QUANDLE COLORING QUIVERS OF \((p, 2)\)-TORUS LINKS

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Abstract. A quandle coloring quiver is a quiver structure, introduced by Karina Cho and Sam Nelson, which is defined on the set of quandle colorings of an oriented knot or link by a finite quandle. We study quandle coloring quivers of \((p, 2)\)-torus knots and links with respect to dihedral quandles.

1. Introduction

A quandle \([7, 12]\) is a non-associate algebraic structure whose definition is motivated by knot theory, and it provides an axiomatization of the Reidemeister moves for link diagrams. The fundamental quandle of a link is a quandle that can be computed from any diagram of the link. Joyce \([7]\) and Matveev \([12]\) proved that the fundamental quandle is a stronger invariant of knots and links than the knot group. Quandles have been used to define other useful invariants of oriented links (see \([1, 2, 3, 4, 5]\) and others). For example, the quandle counting invariant \([4]\) is the size of the set of all homomorphisms from the fundamental quandle of an oriented link to a finite quandle. More recently, Cho and Nelson \([3]\) introduced the notion of a quandle coloring quiver of an oriented link \(K\) with respect to a finite quandle \(X\), which is a directed graph that allows multiple edges and loops, and enhances the quandle counting invariant of \(K\) by \(X\).

In this paper, we study quandle coloring quivers of \((p, 2)\)-torus links by dihedral quandles. The paper is organized as follows. In Sec. 2, we review some basic concepts about torus links and quandles. In Sec. 2.2, we also prove a couple of statements about the coloring space of \((p, 2)\)-torus links by dihedral quandles, as they will be needed later in the paper. Sec. 3 is the heart of the paper; here we review the definition of quandle coloring quivers of oriented links with respect to finite quandles and study the quivers of \((p, 2)\)-torus links with respect to dihedral quandles.

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2. Preliminaries

In order to have a self-contained paper, we begin by reviewing relevant definitions and concepts.

2.1. Basic definitions. A knot or link $K$ is called a $(p,q)$-torus knot or link if it lies, without any points of intersection, on the surface of a trivial torus in $\mathbb{R}^3$. The integers $p$ and $q$ represent the number of times $K$ wraps around the meridian and, respectively, the longitude of the torus. We denote a $(p,q)$-torus knot or link by $T(p,q)$. It is known that $T(p,q)$ is a knot if and only if $\gcd(p,q) = 1$ and $p$ and $q$ are not equal to 1 simultaneously, and that it is a link with $d$ components if and only if $\gcd(p,q) = d$. Moreover, $T(p,q)$ is the trivial knot if and only if either $p$ or $q$ is equal to 1 or $-1$. We use the term link to refer generically to both knots and links. It is also known that the $(p,q)$-torus link is isotopic to the $(q,p)$-torus link. The $(-p,-q)$-torus link is the $(p,q)$-torus link with reverse orientation, and torus links are invertible. For simplicity, we will assume that $p,q > 0$. The mirror image of $T(p,q)$ is $T(p,-q)$ and a non-trivial torus-knot is chiral.

It is known that the crossing number of a torus link with $p,q > 0$ is given by $c = \min((p-1)q,(q-1)p)$. The link $T(p,q)$ can also be represented as the closure of the braid $(\sigma_1\sigma_2\ldots\sigma_{p-1})^q$. Since $T(p,q)$ is isotopic to $T(q,p)$, this torus link has also a diagram which is the closure of the braid $(\sigma_1\sigma_2\ldots\sigma_{q-1})^p$.

The simplest nontrivial torus knot is the trefoil knot, $T(3,2)$, depicted in Fig. 1. For more details on torus knots and links, we refer the reader to the books [9, 10, 11, 13].

![Figure 1. Oriented torus knot $T(3,2)$](image)

The term quandle is attributed to Joyce in his 1982 work [7] as an algebraic structure whose axioms are motivated by the Reidemeister moves.

A quandle [7, 8, 12] is a non-empty set $\mathcal{X}$ together with a binary operation $\triangleright: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following axioms:

1. For all $x \in \mathcal{X}$, $x \triangleright x = x$.
2. For all $y \in \mathcal{X}$, the map $\beta_y : \mathcal{X} \to \mathcal{X}$ defined by $\beta_y(x) = x \triangleright y$ is invertible.
For all \( x, y, z \in \mathcal{X} \), \((x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)\).

In this case, we use the notation \((\mathcal{X}, \triangleright)\) or \((\mathcal{X}, \triangleright_X)\), if we need to be specific about the quandle operation.

A quandle homomorphism [6] between two quandles \((\mathcal{X}, \triangleright_X)\) and \((\mathcal{Y}, \triangleright_Y)\) is a map \(f: \mathcal{X} \to \mathcal{Y}\) such that \(f(a \triangleright_X b) = f(a) \triangleright_Y f(b)\), for any \(a, b \in \mathcal{X}\).

It is not hard to see that the composition of quandle homomorphisms is again a quandle homomorphism. We note that the second axiom of a quandle implies that for all \(y \in \mathcal{X}\), the map \(\beta_y\) is an automorphism of \(\mathcal{X}\). Moreover, by the first axiom of a qundle, the map \(\beta_y\) fixes \(y\).

**Example 1.** Let \(n \in \mathbb{Z}, n \geq 2, \mathcal{X} = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}\) and define \(x \triangleright y = (2y - x) \mod n\), for all \(x, y \in \mathbb{Z}_n\). That is, \(x \triangleright y\) is the remainder of \(2y - x\) upon division by \(n\). Then \(\mathbb{Z}_n\) together with this operation \(\triangleright\) is a quandle, called the dihedral quandle (see [4, 14]).

**Example 2.** To every oriented link diagram of \(K\), we can naturally associate its fundamental quandle (also referred to as the knot quandle), denoted by \(Q(K)\). Given a diagram \(D\) of \(K\), its crossings divide \(D\) into arcs and we use a set of labels to label all of these arcs. The fundamental quandle \(Q(D)\) is the quandle generated by the set of arc-labels together with the crossing relations given by the operation \(\triangleright\) explained in Fig. 2.

![Figure 2. Labelling a knot crossing in a fundamental quandle](image)

The axioms of a quandle encode the Reidemeister moves, and thus the fundamental quandle is independent on the diagram \(D\) representing the oriented link \(K\). Hence, \(Q(K) := Q(D)\) is a link invariant. It was shown independently by Joyce [7] and Matveev [12] that the fundamental quandle is a stronger invariant than the fundamental group of a knot or link. In fact, the fundamental quandle is a complete invariant, in the sense that if \(Q(K_1)\) and \(Q(K_2)\) are isomorphic quandles, then the (unoriented) links \(K_1\) and \(K_2\) are equivalent (up to chirality). However, determining whether or not two quandles are isomorphic is not an easy task.

**Remark 3.** We consider the fundamental quandle of a \((p, 2)\)-torus link. By arranging the quandle relations of the fundamental quandle in a particular
way, we can present $Q(T(p,2))$ in a recursive fashion. We know that a $(p,2)$-torus link can be represented as the closure of the braid $\sigma_1^p$, which is also a minimal diagram for the $(p,2)$-torus link. There are $p$ crossings and $p$ arcs labeled $\{x_1, x_2, \ldots, x_p\}$ in such a diagram of $T(p,2)$. Considering the braid $\sigma_1^p$, we label its arcs starting from the bottom right with $x_1$ then moving to the bottom left arc to label it $x_2$, and continuing up along the braid using labels $x_3, x_4, \ldots, x_p$, as shown in Fig. 3. We obtain the following presentation for the fundamental quandle of the $(p,2)$-torus link:

$$Q(T(p,2)) = \langle x_1, x_2, x_3, \ldots, x_p \mid x_2 \triangleright x_1 = x_p, x_1 \triangleright x_p = x_{p-1}, \text{ and } x_i \triangleright x_{i-1} = x_{i-2}, \text{ for all } 3 \leq i \leq p \rangle.$$

![Figure 3. A diagram of a $(p,2)$-torus link](image)

When classifying links, it is useful to count the number of quandle homomorphisms from the fundamental quandle of a link to a fixed quandle. For example, Inoue [6] showed that the number of all quandle homomorphisms from the fundamental quandle of a link to an Alexander quandle [4, 6] has a module structure and that it is determined by the series of the Alexander polynomials of the link.

A quandle coloring of an oriented link $K$ with respect to a finite quandle $\mathcal{X}$ (also called an $\mathcal{X}$-quandle coloring) is given by a quandle homomorphism $\psi : Q(K) \to \mathcal{X}$. In this case, we refer to $\mathcal{X}$ as the coloring quandle. We let $\text{Hom}(Q(K), \mathcal{X})$ be the collection of all $\mathcal{X}$-quandle colorings and we call it the coloring space of $K$ with respect to $\mathcal{X}$. Finally, the cardinality $|\text{Hom}(Q(K), \mathcal{X})|$ of the coloring space is called the quandle counting invariant with respect to $\mathcal{X}$ (see [3]).

When a quandle coloring consists of a single color, we use the term trivial coloring, and make use of the term nontrivial coloring otherwise.
2.2. The coloring space of \((p, 2)\)-torus links with respect to dihedral quandles. In this paper we are concerned with \((p, 2)\)-torus links, and we need to understand the coloring space of these links with respect to dihedral quandles. The following proposition and theorem are our first results.

Proposition 4. Let \(T(p, 2)\) be a torus link and \(\mathbb{Z}_n\) a dihedral quandle. Consider any two fixed and consecutive arcs of \(T(p, 2)\) with labels \(x_1, x_2 \in Q(T(p, 2))\). If \(\psi \in \text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)\), then \(p\psi(x_1) \equiv p\psi(x_2) \pmod{n}\).

Proof. By Remark 3, the fundamental quandle of the torus link \(T(p, 2)\) has the following presentation:

\[
Q(T(p, 2)) = \langle x_1, x_2, x_3, \ldots, x_p \mid x_2 \triangleright x_1 = x_p, x_1 \triangleright x_p = x_{p-1}, \text{ and } x_i \triangleright x_{i-1} = x_{i-2}, \text{ for all } 3 \leq i \leq p \rangle.
\]

For any \(\mathbb{Z}_n\)-quandle coloring \(\psi \in \text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)\) and \(x_i, x_j \in Q(T(p, 2))\), we must have that \(\psi(x_i \triangleright x_j) = \psi(x_i) \triangleright \psi(x_j)\), where \(\triangleright\) and \(\triangleright'\) are the quandle operations for \(Q(T(p, 2))\) and \(\mathbb{Z}_n\), respectively. In particular,

\[
\psi(x_p) = \psi(x_2 \triangleright x_1) = \psi(x_2) \triangleright' \psi(x_1) = (2\psi(x_1) - \psi(x_2)) \pmod{n},
\]

\[
\psi(x_{p-1}) = \psi(x_1 \triangleright x_p) = \psi(x_1) \triangleright' \psi(x_p) = (2\psi(x_p) - \psi(x_1)) \pmod{n}.
\]

Moreover, for all \(3 \leq i \leq p\), the following holds:

\[
\psi(x_{i-2}) = \psi(x_i \triangleright x_{i-1}) = \psi(x_i) \triangleright' \psi(x_{i-1}) = (2\psi(x_{i-1}) - \psi(x_i)) \pmod{n}.
\]

Using these statements inductively, we obtain:

\[
\psi(x_2) = (2\psi(x_3) - \psi(x_4)) \pmod{n} = (2(2\psi(x_4) - \psi(x_5)) - \psi(x_4)) \pmod{n} = (3\psi(x_4) - 2\psi(x_5)) \pmod{n} = (3(2\psi(x_5) - \psi(x_6)) - 2\psi(x_5)) \pmod{n} = (4\psi(x_5) - 3\psi(x_6)) \pmod{n} = \cdots = ((p-1)\psi(x_p) - (p-2)\psi(x_1)) \pmod{n} = ((p-1)(2\psi(x_1) - \psi(x_2)) - (p-2)\psi(x_1)) \pmod{n} = (p\psi(x_1) - (p-1)\psi(x_2)) \pmod{n}.
\]

It follows that \(p\psi(x_2) \equiv p\psi(x_1) \pmod{n}\). \(\square\)

We are now ready to prove the following theorem about the coloring space of \((p, 2)\)-torus links with respect to dihedral quandles, which will be used extensively in Sec. 3.
Theorem 5. For a given torus link $T(p, 2)$ and $\mathbb{Z}_n$ a dihedral quandle, the following holds:

(i) If $\gcd(p, n) = 1$, then $|\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)| = n$ and the coloring space $\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)$ is the collection of all trivial $\mathbb{Z}_n$-quandle colorings of $T(p, 2)$.

(ii) If $\gcd(p, n) = c \neq 1$, then $|\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)| = nc$, and the coloring space $\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)$ is the union of all $n$ trivial quandle colorings together with a collection of $n(c-1)$ nontrivial $\mathbb{Z}_n$-quandle colorings of $T(p, 2)$.

Proof. Let $\psi \in \text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)$. By Proposition 4

$$p\psi(x_1) \equiv p\psi(x_2) \pmod{n}.$$ 

(i) If $\gcd(p, n) = 1$, then the above congruence simplifies to

$$\psi(x_1) \equiv \psi(x_2) \pmod{n}.$$ 

The latter can only occur when $\psi$ maps $x_1$ and $x_2$ to the same color. Using the relations of $Q(T(p, 2))$, as described in Remark 3, this forces the quandle homomorphism $\psi$ to color every arc in the diagram of $T(p, 2)$ the same. Since there are $n$ quandle colorings to choose from, $|\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)| = n$, and $\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)$ is the collection of all trivial quandle colorings of $T(p, 2)$ with respect to $\mathbb{Z}_n$.

(ii) If $\gcd(p, n) = c \neq 1$, then the congruence $p\psi(x_1) \equiv p\psi(x_2) \pmod{n}$ simplifies to the following:

$$\psi(x_1) \equiv \psi(x_2) \pmod{n/c}.$$ 

Then $\psi(x_2) \equiv \psi(x_1) + k\frac{n}{c} \pmod{n/c}$ also holds for all integers $k$, where $0 \leq k \leq c-1$. Hence, for each fixed color $\psi(x_1) \in \mathbb{Z}_n$, there are $c$ possibilities for $\psi(x_2)$ in $\mathbb{Z}_n$ such that $p\psi(x_1) \equiv p\psi(x_2) \pmod{n}$. Since there are $n$ choices for $\psi(x_1) \in \mathbb{Z}_n$, we conclude that there are $nc$ $\mathbb{Z}_n$-quandle colorings of the link $T(p, 2)$. Moreover, since there are $n$ trivial quandle colorings with respect to $\mathbb{Z}_n$, the remaining $n(c-1)$ $\mathbb{Z}_n$-quandle colorings must be nontrivial quandle colorings of the link $T(p, 2)$ with respect to $\mathbb{Z}_n$. 

3. Quandle coloring quivers of $(p, 2)$-torus links with respect to dihedral quandles

A quandle enhancement is an invariant for oriented links that captures the quandle counting invariant and in general is a stronger invariant. For some examples of quandle enhancements, we refer the reader to [1, 2, 4, 5]. More recently, by representing quandle colorings as vertices in a graph, Cho and Nelson [3] imposed a combinatorial structure on the coloring space of a link that led to new quandle enhancements. For example, one of these
enhancements, the quandle coloring quiver, can distinguish links that have the same quandle counting invariant (see [3, Example 6]).

In what follows, we work with graphs that are allowed to have multiple edges and loops. To capture multiple edges in a graph $G$, we use a weight function $c : \{(x, y) \mid x, y \in V(G)\} \to \mathbb{N} \cup \{0\}$, where $V(G)$ is the vertex set of $G$ and $(x, y)$ is any edge connecting vertices $x, y \in V(G)$. We denote the corresponding graph with a weight function $c$ by $(G, c)$. When $c$ is a constant function $c(x, y) = k$, for all $x, y \in V(G)$ and some $k \in \mathbb{N}$, then we use the notation $c = \hat{k}$ for the weight function and $(G, \hat{k})$ for the corresponding graph.

We say that a graph is complete, provided that there is an edge between every pair of distinct vertices and a loop at each vertex; we denote a complete graph with $n$ vertices by $K_n$. The graphs we work with in this paper are all directed, and we use the notation $\xrightarrow{G}$ to denote a graph $G$ with all edges directed both ways.

Given two graphs $G_1$ and $G_2$ with disjoint vertex sets and edge sets, the join of $G_1$ and $G_2$, denoted $G_1 \nabla G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set containing all edges in $G_1$ and $G_2$, as well as all edges that connect vertices of $G_1$ with vertices of $G_2$. Finally, if there is the same number of directed edges from each vertex of the graph $G_2$ to the graph $G_1$ with constant weight function $\hat{k}$ and no directed edges from vertices of $G_1$ to vertices of $G_2$, we use the notation $G_1 \nabla_k G_2$ to denote such a join of graphs.

Let $\mathcal{X}$ be a finite quandle and $K$ an oriented link. For any set of quandle homomorphims $S \subset \text{Hom}(\mathcal{X}, \mathcal{X})$, the associated quandle coloring quiver [3], denoted $Q^S_{\mathcal{X}}(K)$, is the directed graph with a vertex $v_f$ for every element $f \in \text{Hom}(Q(K), \mathcal{X})$ and an edge directed from the vertex $v_f$ to the vertex $v_g$, whenever $g = \phi \circ f$, for some $\phi \in S$. When $S = \text{Hom}(\mathcal{X}, \mathcal{X})$, we denote the associated quiver by $Q_{\mathcal{X}}(K)$, and call it the full $\mathcal{X}$-quandle coloring quiver of $K$.

It was explained in [3] that $Q^S_{\mathcal{X}}(K)$ is a link invariant, in the sense that the quandle coloring quivers associated to diagrams that are related by Reidemeister moves are isomorphic quivers.

**Notation.** We establish some notation for mappings. Consider a function $g : \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ and $\mathcal{Y} = \{y_1, y_2, \ldots, y_m\}$. If $g(x_i) = y_j$, for each $x_i \in \mathcal{X}$, we will write $g$ as $y_{j_1}y_{j_2}y_{j_3} \ldots y_{j_n}$ (omitting the commas when possible). This notation will simplify the formulas for various quandle homomorphisms considered in this paper.

Our goal is to study quandle coloring quivers of $(p, 2)$-torus links with respect to dihedral quandles $\mathbb{Z}_n$. Before we proceed, we need to prove the
following result about the space $\text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n)$ of quandle automorphisms of the dihedral quandle $\mathbb{Z}_n$, where $n \geq 3$.

**Proposition 6.** Consider the dihedral quandle $\mathbb{Z}_n$, when $n \geq 3$. Then $\text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n)$ is a set of size $n^2$ of quandle homomorphisms $\phi_\gamma: \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ that are defined recursively, where $\gamma$ is any $n$-tuple $(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n) \in (\mathbb{Z}_n)^n$, such that

$$\phi_\gamma := \begin{cases} 
\phi_\gamma(0) = \gamma_1, \\
\phi_\gamma(1) = \gamma_2, \\
\phi_\gamma(k) = \phi_\gamma(k-2) \circ \phi_\gamma(k-1), \quad \text{for } 2 \leq k \leq n-1.
\end{cases}$$

**Proof.** The operation of the dihedral quandle $(\mathbb{Z}_n, \triangleright)$ is given by $x \triangleright y = (2y - x) \mod n$, for any $x, y \in \mathbb{Z}_n$. Then, $k \triangleright (k+1) = k+2$, for all $0 \leq k \leq n-3$, where it follows that $(n-2) \triangleright (n-1) = 0$ and $(n-1) \triangleright 0 = 1$. Since any quandle homomorphism $\phi \in \text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n)$ must satisfy $\phi(x \triangleright y) = \phi(x) \triangleright \phi(y)$, for any $x, y \in \mathbb{Z}_n$ (using the same $\triangleright$), then $\phi$ must now satisfy $\phi(k) \triangleright \phi(k+1) = \phi(k \triangleright (k+1)) = \phi(k+2)$, for all $0 \leq k \leq n-3$, further yielding $\phi(n-2) \triangleright \phi(n-1) = \phi(0)$ and $\phi(n-1) \triangleright \phi(0) = \phi(1)$.

Hence, $\phi(0)$ and $\phi(1)$ completely determine $\phi$, and we have the following recursive relation for the image of $(0,1,2,\ldots,n-1)$ under $\phi$:

$$(0,1,2,\ldots,n-1) \xrightarrow{\phi} (\phi(0), \phi(1), \phi(0) \circ \phi(1), \phi(1) \circ \phi(2), \ldots, \phi(n-3) \circ \phi(n-2)).$$

Since there are $n$ ways to assign each of $\phi(0)$ and $\phi(1)$ an element in $\mathbb{Z}_n$, we see that $|\text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n)| = n^2$. \hfill $\square$

**Remark 7.** From the cyclic behavior of the dihedral quandle $\mathbb{Z}_n$ and the proof of Proposition 6, we see that any homomorphism $\phi_\gamma \in \text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n)$, where $\gamma$ is an $n$-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in (\mathbb{Z}_n)^n$, is completely determined by the images $\phi_\gamma(k) = \gamma_{k+1}$ and $\phi_\gamma(k+1) = \gamma_{k+2}$, where if $k = n-1$ then $k+1 = 0$ and if $k = n$ then $k+2 = 1$.

We also remark that using our notation, we can write $\phi_\gamma = \phi_{\gamma_1 \ldots \gamma_n}$.

Using Theorem 5 together with Proposition 6 and Corollary 7, we are ready to state and prove our main results regarding quivers.

**Theorem 8.** Given a torus link $T(p,2)$ and $\mathbb{Z}_n$ a dihedral quandle, where $\text{gcd}(p,n) = 1$, the full $\mathbb{Z}_n$-quandle coloring quiver of $T(p,2)$ is the complete directed graph:

$$Q_{\mathbb{Z}_n}(T(p,2)) = (\rightarrow K_n, \rightarrow).$$

**Proof.** By Theorem 5 we know that $\text{Hom}(Q(T(p,2)), \mathbb{Z}_n)$ is the collection of all $n$ trivial $\mathbb{Z}_n$-quandle colorings of $T(p,2)$. Hence, the quandle coloring quiver $Q_{\mathbb{Z}_n}(T(p,2))$ has $n$ vertices, one for each of the elements in the coloring space $\text{Hom}(Q(T(p,2)), \mathbb{Z}_n)$.
Let $\phi = \phi_{\gamma_1 \gamma_2 \ldots \gamma_n} \in Hom(\mathbb{Z}_n, \mathbb{Z}_n)$ be a quandle homomorphism given by: $0 \xrightarrow{\phi} \gamma_1, 1 \xrightarrow{\phi} \gamma_2, \ldots, (n-1) \xrightarrow{\phi} \gamma_n$. In Proposition 6 we proved that $|Hom(\mathbb{Z}_n, \mathbb{Z}_n)| = n^2$, which was due to the fact that a quandle homomorphism $\phi_{\gamma_1 \gamma_2 \ldots \gamma_n}$ is completely determined by the choices for $\gamma_1$ and $\gamma_2$. In fact, by Remark 7 we know that $\phi_{\gamma_1 \gamma_2 \ldots \gamma_n}$ is determined by the choices for any two consecutive values $\gamma_k$ and $\gamma_{k+1}$, where if $k = n$ then $k + 1 = 1$.

Let $\psi_i \in Hom(Q(\mathcal{T}(p, 2)), \mathbb{Z}_n)$ denote the trivial $\mathbb{Z}_n$-quandle coloring of $\mathcal{T}(p, 2)$ associated to some fixed $i \in \mathbb{Z}_n$ and let $v_{\psi_i}$ be the corresponding vertex in the quandle coloring quiver $Q_{\mathbb{Z}_n}(\mathcal{T}(p, 2))$. Without loss of generality, by the above discussion, we can use that any map $\phi_{\gamma_1 \gamma_2 \ldots \gamma_n}$ is completely determined by the choices for $\gamma_1$ and $\gamma_{i+1}$.

Consider $\phi_{\gamma_1 \ldots \gamma_i \gamma_{i+2} \ldots \gamma_n} \in Hom(\mathbb{Z}_n, \mathbb{Z}_n)$. That is, $\gamma_{i+1} = i$, which means that $i \xrightarrow{\phi} i$ under this map. There are $n$ maps of the form $\phi_{\gamma_1 \ldots \gamma_i \gamma_{i+2} \ldots \gamma_n}$ (determined by the value of $\gamma_i \in \mathbb{Z}_n$) and they all satisfy that

$$\phi_{\gamma_1 \ldots \gamma_i \gamma_{i+2} \ldots \gamma_n} \circ \psi_i = \psi_i.$$ 

In addition, there are no other maps $\phi_{\gamma_1 \gamma_2 \ldots \gamma_n}$ satisfying $\phi_{\gamma_1 \gamma_2 \ldots \gamma_n} \circ \psi_i = \psi_i$. It follows that the quiver vertex $v_{\psi_i}$ has $n$ loops that represent $\psi_i$ being fixed by those $n$ maps.

Let $\psi_j \in Hom(Q(\mathcal{T}(p, 2)), \mathbb{Z}_n)$, where $j \in \mathbb{Z}_n$ is distinct from the fixed value of $i$ above. For similar reasons as above, there are $n$ quandle homomorphisms in $Hom(\mathbb{Z}_n, \mathbb{Z}_n)$ of the form $\phi_{\gamma_1 \ldots \gamma_j \gamma_{i+2} \ldots \gamma_n}$ that send $i \xrightarrow{\phi} j$. Moreover,

$$\phi_{\gamma_1 \ldots \gamma_j \gamma_{i+2} \ldots \gamma_n} \circ \psi_i = \psi_j,$$

and there are no other maps $\phi_{\gamma_1 \gamma_2 \ldots \gamma_n}$ satisfying $\phi_{\gamma_1 \gamma_2 \ldots \gamma_n} \circ \psi_i = \psi_j$. Thus $\psi_i$ is also mapped $n$ times to each of the colorings $\psi_j$. This implies that there are $n$ directed edges from the vertex $v_{\psi_i}$ to each of the vertices $v_{\psi_j}$, for all $j \neq i$.

Hence, the full quandle coloring quiver of $\mathcal{T}(p, 2)$ with respect to $\mathbb{Z}_n$ is a graph with $n$ vertices where every vertex has $n$ directed edges from itself to all other vertices, including itself. Thus, $Q_{\mathbb{Z}_n}(\mathcal{T}(p, 2)) = (\overrightarrow{K_n}, \hat{n})$, whenever $\gcd(p, n) = 1$. \hfill $\square$

**Theorem 9.** Let $\mathcal{T}(p, 2)$ be a torus link and $\mathbb{Z}_n$ a dihedral quandle. If $\gcd(p, n) = c$ where $c$ is prime, then the full quandle coloring quiver of $\mathcal{T}(p, 2)$ with respect to $\mathbb{Z}_n$ is the join of two complete directed graphs:

$$Q_{\mathbb{Z}_n}(\mathcal{T}(p, 2)) = (\overrightarrow{K_n}, \hat{n}) \mathbin{\overset{\text{d}}{\circlearrowright}} (\overrightarrow{K_{n(c-1)}}, \hat{d}),$$

where $d = n/c$, and the two complete subgraphs correspond to the trivial and nontrivial, respectively, $\mathbb{Z}_n$-quandle colorings of $\mathcal{T}(p, 2)$. 

Proof. Using Theorem 5, the quandle coloring space $\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)$ is the union of all $n$ trivial $\mathbb{Z}_n$-quandle colorings, together with $n(c-1)$ nontrivial $\mathbb{Z}_n$-quandle colorings of $T(p, 2)$. Therefore, the quandle coloring quiver $Q_{\mathbb{Z}_n}(T(p, 2))$ contains $n$ vertices corresponding to the trivial $\mathbb{Z}_n$-quandle colorings and $n(c-1)$ vertices corresponding to the nontrivial $\mathbb{Z}_n$-quandle colorings of $T(p, 2)$.

Let $\psi_\sigma = \psi_{\sigma_1\sigma_2\sigma_3...\sigma_p}$ and $\psi_\omega = \psi_{\omega_1\omega_2\omega_3...\omega_p}$ be any two quandle colorings in $\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)$. Our notation means that $\psi_\sigma(x_i) = \sigma_i$ and $\psi_\omega(x_i) = \omega_i$, where $\sigma_i, \omega_i \in \mathbb{Z}_n$ and $x_i$ is a generator of the fundamental quandle $Q(T(p, 2))$.

If both $\psi_\sigma$ and $\psi_\omega$ are trivial $\mathbb{Z}_n$-quandle colorings (possibly the same coloring), we know from the proof of Theorem 9 that there are $n$ directed edges from the quiver vertex $v_{\psi_\sigma}$ to the vertex $v_{\psi_\omega}$ (including $n$ directed loops from the vertex $v_{\psi_\omega}$ to itself, if $\psi_\sigma = \psi_\omega$). It follows that there is a complete directed graph $\left( \overrightarrow{K_n}, n \right)$ associated with the $n$ trivial $\mathbb{Z}_n$-quandle colorings of $T(p, 2)$, as a subgraph of the full quandle coloring quiver $Q_{\mathbb{Z}_n}(T(p, 2))$.

If $\psi_\sigma$ is a nontrivial quandle coloring, then $\psi_\sigma(x_1) = \sigma_1 \neq \sigma_2 = \psi_\sigma(x_2)$, because equivalent consecutive colors in the subscript of $\psi_\sigma$ would impose a trivial $\mathbb{Z}_n$-quandle coloring. By Proposition 4, for any $x_1, x_2 \in Q(T(p, 2))$, a quandle coloring $\psi \in \text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)$ must satisfy the congruence $p\psi(x_1) \equiv p\psi(x_2) \pmod{n}$, which reduces to $\psi(x_1) \equiv \psi(x_2) \pmod{d}$, where $d = \frac{n}{c}$ and $c = \gcd(p, n), c \neq 1$. Since $\psi(x_1), \psi(x_2) \in \mathbb{Z}_n$, we have that for each fixed $\psi(x_1)$, there are $c$ possibilities for $\psi(x_2)$ in $\mathbb{Z}_n$ satisfying the above two congruences; specifically, $\psi(x_2) = \psi(x_1) + kd$, where $k$ is an integer such that $0 \leq k \leq c-1$. Moreover, since $\psi_\sigma(x_1) \neq \psi_\sigma(x_2)$ for a nontrivial quandle coloring $\psi_\sigma$, then for each fixed $\psi_\sigma(x_1)$ there are $c-1$ possibilities for $\psi_\sigma(x_2)$ in $\mathbb{Z}_n$ satisfying $p\psi_\sigma(x_1) \equiv p\psi_\sigma(x_2) \pmod{n}$.

Let $\psi_\sigma$ be a fixed nontrivial quandle coloring in $\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)$. Then $\psi_\sigma = \psi_{\sigma_1(\sigma_1+kd)\sigma_3...\sigma_p}$, for some $1 \leq k \leq c-1$ and $\sigma_1, \sigma_3, ..., \sigma_p \in \mathbb{Z}_n$. Let $\psi_\omega = \psi_{\omega_1(\omega_1+hd)\omega_3...\omega_p}$ be any fixed coloring in $\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)$; that is, $h$ is an integer such that $0 \leq h \leq c-1$, where $\omega_1, \omega_3, ..., \omega_p \in \mathbb{Z}_n$ and $h \neq 0$ if $\psi_\omega$ is a nontrivial quandle coloring.

Now suppose $\phi_{\gamma_1\gamma_2...\gamma_n} \in \text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n)$ satisfies $\phi_{\gamma_1\gamma_2...\gamma_n} \circ \psi_\sigma = \psi_\omega$. In particular, this means that

\begin{align*}
(1) & \quad (\phi_{\gamma_1\gamma_2...\gamma_n} \circ \psi_\sigma)(x_1) = \psi_\omega(x_1) = \omega_1 \\
(2) & \quad (\phi_{\gamma_1\gamma_2...\gamma_n} \circ \psi_\sigma)(x_2) = \psi_\omega(x_2) = \omega_1 + hd.
\end{align*}

Since $\psi_\sigma(x_1) = \sigma_1$ and $\psi_\sigma(x_2) = \sigma_1 + kd$, equations (1) and (2) imply that the following must hold:

\[ \phi_{\gamma_1\gamma_2...\gamma_n}(\sigma_1) = \omega_1 \text{ and } \phi_{\gamma_1\gamma_2...\gamma_n}(\sigma_1 + kd) = \omega_1 + hd. \]
In Remark 7 we showed that a homomorphism \( \phi_{\gamma_1 \gamma_2 \ldots \gamma_n} \in \text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n) \) is determined by two consecutive values in its subscript. Moreover, by fixing the value of \( \omega_1 \), a homomorphism \( \phi_{\gamma_1 \gamma_2 \ldots \omega_1 \tau \ldots \gamma_n} \), with some \( \tau \in \mathbb{Z}_n \), satisfying equation (1) yields:

\[
\phi_{\gamma_1 \gamma_2 \ldots \omega_1 \tau \ldots \gamma_n}(\sigma_1) = \omega_1 \\
\phi_{\gamma_1 \gamma_2 \ldots \omega_1 \tau \ldots \gamma_n}(\sigma_1 + 1) = \tau \\
\phi_{\gamma_1 \gamma_2 \ldots \omega_1 \tau \ldots \gamma_n}(\sigma_1 + 2) = (2\tau - \omega_1) \mod n \\
\phi_{\gamma_1 \gamma_2 \ldots \omega_1 \tau \ldots \gamma_n}(\sigma_1 + 3) = (3\tau - 2\omega_1) \mod n \\
\vdots \\
\phi_{\gamma_1 \gamma_2 \ldots \omega_1 \tau \ldots \gamma_n}(\sigma_1 + kd) = (kd\tau - (kd - 1)\omega_1) \mod n
\]

Hence, a homomorphism \( \phi_{\gamma_1 \gamma_2 \ldots \omega_1 \tau \ldots \gamma_n} \) satisfies both of the equations (1) and (2) if and only if

\[
kd\tau - (kd - 1)\omega_1 \equiv \omega_1 + hd \pmod n.
\]

The latter congruence is equivalent to \( kd\tau \equiv kd\omega_1 + hd \pmod n \), and since \( \gcd(d, n) = d \), this congruence reduces further to:

\[
k\tau \equiv k\omega_1 + h \pmod{n/d}.
\]

Note that \( k, \omega_1, \) and \( h \) are considered fixed in this congruence. Since \( \frac{n}{d} \) is equal to the prime \( c \) and \( 1 \leq k \leq c - 1 \), we have that \( \gcd(k, \frac{n}{d}) = 1 \) and thus the congruence in (4) has solutions \( \tau \). In particular, \( \tau \) has a unique solution modulo \( \frac{n}{d} \). It follows that for the congruence (3), there are \( d \) incongruent solutions modulo \( n \) for \( \tau \). Therefore, for a given nontrivial quandle coloring \( \psi_{\sigma} \), there are \( d \) homomorphisms \( \phi_{\gamma_1 \gamma_2 \ldots \gamma_n} \) such that \( \phi_{\gamma_1 \gamma_2 \ldots \gamma_n} \circ \psi_{\sigma} = \psi_{\omega} \), for all (trivial and nontrivial) \( \mathbb{Z}_n \)-quandle colorings \( \psi_{\omega} \) of \( \mathcal{T}(p, 2) \).

The above reasoning shows that for each vertex \( v_{\psi_{\sigma}} \) in the quiver corresponding to a nontrivial coloring, there are \( d \) directed edges to all of the vertices in the quiver, including to itself. It follows that the quiver \( Q_{\mathbb{Z}_n}(\mathcal{T}(p, 2)) \) contains as a subgraph, a complete directed graph \( \overrightarrow{\mathcal{K}_{(c-1)}} \) associated with the nontrivial quandle colorings that is joined, using the weight function \( \hat{d} \), to the subgraph associated with trivial colorings:

\[
\overrightarrow{\mathcal{K}_{n}} \xrightarrow{\hat{d}} \overrightarrow{\mathcal{K}_{n(c-1)}} \xleftarrow{\hat{d}}
\]

Moreover, there is no map in \( \text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n) \) that sends a fixed value in \( \mathbb{Z}_n \) to two distinct values in \( \mathbb{Z}_n \). That is, there is no homomorphism \( \phi_{\gamma_1 \gamma_2 \ldots \gamma_n} \) in \( \text{Hom}(\mathbb{Z}_n, \mathbb{Z}_n) \) that sends a trivial \( \mathbb{Z}_n \)-quandle coloring of \( \mathcal{T}(p, 2) \) to a non-trivial \( \mathbb{Z}_n \)-quandle coloring of \( \mathcal{T}(p, 2) \). Hence, the joining behavior cannot be reciprocated from \( \overrightarrow{\mathcal{K}_{n}} \) to \( \overrightarrow{\mathcal{K}_{n(c-1)}} \).
This completes the proof of \( Q_{\mathbb{Z}_n}(\mathcal{T}(p, 2)) = \left( \overrightarrow{K_n}, \hat{n} \right) \sqcup_{\hat{d}} \left( \overrightarrow{K_n(c-1)}, \hat{d} \right) \), when \( \gcd(p, n) = c \) and \( c \) is prime.

**Corollary 10.** For a given torus link \( \mathcal{T}(p, 2) \) and a prime \( n \), we have:

- If \( n \nmid p \), then \( Q_{\mathbb{Z}_n}(\mathcal{T}(p, 2)) = \left( \overrightarrow{K_n}, \hat{n} \right) \).
- If \( n \mid p \), then \( Q_{\mathbb{Z}_n}(\mathcal{T}(p, 2)) = \left( \overrightarrow{K_n}, \hat{n} \right) \sqcup_{\hat{1}} \left( \overrightarrow{K_n(n-1)}, \hat{1} \right) \).

**Proof.** The statements follow as particular cases from Theorems 8 and 9. Since \( n \) is prime, if \( n \nmid p \), then \( \gcd(n, p) = 1 \), and Theorem 8 implies the first statement of this corollary. If \( n \mid p \), then \( \gcd(n, p) = n \) where \( n \) is prime, and the application of Theorem 9 yields the second statement above. \( \square \)

The following statement is a direct consequence of Corollary 10.

**Corollary 11.** Let \( n \) be a prime.

- If \( n \nmid p_1 \) and \( n \nmid p_2 \), then the quandle coloring quivers \( Q_{\mathbb{Z}_n}(\mathcal{T}(p_1, 2)) \) and \( Q_{\mathbb{Z}_n}(\mathcal{T}(p_2, 2)) \) are isomorphic.
- If \( n \mid p_1 \) and \( n \mid p_2 \), then the quandle coloring quivers \( Q_{\mathbb{Z}_n}(\mathcal{T}(p_1, 2)) \) and \( Q_{\mathbb{Z}_n}(\mathcal{T}(p_2, 2)) \) are isomorphic.

**Example 12.** As an example, we consider the full \( \mathbb{Z}_3 \)-quandle coloring quiver of the trefoil knot \( \mathcal{T}(3, 2) \). In this case, \( n = p = 3 \), and according to Theorem 9, the quiver \( Q_{\mathbb{Z}_3}(\mathcal{T}(3, 2)) \) has the following form:

\[
Q_{\mathbb{Z}_3}(\mathcal{T}(3, 2)) = \left( \overrightarrow{K_3}, \hat{3} \right) \sqcup_{\hat{1}} \left( \overrightarrow{K_6}, \hat{1} \right).
\]

In Fig. 4 on the left, we show the graph \( \left( \overrightarrow{K_3}, \hat{3} \right) \) associated with the trivial \( \mathbb{Z}_3 \)-quandle colorings of \( \mathcal{T}(3, 2) \) and on the right of the figure, there is the graph \( \left( \overrightarrow{K_6}, \hat{1} \right) \) corresponding to the \( n(c-1) = 3 \cdot 2 \) nontrivial \( \mathbb{Z}_3 \)-quandle colorings of the trefoil knot. Both of these graphs are subgraphs of the full \( \mathbb{Z}_3 \)-quandle coloring quiver of the trefoil knot. In Fig. 5 we present the quandle coloring quiver \( Q_{\mathbb{Z}_3}(\mathcal{T}(3, 2)) \) as the join of these subgraphs by the weight function \( \hat{d} = \hat{1} \): \( \left( \overrightarrow{K_3}, \hat{3} \right) \sqcup_{\hat{1}} \left( \overrightarrow{K_6}, \hat{1} \right) \).

**Remark 13.** Taniguchi [15] studied quandle coloring quivers of links with respect to dihedral quandles, where the focus was on isomorphic quandle coloring quivers. Among other things, it was proved in [15, Theorem 3.3] that when the dihedral quandle \( \mathbb{Z}_n \) is of prime order, then the \( \mathbb{Z}_n \)-quandle coloring quivers of two links \( L_1 \) and \( L_2 \) are isomorphic if and only if the quandle coloring spaces of \( L_1 \) and \( L_2 \) with respect to the dihedral quandle \( \mathbb{Z}_n \) have the same size. That is, it was proved that if \( n \) is prime, then \( Q_{\mathbb{Z}_n}(L_1) \cong Q_{\mathbb{Z}_n}(L_2) \) as quivers if and only if \( |\text{Hom}(Q(L_1), \mathbb{Z}_n)| = |\text{Hom}(Q(L_2), \mathbb{Z}_n)| \). Hence, for \( n \) prime, we can use Taniguchi’s result in
combination with our results in Theorem 5 and Corollary 10 to say something about the full $\mathbb{Z}_n$-quandle coloring quiver of a certain link $L$. For this, we would need to have that $|\text{Hom}(Q(L), \mathbb{Z}_n)| = |\text{Hom}(Q(T(p, 2)), \mathbb{Z}_n)|$, for a particular link $L$, a torus link $T(p, 2)$, and a prime $n$. Specifically, if $n$ is prime and $|\text{Hom}(Q(L), \mathbb{Z}_n)| = n$ for a certain link $L$, then $Q_{\mathbb{Z}_n}(L) \cong (\overrightarrow{K_n}, \hat{n})$. Moreover, if $n$ is prime and $|\text{Hom}(Q(L), \mathbb{Z}_n)| = n^2$, then $Q_{\mathbb{Z}_n}(L) \cong (\overrightarrow{K_n}, \hat{n}) \downarrow \overrightarrow{\hat{1}} (\overrightarrow{K_{n(n-1)}}, \hat{1})$.

Figure 4. The subgraph components of the quiver $Q_{\mathbb{Z}_3}(T(3, 2))$

Figure 5. The quandle coloring quiver $Q_{\mathbb{Z}_3}(T(3, 2))$
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