Abstract
The purpose of this paper is to construct one parameter families of embedded, screw motion invariant minimal surfaces in $\mathbb{R}^3$ which limit to parking garage structures. We construct such surfaces by defining Weierstrass data on the quotient and closing the periods. In the nodal limit, the periods reduce to algebraic balance equations for the locations of the helicoidal nodes. For any configuration of nodes that solve the equations and satisfy a nondegeneracy condition, we regenerate to obtain a family of surfaces near the limit. We thus prove the existence of many new examples of surfaces near the nodal limit, with helicoidal or planar ends. Among these are candidates for genus $g$ helicoids distinct from those currently known. We do not require any symmetry for the solutions of the balance equations, which suggests the existence of helicoidal surfaces only symmetric with respect to screw motions. This introduces new directions for the study and classification of screw motion invariant surfaces.

Keywords Minimal · Screw motion · Noded surface · Helicoid · Weierstrass data · Implicit function theorem

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1 Introduction

In this paper, we study embedded periodic minimal surfaces in $\mathbb{R}^3$ which are invariant under a screw motion (SMIMS). Specifically, we consider one-parameter families of SMIMS, varying the screw motion angle. While some of the most famous minimal surfaces are of this type, there is a relatively small number of known examples. However all known examples limit to a parking garage structure.

The first and standard example of a SMIMS is the helicoid, which is invariant under screw motions of any angle. For almost two hundred years, that was the only known case.

The next family is that of Karcher’s deformations of Scherk’s saddle towers [1]. In this example, as in all others we consider, the screw motion angle corresponds to the angle of
deformation, which we call twisting. As the twist parameter is pushed to its limit in either
direction, the surface degenerates to a parking garage with two helicoidal components.

Another is a deformation of Fischer and Koch’s translation-invariant surfaces (which we abbreviate $FK$ [2]. These surfaces have not been well-studied, but Hoffman and Karcher used Plateau methods to twist the surface. An unpublished Diplom thesis [3] discusses Weierstrass data for the twisted surfaces but does not succeed at solving the period problem. The Plateau construction of Fischer–Koch surfaces only produces embedded surfaces with $2(k + 1)$ ends, where $k \geq 2$ is even. It was hitherto unknown if any analogue to these surfaces exists if $k$ is odd.

The only known SMIMS with planar ends is the deformation of the Callahan–Hoffman–Meeks ($CHM$) surface of [4]. Images suggested that these surfaces also degenerate to four helicoids as the twist parameter approaches its limit (see Fig. 1).

Hoffman, Weber, and Wolf showed the existence of a family of screw motion invariant helicoids with genus 1 in the quotient [5]. The limit as the twist parameter increases to infinity is the genus one helicoid, and numerical evidence suggests that, as the parameter approaches 0, the surface degenerates to a parking garage structure.

Hoffman, Traizet, and White showed the existence of surfaces asymptotic to the helicoid of any genus [6]. It is conjectured that these the are unique minimal surfaces with these properties.

These surfaces can be categorized into three classes, which we can describe by the behavior of the ends. We construct surfaces so the screw motion angle increases away from the parking garage limit. As it does, the winding number about each end [7] can either increases, decreases, or stays the same. The first class consists of surfaces with helicoidal ends whose winding number increases with the screw motion. We call these helicoid-type, as this class includes the genus $g$ helicoids. The second class is that of Scherk-type surfaces. Their ends become less twisted, that is the winding number decreases, and in the known cases eventually become vertical Scherk ends. The third class consists of surfaces with planar ends, whose winding number remains 1.

Traizet and Weber, regenerating from noded surfaces, showed the existence of one-parameter families of SMIMS near the parking garage limit [8]. This method recovers most of the above examples near the limit, as well as new examples. However, their result is restricted to surfaces with nodes in the limit arranged in a straight line.
This limitation motivates our paper, which generalizes the result to surfaces with planar ends and without the straight line assumption. This allows us to recover all known examples near the parking garage limit. However, we prove the existence of new families in each of the three aforementioned classes.

In the Scherk class, this method produces not only the known Fischer–Koch surfaces but also surprising new examples currently not obtainable by Plateau construction.

**Theorem 1.1** For any \( k \geq 2 \), there exist SMIMS of Scherk class with \( 2k + 2 \) ends and genus 1. If \( k \) is even, these correspond to twisted Fischer–Koch surfaces. If \( k \) is odd, they correspond to analogs of FK which lack a vertical straight line.

In the planar class, there exist surfaces distinct from the Calahan-Hoffman-Meeks family.

**Theorem 1.2** There exists a family of SMIMS with planar ends and dihedral symmetry 2 and genus 7.

In the helicoidal class, a new surface of interest has genus 2 and two ends yet has a different limit than the known genus two helicoid.

**Theorem 1.3** There exists a family of SMIMS asymptotic to a helicoid with dihedral symmetry 4.

This suggests the following

**Conjecture 1.4** There exist multiple translation-invariant genus 2 helicoids and multiple genus 2 helicoids.

While most of our examples have dihedral symmetries, there is evidence of non-symmetric surfaces.

**Conjecture 1.5** There exist SMIMS which are only symmetric with respect to screw motions.

We construct SMIMS with an orientation-reversing screw motion \( \sigma \) and with a parking garage limit, which is the case for all known one-parameter families. This raises the following

**Question 1.6** Do all SMIMS limit to a parking garage structure?

**Question 1.7** Are all SMIMS invariant under an orientation-reversing screw motion?

We also do not know which translation invariant minimal surfaces can be twisted into SMIMS. We know that twisting is not possible for Riemann’s minimal surface [9], and we lack definite answers for many others.

## 2 Main results

We now introduce some of the key concepts and our main theorem.

**Definition 2.1** A configuration consists of a collection of points \( p_1, \ldots, p_n \) in the complex plane, where \( n \geq 2 \), together with corresponding charges \( \epsilon_j = \pm 1 \). Consider the forces, \( F_1, \ldots, F_n \) defined by,

\[
F_j = \overline{p_j} + \sum_{k \neq j} \frac{\epsilon_k}{p_j - p_k}.
\]

We say that a configuration is balanced if \( F_j = 0 \) for all \( j \).
Configurations will correspond to the locations and orientations of helicoids in parking garage limits.

**Example 2.2** The only balanced configuration consisting of two points (up to rotation) is \( p_1 = \sqrt{2}, p_2 = -\sqrt{2}, \varepsilon_1 = \varepsilon_2 = -1 \).

**Remark 2.3** We can also consider the balanced one-point configuration, which will correspond to a helicoid.

Note that the property of being balanced is invariant under rotating the configuration by any angle about the origin. Consequently, we may always assume that \( p_1 > 0 \).

**Definition 2.4** We say that configuration is nondegenerate if the \( 2n \times 2n \) matrix

\[
\begin{pmatrix}
\frac{\partial \Re F_j}{\partial x_l} & \frac{\partial \Re F_j}{\partial y_l} \\
\frac{\partial \Im F_j}{\partial x_l} & \frac{\partial \Im F_j}{\partial y_l}
\end{pmatrix},
\]

has rank \( 2n - 1 \) at \( (p_1, \ldots, p_n) \), where \((x_j, y_j) = (\Re p_j, \Im p_j)\) and \( j, l = 1, \ldots, n \).

This is the highest possible rank since the balance equations are rotation-invariant. Let

\[
N = \sum_{j=1}^{n} \varepsilon_j.
\]

Our main theorem is as follows:

**Theorem 2.5** Suppose that \( p_1^o, \ldots, p_n^o \) is a balanced and nondegenerate configuration. Then there exists \( \delta > 0 \) and a one-parameter family \( \{M_t\}_{0 < t < \delta} \) of embedded minimal surfaces such that:

1. The surface \( M_t \) is invariant under the screw motion \( S_t \) with translation part \((0, 0, 2\pi)\) and angle \( 2\pi t \), and the quotient \( M_t / S_t \) has genus \( n - 1 \) and two ends. If \( N \neq 0 \), these ends are helicoidal with winding number \( 1 + Nt \), and if \( N = 0 \) they are planar with vertical normal.
2. The surface is also invariant under an orientation-reversing screw motion \( s_t \) with translation part \((0, 0, \pi)\) and angle \( \pi t \), which exchanges the two ends.
3. There exist vectors \( T_{j,t} \approx \frac{1}{\sqrt{t}} (\Re p_j^o, \Im p_j^o, 0) \) such that, as \( t \to 0 \), \( M_t - T_{j,t} \) converges on compact subsets of \( \mathbb{R}^3 / S_t \) to a right or left helicoid of period \((0, 0, 2\pi)\). The orientation of the helicoid depends on whether \( \varepsilon_j = 1 \) or \(-1\).
4. If we rescale \( M_t \) horizontally by \( \sqrt{t} \), the new surfaces \( M_t \) (no longer minimal) converges (as sets) to the surface \( M_0 \), defined as follows: consider the multi-valued function

\[
f(z) = \sum_{j=1}^{n} \varepsilon_j \arg(z - p_j^o), \quad z \in \mathbb{C} - \{p_1^o, \ldots, p_n^o\}.
\]

\( M_0 \) is the union of the multigraph of \( f \), that of \( f + \pi \), and the vertical lines through each \( p_j \).

**Remark 2.6** The value of \( N \) determines the class of \( M_t \). Indeed, if \( N > 0 \), the helicoidal ends ascend counter-clockwise, and the winding number increases in \( t \), which makes the surface helicoid-type. We say that the surface twists further as \( t \) increases. On the other hand, if
N < 0, then the helicoidal ends ascend clockwise, and we say that it untwist as t increases. These are Scherk-type, and if they exist up to t = 1/N, their ends become vertical Scherk with winding number 0 (Fig. 2).

To simplify calculations, we modify our main theorem to guarantee symmetric surfaces. If a configuration is dihedrally symmetric and all points lie on symmetry lines, we call it simply dihedrally symmetric. With these assumptions, the definitions of balanced and nondegenerate configurations are greatly simplified (see Sect. 9.3).

**Theorem 2.7** Suppose $p_1^0, \ldots, p_n^0$ is a simply dihedrally symmetric configuration that is both balanced and nondegenerate with respect to the symmetry, then the conclusion of Theorem 2.5 holds. The dihedral symmetries of the configuration induce rotational or screw motion symmetries in the surfaces.

We will prove the main result over the next five sections. In Sects. 3–5, we construct a Riemann surface and Weierstrass data corresponding to $M_t/S_t$. Then in Sects. 6 and 7, we solve the major period problem and prove embeddedness. Finally, in Sect. 8, we prove the result for simply dihedral symmetric configurations. Since it is much easier to find symmetric balanced configurations, most of our examples are covered in this section. In Sect. 9, we consider some open questions that our results raise.

### 3 The Riemann surface

#### 3.1 Construction and symmetry

We define a Riemann surface $\Sigma = M_t/S_t$ underlying the quotient of our SMIMS by the screw motion. We construct $\Sigma$ in such a way as to guarantee symmetry with respect to the orientation-reversing screw motion $s_t$. To do this, we join two copies of $\hat{\mathbb{C}}$ by helicoidal necks. Since the quotient of a helicoid by translation is conformally $\mathbb{C} - \{0\}$, a helicoidal neck will be conformally an annulus.
Consider a configuration $p_1, \ldots, p_n \in \mathbb{C}_1$, with corresponding charges $\varepsilon_j$. For each $p_j$, consider the coordinates

$$v_j(z) = z - p_j \quad \text{in a neighborhood of } p_j \in \mathbb{C}_1,$$
$$w_j(z) = z - \overline{p_j} \quad \text{in a neighborhood of } \overline{p_j} \in \mathbb{C}_2.$$  

We use these coordinates to define annuli which correspond to portions of helicoids which we glue together.

Let $\varepsilon$ be a constant small enough so all disks $\{|v_j| < \varepsilon\}$ are disjoint. For each point, we remove the neighborhoods $\{|v_j| < \frac{r_j^2}{\varepsilon}\}$ and $\{|w_j| < \frac{r_j^2}{\varepsilon}\}$, where each $r_j$ is a positive parameter close to 0. We identify the annuli

$$\left\{ \frac{r_j^2}{\varepsilon} < |v_j| < \varepsilon \right\}, \quad \left\{ \frac{r_j^2}{\varepsilon} < |w_j| < \varepsilon \right\}$$

by

$$v_jw_j = -r_j^2.$$  

The negative sign comes from the way we identify portions of helicoids. We want the “upper part” of the helicoid in one copy of $\mathbb{C}$ to be identified with the “lower part” of the other. Note that the helicoidal axes will be the “central” rings of the annuli, corresponding to $\{|v_j| = |w_j| = r_j\}$ (Fig. 3).

This gives us a compact Riemann surface $\Sigma$ of genus $n - 1$. Note that since an annulus is conformally determined by its modulus, $\Sigma$ depends on each $r_j$. As all $r_j$ converge to 0, $\Sigma$ becomes a noded surface consisting of two spheres joined by $n$ nodes.

We now consider the symmetry $\sigma$ of $\Sigma$ which will correspond to $s_t$. Since $s_t^2 = S_t$, we expect $\sigma$ to be its own inverse.
Fig. 4 $B_j$ as a composition of $\gamma_j$ and $\Gamma_j$ paths

**Lemma 3.1** Let $\sigma$ send a point in one copy of $\hat{\Sigma}$ to its conjugate in the other copy. This defines a well-defined symmetry of $\Sigma$.

**Proof** Note that $\sigma$ will send the annulus around each $p_j$ in $\mathbb{C}_1$ to the corresponding annulus around $\overline{p_j}$ in $\mathbb{C}_2$. We need to check that this map is compatible with the identifications at each neck, that is, if

$$v_j(z)w_j(z') = -r_j^2$$

then

$$w_j(\sigma(z))v_j(\sigma(z')) = -r_j^2.$$ 

To this end, observe, for $\frac{r_j^2}{\epsilon} < |v_j(z)| < \epsilon$:

$$w_j(\sigma(z)) = \sigma(z) - \overline{p_j} = \overline{z - p_j} = \overline{v_j(z)},$$

and hence $\frac{r_j^2}{\epsilon} < |w_j(\sigma(z))| < \epsilon$. The same argument shows $v_j(\sigma(z')) = \overline{w_j(z')}$ for $\frac{r_j^2}{\epsilon} < |w_j(z')| < \epsilon$.

We thus have,

$$w_j(\sigma(z))v_j(\sigma(z')) = \overline{v_j(z)w_j(z')} = -r_j^2.$$ 

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**3.2 Homology basis**

Let us now describe a homology basis for $\Sigma$. Consider, for $j = 1, \ldots, n$, the circles $A_j$ defined by $\{ v_j = \epsilon e^{i\epsilon_j s}, s \in [0, 2\pi] \}$. Note that these are not all oriented the same way; this choice is to guarantee that going around an $A$ curve always goes “up a helicoid”, as will be clear in the definition of $dh$. For $j = 2, \ldots, n$ we define the curves $B_j$ as follows (Fig. 4).

Consider the region

$$\Omega = \mathbb{C}_1 - \bigcup_j \{ |z - p_j| < \epsilon \}.$$ 

For $j = 1, \ldots, n$, consider the paths

1. $\gamma_j$ in $\Omega$ going from $v_1 = \epsilon$ to $v_j = \epsilon$ without winding around any necks. To be precise, we cut the region $\Omega$ into a simply-connected domain with paths connecting a basepoint $z_0$ to each point $v_j = -\epsilon$ and require that each $\gamma_j$ lies in this simply-connected domain. This ensures that these paths lie on only one fundamental piece of $M_1$. 

Fig. 5 Visualizing the homology classes in the quotient and on the surface itself

(2) $\Gamma_j$, that consists of the half-circle $v_j = \epsilon e^{i \pi j s}$, $s \in [0, 1]$, followed by the radial
segment $v_j = s$, $s \in [-\epsilon, -\frac{r_j^2}{\epsilon}]$ joining $v_j = \epsilon$ and $w_j = \epsilon (v_j = -\frac{r_j^2}{\epsilon})$. This goes
“half-way up” one of the necks, that is, from one copy of $\mathbb{C}$ to another.

For $j = 2, \ldots, n$, the curves $B_j$ are defined as $\Gamma_1 * \sigma(\gamma_j) * \Gamma_j^{-1} * \gamma_j^{-1}$. Since $v_j(z) = \epsilon \iff w_j(\sigma(z)) = \epsilon$, these are closed curves.

The $A_j$ and $B_j$ curves, for $j = 2, \ldots, n$, form a homology basis for $\Sigma$. In fact, it is almost
a canonical homology basis since $(A_j \cdot B_k) = \epsilon_j \delta_j^k$ for $j, k \geq 2$ (Fig. 5).

4 The height differential

We now define the height differential based on its expected properties. The symmetry of the
surface will imply that its periods are closed.

If $N \neq 0$, we expect $M_t$ to have helicoidal ends, so $dh$ should have simple poles at infinity
with imaginary residues. In the $N = 0$ case, we expect horizontal planar ends, and hence $dh$
cannot have a pole at infinity. These conditions lead to the following

Definition 4.1 We define $dh$ on $\Sigma$ as follows:

- if $N \neq 0$, then $dh$ is a meromorphic 1-form with simple poles at $\infty_1$ and $\infty_2$.
- if $N = 0$, $dh$ is a holomorphic 1-form.

In both cases, $dh$ is uniquely normalized by the period conditions

$$\int_{A_j} dh = 2\pi \quad \forall \ j = 1, \ldots, n.$$ 

The normalization is unique since the periods are fixed for all $j$ and there is at most one
simple pole in each copy of $\mathbb{C}$, whose residue is determined by the residue theorem. When
$N \neq 0$, we have:

$$\text{Res}_{\infty_1} dh = - \text{Res}_{\infty_2} dh = - \frac{1}{2\pi i} \sum_{j=1}^{n} 2\pi \epsilon_j = Ni.$$ 

Note that when $N = 0$, $dh$ could have zeros at infinity, as is the case for dihedrally
symmetric configurations.

We now show that $dh$ has the desired symmetry and closed periods.
Lemma 4.2 The height differential has the following symmetry: $\sigma^* dh = \overline{dh}$.

Proof To see this we will show the meromorphic forms $\sigma^* dh$ and $dh$ have the same poles, residues, and periods. Since they only have at most simple poles, this implies the forms are equal. Note first that $\sigma(A_j)$ is described by

$$w_j = e^{i\varepsilon_j s},$$

i.e. $v_j = \frac{-r_j^2}{\varepsilon} e^{i\varepsilon_j s}$, for $s \in [0, 2\pi]$, and is homologous to $A_j$. Thus, we have:

$$\int_{A_j} \sigma^* dh = \int_{\sigma(A_j)} \overline{dh} = \int_{A_j} dh = \int_{A_j} dh.$$

Finally, $\text{Res}_{\infty} \sigma^* dh = \text{Res}_{\infty} \overline{dh} = \text{Res}_{\infty} dh$, which completes the proof. \(\square\)

We now consider the period condition for the height differential:

Lemma 4.3 The $B$-periods of $dh$ are pure imaginary.

Proof By the definition of $B_j$, we have:

$$\int_{B_j} dh = \int_{\Gamma_1} dh + \int_{\sigma(y_j)} dh - \int_{\Gamma_j} dh - \int_{y_j} dh.$$

Now, the sum,

$$\int_{\sigma(y_j)} dh - \int_{y_j} dh = \int_{y_j} \sigma^* dh - \int_{y_j} dh = \int_{y_j} (\overline{dh} - dh)$$

is imaginary. Moreover, $\Gamma_j + \sigma(\Gamma_j)$ is a closed loop winding once around the $p_j$ neck and is hence homologous to $A_j$ (Fig. 6). Thus,

$$\text{Re} \int_{\Gamma_j} dh = \frac{1}{2} \left( \int_{\Gamma_j} dh + \int_{\sigma(\Gamma_j)} dh \right) = \frac{1}{2} \int_{A_j} dh = \pi.$$

Thus,

$$\text{Re} \left( \int_{\Gamma_1} dh - \int_{\Gamma_j} dh \right) = \pi - \pi = 0.$$

Since we will be looking at the noded limit, we consider the limit of $dh$ as all annuli shrink to nodes. A similar argument to Traizet’s in [10] proves the following

Lemma 4.4 When all $r_j$ converge to 0, $dh$ converges on compact subsets of $\Omega - \{p_i\}$ to the meromorphic 1-form $dh_0$ defined by

$$dh_0 = \sum_{j=1}^{n} \frac{-i\varepsilon_j dz}{z - p_j}.$$
5 The Gauss map

The Gauss map on $\Sigma$ will be multivalued because of the screw motion. So we construct it by first defining its well-defined logarithmic differential $\omega = \frac{dG}{G}$, then integrate and exponentiate. In the process, we reduce the number of parameters.

5.1 The logarithmic differential

We orient the surface so that $G$ is infinite at $\infty_1$ and at the zeros of $dh$ in $\Omega$. So $\omega$ will have simple poles at the corresponding points, with the proper residues.

Because the genus of $\Sigma$ is $n - 1$, we use the degree of its divisor $\text{deg}(dh) = 2(n - 1) - 2 = 2n - 4$ to compute the number of zeros of $dh$ in both cases. Note that in either case, the symmetry $\sigma^*dh = \overline{dh}$ tells us that for each zero in $C_1$, its conjugate in $C_2$ is also a zero of $dh$.

When $N \neq 0$, $dh$ has two simple poles, so we have $2n - 4 = \text{deg}(dh)_0 - 2$, where $(dh)_0$ is the divisor of zeros of $dh$. Hence $dh$ has $2n - 2$ zeros, with $n - 1$ zeros $q_1, \ldots, q_{n-1}$ in $C_1$.

When $N = 0$, since $dh$ is holomorphic, it will have $2n - 4$ zeros and hence $n - 2$ zeros, $q_1, \ldots, q_{n-2}$ in $\hat{C}_1$.

**Definition 5.1** We define the logarithmic differential $\omega$ of $G$ as the unique meromorphic 1-form on $\Sigma$ with simple poles at each zero of $dh$ in $C_1$ (resp. $C_2$) with residue -1 (resp. +1) and two simple poles at $\infty_1, \infty_2$, normalized by the periods,

$$\int_{A_j} \omega = 2\pi i (\varepsilon_j + t) \quad \forall j = 1, \ldots, n.$$  

If any of the above poles overlap, the residues of $\omega$ at those points are added.

These periods incorporate the orientation of the helicoids (in the $\varepsilon_j$ term) and the screw motion rotation (recall that $S_t$ rotates the surface by $2\pi t$), which gives $G$ its multivaluedness.
We can compute the residues of $\omega$ at infinity using the residue theorem in $\mathbb{C}_1$:

- If $N \neq 0$, $dh$ has $n - 1$ zeros, and we have,
  \[
  \sum_{j=1}^{n} \varepsilon_j \int_{A_j} \omega = 2\pi i \left( \text{Res}_{\infty} \omega + \sum_{k=1}^{n-1} \text{Res}_{q_k} \omega \right),
  \]
  whence,
  \[
  \text{Res}_{\infty} \omega = n - 1 - \sum_{j=1}^{n} (1 + t\varepsilon_j) = -1 - Nt = -\text{Res}_{\infty} \omega.
  \]

- If $N = 0$, $dh$ has $n - 2$ zeros, and the same calculation gives us,
  \[
  \text{Res}_{\infty} \omega = -\text{Res}_{\infty} \omega = n - 2 - \sum_{j=1}^{n} (1 + t\varepsilon_j) = -2.
  \]

The same idea as that of Lemma 4.2 shows that $\omega$ is also symmetric with respect to $\sigma$:

**Lemma 5.2** The logarithmic differential satisfies $\sigma^* \omega = -\bar{\omega}$.

The argument from [10] also proves the following

**Lemma 5.3** When all $r_j \to 0$, $\omega$ converges on compact subsets of $\mathbb{C}_1 - \{p_j, q_k\}$ to the meromorphic 1-form $\omega_0$ on $\mathbb{C}_1$ given by:

\[
\omega_0 = \sum_{j=1}^{n} \frac{(1 + t\varepsilon_j)dz}{z - p_j} - \sum_{k} \frac{dz}{z - q_k},
\]
where $k$ is summed to $\deg(dh)_0$.

We wish for the Gauss map to have no multivaluation along the $B$ cycles, so we want the periods of $\omega$ to be integral multiples of $2\pi i$.

**Proposition 5.4** For $r_1$ small enough, there exist unique $r_2, \ldots, r_n$, depending continuously on $r_1$ and on $\{p_j\}$ and $t$, such that,

\[
\int_{B_j} \omega = 0 \mod 2\pi i.
\]

**Proof** We first compute,

\[
B_j + \sigma(B_j) = \Gamma_1 + \sigma(\gamma_j) - \Gamma_j - \gamma_j + \sigma(\Gamma_1) + \sigma^2(\gamma_j) - \sigma(\Gamma_j) - \sigma(\gamma_j)
= \Gamma_1 + \sigma(\Gamma_1) - (\Gamma_j + \sigma(\Gamma_j)) = A_1 - A_j.
\]

Note that we can deform the $B$ curves if necessary, without changing their homology classes, to avoid poles of $\omega$. This possibly changes the integral of $B_j$ by a multiple of $2\pi i$. The integral of $\sigma(B_j)$ will change by that same multiple by symmetry and thus $\int_{B_j + \sigma(B_j)} \omega$ changes by a multiple of $4\pi i$. Whence,
\[ \int_{B_j} \omega - \int_{B_j} \overline{\omega} = \int_{B_j} \omega + \int_{\sigma(B_j)} \omega = \int_{A_1} \omega - \int_{A_j} \omega + \text{residues of } \omega = 2\pi i (\varepsilon_1 + t) - 2\pi i (\varepsilon_j + t) + \text{residues of } \omega = 0 \mod 4\pi i. \]

Thus \( \text{Im} \int_{B_j} \omega = 0 \mod 2\pi \), as desired. To compute the real part, we use the following

**Lemma 5.5** For \( r_1, \ldots, r_n \) close enough to 0, the B-periods of \( \omega \) satisfy,

\[ \int_{B_j} \omega = 2(1 + \varepsilon_1 t) \log r_1 - 2(1 + \varepsilon_j t) \log r_j + \text{analytic} \]

where analytic means a bounded analytic function of all \( p_j, r_j, \) and \( t \).

**Proof** Note that the integral of \( \omega \) over \( \gamma_j \) and \( \sigma(\gamma_j) \) is already an analytic function of the parameters.

Since \( \Gamma_j \) is a curve from \( v_j = \varepsilon \) to \( v_j = -r_j^2 \varepsilon \), we can use a result from [10] to obtain the following estimate:

\[ \int_{\Gamma_j} \omega = (1 + \varepsilon_j t) \log \frac{-r_j^2}{\varepsilon^2} + \text{analytic} = 2(1 + \varepsilon_j t) \log r_j + \text{analytic}, \quad (1) \]

which proves the Lemma. \( \square \)

Consider the renormalized periods:

\[ \mathcal{F}_j = \frac{1}{\log r_1} \Re \int_{B_j} \omega. \]

We wish for these periods to vanish for \( j = 2, \ldots, n \), and we use the implicit function theorem to find values of \( r_2, \ldots, r_n \) that make this true. The functions \( \mathcal{F}_j \) extend continuously to 0 but not smoothly, so we make the standard substitution

\[ r_j = \exp \left( \frac{-s_j}{\tau^2} \right), \]

where \( s_1 = 1 \), and \( \tau \) depends on \( t \) and \( r_1 \). Note that \( s_j = -\tau^2 \log r_j > 0 \). We will describe later how \( r_1 \) and \( \tau \) will depend on \( t \). Thus, we have:

\[ \mathcal{F}_j = \tau^2 \left( 2(1 + t\varepsilon_1) \log \exp \left( \frac{1}{\tau^2} \right) - 2(1 + t\varepsilon_1) \log \exp \left( \frac{s_j}{\tau^2} \right) + \text{smooth} \right) = 2(1 + t\varepsilon_1) - 2s_j(1 + t\varepsilon_j) + \tau^2 \cdot \text{smooth}, \]

where smooth is a bounded smooth function of all \( p_j \), all \( s_j \) for \( j \geq 2, t, \) and \( \tau \).

Thus, \( \mathcal{F}_j \), as a function of \( \tau, s_2, \ldots, s_n \), extends smoothly to \( \tau = 0 \), and

\[ \mathcal{F}_j |_{\tau=0} = 2(1 + t\varepsilon_1) - 2s_j(1 + t\varepsilon_j). \]

When \( \tau = 0 \) and for \( t \) sufficiently small, we can solve \( \mathcal{F}_j = 0 \), and the derivative of \( (\mathcal{F}_2, \ldots, \mathcal{F}_n) \) with respect to \( s_2, \ldots, s_n \) is invertible. Thus, we can apply the implicit function theorem to guarantee solutions near \( \tau = 0 \), and all \( s_j \) are positive near the limit. We thus also obtain positive \( r_2, \ldots, r_n \) solving the period problem. \( \square \)
5.2 Integrating

We now obtain the Gauss map by integrating. Since $G$ is multivalued on $\Sigma$, we introduce a covering space on which it is well-defined. Let $\hat{\Sigma} \xrightarrow{\pi} \Sigma$ be the universal cover, and let $z_0$ be such that $\pi(z_0) = p_1 + \epsilon$. Now consider the quotient $\hat{\Sigma}$ of $\Sigma$ under the identification $z \sim z'$ if $\pi(z) = \pi(z')$ and $h(z) = h(z')$, where $h(z) = \Re \int_{z_0}^{z} dh$. Here we refer to pullbacks of $dh$ and $\omega$ by the same names.

We define the map $\hat{\sigma}$ of $\hat{\Sigma}$ as follows: for each $z$, $\hat{\sigma}(z)$ is the unique element of $\pi^{-1}(\sigma(\pi(z)))$ such that $h(\hat{\sigma}(z)) = h(z) + \pi$. We wish to define $M_t$ via an immersion on $\hat{\Sigma}$ and for $\hat{\sigma}$ to correspond to $s_t$.

**Definition 5.6** Let

$$G(z) = \Lambda i \exp \left( \int_{z_0}^{z} \omega \right) \quad z \in \hat{\Sigma},$$

where $\Lambda$ is a real constant making $\hat{\sigma}^* G = -\frac{e^{\pi it}}{G}$. This defines a meromorphic function on $\hat{\Sigma}$ with the property that $\hat{\sigma}^2 * G = e^{2\pi it} G$.

Note that $G$ is well-defined since every closed curve of $\hat{\Sigma}$ corresponds to the lift of a closed curve $\alpha$ of $\Sigma$ that satisfies $\Re \int_{\alpha} dh = 0$. These, however are composed of $A$ and $B$ curves with as many $A$ curves going up as down (cf. Sect. 3.2). By the definition of $\omega$ and by Proposition 5.4, $\int_{\alpha} \omega = 0 \mod 2\pi i$ for any such curve.

**Lemma 5.7** There exists a unique $\Lambda > 0$, depending on the parameters, such that $\hat{\sigma}^* G = -\frac{e^{\pi it}}{G}$.

**Proof** Using the symmetry of $\omega$, we get, for any $z \in \Sigma$:

$$\hat{\sigma}^* G(z) = \Lambda i \exp \left( \int_{z_0}^{\hat{\sigma}(z_0)} \omega + \int_{\hat{\sigma}(z_0)}^{\hat{\sigma}(z)} \omega \right)$$

$$= \exp \left( \int_{z_0}^{\hat{\sigma}(z_0)} \omega \right) \cdot \Lambda i \exp \left( \int_{z_0}^{z} \hat{\sigma}^* \omega \right)$$

$$= I \Lambda i \exp \left( \int_{z_0}^{z} \omega \right)$$

$$= \frac{\Lambda^2 I}{\Lambda i \exp \left( \int_{z_0}^{\hat{\sigma}(z_0)} \omega \right)} = \frac{\Lambda^2 I}{G(z)},$$

where $I = \exp \left( \int_{z_0}^{\hat{\sigma}(z_0)} \omega \right)$. To calculate the value of $I$, note that given our choice of $z_0$, we have, modulo $2\pi i$,

$$\int_{z_0}^{\hat{\sigma}(z_0)} \omega = \int_{z_0}^{\sigma(z_0)} \omega = \int_{\Gamma_1} \omega.$$
\[
\frac{1}{2i} \left( \int_{\Gamma_1} \omega + \int_{\Gamma_1} \sigma^* \omega \right)
= \frac{1}{2i} \int_{A_1} \omega = (\varepsilon_1 + t) \pi
\]

Hence, \( \text{mod } 2\pi i \), we have \( \int_{z_0} \hat{\sigma}(z_0) \omega = c + \pi i \varepsilon_1 + \pi i t \), where \( c = \text{Re} \int_{z_0} \hat{\sigma}(z_0) \omega \). Thus, \( I = -Ke^{\pi it} \), where \( K = e^c > 0 \) depends only on our choice of \( z_0 \). Thus, in order to ensure that \( \hat{\sigma}^* G = -e^{\pi it} G \), we need \( \Lambda = K^{-1/2} \).

We can estimate \( \Lambda \) near the limit \( r_1 \to 0 \) using Eq. (1):

\[
\Lambda = K^{-1/2} = \exp \left( -\frac{1}{2} \text{Re} \int_{\Gamma_1} \omega \right)
= \exp \left( -\frac{1}{2} (1 + \varepsilon_1 t) \log r_1^2 + \text{analytic} \right)
= \mathcal{O} \left( \frac{1}{r_1^{1+\varepsilon_1 t}} \right).
\]

Finally, we consider the limit of \( G \) as \( \tau \to 0 \). When calculating the limit, it will serve us to return to \( \Sigma \) and to consider \( G \) as a multivalued function by integrating from \( z_0 = p_1 + \epsilon \).

It has the following multivaluation: \( \gamma \) is any cycle on \( \Sigma \) with \( \frac{1}{2\pi} \text{Re} \int_{\gamma} dh = k \in \mathbb{Z} \), then analytic continuation of \( G \) along \( \gamma \) multiplies \( G \) by \( e^{2k\pi it} \). Considering the limit of \( \omega \), we obtain,

**Lemma 5.8** On compact subsets of \( \mathbb{C}_1 - \{p_j, q_k\} \), we have, as \( \tau \to 0 \),

\[
G \sim \Lambda i \frac{G_0(z)}{G_0(z_0)},
\]

where

\[
G_0(z) = \prod_{j=1}^n \frac{(z - p_j)^{1+i\varepsilon_j}}{\prod_k (z - q_k)}.
\]

Both \( G \) and \( G_0 \) are multi-valued, but their multi-valuation is the same given analytic continuations along the same paths. We will normalize \( t \) so that it will vanish when \( \tau = 0 \), making \( G \) and \( G_0 \) single-valued. Now, in the limit, \( G_0 \) has the same zeros and poles as the function \( \frac{dz}{dh_0} \), so we obtain,

\[
G \sim \Lambda c_0 \frac{dz}{dh_0}, \quad \text{where } c_0 = \sum_{j=1}^n \frac{\varepsilon_j}{z_0 - p_j}.
\]

We can guarantee that \( c_0 \neq 0 \) by making \( \epsilon \) small enough to have \( dh_0(z_0) = dh_0(p_1 + \epsilon) \neq 0 \).

### 6 Horizontal period problem

We now consider the horizontal period problem and use the implicit function theorem to guarantee that the periods are closed in a neighborhood of \( \tau = 0 \). Let

\[
\mu = \frac{1}{2} \left( G^{-1}dh - Gdh \right) = dx_1 + idx_2
\]
be the horizontal differential. This form is multivalued, so its integral on a closed curve is not homology invariant.

Consider the curves $C_j$ of $\Sigma_1$, obtained by composing the following paths (Fig. 7):

1. The curve $A_1$,
2. the path $\gamma_j$ from $v_1 = \epsilon$ to $v_j = \epsilon$,
3. the curve $A_j^{-1}$,
4. the path $\gamma_j^{-1}$.

Note that $\int_{B_j} dh = \int_{C_j} dh = 0 \ \forall j \geq 2$,

so these curves all lift to closed curves of $\hat{\Sigma}$, which we call $\hat{B}_j$ and $\hat{C}_j$.

The same argument as Proposition 2 of [8] proves the following

**Proposition 6.1** Suppose the equations

$$\int_{\hat{B}_j} \mu = \int_{\hat{C}_j} \mu = 0 \ \forall j \geq 2,$$

are satisfied. Then there exists a screw motion $S_t$ of angle $2\pi t$ and translation part $(0, 0, 2\pi)$ such that $Re \int_0^z \phi$ is well-defined modulo $S_t$, where $\phi = (\phi_1, \phi_2, \phi_3)$ are the components of the Weierstrass formula.

We now reduce the period problem to only $C$-periods.

**Proposition 6.2** For $t < 1$, the $B$-periods vanish if and only if the $C$-periods do.

**Proof** We first compute $\hat{\sigma}^* \mu$. Since $\hat{\sigma}^* dh = \overline{dh}$ and $\hat{\sigma}^* G = -\frac{e^{\pi it}}{G}$, we conclude that,

$$\hat{\sigma}^* \mu = \hat{\sigma}^* \left( G^{-1} dh - G dh \right) = \frac{1}{-e^{\pi it}} \frac{dh}{G} - \frac{e^{\pi it}}{G} dh = e^{\pi it} \mu.$$

Now note that since $\int_{A_j} dh = 2\pi$ for any $j$, $\hat{C}_j = \hat{A}_1 * \hat{\sigma}^2(\gamma_j) * \hat{A}_j^{-1} * \gamma_j^{-1}$. By the same reasoning, $\int_{\gamma_j^{-1}} dh = \pi$ implies that $\hat{B}_j = \hat{\Gamma}_1 * \hat{\sigma}(\gamma_j) * \hat{\Gamma}_j^{-1} * \gamma_j^{-1}$. Finally, since $\hat{\Gamma}_j * \hat{\sigma}(\hat{\Gamma}_j) = \hat{A}_j, \hat{B}_j * \hat{\sigma}(\hat{B}_j)$ is homologous to $\hat{C}_j$. We thus obtain:

$$\int_{\hat{C}_j} \mu = \int_{\hat{B}_j} \mu + \int_{\hat{\sigma}(\hat{B}_j)} \mu = \int_{\hat{B}_j} \mu + \int_{\hat{B}_j} e^{\pi it} \mu = (1 + e^{\pi it}) \int_{\hat{B}_j} \mu.$$

We now reduce the period problem to only $C$-periods.
We conclude that for all $t < 1$, $\int_{\hat{B}_j} \mu = 0$ if and only if $\int_{\hat{C}_j} \mu = 0$. 

\[ \int_{\hat{B}_j} \mu = 0 \text{ if and only if } \int_{\hat{C}_j} \mu = 0. \]

\[ \blacksquare \]

**Proposition 6.3** If we set $t = \frac{4}{|c_0|^2} e^{-2/\tau}$, then, as $\tau \to 0$, the rescaled C-periods have the following limit:

\[
\lim_{\tau \to 0} \Lambda \int_{\hat{C}_j} \mu = \frac{4\pi i}{c_0} \left( \sum_{k \neq 1} \frac{\epsilon_k}{p_1 - p_k} - \sum_{k \neq j} \frac{\epsilon_k}{p_j - p_k} + p_1 - p_j \right)
\]

**Proof** For this proof, we return to the context of $\Sigma_1$, where $\mu$ is multivalued and use that $\int_{\hat{C}_j} \mu = \int_{C_j} \mu$. We now adapt an argument of [8], that is, we estimate the integrals for $G^{-1}dh$ and $Gdh$ separately, as $\tau \to 0$.

Since $C_j$ is in $\Omega$, we can use the limit of $G^{-1}dh$ in $C_1$:

\[
\lim_{\tau \to 0} \Lambda \int_{C_j} G^{-1}dh = \int_{C_j} \frac{1}{c_0} \frac{dh_0^2}{dz} = \frac{4\pi i}{c_0} \left( \epsilon_1 \text{Res}_{p_1} \frac{dh_0^2}{dz} - \epsilon_j \text{Res}_{p_j} \frac{dh_0^2}{dz} \right).
\]

We now compute the residues:

\[
\text{Res}_{p_j} \frac{dh_0^2}{dz} = \text{Res}_{p_j} \left( \sum_{k=1}^n \frac{-\epsilon_k i}{z - p_k} \right)^2 = -\text{Res}_{p_j} \left( \frac{1}{(z - p_j)^2} + \frac{2}{z - p_j} \sum_{k \neq j} \frac{\epsilon_j \epsilon_k}{z - p_k} + \sum_{k \neq j} \frac{\epsilon_k \epsilon_l}{(z - p_k)(z - p_l)} \right)
\]

\[ = -2 \sum_{k \neq j} \frac{\epsilon_j \epsilon_k}{(p_j - p_k)}, \]

and conclude,

\[
\lim_{\tau \to 0} \Lambda \int_{C_j} G^{-1}dh = -\frac{4\pi i}{c_0} \left( \sum_{k \neq 1} \frac{\epsilon_k}{(p_1 - p_k)} - \sum_{k \neq j} \frac{\epsilon_k}{(p_j - p_k)} \right)
\]

Estimating $\Lambda \int_{C_j} Gdh$ is more difficult since $\Lambda Gdh \to \Lambda^2 c_0 dz$ becomes infinite, so we will compare it with the integral of $\Lambda G^{-1}dh$. We wish to use the more symmetric path $A_1 - A_j$ homologous to $C_j$, but since $G$ is not single-valued, the integral is not homology invariant. We thus use the corrective (multivalued) factor defined on $C_1$ by:

\[ \psi(z) = (z - p_1)^{-\epsilon_1} (z - p_j)^{-\epsilon_j}. \]

The form $\psi Gdh$ is single-valued on $C_j$, so the integral will be the same as that on $A_1 - A_j$. Hence,

\[
\int_{C_j} \psi Gdh = \int_{A_1} \psi Gdh - \int_{A_j} \psi Gdh.
\]

Note also, for any $j$, that for $z \in A_j$,

\[ \psi(\sigma(z)) = \psi\left(\frac{-r_j^2}{z - p_j} + p_j\right) \sim r_j^\pm 1 \to 1. \]
Using that $\sigma(A_j) \sim A_j$, we obtain,
\[ \Lambda \int_{A_j} \psi Gdh = \Lambda \int_{\sigma(A_j)} \sigma^*(\psi Gdh) = \Lambda \int_{A_j} -\frac{e^{\pi i t}}{G} \sigma \frac{dh}{dh} \to \lim_{\tau \to 0} -\Lambda \int_{A_j} G^{-1}dh \]
as $\tau \to 0$. Hence,
\[ \lim_{\tau \to 0} \Lambda \int_{C_j} \psi Gdh = -\lim_{\tau \to 0} \Lambda \int_{C_j} G^{-1}dh, \]
which we know from the above computation.

The computation of the integral of $(1 - \psi)Gdh$ is also the same as in [8]:
\[ \lim_{\tau \to 0} \Lambda \int_{C_j} (1 - \psi)Gdh = \lim_{\tau \to 0} -\Lambda t \int_{C_j} Gdh \frac{(z - p_1)^{-\varepsilon_1}(z - p_j)^{-\varepsilon_j})'}{t} \]
\[ = \lim_{\tau \to 0} (-\Lambda^2 t)c_0 \int_{C_j} \log((z - p_1)^{-\varepsilon_1}(z - p_j)^{-\varepsilon_j}) dz \]
\[ = -\lim_{\tau \to 0} (\Lambda^2 t)c_0 \cdot 2\pi i (p_1 - p_j). \]

We now express determine $t$ as a smooth function of $\tau$ in order to guarantee the above limit exists and is nonzero. By the estimate (2) and the definition of $\tau$, our choice of $t$ guarantees that $\lim_{\tau \to 0} \Lambda^2 t = \frac{4}{|c_0|^2}$. We thus obtain,
\[ \lim_{\tau \to 0} \Lambda \int_{C_j} (1 - \psi)Gdh = -\frac{8\pi i}{c_0}(p_1 - p_j). \]

Hence, we have:
\[ \lim_{\tau \to 0} \Lambda \int_{C_j} \mu = \frac{4\pi i}{c_0} \left( \sum_{k \neq 1} \frac{\varepsilon_k}{p_1 - p_k} - \sum_{k \neq j} \frac{\varepsilon_k}{p_j - p_k} + p_1 - p_j \right) \]
\[ \square \]

**Proposition 6.4** Suppose $\{(p_j^0, \varepsilon_j)\}$ is a balanced configuration In a neighborhood of $\tau = 0$, there exist smooth functions $p_1(\tau), \ldots, p_n(\tau)$ solving the horizontal period problem, with $p_j(0) = p_j^0$.

**Proof** We define the renormalized periods
\[ \mathcal{F}_j = \frac{c_0}{4\pi i} \int_{C_j} \mu \]
for $j \geq 2$. Since the $C_j$ curves lie in a compact subset of $C_1 - \{p_j\}$ and since the forms converge smoothly, we conclude that these periods extend smoothly to $\tau = 0$. Since
\[ \lim_{\tau \to 0} \mathcal{F}_j = \mathcal{F}_1 - \mathcal{F}_j \]
when $\tau = 0$, the period problem amounts to all $\mathcal{F}_j$ being equal. Translating the points in the configuration by the same amount allows us to assume without loss that the period problem is equivalent to $\mathcal{F}_j = 0$ for all $j$.

As mentioned in Sect. 2, we can assume that $p_1$ is real and consider $F = (F_1, \ldots, F_n)$ as $2n$ real functions in $2n - 1$ variables. Note that there is a linear dependence of the derivative...
vectors of $F$ coming from rotation (which allows us to write $\frac{\partial \Re F_1}{\partial p}$ in terms of the other derivatives). Therefore, by the implicit function theorem, if the configuration is nondegenerate, then there exist unique parameters $x_j = \Re p_j$, $y_j = \Im p_j$ as functions of $\tau$ that solve the period problem in a neighborhood of $\tau = 0$. 

\section{7 Proof of embeddedness}

It remains to show that the SMIMS $M_t$ corresponding to a balanced configuration are embedded near the limit. To this end, we modify proof in [8], and show that the surface is embedded in three overlapping regions. The major differences are that we do not assume the existence of any straight lines, we use a different conformal parameter, and we only use the symmetry $\sigma$.

The original argument shows that after rescaling and translating, the image of $\Omega$ (see Sect. 3.2) converges to the graph of the multivalued function

$$f(z) = \sum_{j=1}^{n} \varepsilon_j \arg(z - p_j).$$

The symmetry implies that on $\sigma(\Omega)$, the images converges to the graph of $f(z) + \pi$. These are both embedded and will not intersect each other.

We now consider each region $\frac{r_j}{c} < |z - p_j| \leq c r_j$, where $c >> 1$. Traizet and Weber use a parameter for slit region of their construction, but we use the following parameter for the annular regions:

$$\zeta = \frac{z - p_j}{r_j}.$$

In the region $\frac{r_j}{c} \leq |\zeta| \leq c$ of $\mathbb{C}_1$, we have,

$$\lim_{\tau \to 0} d\mu = -i \varepsilon_j d\zeta, \quad \lim_{\tau \to 0} \omega = \frac{d\zeta}{\zeta},$$

where the convergence is uniform on compact subsets of $0 < |\zeta| < c$. To get $G$, we integrate,

$$\lim_{\tau \to 0} G = \Lambda \exp \left( \int_{\zeta}^{\infty} \frac{du}{u} \right) = \Lambda \zeta,$$

where $\Lambda = \exp \left( -\frac{1}{2} \Re \int_{1}^{\infty} \frac{d\zeta}{\zeta} \right) = 1$, as calculated in the proof of Lemma 5.7 (here the basepoint of integration is $z_0 = p_j + r_j$). These are the Weierstrass data of a helicoid, right or left depending on if $\varepsilon_j = 1$ or $-1$. Therefore, the image of $|z - p_j| \leq c$ is embedded near the limit.

It remains to show that the image is embedded for $cr_j < |z - p_j| < \epsilon$. Using the estimate

$$|G(z)| = \left( \frac{|z - p_j|}{r_j} \right)^{1+\varepsilon_j} \cdot O(1),$$

we can assume $c$ is large enough to make $|G| > 1$, and hence the function $(X_1, X_2)$ is regular. The (multivalued) image of this annulus is an infinite periodic strip bounded by two curves close to helices (see the previous two cases) and thus have monotone height. We claim that the intersection of this image with any horizontal plane consists of a simple curve joining each boundary.
This intersection is always non-empty, and there cannot be a null-homotopic loop, by the maximum principle. There can also not be a curve with two endpoints on the same boundary as its height is monotone. Therefore, there must be a curve joining the two boundaries, and it must be unique, for the same reasons as the cases we just excluded.

We now claim that this curve must be simple. Indeed, if it intersects itself, then continuing vertically there must be an annular region where the surface intersects itself. Because $|G| > 1$, this region must narrow as $h$ either increases or decreases. But since the strip is periodic, it must stop narrowing. Moreover, since $|G| > 1$, this will only happen at a cone point, contradiction the assumption that $(X_1, X_2)$ is regular.

Since the cross sections with respect to each horizontal plane consist of a unique simple curve, we conclude that this region is embedded.

We conclude that the entire surface is embedded, and the same argument as in [8] shows that the rescaled surface converges to the multigraph described above.

### 8 Applying Theorem 2.5 directly

#### 8.1 The simplest new application

We can directly apply Theorem 2.5 without any symmetry assumptions. To illustrate, let us consider the simplest example outside the scope of [8]. The configuration consists of three noncolinear points, all with charge $-1$.

If all points are third roots of unity, that is powers of $\eta := e^{2\pi i/3}$, then they solve the balance equations:

$$F_1 = 1 + \frac{1}{1 - \eta} + \frac{1}{\eta^2 - \eta} = 1 + \frac{-1 + \eta}{1 - \eta} = 0,$$

$$F_2 = \bar{\eta} F_1 = 0, \quad F_3 = \eta^2 F_1 = 0.$$

We can easily compute the nondegeneracy matrix of Definition 2.4 by utilizing the symmetry and the Wirtinger derivatives. For instance, we compute:

$$\frac{\partial F_1}{\partial p_1} = \frac{1}{(p_1 - p_2)^2} + \frac{1}{(p_1 - p_3)^2} = \frac{1}{(1 - \eta)^2} + \frac{1}{\eta(1 - \eta)^2}$$

$$= \frac{1}{-3\eta} + \frac{1}{-3\eta^2} = -\frac{1}{3},$$

$$\frac{\partial F_1}{\partial p_2} = \frac{\partial F_1}{\partial p_3} = \frac{1}{3\eta} = \frac{1}{6} - \frac{1}{2\sqrt{3}}i.$$

By the symmetry and the cyclic nature of the configuration, and using

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}},$$

$$\frac{\partial f}{\partial y} = -i \left( \frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}} \right),$$
we compute the matrix:
\[
\begin{pmatrix}
\frac{4}{3} & -\frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\
-\frac{1}{6} & \frac{5}{6} & \frac{1}{3} & \frac{1}{3} & \frac{2\sqrt{3}}{2\sqrt{3}} & 0 \\
-\frac{1}{6} & \frac{1}{3} & \frac{5}{6} & -\frac{1}{3} & -\frac{2\sqrt{3}}{2\sqrt{3}} & 0 \\
0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\
0 & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}
\]

This matrix indeed has rank 5 \((2n - 1)\); so the configuration is nondegenerate, and the theorem applies.

**Remark 8.1** This configuration corresponds to the 6-ended Karcher–Scherk surface. Interestingly, Theorem 2.5 applies directly to all \(KS\) examples except to the one with 14 ends, that is, with a configuration consisting of 7 points. For this example, the nondegeneracy matrix has rank \(12 = 2n - 2\). We will see in the next section that if we assume dihedral symmetries, we can apply Theorem 2.7 to generate all \(KS\) surfaces near the limit.

### 8.2 Nonsymmetric configurations

All known SMIMS admit more symmetries than screw motions, and we dedicate the next section to section to generating symmetric surfaces. However, Theorem 2.5 applies to balanced and nondegenerate configurations that are not symmetric. Due to a connection with fluid dynamics (see Sect. 10), some applied mathematicians and physicists have used algebraic and numerical methods to find balanced configurations [11].

The balanced configurations they find have a small overlap with ours, mostly obtaining less symmetric solutions which would correspond to higher genus surfaces. For example, there is a balanced configuration with no dihedral symmetry. Numerical evidence suggests this configuration is nondegenerate, in which case there is a SMIMS which is only symmetric under screw motions (see Conjecture 1.5) (Fig. 8).

### 9 Helicoidal surfaces with dihedral symmetries

We now restrict our attention to configurations which are dihedrally symmetric. The dihedral symmetries induce symmetries of the minimal surface, and they simplify the balance equations significantly. We will use these symmetries to construct one-parameter families of symmetric surfaces.

#### 9.1 Dihedral symmetries

We first consider some of the symmetries induced by dihedrally symmetric configurations.

**Definition 9.1** We call a configuration \(k\)-fold dihedrally symmetric if it is invariant under the dihedral group \(D_k\) of order \(2k\) and, if \(p_i = \eta(p_j)\) for some \(\eta \in D_k\), then \(\varepsilon_i = \varepsilon_j\).

We say that a Riemann surface \(\Sigma\) constructed from a configuration as in Sect. 3.1 is dihedrally symmetric if the configuration is dihedrally symmetric and if \(p_i = \eta(p_j) \Rightarrow r_i = r_j\).
Remark 9.2  We will assume that $k$ is the greatest integer for which a configuration is symmetric.

Remark 9.3  For simplicity of notation, we will assume that our dihedrally symmetric configurations contain a point $p_0$ at the origin with charge $\varepsilon_0$. If $\varepsilon_0 = 0$, then there is no central helicoid, and the point does not factor into the count of $n$.

We desire to construct symmetric SMIMS $M_t$ from dihedrally symmetric $\Sigma_1 = M_t / S_t$. The symmetries of the configuration induce symmetries of $\Sigma$ and finally of the surface itself, which we describe below. We will focus on two symmetries which generate $D_k$, namely a reflection and a rotation.

Let $\rho$ be the symmetry of $\Sigma$ defined by conjugation in each copy of $\hat{C}$ and let $\theta$ be the rotation of angle $\frac{2\pi}{k}$ about the origin, counter-clockwise in $C_1$ and clockwise in $C_2$.

Proposition 9.4  The symmetries $\rho$ and $\theta$ are well-defined on $\Sigma$. The map $\rho$ induces a rotation across a horizontal line on the surface, and $\theta$ induces a vertical screw motion of angle $\frac{2\pi}{k} (1 + \varepsilon_0)$, with translation $\frac{2\varepsilon_0 \pi}{k}$.

Proof  A similar argument to the proof of Lemma 3.1 shows that these maps are well-defined. Indeed, they both send each helicoidal neck to another of the same size and are compatible with identifications.

Now a similar argument to the proof of Lemmas 4.2 and 5.7 shows the following:

$$\rho^* dh = -d\bar{h}, \quad \rho^* \mu = \bar{\mu},$$

which signifies that $\rho$ is indeed a rotation about a horizontal line.

It also shows that

$$\theta^* dh = dh, \quad \theta^* \omega = \omega.$$
We compute,

\[ G(\theta(z)) = \Lambda i \exp \left( \int_{z_0}^{\theta(z_0)} + \int_{\theta(z_0)}^{\theta(z)} \omega \right) \]

\[ = \exp \left( \int_{z_0}^{\theta(z_0)} \right) \cdot \Lambda i \exp \left( \int_{z_0}^{z} \theta^* \omega \right) \]

\[ = J \cdot \Lambda i \exp \left( \int_{z_0}^{z} \omega \right) = J \cdot G(z), \]

where \( J = \exp \left( \int_{z_0}^{\theta(z_0)} \right). \) To calculate \( J \), we again use the symmetries of \( \omega \). Consider the circle of radius \( |z_0| \) about the origin in \( \mathbb{C}_1 \) and let \( \gamma \) be the arc on this circle from \( z_0 \) to \( \theta(z_0) \). Since the poles of \( \omega \) (the zeros of \( dh \)) come in multiples of \( k \) away from the origin, any possible poles in the region \( 0 < |z| < |z_0| \) will only change the integral \( \int_{|z|=|z_0|} \omega \) by a multiple of \( 2\pi i k \). Thus, mod \( 2\pi i k \), we have:

\[ \int_{|z|=|z_0|} \omega = \int_{\gamma} \omega + \int_{\theta(\gamma)} \omega + \cdots + \int_{\theta^{k-1}(\gamma)} \omega \]

\[ = \int_{\gamma} \omega + \int_{\gamma} \theta^* \omega + \cdots + \int_{\gamma} (\theta^*)^{k-1} \omega \]

\[ = k \int_{\gamma} \omega \]

Hence \( \int_{\gamma} \omega = \frac{2\pi i}{k} \text{Res}_0 \omega \mod 2\pi i. \) We compute this residue in the different cases. When \( \epsilon_0 = 0 \), the origin is a point of \( \Sigma \), and we have,

\[ \int_{|z|=|z_0|} \omega = 2\pi i \text{Res}_0 \omega = -2\pi i \text{ord}_0(dh). \]

But since \( dh \) is symmetric with respect to \( \theta \), which has order \( k \), \( \text{ord}_0(dh) = k - 1 \mod k \), so mod \( 2\pi i k \), we have,

\[ \int_{|z|=|z_0|} \omega = -2\pi i (k - 1) = 2\pi i (1 + \epsilon_0 t). \]

If \( \epsilon_0 \neq 0 \) and the configuration includes a central helicoid then (again, mod \( 2\pi i k \)),

\[ \int_{|z|=|z_0|} \omega = \epsilon_0 \int_{A_0} \omega = 2\pi i (1 + \epsilon_0 t). \]

We conclude that, in all cases,

\[ J = e^{2\pi i (1+\epsilon_0 t)/k}. \]

Therefore, \( \theta^*\mu = e^{2\pi i (1+\epsilon_0 t)/k} \mu \), from which we conclude that \( \theta \) is a screw motion with the desired angle. To calculate the translational part, we use the same idea:

\[ \text{Re} \int_{z_0}^{\theta(z_0)} dh = \text{Re} \int_{\gamma} dh = \frac{1}{k} \text{Re} \int_{|z|=|z_0|} dh = \frac{2\pi \epsilon_0}{k}. \]

\( \square \)
Remark 9.5 All dihedral reflections being compositions of $\rho$ with powers of $\theta$ also correspond to reflections about horizontal lines. When $\varepsilon_0 \neq 0$, these horizontal lines will have different heights than that of $\rho$.

Corollary 9.6 $M_t$ also has a family of rotations about normal symmetry lines.

Proof We obtain these by composing each $\rho$ symmetry with the screw motion $\sigma$. Since they are both orientation-reversing, we obtain an orientation-preserving rotation. $\square$

Corollary 9.7 If $k$ is even, then the surface is invariant under a vertical rotation of angle $\pi$. If $\varepsilon_0 \neq 0$, then the fixed point set of this rotation is the central helicoidal axis, which is a vertical line.

Proof We define this rotation in terms of $\sigma$ and $\theta$. Now $\theta^{k/2}$ translates by $\pi \varepsilon_0$ and rotates by $\pi (1 + \varepsilon_0 t)$. If $\varepsilon_0 = 0$ then we have our desired rotation. Otherwise, recall that $\sigma$ translates by $\pi$ and rotates by $\pi t$. Therefore the map $\sigma^{-\varepsilon_0} \circ \theta^{k/2}$ is a rotation by $\pi$.

We can determine fixed points by looking at $\Sigma$. If $\varepsilon_0 = 0$, only the origins will be fixed. If $\varepsilon_0 \neq 0$, this map will send a point $z$ in one copy of $\mathbb{C}$ to $-\bar{z}$ in the other. The only possible fixed points must be near the origin and solve the equation,

$$v_0(z)w_0(-\bar{z}) = -r_0^2,$$

that is the points of the central helicoidal axis, $\{ |z| = r_0 \}$. $\square$

9.2 A different quotient

Before considering specific examples of dihedrally symmetric surfaces, it is useful to consider a more natural way of quotienting $M_t$, which is helpful for identifying our surfaces with ones already known.

Every helicoidal surface we can make with $k$-fold dihedral symmetry has a corresponding quotient with $2|N|$ helicoidal ends and the same $k$-fold symmetry. It is obtained as follows.

Recall that our surface $\Sigma = M_t / S_t$ has two helicoidal ends each with winding number $1 + Nt$. Consider now its $|N|$-fold unbranched cover $\tilde{\Sigma} = M_t / S_t^{|N|}$. Note that $\tilde{\Sigma}$ has $2|N|$ ends, each with winding number $1 + Nt$, and it has $S_t$ and $\theta$ as symmetries.

Now consider the screw motion $\tilde{S}_t$ consisting of a rotation of $2\pi |N|(t + 1)$ and translation by $\frac{2|N|\pi}{k}$. We verify that this is a symmetry of $\tilde{\Sigma}$ by writing it as a composition of powers of $S_t$ and $\theta$. There are three cases to consider:

- $|N| = mk$ for some $m \in \mathbb{N}$: in this case, $\varepsilon_0 = 0$, and $\tilde{S}_t = \sigma^{|N|} \circ \theta$,
- $|N| = mk + 1$: in this case, $\tilde{S}_t = \sigma^{mk} \circ \theta^{\varepsilon_0}$,
- $|N| = mk - 1$: in this case, $\tilde{S}_t = \sigma^{mk} \circ \theta^{-\varepsilon_0}$.

We call $\Sigma' := \tilde{\Sigma} / \tilde{S}_t$ the natural quotient corresponding to $M_t$. $\tilde{S}_t$ only identifies parts of the same end, so $\Sigma'$ has $2|N|$ ends, of winding number $\frac{1 + Nt}{k}$, and $\tilde{\Sigma}$ is a $k$-fold branched cover. This makes $\Sigma'$ the smallest quotient of $M_t$ with $2|N|$ ends. This makes it useful for classifying dihedrally symmetric helicoidal surfaces. It is also the natural identification used to describe twisted versions of the well-known Scherck and Fischer–Koch surfaces.

We can calculate the genera of $\tilde{\Sigma}$ and $\Sigma'$ using Riemann-Hurwitz. Recall that $\Sigma$ has genus $n - 1$ and that the quotient of $\tilde{\Sigma}$ by $S_t$ is $|N|$-fold unbranched. So the genus $\tilde{g}$ of $\tilde{\Sigma}$ is given by

$$\tilde{g} = |N|(n - 2) + 1.$$
Fig. 9 Two simply dihedral configurations

Now the quotient of $\tilde{\Sigma}$ by $\hat{S}_r$ is a $k-$fold branched cover, branched at each end, with index $k$. Thus, again by Riemann-Hurwitz, we have:

$$\tilde{g} = k(g' - 1) + 1 + |N|(k - 1),$$

where $g'$ is the genus of $M'_t$. We thus obtain,

$$g' = \frac{|N|(n - k - 1)}{k} + 1. \tag{3}$$

**Remark 9.8** If $N = 0$, there is also a different quotient that corresponds to those of known examples, such $CHM$. This is the quotient with respect to the map $\Sigma' = M_t / (S_t \circ \theta)$. While it is more natural, it has the same genus and number of ends as $\Sigma$.

### 9.3 Regenerating symmetric surfaces

We now adapt Theorem 2.5 to families of dihedrally symmetric surfaces. If we assume symmetries, then the period problems are greatly simplified. This translates to simpler balance equations, which are much easier to solve.

**Definition 9.9** We say that a symmetric configuration has simple dihedral symmetry of order $k$ if its points lie on the axes of dihedral symmetry. To be precise, it consists of $n_1$ points on the positive real axis, $n_2$ (possibly 0) points on the ray $\text{Arg}(z) = \frac{\pi}{k}$, and $n_0 = 0$ or 1 point at the origin, and their images under $D_k$ (Fig. 9).

**Remark 9.10** A configuration with simple dihedral symmetry consists of $k(n_1 + n_2) + n_0$ points.

Under this assumption, we can simplify the balance equations of Definition 2.1. Since balance equations are symmetric under rotation, equations corresponding to symmetric points are equivalent. The symmetry also implies that $F_0 = 0$ in any dihedral configuration, and thus we have $n_1 + n_2$ real equations. Note that there are also $n_1 + n_2$ variables since the points of the configuration are of the form

$$p_{1j} e^{\frac{m \pi i}{k}}, \quad p_{2j} e^{\frac{\pi i}{k}} e^{\frac{m \pi i}{k}}, \quad j = 1, \ldots, n_1, \quad p_0 = 0,$$

where $p_{11} < \cdots < p_{1n_1}, p_{21} < \cdots < p_{2n_2}, \phi = e^{2\pi i / k}$, and $m = 0, \ldots, k - 1$. 

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The balance equations simplify to the following sums vanishing:

$$F_{ij} = p_{1j} + \sum_{m=1}^{k-1} \frac{\varepsilon_{1j}}{p_{1j} - p_{1j}' \varphi^m} + \sum_{l \neq j}^{k} \sum_{m=1}^{k} \frac{\varepsilon_{1l}}{p_{1l} - p_{1l}' \varphi^m} + \sum_{l=1}^{n_2} \sum_{m=1}^{k} \frac{\varepsilon_{2l}}{p_{1j} - p_{1j}' e^{\frac{i\pi}{k}} \varphi^m} + \frac{\varepsilon_0}{p_{1j}}$$

$$= p_{1j} + \frac{(k - 1)\varepsilon_{1j} + 2\varepsilon_0}{2p_{1j}} + \sum_{l \neq j}^{k} \frac{\varepsilon_{1l} k p_{1j}^{-k-1}}{p_{1j}' - p_{1j}' k} + \sum_{l=1}^{n_2} \frac{\varepsilon_{2l} k p_{1j}^{-k-1}}{p_{1j} - p_{1j}'},$$

$$F_{2j} = p_{2j} + \frac{(k - 1)\varepsilon_{2j} + 2\varepsilon_0}{2p_{2j}} + \sum_{l \neq j}^{k} \frac{\varepsilon_{2l} k p_{2j}^{-k-1}}{p_{2j}' - p_{2j}' k} + \sum_{l=1}^{n_1} \frac{\varepsilon_{1l} k p_{2j}^{-k-1}}{p_{2j} - p_{2j}'},$$

where we multiply $F_{2j}$ by $e^{\frac{i\pi}{k}}$ to make it real. The simplifications come from the following elementary relations:

$$\sum_{m=1}^{k-1} \frac{1}{1 - \varphi^m} = \frac{k - 1}{2},$$

$$\sum_{m=1}^{k} \frac{1}{x - \varphi^m y} = \frac{k \lambda^{-1}}{\lambda - y},$$

**Definition 9.11** A simply dihedrally symmetric configuration is symmetrically nondegenerate if the Jacobian matrix \(\frac{\partial F_{ij}}{\partial p_{ij}}\) is invertible. Its terms are given by:

$$\frac{\partial F_{ij}}{\partial p_{ij}} = 1 - \frac{(k - 1)\varepsilon_{ij} + 2\varepsilon_0}{2p_{ij}^2} + \sum_{(i', j') \neq (i, j)} k p_{ij}^{2k-2} \pm \varepsilon_{i'j'} k (k - 1) p_{ij}^{k-2} p_{i'j'}^{k-1},$$

$$\frac{\partial F_{ij}}{\partial p_{i'j'}} = \pm \varepsilon_{i'j'} k^2 p_{ij}^{k-1} p_{i'j'}^{k-1},$$

where the \(\pm\) symbol reflects whether \(i = i'\) or not.

We can repeat the construction of Sect. 3, assuming the points and the radii \(r_j\) are symmetric.

There are \(n_1 + n_2\) real parameters determining the points, and there are also \(n_1 + n_2 + n_0\) real parameters \(r_{ij}, i = 1, 2, j = 1, \ldots, n_1\) (and possibly \(r_0\)), that correspond to the radii of identification.

Recall that \(r_{11}\) is determined by the condition \(\lim_{r \to 0} \Lambda = \frac{A}{|c_0|^2}\) and hence will not be a free parameter near the limit. We thus expect the period problem to reduce to \(2(n_1 + n_2 + n_0 - 1)\) real equations, one for each remaining parameter. This is indeed the case:

**Proposition 9.12** For a configuration with simple dihedral symmetry, the period problem consists of \(n_1 + n_2 + n_0 - 1\) B-periods of \(\omega\) and \(n_1 + n_2\) real C-periods.

**Proof** Note that there are a total of \(2(n - 1) = 2k(n_1 + n_2) + n_0 - 2\) paths over which to integrate. To reduce the number of periods we write all period paths as compositions of a small number of curves and their images under symmetries. This will also help us to reduce the C-periods to real equations.

Recall the definitions of the curves \(B_j\) and \(C_j\) defined in Sects. 3.2 and 6. These were all defined in terms of two points \(p_1\) and \(p_j\). Consider the following subset of curves:
• $B_{1j}$, corresponding to the points $p_{11}$ and $p_{1j}$, where $j \geq 2$,
• $B_{2j}$, corresponding to the points $p_{11}$ and $p_{2j}e^{\pi j/k}$, where $j \geq 1$,
• $B_{11}$, corresponding to the points $p_{11}$ and $p_{11}e^{2\pi j/k}$,
• $B_0$, corresponding to the points $p_{11}$ and the origin, if it is in the configuration,
• the corresponding $C$-curves, using the same indices.

The $B$-periods of $\omega$ are already real, and it is a simple matter to show that all $B$ paths are compositions of the curves $B_{11}, \ldots, B_{1n_1}, B_{21}, \ldots, B_{2n_2}, B_0$ (if there is a central helicoid), and their images under dihedral symmetries. Moreover, the period over $B_{11}$ always vanishes for a simple dihedral configuration. Indeed, the sum $B_{11} + \theta(B_{11}) + \cdots + \theta^{k-1}(B_{11})$ is homologous to a sum of $A$-curves, which shows the period is closed.

Therefore, if the periods over the remaining $n_1 + n_2 + n_0 - 1$ curves vanish then all $B$-periods are closed.

The same idea as above applies to the $C$-periods: if the horizontal periods over the curves $C_{11}, \ldots, C_{1n_1}, C_{21}, \ldots, C_{2n_2},$ and $C_0$ vanish, then all do. The set of $C$-periods that need to vanish for the period problem to be solved depends on the values of $n_2$ and $n_0$, as shown in the table below:

| $n_0$ | $n_2$ | $C_{11}$ | $C_{1j}$, $j \geq 2$ | $C_{21}$ | $C_{2j}$ | $C_0$ |
|------|------|---------|----------------|---------|---------|-------|
| 0    | 0    | $\Im$   | $\Im$          | $\times$| $\times$| $\times$|
| 0    | $> 0$| $\Im$   | $\Re, \Im$     | $\Re$   | $\times$| $\times$|
| 1    | 0    | $\times$| $\times$       | $\Im$   | $\times$| $\times$|
| 1    | $> 0$| $\times$| $\Re$          | $\Re$   | $\Im$   | $\times$|

Here, an $\times$ means that the period does already vanishes or will vanish when all others do.

We briefly explain these simplifications:

• The same argument as in [8] shows that all periods $C_{1j}$, for $j \geq 2$, and $C_0$ are pure imaginary, so only the imaginary part needs to vanish.

• Consider the curves $C_j'$, $j = 2, \ldots, n_2$ corresponding to points $p_{21}$ and $p_{2j}$. Again using dihedral symmetries, we can show that all periods over these curves are real multiples of $e^{\pi (1+\epsilon_0)/k}i$. Since all $C_{2j}$ curves are compositions of the curve $C_{21}$ and $C_j'$, we conclude that if the period over $C_{21}$ vanishes, then we need only show that the real part of those over $C_{2j}$ vanish.

• Since 0 lies on the same line as the $p_2$ points, if the $C_0$ period is 0, then the $C_{21}$ period is of the same form as the other $C_{2j}$, and it suffice for the real part to vanish.

• Finally, if $n_2$ and $n_0$ are not both 0, then by symmetry, the vanishing of the $C_{21}$ or $C_0$ periods implies the $C_{11}$ periods also vanish. If $n_2 = n_0 = 0$, then dihedral symmetry shows the period is a real multiple of $e^{\pi j/k}$. This implies that if its imaginary part vanishes, so does the whole period. \(\Box\)

We can now solve the period problems:

**Proposition 9.13** Suppose $\{p_{ij}^0\}$ is a simply dihedrally symmetric configuration that is balanced and symmetrically nondegenerate. Then for $\tau$ sufficiently close to 0, there exist values of $r_{ij}$ and $p_{ij}$ solving the period problem.

**Proof** The same argument as that of Proposition 5.4, applied to the smaller number of $B$ curves, guarantees values of $r_{ij}$ (in terms of the other parameters) solving the $B$-periods. Likewise, the proof of Proposition 6.3 shows that if the configuration is balanced, the $C$-periods are closed in the limit. So it remains to show that if the configuration is symmetrically nondegenerate then we can apply the implicit function theorem to obtain symmetric surfaces.
First note that we only need to apply the implicit function theorem to the \( n_1 + n_2 \) real \( C \)-periods described in Proposition 9.12. There are four cases to consider, but we show the details only for the case \( n_2 > 0 \) and \( n_0 = 0 \). The other three cases are simpler and use the same reasoning.

Recall that

\[
\lim_{\tau \to 0} \Lambda \int_{C_j} \mu = \frac{4\pi i}{c_0} (F_1 - F_j).
\]

Dividing by the coefficient and taking the conjugate, the periods limit to the following functions:

\[
\text{Re}(F_{11} - F_{1j}), \quad \text{Re}(F_{11} - e^{-\pi i/k} F_{21}), \quad \text{Im}(F_{11} - e^{-\pi i/k} F_{21}), \quad \text{Im}(F_{11} - e^{-\pi i/k} F_{2j}),
\]

for all appropriate \( j \). Since the \( F_{ij} \) are all real, these simplify to

\[
F_{11} - F_{1j}, \quad F_{11}, \quad \sin(\pi i/k) F_{21}, \quad \sin(\pi i/k) F_{2j}.
\]

The Jacobian corresponding to these functions is clearly nonsingular if and only if the configuration is symmetrically nondegenerate. We conclude that the implicit function theorem applies to guarantee all periods vanish near the limit.

\( \Box \)

Theorem 2.7 then follows by assembling the above results.

9.4 Some solutions

We first consider two simple cases, which have the same proof.

**Proposition 9.14** (Karcher–Scherk case) Given \( k \geq 2 \), consider the configuration of \( k \) points given by \( p_{11} = \sqrt{\frac{k-1}{2}} \) and \( \epsilon_{11} = -1 \). This configuration is balanced and nondegenerate.

**Remark 9.15** The quotient \( \Sigma' \) has genus 0 and \( 2k \) ends, and we can verify that this surface is indeed a twisted Scherk tower using the straight lines and the Plateau problem.

**Proposition 9.16** (Twisted Fischer–Koch surfaces) Given \( k \geq 2 \), consider the configuration of \( k+1 \) points given by \( p_{11} = \sqrt{\frac{k+1}{2}} \), \( p_0 = 0 \), and \( \epsilon_{11} = \epsilon_0 = -1 \). Consider also, for \( k \geq 4 \), the configuration of \( k+1 \) points given by \( p_{11} = \sqrt{\frac{k-3}{2}} \), \( p_0 = 0 \), \( \epsilon_{11} = -1 \), and \( \epsilon_0 = 1 \). These configurations are balanced and nondegenerate.

The corresponding SMIMS all have genus 1 and \( 2(k+1) \) or \( 2k+2 \) ends in the quotient.

**Remark 9.17** (\( \epsilon_0 = -1 \)) If \( k \) is even, these indeed correspond to twisted versions of known Fischer–Koch surfaces. Indeed, they solve the same Plateau problem, as can be seen by considering the horizontal and vertical straight lines fixed by the symmetries described in Sect. 9.1. These will have \( 2(k+1) \) ends.

If \( k \) is odd, however, we obtain surfaces similar to the known Fischer–Koch but which do not have a central straight line. These are the first known embedded FK surfaces with \( 4m \) ends \( (m \geq 2) \), and the absence of a vertical line explains why Plateau methods have failed to produce these surfaces. Numerical evidence suggests that these can be untwisted until they have vertical Scherk ends (Fig. 10).
Remark 9.18 \((\varepsilon_0 = 1)\) We will see in the proof why we need \(k \geq 4\) in this case, but this restriction is meaningful for the surfaces.

When \(k\) is even, these configurations correspond to the other parking garage limit of FK surfaces, albeit mirrored and rescaled. Indeed, if \(k\) is the order of symmetry for \(\varepsilon_0 = -1\), then the other limit will have a symmetry of order \(k + 2\). We can again use straight lines to show that this is the case.

If \(k\) is odd, numerical evidence suggests that the same is true, although a proof using Plateau solutions would be difficult (Fig. 11).

\textbf{Proof} We need only solve the following:

\[0 = F_{11} = p_{11} + \frac{\varepsilon_0}{p_{11}} - \frac{k - 1}{2p_{11}}.\]

This equation is solved by \(p_{11} = \sqrt{\frac{k-1-2\varepsilon_0}{2}}\), when this value exists and is positive. We thus have solutions for any \(k \geq 2\) if \(\varepsilon_0 = 0\) or \(-1\) and for \(k \geq 4\) if \(\varepsilon_0 = 1\).

For the configuration to be nondegenerate, we need only \(\frac{\partial F_{11}}{\partial p_{11}} \neq 0\). But, we have,

\[\frac{\partial F_{11}}{\partial p_{11}} = 1 + \frac{k - 1 - 2\varepsilon_0}{2p_{11}^2} > 0\]

\(\square\)

Remark 9.19 It turns out that the only dihedrally symmetric configurations consisting of points with the same modulus and possibly one at the origin are the three above and the \((+ - +)\) case corresponding to the genus 1 helicoid discussed in [8].

The following is a generalization of a result from [8] about symmetric configurations with only negative charges (see Fig. 9b).
Proposition 9.20 For any \( n \geq 1 \) and any \( k \geq 2 \), there exists a unique simply dihedrally symmetric configuration with \( \varepsilon_{1j} = -1 \), \( j = 1, \ldots, n \) and \( \varepsilon_0 = 0 \) or \(-1\), which is balanced and nondegenerate. The same is true for \( \varepsilon_0 = 1 \) as long as \( k \geq 4 \).

Proof Let \( p_j = p_{1j} \) for ease of notation, with \( p_1 < p_2 < \cdots < p_n \). Then the balance equations become:

\[
F_j = p_j - \frac{k - 1 - 2\varepsilon_0}{2p_j} - \sum_{l \neq j} \frac{k p_j^{k-1}}{p_j^k - p_l^k}.
\]

To show that there exists a unique balanced configuration, we note that the vector \((F_1, F_2, \ldots, F_n)\) has as scalar potential the function

\[
V = \sum_{j=1}^{n} \frac{p_j^2}{2} - \frac{k - 1 - 2\varepsilon_0}{2} \log p_j - \sum_{l \neq m} \log |p_j^k - p_m^k|
\]

on the convex set \( C := \{0 < p_1 < \cdots < p_n\} \). That is, \( \frac{\partial V}{\partial p_j} = F_j \) for each \( j \).

We show first that \( V \) is infinite on the boundary of \( C \). As \( p_1 \to 0 \), the term \( \frac{k - 1 - 2\varepsilon_0}{2} \log p_1 \) goes to \( \infty \), given the hypotheses of the proposition. If \( p_j \to p_{j+1} \) for \( j = 1, \ldots, n-1 \), or \( n-1 \), the terms \( \log |p_j^k - p_{j+1}^k| \) will also approach \( \infty \), and as \( p_n \to \infty \), the terms \( \frac{p_n^2}{2} - \sum_{j \neq n} \log |p_j^k - p_n^k| \) go to \( \infty \). Note that with any combination of these boundary possibilities, the limits do not change. Therefore \( V \to \infty \) on the boundary of \( C \).

It is easy to verify that \( V \) is strictly convex since each of its summands is. Therefore, \( V \) has a unique minimum on \( C \). Since \( V \) is a scalar potential of the balance functions, this minimum occurs at a configuration on which all \( F_j \) vanish, that is, a balanced configuration.

Now the Hessian of \( V \) is precisely the Jacobian of the balance functions. Since \( V \) is strictly convex, we conclude that this configuration is nondegenerate.

\( \square \)
Remark 9.21 When $k = 2$, the solutions are the positive roots of the $n^{th}$ Hermite polynomials, as shown in [8]. This method fails to prove the existence of balanced configurations corresponding to the indefinite case of [8] with higher symmetry, and numerical evidence indicates that they do not exist except for a handful of exceptions.

Proposition 9.22 (Twisted CHM surfaces) Consider the configuration of $2k + n_0$ points given by $\varepsilon_{11} = 1$, $\varepsilon_{21} = -1$, and $k \geq 2$ for $\varepsilon_0 = 0$ or $-1$, and $k \geq 4$ for $\varepsilon_0 = 1$. Then there exist unique positive values of $p_{11}$ and $p_{21}$ such that the configuration is balanced and non-degenerate.

Proof To simplify our calculations, we find solutions to the equations $p_{11} F_{11} = p_{21} F_{21} = 0$, which are equivalent to the balance equations. We have,

\[
p_{11} F_{11} = p_{21}^2 + \frac{k - 1 + 2\varepsilon_0}{2} - \frac{k p_{11}^k}{p_{11}^k + p_{21}^k}, \]

\[
p_{21} F_{21} = p_{21}^2 + \frac{-(k - 1) + 2\varepsilon_0}{2} + \frac{k p_{21}^k}{p_{21}^k + p_{11}^k}, \]

\[
p_{11} F_{11} - p_{21} F_{21} = p_{11}^2 - p_{21}^2 + k - 1 - \frac{k(p_{11}^k + p_{21}^k)}{p_{11}^k + p_{21}^k}. \]

Therefore any solution must satisfy $p_{11}^2 - p_{21}^2 - 1 = 0$. If we make the substitution $p_{11} = \sqrt{p_{21}^2 + 1}$, then the system will be solved if and only if the function $p_{11} F_{11}$ vanishes. With this substitution, we obtain,

\[
p_{11} F_{11} = p_{21}^2 + 1 + \frac{k - 1 + 2\varepsilon_0}{2} - \frac{k \sqrt{p_{21}^2 + 1}^k}{\sqrt{p_{21}^2 + 1}^k + p_{21}^k}, \]

which is continuous and increasing for $p_{21} > 0$.

Note that as $p_{21} \to \infty$, we have $p_{11} F_{11} \to \infty$ since the last term is bounded. As $p_{21} \to 0$, we have

\[
p_{11} F_{11} \to 1 + \frac{k - 1 + 2\varepsilon_0}{2} - k = -\frac{(k - 1) + 2\varepsilon_0}{2}. \]

Given the hypotheses of the proposition, that is that $k \geq 2$ when $\varepsilon_0 = 0$ or $-1$ and $k \geq 4$ when $\varepsilon_0 = 1$, this value is negative. Therefore, there exists a unique value of $p_{21}$ making $p_{11} F_{11} = 0$.

We conclude that there exists a unique balanced configuration.

To see that the configuration is nondegenerate, we consider the simpler matrix \((p_{ij}'', \frac{\partial (p_{ji} F_{ij})}{\partial p_{ji}''})\), which for balanced configurations is singular if and only if the Jacobian is. Its entries are as follows:

\[
p_{11} \frac{\partial (p_{11} F_{11})}{\partial p_{11}} = 2p_{11}^2 - \frac{k^2 p_{11}^k p_{21}^k}{(p_{11}^k + p_{21}^k)^2}, \]

\[
p_{21} \frac{\partial (p_{11} F_{11})}{\partial p_{21}} = \frac{-k^2 p_{11}^k p_{21}^k}{(p_{11}^k + p_{21}^k)^2}. \]
\[ p_{11} \frac{\partial (p_{21} F_{21})}{\partial p_{11}} = \frac{k^2 p_{11}^k p_{21}^k}{(p_{11}^k + p_{21}^k)^2}, \]
\[ p_{21} \frac{\partial (p_{21} F_{21})}{\partial p_{21}} = 2p_{21}^2 + \frac{k^2 p_{11}^k p_{21}^k}{(p_{11}^k + p_{21}^k)^2}. \]

The determinant of this matrix simplifies to,
\[
\det \left( \frac{\partial p_{i j} F_{i j}}{\partial p_{i j}} \right) = 4p_{11}^2 p_{21}^2 + 2k^2 p_{11}^k p_{21}^k (p_{11}^2 - p_{21}^2) \]
\[
= 4p_{11}^2 p_{21}^2 + 2k^2 p_{11}^k p_{21}^k (p_{11}^2 + p_{21}^2)^2 > 0,
\]
as \( p_{21}^2 - p_{11}^2 = 1 \) for our balanced configuration. Therefore, the configuration is nondegenerate.

**Remark 9.23** The surfaces corresponding to \( \varepsilon_0 = 0 \) correspond to the Callahan–Hoffman–Meeks surfaces. The surfaces corresponding to \( \varepsilon_0 \neq 0 \) are new, and they are of interest to us. As they are the simplest surfaces we know to have two helicoidal ends in the natural quotient, besides the helicoid and genus \( g \) helicoids. Indeed, in these cases, \( N = \varepsilon_0 \), and by Eq. (3) the surfaces have genus 2. We name these cases CHM− for \( \varepsilon_0 = -1 \) \((k = 2)\) and CHM+ for \( \varepsilon_0 = 1 \) \((k = 4)\) and will discuss them in the next section. However, we mention here that these configurations can be described algebraically. For instance, the CHM− configuration is described by
\[ p_{11} = \frac{1}{2} + \frac{\sqrt{3}}{2} \text{ and } p_{21} = \frac{\sqrt{3}}{4}. \]

We believe these techniques can also prove the existence of other families of symmetric surfaces. A similar configuration of interest is the following, verifiable by computer algebra system:

**Proposition 9.24** For \( k = 2 \), there exists a balanced and nondegenerate configuration with \( n_1 = 3 \), \( n_2 = 1 \), and \( n_0 = 0 \), with \( \varepsilon_{11} = \varepsilon_{13} = 1, \varepsilon_{12} = \varepsilon_{21} = -1 \).

**Remark 9.25** The configuration above corresponds to the surface of Theorem 1.2, which we name CHM7 since it has planar ends and genus 7 in the quotient.

### 10 Additional questions

#### 10.1 Connection with fluid dynamics

The balance equations for our helicoidal surfaces are of interest in the field of fluid dynamics. Indeed, the equations of motion of a configuration \( n \) vortices with circulations \( \Gamma_j \), at positions \( p_j \) are given by:
\[
\frac{dp_j}{dt} = \frac{1}{2\pi i} \sum_{k \neq j} \frac{\Gamma_k}{p_j - p_i}.
\]

If we consider vortices with the same circulation, up to sign, and which are rotating at constant angular velocity \( \frac{1}{2\pi} \), the equations become,
\[
p_j = \sum_{k \neq j} \frac{\Gamma_k}{p_j - p_i}.
\]
Finally, after rescaling the configuration, we may assume that all vorticities are $\Gamma_j = -\epsilon_j = \pm 1$, and we arrive at our balance equations.

This begs the question, is there a meaningful connection between SMIMS that limit to parking garage structures and vortices?

### 10.2 Long-term behavior

The three new families mentioned in Remarks 9.23 and 9.25 illustrate the value of understanding long-term behavior of SMIMS.

The $CHM^+$ family is that mentioned in Theorem 1.3. As discussed in Remark 9.23, it has genus 2 and two helicoidal ends in the natural quotient, and since $N = 1 > 0$, it is in the helicoid class. However, the parking garage limit is different than that of the standard genus...
2 helicoid. So if it can be twisted to a limit, like the known genus $g$ helicoids, we expect it to limit to a genus 2 helicoid distinct from the known one (Fig. 12).

On the other hand, the $CHM$ family is Scherk-type and untwists in $t$. Since it has only two ends in the natural quotient, we know that it cannot untwist to a translation-invariant surface with only two parallel Scherk ends. It thus stops untwisting before the ends become vertical, but we do not know what obstructs further untwisting.

Finally, in the planar class, if the $CHM7$ surface can be untwisted, it could become another translation-invariant surface with two ends in the quotient (Fig. 13).

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Karcher, H.: Embedded minimal surfaces derived from Scherk’s examples. Manuscripta Math. 62, 83–114 (1988)
2. Fischer, W., Koch, E.: On 3-periodic minimal surfaces with non-cubic symmetry. Zeitschrift für Kristallographie 183, 129–152 (1988)
3. Lynker, A.: Einfach periodische elliptische minimallächen. Diplomarbeit Bonn (1993)
4. Callahan, M., Hoffman, D., Karcher, H.: A family of singly-periodic minimal surfaces invariant under a screw motion. Exp. Math. 2, 157–182 (1993)
5. Weber, M., Hoffman, D., Wolf, M.: An embedded genus-one helicoid. Ann. Math. 169(2), 347–448 (2009)
6. Hoffman, D., Traizet, M., White, B.: Helicoidal minimal surfaces of prescribed genus. Acta Mathematica 216, 217–323 (2016)
7. Meeks, W.H., III., Rosenberg, H.: The geometry of periodic minimal surfaces. Comment. Math. Helvetici 68, 538–578 (1993)
8. Traizet, M., Weber, M.: Hermite polynomials and helicoidal minimal surfaces. Inv. Math. 161, 113–149 (2005)
9. Choe, J., Soret, M.: Nonexistence of certain complete minimal surfaces with planar ends. Commentarii Mathematici Helvetici 75, 189–199 (2000)
10. Traizet, M.: An embedded minimal surface with no symmetries. J. Differ. Geom. 60, 103–153 (2002)
11. Aref, H.: Vortices and polynomials. Fluid Dyn. Res. 39, 5–23 (2007)

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