R–torsion and linking numbers from simplicial abelian gauge theories

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Abstract

Simplicial versions of topological abelian gauge theories are constructed which reproduce the continuum expressions for the partition function and Wilson expectation value of linked loops, expressible in terms of R–torsion and linking numbers respectively. The new feature which makes this possible is the introduction of simplicial fields (cochains) associated with the dual triangulation of the background manifold, as well as with the triangulation itself. This doubling of fields, reminiscent of lattice fermion doubling, is required because the natural simplicial analogue of the Hodge star operator maps between cochains of a triangulation and cochains of the dual triangulation. The simplicial analogue of Hodge–de Rham theory is developed, along with a natural simplicial framework for considering linking numbers of framed loops. When the loops represent torsion elements of the homology of the manifold then \( \mathbb{Q}/\mathbb{Z} \)-valued torsion pairings appear in place of linking numbers for certain discrete values of the coupling parameter of the theory.
1 Introduction

In this paper we consider the problem of discretising abelian Chern–Simons gauge theory on closed 3-manifolds, and its generalisation to arbitrary odd-dimensional closed manifolds, in such a way that the topological objects of interest in the theories are equal to the corresponding objects in the discrete theories. The topological objects of interest are the partition function and Wilson vacuum expectation values (v.e.v.’s) of linked loops. As is well-known, the partition functions (suitably gauge-fixed and analytically regularised) can be expressed in terms of analytic Ray–Singer torsion [Sc], while the Wilson v.e.v.’s of linked loops (and their generalisations to higher dimensions) can be expressed in terms of the linking numbers of the loops [Po, Wit1, §2].

The Ray–Singer torsion [RS] of a manifold has a combinatorial analogue, namely the Reidemeister–Franz torsion [Re, Fra, Mi, RS], also known as the R–torsion, defined via a triangulation of the manifold. The cochain complex of the triangulation (with values in \( \mathbb{R} \), or more generally in a flat vector bundle) plays a role in the construction of the R–torsion which is analogous to the role of the de Rham complex in the construction of the Ray–Singer torsion. These torsions are in fact known to be equal [Miû, Ch]. This leads us to consider discrete versions of abelian Chern–Simons theory (and its generalisations) where the fields are cochains of a triangulation of the background manifold. The aim is to find such a theory where the partition function is expressible in terms of R–torsion (in place of Ray–Singer torsion), and the Wilson v.e.v.’s of linked loops are expressible in terms of the linking numbers, as in the continuum theory.

Discrete versions of abelian Chern–Simons theory have been considered previously by a number of authors [ES, FGP], including setups where the gauge field is taken to be a cochain of a triangulation [KM, BR, AlSc], although it seems that no explicit attempt has been made previously to carry out the aims outlined above. The new feature of our approach, which allows to carry out these aims, is the introduction of

\footnote{These properties have been checked in [CM], where the abelian Chern–Simons theory on \( S^3 \) was explicitly solved.}

\footnote{A way of obtaining linking numbers from a lattice version of abelian Chern-Simons theory is
simplicial fields (cochains) associated with the dual of the triangulation, as well as with the triangulation itself\(^3\). The motivation behind this is intimately connected with the nature of the simplicial analogue of the Hodge star operator. In the continuum abelian Chern–Simons theory the Hodge star operator, and its interrelationship with the exterior derivative (and the adjoint of the exterior derivative), play a crucial role in deriving the expressions for the partition function and Wilson v.e.v.'s of loops. If the same evaluation of the partition function in a cochain version of the theory is to lead to an analogous expression with R–torsion in place of Ray–Singer torsion then there must be a simplicial analogue of the Hodge star operator such that its interrelationship with the cochain action functional and cochain derivative (simplicial analogue of the exterior derivative) are the same as in the continuum case. In fact there is a natural simplicial analogue of the Hodge star operator, namely the duality operator mapping between cochains of the triangulation and cochains of the dual of the triangulation. (The cochains we are considering are over the reals.) After taking into account the fact that this operator maps between different cochain complexes, its interrelationship with the derivative map for cochains of the triangulation and the derivative map for cochains of the dual triangulation is completely analogous to the interrelationships in the continuum case. (We will see this explicitly in proposition 2.4.) This indicates that if a cochain version of abelian Chern–Simons theory is to have its partition function expressible in terms of R–torsion in the desired way then the theory should involve cochains of the dual triangulation as well as of the triangulation itself. This is reminiscent of the field doubling in lattice fermion theories (see e.g. \([BJ]\)).

Motivated by these considerations we consider discretisation of the double abelian Chern–Simons theory (and its generalisation to higher dimensions) with two independent gauge fields, such that in the discrete version one gauge field is a cochain of the triangulation and the other is a cochain of the dual triangulation. We show discussed in the recent preprint ref. \([FGP]\), although the setup and techniques used there are quite different from ours.

\(^3\)The dual triangulation has appeared previously in connection with discrete abelian \(\mathbb{Z}_p\) field theories, although not in the way that we describe here; see \([Ra]\) and the ref.'s therein. Duality has previously played a key role in a variety of lattice field theories, see e.g. \([Ca, FM]\).
that, after making a certain “twist” in the action functional which couples the fields but leaves the expressions for the partition function and Wilson v.e.v.’s of loops unchanged\textsuperscript{4}, there is a canonical discretisation (i.e. cochain version) of this theory such that the interrelationship between the discrete (i.e. simplicial) action functional, the simplicial Hodge star operator (i.e. the duality operator) and the cochain derivatives is the same as in the continuum case. As a consequence the evaluation of the partition function gives an expression involving R–torsion in the desired way, reproducing the continuum expression\textsuperscript{5}. We consider Wilson v.e.v.’s of framed linked loops (and their generalisation to higher dimensions) which fit into this discrete setup in a natural way: The framed loops are taken to be ribbons with one boundary being an edge loop in the triangulation and the other boundary being an edge loop in the dual triangulation. (We call these framed loops simplicial framings of edge loops.) We find that the Wilson v.e.v.’s of these linked, simplicially framed loops can be expressed in terms of the linking numbers of the loops, again reproducing the continuum expression\textsuperscript{6}.

The key tool in constructing the discretisation is the Whitney map \cite{Wh}, a canonical map from cochains to differential forms on the background manifold\textsuperscript{7}. This map has previously been used to construct simplicial versions of quantum field theories in \cite{AlZe}, and was used to construct a simplicial version of the abelian Chern–Simons theory in \cite[§5]{AlSc}. In both cases the emphasis was on proving convergence of the simplicial theory to the continuum limit –the problem of reproducing topological quantities was not addressed. There is a crucial difference between the way the Whitney map is used in \cite{AlZe, AlSc} and the way we use it here: We embed the cochains of the triangulation and the cochains of its dual into the space of cochains of the

\textsuperscript{4}Except for the removal of a metric-dependent phase in the partition function.

\textsuperscript{5}The continuum expression is reproduced when the value of the coupling parameter is \(\lambda = 1\). For \(\lambda \neq 1\) a triangulation-dependent renormalisation of \(\lambda\) is required to reproduce the continuum expression.

\textsuperscript{6}This is true for all values of the coupling parameter \(\lambda\) –no triangulation-dependent renormalisation is required in this case.

\textsuperscript{7}This map played a key role in establishing the equality between R–torsion and Ray–Singer torsion \cite{Dr, Mil}. 

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barycentric subdivision of the triangulation, and then use the Whitney map on the 
cochains of the barycentric subdivision to obtain differential forms on the background 
manifold. (In [AlZe, AlSc] the Whitney map was used directly on the cochains of the 
triangulation –the dual triangulation and barycentric subdivision did not enter.)

It turns out that the doubled, “twisted” abelian Chern–Simons theory which we 
discretise is identical to the abelian BF–theory for two independent gauge fields. 
Thus we end up with a canonical discrete version of abelian BF–theory for which the 
partition function and Wilson v.e.v.’s of linked framed loops are expressed in terms 
of \( R \)–torsion and linking numbers respectively, and are equal to the corresponding 
objects in the continuum theory. (Note that this equality does not require a continuum 
limit –it holds exactly for each triangulation of the background manifold).

The linking numbers \( \text{lk}(\gamma^{(1)}, \gamma^{(2)}) \) of bounding loops \( \gamma^{(1)}, \gamma^{(2)} \) enter in the expres-
sions for the Wilson v.e.v.’s of loops through factors of the form

\[
\exp \left( \frac{i\pi^2}{\lambda} n_1 n_2 \text{lk}(\gamma^{(1)}, \gamma^{(2)}) \right)
\]

where \( \lambda \) is the coupling parameter of the theory and \( n_1 \) and \( n_2 \) are integers. These 
expressions are trivial when the coupling parameter takes the discrete values \( \lambda = \frac{\pi}{2p} \), 
\( p \in \mathbb{Z} \). We show that non-trivial expressions for the Wilson v.e.v.’s can be obtained 
at these discrete values of \( \lambda \) in the more general case where the loops may represent 
torsion elements of the \( \mathbb{Z} \)–homology of the manifold. (Bounding loops represent the 
zero-element in this homology.) There is a canonical \( \mathbb{Q}/\mathbb{Z} \)–valued pairing between 
torsion elements (of appropriate degree) in the \( \mathbb{Z} \)–homology of \( M \), and we show 
(proposition 6.1) that the pairing of torsion elements represented by loops can be 
expressed as a generalisation of linking number. As a consequence \( \mathbb{Q}/\mathbb{Z} \)–valued 
torsion pairings appear in place of \( \mathbb{Z} \)–valued linking numbers in the expressions for 
the Wilson v.e.v.’s in this case.

The mathematical results which we require are derived in \( \S \)2–3. In \( \S \)2 the simplicial 
analogue of Hodge–de Rham theory is developed, culminating in the main result,
theorem 2.6, which gives a natural simplicial analogue of the following basic formula:

\[ \int_M d\omega \wedge \tau = \langle *d\omega, \tau \rangle \]  

(1.1)

where \( \omega \) and \( \tau \) are differential forms on a closed manifold \( M \) and the inner product \( \langle \cdot, \cdot \rangle \) and Hodge star operator \( * \) are determined by a metric on \( M \). In §3 a natural simplicial framework for considering linking numbers of framed loops is developed.

The first main result, theorem 3.3, gives a formula for the linking number of an edge loop in a triangulation \( K \) and an edge loop in the dual triangulation \( \hat{K} \). More generally, this result is a formula for the linking number \( \text{lk}(f_K, g_{\hat{K}}) \) of a simplicial map \( f_K \) from a triangulated closed manifold of dimension \( p \) and a dual-simplicial map \( g_{\hat{K}} \) from a triangulated closed manifold of dimension \( q \) into a triangulated manifold \( M \) of dimension \( n = p + q + 1 \). A particular class of framings of edge loops –namely the simplicial framings described above– is introduced, and it is proved that such framings always exist (theorem 3.6). This is used in proposition 3.9 to give a formula for the linking number of edge loops in \( K \) in the simplicial setup: The linking number is expressed in terms of disjoint simplicial framings of the loops, and is shown to be independent of the choice of framings. (This is the formula which appears in the expression for the Wilson v.e.v.’s of linked framed loops in the discrete abelian gauge theory). In §4 we review the formal evaluation of the partition function and Wilson v.e.v.’s of loops in abelian Chern–Simons theory and its associated BF theory (including features associated with the moduli space of flat \( U(1) \) gauge fields which appear when \( \pi_1(M) \) is non-trivial). In §5 we apply the results of §2–3 to construct the simplicial version of the abelian BF gauge theory described above, and show that it produces the R–torsion and linking numbers as claimed, leading to agreement with the continuum expressions for the partition function and v.e.v.’s obtained in §4. A group of simplicial gauge transformations is introduced which enables the features of the continuum theory involving the modulilspace of flat \( U(1) \) gauge fields to also be reproduced in the simplicial theory. In §6 we consider Wilson v.e.v.’s of loops representing torsion elements of the homology of \( M \), and show that in this case for certain discrete values of the coupling parameter the v.e.v. is a purely homological
quantity, expressible in terms of torsion pairings (as described above).

Background on the mathematical objects and constructions appearing in this paper can be found for example in [DFN] [Mun].

The techniques and results of this paper are of a general nature and we expect them to have application in a variety contexts. These may include lattice fermion doubling [B], a new approach to lattice gauge theory based on non-commutative geometry [BBLT-S] developing discrete versions of theories for topologically massive (abelian) gauge fields [DJT], anyons and high T superconductivity [Wil, Po], quantum gravity in the loop variable approach [8] and reproducing the S–duality in abelian gauge theory demonstrated by Witten [Wit2]. Since abelian Chern–Simons theory can be considered as the weak coupling limit of the non-abelian theory this paper may provide a pointer to how to discretise the non-abelian Chern–Simons theory, or rather its associated BF theory. The aim in this case would be to reproduce the combinatorial 3-manifold invariant of Turaev and Viro [TV] from the partition function, and the (generalised) Jones polynomial and related knot invariants from the Wilson v.e.v.’s (see [CC-RFM] for a discussion of these in the continuum case).

2 Simplicial analogue of Hodge–de Rham theory

In this section we develop a simplicial analogue of Hodge–de Rham theory in which the simplicial analogue of the Hodge star operator is the duality operator between cochains of a triangulation and cochains of the dual triangulation (over the reals).

Let $M$ be a smooth closed oriented manifold of dimension $n$, and let $K$ be a simplicial complex smoothly triangulating $M$. If $\{v_0, v_1, \ldots, v_p\}$ are the vertices of a $p$-simplex in $K$ we denote the oriented $p$-simplex with orientation specified by the

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8 The possible applicability of our work in this context was pointed out by Prof. A.P. Balachandran in a discussion.

9 Linking numbers have recently appeared in this context [As] and work aiming at a lattice formulation is also in progress [FGP].

10 I thank Prof. G. Segal for pointing out this reference.
ordering $v_0, \ldots, v_p$ by $\alpha = \alpha^{(p)} = [v_0, \ldots, v_p]$, and denote the closed cell in $M$ corresponding to $\alpha$ by $|\alpha|$. We write $\alpha^{(p)} < \beta^{(q)}$ to denote that the $p$-simplex $\alpha^{(p)}$ is a face in the $q$-simplex $\beta^{(q)}$ ($p < q$). The vectorspace $C_p(K)$ is the space of formal finite linear combinations ("$p$-chains") of oriented $p$-simplices of $K$ over the reals with the rule that, as an element of $C_p(K)$, an oriented $p$-simplex changes sign under a change of orientation, i.e. $[v_{\tau(0)}, \ldots, v_{\tau(p)}] = (-1)^\tau [v_0, \ldots, v_p]$ for permutation $\tau$ of $\{0, \ldots, p\}$. The vectorspace $C^p(K)$ of $p$-cochains is the dual of $C_p(K)$. There is a canonical inner product in $C_p(K)$ determined by requiring that the oriented $p$-simplices be orthonormal; this gives an identification $C_p(K) \cong C^p(K)$ and we will consider oriented $p$-simplices as elements of $C^p(K)$ as well as $C_p(K)$. The boundary operator $\partial^K : C_p(K) \to C_{p-1}(K)$ is the linear operator which maps an oriented $p$-simplex $\alpha^{(p)}$ to the sum of its $(p-1)$-faces with orientations induced by the orientation of $\alpha^{(p)}$. The coboundary operator $d^K : C^p(K) \to C^{p+1}(K)$ is the adjoint of $\partial^K$; thus $d^K [v_0, \ldots, v_p] = \sum_v [v, v_0, \ldots, v_p]$ where the sum is over all vertices $v$ such that $[v, v_0, \ldots, v_p]$ is a $(p+1)$-simplex. These operators have the property $\partial^K \partial^K = 0$ and $d^K d^K = 0$.

The star $\text{St}(\alpha)$ of a $p$-simplex $\alpha$ is the open region in $M$ consisting of the union of the interiors of all simplices (of any degree) which contain $\alpha$ as a face. Recall that the barycentric coordinate function $\mu_v : M \to \mathbb{R}$ corresponding to a vertex $v$ of $K$ is the function equal to one at $v \in M$, decreasing linearly (in a suitable sense) to zero at the boundary of $\text{St}(v)$ and vanishing outside $\text{St}(v)$. It has the properties

$$\sum_v \mu_v = 1 \quad \text{and} \quad \sum_{v \in \alpha} \mu_v(y) = 1 \quad \forall y \in |\alpha| \quad \forall \alpha \in K \quad (2.1)$$

Let $\Omega^p(M)$ denote the vectorspace of $\mathbb{R}$--valued $p$-forms on $M$ which are smooth on the complement of a set of measure zero, and let $d : \Omega^p(M) \to \Omega^{p+1}(M)$ denote the exterior derivative. The Whitney map $W^K : C^p(K) \to \Omega^p(M)$ is the linear map defined by $[\text{Wh}]$ $[\text{Do}]$

$$W^K(\alpha) = p! \sum_{i=0}^p (-1)^i \mu_i d\mu_0 \wedge \cdots \wedge d\mu_{i-1} \wedge d\mu_{i+1} \wedge \cdots \wedge d\mu_p \quad \text{for} \quad \alpha = [v_0, \ldots, v_p] \quad (2.2)$$

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\( W^K(v) = \mu_v \) where \( \mu_i := \mu_{v_i} \). The functions \( \mu_v \) are smooth on the complement of the \((n-1)\)-skeleton of \( K \), a set of measure zero in \( M \), so the same is true of \( W^K(\alpha) \). Clearly \( W^K \) vanishes outside of \( \text{St}(\alpha) \) and on the boundary of \( \text{St}(\alpha) \). Recall that the de Rham map \( A^K : \Omega^p(M) \to C^p(K) \) is the linear map defined by \( < A^K(\omega), \alpha > = \int_{|\alpha|} \omega \) for each oriented \( p \)-simplex \( \alpha \in K \). The maps \( W^K \) and \( A^K \) have the properties

\[
A^K W^K = \text{Id} \quad \text{(the identity)} \quad \int_{|\beta|} W^K(\alpha) = < \alpha, \beta > \quad (2.3)
\]
\[
d W^K = W^K d^K \quad \quad d^K A^K = A^K d \quad (2.4)
\]
(see [Wh], [Do]). It follows from (2.3) that \( W^K \) is injective.

**Simplicial wedge product.** We define the bilinear product \( \wedge^K : C^p(K) \times C^q(K) \to C^{p+q}(K) \) by

\[
x \wedge^K y := A^K(W^K(x) \wedge W^K(y)) \quad (2.5)
\]

The following properties follow easily from (2.3)–(2.4):

(i) Skewsymmetry: \( x \wedge^K y = (-1)^{pq} y \wedge^K x \)

(ii) Leibniz rule: \( d^K(x \wedge^K y) = d^K x \wedge^K y + (-1)^p x \wedge^K d^K y \)

However, as we will see below, \( \wedge^K \) is not associative. This product, with \( K \) replaced by its barycentric subdivision \( BK \), will play a key role in proving our results in the following. It has an explicit description given in the proposition below, which will allow us to translate differential-geometric problems into combinatorial ones.

**Convention 2.1.** If \( \{v_0, \ldots, v_p\} \) are vertices in \( K \) but are not the vertices of a \( p \)-simplex then we set \( [v_0, \ldots, v_p] := 0 \) in \( C^p(K) \).

With this convention we have

**Proposition 2.2.** \( [v_0, \ldots, v_p] \wedge^K [w_0, \ldots, w_q] = 0 \) if the vertices \( \{v_0, \ldots, v_p\} \) and \( \{w_0, \ldots, w_q\} \) do not have precisely one element in common, and

\[
[v_0, \ldots, v_p] \wedge^K [v_p, \ldots, v_{p+q}] = \frac{p! q!}{(p + q + 1)!} [v_0, \ldots, v_{p+q}] . \quad (2.6)
\]

**Proof.** The proposition obviously holds if either \( \alpha = [v_0, \ldots, v_p] \) or \( \beta = [w_0, \ldots, w_q] \) are not simplices in \( K \). Assume that \( \alpha \) and \( \beta \) are simplices. If the sets \( \{v_0, \ldots, v_p\} \)
and \{w_0, \ldots, w_q\} have no common elements, or have one element in common but their union is not the vertices of a \((p+q)\)-simplex, then \(\text{St}(\alpha) \cap \text{St}(\beta)\) is disjoint from all \((p+q)\)-simplices of \(K\), so the restriction of \(W^K(\alpha) \wedge W^K(\beta)\) to any \((p+q)\)-simplex of \(K\) (considered as a region in \(M\)) vanishes, hence \(\alpha \wedge^K \beta = 0\). If the sets have two or more elements in common then \(W^K(\alpha) \wedge W^K(\beta)\) is a sum of terms with each term containing a factor \(d\mu_i \wedge d\mu_i\) where \(\mu_i\) is the barycentric coordinate function of one of the shared vertices. This vanishes, hence \(\alpha \wedge^K \beta = 0\). When \(\beta = [v_p, \ldots, v_{p+q}]\) the only possibility for a \((p+q)\)-simplex meeting \(\text{St}(\alpha) \cap \text{St}(\beta)\) is \([v_0, \ldots, v_{p+q}]\), so \([v_0, \ldots, v_p] \wedge^K [v_p, \ldots, v_{p+q}]\) is proportional to \([v_0, \ldots, v_{p+q}]\). To complete the proof we calculate

\[
< [v_0, \ldots, v_p] \wedge^K [v_p, \ldots, v_{p+q}], [v_0, \ldots, v_{p+q}] > = \int_{[v_0,\ldots,v_{p+q}]} W^K([v_0,\ldots,v_p]) \wedge W^K([v_p,\ldots,v_{p+q}])
\]

\[
= \int_{\{0 \leq \mu_i \leq 1, 0 \leq \mu_1 + \ldots + \mu_{p+q} \leq 1\}} plq! \mu_1 \wedge \cdots \wedge d\mu_{p+q}
\]

\[
= \frac{plq!}{(p+q+1)!}
\]

The proposition shows that \(\wedge^K\) is an “antisymmetrisation” of the usual cup product. In this form it has appeared previously in the literature in connection with simplicial discretisation of abelian Chern-Simons theory \[BR\] \[AlSc\]. It also follows that \(\wedge^K\) is non-associative: For \(x \in \mathcal{C}_p(K), y \in \mathcal{C}_q(K), z \in \mathcal{C}_r(K)\)

\[
x \wedge^K (y \wedge^K z) = \frac{(p+q+1)}{(r+q+1)} (x \wedge^K y) \wedge^K z
\]

(2.7) is an easy consequence of (2.6).

Let \(BK\) denote the barycentric subdivision of \(K\) and let \(\tilde{K}\) denote the dual triangulation, i.e. the cell decomposition of \(M\) dual to \(K\). These will play central roles in the following, and can be characterised as follows. The vertices of \(BK\) are \(\{\tilde{\alpha} \mid \alpha \in K\}\) where \(\tilde{\alpha}\) denotes the barycenter of the simplex \(\alpha \in K\). The oriented \(p\)-simplices of \(BK\) are

\[
\{[\tilde{\alpha}^{(q_0)}, \ldots, \tilde{\alpha}^{(q_p)}] \mid \alpha^{(q_0)} < \alpha^{(q_1)} < \cdots < \alpha^{(q_p)} \in K\}.
\]
For each oriented $p$-simplex $\alpha^{(p)} \in K$ we denote the oriented $(n - p)$-cell in $M$ dual to $\alpha^{(p)}$ by $\hat{\alpha}^{(p)}$. It is the cell corresponding to the union of all $(n - p)$-simplices in $BK$ of the form $[\hat{\alpha}^{(p)}, \ldots, \hat{\alpha}^{(n)}]$ where $\alpha^{(p)} < \alpha^{(p+1)} < \cdots < \alpha^{(n)} \in K$. The orientation of $\hat{\alpha}^{(p)}$ is uniquely determined by the orientations of $\alpha^{(p)}$ and $M$ as follows. If $\alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(p)} < \cdots < \alpha^{(n)} \in K$ is such that $[\hat{\alpha}^{(0)}, \ldots, \hat{\alpha}^{(p)}]$ is compatible with the orientation of $\alpha^{(p)}$ and $[\hat{\alpha}^{(0)}, \ldots, \hat{\alpha}^{(n)}]$ is compatible with the orientation of $M$ then the orientation of $\hat{\alpha}^{(p)}$ is compatible with that of $[\hat{\alpha}^{(p)}, \ldots, \hat{\alpha}^{(n)}]$. The dual triangulation is the cell decomposition of $M$ given by $\tilde{K} = \{ \hat{\alpha} \mid \alpha \in K \}$.

The complexes $\{C_*(BK), \partial BK\}, \{C^*(BK), d_{BK}\}$ and $\{C_*(\tilde{K}), \partial \tilde{K}\}, \{C^*(\tilde{K}), d_{\tilde{K}}\}$ are defined analogously to $\{C_*(K), \partial K\}, \{C^*(K), d_K\}$. We define injective linear maps

$$B : C^p(K) \hookrightarrow C^p(BK), \quad B : C^p(\tilde{K}) \hookrightarrow C^p(BK)$$

as follows. For oriented $p$-simplex $\alpha^{(p)} \in K$ we set

$$B\alpha^{(p)} := \frac{1}{N(\alpha^{(p)})} \sum_{\alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(p)}} \pm [\hat{\alpha}^{(0)}, \hat{\alpha}^{(1)}, \ldots, \hat{\alpha}^{(p)}]$$

(2.9)

where $N(\alpha^{(p)}) (= p!)$ is the number of terms in the sum and the sign $\pm$ is chosen so that $\pm [\hat{\alpha}^{(0)}, \ldots, \hat{\alpha}^{(p)}]$ has orientation compatible with $\alpha^{(p)}$, and set

$$B\hat{\alpha}^{(p)} := \frac{1}{N(\alpha^{(p)})} \sum_{\alpha^{(p)} < \alpha^{(p+1)} < \cdots < \alpha^{(n)}} \pm [\hat{\alpha}^{(p)}, \hat{\alpha}^{(p+1)}, \ldots, \hat{\alpha}^{(n)}]$$

(2.10)

where $N(\alpha^{(p)})$ is the number of terms in the sum and the sign $\pm$ is chosen so that $\pm [\hat{\alpha}^{(p)}, \ldots, \hat{\alpha}^{(n)}]$ has orientation compatible with $\hat{\alpha}^{(p)}$. These maps are orthogonality-preserving, i.e. $< x_1, x_2 > = 0$ implies $< Bx_1, Bx_2 > = 0$, and the images of $C^p(K)$ and $C^p(\tilde{K})$ under $B$ are orthogonal in $C^p(BK)$.

**Simplicial Hodge star operator.** We define the linear operators $*^K : C^p(K) \rightarrow C^{n-p}(\tilde{K})$ and $*^{\tilde{K}} : C^q(\tilde{K}) \rightarrow C^{n-q}(K)$ by

$$< *^K x, y > := \frac{(n + 1)!}{p!(n - p)!} \int_M W^{BK}(Bx) \wedge W^{BK}(By)$$

(2.11)

$$< *^{\tilde{K}} y, x > := \frac{(n + 1)!}{p!(n - p)!} \int_M W^{BK}(By) \wedge W^{BK}(Bx)$$

(2.12)
for \( x \in C^p(K), y \in C^{n-p}(\hat{K}) \). These definitions are analogous to the definition of the Hodge star operator on differential forms (modulo the numerical factors). The operator \( \ast^K \) coincides with the usual duality operator:

**Proposition 2.3.** Let \( \alpha^{(p)} \in K \) be an oriented \( p \)-simplex with dual cell \( \widehat{\alpha^{(p)}} \in \hat{K} \). Then

\[
\ast^K \alpha^{(p)} = \widehat{\alpha^{(p)}} \quad \text{and} \quad \ast^K (\alpha^{(p)}) = (-1)^{p(n+1)} \alpha^{(p)}.
\]

**Proof.** Using (2.4) the operator \( \ast^K \) can be expressed in terms of the simplicial wedge product for \( C^*(BK) \):

\[
< \ast^K x, y > = \frac{(n+1)!}{p!(n-p)!} < (Bx) \wedge^K (By), [M]_{BK} >
\]

where \([M]_{BK}\) is the orientation \( n \)-cocycle of \( M \) in \( C^n(BK) \), i.e. the sum of all \( n \)-simplices in \( BK \) with orientations induced by that of \( M \). The proposition is now an easy consequence of proposition 2.2 (with \( K \) replaced by \( BK \)) together with the characterisations of \( BK \) and \( \hat{K} \) given above and the definition of the maps \( B \).

**Proposition 2.4.**

(i) \((\ast^K)^* = (\ast^K)^{-1}\) and \((\ast^K)^* = (\ast^K)^{-1}\)

(ii) \(\ast^K_{n-p} \ast^K_p = (-1)^{p(n+1)} \text{Id} \) and \(\ast^K_{n-q} \ast^K_q = (-1)^{q(n+1)} \text{Id} \)

(iii) \((d^K_p)^* = (-1)^{np+1} \ast^K_{n-p} d^K_{n-p-1} \ast^K_{p+1} \) and \((d^K_q)^* = (-1)^{nq+1} \ast^K_{n-q} d^K_{n-q-1} \ast^K_{q+1} \)

**Proof.** Parts (i) and (ii) are immediate consequences of proposition 2.3. To prove (iii) we derive a formula for \( (d^K)^* = \partial^K \). For \( \alpha^{(p)} = [v_0, \ldots, v_p] \in K \) using proposition 2.3 we find

\[
\partial^K \ast^K \alpha^{(p)} = \partial^K \widehat{\alpha^{(p)}} = \sum_{\alpha^{(p+1)} > \alpha^{(p)}} \alpha^{(p+1)} = \ast^K \left( \sum_{\alpha^{(p+1)} > \alpha^{(p)}} \alpha^{(p+1)} \right)
\]

\[
= \ast^K \left( \sum_{v_{p+1}} [v_0, \ldots, v_p, v_{p+1}] \right) = (-1)^{p+1} \ast^K \left( \sum_{v_{p+1}} [v_{p+1}, v_0, \ldots, v_p] \right)
\]

\[
= (-1)^{p+1} \ast^K d^K \alpha^{(p)}
\]

i.e. \((d^K_{n-p-1})^* \ast^K = \partial^K_{n-p} \ast^K = \ast^K_{p+1} d^K_p \). Part (iii) follows easily from this formula and (ii) and (i).
After taking into account the fact that $\ast^K$ and $\ast^{\tilde{K}}$ map between different cochain complexes the formulae in proposition 2.4 are completely analogous to the formulae relating the Hodge star operator on differential forms to the exterior derivative and its adjoint. A simple consequence of this proposition is Poincare duality: the operator $\ast^K$ provides an isomorphism between the cohomology spaces of $d^K$ and $d^{\tilde{K}}$. Of course, this is already well-known since $\ast^K$ coincides with the usual duality operator. The novel aspect of the preceding is that we have seen (in proposition 2.3) how $\ast^K$ arises as a simplicial analogue of the Hodge star operator, which in turn shows how Poincare duality arises as the simplicial analogue of Hodge duality (via proposition 2.4). In contrast, the standard construction of the duality operator (see e.g. [Mun, §67]) is via the cup and cap products, which do not have natural analogues in Hodge–de Rham theory. (For example, cup and cap products require an ordering of the vertices of $K$).

Our aim in the remainder of this section is to derive a simplicial analogue of the basic Hodge–de Rham theoretic formula (1.1). This is theorem 2.6 below, which will allow us to construct a simplicial action functional with the desired properties in §5.

Proposition 2.5. Consider

$$< d^K[v_0, \ldots, v_p] \wedge^K [w_{p+1}, \ldots, w_n], [M]_K >$$

where $[M]_K$ is the orientation $n$-cocycle of $M$ in $C^n(K)$. This equals $\pm (-1)^p+1 \left[ \frac{n+1}{p+1} \right] (-1)$ if $[v_0, \ldots, v_p, w_{p+1}, \ldots, w_n]$ is an $n$-simplex in $K$, with sign $\pm$ determined by its orientation relative to $M$, and vanishes otherwise.

Proof. Since $d^K[v_0, \ldots, v_p] = \sum_{w_{p+1}} [v_{p+1}, v_0, \ldots, v_p]$ we see from proposition 2.2 that (2.13) is potentially non-vanishing in the following two cases: (i) $\{v_0, \ldots, v_p\}$ and $\{w_{p+1}, \ldots, w_n\}$ have precisely one element in common, or (ii) $\{v_0, \ldots, v_p, w_{p+1}, \ldots, w_n\}$ are the vertices of an $n$-simplex in $K$. We must show that (2.15) vanishes in case (i). In this case we can assume without loss of generality that $[w_{p+1}, \ldots, w_n] = [v_p, v_{p+1}, \ldots, v_{p-1}]$. Then, by proposition 2.2,

$$d^K[v_0, \ldots, v_p] \wedge^K [v_p, \ldots, v_{n-1}] = \sum_w [w, v_0, \ldots, v_p] \wedge^K [v_p, \ldots, v_{n-1}] = \frac{(p+1)!(n-p-1)!}{(n+1)!} \sum_w [w, v_0, \ldots, v_{n-1}]$$
and
\[ [v_0, \ldots, v_p] \wedge^K d^K[v_p, \ldots, v_{n-1}] = \sum_w [v_0, \ldots, v_p] \wedge^K [w, v_p, \ldots, v_{n-1}] \]
\[ = (-1)^p (p+1)!(n-p-1)! \left( \frac{n-1}{n+1} \right) \sum_w [w, v_0, \ldots, v_{n-1}] \]
so that, using the Leibniz rule, we get
\[ d^K[v_0, \ldots, v_p] \wedge^K [v_p, \ldots, v_{n-1}] = \frac{1}{2} d^K([v_0, \ldots, v_p] \wedge^K [v_p, \ldots, v_{n-1}]). \]

It follows that
\[ (2.15) = \frac{1}{2} < [v_0, \ldots, v_p] \wedge^K [v_p, \ldots, v_{n-1}], (d^K)^*[M]_K > = 0 \]
since \((d^K)^*[M]_K = \partial^K[M]_K = 0\). To complete the proof we calculate (2.15) in case (ii):
\[ < d^K[v_0, \ldots, v_p] \wedge^K [v_{p+1}, \ldots, v_n], [M]_K > \]
\[ = \sum_{k=1}^{n-p} < [v_{p+k}, v_0, \ldots, v_p] \wedge^K [v_{p+1}, \ldots, v_n], [M]_K > \]
\[ = \sum_{k=1}^{n-p} (-1)^{p+1} \frac{(p+1)!(n-p-1)!}{(n+1)!} < [v_0, \ldots, v_n], [M]_K > \]
\[ = \pm (-1)^{p+1} \frac{(p+1)!(n-p)!}{(n+1)!} \]

We define the bilinear functional \( S : C^p(K) \times C^{n-p-1}(\hat{K}) \to \mathbb{R} \) by
\[ S(x, y) = \left[ \begin{array}{c} x^p+1 \\ p+1 \end{array} \right] \int_M dW^{BK}(Bx) \wedge W^{BK}(By) , \quad (x, y) \in C^p(K) \times C^{n-p-1}(\hat{K}) \]
\[ (2.16) \]
where \( \left[ \begin{array}{c} x^p+1 \\ p+1 \end{array} \right] = \binom{n+1}{p+1} \), and define the linear maps \( T^K_p : C^p(K) \to C^{n-p-1}(\hat{K}) \)
and \( T^K_{n-p-1} = (T^K_p)^*: C^{n-p-1}(\hat{K}) \to C^p(K) \) by
\[ S(x, y) = < T^K_p x, y > = < x, T^K_{n-p-1} y > . \]
\[ (2.17) \]

**Theorem 2.6.**
\[ T^K = ^*K d^K \quad \text{and} \quad T^K_{n-p-1} = (-1)^{np+1} \hat{K} d^{\hat{K}} \]

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Corollary 2.7.

\[ T^K \hat{T}K = (d^K)^* d^K \quad \text{and} \quad T^K \hat{T}K = (d^K)^* d^K \]

Proof. We must show that

\begin{equation}
\int_{M} dW^{BK}(B\alpha^{(p)}) \wedge W^{BK}(B(\beta^{(p+1)})) = < *^K d^K \alpha^{(p)}, \beta^{(p+1)} > \tag{2.18}
\end{equation}

for arbitrary oriented \( p \)-simplex \( \alpha^{(p)} \) and \( (p+1) \)-simplex \( \beta^{(p+1)} \) in \( K \). The theorem and its corollary then follow easily from proposition 2.4. Using (2.4) the l.h.s. of (2.18) can be expressed in terms of the simplicial wedge product for \( C^\ast(BK) \):

\[ \text{l.h.s.} = \int_{M} dW^{BK}(B\alpha^{(p)}) \wedge W^{BK}(B\beta^{(p+1)}), [M]_{BK} > \]

It follows from proposition 2.5 (with \( K \) replaced by \( BK \)) that for sequences of simplices \( \alpha^{(0)} < \ldots < \alpha^{(p)} \in K \) and \( \beta^{p+1} < \ldots < \beta^{(n)} \in K \) the quantity

\[ < d^{BK}[\tilde{\alpha}^{(0)}, \ldots, \tilde{\alpha}^{(p)}] \wedge^{BK} [\tilde{\beta}^{(p+1)}, \ldots, \tilde{\beta}^{(n)}], [M]_{BK} > \]

vanishes unless \( \alpha^{(p)} < \beta^{(p+1)} \), and is equal to \( \pm (-1)^{p+1} \left( \binom{n+1}{p+1} \right)^{-1} \) in the latter case, so from the definition of the maps \( B \) we get

\[ \text{l.h.s.} = \begin{cases} 
\pm (-1)^{p+1} & \text{for} \quad \alpha^{(p)} < \beta^{(p+1)} \\
0 & \text{otherwise} 
\end{cases} \tag{2.19} \]

where the sign \( \pm \) depends on whether \( \alpha^{(p)} \) has the orientation induced by \( \beta^{(p+1)} \) or not. To complete the proof we show that this is equal to the r.h.s. of (2.18). Setting \( \alpha^{(p)} = [v_0, \ldots, v_p] \) we find

\[ < *^K d^K \alpha^{(p)}, \beta^{(p+1)} > = < d^K \alpha^{(p)}, \beta^{(p+1)} > = \sum_{v_{p+1}} < [v_{p+1}, v_0, \ldots, v_p], \beta^{(p+1)} > = (-1)^{p+1} \sum_{v_{p+1}} < [v_0, \ldots, v_p, v_{p+1}], \beta^{(p+1)} > = (2.19) \]

Remark 2.8. The Whitney map generalises to a map between cochains and forms with coefficients in a flat vector bundle \([\text{Müll}]\). The constructions and results of this section generalise in an obvious way to that setting.
Remark 2.9. Although we work exclusively with triangulations, our techniques and results here and in the following also go through for more general polyhedral decompositions $K$ of $M$, in particular for cubic decompositions. In these cases the barycentric subdivision and dual of the decomposition of $M$ can be constructed in an analogous way to the simplicial case. The resulting barycentric subdivision $BK$ will necessarily be a triangulation (even though $K$ is not), so the Whitney map $W^{BK}$ on which our constructions are based continues to be well-defined, and it is straightforward to check that our constructions and results continue to hold.

3 Linking numbers in a simplicial framework

Recall the setup of the preceding section: $K$ is a simplicial complex triangulating $M$ ($\dim M = n$); $BK$ denotes its barycentric subdivision and $\hat{K}$ denotes the dual triangulation. In this section the smoothness requirement on $M$ can be dropped, and all maps that we consider are continuous unless otherwise stated. Let $N_1$ and $N_2$ be closed oriented manifolds with $\dim N_1 = p \leq n - 1$, $\dim N_2 = n - p - 1$, triangulated by simplicial complexes $L_1$ and $L_2$ respectively. Let $f_K : L_1 \rightarrow K$ be a simplicial map, i.e. a map $f_K : N_1 \rightarrow M$ which maps each simplex of $L_1$ linearly onto a simplex of $K$. Let $g_{\hat{K}} : \hat{L}_2 \rightarrow \hat{K}$ be a dual-simplicial map, i.e. a simplicial map $g_{\hat{K}} : BL_2 \rightarrow BK$ which maps each cell in $\hat{L}_2$ (considered as a union of simplices in $BL_2$) onto a cell in $\hat{K}$. (This determines a map $g_{\hat{K}} : N_2 \rightarrow M$.) In this section we show that in the simplicial framework there is a natural expression $\text{lk}(f_K, g_{\hat{K}})$ ((3.3) below) for the linking number of $f_K$ and $g_{\hat{K}}$ when these maps are bounding (as defined below), and derive a formula for $\text{lk}(f_K, g_{\hat{K}})$ in terms of the ingredients of the simplicial Hodge–de Rham theory developed in the last section (theorem 3.3). We go on to study the linking number $\text{lk}(f_K, g_K)$ of $f_K$ with a simplicial map $g_K : L_2 \rightarrow K$ via the introduction of simplicial framings. A simplicial framing of $g_K$ is essentially a homotopy $g$ from $g_K$ to a dual-simplicial map $g_{\hat{K}} : \hat{L}_2 \rightarrow \hat{K}$, $N_2 \rightarrow M$ where $L_2'$ is another simplicial complex triangulating $N_2$. The point is that although a priori there is no natural expression for the linking number of $f_K$ and $g_K$ in the simplicial
framework, we do obtain a natural expression for this linking number in terms of a
simplicial framing $g$ of $g_K$, namely $\text{lk}(f_K, g_{\hat{K}})$ where $g_{\hat{K}}$ is the dual-simplicial map
homotopic to $g_K$ via $g$, provided that the images of $f_K$ and $g$ are disjoint in $M$.
We prove that simplicial framings always exist in the main case of physical interest,
namely when $\text{dim} M = 3$ and $N = S^1$ (theorem 3.6), and that in this case if $f_K$ and
$g_K$ have disjoint images in $M$ then there is a simplicial framing $g$ of $g_K$ such that
$f_K$ and $g$ have disjoint images in $M$. Intuitively one would expect $\text{lk}(f_K, g_{\hat{K}})$ to be
independent of the choice of simplicial framing $g$ of $g_K$ with image disjoint from that
of $f$ in $M$ (provided that such a framing actually exists), thus providing a definition
for $\text{lk}(f_K, g_K)$. We prove in proposition 3.9 that this is in fact the case.

To illustrate and motivate the general expression (3.3) below for the linking num-
ber we first consider the case where $\text{dim} M = 3$, $N_1 = N_2 = S^1$ and $f_K$ and $g_{\hat{K}}$ are
embeddings of $S^1$ in $M$. Then $g_{\hat{K}}(S^1)$ is a union of 1-cells in $\hat{K}$, each of which is the
dual of a 2-simplex in $K$ as in the figure below:

Thus if $D$ is a 2-dimensional surface in $M$ made up of a union of 2-simplices of $K$
(considered as 2-cells in $M$) then $g_{\hat{K}}(S^1)$ intersects $D$ transversely (if at all) at the
barycenters of 2-simplices. Consider now the situation where $f_K(S^1)$ bounds such a
surface $D$ as illustrated below:
The linking number of $f_K(S^1)$ and $g_{\widehat{K}}(S^1)$ is now given by (see [31, p.229]):

$$\text{lk}(f_K, g_{\widehat{K}}) = \sum_{\bar{\alpha}^{(2)} \in D \cap g_{\widehat{K}}(S^1)} \pm 1 \quad (3.1)$$

with sign determined as follows. The orientation of $f_K(S^1)$ induces an orientation for $D$, and thereby for each 2-simplex $\alpha^{(2)} \in D$; this together with the orientation of $M$ determines an orientation for $\bar{\alpha}^{(2)}$. If $\bar{\alpha}^{(2)} \in D \cap g_{\widehat{K}}(S^1)$ then the sign for the corresponding term in (3.1) is positive if $\bar{\alpha}^{(2)}$ has orientation compatible with $g_{\widehat{K}}(S^1)$, and negative otherwise. A succinct expression for the linking number (3.1) in the simplicial framework can now be obtained as follows. Let $D \in C_2(K)$ denote the sum of the oriented 2-simplices making up $D$, then $*^K D \in C_1(\widehat{K})$ is the sum of the oriented 1-cells in $\widehat{K}$ (with orientation determined by $D$ and $M$) which intersect $D$ transversely as shown in the figure below.

Let $g_{\widehat{K}} \in C_1(\widehat{K})$ denote the sum of the oriented 1-cells in $\widehat{K}$ which make up $g_{\widehat{K}}(S^1)$, then (3.1) can be rewritten as

$$\text{lk}(f_K, g_{\widehat{K}}) = \langle *^K D, g_{\widehat{K}} \rangle \quad (3.2)$$
We now show how (3.2) generalises for the general simplicial map $f_K : L_1 \to K$, $N_1 \to M$ and dual-simplicial map $g_\hat K : \hat L_2 \to \hat K$, $N_2 \to M$ introduced at the beginning of this section. Let $[N_1]_{L_1} \in C_p(L_1)$ and $[N_2]_{\hat L_2} \in C_p(\hat L_2)$ denote the orientation cycles of $N_1$ and $N_2$ determined by $L_1$ and $\hat L_2$ respectively (i.e. the sum of all top degree simplices (dual cells) with orientations induced by that of $N_1$ ($N_2$)). The maps $f_K$ and $g_\hat K$ induce chain maps $f_K# : C_*(L_1) \to C_*(K)$ and $g_\hat K# : C_*(\hat L_2) \to C_*(\hat K)$, defined by $f_K#(\alpha^{(i)}) = f_K(\alpha^{(i)})$ if $f_K(\alpha^{(i)})$ is of degree $i$ and $f_K#(\alpha^{(i)}) = 0$ otherwise, and similarly for $g_\hat K#$. For notational simplicity we set $f_K = f_K#([N_1]_{L_1}) \in C_p(K)$ and $g_\hat K = g_\hat K#([N_2]_{\hat L_2}) \in C_{n-p-1}(\hat K)$.

**Definition 3.1.** $f_K$ and $g_\hat K$ above are **bounding** if there are oriented manifolds $D_1$ and $D_2$ such that for $i = 1, 2$

(i) $\partial D_i = N_i$ and there is a simplicial complex triangulating $D_i$ with $L_i$ as a subcomplex.

(ii) There are maps $h_i : D_i \to M$ with $h_1|_{\partial D_1} = f_K$ and $h_2|_{\partial D_2} = g_\hat K$.

**Lemma 3.2.** Assume $f_K : L_1 \to K$ above is bounding, with $h_1 : D_1 \to M$ as in definition 3.1. Then there is a simplicial complex $L'_1$ triangulating $D_1$ with $L_1$ as a subcomplex, and a simplicial map $h'_1 : L'_1 \to K$, $N_1 \to M$ coinciding with $f_K$ on $L_1$. Setting $h'_1 := h'_1#([D_1]_{L'_1}) \in C_{p+1}(K)$ we have $\partial^K h'_1 = f_K$.

**Proof.** The generalised barycentric subdivision procedure described in [Mun, §16] provides $L'_1$ and $h'_1$ (a simplicial approximation to $h_1$) with the required properties. It immediately follows that $\partial^K h'_1 = h'_1#(\partial^L_1[D_1]_{L'_1}) = f_K$. ■

If $f_K$ is bounding and $h'_1$ is as in lemma 3.2 then the expression (3.2) for the linking number generalises to

$$\text{lk}(f_K, g_\hat K) = \langle \ast^K h'_1, g_\hat K \rangle \quad (3.3)$$

This is clearly integer valued (since $h'_1$ and $g_\hat K$ are chains over $\mathbb{Z}$) and can easily be rewritten in an analogous way to (3.1) (we omit the details).

A priori the expression (3.3) depends on the choice of $h'_1$ and does not require $g_\hat K$ to be bounding. However, the following theorem shows that $\text{lk}(f_K, g_\hat K)$ is independent
of the choice of $h'_1$ when both $f_K$ and $g_{\tilde{K}}$ are bounding.

**Theorem 3.3.** If $f_K$ and $g_{\tilde{K}}$ above are bounding then

$$\text{lk}(f_K, g_{\tilde{K}}) = < f_K, (^K d^K)^{-1} g_{\tilde{K}} > .$$

(3.4)

The proof requires the following

**Lemma 3.4.** Assume $g_{\tilde{K}} : \tilde{L}_2 \rightarrow \tilde{K}$ above is bounding, with $h_2 : D_2 \rightarrow M$ as in definition 3.1. Then there is a simplicial complex $L'_2$ triangulating $D_2$ with $BL_2$ as a subcomplex, and a simplicial map $h'_2 : L'_2 \rightarrow BK$, $D_2 \rightarrow M$ with the following properties: (i) $h'_2 |_{\partial D_2} = g_{\tilde{K}}$. (ii) $h'_2$ maps the j-skeleton of $L'_2$ (considered as a subspace of $D_2$) into the j-skeleton of $\tilde{K}$ (considered as a subspace of $M$) $\forall j = 0, 1, \ldots, n-p$. (iii) The chain $h'_2 \equiv h'_2([D_2]_{L'_2}) \in C_{n-p}(BK)$ can be considered as a chain in $C_{n-p}(\tilde{K})$, and $\partial h'_2 \equiv g_{\tilde{K}}$.

**Proof.** The cellular approximation theorem (see e.g. [FR, §2.3.2]) provides a map $h''_2 : D_2 \rightarrow M$ mapping the j-skeleton of the given triangulation of $D_2$ into the j-skeleton of $\tilde{K}$ $\forall j = 0, 1, \ldots, n-p$ with $h''_2 |_{\partial D_2} = g_{\tilde{K}}$ ($h''_2$ is a cellular approximation to $h_2$). The generalised barycentric subdivision procedure [Mun, §16] then provides a simplicial complex $L'_2$ triangulating $D_2$ with $BL_2$ as a subcomplex, and a simplicial map $h'_2 : L'_2 \rightarrow BK$ (a simplicial approximation to $h''_2$) coinciding with $g_{\tilde{K}}$ on $BL_2$. Clearly $h'_2$ must also map the j-skeleton of $L'_2$ into the j-skeleton of $\tilde{K}$ $\forall j = 0, 1, \ldots, n-p$. This shows (i) and (ii). Let $\alpha(p)$ be an arbitrary $(n-p)$-dimensional oriented dual cell in $\tilde{K}$, with $|\alpha(p)|$ the corresponding closed cell in $M$. Using (ii) and the fact that $h'_2$ is continuous it is quite easy to see that the restricted map $h'_2 : (h'_2)^{-1}(|\alpha(p)|) \rightarrow |\alpha(p)|$ has a well-defined degree $\text{deg}(h'_2; |\alpha(p)|)$ when $(h'_2)^{-1}(|\alpha(p)|) \subset D_2$ is given the orientation induced by $D_2$ and $|\alpha(p)| \subset M$ the orientation induced by $\alpha(p)$. (We set $\text{deg}(h'_2; |\alpha(p)|) = 0$ if $(h'_2)^{-1}(|\alpha(p)|)$ is empty.) Then for arbitrary $(n-p)$-simplex $\beta = [\widetilde{\alpha(p)}, \widetilde{\alpha(p+1)}, \ldots, \widetilde{\alpha(n)}]$ in $BK$ contained in $|\alpha(p)|$ with orientation compatible with $\alpha(p)$ we have $< h'_2, \beta > = \text{deg}(h'_2; |\alpha(p)|)$ independent of $\beta$, so $h'_2$ can indeed be considered as a chain in $C_{n-p}(\tilde{K})$ (i.e. $h'_2 = \sum_{\alpha(p)} \text{deg}(h'_2; |\alpha(p)|) \alpha(p)$). Since $\partial BK h'_2([D_2]_{L'_2}) = h'_2([L'_2]_{D_2}) h'_2([N_{2BL_2}]_{L'_2})$ it is clear from (i) that $\partial h'_2 = g_{\tilde{K}}$. ■
Proof of theorem 3.3. By proposition 2.4 the r.h.s. of (3.4) is well-defined if

\[
<((\ast^K d^K)^{-1} f_K, g_K) = (-1)^{np+1} <(\ast^{\tilde{K}} d^{\tilde{K}})^{-1} f_{\tilde{K}}, g_{\tilde{K}}>
\]  

is well-defined. To see that (3.5) is well-defined note that by lemma 3.2 and proposition 2.4 the equation \(\ast^{\tilde{K}} d^{\tilde{K}} y = f_{\tilde{K}}\) has a solution \(y = (-1)^{np+1} \ast^K h_1',\) and that if \(y'\) is another solution then by lemma 3.4

\[
<y - y', g_{\tilde{K}} > = < y - y', \partial^{\tilde{K}} h_2' > = <(\ast^{\tilde{K}})^{-1}(\ast^{\tilde{K}} d^{\tilde{K}}(y - y')) , h_2' > = 0.
\]

Hence the r.h.s. of (3.4) is well-defined, and can be written as

\[
< f_K, (\ast^K d^K)^{-1} g_{\tilde{K}} > (3.5) = < y, g_{\tilde{K}} > = < \ast^K h_1', g_{\tilde{K}} > (3.3) = \text{lk}(f_K, g_{\tilde{K}}) \quad \blacksquare
\]

In the remainder of this section we study the linking number of \(f_K\) with another simplicial map \(g_K : L_2 \to K, N_2 \to M.\) In this case the expression (3.3) is not well-defined. (Also, \(h_1'(D_1)\) and \(g_K(N_2)\) need not intersect transversely in \(M.\) In order to obtain an expression for the linking number within the simplicial framework in this case we introduce a particular class of framings:

**Definition 3.5.** Let \(L\) be a simplicial complex triangulating a manifold \(N\) and let \(\gamma_K : L \to K, N \to M\) be a simplicial map. A **simplicial framing** of \(\gamma_K\) is a homotopy \(\gamma : N \times [0,1] \to M\) from \(\gamma_K\) to a dual-simplicial map \(\gamma_{\tilde{K}} : \tilde{L} \to \tilde{K}, N \to M\) where \(L'\) is another simplicial complex triangulating \(N,\) together with a simplicial complex \(L''\) triangulating \(N \times [0,1]\) with \(BL\) and \(BL'\) as subcomplexes.

Intuitively, if \(g : N_2 \times [0,1] \to M\) is a simplicial framing of \(g_K\) and \(f_K(N_1)\) and \(g(N_2 \times [0,1])\) are disjoint in \(M\) then the linking number of \(f_K\) and \(g_K\) is \(\text{lk}(f_K, g_{\tilde{K}})\) (where \(g_{\tilde{K}} := g|_{N_2 \times \{1\}}\). We must verify that this really is independent of the choice of simplicial framing \(g\) for \(g_K\). But first, to show that this is relevant, we must show that simplicial framings actually exist. We will show that they always exist in the main case of physical interest, namely when \(\dim M = 3\) and \(N = S^1.\) It is possible that they always exist in more general circumstances but at present we have not proved this.
**Theorem 3.6.** If \( \dim M = 3 \) and \( L \) is a simplicial complex triangulating \( S^1 \) then every simplicial map \( \gamma_K : L \to K, S^1 \to M \) has a simplicial framing.

**Proof.** We will show that a simplicial framing for \( \gamma_K \) can be constructed after choosing a sequence of 3-simplices associated with \( \gamma_K(S^1) \) as indicated in the figure below.

![Diagram](image)

(a) A segment of a sequence of 3-simplices of \( K \) "wrapping around" \( \gamma^\wedge_K(S^1) \).

(b) The corresponding segment of \( \gamma(S^1 \times [0,1]) \) where \( \gamma \) is the induced simplicial framing of \( \gamma^\wedge_K \). Each segment \( \bullet \cdots \bullet \) is a 1-cell in \( \hat{K} \).

**Step 1.** We take \( S^1 \) to be \([0,1]\) with endpoints identified so that the vertices of the triangulation \( L \) are \( 0 = v_1 < v_2 < \cdots < v_N < v_{N+1} = 1 \) (\( v_1 = v_{N+1} \) in \( S^1 \)). Define \( 1 \leq i_1 < i_2 < \cdots < i_r < i_{r+1} \leq N + 1 \) by the condition that all \( v_i \) with \( i_j \leq i < i_{j+1} \) are mapped by \( \gamma_K \) to the same vertex \( w_j \) in \( K \), with \( w_j \neq w_{j+1} \) \( \forall j \leq r \) and \( w_{r+1} = w_1 \). Then \([w_j, w_{j+1}]\) is a 1-simplex in \( K \) \( \forall j = 1, \ldots, r \) and \( \gamma_K(S^1) \) is their union (considered as 1-cells in \( M \)). Choose a 3-simplex \( \alpha^{(3)} \) of \( K \) in \( \overline{\St([w_r, w_1])} \). Choose a sequence of 3-simplices \( \{\alpha^{(3)}_{i_l}\}_{l=1, \ldots, l_1} \) in \( \overline{\St([w_1])} \) such that \( \alpha^{(3)}_{i_1} = \alpha^{(3)} \) and \( \alpha^{(3)}_{i_l} \) is contained in \( \overline{\St([w_1, w_2])} \) and \( \alpha^{(3)}_{i_l} \cap \alpha^{(3)}_{i_{l+1}} \) is a 2-simplex \( \alpha^{(2)}_{i_l} \) \( \forall l = 1, \ldots, l_1 - 1 \). This is possible because otherwise \( \overline{\St([w_1])} - \overline{\St([w_r, w_1])} \) would consist of two or more components meeting only at points or line segments, in contradiction to the
fact that $\text{St}(w_1) - \text{St}([w_r, w_1])$ is homeomorphic to the closed ball $D^3$.) By discarding elements in the sequence if necessary, we can assume that $\{\alpha^{(3)}_{jl}\}_{l=2, \ldots, l_r-1}$ are all distinct and contained in $\text{St}(w_1) - \{\text{St}([w_r, w_1]) \cup \text{St}([w_1, w_2])\}$. Proceeding inductively for $j = 2, \ldots, r$ we choose a sequence of 3-simplices $\{\alpha^{(3)}_{jl}\}_{l=1, \ldots, l_r}$ such that $\alpha^{(3)}_{jl} = \alpha^{(3)}_{(j-1)l_{(j-1)}}$, $\alpha^{(3)}_{jl} \subset \text{St}([w_j, w_{j+1}])$, $\{\alpha^{(3)}_{jl}\}_{l=2, \ldots, l_r-1}$ are all distinct and contained in $\text{St}(w_j) - \{\text{St}([w_{j-1}, w_j]) \cup \text{St}([w_j, w_{j+1}])\}$ and $\alpha^{(3)}_{jl} \cap \alpha^{(3)}_{j(l+1)}$ is a 2-simplex $\alpha^{(2)}_{jl} \forall l = 1, \ldots, l_r-1$. (Again this is possible because otherwise we obtain a contradiction to the fact that $\text{St}([w_r, w_1])$ is homeomorphic to $D^3$.) Finally, choose a collection $\{x_{jl} \mid l=1, \ldots, l_j-1; j=1, \ldots, r+1\}$ of distinct points in $[0,1)$ so that $x_{1l} = 0$, $x_{jl} < x_{jl'}$ for $j < j'$ and $x_{jl} < x_{jl'}$ for $l < l'$. Set $x_{jl} := x_{j(l+1)}$, $y_{jl} := \frac{1}{2}(x_{jl} + x_{j(l+1)}) \forall l = 1, \ldots, l_j - 1 \forall j = 1, \ldots, r+1$. Taking $S^1$ to be $[0,1]$ with endpoints identified let $L'$ be the triangulation of $S^1$ with vertices $\{y_{jl} \mid l=1, \ldots, l_j-1; j=1, \ldots, r+1\}$; then $BL'$ has vertices $\{x_{jl}, y_{jl}\}$ and $L'$ has vertices $\{x_{jl}\}$. A simplicial map $\gamma_K : BL' \to BK$ is now defined by $\gamma_K(x_{jl}) := \tilde{\alpha}^{(3)}_{jl}$ and $\gamma_K(y_{jl}) := \tilde{\alpha}^{(2)}_{jl}$. This determines a dual-simplicial map $\gamma_K : \tilde{L'} \to \tilde{K}$ since the dual cell $[\tilde{y}_{jl}] = [x_{jl}, y_{jl}] \cup [y_{jl}, x_{j(l+1)}]$ in $\tilde{L'}$ is mapped by $\gamma_K$ to the dual cell $[\tilde{\alpha}^{(3)}_{jl}, \tilde{\alpha}^{(2)}_{jl}] \cup [\tilde{\alpha}^{(3)}_{jl}, \tilde{\alpha}^{(2)}_{j(l+1)}] = \tilde{\alpha}^{(2)}_{jl}$ in $\tilde{K}$.

**Step 2.** Let $u'_{ij}$ denote the centerpoint of $[u_{ij}, u_{ij+1}] \forall j = 1, \ldots, r$ and define the triangulation $L_2$ of $S^1 \times [\frac{1}{2}, 1]$ as indicated in the figures below:
A simplicial map $\gamma_{\frac{1}{2}}: L_{\frac{1}{2}} \rightarrow BK$ is now defined by $\gamma_{\frac{1}{2}}(x_{jl}) := \gamma_{K}(x_{jl})$, $\gamma_{\frac{1}{2}}(y_{jl}) = \gamma_{K}(y_{jl})$, $\gamma_{\frac{1}{2}}(v_{i}) = \gamma_{K}(v_{i}) = w_{j}$ and $\gamma_{\frac{1}{2}}(u'_{i}) =$ the centerpoint of $[w_{j}, w_{j+1}]$. (It is easily checked that with the choices above $\gamma_{\frac{1}{2}}$ maps 3-simplices of $L_{\frac{1}{2}}$ onto 3-simplices of $BK$.) Let $u_{i}$ denote the centerpoint of $[v_{i}, v_{i+1}] \forall i = 1, \ldots, N$ and define the the triangulation $L_{0}$ of $S^{1} \times [0, \frac{1}{2}]$ as indicated in the figure below (with $v_{i+1} := v_{1}$):

A simplicial map $\gamma_{0}: L_{0} \rightarrow BK$ is now defined by $\gamma_{0}(v_{i}) = \gamma_{K}(v_{i})$, $\gamma_{0}(u'_{i}) = \gamma_{\frac{1}{2}}(u_{i})$ and $\gamma_{0}(u_{i}) =$ the centerpoint between $\gamma_{K}(v_{i})$ and $\gamma_{K}(v_{i+1})$. Clearly $L_{0}$ and $L_{\frac{1}{2}}$ coincide on $S^{1} \times \{\frac{1}{2}\}$, as do $\gamma_{0}$ and $\gamma_{\frac{1}{2}}$, thus determining a triangulation $L'' = L_{0} \ast L_{\frac{1}{2}}$ of $S^{1} \times [0, 1]$ and a simplicial map $\gamma: L'' \rightarrow BK$, and thereby a map $\gamma: S^{1} \times [0, 1] \rightarrow M$. It is clear from the constructions that $\gamma$ and $L''$ satisfy definition 3.5.

Let $f_{K}: L_{1} \rightarrow K$, $N_{1} \rightarrow M$ and $g: L_{2} \rightarrow K$, $N_{2} \rightarrow M$ be simplicial maps as above ($\dim N_{1} = p$, $\dim N_{2} = n - p - 1$). Let $St_{BK}(f_{K}(N_{1}))$ denote the open region of $M$ consisting of the union of all $St_{BK}(v)$ where $v$ is a vertex of $BK$ contained in $f_{K}(N_{1})$, and let $St_{BK}(g_{K}(N_{2}))$ be defined similarly.

**Definition 3.6.** A simplicial framing $g$ for $g_{K}$ is strongly disjoint from $f_{K}$ if the image of $g$ in $M$ is disjoint from $St_{BK}(f_{K}(N_{1})).$
Observation 3.8. (i) $f_K(N_1) \subseteq \text{St}_{BK}(f_K(N_1))$. (ii) $M - \text{St}_{BK}(f_K(N_1))$ has a cell decomposition given by $\tilde{K}$ with the cells $\{v | v \text{ vertex in } K \text{ contained in } f_K(N_1)\}$ removed. (iii) If $f_K(N_1) \cap g_K(N_2) = \emptyset$ then $\text{St}_{BK}(f_K(N_1)) \cap \text{St}_{BK}(g_K(N_2)) = \emptyset$. (iv) If $\dim M = 3$, $N_1 = N_2 = S^1$, $f_K(S^1) \cap g_K(S^1) = \emptyset$ and $g$ is a simplicial framing for $g_K$ constructed as in the proof of theorem 3.6 then $g$ is strongly disjoint from $f_K$.

Proposition 3.9. Let $f_K$ and $g_K$ be simplicial maps as above, and assume that they are bounding with $f_K(N_1) \cap g_K(N_2) = \emptyset$. If there exists a simplicial framing $g$ for $g_K$ strongly disjoint from $f_K$ (as is always the case when $\dim M = 3$ and $N_1 = N_2 = S^1$, cf. theorem 3.6 and observation 3.8(iv)) then $\text{lk}(f_K, g_K) := \text{lk}(f_K, g_{\tilde{K}})$ is independent of the choice of such a framing $g$. (Here $g_{\tilde{K}}$ is the dual-simplicial map determined by $g$ as specified in definition 3.5. Note that $g_{\tilde{K}}$ is bounding since $g_K$ is, so $\text{lk}(f_K, g_{\tilde{K}})$ is well-defined.)

Proof. Set $M_f := M - \text{St}_{BK}(f_K(N_1))$. Let $\bar{g}$ be another simplicial framing for $g_K$ which is also strongly disjoint from $f_K$. We can assume $\bar{g}$ is a map $\bar{g} : N_2 \times [-1, 0] \to M_f$ with $g_K = \bar{g}|_{N_2 \times \{0\}}$ and $\bar{g}_{\tilde{K}} = \bar{g}|_{N_2 \times \{-1\}} : N_2 \to M_f$, $\tilde{L}'_2 \to \tilde{K}$ a dual-simplicial map for some simplicial complex $\tilde{L}'_2$ triangulating $N_2$. Let $L''_2$ and $\tilde{L}_2''$ be the triangulations of $N \times \{0, 1\}$ and $N \times [-1, 0]$ associated with $g$ and $\bar{g}$ respectively, as specified by definition 3.5. This data determines a dual-simplicial map $\bar{g}_{\tilde{K}} \cup g_{\tilde{K}} : \tilde{L}_2' \cup \tilde{L}_2'' \to \tilde{K}$, $N_2 \times \{-1, 1\} \to M_f$ which is bounding: The map $h_2 : N_2 \times [-1, 1] \to M_f$ required by definition 3.1 is defined by $h_2|_{N_2 \times \{-1, 0\}} = \bar{g}$ and $h_2|_{N_2 \times \{0, 1\}} = g$, and the triangulation of $D_2 = N_2 \times [-1, 1]$ is $L''_2 \cup L''_2$. Lemma 3.4 continues to hold in this case with $M$ replaced by $M_f$. (The proof goes through unchanged due to observation 3.8(ii).) Thus we have a triangulation $L''_2$ (denoted $L'_2$ in lemma 3.4) of $N_2 \times [-1, 1]$ and a simplicial map $h'_2 : L''_2 \to BK$, $N_2 \times [-1, 1] \to M_f$ such that $h'_2 \in C_{n-p}(BK)$ can be considered as a chain in $C_{n-p}(\tilde{K})$ and

$$\partial^{\tilde{K}} h'_2 = g_{\tilde{K}} - \bar{g}_{\tilde{K}}.$$  \hspace{1cm} (3.6)

It follows from (3.6), theorem 3.3 and proposition 2.4 that (modulo an irrelevant sign)

$$\text{lk}(f_K, g_{\tilde{K}}) - \text{lk}(f_K, \bar{g}_{\tilde{K}}) = < f_{\tilde{K}}, (\ast^K d^K)^{-1}(g_{\tilde{K}} - \bar{g}_{\tilde{K}}) >$$

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\[ \pm < f_K, (\partial \hat{K} + K)^{-1} \partial \hat{K} h'_2 > \]
\[ \pm < f_K, \ast \hat{K} h'_2 > \]  
(3.7)

Since \( h'_2(N_2 \times [-1, 1]) \subseteq M - \text{St}_{BK}(f_K(N_1)) \) the vertices of \( BK \) contained in \( h'_2(N_2 \times [-1, 1]) \) are disjoint from the vertices of \( BK \) contained in \( f_K(N_1) \) so (3.7) vanishes. 

Finally, if \( M \) has odd dimension \( n = 2p + 1 \) and \( N \) is a closed oriented \( p \)-dimensional manifold triangulated by a simplicial complex \( L \) then each simplicial framing \( f \) for a simplicial map \( f_K : L \to M, N \to M \) has a self-linking number \( \text{lk}(f) := \text{lk}(f_K, f_{\hat{K}}) \).

### 4 Continuum abelian Chern–Simons gauge theory

In this section we review the evaluation of the partition function \([Sc]\) \([Wit]\) \([AdSc]\) and Wilson v.e.v.’s of linked loops \([Po]\) \([Wit]\) in the continuum abelian Chern–Simons theory on closed 3-manifolds (and its generalisation to arbitrary closed odd-dimensional manifolds). Our treatment includes certain features involving the moduli space of flat \( U(1) \) connections (when \( \pi_1(M) \) is non-trivial) not contained in the original treatments\(^{11}\).

**The Setup.** Let \( P \) be a flat principal \( U(1) \) bundle over a closed oriented 3-manifold \( M \). (All bundles, manifolds, maps etc. in this section are smooth.) The connections on \( P \) form an affine vectorspace \( C \) modelled on the space of 1-forms with values in the adjoint bundle, which is necessarily trivial, so given \( A_0 \in C \) the elements \( A \) of \( C \) are \( A = A_0 + 2\pi i \omega, \omega \in \Omega^1(M) \). The abelian Chern–Simons action functional on \( C \) is defined by

\[ i\lambda S(A_0 + 2\pi i \omega) := i\lambda \int_M d\omega \wedge \omega \]  
(4.1)

\(^{11}\)While we were finishing the writeup of this paper a preprint by M. Manoliu \([Ma]\) appeared in which abelian Chern–Simons theory is constructed as a topological field theory (in the sense of Witten–Atiyah) via geometric quantisation. This preprint also contains a comprehensive treatment of the standard path–integral evaluation of the partition function, along similar lines to the present review.
for flat connection $A_0$, where $\lambda \in \mathbb{R}_+$ is the coupling parameter. This is independent of the choice of flat connection $A_0$ by Stokes theorem, since if $A'_0$ is another flat connection then $d(A'_0 - A_0) = 0$. The group $\mathcal{G}$ of gauge transformations can be identified with the maps $g: M \to U(1)$, and acts on $\mathcal{C}$ by $g \cdot A = A + gdg^{-1}$. Locally $g \in \mathcal{G}$ is of the form $g = e^{2\pi i f}$ for $\mathbb{R}$–valued function $f$, leading to $gdg^{-1} = -2\pi idf$, so $d(gdg^{-1}) = 0$ globally and it follows from Stokes theorem that the action functional (4.1) is gauge–invariant. However, if $\pi_1(M)$ is non–trivial and $g$ has non–zero winding number around loops in $M$ then globally $f$ is only defined modulo $\mathbb{Z}$, since the integral of $(-2\pi i)^{-1}gdg^{-1}$ around a cycle in $M$ equals the winding number of $g$ around the cycle. Thus $(-2\pi i)^{-1}gdg^{-1}$ represents an element of $H^1_{dR}(M)_\mathbb{Z}$, where we are using

\[ \text{Definition 4.1. } H^1_{dR}(M)_\mathbb{Z} \text{ is the subgroup of } H^1_{dR}(M) \text{ consisting of the elements whose integrals around the cycles of } M \text{ are } \mathbb{Z}–\text{valued. (Here } H^q_{dR}(M) \text{ denotes the } q\text{'th de Rham cohomology group.)} \]

In the following we will use the fact that the de Rham map gives an isomorphism $H^1_{dR}(M)_\mathbb{Z} \cong \text{Free } H^1(M, \mathbb{Z}) = H^1(M, \mathbb{Z})$. (The last equality is the standard fact that $H^1(M, \mathbb{Z})$ has no torsion; this is clear for example from (6.1) in §6.)

Pick a metric on $M$, let $\mathcal{H}^q(M)$ denote the space of harmonic $q$–forms and let $\phi_q: \mathcal{H}^q(M) \xrightarrow{\cong} H^q_{dR}(M)$ denote the isomorphism induced by the projection map $\ker(d_q) \to H^q_{dR}(M)$. The homomorphism $\mathcal{G} \to H^1_{dR}(M)_\mathbb{Z}$, $g \mapsto (-2\pi i)^{-1}gdg^{-1}$ induces an isomorphism of the homotopy equivalence classes of $\mathcal{G}$ onto $H^1_{dR}(M)_\mathbb{Z}$. This has the following consequences:

(i) $\mathcal{G} \cong \exp(2\pi i\Omega^0(M)) \times H^1_{dR}(M)_\mathbb{Z}$.

(ii) The orbit of $\mathcal{G}$ through $A \in \mathcal{C}$ is

\[ \mathcal{G} \cdot A \cong A + 2\pi i(d_0\Omega^0(M) \times H^1_{dR}(M)_\mathbb{Z}) \tag{4.2} \]

and $\mathcal{G}/\tilde{U}(1) \to \mathcal{G} \cdot A$, $g \mapsto g \cdot A$ is bijective, where $\tilde{U}(1)$ denotes the gauge transformations which are constant on the connected components of $M$.

\[ \text{12} \text{This detail was omitted in the original treatments, where the gauge transformations were restricted to those of the form } A \mapsto A + 2\pi idf. \]
(iii) Using the map $\phi_1$ the orbit space of $C = A_0 + 2\pi i \Omega^1(M)$ can be identified with

$$C/\mathcal{G} \cong A_0 + 2\pi i \left( \frac{H_{dR}^1(M)}{H_{dR}^1(M)_{\mathbb{Z}}} \right) \oplus \ker(d_1)$$

(4.3)

and the moduli space of the flat $U(1)$ connections $\mathcal{F} = A_0 + 2\pi i \ker(d_1)$ can be identified with

$$\mathcal{F}/\mathcal{G} \cong A_0 + 2\pi i \left( \frac{H_{dR}^1(M)}{H_{dR}^1(M)_{\mathbb{Z}}} \right).$$

(4.4)

Since $H_{dR}^1(M) \cong \mathbb{R}^{\text{dim} H^1}$ and $H_{dR}^1(M)_{\mathbb{Z}} \cong \mathbb{Z}^{\text{dim} H^1}$ (where $\text{dim} H^q \equiv \text{dim} H^q_{dR}(M)$) we have

$$\mathcal{F}/\mathcal{G} \cong T^{\text{dim} H^1}$$

(4.5)

where $T^N \equiv \mathbb{R}^N/\mathbb{Z}^N$ is the $N$–torus. Thus $\mathcal{F}/\mathcal{G}$ is a closed manifold with finite volume determined by a choice of volume element for $H_{dR}^1(M)$.

In the following, if $L : V \rightarrow W$ is a linear map between vectorspaces with inner products we will use the notation $|\det'(L)| := \det'(L^*L)^{1/2}$ where $\det'(L^*L)$ is the product of the non–zero eigenvalues of $L^*L$. If $L$ is an isomorphism then $|\det(L)| = |\det'(L)|$ depends only on the volume elements on $V$ and $W$. The metric on $M$ induces metrics on $\Omega^q(M)$, $\mathcal{H}^q(M)$ and $\mathcal{G}$. We choose volume elements for $H^q_{dR}(M)$ for $q = 0, \ldots, 3$ (then $|\det(\phi_q)|$ is defined), and equip $C$ with the metric determined by requiring that the map $\Omega^1(M) \rightarrow C$, $\omega \mapsto A_0 + 2\pi i \omega$ be an isometry.

The partition function. This is the formal object

$$Z(M, \lambda) := \frac{1}{V(\mathcal{G})} \int_C DAD e^{-i\lambda S(A)} = \frac{1}{V(\mathcal{G})} \int_{C/\mathcal{G}} D[A]V([A]) e^{-i\lambda S([A])}$$

(4.6)

where $V(\mathcal{G})$ and $V([A])$ are the formal volumes of $\mathcal{G}$ and $[A] = \mathcal{G} \cdot A$ respectively. It follows from (i) and (ii) above that formally

$$V([A]) = V(\mathcal{G})V(U(1))^{-1}|\det'(d_0)| = V(\mathcal{G})|\det(\phi_0)||\det'(d_0)|$$

(4.7)

Writing the action functional $\mathcal{I}$ as $i\lambda S(A_0 + 2\pi i \omega) = i\lambda < *d_1 \omega, \omega >$ and using $\mathcal{I}$ and (iii) above we find

$$Z(M, \lambda) = V(\mathcal{F}/\mathcal{G})|\det(\phi_0)||\det(\phi_1)|^{-1}|\det'(d_0)| \det \left( \frac{i\lambda}{\pi} *d_1 \right)^{-1/2}$$

(4.8)
where $V(\mathcal{F}/\mathcal{G})$ is determined by the volume element on $H^1_{dR}(M)$ via (4.4), independent of the metric on $M$. Evaluating the determinants in (4.8) via zeta–regularisation as in [AdSe] and using Hodge duality gives

$$Z(M, \lambda) = e^{-\frac{\pi}{4}\eta(*d_1)} \left(\frac{\lambda}{\pi}\right)^{-\dim H^0/2 + \dim H^1/2 - \dim H^0/2 + \dim H^1/2} V(\mathcal{F}/\mathcal{G}) \tau_{RS}(M, d)^{1/2} \left(\frac{a_0}{a_1}\right)^{1/2}$$

(4.9)

where $\eta(*d_1)$ is the analytic continuation to zero of the eta–function of $\star d_1$ and

$$\tau_{RS}(M, d) = \prod_{q=0}^3 \left( |\det(\phi_q)||\det'(d_q)| \right)^{(-1)^q}$$

(4.10)

(with $|\det'(d_3)|\equiv 1$) is the Ray–Singer torsion of $(M, d)$, depending on the choices of volume elements for the spaces $H^q_{dR}(M)$ but independent of the choice of metric on $M$ [RS], and where for $q = 0, 1$ we have set

$$a_q := |\det(\phi_q)||\det(\phi_{3-q})|$$

(4.11)

The quantities $a_q$ are independent of the metric on $M$. To see this we choose an inner product in $H^{3-q}_{dR}(M)$ which reproduces the volume element and define $E_q : H^q(M) \rightarrow H^{3-q}(M)$ by $\langle E_q(a), b \rangle := \int_M a \wedge b$, then $|\det(E_q)|$ is obviously metric–independent. But on the other hand $E_q$ coincides with the map

$$E_q : H^q_{dR}(M) \xrightarrow{\phi_q^{-1}} \mathcal{H}^q(M) \xrightarrow{\star} \mathcal{H}^{3-q}(M) \xrightarrow{(\phi_{3-q})^\ast} H^{3-q}(M)$$

since

$$\int_M a \wedge b = \int_M \phi_q^{-1}(a) \wedge \phi_{3-q}^{-1}(b) = \langle (\phi_{3-q})^\ast(*\phi^{-1}_q)(a), b \rangle$$

so we have $|\det(E_q)|^{-1} = |\det(\phi_q)||\det(\phi_{3-q})| = a_q$. (This also shows that given a volume element for $H^q_{dR}(M)$ we can choose the volume element for $H^{3-q}_{dR}(M)$ so that $a_q = 1$. Thus with appropriate choices of volume elements for $H^q_{dR}(M)$ the factor $\frac{a_q}{a_1}$ drops out of (4.9).) Thus the metric–dependence of the final expression (4.8) for $Z(M, \lambda)$ enters only through $\eta(*d_1)$ in the phase, so $|Z(M, \lambda)|$ is a topological invariant.
The Wilson vacuum expectation value (v.e.v.). This is the formal object

\[
< W(\gamma^{(1)}, \ldots, \gamma^{(r)} ; n_1, \ldots, n_r) >_\lambda := \frac{\int \mathcal{D}A \left[ \prod_{j=1}^r \Phi(A, \gamma^{(j)}, n_j) \right] e^{-i\lambda S(A)}}{\int \mathcal{D}A e^{-i\lambda S(A)}}
\]

(4.12)

where the \( \gamma^{(j)} : S^1 \to M \) are maps which are bounding (i.e. there are maps \( h^{(j)} : D_j \to M \) restricting to \( \gamma^{(j)} \) on \( \partial D_j = S^1 \)) with images \( \gamma^{(j)}(S^1) \) mutually disjoint in \( M \), and \( \Phi(A, \gamma^{(j)}, n_j) \) is the monodromy of \( A \) around \( \gamma^{(j)} \) in the representation of \( U(1) \) on \( C \) corresponding to \( n_j \in \mathbb{Z} \). The evaluation of (4.12) proceeds heuristically as follows.\(^\text{13}\) For \( l = 1, \ldots, r \) let \( \eta^{(j)} \) be the distribution given by \( \int_{S^1} \gamma^{(j)*} \omega = \int_M \omega \wedge \eta^{(j)} \quad \forall \omega \in \Omega^1(M) \). We will treat \( \eta^{(j)} \) heuristically as a 2-form on \( M \) vanishing away from \( \gamma^{(j)}(S^1) \), then \( \eta^{(j)} = d\omega^{(j)} \) where \( \omega^{(j)} \) is the 1-form distribution given by \( \int_{D_j} h^{(j)*} \tau = \int_M \tau \wedge \omega^{(j)} \quad \forall \tau \in \Omega^2(M) \). Writing \( i\lambda S(A_0 + 2\pi i\omega) = i\lambda < *d_1 \omega, \omega > \) and

\[
\Phi(A_0 + 2\pi i\omega, \gamma^{(j)}, n_j) = \Phi(A_0, \gamma^{(j)}, n_j) \exp(2\pi in_j \int_{S^1} \gamma^{(j)*} \omega) = \Phi(A_0, \gamma^{(j)}, n_j) \exp(2\pi in_j < *\eta^{(j)}, \omega >)
\]

the formal evaluation of (4.12) gives

\[
< W(\gamma^{(1)}, \ldots, \gamma^{(r)} ; n_1, \ldots, n_r) >_\lambda = \left( \frac{1}{V(F/G)} \int_{F/G} \mathcal{D}[A] \prod_{l=1}^r \Phi([A], \gamma^{(l)}, n_l) \right) \exp \left( \frac{i\pi^2}{\lambda} \sum_{j m} n_j n_m < *\eta^{(j)}, (*d_1)^{-1} \ast \eta^{(m)} > \right)
\]

\[
= \exp \left( \frac{i\pi^2}{\lambda} \sum_{j m} n_j n_m \text{lk}(\gamma^{(j)}, \gamma^{(m)}) \right)
\]

(4.13)

where we have used the facts that

\[
< *\eta^{(j)}, (*d_1)^{-1} \ast \eta^{(m)} > = \int_M \omega^{(m)} \wedge \eta^{(j)} = \text{lk}(\gamma^{(j)}, \gamma^{(m)})
\]

is the linking number of \( \gamma^{(j)} \) and \( \gamma^{(m)} \) (see p.230 of [34]), and \( \Phi(A, \gamma^{(l)}, n_l) = 1 \) for flat connection \( A \) since the \( \gamma^{(l)} \)'s are bounding and thereby trivial in \( \pi_1(M) \). In §6 we will consider a situation where the integrand in the integral over \( F/G \) in (4.13) is a

\(^{13}\)A mathematically rigorous model for \( < W >_\lambda \) leading to the same final expression in terms of linking numbers as below has been constructed in ref. [34].

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non-trivial constant.) Finally, to obtain a well-defined expression for $< W >$ the self-linking numbers $\text{lk}(\gamma^{(j)}, \gamma^{(j)})$ in (4.13) must be regularised “by hand” by introducing a framing of each $\gamma^{(j)}$. Note that the final expression (4.13) is trivial when $\lambda = \frac{\pi p}{2}$, $p \in \mathbb{Z}$. In §6 we will see how $< W >$ is non-trivial for these discrete values of $\lambda$ when the $\gamma^{(j)}$’s represent torsion elements in the homology of $M$ over $\mathbb{Z}$. (In the present case the $\gamma^{(j)}$’s represent the trivial element in the homology of $M$ since they are bounding.)

**Framings and abelian BF theory.** The regularisation of self-linking numbers via framings of the $\gamma^{(j)}$’s can be incorporated into the theory from the beginning (as opposed to putting it in by hand at the end) if we work with the doubled, “twisted” version of abelian Chern–Simons theory for two independent connections $A = A_0 + 2\pi i \omega$ and $A' = A_0 + 2\pi i \omega'$ in $\mathcal{C}$ obtained by making a “twist” in the doubled action

$$i\lambda S(A) + i\lambda S(A') = i\lambda \int_M d_1 \omega \wedge \omega + d_1 \omega' \wedge \omega' = i\lambda \left< \begin{pmatrix} \omega \\ \omega' \end{pmatrix}, \begin{pmatrix} *d_1 & 0 \\ 0 & *d_1 \end{pmatrix} \begin{pmatrix} \omega \\ \omega' \end{pmatrix} \right>$$

to obtain

$$i\lambda \tilde{S}(A, A') := i\lambda \left< \begin{pmatrix} \omega \\ \omega' \end{pmatrix}, \begin{pmatrix} 0 & *d_1 \\ *d_1 & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \omega' \end{pmatrix} \right> = 2i\lambda \int_M d\omega \wedge \omega'. \quad (4.14)$$

This is the action functional for the so-called abelian BF gauge theory. It is easy to see that the partition function $\tilde{Z}(M, \lambda)$ of this theory is given by

$$\tilde{Z}(M, \lambda) = |Z(M, \lambda)|^2 \quad (4.15)$$

which is now completely independent of the choice of metric. Let the $\gamma^{(j)}$’s now be framed loops, i.e. smooth maps $\gamma^{(j)} : S^1 \times [0, 1] \to M$, with $\gamma^{(j)}_0 := \gamma^{(j)}|_{S^1 \times \{0\}} : S^1 \to M$ and $\gamma^{(j)}_1 := \gamma^{(j)}|_{S^1 \times \{1\}} : S^1 \to M$ such that $\gamma^{(j)}_0$ (and thereby also $\gamma^{(j)}_1$) is bounding and the images $\gamma^{(j)}_0(S^1)$, $\gamma^{(j)}_1(m)(S^1)$ are mutually disjoint in $M$. The Wilson v.e.v. $< \tilde{W} >$ of the framed loops is now defined in a natural way in this theory by replacing $i\lambda S(A)$

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14 The absence of a complex phase factor in $\tilde{Z}(M, \lambda)$ is because the operator in the quadratic form (4.14) has symmetric spectrum, and therefore vanishing eta–function.
and \( \Phi(A, \gamma^j, n_j) \) in (4.12) by \( i\lambda S(A, A') \) and \( \Phi(A, \gamma^j_0, n_j) \Phi(A', \gamma^j_1, n_j) \) respectively, and integrating (formally) over \( \mathcal{C} \times \mathcal{C} \). Then the formal evaluation of \( \langle \tilde{W} \rangle_\lambda \) leads to (4.13) with \( \text{lk}(\gamma^j_0, \gamma^m_1) \) replaced by \( \text{lk}(\gamma^j_0, \gamma^m_1) + \text{lk}(\gamma^j_1, \gamma^m_0) \). Since \( \text{lk}(\gamma^j_0, \gamma^m_1) = \text{lk}(\gamma^j_0, \gamma^m_0) \) for \( j \neq m \) and \( \text{lk}(\gamma^s_0, \gamma^s_1) = \text{lk}(\gamma^s) \) is the self-linking number of \( \gamma^s \), we obtain

\[
\langle \tilde{W}(\gamma^1, \ldots, \gamma^r; n_1, \ldots, n_r) \rangle_\lambda = \exp \left( \frac{2i\pi^2}{\lambda} \left( \sum_{j \neq m} n_j n_m \text{lk}(\gamma^j_0, \gamma^m_1) + \sum_{s=1}^r n_s^2 \text{lk}(\gamma^s) \right) \right) \tag{4.16}
\]

Along with the feature described above, the abelian BF theory has another advantage over the abelian Chern–Simons theory when it comes to constructing simplicial versions of the theories: The simplicial analogue of the Hodge star operator (i.e. the duality operator) can be incorporated in a natural way into a simplicial version of the abelian BF theory since it has two independent coupled gauge fields. (We will see this explicitly in the next section.) This feature is essential if the same evaluation of the partition function in a simplicial version of the theory is to lead to the same final expression with R–torsion in place of Ray–Singer torsion. Therefore we will not attempt to discretise the abelian Chern–Simons theory directly, but instead construct a natural simplicial version of the abelian BF theory—this is done in the next section.

The abelian Chern–Simons theory and the corresponding abelian BF theory generalise to abelian gauge theories on closed oriented manifolds \( M \) of arbitrary odd dimension with a flat vector bundle \( E_\rho \) over \( M \) determined by an orthogonal representation \( \rho : \pi_1(M) \rightarrow O(N) \). The partition functions of the generalised theories can be formally evaluated by the technique of A. Schwarz, leading to the square root of the Ray–Singer torsion \( \tau_{RS}(M, d^\rho) \) of the twisted de Rham complex \( \{ \Omega^*(M, E_\rho), d^\rho \} \) [Sc15]. Also, when the representation \( \rho \) is trivial the Wilson v.e.v.’s of loops generalise and can be evaluated to obtain an expression in terms of linking numbers as above.

15Strictly speaking, the generalised theories are only generalisations of abelian Chern–Simons–and BF–theories when the group \( \mathcal{G} \) of gauge transformations is replaced in the preceding by its identity-connected component \( \mathcal{G}_0 \), since the field(s) in the generalised theories transform under a gauge transformation by \( \omega \rightarrow \omega + d^\rho \tau \).
5 Simplicial abelian gauge theory

In this section we present a canonical simplicial version of the abelian BF theory with action (4.14). Using the results of §2–3 we show that the formal evaluation of the partition function and Wilson v.e.v.’s of framed loops (for simplicial framings of edge loops in the triangulation $K$) leads to expressions in terms of R–torsion and linking numbers respectively, reproducing the continuum expressions\footnote{The continuum expression for the partition function is reproduced when the value of the coupling parameter is 1; for values $\neq 1$ a triangulation-dependent renormalisation of the coupling parameter is required to reproduce this expression. The continuum expressions for the Wilson v.e.v.’s are reproduced for all values of the coupling parameter (without the need for renormalisation).}

Simplicial action functional. $A_0$ continues to denote the flat connection in $C$ chosen in the last section (the expressions below for the partition function and Wilson v.e.v.’s are independent of the choice of $A_0$). Define the embeddings

\begin{align}
\Psi_K &: C^1(K) \hookrightarrow C \quad x \to A_0 + 2\pi i W^{BK}(Bx) \\
\Psi_{\hat{K}} &: C^1(\hat{K}) \hookrightarrow C \quad y \to A_0 + 2\pi i W^{BK}(By)
\end{align}

(5.1) (5.2)

where the maps $B$ are given by (2.9)–(2.10). We construct the simplicial version of the abelian BF theory with action (4.14) by replacing the space of connections $C \times C$ by the subspace $\Psi_K(C^1(K)) \times \Psi_{\hat{K}}(C^1(\hat{K}))$. This is reminiscent of what happens in gauge fixing\footnote{I thank Prof. A.P. Balachandran for this perspective.} The simplicial action functional is then $i\lambda \tilde{S}_K$, where $\tilde{S}_K$ is the bilinear functional on $C^1(K) \times C^1(\hat{K})$ given by

\[ \tilde{S}_K(x, y) := \tilde{S}(\Psi_K(x), \Psi_{\hat{K}}(y)) = 2 \int_M dW^{BK}(Bx) \wedge W^{BK}(By) \]

(5.3)

By theorem 2.6 we have

\[ \tilde{S}_K(x, y) = \frac{1}{6} < *^K d^K x, y > = \frac{1}{6} < x, *^{\hat{K}} d^{\hat{K}} y > = \frac{1}{12} \left( \begin{array}{c} x \\ y \end{array} \right) , \left( \begin{array}{cc} 0 & *^{\hat{K}} d^{\hat{K}} \\ *^{\hat{K}} d^{\hat{K}} & 0 \end{array} \right) \left( \begin{array}{c} x \\ y \end{array} \right) \]
To avoid the numerical factor in (5.4) we will take \( \lambda' := 12\lambda \) to be our coupling parameter in the simplicial theory.

**Simplicial gauge transformations.** We define a group \( G_K \times G_K \) acting on \( C^1(K) \times C^1(\hat{K}) \) and leaving the simplicial action \( \tilde{S}_K \) invariant as follows. Let \([G]\) denote the collection of homotopy equivalence classes of \( G \) and choose a representative \( h[g] \in [g] \) for each \([g] \in [G]\). Then \( \mathcal{H} \) is a group with multiplication defined by \( h[g_1] \cdot h[g_2] := h[g_1g_2] \), where \( e^{2\pi i W_K(u)} \cdot h[g] \) is defined by the usual multiplication. \( G_K \) is a subset of \( G \), although in general the embedding \( G_K \hookrightarrow G \) is only a homomorphism modulo homotopy equivalence. We define the action of \( h \in G_K \) on \( x \in C^1(K) \) by

\[
h \cdot x = x + A^K_{dR}(\Lambda^{-1} v h^{-1})
\]

where \( A^K_{dR} : \Omega^*(M) \rightarrow \Omega^*(K) \) is the de Rham map. Since \( dK(A^K_{dR}(v h^{-1})) = A^K(d(v h^{-1})) = 0 \) the simplicial action functional is invariant under \( G_K \) due to (5.4). Using (2.4) we get

\[
\left( e^{2\pi i W_K(u)} h[g] \right) \cdot x = h[g] \cdot x + d^K_x u
\]

Combining this with de Rham’s theorem that \( A^K_{dR} \) induces isomorphisms \( H^1_{dR}(M) \cong H^1(K, \mathbb{R}) \) and \( H^1_{dR}(M)_{\mathbb{Z}} \cong H^1(K, \mathbb{Z}) \) we see that the properties (i), (ii) and (iii) of \( \mathcal{G} \) described in the preceding section have analogues for \( \mathcal{G}_K \). Set \( \mathcal{H}^q(K) := \{ x \in C^q(K) \mid d^K_x = (d^K)^* x = 0 \} \) (i.e. the simplicial analogues of the harmonic forms), equipped with the canonical inner product from \( C^q(K) \), and let \( \phi^K_q : \mathcal{H}^q(K) \rightarrow H^q(K, \mathbb{R}) \) denote the isomorphism induced by the projections \( \ker(d^K) \rightarrow H^q(K, \mathbb{R}) \). Equip \( H^q(K, \mathbb{R}) \) with the volume element induced by that of \( H^q_{dR}(M) \) via the de Rham map; then \( |\det(\phi^K_q)| \) is defined. Define \( \mathcal{H}^q(\hat{K}) \), \( \phi^K_{\hat{q}} \) and \( |\det(\phi^K_{\hat{q}})| \) analogously. \( \mathcal{G}_K \) has the following properties:

(i)’ \( \mathcal{G}_K \cong \{ e^{2\pi i W_K(u)} \mid u \in C^0(K) \} \times H^1(K, \mathbb{Z}) \).

(ii)’ The orbit \([x]\) of \( \mathcal{G}_K \) through \( x \in C^1(K) \) can be identified with

\[
\mathcal{G}_K \cdot x \cong x + (d^K_0 C^0(K) \times H^1(K, \mathbb{Z}))
\]
and the map $G_{K}/U(1) \to G_{K} \cdot x$, $h \mapsto h \cdot x$ is bijective, where $U(1)_{K} := \{ e^{2\pi i W_{K}(u)} \mid u \in \ker(d_{0}^{K}) \}$.

(iii) Using the map $\phi_{1}^{K}$ the orbit space of $C^{1}(K)$ can be identified with

$$C^{1}(K)/G_{K} \cong \left( \frac{H^{1}(K, \mathbb{R})}{H^{1}(K, \mathbb{Z})} \right) \oplus \ker(d_{1}^{K}) \perp \quad (5.8)$$

and the orbit space of $F_{K} := \ker(d_{1}^{K})$ can be identified with

$$F_{K}/G_{K} \cong \frac{H^{1}(K, \mathbb{R})}{H^{1}(K, \mathbb{Z})} \quad (5.9)$$

The group $G_{K}$ cannot immediately be defined by analogy with $G_{K}$ because we do not have a Whitney map from $C^{*}(\hat{K})$ to $\Omega^{*}(M)$. To get around this we construct a triangulation $\hat{K}'$ of $M$ such that the vertices and 1-cells of $\hat{K}$ are subsets of the vertices and 1-simplices of $\hat{K}'$, and $d_{0}^{\hat{K}}$ coincides with the map

$$C^{0}(\hat{K}) \hookrightarrow C^{0}(\hat{K}') \xrightarrow{d_{0}^{\hat{K}}} C^{1}(\hat{K}') \to C^{1}(\hat{K}) \quad (5.10)$$

where the last map is the restriction. Then $G_{\hat{K}}$ is defined by replacing $C^{q}(K), W^{K}, d_{0}^{K}$ by $C^{q}(\hat{K}), W^{\hat{K}}', d_{0}^{\hat{K}}$ respectively in the preceding construction of $G_{K}$. The group $G_{\hat{K}}$ and its action on $C^{1}(\hat{K})$ (defined by analogy with (5.5)) then has properties completely analogous to the properties (5.6), (i)', (ii)', (iii)' of $G_{K}$, and by (5.4) the simplicial action functional is invariant under the action of $G_{\hat{K}}$ on $C^{1}(\hat{K})$. Briefly, $\hat{K}'$ is constructed from the barycentric subdivision $B^{K}$ as follows. Each $q$-simplex $\tau^{(q)}$ of $B^{K}$ contained in a 3-simplex $\sigma^{(3)}$ of $K$ with a $(q-1)$-face $\tau^{(q-1)}$ of $\tau^{(q)}$ contained in a 2-face $\sigma^{(2)}$ of $\sigma^{(3)}$ has a “mirror image” $\tau^{(q)'}$ contained in the 3-simplex $\sigma^{(3)'}$ of $K$ which shares $\sigma^{(2)}$ as a 2-face. The union $\tau^{(q)} \cup \tau^{(q)'}$ can be considered as an embedding of the standard $q$-simplex into $M$ in a canonical way, as illustrated in the figure below (after a planar projection):
The union of $\tau^{(3)}$ and $\tau^{(3)'}$ in (a) (the shaded region in (b)) is a 3-simplex in $\hat{K}'$.

The vertices of $\hat{K}'$ are the barycenters of all the simplices in $K$ except the 2-simplices. For $1 \leq q \leq 3$ the $q$-simplices of $\hat{K}'$ (considered as cells in $M$) are the unions $\tau^{(q)} \cup \tau^{(q)'}$ of the $q$-simplices $\tau^{(q)}$, $\tau^{(q)'}$ in $BK$ of the form described above, together with the remaining $q$-simplices of $BK$. (For $q = 3$ there are no remaining 3-simplices.) Note that the 1-cells of $\hat{K}$ arise as the unions $\tau^{(1)} \cup \tau^{(1)'}$. It is straightforward to check that $\hat{K}'$ thus defined is indeed a simplicial complex triangulating $M$ (smoothly on the complement of a set of measure zero) and that (5.10) coincides with $d_{\hat{K}'}^0$ as required.

The construction of $\hat{K}'$ in the analogous 2-dimensional situation is illustrated in the figure below:

*Dotted lines * $= K$  
*Solid lines * $= \hat{K}'$

*Regularisation.* The objects of interest in the simplicial theory can be expressed through mathematically meaningful integrals over the finite-dimensional space $C^1(K) \times C^1(\hat{K})$. These integrals are not convergent to start with however, because the simplicial gauge invariance under $G_K \times G_{\hat{K}}$ leads to the appearance of divergent volumes of
orbits of the simplicial gauge group. Therefore we must proceed heuristically at first (as in the continuum case) and rewrite the integrals as integrals over the simplicial orbit space $C^1(K)/\mathcal{G}_K \times C^1(\hat{K})/\mathcal{G}_{\hat{K}}$. The resulting integrals for the partition function and Wilson v.e.v.’s in the simplicial theory are not convergent a priori, but become convergent when we make the following *regularisation* of the simplicial theory: Let $T$ be the selfadjoint map on $V \equiv C^1(K) \times C^1(\hat{K})$ given by

\[ i\lambda' \tilde{S}_K(v) = i\lambda' < v, Tv > \quad v = (x, y) \quad (5.11) \]

(the explicit expression for $T$ is clear from (5.4)). Then there is an orthogonal decomposition $V = V_+ \oplus V_- \oplus V_0$, $T = T_+ \oplus T_- \oplus T_0$ where $V_+$, $V_-$ and $V_0$ denote the subspaces of positive–, negative– and zero–modes of $T$, and $T_+$, $T_-$ and $T_0$ denote the restrictions of $T$ to these spaces. For $\epsilon \in \mathbb{R}_+$ we define the *regularised* simplicial action functional by

\[ i\lambda' \tilde{S}_K^{(\epsilon)}(v) := < v', ((i\lambda' + \epsilon)T_+ \oplus (i\lambda' - \epsilon)T_-)v' > \quad (5.12) \]

where $v'$ is the projection of $v \in V$ onto $V_+ \oplus V_-$. This is clearly invariant under $\mathcal{G}_K \times \mathcal{G}_{\hat{K}}$ and coincides with $i\lambda' \tilde{S}_K$ in the limit $\epsilon \to 0$. With this regularisation the expressions for the partition function and Wilson v.e.v.’s in the simplicial theory in terms of integrals over the simplicial orbit space are convergent: Due to (5.4), (5.8) and the analogue of (5.8) for $\hat{K}$ these integrals are of the form

\[ I_\epsilon = \int_{V_+ \oplus V_-} Dv \, e^{i<w,v>} e^{-i\lambda' \tilde{S}_K^{(\epsilon)}(v)} \quad (5.13) \]

(modulo overall factors; here $w \in V_+ \oplus V_-)$ which clearly converges for $\epsilon > 0$ due to (5.12). In the limit where the regularisation is lifted we find

\[ \lim_{\epsilon \to 0} I_\epsilon = e^{-\frac{\eta(T)}{4\pi} < w, T^{-1}w>} \left( \frac{\lambda'}{\pi} \right)^{-\zeta(T)/2} |\det'|(T)^{-1/2} \quad (5.14) \]

where $\eta(T) = \dim V_+ - \dim V_-$ and $\zeta(T) = \dim V_+ + \dim V_-$. In the following we evaluate the partition function and Wilson v.e.v.’s for the regularised theory given by (5.12). The regularisation is then lifted (i.e. the limit $\epsilon \to 0$ is taken) in the final expressions.
The partition function. Let $\Psi_K(C^1(K))$ and $\Psi_{\hat{K}}(C^1(\hat{K}))$ be equipped with the metrics determined by requiring that $\Psi_K$ and $\Psi_{\hat{K}}$ (given by (5.1)–(5.2)) be isometries (where $C^1(K)$ and $C^1(\hat{K})$ have the metrics given by their canonical inner products). The partition function is then the formal object

$$Z_K(M, \lambda')(^{(c)}) := \frac{1}{V(G_K \times G_{\hat{K}})} \int_{\Psi_K(C^1(K)) \times \Psi_{\hat{K}}(C^1(\hat{K}))} \mathcal{D}A \mathcal{D}A' e^{-i\lambda' \tilde{S}(\epsilon)(A, A')}$$

$$= \frac{1}{V(G_K V(G_{\hat{K}})} \int_{C^1(K)/\mathcal{G}_K \times C^1(\hat{K})/\mathcal{G}_{\hat{K}}} \mathcal{D}[x] \mathcal{D}[y] V([x])V([y]) e^{-i\lambda' \tilde{S}(\epsilon)(x, y)}$$

(5.15)

where we have implemented the regularisation described above. Using the fact that $\mathcal{G}_K$, $\mathcal{G}_{\hat{K}}$ and their actions on $C^1(K)$, $C^1(\hat{K})$ have analogous properties to $\mathcal{G}$ and its action on $\mathcal{C}$ the divergent volumes $V([x])$ and $V([y])$ can be formally evaluated as in the continuum case. The divergent parts factor out as $V(G_K)$ and $V(G_{\hat{K}})$ and cancel against the overall normalisation factor in (5.15), and the resulting expression for the regularised partition function is finite. Evaluating this expression using (5.4) and then lifting the regularisation (i.e. taking the limit $\epsilon \to 0$) using (5.14) we find (compare with (4.8)):

$$Z_K(M, \lambda') = V(\mathcal{F}_K/\mathcal{G}_K) |\det(\phi_0^K)| |\det(\phi_1^K)|^{-1} |\det'((d_0^K)/\pi)| |\det'((d_1^K)/\pi)|^{-1/2}$$

$$\times V(\mathcal{F}_{\hat{K}}/\mathcal{G}_{\hat{K}}) |\det(\phi_0^{\hat{K}})| |\det(\phi_1^{\hat{K}})|^{-1} |\det'((d_0^{\hat{K}})/\pi)| |\det'((d_1^{\hat{K}})/\pi)|^{-1/2}$$

(5.16)

where $V(\mathcal{F}_K/\mathcal{G}_K)$ is determined by the volume element of $H^1(K, \mathbb{R})$ via (5.9), and $V(\mathcal{F}_{\hat{K}}/\mathcal{G}_{\hat{K}})$ is determined analogously. Since the volume elements of $H^1(K, \mathbb{R})$ and $H^1(\hat{K}, \mathbb{R})$ are determined by the volume elements of $H^1_{dR}(M)$ via the de Rham map, we see by comparing (4.4) and (5.9) (and the analogue of (5.9) for $\hat{K}$) that the de Rham map induces isometries

$$\mathcal{F}_K/\mathcal{G}_K \simeq \mathcal{F}/\mathcal{G} \quad , \quad \mathcal{F}_{\hat{K}}/\mathcal{G}_{\hat{K}} \simeq \mathcal{F}/\mathcal{G}$$

(5.17)

As in the continuum case the absence of a phase factor in (5.16) is because the operator in the quadratic form (5.4) has symmetric spectrum (this is easily seen using proposition 2.4) and therefore vanishing $\eta$. 

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so \( V(\mathcal{F}_K/\mathcal{G}_K) = V(\mathcal{F}_\tilde{K}/\mathcal{G}_\tilde{K}) = V(\mathcal{F}/\mathcal{G}) \). Now using proposition 2.4 we find

\[
\hat{Z}_K(M, \lambda') = \frac{\left(\frac{\lambda'}{\pi}\right)^{\dim H^0 + \dim H^1 + N^K_0 - N^K_1}}{V(\mathcal{F}/\mathcal{G})^2} \tau_R(K, d^K) \frac{a^K_0}{a^K_1} \tag{5.18}
\]

where \( N^K_q \) denotes the number of \( q \)-simplices of \( K \) and

\[
\tau_R(K, d^K) = \prod_{q=0}^3 (|\det(\phi^K_q)| |\det'(d^K_q)|)^{(-1)^q} \tag{5.19}
\]

(with \(|\det'(d^K_q)|=1\)) is the \( R \)-torsion of \( (K, d^K) \), depending on the choices of volume elements for the spaces \( H^q_{dR}(M) \) (via the induced volume elements for the spaces \( H^q(K, \mathbb{R}) \)) but independent of the choice of triangulation \( K \) \[Re, Fra, Mi, RS\], and where for \( q = 0, 1 \) we have set

\[
a^K_q := |\det(\phi^K_q)| |\det(\phi^K_{n-q})|. \tag{5.20}
\]

We claim that

\[
a^K_q = a_q \tag{5.21}
\]

where \( a_q \) is given by \[L, Li\]; in particular \( a^K_q \) is independent of \( K \). This is an application of proposition 4.2 of \[RS\], which states (as a special case) that \( <\omega, \tau> = A_{dR}^K(\omega), (\ast^K)^{-1} A_{dR}^K(\ast \tau) > \) for \( \omega, \tau \in \mathcal{H}^q(M) \). This can be rewritten as \( <\omega, \tau> = A_{dR}^K(\omega), A^K_1 > \) for \( \omega, L_q \tau > \) where \( L_q \) is the map

\[
\begin{align*}
\mathcal{H}^q(M) & \xrightarrow{\ast} \mathcal{H}^{n-q}(M) & H^{n-q}(M) & \xrightarrow{\tilde{\phi}_{n-q}} H^{n-q}(\tilde{K}, \mathbb{R}) & \xrightarrow{(\tilde{\phi}_{n-q})^{-1}} \mathcal{H}^{n-q}(\tilde{K}) \\
& \xrightarrow{(\ast^K)^{-1}} \mathcal{H}^q(M) & H^q(K, \mathbb{R}) & \xrightarrow{(A^K_q)^{-1}} \mathcal{H}^q(M) & \xrightarrow{\phi_q} \mathcal{H}^q(M).
\end{align*}
\]

\( L_q \) must coincide with the identity map, so we see from \(5.22\) that

\[
1 = |\det(L_q)| = |\det(\phi_q)||\det(\phi_{n-q})||\det(\tilde{\phi}_{n-q})|^{-1}|\det(\phi^K_q)|^{-1}
\]

and \(5.21\) follows.

Since \( \tau_R(K, d^K) = \tau_{RS}(M, d) \) \[Mi, Ch\] it follows that the final expression \(5.18\) for \( \hat{Z}_K(M, \lambda') \) coincides with the continuum expression \( \hat{Z}(M, \lambda) = |Z(M, \lambda)|^2 \) given

\[\text{In deriving the exponent for} \ x \text{ in (5.18) we have used the fact that} \ N^K_0 - N^K_1 - N^K_0 + N^K_1 \text{ is the Euler characteristic of} \ M, \text{which vanishes since} \ \dim M \text{is odd.}\]

\[39\]
by the modulus square of \([43]\) except for the triangulation-dependent factors \(N_0^K\) and \(N_1^K\) in the exponent of \(\lambda_\pi\). This \(K\)-dependence can be removed by renormalising \(\lambda'\), i.e. by replacing the bare coupling parameter \(\lambda'\) by the renormalised \(K\)-dependent parameter \(\lambda_K'\) given by

\[
\frac{\lambda_K'}{\pi} := \left(\frac{\lambda'}{\pi}\right)^{\kappa(K)} \quad , \quad \kappa(K) = \left(1 + \frac{N_0^K - N_1^K}{\dim H^1 - \dim H^0}\right)^{-1} \tag{5.23}
\]

Then \(\tilde{Z}_K(M, \lambda_K') = \tilde{Z}(M, \lambda) = |Z(M, \lambda)|^2\).

*Wilson v.e.v.'s of framed loops.* Let \(\gamma_K^{(1)}, \ldots, \gamma_K^{(r)}\) be bounding edge loops in the triangulation \(K\); i.e. each \(\gamma_K^{(j)}\) is a bounding simplicial map \(\gamma_K^{(j)} : L_j \rightarrow K, S^1 \rightarrow M\) (for some simplicial complex \(L_j\) triangulating \(S^1\)), and assume that the images \(\gamma_K^{(j)}(S^1)\) are mutually disjoint in \(M\). By theorem 3.6 and observation 3.8(iv) we can choose a simplicial framing \(\gamma^{(j)}\) for each \(\gamma_K^{(j)}\) such that \(\gamma^{(j)}\) and \(\gamma^{(m)}_K\) are mutually strongly disjoint for \(j \neq m\). Let \(\tilde{\gamma}_K^{(j)} : \hat{L}_j \rightarrow \tilde{K}, S^1 \rightarrow M\) denote the dual-simplicial map homotopic to \(\gamma_K^{(j)}\) via \(\gamma^{(j)}\). Then the Wilson v.e.v. of the framed loops \(\gamma^{(1)}, \ldots, \gamma^{(r)}\) in the simplicial theory is given by the formal expression

\[
< \tilde{W}_K(\gamma^{(1)}, \ldots, \gamma^{(r)}; n_1, \ldots, n_r) >_{\lambda'} \equiv \frac{\int_{\Psi_K(\psi(C^1(K)))} \mathcal{D}A \mathcal{D}A' \left[ \prod_{j=1}^r \Phi(A, \gamma^{(j)}_K, n_j) \Phi(A', \gamma^{(j)}_K, n_j) \right] e^{-i\lambda' \tilde{S}^{(\psi)}(A, A')} \int_{\Psi_K(\psi(C^1(\tilde{K})))} \mathcal{D}A \mathcal{D}A' e^{-i\lambda' \tilde{S}^{(\psi)}(A, A')}}}{\int_{\Psi_K(\psi(C^1(K)))} \mathcal{D}A \mathcal{D}A'} \tag{5.24}
\]

where we have again implemented the regularisation described above. For notational simplicity we set \(\gamma_K^{(j)} := \gamma_K^{(j)}(\{S^1, L_j\} \in C_1(K)\) and define \(\gamma_K^{(j)} \in C_1(\tilde{K})\) analogously. The monodromies appearing in \(\sum_2\) are

\[
\Phi(\Psi_K(x), \gamma_K^{(j)}; n_j) = \Phi(A_0, \gamma_K^{(j)}; n_j) \exp \left(2\pi i n_j \int_{S^1} \gamma_K^{(j)}(W_{BK}(Bx)) \right) = \Phi(A_0, \gamma_K^{(j)}; n_j) \exp \left(2\pi i n_j < x, \gamma_K^{(j)} > \right) \tag{5.25}
\]

where the last equality follows from the definition of the map \(B\) in §2 and the property \(\sum_2\) of \(W_{BK}\), and similarly

\[
\Phi(\Psi_{\tilde{K}}(y), \gamma_{\tilde{K}}^{(j)}; n_j) = \Phi(A_0, \gamma_{\tilde{K}}^{(j)}; n_j) \exp \left(2\pi i n_j < y, \gamma_{\tilde{K}}^{(j)} > \right) \tag{5.26}
\]

\(^{20}\)Our considerations continue to hold in the more general case where the \(\gamma_K^{(j)}\)'s represent torsion elements of the homology of \(M\), as discussed in §6.

40
Note that if \( A \) is a flat connection then \( \Phi(A, \gamma_K^{(j)}, n_j) = \Phi(A, \gamma_K^{(j)}, n_j) \) since \( \gamma_K^{(j)} \) and \( \gamma_K^{(j)} \) are homotopic.

The integrals in (5.24) can be formally rewritten as integrals over the simplicial orbit space as previously. The divergent factors \( V(G_K) \) and \( V(G_{\hat{K}}) \) in the formal volumes of simplicial orbits which appear in the numerator and denominator cancel, and the resulting expression for the regularised Wilson v.e.v. is finite. Evaluating this expression using (5.4), (5.17), (5.14) and proposition 2.4 and lifting the regularisation we find (compare with (4.16)):

\[
\langle \bar{W}_K(\gamma^{(0)}, \ldots, \gamma^{(r)}; n_1, \ldots, n_r) \rangle_{\lambda'} = \left( \frac{1}{V(F/G)} \int_{F/G} \mathcal{D}[A] \prod_{i=1}^{r} \Phi([A], \gamma_K^{(l)}, n_l) \right)^2 \\
\times \exp \left( \frac{2i\pi}{\lambda'} \sum_{j<m} n_j n_m < \gamma^{(j)}_K, (\ast K d^K)^{-1} \gamma^{(m)}_{\hat{K}} > \right) \tag{5.27}
\]

By theorem 3.3 and proposition 3.9 the quantity \( \langle \gamma^{(j)}_K, (\ast K d^K)^{-1} \gamma^{(m)}_{\hat{K}} > \) in (5.27) equals the linking number of \( \gamma^{(j)}_K \) and \( \gamma^{(m)}_{\hat{K}} \) (or the self-linking number if \( j = m \)), and it follows that (5.27) coincides with the continuum expression (4.16) for the v.e.v. (No renormalisation of \( \lambda' \) is required in this case.) This equality continues to hold when \( \gamma_K^{(1)}, \ldots, \gamma_K^{(r)} \) represent torsion elements of the \( \mathbb{Z} \)–homology of \( K \), as discussed in the next section.

The techniques of this section generalise in a straightforward way to construct simplicial versions of the abelian \( BF \) theories associated with the generalisations of abelian Chern–Simons theory discussed at the end of §4, again reproducing the continuum expressions for the partition function and Wilson v.e.v.‘s of generalised, simplicially framed edge loops, expressed in terms of \( R \)–torsion and linking numbers respectively. We omit the details, except to point out that in these theories the simplicial gauge symmetry is simpler than above: The continuum gauge transformations \((\omega, \omega') \mapsto (\omega + d\tau, \omega' + d\tau')\) becomes \((x, y) \mapsto (x + d^K u, y + d^\hat{K} v)\) in the simplicial theory.
6 Torsion pairings from Wilson v.e.v.’s at discrete values of the coupling parameter

In this section we consider the Wilson v.e.v.’s in the case where the loops are no longer required to be bounding but instead represent torsion elements of the $\mathbb{Z}$-homology of $M$. We will see that in this case $\mathbb{Q}/\mathbb{Z}$-valued torsion pairings appear in place of linking numbers in the v.e.v.’s. when the coupling parameter takes the discrete values $\lambda = \frac{2\pi}{2l}$, $l \in \mathbb{Z}$. We begin by briefly reviewing torsion pairings. (A brief accessible review of torsion in (co)homology with illuminating examples is given in §3 of [Fre]. More detailed treatments can be found in [DFN] [BT] [Mun].)

**Torsion pairings.** Let $M$ be a closed oriented $n$-manifold with triangulation $K$ as in §2–3. $M$ need not be smooth, and all maps in the following are continuous. An element $x$ of the (singular) $\mathbb{Z}$-homology of $M$, $H_\ast(M, \mathbb{Z})$, is said to be a torsion element if $kx = 0$ for some non-zero $k \in \mathbb{Z}$. These elements form a subgroup $\text{Tor}H_\ast(M, \mathbb{Z})$. Torsion elements can arise as follows. Let $N$ be a closed oriented $p$-manifold, then the element $[[f]] \in H_p(M, \mathbb{Z})$ represented by a map $f : N \to M$ is a torsion element if there is a map $h : D \to M$ where $D$ is an oriented $(p+1)$-manifold with boundary $\partial D$ consisting of $k$ disjoint copies of $N$ and the restriction of $h$ to $\partial D$ is $f$ on each copy, since in this case $\partial h = kf$ where $f$ and $h$ are the chains over $\mathbb{Z}$ corresponding to $f$ and $h$ respectively.

A standard result concerning torsion (see e.g. §3 of [Fre]) is that there is a canonical isomorphism

$$\text{Tor}H^q(M, \mathbb{Z}) \simeq \text{Hom}(\text{Tor}H_{q-1}(M, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}).$$

(6.1)

defined as follows. Let $x$ be a $(q-1)$-cycle over $\mathbb{Z}$ representing an element $[x] \in \text{Tor}_{q-1}(M, \mathbb{Z})$, then there is a non-zero integer $k$ and a $q$-chain $y$ over $\mathbb{Z}$ such that $\partial y = kx$. Let $c$ be a $q$-cocycle over $\mathbb{Z}$ representing an element $[c] \in \text{Tor}H^q(M, \mathbb{Z})$, then the pairing (6.1) is given by

$$\langle [c], [x] \rangle := \frac{1}{k} < c, y > \pmod{\mathbb{Z}}$$

(6.2)
It is easy to check that \( \frac{1}{k} < c, y > \) depends only on \([c]\) and \([x]\) (mod \(\mathbb{Z}\)) as required. Poincare duality gives \(\text{Tor} H^q(M, \mathbb{Z}) \simeq \text{Tor} H_{n-q}(M, \mathbb{Z})\), allowing (6.1) to be re-expressed (with \(p = n - q\)) as

\[
\text{Tor} H_p(M, \mathbb{Z}) \simeq \text{Hom}(\text{Tor} H_{n-p-1}(M, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}).
\] (6.3)

Thus there is a canonical \(\mathbb{Q}/\mathbb{Z}\)-valued pairing between \(\text{Tor} H_p(M, \mathbb{Z})\) and \(\text{Tor} H_{n-p-1}(M, \mathbb{Z})\), which we call the torsion pairing. This pairing arises as a direct generalisation of linking number as follows. Let \(N_1\) and \(N_2\) be closed oriented manifolds of dimensions \(p\) and \(n - p - 1\) respectively, and for \(i = 1, 2\) let \(f_i : N_i \to M\) and \(h_i : D_i \to M\) be analogous to \(f : N \to M\) and \(h : D \to M\) above, so that \(\partial h_i = k_i f_i\) at the chain level and \([\underline{f_i}] \in \text{Tor} H_p(M, \mathbb{Z}), [\underline{f_j}] \in \text{Tor} H_{n-p-1}(M, \mathbb{Z}).\) The chain \(k_i f_i\) corresponds to the map, denoted \(k_i f_i\), from \(k_i\) copies of \(N_i\) into \(M\) coinciding with \(f_i\) on each copy. The maps \(k_1 f_1\) and \(k_2 f_2\) are bounding and therefore have well-defined linking number.

**Proposition 6.1.** The torsion pairing (6.3) of \([\underline{f_1}]\) and \([\underline{f_2}]\) above is given by

\[
\text{tor}([\underline{f_1}], [\underline{f_2}]) = \frac{1}{k_1 k_2} \text{lk}(k_1 f_1, k_2 f_2) \pmod{\mathbb{Z}}
\] (6.4)

This follows via simplicial approximation from its simplicial version, proposition 6.1’ below.

The considerations above have analogues in the simplicial setting. The duality operator (simplicial Hodge star operator) \(*^K : C^q(K) \to C^{n-q}(\widehat{K}) \cong C_{n-q}(\widehat{K})\) (where the (co)chains are over \(\mathbb{R}\)) restricts to \(*^K : C^q(K, \mathbb{Z}) \to C_{n-q}(\widehat{K}, \mathbb{Z}),\) inducing isomorphisms (Poincare dualities)

\[
*^K : H^q(K, \mathbb{Z}) \xrightarrow{\cong} H_{n-q}(\widehat{K}, \mathbb{Z}) \quad , \quad *^K : \text{Tor} H^q(K, \mathbb{Z}) \xrightarrow{\cong} \text{Tor} H_{n-q}(\widehat{K}, \mathbb{Z})
\]

(These are standard facts which follow easily from proposition 2.4). Thus the simplicial version of (6.3) is

\[
\text{Tor} H_p(K, \mathbb{Z}) \simeq \text{Hom}(\text{Tor} H_{n-p-1}(\widehat{K}, \mathbb{Z}), \mathbb{Q}/\mathbb{Z}).
\] (6.5)

i.e. the \(\mathbb{Q}/\mathbb{Z}\)-valued torsion pairing is between \(\text{Tor} H^p(K, \mathbb{Z})\) and \(\text{Tor} H_{n-p}(\widehat{K}, \mathbb{Z}).\)

Let \(f_K : L_1 \to K, N_1 \to M\) and \(g_K : \widehat{L_2} \to K, N_2 \to M\) be simplicial- and dual-simplicial
maps respectively as in §3, and let $h_1 : D_1 \to M$ and $h_2 : D_2 \to M$ be maps with $\partial D_i$ consisting of $k_i$ copies of $N_i$ ($i = 1, 2$) such that $h_1$ restricts to $f_K$ and $h_2$ restricts to $g_{\hat{K}}$ on each copy of $N_1$ and $N_2$ respectively. Let $f_K \in C_p(K, \mathbb{Z})$ and $\hat{g_K} \in C_{n-p-1}(\hat{K}, \mathbb{Z})$ denote the chains corresponding to $f_K$ and $g_{\hat{K}}$. By the arguments of §3 (see lemmas 3.2 and 3.4) we can assume that $h_1$ and $h_2$ determine elements $h_1 \in C_{p+1}(K, \mathbb{Z})$, $h_2 \in C_{n-p-1}(\hat{K}, \mathbb{Z})$ with $\partial_K h_1 = k_1 f_K$ and $\partial_{\hat{K}} h_2 = k_2 g_{\hat{K}}$. Then $[f_K] \in \text{Tor} H_p(K, \mathbb{Z})$ and $[\hat{g_K}] \in \text{Tor} H_{n-p-1}(\hat{K}, \mathbb{Z})$.

**Proposition 6.1′.** The torsion pairing (6.5) of $[f_K]$ and $[\hat{g_K}]$ above is given by

$$\text{tor}([f_K], [\hat{g_K}]) = \frac{1}{k_1 k_2} \text{lk}(k_1 f_K, k_2 g_{\hat{K}})$$

(6.6)

where $k_1 f_K$ is the map from $k_1$ copies of $N_1$ into $M$ coinciding with $f_K$ on each copy, and $k_2 g_{\hat{K}}$ is defined analogously.

**Proof.** By (6.2),

$$\text{tor}([f_K], [\hat{g_K}]) \equiv \langle (\hat{K})^{-1}[f_K], [\hat{g_K}] \rangle = \frac{1}{k_2} \langle (\hat{K})^{-1} f_K, h_2 \rangle$$

(mod $\mathbb{Z}$). Using proposition 2.4 this can be rewritten as

$$\frac{1}{k_1 k_2} < k_1 f_K, (\hat{K} d K)^{-1} (k_2 g_{\hat{K}}) > \quad (\text{mod } \mathbb{Z})$$

and the proposition follows from theorem 3.3. ■

**The v.e.v.’s of loops representing torsion elements.** The evaluations of the v.e.v.’s of loops $\gamma^{(1)}, \ldots, \gamma^{(r)}$ in §4–5 goes through with obvious modifications in the case where the loops represent torsion elements of the $\mathbb{Z}$–homology of $M$: Choose numbers $k_j$ so that $k_j \gamma^{(j)}$ is bounding, then after dividing by $k_j$ in the appropriate places we can replace $\gamma^{(j)}$ by $k_j \gamma^{(j)}$ in the evaluations of the v.e.v.’s. This results in the linking numbers $\text{lk}(\gamma^{(j)}, \gamma^{(m)})$ being replaced by $\frac{1}{k_j k_m} \text{lk}(k_j \gamma^{(j)}, k_m \gamma^{(m)})$ in the final expressions (1.13), (4.16) and (5.27) for the v.e.v.’s. This now results in non-trivial expressions when the coupling parameter takes the discrete values $\lambda = \frac{\pi}{2l}$, $l \in \mathbb{Z}$ in the abelian Chern–Simons theory, or as $\lambda = \frac{\pi}{2l^'}$ ($l^' = \frac{\pi}{2}$) in the corresponding abelian BF theory (and its simplicial version). It follows from (6.4) and (6.6) that in this case the v.e.v.’s
only depend on the homology classes of the loops, and that the linking numbers get
replaced by the $\mathbb{Q}/\mathbb{Z}$-valued torsion pairings $\text{tor}(\gamma^{(j)}, \gamma^{(m)})$ in the final expressions
for the v.e.v.’s. (The framing considerations go through as before.)

There is one other modification in the final expressions for the v.e.v.’s when the
loops represent torsion elements. It comes from the integral over the modulispace
$\mathcal{F}/\mathcal{G}$ in the expression for the v.e.v.’s (see (4.13)), due to the following

Observation/definition 6.2. If $\gamma : S^1 \to M$ represents a torsion element, i.e. if there
is some map $h : D \to M$ with $\partial D = kS^1$ (i.e. $k$ copies of $S^1$) with $h|_{\partial D} = k\gamma$ then the
monodromy $\Phi(A, \gamma, n)$ is the same for all flat connections $A$. For given $\gamma$ and $n$ this
monodromy therefore depends only on the underlying flat principal $U(1)$ bundle $P$,
and we denote it by $\Phi(P, \gamma, n)$.

To see this let $A$ and $A'$ be flat connections, then $A' = A + 2\pi i \omega$ with $\omega \in \ker(d_1)$
and we have

$$
\int_{S^1} \gamma^*(\omega) = \frac{1}{k} \int_{\partial D = kS^1} (k\gamma)^*(\omega) = \frac{1}{k} \int_{D} h^*(d\omega) = 0
$$

which implies $\Phi(A', \gamma, n) = \Phi(A, \gamma, n)$. (Note also that $\partial h = k\gamma$ implies $\Phi(P, \gamma, n)^k = 1$.) Thus the integral over $\mathcal{F}/\mathcal{G}$ in (4.13) leads in the present case to an overall factor
$\prod_{j=1}^r \Phi(P, \gamma^{(j)}, n_j)$ appearing in the final expression (4.13), while the square of this
factor appears in the final expressions (4.16) and (5.27).

These considerations also go through in the previously mentioned generalisations
of abelian Chern–Simons theory and in the corresponding BF theories.

As discussed in [Fre], torsion phenomena arise in connection with global anomalies
in string theory. This suggests a connection between anomalies and abelian Chern–
Simons theory on 3-dimensional parameter spaces of families of Dirac operators. We
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