ON SHAPIRO’S COMPACTNESS CRITERION FOR COMPOSITION OPERATORS

JOHN R. AKEROYD

ABSTRACT. For any analytic self-map \( \psi \) of \( \mathbb{D} := \{ z : |z| < 1 \} \), J. H. Shapiro has established that the square of the essential norm of the composition operator \( C_\psi \) on the Hardy space \( H^2 \) is precisely \( \limsup_{|w| \to 1^-} N_\psi(w)/(1 - |w|) \); where \( N_\psi \) is the Nevanlinna counting function for \( \psi \). In this paper we show that this quantity is equal to \( \limsup_{|a| \to 1^-} (1 - |a|^2)[1/(1 - \bar{a}\psi)]^2_{H^2} \). This alternative expression provides a link between the one given by Shapiro and earlier measure-theoretic notions. Applications are given.

1. Introduction and Preliminaries

Let \( \mathbb{D} \) denote the unit disk \( \{ z : |z| < 1 \} \) and let \( \psi \) be an analytic function on \( \mathbb{D} \) that maps \( \mathbb{D} \) into itself; a so-called analytic self-map of \( \mathbb{D} \). Then \( \psi \) has nontangential boundary values a.e. (Lebesgue measure) on \( \mathbb{T} := \{ z : |z| = 1 \} \), which we also denote by \( \psi \). Throughout this paper we let \( m \) denote normalized Lebesgue measure on \( \mathbb{T} \) and let \( A \) denote normalized two-dimensional Lebesgue measure on \( \mathbb{D} \). Now, by The Littlewood Subordination Principle (cf., [6], Section 1.3), \( \psi \) gives rise to a bounded composition operator \( C_\psi \) on the Hardy space \( H^2 (:= H^2(\mathbb{D})) \); where \( C_\psi(f) := f \circ \psi \). The Nevanlinna counting function \( N_\psi \) (of \( \psi \)) plays a central role in many results concerning \( C_\psi \) in this and other contexts, and is defined on \( \mathbb{D} \) by

\[ N_\psi(w) := - \sum_{z \in \psi^{-1}(\{w\})} \log |z|; \]

where the sum honors the multiplicity of any zero of \( \psi - w \), and is zero if \( \psi^{-1}(\{w\}) = \emptyset \). Featuring among results involving \( N_\psi \) is a theorem of J. H. Shapiro (cf., [5], Theorem 2.3)
that gives the square of the essential norm of $C_\psi$ on $H^2$ precisely as:

$$\limsup_{|w| \to 1^-} \frac{N_\psi(w)}{1 - |w|}.$$ 

Hence, $C_\psi$ is compact on $H^2$ if and only if $\lim_{|w| \to 1^-} \frac{N_\psi(w)}{1 - |w|} = 0$. In this paper we show, by elementary methods, that

$$\limsup_{|w| \to 1^-} \left| \frac{N_\psi(w)}{1 - |w|} \right| = \limsup_{|a| \to 1^-} \int_\mathbb{T} \frac{1 - |a|^2}{|1 - \bar{a}\xi|^2} \, d\mu_\psi(\xi);$$

see Theorem 2.2. In the case that $C_\psi$ is compact on $H^2$, our alternative expression has close ties to earlier measure-theoretic notions. We now proceed to make this connection clear. As in the introduction of [7], let $\mu_\psi$ be the induced measure of $\psi$, which is defined on Borel subsets $E$ of $\mathbb{D}$ by

$$\mu_\psi(E) = m(\{\zeta \in \mathbb{T} : \psi(\zeta) \in E\}).$$

By the definition of $\mu_\psi$, the statement that $\lim_{|a| \to 1^-} \int_\mathbb{T} \frac{1 - |a|^2}{|1 - \bar{a}\xi|^2} \, d\mu_\psi(\xi) = 0$ translates to:

$$\lim_{|a| \to 1^-} \int_\mathbb{T} \frac{1 - |a|^2}{|1 - \bar{a}\xi|^2} \, d\mu_\psi(\xi) = 0.$$

For $0 < h < 1$ and $\theta_0$ in $[0, 2\pi)$, let

$$S(h, \theta_0) = \{re^{i\theta} : 1 - h \leq r \leq 1 \text{ and } |\theta - \theta_0| \leq h\}.$$

**Lemma 1.1.** Let $\nu$ be a finite, positive Borel measure supported in $\mathbb{D}$. Then the following are equivalent.

i) $\lim_{|a| \to 1^-} \int_\mathbb{T} \frac{1 - |a|^2}{|1 - \bar{a}\xi|^2} \, d\nu(\xi) = 0$.

ii) $\nu(S(h, \theta_0)) = o(h)$.

Proof. By standard measure theory (cf., [4], Chapter 7), (i) implies that $\nu(\mathbb{T}) = 0$, and so does (ii). Therefore, we may reduce to the case that $\nu(\mathbb{T}) = 0$. For $0 < R < 1$, let $\nu_R$ denote the restriction of $\nu$ to the annulus $\{z : R \leq |z| < 1\}$; namely, for any Borel subset $E$ of $\mathbb{D}$, $\nu_R(E) := \nu(E \cap \{z : R \leq |z| < 1\})$. If (i) holds, then, for any $\varepsilon > 0$, there exists $R, 0 < R < 1$, such that

$$\sup_{a \in \mathbb{D}} \int_\mathbb{T} \frac{1 - |a|^2}{|1 - \bar{a}\xi|^2} \, d\nu_R(\xi) < \varepsilon.$$
Thus, by Lemma 3.3 in Chapter VI of [2], there is an absolute constant $A$ such that $\nu_R(S(h, \theta_0)) \leq A \varepsilon h$, for all $h$ and $\theta_0$. Hence, $\nu(S(h, \theta_0)) \leq A \varepsilon h$ for $0 < h \leq 1 - R$ and all $\theta_0$. It follows that $\nu(S(h, \theta_0)) = o(h)$. Conversely, suppose that $\nu(S(h, \theta_0)) = o(h)$. Then, for any $\varepsilon > 0$, there exists $R$, $0 < R < 1$, such that $\nu_R(S(h, \theta_0)) \leq \varepsilon h$, for all $h$ and $\theta_0$. Applying Lemma 3.3 in Chapter VI of [2] once again, there is an absolute constant $B$ such that

$$\sup_{\alpha \in D} \int_1^{-|a|} \frac{1-|a|^2}{|1-\bar{\alpha}\xi|^2} d\nu_R(\xi) < B \varepsilon.$$ 

Since $\frac{1-|a|^2}{|1-\bar{\alpha}\xi|^2} \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$, as $|a| \rightarrow 1^-$, we can now conclude that $\lim_{|a| \rightarrow 1^-} \int_1^{-|a|} \frac{1-|a|^2}{|1-\bar{\alpha}\xi|^2} d\nu(\xi) = 0$. $\square$

Now, by Lemma 1.1 and the discussion preceding it,

$$\lim_{|a| \rightarrow 1^-} \int_T \frac{1-|a|^2}{|1-\bar{a}\psi(\zeta)|^2} dm(\zeta) = 0$$

precisely when $\mu_\psi(S(h, \theta_0)) = o(h)$; which is a well-known necessary and sufficient condition for $C_\psi$ to be compact on $H^2$. This last fact is a rather straightforward exercise using the Weak Convergence Theorem in Section 2.4 of [6] along with the observation that

$$||C_\psi(f)||_{H^2}^2 = \int |f|^2 d\mu_\psi;$$

once again, see the introduction of [7], or [1] for details. In the next section of this paper we show that, indeed, $\limsup_{|w| \rightarrow 1^-} \frac{N_\psi(w)}{|1-|w||} = \limsup_{|a| \rightarrow 1^-} \int_T \frac{1-|a|^2}{|1-\bar{a}\psi(\zeta)|^2} dm(\zeta)$ for any analytic self-map $\psi$ of $\mathbb{D}$; again, see Theorem 2.2. Therefore, by Shapiro’s theorem, the square of the essential norm of $C_\psi$ on $H^2$ equals

$$\limsup_{|a| \rightarrow 1^-} \int_T \frac{1-|a|^2}{|1-\bar{a}\psi(\zeta)|^2} dm(\zeta)$$

(see Corollary 2.3), and $C_\psi$ is compact on $H^2$ precisely when

$$\lim_{|a| \rightarrow 1^-} \int_T \frac{1-|a|^2}{|1-\bar{a}\psi(\zeta)|^2} dm(\zeta) = 0$$

(see Corollary 2.4). Applications of this are given in Corollaries 2.5 and 2.6. We close this section by recalling that B. D. MacCluer has extended the aforementioned measure-theoretic equivalence of the compactness of $C_\psi$ to the more complicated setting of the Hardy spaces of the ball in $\mathbb{C}^n$; cf., [3].
We begin our work here with a lemma, some notation and a few observations concerning the Nevanlinna counting function.

**Lemma 2.1.** For $0 < c < 1$,

\[ (1 - c^2) \sum_{n=1}^{\infty} \frac{c^{2n-2}}{n+1} \to 0, \]

as $c \to 1^-$.  

Proof. Given $\varepsilon > 0$, there is a positive integer $N$ such that $\sum_{n=N}^{\infty} \frac{1}{(n+1)^2} < \varepsilon^2$, and hence:

\[ (1 - c^2) \sum_{n=1}^{\infty} \frac{c^{2n-2}}{n+1} \leq (1 - c^2) \sum_{n=1}^{N-1} \frac{c^{2n-2}}{n+1} + (1 - c^2) \left\{ \sum_{n=N}^{\infty} \frac{1}{c^{4n}} \right\}^{1/2} \left\{ \sum_{n=N}^{\infty} \frac{1}{(n+1)^2} \right\}^{1/2} \]

\[ < (1 - c^2) \sum_{n=1}^{N-1} \frac{c^{2n-2}}{n+1} + \left\{ \sum_{n=N}^{\infty} \frac{1}{(n+1)^2} \right\}^{1/2} < 2\varepsilon, \]

provided $c$ is sufficiently near 1. \(\square\)

For any point $a$ in $\mathbb{D}$, let $\varphi_a$ be the analytic automorphism of the unit disk given by $\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}$; notice that $(\varphi_a \circ \varphi_a)(z) = z$. If $\psi$ is any analytic self-map of $\mathbb{D}$, then it is immediate that $N_\psi(\varphi_a(w)) = N_{\varphi_a \circ \psi}(w)$, and in particular that $N_\psi(a) = N_{\varphi_a \circ \psi}(0)$. Parts of the proof of our main result are reminiscent of work in Section 10.7 of [6].

**Theorem 2.2.** Let $\psi$ be an analytic self-map of $\mathbb{D}$. Then

\[ \limsup_{|w| \to 1^-} \frac{N_\psi(w)}{1 - |w|} = \limsup_{|a| \to 1^-} \int_{\mathbb{T}} \frac{1 - |a|^2}{|1 - \overline{a}\psi(\zeta)|^2} dm(\zeta). \]

Proof. The result clearly holds if $\psi \equiv 0$, and so we proceed under the assumption that $\psi \neq 0$. In what follows, for positive functions $f$ and $g$ of the variable $a$ in $\mathbb{D}$, we write $f(a) \approx g(a)$ to indicate that $f(a)/g(a) \to 1$, as $|a| \to 1^-$. Then, by the Littlewood-Paley Identity and a change of variables formula (cf., [6], pages 178 and 186,
respectively), and with \( c(a) := 1/|1 - \bar{a}\psi(0)|^2 \),

\[
\int_{D} \frac{1 - |a|^2}{|1 - \bar{a}\psi(\zeta)|^2} \, dm(\zeta) = (1 - |a|^2)c(a) + 2 \int_{D} \frac{\bar{a}\psi'(z)}{(1 - \bar{a}\psi(z))^2} \left| \log\left(\frac{1}{|z|}\right) \right| dA(z)
\]

\[
\approx (1 - |a|^2)c(a) + 2 \int_{D} \frac{N_\psi(w)}{1 - |a|^2} |\varphi_a'(w)|^2 dA(w)
\]

\[
= (1 - |a|^2)c(a) + 2 \int_{D} \frac{N_\psi(w)}{1 - |a|^2} \varphi_a'(w)^2 dA(w)
\]

\[
\approx (1 - |a|^2)c(a) + \int_{D} \frac{N_\psi(w)}{1 - |a|} |\varphi_a'(w)|^2 dA(w).
\]

Since \( \sup_{a\in D} c(a) < \infty \), we conclude that

\[
\lim_{\|a\| \to 1^-} \sup \int_{\mathbb{T}} \frac{1 - |a|^2}{|1 - \bar{a}\psi(\zeta)|^2} \, dm(\zeta) = \lim_{\|a\| \to 1^-} \sup \int_{D} \frac{N_\psi(w)}{1 - |a|} |\varphi_a'(w)|^2 dA(w).
\]

What remains to be shown is that

\[
\lim_{\|a\| \to 1^-} \sup \int_{D} \frac{N_\psi(w)}{1 - |a|} |\varphi_a'(w)|^2 dA(w) = \lim_{|w| \to 1^-} \frac{N_\psi(w)}{1 - |w|}.
\]

To this end, first observe that by a change of variables, the sub-averaging property (cf., [6], page 190) and the notes at the beginning of this section,

\[
\int_{D} N_\psi(w) |\varphi_a'(w)|^2 dA(w) = \int_{D} N_\psi(\varphi_a(z)) dA(w)
\]

\[
= \int_{D} N_{\varphi_a \psi}(z) dA(z)
\]

\[
\geq |\varphi_a(\psi(0))|^2 N_{\varphi_a \psi}(0) = |\varphi_a(\psi(0))|^2 N_\psi(a).
\]

Thus, for \( a \neq \psi(0) \),

\[
\frac{N_\psi(a)}{1 - |a|} \leq \frac{1}{|\varphi_a(\psi(0))|^2} \int_{D} \frac{N_\psi(w)}{1 - |a|} |\varphi_a'(w)|^2 dA(w).
\]

Since \( |\varphi_a(\psi(0))| \to 1 \), as \( |a| \to 1^- \), it now follows that

\[
\lim_{\|a\| \to 1^-} \sup \frac{N_\psi(a)}{1 - |a|} \leq \lim_{\|a\| \to 1^-} \sup \int_{D} \frac{N_\psi(w)}{1 - |a|} |\varphi_a'(w)|^2 dA(w).
\]

For the reverse inequality, let \( \beta = \lim_{|w| \to 1^-} \frac{N_\psi(w)}{1 - |w|} \), which we know is finite by the corollary on page 188 of [6]. Then, for any \( \varepsilon > 0 \), there exists \( R, 0 < R < 1 \), such that

\[
N_\psi(w) \leq (\beta + \varepsilon)(1 - |w|),
\]
provided $R \leq |w| < 1$. And, by a change of variables and making use of the fact that 
\[ \varphi_a(z) = (a - z) \sum_{n=0}^{\infty} \bar{a}^n z^n, \] 
we find that 
\[ \int_{D} (1 - |w|^2) |\varphi_a'(w)|^2 dA(w) = 1 - \int_{D} |\varphi_a(z)|^2 dA(z) \]
\[ = (1 - |a|^2) \left[ 1 - (1 - |a|^2) \sum_{n=1}^{\infty} \frac{|a|^{2n-2}}{n+1} \right]. \]
Therefore, since $|\varphi_a'(w)|^2 / (1 - |a|) \to 0$ uniformly on compact subsets of $\mathbb{D}$, as $|a| \to 1^-$, and by Lemma 2.1, we have:
\[ \limsup_{|a| \to 1^-} \int_{D} \frac{N_{\psi}(w)}{1 - |a|} |\varphi_a'(w)|^2 dA(w) \]
\[ \leq \limsup_{|a| \to 1^-} \int_{D} \frac{(\beta + \varepsilon)(1 - |w|)}{1 - |a|} |\varphi_a'(w)|^2 dA(w) \]
\[ = \limsup_{|a| \to 1^-} \int_{D} \frac{(\beta + \varepsilon)(1 - |w|^2)}{1 - |a|^2} |\varphi_a'(w)|^2 dA(w) \]
\[ = \limsup_{|a| \to 1^-} (\beta + \varepsilon) \left[ 1 - (1 - |a|^2) \sum_{n=1}^{\infty} \frac{|a|^{2n-2}}{n+1} \right] \]
\[ = \beta + \varepsilon. \]
Now $\varepsilon > 0$ is arbitrary, and so it follows that
\[ \limsup_{|a| \to 1^-} \int_{D} \frac{N_{\psi}(w)}{1 - |a|} |\varphi_a'(w)|^2 dA(w) \leq \limsup_{|w| \to 1^-} \frac{N_{\psi}(w)}{1 - |w|}; \]
which completes our proof. □

Our next two results follow immediately from Theorem 2.3 in [5], and Theorem 2.2 above.

**Corollary 2.3.** Let $\psi$ be an analytic self-map of $\mathbb{D}$. Then
\[ \limsup_{|a| \to 1^-} \int_{\mathbb{D}} \frac{1 - |a|^2}{|1 - \bar{a}\psi(\zeta)|^2} dm(\zeta) \]
is the square of the essential norm of $C_{\psi}$ on $H^2$. 

Corollary 2.4. Let ψ be an analytic self-map of $\mathbb{D}$. Then the following are equivalent.

i) $C_\psi$ is compact on $H^2$.

ii) $\lim_{|w| \to 1} \frac{N_\psi(w)}{1-|w|} = 0$.

iii) $\lim_{|a| \to 1} \int_{\mathbb{T}} \frac{1-|a|^2}{|1-a\psi(\zeta)|^2} \, dm(\zeta) = 0$.

We close the paper with two applications of the equivalence of (i) and (iii) in Corollary 2.4. The first is well-known (cf., the proposition on page 32 of [6]), the second is less so.

Corollary 2.5. If ψ is a nonconstant inner function that fixes zero, then $C_\psi$ is not compact on $H^2$.

Proof. We first observe that, for $a$ in $\mathbb{D}$ and almost all $\zeta$ in $\mathbb{T}$,

$$\frac{1}{1-\bar{a}\psi(\zeta)} = \sum_{n=0}^{\infty} \bar{a}^n \psi^n(\zeta).$$

And since ψ is a nonconstant inner function that fixes zero, $\{\psi^n\}_{n=0}^{\infty}$ is an orthonomal sequence in $H^2$. Therefore,

$$\int_{\mathbb{T}} \frac{1-|a|^2}{|1-\bar{a}\psi(\zeta)|^2} \, dm(\zeta) = (1-|a|^2) \sum_{n=0}^{\infty} |a|^{2n} ||\psi^n||_{H^2}^2$$

$$= 1,$$

independent of $a$ in $\mathbb{D}$. And so, by Corollary 2.4, $C_\psi$ is not compact on $H^2$. □
Corollary 2.6. Let $\psi$ be an analytic self-map of $\mathbb{D}$. If $\sum_{n=0}^{\infty} ||\psi^n||^2_{H^2}$ converges, then $C_\psi$ is compact on $H^2$.

Proof. By our hypothesis, for any $\varepsilon > 0$, there is a positive integer $N$ such that $\sum_{n=N}^{\infty} ||\psi^n||^2_{H^2} < \varepsilon^2$. Therefore, by the observation at the start of the proof of Corollary 2.5,

$$\left\{ \int_{T} \frac{1-|a|^2}{|1-\bar{a}\psi(\zeta)|^2} \, dm(\zeta) \right\}^{\frac{1}{2}} \leq \sqrt{1-|a|^2} \sum_{n=0}^{\infty} |a|^n ||\psi^n||_{H^2}$$

$$\leq \sqrt{1-|a|^2} \left( \sum_{n=0}^{N-1} |a|^n ||\psi^n||_{H^2} + \left\{ \sum_{n=N}^{\infty} |a|^{2n} \right\}^{\frac{1}{2}} \left\{ \sum_{n=N}^{\infty} ||\psi^n||^2_{H^2} \right\}^{\frac{1}{2}} \right)$$

$$< \sqrt{1-|a|^2} \sum_{n=0}^{N-1} |a|^n ||\psi^n||_{H^2} + \varepsilon < 2\varepsilon,$$

if $|a|$ is sufficiently near 1. Hence, $\lim_{|a| \to 1^{-}} \int_{T} \frac{1-|a|^2}{|1-\bar{a}\psi(\zeta)|^2} \, dm(\zeta) = 0$. Thus, by Corollary 2.4, $C_\psi$ is compact on $H^2$. \(\square\)

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