Abelian covers of hyperbolic surfaces: equidistribution of spectra and infinite volume mixing asymptotics for horocycle flows

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Abstract
We consider Abelian covers of compact hyperbolic surfaces. We establish an asymptotic expansion of the correlations for the horocycle flow on $\mathbb{Z}^d$-covers, thus proving a strong form of Krickeberg mixing. We also prove that the spectral measures around 0 of the Casimir operators on any increasing sequence of finite Abelian covers converge weakly to an absolutely continuous measure.

Keywords: horocycle flow, infinite mixing, Abelian covers, representation theory of $\text{PSL}_2(\mathbb{R})$

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1. Introduction

Let $H$ denote the upper-half plane model of the hyperbolic plane. Any closed Riemann surface $S$ with a hyperbolic structure can be realised as a quotient $\Gamma \backslash H$ of $H$ by a discrete group of isometries $\Gamma$. In turn, $\Gamma$ can be identified with the fundamental group of $S$. The horocycle flow $\{h_t\}_{t \in \mathbb{R}}$ on the unit tangent bundle $T^1S$ of $S$ is the unit speed translation along the stable leaves of the geodesic flow. Identifying $\Gamma$ with a co-compact discrete subgroup of $G = \text{PSL}_2(\mathbb{R})$ acting on $H$ by M"obius transformations, and $T^1S$ with $M = \Gamma \backslash G$, the flow $\{h_t\}_{t \in \mathbb{R}}$ is the action by right-multiplication on $M$ by the upper triangular unipotent subgroup of $G$.

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Qualitative and quantitative properties of the horocycle flow on compact (or finite volume) quotients $M$ have been studied at length and are now well-understood. When the underlying surface has infinite genus, however, much less is known. We will be interested in the case of Abelian covers of compact surfaces $S$. In this setting, the only invariant Radon measures have infinite total mass [23]. Babillot and Ledrappier constructed a family of mutually singular ergodic Radon invariant measures [3], which are quasi-invariant under the geodesic flow and arise from positive eigenfunctions of the Laplacian via the associated conformal measures on the boundary of the hyperbolic plane [2]. By a result of Sarig [24], which applies to a wider class of hyperbolic surfaces, these are the only ergodic Radon invariant measures. Ledrappier and Sarig showed that, of these infinitely many ergodic measures, only the volume (Haar) measure $vol$ satisfies a so-called generalised law of large numbers: a certain Cesaro summation with weights of appropriately rescaled ergodic integrals of any $L^1$ observable converges almost everywhere to a multiple of its expected value [15].

In this paper, we are interested in the mixing properties of the horocycle flow on Abelian covers $M_0$ of $M$. The generalisation of the definition of mixing to the infinite volume setting is not straightforward and has a long history (see, e.g. [16, section 1]). One standard notion of infinite volume mixing is the so-called Krickeberg mixing (or local mixing). In our setting, proving Krickeberg mixing for $\{h_t\}_{t \in \mathbb{R}}$ on $M_0$ means that we can find a scaling rate $\rho(t)$ such that

$$\lim_{t \to \infty} \rho(t) \int_{M_0} (v \circ h_t) \cdot \varpi \, d\ vol = \frac{1}{(2\pi \sigma)^{d/2}} vol(v) \varpi,$$

for all smooth functions $v, w \in \mathcal{C}_c^\infty(M_0)$ with compact support. In probabilistic language, it corresponds to a local limit theorem. Krickeberg mixing and the related local limit theorems have been investigated for several (non-uniformly) hyperbolic systems, see, e.g. [1, 8, 11, 12, 17, 25, 26] and references therein. For the geodesic flow $\{g_t\}_{t \in \mathbb{R}}$ on a $\mathbb{Z}^d$-cover $M_0$ of $M$, Oh and Pan [19] showed that

$$\lim_{t \to \infty} \rho^{d/2} \int_{M_0} (v \circ g_t) \cdot \varpi \, d\ vol = \frac{1}{(2\pi \sigma)^{d/2}} vol(v) \varpi, \tag{1}$$

for all continuous, compactly supported $v, w \in \mathcal{C}_c(M_0)$, where $\sigma > 0$ is a constant depending on $M_0$ only. Later, this result was extended by Pan [20] to the case where $M$ is a non-compact manifold of finite volume.

In a recent work, Dolgopyat et al [9] strengthened the result by Oh and Pan by providing a full asymptotic expansion of the left hand side in (1), under some regularity assumptions on the observables $u$ and $v$. All the works [9, 19, 20] rely on the symbolic representation of the geodesic flow and the spectral theory of the associated transfer operator.

For the horocycle flow, Krickeberg mixing can be derived from [9, theorem 4.1], using the KAK decomposition of $\text{PSL}_2(\mathbb{R})$. In our first main result, theorem A below, we achieve the same result by a direct method. (Indeed, it is also possible to provide an alternative proof of [9, Theorem 4.1] by adapting our approach to the case of the geodesic flow.) One advantage of our method is that it characterises the constant $\sigma$ in (1) in geometric terms. Furthermore, our direct approach clarifies that the leading terms of the correlation function arise from the “merging” of the discrete series of the compact surface into the complementary series weakly appearing in the Abelian cover. The subject of further investigation is to generalise our techniques to obtain precise mixing asymptotics in the case of Abelian covers of finite volume, non-compact surfaces.
Our methods use representation theory of $G$. The crucial difference with the compact (or finite volume) case is the absence of a spectral gap. In fact, the analysis of the asymptotics of the correlations relies on the study of the irreducible representations close to the trivial representation, and hence on the spectrum of the Casimir operator close to 0. For a sequence of increasing finite Abelian covers $S_k$ of a given compact surface $S$, the distribution of the resonances of the Laplace-Beltrami operators on $S_k$ in a neighbourhood of 0 has been studied by Jakobson et al in [13]. They proved that the spectral measures around 0 converge weakly to an absolutely continuous measure. In our second main result, theorem B, we provide an alternative proof of their result.

1.1. Abelian covers of compact hyperbolic surfaces

We now introduce the notation we use in this paper. Let $\Gamma$ be a co-compact lattice in $G = \text{PSL}_2(\mathbb{R})$, and let $M = \Gamma \backslash G$. We can identify $M$ with the unit tangent bundle of the compact orientable hyperbolic surface $S = \Gamma \backslash H$. We denote by $g \geq 2$ its genus.

Denote by $[\Gamma, \Gamma]$ the commutator subgroup of $\Gamma$. Abelian covers $S_0 = \Gamma_0 \backslash H$ of $S$ are in one-to-one correspondence with intermediate subgroups $[\Gamma, \Gamma] \leq \Gamma_0 \leq \Gamma$. The quotient group $G := \Gamma / \Gamma_0$ is a finitely generated Abelian group isomorphic to the product of a free Abelian group of rank $d$, with $0 \leq d \leq 2g$, and a finite Abelian group (the torsion of $G$). For the purpose of this article we may assume, by passing to a finite cover, that $G$ has no torsion and hence $G \simeq \mathbb{Z}^d$.

Let $M_0 = \Gamma_0 \backslash G$. Observe that, since $M_0 \to M$ is a normal cover, the Galois group $G$ acts on $M_0$ by deck transformations; furthermore this action commutes with the action of $G$ by right translations on $M_0 = \Gamma_0 \backslash G$. To avoid trivialities, we shall assume henceforth that $M_0$ is an Abelian cover of $M$ with Galois group $G \simeq \mathbb{Z}^d$, where $1 \leq d \leq 2g$.

1.2. Horocycle flows

The right actions of $G$ on $M$ and on $M_0$ are generated by vector fields which we may identify with elements of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $G$. We let

$$U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}).$$

The horocycle flow $\{h_t\}_{t \in \mathbb{R}}$ on $M_0$ (or on any quotient of $G$) is the homogeneous flow generated by $U$, namely the flow given by

$$h_t(\Gamma_0 \gamma) = \Gamma_0 \gamma \exp(tU).$$

The right actions of $G$ on $M$ and on $M_0$ preserve the volume forms locally given by the Haar measure. We normalise the volume form $\text{dvol}$ so that $\text{dvol}(X, Y, \Theta) = 1$. The volume form $\text{dvol}_0$ on $M_0$ is the pull-back of $\text{dvol}$. As a consequence we obtain unitary representations of $G$ on $L^2(M) := L^2(M, \text{dvol})$ and on $L^2(M_0) := L^2(M_0, \text{dvol}_0)$.

1.3. Infinite volume mixing asymptotics

Our main result, theorem A, provides an asymptotic expansion of the correlations for the horocycle flow between smooth observables with compact support. The first term in the expansion is usually called Krickeberg (or local) mixing.
Theorem A. There exists a constant \( \sigma(\Gamma_0) > 0 \) such that the following holds. Let \( v, w \in \mathcal{C}_c^\infty(M_0) \). There exist \( (c_j)_{j \in \mathbb{N}} \) such that for all \( N \geq 1 \) we have
\[
\langle v \circ h_1, w \rangle = \left( \frac{g - 1}{2} \right)^{\frac{d}{2}} \sigma(\Gamma_0) \frac{\text{vol}(v) \text{vol}(w)}{(\log t)^{\frac{d}{2}}} + \sum_{j=1}^{N} \frac{c_j}{(\log t)^{j+\frac{d}{2}}} + o\left( (\log t)^{-(N+\frac{d}{2})} \right).
\]

The constant \( \sigma(\Gamma_0) \) in theorem A is the determinant of an explicit period matrix of harmonic one-forms, see section 3.5. In the case where \( \Gamma_0 = [\Gamma, \Gamma] \), we have \( \mathcal{G} = H_1(M, \mathbb{Z}) = \mathbb{Z}^{2g} \), and \( \sigma(\Gamma_0) = 1 \).

Remark 1.1. Theorem A will be proved for a larger class of functions, which we call rapidly decaying functions, which, roughly speaking, consists of smooth functions which decay, together with all their derivatives, faster than any polynomial at infinity.

1.4. Equidistribution of spectra of large covers

Let \( \psi : \Gamma \to \mathbb{Z}^d \), with \( d \geq 1 \), be a surjective homomorphism and denote \( \Gamma_0 = \ker(\psi) \).

Let us fix \( d \) strictly increasing sequences \( \{N_1^{(k)}, \ldots, N_d^{(k)}\}_{k \geq 1} \) of natural numbers such that \( \min\{N_1^{(k)}, \ldots, N_d^{(k)}\} \to \infty \) when \( k \to \infty \). We define
\[
\Gamma_k = \psi^{-1}\left( N_1^{(k)} \mathbb{Z} \times \cdots \times N_d^{(k)} \mathbb{Z} \right),
\]
and let \( M_k = \Gamma_k \backslash \text{PSL}_2(\mathbb{R}) \). Clearly, \( M_k \) is a finite Abelian cover of \( M \).

Denote by \( \Box_k \) the Casimir operator on \( L^2(M_k) \). We show that the spectral measures of \( \Box_k \) converge weakly, in a neighbourhood of zero, to an absolutely continuous measure. A general result of this type for the resonances of the Laplacian was first proved by Jakobson et al [13]; here, we give a different proof.

Theorem B. There exists \( \varepsilon > 0 \) and a function \( \zeta \in L^\infty([0, \varepsilon]) \) such that, for every continuous function \( f \) compactly supported on \([0, \varepsilon]\), we have
\[
\lim_{k \to \infty} \frac{1}{|\Gamma_k|} \sum_{\lambda \in \text{Spec}(\Box_k)^{-\infty}[0, \varepsilon]} f(\lambda) = \int_0^\varepsilon f(x) x^{\frac{d}{2}-1} \zeta(x) \, dx.
\]

1.5. Outline of the paper

Mixing rates of the one-parameter subgroup \( \{\exp(tU)\} \) acting on an irreducible unitary representation of \( \text{PSL}_2(\mathbb{R}) \) have been studied by several authors, often under the name “decay of correlations”. The gist of the matter is that mixing rates are uniform for all representations in the unitary dual \( \text{PSL}_2(\mathbb{R}) \) lying outside a (Fell) neighbourhood of the trivial representation. In section 2 we make this statement precise and we prove it using elementary methods inspired by [22].

Since the action of Galois group \( \mathcal{G} \) on \( M_0 \) by deck transformations commutes with the regular representation of \( G \) on \( L^2(M_0) \), these two actions are simultaneously “diagonalisable”. This means that we can decompose the Hilbert space \( L^2(M_0) \) as a Hilbert direct integral of irreducible unitary representations \( \chi \otimes \pi \) of \( \mathcal{G} \times G \) (here \( \chi \) is a character of the Galois group \( \mathcal{G} \) and \( \pi \) an irreducible unitary representations of \( G \)). Thus \( L^2(M_0) = \int_{\mathcal{G} \times G} \chi \otimes \pi \, dm(\chi, \pi) \), where
$m$ is a spectral measure on the product of the group of characters $\hat{G}$ of $G$ and of the unitary dual $\hat{G}$ of $G$. In section 3, we make explicit this decomposition. We show that for any $\chi \in \hat{G}$ the irreducible unitary representations $\pi$ belonging to the continuous part of $G$ occur discretely in the support of the spectral measure $m$. They may be ordered as $\pi_0(\chi), \pi_1(\chi), \ldots$, in increasing values of the Casimir parameter, so that each $\pi_j(\chi)$ depend smoothly on $\chi \in \hat{G}$. Furthermore, among these representations, only $\pi_0(\chi)$ may be close to the trivial representation of $G$ and this may occur only if $\chi$ is close to the trivial character of $\hat{G}$.

Thus correlations of the horocycle flows are finally estimated separating the contributions coming from the representations $\pi_j(\chi)$ far away from the trivial representation—for these the uniform bounds mentioned above apply—from the contributions coming from the representations $\pi_0(\chi)$ for $\chi$ in a neighbourhood $\mathcal{U}$ of trivial character of $\hat{G}$. This last step is accomplished in section 4.

Finally, in section 5, we prove theorem B.

2. Asymptotics of correlations

2.1. Ratner’s theorem

We turn our attention to the problem of estimating the decay of matrix coefficients. For unitary representations of $G = \text{PSL}_2(\mathbb{R})$, upper bounds of the matrix coefficients for analytic vectors have been established by a number of authors [4, 6, 18, 27]. Ratner [22] extended these bounds assuming only that the vectors are Hölder in the direction of the rotation subgroup. We now recall her result in the case of finite Sobolev regularity.

Let $\rho: G \to \mathcal{U}(H)$ be an irreducible unitary representation of $G$ on a complex separable Hilbert space $H$. Such representations are classified, up to unitary equivalence, by a continuous complex parameter $\nu \in i\mathbb{R}_{>0} \cup \{0, 1\} \subset \mathbb{C}$ and a discrete parameter $\{\nu = n - 1 : n \in 2\mathbb{Z}_{>0}\} \times \{\sigma = \pm 1\}$. The first set accounts for the principal ($\nu \in i\mathbb{R}_{>0}$), the complementary ($\nu \in (0, 1)$) series, and the trivial representation ($\nu = 1$); the second set accounts for the holomorphic ($\sigma = 1$) and anti-holomorphic ($\sigma = -1$) discrete series.

We choose the normalisation of the Casimir operator $\Box$ to be

$$\Box = -X^2 - Y^2 + \Theta^2 = -U^2 + 2U\Theta + X - X^2,$$

so that it coincides with the Laplace–Beltrami operator when acting on function pulled back from the hyperbolic plane. The Casimir operator is a generator of the centre of the enveloping algebra of $\mathfrak{sl}_2(\mathbb{R})$ and it acts on the Hilbert space $H = H_\lambda$ as the constant

$$\lambda = \lambda(\nu) := \frac{1 - \nu^2}{4}.$$

We set $\varepsilon(\lambda) = 1$ if $\lambda = 1/4$ and $\varepsilon(\lambda) = 0$ otherwise.

Finally, let $\Delta = \Box - 2\Theta^2 = -(X^2 + Y^2 + \Theta^2)$ and, for any $s > 0$, define the Sobolev space $W^s(H)$ of order $s$ to be the completion of the subspace $\mathfrak{C}^\infty(H)$ of smooth vectors $v \in H$ with respect to $\|v\|_w := \|(\text{Id} + \Delta)^{s/2}v\|$. The space $W^s(H)$ is a Hilbert space with the inner product $\langle v, w \rangle_w = \langle (\text{Id} + \Delta)^{s/2}v, w \rangle$ inducing the norm $\|w\|$ as above.

**Theorem 2.1 ([4, 6, 18, 22, 27]).** There exists an explicit constant $C \geq 1$ such that the following holds. Let $\rho: G \to \mathcal{U}(H)$ be a non-trivial irreducible unitary representation of $G$ of Casimir parameter $\lambda = \lambda(\nu)$, and let $v, w \in W^1(H)$. For all $t \geq 1$ we have

$$|\langle \rho(\exp(tU)).v, w \rangle| \leq C\|v\|_w \|w\|_w t^{1 + \nu} (\log t)^{\varepsilon(\nu)}.$$
For our purposes, for small positive Casimir parameters, we need more precise information on the asymptotics of the matrix coefficients.

In the following, for every \( v \in H \), we will denote
\[
\mathbf{u}^t, v := \rho(\exp(iU))\cdot v.
\]

We now focus on vectors in \( H \) that are eigenfunctions of both the Casimir operator \( \Box \) and the element \( \Theta \). More precisely, let \( \lambda \in [0, 1/4] \) and \( n, m \in \mathbb{Z} \) be fixed, and let \( e_n \) and \( e_m \) be two vectors in \( W^3(H) \) such that \( \Theta e_j = j e_j \) and \( \Box = \lambda e_j \) for \( j \in \{ n, m \} \). Following Ratner’s method, we will prove the following result (the corresponding result for the geodesic flow can be found in [5, lemma 2.30]).

**Theorem 2.2.** Under the assumptions above, there exists \( A = A_{n,m} \) explicitly defined in (12) such that
\[
\| [\mathbf{u}^t, e_n, e_m] - A_{n,m} t^{-1+\nu} \| \leq \frac{C}{t^2} \| e_n \|_{W^5} \cdot \| e_m \|_{W^5} t^{-1}
\]
for some absolute constant \( C \) and for all \( t \geq 1 \).

The remainder of this section is devoted to the proof of theorem 2.2.

### 2.2. Reduction to an ODE

Let us define
\[
Y_{n,m}(t) := [\mathbf{u}^t, e_n, e_m].
\]

It is well-known that the \( e_n \) are analytic vectors and thus \( Y_{n,m}(t) \) is a real analytic function of \( t \). The derivatives with respect to \( t \) are given by
\[
\begin{align*}
Y'_{n,m}(t) &= [\mathbf{u}^t, U e_n, e_m], \\
Y''_{n,m}(t) &= [\mathbf{u}^t, U^2 e_n, e_m].
\end{align*}
\]

The key observation is that the function \( Y_{n,m}(t) \) satisfies a second order linear ODE with constant coefficients, as the next proposition shows.

**Proposition 2.3.** The function \( y(t) = Y_{n,m}(t) \) satisfies the differential equation
\[
t^2 y'' + 3t y' + 4\lambda y = f(t)
\]
for a function \( f(t) = f_{n,m}(t) \), explicitily defined in (11), which is smooth in \( t \) and satisfies
\[
|f(t)| \leq \frac{C}{t} \| e_n \|_{W^5} \cdot \| e_m \|_{W^5} \quad \text{for all } t \geq 1
\]
for an absolute constant \( C \), and
\[
f(0) = 4\lambda Y_{n,m}(0), \quad f'(0) = (4 - \nu^2) Y'_{n,m}(0).
\]
Proof. The last formulas follow immediately from equation (3). A standard computation gives us
\[ \Theta(u', e_m) = u'[\langle Ad_{\exp(tU)}(\Theta) \rangle e_m] = u'[\left( \Theta + tX + \frac{1}{2} t^2 U \right) e_m]. \]

Denoting \( y(t) = Y_{n,m}(t) \), we compute
\[ uny(t) = -\langle u', e_n, \Theta e_m \rangle = \langle \Theta(u', e_n), e_m \rangle = \langle u', (\Theta e_n), e_m \rangle + t\langle u', (Xe_n), e_m \rangle + \frac{1}{2} t^2 \langle u', (Ue_n), e_m \rangle = uny(t) + t\langle u', (Xe_n), e_m \rangle + \frac{1}{2} t^2 y'(t). \]

Define \( q(t) = \langle u', (Xe_n), e_m \rangle \). We rewrite the equation above as
\[ tq(t) = \iota(m - n) y(t) - \frac{1}{2} t^2 y'(t), \]
and, after differentiating, we get
\[ tq'(t) = -q(t) + \iota(m - n) y'(t) - \frac{1}{2} t^2 y''(t) - t y'(t). \]

Repeating for \( q(t) \) the computation we made for \( y(t) \), we obtain
\[ unq(t) = \langle \Theta(u', (Xe_n)), e_m \rangle = \left\langle u', \left[ (\Theta X + tX^2 + \frac{1}{2} t^2 UX) e_n, e_m \right] \right\rangle. \]

Using the identity \( \Theta X = X\Theta - Y = X\Theta - U + \Theta \), the formula above yields
\[ unq(t) = unq(t) - y'(t) + uny(t) + \frac{1}{2} t^2 q'(t) + t\langle u', (X^2 e_n), e_m \rangle. \]

From the assumption \( \square e_n = \lambda e_n \) we get
\[ \lambda y(t) = \langle u', (\square e_n), e_m \rangle = \langle u', \left[ (-U^2 + 2U\Theta + X - X^2) e_n \right], e_m \rangle = -y''(t) + 2uny'(t) + q(t) - \langle u', (X^2 e_n), e_m \rangle. \]

Combining (7) with (6), we obtain
\[ -ty''(t) + 2unty'(t) + tq(t) - \lambda ty(t) = \iota(m - n) q(t) + y'(t) - uny(t) - \frac{1}{2} t^2 q'(t), \]
and, substituting the expressions for \( q(t) \) and \( q'(t) \) from (4) and (5), we deduce
\[ (t^2 + 4t) y''(t) + (3t^2 + 4) y'(t) + 4\lambda ty(t) = 4\iota(m + n) ty'(t) + 2\iota(m + n) y(t) + 4\frac{(m - n)^2}{t} y(t). \]
We can rewrite the above equation as
\[
\left((t^3 + 4) y'\right)'(t) + 4\lambda ty(t) = g(t),
\] (8)
where
\[
g(t) = 4\lambda (m + n) ty'(t) + 2\lambda (m + n)y(t) + 4\frac{(m-n)^2}{t}y(t).
\] (9)

Note that, if \( m \neq n \), then \( y(0) = 0 \) so that \( g \) is bounded for all \( t \). Integraing (8), we have
\[
y'(t) = \frac{1}{t^3 + 4t} \int_0^t h(s) \, ds, \quad \text{where} \quad h(t) = -4\lambda ty(t) + g(t).
\] (10)

Observe that, since \( h \) is bounded, the integral above is well defined and the expression on the right-hand side above converge as \( t \to 0 \).

We now estimate \( h(t) \). Let us first notice that
\[
(1 + |n| + |m| + n^2 + m^2) \left| y(t) \right| \leq \sum_{k + \ell \leq 2} \| e_n \| \| e_m \| \| w^k \| \| w^\ell \|,
\]
where we used (2) and the Cauchy-Schwarz Inequality. Similarly, since there exists an absolute constant \( C \geq 1 \) such that
\[
|n| \cdot \| U e_n \| = \| U \Theta e_n \| \leq C \| e_n \| \| w^k \|,
\]
then, we can bound
\[
(1 + |n| + |m|) \left| y'(t) \right| \leq C \sum_{k + \ell \leq 2} \| e_n \| \| e_m \| \| w^k \| \| w^\ell \|.
\]
Thus, from (9), we deduce
\[
|h(t)| \leq 20C \left( \sum_{k + \ell \leq 2} \| e_n \| \| e_m \| \| w^k \| \| w^\ell \| \right) t \quad \text{for all} \quad t \geq 1.
\]

By (10), for all \( t \geq 1 \) we have
\[
|y'(t)| \leq 10C \left( \sum_{k + \ell \leq 2} \| e_n \| \| e_m \| \| w^k \| \| w^\ell \| \right) t^{-1}.
\]

Rewriting the equation (8) as
\[
t^2 y'' = -4y''' - 3y'' - 4y'/t - 4\lambda y + g(t)/t,
\]
we deduce that, for all \( t \geq 1 \),
\[
|y''(t)| \leq \frac{4}{t^2} |y'''(t)| + \frac{7}{t} |y'(t)| + \frac{|h(t)|}{t^3}
\leq 100C \left( \sum_{k + \ell \leq 2} \| e_n \| \| e_m \| \| w^k \| \| w^\ell \| \right) t^{-2}.
\]

Finally, rewriting once more the equation (8) as
\[ t^2 y'' + 3ty' + 4\lambda y = f(t) \]
with
\[ f(t) = -4y'' - \frac{4}{t}y' + \frac{g(t)}{t} \]
\[ = -4y'' - \frac{4}{t}y' - 4t(m+n)y'(t) - 2t\frac{m+n}{t}y(t) + 4\frac{(m-n)^2}{t^2}y(t) \]
and combining the previous estimates, we conclude the bound
\[ |f(t)| \leq \frac{500C}{t} \left( \sum_{k,t \leq 3} \|e_n\|w^k \|e_m\|w^k \right), \]
which proves the result.

We now use proposition 2.3 to estimate \( Y_{n,m}(t) \) and prove theorem 2.2. Note that \( \nu \in (0, 1] \) for all \( 0 \leq \lambda < 1/4 \).

**Lemma 2.4.** For \( \nu \in (0, 1] \) define
\[ \tilde{f}_{n,m}(t) = 4 (\nu - 1) Y'_{n,m}(t) + 2i(m+n)(2\nu - 1) Y_{n,m}(t) + 4 \frac{(m-n)^2}{t} Y_{n,m}(t), \]
and
\[ A_{n,m} = \frac{1}{2\nu} \left( 5Y'_{n,m}(1) + (4i(m+n) + 1 + \nu) Y_{n,m}(1) - \int_{1}^{\infty} r^{-1-\nu} \tilde{f}_{n,m}(r) \ dr \right). \] (12)

Then, for every \( t \geq 1 \), we have
\[ |Y_{n,m}(t) - A_{n,m}t^{-1+\nu}| \leq \frac{C}{\nu^2} \|e_n\|w^k \|e_m\|w^k t^{-1}. \]

**Proof.** The function \( \tilde{f}_{n,m} \) is uniformly bounded over \([1, \infty)\), hence \( A_{n,m} \) is well-defined.

We consider the initial value problem given by the ODE (3) with the initial conditions \( y(1) = Y_{n,m}(1) = \langle u^t, e_n, e_m \rangle \) and \( y'(1) = Y'_{n,m}(1) = \langle u^t, Ue_n, e_m \rangle \). Its solution is given by the formula
\[ Y_{n,m}(t) = \frac{t^{-1+\nu}}{2\nu} \left( \int_{1}^{t} r^{-\nu} \tilde{f}_{n,m}(r) \ dr + (1 + \nu) Y_{n,m}(1) + Y'_{n,m}(1) \right) \]
\[ + \frac{t^{-1-\nu}}{2\nu} \left( - \int_{1}^{t} r^{-\nu} \tilde{f}_{n,m}(r) \ dr + (\nu - 1) Y_{n,m}(1) - Y'_{n,m}(1) \right). \]
Since \( \nu \in (0, 1] \), the function \( r^{-\nu} f_{n,m}(r) \) is in \( L^1([1, \infty)) \). Thus, integrating by parts the terms \( 4^{\nu'}r'' \) and \(-4i(m+n)\nu'(t)\) appearing in the definition \( (11) \) of \( f_{n,m} \), we can write
\[
\int_1^\infty r^{-\nu} f_{n,m}(r) \, dr = \int_1^\infty r^{-\nu} f_{n,m}(r) \, dr - \int_1^\infty r^{-\nu} f_{n,m}(r) \, dr \\
= 4^{\nu'}(1) + 4^i(m+n)(1) - \int_1^\infty r^{-1-\nu} \tilde{f}_{n,m}(r) \, dr \\
- \int_1^\infty r^{-\nu} f_{n,m}(r) \, dr.
\]
Substituting in the expression for \( Y_{n,m}(t) \), we obtain the inequality
\[
|Y_{n,m}(t) - t^{-1+\nu} A_{n,m}| \leq t^{-1+\nu} \left| \int_1^\infty \frac{r^{-\nu} f_{n,m}(r)}{2\nu} \, dr \right| \\
+ t^{-1-\nu} \left| - \int_0^t \frac{r^{-1} \tilde{F}_{n,m}(r)}{2\nu} \, dr + (\nu - 1) Y_{n,m}(1) - Y_{n,m}'(1) \right|. \tag{13}
\]
Using the bound on \(|f_{n,m}|\) given by proposition 2.3, we conclude that the two summands in the right-hand side of \( (13) \) are both bounded by
\[
\frac{C}{2\nu^2} \|e_n\|_W \|e_m\|_W \|e_n\|_W t^{-1}.
\]
This completes the proof. \( \square \)

We finish this section with a corollary that extends the asymptotics of theorem 2.2 to all \( W^3 \) vectors.

**Corollary 2.5.** There exists a constant \( C \geq 1 \) such that the following holds. Let \( v, w \in W^3(H) \), and let
\[
F_{v,w}(t) = 4(\nu - 1) \langle u', Uv, w \rangle + 2(2\nu - 1) \left( \langle u', \Theta v, w \rangle - \langle u', v, \Theta w \rangle \right) \\
- \frac{4}{t} \left( \langle u', \Theta^2 v, w \rangle + \langle u', v, \Theta^2 w \rangle + 2 \langle u', \Theta v, \Theta w \rangle \right),
\]
and
\[
A(v,w) = \frac{1}{2\nu} \left( 5 \langle u^1, Uv, w \rangle + 4 \left( \langle u^1, \Theta v, w \rangle - \langle u^1, v, \Theta w \rangle \right) \right) \\
+ (1 + \nu) \langle u^1, v, w \rangle - \frac{1}{2\nu} \int_1^\infty r^{-1-\nu} F_{v,w}(r) \, dr \right). \tag{14}
\]
For every \( t \geq 1 \) we have
\[
|\langle u', v, w \rangle - t^{-1+\nu} A(v,w)| \leq \frac{C}{t^3} \|v\|_W \|v\|_W \|w\|_W t^{-1}.
\]
**Proof.** By the representation theory of compact Abelian groups, \( H \) admits an orthonormal basis of analytic eigenvectors \( e_n \) of \( \Theta \) with eigenvalue \( m \), where \( n \in \mathbb{Z} \).
Let us write \( v = \sum_{n \in \mathbb{Z}} v_n e_n \) and \( w = \sum_{n \in \mathbb{Z}} w_n e_n \). From the definition (12) and the bound in proposition 2.3, we deduce that

\[
|A_{n,m}| \leq C \|e_n\| w^2 \|e_m\| w^2,
\]

so that the Cauchy–Schwarz inequality yields

\[
\sum_{n,m \in \mathbb{Z}} |v_n w_m A_{n,m}| \leq C \|v\| w^2 \|w\| w^2.
\]

This proves that

\[
A(v,w) = \sum_{n,m \in \mathbb{Z}} v_n w_m A_{n,m}
\]

is well-defined and the conclusion follows. \( \square \)

3. Harmonic analysis on Abelian covers

3.1. Characters and harmonic one-forms

Let us recall that the Galois group \( \mathcal{G} = \Gamma/\Gamma_0 \) of the cover \( p: S_0 \to S \), as well as of \( M_0 = T^1(S_0) \to M = T^1(S) \), is a finitely generated Abelian group, which is isomorphic to \( \mathbb{Z}^d \) for some \( 1 \leq d \leq 2g \), where \( g \) is the genus of \( S \). The group \( \mathcal{G} \) acts on \( M_0 \) by left-translations, that is, for any \( [\gamma] = \gamma \Gamma_0 \) and any \( x = \Gamma_0 g \in M_0 \) we have \([\gamma].x = \Gamma_0 \gamma g \). Similarly, \( \mathcal{G} \) acts on \( S_0 \) by \([\gamma].j(x) = j([\gamma].x) \), where \( j: M_0 \to S_0 \) is the canonical projection \( j(\Gamma_0 g) = \Gamma_0 g \) \( \mathbb{R} \mathbf{SO}(2, \mathbb{R}) \).

Let us denote by \( \hat{\mathcal{G}} \) the dual of \( \mathcal{G} \), namely the group of characters \( \chi : \mathcal{G} \to U(1) \), which we identify with the group of homomorphisms \( \chi : \Gamma \to U(1) \) such that \( \ker(\chi) \supset \Gamma_0 \). We now describe its Lie algebra in terms of harmonic one-forms.

Recall that every cohomology class has a canonical representative which need to understand the behavior is a harmonic one-form, namely a one-form \( \omega \) such that \( \Delta \omega = 0 \). We denote by \( \mathcal{H} \) the \( d \)-dimensional vector space of harmonic one-forms on \( S \) representing cohomology classes with vanishing periods on the cycles represented by \( \Gamma_0 \).

Let \( x \in M \) be fixed. For every \( \omega \in \mathcal{H} \), define

\[
\chi_\omega : \mathcal{G} \to U(1) \subset \mathbb{C}
\]

\[
[\gamma] \mapsto \exp \left( 2\pi i \int_x^{[\gamma].x} f^* p^\ast \omega \right),
\]

where the integral is taken along a path in \( M_0 \) from \( x \) to \([\gamma].x \). The value \( \chi_\omega([\gamma]) \) does not depend on the path chosen. Indeed, since \( \omega \) vanishes on the cycles represented by elements in \( \Gamma_0 \), the one-form \( p^\ast \omega \) is exact on \( S_0 \) and vanishes on the fibres of the projection \( j: M_0 \to S_0 \). Moreover, for the same reason, \( \chi_\omega \) does not depend on the choice of the point \( x \).

**Lemma 3.1.** For every \( \omega \in \mathcal{H} \), we have \( \chi_\omega \in \hat{\mathcal{G}} \). Moreover, the space \( \mathcal{H} \) is isomorphic to the Lie algebra \( \text{Lie}(\mathcal{G}) \) of the group \( \mathcal{G} \), the map \( \omega \mapsto \chi_\omega \) coincides with the exponential map of \( \hat{\mathcal{G}} \) and induces an isomorphism between \( \hat{\mathcal{G}} \) and \( \mathcal{H}/\mathcal{H}(\mathbb{Z}) \), where \( \mathcal{H}(\mathbb{Z}) \) consists of harmonic one-forms representing elements in \( H^1(S, \mathbb{Z}) \).
Proof. For every \([\gamma_1], [\gamma_2] \in \mathcal{G}\) and \(\omega \in \mathcal{H}\), we have

\[
\chi_\omega ([\gamma_1] \cdot [\gamma_2]) = \exp \left( 2\pi i \int_x^{[\gamma_1] \cdot [\gamma_2], x} f^* p^* \omega \right) \\
= \exp \left( 2\pi i \int_x^{[\gamma_2], x} f^* p^* \omega \right) \exp \left( 2\pi i \int_x^{[\gamma_1], x} f^* p^* \omega \right) \\
= \exp \left( 2\pi i \int_x^{[\gamma], x} f^* p^* \omega \right) \exp \left( 2\pi i \int_x^{[\gamma_1] \cdot [\gamma_2], x} f^* p^* \omega \right) \\
= \chi_\omega ([\gamma_1]) \chi_\omega ([\gamma_2]),
\]

where \(x' = [\gamma_2] \cdot x\). This shows that \(\chi_\omega\) is a character of \(\mathcal{G}\). It is easy to see that, for every \(\omega_1, \omega_2 \in \mathcal{H}\), we have \(\chi_{\omega_1 + \omega_2} = \chi_{\omega_1} \chi_{\omega_2}\), proving that the map \(\omega \mapsto \chi_\omega\) is a homomorphism of the additive Abelian group \(\mathcal{H}\) into multiplicative Abelian group \(\mathcal{G}\). By the definition of the space \(\mathcal{H}\), the differential of this map at the origin \(\omega \mapsto ([\gamma] \mapsto 2\pi i \int_x^{[\gamma], x} f^* p^* \omega)\) is an isomorphism, thus concluding the proof.

3.2. The line bundle associated to a character

Let us fix a fundamental domain \(\mathcal{F} \subset M_0\) for the action of \(\mathcal{G}\) on \(M_0\). Any function \(f : M \to \mathbb{C}\) can be seen as a Section of the trivial bundle \(M = M \times \mathbb{C}\), which could be defined as

\[
M_1 = \mathcal{G} \backslash (M_0 \times \mathbb{C}),
\]

where \(\mathcal{G}\) acts on \(M_0 \times \mathbb{C}\) by \([\gamma], (x, z) = ([\gamma]^{-1} x, \chi([\gamma]) z)\). We generalise the previous definition by allowing a possibly non-trivial holonomy; in other words, given a character \(\chi \in \mathcal{G}\), we define the line bundle \(M_\chi \to M\) associated to \(\chi\) by

\[
M_\chi = \mathcal{G} \backslash (M_0 \times \mathbb{C}),
\]

where \(\mathcal{G}\) acts on \(M_0 \times \mathbb{C}\) by \([\gamma], (x, z) = ([\gamma]^{-1} x, \chi([\gamma]) z)\).

We denote by \(L^2(M_\chi)\) the Hilbert space completion of the space of continuous sections of \(M_\chi\) with respect to the inner product

\[
\langle f_1, f_2 \rangle_{L^2(M_\chi)} := \int_M f_1 \overline{f_2} \, d\text{vol}.
\]

We remark that \(f_1 \overline{f_2}\) is a well-defined function on \(M\) for any two sections \(f_1, f_2\) of \(M_\chi\). Equivalently, we could define a section of \(M_\chi\) as a function \(f\) on \(M_0\) which satisfies the condition \(f([\gamma]^{-1} x) = \chi([\gamma]) f(x)\), and \(L^2(M_\chi)\) as the space of such measurable functions whose square norm \(\int_{\mathcal{F}} |f|^2 \, d\text{vol}\) is finite. By the same token, for every positive integer \(s\), we define the \(W^{2s}\) Sobolev norm of \(f\) as

\[
\|f\|_{W^{2s}(M_\chi)} := \int_{\mathcal{F}} |\Delta^s f| \, d\text{vol},
\]

and the Sobolev norm for other real values of \(s\) can be obtained by interpolation.
Let us give an explicit example, which will be relevant in the following. Let us fix \( x_0 \in \mathcal{F} \). Given \( \chi = \chi_\omega \in \mathcal{G} \), where \( \omega \in \mathcal{H} \), let us define

\[
G_\omega (x) = \exp \left( 2\pi i \int_{x}^{x_0} j^* p^* \omega \right).
\]  

(15)

Then,

\[
G_\omega \left( [\gamma]^{-1} \cdot x \right) = \exp \left( 2\pi i \int_{[\gamma]^{-1} \cdot x}^{x_0} j^* p^* \omega \right) = \exp \left( 2\pi i \int_{x}^{x_0} j^* p^* \omega \right) \exp \left( 2\pi i \int_{x}^{\gamma} j^* p^* \omega \right) = \exp \left( 2\pi i \int_{x}^{\gamma \cdot x} j^* p^* \omega \right) G_\omega (x) = \chi_\omega ([\gamma]) G_\omega (x),
\]

which proves that \( G_\omega \) is a section of \( M_{\chi_\omega} \). Moreover,

\[
\int_{\mathcal{F}} |G_\omega|^2 \, d\text{vol} = \text{vol}(\mathcal{F}) < \infty,
\]

so that \( G_\omega \in L^2(M_{\chi_\omega}) \). Furthermore, it follows from (15) that \( G_\omega \) is real-analytic as a function of \( \omega \in \mathcal{H} \).

Let us compute its derivatives. We identify \( x \in M_0 \) with the point \((x, v) \in T^1(S_0)\). Easy computations show

\[
(\Theta G_\omega) (x) = 2\pi i \Theta \left( \int_{x}^{x_0} j^* p^* \omega \right) G_\omega (x) = 0,
\]

\[
(X G_\omega) (x) = 2\pi i X \left( \int_{x}^{x_0} j^* p^* \omega \right) G_\omega (x) = -2\pi i \left( p^* \omega \right)_x (v) G_\omega (x),
\]

\[
(Y G_\omega) (x) = 2\pi i Y \left( \int_{x}^{x_0} j^* p^* \omega \right) G_\omega (x) = -2\pi i \left( p^* \omega \right)_x (v^\perp) G_\omega (x),
\]

(16)

where \( v^\perp \) is the vector obtained by rotating \( v \) by \( \pi/2 \) in the clockwise direction.

Let now \( s \in L^2(M_\chi) \). Then, the function \( G_\omega \cdot s \) satisfies

\[
(G_\omega \cdot s) \left( [\gamma]^{-1} \cdot x \right) = G_\omega \left( [\gamma]^{-1} \cdot x \right) s \left( [\gamma]^{-1} \cdot x \right) = \chi_\omega ([\gamma]) \chi ([\gamma]) G_\omega (x) s (x) = \left( \chi \chi_\omega \right) ([\gamma]) (G_\omega \cdot s) (x),
\]

so that it is a section of \( M_{\chi \chi_\omega} \). Moreover, it is also clear that

\[
\|G_\omega \cdot s\|_{L^2(M_{\chi \chi_\omega})} = \|s\|_{L^2(M_\chi)}.
\]

We obtain the following lemma.

**Lemma 3.2.** The map

\[
I_\omega: s \mapsto G_\omega \cdot s
\]

is a surjective linear isometry between \( L^2(M_\chi) \) and \( L^2(M_{\chi \chi_\omega}) \). Moreover, for every \( r > 0 \), the map \( I_\omega \) is a linear isomorphism between the Sobolev spaces \( W^r(M_\chi) \) and \( W^r(M_{\chi \chi_\omega}) \).
Proof. Since we have already verified the first part, it remains to show that \( I_\omega \) is a bounded map between \( W^r(M) \) and \( W^r(M, \chi_{\omega}) \).

For every \( j \geq 0 \), let us denote \( \|\omega\|_{\psi(j)} \) the \( \psi \)-norm of \( \omega : \mathbb{F} \to T^* \mathbb{F} \). Using (16) we can bound

\[
\|\Theta^j G_\omega\|_{L^\infty(\mathbb{F})}, \|X^j G_\omega\|_{L^\infty(\mathbb{F})}, \|Y^j G_\omega\|_{L^\infty(\mathbb{F})} \leq (2\pi)^{j-1} \|\omega\|_{\psi(j)}.
\]

Therefore, for any integer \( r > 0 \) and every \( j_1, j_2, j_3 \in \mathbb{Z}_{\geq 0} \) such that \( j_1 + j_2 + j_3 = r \), we have

\[
\|X^{j_1} Y^{j_2} \Theta^{j_3}(G_\omega \cdot s)\|_{L^2(M_\chi \omega)} \leq (2\pi)^r \|\omega\|_{\psi(r)} \left( \sum_{j_1 + j_2 + j_3 \leq r} \|X^{j_1} Y^{j_2} \Theta^{j_3}s\|_{L^2(M_\chi)} \right)
\]

\[
\leq C_\omega(r) \|s\|_{W^r(M_\chi)}
\]

for some constant \( C_\omega(r) > 0 \) depending only on \( \omega \) and \( r \). This completes the proof for integers \( s \), the general case follows from interpolation. \( \square \)

3.3. Rapidly decaying functions

We now define the space of functions we are interested in. Let us fix a norm \( \|\cdot\| \) in the first homology \( H_1(\mathbb{S}, \mathbb{R}) \). Since \( \mathcal{H} \) is a rank-\( d \) sublattice of \( H_1(\mathbb{S}, \mathbb{Z}) \), we have that

\[
\sum_{[\gamma] \in \mathcal{H}} \|\gamma\|^{\frac{1}{d} - \epsilon} < \infty, \quad \text{for every} \quad \epsilon > 0.
\]

We define the space \( \mathcal{R} \) of what we call rapidly decaying functions as the subspace of \( \mathcal{H}^\infty(M_\emptyset) \) consisting of smooth functions \( f \) on \( M_\emptyset \) such that

\[
\sum_{[\gamma] \in \mathcal{H}} \|\gamma\|^{\frac{1}{d} + \epsilon} \int_{[\gamma]} |D^k f(x)|^2 \text{vol}(x) < \infty,
\]

for every \( k, n \in \mathbb{Z}_{\geq 0} \) and \( \epsilon > 0 \),

where \( |D^k f(x)| \) is any norm of the \( n \)th derivative \( D^k f(x) : T_x M_\emptyset \otimes \cdots \otimes T_x M_\emptyset \to \mathbb{C} \) of \( f \) at \( x \in M_\emptyset \). In particular, we observe that \( \mathcal{R} \) contains the space \( \mathcal{H}^\infty(M_\emptyset) \) of smooth, compactly supported functions since, for these functions, only finitely many terms in the sum above are non-zero.

For any \( \chi \in \mathcal{H} \) and \( f \in \mathcal{R} \), let us define

\[
\hat{f}(x | \chi) := \sum_{[\gamma] \in \mathcal{H}} f([\gamma], x) \chi([\gamma]).
\]

By the Cauchy–Schwarz Inequality, for every \( n \in \mathbb{Z}_{\geq 0} \) and \( \epsilon > 0 \) we have

\[
|D^k \hat{f}(x | \chi)|^2 \leq \left( \sum_{[\gamma] \in \mathcal{H}} |D^k f([\gamma], x)\chi([\gamma])|^2 \right) \left( \sum_{[\gamma] \in \mathcal{H}} \|\gamma\|^{\frac{1}{d} + \epsilon} \right) \left( \sum_{[\gamma] \in \mathcal{H}} \|\gamma\|^{\frac{1}{d} - \epsilon} \right),
\]
and, from the definition of $\mathcal{R}$, it follows that
\[
\int_{\mathcal{F}} |\hat{D}^{\prime}f(x | \chi)|^2 \text{dvol} \\
\leq \left( \sum_{[\gamma] \in \mathcal{G}} \| [\gamma] \|^{d+\varepsilon} \int_{[\gamma], \mathcal{F}} |\hat{D}^{\prime}f|^2 \text{dvol} \right) \left( \sum_{[\gamma] \in \mathcal{G}} \| [\gamma] \|^{-d-\varepsilon} \right) < \infty.
\]
Moreover, for every $[\gamma] \in \mathcal{G}$, we have
\[
\hat{f}(\gamma^{-1}x | \chi) = \sum_{[\sigma] \in \mathcal{G}} f([\sigma], \gamma^{-1}x) \chi([\sigma]) = \sum_{[\sigma] \in \mathcal{G}} f([\sigma], x) \chi([\sigma]) \\
= \chi([\gamma]) \sum_{[\sigma] \in \mathcal{G}} f([\sigma], x) \chi([\sigma]) = \chi([\gamma]) \hat{f}(x | \chi).
\]
Thus, we obtain a well-defined map
\[
\pi_\chi : \mathcal{R} \to \bigcap_{\gamma \geq 0} W^r(M_\chi) \subset L^2(M_\chi).
\]
Furthermore, the map $\pi_\chi$ is $G$-equivariant: if we set $(\rho(g)f)(x) = f(xg)$, then we have $\pi_\chi(\rho(g)f)(x) = \hat{f}(xg | \chi)$.

By lemma 3.1, we can identify $\hat{\mathcal{G}}$ with the torus $\mathcal{H}/\mathcal{H}(\mathbb{Z})$. Parseval Identity yields
\[
\|f\|^2_{L^2(M_0)} = \int_{\mathcal{F}} \sum_{[\gamma] \in \mathcal{G}} |f([\gamma], x)|^2 = \int_{\hat{\mathcal{G}}} \left( \sum_{[\gamma] \in \mathcal{G}} \int_{\mathcal{F}} |\hat{f}(x | \chi)|^2 \text{dvol} \right) \text{d}\omega,
\]
hence, for any $f_1, f_2 \in \mathcal{R}$, it follows that
\[
\langle f_1, f_2 \rangle = \int_{\hat{\mathcal{G}}} \langle \pi_\chi(T f_1), \pi_\chi(T f_2) \rangle_{L^2(M_\chi)} \text{d}\omega.
\]
By density of $C^\infty_c(M_0)$, and hence of $\mathcal{R}$, in $L^2(M_0)$, we deduce the following lemma, which the reader can compare with, e.g. [7, theorem 4.1].

**Lemma 3.3.** We have the following $G$-equivariant isometric direct integral decomposition:
\[
L^2(M_0) = \int_{\hat{\mathcal{G}}} L^2(M_\chi) \text{d}\omega, \quad f = \int_{\hat{\mathcal{G}}} \pi_\chi(f) \text{d}\omega.
\]

### 3.4. Twisted $G$-actions

Let $1 = \chi_0 \in \hat{\mathcal{G}}$ be the trivial character. For any $f \in \mathcal{R}$ and $\omega \in \mathcal{H}$, we have
\[
(\pi_1 \circ I_{-\omega})(f)(x) = \pi_1(G_{-\omega} \cdot f)(x) = \sum_{[\gamma] \in \mathcal{G}} (G_{-\omega} \cdot f)([\gamma], x) \\
= \sum_{[\gamma] \in \mathcal{G}} \chi_\omega([\gamma]) G_{-\omega}(x)f([\gamma], x) = (I_{-\omega} \circ \pi_{\chi_\omega})(f)(x).
\]
In particular, by lemma 3.2, we deduce that for every \( f_1, f_2 \in \mathcal{R} \) and \( \chi = \chi_\omega \in \mathcal{G} \),
\[
\langle \pi_{\chi_\omega}(f_1), \pi_{\chi_\omega}(f_2) \rangle_{L^2(M, \mu_\omega)} = \langle \pi_\Omega(G_{-\omega} \cdot f_1), \pi_\Omega(G_{-\omega} \cdot f_2) \rangle_{L^2(M)}.
\]
(17)

In the following, for any \( f \in \mathcal{R} \), we will write \( \Omega_\omega(f) = \pi_\Omega(G_{-\omega} \cdot f) \in L^2(M) \).

From lemma 3.3 and (17), it follows that we can express the correlations of two functions \( f_1, f_2 \in \mathcal{R} \) as an integral of correlations
\[
\langle f_1, f_2 \rangle_{L^2(M, \mu_\omega)} = \int_{\mathcal{G}} \langle \pi_{\chi_\omega}(f_1), \pi_{\chi_\omega}(f_2) \rangle_{L^2(M, \mu_\omega)} \, d\omega
\]
\[
= \int_{\mathcal{G}} \langle \Omega_\omega(f_1), \Omega_\omega(f_2) \rangle_{L^2(M)} \, d\omega
\]
on the same Hilbert space \( H = L^2(M) \). The corresponding unitary representation \( \rho_\omega : G \to \mathcal{U}(H) \) of \( G = \text{PSL}_2(\mathbb{R}) \) on \( H \) is defined so that for all \( W \in \mathfrak{sl}_2(\mathbb{R}) \) we have
\[
W_\omega(\Omega_\omega(f)) = \Omega_\omega(Wf), \quad \text{where } W_\omega = d\rho_\omega(W).
\]

In the following, we will denote by \( H_\omega \) the datum of the Hilbert space \( H = L^2(M) \) together with the unitary representation \( \rho_\omega \).

Let us write explicit formulas for the generators \( \Theta_\omega, X_\omega, Y_\omega \). For every \( W \in \mathfrak{sl}_2(\mathbb{R}) \), since \( \pi_\Omega \) is \( G \)-equivariant, we have
\[
\Omega_\omega(Wf) = \pi_\Omega(G_{-\omega} \cdot Wf) = \pi_\Omega(W(G_{-\omega} \cdot f) - (W G_{-\omega}) \cdot f)
\]
\[
= W(\Omega_\omega(f)) - \pi_\Omega((W G_{-\omega}) \cdot f),
\]
thus, using (16) to compute the derivatives of \( G_{-\omega} \) in the right-hand side above, we obtain
\[
\Theta_\omega f(x) = \Theta f(x),
\]
\[
X_\omega f(x) = X f(x) - 2\pi \imath \omega_z(v) f(x),
\]
\[
Y_\omega f(x) = Y f(x) - 2\pi \imath \omega_z(v^1) f(x),
\]
for any \( f \in H \) and \( x = (z, \nu) \in M = T^1(S) \). In other words, we obtained the following lemma.

**Lemma 3.4.** For every \( f_1, f_2 \in \mathcal{R} \), there exist two associated smooth family of vectors \( \omega \mapsto \Omega_\omega(f_j) \in H = L^2(M) \) for \( j = 1, 2 \) such that for every \( t \in \mathbb{R} \)
\[
\langle f_1 \circ h_t f_2 \rangle_{L^2(M, \mu_\omega)} = \int_{\mathcal{G}} \langle \rho_\omega(\exp(tU)), \Omega_\omega(f_1), \Omega_\omega(f_2) \rangle_H \, d\omega,
\]
where the differential \( d\rho_\omega \) of the action \( \rho_\omega : G \to \mathcal{U}(H) \) is defined by (18).

We now verify that, for every fixed \( t \in \mathbb{R} \), the integrand function
\[
\omega \mapsto \langle \Omega_\omega(f_1), \Omega_\omega(f_2) \rangle_{L^2(M)}
\]
is smooth, where we denote \( \Omega_\omega = \rho_\omega(\exp(tU)) \).

**Lemma 3.5.** Let \( f \in \mathcal{R} \) be fixed. Then, \( \omega \mapsto \Omega_\omega(f) \) is a \( \mathcal{C}^\infty \) map from \( \mathcal{H} \) to \( W(H_\omega) \) for every \( r \geq 0 \).
We prove that the map \( \omega \mapsto \Omega_\omega(f) \) from \( \mathcal{H} \) to \( H \) is \( C^\infty \), the case \( r > 0 \) can be proved in a similar way using (18) and (16).

For every \( n \in \mathbb{Z}_{\geq 0} \), we have

\[
\left| \frac{d^n}{(d\omega)^n} G_{-\omega}([\gamma], x) \right| \leq (1 + \| \gamma \|)^n.
\]

Therefore, the series

\[
\sum_{[\gamma] \in \mathcal{G}} \left| \frac{d^n}{(d\omega)^n} (G_{-\omega} \cdot f)([\gamma], x) \right|^2_{L^2(M_{\kappa \omega})} \leq \sum_{[\gamma] \in \mathcal{G}} (1 + \| \gamma \|)^n \| f([\gamma], x) \|^2_{L^2(M_{\kappa \omega})}
\]

converges, since \( f \) is a rapidly decaying function according to section 3.3. This proves that \( \Omega_\omega(f) \) is \( n \) times differentiable with respect to \( \omega \) and

\[
\frac{d^n \Omega_\omega(f)}{(d\omega)^n} = \sum_{[\gamma] \in \mathcal{G}} \left( \frac{d^n}{(d\omega)^n} G_{-\omega}([\gamma], x) \right) f([\gamma], x),
\]

which completes the proof. \( \square \)

**Lemma 3.6.** Let \( \omega \mapsto \mathbf{f}_\omega \) be a family of smooth vectors in \( W(H_{\omega}) \). Then, \( \omega \mapsto \mathbf{u}_{\omega,f_\omega} \) is a smooth map from \( \mathcal{H} \) to \( W(H_{\omega}) \), and for every \( n \in \mathbb{Z}_{\geq 0} \) and for every \( t \geq 1 \), we have

\[
\left| \frac{d^n}{(d\omega)^n} \mathbf{u}_{\omega,f_\omega} \right| \leq C_n(\omega) \left( 1 + \log t \right)^n \max_{k \leq n} \left| \frac{d^k f_{\omega}}{(d\omega)^k} \right|
\]

for some constant \( C_n(\omega) \) independent of \( t \).

**Proof.** From the definition of the twisted action \( \rho_{\omega} \), we can write

\[
\mathbf{u}_{\omega,f_\omega} = f_\omega \circ h_t \cdot \exp \left( 2\pi i \int_x^{h_t(x)} j^* p^* \omega \right).
\]

Thus, there exists a constant \( C_n \) and \( k \leq n \) so that

\[
\left| \frac{d^n}{(d\omega)^n} \mathbf{u}_{\omega,f_\omega} \right| \leq C_n \left| \frac{d^{n-k} f_{\omega}}{(d\omega)^{n-k}} \right| \left| \frac{d^k}{(d\omega)^k} \exp \left( 2\pi i \int_x^{h_t(x)} j^* p^* \omega \right) \right|_{\infty} \leq C_n \left| \frac{d^{n-k} f_{\omega}}{(d\omega)^{n-k}} \right| \left| \int_x^{h_t(x)} j^* p^* \omega \right|_{\infty}^k.
\]

It remains to bound the integral \( \int_x^{h_t(x)} j^* p^* \omega \). The commutation relation between geodesic and horocycle flow tells us that \( g_{\log t}(h_t(x)) = h_t(g_{\log t}(x)) \), where \( g_t \) denotes the geodesic flow. Since \( p^* \omega \) is exact on the cover \( S_0 \), Stokes’ Theorem yields

\[
\int_x^{h_t(x)} j^* p^* \omega = \int_x^{h_t(x)} j^* p^* \omega + \int_{g_{\log t}(x)}^{h_t(g_{\log t}(x))} j^* p^* \omega = \int_{h_t(x)}^{h_t(g_{\log t}(x))} j^* p^* \omega.
\]
Since all paths in the right hand side above have length at most $1 + \log t$, we deduce the bound

$$\left| \int_{x}^{h(x)} p^* \omega \right| \leq 2 \|\omega\|_{\infty} (1 + \log t),$$

which completes the proof. $\square$

In section 2, we strengthened Ratner’s result to obtain precise estimates of the asymptotics of the correlations above, depending on the unitary representation. We now need to understand the behaviour of the spectrum of the associated Casimir operator, which is the subject of the remainder of this section.

3.5. The spectra around zero

As above, we denote by $H_\omega$ the space $H = L^2(M)$ equipped with the action $\rho_\omega$ of $G$. By the standard theory of unitary representations of $G$, every Hilbert space $H_\omega$ decomposes into irreducible representations $H_{\omega,\lambda(\omega)}$, and on each of these subspaces the Casimir operator

$$\Box_\omega = -X_\omega^2 - Y_\omega^2 + \Theta_\omega^2$$

acts as multiplication by the real number $\lambda(\omega) \in \mathbb{R}$. If $\lambda(\omega) > 0$, the space $H_{\omega,\lambda(\omega)}$ contains an invariant vector by $\Theta_\omega = \Theta$. Therefore, in order to study the positive parameters $\lambda(\omega)$, we can study the eigenvalues of the twisted Laplace-Beltrami operator $\Delta_{S,\omega} = d_{\omega}^* d_{\omega} + d_{\omega} d_{\omega}^*$. Here

$$d_{\omega} = I_{\omega} \circ d \circ I_{-\omega} = d + 2\pi t \omega \wedge$$

is the twisted differential operator and $d_{\omega}^*$ its $L^2$ adjoint (see the proof of proposition 3.7 below).

Recall that the maps $I_{\pm\omega}$ are given by multiplication by $G_{\pm\omega}$, which is a holomorphic function in the parameter $\omega \in H$. Thus, the family of elliptic operators $\omega \mapsto \Delta_{S,\omega}$ is holomorphic. By classical results in perturbation theory of selfadjoint operators (see, e.g. [13, p. 392]) the eigenvalues $0 \leq \lambda_0(\omega) \leq \lambda_1(\omega) \leq \cdots$ of $\Delta_{S,\omega}$ are real analytic functions of $\omega$ in a neighbourhood of $0$.

We will now focus on the first eigenvalue $\lambda_0(\omega)$ and we will show that $\lambda_0(\omega) \geq 0$ with equality if and only if $\omega = 0$. Recall that we denoted by $g$ the genus of the surface $S$.

**Proposition 3.7.** For every $\eta, \zeta \in H$, we have

$$D\lambda_0(0)(\eta) = 0, \quad \text{and} \quad D^2\lambda_0(0)(\eta, \zeta) = \frac{2\pi}{g-1} (\eta, \zeta).$$

In particular, the bilinear form $D^2\lambda_0(0)$ is positive definite.

**Proof.** Given a real valued harmonic one-form $\omega \in H$ we consider the twisted differential operator defined above

$$d_{\omega} : \Omega^*(S) \to \Omega^*(S), \quad d_{\omega} := d + 2\pi i \omega \wedge.$$

As $\omega$ is closed, $d_{\omega}^2 = 0$. The $L^2$ adjoint of $d_{\omega}$ on $\Omega^*(S)$ is given by

$$d_{\omega}^* = d^* + (2\pi i \omega \wedge)^\dagger = d^* - * (2\pi i \omega \wedge *), \quad \text{on} \ \Omega^*(S)$$
where \( d^* \equiv -\ast d\ast \) is the usual co-differential and \( \ast \) is usual Hodge operator verifying

\[
\langle \eta, \theta \rangle = \int_S \eta \wedge \ast \theta, \quad \text{for all } \eta, \theta \in \Omega^k(S).
\]

We can now define the twisted Laplace-Beltrami operator on \( \Omega^*(S) \) by setting

\[
\Delta_\omega = d_\omega \ast d^*_\omega + d^*_\omega \ast d_\omega.
\]

Let \( f_\omega \in L^2(S), \ \omega \in U \subset \mathcal{H} \), be any smooth family of normalised eigenfunctions of the Laplace-Beltrami operator \( \Delta_\omega = d^*_\omega \ast d_\omega \) with simple eigenvalue \( \lambda_\omega \):

\[
\Delta_\omega f_\omega = \lambda_\omega f_\omega, \quad \forall \omega : \|f_\omega\| = 1,
\]

and set \( h(\omega) = \langle \Delta_\omega f_\omega, f_\omega \rangle = \langle d_\omega f_\omega, d^*_\omega f_\omega \rangle \). Differentiating the identity \( h(\omega) = \lambda_\omega \) we have, for any \( \eta \in \mathcal{H} \)

\[
D\lambda_\omega (\eta) = 2\Re(2\pi i \eta, d_\omega f_\omega)
\]

and

\[
\frac{1}{2}D^2\lambda_\omega (\eta, \zeta) = 2\Re(2\pi i \eta, 2\pi i \eta, \zeta) + 2\Re(2\pi i Df_\omega (\zeta) \eta, d_\omega f_\omega) + 2\Re(2\pi i D\eta, d_\omega Df_\omega (\zeta)).
\]

In particular when \( \omega = 0 \) we can take \( f_0 = (4\pi (g-1))^{-1/2} \), where \( 4\pi (g-1) \) is the area of \( S \). Then the second terms of the above identity vanish because \( df_0 = 0 \). The third term also vanishes because \( \langle 2\pi i \eta, dDf_0 (\zeta) \rangle = \langle 2\pi i \eta, d^* \eta, Df_0 (\zeta) \rangle \) and the harmonicity of \( \eta \) implies that \( \eta \) is co-closed, i.e. \( d^* \eta = 0 \). Then the latter identity becomes

\[
\frac{1}{2}D^2\lambda_0 (\eta, \zeta) = \frac{\pi}{g-1} \langle \eta, \zeta \rangle
\]

since \( \eta \) and \( \zeta \) are real.

We will also need the following result on the eigenvalues \( \lambda_j(\omega) \).

**Lemma 3.8.** The value \( 0 = \lambda_0(0) \) is a global minimum for \( \lambda_j(\omega) \). Moreover, there exists \( \delta > 0 \) such that \( \lambda_j(\omega) > \delta \) for all \( j \geq 1 \) and all \( \omega \in \mathcal{H} \).

**Proof.** Both assertions follow from the min-max principle, see [21, p 289].

### 4. Infinite volume mixing asymptotics

This section is devoted to the proof of theorem A. We start with a technical result.

#### 4.1. A “stationary phase” estimate

The following lemma is a standard estimate of a certain type of integrals, whose asymptotics is controlled by the behaviour of the exponent in the integrand function near the stationary point. We present a proof for the sake of completeness, see also [21, lemma 2.3].
Lemma 4.1. Let $V$ be a $d$-dimensional vector space and let $U$ be an open neighbourhood of $0$. Let $v: U \to \mathbb{R}$ be a smooth function such that $v(0) = 0$ and $v(\xi) < 0$ for $\xi \neq 0$. Assume that $H = -D^2 v(0)$ is positive definite. Let $a: U \to \mathbb{R}$ be a $C^\infty$ function. Then, there exist constants $(c_j)_{j \in \mathbb{N}}$ such that for all $N \geq 1$ we have

$$
\int_U e^{Tv(\xi)} a(\xi) \, d\xi = \frac{(2\pi)^\frac{d}{2}}{T^\frac{d}{2} \sqrt{\det H}} a(0) + \sum_{j=1}^N \frac{c_j}{T^{j+d}} + o\left(T^{-N-\frac{d}{2}}\right).
$$

Proof. By assumption, $0$ is a non-degenerate critical point of $v$. Therefore, by the Morse Lemma, there exist open neighbourhoods $U', W \subseteq U$ of $0$ and a smooth diffeomorphism $\rho: W' \to U'$, $\rho(\tilde{\theta}) = \xi$, with $\rho(0) = 0$ and $\det D\rho(0) = 1$, such that

$$
v(\xi) = v\left(\rho\left(\tilde{\theta}\right)\right) = \frac{1}{2} D^2 v(0) \left(\tilde{\theta}, \tilde{\theta}\right).
$$

Without loss of generality, up to choosing a smaller $W'$, we can assume that $\det D\rho(\tilde{\theta}) > 0$ on $W'$. Applying $\rho$, followed by another linear change of variable, we get

$$
\int_U e^{Tv(\xi)} a(\xi) \, d\xi = \int_{W'} e^{-\frac{2\pi}{T^d} T^d (\tilde{\theta}, \tilde{\theta})} (a \circ \rho) \left(\tilde{\theta}\right) \det D\rho \left(\tilde{\theta}\right) \, d\tilde{\theta}
$$

$$
= \frac{1}{\sqrt{\det H}} \int_{W'} e^{-\frac{2\pi}{T^d} T^d (\tilde{\theta}, \tilde{\theta})} b(\tilde{\theta}) \, d\tilde{\theta},
$$

where $W \subseteq U$ is an open neighbourhood of $0$ and $b$ is a smooth function such that $b(0) = a(0)$. Therefore, we can find $\alpha, \delta > 0$, which depend only on $v$ and $a$, such that

$$
\int_U e^{Tv(\xi)} a(\xi) \, d\xi = \frac{1}{\sqrt{\det H}} \int_{\|\theta\| \leq \delta} e^{-\frac{2\pi}{T^d} T^d \|\theta\|^2} b(\theta) \, d\theta + O\left(e^{-\alpha T}\right),
$$

where we used the fact that $v(\xi) < 0$ for all $\xi \neq 0$.

In order to complete the proof, we are left to estimate the term

$$
\frac{1}{\sqrt{\det H}} \int_{\|\theta\| \leq \delta} e^{-\frac{2\pi}{T^d} T^d \|\theta\|^2} b(\theta) \, d\theta = \frac{1}{T^d \sqrt{\det H}} \int_{\|\theta\| \leq \delta \sqrt{T}} e^{-\frac{2\pi}{T^d} T^d \|\theta\|^2} b\left(\frac{\theta}{\sqrt{T}}\right) \, d\theta.
$$

(19)

For any $N \geq 1$, we can write the Taylor expansion of $b$ as

$$
b\left(\frac{\theta}{\sqrt{T}}\right) = b(0) + \sum_{|k| \leq N} \frac{D^k b(0)}{k!} \frac{\theta^k}{T^{\frac{|k|}{2}}} + R_N\left(\frac{\theta}{\sqrt{T}}\right),
$$

where $k = (k_1, \ldots, k_d) \in \mathbb{Z}_{>0}^d$ is a multi-index, $|k| = k_1 + \cdots + k_d$, and the last term satisfies

$$
\left|R_N\left(\frac{\theta}{\sqrt{T}}\right)\right| \leq C_{a, \nu} \frac{\theta^{k_1} \cdots \theta^{k_d}}{(N+1)! T^{\frac{|k|}{2}}},
$$

(20)
for a constant $C_{a,\nu}$ depending on $b$, and hence on $a$ and $\nu$ only. Plugging this expansion into (19), and recalling $b(0) = a(0)$, we get

$$
\frac{1}{\sqrt{\det H}} \int_{\|\theta\| \leq \delta} e^{-\frac{1}{2} \|\theta\|^2} b(\theta) \, d\theta = \frac{a(0)}{T^2} \frac{\sqrt{\det H}}{\sqrt{\det H}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \|\theta\|^2} d\theta \\
+ \sum_{|k| \leq N} \frac{D^k b(0)}{k! T^{\frac{1}{2}+|k|} \sqrt{\det H}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \|\theta\|^2} \theta^k \, d\theta \\
+ \frac{1}{T^2} \frac{\sqrt{\det H}}{\sqrt{\det H}} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \|\theta\|^2} R_N \left( \frac{\theta}{\sqrt{T}} \right) \, d\theta + o \left( e^{-\sqrt{T}} \right).
$$

The conclusion follows from the bound (20) and the fact that the terms with $|k|$ odd do not appear, since the integrals $\int_{-\infty}^{\infty} e^{-\frac{1}{2} \|\theta\|^2} \theta^k \, d\theta$ vanish for $|k|$ odd. \hfill \qed

4.2. Proof of theorem A

Let us fix $f_1, f_2 \in \mathcal{R}$. By lemmas 3.4 and 3.5, we can write

$$
\langle f_1 \circ h_1, f_2 \rangle_{L^2(M_b)} = \int_{\mathcal{G}} (\mathbf{w}_\omega, \Omega_{\omega}(f_1), \Omega_{\omega}(f_2))_{H_{\omega}} \, d\omega,
$$

where $\Omega_{\omega}(f_j)$ are smooth families of vectors in $W^r(H_{\omega})$ for every $r \geq 0$. We now show that the main contribution in the integral above comes from the projection on the subspaces $H_{\omega, \lambda_0(\omega)}$ corresponding to the smallest positive eigenvalue $\lambda_0(\omega)$ of the twisted Laplacian $\Delta_{S, \omega}$.

For any $\omega \in \mathcal{H}$, we have an orthogonal decomposition into invariant subspaces

$$
H_{\omega} = H_{\omega, \lambda_0(\omega)} \oplus H_{\omega, \lambda_0(\omega)}^\perp,
$$

where the Casimir operator $\Box_{\omega}$ acts on $H_{\omega, \lambda_0(\omega)}$ as the constant $\lambda_0(\omega)$, and $H_{\omega, \lambda_0(\omega)}^\perp$ is the orthogonal complement of $H_{\omega, \lambda_0(\omega)}$. According to this decomposition, we write

$$
\Omega_{\omega}(f_j) = \Omega_{\lambda_0(\omega)}(f_j) + \Omega_{\omega}^\perp(f_j), \quad \text{where} \quad \Omega_{\lambda_0(\omega)}(f_j) \in W^3(H_{\omega, \lambda_0(\omega)}),
$$

and

$$
\Omega_{\omega}^\perp(f_j) \in W^3(H_{\omega, \lambda_0(\omega)}^\perp).
$$

By lemma 3.8, there exists $\delta > 0$ such that the spectrum of $\Box_{\omega}$ restricted to $H_{\omega, \lambda_0(\omega)}^\perp$ does not intersect the interval $[0, \delta)$. Thus, by theorem 2.1, up to changing $\delta > 0$, we deduce

$$
\left\| \mathbf{w}_\omega, \Omega_{\omega}^\perp(f_1), \Omega_{\omega}^\perp(f_2) \right\|_{H_{\omega}} \leq C \left\| \Omega_{\omega}(f_1) \right\|_{W^3 \| \Omega_{\omega}(f_1) \|_{W^3 t^{-\delta}},
$$

for some absolute constant $C > 0$. This implies that

$$
\langle f_1 \circ h_1, f_2 \rangle_{L^2(M_b)} = \int_{\mathcal{G}} (\mathbf{w}_\omega, \Omega_{\lambda_0(\omega)}(f_1), \Omega_{\lambda_0(\omega)}(f_2))_{H_{\omega}} \, d\omega \leq C \|f_1\|_{W^3} \|f_2\|_{W^3 t^{-\delta}}.
$$

We now focus on the space $H_{\omega, \lambda_0(\omega)}$ and we show that we can reduce to consider a neighbourhood of $0 \in \mathcal{G}$. By the representation theory of compact Abelian groups, $H_{\omega, \lambda_0(\omega)}$ admits an orthonormal basis of eigenvectors $e_j(\omega)$ of $\Theta_{\omega}$, with eigenvalue $j \in \mathbb{Z}$. Any such element $e_j(\omega)$ is an eigenvector of the elliptic selfadjoint operator $\Delta_{\omega} = \Box_{\omega} - 2\Theta_{\omega}^2$, hence, by classical
results in perturbation theory (see, e.g. [14, p. 392]), we can choose these bases to depend analytically in \( \omega \) in a neighbourhood \( U \subset \mathbb{G} \) of 0, so that \( \omega \mapsto \Omega^{\lambda_0(\omega)}(f_j) \), for \( j = 1, 2 \), are analytic families of smooth vectors. Since \( 0 = \lambda_0(0) \) is a global minimum, again by lemma 3.5, up to changing the constants \( C \) and \( \delta \), we obtain

\[
\left| \langle f_1 \circ h_2, f_2 \rangle_{L^2(M_0)} - \int_U \langle u_{\omega}, \Omega^{\lambda_0(\omega)}(f_1), \Omega^{\lambda_0(\omega)}(f_2) \rangle_{H_{\omega}} \, d\omega \right| \leq C \| f_1 \|_{W^1} \| f_2 \|_{W^1} r^{-\delta}.
\]  

Moreover, we can assume that \( \lambda_0(\omega) < 1/4 \) for all \( \omega \in U \).

Finally, we now show that we can use the result of section 2, together with the “stationary phase” estimate of lemma 4.1. Combining (21) with corollary 2.5 and the Cauchy-Schwarz Inequality, we obtain

\[
\left| \langle f_1 \circ h_2, f_2 \rangle_{L^2(M_0)} - \int_U A_{\omega}(f_1, f_2) r^{-1+\nu(\omega)} \, d\omega \right| \leq C \| f_1 \|_{W^1} \| f_2 \|_{W^1} r^{-\delta},
\]  

where \( \nu_0(\omega) = \sqrt{1 - 4\lambda_0(\omega)} \) and \( A_{\omega}(f_1, f_2) \) is defined in (14) in corollary 2.5. In order to conclude, we show that we can apply lemma 4.1 to the integral above, which can be rewritten as

\[
\int_U A_{\omega}(f_1, f_2) r^{-1+\nu(\omega)} \, d\omega = \int_U A_{\omega}(f_1, f_2) e^{(\log t)(-1+\nu(\omega))} \, d\omega.
\]  

**Lemma 4.2.** The map \( \omega \mapsto A_{\omega}(f_1, f_2) \) is a smooth map from \( \mathcal{H} \) to \( \mathbb{C} \).

**Proof.** By lemma 3.5 and the previous discussion, \( v(\omega) := \Omega^{\lambda_0(\omega)}(f_1) \) and \( w(\omega) := \Omega^{\lambda_0(\omega)}(f_2) \) are smooth families of vectors in \( W'(H_{\omega}) \) for every \( \omega \geq 0 \). By the definition (14) of \( A_{\omega}(f_1, f_2) \) from corollary 2.5, we need to verify that the derivatives of the function

\[
r^{-1-\nu(\omega)} F_{v(\omega), w(\omega)}(r)
\]

are integrable over \([1, \infty)\). This follows from lemma 3.6 and the bound

\[
\left| \frac{d^n}{d\omega^n} r^{-1-\nu(\omega)} \right| \leq C_n(\omega) (1 + \log r)^n r^{-1-\nu(\omega)},
\]

for some constant \( C_n(\omega) \) independent of \( \omega \geq 1 \).

Finally, by proposition 3.7, we have that

\[
D(-1 + \nu_0(\omega))(0) = -D\left( \sqrt{1 - 4\lambda_0(\omega)} \right)(0) = -2D\lambda_0(0) = 0,
\]

and also that

\[
H := D^2(1 - \nu_0(\omega))(0) = -D^2\left( \sqrt{1 - 4\lambda_0(\omega)} \right) = 2D^2\lambda_0(0)
\]
is positive definite. Therefore, by (22) and by lemma 4.1, there exists constants \((c_j)_{j \in \mathbb{N}}\) such that for every \(N \geq 1\) we can write

\[
\langle f_1 \circ h, f_2 \rangle_{L^2(M_k)} = \frac{(2\pi)^{\frac{d}{2}}}{\sqrt{\det \mathcal{H}}} A_0(f_1, f_2) + \sum_{j=1}^{N} \frac{c_j}{(\log t)^{\frac{d+2}{2}}} + o \left( (\log t)^{-N-\frac{d}{2}} \right)
\]

\[
= \left( \frac{g-1}{2} \right)^{\frac{d}{2}} \sigma(\Gamma_0) A_0(f_1, f_2) (\log t)^{\frac{d}{2}} + \sum_{j=1}^{N} \frac{c_j}{(\log t)^{\frac{d+2}{2}}} + o \left( (\log t)^{-N-\frac{d}{2}} \right),
\]

where \(\sigma(\Gamma_0)\) is the determinant of the matrix associated to the bilinear form \(\langle \eta, \zeta \rangle\) on the space \(\mathcal{H}\) of harmonic one-forms.

Finally,

\[
A_0(f_1, f_2) = \lim_{t \to \infty} \langle \Omega_0(f_1 \circ h_t), \Omega_0(f_2) \rangle_{\mathcal{H}_0}
\]

\[
= \int_M \Omega_0(f_1) \text{dvol} \cdot \int_M \overline{\Omega_0(f_2)} \text{dvol}
\]

\[
= \int_M \pi_1(f_1) \text{dvol} \cdot \int_M \overline{\pi_1(f_2)} \text{dvol} = \int_{M_0} f_1 \text{dvol} \cdot \int_{M_0} f_2 \text{dvol},
\]

which completes the proof of theorem A.

5. Proof of theorem B

In this section we prove theorem B using the theory we developed in section 3.

Let us denote by \(\mathcal{G}_k\) the Galois group \(\Gamma / \Gamma_k\) of the cover \(p_k : M_k \to M\). The surjective map \(\psi : \Gamma \to \mathbb{Z}^d\), as in the assumptions, induces an isomorphism of finite Abelian groups

\[
\mathcal{G}_k \cong \mathbb{Z} / \left( N_1^{(k)} \mathbb{Z} \right) \times \cdots \times \left( N_d^{(k)} \mathbb{Z} \right).
\]

By lemma 3.1, for any \(\chi \in \hat{\mathcal{G}}_k\), there exist \(\omega \in \mathcal{H}\) such that

\[
\chi([\gamma]) = \exp \left( 2\pi i \int_{\gamma} j_k p_k^* \omega \right),
\]

where \(j_k : M_k = T^1(S_k) \to S_k\) is the canonical projection. More precisely, the same argument as in lemma 3.1 shows that \(\hat{\mathcal{G}}_k\) coincides with the exponential of \(\mathcal{H}(k) / \mathcal{H}(\mathbb{Z})\), where \(\mathcal{H}(k)\) is the \(\mathbb{Z}\)-module of all harmonic forms \(\omega\) such that

\[
\int_{\gamma} j_k p_k^* \omega \in \mathbb{Z} \quad \text{for all } [\gamma] \in \hat{\mathcal{G}}_k.
\]

Notice that

\[
\mathcal{H}(k) / \mathcal{H}(\mathbb{Z}) \cong \left( 1 / N_1^{(k)} \right) \mathbb{Z} / \mathbb{Z} \times \cdots \times \left( 1 / N_d^{(k)} \right) \mathbb{Z} / \mathbb{Z}.
\]

As in section 3.2, let us denote by \(L^2(M_k)\) the space of \(L^2\)-sections of the line bundle with holonomy \(\chi\).
Lemma 5.1. We have

\[ L^2(M_k) = \bigoplus_{\omega \in \mathcal{H}(k) / \mathcal{H}(\mathbb{Z})} L^2(M_{\chi, \omega}), \]

in particular

\[ \text{Spec}(\square_k) \cap [0, \infty) = \bigcup \{ \lambda_n(\omega) : n \in \mathbb{N}, \omega \in \mathcal{H}(k) / \mathcal{H}(\mathbb{Z}) \}. \]

Proof. Let \( N_j := |\mathcal{O}| = N^{(k)}_1 \cdots N^{(k)}_d \). It is not hard to see that the map

\[ \Phi : \bigoplus_{\omega \in \mathcal{H}(k) / \mathcal{H}(\mathbb{Z})} L^2(M_{\chi, \omega}) \rightarrow L^2(M_k) \]

\[ (s_\omega)_{\omega \in \mathcal{H}(k) / \mathcal{H}(\mathbb{Z})} \mapsto \frac{1}{N_j} \sum_{\omega \in \mathcal{H}(k) / \mathcal{H}(\mathbb{Z})} s_\omega \]

is a well-defined isometry whose image contains the dense subspace of smooth functions \( \mathcal{C}^\infty(M_k) \), from which the conclusion follows.

By Lemma 3.8, there exists \( \delta > 0 \) such that \( \lambda_0(\omega) \) is analytic for all \( \omega \) such that \( |\lambda_0(\omega)| < \delta \), and \( \lambda_n(\omega) > \delta \) for all \( n \geq 1 \) and all \( \omega \in \mathcal{H}(k) / \mathcal{H}(\mathbb{Z}) \). In particular,

\[ \text{Spec}(\square_k) \cap [0, \delta] = \{ \lambda_0(\omega) : \omega \in \mathcal{H}(k) / \mathcal{H}(\mathbb{Z}) \} \cap [0, \delta]. \]

Let \( f \) be a continuous function compactly supported in \([0, \delta]\). We have

\[ \frac{1}{|\mathcal{O}| / \mathbb{Z}} \sum_{\omega \in \text{Spec}(\square_k) \cap [0, \delta]} f(\lambda) = \frac{1}{|\mathcal{O}| / \mathbb{Z}} \sum_{\omega \in \mathcal{H}(k) / \mathcal{H}(\mathbb{Z})} f(\lambda_0(\omega)), \]

so that, by equidistribution of the points \( \omega \in \mathcal{H}(k) / \mathcal{H}(\mathbb{Z}) \) in the torus \( \mathcal{H} / \mathcal{H}(\mathbb{Z}) \),

\[ \lim_{k \rightarrow \infty} \frac{1}{|\mathcal{O}| / \mathbb{Z}} \sum_{\omega \in \text{Spec}(\square_k) \cap [0, \delta]} f(\lambda) = \int_{\mathcal{H} / \mathcal{H}(\mathbb{Z})} f(\lambda_0(\omega)) d\omega \]

\[ = \int_{\lambda_0^{-1}(\mathcal{O}) / \mathbb{Z}} f(\lambda_0(\omega)) d\omega. \]

Let us rewrite the integral on the right. By the Morse Lemma (as in the proof of lemma 4.1), up to choosing a smaller \( \delta \), we can find a change of variable \( \omega = \Psi(y) \), with \( y \in \mathbb{R}^d \), such that \( \lambda_0(\Psi(y)) = y_1^2 + \cdots + y_d^2 \). In particular

\[ \int_{\lambda_0^{-1}(\mathcal{O}) / \mathbb{Z}} f(\lambda_0(\omega)) d\omega = \int_{\Psi^{-1}(\lambda_0^{-1}(\mathcal{O}))} f(y_1^2 + \cdots + y_d^2) \Psi'(y) dy \]

\[ = \int_{|y|^2 < \delta} f(y_1^2 + \cdots + y_d^2) \tilde{\Psi}(y) dy, \]

where

\[ \tilde{\Psi}(y) = \Psi'(y) 1_{\Psi^{-1}(\lambda_0^{-1}(\mathcal{O}))}(y). \]
Passing to polar coordinates, we can write

$$\int_{|y|<\delta} f \left( y_1^2 + \cdots + y_d^2 \right) \tilde{\Psi}(y) \, dy = \int_0^{\sqrt{\delta}} f \left( r^2 \right) r^{d-1} \zeta(r) \, dr,$$

for a function $\zeta \in L^\infty([0,\delta])$. Finally, setting $x = r^2$, we conclude

$$\lim_{k \to \infty} \frac{1}{[\Gamma / \Gamma_k]} \sum_{\lambda \in \text{Spec}(\mathbb{C}_k) \cap [0,\delta]} f(\lambda) = \int_0^{\delta} f(x) x^{\frac{d}{2} - 1} \frac{\zeta(\sqrt{x})}{2} \, dx.$$

Thus, the proof of theorem B is complete.

**Data availability statement**

No new data were created or analysed in this study.

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