case, Bob needs \((N - 1)\) successive CNOT and one Hadamard gate so that he gains

\[
R_m = \frac{N}{(N - 1)(N - 1)t_c + t_h}
\]

(22)

bits per time and particle. Now, if we assume that both \(N\) and \(N\) are very large and all basic gates operate in an equal time interval, i.e. \(t_c \sim t_h \sim t_p/4 \sim t_u/N \equiv t\), then \(R_x = 2 \log_2 (2N)/Nt\) and \(R_p = R_m = 1/Nt\). Therefore, as is seen, at large \(N\)'s, if \(N = N\) is considered, i.e. dimensions of Hilbert spaces are identical, then our protocol is more efficient than both the pairwise and the maximally entangled cases with a logarithmic factor.

Furthermore, other degrees of freedom such as spin, polarization and so on can be added to our protocol in order to obtain a more powerful dense coding. Here, for instance, we consider that each particle can also have the spin \(S\). To introduce spin into our protocol, it is sufficient to assume that the source emits entangled pair of particles not only with vanishing total momentum but also with zero total spin. Therefore, the quantum state of the system would be

\[
|\psi_1\rangle_{xs} = \frac{1}{\sqrt{2N(2S+1)}} \sum_{n=1}^{N} (|n, -n\rangle + | -n, n\rangle)
\times \sum_{s=0}^{2S} (\pm 1^s)|((S - s), -(S - s))\]

(23)

which is simply a tensor product of position and spin states of the system. Therefore, now Alice is capable of sending \(2 \log_2 [2N(2S + 1)]\) bits per particle.

In summary, we have proposed another theoretical extension of dense coding protocol by using entangled spatial states of the two particles shared between two parties. Our construction is based on using the well known Hadamard matrices for building orthogonal states, required encoding and decoding operators, and hence is subject to their intrinsic characteristics. Furthermore, we have given a typical proposition for approaching to realization of the scheme. For this mean, some basic operations have been introduced and it is shown that whole the scheme can be established based upon them.

By comparing our scheme with some previously proposed multi-qubit protocols, it is shown that the rate of classical information gain in our case is better than them with a logarithmic factor. Also we have shown that considering internal degrees of freedom, like spin, strengthens the scheme in a straightforward manner.

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