A versatile entanglement synthesizer in the spatial domain

David Barral,† Nadia Belabas,‡ Kamel Bencheikh,¶ Juan Ariel Levenson,† Mattia Walschaers,¶ Valentina Parigi,‡ and Nicolas Treps¶

†Centre de Nanosciences et de Nanotechnologies C2N, CNRS, Université Paris-Saclay, 10 boulevard Thomas Gobert, 91120 Palaiseau, France
‡Laboratoire Kastler Brossel, Sorbonne Université, CNRS, ENS-PSL Research University, Collège de France, 4 place Jussieu, F-75252 Paris, France

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Multimode entanglement is an essential resource for quantum information in continuous-variable systems. Quantum light-based mainstream technologies will arguably not be built upon table-top bulk optics-based setups. Integrated optics is a leading substrate technology for real-world light-based quantum information technologies. Sequential bulk optics-like proposals based on cascaded interferometers are not scalable with the current state-of-the-art low-loss materials used for continuous variables. In this work we analyze the multimode continuous-variable entanglement capabilities of a compact currently-available integrated device without bulk-optics analogous: the array of nonlinear waveguides. We demonstrate that this simple and compact structure, together with a reconfigurable input pump distribution and multimode coherent detection of the output modes, is a versatile entanglement synthesizer in the spatial domain. We demonstrate this versatility through analytical and numerically optimized examples of multimode squeezing, entanglement, and cluster state generation in different encodings. Our results establish back spatial encoding as a contender in the game of continuous-variable quantum information processing.

I. INTRODUCTION

Two key phenomena underpin current quantum technologies: quantum superposition and quantum correlations—entanglement—[1]. The paradigmatic example of entanglement is the case of two spatially separated quantum particles that have both maximally correlated momenta and maximally anticorrelated positions [2]. Position and momentum are continuous variables (CV), i.e. variables that take a continuous spectrum of eigenvalues [3]. In the optical domain CV-based quantum information can be encoded in the fluctuations of the electromagnetic field quadratures. Features like deterministic resources, unconditional operations and near-unity efficiency homodyne detectors make CV a powerful framework for the development of quantum technologies [4]. Remarkably, entanglement between more than two parties is also possible. Particularly, large-scale CV entangled states are the resources of a promising class of quantum computing, measurement-based quantum computing (MBQC) [5,6].

Multimode squeezed states have been predicted in arrays of nonlinear waveguides in the last few years [21,22]. Bipartite and tripartite CV entanglement have been demonstrated in periodically poled lithium niobate (PPLN) arrays in the last few years [21,22]. Bipartite and tripartite CV entanglement have been predicted in arrays of nonlinear waveguides in the spontaneous (SPDC) and stimulated parametric down-
contribution in the multimode coherent detection, enable on-demand programmability of the operation. Furthermore, a suitable engineering of the coupling profile and phase-matching of the ANWs can optimize a specific operation. We demonstrate the versatility of our approach in the production of multimode squeezing, multipartite entanglement and cluster states, through analytical and numerical solutions. Remarkably, since the encoding of quantum information is mode basis-dependent, we present two cluster-state-generation operation modes: one producing entanglement among individual modes of the array suitable for quantum networks, and other producing entanglement among eigenmodes of the array suitable for MBQC.

The article is organized as follows: in the next section we introduce the array of nonlinear waveguides. In section III we present the linear and nonlinear supermode bases, develop the propagation equations in both, enabling us to draw mathematic parallels of our approach with frequency comb-based CV, and present general solutions. In section IV we analyze a number of strategies to produce spatial multimode squeezed states: namely engineering of the coupling profile, the pump profile, the phase-matching and the detection. In section V we exhibit the generation of multimode squeezing in ANWs for some emblematic cases. In section VI we analyze the generation of bipartite and multipartite entanglement in arrays of nonlinear waveguides. In section VII we focus on the generation of cluster states and present two operation modes to encode them. Finally, in section VIII we discuss the features of the different modal-basis encoding of quantum information, we present future research directions following our results and analyze the feasibility of our approach.

II. THE ARRAY OF NONLINEAR WAVEGUIDES

The array of nonlinear waveguides consists of $N$ of identical $\chi^{(2)}$ waveguides in which degenerate SPDC and evanescent coupling between the generated fields take place. The array can be made up of, for instance, PPLN waveguides as sketched in Figure 1. In each waveguide, an input harmonic field at frequency $\omega_s$ is type-0 down-converted into a signal field at frequency $\omega_s$. We consider pump undepletion with $\alpha_{h,j}$ a strong coherent pump field propagating in the $j$th waveguide. We consider the phase matching condition $\Delta\beta = \beta(\omega_h) - 2\beta(\omega_s) = 0$, with $\beta(\omega_{h,s})$ the propagation constant at frequency $\omega_{h,s}$, is fulfilled only in the coupling zone. The energy of the signal modes propagating in each waveguide is exchanged between the coupled waveguides through evanescent waves, whereas the interplay of the second harmonic waves is negligible for the considered propagation lengths due to their high confinement into the guiding region. We consider a general inhomogeneous array of $N$ identical waveguides and continuous-wave propagating fields. The physical processes involved are described by the following system of equations [25]

$$\frac{d\hat{A}}{dz} = iC_0(f_{j-1}\hat{A}_{j-1} + f_j\hat{A}_{j+1}) + 2im\hat{a}_j^\dagger$$

where $\hat{A}_0 = 0$ and $\hat{A}_{N+1} = 0$, $f_0 = f_N = 0$ and $j = 1, \ldots, N$ is the individual mode index. $\hat{A}_j = \hat{A}_j(z, \omega_s)$ are monochromatic slowly-varying amplitude annihilation operators of signal (s) photons corresponding to the $j$th waveguide—individual mode basis—where

$$[\hat{A}_j(z, \omega), \hat{A}_{j'}^\dagger(z', \omega')] = \delta(z - z')\delta(\omega - \omega')\delta_{j,j'}.$$
\[ \eta_j = g_0 \alpha_{n,j} \] is the effective nonlinear coupling constant corresponding to the \( j \)th waveguide, with \( g \) the nonlinear constant proportional to \( \chi^{(2)} \) and the spatial overlap of the signal and harmonic fields in each waveguide. These parameters can be tuned by means of a suitable set of pump phase and amplitude at each waveguide. \( C_j = C_0 f_j \) is the linear coupling constant between modes \( j \) and \( j+1 \), and \( z \) is the coordinate along the direction of propagation. Both the coupling and nonlinear constants depend on the signal frequency set, \( C_0 = C_0(\omega_s) \) and \( g \equiv g(\omega_s) \), and they are taken as real without loss of generality.

Since we are interested in CV squeezing and entanglement, we will use also along the paper the field quadratures \( \hat{x}_j, \hat{y}_j \), where \( \hat{x}_j = (A_j + A_j^\dagger) \) and \( \hat{y}_j = i(A_j^\dagger - A_j) \) are, respectively, the orthogonal amplitude and phase quadratures corresponding to a signal optical mode \( A_j \).

The system of equations (1) in terms of the individual modes quadratures can be rewritten in compact form as

\[ \frac{d\xi}{dz} = \Delta(z) \xi, \]  

(3)

where \( \Delta(z) \) is a 2\( N \times 2\( N \) matrix of coefficients and \( \xi = (\hat{x}_1, \ldots, \hat{x}_N, \hat{y}_1, \ldots, \hat{y}_N)^T \). In general, either Equation (1) or Equation (3) can be solved numerically for a specific set of parameters \( C_j, \eta_j \) and \( N \), or even analytically if \( N \) is small. However, it is difficult to gain physical insight from numerical or low-dimension analytical solutions due to the increasing complexity of the system with the number of waveguides. There are two additional approaches that enlighten the problem of propagation in ANWs. The first one is based on the use of the eigenmodes of the corresponding linear array of waveguides – the linear (propagation) supermodes – which are squeezed and coupled through the nonlinearity. The second is based on the eigenmodes of the nonlinear system – the nonlinear (squeezing) supermodes, which are squeezed and fully decoupled but \( z \)-dependent. The linear supermode basis presents analytical solutions independently of the dimension \( N \) for specific pump-field distributions and, in some cases, both linear and nonlinear supermodes are degenerate up to local phases. Furthermore, the relationship between these two bases allows drawing mathematical parallels with frequency modes SPDC. Below we introduce both linear and nonlinear supermode bases, work out the corresponding propagation equations, and give the general solution to the propagation problem.

### III. PROPAGATION EQUATIONS

In the following paragraphs we introduce the general solutions to the propagation in ANWs in both the complex and quadratures representation of the optical fields. Both representations are equivalent and complementary, each showing specific features of the generated quantum states. The first representation presents the joint-spatial supermode distribution which yields clues about possible analytical solutions that we introduce in section IV. The solutions in this representation give the relative downconversion gains of the nonlinear interaction. The quadratures representation yields direct information on the quantum noise properties of the downconverted light.

#### A. Complex optical fields

Considering coupling only between nearest-neighbour waveguides, a linear waveguide array (Equation (1) with \( \eta_j = 0 \)) presents supermodes \( \hat{A}_{S,k} \), i.e. propagation eigenmodes. In general, any linear waveguide array is represented by a Hermitian tridiagonal matrix – Jacobi matrix – with non-negative entries and thus by a set of non-degenerate eigenvalues and eigenvectors given in terms of orthogonal polynomials. These eigenvectors (linear supermodes) form a basis and are represented by a matrix \( M \) with elements \( M_{m,j} \). The individual and supermode basis are related by

\[ \hat{A}_{S,k} = \sum_{j=1}^{N} M_{k,j} \hat{A}_j, \quad \hat{A}_j = \sum_{k=1}^{N} M_{j,k} \hat{A}_{S,k}. \]  

(4)

The supermodes are orthonormal

\[ \sum_{j=1}^{N} M_{k,j} M_{m,j} = \sum_{j=1}^{N} M_{j,k} M_{j,m} = \delta_{k,m}, \]  

(5)

and the spectrum of eigenvalues is \( \lambda_k \). We consider here a homogeneous coupling along propagation \( C_j \neq C_j(z) \). In the case of \( C_j = C_j(z) \) the eigenmode basis is local with \( M = M(z) \). Equation (1) for the nonlinear waveguide array in the supermode basis can be written as

\[ \frac{d\hat{A}_{S,k}}{dz} = i \sum_{j=1}^{N} \sum_{m=1}^{N} M_{k,j} M_{m,j} (\lambda_m \hat{A}_m + 2\eta_j \hat{A}_m^\dagger), \]  

(6)

where we have used the eigenvalue condition \( C_0(f_{j-1}M_{m,j-1} + f_j M_{m,j+1}) = \lambda_m M_{m,j} \). Using slowly-varying supermode amplitudes \( \hat{B}_{S,k} = \hat{A}_{S,k} e^{-i\lambda_k z} \), and the orthogonality of the supermodes Equation (6), the following propagation equation is obtained

\[ \frac{d\hat{B}_{S,k}}{dz} = 2i \sum_{j=1}^{N} \sum_{m=1}^{N} M_{k,j} M_{m,j} \eta_j \hat{B}_{S,m}^\dagger e^{-i(\lambda_m + \lambda_k) z}. \]  

(7)

The momentum operator in the interaction picture which produces Equation (7) by means of the Heisenberg equations \( d\hat{B}_{S,k}/dz = (-i/h)\hat{B}_{S,k}, \hat{M}_S \) is thus

\[ \hat{M}_S = i\hbar \frac{\eta_j}{2} \sum_{k,m=1}^{N} \mathcal{L}_{k,m}(z) \hat{B}_{S,k}^\dagger \hat{B}_{S,m} + \text{H.c.}, \]  

(8)
with \([\hat{B}_{S,k}(z), \hat{B}_{S,m}^\dagger(z')] = \delta(z - z') \delta_{k,m}\). The coupling matrix \(\mathcal{L}(z)\) is the local joint-spatial supermode distribution of the ANW and its elements are given by

\[
\mathcal{L}_{k,m}(z) = 2i \sum_{j=1}^{N} \eta_j |M_k,j| M_{m,j} e^{i(\phi_j - (\lambda_k + \lambda_m)z)},
\]

with \(\eta_j = |\eta_j| e^{i\phi_j}\) and \(\bar{\eta}\) an arbitrary parameter, e.g. the highest \(|\eta_j|\). \(\eta_j\) can also be a function of the position \(\eta_j \equiv \eta_j(z) = g(z)\omega_{k,j}\) [28]. \(\mathcal{L}(z)\) is a complex symmetric matrix which gathers the information about the spatial shape of the pump, i.e. amplitudes and phases in each waveguide, and the signal supermodes coupling. Finally, the Heisenberg equations in the linear supermode basis can be simply written as

\[
\frac{d\hat{B}_{S,k}}{dz} = \hat{\eta} \sum_{m=1}^{N} \mathcal{L}_{k,m}(z) \hat{B}_{S,m}^\dagger,
\]

The general solution to this equation can be obtained by diagonalization. In general, the propagation supermode basis does not diagonalize the propagation in the ANWs, but Equation (10) presents analytical solutions independently of the dimension \(N\) for specific pump-field distributions as we show in section [14]. A feature of the ANWs is that the evanescent coupling produces a phase mismatch between the pump and the generated signal waves which results in a \(z\)-dependent interaction, in such a way that the eigenmodes of the nonlinear system (nonlinear supermodes) are local. This coupling-based phase mismatch affects the amount of squeezing and entanglement generated in the ANWs. The nonlinear supermode basis displays modes independently squeezed and helps to quantify the amount of nonclassicality generated in the array at different propagation distances.

The joint-spatial supermode distribution \(\mathcal{L}_{k,m}(z)\) given by Equation (6) is a complex symmetric matrix which can be diagonalized by a congruence transforma-

| Function | Frequency combs [20] | ANWs |
|----------|-----------------|------|
| Individual modes | \(\tilde{a}_j\) | \(\hat{A}_j\) |
| Generator indiv. mod. | \(\hat{\mathcal{H}} = i\hbar \sum_{j=1}^{N} \hat{L}_{j,k} \tilde{a}_j^\dagger \tilde{a}_k^\dagger + \text{H.c.}\) | \(\hat{\mathcal{M}} = \hbar \sum_{j=1}^{N} \{C_0(f_j \hat{A}_{j+1}^\dagger + f_{j-1} \hat{A}_{j-1}^\dagger) + \eta_j \hat{A}_j^2 + \text{H.c.}\}\) |
| Linear supermodes | \(\hat{B}_{S,k} = \hat{A}_j e^{-i\lambda_k z} = \sum_{j=1}^{N} \eta_j M_{k,j} \hat{A}_j e^{-i\lambda_k z}\) | \(\hat{\mathcal{M}}^E = i\hbar \sum_{k,m=1}^{N} \mathcal{L}_{k,m}(z) \hat{B}_{S,k}^\dagger \hat{B}_{S,m}^\dagger + \text{H.c.}\) |
| Generator lin. superm. | | \(\hat{\mathcal{L}}_{k,m}(z) = 2i \sum_{j=1}^{N} \eta_j M_{k,j} M_{m,j} e^{i(\phi_j - (\lambda_k + \lambda_m)z)}\) |
| Coupling matrix | \(\hat{\mathcal{L}}_{j,k} = \text{sinc}(\omega_j, \omega_k)\alpha(\omega_j + \omega_k)\) | \(\hat{\mathcal{C}}_{S,m} = \sum_{j,k=1}^{N} (\Lambda_{m,j} M_{k,j} e^{i(\phi_j - (\lambda_k + \lambda_m)z)}) \hat{A}_j\) |
| Nonlinear supermodes | \(\hat{\mathcal{B}}_k = \sum_{j=1}^{N} V_{k,j}^\dagger \tilde{a}_j\) | \(\hat{\mathcal{M}} = i\hbar \sum_{m=1}^{N} \Lambda_{m,m}(z)(\hat{C}_{S,m}^\dagger)^2 + \text{H.c.}\) |
| Gen. nonl. sup. basis | \(\hat{\mathcal{H}}_S = i\hbar \sum_{k,m=1}^{N} \Lambda_{k,m}(z)(\hat{C}_{S,m}^\dagger)^2 + \text{H.c.}\) | |
This expression encapsulates the mechanisms at play in the ANWs: the evanescent coupling generates the linear supermodes \((M_{k,j})\) which get a phase due to propagation \((\lambda_k z)\) and the nonlinearity couples them locally \((T_{m,k}^\dagger(z)).\) In terms of the individual modes, the solution to the nonlinear system is

\[
\hat{A}_j(z) = \sum_{k,m,j'=1}^N (M_{j,k} \gamma_{k,m}(z) M_{m,j'} e^{i\lambda_k z}) \{\cosh[\tilde{r}_m(z)] \hat{A}_{j'}(0) + \sinh[\tilde{r}_m(z)] \hat{A}_{j'}(0)\}. \tag{17}
\]

Equations (15 - 17) find direct application in DV, for instance in driven quantum walks \([21, 22, 29].\) Remarkably, this kind of equations appears also in the context of SPDC in frequency combs \([30].\) In frequency combs, the individual modes are a discrete set of \(N\) frequency modes \(\hat{A}_j\) that are nonlinearly coupled in a bulk crystal with a quadratic nonlinearity. The diagonalization of the corresponding coupling matrix \(\hat{L}\) produces a set of nonlinear supermodes \(\tilde{b}_k,\) whose eigenvalues \(\tilde{\eta} \Lambda_{k,j}\) are proportional to SPDC gains. Table I (left) shows the main elements involved in frequency-comb SPDC and the related Hamiltonian in the individual \(\mathcal{H}\) and nonlinear supermode \(\mathcal{H}_S\) basis. In contrast, in ANWs the evanescent coupling between the individual modes \(\hat{A}_j\) generates the linear supermodes \(\hat{A}_{S,k}\) and the nonlinear coupling mediated by the pump fields mix them. Table I (right) shows the main elements involved in spatial ANWs and the related momenta in the individual \(\mathcal{M}\), linear supermode basis \(\mathcal{M}_L\), and nonlinear supermode \(\mathcal{M}_S\) basis. The coupling matrix \(\mathcal{L}(z)\) is defined here in the linear supermode basis and its diagonalization also produces a set of nonlinear supermodes \(C_{S,m}.\) The consequence of the double diagonalization is that the nonlinear supermodes are \(z\)-dependent –local–. At each propagation plane \(z\) a different set of nonlinear supermodes diagonalizes Equation (9), with eigenvalues \(\tilde{n} \int \Lambda_{m,m}(z') dz'\) corresponding to SPDC gains. This feature is the main conceptual difference between frequency combs and spatial ANWs. It makes the system very complex but, equally, highly versatile for the generation of multimode quantum states.

**B. Quadratures of the optical fields**

In terms of individual modes quadratures \(\hat{x}_j, \hat{y}_j\), the full evolution of the system is obtained by solving Equation (3). The formal solution of this equation is given by

\[
\xi(z) = S(z) \xi(0), \tag{18}
\]

with \(S(z) = \exp\{\int_0^z \Delta(z') dz'\}.\) \(S(z)\) is a symplectic matrix which contains all the information about the propagation of the quantum state of the system. We can apply on it a Bloch-Messiah decomposition as follows \([15]\)

\[
S(z) = R_1(z)K(z)R_2(z), \tag{19}
\]

where \(R_1(z)\) and \(R_2(z)\) are both orthogonal and symplectic matrices and \(K(z) = \text{diag}\{e^{r_1(z)}, e^{r_2(z)}, \ldots, e^{r_N(z)}\}\) is a phase-squeezed diagonal matrix. The spatial profiles of the local nonlinear supermodes are the columns of the unitary matrix \(U\) appearing in the complex representation of \(R_1\)

\[
R_1^{(c)} = DR_1D^\dagger = \text{diag}\{U, U^*\}, \tag{20}
\]

where

\[
D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i1 \\ i1 & -i1 \end{pmatrix} \tag{21}
\]

with \(I = \text{diag}\{1, 1, \ldots, 1\}\) the identity matrix. The local nonlinear supermodes are the same as those obtained by the Autonne-Takagi factorization and thus \(K(z)\) is given by \([43, 45]\)

\[
K(z) = \exp\{\tilde{n} \int_0^z \Lambda(z') dz'\} \begin{pmatrix} 0 & 0 \\ 0 & -\int_0^z \Lambda(z') dz' \end{pmatrix}. \tag{22}
\]

Therefore we can identify the squeezing parameters as \(r_m(z) = \tilde{r}_m(z) = \tilde{n} \int_0^z \Lambda_{m,m}(z') dz'.\)

The quantum states generated in ANWs are Gaussian. The most interesting observables in Gaussian CV are the second-order moments of the quadrature operators, properly arranged in the covariance matrix \(V\) \([22]\). The elements of this matrix can be efficiently measured by means of homodyne detection. For a quantum state initially in the vacuum, the covariance matrix at any plane \(z\) is given by \(V(z) = \mathcal{S}(z)^T(z),\) with \(1\) the value of the shot noise related to each quadrature in our notation. Evolution of variances \(V(\xi_i, \xi_j)\) and quantum correlations \(V(\xi_i, \xi_j)\) can be obtained at any length from the elements of this matrix. The covariance matrix can also be computed from the Bloch-Messiah decomposition as

\[
V(z) = R_1(z)K^2(z)R_1^T(z). \tag{23}
\]

Thus, \(K^2(z)\) is the covariance matrix in the nonlinear supermode basis and \(R_1(z)\) the symplectic transformation matrix between the individual and nonlinear supermode basis [equivalent to Equation (16) for complex fields]. The \(mth\) nonlinear supermode is squeezed and thus non-classical if \(K^2_{m+N}(z) = e^{-2r_m(z)} < 1,\) and the smallest value of \(K^2_{m+N}(z)\) is called the generalized squeezed variance and it is a measure of the nonclassicality of the quantum state \([47]\).

Note that the above approaches are equivalent \([48]\). Whereas the first method is applied to the complex joint-spatial supermode distribution, is numerically easier to compute and give the relative downconversion gains, the second method is applied to the propagator in the symplectic form and enables working out directly the noise properties of the quantum state as we show along the following sections.
IV. ENGINEERING TOOLBOX FOR PRODUCTION AND DETECTION OF MULTIMODE SQUEEZING

The class of ANWs which we introduced in section 1 presents a number of parameters that can be engineered for a desired operation. There are two types of parameters according to our ability to reconfigure them: passive and active parameters. The evanescent coupling profile \( \vec{f} = (f_1, \ldots, f_N) \), the length of the sample \( L \), the number of waveguides \( N \) – and notably its parity – and the phase-matching \([28]\) belong to the first kind. They can not be tuned once the sample is fabricated. By constrast, the power and phase pump profile, given respectively by \( \vec{\eta} = (|\eta_1|, |\eta_N|) \) and \( \vec{\phi} = (\arg(\eta_1), \ldots, \arg(\eta_N)) \), the coupling strength \( C_0(\omega_c) \) and the basis of detection can be set according to a required operation or encoding of information. We introduce in this section some engineering strategies related to five parameters that can be used to produce and detect a desired multimode squeezed state.

A. Coupling profile engineering

As introduced in the previous section, every set of nearest-neighbour coupled waveguides has a family of propagation supermodes given by a matrix \( M \). The slowly varying amplitude corresponding to the \( k \)th supermode propagates along the array with a propagation constant \( \lambda_k \). Each family of linear supermodes is characterized by a coupling profile \( \vec{f} \). The engineering of this profile enables a specific operation or logic gate \([32]\). A number of outstanding demonstrations with optical lattices has been exhibited over the last years \([33–35]\). Very recently, the production of topologically protected quantum states in a Su-Schrieffer-Heeger lattice has been demonstrated \([36]\).

A summary of properties of the supermodes can be found in ref. \([31]\). Particularly, every family of supermodes corresponding to an array of identical waveguides fulfill the following relations

\[
\lambda_m = -\lambda_{N+1-m}, \quad (24)
\]

\[
M_{N+1-m,j} = (-1)^{j+1} M_{m,j}. \quad (25)
\]

Applying these relations in the orthonormalization condition Equation \([5]\), we get the following modified orthonormality conditions

\[
\sum_{j=1}^{N} (-1)^{j+1} M_{k,j} M_{m,j} = \delta_{k,N+1-m}, \quad (26)
\]

\[
\sum_{2 \leq 2j \leq N} M_{k,2j} M_{m,2j} = \frac{1}{2} (\delta_{k,m} - \delta_{k,N+1-m}), \quad (27)
\]

\[
\sum_{1 \leq 2j-1 \leq N} M_{k,2j-1} M_{m,2j-1} = \frac{1}{2} (\delta_{k,m} + \delta_{k,N+1-m}). \quad (28)
\]

These relations are general and, notably, they can help us to decide how to configure the pump to obtain simple analytical solutions through Equation \([9]\).

We exhibit below two paradigmatic symmetric arrays with very interesting features: the homogeneous and the parabolic profile arrays. We display their supermodes and properties below.

1. Homogeneous profile array

The homogeneous linear array exhibits a constant coupling between waveguides \( f_j = 1 \). The supermodes are orthonormal Chebyshev polynomials that can be written in terms of simple trigonometric functions as \([37]\).

\[
M_{k,j} = M_{j,k} \equiv \frac{\sin(\frac{jk\pi}{N+1})}{\sqrt{\sum_{j'=1}^{N} \sin^2(\frac{j'k\pi}{N+1})}}. \quad (29)
\]

The Chebyshev supermodes for \( N = 5 \) waveguides are sketched in Figure 2. The spectrum of its eigenvalues is given by

\[
\lambda_k = 2C_0 \cos\left(\frac{k\pi}{N+1}\right), \quad (30)
\]

which are the propagation constants related to each supermode.

2. Parabolic profile array

The parabolic linear array exhibits a coupling between waveguides given by the profile \( f_j = \sqrt{j(N-j)/2} \). The supermodes are orthonormal Krawtchouk polynomials that can be written in terms of Jacobi polynomials as \([27, 38]\).

\[
M_{k,j} = 2^{j-\frac{N+1}{2}} \sqrt{(j-1)!(N-j)!/(k-1)!(N-k)!} \binom{N}{j}^{N-k-1}_{k-1}(0) \equiv M_{j,k}. \quad (31)
\]

The Krawtchouk supermodes for \( N = 6 \) waveguides are sketched in Figure 3. The eigenvalues are equally spaced and given by

\[
\lambda_k = \frac{N - 2k + 1}{2} C_0. \quad (32)
\]

The continuous limit \((N \to \infty)\) of these discrete eigenfunctions are the Hermite-Gaussian functions \([39]\). Remarkably, a parametric generalization of this set of supermodes, so-called para-Krawtchouk supermodes, allows fractional revivals and thus generalizes beam splitters–or directional couplers– to \( N \) dimensions \([27]\).
B. Pump profile engineering

Suitable manipulation of individual power and phase pump fields by means of off-the-shelf elements as fiber attenuators and phase shifters, and input into the ANWs through V-groove arrays, enables an on-demand pump distribution engineering.

The pump profile couples the propagation supermodes generating the joint-spatial supermode distribution Equation (9). In general, this generates complicated connections between the linear supermodes. However, the orthogonality and symmetry properties of the linear supermodes lead to simple analytical solutions in some cases. An outstanding simplification of the system is obtained when pumping all the waveguides with the same power $|\eta_j| = \text{constant}$. From now on, we refer to this as a flat pump profile. Another simplified solution is obtained when pumping only the even or odd waveguides, or when pumping only the central waveguide in an odd ANWs. Below we give the joint-spatial supermode distributions obtained with these input configurations and the analytical solutions to the Heisenberg Equations (10) in the simplest cases.

1. Flat pump profile: uniform phase

When all waveguides are equally pumped such that $|\eta_j| = |\eta| = \tilde{\eta}$ and $\phi_j = \phi$, the local joint-spatial supermode distribution Equation (9) is notably simplified to

$$\mathcal{L}_{k,m}(z) = 2i \delta_{k,m} e^{i(\lambda_k + \lambda_m)z},$$  \hspace{1cm} (33)

where we have used the orthonormality of the linear supermodes Equation (9). This pump configuration diagonalizes the momentum and the following Heisenberg equations are obtained

$$\frac{d\hat{B}_{S,k}}{dz} = 2i\tilde{\eta} e^{i(\phi - 2\lambda_k)z} \hat{B}_{S,k}^\dagger,$$  \hspace{1cm} (34)

The solutions are

$$\hat{B}_{S,k} = \{ \cos(\hat{F}_kz) \hat{B}_{S,k}(0) + i\sin(\hat{F}_kz) \mathcal{L}_{k,0}(0) + 2|\eta| e^{i\phi} \hat{B}_{S,k}(0) \} e^{-i\lambda_k z},$$  \hspace{1cm} (35)

with $\hat{F}_k = \sqrt{\lambda_k^2 - 4|\eta|^2}$. For typical evanescent coupling, nonlinearities and pump powers found in quadratic ANWs $|\lambda_k| > 2|\eta|$ and thus $\hat{F}_k \in \mathbb{R}$. We consider cases only in this power regime in the remainder of the article. Equation (35) simplifies into Equations (6)-(7) of ref. 69 for the nonlinear directional coupler ($N = 2$). The supermodes evolution is similar to the one found there for the individual modes: the power of the SPDC supermode periodically oscillates between a maximum and zero with oscillation periods $L_k = \pi/(2\hat{F}_k)$.

It is interesting to note that waveguide arrays with odd number of waveguides $N$ exhibit a zero supermode.

FIG. 2. Sketch of the Chebyshev supermodes related to an array of linear waveguides with a homogeneous coupling profile and $N = 5$. The horizontal axis stands for the individual modes. The propagation constants corresponding to each supermode are $\lambda = \{\sqrt{3}C_0, C_0, 0, -C_0, -\sqrt{3}C_0\}$. $k = 3$ is the zero supermode.

FIG. 3. Sketch of the Krawtchouk supermodes related to an array of linear waveguides with a parabolic coupling profile and $N = 6$. The horizontal axis stands for the individual modes. The propagation constants corresponding to each supermode are $\lambda = \{5C_0/2, 3C_0/2, C_0/2, -C_0/2, -3C_0/2, -5C_0/2\}$.
\( l = (N + 1)/2 \). This is a propagation eigenmode with zero eigenvalue \( \lambda_l = 0 \) in the slowly-varying amplitude approximation \(31\). The oscillation period of the zero-supermode is imaginary \( L_l = \pi/(4|\eta|) \), thus leading to hyperbolic solutions as

\[
\hat{B}_{S,l}(z) = \cosh(2|\eta|z)\hat{B}_{S,l}(0) + i e^{i\phi}\sinh(2|\eta|z)\hat{B}_{S,l}^\dagger(0).
\]

(36)

Note that Equations (35) and (36) are respectively the solutions of a non phasematched and perfectly phasematched degenerate parametric amplifiers \(40\). Hence, after a small number of coupling lengths the zero supermode will be dominant—efficiently built-up—with respect to the other propagating supermodes. The higher the value of \( F_k \) for the supermodes \( k \neq l \), the larger the difference of squeezing between the zero and the side supermodes. We will go in depth into this in the next section.

The supermode solution Equation (35) can be written in the individual mode basis as the following Bogolyubov transformations

\[
\hat{A}_j(z) = \sum_{j' = 1}^N [\hat{U}_{j,j'}(z)\hat{A}_{j'}(0) + \hat{V}_{j,j'}(z)\hat{A}_{j'}^\dagger(0)],
\]

(37)

where

\[
\hat{U}_{j,j'}(z) = \sum_{k = 1}^N M_{j,k}M_{k,j'}[\cos(F_kz) + i \lambda_k F_k \sin(F_kz)],
\]

\[
\hat{V}_{j,j'}(z) = \sum_{k = 1}^N M_{j,k}M_{k,j'}[2i|\eta|e^{i\phi} \sin(F_kz)],
\]

(38)

with \(|\hat{U}_{j,j'}(z)|^2 - |\hat{V}_{j,j'}(z)|^2 = 1\). Note that for \( |\eta| = 0 \), \( \hat{U}_{j,j'}(z) = U_{j,j'}(z) \) and \( \hat{V}_{j,j'}(z) = 0 \), with \( U_{j,j'}(z) \equiv \sum_{k = 1}^N M_{j,k}M_{k,j}e^{i\lambda_k z} \) the solution corresponding to the linear array. From these equations is straightforward to obtain the elements of the covariance matrix \( V(z) \), which read as follows

\[
V(x_i, x_j) = \sum_{k = 1}^N \frac{M_{i,k}M_{j,k}}{F_k^2}[\lambda_k^2 - 4|\eta|^2 \cos(2F_kz) - 4|\eta| \sin(F_kz)[F_k \sin(\phi) \cos(F_kz) + \lambda_k \cos(\phi) \sin(F_kz)]}.
\]

\[
V(y_i, y_j) = \sum_{k = 1}^N \frac{M_{i,k}M_{j,k}}{F_k^2}[\lambda_k^2 - 4|\eta|^2 \cos(2F_kz) + 4|\eta| \sin(F_kz)[F_k \sin(\phi) \cos(F_kz) + \lambda_k \cos(\phi) \sin(F_kz)]}.
\]

\[
V(x_i, y_j) = \sum_{k = 1}^N \frac{M_{i,k}M_{j,k}}{F_k^2} \times
\]

\[
4|\eta| \sin(F_kz)[F_k \cos(\phi) \cos(F_kz) - \lambda_k \sin(\phi) \sin(F_kz)].
\]

(39)

This configuration generates quantum correlations between the individual modes—off-diagonal components of the covariance matrix (as shown in Figure 3)—, and hence entanglement in that basis.

Remarkably, the results displayed in this section are general for any ANWs—any evanescent coupling profile \( f^\pi \)—since they are based only on the orthonormality of the supermodes. Equations (39) remain valid for any number of waveguides \( N \) or propagation distance \( z \). Thus they are a valuable tool which we use in the following sections to engineer suitably multimode quantum states with specific features.

2. Flat pump profile: alternating \( \pi \) phase

For an homogeneous or a parabolic coupling profile with \( N \) waveguides equally pumped such that \( |\eta_j| = |\eta| = \eta \) and an alternating phase \( \phi_j = (j + 1)\pi + \phi \), the joint-spatial supermode matrix Equation (9) is notably simplified to

\[
L_{k,m}(z) = 2i \delta_{k,N+1,m}e^{i\phi-(\lambda_k+\lambda_m)z}.
\]

(40)

via Equation (26). This pump configuration anticorrelates the momentum and the following Heisenberg equations are obtained

\[
\frac{d\hat{B}_{S,k}}{dz} = 2i\eta e^{i\phi} \hat{B}_{S,N+1-k}^\dagger.
\]

(41)

The downconversion gains are proportional to \( 2|\eta| \). The solution to Equation (41) is

\[
\hat{B}_{S,k}(z) = \cosh(2|\eta|z)\hat{B}_{S,k}(0) + i e^{i\phi}\sinh(2|\eta|z)\hat{B}_{S,k}^\dagger(0),
\]

(42)

Note that this is the solution of a perfectly phase-matched nondegenerate parametric amplifier \(40\). The supermode solution Equation (42) can be written in the individual mode basis as the following transformation

\[
\hat{A}_j(z) = \sum_{j' = 1}^N U_{j,j'}(z)\times
\]

\[
[cosh(2|\eta|z)\hat{A}_{j'}(0) + (1)^{j'-1}ie^{i\phi}\sinh(2|\eta|z)\hat{A}_{j'}^\dagger(0)],
\]

(43)

where we have used Equations (25) and (26). The solution is thus decoupled in this configuration: input single-mode squeezed states of light squeezed along the axis \((j' + 1)\pi + \phi \) propagate in the corresponding linear array with propagation matrix \( U_{j,j'}(z) \). From this equation, after a long but straightforward calculation, we obtain the elements of the covariance matrix \( V(z) \), which read as follows

\[
V(x_i, x_j) = [cosh(4|\eta|z)\hat{A}_{j'}(0) + (1)^j\sin(4|\eta|z)\delta_{i,j},
\]

\[
V(y_i, y_j) = [cosh(4|\eta|z) - (1)^j\sin(4|\eta|z)\delta_{i,j},
\]

\[
V(x_i, y_j) = (1)^j \cos(\phi)\sinh(4|\eta|z)\delta_{i,j}.
\]

(44)

Then, in this case quantum correlations are efficiently generated in the supermode basis but they disappear in
the individual mode basis—no off-diagonal elements of the covariance matrix (Figure 5b). The device produces thus independent squeezed fields. The results obtained in this section are general for any coupling profile \( f \) since they rely on Equations (27) - (28) only. Notably, this is an interesting regime for discrete variables since \( N \)-dimensional two-photon NOON states are generated [25].

3. Flat pump profile: any alternating phase

Both cases analyzed in sections IV.B.1 and IV.B.2 are encompassed through the use of Equations (27) - (28). In the case of an array composed of \( N \) waveguides equally pumped such that \( |\eta_j| = |\eta| = \tilde{\eta} \) and alternating phases \( \phi_{2j} \) and \( \phi_{2j-1} \), the joint-spatial supermode matrix Equation (9) is notably simplified to

\[
\tilde{L}_{k,m}(z) = 2ie^{i\Delta \phi^+} |\cos (\Delta \phi^-) e^{-2i\lambda_k z} \delta_{k,m} - i \sin (\Delta \phi^-) \delta_{k,N+1-m}|, \tag{45}
\]

with \( \Delta \phi^\pm = (\phi_{2j} \pm \phi_{2j-1})/2 \). Thus the system oscillates between Equations (33) and (40) for a general phase difference \( \Delta \phi^- \). Particularly, for \( \phi_{2j} = \phi_0 + \pi/2 \) and \( \phi_{2j-1} = \phi \) both the diagonal and antidiagonal terms have the same weight such as

\[
\tilde{L}_{k,m}(z) = \sqrt{2} i e^{i(\phi+\pi/4)} [e^{-2i\lambda_k z} \delta_{k,m} - i \delta_{k,N+1-m}]. \tag{46}
\]

The solution will present then both oscillatory and hyperbolic terms. More light is shed on the features that this configuration produces in section V.

4. Pumping only the even or odd waveguides

Another simplified joint-spatial supermode matrix is obtained if either even waveguides only \( (|\eta_{2j}| = \tilde{\eta}, |\eta_{2j-1}| = 0 \) and \( \phi_{2j} = \phi \) or odd waveguides only \( (|\eta_{2j-1}| = \tilde{\eta}, |\eta_{2j}| = 0 \) and \( \phi_{2j-1} = \phi \) are pumped, such that

\[
\tilde{L}_{k,m}(z) = ie^{i\phi} [e^{-2i\lambda_k z} \delta_{k,m} \pm i \delta_{k,N+1-m}], \tag{47}
\]

with plus for odd and minus for an even pump profile through Equations (27) - (28). Notably, in the case of an array made up of an odd number of waveguides, the solution for the zero supermode \( k = l \) is Equation (30) for an odd input pump and \( B_{z,i}(z) = 0 \) for an even input pump, removing the hyperbolic solution.

5. Pumping the central waveguide in an odd ANWs

A common and simple way of pumping an odd ANWs is to inject the pump only in the central waveguide \( j = l \equiv (N+1)/2 \) [22]. The following joint-spatial supermode distribution is then obtained

\[
\tilde{L}_{k,m}(z) = 2ie^{i\phi_0} M_{k,l} M_{m,l} e^{-i(\lambda_k + \lambda_m) z}. \tag{48}
\]

The elements of the zero supermode related to odd symmetric arrays as those above introduced have zeros in the even elements, i.e. \( M_{k,l} = 0 \) for \( k \) even. Thus, only odd supermodes are produced in the ANWs under this configuration. For instance, for \( N=5 \) and a homogeneous coupling profile we obtain as approximated solutions Equation (30) for the zero supermode \( (l = 3) \) and Equation (42) for the \( k = 1, 5 \) side supermodes after rescaling \( |\eta| \) to \( |\eta|/l \). Figure 5c shows the covariance matrix in the individual mode basis related to this pump configuration in an ANWs with a homogeneous coupling profile.

The above five cases exhibit the versatility of the ANWs through pump engineering and help us to understand what happens along propagation in these devices.

C. Phase-matching technique

A common phase-matching technique for efficient frequency conversion in \( \chi^{(2)} \) nonlinear waveguides is obtained through wavevector quasi-phase matching (\( \Delta \beta \)-QPM). A standard implementation of \( \Delta \beta \)-QPM is periodic inversion of the second-order susceptibility \( \chi^{(2)} \) with period \( \Delta \beta = 2\pi/\Delta \beta \), like for instance in PPLN waveguides [41]. However, in the case of waveguide arrays, a second cause of phase mismatch—the coupling—is present, as shown in Equation (6). In this case a similar strategy can be used to phase match specific supermodes through a second periodical inversion \( \Lambda_C(k') \)–coupling quasi-phase matching (C-QPM)– [28]. This slow modulation will match the propagation constant \( \lambda_{k'} \) of the \( k' \)th slowly varying supermode amplitude. We consider, for instance, a homogeneous coupling profile where \( \lambda_{k'} = -\lambda_{N+1-k'} = 2C_0 |k'\pi/(N+1)| \). In this case, the periodical inversion–coupling period–can be set as \( \Lambda_C(k') = \pi/|\lambda_{k'}| \), thus phase matching the \( k' \)th and \((N+1-k')\)th supermodes. Equation (9) is then written as

\[
\tilde{L}_{k,m}(z) \approx \frac{8i}{\pi} \sum_{j=1}^{N} |\eta_j| \tilde{\eta} \frac{M_{k,j} M_{m,j}}{\cos (2k \pi z)} e^{i(\phi_j - (\lambda_k + \lambda_m) z)}, \tag{49}
\]

where we have used the first-order Fourier series of square-wave C-QPM domains with duty cycles of 50%. Thus, using a flat pump profile Equation (49) is simplified to

\[
\tilde{L}_{k,m}(z) \approx \frac{8i}{\pi} \cos (2\lambda \pi z) \delta_{k,m} e^{i(\phi - (\lambda_k + \lambda_m) z)}, \tag{50}
\]
and the Heisenberg equations read
\[ \frac{d\hat{B}_{S,k}}{dz} \approx \frac{4i\eta}{\pi} e^{i\phi} \hat{B}_{S,k} \] \( k = k', N + 1 - k' \),
\[ \frac{d\hat{B}_{S,k}}{dz} \approx \frac{4i\eta}{\pi} e^{i(\phi-2\lambda_k z)} \hat{B}_{S,k} \] \( k \neq k', N + 1 - k' \).

Hyperbolic solutions as Equation (36) are obtained for the \( k \)'th and \( (N + 1 - k \)'th supermodes and oscillatory solutions like Equation (35) for the other supermodes. Note that the gains are reduced by a factor \( 2/\pi \) in comparison with the no C-QPM case which can be compensated with a \( \pi/2 \) longer propagation distance.

This powerful technique allows to control the supermodes efficiently building up. In terms of individual-modes entanglement it could be interesting to build up supermodes but with light in all the individual modes, unlike the zero supermode in the cases presented in section [V].A where only odd waveguides are populated. Remarkably, in the case of parabolic arrays with an even number of waveguides all the supermodes can be efficiently built-up. This interesting case will be presented elsewhere.

D. Balanced homodyne detection

The measurement of quantum noise variances and correlations is carried out by multimode balanced homodyne detection (BHD). In a fully fibered approach the multimode squeezed state generated in the array is collected into optical fibers through a V-groove array. A laser at signal frequency is demultiplexed into a number of individual optical fibers with fiber attenuators and phase shifters and individually mixed with the output SPDC through 3 dB fibered beam splitters as sketched in Figure 4. Each pair of mixed signals is sent to a BHD where the current of each photodiode is subtracted and suitably amplified.

In this section we would like to point out the different modes of operation related to the spatial profile of the LO in the multimode BHD. Access to the quantum information encoded in the individual or any of the supermode bases will indeed depend on a suitable BHD [42]. In the individual mode basis, at least two independent fibered BHDs are necessary in order to completely characterize any multimode quantum state. The variance measured in each mode and the quantum correlations between any pair of modes allow to reconstruct the full covariance matrix associated to the generated quantum state. Then, if a quantum information protocol is performed, it is necessary to use the same number of LOs as the number of involved modes in a multimode BHD.

The detection in the propagation supermode basis is based on a reconfigurable spatial local oscillator—a LO shaper—. This LO can be set in any of the elements of a supermode basis enabling the measurement of the full covariance matrix in that basis. This measurement method can be implemented in three ways. The first approach is based on a fibered multimode BHD with common phase and amplitude references, i.e phase and amplitude locking, to emulate the spatial profile of given supermode in an array of fibers [13]. In this case, the quantum noise measured at each individual BHD is equal and corresponds to the noise of the supermode. The second approach is based on postprocessing the results obtained in the individual mode basis in order to emulate the LO shaper transformation [11]. These two measurement methods are discussed in more detail in section VII. Finally, a bulk-optics-based approach is to directly image the output of the ANWs and mix it with a suitable shaped LO in a bulk beam splitter and one BHD. In this case a shaped LO with a controllable global phase to respectively match every array supermode and select the quadrature to be measured is necessary to retrieve all the information of the quantum state [13] [44]. Note that the supermodes of the nonlinear system are different from the propagation supermodes of the linear system as discussed in section [III]. In that case, it is also possible to apply these detection approaches. However, each pump configuration produces a different nonlinear supermode basis. Thus, the detection basis has to be reconfigured for every pump distribution.

V. MULTIMODE SQUEEZING

The ANWs is a natural platform for generating multimode squeezing due to the distributed coupling and nonlinearity only accessible to guided-wave nonlinear components. Along the next paragraphs we analyze some representative cases of multimode squeezing generated
in ANWs through the covariance matrix and the Bloch-Messiah decomposition introduced in section IV.

We begin with Figure 5 which exhibits covariance matrices in the individual mode basis $V(z)$ (Figures 5a-5c), Bloch-Messiah’s transformation matrices $R_1(z)$ (Figures 5d-5f), and diagonal covariance matrices in the nonlinear supermode mode basis $K^2(z)$ (Figures 5g-5i) for a five-waveguides homogeneous coupling-profile ANWs. The upper row displays the results obtained for a flat pump profile with a uniform phase: Equation (39) with $\phi = -\pi/2$. The central row displays the results obtained for a flat pump profile with an alternative $\pi$ phase: Equation (44) with $\phi = -\pi/2$. The lower row displays the results obtained pumping only the central waveguide. We applied Equation (48) into Equation (10) and solved numerically for $\phi_i = -\pi/2$. We set typical parameters in PPLN waveguides: $C_0 = 0.24 \text{ mm}^{-1}$, $\eta = 0.015 \text{ mm}^{-1}$ and $z = 20 \text{ mm}$. 

FIG. 5. Covariance matrices in the individual mode basis $V(z)$ (a-c), Bloch-Messiah’s transformation matrices $R_1(z)$ (d-f), and diagonal covariance matrices in the nonlinear supermode mode basis $K^2(z)$ (g-i) for a five-waveguides homogeneous coupling-profile ANWs. The upper row displays the results obtained for a flat pump profile with a uniform phase: Equation (39) with $\phi = -\pi/2$. The central row displays the results obtained for a flat pump profile with an alternative $\pi$ phase: Equation (44) with $\phi = -\pi/2$. The lower row displays the results obtained pumping only the central waveguide. We applied Equation (48) into Equation (10) and solved numerically for $\phi_i = -\pi/2$. We set typical parameters in PPLN waveguides: $C_0 = 0.24 \text{ mm}^{-1}$, $\eta = 0.015 \text{ mm}^{-1}$ and $z = 20 \text{ mm}$. 

We begin with Figure 5 which exhibits covariance matrices in the individual mode basis $V(z)$ (Figures 5a-5c), Bloch-Messiah’s transformation matrices $R_1(z)$ (Figures 5d-5f), and diagonal covariance matrices in the nonlinear supermode mode basis $K^2(z)$ (Figures 5g-5i) at a given propagation distance $z$. We display some of the cases analyzed in Section IV in a 5-waveguides homogeneous coupling-profile ANWs. Figures 5a and 5b display respectively the results for a flat pump profile and uniform phase, where strong quantum correlations between specific quadratures of the fields are generated, and for a flat pump profile and alternating $\pi$ phase, where single-mode squeezing is generated but not correlations in the individual basis. Figures 5g and 5h display the respective diagonal covariance matrix in the supermode basis, and Figures 5i and 5j the respective transformations between individual and supermode bases. Likewise, Figure 5k display the covariance matrix obtained when pump-
ing only the central waveguide. This case resembles that shown in Figure 5a, with a similar topology of quantum correlations, but different strength and sign. Figures 5 and 6 display the diagonal covariance matrix $K^2(z)$ and the transformation between bases $R_j(z)$, making more obvious the difference with the flat pump case. Overall, these figures show the versatility of our approach, offering different multimode squeezing features for different input pump profiles.

Figure 6 shows the evolution of noise squeezing ($K^2_{m+N}(z) < 1$) of the five nonlinear supermodes for a flat pump profile in a $N = 5$ ANWs. We show the effect of the coupling profile, the value of the coupling constant $C$ and the relative pump phase $\Delta \phi_1$ (Equation 45) on $K^2(z)$. Figure 6a and 6b show the result for a homogeneous and parabolic coupling profile, respectively, and low coupling. Figure 6c shows the result for a homogeneous coupling profile and high coupling. The squeezed eigenvalues are degenerate two by two for the side supermodes. The zero supermode ($k \equiv l = 3$, Fig. 4) is the only one nondegenerate and it is always efficiently buildup and squeezed, independently of $\Delta \phi_1$ (solid, green). Full degeneracy and efficient squeezing –hyperbolic– is obtained for all the supermodes for $\Delta \phi_1 = \pi/2$ (solid, green), $\Delta \phi_1 = 0$ produces oscillatory squeezing (solid, blue and orange) in the side supermodes which decreases as the coupling $C$ increases (Fig. 6a-6c). Notably, for intermediate cases $\Delta \phi_1 = \pi/8$ (dotted), $\pi/4$ (dashed), $3\pi/8$ (dot-dashed), squeezing builds up smoothly for the side supermodes and it approaches degeneracy for long propagation distances, whereas at short distances it is disturbed by the oscillatory part of Equation (45). However, this disturbance is important since it mixes the individual downconverted modes and thus trigger quantum correlations in the individual basis.

We discuss the generation of entanglement in these configurations in section VI.B. We also display the different features obtained for homogeneous and parabolic coupling profiles (Fig. 4, 6). For $\Delta \phi_1 = 0$, there are certain lengths for the parabolic coupling profile where only the zero supermode survives due to the equal spacing between the supermode propagation constants. Remarkably, for even number of waveguides and a parabolic coupling profile (not shown), there are propagation distances where destructive interference destroys all the SPDC generated light due to a evolving phase mismatch that periodically switches the system from downconversion to upconversion. Recently, bipartite entanglement between non-coupled pump fields has been demonstrated through this effect for two waveguides in the optical parametric amplification and second harmonic generation regimes [19, 69]. Thus, this effect can also produce multipartite entanglement between non-interacting fields. Finally, we outline that the parabolic-coupling profile excited with a flat pump profile represents the spatial analogous case to frequency comb pumped with a Gaussian spectral shape since the Krawtchouk supermodes are Hermite-Gaussian functions in the continuous limit [19].

Figure 7 shows the evolution of noise squeezing for a single pump in the central waveguide $|\eta_j| = |\eta| \delta_{j,1}$ of a homogeneous coupling profile $N = 5$ ANWs. Notably, in this case only three nonlinear supermodes are present, with the other two in vacuum state along propagation. The squeezing increases hyperbolically with an oscillatory modulation. Note that in this case there is
The Bloch-Messiah decomposition evidences the squeezing arising from this effect, as exhibited in Figure 7. The period shown in the figure agrees with that calculated $z_p = 2\pi/\sqrt{3}C_0 = 15.1$ mm. The level of squeezing is lower than that obtained in Figure 6 at the same distance since we use the same input pump power per waveguide, but the total power available per individual mode is 1/5.

Finally, we would like to point out one small difference between the diagonalization via linear supermodes and that obtained with the Bloch-Messiah decomposition for a flat pump profile and uniform phase (Section IV.B.1). Both basis exhibit the same levels of squeezing, but different spatial profile. The spatial shape of the nonlinear zero supermode ($k = l$) obtained from Equation (20) coincides with that calculated with Equation (36), but the nonlinear side supermodes ($k \neq l$) are slightly different from those obtained through Equation (35) changing with propagation. The cause of this disagreement is that this configuration diagonalizes the system up to a local phase rotation, i.e. the quadratures of the linear supermodes are not at the maximum and minimum of the squeezing ellipse, whereas the Bloch-Messiah decomposition mixes the propagation supermodes to obtain a fully diagonal covariance matrix. Suitable rotations (phase shifts) in the phase space related to each supermode can however diagonalize the linear supermode covariance matrix $V$. From Equation (35), we can straightforwardly calculate the covariance matrix $V$ related to the uncoupled $k$th linear supermode. A rotation in the $k$th supermode phase space of an angle

$$\theta_k = \frac{1}{2} \arctan \left[ \frac{2V(x_s,k, y_s,k)}{V(y_s,k, y_s,k) - V(x_s,k, x_s,k)} \right] + \frac{\pi}{2},$$

diagonalizes the covariance matrix obtaining

$$V(x'_s,k, y'_s,k) = \frac{V(x_s,k, x_s,k) + V(y_s,k, y_s,k)}{2} + \frac{\sqrt{(V(y_s,k, y_s,k) - V(x_s,k, x_s,k))^2 + 4V(x_s,k, y_s,k)^2}}{2},$$

$$V(y'_s,k, y'_s,k) = \frac{V(x_s,k, x_s,k) + V(y_s,k, y_s,k)}{2} - \frac{\sqrt{(V(y_s,k, y_s,k) - V(x_s,k, x_s,k))^2 + 4V(x_s,k, y_s,k)^2}}{2}. \quad (52)$$

This diagonal matrix is the same as $K^2(z)$ obtained by a Bloch-Messiah decomposition. $R_1(z)$ can be factorized thus as $R_1(z) = MR(\tilde{\theta})$, with

$$R(\tilde{\theta}) = \begin{pmatrix} \cos(\tilde{\theta}) & \sin(\tilde{\theta}) \\ -\sin(\tilde{\theta}) & \cos(\tilde{\theta}) \end{pmatrix}, \quad (53)$$

and $\cos(\tilde{\theta}) = \text{diag}\{\cos(\theta_1), \ldots, \cos(\theta_k), \ldots, \cos(\theta_N)\}$ [equally for $\sin(\tilde{\theta})$]. From the point of view of the experiment, this phase does not make any difference since the local oscillator of the balanced homodyne detector will sweep the entire squeezing ellipse. However, in terms of insight, the linear supermodes approach is far more powerful than Bloch-Messiah’s one since the spatial profile is invariant along propagation and thus the $k$th supermode squeezing can be measured with a LO excited with a spatial profile $\{N_{k,1}, N_{k,2}, \ldots, N_{k,N}\}$.

Let us exhibit an example for the sake of clarification. For a pump phase profile $\phi = 0$, the covariance matrix elements in the propagation supermode basis are

$$V(x_s,k, x_s,k) = [\cosh(r_k) + \sinh(r_k) \cos(2F_kz)]e^{-r_k},$$
$$V(y_s,k, y_s,k) = [\cosh(r_k) - \sinh(r_k) \cos(2F_kz)]e^{r_k},$$
$$V(x_s,k, y_s,k) = \sinh(r_k) \sin(2F_kz), \quad (54)$$

FIG. 7. Evolution of nonlinear supermode squeezing $K^2_{m,l}(z)$ in a five-waveguides homogeneous coupling-profile nonlinear array pumping only the central waveguide. 3 dB squeezing level in dotted, gray. $C_0 = 0.24$ mm$^{-1}$. $\eta = 0.015$ mm$^{-1}$. No direct correlation between linear and nonlinear supermodes anymore. However, the nonlinear supermode squeezing exhibited in this case can be also explained in terms of the linear supermodes as in Section IV.B.5. The leading terms of the zero ($k = 3$) and side supermodes ($k = 1, 5$) equations correspond to degenerate and non-degenerate parametric amplifiers, respectively, leading to hyperbolic squeezing. First-order terms introduce a $z$-dependent coupling between the zero and the side supermodes with period $z_p \approx 2\pi/|\lambda_{1(5)}|$ for $|\lambda_{1(5)}| \gg 2|\eta|$. The main difference here with respect to the flat pump profile case is that in this case the linear supermodes do not evolve independently but together, leading to coupling.
VI. BIPARTITE AND MULTIPARTITE ENTANGLEMENT

Once $V$ is known, the amount of CV entanglement in bipartite splittings of the system is easily quantified through the Peres-Horodecki-Simon (PHS) criterion, which establishes that a quantum state is entangled if the partially transposed density matrix is non-positive. In terms of continuous variables, the entanglement witness is $\nu_− < 1$, with $\nu_−$ the minimum eigenvalue of the partial transpose of the covariance matrix with respect to a subsystem $j$, $V^T j$. The closer the value of $\nu_−$ to zero, the higher the entanglement between two optical modes (individual or collective).

Measuring multipartite full inseparability in CV systems requires the simultaneous fulfillment of a set of conditions which leads to genuine multipartite entanglement when pure states are involved [51]. This criterion, known as van Loock - Furusawa inequalities, can be easily calculated from the elements of the covariance matrix $V$. Full $m$-partite inseparability is guaranteed if the following $m - 1$ inequalities are simultaneously violated [51].

$$V \text{LF}_j \equiv V[x_j(\theta_j) - x_{j+1}(\theta_{j+1})] + V[x_j(\theta_j + \pi/2) + x_{j+1}(\theta_{j+1} + \pi/2) + \sum_{k \neq j,j+1} G_k x_k(\theta_k + \pi/2)] \geq 4, \quad (55)$$

where $\hat{x}_j(\theta) = \hat{x}_j \cos(\theta_j) + \hat{y}_j \sin(\theta_j)$ are generalized quadratures which fulfill $[\hat{x}_j(\theta), \hat{x}'_j(\theta + \pi/2)] = i \delta_{j,j'}$, $\theta_j$ is the measurement phase corresponding to the $j$th local oscillator, $G_1, \ldots, G_N$ are $N$ real parameters corresponding to electronic gains in multimode BHD which are set by optimization, and

$$V(\sum_j l_j \xi_j) \equiv \sum_j l_j^2 V(\xi_j, \xi_j) + \sum_{i \neq j} l_i l_j V(\xi_i, \xi_j) \quad (56)$$

where $l_j$ is a set of real numbers. $\tilde{\theta} = (\theta_1, \ldots, \theta_N)$ and $\tilde{G} = (G_1, \ldots, G_N)$ stand, respectively, for the local oscillator phase and gain profiles. Note that the use of generalized quadratures is related to the generation of amplitude-phase quantum correlations in coupled waveguides, in contrast to bulk beam splitters which generate amplitude-amplitude and phase-phase quantum correlations. Equation (55) can be used to witness entanglement in real time in any basis where all the modes are measured simultaneously. In our case this is only fulfilled in the individual mode basis. Indeed, in supermode bases only one supermode can be measured in one go. Other multipartite entanglement witness are based directly on the measurement of the full covariance matrix [52, 53]. In that case any multipartite entanglement can be assessed in any basis. Multipartite entangled states are a key resource in quantum key distribution networks, obtaining higher secret key rates with respect to bipartite entangled states [54].

Below we analyze the generation of multipartite entanglement in the individual mode basis. We show two remarkable methods of production of bipartite and multipartite entanglement in quadratic waveguide arrays through SPDC. The first approach is based on the zero propagation supermode of the array. It is shown that efficient and scalable multipartite entanglement among the single-mode elements which participate in the zero supermode is obtained in arrays made up of an odd number of waveguides when a flat pump profile is used. The second approach is based on optimization of both pump profile and BHD parameters in order to maximize the generation of multipartite entanglement among all the propagating modes in the array.

A. Zero supermode-based entanglement

We present as an example of the capabilities of ANWs bipartite and multipartite entanglement in a homogeneous coupling profile array with an odd number of waveguides $N$ and a flat pump distribution (section IV.B.1). All the results presented below are obtained analytically using the propagation solutions Equations (39).

Figure 8 displays the evolution of entanglement in bipartite splittings of the three prevalent individual modes
obtained in the above configuration. These are the elements of the zero supermode. For \( N = 5 \), they are the odd modes \( j = 1, 3 \) and 5 (Figure 2). Bipartite splittings made up of single-mode fields \( 1 - 3 (3 - 5) \) and \( 1 - 5 \) are shown respectively in solid blue and yellow. Bipartite splittings made up of multimode fields \( 1 - \{3, 5\} \{5 - \{1, 3\}\} \) and \( 3 - \{1, 5\} \) respectively in dashed blue and yellow. The single mode and multimode asymptotic entanglement at long distances (or large coupling) \( \hat{\nu}_- \rightarrow 1/\sqrt{3} \) and \( \hat{\nu}_- \rightarrow 0 \), respectively. This entanglement is related mainly to the zero supermode \( k \equiv l = 3 \) buildup, but modulated due to the presence of two side supermodes \( k = 1, 5 \) (see [24] for a comparison with the second harmonic generation (SHG) case). The higher the coupling \( C \), the lower the effect of the supermodes \( k \neq l \). Notably, we have found that bipartite entanglement is always present independently of the number \( N \) of waveguides in the array. If bipartitions of three modes are considered, the entanglement increases continuously with propagation: \( \hat{\nu}_- \rightarrow 0 \).

Below we analyze multipartite entanglement for an odd number \( N \) of waveguides using the above configuration. Here, we found scalable multipartite entanglement between the odd \( (N + 1)/2 \) individual modes which compose the zero supermode using a local oscillator phase profile \( \vec{\theta} = \{0, 3\pi/2, 0.5 \pi/2, \ldots\} \) and mapping \( 2j - 1 \rightarrow j \) in Equation (55). Figure 3 (color) shows two, three and four inequalities for arrays with, respectively, three \( (N = 5) \), four \( (N = 7) \) and five \( (N = 9) \) propagating modes. We also show bipartite entanglement \( (N = 3) \) for comparison. The optimized violation \( \text{VLF} < 4 \) (our notation) of two, three and four inequalities – Equations (43) of ref. [51] – guarantees full inseparability. Since we deal with pure states the propagating signal modes are genuinely multipartite entangled. The downconverted signal fields thus exhibit multipartite entanglement at any \( z \) independently of the number of propagating modes. The number of entangled fields scales linearly with the number of waveguides. This result is obtained through minimization of Equations (55) through the gain profile \( \vec{G} \). Remarkably, an asymptotic lower bound on the violations in SPDC is obtained for \( \vec{G} = 0 \) in the limit of large coupling \( (C_0 \rightarrow \infty) \). Figure 4 displays in solid and dashed gray the violations in this limit for \( \vec{G} \neq 0 \) and \( \vec{G} = 0 \), respectively. We have found the following degenerate equation for the \( l - 1 \) inseparability conditions in this limit (appendix A)

\[
\text{VLF}(\vec{G}) \leq \text{VLF}(\vec{G} = 0) = 4\left(\frac{N - 1 + 2e^{-4|\eta|z}}{N + 1}\right) < 4, \quad (57)
\]

\( \forall z > 0 \) and odd \( N \). Noticeably, the asymptotic violation of the inequalities \( \lim_{z \rightarrow \infty} \text{VLF}(\vec{G} = 0) \) is the same as that obtained in SHG when a zero supermode is excited at the input of the ANW [24]. Unlike the SHG case where only the zero supermode is present, here the nonzero \( (k \neq l) \) supermodes are involved in the production of entanglement increasing the violation of the inequalities Equation (55) through the use of optimized gains \( \vec{G} \neq 0 \). Indeed, as the number of involved modes increases, the difference between optimized (Figure 4 solid gray) and non-optimized (Figure 4 dashed gray) solutions also increases. Notably, this configuration is very

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**FIG. 8.** Evolution of entanglement in bipartite splittings of the three prevalent modes obtained with a flat pump and homogeneous phase for a five-waveguides array. \( \hat{\nu}_- < 1 \) indicates entanglement. Bipartite splittings made up of single-mode fields \( 1 - 3 (3 - 5) \) and \( 1 - 5 \) respectively in solid blue and yellow. Bipartite splittings made up of multimode fields \( 1 - \{3, 5\} \{5 - \{1, 3\}\} \) and \( 3 - \{1, 5\} \) respectively in dashed blue and yellow. \( C_0 = 0.70 \text{ mm}^{-1} \), \( \eta = 0.015 \text{ mm}^{-1} \).

**FIG. 9.** Evolution of multipartite entanglement. Optimized van Loock - Furusawa (VLF) inequalities in color. Simultaneous values under the threshold value VLF=4 imply CV tripartite entanglement (two inequalities, \( N=5 \), dotted), quadrupartite entanglement (three inequalities –two degenerate–, \( N=7 \), dot-dashed) and pentapartite entanglement (four inequalities –degenerate two by two –, \( N=9 \), solid). We also show bipartite entanglement for comparison (one inequality, \( N=3 \), dashed). The optimized inequalities in the limit \( C_0 \rightarrow \infty \) are shown in solid gray. The inequalities for \( \vec{G} = 0 \) are shown in dashed gray. \( C_0 = 0.70 \text{ mm}^{-1} \), \( \eta = 0.015 \text{ mm}^{-1} \).
appealing for the generation of scalable multipartite entanglement since it relies on coupling $C_0$ and nonlinearity $g$ within the array, but not on specific values of these parameters. Note that in the case of three waveguides, quantum steering (VLF < 2) is accomplished due to the $k \neq l$ supermodes modulation: the lower the coupling $C$, the larger the violation. The asymptotic behavior exhibited in Figure 9 appears as a consequence of mixing of the considered fields present in the odd channels with vacuum fields present in the neglected even channels. This weakening of quantum correlations as the number of modes increases due to additional vacuum contributions is expected and consistent with what is also found in bulk-optics approaches. In fact, the number of vacuum modes is usually much larger than the number of squeezed inputs preventing scalability.

B. Optimized versus non-optimized multipartite entanglement

In general, beyond the zero-supermode, there is a pump amplitude $\eta$ and phase $\phi$ profile, and a set of local oscillator phases $\theta$ and gains $\vec{G}$, which optimize the violation of Equations (55) for a given set of fixed parameters of the array $\{f, N, z\}$. Or in other words, we can prepare the state in such a way that all the SPDC modes are multipartite entangled and not only a subset. Below we exhibit two examples of non-optimized and optimized multipartite entanglement between all the modes propagating in the array.

To simplify the analysis we take the same input power in each waveguide $|\eta_j| = |\eta|$. Firstly, as an example of a non-optimized procedure we focus on the cases shown in Figure 8. $N = 5$ ANWs with a homogeneous coupling profile and a flat pump power distribution. We analyze the impact of the pump phase difference $\Delta \phi^-$ on the multipartite entanglement. In this case we tune the BHD parameters, i.e., the local oscillator phases $\theta$ and the electronic gains $\vec{G}$. We use the sum of the four inequalities $F_M(\vec{G}, \vec{\theta}) = \sum_{j=1}^{4} VLF_j$ as the fitness function to optimize. We use an evolution-strategy algorithm to tackle the optimization problems found along the paper. Our optimization algorithm adjusts 10 parameters to find the minimum of $F_M$. Figure 10 shows two by two degenerate inequalities found for different values of $\Delta \phi^-$. Note that the case $\Delta \phi^- = \pi/2$ is fully degenerate and does not present entanglement at any distance as expected by inspection of Equations (44). The other cases present regions where fully multipartite entanglement is accomplished, being $\Delta \phi^- = 0$ the most favorable case.

Now, we study the same ANWs at a fixed length $z = 30$ mm where we can additionally tune the individual pump phases $\phi$. Note that there is no entanglement at this distance for any $\Delta \phi^-$ as shown in Figure 10. We use again the sum of the four inequalities $F_M(\vec{\phi}, \vec{G}, \vec{\theta})$ as the fitness function to optimize, now with 5 parameters extra related to the pump phases. Figure 11 shows the four inequalities for five propagating modes for different values of power per waveguide $\eta$, among them $\eta = 0.015$ mm$^{-1}$, the case shown in Figure 10. Genuine multipartite entanglement is obtained for any value of $\eta$. Remarkably, the optimized simultaneous violation of the four inequalities at $\eta = 0.015$ mm$^{-1}$ exhibits the versatility of our approach. Note that, as in Figure 10, the optimized inequalities abide the symmetry forced by the coupling profile and the homogeneous distribution of pump power: $VLF_2 \approx VLF_4$ (orange), and $VLF_1 \approx VLF_4$ (blue).

We would like to finish by noting that the above optimization procedure represents a lower bound on the violations based on the fitness function we have chosen. There can be nevertheless other sets of parameters which present larger violations of the VLF inequalities.

VII. CLUSTER STATES

An ideal CV cluster state is a simultaneous eigenstate of specific quadrature combinations called nullifiers. Cluster states are associated with a graph or adjacency matrix $B$. The nodes of the graph represent the modes of the cluster state in a given basis, and the vertices the entanglement connections among the nodes. Moreover, the label of the modes that are part of the cluster can be suitably set to maximize the entanglement between
FIG. 11. Multipartite entanglement versus flat pump-profile power in a 5-waveguides ANWs. Optimized van Loock - Furusawa (VLF) inequalities through $F_M(\phi, \vec{G}, \vec{\theta})$. Simultaneous values under the threshold value $VLF=4$ (black) imply CV pentapartite entanglement (four inequalities, $N=5$). $j=1$ (dotted blue), $j=2$ (dotted orange), $j=3$ (solid orange) and $j=4$ (solid blue). $C_0 = 0.24 \text{ mm}^{-1}$. $z = 30 \text{ mm}$.

nodes. The nullifiers are given by

$$\hat{\delta}_i \equiv \hat{x}_i(\theta_i + \pi/2) - \sum_{l=1}^{N} B_{il} \hat{x}_l(\theta_l) \quad \forall i = 1, \ldots, N,$$

(58)

where $B$ is the graph associated to the cluster and $\hat{x}_i(\theta_i)$ is the $i$th generalized quadrature in a given basis. In our case the cluster states can be encoded in the individual mode basis, in the linear supermode basis, the nonlinear supermode basis, or in any other basis. Cluster states are the resources of CV measurement-based quantum computing (MBQC) [57]. The computation relies on the availability of a large multimode entangled state on which a specific sequence of measurements is performed. The choice of natural or exotic bases widens the range of application in MBQC [58]. The variance of the nullifiers tend to zero in the limit of infinite squeezing. Experimentally, a cluster state can be certified if two conditions are satisfied: i) the noise of a set of normalized nullifiers lies below shot noise

$$V(\bar{\delta}_i) < 1 \quad \forall i = 1, \ldots, N,$$

(59)

where $\bar{\delta}_i \equiv \delta_i / \sqrt{1 + n(i)}$ is the normalized nullifier and $n(i)$ is the number of nearest neighbours to the $i$th node of the cluster, and ii) the cluster state is fully inseparable, i.e. it violates a set of VLF inequalities [51, 59].

As the encoding of quantum information is mode basis-dependent, we exhibit the versatility of our platform by presenting two cluster-state-generation operation modes: one producing entanglement among individual modes of the array suitable for quantum networks, and other producing entanglement among nonlinear supermodes of the array suitable for MBQC. In the first case, we exhibit how
linear cluster states are produced naturally in a flat pump configuration and how optimized configurations can produce other types of clusters. In the second case, we display how any class of supermode cluster can be created or simulated by suitable selection of pump and detection parameters.

A. Individual mode basis

1. Flat pump profile

Notably, in the context of MBQC, a linear 4-mode cluster state is a sufficient resource for an arbitrary single-mode Gaussian unitary \[ U \]. Hence, linear cluster states represent key resources in this domain. The adjacency matrix \( B_{lin} \) corresponding to a linear cluster is the same as that related to the coupling in an homogeneous array when the encoding node \( i \) is used. Thus, the ANWs can be a natural platform for the generation of this class of cluster states. Below we exhibit the use of the analytical solutions Equations (39) in the generation of linear cluster states. For instance, the linear cluster for \( N = 5 \) modes is depicted in Figure 14 with normalized nullifiers given by

\[
\begin{align*}
\bar{\delta}_1 &= \frac{y_1(\theta_1) - x_2(\theta_2)}{\sqrt{2}}, \\
\bar{\delta}_2 &= \frac{y_2(\theta_2) - x_1(\theta_1) - x_3(\theta_3)}{\sqrt{3}}, \\
\bar{\delta}_3 &= \frac{y_3(\theta_3) - x_2(\theta_2) - x_4(\theta_4)}{\sqrt{3}}, \\
\bar{\delta}_4 &= \frac{y_4(\theta_4) - x_3(\theta_3) - x_5(\theta_5)}{\sqrt{3}}, \\
\bar{\delta}_5 &= \frac{y_5(\theta_5) - x_4(\theta_4)}{\sqrt{2}},
\end{align*}
\]

where we have related the \( i \)th node of the cluster with the \( j \)th individual mode of the ANWs \((i = j = 1, \ldots, 5)\). The full inseparability of the cluster nodes can also be assessed by means of specific VLF inequalities. In our linear cluster, due to the use of normalized nullifiers, they are the following \( N - 1 \) inequalities

\[
V(\bar{\delta}_i) + V(\bar{\delta}_{i+1}) \geq \begin{cases} 
\sqrt{\frac{8}{3}} & \text{for } i = 1, N - 1, \\
\frac{4}{\sqrt{3}} & \text{for } i = 2, \ldots, N - 2.
\end{cases}
\]

(61)

Thus, simultaneous values of \( V(\bar{\delta}_i) < 2/3 \) ensure the production of a linear cluster.

Figure 12 maps the nullifier variances characterizing a \( N = 5 \) linear cluster state produced in an ANWs with homogeneous coupling and propagation length \( z = 20 \) mm. Due to the symmetry of the system the nullifiers are degenerate two by two, but the \( i \)th nullifier: \( V(\bar{\delta}_i) = V(\bar{\delta}_{i+1}) \) (Figure 12a), \( V(\bar{\delta}_2) = V(\bar{\delta}_4) \) (Figure 12b), and \( V(\bar{\delta}_3) \) (Figure 12c). The contour plots display common areas fulfilling the condition \( V(\bar{\delta}_i) < 2/3 \) (blue areas). For instance, for \(|\eta| = 0.06 \text{ mm}^{-1}\) and \( C_0 = 0.16 \text{ mm}^{-1}\), we get \( V(\bar{\delta}_{1(5)}) = 0.34, V(\bar{\delta}_{2(4)}) = 0.42, \) and \( V(\bar{\delta}_3) = 0.40 \). These values are of the order of those obtained in the frequency domain with frequency combs \[ \text{[61]}. \]

In order to gain insight about the scalability of this configuration, Figure 13 pictures the evolution along propagation of the nullifier variances related to linear cluster states made up of \( N = 5 \) (Figure 13a) and \( N = 15 \) (Figure 13b) modes. Now, we optimize the amount of power per waveguide \( \eta \) for a given coupling constant. We use the sum of the five (fifteen) nullifier variances \( F_C(\eta) = \sum_{i=1}^{N} V(\bar{\delta}_i) \) at each \( z \) as the fitness function to optimize. As commented above, the nullifier variances are degenerate due to the symmetry of the system. Remarkably, the linear cluster condition \( V(\bar{\delta}_i) < 2/3 \) is fulfilled in both cases for a large range of distances. In order to connect Figures 12 and 13 we have marked as a black dot in Figures 12a, b and c, the coordinates \((C, \eta) = (0.08, 0.033) \text{ mm}^{-1}\) corresponding to the variances of the nullifiers at \( z = 20 \) mm shown in Figure
The maxima values of $\eta$ used in the optimization are 0.038 and 0.035 mm$^{-1}$ for $N = 5$ and 15, respectively. These values are attainable with current technology $[62, 63]$. Note that the constant coupling is wavelength dependent $C_0 = C_0(\omega_i)$. Thus for a fixed ANW length, modifying the operating wavelength $\lambda_s$, we can access to more favorable conditions to obtain multipartite entanglement. This is clearly shown comparing the nullifiers for a $N = 5$ linear cluster obtained at $z = 30$ mm with two different values of $C_0$. Whereas for a low coupling the inseparability condition is not fulfilled ($C_0 = 0.08$ mm$^{-1}$, Figure 13a), for a higher coupling the condition is indeed fulfilled ($C_0 = 0.24$ mm$^{-1}$, Table I).

We have demonstrated the production of linear cluster states with our analytical solutions Equations (61). However, the parameter space of the full approach is much larger than that corresponding to this special case. This enables the optimized generation of linear and other classes of cluster states. Below, we display the versatility of this approach with a number of examples.

### 2. Optimized cluster state generation

Table II details the optimized nullifier variances obtained for the $N = 5$ modes cluster states shown in Figure 14: a) linear, b) pentagon, c) star, d) square pyramid, and e) maximally connected pentagon—or Greenberger-Horne-Zeilinger (GHZ) state—. We have set the relation between nodes and individual modes as in the previous section $(i = j = 1, \ldots, 5)$. We take a homogeneous coupling profile $\vec{f} = \bar{1}$ with $C_0 = 0.24$ mm$^{-1}$ and a fixed length $z = 30$ mm. We use the sum of the five nullifier variances $F_C(\vec{\eta}, \vec{\phi}, \vec{\theta}) = \sum_{i=1}^{5} V(\vec{\delta}_i)$ with 15 free parameters as the fitness function to minimize. Remarkably, we have found realistic set of parameters $\{\vec{\eta}, \vec{\phi}, \vec{\theta}\}$ where the clusters are generated in the five analyzed cases. Note that star-shaped and GHZ clusters are related by only $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5$. The maxima values of $\eta_1$, $\phi_1$, $\theta_1$ maximally connected pentagon— or Greenberger-Horne-Zeilinger (GHZ) state—. The second approach is based on the freedom of distributing the degree of squeezing among the cluster modes $\vec{K}^2(\vec{r}, \vec{\phi}, \vec{\theta}) = \vec{O}(\vec{\phi})\vec{K}^2(\vec{r}, \vec{\phi}, \vec{\theta})^{\top}$. Thus, setting the nonlinear supermodes as the basis the cluster is built from, i.e. $\vec{K}^2(\vec{r}, \vec{\phi}, \vec{\theta})$, the cluster and the ANWs covariance matrices are related by

$$V_C = S_C(\bar{\phi})\bar{V}(\bar{\phi}) S_C^{\top},$$

where $S_C$ stands for the symplectic and orthogonal transformation which produces the cluster Equation (58), and $\vec{K}^2(\vec{r})$ a symplectic diagonal matrix which stands for phase-squeezed states where the elements of $\vec{r}$ are the squeezing parameters corresponding to each mode. A symmetric cluster transformation $S_C$ can be obtained from the adjacency matrix $B$ as follows $[58]$

$$S_C = \left(\begin{array}{cc} X_s & -Y_s \\ Y_s & X_s \end{array}\right),$$

with $X_s = (B^2 + I)^{-1/2}$ and $Y_s = BX_s$. Note that the unitary related to the cluster shape $S_C$ is defined up to a transformation $\bar{S}_C(\vec{\phi}) = S_C \bar{O}(\vec{\phi})$, where

$$\bar{O}(\vec{\phi}) = \begin{pmatrix} O(\vec{\phi}) & 0 \\ 0 & O(\vec{\phi}) \end{pmatrix},$$

and $O(\vec{\phi})$ is an N-dimensional orthogonal matrix which can be parametrized for instance by $N(N-1)/2$ generalized Euler angles $\vec{\phi}$. This degree of freedom is related to the freedom of squeezing among the cluster modes $\vec{K}^2(\vec{r}, \vec{\phi})$. The first approach is to use a LO with a spatial profile given by the complex representation of $S_C(\vec{\phi})$ as in Equation (62). The first approach is to use a LO with a spatial profile given by the complex representation of $S_C(\vec{\phi})$ as in Equation (62). This can be carried out by means of a single-mode bulk BHD or a multimode fibered BHD which mix the multimode SPDC light with a spatially multimode LO $[13]$. Outstandingly, using as resource a fully inseparable quantum state as those shown in Figures 10 and 11 any cluster state can be realized with a suitable LO shaping. The second approach is based on the emulation of the statistics of a given cluster state. This is carried out in a multimode fibered BHD with independent single-mode LOs and postprocessing by computer the photocurrents coming from every detector $[11]$. In this case $S_{LO}(\vec{\phi}, \vec{z})$ is decomposed as

$$S_{LO}(\vec{\phi}, \vec{z}) = \bar{O}_{post}(\vec{\theta}) D_{LO}(\vec{\theta}).$$

with $D_{LO}(\vec{\theta}) = R(\vec{\theta})$ is the LO phase profile as defined in Equation (53) but applied on individual modes, and $\bar{O}_{post}(\vec{\theta})$ is an orthogonal matrix associated to the post-processing gains defined as in Equation (64).
FIG. 14. Some 5-modes graphs. a) Linear, b) pentagon, c) star, d) square pyramid, and e) maximally connected pentagon (GHZ).

| Graph     | Nullifiers \(\{V(\delta_i)\}\) | \(\tilde{\eta} \times 10^2\) mm\(^{-1}\) | \(\phi/\pi\) | \(\theta/\pi\) |
|------------|----------------------------------|-------------------------------------------|--------------|--------------|
| Linear     | \{0.20, 0.39, 0.37, 0.38, 0.20\} | 9.2, 8.9, 9.1, 9.1, 9.2 \(\pm 0.05\) | \(-0.50 \pm \{1.1, 1.1, 1.1\}\) | \(\{0, 0, 0, 0, 0\}\) |
| Pentagon   | \{0.59, 0.73, 0.09, 0.34, 0.11\} | 8.7, 4.9, 3.4, 1.9, 8.7 \(\pm 0.05\) | \{1.50, 0.87, 1.06, 1.34, 0.63\} | \{0.60, 0.19, -1.00, -0.88, 0.20\} |
| Star       | \{0.36, 0.21, 0.72, 0.45, 0.37\} | 3.6, 1.7, 2.8, 3.6, 3.8 \(\pm 0.05\) | \{0.04, -1.02, 0.98, 0.18, 0.87\} | \{0.86, -0.71, 0.25, 0.27, -0.43\} |
| Pyramid    | \{0.33, 0.34, 0.23, 0.30, 0.64\} | 1.0, 7.3, 1.3, 7.7, 3.2 \(\pm 0.05\) | \{0.04, 0.17, 0.17, 0.56, 0.63\} | \{-0.18, 0.39, -0.69, 0.21\} |
| GHZ        | \{0.36, 0.21, 0.72, 0.45, 0.37\} | 3.6, 1.7, 2.8, 3.6, 3.8 \(\pm 0.05\) | \{0.04, -1.02, 0.98, 0.18, 0.87\} | \{1.36, -0.21, 0.75, 0.77, 0.07\} |

TABLE II. Individual-mode-basis cluster state generation in a 5-waveguides ANWs with homogeneous coupling profile for linear, pentagon, star, square pyramid and GHZ graphs. We show the value of the nullifiers \(\{V(\delta_i)\}\), pump power profile \(\tilde{\eta}\), pump phase profile \(\phi\) and local oscillator phase profile \(\theta\). Simultaneous values of \(V(\delta_i)\) under the shot noise threshold \(V(\delta_i) = 1\) are a signature of cluster generation. \(C_0 = 0.24\) mm\(^{-1}\). \(z = 30\) mm.

choose to minimize the following fitness function with 24 free parameters

\[
F_P(\tilde{\eta}, \phi, \varphi, \theta) = \left\| S_{LO}(\varphi, z) - O_{post}(\theta) D_{LO}(\theta) \right\|
\]

(67)

The norm is a standard matrix norm as the Frobenious norm \(\|A\|^2 = \sum_{i,j} |A_{i,j}|^2\). We obtain the following nullifiers in the nonlinear supermode basis: \(V(\delta_1) = 0.31\), \(V(\delta_2) = 0.34\), \(V(\delta_3) = 0.30\) and \(V(\delta_4) = 0.37\). The virtual nullifiers thus produced violate also the three inseparability conditions \(V(\delta_i) + V(\delta_i) \geq \sqrt{2}\) for \(i = 2, \ldots, 4\). The parameters obtained through minimization of Equation (67) are shown in the appendix C.

Thus, any N-dimensional cluster state can be directly measured with a suitable shaped LO or approximately emulated by postprocessing, and even a large class of Gaussian computations can be performed exchanging \(S_C\) by \(S'_C = U_{comp} S_C\), with \(U_{comp}\) the orthogonal matrix associated to the required computation \[68\], \[65\].

To end this section we would like to note that weighted bipartite CV cluster states appear naturally in the supermode basis due to the form of the multimode squeezing momentum operator Equation \[6\] \[34\]. The analysis of the clusters produced in this way will be presented elsewhere.

VIII. DISCUSSION AND PERSPECTIVES

We conclude with a few comments about the encoding and processing of information in different mode basis, the range of application and the feasibility of ANWs-based quantum information processing. Firstly, we would like to compare both coding schemes of quantum information we have introduced above, the individual mode basis and the nonlinear supermode basis, in terms of their respective abilities in quantum information processing and quantum networks. Table \[III\] summarizes the main attributes of both approaches. In the individual-mode basis the quantum network is physically yielded by the array with spatially distant nodes, which allows simultaneous access to all the nodes of the network, whereas in the nonlinear supermode basis the quantum network is produced by LO shaping with each node localized in the full optical profile at the output of the array, allowing only access to one node of the network at a time, or virtually by postprocessing. The individual-mode nodes are addressed by independent BHDs, whereas the squeezed-supermode nodes are measured with a shaped LO or postprocessed in a suitable basis measuring the statistics associated to the each node of the quantum network. The resource of both quantum networks is entanglement. However, the postprocessing approach allows
the mixing the results of single-mode squeezed states as those obtained in section IV.B.2. A last comment should be made about the usefulness of both approaches in terms of MBQC. The postprocessing approach is not universal [66]. The most general Gaussian operation can not be implemented with it. Ancillary squeezed states are in general necessary to implement universal operations. There is however a case where universal Gaussian operations are accomplished: if the transformation matrix $R_1(z)$ coincides with the matrix $S_C$ related to a resource cluster state, like a linear cluster for single-mode operations or a square cluster for multimode operations [60]. Indeed, we have demonstrated in section VII a that linear cluster states are directly generated in ANWs, thus single-mode universal operations are possible using that configuration. The two-dimension cluster states required for universal multimode MBQC have been recently demonstrated by time-domain multiplexing [15] [16]. The level of squeezing necessary to implement fault-tolerant MBQC is nevertheless far from technologically available [60]. Fault-tolerant MBQC is potentially realizable with lower squeezing thresholds by using cluster states of higher dimension [67]. It is important to remark that the spatial encoding shown here can be multiplexed in frequency and time in the pulsed regime. ANWs thus represent a potential platform to implement that technology. It is foreseeable that future MBQC will be based on multiplexing the CV modes in space, time, frequency or angular momentum, and on integration on chip [68].

Secondly, we would like to disclose some possible research directions which follow from this work. The first is emulation of quantum complex networks [17]. The dynamics of an ensemble of quantum harmonic oscillators linked according to a specific topology can be mapped to our multimode platform through the symplectic propagator in Equation (18). The temporal evolution of a quantum network $S_{net}(t)$ is directly mapped to our propagator $S(z)$, which can be experimentally realized by adequate pump profile optimization and multimode BHD. The tunability our approach offers enables the study of different network topologies with a single setup. Another interesting feature of integrated ANWs is the possibility to include non-Gaussian operations on the quantum state, cornerstone of quantum advantage in CV-MBQC. Single-photon subtraction can be indeed implemented by means of introducing defects in the array, i.e. a weakly coupled single waveguide in between two ANWs in such a way that the detection of a single photon de-Gaussifies and entangles the propagating quantum states related to each array [69] [70]. Additionally, besides the practical applications in CV previously discussed such as quantum secret sharing and MBQC, an appealing exploitation of the ANWs in DV is Gaussian boson sampling (GBS) [71]. As every $N$-mode Gaussian state generated in the array can be decomposed by Bloch-Messiah into $N$ single-mode squeezers in between two linear interferometers, the sampled photon pattern at the output of the ANWs enables the computation of the hafnian of the matrix which characterizes the quantum state. Remarkably, molecular vibronic spectra can be calculated using GBS and $N$ single-mode input coherent states [72]. Optical parametric amplification in ANWs is a suitable platform for this kind of simulation as the transformation related to vibrational transitions can be mapped into it. Interestingly, molecules with structural changes but no displacement can be directly simulated with SPDC, as recently shown for the Tropolone $C_7H_6O_2$ [73]. An adequate pump profile configuration can thus be enough to estimate the Franck-Condor factors associated with a given transition in such a family of molecules. Furthermore, our integrated platform offers the possibility to simulate the set of synthetic molecules associated to a given pump-array configuration $\{f, \eta, \phi, N, z\}$. This can be used as a benchmarking tool to quantify the enhancement of the quantum simulation over classical approximation strategies [73] [74].

Finally, the influence of losses on the CV entanglement can be included in our analysis by inserting fictitious beam splitters with effective transmissivity $\sqrt{T}$. The covariance matrix of these realistic quantum states $V^R$ is easily found as $V^R(\xi, \xi) = T V^I(\xi, \xi) + (1 - T) \delta_{ij}$, where $V^I$ is computed from the lossless covariance matrix and $\delta$ stands for the Kronecker delta. These values are included in the covariance matrix by means of $T_i(\gamma_i, z) = e^{-\gamma_i z}$. Propagation losses have a small impact on squeezing and entanglement assuming typical values in PPLN waveguides ($\leq 0.14$ dB cm$^{-1}$) [20] [62]. Nonlinearities as high as $g = 20 \times 10^{-4}$ mm$^{-1}$ mW$^{-1/2}$ and $g = 49 \times 10^{-4}$ mm$^{-1}$ mW$^{-1/2}$, and coupled pump powers ranging from tens to few hundreds milliwatts, have been recently shown in soft proton exchange and ridge PPLN waveguides [62] [63]. A squeezing level as high as -6 dB in cw has been estimated inside a PPLN chip [63]. Furthermore, nanophotonic PPLN waveguides promise to increase the nonlinear efficiency one order of magnitude [75] [76]. The pump profile engineering can be realized by means of off-the-shelf elements such as fiber attenuators, phase shifters and V-groove arrays, or by means of active elements in electrooptics materials such as LN [20]. An overall squeezing detection efficiency of 71% has been re-

| Individual modes | Nonlinear supermodes |
|------------------|----------------------|
| Quantum network  | Real                 |
| Network nodes    | Real or virtual      |
| Simultaneous access to nodes | Distributed |
| Access to network vertices | Localized |
|                  | Yes                  |
|                  | Independent LO       |
|                  | No                   |
|                  | Shaped LO or postprocessing |

TABLE III. Features of individual mode-based and nonlinear supermode-based approaches for quantum information processing in ANWs.
ently reported [62]. Anti-reflection coating on the chip output facet and balanced-homodyne-detector photodiodes with almost 100% quantum efficiency are expected to eventually step the detection efficiency up to 99% [20]. The use of V-groove arrays to fiber the output light can also lead to balanced-homodyne-detector spatial mode-matching visibilities of 99%.

We have demonstrated that the array of nonlinear waveguides is a versatile synthesizer of spatial multimode squeezing and multipartite entanglement. Features as scalability, reconfigurability, subwavelength stability, reproducibility and low-cost make this platform an appealing quantum technology. All the above hallmarks represent a boost in the spatial encoding domain with respect to previous bulk-optics-based multipartite-entanglement approaches. The analysis here carried out demonstrates that the array of nonlinear waveguides is a competitive contender in quantum communication, quantum computing and quantum simulation.

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Appendix A

We exhibit in this appendix how we obtained Equation (57). The covariance matrix elements Equations (39) for an array with odd number $N$ of waveguides in the limit of large coupling ($C_0 \to \infty$) is

$$V(x_i, x_j) = V(y_i, y_j) \rightarrow \delta_{i,j} - 2M_{i,l}M_{j,l} \sinh^2(2|\eta|z),$$

$$V(x_i, y_j) \rightarrow \pm M_{i,l}M_{j,l} \sinh(4|\eta|z),$$

where the $(+, -)$ signs are obtained setting respectively $\phi = (0, \pi)$, and $l = (N+1)/2$ is the index corresponding to the zero supermode. Applying this result into the general expression for the VLF inequalities Equation (55) without optimization ($\vec{G} = 0$) and using a $3\pi/2$ rotated quadrature for one of the involved modes (for instance, $\theta_j = 0$, $\theta_{j+1} = 3\pi/2$), we obtain

$$\text{VLF}_j(\vec{G} = \vec{0}) = 4 - 2(M^2_{j,l} + M^2_{j+1,l})$$

$$+ (M_{j,l} \pm M_{j+1,l})^2 e^{4|\eta|z} + (M_{j,l} \mp M_{j+1,l})^2 e^{-4|\eta|z} \geq 4.$$  

The best scenario in terms of violation of these inequalities corresponds to the case $M_{j,l} = \mp M_{j+1,l}$, for which we obtain

$$\text{VLF}_j(\vec{G} = \vec{0}) = 4 - 4M^2_{j,l}(1 - e^{-4|\eta|z}) < 4 \quad \forall z > 0.$$  

We note that the same result is obtained for pump phase profiles $\phi = (\pi/2, 3\pi/2)$ using generalized quadratures with $\theta_j = \theta_{j+1} = 0$.

Particularly, the coefficients of the zero supermode in an array with homogeneous coupling profile are given by

$$M_{j,l} = \frac{\sin(2\pi l)}{\sqrt{l}} = \sqrt{\frac{2}{(N+1)}} \sin(\frac{j\pi}{2}).$$

The $l$ odd elements of this vector satisfy $M_{2j-1,l} = -M_{2j+1,l}$, which maximizes the violation of the separability conditions for $\phi = 0$. Thus, taking into account only the odd elements of the zero supermode, we obtain a degenerate expression for the $l - 1$ inseparability conditions

$$\text{VLF}(\vec{G} = \vec{0}) = 4\left(\frac{N - 1 + 2e^{-4|\eta|z}}{N + 1}\right) < 4 \quad \forall z > 0.$$  

Finally, the use of a gain profile $\vec{G} \neq \vec{0}$ can only improve the above result. Then, we can write

$$\text{VLF}(\vec{G}) \leq \text{VLF}(\vec{G} = \vec{0}) < 4.$$  

Appendix B

Below we define the five normalized nullifiers and four VLF inequalities corresponding to the 5-mode cluster states exhibited in Figure 14 and Table II. The upper bounds for complete inseparability are slightly different from the usual ones because of the use of normalized nullifiers [14, 15].

i) Pentagon

$$\delta_i = \frac{y_i(\theta_i) - x_{i+1}(\theta_{i+1}) + x_{i-1}(\theta_{i-1})}{\sqrt{3}}.$$

$$V(\delta_i) + V(\delta_{i+1}) \geq \frac{4}{3} \quad \text{for} \quad i = 1, \ldots, 4.$$  

with $x_0(\theta_0) \equiv x_5(\theta_5)$ and $x_6(\theta_6) \equiv x_1(\theta_1)$.

ii) Star

$$\delta_i = \frac{y_i(\theta_i) - x_3(\theta_3)}{\sqrt{2}} \quad \text{for} \quad i \neq 3,$$

$$\delta_3 = \frac{y_3(\theta_3) - \sum_{i \neq 3} x_i(\theta_i)}{\sqrt{5}}.$$

$$V(\delta_i) + V(\delta_3) \geq \sqrt{\frac{8}{5}} \quad \text{for} \quad i \neq 3.$$
iii) Square Pyramid

\[ \tilde{\delta}_1 = y_1(\theta_1) - \left[x_2(\theta_2) + x_3(\theta_3) + x_5(\theta_5)\right], \]
\[ \tilde{\delta}_2 = y_2(\theta_2) - \left[x_1(\theta_1) + x_3(\theta_3) + x_4(\theta_4)\right], \]
\[ \tilde{\delta}_3 = \frac{y_3(\theta_3) - \sum_{i \neq 3} x_i(\theta_i)}{\sqrt{5}}, \]
\[ \tilde{\delta}_4 = \frac{y_4(\theta_4) - y_1(\theta_1)}{\sqrt{2}}, \]
\[ \tilde{\delta}_5 = \frac{y_5(\theta_5) - y_2(\theta_2)}{\sqrt{2}}, \]
\[ V(\tilde{\delta}_i) + V(\tilde{\delta}_3) \geq \sqrt{\frac{8}{5}} \text{ for } i = 4, 5. \]

We have used here the fact that linear combinations of nullifiers are also nullifiers in order to define \( \tilde{\delta}_{4,5}. \) Due to the symmetry of the cluster, the four VLF inequalities are degenerate two by two.

iv) GHZ

The GHZ cluster state is equivalent to the star cluster by means of a \( \pi/2 \) LO rotation for all modes \( i \neq 3 \) as demonstrated in ref. [2]. Applying this labelling we obtain

\[ \tilde{\delta}_i = \frac{x_i(\theta_i) - x_3(\theta_3)}{\sqrt{2}} \text{ for } i \neq 3, \]
\[ \tilde{\delta}_3 = \frac{\sum_{i=1}^{5} y_i(\theta_i)}{\sqrt{5}}, \]
\[ V(\tilde{\delta}_i) + V(\tilde{\delta}_3) \geq \sqrt{\frac{8}{5}} \text{ for } i \neq 3. \]

Thus, a GHZ cluster state is generated using the same set of parameters as that obtained for the star cluster with a \( \pi/2 \) LO rotation in all the modes except the mode 3.

Appendix C

Below, we display the matrices and parameters obtained for the emulation of the statistics associated to a 4-mode T-shape cluster state. The elements of the symmetric cluster transformation \( S_C \) are

\[ X_s = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ 0 & -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{pmatrix}, \]
\[ Y_s = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \]

We have obtained \( F_p = 0.17 \) for the fitness function. The optimized pump profile is given by

\[ \bar{\eta} = 0.015 \times \{1.08, 1.45, 1.78, 0.29\} \text{ mm}^{-1}, \]
\[ \bar{\phi} = -\pi \times \{0.02, 0.42, 0.07, -1.18\}. \]

The free and postprocessing orthogonal matrices are given by

\[ \bar{O} = \begin{pmatrix} -0.12 & -0.36 & 0.87 & 0.31 \\ -0.06 & 0.85 & 0.17 & 0.49 \\ -0.99 & -0.03 & -0.14 & -0.02 \\ 0.05 & -0.38 & -0.43 & 0.82 \end{pmatrix}, \]
\[ \bar{O}_{post} = \begin{pmatrix} 0.13 & -0.75 & 0.03 & 0.65 \\ 0.67 & 0.46 & 0.44 & 0.38 \\ -0.73 & 0.33 & 0.33 & 0.51 \\ 0.07 & 0.35 & -0.83 & 0.42 \end{pmatrix}. \]

Finally, the LO phase profile is

\[ \bar{\theta} = -\pi \times \{0.32, 0.19, 0.34, 0.48\}. \]

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