Kappa Snyder deformations of Minkowski spacetime, realizations and Hopf algebra

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Abstract

We present Lie-algebraic deformations of Minkowski space with undeformed Poincaré algebra. These deformations interpolate between Snyder and κ-Minkowski space. We find realizations of noncommutative coordinates in terms of commutative coordinates and derivatives. By introducing modules, it is shown that although deformed and undeformed structures are not isomorphic at the level of vector spaces, they are however isomorphic at the level of Hopf algebraic action on corresponding modules. Invariants and tensors with respect to Lorentz algebra are discussed. A general mapping from κ-deformed Snyder to Snyder space is constructed. Deformed Leibniz rule, the Hopf structure and star product are found. Special cases, particularly Snyder and κ-Minkowski in Maggiore-type realizations are discussed. The same generalized Hopf algebraic structures are as well considered in the case of an arbitrary allowable kind of realisation and results are given perturbatively up to second order in deformation parameters.

I Introduction

Current progress in high energy physics in considerable part very much relies on concepts and ideas which come from the scope of noncommutative (NC) physics. These concepts embody a modification in a description of spacetime as understood from the point of view of standard commutative field theories, where it is considered

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as a nondiscrete continuum, i.e. continuous, nondiscretized geometric structure. A signal for the possible modification of spacetime structure emerges through the appearance of a new fundamental length scale in physics. This new fundamental length scale, known as Planck length [1],[2], appears within two different and highly significant physical contexts. The first one comes from loop quantum gravity in which the Planck length plays a fundamental role. There, a process of quantization leads to the area and volume operators having discrete spectra, with minimal values of the corresponding eigenvalues being proportional to the square and cube of Planck length, respectively [3],[4]. The second important context where the signal for the existence of a new fundamental length scale can be found is related to certain observations of ultra-high energy cosmic rays which seem to contradict the usual understanding of some astrophysical processes like, for example, electron-positron production in collisions of high energy photons. It turns out that deviations observed in these processes can be explained by modifying dispersion relation in such a way as to incorporate the fundamental length scale [5],[6],[7],[8]. The appearance of this new fundamental length scale has a far reaching consequence for the spacetime structure because at this scale the spacetime structure becomes discretized and fuzzy and thus most conveniently described in terms of noncommutative geometry. NC spacetime has also been revived in the paper by Seiberg and Witten [9] where NC manifold emerged in a certain low energy limit of open strings moving in the background of a two form gauge field. Recently, a $\kappa$-Minkowski NC space in bicrossproduct basis was shown to emerge from consideration of a NC differential structure on the (pseudo)-Riemannian manifold [10],[11].

As a result of different approaches to quantum gravity, various phenomenological models emerged, whose aim is to predict a phenomenology that tests and searches for specific properties the fundamental theory of quantum gravity might have. Among them the most interesting are Lorentz invariance violation (LIV) models [12],[13] and the models of the so called doubly special relativity theory (DSR) [14],[15],[16]. In LIV models the observer independence is explicitly violated and a preferred frame is singled out. Preferred frame in this case is typically thought to be the rest frame of the cosmic microwave background radiation. On the other side, in DSR models postulates of relativity may be reformulated so as to avoid a necessity for singling out a preferred frame. It is this set of models that provide a kinematical theory within
which a Planck length is incorporated as a new fundamental invariant, besides that of
the speed of light $c$. Currently there is no unique view about the source where doubly
special relativity could possibly originate from. While the motivation for DSR comes
originally from loop quantum gravity, there are some arguments suggesting that it
may as well emerge in a form of a theory described in terms of what is known as
\(\kappa\)-Poincaré algebra, after taking an appropriate low energy limit of quantum gravity.
Most of considerations on DSR has been made within the framework of \(\kappa\)-Poincaré
algebra, a type of quantum Hopf algebra where the algebraic symmetry properties
are considered as necessarily emerging from a deformation of Poincaré and even
Lorentz symmetry \([17],[18],[19],[20],[21],[22]\). In some recent considerations it is
indicated that a doubly special relativity framework does not necessarily require
a deformation of the Lorentz group, neither its action. As example, the minimal
canonical covariantisation of the usual \(\kappa\)-Minkowski model was shown in \([23]\) to give
rise to a covariant \(\kappa\)-Minkowski spacetime, preserving Lorentz symmetry. Another
analysis which goes along the similar track was put forward in \([24]\) by requiring
the parameter of deformation to transform as a vector under Lorentz generators.
Certain aspects of DSR models are however currently under debate \([25],[26],[27]\).

In this paper, we shall be interested in still wider class of deformations of \(\kappa\)-
Minkowski space that can be described within a generalization of \(\kappa\)-Poincaré algebra.
\(\kappa\)-Poincaré algebra alone is an algebra that describes in a direct way only the energy-
momentum sector of the physical theory. This means that this algebra is specified
by commutation relations between linear and angular momenta only. Although this
sector alone is insufficient to set up physical theory, the Hopf algebra structure makes
it possible to extend the energy-momentum algebra to the algebra of spacetime.

It is known \([22]\) that there exists a transformation which maps \(\kappa\)-Minkowski
spacetime into spacetime with noncommutative structure described by the algebra
first introduced by Snyder \([28]\). Although this map is not an isomorphism, Snyder
algebra itself is particularly interesting since it is compatible with Lorentz symme-
try and it also provides \([22]\) configuration space consistent with DSR and thus can
be used to construct the second order particle Lagrangian. The last observation
makes possible to define physical four-momenta determined by the particle dynam-
ics. This would be significant step forward because the theoretical development in
this area has been mainly kinematical so far. One such dynamical picture has been
given recently in [29] where it was shown that reparametrisation symmetry of the proposed Lagrangian, together with the appropriate change of variables and conveniently chosen gauge fixing conditions, leads to an algebra which is a combination of $\kappa$-Minkowski and Snyder algebra. This generalized type of algebra describing noncommutative structure of Minkowski spacetime is shown to be consistent with the Magueijo-Smolin dispersion relation [16]. This type of NC space is also considered in [30]. It has to be stressed that NC spaces in neither of these papers are of Lie-algebra type.

In order to fill this gap, in the present paper we unify $\kappa$-Minkowski and Snyder space in a more general NC space which is of a Lie-algebra type and, in addition, is characterized by the undeformed Poincaré algebra and deformed coalgebra. In other words, we shall consider a type of NC space which interpolates between $\kappa$-Minkowski space and Snyder space in a Lie-algebraic way and has all deformations contained within the coalgebraic sector. First such example of NC space with undeformed Poincaré algebra, but with deformed coalgebra is given by Snyder [28]. Some other types of NC spaces are also considered within the approach in which the Poincaré algebra is undeformed and coalgebra deformed, in particular the type of NC space with $\kappa$-deformation [21],[22],[31],[32],[33],[34],[35]. Here we present a broad class of Lie-algebra type deformations with the same property of having deformed coalgebra, but undeformed algebra. The investigations carried out in this paper are along the track of developing general techniques of calculations, applicable for a widest possible class of NC spaces and as such are a continuation of the work done in a series of previous papers [31],[32],[33],[36],[37],[38],[39]. They are in particular a continuation of Snyder space analysis undertaken in [40],[41]. The methods used in these investigations were taken over from the Fock space analysis carried out in [42],[43].

The plan of paper is as follows. In section 2 we introduce a type of deformations of Minkowski space that have a structure of a Lie algebra and which interpolate between $\kappa$-type of deformations and deformations of the Snyder type. In section 3 we analyze realizations of NC space in terms of operators belonging to the undeformed Heisenberg-Weyl algebra. The introduction of modules in section 4 makes it possible to establish an isomorphism between deformed and undeformed structures, which otherwise does not exist. This is due to the fact that in the case of kappa-Snyder
type of deformation, deformed algebra of noncommutative functions is much larger than the corresponding undeformed algebra of commutative functions, making any isomorphism between these two impossible. Described situation can however be overcome if one introduces module for the envelopes of the deformed and undeformed Heisenberg algebras, respectively, and looks at their action on the unit element in the module. Then corresponding projections appear to be isomorphic to each other. In section 5 we tackle the issue of the way in which general invariants and tensors can be constructed out of NC coordinates. Section 6 is devoted to an analysis of the effects which these deformations have on the Hopf structure of the symmetry algebra and after that, in section 7 we specialize the general results obtained to some interesting special cases, such as \( \kappa \)-Minkowski space and Snyder space. In addition, the most general case of realization, consistent with all Jacobi identities and given algebra, is analysed perturbatively up to second order in deformation parameters and resulting coalgebraic structures, such as coproducts and antipodes, are also given up to the same order. We end the paper with some discussion.

II Noncommutative coordinates and Poincaré algebra

We are considering a Lie algebra type of noncommutative (NC) space generated by the coordinates \( \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1} \), satisfying the commutation relations

\[
[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) + s M_{\mu\nu},
\]

where indices \( \mu, \nu = 0, 1 \ldots, n - 1 \) and \( a_0, a_1, \ldots, a_{n-1} \) are components of an \( n \)-vector \( a \) in Minkowski space whose metric signature is \( \eta_{\mu\nu} = \text{diag}(-1, 1, \cdots, 1) \). The quantities \( a_\mu \) and \( s \) are deformation parameters which measure a degree of deviation from standard commutativity. They are related to length scale characteristic for distances where it is supposed that noncommutative character of space-time becomes important. When parameter \( s \) is set to zero, noncommutativity \( \Rightarrow \) reduces to covariant version of \( \kappa \)-deformation, while in the case that all components of a \( n \)-vector \( a \) are set to 0, we get the type of NC space considered for the first time by Snyder [28]. The NC space of this type has been analyzed in the literature from different points of view [40],[41],[44],[45],[46],[47],[48],[49],[50].
The symmetry of the deformed space (1) is assumed to be described by an undeformed Poincaré algebra, which is generated by generators $M_{\mu\nu}$ of the Lorentz algebra and generators $D_\mu$ of translations. This means that generators $M_{\mu\nu}$, $M_{\mu\nu} = -M_{\nu\mu}$, satisfy the standard, undeformed commutation relations,

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\mu\rho} M_{\nu\lambda},$$  \hspace{1cm} (2)

with the identical statement as well being true for the generators $D_\mu$,

$$[D_\mu, D_\nu] = 0, \hspace{1cm} (3)$$

$$[M_{\mu\nu}, D_\lambda] = \eta_{\nu\lambda} D_\mu - \eta_{\mu\lambda} D_\nu. \hspace{1cm} (4)$$

The undeformed Poincaré algebra, Eqs.(2),(3) and (4) define the energy-momentum sector of the theory considered. However, full description requires space-time sector as well. Thus, it is of interest to extend the algebra (2), (3) and (4) so as to include NC coordinates $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1}$, and to consider the action of Poincaré generators on NC space (1),

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda} - i (a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda}).$$  \hspace{1cm} (5)

The main point is that commutation relations (1), (2) and (5) define a Lie algebra generated by Lorentz generators $M_{\mu\nu}$ and $\hat{x}_\lambda$. The necessary and sufficient condition for consistency of an extended algebra, which includes generators $M_{\mu\nu}$, $D_\mu$ and $\hat{x}_\lambda$, is that Jacobi identity holds for all combinations of the generators $M_{\mu\nu}$, $D_\mu$ and $\hat{x}_\lambda$. Particularly, the algebra generated by $D_\mu$ and $\hat{x}_\nu$ is a deformed Heisenberg-Weyl algebra and we require that this algebra has to be of the form,

$$[D_\mu, \hat{x}_\nu] = \Phi_{\mu\nu}(D),$$  \hspace{1cm} (6)

where $\Phi_{\mu\nu}(D)$ are functions of generators $D_\mu$, which are required to satisfy the boundary condition $\Phi_{\mu\nu}(0) = \eta_{\mu\nu}$ and still be consistent with Eq.(1) and relevant Jacobi identities, as explained below. This condition means that deformed NC space, together with the corresponding coordinates, reduces to ordinary commutative space in the limiting case of vanishing deformation parameters, $a_\mu, s \to 0$.

One certain type of noncommutativity, which interpolates between Snyder space and $\kappa$-Minkowski space, has already been investigated in [29], [30] in the context of...
Lagrangian particle dynamics. However, in these papers algebra generated by NC coordinates and Lorentz generators is not linear and is not closed in the generators of the algebra. Thus, it is not of Lie-algebra type. In contrast to this, here we consider an algebra \((1),(2),(5)\), which is of Lie-algebra type, that is, an algebra which is linear in all generators and Jacobi identities are satisfied for all combinations of generators of the algebra. Besides that, it is important to note that, once having relations \((1)\) and \((2)\), there exists only one possible choice for the commutation relation between \(M_{\mu\nu}\) and \(\hat{x}_\lambda\), which is consistent with Jacobi identities and makes Lie algebra to close, and this unique choice is given by the commutation relation \((5)\).

In subsequent considerations we shall be interested in possible realizations of the space-time algebra \((1)\) in terms of canonical commutative spacetime coordinates \(X_\mu\),

\[ [X_\mu, X_\nu] = 0, \]  

which, in addition, with derivatives \(D_\mu \equiv \frac{\partial}{\partial X_\mu} \) close the undeformed Heisenberg algebra,

\[ [D_\mu, X_\nu] = \eta_{\mu\nu}. \]  

Thus, our aim is to find a class of covariant \(\Phi_{\alpha\mu}(D)\) realizations,

\[ \hat{x}_\mu = X^\alpha \Phi_{\alpha\mu}(D), \]  

satisfying the boundary conditions \(\Phi_{\alpha\mu}(0) = \eta_{\alpha\mu}\) and commutation relations \((1)\) and \((5)\). In order to complete this task, we introduce the standard coordinate representation of the Lorentz generators \(M_{\mu\nu}\),

\[ M_{\mu\nu} = X_\mu D_\nu - X_\nu D_\mu. \]  

All other commutation relations, defining the extended algebra, are then automatically satisfied, as well as all Jacobi identities among \(\hat{x}_\mu, M_{\mu\nu},\) and \(D_\mu\). This is assured by the construction \((9)\) and \((10)\).

As a final remark in this section, it is interesting to observe that the trilinear commutation relation among NC coordinates has a particularly simple form,

\[ [[\hat{x}_\mu, \hat{x}_\nu], \hat{x}_\lambda] = a_\lambda (a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) + s (\hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda}), \]  

which shows that on the right hand side Lorentz generators completely drop out. Described property of trilinear relations is relevant and significant only when \(s \neq \)
0, since otherwise it makes no difference nor gives any new information. Hence, when \( s \neq 0 \), it has important consequences for the relationship between algebras of commutative and noncommutative functions, particularly for the properties of the map that acts between the two. The problem of existing isomorphism is of special interest in this regard. While, for example, in the case of \( \kappa \)-deformation the relationship between deformed and undeformed algebras of functions is rather clear, the situation when \( s \neq 0 \) is to a certain extent relatively dim. What lies behind this statement is that in the case of \( \kappa \)-deformation, it is easy to establish the isomorphism between deformed and undeformed algebras, while for the more general case of \( \kappa \)-Snyder deformation, when \( s \neq 0 \), the situation gets more complicated. It does not mean that in the case when \( s \neq 0 \) isomorphism cannot be established. It can, but the things are not so easy and straightforward. We shall continue to elaborate on these issues in section 4, where we shall show how this isomorphism can be constructed.

To better understand the nature of these mutual relations, it is desirable to introduce necessary notions and ingredients. Thus, let us define an enveloping algebra \( \hat{A}_x \) as free algebra generated by \( \hat{x}_\mu \) and divided by the ideal generated with trilinear relations, Eq.(11). The algebra \( \hat{A}_x \) contains unit element 1. We note that in the case when \( s \neq 0 \), two monomials \( \hat{x}_\mu \hat{x}_\nu \) and \( \hat{x}_\nu \hat{x}_\mu \), with \( \mu \neq \nu \), are algebraically independent in the algebra \( \hat{A}_x \). Furthermore, if \( \mu \neq \nu \neq \lambda \neq \mu \) (mutually different) among 6 monomials: \( \hat{x}_\mu \hat{x}_\nu \hat{x}_\lambda \), \( \hat{x}_\mu \hat{x}_\lambda \hat{x}_\nu \), \( \hat{x}_\nu \hat{x}_\mu \hat{x}_\lambda \), \( \hat{x}_\nu \hat{x}_\lambda \hat{x}_\mu \), \( \hat{x}_\lambda \hat{x}_\mu \hat{x}_\nu \) and \( \hat{x}_\lambda \hat{x}_\nu \hat{x}_\mu \), there are two relations:

\[
[[\hat{x}_\mu, \hat{x}_\nu], \hat{x}_\lambda] = a_\lambda(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) + s(\hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda}), \tag{12}
\]

\[
[[\hat{x}_\mu, \hat{x}_\lambda], \hat{x}_\nu] = a_\nu(a_\mu \hat{x}_\lambda - a_\lambda \hat{x}_\mu) + s(\hat{x}_\mu \eta_{\lambda\nu} - \hat{x}_\lambda \eta_{\mu\nu}). \tag{13}
\]

The third relation \([[[\hat{x}_\mu, \hat{x}_\lambda], \hat{x}_\nu]]\) follows from above two relations by Jacobi identity. Thus, among these 6 monomials there are only four of them that are linearly independent. In special case \( \mu = \nu \neq \lambda \) there is only one relation

\[
[[\hat{x}_\mu, \hat{x}_\lambda], \hat{x}_\mu] = a_\mu(a_\mu \hat{x}_\lambda - a_\lambda \hat{x}_\mu) - s\hat{x}_\lambda \eta_{\mu\mu} \tag{14}
\]

and there are only 2 linearly independent monomials. We point out that in the case when \( s \neq 0 \) the algebra \( \hat{A}_x \) is larger than the algebra \( A_X \) generated by commutative
generators $X_\mu$ and thus there is no isomorphism between $\hat{A}_x$ and $A_X$ at the level of vector spaces.

These things will be further elaborated in Section 4, where their proper understanding will lead to realization of an isomorphism between deformed and undeformed algebraic structures. In the next section we turn to problem of finding an explicit $\Phi_{\mu\nu}(D)$ realizations [3].

III Realizations of NC coordinates

Let us define general covariant realizations:

$$\hat{x}_\mu = X_\mu \varphi + i(aX) (D_\mu \beta_1 + ia_\mu D^2 \beta_2) + i(XD) (a_\mu \gamma_1 + i(a^2 - s)D_\mu \gamma_2), \quad (15)$$

where $\varphi$, $\beta_i$ and $\gamma_i$ are functions of $A = ia_\alpha D^\alpha$ and $B = (a^2 - s)D_\alpha D^\alpha$. We further impose the boundary condition that $\varphi(0,0) = 1$ as the parameters of deformation $a_\mu \to 0$ and $s \to 0$. In this way we assure that $\hat{x}_\mu$ reduce to ordinary commutative coordinates in the limit of vanishing deformation.

It can be shown that Eq.(15) requires the following set of equations to be satisfied,

$$\frac{\partial \varphi}{\partial A} = -1, \quad \frac{\partial \gamma_2}{\partial A} = 0, \quad \beta_1 = 1, \quad \beta_2 = 0, \quad \gamma_1 = 0. \quad (16)$$

Besides that, the commutation relation (11) leads to the additional two equations,

$$\varphi(\frac{\partial \varphi}{\partial A} + 1) = 0,$$

$$(a^2 - s)[2(\varphi + A) \frac{\partial \varphi}{\partial B} - \gamma_2(A \frac{\partial \varphi}{\partial A} + 2B \frac{\partial \varphi}{\partial B}) + \gamma_2 \varphi] - a_\mu^2 \frac{\partial \varphi}{\partial A} - s = 0. \quad (17)$$

The important result of this paper is that all above required conditions are solved by a general form of realization which in a compact form can be written as

$$\hat{x}_\mu = X_\mu (-A + f(B)) + i(aX)D_\mu - (a^2 - s)(XD)D_\mu \gamma_2, \quad (18)$$

where $\gamma_2$ is necessarily restricted to be

$$\gamma_2 = -\frac{1 + 2f(B) \frac{\partial f(B)}{\partial B}}{f(B) - 2B \frac{\partial f(B)}{\partial B}}, \quad (19)$$
From the above relation we see that $\gamma_2$ is not an independent function, but instead is related to generally an arbitrary function $f(B)$, which has to satisfy the boundary condition $f(0) = 1$. Also, it is readily seen from the realization (18) that $\varphi$ in (15) is given by $\varphi = -A + f(B)$. Various choices of the function $f(B)$ lead to different realizations of NC spacetime algebra (1). Two particularly interesting cases are for $f(B) = 1$ and $f(B) = \sqrt{1-B}$. The later one, when $f(B) = \sqrt{1-B}$, will be given special attention since it allows for an exact and complete analysis resulting in coproducts and antipodes for the generators of Poincaré algebra, which are contained in and define a Hopf algebraic structure in that particular case. Although the exact treatment is reserved for the case $f(B) = \sqrt{1-B}$ only, we shall be interested in other realizations as well and, in particular, in the analysis of the most general case, which will be tackled perturbatively.

It is now straightforward to show that the deformed Heisenberg-Weyl algebra (6) takes the form

$$[D_\mu, \hat{x}_\nu] = \eta_{\mu\nu}(-A + f(B)) + ia_\mu D_\nu - (a^2 - s)D_\mu D_\nu \gamma_2$$

and that the Lorentz generators $M_{\mu\nu}$ can be expressed in terms of NC coordinates as

$$M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu) \frac{1}{\varphi}. \quad (21)$$

We also point out that in the special case when realization of NC space (1) is characterized by the function $f(B) = \sqrt{1-B}$, there exists a unique element $Z$ satisfying:

$$[Z^{-1}, \hat{x}_\mu] = -ia_\mu Z^{-1} + sD_\mu, \quad [Z, \hat{x}_\mu] = ia_\mu Z - sD_\mu Z^2. \quad (22)$$

and also

$$\hat{x}_\mu Z \hat{x}_\nu = \hat{x}_\nu Z \hat{x}_\mu, \quad [Z, D_\mu] = 0. \quad (23)$$

The element $Z$ is a generalized shift operator [32] and its expression in terms of $D_\mu$ can be shown to have the form

$$Z^{-1} = -A + \sqrt{1-B}. \quad (24)$$

As a consequence, the Lorentz generators can be expressed in terms of $Z$ as

$$M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu)Z, \quad (25)$$
and one can also show that the relation

\[ [Z^{-1}, M_{\mu\nu}] = -i(a_\mu D_\nu - a_\nu D_\mu) \quad (26) \]

holds. In what follows we shall mainly be concerned with the realizations determined by \( f(B) = \sqrt{1 - B} \), but shall consider other realizations as well. While the realization defined by \( f(B) = \sqrt{1 - B} \) can be treated exactly, the other realizations are very difficult to treat in that manner, although some types of realizations admit exact treatment under some special circumstances. For example, when the deformation of spacetime is of Snyder-type, then it is possible to carry out an exact analysis not only for the function \( f(B) = \sqrt{1 - B} \), but for the function \( f(B) = 1 \), as well. Due to technical difficulties related to exact treatment of the general type of realization, the final part of the paper will be devoted to perturbative treatment of that case, which is determined by the form of function \( f \) that is consistent with all imposed requirements, but otherwise arbitrary. This perturbative treatment of the general case of realization will result with all necessary Hopf-algebraic ingredients, determined up to second order in the deformation parameters \( a \) and \( s \).

IV Enveloping algebra modules

In this section we investigate modifications which affect the algebra of functions upon deforming the spacetime structure. Since the algebra \([1]\) mixes spacetime coordinates with Lorentz generators, the algebraic structure obtained upon deformation is much richer than the original one. As a consequence, these two structures will not be isomorphic to each other. This is somewhat unpleasant situation because it does not allow us to make full correspondence between commutative and noncommutative algebras, the feature which is anyway common for Moyal and pure \( \kappa \)-deformation \((s = 0)\). It would be thus convenient to find a way out of this situation and to look for the means by which the isomorphism could be again established. Before being able to do that, we have to recapitulate some basic notions from previous sections and to introduce few new ones. Firstly, it is assumed in our considerations that noncommutative functions are infinitely derivable, i.e. that they belong to the set \( C^\infty(\mathbb{R}^{1,3}) \) of infinitely derivable functions defined on Minkowskian spacetime manifold. As already pointed in previous section, they can be arranged into an algebra,
whose multiplication operation is given by the standard multiplication of functions, to form an algebra of functions in commutative coordinates, $\mathcal{A}_X$. This algebra can also be considered as the commutative enveloping algebra in $X_\mu$. In the similar way, as already defined in section 2, with $\hat{A}_\hat{x}$ we denote the noncommutative enveloping algebra in $\hat{x}_\mu$, which is set to represent the algebra of functions in noncommutative coordinates. Besides these two kinds of algebras, $\mathcal{A}_X$ and $\hat{A}_\hat{x}$, there is still a third type of algebra that is relevant in our discussions and that one is denoted by $\mathcal{A}_X,^\star$ and represents a noncommutative algebra of functions in commutative coordinates, but with the $\star$-product having the role of a multiplication operation. We shall return to the algebra $\mathcal{A}_X,^\star$ and definition of the star product shortly, after we introduce all other necessary conceptual pieces in the scheme we work in.

Let us further denote by $H \equiv H(X_\mu, D_\mu)$ an undeformed Heisenberg algebra generated by $X_\mu, D_\mu$ and defined by equations (3),(7),(8). The corresponding universal enveloping algebra can then be denoted by $U(H)$. 

Let us next introduce the unit element $1 \in \mathcal{A}_X$ and define the action of Poincaré generators $D_\mu$ and $M_{\mu\nu}$ on $1$ as

$$D_\mu \triangleright 1 = 0, \quad M_{\mu\nu} \triangleright 1 = 0.$$  \hspace{1cm} (27)

Hence, the action of Poincaré algebra $\mathcal{P}$, generated by $D_\mu$ and $M_{\mu\nu}$, on $1$ is: $\mathcal{P} \triangleright 1 = 0$. The unit element $1$ is here assumed to be a unit element in the algebra $\mathcal{A}_X$, understood as a module for the algebra $\mathcal{A}_X$ (i.e. $\mathcal{A}_X$ is $U(H)$ module).

Bearing this in mind, we can define the action of $\mathcal{A}_X$ on $1$ as a Fock-like vector space denoted by $\mathcal{A}_X \triangleright 1$, with the property

$$X_\mu \triangleright 1 = X_\mu \quad \text{and consequently} \quad \phi(X) \triangleright 1 = \phi(X),$$ \hspace{1cm} (28)

satisfied for any $\phi(X) \in \mathcal{A}_X$. Hence $\mathcal{A}_X \triangleright 1 = \mathcal{A}_X$. Similarly we define the action $U(H) \triangleright 1$. Then the $U(H)$ module is defined by

$$U(H) \triangleright \mathcal{A}_X = U(H)\mathcal{A}_X \triangleright 1 \equiv \mathcal{A}_X.$$ \hspace{1cm} (29)

By following the same steps, we introduce deformed Heisenberg algebra $\hat{H}(\hat{x}_\mu, D_\mu)$, defined by equations (1),(3),(6). The corresponding universal enveloping algebra can then be denoted with $U(\hat{H})$. As defined earlier, the noncommutative enveloping algebra in $\hat{x}_\mu$ is denoted by $\hat{A}_\hat{\hat{x}}$. The action of $D_\mu$ and $M_{\mu\nu}$ is given by (27), so that we can also define the action of $\hat{A}_\hat{\hat{x}}$ on $1$ as a Fock-like vector space denoted by $\hat{A}_\hat{\hat{x}} \triangleright 1$. 

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In the previous section, we have seen that the commutation relations (1) admit a class of realizations of the form

\[ \hat{x}_\mu = X^\alpha \Phi_{\alpha\mu}(D). \]

The commutation relations (1), (6) and the general form (9) of realizations imply that we have

\[
[\hat{x}_\mu, \hat{x}_\nu] \triangleright 1 = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) \triangleright 1,
\]

as well as

\[ [D_\mu, \hat{x}_\nu] \triangleright 1 = \eta_{\mu\nu} \triangleright 1. \]

and

\[ \hat{x}_\mu \triangleright 1 = X_\mu. \]

We point out that by shifting the derivative \( D_\mu \) to the most right in the element of \( \mathcal{U}(\hat{H}) \), acting on 1, and by using (31) and Jacobi identities, we can establish the equivalence

\[ \mathcal{U}(\hat{H}) \triangleright 1 \equiv \hat{A}_x \triangleright 1. \]

In the similar way as done with the monomial (32), it is also possible to calculate the action on unit element 1 by an arbitrary monomial in noncommutative coordinates. All that is necessary to do this is to replace NC coordinates according to (9) and then, by successively applying the commutation relation (8), shift the derivative \( D_\mu \) to the most right, to get

\[
\hat{x}_{\mu_{\Pi(1)}} \ldots \hat{x}_{\mu_{\Pi(m)}} \triangleright 1 = X_{\mu_1} \ldots X_{\mu_m} + P_{(m-1),\Pi}(X)
\]

where \( P_{(m-1),\Pi}(X) \) is a polynomial in \( X_\mu \) of order \( (m-1) \), \( m \in \mathbb{N} \) and \( \Pi \) is permutation, \( \Pi \in S_m \).

It is obvious from (34) that the space \( \hat{A}_x \triangleright 1 \) is contained within \( \mathcal{A}_X \). However, to show that opposite is also true, i.e. that \( \mathcal{A}_X \) is contained within \( \hat{A}_x \triangleright 1 \), we have to invoke the inverse of the transformation (9). It is assumed that the inverse of (9) exists, so that it can generally be written in the form

\[ X_\mu = \hat{x}^\alpha (\Phi^{-1})_{\alpha\mu}(D). \]

For example, for the case of realization (18), the corresponding inverse will be given in the next section (see Eq.(52)). Now, by taking the arbitrary element of \( \mathcal{A}_X \),
replacing all commutative coordinates according to prescription (35) and by shifting all derivatives to the most right, we regularly finish with some element in $\hat{A}_x \triangleright 1$, showing that $A_X$ is indeed contained within $\hat{A}_x \triangleright 1$. Thus, as the conclusion, we have that the space $\hat{A}_x \triangleright 1$ is isomorphic to $A_X$, as a vector space.

Hence, the space $\hat{A}_x$ is larger than space $\hat{A}_x \triangleright 1$ and the space $\hat{A}_x \triangleright 1$ is isomorphic to $A_X$ as a vector space. (Note that $\hat{\phi}(\hat{x}) \triangleright 1$ can be identified with $\hat{\phi}(\hat{x})|0 >$ in analogy with Fock-like space). Overall consistency follows from the Jacobi identities and, in addition, the corresponding $U(\hat{H})$ module can then be defined by:

$$U(\hat{H}) \triangleright (\hat{A}_x \triangleright 1) = (U(\hat{H})\hat{A}_x) \triangleright 1 = \hat{A}_x \triangleright 1$$

and generally we can write

$$\hat{\phi}(\hat{x}) \triangleright 1 = \phi(X),$$

for any $\hat{\phi}(\hat{x}) \in \hat{A}_x$ and for $\hat{x}$ given by (9).

Let us now turn to definition of the star product, as promised before. Thus, if we have two associations of the form (37), namely, $\hat{\phi}(\hat{x}) \triangleright 1 = \phi(X)$ and $\hat{\psi}(\hat{x}) \triangleright 1 = \psi(X)$, for two noncommutative functions $\hat{\phi}(\hat{x})$ and $\hat{\psi}(\hat{x})$, then the star product is defined by

$$\hat{\phi}(\hat{x})\hat{\psi}(\hat{x}) \triangleright 1 = \hat{\phi}(\hat{x}) \triangleright (\hat{\psi}(\hat{x}) \triangleright 1)$$

$$= \hat{\phi}(\hat{x}) \triangleright \psi(X) = \phi(X) \star \psi(X),$$

where it is understood that $\hat{x}$ is given by (9). The vector space $A_X$ together with the star product (38) constitutes a noncommutative algebra, which we denote by $A_{X,\star}$. The algebra $A_{X,\star}$ is identical to noncommutative algebra $\hat{A}_x \triangleright 1$ and is generally nonassociative, which can be seen by generalizing the definition (38) of the star product to a star product of three fields. In this case we come up with

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x})\hat{\phi}_3(\hat{x}) \triangleright 1 = (\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x})) \triangleright (\hat{\phi}_3(\hat{x}) \triangleright 1) = \hat{\phi}_1(\hat{x}) \triangleright (\hat{\phi}_2(\hat{x}) \triangleright (\hat{\phi}_3(\hat{x}) \triangleright 1))$$

$$= \hat{\phi}_1(\hat{x}) \triangleright (\phi_2(X) \star \phi_3(X)) = \hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) \triangleright \phi_3(X) = \phi_1(X) \star (\phi_2(X) \star \phi_3(X))$$

and specifically, when looking a product of NC coordinates, we have

$$\hat{x}_1\hat{x}_2\hat{x}_3 \triangleright 1 = \hat{x}_1 \triangleright (\hat{x}_2 \triangleright (\hat{x}_3 \triangleright 1)) = X_1 \star (X_2 \star X_3).$$

showing that if $s \neq 0$, the star product is generally non-associative [49, [1], [50], [51].
In addition, it should be noted that translation generators $D_\mu$ act on the elements of $\hat{A}_\hat{x}$ as

$$D_\mu \hat{\phi}(\hat{x}) \triangleright 1 = D_\mu \triangleright (\hat{\phi}(\hat{x}) \triangleright 1) = D_\mu \triangleright \phi(X) = \frac{\partial\phi(X)}{\partial X^\mu}$$

and

$$D_\mu \hat{\psi}(\hat{x}) \triangleright 1 = D_\mu \triangleright (\hat{\phi}(\hat{x})\hat{\psi}(\hat{x}) \triangleright 1) = D_\mu \triangleright (\phi(X) \star \psi(X))$$

and that the action of Lorentz generators $M_{\mu\nu}$ is defined by

$$M_{\mu\nu} \hat{\phi}(\hat{x}) \triangleright 1 = M_{\mu\nu} \triangleright (\hat{\phi}(\hat{x}) \triangleright 1) = M_{\mu\nu} \triangleright \phi(X),$$

$$M_{\mu\nu} \hat{\psi}(\hat{x}) \triangleright 1 = M_{\mu\nu} \triangleright (\hat{\phi}(\hat{x})\hat{\psi}(\hat{x}) \triangleright 1) = M_{\mu\nu} \triangleright (\phi(X) \star \psi(X)).$$

Finally, we define another action $\circ$ on the same unit element 1, defined by

$$D_\mu \circ 1 = 0, \quad M_{\mu\nu} \circ 1 = 0$$

and

$$\hat{x}_\mu \circ 1 = \hat{x}_\mu,$$  \hspace{1cm} (41)

where the last relation provides a construction of Fock-like space $\hat{A}_\hat{x} \circ 1$. The action denoted by $\circ$ implicitly includes the information that, besides definition (40), one has to use the inverse transformation (35) in calculating the expressions of the form $\Phi(X(\hat{x}, D)) \circ 1$, in the same way as the action $\triangleright$ implicitly includes the instruction of using direct transformation (9), when calculating the expressions such as $\hat{\Phi}(\hat{x}) \triangleright 1$.

In this way, by using the inverse map (35), it is easy to show that

$$X_\mu \circ 1 = \hat{x}^\alpha (\Phi^{-1})_{\alpha\mu}(D) \circ 1 = \hat{x}^\alpha \circ ((\Phi^{-1})_{\alpha\mu}(D) \circ 1) = \hat{x}_\mu \circ 1 = \hat{x}_\mu.$$  \hspace{1cm} (42)

If we assume that the relation

$$\hat{\Phi}(\hat{x}) \triangleright 1 = \Phi(X)$$

is satisfied, then Eq.(42) could possibly tempt one to conclude that

$$\Phi(X(\hat{x}, D)) \circ 1 = \hat{\Phi}(\hat{x}).$$  \hspace{1cm} (43)

However, this is not generally true, since already for monomial including the product of two coordinates, this relation fails to hold. For example, in the general case when $s \neq 0$, we have

$$X_\mu X_\nu \circ 1 = (\hat{x}^\alpha (\Phi^{-1})_{\alpha\mu}(D))(\hat{x}^\beta (\Phi^{-1})_{\beta\nu}(D)) \circ 1$$

$$= (\hat{x}^\alpha (\Phi^{-1})_{\alpha\mu}(D))\hat{x}_\nu \circ 1 = (\hat{x}_\mu \hat{x}_\nu + P_{\mu\nu}(\hat{x})) \circ 1,$$
where $P_{\mu\nu}(\hat{x})$ is some polynomial linear in $\hat{x}$. It is only in the special case when $s = 0$ that the above relation takes the form (43). That this is so can be inferred by inserting the explicit form (52) of the inverse map (35) into l.h.s. of the above expression and proceed further according to definition (40).

Although relation (43) is not valid generally, the correct relationship between commutative and noncommutative functions can be stated in the following way: If the function $\Phi(\hat{x})$ from $A_{\hat{x}}$ satisfies the relation

$$\Phi(\hat{x}) \triangleright 1 = \Phi(X),$$

for some $\Phi(X)$ in $A_X$, then use of the inverse map (35) leads generally not to (43), but instead leads to

$$\Phi(X(D)) \circ 1 = \Phi(\hat{x}) \circ 1,$$

showing that generally $\Phi(\hat{x}) \circ 1 \neq \Phi(\hat{x})$. An example that can serve to illustrate this last statement is provided by analysing the action

$$[\hat{x}_\mu, \hat{x}_\nu] \circ 1 = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu).$$

A simple analysis shows that relation $[\hat{x}_\mu, \hat{x}_\nu] \circ 1 = [\hat{x}_\mu, \hat{x}_\nu]$ is fulfilled only for $s = 0$, while in the general case when $s \neq 0$, it is not satisfied.

Described relationship thus establishes the isomorphism between spaces $A_{\hat{x}} \circ 1$ and $A_{\hat{x}} \triangleright 1$ at the level of vector space. The space $A_{\hat{x}} \circ 1$ as a vector space is thus isomorphic to $A_{\hat{x}} \triangleright 1$, since

$$\hat{x}_{\Pi(1)} \ldots \hat{x}_{\Pi(m)} \circ 1 = \hat{x}_{\mu_1} \ldots \hat{x}_{\mu_m} \circ 1 + \hat{P}_{(m-1), \Pi}(\hat{x}) \circ 1,$$

where $\hat{P}_{(m-1), \Pi}(\hat{x})$ is a polynomial in $\hat{x}_\mu$ of order $(m - 1)$, $m \in N$ and $\Pi$ is permutation, $\Pi \in S_m$. Hence, the algebra $A_{\hat{x}}$ is larger than space $A_{\hat{x}} \circ 1$ and the space $A_{\hat{x}} \triangleright 1$ is isomorphic to $A_X$ as a vector space.

We briefly summarise the basic results of this section. We have shown that although the vector spaces $A_X$ and $A_{\hat{x}}$ of commutative and noncommutative functions are not isomorphic to each other, it is possible to find vector spaces that are isomorphic to the space $A_X$. These are given by the actions $A_{\hat{x}} \triangleright 1$ and $A_{\hat{x}} \circ 1$ of the algebra $A_{\hat{x}}$ to the unit element 1. We emphasize that given a specific realization $\Phi_{\alpha\mu}(D)$, Eq. (9), of noncommutative coordinates, the necessary and sufficient
condition to have the above isomorphism realised at the level of vector spaces is the existance of the inverse map \((\Phi^{-1})_{\alpha\mu}(D)\) in (35). Since the realizations [18] that are of interest to us in this paper have a well defined inverse map (see Eq. (52) below), we can regularly establish the isomorphism in our case. This isomorphism is compactly described by the two mutually entangled relations (44) and (45).

As an example, instead of (6), the most general relations including coordinates and derivatives can be written in the form

\[
\hat{\partial}_i \hat{x}_j - q_{ij} \hat{x}_i \hat{\partial}_i = \Phi_{ij}(\hat{\partial}),
\]

(47)
describing the \(q\)-deformation of the Heisenberg algebra. For \(\Phi_{ij} = \delta_{ij}\), these relations were classified in [52]. In this case, the choice with \(q_{ij} = \pm 1\) leads to Heisenberg or Clifford algebra, or in other words to Bose or Fermi statistics, respectively. The situation with generic \(q_{ij}\) leads to infinite statistics algebras [53]. In this example there is no mapping at all between \(A_X\) and \(\hat{A}_\hat{x}\) and noncommutative coordinates and derivatives cannot be expressed in terms of commutative ones, showing that the \(q\)-deformed Heisenberg algebra is much larger than the undeformed Heisenberg algebra. Unlike the situation we have in this paper, in the case with generic \(q_{ij}\) the isomorphism cannot be established even between \(\hat{A}_\hat{x} \circ 1\) and \(A_X\).

V Invariants under Lorentz algebra

As in the ordinary commutative Minkowski space, here we can also take the operator \(P^2 = P_\alpha P^\alpha = -D^2\) as a Casimir operator, playing the role of an invariant in noncommutative Minkowski space. In doing this, we introduced the momentum operator \(P_\mu = -iD_\mu\). In this case, arbitrary function \(F(P^2)\) of Casimir also plays the role of invariant, namely \([M_{\mu\nu}, F(P^2)] = 0\). However, unlike the ordinary Minkowski space, in NC case we have a freedom to introduce still another invariant by generalizing the standard notion of d’Alambertian operator to the generalized one required to satisfy

\[
[M_{\mu\nu}, \Box] = 0,
\]

(48)
\[
[\Box, \hat{x}_\mu] = 2D_\mu.
\]

(49)
The general form of the generalized d’Alambertian operator $\Box$, valid for the large class of realizations (18), which are characterized by an arbitrary function $f(B)$, can be written in a compact form as

$$\Box = \frac{1}{a^2 - s} \int_0^B \frac{dt}{f(t) - t \gamma_2(t)},$$

(50)

where $\gamma_2(t)$ is given in (19). Due to the presence of the Lorentz invariance in NC Minkowski space (1), the basic dispersion relation is undeformed, i.e. it reads $P^2 + m^2 = 0$ for all $f(B)$. Specially, for $f(t) = \sqrt{1 - t}$, we have $\gamma_2(t) = 0$ and, consequently, the generalized d’Alambertian is given by

$$\Box = \frac{2(1 - \sqrt{1 - B})}{a^2 - s}.$$

(51)

It is easy to check that in the limit $a, s \rightarrow 0$, we have the standard result, $\Box \rightarrow D^2$, valid in undeformed Minkowski space.

Lorentz symmetry provides us with the possibility of constructing the invariants. In most general situation, for a given realization $\Phi_{\mu \nu}$, Eq.(18), Lorentz invariants can as well be constructed out of NC coordinates $\hat{x}_\mu$.

To proceed further with the construction of invariants in NC coordinates for a given realization $\Phi_{\mu \nu}$, it is also of interest to write down the inverse of realization (18), namely,

$$X_\mu = [\hat{x}_\mu - i(a\hat{x})] \frac{1}{f(B) - B \gamma_2} D_\mu + (a^2 - s)(\hat{x} D) \frac{1}{f(B) - B \gamma_2} D_\mu \gamma_2] \frac{1}{-A + f(B)}.$$  

(52)

Since we know how to construct invariants out of commutative coordinates and derivatives, namely, $X_\mu$ and $D_\mu$, relation (52) ensures that the similar construction can be carried out in terms of NC coordinates $\hat{x}_\mu$. The same construction also applies to tensors. All that is required is that the general invariants and tensors, expressed in terms of $X_\mu$ and $D_\mu$, have to be transformed into corresponding invariants and tensors in NC coordinates $\hat{x}_\mu$. The same holds for the invariants. For example, following the described pattern, we can
construct the second order invariant in NC coordinates in a following way. Knowing that the object \( X^2 = X_\alpha X^\alpha \) is a Lorentz second order invariant, \([M_{\mu\nu}, X_\alpha X^\alpha] = 0\), the corresponding second order invariant \( \hat{I}_2 \) in NC coordinates can be introduced as \( \hat{I}_2 = X_\alpha X^\alpha \circ 1 \). After use has been made of (52), simple calculation gives \( \hat{I}_2 \) expressed in terms of NC coordinates, \( \hat{I}_2 = \hat{x}_\alpha \hat{x}^\alpha \circ 1 - i(n - 1)a_\alpha \hat{x}^\alpha \circ 1 \). It is now easy to check that the action of Lorentz generators on \( \hat{I}_2 \) gives \( M_{\mu\nu} \circ \hat{I}_2 = 0 \), confirming the validity of the construction.

**Remark**

It is important to realize that NC space with the type of noncommutativity (1) can be mapped to Snyder space with the help of transformation

\[
\hat{x}_\mu = \hat{x}_\mu - ia^\alpha M_{\alpha\mu},
\]

(53)

generalizing the transformation used in [22] to map \( \kappa \)-deformed space to Snyder space. After applying this transformation, we get

\[
[\hat{x}_\mu, \hat{x}_\nu] = (s - a^2)M_{\mu\nu},
\]

(54)

\[
[M_{\mu\nu}, \hat{x}_\lambda] = \eta_{\nu\lambda} \hat{x}_\mu - \eta_{\mu\lambda} \hat{x}_\nu.
\]

(55)

The mapping (53) between spaces with symplectic structures of \( \kappa \)-Snyder and Snyder type, respectively, has properties that obviously depend on the mutual relations between the deformation parameters. This can best be seen when the question of isomorphism is raised. Hence, for \( s = 0 \), the above spaces are not isomorphic, while for \( s \neq 0 \), with an additional condition that \( a^2 \neq s \), these spaces are isomorphic. Finally, for \( a^2 = s \) there is no isomorphism and this situation represents a singular point, with an effective Snyder deformation parameter equal to zero, i.e. leading to commutative geometry.

The Lorentz generators are expressed in terms of this new coordinates as

\[
M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu)\frac{1}{f(B)},
\]

(56)

and \( \hat{x}_\mu \) alone, allows the representation

\[
\hat{x}_\mu = X_\mu f(B) - (a^2 - s)(XD)D_\mu \gamma_2,
\]

(57)
in accordance with (15). The results, starting with the mapping (53) and all down through Eq.(57), are valid not only for the particularly interesting choice \( f(B) = \sqrt{1-B} \), but instead are valid for an arbitrary function satisfying the boundary condition \( f(0) = 1 \).

VI Leibniz rule and Hopf algebra

The symmetry underlying deformed Minkowski space, characterized by the commutation relations (1), is the deformed Poincaré symmetry which can most conveniently be described in terms of quantum Hopf algebra. As was seen in relations (2),(3) and (4), the algebraic sector of this deformed symmetry is the same as that of undeformed Poincaré algebra. However, the action of Poincaré generators on the deformed Minkowski space is deformed, so that the whole deformation is contained in the coalgebraic sector. This means that the Leibniz rules, which describe the action of \( M_{\mu\nu} \) and \( D_\mu \) generators, will no more have the standard form, but instead will be deformed and will depend on a given \( \Phi_{\mu\nu} \) realization.

Generally we find that in a given \( \Phi_{\mu\nu} \) realization we can write

\[
e^{ik\hat{x}} \triangleright 1 = e^{iK_\mu(k)X^\mu}
\]

and

\[
e^{ik\hat{x}} \triangleright e^{iqX} = e^{iP_\mu(k,q)X^\mu},
\]

where the action on the unit element is defined in section 4, Eqs.(27),(37) and \( k\hat{x} = k^\alpha X^\beta \Phi_{\beta\alpha}(D) \). The quantities \( K_\mu(k) \) are readily identified as \( K_\mu(k) = P_\mu(k,0) \) and \( P_\mu(k,q) \) can be found by calculating the expression

\[
P_\mu(k,-iD) = e^{-ik\hat{x}}(-iD_\mu)e^{ik\hat{x}},
\]

where it is assumed that at the end of calculation the identification \( q = -iD \) has to be made. One way to explicitly evaluate the above expression is by using the standard \( ad \)-expansion perturbatively, order by order. To avoid this tedious procedure, we can turn to much more elegant method to obtain the quantity \( P_\mu(k,-iD) \). This consists in writing the differential equation

\[
\frac{dP_\mu^{(t)}}{dt}(k,-iD) = \Phi_{\mu\alpha}(iP^{(t)}(k,-iD))k^\alpha,
\]
satisfied by the family of operators \( P^{(t)}_{\mu}(k; -iD) \), defined as

\[
P^{(t)}_{\mu}(k; -iD) = e^{-itk\hat{x}}(-iD_{\mu})e^{itk\hat{x}}, \quad 0 \leq t \leq 1, \tag{62}
\]

and parametrized with the free parameter \( t \) which belongs to the interval \( 0 \leq t \leq 1 \). The family of operators (62) represents the generalization of the quantity \( P^{(0)}_{\mu}(k; -iD) \), determined by (60), namely, \( P^{(0)}_{\mu}(k; -iD) = P^{(1)}_{\mu}(k; -iD) \). Note also that solutions to differential equation (61) have to satisfy the boundary condition \( P^{(0)}_{\mu}(k; -iD) = -iD_{\mu} \equiv q_{\mu} \). The function \( \Phi_{\mu\alpha}(D) \) in (61) is deduced from (18) and reads as

\[
\Phi_{\mu\alpha}(D) = \eta_{\mu\alpha}(-A + f(B)) + ia_{\mu}D_{\alpha} - (a^2 - s)D_{\mu}D_{\alpha}\gamma_2. \tag{63}
\]

All results so far are written for the most general type of realizations. A complete solution requires integration of Eq.(61), which may not generally be so easy problem to handle and cannot be solved exactly for an arbitrary admissible function \( f(B) \). We shall take care of this most general case in the final part of the paper, where the perturbative solution to Eq.(61) will be found, valid through the second order in deformation parameters. There are however few exceptional choices for the function \( f(B) \) that allow for an exact solution. One example of such case is the function \( f(B) = \sqrt{1 - B} \), which anyway appers frequently in the literature. For this particular case, that will below be solved exactly, we have \( \gamma_2 = 0 \) and consequently Eq.(61) reads

\[
dP^{(t)}_{\mu}(k; q) = k_{\mu}\left[ aP^{(t)} + \sqrt{1 + (a^2 - s)\left(P^{(t)}\right)^2} \right] - a_{\mu}kP^{(t)}, \tag{64}
\]

where we have used an abbreviation \( P^{(t)}_{\mu} \equiv P^{(t)}_{\mu}(k; -iD) \). The exact solution to differential equation (64), which obeys the required boundary conditions, looks as

\[
P^{(t)}_{\mu}(k; q) = q_{\mu} + \left( k_{\mu}Z^{-1}(q) - a_{\mu}(kq) \right) \frac{\sinh(tW)}{W} \tag{65}
\]

\[
+ \left[ (k_{\mu}(ak) - a_{\mu}k^2)Z^{-1}(q) + a_{\mu}(ak)(kq) - sk_{\mu}(kq) \right] \frac{\cosh(tW) - 1}{W^2}. \]

In the above expression we have introduced the following abbreviations,

\[
W = \sqrt{(ak)^2 - sk^2}, \tag{66}
\]

\[
Z^{-1}(q) = (aq) + \sqrt{1 + (a^2 - s)q^2} \tag{67}
\]
and it is understood that quantities like \((kq)\) mean the scalar product in a Minkowski space with signature \(\eta_{\mu\nu} = \text{diag}(-1,1,\cdots,1)\). Now that we have \(P^{(t)}_\mu(k,q)\), the required quantity \(P_\mu(k,q)\) simply follows by setting \(t = 1\) and finally we also get
\[
K_\mu(k) = \left[ k_\mu(ak) - a_\mu k^2 \right] \frac{\cosh W - 1}{W^2} + k_\mu \frac{\sinh W}{W}. \tag{68}
\]

We can now write the star product between arbitrary two plane waves in the algebra \(A_{X,*}\) as follows,
\[
e^{ikX} \star e^{iqX} \equiv e^{iK^{-1}(k)\hat{x} \triangleright e^{iqX}} = e^{iD_\mu(k,q)X^\mu}, \tag{69}
\]
where
\[
D_\mu(k,q) = P_\mu(K^{-1}(k),q), \tag{70}
\]
with \(K^{-1}(k)\) being the inverse of the transformation (68). In writing the star product (69), we have applied Eq. (58) and the definition (38) of the star product. It is further possible to show that quantities \(Z^{-1}(k)\) and \(\Box(k)\) can be expressed in terms of quantity \(K^{-1}(k)\) as
\[
Z^{-1}(k) \equiv (ak) + \sqrt{1 + (a^2 - s)k^2} = \cosh W(K^{-1}(k)) + aK^{-1}(k)\frac{\sinh W(K^{-1}(k))}{W(K^{-1}(k))}, \tag{71}
\]
\[
\Box(k) \equiv \frac{2}{a^2 - s} \left[ 1 - \sqrt{1 + (a^2 - s)k^2} \right] = 2(K^{-1}(k))^2 \frac{1 - \cosh W(K^{-1}(k))}{W^2(K^{-1}(k))}, \tag{72}
\]
where \(W(K^{-1}(k))\) is given by (66), or explicitly
\[
W(K^{-1}(k)) = \sqrt{(aK^{-1}(k))^2 - s(K^{-1}(k))^2}. \tag{73}
\]

With the exact solutions (65) and (68), corresponding to realization \(f(B) = \sqrt{1-B}\), it is possible to determine all the ingredients that define Hopf algebra in the case of that particular realization. As a first step, the function \(D_\mu(k,q)\) determines \(54\) a deformed Leibniz rule and the corresponding coproduct \(\Delta D_\mu\) in the following way,
\[
\Delta D_\mu = iD_\mu(-iD \otimes 1, 1 \otimes (-iD)). \tag{74}
\]
Relations (71) and (72) are useful in obtaining the expression for the coproduct. However, in the general case of deformation, when both parameters \(a_\mu\) and \(s\) are
Since in the case of a pure Snyder deformation \((a = 0)\), the coproduct for Lorentz generators is undeformed, in the case of general deformation \((a, s \neq 0)\), the same coproduct will be identical as in the case of pure \(\kappa\)-deformation,

\[
\begin{align*}
\Delta M_{\mu\nu} &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\
&+ i a_\mu \left( D^\lambda - \frac{ia^\lambda}{2} \Box \right) Z \otimes M_{\lambda\nu} - i a_\nu \left( D^\lambda - \frac{ia^\lambda}{2} \Box \right) Z \otimes M_{\lambda\mu}.
\end{align*}
\]

As we shall see at the end of the paper, this result that is above established in the special case of realization when \(f(B) = \sqrt{1 - B}\) and which says that the coproduct for Lorentz generators in the case of general deformation \((a, s \neq 0)\) is the same as the coproduct for Lorentz generators in the case of \(\kappa\)-deformation is, in fact, a most general result, valid for all realizations, i.e. for all functions \(f(B)\), consistent with imposed requirements, because, for Snyder deformation, the coproduct for Lorentz generators is undeformed \([41]\), no matter which realization is used. The same type of reasoning applies when one is concerned with obtaining the antipodes for Lorentz generators \([24]\). Namely, since it is known \([41]\) that the antipodes for Lorentz generators are also undeformed in the case of pure Snyder deformation \((a = 0)\), the same antipodes in the case of general, i.e. \(\kappa\)-Snyder deformation \((a, s \neq 0)\), will be identical to the antipodes for Lorentz generators in a pure \(\kappa\)-deformation. This statement, like the previous one, is true not only for the realization \(f(B) = \sqrt{1 - B}\), but for all realizations satisfying the required conditions. As far as the antipodes for translation generators are concerned, the function \(D_\mu(k, q)\) also plays a crucial role, since these antipodes can immediately be obtained by solving the conditions,

\[
D_\mu(S(k), k) = D_\mu(k, S(k)) = 0.
\]
Now that we have a coproduct, it is a straightforward procedure \cite{31,33} to construct a star product between arbitrary two functions $f$ and $g$ of commuting coordinates, generalizing in this way relation (69) that holds for plane waves. Thus, the general result for the star product, valid for the NC space (1), has the form

$$ (f \star g)(X) = \lim_{Y \to X} \lim_{Z \to X} e^{X_\alpha \left[ iD^\alpha (-iD_Y, -iD_Z) - D_0^\alpha - D_2 \right]} f(Y)g(Z). \quad (78) $$

Although star product is a binary operation acting on the algebra of functions defined on the ordinary commutative space, it encodes features that reflect noncommutative nature of space (1).

Following the line set up in section 4, it is worth noting that relation (58) gives a suitable example of the vector space-level isomorphism established in section 4 between the algebras $A_X$ and $\hat{A}_x \triangleright 1$. Particularly, from (58) it follows that

$$ e^{iK_\nu(k)X^\nu} \circ 1 = e^{ik\hat{x}} \circ 1, \quad (79) $$

being in accordance with relations (44) and (45), that is two relations that form the basis of described vector space-level isomorphism. As already established before, for $s \neq 0$ the algebra $\hat{A}_x \triangleright 1$ is nonassociative. In some considerations that deal with noncommutative Snyder space ($a = 0$), which is also characterized by the nonassociative star product, there were attempts to enlarge the original noncommutative spacetime by including Lorentz generators as well, with the aim of getting an associative star product \cite{50,51}. In the described setting the coordinates $M_{\mu\nu}$ are interpreted as coordinates describing extra dimensions and the Snyder space with nonassociative algebra structure is considered as a subspace of a bigger noncommutative space, which is generated by the coordinates $(\hat{x}_\mu, M_{\mu\lambda})$ and admits associative star product \cite{50,51}. In the following section we shall specialize the general results obtained so far to four particularly interesting special cases.

**VII Special cases**

**VII.1 1. case ($s = a^2$)**

In this case, NC commutation relations take on the form

$$ [\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) + a^2 M_{\mu\nu}. \quad (80) $$
Since we now have \( f(B) = f(0) = 1 \), the generalized shift operator becomes \( Z^{-1} = 1 - A \) and the realizations \( \text{(18) and (25)} \) for NC coordinates and Lorentz generators, respectively, take on a simpler form, namely,

\[
\hat{x}_\mu = X_\mu(1 - A) + i(aX)D_\mu, \\
M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu) \frac{1}{1 - A}.
\]

In addition, the generalized d’Alambertian operator becomes a standard one, \( \Box = D^2 \), and deformed Heisenberg-Weyl algebra \( \text{(20)} \) reduces to

\[
[D_\mu, \hat{x}_\nu] = \eta_{\mu\nu}(1 - A) + ia_\mu D_\nu.
\]

Relations \( \text{(22) and (23)} \), that include generalized shift operator, also change in an appropriate way. Particularly, we have

\[
[1 - A, \hat{x}_\mu] = -ia_\mu(1 - A) + a^2 D_\mu.
\]

We see from Eq.(75) that the coproduct for this case also simplifies since the term with \((a^2 - s)\) drops out.

**VII.2 2. case \( (a = 0) \)**

When \( a^2 = 0 \), we have a Snyder type of noncommutativity,

\[
[\hat{x}_\mu, \hat{x}_\nu] = sM_{\mu\nu}.
\]

In this situation, our realization \( \text{(18)} \) reduces precisely to that obtained in \[40,41].\]

For a special choice when \( f(B) = 1 \), we have the realization

\[
\hat{x}_\mu = X_\mu - s(XD)D_\mu,
\]

which is the case that was also considered in \[55].\] The solution to Eq.(61) for \( f(B) = 1 \) and \( a = 0 \) leads to the following coproduct for translation generators,

\[
\Delta D_\mu = \frac{1}{1 + sD_\nu \otimes D^\nu} \left( D_\mu \otimes 1 + \frac{s}{1 + \sqrt{1 + sD^2 \otimes D_\mu}} D_\mu D_\nu \otimes D^\nu + \sqrt{1 + sD^2 \otimes D_\mu} \right).
\]
In other interesting situation, when \( f(B) = \sqrt{1-B} \), the general result (18) reduces to
\[
\hat{x}_\mu = X_\mu \sqrt{1 + sD^2}.
\] (88)
This choice of \( f(B) \) is the one for which most of our results, through all over the paper, are obtained and which is one of the main objects of our investigations. It is also considered by Maggiore [56]. For this case when \( f(B) = \sqrt{1-B} \), the exact result for the coproduct (70) can be obtained and it is given by
\[
\Delta D_\mu = D_\mu \otimes Z^{-1} + 1 \otimes D_\mu + sD_\mu D_\alpha \frac{1}{Z^{-1} + 1} \otimes D^\alpha,
\] (89)
where
\[
Z^{-1} = \sqrt{1 + sD^2}.
\] (90)
Relations (87) and (89) appear to be equivalent to relations of Ref. [49] which give the rules for adding of momenta and are obtained by considering a momentum addition law on the corresponding momentum space given by a coset.

As indicated earlier, Snyder deformation \((a = 0)\) has a noteworthy property that, no matter of the realization used, the coproduct for Lorentz generators is undeformed,
\[
\Delta M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}.
\] (91)
An immediate consequence of this property is that the corresponding antipodes are also undeformed,
\[
S(D_\mu) = -D_\mu, \quad S(M_{\mu\nu}) = -M_{\mu\nu}.
\] (92)
First one of these relations can alternatively be confirmed by solving conditions (77) for two special cases of coproducts, for the coproduct (87) corresponding to realization (86) or for the coproduct (89) corresponding to realization (88). In view of conditions (77), both of these coproducts lead to the same result for antipode and even more, every admissible realization, for the case of Snyder deformation \((a = 0)\), leads to this same result, \( S(D_\mu) = -D_\mu \).
VII.3  3. case \((s = 0)\)

The situation when parameter \(s\) is equal to zero corresponds to \(\kappa\)-deformed space investigated in \cite{32,33}. The generalized d’Alambertian operator is now given as

\[
\Box = \frac{2}{a^2}(1 - \sqrt{1 - a^2D^2}),
\]

and the general form \cite{18} for the realizations now reduces to

\[
\hat{x}_\mu = X_\mu \left(-A + \sqrt{1 - B}\right) + i(aX) D_\mu,
\]

where \(B = a^2D^2\). The Lorentz generators can be expressed as

\[
M_{\mu\nu} = (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu)Z
\]

and deformed Heisenberg-Weyl algebra \cite{20} takes on the form

\[
[D_\mu, \hat{x}_\nu] = \eta_{\mu\nu} Z^{-1} + ia_{\mu} D_\nu.
\]

In the case of \(\kappa\)-deformed space, we can also write the exact result for the coproduct, which in a closed form looks as

\[
\Delta D_\mu = D_\mu \otimes Z^{-1} + 1 \otimes D_\mu + ia_{\mu} (D_\alpha Z) \otimes D^\alpha - \frac{ia_{\mu}}{2} \Box Z \otimes iaD,
\]

where the generalized shift operator \cite{24} is here specialized to

\[
Z^{-1} = -iaD + \sqrt{1 - a^2D^2}.
\]

This operator has the following useful properties, with first of them expressing the coproduct for the operator \(Z\),

\[
\Delta Z = Z \otimes Z,
\]

\[
\hat{x}_\mu Z \hat{x}_\nu = \hat{x}_\nu Z \hat{x}_\mu.
\]

The coproducts for Lorentz generators in this case are given in relation \cite{26} and antipodes for both, translation and Lorentz generators, can also be expressed in a closed form \cite{24}.
VII.4 4. case (Perturbative results up to $a^2$ and $s$)

In this subsection we shall treat perturbatively the most general case of deformation $(a, s \neq 0)$ for an arbitrary admissible realization,

$$\hat{x}_\mu = X_\mu(-A + f(B)) + i(aX)D_\mu - (a^2 - s)(XD)D_\mu\gamma_2,$$

(101)

where use will be made of the function

$$f(B) = 1 - uB + \mathcal{O}(a^3) = 1 - u(a^2 - s)D^2 + \mathcal{O}(a^3),$$

(102)

expanded in a Taylor series up to second order in deformation parameters $a$ and $s$. Here the parameter $u$ plays the role of characterizing the realization we are working with. Thus, the results for the function $f(B) = \sqrt{1 - B}$, valid up to second order in $a$ and $s$, will be reproduced for $u = \frac{1}{2}$, while the results for the function $f(B) = 1$, valid within the same order, will be reproduced for $u = 0$. The same procedure which was carried out in section 6 for the function $f(B) = \sqrt{1 - B}$, results now in the expression

$$P_\mu(k, q) = q_\mu + k_\mu(1 + aq) - a_\mu(kq) - (1 - 2u)(a^2 - s)q_\mu(kq) + u(a^2 - s)k_\mu q^2$$

$$+ \frac{1}{2}[k_\mu(ak) - a_\mu k^2](1 + aq) + \frac{1}{2}[a_\mu(ak) - k_\mu a^2](kq)$$

$$- \frac{1}{2}(a^2 - s)[(1 - 4u)k_\mu(kq) + (1 - 2u)q_\mu k^2] + \frac{1}{6}k_\mu[(ak)^2 - a^2 k^2]$$

$$- \frac{1}{3}(1 - 3u)(a^2 - s)k_\mu k^2 + \mathcal{O}(a^3),$$

(103)

as the solution to Eq.(61) for the case of function (102). This also gives

$$K_\mu = k_\mu[1 + \frac{1}{2}(ak) + \frac{1}{6}(ak)^2 - \frac{1}{6}a^2 k^2 - \frac{1}{3}(1 - 3u)(a^2 - s)k^2] - \frac{1}{2}a_\mu k^2 + \mathcal{O}(a^3),$$

(104)

as an adequate counterpart to the quantity (68). The inverse transformation of (104) looks as

$$K^{-1}_\mu(k) = k_\mu[1 - \frac{1}{2}(ak) + \frac{1}{3}(ak)^2 - \frac{1}{12}a^2 k^2 + \frac{1}{3}(1 - 3u)(a^2 - s)k^2]$$

$$+ \frac{1}{2}a_\mu k^2 - \frac{1}{4}a_\mu(ak) k^2 + \mathcal{O}(a^3).$$

(105)
According to Eq.(70), these last results immediately yield the relation
\[ \mathcal{D}_\mu(k, q) = q_\mu[1 - (1 - 2u)(a^2 - s)(kq) - \frac{1}{2}(1 - 2u)(a^2 - s)k^2] \]
\[ + \ k_\mu[1 + (aq) + u(a^2 - s)q^2 - \frac{1}{2}a^2(kq) - \frac{1}{2}(1 - 4u)(a^2 - s)(kq)] \]
\[ + \ a_\mu[(kq)(ak - 1) - \frac{1}{2}(aq)k^2] + \mathcal{O}(a^3), \tag{106} \]
which in itself comprises a deformed momentum addition rule and a deformed co-
product for translation generators of Poincaré algebra. As already pointed out
earlier, due to properties of a pure Snyder deformation \( a = 0 \), the coproduct for
Lorentz generators in the currently considered case \( a, s \neq 0, \ f(B) = 1 - uB \)
will be the same as in the corresponding case of a pure \( \kappa \)-deformation. The same
holds for the antipodes of Lorentz generators. On the other side, the antipodes for
translation generators can be obtained in a straightforward manner by solving the
conditions \( (77) \), imposed on the function \( D_\mu(k, q) \) in Eq.(106). This gives
\[ S(k_i) = -k_i[1 + a_0k_0 + (a_0k_0)^2 - \frac{1}{2}a_0^2k_i^2] + \mathcal{O}(a^3), \]
\[ S(k_0) = -k_0(1 - a_0^2 \sum_{i=1}^{n-1} k_i^2) - a_0 \sum_{i=1}^{n-1} k_i^2 + \mathcal{O}(a^3), \tag{107} \]
with the property
\[ (S(k))^2 = -(S(k_0))^2 + \sum_{i=1}^{n-1} (S(k_i))^2 = k^2, \tag{108} \]
which is, of course, valid within the second order in \( a \) and \( s \).

We briefly show how the results obtained can be used to construct a field theory
for free, as well as for interacting field theory. As for toy model, we consider a scalar
field theory with mass and cubic interaction terms.

Having in mind Eqs.(58) and (59), we have the following associations,
\[ e^{iK^{-1}(k)\hat{\epsilon}} \triangleright 1 = e^{ikX} \tag{109} \]
and
\[ e^{iK^{-1}(k)\hat{\epsilon}} \triangleright e^{iqX} = e^{i\mathcal{D}_\mu(k^{-1}(k), q)X^\mu} = e^{i\mathcal{D}_\mu(k, q)X^\mu}, \tag{110} \]
as well as
\[ D_\mu e^{iK^{-1}(k)\hat{\epsilon}} \triangleright 1 = D_\mu e^{ikX} = i\kappa_\mu e^{ikX}. \tag{111} \]
From the definition of the star product (38), we can write
\[
\hat{\phi}(\hat{x})\hat{\phi}(\hat{x}) \triangleright 1 = \phi(X) \star \phi(X),
\]
(112)
\[
\hat{\phi}(\hat{x})\hat{\phi}(\hat{x})\hat{\phi}(\hat{x}) \triangleright 1 = \phi(X) \star (\phi(X) \star \phi(X)).
\]
(113)

We also assume that functions in NC coordinates and functions in commutative
coordinates have the following Fourier transforms, respectively,
\[
\hat{\phi}(\hat{x}) = \int \frac{d^n k}{(2\pi)^n} \hat{\phi}(k) e^{iK^{-1}(k)\hat{x}},
\]
(114)
\[
\phi(X) = \int \frac{d^n k}{(2\pi)^n} \hat{\phi}(k) e^{ikX},
\]
(115)
so that they can mutually be related through the association \(\hat{\phi}(\hat{x}) \triangleright 1 = \phi(X)\). That
this is really the case, can easily be inferred from Eq. (109).

The action for interacting massive scalar field on spacetime (1) can
then be obtained by the projection on the unit element as follows,
\[
S[\phi] = \int d^n X \left( \frac{1}{2} (D_\mu \hat{\phi} D^\mu \hat{\phi} + m^2 \hat{\phi}^2) + \frac{\xi}{3!} \hat{\phi}^3 \right) \triangleright 1
\]
\[
= \frac{1}{2} \int d^n X (D_\mu \phi) \star (D^\mu \phi) + \frac{m^2}{2} \int d^n X \phi \star \phi + \frac{\xi}{3!} \int d^n X \phi \star (\phi \star \phi).
\]
(116)

By making use of Fourier transforms and associations given in Eqs. (110) and (111),
various terms appearing in (116) are calculated as follows
\[
D_\mu \hat{\phi}(\hat{x}) D^\mu \hat{\phi}(\hat{x}) \triangleright 1 = D_\mu \hat{\phi}(\hat{x}) \triangleright \left( D^\mu \hat{\phi}(\hat{x}) \triangleright 1 \right) = (D_\mu \phi) \star (D^\mu \phi)
= \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \hat{\phi}(k) \hat{\phi}(q) \left( q^2 - qD \right) e^{iD(k,q)X},
\]
(117)
\[
\hat{\phi}(\hat{x})^2 \triangleright 1 = \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \hat{\phi}(k) \hat{\phi}(q) e^{iK^{-1}(k)\hat{x}} \triangleright \left( e^{iK^{-1}(q)\hat{x}} \triangleright 1 \right)
= \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \hat{\phi}(k) \hat{\phi}(q) e^{iD_\mu(k,q)X^\mu},
\]
(118)
\[ \hat{\phi}(\hat{x})^3 \triangleright 1 = \int \frac{d^np}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \hat{\phi}(p) \hat{\phi}(k) \hat{\phi}(q) e^{iK^{-1}(k)\hat{x}} \triangleright \left( e^{iK^{-1}(q)\hat{x}} \triangleright 1 \right) \]

\[ = \int \frac{d^np}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \hat{\phi}(p) \hat{\phi}(k) \hat{\phi}(q) e^{iK^{-1}(p)\hat{x}} \triangleright e^{iP\mu(K^{-1}(k),q)X^\mu} \]

\[ = \int \frac{d^np}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \hat{\phi}(p) \hat{\phi}(k) \hat{\phi}(q) e^{iP\mu(K^{-1}(p),P(K^{-1}(k),q))X^\mu} \]

\[ = \int \frac{d^np}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \hat{\phi}(p) \hat{\phi}(k) \hat{\phi}(q)e^{iD\mu(p,D(k,q))X^\mu}, \quad (119) \]

where \( K^{-1}(k), D(k, q) \) are given in Eqs. (105) and (106), respectively. Note the appearance of nested terms like \( D_{\mu, \nu}(K^{-1}(k), q) X^\mu \). These can be calculated by applying Eq. (106) repeatedly. As pointed out before, relations (105) and (106) represent the most general case of deformation \((a, s \neq 0 \text{ and parameter } u \text{ is arbitrary})\) and they are obtained perturbatively up to second order in deformation parameters. In obtaining Eqs. (117), (118) and (119), we have used relation (70) together with relations (109), (110) and (111).

The action (116) can be expanded by making use of Eq. (106) and the above results. For transparency, we look at somewhat simpler case when \( a = 0 \). Then the expansion up to linear order in parameter \( s \) leads to a standard action with the additional correction terms,

\[ S[\phi] = \int d^n X \ L(\phi, D_\mu \phi, D_\mu D_\nu \phi) \]

\[ = \frac{1}{2} \int d^n X (D_\mu \phi)(D^\mu \phi) + \frac{m^2}{2} \int d^n X \phi^2 + \frac{s}{3!} \int d^n X \phi^3 \]

\[ + \frac{s}{4} \int d^n X (X_\mu D_\mu D_\lambda \phi) D^\lambda D^2 \phi + \frac{s}{4} \int d^n X (X_\mu D_\mu D_\nu D_\lambda \phi) D^\nu D^\lambda \phi \]

\[ + \frac{s}{4} \int d^n X (X_\mu D_\mu D_\nu \phi) D^\nu \phi + \frac{m^2}{4} \int d^n X (X_\mu D_\mu D_\nu \phi) D^\nu \phi \]

\[ + \frac{\xi}{4} \int d^n X \phi (X_\mu D_\mu \phi) D^2 \phi + \frac{\xi}{4} \int d^n X \phi(X_\mu D_\mu D_\nu \phi) D^\nu \phi \]

\[ + \frac{\xi}{3!} \int d^n X (X_\mu D_\mu \phi)(D_\nu \phi) D^\nu \phi + O(s^2). \quad (120) \]

**VIII Conclusion**

In summary, the focus of our analysis was directed toward \( \kappa \)-Snyder deformation of Minkowski spacetime. This deformation is of a Lie algebra type and it includes
features of both, a pure $\kappa$-deformation and of pure Snyder deformation, at the same time. It in fact interpolates between the two in a smooth way, broadening a possible range of deformations, thus making features resulting from that extension more likely to correspond to and to fit within the scope of what is really happening at the Planck scale level. The analysis is further made of the impact that these deformations have on the Hopf algebraic structure of the symmetry algebra underlying Minkowski space. Particularly, the nature of the relationship between algebraic structures, the original, undeformed one and the one resulting from a deformation of an underlying spacetime geometry is considered. Although these algebraic structures are not isomorphic to each other, we have however shown that this situation can be overcome by introducing a notion of module for the corresponding universal enveloping algebras. A special thing about introducing a module is that when we take a unit element in the module and project the enveloping algebras of the deformed and undeformed Heisenberg algebras to the unit element in the module, respectively, we arrive at the conclusion that the resulting structures are isomorphic to each other. This path appears to be the right way in which the isomorphism can again be established, even in the most general case of deformation, when both deformation parameters are different from zero ($a, s \neq 0$). We have further investigated the way in which a construction of tensors and invariants, in terms of NC coordinates, should be modified in order for it to be compatible with Lorentz symmetry and to avoid all inconsistencies that could possibly arise on the way. Deformations that have been studied are further found to completely fit within the framework of a quantum Hopf algebra. They are characterized by the common feature that the algebraic sector of the Hopf algebra, which is described by the Poincaré algebra, is undeformed, while, on the other side, the corresponding coalgebraic sector is affected by deformations. Deformation of the coalgebra manifests in a form of having the modified coproducts for Poincaré generators, which in turn tell us to which extent the corresponding Leibniz rules are deformed in comparison to standard Leibniz rules. Since the coproduct is related to a star product, we were also able to write down how star product looks like for NC spaces characterized by the general class of deformations of type (1). We have also found many different classes of realizations of NC space (1) and specialized obtained results to some specific cases of particular interest, including the perturbative analysis of the most general case of realization,
valid up to second order in deformation parameters.

The point that might be important to emphasize is that the realizations that we have been working with in this paper are not hermitian ones, but they could be made hermitian. This procedure of hermitization has led in the case of pure $\kappa$-space to a very important result satisfied by the star product corresponding to hermitian realization, namely, that under the integration sign, star product can be replaced by the ordinary multiplication operation. Consequences that hermitization process has for the trace and cyclic properties of an invariant integral have also been discussed for $\kappa$-space. We would expect a similar kind of results emerging in $\kappa$-Snyder space as well, if the similar process of hermitization was carried out there. These matters and particularly the issue of the trace property of an invariant integral defined on noncommutative spacetime are highly relevant in building field theories and gauge theories on these noncommutative manifolds. It hence remains challenging to address the problem of invariant integration on $\kappa$-Snyder deformed space and to construct a scalar field theory, following the work that was previously done in the context of $\kappa$-deformation, as well as in the context of Snyder deformation.

There is also a wide range of physically appealing questions which could be expected to originate from the modified geometry at the Planck scale, which reveals itself through a noncommutativity of spacetime coordinates. Some of these questions are related to investigations of the effects that noncommutativity has on dispersion relations, black hole horizons, and Casimir energy, the issues that have already been analysed in the context of $\kappa$-type noncommutativity. However, since in our approach Poincaré algebra is undeformed, dispersion relation will also be undeformed, being in line with the standard dispersion relation, $P^2 + m^2 = 0$. In case we treated derivative as not being vector-like, we would get modified dispersion relations. In order to formulate field theory for the case of $\kappa$-Snyder deformation and to investigate the impacts that deformation has on particle statistics, as well as on certain important physical properties such as Lorentz and CPT invariance, it is necessary to find Drinfeld twist, twisted flip operator, and $R$-matrix, that are relevant on spacetime with $\kappa$-Snyder deformation. The issue of proper construction of differential forms would also be of significant importance. The most of these issues will be addressed in the forthcoming
papers, particularly the issues related to field theory for scalar fields and its twisted
statistics properties, as a natural continuation of our investigations put forward in
previous papers [65],[67]. At the Planck scale it is also not clear at all if the Lorentz
invariance has remained intact, or is it violated. From the theoretical point of view,
the problem of Lorentz invariance, in conditions regulated by $\kappa$-Snyder deformation,
can also be accessed by assuming that the deformation parameter $a$, instead of being
fixed, transforms as a $n$-vector under Lorentz algebra, which is a kind of approach
already considered in [24].

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