# Mackey profunctors

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Introduction.

The so-called “Mackey functors” associated to a finite group $G$ have long been a standard tool both in group theory and in algebraic topology, where they appear as natural coefficients for $G$-equivariant cohomology theories. The reader can find modern introductions to Mackey functor in the topological context e.g. in [LMS], [M], [tD], or a more algebraic treatment in [T]. A derived version of the theory has been suggested in [K2].

In topological applications, the group $G$ does not have to be finite — one can generalize the notion of a Mackey functor to allow $G$ to be an arbitrary compact Lie group equipped with its standard topology.

The goal of the present paper is to suggest another generalization. We do not equip $G$ with any topology, but we do not require it to be finite — we only require that $G$ is finitely generated.

We should note that formally, the definition of a $G$-Mackey functor depends not on the group $G$ but on the category of finite $G$-sets. Thus already in the original definition, one can allow $G$ to be infinite — it is only $G$-sets we consider that have to be finite. However, the finer points of the theory break down, and the resulting category of $G$-Mackey functors seems not to
be the right thing to consider. What we suggest in the present paper is an alternative theory of “Mackey profunctors” that seems to behave better and preserve most of the nice properties of Mackey functors for finite groups. In fact, our theory is a “profinite completion” of the usual theory — roughly speaking, giving a $G$-Mackey profunctor is equivalent to giving a system of $W$-Mackey profunctors for all finite quotients $W$ of the group $G$, related by some natural compatibility maps. This allows to transfer most of the finite theory to our profinite case without much effort.

We note that both the naive generalization of Mackey functors and our theory of Mackey profunctors only depend on the profinite completion $\hat{G}$ of the group $G$. This is still a compact group with respect to its natural profinite topology. However, the topology is very different from the standard topology on Lie groups, and the theory we develop also looks differently — in effect, it is completely orthogonal to the theory of Mackey functors for compact Lie groups. At present, we do not know whether there is a general context that unifies the two.

Formally, the theory developed in the paper is self-contained, and the whole paper could be considered simply an extended exercise in homological algebra. However, given the length of the exercise, we should say at least a couple of words about motivations. These are twofold. Firstly, one can wonder whether there is a meaningful stable equivariant homotopy theory for profinite groups, and treat the theory of Mackey profunctors as a first step in this direction — a sort of homological approximation of the full homotopical theory. This point of view has been suggested recently by C. Barwick in [B], and it looks extremely interesting. Our motivations are more mundane and have to with Topological Cyclic Homology of [BHM]. Here the relevant topological group has long been understood to be the unit circle $S^1$. In practice, what one considers is not the full unit circle but all its finite subgroups — that is, the subgroups $\mathbb{Z}/n\mathbb{Z} \subset S^1$ formed by roots of unity of order $n$, $n \geq 1$. However, one can package the same groups differently — they are also finite quotients of the infinite cyclic group $\mathbb{Z}$. Does this have any relevance for Topological Cyclic Homology? At present, we do not know the answer, but we strongly suspect that it is positive. In particular, a compatible system of $\mathbb{Z}/n\mathbb{Z}$-Mackey functors generated by a $\mathbb{Z}$-Mackey profunctor in our sense looks quite similar to the projective system used to define the spectrum $TR(A)$ associated to a ring spectrum $A$.

In spite of the fact that our main interest is in the group $\mathbb{Z}$, we found out that the theory in full generality is actually more transparent (in particular, the fact that $\mathbb{Z}$ is commutative obscures things). Thus we start with a completely arbitrary group $G$. At some point in the story, we need to
require it to be finitely generated, but this is the only restriction. We develop profinite counterparts both for the classic theory of Mackey functors and for its derived version constructed in [K2].

The paper is organized as follows. Section 1 contains preliminaries and notation. Section 2 is a brief recapitulation of the standard theory of Mackey functors. Section 3 introduces $G$-Mackey profunctors and contains the beginning of the theory, up to the point when we need to go to the derived setting. In particular, we introduce the notion of a “normal system”, an axiomatization of a compatible family of $W$-Mackey functors for all the finite quotients $W$ of the group $G$. To go to the derived setting, we use some technology developed in [K2]. In Section 4, we recall this technology and we develop it further (in particular, Subsection 4.3 is new). Section 5 is also new — while its main result is a version of a result proved in [K2], we give an alternative proof that is easier and cleaner. In Section 6, we start developing the theory of derived Mackey profunctors. We use both the technology of [K2] and new tools created for this paper. These new tools also allow to clean up and strengthen some of the results of [K2]: we take the opportunity to do so in Section 7. Then in Section 8, we continue with derived Mackey profunctors, and we prove the results that depend on a derived version of the notion of a normal system. Finally, in Section 9, we show how our abstract machinery works in the particular case $G = Z$, the infinite cyclic group.

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1 Preliminaries.

We start with recalling some facts about combinatorics of simplicial sets and homology of small categories that we will need in the paper. All the material is quite standard; we mostly include it to fix notation.
1.1 Homology of small categories. For any two objects \( c, c' \in C \) of a category \( C \), we will denote by \( C(c, c') \) the set of maps in \( C \) from \( c \) to \( c' \). For any object \( c \in C \) in a category \( C \), we will denote by \( C/c \) the category of objects \( \tilde{c} \in C \) equipped with a map \( \tilde{c} \to c \). Any map \( f : c \to c' \) in \( C \) induces a functor

\[
(1.1) \quad f_! : C/c \to C/c'
\]

sending a map \( \tilde{c} \to c \) to its composition with \( f \), and if \( C \) has fibered products,

\[
(1.2) \quad f^* : C/c' \to C/c
\]

sending a map \( \tilde{c} \to c' \) to the natural projection \( \tilde{c} \times_{c'} c \to c \).

For any category \( C \), an inverse system \( \{c_i\} \) of objects in \( C \) is a collection of objects \( c_i \in C \), \( i \geq 1 \), equipped with transition maps \( c_{i+1} \to c_i \).

For any small category \( C \) and ring \( R \), we denote by \( \text{Fun}(C, R) \) the category of functors from \( C \) to the category of \( R \)-modules. For any \( R \)-module \( M \), we will denote by \( M_C \in \text{Fun}(C, R) \) the constant functor with value \( M \), and we will sometimes shorten it to \( M \) if \( C \) is clear from the context. For any object \( c \in C \), we will denote by \( M_c \in \text{Fun}(C, R) \) the representable functor given by

\[
(1.3) \quad M_c(c') = M[C(c, c')],
\]

the sum of copies of \( M \) numbered by elements in the set of maps \( C(c, c') \).

The category \( \text{Fun}(C, R) \) is abelian. We denote its derived category by \( D(C, R) \). If \( C \) is the point category \( \text{pt} \), so that \( \text{Fun}(\text{pt}, R) \) is the category of \( R \)-modules, we shorten the notation by setting \( D(R) = D(\text{pt}, R) \), the derived category of \( R \)-modules. Representable objects \( M_c \) of \((1.3)\) generate the category \( D(C, R) \) in the sense that every object \( E \in D(C, R) \) can be represented by an \( h \)-projective complex whose terms are sums of objects of the form \((1.3)\).

For any functor \( \gamma : C \to C' \) between small categories, composition with \( \gamma \) gives a pullback functor \( \gamma^* : \text{Fun}(C', R) \to \text{Fun}(C, R) \). It has a left and a right-adjoint functor known as the left and the right Kan extension; we denote them by \( \gamma_!, \gamma_* : \text{Fun}(C, R) \to \text{Fun}(C', R) \). The derived functors \( L^\ast \gamma_!, R^\ast \gamma_* : D(C, R) \to D(C', R) \) are left resp. right-adjoint to the pullback functor \( \gamma^* : D(C', R) \to D(C, R) \). If \( C' \) is the point category \( \text{pt} \), and \( \tau : C \to \text{pt} \) is the tautological projection, then the Kan extension functors \( \tau_!, \tau_* \) are just the colimit and limit over the small category \( C \). We denote

\[
C_*(C, E) = L^\ast \tau_! E \in D(R)
\]
for any $E \in \text{Fun}(C, R)$. The homology groups $H_*(C, E)$ of this complex are by definition the homology groups of the category $C$ with coefficients in $E$.

In some situations, computing the derived Kan extensions is easy. For any example, for any $c \in C$, $\gamma : C \to C'$ and $R$-module $M$, we have

\begin{equation}
L^* \gamma_! M_c \cong M_{\gamma(c)},
\end{equation}

where $M_c, M_{\gamma(c)}$ are the representable functors. Another case is a functor $\gamma : C \to C'$ that admits a right-adjoint $\delta : C' \to C$. Then we have a natural isomorphism

\begin{equation}
\gamma_! \cong \delta^*,
\end{equation}

and in particular, $\gamma_!$ is an exact functor, so that we also have an isomorphism

\begin{equation}
L^* \gamma_! \cong \delta^*
\end{equation}

on the level of derived categories.

Given a small category $C$, two algebras $R_1, R_2$ flat over a commutative ring $k$, and two objects $E_1 \in \text{Fun}(C, R_1), E_2 \in \text{Fun}(C, R_2)$, we denote by $E_1 \otimes_k E_2 \in \text{Fun}(C, R_1 \otimes_k R_2)$ their pointwise tensor product, and we use the same notation for the derived pointwise tensor product of objects in the derived categories $D(C, R_1), D(C, R_2)$. If $k$ is clear from context, we drop it from notation. If we have two small categories $C_1, C_2$, and objects $E_1 \in D(C, R_1), E_2 \in D(C, R_2)$, then their \textit{box product} is given by

\[ E_1 \boxtimes_k E_2 = \pi_1^* E_1 \otimes_k \pi_2^* E_2 \in D(C_1 \times C_2, R_1 \otimes_k R_2), \]

where $\pi_1, \pi_2$ are projections from $C_1 \times C_2$ to $C_1$ resp. $C_2$. Again, if $k$ is clear from context, we drop it from notation.

Assume given two small categories $C, C_1$, a ring $R$, and an object $T \in \text{Fun}(C_1, \mathbb{Z})$. Define a functor $j_T^C : D(C, R) \to D(C \times C_1, R)$ by setting

\begin{equation}
J^C_T(E) = E \boxtimes T.
\end{equation}

If we have another small category $C'$ and a functor $\gamma : C' \to C$, we have an obvious isomorphism

\begin{equation}
(\gamma \times \text{id})^* \circ J^C_T \cong J^{C'}_{\gamma*} \circ \gamma^*.
\end{equation}

We will need the following easy result.
Lemma 1.1. Assume that for any object $c \in C_1$, $T(c)$ is a finitely generated flat $\mathbb{Z}$-module. Then for any small category $\mathcal{C}$, the functor $j^T_\mathcal{C}$ of (1.6) has a left-adjoint $l^T_\mathcal{C}$: $\mathcal{D}(\mathcal{C} \times C_1, R) \to \mathcal{D}(\mathcal{C}, R)$, and for any functor $\gamma : \mathcal{C}' \to \mathcal{C}$ between small categories, the base change map

\[ l^T_{\mathcal{C}'} \circ (\gamma \times \text{id})^* \to \gamma^* \circ l^T_{\mathcal{C}} \]

induced by the isomorphism (1.7) is itself an isomorphism.

Proof. Since the category $\mathcal{D}(\mathcal{C} \times C_1, R)$ is generated by objects $M_{c \times c_1}$ of (1.3), $c \times c_1 \in \mathcal{C} \times C_1$, $M$ an $R$-module, it suffices to construct $l^T_{\mathcal{C}}$ for such objects. By adjunction, it is given by

\[ l^T_{\mathcal{C}}(M_{c \times c_1}) = M_c \otimes T(c_1). \]

To prove that (1.8) is an isomorphism, it again suffices to check it after evaluating at an object $M_{c \times c_1}$, and moreover, it suffices to consider the case when $\mathcal{C}' = \text{pt}$ is a point category. In this case, the claim immediately follows from (1.3) and (1.9). \[ \square \]

1.2 Fibrations and cofibrations. Throughout the paper, we will heavily use the machinery of [SGA]. To fix the terminology, here are the basic ingredients.

- A morphism $f : c_1 \to c_2$ in a category $\mathcal{C}'$ is cartesian with respect to a functor $\gamma : \mathcal{C}' \to \mathcal{C}$ if any $f' : c'_1 \to c_2$ with $\pi(f) = \pi(f')$ factors uniquely as $f' = p \circ f$ with invertible $\pi(p)$.

- A functor $\pi : \mathcal{C}' \to \mathcal{C}$ is a prefibration if for any $c \in \mathcal{C}'$ and any morphism $f : b \to \pi(c)$ in $\mathcal{C}$, there exists a cartesian $f' : b' \to c$ with $\pi(f') = f$.

- A prefibration is a fibration if the composition of any two cartesian maps is cartesian.

- For any map $f : b' \to b$ in $\mathcal{C}$, the associated transition functor $f^* : \pi^{-1}(b) \to \pi^{-1}(b')$ between fibers of a fibration $\pi : \mathcal{C}' \to \mathcal{C}$ sends $c \in \pi^{-1}(b)$ to the source $c'$ of the cartesian map $f' : c' \to c$ with $\pi(f') = f$ (one checks that this is functorial, and $f^*$ is well-defined up to a canonical isomorphism).

- For any two fibrations $\gamma_1 : \mathcal{C}_1 \to \mathcal{C}$, $\gamma_2 : \mathcal{C}_2 \to \mathcal{C}$, a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ such that $\gamma_1 \simeq \gamma_2 \circ F$ is cartesian if it sends cartesian maps to cartesian maps.
The notions of a cocartesian map, a cofibration $\pi : C' \to C$, a transition functor $f_* : \pi^{-1}(b') \to \pi^{-1}(b)$, and a cocartesian functors are obtained by passing to opposite categories.

For any cartesian square

$$\begin{array}{ccc}
C'_1 & \xrightarrow{\gamma'_1} & C' \\
\downarrow{\nu_1} & & \Downarrow{\nu} \\
C_1 & \xrightarrow{\gamma_1} & C
\end{array}$$

(1.10)

of small categories, if $\gamma$ is a fibration or a cofibration, then $\gamma'$ is also a fibration resp. a cofibration. Moreover, for any cartesian functor $F : C_1 \to C_2$ between fibrations $\gamma_1 : C_1 \to C$, $\gamma_2 : C_2 \to C$, and an arbitrary functor $\nu : C' \to C$, we can form the commutative diagram

$$\begin{array}{ccc}
C'_1 & \xrightarrow{F'} & C'_2 & \xrightarrow{\gamma'_2} & C' \\
\downarrow{\nu_1} & & \Downarrow{\nu_2} & & \downarrow{\nu} \\
C_1 & \xrightarrow{F} & C_2 & \xrightarrow{\gamma_2} & C
\end{array}$$

(1.11)

with cartesian squares. Then $\gamma'_2$, $\gamma'_1 = F' \circ \gamma'_2$ are fibrations, the functor $F'$ is cartesian, and for any ring $R$, the isomorphism $F'^* \circ \nu'_2 \cong \nu'_1 \circ F'^*$ induces by adjunction a base change map

$$\nu_2^* \circ R^* F'_* \to R^* F'^* \circ \nu_1^*.$$ (1.12)

Dually, if $\gamma_1$, $\gamma_2$ are cofibrations, and $F$ is cocartesian, we have the base change map

$$L^* F'^* \circ \nu_1^* \to \nu_2^* \circ L^* F_1.$$ (1.13)

We prove the following easy result for which we could not find a convenient reference.

**Lemma 1.2.** If $F$ in (1.11) is a cartesian functor between fibrations, then the base change map (1.12) is an isomorphism, and if $F$ is a cocartesian functor between cofibrations, then (1.13) is an isomorphism.

**Proof.** We will prove the claim for cofibrations; the proof for fibrations is dual. First of all, it clearly suffices to prove the claim when $C' = \text{pt}$, and $\nu$ is the embedding onto an object $c' \in C$. Next, the category Fun($C_1$, $R$) is
generated by representable functors (1.3), so it suffices to prove that for any 
c ∈ C and any R-module M, the natural map
\[ \nu_2^* L^* F_i M_c \rightarrow L^* F_i \nu_1^* M_c \]
is an isomorphism. By (1.4), we have \( L^* F_i M_c \cong M_{F(c)} \), and since \( \gamma_1, \gamma_2 \) are 
cofibrations, (1.3) gives canonical identifications
\[ \nu_1^* M_c \cong \bigoplus_{f \in C(\gamma_1(c), c')} M_{f_*(c)}, \quad \nu_2^* M_{F(c)} \cong \bigoplus_{f \in C(\gamma_1(c), c')} M_{f_*(F(c))}. \]
Since \( F \) is cocartesian, we have \( f_*(F(c)) \cong F(f_*(c)) \), so that to finish the 
proof, it remains to notice that \( L^* F_i M_{f_*(c)} \cong M_{F(f_*(c))} \) by (1.4).

In particular, by taking \( C_2 = C \), we can apply Lemma (1.2) to any cartesian diagram (1.10). This shows that for any cofibration \( \gamma : C' \rightarrow C \) and any 
\( E \in \mathcal{D}(C', R) \), the value of the Kan extension \( L^* \gamma_! E \) at some object \( c \in C \)
can be expressed as
\[ L^* \gamma_! E(c) \cong H_*(\gamma^{-1}(c), E), \]
where \( \gamma^{-1}(c) \subset C' \) is the fiber of the cofibration \( \gamma \) over \( c \).

If \( \gamma : C' \rightarrow C \) is not a cofibration, then it can still be factorized as
\[ C' \xrightarrow{e} \gamma \backslash C \xrightarrow{t} C, \tag{1.14} \]
where \( \gamma \backslash C \) is the comma-category of the functor \( \gamma \) (that is, the category 
of triples \( \langle c', c, f \rangle \), \( c' \in C', c \in C, f : \gamma(c') \rightarrow c \)), \( t \) is the natural projection 
sending \( \langle c', c, f \rangle \) to \( c \), and \( e \) is the natural embedding sending \( c' \) to 
\( \langle c', \gamma(c), \text{id} \rangle \). Then \( t \) is a cofibration, while \( e \) has a right-adjoint \( s : \gamma \backslash C \rightarrow C' \)
sending \( \langle c', c, f \rangle \) to \( c' \). Therefore by (1.5), we have
\[ L^* \gamma_! \cong L^* t_! \circ L^* e_! \cong L^* t_! \circ s^*. \]
In particular, if \( M \) is an R-module, and \( M_{C'} \in \text{Fun}(C', R) \) is the constant 
functor with value \( M \), then \( s^* M_{C'} \cong M_{\gamma \backslash C} \in \text{Fun}(\gamma \backslash C, R) \), and we have
\[ L^* \gamma_! M \cong L^* t_! M. \tag{1.15} \]
For example, let \( C' = \text{pt} \), the point category, embedded by \( \gamma \) to an object 
\( c \in C \). Then (1.14) reads as
\[ \begin{aligned}
\text{pt} & \longrightarrow c \backslash C \xrightarrow{t} C,
\end{aligned} \]
where \( c \backslash C \) is the category of objects \( c' \in C \) equipped with a map \( c \rightarrow c' \), 
and \( t \) is the tautological projection forgetting the map. In this case, the 
cofibration \( t \) is discrete, so that \( t_! \) is an exact functor, and we have
\[ L^* \gamma_! M \cong \gamma_! M \cong t_! M \cong M_c, \tag{1.16} \]
where \( M_c \) is the representable functor (1.3).
1.3 Pointed sets. We will denote by Sets the category of sets, with $\emptyset$ being the empty set and $\text{pt}$ being the one-element set. A pointed set is a set $S$ equipped with a distinguished element $o \in S$. We will denote the category of pointed sets by $\text{Sets}_+$. For any set $S$ equipped with a subset $A \subset S$, we denote by $S/A \in \text{Sets}_+$ the pointed set obtained by the pushout square

$$
\begin{array}{ccc}
A & \longrightarrow & S \\
\downarrow & & \downarrow \\
\text{pt} & \xrightarrow{i} & S/A,
\end{array}
$$

with the distinguished element $i(pt) \in S/A$. For example, for any set $S$, the quotient $S_+ = S/\emptyset$ is given by

$$(1.17) \quad S_+ = S \sqcup \{o\},$$

that is, the union of $S$ with a new element $o$ which we take as distinguished. The functor $\text{Sets} \to \text{Sets}_+$ sending $S$ to $S_+$ is left-adjoint to the forgetful functor $\text{Sets}_+ \to \text{Sets}$. The category $\text{Sets}_+$ has finite coproducts given by

$$(1.18) \quad S \vee S' = (S \sqcup S')/\{o,o'\},$$

and we have $(S \sqcup S')_+ = S_+ \vee S'_+$ for any two sets $S,S' \in \text{Sets}$. The smash product $S \wedge S'$ of two pointed sets $S,S' \in \text{Sets}_+$ is given by

$$(1.19) \quad S \wedge S' = (S \times S')/((\{o\} \times S') \cup (S \times \{o'\})), $$

and we have $(S \times S')_+ = S_+ \wedge S'_+$. The one-point set $\emptyset_+$ is both the initial and the terminal object of $\text{Sets}_+$, and for any $S \in \text{Sets}_+$, we have

$$(1.20) \quad S \wedge \emptyset_+ = \emptyset_+.$$

1.4 Simplicial objects. As usual, we denote by $\Delta$ the category of non-empty finite totally ordered sets, – or in other words, finite ordinals, – with $[n] \in \Delta$ being the set with $n$ elements, $n \geq 1$. A simplicial object in a category $\mathcal{C}$ is a functor $\Delta^o \to \mathcal{C}$ from the opposite category $\Delta^o$. More generally, an $n$-simplicial object is a functor $(\Delta^o)^n \to \mathcal{C}$ from the $n$-fold self-product $(\Delta^o)^n = \Delta^o \times \cdots \times \Delta^o$; we extend it to $n = 0$ by setting $(\Delta^o)^0 = \text{pt}$, so that a 0-simplicial object is just an object in $\mathcal{C}$. We denote the category of $n$-simplicial objects in $\mathcal{C}$ by $(\Delta^o)^n \mathcal{C}$. For any object $c \in \mathcal{C}$ and any $n$-simplicial object $c' \in (\Delta^o)^n \mathcal{C}$, we denote by $\mathcal{C}(c,c') \in (\Delta^o)^n \mathcal{C}$ Sets the $n$-simplicial set given by

$$(1.21) \quad \mathcal{C}(c,c')([m]) = \mathcal{C}(c,c'([n])), \quad [m] \in (\Delta^o)^n.$$
For any abelian category $\mathcal{E}$, we have the Dold-Kan equivalence

$$\Delta^\circ \mathcal{E} \cong C_{\geq 0}(\mathcal{E})$$

between $\Delta^\circ \mathcal{E}$ and the category $C_{\geq 0}(\mathcal{E})$ of chain complexes in $\mathcal{E}$ concentrated in non-negative homological degrees. The equivalence sends a simplicial object $E \in \Delta^\circ \mathcal{E}$ to its normalized chain complex $C_*(E)$. For any ring $R$ and any $E \in \text{Fun}(\Delta^\circ, R)$, the complex $C_*(E)$ is quasiisomorphic to the homology complex $C_*(\Delta^\circ, E)$.

For any simplicial set $S \in \Delta^\circ \text{Sets}$ and any abelian group $M$, we have the standard normalized chain complex

$$C_*(X, M)$$

that computes the homology of $X$ with coefficients in $M$. If we consider the functor $M[X] \in \text{Fun}(\Delta^\circ, \mathbb{Z})$ given by

$$M[X]([n]) = M[X([n])],$$

then the complex (1.23) corresponds to $M[X]$ under the Dold-Kan equivalence (1.22). In particular, it computes the homology of the category $\Delta^\circ$ with coefficients in $M[X]$, so that we have a natural quasiisomorphism

$$C_*(X, M) \cong C_*(\Delta^\circ, M[X]).$$

For an $n$-simplicial set $X$, there are two ways to define homology. Firstly, we can consider the restriction $\delta^\ast X \in \Delta^\circ \text{Sets}$ with respect to the diagonal embedding $\delta : \Delta^\circ \to (\Delta^\circ)^n$, and set

$$C_*(X, M) = C_*(\delta^\ast X, M).$$

Secondly, we can take

$$C_*(X, M) = C_*((\Delta^\circ)^n, M[X]),$$

where $M[X]$ has the same meaning as in (1.24). The two definitions are the same up to a quasiisomorphism: for any ring $R$ and any $E \in \text{Fun}((\Delta^\circ)^n, R)$, the adjunction map gives a canonical quasiisomorphism

$$C_*(\Delta^\circ, \delta^\ast E) \to C_*(((\Delta^\circ)^n, E).$$

In particular, for any two simplicial sets $X_1, X_2 \in \Delta^\circ \text{Sets}$, their box product $X_1 \boxtimes X_2$ is a 2-simplicial set given by

$$X_1 \boxtimes X_2([n_1] \times [n_2]) = X_1([n_1]) \times X_2([n_2]),$$
the restriction $\delta^*(X_1 \boxtimes X_2)$ is the pointwise product $X_1 \times X_2$, and (1.25) gives the Künneth quasiisomorphism

$$C_\ast(X_1 \times X_2, \mathbb{Z}) \cong C_\ast(X_1 \boxtimes X_2, \mathbb{Z}) = C_1(X_1, \mathbb{Z}) \otimes C_2(X_2, \mathbb{Z}).$$

For pointed simplicial set $X \in \Delta^o \text{Sets}_+$, the reduced chain complex is given by

$$C_\ast(X, \mathbb{Z}) = C_\ast(Z, \mathbb{Z}) / Z \cdot o,$$

where $o \in X([1])$ is the distinguished element. For reduced chain complexes, the Künneth formula reads as

$$C_\ast(X_1 \wedge X_2, \mathbb{Z}) \cong C_\ast(X_1, \mathbb{Z}) \otimes C_\ast(X_2, \mathbb{Z}),$$

where $X_1 \wedge X_2$ is the pointwise smash product of pointed simplicial sets $X_1, X_2 \in \Delta^o \text{Sets}_+$.

1.5 Contractible simplicial sets. Denote by $\Delta_+ \subset \Delta$ the subcategory of non-empty finite ordinals and maps between them that send the terminal element to the terminal element. The embedding $\Delta_+ \subset \Delta$ has a left-adjoint functor $\kappa : \Delta \to \Delta_+$ that adds a new terminal element to an ordinal. We will denote by $[n]_+ \in \Delta_+$, $n \geq 0$ the ordinal with $n + 1$ elements, so that $\kappa([n]) = [n]_+$.

We say that a 0-simplicial set $S \in \text{Sets}$ is contractible if $S = \text{pt}$, and we introduce the following inductive definition.

**Definition 1.3.** For any $n \geq 1$, an $n$-simplicial set $X \in (\Delta^o)^n \text{Sets}$ is contractible if we have

$$X \cong (\kappa \times \text{id}^{n-1})^* \bar{X}$$

for some functor $\bar{X} : \Delta_+^o \times (\Delta^o)^{n-1} \to \text{Sets}$ called an extension of $X$, and the restriction of $\bar{X}$ to $[0]_+ \times (\Delta^o)^{n-1} \subset \Delta_+^o \times (\Delta^o)^{n-1}$ is a contractible $(n - 1)$-simplicial set.

We say that a pointed $n$-simplicial set is contractible if it becomes contractible after forgetting the distinguished point, and we note that the product of contractible $n$-simplicial sets and the smash-product of contractible pointed $n$-simplicial sets is obviously contractible. Moreover, by (1.20), the smash product $S \wedge X$ of a contractible pointed $n$-simplicial $X$ and a pointed set $S$ is automatically contractible.

**Lemma 1.4.** For any contractible $n$-simplicial set $X$, we have a natural isomorphism

$$C_\ast(X, \mathbb{Z}) \cong \mathbb{Z}.$$
Proof. Applying (1.25) inductively, we see that it suffices to prove that for any \( E \in \text{Fun}(\Delta^+_+, \mathbb{Z}) \), the natural map

\[
C_*(\Delta^o, \kappa^* E) \to E([0]_+)
\]

is a quasiisomorphism. Since \( \kappa : \Delta^o \to \Delta^+_o \) is right-adjoint to the embedding \( \Delta^+_o \subset \Delta^o \), we have

\[
C_*(\Delta^o, \kappa^* E) \cong C_*(\Delta^+_o, E),
\]

and since \([0]_+ \in \Delta^+_o\) is the terminal object, the claim is obvious. \(\square\)

We will need the following standard examples of contractible \( n \)-simplicial sets.

**Example 1.5.** For any \( n \geq 0 \), the elementary \( n \)-simplex \( I_n \in \Delta^o \text{Sets} \) given by

\[
I_n([m]) = \Delta([m], [n + 1]), \quad [m] \in \Delta^o
\]

is contractible (\( I_n \in \Delta^o \text{Sets} \) is given by \( I_n([m]_+) = \Delta_+([m]_+, [n]_+) \)).

**Example 1.6.** Let \( \rho : \Delta \to \text{Sets} \) be the forgetful functor taking an ordinal to itself considered simply as a set, and extend it to the forgetful functor \( \rho_+ : \Delta_+ \to \text{Sets}_+ \) by taking the terminal element as the distinguished point. For any set \( S \), let \( ES \in \Delta^o \text{Sets} \) be the simplicial set given by

\[
ES([n]) = S^n = \text{Sets}(\rho([n]), S).
\]

Then as soon as \( S \) is not empty, \( ES \) is contractible – to obtain an extension \( \tilde{ES} \in \Delta^+_o \text{Sets} \), let

\[
\tilde{ES}([n]_+) = \text{Sets}_+(\rho_+([n]_+), S),
\]

where we choose an arbitrary element in \( S \) and treat it as a pointed set.

**Example 1.7.** For any simplicial set \( X \in \Delta^o \text{Sets} \), let \( C(X) \) be the 2-simplicial pointed set obtained by pushout square

\[
\begin{array}{ccc}
X \boxtimes \{s, t\} & \longrightarrow & \{s, t\} \\
\downarrow & & \downarrow \\
X \boxtimes I & \longrightarrow & C(X),
\end{array}
\]

where \( I = I_1 \in \Delta^o \text{Sets} \) is the simplicial interval, \( \{s, t\} \subset I \) is the constant simplicial set formed by its two ends, and we choose \( s \in \{s, t\} \subset C(X) \) as
the distinguished point. Then if $X$ is contractible, $C(X)$ is also contractible. Indeed, an extension $\tilde{C}(X) : \Delta^o_+ \times \Delta^o \to \text{Sets}$ is given by the same square with $X$ replaced by its extension $\tilde{X}$, and since $\tilde{X}([0]_+)$ must be the one-element set $\text{pt}$, the restriction of $\tilde{C}(X)$ to $[0]_+ \times \Delta^o$ is the interval $I$.

**Remark 1.8.** Example [17] is the reason we need to bother with $n$-simplicial sets – I do not know whether one can define a contractible $C(X) \in \Delta^o \text{Sets}_+$ with the same properties.

### 1.6 Triangulated categories

Finally, we need some results on triangulated categories. It is worthwhile to mention that as in [K2], our notion of a triangulated category is the original one of Verdier, with derived categories $D(C, R)$ giving examples. We also assume known the notion of a $t$-structure on a triangulated category $D$ introduced in [BBD], and the fact that such a structure is completely defined by the subcategories $D^{\leq i} \subset D$, $i$ an integer. The *standard t-structure* on $D = D(C, R)$ is obtained by taking as $D^{\leq i}(C, R)$ the full subcategory spanned complexes concentrated in cohomological degrees $\leq i$. For any $t$-structure, we denote by $D^- \subset D$ the union of all the full subcategories $D^{\leq i} \subset D$.

For an inverse system $E_i$ of objects in a triangulated category $D$ with infinite products, the *telescope* $\text{Tel}(E_i)$ is defined by the distinguished triangle

$$\text{Tel}(E_i) \to \prod_i E_i \xrightarrow{id - t} \prod_i E_i \to \text{Tel}(E_i),$$

where $t$ is the product of the transition maps $E_{i+1} \to E_i$. If we throw away a finite number of terms in the inverse system, the telescope does not change, and if the inverse system is a constant one with value $E$, then its telescope is also isomorphic to $E$. If $D = D(C, R)$, then the inverse system $E_i$ comes from an inverse system of complexes in $\text{Fun}(C, R)$, and we have

$$\text{Tel}(E_i) \cong \lim_{\leftarrow}^* E_i,$$

where $\lim_{\leftarrow}^*$ in the right-hand side is the derived functor of the inverse limit.

Analogously, for a direct system $E_i$ of objects in $C$, its telescope $\text{tel}(E_i)$ is defined by the distinguished triangle

$$\bigoplus_i E_i \xrightarrow{id - t} \bigoplus_i E_i \to \text{tel}(E_i) \to ,$$

where $t$ is the sum of the transition maps $E_i \to E_{i+1}$. It enjoys the properties similar to Tel. Moreover, if $D = D(R)$ is the derived category of modules
over a ring $R$ – or more generally, the derived category of an abelian category satisfying $AB5$ – then the homology objects $\mathcal{H}_*(\text{tel}(E_i))$ are given by

$$\mathcal{H}_*(\text{tel}(E_i)) = \lim_{\to} \mathcal{H}_*(E_i),$$

where $\mathcal{H}_*(E_i)$ are the homology objects of $E_i$.

We will also need one more technical result. Assume given two triangulated categories $D_1$, $D_2$, and a pair of triangulated functors $\varphi : D_2 \to D_1$, $\rho : D_1 \to D_2$ such that $\rho$ is right-adjoint to $\varphi$. Moreover, assume given full triangulated subcategories $D'_1 \subset D_1$, $D'_2 \subset D_2$ such that the quotients $D_1/D'_1$, $D_2/D'_2$ are well-defined, and denote by $q_1 : D_1 \to D_1/D'_1$, $q_2 : D_1 \to D_1/D'_1$ the quotient functors. Finally, assume that $D'_1 \subset D_1$ is left-admissible – that is, $q_1$ has a right-adjoint functor $r : D_1/D'_1 \to D_1$.

**Lemma 1.9.** In the assumptions above, assume that $\varphi : D_2 \to D_1$ sends $D'_2 \subset D_2$ into $D'_1 \subset D_1$, thus induces a functor

$$\overline{\varphi} : D_2/D'_2 \to D_1/D'_1.$$

Then the functor

$$\overline{\rho} = q_2 \circ \rho \circ r : D_1/D'_1 \to D_2/D'_2$$

is right-adjoint to the functor $\overline{\varphi}$.

**Proof.** We note that if the projection $q_2$ has a left-adjoint functor $l : D_2/D'_2 \to D_2$, the statement is obvious: $\overline{\rho}$ is right-adjoint to $q_1 \circ \overline{\varphi} \circ l \cong \varphi \circ q_2 \circ l$, and since $q_2$ is the localization functor, $l$ must be fully faithful, so that $q_2 \circ l \cong \text{Id}$.

In the general case, note that for an arbitrary object $E \in D_2$, we have a natural map

$$\begin{array}{c}
q_2(E) \xrightarrow{q_2(a(\varphi))} (q_2 \circ \rho \circ \varphi)(E) \xrightarrow{(q_2 \circ \rho \circ a(q_1))} (q_2 \circ \rho \circ r \circ q_1 \circ \varphi)(E)
\end{array}$$

where $a(\varphi) : \text{Id} \to \rho \circ \varphi$, $a(q_2) : \text{Id} \to r \circ q_1$ are the adjunction maps, and by definition, we have

$$(q_2 \circ \rho \circ r \circ q_1 \circ \varphi)(E) \cong (q_2 \circ \rho \circ r \circ \overline{\varphi} \circ q_2)(E) = (\overline{\varphi} \circ \overline{\varphi})(q_2(E)).$$

The map (1.29) is functorial in $E$, and since $q_2$ is a localization functor, it induces a map

$$a(\overline{\varphi}) : \text{Id} \to \overline{\rho} \circ \overline{\varphi}.$$
Analogously, for any $E \in D_1$, we have a diagram

$$(\varphi \circ \rho \circ r \circ q_1)(E) \xrightarrow{a(\rho)} (r \circ q_1)(E) \xleftarrow{a(q_1)} E,$$

where $a(\rho) : \rho \circ \varphi \to \text{Id}$ is the adjunction map, and by the definition of Verdier localization, it induces a map

$$a(\mathfrak{p}) : \mathfrak{p} \circ \mathfrak{p} \to \text{Id}.$$ 

To prove that the maps $a(\varphi)$ and $a(\mathfrak{p})$ define an adjunction between $\varphi$ and $\mathfrak{p}$, it remains to check that the composition morphisms

$$\mathfrak{p} \xrightarrow{a(\mathfrak{p})} \mathfrak{p} \circ \mathfrak{p} \circ \mathfrak{p} \xrightarrow{\mathfrak{p}(a(\varphi))} \mathfrak{p}, \quad \mathfrak{p} \circ \mathfrak{p} \circ \mathfrak{p} \xrightarrow{\mathfrak{p}(a(\mathfrak{p}))} \mathfrak{p} \circ \mathfrak{p} \circ \mathfrak{p} \xrightarrow{a(\varphi)} \mathfrak{p},$$

are equal to the identity maps. This is a straightforward exercise that we leave to the reader. 

\[ \square \]

2 Recollection on Mackey functors.

We will also need some facts from the classical theory of Mackey functors; this is reasonably well-known, but we include a brief sketch for the convenience of the reader. We follow the exposition in [K2, Section 2].

2.1 Definitions. For any group $G$, let $B^G$ be the category with the same objects as $\Gamma_G$, the category of finite $G$-sets, and with Hom-sets given by

$$(2.1) \quad B^G(S_1, S_2) = \mathbb{Z}[\text{Iso}(\Gamma_G/S_1 \times S_2)]/([S \sqcup S'] - [S] - [S']).$$

That is, $B^G(S_1, S_2)$ is the free abelian group spanned by isomorphism classes of diagrams

$$(2.2) \quad S_1 \longleftarrow S \longrightarrow S_2$$

in $\Gamma_G$, modulo the relations $[S \sqcup S'] = [S] + [S']$, where $\sqcup$ stands for disjoint union. Composition of diagrams is defined by taking fibered products. One checks easily that $B^G$ is an additive category, with direct sum given by disjoint union of $G$-sets.

Definition 2.1. A $G$-Mackey functor with values in some ring $R$ is an additive functor from $B^G$ to the category of $R$-modules. The category of $R$-valued $G$-Mackey functors is denoted by $\mathcal{M}(G, R)$. 

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Example 2.2. For any \( S \in \Gamma_G \), let \( A(S) = B^G(\text{pt}, S) \), where \( \text{pt} \) is the one-point set with the trivial \( G \)-action. Then \( A \) is a \( \mathbb{Z} \)-valued Mackey functor known as *Burnside Mackey functor*. Explicitly, we have

\[
A(S) = \mathbb{Z}[\text{Iso}(\Gamma_G/S)]/([S \sqcup S'] - [S] - [S']).
\]

We further note that if \( S = \text{pt} \), then \( A(\text{pt}) = B^G(\text{pt}, \text{pt}) \) is a ring, known as the *Burnside ring* of the group \( G \) and denoted \( A^G \). As a group, \( A^G \) is freely generated by the set of isomorphisms classes of \( G \)-orbits, that is, \( G \)-sets of the form \( G/H, H \subset G \) a cofinite subgroup. The product is induced by the cartesian product of orbits. In particular, it is obviously commutative.

Moreover, for a \( G \)-orbit \( S = G/H \), we have a natural equivalence

\[
\Gamma_G/S \cong \Gamma_H,
\]

so that \( A(G/H) = A^H \), the Burnside ring of the group \( H \). For any ring \( R \), the Burnside ring \( A^G \) maps naturally to the center of the category \( \mathcal{M}(G, R) \) – that is, for any \( M \in \mathcal{M}(G, R) \) and any \( a \in A^G \), we have a natural map \( a : M \to M \). Explicitly, if \( a = [S_0] \in A^G \) is the class of a \( G \)-set \( S_0 \in \Gamma_G \), then for any \( S \in \Gamma_G \), the map \( a(S) : M(S) \to M(S) \) corresponds to the diagram

\[
\begin{array}{cccc}
S & \leftarrow & S \times S_0 & \rightarrow & S, \\
& f & & f & \\
& & & & \\
\end{array}
\]

where \( f : S \times S_0 \to S \) is the natural projection.

A useful alternative description of Mackey functors is due to Lindner [L]. Let \( Q\Gamma_G \) be the category with the same objects as \( \Gamma_G \), and with morphisms given by isomorphism classes of diagrams (2.2). Then by definition, both \( \Gamma_G \) and the opposite category \( \Gamma_G^o \) are subcategories in \( Q\Gamma_G \): objects are the same, and a diagram (2.2) defines a map in \( \Gamma_G \subset Q\Gamma_G \) resp. \( \Gamma_G^o \subset Q\Gamma_G \) if and only if the map \( S \to S_1 \) resp. \( S \to S_2 \) is an isomorphism.

**Definition 2.3.** A functor \( M \) from \( \Gamma_G^o \) to an additive category is additive if the natural map

\[
M(S_1 \sqcup S_2) \to M(S_1) \oplus M(S_2)
\]

is an isomorphism for any \( S_1, S_2 \in \Gamma_G \). A functor \( M \) from \( Q\Gamma_G \) is additive if so is its restriction to \( \Gamma_G^o \subset Q\Gamma_G \).

Then for any ring \( R \), the full subcategory

\[
\text{Fun}_{\text{add}}(Q\Gamma_G, R) \subset \text{Fun}(Q\Gamma_G, R)
\]
spanned by additive functors is naturally identified with \( \mathcal{M}(G, R) \), and the embedding has a left-adjoint additivization functor

\[
\text{Add} : \text{Fun}(Q\Gamma_G, R) \to \text{Fun}_{\text{add}}(Q\Gamma_G, R) \cong \mathcal{M}(G, R).
\]

The maps (2.5) for all \( S_1, S_2 \in \Gamma_G \) can be bundled together into a single map of functors

\[
s^* : p_1^* \oplus p_2^*,
\]

where we denote by \( p_1, p_2 : Q\Gamma_G \times Q\Gamma_G \to Q\Gamma_G \) the natural projections, and we denote by \( s : Q\Gamma_G \times Q\Gamma_G \to Q\Gamma_G \) the disjoint union functor. Then \( M \in \text{Fun}(Q\Gamma_G, R) \) is additive if and only if the map \( s^* M \to p_1^* M \oplus p_2^* M \) is an isomorphism. We also note that \( s \) is right and left-adjoint to the diagonal embedding \( \delta : Q\Gamma_G \to Q\Gamma_G \times Q\Gamma_G \), and the projections \( p_1, p_2 \) are right and left-adjoint to the embeddings \( i_1, i_2 : Q\Gamma_G \to Q\Gamma_G \times Q\Gamma_G \) sending \( S \) to \( S \times \emptyset \) resp. \( \emptyset \times S \). Thus \( s^* \cong \delta^*, p_1^* \cong i_1^*, p_2^* \cong i_2^* \), and saying that (2.5) is an isomorphism is equivalent to saying that the natural map

\[
\delta_! M \to i_1! M \oplus i_2! M
\]

is an isomorphism.

It is clear from Lindner’s description that an \( R \)-valued Mackey functor \( M \in \mathcal{M}(G, R) \) is completely defined by the following data:

(i) an \( R \)-module \( M(S) \) for any finite \( G \)-set \( S \in \Gamma_G \), and

(ii) two maps

\[
(2.8) \quad f_s : M(S) \to M(S'), \quad f^* : M(S') \to M(S)
\]

for any map \( f : S \to S' \) in \( \Gamma_G \).

By additivity, it suffices to specify \( M(S) \) and \( f_s, f^* \) for \( G \)-orbits \( S = G/H \). Traditionally, \( M(G/H) \) is denoted by \( M^H \). The maps \( f_s \) and \( f^* \) should satisfy some compatibility conditions encoded in the structure of the category \( Q\Gamma_G \). Explicitly, for any two maps \( f' : G/H' \to G/H, f'' : G/H'' \to G/H \) induced by embeddings \( H', H'' \subset H \), the fibered product \( (G/H') \times_{G/H} (G/H'') \) decomposes into a disjoint union

\[
(2.9) \quad (G/H') \times_{G/H} (G/H'') = \bigsqcup_{s \in S} G/H_s
\]

of \( G \)-orbits indexed by the finite set \( S = H' \setminus H/H'' \), and we must have

\[
(2.10) \quad f'^* \circ f''_* = \sum_{s \in S} \tilde{f}'_* \circ \tilde{f}''_* ,
\]

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where $f'_s : G/H_s \to G/H''$, $f''_s : G/H_s \to G/H'$ are projections of the component $G/H_s$ of the decomposition (2.9). This is known as the double coset formula.

2.2 Fixed points. Assume now given a subgroup $H \subset G$, let $N_H \subset G$ be its normalizer, and let $W = N_H/H$. Then for any finite $G$-set $S \in \Gamma_G$, the subset $S^H \subset S$ of $H$-fixed points is naturally a finite $W$-set. Sending $S \in \Gamma_G$ to $S^H$ gives a functor

$$
(2.11) \quad \varphi^H : \Gamma_G \to \Gamma_W.
$$

This functor is left-exact, thus extends to a functor $Q(\varphi^H) : Q\Gamma_G \to Q\Gamma_W$. Since $Q(\varphi^H)$ obviously commutes with the functors $\delta$, $i_1$, $i_2$ of (2.7), the corresponding left Kan extension functor

$$
Q(\varphi^H) ! : \text{Fun}(Q\Gamma_G, R) \to \text{Fun}(Q\Gamma_W, R)
$$

preserves additivity in the sense of Definition 2.3. By definition, the geometric fixed points functor

$$
\Phi^H : \mathcal{M}(G, R) = \text{Fun}_{add}(Q\Gamma_G, R) \to \mathcal{M}(W, R) = \text{Fun}_{add}(Q\Gamma_W, R)
$$

is induced by the functor $Q(\varphi^H) !$.

On the other hand, one can simply restrict a $G$-action on a set $S$ to an action of the subgroup $H \subset G$; this gives a left-exact functor $\rho^H : \Gamma_G \to \Gamma_H$ and its extension $Q(\rho^H) : Q\Gamma_G \to \Gamma_H$. The corresponding left Kan extension functor $Q(\rho^H) !$ also preserves additivity and induces the categorical fixed points functor

$$
\Psi^H = Q(\rho^H) ! : \mathcal{M}(G, R) \to \mathcal{M}(H, R).
$$

We note that for two subgroups $H' \subset H \subset G$, we have an obvious isomorphism

$$
(2.12) \quad \varphi^{H'} \circ \rho^H \cong \rho^{W_H} \circ \varphi^{H'},
$$

where we denote $W' = N_{H'}/H'$ and $W_H = (N_{H'} \cap H)/H' \subset W'$, and it induces an isomorphism

$$
(2.13) \quad \Phi^{H'} \circ \Psi^H \cong \Psi^{W_H} \circ \Phi^{H'}.
$$

We also note that for any subgroup $H \subset G$, the centralizer $Z_H \subset G$ of the group $H$ acts on the functor $\rho^H$, thus on $\Psi^H$, so that it can be promoted to a functor

$$
(2.14) \quad \tilde{\Psi}^H : \mathcal{M}(G, R) \to \mathcal{M}(H, R[Z_H]),
$$

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where $R[Z_H]$ is the group algebra of the centralizer. If $H \subset G$ is cofinite, then the functor $\rho^H$ has a left-adjoint functor

$$\gamma_H : \Gamma_H \to \Gamma_G \tag{2.15}$$

given by the identification \[^{2.3}\] composed with the forgetful functor (explicitly, we have $\gamma_H(S) = (G \times S)/H$). The functor $\gamma_H$ is also left-exact, and the corresponding functor $Q(\gamma_H) : Q\Gamma_H \to Q\Gamma_G$ is adjoint to $Q(\rho^H)$ both on the left and on the right. Thus we have

$$\Psi^H = Q(\rho^H)! \cong Q(\gamma_H)^*,$$

and this allows to compute $\Psi^H$ rather explicitly (in particular, it is an exact functor).

Geometric fixed points are usually much more difficult to compute explicitly, and we will only give one result in this direction. Denote by

$$\text{Infl}^H = Q(\phi^H)^* : \mathcal{M}(W, R) \to \mathcal{M}(G, R) \tag{2.16}$$

the inflation functor right-adjoint to $\Phi^H$. Assume that the subgroup $H \subset G$ is normal, so that $W = G/N$, and Let

$$\lambda_N : \Gamma_{G/N} \to \Gamma_G \tag{2.17}$$

be the fully faithful embedding sending a $G/N$-set $S$ to the same set on which $G$ acts via the map $G \to G/N$. For every $G$-orbit $S$, we have one of the two alternatives: either $S^N$ is empty, or $S^N = S$, and $S$ lies in the image of the embedding $\lambda_N$. Let

$$Q(\lambda_N) : Q\Gamma_{G/H} \to Q\Gamma_G$$

be the functor induced by the embedding $\lambda_N$, and let

$$\mathcal{M}_N(G, R) \subset \mathcal{M}(G, R)$$

be the full subcategory spanned by $M \in \mathcal{M}(G, R)$ such that $M(S) = 0$ for any $S \in \Gamma_G$ with empty $S^N$.

**Lemma 2.4.** The inflation functor $\text{Infl}^N$ is fully faithful and identifies the category $\mathcal{M}(W, R)$ with the full subcategory $\mathcal{M}_N(G, R) \subset \mathcal{M}(G, R)$, with inverse equivalence given by $Q(\lambda_N)^*$. For any $M \in \mathcal{M}(G, R)$, the adjunction map

$$M \to \text{Infl}^N \Phi^N M \tag{2.18}$$

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is surjective, and for any $S \in \Gamma_{G/N}$, we have a short exact sequence

$$\bigoplus_{f:S' \to S} M(S') \xrightarrow{\sum f_*} M(S) \xrightarrow{\Phi^N(M)(S)} 0,$$

where $f_*$ is as in (2.8), and the sum is over all maps $f : S' \to S$ in $\Gamma_G$ such that $S'$ has no elements fixed under $N \subset G$.

**Remark 2.5.** Note that by additivity of $M \in \text{Fun}(Q\Gamma_G, R)$, the image of the map $\sum f_*$ in (2.19) is not only the sum of the images of individual maps $f_*$, it is actually the union of these images.

**Proof.** The functor $\lambda_N : \Gamma_{G/N} \to \Gamma_G$ is a fully faithful embedding, and $\rho_N$ is by definition right-adjoint to this embedding. Therefore for any $M \in \mathcal{M}(W, R)$, we have an adjunction map

$$\mathcal{M} \to \lambda_N^* \varphi^N \mathcal{M},$$

where $\mathcal{M} \in \text{Fun}(\Gamma_W, R)$ is the restriction of the Mackey functor $M$ to $\Gamma_W \subset Q\Gamma_W$, and analogously, for any $M \in \mathcal{M}(G, R)$, we have an adjunction map

$$\varphi^N \lambda_N^* \mathcal{M} \to \mathcal{M},$$

where $\mathcal{M}$ is obtained by restriction to $\Gamma_G \subset Q\Gamma_G$. Moreover, the map (2.20) is an isomorphism, and it is compatible with the maps $f_*$ of (2.8), so that it gives an isomorphism

$$M \cong Q(\lambda_N)^* Q(\varphi^N)^* M = Q(\lambda_N)^* \text{Infl}^N M.$$ 

In general, the map (2.21) does not commute with the maps $f_*$, but it clearly does so if $M$ lies in $\mathcal{M}_N(G, R) \subset \mathcal{M}(G, R)$, and in this case, it is an isomorphism. Since by definition, $\text{Infl}^N = Q(\varphi^N)^*$ sends $\mathcal{M}(W, R)$ into $\mathcal{M}_N(G, R)$, we conclude that $\text{Infl}^N$ and $Q\lambda_N^*$ are indeed inverse equivalences between $\mathcal{M}(W, R)$ and $\mathcal{M}_N(G, R)$. Now, the cokernel of the map $\sum f_*$ in (2.19) is clearly functorial in $S$ and trivial for $S$ with empty $S^N$, thus defines a functor $\Phi : \mathcal{M}(G, R) \to \mathcal{M}_N(G, R)$ equipped with a map $\text{Id} \to \Phi$. This map becomes an isomorphism after restricting to $\mathcal{M}_N(G, R)$, thus gives an adjunction between $\Phi$ and the embedding $\mathcal{M}_N(G, R) \subset \mathcal{M}(G, R)$. This identifies $\Phi$ and $\Phi^N$. \[\square\]
2.3 Products. Assume now given two rings $R, R'$. Then we can define a natural tensor product functor

\[(2.22) \quad \mathcal{M}(G, R) \times \mathcal{M}(G, R') \to \mathcal{M}(G, R \otimes R').\]

To do it, use Lindner’s description. Cartesian product of $G$-sets commutes with fibered products, thus extends to a functor

\[(2.23) \quad m : Q\Gamma_G \times Q\Gamma_G \to Q\Gamma_G.\]

The product $M \circ M'$ of two Mackey functors $M \in \mathcal{M}(G, R)$, $M' \in \mathcal{M}(G, R')$ is then defined by

\[(2.24) \quad M \circ M' = \text{Add}(m(M \boxtimes M')).\]

This is clearly associative and commutative in the obvious sense. Moreover, for any subgroup $H \subset G$, the functors $Q\rho^H : Q\Gamma_G \to Q\Gamma_H$, $Q\varphi^H : Q\Gamma_G \to Q\Gamma_W$ commute with $m$; therefore the fixed points functors $\Psi^H$, $\Phi^H$ are tensor functors.

For any three Mackey functors $M_1 \in \mathcal{M}(G, R_1)$, $M_2 \in \mathcal{M}(G, R_2)$, $M \in \mathcal{M}(G, R_1 \otimes R_2)$, a map $M_1 \circ M_2 \to M$ by adjunction corresponds to a map

\[(2.25) \quad \tilde{\mu} : M_1 \boxtimes M_2 \to m^*M.\]

To see such maps more explicitly, one can first restrict to the subcategory $\Gamma^o_G \subset Q\Gamma_G$. Then the functor $m$ is left-adjoint to the diagonal embedding $\delta : \Gamma^o_G \to \Gamma^o_G \times \Gamma^o_G$, and again by adjunction, \((2.25)\) induces a map

\[\mu : M_1 \otimes M_2 = \delta^*(M_1 \boxtimes M_2) \to M\]

in $\text{Fun}(\Gamma^o_G, R_1 \otimes R_2)$. In other words, we have a map

\[(2.26) \quad \mu : M_1^H \otimes M_2^H \to M^H\]

for any cofinite subgroup $H \subset G$, and for any two cofinite subgroups $H_1, H_2 \subset G$ and a $G$-map $f : G/H_1 \to G/H_2$, we have

\[(2.27) \quad \mu(f^*(a_1) \otimes f^*(a_2)) = f^*\mu(a_1 \otimes a_2), \quad a_1 \in M_1^{H_2}, a_2 \in M_2^{H_2}.\]

Lemma 2.6. A collection of maps \((2.26)\) satisfying \((2.27)\) define a map \((2.25)\) if and only if we also have

\[(2.28) \quad f_*(\mu(a_1 \otimes f^*(a_2))) = \mu(f_*(a_1) \otimes a_2), \quad a_1 \in M_1^{H_1'}, a_2 \in M_2^{H_2'},
\]

\[f_*(\mu(f^*(a_1) \otimes a_2)) = \mu(a_1 \otimes f_*(a_2)), \quad a_1 \in M_1^{H_2'}, a_2 \in M_2^{H_1'},\]

for any two cofinite subgroups $H_1, H_2 \subset G$ and a map $f : G/H_1 \to G/H_2$. 

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Proof. By adjunction, the maps (2.26) define a map (2.25) in the category $\text{Fun}(\Gamma_G \times \Gamma_G^I, R)$; explicitly, it is given by

\[(2.29)\] $$\tilde{\mu}(b_1 \otimes b_2) = \mu(p_1^*(b_1) \otimes p_2^*(b_2))$$

for any cofinite $H_1, H_2 \subset G$ and any elements $b_1 \in M_1(G/H_1), b_2 \in M_2(G/H_2)$, where $p_1 : (G/H_1) \times (G/H_2) \to G/H_1, p_2 : (G/H_1) \times (G/H_2) \to G/H_2$ are the natural projections. This map $\tilde{\mu}$ is a map in $\text{Fun}(Q\Gamma_G \times Q\Gamma_G, R)$ if and only if for any cofinite $H'_1 \subset G$, a map $g : G/H'_1 \to G/H_1$, and an element $b_1 \in M_1(G/H'_1), b_2 \in M_2(G/H_2)$, we have

\[(2.30)\] $$\tilde{\mu}(g_* b_1 \otimes b_2) = (g \times \text{id})_* (\tilde{\mu}(b_1 \otimes b_2)),$$

and similarly for the product in the opposite order. Combining (2.29) and (2.30) gives

\[(2.31)\] $$\mu(p_1^* g_* (b_1) \otimes p_2^*(b_2)) = (g \times \text{id})_* \mu(p_1^*(b_1) \otimes p_2^*(b_2)),$$

where $p_2' : (G/H'_1) \times (G/H_2) \to G/H_2$ is the natural projection. Since $p_1^* \circ g_* = (g \times \text{id})_* \circ p_1^*$ and $p_2' = p_2 \circ (g \times \text{id})$, (2.31) is exactly the first equation in (2.28) with $f = g \times \text{id}, a_1 = p_1^* b_1, a_2 = p_2^* b_2$. Thus (2.29) implies that $\tilde{\mu}$ is a map in $\text{Fun}(Q\Gamma_G \times Q\Gamma_G, R)$. Conversely, $\mu$ is expressed in terms of $\tilde{\mu}$ by

$$\mu(a_1 \otimes a_2) = \delta^* \tilde{\mu}(a_1 \otimes a_2), \quad a_1, a_2 \in M^H,$$

where $\delta : G/H \to (G/H \times G/H)$ is the diagonal embedding, and since $\delta^* \circ (g \times \text{id})_* = g_* \circ \delta^*$, the equation (2.30) together with the opposite equation imply (2.25).

In particular, if our base ring $R$ is commutative, we have a natural product map $R \otimes R \to R$, so that the product (2.22) induces a symmetric tensor product on the category $\mathcal{M}(G, R)$. A Green functor is a ring object in the category $\mathcal{M}(G, R)$. Explicitly, a Green functor structure on a Mackey functor $M$ is given by a ring structure on every $M^H, H \subset G$ cofinite; the maps $f^*$ are ring maps, and the maps $f_*$ satisfy the projection formula (2.28). For example, the Burnside Mackey functor $\mathcal{A}$ of Example 2.2 is a Green functor (in fact, $\mathcal{A}$ is the unit object for the product on $\mathcal{M}(G, \mathbb{Z})$, and more generally, the unit object for the product (2.22)).
3 Mackey profunctors.

3.1 Definitions. In the classical theory of Mackey functors, one always assumes that the group $G$ is finite (or compact, in the topological versions of the theory). We do not do so since there is no formal need: it is only the $G$-sets that have to be finite (otherwise $B^G(\cdot,\cdot)$ of (2.1) would be identically 0). However, for an infinite group $G$, there is the following useful alternative version of the category $\mathcal{M}(G,R)$.

**Definition 3.1.** For any group $G$, a $G$-set $S$ is admissible if

(i) for any element $s \in S$, its stabilizer $G_s \subset G$ is cofinite, and

(ii) for any cofinite subgroup $H \subset S$, its fixed point set $S^H \subset S$ is finite.

Denote the category of admissible $G$-sets by $\hat{\Gamma}_G$, and note that it is a small category (with the number of isomorphism classes of objects depending on the cardinality of $G$). A subset of an admissible $G$-set and the product of two admissible $G$-set is admissible, so that we can form the quotient category $Q\hat{\Gamma}_G$ and consider the functor category $\text{Fun}(Q\hat{\Gamma}_G, R)$ for any ring $R$. As in the case of finite sets, we have natural embeddings $\hat{\Gamma}_G \subset Q\hat{\Gamma}_G$, $\hat{\Gamma}_G^\circ \subset Q\hat{\Gamma}_G$.

**Definition 3.2.** A functor $M$ from $\hat{\Gamma}_G^\circ$ to an additive category with arbitrary products is additive if for any admissible $G$-set $S \in \hat{\Gamma}_G$, with the quotient map $q : S \rightarrow S/G$, the natural map

$$M(S) \rightarrow \prod_{s \in S/G} M(q^{-1}(s))$$

induced by the decomposition $S = \coprod_{s \in S/G} q^{-1}(s)$ is an isomorphism. An $R$-valued $G$-Mackey profunctor $M$ is a functor from $Q\hat{\Gamma}_G$ to the category of $R$-modules whose restriction to $\hat{\Gamma}_G^\circ$ is additive.

We will denote the category of $R$-valued $G$-Mackey profunctors by

$$\hat{\mathcal{M}}(G, R) \subset \text{Fun}(Q\hat{\Gamma}_G, R).$$

We note right away that since the abelian category of $R$-modules satisfies $AB4^*$, additivity is preserved under kernels and cokernels; thus $\mathcal{M}(G, R)$ is also abelian and satisfies $AB4^*$. 

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By definition, a finite $G$-set is admissible, and additivity in the sense of Definition 3.2 obviously implies additivity in the sense of Definition 2.3, so that we have an embedding $\Gamma_G \subset \hat{\Gamma}_G$ and a restriction functor

$$\hat{\mathcal{M}}(G, R) \to \mathcal{M}(G, R).$$

If $G$ is finite, the restriction functor is an equivalence. For some infinite groups – for example, for the group $G = \mathbb{Z}_p$ of $p$-adic integers – the restriction functor, while not an equivalence, is at least fully faithful. In general, it is not even that.

**Example 3.3.** Let $\hat{\mathcal{B}}^G$ be an additive category whose objects are admissible $G$-sets, and whose Hom-sets are given by (2.1) with $\Gamma_G$ replaced by $\hat{\Gamma}_G$. Then for any $S \in \hat{\Gamma}_G$, the representable functor $\mathcal{B}^G(S, -)$ is a functor from $Q\hat{\Gamma}_G$ to abelian groups, and it is obviously additive in the sense of Definition 3.2. Thus it gives a $\mathbb{Z}$-valued $G$-Mackey profunctor, and we obtain an embedding

$$\hat{\mathcal{B}}^G \subset \hat{\mathcal{M}}(G, \mathbb{Z}).$$

This embedding is fully faithful.

In particular, Example 2.2 generalizes to Mackey profunctors, so that we have the completed Burnside ring $\hat{\mathcal{A}}^G = \hat{\mathcal{B}}^G(pt, pt)$ and the corresponding $G$-Mackey profunctor $\hat{\mathcal{A}} \in \hat{\mathcal{M}}(G, \mathbb{Z})$. As in the case of usual Mackey functor, $\hat{\mathcal{A}}^G$ maps into the center of the category $\hat{\mathcal{M}}(G, R)$ for any ring $R$. In fact, the full subcategory in $\hat{\mathcal{B}}^G$ spanned by finite $G$-sets can be alternatively described as follows. For any cofinite subgroup $H \subset G$, classes of $G$-sets $S \in \Gamma_G$ with empty $S^H$ form a two-side ideal $\mathcal{B}_H \subset \mathcal{B}_G$. Then for any $S_1, S_2 \in \Gamma_G$, we have

$$\hat{\mathcal{B}}^G(S_1, S_2) = \lim_{\mathcal{B}} \mathcal{B}^G(S_1, S_2)/\mathcal{B}_H^G(S_1, S_2),$$

where the limit is taken over all cofinite subgroups $H \subset G$. Explicitly, we have

$$\hat{\mathcal{B}}^G(S_1, S_2) = \prod_{H \subset G} \mathbb{Z}[(S_1 \times S_2)^H/W_H],$$

where the product is taken over all conjugacy classes of cofinite subgroups $H \subset G$, and for any such $H$, $W_H = N_H/H$ is the quotient of the normalizer.
$N_H \subset G$ by the subgroup $H \subset N_H \subset$. In particular, the completed Burnside ring is indeed a completion of the usual Burnside ring – we have

\[(3.4) \quad \hat{A}^G = \prod_{H \subset G} \mathbb{Z},\]

where the product is over all conjugacy classes of cofinite subgroups $H \subset G$. For the usual Burnside ring $\hat{A}^G$, the formula is the same, but the product is replaced with the sum.

### 3.2 Normal systems.

For any cofinite subgroup $H \subset G$ and any admissible $G$-set $S \in \hat{\Gamma}_G$, the fixed point set $S^H$ is by definition finite, so that as in the usual Mackey functors case, we have a natural left-exact functor $\varphi^H : \hat{\Gamma}_G \to \Gamma_W$ and the corresponding adjoint pair of functors

\[(3.5) \quad \Phi^H = Q\varphi^H : \text{Fun}(\hat{\Gamma}_G, R) \to \text{Fun}(\Gamma_W, R),
\]

\[\text{Infl}^H = Q\varphi^{H*} : \text{Fun}(\Gamma_W, R) \to \text{Fun}(\hat{\Gamma}_G, R).\]

For the same reasons as in the usual case, these functors preserve additivity, thus induce functors between categories of $G$-Mackey profunctors. For any $M \in \hat{\mathcal{M}}(G, R)$ and any pair $N \subset N' \subset G$ of cofinite normal subgroup, we have a natural isomorphism

\[(3.6) \quad \Phi^{N'} M \cong \Phi^{N'/N} \Phi^N M.\]

Thus every Mackey profunctor $M \in \hat{\mathcal{M}}(G, R)$ induces a collection of Mackey functors $\Phi^N M$ related by the isomorphisms (3.6).

To axiomatize the situation, denote by $\mathcal{N}(G)$ the partially ordered set of cofinite normal subgroups $N \subset G$, ordered by inclusion, treat it as a small category in the usual way, and let

$$\overline{\Gamma}_G \subset \Gamma_G \times \mathcal{N}(G)^\circ$$

be the full subcategory spanned by pairs $(S, N)$ such that $N$ acts trivially on $S$. Thus explicitly, morphisms from $(S, N)$ to $(S', N')$ are pairs of a $G$-equivariant map $f : S \to S'$ and an inclusion $N' \subset N$. We have a natural forgetful functor

\[(3.7) \quad \nu : \overline{\Gamma}_G \to \mathcal{N}(G)^\circ, \quad (S, N) \mapsto N.\]

This functor is a fibration, with fiber over $N \in \mathcal{N}(G)$ equivalent to $\Gamma_W$, $W = G/N$, and with transition functors $\varphi^{N'/N}$, $N \subset N' \subset G$. 

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Now let $Q\Gamma G$ be the category with the same objects as $\Gamma G$, with morphisms from $⟨S, N⟩$ to $⟨S', N'⟩$ given by isomorphism classes of diagrams

$$\langle S, N⟩ \leftarrow \langle \tilde{S}, N' \rangle \rightarrow \langle S', N' \rangle$$

in $\Gamma G$, and with compositions given by pullbacks (we note that the necessary pullbacks in $\Gamma G$ do exist). Then the fibration (3.7) defines a cofibration

$$Q\Gamma G \rightarrow N(G), \quad ⟨S, N⟩ \mapsto N,$$

with fibers $Q\Gamma W$ and transition functors $Q(\varphi^{N'/N})$. Explicitly, an object $M \in \text{Fun}(Q\Gamma G, R)$ is given by a collection of objects $M_N \in \text{Fun}(Q\Gamma W, R)$, $N \in N(G)$, $W = G/N$, and transition morphisms

$$\Phi^{N'/N} M_N \rightarrow M_{N'}$$

for any $N, N' \in N(G)$, $N \subset N'$.

**Definition 3.4.** For any group $G$, a normal system of $G$-Mackey functors is an object $M \in \text{Fun}(Q\Gamma G, R)$ such that for any $N \in N(G)$, $W = G/N$, the object $M_N \in \text{Fun}(\Gamma W, R)$ is additive in the sense of Definition 2.3, and for any $N, N' \in N(G)$, $N \subset N'$, the transition morphism (3.8) is an isomorphism.

Normal systems obviously form an additive category with arbitrary sums; we denote it by $\mathcal{N}(G, R)$. Moreover, we have a natural functor

$$\varphi : N(G) \times \hat{\Gamma} G \rightarrow \Gamma G$$

given by $\varphi(N \times S) = ⟨S^N, N⟩$, and it defines a functor $Q(\varphi) : N(G) \times Q\hat{\Gamma} G \rightarrow Q\Gamma G$. If we denote by $p : N(G) \times Q\hat{\Gamma} G \rightarrow Q\Gamma G$ the natural projection, then for any Mackey profunctor $M \in \hat{\mathcal{M}}(G, R)$, $\varphi p^* M \in \text{Fun}(Q\Gamma G, R)$ is a normal system in the sense of Definition 3.4 so that we have a natural functor

$$\Phi = Q(\varphi)! \circ p^* : \hat{\mathcal{M}}(G, R) \rightarrow \mathcal{N}(G, R).$$

with a right-adjoint functor

$$\text{Infl} = p_* \circ Q(\varphi)^* : \mathcal{N}(G, R) \rightarrow \hat{\mathcal{M}}(G, R).$$

Explicitly, for any $M \in \hat{\mathcal{M}}(G, R)$, we have $\Phi (M)_N \cong \Phi^N M$, and the transition maps (3.8) are the isomorphisms (3.6). Conversely, for any normal system $M = \{M_N\} \in \mathcal{N}(G, R)$, we have

$$\text{Infl} M = \lim_{\xi} \text{Infl}^N M_N,$$

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where the limit is over $N \in \mathbb{N}(G)$ and with respect to the transition maps adjoint to (3.8). We note that by Lemma 2.4 all the transition maps in the inverse system $\text{Infl}^N M_N$ are surjective. For any $M \in \hat{\mathcal{M}}(G, R)$, the adjunction map $M \to \text{Infl}(\Phi(M))$ is the natural map

$$M \to \text{Infl}(\Phi(M)) \cong \lim_{\mathcal{K}} \text{Infl}^N \Phi^N M,$$

induced by the adjunction maps (2.18).

**Proposition 3.5.**

(i) For any normal cofinite subgroup $N \subset G$, the inflation functor $\text{Infl}^N$ is fully faithful, and the inflation functor $\text{Infl}$ of (3.10) is also fully faithful.

(ii) For any $R$-valued $G$-Mackey profunctor $M \in \hat{\mathcal{M}}(G, R)$, the natural map (3.11) is surjective.

(iii) If $\Phi(M) = 0$ for some $M \in \hat{\mathcal{M}}(G, R)$, then $M = 0$.

**Proof.** For any normal cofinite $N \subset G$ with the quotient $W = G/N$, let $\hat{\mathcal{M}}_N(G, R) \subset \hat{\mathcal{M}}(G, R)$ be the full subcategory spanned by Mackey profunctors $M$ such that $M(S) = 0$ whenever $S^N = \emptyset$. Then as in the proof of Lemma 2.4, $\hat{\mathcal{M}}_N(G, R)$ is equivalent to $\mathcal{M}(W, R)$, with an equivalence given by restriction to $\Gamma_W \subset \hat{\Gamma}_G$, and the inverse equivalence given by $\text{Infl}^N$. In particular, $\text{Infl}^N$ is fully faithful. Moreover, again as in the proof of Lemma 2.4, for any $M \in \hat{\mathcal{M}}(G, R)$ we have exact sequences (2.19), where $S$ and $S'$ are admissible $G$-sets.

To prove that $\text{Infl}$ is fully faithful, we have to prove that the adjunction map $\Phi \circ \text{Infl} \to \text{Id}$ is an isomorphism. In other words, for any normal system $M = \{M_N\} \in \mathcal{N}(G, R)$ with $\hat{M} = \text{Infl}(M)$, and any normal cofinite subgroup $\mathcal{N} \subset G$ with quotient $W = G/N$, we have to show that the map $\Phi^\mathcal{N}(\hat{M}) \to M_{\mathcal{N}}$ is an isomorphism. By (2.19), this is equivalent to proving that for any $S \in \Gamma_W \subset \hat{\Gamma}_G$, the kernel of the natural surjective map

$$\hat{M}(S) \to M_{\mathcal{N}}(S)$$

is the union of the images of the maps $f_s$, with $f$ as in (2.19). Indeed, by definition, we have

$$\hat{M}(S) = \lim_{\mathcal{N}} M_N(S),$$

where the limit is taken over all cofinite normal subgroups $N \subset G$, and by (2.19), for any $N' \subset N \subset G$, the kernel of the map $M_{N'}(S) \to M_N(S) \cong$
\(\Phi^{N/N'}(M_{N'})\) consists of elements of the form \(f^N_s(m_N)\), \(m_N \in M_{N'}(S_N)\), \(f : S_N \to S\), where \(S_N \in \hat{\Gamma}_G\) is an admissible \(G\)-set with no elements fixed under \(N\). Then by induction on normal cofinite subgroups \(N \subset \overline{N}\), any element \(\tilde{m}\) in the kernel of the map (3.12) can be represented as a series

\[(3.13) \quad \tilde{m} = \sum_{N \in N} f^N_s(\tilde{m}_N)\]

for some \(\tilde{m}_N \in \hat{M}(S_N)\), \(f^N : S_N \to S\), \(S_N^N = \emptyset\). But then the union

\[(3.14) \quad S' = \prod_N S_N\]

is an admissible \(G\)-set, with a natural map \(f : S' \to S\) and no elements fixed under \(\overline{N}\), and \(\tilde{m}\) is the image of the element

\[\left(\prod_N \tilde{m}_N\right) \in \hat{M}(S')\]

under the map \(f_s\).

The argument for (ii) is exactly the same: for any admissible \(G\)-set \(S\), every element \(\tilde{m}\) in the target of the map (3.11) evaluated at \(S\) can be represented as a series (3.13), with \(\overline{N} = G\), and if we take \(S'\) as in (3.14), with the map \(f : S' \to S\), then \(\tilde{m}\) is the image of the element

\[f_s\left(\prod_N \tilde{m}_N\right) \in M(S')\]

under the canonical map (3.11).

Finally, for (iii), assume given \(M \in \hat{M}(G,R)\) with trivial \(\Phi(M)\). Then by (2.19), for any \(S \in \hat{\Gamma}_G\), any element \(m \in M(S)\), and any normal cofinite subgroup \(N \subset G\), there exists a map \(f : S_N \to S\) in \(\hat{\Gamma}_G\) and an element \(m_N \in M(S_N)\) such that \(m = f_s(m_N)\), while \(S_N^N\) is empty. By induction, we can choose a decreasing sequence of cofinite normal subgroups \(N_i \subset G\), \(i \geq 1\), admissible \(G\)-sets \(S_i \in \hat{\Gamma}_G\) and elements \(m_i \in M(S_i)\) such that \(N_{i+1} \subset N_i\), the intersection of all the subgroups \(N_i\) is empty, \(S_i^N\) is empty, and \(f_i^i(m_i) = m_{i-1}\) for some maps \(f^i : S_i \to S_{i-1}\), \(i \geq 1\), where we let \(S_0 = S\) and \(m_0 = m\). Note that since \(\cap N_i = \emptyset\), \(S_i^N\) is empty for any fixed cofinite normal subgroup \(N \subset G\) and almost all \(i\). Therefore the union

\[S' = \prod_{i \geq 1} S_i\]
is an admissible $G$-set, and so is $S'_0 = S' \sqcup S$. We have the natural projection $p : S'_0 \to S$ and two maps $i, f : S' \to S'_0$ – the natural embedding $i$, and the disjoint union $f$ of all the maps $f^i$. Moreover, we have $p \circ f = p \circ i$. It remains to notice that if we let

$$\tilde{m} = \prod_{i \geq 0} m_i \in \prod_{i \geq 0} M(S_i) = M(S'_0),$$

then we have $f_*(\tilde{m}) = \tilde{m}$ and

$$m = p_*(\tilde{m}) - p_*(i_*(\tilde{m})) = p_*(f_*(\tilde{m})) - (p \circ i)_*(\tilde{m}) =$$

$$(p \circ f)_*(\tilde{m}) - (p \circ i)_*(\tilde{m}) = 0.$$

Since $S \in \widehat{\Gamma}_G$ was arbitrary, this means that $M = 0$. \qed

### 3.3 Separated profunctors.

We note that Proposition 3.5 (iii) does not imply that the canonical map (3.11) is always an isomorphism – a priori, the functor $\Phi$ is only right-exact, so that the kernel of the map (3.11) can have a non-trivial image under this functor. We do not know whether this can happen, and for which groups. Therefore we introduce the following.

**Definition 3.6.** An $R$-valued $G$-Mackey profunctor $M$ is **separated** if the natural surjective map (3.11) is an isomorphism.

We denote by $\widehat{M}_s(G, R) \subset \widehat{M}(G, R)$ the full subcategory spanned by separated $G$-Mackey profunctors. By Proposition 3.5 (i), the inflation functor $\text{Infl}$ induces an equivalence between $\widehat{M}_s(G, R)$ and the category $\mathcal{N}(G, R)$ of normal systems in the sense of Definition 3.4. The subcategory $\widehat{M}_s(G, R)$ is closed under subobjects (indeed, in the definition, it suffices to require that the map (3.11) is injective, and this property is inherited by subobjects).

We do not know whether $\widehat{M}_s(G, R) \subset \mathcal{N}(G, R)$ is closed under quotients or extensions.

**Remark 3.7.** Note that while $\widehat{M}_s(G, R) \cong \mathcal{N}(G, R)$ has infinite sums, they do not commute with evaluation – for any $S \in \widehat{\Gamma}_G$ and $M_i \in \widehat{M}(G, R)$, $i \geq 0$, the map

$$\bigoplus_i M_i(S) \to \left( \bigoplus_i M_i \right)(S)$$

is not an isomorphism (the right-hand side is a certain completion of the left-hand side).
**Definition 3.8.** The *canonical filtration* on a separated $G$-Mackey profunctor $M \in \hat{\mathcal{M}}_s(G, R)$ is the decreasing filtration $F^N M$ indexed by normal cofinite subgroups $N \subset G$ such that $F^N M \subset M$ is the kernel of the adjunction map

$$M \to \text{Infl}^N \Phi^N M.$$ 

By Proposition 3.5, a separated Mackey profunctor is automatically complete with respect to the canonical filtration.

**Lemma 3.9.** The embedding $\hat{\mathcal{M}}_s(G, R) \subset \text{Fun}(Q\hat{\Gamma}_G, R)$ admits a left-adjoint additivization functor $\text{Add} : \text{Fun}(Q\hat{\Gamma}_G, R) \to \hat{\mathcal{M}}^s(G, R)$.

**Proof.** For any two normal subgroups $N' \subset N \subset G$, the fixed points functor $\Phi^N_{N'}$ preserves additivity, thus commutes with the addivization functor $\text{Add}$. Therefore for any object $E \in \text{Fun}(Q\hat{\Gamma}_G, R)$, we can apply addivization fiberwise with respect to the projection $Q\hat{\Gamma}_G \to N(G)$ and obtain an object $\text{Add}(E) \in \text{Fun}(Q\hat{\Gamma}_G, R)$ such that $\text{Add}(E)_N$ is additive for any $N \subset G$. Moreover, if the transition morphisms (3.8) for $E$ were isomorphisms, then the transition morphisms for $\text{Add}(E)$ are also isomorphisms, so that $\text{Add}(E)$ is a normal system. Now to define the additivization functor $\hat{\mathcal{M}}(G, R) \to \hat{\mathcal{M}}_s(G, R) \cong \hat{\mathcal{N}}(G, R)$, it suffices to set

$$\text{Add}(E) = \text{Infl}(\text{Add}(\Phi(E)))$$

for any $E \in \text{Fun}(Q\hat{\Gamma}_G, R)$. We then have a natural morphism $\alpha_E : E \to \text{Add}(E)$, $\text{Add}(E)$ is additive, and $\text{Add}(\alpha_E)$ is an isomorphism for any $E$, so that $\text{Add}$ is indeed left-adjoint to the embedding $\hat{\mathcal{M}}_s(G, R) \subset \hat{\mathcal{M}}(G, R)$.

We can now extend the material of Subsection 2.2 and Subsection 2.3 to Mackey profunctors. First, we note that as in the Subsection 2.3, we have the adjoint pair of functors

$$\rho^H : \hat{\Gamma}_G \to \hat{\Gamma}_H, \quad \gamma^H : \hat{\Gamma}_H \to \hat{\Gamma}_H,$$

for any cofinite subgroup $H \subset G$, and the functor

$$Qp^H_i \cong Q\gamma^H_i : \text{Fun}(\hat{\Gamma}_G, R) \to \text{Fun}(\hat{\Gamma}_H, R)$$

obviously preserves additivity, thus induces a categorical fixed points functor

$$\Psi^H : \hat{\mathcal{M}}(G, R) \to \hat{\mathcal{M}}(H, R).$$
As in (2.14), the functor $\Psi^H$ can be refined to a functor $\tilde{\Psi}^H : \hat{\mathcal{M}}(G, R) \to \hat{\mathcal{M}}(H, R[Z_H])$. We also have the isomorphism (2.13). Therefore in particular, the functors $\Psi^H$ and $\tilde{\Psi}^H$ send separated Mackey profunctors to separated ones.

Moreover, even if a subgroup $H \subset G$ is not cofinite, we still have a well-defined functor $\varphi^H : \hat{\Gamma}_G \to \hat{\Gamma}_W$ of (2.11), where $W = N_H / H$, and $N_H \subset G$ is the normalizer of $H \subset G$. Therefore we have the inflation functor

$$\text{Infl}^H = Q(\varphi^H)^* : \hat{\mathcal{M}}(W, R) \to \hat{\mathcal{M}}(G, R).$$

We also have the inflation functor $\text{Infl}^H : \mathcal{N}(W, R) \to \mathcal{N}(G, R)$, and it commutes with the functor $\text{Infl}$ of Proposition 3.3 so that $\text{Infl}^H$ sends separated Mackey profunctors to separated ones. On the subcategories of separated Mackey profunctors, $\text{Infl}^H$ has a left-adjoint geometric fixed points functor

$$\Phi^H = \text{Add} \circ S(\varphi^H)_! : \hat{\mathcal{M}}_s(G, R) \to \hat{\mathcal{M}}_s(W, R),$$

where $\text{Add}$ is the additivization functor provided by Lemma 3.9. The isomorphism (2.13) also holds, as long as $H \subset G$ is cofinite.

Finally, we note that for any rings $R_1, R_2$, (2.24) with the addivization functor $\text{Add}$ provided by Lemma 3.9 defines an associative commutative product

$$(3.17) \quad \hat{\mathcal{M}}_s(G, R_1) \times \hat{\mathcal{M}}_s(G, R_1) \to \hat{\mathcal{M}}_s(G, R_1 \otimes R_2).$$

The fixed point functors $\Phi^H$, $\Psi^H$ are tensor functors with respect to this product, and the completed Burnside Mackey functor $\hat{A} \in \hat{\mathcal{M}}_s(G, \mathbb{Z})$ is the unit object. If $R_1 = R_2 = R$ is a commutative ring, we can compose the product with the natural functor induced by the product map $R \otimes R \to R$, and this turns $\hat{\mathcal{M}}_s(G, R)$ into a symmetric monoidal unital category.

4 Generalities on the $S$-construction.

To proceed further, we need to recall and generalize slightly a certain construction introduced in [K2 Section 4]. It bears some similarity to Waldhausen’s $S$-construction used in the definition of algebraic $K$-theory, so we denote it by the same letter. In fact, one can introduce an even more general construction that would include both Waldhausen’s construction and the construction of [K2 Section 4] as particular cases. However, we will not do it here; we restrict our attention to what is really needed for the theory of Mackey functors.
4.1 The constructions. We start with the following general definition.

**Definition 4.1.** A class of morphisms $I$ in a category $C$ is *admissible* if it is closed under compositions, contains all identity maps, and for any two maps $f : c_1 \to c$, $i : c_2 \to c$ in $C$ with $i$ in $I$, there exists a cartesian square

\[
\begin{array}{ccc}
c_1 & \xrightarrow{f'} & c_2 \\
i & \downarrow & \downarrow i \\
c_1 & \xrightarrow{f} & c
\end{array}
\]

in $C$ with $i'$ in $I$. Given an admissible class $I$, we denote by $C_I \subset C$ the subcategory with the same objects, and those morphisms between them that lie in the class $I$.

**Example 4.2.** If a category $C$ has all fibered products, then the class of all maps is admissible. We will denote this class by $\text{Id}$.

**Example 4.3.** In Example 4.2, we can also let $I$ consist of all monomorphisms in $C$. We denote this class by $\text{Inj}$.

**Definition 4.4.** Assume given a small category $C$ with an admissible class of maps $I$. For any two objects $c, c' \in C$, the *category* $Q_I C(c, c')$ is the category of diagrams

\[
\begin{array}{ccc}
c & \xleftarrow{i} & \bar{c} \\
& \xrightarrow{f} & c'
\end{array}
\]

in $C$, with $i$ in $I$, and isomorphisms between such diagrams. The *category* $Q_I C$ is the category with the same objects as $C$, with morphisms from $c$ to $c'$ given by isomorphism classes of objects of the category $Q_I C(c, c')$, and with composition given by pullbacks.

To simplify notation, in the situation of Example 4.2 we will denote $Q C(-, -) = Q_{\text{Id}} C(-, -)$ and $Q C = Q_{\text{Id}} C$.

**Example 4.5.** Consider the situation of Example 4.2 with $C = \Gamma_G$, the category of finite $G$-sets for some group $G$. Then $Q C = Q \Gamma_G$ is the category considered in Section 2.

Note that the categories $Q_I C(c, c')$ of Definition 4.1 fit together to form a 2-category $Q_I C$; the category $Q_I C$ is only a truncation of this 2-category.
In general, this truncation loses some information, and the resulting category may not be the right one to consider. In particular, this is the case for the category \( Q\Gamma G \) – diagrams (2.2) can have non-trivial automorphisms, and when we pass to \( Q\Gamma G \), we completely forget them. Thus when considering derived Mackey functors, it would be more natural to work not with functors from \( Q\Gamma G \) to complexes of modules over a ring \( R \), but with functors from \( Q\Gamma G \) to such complexes. Of course, to do this, one has to define functors from a 2-category to complexes in the correct way. This has been done in [K2], in several different ways that are all proved to be equivalent. The resulting category \( DM(G,R) \) of “derived Mackey functors” of [K2] indeed behaves better than the derived category \( D(M(G,R)) \) of the abelian category \( M(G,R) \).

In the present paper, we will adopt the least technically cumbersome of the approaches of [K2], namely, that of [K2 Subsection 4.2]. Let us recall the construction and its main properties (for an heuristic description and motivation, see [K2 Subsection 4.1]).

The construction uses the simplicial technology overviewed in Subsection 1.4. Recall that any object \([n] \in \Delta \) in the category \( \Delta \) of finite non-empty ordinals can be treated as a small category: a functor \( f : [n] \rightarrow C \) to some category \( C \) is the same thing as a diagram

\[
\begin{array}{cccc}
c_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-1}} c_n \\
\end{array}
\]

in \( C \). A morphism \( \alpha : f \rightarrow f' \) between two such functors is given by a commutative diagram

\[
\begin{array}{cccc}
c_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-1}} c_n \\
\downarrow{\alpha_1} & & & \downarrow{\alpha_n} \\
c'_1 & \xrightarrow{f'_1} & \cdots & \xrightarrow{f'_{n-1}} c'_{n'} \\
\end{array}
\]

where the bottom row represents \( f' \).

**Definition 4.6.** For any small category \( C \) equipped with an admissible class of morphisms \( I \), the category \( S_I C \) is given by the following:

(i) objects are pairs \( ([n], f) \) of \( [n] \in \Delta \) and a functor \( f : [n] \rightarrow C \),

(ii) morphisms from \( ([n], f) \) to \( ([n'], f') \) are given by a pair of a morphism \( \varphi : [n] \rightarrow [n'] \) and a morphism \( \alpha : f' \circ \varphi \rightarrow f \) such that for any
The commutative square

\[
\begin{array}{ccc}
\varphi(i) & \longrightarrow & \varphi(j) \\
\downarrow & & \downarrow \\
\alpha_i & \longrightarrow & \alpha_j \\
\end{array}
\]

induced by \((4.4)\) is cartesian, and the maps \(\alpha_i, \alpha_j\) are in \(I\).

As in Definition 4.4, if \(I = \text{Id}\) is the class of all maps, we will drop it from notation and denote \(SC = S_{\text{Id}}C\).

By definition, the category \(S_I C\) is equipped with a forgetful functor \(\pi : S_I C \to \Delta\) sending \(\langle [n], f \rangle\) to \([n]\). This functor is a fibration. Its fiber \((S_I C)_{[1]}\) over \([1] \in \Delta\) is naturally identified with the category \(C_f^I\) opposite to the category \(C_f\). For \(n \geq 2\), the fiber \((S_I C)_{[n]}\) is opposite to the category of diagrams \((4.3)\), with morphisms given by diagrams \((4.4)\), with \(\alpha_i\) in \(I\) and cartesian squares. Note that by Definition 4.4 it suffices to require that \(\alpha_n\) is in \(I\).

Note that for any two objects \(c, c' \in C = (S_I C)_{[1]}\), the category \(Q_I C(c, c')\) of Definition 4.4 can be described in terms of the category \(S_I C\) in the following way. Consider the diagram

\[
\begin{array}{ccc}
[1] & \xrightarrow{s} & [2] \\
\downarrow & & \downarrow t \\
[1] & \xleftarrow{t} & [1]
\end{array}
\]

in \(\Delta\), where \(s, t : [1] \to [2]\) send the unique element in \([1] \in \Delta\) to the initial resp. terminal object of the ordinal \([2] \in \Delta\). Then \(Q_I C(c, c')\) is equivalent to the category of diagrams

\[
c \xrightarrow{\tilde{s}} c_{[2]} \xleftarrow{\tilde{t}} c'
\]

in \(S_I C\) that are sent to \((4.6)\) by the projection \(\pi : S_I C \to \Delta\), and such that \(\tilde{t}\) is cartesian with respect to \(\pi\). The equivalence sends an object in \(Q_I C(c, c')\) represented by a diagram \((4.2)\) to the diagram

\[
c \longrightarrow [\overline{c} \to c'] \xleftarrow{c'}
\]

where \(\overline{c} \to c'\) is an object in \((S_I C)_{[2]}\) represented by the arrow \(\overline{c} \to c'\).

Definition 4.7. A map \(f : [n] \to [n']\) in the category \(\Delta\) is \textit{special} if it sends the terminal element in \([n]\) to the terminal element in \([n']\). For small category \(C\) with an admissible class of maps \(I\), a morphism \(\langle f, \alpha \rangle : \langle [n], f \rangle \to \langle [n'], f' \rangle\) is \textit{special} if \(f = \pi(\langle f, \alpha \rangle)\) is special and \(\alpha_n : c'_{n'} \to c_n\) is an isomorphism (that is \(\langle f, \alpha \rangle\) is cartesian with respect to the fibration \(\pi : S_I C \to \Delta\)).
**Definition 4.8.** For any ring $R$, an object $M \in \mathcal{D}(S_I\mathcal{C}, R)$ is **special** if it can be represented by such a complex $M_\ast$ in $\text{Fun}(S_I\mathcal{C}, R)$ that $M_\ast(f)$ is a quasiisomorphism for any special map $f$ in $S\mathcal{C}$. The full subcategory in $\mathcal{D}(S_I\mathcal{C}, R)$ spanned by special objects is denoted by $\mathcal{D}S_I(C, R)$.

**Remark 4.9.** The notion of a special map in Definition 4.7 is different from the one used in [K2, Subsection 4.2]. However, the resulting category $\mathcal{D}S_I(C, R)$ is clearly the same.

For any $M \in \mathcal{D}S_I(C, R)$ and any object $c \in \mathcal{C}$, we can evaluate $M$ at $c$ by considering $c$ as an object of $[S_I\mathcal{C}][1] \subset S_I\mathcal{C}$; this gives an object $M(c) \in \mathcal{D}(R)$. For any two objects $c, c' \in \mathcal{C}$, an object in $Q_I\mathcal{C}(c, c')$ represented by a digram (4.2) defines a natural map from $M(c)$ to $M(c')$ – namely, the composition map

\[
(4.8) \quad M(c) \xrightarrow{M(\xi)} M(\xi c \rightarrow c') \xrightarrow{M(\eta)^{-1}} M(c')
\]

induced by the corresponding diagram (4.7). If $f$ resp. $i$ in (4.2) is the identity map, then we will denote the map (4.8) by $i^*$ resp. $f^*$; in general, (4.8) is the composition $f^* \circ i^*$. This is compatible with the composition of diagrams, so that every map $M \in \mathcal{D}S_I(C, R)$ induces a functor from $Q_I\mathcal{C}$ to the derived category $\mathcal{D}(R)$.

This construction can be refined in the following way. For any object $c \in S_I\mathcal{C}$ represented by a diagram (4.3), let $q(c) = c_n$, and for any morphism $f = \langle f, \alpha \rangle : c \rightarrow c'$ in $S_I\mathcal{C}, c \in (S_I\mathcal{C})[n], c' \in (S_I\mathcal{C})[n']$, let $q(f) \in Q_I\mathcal{C}(c_n, c'_n)$ be the diagram

\[
(4.9) \quad c_n \xleftarrow{\alpha} c'_n \xrightarrow{f(n)} c'_{n'},
\]

where the map on the right is the composition of the natural maps in the diagram (4.3). Then sending $c$ to $q(c)$ and $f$ to the isomorphism class of $q(f)$ defines a functor

\[
(4.10) \quad q : S_I\mathcal{C} \rightarrow Q_I\mathcal{C}
\]

such that $q(f)$ is invertible for any special morphism $f$ in $S_I\mathcal{C}$. Therefore for any ring $R$ and any $M \in \mathcal{D}(Q_I\mathcal{C}, R)$, the pullback $q^*M \in \mathcal{D}(S_IQ, R)$ is special, so that we have a natural pullback functor

\[
(4.11) \quad q^* : \mathcal{D}(Q_I\mathcal{C}, R) \rightarrow \mathcal{D}S_I(C, R).
\]
Lemma 4.10. The truncation functors with respect to the standard $t$-structure on $\mathcal{D}(S_I\mathcal{C}, R)$ send special objects to special objects, thus induce a $t$-structure on $\mathcal{D}^{\leq i}(S_I\mathcal{C}, R) \subset \mathcal{D}(S_I\mathcal{C}, R)$ given by

$$\mathcal{D}^{\leq i}(S_I\mathcal{C}, R) = \mathcal{D}S_I\mathcal{C}(C, R) \cap \mathcal{D}^{\leq i}(S_I\mathcal{C}, R) \subset \mathcal{D}(S_I\mathcal{C}, R)$$

for any integer $i$. The functor (4.11) gives an equivalence

$$\text{Fun}(Q_{I}\mathcal{C}, R) \cong \text{Fun}(S_I\mathcal{C}, R) \cap \mathcal{D}S_I\mathcal{C}(C, R) \subset \mathcal{D}(S_I\mathcal{C}, R)$$

between the heart of this $t$-structure and the category $\text{Fun}(Q_{I}\mathcal{C}, R)$.

Proof. Clear. □

Definition 4.11. Assume given two small categories $\mathcal{C}$, $\mathcal{C}'$ with admissible classes of maps $I$, $I'$. A morphism $\varphi : (\mathcal{C}, I) \to (\mathcal{C}', I')$ is a functor $\varphi : \mathcal{C} \to \mathcal{C}'$ that sends morphisms in $I$ to morphisms in $I'$ and cartesian squares (4.1) in $\mathcal{C}$ to cartesian squares in $\mathcal{C}'$. For any two such morphisms $\varphi, \varphi' : \mathcal{C} \to \mathcal{C}'$, a map $\alpha : \varphi \to \varphi'$ is $Q$-compatible if for any object $c \in \mathcal{C}$, $\alpha : \varphi(c) \to \varphi'(c)$ is in $I'$, and for any morphism $f : c \to c'$ in $\mathcal{C}$, the commutative square

$$\begin{array}{ccc}
\varphi(c) & \xrightarrow{\alpha} & \varphi'(c) \\
\varphi(f) \downarrow & & \downarrow \varphi'(f) \\
\varphi(c') & \xrightarrow{\alpha} & \varphi'(c')
\end{array}$$

is cartesian.

Example 4.12. Assume given a morphism $\varphi : (\mathcal{C}, I) \to (\mathcal{C}', I')$ in the sense of Definition 4.10, $f : c \to c'$ in a small category $\mathcal{C}$ with fibered products. Then the functors $f_1, f^*$ of (1.1), (1.2) both give morphisms between $(\mathcal{C}/c, \text{Id})$ and $(\mathcal{C}/c', \text{Id})$, and both adjunction maps $\text{Id} \to f^* \circ f_1, f_1 \circ f^* \to \text{Id}$ are $Q$-compatible.

Lemma 4.13. (i) A morphism $\varphi : (\mathcal{C}, I) \to (\mathcal{C}', I')$ in the sense of Definition 4.11 induces a functor $S(\varphi) : S_I\mathcal{C} \to S_{I'}\mathcal{C}'$. This functor commutes with projections to $\Delta$ and sends special maps to special maps, thus induces a pullback functor

$$S(\varphi)^* : \mathcal{D}S_{I'}(C', R) \to \mathcal{D}S_I(C, R)$$

for any ring $R$. Moreover, $\varphi$ induces a functor $Q(\varphi) : Q_I\mathcal{C} \to Q_{I'}\mathcal{C}'$, and we have $q \circ S(\varphi) \cong Q(\varphi) \circ q$. 

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(ii) For any two such morphisms \( \varphi, \varphi' : \langle C, I \rangle \to \langle C', I' \rangle \), a \( Q \)-compatible map \( \alpha : \varphi \to \varphi' \) induces maps of functors \( S(\alpha) : S(\varphi) \to S(\varphi') \), \( Q(\alpha) : Q(\varphi) \to Q(\varphi') \).

Proof. Clear. \( \square \)

4.2 Basic properties. On the level of full derived categories, the functor \( q^* \) of (4.11) is usually not an equivalence – \( DS_I(C, R) \) is an example of a triangulated category with a \( t \)-structure that is not equivalent to the derived category of its heart. From our point of view, it is \( DS_I(C, R) \) that is the right category to consider, and it is the correct replacement of the derived category \( D(Q_I C, R) \).

To study it, we need one result essentially proved in [K2, Section 4]. Denote by \( \tilde{S}_I C \) the category of diagrams

\[
c_1 \leftarrow s_1 \quad c_0 \quad s_2 \rightarrow c_2
\]

in \( S_I C \) with special maps \( s_1, s_2 \), and let \( \pi_1, \pi_2 : \tilde{S}_I C \to S_I C \) be the projections sending such a diagram to \( c_1 \) resp. \( c_2 \). Both projections are cofibrations, and we have the natural endofunctor

\[
L \pi_2! \pi_1^* : D(S_I C, R) \to D(S_I C, R)
\]

of the derived category \( D(S_I C, R) \). Moreover, cofibrations \( \pi_1, \pi_2 \) admit a common section \( \lambda : S_I C \to \tilde{S}_I C \) sending \( c \) to a diagram (4.12) with \( c_1 = c_2 = c_0 = c \), \( s_1 = s_2 = \text{id} \), and the adjunction map \( L^* \lambda_! \circ \lambda^* \to \text{id} \) induces a natural map

\[
a : \text{id} \cong L^* \pi_2! \circ L^* \lambda_! \circ \lambda^* \circ \pi_1^* \to L^* \pi_2! \pi_1^*.
\]

Lemma 4.14. Assume given a small category \( C \) and an admissible class of morphisms \( I \). Then the functor (4.13) takes values in \( DS_I(C, R) \subset D(S_I C, R) \) and induces a functor

\[
Sp : D(S_I C, R) \to DS_I(C, R)
\]

left-adjoint to the embedding \( DS_I(C, R) \hookrightarrow D(S_I C, R) \), with the adjunction induced by the natural map (4.14).

Proof. Say that a map \( f : [n] \to [n'] \) is co-special if it identifies the ordinal \( [n] \) with an initial segment of the ordinal \( [n'] \), and say that a map \( f \) in \( S_I C \)
is co-special if so is its image \( \pi(f) \) with respect to the canonical projection \( \pi : S_I C \to \Delta \). Then by the same argument as in \cite[Lemma 4.8]{K2}, the classes of special and co-special maps form a complementary pair on \( S_I C \) in the sense of \cite[Definition 4.3]{K2}, and the result immediately follows from \cite[Lemma 4.6]{K2}. □

**Corollary 4.15.** For any morphism \( \varphi : (C, I) \to (C', I') \) in the sense of Definition 4.11 and any ring \( R \), the pullback functor \( S(\varphi)^* \) of Lemma 4.13 admits a left-adjoint functor

\[
S(\varphi)_! : \mathcal{D}(C, R) \to \mathcal{D}(C', R).
\]

**Proof.** It suffices to set \( S(\varphi)_! = \text{Sp} \circ \text{L}^* S(\varphi)_! \), where the functor \( \text{Sp} \) is provided by Lemma 4.14 and \( \text{L}^* S(\varphi)_! : \mathcal{D}(S_I C, R) \to \mathcal{D}(S_I C', R) \) is the derived left Kan extension functor with respect to \( S(\varphi) : S_I C \to S_I C' \).

In some special cases, \( S(\varphi)_! \) is easy to compute. In particular, assume given two categories \( C, C' \) with fibered products, equip them with admissible classes \( \text{Id} \) of all morphisms, and assume given a pair of adjoint functors \( \varphi : C' \to C, \psi : C \to C' \) such that both send fibered products to fibered products, and both adjunction maps \( \text{Id} \to \psi \circ \varphi, \varphi \circ \psi \to \text{Id} \) are \( Q \)-compatible in the sense of Definition 4.11 (such a situation occurs, for instance, in Example 4.12). Then by Lemma 4.13 the adjunction maps induce natural maps

\[
S(\psi) \circ S(\varphi) \to \text{Id}, \quad \text{Id} \to S(\varphi) \circ S(\psi),
\]

and these maps give an adjunction between \( S(\varphi) \) and \( S(\psi) \), so that we have a natural isomorphism

\[
(4.15) \quad S(\psi)_! \cong S(\varphi)^* : \mathcal{D}(C, R) \to \mathcal{D}(C', R).
\]

The specialization functor \( \text{Sp} \) does not enter into the picture at all.

In the general case, to control \( S(\varphi)_! \), it is useful to have a reasonably explicit description of objects \( \text{Sp}(F), F \in \mathcal{D}(S_I C, R) \). Such a description is in fact provided in \cite[Lemma 4.6]{K2}. We will need it in one particular case – namely, for objects of the form \( \text{Sp}(M_c) \in \mathcal{D}(S_I C, R) \), where \( M \) is an \( R \)-module, \( c \in C^0 \cong (S_I C)[1] \subseteq S_I C \) is an object of \( C \) considered as an object of \( S_I C \), and \( M_c \in \text{Fun}(S_I C, R) \) is the representable functor \( 1,3 \). To obtain such a description, note that the categories \( Q_I C(c, \cdot) \) fit together into a cofibration

\[
(4.16) \quad \rho_c : Q_I C \to S_I C
\]
whose fiber over \( c' \in S_I \mathcal{C} \) is the category \( Q_I \mathcal{C}(c, q(c')) \), and whose transition functor corresponding to a morphism \( f \) is given by the composition with the diagram \( q(f) \) of (4.9).

**Lemma 4.16.** For any \( c \in \mathcal{C} \) and any \( R \)-module \( M \), we have a natural identification

\[
\text{Sp}(M_c) \cong L^* \rho_c M,
\]

where \( \rho_c \) is the cofibration (4.16), and \( M \in \text{Fun}(Q_I^c \mathcal{C}, R) \) is the constant functor with value \( M \).

We note that by definition, for any special map \( f \) in \( S_I \mathcal{C} \), the transition functor of the cofibration \( \rho_c \) is an equivalence of categories, so that by base change, the right-hand side of (4.17) is a special object in \( D(S_I \mathcal{C}, R) \). Explicitly, for any \( c' \in S_I \mathcal{C} \), we have a natural identification

\[
\text{Sp}(M_c)(c') \cong H_0(Q_I \mathcal{C}(q(c), q(q(c'))), M),
\]

where the right-hand side denotes the homology of \( Q_I \mathcal{C}(q(c), q(c')) \) with coefficients in the constant functor with value \( M \). We also note that by adjunction, we have a natural map \( M_c \rightarrow q^* M_{q(c)} \), and since \( q^* M_{q(c)} \) is special, it factors through a map

\[
\text{Sp}(M_c) \rightarrow q^* M_{q(c)}.
\]

In terms of the identification (4.18), this map becomes the map

\[
H_0(Q_I \mathcal{C}(q(c), q(q(c'))), M) \cong M[Q_I \mathcal{C}(q(c), q(q(c')))]
\]

corresponding to the identification of \( Q_I \mathcal{C}(q(c), q(q(c')) \) with the set of isomorphism classes of objects in the groupoid \( Q_I(q(c), q(q(c')) \). In particular, (4.19) identifies \( q^* M_{q(c)} \) with the truncation at 0 of the object \( \text{Sp}(M_c) \) with respect to the standard \( t \)-structure on \( DS_I(\mathcal{C}, R) \).

**Proof of Lemma 4.16.** As shown in the proof of [K2, Lemma 4.6], to compute the specialization functor (4.13), we can replace \( S_I \mathcal{C} \) with its subcategory \( \overline{S}_I \mathcal{C} \) spanned by diagrams (4.12) with \( c_0 \) lying in \( (S_I \mathcal{C})_{[1]} \subset S_I \mathcal{C} \). To apply this to \( M_c \), use its explicit description (1.16). Then by base change, we obtain an isomorphism

\[
\text{Sp}(M_c) \cong L^* \tilde{\rho}_c M,
\]

where \( \tilde{\rho}_c : \tilde{Q}_I \rightarrow S_I \mathcal{C} \) is a cofibration whose fiber over \( c' \in S_I \mathcal{C} \) is the category of diagrams

\[
c \xrightarrow{p} \tilde{c} \xleftarrow{i} q(c')
\]

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in $S_jC$ with special $i$ and co-special $p$ (the meaning of co-special is the same as in the proof of Lemma 4.14). But the category $Q^c_I$ is actually embedded into $\tilde{Q}^c_I$, with the embedding sending a diagram (4.2) to the corresponding diagram (4.7). It remains to notice that the embedding $i: Q^c_I \rightarrow \tilde{Q}^c_I$ admits a right-adjoint functor, so that $L^*i^!M \cong M$, and $L^*\tilde{\rho}_c^!M \cong L^*\rho_c^!M$. □

Lemma 4.16 has one immediate corollary. Say that a pair $\langle C, I \rangle$ of a small category $C$ and an admissible class of morphisms $I$ is discrete if all the categories $Q_I(c, c')$ of Definition 4.4 are essentially discrete categories (that is, groupoids with trivial automorphism groups).

**Corollary 4.17.** Assume given a small category $C$ and a class of morphisms $I$, and assume that the pair $\langle C, I \rangle$ is discrete. Then the functor $q^*$ of (4.11) is an equivalence of categories.

**Proof.** Under the assumptions of the Corollary, the cofibration $\rho_c$ of (4.16) is actually discrete for any $c \in C$. Therefore by base change, the functors $\rho_c^!$ is exact, and we have

$$\text{Sp}(M_c) \cong L^*\rho_c^!M \cong \rho_c^!M$$

for any $R$-module $M$. Moreover, it is easy to see that

$$\rho_c^!M \cong q^*\tilde{c},$$

where $\tilde{c}$ denotes $c$ considered as an object in $Q_I C$. But by adjunction, we have

$$L^*q_!\text{Sp}(M_c) \cong M_{\tilde{c}},$$

so that the adjunction map $L^*qq^*E \rightarrow E$ is an isomorphism for an object $E \in D(Q_I C, R)$ of the form $E = M_{\tilde{c}}$, $c \in C$. Since such objects generate the category $D(Q_I C, R)$, we have $L^*qq^* \cong \text{Id}$, so that $q^*$ is fully faithful. Since objects of the form $\text{Sp}(M_c)$, $c \in C$, generate the category $D_S(I(C, R))$, the functor (4.11) is also essentially surjective. □

### 4.3 A base change lemma

We will also need one general result that in some cases greatly simplifies the computation of the functors $S(\varphi)_!$ provided by Corollary 4.15.

The setup is the following. Assume given a small category $C$ with fibered products and a full subcategory $C' \subset C$ closed under fibered products. Moreover, assume that the embedding $C' \subset C$ admits a right-adjoint functor

$$\varphi: C \rightarrow C'.$$
Then by adjunction, \( \varphi \) automatically sends fibered products to fibered products, thus gives a morphism \( \varphi : \langle C, \text{Id} \rangle \rightarrow \langle C', \text{Id} \rangle \) in the sense of Definition 4.11.

Moreover, assume also given an admissible class of maps \( I \) in \( C \). Since \( C' \subseteq C \) is closed under taking fibered products, \( I' = I \cap C' \) is then an admissible class of maps in \( C' \), with \( C'I' = C' \cap C_I \subseteq C_I \). Assume further that \( \varphi \) sends morphisms in \( I \) to morphisms in \( I' \), so that it induces a functor \( \varphi : C_I \rightarrow C_I' \) and a morphism \( \varphi : \langle C, I \rangle \rightarrow \langle C', I' \rangle \). In addition, assume that \( \langle C, I \rangle \), and therefore also \( \langle C', I' \rangle \), is discrete in the sense of Corollary 4.17, so that we have natural equivalences of categories

\[
DS_I(C, R) \cong D(Q IC, R), \quad DS_I'(C', R) \cong D(Q I'C', R).
\]

**Remark 4.18.** If \( I = \text{Inj} \) is the class of all monomorphisms in \( C \), as in Example 4.3, then all these assumption on \( I \) are automatically satisfied.

Then we have tautological embedding morphisms \( i : \langle C, I \rangle \rightarrow \langle C, \text{Id} \rangle \), \( i : \langle C', I' \rangle \rightarrow \langle C', \text{Id} \rangle \), and \( i \circ \varphi \cong \varphi \circ i \), so that we have an isomorphism

\[
S(\varphi)^* \circ S(i)^* \cong S(i)^* \circ S(\varphi)^*
\]

of the corresponding pullback functors of Lemma 4.13. By adjunction, it induces a base change map

\[
(4.20) \quad S(\varphi)_! \circ S(i)^* \rightarrow S(i)^* \circ S(\varphi)_!,
\]

where the adjoint functors \( S(\varphi)_! \) are those of Corollary 4.15.

**Proposition 4.19.** Under the assumptions above, assume further that for any \( c \in C \), the adjunction map \( \varphi(c) \rightarrow c \) is in the class \( I \). Then the base change map (4.20) is an isomorphism.

Before we prove this, we need a technical lemma. Fix an object \( c \in C \), and let \( (C/c)_I \) be the category of objects \( c' \in C \) equipped with a map \( f : c' \rightarrow c \), with morphisms from \( f_1 : c'_1 \rightarrow c \) to \( f_2 : c'_2 \rightarrow c \) given by morphisms \( i : c'_1 \rightarrow c'_2 \) in the class \( I \) such that \( f_2 \circ i = f_1 \). Then we have a natural forgetful functor \( (C/c)_I \rightarrow C_I \). Composing the opposite functor with the obvious embedding \( C_I^c \rightarrow Q IC \), we obtain a functor

\[
j_c : (C/c)_I^c \rightarrow Q IC.
\]

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Lemma 4.20. For any $c \in \mathcal{C}$ and any $R$-module $M$, we have a natural isomorphism

$$S(i)^* \mathcal{S}(M_c) \cong L^* j_{ct} M \in DS_1(\mathcal{C}, R) \cong \mathcal{D}(Q_I \mathcal{C}, R),$$

where in the right-hand side, $M \in \text{Fun}((\mathcal{C}/c)^{I_0}, R)$ stands for the constant functor with value $M$.

Proof. To compute $L^* j_{ct} M$, we can use the decomposition (1.14) and the isomorphism (1.15). Spelling out the definition, we check that the comma-category $j_c \backslash Q_I \mathcal{C}$ is the category of diagrams

$$(4.21) \quad \begin{array}{ccc}
  c & \leftrightarrow & c_1 \\
  & \alpha_1 \uparrow \\
  & \downarrow \\
  c & \leftrightarrow & c_3
\end{array}$$

in $\mathcal{C}$, with $i$ in the class $I$, and morphisms between such diagrams are given by commutative diagrams

$$\begin{array}{ccc}
  c & \leftrightarrow & c_1 \\
  & \alpha_1 \uparrow \\
  & \downarrow \\
  c & \leftrightarrow & c_3
\end{array} \quad \begin{array}{ccc}
  c & \leftrightarrow & c_1' \\
  & \alpha_1' \uparrow \\
  & \downarrow \\
  c & \leftrightarrow & c_3'
\end{array}$$

in $\mathcal{C}$ such that $\alpha_1$ is in $I$ and $\alpha_2$ is an isomorphism. The cofibration $t : j_c \backslash Q_I \mathcal{C} \to Q_I \mathcal{C}$ sends a diagram (4.21) to $c_3 \in Q_I \mathcal{C}$.

Inside the category $j_c \backslash Q_I \mathcal{C}$, we have a full subcategory $j_c \backslash Q_I \mathcal{C}$ spanned by diagrams (4.21) with invertible $i$, and the embedding $j_c \backslash Q_I \mathcal{C} \subset j_c \backslash Q_I \mathcal{C}$ has a right-adjoint $\rho : j_c \backslash Q_I \mathcal{C} \to j_c \backslash Q_I \mathcal{C}$. Therefore we have canonical identifications

$$L^* j_{ct} M \cong L^* t_M \cong L^* \rho^* M \cong L^* t_M,$$

where $\overline{t}$ is the restriction of the projection $t$ to $j_c \backslash Q_I \mathcal{C}$. It remains to notice that we have a cartesian square

$$\begin{array}{ccc}
  Q_I \mathcal{C} & \longrightarrow & j_c \backslash Q_I \mathcal{C} \\
  \rho & \downarrow & \overline{t} \\
  S^I \mathcal{C} & \longrightarrow & Q_I \mathcal{C}
\end{array}$$

of categories and functors, and apply Lemma 4.16 and the base change isomorphism (1.13). □

Proof of Proposition 4.19. We have to prove that for any $E \in \mathcal{D}(SC, R)$, the map

$$S(\varphi)_! S(i)^* \mathcal{S}(E) \to S(i)^* S(\varphi)_! \mathcal{S}(M)$$

(4.22)
induced by (4.20) is an isomorphism. Since the derived category $D(SC, R)$ is generated by objects of the form $M_c, c \in C, M$ an $R$-module, it suffices to prove this for $E = M_c$. By adjunction, we have $S(\varphi) \cdot Sp(M_c) \cong Sp(M_{\varphi(c)})$, and under the identification of Lemma 4.20 (4.22) becomes the natural map

$$L^*\varphi L^*j_c M \rightarrow L^*j_{\varphi(c)} M.$$ 

Moreover, the functor $\varphi$ induces a natural functor $\varphi_c : (C/c)_{I}^o \rightarrow (C'/\varphi(c))_{I'}^o$ such that $\varphi \circ j_c \cong j_{\varphi(c)} \circ \varphi_c$, so that it suffices to prove that $L^*\varphi_c M$ is the constant functor with value $M$.

But the assumptions of the proposition imply that if we denote by $\iota : C' \rightarrow C$ the embedding functor, then the adjunction maps (4.23)

$$\text{id} \rightarrow \iota \circ \varphi, \quad \varphi \circ \iota \rightarrow \text{id}$$

both lie pointwise in the class $I$. Therefore they also induce an adjunction between the embedding $C'_{I'} \subset C_I$ and the functor $\varphi : C_I \rightarrow C'_{I'}$. When we pass to the opposite categories, the embedding $C'^o_{I'} \subset C^o_I$ becomes right-adjoint to $\varphi : C^o_I \rightarrow C'^o_{I'}$.

It remains to notice that the embedding $\iota : C'_{I'} \rightarrow C^o_I$ extends to a functor $\iota_c : (C'/\varphi(c))_{I'}^o \rightarrow (C/c)_{I}^o$ sending $f : c' \rightarrow \varphi(c)$ to its composition with the adjunction map $\varphi(c) \rightarrow c$. Moreover, the adjunction maps (4.23) induce maps of functors

$$\text{id} \rightarrow \iota_c \circ \varphi_c, \quad \varphi_c \circ \iota_c \rightarrow \text{id},$$

and these maps give an adjunction between $\varphi_c$ and $\iota_c$. Thus $L^*\varphi_c M \cong \iota_c^* M \cong M$ is indeed the constant functor. □

5 Additivization.

We will also need one result concerning a more special situation – that of a small category $C$ that admits finite coproducts.

5.1 The setup. We start with a formal definition and the statement.

Definition 5.1. A pair $(C, I)$ of a small category $C$ and an admissible class of morphisms $I$ in $C$ has finite coproducts if the following holds.

(i) The category $C$ has finite coproducts and an initial object $\emptyset \in C$.

(ii) For any $c \in C$, the sole morphism $\emptyset \rightarrow c$ is a monomorphism and lies in the class $I$. 

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(iii) The coproduct of two morphisms in the class $I$ is in the class $I$, and the coproduct of two cartesian squares \([4.1]\) in $C$ is cartesian.

**Definition 5.2.** Assume that a small category $C$ has finite coproducts, and an admissible class of morphisms $I$ in $C$ is compatible with coproducts in the sense of Definition \([5.1]\). For any ring $R$, an object $E \in DS_I(C, R)$ is **additive** if for any two objects $c_1, c_2 \in C$ with the natural maps $i_1 : c_1 \to c_1 \sqcup c_2$, $i_2 : c_2 \to c_1 \sqcup c_2$, the natural map

\[
M(c_1 \sqcup c_2) \xrightarrow{i_1^* \oplus i_2^*} M(c_1) \oplus M(c_2)
\]

is an isomorphism. The full subcategory in $DS_I(C, R)$ spanned by additive objects is denoted by $DS_I^{add}(C, R) \subset DS_I(C, R)$.

**Lemma 5.3.** In the situation of Definition \([5.2]\), an object $M \in DS_I(C, R)$ is additive if and only if the map \([5.1]\) is an isomorphism whenever $c_1 \cong c_2$.

**Proof.** Definition \([5.1]\)(ii) and (iii) immediately imply that for any two objects $c_1, c_2 \in C$ with coproduct $c = c_1 \sqcup c_2$, the natural maps $i_1 : c_1 \to c$, $i_2 : c_2 \to c$ are in the class $I$. Moreover, Definition \([5.1]\)(ii) implies that we have $\emptyset \times_c \emptyset = \emptyset$ for any $c' \in C$, and then Definition \([5.1]\)(iii) implies that $c_1 \times_c c_1 \cong c_1$ and $c_2 \times_c c_2 \cong c_2$ (in other words, $i_1$ and $i_2$ are monomorphisms). Therefore for any $M \in DS_I(C, R)$, the endomorphisms $i_1^* \circ i_1 : M(c_1) \to M(c_1)$, $i_2^* \circ i_2 : M(c_2) \to M(c_2)$ are equal to the identity maps. Moreover, if we consider the map $i = i_1 \sqcup i_2 : c = c_1 \sqcup c_2 \to c \sqcup c$,

then $i^* \circ i : M(c) \to M(c)$ is also the identity map. Thus the compositions

\[
p_1 = i_1^* \circ i_1^* : M(c) \to M(c),
\]

\[
p_2 = i_2^* \circ i_2^* : M(c) \to M(c),
\]

\[
p = i^* \circ i^* : M(c \sqcup c) \to M(c \sqcup c),
\]

are idempotent endomorphisms, with images $M(c_1)$, $M(c_2)$, $M(c)$. But the natural map $M(c \sqcup c) \to M(c) \oplus M(c)$ intertwines $p$ with $p_1 \oplus p_2$. Therefore if this map is an isomorphism, then so is the map \([5.1]\). \(\square\)

The additivity condition is obviously preserved under truncation functors with respect to the $t$-structure of Lemma \([4.10]\) on $DS_I(C, R)$ (and those are by definition the truncation functors with respect to the standard $t$-structure on $D(S_I C, R)$). Therefore the subcategory $DS_I^{add}(C, R) \subset DS_I(C, R)$ inherits a natural $t$-structure.

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Proposition 5.4. In the situation of Definition 5.2, the embedding
\[ DS_{I}^{\text{add}}(C, R) \subset DS_{I}(C, R) \]
admits a left-adjoint additivization functor
\[ \text{Add} : DS_{I}(C, R) \to DS_{I}^{\text{add}}(C, R) \]
that is right-exact with the respect to the natural t-structures.

One example of a category with coproducts is the category \( \Gamma_{G} \) of finite \( G \)-sets for some group \( G \), and the class of all maps is compatible with coproducts in the sense of Definition 5.1. In this case, Proposition 5.4 has been essentially proved in [K2]. However, the proof is pretty roundabout, and the details are hard to trace. Therefore we reprove Proposition 5.4 from scratch using the additivization technique of [K1, Subsection 3.2] (which is essentially due to T. Pirashvili). The proof is split into a sequence of lemmas and takes up the rest of this section.

5.2 Finite sets. We first consider the case \( C = \Gamma \), the category of finite sets, with the class \( \text{Inj} \) of all monomorphisms as in Example 4.3. These obviously satisfy all the assumptions of Definition 4.1. The category \( Q_{\text{Inj}} \Gamma \) is naturally equivalent to the category \( \Gamma_{+} \) of finite pointed sets, that is, finite sets with a distinguished element. The equivalence sends a pointed set \( S \) with the distinguished element \( o \in S \) to the complement \( S' = S \setminus \{o\} \), and a pointed map \( f : S \to S' \) goes to the diagram
\[
\begin{array}{ccc}
S & \leftarrow & f^{-1}(S') \\
\downarrow & & \downarrow \\
S' & \rightarrow & S'
\end{array}
\]
(5.2)
Moreover, the pair \( (\Gamma, \text{Inj}) \) is discrete. Thus Corollary 4.17 applies to \( (\Gamma, \text{Inj}) \), so that for any ring \( R \), we have a canonical equivalence
\[ DS_{\text{Inj}}(\Gamma, R) \cong D(\Gamma_{+}, R). \]
For any integer \( n \geq 0 \), let us denote by \( [n] \in \Gamma \) the set with \( n \) elements, and in keeping with our convention (1.17), we denote by \( [n]_{+} \in \Gamma_{+} \) the pointed set with \( n \) non-distiguished elements. Moreover, let \( j_{n} : \text{pt} \to \Gamma_{+} \) be the embedding onto \( [n]_{+} \), and let \( T_{n} = j_{n}!Z = Z_{[n]_{+}} \in \text{Fun}(\Gamma_{+}, Z) \) be the object represented by \( [n]_{+} \in \Gamma \). Then since \( [0]_{+} \) is canonically a retract of \( [1]_{+} \), \( T_{0} \) is canonically a retract of \( T_{1} \), so that we have a canonical direct sum decomposition
\[ T_{1} = T_{0} \oplus T \]
(5.3)
for a certain object $T \in \text{Fun}(\Gamma_+, \mathbb{Z})$. Explicitly, $T$ is given by

\[(5.4) \quad T(S) = \mathbb{Z}[S]/\mathbb{Z} \cdot o, \quad S \in \Gamma_+,\]

where $o \in S$ is the distinguished element. Now consider the functor

\[j^T : \mathcal{D}(R) \to \mathcal{D}(\Gamma_+, R), \quad j^T(M) = M \boxtimes T\]

of (1.6), with $\mathcal{C} = \text{pt}$ being the point category, and $\mathcal{C}_1 = \Gamma_+$. Then the decomposition (5.3) induces an isomorphism

\[(5.5) \quad j_1! \cong j_0! \oplus j^T,\]

so that $j^T$ has an obvious right-adjoint functor $r^T : \mathcal{D}(\Gamma_+, R) \to \mathcal{D}(R)$ sending $E \in \mathcal{D}(\Gamma_+, R)$ to the direct summand of $E([1]_+)$ complementary to its direct summand $E([0]_+)$. Moreover, by (5.4), $T$ satisfies the assumptions of Lemma 1.1, so that $j^T$ has a left-adjoint functor

\[(5.6) \quad l^T : \mathcal{D}(\Gamma_+, R) \to \mathcal{D}(R).\]

Note that if we let $T' = j_1^* \mathbb{Z} \in \text{Fun}(\Gamma_+, \mathbb{Z})$, then the isomorphism $T([1]_+) \cong \mathbb{Z}$ extends by adjunction to a map $\varepsilon : T \to T'$. This induces a functorial map

\[j^T \to j^{T'} \cong j_1^*,\]

and by adjunction, we obtain a natural map

\[(5.7) \quad \varepsilon : j_1^* \to l^T.\]

**Lemma 5.5.** Proposition 5.4 holds for the pair $(\Gamma, \text{Inj})$. Moreover, the functor $j^T$ induces an equivalence

\[j^T : \mathcal{D}(R) \cong \mathcal{D}S^{\text{add}}_{\text{Inj}}(\Gamma, R) \subset \mathcal{D}S_{\text{Inj}}(\Gamma, R) \cong \mathcal{D}(\Gamma_+, R),\]

and the addivization functor

\[\text{Add} : \mathcal{D}(\Gamma_+, R) \to \mathcal{D}(R) = \mathcal{D}S^{\text{add}}_{\text{Inj}}(\Gamma, R)\]

coincides with the functor $l^T$ of (5.6).

**Proof.** One observes immediately that $r^T \circ j^T = \text{Id}$, so that $j^T$ is a fully faithful embedding, and moreover, an object $E \in \mathcal{D}(\Gamma_+, R)$ is additive if and only the adjunction map $j^T r^T(E) \to E$ is an isomorphism. This proves
the first claim. The second immediately follows from the definition, since \( l^T \) is left-adjoint to \( j^T \).

Note that more generally, given a small category \( C \), we have a natural functor

\[
j^T_C : D(C, R) \to D(C \times \Gamma_+, R)
\]

of (1.6), and by Lemma 1.1 it has a left-adjoint functor \( l^T_C \). If we denote by \( j_1 : C \to C \times \Gamma_+ \) the embedding sending \( c \in C \) to \( c \times [1]_+ \), then we have a natural map

\[
j_1^* \to l^T_C
\]

defined by the same procedure as the map (5.7).

5.3 The general case. Now consider an arbitrary small category \( C \) equipped with an admissible class of morphisms \( I \). Then on one hand, we can consider the product \( C \times \Gamma \), with the admissible class of maps \( I \times \text{Inj} \), and obtain the corresponding category

\[
D S_I \times \text{Inj}(C \times \Gamma, R)
\]

for any ring \( R \). On the other hand, we can consider the derived category \( D(S_I C \times \Gamma_+, R) \). Note that any object \( E \in D(S_I C \times \Gamma_+, R) \) gives in particular a functor from \( S_I C \) to the derived category \( D(\Gamma_+, R) \). Say that \( E \) is special if it sends special maps in \( S_I C \) to invertible maps, and let

\[
D S_I(C, \Gamma_+, R) \subset D(S_I C \times \Gamma_+, R)
\]

be the full subcategory spanned by special objects. Then the functor \( q \) of (4.10) defines a pullback functor

\[
q^* : D S_I(C, \Gamma_+, R) \to D S_I \times \text{Inj}(C \times \Gamma, R),
\]

and we observe the following.

**Lemma 5.6.** The functor \( q^* \) of (5.10) is an equivalence of categories.

**Proof.** Same as Corollary 4.17.

We can now define the additivization functor required in Proposition 5.4. Assume that the pair \( (C, I) \) has finite coproducts in the sense of Definition 5.1. Then sending an object \( c \times S \in C \times \Gamma \) to the coproduct of copies of the object \( c \in C \) numbered by elements of the finite set \( S \) gives a functor
\[ m : \mathcal{C} \times \Gamma \to \mathcal{C}, \text{ and the conditions of Definition 5.1 insure that } m \text{ is a morphism} \]

\[ m : \langle \mathcal{C} \times \Gamma, I \times \text{lnj} \rangle \to \langle \mathcal{C}, I \rangle \]

in the sense of Definition 4.11. Therefore we have a pullback functor

\[ S(m)^* : \mathcal{DS}_I(C, R) \to \mathcal{DS}_{I \times \text{lnj}}(\mathcal{C} \times \Gamma, R) \cong \mathcal{DS}_I(C, \Gamma_+, R). \]

We also have the embedding \( e : \mathcal{C} \to \mathcal{C} \times \Gamma \) sending \( c \in \mathcal{C} \) to \( c \times [1] \), and \( e \) is a morphism

\[ e : \langle \mathcal{C}, I \rangle \to \langle \mathcal{C} \times \Gamma, I \times \text{lnj} \rangle, \]

so that we have a pullback functor

\[ S(e)^* : \mathcal{DS}_{I \times \text{lnj}}(\mathcal{C} \times \Gamma, R) \to \mathcal{DS}_I(\mathcal{C}, R). \]

Since \( m \circ e \cong \text{Id} \), the composition \( S(e)^* \circ S(m)^* \) is the identity functor.

On the other hand, the functor \( j_{\mathcal{I}C}^T : \mathcal{D}(\mathcal{S}_I \mathcal{C}, R) \to \mathcal{D}(\mathcal{S}_I \mathcal{C} \times \Gamma_+, R) \) obviously sends \( \mathcal{DS}_I(\mathcal{C}, R) \) into \( \mathcal{DS}_I(\mathcal{C}, \Gamma_+, R) \), and by Lemma 1.1, the adjoint functor \( l_{\mathcal{I}C}^T \) also sends special objects into special objects, thus induces a functor

\[ L^T : \mathcal{DS}_I(\mathcal{C}, \Gamma_+, R) \to \mathcal{DS}_I(\mathcal{C}, R). \]

We define an endofunctor \( \text{Add} \) of the category \( \mathcal{DS}_I(\mathcal{C}, R) \) by

\[ \text{Add} = L^T \circ S(m)^*. \]

We note that this functor is right-exact with respect to the standard \( t \)-structures. Moreover, the embedding \( j_1 \) of (5.3) coincides with the composite

\[ q \circ S(e) : \mathcal{S}_I \mathcal{C} \to \mathcal{C} \times \Gamma_+, \]

so that \( j_1^* \circ S(m)^* \cong S(e)^* \circ S(m)^* \) is the identity functor, and the canonical map (5.3) induces a natural map

\[ (5.13) \quad \varepsilon : \text{Id} \cong j_1^* \circ S(m)^* \to \text{Add}. \]

Then Proposition 5.4 immediately follows from the following fact.

**Lemma 5.7.** For any \( E \in \mathcal{DS}_I(\mathcal{C}, R) \), \( \text{Add}(E) \in \mathcal{DS}_I(\mathcal{C}, R) \) is additive in the sense of Definition 5.2, and if \( E \) were already additive, then the natural map \( \varepsilon : E \to \text{Add}(E) \) of (5.13) is an isomorphism.
Proof. For any object \( c \in \mathcal{C} \), restricting the morphism \( m \) of (5.11) to \( \{ c \} \times \Gamma \subset \mathcal{C} \times \Gamma \) gives a morphism

\[
m_c : (\Gamma, \text{Inj}) \to (\mathcal{C}, I),
\]

and by Lemma 5.3, \( E \in \mathcal{DS}_I(\mathcal{C}, R) \) is additive if and only if so are all the restrictions \( S(m_c)^*E \). By Lemma 1.1, the additivization functor \( \text{Add} \) commutes with the restriction functors \( S(m_c)^* \). We conclude that both claims of the lemma can be checked after applying the functors \( S(m_c)^* \), \( c \in \mathcal{C} \). In other words, we may assume right away that \( (\mathcal{C}, I) \cong (\Gamma, \text{Inj}) \). Then we can identify \( \mathcal{DS}_I(\mathcal{C}, R) \cong \mathcal{D}(\Gamma^+, R) \), the product functor \( m \) becomes the smash product functor

\[
m : \Gamma^+ \times \Gamma^+ \to \Gamma^+
\]
sending \( [n]^+ \times [n']^+ \) to their smash product \( [n]^+ \wedge [n']^+ \cong [nn']^+ \) of (1.19), and by Lemma 5.5 it suffices to prove that we have

\[
L^T \circ m^* \cong j^T \circ l^T,
\]

where \( L^T : \mathcal{D}(\Gamma^+ \times \Gamma^+, R) \to \mathcal{D}(\Gamma^+, R) \) is the functor \( l^T \) applied along the first variable. By adjunction, this is equivalent to checking that \( m_* \circ j^T \cong j^T \circ r^T \), or in other words, that

\[
(5.14) \quad m_*(T \boxtimes E) \cong T \otimes r^T(E)
\]

for any \( E \in \text{Fun}(\Gamma^+, R) \), where \( r^T \) is the right-adjoint functor to \( j^T \). To check this, it is convenient to use the equivalence

\[
P : \text{Fun}(\Gamma^+, R) \cong \text{Fun}(\Gamma^-, R),
\]

where \( \Gamma^- \) is the category of finite sets and surjective maps between them (with the same notation as here, this statement is proved in [K1, Lemma 3.10], and wrongly asserted to belong to folklore in [K1, Remark 3.11] – in fact it is also explicitly due to Pirashvili [P]). The equivalence sends \( T \) to the object \( t \in \text{Fun}(\Gamma^-, \mathbb{Z}) \) given by \( t([1]) = \mathbb{Z} \) and \( t([n]) = 0 \) for \( n \neq 1 \). The functor \( r^T \) becomes evaluation at \( [1] \), and \( j^T \) becomes the embedding sending \( M \) to \( M \otimes t \). And crucially, as shown in [K1, Lemma 3.18], the functor \( m_* \) becomes the restriction to the diagonal \( \Gamma^- \to \Gamma^- \times \Gamma^- \), so that (5.14) reads as

\[
t \otimes P(E) \cong t \otimes (P(E)([1])).
\]

Since \( t([n]) = 0 \) for \( n \neq 1 \), this is obvious. \qed

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6 Derived Mackey profunctors.

We can now develop the derived version of the theory of Section 3. We assume given a group \( G \), and we consider the category \( \Gamma_G \) of finite \( G \)-sets. It has fibered products, so that Example 4.2 and Example 4.3 both apply, and produce two classes of admissible maps in \( \Gamma_G \) – all maps, and injective maps.

6.1 Pointed sets. We first consider the second class – that is, we equip \( \Gamma_G \) with the admissible class \( \text{Inj} \) of all injective maps. The category \( Q_{\text{Inj}} \Gamma_G \) of Definition 4.4 is then naturally equivalent to the category \( \Gamma_{G+} \) of finite pointed \( G \)-sets – that is, finite \( G \)-sets equipped with a distinguished \( G \)-fixed element. The equivalence is the same as in the case of the category of finite sets: it sends a pointed \( G \)-set \( S \) with the distinguished element \( o \in S \) to the complement \( S = S \setminus \{o\} \), and a pointed map \( f : S \to S' \) goes to the diagram (5.2). We have a natural embedding \( \Gamma_G \to \Gamma_{G+} \) sending a finite \( G \)-set \( S \) to the set \( S+ \) of (1.17), with \( G \) acting trivially on the added distinguished point \( o \). The category \( \Gamma_{G+} \) has finite coproducts given by (1.18), and every \( G \)-set \( S \) decomposes as

\[
S = [G/H_1]+ \vee \cdots \vee [G/H_n]+ = \left( \coprod_{1 \leq i \leq n} [G/H_i] \right) _+ ,
\]

where \( H_i \subseteq G, 1 \leq i \leq n \) are cofinite subgroups, and \( [G/H_i] \) are the corresponding \( G \)-orbits. For any \( S_1 \in \Gamma_{G+} \), we have a natural identification

\[
\Gamma_{G+}(S, S_1) = \prod_i \Gamma_{G+}([G/H_i]+, S_1) = \prod_i S_{1}^{H_i} ,
\]

where \( S_{1}^{H_i} \subseteq S_1 \) is the subset of \( H_i \)-fixed points. The pair \( (\Gamma_G, \text{Inj}) \) is discrete in the sense of Corollary 4.15, so that we have a canonical equivalence

\[
\mathcal{D}S_{\text{Inj}}(\Gamma_G, R) \cong \mathcal{D}(\Gamma_{G+}, R)
\]

for any ring \( R \).

Analogously, let \( \hat{\Gamma}_G \) be the category of \( G \)-sets admissible in the sense of Definition 3.11 with the admissible class \( \text{Inj} \) of all monomorphisms. Then the same equivalence identifies \( Q_{\text{Inj}} \hat{\Gamma}_G \) with the category \( \hat{\Gamma}_{G+} \) of admissible pointed \( G \)-sets. The category \( \hat{\Gamma}_{G+} \) also has coproducts, and any \( S \in \hat{\Gamma}_{G+} \) has a decomposition (6.1) although possibly with an infinite number of terms.
We have a natural embedding $\hat{\Gamma}_G \subset \hat{\Gamma}_{G+}$, $S \mapsto S_+$. The pair $\langle \hat{\Gamma}_G, \text{Inj} \rangle$ is discrete in the sense of Corollary 4.17 so that we have an equivalence
\[
\mathcal{D}S_{\text{Inj}}(\hat{\Gamma}_G, R) \cong \mathcal{D}(\hat{\Gamma}_{G+}, R)
\]
for any ring $R$.

Now fix a cofinite subgroup $H \subset G$, with the normalizer $N_H \subset G$ and the quotient $W = N_H/H$. Then the fixed points functor $\varphi^H$ of (2.11) tautologically extends to a functor
\[
\varphi^H : \hat{\Gamma}_{G+} \rightarrow \Gamma_{W+}
\]
sending $S$ to $S^N$. This functor preserves smash products, that is, we have
\[
(S_1 \wedge S_2)^H = S_1^H \wedge S_2^H.
\]
for any $S_1, S_2 \in \hat{\Gamma}_{G+}$. Moreover, assume that $H = N$ is normal, so that $N_H = G$ and $W$ is the quotient $G/N$. Then $\varphi^N$ has a left-adjoint $\lambda^N : \Gamma_{W+} \rightarrow \hat{\Gamma}_{G+}$ sending a $W$-set $S$ to the same same on which $G$ acts via the quotient map $G \rightarrow W$. Thus the pullback functor $\lambda^N*$ is right-adjoint to the pullback functor $\varphi^N*$, and since $\varphi^N \circ \lambda^N$ is obviously the identity functor, $\varphi^N*$ is a full embedding.

We will also need an explicit description of the left-adjoint functor
\[
L \varphi^N_! : \mathcal{D}(\hat{\Gamma}_{G+}, R) \rightarrow \mathcal{D}(\Gamma_{W+}, R).
\]
To obtain it, we use simplicial combinatorics of Subsection 1.4 and Subsection 1.5. So, we consider $n$-simplicial pointed admissible $G$-sets – that is, functors $X : (\Delta^n)_o \rightarrow \hat{\Gamma}_{G+}$ from the $n$-fold self-product $\hat{\Delta}^n$ to the category $\hat{\Gamma}_{G+}$.

**Definition 6.1.** The homology complex $C_*(X, E)$ of an $n$-simplicial pointed admissible $G$-set $X$ with coefficients in an object $E \in \mathcal{D}(\hat{\Gamma}_{G+}, R)$ is the complex
\[
C_*(X, E) = C_*(\hat{\Delta}^n, X^*E),
\]
where $X^* : \mathcal{D}(\hat{\Gamma}_{G+}, R) \rightarrow \mathcal{D}(\hat{\Delta}^n, R)$ is the pullback functor associated to $X : \hat{\Delta}^n \rightarrow \hat{\Gamma}_{G+}$.

**Lemma 6.2.** Assume given a pointed $n$-simplicial admissible $G$-set $X$ such that for any cofinite subgroup $H \subset G$, the fixed point set $X^H$ is contractible in the sense of Definition 1.3. Then for any $E \in \mathcal{D}(\hat{\Gamma}_{G+}, R)$, the map
\[
E([0]_+) \rightarrow C_*(X, E)
\]
induced by the distinguished point embedding \([0]_+ \to X\) is a quasiisomorphism.

Proof. Instead of an arbitrary \(E\), it suffices to consider the generators \(M_S\) of the category \(\mathcal{D}(\hat{\Gamma}_G+,R)\) given by (1.16), \(M\) an \(R\)-module, \(S \in \hat{\Gamma}_G+\) an admissible pointed \(G\)-set. Fix \(M\) and \(S\), and consider the \(n\)-simplicial set \(\hat{\Gamma}_G+(S,X)\) of (1.21). Then by (1.16), we have

\[
C_\ast(X,M_S) \cong C_\ast(\hat{\Gamma}_G+(S,X),\mathbb{Z}) \otimes M,
\]

the homology of the \(n\)-simplicial set \(\hat{\Gamma}_G+(S,X)\) with coefficients in \(M\). But \(S\) has the decomposition (6.1), and (6.2) gives an isomorphism

\[
\hat{\Gamma}_G+(S,X) \cong \prod_i X^{H_i}.
\]

Since all the terms \(X^{H_i}\) in the product are by assumption contractible, the product itself is contractible, and we are done by Lemma 1.4.

Definition 6.3. For any normal cofinite subgroup \(N \subset G\), an \(n\)-simplicial pointed \(G\)-set \(X \in (\Delta^n)^n\hat{\Gamma}_G+\) is \(N\)-adapted if

(i) the fixed points subset \(X^N\) is isomorphic to the constant \(n\)-simplicial set pointed \([1]_+\) with the trivial \(G\)-action, and

(ii) for any cofinite subgroup \(H \subset G\) not containing \(N\), the fixed points subset \(X^H \subset X\) is contractible in the sense of Definition 1.3.

Lemma 6.4. For any cofinite normal subgroup \(N \subset G\), there exists an admissible \(2\)-simplicial pointed \(G\)-set \(S\) that is \(N\)-adapted in the sense of Definition 6.3.

Proof. Let

\[
S = \prod_{N \subset G} [G/N]
\]

be the disjoint union of all \(G\)-orbits \([G/N]\), \(N \subset G\) a cofinite normal subgroup. Then \(S\) is an admissible \(G\)-set. Fix a cofinite normal subgroup \(N \subset G\), and let

\[
S_N = S \setminus S^N.
\]

Then \((S_N)^N\) is empty, and \((S_N)^H\) is not empty for any cofinite subgroup \(H \subset G\) not containing \(N\). Therefore the 2-simplicial set \(C(ES)\) obtained by combining Example 1.6 and Example 1.7 satisfies all the assumptions.

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Lemma 6.5. For any $n$-simplicial pointed admissible $G$-set $X$ adapted to $N$, any $S \in \hat{\Gamma}_{G+}$, and any $E \in D(\hat{\Gamma}_{G+}, R)$, the natural map

$$(6.7) \quad C_\ast(S^N \wedge X, E) \to C_\ast(S \wedge X, E)$$

induced by the embedding $S^N \to S$ is a quasiisomorphism.

Proof. As in the proof of Lemma 6.2, we may assume that $E = M_{S'}$, $M$ an $R$-module, $S' \in \Gamma_{G+}$ an admissible pointed $G$-set. Fix $M$ and $S'$, and for any $n$-simplicial pointed admissible $G$-set $Y$, consider the $n$-simplicial set $\hat{\Gamma}_{G+}(S', Y)$ of (1.21). Then by (1.16), we have

$$(6.8) \quad C_\ast(Y, M_{S'}) \cong C_\ast(\hat{\Gamma}_{G+}(S', Y), Z) \otimes M.$$ 

The pointed set $S'$ has the decomposition (6.1), and (6.2) gives an isomorphism

$$\hat{\Gamma}_{G+}(S', Y) \cong \prod_i Y_{H_i}.$$ 

Since $S$ is admissible, all but a finite number $H_1, \ldots, H_n$ of subgroups $H_i \subset G$ do not contain $N \subset G$. Denote by $Y$ the product of the terms corresponding to these subgroups, so that we have

$$(6.9) \quad \hat{\Gamma}_{G+}(S', Y) \cong Y \times \prod_{i=1}^n Y_{H_i}.$$ 

Then by the K"unneth formula, (6.8) gives a quasiisomorphism

$$(6.10) \quad C_\ast(Y, M_S) \cong M \otimes C_\ast(Y, Z) \otimes \bigotimes_{i=1}^n C_\ast(Y_{H_i}, Z).$$ 

Now, by Definition 6.3 (ii), for every cofinite $H \subset G$ not containing $N$, the fixed point set $X^H$ is contractible. Thus if we take $Y = S \wedge X$, then for any such $H$,

$$Y^H = (S \wedge X)^H = S^H \wedge X^H$$

is also contractible, and then $Y$, being the product of contractible pointed $n$-simplicial sets, is itself contractible. Then by Lemma 6.2, (6.10) reduces to a quasiisomorphism

$$C_\ast(S \wedge X, M_S) \cong M \otimes \bigotimes_{i=1}^n C_\ast(S^{H_i} \wedge X^{H_i}, Z),$$ 

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where the product is over those subgroups $H_i \subset G$ in (6.1) that contain $N$. But for such subgroups, $S^{H_i} \subset S$ lies in $S^N \subset S$. Therefore if we do the same reduction for $Y = S^N \land X$, then the result is exactly the same. \square

Now fix an $N$-adapted 2-simplicial admissible pointed $G$-set $X$ – for example, the one provided by Lemma 6.4 – and consider the functor

$$m_X : (\Delta^o)^2 \times \widehat{\Gamma}_{G^+} \to \widehat{\Gamma}_{G^+}$$

given by

$$m_X([n_1] \times [n_2] \times S) = X([n_1] \times [n_2]) \land S.$$ Let $p : (\Delta^o)^2 \times \widehat{\Gamma}_{G^+} \to \widehat{\Gamma}_{G^+}$ be the natural projection, and for any $E \in \mathcal{D}(\widehat{\Gamma}_{G^+}, R)$, let

$$\text{Av}_X(E) = L^* p \cdot m_X^* E.$$ Then $\text{Av}_X$ is an endofunctor of the category $\mathcal{D}(\widehat{\Gamma}_{G^+}, R)$, a sort of “averaging” over the pointed 2-simplicial $G$-set $X$. By base change, for any $S \in \widehat{\Gamma}_{G^+}$ and any $E \in \mathcal{D}(\widehat{\Gamma}_{G^+}, R)$, we have

$$\text{Av}_X(E)(S) \cong C_*(S \land X, E).$$

The natural embedding $[1]_+ \cong X^N \to X$ induces a functorial map $S \to S \land X$, so that we have a functorial map

$$\text{Id} \to \text{Av}_X.$$ Moreover, define a functor $\Phi^N : \mathcal{D}(\widehat{\Gamma}_{G^+}, R) \to \mathcal{D}(\widehat{\Gamma}_{W^+}, R)$ by

$$\Phi^N = \lambda^N \circ \text{Av}_X.$$ Then for any $S \in \widehat{\Gamma}_{W^+}$ and any $E \in \mathcal{D}(\widehat{\Gamma}_{G^+}, R)$, we have

$$\Phi^N(E)(S) = C_*(\lambda^N(S) \land X, E),$$

and for any $S \in \widehat{\Gamma}_{G^+}$ and any $E \in \mathcal{D}(\widehat{\Gamma}_{G^+}, R)$, we have

$$\varphi^N \Phi^N(E)(S) = C_*(S^N \land X, E).$$

The map $S^N \to S$ induces a map $\varphi^N \Phi^N \to \text{Av}_X$, and by Lemma 6.5 this map is a quasiisomorphism. Thus (6.11) induces by adjunction a map

$$L^* \varphi^N \to \Phi^N.$$

Proposition 6.6. The map (6.13) is an isomorphism of functors.
Proof. As in Lemma 6.2, we may assume $E = M_S$, $M$ an $R$-module, $S \in \hat{\Gamma}_{G^+}$ an admissible $G$-set with decomposition (6.1). Then by adjunction,

$$L^* \varphi_i^N M_S = M_{\varphi_i^N(S)} = M_{SN},$$

so that for any $\tilde{S} \in \Gamma_{W^+}$, we have

$$L^* \varphi_i^N M_S(\tilde{S}) = M \otimes \mathbb{Z} \left[ \Gamma_{W^+}(SN, \tilde{S}) \right] = M \otimes \bigotimes_i \mathbb{Z}[S^{H_i}],$$

where the product is taken over all subgroups $H_i \subset G$ in (6.1) that contain $N$. On the other hand, the value $\Phi^N(M_S)(\tilde{S})$ of the functor $\Phi^N$ can be computed by (6.12). Then as in the proof of Lemma 6.5, we have

$$\Phi^N(M_S)(\tilde{S}) \cong M \otimes \bigotimes_i C_i(\tilde{S}^{H_i} \wedge X^{H_i}, \mathbb{Z}),$$

where again, the product is taken over all subgroups $H_i \subset G$ in (6.1) that contain $N$. To finish the proof, it remains to notice that for each of these subgroups, we have

$$\tilde{S}^{H_i} \cong \tilde{S}^{H_i} \wedge [1]^+ \cong \tilde{S}^{H_i} \wedge X^{H_i}$$

by Definition 6.3 (i). □

6.2 Profunctors. Now equip the category $\Gamma_G$ with the admissible class of all maps, fix a ring $R$, and consider the category $\mathcal{DS}(\Gamma_G, R)$ of Definition 4.8. We have an embedding $\Gamma_G \cong (\mathcal{ST}_G)[1] \subset \mathcal{ST}_G$ and the corresponding restriction functor $\mathcal{DS}(\Gamma_G, R) \to \mathcal{D}(\Gamma_G, R)$. Note that every object $E$ in $\mathcal{D}(\Gamma_G, R)$ defines in particular a functor from $\Gamma_G$ to the additive category $\mathcal{D}(R)$.

Definition 6.7. An $R$-valued derived $G$-Mackey functor is an object $E \in \mathcal{DS}(\Gamma_G, R)$ such that the corresponding functor $\Gamma_G \to \mathcal{D}(R)$ is additive in the sense of Definition 2.3. The full subcategory in $\mathcal{DS}(\Gamma_G, R)$ spanned by derived Mackey functors is denoted by $\mathcal{DM}(G, R) \subset \mathcal{DS}(\Gamma_G, R)$.

We note that the pair $(\Gamma_G, \text{Id})$ has finite coproducts in the sense of Definition 5.1 and the additivity conditions of Definition 2.3 and Definition 5.2 are identically the same. Therefore we have

$$\mathcal{DM}(G, R) = \mathcal{DS}^{add}(\Gamma_G, R) \subset \mathcal{DS}(\Gamma_G, R).$$
and Proposition 5.4 insures that the embedding $\mathcal{D} \mathcal{M}(G,R) \subset \mathcal{D}\mathcal{S}(\Gamma_G,R)$ admits a left-adjoint additivization functor

(6.14) \quad \text{Add} : \mathcal{D}\mathcal{S}(\Gamma_G,R) \to \mathcal{D} \mathcal{M}(G,R).

The triangulated subcategory $\mathcal{D} \mathcal{M}(G,R) \subset \mathcal{D}\mathcal{S}(\Gamma_G,R)$ inherits a natural $t$-structure, and the additivization functor $\text{Add}$ is right-exact with respect the natural $t$-structures. The induced functor on the hearts $t$-structures is the additivization functor (2.6).

Analogously, consider the category $\widehat{\Gamma}_G$ of $G$-sets admissible in the sense of Definition 3.1. It also has fibered products, so that for every ring $R$, we have the category $\mathcal{D}\mathcal{S}(\widehat{\Gamma}_G,R)$ and the restriction functor

\[ \mathcal{D}\mathcal{S}(\widehat{\Gamma}_G,R) \to \mathcal{D}(\widehat{\Gamma}_G,R). \]

Every object $E \in \mathcal{D}(\widehat{\Gamma}_G,R)$ in turns gives a functor $\widehat{\Gamma}_G \to \mathcal{D}(R)$.

**Definition 6.8.** An $R$-valued derived $G$-Mackey profunctor is an object $E \in \mathcal{D}\mathcal{S}(\widehat{\Gamma}_G,R)$ such that the corresponding functor $\widehat{\Gamma}_G \to \mathcal{D}(R)$ is additive in the sense of Definition 3.2. The subcategory of derived Mackey profunctors is denoted by $\widehat{\mathcal{D}} \mathcal{M}(G,R) \subset \mathcal{D}\mathcal{S}(\widehat{\Gamma}_G,R)$.

The additivity condition of Definition 3.2 is preserved under truncation with respect to the natural $t$-structure on $\mathcal{D}\mathcal{S}(\widehat{\Gamma}_G,R)$, so that $\widehat{\mathcal{D}} \mathcal{M}(G,R) \subset \mathcal{D}\mathcal{S}(\widehat{\Gamma}_G,R)$ inherits a natural $t$-structure. We denote by

$$\widehat{\mathcal{D}} \mathcal{M}^< (G,R) = \bigcup_i \widehat{\mathcal{D}} \mathcal{M}^{<i} (G,R)$$

the full subcategory of objects bounded from above with respect to this $t$-structure. The pullback functor $q^*$ of (1.11) commutes with restrictions to $\widehat{\Gamma}_G^0$, so that we have a natural functor

$$q^* : \mathcal{D}(\widehat{\mathcal{M}}(G,R)) \to \widehat{\mathcal{D}} \mathcal{M}(G,R),$$

where $\mathcal{D}(\widehat{\mathcal{M}}(G,R))$ is the derived category of the abelian category $\widehat{\mathcal{M}}(G,R)$ of $R$-valued $G$-Mackey profunctors of Definition 3.2. By Lemma 4.10, this induces a fully faithful embedding

(6.15) \quad \widehat{\mathcal{M}}(G,R) \subset \widehat{\mathcal{D}} \mathcal{M}^{<0} (G,R) \subset \widehat{\mathcal{D}} \mathcal{M}(G,R)$

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that identifies the category $\widehat{\mathcal{M}}(G, R)$ with the heart of the natural $t$-structure on $\widehat{\mathcal{D}\mathcal{M}}(G, R)$. We denote by

$$\tau : \widehat{\mathcal{D}\mathcal{M}}^{-0}(G, R) \to \widehat{\mathcal{M}}(G, R)$$

the associated truncation functor.

As in the non-derived case, if $G$ is finite, all admissible $G$-sets are also finite, so that $\Gamma_G = \widehat{\Gamma}_G$ and $\mathcal{D}\mathcal{M}(G, R) = \mathcal{M}(G, R)$.

In the general case, we still have an embedding $\Gamma_G \to \widehat{\Gamma}_G$ and the restriction functor $\mathcal{D}S(\widehat{\Gamma}_G, R) \to \mathcal{D}S(\Gamma_G, R)$. This clearly sends objects additive in the sense of Definition 3.2 to objects additive in the sense of Definition 2.3 so that we have a natural forgetful functor

$$\mathcal{D}\mathcal{M}(G, R) \to \mathcal{M}(G, R).$$

This is compatible with the natural $t$-structures and induces the restriction functor (3.2) on their hearts.

The identity functor $\Gamma_G \to \Gamma_G$ defines a morphism

$$i : (\Gamma_G, \text{Inj}) \to (\Gamma_G, \text{Id})$$

in the sense of Definition 4.11 so that we have a restriction functor

$$S(i)^* : \mathcal{D}\mathcal{M}(G, R) \to \mathcal{D}\mathcal{S}_{\text{Inj}}(\Gamma_G, R) \cong \mathcal{D}(\Gamma_G^+, R).$$

Analogously, we have a morphism $i : (\widehat{\Gamma}_G, \text{Inj}) \to (\widehat{\Gamma}_G, \text{Id})$ and a restriction functor

$$\widehat{S}(i)^* : \mathcal{D}\widehat{\mathcal{M}}(G, R) \to \mathcal{D}\widehat{\mathcal{S}}_{\text{Inj}}(\widehat{\Gamma}_G, R) \cong \mathcal{D}(\widehat{\Gamma}_G^+, R).$$

**Definition 6.9.** The homology complex $C_*(X, E)$ of an $n$-simplicial pointed admissible $G$-set $X$ with coefficients in a derived Mackey profunctor $E \in \mathcal{D}\mathcal{M}(G, R)$ is the complex

$$C_*(X, E) = C_*(X, S(i)^* E),$$

where $S(i)^*$ is the restriction functor (6.20), and $C_*(X, S(i)^* E)$ is the homology complex of Definition 6.1.

While the homology complex of Definition 6.1 is obviously covariant with respect to $X$, the homology complex of Definition 6.9 is also contravariant: we have a natural map

$$f^* : C_*(X', E) \to C_*(X, E)$$
for any $E \in \hat{\mathcal{DM}}(G,R)$ and any map $f : X \to X'$ of $n$-simplicial pointed admissible $G$-sets. It also enjoys the following continuity properties that we will need later.

**Lemma 6.10.** (i) Assume given an $n$-simplicial pointed admissible $G$-set $X$ and an inverse system $\{E_i\}$ of derived Mackey profunctors $E_i \in \hat{\mathcal{DM}}(G,R)$. Then we have a natural identification

$$C_*(X, \text{Tel}(E_i)) \cong \text{Tel}(C_*(X,E_i)),$$

where Tel is the telescope of (1.26).

(ii) Assume given a derived Mackey profunctor $E \in \hat{\mathcal{DM}}(G,R)$ and an inverse system of subsets $X_{i+1} \subset X_i \subset X$ of an $n$-simplicial pointed admissible $G$-set $X$ with intersection $\overline{X} = \cap_i X_i$. Then we have a natural identification

$$C_*(\overline{X}, E) \cong \text{Tel}(C_*(X_i,E)).$$

**Proof.** For (i), note that we have $\text{Tel}(X^*E_i) \cong X^*\text{Tel}(E_i)$, and then note that for an inverse system $M_i$ of objects in $\mathcal{D}^{\leq n}(\Delta^n, R)$, with some fixed integer $n$ independent of $i$, we have

$$C_*(\Delta^n, \text{Tel}(M_i)) \cong \text{Tel}(C_*(\Delta^n,M_i)).$$

For (ii), observe that we have $\text{Tel}(X_i^*E) \cong \overline{X}^*E$ by (3.1), and use the same argument. $\square$

Assume now given a subgroup $H \subset G$, with normalizer $N_H \subset G$ and the quotient $W = N_H/H$. Then the functor $\varphi^H$ of (2.11) defines a morphism

$$\varphi^H : (\hat{\Gamma}_G, \text{Id}) \to (\hat{\Gamma}_W, \text{Id})$$

in the sense of Definition 4.11, so that we have a pullback functor

$$S(\varphi^H)^* : \mathcal{DS}(\hat{\Gamma}_W, R) \to \mathcal{DS}(\hat{\Gamma}_G, R).$$

This functor clearly preserves additivity in the sense of Definition 3.2, thus sends a derived Mackey profunctor to a derived Mackey profunctor. We also have the left-adjoint functor

$$S(\varphi^H)_! : \mathcal{DS}(\hat{\Gamma}_G, R) \to \mathcal{DS}(\hat{\Gamma}_W, R)$$

provided by Corollary 4.15.
Definition 6.11. The inflation functor

\[ \text{Infl}^H : \hat{\mathcal{DM}}(W, R) \to \hat{\mathcal{DM}}(G, R) \]

is the functor induced by the pullback functor \((6.22)\).

For any normal subgroup \(N \subset G\) with the quotient \(W = G/N\), say that an object \(E \in \hat{\mathcal{DS}}(\hat{\Gamma}G, R)\) is supported at \(N\) if for any admissible \(G\)-set \(S\), the natural map \(E(S) \to E(S^N)\) is a quasiisomorphism. Then by definition, \(\text{Infl}^N\) factors through the full subcategory

\[ (6.24) \quad \hat{\mathcal{DM}}_N(G, R) \subset \hat{\mathcal{DM}}(G, R) \]

spanned by objects \(E \in \hat{\mathcal{DM}}(G, R)\) supported at \(N\). Note that by \((3.1)\), a derived Mackey profunctor \(E \in \hat{\mathcal{DM}}(G, R)\) is supported at \(N\) if and only if \(M([G/H]) = 0\) for any cofinite \(H \subset G\) not contained in \(N \subset G\).

Lemma 6.12. For any normal subgroup \(N \subset G\), the inflation functor induces an equivalence

\[ \text{Infl}^N : \hat{\mathcal{DM}}(W, R) \cong \hat{\mathcal{DM}}_N(G, R). \]

Proof. More generally, we will prove that \(S(\varphi^N)^*\) induces an equivalence between \(\mathcal{DS}(\hat{\Gamma}W, R)\) and the full subcategory in \(\mathcal{DS}(\hat{\Gamma}G, R)\) spanned by objects supported at \(N\). This is equivalent to proving that the adjunction map

\[ S(\varphi^N)_! S(\varphi^N)^* : E \to E \]

is an isomorphism for any \(E \in \mathcal{DS}(\hat{\Gamma}W, R)\), and the adjunction map

\[ E \to S(\varphi^N)^* S(\varphi^N)_! E \]

is an isomorphism for any \(E \in \mathcal{DS}(\hat{\Gamma}G, R)\) supported at \(N\). But by Proposition \((4.19)\) we have

\[ (6.25) \quad S(i)^* \circ S(\varphi^N)_! \cong L^* \varphi^N_i \circ S(i)^*, \]

where \(S(i)^*\) are the canonical restriction functors \((6.19), (6.20)\), and \(L^* \varphi^N_i\) is the functor \((6.6)\). Since we obviously also have \(S(i)^* \circ S(\varphi^N)^* \cong \varphi^{N*} \circ S(i)^*\), it suffices to prove both claims after restriction to \(\hat{\Gamma}G_+\). But then, it is equivalent to proving that \(\varphi^{N*}\) induces an equivalence between \(\mathcal{D}(\hat{\Gamma}W_+, R)\) and the full subcategory in \(\mathcal{D}(\hat{\Gamma}G_+, R)\) spanned by objects supported at \(N\). Since \(\varphi^N\) has a fully faithful left adjoint \(\lambda^N\), this is obvious. \(\square\)
Now assume that the subgroup $H \subset G$ is cofinite. Then we have the adjoint pair of functors $\rho^H, \gamma^H$ of $\text{(3.15)}$, and we note that this is in fact a situation of Example $\text{4.12}$ with $\mathcal{C} = \hat{\Gamma}_G$, $c = [G/H]$, $c' = [G/G]$, and $f : c \to c'$ the natural projection. Indeed, for any a $G$-set $S$ equipped with a map $f : S \to [G/H]$, the preimage $f^{-1}(e) \subset S$ of the image $e \in [G/H]$ of the unity element $e \in G$ is an $H$-set, and sending $f : S \to [G/H]$ to $f^{-1}(e)$ gives an equivalence of categories

$$\hat{\Gamma}_G / [G/H] \cong \hat{\Gamma}_H.$$ 

Under this equivalence, $f^*$ of Example $\text{4.12}$ becomes the restriction functor $\rho^H$, and $f_!$ becomes its left-adjoint functor $\gamma^H$. Then $\text{(4.15)}$ becomes the isomorphism

$$\text{(6.26)} \quad S(\gamma^H)^* \cong S(\rho^H)_!,$$

and this functor preserves additivity in the sense of Definition $\text{3.2}$.

**Definition 6.13.** The categorical fixed points functor

$$\Psi^H : \hat{\mathcal{DM}}(G, R) \to \hat{\mathcal{DM}}(H, R)$$

is the functor induced by the functor $\text{(6.26)}$.

As in the underived situation, the centralizer $Z_H \subset G$ of any cofinite group $H \subset G$ acts on the functor $\rho^H$ and its adjoint $\gamma^H$, thus on $\Psi^H = S(\gamma^H)^*$, so that it can be promoted to a functor

$$\text{(6.27)} \quad \tilde{\Psi}^H : \mathcal{M}(G, R) \to \mathcal{M}(H, R[Z_H]),$$

where $R[Z_H]$ is the group algebra of the centralizer.

Now we note that for any cofinite subgroup $H \subset G$ with normalizer $N_H \subset G$ and quotient $W_H = N_H / N$, we have $\varphi^H \cong \varphi^H \circ \rho^{N_H}$, so that

$$\text{(6.28)} \quad S(\varphi^H)_! \cong S(\varphi^H)_! \circ S(\rho^{N_H})_! \cong S(\varphi^H)_! \circ S(\gamma^{N_H})^*.$$ 

**Lemma 6.14.** For any cofinite normal subgroup $N \subset G$ with the quotient $W = G/N$, the functor $S(\varphi^N)_!$ of $\text{(6.23)}$ sends $\hat{\mathcal{DM}}(G, R) \subset \mathcal{DS}(\hat{\Gamma}_G, R)$ to $\mathcal{DM}(W, R) \subset \mathcal{DS}(\Gamma_W, R)$.

**Proof.** By $\text{(6.28)}$, we may replace $G$ with $N_H$ – in other words, we may assume right away that $H = N \subset G$ is normal, and $W = G/N$. Then as
in the proof of Lemma 6.12, Proposition 4.19 provides a canonical isomorphism (6.25), and by Proposition 6.6, for any \( E \in \mathcal{DS}^{\hat{\Gamma}_G, R} \), the values \( S(\varphi^N) : (E)(S) \), \( S \in \Gamma_W \) can be computed by (6.12), where the right-hand side is the homology complex of Definition 6.9. Then the claim becomes obvious – we have

\[
(S \sqcup S')_+ \land X \cong (S_+ \land X) \lor (S'_+ \land X)
\]

for any two \( G \)-sets \( S_1, S_2 \).

\[\square\]

**Definition 6.15.** The *geometric fixed points functor*

\[
\Phi^H : \hat{\mathcal{DM}}(G, R) \to \mathcal{DM}(W, R)
\]

is the functor induced by the functor (6.23).

By definition, the functor \( \Phi^H \) is left-adjoint to the inflation functor \( \text{Infl}^H \) of Definition 6.11. For any subgroup \( H' \subset H \subset G \), the isomorphism (2.12) induces an isomorphism

\[
(6.29) \quad \Phi^{H'} \circ \Psi^H \cong \Psi^{W_{H'}} \circ \Phi^H,'
\]

a derived version of the isomorphism (2.13).

**Lemma 6.16.** The functors \( \text{Infl}^H \) of Definition 6.11, \( \Psi^H \) of Definition 6.13 are exact with respect to the standard t-structures, and induce the functors \( \text{Infl}^H \), \( \Psi^H \) of (3.5) resp. (3.16) on their hearts. The functor \( \Phi^H \) of Definition 6.15 is right-exact with respect to the standard t-structures, and induces the functor \( \Phi^H \) of (3.5) on their hearts (that is, we have \( \tau \circ \Phi^H \cong \Phi^H \circ \tau \), where \( \tau \) is the truncation functor (6.16)).

**Proof.** The first claim is obvious, the second immediately follows by adjunction. \(\square\)

7 Mackey functors and representations.

If the group \( G \) is finite, then any admissible \( G \)-set \( S \) is finite, so that derived Mackey profunctors are the same as derived Mackey functors studied in [K2]. However, most of the proofs in [K2] are rather involved. Throughout this section, we assume that \( G \) is a finite group, and we reprove some of the results from [K2] using the technology we have developed, especially Proposition 6.6.
7.1 Fixed points as representations. Fix a finite group $G$. For any subgroup $H \subset G$, denote

\[
\Phi^H = \Phi^H \circ \Psi^H : \mathcal{D}M(G, R) \to \mathcal{D}(R),
\]

and let

\[
\Phi_q = \bigoplus_{H \subset G} \Phi^H : \mathcal{D}M(G, R) \to \prod_{H \subset G} \mathcal{D}(R)
\]

be the sum of the functors $\Phi^H$ over all conjugacy classes of subgroups $H \subset G$.

**Lemma 7.1.** (i) If the image $\Phi_q(M)$ of a derived Mackey functor $M \in \mathcal{D}M(G, R)$ under the functor (7.2) lies in $\prod_H \mathcal{D}^{\leq n}(R)$ for some integer $n$, then $M$ lies in $\mathcal{D}M^{\leq n}(G, R) \subset \mathcal{D}M(G, R)$.

(ii) The functor $\Phi_q$ of (7.2) is conservative (that is, if $\Phi_q(f)$ is invertible, then $f$ is invertible).

**Proof.** For (i), let $M' = \tau_{> n} M$ be the truncation of $M$ with respect to the standard $t$-structure on $\mathcal{D}M(G, R)$. We need to prove that $M' = 0$. By (6.29) and induction on cardinality of $G$, we may assume that $\Psi^H M' = 0$ for any proper subgroup $H \subset G$. Then $M'$ is supported at $G$, and by Lemma 6.12, $M' \cong \text{Infl}^G(\Phi^G(M'))$ and $\Phi^G(M') \cong M'([G/G])$. In particular, $\Phi^G(M')$ is non-trivial only in homological degrees $> n$. Since the functor $\Phi^G$ is left-exact with respect to the standard $t$-structure, the natural map $\Phi^G M \to \Phi^G M'$ is an isomorphism in these homological degrees, and since by assumption $\Phi^G M = \Phi^G M'$ lies in $\mathcal{D}^{\leq n}(R)$, we have $\Phi^G M = 0$ and $M' \cong \text{Infl}^G(\Phi^G(M')) = 0$.

For (ii), let $M$ be the cone of the map $f$. Then $\Phi_q(M) = 0$, so that $M$ must lie in $\mathcal{D}M^{\leq n}(G, R)$ for any integer $n$. Therefore $M = 0$, and $f$ is invertible. □

By (6.29), we have $\Phi^H \cong \Psi^{(e)} \circ \Phi^H$, where $\{e\} \subset W_H$ is the trivial subgroup in the quotient $W_H = N_H/H$, $N_H \subset G$ the normalizer of $H$. We can promote $\Psi^{(e)}$ to the functor $\tilde{\Psi}^{(e)}$ of (5.27), and this promotes $\Phi^H$ to a functor

\[
\tilde{\Phi}^H = \tilde{\Psi}^{(e)} \circ \Phi^H : \mathcal{D}M(G, R) \to \mathcal{D}(R[W_H]).
\]

**Lemma 7.2.** For any $H \subset G$, the functor $\tilde{\Phi}^H$ of (7.3) admits a fully faithful right-adjoint functor

$$R_H : \mathcal{D}(R[H]) \to \mathcal{D}M(H, R).$$
Proof. The functor $\overline{\Phi}^H = S(\varphi^H \circ \rho^H)$ has an obvious right-adjoint $\overline{R}^H = S(\varphi^H \circ \rho^H)^*$. Explicitly, if we consider the object $T_H \in M(G, Z)$ given by $T_H(S) = Z[S^H], \quad S \in \Gamma_G,$ then we have $\overline{R}^H(M) = M \otimes T$ for any $M \in D(R)$. To promote it to a right-adjoint functor $R^H$, observe that $W = W_H$ acts on $T$, so that it can be considered as an object $T \in D(pt_W \times S\Gamma_G; Z)$, where $pt_W$ is the groupoid with one object with automorphism group $W$. Then $R^H$ is given by

$$R^H(M) = R^*\pi_2*(\pi_1^*M \otimes T),$$

where $\pi_1 : pt_W \times S\Gamma_G \to pt_W$, $\pi_2 : pt_W \times S\Gamma_G \to S\Gamma_G$ are the natural projections. Explicitly, for any $S \in \Gamma_G$, we have

$$R^H(M)(S) \cong C^*(W, M \otimes T(S)) = C^*(W, M[S^H]).$$

This implies that for any $S \in \Gamma_H$, we have

$$R^H(M)(S) \cong C^*(W, M \otimes T(S)) = C^*(W, M[S^H]).$$

However, points in the orbit $G/H$ fixed under $H$ correspond to elements $g \in G$ that normalize $H$, so that we have a natural identification $[G/H]^H \cong [N_H/H]^H \cong W$. Then by the definition of the functor $\gamma^H$, we have

$$\gamma^H(S)^H \cong S^H \times W$$

for any $S \in \Gamma_H$. Therefore (7.6) implies that $\Psi^H(R^H(M)) \in D\mathcal{M}(H, R)$ is supported at $H$, and

$$(\Psi^H \circ R^H)(M)([H/H]) \cong C^*(W, M[W]) \cong M,$$

so that $\overline{\Phi}^H(R^H(M)) = \Phi^H(\Psi^H(R^H(M))) \cong M$. Thus the adjunction map $\overline{\Phi}^H(R^H(M)) \to M$ is an isomorphism. □

**Lemma 7.3.** Any object $M \in D\mathcal{M}(G, R)$ is a finite iterated extension of objects of the form $R^H(M_H)$, $M_H \in D(R[W_H]), H \subset G$ a subgroup.

**Proof.** For any $M \in D\mathcal{M}(G, R)$, denote by $\text{Supp}(M)$ the set of all conjugacy classes of subgroups $H \subset G$ such that $M([G/H]) \neq 0$ for some subgroup $H' \subset H$. Take an object $M \in D\mathcal{M}(G, R)$, and choose subgroup $H \in \text{Supp}(M)$ that is minimal by inclusion – that is, no proper
subgroup $H' \subset H$ lies in Supp$(M)$. Let $M'$ be the cone of the adjunction map $M \to R_H(\Phi^H(M))$. Then by (7.5), Supp$(R_H(\Phi^H(M)))$ is contained in the set of subgroups $H' \supset H$, and in particular, Supp$(R_H(\Phi^H(M)))$ lies inside Supp$(M)$. Then Supp$(M')$ is also contained in Supp$(M)$. However, since $R_H$ is fully faithful, $H$ does not lie in Supp$(M')$, so that the inclusion Supp$(M') \subset$ Supp$(M)$ is strict. By induction on the cardinality of Supp$(M)$, this finishes the proof. □

7.2 Maximal Tate cohomology. By Lemma 7.3 and Lemma 7.2, the category $\mathcal{D}M(G, R)$ is an iterated extension of full subcategories $\mathcal{D}(\mathbb{Z}H)$, $H \subset G$, $W_H = N_H/H$. To describe the gluing data between these subcategories, one needs to compute the composition functors

$$E^H_{H'} = \tilde{\Phi}^{H'} \circ R_H : \mathcal{D}(R[H]) \to \mathcal{D}(R[H'])$$

for pairs of different subgroups $H, H' \subset G$. This can be done using a certain natural generalization of Tate cohomology of finite groups.

Definition 7.4. A finitely generated $\mathbb{Z}[G]$-module $M$ is induced if $M = i_H(M')$, where $H \subset G$ is a proper subgroup, $M'$ is a finitely generated $\mathbb{Z}[H]$-module, and $i_H : \mathbb{Z}[H]$-mod $\to \mathbb{Z}[G]$-mod is the induction functor left-adjoint to the obvious restriction functor $\mathbb{Z}[G]$-mod $\to \mathbb{Z}[H]$-mod. For any finitely generated $\mathbb{Z}[G]$-module $M$, the maximal Tate cohomology groups $\tilde{H}^*(G, M)$ are given by

$$\tilde{H}^*(G, M) = \mathrm{RHom}^*_{\mathcal{D}(\mathbb{Z}[G])}(\mathbb{Z}, M),$$

where $\mathcal{D}(\mathbb{Z}[G])$ is the bounded derived category of the category of finitely generated $\mathbb{Z}[G]$-modules, and $\mathcal{D}(\mathbb{Z}[G]) \subset \mathcal{D}(\mathbb{Z}[G])$ is the smallest Karoubi-closed full triangulated subcategory containing all induced modules.

Remark 7.5. The difference with the usual Tate cohomology is that one takes the quotient not by the subcategory $\mathcal{D}^b(\mathbb{Z}[G])$ of perfect complexes of $\mathbb{Z}[G]$-modules but by the larger subcategory $\mathcal{D}(\mathbb{Z}[G]) \subset \mathcal{D}(\mathbb{Z}[G])$ of induced modules.

We note that since the category $\mathcal{D}(\mathbb{Z}[G])$ is small, taking the quotient in Definition 7.4 presents no problem. In effect, one considers the category $I$ of objects $V \in \mathcal{D}(\mathbb{Z}[G])$ equipped with a map $V \to \mathbb{Z}$ whose cone lies in $\mathcal{D}(\mathbb{Z}[G])$, and one has

$$\tilde{H}^i(G, M) = \lim_{V \in I} \mathrm{Hom}(V, M[i]) = \lim_{V \in I} H^i(G, M \otimes V^*), \quad i \in \mathbb{Z},$$

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where $V^* = \text{Hom}(V, \mathbb{Z}) \in D^b_f(\mathbb{Z}[G])$ is the object dual to $V$. Since the index category $I$ is filtered, the limit in the right-hand side of (7.8) is an exact functor. One can then use (7.8) to define maximal Tate cohomology with coefficients.

**Definition 7.6.** For any $M \in D(R[G])$, the maximal Tate cohomology modules $\check{H}^*(G, M)$ are given by

$$\check{H}^*(G, M) = \lim_{V \in I \rightarrow} H^*(G, M \otimes V^*),$$

where $I$ and $V^*$ are as in (7.8).

One can further refine this to obtain a maximal Tate cohomology object $\check{C}^*(G, M) \in D(R)$ with homology modules $\check{H}^*(G, M)$. To do this, it is convenient to use the following technical gadget. Assume given a complex $V_\cdot \in C_{\geq 0}(\mathbb{Z}[G]\text{-mod})$ of $\mathbb{Z}[G]$-modules concentrated in non-negative homological degrees, and let $V \in \text{Fun}(\Delta^o, \mathbb{Z}[G]) \cong \text{Fun}(pt \times \Delta^o, \mathbb{Z})$ be the simplicial $\mathbb{Z}[G]$-module corresponding to $V_\cdot$ under the Dold-Kan equivalence (1.22). Then for any $M \in D(R)$, we can consider the object

$$\check{C}^*(G, V_\cdot, M) = C_\cdot(\Delta^o, R^* \pi_2^*(V \otimes \pi_1^! M)) \in D(R),$$

where $\pi_1 : pt \times \Delta^o \rightarrow pt$, $\pi_2 : pt \times \Delta^o \rightarrow \Delta^o$ are the natural projections. This object is functorial in $M$ and in $V_\cdot$—for any two complexes $V_\cdot, V'_\cdot$, any map $f : V_\cdot \rightarrow V'_\cdot$ induces a functorial map

$$\check{C}^*(G, V_\cdot, -) \rightarrow \check{C}^*(G, V'_\cdot, -).$$

**Definition 7.7.** A complex $P_\cdot$ of finitely generated $\mathbb{Z}[G]$-modules is maximally adapted if

(i) $P_i = 0$ for $i < 0$, $P_0 = \mathbb{Z}$, and for every $i \geq 1$, $P_i$ is flat over $\mathbb{Z}$ and induced in the sense of Definition 7.4, and

(ii) for any proper subgroup $H \subset G$, the complex $P_\cdot$ is contractible as a complex of $\mathbb{Z}[H]$-modules.

**Example 7.8.** For any $n$-simplicial admissible pointed $G$-set $X$ that is $G$-adapted in the sense of Definition 6.3, the complex $P_\cdot = C_\cdot(X, \mathbb{Z})$ with its natural $G$-action is maximally adapted in the sense of Definition 7.7.
Lemma 7.9. For any maximally adapted complex $P_\cdot$, and any object $M \in \mathcal{D}(R[G])$, the homology modules of the object $\check{C}^*(G, P_\cdot, M)$ of (7.9) are naturally identified with the maximal Tate cohomology modules $\check{H}^*(G, M)$ of Definition 7.6. For any two maximally adapted complexes $P_\cdot, P'_\cdot$ and any map $f : P_\cdot \to P'_\cdot$, identical in degree 0, this identification commutes with the map (7.10).

Proof. For any object $E \in \mathcal{D}(\Delta^\circ, R)$ represented by a complex of simplicial $R$-modules, we can apply the Dold-Kan equivalence termwise and obtain a bicomplex $E_{\cdot \cdot}$. The sum-total complex of this bicomplex then represents the homology object $C^*(\Delta^\circ, E)$. If the bicomplex is of finite length in either of the two directions, then the sum-total complex is the same as the product-total complex. Therefore for a complex $V_\cdot \in C_{\geq 0}(\mathbb{Z}[G]-\text{mod})$ of finite length, we have a natural isomorphism

$$\check{C}^*(G, V_\cdot, M) \cong C^*(G, M \otimes C^*(\Delta^\circ, V)) \cong C^*(G, M \otimes V_\cdot).$$

For a general complex $V_\cdot$, we can consider its stupid filtration $F^iV_\cdot$ given by

$$F^iV_i = \begin{cases} V_i, & i \leq l, \\ 0, & i > l. \end{cases}$$

Then $V_\cdot = \lim \limits_{\leftarrow} F^iV_\cdot$, and since homology commutes with direct limits, we have an identification

$$(7.11) \quad \check{C}^*(G, V_\cdot, M) \cong \lim \limits_{\leftarrow} C^*(G, M \otimes F^iV_\cdot).$$

Since the limit is filtered, its homology modules are then naturally identified with the corresponding direct limits of the homology groups $H^*(G, M \otimes F^iV_\cdot)$. Now take a maximally adapted complex $P_\cdot$, and consider the double limit

$$\lim \lim \limits_{\leftarrow} H^*(G, M \otimes F^iP_\cdot \otimes V^\cdot).$$

By Definition (7.7) (i), we have $\check{H}^*(G, M \otimes P_i) = 0$ for any $i \geq 1$, so that the natural embedding

$$\lim \limits_{\leftarrow} H^*(G, M \otimes F^iP_\cdot) \to \lim \lim \limits_{\leftarrow} H^*(G, M \otimes F^iP_\cdot \otimes V^\cdot)$$

is an isomorphism. By Definition (7.7) (ii), the limit (7.11) vanishes for any induced $M = i_H(M')$, $M' \in \mathcal{D}(R[H])$, $H \subset G$ a proper subgroup. By the
projection formula, for any \( M \in \mathcal{D}(R[G]) \) and \( V' \in \mathcal{D}_f^b(Z[G]) \), the product \( M \otimes V \) is a direct summand of a finite iterated extension of objects of this type, so that the natural embedding
\[
\lim_{\nu \leq l} H^*(G, M \otimes V^*) \to \lim_{\nu \leq l} H^*(G, M \otimes F^j P_\ast \otimes V^*)
\]
induced by the embedding \( Z = P_0 \to P_\ast \) is also an isomorphism. This finishes the proof. \( \Box \)

Now note that for any two maximally adapted complexes \( P_\ast, P'_\ast \), their product \( P_\ast \otimes P'_\ast \) is also maximally adapted, and the embeddings \( Z = P_0 \to P_\ast, Z = P'_0 \to P'_\ast \) induce natural maps
\[
P_\ast \to P_\ast \otimes P'_\ast, \quad P'_\ast \to P_\ast \otimes P'_\ast.
\]
The corresponding maps \((7.10)\) then provide functorial quasiisomorphisms
\[
\tilde{C}^*(G, P_\ast, M) \cong \tilde{C}^*(G, P_\ast \otimes P'_\ast, M) \cong \tilde{C}^*(G, P'_\ast, M).
\]
Thus up to a quasiisomorphism, \( C_\ast(G, P_\ast, -) \) does not depend on the choice of a maximally adapted complex \( P_\ast \), so that we can drop it from notation and obtain a maximal Tate cohomology functor
\[
(7.12) \quad \tilde{C}^*(G, -) : \mathcal{D}(R[G]) \to \mathcal{D}(R).
\]
Strictly speaking, this functor is only defined up to a non-canonical isomorphism, but we will ignore this.

Finally, we record one further refinement of maximal Tate cohomology. For any subgroup \( H \subset G \) and any \( M \in \mathcal{D}(R[G]) \), the cohomology complex \( C^*(H, R) \) carries a natural action of the group \( W_H = N_H/H, N_H \subset G \) the normalizer of \( H \subset G \), so that \( C^*(H, -) \) is a functor from \( \mathcal{D}(R[G]) \) to \( \mathcal{D}(R[W_H]) \). To obtain a similar maximal Tate cohomology functor
\[
(7.13) \quad \tilde{C}^*(G, -) : \mathcal{D}(R[G]) \to \mathcal{D}(R[W_H]),
\]
it suffices to choose a maximally adapted complex \( P_\ast \) of \( H \)-modules such that the \( H \)-action is extended to an action of \( N_H \). Choosing an \( H \)-adapted pointed \( n \)-simplicial \( N_H \)-set \( X \) in Example \( 7.8 \) does the job.

### 7.3 Gluing data

We can now describe the gluing functors \( E_{H'}^H \) of \((7.7)\). For any two subgroups \( H, H' \subset G \), denote
\[
(7.14) \quad c(H', H) = \text{Hom}_G([G/H], [G/H']) \cong [G/H']^H.
\]
The group \( W_H = \text{Aut}_G([G/H]) \) acts on this set on the left, and the group \( W'_H = \text{Aut}_G([G/H']) \) acts on the right. Denote by
\[
(7.15) \quad \overline{\tau}(H', H) = W_H \backslash c(H', H)/W_{H'}
\]
the double quotient. Moreover, assume that \( H' \subset G \) lies in the normalizer \( N_H \subset G \), and consider the intersection \( N^H_{H'} = N_H \cap N_{H'} \subset G \). Then \( H' \) lies in \( N^H_{H'} \) as a normal subgroup, so that we can consider the subgroup \( W^H_{H'} = N^H_{H'}/H' \subset W_{H'} = N_{H'}/H' \), and the associated induction functor
\[
(7.16) \quad \iota : \mathcal{D}(R[W^H_{H'}]) \to \mathcal{D}(R[W_{H'}]).
\]
By definition, the functor \( \tilde{\Phi}^{H'} \) only depends on \( H' \) up to a conjugation, so that the following results gives a complete description of the functors \( E^H_{H'} \).

**Proposition 7.10.** For any two subgroups \( H, H' \subset G \), \( H \neq H' \), and an object \( M \in \mathcal{D}(R[W_H]) \), \( E^H_{H'}(M) = 0 \) unless \( H' \) is conjugate to a subgroup in \( N_H \) containing \( H \subset N_H \). If \( H \subset H' \subset N_H \), then we have a natural isomorphism
\[
E^H_{H'}(M) \cong \iota(\hat{C}^*(H'/H, M)) [\overline{\tau}(H', H)],
\]
where \( \hat{C}^*(-,-) \) is the refined maximal Tate cohomology functor \( (7.13) \), \( \iota \) is the induction functor \( (7.16) \), and \( \overline{\tau}(H', H) \) is the finite set \( (7.15) \).

**Proof.** Take some \( M \in \mathcal{D}(R[W_H]) \). Assume first that \( H' = G \), so that \( W_{H'} \) is trivial, and \( (7.13) \) reduces to \( (7.12) \). Choose a 2-simplicial finite pointed \( G \)-set \( X \) that is \( G \)-adapted in the sense of Definition 6.3. Then \( R_H(M) \) is given by \( (7.4) \), and as in the proof of Lemma 6.14 we have a natural identification
\[
(7.17) \quad E^H_G(M) \cong C_*(X, R^\pi_{2*}(\pi^*_1 M \otimes T_H)) \otimes T_H).
\]
Note that by definition, we have \( T_H = S(\varphi^H)^* T \), where \( T \in \mathcal{M}(W_H, \mathbb{Z}) \) sends a \( W_H \)-set \( S \) to \( \mathbb{Z}[S] \). Therefore by base change, \( (7.17) \) yields an identification
\[
(7.18) \quad E^H_G(M) \cong C_*(Y, R^\pi_{2*}(\pi^*_1 M \otimes T)),
\]
where we denote \( Y = X^H \). If \( H \subset G \) is not normal, so that \( N_H \subset G \) is a proper subgroup, then for every subgroup \( H_0 \subset N_H \) containing \( H \), the fixed point set \( Y_{H_0/H} = X^{H_0} \) is contractible by Definition 6.3 (ii), and \( E^H_G(M) = 0 \) by Lemma 6.2. If \( H \subset G \) is normal, so that \( G = N_H \), then \( Y \)}
is a $W_H$-adapted 2-simplicial pointed $W_H$-set. By Definition 6.1 and (1.25),

the right-hand side of (7.18) is then given by

(7.19) \[ C_\ast(X, R'\pi_2^\ast(\pi_1^\ast M \otimes T)) \cong C_\ast(\Delta^0, R'\pi_2^\ast(\pi_1^\ast M \otimes \delta^Y T)). \]

But by definition, the simplicial $\mathbb{Z}[W_H]$-module $\delta^Y T$ corresponds to the
maximally adapted complex $P_\ast = C_\ast(Y, \mathbb{Z})$ of Example 7.8, so that the
right-hand side of (7.19) is exactly (7.9).

In the general case, recall that by (2.3), the category $\Gamma_H$ is naturally
identified with the category of finite $G$-sets $S$ equipped with a $G$-equivariant
map $S \to [G/H']$. Then every element $f : [G/H] \to [G/H']$ of the set
$c(H', H)$ of (7.14) defines in particular an object in $\Gamma_{H'}$. This object is an
orbit of the form $[H'/H_f]$, where the subgroup $H_f \subset H' \subset G$ is conjugate
to $H$ in $G$, and well-defined up to conjugation in $H'$. For any $S \in \Gamma_{H'}$, we
then have a natural identification

\[(\gamma^{H'}(S))^H \cong \prod_{f \in c(H', H)} S^{H_f}, \]

and by definition, this gives an isomorphism

\[ \Psi^{H'} T_H \cong \bigoplus_{f \in c(H', H)} T_{H_f}, \]

where $T_{H_f} \in \mathcal{M}(H', \mathbb{Z})$ corresponds to the subgroup $H_f \subset H'$. The group
$W_H$ acts on $C(H', H)$, and then for any $f \in c(H', H)$, its stabilizer $W_{H_f} = \text{Aut}_{H'}([H'/H_f]) \subset W_H$ acts on $T_{H_f}$. Therefore by base change, we have

(7.20) \[ E_{H'}^H \cong \bigoplus_{f \in W_{H'} \setminus c(H', H)} \tilde{E}^H_{H_f}, \]

where $\tilde{E}^H_{H_f}$ are the functors (7.17) for $H_f$ considered as a subgroup in $H'$. If
$H_f \subset H'$ is not normal – that is, $H'$ is not conjugate to a subgroup in $N_H$
– then the right-hand side of (7.20) vanishes. In the case $H \subset H' \subset N_H$,
the group $W_{H'}$ acts on the quotient $W_H \setminus c(H', H)$, and the stabilizer of any
element is conjugate to $W_{H'}^H \subset W_{H'}$. Therefore (7.20) can be rewritten as

\[ E_{H'}^H \cong \iota \left( \tilde{E}^H_{H'} \right) [\pi(H', H)], \]

where $\iota$ is the induction functor (7.16), and $\pi(H', H)$ is the finite set (7.15).

Plugging in the expression for $\tilde{E}^H_{H'}$, we get the claim. \(\square\)
7.4 Autoequivalences. Proposition 7.10 is essentially a reformulation of \[K2\] Corollary 7.10. The full story in \[K2\] is considerably more elaborate. Roughly speaking, one refines the gluing data description obtained in Proposition 7.10 to a certain DG coalgebra, and then uses it to give a full description of the category $\mathcal{DM}(G, R)$. We will not need this. However, we will prove one useful corollary of Proposition 7.10.

Consider the full subcategory $\mathcal{DS}(\Gamma G, \Gamma G+, R) \subset \mathcal{D}(S\Gamma G \times \Gamma G+, R)$ spanned by special objects, as in (5.9). As in (5.10), we have a natural equivalence $\mathcal{DS}(\Gamma G, \Gamma G+, R) \cong \mathcal{DS}_{\text{id} \times \text{inj}}(\Gamma G \times \Gamma G, R)$.

As in (5.11), the cartesian product functor $m : \Gamma G \times \Gamma G \rightarrow \Gamma G$ is a morphism in the sense of Definition 4.11, so that we have the pullback functor $S(m)^* : \mathcal{DS}(\Gamma G, R) \rightarrow \mathcal{DS}(\Gamma G, \Gamma G+, R)$.

Moreover, for any simplicial pointed finite $G$-set $X : \Delta^o \rightarrow \Gamma G+$, we have the pullback functor $(\text{id} \times X)^* : \mathcal{DS}(\Gamma G, \Gamma G+, R) \rightarrow \mathcal{DS}(\Gamma G, \Delta^o, R)$, where $\mathcal{DS}(\Gamma G, \Delta^o, R) \subset \mathcal{D}(S\Gamma G \times \Delta^o, R)$ is again defined as in (5.9).

**Definition 7.11.** The smash product $M \wedge X$ of a derived Mackey functor $M \in \mathcal{DM}(G, R)$ and a simplicial finite pointed $G$-set $X \in \Delta^o \Gamma G+$ is given by

$$M \wedge X = L^\pi (\text{id} \times X)^* S(m)^* M,$$

where $\pi : \Delta^o \times S\Gamma G \rightarrow S\Gamma G$ is the natural projection.

Let us record some of the obvious properties of the smash product of Definition 7.11. First of all, if $G$ is trivial, so that $\mathcal{DM}(G, R) \cong \mathcal{D}(R)$, then we have

$$M \wedge X \cong M \otimes \overline{C}_*(X, \mathbb{Z}),$$

where $\overline{C}_*(X, \mathbb{Z}) = C_*(X, \mathbb{Z})/\mathbb{Z} \cdot \{o\}$ is the reduced chain complex of the pointed simplicial set $X$. In the general case, by base change, we have a natural identification

$$(M \wedge X)(S) \cong C_*(X \wedge S_+, M)$$
for any $S \in \Gamma_G$, where the right-hand side is as in Definition 6.1. This is additive in $S$, so that for any $M \in \mathcal{D}\mathcal{M}(G,R)$ and any $X \in \Delta^o\Gamma_{G^+}$, the smash product $M \wedge X$ is also a derived Mackey functor. The Künneth formula provides a natural quasiisomorphism

$$(M \wedge X_1) \wedge X_2 \approx M \wedge (X_1 \wedge X_2)$$

for any $X_1, X_2 \in \Delta^o\Gamma_{G^+}$. For any subgroup $H \subset G$, the obvious isomorphism $\rho^H(X_1 \wedge X_2) \cong \rho^H(X_1) \wedge \rho^H(X_2)$ induces by adjunction a projection formula isomorphism

$$\gamma^H(\rho^H(X) \wedge S_+) \cong X \wedge \gamma^H(S)_+, \quad \text{and this induces a natural functorial isomorphism}$$

$$\Psi^H(M \wedge X) \cong \Psi^H(M) \wedge \rho^H(X).$$

This isomorphism respects the $W_H$-action, thus gives an isomorphism

$$(7.22) \quad \tilde{\Psi}^H(M \wedge X) \cong \tilde{\Psi}^H(M) \wedge \rho^H(X)$$

of extended functors of (2.14). Also, we have $\varphi^H(X \wedge S_+) \cong X^H \wedge \varphi^H(S)_+$, and this induces a functorial isomorphism

$$\text{Infl}^H(M \wedge X) \cong \text{Infl}^H(M \wedge X^H).$$

By adjunction, we obtain an isomorphism

$$(7.23) \quad \Phi^H(M \wedge X) \cong \Phi^H(M) \wedge X^H$$

for any $M \in \mathcal{D}\mathcal{M}(G,R)$ and any $X \in \Delta^o\Gamma_{G^+}$.

**Definition 7.12.** A simplicial finite pointed $G$-set $X \in \Delta^o\Gamma_{G^+}$ is homologically adapted to a normal subgroup $N \subset G$ if for any subgroup $H \subset G$ containing $N$, the reduced chain complex $\overline{C}_\ast( X^H , \mathbb{Z} )$ is quasiisomorphic to $\mathbb{Z}$, while for any subgroup $H \subset G$ not containing $N$, the complex $\overline{C}_\ast( X^H , \mathbb{Z} )$ is acyclic.

**Example 7.13.** For any $n$-simplicial finite pointed $G$-set $X$ adapted to $N$ in the sense of Definition 6.3, the diagonal $\delta^\ast X$ is homologically adapted to $N$ in the sense of Definition 7.12.
Lemma 7.14. Assume given a normal subgroup \( N \) and a simplicial finite pointed \( G \)-set \( X \in \Delta^o \Gamma_G^+ \) homologically adapted to \( N \) in the sense of Definition 7.12. Then for any \( E \in \mathcal{DM}(G,R) \), we have a natural isomorphism
\[
X \wedge E \cong \text{Infl}^N(\Phi^N(E)).
\]

Proof. Since \( X^G \) has non-trivial homology, it is not empty, so that we have a nontrivial \( G \)-equivariant map \( i : [1]_+ \to X \). This map then automatically induces a quasiisomorphism \( \mathbb{Z} \cong \mathcal{C}_*(X^H, \mathbb{Z}) \) for any \( H \subset G \) containing \( N \). Consider the natural map
\[
\tilde{i} : E \cong [1]_+ \wedge E \to X \wedge E
\]
induced by \( i \). Then by (7.23), \( X \wedge E \) lies inside the subcategory \( \mathcal{DM}_N(G,R) \), so that \( \tilde{i} \) factors through a map
\[
\text{Infl}^N(\Phi^N(E)) \to X \wedge E,
\]
and again by (7.23), this map becomes an isomorphism after applying the conservative functor \( \Phi_* \) of Lemma 7.1. □

Definition 7.15. A simplicial finite pointed \( G \)-set \( X \in \Delta^o \Gamma_G^+ \) is a homological sphere if for any subgroup \( H \subset G \), the reduced chain complex \( \mathcal{C}_*(X^H, \mathbb{Z}) \) is quasiisomorphic to \( \mathbb{Z} \) placed in some degree \( d_H \).

Example 7.16. For any finite set \( S \), let \( X(S) = I^S / \overline{I^S} \) be the quotient of the product of copies of the simplicial interval \( I \) numbered by elements \( s \in S \) by its boundary
\[
\overline{I^S} = \coprod_{s \in S} \{ s \} \times I^{S \backslash \{ s \}} \cup \coprod_{t \in S} \{ t \} \times I^{S \backslash \{ t \}},
\]
where \( s, t \in I \) are the endpoints. Then \( \mathcal{C}_*(X(S), \mathbb{Z}) \) is \( \mathbb{Z} \) placed in degree \( |S| \), the cardinality of \( S \). Let \( G \) act on \( X(G) \) via its action on itself. Then for any subgroup \( H \subset G \), we have \( X(G)^H \cong X(G/H) \), so that \( X(G) \) is a homological sphere in the sense of Definition 7.15 (with \( d_H = |G/H| \), the cardinality of the orbit \( G/H \)).

Example 7.17. More generally, for any finite-dimensional representation \( V \) of the group \( G \) over \( \mathbb{R} \), consider its one-point compactification \( S_V \), choose its finite triangulation compatible with the \( G \)-action, and let \( X \) be the corresponding simplicial pointed \( G \)-set. Then \( X \) is a homological sphere, with \( d_H = \dim \mathbb{R} V^H \). If \( V = \mathbb{R}[G] \) is the regular representation, one recovers \( X(G) \) of Example 7.16.
Proposition 7.18. Assume given a simplicial finite pointed $G$-set $X \in \Delta^\circ \Gamma_{G^+}$, and assume that $X$ is a homological sphere in the sense of Definition 7.15. Then the endofunctor $E_X$ of the category $\mathcal{DM}(G,R)$ sending $M$ to $M \wedge X$ is an autoequivalence.

Proof. For any subgroup $H \subset G$ and $M \in \mathcal{DM}(G,R)$, the isomorphisms (7.22), (7.23) and (7.21) provide a functorial isomorphism $\tilde{\Phi}^H(M \wedge X) \cong \tilde{\Phi}^M(M)[d_H]$, where $\tilde{\Phi}^H$ is the functor (7.3). By adjunction, we obtain functorial base change maps

$$\text{(7.24)} \quad R_H(M) \wedge X \to R_H(M)[d_H], \quad M \in \mathcal{D}(R[W_H]),$$

where $R_H$ is the adjoint functor of Lemma 7.2. Applying the functors $\tilde{\Phi}^{H'}$, $H' \subset G$ another subgroups, we obtain functorial maps

$$\text{(7.25)} \quad E^{H'}_H(M)[d_H'] \to E^{H'}_H(M)[d_H]$$

where we have used the identification $E^{H'}_H(M)[d_H'] \cong \tilde{\Phi}^{H'}(R_H(M) \wedge X)$ provided by (7.23) and (7.21). Assume for the moment that all the maps (7.25) are isomorphisms. Then by Lemma 7.1 (ii), the maps (7.24) are also isomorphisms, so that $E_X$ preserves the full subcategories $R_H(\mathcal{D}(R[H])) \subset \mathcal{DM}(G,R)$ of Lemma 7.3, and induces a homological shift on each of these subcategories. Moreover, by adjunction, $E_X$ is fully faithful on objects of the form $R_H(M)$, $H \subset G$, $M \in \mathcal{D}(R[H])$. Then by Lemma 7.3 it is fully faithful on the whole $\mathcal{DM}(G,R)$, and since homological shifts are autoequivalences, it is essentially surjective, again by Lemma 7.3.

It remains to prove that all the functorial maps (7.25) are isomorphisms. By Proposition 7.10 it suffices to consider the case when $H$ lies in $H'$ as a normal subgroup, and in this case, the cone of the map (7.25) is a sum of terms of the form

$$\tilde{C}^*(H'/H, M \otimes \overline{C}_*(X^H/X^{H'}, \mathbb{Z})).$$

But since the finite simplicial pointed $W_H$-set $X^H/X^{H'}$ has no non-trivial points fixed under $H'/H \subset W_H$, the reduced chain homology complex $\overline{C}_*(X^H/X^{H'}, \mathbb{Z})$ is induced in the sense of Definition 7.3, and the maximal Tate cohomology in question indeed vanishes. \qed

8 Derived normal systems.

To proceed further, we need to describe a derived version of the theory of normal systems introduced in Subsection 3.2.
8.1 The comparison theorem. Consider the fibration $\nu$ of (3.7), and recall that its fibers are identified with the categories $\Gamma_{W}$, $W = G/N$, $N \in \mathcal{N}(G)$ a normal cofinite subgroup in $G$, while the transition functors are the functors $\varphi^{N'/N}$, $N \subset N' \subset G$. Note that the $S$-construction of Definition 4.6 can be applied fiberwise to the fibration $\Gamma_{G}/N(G)^{o}$. Namely, let $S(\Gamma_{G}/N(G)^{o})$ be the category of pairs $\langle [n], f \rangle$ of $[n] \in \Delta$ and a functor $f : [n] \rightarrow \hat{\Gamma}_{G}$ such that $\nu \circ f : [n] \rightarrow N(G)^{o}$ is the constant functor onto an object $N \in N(G)^{o}$, with morphisms from $\langle [n], f \rangle$ to $\langle [n'], f' \rangle$ given by a pair of a morphism $\varphi : [n] \rightarrow [n']$ and a morphism $\alpha : f' \circ \varphi \rightarrow f$ such that for any $i, j \in [n]$, the commutative square (4.5) induced by (4.4) is cartesian in $\Gamma$. Then $\nu$ induces a forgetful functor
\[ S(\nu) : S(\Gamma_{G}/N(G)^{o}) \rightarrow N(G). \]

This functor is a cofibration whose fibers are the categories $S\Gamma_{W}$, and whose transition functors are functors $S(\varphi^{N'/N})$. Say that a morphism $f$ in the category $S(\Gamma_{G}/N(G)^{o})$ is special if $S(\nu)(f)$ is invertible, and $f$ is special in the sense of Definition 4.8 when considered as a morphism in a fiber of the cofibration $S(\nu)$. Say that an object $E \in \mathcal{D}(S\Gamma_{G}, R)$ is special if it can be represented by complex $E$, such that $E.(f)$ is a quasiisomorphism for any special morphism $f$, and denote by
\[ \mathcal{D}S(\Gamma_{G}/N(G)^{o}, R) \subset \mathcal{D}(S(\Gamma_{G}/N(G)^{o}), R) \]
the full subcategory spanned by special objects. Then every special object $E \in \mathcal{D}S(\Gamma_{G}/N(G)^{o}, R)$ defines a collection of objects $E_{N} \in \mathcal{D}S(\hat{\Gamma}_{G}, R)$, $N \in N(G)$, $W = G/N$, and transition morphisms
\[ S(\varphi^{N'/N})_{!}E_{N} \rightarrow E_{N'}, \]
for any $N, N' \in N(G)$, $N \subset N'$.

Definition 8.1. A derived normal system of $G$-Mackey profunctors is a special object $E \in \mathcal{D}S(\Gamma_{G}/N(G)^{o}, R)$ such that for any $N \in N(G)$, $W = G/N$, the object $E_{N} \in \mathcal{D}S(\hat{\Gamma}_{W}, R)$ lies in $\mathcal{D}M(W, R) \subset \mathcal{D}S(\hat{\Gamma}_{W}, R)$, and for any $N, N' \in N(G)$, $N \subset N'$, the map
\[ \Phi^{N'/N}E_{N} \rightarrow E_{N'}, \]
induced by (8.2) is an isomorphism.
By definition, derived normal systems form a full triangulated subcategory in \( \mathcal{D} \mathcal{S}(\hat{\Gamma}_G/N(G)^o, R) \); we denote this category by \( \mathcal{D} \mathcal{N}(G, R) \). For any integer \( i \), we let

\[
\mathcal{D} \mathcal{N}^{\leq i}(G, R) = \mathcal{D} \mathcal{N}(G, R) \cap \mathcal{D} \mathcal{S}^{\leq i}(\hat{\Gamma}_G/N(G)^o, R) \subset \mathcal{D} \mathcal{S}(\hat{\Gamma}_G/N(G)^o, R).
\]

We note that since the functors \( \Phi^N/N \) are only right-exact with respect to the standard \( t \)-structure, these subcategories do not automatically give a \( t \)-structure on \( \mathcal{D} \mathcal{N}(G, R) \). Nevertheless, they are perfectly well-defined. So is the subcategory

\[
\mathcal{D} \mathcal{N}^- (G, R) = \bigcup_i \mathcal{D} \mathcal{N}^{\leq i}(G, R) \subset \mathcal{D} \mathcal{N}(G, R),
\]

and the truncation functor \( \tau \) of (6.16) induces a natural functor

\[
\tau : \mathcal{D} \mathcal{N}^{\leq 0}(G, R) \to \mathcal{N}(G, R).
\]

Now let \( \tilde{S}(\hat{\Gamma}_G/N(G)^o) \) be the category of diagrams (4.12) in \( S(\hat{\Gamma}_G/N(G)^o) \) with special \( s_1, s_2 \), and note that the projections \( \pi_1, \pi_2 : \tilde{S}(\hat{\Gamma}_G/N(G)^o) \to S(\hat{\Gamma}_G/N(G)^o) \) are cocartesian over \( N(G) \). Therefore by Lemma 1.2 (4.13) provides a functor

\[
\mathcal{S}p : \mathcal{D}(S(\hat{\Gamma}_G/N(G)^o), R) \to \mathcal{D} \mathcal{S}(\hat{\Gamma}_G/N(G)^o, R)
\]

left-adjoint to the embedding (8.1) that induces the specialization functor of Lemma 4.14 on the fiber over every \( N \in N(G)^o \). Then the functor \( \varphi \) of (3.9) induces a functor

\[
\mathcal{S}(\varphi) : S(\hat{\Gamma}_G \times N(G) \to S(\hat{\Gamma}_G/N(G)^o),
\]

and \( \mathcal{S}(\varphi) \) induces a pair of adjoint functors

\[
\Phi = \mathcal{S}(\varphi)_! \circ p^* : \mathcal{D} \mathcal{S}(\hat{\Gamma}_G, R) \to \mathcal{D} \mathcal{S}(\hat{\Gamma}_G/N(G), R),
\]

\[
\mathcal{I}nf = R_! p_* \circ \mathcal{S}(\varphi)^* : \mathcal{D} \mathcal{S}(\hat{\Gamma}_G/N(G), R) \to \mathcal{D} \mathcal{S}(\hat{\Gamma}_G, R).
\]
Explicitly, for any $E \in \mathcal{D}S(\hat{\Gamma}_G, R)$, we have $\Phi(E)_N = S(\varphi^N)_i(E)$ for any cofinite normal $N \subset G$, and the transition morphisms (8.2) are induced by the natural isomorphisms $\varphi^{N'/N} \circ \varphi^N \cong \varphi^{N'}$. Thus by Lemma [6.14] for any $E \in \mathcal{D}M(G, R) \subset \mathcal{D}S(\hat{\Gamma}_G, R)$, $\Phi(E)$ is a derived normal system in the sense of Definition [8.1], so that the functors $\text{Infl}$ and $\Phi$ of (8.5) induce a pair of adjoint functors between the categories $\mathcal{D}M(G, R)$ and $\mathcal{D}N(G, R)$.

**Proposition 8.2.** Assume that the group $G$ is finitely generated. Then for any integer $i$, the functors (8.5) induce a pair of inverse equivalences between the triangulated categories $\mathcal{D}M^{-\leq i}(G, R)$ and $\mathcal{D}N^{-\leq i}(G, R)$, and they also induce equivalences between $\mathcal{D}M^{-}(G, R)$ and $\mathcal{D}N^{-}(G, R)$.

**Proof.** Since both functors of (8.5) are triangulated, in particular commute with shifts, it suffices to consider the case $i = 0$. By Lemma [6.16] the functor $\Phi$ of (8.5) sends $\mathcal{D}M^{-\leq 0}(G, R)$ into $\mathcal{D}N^{-\leq 0}(G, R)$. By definition, for any $E \in \mathcal{D}N^{-\leq 0}(G, R)$ with components $E_N$, $N \in N(G)$, and any admissible $G$-set $S \in \hat{\Gamma}_G$, the value of the inflation $\text{Infl}(E)$ at $S$ is given by

\[
\text{Infl}(E)(S) = \lim_{\xymatrix{N \ar[r]^-{q} & S_N}}^\leftarrow E_N(S_N),
\]

where $\lim^\leftarrow$ in the right-hand is the derived inverse limit functor, taken over $N \in N(G)$. Thus in general, even for $E \in \mathcal{D}N^{-\leq 0}(G, R)$, $\text{Infl}(E)$ does not have to lie in $\mathcal{D}M^{-\leq 0}(G, R)$. However, since the group $G$ is finitely generated, for any integer $i \geq 1$, there exists at most a finite number of cofinite normal subgroups $N \subset G$ with $|G/N| \leq i$, so that their intersection $N_i \subset G$ is also a normal cofinite subgroup. The subset $N(G)$ formed by the subgroups $N_i$ is cofinal, so that we have

\[
\lim_{\xymatrix{N \ar[r]^-{q} & S_N}}^\leftarrow E_N(S_N) \cong \lim_{\xymatrix{i \ar[r]^-{\tau} & S_{N_i}}}^\leftarrow E_{N_i}(S_{N_i}).
\]

By (1.27), the limit in the right-hand side is isomorphic to the telescope $\text{Tel}(E_{N_i}(S_{N_i}))$ of the inverse system $E_{N_i}(S_{N_i})$. Therefore $\text{Infl}$ at least sends $\mathcal{D}N^{-\leq 0}(G, R)$ into $\mathcal{D}M^{-1}(G, R)$. Moreover, for any $E \in \mathcal{D}N^{-\leq 0}(G, R)$, the truncation at 1 of $\text{Infl}(E)(S)$ is given by

\[
R^1 \lim_{\xymatrix{i \ar[r]^-{\tau} & S_{N_i}}}^\leftarrow E_i(S_{N_i}),
\]

where we denote $E_i = \tau(E)_{N_i}$, and $\tau(E) \in N(G, R)$ is obtained by the truncation functor (8.3). The projective system $E_i(S_{N_i})$, $i \geq 1$ is a part.
of the natural projective system \((3.10)\) for the normal system \(\tau(E)\). In particular, its transition maps are surjective. Therefore the group \((8.8)\) vanishes, so that for a finitely generated group \(G\), we have

\[
\text{Inf} \left( D\mathcal{N}^{\leq 0}(G,R) \right) \subset \widehat{D\mathcal{M}}^{\leq 0}(G,R) \subset \widehat{D\mathcal{M}}(G,R).
\]

Moreover, as in the proof of Lemma \((6.14)\), we have

\[
(8.9) \quad \Phi^N(E)(S) \cong C_\ast(S_+ \wedge X_N, E)
\]

for any \(N \in \mathbb{N}(G)\), \(W = G/N\), \(S \in \Gamma_W \subset \widehat{\Gamma}_G\), \(E \in \widehat{D\mathcal{M}}(G,R)\), \(X_N\) an \(N\)-adapted pointed admissible \(n\)-simplicial \(G\)-set in the sense of Definition \((6.3)\). By Lemma \((6.10)\) (i), for any \(E \in D\mathcal{N}^{\leq 0}(G,R)\), we have

\[
C_\ast(S_+ \wedge X_N, \text{Tel}(\text{Inf}^N_i E_N)) \cong \text{Tel}(C_\ast(S_+ \wedge X_N, \text{Inf}^N_i E_N)),
\]

so that this implies that

\[
\Phi^N(\text{Inf}(E)) \cong \text{Tel}(\Phi^N(\text{Inf}^N_i E_N)),
\]

and as soon as \(N_i \subset G\) becomes contained in \(N \subset G\), the inverse system in the right-hand side becomes the constant system with value \(E_N\). Thus the adjunction map

\[
\Phi(\text{Inf}(E)) \to E
\]

is an isomorphism for any \(E \in D\mathcal{N}^{\leq 0}(G,R)\).

To finish the proof, we have to prove that the adjunction map \(E \to \text{Inf}(\Phi(E))\) is an isomorphism for any \(E \in \widehat{D\mathcal{M}}^{\leq 0}(G,R)\). Choose adapted 2-simplicial sets \(X_N = C(ES_N)\) for all cofinite normal subgroups \(N \subset G\) as in Lemma \((6.4)\), so that effectively, we have

\[
X_{N'} \subset X_N
\]

whenever \(N' \subset N\), and we have \([1]_+ \cong \bigcap_N X_N\). Then \((8.9)\) gives an isomorphism

\[
\text{Inf}^N(\Phi^N(E))(S) \cong C_\ast(S_+ \wedge X_N, E)
\]

for any \(S \in \widehat{\Gamma}_G\) and \(E \in \widehat{D\mathcal{M}}(G,R)\), and by \((8.6)\) and \((8.7)\), this gives an isomorphism

\[
\text{Inf}(\Phi(E))(S) = \lim_{\leftarrow} C_\ast(S_+ \wedge X_i, E),
\]

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where we denote $X_i = X_{N_i}$. Then by Lemma 6.10(ii), for $E \in \widehat{DM}^{\leq 0}(G, R)$ we have

$$\lim_{\leftarrow} C_*(S \wedge X_i, E) \cong C_*(\cap_i (S_+ \wedge X_i), E) = C_*(S_+ \wedge \cap_i X_i, E),$$

and since $\cap_i X_i$ is the constant 2-simplicial set $[1]_+$, the right-hand side is naturally identified with $E(S)$. □

8.2 Corollaries. Comparing Proposition 8.2 and Proposition 3.5, we see that the situation in the derived case is better: every derived Mackey profunctor bounded from above is separated in the derived sense. In particular, this is true for a Mackey profunctor $E$ considered as a derived Mackey profunctor via the embedding (6.15), even if $M$ is not separated in the sense of Definition 3.6. The reason for the discrepancy is that the fixed points functor $\Phi$ is only exact on the right, while the inverse limit functor is only exact on the left. Thus even for $E \in \widehat{M}(G, R) \subset \widehat{DM}(G, R)$, $\text{Infl}(\Phi(E))$ can contain contributions coming from the derived functor $R^1 \lim_\leftarrow$ that do not appear in (3.11). Thus we could expect the derived theory to allow us to improve results in the underived setting. Here is a first example.

Lemma 8.3. For any finitely generated group $G$, the natural embedding

$$\widehat{DM}^-(G, R) \subset DS^-((\hat{\Gamma}_G), R)$$

admits a left-adjoint additivization functor

$$\text{Add} : DS^-((\hat{\Gamma}_G), R) \to \widehat{DM}^-(G, R).$$

This functor is right-exact with respect to the standard $t$-structures and induces the functor

$$(8.10) \quad \text{Add} = \tau \circ \text{Add} \circ q^* : \text{Fun}(Q(\hat{\Gamma}_G), R) \to \widehat{M}(G, R)$$

left-adjoint to the embedding $\widehat{M}(G, R) \subset \text{Fun}(Q(\hat{\Gamma}_G), R)$.

Proof. If $G$ is finite, so that $\widehat{DM}(G, R) = DM(G, R)$, take the functor (6.14) provided by Proposition 5.4. In the general case, applying Add fiberwise to the fibration $S\hat{\Gamma}_G \to N(G)$ gives a functor $\text{Add} : DS^-((S\hat{\Gamma}_G), N(G), R) \to DS^-((S\hat{\Gamma}_G), N(G), R)$, and as in Lemma 3.9 the functor

$$\text{Add} = \text{Infl} \circ \text{Add} \circ \Phi : DS^-((\hat{\Gamma}_G), R) \to \widehat{DM}^-(G, R).$$

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does the job. By construction, it is right-exact with respect to the standard\r
\`t-structures, so that by adjunction, \(^{(8.10)}\) indeed defines a functor adjoint\r
to the embedding \(\hat{\mathcal{M}}(G, R) \subset \text{Fun}(Q \hat{\Gamma}_G, R)\).
\[\square\]

As a consequence of this, for finitely generated groups, all the material\r
of Subsection 3.3 that depends on Lemma 3.9 can be now generalized to\r
arbitrary Mackey profunctors. This concerns the categorical fixed points\r
functor \(\Psi^H\) with its refinement \(\tilde{\Psi}^H\), the inflation functor \(\text{Infl}^H\), the geometric\r
fixed points functor \(\Phi^H\), and the product \(^{(3.17)}\).

We can now give the derived versions of all this material. Firstly, we\r
\ldots
and since $\hat{A}_q$ is the unit object for the tensor product, $\hat{A}^G_q$ acts on any $E \cong \hat{A}_q \otimes E \in \hat{D}(G, R)$ – more precisely, we have an action map

\[(8.12) \quad \hat{A}^G_q \otimes E \to E\]

functorial with respect to $E$.

Another thing we can now extend to derived Mackey profunctors is the notion of a smash product of Definition 7.11.

**Definition 8.4.** Assume that the group $G$ is finitely generated. The smash product $M \wedge X$ of a derived Mackey profunctor $M \in \hat{D}(G, R)$ and a simplicial admissible pointed $G$-set $X \in \Delta^o \hat{\Gamma}_G$ is given by

\[M \wedge X = \text{Add}(L^* \pi_1((\text{id} \times X)^* S(m)^* M)) \in \hat{D}(G, R),\]

where $\pi_1, (\text{id} \times X)^* S(m)^*$ are as in Definition 7.11.

**Remark 8.5.** There is a relation between smash products and the Mackey functor product (8.11) similar to (7.21), but since in this paper, we do not have enough technology to explore it properly, we will return to it elsewhere.

Proposition 7.18 also generalizes to Mackey profunctors but with a certain twist. Assume given a sequence $\{d_q\}$ of integers numbered by cofinite subgroups $H \subset G$, and assume that the sequence is non-decreasing in the following sense: for any $H \subset H' \subset G$, we have $d_H \geq d_{H'}$. For any integer $n$, let

\[(8.13) \quad D^\wedge \lesssim_{n-d_q} (G, R) \subset D^\wedge (G, R) \subset D\hat{D}(G, R)\]

be the full subcategory spanned by objects $M \in \hat{D}(G, R)$ such that $M([G/H])$ lies in $D^\lesssim_{n-d_H} (R)$ for any cofinite subgroup $H \subset G$.

**Lemma 8.6.** Assume that the group $G$ is finitely generated. Then an object $M \in \hat{D}(G, R)$ lies in the subcategory (8.13) if and only if for any cofinite subgroup $H \subset G$, $\overline{\Phi}^H(M) = \Phi^H(\Psi^H(M))$ lies in $D^\lesssim_{n-d_H} (R) \subset D(R)$.

**Proof.** Since the sequence $\{d_q\}$ is non-decreasing, $M$ lies in the subcategory (8.13) if and only if $\Psi^H(M)$ lies in $D^\wedge \lesssim_{n-d_H} (H, R)$ for any cofinite subgroup $H \subset G$. If this happens, $\overline{\Phi}^H(M)$ lies in $D^\lesssim_{n-d_H} (R)$ since $\Phi^H$ is left-exact with respect to the standard $t$-structures. Conversely, if
If \( \Phi^H(M) \in \mathcal{D}^{\leq n-d_H}(R) \) for any cofinite \( H' \subset H \), then \( \Phi(\Psi^H(M)) \) lies in \( \mathcal{D}N^{\leq n-d_H}(H, R) \) by Lemma 7.1 (i), and then \( \Psi^H(M) \cong \text{Infl}(\Phi(\Psi^H(M))) \) lies in \( \mathcal{D}M^{\leq n-d_H}(H, R) \) by Proposition 8.2.

I do not know whether for a general non-decreasing sequence \( \{d_\ast\} \), the subcategories (8.13) for all integers \( n \) define a \( t \)-structure on \( \mathcal{D}M(G, R) \). However, note the following. The notion of a homological sphere of Definition 7.15 extends literally to infinite groups: a simplicial pointed admissible \( G \)-set \( X \in \Delta^{\text{op}} \hat{\Gamma}_G^+ \) is a homological sphere if \( C_q(X^H, \mathbb{Z}) \cong \mathbb{Z}[d_H] \) for any cofinite subgroup \( H \subset G \). Moreover, if one modifies Example 7.16 by letting (8.14)

\[
X(G) = \lim_{H \supset G} X(G/H),
\]

where the limit is taken over all cofinite subgroups \( H \subset G \), then \( X(G) \) is a homological sphere, with \( d_H = |G/H| \).

**Lemma 8.7.** Assume that the group \( G \) is finitely generated. Assume given a homological sphere \( X \in \Delta^{\text{op}} \hat{\Gamma}_G^+ \), with the sequence of degrees \( \{d_\ast\} \). Then for any integer \( n \), the functor \( E_X, M \mapsto M \wedge X \) is an equivalence of categories between \( \mathcal{D}M^{\leq n}(G, R) \) and \( \mathcal{D}M^{\leq n-d_\ast}(G, R) \).

**Proof.** If the group is finite, this is Proposition 7.18. In the general case, apply Proposition 8.2 and Lemma 8.6.

As a consequence of this, the subcategories (8.13) do form a \( t \)-structure on \( \mathcal{D}M(G, R) \) if the sequence \( \{d_\ast\} \) corresponds to a homological sphere (for example, one can take \( d_H = |G/H| \), with the sphere (8.14)). However, in this case, the subcategory

\[
\bigcup_{n} \mathcal{D}M^{\leq n-d_\ast}(G, R) \subset \mathcal{D}M^-(G, R) \subset \mathcal{D}M(G, R)
\]

equivalent to \( \mathcal{D}M^-(G, R) \) need not coincide with the whole \( \mathcal{D}M(G, R) \). I do not know whether the functor \( E_X \) is an autoequivalence of the whole \( \mathcal{D}M(G, R) \).

Another consequence of Lemma 8.3 is the following result that turns out to be very useful for constructing Mackey profunctors.

**Definition 8.8.** An \( R \)-valued derived Mackey functor \( M \in \mathcal{D}M(G, R) \) is **locally finitely supported** if \( M \) lies in \( \mathcal{D}M^{\leq n}(G, R) \) for some integer \( n \), and for any integer \( m \), \( M([G/H]) \) lies in \( \mathcal{D}^{\leq m}(R) \) for all but a finite number of cofinite subgroups \( H \subset G \).

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Lemma 8.9. Assume that the group $G$ is finitely generated. Then for any integer $n$ and any ring $R$, the forgetful functor

$$\widehat{DM} \leq n(G, R) \rightarrow DM \leq n(G, R)$$

induced by the forgetful functor (6.17) has a left-adjoint completion functor

$$DM \leq n(G, R) \rightarrow \widehat{DM} \leq n(G, R).$$

Moreover, if $M \in DM(G, R)$ is locally finitely supported in the sense of Definition 8.8, then the adjunction map $M \rightarrow \widehat{M}$ from $M$ to its completion $\widehat{M}$ is an isomorphism.

In other words, for finitely generated groups, locally finitely supported derived Mackey functors canonically extend to derived Mackey profunctors.

Proof. The completion functor is the composition $Add \circ S(i)_!$, where $i : \Gamma_G \rightarrow \hat{\Gamma}_G$ is the natural embedding. Explicitly, for any $M \in DM \leq n(G, R)$, its completion $\widehat{M}$ is given by

$$\widehat{M} = \lim_{\leftarrow N} \Phi^N_M,$$

where the derived inverse limit in the right-hand side is taken over all cofinite normal subgroups $N \subset G$, and $\Phi^N$ is the geometric fixed points functor $S(\phi^N)_! : DM(G, R) \rightarrow DM(W, R)$, $W = G/N$. If $M$ is finitely supported – that is, $M([G/H]) = 0$ for all but a finite number of cofinite subgroups $H \subset G$ – then the inverse system stabilizes at a finite step, and we have $M \cong S(i)_! \widehat{M}$, as required. To extend the argument to a locally finitely supported derived Mackey functor $M$, note that the completion functor is left-exact with respect to the standard $t$-structure, and that by Definition 8.8, for any integer $m$, the truncation $\tau_{\geq m}M$ with respect to the standard $t$-structure is a finitely supported derived Mackey functor. \hfill \square

8.3 Computations. To finish the section, we will now describe explicitly the derived versions of the profunctors $\widehat{B}^G(S, -)$ of Example 3.3 (and in particular, we will compute the derived Burnside ring $\widehat{A}^G$).

For any admissible $G$-set $S$, consider the derived Mackey profunctor

$$Add(Sp(\mathbb{Z}, S)) \in DM^{-\leq 0}(G, \mathbb{Z}),$$
where $Z_S \in \text{Fun}(S \hat{\Gamma}_G, \mathbb{Z})$ is the functor represented by $S \in \hat{\Gamma}_G \subset S \hat{\Gamma}_G$, $\text{Sp}$ is the specialization functor of Lemma 4.14 and $\text{Add}$ is the addivization functor of Lemma 8.3. For any two admissible $G$-sets $S, S' \in \hat{\Gamma}_G$, let

$$
\hat{B}_G^*(S, S') = \text{Add}(\text{Sp}(Z_S))(S').
$$

Note that by adjunction, we have

$$
\hat{B}_G^*(S, S') = \text{Hom}^*(\text{Add}(\text{Sp}(Z_S)), \text{Add}(\text{Sp}(Z_{S'}))),
$$

so that we have natural associative product maps

$$
\hat{B}_G^*(S_1, S_2) \otimes \hat{B}_G^*(S_2, S_3) \to \hat{B}_G^*(S_1, S_3).
$$

These maps can be refined so that $\hat{B}_G^*(-, -)$ becomes an $A_\infty$-category, but we will not need this; we refer an interested reader to [K2, Subsection 1.6]. However, we note that by adjunction, we have a natural map

$$
\hat{B}_G^*(S_1, S_2) \to \hat{B}_G^*(S_1, S_2)
$$

for any $S_1, S_2 \in \hat{\Gamma}_G$, where the right-hand side is as in (3.3), and these maps are multiplicative with respect to the product (8.16). In particular, setting $S_1 = S_2 = \text{pt}$, we obtain a natural map of DG rings

$$
\hat{A}_G^* \to \hat{A}_G^*
$$

where $\hat{A}_G^*$ is the completed Burnside ring of (3.4).

**Proposition 8.10.** For any finitely generated group $G$ and any admissible $G$-sets $S, S' \in \hat{\Gamma}_G$, we have

$$
\hat{B}_G^*(S, S') \cong \prod_{H \subset G} C_*(W_H, \mathbb{Z}[(S \times S')^H]),
$$

where the product is over all the conjugacy classes of cofinite subgroups $H \subset G$, $W_H = \text{Aut}_G([G/H])$ is the quotient $N_H/H$, as in (3.3), and $C_*(W_H, -)$ is the homology complex of the group $W_H$ with coefficients in the free abelian group $\mathbb{Z}[(S_1 \times S_2)^H]$ spanned by the fixed point set $(S_1 \times S_2)^H$. In particular, $\hat{B}_G^*(S_1, S_2)$ lies in $\mathcal{D}^{\leq 0}(\mathbb{Z})$, and the map (8.17) identifies its truncation at 0 with $\hat{B}_G^*(S_1, S_2)$ of (3.3).
Proof. The claim is clearly compatible with the functors $\Phi$ and $\text{Infl}$ of Proposition 8.2, so it suffices to consider the case when the group $G$ is finite. Then the addivization functor $\text{Add}$ in (8.15) is the functor of Proposition 5.4. Thus to evaluate the right-hand side of (8.15), we need to restrict $\text{Sp}(Z_S)$ via the map $m_{S'} : \Gamma \to \Gamma G$ and then apply the functor $L T$. By Lemma 4.16, we have

$$\text{Sp}(Z_S) = L^* \rho_S \mathbb{Z},$$

where $\rho_S$ is the natural cofibration (4.16) with $c = S$. Thus if denote by $\tilde{\rho} : \tilde{Q} \to \Gamma_+$ the fibration obtained by the cartesian square

$$
\begin{array}{ccc}
\tilde{Q} & \longrightarrow & Q^S \Gamma G \\
\tilde{\rho} \downarrow & & \rho_S \downarrow \\
S_{\text{Inj}} \Gamma & \longrightarrow & S \Gamma G,
\end{array}
$$

then $\tilde{Q}$ descends to a fibration $\rho : Q \to \Gamma_+ \cong Q_{\text{Inj}} \Gamma$, in the sense that there is a cartesian square

$$
\begin{array}{ccc}
\tilde{Q} & \longrightarrow & Q \\
\tilde{\rho} \downarrow & & \downarrow \rho \\
S_{\text{Inj}} \Gamma & \longrightarrow & \Gamma_+,
\end{array}
$$

and we have a natural identification

$$\tilde{B}^G_* (S, S') \cong L T L^* \rho_i \mathbb{Z}.$$

Explicitly, the category $Q$ is the category of triples $(S_1, S_2, f)$ of a finite set $S \in \Gamma$, a finite $G$-set $S_2 \in \Gamma_G$, and a map $f : S_2 \to S_1 \times S \times S'$. Morphisms from $(S_1, S_2, f)$ to $(S'_1, S'_2, f')$ are given by pair of a map $S_1 \to S'_1$ in $\Gamma_+ \cong Q_{\text{Inj}} \Gamma$ represented by a diagram

$$(8.20) \quad S_1 \xleftarrow{i} \tilde{S}_1 \xrightarrow{g_1} S'_1$$

in $\Gamma$ with injective $i$, and a map $g_2 : S'_2 \to S_2$ in $\Gamma_G$ that fit into a commutative diagram

$$
\begin{array}{ccc}
S_2 & \xleftarrow{g_2} & S'_2 \\
\downarrow f & & \downarrow \quad \\
S_1 \times S \times S' & \xleftarrow{i \times \text{id} \times \text{id}} & \tilde{S} \times S \times S' & \xrightarrow{g_1 \times \text{id} \times \text{id}} & S'_1 \times S \times S'
\end{array}
$$
in $\Gamma_G$ with cartesian square on the left. The projection $\rho$ sends $\langle S_1, S_2, f \rangle$ to $S_1 \in \Gamma$.

Moreover, let $\mathcal{R}$ be the category of finite $G$-sets $\overline{S}$ equipped with a morphism $\overline{f} : \overline{S} \to S \times S'$, with maps from $\overline{f} : \overline{S} \to S \times S'$ to $\overline{f} : \overline{S} \to S \times S'$ given by injective maps $g : \overline{S}' \to \overline{S}$ such that $\overline{f} \circ g = \overline{f}$. Then for any $\langle S_1, S_2, f \rangle \in \mathcal{Q}$, we can compose $f$ with the natural projection $S_1 \times S_2 \times S' \to S \times S'$ to obtain a map $\overline{f} : S_2 \to S \times S'$, and sending $\langle S_1, S_2, f \rangle$ to $\overline{f}$ and sending a map $\langle i, g_1, g_2 \rangle$ to $g_2$ gives a functor $\mathcal{Q} \to \mathcal{R}$. This functor has a left-adjoint $\nu : \mathcal{R} \to \mathcal{Q}$ that sends a finite $G$-set $S$ with a map $f : S \to S \times S'$ to the triple $\langle \overline{S}/G, \overline{S}, q \times f \rangle$, where $q : S \to \overline{S}/G$ is the quotient map. Therefore $L \nu_! Z = Z$, and we have

$$\widehat{B}^G, (S, S') \cong L^T (L^* \tau_! Z),$$

where $\tau = \rho \circ \nu : \mathcal{R} \to \Gamma_+$ sends $\overline{S}$ with a map $\overline{f} : \overline{S} \to S \times S'$ to the quotient $\overline{S}/G$. By definition, $L^T \circ L^* \tau_!$ is left-adjoint to the composition $\tau^* \circ j^T$ where $j^T$ is the functor of (5.5). Explicitly, this composition sends an abelian group $M$ to $M \otimes \tau^* T \in \text{Fun}(\mathcal{R}, Z)$. But the functor $\tau : \mathcal{R} \to \Gamma_+$ factors through the subcategory $\Gamma^\text{inj} \subset \Gamma_+$ of finite sets and maps between them represented by diagrams (8.20) with invertible $g_1$, and the functor $\tau : \mathcal{R} \to \Gamma^\text{inj}_+$ is a cofibration. If we let $O = \tau^{-1}([1])$ be the fiber of this cofibration over $[1] \in \Gamma^\text{inj}_+$ then the cartesian square

$$\begin{array}{ccc}
O & \xrightarrow{\eta} & \mathcal{R} \\
\downarrow & & \downarrow \tau \\
\text{pt} & \longrightarrow & \Gamma^\text{inj}_+
\end{array}$$

induces a base change isomorphism $\tau^*(j^T(M)) = M \otimes \tau^* T \cong L^* \eta_* M$, so that by adjunction, we have

$$L^T (L^* \tau_! Z) \cong C_*(O, Z).$$

It remains to notice that by definition, $O$ is the groupoid of all $G$-orbits $[G/H]$ equipped with a map $[G/H] \to S \times S'$, and giving such a map is equivalent to giving and element $s \in (S \times S')^H$. Therefore $C_*(O, Z)$ is exactly the same as the right-hand side of (8.19). This finishes the proof. $\square$

In particular, we see that the derived Burnside ring $\widehat{A}^G_*$ is given by

$$\widehat{A}^G_* \cong \prod_{H \subset G} C_*(W_H, Z),$$

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where as in [8.19] and [3.4], the product is over all conjugacy classes of cofinite subgroups \( H \subset G \). Comparing this to (3.4), we see that in degree 0, the homology of \( \hat{A}^G \), is exactly the underived completed Burnside ring \( \hat{A}^G \).

## 9 The cyclic group case.

We finish the paper by showing how all the abstract machinery that we have developed works in the particular case \( G = \mathbb{Z} \), the infinite cyclic group.

### 9.1 Overview.

All cofinite subgroups in \( \mathbb{Z} \) are of the form \( l\mathbb{Z} \subset \mathbb{Z} \), \( l \geq 1 \).

To simplify notation, denote

\[
M_l = M([\mathbb{Z}/l\mathbb{Z}])
\]

for any \( \mathbb{Z} \)-Mackey functor \( M \in \mathcal{M}(\mathbb{Z}, R) \). For any \( l \geq 1 \), \( M_l \) carries a natural action of the quotient group \( \mathbb{Z}/l\mathbb{Z} \), and we denote the generator of this group by \( \sigma \). For any \( l, l' \geq 1 \), the quotient map \( q : \mathbb{Z}/ll'\mathbb{Z} \rightarrow \mathbb{Z}/l\mathbb{Z} \) induces canonical maps

\[
v_{l', l} = q_* : M_{l'} \rightarrow M_l, \quad f_{l', l} = q^* : M_l \rightarrow M_{l'}.
\]

**Lemma 9.1.** The category \( \mathcal{M}(\mathbb{Z}, R) \) is equivalent to the category of collections of \( R[\mathbb{Z}/l\mathbb{Z}] \)-modules \( M_l, l \geq 1 \), equipped with maps (9.2) satisfying

\[
\begin{align*}
v_{l', l} &= v_{l', l} \circ \sigma^l, \\
f_{l', l} &= \sigma^l \circ f_{l', l}, \\
v_{l'', l'} &= v_{l', l} \circ v_{l'', l'}, \\
f_{l'', l'} &= f_{l'', l'} \circ f_{l', l}, \\
f_{l'', l'} \circ v_{l', l} &= v_{l''/l, l'} \circ f_{l''/l, l'} & \text{if } l'/l \text{ and } l''/l \text{ are coprime} \\
t_{l', l} &= f_{l', l} \circ v_{l', l},
\end{align*}
\]

where \( t_{l', l} = 1 + \sigma^l + \sigma^{2l} + \cdots + \sigma^{(l'-1)l} \) is the averaging over the subgroup \( \mathbb{Z}/l'\mathbb{Z} \subset \mathbb{Z}/ll'\mathbb{Z} \).

**Proof.** The first four equations represent the functoriality of the maps \( f_* \), \( f^* \) of Subsection 2.1 and the last two equations express the double coset formula (2.10). \( \square \)
A $\mathbb{Z}$-Mackey profunctor is in particular a $\mathbb{Z}$-Mackey functor, so that we will still use the notation $M_l$, and we still have the canonical maps \([9.2]\). For any $l \geq 1$, we denote
\[
\gamma_l = \gamma^{l\mathbb{Z}} : \hat{\Gamma}_\mathbb{Z} \to \hat{\Gamma}_\mathbb{Z}, \\
\Psi^l = \Psi^{l\mathbb{Z}} : \hat{M}(\mathbb{Z}, R) \to \hat{M}(\mathbb{Z}, R).
\]
Of course, $l\mathbb{Z} \subset \mathbb{Z}$ is canonically isomorphic to $\mathbb{Z}$ as an abstract group, so that $\Psi^l$ is an endofunctor of the category $\hat{M}(\mathbb{Z}, R)$. For any $l, l' \geq 1$, we obviously have $\Psi^l \circ \Psi^{l'} = \Psi^{ll'}$.

The completed Burnside ring $\hat{\mathbb{A}}^{\mathbb{Z}}$ is easy to compute. As a group, it is given by
\[
\hat{\mathbb{A}}^{\mathbb{Z}} = \mathbb{Z}\langle \varepsilon_1, \varepsilon_2, \ldots \rangle,
\]
the group of infinite linear combinations of generators $\varepsilon_i$ numbered by all integers $i \geq 1$. The generator $\varepsilon_i$ corresponds to the $\mathbb{Z}$-orbit $\mathbb{Z}/i\mathbb{Z}$. The product in $\hat{\mathbb{A}}^{\mathbb{Z}}$ is given by
\[
\varepsilon_i \varepsilon_j = \frac{ij}{\{i, j\}} \varepsilon_{\{i, j\}},
\]
where $\{i, j\}$ stands for the least common multiple of $i$ and $j$. In particular, we have
\[
\varepsilon_i^2 = i \varepsilon_i.
\]
The element $\varepsilon_1 \in \hat{\mathbb{A}}^{\mathbb{Z}}$ is the unit of the ring $\hat{\mathbb{A}}^{\mathbb{Z}}$. The Burnside ring acts on any $M \in \hat{M}(\mathbb{Z}, R)$, and the action of the generators $\varepsilon_i$ is given by \([2.4]\). In particular, for any $l, l' \geq 1$ and any $M \in \hat{M}(\mathbb{Z}, R)$, we have
\[
\varepsilon_{ll'} = v_{l,l'} \circ f_{l,l'} : M_l \to M_l,
\]
where $v_{l,l'}$ and $f_{l,l'}$ are the natural maps \([9.2]\).

**Remark 9.2.** A reader might notice that the ring $\hat{\mathbb{A}}^{\mathbb{Z}}$ in fact coincides with the universal Witt vectors ring $\mathbb{W}(\mathbb{Z})$. We will explore this coincidence elsewhere.

All subgroups in $\mathbb{Z}$ are normal, so that for any separated $M \in \hat{M}_s(\mathbb{Z}, R)$, the terms in the canonical filtration \([3.11]\) are numbered by positive integers $l \geq 1$, and we have $F^{l\mathbb{Z}}M \subset F^{l'\mathbb{Z}}M$ whenever $l'$ is divisible by $l$. It is convenient to consider a slightly different filtration on $M$ by letting
\[
F^l M = \cap_{n \leq l} F^{n\mathbb{Z}} M \subset M
\]
for any \( l \geq 1 \). Then \( F^{l'} M \subset F^l M \) whenever \( l' \geq l \), \( F^{l \mathbb{Z}} M \supset F^l M \supset F^{l \mathbb{Z}} M \), and \( M \) is automatically complete with respect to the filtration \( F^l M \).

By definition, for any group \( G \), we have \( \hat{\Gamma}_G = \hat{\Gamma}_{\hat{G}} \), where \( \hat{G} \) is the profinite completion of the group \( G \). In particular, \( \mathcal{M}(\mathbb{Z}, R) = \mathcal{M}(\hat{\mathbb{Z}}, R) \). Note that

\[
\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p,
\]

where the product is over all primes \( p \). For every prime \( p \), let

\[
(9.9) \quad \mathbb{Z}'_p = \prod_{l \neq p} \mathbb{Z}_l,
\]

where the product is over all primes distinct from \( p \). Then we have \( \hat{\mathbb{Z}} = \mathbb{Z}_p \times \mathbb{Z}'_p \). Denote by

\[
(9.10) \quad \Phi^{(p)} = \Phi^{\mathbb{Z}'_p} : \hat{\mathcal{M}}(\mathbb{Z}, R) = \hat{\mathcal{M}}(\hat{\mathbb{Z}}, R) \to \hat{\mathcal{M}}(\mathbb{Z}_p, R)
\]

the geometric fixed points functor with respect to the embedding \( \mathbb{Z}'_p \subset \hat{\mathbb{Z}} \), and let \( M^{(p)} = \Phi^{(p)}(M) \) for any \( M \in \hat{\mathcal{M}}(\mathbb{Z}, R) \).

**Definition 9.3.** The \( \mathbb{Z}_p \)-Mackey profunctor \( M^{(p)} \in \hat{\mathcal{M}}(\mathbb{Z}_p, R) \) is called the \( p \)-typical part of the \( \mathbb{Z} \)-Mackey profunctor \( M \in \hat{\mathcal{M}}(\mathbb{Z}, R) \).

Cofinite subgroups in \( \mathbb{Z}_p \) are of the form \( \mathbb{Z}_p \subset \mathbb{Z}_p, \ q = p^{n-1}, \ n \geq 1 \); as before, we simplify notation and denote

\[
M_n = M(\mathbb{Z}_p/q\mathbb{Z}_p)
\]

and \( \Psi^n = \Psi^{q\mathbb{Z}_p} \) for any \( M \in \hat{\mathcal{M}}(\mathbb{Z}_p, R) \) and any such \( q \). Since \( q\mathbb{Z}_p \) is abstractly isomorphic to \( \mathbb{Z}_p \), we can treat \( \Psi^n \) as an endofunctor of the category \( \mathcal{M}(\mathbb{Z}_p, R) \). We further simplify notation by setting \( \Psi = \Psi^1 \). We have natural isomorphisms \( \Psi^n \cong (\Psi^1)^n \), so that the notation is consistent. If \( M \) is separated, we also renumber the canonical filtration (3.11) on \( M \) by setting

\[
(9.11) \quad F^n M = F^{n\mathbb{Z}_p} M
\]

for any \( n \geq 1 \) and \( q = p^n \).
9.2 The \( p \)-typical decomposition. It turns out that under some assumptions, a \( \mathbb{Z} \)-Mackey profunctor \( M \) is completely determined by the \( p \)-typical parts of the functors \( \Psi^l(M) \), \( l \geq 1 \) not divisible by \( p \).

Namely, fix a prime \( p \), and assume that the base ring \( R \) is \( p \)-local – that it, all integers \( i \geq 1 \) coprime to \( p \) are invertible in \( R \). Then the completed Burnside ring \( \hat{A}^{\mathbb{Z}} \) is given by (9.5), and in particular, \( \frac{1}{1} \varepsilon_i \) with \( i \) coprime to \( p \) is a well-defined idempotent in \( R \otimes \hat{A}^{\mathbb{Z}} \). Any \( \mathbb{Z} \)-Mackey functor \( M \in \mathcal{M}(\mathbb{Z}, R) \) comes equipped with an action of \( \hat{A}^{\mathbb{Z}} \otimes R \), thus with commuting idempotent endomorphisms \( \frac{1}{1} \varepsilon_i \). Moreover, for any \( l \) prime to \( p \), \( \hat{A}^{\mathbb{Z}} \otimes R \) contains a well-defined idempotent element

\[
\varepsilon_l = \frac{1}{1} \varepsilon_l \cdot \prod_{i \text{ does not divide } l} \left( 1 - \frac{1}{1} \varepsilon_i \right),
\]

where the product is over \( i \) prime to \( p \). We have

\[
1 = \sum_{l \text{ prime to } p} \varepsilon_l,
\]

so that for any \( M \in \hat{M}(\mathbb{Z}, R) \), we have a canonical decomposition

\[
M = \prod_{l \text{ prime to } p} M_{(l)},
\]

where \( M_{(l)} \) is the image of the idempotent \( \varepsilon_l \). We will say that \( M \in \hat{M}(\mathbb{Z}, R) \) is type \( l \) if \( M = M_{(l)} \), and we will denote by \( \hat{M}_l(\mathbb{Z}, R) \) the full subcategory spanned by Mackey profunctors of type \( l \). We then have the natural decomposition

\[
\hat{M}(\mathbb{Z}, R) \cong \prod_{l \text{ prime to } p} \hat{M}_l(\mathbb{Z}, R).
\]

Now note that for any \( l \geq 1 \) not divisible by \( p \), (2.13) with \( H = \mathbb{Z}_p' \subset \hat{Z} \) and \( H' = l\mathbb{Z}_p' \subset H \) gives an isomorphism

\[
(\Psi^l M)^{(p)} = \Phi(p) \Psi^l M \cong \Psi^l \Phi^l \mathbb{Z}_p',
\]

and \( \Psi^l \mathbb{Z}_p \) in the right-hand side can be promoted to the functor \( \Psi^l \mathbb{Z}_p \) of (2.15). In particular, \( (\Psi^l M)^{(p)} \) carries a natural action of the cyclic group \( \mathbb{Z}/l\mathbb{Z} \subset \mathbb{Z}_p \subset \mathbb{Z}_p \times \mathbb{Z}/l\mathbb{Z} = \hat{Z}/l\mathbb{Z}' \), so that we can promote the functor \( \Phi(p) \circ \Psi^l \) to a functor

\[
\Phi_{(l)}^{(p)} : \hat{M}(\mathbb{Z}, R) \to \hat{M}(\mathbb{Z}_p, R[\mathbb{Z}/l\mathbb{Z}]).
\]
Proposition 9.4. Assume that $R$ is $p$-local. Then for any $l$ prime to $p$, the functor $\Phi^{(p)}_{(l)}$ of (9.15) induces an equivalence

$$\Phi^{(p)}_{(l)} : \hat{\mathcal{M}}_l(Z, R) \cong \hat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/l\mathbb{Z}]),$$

where $\hat{\mathcal{M}}_l(Z, R) \subset \hat{\mathcal{M}}(\mathbb{Z}, R)$ is the term of the decomposition (9.14) corresponding to $l$.

**Proof.** Since $R$ is $p$-local, for any $l \geq 1$ not divisible by $p$, the functor $R_l = R_{(e)} : \mathcal{D}(R[\mathbb{Z}/l\mathbb{Z}]) \to \mathcal{M}(\mathbb{Z}/l\mathbb{Z}, R)$ provided by Lemma 7.2 is exact with respect to the standard $t$-structure, thus induces a functor from $R[\mathbb{Z}/l\mathbb{Z}]$-modules to $\mathcal{M}(\mathbb{Z}/l\mathbb{Z}, R)$ right-adjoint to the restriction functor $\Psi^{(e)}$. Explicitly, we have

$$R_l(M)_n = M_{\sigma^n/n} \cong M^e^{l/n}$$

for any $R[\mathbb{Z}/l\mathbb{Z}]$-module $M$ and divisor $n$ of the integer $l$, where $\sigma : M \to M$ generates the action of $\mathbb{Z}/l\mathbb{Z}$. The functor is fully faithful. Applying this functor pointwise over $Q\mathbb{Z}_p$, we obtain a functor

$$\tilde{R}_l : \hat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/l\mathbb{Z}]) \to \hat{\mathcal{M}}(\mathbb{Z}_p \times (\mathbb{Z}/l\mathbb{Z}), R)$$

right-adjoint to the functor $\tilde{\Psi}^l$, and moreover, we have $\tilde{R}_l \circ \tilde{\Psi}^l \cong \text{Id}$, so that $\tilde{R}_l$ is fully faithful. Composing it with the inflation functor $\text{Infl}^{\mathbb{Z}_p}$, we obtain a fully faithful embedding

$$\overline{R}_l : \hat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/l\mathbb{Z}]) \to \hat{\mathcal{M}}(\mathbb{Z}, R)$$

right-adjoint to the functor $\Phi^{(p)}_{(l)}$ of (9.15). By (9.17) and (9.7), for any $E \in \hat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/l\mathbb{Z}])$ and any $n \geq 1$ prime to $p$, the generator $\varepsilon_n$ of the completed Burnside ring $\hat{\mathcal{A}}^2$ acts on $\overline{R}_l(E)$ by $n \text{Id}$ if $n$ divides $l$ and by $0$ otherwise. By (9.12), this means that $\overline{R}_l(E)$ is of type $l$, that is, the functor $\overline{R}_l$ factors through a fully faithful embedding

$$\overline{R}_l : \hat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/l\mathbb{Z}]) \to \hat{\mathcal{M}}_l(\mathbb{Z}, R).$$

This embedding is then automatically right-adjoint to the functor (9.16), so to finish the proof, it suffices to check that $\overline{R}_l$ is essentially surjective. By (9.14), we may as well prove that the fully faithful functor

$$\overline{R} = \prod_{l \text{ prime to } p} \overline{R}_l : \prod_{l \text{ prime to } p} \hat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/l\mathbb{Z}]) \to \hat{\mathcal{M}}_l(\mathbb{Z}, R)$$

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is essentially surjective. By construction, this functor is right-adjoint to
\[ \Phi^p = \prod_{l \text{ prime to } p} \Phi_l : \widehat{\mathcal{M}}(\mathbb{Z}, R) \to \prod_{l \text{ prime to } p} \widehat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/l\mathbb{Z}]), \]
so it suffices to prove that the adjunction map \( a : \text{Id} \to R^p \circ \Phi^p \) is an isomorphism. Moreover, since \( R^p \) is fully faithful, we know that \( \Phi^p(a) = \text{id} \), so it suffices to check that the functor \( \Phi^p \) is conservative. However, we can extend it to a functor
\[ (9.18) \quad \Phi^p : \widehat{\mathcal{D}}\mathcal{M}^{\leq 0}(\mathbb{Z}, R) \to \prod_{l \text{ prime to } p} \widehat{\mathcal{D}}\mathcal{M}^{\leq 0}(\mathbb{Z}_p, R[\mathbb{Z}/l\mathbb{Z}]), \]
with the same definition, and note that since \( R \) is \( p \)-local, the extended functor is exact with respect to the standard \( t \)-structures. Therefore it suffices to check that the extended functor \( \Phi^p \) is conservative. This is clear: Proposition 8.2 and Lemma 7.1 (ii) show that the functor \( \Phi_q^p = \prod_{n \geq 1} \Phi_{(l^n)}(M) \) is conservative, and \( \Phi^p \) factors through \( \Phi^q \).

We note that (9.17) allows to describe the decomposition (9.13) rather explicitly. In particular, for any \( M \in \widehat{\mathcal{M}}(\mathbb{Z}, R) \) and any integer \( n \geq 1 \) of the form \( n = lp^m, l \text{ prime to } p \), we have a natural decomposition
\[ M_n \cong \prod_{l' \text{ prime to } p} \left( \Phi_{(l^n)}(M) \right)_{\sigma_l}, \]
where \( \sigma_l \) in the right-hand side is the generator of the group \( \mathbb{Z}/l'n\mathbb{Z} \), and it acts on \( \Phi_{(l^n)}(M) \in \widehat{\mathcal{M}}(\mathbb{Z}_p, R[\mathbb{Z}/l'n\mathbb{Z}]) \) via the group algebra \( R[\mathbb{Z}/l'n\mathbb{Z}] \).

**Lemma 9.5.** Assume that \( R \) is \( p \)-local, and assume that a \( \mathbb{Z} \)-Mackey pro-functor \( M \in \widehat{\mathcal{M}}(\mathbb{Z}, R) \) is separated in the sense of Definition 3.6. Then the canonical filtration (9.8) on \( M \) induces the canonical filtration (9.11) on \( M(1) = M(1) \) by
\begin{align*}
(F^m M) &\cong (F^{p^m} M)_{(1)},
\end{align*}
and we have
\[ (\text{gr}_F^m M)_{(1)} = \begin{cases} 
\text{gr}_F^m M, & m = p^n, \\
0, & \text{otherwise}.
\end{cases} \]

**Proof.** Immediately follows from (9.19). \( \square \)
9.3 Fixed points data. On the level of derived Mackey profunctors, we still have an identification 
\( \hat{DM}(\mathbb{Z}, R) \cong \hat{DM}(\hat{\mathbb{Z}}, R) \), so that taking the \( p \)-typical part of a derived Mackey profunctor makes sense (at least for derived Mackey profunctors bounded from above). Moreover, by Proposition 8.10, the elements \( \varepsilon_l \in \hat{A}^\mathbb{Z} \), \( l \) prime to \( p \) lift to elements in the degree-0 homology of the derived completed Burnside ring \( \hat{A}^\mathbb{Z}_* \), thus act on any object \( E \in \hat{DM}(\mathbb{Z}, R) \) by (8.12). Thus it is reasonable to expect that Proposition 9.4 has a derived version.

However, it turns out that if we restrict our attention to \( \hat{DM}^-(\mathbb{Z}, R) \subset \hat{DM}(\mathbb{Z}, R) \), we can do much better: there is an alternative model of the category \( \hat{DM}^-(\mathbb{Z}, R) \) that is reasonably simple, requires no assumptions on the ring \( R \), and in the \( p \)-local case, makes the \( p \)-typical decomposition of Proposition 9.4 completely obvious.

To describe this model, denote by \( I \) the groupoid of all finite \( \mathbb{Z} \)-orbits and isomorphisms between them. We have a decomposition
\[
I = \coprod_{n \geq 1} \text{pt}_n,
\]
where \( \text{pt}_n \) is a groupoid with one object with automorphism group \( \mathbb{Z}/n\mathbb{Z} \). For any \( M \in \mathcal{D}(I, R) \) and \( n \geq 1 \), we denote by
\[
M_n \in \mathcal{D}(\text{pt}_n, R) \cong \mathcal{D}(R[Z/n\mathbb{Z}])
\]
its restriction to \( \text{pt}_n \subset I \). For any prime \( p \), let \( I_p \subset I \) be the full subcategory given by
\[
(9.20) \quad I_p = \coprod_{n \geq 1} \text{pt}_{np} \subset I,
\]
and let \( I_* \) be the disjoint union of the categories \( I_p \) over all primes. Denote by
\[
(9.21) \quad i : I_* \to I
\]
the functor that acts on \( I_p \) by the embedding (9.20), and let
\[
(9.22) \quad \pi : I_* \to I
\]
be the functor that acts on \( I_p \) by the union of the natural projections \( \text{pt}_{np} \to \text{pt}_n \) corresponding to the quotient maps \( \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \). For any ring \( R \), any object \( M \in \text{Fun}(I, R) \), and any integer \( n \geq 1 \) we denote by \( M^n \) the restriction of \( M \) to \( \text{pt}_n \subset I \), and for any prime \( p \) and an object \( M \in \text{Fun}(I, R) \), we denote by \( M^{p,n} \) the restriction of \( M \) to \( \text{pt}_{np} \subset I_p \subset I \).
**Definition 9.6.** For any prime $p$ and integer $n \geq 1$, a $\mathbb{Z}[\mathbb{Z}/pn\mathbb{Z}]$-module $M$ is $p$-adapted if it gives a finitely generated projective $\mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$-module after restriction to $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{Z}/pn\mathbb{Z}$. A functor $M \in \text{Fun}(I, \mathbb{Z})$ is adapted if for any $p$, $n$, its component $M^{p,n}$ is a $p$-adapted $\mathbb{Z}[\mathbb{Z}/np\mathbb{Z}]$-module. An adapted complex is an acyclic complex $P_\bullet$ in $\text{Fun}(I, \mathbb{Z})$ such that $P_i = 0$ for $i < 0$, $P_0$ is the constant functor $\mathbb{Z}$, and $P_i$ is adapted for $i > 0$.

**Definition 9.7.** An $R$-valued fixed points datum with respect to $P_\bullet$ is a pair $\langle M_\bullet, \alpha \rangle$ of a complex $M_\bullet$ in $\text{Fun}(I, R)$ and a map

\begin{equation}
\alpha : \pi^* M_\bullet \to i^* M_\bullet \otimes P_\bullet,
\end{equation}

where $i, \pi : I \to I$ are projections (9.21) and (9.22). A map of fixed points data from $\langle M_\bullet, \alpha \rangle$ to $\langle M'_\bullet, \alpha' \rangle$ is a map of complexes $f : M_\bullet \to M'_\bullet$ such that $\alpha' \circ \pi^* f = (i^* f \otimes \text{id}) \circ \alpha$, and the category of $R$-valued fixed points data with respect to $P_\bullet$ is denoted by $C_\alpha(P_\bullet, R)$.

We say that an object $\langle M_\bullet, \alpha \rangle$ in $C_\alpha(P_\bullet, R)$ is acyclic if the underlying complex $M_\bullet$ is acyclic, and we say that a map in $C_\alpha(P_\bullet, R)$ is a quasiisomorphism if it is a quasiisomorphism of the underlying complexes. Inverting quasiisomorphisms in $C_\alpha(P_\bullet, R)$, one obtains the derived category of $R$-valued fixed points data that we denote by $D_\alpha(P_\bullet, R)$. We note that this localization procedure presents no set-theoretical problems, by the same argument as in [K2, Lemma 1.7].

We can also do the localization in two steps. First, we define the category $H_\alpha(P_\bullet, R)$ of $R$-valued fixed points data and chain-homotopy classes of morphisms between them. This is a triangulated category. Then we take its Verdier localization with respect to the full subcategory spanned by acyclic complexes. This shows that the category $D_\alpha(P_\bullet, R)$ is triangulated. The obvious forgetful functor

\begin{equation}
\eta : D_\alpha(P_\bullet, R) \to D(I, R), \quad \langle M_\bullet, \alpha \rangle \mapsto M_\bullet,
\end{equation}

is a triangulated functor. We note that it is conservative (that is, for any morphism $f$ in $D_\alpha(P_\bullet, R)$, $\eta(f)$ is invertible if and only if $f$ is invertible).

**Remark 9.8.** Usually, another way to construct and study unbounded derived categories is to use model structures in the sense of Quillen. This does not seem to work for the category $C_\alpha(P_\bullet, R)$. Indeed, it is a very simple example of the category of DG comodules over a DG coalgebra, and those are not known to possess reasonable model structures.
Lemma 9.9. The forgetful functor $\eta$ of (9.24) has a right-adjoint functor $\rho : \mathcal{D}(I, R) \to \mathcal{D}^\alpha(R)$.

Proof. Let $H(I, R)$ be the triangulated category of complexes in $\text{Fun}(I, R)$ and chain-homotopy classes of maps between them, so that $\mathcal{D}(I, R)$ is the quotient of $H(I, R)$ by the subcategory of acyclic complexes. Then the forgetful functor $\eta$ is induced by the forgetful functor $\eta^h : H^\alpha(P_\ast, R) \to H(I, R)$, and this has an obvious right-adjoint $\rho^h : H(I, R) \to H^\alpha(P_\ast, R)$ given by

\begin{equation}
\rho^h(M) = M \oplus \pi_*(i^* M \otimes P_\ast).
\end{equation}

Moreover, the quotient functor $q : H(I, R) \to \mathcal{D}(I, R)$ has a right-adjoint fully faithful functor $r : \mathcal{D}(I, R) \to H(I, R)$ sending $M \in \mathcal{D}(I, R)$ to its $h$-injective representative in $H(I, R)$. Thus we are in the situation of Lemma 1.9, so that the functor $\rho = q \circ \rho^h \circ r : \mathcal{D}(I, R) \to \mathcal{D}^\alpha(P_\ast, R)$ with $q : H^\alpha(P_\ast, R) \to \mathcal{D}^\alpha(P_\ast, R)$ being the natural projection, is right-adjoint to $\eta$. \hfill \Box

Now denote by $R = \eta \circ \rho : \mathcal{D}(I, R) \to \mathcal{D}(I, R)$ the composition of the forgetful functor $\eta$ of (9.24) with its right-adjoint $\rho$ provided by Lemma 9.9. Note that by (9.25), the adjunction map $R \to \text{Id}$ fits into a functorial exact triangle

\begin{equation}
\begin{array}{ccc}
\overline{R} & \longrightarrow & R \\
& & \longrightarrow \\
& & \text{Id} \longrightarrow
\end{array}
\end{equation}

where $\overline{R} : \mathcal{D}(I, R) \to \mathcal{D}(I, R)$ is given by

\begin{equation}
\overline{R}(M) = \pi_*(i^* r(M) \otimes P_\ast).
\end{equation}

Lemma 9.10. For any $M \in \mathcal{D}(I, R)$, we have

\begin{equation}
\overline{R}(M)^n \cong \bigoplus_{p \text{ a prime}} \check{C}^\ast(\mathbb{Z}/p\mathbb{Z}, M^{np}),
\end{equation}

where $\overline{R}$ is the functor of (9.26), and $\check{C}^\ast(\mathbb{Z}/p\mathbb{Z}, -)$ in the right-hand side are the maximal Tate cohomology objects (7.12). Moreover, we have $\overline{R} \circ \overline{R} = 0$, and every object in $\mathcal{D}^\alpha(P_\ast, R)$ is a cone of objects of the form $\rho(M)$, $M \in \mathcal{D}(I, R)$.
Proof. Note that for any \( n \geq 1 \), any \( h \)-injective complex \( E \) of \( R[\mathbb{Z}/n\mathbb{Z}] \)-modules, and any finite-length complex \( K \) of \( \mathbb{Z}/n\mathbb{Z} \)-modules that are finitely generated and flat over \( \mathbb{Z} \), the product \( E \otimes K \) is \( h \)-injective. Therefore for each term \( F^l P \) of the stupid filtration on the adapted complex \( P \), the product \( i^* r(M) \otimes F^l P \) is \( h \)-injective. Since the functor \( \pi_* \) commutes with filtered direct limits, (9.26) implies that

\[
\overline{R}(M) \cong \lim_{\downarrow} R^* \pi_* (i^* M \otimes F^l P).
\]

For any \( n \) and \( p \), the complex \( P^{p,n} \) considered as a complex of \( \mathbb{Z}/p\mathbb{Z} \)-modules is obviously maximally adapted in the sense of Definition 7.7, so that (9.27) follows from (9.26). To prove that \( \overline{R} \circ \overline{R} = 0 \), it then suffices to check that

\[
\check{C}^* (\mathbb{Z}/p\mathbb{Z}, \check{C}^* (\mathbb{Z}/p\mathbb{Z}, M)) = 0
\]

for any two primes \( p, p' \), and any \( M \in \mathcal{D}(R[\mathbb{Z}/p\mathbb{Z}]) \). For any integer \( n \geq 2 \), the trivial representation \( \mathbb{Z} \) of the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) has a standard 2-periodic projective resolution \( \hat{P} \). Using this resolution, one immediately observes that the cohomology algebra \( H^* (\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \) is the algebra \( \mathbb{Z}[u] \) with a single generator \( u \) of degree 2, subject to a single relation \( nu = 0 \). Thus for any \( M \in \mathcal{D}(R[\mathbb{Z}/n\mathbb{Z}]) \), we have a natural map

\[
u \colon M \to M[2].
\]

If \( n = p \) is a prime, then the cone of the augmentation map \( \hat{P} \to \mathbb{Z} \) is maximally adapted in the sense of Definition 7.7. Using this complex to compute maximal Tate cohomology, we conclude that for any \( M \in \mathcal{D}(R[\mathbb{Z}/p\mathbb{Z}]) \), we have

\[
\check{C}^* (\mathbb{Z}/p\mathbb{Z}, M) \cong \lim_{\downarrow} \check{C}^* (\mathbb{Z}/p\mathbb{Z}, M)[2l],
\]

where the limit is taken with respect to the map (9.30). This shows that \( \nu C^* (\mathbb{Z}/p\mathbb{Z}, M) \) is \( p \)-local, and trivial if \( p \) is invertible in \( M \). Thus (9.29) trivially holds if \( p \neq p' \), and we may assume that \( p = p' \). The natural map

\[
\pi : H^* (\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \to H^* (\mathbb{Z}/p^2\mathbb{Z})
\]

induced by the quotient map \( \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \) sends the generator \( u \in H^2 (\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \) to \( pu \in H^2 (\mathbb{Z}/p^2\mathbb{Z}, \mathbb{Z}) \), so that \( \pi(u)^2 = 0 \). Therefore for any \( M \in \mathcal{D}(R[\mathbb{Z}/p\mathbb{Z}]) \) of the form \( M = C^* (\mathbb{Z}/p\mathbb{Z}, M') \), \( M' \in \mathcal{D}(R[\mathbb{Z}/p^2\mathbb{Z}]) \), the
natural map (9.30) squares to 0. The same is then true for arbitrary sums of objects of this type. Then by (9.31), we have

$$C^* \left( \bigoplus_{l \geq 0} C^*(\mathbb{Z}/p^l \mathbb{Z}, M) [2l] \right) = 0$$

for any $M \in \mathcal{D}(R[\mathbb{Z}/p^2 \mathbb{Z}])$. To deduce (9.29), evaluate its left-hand side by (9.31) and compute the limit by the telescope construction (1.28).

Finally, for the last claim, take an object $E \in \mathcal{D}(\alpha(I, R))$, and let $E$ be the cone of the adjunction map $E \to \rho(\eta(E))$. Then $E(\eta(E)) = \overline{R} \circ \overline{R}(E) = 0$, so that the adjunction map $E \to \rho(\eta(E))$ becomes an isomorphism after applying $\eta$. Since $\eta$ is conservative, $E \cong \rho(\eta(E))$. \hfill \Box

**Lemma 9.11.** The category $\mathcal{D}(P, R)$ does not depend on the choice of the adapted complex $P$.

**Proof.** Any map $f : P \to P'$ between adapted complexes $P, P'$ compatible with augmentations induces a functor $f_* : C^*(P, R) \to C^*(P', R)$ given by

$$f_*(\langle M, \alpha \rangle) = \langle M, (\text{id} \otimes f) \circ \alpha \rangle.$$ 

This functor commutes with the forgetful functors $\eta$. Moreover, by (1.27), the base change map

$$f_* \circ \rho \to \rho \circ f_*$$

induced by the isomorphism $\eta \circ f_* \cong f_* \circ \eta$ is also an isomorphism. Then by adjunction, $f_*$ is fully faithful on the full triangulated subcategory in $\mathcal{D}(P, R)$ spanned by $\rho(\mathcal{D}(I, R))$, and by Lemma 9.10 this is the whole category $\mathcal{D}(P, R)$. Moreover, its essential image contains the full subcategory in $\mathcal{D}(P', R)$ spanned by $\rho(\mathcal{D}(I, R))$, and by Lemma 9.10 this is the whole category $\mathcal{D}(P', R)$. Therefore $f_*$ is an equivalence of categories. To finish the proof, it suffices to note that if $\overline{P}$ is a projective resolution $P$ of the constant functor $\mathbb{Z} \in \text{Fun}(I, \mathbb{Z})$, then the cone $\overline{P}$ of the augmentation map $\overline{P} \to \mathbb{Z}$ is obviously an adapted complex, and for any other adapted complex $P'$, we have a map $f : P \to P'$.

Note that because of Lemma 9.11 it is safe to drop $P$ from notation and denote $\mathcal{D}(R) = \mathcal{D}(P, R)$.

Assume now given a subgroup $H \subset \hat{\mathbb{Z}}$, recall that the groupoid $I$ is the groupoid of finite $\hat{\mathbb{Z}}$-orbits, and let $I^H \subset I$ be the full subcategory spanned by orbits on which $H$ acts trivially. Say that an object $M \in \mathcal{D}(I, R)$ is
supported at $H$ if $M(i) = 0$ for any $i \in I \setminus I^H$, and denote by $\mathcal{D}_H(I, R) \subset \mathcal{D}(I, R)$ the full subcategory spanned by objects supported at $H$. Then the subcategory $\mathcal{D}_H(I, R) \subset \mathcal{D}(I, R)$ is canonically a direct summand, and the embedding $\mathcal{D}_H(I, R) \subset \mathcal{D}(I, R)$ has an obvious left-adjoint functor

\[
\varphi^H : \mathcal{D}(I, R) \to \mathcal{D}_H(I, R).
\]

Explicitly, $\varphi^H(M)(i) = M(i)$ if $i \in I$ lies in $I^H \subset I$, and 0 otherwise. Moreover, let $\mathcal{D}_H^a(R) \subset \mathcal{D}^a(R)$ be the full subcategory spanned by objects $M$ such that $\eta(M) \in \mathcal{D}(I, R)$ is supported at $H$. Then by Lemma 9.10, the adjoint functor $\rho$ of Lemma 9.9 sends $\mathcal{D}_H(I, R) \subset \mathcal{D}(I, R)$ into $\mathcal{D}_H^a(R) \subset \mathcal{D}^a(I, R)$, and the functor (9.32) defines a functor

\[
\varphi^H : \mathcal{D}^a(R) \to \mathcal{D}_H^a(R)
\]

left-adjoint to the embedding $\mathcal{D}_H^a(R) \subset \mathcal{D}^a(R)$.

### 9.4 Derived correspondence.

Now we go back to $\widehat{\mathcal{D}M}(\mathbb{Z}, R)$, the category of $R$-valued derived $\mathbb{Z}$-Mackey profunctors. We will use the same shorthand notation (9.1), (9.4), (9.10), where $\Psi$ and $\Phi$ now stand for fixed points functors on the derived level. Moreover, for any $n \geq 1$, we denote

\[
\Phi^n = \Phi^{n\mathbb{Z}} : \widehat{\mathcal{D}M}(\mathbb{Z}, R) \to \mathcal{D}M(\mathbb{Z}/n\mathbb{Z}, R),
\]

and we let

\[
\widetilde{\Phi}^n = \Psi^{(e)} \circ \Phi^n : \widehat{\mathcal{D}M}(\mathbb{Z}, R) \to \mathcal{D}(R[\mathbb{Z}/n\mathbb{Z}]),
\]

where as in (7.3), $\{e\} \subset \mathbb{Z}/n\mathbb{Z}$ is the trivial group. Let

\[
R^n = \text{Inf}^{n\mathbb{Z}} \circ R_{\{e\}} : \mathcal{D}(R[\mathbb{Z}/n\mathbb{Z}]) \to \widehat{\mathcal{D}M}(\mathbb{Z}, R)
\]

be the right-adjoint functor to $\widetilde{\Phi}^n$, where $R_{\{e\}}$ is as in Lemma 7.2 and $\{e\} \subset \mathbb{Z}/n\mathbb{Z}$ is again the trivial group. Note that for any $M \in \mathcal{D}(R[\mathbb{Z}/n\mathbb{Z}])$, $R^n(M)$ is supported at $n\mathbb{Z} \subset \mathbb{Z}$.

**Lemma 9.12.** For any $n, l \geq 1$ and $M \in \mathcal{D}(R[\mathbb{Z}/n\mathbb{Z}])$, we have

\[
\widetilde{\Phi}^l(R^n(M)) \cong \begin{cases} 
M, & l = n, \\
\hat{C}^*(\mathbb{Z}/p\mathbb{Z}, M), & n = pl, \ p \text{ a prime}, \\
0, & \text{otherwise}.
\end{cases}
\]
Proof. Since $R^n(M)$ is supported at $n\mathbb{Z} \subset \mathbb{Z}$, $\tilde{\Phi}(R^n(M))$ vanishes unless $l$ divides $n$, and in this case, $\tilde{\Phi}(R^n(M))$ is given by Proposition 7.10 with $G = \mathbb{Z}/n\mathbb{Z}$. It remains to note that by [K2, Lemma 7.15 (ii)], the maximal Tate cohomology $\hat{C}^r(\mathbb{Z}/m\mathbb{Z}, M)$ with any coefficients vanishes unless $m$ is a prime. □

Taken together, the functors $\tilde{\Phi}^n$ define a functor

\[(9.34) \quad \tilde{\Phi} : \hat{DM}(\mathbb{Z}, R) \to \mathcal{D}(I, R)\]

such that $\tilde{\Phi}(M)^n = \tilde{\Phi}^n(M)$, and the functors $R^n$ together provide its right-adjoint $\tilde{R} = \prod_n R^n : \mathcal{D}(I, R) \to \hat{DM}(\mathbb{Z}, R)$.

Now for any $n \geq 1$, choose an admissible pointed 2-simplicial $\mathbb{Z}$-set $X_n$ that is adapted to $n\mathbb{Z} \subset \mathbb{Z}$ in the sense of Definition 6.3. For any prime $p$, let $X_n^p = X_{np\mathbb{Z}}^p \subset X_n$. Denote

\[(9.35) \quad P_{p,n}^i = \overline{C}_*(X_n^p/\mathbb{Z}),\]

and let $P$ be the complex in $\text{Fun}(I, \mathbb{Z})$ whose value at $\text{pt}_{np} \subset I_p \subset I$ is given by the complex $P_{p,n}^i$.

**Lemma 9.13.** The complex $P$ is adapted in the sense of Definition 9.6.

Proof. For any $[m] \in \Delta^o \times \Delta^o$, if an element $x \in X_n^p([m])$ is fixed under $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{Z}/pn\mathbb{Z}$, it is also fixed under $n\mathbb{Z} \subset \mathbb{Z}$, and then it must lie in $\iota([1]_+)$ by Definition 6.3 (i). Therefore $\mathbb{Z}/p\mathbb{Z} \subset \mathbb{Z}/np\mathbb{Z}$ acts freely on the quotient $X_n^p/\mathbb{Z}$, so that $P_{i,n}^p$ is $p$-adapted for any $i \geq 1$. By construction, $P_0^p$ is $\overline{C}_*(\mathbb{Z}) = \mathbb{Z}$. To finish the proof, we note that the pointed $n$-simplicial set $X_n^p$ is contractible by Definition 6.3 (ii), so that the right-hand side of (9.35) is an acyclic complex. □

We can now define our comparison functor from $\hat{DM}(\mathbb{Z}, R)$ to the category $\mathcal{D}^\alpha(R)$ of fixed points data. We will denote it by

\[(9.36) \quad \nu : \hat{DM}(\mathbb{Z}, R) \to \mathcal{D}^\alpha(R).\]

By Lemma 9.11, we are free to choose any adapted complex $P$ to define the category of fixed points data $\mathcal{D}^\alpha(R)$; we will use the one provided by Lemma 9.13. For any $E \in \hat{DM}(\mathbb{Z}, R)$ and any integer $n \geq 1$, let

\[(9.37) \quad \nu(E)_n = \overline{C}_*(X_n \wedge [\mathbb{Z}/n\mathbb{Z}]_+, E).\]
The group $\mathbb{Z}/n\mathbb{Z}$ acts on $X_n \times [\mathbb{Z}/n\mathbb{Z}]_+$ via the second factor, so that $\nu(E)_n$ is a complex of $R[\mathbb{Z}/n\mathbb{Z}]$-modules. Taking together $\nu(E)_n$ for all $n \geq 1$, we obtain a natural complex $\nu(E)$ in $\text{Fun}(I,R)$.

To promote $\nu(E)$ to an object in $\mathcal{D}^\alpha(R)$, we need to construct $(\mathbb{Z}/pn\mathbb{Z})$-equivariant maps

$$\alpha_{p,n} : \nu(E)_n \to P_{p,n}^p \otimes \nu(E)_{np}$$

for all primes $p$ and integers $n \geq 1$. Note that the natural inclusion $\iota : [1]_+ \to X_n^p$ defines $(\mathbb{Z}/pn\mathbb{Z})$-equivariant morphisms

$$\iota_{p,n} : E_{np} = \mathcal{C}_*,([\mathbb{Z}/np\mathbb{Z}]_+,E) = \mathcal{C}_*,([1]_+ \wedge [\mathbb{Z}/np\mathbb{Z}]_+,E) \to \nu(E)_{np},$$

so that it suffices to construct morphisms

$$\alpha_{p,n} = \iota_{p,n} \circ \alpha_{p,n},$$

and take $\alpha_{p,n} = \iota_{p,n} \circ \alpha_{p,n}$. By definition, the $\mathbb{Z}$-action on $X_n^p$ factors through $\mathbb{Z}/pn\mathbb{Z}$, and the quotient $(X_n^p \wedge [\mathbb{Z}/pn\mathbb{Z}]_+)/((\mathbb{Z}/pn\mathbb{Z})$ of the product $X_n^p \wedge [\mathbb{Z}/pn\mathbb{Z}]_+$ by the diagonal $(\mathbb{Z}/pn\mathbb{Z})$-action is of course isomorphic to $X_n^p$. Thus the embedding $X_n^p \subset X_n$ induces a natural map

$$X_n^p \wedge [\mathbb{Z}/pn\mathbb{Z}]_+ \quad \overline{q} \quad X_n^p \quad \longrightarrow \quad X_n,$$

where $\overline{q}$ is the quotient map. If we let $\mathbb{Z}$ act on $X_n^p \wedge [\mathbb{Z}/pn\mathbb{Z}]_+$ via the second factor, then this map is $\mathbb{Z}$-equivariant. Taking its product with the natural projection

$$X_n^p \wedge [\mathbb{Z}/pn\mathbb{Z}]_+ \quad \longrightarrow \quad [\mathbb{Z}/pn\mathbb{Z}]_+ \quad \longrightarrow \quad [\mathbb{Z}/n\mathbb{Z}]_+,$$

we obtain a map

$$a_{p,n} : X_n^p \wedge [\mathbb{Z}/pn\mathbb{Z}]_+ \to X_n \wedge [\mathbb{Z}/n\mathbb{Z}]_+.$$

By definition, under this map, the diagonal $(\mathbb{Z}/pn\mathbb{Z})$-action on the left goes to the action of $\mathbb{Z}/pn\mathbb{Z}$ on the right via the second factor, and conversely, the action via the second factor on the left goes to the diagonal action on the right. To construct the morphism $\overline{\alpha}_{p,n}$ of (9.38), it remains to use the natural identification

$$\mathcal{C}_*,(X_n^p \wedge [\mathbb{Z}/pn\mathbb{Z}]_+,E) \cong \mathcal{C}_*,(X_n^p,\mathbb{Z}) \otimes \mathcal{C}_*,([\mathbb{Z}/pn\mathbb{Z}]_+,E) \cong P_{p,n}^p \otimes E_{pn},$$

and take as $\overline{\alpha}_{p,n}$ the natural morphism $a_{p,n}^*$ of (6.21).

We note that by construction, for any subgroup $H \subset \hat{\mathbb{Z}}$ with the quotient $W = \hat{\mathbb{Z}}/H$, the composition $\varphi^H \circ \nu$ of the comparison functor $\nu$ of (9.36)
with the restriction functor $\varphi^H$ of (9.33) factors through the fixed points functor $\Phi^H$, so that $\nu$ induces a natural functor

$$\nu: \hat{D}\mathcal{M}(W, R) \to \mathcal{D}_H^a(R)$$

such that $\nu \circ \Phi^H \cong \varphi^H \circ \nu$.

**Proposition 9.14.** (i) If $H = n\widehat{\mathbb{Z}} \subset \widehat{\mathbb{Z}}$ is a cofinite subgroup, then the functor $\nu$ of (9.39) is an equivalence of categories.

(ii) For any subgroup $H \subset \widehat{\mathbb{Z}}$ with the quotient $W = \widehat{\mathbb{Z}}/H$, and any integer $l$, let $\mathcal{D}_H^a(R)^{\leq l} \subset \mathcal{D}_H^a(R)$ be the full subcategory spanned by objects $M$ such that $\nu(M)$ lies in $\mathcal{D}^{\leq l}(I, R) \subset \mathcal{D}(I, R)$. Then the functor $\nu$ of (9.39) induces an equivalence between $\hat{D}\mathcal{M}^{\leq l}(W, R)$ and $\mathcal{D}_H^a(R)^{\leq l}$.

**Proof.** As in the proof of Lemma 6.14, Proposition 6.6 provides an isomorphism of functors

$$\tilde{\Phi} \cong \eta \circ \nu: \hat{D}\mathcal{M}(\mathbb{Z}, R) \to \mathcal{D}(I, R),$$

where $\nu$ is the functor (9.36). It induces a base change map

$$\nu \circ \tilde{R} \to \rho,$$

and by comparing Lemma 9.10 and Lemma 9.12 we see that this map is also an isomorphism. For any subgroup $H \subset \widehat{\mathbb{Z}}$, the functor $\tilde{\Phi}$ induces a functor $\tilde{\Phi}: \hat{D}\mathcal{M}(W, R) \to \mathcal{D}_H(I, R)$, the functor $\tilde{R}$ induces a right-adjoint functor $\tilde{\mathcal{R}}: \mathcal{D}_H(I, R) \to \hat{D}\mathcal{M}(W, R)$, and we have induced isomorphisms $\tilde{\Phi} \cong \eta \circ \nu$, $\nu \circ \tilde{R} \cong \rho$.

Assume first that the subgroup $H$ is cofinite, that is, $H = n\widehat{\mathbb{Z}} \subset \widehat{\mathbb{Z}}$ for some $n \geq 1$, and $W = \mathbb{Z}/n\mathbb{Z}$. Denote by $\hat{D}\mathcal{M}(W, R)^{fr} \subset \hat{D}\mathcal{M}(W, R)$, $\mathcal{D}_H^a(R)^{fr} \subset \mathcal{D}_H^a(R)$ the full triangulated subcategories generated by images of the functors $\mathcal{R}$ resp. $\mathcal{R}$. Then by adjunction, $\nu$ induces a fully faithful functor

$$\hat{D}\mathcal{M}^a(H, \mathbb{Z})^{fr} \to \mathcal{D}_H^a(R)^{fr}.$$

But Lemma 9.10 implies that $\mathcal{D}_H^a(R)^{fr} = \mathcal{D}_H^a(R)$, so that $\nu$ is essentially surjective, and by Lemma 7.1 (ii), the functor $\tilde{\Phi}: \hat{D}\mathcal{M}(W, R) \to \mathcal{D}_H(I, R)$ is conservative, so that the same argument shows that $\hat{D}\mathcal{M}(W, R)^{fr} = \hat{D}\mathcal{M}(W, R)$. Therefore $\nu$ is fully faithful on the whole of $\hat{D}\mathcal{M}(W, R)$. This proves (i).
For (ii), note that for a cofinite subgroup $H \subset \hat{\mathbb{Z}}$, the statement immediately follows from Lemma 7.1 (i). In the general case, it remains to apply Proposition 8.2.

To see how Proposition 9.14 implies the $p$-typical decomposition of Subsection 9.2, note that for any $l \geq 1$, the functor $\gamma_l$ of (9.4) induces a functor $\gamma_l^*$ lifts to a natural functor

$$\gamma_l^* : \mathcal{D}^\alpha(R) \to \mathcal{D}^\alpha(R),$$

and we have $\gamma_l \circ \nu \cong \nu \circ \Psi^l$. Moreover, fix a prime $p$, and take $H = \mathbb{Z}_p'$, so that $I^H \subset I$ is the union of groupoids $pt_q \subset I$, $q = p^m$, $m \geq 0$. Then for any $l$ prime to $p$, the functor $\gamma_l^*$ can be refined to a functor

$$\widetilde{\gamma}_l : I^H \times pt_l \to I^H$$

sending $pt_q \times pt_l$ to $pt_{ql} \subset I^H$, and this lifts to a functor

$$(9.40) \quad \widetilde{\gamma}_l^* : \mathcal{D}_H^\alpha(R) \to \mathcal{D}_H^\alpha(R[Z/l\mathbb{Z}]).$$

We then have a natural isomorphism

$$(9.41) \quad \nu \circ \Phi_{(l)}^{(p)} \cong \widetilde{\gamma}_l^* \circ \nu,$$

where $\Phi_{(l)}^{(p)}$, $l$ prime to $p$ are the functors (9.18).

**Corollary 9.15.** Assume that the ring $R$ is $p$-local. Then for any integer $n$, the functor

$$\Phi_{(p)}^* : \widehat{\mathcal{D}}M^{\leq n}(\mathbb{Z}, R) \to \prod_{l \text{ prime to } p} \widehat{\mathcal{D}}M^{\leq n}(\mathbb{Z}_p, R[Z/l\mathbb{Z}])$$

of (9.18) is an equivalence of categories.

**Proof.** Choose an adapted complex $P_\alpha$, and let $\mathcal{P}_\alpha \subset P_\alpha$ be the sum of its components $P_{p,n}$ (in other words, we remove all the components $P_{p',n}$ with $p' \neq p$). Then if we consider the category $C^\alpha(\mathcal{P}_\alpha, R)$, the only non-trivial components of the map $\alpha$ of (9.23) relate $M^\alpha_n$, $n \geq 1$, and $M^\alpha_{p,n}$. Therefore any object $(\langle M, \alpha \rangle)$ of $C^\alpha(\mathcal{P}_\alpha, R)$ splits into a direct sum of objects supported at

$$\gamma_l(I^H) = \coprod_{m \geq 0} pt_{p^m} \subset I$$

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over all $l \geq 1$ prime to $p$, and the pullback functors $\widetilde{\gamma}_l^*$ of \eqref{eq:9.40} induce an equivalence of categories

$$
\mathcal{D}^\alpha(\mathcal{P}_*, R) \cong \prod_{l \text{ prime to } p} \mathcal{D}^\alpha_l(\mathcal{P}_*, R[\mathbb{Z}/l\mathbb{Z}]).
$$

By Proposition \ref{prop:9.14} and \eqref{eq:9.41}, to finish the proof, it remains to show that the natural functor $\mathcal{D}^\alpha(\mathcal{P}_*, R) \to \mathcal{D}^\alpha(P_*, R) = \mathcal{D}^\alpha(R)$ induced by the embedding $\mathcal{P}_* \subset P_*$ is an equivalence of categories. The argument for this is exactly the same as in Lemma \ref{lem:9.11}; one simply needs to observe that since the ring $R$ is $p$-local, the only non-trivial Tate cohomology terms in the right-hand side of \eqref{eq:9.27} are those with our fixed prime $p$. \hfill \Box

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