Floquet-Drude Conductivity

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This letter presents a generalization of the Drude conductivity for systems which are exposed to periodic driving. The probe bias is treated perturbatively by using the Kubo formula, whereas the external driving is included non-perturbatively using the Floquet theory. Using a new type of four-times Green’s functions disorder is approached diagrammatically, yielding a fully analytical expression for the Floquet-Drude conductivity. Furthermore, the Floquet Fermi’s golden rule is generalized to \(t'\)-Floquet states, connecting the Floquet-Dyson series with scattering theory for Floquet states. Our formalism allows for a direct application to numerous systems e.g. graphene or spin-orbit systems.

Introduction.—Paul Drude published his theory of electric transport in metals as long ago as 1900 [1, 2], which is today known as Drude theory. To the present day several approaches have been developed to deepen the understanding of the microscopic mechanisms occurring in charge transport, including scattering theory using Fermi’s golden rule [3, 4] or quantum corrections to the Drude conductivity. The latter covers weak (anti-)localization [5, 6] in the form of geometry or spin dependent corrections [7–15]. In contrast to studies of static systems the development of lasers and masers generated a rising activity on explicitly time-dependent Hamiltonians, where the external field cannot be considered a small perturbation [16]. In the most recent decade, owing to the possibility of changing the topology of a system by means of external driving, the investigation of transport in driven systems increased [17–26]. This includes transport in driven systems [27, 28], either with or without disorder [29–31], or the photo-voltaic Hall effect [19]. Most works studying the renormalization of conductivity, due to an external driving, use a perturbative approach regarding the external driving [20, 22]. In this letter we present a new general formalism that allows the determination of the Drude conductivity in the presence of a non-perturbative external driving. We unify linear response theory and Floquet formalism to account for the probe bias and an external driving. Using a new type of four-times Green’s function formalism we derive both a Floquet-Dyson series and a generalized Floquet Fermi’s golden rule. To prove the consistency, both are shown to yield the same scattering time, a link that was missing so far. Finally, we present a closed analytical form for the Floquet-Drude conductivity and apply the developed theory to a 2DEG with circularly polarized external driving. The analysis shows that previous results have been overestimating the effect of the driving on the conductivity.

Kubo formula for periodically driven systems.—Our first aim is to express the linear conductivity using Floquet states [16, 32, 33] \(|\psi_\alpha(t)\rangle = \exp(-i\frac{\varepsilon_\alpha t}{\hbar})|u_\alpha(t)\rangle\), where \(\alpha\) is labeling a discrete set of quantum numbers and \(\varepsilon_\alpha\) are the corresponding quasienergies [34, 35]. The periodicity of the Floquet functions, which are eigenstates of the Floquet Hamiltonian \(H_F(t) = H(t) - i\hbar\partial_t\), allow for the Fourier expansion \(|u_\alpha(t)\rangle = \sum_n e^{-i\varepsilon_n t}\langle u_\alpha|_n\rangle\) with \(\Omega\) being the frequency of the external driving. The probe bias, with the corresponding vector potential \(A(q, \omega)\), is treated perturbatively in linear response theory, i.e., using the Kubo formula in momentum space [36]

\[
\langle J^a(q, \omega) \rangle = \frac{i}{\hbar \Omega} \sum_{n} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt \left[ e^{i\omega t} \Theta(t - t') \times \langle [J^a(q, t'), J^b(-q, t')] A^b(q, t') \rangle - \frac{e^2}{m} A^a(q, \omega) \right],
\]

with \(a, b \in \{x, y, z\}\), \(\omega\) being the frequency of the response current, \(\mathcal{V}\) the volume of the system, \(e < 0\) the electron charge, \(\Theta(\cdot)\) the Heaviside function, \(m\) the effective electron mass, \(n\) the electron density. \(\langle \cdot \rangle\) denotes the statistical average with respect to the system’s state which will in the presence of external driving not be in equilibrium. However, in what follows we shall assume the system to be in a stationary state so that occupation numbers of Floquet states are time-independent [18, 27, 37–41].

\[
\langle J^a(q, \omega) \rangle = \sum_{\beta} \int_{-\infty}^{\infty} d\omega' \tilde{\sigma}^{ab}(q, \omega, \omega') E^b(q, \omega').
\]

Thus far, the conductivity tensor \(\tilde{\sigma}^{ab}\) depends on the bias frequency, the momentum \(q\) and on the resulting current.
frequency $\omega$,
\[
\sigma^{ab}(q,\omega,\omega') = \frac{i}{\hbar \omega_0} \sum_{\alpha \beta} \sum_{n_1n_4=-\infty}^{\infty} (f_{\alpha} - f_{\beta}) \times \langle u^{a}_{n_1}\alpha | j^a(q) | u^{b}_{n_4}\beta \rangle \langle u^{b}_{n_3}\beta | j^b(-q) | u^{a}_{n_2}\alpha \rangle \times \delta(\omega + (n_1 - n_2 + n_3 - n_4)\Omega - \omega') + \frac{ie^2\hbar}{m\omega} \delta_{ab}(\omega - \omega') ,
\]
with the single particle current operator $j^i$ and the distribution function $f_{\alpha,\beta} = \langle c_{\alpha,\beta}^\dagger c_{\alpha,\beta} \rangle$. In what follows we limit our calculations to a position independent driving field, i.e., $\mathbf{q} = 0$. Since we are ultimately interested in the DC conductivity, Eq. (2) simplifies to
\[
\langle J^a(q,\omega) \rangle = \sum_{\alpha \beta} \sigma^{ab}(q,\omega) E^b(q,\omega),
\]
as shown in the supplemental material (SM). This allows us to express the real part of the longitudinal DC conductivity as
\[
\mathrm{Re} \omega \to 0 [\sigma^{xx}(0,\omega)] = \frac{\pi \hbar}{2} \left( \frac{e}{m} \right)^2 \int d\varepsilon \left( -\frac{\partial f}{\partial \varepsilon} \right) \sigma(\varepsilon)
\]
along with the definition
\[
\sigma(\varepsilon) = \sum_{n,n'=-\infty}^{\infty} \sum_{\alpha \beta} \langle u^{a}_{n}\alpha | p^x | u^{b}_{n'}\beta \rangle \langle u^{b}_{n'}\beta | p^x | u^{a}_{n}\alpha \rangle \times \delta(\varepsilon - \varepsilon_\alpha) \delta(\varepsilon - \varepsilon_\beta),
\]
and the abbreviation $\mathrm{Re}_{\omega \to 0}[]$ for $\lim_{\omega \to 0} \mathrm{Re}[]$. A similar result has already been derived in Ref. [18] using the Keldysh framework. In derivating Eq. (4) one requires the difference of two quasienergies to always be smaller than the photon energy of the external driving. Thus far, we have not used the convenient choice of the quasienergy [42] to be in $[-\Omega/2,\hbar \Omega/2)$, however we must choose a suitable, possibly momentum dependent function $\lambda$ such that
\[
\forall \lambda : \lambda - \frac{\hbar \Omega}{2} \leq \varepsilon_\alpha < \lambda + \frac{\hbar \Omega}{2}.
\]

Four-times Floquet Green’s function and conductivity.—In this section we set up a formalism using four-times Green’s functions to express the result of the foregoing section for the conductivity in terms of Green’s functions. These are the building blocks for the Floquet-Dyson equation. First, we define a $tt'$-state for the $\ell$-th Floquet zone [16, 43-45]
\[
|\psi_{\alpha}^\ell(t,t')\rangle = e^{-\frac{i}{\hbar} (\varepsilon_\alpha + \ell \Omega) t} |u_{\alpha}(t')\rangle e^{i\alpha t},
\]
recovering for $t = t'$ the Floquet state solution $|\psi_{\alpha}(t)\rangle$ of the time-dependent Schrödinger equation. From these a bare four-times Green’s function is constructed
\[
G_{0}^{r,s}(t_1, t_2, t'_1, t'_2) = \mp i \Theta(\pm (t_1 - t_2))
\]
\[
\times \frac{1}{T} \sum_{\ell = -\infty}^{\infty} \sum_{\alpha} \langle \psi_{\alpha}^r(t_1,t'_1) | \psi_{\alpha}^s(t_2,t'_2) \rangle
\]
with $T = 2\pi/\Omega$. This propagator fulfills
\[
\left( i \partial_{t_1} - \frac{1}{\hbar} H_F(t'_1) \right) G_{0}^{r,s}(t_1, t_2, t'_1, t'_2) = \delta(t_1 - t_2) \sum_{\ell = -\infty}^{\infty} \delta(t'_1 - t'_2 + sT) \mathbb{1}.
\]
Fourier transforming the Green’s function and expanding the Floquet function into a Fourier series yields
\[
G_{0}^{r,s}(\varepsilon, t_1, t'_2) = \frac{1}{T} \sum_{n,n'=\infty}^{\infty} G_{0}^{r,s}(\varepsilon, n, n') \times e^{-i n \Omega t_1} e^{i n' \Omega t_2}.
\]
Generalizing the completeness relation of the Floquet functions to different times
\[
\sum_{\alpha} |u_{\alpha}(t'_1)\rangle \langle u_{\alpha}(t'_2)\rangle \sum_{\ell = -\infty}^{\infty} \delta(t'_1 - t'_2 + \ell T)
\]
\[
= \sum_{\ell = -\infty}^{\infty} \delta(t'_1 - t'_2 + \ell T)
\]
gives particular insight to the Lehmann representation of the four-times Floquet Green’s function:
\[
G_{0}^{r,s}(\varepsilon, t'_1, t'_2) = \frac{1}{T} \sum_{\ell = -\infty}^{\infty} \delta(t'_1 - t'_2 + \ell T) \frac{1}{\varepsilon - \frac{1}{\hbar} \Omega F(t'_1)}.
\]
One can show that the Fourier coefficients [46, 47]
\[
G_{0}^{r,s}(\varepsilon, n, n') = \sum_{\ell = -\infty}^{\infty} \sum_{\alpha} |u_{\alpha}^{n+\epsilon}\rangle \langle u_{\alpha}^{n'+\epsilon}| \frac{1}{\varepsilon - \frac{1}{\hbar} \varepsilon_\alpha - n\Omega \pm i 0^+},
\]
are equal to the inverse of the Floquet matrix [27, 29, 30, 48-53]. The periodicity of the Floquet eigenstates [54, 55] suggests defining the unitary transformation $T$ as in Sec. V of Ref. [56] which diagonalizes the Green’s function
\[
(D(\varepsilon))_{\alpha\beta} = (T^\dagger G_{0}^{r,s}(\varepsilon) T)_{\alpha\beta}
\]
\[
= \frac{1}{\varepsilon - \frac{1}{\hbar} \varepsilon_\alpha - n\Omega \pm i 0^+}
\]
where we denoted the matrix spanned by the Fourier components of the Green’s function as $(G_{0}^{r,s}(\varepsilon))_{nn'} = G_{0}^{r,s}(\varepsilon, n, n')$. The Green’s function defined in Eq. (10)
is used to express the conductivity from Eq. (4) as
\[
\text{Re} \left[ \sigma_{xx}(0, \omega) \right] = -\frac{1}{4\pi v} \left( \frac{e}{m} \right)^2 \int_{\lambda^-}^{\lambda^+} d\epsilon \left( -\frac{\partial f}{\partial \epsilon} \right) \int_0^T dt_1' \int_0^T dt_2' \text{tr} \left[ p^r \left( G_0^r(\epsilon, t_1', t_2') - G_0^a(\epsilon, t_1', t_2') \right) \right] .
\]  
(16)

Floquet-Dyson equation and generalized Floquet Fermi’s golden rule.— The focus of this section is to formulate a perturbative approach to include disorder in the expression of the conductivity described by bare propagators, i.e., Eq. (16). In the following we will use the notation
\[
\langle x | G^{r,a}(t_1, t_2, t_1', t_2') | x' \rangle \equiv G^{r,a}(t_1, t_2, x, x', t_1', t_2') ,
\]
(17)
for the matrix elements of the Green’s function in real space. The Green’s function for the system with an impurity potential \( V(x_1, t_1, t_1') \) at site \( x_1 \) is supposed to fulfill
\[
\left( i\partial_t - \frac{1}{\hbar} \mathcal{H}_F(x_1, t_1, t_1') \right) G^{r,a}_p(t_1, t_2, x_1, x_2, t_1', t_2') = \delta(t_1 - t_2) \delta(x_1 - x_2) \sum_{s=-\infty}^{\infty} \delta(t_1' - t_2' + sT) \mathbb{1}
\]
(18)
with \( \mathcal{H}_F(x_1, t_1, t_1') = H_F(x_1, t_1') + V(x_1, t_1, t_1') \). We limit the calculation presented here to the time-independent potential \( V(x) \), as including an explicit time-dependence is straightforward (see SM). Following the standard steps [36, 57–59], we can derive a recursive integral expansion, i.e., a Dyson series, for the Green’s function, in case of a system perturbed by impurities
\[
G^{r,a}_p(\epsilon, x_1, x_2, n, n') = G^{r,a}_0(\epsilon, x_1, x_2, n, n') + \frac{1}{\hbar} \int dx_3 \sum_{n_1=-\infty}^{\infty} G^{r,a}_p(\epsilon, x_1, x_3, n, n_1) V(x) \sum_{n_2=-\infty}^{\infty} G^{r,a}_0(\epsilon, x_2, n_1, n_2) .
\]  
(19)

The impurity potential \( V(x) = \sum_i^{N_{\text{imp}}} v(x - r_i) \) is assumed to be a Gaussian random potential, which is uncorrelated such that the impurity average yields \( \langle v(x) v(x') \rangle_{\text{imp}} = \mathcal{V}_{\text{imp}} \delta(\mathbf{x} - \mathbf{x}') \Leftrightarrow v(k) = \mathcal{V}_{\text{imp}} \) and \( \langle v(x) \rangle_{\text{imp}} = 0 \) (white noise) [58, 59]. Fourier transforming Eq. (19) into momentum space and performing a disorder average one arrives at the expression for the disorder averaged Green’s function
\[
G^{r,a}(\epsilon, k) = G^{r,a}_0(\epsilon, k) + G^{r,a}_0(\epsilon, k) \sum_{n=1}^{n_{\text{IBA}}} \left[ \Sigma^{r,a}(\epsilon, k) G^{r,a}_0(\epsilon, k) \right]^{n-1}
\]
(20)
where the self-energy \( \Sigma^{r,a}(\epsilon, k) \) is the sum over all irreducible diagrams. Applying the transformation \( T \), the solution of the recursive Eq. (20) in the eigenbasis is governed by
\[
T^\dagger(k) G^{r,a}(\epsilon, k) T(k) = \left[ D(\epsilon, k) + T^\dagger(k) \Sigma^{r,a}(\epsilon, k) T(k) \right]^{-1}
\]
(21)
with the diagonal matrix \( D(\epsilon, k) \) given in Eq. (14). The difference of the retarded and advanced self-energy in the first order Born approximation (IBA) can be related to a scattering time derived within the framework of the Floquet Fermi’s Golden rule for \( tt' \)-states (7) as
\[
\left( \frac{1}{\tau(\epsilon, k)} \right)_{\alpha\beta}^{nn'} = i \left( T^\dagger(k) (\Sigma^{r}_{\text{IBA}}(\epsilon, k) - \Sigma^{a}_{\text{IBA}}(\epsilon, k)) T(k) \right)_{\alpha\beta}^{nn'}
\]
(22)
To our knowledge, this remarkable connection has not been established before. The related scattering rates are given by
\[
\Gamma_{\alpha\beta}^{nn'}(\mathbf{k}, \mathbf{k}') = \frac{2\pi}{\hbar} V^2 \sum_{\gamma} c_{\alpha\gamma}^{n}(\mathbf{k}, \mathbf{k}') (c_{\beta\gamma}^{n'}(\mathbf{k}, \mathbf{k}'))^* \times \delta(\epsilon - \epsilon(\mathbf{k}'))
\]
(23)
with \( c_{\alpha\beta}^{n}(\mathbf{k}, \mathbf{k}') = \sum_{m=-\infty}^{\infty} (u_{\alpha}^{m+n}(\mathbf{k}))^* u_{\beta}^{m}(\mathbf{k}') \). One can provide a connection to previous studies by setting \( n = n' = 0 \) in Eq. (23), which results in the scattering rates given in Ref. [17, 42]. However, in general the relaxation rate matrix is not diagonal in the Floquet space nor can it be reduced to the \((nn') = (00)\) element only, as will be discussed later on. Also, one should notice that only on the diagonal is the difference of the retarded and advanced self energy equal to the imaginary part of the retarded self-energy.

Floquet-Drude conductivity.—Now, let us consider only the self-energy corrections during the disorder average and perform the time integration in Eq. (16). The disorder is not supposed to change the eigenenergies of the bare system, hence we drop all off-diagonal elements of the self energy. This allows us to proceed analytically and ultimately express the conductivity in a com-
pact way,

\[
\text{Re}_{\omega \to 0} [\sigma^{xx}(0, \omega)] = \frac{\hbar}{4\pi \Omega} \left( \frac{e}{m} \right)^2 \int d\varepsilon \left( \frac{\partial f}{\partial \varepsilon} \right)_{\varepsilon=0}^{\Omega/2} \int_{-\Omega/2}^{\Omega/2} d\varepsilon \left( \frac{1}{2\tau(\varepsilon, \mathbf{k})} \right)_{\alpha \alpha}^{nn} \right)
\]

\times \left[ \left( \frac{1}{\hbar^2} \varepsilon - \frac{1}{\hbar^2} \varepsilon_0 - n\Omega \right)^2 + \left( \frac{1}{2\tau(\varepsilon, \mathbf{k})} \right)_{\alpha \alpha}^{nn} \right]^{-2}.
\]

Application to a 2DEG.— For a simple non-trivial application of the method described above we chose a 2DEG from a direct semiconductor close to the Γ-point under illumination with circularly polarized light. The effective model for the lowest s-type conduction band is given by a parabolic Hamiltonian \( H = p^2 / 2m \). The vector potential of the radiation

\[
\mathbf{A}(t) = A \cos(\Omega t) \hat{x} + A \sin(\Omega t) \hat{y}
\]

is coupled to the momentum via minimal coupling leading to the time-dependent Hamiltonian

\[
H(t) = \frac{\hbar^2}{2m} \left[ \mathbf{k}^2 + \gamma^2 + 2\gamma(k_x \cos(\Omega t) + k_y \sin(\Omega t)) \right]
\]

with the light parameter \( \gamma = eA/\hbar \). The solution of the time-dependent Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{k}, t) = H(t) \Psi(\mathbf{k}, t)
\]

was already given by Kibis [25]

\[
\Psi(\mathbf{k}, t) = e^{-i\mathbf{k} \cdot \mathbf{u} t} \sum_{n=-\infty}^{\infty} u_n(\mathbf{k}) e^{-in\Omega t}
\]

where the quasienergy and the Fourier components are given by

\[
\epsilon_\mathbf{k} = \frac{\hbar^2}{2m} (\mathbf{k}^2 + \gamma^2),
\]

\[
u_n(\mathbf{k}) = J_n \left( \frac{\hbar^2 \gamma \mathbf{k}}{2m \hbar \Omega} \right) e^{-in(\phi+\pi/2)}
\]

with \( \mathbf{k} = k(\cos(\phi), \sin(\phi)) \). As one can demonstrate by solving the driven tight binding model for the square lattice with lattice constant \( a \) (see SM), the effective mass \( m \) is also \( \gamma \) dependent. Thus, we change \( m \to m(\gamma) = m/J_0(\alpha \gamma) \). To evaluate the expression for the conductivity, one has to specify the distribution function further. In the off resonant regime, absorption of photons is suppressed, hence, a Fermi distribution can be assumed. However, since the parabolic spectrum is unbounded, it is not obvious how to set the Fermi energy \( \varepsilon_F \) for the driven parabolic spectrum [42]. Here, we truncate the momentum range to set the Fermi energy (for further discussion see SM). Evaluating Eq. (22) yields for the scattering time on the diagonal

\[
\left( \frac{1}{\tau(\varepsilon_F, \mathbf{k})} \right)^{nn} = \frac{\gamma_{\text{imp}}(\gamma)}{\hbar^3} \sum_{m=-\infty}^{\infty} J^2_{m+n}(z_k) J^2_m(z_\varepsilon),
\]

together with

\[
z_k = \frac{\hbar^2 \gamma \mathbf{k}}{m(\gamma) \hbar \Omega}, \quad z_\varepsilon = \frac{\hbar^2 \gamma \sqrt{2m(\gamma)\varepsilon_F / \hbar^2}}{m(\gamma) \hbar \Omega}.
\]

We can further disregard all pairings of only retarded or only advanced Green’s functions in Eq. (24), since they give a contribution of the order of \( 1/(\varepsilon_F \tau_0) \) with \( \tau_0 \) being the scattering time of the undriven system [59]. Aside from the relation between the Fermi energy and the relaxation rate, in the driven case one finds an important relation between relaxation rate and driving frequency. This ratio controls whether or not only the \( n = 0 \) element in Eq. (31) is significant: If \( \Omega \tau_0 \gg 1 \), the broadening of the nonzero Floquet modes is small enough such that the leaking into the central Floquet zone is negligibly small. The theory presented here also makes the regime accessible where \( \Omega \tau_0 \simeq 1 \). In that case, the nonzero Floquet modes contribute significantly (for further details see SM). But even in the off resonant regime, \( \Omega \tau \gg 1 \), previous studies [20] overestimate the effect of circular driving as will be shown in the following. The central entry of the product of the retarded Green’s function with an advanced one is

\[
\left[ G^r(\varepsilon_F, \mathbf{k}) G^a(\varepsilon_F, \mathbf{k}) \right]^{00} = \frac{1}{(\varepsilon_F - \varepsilon_k)^2 + (2\tau(\varepsilon_F, \mathbf{k}))^2} \approx 2\pi \hbar (\tau(\varepsilon_F, \mathbf{k}))^{00} \delta(\varepsilon_F - \varepsilon_k).
\]

Applying these simplifications to Eq. (24), one arrives at the conductivity

\[
\left[ \rho_{xx}(0, \omega) \right]^{00} = \frac{1}{2\Omega^4 \pi \left( \frac{e^2}{m(\gamma)} \right) k^2_F (\tau(\varepsilon_F, \mathbf{k}))^{00},
\]

where the scattering time is evaluated at the Fermi energy and Fermi wave vector \( k_F = \sqrt{2m(\gamma)\varepsilon_F / \hbar} \),

\[
(\tau(\varepsilon_F, \mathbf{k}))^{00} = \left( \frac{\gamma_{\text{imp}}(\gamma)}{\hbar^3} \sum_{l=-\infty}^{\infty} J^2_l(z_\varepsilon) \right)^{-1}.
\]

Hence, the ratio between conductivity without driving and dressed conductivity is given by

\[
\left[ \rho_{xx}(0, \omega) \right]^{00} \left|_{\gamma=0} \right. \left. = \frac{J_0(\alpha \gamma)}{\sum_{l=-\infty}^{\infty} J^2_l(z_\varepsilon)}. \right.
\]
In Fig. 1 we present the conductivity of a 2DEG irradiated by a circularly ($\sigma_c$) polarized light of intensity $I$. For comparison also the results in case of linearly polarized light are shown. The applied approximations above allow for a direct comparison with existing theories on the topic of conductivity in driven systems. The central entry of the scattering time used in Eq. (35) is equal to the result which one yields from the Floquet Fermi’s golden rule [17, 25, 42] (proof in the SM). It is used e.g. by the authors of Ref. [20] to calculate the conductivity. However, the equation ibid overestimates the effect of the driving for the latter.

\[
\sigma \propto \frac{E_F}{\hbar \omega} \left( \frac{\omega}{\hbar} \right)^{2} \text{with } \sigma = \sqrt{10/10^{12}} \text{ is chosen.}
\]

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Supplemental Material: Floquet-Drude conductivity

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FLOQUET KUBO FORMULA FOR THE LINEAR CONDUCTIVITY

In a driven system the current is not simply a product of resistance and electric field, see Eq. (2) together with Eq. (3) in the main article. Rather, the conductivity depends on both the frequency spectrum of the bias, \( \omega' \) as well as a response frequency \( \omega \). Furthermore, we introduce

\[
\omega = \tilde{\omega} + p\Omega \iff |\tilde{\omega}| \leq \Omega/2 \quad \text{with} \quad p \in \mathbb{Z}, \tag{1}
\]

and require that the electric field \( E^b \) depends only on frequencies \( |\tilde{\omega}'| \leq \Omega/2 \). With this, Eq. (2) of the main article becomes

\[
\langle J^a(q, \tilde{\omega} + p\Omega) \rangle = \sum_b \int_{-\Omega/2}^{\Omega/2} d\tilde{\omega}' \sigma^{ab}(q, \tilde{\omega} + p\Omega, \tilde{\omega}') E^b(q, \tilde{\omega}') \tag{2}
\]

where the conductivity tensor is given by

\[
\sigma^{ab}(q, \tilde{\omega} + p\Omega, \tilde{\omega}') = \frac{i}{\hbar(\tilde{\omega} + p\Omega)} \Im \left\{ \sum_{\alpha\beta} \sum_{n_1-n_4=-\infty}^{\infty} \frac{f_\alpha - f_\beta}{\tilde{\omega} + p\Omega + \frac{1}{\hbar}(\epsilon_\alpha - \epsilon_\beta) + (n_1 - n_2)\Omega + i0^+} \right. \\
\times \langle u^n_{\alpha1} \mid j^f(q) \mid u^n_{\beta2} \rangle \langle u^n_{\alpha3} \mid j^f(-q) \mid u^n_{\beta4} \rangle \\
\times \delta(\tilde{\omega} + p\Omega + (n_1 - n_2 + n_3 - n_4)\Omega - \tilde{\omega}') \\
+ i \frac{e^2 n}{m(\tilde{\omega} + p\Omega)} \delta_{\ell j} \delta(\tilde{\omega} + p\Omega - \tilde{\omega}') \tag{3}
\]

The argument of the \( \delta \)-distribution of the first term can become zero if and only if

\[
n_1 - n_2 + n_3 - n_4 = -p. \tag{4}
\]

Since we are ultimately interested in the DC limit, we consider only the case where \( p = 0 \). Taking only the real part of the longitudinal conductivity one yields

\[
\text{Re} \left[ \sigma^{xx}(0, \tilde{\omega}) \right] = \frac{\pi}{\hbar^2} \left( \frac{e}{m} \right)^2 \sum_{n,n'} \sum_{\alpha\beta} \left[ \frac{f_\alpha - f_\beta}{\hbar \omega} \langle u^n_\alpha | p^x | u^n_\beta \rangle \langle u^{n'}_\beta | p^x | u^{n'}_\alpha \rangle \delta(\omega + \frac{1}{\hbar}(\epsilon_\alpha - \epsilon_\beta)) \right]. \tag{5}
\]

In the limit of \( \tilde{\omega} \to 0 \) one ends up with Eq. (4) of the main article.
DYSON SERIES FOR TIME-DEPENDENT PERTURBATIONS

Here, we use the same notation as in Eqs. (17) of the main article. As already shown in the latter, the bare Green’s function \( G_0 \) fulfills the equation

\[
\left( i \partial_{t_1} - \frac{1}{\hbar} H_F(x_1, t_1') \right) G_0^r, a(t_1, t_2, x_1, x_2, t_1', t_2') = \delta(t_1 - t_2) \delta(x_1 - x_2) \sum_{s=-\infty}^{\infty} \delta(t_1' - t_2' + sT) 1 ,
\]

(6)

with the definition of the Floquet Hamiltonian for the unperturbed system

\[
H_F^0 (x_1, t_1') = H(x_1, t_1') - i\hbar \partial_{t_1}
\]

(7)

The Hamiltonian for the perturbed one has the form

\[
\mathcal{H}_F(x_1, t_1, t_1') = H_F^0 (x_1, t_1') + V(x_1, t_1, t_1')
\]

(8)

where we stress that the dependency on \( t_1 \) is fully kept by the potential. Obviously, the bare Green’s function \( G_0 \) fulfills

\[
\left( i \partial_{t_1} - \frac{1}{\hbar} \mathcal{H}_F(x_1, t_1, t_1') \right) G_0^r, a(t_1, t_2, x_1, x_2, t_1', t_2') = \delta(t_1 - t_2) \delta(x_1 - x_2) \sum_{s=-\infty}^{\infty} \delta(t_1' - t_2' + sT) 1 .
\]

(9)

We are interested in the Green’s function of the perturbed system \( G_p \) being a solution of

\[
\left( i \partial_{t_1} - \frac{1}{\hbar} \mathcal{H}_F(x_1, t_1, t_1') \right) G_p^r, a(t_1, t_2, x_1, x_2, t_1', t_2') = \delta(t_1 - t_2) \delta(x_1 - x_2) \sum_{s=-\infty}^{\infty} \delta(t_1' - t_2' + sT) 1 .
\]

(10)

Equating Eq. (9) and Eq. (10) we get

\[
\left( i \partial_{t_1} - \frac{1}{\hbar} \mathcal{H}_F(x_1, t_1, t_1') \right) G_p^r, a(t_1, t_2, x_1, x_2, t_1', t_2') = \left( i \partial_{t_1} - \frac{1}{\hbar} \mathcal{H}_F(x_1, t_1, t_1') \right) G_0^r, a(t_1, t_2, x_1, x_2, t_1', t_2') + \frac{1}{\hbar} V(x_1, t_1, t_1') G_0^r, a(t_1, t_2, x_1, x_2, t_1', t_2') .
\]

(11)

The bare Green’s function is periodic in both \( t_1' \), and \( t_2' \),

\[
G_0^r, a(t, t_2, x, x_2, t_1' + T, t_2' + T) = G_0^r, a(t, t_2, x, x_2, t_1', t_2') .
\]

(12)
Therefore, without loss of generality one can use the restriction
\[ t'_1, t'_2 \in \left[ -\frac{T}{2}, \frac{T}{2} \right] . \] (13)

Making use of the periodicity of the Green’s function and requiring that the potential is as well periodic in the second time argument
\[ V(x, t_1, t'_1 + T) = V(x, t_1, t'_1) \] (14)

one can show that
\[
\int_{Vx} \int_{-\infty}^{\infty} dt \int_{-T/2}^{T/2} dt' \delta(t_1 - t) \delta(x_1 - x) \sum_{s=-\infty}^{\infty} \delta(t'_1 - t' + sT) V(x, t, t') G_{0}^{r,a}(t, t_2, x, x_2, t', t'_2) \]
\[
= \int_{-T/2}^{T/2} dt' \sum_{s=-\infty}^{\infty} \delta(t'_1 - t' + sT) V(x_1, t_1, t') G_{0}^{r,a}(t_1, t_2, x_1, x_2, t', t'_2) \]
\[ = V(x_1, t_1, t'_1) G_{0}^{r,a}(t_1, t_2, x_1, x_2, t'_1, t'_2) \] (15)
\[ = V(x_1, t_1, t'_1) G_{0}^{r,a}(t_1, t_2, x_1, x_2, t'_1, t'_2) \] (16)

where in the last step we have used that the argument of the delta-distribution can only be zero if \( s = 0 \). Comparing this equation with Eq. (11) one finds a Dyson expansion for the Green’s function of the perturbed system
\[
G_{p}^{r,a}(t_1, t_2, x_1, x_2, t'_1, t'_2) = G_{0}^{r,a}(t_1, t_2, x_1, x_2, t'_1, t'_2) +
\]
\[
\frac{1}{\hbar} \int_{Vx} \int_{-\infty}^{\infty} dt \int_{-T/2}^{T/2} dt' G_{p}^{r,a}(t_1, t, x_1, x, t'_1, t') V(x, t, t') G_{0}^{r,a}(t, t_2, x, x_2, t', t'_2) . \] (17)

If one assumes that the potential depends only on the periodic time component
\[ V(x, t, t') = V(x, t') \quad \Leftrightarrow \quad H_{F}(x, t, t') = H_{F}(x, t') \] (18)

the Green’s function depends only on the difference \( t_1 - t_2 \),
\[ G_{p}^{r,a}(t_1, t_2, x_1, x_2, t'_1, t'_2) = G_{p}^{r,a}(t_1 - t_2, x_1, x_2, t'_1, t'_2) . \] (19)

Applying Fourier transform on Eq. (17) with respect to \( t_1 - t_2 \), one yields in energy space
\[
G_{p}^{r,a}(\varepsilon, x_1, x_2, t'_1, t'_2) = G_{0}^{r,a}(\varepsilon, x_1, x_2, t'_1, t'_2) +
\]
\[
\frac{1}{\hbar} \int_{Vx} \int_{-T/2}^{T/2} dt' G_{p}^{r,a}(\varepsilon, x_1, x, t'_1, t') V(x, t') G_{0}^{r,a}(\varepsilon, x, x_2, t', t'_2) , \] (20)
where the explicit form of the bare Green’s function is given by

$$G_{r,a}^0(\varepsilon, x_1, x_2, t_1', t_2') = \frac{1}{T} \sum_{r=-\infty}^{\infty} \sum_{\alpha} \frac{u_{\alpha}(x_1, t_1')(u_{\alpha}(x_2, t_2'))^*}{\frac{1}{\hbar} \varepsilon - \frac{1}{\hbar} \varepsilon_{\alpha} - r \Omega \pm i 0} e^{ir\Omega(t_1'-t_2')}$$

(21)

$$= \frac{1}{T} \sum_{n,n'} G_{r,a}^0(\varepsilon, x_1, x_2, n, n') e^{-in\Omega t'_1} e^{in\Omega t'_2}.$$  

(22)

The Fourier coefficients are given by

$$G_{r,a}^0(\varepsilon, x_1, x_2, n, n') = \sum_{r=-\infty}^{\infty} \sum_{\alpha} \frac{u_{\alpha}^n(x_1)(u_{\alpha}^{n'}(x_2))^*}{\frac{1}{\hbar} \varepsilon - \frac{1}{\hbar} \varepsilon_{\alpha} - r \Omega \pm i 0}$$

(23)

where we used the shortened notation

$$u_{\alpha}(x_1, t_1') \equiv \langle x_1 | u_{\alpha}(t_1') \rangle, \quad u_{\alpha}^n(x_1) \equiv \langle x_1 | u_{\alpha}^n \rangle.$$  

(24)

Since we required the potential to be periodic in the second time argument it can be expanded in a Fourier series,

$$V(x, t') = \sum_{n=-\infty}^{\infty} V_n(x) e^{-in\Omega t'}.$$  

(25)

This allows to rewrite Eq. (20) and perform the remaining time integration,

$$G_{r,a}^p(\varepsilon, x_1, x_2, n, n') = G_{r,a}^r(\varepsilon, x_1, x_2, n, n') + 
\frac{1}{\hbar} \int_{V_k} dx \sum_{n_1,n_2} G_{r,a}^p(\varepsilon, x_1, x, n, n_1) V_{n_1-n_2}(x) G_{r,a}^0(\varepsilon, x, x_2, n_2, n')$$

(26)

\underline{*-FORMALISM*}

\underline{Separating the Periodic from the Aperiodic Time Dependence}

In the \textit{t-t’}-formalism one starts from the Floquet states $|\psi_{\alpha}(t)\rangle = \exp\left(-i\frac{\varepsilon_{\alpha} t}{\hbar}\right) |u_{\alpha}(t)\rangle$ but formally discriminates the time dependence of the exponential from periodic time dependence as

$$|\psi_{\alpha}(t', t)\rangle = e^{-i\varepsilon_{\alpha} t} |u_{\alpha}(t')\rangle$$

(27)

where obviously

$$|\psi_{\alpha}(t, t)\rangle = |\psi_{\alpha}(t)\rangle.$$  

(28)
The advantage of this artifice lies in the fact that the evolution of the states as a function of $t$ is governed by the operator

$$U_F(t', t) = e^{-\frac{i}{\hbar}H_F(t')t}, \quad (29)$$

i.e.

$$|\psi_\alpha(t', t)\rangle = U_F(t', t - t_0)|\psi_\alpha(t', t_0)\rangle = e^{-\frac{i}{\hbar}\epsilon_\alpha t_0}e^{-\frac{i}{\hbar}H_F(t' - t_0)}|u_\alpha(t')\rangle = e^{-\frac{i}{\hbar}\epsilon_\alpha t}|u_\alpha(t')\rangle, \quad (30)$$

which avoids any time ordering.

On the space of all states depending periodically with period $T$ on a parameter $t'$ having dimension of time, we define the scalar product

$$(\varphi|\chi) = \frac{1}{T} \int_0^T dt' \langle \varphi(t')|\chi(t')\rangle = \frac{1}{T} \int_0^T dt' \langle \varphi|t'\rangle\langle t'|\chi\rangle \quad (32)$$

which differs from the scalar product introduced by Sambe[1]

$$\langle\langle \varphi(t)|\chi(t)\rangle\rangle = \frac{1}{T} \int_0^T dt \langle \varphi(t)|\chi(t)\rangle \quad (33)$$

by a factor $1/T$. The notation

$$\langle t'|\psi \rangle := |\psi(t')\rangle \quad (34)$$

suggests to consider $t'$ as a coordinate rather than a time parameter. The corresponding operator $\hat{t}'$ can be defined to act multiplicatively on the above wave functions,

$$\hat{t}'|\psi(t')\rangle = \langle t'|\hat{t}'|\psi\rangle = t'|\psi(t')\rangle, \quad (35)$$

and the canonically conjugate operator is

$$\hat{w} := -i\hbar\partial_{t'} = H_F(t') - h(t') \quad \Rightarrow \quad [\hat{w}, \hat{t}'] = -i\hbar \quad (36)$$

with a complete system of orthonormalized periodic eigenfunctions

$$\langle t'|l\rangle = e^{-\frac{i}{\hbar}\Omega t'}, \quad \hat{w}|l\rangle = i\hbar\Omega|l\rangle, \quad (k|l) = \delta_{kl}, \quad \text{with} \quad k,l \in \mathbb{Z}, \quad (37)$$

$$\sum_{l=-\infty}^{\infty} \langle t'_1|l\rangle\langle l|t'_2\rangle = T \sum_{s=-\infty}^{\infty} \delta(t'_1 - t'_2 + sT) = \langle t'_1|t'_2\rangle. \quad (38)$$
In obtaining the completeness relation we have taken into account the Fourier expansion of
the Dirac comb,

\[ \sum_{r=-\infty}^{\infty} e^{i r \Omega t} = T \sum_{s=-\infty}^{\infty} \delta(t + sT) . \]  

(39)

Switching between the two pertaining representations amounts, up to signs and prefactors,
in the usual Fourier expansion,

\[ \langle l | \psi \rangle = \frac{1}{T} \int_0^T dt' \langle l | t' \rangle \langle t' | \psi \rangle = \frac{1}{T} \int_0^T dt' e^{i \Omega t'} \langle t' | \psi \rangle \]  

(40)

\[ \Leftrightarrow \langle t' | \psi \rangle = \sum_{l=-\infty}^{\infty} \langle t' | l \rangle \langle l | \psi \rangle = \sum_{l=-\infty}^{\infty} e^{-i \Omega t} \langle l | \psi \rangle . \]  

(41)

Finally, the analogs of the wave functions \( \psi_\alpha(q, t) = \langle q | \psi_\alpha(t) \rangle \) read in the \( t-t' \)-formalism

\[ \psi_\alpha(q, t', t) = \langle q | \psi_\alpha(t', t) \rangle = \langle q, t' | \psi_\alpha(t) \rangle . \]  

(42)

**Field Operators and One-Particle Green’s Functions**

Generalizing the states (27) we define

\[ | \phi_\alpha^r(t', t) \rangle = e^{i r \Omega(t'-t)} | \psi_\alpha(t', t) \rangle = e^{-\frac{i}{\hbar}(\epsilon_\alpha + r \hbar \Omega)t} e^{i r \Omega t'} | u_\alpha(t') \rangle \]  

(43)

with

\[ \phi_\alpha^r(q, t', t) = \langle q | \phi_\alpha^r(t', t) \rangle = \langle q, t' | \phi_\alpha^r(t) \rangle \]  

(44)

and the simple properties

\[ | \phi_\alpha^r(t', t) \rangle = U_F(t', t - t_0) | \phi_\alpha^r(t', t_0) \rangle , \]  

(45)

\[ \langle \phi_\alpha^\beta(t) | \phi_\alpha^\beta(t) \rangle = \delta_{\alpha \beta} \delta_{rs} , \]  

(46)

\[ \sum_\alpha \sum_{r=-\infty}^{\infty} | \phi_\alpha^r(t_1', t) \rangle \langle \phi_\alpha^r(t_2', t) | = 1 T \sum_{s=-\infty}^{\infty} \delta(t_1' - t_2' + sT) . \]  

(47)

In second quantization this allows to define a system of creation and annihilation operators
\( b_{\alpha r}(t) \), \( b_{\alpha r}^\dagger(t) \) with

\[ | \phi_\alpha^r(t) \rangle = b_{\alpha r}^\dagger(t) | 0 \rangle , \quad b_{\alpha r}(t) | 0 \rangle = 0 \]  

(48)

and

\[ [b_{\alpha r}(t), b_{\beta s}^\dagger(t)]_\pm = \delta_{\alpha \beta} \delta_{rs} , \quad [b_{\alpha r}(t), b_{\beta s}(t)]_\pm = [b_{\alpha r}^\dagger(t), b_{\beta s}^\dagger(t)]_\pm = 0 . \]  

(49)
Field operators can be constructed as
\[ \Phi(q, t', t) = \sum_{\alpha} \sum_{r=\pm \infty} \phi_{\alpha}(q, t', t)b_{\alpha r}(t) \] (50)

fulfilling
\[ [\Phi(q_1, t_1', t), \Phi^\dagger(q_1, t_2', t)]_\pm = \delta(q_1 - q_2)T \sum_{s=\pm \infty} \delta(t_1' - t_2' + sT) \] (51)

with again all other (anti-)commutators at equal times \( t \) being zero, and
\[ \langle q_2, t_2'|\Phi^\dagger(q_1, t_1', t)|0 \rangle = \delta(q_1 - q_2)T \sum_{s=\pm \infty} \delta(t_1' - t_2' + sT). \] (52)

The Floquet Hamiltonian \( H_F \) can be formulated as
\[ H_F(t) = \sum_{\alpha} \sum_{r=\pm \infty} (\epsilon_{\alpha} + r\hbar\Omega)b_{\alpha r}^\dagger(t)b_{\alpha r}(t) \] (53)

and is neither bounded form below nor from above. Going over to the Heisenberg picture,
\[ \Phi_H(q, t', t) = U_F^\dagger(t', t)\Phi(q, t', t)U_F(t', t) = \sum_{\alpha} \sum_{r=\pm \infty} \phi_{\alpha}(q, t', t)b_{\alpha r}(0), \] (54)

we yield the retarded/advanced one-particle Green’s function
\[ \mathcal{G}^{r/a}(q_1, t_1', t_1, q_2, t_2', t_2) \]
\[ = \mp i\Theta(\pm(t_1 - t_2)) \frac{1}{T} \left\langle \left[ \Phi_H(q_1, t_1', t_1), \Phi_H^\dagger(q_2, t_2', t_2) \right] \right\rangle \] (55)

\[ = \mp i\Theta(\pm(t_1 - t_2)) \frac{1}{T} \sum_{r=\pm \infty} \sum_{\alpha} \phi_{\alpha r}(q_1, t_1', t_1)(\phi_{\alpha r}^\dagger(q_2, t_2', t_2))^* \] (56)

\[ = \mp i\Theta(\pm(t_1 - t_2)) \frac{1}{T} \sum_{r=\pm \infty} \sum_{\alpha} \left[ e^{-\frac{i}{\hbar}(\epsilon_{\alpha} + r\hbar\Omega)(t_1 - t_2)} \right] \cdot \langle q_1|u_{\alpha}(t_1')\rangle\langle u_{\alpha}(t_2')|q_2\rangle e^{ir\Omega(t_1' - t_2')} \] (57)

or, formulated as a Green’s operator,
\[ \hat{\mathcal{G}}^{r/a}(t_1', t_1, t_2', t_2) \]
\[ = \mp i\Theta(\pm(t_1 - t_2)) \frac{1}{T} \sum_{r=\pm \infty} \sum_{\alpha} \left| \phi_{\alpha r}^\dagger(t_1', t_1)\phi_{\alpha r}(t_2', t_2) \right| \] (58)

\[ = \mp i\Theta(\pm(t_1 - t_2)) \frac{1}{T} \sum_{r=\pm \infty} \sum_{\alpha} \left[ e^{-\frac{i}{\hbar}(\epsilon_{\alpha} + r\hbar\Omega)(t_1 - t_2)} \right] \cdot \langle u_{\alpha}(t_1')\rangle\langle u_{\alpha}(t_2')|e^{ir\Omega(t_1' - t_2')} \] (59)
These quantities have the significant property
\[
\left( i\partial_t - \frac{1}{\hbar} H_F(t_1) \right) \hat{G}^{r/a}(t_1, t_2, t'_1, t'_2) = \delta(t_1 - t_2) \sum_{\alpha} |u_{\alpha}(t'_1)\rangle \langle u_{\alpha}(t'_2)| \frac{1}{T} \sum_{r=-\infty}^{\infty} e^{i\omega(t'_1-t'_2)}
\]
\[
= \delta(t_1 - t_2) \mathbb{1} \sum_{s=-\infty}^{\infty} \delta(t'_1 - t'_2 + sT)
\]
(60)
where we have used Eqs. (39), and the completeness relation of the Floquet functions \(|u_{\alpha}(t)\rangle\).

As the expressions (58), (59) depend only on the difference \(t_1 - t_2\) and are periodic in \(t'_1, t'_2\)
we can go over to Fourier components as
\[
\hat{G}^{r/a}(\omega, n_1, n_2) = \int_{-\infty}^{\infty} dt e^{i\omega t} \sum_{n_1, n_2} e^{-i n_1 \Omega t'_1 + i n_2 \Omega t'_2} \hat{G}^{r/a}(t_1, t_2, 0)
\]
(61)
\[
= \frac{1}{T} \sum_{r=-\infty}^{\infty} \sum_{\alpha} \frac{\langle u_{\alpha}^{n_1+r} | u_{\alpha}^{n_2+r} \rangle}{\omega - \frac{\hbar i}{\hbar}(\varepsilon_\alpha + r\hbar \Omega) \pm i0^+}
\]
(62)
where the last line follows from
\[
\Theta(\pm t) = \pm i \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega \pm i0^+}.
\]
(63)
Moreover, with the spectral density
\[
A(q_1, t'_1, t_1, q_2, t'_2, t_2) = \frac{1}{2\pi T} \left\langle \left[ \Phi_H(q_1, t'_1, t_1), \Phi_H^\dagger(q_2, t'_2, t_2) \right] \right\rangle
\]
(64)
\[
= \frac{1}{2\pi T} \sum_{r=-\infty}^{\infty} \sum_{\alpha} \left[ e^{-\frac{\hbar}{\hbar}(\varepsilon_\alpha + r\hbar \Omega)(t_1-t_2)} \right.
\]
\[
\left. \cdot \langle q_1 | u_{\alpha}(t'_1) \rangle \langle u_{\alpha}(t'_2) | q_2 \rangle e^{i\omega(t'_1-t'_2)} \right],
\]
(65)
having Fourier components
\[
A(\omega, q_1, t'_1, q_2, t'_2) = \frac{1}{T} \sum_{r=-\infty}^{\infty} \sum_{\alpha} \left[ \delta \left( \omega - (\varepsilon_\alpha + r\hbar \Omega) \right) \right.
\]
\[
\left. \cdot \langle q_1 | u_{\alpha}(t'_1) \rangle \langle u_{\alpha}(t'_2) | q_2 \rangle e^{i\omega(t'_1-t'_2)} \right],
\]
(66)
we obtain the familiar Lehmann representation of the Green’s function,
\[
\hat{G}^{r/a}(\omega, q_1, t'_1, q_2, t'_2) = \int_{-\infty}^{\infty} d\omega' \frac{A(\omega, q_1, t'_1, q_2, t'_2)}{\omega - \omega' \pm i0^+}.
\]
(67)
In summary, when treating \(t'\) not as a time parameter but rather as a state coordinate, the
remaining time evolution in \(t\) is governed by the Floquet Hamiltonian being independent
of \(t\). Thus, we are left with an effectively time-independent Hamiltonian, and many formal
manipulations known for such a situation work just in the same way. Note, however, that
(i) the physical case is still requires \(t = t'\), and (ii) the Floquet Hamiltonian (53) fails to be
bounded from below.
The aim of this section is to relate the self-energy in first order Born approximation (1BA) to the scattering time given by Fermi’s golden rule \([2–6]\). This requires Fermi’s golden rule to be applicable to \(tt'\)-Floquet states. To do so, let us first recall Fermi’s golden rule for Floquet states.

**Floquet Fermi’s Golden Rule**

A generalization of Fermi’s golden rule to time periodic Hamiltonians, i.e. the Floquet Fermi golden rule, was already derived by Kitagawa \textit{et. al.} in Ref. \([7]\). However, a detailed derivation and discussion of the “Scattering theory for Floquet-Bloch states”, is given in Ref. \([8]\). The Floquet Fermi’s golden rule was used by O.V. Kibis in Ref. \([9]\) in order to explain the suppression of backscattering of conduction electrons in presence of a high-frequency electric field. In regards to Fermi’s golden rule for the \(tt'\)-Floquet states, the derivation of the Floquet Fermi’s golden rule is presented here in detail. It is assumed that the solution of the time-dependent Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} |\psi_\alpha(t)\rangle = H(t) |\psi_\alpha(t)\rangle
\]

and the corresponding time evolution operator \(U_0(t,t_0)\) are known. In presence of a time-dependent perturbation \(V(t)\) the Schrödinger equation becomes

\[
i\hbar \frac{\partial}{\partial t} |\Psi_\alpha(t)\rangle = \left[H(t) + V(t)\right] |\Psi_\alpha(t)\rangle.
\]

The potential \(V(t)\) is switched on at a reference time \(t_0\) such that the solutions of the Schrödinger equation coincide for times \(t \leq t_0\)

\[
|\psi_\alpha(t)\rangle = |\Psi_\alpha(t)\rangle \quad \text{for} \quad t \leq t_0.
\]

At times \(t \leq t_0\) the particle is assumed to be in an eigenstate of the unperturbed Hamiltonian. Standard perturbation theory leads to the transition amplitude

\[
\langle \psi_\beta(t) | \Psi_\alpha(t) \rangle = \delta_{\alpha\beta} + \frac{1}{i\hbar} \int_{t_0}^{t} dt' \langle \psi_\beta(t') | V(t') | \psi_\alpha(t') \rangle
\]
up to first order in the potential. Without loss of generality $t_0$ can be set to zero and for $\alpha \neq \beta$ the first nontrivial order of Eq. (71) is given by

$$a_{\alpha\beta}(t) = -\frac{i}{\hbar} \int_0^t dt' \langle \psi_\beta(t') | V(t') | \psi_\alpha(t') \rangle .$$

(72)

This formula, the Floquet Fermi’s golden rule, is equal to Eq. (10) of Ref. [9]. To proceed further, scattering from a Floquet state into a state with constant energy

$$|\psi_\alpha(\varepsilon, t)\rangle = e^{-\frac{i}{\hbar} \varepsilon t} |u_\alpha(t)\rangle$$

(73)

is considered. The quasienergy $\varepsilon$ is independent of the quantum number. Hence, this state is not an eigenstate of the Hamiltonian, nevertheless it fulfills

$$\langle \psi_\alpha(t) | \psi_\beta(\varepsilon, t) \rangle = \delta_{\alpha\beta} e^{\frac{i}{\hbar} (\varepsilon_\alpha - \varepsilon) t} .$$

(74)

Consequently, Eq. (72) remains valid if the final state is $|\psi_\alpha(\varepsilon, t)\rangle$. Now, consider a scattering event from a Floquet state $\psi_\alpha(k', t)$ into a state with constant energy $\varepsilon$,

$$\psi_\alpha(k', t) = e^{-\frac{i}{\hbar} \varepsilon_\alpha(k') t} u_\alpha(k', t) \rightarrow e^{-\frac{i}{\hbar} \varepsilon t} u_\beta(k, t) .$$

(75)

If the perturbation $V(t)$ is time-independent Eq. (72) becomes

$$a_{\alpha\beta}(k, k', t) = -\frac{i}{\hbar} V_{kk'} \sum_{nn'=\cdots}^\infty \int_0^t dt' e^{\frac{i}{\hbar} (\varepsilon_\alpha(k') - (n-n')\hbar\Omega) t'} (u_{\beta n'}(k))^* u_{\alpha n}(k')$$

(76)

with $V_{kk'} = \langle \varphi_{k,r} | V(r) | \varphi_{k',r} \rangle$ where $\varphi_{k,r} = \exp(-ik \cdot r)/\sqrt{Q}$. Shifting $t'$ by $-t/2$ yields the probability density

$$|a_{\alpha\beta}(k, k', t)|^2 = \frac{V_{kk'}^2}{\hbar^2} \left| \sum_{nn'=\cdots}^\infty e^{\frac{i}{\hbar} (\varepsilon_\alpha(k') - (n-n')\hbar\Omega) t'} (u_{\beta n'}(k))^* u_{\alpha n}(k') \right|^2 .$$

(77)

In the long time limit $t \rightarrow \infty$ this simplifies to

$$|a_{\alpha\beta}(k, k', t)|^2 = 4\pi^2 V_{kk'}^2 \left| \sum_{nn'=\cdots}^\infty (u_{\beta n'}(k))^* u_{\alpha n}(k') \delta(\varepsilon_\alpha(k') - (n-n')\hbar\Omega) \right|^2 .$$

(78)
The quasienergies $\varepsilon$ and $\varepsilon_\alpha(k)$ are chosen to be in the central Floquet zone such that

$$\forall k : |\varepsilon - \varepsilon_\alpha(k)| \leq \hbar \Omega,$$

$$\delta(\varepsilon - \varepsilon_\alpha - n\hbar\Omega) \delta(\varepsilon - \varepsilon_\alpha - m\hbar\Omega) = \delta^2(\varepsilon - \varepsilon_\alpha(k) - n\hbar\Omega) \delta_{nm}.$$  

Hence, Eq. (78) becomes

$$|a_{\alpha\beta}(k, k', t)|^2 = 4\pi^2 V_{kk'}^2 \sum_{n=-\infty}^{\infty} c_{\beta\alpha}^{-n}(k, k')(c_{\beta\alpha}^{-n}(k, k'))^* \delta^2(\varepsilon - \varepsilon_\alpha(k') - n\hbar\Omega).$$

with

$$c_{\alpha\beta}^{n}(k, k') \equiv \sum_{m=-\infty}^{\infty} (u_m^{m+n}(k))^* u_m^{m}(k').$$

The square of the delta-distribution can be rewritten as [9]

$$\delta^2(\varepsilon) = \delta(\varepsilon)\delta(0) = \frac{\delta(\varepsilon)}{2\pi\hbar} \lim_{t \to \infty} \int_{-t/2}^{t/2} dt' e^{\pm i0t'} = \frac{\delta(\varepsilon)t}{2\pi\hbar}.$$  

The transition probability is then

$$\Gamma_{\alpha\beta}(k, k') \equiv \frac{d|a_{\alpha\beta}(k, k', t)|^2}{dt}$$

$$= \frac{2\pi}{\hbar} V_{kk'}^2 \sum_{n=-\infty}^{\infty} c_{\beta\alpha}^{-n}(k, k')(c_{\beta\alpha}^{-n}(k, k'))^* \delta(\varepsilon - \varepsilon_\alpha(k') - n\hbar\Omega).$$

The delta-distribution can only have support if $n = 0$. Performing an impurity average according to the main article, leads to $\langle V_{kk'}^2 \rangle_{\text{imp}} = V_{\text{imp}}$ such that

$$\langle \Gamma_{\alpha\beta}(k, k') \rangle_{\text{imp}} = \langle \Gamma_{\alpha\beta}(k, k') \rangle_{\text{imp}}$$

$$= \frac{2\pi}{\hbar} V_{\text{imp}} |c_{\beta\alpha}^{0}(k, k')|^2 \delta(\varepsilon - \varepsilon_\alpha(k')).$$

The scattering time is then governed by the sum over all initial states and the sum over all momenta

$$\frac{1}{\tau_{\beta}(\varepsilon, k)} = \frac{1}{V_{kk'}} \sum_{k'} \sum_{\alpha} \langle \Gamma_{\alpha\beta}(k, k') \rangle_{\text{imp}}$$

$$= \frac{2\pi}{\hbar} V_{\text{imp}} \frac{1}{V_{kk'}} \sum_{k'} \sum_{\alpha} |c_{\beta\alpha}^{0}(k, k')|^2 \delta(\varepsilon - \varepsilon_\alpha(k')).$$

The last equation is the Floquet Fermi’s golden rule.
Fermi’s Golden Rule for $tt'$-Floquet States

In the following the steps of the derivation of the Fermi’s Golden Rule for $tt'$-Floquet states are similar to the one applied in Refs. [5, 9]. The difference lies in the use of the $tt'$-Floquet states (see Eq. (7) in the main article) instead of the Floquet states. A $tt'$-state fulfills

$$i\hbar \frac{\partial}{\partial t} |\psi_{\ell}^{\alpha}(t,t')\rangle = H_F(t')|\psi_{\ell}^{\alpha}(t,t')\rangle .$$

(90)

The corresponding time-evolution operator fulfilling this Schrödinger equation is given by

$$U_0(t,t_0,t') = e^{-\frac{i}{\hbar}H_F(t') \cdot (t-t_0)} .$$

(91)

If a perturbation is switched on at time $t_0$ the Schrödinger equation becomes

$$i\hbar \frac{\partial}{\partial t} |\Psi_{\ell}^{\alpha}(t,t')\rangle = [H_F(t') + V(t,t')]|\Psi_{\ell}^{\alpha}(t,t')\rangle$$

(92)

with the boundary condition $|\psi_{\ell}^{\alpha}(t,t')\rangle = |\Psi_{\ell}^{\alpha}(t,t')\rangle$ for $t \leq t_0$. Changing into the interaction picture with

$$|\Psi_{\ell}^{\alpha}(t,t')\rangle_I = U_0^\dagger(t_0,t_0,t')|\Psi_{\ell}^{\alpha}(t,t')\rangle ,$$

$$V_I(t,t') = U_0^\dagger(t_0,t_0,t')V(t,t')U_0(t,t_0,t'),$$

(93) (94)

one finds up to first order in the potential $V$

$$|\Psi_{\ell}^{\alpha}(t,t')\rangle_I \approx |\psi_{\ell}^{\alpha}(t_0,t')\rangle + \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 V_I(t_1,t')|\psi_{\ell}^{\alpha}(t_0,t')\rangle .$$

(95)

$$\langle \psi_{\beta}^{\ell'}(t,t'')|\Psi_{\alpha}^{\ell}(t,t')\rangle = \langle \psi_{\beta}^{\ell'}(t,t'')|\psi_{\alpha}^{\ell}(t,t')\rangle$$

$$+ \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \langle \psi_{\beta}^{\ell'}(t_1,t'')|V(t_1,t')|\psi_{\alpha}^{\ell}(t_1,t')\rangle .$$

(96) (97)

In the next step let us consider the matrix element where the $tt'$-Floquet states have the same time dependence but different Floquet indices,

$$a_{\ell\beta}(t,t') = \sum_{n=-\infty}^{\infty} a_{\ell\beta}(t,n) e^{in\Omega t'}$$

$$= \langle \psi_{\beta}^{\ell}(t,t')|\psi_{\alpha}^{\ell}(t,t')\rangle$$

$$\approx \delta_{\alpha\beta} e^{\Omega(t-t')(t'-t)} + \frac{1}{i\hbar} \int_{t_0}^{t} dt_1 \langle \psi_{\beta}^{\ell}(t_1,t')|V(t_1,t')|\psi_{\alpha}^{\ell}(t_1,t')\rangle .$$

(98) (99) (100)
The Fourier coefficients for a perturbation, which is time-independent in the second time argument, are governed by

\[ a_{\alpha\beta}^{\ell\ell'}(t, n) = \frac{1}{T} \int_0^T dt' a_{\alpha\beta}^{\ell\ell'}(t, t') e^{i\Omega t'} \]

\[ = \delta_{\alpha\beta} \delta_{n, -\ell'} e^{-i\Omega t} + \frac{1}{i\hbar} \int_{t_0}^t dt_1 e^{i\left(\varepsilon_{\alpha\beta} + (\ell - \ell')\hbar\Omega\right)t_1} \sum_{m=-\infty}^{\infty} \langle u_{\alpha}^{m+\ell+n} | V(t_1) | u_{\beta}^{m+\ell'} \rangle . \]

We see that the transition amplitude is only a function of the difference of the Floquet indices, \( a_{\alpha\beta}(t, t') = a_{\alpha\beta}^{(t-t')} \). Analogue to the last section, \( t_0 \) can be set to zero and for \( \alpha \neq \beta \) Eq. (100) simplifies to

\[ a_{\alpha\beta}^{\ell\ell'}(t, t') = -\frac{i}{\hbar} \int_0^t dt_1 \langle \psi_{\beta}^\ell(t_1, t') | V(t_1, t') | \psi_{\alpha}^\ell(t_1, t') \rangle . \]

Now, let us assume a scattering event from a \( tt'\)-Floquet state into another \( tt'\)-Floquet state with constant quasienergy, given by

\[ |\psi_{\alpha}^\ell(\varepsilon, t, t')\rangle \equiv e^{-\frac{i}{\hbar}(\varepsilon + \ell\hbar\Omega)t} u_{\alpha}(t, t') e^{i\ell\Omega t'} . \]

The quasienergy is independent of the quantum number. This state is not an eigenstate of the Hamiltonian, nevertheless it fulfills

\[ \langle \psi_{\alpha}^\ell(t, t') | \psi_{\beta}^\ell(\varepsilon, t, t') \rangle = \delta_{\alpha\beta} e^{-\frac{i}{\hbar}(\varepsilon - \varepsilon_{\alpha} + (t' - t)\hbar\Omega)t} e^{i\ell\Omega(t' - t)} . \]

Hence, Eq. (103) remains valid if the final state is of the same form as in Eq. (104). Consider now a scattering event from a \( tt'\)-Floquet state \( \psi_{\alpha}^\ell(k', t, t') \) into a state with constant energy \( \varepsilon \),

\[ \psi_{\alpha}^\ell(k', t, t') = e^{-\frac{i}{\hbar}(\varepsilon_{\alpha}(k') + \ell\hbar\Omega)t} u_{\alpha}(k', t') e^{i\ell\Omega t'} \sim e^{-\frac{i}{\hbar}(\varepsilon + \ell'\hbar\Omega)t} u_{\beta}(k, t') e^{i\ell'\Omega t'} . \]
The Fourier coefficient of the matrix element for a scattering as in Eq. (106) is for a time-independent perturbation given by

\[
a_{\alpha\beta}^{n\ell}(k, k', t, n) = -i \frac{V_{kk'}}{\hbar} \int_0^t dt' e^{i(\varepsilon - \varepsilon_n(k') - (\ell - \ell')\hbar t')}
\times \sum_{m=-\infty}^{\infty} (u_{\ell\beta}^m + \ell + n(k)) u_{\alpha\beta}^m (k') \\
= -i \frac{V_{kk'}}{\hbar} \int_0^t dt' e^{i(\varepsilon - \varepsilon_n(k') - (\ell - \ell')\hbar t')} c_{\ell\alpha}^{\ell - \ell' + n}(k, k') .
\]

(107)

In the last step the definition given in Eq. (82) has been used. This allows for a definition of the transition probability matrix

\[
\left( A_{\alpha\beta}^{\ell j j'}(k, k', t) \right)_{n,n'} \equiv \sum_{\gamma} a_{\gamma\alpha}^{\ell j}(k, k', t, n) (a_{\gamma\beta}^{j j'}(k, k', t, n'))^* .
\]

(109)

Equivalently to Eq. (77), in the limit \( t \to \infty \) the transition probability matrix becomes

\[
\left( A_{\alpha\beta}^{\ell j j'}(k, k', t) \right)_{n,n'} = 4\pi^2 V_{kk'}^2 \sum_{\gamma} \xi_{\ell-\ell'+n}(k, k') \delta(\varepsilon - \varepsilon_{\gamma}(k') - (\ell - \ell')\hbar) \\
\times (c_{\beta\gamma}^{j - j + n}(k, k'))^* \delta(\varepsilon - \varepsilon_{\gamma}(k') - (j - j')\hbar) .
\]

(110)

Since the quasienergies are always defined to be in the central Floquet zone, c.f. Eq. (80), the probability matrix simplifies to

\[
\left( A_{\alpha\beta}^{\ell j j'}(k, k', t) \right)_{n,n'} = 4\pi^2 V_{kk'}^2 \sum_{\gamma} \xi_n^{\ell}(k, k') (c_{\beta\gamma}^{j j'}(k, k'))^* \delta(\varepsilon - \varepsilon_{\gamma}(k')) .
\]

(111)

Using Eq. (83) and performing the time derivative of each matrix element yields

\[
\Gamma_{n,n'}^{\alpha\beta}(k, k') \equiv \frac{d \left( A_{\alpha\beta}^{\ell j j'}(k, k', t) \right)_{n,n'}}{dt} \\
= \frac{2\pi}{\hbar} V_{kk'}^2 \sum_{\gamma} \xi_n^{\ell}(k, k') (c_{\beta\gamma}^{j j'}(k, k'))^* \delta(\varepsilon - \varepsilon_{\gamma}(k')) .
\]

(113)

Finally, one can perform an impurity average and identify \( \langle V_{kk'}^2 \rangle_{\text{imp}} = V_{\text{imp}} \). Summing the rate over all momenta one gets the inverse scattering time matrix,

\[
\left( \frac{1}{\tau(\varepsilon, k)} \right)^n_{\alpha\beta} = \frac{1}{V_{kk'}} \sum_{k'} \langle \Gamma_{n,n'}^{\alpha\beta}(k, k') \rangle_{\text{imp}} \\
= \frac{2\pi}{\hbar} V_{\text{imp}} \frac{1}{V_{kk'}} \sum_{k'} \sum_{\gamma} \xi_n^{\ell}(k, k') (c_{\beta\gamma}^{j j'}(k, k'))^* \delta(\varepsilon - \varepsilon_{\gamma}(k')) \\
= i \left( T^\dagger(k) \left( \Sigma_{\text{IBA}}^{\alpha\beta}(\varepsilon, k) - \Sigma_{\text{IIBA}}^{\alpha\beta}(\varepsilon, k) \right) T(k) \right)^n_{\alpha\beta} .
\]

(116)
This expression is equal to the result derived from the Dyson series for the Floquet Green’s function. Remarkably, the central entry of the scattering time for the $tt'$ Floquet states is equal to the Floquet Fermi’s golden rule given in Eq. (88) and Refs. [7, 8].

**DEFINITION OF THE FLOQUET ZONE**

In the following we would like focus on the a parabolic spectrum and describe the appropriate choice of the function $\lambda$, which defines the boundary for the quasienergy $\varepsilon_\alpha$,

$$\forall_\alpha : \lambda - \frac{\hbar\Omega}{2} \leq \varepsilon_\alpha < \lambda + \frac{\hbar\Omega}{2} .$$

(117)

Since the spectrum is not bounded, one has to choose the Floquet zone as indicated in Fig. 1. This limits the validity of the calculation to the $\Omega\tau_0 \gg 1$ regime as will be clear in the following. However, this limitation is only a peculiarity of the unbounded spectrum: In the derivation of the main text we defined the quasienergies to fulfill

$$\forall_{k,k'} : |\varepsilon_\alpha(k) - \varepsilon_\beta(k')| < \hbar\Omega .$$

(118)

As a consequence, in a system with a single band the condition forces the band width to be smaller than $\hbar\Omega$. Obviously this cannot be fulfilled by the parabolic spectrum. In the later case, the momentum range where the quasi-Fermi energy is defined has to be truncated, as depicted with a red line in Fig. 1: $k_1$ and $k_2$, are functions of the driving frequency $\Omega$. For decreasing $\Omega$ the momenta $k_1$ and $k_2$ move closer together. If the momentum range

![Figure 1. The Floquet zones are chosen to wrap around the parabolas. The quasi-Fermi energy is only defined in a certain momentum range, i.e., $k \in [k_1, k_2)$, in the central Floquet zone.](image)
$k \in [k_1, k_2)$ is of the order of the broadening of the Green’s function, the truncation leads to an incorrect result for the conductivity. If $\Omega \tau_0 \gg 1$, $\tau_0$ being the scattering time of the undriven system, the broadening of the nonzero Floquet modes is small enough such that the leaking into the central Floquet zone is negligibly small, compare Fig. 2. If $\Omega \tau_0 \simeq 1$,

Figure 2. The peaks show the broadening of the Floquet bands caused by the scattering time. The blue shaded area is the central Floquet zone. If $\Omega \tau_0 \gg 1$, the leaking of the nonzero Floquet modes (red curves) into the central Floquet zone is negligibly small.

Figure 3. The peaks show the broadening of the Floquet bands caused by the scattering time. The blue shaded area is the central Floquet zone. If $\Omega \tau_0 \simeq 1$, the nonzero Floquet modes are leaking into the central Floquet zone. The red shaded area marks the contribution of the minus one Floquet band to the conductivity.

the nonzero modes give a significant contribution to the conductivity as depicted in Fig. 3. In a system which rigorously fulfills Eq. (118) this constitutes no limitation.
In this section we derive a fully analytic solution of the time-dependent Schrödinger equation for the quadratic lattice with time-periodic driving and compare it with the results for the parabolic dispersion. We define the two lattice vectors of the square lattice with a lattice constant \( a \) as

\[
\mathbf{a}_1 = a \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = a \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

We choose the vector potential as

\[
\mathbf{A} = \begin{pmatrix} A_x \sin(\Omega t) \\ A_y \cos(\Omega t) \end{pmatrix}
\]

which allows us to tune the polarization between linear, elliptic and circular for an appropriate choices of amplitudes \( A_i \). The time-dependent tight-binding Hamiltonian with hopping parameter \( g \) and a limitation to nearest neighbour hopping is this given by

\[
H(t) = -ge^{i\mathbf{k} \cdot \mathbf{a}_1}e^{i\frac{\mathbf{A}}{\hbar} \cdot \mathbf{a}_1} + e^{i\mathbf{k} \cdot \mathbf{a}_2}e^{i\frac{\mathbf{A}}{\hbar} \cdot \mathbf{a}_2} + \text{H.C.}.
\]

In the following we will make use of the identities

\[
\int dt e^{i\gamma \sin(\Omega t)} = \sum_{n=-\infty}^{\infty} J_n(\gamma) \int dt e^{in\Omega t} = \sum_{n \neq 0} \frac{J_n(\gamma)}{in\Omega} e^{in\Omega t} + J_0(\gamma)t,
\]

\[
\int dt e^{i\gamma \cos(\Omega t)} = 2 \sum_{n=1}^{\infty} \frac{inJ_n(\gamma)}{n\Omega} \sin(n\Omega t) + J_0(\gamma)t,
\]

which are based on the Jacobi-Anger expansion. In order to solve the time-dependent Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} \psi_k(t) = H(t)\psi_k(t)
\]

we choose the ansatz

\[
\psi_k(t) = e^{-\frac{i}{\hbar}F(t)} \quad \text{with} \quad F(t) = \int dt H(t).
\]

Integrating the Hamiltonian (121) yields

\[
F(t) = -2g \left[ J_0(\gamma_x) \cos(k_xa) + J_0(\gamma_y) \cos(k_ya) \right] t
\]

\[
- ge^{ik_xa} \sum_{n \neq 0} \frac{J_n(\gamma_x)}{in\Omega} e^{in\Omega t} - ge^{-ik_xa} \sum_{n \neq 0} \frac{J_n(\gamma_x)}{-in\Omega} e^{-in\Omega t}
\]

\[
- 2ge^{ik_ya} \sum_{n=1}^{\infty} \frac{inJ_n(\gamma_y)}{n\Omega} \sin(n\Omega t) - 2ge^{-ik_ya} \sum_{n=1}^{\infty} \frac{(-i)^nJ_n(\gamma_y)}{n\Omega} \sin(n\Omega t).
\]
The light parameters are defined by $\gamma_i = eA_i/\hbar$. The quasienergy is the non-oscillatory part of $F(t)$, thus

$$
\epsilon = -2g \left[ J_0(a\gamma_x) \cos(k_xa) + J_0(a\gamma_y) \cos(k_ya) \right].
$$

(127)

To make contact with the parabolic spectrum we expand around the $\Gamma$ point,

$$
\epsilon_T = -2g \left[ J_0(a\gamma_x) + J_0(a\gamma_y) \right] + \frac{\hbar^2 (J_0(a\gamma_x)k_x^2 + J_0(a\gamma_y)k_y^2)}{2m}
$$

$$
= -2g \left[ J_0(a\gamma_x) + J_0(a\gamma_y) \right] + \frac{\hbar^2 k_x^2}{2m_d(\gamma_x)} + \frac{\hbar^2 k_y^2}{2m_d(\gamma_y)}
$$

(128)

(129)

where the effective mass $m = \hbar^2/(2ga^2)$ of the undriven system got renormalized due to the driving, $m_d(\gamma_i) = m/J_0(a\gamma_i)$. It is worth noticing that if the light parameter $\gamma_i$ is close to a zero of the Bessel function the mass $m_d(\gamma_i)$ diverges and the expansion around the $\Gamma$ point is not applicable anymore. Moreover, for rather high intensities this shift of the parabola in Eq. (129) can significantly change the relative position to the Fermi energy. To go beyond a small renormalization of the static effective mass, i.e., if $a\gamma_i$ is close to the zero of $J_0$ or even larger, $a\gamma_i \gtrsim 2.4$, a realistic multiband model has to be applied.
[9] O. V. Kibis, *EPL (Europhysics Letters)* **107**, 57003 (2014).