I. INTRODUCTION

Random matrix theories have been successfully applied to the energy spectra of not only chaotic systems but also quantum spin systems. If a given Hamiltonian is integrable by the Bethe ansatz, the level-spacing distribution should be described by the Poisson distribution:

$$P_{\text{Poi}}(s) = \exp(-s).$$

If it is not integrable, the level-spacing distribution should be given by the Wigner distribution:

$$P_{\text{Wig}}(s) = \pi s \exp\left(-\frac{\pi s^2}{4}\right).$$

These behaviors of the level-spacing distribution have been observed in one-dimensional (1D) and 2D quantum spin systems. They are also confirmed for strongly correlated systems and applied to the recent study of quantum dots. The numerical observations are important. In fact, for the quantum spin systems, there has been no theoretical or analytical derivation of the suggested behaviors of the level-spacing distribution. Random matrix theories do not necessarily extend to them. However, it seems that no counterexample has been shown explicitly for quantum spin systems. The suggested behaviors have been numerically confirmed for quite a large number of quantum spin systems and statistical lattice systems. Moreover, the behaviors are often considered as empirical rules, by which we can practically determine whether a given lattice system is integrable or not. Thus, it could be nontrivial if one would find such a quantum spin system that does not show the standard behaviors of the level-spacing distribution.

Recently, it has been found that an extraordinary symmetry appears for special cases of the integrable XXZ spin chain: the XXZ Hamiltonian commutes with the $sl_2$ loop algebra at some particular values of the XXZ coupling associated with roots of unity. The loop algebra is an infinite-dimensional Lie algebra, which holds only at the particular values of the XXZ coupling and not for generic values. It is suggested numerically that the standard Bethe ansatz does not hold for the special cases. Furthermore, the dimensions of degenerate eigenspaces of the loop algebra are given by some exponential functions of the system size and they can be extremely large. It should therefore be nontrivial how the large degeneracies are resolved by nonintegrable perturbative terms in the spin Hamiltonian. Thus, the $sl_2$ loop algebra symmetry may motivate us to reconsider not only the Bethe ansatz solvability of the integrable XXZ chain but also the standard statistical behaviors of energy levels for various XXZ chains close to the integrable one. We note that some level crossings of the $sl_2$ loop algebra are shown in Fig. 1 of Ref. [15].

In this paper, we discuss level-spacing distributions for the spin $\frac{1}{2}$ XXZ chains on finite sites under periodic boundary conditions. We mainly discuss nonintegrable cases, although the present study has been originally motivated by the $sl_2$ loop algebra symmetry of the integrable XXZ chain. We first confirm that integrable spin chains show the Poisson distribution for generic cases. Here we exclude the special cases related to roots of unity. Then, we consider the next-nearest-neighbor (NNN) coupled Heisenberg spin chain (or XXX spin chain), which is nonintegrable. We now note that the XXX chain with NNN couplings has the Wigner distribution, as shown in Ref. [3]. For the XXZ spin chains with NNN couplings, however, we find that the level-spacing distributions are not given by the standard Wigner distribution. The observation should be nontrivial since the systems are nonintegrable. Finally, we discuss possible reasons why the non-Wigner distribution is obtained for the NNN coupled XXZ spin chains.
II. NUMERICAL PROCEDURE

The Hamiltonian matrices can be separated into some sectors; in each sector, the eigenstates have the same quantum numbers. This is because the eigenvalues with different symmetries are uncorrelated. The XXZ chains have three trivial symmetries, namely, reflections, translations, and spin rotations around the $z$ axis. Therefore, to desymmetrize the Hamiltonians, we consider three quantum numbers: parities, momenta, and total $S^z$. We calculate the eigenvalues of the largest sectors. The largest sector of each Hamiltonian has 440 eigenvalues for the lattice size $L = 16$. To calculate the eigenvalues, we use standard numerical methods, which are contained in the LAPACK library.

To find universal statistical properties of the Hamiltonians, one has to deal with unfolded eigenvalues instead of raw eigenvalues. The unfolded eigenvalues are renormalized values, whose local density of states is equal to unity everywhere in the spectrum. In this paper, the unfolded eigenvalues are obtained from the raw eigenvalues. The unfolded eigenvalues are renormalized on

$$ n(E) = \sum_{i=1}^{N} \theta(E - E_i). \quad (3) $$

Here $\theta(E)$ is the step function and $N$ is the number of the eigenvalues. We choose some points of coordinates: $(E_i, n(E_i))$ for $i = 1, 21, 41, \ldots, N$. The average of integrated density of states $\langle n(E) \rangle$ is approximated by the spline interpolation through the chosen points. The unfolded eigenvalues are defined as

$$ x_i = \langle n(E_i) \rangle. \quad (4) $$

The level-spacing distributions are given by the probability function $P(s)$, where $s = x_{i+1} - x_i$.

III. NEXT-NEAREST-NEIGHBOR COUPLED XXZ CHAIN

Let us introduce the spin $1/2$ XXZ chain with NNN couplings on $L$ sites by

$$ H = J_1 \sum_{i=1}^{L} (S^x_i S^x_{i+1} + S^y_i S^y_{i+1} + \Delta_1 S^z_i S^z_{i+1}) \quad (5) $$

$$ + J_2 \sum_{i=1}^{L} \left[ \alpha \left( S^x_i S^x_{i+2} + S^y_i S^y_{i+2} \right) + \Delta_2 S^z_i S^z_{i+2} \right], $$

where $S^a = (1/2)\sigma^a$ and $(\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices; periodic boundary conditions are imposed. For simplicity, we put $J_1 = 1$ hereafter in the paper. The Hamiltonian is nonintegrable for $J_2 \neq 0$, while it is integrable for $J_2 = 0$.

Let us confirm the Poissonian behavior for the generic case of the integrable XXZ spin chain. When $J_2 = 0$, the level-spacing distribution $P(s)$ mostly shows the Poisson distribution as shown in Fig. 1. We confirmed numerically the standard result for some generic values of the XXZ coupling $\Delta_1$ ($0 \leq \Delta_1 < 1$). We note that we exclude the special values of the XXZ coupling that are given by $\Delta_1 = \cos(m\pi/N)$ for some integers $m$ and $N$, where $\Delta_1$ is related to a root of unity, $q$, through the relation $\Delta_1 = (q + 1/q)/2$.

In nonintegrable cases ($J_2 \neq 0$), however, $P(s)$ shows a non-Wigner distribution against expectations. The non-Wigner behavior of the NNN coupled XXZ spin chains is generic. We have calculated $P(s)$ for various combinations of $J_2$ and $\Delta_2$ when $\alpha = 1$ and for various values of $\Delta_2$ when $\alpha = 0$. Here, $\Delta_1 = 1$ for simplicity. When $\alpha = 1$ and $\Delta_2 = 0.98$, $P(s)$ looks similar to the Poisson distribution rather than the Wigner distribution for any value of $J_2$, as shown in Figs. 2(a) and 2(c). The Hamiltonian is close to but not exactly the same as the Heisenberg chain, when $\Delta_2 = 0.98$. We find such a Poisson-like distribution when $\Delta_2$ is very close to 1.0 such as more than about 0.95. For the NNN coupled Heisenberg chain ($\Delta_2 = 1$), however, it was shown that $P(s)$ has the Wigner distribution.

When $\Delta_2$ decreases, $P(s)$ becomes a non-Poisson distribution as shown in Figs. 2(b) and 2(d). However, it is not a Wigner-like distribution, either. The level-spacing distribution $P(s)$ strongly depends on $\Delta_2$. On the other hand, it seems that $P(s)$ is almost independent of $J_2$.

When only one parameter is nonzero, namely, for the case of either $\alpha = 0$ or $\Delta_2 = 0$, the nonzero parameter does not change $P(s)$ very much. In that case, $P(s)$ is given by neither a Poisson-like nor Wigner-like distribution.

For all the investigated nonintegrable cases except for the case where $\Delta_2$ is close to 1.0, $P(s)$ can be approximated by the arithmetic average of the Poisson
and Wigner distributions:

\[ P_{\text{av}}(s) = \frac{P_{\text{Poi}}(s) + P_{\text{Wig}}(s)}{2}, \tag{6} \]

as shown in Fig. 3.

Let us remark on the special case of the integrable XXZ chain related to a root of unity. At the special values of \( \Delta_1 \), we find that \( P(s) \) shows a novel peak at \( s = 0 \) as shown in Fig. 4. For \( L = 16 \), large degeneracies remain for \( \Delta_1 = \cos(\pi/i), \ i = 2, 3, 4 \), even after the desymmetrization procedure mentioned in Sec. III is performed. We recall that for generic values of \( \Delta_1 \), no degeneracy remains after the desymmetrization procedure is completed and the Poisson distribution is obtained as shown in Fig. 1. The appearance of the peak in \( P(s) \) at \( s = 0 \) should be consistent with the \( sl_2 \) loop algebra symmetry, which holds for the transfer matrices of the XXZ and XYZ spin chains only at the special values of \( \Delta_1 \). The result could be related to the observation in Ref. 3 that the eigenvalue spacing distribution \( P(s) \) has a peak at \( s = 0 \) for the transfer matrix of the eight-vertex model under the free-fermion conditions. Details should be discussed in forthcoming papers.

**IV. DISCUSSION**

Let us discuss some possible reasons why the non-integrable models give the non-Wigner distributions. We may consider two reasons: extra symmetries or finite-size effects.

The non-Wigner distributions, such as shown in Figs. 2 and 3, suggest that the Hamiltonian should have some extra symmetries. It must be a possible standard interpretation according to Refs. 4 and 8. However, for the NNN coupled XXZ chains, it is not clear whether there indeed exist some extra symmetries other than reflections, translations, and spin rotations around the \( z \) axis.

We may also consider some finite-size effects, since the distribution \( P(s) \) is always obtained for some finite systems. All the calculations in this paper are performed on 16-site chains. Recall that the largest sector of the NNN coupled XXZ Hamiltonian has 440 eigenvalues. The number is not small. If the level-spacing distribution calculated on hundreds-site or thousands-site chains could give a Wigner-like distribution, then the finite-size effects should be nontrivial.
FIG. 3: Level-spacing distributions of NNN coupled chain for $L = 16$, $J_1 = \Delta_1 = J_2 = 1$, (a) $\alpha = 0$, $\Delta_2 = 1$; (b) $\alpha = 1$, $\Delta_2 = 0$. Broken lines, the Poisson distribution; dotted lines, the Wigner distribution; long dashed lines, the arithmetic average of the Poisson and Wigner distributions. There is no degeneracy for both (a) and (b) [$P(s) = 0$ at $s = 0$].

FIG. 4: Level-spacing distribution of integrable XXZ chain ($J_2 = 0$) for $L = 16$, $J_1 = 1$, $\Delta_1 = \cos(\pi/3) = 0.5$. Broken line shows the Poisson distribution. The peak at $s = 0$ is given by degenerate eigenvalues.

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1 G. Montambaux, D. Poilblanc, J. Bellissard, and C. Sire, Phys. Rev. Lett. 70, 497 (1993).
2 T. C. Hsu and J. C. Anglès d’Auriac, Phys. Rev. B 47, 14291 (1993).
3 D. Poilblanc, T. Ziman, J. Bellissard, F. Mila, and G. Montambaux, Europhys. Lett. 22, 537 (1993).
4 P. van Ede van der Pals and P. Gaspard, Phys. Rev. E 49, 79 (1994).
5 H. Meyer, J. C. Anglès d’Auriac, and H. Bruus, J. Phys. A 29, L483 (1996).
6 H. Meyer, J. C. Anglès d’Auriac, and J. M. Maillard, Phys. Rev. E 55, 5380 (1997).
7 H. Meyer and J. C. Anglès d’Auriac, Phys. Rev. E 55, 6608 (1997).
8 J. C. Anglès d’Auriac and F. Iglói, Phys. Rev. E 58, 241 (1998).
9 J. C. Anglès d’Auriac and J. M. Maillard, Physica A 321, 325 (2003).
10 M. Faas, B. D. Simons, X. Zotos, and B. L. Altshuler, Phys. Rev B 48, 5439 (1993).
11 K. Held, E. Eisenberg, and B. L. Altshuler, Phys. Rev. Lett. 90, 106802 (2003).
12 T. Deguchi, K. Fabricius, and B. M. McCoy, J. Stat. Phys. 102, 701 (2001).
13 K. Fabricius and B. M. McCoy, in MathPhys Odyssey 2001, edited by M. Kashiwara and T. Miwa (Birkhäuser, Boston, 2002), p. 119.
14 T. Deguchi, J. Phys. A 35, 879 (2002).
15 K. Kudo and T. Deguchi, J. Phys. Soc. Jpn. 72, 1599 (2003).