Efficient sharing of a continuous-variable quantum secret

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We propose an efficient scheme for sharing a continuous variable quantum secret using passive optical interferometry and squeezers: this efficiency is achieved by showing that a maximum of two squeezers is required to replicate the secret state, and we obtain the cheapest configuration in terms of total squeezing cost. Squeezing is a cost for the dealer of the secret as well as for the receivers, and we quantify limitations to the fidelity of the replicated secret state in terms of the squeezing employed by the dealer.

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I. INTRODUCTION

Secret sharing (SS) is an important cryptosystem protocol for dealing secret information to a set of players, not all of whom can be trusted. The encoded secret can only be replicated (or, equivalently, reconstructed) if certain subsets of players collaborate, and these subsets are referred to as the access structure. The remaining subsets comprise the adversary structure, and the sets are referred to as the access structure; other SS schemes can be constructed via a secret structure; other SS schemes can be constructed via an adversary structure. The remaining subsets comprise the adversary structure, and the protocol denies the adversary structure any information about the secret. The underpinning scheme for arbitrary SS is (k,n)-threshold SS, which involves n players, and any subset of k players constitutes a valid set in the access structure; other SS schemes can be constructed via threshold SS, for example by distributing unequal numbers of shares to players. Although quantum secret sharing (QSS) was first introduced as a method to transmit classical information in a hostile environment with quantum-enhanced security, QSS was subsequently established as a quantum analogue to Shamir’s secret sharing described above and we use the term QSS to refer to the latter approach. QSS provides a valuable protocol in quantum communication but also is important as an error correction scheme.

Here we are concerned with continuous-variable (CV) QSS. Quantum information protocols and tasks are now studied both as discrete-variable, qubit-based (or qudit-based) protocols and tasks, and as CV realizations. CV quantum information protocols are generally realized in optical systems and exploit advanced optical tools, such as the generation of squeezed light and ability to count single photons, as well as the low rate of decoherence for optical systems. These recent demonstrations of CV unconditional quantum teleportation are an excellent example of the capabilities of CV quantum information processes in optical systems. Moreover, the technology for this CV quantum teleportation is not very different from the techniques required for CV QSS.

The original proposal for CV QSS established a general method for CV QSS, and for (k,n)-threshold schemes in particular, using interferometry involving passive optical elements (mirrors, beam splitters and phase shifters), active elements (squeezers) and homodyne detectors. A (2,3)-threshold scheme was proposed involving a single squeezer, thereby suggesting an experiment that is within the reach of current technology. The original proposal of how to perform the general (k,n) scheme was complicated, though, by the need for an increasing number of squeezers in the interferometer. A practical realization of threshold-QSS would need to minimize the number of optical squeezers as the number of players increases.

Here we establish that, for any number of players n and any threshold level k for the number of collaborators to be in the access structure, the total number of squeezers needed by the collaborating players does not exceed two. This remarkable result informs us that at most two squeezers are required for an arbitrary number of players n. In particular, to replicate the secret state, the collaborating players require access to an interferometer with k channels but only two active components (i.e., squeezers). This analysis also allows us to determine the total amount of squeezing required in a two-squeezer threshold-QSS protocol: the analysis is important because the degree of squeezing required for the protocol can be regarded as an effective cost for the procedure.

The second major concern of this paper is the extent to which it is possible to achieve the goals of the CV QSS protocol with finite physical resources. For the protocol to work perfectly, the dealer needs access to ancillary states prepared with infinite squeezing; as this is not physically possible, we analyze the effects of finite squeezing, which imposes limitations on the fidelity of the replicated secret state.

The paper is organized as follows: in Section we summarize the CV QSS protocol for threshold schemes. In Section we describe efficient replication of the secret state, which requires the minimal number of squeezing elements and minimal overall squeezing. The total amount of squeezing is discussed in Section and we conclude in Section.
II. THRESHOLD QSS WITH FINITE RESOURCES

Fig. 1. The optical \((k = 4, n = 7)\) QSS threshold scheme: the dealer encodes the secret via an active interferometer (AI) by mixing it with \(n - 1\) ancillary states, transmits the resulting \(n\) shares to the players, and any \(k\) players employ a second interferometer to replicate the secret state. The interferometers are active, meaning that they employ both passive optical devices and energy-consuming squeezers.

The optical \((k, n)\) threshold scheme is sketched in Fig. 1. A dealer holds a pure secret state \(|\psi\rangle\) realized in a single mode of the electromagnetic field and encodes the secret as an \(n\)-mode entangled state \(|\Psi\rangle\) by mixing with \(n - 1\) ancillary modes in an \(n\)-channel active interferometer, where the term active refers to one- or two-mode squeezers. The dealer then sends one output, or “share,” to each of the players, and at least \(k\) players must combine their shares in an active interferometer to replicate the secret state. However, the no-cloning theorem requires that no threshold scheme exists for \(n \geq 2k\). Also any threshold scheme with \(n < 2k - 1\) can be obtained from the \((k, 2k - 1)\) scheme by discarding \(2k - 1 - n\) shares. Therefore, we concentrate on the \((k, 2k - 1)\) threshold scheme.

A. Entanglement of the secret state

The secret is a state \(|\psi\rangle \in \mathbb{H}^{(1)} \sim \mathcal{L}^2(\mathbb{R})\) with wave function \(\psi(x) = \langle x|\psi\rangle\). Let \(\mathbb{H}^{(n)}\) be the tensor product of \(n = 2k - 1\) copies of \(\mathbb{H}^{(1)}\), one copy of which is owned by each player. The idea is to entangle the information among states of \(\mathbb{H}^{(n)}\) such that any \(k\) players can cooperate to untangle the information but any smaller number is unable to do so.

The Hilbert space \(\mathbb{H}^{(n)}\) is the space \(\mathcal{L}^2(\mathbb{R}^n)\) of square integrable wave functions on \(\mathbb{R}^n\). Thus, if \(\mathbb{F}^n\) denotes the linear space of coordinate functions for \(\mathbb{R}^n\), then choosing a system of Euclidean coordinates \((x_1, \ldots, x_n)\) for any vector \(x \in \mathbb{R}^n\) is equivalent to picking an orthonormal basis \((f_1, \ldots, f_n)\) for \(\mathbb{F}^n\) such that

\[
 f_i(x) = x_i. \tag{1}
\]

We denote the inner product of these coordinate functions by \(f_i \cdot f_j = \delta_{ij}\).

Suppose the dealer starts with an unentangled tensor

\[
 |\Psi\rangle = |\psi\rangle \otimes |\varphi_a\rangle \otimes \cdots \otimes |\varphi_a\rangle, \tag{2}
\]

of the secret state \(|\psi\rangle\), with \(k - 1\) copies of a state \(|\varphi_a\rangle\) and \(k - 1\) copies of a state \(|\varphi_1/\alpha\rangle\), where

\[
 \varphi_a(x) = \langle x|\varphi_a\rangle = (\pi a^2)^{-1/4} e^{-x^2/2a^2}. \tag{3}
\]

Write this state

\[
 |\Psi\rangle = \int dx^n \Psi(x) |x_1\rangle \otimes \cdots \otimes |x_n\rangle, \tag{4}
\]

where

\[
 \Psi(x) = \psi(x_1) \prod_{i=2}^k \varphi_a(x_i) \prod_{i=k+1}^n \varphi_1/\alpha(x_i). \tag{5}
\]

The dealer then entangles the secret state by a linear canonical point transformation

\[
 f_i \to g_i = \sum_j g_{ij} f_j, \tag{6}
\]

in which the orthogonal (Euclidean) coordinate functions \(\{f_i\}\) are replaced by a general linear system \(\{g_i\}\) for which \(g_i(x) = \sum_j g_{ij} f_j(x) = \sum_j g_{ij} x_j\). The corresponding unitary transformation of \(\mathbb{H}^{(n)}\) then maps the state \(|\Psi\rangle\) to

\[
 |\Psi_g\rangle = |\det g|^{1/2} \int dx^n \Psi(x) |g_1(x)\rangle \otimes \cdots \otimes |g_n(x)\rangle. \tag{7}
\]

For it to be possible for any subset of \(k\) players to reconstruct the secret state, certain conditions must be respected. These conditions become apparent when we consider the replication algorithm.

B. The replication algorithm

In replicating the secret state, it is convenient to identify three subspaces of coordinates; i.e., express \(\mathbb{F}^n\) as a direct sum of three mutually orthogonal subspaces

\[
 \mathbb{F}^n = X \oplus Y \oplus Z, \tag{8}
\]
where $\mathbb{X}$ is the one-dimensional space spanned by $f_1$, and $\mathbb{Y}$ and $\mathbb{Z}$ are the $(k-1)$-dimensional spaces spanned, respectively, by $\{f_2, \ldots, f_k\}$ and $\{f_{k+1}, \ldots, f_n\}$. Thus, we relabel the $\{x_i\}$ coordinates as $(x, y_i, z_i)$ coordinates with

$$ x = x_1, \quad y_i = x_{i-1}, \quad z_i = x_{k+i}, \quad i = 2, \ldots, k. \quad (9) $$

The wave function $\Psi$ is then

$$ \Psi(x) = \psi(x) \prod_{i=1}^{k-1} \varphi_\alpha(y_i)\varphi_{1/\alpha}(z_i). \quad (10) $$

It will be understood in the following that all $n$ players know any player transformation in which $f_i \rightarrow g_i$. We then impose the requirement that this transformation is such that the components of any $k$ basis vectors of the set $\{g_i\}$ that lie in the subspace $\mathbb{X} \oplus \mathbb{Y} \subset \mathbb{F}^m$ are linearly independent and hence span this subspace.

Without loss of generality, we may suppose that the first $k$ players form the collaborating set. These players are able to make any transformation of the states in the subset of Hilbert spaces accessible to them. However, we will restrict the transformations they can make to those corresponding to general linear coordinate transformations, as defined above. Let us suppose they make the transformation

$$ g_i \rightarrow \xi_i = \sum_j \xi_{ij} f_j \quad (11) $$

with the understanding that $\xi_i = g_i$ for all $i > k$.

The orthogonal decomposition of $\mathbb{F}^n$ given by Eq. (5) now defines a corresponding decomposition of every $\xi_i$ vector as a sum of three mutually orthogonal vectors

$$ \xi_i = \alpha_i + \beta_i + \gamma_i. \quad (12) $$

Equivalently, we can write

$$ \xi_i(x) = \alpha_i x + \sum_j \beta_{ij} y_j + \sum_j \gamma_{ij} z_j. \quad (13) $$

We now claim that a transformation $g_i \rightarrow \xi_i$ which is such that

$$ \alpha_1 = 1, \quad \beta_1 = 0, \quad \alpha_{i+1} = \alpha_{k+i}, \quad \beta_{i+1} = \beta_{k+i}, \quad i = 1, \ldots, k-1. \quad (14) $$

replicates the secret for sufficiently large values of the parameter $a$. We demonstrate this result explicitly for the simple case in which $k = 2$ and $n = 3$.

For the $k = 2$, $n = 3$ case $(\xi_1, \xi_2, \xi_3)$ will have expansions of the form

$$ \xi_1(x) = x + \gamma_1 z, \quad (15) $$

$$ \xi_2(x) = \alpha x + \beta y + \gamma_2 z, \quad (16) $$

$$ \xi_3(x) = \alpha x + \beta y + \gamma_3 z, \quad (17) $$

and $|\Psi_\xi\rangle$ will be given by

\[
|\Psi_\xi\rangle = \frac{|\beta(\gamma_2 - \gamma_3)|^{1/2}}{\pi^{1/2}} \int \psi(x) \exp \left[ - \frac{1}{2a^2} y^2 - \frac{a^2}{2} z^2 \right] \times |x + \gamma_1 z\rangle \otimes |ax + \beta y + \gamma_2 z\rangle \otimes |ax + \beta y + \gamma_3 z\rangle \times dx \, dy \, dz. \quad (18)
\]

By a change of the variable $x$ to $x - \gamma_1 z$, we then have

\[
|\Psi_\xi\rangle = \frac{|\beta(\gamma_2 - \gamma_3)|^{1/2}}{\pi^{1/2}} \int \psi(x - \gamma_1 z) \times \exp \left[ - \frac{1}{2a^2} y^2 - \frac{a^2}{2} z^2 \right] \times |x\rangle \otimes |ax + \beta y + \gamma_2 z\rangle \otimes |ax + \beta y + \gamma_3 z\rangle \times dx \, dy \, dz. \quad (19)
\]

Now observe that if $a$ is sufficiently large that $\psi(x - \gamma_1 z) \approx \psi(x)$ for all values of $z$ for which $\exp[-a^2 z^2/2]$ is non-negligible, then

\[
\psi(x - \gamma_1 z) \approx \psi(x) \exp \left[ - \frac{a^2}{2} z^2 \right]. \quad (20)
\]

Moreover, this approximation becomes precise to any desired level of accuracy for sufficiently large values of $a$. By a second change of variables,

$$ x \rightarrow x, \quad \beta y \rightarrow \beta y - \alpha x, \quad (21) $$

we also have

\[
|\Psi_\xi\rangle = \frac{|\beta(\gamma_2 - \gamma_3)|^{1/2}}{\pi^{1/2}} \int \psi(x) \times \exp \left[ - \frac{1}{2a^2} (y - \frac{\alpha}{\beta} x)^2 - \frac{a^2}{2} z^2 \right] \times |x\rangle \otimes |\beta y + \gamma_2 z\rangle \otimes |\beta y + \gamma_3 z\rangle \times dx \, dy \, dz \quad (22)
\]

Now for $a$ sufficiently large that $\exp \left[ - \frac{1}{2a^2} (y - \frac{\alpha}{\beta} x)^2 \right] \approx \exp \left[ - \frac{1}{2a^2} y^2 \right]$ for all values of $x$ for which $\psi(x)$ is non-negligible, we have

\[
|\Psi_\xi\rangle \approx \frac{|\beta(\gamma_2 - \gamma_3)|^{1/2}}{\pi^{1/2}} \int \psi(x) \exp \left[ - \frac{1}{2a^2} y^2 - \frac{a^2}{2} z^2 \right] \times |x\rangle \otimes |\beta y + \gamma_2 z\rangle \otimes |\beta y + \gamma_3 z\rangle \times dx \, dy \, dz = |\psi\rangle \otimes |\Phi\rangle, \quad (23)
\]

where $|\Phi\rangle$ is the entangled state

\[
|\Phi\rangle = \frac{|\beta(\gamma_2 - \gamma_3)|^{1/2}}{\pi^{1/2}} \int \exp \left[ - \frac{1}{2a^2} y^2 - \frac{a^2}{2} z^2 \right] \times |\beta y + \gamma_2 z\rangle \otimes |\beta y + \gamma_3 z\rangle \times dy \, dz \quad (24)
\]

The generalization of the proof to larger values of $k$ is straightforward.
C. Fidelity of the secret sharing scheme

As we have seen, the CV QSS scheme works perfectly only for $a \to \infty$ in Eq. (3). In this case the dealer has infinitely squeezed ancillary states with which to entangle the secret state $|\psi\rangle$. The situation is similar to CV quantum teleportation [16], where an ideal EPR pair (which is a two-mode infinitely squeezed vacuum) is required for the protocol to work perfectly. However, with some loss of fidelity the scheme can be adapted to a realistic, finitesqueezing situation [17]. In CV QSS, finite squeezing implies that the secret state can only be approximately replicated because there is entanglement between the secret state and the shares in both the access structure and the adversary structure, which limits the fidelity of the replicated state with respect to the original secret state. Also entanglement with the adversarial shares allows some information about the secret state to escape. These compromises to CV QSS are reduced by increasing the degree of squeezing.

A detailed analysis reveals that the reduced density operator $\hat{\rho}$ of the replicated secret is related to the original density operator $\hat{\rho} = |\psi\rangle \langle \psi|$ by

$$
\rho' (x', x) = \langle x | \rho' | x \rangle = \frac{a}{\sqrt{\pi v}} \exp \left[ - \frac{a^2 (x - x')^2}{4a^2} \right] \times \int_{\mathbb{R}} \rho(x - y, x' - y) \exp \left[ - \frac{a^2 y^2}{v^2} \right] dy. \quad (25)
$$

Here $v$ is the norm of the vector $\gamma_1$ in Eq. (12) and $u^2 = \sum_{i=1}^{k-1} u_i^2$, where $\{u_i\}$ are the coefficients of the expansion $\alpha_j = \sum_{i=1}^{k-1} u_i \beta_{ji}$, $j = 2, \ldots, k$. The parameters $u$ and $v$ quantify the degree to which the secret state has been degraded for a given $a$ by encoding and decoding. Perfect replication corresponds to $u = 0$ and $v = 0$, which is in general unachievable. The degradation is symmetric under the exchange of $u \leftrightarrow v$.

Eq. (25) shows that the effect of using finite squeezing for the encoding procedure is twofold. First, the Gaussian factor in front of the integral in Eq. (25) suppresses off-diagonal elements of the density operator for $x - x' \gg 2a/u$ implying decoherence. Second, the density operator element of the replicated secret is a convolution of the original density operator with a Gaussian. The larger $a$ is, the more accurately is the secret state replicated; in the limit $a \to \infty$, it is perfectly replicated.

The replication fidelity of the system can be characterized by evaluating $F = \langle \psi | \hat{\rho}' | \psi \rangle$ for some standard secret state $|\psi\rangle$. For an arbitrary coherent state as the secret, the fidelity is given by the function

$$
F = 1 + (u^2 + v^2)/2a^2 + u^2v^2/4a^4)^{-1/2}, \quad (26)
$$

The dependence of $F$ on $r = \ln a$ for some particular values of $u$ and $v$ can be seen in Fig. 2. The fidelity tends to unity for large squeezing ($a \to \infty$, $r \to \infty$) and to zero for large antisqueezing ($a \to 0$, $r \to -\infty$). The fidelity for $r = 0$ corresponds to the case when the ancillary states are all vacuum states.

![Fig. 2: The fidelity $F$ versus the squeezing parameter $r = \ln a$ for an arbitrary coherent state as the secret. Two cases are presented: (1) $u = 0.5$ and $v = 1$ (solid line) and (2) $u = 3$ and $v = 5$ (dashed line).](image)

III. EFFICIENT REPLICATION

In the previous section we have established a replication protocol for the access structure; here we seek the most efficient protocol, which minimizes the total number of squeezers (expensive components in an active interferometer) required. In the following we show that by a suitable choice of a particular disentangling transformation, it is possible to reduce the total number of squeezers required to no more than two.

Let $\xi_i \to \zeta_i$ denote the orthogonal projection of $\xi_i \in \mathbb{C}^n$ to the subspace $X \oplus Y \subset \mathbb{C}^n$ so that

$$
\xi_i(x) = \zeta_i(x) + \sum_j \gamma_{ij} z_j, \\
\zeta_i(x) = \alpha_i x + \sum_j \beta_{ij} y_j. \quad (27)
$$

Claim: A transformation $g_i \to \xi_i = \alpha_i + \beta_i + \gamma_i$, with $\alpha_i \in X$, $\beta_i \in Y$, and $\gamma_i \in \mathbb{Z}$, which leaves the coordinates $\xi_i = g_i$ unchanged for $i = k + 1, \ldots, n$ and is such that

$$
\alpha_1 = 1, \quad \beta_1 = 0, \\
\text{span}(\zeta_2, \ldots, \zeta_k) = \text{span}(\zeta_{k+1}, \ldots, \zeta_n), \quad (28)
$$

disentangles the secret state for sufficiently large values of the parameter $a$.

To prove this claim, we show, by a change of variables that, for sufficiently large values of $a$, the state

$$
\Psi_\xi = \frac{\det \xi}{\xi} \int dx \psi(x) \prod_{i=2}^{k} \frac{1}{(\pi)^{1/2}} \int dy_i dy_z \\
\times \exp \left[ - \frac{1}{2a^2} y_i^2 - \frac{a^2}{2} z_i^2 \right] \\
\times |x + \gamma_1(x)\rangle \otimes |\zeta_2(x) + \gamma_2(x)\rangle \\
\otimes \cdots \otimes |\zeta_n(x) + \gamma_n(x)\rangle 
$$

(29)
defined by the transformation \( g_i \to \xi_i \), is expressible in the form
\[
|\Psi_\xi\rangle = |\psi\rangle \otimes |\Phi\rangle
\] 
with
\[
|\Phi\rangle = \left[ \det \xi \right]^{1/2} \frac{1}{\sqrt{\pi}} \int \frac{dy_i dz_i}{2!} \exp \left[ -\frac{1}{2a^2} y_i^2 - \frac{a^2}{2} z_i^2 \right] 
\]
\[
\times |\beta_2(x) + \gamma_2(x)\rangle \otimes \cdots \otimes |\beta_n(x) + \gamma_n(x)\rangle .
\]

Next observe that, since the vectors \( \{\zeta_{i+1}, \ldots, \zeta_n\} \) are linear combinations of the vectors \( \{\zeta_1, \ldots, \zeta_k\} \), the change of variables given by the projection \( \zeta_i \to \beta_i \), for \( i = 2, \ldots, k \), results in the corresponding projections \( \zeta_i \to \beta_i \) for \( i = k + 1, \ldots, n \). Now, if \( \beta_i \) is defined such that
\[
\sum_j \beta_i^j \beta_{jk} = \delta_{ik},
\]
then the projection \( \zeta_i \to \beta_i \), for \( i = 2, \ldots, k \), corresponds to the coordinate transformation \( y_i \to y_i - (\sum_j \beta_i^j \alpha_j) x \).

Thus, if \( a \) is sufficiently large that \( \exp \left[ -\frac{1}{2a^2} (y_i - (\sum_j \beta_i^j \alpha_j) x)^2 \right] \approx \exp \left[ -\frac{1}{2a^2} y_i^2 \right] \) for all values of \( x \) for which \( \psi(x) \) is non-negligible, we obtain Eq. (28).

Now, let the vectors \( g_i \) defining the encoded state \( |\Psi_g\rangle \) by the linear transformation (6) have decomposition, parallel to that given by Eq. (27),
\[
g_i = \kappa_i + \lambda_i, \quad i = 1, \ldots, k,
\]
\[
g_i = \xi_i = \zeta_i + \gamma_i, \quad i = k + 1, \ldots, n,
\]
with \( \kappa_i \in \mathbb{X} \oplus \mathbb{Y} \) and \( \lambda_i \in \mathbb{Z} \), respectively. And let \( T \) denote a transformation
\[
g_i \to \xi_i = \sum_{j=1}^k T_{ij} g_j, \quad i = 1, \ldots, k
\]
such that the vectors
\[
\zeta_i = \sum_{j=1}^k T_{ij} \kappa_j, \quad i = 1, \ldots, k
\]
satisfy the disentanglement criteria (28).

The condition that the vectors \( \zeta_2, \ldots, \zeta_k \) span the same subspace of \( \mathbb{X} \oplus \mathbb{Y} \) as do \( \zeta_{k+1}, \ldots, \zeta_n \) can be satisfied by requiring that both sets are orthogonal to a common vector \( v \in \mathbb{X} \oplus \mathbb{Y} \). Thus, if \( v \in \mathbb{X} \oplus \mathbb{Y} \) is a vector defined such that
\[
v \cdot \zeta_i = 0 \quad i > k,
\]
the transformation \( T \) is required to satisfy the equation
\[
v \cdot \zeta_i = \sum_{j=1}^k T_{ij} v \cdot \kappa_j = 0, \quad \forall i = 2, \ldots, k.
\]

To satisfy the first condition of Eq. (28), \( T \) should also be such that
\[
\zeta_1 = \sum_{j=1}^k T_{ij} \kappa_j = f_1
\]
so that \( \zeta_1(x) = x \).

Eq. (30) implies that the first row of the matrix \( T \) is the row vector \( a = (a_1, a_2, \ldots, a_k) \) whose components are the coefficients in the expansion \( f_1 = \sum_{j=1}^k a_j \kappa_j \); i.e., \( T_{1j} = a_j \). The remaining rows can be defined as a set of orthogonal row vectors \( \{T_i; i = 2, \ldots, k\} \), all of which are orthogonal to the unit row vector \( W_1 \) whose components are given by
\[
W_{ij} = \frac{v \cdot \kappa_j}{\sqrt{\sum_{i=1}^k (v \cdot \kappa_i)^2}}.
\]
The orthogonality of the vectors \( \{T_i; i > 1\} \) to \( W_1 \) then ensures that \( \sum_{i=1}^k T_{ij} W_{ij} = 0 \) for \( i > 1 \) and that the condition (28) is satisfied. The norms of the orthogonal vectors \( \{T_i; i > 1\} \) are arbitrary and can be chosen to minimize the cost of the transformation. We find (cf. following section) that it is convenient to choose all but one of these vectors (e.g., the vector \( T_2 \)) to be normalized to unity. Denoting the norm of the vector \( T_2 \) by \( \gamma \), we then have
\[
T_{ij} = a_j, \quad \gamma W_{ij}, \quad i > 2,
\]
where \( W_{ij} \) is an orthogonal matrix.

As remarked above, an orthogonal transformation of the collaborating players' states can be achieved with passive elements. However, the replacement of the first row of \( W \) by the vector \( a \) in forming the matrix \( T \), means that the resulting transformation involves squeezing operations and hence a need for active elements. As we now show, the transformation defined by \( T \) can be achieved with just two squeezer.

Choose the vector \( W_2 \) to lie in the span of the vectors \( a \) and \( W_1 \). It then follows that \( a = \cdots \)
\[ \alpha W_1 + \beta W_2 \text{ and} \]
\[ T = \begin{pmatrix} \alpha & \beta & 0 & \ldots & 0 \\ 0 & \gamma & 0 & \ldots & 0 \\ 0 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & I \end{pmatrix} \]
\[ W \equiv VW, \quad (42) \]

with a free parameter \( \gamma \neq 0 \). This parameter can be adjusted, according to the criteria outlined in Sec. IV to minimize the demands on the squeezing resources. The \( \text{GL}(k, \mathbb{R}) \) matrix \( V \) can now be factored as \( V = XV_dY \), with \( X \) and \( Y \) orthogonal matrices and
\[ V_d = \text{diag}(v_1, v_2, 1, 1, \ldots, 1). \quad (43) \]
The complete transformation \( T \) then assumes the simple form
\[ T = VW = XV_dYW = XV_dZ, \quad (44) \]
with both \( X \) and \( Z \) orthogonal matrices.

The disentangling transformation represented by the matrix \( T \) is now achieved by a sequence of three transformations: the first transformation, represented by the orthogonal matrix \( Z \), is achieved by a passive interferometer consisting of only beam splitters and phase shifters; the transformation represented by the diagonal matrix \( V_d \) is given by single-mode \( \text{Sp}(1, \mathbb{R}) \) squeezers on the first two modes, with squeezing parameters \( r_1 = \ln v_1 \) and \( r_2 = \ln v_2 \); finally, the transformation corresponding to the matrix \( X \) is given by a two-mode beam splitter (see Fig. 3). Hence the number of active optical elements (squeezers) is reduced to two.

![Fig. 3: The general scheme of an interferometer used by the players to decode the secret state. The passive \( k \)-port interferometer is followed by two independent single-mode squeezers, and the last step is a passive two-mode interferometer that yields the secret at one output port.](image)

**IV. TOTAL AMOUNT OF SQUEEZING**

It is of interest not only to consider the number of active optical elements necessary for the replication part of QSS, but also the total amount of squeezing \( R \). It is natural to define this quantity as the sum of magnitudes of squeezing parameters corresponding to the two squeezers, i.e.,
\[ R = |r_1| + |r_2| = |\ln v_1| + |\ln v_2|, \quad (45) \]

which can be minimized by a judicious choice of \( \gamma \) in Eq. (42).

We can express \( R \) as \( R = \frac{1}{2}(|\ln \lambda_1| + |\ln \lambda_2|) \), where \( \lambda_{1,2} \) are the eigenvalues of the symmetric matrix \( V'\hat{V}' \) with \( V' = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \), and \( \hat{V}' \) the transpose of \( V' \). A simple calculation shows that the eigenvalues are
\[ \lambda_{1,2} = \frac{1}{2} \left[ (\alpha^2 + \beta^2 + \gamma^2 \pm \sqrt{(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\alpha^2\beta^2}) \right]. \quad (46) \]

Depending on \( \gamma \), the total amount of squeezing \( R \) is either (i) \( R = \frac{1}{2} |\ln(\lambda_1\lambda_2)| \) (if both \( \ln \lambda_1 \) and \( \ln \lambda_2 \) have the same sign) or (ii) \( R = \frac{1}{2} |\ln(\lambda_1/\lambda_2)| \) (if \( \ln \lambda_1 \) and \( \ln \lambda_2 \) have different signs). We seek \( \gamma \) that minimizes \( R \), which can occur for either case (i) or (ii), so both must be checked. We define the quantity \( \kappa \equiv (1-\alpha^2-\beta^2)/(1-\alpha^2) \) and have:

1. The minimum value of \( R(\gamma) \) is \( R_{\min} = |\ln(\kappa)| \) and occurs for \( \gamma_0 = \sqrt{\kappa} \) in the following situations:
   \[ \alpha^2 + \beta^2 < 1 \text{ and } \alpha^2 + \beta^2 < \kappa \]
   \[ \alpha^2 + \beta^2 > 1 + \beta^2/\alpha^2 \text{ and } \alpha^2 + \beta^2 > \kappa \]

2. The minimum value of \( R(\gamma) \) is \( R_{\min} = \ln(\sqrt{\alpha^2 + \beta^2 + |\beta|/|\alpha|}) \) and occurs for \( \gamma_0 = \sqrt{\alpha^2 + \beta^2} \) in the following situations:
   \[ 1 \leq \alpha^2 + \beta^2 \leq 1 + \beta^2/\alpha^2 \]
   \[ \kappa \leq \alpha^2 + \beta^2 \leq 1 \]
   \[ 1 + \beta^2/\alpha^2 \leq \alpha^2 + \beta^2 \leq \kappa \]

The strategy for a collaborating group of players to minimize the squeezing resources for the replication of the secret state is the following: for given \( \alpha \) and \( \beta \), the players calculate the value of \( \kappa \) and decide which of the two cases (i) or (ii) occurs. Then they find the value \( \gamma_0 \) and construct the matrix \( T \) in Eq. (42) and from this, the corresponding active interferometer that contains only two squeezers with a minimum total amount of squeezing equal to \( R_{\min} \).

**V. CONCLUSION**

We have shown that the replication procedure in optical continuous-variable quantum secret sharing can be achieved with a small number (at most two) of squeezing elements for any authorized group of players. In particular, we have demonstrated this for the QSS threshold schemes. We have quantified the total amount of squeezing defined as the sum of absolute values of the single-mode squeezing parameters, and found its minimum value analytically. We have also seen that in the realistic situation when the dealer has only finite squeezing resources available, the density operator of the replicated secret becomes a Gaussian convolution of the original secret state.
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