Massive chiral random matrix ensembles at $\beta = 1 \& 4$ : Finite-volume QCD partition functions

Taro Nagao
Department of Physics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

Shinsuke M. Nishigaki
Department of Physics, Faculty of Science, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152-8551, Japan

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In a deep-infrared (ergodic) regime, QCD coupled to massive pseudoreal and real quarks are described by chiral orthogonal and symplectic ensembles of random matrices. Using this correspondence, general expressions for the QCD partition functions are derived in terms of microscopically rescaled mass variables. In limited cases, correlation functions of Dirac eigenvalues and distributions of the smallest Dirac eigenvalue are given as ratios of these partition functions. When all masses are degenerate, our results reproduce the known expressions for the partition functions of zero-dimensional $\sigma$ models.

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The low energy dynamics of quantum chromodynamics is dictated by confinement of colored particles and spontaneous breakdown of the chiral symmetry $\mathbb{Z}_2$. While the vector part of the flavor symmetry in a vectorial theory such as QCD is protected by Vafa-Witten theorem, its axial part is considered to be maximally broken by the quark condensate. Accordingly the pattern of global symmetry breaking of QCD coupled to $n \geq 2$ massless quarks falls onto either of the following three classes $\mathbb{Z}_2^3$:

\begin{align*}
N_c \geq 3, \text{ fund.} & : SU(n) \times SU(n) \rightarrow SU(n), \\
N_c = 2, \text{ fund.} & : SU(2n) \rightarrow Sp(2n), \\
N_c \geq 2, \text{ adj.} & : SU(n) \rightarrow SO(n),
\end{align*}

where ‘fund.’ and ‘adj.’ stand for the representations of the $SU(N_c)$ gauge group to which quark fields belong. These three classes are assigned Dyson indices $\beta = 2, 1, 4$, respectively, according to the anti-unitary symmetry of associated Euclidean Dirac operators $\mathbb{Z}_2^3$.

In the vicinity of the chiral limit where the quark masses $m$ are much smaller than the typical hadronic scale $\Lambda_{QCD}$, the theory is effectively described by a non-linear $\sigma$ model over the coset manifold, without involving gluons and quarks that are confined. The effective Lagrangian consists of terms made of the Goldstone pion field $U(x)$ and the quark mass matrix, consistent with the flavor symmetry $\mathbb{Z}_2^3$. Furthermore, in the ‘ergodic regime’ where the dimension $L$ of the system is much smaller than the pion Compton length $\frac{1}{m_a} \sim \frac{1}{\sqrt{\Lambda_{QCD}}}$, only the zero mode of $U(x)$ dominates, so that the effective ‘finite-volume’ partition function in the presence of the $\theta$ angle simplifies into a finite-dimensional integral: $\mathbb{Z}^4(\theta; M) = \int_{SU(n)} dU \exp \left( \text{Re tr} e^{i\theta/(N_c n)} M U U^T \right)$.

Here the rescaled mass matrices $M$ are

\begin{align*}
M &= \text{diag} \left( \mu_1, \ldots, \mu_n \right) \quad (\beta = 2, 4), \\
M &= \text{diag} \left( \mu_1, \ldots, \mu_n \right) \otimes J \quad (\beta = 1),
\end{align*}

and $\Sigma$ stands for the quark condensate in the chiral limit. The integrals are extended from cosets to full unitary groups (for $\beta = 1, 4$), and their Haar measures $dU$ are normalized such that $\int dU = 1$. After Fourier transformation, the partition function in a sector with a topological charge $\nu$ takes the form:

\begin{align*}
Z^2(\nu; M) &= \int_{U(n)} dU \det -\nu U \exp \left( \text{Re tr} M U U^T \right), \\
Z^4(\nu; M) &= \int_{U(2n)} dU \det -\nu U \exp \left( \frac{1}{2} \text{Re tr} M U U^T \right), \\
Z^4(\nu; M) &= \int_{U(n)} dU \det -2\nu U \exp \left( \text{Re tr} M U U^T \right).
\end{align*}

For the $\beta = 4$ case, $\nu$ is understood to be substituted by $\nu N_c$. With a help of Itzykson-Zuber-like formula, an explicit form of the integral, which can be considered as a ‘matrix Bessel function’, was obtained for $\beta = 2$:

\begin{align*}
Z^2(\mu_1, \ldots, \mu_n) = 2 \frac{n^{n-1}}{\pi} \prod_{k=0}^{n-1} \frac{\det \Delta_{\mu_1, \ldots, \mu_n} \mu_i^{-1} I_{\nu+j-1}(\mu_i)}{\Delta(\mu_1, \ldots, \mu_n)^2}, \\
\Delta(x_1, \ldots, x_n) &\equiv \prod_{1 \leq i < j \leq n} (x_i - x_j).
\end{align*}
and namely for identical $\mu_i$'s \[Z^{(2)}_\nu(\mu, \ldots, \mu) = \det_{-n+\frac{1}{2} \leq i, j \leq -n+\frac{1}{2}} I_{\nu+j-i}(\mu), \tag{7}\]

On the other hand, due to technical difficulties, only the case with completely degenerated masses was worked out for $\beta = 1$ and 4 \[Z^{(1)}_\nu(\mu, \ldots, \mu) = \frac{1}{(2n-1)!!} \text{Pf}_{-n+\frac{1}{2} \leq i, j \leq -n+\frac{1}{2}} (j-i)I_{\nu+j+i}(\mu), \tag{8}\]

$Z^{(4)}_\nu(\mu, \ldots, \mu) = (n-1)!! \text{Pf}(A) \quad (n: \text{even}) \tag{9a}$

$= n!! \text{Pf} \left( \begin{array}{cc} A & b \\ -b^T & 0 \end{array} \right) \quad (n: \text{odd}), \tag{9b}$

$\quad A^{ij} = \sum_{k=-\infty}^{\infty} \frac{1}{2k+1} I_{\nu+i+k+\frac{1}{2}}(\mu)I_{\nu+j-k-\frac{1}{2}}(\mu),$ $b^i = I_{\nu+i}(\mu), \quad -\frac{n-1}{2} \leq i, j \leq \frac{n-1}{2}.$

Although the above expressions are primarily valid for $n \geq 2$, they can be extended to $n = 1$, as a separate consideration for this mass-gapped case leads to \[Z^{(3)}_\nu(\mu) = I_{\nu}(\mu), \tag{10}\]

irrespective of the values of $\beta$. Once the partition functions are computed in a closed form as the above, physical quantities such as the topological susceptibility

$$\langle \nu^2 \rangle = \frac{\sum_i \nu^2 Z^{(3)}_\nu(\{\mu\})}{\sum_i Z^{(3)}_\nu(\{\mu\})} \tag{11}$$

are expressed in terms of them.

An important observation made by Verbaarschot and collaborators \[\text{[3]}\] is that the finite-volume partition functions \[\text{[3]}\] can as well be derived from models much simpler than QCD, Chiral Random Matrix Ensembles (\(\chi\)RMEs, for a recent review see Ref. \[\text{[3]}\]):

$$Z^{(3)}_\nu(\{m\}) = \int dW e^{-\beta \text{tr} V(W^*W)} \prod_{i=1}^{n} \text{det} \left( \begin{array}{cc} m_i & W \\ -W^* & m_i \end{array} \right), \tag{12}$$

where the integrals are over complex, real, and quaternion real $N \times (N+n)$ matrices $W$ for $\beta = 2, 1, 4$, respectively, and it is understood for $\beta = 4$ that twofold degenerated eigenvalues in the determinant are only counted once. Their proofs consist of the "color-flavor transformation" \[\text{[2]}\] that converts the integration variables into $n \times n$ matrices and the saddle point method under which

$$N \to \infty, \quad \mu_i \to 0, \quad \mu_i \equiv \pi \rho(0)m_i: \text{fixed.} \tag{13}$$

Here $\rho(0)$ stands for the spectral density of the random matrix $D = \left( \begin{array}{cc} 0 & W \\ -W^* & 0 \end{array} \right)$:

$$\rho(\lambda) = (\text{tr} D\delta(\lambda - iD)), \tag{14}$$

at the origin. The $\chi$RME is motivated by the microscopic theory (Euclidean QCD) on a lattice, with a crude simplification of replacing matrix elements of the anti-Hermitian Dirac operator $D = (\partial_\mu + iA_\mu)\gamma_\mu$ by random numbers $D$ generated according to the weight $e^{-\beta \text{tr} V(W^*W)}$. Under this correspondence, the microscopic limit \[\text{[3]}\] is equivalent to Leutwyler-Smilga limit \[\text{[3]}\], since the size $N$ of the matrix $W$ is interpreted as the number of cites $L^4$ of the lattice on which QCD is discretized, and the Dirac spectral density at zero virtuality $\rho(0)$ is related to the quark condensate by Banks-Casher relation $\Sigma = \pi \rho(0)/L^4 \tag{13}$. These $\chi$RMEs are technically more suited for the computation of microscopic spectral correlations of Dirac operators $D \sim D$, than the zero-dimensional $\sigma$ models \[\text{[3]}\] where inevitable introduction of a probe quark pair (partial quenching) \[\text{[4, 15]}\] brings forth additional complication. In the chiral limit $\mu \equiv 0$, the eigenvalue correlation functions \[\text{[16, 23]}\] as well as the smallest eigenvalue distributions \[\text{[21–24]}\] have been computed for $\chi$RMEs with all three values of $\beta$. On the other hand, in the presence of finite and generic $\mu$'s, these quantities have so far been analytically computed only for the $\beta = 2$ case \[\text{[22, 23]}\]. For $\beta = 4$, the smallest eigenvalue distribution in the presence of $\mu$'s was numerically computed in Ref. \[\text{[16]}\], and the partition function with four degenerate $\mu$'s was analytically treated in Ref. \[\text{[23]}\]. We shall treat the remaining cases, chiral orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) ensembles with finite mass parameters. This Letter is devoted to the computation of partition functions of these ensembles. As corollaries, we provide closed expressions for the smallest eigenvalue distribution for $\beta = 1$ and odd $\nu$, and the microscopic eigenvalue correlation functions for $\beta = 4$. The microscopic eigenvalue correlation functions for the $\beta = 1$ case will be presented in a separate publication \[\text{[31]}\].

We start by expressing the partition function \[\text{[3]}\] of the $\chi$RME in terms of eigenvalues $x_i = \lambda_i^\beta$ of the positive definite matrix $W^*W$ (up to a constant independent of $m$ and $\nu$):

$$Z^{(3)}_\nu(\{m\}) = \left( \prod_{i=1}^{n} m_i^\nu \right) \Xi^{(3)}_\nu(\{m\}), \tag{15}$$

$$\Xi^{(3)}_\nu(\{m\}) = \int_0^\infty \prod_{j=1}^{N} (dx_j w(x_j; \{m\}) \mid \Delta(x_1, \ldots, x_N)^{\beta}, \tag{16}$$

$$w(x; \{m\}) = e^{-\beta V(x)} x^{\beta(\nu+1)/2-1} \prod_{i=1}^{n} (x - m_i^2). \tag{16}$$
Since the partition function is even under $\nu \to -\nu$, we have set $\nu$ non-negative integer, without loss of generality. We note that all spectral correlation functions of the orthogonal and symplectic ensembles can be constructed from the scalar kernel of the unitary ensemble sharing the same weight function $w(x)$ [12, 13]. Since the scalar kernel in the microscopic limit (19) is known to be insensitive to the details of the potential $V(x)$ either in the absence [34, 35] or in the presence of finite $\mu$’s [33], it suffices to concentrate onto Laguerre (chiral Gaussian) ensembles, $V(x) = x$. This choice leads to

$$\rho(\lambda) = \frac{2}{\pi} \sqrt{2N - \lambda^2}. \quad (17)$$

We sketch the skew-orthogonal polynomial method [30] as employed in Ref. [24], to which we leave details. As the final result in the microscopic limit is insensitive to the parity of $N$, we consider only even $N$ henceforth.

### A. orthogonal ensemble

We use the identity

$$\Delta(x_1, \ldots, x_N) \prod_{j=1}^{N} \prod_{n=1}^{n} (x_j - x_{N+i}) = \frac{\det_{1\leq i,j \leq N+n} R_i(x_j)}{\Delta(x_{N+1}, \ldots, x_{N+n})} \prod_{n=1}^{N} R_i(x_j), \quad (18)$$

where $R_i(x)$ is an arbitrary monic polynomial of the $i$-th order, and $x_{N+i} \equiv -m_i^2 \leq 0$. We take $\{R_i(x)\}$ to be skew-orthogonal

$$\langle R_{2i} R_{2j+1} \rangle_R = -(\langle R_{2j+1} R_{2i} \rangle_R \equiv h_i \delta_{ij}, \quad \text{others} = 0, \quad (19)$$

with respect to the antisymmetric product

$$\langle f, g \rangle_R = \int_{0}^{\infty} dx x^{\nu-1} e^{-x} g(x) \int_{0}^{\infty} dy y^{\nu-1} e^{-y} f(y) - \langle f \leftrightarrow g \rangle. \quad (20)$$

When eq. (18) is integrated over $x_1, \ldots, x_N$ with the weight $\prod_{n=1}^{N} (e^{-x_i} x_i^{-\nu+1})$ in a cell $0 \leq x_1 \leq x_2 \leq \cdots \leq x_N$, it can be neatly expressed as a Pfaffian, due to the skew orthogonality [19, 24]:

$$\Xi_{\nu}^{(1)}(\{m\}) = \left( \prod_{k=0}^{n} h_k \right) \frac{\text{Pf}(F)}{\Delta(m_1^2, \ldots, m_n^2)} \quad (n:\text{even}) \quad (21a)$$

$$= \left( \prod_{k=0}^{n} h_k \right) \frac{\text{Pf}(F)}{\Delta(m_1^2, \ldots, m_n^2)} \quad (n:\text{odd}), \quad (21b)$$

where

$$F^{ij} = \sum_{k=0}^{n} \frac{R_{2k}(-m_k^2) R_{2k+1}(-m_k^2)}{h_k} = (i \leftrightarrow j), \quad R^i = R_{N+n-1}(-m_i^2), \quad 1 \leq i,j \leq n.$$

Explicit forms of the monic skew-orthogonal polynomials and their norms are known as [37]:

$$R_{2k}(x) = -\frac{(2k)!}{2^{2k+1}} \frac{d}{dx} L_{2k+1}^{(\nu-1)}(2x),$$

$$R_{2k+1}(x) = -\frac{(2k+1)!}{2^{2k+1}} \frac{d}{dx} L_{2k+1}^{(\nu-1)}(2x) - \frac{(2k)!}{2^{2k+2}} (2k+\nu) \frac{d}{dx} L_{2k}^{(\nu-1)}(2x),$$

$$h_k = 2^{-4k-\nu(2k)!/(2k+\nu)!}. \quad (22)$$

In the microscopic limit [13] with $\mu_i = 2\sqrt{N m_i}$ fixed, the sum over the indices $k$ becomes an integral, and Laguerre polynomials approach modified Bessel functions:

$$L_{k}^{(\nu)}(x) = \left( \frac{k}{-x} \right)^{\frac{\nu}{2}} I_{\alpha}(2\sqrt{-kx}) \quad (x = O(\frac{1}{k}) < 0). \quad (23)$$

Then the partition function is expressed as

$$Z_{\nu}^{(1)}(\{\mu\}) = \left( \prod_{i=1}^{n} \mu_i^{\nu} \right) \xi_{\nu}^{(1)}(\{\mu\}),$$

$$\xi_{\nu}^{(1)}(\{\mu\}) = c_n \frac{\text{Pf}(f)}{\Delta(\mu_1^2, \ldots, \mu_n^2)} \quad (n:\text{even}), \quad (24a)$$

$$= c_n \frac{\text{Pf}(f)}{\Delta(\mu_1^2, \ldots, \mu_n^2)} \quad (n:\text{odd}), \quad (24b)$$

where

$$c_n = (-1)^{n(n-1)/2} 2^{n^2-1} (n-1)! \prod_{k=0}^{n-2} (2k+1)! (n:\text{even}),$$

$$= (-1)^{n(n-1)/2} 2^{(n-1)(n+1)/2} (n-1)! \prod_{k=0}^{n-2} (2k+1)! (n:\text{odd}),$$

$$f^{ij} = \int_{0}^{1} dt t^{2} \frac{I_{\nu-1}(t \mu_i) I_\nu(t \mu_j)}{\mu_i^{\nu} \mu_j^{\nu}} - (i \leftrightarrow j),$$

$$r^{i} = \frac{I_\nu(\mu_i)}{\mu_i^{\nu}}, \quad 1 \leq i,j \leq n.$$

The constant $c_n$ is conveniently determined as the above by requiring the small-$\mu$ behavior be in accord with that of the zero-dimensional $\sigma$ model,

$$Z_{\nu}^{(1)}(\mu_1, \ldots, \mu_n) \simeq \left( \frac{\mu}{2} \right)^{n(n-1)/2} \prod_{k=0}^{n-1} (2k)! \quad (\mu \ll 1). \quad (25)$$
B. symplectic ensemble

We concentrate on the case with an even \( n (= 2a) \) number of flavors and pairwise degenerated mass parameters, corresponding to adjoint Dirac fermions in the QCD context.

We use the identity

\[
\Delta(x_1, \ldots, x_N) = \prod_{j=1}^N \prod_{i=1}^a (x_j - x_{N+i})^2
\]

where

\[
\begin{align*}
F^{ij} &= \sum_{k=0}^{N+1} \frac{Q_{2k}(-m_i^2)Q_{2k+1}(-m_j^2) - (i \leftrightarrow j)}{h_k}, \\
Q^i &= Q_{2N+a-1}(-m_i^2), \quad 1 \leq i, j \leq a.
\end{align*}
\]

Explicit forms of the monic skew-orthogonal polynomials and their norms are known as \([37]\):

\[
\begin{align*}
Q_{2k}(x) &= \frac{k!(k + \nu)!}{2^{2k}} \sum_{l=0}^{k} \frac{(2l - 1)!!}{2^l(l + \nu)!} L_{2l}^{(2\nu)}(4x), \\
Q_{2k+1}(x) &= -\frac{(2k + 1)!!}{2^{k+1} k!} L_{2k+1}^{(2\nu)}(4x), \\
h_k &= 2^{-8k-4\nu-4}(2k + 1)!(2k + 2\nu + 1)!
\end{align*}
\]

In the microscopic limit with \( \mu_i = 2\sqrt{2N}m_i \) fixed, the sums over the indices \( k \) and \( l \) become integrals, and Laguerre polynomials approach modified Bessel functions.

Then the partition function is expressed as

\[
Z_\nu^{(4)}(\{\mu\}) = \left( \prod_{i=1}^a \mu_i^{2\nu} \right) \xi_\nu^{(4)}(\{\mu\}),
\]

\[
\begin{align*}
\xi_\nu^{(4)}(\{\mu\}) &= c_\mu \frac{\text{Pf}(f)}{\Delta(\mu_1^2, \ldots, \mu_a^2)} \quad (a : \text{even}) \quad (31a) \\
\xi_\nu^{(4)}(\{\mu\}) &= c_\mu \frac{\text{Pf}\left( \frac{f}{-q^T} \right) q^T}{\Delta(\mu_1^2, \ldots, \mu_a^2)} \quad (a : \text{odd}) \quad (31b)
\end{align*}
\]

where

\[
\begin{align*}
c_\mu &= (-1)^{a(a-1)/2} \prod_{k=0}^{a-1} (2k + 1)!
\end{align*}
\]

When eq.(26) is integrated over \( x_1, \ldots, x_N \) with the weight \( \prod_{i=1}^a e^{-4x_i^2} \), it can be neatly expressed as a Pfaffian, due to the skew-orthogonality \([27], [28]\):

\[
\begin{align*}
\xi_\nu^{(4)}(\{m\}) &= \left( \prod_{i=1}^{a+1} \frac{F^{ij} h_i}{\Delta(m_1^2, \ldots, m_a^2)} \right) \text{Pf}(F) \quad (a : \text{even}) \quad (29a) \\
&= \left( \prod_{i=1}^{a+1} \frac{F^{ij}}{\Delta(m_1^2, \ldots, m_a^2)} \right) \text{Pf}\left( \begin{array}{c}
F^{ij} \\
Q^i
\end{array} \right) \quad (a : \text{odd}) \quad (29b)
\end{align*}
\]

with respect to the antisymmetric product

\[
\langle f, g \rangle_Q = \int_0^{\infty} dx \, x^{2\nu+1} e^{-4x} (f(x)g'(x) - f'(x)g(x)).
\]

The constant \( c_\mu \) is conveniently determined as the above by requiring the small-\( \mu \) behavior be in accord with that of the zero-dimensional \( \sigma \) model,

\[
Z_\nu^{(4)}(\mu, \ldots, \mu) \approx \mu^{2a^2} \prod_{k=0}^{a-1} \frac{(2k + 1)!!}{(2k + 2\nu + 1)!} \quad (\mu \ll 1) \quad (32)
\]

It remains to confirm whether the above expressions for the RME partition functions agree with those of the zero-dimensional \( \sigma \) models. By taking all \( \mu \)'s to be identical, we obtain for \( \beta = 1 \):
\[
Z^{(1)}_\nu(\mu_1, \ldots, \mu_n) = \tilde{c}_n \frac{\text{Pf}(\tilde{f})}{\mu^{(n/2-1)n}} (n: \text{even}) \quad (33a)
\]
\[
\quad = \tilde{c}_n \frac{\text{Pf} \left( \begin{array}{c} \tilde{f} \\ -\tilde{q}^T \\ 0 \end{array} \right)}{\mu^{(n-1)n/2}} (n: \text{odd}), \quad (33b)
\]
where
\[
\tilde{c}_n = (-1)^{n(n-1)/2} \prod_{k=0}^{n-1} \frac{(2k + n - 1)!}{(2k)!} (n: \text{even})
\]
\[
= (-1)^{n(n-1)/2} 2^{(n-1)(3n+1)} \prod_{k=0}^{n-3} \frac{(2k + n)!}{(2k + 1)!} (n: \text{odd}),
\]
\[
\tilde{f}^{ij} = \int_0^1 dt t^{i+j+2} I_{\nu-1}(\mu t) I_{\nu+j}(\mu) - (i \leftrightarrow j),
\]
\[
\tilde{q}^i = I_{\nu+i}(\mu), \quad 0 \leq i, j \leq n - 1,
\]
and for \( \beta = 4 \):
\[
Z^{(4)}_\nu(\mu_1, \ldots, \mu_n) = \tilde{c}_a \frac{\text{Pf}(\tilde{f})}{\mu^{(a-1)n/2}} (a: \text{even}) \quad (34a)
\]
\[
\quad = \tilde{c}_a \frac{\text{Pf} \left( \begin{array}{c} \tilde{f} \\ -\tilde{q}^T \\ 0 \end{array} \right)}{\mu^{(a-1)n/2}} (a: \text{odd}), \quad (34b)
\]
where
\[
\tilde{c}_a = (-1)^{a(a-1)/2} \prod_{k=0}^{a-1} \frac{(2k + 1)!}{k!},
\]
\[
\tilde{f}^{ij} = \int_0^1 dt t^{i+j+1} I_{2\nu+i}(2t) \int_0^1 du u I_{2\nu+j}(2tu) - (i \leftrightarrow j),
\]
\[
\tilde{q}^i = \int_0^1 dt t^i I_{2\nu+i}(2t), \quad 0 \leq i, j \leq a - 1.
\]

We have numerically checked that, despite the appearances, the above expressions are identical to eqs. (33) and (34). Together with the \( \beta = 2 \) case that has previously been confirmed, they explicitly show the equivalence between the\( \chi \)RMEs and the \( \sigma \) models in Leutwyler-Smilga limit.

**C. smallest eigenvalue distribution**

The probability of finding no eigenvalue in the interval \( 0 \leq x < s \) is given by
\[
E^{(\beta)}_\nu(s; \{m\}) = \frac{\int_s^\infty \prod_{j=1}^N (dx_j w(x_j; \{m\})) |\Delta(\{x\})|^{\beta}}{\int_0^\infty \prod_{j=1}^N (dx_j w(x_j; \{m\})) |\Delta(\{x\})|^{\beta}}.
\]

The integral domain in the numerator can be traded to \([0, \infty)\) with the weight function shifted by \( s, w(x + s; \{m\}) \). If the exponent in the weight function \( \frac{1}{2}(\nu + 1) - 1 \) is an integer (excluding the case with \( \beta = 1 \) and \( \nu \) even), we can utilize the ‘flavor-topology duality’ [39]
\[
\Xi^{(\beta)}(m_1, \ldots, m_n) = \Xi^{(\beta)}_{\frac{1}{2}-1}(m_1, \ldots, m_n, 0, \ldots, 0), \quad (36)
\]
to express \( E^{(\beta)}_\nu(s; \{m\}) \) in terms of the partition functions:
\[
E^{(\beta)}_\nu(s; \{m\}) = e^{-N\bar{\beta}s} \times \Xi^{(\beta)}_{\frac{1}{2}-1}(\sqrt{s + m_1}, \ldots, \sqrt{s + m_n}, \sqrt{n}, \ldots, \sqrt{n}).
\]

Now we change the picture back from Laguerre to chiral Gaussian, and take the microscopic limit with \( \xi = \pi \rho(0)/\sqrt{s} = 2\sqrt{2N} \) and \( \mu_2 = 2\sqrt{2N} \) fixed. The distribution of the smallest eigenvalue of chiral random matrices is then given by the first \( \xi \)-derivative of \( E^{(\beta)}_\nu \):
\[
P^{(\beta)}_\nu(\xi; \{\mu\}) = -\frac{\partial}{\partial \xi} \left\{ e^{-(\beta/8)\xi^2} \times \Xi^{(\beta)}_{\frac{1}{2}-1}(\sqrt{\xi^2 + 2}, \ldots, \sqrt{\xi^2 + 2}, \xi, \ldots, \xi) \right\}.
\]

For \( \beta = 1 \) and \( \nu \) odd, eqs. (38) and (24) suffice to express the smallest eigenvalue distribution in a closed form. This prediction should in future be put in comparison with lattice QCD simulations with overlap dynamical quarks.

On the other hand, for \( \beta = 4 \), the partition function in the numerator falls out of the range of this Letter, as the number of additional flavors is odd. A different formalism based on Fredholm determinant [40] might be needed in order to overcome this limitation.

**D. correlation function**

In the case of even \( \beta \), a \( p \)-level correlation function
\[
\rho(\lambda_1, \ldots, \lambda_p; \{\mu\}) = \left\langle \prod_{k=1}^p \text{tr} \delta(\lambda_k - iD) \right\rangle
\]
is expressed by construction as a ratio of partition functions with \( n \) and \( n + \beta p \) flavors [38, 41]. After taking the microscopic limit, in the \( \beta = 4 \) case with \( p \) pairs of doubly degenerated masses, it reads:
\[
\rho^{(4)}_\nu(\zeta_1, \ldots, \zeta_p; \{\mu\}) = C_{a,\nu}^{(p)} \Delta(\zeta_1^2, \ldots, \zeta_p^2)^4 \prod_{k=1}^p \left( \frac{\zeta_k^2}{\mu_k^2} + \mu_k^2 \right)^2 \frac{Z^{(4)}_\nu(\mu_1, \mu_1, \ldots, \mu_a, \mu_a, i\zeta_1, \ldots, i\zeta_p, \ldots, i\zeta_p)}{Z^{(4)}_\nu(\mu_1, \mu_1, \ldots, \mu_a, \mu_a)}.\]

(40)

As our derivation of the partition function (31) is valid as well for negative values of \(m_2^2\) or \(\mu_2^2\), the above relationship suffices to express any \(p\)-level correlation function in a closed form, up to an overall constant independent of \(\mu\)’s.

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* e-mail addresses:
  naga@sphinx.phys.sci.osaka-u.ac.jp
  nishigak@th.phys.titech.ac.jp

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