Exploring the 2-D Bateman dual system Hamiltonian: a first approach

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Abstract. Dissipation is of major interest in physics and engineering in all energy scales and, hence, it devoted a large amount of investigation. For quantum systems, an usual model for study is the Bateman dual system. Here we use the same model to explore dissipation in angular momentum instead of energy. For that matter, we start with an effective 2-D Bateman model in polar (cylindrical) coordinates and we find the expression for the Hamiltonian. Two constants of motion are found related to modified angular momenta and we test if they are in involution with the Hamiltonian, i.e., if they are integrals of motion. This is in order to verify that the corresponding quantum versions of these constants are good candidates to be a complete set of commuting observables.

1. Introduction
Dissipation of energy is a main matter in several fields in engineering and physics. Not only for handle with practical designs but for a correct model of the phenomena as well. The subject has devoted a large amount of papers and many aspects remain unclear. For physicists and mathematicians the study is related to simple models, even in the classical mechanics case. In quantum mechanics the introduction of dissipation break the probability conservation and, hence, usual approaches are insufficient. Non-unitary or extended models are proposed to obtain insight in the $\hbar$ scale. For a view of the subject a new version of [1] recently appeared.

Between the several approaches to the subject in the quantum case the Bateman dual system is widely studied [2]. This model uses dual harmonic oscillators, one with linear dissipation and the other with an absorbing term of the same amount. Hence the energy that one system loose is the gain for the other and the whole system preserves the total amount of energy. The Lagrangian of this dual system is

$$L = m (\dot{x} \dot{y}) + \frac{\gamma}{2} (x \dot{y} - \dot{x} y) - m \omega^2 xy$$

(1)

which gives place to the Newtonian equations of dissipation/absorptio one dimension harmonic oscillators. Dots on the variables denote time derivative. The $x$ variable is for the dissipative oscillator, meanwhile the $y$ is for the amplified or absorptive one. It is important to say that the dissipative term is proportional to the velocity and corresponds to the simplest case. The case where the friction is

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constant, the so-called Coulomb case, is harder to solve, even when the function is simpler. As we are interested in quantum mechanics the usual operator to quantize is the Hamiltonian, for (1) it is

\[ H = \frac{p_x p_y}{m} + \frac{\gamma}{2} (y p_y - x p_x) + \left( m \omega^2 - \frac{\gamma^2}{4 m} \right) x y \]  

(2)

Even when the present case and in the rest of the article we are considering classical quantities the purpose is to analyse the quantum case and hence the Weyl rule of ordering the dynamical variables should be considered. As usual the system contains an effective frequency, \( m \omega^2 - \frac{\gamma^2}{4 m} \), that rules the damped, under-damped and critically damped cases. Here we consider the under-damped case, that is, we have oscillations always.

1.1 Motivation
In the vast majority of the literature the energy is the main variable analysed [1]. However, for several many-body systems, the angular momenta are the main physical quantity subject to measurement and to dissipation as well. For the light, for instance, the existence of orbital angular momentum [3] makes the subject of theoretical interests, without lack of experimental possibilities.

The paper is organized as follows: in the next section we propose the Lagrangian and the Hamiltonian for the effective 2-D system. In the next sub-section we discuss the transformation to cylindrical coordinates for both the Lagrangian and the Hamiltonian, followed by the discussion of the constants of motion. The conclusions are presented in section 3.

2. Building a 2-D Bateman dual model

2.1. The Lagrangian and the Hamiltonian proposed in Cartesian coordinates
The main idea is to introduce a 3-D model that allows the existence of angular momentum for each oscillator, but such a goal is full of unclear terms that make the interpretation hard [4]. A systematic approach start with a 2-D confined system to the cylindrical coordinates, this left only the angular momentum \( z \) component to analyse. The conservation of this component is confirmed in the dissipation-free case as explained below. Hence a natural way to generalize the system is to consider a Lagrangian with the form

\[ L(x, y, z, u, v, w, \dot{x}, \dot{y}, \dot{z}, \dot{u}, \dot{v}, \dot{w}) = m (\dot{x} u + \dot{y} v) + \frac{\gamma}{2} (x \dot{u} - x \dot{u} + y \dot{v} - y \dot{v}) - m \omega^2 (x u - y v) \]  

(3)

with the variables \( x, y \) and \( z \) to the dissipated oscillator and the \( u, v \) and \( w \) to the absorbing part. In the rest of the article we shall refer only the in planes \( x-y \) and \( u-v \) coordinates and the corresponding velocities and momenta. Here we are pairing the \( x \) with the \( u \) and the \( y \) with the \( v \) for the duals systems and they represent damped/absorbing oscillators in the Newtonian case. The Hamiltonian is obtained in the usual way as the Legendre transformation of the generalized velocities as

\[ H = \frac{p_x p_u}{m} + \frac{\gamma}{2} (p_x x - p_u u) + \left( m \omega^2 - \frac{\gamma^2}{4 m} \right) x u + \frac{p_y p_v}{m} + \frac{\gamma}{2} (p_y y - p_v v) + \left( m \omega^2 - \frac{\gamma^2}{4 m} \right) y v \]  

(4)

This Hamiltonian can be solved analytically, in the quantum case, with the Feshbach- Tikochinsky method [5], but it is not our interest to find solutions in the Cartesian form. We are looking for the solutions in the angular momentum variables, hence we transform to polar coordinates. Notice that the system lies in an 8-dimensional phase space (plus the additional coordinates that allow the existence of angular momentum as a wedge product).

2.2. Transformation to angular variables
The procedure we follow to change to the new variables is the standard one, except that the crossing terms in the kinetic energy produces the effect that the momenta from one oscillator are generated by
the generalized velocities of the other, even in the friction-free case. Here we summarize and discuss the results.

The notation we use is the following:

\[ x = r \cos \theta, \quad y = r \sin \theta \] (5)

for the dissipative subsystem and

\[ u = \rho \cos \psi, \quad v = \rho \sin \psi \] (6)

for the absorptive, amplified, oscillator. The Hamiltonian obtained in these coordinates is

\[
H = \frac{\cos(\theta - \psi)P_r P_\rho}{m} + \frac{\sin(\theta - \psi)P_r P_\psi}{m \rho} - \frac{\sin(\theta - \psi)P_\theta P_\rho}{m r} + \frac{\cos(\theta - \psi)P_\theta P_\psi}{m r \rho} \\
+ \frac{\gamma}{2} \left( \rho P_\rho - r P_r \right) + \left( m \omega^2 \cos(\theta - \psi) - \frac{\gamma^2}{4 m} \right) r \rho
\] (7)

which depends on all the momenta and coordinates, i.e., from the eight effective variables. In order to understand how to simplify the problem we obtain insight from the drag-free case. The Hamiltonian obtained from the decoupled Lagrangian is

\[
H_0 = \frac{P_r P_\rho}{m} + \left( \frac{P_r P_\psi}{m \rho} - \frac{P_\theta P_\rho}{m r} \right) + \frac{P_\theta P_\psi}{m r \rho} + (m \omega^2) r \rho
\] (8)

Notice that the dual system is decoupled now and the dynamics is the same for both oscillators. From the equation of Hamilton we obtain that

\[
\frac{d}{dt} r^2 \dot{\theta} = 0
\] (9)

i.e., it is a constant of motion. We shall call it as

\[ \kappa_1 = \left( r^2 \dot{\theta} \right) \] (10)

The same happens for the other oscillator,

\[ \kappa_2 = \left( \rho^2 \dot{\psi} \right) \] (11)

These quantities correspond to the \( z \) component of the angular momenta and its time derivative to the centripetal acceleration. Now, we can rewrite the Hamiltonian in term of these constants of motion and obtain

\[
H = \frac{P_r P_\rho}{m} - \frac{\kappa_1 \kappa_2}{m r \rho} + \frac{\gamma}{2} \left( \rho P_\rho - r P_r \right) + (m \omega^2) r \rho
\] (12)

Notice that it corresponds to a 2-D harmonic oscillator in polar coordinates if \( \rho = r \) and, the usual effective potential,

\[
V_{\text{eff}}(r) = \frac{p_\theta^2}{2 m r^2} + \frac{m \omega^2}{2} r^2
\] (13)

appears. With this in mind we find, from the equations of motion, that the Hamiltonian written in equation (7) have the following constants of motion,

\[
\frac{d}{dt} \left( \log(r^2 \dot{\theta}) + \langle \gamma/m \rangle t \right) = 0
\] (14)

which we renamed as

\[ \kappa_3 = \left( \log(r^2 \dot{\theta}) + \langle \gamma/m \rangle t \right) \] (15)

For the dual oscillator we find

\[
\frac{d}{dt} \left( \log(\rho^2 \dot{\psi}) - \langle \gamma/m \rangle t \right) = 0
\] (16)
which we recast as

\[ \kappa_4 = (\log(\rho^2 \psi - (\ell'/m)t)). \]  

These constants are explicitly time dependent, hence it is not obvious if they are truly integrals of motion or isolating integrals. In order to analyze this aspect we check the full time equation in term of the Poisson bracket

\[ \frac{dl_k}{dt} = \frac{\partial l_k}{\partial t} + [H, l_k] \]  

for a constant \( l_k \), and the bracket

\[ [A, B] = \sum_{i=1}^{n} \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \]

For the constants given in (15) and (17), we obtain that

\[ \frac{d\kappa_3}{dx} = 0. \]

Using these facts we can recast the Hamiltonian as

\[ H = \frac{P_r P_\rho}{m \cos(\theta - \psi)} - \frac{m e^{\kappa_3+\kappa_4}}{\cos(\theta - \psi) r \rho} + \frac{\gamma}{2} (\rho P_\rho - r P_r) + \left( m \omega^2 \cos(\theta - \psi) - \frac{\gamma^2}{4m} \right) r \rho \]  

(19)

Even more, this Hamiltonian implies that the equations for the time derivative of the angles are equal to zero. Hence the angles are constants times time (plus a phase). In the case we have synchronous oscillators the Hamiltonian have a simpler form

\[ H = \frac{P_r P_\rho}{m} - \frac{m e^{\kappa_3+\kappa_4}}{r \rho} + \frac{\gamma}{2} (\rho P_\rho - r P_r) + \left( m \omega^2 - \frac{\gamma^2}{4m} \right) r \rho \]  

(20)

This equation recovers the crossed terms for the damped/absorptive term and an effective potential, with a minus sign in the momentum angular part. This allows to have hard bounded states and a possible family of solutions for the same angular momentum.

3. Conclusions

In this work a proposal of a Bateman dual system that allows the study of angular momentum and their possible decay or modification under the presence of dissipation of energy is presented. The proper coordinates to handle this situation are curvilinear ones, in particular the cylindrical ones. We selected them due to simplicity, the proper 3-D spherical coordinates are harder to handle and unclear to interpret the equations of motion [4]. The Hamiltonian we arrived is depicted in equation (20) with a couple of constants of motion included and explicitly written in (15) and (17). They are linear time dependent constant, but together with the Poisson bracket they are shown to be integrals of the motion. Hence, Hamiltonian in equation (20) is a good start to the study of angular momentum.

References

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