The behavior of solution function of the fractional differential equations using modified homotopy perturbation method

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Abstract. The discourse regarding fractional calculus, in particular those related to fractional differential equation, is still continue to attract researcher attention. Previous studies have elaborated on the variation of fractional differential equation models. This study aims to uncover the problem of convergence of the solution function sequence related to the order of fractional differential equation. Firstly, this study presents how to find a solution for the model by using a Modified Homotopy Perturbation Method as the improvement of Homotopy Perturbation Method. Furthermore, the solution function with the sequence of fractional order is drawn by Maple. Using the geometrical analysis, the result of this study shows that if fractional order sequence is convergent to $\alpha$, then sequence of its solution function will be convergent to a solution function of fractional differential equation with order $\alpha$.

1. Introduction

The literature of the fractional calculus problem has emerged since the discovery of the conventional derivative concept with a natural numbered order by Leibnitz. However, recently the discourse seems to be stagnant with unclear reasons [1,2]. Although some mathematicians such as Caputo, Riemann-Liouville, and Grundwald-Letnikov discovered the formula for determining fractional derivatives, the development of fractional calculus is still limited. This is probably due to the many non-equivalent definitions of fractional derivatives [3,4]. Conditions like that until the middle of the 20th century.

In the 21st century, the topic about fractional differential equation model has become popular in the literature of fractional derivatives. Fractional differential equation model is a differential equation that contains the derivative of a fractional order. Similar to the differential equation of natural number order, the type of equation is divided into linear and nonlinear fractional differential equations. One of the nonlinear fractional equation is Riccati fractional differential equation. The development of fractional differential equation has emerged particularly related to the problem of convergence of the solution function sequence [5,6]. It turns out that many models in interdisciplinary cases could be easier to be expressed in the form of fractional differential equation model, such as in the field of finance [7]. Moreover, the problem of potential energy and fluid mechanics such as viscosity and surface tension are the other examples of cases that could be solved by fractional derivatives [8,9].

Various methods have been developed to find solutions of fractional differential equations, including Homotopy Analysis Method (HAM) which was firstly devised by Shijun [10] and Homotopy Perturbation Method (HPM) introduced by He [11]. Those methods combine perturbation and homotopy technique in order to eliminate small parameter in the equations which solve the limitation of the existing perturbation method. Moreover, those approaches are modified by Odibat and Momani [12] into
Modified Homotopy Perturbation Method (MHPM) to create a more efficient solution approach which move within rapid iteration process to achieve higher speed of convergence in order to get faster solution function [13,14].

This paper discusses the Modified Homotopy Perturbation method which is used to find the solution of a fractional differential equation and to analyze the convergence and behavior of the solution function sequences. The results show that the sequence of order of the fractional differential equation converging to a number, will cause the sequence of solution functions to converge to the solution function of the fractional differential equation with order is that number.

2. Method
In this section, we discuss the basic concepts of fractional derivatives followed by fractional differential equations and the methods that are used to find solutions.

2.1 Fractional Derivative
Riemann-Liouville (1859) defined fractional derivative with order \( \alpha \) as follows, [2,3]
\[
D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t f(\xi) (t-\xi)^{-(n-\alpha+1)} d\xi
\]
with \( n-1 \leq \alpha < n \) or \( n-1 = \lfloor \alpha \rfloor \).

On the other hand, Grunwald-Letnikov (1867) defined fractional derivative of \( f(t) \) with order \( \alpha \) in interval \([a, b]\) with
\[
D_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^n} \sum_{i=0}^{n} (-1)^i \frac{\Gamma(\alpha+1)}{\Gamma(i+1) \Gamma(\alpha-i+1)} f(t-ih)
\]
where \( n = \left\lfloor \frac{b-a}{h} \right\rfloor \). [2,3].

Different from Riemann-Liouville and Grunwald-Letnikov, based on Caputo (1969), definition of fractional derivative as follows: Let \( \alpha \) is a real number, and \( n-1 < \alpha \leq n \) where \( n \) is natural number. Fractional derivative of order \( \alpha \) of \( f(t) \) to \( x \) is:
\[
D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t f(\xi) (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi.
\]

From the three formulas above, specifically obtained that if \( f(t) = t^p \) then \( \alpha \)-th derivative of \( f(t) \), or derivative of \( f(t) \) with fractional order \( \alpha \) as follows:
\[
D_t^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}.
\]

The results of the Riemann-Liouville formula will be different from the results of the Caputo formula if \( f(t) \) is a constant function, where the result of \( D_t^\alpha f(t) \) by Caputo is zero, while according to Liouville is not zero.

2.2 Fractional Differential Equation
The general form of the fractional differential equation model with order \( \alpha \) is:
\[
D^\alpha u(t) + L(u(t)) + N(u(t)) = A(t), \quad t > 0, \quad m - 1 < \alpha \leq m
\]
where $D^\alpha$ is operator of fractional derivative with order $\alpha$, $L$ is linear operator, $N$ is nonlinear operator, and $A(t)$ is a function of $t$. The specific form of model (1) is known as Riccati fractional differential model, presented as follow:

$$\frac{d^\alpha u}{dt^\alpha} + a u(t) + b u^2(t) = A(t)$$

(2)

with $0 < \alpha \leq 1$, $t > 0$, $a$, $b$ the real constant, and the initial condition is $u(0) = 0$.

In case $b = 0$, model (2) will become a fluid relaxation problem model, in particular for $A(t)$ is an exponent function, the equation will become a viscosity model [9].

### 2.3 Method to finding Solution

In this section, the algorithm of Homotopy Perturbation Methods combining perturbation and homotopy technique will be presented. Given a nonlinear fractional differential equation as follows:

$$L(u) + N(u) = f(r), r \in \Omega,$$

(3)

with the initial condition:

$$B \left( u, \frac{\partial u}{\partial n} \right) = 0 , r \in \Gamma$$

where $L$ is linear differential operator, $N$ is nonlinear differential operator, $f(r)$ is a known analytic function, $B$ is a boundary, $n$ is the unit outward normal and $\Gamma$ is the boundary of the domain $\Omega$.

In this Homotopy Perturbation Method, homotopy is defined as

$$v(r,p): \Omega \times [0,1] \rightarrow \mathbb{R},$$

which fulfil

$$H(v,p) = (1 - p) [L(v) - L(u_0)] + p [L(v) + N(v) - f(r)] = 0$$

(4)

or

$$H(v,p) = L(v) - L(u_0) + p L(u_0) + p [N(v) - f(r)] = 0$$

(5)

where $r \in \Omega$ and $p \in [0,1]$ are the attached parameters, $u_0$ is initial approach value which fulfil the initial condition. From equations (4) and (5), it is obtained

$$H(v,0) = L(v) - L(u_0) = 0 \text{ dan } H(v,1) = L(v) + N(v) - f(r) = 0.$$  

(6)

The adjustment process of $p$ from 0 to 1 is equal to the change of $v(r,p)$ from $u_0(r)$ into $u(r)$. This process is known as deformation while $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ is called as homotopic. He [11] assumes that the solutions of (5) and (6) can be expressed as the power series of $p$:

$$v = v_0 + p v_1 + p^2 v_2 + \cdots.$$  

Therefore, the approach solution of (3) is

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots.$$  

Furthermore, HAM method is modified by Odibat and Momani [12] into Modified Homotopy Perturbation Method (MHPM) by including $u^m$ into both sides of the Homotopy equation (5). Let the Fractional Differential Equation Model is given below:

$$D^\alpha u(t) + L(u(t)) + N(u(t)) = A(t), \; t > 0, \quad m - 1 < \alpha \leq m$$

(7)

where $D^\alpha$ Caputo fractional derivative with order $\alpha$, $L$ is linear operator, $N$ is nonlinear operator, and $A(t)$ is a function of $t$. The initial condition is:

$$u^k(0) = c_k, \quad k = 0,1,2,\ldots,m - 1.$$  

(8)

From (7), we obtained the Homotopy equation (9) and (11).
\[ u^m + L(u) - A(t) = p [u^m - N(u) - D^α u], \quad p \in [0,1] \]  
(9)

or

\[ u^m - A(t) = p [u^m - L(u) - N(u) - D^α u], \quad p \in [0,1]. \]  
(10)

Hence the solution of (4) and (5) in form of \( p \)-power series is

\[ u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots. \]  
(11)

By substituting (9) to (10) and take \( p = 1 \), then solution function of (6) will be:

\[ u(t) = \sum_{n=0}^{\infty} u_n(t), \]  
(12)

where \( u_n(t) \) is obtained from integral result of derivative function as follows on equation (13).

\[
\begin{align*}
\frac{d^m u_0}{dt^m} &= A(t), \quad u^k(0) = c_k \\
\frac{d^m u_1}{dt^m} &= \frac{d^m u_0}{dt^m} - L_0(u_0) - N_0(u_0) - D^α u_0, \quad u^k(0) = c_k \\
\frac{d^m u_2}{dt^m} &= \frac{d^m u_1}{dt^m} - L_1(u_0, u_1) - N_1(u_0, u_1) - D^α u_0, \quad u^k(0) = c_k \\
&\vdots
\end{align*}
\]  
(13)

3. Result and Discussion

In this section the method to find a Fractional Differential Equation Model solution will be provided using the Modified Homotopy Perturbation Method as described above, and continued by analyzing the convergence of the solution function sequence.

3.1. Solution Function

Again, we see the general form of Nonlinear Fractional Differential Equation

\[ \frac{d^a u}{dt^a} + a u(t) + b u^2(t) = A(t). \]

If we take \( a = -1 \), \( b = 1 \), and \( A(t) = 2t \), the equation become

\[ \frac{d^a u}{dt^a} - u(t) + u^2(t) = 2t, \quad t > 0, \]  
(14)

with initial conditions \( u(0) = 0 \).

From (7), it means \( L(u(t)) = -u \) and \( N(u(t)) = u^2 \).

Based on Modified Homotopy Perturbation Method, homotopy equation (10) became

\[ u' - 2t = p[u' + u - u^2 - D^α u], \quad p \in [0,1]. \]

So, the solution of this equation is

\[ u = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \cdots. \]

By substituting the basic assumptions and initial conditions for the homotopy equation obtained

\[ u_0' = 2t, \quad u_0(0) = 0 \]
\[ u_1' = u_0' + u_0 - u_0^2 - D^α u_0, \quad u_1(0) = 0 \]
\[ u_2' = u_1' + u_1 - 2u_0 u_1 - D^α u_1, \quad u_2(0) = 0 \]
\[ u_3' = u_2' + u_2 - 2u_0 u_2 - u_1^2 - D^α u_2, \quad u_3(0) = 0. \]
Then, it is integrated so that it can be obtained
\[
\begin{align*}
  u_0 &= t^2, \\
  u_1 &= t^2 + \frac{t^3}{3} - \frac{t^5}{5} - \frac{2t^{3-a}}{\Gamma(4-a)} \\
  u_2 &= t^2 + \frac{2t^3}{3} + \frac{t^4}{12} - \frac{5t^5}{90} + \frac{13t^6}{90} + \frac{t^8}{20} - \frac{4t^{3-a}}{\Gamma(4-a)} - \frac{2t^{4-a}}{\Gamma(5-a)} + \frac{24t^{6-a}}{\Gamma(7-a)} + \frac{2t^{4-2a}}{\Gamma(5-2a)} \\
  &\quad + \frac{4t^{7-2a}}{(7-2a)\Gamma(4-a)} + \cdots.
\end{align*}
\]

Therefore, solution function of equation (14):
\[
  u(t) = u_0(t) + u_1(t) + u_2(t) + \cdots
  = 3t^2 + t^3 + \frac{t^4}{12} - \frac{5t^5}{90} + \frac{13t^6}{90} + \frac{t^8}{20} - \frac{6t^{3-a}}{\Gamma(4-a)} - \frac{2t^{4-a}}{\Gamma(5-a)} + \frac{24t^{6-a}}{\Gamma(7-a)}
  + \frac{2t^{4-2a}}{\Gamma(5-2a)} + \frac{4t^{7-2a}}{(7-2a)\Gamma(4-a)} + \cdots.
\]

Furthermore for the second example, given a linear fractional differential equation
\[
  \frac{d^\alpha u}{dt^\alpha} = 1 - u(t), \quad t > 0, \quad 0 < \alpha \leq 1
\]
with initial condition \( u(0) = 0 \). We have \( A(t) = 1, \ L(u(t)) = u \), and \( N(u(t)) = 0 \).

Based on Modified Homotopy Perturbation Method, homotopy equation is:
\[
  u' - 1 = p[u' - u - D^\alpha u].
\]

With the similar method, the equation was obtained solution as follow:
\[
  u(t) = u_0(t) + u_1(t) + u_2(t) + u_3(t) + \cdots
  = 4t - 3t^2 + \frac{4t^3}{3!} - \frac{t^4}{4} - \frac{6t^{2-a}}{\Gamma(3-a)} + \frac{8t^{3-a}}{\Gamma(4-a)} - \frac{3t^{4-a}}{\Gamma(5-a)} + \frac{4t^{3-2a}}{\Gamma(4-2a)}
  - \frac{3t^{4-2a}}{\Gamma(5-2a)} + \frac{t^{4-3a}}{\Gamma(5-3a)} + \cdots.
\]

### 3.2 Convergence Analysis

In this section, convergence analysis of solution function sequence will be elaborated in order to capture the behavioral change of solution function graph based on the change of fractional order. It is developed according to two examples in the previous section, which are solution of linear and nonlinear fractional differential equations.

Initially, it is given a fractional order of a number sequence \( (\alpha_n) = \left( \frac{n}{n+1} \right) \). It is clear that this sequence is converge to \( \alpha = 1 \) because for any \( \epsilon > 0 \) we can choose natural number \( N \geq \frac{1}{\epsilon} \) such that if \( n > N \) then \( |\alpha_n - \alpha| = \left| \frac{n}{n+1} - 1 \right| = \frac{1}{n} < \frac{1}{N} \leq \frac{1}{\frac{1}{\epsilon}} = \epsilon \).
Then, Figure 1 presents the graph of the solution function of nonlinear fractional differential equation with order $\alpha_n$

$$\frac{d^{\alpha_n} u}{dt^{\alpha_n}} - u(t) + u^2(t) = 2t$$

as in (14) with solution of $g_n(t)$ in (15).

![Figure 1](image)

**Figure 1.** Graphs of solution function of nonlinear FDE (14) with $\alpha_n = \frac{n}{n+1}$

Hence based on Figure 1, it can be seen that when $\alpha_n$ is closer to 1, the solution function graph will be closer to nonlinear FDE solution function graph with 1 order.

Similarly, figure 2 presents graph for the second case of Linear FDE in (16) with the solution of (17).

![Figure 2](image)

**Figure 2.** Graphs of solution function of linear FDE (16) with $\alpha_n = \frac{n}{n+1}$

Hence, it can be seen that the solution functions sequence moves according to the sequence of fractional orders, starting from red graph to black graph as a convergence function.

4. Conclusion
Based on what has been discussed above, the conclusion is that the Modified Homotopy Perturbation Method guarantees the existence of a solution of Fractional Differential Equation model. The second conclusion is that if the fractional order sequence is converges to $\alpha$, then the sequence of solution functions is also converges to solution function of Fractional Differential Equation with order $\alpha$.

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References

[1] Capelas O E, and Machado J A T 2014 A Review of Definitions for Fractional Derivatives and Integral Mathematical Problems in Engineering Journal 238459

[2] Podlubny I 1999 Fractional Differential Equations (New York: Academic Press)

[3] Kisela T 2008 Fractional Differential Equations and Their Application Ph. D Thesis (Brno: Brno University of Technology)

[4] Podlubny A, Chechkin T, Skovranek Y Q Chen, and B M J Vinagre 2009 Matrix approach to discrete fractional calculus II: Partial fractional differential equations Journal of Computational Physics 228 3137–3153

[5] Rusyaman E, Parmikanti K, Chaerani D, Asefan, and Irianingsih I 2018 The convergence of the order sequence and the solution function sequence on fractional partial differential equation Journal of Physics: Conference Series 983 012087

[6] Ruzitalab, Ahmad, Farahi M H, and Erjaee G 2018 Generalized convergence analysis of the fractional order systems De Gruyter Open Phys 16 404–411

[7] Sumiati I, Rusyaman E, and Sukono 2019 Black-Scholes Equation Solution Using Laplace-Adomian Decomposition Method IAENG Int. Journal of Computer Science 46 1-6

[8] Rusyaman E 2017 The effect of Surface Presure and Elasticity to the Surface Minimum Energy with Fractional Order Journal of Engineering and Applied Science 12 4851-4855

[9] Rusyaman E, Parmikanti K, Chaerani D, and Supian S 2020 The Fractional Relationship between Viscosity and Surface Tension on Lubricating Oils IAENG - International Journal of Applied Mathematics 50 1

[10] Shijun L 1998 Homotopy analysis method: a new analytic method for nonlinear problems Applied Mathematics and Mechanics 19 957-962

[11] He J H 1999 Homotopy perturbation technique ComputerMethods in Applied Mathematics and Engineering 178 257–262

[12] Odibat Z, and Momani S 2008 Modified Homotopy Perturbation Method: Application to Quadratic Riccati Differential Equation of Fractional Order Chaos Solitons and Fractals 36 167-174

[13] Jafari H, and Seifi S 2009 Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation Communications in Nonlinear Science and Numerical Simulation 14 2006–2012

[14] Hemeda A A 2014 Modified Homotopy Perturbation Method for Solving Fractional Differential Equations Journal of Applied Mathematics 2014 594245