Separability of the Hamilton–Jacobi and Klein–Gordon equations in Kerr–de Sitter metrics

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Abstract
We study separability of the Hamilton–Jacobi and massive Klein–Gordon equations in the general Kerr–de Sitter spacetime in higher dimensions. Complete separation of both equations is carried out in $2n + 1$ spacetime dimensions with all $n$ rotation parameters equal, in which case the rotational symmetry group is enlarged from $(U(1))^n$ to $U(n)$. We explicitly construct the additional Killing vectors associated with the enlarged symmetry group which permit separation. We also derive first-order equations of motion for particles in these backgrounds and examine some of their properties.

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1. Introduction

Solutions of the vacuum Einstein equations describing black-hole solutions in higher dimensions are currently of great interest. This is mainly due to a number of recent developments in high energy physics. Models of spacetimes with large extra dimensions have been proposed to deal with several questions arising in modern particle phenomenology (e.g., the hierarchy problem) [1–3]. These models allow for the existence of higher dimensional black holes which can be described classically. Also of interest in these models is the possibility of mini black-hole production in high energy particle colliders which, if they occur, provide a window into non-perturbative gravitational physics [4, 5].

Superstring and M-theory, which call for additional spacetime dimensions, naturally incorporate black-hole solutions in higher dimensions (10 or 11). P-branes present in these theories can also support black holes, thereby making black-hole solutions in an intermediate number of dimensions physically interesting as well. Black-hole solutions in superstring theory are particularly relevant since they can be described as solitonic objects. They provide important keys to understanding strongly coupled non-perturbative phenomena which cannot be ignored at the Planck/string scale [6, 7].
Astrophysically relevant black-hole spacetimes are, to a very good approximation, described by the Kerr metric [8]. One generalization of the Kerr metric to higher dimensions is given by the Myers–Perry construction [9]. With interest now in a nonzero cosmological constant, it is worth studying spacetimes describing rotating black holes with a cosmological constant. Another motivation for including a cosmological constant is driven by the AdS/CFT correspondence. The study of black holes in an anti-de Sitter background could give rise to interesting descriptions in terms of the conformal field theory on the boundary leading to better understanding of the correspondence [10, 11]. The general Kerr–de Sitter metrics describing rotating black holes in the presence of a cosmological constant have been constructed explicitly in [12, 13].

In this paper, we study the separability of the Hamilton–Jacobi equation in these spacetimes, which can be used to describe the motion of classical massive and massless particles (including photons). We also investigate the separability of the Klein–Gordon equation describing a spinless field propagating in this background. For both equations, separation is possible in some special cases due to the enlargement of the dynamical symmetry group underlying these metrics. We construct the separation of both equations explicitly in these cases. We also construct Killing vectors, which exist due to the additional symmetry, and which permit the separation of these equations. We also derive and study equations of motion for particles in these spacetimes.

2. Construction and overview of the Kerr–de Sitter metrics

A remarkable property of the Kerr metric is that it can be written in the so-called Kerr–Schild [14] form, where the metric $g_{\mu\nu}$ is given exactly by its linear approximation around the flat metric $\eta_{\mu\nu}$ as follows:

$$ ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = \eta_{\mu\nu} \, dx^\mu \, dx^\nu + \frac{2M}{U} (k_\mu \, dx^\mu)^2, $$

(2.1)

where $k_\mu$ is null and geodesic with respect to both the full metric $g_{\mu\nu}$ and the flat metric $\eta_{\mu\nu}$.

The Kerr–de Sitter metrics in all dimensions are obtained in [12] by using the de Sitter metric instead of the flat background $\eta_{\mu\nu}$, with coordinates chosen appropriately to allow for the incorporation of the Kerr metric via the null geodesic vectors $k_\mu$. We quickly review the construction here.

We introduce $n = [D/2]$ coordinates $\mu_i$ subject to the constraint

$$ \sum_{i=1}^n \mu_i^2 = 1, $$

(2.2)

together with $N = [(D-1)/2]$ azimuthal angular coordinates $\phi_i$, the radial coordinate $r$ and the time coordinate $t$. When the total spacetime dimension $D$ is odd, $D = 2n + 1 = 2N + 1$, there are $n$ azimuthal coordinates $\phi_i$, each with period $2\pi$. If $D$ is even, $D = 2n = 2N + 2$, there are only $N = n - 1$ azimuthal coordinates $\phi_i$. Define $\epsilon$ to be 1 for even $D$, and 0 for odd $D$.

The Kerr–de Sitter metrics $ds^2$ in $D$ dimensions satisfy the Einstein equation

$$ R_{\mu\nu} = (D - 1) \lambda g_{\mu\nu}. $$

(2.3)

Define $W$ and $F$ as follows:

$$ W \equiv \sum_{i=1}^n \frac{\mu_i^2}{1 + \lambda a_i^2}, \quad F \equiv \frac{r^2}{1 - \lambda r^2} \sum_{i=1}^n \frac{\mu_i^2}{r^2 + a_i^2}. $$

(2.4)
In $D$ dimensions, the Kerr–de Sitter metrics are given by
\[ ds^2 = d\theta^2 + \frac{2M}{U} (k_\mu \, dx^\mu)^2, \] (2.5)
where the de Sitter metric $ds^2$, the null vector $k_\mu$, and the function $U$ are now given by
\[
d\bar{s}^2 = -W(1 - \lambda r^2) \, dt^2 + F \, dr^2 + \sum_{i=1}^{n-\epsilon} \frac{r^2 + a_i^2}{1 + \lambda a_i^2} \, d\mu_i^2 + \sum_{i=1}^{n-\epsilon} \frac{r^2 + a_i^2}{1 + \lambda a_i^2} \, d\phi_i^2 \]
\[ + \frac{\lambda}{W(1 - \lambda r^2)} \left( \sum_{i=1}^{n-\epsilon} \frac{(r^2 + a_i^2) \, d\mu_i \, d\mu_i}{1 + \lambda a_i^2} \right)^2, \] (2.6)
\[ k_\mu \, dx^\mu = W \, dt + F \, dr - \sum_{i=1}^{n-\epsilon} \frac{a_i \mu_i^2}{1 + \lambda a_i^2} \, d\phi_i, \] (2.7)
\[ U = r^n \sum_{i=1}^{n-\epsilon} \mu_i^2 \frac{1}{r^3 + a_i^2} \prod_{i=1}^{n-\epsilon} (r^2 + a_i^2). \] (2.8)

In the even-dimensional case, where there is no azimuthal coordinate $\phi_i$, there is also no associated rotation parameter; i.e., $a_n = 0$. Note that the null vector corresponding to the null 1-form is
\[ k^{\mu} \partial_\mu = -\frac{1}{1 - \lambda r^2} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} - \sum_{i=1}^{n-\epsilon} \frac{a_i}{r^2 + a_i^2} \frac{\partial}{\partial \phi_i}. \] (2.9)
This is easily obtained by using the background metric to raise and lower indices rather than the full metric, since $k$ is null with respect to both metrics.

For the purposes of analysing the equations of motion and the Klein–Gordon equation, it is very convenient to work with the metric expressed in Boyer–Lindquist coordinates. In these coordinates there are no cross terms involving the differential $dr$. In both even and odd dimensions, the Boyer–Lindquist form is obtained by means of the following coordinate transformation:
\[ dr = d\tau + \frac{2M}{(1 - \lambda r^2)(V - 2M)} \, d\phi = d\phi - \lambda a_i \, d\tau + \frac{2M a_i \, dr}{(r^2 + a_i^2)(V - 2M)}. \] (2.10)

In Boyer–Lindquist coordinates in $D$ dimensions, the Kerr–de Sitter metrics are given by
\[
d\bar{s}^2 = -W(1 - \lambda r^2) \, d\tau^2 + \frac{U \, dr^2}{V - 2M} + \frac{2M}{U} \left( \frac{1}{(r^2 + a_i^2)^2} \cdot \sum_{i=1}^{n-\epsilon} \frac{a_i \mu_i^2}{1 + \lambda a_i^2} \right)^2 \]
\[ + \sum_{i=1}^{n-\epsilon} \frac{r^2 + a_i^2}{1 + \lambda a_i^2} \, d\mu_i^2 + \sum_{i=1}^{n-\epsilon} \frac{r^2 + a_i^2}{1 + \lambda a_i^2} \, d\phi_i \left( \mu_i - \lambda a_i \, d\tau \right)^2 \]
\[ + \frac{\lambda}{W(1 - \lambda r^2)} \left( \sum_{i=1}^{n-\epsilon} \frac{(r^2 + a_i^2) \, d\mu_i \, d\mu_i}{1 + \lambda a_i^2} \right)^2, \] (2.11)
where $V$ is defined here by
\[ V = r^{n-2} \left(1 - \lambda r^2\right) \prod_{i=1}^{n-\epsilon} (r^2 + a_i^2) = \frac{U}{F}. \] (2.12)
Note that obviously $a_n = 0$ in the even-dimensional case, as there is no rotation associated with the last direction.
3. Obtaining the inverse metric

Note that the metric is block diagonal in the \((\mu_i)\) and the \((r, \tau, \varphi_i)\) sectors and so can be inverted separately.

To deal with the \((r, \tau, \varphi_i)\) sector, the most efficient method is to use the Kerr–Schild construction of the metric. From (2.1) and using the fact that \(k\) is null, we can write

\[
g^{\mu\nu} = \eta^{\mu\nu} - \frac{2M}{U} k^{\mu} k^{\nu},
\]

where \(\eta\) here is the de Sitter metric rather than the flat metric, and we raise and lower indices with \(\eta\). Since the null vector \(k\) has no components in the \(\mu_i\) sector, we can regard the above equation as holding true in the \((r, \tau, \varphi_i)\) sector with \(k\) null here as well. Then we can explicitly perform the coordinate transformation (2.10) (or rather its inverse) on the raised metric to obtain the components of \(g^{\mu\nu}\) in Boyer–Lindquist coordinates in the \((r, \tau, \varphi_i)\) sector.

We get the following components for the \((r, \tau, \varphi_i)\) sector of \(g^{\mu\nu}\):

\[
\begin{align*}
    g^{rr} &= g^{\tau r} = 0, \\
    g^{\tau\tau} &= \frac{V - 2M}{U}, \\
    g^{\varphi_i\varphi_j} &= \frac{4M^2}{U(1 - \lambda r^2)^2(V - 2M)}, \\
    g^{\varphi_i\varphi_j} &= \lambda a_i Q - \frac{4M^2 a_i (1 + \lambda a_i^2)}{U(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)} - \frac{2M a_i}{U(1 - \lambda r^2)(r^2 + a_i^2)}, \\
    g^{ij} &= \frac{(1 + \lambda a_i^2)}{(r^2 + a_i^2)\mu_i} \delta^{ij} + \lambda^2 a_i a_j Q + \frac{4M^2 a_i a_j (1 + \lambda a_i^2)(1 + \lambda a_j^2)}{U(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)(r^2 + a_j^2)},
\end{align*}
\]

where \(Q\) and \(Q^{ij}\) are defined to be

\[
\begin{align*}
    Q &= -\frac{1}{W(1 - \lambda r^2)} - \frac{2M}{U} \frac{1}{(1 - \lambda r^2)^2}, \\
    Q^{ij} &= \frac{-4M^2 \lambda a_i a_j \left[(1 + \lambda a_i^2)(r^2 + a_j^2) + (1 + \lambda a_j^2)(r^2 + a_i^2)\right]}{U(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)(r^2 + a_j^2)} \\
    &\quad - \frac{2M}{U} \frac{a_i a_j}{(r^2 + a_i^2)(r^2 + a_j^2)} - \frac{2M \lambda a_i a_j}{U(1 - \lambda r^2)} \left[\frac{1}{(r^2 + a_i^2)} + \frac{1}{(r^2 + a_j^2)}\right] \\
    &\quad + \frac{4M^2 a_i a_j [(1 + \lambda a_i^2) + (1 + \lambda a_j^2)]}{U(1 - \lambda r^2)^2(V - 2M)(r^2 + a_i^2)(r^2 + a_j^2)}.
\end{align*}
\]

These results were compared to previously known ones in the case of \(\lambda = 0\) and showed agreement [15]. Also, we used the GRTensor package for Maple explicitly to check that this is the correct inverse metric [16].

Note that both the functions \(W\) and \(U\) depend explicitly on the \(\mu_i\). Unless the \((r, \tau, \varphi_i)\) sector can be decoupled from the \(\mu\) sector, complete separation is unlikely. If however, all the \(a_i\) are equal, then the functions \(W\) and \(U\) are no longer \(\mu\) dependent (taking the constraint into account). With unequal values of the rotation parameters \(a_i\), separation does not seem to be possible in this coordinate system, and it is likely that a different coordinate system might be needed to analyse separability in those cases. We will consider the case where all rotation parameters are equal: \(a_i = a\). Then we explicitly show separability. Note that since \(a_n = 0\)...
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by definition for even-dimensional cases, we will restrict our attention to odd-dimensional spaces. In the discussions that follow, we explicitly set all rotation parameters equal, and assume that the spacetime dimensionality is odd.

Note that the \( \mu \) sector metric is completely diagonal upon assuming that the rotation parameters are equal and upon imposing the constraint. Consider the last term in equation (2.11) in the case of odd dimensions with all \( a_i = a \). In this case the term reads

\[
\frac{\lambda}{W(1 - \lambda r^2)} \frac{(r^2 + a^2)}{(1 + \lambda a^2)} \left( \sum_{i=1}^{n} \mu_i d\mu_i \right)^2.
\]

(3.5)

However, by differentiating the constraint (2.2) we get \( \sum_{i} \mu_i d\mu_i = 0 \). Hence upon imposing the constraint this term vanishes from the metric, and the corresponding term vanishes from the inverse metric (and thus in the Hamilton–Jacobi equation.)

Now that the \( \mu_i \) are constrained by (2.2); we can use independent coordinates. Since the constraint describes a unit \((n - 1)\) sphere in \( \mu \) space, the natural choice is to use spherical polar coordinates. We write

\[
\mu_i = \left( \prod_{j=1}^{n-i} \sin \theta_j \right) \cos \theta_{n-i+1},
\]

(3.6)

with the understanding that the product is 1 when \( i = n \) and that \( \theta_n = 0 \). The \( \mu \) sector metric can then be written as

\[
d\mathbf{s}_\mu^2 = \frac{r^2 + a^2}{1 + \lambda a^2} \sum_{i=1}^{n-1} \left( \prod_{j=1}^{i-1} \sin^2 \theta_j \right) d\theta_i^2,
\]

(3.7)

again with the understanding that the product is 1 when \( i = 1 \). This diagonal metric can be easily inverted to give

\[
g_{\theta\theta_i} = \frac{(1 + \lambda a^2)}{(r^2 + a^2) \left( \prod_{k=1}^{i-1} \sin^2 \theta_k \right)} \delta_{ij}.
\]

(3.8)

4. The Hamilton–Jacobi equation and separation

The Hamilton–Jacobi equation in a curved background is given by

\[
-\frac{\partial S}{\partial l} = H = \frac{1}{2} g_{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu},
\]

(4.1)

where \( S \) is the action associated with the particle and \( l \) is some affine parameter along the worldline of the particle. Note that this treatment also accommodates the case of massless particles, where the trajectory cannot be parametrized by proper time.

Using (3.2) and (3.8), we write the Hamilton–Jacobi equation in odd dimensions with all rotation parameters equal as

\[
-2 \frac{\partial S}{\partial l} = Q \left[ \frac{\partial S}{\partial \tau} + \lambda a \sum_{i=1}^{n} \frac{\partial S}{\partial \psi_i} \right]^2 + \frac{4M^2}{U(1 - \lambda r^2)^2(V - 2M)} \left[ \frac{\partial S}{\partial \tau} - a \frac{\partial S}{\partial \psi_i} \right]^2
\]

\[
- \frac{4M}{U(1 - \lambda r^2)(r^2 + a^2)} \sum_{i=1}^{n} \frac{\partial S}{\partial \tau} \frac{\partial S}{\partial \psi_i} + \frac{V - 2M}{U} \left( \frac{\partial S}{\partial \tau} \right)^2
\]

\[
- \frac{8M^2}{U(1 - \lambda r^2)^2(V - 2M)} \left( \frac{\partial S}{\partial \tau} \right)^2
\]
\begin{equation}
+ \sum_{i=1}^{n} Q^{ij} \frac{\partial S}{\partial \varphi_i} \frac{\partial S}{\partial \varphi_j} + \frac{(1 + \lambda a^2)}{(r^2 + a^2)} \sum_{i=1}^{n} \frac{1}{\mu_i} \left( \frac{\partial S}{\partial \varphi_i} \right)^2 \\
+ \frac{(1 + \lambda a^2)}{(r^2 + a^2)} \sum_{i=1}^{n-1} \frac{1}{\prod_{k=1}^{i-1} \sin^2 \theta_k} \left( \frac{\partial S}{\partial \theta_i} \right)^2 .
\end{equation}

Note that here the \( \mu_i \) are not coordinates, but simply notation defined by (3.6). The set of coordinates relevant to the problem is \((r, \varphi_i, \theta_i)\). Also note that the functions \( U, W, Q \) and \( Q^{ij} \) are all now independent of the \( \theta_i \); i.e., in the Hamilton–Jacobi equation, the \( r \) sector has completely decoupled from the \( \theta_i \) sector.

Now we can attempt a separation of coordinates as follows. Let

\begin{equation}
S = \frac{1}{2} m^2 l - E \tau + \sum_{i=1}^{n} L_i \varphi_i + S_r (r) + \sum_{i=1}^{n-1} S_{\theta_i} (\theta_i) .
\end{equation}

\( \tau \) and \( \varphi_i \) are cyclic coordinates, so their conjugate momenta are conserved. The conserved quantity associated with time translation is the energy \( E \), and those with rotation in the \( \varphi_i \) are the corresponding angular momenta \( L_i \), all of which are conserved. Applying this ansatz to (4.2), we can separate the overall \( \theta \) dependence as

\begin{equation}
J_i^2 = \sum_{i=1}^{n} \left[ \frac{L_i^2}{\prod_{k=1}^{i-1} \sin^2 \theta_k \cos^2 \theta_{n-i+1}} \right] + \sum_{i=1}^{n-1} \frac{1}{\prod_{k=1}^{i-1} \sin^2 \theta_k} \left( \frac{dS_{\theta_i}}{d\theta_i} \right)^2 ,
\end{equation}

where \( J_i^2 \) is a constant. The separated \( r \) equation is

\begin{equation}
K = m^2 (r^2 + a^2) + Q (r^2 + a^2) \left[ -E + \lambda a \sum_{i=1}^{n} L_i \right]^2 + \frac{4 MaE}{U(1 - \lambda r^2)} \sum_{i=1}^{n} L_i \\
+ \frac{4 M^2 (r^2 + a^2)}{U(1 - \lambda r^2)(V - 2M)} \left[ E + \frac{a(1 + \lambda a^2)}{r^2 + a^2} \sum_{i=1}^{n} L_i \right]^2 + \frac{(V - 2M)(r^2 + a^2)}{U} \\
\times \left[ \frac{dS_r}{dr} \right] - \frac{8 M^2 E^2 (r^2 + a^2)}{U(1 - \lambda r^2)(V - 2M)} + (r^2 + a^2) \sum_{i,j=1}^{n} Q^{ij} L_i L_j ,
\end{equation}

where this separation constant is \( K = -(1 + \lambda a^2) J_i^2 \). At this point the \((r, \tau, \varphi_i)\) coordinates have been separated. To show complete separation of the Hamilton–Jacobi equation we analyse the \( \theta \) sector (4.4).

The pattern here is that of a Hamiltonian of a classical (non-relativistic) particle on the unit \((n - 1)\) \( \mu \)-sphere, with some potential dependent on the squares of the \( \mu_i \). This can easily be additively separated following the usual procedure, one angle at a time, and the pattern continues for all integers \( n \geq 2 \).

The separation has the following inductive form for \( k = 1, \ldots, n - 2 \):

\begin{equation}
J_k^2 \sin^2 \theta_k = \frac{L_{n-k+1}^2 \sin^2 \theta_k}{\cos^2 \theta_k} \sin^2 \theta_k \left( \frac{dS_{\theta_k}}{d\theta_k} \right)^2 = J_{k+1}^2 ,
\end{equation}

\begin{equation}
J_{k+1}^2 = \sum_{i=k+1}^{n} \left( \frac{L_{n-i+1}^2}{\prod_{j=k+1}^{i} \sin^2 \theta_j} \right) \cos^2 \theta_i + \sum_{i=k+1}^{n-1} \frac{1}{\prod_{k=1}^{i-1} \sin^2 \theta_j} \left( \frac{dS_{\theta_i}}{d\theta_i} \right)^2 .
\end{equation}

The final step of separation gives

\begin{equation}
J_{n-1}^2 = \frac{L_{2}^2}{\cos^2 \theta_{n-1}} + \frac{L_{1}^2}{\sin^2 \theta_{n-1}} + \left( \frac{dS_{\theta_{n-1}}}{d\theta_{n-1}} \right)^2 .
\end{equation}
Thus, the Hamilton–Jacobi equation in odd-dimensional Kerr–de Sitter space with all rotation parameters $a_i = a$ has the general separation
\[ S = \frac{1}{2} m^2 l - E \tau + \sum_{i=1}^{n} L_i \psi_i + S_e (r) + \sum_{i=1}^{n-1} S_{\theta_i} (\theta_i), \]
where the $\theta_i$ are the spherical polar coordinates on the unit $(n - 1)$ sphere. $S_e (r)$ can be obtained by quadratures from (4.5), and the $S_{\theta_i}$ again by quadratures from (4.6) and (4.7).

5. The equations of motion

5.1. Derivation of the equations of motion

To derive the equations of motion, we will write the separated action $S$ from the Hamilton–Jacobi equation in the following form:
\[ S = \frac{1}{2} m^2 l - E \tau + \sum_{i=1}^{n} L_i \psi_i + \int R(r') \sqrt{r'} \, dr' + \sum_{i=1}^{n-1} \int \sqrt{\Theta_i (\theta_i')} \, d\theta_i', \]
where
\[ \Theta_k = J_k^2 - \frac{L_{n-k+1}^2}{\sin^2 \theta_k}, \quad k = 1, \ldots, n - 1, \]
\[ R = -J_1^2 \frac{(1 + \lambda a^2) U}{(V - 2M)(r^2 + a^2)} - \frac{QU}{(V - 2M)} \left[ -E + \lambda a \sum_{i=1}^{n} L_i \right]^2 \]
\[ - m^2 \frac{U}{(V - 2M)} - \frac{4M^2}{(1 - \lambda r^2)(V - 2M)^2} \left[ E + \frac{a(1 + \lambda a^2)}{r^2 + a^2} \sum_{i=1}^{n} L_i \right]^2 \]
\[ - \frac{4MaE}{(V - 2M)(r^2 + a^2)} \sum_{i=1}^{n} L_i - \frac{8M^2 E^2}{(1 - \lambda r^2)(V - 2M)^2} \]
\[ - \frac{U}{(V - 2M)} \sum_{i,j=1}^{n} Q^{ij} L_i L_j, \]
where $Q$ and $Q^{ij}$ are functions of $r$ given in (3.4) (with all $a_i = a$). For convenience, we define $J_n^2 = L_n^2$. (Note that $J_n^2$ is obviously not a new conserved quantity. It is simply written this way to facilitate the inductive definition given above for $\Theta_{n-1}$.)

To obtain the equations of motion, we differentiate $S$ with respect to the parameters $m^2, E, L_i, J_i^2$ and set these derivatives to equal other constants of motion. However, we can set all these new constants of motion to zero (following from freedom in the choice of origin for the corresponding coordinates, or alternatively by changing the constants of integration). Following this procedure, we get the following equations of motion:
\[ \frac{dr}{d\tau} = \frac{(V - 2M) \sqrt{R}}{U} \]
\[ \frac{d\theta_i}{d\tau} = \frac{(1 + \lambda a^2) \sqrt{\Theta_i}}{(r^2 + a^2) \left( \prod_{j=1}^{n} \sin^2 \theta_j \right)} \quad i = 1, \ldots, n - 1 \]
\[
\frac{d\tau}{dt} = 2Q(r^2 + a^2) \left( E + \lambda a \sum_{i=1}^{n} L_i \right) - \frac{4Ma}{U(1-\lambda r^2)} \sum_{i=1}^{n} L_i - \frac{8M^2(r^2 + a^2)}{(1-\lambda r^2)^2(V-2M)} \times \left( E + \alpha(1 + \lambda a^2) \sum_{i=1}^{n} L_i \right) + \frac{16M^2E(r^2 + a^2)}{U(1-\lambda r^2)(V-2M)}. \tag{5.4}
\]

We can obtain \(n\) more equations of motion which give the \(\frac{d\chi}{d\tau}\) in terms of the \(r, \theta_j\) coordinates by differentiating \(S\) with respect to the angular momenta \(L_i\). However, these equations are not particularly illuminating, but can be written explicitly if necessary by following this procedure.

### 5.2. Analysis of the radial equation

The worldline of particles in the Kerr–de Sitter backgrounds considered above are completely specified by the values of the conserved quantities \(E, L_i, J^2\), and by the initial values of the coordinates. We will consider particle motion in the black-hole exterior. Allowed regions of particle motion necessarily need to have positive value for the quantity \(R\), owing to equation (5.4). We determine some of the possibilities of the allowed motion.

At large \(r\), the dominant contribution to \(R\), in the case of \(\lambda = 0\), is \(E^2 - m^2\). Thus we can say that for \(E^2 < m^2\), we cannot have unbounded orbits, whereas for \(E^2 > m^2\), such orbits are possible. For the case of nonzero \(\lambda\), the dominant term at large \(r\) in \(R\) (or rather the slowest decaying term) is \(m^2/\sqrt{\lambda}\). Thus in the case of the Kerr–anti-de Sitter background, only bound orbits are possible, whereas in the Kerr–de Sitter backgrounds, both unbounded and bound orbits may be possible.

In order to study the radial motion of particles in these metrics, it is useful to cast the radial equation of motion into a different form. Decompose \(R\) as a quadratic in \(E\) as follows:

\[
R = \alpha E^2 - 2\beta E + \gamma, \tag{5.5}
\]

where

\[
\alpha = -\frac{QU}{V-2M} - \frac{4M^2}{(1-\lambda r^2)^2(V-2M)^2} - \frac{8M^2}{(1-\lambda r^2)(V-2M)^2},
\]

\[
\beta = \left( \frac{QU\lambda a}{V-2M} + \frac{4M^2a(1+\lambda a^2)}{(1-\lambda r^2)^2(V-2M)^2(r^2 + a^2)} + \frac{2Ma}{(V-2M)(r^2 + a^2)} \right) \sum_{i=1}^{n} L_i,
\]

\[
\gamma = \frac{J^2(1 + \lambda a^2)U}{(V-2M)(r^2 + a^2)} - \frac{QU\lambda^2 a^2}{V-2M} \left( \sum_{i=1}^{n} L_i \right)^2 - \frac{M^2 U}{V-2M}
\]

\[
- \frac{4M^2a^2(1+\lambda a^2)^2}{(1-\lambda r^2)^2(V-2M)^2(r^2 + a^2)^2} \left( \sum_{i=1}^{n} L_i \right)^2 - \frac{U}{V-2M} \sum_{ij=1}^{n} Qij L_i L_j. \tag{5.6}
\]

The turning points for trajectories in the radial motion (defined by the condition \(R = 0\)) are given by \(E = V_\pm\) where

\[
V_\pm = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha \gamma}}{\alpha}. \tag{5.7}
\]

These functions, called the effective potentials [15], determine allowed regions of motion. In this form, the radial equation is much more suitable for detailed numerical analysis for specific values of parameters.
5.3. Analysis of the angular equations

Another class of interesting motions possible describes motion at a constant value of $\theta_i$. These motions are described by the simultaneous equations

$$\frac{\Theta_i}{\sin^4 \theta_i} \frac{d\Theta_i}{d\theta_i} (\theta_i = \theta_i) = 0,$$

where $\theta_i$ is the constant value of $\theta_i$ along this trajectory. These equations can be explicitly solved to give the relations

$$J_{2i}^2 + 1 \sin^4 \theta_i = L_{n-i-i}^2 - i - 1 \cos^4 \theta_i,$$

$$J_i^2 = J_{2i}^2 + 1 \sin^2 \theta_i + L_{n-i-i}^2 - i + 1 \cos^2 \theta_i, \quad i = 1, \ldots, n - 1,$$

where, as before, $J_{2n}^2 = L_{1}^2$. Note that if $\theta_i = 0$, then $J_{2+n}^2 = 0$, and if $\theta_i = \pi/2$, then $L_{n-1-i}^2 = 0$.

Exercising $\Theta_k$ in the general case, $\theta_k = 0$ can only be reached if $J_k^2 = 0$, and $\theta_k = \pi/2$ can only be reached if $L_{n-k+1}^2 = 0$. The orbit will be completely in the subspace $\theta_i = 0$ only if $J_i^2 = L_{n-i+1}^2$, and will be completely in the subspace $\theta_i = \pi/2$ only if $J_i^2 = J_{2+i}^2$.

Again these equations are in a form suitable for numerical analysis for specific values of the black hole and particle parameters.

6. Dynamical symmetry

The general class of metrics discussed here is stationary and ‘axisymmetric’; i.e., $\partial/\partial \tau$ and $\partial/\partial \phi_i$ are Killing vectors and have associated conserved quantities, $-E$ and $L_i$. In general if $\xi$ is a Killing vector, then $\xi^\mu p_\mu$ is a conserved quantity, where $p$ is the momentum. Note that this quantity is first order in the momenta.

With the assumption of odd dimensions and equality of all the $a_i$, the spacetime acquires additional dynamical symmetry and more Killing vectors are generated. By setting the rotation parameters $a_i$ equal, we have complete symmetry between the various planes of rotation, and we can ‘rotate’ one into another. The vectors that generate these transformations are the required Killing vectors. We will construct these explicitly. Parametrize the rotation planes as follows:

$$x_i = r \mu_i \cos \phi_i = r \left( \prod_{j=1}^{n-i-j} \sin \theta_j \right) \cos \theta_{n-i+1} \cos \phi_i,$$

$$y_i = r \mu_i \sin \phi_i = r \left( \prod_{j=1}^{n-i-j} \sin \theta_j \right) \cos \theta_{n-i+1} \sin \phi_i,$$

again with the understanding that the product equals 1 when $i = n$ and that $\theta_n = 0$.

Define the rotation generators on the planes as

$$L_{ab} = a \partial_b - b \partial_a,$$

where $a$ and $b$ can be any $x^i$ or $y^i$. The case of $a = x^i$, $b = y^i$ for the same $i$ is not interesting, as it simply represents rotation in $\phi_i$, which is already known to generate a Killing vector. The $L_{ab}$ themselves are obviously not Killing vectors (apart from the trivial cases just mentioned), but the combinations

$$\xi_{ij} = L_{x^i y^j} + L_{y^i y^j}, \quad \rho_{ij} = L_{x^i y^j} + L_{x^j y^i},$$

(6.3)
are Killing vectors. Explicit expressions for these in polar coordinates in the case of \( n = 2 \) can be found in [17, 18].

These additional Killing vectors exist, since the symmetry of the spacetime has been greatly enhanced by the equality of the rotation parameters. The \((U(1))^n\) spatial rotation symmetry, where each \( U(1) \) is the rotational symmetry in one of the planes, has been increased to a \( U(n) \) symmetry. This follows from the fact that we now have the additional symmetry of being able to rotate planes into one another.

The separation constants \( K \) in (4.5) and \( J_i^2 \) in (4.4) are conserved quantities that are quadratic in the associated momenta. So these quantities must be derived from a rank two Killing tensor \( K_{\mu\nu} \) [19]. We will work with the \( J_i^2 \). (We can ignore \( K \) since it only differs from \( J_i^2 \) by a constant factor.) Any conserved quantity \( A \) that is second order in momenta is constructed from a Killing tensor as

\[
A = K_{\mu\nu} p_\mu p_\nu = K_{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}.
\]  

(6.4)

Since the Hamilton–Jacobi equation can be fully separated, we should be able to construct Killing tensors explicitly. It turns out however that these Killing tensors are not irreducible; i.e., they can be constructed as linear combinations of tensor products of the Killing vectors present due to the increased symmetry.

Comparing (4.4), (4.6) and (4.7) with (6.4), where the conserved quantities are \( J_i^2 \), we can obtain the following Killing tensors:

\[
K_{\mu\nu}^{n-1} = \frac{1}{\sin^2 \theta_{n-1}} \delta_\mu^{\phi_1} \delta_\nu^{\phi_1} + \frac{1}{\cos^2 \theta_{n-1}} \delta_\mu^{\phi_2} \delta_\nu^{\phi_2} + \delta_\mu^{\theta_1} \delta_\nu^{\theta_1},
\]

\[
K_{\mu\nu}^k = \frac{1}{\sin^2 \theta_k} K_{\mu\nu}^{k+1} + \frac{1}{\cos^2 \theta_k} \delta_\mu^{\phi_{k+1}} \delta_\nu^{\phi_{k+1}} + \delta_\mu^{\theta_{k+1}} \delta_\nu^{\theta_{k+1}}, \quad k = 1, \ldots, n - 2,
\]

(6.5)

which can be written as

\[
K_{\mu\nu}^{n-1} = \sum_{i=1}^{k+1} \partial_{\phi_i} \otimes \partial_{\phi_i} - \sum_{i=1}^{k+1} \sum_{j=1}^{i-1} \text{sym}(\partial_{\phi_i} \otimes \partial_{\phi_j}) + \sum_{i=1}^{k+1} \sum_{j=1}^{i-1} \partial_{\xi_{ij}} \otimes \partial_{\xi_{ij}} + \sum_{i=1}^{k+1} \sum_{j=1}^{i-1} \partial_{\rho_{ij}} \otimes \partial_{\rho_{ij}},
\]

\[
k = 1, \ldots, n - 1,
\]

(6.6)

where \( J_i^2 = K_{i\mu}^{\rho_{\mu} \rho_{\nu}} \).

Therefore, as we can see from the form of the Killing tensors, they can explicitly be obtained from quadratic combinations of the Killing vectors \( \partial_{\phi_i}, \xi_{ij} \) and \( \rho_{ij} \).

This is a demonstration of the fact that in this case separation of the Hamilton–Jacobi equation is possible due to the enlargement of the symmetry group in the case of all \( a_i = a \).

7. The scalar field equation

Consider a scalar field \( \Psi \) with the action

\[
S[\Psi] = -\frac{1}{2} \int d^Dx \sqrt{-g}((\nabla \Psi)^2 + \alpha R \Psi^2 + m^2 \Psi^2),
\]

(7.1)

where we have included a curvature-dependent coupling. However, in the Kerr–(anti) de Sitter background, \( R = \lambda \) is constant. As a result we can trade off the curvature coupling for a different mass term. So it is sufficient to study the massive Klein–Gordon equation in this background. We will simply set \( \alpha = 0 \) in the following. Variation of the action leads to the Klein–Gordon equation

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) = m^2 \Psi.
\]

(7.2)
As discussed by Carter [20], the assumption of separability of the Klein–Gordon equation usually implies separability of the Hamilton–Jacobi equation. Conversely, if the Hamilton–Jacobi equation does not separate, the Klein–Gordon equation seems unlikely to separate. We can also see this explicitly (as in the case of the Hamilton–Jacobi equation), since the Jacobi equation does not separate, the Klein–Gordon equation seems unlikely to separate. We usually implies separability of the Hamilton–Jacobi equation. Conversely, if the Hamilton–Jacobi equation does not separate, the Klein–Gordon equation seems unlikely to separate. We calculate the determinant of the metric to be

$$ g = - \frac{r^2(r^2 + a^2)^{2n-2}}{(1 + \lambda a^2)^{2n}} \prod_{j=1}^{n-1} \sin^{4n-4j-2} \theta_j \cos^2 \theta_j. $$

(7.3)

For convenience we write

$$ P = \frac{r^2(r^2 + a^2)^{2n-2}}{(1 + \lambda a^2)^{2n}}, \quad A = \prod_{j=1}^{n-1} \sin^{4n-4j-2} \theta_j \cos^2 \theta_j. $$

(7.4)

Then the Klein–Gordon equation in this background (7.2) becomes

$$ m^2 \Psi = Q \left[ \frac{\partial}{\partial \tau} + \lambda a \sum_{i=1}^{n} \frac{\partial}{\partial \varphi_i} \right]^2 \Psi + \frac{4M^2}{U(1 - \lambda r^2)(V - 2M)} \left[ \frac{\partial}{\partial \tau} - \frac{a(1 + \lambda a^2)}{r^2 + a^2} \sum_{i=1}^{n} \frac{\partial}{\partial \varphi_i} \right]^2 \Psi $$

$$ + \frac{4M}{U(1 - \lambda r^2)(r^2 + a^2)} \sum_{i=1}^{n} \frac{\partial^2 \Psi}{\partial \tau \partial \varphi_i} - \frac{8M^2}{U(1 - \lambda r^2)(V - 2M)} \left( \frac{\partial^2 \Psi}{\partial \tau^2} \right) $$

$$ + \sum_{i,j=1}^{n} Q^{ij} \frac{\partial^2 \Psi}{\partial \varphi_i \partial \varphi_j} + \left( \frac{1 + \lambda a^2}{r^2 + a^2} \right) \sum_{i=1}^{n} \frac{1}{\mu_i^2} \left( \frac{\partial^2 \Psi}{\partial \varphi_i^2} \right) $$

$$ + \frac{1}{\sqrt{P}} \partial_r \left( \sqrt{P} \frac{(V - 2M)}{U} \partial \Psi \right) + \frac{1}{\sqrt{A}} \sum_{i,j=1}^{n} \partial_{\theta_i} \left( \sqrt{A} g^{\theta_i \theta_j} \frac{\partial \Psi}{\partial \theta_j} \right). $$

(7.5)

We attempt the usual multiplicative separation for $\Psi$ in the following form:

$$ \Psi = e^{-iE \tau} e^{i \sum_{1}^{n} L_i \varphi_i(\theta_1, \ldots, \theta_{n-1})} \Phi_{r}(r). $$

(7.6)

Then the Klein–Gordon equation simplifies to give the following ordinary differential equation in $r$ for $\Phi_{r}(r)$:

$$ m^2 \Phi_{r} = - Q \left[ E - \lambda a \sum_{i=1}^{n} L_i \right]^2 \Phi_{r} - \frac{4M^2}{U(1 - \lambda r^2)(V - 2M)} \left[ E + \frac{a(1 + \lambda a^2)}{r^2 + a^2} \sum_{i=1}^{n} L_i \right]^2 \Phi_{r} $$

$$ - \frac{4MaE}{U(1 - \lambda r^2)(r^2 + a^2)} \sum_{i=1}^{n} L_i \Phi_{r} + \frac{8M^2E^2}{U(1 - \lambda r^2)(V - 2M)} \Phi_{r} $$

$$ - \sum_{i,j=1}^{n} Q^{ij} L_i L_j \Phi_{r} + \frac{1}{\sqrt{P}} \frac{d}{dr} \left( \sqrt{P} \frac{(V - 2M)}{U} \frac{d \Phi_{r}}{dr} \right) + \left( \frac{1 + \lambda a^2}{r^2 + a^2} \right) K_{1} \Phi_{r}. $$

(7.7)

We have separated all the $\theta_i$ dependence into the separation constant $K_1$ given by

$$ K_1 = \frac{1}{\Psi_0} \sum_{i=1}^{n} \left[ - \frac{L_i^2}{\mu_i^2} \right] + \sum_{i=1}^{n-1} \frac{1}{\Psi_0 \sqrt{A}} \partial_{\theta_i} \left( \sqrt{A} g^{\theta_i \theta_j} \frac{\partial \Psi}{\partial \theta_j} \right), $$

(7.8)
where we have used the fact that $g^{\theta\theta}$ is diagonal, and that the $\mu_i$ are functions of the $\theta_j$ given by (3.6).

Equation (7.7) separates the $r$ dependence of the Klein–Gordon equation, and gives the function $\Phi_\mu(r)$ when the differential equation is solved. We can also completely separate the $\theta_i$ sector. Again, assume a multiplicative separation of the form

$$\Psi_\theta = \Phi_\theta(\theta_1) \ldots \Phi_{\theta_{n-1}}(\theta_{n-1}).$$

The $\theta$ separation then reads

$$K_1 = \sum_{i=1}^{k-1} A_i + \frac{K_k}{\prod_{j=1}^{k-1} \sin^2 \theta_j}, \quad k = 1, \ldots, n - 1, \quad (7.10)$$

where

$$A_i = \frac{1}{\Phi_{\theta_i} \cos \theta_i \sin^{2n-2i-1} \theta_i \prod_{k=i}^{i-1} \sin^2 \theta_j} \frac{d}{d \theta_i} (\cos \theta_i \sin^{2n-2i-1} \theta_i \frac{d \Phi_{\theta_i}}{d \theta_i})$$

$$= \frac{L_{n-k+1}^2}{\cos^2 \theta_i \prod_{j=1}^{n-k-1} \sin^2 \theta_j}. \quad (7.11)$$

Then we inductively have the complete separation of the $\theta_i$ dependence as

$$K_k = \frac{K_{k+1}}{\sin^2 \theta_k} \left( \frac{L_{n-k+1}^2}{\cos^2 \theta_k} + \frac{1}{\Phi_{\theta_k} \sin^2 \theta_k} \frac{d}{d \theta_k} (\cos \theta_k \sin \theta_k \frac{d \Phi_{\theta_k}}{d \theta_k}) \right), \quad (7.12)$$

where $k = 1, \ldots, n - 1$, and we use the convention $K_n = -L_1^2$.

As a result we can write the complete separation of the Klein–Gordon equation (7.5) in the Kerr–de Sitter background in odd dimensions with all rotation parameters equal as

$$\Psi = e^{-iEt} e^{i \sum \lambda_i \Phi_{\theta_i}(\theta_1) \ldots \Phi_{\theta_{n-1}}(\theta_{n-1}) \Phi_\mu(r)}, \quad (7.13)$$

where $\Phi(r)$ is obtained from (7.7), and the $\Phi_{\theta_i}$ are the decomposition of the $\mu$ sector into eigenmodes in independent coordinates $\theta_i$ on the $\mu$ sphere.

Note that the separation of the Klein–Gordon equation in this geometry is again due to the fact that the symmetry of the space has been enlarged. (We can explicitly see the role of the Killing vectors again in the separation of the $r$ equation from the $\theta$ sector in a very similar fashion to that in the Hamilton–Jacobi equation [20]).

8. Conclusions

We studied the separability properties of the Hamilton–Jacobi and the Klein–Gordon equations in the Kerr–de Sitter backgrounds. Separation in Boyer–Lindquist coordinates seems to be possible only for the case of an odd number of spacetime dimensions with all rotation parameters equal. This is possible due to the enlarged dynamical symmetry of the spacetime. We derive expressions for the Killing vectors that correspond to the additional symmetries. We also show that integrals of motion are obtained from reducible Killing tensors, which
are themselves constructed from the angular Killing vectors. Thus, we demonstrate the separability of the Hamilton–Jacobi and the Klein–Gordon equations as a direct consequence of the enhancement of symmetry. We also derive first-order equations of motion for classical particles in these backgrounds, and analyse the properties of some special trajectories.

Future work in this direction could include finding a suitable coordinate system to permit possible separation in an even number of spacetime dimensions. Different coordinates might also be required to study the cases of unequal rotation parameters, since separation does not seem likely in Boyer–Lindquist coordinates. The study of higher-spin field equations in these backgrounds could also prove to be of great interest, particularly in the context of string theory. Explicit numerical study of the equations of motion for specific values of the black-hole parameters could lead to interesting results.

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