An Uncertain Allocation Model in Data Envelopment Analysis: A Case in the Iranian Stock Market

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Abstract
Data envelopment analysis (DEA) can be employed for investigating operation and evaluation of units as one of the most important concerns of managers. DEA is a linear programming technique for calculating relative performance of decision-making units (DMUs) with multiple inputs and outputs. Although all inputs and outputs are considered as certain items in these models, there are uncertain items in the real word and existing interference between these two concepts will result in uncertain models. Allocation models were studied in an uncertain environment with belief degree based uncertain input costs and output prices. Belief degree based uncertainty is useful for cases where there is no historical information on an uncertain event. Utilizing the uncertain entropy model as a second objective function, the cost and revenue models showed an optimal performance with a maximum dispersion rate in their constituent components. As a solution methodology, the uncertain allocation models were separately converted into crisp models by expect value (EV) and expected value and chance-constrained (EVCC) methods. A practical example from the Iranian Stock Market was also presented to evaluate the performance of the new model.

Keywords: Data envelopment analysis; Uncertainty theory; Belief degree; Stock market.

1. Introduction
Data Envelopment Analysis (DEA) is recognized as a robust analytical tool extensively utilized in measuring the relative efficiency of a group of Decision Making Units (DMUs) with multiple inputs and outputs. The DEA models originally developed by Charens et al. [1] within a printed-paper named CCR. They expanded the nonparametric method introduced by Farrell [2] to gauge DMUs with multiple inputs and outputs CCR makes use of mathematical optimization in linear programming formula to measure the performance of a DMU relative to a set of DMUs. Afterward, Banker et al. [3] introduced the BCC model. In addition to CCR and BCC, there are several models that discuss DEA from several perspectives: RAM by Cooper et al. [4], slack-adjust by Sueyoshi [5], additive model by Seiford et al. [6], SBM model by Tone [7], and FDH model by Deprins et al. [8], all of which are DEA basic models.

In classical DEA models, DMUs are evaluated by considering input and output values in order to measure rational efficiency as compared to different DMUs; eventually, the measure to which rational efficiency belong is obtained (0, 1). DEA is used for measuring and analyzing some concepts such as cost and revenue efficiencies [9]. In fact, one of the most important aspects of product analysis and organization is to measure cost and revenue efficiencies [2]. Real efficiency models for calculating cost efficiency search for a unit that consumes least cost for buying inputs not more than the inputs of the units under investigation for producing outputs equal to the outputs of the units under investigation. Furthermore, a revenue efficiency model searches for a unit which gain the best
revenue from selling outputs greater than outputs of the units under investigation by consuming the inputs equal to the inputs of the units under investigation.

The method for experimental evaluation of cost and revenue efficiencies was first developed by Färe et al. [10]. The cost and revenue models introduced by Färe et al. [10] not only need inputs and outputs but every price can be different in every unit. This difference may limit applications of this model. Their model was based on some sampling hypothesis. The inputs should be homogenous and the prices should be available and specified. Because of changes in the process or in input specifications, the techniques and inputs in big and small organizations may be different. As a result, inputs and their costs can also be different. Accordingly, the technical structure of DEA models can be more sophisticated [9].

According to Jamshidi et al. [11], the classical DEA models assumed that inputs and outputs are represented by precise values. But in many real world systems like banking, insurance, and other financial systems, the inputs and outputs are not precise and cannot be measured exactly. Considering this difficulty, many researchers tried to formulate the DEA problems with different hypotheses. The possible hypothesis is the earliest principles which may be used to build stochastic DEA models. Sengupta [12] summed up the stochastic DEA model utilizing expected value. Moreover, Banker [13] consolidated applied mathematics elements underneath DEA in order to develop a statistical method. Several papers utilized chance-constrained programming to DEA so as to introduce stochastic varieties to information [14,4,15,16]. Fuzzy outlook is another theory in which the hypothesis has been incorporated to cope with the uncertainty in DEA. United with the DEA innovator, Cooper et al. [17,18,19] introduced a technique to deal with inaccurate information such as moderate data, adjectival data and ratio moderate data in DEA. Furthermore, Kao and Liu [20] designed a technique to discover the membership function of fuzzy performance marks when each input and output are fuzzy numbers. Entani et al. [21] proposed an interval potency DEA model by pessimistic and idealistic values. Several researchers have introduced the possibility measure into DEA [22,23,24]. As confirmed by several studies, human uncertainty does not come with the same fuzziness. Because of the shortages of Fuzzy Theory, Liu [25] introduced Uncertain Theory and refined it in 2010 as an understandable mathematical structure for confronting uncertainty in data which serves as a strong alternative to the probability theory when one has to restrict the information in the face of insufficient trusted data. The belief degree function is associated with an underlying concept of this theory built according to the experts’ opinion [26].

Optimization problems, including uncertain data, can be even more interesting and realistic in uncertain environments with uncertain values for parameters and even variables [27]. To tackle such problems with uncertain parameters, any approach based on randomness, fuzzy theory, stochastic programming, probability theory, and so on can be applied in the face of historical information of the parameters.

In such cases, the uncertain manner of the problem is estimated from the historical data as the probability function, random number, fuzzy number, etc [25]. On the other hand, for cases in which no historical information for an uncertain event exists, uncertainty theory based on belief degree has been applied to solve the problem.

This uncertainty theory can be explained by a simple example. Consider a bridge and strength. At first, it is assumed that no destructive experiment is allowed for the bridge. Thus, there is no sample regarding the strength of the bridge. In this case, no statistical methods exist for estimating its probability distribution. Therefore, there is choice but to invite bridge engineers to evaluate the belief degrees about the bridge’s strength [28]. Some basic concepts of the belief degree-based uncertainty theory will be explained in Section 2, and a complete study of this topic can be found in Liu [25].

The belief degree depends heavily on personal knowledge (even including preferences) concerning the event. When the personal knowledge changes, the belief degree changes as well. Different people may produce different belief degrees. The question is which belief degree is correct, a question which may be answered as follows: All belief degrees are wrong, but some are useful [28].

A belief degree becomes “correct” only when it is close enough to the frequency of the indeterminate quantity which, however, does not usually occur. Numerous surveys demonstrated that human beings usually estimate a much wider range of values than the object actually takes. This human conservatism makes the belief degrees deviate far from the frequency. Thus, all belief degrees are
wrong compared with the frequency [25]. Nevertheless, it is undeniable that these belief degrees are indeed helpful for decision-making. Moreover, as determined by the choice of $\alpha$, there is a risk that DMUs would not be efficient even when their condition is satisfied [28].

Wen et al. [29] applied the uncertain theory for the first time to rewrite the DEA model in uncertainty condition and then published a paper on the sensitivity and stability of the additive model in terms of uncertainty. Wen et al. [30] also introduced a new Additive model with uncertain inputs and outputs. Additionally, Wen et al. [31] developed the DEA model with uncertainty index ranking for criteria. Lio et al. [32] also published a paper to evaluate DMU with uncertain inputs and outputs. Jamshidi et al. [11] developed the RUSSEL DEA model with uncertain inputs and outputs. Jamshidi et al. [26] applied the slacks-based measure (SBM) model in an uncertain environment where the uncertain inputs and outputs are belief degree-based uncertainty.

These studies are nevertheless insufficient for describing programming with uncertain data [33, 34, 35, 36, 37, 38, 27, 39] and new models are often introduced to create a new method. According to Shannon entropy, Liu [40] introduced the concept of entropy for uncertain variables for the first time to determine the uncertainty of uncertain variables resulting from information deficiency. Clearly, entropy functions are used as a tool for measuring dispersion in allocation models. Chen and Dai [41], Dai and Chen [42] investigated the maximum entropy of uncertainty distribution for uncertain variables. They presented the entropy of a function of an uncertain variable. In this study, input costs and output prices are considered as uncertain variables and then two different models are proposed to convert the new model into a crisp model to deal with the uncertainty problem. Thereafter, uncertain allocation models are used in a practical example in the Iranian Stock Market to find a stock portfolio with the maximum return.

Based on the literature of the uncertain DEA problems, no serious study has been performed on incorporating entropy and belief degree-based uncertain DEA models. Actually, entropy measures the degree of uncertainty. This study for the first time aims at applying an uncertain environment in allocation models. Since entropy helps diversification of our model, another function is considered as a second objective function. This in turn resulted in the best and most diversification of allocation models. Considering the maximum entropy in allocation models help us to achieve highest revenue and the least cost with a maximum dispersal measure. This new multi-objective models will result in the efficiency of units which is considered more dispersed in all processes. The parameters of the objective function and constraints are considered to be of zigzag uncertainty variables. Two approaches of expect value (EV) and expected value and chance-constrained (EVCC) methods are developed for the uncertain allocation problems. A real practical example from the Iranian Stock Market was also presented to evaluate the performance of the new model. Moreover, the new multi-objective uncertain allocation models will develop the power of managerial decisions and a clear way for choosing components of allocation models.

The paper is organized as follows. Some preliminary knowledge on uncertainty theory is reviewed in Section 2. Entropy function in an uncertain space is introduced in Section 3. Basic concepts of allocation DEA models are introduced in Section 4. Some new uncertain DEA models are introduced in Section 5 and their new structures are proved. Section 6 concerns crisp equivalents of DEA models. Entropy-based allocation models with uncertain variables are introduced in Section 7. Weighted method which is solely a multi-objective model is introduced in Section 8. Finally, an applied example on Iranian Stock Market is solved by allocation uncertain models.

2. Preliminaries

Here, discuss basic concept and present uncertain variables [25]. Let $\Gamma$ be a nonempty set, and $L$ an $\sigma$-algebra over $\Gamma$. Each element $\Lambda \in L$ is called an event. A set function $M(\Lambda) \in [0,1]$ is known as an uncertain measure if it satisfies the following three axioms [25]:

1) $M(\Omega) = 1$ for the universal set $\Omega$.

2) $M(\Lambda) + M(\Lambda^c) = 1$ for any event $\Lambda$.

3) For every countable subadditive of events, $\{\Lambda_i\}$ we have $M(\bigcup_{i=1}^{\infty} \Lambda_i) \leq \sum_{i=1}^{\infty} M(\Lambda_i)$. 


**Definition 1** [25]. The set function $\mathcal{M}$ called an uncertain measure if it contents the duality, normality, and subadditivity axioms.

The uncertain measure has the following attributes:

i. $\mathcal{M}\{\emptyset\} = 0$.

ii. $0 \leq \mathcal{M}\{\Lambda\} \leq 1$ for any event $\Lambda$.

iii. $\mathcal{M}\{\Lambda_i\} \leq \mathcal{M}\{\Lambda_j\}$ for any events $\Lambda_i \subset \Lambda_j$.

The triplet $(\Gamma, L, \mathcal{M})$ called an uncertainty space. In order to define product uncertain measure, Liu [43] proposed the fourth axiom as follow:

Let $(\Gamma_k, L_k, \mathcal{M}_k)$ be uncertainty space for $k=1, 2, ..., n$, then the product uncertain measure is an uncertain measure satisfying

$$\mathcal{M}\left(\prod_{k=1}^{n}\Lambda_k\right) = \prod_{k=1}^{n}\mathcal{M}_k\{\Lambda_k\}.$$

**Definition 2** [25]. An uncertain variable is a measurable function $\xi$ from an uncertainty space $(\Gamma, L, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set $B$ of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

An uncertainty distribution function is used to characterize an uncertain variable and is defined as follows [25].

$$\varphi(x) = \mathcal{M}\{\xi \leq x\} \quad \forall x, x \in \mathcal{R} \quad (1)$$

**Theorem 1** [44]. A function $\Phi(x) : \mathcal{R} \rightarrow [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.

**Point 1.** The zigzag uncertainty distribution is an uncertain variable $\xi$ is shown with $Z(a, b, c)$ expressed as follows:

$$\varphi(x) = \begin{cases} 0 & x \leq a \\ \frac{(x-a)}{2(b-a)} & a \leq x \leq b \\ \frac{(x+c-2b)}{2(c-b)} & b \leq x \leq c \\ 1 & x \geq c. \end{cases} \quad (2)$$

where $a, b, c$ are real numbers with $a < b < c$.

**Definition 3.** An uncertainty distribution $\varphi$ is said to be regular if its inverse function $\varphi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$ for instance the uncertain distributions are called in (2).

**Point 2.** The inverse uncertainty distribution of zigzag uncertain variable $Z(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha-1)c, & \alpha \geq 0.5 \end{cases} \quad (3)$$

**Definition 4** [28]. The uncertain variables $\xi_1, \xi_2, ..., \xi_n$ are considered to be independent if

$$\mathcal{M}\left(\bigcap_{i=1}^{n}(\xi_i \in B_i)\right) = \mathcal{M}\left(\bigcap_{i=1}^{n}\xi_i \in B_i\right).$$
for any Borel sets $B_1, B_2, \ldots, B_n$.

**Definition 5** [28]. The uncertain variable $\xi_1, \xi_2, \ldots, \xi_n$ are independent if and only if

$$
\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}
$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

**Theorem 2** [28]. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables, and $f_1, f_2, \ldots, f_n$ measurable functions. Then $f_1(\xi_1), f_2(\xi_2), \ldots, f_n(\xi_n)$ are independent uncertain variables.

**Theorem 3** [28]. Let $\xi_1, \xi_2, \ldots, \xi_n$ be variables with independent uncertainty with regular uncertainty distributions $\phi_1, \phi_2, \ldots, \phi_n$ respectively. If $f$ is a strictly increasing function, so $\xi = f(\xi_1, \ldots, \xi_n)$ is an uncertain variable with inverse uncertain distribution $\psi^{(-1)} = f(\phi_1^{-1}(\alpha), \phi_2^{-1}(\alpha), \ldots, \phi_n^{-1}(\alpha))$.

**Theorem 4** [28]. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\phi_1, \phi_2, \ldots, \phi_n$ respectively. If $f$ is a strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_{m+n}$, then $\xi = f(\xi_1, \ldots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution

$$
\phi^{-1}(\alpha) = f(\phi_1^{-1}(\alpha), \ldots, \phi_m^{-1}(\alpha), \phi_{m+1}^{-1}(1-\alpha), \ldots, \phi_n^{-1}(1-\alpha))
$$

**Definition 6** [28]. If $\xi$ be an uncertain variable. So the expected value of $\xi$ is defined by

$$
E[\xi] = \int_{0}^{\infty} \mathcal{M}(\xi \geq x)dx - \int_{-\infty}^{0} \mathcal{M}(\xi \leq x)dx
$$

provided that at least one of the two integral is finite.

For the couple uncertain variable and distribution $(\xi, \phi)$ have some formulas explained as follow:

$$
E[\xi] = \int_{0}^{\infty} (1-\phi(x))dx - \int_{-\infty}^{0} \phi(x)dx
$$

(5)

**Point 3.** The expected value for variable $\xi$ with zigzag uncertain distribution is defined as follow:

$$
E[\xi] = \frac{a+2b+c}{4}
$$

(7)

**Theorem 5** [28]. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\phi_1, \phi_2, \ldots, \phi_n$, respectively. If $f$ is a strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_n$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_{m+n}$, then $\xi = f(\xi_1, \ldots, \xi_n)$ has an expected value

$$
E[\xi] = \int_{0}^{1} f(\phi_1^{-1}(\alpha), \ldots, \phi_m^{-1}(\alpha), \phi_{m+1}^{-1}(1-\alpha), \ldots, \phi_n^{-1}(1-\alpha))d\alpha.
$$

(8)

**Theorem 6** [28]. Let $\xi$ and $\eta$ be independent uncertain variables with finite expected value. Then for any real numbers $a$ and $b$ have, $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$. 

5
Theorem 7 [28]. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\varphi_1, \varphi_2, \ldots, \varphi_n$, respectively. If the function $f(\xi_1, \ldots, \xi_n)$ is a strictly increasing with respect to $\xi_1, \xi_2, \ldots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_{mn}$, then

$$\mathcal{M}\{f(\xi_1, \ldots, \xi_n) \leq 0\} \geq \alpha$$

if and only if

$$f(\phi_1^{-1}(\alpha), \ldots, \phi_m^{-1}(\alpha), \phi_{m+1}^{-1}(1-\alpha), \ldots, \phi_n^{-1}(1-\alpha)) \leq 0.$$

Theorem 8 [28]. Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$. Then

$$E[\xi] = \int_0^1 \varphi^{-1}(\alpha) d\alpha$$

(9)

3. Entropy function in uncertain variable

The primary definition of entropy is presented in this section to specify the uncertainty of uncertain variable.

Definition 7 [28]. Suppose that $\xi$ is an uncertain variable with uncertainty distribution $\Phi$. Then its entropy is defined as follows:

$$H[\xi] = \int_{-\infty}^{\infty} S(\Phi(x)) dx$$

(10)

where $S(t) = -t \ln t - (1-t) \ln (1-t)$.

It is easy to verify that $S(t)$ is a symmetric function about $t = 0.5$, strictly increasing on the interval $[0, 0.5]$, strictly decreasing on the interval $[0.5, 1]$, and reaches its unique maximum $\ln 2$ at $t = 0.5$.

Point 4. Let $\xi$ be a zigzag uncertain variable $\xi \sim Z(a, b, c)$. Then its entropy is

$$H[\xi] = \frac{c-a}{2}$$

(11)

Theorem 9. Let $\xi$ be an uncertain variable. Then $H[\xi] \geq 0$ and equality holds if $\xi$ is essentially a constant.

Theorem 10. Let $\xi$ be an uncertain variable, and let $c$ be a real number, then

$$H[\xi + c] = H[\xi]$$

Theorem 11 [42]. Let $\xi$ be an uncertain variable with regular uncertainty distribution $\Phi$. Then

$$H[\xi] = \int_0^1 \left( \Phi^{-1}(\alpha) \ln \left( \frac{\alpha}{1-\alpha} \right) \right) d\alpha$$

(13)

Theorem 12 [42]. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\varphi_1, \varphi_2, \ldots, \varphi_n$, respectively. If $f(x_1, \ldots, x_n)$ is strictly increasing with respect to $x_1, \ldots, x_m$...
and strictly decreasing with respect to \( x_{m+1}, \ldots, x_{m+n} \), then the uncertain variable \( \xi = f(\xi_1, \ldots, \xi_n) \) has an entropy

\[
H[\xi] = \int_0^1 f \left( \Phi_m^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_m^{-1}(1-\alpha), \ldots, \Phi_m^{-1}(1-\alpha) \right) \ln \left( \frac{1}{1-\alpha} \right) d\alpha
\]

(14)

**Theorem 13** [42]. Let \( \xi \) and \( \eta \) be independent uncertain variables. Then for any real numbers \( a \) and \( b \), we have

\[
H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta]
\]

(15)

### 4. DEA models

Assume that there are \( n \) DMUs to be evaluated, each consisting of \( x_i (i = 1, \ldots, m) \) as input vector and \( y_j (r = 1, \ldots, s) \) as output vector. Also, \( c = (c_1, \ldots, c_m) \) for \( (i = 1, \ldots, m) \) and \( p = (p_1, \ldots, p_m) \) for \( (r = 1, \ldots, s) \), respectively, represent the input costs and output prices. Assume that \( DMU_0 \) is an evaluated unit. The cost efficiency model searches for a unit which consumes least cost for buying inputs not more than the inputs in the units under investigation for producing outputs equal to the outputs of the units under investigation. The cost efficiency model is defined as follows [7]:

\[
\mathbf{cx}^* = \min_{x, \lambda} cx = \min \sum_{i=1}^m c_{io} x_i
\]

s.t. \( x_i \geq \sum_{j=1}^n x_j \lambda_j \quad i = 1, \ldots, m \)

\( y_{ro} \leq \sum_{j=1}^n y_{oj} \lambda_j \quad r = 1, \ldots, s \)

\( x_i \geq 0 \quad i = 1, \ldots, m \)

\( \lambda_j \geq 0 \quad j = 1, \ldots, n. \)

(16)

Assuming an optimal solution of \((x^*, \lambda^*)\), the cost efficiency ratio is defined as:

\[
E_c = \frac{cx^*}{cx_0}
\]

(17)

according to the equation (17), \( 0 \leq E_c \leq 1 \).

**Definition 8.** \( DMU_0 \) is cost efficient if and only if \( E_c = 1 \).

In addition, the revenue efficiency model also searches for a unit which gains the best revenue from selling the outputs greater than outputs of the units under investigation and consumes the inputs equal to the inputs of the units under investigation. The revenue efficiency model is defined as follows [7]:

\[
\mathbf{py}^* = \max_{y, \lambda} py = \max \sum_{r=1}^i p_{ro} y_r
\]

s.t. \( x_{ro} \geq \sum_{j=1}^n x_{oj} \lambda_j \quad i = 1, \ldots, m \)
\[ y_r \leq \sum_{j=1}^{n} y_{o_j} \lambda_j \quad r = 1, \ldots, s \]
\[ y_r \geq 0 \quad r = 1, \ldots, s \]
\[ \lambda_j \geq 0 \quad j = 1, \ldots, n. \]

Assuming an optimal solution of \((y^*, \lambda^*)\) the revenue efficiency ratio is defined as:
\[ E_R = \frac{p_y^*}{p_y} \]

according to the equation (19), \(0 \leq E_R \leq 1\).

**Definition 9.** DMU \(o\) is revenue efficient if and only if \(E_R = 1\).

5. Uncertain DEA model

The allocation model requires inputs and outputs equipped with precise data. Nevertheless, in real-world situations, inputs and outputs may be unstable and complicated and, therefore, cannot be measured in an accurate manner. Consequently, this conflict results in the investigation of uncertain DEA models. Decision-makers in real-world situations make their decisions in the indeterminacy state. To model indeterminacy, there exist two mathematical systems, one the probability theory [45] and the other the uncertainty theory [25]. If there exists frequency in phenomena, the probability theory is employed; otherwise, the uncertain theory can be a powerful technique for resolving the drawback with no sample, using the personal belief degree. For this purpose, skilled consultants and experts should be invited to measure the belief degree. Belief degree based uncertainty is useful for cases where there is no historical information on an uncertain event. For example, both costs and prices may be unstable and complex in the stock market. Therefore, uncertain DEA models should be used for discovering the higher efficiency of two people with two different stock portfolios in terms of cost and revenue. Throughout this approach, we aimed to introduce an allocation model with uncertain inputs and outputs referred to as the uncertain an allocation model. First, new symbols and notations are presented:

\[ \tilde{x}_k = (\tilde{x}_{1k}, \tilde{x}_{2k}, \ldots, \tilde{x}_{mk}) : \text{the uncertain inputs vector of } DMU_k, \ k = 1, 2, \ldots, n, \]
\[ \varphi_{ik}(x) : \text{the uncertainty distribution of } \tilde{x}_{ik}, \ k = 1, 2, \ldots, n, i = 1, 2, \ldots, m, \]
\[ \tilde{y}_k = (\tilde{y}_{1k}, \ldots, \tilde{y}_{rk}) : \text{the uncertain outputs vector of } DMU_k, \ k = 1, 2, \ldots, n, \]
\[ \psi_{rk}(x) : \text{the uncertainty distribution of } \tilde{y}_{rk}, \ k = 1, 2, \ldots, n, r = 1, 2, \ldots, s, \]
\[ \alpha : \text{is a predetermined confidence level; } \]
\[ \mathcal{M} : \text{the uncertainty measure expressed in section 2; } \]

Now, suppose that the input costs, \(c_{ij}\) and the input vectors, \(x_{ij}\), for \((i = 1, \ldots, m)\), \((j = 1, \ldots, n)\) and the output vector \(y_{oj} (r = 1, \ldots, s)\), in Model (16) are uncertain variables which are represented by \(\tilde{c}_{oj}, \tilde{x}_{ij}\) and \(\tilde{y}_{oj}\), respectively. Accordingly, the uncertain cost efficiency model can be rewritten as follows:
\[ \tilde{c}x^* = \min_{x, \lambda} \tilde{c}x = \min \sum_{i=1}^{m} \tilde{c}_{io} x_i \]

subject to:

\[ x_i \geq \sum_{j=1}^{n} \bar{x}_{ij} \lambda_j \quad i = 1, \ldots, m \]  

\[ \bar{y}_{ro} \leq \sum_{j=1}^{n} \bar{y}_{oj} \lambda_j \quad r = 1, \ldots, s \]

\[ \lambda_j \geq 0, x_i \geq 0 \quad j = 1, \ldots, n, i = 1, \ldots, m. \]

Suppose also that the output prices \( p_{rj} \) and the output vectors \( y_{oj} \) for \( (r = 1, \ldots, s), (j = 1, \ldots, n) \) and the input vector \( x_{ij} \) \( (i = 1, \ldots, m) \), in Model (18) are uncertain variables which are represented by \( \tilde{p}_{rj}, \tilde{y}_{oj} \) and \( \tilde{x}_{ij} \), respectively. Accordingly, the uncertain cost efficiency model can be rewritten as follows:

\[ \tilde{p}y^* = \max_{y, \lambda} \tilde{p}y = \max \sum_{r=1}^{s} \tilde{p}_{rj} y_r \]

subject to:

\[ \ddot{x}_{io} \geq \sum_{j=1}^{n} \tilde{x}_{ij} \lambda_j \quad i = 1, \ldots, m \]  

\[ y_{rj} \leq \sum_{j=1}^{n} \tilde{y}_{oj} \lambda_j \quad r = 1, \ldots, s \]

\[ \lambda_j \geq 0, y_{rj} \geq 0 \quad j = 1, \ldots, n, r = 1, \ldots, s. \]

6. Crisp equivalents of the model

To deal with the uncertainty problem, two different methods are presented to convert it into a crisp model:

1. Expected Value (EV) method,

2. Expected Value and Chance-Constrained (EVCC) method.

6.1. EV method

In this section, the uncertain cost model is transformed to a crisp model using EV method.

6.1.1. Optimization of cost model

The uncertain cost efficiency model (20) is introduced as follows using EV method:

\[ \theta = E[\tilde{c}x^*] = \min_{x, \lambda} E[\tilde{c}x] = \min \sum_{i=1}^{m} E[\tilde{c}_{io} x_i] \]

subject to:

\[ E\left[ \sum_{j=1}^{n} \bar{x}_{ij} \lambda_j - x_i \right] \leq 0 \quad i = 1, \ldots, m \]  

\[ E\left[ \bar{y}_{ro} - \sum_{j=1}^{n} \bar{y}_{oj} \lambda_j \right] \leq 0 \quad r = 1, \ldots, s \]

\[ \lambda_j \geq 0, x_i \geq 0 \quad j = 1, \ldots, n, i = 1, \ldots, m. \]
Definition 10 (Efficiency). In the model (22), DMU₀ is efficient if and only if \( \theta^* = 1 \), where \( \theta^* \) is the optimal value of (22).

Theorem 14. Assume that the input costs \( \tilde{c}_i, \tilde{c}_2, \ldots, \tilde{c}_m \), the inputs \( \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m \) and the outputs \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_m \) are independent uncertain variables respectively with uncertainty distributions of \( \Phi_i, \Phi_2, \ldots, \Phi_m \) and \( \psi_i, \psi_2, \ldots, \psi_m \) and \( \chi_1, \chi_2, \ldots, \chi_m \) where \( i = 1, 2, \ldots, m \) and \( r = 1, \ldots, s \). Then the uncertain programming model (22) will be equivalent to the following model:

\[
\min \sum_{i=1}^{m} x_i \int_{0}^{1} \Phi_i^{-1}(\alpha) d\alpha
\]

s.t.
\[
\sum_{j=1}^{n} \lambda_j \int_{0}^{1} \psi_j^{-1}(\alpha) d\alpha - x_i \leq 0 \quad i = 1, \ldots, m \\
\int_{0}^{1} \chi_{ro}^{-1}(\alpha) d\alpha - \sum_{j=1}^{n} \lambda_j \int_{0}^{1} \chi_{ro}^{-1}(1-\alpha) d\alpha \leq 0 \quad r = 1, \ldots, s \\
\lambda_j \geq 0, x_i \geq 0 \quad j = 1, \ldots, n, i = 1, \ldots, m
\]

Proof: According of Theorem (6), the objective function in (22) is rewritten as follows:

\[
E \left[ \sum_{i=1}^{m} \tilde{c}_i x_i \right] = \sum_{i=1}^{m} E[\tilde{c}_i x_i] = \sum_{i=1}^{m} x_i E[\tilde{c}_i]
\]

The function \( \sum_{i=1}^{m} x_i E[\tilde{c}_i] \) is strictly increasing with respect to \( \tilde{c}_i \) for each \( i \). According to Theorem (8), we have:

\[
\sum_{i=1}^{m} x_i E[\tilde{c}_i] = \sum_{i=1}^{m} x_i \int_{0}^{1} \Phi_i^{-1}(\alpha) d\alpha
\]

As a result, the objective function is proved. Now, according to Theorem (8):

\[
E[a \tilde{\xi}] = \int_{0}^{1} a \Phi^{-1}(\alpha) d\alpha = a \int_{0}^{1} \Phi^{-1}(\alpha) d\alpha = a E[\tilde{\xi}]
\]

Then let us to prove the constraint as follows:

According to Theorem (6), the first constraint of the model (22) is rewritten as follows:

\[
E \left[ \sum_{j=1}^{n} \tilde{y}_j \lambda_j - x_i \right] = E \left[ \sum_{j=1}^{n} \tilde{y}_j \lambda_j \right] - x_i = \sum_{j=1}^{n} \lambda_j E[\tilde{y}_j] - x_i = \sum_{j=1}^{n} \lambda_j \int_{0}^{1} \psi_j^{-1}(\alpha) d\alpha - x_i
\]

The function \( \tilde{y}_ro - \sum_{j=1}^{n} \tilde{y}_j \lambda_j \) is strictly increasing with respect to \( \tilde{y}_ro \) and strictly decreasing with respect to \( \tilde{y}_j \) for each \( r, r=1, \ldots, s \) and \( j, j=1, \ldots, n \). According to theorems (6) and (8), we have:

\[
E \left[ \tilde{y}_ro - \sum_{j=1}^{n} \tilde{y}_j \lambda_j \right] = E[\tilde{y}_ro] - E \left[ \sum_{j=1}^{n} \tilde{y}_j \lambda_j \right] = \int_{0}^{1} \chi_{ro}^{-1}(\alpha) d\alpha - \sum_{j=1}^{n} \lambda_j \int_{0}^{1} \chi_{ro}^{-1}(1-\alpha) d\alpha
\]
The theorem is therefore proved.

### 6.1.2. Optimization of revenue model

Using the EV method which was explained in the previous section, the uncertain revenue model (21) is introduced as follows:

\[
\theta = E[\hat{p}y^*] = \max_{y, \alpha} E[\hat{p}y] = \max_{y, \alpha} E\left[\sum_{r=1}^{s} \hat{p}_r y_r \right]
\]

s.t.

\[
E\left[\sum_{j=1}^{n} \hat{x}_j \lambda_j - \hat{x}_io \right] \leq 0 \quad i = 1, \ldots, m
\]

\[
E\left[y_r - \sum_{j=1}^{n} \hat{y}_j \lambda_j \right] \leq 0 \quad r = 1, \ldots, s
\]

\[
\lambda_j \geq 0, y_r \geq 0.
\]

**Definition 11** (Efficiency). In the model (24), DMU\(_o\) is efficient if and only if \(\theta^* = 1\), where \(\theta^*\) is the optimal value of (24).

**Theorem 15.** Assume that the output prices \(\tilde{p}_{i1}, \tilde{p}_{i2}, \ldots, \tilde{p}_{im}\), the inputs \(\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots, \tilde{x}_{im}\) and the outputs \(\tilde{y}_{r1}, \tilde{y}_{r2}, \ldots, \tilde{y}_{rn}\) are independent uncertain variables respectively with uncertainty distributions \(\tau_{i1}, \tau_{i2}, \ldots, \tau_{im}\), \(\psi_{i1}, \psi_{i2}, \ldots, \psi_{im}\) and \(\chi_{r1}, \chi_{r2}, \ldots, \chi_{rn}\) where \(i=1,2,\ldots,m\) and \(r=1,\ldots,s\). Then the uncertain programming model (24) will be equivalent to the following model:

\[
\text{max} \sum_{r=1}^{s} y_r \int_{0}^{1} \tau_{rj}^{-1}(\alpha) d\alpha
\]

s.t

\[
\sum_{j=1}^{n} \lambda_j \int_{0}^{1} \psi_{ij}^{-1} (\alpha) - \int_{0}^{1} \psi_{io}^{-1} (1-\alpha) d\alpha \leq 0 \quad i = 1,2,\ldots,m
\]

\[
y_r - \sum_{j=1}^{n} \lambda_j \int_{0}^{1} \chi_{rj}^{-1} (1-\alpha) d\alpha \leq 0 \quad r = 1,\ldots,s
\]

\[
\lambda_j \geq 0, y_r \geq 0
\]

**Proof:** According to Theorem (6), the objective function in (24) is rewritten as follows:

\[
E\left[\sum_{r=1}^{s} \tilde{p}_r y_r \right] = \sum_{r=1}^{s} E[\tilde{p}_r y_r] = \sum_{r=1}^{s} y_r E[\tilde{p}_r]
\]

The function \(\sum_{r=1}^{s} y_r E[\tilde{p}_r]\) is strictly increasing with respect to \(\tilde{p}_r\) for each \(r\). According to Theorem (8), we have:

\[
\sum_{r=1}^{s} y_r E[\tilde{p}_r] = \sum_{r=1}^{s} y_r \int_{0}^{1} \tau_{rj}^{-1}(\alpha) d\alpha
\]
Thus, the objective function is proved. Now, according to Theorem (8):

\[
E[a\xi] = \int_0^1 a\Psi^{-1}(\alpha) d\alpha = a\int_0^1 \Psi^{-1}(\alpha) d\alpha = aE[\xi]
\]

Then let us to prove the constraint as follows:

According to Theorem (6), the first constraint of the model (24) is rewritten as follows:

\[
E\left[y_r - \sum_{j=1}^n \bar{y}_{rj} \lambda_j\right] = y_r - E\left[ \sum_{j=1}^n \bar{y}_{rj} \lambda_j\right] = y_r - \sum_{j=1}^n E\left[ \bar{y}_{rj}\right] \lambda_j
\]

The function \(y_r - \sum_{j=1}^n E\left[ \bar{y}_{rj}\right] \lambda_j\) is strictly increasing with respect to \(-\bar{y}_{rj}\) for each \(r=1, \ldots, s\) and \(j=1, \ldots, n\). According to Theorem (8), we have:

\[
y_r - \sum_{j=1}^n E\left[ \bar{y}_{rj}\right] \lambda_j = y_r - \sum_{j=1}^n \lambda_j \int_0^1 \chi^{-1}_{rj}(1-\alpha) d\alpha, \quad \forall r, r=1, \ldots, s.
\]

The function \(\sum_{j=1}^n \bar{x}_{ij} \lambda_j - \bar{x}_{io}\) is strictly increasing with respect to \(\bar{x}_{ij}\) and strictly decreasing with respect to \(-\bar{x}_{io}\) for each \(i=1, \ldots, m\) and \(j=1, \ldots, n\). According to theorems (6) and (8), we have:

\[
E\left[ \sum_{j=1}^n \bar{x}_{ij} \lambda_j - \bar{x}_{io}\right] = \sum_{j=1}^n \lambda_j E\left[ \bar{x}_{ij}\right] - E\left[ \bar{x}_{io}\right] = \sum_{j=1}^n \lambda_j \int_0^1 \Psi^{-1}_i(\alpha) d\alpha - \int_0^1 \Psi^{-1}_io(1-\alpha) d\alpha, \quad i=1, \ldots, m
\]

The theorem is therefore proved.

6.2. EVCC method
Using EVCC method, the uncertain cost and revenue models are converted into crisp models and the new crisp models are solved with the help of specific software.

6.2.1. Optimization of cost model
The uncertain cost model (20) is converted to a crisp model using EVCC method as follows:

\[
E[\tilde{c}x^*] = \min_{x,i} E[\tilde{c}x] = \min_{x,i} E\left[ \sum_{i=1}^m \tilde{c}_{io} x_i \right]
\]

s.t. \( \mathcal{M}\left\{ \sum_{j=1}^n \bar{x}_{ij} \lambda_j - x_i \leq 0 \right\} \geq \alpha \quad i=1, 2, \ldots, m \)

\( \mathcal{M}\left\{ \tilde{y}_{io} - \sum_{j=1}^n \tilde{y}_{rj} \lambda_j \leq 0 \right\} \geq \alpha \quad r=1, \ldots, s \)

\( \lambda_j \geq 0, x_i \geq 0 \quad j=1, \ldots, n, i=1, \ldots, m \).

Definition 12. A vector \((x_i, \lambda_j) \geq 0\) where \(j=1, \ldots, n, i=1, \ldots, m\), is called a feasible solution to the uncertain programming model (26) if:
\[ M \left\{ \sum_{j=1}^{n} \tilde{x}_{ij} \tilde{\lambda}_j - x_i \leq 0 \right\} \geq \alpha \quad i = 1, \ldots, m. \]

\[ M \left\{ \tilde{y}_{ro} - \sum_{j=1}^{n} \tilde{y}_{j} \tilde{\lambda}_j \leq 0 \right\} \geq \alpha \quad r = 1, \ldots, s. \]

**Definition 13.** A feasible solution \((x_i^*, \lambda_j^*)\) is called an expected optimal solution to the uncertain programming model (26) if:

\[ E \left[ \sum_{i=1}^{m} \tilde{c}_i \tilde{x}_i^* \right] \leq E \left[ \sum_{i=1}^{m} \tilde{c}_i x_i \right] \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \quad (27) \]

for any solution \((x, \lambda_j)\).

**Definition 14.** A larger optimal objective value in model (26) means a more efficient DMU_o.

**Theorem 16.** Assume that the input costs \(\tilde{c}_{i1}, \tilde{c}_{i2}, \ldots, \tilde{c}_{im}\), the inputs \(\tilde{x}_{i1}, \tilde{x}_{i2}, \ldots, \tilde{x}_{im}\) and the outputs \(\tilde{y}_{r1}, \tilde{y}_{r2}, \ldots, \tilde{y}_{rm}\) are independent uncertain variables respectively with uncertainty distributions of \(\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{im}\), \(\psi_{r1}, \psi_{r2}, \ldots, \psi_{rm}\) and \(\chi_{i1}, \chi_{i2}, \ldots, \chi_{in}\) where \(i = 1, 2, \ldots, m\) and \(r = 1, \ldots, s\). Then, the uncertain programming model (26) will be equivalent to the following model:

\[
\begin{align*}
\min & \quad \sum_{i=1}^{m} \int_{0}^{1} \Phi_{iw}^{-1}(\alpha) d\alpha \\
\text{s.t.} \quad & \quad \sum_{j=1}^{n} \psi_{ij}^{-1}(\alpha) \tilde{\lambda}_j - x_i \leq 0 \quad i = 1, 2, \ldots, m, \\
& \quad \chi_{ro}^{-1}(\alpha) - \sum_{j=1}^{n} \chi_{ij}^{-1}(1-\alpha) \tilde{\lambda}_j \leq 0 \quad r = 1, \ldots, s, \\
& \quad \tilde{\lambda}_j \geq 0, x_i \geq 0 \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \quad (28)
\end{align*}
\]

**Proof:** Equivalency of objective function was proved in the proof of Theorem 15. To prove the equivalency of the constraints, the function \(\sum_{j=1}^{n} \tilde{x}_{ij} \tilde{\lambda}_j - x_i\) is strictly increasing with respect to \(\tilde{x}_{ij}\) for each \(i = 1, \ldots, m\) and \(j = 1, \ldots, n\). According to Theorem (7):

\[ M \left\{ \sum_{j=1}^{n} \tilde{x}_{ij} \tilde{\lambda}_j - x_i \leq 0 \right\} \geq \alpha \quad \Leftrightarrow \quad \sum_{j=1}^{n} \psi_{ij}^{-1}(\alpha) \tilde{\lambda}_j - x_i \leq 0, \quad \forall j, i = 1, \ldots, n, \quad i = 1, \ldots, m, \]

Also, the function \(\tilde{y}_{ro} - \sum_{j=1}^{n} \tilde{y}_{j} \tilde{\lambda}_j\) is strictly increasing with respect to \(\tilde{y}_{ro}\) and strictly decreasing with respect to \(\tilde{y}_{j}\) for each \(r = 1, \ldots, s\) and \(j = 1, \ldots, n\). According to Theorem (7):
\[ \mathcal{M}\left\{ \bar{y}_r - \sum_{j=1}^{n} \bar{y}_j \lambda_j \leq 0 \right\} \geq \alpha \iff \chi_{\alpha}^{-1}(\alpha) - \sum_{j=1}^{n} \chi_{\alpha}^{-1}(1-\alpha) \lambda_j \leq 0, \quad j = 1, \ldots, n, r = 1, \ldots, s \]

Therefore, the theorem is proved.

6.2.2. Optimization of revenue model

The uncertain revenue model (21) is converted to a crisp model using the EVCC method:

\[ E[\tilde{\delta}y^*] = \max_{y, \alpha} E[\tilde{\delta}y] = \max E \left[ \sum_{r=1}^{s} \tilde{\delta}_r y_r \right] \]

s.t. \[ \mathcal{M}\left\{ \sum_{j=1}^{n} \bar{x}_j \lambda_j - \bar{x}_i \leq 0 \right\} \geq \alpha \quad i = 1, 2, \ldots, m, \]
\[ \mathcal{M}\left\{ y_r - \sum_{j=1}^{n} \bar{y}_j \lambda_j \leq 0 \right\} \geq \alpha \quad r = 1, \ldots, s, \]
\[ \lambda_j \geq 0, y_r \geq 0 \quad j = 1, \ldots, n, r = 1, \ldots, s \]

Definition 15. A vector \( (y_r, \lambda_j) \geq 0 \) where \( j = 1, \ldots, n, r = 1, \ldots, s \), is called a feasible solution to the uncertain programming model (29) if

\[ \mathcal{M}\left\{ \sum_{j=1}^{n} \bar{x}_j \lambda_j - \bar{x}_i \leq 0 \right\} \geq \alpha \quad i = 1, 2, \ldots, m, \]
\[ \mathcal{M}\left\{ y_r - \sum_{j=1}^{n} \bar{y}_j \lambda_j \leq 0 \right\} \geq \alpha \quad r = 1, \ldots, s, \]

Definition 16. A feasible solution \( (y_r^*, \lambda_j^*) \) is called an expected optimal solution to the uncertain programming model (29) if:

\[ E \left[ \sum_{r=1}^{s} \tilde{\delta}_r y_r^* \right] \geq E \left[ \sum_{r=1}^{s} \tilde{\delta}_r y_r \right] \quad j = 1, \ldots, n, r = 1, \ldots, s \]

(30)

for any solution \( (y_r, \lambda_j) \).

Definition 17. A larger optimal objective value in model (29) means a more efficient DMU.

Theorem 17. Assume that the output prices \( \tilde{\delta}_{i1}, \tilde{\delta}_{i2}, \ldots, \tilde{\delta}_{in} \), the inputs \( \bar{x}_{i1}, \bar{x}_{i2}, \ldots, \bar{x}_{in} \) and the outputs \( \bar{y}_{i1}, \bar{y}_{i2}, \ldots, \bar{y}_{im} \) are independent uncertain variables respectively with uncertainty distributions of \( \tau_{i1}, \tau_{i2}, \ldots, \tau_{in} \), \( \Psi_{i1}, \Psi_{i2}, \ldots, \Psi_{in} \) and \( \chi_{r1}, \chi_{r2}, \ldots, \chi_{rn} \) where \( i = 1, 2, \ldots, m \) and \( r = 1, \ldots, s \). Then the uncertain programming model (29) will be equivalent to the following model:
\[
\max \sum_{r=1}^{s} y_r \int_{0}^{\tau} \hat{y}_r \left( \alpha \right) d\alpha
\]

s.t. \[
\sum_{j=1}^{r} \hat{y}_{ij} \psi_{ij}^{-1}(\alpha) - \psi_{ij}^{-1}(1-\alpha) \leq 0 \quad i = 1, 2, \ldots, m, \]
\[
y_r - \sum_{j=1}^{n} \hat{y}_{ij} \left( 1-\alpha \right) \hat{\lambda}_j \leq 0 \quad r = 1, \ldots, s, \]
\[
\hat{\lambda}_j \geq 0, y_r \geq 0 \quad j = 1, \ldots, n, r = 1, \ldots, s
\]

(31)

**Proof:** Equivalency of objective function was proved in the proof of Theorem 15. To prove the equivalency of the constraints, the function \[
\sum_{j=1}^{r} \hat{y}_{ij} \hat{\lambda}_j - \hat{x}_{io} \quad \text{is strictly increasing with respect to } \hat{x}_{ij}
\]
and strictly decreasing with respect to \(-\hat{x}_{io}\) for each \(i=1, \ldots, m, \text{ and } j=1, \ldots, n.\) According to Theorem (7), we have:

\[
\mathcal{M} \left\{ \sum_{j=1}^{r} \hat{y}_{ij} \hat{\lambda}_j - \hat{x}_{io} \leq 0 \right\} \geq \alpha \quad \iff \quad \mathcal{M} \left\{ \sum_{j=1}^{n} \hat{x}_{ij} \hat{\lambda}_j - \hat{x}_{io} \leq 0 \right\} \geq \alpha \quad \iff \quad \sum_{j=1}^{n} \hat{y}_{ij} \psi_{ij}^{-1}(\alpha) - \psi_{ij}^{-1}(1-\alpha) \leq 0, \quad j = 1, \ldots, n, i = 1, \ldots, m,
\]

Also, the function \(y_r - \sum_{j=1}^{n} \hat{y}_{ij} \hat{\lambda}_j\) is strictly decreasing with respect to \(-\hat{y}_{ij}\) for each \(r=1, \ldots, s, \text{ and } j=1, \ldots, n.\) According to Theorem (7) we have:

\[
\mathcal{M} \left\{ y_r - \sum_{j=1}^{n} \hat{y}_{ij} \hat{\lambda}_j \leq 0 \right\} \geq \alpha \quad \iff \quad y_r - \sum_{j=1}^{n} \hat{y}_{ij} \left( 1-\alpha \right) \hat{\lambda}_j \leq 0, \quad j = 1, \ldots, n, r = 1, \ldots, s.
\]

Proof is completed.

**7. Entropy-based Allocation models with uncertain variables**

According to the literature on uncertain DEA, there is no study on the use of entropy in models assigned by uncertain data. Entropy is applied to provide a quantitative measure for the degree of uncertainty. According to Shannon entropy [46], Liu [47] introduced the concept of entropy for uncertain variables for the first time to determine the uncertainty of uncertain variables resulting from information deficiency. Chen and Dai [41] and Dai and Chen [42] investigated the principle of maximum entropy of uncertainty distribution for uncertain variables. They presented the entropy of a function for uncertain variables. Considering the maximum entropy in allocation models help us to achieve highest revenue and the least cost with a maximum dispersal measure. This model will result in the efficiency of units which is considered more dispersed in all processes. Moreover, it will develop the power of managerial decisions.

Now, the model (13) is rewritten as a second objective function using the models (23), (25), (28) and (31) as follow. By utilizing Theorem (12) and (13) to the objective function of model (23), we define the following uncertain entropy function.

**Lemma 1.** Suppose \(\tilde{c}_{i1}, \tilde{c}_{i2}, \ldots, \tilde{c}_{im}\) is an uncertain variable with regular uncertainty distribution \(\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{in}\) for \((i=1, \ldots, m)\). If \(f : \mathcal{R}^{n} \to \mathcal{R}\) is a strictly increasing function with respect to \(x_{i}, i=1, \ldots, m\), then the uncertain function \(f(x, \tilde{c})\) has an entropy,
\[ H(x, \tilde{c}) = \sum_{i=1}^{m} x_i \int_{0}^{1} \phi_{i0}^{-1}(\alpha) \ln \left( \frac{\alpha}{1-\alpha} \right) d\alpha \]  

(32)

**Proof:** Since \( \tilde{c}_{i0} \) has a regular uncertain distribution \( \phi_{i0} \) we obtain

\[ H(x, \tilde{c}) = \sum_{i=1}^{m} x_i \int_{0}^{\infty} f (\phi_{i0}(x)) \, dx \]

From Theorem (13), this equality can be rewritten as:

\[ H(x, \tilde{c}) = \sum_{i=1}^{m} x_i \int_{0}^{\infty} \phi_{i0}(x) \, f' (\alpha) \, d\alpha \, dx + \sum_{i=1}^{m} x_i \int_{0}^{\infty} \phi_{i0}(x) \, -f' (\alpha) \, d\alpha \, dx \]

where \( f' (\alpha) = (-\alpha \ln \alpha - (1-\alpha) \ln(1-\alpha))' = -\ln \left( \frac{\alpha}{1-\alpha} \right) \).

By applying the Fubini Theorem \([48]\) to the above function, we obtain

\[ H(x, \tilde{c}) = \sum_{i=1}^{m} x_i \int_{0}^{1} \phi_{i0}^{-1}(\alpha) \, f' (\alpha) \, d\alpha \, dx + \sum_{i=1}^{m} x_i \int_{0}^{1} \phi_{i0}^{-1}(\alpha) \, -f' (\alpha) \, d\alpha \, dx \]

\[ = \sum_{i=1}^{m} x_i \int_{0}^{1} \phi_{i0}^{-1}(\alpha) \, f' (\alpha) \, d\alpha = \sum_{i=1}^{m} x_i \int_{0}^{1} \phi_{i0}^{-1}(\alpha) \ln \left( \frac{\alpha}{1-\alpha} \right) d\alpha \]

**Lemma 2.** Suppose \( \tilde{p}_{r1}, \tilde{p}_{r2}, \ldots, \tilde{p}_{rn} \) is an uncertain variable with regular uncertainty distribution \( \tau_{r1}, \tau_{r2}, \ldots, \tau_{rn} \) for \( r=1, \ldots, s \). If \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a strictly increasing function with respect to \( y_{r}, r=1, \ldots, s \), then the uncertain function \( f(y, \tilde{p}) \) has an entropy,

\[ H(y, \tilde{p}) = \sum_{r=1}^{s} y_{r} \int_{0}^{\infty} \tau_{r0}^{-1}(\alpha) \ln \left( \frac{\alpha}{1-\alpha} \right) d\alpha \]  

(33)

**Proof:** Since \( \tilde{p}_{r0} \) has a regular uncertain distribution \( \tau_{r0} \) we obtain

\[ H(y, \tilde{p}) = \sum_{r=1}^{s} y_{r} \int_{0}^{\infty} f (\tau_{r0}(x)) \, dx \]

From Theorem (13), this equality can be rewritten as:

\[ H(y, \tilde{p}) = \sum_{r=1}^{s} y_{r} \int_{0}^{1} f' (\alpha) \, d\alpha \, dx + \sum_{r=1}^{s} y_{r} \int_{0}^{1} f' (\alpha) \, -f' (\alpha) \, d\alpha \, dx \]

Where \( f'(\alpha) = (-\alpha \ln \alpha - (1-\alpha) \ln(1-\alpha))' = -\ln \left( \frac{\alpha}{1-\alpha} \right) \)

\[ f' (\alpha) = (-\alpha \ln \alpha - (1-\alpha) \ln(1-\alpha))' = -\ln \left( \frac{\alpha}{1-\alpha} \right) \]

By applying the Fubini Theorem \([48]\) to the above function, we obtain

\[ H(y, \tilde{p}) = \sum_{r=1}^{s} y_{r} \int_{0}^{1} f' (\alpha) \, d\alpha \, dx + \sum_{r=1}^{s} y_{r} \int_{0}^{1} f' (\alpha) \, -f' (\alpha) \, d\alpha \, dx \]

\[ = \sum_{r=1}^{s} y_{r} \int_{0}^{1} \tau_{r0}^{-1}(\alpha) \, f' (\alpha) \, d\alpha = \sum_{r=1}^{s} y_{r} \int_{0}^{1} \tau_{r0}^{-1}(\alpha) \ln \left( \frac{\alpha}{1-\alpha} \right) d\alpha \]

According to above discussions, the allocative efficiency models are presented below. First, the new multi-objective cost efficiency model consisting of an uncertain cost model and an uncertain entropy function is explained.
Note 1. All multi-objective problems are solved by the weighting method. Accordingly, the multi-objective problems are converted into single-objective ones. The coefficients $w_1$ and $w_2$ are considered for improving the flexibility of our model to be used by decision makers for determining the importance of objective functions as $w_1+w_2=1$. The values of these functions are determined before solving problems by decision makers. Equal weights ($w_1=w_2$) means equal attention of decision makers to both functions.

The uncertain multi-objective cost efficiency model is introduced as follows using EVCC method. EVCC method is used for converting the new model to a crisp model.

\[
Z_1 = \min \sum_{i=1}^{m} x_i \int_{0}^{1} \Phi_{i_{w}}^{-1}(\alpha) \, d\alpha
\]
\[
Z_2 = \max \sum_{i=1}^{m} x_i \int_{0}^{1} \Phi_{i_{w}}^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \, d\alpha
\]
\[\text{s.t} \]
\[
\sum_{j=1}^{n} \psi_{ij}^{-1}(\alpha) \lambda_j - x_i \leq 0 \quad i = 1, \ldots, m
\]
\[
\chi_{r_{w}}^{-1}(\alpha) - \sum_{j=1}^{n} \chi_{r_{ij}}^{-1}(1-\alpha) \lambda_j \leq 0 \quad r = 1, \ldots, s
\]
\[
\lambda_j \geq 0 \quad x_i \geq 0 \quad j = 1, \ldots, n, \quad i = 1, \ldots, m
\]

Using the weighting method which was explained in the previous, the uncertain multi-objective model (34) is rewritten as follows:

\[
\text{Min}(w_1 Z_1 - w_2 Z_2)
\]
\[\text{s.t} \]
\[
\sum_{j=1}^{n} \psi_{ij}^{-1}(\alpha) \lambda_j - x_i \leq 0 \quad i = 1, \ldots, m
\]
\[
\chi_{r_{w}}^{-1}(\alpha) - \sum_{j=1}^{n} \chi_{r_{ij}}^{-1}(1-\alpha) \lambda_j \leq 0 \quad r = 1, \ldots, s
\]
\[
\lambda_j \geq 0 \quad x_i \geq 0 \quad j = 1, \ldots, n, \quad i = 1, \ldots, m
\]

Definition 19. The greater the optimal objective value in (35) is, the more efficient $DMU_o$ is ranked.

The new multi-objective revenue efficiency model consisting of an uncertain revenue model as first objective function and an uncertain entropy as second objective function is then introduced. The uncertain multi-objective revenue efficiency model is introduced as follows using EVCC method. Again, EVCC method is used for converting the uncertain form to crisp form.

\[
Z_1 = \max \sum_{r=1}^{s} y_r \int_{0}^{1} \Psi_{r_{w}}^{-1}(\alpha) \, d\alpha
\]
\[
Z_2 = \max \sum_{r=1}^{s} y_r \int_{0}^{1} \Psi_{r_{w}}^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \, d\alpha
\]
\[\text{s.t} \]

\[
\sum_{j=1}^{n} \lambda_j \psi_j^{-1}(\alpha) - \psi_i^{-1}(1-\alpha) \leq 0 \quad i = 1, \ldots, m
\]
\[
y_r - \sum_{j=1}^{n} \chi_j^{-1}(1-\alpha) \lambda_j \leq 0 \quad r = 1, \ldots, s
\]
\[
\lambda_j \geq 0, y_r \geq 0 \quad j = 1, \ldots, n, r = 1, \ldots, s
\]

Using the weighting method, the uncertain multi-objective model (36) is rewritten as follows:

Min\((-w_1Z_1 - w_2Z_2)\)

s.t

\[
\sum_{j=1}^{n} \lambda_j \psi_j^{-1}(\alpha) - \psi_i^{-1}(1-\alpha) \leq 0 \quad i = 1, \ldots, m
\]
\[
y_r - \sum_{j=1}^{n} \chi_j^{-1}(1-\alpha) \lambda_j \leq 0 \quad r = 1, \ldots, s
\]
\[
\lambda_j \geq 0, y_r \geq 0 \quad j = 1, \ldots, n, r = 1, \ldots, s
\]

Definition 20. The greater the optimal objective value in (37) is, the more efficient \(DMU_0\) is ranked.

Using the uncertain entropy model as a second objective function, we will be able to achieve an optimal performance in cost and revenue models with a maximum dispersion rate in their constituent components. As a solution methodology, the uncertain allocation models were converted to crisp models using two EV approaches. The uncertain multi-objective cost efficiency model is introduced as follows using EV method. EV method is used for converting the new model to a crisp model.

\[
Z_1 = \min \sum_{i=1}^{m} x_i \int_0^{1} \Phi_i^{-1}(\alpha) d\alpha
\]
\[
Z_2 = \max \sum_{i=1}^{m} x_i \int_0^{1} \Phi_i^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} d\alpha
\]

s.t

\[
\sum_{j=1}^{n} \int_{0}^{1} \psi_j^{-1}(\alpha) \lambda_j d\alpha - x_i \leq 0 \quad i = 1, \ldots, m
\]
\[
\int_{0}^{1} \chi_i^{-1}(\alpha) d\alpha - \sum_{j=1}^{n} \int_{0}^{1} \chi_j^{-1}(1-\alpha) \lambda_j d\alpha \leq 0 \quad r = 1, \ldots, s
\]
\[
\lambda_j \geq 0, x_i \geq 0 \quad j = 1, \ldots, n, i = 1, \ldots, m
\]

Using the weighting method, the uncertain multi-objective model (38) is rewritten as follows:

Min\((w_1Z_1 - w_2Z_2)\)

s.t

\[
\sum_{j=1}^{n} \int_{0}^{1} \psi_j^{-1}(\alpha) \lambda_j d\alpha - x_i \leq 0 \quad i = 1, \ldots, m
\]
\[
\int_{0}^{1} \chi_i^{-1}(\alpha) d\alpha - \sum_{j=1}^{n} \int_{0}^{1} \chi_j^{-1}(1-\alpha) \lambda_j d\alpha \leq 0 \quad r = 1, \ldots, s
\]
\[
\lambda_j \geq 0, x_i \geq 0 \quad j = 1, \ldots, n, i = 1, \ldots, m
\]
Definition 21 (Efficiency). DMU$_o$ is efficient if and only if the optimal solution in (39) can achieve a value of 1.

Using the EV method, the multi-objective revenue efficiency model consisting of an uncertain revenue model and an uncertain entropy is introduced as follows:

\[
Z_1 = \max \sum_{r=1}^s y_r \int_0^1 \Psi_{ro}^{-1}(\alpha) d\alpha
\]

\[
Z_2 = \max \sum_{r=1}^s y_r \int_0^1 \Psi_{ro}^{-1}(\alpha) \ln \left( \frac{\alpha}{1-\alpha} \right) d\alpha
\]

s.t

\[
\sum_{j=1}^n \lambda_j \int_0^1 \psi_{ij}^{-1}(\alpha) d\alpha - \int_0^1 \psi_{io}^{-1}(1-\alpha) d\alpha \leq 0 \quad i = 1, \ldots, m
\]

\[
y_r - \sum_{j=1}^n \lambda_j \int_0^1 \chi_{rj}^{-1}(1-\alpha) \leq 0 \quad r = 1, \ldots, s
\]

\[
\lambda_j \geq 0, \quad y_r \geq 0 \quad j = 1, \ldots, n, \quad r = 1, \ldots, s.
\]

Using the weighting method, the uncertain multi-objective model (38) is rewritten as follows:

Min$(-w_1Z_1 - w_2Z_2)$

s.t

\[
\sum_{j=1}^n \lambda_j \int_0^1 \psi_{ij}^{-1}(\alpha) d\alpha - \int_0^1 \psi_{io}^{-1}(1-\alpha) d\alpha \leq 0 \quad i = 1, \ldots, m
\]

\[
y_r - \sum_{j=1}^n \lambda_j \int_0^1 \chi_{rj}^{-1}(1-\alpha) \leq 0 \quad r = 1, \ldots, s
\]

\[
\lambda_j \geq 0, \quad y_r \geq 0 \quad j = 1, \ldots, n, \quad r = 1, \ldots, s.
\]

Definition 22 (Efficiency). DMU$_o$ will be efficient if and only if the optimal solution in (41) can achieve a value of 1.

8. Practical Example

The accuracy of the above-mentioned models is examined using a practical example. There is a well-known direct relationship between business in stock market and market forecasting. In the meantime, the use of beliefs and opinions of experts in the field of buying and selling stocks is of great importance to obtain maximum revenue at minimum cost. Therefore, the use of belief theory in the stock market to take into account most efficient suggestions will be helpful in this business. Tables 1 and 2 list data of 25 stockbrokers for buying and selling a same stock portfolio with different prices. Considering that the amounts and numbers of stock are equal and the buying and selling prices are uncertain variables (Tables 3 and 4), the efficiency of cost and revenue models in the stock portfolio is explained.

Please Insert Table 1 about here.
Please Insert Table 2 about here.
Please Insert Table 3 about here.
Please Insert Table 4 about here.

The results obtained from Model (35) with different $w_1$ and $w_2$ are presented in Table 5.
According to the results, none of cost efficient DMUs are equal to 1. According to Definition (19), a larger optimal objective value means a more efficient $DMU_0$. Therefore, according to Model (35):

If $w_1 = 0.6, w_2 = 0.4$ then $DMU_{17}$ is efficient,
If $w_1 = 0.7, w_2 = 0.3$ then $DMU_8$ is efficient,
If $w_1 = 0.8, w_2 = 0.2$ then $DMU_8$ is efficient,
If $w_1 = 0.9, w_2 = 0.1$ then $DMU_8$ is efficient.

The results obtained from Model (39) with different $w_1$ and $w_2$ are shown in Table 6.

As clearly seen, the cost efficient DMUs are equal to 1. According to Definition (21) and Model (39):

1) If $w_1 = 0.6, w_2 = 0.4$ then $DMU_{17}$ is efficient,
2) If $w_1 = 0.7, w_2 = 0.3$ then $DMU_8$ is efficient,
3) If $w_1 = 0.8, w_2 = 0.2$ then $DMU_8$, $DMU_7$, $DMU_8$, $DMU_{11}$, $DMU_{16}$ and $DMU_{17}$ are efficient.

Table 7 presents the results obtained from Model (37) with different $w_1$ and $w_2$.

According to the result, none of revenue efficient DMUs are equal to 1. According to Definition (20), a larger optimal objective value means a more efficient $DMU_0$. Therefore, according to Model (37):

1) If $w_1 = 0.6, w_2 = 0.4$ then $DMU_6$, $DMU_7$, $DMU_8$, $DMU_{17}$, $DMU_{19}$, $DMU_{20}$ and $DMU_{21}$ are efficient.
2) If $w_1 = 0.5, w_2 = 0.5$ then $DMU_6$, $DMU_7$, $DMU_8$, $DMU_{17}$, $DMU_{19}$, $DMU_{20}$ and $DMU_{21}$ are efficient.

The results obtained from Model (41) with different $w_1$ and $w_2$ are presented in Table 8.

It seems that $DMU_8$ and $DMU_{17}$ have the best performance for selling a same stock portfolio with different prices.
According to the above results, DMU8 and DMU17 seem to be efficient in most models. These two DMUs showed the best performance in both cost and revenue models for buying a selling a same stock portfolio with different prices.

9. Conclusion

This paper aimed at presenting an uncertain allocation model with inherent complexity of uncertain models. Due to the complexity of the new models, two methods were proposed to convert the uncertain models into crisp models. Finally, an applied example regarding the Iranian Stock Market was discussed to examine the performance of the new models. For this purpose, 25 stockbrokers were selected to determine buying and selling prices of a same stock portfolio with different prices in the cost and revenue models. The amounts and numbers of stock were considered to be equal. The buying and selling prices were also considered as uncertain variables. These models help managers in choosing the best portfolio in the stock market. Instead of the expected value in objective functions with uncertain variables, the proposed variance can be taken into account in future studies. The problem presented in this paper can be also studied with normal uncertain variables.

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Tables:

Table 1. DMUs with three uncertain inputs (x_{ij}).

| DMU_i | Iran-Khodro     | Saderat Bank     | Pars Petrochemical |
|-------|-----------------|------------------|--------------------|
| DMU1  | Z(23200,24300,26400) | Z(26300,27400,28600) | Z(19000,19900,20200) |
| DMU2  | Z(20000,22000,240000) | Z(27600,28900,290000) | Z(18500,19300,21200) |
| DMU3  | Z(21000,22500,24300) | Z(28300,29500,31200) | Z(22000,23200,24100) |
| DMU4  | Z(21500,22700,23300) | Z(31200,32700,34200) | Z(17100,17900,18600) |
| DMU5  | Z(22900,25000,26300) | Z(33100,34300,36100) | Z(22300,23300,24000) |
| DMU6  | Z(21200,23200,24500) | Z(25230,25740,26150) | Z(18340,18770,18960) |
| DMU7  | Z(22300,23300,24900) | Z(26150,26630,26970) | Z(17000,17300,17700) |
| DMU8  | Z(21700,22900,23700) | Z(27200,27730,28050) | Z(18100,18900,19150) |
| DMU9  | Z(23100,23800,24100) | Z(27330,27520,27740) | Z(19200,19740,20000) |
| DMU10 | Z(21100,21700,22300) | Z(28130,28430,28720) | Z(22350,22980,23110) |
| DMU11 | Z(22400,23200,24300) | Z(29220,29780,30050) | Z(18100,18820,19100) |
| DMU12 | Z(23200,23900,24500) | Z(26130,26740,26970) | Z(19230,19490,19990) |
| DMU13 | Z(22200,22700,23200) | Z(26000,26370,26700) | Z(21340,21780,22000) |
| DMU14 | Z(23100,23500,23900) | Z(28240,28530,28780) | Z(22270,22520,22990) |
Table 2. DMUs with three uncertain outputs ($y_{ij}$).

| DMU  | Iran-Khodro          | Saderat Bank         | Pars Petrochemical     |
|------|----------------------|----------------------|------------------------|
| DMU1 | $Z(17500,18100,19600)$ | $Z(23500,25000,26100)$ | $Z(12000,12500,13200)$ |
| DMU2 | $Z(15100,16300,20000)$ | $Z(25000,26100,27300)$ | $Z(13400,17800,18400)$ |
| DMU3 | $Z(19000,20000,20900)$ | $Z(24200,25700,27600)$ | $Z(17300,18400,20000)$ |
| DMU4 | $Z(16000,17300,20000)$ | $Z(28100,29300,30300)$ | $Z(13000,13400,14100)$ |
| DMU5 | $Z(18400,19200,24000)$ | $Z(24000,25100,26500)$ | $Z(17300,18500,20500)$ |
| DMU6 | $Z(16200,16900,17200)$ | $Z(26230,26470,26580)$ | $Z(14110,14730,15100)$ |
| DMU7 | $Z(19100,19700,20200)$ | $Z(23130,23640,23830)$ | $Z(12410,12800,13000)$ |
| DMU8 | $Z(17100,17700,18300)$ | $Z(26430,26790,27100)$ | $Z(17230,17490,17990)$ |
| DMU9 | $Z(15100,15650,16100)$ | $Z(28130,28740,28970)$ | $Z(13210,13840,14050)$ |
| DMU10| $Z(18050,18250,18530)$ | $Z(23330,23520,23740)$ | $Z(15310,15660,15960)$ |
| DMU11| $Z(17300,17750,17960)$ | $Z(28240,28530,28780)$ | $Z(14340,14780,15000)$ |
| DMU12| $Z(16000,16300,16800)$ | $Z(24300,24800,24990)$ | $Z(12230,12540,12870)$ |
| DMU13| $Z(19150,19400,20000)$ | $Z(25170,25660,25820)$ | $Z(13180,13900,14000)$ |
| DMU14| $Z(15200,15900,1650)$ | $Z(24230,24740,25150)$ | $Z(15700,15900,16150)$ |
| DMU15| $Z(18300,19300,20900)$ | $Z(24200,24730,25050)$ | $Z(12730,13000,13150)$ |
| DMU16| $Z(19400,20200,21300)$ | $Z(23000,23370,23700)$ | $Z(16380,16980,17400)$ |
| DMU  | Iran-Khodro          | Saderat Bank          | Pars Petrochemical    |
|------|----------------------|-----------------------|-----------------------|
| DMU1 | Z(2603,2610,2715)    | Z(950,1000,1100)      | Z(30000,32000,34000)  |
| DMU2 | Z(2500,2520,2580)    | Z(900,950,990)        | Z(28000,29000,30000)  |
| DMU3 | Z(2450,2500,2530)    | Z(800,900,910)        | Z(27000,27500,27900)  |
| DMU4 | Z(2300,2310,2340)    | Z(810,820,850)        | Z(26000,26500,26700)  |
| DMU5 | Z(2200,2250,2290)    | Z(840,850,869)        | Z(25000,25500,25800)  |
| DMU6 | Z(2000,2100,2200)    | Z(890,899,910)        | Z(24000,24900,25100)  |
| DMU7 | Z(2190,2220,2240)    | Z(800,810,830)        | Z(28500,28600,29100)  |
| DMU8 | Z(2199,2210,2299)    | Z(830,840,850)        | Z(29100,29250,29380)  |
| DMU9 | Z(2090,2099,2115)    | Z(840,850,860)        | Z(29500,29900,30000)  |
| DMU10| Z(2100,2150,2180)    | Z(830,890,920)        | Z(30000,31000,31990)  |
| DMU11| Z(2110,2120,2130)    | Z(910,960,990)        | Z(31000,32000,32850)  |
| DMU12| Z(2122,2139,2169)    | Z(990,1010,1030)      | Z(32000,32800,33500)  |
| DMU13| Z(2220,2270,2299)    | Z(1000,1019,1038)     | Z(33000,33790,33990)  |
| DMU14| Z(2000,2050,2091)    | Z(1100,1200,1290)     | Z(33000,33990,34100)  |
| DMU15| Z(1900,1960,1999)    | Z(1090,1099,1120)     | Z(34000,34900,35200)  |
| DMU16| Z(2070,2080,2090)    | Z(1190,1200,1210)     | Z(35000,35990,36590)  |
| DMU17| Z(2120,2160,2190)    | Z(1210,1219,1229)     | Z(35900,36200,36990)  |
| DMU18| Z(2198,2210,2250)    | Z(1290,1231,1391)     | Z(36000,36900,37000)  |
| DMU19| Z(2260,2290,2300)    | Z(1391,1399,1410)     | Z(37000,37500,37990)  |
| DMU20| Z(2310,2321,2347)    | Z(1290,1310,1380)     | Z(37700,38100,38800)  |

Table 3. Input costs for DMUs ($c_{ij}$).
Table 4. The output prices for DMUs ($p_{ij}$).

| DMU  | Iran-Khodro       | Saderat Bank         | Pars Petrochemical |
|------|-------------------|----------------------|--------------------|
| DMU1 | Z(17500, 18100, 19600) | Z(25000, 26100, 26100) | Z(12000, 12500, 13200) |
| DMU2 | Z(15100, 16300, 20000) | Z(25000, 26100, 27300) | Z(13000, 17800, 18400) |
| DMU3 | Z(19000, 20000, 20900) | Z(24200, 25700, 27600) | Z(17300, 18400, 20000) |
| DMU4 | Z(16000, 17300, 20000) | Z(28100, 29300, 30300) | Z(13000, 13400, 14100) |
| DMU5 | Z(18400, 19200, 24000) | Z(24000, 25100, 26500) | Z(17300, 18500, 20500) |
| DMU6 | Z(16200, 16900, 17200) | Z(26230, 26470, 26580) | Z(14110, 14730, 15100) |
| DMU7 | Z(19100, 19870, 20200) | Z(23130, 23640, 23830) | Z(12410, 12800, 13000) |
| DMU8 | Z(17100, 17700, 18300) | Z(26430, 26790, 27100) | Z(17230, 17490, 17990) |
| DMU9 | Z(15100, 15650, 16100) | Z(28130, 28740, 28970) | Z(13210, 13840, 14050) |
| DMU10| Z(18050, 18250, 18530) | Z(23330, 23520, 23740) | Z(15310, 15660, 15960) |
| DMU11| Z(17300, 17750, 17960) | Z(28240, 28530, 28780) | Z(14340, 14780, 15000) |
| DMU12| Z(16000, 16300, 16800) | Z(24300, 24800, 24990) | Z(12230, 12540, 12870) |
| DMU13| Z(19150, 19400, 20000) | Z(25150, 25660, 25820) | Z(13180, 13900, 14000) |
| DMU14| Z(15200, 15900, 16500) | Z(24230, 24740, 25150) | Z(15700, 15900, 16150) |
| DMU15| Z(18300, 19300, 20900) | Z(24200, 24730, 25050) | Z(12730, 13000, 13150) |
| DMU16| Z(19400, 20200, 21300) | Z(23000, 23370, 23700) | Z(16380, 16980, 17400) |
| DMU17| Z(16100, 16500, 16900) | Z(22200, 22730, 23050) | Z(17340, 17770, 18960) |
| DMU18| Z(18300, 18900, 19200) | Z(23000, 23390, 23930) | Z(13170, 13890, 14220) |
| DMU19| Z(17150, 17230, 17780) | Z(28330, 28560, 28810) | Z(14430, 14730, 15150) |
| DMUi  | DMU1   | DMU2   | DMU3   | DMU4   | DMU5   |
|-------|--------|--------|--------|--------|--------|
|       | w₁ = 0.6 | w₂ = 0.4 |        |        |        |
|       | 0.149   | 0.168  | 0.156  | 0.170  | 0.150  |
|       | w₁ = 0.7 | w₂ = 0.3 |        |        |        |
|       | 0.324   | 0.370  | 0.342  | 0.375  | 0.330  |
|       | w₁ = 0.8 | w₂ = 0.2 |        |        |        |
|       | 0.504   | 0.571  | 0.527  | 0.581  | 0.513  |
|       | w₁ = 0.9 | w₂ = 0.1 |        |        |        |
|       | 0.687   | 0.770  | 0.718  | 0.790  | 0.694  |
|       | DMU6   | DMU7   | DMU8   | DMU9   | DMU10  |
|       | w₁ = 0.6 | w₂ = 0.4 |        |        |        |
|       | 0.164   | 0.170  | 0.172  | 0.158  | 0.139  |
|       | w₁ = 0.7 | w₂ = 0.3 |        |        |        |
|       | 0.361   | 0.376  | 0.380  | 0.347  | 0.303  |
|       | w₁ = 0.8 | w₂ = 0.2 |        |        |        |
|       | 0.556   | 0.558  | 0.585  | 0.539  | 0.471  |
|       | w₁ = 0.9 | w₂ = 0.1 |        |        |        |
|       | 0.753   | 0.790  | 0.799  | 0.728  | 0.640  |
|       | DMU11  | DMU12  | DMU13  | DMU14  | DMU15  |
|       | w₁ = 0.6 | w₂ = 0.4 |        |        |        |
|       | 0.168   | 0.144  | 0.146  | 0.133  | 0.130  |
|       | w₁ = 0.7 | w₂ = 0.3 |        |        |        |
|       | 0.370   | 0.314  | 0.320  | 0.293  | 0.285  |

Table 5. The result of evaluating the cost efficiency with model (35) (α=0.5).
Table 6. The result of evaluating the cost efficiency with model \((39) (\alpha=0.5)\).

| DMUi | DMU1 | DMU2 | DMU3 | DMU4 | DMU5 |
|------|------|------|------|------|------|
| \(w_1 = 0.8\) \(w_2 = 0.2\) | 0.575 | 0.488 | 0.496 | 0.455 | 0.442 |
| \(w_1 = 0.9\) \(w_2 = 0.1\) | 0.777 | 0.664 | 0.675 | 0.618 | 0.601 |
| \(w_1 = 0.6\) \(w_2 = 0.4\) | 0.168 | 0.173 | 0.142 | 0.160 | 0.128 |
| \(w_1 = 0.7\) \(w_2 = 0.3\) | 0.370 | 0.377 | 0.310 | 0.352 | 0.282 |
| \(w_1 = 0.8\) \(w_2 = 0.2\) | 0.574 | 0.584 | 0.481 | 0.545 | 0.438 |
| \(w_1 = 0.9\) \(w_2 = 0.1\) | 0.781 | 0.795 | 0.654 | 0.742 | 0.595 |
| \(w_1 = 0.6\) \(w_2 = 0.4\) | 0.133 | 0.155 | 0.150 | 0.135 | 0.132 |
| \(w_1 = 0.7\) \(w_2 = 0.3\) | 0.291 | 0.340 | 0.329 | 0.294 | 0.289 |
| \(w_1 = 0.8\) \(w_2 = 0.2\) | 0.451 | 0.528 | 0.510 | 0.456 | 0.450 |
| \(w_1 = 0.9\) \(w_2 = 0.1\) | 0.617 | 0.718 | 0.693 | 0.619 | 0.611 |

| DMUi | DMU6 | DMU7 | DMU8 | DMU9 | DMU10 |
|------|------|------|------|------|-------|
| \(w_1 = 0.6\) \(w_2 = 0.4\) | 0.262 | 0.287 | 0.274 | 0.300 | 0.267 |
| \(w_1 = 0.7\) \(w_2 = 0.3\) | 0.570 | 0.628 | 0.600 | 0.656 | 0.587 |
| \(w_1 = 0.8\) \(w_2 = 0.2\) | 0.882 | 0.965 | 0.921 | 1 | 0.907 |
| \(w_1 = 0.6\) \(w_2 = 0.4\) | 0.285 | 0.297 | 0.301 | 0.275 | 0.244 |
| \(w_1 = 0.7\) \(w_2 = 0.3\) | 0.627 | 0.651 | 0.663 | 0.605 | 0.532 |
| \(w_1 = 0.8\) \(w_2 = 0.2\) | 0.966 | 1 | 1 | 0.939 | 0.826 |
| \(w_1 = 0.6\) \(w_2 = 0.4\) | 0.294 | 0.252 | 0.255 | 0.233 | 0.228 |
| \(w_1 = 0.7\) \(w_2 = 0.3\) | 0.645 | 0.548 | 0.559 | 0.511 | 0.500 |
| \(w_1 = 0.8\) \(w_2 = 0.2\) | 1 | 0.850 | 0.867 | 0.791 | 0.773 |
| \(w_1 = 0.6\) \(w_2 = 0.4\) | 0.298 | 0.302 | 0.248 | 0.280 | 0.225 |
### Table 7. The result of evaluating the revenue efficiency with model (37) (α=0.5).

| DMU_i | DMU1 | DMU2 | DMU3 | DMU4 | DMU5 |
|-------|------|------|------|------|------|
| w_1 = 0.4 | 0.51 | 0.59 | 0.63 | 0.59 | 0.57 |
| w_2 = 0.6 | 0.71 | 0.71 | 0.71 | 0.53 | 0.58 |
| w_1 = 0.5 | 0.82 | 0.82 | 0.82 | 0.67 | 0.73 |
| w_2 = 0.5 | 0.57 | 0.50 | 0.54 | 0.55 | 0.55 |
| w_1 = 0.5 | 0.74 | 0.62 | 0.69 | 0.69 | 0.71 |
| w_2 = 0.5 | 0.62 | 0.71 | 0.53 | 0.71 | 0.71 |
| w_1 = 0.5 | 0.73 | 0.82 | 0.68 | 0.82 | 0.82 |
| w_2 = 0.5 | 0.98 | 0.89 | 0.90 | 0.915 | 0.87 |
| w_1 = 0.5 | 0.92 | 1.0 | 1.0 | 0.955 | 0.977 |
| w_2 = 0.5 | 0.98 | 0.92 | 0.83 | 0.93 | 0.93 |
| w_1 = 0.6 | 0.915 | 0.885 | 0.97 | 0.87 | 0.905 |
| w_2 = 0.7 | 0.963 | 1.0 | 0.946 | 0.963 | 0.963 |
| w_1 = 0.6 | 1.0 | 0.963 | 1.0 | 0.946 | 0.963 |
| w_2 = 0.5 | 0.98 | 0.89 | 0.90 | 0.92 | 0.88 |
| w_1 = 0.7 | 0.905 | 0.85 | 0.89 | 0.89 | 0.86 |
| w_2 = 0.6 | 1.0 | 0.937 | 0.963 | 0.955 | 0.946 |

### Table 8. The result of evaluating the revenue efficiency with model (41) (α=0.5).

| DMU_i | DMU1 | DMU2 | DMU3 | DMU4 | DMU5 |
|-------|------|------|------|------|------|
| w_1 = 0.3 | 0.81 | 0.95 | 1.0 | 0.90 | 0.93 |
| w_2 = 0.7 | 0.83 | 0.93 | 0.81 | 0.915 | 0.87 |
| w_1 = 0.4 | 0.92 | 1.0 | 1.0 | 0.955 | 0.977 |
| w_2 = 0.5 | 0.98 | 0.92 | 0.83 | 0.93 | 0.93 |
| w_1 = 0.4 | 0.915 | 0.885 | 0.97 | 0.87 | 0.905 |
| w_2 = 0.6 | 1.0 | 0.963 | 1.0 | 0.946 | 0.963 |
| w_1 = 0.5 | 0.98 | 0.89 | 0.90 | 0.92 | 0.88 |
| w_2 = 0.7 | 0.905 | 0.85 | 0.89 | 0.89 | 0.86 |
| w_1 = 0.5 | 1.0 | 0.937 | 0.963 | 0.955 | 0.946 |
| w_2 = 0.5 | 0.98 | 0.90 | 0.98 | 0.92 | 0.92 |
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