A DISPERSIVE ESTIMATE FOR THE LINEAR WAVE EQUATION WITH AN ELECTROMAGNETIC POTENTIAL

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Abstract. We consider radial solutions to the Cauchy problem for the linear wave equation with a small short–range electromagnetic potential (the “square version” of the massless Dirac equation with a potential) and zero initial data. We prove two a priori estimates that imply, in particular, a dispersive estimate.

1. Introduction

In this paper, we investigate the dispersive properties of the linear wave equation with an electromagnetic potential, that is

\[ \square_A u = F \quad (t, x) \in [0, \infty) \times \mathbb{R}^3, \]

where

\[ x = (x_1, x_2, x_3), \quad r = |x|, \]

\[ \square_A = \square - A \cdot \nabla_{t,r}, \]

\[ \square = \partial_t^2 - \Delta = \partial_t^2 - (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2), \]

\[ \nabla_{t,r} = \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial r} \right). \]

The fact that the potential \( A = A(t, x) \) is electromagnetic means that \( A \in i\mathbb{R} \times i\mathbb{R} \), where \( i \) is the imaginary unit. This will play a crucial role in the development of the proof, since electromagnetic potential are gauge invariant (see what follows).

We restrict ourselves to radial solutions \( u = u(t, r) \), with \( F = F(t, r) \) and

\[ A = A(t, r) = \left( \frac{A_0(t, r)}{A_1(t, r)} \right), \quad A_0, A_1 \in i\mathbb{R}. \]

We assume further that the potential decreases sufficiently rapidly when \( r \) approaches infinity; more precisely, we suppose that

\[ \sum_{j \in \mathbb{Z}} 2^{-j} (2^{-j})^{\varepsilon_A} \| \varphi_j A \|_{L^p} \leq \delta_A \]

(that is, \( A \) is a short–range potential), where \( \varepsilon_A > 0, \delta_A \) is a sufficiently small positive constant independent of \( r \) (see Section 2) and the sequence \( \{\varphi_j\}_{j \in \mathbb{Z}} \) is a Paley–Littlewood partition of unity, which means that \( \varphi_j(r) = \varphi(2^j r) \) and \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) (\( \mathbb{R}^+ \) is the set of all non–negative real numbers) is a function so that

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a) \( \text{supp } \varphi = \{ r \in \mathbb{R} : 2^{-1} \leq r \leq 2 \} \);

b) \( \varphi(r) > 0 \) for \( 2^{-1} < r < 2 \);

c) \( \sum_{j \in \mathbb{Z}} \varphi(2^j r) = 1 \) for each \( r \in \mathbb{R}^+ \).

In other words, \( \sum_{j \in \mathbb{Z}} \varphi_j(r) = 1 \) for all \( r \in \mathbb{R}^+ \) and

\[
(1.8) \quad \text{supp } \varphi_j = \{ r \in \mathbb{R} : 2^{-j-1} \leq r \leq 2^{-j+1} \}.
\]

It is well-known that there exists a unique global solution to the Cauchy problem

\[
(1.9) \quad \left\{ \begin{array}{l}
\Box_A u = F \\
(t, x) \in [0, \infty[ \times \mathbb{R}^3, \\
u(0, x) = \partial_t u(0, x) = 0 \\
x \in \mathbb{R}^3;
\end{array} \right.
\]

in particular, this fact holds for the smaller class of radial solutions, that is for the problem

\[
(1.10) \quad \left\{ \begin{array}{l}
\Box_A u = F \\
(t, r) \in [0, \infty[ \times \mathbb{R}^+, \\
u(0, r) = \partial_t u(0, r) = 0 \\
r \in \mathbb{R}^+.
\end{array} \right.
\]

Let introduce the change of coordinates

\[
(1.11) \quad \tau_{\pm} = \frac{t \pm r}{2}
\]

and the standard notation \( \langle s \rangle = \sqrt{1 + s^2} \); our main result can be expressed as follows.

**Theorem 1.1.** Let \( u \) be a radial solution to (1.9), i.e. a solution to (1.10), where \( A = A(t, r) \) is an electromagnetic potential satisfying (1.7) for some \( \delta_A > 0 \) and \( \epsilon_A > 0 \). Then, for every \( \epsilon > 0 \), there exist two positive constants \( \delta \) and \( C \) (depending on \( \epsilon \)) such that for each \( \delta_A \in ]0, \delta] \), one has

\[
(1.12) \quad ||\tau_{\pm} u||_{L^\infty_t \mathbb{R}^3} \leq C ||\tau_{\pm} r^2 \langle r \rangle \epsilon F||_{L^\infty_t \mathbb{R}^3}.
\]

Let introduce the differential operators

\[
(1.13) \quad \nabla_{\pm} = \partial_t \pm \partial_r.
\]

The proof of the previous a priori estimate follows easily from the following one.

**Lemma 1.1.** Under the same conditions of Theorem 1.1, for every \( \epsilon > 0 \), there exist two positive constants \( \delta \) and \( C \) (depending on \( \epsilon \)) such that for each \( \delta_A \in ]0, \delta] \), one has

\[
(1.14) \quad ||\tau_{\pm} r \nabla_- u||_{L^\infty_t \mathbb{R}^3} \leq C ||\tau_{\pm} r^2 \langle r \rangle \epsilon F||_{L^\infty_t \mathbb{R}^3}.
\]

An immediate consequence of Theorem 1.1 is the following dispersive estimate.

**Corollary 1.1.** Under the same conditions of Theorem 1.1, for every \( \epsilon > 0 \), there exist two positive constants \( \delta \) and \( C \) (depending on \( \epsilon \)) such that for each \( \delta_A \in ]0, \delta] \), one has

\[
(1.15) \quad |u(t, x)| \leq C \frac{1}{t} ||\tau_{\pm} r^2 \langle r \rangle \epsilon F||_{L^\infty_t \mathbb{R}^3}
\]

for every \( t > 0 \).

The idea to prove the lemma is the following. First of all, the potential term in (1.10) can be thought as part of the forcing term, that is \( \Box_A u = F \) can be viewed as

\[
(1.16) \quad \Box u = F_1 + A \cdot \nabla_{t,r} u.
\]

Then we can rewrite this equation in terms of \( \tau_{\pm} \) and \( \nabla_{\pm} \) (see Section 2), obtaining

\[
(1.17) \quad \nabla_+ \nabla_- v = G.
\]
where
\[(1.18)\] 
\[v(t, r) \doteq ru(t, r) \quad \text{and} \quad G(t, r) \doteq rF_1(t, r).\]

This last equation can be easily integrated to obtain a relatively simple explicit representation of \((\nabla v)(\tau^+, \tau^-)\) in terms of \(G\).

Another fundamental step consists in taking advantage of the gauge invariance property of the electromagnetic potential \(A\), which means that, set
\[(1.19)\] 
\[A_\pm \doteq \frac{A_0 \pm A_1}{2},\]
we can assume, with no loss of generality, that \(A_+ \equiv 0\) (see [2], p. 34). This implies that
\[(1.20)\] 
\[A \cdot \nabla_{t, r} u = A_- \nabla_- u + A_+ \nabla_+ u = A_- \nabla_- u\]
and hence
\[(1.21)\] 
\[F_1 = F + A_- \nabla_- u,\]
thus
\[(1.22)\] 
\[G = rF + rA_- \nabla_- u = rF + A_- \nabla_- v + \frac{1}{r}A_- v.\]

Obviously, one has
\[(1.23)\] 
\[\sum_{j \in \mathbb{Z}} 2^{-j} (2^{-j})^\varepsilon A ||\varphi_j A_-||_{L_\infty} \leq \delta_A.\]

These simplifications, combined with technical Lemma 2.1 and the estimate of Lemma 2.2, allow us to easily obtain Lemma 1.1 and Theorem 1.1.

The dispersive properties of evolution equations are important for their physical meaning and, consequently, they have been deeply studied, though the problem in its generality is still open. The dispersive estimate obtained in Corollary 1.1 provides the natural decay rate, that is the same rate one has for the non–perturbed wave equation (see [11, 13]), i.e. a \(t^{-(n-1)/2}\) decay in time, where \(n\) is the space dimension (in our case, \(n = 3\)). The generalization to the case of a potential–like perturbation has been considered widely (see [1, 3, 4, 5, 7, 10, 14, 16, 17, 18, 19]), also for the Schrödinger equation (see [8, 9, 12, 15]). Recently, D’Ancona and Fanelli have considered in [6] the case
\[(1.24)\] 
\[\begin{cases} 
\partial_t^2 u(t, x) + Hu = 0, \\
u(0, x) = 0, \quad \partial_t u(0, x) = g(x),
\end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3,
\]
where
\[(1.25)\] 
\[H \doteq -(\nabla + iA(x))^2 + B(x),\]
\[(1.26)\] 
\[A : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad B : \mathbb{R}^3 \rightarrow \mathbb{R}.
\]
Under suitable condition on \(A, \nabla A\) and \(B\), in particular
\[(1.27)\] 
\[|A(x)| \leq \frac{C_0}{r\langle r\rangle(1 + |\lg r|)^\beta}, \quad \sum_{j=1}^3 |\partial_j A_j(x)| + |B(x)| \leq \frac{C_0}{r^2(1 + |\lg r|)^\beta},\]
with $C_0 > 0$ sufficiently small, $\beta > 1$ and $r = |x|$, they have obtained the dispersive estimate
\begin{equation}
|u(t,x)| \leq C \sum_{j \geq 0} 2^{2j} \|w_j^{1/2} \varphi_j(\sqrt{H})g\|_{L^2},
\end{equation}
where $w_\beta = r(1 + |\log r|)^\beta$ and $(\varphi_j)_{j \geq 0}$ is a non–homogeneous Paley–Littlewood partition of unity on $\mathbb{R}^3$.

In this paper, restricting ourselves to radial solutions, we are able to obtain the result in Corollary \[1.1\] which is optimal from the point of view of the estimate decay rate $t^{-1}$ and improve essentially the assumptions on the potential, assuming weaker condition \[1.7\] instead of \[1.27\].

2. **A priori estimates**

First of all, we reformulate our problem taking advantage of the radiality of the solution $u$ to \[1.10\]. Indeed, since $\Delta S^2 u(t,r) = 0$ and $v = ru$, we have
\begin{equation}
\Box u(t,r) = (\partial_t^2 - \Delta_x) u = \left( \partial_t^2 - \partial_r^2 - \frac{2}{r} \partial_r - \frac{1}{r^2} \Delta_{S^2} \right) u(t,r)
\end{equation}
\begin{equation}
= \frac{1}{r} \partial_r^2 v(t,r) - \frac{1}{r} \partial^2_r \psi(t,r)
\end{equation}
\begin{equation}
= \frac{1}{r} \nabla_+ \nabla_- v(t,r) = \frac{1}{r} \nabla_- ^2 v(t,r).
\end{equation}
Recalling \[1.18\] and \[1.21\], we get that the equation in \[1.10\] is equivalent to
\begin{equation}
\nabla_+ \nabla_- v = G.
\end{equation}
Let us notice that the support of $u(t,r)$ is contained in the domain \{ $(t,r) \in \mathbb{R}^2 : r > 0, t > r$ \}, therefore we have
\begin{equation}
\text{supp } v(\tau_+, \tau_-) \subseteq \{ (\tau_+, \tau_-) \in \mathbb{R}^2 : \tau_- > 0, \tau_+ > \tau_- \}.
\end{equation}

From this fact, we get
\begin{equation}
\nabla_- v(\tau_+, \tau_-) = \nabla_- v(\tau_-, \tau_-) + \int_{\tau_-}^{\tau_+} G(s, \tau_-) ds = \int_{\tau_-}^{\tau_+} G(s, \tau_-) ds.
\end{equation}
Let us observe that, for each $s \in [\tau_-, \tau_+]$, we have
\begin{equation}
s \leq \tau_+, \quad s - \tau_- \leq \tau_+ - \tau_- = r,
\end{equation}
hence
\begin{equation}
\left| \int_{\tau_-}^{\tau_+} G(s, \tau_-) ds \right| \leq \int_{\tau_-}^{\tau_+} \frac{s(s - \tau_-)^\varepsilon |G(s, \tau_-)|}{\langle s \rangle \langle s - \tau_- \rangle^{-\varepsilon}} ds \leq \| \tau_+ \langle r \rangle^\varepsilon G \|_{L^\infty_r} \int_{\tau_-}^{\tau_+} \langle s \rangle \langle s - \tau_- \rangle^{-\varepsilon} ds
\end{equation}
for every $\varepsilon > 0$. Applying lemma \[2.21\] (see the end of this section), we conclude
\begin{equation}
\tau_+ |\nabla_- v(\tau_+, \tau_-)| \leq Cr \| \tau_+ \langle r \rangle^\varepsilon G \|_{L^\infty_r};
\end{equation}
recalling that $G$ satisfies (1.22), we obtain
\begin{equation}
\tau_+ |\nabla_ v (\tau_+, \tau_-)| \leq C r\left(\|\tau_+ \langle r \rangle^{\varepsilon} A_+ \nabla_ v\|_{L^\infty_t}, + \|\tau_+ \langle r \rangle^{\varepsilon} r^{-1} A_- v\|_{L^\infty_t}
+ \|\tau_+ \langle r \rangle^{\varepsilon} r F\|_{L^\infty_t}\right).
\end{equation}

(2.9)

Now, if we choose for the moment $\varepsilon \leq \varepsilon_A$, we have
\begin{equation}
r\langle r \rangle^{\varepsilon} \varphi_j (r) |A_- (t, r)| \leq C 2^{-j} (2^{-j})^{\varepsilon_A} \|\varphi_j A_-\|_{L^\infty_t}
\end{equation}

(2.10)

(here and in the following, we assume that $C = C(\varepsilon) > 0$ could change time by time), thus
\begin{equation}
r\|\tau_+ \langle r \rangle^{\varepsilon} A_- \nabla_ v\|_{L^\infty_t} \leq C \|\tau_+ \nabla_ v\|_{L^\infty_t} \sum_{j \in \mathbb{Z}} 2^{-j} (2^{-j})^{\varepsilon_A} \|\varphi_j A_-\|_{L^\infty_t}
\end{equation}

(2.11)

\[\leq C \delta_A \|\tau_+ \nabla_ v\|_{L^\infty_t},\]

where we have used the fact that $(\varphi_j)_{j \in \mathbb{Z}}$ is a Paley–Littlewood partition of unity and property (1.23). Moreover, $v(\tau_+, \tau_+) = 0$ because of (2.6), whence
\begin{equation}
v(\tau_+, \tau_-) = - \int_{\tau_-}^{\tau_+} \nabla_ v (\tau_+, s) \, ds
\end{equation}

(2.12)

and consequently
\begin{equation}
|v(\tau_+, \tau_-)| \leq \int_{\tau_-}^{\tau_+} |\nabla_ v (\tau_+, s)| \, ds \leq r \|\nabla_ v\|_{L^\infty_t}.
\end{equation}

(2.13)

Thus we have
\begin{equation}
\langle r \rangle^{\varepsilon} \varphi_j (r) |A_- (t, r) v(\tau_+, \tau_-)| \leq C 2^{-j} (2^{-j})^{\varepsilon_A} \|\varphi_j A_-\|_{L^\infty_t} \|\nabla_ v\|_{L^\infty_t},
\end{equation}

(2.14)

which implies
\begin{equation}
r\|\tau_+ \langle r \rangle^{\varepsilon} r^{-1} A_- v\|_{L^\infty_t} \leq C \delta_A \|\tau_+ \nabla_ v\|_{L^\infty_t}.
\end{equation}

(2.15)

Using (2.11) and (2.15) in (2.9), we deduce
\begin{equation}
|\tau_+ \nabla_ v\|_{L^\infty_t} \leq C \|\tau_+ r^2 \langle r \rangle^{\varepsilon} F\|_{L^\infty_t},
\end{equation}

(2.16)

provided $\delta_A$ is sufficiently small. For instance, one can take $\delta_A$ such that $3C^2 \delta_A \leq 1$.

From the definition of $v$, we have
\begin{equation}
r \nabla_- u = \nabla_- v + u
\end{equation}

(2.17)

and hence
\begin{equation}
|\tau_+ r \nabla_- u| \leq |\tau_+ \nabla_- v| + |\tau_+ u|,
\end{equation}

(2.18)

so
\begin{equation}
|\tau_+ r \nabla_- u|_{L^\infty_t} \leq |\tau_+ \nabla_- v|_{L^\infty_t} + |\tau_+ u|_{L^\infty_t}
\leq C \left(\|\tau_+ r^2 \langle r \rangle^{\varepsilon} F\|_{L^\infty_t} + \|\tau_+ r^2 \langle r \rangle^{\varepsilon} F_1\|_{L^\infty_t}\right),
\end{equation}

(2.19)
Lemma 2.1. and also Theorem 1.1 is proven.

\[ r^2(r^\varepsilon)|F_1| \leq r^2(r^\varepsilon)|A_-| \cdot |\nabla_\varepsilon u| + r^2(r^\varepsilon)|F| \]

(2.19) \[ \leq \left( \sum_{j \in \mathbb{Z}} r(r^\varepsilon A_j |A_-|) \right) r|\nabla_\varepsilon u| + r^2(r^\varepsilon)|F| \]

\[ \leq C\delta A|\nabla_\varepsilon u| + r^2(r^\varepsilon)|F|, \]

thus

(2.20) \[ \| \tau_+ r^2(r^\varepsilon F_1) \|_{L^\infty_{t,r}} \leq C\delta A\| \tau_+ r\nabla_\varepsilon u\|_{L^\infty_{t,r}} + \| \tau_+ r^2(r^\varepsilon F) \|_{L^\infty_{t,r}} \]

and consequently

(2.21) \[ \| \tau_+ r\nabla_\varepsilon u\|_{L^\infty_{t,r}} \leq C\| \tau_+ r^2(r^\varepsilon F) \|_{L^\infty_{t,r}} \]

provided \( \delta_A > 0 \) small enough, that is Lemma 1.1.

Now we use the fact that, because of (2.17), we have

(2.22) \[ |\tau_+ u| \leq |\tau_+ r\nabla_\varepsilon u| + |\tau_+ \nabla v|; \]

combining this estimate with (2.19) and (2.20), we finally conclude

(2.23) \[ \| \tau_+ u \|_{L^\infty_{t,r}} \leq C\| \tau_+ r^2(r^\varepsilon F) \|_{L^\infty_{t,r}}, \]

and also Theorem 1.1 is proven.

**Lemma 2.1.** For each \( \varepsilon > 0 \), there exists a positive constant \( C = C(\varepsilon) \) such that

\[ \int_{\tau_-}^{\tau_+} \langle s \rangle^{-1} \langle s - \tau_- \rangle^{-\varepsilon} ds \leq \frac{C r}{\tau_+} \]

**Proof.** We distinguish two cases.

**Case 1:** \( \tau_+ \geq 2 \tau_- \). Let us notice that since \( r = \tau_+ - \tau_- \geq \tau_+/2 \), in this case it is sufficient to prove that

(2.24) \[ \int_{\tau_-}^{\tau_+} \langle s \rangle^{-1} \langle s - \tau_- \rangle^{-\varepsilon} ds \leq C_0(\varepsilon). \]

We observe that \( s - \tau_- \geq s/2 \) provided \( s \geq 2 \tau_- \), so

\[ \int_{\tau_-}^{\tau_+} \langle s \rangle^{-1} \langle s - \tau_- \rangle^{-\varepsilon} ds \leq \int_{\tau_-}^{\tau_- + 1} \langle s \rangle^{-1} ds + 2^\varepsilon \int_{\tau_- + 1}^{\tau_+ + 1} s^{-(1+\varepsilon)} ds \]

\[ \leq \frac{1}{\tau_-} + 2^\varepsilon \int_1^{\infty} s^{-(1+\varepsilon)} ds \]

\[ \leq 1 + C_1(\varepsilon). \]

**Case 2:** \( \tau_+ < 2 \tau_- \). We use the estimates \( \langle s \rangle^{-1} \leq 2/\tau_+ \) and \( \langle s - \tau_- \rangle^{-\varepsilon} \leq 1 \) to get

(2.25) \[ \int_{\tau_-}^{\tau_+} \langle s \rangle^{-1} \langle s - \tau_- \rangle^{-\varepsilon} ds \leq \frac{2}{\tau_+} (\tau_+ - \tau_-) = \frac{2r}{\tau_+}. \]

This concludes the proof. \( \blacksquare \)

In the case \( A \equiv 0 \) (non–perturbed equation), we have the following version of the estimate in Theorem [11]. It consists in a slight modification of estimate (1.8) shown in [7], p. 2269.
Lemma 2.2. Let $u$ be the solution to
\begin{equation}
\begin{aligned}
\Box u &= F \\
u(0, r) &= \partial_t u(0, r) = 0 \\
(t, r) &\in [0, \infty) \times \mathbb{R}^+, \\
r &\in \mathbb{R}^+.
\end{aligned}
\end{equation}
Then, for every $\varepsilon > 0$, there exists $C > 0$ such that
\begin{equation}
\left\| \tau_+ \frac{r}{t} u \right\|_{L^\infty_t L^\infty_r} \leq C \left\| \tau_+ \frac{r^2}{t} \frac{1}{r^{\varepsilon}} F \right\|_{L^\infty_t L^\infty_r}.
\end{equation}

**Proof.** Let notice that $u$ is the solution to (1.10) with $A \equiv 0$. Then, from (2.9), we have
\begin{equation}
\tau_+ |\nabla - v| (\tau_+, \tau_-) \leq C \left\| \tau_+ \frac{r^2}{t} \frac{1}{r^{\varepsilon}} F \right\|_{L^\infty_t L^\infty_r},
\end{equation}
where $v = ru$. Using (2.13), we deduce
\begin{equation}
\tau_+ |u| = \tau_+ |v| r^{-1} \leq \left\| \tau_+ \nabla - v \right\|_{L^\infty_t L^\infty_r}
\end{equation}
and hence the claim. ■

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