Electric-Magnetic Duality and WDVV Equations

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We consider the associativity (or WDVV) equations in the form they appear in Seiberg-Witten theory and prove that they are covariant under generic electric-magnetic duality transformations. We discuss the consequences of this covariance from various perspectives.

1 Introduction

Duality transformations play an important role in modern theoretical physics. In Seiberg-Witten theory electric-magnetic duality (for a recent review of electric-magnetic duality see and references therein) is a basic ingredient in obtaining the exact form of the low-energy effective action. Hence, duality is a crucial tool in studying non-perturbative physics. Any truly non-perturbative result should be consistent with electric-magnetic duality.

Based on electric-magnetic duality, Seiberg-Witten theory enables the determination of the holomorphic function $F(a)$ in terms of which the low-energy effective action is encoded. Here $a$ denotes complex fields associated with the Cartan subalgebra of the gauge group. The function $F$ plays the role of a prepotential for the corresponding special Kähler geometry. The construction involves an auxiliary complex curve, whose moduli space of complex structures is identified with the special Kähler space with $a$ playing the role of local

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1Strictly speaking ‘special geometry’ refers to the Kähler geometry associated with locally $\mathcal{N}=2$ supersymmetric Yang-Mills theories coupled to Poincaré supergravity. In the rigid supersymmetry context one sometimes uses the term ‘rigid special geometry’. Here we do not make this distinction.
coordinates. This construction can be cast in terms of an integrable system \(\mathcal{F}(a)\), identifying \(\mathcal{F}(a)\) with (the logarithm of) a tau-function of the so-called quasiclassical or universal Whitham hierarchy \(\mathcal{F}\), which satisfies a set of nontrivial differential equations (see, for example, \(\mathcal{F}\) and references therein for the details of this correspondence).

An intriguing example of these equations is the set of associativity or Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations originally found in the context of 2D topological theories \(\mathcal{F}\). In \(\mathcal{F}\) it was established that the function \(\mathcal{F}(a)\) associated with the exact solution for pure \(SU(N)\) Yang-Mills theory \(\mathcal{F}\), satisfies the set of associativity equations,

\[
\mathcal{F}_i \cdot \mathcal{F}_j^{-1} \cdot \mathcal{F}_k = \mathcal{F}_k \cdot \mathcal{F}_j^{-1} \cdot \mathcal{F}_i \quad \forall i, j, k ,
\]

written in terms of the matrices \(\mathcal{F}_i\) whose matrix elements are the third derivatives of \(\mathcal{F}(a)\),

\[
\mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} .
\]

Later it was shown that \(\mathcal{F}\) holds for all known Seiberg-Witten solutions based on hyperelliptic auxiliary curves \(\mathcal{F}\) (for recent progress for the case of non-hyperelliptic curves, see \(\mathcal{F}\)). A crucial ingredient of this result is that the \(\mathcal{F}_{ijk}\) determine the structure constants of the algebra constructed from holomorphic differentials on the associated complex curve. Because the effective low-energy theory is subject to electric-magnetic duality, the WDVV equations \(\mathcal{F}\) should respect this duality.

We emphasize that one should clearly distinguish between the special Kähler metric, constructed from second derivatives of \(\mathcal{F}\), and the so-called metric of the topological theory \(\mathcal{F}\), constructed from the third derivatives of \(\mathcal{F}\). In the context of \(\mathcal{F}\) it seems clear that the “topological metric” plays no role and neither does the (non-holomorphic) special Kähler metric, while the matrix \(\mathcal{F}_j\) can be regarded as the inverse of a (holomorphic) metric \(\mathcal{F}_j\). However, \(\mathcal{F}_j\) can equally well be replaced by the inverse of any linear combination of the matrices \(\mathcal{F}_j\) (we prefer not to interpret the inverse of any linear combination of the \(\mathcal{F}_j\) as a metric). So the issue of a metric is not immediately obvious in the context of \(\mathcal{F}\).

In this note we demonstrate that the WDVV equations \(\mathcal{F}\) are covariant under generic electric-magnetic duality transformations \(\mathcal{F}\). Specifically, we prove that if a holomorphic function \(\mathcal{F}\) satisfies the WDVV equations, so does its dual function. Technically the proof is simple and it is not restricted to the Seiberg-Witten case; it is applicable to any solution of WDVV equations and thus, hopefully, to any tau-function of the Whitham hierarchy for an allowed class of duality transformations. In sect. \(\mathcal{F}\) we review the associativity equations and introduce the dual holomorphic function for the duality that interchanges electric and magnetic charges. In sect. \(\mathcal{F}\) we prove that upon generic duality transformations, the dual function also satisfies the WDVV equations.

Sect. \(\mathcal{F}\) contains some discussion and outlook.

\[\text{In this work it was also shown that }\mathcal{F}\text{ is not satisfied by the function }\mathcal{F}(a)\text{ for softly broken }\mathcal{F}\text{, a phenomenon that is not yet completely understood.}\]

\[\text{The covariance of the WDVV equations under electric-magnetic duality was first noticed in }\mathcal{F}\text{. Restricted duality transformations were considered in }\mathcal{F}\text{, but the results reported there are not fully in agreement with the results of this note.}\]
2 WDVV equations for the dual prepotential

In Seiberg-Witten theory the second derivatives of $F(a)$ are identified with the period matrix of the corresponding auxiliary complex curve,

$$T_{ij} = \frac{\partial^2 F}{\partial a_i \partial a_j}. \quad (3)$$

It’s imaginary part is equal to the Kähler metric, as follows directly from the Kähler potential,

$$K(a, \bar{a}) = \text{Im} \sum_i a_i \frac{\partial F}{\partial a_i}.$$

The derivatives of the period matrix, $\partial T_{ij}/\partial a_k = T_{ijk}$, are the (totally symmetric) holomorphic tensors that appear in the WDVV equation (1). We will assume that the matrices $\|F_i\|$ and/or a linear combination thereof is nonsingular. In contrast to 2D topological field theory where one linear combination of the matrices $\|F_i\|$ in (1) can always be chosen constant [6, 13], this is not so for Seiberg-Witten theory.

Let us now consider the electric-magnetic duality transformation

$$a_i \rightarrow a^D_i = \frac{\partial F}{\partial a_i}, \quad a^D_i \rightarrow -a_i = -\frac{\partial F^D(a^D)}{\partial a^D_i}, \quad (5)$$

with the dual function $F^D(a^D)$. As is well-known, this transformation is effected by a Legendre transform,

$$F^D(a^D) = F(a) - \sum_i a_i a^D_i.$$

Obviously we have

$$\frac{\partial a^D_i}{\partial a^D_j} = \frac{\partial^2 F}{\partial a_i \partial a_j} = T_{ij}, \quad \frac{\partial a_i}{\partial a^D_j} = -\frac{\partial^2 F^D}{\partial a^D_i \partial a^D_j} = -T_{ij}^D, \quad (7)$$

so that the dual period matrix $T_{ij}^D$ equals minus the inverse of the original period matrix $T_{ij}$, i.e.,

$$\sum_j T_{ij}^D T_{jk} = -\delta_{ik}. \quad (8)$$

Now consider

$$\|F_i^D\|_{jk} \equiv F_{ijk}^D = -\frac{\partial T_{ij}^D}{\partial a^D_k} = \frac{\partial^3 F^D}{\partial a^D_i \partial a^D_j \partial a^D_k}. \quad (9)$$

It directly follows that

$$\frac{\partial T_{ij}^D}{\partial a^D_k} = \sum_{l,m,n} T_{il}^D \frac{\partial T_{mn}^D}{\partial a_l} T_{nj}^D \frac{\partial a_l}{\partial a_k}. \quad (10)$$

Consequently $F_{ijk}$ transforms simply as,

$$\frac{\partial^3 F^D}{\partial a^D_i \partial a^D_j \partial a^D_k} = \sum_{l,m,n} \frac{\partial^3 F}{\partial a_l \partial a_m \partial a_n} \frac{\partial a_l}{\partial a^D_i} \frac{\partial a_m}{\partial a^D_j} \frac{\partial a_n}{\partial a^D_k}, \quad (11)$$

or, in matrix form,

$$\|F^D\| = \sum_j \frac{\partial a_j}{\partial a^D_j} \|T^D \cdot F \cdot T^D\|.$$

From this result it is obvious that the equations (1) are valid for the dual function $F^D(a^D)$, because

$$F^D_i \cdot (F^D_j)^{-1} \cdot F^D_k - (i \leftrightarrow k) = \sum_{l,m,n} \frac{\partial a_j}{\partial a^D_m} \frac{\partial a_k}{\partial a^D_n} \frac{\partial a^D_l}{\partial a^D_i} \left[ F_m \cdot F_i^{-1} \cdot F_n - (m \leftrightarrow n) \right] = 0, \quad (13)$$

where on the right-hand side we made use of (5) for all $l, m, n$.\footnote{The reader may appreciate the following equation,}

$$\frac{\partial T_{ij}^D}{\partial a^D_l} = T_{ik}^D T_{lj}^D.$$(10)
3 Generic duality transformations

The same logic can be applied to generic electric-magnetic duality transformations forming an arithmetic subgroup of $Sp(2r, \mathbb{R})$, which generalize the special duality transformation given in formula (5). Here $r$ is the rank of the gauge group. At the perturbative level these transformations are continuous. The covariance properties that we are about to establish do not depend on this feature.

In the dual basis (denoted by the superscript $S$), we have new variables $a^S$ and a new function $F^S(a^S)$, defined by (14, 15),

\[ a^S = U \cdot a + Z \cdot \left( \frac{\partial F}{\partial a} \right), \]

\[ \left( \frac{\partial F^S}{\partial a^S} \right) = V \cdot \left( \frac{\partial F}{\partial a} \right) + W \cdot a, \]

where the $r \times r$ matrices $U$, $V$, $W$, $Z$ combine into an $Sp(2r, \mathbb{R})$ matrix, by virtue of the relations

\[ U^t \cdot V - W^t \cdot Z = V^t \cdot U \quad \text{and} \quad Z^t \cdot V = V^t \cdot Z. \]

The result analogous to (6) reads,

\[ F^S(a^S) = F(a) + \frac{1}{2} \left( a^t \cdot U^t \cdot W \cdot a + a^t \cdot W^t \cdot Z \cdot \left( \frac{\partial F}{\partial a} \right) + \frac{1}{2} \left( \frac{\partial F^S}{\partial a^S} \right)^t \cdot Z^t \cdot V \cdot \left( \frac{\partial F}{\partial a} \right). \]

Observe that this represents only a (partial) Legendre transform when $U^t \cdot W = Z^t \cdot V = 0$.

From these results one proves that the period matrix, again defined by (3), and its dual counterpart,

\[ \frac{\partial^2 F^S}{\partial a^S_i \partial a^S_j} = T^S_{ij}, \]

are related by

\[ T^S = (V \cdot T + W) \cdot S^{-1}(T). \]

The special Kähler metric associated with the Kähler potential (4),

\[ G_{ij} = \frac{\partial^2 K(a, \bar{a})}{\partial a_i \partial \bar{a}_j} = \text{Im} \ T_{ij}, \]

transforms as

\[ G^S = [S^t]^{-1}(\bar{T}) \cdot G \cdot S^{-1}(T). \]

Here the matrix $S(T)$ is defined by

\[ S_{ij}(T) = \frac{\partial a^S_i}{\partial a_j} = \|U + Z \cdot T\|_{ij}. \]

Now we wish to demonstrate that the third derivatives of $F$ and $F^S$ remain related just as in (11), i.e.,

\[ F^S_{ijk} = \sum_{l,m,n} F_{lmn} (S^{-1})_{li} (S^{-1})_{mj} (S^{-1})_{nk}. \]

or,

\[ \frac{\partial^3 F^S}{\partial a^S_i \partial a^S_j \partial a^S_k} = \sum_{l,m,n} \frac{\partial^3 F}{\partial a_i \partial a_m \partial a_n} \frac{\partial a_i}{\partial a^S_l} \frac{\partial a_j}{\partial a^S_m} \frac{\partial a_k}{\partial a^S_n}, \]
This result is known from the literature [15] and we will briefly review its proof. First one shows that
\[
\delta T^S = (V - (V \cdot T + W) \cdot S^{-1} \cdot Z) \cdot \delta T \cdot S^{-1} = (S')^{-1} \cdot \delta T' \cdot S^{-1}.
\] (24)

This result (24) follows directly from the equations (18), (21) and (15). Likewise one shows that \( S^{-1}(T) \cdot Z \) is a symmetric matrix.

Replacing the variation in (24) by a derivative with respect to \( a^S \) and using \( F_{ijk} = \partial T_{ij}/\partial a^k \) and (21), one readily proves the validity of (22). Along the same line as in sect. 3, this then leads to the conclusion that the WDVV equations (1) remain invariant under general duality transformations (18), so that the function \( F^S(a^S) \) satisfies
\[
F^S_i \cdot (F^S)^{-1} \cdot F^S_k = F^S_i \cdot (F^S)^{-1} \cdot F^S_k.
\] (25)

provided the WDVV equations were valid for the original function \( F \). Upon setting \( U = V = 0 \) and \( Z = -W = 1 \), the reader can also verify that the results of the previous section are reproduced. Finally, we note that most of the transformations of this section concern only the holomorphic sector of the theory, where the symplectic transformations can be extended to complex-valued matrices. While this would bring one outside the strict context of electric-magnetic duality, this extension may have some relevance in the context of integrable models.

4 Discussion

According to electric-magnetic duality two dual holomorphic functions, \( F \) and \( F^S \), describe the same system and thus belong to the same equivalence class. As we stressed, this duality is therefore at the basis of the Seiberg-Witten theory. Consequently it follows that physically relevant results, when expressible directly in terms of the function \( F \), should hold for all representatives of the equivalence class. Specifically, \( F \) and \( F^S \) should both satisfy the corresponding relations. Therefore it follows that the associativity equations should hold for all representatives of a given equivalence class. In this note we have shown this to be the case as the associativity equations (1) are simply covariant under generic duality transformations.

We should point out here that versions of the associativity equations in Seiberg-Witten theory different from (1) have appeared in the literature (in particular, see [16]). The incorrectness of these versions can be deduced from their lack of covariance with respect to electric-magnetic duality; hence it should not come as a surprise that they have meanwhile been rejected on other grounds as well.

The issue of the metric that appears in the associativity equations remains a confusing one. We have already stressed that the special Kähler metric has nothing in common with the “metric” in the context of the 2D topological theory that underlies the original WDVV equations. The latter is related to the third derivative of some function and can be chosen constant. Note that the extra condition of the constancy is not preserved under duality, so that duality seems to take us out of the class of topological solutions in the sense of [13]. The first metric, on the other hand, is related to the second derivative and it is non-holomorphic and transforms non-holomorphically under duality (cf. (20)). In contradistinction with the above, the metric in the context of the associativity equations (1) is clearly non-constant and holomorphic. To clarify this issue is obviously relevant.
Our result goes beyond Seiberg-Witten theory because it applies to all cases where the WDVV equations are valid, irrespective of whether one can identify proper arguments for the relevance of electric-magnetic duality for the cases at hand. All Seiberg-Witten solutions are related to integrable systems, where the function $\mathcal{F}$ is the logarithm of the tau-function of the universal Whitham hierarchy (restricted to a finite set of variables). However, not all tau-functions correspond to Seiberg-Witten solutions (as far as we know), and some of those nevertheless satisfy the WDVV equations (1). Hence, by applying duality transformations we obtain other tau-functions satisfying the WDVV equations, without having an a priori understanding as to why the duality constitutes an equivalence relation for these tau-functions. Duality transformations with $U^t \cdot W = Z^t \cdot V = 0$, where (16) takes the form of a (partial) Legendre transform, may be of particular importance in the context of the Whitham hierarchies.

Yet another issue concerns the relation of WDVV equations in Seiberg-Witten theory with the geometry of moduli spaces of Riemann surfaces and integrable systems. Certainly dual period matrices are not distinguishable from the point of view of the geometry of complex curves. They are equivalent and the corresponding equivalence of the associativity equations is a consequence of this fact. On the other hand, it is well-known (see, for example, [13, 17]) that when two different functions $F(a)$ and $\tilde{F}(\tilde{a})$ satisfy

$$\frac{\partial^2 F}{\partial a_i \partial a_j} = \frac{\partial^2 \tilde{F}}{\partial \tilde{a}_i \partial \tilde{a}_j},$$

and $F(a)$ is a solution to WDVV equations, then $\tilde{F}(\tilde{a})$ is trivially a solution to the same equations. In this note we extended this equivalence to the case of functions whose second derivatives (i.e. their period matrix) are related by duality transformations. Observe that, while representing the same geometry and belonging to the same equivalence class, the two functions which solve the WDVV equations are in general completely different as functions depending on their respective arguments.

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References

[1] N. Seiberg and E. Witten, Nucl.Phys. B426 (1994) 19, hep-th/9407087; Nucl.Phys. B431 (1994) 484, hep-th/9408099.

[2] B. de Wit, Electric-magnetic Dualities in Supergravity, proc. Thirty Years of Supersymmetry, October 2000, to appear in Nucl. Phys. Proc. Suppl., hep-th/0103081.
[3] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B355 (1995) 466, hep-th/9505035.

[4] I. Krichever, Comm. Pure Appl. Math. 47 (1994) 437, hep-th/9205110.

[5] A. Marshakov, Seiberg-Witten theory and integrable systems, World Scientific, 1999.

[6] E. Witten, Nucl. Phys. B340 (1990) 281; R. Dijkgraaf, H. Verlinde and E. Verlinde, Nucl. Phys. B352 (1991) 59.

[7] A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B389 (1996) 43, hep-th/9607103.

[8] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. 344B (1995) 169; hep-th/9411048; hep-th/9412158.

[9] A. Marshakov, A. Mironov and A. Morozov, Int. J. Mod. Phys. A15 (2000) 1157, hep-th/9701123;

[10] L.K. Hoevenaars, P.H.M. Kersten and R. Martini, Phys. Lett. B503 (2001) 189, hep-th/0012133.

[11] B. de Wit, B. Kleijn and S. Vandoren, Fortsch. Phys. 47 (1999) 317-323, hep-th/9801039.

[12] Y. Ohta, J. Math. Phys. 41 (2000) 6042-6047, hep-th/9905120.

[13] B. Dubrovin, Geometry of 2-D topological field theories, in: Integrable Systems and Quantum Groups (Montecatini Terme, 1993), Lecture Notes in Math. 1620, Springer, Berlin, 1996, 120-348, hep-th/9407018.

[14] B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89; S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. A4 (1989) 89.

[15] B. de Wit, F. Vandehey PEN and A. Van Proeyen, Nucl. Phys. B400 (1993) 463, hep-th/9210068.

[16] G. Bonelli and M. Matone, Phys. Rev. Lett. 77 (1996) 4712, hep-th/9605090.

[17] A. Mironov and A. Morozov, Phys. Lett. B424 (1998), 48, hep-th/9712177.