Bounds of Multiplicative Character Sums with Shifted Primes

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Abstract

For integer $q$, let $\chi$ be a primitive multiplicative character (mod $q$). For integer $a$ coprime to $q$, we obtain a new bound for the sums

$$\sum_{n\leq N} \Lambda(n)\chi(n+a),$$

where $\Lambda(n)$ is the von Mangoldt function. This bound improves and extends the range of a result of Friedlander, Gong and Shparlinski.

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1 Introduction

Let $q$ be an arbitrary positive integer and let $\chi$ be a primitive non-principal multiplicative character (mod $q$). Our goal is to estimate character sums of the form

$$S_a(q;N) = \sum_{n\leq N} \Lambda(n)\chi(n+a),$$  \hspace{1cm} (1)

where $a$ is an integer relatively prime to $q$ and as usual,

$$\Lambda(n) = \begin{cases} 
\log p, & \text{if } n \text{ is a power of a prime } p, \\
0, & \text{otherwise}, 
\end{cases}$$

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is the von Mangoldt function. For prime modulus $q$, Karatsuba [8] has given a nontrivial estimate of the sums $S_a(q; N)$ in the range $N > q^{1/2+\varepsilon}$. Recently, much more general sums over primes have been considered by Fouvry, Kowalski and Michel [4]. A special case of their general result (see [4, Corollary 1.12]) gives nontrivial bounds for character sums to prime modulus $q$, with a very general class of rational functions over primes, which is nontrivial provided $N > q^{3/4+\varepsilon}$. Rakhmonov [9, 10] has shown that nontrivial cancellations in the sums $S_a(q; N)$ also occur in the more difficult case of general modulus $q$, but only in the narrower range $N > q^{1+\varepsilon}$. This range has been extended in [5] to $N > q^{8/9+\varepsilon}$, where the bound

$$|S_a(q; N)| \leq (N^{7/8}q^{1/9} + N^{33/32}q^{-1/18})q^{o(1)},$$

is given for $N \leq q^{16/9}$ (since for $N > q^{16/9}$, the result of [10] already produces a strong estimate). Here we give a further improvement.

**Theorem 1.** For $N \leq q$, we have

$$|S_a(q; N)| \leq \left(Nq^{-1/24} + q^{5/42}N^{6/7}\right)q^{o(1)}.$$
which we deal with using techniques of Burgess [1,2]. To bound the bilinear forms (4), we apply the Cauchy-Schwartz inequality, interchange summation and complete the resulting sums. This allows us to reduce the problem to bounding the sums

\[
\sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi \left( 1 + \frac{b}{n} \right) e^{2\pi i \lambda n / q},
\]

which are dealt with using ideas based on Burgess [1]. Finally, we note that our new bound for the sums (3) when combined with the argument from [5] improves on the bound (2) although not the range of $N$ for which this bound becomes nontrivial. Our new bound for the bilinear forms (4) is what increases the range from $N > q^{8/9 + o(1)}$ to $N > q^{5/6 + o(1)}$ for which the bound for $|S_a(q;N)|$ becomes nontrivial.

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2 Preliminaries

As in [5] our basic tool is the Vaughan identity [12].

Lemma 2. For any complex-valued function $f(n)$ and any real numbers $U, V > 1$ with $UV \leq N$, we have

\[
S_a(q;N) \ll \Sigma_1 + \Sigma_2 + \Sigma_3 + |\Sigma_4|,
\]
where

\[
\begin{align*}
\Sigma_1 &= \left| \sum_{n \leq U} \Lambda(n) f(n) \right|, \\
\Sigma_2 &= (\log UV) \sum_{v \leq UV} \left| \sum_{s \leq N/v} f(sv) \right|, \\
\Sigma_3 &= (\log N) \sum_{v \leq V} \max_{w \geq 1} \left| \sum_{w \leq s \leq N/v} f(sv) \right|, \\
\Sigma_4 &= \sum_{k \ell \leq N} \Lambda(\ell) \sum_{d \mid k, \ell \leq V} \mu(d) f(k\ell).
\end{align*}
\]

where \(\mu(d)\) denotes the M"{o}bius function, defined by

\[
\mu(d) = \begin{cases} 
(-1)^{\omega(d)}, & \text{if } n \text{ squarefree}, \\
0, & \text{otherwise},
\end{cases}
\]

and \(\omega(d)\) counts the number of distinct prime factors of \(d\).

3 Pólya-Vinogradov Bound

The following is [5, Lemma 4].

**Lemma 3.** For any integers \(d, M, N, a\) with \((a, q) = 1\) and any primitive character \(\chi \pmod{q}\) we have

\[
\left| \sum_{M < n \leq M+N} \chi(dn + a) \right| \leq (d, q) \frac{N}{q^{1/2}} + q^{1/2+o(1)}.
\]

Lemma 3 was used to show [5, Lemma 5].

**Lemma 4.** For any integers \(M, N, a\) with \((a, q) = 1\) and any primitive character \(\chi \pmod{q}\) we have

\[
\left| \sum_{M < n \leq M+N \atop (n, q) = 1} \chi(n + a) \right| \leq q^{1/2+o(1)} + Nq^{-1/2}.
\]
4 Burgess Bounds

In [5], the Burgess bound for the sums
\[ \sum_{v_1,\ldots,v_{2r}=1}^{V} \left| \sum_{x=1}^{q} \chi \left( \prod_{i=1}^{r} (x + v_i) \right) \overline{\chi} \left( \prod_{i=r+1}^{2r} (x + v_i) \right) \right|, \quad r = 2, 3, \]
was used to improve on Lemma 4 for small values of \( N \). We a give further improvement by using the methods of Burgess [1, 2] to bound the sums
\[ \sum_{v_1,\ldots,v_{2r}=1}^{V} \left| \sum_{x=1}^{q} \chi \left( \prod_{i=1}^{r} (x + dv_i) \right) \overline{\chi} \left( \prod_{i=r+1}^{2r} (x + dv_i) \right) \right|, \quad r = 2, 3, \] (6)
which will then be used with techniques from [5] to obtain new bounds for sums of the form
\[ \sum_{n \leq N \atop (n,q) = 1} \chi(n + a). \]

4.1 The case \( r=2 \)

We use a special case of [1, Lemma 7],

**Lemma 5.** For integer \( q \) let \( \chi \) be a primitive character (mod \( q \)) and let
\[ f_1(x) = (x - dv_1)(x - dv_2), \quad f_2(x) = (x - dv_3)(x - dv_4). \]
Suppose at least 3 of \( v_1, v_2, v_3, v_4 \) are distinct and define
\[ A_i = \prod_{j \neq i} (dv_i - dv_j). \]

Then we have
\[ \left| \sum_{x=1}^{q} \chi(f_1(x))\overline{\chi}(f_2(x)) \right| \leq 8^{\omega(q)} q^{1/2}(q, A_i), \]
for some \( A_i \neq 0 \) with \( 1 \leq i \leq 4 \), where \( \omega(q) \) counts the number of distinct prime factors of \( q \).

We use Lemma 5 and the proof of [1, Lemma 8] to show,
Lemma 6. For any primitive character $\chi$ modulo $q$ and any positive integer $V$ we have,

$$\sum_{v_1,\ldots,v_4=1}^{V} \left| \sum_{x=1}^{q} \chi \left( \prod_{i=1}^{2} (x + dv_i) \right) \overline{\chi} \left( \prod_{i=3}^{4} (x + dv_i) \right) \right| \leq \left( V^2 q + (d, q)^3 q^{1/2} V^4 \right) q^{o(1)}.$$ 

Proof. We divide the outer summation of

$$\sum_{v_1,\ldots,v_4=1}^{V} \left| \sum_{x=1}^{q} \chi \left( \prod_{i=1}^{2} (x + dv_i) \right) \overline{\chi} \left( \prod_{i=3}^{4} (x + dv_i) \right) \right|,$$

into two sets. In the first set we put all $v_1, v_2, v_3, v_4$ which contain at most 2 distinct numbers and we put the remaining $v_1, v_2, v_3, v_4$ into the second set. The number of elements in the first set is $\ll V^2$ and for these sets we estimate the inner sum trivially. This gives

$$qV^2 + \sum_{v_1,\ldots,v_4=1}^{V'} \left| \sum_{x=1}^{q} \chi \left( \prod_{i=1}^{2} (x + dv_i) \right) \overline{\chi} \left( \prod_{i=3}^{4} (x + dv_i) \right) \right| \ll$$

where the last sum is restricted to $v_1, v_2, v_3, v_4$ which contain at least 3 distinct numbers. With notation as in Lemma 5, we have

$$\sum_{v_1,\ldots,v_4=1}^{V'} \left| \sum_{x=1}^{q} \chi \left( f_i(x) \right) \overline{\chi} \left( f_2(x) \right) \right| \leq q^{1/2 + o(1)} \sum_{v_1,\ldots,v_4=1}^{V'} \sum_{A_i \neq 0} (A_i, q).$$

Since $A_i = \prod_{i \neq j} (dv_i - dv_j) = d^3 \prod_{i \neq j} (v_i - v_j) = d^3 A'_i$, we have

$$\sum_{v_1,\ldots,v_4=1}^{V'} \sum_{A_i \neq 0} (A_i, q) \leq (d^3, q) \sum_{v_1,\ldots,v_4=1}^{V'} \sum_{A_i \neq 0} (A'_i, q),$$

and in [1, Lemma 8] it is shown

$$\sum_{v_1,\ldots,v_4=1}^{V'} \sum_{A_i \neq 0} (A'_i, q) \leq V^4 q^{o(1)},$$

from which the result follows. \qed
Using Lemma 6 in the proof of [5, Lemma 10] we get,

**Lemma 7.** For any primitive character \( \chi \) (mod \( q \)) and integers \( M, N, a \) and \( d \) satisfying

\[
N \leq q^{5/8}d^{-5/4}, \quad d \leq q^{1/6}, \quad (a, q) = 1,
\]
we have

\[
\left| \sum_{M < n \leq M + N} \chi(dn + a) \right| \leq q^{3/16 + o(1)}d^{3/8}N^{1/2}.
\]

**Proof.** We proceed by induction on \( N \). Since the result is trivial for \( N \leq q^{3/8} \), this forms the basis of the induction. We define

\[
U = [0, 0.25Nd^{3/2}q^{-1/4}], \quad V = [0, 0.25d^{-3/2}q^{1/4}],
\]
and let

\[
\mathcal{U} = \{ 1 \leq u \leq U : (u, dq) = 1 \}, \quad \mathcal{V} = \{ 1 \leq v \leq V : (v, q) = 1 \}.
\]

By the inductive assumption, for any \( \varepsilon > 0 \) and integer \( h \leq UV < N \) we have

\[
\left| \sum_{M < n \leq M + N} \chi(dn + a) \right| \leq \left| \sum_{M < n \leq M + N} \chi(d(n + h) + a) \right| + 2q^{3/16 + \varepsilon}d^{3/8}h^{1/2},
\]
for sufficiently large \( q \). Hence

\[
\left| \sum_{M < n \leq M + N} \chi(dn + a) \right| \leq \frac{1}{\#\mathcal{U} \#\mathcal{V}}|W| + 2q^{3/16 + \varepsilon}d^{3/8}(UV)^{1/2},
\]
where

\[
W = \sum_{u \in \mathcal{U}} \sum_{v \in \mathcal{V}} \sum_{M < n \leq M + N} \chi(d(n + uv) + a) = \sum_{u \in \mathcal{U}} \chi(u) \sum_{M < n \leq M + N} \sum_{v \in \mathcal{V}} \chi((dn + a)u^{-1} + dv).
\]
We have

\[
|W| \leq \sum_{x=1}^{q} \nu(x) \left| \sum_{v \in \mathcal{V}} \chi(x + dv) \right|,
\]
where \( \nu(x) \) is the number of representations \( x \equiv (dn + a)u^{-1} \pmod{q} \) with \( M < n \leq M + N \) and \( u \in U \). Two applications of Hölder’s inequality gives

\[
|W|^4 \leq \left( \sum_{x=1}^{q} \nu^2(x) \right)^{1/2} \left( \sum_{x=1}^{q} \nu(x) \right)^{1/2} \left( \sum_{x=1}^{q} \sum_{v \in \mathcal{V}} \chi(x + dv) \right)^{1/2}.
\]

From the proof of [5, Lemma 7] we have

\[
\sum_{x=1}^{q} \nu(x) = N \#U, \quad \sum_{x=1}^{q} \nu^2(x) \leq \left( \frac{dNU}{q} + 1 \right) NUq^{o(1)},
\]

and by Lemma 6

\[
\sum_{x=1}^{q} \left| \sum_{v \in \mathcal{V}} \chi(x + dv) \right|^4 = \sum_{v_1, \ldots, v_4 \in \mathcal{V}} \sum_{x=1}^{q} \chi \left( \prod_{i=1}^{2}(x + dv_i) \right) \chi \left( \prod_{i=3}^{4}(x + dv_i) \right) \leq V^2 q^{1+o(1)},
\]

since \( V \leq d^{-3/2} q^{1/4} \). Combining the above bounds gives

\[
|W|^4 \leq \left( \frac{dNU}{q} + 1 \right) NU(N \#U)^2 V^2 q^{1+o(1)},
\]

and since

\[
\#U = Uq^{o(1)}, \quad \#\mathcal{V} = Vq^{o(1)},
\]

we have

\[
\left| \sum_{M<n \leq M+N} \chi(dn + a) \right| \leq \left( \frac{d^{1/4}N}{V^{1/2}} + \frac{q^{1/4}N^{3/4}}{U^{1/4}V^{1/2}} \right) q^{o(1)} + 2q^{3/16 + \varepsilon} d^{3/8} (UV)^{1/2}.
\]

Recalling the choice of \( U \) and \( V \) we get

\[
\left| \sum_{M<n \leq M+N} \chi(dn + a) \right| \leq \left( \frac{dN}{q^{3/8}} + q^{3/16} d^{3/8} N^{1/2} \right) q^{o(1)} + \frac{1}{2} q^{3/16 + \varepsilon} d^{3/8} N^{1/2},
\]
and since by assumption,
\[ N \leq q^{5/8}d^{-5/4}, \]
we get for sufficiently large \( q \)
\[
\left| \sum_{M < n \leq M+N} \chi(dn + a) \right| \leq q^{3/16}d^{3/8}N^{1/2}q^{o(1)} + \frac{1}{2}q^{3/16+\varepsilon}d^{3/8}N^{1/2} \\
\leq q^{3/16+\varepsilon}d^{3/8}N^{1/2}.
\]

**Lemma 8.** Let \( \chi \) be a primitive character \((\text{mod } q)\) and suppose \((a, q) = 1\), then for \( N \leq q^{43/72} \) we have
\[
\left| \sum_{M < n \leq M+N} \chi(n + a) \right| \leq q^{3/16+o(1)}N^{1/2}.
\]

**Proof.** We have
\[
\left| \sum_{M < n \leq M+N} \chi(n + a) \right| = \left| \sum_{d|q} \mu(d) \sum_{M/d < n \leq (M+N)/d} \chi(dn + a) \right| \\
\leq \sum_{d|q} \left| \sum_{M/d < n \leq (M+N)/d} \chi(dn + a) \right|.
\]
Let
\[
Z = \left\lfloor \frac{N^{1/2}}{q^{-3/16}} \right\rfloor,
\]
then by Lemma 7 we have
\[
\sum_{d | q \atop d \leq Z} \sum_{M/d < n \leq (M+N)/d} \chi(dn + a) = \sum_{d | q \atop d \leq Z} \sum_{M/d < n \leq (M+N)/d} \chi(dn + a) + \sum_{d | q \atop d > Z} \sum_{M/d < n \leq (M+N)/d} \chi(dn + a) \leq \sum_{d | q \atop d \leq Z} q^{3/16 + o(1)} d^{-1/8} N^{1/2} + \sum_{d | q \atop d > Z} \frac{N}{d}.
\]
By choice of \(Z\) we get
\[
\sum_{d | q \atop d \leq Z} q^{3/16 + o(1)} d^{-1/8} N^{1/2} + \sum_{d | q \atop d > Z} \frac{N}{d} \leq \left( q^{3/16} N^{1/2} + \frac{N}{Z} \right)^{o(1)} \leq q^{3/16 + o(1)} N^{1/2},
\]
which gives the desired bound. It remains to check that the conditions of Lemma 7 are satisfied. For each \(d | q\) with \(d \leq Z\), we need
\[
\frac{N}{d} \leq q^{5/8} d^{-5/4}, \quad d \leq q^{1/6}.
\]
Recalling the choice of \(Z\), we see this is satisfied for \(N \leq q^{43/72}\).

4.2 The case \(r=3\)

Throughout this section we let
\[
f_1(x) = (x + dv_1)(x + dv_2)(x + dv_3), \quad f_1(x) = (x + dv_4)(x + dv_5)(x + dv_6), \quad (7)
\]
and
\[
F(x) = f'_1(x)f_2(x) - f_1(x)f'_2(x), \quad (8)
\]
and write \(v = (v_1, \ldots, v_6)\). We follow the argument of Burgess [2] to give an upper bound for the cardinality of the set
\[
\mathcal{A}(s, s') = \{v : 0 < v_i \leq V, \text{ there exists an } x \text{ such that } (s, f_1(x)f_2(x)) = 1, s | F(x), s | F'(x), s' | F''(x)\},
\]
which will then be combined with the proof of [2, Theorem 2] to bound the sums (6). The proof of the following Lemma is the same as [2, Lemma 3].
Lemma 9. Let $s'|s$ and consider the equations

$$(\lambda, s) = 1, \quad (f_1(-t), s/s') = 1,$$  \hspace{1cm} (9)

$$6(f_1(X) + \lambda f_2(X)) \equiv 6(1 + \lambda)(X + t)^3 \pmod{s},$$  \hspace{1cm} (10)

$$6(1 + \lambda) \equiv 0 \pmod{s'}. \hspace{1cm} (11)$$

Let

$$\mathcal{A}_1(s, s') = \{v, \lambda, t : 0 < v_i \leq V, v_i \neq v_1, \ i \geq 2, \ 0 < \lambda \leq s, \ 0 < t \leq s/s', \ (9), \ (10), \ (11)\},$$

then we have

$$\#\mathcal{A}(s, s') \ll V^3 + \#\mathcal{A}_1(s, s').$$

We next make the substitutions

$$Y = X + dv_1,$$
$$V_i = v_i - v_1, \quad i \geq 2,$$
$$T = t - dv_1 \pmod{s/s'},$$

so that

$$f_1(X) = Y(Y + dV_2)(Y + dV_3) = Y^3 + d(V_2 + V_3)Y^2 + d^2V_2V_3Y$$
$$= g_1(Y),$$  \hspace{1cm} (13)

$$f_2(X) = (Y + dV_4)(Y + dV_5)(Y + dV_6) = Y^3 + d\sigma_1Y^2 + d^2\sigma_2Y + d^3\sigma_3$$
$$= g_2(Y), \hspace{1cm} (14)$$

where

$$\sigma_1 = V_4 + V_5 + V_6,$$
$$\sigma_2 = V_4V_5 + V_4V_6 + V_5V_6,$$
$$\sigma_3 = V_4V_5V_6,$$

(15)

and we see that (10) becomes

$$6(g_1(Y) + \lambda g_2(Y)) \equiv 6(1 + \lambda)(Y + T)^3 \pmod{s}. \hspace{1cm} (16)$$

The proof of the following Lemma follows that of [2, Lemma 4].
Lemma 10. With notation as in \((12)\) and \((15)\), consider the equations
\[
(s/s', T) = 1, \quad (s/s', T - dM_3) = 1, \quad (17)
\]
\[
6d^2T^3(V_3^2 - \sigma_1 V_3 + \sigma_2) - 18d^3\sigma_3 T^2 + 18d^4V_3\sigma_3 T - 6d^5V_3^2\sigma_3 \equiv 0 \pmod{s}, \quad (18)
\]
\[
6d^3\sigma_3 \equiv 0 \pmod{s'}, \quad (19)
\]
and let
\[
\mathcal{A}_2(s, s') = \{(V_3, V_4, V_5, V_6, T) : 0 < |M_i| \leq V, \quad 0 < T \leq s/s', \quad (17), \quad (18), \quad (19)\}.
\]
Then we have
\[
\#\mathcal{A}_1(s, s') \ll (d, s)V(1 + V/q)\#\mathcal{A}_2(s, s').
\]

Proof. We first note that \((9)\) and \((12)\) imply \((17)\). Let
\[
\mathcal{B}_1 = \{(V_2, V_3, V_4, V_5, V_6, T) : 0 < |V_i| \leq V, \quad 0 < \lambda \leq s, \quad (\lambda, s) = 1, \quad 0 < T \leq s/s', \quad (11), \quad (16), \quad (17)\},
\]
so that
\[
\#\mathcal{A}_1(s, s') \leq V\#\mathcal{B}_1.
\]
Using \((13)\) and \((14)\) and considering common powers of \(Y\) in \((16)\) we get
\[
6d(V_2 + V_3 + \lambda \sigma_1) \equiv 18(1 + \lambda)T \pmod{s}, \quad (20)
\]
\[
6d^2(v_2 V_3 + \lambda \sigma_2) \equiv 18(1 + \lambda)T^2 \pmod{s}, \quad (21)
\]
\[
6d^3\lambda \sigma_3 \equiv 6(1 + \lambda)T^3 \pmod{s}. \quad (22)
\]
By \((20)\) we see that
\[
6dV_2 \equiv 18(1 + \lambda)T - dV_3 - d\lambda \sigma_1 \pmod{s},
\]
which has \(O((d, s)(1 + V/q))\) solutions in \(V_2\). The equations \((11)\) and \((22)\) imply that
\[
6d^3\sigma_3 \equiv 0 \pmod{s'},
\]
\[
6(1 + \lambda) \equiv 0 \pmod{s'},
\]
and
\[
6\lambda(d^3\sigma_3 - T^3) \equiv 6T^3 \pmod{s}. \quad (23)
\]
Since \((T, s/s') = 1\) by the above equations, there are \(O(1)\) possible values of \(\lambda\). Finally combining \((20)\), \((21)\) and \((23)\) gives \((18)\). \(\square\)
The following is [2, Lemma 2].

**Lemma 11.** For any integer $s$ and polynomial $G(X)$ with integer coefficients, we have

$$\# \{0 \leq x < s, \ G(x) \equiv 0 \pmod{s}, \ (s, G'(x))|6\} \leq s^{o(1)},$$

where the term $o(1)$ depends only on the degree of $G$.

The proof of the following Lemma follows that of [2, Lemma 5].

**Lemma 12.** For $s''|(s/s')$ consider the equations

$$\begin{align*}
(s, 6d^3\sigma_3) &= s's'', \quad (24) \\
6d^2(V_3^2 - \sigma_1 V_3 + \sigma_2) &\equiv 0 \pmod{s}, \quad (25)
\end{align*}$$

and let

$$\mathcal{A}_3(s, s', s'') = \{(V_3, V_4, V_5, V_6) : 0 < |V_i| \leq V, \ (24), \ (25)\}.$$

Then we have

$$\#\mathcal{A}_2(s, s') \leq s^{o(1)} \sum_{s''|(s/s')} s'' \#\mathcal{A}_3(s, s', s'').$$

**Proof.** For $s''|(s/s')$, let

$$\mathcal{A}'_3(s, s', s'') = \{(V_3, V_4, V_5, V_6, T) \in \mathcal{A}_2(s, s') : (s, 6d^3\sigma_3) = s's''\},$$

so that

$$\#\mathcal{A}_2(s, s') = \sum_{s''|(s/s')} \#\mathcal{A}'_3(s, s', s''). \quad (26)$$

Let $S = (s', s/s')$, so that $(s'/S, s/s') = 1$. For elements of $\mathcal{A}_3(s, s', s'')$, since

$$6d^3\sigma_3 \equiv 0 \pmod{Ss''}, \quad (27)$$

we have by (17), (18) and (24)

$$6d^2(V_3^2 - \sigma_1 V_3 + \sigma_2) \equiv 0 \pmod{Ss''}, \quad (28)$$

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hence (18) implies that
\[ 6d^2(V_3^2 - \sigma_1 V_3 + \sigma_2) T^3 - \frac{18d^3 \sigma_3}{S''} T^2 + \frac{18d^4 \sigma_3 V_3^2}{S''} T - \frac{6d^5 \sigma_3 V_3^2}{S''} \equiv 0 \pmod{s/(s's'')} . \] (29)

Let
\[ G(T) = \frac{6d^2(V_3^2 - \sigma_1 V_3 + \sigma_2)}{S''} T^3 - \frac{18d^3 \sigma_3}{S''} T^2 + \frac{18d^4 \sigma_3 V_3^2}{S''} T - \frac{6d^5 \sigma_3 V_3^2}{S''} , \]
so that
\[ 3G(T) - TG'(T) = -\frac{18d^3 \sigma_3}{S''}(T - dV_3)^2. \]
Writing \(6d^3 \sigma_3 = s's''\sigma'\) with \((\sigma', s) = 1\), we see from (17) that for some integer \(y\) with \((y, s/s') = 1\) that
\[ 3G(T) - TG'(T) = \frac{3s'}{S} y. \]

If \(T_0\) is a root of \(G(T) \pmod{s/(s's'')}\) then since \((s'/S, s/s') = 1\) we have
\[ (G'(T_0), s/(s's''))|6, \]
hence from Lemma [11] the number of possible values for \(T\) is \(\ll s''s^{o(1)}\). Finally (28) implies
\[ 6d^2(V_3^2 - \sigma_1 V_3 + \sigma_2) \equiv 0 \pmod{s''}, \]
and the result follows from (26).

\[ \text{Lemma 13.} \quad \text{With notation as in Lemma [12], for integers } s, s', s'' \text{ satisfying } s'|s \text{ and } s''|s/s' \text{ we have} \]
\[ \#A_3(s, s', s'') \leq (d^3, s)V^4s^{o(1)}/(s's''). \]

\[ \text{Proof.} \quad \text{Bounding the number of solutions to the equation (25) trivially and recalling the definition of } \sigma_3 \text{ from (15), we see that} \]
\[ \#A_3(s, s', s'') \leq V\#\{ (V_4, V_5, V_6) : 0 < |M_i| \leq V, (s, 6d^3 V_4 V_5 V_6) = s's'' \}. \] (30)
Writing \( s = (d^3, s)s_1, d^3 = (d^3, s)d_1 \), we see that

\[(s, 6d^3V_4V_5V_6) = s's'' , \]

implies

\[(s_1, 6V_4V_5V_6) = s's''/(d^3, s). \]

For integers \( s_1, s_2, s_3 \), let

\[ \mathcal{A}_4(s_1, s_2, s_3) = \{ V_4, V_5, V_6 : 0 < |V_i| \leq V, s_1|6V_4, s_2|6V_5, s_3|6V_6 \}, \]

so that from (30)

\[ \# \mathcal{A}_3(s, s', s'') \leq V \sum_{s_1 s_2 s_3 = s's''/(d^3, s)} \mathcal{A}_4(s_1, s_2, s_3). \]

Since

\[ \mathcal{A}_4(s_1, s_2, s_3) \ll \frac{V^3}{s_1 s_2 s_3} = \frac{(d^3, s)V^3}{s's''}, \]

we see that

\[ \# \mathcal{A}_3(s, s', s'') \leq \frac{(d^3, s)V^4 s^{o(1)}}{s's''}. \]

Combining the above results we get

**Lemma 14.** Let \( s'|s \) and

\[ \mathcal{A}(s, s') = \{ dv : 0 < v_i \leq V, there \ exists \ an \ x \ such \ that \]

\[(s, f_1(x)f_2(x)) = 1, s|F(x), s'|F''(x) \}. \]

Then

\[ \# \mathcal{A}(s, s') \leq (d, s)^4 \left( \frac{V^6}{ss'} + \frac{V^5}{s'} \right) q^{o(1)} + V^3. \]

**Proof.** From Lemma 9, Lemma 10, Lemma 12 we see that

\[ \# \mathcal{A}(s, s') \leq V^3 + (d, s) \left( 1 + \frac{V}{s} \right) V \sum_{s''|s/s'} s'' \# \mathcal{A}_3(s, s', s''), \]

and from Lemma 13 we have

\[ \sum_{s''|s/s'} s'' \# \mathcal{A}_3(s, s', s'') \leq \frac{s^{o(1)}(d^3, s)V^4}{s'}, \]

which gives the desired result.

\[ \square \]
For integer $q$, we define the numbers $h_1(q), h_2(q), h_3(q)$ as in [2],

\[ h_1(q)^2 = \text{smallest square divisible by } q, \]
\[ h_2(q)^3 = \text{smallest cube divisible by } q, \]
\[ h_3(q) = \text{product of distinct prime factors of } q. \]  

The following is [2, Theorem 2].

**Lemma 15.** Let $\chi$ be a primitive character mod $q$ and let

\[ q = q_0 q_1 q_2 q_3, \]

where the $q_i$ are pairwise coprime. Let the integers $l_0, l_1, l_2$ satisfy

\[ l_0 | h_1(q_0)/h_3(q_0), \quad l_1 | h_2(q_1)/h_3(q_1), \quad l_2 | h_2(q_2)/h_3(q_2), \]

and consider the equations

\[ l_0 h_1(q_1 q_2 q_3) | F(x), \quad (F(x), h_1(q_0)) = l_0, \]
\[ l_1 h_2(q_2 q_3) | F'(x), \quad (F'(x), h_2(q_1)) = l_1, \]
\[ l_2 h_2(q_3) | F''(x), \quad (F''(x), h_2(q_2)) = l_2, \]

and let

\[ C = C(l_0, l_1, l_2, q_0, q_1, q_2, q_3) = \{1 \leq x \leq q : (34), (35), (36)\}. \]

Then we have

\[ \left| \sum_{x \in C} \chi(f_1(x)) \overline{\chi}(f_2(x)) \right| \leq q^{1/2+o(1)} \frac{(q_2 q_3 l_1)^{1/2} l_2}{h_2(q_2)}. \]

**Lemma 16.** For any primitive character $\chi$ modulo $q$ and any integer $V < q^{1/6}d^{-2}$, we have

\[ \sum_{V} \left| \sum_{x=1}^{q} \chi \left( \prod_{i=1}^{3} (x + dv_i) \right) \overline{\chi} \left( \prod_{i=4}^{6} (x + dv_i) \right) \right| \leq V^3 q^{1+o(1)}. \]
Proof. With notation as above and in (7) and (8)
\[
\sum_{v_1, \ldots, v_6=1} V \sum_{i=1}^{q} \chi \left( \prod_{i=1}^{3} (x + dv_i) \right) \overline{\chi} \left( \prod_{i=4}^{6} (x + dv_i) \right) \leq \sum_{d, l_1} \left| \sum_{c \in C} \chi(f_1(x)) \overline{\chi}(f_2(x)) \right|
\]
where the last sum is extended over all \(q_1, q_2, q_3, q_4\) and \(l_0, l_1, l_2\) satisfying the conditions of Lemma 15. Hence by Lemma 14 for some fixed \(q_1, \ldots, q_4\) satisfying (32) and \(l_0, l_1, l_2\) satisfying (33),
\[
\sum_{v_1, \ldots, v_6=1} V \sum_{i=1}^{q} \chi \left( \prod_{i=1}^{3} (x + dv_i) \right) \overline{\chi} \left( \prod_{i=4}^{6} (x + dv_i) \right) \leq \\
\left( (q, d)^4 \frac{V^6}{l_1 h_2(q_2 q_3) l_2 h_2(q_3)} + \frac{V^5}{l_2 h_2(q_3)} \right) q + V^3 \left( \frac{qq_2 q_3 l_1}{h_2(q_2)} \right)^{1/2} l_2 q^{o(1)} \leq \\
\left( (q, d)^4 V^6 q^{1/2} + (q, d)^4 V^5 q^{2/3} + V^3 q \right) q^{o(1)},
\]
from the definition of \(l_i, h_i, q_i\). The result follows since the term \(V^3 q\) dominates for \(V \leq q^{1/6} d^{-2}\).

\[\square\]

Lemma 17. For any primitive character \(\chi\) modulo \(q\) and integers \(M, N, d\) and \(a\) satisfying
\[
N \leq q^{7/12} d^{-3/2}, \quad d \leq q^{1/12}, \quad (a, q) = 1,
\]
we have
\[
\left| \sum_{M < n \leq M+N} \chi(dn + a) \right| \leq q^{1/9 + o(1)} d^{2/3} N^{2/3}.
\]

Proof. Using the same argument from Lemma 14 we proceed by induction on \(N\). Since the result is trivial for \(N \leq q^{1/3}\), this forms the basis of our induction. Define
\[
U = [0.5 N d^2 q^{1/6}], \quad V = [0.5 d^{-2} q^{1/6}],
\]
and let
\[
\mathcal{U} = \{ 1 \leq u \leq U : (a, dq) = 1 \}, \quad \mathcal{V} = \{ 1 \leq v \leq V : (v, q) = 1 \}.
\]
Fix \(\varepsilon > 0\), by the inductive hypothesis, for any integer \(h \leq UV < N\) we have
\[
\left| \sum_{M < n \leq M+N} \chi(dn + a) \right| \leq \left| \sum_{M < n \leq M+N} \chi(d(n + h) + a) \right| + 2d^{1/9 + \varepsilon} h^{2/3},
\]
and
for sufficiently large $q$. Hence
\[
\left| \sum_{M<n\leq M+N} \chi(dn+a) \right| \leq \frac{1}{\#U\#V} |W| + 2q^{1/9+\varepsilon}d^{2/3}(UV)^{2/3},
\]
where
\[
W = \sum_{u\in U} \sum_{v\in V} \sum_{M<n\leq M+N} \chi(d(n+uv)+a) = \sum_{u\in U} \chi(u) \sum_{M<n\leq M+N} \sum_{v\in V} \chi((dn+a)u^{-1}+dv).
\]
We have
\[
|W| \leq \sum_{x=1}^q \nu(x) \left| \sum_{v\in V} \chi(x+dv) \right|,
\]
where $\nu(x)$ is the number of representations $x \equiv (dn+a)u^{-1} \pmod{q}$ with $M < n \leq M+N$ and $u \in U$. Two applications of Hölder’s inequality gives,
\[
|W|^6 \leq \left( \sum_{x=1}^q \nu^2(x) \right) \left( \sum_{x=1}^q \nu(x) \right)^4 \left| \sum_{x=1}^q \sum_{v\in V} \chi(x+dv) \right|^6.
\]
As in Lemma 7
\[
\sum_{x=1}^q \nu(x) = N\#U, \quad \sum_{x=1}^q \nu^2(x) \leq \left( \frac{dNU}{q} + 1 \right) NUq^{o(1)},
\]
and by Lemma 16
\[
\sum_{x=1}^q \left| \sum_{v\in V} \chi(x+dv) \right|^3 = \sum_{v_1,\ldots,v_4 \in V} \sum_{x=1}^q \chi \left( \prod_{i=1}^3 (x+dv_i) \right) \chi \left( \prod_{i=4}^6 (x+dv_i) \right) \\
\quad \leq \sum_{v_1,\ldots,v_4 \in V} \sum_{x=1}^q \chi \left( \prod_{i=1}^2 (x+dv_i) \right) \chi \left( \prod_{i=3}^4 (x+dv_i) \right) \\
\quad \leq V^2q^{1+o(1)}.
\]
The above bounds give
\[
|W|^6 \leq \left( \frac{dNU}{q} + 1 \right) NUq^{o(1)}(N\#U)^4 \left( V^3q \right) q^{o(1)},
\]
so that

$$\sum_{M<n\leq M+N} \chi(n+a) \leq \left( \frac{d^{1/6}N}{V^{1/2}} + \frac{q^{1/6}N^{5/6}}{U^{1/6}V^{1/2}} \right) q^{o(1)} + 2q^{1/9+\varepsilon}d^{2/3}(UV)^{2/3}.$$ 

Recalling the choice of $U$ and $V$ we get

$$\sum_{M<n\leq M+N} \chi(n+a) \leq \frac{d^{7/6}N}{q^{1/12+o(1)}} + q^{1/9+o(1)}d^{2/3}N^{2/3} + \frac{2}{5}q^{1/9+\varepsilon}d^{2/3}N^{2/3},$$

and since

$$\frac{d^{7/6}N}{q^{1/12}} \leq q^{1/9}d^{2/3}N^{2/3} \text{ when } dN \leq q^{13/24},$$

we have by assumption on $N$ and $d$

$$\sum_{M<n\leq M+N} \chi(n+a) \leq q^{1/9+o(1)}d^{2/3}N^{2/3} + \frac{2}{5}q^{1/9+\varepsilon}d^{2/3}N^{2/3}$$

$$\leq q^{1/9+\varepsilon}d^{2/3}N^{2/3},$$

for sufficiently large $q$. \hfill \Box

Using Lemma 17 as in the proof of Lemma 8 we get,

**Lemma 18.** Let $\chi$ be a primitive character (mod $q$) and suppose $(a,q) = 1$, then for $N \leq q^{23/42}$ we have

$$\left| \sum_{M<n\leq M+N \atop (n,q)=1} \chi(n+a) \right| \leq q^{1/9+o(1)}N^{2/3}.$$ 

**Proof.** We have

$$\left| \sum_{M<n\leq M+N \atop (n,q)=1} \chi(n+a) \right| = \left| \sum_{d|q} \mu(d) \sum_{M/d<n\leq (M+N)/d} \chi(dn+a) \right|$$

$$\leq \sum_{d|q} \left| \sum_{M/d<n\leq (M+N)/d} \chi(dn+a) \right|.$$
Let
\[ Z = \left\lfloor \frac{N^{1/3}}{q^{1/9}} \right\rfloor, \]
then by Lemma 7 we have
\[
\sum_{d \mid q \atop d \leq Z} \sum_{M/d < n \leq (M+N)/d} \chi(dn + a) = \]
\[
\left| \sum_{d \mid q \atop d \leq Z} \sum_{M/d < n \leq (M+N)/d} \chi(dn + a) \right| + \left| \sum_{d \mid q \atop d > Z} \sum_{M/d < n \leq (M+N)/d} \chi(dn + a) \right| \leq \sum_{d \mid q \atop d \leq Z} q^{1/9 + o(1)} N^{2/3} + \sum_{d \mid q \atop d > Z} \frac{N}{d}.\]

Since by choice of \( Z \)
\[
\sum_{d \mid q \atop d \leq Z} q^{1/9 + o(1)} N^{2/3} + \sum_{d \mid q \atop d > Z} \frac{N}{d} \leq \left( q^{1/9} N^{2/3} + \frac{N}{Z} \right) q^{o(1)} \leq q^{1/9 + o(1)} N^{2/3},\]
we get the desired bound. It remains to check that the conditions of Lemma 7 are satisfied. For each \( d \mid q \) with \( d \leq Z \) we need
\[
\frac{N}{d} \leq q^{7/12} d^{-3/2}, \quad d \leq q^{1/12},\]
and from the choice of \( Z \), this is satisfied for \( N \leq q^{23/42} \).

\[ \square \]

5 Bilinear Character Sums

Lemma 19. Let \( \chi \) be a primitive character \( \pmod{q} \). Then for integers \( u_1, u_2, \lambda \) we have
\[
\left| \sum_{n=1}^{q} \chi(n + u_1) \overline{\chi}(n + u_2) e^{2\pi i \lambda n/q} \right| = \left| \sum_{n=1}^{q} \chi(n + \lambda) \overline{\chi}(n) e^{2\pi i (u_1-u_2)n/q} \right|.\]
Proof. Let
\[ \tau(\chi) = \sum_{n=1}^{q} \chi(n)e^{2\pi in/q}, \]
be the Gauss sum, so that
\[ |\tau(\chi)| = q^{1/2} \quad \text{and} \quad \sum_{n=1}^{q} \chi(n)e^{2\pi i an/q} = \chi(a)\tau(\chi). \]
Writing
\[ \chi(n + u_1) = \frac{1}{\tau(\chi)} \sum_{\lambda_1=1}^{q} \overline{\chi}(\lambda_1)e^{2\pi i (n + u_1)\lambda_1/q}, \]
and
\[ \overline{\chi}(n + u_1) = \frac{1}{\tau(\chi)} \sum_{\lambda_2=1}^{q} \chi(\lambda_2)e^{2\pi i (n + u_2)\lambda_2/q}, \]
we have
\[ \sum_{n=1}^{q} \chi(n + u_1)\overline{\chi}(n + u_2)e^{2\pi i \lambda n/q} = \]
\[ \frac{1}{\tau(\chi)\tau'(\chi)} \sum_{\lambda_1=1}^{q} \sum_{\lambda_2=1}^{q} \overline{\chi}(\lambda_1)e^{2\pi i \lambda_1 u_2/q} \chi(\lambda_2)e^{2\pi i \lambda_2 u_2/q} \sum_{n=1}^{q} e^{2\pi i n(\lambda + \lambda_1 + \lambda_2)}, \]
since
\[ \sum_{\lambda_1=1}^{q} \sum_{\lambda_2=1}^{q} \overline{\chi}(\lambda_1)e^{2\pi i \lambda_1 u_2/q} \chi(\lambda_2)e^{2\pi i \lambda_2 u_2/q} \sum_{n=1}^{q} e^{2\pi i n(\lambda + \lambda_1 + \lambda_2)} = \]
\[ \chi(-1)e^{-2\pi i u_2\lambda/q} \sum_{\lambda_1=1}^{q} \chi(\lambda_1 + \lambda)\overline{\chi}(\lambda_1)e^{2\pi i \lambda_1(\lambda_1 - u_2)/q}, \]
we have
\[ \left| \sum_{n=1}^{q} \chi(n + u_1)\overline{\chi}(n + u_2)e^{2\pi i \lambda n/q} \right| = \left| \frac{q}{\tau(\chi)} \right| \left| \sum_{n=1}^{q} \chi(n + \lambda)\overline{\chi}(n)e^{2\pi i n(u_1 - u_2)/q} \right| \]
\[ = \left| \sum_{n=1}^{q} \chi(n + \lambda)\overline{\chi}(n)e^{2\pi i n(u_1 - u_2)/q} \right|. \]
Lemma 20. Let $\chi$ be a primitive character (mod $q$), then for integers $b, \lambda$ with $b \not\equiv 0 \pmod{q}$ we have

$$\left| \sum_{\substack{n=1 \\ (n,q) = 1}}^q \chi \left( 1 + \frac{b}{n} \right) e^{2\pi i \lambda n/q} \right| \leq (b,q) q^{1/2+o(1)}.$$ 

Proof. Consider first when $\lambda \equiv 0 \pmod{q}$. Then from Lemma 19 we have

$$\left| \sum_{\substack{n=1 \\ (n,q) = 1}}^q \chi \left( 1 + \frac{b}{n} \right) e^{2\pi i \lambda n/q} \right| = \left| \sum_{n=1}^q |\chi(n)| e^{2\pi i bn/q} \right| = \left| \sum_{n=1}^q e^{2\pi i bn/q} \right|,$$

and from [7, Equation 3.5] we have

$$\left| \sum_{\substack{n=1 \\ (n,q) = 1}}^q e^{2\pi i bn/q} \right| \ll (b,q),$$

so that

$$\left| \sum_{\substack{n=1 \\ (n,q) = 1}}^q \chi \left( 1 + \frac{b}{n} \right) e^{2\pi i \lambda n/q} \right| \ll (b,q) \leq (b,q) q^{1/2+o(1)}.$$ 

Next consider when $\lambda \not\equiv 0 \pmod{q}$. We first note that if $\chi$ is a character (mod $p$), with $p$ prime, then we have from the Weil bound, see [11, Theorem 2G]

$$\left| \sum_{\substack{n=1 \\ (n,p) = 1}}^p \chi \left( 1 + \frac{b}{n} \right) e^{2\pi i \lambda n/p} \right| \ll p^{1/2}.$$ 

For $p$ prime and integers $\lambda, b, c, \alpha$, let $N(\lambda, b, c, p^\alpha)$ denote the number of solutions to the congruence

$$\lambda n^2 \equiv cb \pmod{p^\alpha}, \quad 1 \leq n \leq p^\alpha, \quad (n,p) = 1. \quad (37)$$ 

Then we have

$$N(\lambda, b, c, p^\alpha) \leq 4(\lambda, p^\alpha), \quad (38)$$ 

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since if there exists a solution $n$ to (37) then we must have $(\lambda, p^\alpha) = (cd, p^\alpha)$ so that we arrive at the congruence

$$n^2 \equiv a \pmod{p^\alpha/(p^\alpha, \lambda)},$$

for some integer $a$ with $(a, p) = 1$. Since there are at most 4 solutions to (39) we get (38). Suppose $q = p^{2\alpha}$ is an even prime power and let $c$ be defined by

$$\chi(1 + p^\alpha) = e^{2\pi ic/p^\alpha},$$

then from the argument of [1, Lemma 2] (see also [7, Lemma 12.2]) we have by (38)

$$\left| \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi \left( 1 + \frac{b}{n} \right) e^{2\pi i \lambda n/q} \right| \ll \alpha N(\lambda, b, c) \ll (\lambda, q)q^{1/2}.$$

Suppose next $q = p^{2\alpha+1}$ is an odd prime power, with $p > 2$. Let $c$ be defined by

$$\chi(1 + p^{\alpha+1}) = e^{2\pi ic/p^\alpha},$$

then from the argument of [1, Lemma 4] (see also [7, Lemma 12.3])

$$\left| \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi \left( 1 + \frac{b}{n} \right) e^{2\pi i \lambda n/q} \right| \ll p^{(2\alpha+1)/2}(\lambda, b, c, p^\alpha) + p^\alpha N(\lambda, b, c, q^{\alpha+1}) \ll (\lambda, q)q^{1/2}.$$

Finally if $q = 2^{2\alpha+1}$, then from the argument of [1, Lemma 3]

$$\left| \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi \left( 1 + \frac{b}{n} \right) e^{2\pi i \lambda n/q} \right| \ll 2^{1/2}2^\alpha N(\lambda, b, c, p^\alpha) \ll (\lambda, q)q^{1/2}.$$

Combining the above bounds gives the desired result when $q$ is a prime power. For the general case, suppose $\chi$ is a primitive character (mod $q$)
and let \( q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) be the prime factorization of \( q \). By the Chinese Remainder Theorem we have

\[
\chi = \chi_1 \chi_2 \cdots \chi_k,
\]

where each \( \chi_i \) is a primitive character \((\text{mod} \ p_i^{\alpha_i})\). Let \( q_i = q/p_i^{\alpha_i} \), then by the above bounds and another application of the Chinese remainder theorem (see [7, Equation 12.21]), for some absolute constant \( C \)

\[
\sum_{n=1}^{q} \left| \chi \left( 1 + \frac{b}{n} \right) e^{2\pi i \lambda n/q} \right| =
\]

\[
\sum_{n=1}^{p_1^{\alpha_1}} \cdots \sum_{n=1}^{p_k^{\alpha_k}} \chi_1 \left( 1 + \frac{b}{\sum_{i=1}^{k} n_i q_i} \right) e^{2\pi i \lambda n_1/p_1^{\alpha_1}} \cdots \chi_k \left( 1 + \frac{b}{\sum_{i=1}^{k} n_i q_i} \right) e^{2\pi i \lambda n_k/p_k^{\alpha_k}} \leq \prod_{i=1}^{k} \left| \sum_{n=1}^{p_i^{\alpha_i}} \chi_i \left( 1 + \frac{b}{n_i q_i} \right) e^{2\pi i \lambda n_i/p_i^{\alpha_i}} \right| \leq \prod_{i=1}^{k} C(\lambda, p_i^{\alpha_i}) p_i^{\alpha_i/2} \leq (\lambda, q) q^{1/2 + o(1)},
\]

and the result follows from Lemma 19.

**Lemma 21.** Let \( K, L \) be natural numbers and for any two sequences \((\alpha_k)_{k=1}^{K}\) and \((\beta_\ell)_{\ell=1}^{L}\) of complex numbers supported on integers coprime to \( q \) and any integer \( a \) coprime to \( q \), let

\[
W = \sum_{k \leq K} \sum_{\ell \leq L} \alpha_k \beta_\ell \chi(k\ell + a).
\]

Then

\[
W \leq AB \left( KL^{1/2} + q^{1/4} K^{1/2} L + \frac{KL}{q^{1/4}} \right) q^{o(1)},
\]

where

\[
A = \max_{k \leq K} |\alpha_k| \quad \text{and} \quad B = \max_{\ell \leq L} |\beta_\ell|.
\]
Proof. By the Cauchy-Schwartz inequality

\[ |W|^2 \leq A^2 K \sum_{k \leq K} \left| \sum_{\ell \leq L} \beta_\ell \chi(k\ell + a) \right|^2 \]

\[ \leq A^2 B^2 K^2 L + \left| \sum_{k \leq K} \sum_{\ell_1, \ell_2 \leq L} \beta_{\ell_1} \overline{\beta_{\ell_2}} \chi(k\ell_1 + a) \overline{\chi(k\ell_2 + a)} \right|. \]

Let

\[ W_1 = \sum_{k \leq K} \sum_{\ell_1, \ell_2 \leq L} \beta_{\ell_1} \overline{\beta_{\ell_2}} \chi(k\ell_1 + a) \overline{\chi(k\ell_2 + a)}, \]

then we have

\[ |W_1| \leq \frac{B^2}{q} \sum_{\ell_1 < \ell_2 \leq L} \left| \sum_{s=1}^q \sum_{k \leq K} e^{-2\pi is\ell_1} \sum_{\lambda=1}^q \chi(\lambda + a\ell_1^{-1}) \overline{\chi(\lambda + a\ell_2^{-1})} e^{2\pi i s\lambda/q} \right| \]

\[ \leq \frac{B^2}{q} \sum_{\ell_1 < \ell_2 \leq L} \left| \sum_{s=1}^q \sum_{k \leq K} e^{-2\pi is\ell_1} \sum_{\lambda=1}^q \chi(\lambda + a\ell_1^{-1}) \overline{\chi(\lambda + a\ell_2^{-1})} e^{2\pi i s\lambda/q} \right|. \]

By Lemma 20

\[ \sum_{\ell_1 < \ell_2 \leq L} \left( \sum_{s=1}^q \sum_{k \leq K} e^{-2\pi is\ell_1} \sum_{\lambda=1}^q \chi(\lambda + a\ell_1^{-1}) \overline{\chi(\lambda + a\ell_2^{-1})} e^{2\pi i s\lambda/q} \right) \ll \]

\[ \sum_{\ell_1 < \ell_2 \leq L} \sum_{s=1}^q \min \left( K, \frac{1}{||s/q||} \right)(\ell_1 - \ell_2, q)^{1/2+o(1)}, \]

and since

\[ \sum_{\ell_1, \ell_2 \leq L} \sum_{\ell_1 \neq \ell_2} (\ell_1 - \ell_2, q) \ll \sum_{\ell \leq L} \sum_{\ell_1, \ell_2 \leq L} \sum_{\ell_1 < \ell_2} (\ell, q) \leq L \sum_{d|\ell} \sum_{d|\ell} \sum_{\ell_1 - \ell_2 = \ell} 1 \leq L^2 q^{o(1)}, \]
we get

\[|W_1| \leq \frac{B^2}{q} \left( \sum_{s=1}^{q} \min \left( K, \frac{1}{||s/q||} \right) \right) q^{1/2+o(1)} L^2 \leq B^2 \left( 1 + \frac{K}{q} \right) q^{1/2+o(1)} L^2,\]

so that

\[|W|^2 \leq A^2 B^2 K \left( KL + \left( 1 + \frac{K}{q} \right) q^{1/2+o(1)} L^2 \right).\]

Next, we use an idea of Garaev [6] to derive a variant of Lemma 21 in which the summation limits over \(\ell\) depend on the parameter \(k\).

**Lemma 22.** Let \(K, L\) be natural numbers and let the sequences \((L_k)_{k=1}^{K}\) and \((M_k)_{k=1}^{K}\) of nonnegative integers be such that \(M_k < L_k \leq L\) for each \(k\). For any two sequences \((\alpha_k)_{k=1}^{K}\) and \((\beta_\ell)_{\ell=1}^{L}\) of complex numbers supported on integers coprime to \(q\) and for any integer \(a\) coprime to \(q\), let

\[
\widetilde{W} = \sum_{k \leq K} \sum_{M_k < \ell \leq L_k} \alpha_k \beta_\ell \chi(k\ell + a).
\]

Then

\[
\widetilde{W} \ll (KL^{1/2} + (1 + K^{1/2} q^{-1/2}) q^{1/4} K^{1/2} L) (Lq)^{o(1)},
\]

where

\[
A = \max_{k \leq K} |\alpha_k| \quad \text{and} \quad B = \max_{\ell \leq L} |\beta_\ell|.
\]

**Proof.** For real \(z\) we denote

\[
e_L(z) = \exp(2\pi iz/L).
\]

For each inner sum, using the orthogonality of exponential functions, we have

\[
\sum_{M_k < \ell \leq L_k} \beta_\ell \chi(k\ell + a) = \sum_{\ell \leq L} \sum_{M_k < s \leq L_k} \beta_\ell \chi(k\ell + a) \cdot \frac{1}{L} \sum_{-\frac{1}{2} L < r \leq \frac{1}{2} L} e_L(r(\ell - s))
\]

\[
= \frac{1}{L} \sum_{-\frac{1}{2} L < r \leq \frac{1}{2} L} \sum_{M_k < s \leq L_k} e_L(-rs) \sum_{\ell \leq L} \beta_\ell e_L(r\ell) \chi(k\ell + a).
\]
In view of [7, Bound (8.6)], for each \( k \leq K \) and every integer \( r \) such that \( |r| \leq \frac{1}{2} L \) we can write

\[
\sum_{M_k < s \leq L_k} e_L(-rs) = \sum_{s \leq L_k} e_L(-rs) - \sum_{s \leq M_k} e_L(-rs) = \eta_{k,r} \frac{L}{|r| + 1},
\]

for some complex number \( \eta_{k,r} \ll 1 \). Thus, if we put \( \tilde{\alpha}_{k,r} = \alpha_k \eta_{k,r} \) and \( \tilde{\beta}_{\ell,r} = \beta_\ell e_L(r\ell) \), it follows that

\[
\sum_{K_0 < k \leq K} \sum_{M_k < \ell \leq L_k} \alpha_k \beta_\ell \chi(k\ell + a) = \sum_{-\frac{1}{2} L < r \leq \frac{1}{2} L} \frac{1}{|r| + 1} \sum_{k \leq K} \sum_{\ell \leq L} \tilde{\alpha}_{k,r} \tilde{\beta}_{\ell,r} \chi(k\ell + a).
\]

Applying Lemma 21 with the sequences \( (\tilde{\alpha}_{k,r})_{k=1}^K \) and \( (\tilde{\beta}_{\ell,r})_{\ell=1}^L \), and noting that

\[
\sum_{-\frac{1}{2} L < r \leq \frac{1}{2} L} \frac{1}{|r| + 1} \ll \log L,
\]

we derive the stated bound.

6 Proof of Theorem 1

Considering the sum

\[
S_a(q; N) = \sum_{n \leq N} \Lambda(n) \chi(n + a),
\]

we apply Lemma 2 with

\[
f(n) = \begin{cases} 
\chi(n + a) & \text{if } (n, q) = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

For \( \Sigma_1 \) in Lemma 2 we apply the trivial estimate,

\[
\Sigma_1 = \left| \sum_{n \leq U} \Lambda(n) f(n) \right| \ll U.
\]
6.1 The sum $\Sigma_2$

We have

$$\Sigma_2 = (\log UV) \sum_{v \leq UV \atop (v,q)=1} \sum_{s \leq N/v \atop (s,q)=1} \chi(sv + a) = (\log UV) \sum_{v \leq UV \atop (v,q)=1} \sum_{s \leq N/v \atop (s,q)=1} \chi(s + av^{-1}).$$

By Lemma 3 since $N \leq q$

$$\sum_{v \leq Nq^{-43/72} \atop (v,q)=1} \sum_{s \leq N/v \atop (s,q)=1} \chi(s + av^{-1}) \leq \sum_{v \leq Nq^{-43/72} \atop (v,q)=1} q^{1/2 + o(1)} \leq Nq^{-7/24 + o(1)},$$

by Lemma 8

$$\sum_{Nq^{-43/72} < v \leq Nq^{-11/24} \atop (v,q)=1} \sum_{s \leq N/v \atop (s,q)=1} \chi(s + av^{-1}) \leq q^{3/16 + o(1)} N^{1/2} \left( \sum_{Nq^{-43/72} < v \leq Nq^{-11/24}} v^{-1/2} \right) \leq Nq^{-1/24 + o(1)},$$

and by Lemma 18

$$\sum_{Nq^{-11/24} < v \leq UV \atop (v,q)=1} \sum_{s \leq N/v \atop (s,q)=1} \chi(s + av^{-1}) \leq q^{1/9 + o(1)} N^{2/3} \left( \sum_{Nq^{-11/24} < v \leq UV} v^{-2/3} \right) \leq q^{1/9 + o(1)} N^{2/3}(UV)^{1/3}.$$

Combining the above bounds gives

$$\Sigma_2 \leq \left( Nq^{-1/24 + o(1)} + q^{1/9} N^{2/3}(UV)^{1/3} \right) q^{o(1)}.$$
6.2 The sum $\Sigma_3$

As above, we get

$$\Sigma_3 = \left( \log N \right) \sum_{v \leq V} \max_{w \geq 1} \left| \sum_{w \leq s \leq N/v} \chi(s + av^{-1}) \right| \leq \left( Nq^{-1/24 + o(1)} + q^{1/9} N^{2/3} V^{1/3} \right) q^{o(1)}.$$  

6.3 The sum $\Sigma_4$

For the sum $\Sigma_4$, we have

$$\Sigma_4 = \sum_{U < k \leq \frac{N}{V}} \sum_{\gcd(k, q) = 1} \Lambda(k) \sum_{V < \ell \leq N/k} A(\ell) \chi(k\ell + a),$$

where

$$A(\ell) = \sum_{d | \ell, d \leq V} \mu(d), \quad \gcd(\ell, q) = 1,$$

and

$$A(\ell) = 0, \quad \gcd(\ell, q) > 1.$$  

Note that

$$\Lambda(k) \leq \log k \leq k^{o(1)} \quad \text{and} \quad |A(\ell)| \leq \tau(\ell) \leq \ell^{o(1)}.$$  

We separate the sum $\Sigma_4$ into $O(\log N)$ sums of the form

$$W(K) = \sum_{K < k \leq 2K} \Lambda(k) \sum_{V < \ell \leq N/k} A(\ell) \chi(k\ell + a),$$

where $U \leq K \leq N/V$. By Lemma 22 we have

$$W(K) \leq \left( K^{1/2} N^{1/2} + (1 + K^{1/2} q^{-1/2}) q^{1/4} K^{-1/2} N \right) q^{o(1)}$$

$$\leq K^{1/2} N^{1/2} + q^{1/4} K^{-1/2} N + Nq^{-1/4 + o(1)},$$

so that summing over the $O(\log N)$ values of $U \leq K \leq N^{-1}$ gives

$$\Sigma_4 \leq \left( N V^{-1/2} + q^{1/4} N U^{-1/2} + Nq^{-1/4} \right) (Nq)^{o(1)}.$$  

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6.4 Concluding the Proof

Combining the estimates for $\Sigma_1, \ldots, \Sigma_4$ gives

$$S_a(q; N) \leq (Nq^{-1/24+o(1)} + U + NV^{-1/2} + q^{1/4}NU^{-1/2} + q^{1/9}N^{2/3}(UV)^{1/3}) (Nq)^{o(1)}.$$ 

We choose $U = q^{1/2}V$ to balance the terms $NV^{-1/2}$ and $q^{1/4}NU^{-1/2}$ which gives

$$S_a(q; N) \leq (Nq^{-1/24+o(1)} + U + NV^{-1/2} + q^{5/18}N^{2/3}V^{2/3}) (Nq)^{o(1)}.$$ 

Choosing $V = N^{2/7}q^{-5/21}$ to balance the terms $NV^{-1/2}$ and $q^{5/18}N^{2/3}V^{2/3}$ we get

$$S_a(q; N) \leq (Nq^{-1/24+o(1)} + q^{11/42}N^{2/7} + q^{5/42}N^{6/7}) (Nq)^{o(1)}.$$ 

We have $U \geq V \geq 1$ when $N \geq q^{5/6}$, which is when the term $q^{5/42}N^{6/7}$ becomes nontrivial. Also we need

$$UV = q^{1/42}N^{4/7} \leq N$$

which is satisfied for $N \geq q^{1/18}$ which we may suppose since otherwise the bound is trivial. Finally we note that we may remove the middle term, since it is dominated by the last term for $N \geq q^{1/4}$.

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