ASYMPTOTIC FOURIER AND LAPLACE TRANSFORMS FOR VECTOR-VALUED HYPERFUNCTIONS

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Abstract. We study Fourier and Laplace transforms for Fourier hyperfunctions with values in a complex locally convex Hausdorff space. Since any hyperfunction with values in a wide class of locally convex Hausdorff spaces can be extended to a Fourier hyperfunction, this gives simple notions of asymptotic Fourier and Laplace transforms for vector-valued hyperfunctions, which improves the existing models of Komatsu, Bäumer, Lumer and Neubrander and Langenbruch.

1. Introduction

The Fourier and Laplace transforms are frequently used in engineering, physics and analysis, in particular, in the theory of partial differential and convolution equations and in the theory of semigroups, see the works of Paley and Wiener [63], Titchmarsh [72], Schwartz [68, 69], Bracewell [8], Hörmander [20], Doetsch [14–16], Widder [73], Sebastião e Silva [70], Küčera [47], Ouchi [62], Zharniţov [76], Kunstmann [48], Arendt, Batty, Hieber and Neubrander [2] and the references given in Deakin [10].

The main obstacle to apply the Fourier or Laplace transform is that they require exponential bounds, for instance, for the Laplace transform

$$L(f)(\zeta) := \int_0^\infty f(t)e^{-t\zeta}dt$$

of a locally integrable function $f \in L^1_{loc}([0, \infty[)$ to converge (absolutely on the right halfplane), it is needed that $|f(t)| \leq Ce^{Ht}$ on $[0, \infty[$ for some constants $C, H > 0$. In order to circumvent this difficulty and to use these tools with no restrictions on the growth of the functions, several different approaches have been made to extend the transforms to asymptotic versions without losing the properties that are necessary in applications.

Vignaux [73] introduced in 1939 the concept of an asymptotic Laplace transform which was further investigated by Ditkin [13], Berg [8], Lyubich [56] and Deakin [11]. Based on these works, Bäumer [3, 4], Lumer and Neubrander [53, 54] and Mihai [58] developed and investigated an asymptotic Laplace transform $\Sigma_{LN}$ on the space $L^1_{loc}([0, \infty[, E)$ of locally Bochner integrable functions on $[0, \infty[$ with values in a complex Banach space $(E, \|\cdot\|_E)$ by studying the asymptotic behaviour of the
local Laplace transforms

\[ L_j(f)(\zeta) := \int_0^j f(t)e^{-t\zeta}dt, \quad j \to \infty. \]  

(1)

Another approach is due to Komatsu \[\text{[32–37]}\]. Let \( \mathcal{O}(\Omega, E) \) be the space of holomorphic functions on an open set \( \Omega \subset \mathbb{C} \) with values in a complex locally convex Hausdorff space \( E \). Denoting by \( \mathcal{D} := \mathbb{C} \cup \{ \infty e^{i\varphi} \mid |\varphi| \leq \pi \} \) the radial compactification of \( \mathbb{C} \), he defined for \( a \in \mathbb{R} \) and a complex Banach space \( E \) a Laplace transform on the space \( \mathcal{H}^{\exp}(\mathcal{D} \setminus [a, \infty], E) \) of \( E \)-valued holomorphic functions \( F \) on \( \mathbb{C} \setminus [a, \infty[ \) which are of exponential type on each closed sector \( \Sigma \subset \mathbb{C} \setminus [a, \infty[ \), i.e. for each \( \Sigma \) there are some constants \( C, H > 0 \) such that \( \| F(z) \|_E \leq Ce^{H|z|} \) for all \( z \in \Sigma \). This Laplace transform is given by

\[ \mathcal{L}_{\text{Kom}, a}(F)(\zeta) := \int_{\Gamma_a} F(z)e^{-z\zeta}dz \]  

and converges near the half-circle \( S_\infty := \{ \infty e^{i\varphi} \mid |\varphi| < \frac{\pi}{2} \} \) at \( \infty \) for \( F \in \mathcal{H}^{\exp}(\mathcal{D} \setminus [a, \infty], E) \) where \( \Gamma_a \) is a path composed of the ray from \( \infty e^{ia} \), \( -\frac{\pi}{2} < a < 0 \), to a point \( c < a \) and the ray from \( c \) to \( \infty e^{i\beta} \), \( 0 < \beta < \frac{\pi}{2} \) (see \[\text{[32]}\, \text{p. 216}]\).

![Figure 1. Path \( \Gamma_a \) for \( a \in \mathbb{R} \)](image-url)
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\[ \mathcal{L}_{\text{Kom}}(f) := \left[ (\mathcal{L}_{\text{Kom,0}} \circ \rho^1_n) (f) \right], \]

is a linear isomorphism where \( \mathcal{L}_{\text{Kom}}^\text{exp}_n(E) \) is the space of exponentially decreasing holomorphic germs near \( S_\infty \) (see Section 8 for the precise definitions of the spaces \( \mathcal{L}_{\text{Kom}}^\text{exp}_n(E) \) and \( \mathcal{L}_{\text{Kom}}^\text{exp}(E) \)). By \[54, \text{Eq. (15), p. 157}\] the two asymptotic Laplace transforms \( \mathcal{L}_{\text{LN}} \) and \( \mathcal{L}_{\text{Kom}} \) coincide on \( L^1_{\text{loc}}([0, \infty[^{-},E) \). The advantage of Komatsu’s asymptotic Laplace transform is that it can be applied to any Banach-valued hyperfunction with support in \([0, \infty[^{-}\) and thus to a wider class of (generalised) functions, e.g. to distributions \( \mathcal{D}'([0, \infty[^{-},E) \) with support in \([0, \infty[^{-}\) too.

A third approach is due to Langenbruch \[50\]. Distilling from the discussions in \[54, \text{Section 4}, \[54, \text{Section 2}\] (we may add \[3, \text{p. 80-81}\]) and \[49, \text{p. 429}\] on the appropriate way to define an asymptotic Laplace transform, he states in \[50, \text{p. 42}\] that a satisfactory theory should meet the following conditions:

(I) The model contains a wide class of generalized functions and is based on an elementary version of the Laplace transform defined on a space of generalized functions which has a simple topological structure.

(II) For (generalized) functions with compact support, the Laplace transform should coincide with the Fourier–Laplace transform. Moreover, the Laplace transform should be compatible with convolution and multiplication by (a large class of) functions.

(III) The Laplace transform should be asymptotic, i.e. in applications calculations should be needed near \( S_\infty \) only.

One of the reasons to include condition (III) is Ōuchi’s characterisation of the existence of hyperfunction fundamental solutions for abstract Cauchy problems in locally convex spaces \[61, \text{Theorem 2, p. 139}\] (see Komatsu’s \[54, \text{Theorem 4, p. 221}\] and Domański’s and Langenbruch’s \[18, \text{Theorem 4.6, p. 262}\] as well). Lumer and Neuberger’s asymptotic Laplace transform \( \mathcal{L}_{\text{LN}} \) does not fulfil condition (I) since it is restricted to \( L^1_{\text{loc}}([0, \infty[^{-},E) \) even though it is sketched in \[54, \text{Section 4}\] how to extend \( \mathcal{L}_{\text{LN}} \) to \( \mathcal{B}([0, \infty[^{-},E) \) by first interpreting \( \mathcal{B} \) for hyperfunctions with compact support and then using the flabbiness of the sheaf of Banach-valued hyperfunctions. Komatsu’s asymptotic Laplace transform \( \mathcal{L}_{\text{Kom}} \) fulfils condition (II) by \[54, \text{Theorem 3.3, p. 818}\] (compact support), \[33, \text{Section 3, 34, \text{Section 4, 5}}\] (convolution), \[32, \text{Theorem 3.3, p. 818}\] (compact support), \[33, \text{Section 3}, \[35, \text{Section 4, 5}\] (multiplication) and (III) as discussed above. But it only fulfils (I) partially because the spaces \( \mathcal{H}^\text{exp}(.,E) \) have no simple topological structure (we even did not specify a topology on them).

Langenbruch’s approach fulfils all three conditions. We only describe it briefly and go into the details in Section 8. Instead of exponentially increasing holomorphic functions Langenbruch uses exponentially decreasing functions of type \( -\infty \) as key components. Namely, he considers the space

\[ \mathcal{G}([0, \infty[^{-}) := \mathcal{H}_{-\infty}(\overline{\mathbb{C}} \setminus [0, \infty[^{-})/\mathcal{H}_{-\infty}([0, \infty[^{-})], \]

of exponentially decreasing hyperfunctions of type \( -\infty \) supported in \([0, \infty[^{-}\) where for \( K = [0, \infty[^{-}\) or \( K = \emptyset \)

\[ \mathcal{H}_{-\infty}(\overline{\mathbb{C}} \setminus K) := \{ f \in \mathcal{O}(\mathbb{C} \setminus K) \mid \forall n \in \mathbb{N} : \sup_{z \in W_n(K)} |f(z)|e^{\Re(z)} < \infty \}
\]

with \( \overline{\mathbb{C}} := [-\infty, \infty[^{-} + i\mathbb{R} \) and

\[ W_n(K) := \{ z \in \mathbb{C} \mid |\text{Im}(z)| \leq n, \text{dist}(z, K \cap \mathbb{C}) > \frac{1}{n} \} \).

The Laplace transform \( \mathcal{L} \) of an element \( f = [F] \in \mathcal{G}([0, \infty[^{-}) \) is then defined by formula \[2\] where the path \( \Gamma_0 \) is replaced by the path \( \gamma_{[0,\infty[^{-}} \) depicted below.
The Laplace transform
\[ L : \mathcal{G}([0, \infty]) \to \mathcal{L}G_{[0, \infty]} \]
is a topological isomorphism by [50, Theorem 4.1, p. 53] where the Laplace range \( \mathcal{L}G_{[0, \infty]} \) is given by
\[ \mathcal{L}G_{[0, \infty]} = \{ f \in \mathcal{O}(\mathbb{C}) | \forall k \in \mathbb{N} : \sup_{\text{Re}(z) \geq -k} |f(z)|e^{-\frac{1}{k}|z|} < \infty \}. \]

Every hyperfunction with support in \([0, \infty]\) can be extended to an exponentially decreasing hyperfunction of type \(-\infty\) supported in \([0, \infty]\) and such an extension is unique only up to \( \mathcal{G}(\{\infty\}) \) by [50, Theorem 5.1, p. 54], i.e. the canonical (restriction) map
\[ R_+: \mathcal{H}_{-\infty}(\mathbb{C} \setminus \{0, \infty\}) \to \mathcal{B}([0, \infty]), [F] \mapsto [F], \]
is a linear isomorphism where \( \mathcal{H}_{-\infty}(\mathbb{C} \setminus \{0, \infty\}) :\text{=} \mathcal{H}_{-\infty}(\mathbb{C} \setminus \{0, \infty\}) \cap \mathcal{O}(\mathbb{C}) \) and \( \mathcal{G}(\{\infty\}) :\text{=} \mathcal{H}_{-\infty}(\mathbb{C} \setminus \{0, \infty\}) \cap \mathcal{O}(\mathbb{C}) \) is the space of exponentially decreasing hyperfunctions of type \(-\infty\) supported at \( \infty \). The Laplace transform
\[ \mathcal{L}: \mathcal{G}(\{\infty\}) \to \mathcal{L}G_{\{\infty\}} \]
is also a topological isomorphism by [50, Proposition 5.2, p. 55] where
\[ \mathcal{L}G_{\{\infty\}} := \{ f \in \mathcal{O}(\mathbb{C}) | \forall k \in \mathbb{N} : \sup_{\text{Re}(z) \geq -k} |f(z)|e^{k|\text{Re}(z)|+\frac{1}{k}|z|} < \infty \}. \]

Combining these results, Langenbruch’s asymptotic Laplace transform
\[ \mathcal{L}_B : B([0, \infty], \mathbb{C}) \to \mathcal{L}G'_{[0, \infty]} / \mathcal{L}G'_{\{\infty\}}, \mathcal{L}_B(f) :\text{=} \left[ (\mathcal{L} \circ R_+^{-1})(f) \right], \]
is a linear isomorphism by [50, Theorem 5.3, p. 55].

This asymptotic Laplace transform fulfills condition (I). In particular, the spaces \( \mathcal{H}_{-\infty}(\mathbb{C} \setminus K) \) are nuclear Fréchet spaces for \( K = \emptyset, \{\infty\} \) and \([0, \infty]\). It fulfills condition (II) by [50, Section 5] and also condition (III) because it coincides up to a linear isomorphism with Komatsu’s asymptotic Laplace transform \( \mathcal{L}_{\text{Kom}} \) for \( E = \mathbb{C} \) by [50, Theorem 6.3, p. 59].

By using the \( \pi \)-tensor product for nuclear Fréchet spaces, Langenbruch’s asymptotic Laplace transform may be extended to the space \( B([0, \infty], E) \) of \( E \)-valued hyperfunctions with support in \([0, \infty]\) where \( E \) is a complex Fréchet space (see [50, p. 61]). This is needed for applications to the abstract Cauchy problem in Fréchet spaces (see [52, Section 7]). Since abstract Cauchy problems are of interest in more general locally convex spaces \( E \), like distributions, as well (see [1–18, 23, 24, 31, 38, 52, 61, 62, 71] and the references therein), we may add a fourth condition that a satisfactory theory of Laplace transforms should meet:
(IV) The Laplace transform should be applicable to (generalized) functions with values in a wide class of locally convex Hausdorff spaces, containing most of the common spaces in analysis.

This fourth condition is only partially fulfilled by the asymptotic Laplace transforms \( \mathcal{L}_{\text{LN}}, \mathcal{L}_{\text{Kom}} \) and \( \mathcal{L}_B \) which are restricted to Banach and Fréchet spaces, respectively.

It is indicated in [54, p. 42] that instead of Komatsu’s Laplace hyperfunctions (or exponentially decreasing hyperfunctions of type \(-\infty\)) other hyperfunctions may be used to develop a satisfactory theory, like Kawai’s Fourier hyperfunctions [30]. Saburi’s modified Fourier hyperfunctions [66], Kunstmann’s bounded hyperfunctions [49] or, we may add, Park’s and Morimoto’s Fourier ultra-hyperfunctions [64].

Kawai’s Fourier hyperfunctions are the route we take in the present paper. They were introduced by Kawai [30] in the complex-valued case and then extended by Ito and Nagamachi to Fourier hyperfunctions with values in a separable Hilbert space [23, 24] (in a general Hilbert space [22]), by Junker to Fourier hyperfunctions with values in a Fréchet space [28] and then beyond Fréchet spaces by us in [29, 41].

Following in Langenbruch’s footsteps [50], we develop a theory that satisfies all four conditions. In Section 2 we recall the necessary notation and results from the theory of vector-valued Fourier hyperfunctions that are needed. In Section 3 and Section 4 we study the Fourier transform of vector-valued Fourier hyperfunctions supported in extended half-lines and real compact sets, respectively. In Section 5 we introduce our asymptotic Fourier transform for vector-valued hyperfunctions and show that it coincides (up to a linear isomorphism) with Langenbruch’s asymptotic Fourier transform in the case \( E = C \) in Section 6. We use our Fourier transform to define and develop our Laplace and asymptotic Laplace transforms \( \mathcal{L}_B \) in Section 7. In Section 8 we clarify the relation between the asymptotic Laplace transforms considered so far. Namely, we show that \( \mathcal{L}_B \) and \( \mathcal{L}_B \) coincide (up to a linear isomorphism) for complex Fréchet spaces \( E \), that both coincide with \( \mathcal{L}_{\text{Kom}} \) for complex Banach spaces \( E \) and thus all three coincide on \( L^1_{\text{loc}}([0, \infty], E) \) with \( \mathcal{L}_{\text{LN}} \).

2. Notation and Preliminaries

In the following \( E \) is always a locally convex Hausdorff space over \( \mathbb{C} \) equipped with a directed system of seminorms \( (\rho_n)_{n \in \mathbb{N}} \), short, \( E \) is a \( \mathbb{C}\)-lcHs. We set \( (\rho_n)_{n \in \mathbb{N}} := \{ \cdot \mid \} \) if \( E = \mathbb{C} \) and \( \cdot \mid \) the Euclidean norm on \( \mathbb{C} \). We recall that the space \( E \) is called locally complete if every closed disk \( D < E \) is a Banach disk (see [27, 10.2.1 Proposition, p. 197]). In particular, every sequentially complete \( \mathbb{C}\)-lcHs is locally complete. Further, we denote by \( L(F, E) \) the space of continuous linear maps from a \( \mathbb{C}\)-lcHs \( F \) to \( E \) and we sometimes use the notion \( (T, f) := T(f), f \in F \), for \( T \in L(F, E) \). If \( E = \mathbb{C} \), we write \( F^\prime := L(F, \mathbb{C}) \) for the dual space of \( F \). If \( F \) and \( E \) are (linearly) topologically isomorphic, we write \( F \cong E \). We denote by \( L_t(F, E) \) the space \( L(F, E) \) equipped with the locally convex topology of uniform convergence on the absolutely convex compact subsets of \( F \) if \( t = \kappa \), and on the bounded subsets of \( F \) if \( t = h \). We recall that the \( \varepsilon \)-product of Schwartz is defined by \( f \varepsilon E := L_t(F^\prime, E) \) where \( L(F^\prime, E) \) is equipped with the topology of uniform convergence on the equicontinuous subsets of \( F^\prime \).

If \( F(\Omega, E) \) is a space of (equivalence classes) of functions from a set \( \Omega \) to \( E \), we use the convention \( F(\Omega) := F(\Omega, \mathbb{C}) \). We denote by \( \mathcal{O}(\Omega, E) \) the space of \( E \)-valued holomorphic functions on an open set \( \Omega \subset \mathbb{C} \) and by \( \mathcal{C}^\infty(\Omega, E) \) the space of \( E \)-valued infinitely continuously partially differentiable functions on an open set \( \Omega \subset \mathbb{R}^2 = \mathbb{C} \). We denote by \( \partial^\beta f \) the partial derivative of \( f \in \mathcal{C}^\infty(\Omega, E) \) for a multiindex \( \beta \in \mathbb{N}_0^2 \). If \( E \) is locally complete, then a function \( f \in \mathcal{O}(\Omega, E) \) is infinitely complex differentiable on \( \Omega \) and we write \( \partial^n f := f^{(n)} \) for its \( n \)th complex derivative where \( n \in \mathbb{N}_0 \). If we want to emphasize that we consider derivatives w.r.t. a complex variable, we write
\[ \partial^n f \] instead of just \( \partial^n f \). We use the symbol \( \mathcal{C}(\Omega, E) \) for the space of \( E \)-valued continuous functions on a set \( \Omega \subset \mathbb{C} \). We denote by \( \overline{\mathbb{R}} := \mathbb{R} \cup \{ \pm \infty \} \) the two-point compactification of \( \mathbb{R} \) and set \( \overline{\mathbb{C}} := \mathbb{C} + i\mathbb{R} \). We define the distance of \( z \in \mathbb{C} \) to a set \( M \subset \mathbb{C} \) w.r.t. \( \cdot \) via \( d(z, M) := \inf_{w \in M} |z - w| \) if \( M \neq \emptyset \), and \( d(z, M) := \infty \) if \( M = \emptyset \).

For a compact set \( K \subset \overline{\mathbb{R}} \) and \( c > 0 \) we define the sets

\[
U_c(K) := \{ z \in \mathbb{C} | d(z, K \cap \mathbb{C}) < c \}
\]

\[
\cup_{\emptyset} \cup_{\{-\infty, -\infty, \infty, \infty, \infty, \infty, \infty, \infty \} - c, c}
\]

\[
\cup_{\emptyset} \cup_{\{-\infty, -\infty, \infty, \infty, \infty, \infty, \infty, \infty \} - c, c}
\]

and for \( n \in \mathbb{N} \)

\[
S_n(K) := \left( C \setminus U_n(K) \right) \cap \{ z \in \mathbb{C} | ||\text{Im}(z)|| < n \}.
\]

We remark that \( S_n(K) \) coincides with \( W_n(K) \) from the introduction for \( K = \emptyset \), \([0, \infty] \).

\[2.1. \textbf{Definition.} \]

Let \( K \subset \overline{\mathbb{R}} \) be compact. The space of rapidly decreasing holomorphic germs near \( K \) is defined as the inductive limit

\[ \mathcal{P}_*(K) := \lim_{n \to \infty} \mathcal{O}_n(U_n(K)) \]

with restrictions as linking and spectral maps where

\[ \mathcal{O}_n(U_n(K)) := \{ f \in \mathcal{O}(U_n(K)) \cap C(U_n(K)) | ||f||_{n,K} := \sup_{z \in U_n(K)} |f(z)|e^{-\frac{1}{\beta}|\text{Re}(z)|} < \infty \} \]

if \( K \neq \emptyset \). Further, we set \( \mathcal{P}_*(\emptyset) := 0 \).

\( \mathcal{P}_*(K) \) is a DFS-space by [30, p. 469] resp. [28, 1.11 Satz, p. 11]. Another common symbol for \( \mathcal{P}_*(K) \) is \( \mathcal{Q}(K) \) and for the special case \( \mathcal{P}_*(\mathbb{R}) \) the symbol \( \mathcal{P} \) is used as well (see [30, Definition 1.1.3, p. 468-469]). We remark that \( \mathcal{P}_*(K) \equiv \mathcal{A}(K) \) if \( K \subset \mathbb{R} \) where \( \mathcal{A}(K) \) is the space of germs of real analytic functions on \( K \) with its inductive limit topology.

\[2.2. \textbf{Definition (33, 3.2 Definition, p. 12-13).} \]

Let \( E \) be a \( \mathbb{C} \)-lcHs and \( K \subset \overline{\mathbb{R}} \) be compact.

\( a) \) The space of vector-valued slowly increasing infinitely continuously partially differentiable functions outside \( K \) is defined as

\[ \mathcal{C}^{exp}(\overline{\mathbb{C}} \setminus K, E) := \{ f \in \mathcal{C}^{\infty}(\overline{\mathbb{C}} \setminus K, E) | \forall n \in \mathbb{N}, m \in \mathbb{N}_0, \alpha \in \mathbb{A} : ||f||_{n,m,\alpha,K} < \infty \} \]

where

\[ ||f||_{n,m,\alpha,K} := \sup_{z \in S_{n}(K)} p_\alpha(\partial^\beta f(z))e^{-\frac{1}{\beta}|\text{Re}(z)|}. \]

\( b) \) The space of vector-valued slowly increasing holomorphic functions outside \( K \) is defined as

\[ \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K, E) := \{ f \in \mathcal{O}(\overline{\mathbb{C}} \setminus K, E) | \forall n \in \mathbb{N}, \alpha \in \mathbb{A} : ||f||_{n,\alpha,K} < \infty \} \]

where

\[ ||f||_{n,\alpha,K} := \sup_{z \in S_{n}(K)} p_\alpha(f(z))e^{-\frac{1}{\beta}|\text{Re}(z)|}. \]

Furthermore, we denote by \( ||f||_{n,\alpha,K}^{\hat{\cdot}} \) the usual seminorms on the quotient space

\[ bv_K(E) := \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus K, E)/\mathcal{O}^{exp}(\overline{\mathbb{C}}, E). \]
In the Banach-valued, particularly, scalar-valued, case the subscript $\alpha$ in the notation of the seminorms is omitted.

We note that $S_1(\mathbb{R}) = \varnothing$ and $\|f\|_{1,m,0,\mathbb{R}} = -\infty = \|f\|_{1,m,0,\mathbb{R}}$ for any $f: \mathbb{C} \setminus \mathbb{R} \to E$, $m \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$. Other common symbols for the spaces $\mathcal{E}^{\exp}(\mathbb{C} \setminus K, E)$ resp. $\mathcal{O}^{\exp}(\mathbb{C} \setminus K, E)$ are $\mathcal{E}(\mathbb{C} \setminus K, E)$ resp. $\mathcal{O}(\mathbb{C} \setminus K, E)$ (see [28, 1.2 Definition, p. 5]).

2.3. Definition ([strictly) admissible, [39, p. 55]). A $\mathcal{C}$-lcHs $E$ is called admissible, if the Cauchy-Riemann operator

$$\mathcal{E}^{\exp}(\mathbb{C} \setminus K, E) \to \mathcal{E}^{\exp}(\mathbb{C} \setminus K, E)$$

is surjective for any compact set $K \subset \mathbb{R}$. $E$ is called strictly admissible if $E$ is admissible and if, in addition,

$$\mathcal{E}^{\infty}(\Omega, E) \to \mathcal{C}^{\infty}(\Omega, E)$$

is surjective for any open set $\Omega \subset \mathbb{C}$.

If $E$ is strictly admissible and sequentially complete, then the sheaf of $E$-valued Fourier hyperfunctions is flabby and can be represented on the one hand by boundary values of exponentially slowly increasing holomorphic functions and on the other by equivalence classes of $E$-valued $\mathcal{P}_\ast$-functionals (see [41, Theorem 5.9, p. 33]). In particular, its subsheaf of $E$-valued hyperfunctions is flabby under this condition as well. Moreover, we may regard $\text{bv}_K(E)$ as the space of $E$-valued Fourier hyperfunctions with support in $K \subset \mathbb{R}$ under this condition by [41, 5.11 Lemma, p. 44].

2.4. Theorem ([39, 5.25 Theorem, p. 98]). If

a) $E$ is a $\mathcal{C}$-Fréchet space, or if

b) $E := F_b^\prime$ where $F$ is a $\mathcal{C}$-Fréchet space satisfying (DN), or if

c) $E$ is a complex ultrabornological $\mathcal{P}_\ast$-space satisfying (PA),

then $E$ is strictly admissible.

The definitions of the topological invariants (DN) and (PA) are given in [57, Chap. 29, Definition, p. 359] and [5, Section 4, Eq. (24), p. 577], respectively. Besides every $\mathcal{C}$-Fréchet space, the theorem above covers the space $E = S(\mathbb{R}^d)^\prime$ of tempered distributions, the space $\mathcal{D}(V)^\prime$ of distributions and the space $\mathcal{D}(V)^\prime$ of ultradistributions of Beurling type and many more spaces given in [17, 14, p. 1116], [11, Example 4.4, p. 14-15] and [74]. Due to [17, Theorem 6.9, p. 1125] another sufficient condition for the flabbiness of the sheaf of $E$-valued hyperfunctions is that $E$ is complete and 2-admissible which means that the 2-dimensional Laplace operator

$$\Delta: \mathcal{C}^{\infty}(\Omega, E) \to \mathcal{C}^{\infty}(\Omega, E)$$

is surjective for any open set $\Omega \subset \mathbb{R}^2$ (see [17, p. 1112]). If $E$ is an ultrabornological $\mathcal{P}_\ast$-space, then the three conditions strictly admissible, 2-admissible and (PA) are equivalent and actually necessary for a reasonable theory of $E$-valued (Fourier) hyperfunctions by [17, Theorem 8.9, p. 1139] and [11, Theorem 5.12, p. 45-46].

The introduced spaces $\text{bv}_K(E)$ and $\mathcal{P}_\ast(K)$ are connected by duality in the following way, which is a generalisation of [33, 4.1 Theorem, p. 41] where $E$ is complete, and the idea of its proof comes from [50, Theorem 3.3, p. 85-86] where $K = [a, \infty]$, $a \in \mathbb{R}$, and $E = \mathbb{C}$.

2.5. Theorem ([33, 3.15 Corollary (i), (iii), p. 21]). Let $E$ be a $\mathcal{C}$-lcHs and $K \subset \mathbb{R}$ a non-empty compact set. If

(i) $K \subset \mathbb{R}$ and $E$ is locally complete, or if

(ii) $E$ is sequentially complete,
then the map

\[ \mathcal{H}_K : bv_K(E) \to L_b(\mathcal{P}_s(K), E), \]

\[ \mathcal{H}_K([F])(\varphi) := \int_{\gamma_{K,n}} F(\zeta)\varphi(\zeta) d\zeta, \]

for \([F] \in bv_K(E)\) and \(\varphi \in \mathcal{O}_b(U_n(K))\), \(n \in \mathbb{N}\), is a topological isomorphism where \(\gamma_{K,n}\) is a suitable path around \(K\) in \(U_n(K)\) and the integral is a Pettis integral. If \(K\) is an interval, we may choose \(\gamma_{K,n}\) as the boundary of \(U_{\frac{1}{n}}(K)\), \(0 < c < \frac{1}{n}\), with clockwise orientation. Its inverse

\[ \mathcal{H}_K^{-1} : L_b(\mathcal{P}_s(K), E) \to bv_K(E) \]

is given by

\[ \mathcal{H}_K^{-1}(T) = \left[ C \setminus K \ni z \mapsto \frac{i}{2\pi} \left(T, \frac{e^{-(z-\cdot)^2}}{z - \cdot} \right) \right], \quad T \in L_b(\mathcal{P}_s(K), E). \]

We note that in \([60, 3.15\text{ Corollary (i)}, (iii), p. 21}] the orientation of \(\gamma_{K,n}\) is chosen to be counterclockwise but that only changes the sign. For our purpose the clockwise orientation is more suitable. Let

\[ B(A, E) := \mathcal{O}(C \setminus A, E)/\mathcal{O}(C, E) \]

be the space of hyperfunctions with values in a \(C\)-lcHs \(E\) and support in a closed set \(A \subset \mathbb{R}\). If \(K \subset \mathbb{R}\) is compact, then we equip \(\mathcal{O}(C \setminus K, E)\) resp. \(\mathcal{O}(C, E)\) with the topology of uniform convergence on compact subsets of \(C \setminus K\) resp. \(C\) and \(B(K, E)\) with the induced quotient topology. We note the following well-known corollary of the theorem above for the space \(B(K, E)\) whose \(E\)-valued version is unfortunately not contained in literature (to the best of our knowledge). The \(C\)-valued version can be found in \([60, \text{Theorem 2.1.3}, p. 25}]\.

2.6. Corollary. Let \(E\) be a locally complete \(C\)-lcHs and \(K \subset \mathbb{R}\) a non-empty compact set. Then \(\mathcal{H}_K : B(K, E) \to L_b(\mathcal{A}(K), E)\) is a topological isomorphism, with inverse \(\mathcal{H}_K^{-1}\) as above.

Proof. We only sketch the proof. The continuity and injectivity of \(\mathcal{H}_K\) are easily checked (like in the proof of Theorem 2.5 in \([60, \text{Theorem 2.1.3}, p. 25}]\), the surjectivity follows directly from Theorem 2.3 and \(\mathcal{A}(K) \equiv \mathcal{P}_s(K)\) for \(K \subset \mathbb{R}\) and actually gives that every \(f \in B(K, E)\) has a representing function in \(\mathcal{O}^{cp}(C \setminus K, E)\). The continuity of \(\mathcal{H}_K^{-1}\) follows like in the proof of Theorem 2.5. \(\Box\)

In the proof of \([60, \text{Theorem 2.1.3}, p. 25}]\ the inverse of \(\mathcal{H}_K\) is given in \([60, \text{Eq. (1.6)}, p. 26}]\ as

\[ \mathcal{H}_K^{-1}(T) = \left[ C \setminus K \ni z \mapsto \frac{i}{2\pi} \left(T, \frac{1}{z - \cdot} \right) \right], \quad T \in L_b(\mathcal{A}(K), E). \]

This is no contradiction to our inverse because the difference

\[ C \setminus K \ni z \mapsto \frac{i}{2\pi} \left(T, \frac{e^{-(z-\cdot)^2}}{z - \cdot} \right) - \frac{i}{2\pi} \left(T, \frac{1}{z - \cdot} \right) \]

extends to a function in \(\mathcal{O}(C, E)\) (cf. \([60, \text{Eq. (2.9)}, p. 47}]\).

Next, we define the Fourier transform on \(L_b(\mathcal{P}_s(\mathbb{R}), E)\) which goes back to Kawai \([30] in the case \(E = \mathbb{C}\). By \([30, \text{Proposition 3.2.4}, p. 483}]\ (cf. \([29, \text{Proposition 8.2.2}, p. 376}]\) the Fourier transform \(F : \mathcal{P}_s(\mathbb{R}) \to \mathcal{P}_s(\mathbb{R})\) defined by

\[ F(\varphi)(\zeta) := \hat{\varphi}(\zeta) := \int_{\mathbb{R}} \varphi(x)e^{-ix\zeta}dx, \quad \varphi \in \mathcal{O}_b(U_n(\mathbb{R})), \quad \zeta \in U_k(\mathbb{R}), \quad k > n, \]
is a topological isomorphism and \( \overline{\varphi} \in \mathcal{O}_b(U_k(\mathbb{R})) \) with \( k > n \) for \( \varphi \in \mathcal{O}_b(U_n(\mathbb{R})) \).

Here we follow the convention from [29, Proposition 8.2.2, p. 376] and use \( e^{-ix\xi} \) in the definition of \( F \) instead of \( e^{ix\xi} \) as in [30, Proposition 3.2.4, p. 483]. The Fourier transform on \( L_0(P_*(\mathbb{R}), E) \) is now defined by transposition and we obtain (cf. [28, 3.14 Folgerung, p. 46], [30, Definition 3.2.5, p. 483], [39, 4.6 Theorem, p. 53] and [11, 3.10 Corollary, p. 13]):

2.7. **Theorem.** Let \( E \) be a \( \mathcal{C} \)-lcHs. The Fourier transform \( F_* : L_0(P_*(\mathbb{R}), E) \to L_0(P_*(\mathbb{R}), E) \) defined by

\[
F_*(T)(\varphi) := (T, F(\varphi)), \quad T \in L(P_*(\mathbb{R}), E), \quad \varphi \in P_*(\mathbb{R}),
\]

is a topological isomorphism with inverse given by

\[
F_*^{-1}(T)(\varphi) = (T, F_*^{-1}(\varphi)), \quad T \in L(P_*(\mathbb{R}), E), \quad \varphi \in P_*(\mathbb{R}).
\]

Thus we have the following reasonable definition of the Fourier transform on \( b\mathbb{R}(E) \).

2.8. **Corollary.** Let \( E \) be a sequentially complete \( \mathcal{C} \)-lcHs. The Fourier transform \( F_* : b\mathbb{R}(E) \to b\mathbb{R}(E) \) defined by \( F_* := \mathcal{H}^{-1}_E \circ F_* \circ \mathcal{H}_E^{-1} \) is a topological isomorphism with inverse given by \( F_*^{-1} = \mathcal{H}_E^{-1} \circ F_*^{-1} \circ \mathcal{H}_E^{-1} \).

3. **Fourier transform of Fourier hyperfunctions with support in an extended half-line**

In this section we study the Fourier transform \( F_K \) of vector-valued Fourier hyperfunctions with support in the extended half-lines \( K = [a, \infty) \) or \( K = [-\infty, a] \) with \( a \in \mathbb{R} \) but \( K \neq \mathbb{R} \). We will use the duality given by the map \( \mathcal{H}_K \) and Kawai’s Fourier transform \( F_\mathbb{R} \) on the whole extended real line \( \mathbb{R} \). To characterise the range of \( F_K \) we need the supporting function of \( K \) and some of its properties, whose simple proof we omit.

3.1. **Proposition.** For a compact interval \( K \subseteq \mathbb{R} \) such that \( K \neq \pm \{\infty\} \) and \( \eta \in \mathbb{R} \) we define the supporting function

\[
H_K(\eta) := \sup_{x \in K} x\eta.
\]

Let \( a, b, c, d \in \mathbb{R} \). Then we have

a) \( H_{[a,b]}(\eta) = \max(a\eta, b\eta) \) for \( \eta \in \mathbb{R} \),
b) \( H_{[a,\infty)}(\eta) = a\eta - a|\eta| \) for \( \eta < 0 \),
c) \( H_{[-\infty,a)}(\eta) = a\eta + a|\eta| \) for \( \eta > 0 \),
d) \( H_{[a,b]}(\eta) + H_{[c,d]}(\eta) = H_{[a,b]}(\eta) + H_{[c,d]}(\eta) \) for \( \eta \in \mathbb{R} \).

3.2. **Definition.** Let \( E \) be a \( \mathcal{C} \)-lcHs and \( \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\} \).

a) For \( -\infty < a \leq \infty \) we define the space

\[
\mathcal{FO}_{[a,\infty)}(E) := \{f \in \mathcal{O}(-\mathbb{H}, E) \mid \forall k \in \mathbb{N}, \alpha \in \mathbb{R} : |f|_{[k,\alpha,[a,\infty)} < \infty \}
\]

where

\[
|f|_{[k,\alpha,[a,\infty]} := \sup_{\Im(z) \leq -\alpha} p_\alpha(f(z))e^{-\frac{1}{2}|z|^2}w_\alpha^*(\Im(z))
\]

and for \( x < 0 \)

\[
w_\alpha^*(x) := \begin{cases} H_{[a,\infty)}(x) = -a|x| & , \alpha \neq \infty, \\ H_{[k,\infty)}(x) = -k|x| & , \alpha = \infty. \end{cases}
\]
b) For \(-\infty < a < \infty\) we define the space
\[ \mathcal{FO}_{[-\infty,a]}(E) := \{ f \in \mathcal{O}(H, E) \mid \forall k \in \mathbb{N}, \alpha \in \mathbb{A} : |f|_{k,\alpha,[-\infty,a]} < \infty \}, \]
where
\[ |f|_{k,\alpha,[-\infty,a]} := \sup_{\text{Im}(z) > \frac{\pi}{2}} p_{\alpha}(f(z)) e^{-\frac{\pi}{2}|z| - \mu_{\alpha} \text{Im}(z)} \]
and for \( x > 0 \)
\[ w_{\alpha}^{-}(x) := \begin{cases} H[\alpha,\alpha]^{-1}(x) = a|x| & , a \neq -\infty, \\
H[\alpha,-\alpha]^{-1}(x) = -k|x| & , a = -\infty. \end{cases} \]

We may consider the elements of \( \mathcal{FO}_{[a,\infty]}(E) \) resp. \( \mathcal{FO}_{[-\infty,a]}(E) \) as elements of \( \mathcal{O}^{exp}(\mathbb{C} \setminus \mathbb{R}, E) \) by trivial extension.

3.3. Proposition. Let \( E \) be a \( \mathbb{C}\text{-lcHs.} \)

a) Let \( a \in \mathbb{R} \cup \{ \infty \}, f \in \mathcal{FO}_{[a,\infty]}(E) \) and set \( f(z) := 0 \) for \( \text{Im}(z) > 0 \). Then \( f \in \mathcal{O}^{exp}(\mathbb{C} \setminus \mathbb{R}, E) \).

b) Let \( a \in \mathbb{R} \cup \{ -\infty \}, f \in \mathcal{FO}_{[-\infty,a]}(E) \) and set \( f(z) := 0 \) for \( \text{Im}(z) < 0 \). Then \( f \in \mathcal{O}^{exp}(\mathbb{C} \setminus \mathbb{R}, E) \).

Proof. Let \( K = [a,\infty] \) or \( K = [-\infty,a] \) but \( K \neq \mathbb{R} \). In both parts the extended \( f \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R} \) and
\[ \|f\|_{k,\alpha,\mathbb{R}} \leq \begin{cases} e^{a|k|} |f|_{k,\alpha,K} & , a \in \mathbb{R}, \\
|k^{2} |f|_{k,\alpha,K} & , a = \pm \infty, \end{cases} \]
for \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{A} \). \( \square \)

3.4. Lemma. Let \( E \) be a sequentially complete \( \mathbb{C}\text{-lcHs, } a \in \mathbb{R} \text{ and } K := [a,\infty] \) or \( K := [-\infty,a] \) but \( K \neq \mathbb{R} \). Then the map
\[ \mathcal{F} : \mathcal{BV}_{K}(E) \to \mathcal{FO}_{K}(E), \]
\[ \mathcal{F}([F])(z) := \langle \mathcal{H}_{K}([F]), e^{-iz} \rangle = \int \gamma_{K} F(z) e^{-iz} dz, \]
where \( \gamma_{K} \) is the path along the boundary of \( U_{1/4}(K) \) with clockwise orientation, does not depend on the choice of \( c > 0 \), is well-defined and continuous. Further, the equations
\[ \mathcal{H}_{K}([\mathcal{F}([F])]) = -\mathcal{F}([\mathcal{H}_{[a,\infty]}([F])) \text{ and } \mathcal{H}_{K}([\mathcal{F}([F])]) = \mathcal{F}([\mathcal{H}_{[-\infty,a]}([F])]) \] (3)
hold on \( \mathcal{P}_{c}(\mathbb{R}) \) where \( \mathcal{F}([F]) \) is considered as an element of \( \mathcal{O}^{exp}(\mathbb{C} \setminus \mathbb{R}, E) \) in these equations according to Proposition 3.3.

Proof. Let \( K := [a,\infty], F \in \mathcal{O}^{exp}(\mathbb{C} \setminus [a,\infty], E) \) and \( \alpha \in \mathbb{A} \). Let \( a \in \mathbb{R} \) and define \( a_{+} := \max(0,a) \). For \( k \in \mathbb{N} \) let \( \zeta \in \mathbb{C} \) such that \( \text{Im}(\zeta) \leq -\frac{1}{k} \). We choose \( c := \frac{1}{2k} \) and get \( \frac{1}{4(k+1)} < c < 2(k+1) \) as well as
\[ a, \text{ Im}(\zeta) \leq -a|\text{Im}(\zeta)| \text{ and } \sup_{z \in \gamma_{K}} e^{\text{Re}(z)} \text{Im}(\zeta) = e^{-\text{inf}_{z \in \gamma_{K}} \text{Re}(z)} |\text{Im}(\zeta)| = e^{-(a-c)|\text{Im}(\zeta)|} \]
(4)
since \( \text{Im}(\zeta) < 0 \), where \( \gamma_{K} \) is the part of \( \gamma_{K} \) as depicted in Figure 3.
Furthermore, we have
\[
\frac{1}{2(k+1)} + \text{Im}(\zeta) \leq \frac{1}{2(k+1)} - \frac{1}{k} = -\frac{k+2}{2k(k+1)} < 0. \tag{5}
\]
Moreover, we obtain
\[
p_0(\mathcal{F}[\mathcal{F}^\alpha](\zeta)) = p_0\left(\int_{\gamma_K} F(z) e^{-iz\zeta} dz\right) \leq (\pi c + 2(a_+ - a)) \sup_{z \in \gamma_K} p_0(F(z)) e^{-Re(iz\zeta)} + \int_{a_+}^{\infty} p_0(F(t + ic)) e^{-Re(i(t+ic)\zeta)} dt \\
+ \int_{a_+}^{\infty} p_0(F(t - ic)) e^{-Re(i(t-ic)\zeta)} dt \leq C_0 \|F\|_{2(k+1),a_+} \|e^{-\frac{1}{2(k+1)}|Re(z)|}\sup_{z \in \gamma} e^{|Re(z)| + |Im(z)|} Re(\zeta) \|
+ \|F\|_{2(k+1),a_+} \int_{a_+}^{\infty} e^{\frac{1}{2(k+1)}|Im(z)|} Re(\zeta) dt \\
+ \|F\|_{2(k+1),a_+} \int_{a_+}^{\infty} e^{\frac{1}{2(k+1)}|Im(z)|} Re(\zeta) dt \\
\leq \|F\|_{2(k+1),a_+} C_0 e^{\frac{1}{2(k+1)} \max \{|a-c|, a_+, d(c-a)|\} |Im(\zeta)|} e^{|Re(\zeta)|} \\
+ 2\|F\|_{2(k+1),a_+} e^{|Re(\zeta)|} \int_{a_+}^{\infty} e^{\frac{1}{2(k+1)}|Im(\zeta)|} Re(\zeta) dt \leq \|F\|_{2(k+1),a_+} \left(C_1 e^{(c-a)|Im(\zeta)|} + \frac{2}{2(k+1)} + |Im(\zeta)|\right) e^{\frac{1}{2(k+1)}|Im(\zeta)|} a_+ \\
\leq \|F\|_{2(k+1),a_+} \left(C_1 + \frac{2}{2(k+1)} + |Im(\zeta)|\right) e^{\frac{1}{2(k+1)}|Im(\zeta)|} a_+ \\
\leq \|F\|_{2(k+1),a_+} \left(C_1 + \frac{4k(k+1)}{k+2} e^{\frac{1}{2(k+1)}|Im(\zeta)|} a_+ \right) e^{2(c-a)|Im(\zeta)|} \\
= C_2 \|F\|_{2(k+1),a_+} e^{2(c-a)|Im(\zeta)|} \tag{6}
\]

**Figure 3.** Path $\gamma_K$ for $K = [a, \infty], a \in \mathbb{R}$
If we choose $k$ our Fourier transform is well-defined and continuous.

Moreover, the definition of the Fourier transform does not depend on $c > 0$ or the choice of the representative by virtue of Cauchy’s integral theorem. Hence the estimate (6) we have for every $a \in \mathbb{R}$.

Now, we consider the case $a = \infty$. We replace Figure 3 by:

![Figure 4. Path $\gamma_K$ for $K = \{\infty\}$](image)

If we choose $k$ like before, we get

$$\sup_{z \in \gamma_K} e^{\Re(z)} \Im(z) = e^{-\frac{1}{2} \Im(z)}$$

and

$$p_\alpha(\mathcal{F}(F)(\zeta)) \leq \|F\|_{2(\kappa+1), \alpha, (\infty)} \left( \frac{2}{c} \left( \frac{1}{2\kappa + 1} + \frac{k - 1}{2} \right) e^{\frac{1}{2} \Im(z)} \right)$$

Moreover, the definition of the Fourier transform does not depend on $c > 0$ or the choice of the representative by virtue of Cauchy’s integral theorem. Hence the Fourier transform is well-defined and continuous.

Let us turn to (3). We only consider the case $K = [a, \infty]$ with $a \in \mathbb{R}$. By Proposition 3.3 we regard $\mathcal{F}(F')$ as an element of $\mathcal{O}^{\exp}(\mathbb{C} \setminus \mathbb{R}, E)$. Let $n \in \mathbb{N}$ and $\varphi \in \mathcal{O}_n(U_n(\mathbb{R}))$. We choose $\delta > 0$ and $k \in \mathbb{N}$ such that $\frac{1}{k} \leq \delta < \frac{1}{n}$. Due to the estimate (6) we have for every $\varepsilon' \in E'$

$$\int_{-\infty}^{\infty} \int_{-\pi/2}^{\pi/2} \left| (-1)^2 c e^{it \varepsilon'} e^{i(t + ic)} \varphi(s - i\delta) \right| dt$$

$$+ \int_{a}^{\infty} \left| (-1)^2 c e^{i(t + ic)} \varphi(s - i\delta) \right| ds$$

$$\leq C_2 \|\varphi\|_{2(\kappa+1), [a, \infty]} \int_{-\infty}^{\infty} e^{\frac{i}{k}|s-i\delta|} |\varphi(s - i\delta)| ds$$

$$\leq 2C_2 e^{\frac{i}{k}-\alpha \delta} \|\varphi\|_{2(\kappa+1), [a, \infty]} \|\varphi\|_{n, \mathbb{R}} \int_{0}^{\infty} e^{\frac{i}{k}(|\varphi| - 1)} d\varepsilon'$$

$$= 2C_2 e^{\frac{i}{k}-\alpha \delta} \|\varphi\|_{2(\kappa+1), [a, \infty]} \|\varphi\|_{n, \mathbb{R}} \frac{1}{n} < \infty.$$
It follows from the Fubini-Tonelli theorem that we may change the order of integration and obtain
\[
\langle e', \mathcal{H}_K([F]) \rangle(\varphi) = \mathcal{H}_K([F])(\varphi) = -\int_{\gamma[a, \infty)} \int_{\text{Im}(\zeta) = -\delta} (e' \circ F)(z)e^{-iz\zeta} \varphi(\zeta) dz d\zeta = -\int_{\gamma[a, \infty)} (e' \circ F)(z) \int_{\text{Im}(\zeta) = -\delta} \varphi(\zeta)e^{-iz\zeta} d\zeta dz
\]
for all \( e' \in E' \), where we used Cauchy’s integral theorem in the fourth equation. Thus the Hahn-Banach theorem implies the first equation in (3). The proof in the case \( K = [-\infty, a] \) is analogous. \( \square \)

If we want to emphasize that \( F \) depends on \( K \), we write \( F_K \) instead.

3.5. Theorem. Let \( E \) be a sequentially complete \( \mathbb{C} \)-left HS, \( a \in \mathbb{R} \) and \( K := [a, \infty] \) or \( K := [-\infty, a] \) but \( K \neq \mathbb{R} \). Then the Fourier transform \( F \) in Lemma 3.4.2 is a topological isomorphism.

Proof. Let \( F \in \mathcal{O}^{exp}(\mathbb{C} \setminus K, E) \) and \( \mathcal{F}([F]) = 0 \). By Proposition 3.3 we consider \( \mathcal{F}([F]) \) as an element of \( \mathcal{O}^{exp}(\mathbb{C} \setminus K, E) \). By virtue of (3) we obtain
\[
0 = \mathcal{H}_K([\mathcal{F}([F])])(\varphi) = \langle \mathcal{F}_*([\mathcal{H}_K([F])]), \varphi \rangle, \varphi \in \mathcal{P}_s(\mathbb{R}),
\]
implying \( [F] = 0 \) since \( \mathcal{F}_* \) and \( \mathcal{H}_K \) are isomorphisms. Therefore the Fourier transform \( F \) is injective.

The idea how to prove the surjectivity of \( F \) can be found in the proof of [31, Theorem 3.3.2, p. 485-487] and we adapt the notation that is used there. Let \( f \in \mathcal{F}O_K(E) \) and again we consider \( f \) as an element of \( \mathcal{O}^{exp}(\mathbb{C} \setminus K, E) \) by Proposition 3.3. We define \( \nu := \mathcal{F}_*^{-1}(\mathcal{H}_K([f])) \in L(\mathcal{P}_s(\mathbb{R}), E) \) and next we show the following:
\[
\forall \ a \in \mathbb{R}, \ n \in \mathbb{N}, \exists \ k \in \mathbb{N}, \ C > 0 \ \forall \ \varphi \in \mathcal{O}_n(U_n(\mathbb{R})): \ p_n(\nu(\varphi)) \leq C \varphi_{[k, a, K]} \leq \varphi_{n, K} \quad (7)
\]
Since \( \mathcal{P}_s(K) \) is dense in \( \mathcal{P}_s(\mathbb{R}) \) by [31, Theorem 2.2.1, p. 474] (see the correction of its proof in [69, Remark, p. 247-248]), this implies that there is an unique, well-defined extension \( \tilde{\nu} \in L(\mathcal{P}_s(\mathbb{R}), E) \) of \( \nu \) given by \( \tilde{\nu}(\varphi) := \lim_{m \to \infty} \nu(\varphi_m) \) where \( (\varphi_m)_{m \in \mathbb{N}} \) is a net in \( \mathcal{P}_s(\mathbb{R}) \) converging to \( \varphi \in \mathcal{P}_s(K) \) due to [27, 3.4.2 Theorem, p. 61].

We choose \( -\frac{1}{n} < \varepsilon < \frac{1}{n} \) and have for \( \varphi \in \mathcal{O}_n(U_n(\mathbb{R})) \)
\[
\mathcal{F}^{-1}\varphi(\zeta) = \frac{1}{2\pi \text{Im}(\zeta)v} \int \varphi(z)e^{iz\zeta} dz.
\]
Due to Cauchy’s integral theorem and the growth of \( \varphi \) the definition of \( \mathcal{F}^{-1}\varphi(\zeta) \) does not depend on the choice of \( \varepsilon \). Let \( K := [a, \infty] \). By virtue of Proposition 3.3 we consider \( f \) as an element of \( \mathcal{O}^{exp}(\mathbb{C} \setminus K, E) \) and we have for \( \delta > 0 \) by Theorem 2.7.4
\[
\nu(\varphi) = \mathcal{F}_*^{-1}(\mathcal{H}_K([f]))(\varphi) = \mathcal{H}_K([f])(\mathcal{F}^{-1}\varphi) = \int_{\gamma} f(\zeta)\mathcal{F}^{-1}\varphi(\zeta) d\zeta
\]
\[
= -\frac{1}{2\pi} \int_{\text{Im}(\zeta)=-\delta}^{\infty} f(\zeta) \int_{\text{Im}(\zeta)=\delta}^{-\infty} \varphi(z)e^{iz\zeta} dz \, d\zeta.
\]

First, we consider the case \( a \in \mathbb{R} \). We choose \( \delta, \delta' > 0 \) such that
\[
(\delta')^2 + \delta^2 < \frac{1}{n^2}, \quad \text{implying} \quad 0 < \delta, \delta' < \frac{1}{n},
\]
and define
\[
I_+ := \int_{0-i\delta}^{\infty-i\delta} f(\zeta) \int_{\text{Im}(\zeta)=\delta}^{-\infty} \varphi(z)e^{iz\zeta} dz \, d\zeta \quad \text{and} \quad I_- := \int_{-\infty+i\delta}^{-\infty-i\delta} f(\zeta) \int_{\text{Im}(\zeta)=-\delta}^{\infty} \varphi(z)e^{iz\zeta} dz \, d\zeta.
\]
Furthermore, we set
\[
J_{++} := \int_{0-i\delta}^{\infty-i\delta} f(\zeta) \int_{a-\delta'+i\delta}^{\infty} \varphi(z)e^{iz\zeta} dz \, d\zeta \quad \text{and} \quad J_{+-} := \int_{0-i\delta}^{-\infty+i\delta} f(\zeta) \int_{a-\delta'+i\delta}^{\infty} \varphi(z)e^{iz\zeta} dz \, d\zeta
\]
as well as
\[
J_{-+} := \int_{-\infty+i\delta}^{-\infty-i\delta} f(\zeta) \int_{a-\delta'+i\delta}^{\infty} \varphi(z)e^{iz\zeta} dz \, d\zeta \quad \text{and} \quad J_{--} := \int_{-\infty+i\delta}^{\infty-i\delta} f(\zeta) \int_{a-\delta'+i\delta}^{\infty} \varphi(z)e^{iz\zeta} dz \, d\zeta.
\]
Then we have \(-2\pi \nu(\varphi) = I_+ + I_-\) and \(I_+ = J_{++} + J_{+-}\) as well as \(I_- = J_{-+} + J_{--}\). Now, we set \( a_0 := \max(0, a - \delta') \) and choose \( k \in \mathbb{N} \) such that
\[
\frac{1}{k} < \delta.
\]
We obtain the following estimates for \( \alpha \in \mathbb{R} \)
\[
p_\alpha(J_{++})
\leq \int_{0-i\delta}^{\infty-i\delta} \int_{a-\delta'}^{\infty} |\varphi(s+i\delta)|e^{\text{Re}(i(s+i\delta)(t-i\delta))} ds \, dt
\leq \int_{0-i\delta}^{\infty-i\delta} \int_{a-\delta'}^{\infty} ||\varphi||_{n,K} e^{\frac{1}{n} |s|} e^{\delta - i\delta} ds \, dt
\leq e^{\frac{1}{n} \delta} ||\varphi||_{n,K} \int_{0-i\delta}^{\infty-i\delta} e^{\left(\frac{1}{n} - \frac{1}{\delta}\right)s} ds \, dt
\leq e^{\frac{1}{n} \delta} ||\varphi||_{n,K} \int_{0-i\delta}^{\infty-i\delta} e^{\left(\frac{1}{n} - \frac{1}{\delta}\right)s} ds \, dt
\leq e^{\frac{1}{n} \delta} \int_{0-i\delta}^{\infty-i\delta} e^{\left(\frac{1}{n} - \frac{1}{\delta}\right)s} ds \, dt
\leq e^{\frac{1}{n} \delta} ||\varphi||_{n,K} \int_{0-i\delta}^{\infty-i\delta} e^{\left(\frac{1}{n} - \frac{1}{\delta}\right)s} ds \, dt
\]
and analogously
\[
p_\alpha(J_{+-})
\leq e^{\frac{1}{n} \delta} ||\varphi||_{n,K} \int_{0-i\delta}^{\infty-i\delta} e^{\left(\frac{1}{n} - \frac{1}{\delta}\right)s} ds \, dt
\leq e^{\frac{1}{n} \delta} ||\varphi||_{n,K} \int_{0-i\delta}^{\infty-i\delta} e^{\left(\frac{1}{n} - \frac{1}{\delta}\right)s} ds \, dt
\]
\[= e^{\left(\frac{1}{n} - a\right)\delta} \|f\|_{k,\alpha,K} \varphi\|_{n,K} \int_{\infty}^{0} e^{(\frac{1}{n} - a)\delta} s ds \]
\[\int_{0}^{\infty} e^{\left(\frac{1}{n} - a\right)\delta} s ds \]

Let \( \eta > 0 \) and set

\[J(-\eta; I) := \int_{0-i\eta}^{\infty-i\eta} f(\zeta) J_I(\zeta) d\zeta \quad \text{and} \quad J(-\eta; II) := \int_{-\infty-i\eta}^{0-i\eta} f(\zeta) J_{II}(\zeta) d\zeta \]

where

\[J_I(\zeta) := \int_{-\infty-i\delta}^{a-i\delta+i\delta} \varphi(z) e^{iz\zeta} dz \quad \text{and} \quad J_{II}(\zeta) := \int_{-\infty-i\delta}^{a+i\delta} \varphi(z) e^{iz\zeta} dz \]

as well as

\[J(\downarrow; I) := \int_{-\infty-i\delta}^{-i\eta} f(\zeta) J_{I}(\zeta) d\zeta \quad \text{and} \quad J(\uparrow; II) := \int_{-i\eta}^{\infty-i\eta} f(\zeta) J_{II}(\zeta) d\zeta.\]

We remark that \( J_+ = J(-\delta; I) \) and \( J_- = J(-\delta; II) \).

\[\text{Figure 5. Paths of integration in } \zeta\text{-plane for } K = [a, \infty) \text{ (cf. [30, p. 487])} \]

\[\text{Figure 6. Paths of integration in } z\text{-plane for } K = [a, \infty), \ a \in \mathbb{R} \text{ (cf. [30, p. 487])} \]
Next, we show that:

(i) \( \lim_{\eta \to \infty} J(-\eta; I) = \lim_{\eta \to \infty} J(-\eta; II) = 0 \)

(ii) \( J(-\eta; I) + J(\uparrow; I) = J_{+} \) and \( J(-\eta; II) + J(\uparrow; I) = J_{+} \) for \( \eta > \delta \)

(i) We choose \( k \in \mathbb{N} \) such that

\[ \frac{1}{k} < \delta' . \]

Further, we observe that for \( b_0 \geq 0 \)

\[- b_0|x| \leq b_0x, \quad x \in \mathbb{R} . \]

Then we get the following estimates for \( \eta > \delta > \frac{1}{k} \)

\[ p_{\alpha}(J(-\eta; I)) \leq \int_{0}^{\infty} |f(k,\alpha, K) e^{\frac{1}{\alpha} t - i\eta}| - a_{\eta} \int_{-\infty}^{a_{\delta'}} \| \varphi \|_{n, \mathbb{R}} e^{-\frac{1}{\alpha} t} e^{a_{\eta} - \delta s} ds \, dt \]

\[ \leq e^{\left(\frac{1}{\alpha} - a\right)\eta} |f(k,\alpha, K) \| \varphi \|_{n, \mathbb{R}} \int_{0}^{\infty} e\left(\frac{1}{\alpha} - a\right) t dt \int_{-\infty}^{a_{\delta'}} e\left(\frac{1}{\alpha} + \eta\right) s ds \]

\[ = e^{\frac{1}{\alpha} (a - \delta')} |f(k,\alpha, K) \| \varphi \|_{n, \mathbb{R}} \frac{1}{(\delta - \frac{1}{\alpha}) (1 + \eta)} e\left(\frac{1}{\alpha} - \delta'\right) \eta \]

\[ \to 0, \quad \eta \to \infty , \]

and

\[ p_{\alpha}(J(-\eta; II)) \leq \int_{-\infty}^{\infty} |f(k,\alpha, K) e^{\frac{1}{\alpha} t - i\eta}| - a_{\eta} \int_{-\infty}^{a_{\delta'}} \| \varphi \|_{n, \mathbb{R}} e^{-\frac{1}{\alpha} t} e^{a_{\eta} + \delta s} ds \, dt \]

\[ \leq e^{\left(\frac{1}{\alpha} - a\right)\eta} |f(k,\alpha, K) \| \varphi \|_{n, \mathbb{R}} \int_{-\infty}^{\infty} e\left(\frac{1}{\alpha} - a\right) t dt \int_{-\infty}^{a_{\delta'}} e\left(\frac{1}{\alpha} + \eta\right) s ds \]

\[ = e^{\frac{1}{\alpha} (a - \delta')} |f(k,\alpha, K) \| \varphi \|_{n, \mathbb{R}} \frac{1}{(\delta - \frac{1}{\alpha}) (1 + \eta)} e\left(\frac{1}{\alpha} - \delta'\right) \eta \]

\[ \to 0, \quad \eta \to \infty , \]

for every \( \alpha \in \mathbb{A} \), implying statement (i).

(ii) The function defined by \( G_0(\zeta) := f(\zeta) J_I(\zeta) \) is holomorphic on the lower halfplane and with the path \( \gamma_0 \) from Figure 5 we have

\[ p_{\alpha}\left( \int_{\gamma_0} G_0(\zeta) d\zeta \right) = p_{\alpha}\left( - \int_{\delta}^{\eta} G_0(r - it) \cdot (-i) dt \right) \]

\[ \leq e^{\frac{1}{\alpha} \int_{\delta}^{\eta} |f(k,\alpha, K) \| \varphi \|_{n, \mathbb{R}} \int_{\delta}^{\eta} e^{\frac{1}{\alpha} |t|} e^{\delta t - \delta s} ds \, dt \]

\[ \leq e^{\left(\frac{1}{\alpha} - a\right)\eta} |f(k,\alpha, K) \| \varphi \|_{n, \mathbb{R}} \int_{\delta}^{\eta} e\left(\frac{1}{\alpha} - a\right) t dt \int_{\delta}^{\eta} e\left(\frac{1}{\alpha} + \eta\right) s ds \]

\[ \leq e\left(\frac{1}{\alpha} - \delta\right) |f(k,\alpha, K) \| \varphi \|_{n, \mathbb{R}} \frac{1}{(\delta - \frac{1}{\alpha}) (1 + \eta)} e\left(\frac{1}{\alpha} - \delta\right) \eta \]

\[ \to 0, \quad r \to \infty \]
for all \( \alpha \in \mathfrak{A} \). The function defined by \( G_1(\zeta) := f(\zeta)J_{II}(\zeta) \) is also holomorphic on the lower halfplane and like above we have

\[
p_a(\int_{\gamma_1} G_1(\zeta)d\zeta) = p_a(\int_{-\delta}^{\delta} G_1(-r - it) \cdot (-i)dt) \leq (\eta - \delta) \frac{1}{\eta} e^{\frac{\eta + \delta}{2n}} \| f \|_{\mathcal{K}, a, K} \| \varphi \|_{n, \mathcal{R}} e^{(\frac{\eta - \delta}{n})r} \to 0, \ r \to \infty.
\]

Hence Cauchy’s integral theorem proves claim (ii).

Further, we have

\[
J_+ + J_- - (J(-\eta; I) + J(-\eta; I)) = \frac{-i\eta}{2} \int_{-\delta}^{\delta} G_0(\zeta)d\zeta = \frac{-i\eta}{2} \int_{-\delta}^{\delta} G_1(\zeta)d\zeta.
\]

Now, we claim that

\[
J_I(\zeta) - J_{II}(\zeta) = \int_{a - \delta - i\delta}^{a - \delta + i\delta} \varphi(z) e^{iz\zeta}dz =: J_{III}(\zeta), \ \zeta \in [-i\eta, -i\delta]. \quad (14)
\]

Since for \(-\eta < a - \delta' \) and the path \( \gamma_2 \) from Figure 6 it holds that

\[
\left| \int_{\gamma_2} \varphi(z) e^{iz\zeta}dz \right| = \left| - \int_{-\delta}^{\delta} \varphi(-r + it) e^{(r - \eta)it}dz \right| \leq \int_{-\delta}^{\delta} \| \varphi \|_{n, \mathcal{R}} e^{\frac{\eta}{\eta + \delta}} - \eta \mathcal{R}(\zeta) dt \leq 2\delta \| \varphi \|_{n, \mathcal{R}} e^{\frac{(\eta)(\frac{\rho}{\alpha})}{\rho}} \to 0, \ r \to \infty,
\]

the claim follows again by Cauchy’s theorem. Thus we get

\[
J(\zeta; I) + J(\zeta; II) = \int_{-\delta}^{\delta} f(\zeta)J_{III}(\zeta)d\zeta =: J(-\eta; III),
\]

yielding

\[
J_+ + J_- - (J(-\eta; I) + J(-\eta; II)) = (J(\zeta; I) + J(\zeta; II)) = J(-\eta; III)
\]

and

\[
J_+ + J_- = \lim_{\eta \to 0^+} (J_+ + J_- - (J(-\eta; I) + J(-\eta; II))) = \lim_{\eta \to 0^+} J(-\eta; III). \quad (15)
\]
Now, we estimate the right-hand side of this equation and obtain for all \( \alpha \in \mathfrak{A} \)
\[
 p_\alpha(J(-\eta; III)) = p_\alpha\left( \int_{-\delta}^{\delta} f(-it) \cdot (-i) \int_{-\delta}^{\delta} \varphi(\alpha - \delta' + is)e^{i(\alpha - \delta' + is)(-it)}i\,ds \,dt \right)
\leq \int_{-\delta}^{\delta} |f|_{k,\alpha,K} e^{\frac{1}{k}t-at} \int_{-\delta}^{\delta} \|\varphi\|_{n,K} e^{-\frac{1}{k}(\alpha - \delta')t}i\,ds \,dt
\leq 2\delta|f|_{k,\alpha,K}\|\varphi\|_{n,K} \int_{-\delta}^{\delta} \frac{1}{\delta - \delta'} (e^{\frac{1}{k}(\alpha - \delta')t} - e^{\frac{1}{k}(\alpha - \delta')\eta}) dt
= 2\delta|f|_{k,\alpha,K}\|\varphi\|_{n,K} \frac{2\delta}{\delta - \delta'} (e^{\frac{1}{k}(\alpha - \delta')\eta} - e^{\frac{1}{k}(\alpha - \delta')})
\leq 2\delta/e^{\frac{1}{k}(\alpha - \delta')\eta}|f|_{k,\alpha,K}\|\varphi\|_{n,K}, \eta \to \infty. \tag{16}
\]

Therefore we get for all \( \alpha \in \mathfrak{A} \)
\[
2\pi p_\alpha(\nu(\varphi)) = p_\alpha(J_{++} + J_{--} + J_+ + J_-) \leq p_\alpha(J_{++}) + p_\alpha(J_{--}) + p_\alpha(J_+ + J_-)
\leq p_\alpha(J_{++}) + p_\alpha(J_{--}) + p_\alpha(\lim_{\eta \to \infty} J(-\eta; III))
\leq 2\delta/e^{\frac{1}{k}(\alpha - \delta')\eta}|f|_{k,\alpha,K}\|\varphi\|_{n,K} + 2\delta/e^{\frac{1}{k}(\alpha - \delta')\eta}|f|_{k,\alpha,K}\|\varphi\|_{n,K}.
\]

Now, we consider the case \( a = \infty \). The proof is quite similar if we replace Figure 6 by

![Figure 7. Paths of integration in z-plane for K = {\infty}](image)

and thus replace in the definition of the J-integrals \( a - \delta' \) by \( k - \delta' \). We obtain for the choice \( \frac{1}{k} < \delta, \delta' < \frac{1}{n} \) and \( \alpha \in \mathfrak{A} \) the estimate
\[
p_\alpha(J_{++}) \leq \frac{1}{(\delta - \frac{1}{k})(\frac{1}{n} - \delta)} e^{(\delta - \frac{1}{k})(k - \delta')} e^{(\frac{1}{k} - \delta')\delta} |f|_{k,\alpha,K}\|\varphi\|_{n,K}
\]
and the same for \( p_\alpha(J_{--}). \) Furthermore, for \( \eta \geq \delta \) and \( \alpha \in \mathfrak{A} \)
\[
p_\alpha(J(-\eta; I)) \leq e^{\frac{1}{k}(k - \delta')\eta} |f|_{k,\alpha,K}\|\varphi\|_{n,K} \frac{1}{(\delta - \frac{1}{k})(\frac{1}{n} + \eta)} e^{(\frac{1}{k} - \delta')\eta}
\to 0, \eta \to \infty,
\]
and the same for \( p_\alpha(J(-\eta; II)) \). Moreover, we have for \( \alpha \in \mathfrak{A} \)
\[
p_\alpha \int_{\gamma_0} G_0(\zeta) d\zeta \leq (\eta - \delta) \frac{1}{n + \delta} e^{\frac{\eta}{n + \delta}} \| f_{k,\alpha,K} \|_n, \varphi e^{(\frac{1}{n + \delta})r} \rightarrow 0, r \rightarrow \infty,
\]
and the same for \( p_\alpha(\int_{\gamma_0} G_1(\zeta) d\zeta) \). The proof of \( J_I(\zeta) - J_{II}(\zeta) = J_{III}(\zeta) \) for all \( \zeta \in [-\eta, -i\delta] \) needs no adjustment and we still have
\[
\lim_{\eta \rightarrow \infty} p_\alpha(J(-\eta; III)) \leq \frac{2\delta}{\delta' - \frac{1}{r}} e^{\frac{1}{\delta' - \frac{1}{r}}((\frac{1}{\delta' - \frac{1}{r}}) - \frac{1}{\delta'})} \| f_{k,\alpha,K} \|_n, \varphi
\]
Altogether we get for \( \alpha \in \mathfrak{A} \)
\[
2\pi p_\alpha(\nu(\varphi)) \leq \frac{2}{(\delta - \frac{1}{r})(\frac{1}{\delta} - \frac{1}{r})} e^{\frac{1}{\delta - \frac{1}{r}}((\frac{1}{\delta - \frac{1}{r}}) - \frac{1}{\delta'})} \| f_{k,\alpha,K} \|_n, \varphi
\]
and thus (4) in both cases.

Now, we consider \( \mathcal{F}(\mathcal{H}_K^{-1}(\varphi)) \) as an element of \( \mathcal{O}^{\text{exp}}(\mathbb{C}, E) \) by Proposition 3.3. We get for every \( \varphi \in \mathcal{P}_*(\mathbb{R}) \)
\[
\mathcal{H}_K^{-1}[\mathcal{F}(\mathcal{H}_K^{-1}(\varphi))] = ((\mathcal{F} \circ \mathcal{H}_K \circ \mathcal{H}_K^{-1})(\varphi)) = (\mathcal{F}, \varphi) = (\mathcal{F}, \varphi)
\]
\[
= (\nu, \varphi) = (\mathcal{F}^{-1}(\mathcal{H}_K^{-1}(f)), \varphi) = \mathcal{H}_K^{-1}(f)(\varphi),
\]
implying \( \mathcal{F}(\mathcal{H}_K^{-1}(\varphi)) = f \in \mathcal{O}^{\text{exp}}(\mathbb{C}, E) \) because \( \mathcal{H}_K \) is an isomorphism. Since \( \mathcal{F}(\mathcal{H}_K^{-1}(\varphi)) - f \) is in particular an entire function and
\[
\mathcal{F}(\mathcal{H}_K^{-1}(\varphi)) - f = 0
\]
on the upper halfplane, we obtain \( f = \mathcal{F}(\mathcal{H}_K^{-1}(\varphi)) \) by the identity theorem. Thus the Fourier transform \( \mathcal{F} \) is surjective.

Let \( \beta \in \mathfrak{A} \) and \( q \in \mathbb{N} \). Due to the continuity of \( \mathcal{H}_K^{-1} \) there are a bounded set \( M \subset \mathcal{P}_*(K), \alpha \in \mathfrak{A} \) and \( C_0 > 0 \) such that
\[
\left\| \mathcal{F}^{-1}(f) \right\|_{q,\beta,K}^\lambda = \left\| \mathcal{H}_K^{-1}(\varphi) \right\|_{q,\beta,K}^\lambda \leq C_0 \sup_{\varphi \in M} p_\alpha(\varphi).
\]
Since \( \mathcal{P}_*(K) \) is a DFS-space, there are \( n \in \mathbb{N} \) and \( \lambda > 0 \) such that \( M \subset \lambda B_n \) where \( B_n \) is the closed unit ball of \( \mathcal{C}_0(U_n(K)) \) by [32] Proposition 25.19 (2), p. 303. Let \( (\varphi_m)_{m \in \mathbb{N}} \) be a net in \( \mathcal{P}_*(\mathbb{R}) \) converging to \( \varphi \in M \). Then there are \( k \in \mathbb{N} \) and \( C > 0 \) such that
\[
p_\alpha(\varphi) = \lim_{m \in \mathbb{N}} p_\alpha(\nu(\varphi_m)) \leq C \| f_{k,\alpha,K} \|_n, \varphi = C \| f_{k,\alpha,K} \|_n, K,
\]
implying
\[
\left\| \mathcal{F}^{-1}(f) \right\|_{q,\beta,K}^\lambda \leq C_0 C \| f_{k,\alpha,K} \|_n, \varphi = C_0 C \| f_{k,\alpha,K} \|_n, K.
\]
Thus \( \mathcal{F}^{-1} \) is continuous and by Lemma 3.4 \( \mathcal{F} \) as well. The proof for the remaining case \( K = [-\infty, a] \) is analogous. \( \square \)

Theorem 3.5 improves [31] Theorem 3.3.1, 3.3.2, p. 485] (in one variable) since the latter theorem only covers the case \( u = 0 \) and \( E = \mathbb{C} \) and, more importantly, only shows that \( \mathcal{F} \) is a linear isomorphism. We kept track of the estimates to show that \( \mathcal{F} \) is actually a topological isomorphism.

In Section 5 we treat the asymptotic Fourier transform on the space of hyperfunctions \( \mathcal{B}(\mathbb{R}, E) \) which also needs a characterisation of the Fourier transform of
Fourier hyperfunctions with support in the union of two disjoint extended half-
lines.

3.6. Definition. Let $E$ be a $\mathbb{C}$-lcHs, $-\infty \leq a < b \leq \infty$ and set $f + g := f$ on $\mathbb{R}$ resp. $f + g := g$ on $-\mathbb{R}$ for $f \in \mathcal{FO}_{[a,b]}(E)$ and $g \in \mathcal{FO}_{[b,\infty]}(E)$. We define the space
\[ \mathcal{FO}_{[a,b]}(E) := \mathcal{FO}_{[\infty,\infty]}(E) \oplus \mathcal{HO}_{[b,\infty]}(E). \]

3.7. Theorem. Let $E$ be a sequentially complete $\mathbb{C}$-lcHs and $-\infty \leq a < b \leq \infty$. Then the Fourier transform
\[ \mathcal{F}_{[b]} : \mathcal{HO}_{[a,b]}(E) \to \mathcal{HO}_{[\infty,\infty]}(E), \]
\[ \mathcal{F}([F])(\zeta) := \mathcal{F}_{[\infty,\infty]}([F])(\zeta) + \mathcal{F}_{[b,\infty]}([F])(\zeta), \quad \zeta \in \mathbb{C} \setminus \mathbb{R}, \]
with $0 < c < \frac{b-a}{2}$ is a topological isomorphism and
\[ \mathcal{H}_{\mathbb{R}}([\mathcal{F}([F])]) = \mathcal{F}(\mathcal{H}_{\mathbb{R}}([F])) - \mathcal{F}(\mathcal{H}_{\mathbb{R}}([F])), \quad (\text{18}) \]
Proof. $\mathcal{F}$ being well-defined and its continuity follow from adjusting the proof of Lemma \[ \text{3.3} \]
Namely, if $a = -\infty$ and $b = \infty$, we have to adjust $c$ and choose $\varepsilon = \frac{1}{2m}$
with $m \in \mathbb{N}$, $m \geq \frac{1}{(b-a)^3}$, which guarantees that $\gamma_{[b,\infty]}$ does not intersect $[-a, a]$ and $\gamma_{[\infty, \infty]}$ does not intersect $[b, \infty]$. Then we have
\[ \mathcal{F}_{[b,\infty]}([F])(\zeta) \leq C_2 \|F\|_{2m(k+1), \alpha} \|\cdot\|_{[b,\infty]} \|\cdot\|_{[\alpha, \infty]} \Im(\zeta) \leq b \Im(\zeta), \quad \Im(\zeta) < 0, \]
and
\[ \mathcal{F}_{[\infty,\infty]}([F])(\zeta) \leq C_2 \|F\|_{2m(k+1), \alpha} \|\cdot\|_{[\infty,\infty]} \|\cdot\|_{[\alpha, \infty]} \Im(\zeta) \leq b \Im(\zeta), \quad \Im(\zeta) > 0, \]
for every $\alpha \in \mathbb{R}$ even if $F \in \mathcal{O}_{\exp}([\infty, \infty] \cup [b, \infty]), E)$. If $a = -\infty$ or $b = \infty$, no adjustment in the choice of $c$ is needed in the proof of Lemma \[ \text{3.3} \]
The proof of Lemma \[ \text{3.3} \]
shows that the equations
\[ \mathcal{H}_{\mathbb{R}}([\mathcal{F}_{[\infty,\infty]}([F])]) = \mathcal{F}(\mathcal{H}_{\mathbb{R}}([F])), \]
\[ \mathcal{H}_{\mathbb{R}}([\mathcal{F}_{[b,\infty]}([F])]) = -\mathcal{F}(\mathcal{H}_{\mathbb{R}}([F])), \quad (\text{19}) \]
even hold for $F \in \mathcal{O}_{\exp}([\infty, \infty] \cup [b, \infty]), E)$. This implies that (18) is valid as $\mathcal{F} = \mathcal{F}_{[\infty,\infty]} + \mathcal{F}_{[b,\infty]}$.

Let us turn to injectivity. Let $F \in \mathcal{O}_{\exp}([\infty, \infty] \cup [b, \infty]), E)$ such that $\mathcal{F}([F]) = 0$. This implies that $\mathcal{F}_{[\infty,\infty]}([F]) = 0$ and $\mathcal{F}_{[b,\infty]}([F]) = 0$ by definition of $\mathcal{F}$. It follows from the equations above that
\[ 0 = \mathcal{H}_{\mathbb{R}}([\mathcal{F}_{[\infty,\infty]}([F])])(\varphi) = (\mathcal{F}(\mathcal{H}_{\mathbb{R}}([F])), \varphi), \quad \varphi \in \mathcal{P}(\mathbb{R}), \]
and
\[ 0 = -\mathcal{H}_{\mathbb{R}}([\mathcal{F}_{[b,\infty]}([F])])(\varphi) = (\mathcal{F}(\mathcal{H}_{\mathbb{R}}([F])), \varphi), \quad \varphi \in \mathcal{P}(\mathbb{R}). \]
Since $\mathcal{F}$ is an isomorphism, we obtain
\[ (\mathcal{H}_{\mathbb{R}}([F]), \varphi) = 0 \quad \text{and} \quad (\mathcal{H}_{\mathbb{R}}([F]), \varphi) = 0, \varphi \in \mathcal{P}(\mathbb{R}). \]
Summing both equations, we have
\[ 0 = (\mathcal{H}_{\mathbb{R}}([F]), \varphi) + (\mathcal{H}_{\mathbb{R}}([F]), \varphi) = (\mathcal{H}_{\mathbb{R}}([F], \varphi), \varphi), \quad \varphi \in \mathcal{P}(\mathbb{R}). \]
The map $\mathcal{H}_{\mathbb{R}}([F], \varphi) : b\mathcal{H}_{[\infty,\infty]} \to L_2(\mathcal{P}(\mathbb{R}), \mathcal{H}_{[\infty,\infty]} \cup [b, \infty]), E) \to \mathcal{P}(\mathbb{R})$ is a topological isomorphism by Theorem \[ \text{2.5} \] and $\mathcal{P}(\mathbb{R})$ is dense in $\mathcal{P}(\mathbb{R})$, $\mathbb{R}$ by \[ \text{3.3} \] Theorem 2.2.1, p. 474. Thus we obtain from the equation above that $F \in \mathcal{O}_{\exp}([\infty, \infty]), E)$, which implies the injectivity of $\mathcal{F}$. Next, we prove that $\mathcal{F}$ is surjective with continuous inverse. Let $f = f_1 + f_2 \in \mathcal{FO}_{[\infty,\infty]}(E) \oplus \mathcal{FO}_{[b,\infty]}(E)$. Due to Theorem \[ \text{3.3} \] we have $g_1 := \mathcal{F}_{[\infty,\infty]}^{-1}(f_1) \in \mathcal{b}_{[\infty,\infty]}(E)$ and $g_2 := \mathcal{F}_{[b,\infty]}^{-1}(f_2) \in \mathcal{b}_{[b,\infty]}(E)$. Then $g_1 + g_2 \in \mathcal{b}_{[\infty,\infty]}(E)$ and $\mathcal{F}(g_1 + g_2)(\zeta) = \mathcal{F}_{[\infty,\infty]}(g_1 + g_2)(\zeta) + \mathcal{F}_{[b,\infty]}(g_1 + g_2)(\zeta).$
For \( \zeta \in \mathbb{C} \setminus \mathbb{R} \). Hence \( \mathcal{F}^{-1}(f) = \mathcal{F}^{-1}_{[a,b]}(f_1) + \mathcal{F}^{-1}_{[b,\infty]}(f_2) \), yielding the surjectivity of \( \mathcal{F} \) and the continuity of \( \mathcal{F}^{-1} \).

4. Fourier transform of Fourier hyperfunctions with real compact support

In this section we take a look at the Fourier transform \( \mathcal{F}_K \) for a real compact set \( K \). We want to characterise the range of the Fourier transform on \( \mathfrak{b}v_K(E) \) for an lcHs \( E \). The supporting function \( H_K \) is defined for general real compact sets \( K \) as well but it cannot distinguish a compact set \( K_1 \subset \mathbb{R} \) from a compact set \( K_2 \subset \mathbb{R} \) if their convex hulls coincide. Therefore we restrict our considerations to real compact convex sets, i.e. to intervals \( K = [a,b] \subset \mathbb{R} \). Here a corresponding Paley-Wiener theorem [24, Theorem 8.1.1, p. 368-369] is already known for \( K \) sets. We want to characterise the range of the Fourier transform on \( \mathcal{F}_E \mathfrak{O}_K(E) \),

\[
\mathcal{F}([F])(\zeta) = \mathcal{F}_{[a,\infty]}([F])(\zeta), \quad \text{Im}(\zeta) > 0,
\]

hold for \( F \in \mathfrak{O}_{\text{exp}}(\mathbb{C} \setminus K, E) \).

**Proof.** Let \( F \in \mathfrak{O}_{\text{exp}}(\mathbb{C} \setminus K, E) \), \( k \in \mathbb{N} \), \( a \in \mathfrak{A} \) and \( \zeta \in \mathbb{C} \). We choose \( c := \frac{1}{k+1} \) and get

\[
\sup_{z \in y_K} e^{\text{Re}(z) \text{Im}(\zeta)} = e^{\sup_{z \in y_K} \text{Re}(z) \text{Im}(\zeta)} \leq e^{\frac{1}{k+1} \text{Im}(\zeta)} + e^{\frac{1}{k+1} \text{Im}(\zeta)} 
\]

by Proposition 3.1.
We obtain
\[ p_α(ℒ[F])_k,α,K = p_α\left(\int_{γ_K} F(z)e^{-izδ}dz\right) \leq 2(πc + b - a) \sup_{z ∈ γ_K} p_α(F(z))e^{-Re(izδ)} \]
\[ \leq C_0\|F\|_{2(k+1),α,K} \sup_{z ∈ γ_K} e^{πc+1/2}\|Re(z)\| \sup_{z ∈ γ_K} e^{Re(z)Im(\zeta)+Im(z)Re(\zeta)} \]
\[ \leq \|F\|_{2(k+1),α,K} C_0 e^{\sup_{\{a-c,|b+c|\}}Re(\zeta)} \|Re(z)\| \sup_{z ∈ γ_K} e^{Re(z)Im(\zeta)} \]
\[ ≤ C_1\|F\|_{2(k+1),α,K} e^{2e|z|_E H(z)(Im(\zeta))} \]
and therefore
\[ \|ℒ[F]\|_{k,α,K} ≤ C_2\|F\|_{2(k+1),α,K}. \]
Moreover, the definition of the Fourier transform does not depend on \( c > 0 \) or the choice of the representative by virtue of Cauchy’s integral theorem. Hence the Fourier transform is well-defined and continuous.

Due to Cauchy’s integral theorem we can deform the path of integration for \( F ∈ 𝒪^{ex}(ℂ \setminus [a, b], E) \) and obtain the remaining equations (20).

\[ \Box \]

4.3. Theorem. Let \( E \) be a locally complete \( C\)-analytic. Then the Fourier transform \( 𝒟 \) in Lemma 4.2 is a topological isomorphism.

Proof. Let \( F ∈ 𝒪^{ex}(ℂ \setminus [a, b], E) \) and \( 𝒟_{[a, b]}(人都) = 0 \). By equation (20) we get \( 𝒟_{[a, b]}(人都) = 0 \) and hence \( 𝒟_{[a, b]} \) is injective by Theorem 3.5.

We prove the surjectivity of the Fourier transform like in Theorem 3.5. Let \( K := [a, b], f ∈ 𝒪_K(E) \) and set \( f_- := f \) on the lower and \( f_+ := 0 \) on the upper halfplane. We define \( 𝜈 := 𝒟^{-1}(人都) \) \( \in L( Patreon(\mathbb R), E) \) and next we show that (7) holds for \( K = [a, b] \). We choose \(-\frac{1}{\pi} < \overline{c} < \frac{1}{\pi} \) and have for \( 𝜈 ∈ 𝒪_n(U_n(\mathbb R)) \) and \( δ > 0 \)
\[ 𝜈(φ) = -\frac{1}{2\pi} \int_{\text{Im}(\zeta) = -δ} \int_{\text{Im}(z) = \overline{c}} f(\zeta) φ(z)e^{izδ}dz dζ. \]
like in Theorem 3.5. We choose \( δ, δ' > 0 \) according to (8) and define \( I_+ \) and \( I_- \) like before. Furthermore, we set
\[ J_{++} := \int_{0-\i δ}^{0+\i δ} f(\zeta) \int_{-\i δ+i δ}^{\i δ+i δ} φ(z)e^{izδ}dz dζ \]
and
\[ J_{++} := \int_{0+\i δ}^{0-\i δ} f(\zeta) \int_{-\i δ-i δ}^{\i δ-i δ} φ(z)e^{izδ}dz dζ. \]
as well as for \( y \in \mathbb{R} \) and \( x \in \{a, b\} \)

\[
J(-y; I_a) := \int_{0-iy}^{\infty-iy} f(\zeta) J_{I_a}(\zeta) d\zeta \quad \text{and} \quad J(-y; I_b) := \int_{-\infty-iy}^{0-iy} f(\zeta) J_{I_b}(\zeta) d\zeta
\]

and

\[
J_{I_a}(\zeta) := \int_{-\infty+i\delta}^{a-i\delta} \varphi(z) e^{iz\zeta} dz \quad \text{and} \quad J_{I_b}(\zeta) := \int_{-\infty-i\delta}^{0-i\delta} \varphi(z) e^{iz\zeta} dz
\]

plus

\[
J_{II_a}(\zeta) := \int_{b+i\delta}^{a+i\delta} \varphi(z) e^{iz\zeta} dz \quad \text{and} \quad J_{II_b}(\zeta) := \int_{b-i\delta}^{a-i\delta} \varphi(z) e^{iz\zeta} dz.
\]

Then we have \(-2\pi i(\varphi) = I_+ + I_- \) and \( I_+ = J_{++} + J(-\delta; I_b) + J(\delta; I_a) \) as well as \( I_- = J_{--} + J(-\delta; I_I) + J(\delta; I_a) \). If we choose \( k \in \mathbb{N} \) such that \( \frac{1}{k} < \delta \), we obtain for \( \alpha \in \mathbb{A} \)

\[
p_{\alpha}(J_{++}) \leq \int_0^\infty |f|_{k, \alpha, \mathcal{K}} e^{\frac{k}{a} |t-\delta|+\mathcal{H}_K(-\delta)} \int_{a-\delta'}^{b+\delta'} \|\mathcal{F}\|_{n, \mathcal{K}} e^{-\frac{k}{a} |s|} e^{\delta-t\delta} ds \, dt
\]

\[
\leq \frac{1}{\delta - \frac{1}{k}} (b - a + 2\delta') e^{\left(\frac{k}{a} + \delta\right) (\max(|a|, |b|) + \delta')} e^{\left(\frac{k}{a} - a\right) \delta} |f|_{k, \alpha, \mathcal{K}}\|\mathcal{F}\|_{n, \mathcal{K}}
\]

(22)

and an analogous estimate for \( J_{--} \).

Moreover, we set

\[
J(\uparrow) := \int_{-\delta}^{i\delta} f(\zeta) J_{I_a}(\zeta) d\zeta \quad \text{and} \quad J(\downarrow) := \int_{-i\delta}^{i\delta} f(\zeta) J_{I_b}(\zeta) d\zeta.
\]

**Figure 9.** Some paths of integration in \( \zeta \)-plane for \( K = [a, b] \) and \( x = b \)
Since it holds for the path $\gamma_0$ from Figure 10 that

$$p_m \left( \int f(\zeta) J_{I_k}(\zeta) d\zeta \right) \leq \int \left| f \right|_{k,\alpha,K} e^{-\frac{1}{\delta}|t+k|} H_K(t) \int_0^\infty \|\varphi\|_{n,K} e^{-\frac{1}{\delta}|s|} e^{-\tau^2+\delta} ds dt$$

$$\leq 2 \left| f \right|_{k,\alpha,K} \|\varphi\|_{n,K} \left( \frac{1}{\delta} \right)^{\frac{1}{\delta}} \int_0^\infty e^{-\frac{1}{\delta}|t+\max(1,\delta)|} dt \int_0^\infty e^{-\delta^{\frac{1}{\delta}}|s|} ds$$

$$\leq \frac{4\delta^2}{\delta} \left| f \right|_{k,\alpha,K} \|\varphi\|_{n,K} \left( \frac{1}{\delta} \right)^{\frac{1}{\delta}} \int_0^\infty e^{-\delta^{\frac{1}{\delta}}|s|} ds$$

$$\to 0, \quad r \to \infty,$$

we get that $J(-\delta; I_b) = J(\delta; I_b) + J(\uparrow)$ and analogously $J(-\delta; I_I) = J(\delta; I_I) + J(\downarrow)$ by Cauchy’s integral theorem. Like in (14) we have by Cauchy’s integral theorem

$$J(\uparrow) + J(\downarrow) = \int_{-\delta}^{\delta} f(\zeta) J_{II_b}(\zeta) d\zeta$$

where

$$J_{II_b}(\zeta) := \int_{b+\delta-i\delta}^{b+\delta+i\delta} \varphi(z) e^{iz\zeta} dz.$$

Further, we estimate

$$p_m(J(\uparrow) + J(\downarrow)) \leq \int \left| f \right|_{k,\alpha,K} e^{-\frac{1}{\delta}|t+k|} H_K(t) \int_0^\infty \|\varphi\|_{n,K} e^{-\frac{1}{\delta}|s|} e^{-\tau^2+\delta} ds dt$$

$$\leq 4\delta^2 \left( \frac{1}{\delta} \right)^{\frac{1}{\delta}} \left| f \right|_{k,\alpha,K} \|\varphi\|_{n,K}.$$

Altogether we obtain

$$-2\pi r(\varphi) = J_{\uparrow} + J(-\delta; I_b) + J(-\delta; I_I) + J(-\delta; I_P) + J(\delta; I_P) + J(\delta; I_I)$$

$$= J_{\uparrow} + J(\uparrow) + J(-\delta; I_b) + J(-\delta; I_I) + J(-\delta; I_P) + J(\delta; I_I) + J(\delta; I_P) + J(\downarrow) + J(\downarrow)$$

$$= J_{\uparrow} + J(\uparrow) + J(-\delta; I_b) + J(-\delta; I_I) + J(-\delta; I_P) + J(\delta; I_I) + J(\delta; I_P) + J(\downarrow) + J(\downarrow) + J(\downarrow) + J(\downarrow).$$
By the same method as in Theorem 3.5 we get that
\[ J(-\delta; I_a) + J(-\delta; II_a) = \lim_{\eta \to \infty} \int_{-i\delta}^{i\delta} f(\zeta) \varphi(z) e^{iz\zeta} d\zeta =: \lim_{\eta \to \infty} J(-\eta; II_a) \]
and
\[ J(\delta; I_a) + J(\delta; II_b) = \lim_{\eta \to \infty} \int_{-i\delta}^{i\delta} f(\zeta) J_{II_a}(\zeta) d\zeta =: \lim_{\eta \to \infty} J(\eta; II_b). \]
Thus \( p_\alpha(J(-\delta; I_a) + J(-\delta; II_a)) \) and \( p_\alpha(J(\delta; I_a) + J(\delta; II_b)) \) for \( \alpha \in \mathfrak{A} \) can be estimated by the right-hand side of (10) with \( K = [a, b] \), using that \( H_K(-t) = -at \) and \( H_K(t) = bt \) for \( t > 0 \). Combining the estimates (22), (23), (10), (16), we get (7). Since \( \mathcal{H}_{[a,b]}(\nu) = \mathcal{H}_{[a,\infty]}(\nu) \) holds by [46, Eq. (11), p. 20] for the unique extension \( \nu \in L(P_\star([a,b], E)) \) of \( \nu = F_\alpha^{-1}(\mathcal{H}(f_\star)) \), it follows by Theorem 3.5 that
\[ \mathcal{F}_{[a,b]}(\mathcal{H}_{[a,b]}(\nu)) = \mathcal{F}_{[a,\infty]}(\mathcal{H}_{[a,\infty]}(\nu)) = f_\star \]
holds on the lower halfplane. The left-hand side is an entire function and thus coincides with \( f \) on \( \mathbb{C} \) by the identity theorem, implying the surjectivity of \( \mathcal{F}_{[a,b]} \). The continuity of the inverse \( \mathcal{F}_{[a,b]}^{-1} \) is a consequence of (7) like before.

4.4. Remark. a) In short we express the inverse of \( \mathcal{F}_K \) for a non-empty compact interval \( K \subset \mathbb{R} \) as
\[ \mathcal{F}_{[-\infty,0]} = -\mathcal{H}_{[-\infty,0]} \circ \mathcal{F}_\star \circ \mathcal{H}_{\mathbb{R}} \quad \text{and} \quad \mathcal{F}_{[-\infty,b]} = \mathcal{H}_{[-\infty,b]} \circ \mathcal{F}_\star \circ \mathcal{H}_{\mathbb{R}} \]
for \(-\infty < a \leq b < \infty \), resp. as
\[ \mathcal{F}_{[a,b]}(f) = (\mathcal{H}_{[a,b]} \circ \mathcal{F}_\star)(f_\star) \]
for \(-\infty < a \leq b < \infty \) where \( f_\star := f \) on the upper halfplane and \( f_\star := 0 \) on the lower halfplane, since there is a unique extension of \( \mathcal{F}_\star^{-1}(\mathcal{H}_{\mathbb{R}}([f])) \) resp. \( \mathcal{F}_\star^{-1}(\mathcal{H}_{\mathbb{R}}([f_\star])) \) to \( L(P_\star(K), E) \) for every \( f \in \mathcal{FO}(E) \).

b) Let \( E \) be sequentially complete. If \( E \) is strictly admissible, then the sheaf of \( E \)-valued Fourier hyperfunctions is flabby by [41, Theorem 5.9 b), p. 33]. Hence for \( f \in bv_\nu(E) \) and \( a \in \mathbb{R} \) there are \( f_1 \in bv_{\nu,[a,\infty)}(E) \) and \( f_2 \in bv_{\nu,[a,\infty)}(E) \) such that \( f = f_1 + f_2 \) by [24, Lemma 1.4.4, p. 36]. Due to [16, Eq. (6), p. 14] we have
\[ \mathcal{H}_{\mathbb{R}}(f_1) = \mathcal{H}_{[-\infty,a]}(f_1) \quad \text{and} \quad \mathcal{H}_{\mathbb{R}}(f_2) = \mathcal{H}_{[a,\infty)}(f_2) \]
on \( P_\star(E) \). Thus we get
\[ \mathcal{F}_\star \circ \mathcal{H}_{\mathbb{R}}(f) = (\mathcal{F}_\star \circ \mathcal{H}_{\mathbb{R}})(f_1) + (\mathcal{F}_\star \circ \mathcal{H}_{\mathbb{R}})(f_2) \]
\[ = (\mathcal{F}_\star \circ \mathcal{H}_{[-\infty,a]})(f_1) + (\mathcal{F}_\star \circ \mathcal{H}_{[a,\infty)})(f_2) \]
\[ = \mathcal{H}_{\mathbb{R}}([\mathcal{F}_{[-\infty,a]}(f_1)]) - \mathcal{H}_{\mathbb{R}}([\mathcal{F}_{[a,\infty)}(f_2)]). \]
Therefore we may write the Fourier transform of \( f \) by Corollary 2.3 as
\[ \mathcal{F}_\mathbb{R}(f) = (\mathcal{H}_{\mathbb{R}}^{-1} \circ \mathcal{F}_\star \circ \mathcal{H}_{\mathbb{R}})(f) = [\mathcal{F}_{[-\infty,a]}(f_1) - \mathcal{F}_{[a,\infty)}(f_2)] \]
keeping Proposition 3.3 in mind (cf. [33, Theorem 3.2.6, p. 483, Definition 3.2.7, p. 484] for \( a = 0 \) and \( E = \mathbb{C} \)). If \( f \in bv_\nu(E) \), such that \( K \subset [a,b] \subset \mathbb{R} \) compact, we get by (20) that
\[ \mathcal{F}_{[a,b]}(f) = [\mathcal{F}_{[-\infty,b]}(f)] = [\mathcal{F}_{[a,b]}(f)_+] = -[\mathcal{F}_{[a,b]}(f)] = -(\mathcal{F}_{[a,b]}(f)_-). \]
We close this section with a discussion of some standard operations and their connection to the Fourier transform \( \mathcal{F}_K \) for compact intervals \( \emptyset \neq K \subset \mathbb{R} \). We have the following counterpart of (50) Example 2.5, p. 47, which follows from a simple calculation.
4.5. Proposition. Let $E$ be a $\mathbb{C}$-lcHs and $\emptyset \neq K \subseteq \mathbb{R}$ a compact interval. Let $\tau_h([F]) := [F(-h)]$, $h \in \mathbb{R}$, be the shift operator from $bv_K(E)$ to $bv_{h+K}(E)$. If

(i) $K \subset \mathbb{R}$ and $E$ is locally complete, or if

(ii) $E$ is sequentially complete,

then we have for $[F] \in bv_K(E)$

$$\mathcal{F}_{h+K}(\tau_h([F])) = e^{-ih \cdot (\cdot)} \mathcal{F}_K([F]).$$

Next, we transfer [54, Example 2.6, p. 48] and [54, Proposition 2.9, p. 49] to our setting.

4.6. Proposition. Let $E$ be a sequentially complete $\mathbb{C}$-lcHs, $\emptyset \neq K \subseteq \mathbb{R}$ a compact interval and $P(-i\partial) := \sum_{k=0}^{\infty} \frac{c_k}{k!} (-i\partial)^k$ where $(c_k) \in \mathbb{C}$ and $P$ is of exponential type 0, i.e.

$$\forall \varepsilon > 0 \exists C > 0 \forall k \in \mathbb{N}_0: |c_k| = |P(c_k)(0)| \leq C \varepsilon^k.$$

Then $P(-i\partial)$ resp. $P(-i\partial)([F]) := [P(-i\partial)F]$ and the multiplication operator $M_P(F) := PF$ resp. $M_P([F]) := [PF]$ are well-defined continuous linear operators on $\mathcal{O}^{exp}(\mathbb{C} \setminus K, E)$ and $\mathcal{O}^{exp}(\mathbb{C}, E)$ resp. $bv_K(E)$, and for $[F] \in bv_K(E)$ we have

$$\mathcal{F}_K(P(-i\partial)[F])(\zeta) = P(\zeta) \mathcal{F}_K([F])(\zeta) \quad (24)$$

and

$$\mathcal{F}_K([F])(\zeta) = P(\zeta) \mathcal{F}_K([F])(\zeta) \quad (25)$$

for $\text{Im}(\zeta) < 0$ if $K = [a, \infty)$, for $\text{Im}(\zeta) > 0$ if $K = [-\infty, a)$, and for $\zeta \in \mathbb{C}$ if $K \subset \mathbb{R}$, respectively.

Proof. Let $F \in \mathcal{O}^{exp}(\mathbb{C} \setminus K, E)$ and $n \in \mathbb{N}$. We choose $0 < r < \frac{1}{2n}$ and remark that

$$-\frac{1}{n} |\text{Re}(z)| \leq -\frac{1}{n} |\text{Re}(\zeta)| + \frac{1}{n} \text{Re}(z) - \text{Re}(\zeta) | \leq -\frac{1}{2n} |\text{Re}(\zeta)| + \frac{r}{n}, \quad z, \zeta \in \mathbb{C}, |\zeta - z| = r.$$

By Cauchy’s inequality [44, Corollary 5.3 a), p. 263] we have for $k \in \mathbb{N}_0$ and $\alpha \in \mathfrak{A}$

$$p_\alpha(\partial^k F(z)) \leq \frac{k!}{r^k} \sup_{\zeta \in \mathbb{C}, |\zeta - z| = r} p_\alpha(F(\zeta)), \quad z \in S_n(K),$$

which implies

$$\sup_{z \in S_n(K)} p_\alpha(\partial^k F(z)) e^{-\frac{1}{n} |\text{Re}(z)|} \leq \sup_{z \in S_n(K)} \frac{k!}{r^k} \sup_{\zeta \in \mathbb{C}, |\zeta - z| = r} p_\alpha(F(\zeta)) e^{-\frac{1}{n} |\text{Re}(\zeta)|} \leq e^{\frac{k!}{r^k}} \sup_{\zeta \in S_n(K)} p_\alpha(F(\zeta)) e^{-\frac{1}{n} |\text{Re}(\zeta)|} = e^{\frac{k!}{r^k}} \|F\|_{2n_\alpha, K}.$$  

Since $P$ is of exponential type 0, we have for $0 < \varepsilon < r$ and $m, l \in \mathbb{N}_0$ with $m \geq l$

$$\left\| \sum_{k=l}^{m} \frac{c_k}{k!} (-i\partial)^k F \right\|_{n_\alpha, K} \leq \sup_{z \in S_n(K)} \sum_{k=l}^{m} \frac{|c_k|}{k!} p_\alpha(\partial^k F(z)) e^{-\frac{1}{n} |\text{Re}(z)|} \leq C e^{\frac{k!}{r^k}} \sum_{k=l}^{m} \left( \frac{\varepsilon}{r} \right)^k \|F\|_{2n_\alpha, K},$$

which implies that the sum defining $P(-i\partial)$ converges on $\mathcal{O}^{exp}(\mathbb{C} \setminus K, E)$ because this space is sequentially complete due to the sequential completeness of $E$ and $\mathcal{O}^{exp}(\mathbb{C} \setminus K)$ combined with $\mathcal{O}^{exp}(\mathbb{C} \setminus K, E) \cong \mathcal{O}^{exp}(\mathbb{C} \setminus K) \varepsilon E$ by [44, Remark 3.4 b), p. 8]. Moreover, we deduce that

$$\|P(-i\partial) F\|_{n_\alpha, K} \leq C e^{\frac{k!}{r^k}} \left( \frac{1}{1 - \varepsilon} \right) \|F\|_{2n_\alpha, K}.$$
yielding that $P(-i\partial)$ is a continuous linear operator on $O^{exp}(\overline{\mathbb{C}} \setminus K, E)$. Analogously, we obtain that $P(-i\partial)$ is a continuous linear operator on $O^{exp}(\overline{\mathbb{C}}, E)$. Hence $P(-i\partial)$ is a well-defined continuous linear operator on $bV_{K}(E)$ as well.

Let us turn to (24). If $P(-i\partial)$ is of finite order, then
\[
\langle e', \mathcal{F}_{K}(P(-i\partial)[F])(\zeta) \rangle = \mathcal{F}_{K}(P(-i\partial)[e' \circ F])(\zeta) = P(\zeta)\mathcal{F}_{K}([e' \circ F])(\zeta) = \langle e', P(\zeta)\mathcal{F}_{K}([F])(\zeta) \rangle, \quad e' \in E'
\]
holds by partial integration, which implies (24) by the Hahn-Banach theorem. If $P(-i\partial)$ is of infinite order, then the sum defining $P(-i\partial)$ converges on $bV_{K}(E)$. In combination with the continuity of $\mathcal{F}_{K}$ this proves (24).

The proof in the case of the multiplication operator $M_{P}$ is quite similar. Since $P$ is of exponential type 0, we have for $0 < \varepsilon < \frac{1}{2n}$ and $m, l \in \mathbb{N}_{0}$ with $m \geq l$
\[
\left\| \sum_{k_{1}=0}^{m} \frac{c_{k}}{k!}(\zeta \partial)^{k} F \right\|_{n,\alpha,K} \leq C_{\varepsilon} \sup \left\| \sum_{k_{1}=0}^{m} \sum_{k_{2}=0}^{l} \frac{k^{k}}{k!} e^{z \frac{i}{2} \text{Re}(z)} p_{\alpha}(F(z)) e^{-\frac{i}{2} \text{Re}(z)} \right\|_{2n,\alpha,K}
\]
\[\leq C_{\varepsilon} e^{\frac{1}{2}} \sum_{k_{1}=0}^{m} \sum_{k_{2}=0}^{l} \frac{k^{k}}{k!} e^{-\frac{i}{2} \text{Re}(z)} \| F \|_{2n,\alpha,K}
\]
\[\leq C_{\varepsilon} e^{\frac{1}{2}} \sum_{k_{1}=0}^{m} \sum_{k_{2}=0}^{l} k!(2n)^{k} \| F \|_{2n,\alpha,K} = C_{\varepsilon} e^{\frac{1}{2}} \sum_{k_{1}=0}^{m} (2\varepsilon n)^{k} \| F \|_{2n,\alpha,K},
\]
which yields that the sum defining $P$ converges on $O^{exp}(\overline{\mathbb{C}} \setminus K, E)$ as above. Further, we derive that for $\varepsilon < \frac{1}{2n}$
\[
\left\| PF \right\|_{n,\alpha,K} \leq C_{\varepsilon} e^{\frac{1}{2}} \frac{1}{1 - 2\varepsilon n} \| F \|_{2n,\alpha,K},
\]
implying that $M_{P}$ is a continuous linear operator on $O^{exp}(\overline{\mathbb{C}} \setminus K, E)$. Analogously, we obtain that $M_{P}$ is a continuous linear operator on $O^{exp}(\overline{\mathbb{C}}, E)$ and hence a well-defined continuous linear operator on $bV_{K}(E)$ as well.

Let us turn to (25). If $P$ is a polynomial, then
\[
\langle e', \mathcal{F}_{K}(P[F])(\zeta) \rangle = \mathcal{F}_{K}(P[e' \circ F])(\zeta) = P(i\partial)\mathcal{F}_{K}([e' \circ F])(\zeta) = \langle e', P(i\partial)\mathcal{F}_{K}([F])(\zeta) \rangle, \quad e' \in E'
\]
holds by differentiation w.r.t. the parameter $\zeta$, which implies (25) by the Hahn-Banach theorem. If $P$ is not a polynomial, then we have for its Taylor series ($P_{j}$) and $g := \mathcal{F}_{K}([F])$ by Cauchy’s inequality with $0 < \varepsilon < r < \frac{1}{2n}$
\[
p_{\alpha}(\sum_{k_{1}=0}^{m} \frac{c_{k}}{k!}(i\partial)^{k} g(\zeta)) \leq C_{z} \sum_{k_{1}=0}^{m} \frac{1}{r} \sup_{z \in C,|z| = |r|} p_{\alpha}(g(z)),
\]
which implies that $P_{j}(i\partial)g$ converges to $P(i\partial)g$ as $j \to \infty$ in the sequentially complete space $O([-\infty, E]$ if $K = [a, \infty)$, in $O(\infty, E]$ if $K = [-\infty, a]$ and in $O(\overline{\mathbb{C}}, E)$ if $K \subset \mathbb{R}$ with respect to the topology of uniform convergence on compact subsets, respectively. In combination with the continuity of $\mathcal{F}_{K}$ this proves (25).

In [50, Proposition 2.9, p. 49] it is allowed that the $P$ in (25) is an entire function of exponential type because there the test functions (see Definition 4.1 for the introduction) are exponentially decreasing of type $-\infty$ and not exponentially increasing like ours. At the end of this section we treat convolutions.

4.7. Theorem. Let $(E_{1}, p_{\alpha})_{\alpha \in A}$, $(E_{2}, p_{\beta})_{\beta \in B}$ and $(E_{2}, p_{\omega})_{\omega \in \Omega}$ be sequentially complete $C$-IcHs such that a canonical bilinear map $: E_{1} \times E_{2} \to E$ is defined with the property

\[
\forall \alpha \in A \exists \beta \in B, \omega \in \Omega, D > 0 \forall x \in E_{1}, y \in E_{2}: p_{\alpha}(x \cdot y) \leq Dp_{\beta}(x)p_{\omega}(y). \tag{26}
\]
Let $a, b, c, d \in \mathbb{R}$ and $a, c \neq -\infty$ or $b, d \neq \infty$. 


a) If \( f \in \mathcal{FO}_{[a,b]}(E_1) \) and \( g \in \mathcal{FO}_{[c,d]}(E_2) \), then \( fg \in \mathcal{FO}_{[a+c,b+d]}(E) \) where \( fg \) is defined by \( (fg)(z) := f(z) \cdot g(z) \).

b) We define the convolution
\[
* : \mathcal{bv}_{[a,b]}(E_1) \times \mathcal{bv}_{[c,d]}(E_2) \to \mathcal{bv}_{[a+c,b+d]}(E), \quad f \ast g := \mathcal{F}^{-1}(\mathcal{F}(f) \mathcal{F}(g)).
\]

The convolution is well-defined, bilinear and continuous.

c) If \( E_1 = E_2 \) and \( \cdot : E_1 \times E_2 \to E \) is commutative, then the convolution is commutative as well. If \( E = E_1 = E_2 = \mathbb{C} \) and \( \cdot : E_1 \times E_2 \to E \) is the multiplication, then the convolution is associative.

Proof. a) Let \( a, c \neq -\infty \) and \( b, d = \infty \). Let \( f \in \mathcal{FO}_{[a,\infty]}(E_1) \) and \( g \in \mathcal{FO}_{[c,\infty]}(E_2) \).

Due to the bilinearity of the map \( \cdot : E_1 \times E_2 \to E \) we get for \( z \in \mathbb{H} \) and \( h \in \mathbb{C}, h \neq 0 \),
\[
\begin{align*}
f(z+h) & \cdot g(z+h) - f(z) \cdot g(z) \\
& = \left( \frac{f(z+h) - f(z)}{h} \right) \cdot g(z+h) + f'(z) \cdot (g(z+h) - g(z)) \\
& = f(z) \left( \frac{g(z+h) - g(z)}{h} - g'(z) \right).
\end{align*}
\]

This implies that \( fg \in \mathcal{O}(\mathbb{H}) \) by (20). First, we consider the cases \( a, c \in \mathbb{R} \) or \( a = c = \infty \). Let \( \alpha \in \mathbb{R} \) and \( k \in \mathbb{N} \). Due to (20) there are \( \beta \in \mathbb{B}, \omega \in \Omega \) and \( D > 0 \) such that
\[
\|fg\|_{k,\alpha,[a+c,\infty]} \leq D \|f\|_{2k,\beta,[a,\infty]}\|g\|_{2k,\omega,[c,\infty]}.
\]

Now, let \( a := \infty \) or \( c := \infty \) but \( a \neq c \). W.l.o.g. \( a = \infty \). Then there is \( n \in \mathbb{N} \) such that \( k + |c| \leq n \) and
\[
\|fg\|_{k,\alpha,(\infty)} \leq D \|f\|_{2n,\beta,(\infty)}\|g\|_{2n,\omega,(\infty)}.
\]

Hence \( fg \in \mathcal{FO}_{[a+c,\infty]}(E) \). The proof in the other cases is analogous.

b) The convolution is well-defined by part a) and Theorem 4.6 resp. Theorem 4.7. The bilinearity is due to the bilinearity of the map \( \cdot : E_1 \times E_2 \to E \). The continuity follows directly from (20) and Theorem 4.6 resp. Theorem 4.8.

c) This is obviously true. \( \square \)

4.8. Remark. In the following cases condition (20) is fulfilled.

a) Let \( E_1 = E_2 \) and \( E \) be complex Banach spaces and \( a : E_1 \times E_1 \to E \) a continuous bilinear form. Define \( x \cdot y := a(x, y) \) for \( x, y \in E_1 \). If \( a \) is symmetric as well, then \( \cdot \) is commutative.

b) Let \( E_2, E \) be complex Banach spaces and \( E_1 := L(E_2, E) \). Define \( x : y := x(y) \) for \( x \in E_1, y \in E_2 \).

c) Let \( X, Y, Z \) be complex Banach spaces, \( E_1 := L(X, Y) \), \( E_2 := L(Y, X) \) and \( E := L(Y, Z) \). Define \( x : y := x \circ y \) for \( x \in E_1, y \in E_2 \).

If one of the Fourier hyperfunctions has real compact support, we may define the convolution in a more common way.

4.9. Corollary. Let \( (E, (p_\alpha)_{\alpha \in \mathbb{A}}), (E_1, (p_\beta)_{\beta \in \mathbb{B}}) \) and \( (E_2, (p_\omega)_{\omega \in \Omega}) \) be sequentially complete \( \mathbb{C}\)-cHs such that a canonical bilinear map \( \cdot : E_1 \times E_2 \to E \) is defined with the property
\[
\forall \alpha \in \mathbb{A} \exists \beta \in \mathbb{B}, \omega \in \Omega, D > 0 \forall x \in E_1, y \in E_2 : p_\alpha(x \cdot y) \leq Dp_\beta(x)p_\omega(y).
\]

Let \( [a, b] \subset \mathbb{R} \) and \( [c, d] \subset \mathbb{R} \). We set
\[
\otimes : \mathcal{bv}_{[a,b]}(E_1) \times \mathcal{bv}_{[c,d]}(E_2) \to \mathcal{bv}_{[a+c,b+d]}(E), \quad [F] \otimes [G] := \left[ z \mapsto \int_{[a,b] \cdot [c,d]} F(w)G(z-w)dw \right],
\]
\[
\mathcal{F}^{-1}(\mathcal{F}(f) \mathcal{F}(g)) \in \mathcal{O}(\mathbb{H}) \text{ for all } f, g \in \mathcal{FO}_{[a,\infty]}(E_1) \text{ and } \mathcal{FO}_{[c,\infty]}(E_2).
\]
where for \( z \in S_n([a + c, b + d]) \), \( n \in \mathbb{N} \), the path \( \gamma_{[a,b],2n} \) goes along the boundary of \( U_{2n}([a,b]) \) with clockwise orientation. Then we have

\[
[F] \oplus [G] = [F] \ast [G]
\]

for \([F] \in \text{bv}_{[a,b]}(E_1)\) and \([G] \in \text{bv}_{[c,d]}(E_2)\).

**Proof.** We restrict our considerations to the case that \([c,d] \neq \pm \{\infty\} \). The other case is similar.

(i) Let \([F] \in \text{bv}_{[a,b]}(E_1)\), \([G] \in \text{bv}_{[c,d]}(E_2)\), \( n \in \mathbb{N} \) and \( \frac{1}{2n} < \frac{1}{n} \). Then

\[
 f: \left( U_{n}(\{a,b\}) \right) \times S_n([a+c,b+d]) \to E, \quad f(w,z) := F(w)G(z-w),
\]

is well-defined, i.e. \( z - w \in C \setminus [c,d] \) for all \( w \in V := U_{n}(\{a,b\}) \setminus U_{3n}([a,b]) \) and \( z \in U := S_n([a+c,b+d]) \). Let \( w \in V \) and \( z \in U \) such that \( y := \text{Im}(z) = \text{Im}(w) \). This can only happen if \( y = \text{Im}(z) \in [\frac{1}{n}, 0] \). Then \( z = a + c - \sqrt{r^2 - y^2} + iy \) or \( z = b + d + \sqrt{r^2 - y^2} + iy \) for some \( r > \frac{1}{n} \). If \( w = t + iy \) with some \( t \in [a,b] \), then

\[
z - w = c + a - t - \sqrt{r^2 - y^2} < c \quad \text{or} \quad z - w = d + b - t + \sqrt{r^2 - y^2} > d.
\]

If \( w = a - \sqrt{r^2 - y^2} + iy \) for some \( \frac{1}{3n} < R < \frac{1}{n} \), then \( R < r \) and

\[
z - w = c + \sqrt{r^2 - y^2} - \sqrt{r^2 - y^2} < c \quad \text{or} \quad z - w = d + (b - a) + \sqrt{r^2 - y^2} + \sqrt{r^2 - y^2} > d.
\]

If \( w = b + \sqrt{r^2 - y^2} + iy \) for some \( \frac{1}{3n} < R < \frac{1}{n} \), then

\[
z - w = c + (a - b) + \sqrt{r^2 - y^2} - \sqrt{r^2 - y^2} < c \quad \text{or} \quad z - w = d + \sqrt{r^2 - y^2} - \sqrt{r^2 - y^2} > d.
\]

Hence we have \( z - w \in \mathbb{C} \setminus \mathbb{R} \) or \( z - w \in \mathbb{R} \setminus [c,d] \) for all \( w \in V \) and \( z \in U \), i.e. \( z - w \in \mathbb{C} \setminus [c,d] \).

(ii) It follows from (26) that \( f(w,\cdot) \) is holomorphic on \( U \) for all \( w \in V \), and that \( f(\cdot, z) \) is holomorphic on \( V \) for all \( z \in U \). Therefore

\[
 z \mapsto \int_{\gamma_{[a,b],2n}} F(w)G(z-w)dw
\]

is holomorphic on \( U = S_n([a+c,b+d]) \) by differentiation under the Pettis integral w.r.t. the parameter \( z \) (see [44, Lemma 4.8 b), p. 259]). Due to Cauchy’s integral theorem

\[
 \int_{\gamma_{[a,b],2n}} F(w)G(z-w)dw = \int_{\gamma_{[a,b],2k}} F(w)G(z-w)dw
\]

for \( z \in S_n([a+c,b+d]) \) and \( k \in \mathbb{N} \) with \( k > n \). Thus we get a well-defined holomorphic function \( F \otimes G: \mathbb{C} \setminus [a+c,b+d] \to E \) by setting

\[
(F \otimes G)(z) := \int_{\gamma_{[a,b],2n}} F(w)G(z-w)dw, \quad z \in S_n([a+c,b+d]),
\]

for \( n \in \mathbb{N} \).

(iii) Next, we show that \( F \otimes G \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus [a+c,b+d], E) \). We claim that \( z - \gamma_{[a,b],2n} \in S_{2n}([c,d]) \) for every \( z \in S_n([a+c,b+d]) \). Let the path \( \gamma_{[a,b],2n} \) be parametrised by \([0,1]\).

(iii.1) We note that

\[
|\text{Im}(z - \gamma_{[a,b],2n}(t))| < n + \frac{1}{2n} < 2n
\]

for all \( z \in S_n([a+c,b+d]) \) and \( t \in [0,1] \). If \( z \in S_n([a+c,b+d]) \) with \( |\text{Im}(z)| > \frac{1}{n} \), then

\[
|\text{Im}(z - \gamma_{[a,b],2n}(t))| > \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}
\]

for all \( t \in [0,1] \).
(iii.2) Let $z \in S_n([a + c, b + d])$ with $|y| = |\Im(z)| \leq \frac{1}{n}$. Then we have $\text{Re}(z) < a + c$ or $\text{Re}(z) > b + d$. In particular, there is $r > \frac{1}{n}$ such that

$$z = a + c - \sqrt{r^2 - y^2} + iy \quad \text{or} \quad z = b + d + \sqrt{r^2 - y^2} + iy.$$ 

Hence we have

$$z = a + c - \sqrt{r^2 - y^2} + iy.$$ 

and thus

$$z - \gamma_{[a,b],2n}(t) = c + a - t_0 \sqrt{r^2 - y^2} + i\left(y + \frac{1}{2n}\right)$$

and thus $\text{Re}(z - \gamma_{[a,b],2n}(t)) < c$. This implies

$$d(z - \gamma_{[a,b],2n}(t), [c, d] \cap \mathbb{C}) = |z - \gamma_{[a,b],2n}(t) - c| = \sqrt{(a - t_0)^2 - 2(a - t_0)\sqrt{r^2 - y^2} + r^2 - y^2 + y^2 + y^2 + y^2 + \frac{1}{4n^2}}$$

$$\geq \sqrt{r^2 - y^2 + \frac{1}{4n^2}} + \frac{1}{2n}$$

If $\text{Re}(z) > b + d$, then

$$z - \gamma_{[a,b],2n}(t) = d + b - t_0 + \sqrt{r^2 - y^2} + i\left(y + \frac{1}{2n}\right)$$

and thus $\text{Re}(z - \gamma_{[a,b],2n}(t)) > d$. This implies

$$d(z - \gamma_{[a,b],2n}(t), [c, d] \cap \mathbb{C}) = |z - \gamma_{[a,b],2n}(t) - d| = \sqrt{(b - t_0)^2 + 2(b - t_0)\sqrt{r^2 - y^2} + r^2 - y^2 + y^2 + y^2 + \frac{1}{4n^2}}$$

$$\geq \sqrt{r^2 - y^2 + \frac{1}{4n^2}} > \frac{1}{2n}.$$ 

(iii.2.1) Let $t \in [0, 1]$ such that $\gamma_{[a,b],2n}(t) = t_0 \pm \frac{1}{2n}$ for some $t_0 \in [a,b]$. If $\text{Re}(z) < a + c$, then

$$z - \gamma_{[a,b],2n}(t) = c + a - t_0 - \sqrt{r^2 - y^2} + i\left(y = \frac{1}{2n}\right)$$

If $|y - y_1| \leq \frac{1}{2n}$, then

$$|y_1 - y|^2 = |y_1 - y| |y_1 + y| < \frac{1}{2n}\left(\frac{1}{2n} + \frac{1}{n}\right) = \frac{3}{4n^2},$$

which yields

$$\frac{1}{4n^2} - r^2 < \frac{1}{4n^2} - \frac{1}{n} = \frac{3}{4n^2} < y_1^2 - y^2$$

and thus

$$\sqrt{\frac{1}{4n^2} - y_1^2} < \sqrt{r^2 - y^2} < 0.$$ 

Hence we have $\text{Re}(z - \gamma_{[a,b],2n}(t)) < c$ and

$$d(z - \gamma_{[a,b],2n}(t), [c, d] \cap \mathbb{C}) = |z - \gamma_{[a,b],2n}(t) - c| = \sqrt{\left(\frac{1}{4n^2} - y_1^2 - \sqrt{r^2 - y^2}\right)^2 + (y - y_1)^2}$$

$$= \sqrt{\frac{1}{4n^2} - 2\sqrt{\frac{1}{4n^2} - y_1^2} + r^2 - 2yy_1}.$$
if Re(z) < a + c and |y − y1| ≤ 1/2n. We claim that
\[-2\sqrt{\frac{r}{4n^2}} - y_1^2\sqrt{r^2 - y^2} + r^2 - 2yy_1 > 0,\]  
which then implies d(−γ[a,b]2n(t), [c, d] ∩ C) > 1/2n. The inequality (27) is equivalent to
\[r^4 - 4yy_1r^2 + 4y_1^2y^2 > 4\left(\frac{r^2}{4n^2} - \frac{y^2}{4n^2} - y_1^2r^2 + y_1^2y^2\right),\]  
which is again equivalent to
\[0 < r^4 - \frac{r^2}{n^2} - 4yy_1r^2 + \frac{y^2}{n^2} + 4y_1^2r^2 = r^4 - \frac{r^2}{n^2} + h(y_1).\]
We note that h′(y1) = 8r^2y_1 - 4r^2y, h′′(y1) = 0 and h''(y1) = 8r^2 > 0, yielding
\[r^4 - \frac{r^2}{n^2} + h(y_1) ≥ r^4 - \frac{r^2}{n^2} + h(\frac{y}{2}) = r^4 - \frac{r^2}{n^2} - \frac{1}{n^2} - 2y_1y^2 = (r^2 - \frac{1}{n})^2 - 2y_1y^2 > 0\]  
as r > 1/2n and |y| ≤ 1/n. Thus our claim (27) is true.

If Re(z) > b + d, then
\[z - γ[a,b]2n(t) = d + b - a + \sqrt{r^2 - y^2} + \sqrt{\frac{1}{4n^2} + y_1^2} + i(y - y_1),\]  
which implies Re(z) − γ[a,b]2n(t) > d and
\[d(−γ[a,b]2n(t), [c, d] ∩ C) = |z − γ[a,b]2n(t) − d| = \sqrt{(b - a + \sqrt{r^2 - y^2} + \sqrt{\frac{1}{4n^2} - y_1^2})^2 + (y - y_1)^2} \geq \sqrt{r^2 - y^2 + \frac{1}{4n^2} - y_1^2 + y^2 - 2yy_1 + y_1^2} = \sqrt{\frac{1}{4n^2} + r^2 - 2yy_1} > \frac{1}{2n}\]  
because
\[r^2 - 2yy_1 > r^2 - 2\frac{1}{n^2} = r^2 - \frac{1}{n^2} > 0\]  
as r > 1/2n.

Similarly, we can handle the case that γ[a,b]2n(t) = b + \sqrt{\frac{1}{4n^2} - y^2} + iy_1 for some |y| < 1/2n and t ∈ [0, 1]. Altogether we obtain z − γ[a,b]2n ∈ S_2n([c, d]) for every z ∈ S_2n([a + c, b + d]).

Let α ∈ A. We observe that
\[-\frac{1}{n} |Re(z)| ≤ \frac{1}{2n} |Re(z - γ[a,b]2n(t))| + \frac{1}{2n} \max\left(|a - \frac{1}{2n}|, |b + \frac{1}{2n}|\right) \geq -\frac{1}{2n} |Re(z - γ[a,b]2n(t))| + C_{n,a,b}.\]
Denoting by by ℓ(γ[a,b]2n) := \int_0^1 |γ[a,b]2n(t)| dt the length of the path γ[a,b]2n, we get for z ∈ S_2n([a + c, b + d])
\[p_n((F ⊗ G)(z))e^{-\frac{1}{n}|Re(z)|} = p_n\left(\int_{γ[a,b]2n} F(w)G(z - w)dw\right)e^{-\frac{1}{n}|Re(z)|} \leq D(ℓ(γ[a,b]2n)) sup_{t ∈ [0, 1]} p_n(F(γ[a,b]2n(t))) sup_{t ∈ [0, 1]} p_n(G(z - γ[a,b]2n(t)))e^{-\frac{1}{n}|Re(z)|}\]  
for
\[ \leq D(\gamma_{[a,b],2n})e^{C_{n,a,b}} \sup_{t \in [0,1]} p(t)(F(\gamma_{[a,b],2n}(t)))\|G\|_{2n,\omega, [c,d]}. \]

Hence we obtain \( F \otimes G \in O^{exp}(\mathcal{T} \setminus [a + c, b + d], E) \).

**(iv)** Let \( \zeta \in \mathbb{H} \) if \( -\infty \in [a + c, b + d] \), or \( \zeta \in -\mathbb{H} \) if \( \infty \in [a + c, b + d] \), or \( \zeta \in \mathbb{C} \) if \( [a + c, b + d] \subset \mathbb{R} \), respectively. We choose \( \frac{1}{n} < \varepsilon < n \) and let \( w \in \gamma_{[a,b],2n} \). We claim that

\[
\int_{\gamma_{[a,c+b+d],\pi}} G(z-w)e^{-i(z-w)\zeta}dz = F_{[c,d]}([G]) \zeta.
\]

(Due to Theorem 3.3 resp. Theorem 4.3 and Cauchy’s integral theorem we only need to prove that there are \( z_1, z_2 \in \gamma_{[a+c,b+d],\pi} \) such that \( z_1 - w < c \) if \( c > -\infty \), and \( z_2 - w > d \) if \( d < \infty \), which implies that \( \gamma_{[a+c,b+d],\pi} - \gamma_{w} \) encircles \([c,d]\).) We only consider the case \( c \in \mathbb{R} \) and \( d = \infty \). The other cases are similar. If \( w = t + i\frac{\pi}{2n} \) for some \( t \in [a,b] \), we choose \( z_1 := a + c + \frac{1}{2}e^{i\theta} \) where \( \theta \in \mathbb{R} \). \( \frac{\pi}{2n} \) is the unique solution of \( \sin(\theta) = \pm \frac{\pi}{2n} \), and get \( z_1 - w = c + a - t + \frac{1}{2} \cos(\theta) < c \). If \( w = a - \sqrt{\frac{1}{4n^2} - y^2} + iy \) or \( w = b + \sqrt{\frac{1}{4n^2} - y^2} + iy \) for some \( |y| < \frac{1}{2n} \), then we choose \( z_1 := a + c - \sqrt{\frac{1}{4n^2} - y^2} + iy \) and get \( z_1 - w = c + \sqrt{\frac{1}{4n^2} - y^2} - \sqrt{\frac{1}{4n^2} - y^2} < c \) or \( z_1 - w = c + a - b - \sqrt{\frac{1}{4n^2} - y^2} - \sqrt{\frac{1}{4n^2} - y^2} < c \).

Moreover, we have for every \( \epsilon' \in E' \) that \( \epsilon'(F(w) \cdot) \in E'_2 \) by (29), which implies

\[
\epsilon'(F(w)G(z-w)e^{-i(z-w)\zeta}dz) = \epsilon(F(w)) \int_{\gamma_{[a,c+b+d],\pi}} G(z-w)e^{-i(z-w)\zeta}dz = \epsilon(F(w)) \int_{\gamma_{[a,c+b+d],\pi}} \epsilon'(G(z-w)e^{-i(z-w)\zeta})dz.
\]

by the definition of Pettis integrability. It follows from the Hahn-Banach theorem that

\[
\int_{\gamma_{[a,c+b+d],\pi}} F(w)G(z-w)e^{-i(z-w)\zeta}dz = F(w) \int_{\gamma_{[a,c+b+d],\pi}} G(z-w)e^{-i(z-w)\zeta}dz.
\]

Analogously it follows that

\[
\int_{\gamma_{[a,b],2n}} F(w)F_{[c,d]}([G]) \zeta e^{-iw\zeta}dw = F_{[a,b][F]}([F]) \zeta F_{[c,d]}([G]) \zeta,
\]

using that \( \epsilon'(G_{[c,d]}([G]) \zeta) \in E'_1 \) by (26) for every \( \epsilon' \in E' \) and Theorem 3.3 resp. Theorem 4.3.

We obtain by the Fubini-Tonelli theorem and Theorem 3.3 resp. Theorem 4.3 again that

\[
\int_{\gamma_{[a,c+b+d],\pi}} F_{[a+c,b+d]}([F \otimes G]) \zeta = \int_{\gamma_{[a,c+b+d],\pi}} (F \otimes G) (z)e^{-iz\zeta}dz = \int_{\gamma_{[a,c+b+d],\pi}} F(w)G(z-w)dwe^{-iz\zeta}dz = \int_{\gamma_{[a,b],2n}} F(w) \int_{\gamma_{[c,b+d],\pi}} G(z-w)e^{-i(z-w)\zeta}dz e^{-iw\zeta}dw
\]

(29)
more precisely, \( \tilde{\implying} [F] \otimes [G] = [F] \ast [G] \) by Theorem 4.7. Moreover, it follows from this equation that \([F] \otimes [G]\) does not depend on the representing functions \(F\) and \(G\).

4.10. Remark. Let \(E, E_1\) and \(E_2\) be sequentially complete \(\mathbb{C}\)-lcHs, \(E_2\) strictly admissible, \(\cdot : E_1 \times E_2 \to E\) a canonical bilinear map with \([25]\) and \([a, b] \in \mathbb{R}\). Then the sheaf of \(E_2\)-valued Fourier hyperfunctions is flabby by \([41\text{, Theorem 5.9 b)}\), p. 33] and for \(g \in bv_{[\infty, 0]}(E_2)\) there are \(g_1 \in bv_{[-\infty, 0]}(E_2)\) and \(g_2 \in bv_{[0, \infty]}(E_2)\) such that \(g = g_1 + g_2\) by \([24\text{, Lemma 1.4.4, p. 36}]\). Hence we may define the convolution

\[
f \otimes g = f \otimes g_1 + f \otimes g_2 \in bv_{[-\infty, 0]}(E) + bv_{[0, \infty]}(E)\]

for \(f \in bv_{[a, b]}(E_1)\) by Corollary 4.9.

4.11. Example. Let \(F(z) := \frac{-1}{2\pi i z}\) for \(z \neq 0\). Then \(F \in \mathcal{O}^{\ell^p} (\mathbb{C} \setminus \{0\})\) and we call \(\delta_0 := [F] \in bv_{(0)}\) Dirac hyperfunction (see \([29\text{, Examples 1.1.5 b)}\), p. 15]). Due to \((20)\) and Cauchy’s integral formula we get \(\mathcal{F}_{(0)}(\delta_0) = 1\ on \ \mathbb{C}, \ \mathcal{F}_{[-\infty, 0]}(\delta_0) = 1\ on\ the\ upper\ halfplane\ and\ \mathcal{F}_{[0, \infty]}(\delta_0) = 1\ on\ the\ lower\ halfplane.\ Hence\ \delta_0\ is\ the\ neutral\ element\ of\ the\ convolution.

5. ASYMPTOTIC FOURIER TRANSFORM

The results of the preceding sections allow us to define an asymptotic Fourier transform on the space \(\mathcal{B}(A, E)\) of \(E\)-valued hyperfunctions with support in a closed interval \(A \subset \mathbb{R}\).

5.1. Theorem. Let \(E\) be an admissible \(\mathbb{C}\)-lcHs and \(a \in \mathbb{R}\). Then the canonical (restriction) maps

\[
\mathcal{R}_{[a, \infty]} : \mathcal{O}^{\ell^p} (\mathbb{C} \setminus [a, \infty], E) / \mathcal{O}^{\ell^p} (\mathbb{C} \setminus \{ \infty \}, E) \to \mathcal{B}([a, \infty], E), \quad [F] \mapsto [F],
\]

and

\[
\mathcal{R}_{[-\infty, a]} : \mathcal{O}^{\ell^p} (\mathbb{C} \setminus [-\infty, a], E) / \mathcal{O}^{\ell^p} (\mathbb{C} \setminus \{ -\infty \}, E) \to \mathcal{B}([-\infty, a], E), \quad [F] \mapsto [F],
\]

are linear isomorphisms.

Proof. Let \(F \in \mathcal{O}^{\ell^p} (\mathbb{C} \setminus [a, \infty], E)\) such that \(\mathcal{R}_{[a, \infty]}([F]) = 0\). This means that \(F \in \mathcal{O}(\mathbb{C}, E)\) and thus \(F \in \mathcal{O}^{\ell^p} (\mathbb{C} \setminus \{ \infty \}, E)\), yielding \([F] = 0\). Hence \(\mathcal{R}_{[a, \infty]}\) is injective.

Let \([F] \in \mathcal{B}([a, \infty], E)\) and fix \(\varepsilon > 0\). Then \(F \in \mathcal{O}(\mathbb{C} \setminus \{ a - \varepsilon, \infty \}, E)\) and by \([41\text{, Lemma, p. 25}]\) there is \(\tilde{F} \in \mathcal{O}^{\ell^p} (\mathbb{C} \setminus [a - \varepsilon, \infty], E)\) such that \(\tilde{F} - F \in \mathcal{O}(\mathbb{C}, E)\), more precisely, \(\tilde{F} - F\) extends to a function \(G \in \mathcal{O}(\mathbb{C}, E)\). Now, we extend \(\tilde{F}\) by \(\tilde{F}(z) := G(z) + F(z)\) on \([a - \varepsilon, a]\) and obtain \(\tilde{F} \in \mathcal{O}^{\ell^p} (\mathbb{C} \setminus [a, \infty], E)\) with \(\mathcal{R}_{[a, \infty]}([\tilde{F}]) = [F]\) by this extension. Thus \(\mathcal{R}_{[a, \infty]}\) is surjective. The proof for \(\mathcal{R}_{[-\infty, a]}\) is analogous.

5.2. Theorem. Let \(E\) be an admissible sequentially complete \(\mathbb{C}\)-lcHs and \(A := [a, \infty]\) or \(A := [-\infty, a]\) for some \(a \in \mathbb{R}\). Then the asymptotic one-sided Fourier transform

\[
\mathcal{F}^R_A : \mathcal{B}(A, E) \to \mathcal{F} \mathcal{O}(E) \big/ \mathcal{F} \mathcal{O}(\overline{\mathbb{C}} \setminus A), \quad \mathcal{F}^R_A(f) := [(F \circ \mathcal{R}^{-1}_A)(f)],
\]

is a linear isomorphism, where the closure \(\overline{A}\) is taken in \(\mathbb{R}\).

Proof. This follows directly from Theorem 5.1 and Theorem 5.5.
5.3. **Theorem.** Let $E$ be a locally complete $\mathbb{C}$-lcHs and $K \subset \mathbb{R}$ a non-empty compact set. Then the canonical (restriction) map

$$\mathcal{R}_K^*: \mathcal{B}(K,E) \to \mathcal{B}(K,E), \ [F] \mapsto [F],$$

is a topological isomorphism and its inverse $\mathcal{R}_K^{-1}: \mathcal{B}(K,E) \to \mathcal{B}(K,E)$ is given by $\mathcal{R}_K^{-1}([F]) = [\Psi_K([F])]$ where

$$\Psi_K([F]): \mathbb{C} \setminus K \to E, \ \Psi_K([F])(z) := \frac{i}{2\pi} \left\{ K([F]), \frac{e^{-i(z-\cdot)^2}}{z-\cdot} \right\}. $$

**Proof.** The statement is a consequence of Theorem 2.3 and Corollary 2.6. □

5.4. **Corollary.** Let $E$ be a locally complete $\mathbb{C}$-lcHs and $-\infty < a \leq b < \infty$. Then the Fourier transform

$$\mathcal{F}^E_{[a,b]}: \mathcal{B}([a,b], E) \to \mathcal{F}_{[a,b]}(E), \ \mathcal{F}^E_{[a,b]}([F]) := (\mathcal{F}_{[a,b]} \circ \mathcal{R}^{-1}_{[a,b]})([F]),$$

is a topological isomorphism and

$$\mathcal{F}^E_{[a,b]}([F])(\zeta) = \mathcal{F}_{[a,b]}([\Psi([F])])(\zeta) = (\mathcal{H}_{[a,b]}([F]), e^{-i(\cdot)^2}), \ \zeta \in \mathbb{C}. \ (31)$$

**Proof.** This follows from Theorem 5.3 and Theorem 5.5. □

5.5. **Proposition.** Let $E$ be a strictly admissible sequentially complete $\mathbb{C}$-lcHs and $-\infty < a < b < \infty$. Then the following holds:

a) For every $f \in \mathcal{B}([a,\infty[, E)$ there are $f_1 \in \mathcal{B}([a,b], E)$ and $f_2 \in \mathcal{B}([b,\infty[, E)$ such that $f = f_1 + f_2$ and

$$\mathcal{F}^E_{[a,\infty]}(f) = [\mathcal{F}^E_{[a,b]}(f_1)|_{m=0}] + \mathcal{F}^E_{[b,\infty]}(f_2).$$

b) For every $f \in \mathcal{B}([-\infty,b], E)$ there are $f_1 \in \mathcal{B}([a,b], E)$ and $f_2 \in \mathcal{B}([-\infty,a], E)$ such that $f = f_1 + f_2$ and

$$\mathcal{F}^E_{[-\infty,b]}(f) = [\mathcal{F}^E_{[a,b]}(f_1)|_{m>0}] + \mathcal{F}^E_{[-\infty,a]}(f_2).$$

**Proof.** Let us start with part a). Since $E$ is strictly admissible, the sheaf of $E$-valued hyperfunctions is flabby by [11, Theorem 5.9 b), p. 33] and thus for every $f \in \mathcal{B}([a,\infty[, E)$ there are $f_1 \in \mathcal{B}([a,b], E)$ and $f_2 \in \mathcal{B}([b,\infty[, E)$ such that $f = f_1 + f_2$ by [29, Lemma 1.4.4, p. 36]. Moreover, we have $\mathcal{H}_{[a,\infty]}(f_1) = \mathcal{H}_{[a,b]}(f_1)$ and $\mathcal{H}_{[a,\infty]}(f_2) = \mathcal{H}_{[b,\infty]}(f_2)$ due to [14, Eq. (6), p. 14], which implies

$$\mathcal{F}^E_{[a,\infty]}(f_1) = [\mathcal{F}^E_{[a,b]}(f_1)|_{m=0}] \quad \text{and} \quad \mathcal{F}^E_{[a,\infty]}(f_2) = [\mathcal{F}^E_{[a,b]}(f_1)|_{m>0}] + \mathcal{F}^E_{[b,\infty]}(f_2).$$

by (31) and the definition of the asymptotic Fourier transforms. We deduce that

$$\mathcal{F}^E_{[a,\infty]}(f) = [\mathcal{F}^E_{[a,b]}(f_1)|_{m>0}] + \mathcal{F}^E_{[b,\infty]}(f_2) = \mathcal{F}^E_{[a,b]}(f_1) + \mathcal{F}^E_{[a,b]}(f_2) = \mathcal{F}^E_{[a,b]}(f).$$

The proof of part b) is analogous. □

5.6. **Theorem.** Let $E$ be an admissible sequentially complete $\mathbb{C}$-lcHs. Then the canonical (restriction) map

$$\mathcal{R}_\mathbb{R}: \mathcal{O}^{exp}(\mathbb{C} \setminus \mathbb{R}, E)/\mathcal{O}^{exp}(\mathbb{C} \setminus (-\infty), E) \to \mathcal{B}(\mathbb{R}, E), \ [F] \mapsto [F],$$

and the asymptotic Fourier transform

$$\mathcal{F}_\mathbb{R}: \mathcal{B}(\mathbb{R}, E) \to \mathcal{O}^{exp}(\mathbb{C} \setminus \mathbb{R}, E)/\mathcal{O}^{exp}(\mathbb{C} \setminus (-\infty), \mathcal{O}(\mathbb{R}, E)),$$

$$\mathcal{F}_\mathbb{R}(f) := ([\mathcal{R}_\mathbb{R} \circ \mathcal{R}_\mathbb{R}^{-1}](f));$$

are linear isomorphisms.
Proof. (i) We consider \( \mathcal{R}_R \). Let \( F \in \mathcal{O}^{exp}(\overline{C} \setminus \mathbb{R}, E) \) such that \( \mathcal{R}_R([F]) = 0 \), i.e. \( F \in \mathcal{O}(C, E) \). We deduce that \( F \in \mathcal{O}^{exp}(\overline{C} \setminus \{ \pm \infty \}, E) \), resulting in \([F]=0\). Thus \( \mathcal{R}_R \) is injective.

Let \([F]\in \mathcal{B}(\mathbb{R}, E)\). Then \( F \in \mathcal{O}(C \setminus \mathbb{R}, E) \) and by [11], 5.7 Lemma, p. 25 there is \( \tilde{F} \in \mathcal{O}^{exp}(\overline{C} \setminus \mathbb{R}, E) \) such that \( \tilde{F} - F \in \mathcal{O}(C, E) \). It follows that \( \mathcal{R}_R([\tilde{F}]) = [F] \), meaning that \( \mathcal{R}_R \) is surjective.

(ii) Next, we prove \( \mathcal{O}^{exp}(\overline{C}, E) \cap \mathcal{F}\mathcal{O}_{(\pm \infty)}(E) = \{ 0 \} \). The proof is similar to the one of part (a) of the proof of [74], Proposition 3.3, p. 50. Let \( f \in \mathcal{O}^{exp}(\overline{C}, E) \cap \mathcal{F}\mathcal{O}_{(\pm \infty)}(E) \). This implies

\[
\sup_{z \in \mathbb{C}} p_{\alpha}(f(z)) e^{-\frac{1}{2}|z-k| |\text{Im}(z)|} \\
\leq |f|_{k, \alpha, (-\infty)} + |f|_{k, \alpha, (\infty)} + \sup_{|\text{Im}(z)|<\infty} p_{\alpha}(f(z)) e^{-\frac{1}{2}|z-k| |\text{Im}(z)|} \\
\leq |f|_{k, \alpha, (-\infty)} + |f|_{k, \alpha, (\infty)} + \|f\|_{k, \alpha, (\mathbb{C})}
\]

for every \( \alpha \in \mathfrak{A} \) and \( k \in \mathbb{N} \). Hence we have

\[
\forall \alpha \in \mathfrak{A}, k \in \mathbb{N} \exists C > 0 \forall z \in \mathbb{C} : p_{\alpha}(f(z)) \leq C e^{\frac{1}{2}|z-k| \text{Re}(z)},
\]

Let \( e' \in E' \). Due to the Paley-Wiener theorem (see [63], 19.3 Theorem, p. 375) applied to \( e'(f(i\cdot)) \) there is \( F_{e', k} \in L^2(-\frac{1}{k}, \frac{1}{k}) \) such that

\[
(e' \circ f)(iz) = \int F_{e', k}(t) e^{itz} dt, \ z \in \mathbb{C},
\]

for every \( k \in \mathbb{N} \). We derive that \( e' \circ f = 0 \) and the Hahn-Banach theorem yields \( f = 0 \).

(iii) Let \( f \in \mathcal{B}(\mathbb{R}, E) \) such that \( \mathcal{F}_R(f) = 0 = \mathcal{O}^{exp}(\overline{C}, E) \oplus \mathcal{F}\mathcal{O}_{(\pm \infty)}(E) \). Let \( F \in \mathcal{F}_R(f) \). Then there are \( F_1 \in \mathcal{O}^{exp}(\overline{C}, E) \) and \( F_2 \in \mathcal{F}\mathcal{O}_{(\pm \infty)}(E) \) such that \( F = F_1 + F_2 \). Due to Theorem 3.7 there is \( g \in \mathcal{b}\mathcal{v}_{(\pm \infty)}(E) \) such that \( F_2 = \mathcal{F}_{(\pm \infty)}(g) \). Taking equivalence classes in \( \mathcal{b}\mathcal{v}_{(\pm \infty)}(E) \), we get

\[
(\mathcal{H}_R^{-1} \circ \mathcal{F}_* \circ \mathcal{H}_R)(\mathcal{R}_R^{-1}(f)) = \mathcal{F}_R(\mathcal{R}_R^{-1}(f)) = [F_1 + F_2] = [F_2] = [\mathcal{F}_{(\pm \infty)}(g)]
\]

which implies

\[
\mathcal{H}_R(\mathcal{R}_R^{-1}(f)) = \mathcal{H}_{(\pm \infty)}(g) - \mathcal{H}_{(\infty)}(g) \in L(\mathcal{P}(\{ \pm \infty \}), E).
\]

By Theorem 2.5 and [14], Eq. (6), p. 14 there is \( h \in \mathcal{b}\mathcal{v}_{(\pm \infty)}(E) \) such that

\[
\mathcal{H}_{(\pm \infty)}(h) = \mathcal{H}_{(\pm \infty)}(g) - \mathcal{H}_{(\infty)}(g) \quad \text{and} \quad \mathcal{H}_{(\pm \infty)}(h) = \mathcal{H}_R(h),
\]

yielding \( \mathcal{H}_R(\mathcal{R}_R^{-1}(f)) = \mathcal{H}_R(h) \). From the injectivity of \( \mathcal{H}_R \) on \( \mathcal{b}\mathcal{v}_{(\pm \infty)}(E) \) it follows that \( \mathcal{R}_R^{-1}(f) = h \). We derive from part (i) that \( f = 0 \), i.e. \( \mathcal{F}_R(f) \) is injective.

Let \( F \in \mathcal{O}^{exp}(\overline{C} \setminus \mathbb{R}, E) \). Since \( \mathcal{F}_R \circ \mathcal{b}\mathcal{v}_{(\pm \infty)}(E) \rightarrow \mathcal{b}\mathcal{v}_{(\pm \infty)}(E) \) is surjective by Corollary 2.8 there is \( g = [G] \in \mathcal{b}\mathcal{v}_{(\pm \infty)}(E) \) such that \( \mathcal{F}_R(g) = F + \mathcal{O}^{exp}(\overline{C}, E) \). From the surjectivity of \( \mathcal{R}_R^{-1} \) we deduce that there is \( h \in \mathcal{B}(\mathbb{R}, E) \) such that \( H = G \in \mathcal{O}^{exp}(\overline{C} \setminus \{ \pm \infty \}, E) \) for \( [H] = \mathcal{R}_R^{-1}(h) \). We note that by [14], Eq. (6), p. 14

\[
\mathcal{H}_R([H - G]) = \mathcal{H}_{(\pm \infty)}([H - G]) = \mathcal{H}_{(\infty)}([H - G]) + \mathcal{H}_{(\infty)}([H - G])
\]

and so

\[
\mathcal{F}_R([H - G]) = (\mathcal{H}_R^{-1} \circ \mathcal{F}_* \circ \mathcal{H}_R)([H - G])
\]

\[
= (\mathcal{H}_R^{-1} \circ \mathcal{F}_* \circ \mathcal{H}_{(\infty)})([H - G]) + (\mathcal{H}_R^{-1} \circ \mathcal{F}_* \circ \mathcal{H}_{(\infty)})([H - G])
\]
\[ \mathcal{F}(\mathbb{C} \times \mathbb{R}) = \mathcal{F}(\mathbb{C} \times \mathbb{R}) + \mathcal{O}(\mathbb{C} \times \mathbb{R}). \]

Thus there is \( P \in \mathcal{O}(\mathbb{C} \times \mathbb{R}) \) such that \( \mathcal{F}(\mathbb{C} \times \mathbb{R}) = P + \mathcal{O}(\mathbb{C} \times \mathbb{R}) \), which guarantees that our map is well-defined as well. We conclude that

\[ \mathcal{F}(\mathbb{R}^3(h)) = \mathcal{F}(\mathbb{R}^3(h)) + \mathcal{F}(\mathbb{R}^3(h)) = F + P + \mathcal{O}(\mathbb{C} \times \mathbb{R}), \]

which means

\[ \mathcal{F}(\mathbb{R}^3(h)) = \mathcal{F}(\mathbb{R}^3(h)) + \mathcal{F}(\mathbb{R}^3(h)) = [F]. \]

where the equivalence class is taken in \( \mathcal{O}(\mathbb{C} \times \mathbb{R}) \). This proves that \( \mathcal{F}(\mathbb{R}^3(h)) \) is surjective. \( \square \)

Now, we may transfer the connections of some standard operators and the asymptotic Fourier transform from the preceding section.

5.7. **Proposition.** Let \( E \) be an admissible sequentially complete \( \mathbb{C} \)-leHs, \( A := [a, a] \) or \( A := [a, \infty) \) for some \( a \in \mathbb{R} \). Let \( \gamma_h([F]) := [F(-h)] \), \( h \in \mathbb{R} \), be the shift operator from \( \mathcal{B}(A, E) \) to \( \mathcal{B}(h + A, E) \). Then we have for \( [F] \in \mathcal{B}(A, E) \)

\[ \mathcal{F}(h \cdot A)(\gamma_h([F])) = e^{-i\tau h}[\mathcal{F}(A)](h). \]

**Proof.** This is a consequence of Proposition 5.6 and \( \mathcal{R}(\gamma_h(\mathcal{R}(\gamma_h([F]))) = \gamma_h([F])). \) which implies \( \gamma_h(\mathcal{R}(\gamma_h([F]))) = \mathcal{R}(\gamma_h([F])). \) \( \square \)

5.8. **Proposition.** Let \( E \) be an admissible sequentially complete \( \mathbb{C} \)-leHs, \( A := [a, a] \) or \( A := [a, \infty] \) for some \( a \in \mathbb{R} \) and \( P(-i\partial) = \sum_{k=0}^{\infty} \frac{(-i\partial)^k}{k!} \) where \( (c_k) \subset \mathbb{C} \) and \( P \) is of exponential type 0. Then we have for \( f = [F] \in \mathcal{B}(A, E) \)

\[ \mathcal{F}(P(-i\partial)f) = \mathcal{F}(A)(f) \quad \text{and} \quad \mathcal{F}(Pf) = \mathcal{F}(A)(f) \]

where \( P(-i\partial)f = [P(-i\partial)F] \) and \( Pf := [Pf] \).

**Proof.** First, we remark that \( F \mapsto P(-i\partial)F \) and \( F \mapsto PF \) are well-defined continuous linear operators on \( \mathcal{O}(\mathbb{C} \setminus A, E) \) and \( \mathcal{O}(\mathbb{C}, E) \) as in Proposition 4.8 when these spaces are equipped with the topology of uniform convergence on compact subsets. This implies that \( P(-i\partial)f \) and \( Pf \) are well-defined for \( f \in \mathcal{B}(A, E) \).

The first equation follows from (24) because \( \mathcal{R}(P(-i\partial)) = \mathcal{R}(P(-i\partial)) \) and hence \( P(-i\partial)\mathcal{R}(P(-i\partial)f) = \mathcal{R}(P(-i\partial)f) \) for \( f \in \mathcal{B}(A, E) \) (like in (54) Example 3.8 (b), p. 53).

The second equation follows from (24) since \( \mathcal{R}(P\mathcal{R}(P(-i\partial))f) = Pf \) and thus \( P\mathcal{R}(P(-i\partial)f) = \mathcal{R}(P(-i\partial)(f)) \) (like in (54) Proposition 3.10, p. 53)). \( \square \)

5.9. **Theorem.** Let \( (E, (p_a)_{a \in A}), (E_1, (p_{\beta})_{\beta \in B}) \) and \( (E_2, (p_{\omega})_{\omega \in \Omega}) \) be admissible sequentially complete \( \mathbb{C} \)-leHs such that a canonical bilinear map \( : E_1 \times E_2 \to E \) is defined with the property

\[ \forall \alpha \in A \exists \beta \in B, \omega \in \Omega, D > 0 \forall x \in E_1, y \in E_2 : p_\alpha(x \cdot y) \leq Dp_\beta(x)p_\omega(y). \]

Let \( a, b, c, d \in \mathbb{R} \).

a) If \( f \in \mathcal{O}(E_1) \) and \( g \in \mathcal{O}(E_2) \), or if \( f \in \mathcal{O}(E_1) \) and \( g \in \mathcal{O}(E_2) \), then \( f \in \mathcal{O}(E) \).

b) If \( f \in \mathcal{O}(E_1) \) and \( g \in \mathcal{O}(E_2) \), or if \( f \in \mathcal{O}(E_1) \) and \( g \in \mathcal{O}(E_2) \), then \( f \in \mathcal{O}(E) \).

c) We define the convolution

\[ * : \mathcal{B}([b, c \infty], E_1) \times \mathcal{B}([c, b \infty], E_2) \to \mathcal{B}([a + b, c \infty], E), \]

and

\[ * : \mathcal{B}([a, b - \infty], E_1) \times \mathcal{B}([b, d], E_2) \to \mathcal{B}([a, b + d), E_2), \]
The proof of part b) is analogous.

Proof. The parts a) and b) are just special cases of Theorem 4.7 a). In particular, this implies that if \( \mathcal{F}(f) = \hat{F} \) and \( \mathcal{F}(g) = \hat{G} \) the product \( \mathcal{F}(f) \mathcal{F}(g) := [\hat{F} \cdot \hat{G}] \) is well-defined. The convolutions in c) are well-defined due to a) and b) and Theorem 4.7 a). The rest follows from Theorem 4.7 b) and c).

Let \( E, E_1 \) and \( E_2 \) and \( \cdots : E_1 \times E_2 \rightarrow E \) be as above and \( a, b, c, d \in \mathbb{R} \). We note that the convolution \( f \otimes g \) from Corollary 4.9 is even defined and an element of \( \mathcal{B}((a + c, \infty[, E) \) resp. \( \mathcal{B}([-\infty, b + d], E) \) by part (i) and (ii) of the proof of Corollary 4.9 if \( f \in \mathcal{B}([a, b], E_1) \) and \( g \in \mathcal{B}([c, \infty[, E_2) \) resp. \( g \in \mathcal{B}([-\infty, d][, E_2) \). If \( E = E_1 = E_2 = \mathbb{C} \), then this is the usual definition of the convolution of two hyperfunctions where one of them has real compact support (see [10], Proposition 2.53, p. 141).

5.10. Corollary. Let \( (E, (p_a)_{a \in \mathcal{A}}), (E_1, (p_{b1})_{b_1 \in \mathcal{B}}) \) and \( (E_2, (p_{d2})_{d_2 \in \mathcal{D}}) \) be admissible sequentially complete \( \mathbb{C}\)r-Hs, \( E \) and \( E_1 \) strictly admissible, such that a canonical bilinear map \( \cdot : E_1 \times E_2 \rightarrow E \) is defined with the property

\[
\forall \alpha \in \mathfrak{A} \exists \beta \in \mathfrak{B}, \omega \in \Omega, D > 0 \forall x \in E_1, y \in E_2 : p_\alpha(x \cdot y) \leq D p_\beta(x) p_\omega(y).
\]

Let \( a, b, c, d \in \mathbb{R} \).

a) For every \( f \in \mathcal{B}([a, \infty[, E_1) \) and \( j > a \) there is \( f_j \in \mathcal{B}([a, j], E_1) \) such that \( f - f_j \in \mathcal{B}([j, \infty[, E_1) \).

b) For every \( f \in \mathcal{B}([-\infty, b], E_1) \) and \( j < b \) there is \( f_j \in \mathcal{B}([j, b], E_1) \) such that \( f - f_j \in \mathcal{B}([-\infty, j], E_1) \).

c) We define the convolutions

\[
\otimes^B : \mathcal{B}([a, \infty[, E_1) \times \mathcal{B}([c, \infty[, E_2) \rightarrow \mathcal{B}([a + c, \infty[, E)
\]

by \( (f \otimes^B g)[j] := (f_j \otimes g)[j] \) for every \( j > a \), and

\[
\otimes^B : \mathcal{B}([-\infty, b], E_1) \times \mathcal{B}([-\infty, d], E_2) \rightarrow \mathcal{B}([-\infty, b + d], E)
\]

by \( (f \otimes^B g)[j] := (f_j \otimes g)[j] \) for every \( j < b \). Then we have

\[
f \otimes^B g = f \ast^B g \quad \text{and} \quad \mathcal{F}(f \otimes^B g) = \mathcal{F}(f) \mathcal{F}(g).
\]

Proof. a) Since \( E_1 \) is strictly admissible, the sheaf of \( E_1 \)-valued hyperfunctions is flabby by [11], Theorem 5.9 b), p. 33] and for \( f \in \mathcal{B}([a, \infty[, E_1) \) and \( j > a \) there is \( f_j \in \mathcal{B}([a, j], E_1) \) such that \( f - f_j \in \mathcal{B}([j, \infty[, E_1) \) due to [29], Lemma 1.4.4, p. 36]. The proof of part b) is analogous.

c) Let \( f \in \mathcal{B}([a, \infty[, E_1) \) and \( g \in \mathcal{B}([c, \infty[, E_2) \). It follows from the glueing property of a sheaf (see [1], Property (S2), p. 6]) that by setting \( (f \otimes^B g)[j] := (f_j \otimes g)[j] \) for every \( j > a \), we get a hyperfunction \( f \otimes^B g \in \mathcal{B}([a + c, \infty[, E) \). We deduce from Theorem 5.2 Theorem 4.7 and Corollary 4.9 that

\[
\mathcal{F}(f \otimes^B g) - \mathcal{F}(f) \mathcal{F}(g)
\]

\[
= \mathcal{F}((f - f_j) \otimes^B g) + \mathcal{F}(f_j \otimes^B g) - \mathcal{F}(f) \mathcal{F}(g)
\]

\[
= \mathcal{F}((f - f_j) \otimes^B g) + \mathcal{F}(-f_{j-1}) \otimes^B g - \mathcal{F}(f) \mathcal{F}(g)
\]

\[
= \mathcal{F}((f - f_j) \otimes^B g) + \mathcal{F}(f_j) \mathcal{F}(g) - \mathcal{F}(f) \mathcal{F}(g)
\]

\[
= \mathcal{F}((f - f_j) \otimes^B g) + \mathcal{F}(f_j - f) \mathcal{F}(g).
\]
We note that \( (f - f_j) \otimes^B g \in \mathcal{B}([c + j, \infty[, E]) \), \( f_j - f \in \mathcal{B}([j, \infty[, E_2]) \), which implies that \( \mathcal{F}^B((f - f_j) \otimes^B g) \in \mathcal{F} \mathcal{O}_{[c+j,\infty)}(E) / \mathcal{F} \mathcal{O}_\infty(E), \mathcal{F}^B(f_j - f) \in \mathcal{F} \mathcal{O}_{[j,\infty)}(E_1) / \mathcal{F} \mathcal{O}_\infty(E_1) \) and \( \mathcal{F}^B(g) \in \mathcal{F} \mathcal{O}_{[c,\infty)}(E_2) / \mathcal{F} \mathcal{O}_\infty(E_2) \) by Theorem 5.9. We derive from Theorem 5.7(a) that

\[
\mathcal{F}^B(f_j - f) \mathcal{F}^B(g) \in \mathcal{F} \mathcal{O}_{[c+j,\infty)}(E) / \mathcal{F} \mathcal{O}_\infty(E).
\]

We conclude that \( \mathcal{F}^B(f \otimes^B g) - \mathcal{F}^B(f) \mathcal{F}^B(g) \in \mathcal{F} \mathcal{O}_{[c+j,\infty)}(E) / \mathcal{F} \mathcal{O}_\infty(E) \) for every \( j > a \), yielding

\[
\mathcal{F}^B(f \otimes^B g) = \mathcal{F}^B(f) \mathcal{F}^B(g).
\]

It follows from Theorem 5.9 that \( f \otimes^B g = f \ast^B g \). The other case is analogous. \( \square \)

The idea of the proof above is a modification of the proof of [50, Theorem 5.5, p. 56].

6. Langenbruch’s asymptotic Fourier transform

In this section we study the relation of our asymptotic Fourier transform on \( \mathcal{B}(\mathbb{R}) \) to the one of Langenbruch. We briefly recall the relevant notions and results from [50] (and the introduction).

6.1. Definition ([54], p. 44, 49, 53, 54, 55, 61]). Let \( E \) be a \( \mathbb{C} \)-lcHs and \( K \subset \mathbb{R} \) be compact. We define the space

\[
\mathcal{H}_{-\infty}(\mathbb{C} \setminus K, E) := \{ f \in \mathcal{O}(\mathbb{C} \setminus K, E) \mid \forall n \in \mathbb{N}, \alpha \in \mathfrak{A} : \| f \|_{n, \alpha, K}^{\mathcal{H}_{-\infty}} < \infty \}
\]

where

\[
\| f \|_{n, \alpha, K}^{\mathcal{H}_{-\infty}} := \sup_{z \in S_n(K)} p_\alpha(f(z)) e^{n|\text{Re}(z)|},
\]

and the quotient space

\[
\mathcal{G}(K, E) := \mathcal{H}_{-\infty}(\mathbb{C} \setminus K, E) / \mathcal{H}_{-\infty}(\mathbb{C}, E).
\]

We remark that our definition of the spaces above coincides with Langenbruch’s original definition because \( S_n(K) = W_n(K) \) for \( K = \emptyset, [0, \infty] \), \( S_n(\mathbb{R}) = F_n \) with \( F_n \) from [50], p. 44 and \( \mathcal{H}_{-\infty}(\mathbb{C} \setminus \{ \pm \infty \}, E) = \mathcal{H}_{-\infty}(\mathbb{C} \setminus \mathbb{R}, E) \cap \mathcal{O}(\mathbb{C}, E) \) and \( \mathcal{H}_{-\infty}(\mathbb{C} \setminus \{ \infty \}, E) = \mathcal{H}_{-\infty}(\mathbb{C} \setminus [0, \infty], E) \cap \mathcal{O}(\mathbb{C}, E) \).

6.2. Definition ([54], p. 46, 50]). Let \( E \) be a \( \mathbb{C} \)-lcHs. We define the spaces

\[
\mathcal{H}_j(E) := \{ f \in \mathcal{O}(\mathbb{C}, E) \mid \forall j \in \mathbb{N}, \alpha \in \mathfrak{A} : \| f \|_{j, \alpha}^{\mathcal{H}_j} < \infty \}
\]

where

\[
\| f \|_{j, \alpha}^{\mathcal{H}_j} := \sup_{z \in S_j(\mathbb{C})} p_\alpha(f(z)) e^{j|\text{Re}(z)| + j|\text{Im}(z)|},
\]

and

\[
\mathfrak{G}(\mathbb{R}, E) := \mathcal{O}^{\exp}(\mathbb{C}, E) \quad \text{and} \quad \mathfrak{G}(\{ \pm \infty \}, E) := \mathcal{H}_0(E) \ast \mathcal{H}_j(E).
\]

Let \( K = \mathbb{R} \) or \( K = \{ \pm \infty \} \). By [50], Theorem 2.3, p. 47, Proposition 3.3, p. 50] the Fourier transform

\[
\mathfrak{F} : \mathcal{G}(K) \rightarrow \mathfrak{G}(K), \mathfrak{F}([g])(z) := \int e^{-iz\zeta} g(\zeta) d\zeta, \ z \in \mathbb{C},
\]

where \( \gamma_\mathbb{R} \) is the path along the boundary of \( U_{1/\epsilon}(\mathbb{R}) \) for \( \epsilon > 0 \) with clockwise orientation, is a topological isomorphism (for \( K = \{ \pm \infty \} \) the path may be deformed to the boundary of \( U_{1/\epsilon}(\{ \pm \infty \}) \)).

The canonical (restriction) map

\[
R: \mathcal{H}_{-\infty}(\mathbb{C} \setminus K) / \mathcal{H}_{-\infty}(\mathbb{C} \setminus \{ \pm \infty \}) \rightarrow \mathcal{B}(\mathbb{R}), \ [F] \rightarrow [F],
\]

is a linear isomorphism by [51, Theorem 3.1, p. 49]. The combination of both results gives the following theorem.
6.3. **Theorem** ([54], Theorem 3.4, p. 51). *The asymptotic Fourier transform* 
\[ \mathfrak{F}_B : B(R) \to \mathfrak{S}(R) / \mathfrak{S}(\{ \pm \infty \}) \]. 
\[ \mathfrak{F}_B(f) \equiv [ (\mathfrak{F} \circ R^{-1})(f) ] \] 
*is a linear isomorphism.*

6.4. **Proposition.** Let \( j \in \mathbb{N} \).

a) Then for every \( f \in B([0, \infty[) \) there are \( f_1 \in B([0, j]) \) and \( f_2 \in B([j, \infty[) \) such that \( f = f_1 + f_2 \). For \( h \in \mathfrak{F}_B(f) \) we have for every \( k \in \mathbb{N} \)
\[ \sup_{\text{Im}(z) \leq k} |h(z) - \mathcal{F}_{[0,j]}(f_1)(z)| e^{-\frac{1}{2}|z| + j|\text{Im}(z)|} < \infty \].

b) Then for every \( f \in B([-\infty,0]) \) there are \( f_1 \in B([-j,0]) \) and \( f_2 \in B([-\infty,-j]) \) such that \( f = f_1 + f_2 \). For \( h \in \mathfrak{F}_B(f) \) we have for every \( k \in \mathbb{N} \)
\[ \sup_{\text{Im}(z) \geq -k} |h(z) - \mathcal{F}_{[-j,0]}(f_1)(z)| e^{-\frac{1}{2}|z| - j|\text{Im}(z)|} < \infty \].

**Proof.** a) The sheaf of \( \mathbb{C} \)-valued hyperfunctions is flabby by [67, p. 392] and thus for every \( f \in B([0, \infty[) \) there are \( f_1 \in B([0, j]) \) and \( f_2 \in B([j, \infty[) \) such that \( f = f_1 + f_2 \). Let \( [g] = R^{-1}(f) \). Then we have \( g \in H_{-\infty}(\mathbb{C} \setminus [0, \infty[) \) and \( g - g_1 \in H_{-\infty}(\mathbb{C} \setminus \{ \infty \}) \) for any \( g_1 \) with \( [g_1] = [g] \) by [51]. Corollary 4.2, p. 41 as well as
\[ \mathfrak{F}(g)(z) = \int_{\gamma} e^{iz\zeta} g(\zeta) d\zeta = \int_{\gamma_{[0, \infty[}} e^{iz\zeta} g(\zeta) d\zeta = L(g)(iz) \text{, } z \in \mathbb{C} \text{,} \]
by Cauchy’s integral theorem with the Laplace transform \( L \) from [52]. Let \( h \in \mathfrak{F}_B(f) \). It follows that there are \( g \in H_{-\infty}(\mathbb{C} \setminus [0, \infty[) \) and \( h_1 \in H_{-\infty}(\mathbb{C} \setminus \{ \infty \}) \) such that \( h = \mathfrak{F}(g + h_1) \). Hence we have
\[ \sup_{\text{Im}(z) \leq k} |h(z) - \mathcal{F}_{[0,j]}(f_1)(z)| e^{-\frac{1}{2}|z| + j|\text{Im}(z)|} \]
\[ = \sup_{\text{Im}(z) \leq k} |L(g + h_1)(iz) - \mathcal{F}_{[0,j]}(f_1)(iz)| e^{-\frac{1}{2}|z| + j|\text{Im}(z)|} \]
\[ = \sup_{\text{Re}(z) \geq -k} |L(g + h_1)(z) - \mathcal{F}_{[0,j]}(f_1)(z)| e^{-\frac{1}{2}|z| + j|\text{Re}(z)|} < \infty \]
for every \( k \in \mathbb{N} \) by Proposition [53] because \( L(g + h_1) \in L_B(f) \).

b) As above for every \( f \in B([-\infty,0]) \) there are \( f_1 = [F_1] \in B([-j,0]) \) and \( f_2 = [F_2] \in B([-\infty,-j]) \) such that \( f = f_1 + f_2 \). The map \( [F] \mapsto [F^{-1}] \) is a linear isomorphism \( B([-\infty,0]) \to B((0, \infty[) \) resp. \( H_{-\infty}(\mathbb{C} \setminus [0, \infty[) / H_{-\infty}(\mathbb{C} \setminus \{ \infty \}) \to H_{-\infty}(\mathbb{C} \setminus [0, \infty[) / H_{-\infty}(\mathbb{C} \setminus \{ \infty \}) \). Due to part a) we have for \( [g] = R^{-1}(f) \) that \( g \in H_{-\infty}(\mathbb{C} \setminus [-\infty,0]) \) and \( g - g_1 \in H_{-\infty}(\mathbb{C} \setminus \{ -\infty \}) \) for any \( g_1 \) with \( [g_1] = [g] \) as well as
\[ \mathfrak{F}(g)(z) = \int_{\gamma} e^{iz\zeta} g(\zeta) d\zeta = \int_{\gamma_{[-\infty,0]}} e^{iz\zeta} g(\zeta) d\zeta = \int_{\gamma_{[0, \infty[}} e^{iz\zeta} g(-\zeta) d\zeta = L(g(-))(iz) \text{, } z \in \mathbb{C} \text{,} \]
by Cauchy’s integral theorem. The rest follows from part a).

**\( \square \)**

6.5. **Theorem.** *The restriction map* 
\[ I : \mathfrak{F}(\mathbb{R}) / \mathfrak{S}(\{ \pm \infty \}) \to \mathcal{O}^{exp}(\mathbb{C} \setminus \mathbb{R}) / (\mathcal{O}^{exp}(\mathbb{C}) \oplus \mathcal{F} \mathcal{O}_{(\infty)}) \]. 
\[ [F] \mapsto [F_*] \text{,} \]
*where \( F_* := F \) on \( \mathbb{R} \) and \( F_* := 0 \) on \( -\mathbb{R} \), is a linear isomorphism and \( I \circ \mathfrak{F}_B = \mathcal{F}_B \).*
Proof. The map $I$ is well-defined because $F_+ \in \mathcal{O}^{exp}(\overline{\mathbb{C}} \setminus \mathbb{R})$ for $F \in \mathfrak{G}(\mathbb{R}) = \mathcal{O}^{exp}(\mathbb{C})$ and $F_+ \in \mathcal{F}\mathcal{O}_{\{z\}}$ for $F \in \mathfrak{G}(\{z = \pm \infty\})$.

Now, we prove $I \circ \mathfrak{G}_B = \mathcal{F}\mathcal{G}_B$, which also implies that $I$ is a linear isomorphism since $\mathfrak{G}_B$ and $\mathcal{F}\mathcal{G}_B$ are linear isomorphisms by Theorem 5.3 and Theorem 5.9. Let $f \in \mathcal{B}(\mathbb{R})$ and $j \in \mathbb{N}$. Then there are $f_1 \in \mathcal{B}([-\infty, 0])$ and $f_2 \in \mathcal{B}([0, \infty])$ such that $f = f_1 + f_2$ due to the flabbiness of the sheaf of $\mathcal{C}$-valued hyperfunctions by [67, p. 392] and [29, Lemma 1.4.4, p. 36]. Furthermore, there are $f_{1,1} \in \mathcal{B}([-j, 0])$ and $f_{1,2} \in \mathcal{B}([-\infty, -j])$ such that $f_1 = f_{1,1} + f_{1,2}$ and

$$(\mathcal{F} \circ \mathcal{R}^{-1}_{[-\infty, 0]})(f_1) - \mathcal{F} \circ \mathcal{R}^{-1}_{[-j, 0]}(f_{1,1})|_{\operatorname{Im} > 0} \in \mathcal{F}\mathcal{O}_{[-j, -j]}$$

as well as $f_{2,1} \in \mathcal{B}([0, j])$ and $f_{2,2} \in \mathcal{B}([j, \infty])$ such that $f_2 = f_{2,1} + f_{2,2}$ and

$$(\mathcal{F} \circ \mathcal{R}^{-1}_{[0, \infty]})(f_2) - \mathcal{F} \circ \mathcal{R}^{-1}_{[0, j]}(f_{2,1})|_{\operatorname{Im} < 0} \in \mathcal{F}\mathcal{O}_{[j, \infty]}$$

by Proposition [3.3]. In addition, we obtain

$$\sup_{\operatorname{Im}(z) \geq k} |(\mathfrak{G} \circ R^{-1})(f_1)(z) - \mathcal{F} \circ \mathcal{R}^{-1}_{[-j, 0]}(f_{1,1})(z)|e^{-\frac{1}{2}\sqrt{z+j}\operatorname{Im}(z)} < \infty$$

and

$$\sup_{\operatorname{Im}(z) \leq k} |(\mathfrak{G} \circ R^{-1})(f_2)(z) - \mathcal{F} \circ \mathcal{R}^{-1}_{[0, j]}(f_{2,1})(z)|e^{-\frac{1}{2}\sqrt{z+j}\operatorname{Im}(z)} < \infty$$

for every $k \in \mathbb{N}$ from Proposition [3.4]. We remark that

$$\mathfrak{G}(R^{-1}(f)) = \mathfrak{G}(R^{-1}(f)) + \mathfrak{G}(R^{-1}(f)) - \mathfrak{G}(R^{-1}(f)) - \mathfrak{G}(R^{-1}(f))$$

$$= \mathfrak{G}(R^{-1}(f)) + \mathfrak{G}(R^{-1}(f)) - \mathfrak{G}(R^{-1}(f)) - \mathfrak{G}(R^{-1}(f))$$

$$= \mathfrak{G}(R^{-1}(f)) + \mathfrak{G}(R^{-1}(f)) - \mathfrak{G}(R^{-1}(f)) - \mathfrak{G}(R^{-1}(f))$$

$$= \mathfrak{G}(R^{-1}(f)) + \mathfrak{G}(R^{-1}(f)) - \mathfrak{G}(R^{-1}(f)) - \mathfrak{G}(R^{-1}(f))$$

where $\mathfrak{G}(R^{-1}(f)) := \mathfrak{G}(R^{-1}(f))$ on $-\mathbb{H}$ and $\mathfrak{G}(R^{-1}(f)) := 0$ on $\mathbb{H}$. We deduce with $k = j$ that

$$|\mathfrak{G}(R^{-1}(f))|_{j,(-\infty)} \leq |\mathfrak{G}(R^{-1}(f))|_{j,(-\infty)} + |\mathfrak{G}(R^{-1}(f))|_{j,(-\infty)}$$

and

$$|\mathfrak{G}(R^{-1}(f))|_{j,(-\infty)} \leq |\mathfrak{G}(R^{-1}(f))|_{j,(-\infty)} + |\mathfrak{G}(R^{-1}(f))|_{j,(-\infty)}$$

plus $\mathfrak{G}(R^{-1}(f)) \in \mathfrak{G}\mathfrak{G}(\mathbb{R}) = \mathcal{O}^{exp}(\mathbb{C})$. Hence we conclude

$$\mathfrak{G}(R^{-1}(f)) = \mathfrak{G}(R^{-1}(f)) \in (\mathcal{O}^{exp}(\mathbb{C}) \oplus \mathcal{F}\mathcal{O}_{\{z = \pm \infty\}}),$$

resulting in $I \circ \mathfrak{G}_B = \mathcal{F}\mathcal{G}_B$. \qed
7. LAPLACE ANDASYMPTOTIC LAPLACE TRANSFORM

In this short section we phrase our results on the (asymptotic) Fourier transform in terms of the Laplace transform. We start with the definition of the range spaces.

7.1. Definition. Let $E$ be a $\mathbb{C}$-lcHs.

a) For $-\infty < a \leq \infty$ we define the space

$$\mathcal{LO}_{[a,\infty)}(E) := \{ f \in \mathcal{O}(-iH, E) \mid \forall k \in \mathbb{N}, \alpha \in \mathbb{R} : \| f \|_{k,\alpha,[a,\infty)} < \infty \}$$

where

$$\| f \|_{k,\alpha,[a,\infty)} := \sup_{\text{Re}(z) \geq \frac{a}{k}} p_\alpha(f(z)) e^{-\frac{1}{k}|z|} w_\alpha^*(-\text{Re}(z)).$$

b) For $-\infty \leq a < \infty$ we define the space

$$\mathcal{LO}_{[-\infty,a)}(E) := \{ f \in \mathcal{O}(iH, E) \mid \forall k \in \mathbb{N}, \alpha \in \mathbb{R} : \| f \|_{k,\alpha,[-\infty,a)} < \infty \}$$

where

$$\| f \|_{k,\alpha,[-\infty,a)} := \sup_{\text{Re}(z) \leq -\frac{a}{k}} p_\alpha(f(z)) e^{-\frac{1}{k}|z|} w_\alpha^*(-\text{Re}(z)).$$

c) For $-\infty < a \leq b < \infty$ we define the space

$$\mathcal{LO}_{[a,b]}(E) := \{ f \in \mathcal{O}(\mathbb{C}, E) \mid \forall k \in \mathbb{N}, \alpha \in \mathbb{R} : \| f \|_{k,\alpha,[a,b]} < \infty \}$$

where

$$\| f \|_{k,\alpha,[a,b]} := \sup_{z \in \mathbb{C}} p_\alpha(f(z)) e^{-\frac{1}{k}|z|} H_{\alpha,b}^*(-\text{Re}(z)).$$

7.2. Theorem. Let $E$ be a $\mathbb{C}$-lcHs and $\emptyset \neq K \subseteq \mathbb{R}$ a compact interval. If

(i) $K \subset \mathbb{R}$ and $E$ is locally complete, or if

(ii) $E$ is sequentially complete,

then the map

$$\mathcal{L} : \mathcal{bv}_K(E) \to \mathcal{LO}_K(E), \mathcal{L}([F])(\zeta) := \mathcal{F}([F])(-i\zeta),$$

is a topological isomorphism.

Proof. This follows directly from Theorem 3.3 resp. Theorem 4.3 and $\text{Im}(-i\zeta) = -\text{Re}(\zeta)$. \qed

7.3. Proposition. Let $\emptyset \neq K \subseteq \mathbb{R}$ be a compact interval. Then the spaces $\mathcal{bv}_K$ and $\mathcal{LO}_K$ are nuclear Fréchet spaces.

Proof. For $\mathcal{bv}_K$ this statement follows from [41, Remark 3.4 a), p. 8] and [57, Proposition 28.6, p. 347]. This implies that $\mathcal{LO}_K$ is a nuclear Fréchet space by Theorem 7.2 too. \qed

7.4. Theorem. Let $E$ be an admissible sequentially complete $\mathbb{C}$-lcHs and $A := [-\infty,a]$ or $A := [a,\infty[$ for some $a \in \mathbb{R}$. Then the asymptotic Laplace transform

$$\mathcal{L}_A^B : \mathcal{B}(A,E) \to \mathcal{LO}_A^B(E)/\mathcal{LO}_A^+(E), \mathcal{L}_A^B(f) := [(\mathcal{L} \circ \mathcal{R}_A^-(f)],$$

is a linear isomorphism, where the closure $\overline{A}$ is taken in $\mathbb{R}$.

Proof. It follows directly from Theorem 5.1 and Theorem 7.2 (ii). \qed

Due to the theorem above, Proposition 7.3 and Theorem 2.2, our asymptotic Laplace transform fulfills the conditions (I) and (IV) from the introduction.
7.5. Corollary. Let $E$ be a locally complete $\mathbb{C}$-lcHs and $-\infty < a \leq b < \infty$. Then the Laplace transform

$$
\mathcal{L}^B_{[a,b]} : B([a,b], E) \to \mathcal{L}O_{[a,b]}(E), \mathcal{L}^B_{[a,b]}([F]) := (\mathcal{L}_{[a,b]} \circ R_{[a,b]}^{-1})([F]),
$$

is a topological isomorphism and

$$
\mathcal{L}^B_{[a,b]}([F])(\zeta) = \mathcal{L}_{[a,b]}([\Psi([F])])(\zeta) = (\mathcal{H}_{[a,b]}([F]), e^{-\zeta})\zeta \in \mathbb{C}.
$$

Proof. This holds due to Theorem 5.3 and Theorem 7.2 (i). \qed

7.6. Proposition. Let $E$ be a strictly admissible sequentially complete $\mathbb{C}$-lcHs and $-\infty < a < b < \infty$. Then the following holds:

a) For every $f \in B([a, \infty[, E)$ there are $f_1 \in B([a, b], E)$ and $f_2 \in B([b, \infty[, E)$ such that $f = f_1 + f_2$ and

$$
\mathcal{L}^B_{[a, \infty]}(f) = [\mathcal{L}^B_{[a, b]}(f_1)_{[c > 0]}] + \mathcal{L}^B_{[b, \infty]}(f_2).
$$

b) For every $f \in B([a, b], E)$ there are $f_1 \in B([a, b], E)$ and $f_2 \in B([b, \infty[, E)$ such that $f = f_1 + f_2$ and

$$
\mathcal{L}^B_{[a, b]}(f) = [\mathcal{L}^B_{[a, b]}(f_1)_{[c > 0]}] + \mathcal{L}^B_{[a, \infty]}(f_2).
$$

Proof. The statement is a consequence of Proposition 5.6 and the definition of $\mathcal{L}K$ and $\text{Im}(-i\zeta) = -\text{Re}(\zeta)$. \qed

7.7. Proposition. Let $E$ be a $\mathbb{C}$-lcHs and $\emptyset \neq K \subseteq \mathbb{R}$ a compact interval. Let $\tau_h([F]) := [F(-h)]$, $h \in \mathbb{R}$, be the shift operator from $bv_K(E)$ to $bv_{h+K}(E)$. If

(i) $K \subseteq \mathbb{R}$ and $E$ is locally complete, or if

(ii) $E$ is sequentially complete,

then we have for $[F] \in bv_K(E)$

$$
\mathcal{L}_{h+K}(\tau_h([F])) = e^{-h} \mathcal{L}_K([F]).
$$

Proof. This is a consequence of Proposition 7.6 and the definition of $\mathcal{L}_K$. \qed

7.8. Proposition. Let $E$ be a sequentially complete, $\mathbb{C}$-lcHs and $A := ]-\infty, a]$ or $A := [a, \infty[$ for some $a \in \mathbb{R}$. Let $\tau_h([F]) := [F(-h)]$, $h \in \mathbb{R}$, be the shift operator from $B(A, E)$ to $B(h+ A, E)$. Then we have for $[F] \in B(A, E)$

$$
\mathcal{L}_{h+ A}(\tau_h([F])) = e^{-h} \mathcal{L}_A^B([F]).
$$

Proof. This is a consequence of Proposition 5.7 and the definition of $\mathcal{L}_A^B$. \qed

7.9. Proposition. Let $E$ be a sequentially complete $\mathbb{C}$-lcHs, $\emptyset \neq K \subseteq \mathbb{R}$ a compact interval and $P(\partial) := \sum_{k=0}^{\infty} \frac{c_k}{k!} \partial^k$ where $(c_k) \subseteq \mathbb{C}$ and $P$ is of exponential type 0. Then we have for $[F] \in bv_K(E)$

$$
\mathcal{L}_K(P(\partial)[F])(\zeta) = P(\zeta)\mathcal{L}_K([F])(\zeta) \quad \text{and} \quad \mathcal{L}_K(P[F])(\zeta) = P(-\partial)\mathcal{L}_K([F])(\zeta)
$$

for $\text{Re}(\zeta) > 0$ if $K = [a, \infty[$, for $\text{Re}(\zeta) < 0$ if $K = [-\infty, a]$, and for $\zeta \in \mathbb{C}$ if $K \subseteq \mathbb{R}$, respectively.

Proof. This follows from Proposition 5.6 and the definition of $\mathcal{L}_K$. \qed

7.10. Proposition. Let $E$ be an admissible sequentially complete $\mathbb{C}$-lcHs, $A := ]-\infty, a]$ or $A := [a, \infty[$ for some $a \in \mathbb{R}$ and $P(\partial) := \sum_{k=0}^{\infty} \frac{c_k}{k!} \partial^k$ where $(c_k) \subseteq \mathbb{C}$ and $P$ is of exponential type 0. Then we have for $f \in B(A, E)$

$$
\mathcal{L}_A^B(P(\partial)f) = P\mathcal{L}_A^B(f) \quad \text{and} \quad \mathcal{L}_A^B(Pf) = P(-\partial)\mathcal{L}_A^B(f).
$$

Proof. This follows from Proposition 7.8 and the definition of $\mathcal{L}_A^B$. \qed
7.11 Proposition. Let \((E, (p_\alpha)_{\alpha \in \mathfrak{A}}), (E_1, (p_\beta)_{\beta \in \mathfrak{B}})\) and \((E_2, (p_\omega)_{\omega \in \Omega})\) be admissible sequentially complete \(\mathcal{C}\) Fréchet spaces, \(E\) and \(E_1\) strictly admissible, such that a canonical bilinear map \(\cdot : E_1 \times E_2 \to E\) is defined with the property
\[
\forall \alpha \in \mathfrak{A}, \exists \beta \in \mathfrak{B}, \omega \in \Omega, D > 0 \forall x \in E_1, y \in E_2 : p_\alpha(x \cdot y) \leq D p_\beta(x) p_\omega(y).
\]
Let \(a, b, c, d \in \mathbb{R}\). Then we have for \(f \in \mathcal{B}([a, \infty[, E_1)\) and \(g \in \mathcal{B}([c, \infty[, E_2)\), or for \(f \in \mathcal{B}([a, b], E_1)\) and \(g \in \mathcal{B}([c, d], E_2)\) that
\[
\mathcal{L}^B(f * B^1 g) = \mathcal{L}^B(f \hat{\otimes}^B g) = \mathcal{L}^B(f) \mathcal{L}^B(g).
\]

Proof. This follows from Corollary 5.10 c) and the definition of \(\mathcal{L}^B_E\).

Due to (32), Proposition 7.10 and Proposition 7.11 combined with Theorem 5.9 and Corollary 5.10 our asymptotic Laplace transform fulfills condition (II) for a satisfactory theory of Laplace transforms as well.

8. Langenbruch’s Asymptotic Laplace Transform

In Section 7 we already discussed the relation between our asymptotic Fourier transform and the one Langenbruch on \(\mathcal{B}(\mathbb{R})\). This section is dedicated to the connection of our asymptotic Laplace transform on \(\mathcal{B}([0, \infty[, E)\) with the one of Langenbruch in the case that \(E\) is a \(\mathcal{C}\) Fréchet space and with the ones of Komatsu, Lumer and Neubrander in the case that \(E\) is a \(\mathcal{C}\) Banach space as well. We recall the relevant notions and results from [50] (and the introduction).

8.1 Definition ([50, p. 53, 55, 61]). Let \(E\) be a \(\mathcal{C}\) Fréchet space. We define the spaces
\[
\mathcal{L}_E^{(0, \infty)}(E) := \{ f \in \mathcal{O}(\mathbb{C}, E) \mid \forall k \in \mathbb{N}, \alpha \in \mathfrak{A} : \| f \|_{k, \alpha, [0, \infty)}^E < \infty \}
\] where
\[
\| f \|_{k, \alpha, [0, \infty)}^E := \sup_{R \in \mathbb{Z} \geq k} p_\alpha(f(z)) e^{-\frac{k}{2} |z|},
\]
and
\[
\mathcal{L}_E^{(\infty)}(E) := \{ f \in \mathcal{O}(\mathbb{C}, E) \mid \forall k \in \mathbb{N}, \alpha \in \mathfrak{A} : \| f \|_{k, \alpha, \infty)}^E < \infty \}
\] where
\[
\| f \|_{k, \alpha, \infty)}^E := \sup_{R \in \mathbb{Z} \geq k} p_\alpha(f(z)) e^{k|\Re(z)| + \frac{k}{2} |z|}.
\]

Let \(E\) be a \(\mathcal{C}\) Fréchet space and \(K = [0, \infty)\) or \(K = \{ \infty \}\). By [50, Theorem 4.1, p. 53, Proposition 5.2, p. 55, 61] the Laplace transform
\[
\mathcal{L}^E : \mathcal{L}_E^{(0, \infty)}(E) \to \mathcal{L}_E^{(0, \infty)}(E), \mathcal{L}^E([g])(z) := \int_{\gamma_K} g(\zeta) e^{-\zeta \bar{z}} d\zeta, \ z \in \mathbb{C},
\]
where \(\gamma_K\) is the path along the boundary of \(U_1(K)\) for \(c > 0\) with clockwise orientation, is a topological isomorphism.

The canonical (restriction) map
\[
R_* : \mathcal{H}^{(0, \infty)}(\mathbb{C} \setminus [0, \infty[, E) / \mathcal{H}^{(\infty)}(\mathbb{C} \setminus \{ \infty \}, E) \to \mathcal{B}([0, \infty[, E)
\]
is a linear isomorphism for any \(\mathcal{C}\) Fréchet space \(E\) by [51, Corollary 5.2, p. 42]. The combination of both results gives the following theorem.

8.2 Theorem ([50, Theorem 5.3, p. 55, 61]). Let \(E\) be a \(\mathcal{C}\) Fréchet space. Then the asymptotic Laplace transform
\[
\mathcal{L}_E^B : \mathcal{B}([0, \infty[, E) \to \mathcal{L}_E^{(0, \infty)}(E) / \mathcal{L}_E^{(\infty)}(E), \mathcal{L}_E^B(f) := \left( (\mathcal{L} \circ R_*)^{-1}\right)(f),
\]
is a linear isomorphism.
8.3. **Proposition.** Let $E$ be a $C$-Fréchet space and $j \in \mathbb{N}$. Then for every $f \in B([0, \infty[, E)$ there are $f_1 \in B([0, j], E)$ and $f_2 \in B([j, \infty[, E)$ such that $f = f_1 + f_2$. For $h \in \mathcal{L}_B(f)$ we have for every $k \in \mathbb{N}$ and $\alpha \in \mathfrak{A}$

$$\sup_{\text{Re}(z)\geq k} p_\alpha(h(z) - \mathcal{L}_B^0_{[0,j]}(f_1)(z)) e^{-\frac{1}{2} |z| |\text{Re}(z)|} < \infty.$$ 

**Proof.** First, we observe that the operation $e'(g) := [e' \circ G]$ is well-defined for $g = [G] \in B([0, \infty[, E)$ resp. $g = [G] \in B([0, j], E)$ resp. $g = [G] \in \mathcal{L}_G([0, \infty])[E] / \mathcal{L}_G([j, \infty])[E]$ and any $e' \in E'$. The decomposition $f = f_1 + f_2$ follows from the flabbiness of the sheaf of Fréchet-valued hyperfunctions by [21, Theorem 2.6, p. 14] and [29, Lemma 1.4.4, p. 36]. Let $h \in \mathcal{L}_B(f)$, $e' \in E'$, $f = [F]$, $f_1 = [F_1]$ and $f_2 = [F_2]$. Then $e'(f) = [e' \circ F_1] + [e' \circ F_2]$ and

$$(e' \circ \mathcal{L}_B^0_{[0,j]}(f_1))(z) = \mathcal{L}_B^0_{[0,j]}([ e' \circ F_1 ])(z), \ z \in \mathbb{C},$$

as well as $e' \circ h \in \mathcal{L}_B([e' \circ F])$ because $\mathcal{L}(e' \circ g) = e' \circ \mathcal{L}(g)$ for any $g \in \mathcal{H}_{\infty}(\mathbb{C} \setminus [0, \infty[)$. By [21, Lemma 5.4 (a), p. 56] ($\mathcal{L}_B^0_{[0,j]}([e' \circ F_1])$) corresponds to $\nu_j$ in this lemma due to Corollary 2.6 and we obtain

$$\sup_{\text{Re}(z)\geq k} |(e' \circ h)(z) - \mathcal{L}_B^0_{[0,j]}([ e' \circ F_1 ])(z)| e^{-\frac{1}{2} |z| |\text{Re}(z)|} < \infty$$

every $k \in \mathbb{N}$. Since this holds for any $e' \in E'$, the Mackey theorem implies that

$$\sup_{\text{Re}(z)\geq k} p_\alpha(h(z) - \mathcal{L}_B^0_{[0,j]}([f_1])(z)) e^{-\frac{1}{2} |z| |\text{Re}(z)|} < \infty$$

every $k \in \mathbb{N}$ and $\alpha \in \mathfrak{A}$. \hfill \Box

We come to our main result of this section.

8.4. **Theorem.** Let $E$ be a $C$-Fréchet space. Then the canonical map

$$I_0: \mathcal{L}_G([0, \infty])[E] / \mathcal{L}_G([j, \infty])[E] \to \mathcal{L}_G([0, \infty])[E] / \mathcal{L}_G([j, \infty])[E], \ [F] \mapsto [F],$$

is a linear isomorphism and $I_0 \circ \mathcal{L}_B = \mathcal{L}_B$.

**Proof.** The map $I_0$ is well-defined as $\mathcal{L}_G([0, \infty])[E] \subset \mathcal{L}_G([0, \infty])[E]$ and $\mathcal{L}_G([j, \infty])[E] \subset \mathcal{L}_G([j, \infty])[E]$.

We only need to prove that $I_0 \circ \mathcal{L}_B = \mathcal{L}_B$ since $\mathcal{L}_B$ and $\mathcal{L}_B^0$ are linear isomorphisms by Proposition 7.6 and Theorem 2.3 (a). Let $f \in B([0, \infty[, E)$ and $j \in \mathbb{N}$. Then there are $f_1 \in B([0, j], E)$ and $f_2 \in B([j, \infty[, E)$ such that $f = f_1 + f_2$ and

$$(L \circ \mathcal{R}_{[0, \infty]}^1) (f) - \mathcal{L}_B^0_{[0,j]}(f_1)(z) \in \mathcal{O}_G([j, \infty])[E]$$

as well as

$$\sup_{\text{Re}(z)\geq k} p_\alpha((L \circ \mathcal{R}_{[0, \infty]}^1)(f)(z) - \mathcal{L}_B^0_{[0,j]}(f_1)(z)) e^{-\frac{1}{2} |z| |\text{Re}(z)|} < \infty$$

for all $k \in \mathbb{N}$ and $\alpha \in \mathfrak{A}$ by Proposition 7.6 (a) and Proposition 8.3. Using

$$(L \circ \mathcal{R}_{[0, \infty]}^1) (f)(z) - (L \circ \mathcal{R}_{[0, \infty]}^1)(f)(z) = (L \circ \mathcal{R}_{[0, \infty]}^1)(f)(z) - \mathcal{L}_B^0_{[0,j]}(f_1)(z) - (L \circ \mathcal{R}_{[0, \infty]}^1)(f)(z) - \mathcal{L}_B^0_{[0,j]}(f_1)(z))$$

for $\text{Re}(z) > 0$, this implies with $k = j$ that

$$\| (L \circ \mathcal{R}_{[0, \infty]}^1)(f)(z) - (L \circ \mathcal{R}_{[0, \infty]}^1)(f)(z) \|_{\text{Re}(z) > \infty} < \infty.$$ 

We deduce that $(L \circ \mathcal{R}_{[0, \infty]}^1)(f) - (L \circ \mathcal{R}_{[0, \infty]}^1)(f) \in \mathcal{L}_G([j, \infty])[E]$ and so $I_0 \circ \mathcal{L}_B = \mathcal{L}_B$. \hfill \Box
8.5. Remark. Due to Theorem 5.4 Proposition 7.11 and 5. Theorem 5.5, p. 56] our definition of the convolution of two elements from \( \mathcal{B}([0, \infty]) \) is consistent with the one given by Langenbruch.

Let us turn to Komatsu’s asymptotic Laplace transform. Again, we briefly recall the relevant notions and results. For \( 0 < \varphi < \frac{\pi}{2} \) and \( r \geq 0 \) we set

\[
\Gamma_{r, \varphi} := \{ \rho e^{i\psi} \mid \rho \geq r, |\psi| \leq \varphi \}.
\]

An open set \( U \subset \mathbb{C} \) is called \textit{postsectorial} (see [53, p. 37], [54, p. 150]) if \( \forall \ 0 < \varphi < \frac{\pi}{2} \exists \ r > 0 : \Gamma_{r, \varphi} \subset U \).

![Figure 11. \( \Gamma_{r, \varphi} \) for \( 0 < \varphi < \frac{\pi}{2} \) and \( r \geq 0 \)](image)

8.6. Definition ([50, p. 57]). Let \( (E, \| \cdot \|_E) \) be a \( \mathbb{C} \)-Banach space, \( a \in \mathbb{R} \) and \( U \subset \mathbb{C} \) open and postsectorial. We define

\[
\mathcal{G}_a(U, E) := \{ f \in \mathcal{O}(U, E) \mid \forall \ j \in \mathbb{N}, 0 < \varphi < \frac{\pi}{2}, r > 0, \Gamma_{r, \varphi} \subset U : \| f \|^\text{Kom}_{j, r, \varphi, [a, \infty]} < \infty \}
\]

where

\[
\| f \|^\text{Kom}_{j, r, \varphi, [a, \infty]} := \sup_{z \in \Gamma_{r, \varphi}} \| f(z) \|_{E_E^{-\frac{1}{j}|z|+a|\text{Re}(z)|}},
\]

and

\[
\mathcal{G}_{(\infty)}(U, E) := \{ f \in \mathcal{O}(U, E) \mid \forall \ j \in \mathbb{N}, 0 < \varphi < \frac{\pi}{2}, r > 0, \Gamma_{r, \varphi} \subset U : \| f \|^\text{Kom}_{j, r, \varphi, (\infty)} < \infty \}
\]

where

\[
\| f \|^\text{Kom}_{j, r, \varphi, (\infty)} := \sup_{z \in \Gamma_{r, \varphi}} \| f(z) \|_{E_E^{-\frac{1}{j}|z|+|\text{Re}(z)|}},
\]

as well as

\[
\mathcal{L}_\text{Kom} \mathcal{B}^{\text{exp}}_{[a, \infty]}(E) := \lim_{U} \mathcal{G}_{(a, \infty)}(U, E) \quad \text{and} \quad \mathcal{L}_\text{Kom} \mathcal{B}^{\text{exp}}_{(\infty)}(E) := \lim_{U} \mathcal{G}_{(a, \infty)}(U, E)
\]

where the inductive limits run over all open postsectorial sets \( U \subset \mathbb{C} \).

Komatsu’s asymptotic Laplace transform \( \mathcal{L}_\text{Kom} \) is a linear isomorphism since the canonical (restriction) map

\[
\rho_a : \mathcal{B}^{\text{exp}}_{[a, \infty]}(E)/\mathcal{B}^{\text{exp}}_{(\infty)}(E) \to \mathcal{B}([a, \infty[, E), [F] \mapsto [F],
\]

is a linear isomorphism for \( a \in \mathbb{R} \) by [32, Theorem 1, p. 361] (cf. [32, Theorem 3.5, p. 816], [34, Theorem 2, p. 61] with different proofs) and his Laplace transform

\[
\mathcal{L}_\text{Kom, a} : \mathcal{B}^{\text{exp}}_{[a, \infty]}(E) \to \mathcal{L}_\text{Kom} \mathcal{B}^{\text{exp}}_{[a, \infty]}(E)
\]

for \( a \in \mathbb{R} \cup \{ \infty \} \) as well due to [32, Theorem 3.3, 3.4, p. 815-816] for \( E = \mathbb{C} \) and [35, p. 218] for \( \mathbb{C} \)-Banach spaces \( E \).
8.7. Theorem ([53, Eq. (24), p. 217]). Let \((E, \| \cdot \|_E)\) be a \(C\)-Banach space and \(a \in \mathbb{R}\). Then
\[
\mathcal{L}_{\text{Kom}} : B([a, \infty[, E) \to \mathcal{L}_{\text{Kom}} B^{\text{exp}}_{[a, \infty]}(E)/\mathcal{L}_{\text{Kom}} B^{\text{exp}}_{(\infty)}(E),
\]
\[
\mathcal{L}_{\text{Kom}}(f) := [(\mathcal{L}_{\text{Kom},a} \circ \rho_a^{-1})(f)],
\]
is a linear isomorphism.

We note the following generalisation of [50, Theorem 6.3, p. 59] from \(E = \mathbb{C}\) to general \(C\)-Banach spaces \(E\).

8.8. Theorem. Let \((E, \| \cdot \|_E)\) be a \(C\)-Banach space. Then the canonical map
\[
I_1 : \mathcal{G}_{[0, \infty)}(E)/\mathcal{G}_{(\infty)}(E) \to \mathcal{L}_{\text{Kom}} B^{\text{exp}}_{[0, \infty]}(E)/\mathcal{L}_{\text{Kom}} B^{\text{exp}}_{(\infty)}(E), [F] \mapsto [F],
\]
is a linear isomorphism such that \(I_1 \circ \mathcal{L}_E = \mathcal{L}_{\text{Kom}}\).

Proof. The map \(I_1\) is well-defined due to the canonical inclusions \(\mathcal{G}_{[0, \infty)}(E) \subset \mathcal{L}_{\text{Kom}} B^{\text{exp}}_{[0, \infty]}(E)\) and \(\mathcal{L}_{\text{Kom}} B^{\text{exp}}_{(\infty)}(E) \subset \mathcal{L}_{\text{Kom}} B^{\text{exp}}_{(\infty)}(E)\).

We only need to show that \(I_1 \circ \mathcal{L}_E = \mathcal{L}_{\text{Kom}}\) because \(\mathcal{L}_E\) and \(\mathcal{L}_{\text{Kom}}\) are linear isomorphisms by Theorem 8.2 and Theorem 8.7. We repeat the proof from [51] Theorem 6.3, p. 59 with small modifications. Let \(f \in B([0, \infty[, E)\) and \((\mathcal{L}_{\text{Kom},0} \circ \rho_0^{-1})(f)\) be defined on some postsectorial open set \(U \subset \mathbb{C}\). For \(j \in \mathbb{N}\) there are \(f_1 \in B([0, j], E)\) and \(f_2 \in B([j, \infty[, E)\) such that \(f = f_1 + f_2\) and
\[
(\mathcal{L}_{\text{Kom},0} \circ \rho_0^{-1})(f) - \mathcal{L}^B_{[0,j]}(f_1) \in \mathcal{L}_{\text{Kom}} B^{\text{exp}}_{[j, \infty]}(E)
\]
as well as
\[
\sup_{z \in U, z \geq k} \| E^{-\frac{1}{2}|z|} Re(z) \|_E < \infty
\]
for all \(k \in \mathbb{N}\) by [32, Theorem 3.9, p. 818] combined with [53, p. 218] and Proposition 8.3. We note that
\[
((\mathcal{L}_{\text{Kom},0} \circ \rho_0^{-1})(f)(z) - (\mathcal{L} \circ R^1)(f)(z)) - ((\mathcal{L} \circ R^1)(f)(z) - (\mathcal{L}_{\text{Kom},0} \circ \rho_0^{-1})(f)(z))
\]
for \(z \in U\). Let \(0 < \varphi < \frac{T}{2}\). Then there is \(r > 0\) such that \(G_{r, \varphi} \subset U\) and with \(k = j\)
\[
\| (\mathcal{L}_{\text{Kom},0} \circ \rho_0^{-1})(f) - (\mathcal{L} \circ R^1)(f) \|_{\mathcal{L}_{\text{Kom},0} G_{r, \varphi}}
\]
\[
= \sup_{z \in G_{r, \varphi}} \| (\mathcal{L}_{\text{Kom},0} \circ \rho_0^{-1})(f)(z) - (\mathcal{L} \circ R^1)(f)(z) \|_E e^{-\frac{1}{2}|z|} Re(z) < \infty.
\]
We deduce that \((\mathcal{L}_{\text{Kom},0} \circ \rho_0^{-1})(f) - (\mathcal{L} \circ R^1)(f) \in \mathcal{L}_{\text{Kom}} B^{\text{exp}}_{(\infty)}(E)\) and so \(I_1 \circ \mathcal{L}_E = \mathcal{L}_{\text{Kom}}\). \(\square\)

8.9. Corollary. Let \(E\) be a \(C\)-Banach space. Then the canonical map
\[
I_2 : \mathcal{L}^{0}[0, \infty](E)/\mathcal{L}^{0}(E) \to \mathcal{L}_{\text{Kom}} B^{\text{exp}}_{[0, \infty]}(E)/\mathcal{L}_{\text{Kom}} B^{\text{exp}}_{(\infty)}(E), [F] \mapsto [F],
\]
is a linear isomorphism such that \(I_2 \circ I_0 = I_1\) and \(I_2 \circ \mathcal{L}_E = \mathcal{L}_{\text{Kom}}\).

Proof. The map \(I_2\) is well-defined due to the canonical inclusions \(\mathcal{L}^{0}[0, \infty](E) \subset \mathcal{L}_{\text{Kom}} B^{\text{exp}}_{[0, \infty]}(E)\) and \(\mathcal{L}^{0}(E) \subset \mathcal{L}_{\text{Kom}} B^{\text{exp}}_{(\infty)}(E)\). Clearly, \(I_2 \circ I_0 = I_1\) and the rest of the statement follows from Theorem 8.3 and Theorem 8.8. \(\square\)

It follows from the corollary above that our asymptotic Laplace transform satisfies condition (III) from the introduction as well. In [53, Lumer and Neubrander (cf. Bäumer [3]) introduced an asymptotic Laplace transform \(\mathcal{L}_{0,\text{LN}}\) on the space \(L^1_{\text{loc}}([0, \infty[, E)\) of (equivalence classes) of locally integrable functions on \([0, \infty[\) with values in a \(C\)-Banach space \(E\). They modified their Laplace transform in [54, 2.5 Definition, p. 156] to a Laplace transform \(\mathcal{L}_{\text{LN}}\) because the unmodified
Laplace transform does not fulfil condition (II) from the introduction by [54, Example 2.1, p. 153]. Their modified Laplace transform \( \Sigma_{LN} \) coincides with Komatsu’s Laplace transform on \( L^1_{\text{loc}}([0, \infty[, E) \) by [54, Eq. (15), p. 157]. Here we may regard \( L^1_{\text{loc}}([0, \infty[, E) \) as a linear subspace of \( B([0, \infty[, E) \) as in [29, Theorem 1.3.10, p. 23]. Further, they clarified the relation between their unmodified Laplace transform and the one of Komatsu, namely that \( \Sigma_{\text{Kom}}(f) \in \Sigma_{\text{LN}}(f) \) for all \( f \in L^1_{\text{loc}}([0, \infty[, E) \) by [54, 3.1 Theorem, p. 157]. In combination with Theorem 8.4 and Theorem 8.8 we note the following implication, which generalises [64, Corollary 6.4, p. 60].

8.10. **Corollary.** Let \( E \) be a \( \mathbb{C} \)-Banach space. Then we have

\[
\mathcal{L}^E(f) = \Sigma_0(f) = \Sigma_{\text{Kom}}(f) = \Sigma_{\text{LN}}(f) \in \Sigma_{\text{LN}}(f), \quad f \in L^1_{\text{loc}}([0, \infty[, E).
\]

We omitted the linear isomorphisms \( I_j \) in the equations above.

**Appendix A. An alternative proof of Theorem 8.4**

In this section we give an alternative proof of the map

\[
I_0: \Sigma_{\mathbb{C}^0([0, \infty[, E)}/\Sigma_{\mathbb{C}^0([0, \infty[, E)} \to \mathcal{L}\mathcal{O}_{[0, \infty[, E)}|E)/\mathcal{L}\mathcal{O}_{[0, \infty[, E)}, \quad [F] \mapsto [F],
\]

in our main result Theorem 8.4 of Section 8 is a linear isomorphism if \( E \) is a \( \mathbb{C} \)-Fréchet space.

A.1. **Definition.** Let \( E \) be a \( \mathbb{C} \)-lcHs and \( V_0 := -i\mathbb{H} = \mathbb{C}_{\text{Re} > 0} \). We define the space

\[
\mathcal{E}_\infty(E) := \{ f \in \mathcal{C}^\infty(\mathbb{C}, E) \mid \forall n \in \mathbb{N}, m \in \mathbb{N}_0, \alpha \in \mathbb{A} : |f|_{n,m,\alpha}^\infty < \infty \}
\]

where

\[|f|_{n,m,\alpha}^\infty := \sup_{\operatorname{Re}(z) > -n} p_\alpha(\partial^\beta f(z)) e^{-\frac{1}{2} |z|^\alpha \operatorname{Re}(z)}, \]

and its topological subspaces

\[
\mathcal{E}_{\infty}^\Re(E) := \{ f \in \mathcal{E}_\infty(E) \mid \text{supp } f \subset \overline{V}_0 \}
\]

and

\[
\mathcal{E}_{\infty}^\Im(E) := \{ f \in \mathcal{E}_\infty(E) \mid \text{supp } \overline{f} \subset \overline{V}_0 \}.
\]

An alternative way to prove the surjectivity of \( I_0 \) is to find a solution \( g \in \mathcal{E}_{\infty}^\Re(E) \) of the Cauchy-Riemann equation \( \overline{\partial} g = f \) for a given \( f \in \mathcal{E}_{\infty}^\Re(E) \) in the case that \( E \) is a \( \mathbb{C} \)-Fréchet space. We start with a minimal modification of [54, Lemma 6.1, p. 58]. We set \( V_n := \{ z \in \mathbb{C} \mid \text{Re}(z) > -n \} \) for \( n \in \mathbb{N} \) and

\[
L^2(\Sigma_{\mathbb{C}^0([0, \infty[, E)}) := \{ f \in L^2_{\text{loc}}(\mathbb{C}) \mid \forall n \in \mathbb{N} : |f|^2_n := \int_{V_n} |f(z)|^2 e^{-\frac{1}{2} |z|^\alpha |\text{Re}(z)|}dz < \infty \}.
\]

A.2. **Lemma.** For any \( f \in L^2(\Sigma_{\mathbb{C}^0([0, \infty[, E)}) \) with \( \text{supp } f \subset \overline{V}_0 \) there is \( g \in L^2(\Sigma_{\mathbb{C}^0([0, \infty[, E)}) \) such that \( \overline{\partial} g = f \).

**Proof.** This is just [54, Lemma 6.1, p. 58] with the closure \( \overline{V}_0 \) instead of \( V_0 \). The proof of [54, Lemma 6.1, p. 58] does not change since the condition on the support is only used in part (a) of the proof in the estimate

\[
\int_{\mathbb{C}} |g_n(z)|^2 e^{-\frac{2\imath \alpha}{ \alpha + 1} (1 + |z|^\alpha)^{-2}}dz \leq \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{2\imath \alpha}{ \alpha + 1} |\text{Re}(z)|}dz \leq \int_{V_0} |f(z)|^2 e^{-\frac{2\imath \alpha}{ \alpha + 1} |\text{Re}(z)|}dz
\]

for \( n \in \mathbb{N} \) which does not change if \( V_0 \) is replaced by \( \overline{V}_0 \), where \( g_n \in L^1_{\text{loc}}(\mathbb{C}) \) is a solution of \( \overline{\partial} g_n(z) = f(z)e^{\imath j z} \) for a fixed \( j \in \mathbb{N} \). □
A.3. Corollary. The Cauchy-Riemann operator
\[ \overline{\partial} \mathcal{E}_{\mathcal{F}_0}(\mathbb{C}) \to \mathcal{E}_{\mathcal{F}_0}(\mathbb{C}) \]
is surjective.

Proof. (i) We note that the system of seminorms \((\sup_{n \in \mathbb{N}, m \in \mathbb{N}_0} |\partial^n|, n)\) from \(L^2(\mathcal{G}_0(\infty))\) induces the same topology on \(\mathcal{E}_\infty(\mathbb{C})\) as the system \((|f|_n, n)\) for \(f \in \mathcal{G}(\infty)\). Hence \(\mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\) and \(\mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\) are isomorphic.

(ii) Let \(f \in \mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\). It follows from part (i) that \(f \in L^2(\mathcal{G}_0(\infty))\). Due to Lemma A.2 there is \(g \in L^2(\mathcal{G}_0(\infty))\) such that \(\overline{\partial} g = f\). Since \(\overline{\partial}\) is hypoelliptic and \(f \in \mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\), we obtain that \(g \in \mathcal{C}^\infty(\mathbb{C})\) and \(\sup g, c V_0\). From \(|g|_n < \infty\) and \(\sup_{n \in \mathbb{N}, m \in \mathbb{N}_0} |\partial^n|, n) < \infty\) for all \(n \in \mathbb{N}\) and \(m \in \mathbb{N}_0\) we deduce that \(g \in \mathcal{E}_\infty(\mathbb{C})\) by [13] Lemma 3.6 (a), p. 9. Hence we have \(g \in \mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\), implying the surjectivity of \(\overline{\partial}\).

\[\square\]

A.4. Corollary. Let \(E\) be a \(\mathbb{C}\)-Fréchet space. Then the Cauchy-Riemann operator
\[ \overline{\partial} \mathcal{E}_{\mathcal{F}_0}(E) \to \mathcal{E}_{\mathcal{F}_0}(E) \]
is surjective.

Proof. \(\mathcal{E}_\infty(\mathbb{C})\) is a Fréchet space by [40] Proposition 3.7, p. 240] and thus its closed subspaces \(\mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\) and \(\mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\) as well (this is the reason why we phrased Lemma A.2 with the closure \(V_0\) instead of \(V_0\)). \(\mathcal{E}_\infty(\mathbb{C})\) is nuclear by [13] Theorem 3.1, p. 188] and part (i) of the proof of Corollary A.3] and thus its subspaces \(\mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\) and \(\mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\) too.

By [42] Theorem 14 (i), p. 1524] the maps
\[ S: \mathcal{E}_{\mathcal{F}_0}(\mathbb{C}) \otimes E \to \mathcal{E}_{\mathcal{F}_0}(E), \ u \mapsto (z \mapsto u(\delta_z)) \]
and
\[ S: \mathcal{E}_{\mathcal{F}_0}(\mathbb{C}) \otimes E \to \mathcal{E}_{\mathcal{F}_0}(E), \ u \mapsto (z \mapsto u(\delta_z)) \]
are topological isomorphisms like in [42] Example 16 c), p. 1526] where we only have to observe in addition that \(\sup|\overline{\partial} S(u)| = \sup u(\delta_z) \circ \overline{\partial} E \to \mathcal{E}_{\mathcal{F}_0}(\mathbb{C}) \otimes E\) resp. \(\sup S(u) \subseteq \overline{\partial} E \to \mathcal{E}_{\mathcal{F}_0}(\mathbb{C}) \otimes E\) and that \(\sup|\overline{\partial} (e^s \circ f)| = \sup(e^s \circ \overline{\partial} f) E \to \mathcal{E}_{\mathcal{F}_0}(E)\) resp. \(\sup(e^s \circ \overline{\partial} E \to \mathcal{E}_{\mathcal{F}_0}(E)\) and \(e^s \in E\). This observation follows from \(\delta_z \circ \overline{\partial} = 0\) on \(E \subseteq \mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\) resp. \(\delta_z = 0\) on \(E \subseteq \mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\) for \(z \notin V_0\).

Now, using the nuclearity, we obtain \(\mathcal{E}_{\mathcal{F}_0}(\mathbb{C}) \otimes E \cong \mathcal{E}_{\mathcal{F}_0}(E)\) and \(\mathcal{E}_{\mathcal{F}_0}(\mathbb{C}) \otimes E \cong \mathcal{E}_{\mathcal{F}_0}(E)\) where \(\otimes_E\) denotes the completion of the projective tensor product. Further, the Cauchy-Riemann operator \(\overline{\partial} \mathcal{E}_{\mathcal{F}_0}(\mathbb{C}) \to \mathcal{E}_{\mathcal{F}_0}(\mathbb{C})\) is a linear, continuous and surjective map between Fréchet spaces in the \(\mathbb{C}\)-valued case by Corollary A.3]. It follows like in the proof of [45] Corollary 4.3, p. 14] that the Cauchy-Riemann operator in the \(E\)-valued case is surjective as well.

\[\square\]

A.5. Theorem. Let \(E\) be a \(\mathbb{C}\)-Fréchet space. Then the canonical map
\[ I_0: \mathcal{G}(0, \infty)(E)/\mathcal{G}(\infty)(E) \to \mathcal{L}(0, \infty)(E)/\mathcal{L}(\infty)(E), \ [F] \mapsto [F], \]
is a linear isomorphism.

Proof. The map \(I_0\) is well-defined as \(\mathcal{G}(0, \infty)(E) \subseteq \mathcal{L}(0, \infty)(E)\) and \(\mathcal{G}(\infty)(E) \subseteq \mathcal{L}(\infty)(E)\).
(i) Let \([ F ] \in \mathcal{L}\mathcal{G}_{[0,\infty)}(E)/\mathcal{L}\mathcal{G}_{(\infty)}(E)\) such that \(I_0([F]) = 0\), which means that \(F \in \mathcal{L}\mathcal{G}_{[0,\infty)}(E) \cap \mathcal{L}\mathcal{O}_{(\infty)}(E)\). Let \(k \in \mathbb{N}\), \(\alpha \in \mathfrak{A}\) and set \(n := k + 2\). Then \(-\frac{1}{n} \geq k + \frac{1}{k}\) and we obtain
\[
\| F \|_{k,\alpha, (\infty)}^{\mathcal{L}\mathcal{G}} = \sup_{\text{Re}(z) \geq -k} \left| p_\alpha(F(z))e^{k|\text{Re}(z)|} \right| \\
\leq e^k \sup_{\text{Re}(z) \geq -k} \left| p_\alpha(F(z))e^{k|\text{Re}(z)|} \right| + \sup_{\text{Re}(z) \geq -k} \left| p_\alpha(F(z))e^{k|\text{Re}(z)|} \right|
\]
This implies that \(F \in \mathcal{L}\mathcal{G}_{[0,\infty)}(E)\) and thus the injectivity of \(I_0\).

(ii) Let \([ F ] \in \mathcal{L}\mathcal{O}_{[0,\infty)}(E)/\mathcal{L}\mathcal{O}_{(\infty)}(E)\). We choose a cut-off function \(\varphi \in \mathcal{C}^\infty(\mathbb{C})\) such that \(0 \leq \varphi \leq 1\) on \(\mathbb{C}\), \(\varphi = 1\) near \(\{ z \in \mathbb{C} \mid \text{Re}(z) \geq 1 \}\), \(\varphi = 0\) near \(\{ z \in \mathbb{C} \mid \text{Re}(z) \leq \frac{1}{2} \}\) and
\[
|\partial^\varphi \varphi(z)| \leq C_\beta, \; z \in \mathbb{C}, \; \beta \in \mathbb{N}_0^n,
\]
where \(C_\beta > 0\) only depends on \(\beta\) (see [24, Theorem 1.4.1, Eq. (1.4.2), p. 25]). Then we may regard \(\overline{\mathcal{O}}(\varphi F)\) as an element of \(\mathcal{C}^\infty(E, E)\) with \(\overline{\mathcal{O}}(\varphi F) \in \mathcal{V}_0\) by setting \(\overline{\mathcal{O}}(\varphi F)(z) := 0\) for \(\text{Re}(z) \leq 0\). Further, we note that for \(n \in \mathbb{N}\), \(m \in \mathbb{N}_0^n\) and \(\alpha \in \mathfrak{A}\) it follows from the Leibniz rule that
\[
\overline{\mathcal{O}}(\varphi F)_{m, n, \alpha}^{\mathcal{L}_{\infty}} = \sup_{\text{Re}(z) \geq -n} \left| \partial^\alpha(\overline{\mathcal{O}}(\varphi F)(z))e^{-\frac{1}{2}|z| + n|\text{Re}(z)|} \right|
\]
\[
= \sup_{\text{Re}(z) \geq -n} \left| \partial^\alpha(\overline{\mathcal{O}}(\varphi F)(z))e^{-\frac{1}{2}|z| + n|\text{Re}(z)|} \right|
\]
Next, we observe the relation \(\partial^\beta F(z) = i^\beta \partial^\beta C F(z)\) for \(\text{Re}(z) > 0\) and \(\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^n\) between the real partial derivative \(\partial^\beta F\) and the complex derivative \(\partial^\beta C F\) (see [14, Proposition 7.1, p. 270]). We fix \(0 < r < \frac{1}{2}\) and choose \(k \in \mathbb{N}\) such that \(\frac{1}{k} < \frac{1}{k} - r\) and \(k \geq n\). We remark that
\[
-\frac{1}{n} |z| \leq -\frac{1}{n} |\zeta| + \frac{1}{n} |z - \zeta| \leq -\frac{1}{k} |\zeta| + \frac{r}{n}, \; \zeta, \zeta \in \mathbb{C}, \; |\zeta - z| = r.
\]
By Cauchy’s inequality [12, Proposition 2.5, p. 57] we have for $\beta \in \mathbb{N}_0^2$

$$p_\alpha(\partial^\beta F(z)) = p_\alpha(\partial^\beta_C F(z)) \leq \left|\beta!\right| \sup_{\rho \geq 1} p_\alpha(F(\zeta)),$$  
which implies

$$\sup_{\frac{1}{2} \leq \text{Re}(z) \leq 1} p_\alpha(\partial^\beta F(z))e^{-\frac{\beta !}{|z|^\alpha}} \leq \sup_{\frac{1}{2} \leq \text{Re}(z) \leq 1} \left|\beta!\right| \sup_{\rho \geq 1} p_\alpha(F(\zeta))e^{-\frac{\beta !}{|z|^\alpha}}$$

$$\leq e^{\frac{m!}{m}} \sup_{\text{Re}(z) \geq \frac{1}{2}} p_\alpha(F(\zeta))e^{-\frac{\beta !}{|z|^\alpha}} = e^{\frac{m!}{m}} \frac{m!}{m} \|F\|_{k,0,[0,\infty]}.$$

In combination with [44] we derive that

$$|\overline{\partial}(\varphi F)|_{n,0,\alpha} \leq \frac{m!}{m} \sum_{|\gamma| \leq m+1} C_\gamma \|F\|_{k,0,[0,\infty]} < \infty$$

since $F \in \mathcal{LO}_{[0,\infty]}(E)$, yielding $\overline{\partial}(\varphi F) \in \mathcal{E}_{\text{div}}(E)$. Hence there is $g \in \mathcal{E}_{\text{div}}(E)$ such that $\overline{\partial}g = \overline{\partial}(\varphi F)$ on $\mathbb{C}$ by Corollary [43].

Now, we set $F_1 := \varphi F - g$ and $F_2 := (1 - \varphi)F + g$ and show that $F_1$ may be regarded as an element of $\mathcal{L}_{[0,\infty]}(E)$ by setting $(\varphi F)(z) = 0$ for $\text{Re}(z) \leq 0$ and that $F_2 \in \mathcal{LO}_{[0,\infty]}(E)$. We have $\overline{\partial}F_1 = 0$ on $\mathbb{C}$ and $\overline{\partial}F_2 = 0$ on $-\mathbb{H}$, which implies $F_1 \in \mathcal{O}(\mathbb{C}, E)$ and $F_2 \in \mathcal{O}(\mathbb{C}\setminus E)$. Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ and choose $n \in \mathbb{N}$ such that $n \geq \max(2,k)$. Then we get

$$\|F_1\|_{k,0,[0,\infty]} \leq \frac{m!}{m} e^{\frac{n}{m} \sum |\gamma| \leq m+1} C_\gamma \|F\|_{k,0,[0,\infty]} < \infty$$

because $F \in \mathcal{LO}_{[0,\infty]}(E)$ and $g \in \mathcal{E}_{\text{div}}(E)$. Thus we get $F_1 \in \mathcal{L}_{[0,\infty]}(E)$.

Let us turn to $F_2$. We have

$$\|F_2\|_{k,0,[0,\infty]} \leq \frac{m!}{m} \sum_{|\gamma| \leq m} C_\gamma \|F\|_{k,0,[0,\infty]} < \infty$$

because $F \in \mathcal{LO}_{[0,\infty]}(E)$ and $g \in \mathcal{E}_{\text{div}}(E)$. Hence we have $F_2 \in \mathcal{LO}_{[0,\infty]}(E)$. Altogether, we deduce that $I_0([F_1]) = [F_1] = [F_1 + F_2] = [F]$, which means that $I_0$ is surjective.
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