Induced cosmology on a codimension-2 brane in a conical bulk

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Abstract

We study the cosmology of a 5-dimensional brane, which represents a regularization of a 4-dimensional codimension-2 brane, embedded in a conical bulk. The brane is assumed to be tensional, and to contain a curvature term. Cosmology is obtained by letting the brane move through the bulk, and implementing dynamical junction conditions. Our results shows that, with suitable choices of the parameters, the resulting cosmological dynamics mimics fairly well standard 4-dimensional cosmology.
1 Introduction

For more than ten years now, the appealing idea that our universe is a “brane” embedded in a higher dimensional bulk has been extensively explored, and widely accepted as a serious alternative to standard 4-dimensional cosmological models to solve the longstanding problems that prevents the complete understanding of our universe.

Codimension-1 branes, since the seminal paper by Randall and Sundrum [1], have been proved to be viable in allowing both a low-energy limit that mimics Newtonian gravity, and a compelling cosmological dynamics that can accommodate for inflation, match cosmological observations with respect to scalar and tensor perturbations, as well as opening novel possibilities for addressing both the initial singularity and the late time acceleration problem (for a review on these ad other aspects of Randall-Sundrum cosmology, see, for example, [2, 3]).

Codimension-2 branes are, in some sense, even more interesting, but, unfortunately, also much less “feasible”. In fact, the celebrated ADD mechanism [4] to address the hierarchy problem is viable only with two (or more) extra dimension. In addition, a proposal has been put forward [6] that the vacuum energy of a codimension-2 brane can be “off-loaded” in the bulk. Brane geometry would stay flat, and the energy will generate a deficit angle in the codimension-2 bulk (thus generating a conical singularity at the origin). This idea could explain the absence of a cosmological constant originated from vacuum fluctuations in quantum field theory. In this approach, the current small value observed for the cosmological constant should be generated by some different mechanism, which can be provided by a generalization of this proposal, the so-called Supersymmetric Large Extra Dimension model (SLED) [7]. The idea is that supersymmetry-breaking on the brane at high energy, which do not generate a vacuum energy because of the self-tuning property, induces a supersymmetry-breaking scale in the bulk at a much lower energy which depends on the size of the extra dimensions. The desired size to solve the hierarchy problem within the ADD framework provides an order of magnitude for the bulk SUSY breaking scale that generates, back on the brane, a cosmological constant with the correct order of magnitude. However, this proposal, even in the supersymmetric extension, has met severe criticism [8], because the self-tuning property rely on a tuning of the magnetic flux that stabilize the extra dimensions, which can not be kept stable under a phase transition on the brane.

Even worse, it is not possible to accommodate on a codimension-2 brane any kind of energy-momentum tensor different from the pure tension [9]. So, to study low energy limit, and possibly cosmology, one has to implement some kind of regularization of the 4-dimensional brane. Several regularization has been proposed [10–15], in which linear analysis show that weak gravity has the tensor structure of general relativity, but with the presence of some long-range modulus which should disappear from the spectrum at
the nonlinear level. The key point then, is how to describe a non-trivial dynamics of the regularized brane, and eventually how to derive from this description a viable 4-dimensional cosmology. This is, as one can imagine, a formidable task, because it would require to handle with the complete 6-dimensional dynamics. Some attempts have been made [16–20] using approximation to tackle the problem.

In this paper we study cosmology of a regularized brane in a 6-dimensional conical bulk. The dynamical equations are obtained by letting the regularized brane move through the bulk and implementing dynamical junction conditions [21, 22]. The model we present can be seen as a warped generalization of the conical codimension-2 brane of [11], but, since we are interested in cosmology on the brane, we are not in need of explicitly adding an axion field to recover a 4-dimensional tensional brane (a similar contribution can be obtained by a particular choice of the parameters of the energy-momentum tensor of the brane, see section 3). Furthermore, the present model differs from the standard “rugby-ball” regularization because the radial extra-dimension is non-compact, and the space-time is “capped” only in the inner side of the bulk. This could be a great advantage, because the dynamical behavior of the model should have some peculiar features that cannot be obtained by a low energy KK reduction of a 6-dimensional theory, but could as well leads to completely unacceptable behavior in the regime in which the wrapping of the brane becomes too large. We simply avoid this problem assuming that the brane movement is always close enough to the inner cap (so that even the “late intrinsic time” regime must be assumed to respect this limit); this condition can be achieved by tuning conveniently the deficit angle to a near-critical value (which is the same request to obtain a correct tensorial structure in the linear approximation [11]). With this precaution, and under some assumptions described below, we find that unlike other models studied with similar techniques, 4-dimensional induced cosmology can mimics fairly well the standard cosmological model, with an initial singularity and accelerated expansion at late time. The latter is driven by an effective cosmological constant given by a particular combination of the bulk cosmological constant and the brane tension.

The paper is organized as follows: In section 2 we present the static solution and the set up of the 5-dimensional brane. Then, in section 3 we derive the cosmological equation on the brane with the junction conditions. These equations are studied in section 4, with some assumptions that allows us to solve equations analytically and, where not possible, numerically. Finally, in section 5 we comments on the results obtained in the previous section and draw our conclusions.
2 The static solution

We consider a 6-dimensional space with cosmological constant, in which a 5-dimensional brane is embedded. We assume the brane to be tensional, and to have a curvature term, as well as matter, on it. The action that describes the model is

\[ S = \int d^6x \sqrt{-g} \left( \frac{M^4}{2} R - \Lambda_6 \right) - \int d^5\xi \sqrt{\gamma} \left( \frac{M_3^3}{2} (5) R + \mathcal{L}_{brane} \right). \]  

(2.1)

Capital latin indices \((A, B \ldots)\) run from 0 to 5 (so refer to bulk objects), while greek indices \((\mu, \nu, \ldots)\) run from 0 to 4 (so refer to brane objects). The brane intrinsic metric is \(\gamma_{\mu\nu}\). The equations of motion following from this action, far from the brane, are:

\[ G_{AB} = -\Lambda g_{AB}, \]  

(2.2)

with \(\Lambda = \Lambda_6/M^4\). We do not consider the singular contribution coming from the brane in (2.2), because we will take it into account later via the junction conditions. We seek for a metric which has a flat 4-dimensional submanifold, to be identified with our universe, and the two remaining dimensions having the geometry of a cone \([11]\). The metric is:

\[ ds_6^2 = R^2 ds_4^2 + \left[ \frac{\Lambda}{10} \left( R^2 - \frac{\mu^5}{R^5} \right) \right]^{-1} dR^2 + \beta^2 \ell^2 \left( \frac{R^2 - \mu^5}{R^5} \right) d\chi^2. \]  

(2.3)

with \(ds_4^2 = \eta_{\mu\nu} dx^\mu dx^\nu\). This metric can have a conical singularity in \(R = \mu = R_b\), in addition to the “true” singularity in \(R = 0\). Upon defining the new coordinate \(\rho\) by

\[ d\rho^2 = \left[ \frac{\Lambda \mu}{2} (R - \mu) \right]^{-1} dR^2 \]  

(2.4)

the extradimensional part of the metric, close to the horizon, can be approximated by:

\[ ds_2^2 \approx d\rho^2 + \frac{5}{8} \beta^2 \ell^2 \mu^2 \Lambda \rho^2 d\chi^2 \]  

(2.5)

thus we see that the metric is regular close to the horizon if

\[ \beta^2 \ell^2 = \left( \frac{5}{8} \mu^2 \Lambda \right)^{-1} \]  

(2.6)

The space-time of our model consists of two manifolds described by the metric (2.3), joined together at the radial position of the brane \(R = R_b\):

\[ ds_{6,\text{in}}^2 = z_i^2 ds_4^2 + \left[ \frac{\Lambda_i}{10} \left( z_i^2 - \frac{\mu_i^5}{z_i^3} \right) \right]^{-1} dR^2 + \beta_i^2 \ell_i^2 \left( z_i^2 - \frac{\mu_i^5}{z_i^3} \right) d\chi^2, \]  

\[ ds_{6,\text{out}}^2 = z_o^2 ds_4^2 + \left[ \frac{\Lambda_o}{10} \left( z_o^2 - \frac{\mu_o^5}{z_o^3} \right) \right]^{-1} dR^2 + \beta_o^2 \ell_o^2 \left( z_o^2 - \frac{\mu_o^5}{z_o^3} \right) d\chi^2. \]  

(2.7)
where \( z = R/R_0 + C \). Continuity along the brane directions requires \( z_i = z_0 \), while continuity along the compact extradimensional direction \( \chi \) gives

\[
\mu_\infty = \mu_0, \quad \beta_\infty^2 \ell_i^2 = \beta_0^2 \ell_0^2 = \beta^2 \ell^2.
\]

(2.8)

Then we can, without any loss of generality, set the integration constants \( C = \mu \), so that the space-time “begins” at \( R = 0 \), and \( R_0 = 1 \) (a different choice would just end up in a rescaling of the cosmological constant). So the two parts of the space-time differ only because of the different values of the cosmological constants. Having in mind to study mirage cosmology on the brane by allowing it to move through the radial direction, we can impose the relation (2.6) only for the in part of the space-time, so that it ends smoothly at the position of the horizon. On the other side, the out part of the space-time is allowed to have a deficit angle \( 1 - b \), so that the codimension metric written using the variable \( \rho \) of (2.4) reads:

\[
d_{s^2}_{in, out} = d\rho^2 + (1 - b)^2 \rho^2 d\chi^2.
\]

(2.9)

This fixes the relation between the cosmological constants of the in and out parts of the space-time to be:

\[
\Lambda_i = \frac{\Lambda_0}{(1 - b)^2}
\]

(2.10)

### 3 Cosmological equations on the brane

The presence of matter on the brane, and the movement of the brane itself across the extra-dimension would in principle modify the bulk geometry. Solving exactly this problem would be extremely complicated, so we assume that cosmology is induced on the brane by implementing time-dependent Israel junction condition, while the bulk is not modified by the brane movement [18, 19]. Let us assume then that the brane position is \( R_b \equiv a(\tau) \). The brane embedding is thus described by the relation between the bulk and the brane coordinates \( \xi^a = (\tau, x, \chi) \):

\[
t = t(\tau), \quad R = a(\tau)
\]

(3.1)

the other relations being just identities. So the tangent vectors are trivial, except the timelike one \( u^A_T \), which reads, in the coordinate system we are using:

\[
u^A_T = (\dot{t}, 0, \dot{a}, 0)
\]

(3.2)

where dot indicates derivative with respect to \( \tau \). The normal vector \( n^A \) can be expressed as

\[
n^A = (n^t, 0, n^R, 0).
\]

(3.3)

By using the orthogonality conditions \( g_{AB}u^A u^B = 1 \), \( g_{AB} n^A n^B = -1 \), \( g_{AB} u^A n^B = 0 \) we can express all the unknown functions in terms of the scale factor \( a(\tau) \). Of course, the
orthogonality conditions are different on the two sides of the brane, because of the difference between the \textit{in} and \textit{out} metric. We have
\[
\dot{t} = \frac{\sqrt{\dot{a}^2 + z^2 f_i(z)}}{z \sqrt{f_i(z)}}, \quad n^R = -\frac{\dot{a}^2 + z^2 f_i(z)}{z^2 \sqrt{f_i(z)}}, \quad n^t = -\frac{\dot{a}}{z^2 \sqrt{f_i(z)}}
\]
\[
\dot{t} = \frac{\sqrt{\dot{a}^2 + z^2 f_o(z)}}{z \sqrt{f_o(z)}}, \quad n^R = \sqrt{\dot{a}^2 + z^2 f_o(z)}, \quad n^t = \frac{\dot{a}}{z^2 \sqrt{f_o(z)}}
\]
with
\[
f_{i/o}(z) = \frac{\Lambda_{i/o}}{10} \left( 1 - \mu^5 \frac{z^5}{\Lambda_{i/o}} \right).
\]
the difference is due to the different value of the cosmological constants and to the different orientation of the normal unit vector in the two branches. With the normal unit vector in hands, we can calculate the induced metric \( h_{AB} = g_{AB} - n_A n_B \), and consequently the intrinsic line element on the brane:
\[
ds_5^2 = -d\tau^2 + z^2(\tau) d\chi^2 + \beta^2 \ell^2 \left( z^2(\tau) - \frac{\mu^5}{z^3(\tau)} \right) d\chi^2
\]
which is the same on both side of the by means of (2.8), as expected. Then we can evaluate the extrinsic curvature \( K_{AB} = h^C_A \nabla_C n_B \), noting that derivatives w.r.t. the bulk variables are expressed in terms of derivatives w.r.t. brane variables via the chain relation [18]
\[
\partial_A = g_{AB} \partial_\mu x^B \gamma^{\mu\nu} \partial_\nu.
\]
We find
\[
K_{tt} = \pm z \sqrt{\dot{a}^2 + z^2 f_{i/o}} \left[ \frac{\dot{a}}{z f_{i/o}} - \frac{\dot{a} f_{i/o}'}{2z^2 f_{i/o}^2} + 1 \right]
\]
\[
K_{ij} = \mp z \sqrt{\dot{a}^2 + z^2 f_{i/o}} \delta_{ij}
\]
\[
K_{RR} = \frac{\dot{a}^2}{z^4 f_{i/o} (\dot{a}^2 + f_{i/o})} K_{tt}
\]
\[
K_{Rt} = -\frac{\dot{a}}{z^2 \sqrt{f_{i/o} (\dot{a}^2 + f_{i/o})}} K_{tt}
\]
\[
K_{\chi\chi} = \mp \frac{8 \left( 2z f_{i/o} + z^2 f_{i/o}' \right)}{\mu^2 \Lambda_{i/o}^2} \sqrt{\dot{a}^2 + z^2 f_{i/o}}
\]
where the upper sign refers to the \textit{in} side of the brane, and the prime indicates derivative w.r.t. \( z \). Equations of motion on the brane are obtained by equating the discontinuity of the projected extrinsic curvature across the brane with the energy-momentum contribution on the brane (which, in our case, includes the curvature term):
\[
[K_{\mu \nu}] - [K] \gamma_{\mu \nu} = \frac{1}{M^4} (T_{\mu \nu} - M^3 G_{\mu \nu})
\]
where \([x]\) stands for \(x_o - x_i\), \(K_{\mu\nu} = u^A_\mu u^B_\nu K_{AB}\) and \(K\) is its trace, \(G_{\mu\nu}\) is the intrinsic Einstein tensor as calculated from the intrinsic metric \(\gamma_{\mu\nu}\). The brane energy-momentum tensor needs to be specified. We assume, in addition to the tension contribution, a “perfect fluid-like” form, which is compatible with the symmetry of the space-time:

\[
T_{\mu\nu} = -\lambda \eta_{\mu\nu} + (p.f.) T_{\mu\nu} \quad (p.f.) T_{\mu\nu} = \text{diag} (-\rho, p, p, p, P).
\] (3.10)

Let us stress that with the particular choice of the form of the e.m. tensor (3.10), the cosmological constant on the brane is taken into account separately, so that we can impose \(w > -1\). Of course this is restrictive, since the symmetry of the space-time allows an energy momentum tensor with \(\rho = -p = \lambda\) and \(P = -\lambda'\), thus having a sort of “ring” cosmological constant that can differs from the 4-dimensional one. We will comment more on this in the next section.

Finally, substituting (3.9) in (3.10), after some algebraic manipulation, the equations of motion read (from now on, we drop the subscript \(i/o\), assuming that all quantities are intended to be in the outside, and use (2.10) to express appropriately the correspondent in objects):

\[
\sqrt{H^2 + f} + \sqrt{H^2 + \sigma^2 f} = \frac{2}{M^4} \left( 1 - \left( \frac{\mu}{z} \right)^5 \right) (\rho + \lambda) - 3r_c \frac{4 + (\frac{\mu}{z})^5}{8 - 3 (\frac{\mu}{z})^5} H^2
\] (3.11)

\[
\dot{\rho} + H \left( \frac{4 + 3w}{1 - (\frac{\mu}{z})^5} \right) \rho - \frac{1 + \frac{3}{2} (\frac{\mu}{z})^5}{1 - (\frac{\mu}{z})^5} P = 0
\] (3.12)

\[-\dot{H} \left[ (H^2 + f)^{-\frac{1}{2}} + (H^2 + \sigma^2 f)^{-\frac{1}{2}} \right] - \]

\[-H^2 \left[ \left( 5 - \frac{1 + \frac{3}{2} (\frac{\mu}{z})^5}{1 - (\frac{\mu}{z})^5} \right) \left( (H^2 + f)^{-\frac{1}{2}} + (H^2 + \sigma^2 f)^{-\frac{1}{2}} \right) + 3 \frac{\sqrt{H^2 + \sigma^2 f}}{H^2 + f} \right] - \]

\[-f \left[ 4 (H^2 + f)^{-\frac{1}{2}} + \sigma^2 (H^2 + \sigma^2 f)^{-\frac{1}{2}} + 3 \frac{\sqrt{H^2 + \sigma^2 f}}{H^2 + f} \right] = \]

\[= \frac{1}{M^4} (P - \lambda) + 3r_c (\dot{H} + 4H^2) \]

with \(H = \dot{z}/z\), \(\sigma = (1 - b)^{-1}\), \(r_c = M_5^3/M^4\).

Eq. (3.11) represents the modified Friedman equation that controls the cosmological evolution of the 5-dimensional brane. Notice, however, that \(H\) is the actual 4-dimensional Hubble parameter, since it is obtained from the scale factor that controls the dynamics of the 4-dimensional slice of the brane. Since the brane is wrapped around the azimuthal direction, in order to obtain “sensible” 4-dimensional sources, we must integrate the energy density \(\rho\) over the fifth dimension [18, 19]:

\[
(4)\rho = \int d\chi \sqrt{\gamma_{XX}} \rho = 2\pi \sqrt{\gamma_{XX}} \rho
\] (3.14)
since we assume that neither the metric nor the energy density depend on the azimuthal coordinate. The modified Friedman eq. (3.11) could be cast in a more “conventional” form \( H^2 = f^{(4)\rho} \) by solving it for \( H^2 \) and inserting \((4)\rho\), but its form would be overcomplicated, and we will show that, if one seeks for solutions only in particular regimes, “effective” Friedman equations will regain their simplicity.

Eq. (3.12) is the conservation equation for the energy-momentum tensor (3.10), in which we have imposed an equation of state that relates only energy density and pressure, \( p = w\rho \). The symmetry of the 5-dimensional space-time leads us to include an extra-dimensional component which is undetectable (unless one includes gauge coupling between ordinary matter and extradimensional one), but which modifies the dynamic of the 4-dimensional universe. The presence of such a “dark” term is quite common in braneworld models [23,24] and in our model plays the crucial role of slowing down the expansion of the energy density (in some particular regimes) thus compensating the leakage due to higher dimensionality.

Eq. (3.13) is the fifth component of the 5-dimensional junction conditions, and (by means of the Bianchi identities) it is related to the fifth component of the conservation equation of the energy-momentum tensor. We assume that \( P \) satisfies this equation, which is then a constraint equation for the extradimensional pressure. It is possible to imagine a more complicated scenario in which the extradimensional pressure is not assumed to satisfy a constraint equation, but (maybe more physically) to be related to the energy density by a general equation of state. In this case the system would look like being not compatible, since there would be more equations than degrees of freedom. Actually, a more general extradimensional equation of state would induce a time-dependent tilt in the azimuthal direction, which results in a deformation of the ring shape of the brane. Studying a system like this would be very complicated, and will probably not give an acceptable 4D cosmology because it would be very hard to get a FRW-like 4-dimensional slicing of the 5-brane. This is the reason we assume the brane stays rigid during its movement trough the cone.

The system of eqs. (3.11-3.13) looks very complicated to handle. Nevertheless, it is possible, as already anticipated, to make some assumptions that allows us to simplify it, so to get an analytic solutions. This will be the aim of the next section.

4 Cosmology on the brane

It is more convenient to track back the cosmological evolution of the brane from late to early times. Let’s then first consider what happens at late times. We can guess (assuming that the universe is expanding) that \( a(\tau) \gg \mu \) so that \( f \) will become just proportional to the cosmological constant. In addition, we can assume that the energy density is negligible
with respect to the cosmological constant itself\(^1\). At this point, the Hubble parameter \(H\) will be constant as well. The cosmological equations become:

\[
\sqrt{H_0^2 + \frac{\Lambda}{10}} + \sqrt{H_0^2 + \frac{\sigma^2 \Lambda}{10}} = \frac{\lambda}{4M^4} - \frac{3}{2} r_c H_0^2 \quad (4.1)
\]

\[
-H_0^2 \left[ 4 \left( H_0^2 + \frac{\Lambda}{10} \right)^{-\frac{1}{2}} + 4 \left( H_0^2 + \frac{\sigma^2 \Lambda}{10} \right)^{-\frac{1}{2}} + 3 \sqrt{\frac{H_0^2 + \sigma^2 \Lambda}{H_0^2 + \frac{\Lambda}{10}}} \right] =
\]

\[
-\frac{\Lambda}{10} \left[ 4 \left( H_0^2 + \frac{\Lambda}{10} \right)^{-\frac{1}{2}} + \sigma^2 \left( H_0^2 + \frac{\sigma^2 \Lambda}{10} \right)^{-\frac{1}{2}} + 3 \sqrt{\frac{H_0^2 + \sigma^2 \Lambda}{H_0^2 + \frac{\Lambda}{10}}} \right] \]

\[
= -\frac{\lambda}{M^4} + 12 r_c H_0^2 \quad (4.2)
\]

with the conservation equation trivially satisfied. These are a set of two algebraic equations in the unknown \(H_0\), \(\Lambda\) and \(\lambda\), whose solution\(^2\) gives the curvature in terms of the bulk cosmological constant and the brane tension, and an unavoidable fine-tuning between these last two parameters. A similar situation occurs if we set \(w = -1\), so that the energy density just results in a further contribution to the brane tension. From eq. (3.12) we see that the extradimensional pressure \(P\) is also constant, and can be expressed in terms of the energy density. Then we eventually get eqs. (4.1,4.2) again, with a rescaled brane tension.

On the other side, we can assume that the energy density dominates over cosmological constant and tension, so that we can ignore the latter two. In addition, since the Hubble parameter has to be the same order of magnitude as \(\rho\), we can also ignore cosmological constant when added to \(H^2\). In this way eqs. (3.11),(3.12),(3.13) greatly simplify. We then substitute the extradimensional pressure \(P\) as evaluated from (3.13) in (3.12), so to obtained the modified Friedman equations (from now on we substitute \(\rho/M^4 \rightarrow \rho\), \(P/M^4 \rightarrow P\) and \(\tau \rightarrow t\)):

\[
2H = \frac{\rho}{4} - \frac{3}{2} r_c H^2 \quad (4.3)
\]

\[
\dot{\rho} + H \left[ (4 + 3w) \rho + 3r_c \left( \dot{H} + 4H^2 \right) \right] + 2\dot{H} + 11H^2 \quad (4.4)
\]

Let us stress that the energy density we are considering in these equation is a 5-dimensional energy density. The observable 4-dimensional energy density is obtained by integrating over the compact direction; again, in the limit we are considering \(a(\tau) \gg \mu\) it is easy to see that:

\[
(4) \rho \propto \frac{a}{\Lambda} \rho \quad (4.5)
\]

\(^{1}\)Of course, current observations suggest that we are actually living in a very special time in the evolution of our universe in which matter energy density and cosmological constant are of the same order of magnitude.

We will not pursue any suggestion about the resolution of this “coincidence” problem here.

\(^{2}\)Again, since the system can be cast into a set of two fourth-order equations in \(H_0^2\), an analytical solution could be found, but its (very complicated) form is unimportant here.
Equations (4.3),(4.4) can be further approximated by noting that the Hubble radius can be either much greater or much smaller than the crossover scale \( r_c \), so we will consider the two cases separately:

- **Sub-crossing regime:**
  In this case we have \( r_c H \gg 1 \), so that we can discard terms not proportional to \( r_c \). The equations (4.3),(4.4) can be exactly solved to give:

  \[
  a(t) = \left( \frac{t}{t_0} \right)^{\frac{5}{6(w+2)}} \quad (4.6)
  \]

  \[
  \rho(t) = \rho_0 (a(t))^{-\frac{6}{5}(w+2)} \quad (4.7)
  \]

  Then we can use (3.14) to express the behavior of \( \rho \) with respect to the scale factor, specializing the results for the case of interest of radiation \( (w = 1/3) \) and matter \( (w = 0) \). We have:

  \[
  (4) \rho_r = (4) \rho_{r,0} a^{-\frac{2}{5}} 
  \]

  \[
  (4) \rho_m = (4) \rho_{m,0} a^{-\frac{7}{5}} \quad (4.8)
  \]

- **Super-crossing regime:**
  In this case we have \( r_c H \ll 1 \) so we can drop terms proportional to \( r_c \) in (4.3),(4.4). The solutions are:

  \[
  a(t) = \left( \frac{t}{t_0} \right)^{\frac{10}{24w+43}} \quad (4.10)
  \]

  \[
  \rho(t) = \rho_0 (a(t))^{-\frac{24w+43}{10}} \quad (4.11)
  \]

  which become, for the 4-dimensional radiation and matter energy density:

  \[
  (4) \rho_r = (4) \rho_{r,0} a^{-\frac{41}{10}} 
  \]

  \[
  (4) \rho_m = (4) \rho_{m,0} a^{-\frac{33}{10}} \quad (4.12)
  \]

To analyze the behavior of the brane at early times, we need to revert the approximation we made at the beginning of this section, and assume \( a(t) \ll \mu \). After some algebra, substituting \( P \) from (3.13) in (3.12) as before, the modified Friedman equations can be approximated as:

\[
2H = 2\frac{a}{\mu} \rho - 3r_c H^2 \quad (4.14)
\]

\[
\dot{\rho} + \frac{\mu H}{2a} \left[ \rho + \frac{3}{2} r_c (\dot{H} + 4H^2) - \frac{\mu H}{a} \right] = 0 \quad (4.15)
\]
These equations cannot be solved analytically, so we must turn to numeric. The behavior of the scale factor $a(t)$, the Hubble parameter $H(t)$ and the 4-dimensional energy density $(4)\rho(t)$ are shown in Fig. 1. Notice that, with the approximation $a(t) \ll \mu$ we are using, the 4-dimensional energy density is related with the 5-dimensional one by the relation:

$$(4)\rho \propto \sqrt{\frac{\mu a}{\Lambda}} \rho$$

(4.16)

Figure 1: Plots of the scale factor $a$ (1), the Hubble parameter $H$ (2) and the 4-dimensional energy density $(4)\rho$ (3) as obtained from numerical solutions of eqs. (4.14),(4.15).

5 Comments and conclusions

In this paper we have studied the cosmological properties of a 5-dimensional brane, described by the action (2.1), assumed to be a regularization of a codimension-2 braneworld model. Cosmological evolution is governed by the movement of the brane through the extra-dimension, while 4-dimensional cosmology is obtained by integrating over the fifth compact dimension. The cosmological equations can be specialized to describe different regimes.

Close to the inner cap, evolution of the brane seem to emerge from an initial singularity, much alike what happens in standard cosmology. It is known that extradimensional contribution to the energy-momentum tensor result in an effective negative energy density [25], and codimension-1 models have been proposed [26,27] in which this contribution dominates at early times, thus providing a non-singular brane cosmology. Eqs. (4.14), (4.15) shows that this negative contribution is actually present in our model, but is exactly cancelled by the modified dynamics of the energy density, so that even if the static space-time is non-singular in the origin, still cosmological dynamics is plagued by an initial singularity (the 4-dimensional curvature is roughly $(4)R \propto H^2/a^{3/2}$).
Next, the universe is supposed to enter in an energy-dominated phase. In the present model we assume that different sources dominate at different eras, so single contributions can be considered independently. The presence of a curvature term on the brane indicates that a crossover scale $r_c$ can be identified, which is given by the ratio between 5-dimensional and 6-dimensional gravitational coupling constants, and which should represent the scale at which extradimensional physics become effective. In the DGP scenario [28], these extradimensional effects can provide, at super-crossing scale, for a self-accelerated expansion without the necessity of a cosmological constant [29]. Here the dynamics is quite different. We find that, if energy density dominates at a scale smaller than the crossover scale, cosmology on the brane differs evidently from standard 4-dimensional cosmology. If, on the contrary, radiation and matter eras begin on a large enough scale (or, equivalently, if the crossover scale is small enough), we find (4.12), (4.12) that energy density scales with respect to scale factor with almost the same power-law as standard cosmology, differently for what happens in other examples of induced cosmology on a codimension-2 brane presented in the literature.

Eventually, energy density will drop below the order of magnitude of the cosmological constant and the tension (notice that the model under discussion depends crucially on the presence of a bulk cosmological constant to be dynamically meaningful); at that point, the brane undergoes a phase of de Sitter expansion, with an effective cosmological constant given by the solution of eqs. (4.1), (4.2). These equations impose also an unavoidable fine tuning between bulk cosmological constant and brane tension. It is possible that such a dependence could arise in the process of nucleation of the brane, which is of course fully nonlinear and very difficult to describe. A hint towards this assumption comes from the observation that the effective brane tension can be rescaled by tuning the extradimensional pressure. Still, even at the level of our analysis, it seems that the self-tuning property is lost once we go beyond the static solutions, since the effective cosmological constant on the brane has a non-trivial dependence on the tension.

Summing up, our investigation seems to suggest that a braneworld model embedded in a conical de Sitter bulk could have a viable cosmology, i.e. the evolution of the brane during radiation and matter dominated phase is similar to what happens in standard cosmology, but, at the level of our analysis, the self-tuning property is lost; neither extradimensional contribution are enough to address the initial singularity problem.

In order to address these drawbacks, it would be important to develop a model in which brane movement could modify bulk geometry, so that the deficit angle could become in some sense “dynamical”. This is a formidable problem, as said elsewhere, because it would require a solution of the full 6-dimensional problem, which means tackling a set of non-linear partial differential equations. We are working in this topic, though preliminary
results are not encouraging. A question remains on how reliable is a “mirage cosmology” approximation when applied to a codimension-2 cosmological brane (though regularized). We assume that the probe brane approximation should work as long as the curvature does not blow up. In this spirit, the undesirable singularity should not be taken too seriously, also because in a realistic model the brane should emerge from a nucleation process in the bulk. However, the lack of self-tuning should be a robust prediction, if some other effect does not change the picture drastically (such as supersymmetry).

Another important development would be studying of perturbations around the background we have presented. Perturbations around the static solution give, as stated in section 1, the tensorial structure of 4-dimensional gravity, which has an unwanted scalar degree of freedom that propagate on the brane. It is conjectured that this degree of freedom would be reabsorbed in a nonlinear realization of our codimension-2 model. Perturbations would allow us to identify the exact form of the tree-level graviton exchange between two probe masses in 4-dimensional gravity, which, since 4-dimensional physics is obtained by integrating a time-varying azimuthal dimension, would imply a time-varying Planck mass and observational signatures that would change in different cosmological epochs as well. This could result in a tight constraint on the parameters of the model (and of other regularized codimension-2 brane models) from space-based experiments. Unfortunately, it is not clear how to obtain reliable perturbation equations starting from mirage cosmology; so in order to perform these very interesting investigations, one is again led to the necessity of a full 6-dimensional nonlinear study of brane cosmology.

A simpler, and yet interesting development would be to study the supersymmetric extension of the present model, which could help to improve the fine-tuning problem between the brane tension and the bulk cosmological constant with a suitable dilaton potential. All these aspects will be addressed in forthcoming works.

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