Riemannian geometry as a curved pre-homogeneous geometry

Ercüment Ortaçgil
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Abstract

We define a Riemannian structure as a pre-homogeneous geometric structure with curvature $R$. We show that $R = 0$ if and only if the underlying metric has constant curvature. We define pre-homogeneous geometric structures and pose some problems.

1 Introduction

This note is the continuation of [24], [3], [2] and its main purpose is to carry out the program outlined in the introduction of [2] in the case of Riemannian and affine structures. So we will start by recalling this program in more technical detail than [2].

Let $G$ be a connected Lie transformation group which acts effectively and transitively on a connected and smooth manifold $M$. We call this data a Klein geometry and denote it by $(G, M)$. We fix some $p, q \in M$, $g \in G$ with $g(p) = q$, and define $H_k \overset{def}{=} \{ h \in G \mid j_k(h)^{p,q} = j_k(g)^{p,q}, \ k \geq 0 \}$ where $j_k(g)^{p,q}$ denotes the $k$-jet of the transformation $g$ with source at $p$ and target at $q$. We call $j_k(g)^{p,q}$ a $k$-arrow (from $p$ to $q$). So $H_k$ consists of all $h \in G$ such that the $k$-arrow of $h$ coincides with the $k$-arrow of $g$ (for $k = 0$, this simply means $h(p) = g(p)$). Clearly $\{g\} \subset H_{k+1} \subset H_k$. In [3], we showed that this sequence stabilizes at $\{g\}$ and the smallest integer $m$ with $H_m = \{g\}$ does not depend on $p, q$ and $g$. Therefore, any $g \in G$ is determined by its $m$-arrow $j_m(g)^{p,q}$ for any $p, q$ and this holds in particular for the elements of the stabilizer $H_p \overset{def}{=} \{ g \in G \mid g(p) = p \}$. The integer $m$ is called the geometric order of the Klein geometry $(G, M)$ in [3]. Clearly, $m = 0$ if and only if $G$ acts simply transitively. We showed in [3] that $m$ can be arbitrarily large even if $M$ is compact and actions whose stabilizers are parabolic subgroups are the prototypes of this situation.

There is no curvature in the above picture in the sense that any $m$-arrow $j_m(g)^{p,q}$ integrates uniquely to a local (in fact global) diffeomorphism which is the transformation $g$. Shortly, the groupoid of $m$-arrows integrates to a pseudogroup (in fact to a Lie group). So the question arises how we should “curve” the above picture, that is, what is curvature? However we look at
curvature, it must deform the “symmetric” object \((G, M)\) into a “lumpy” object.
All the existing approaches to the concept of curvature (see, for instance, [14], [29], [6], [8]) circle around this fundamental idea and this note and [2] are no exceptions. It is commonly accepted today (at least in Riemannian geometry, see [15] and the recent work [8] for parabolic geometries) that this lumpy object is a principal bundle \(P \to N\) with structure group \(H \simeq H_p\), dim \(N = \dim M\), together with some extra structure on \(P \to N\), like a torsionfree connection. A different approach is taken in [6] which is more in the spirit of this note and [2]. To explain our lumpy object \(\tilde{G}(N)\), we continue with the setting of the above paragraph and let \(U^{p,p}(r)\) denote the group consisting of \(r\)-arrows of all local diffeomorphisms with source and target at \(p \in M\), \(r \geq 0\). We fix a coordinate system around \(p\) which identifies \(U^{p,p}(r)\) with the jet group \(G_r(n)\) where \(n = \dim M\). Since \(j_m(h)^{p,p}\) determines \(h \in H_p\), (for a technical reason to be explained in Remark 1 and Section 7, we assume here that \(M\) is simply connected), \(H_p\) injects onto a subgroup of \(G_m(n)\) which we denote also by \(H_p\). Since \(h \in H_p\) determines \(j_m(h)^{p,p}\), we also have an injection of \(H_p\) onto a subgroup \(G_{m+1}(n)\) which we denote by \(\varepsilon H_p\). So we have the commutative diagram

\[
\begin{array}{c}
G_{m+1}(n) \\
\cup \\
\varepsilon H_p \\
\pi \\
\downarrow \\
\cup \\
G_m(n) \\
\pi \\
\downarrow \\
H_p
\end{array}
\] (1)

where \(\pi\) is the projection homomorphism induced by the projection of jets and the vertical inclusions depend on the coordinate system around \(p\) which we fixed. The restriction of \(\pi\) to \(\varepsilon H_p\) is a bijection in (1) and \(\varepsilon = (\pi|_{H_p})^{-1}\) so that \(\pi \circ \varepsilon = \text{id}\). It is easy to check that the conjugacy class of \(\varepsilon H_p\) inside \(G_{m+1}(n)\) is independent of the coordinate system around \(p\) and therefore also independent of \(p\) since any two stabilizers are conjugate. We denote this conjugacy class by \(\{G, M, H\}\) where \(H\) denotes some “easy” representative of the isomorphism class of the stabilizers \(H_p, p \in M\). For instance in Euclidean geometry, \(H = O(n)\), \(G = O(n) \times \mathbb{R}^n\) and \(M = \mathbb{R}^n\). We call \(\{G, M, H\}\) the vertex connection of the Klein geometry \((G, M)\). In disguise, the vertex connection \(\{G, M, H\}\) is a PDE for \(\dim M \geq 2\) and ODE for \(\dim M = 1\) and plays a fundamental role in the theory. Two Klein geometries \((G_1, M_1), (G_2, M_2)\) with \(\dim M_1 = \dim M_2\), \(\dim H_1 = \dim H_2\) and \(m_1 = m_2\) may define the same vertex connection \(\{G_1, M_1, H\}\) but \(G_1\) and \(G_2\) may have nonisomorphic Lie algebras, that is, even though \(\{G_1, M_1, H\}\) by definition determines the isomorphism class of \(H\), it does not completely determine \(G\) locally. In Riemannian geometry, there are only three and in affine geometry only one such Lie algebra up to isomorphism. The other extreme case occurs in [2] for \(m = 0\) where all Lie algebras arise.

Now, in search for a geometric structure \(\tilde{G}(N)\) which is the curved analog of \((G, M)\), the main idea of the reference [22] in [2] is to dispense with \((G, M)\) but retain the vertex connection \(\{G, M, H\}\), start with some smooth manifold \(N\) with \(\dim N = \dim M\), and try to reconstruct the action of \(G\) on \(M\) from a transitive Lie groupoid \(\tilde{G}(N)\) of order \(m+1\) on \(N\) with the property that the
conjugacy class of the vertex groups of $G(N)$ is $\{G, M, H\}$ (see Definition 14). As required, $(m+1)$-arrows of $G(N)$ integrate to a pseudogroup if and only if the curvature $R = 0$ so that $R$ is the obstruction to construct the action of $G$ locally. The lift of this pseudogroup globalizes to a Lie group $\tilde{G}$ on the universal covering space $\tilde{N}$ of $N$ and the Klein geometry $(\tilde{G}, \tilde{N})$ defines the vertex connection that we started with. In this way we obtain the well known uniformization theorems for Riemannian and affine structures (Propositions 9, 10). The remarkable fact is that $R$ and the curvature in the formalism of connections on principal bundles are different objects, even in Riemannian geometry, as can be seen from our abstract. We hope that these differences will become more transparent in this note. Another surprising fact is that we mention torsion, the Levi-Civita connection and covariant differentiation a few times in this note only for convinience.

The above idea is worked out in detail for $m = 0$ in [2]. Even in this simplest case (or the most complicated case depending on our view), this approach gives rise to some new possibilities and questions with wide scope and subtlety. The purpose of this note is to show how this program works out for $m = 1$ in the case of Riemannian and affine structures and indicate how it generalizes in a straightforward way to all pre-homogeneous structures of arbitrary geometric order $m$.

This note is organized as follows. In Section 2 we fix some “easy” representatives of the Riemannian and affine vertex connections and derive some elementary formulas. In Section 3 we define a Riemannian structure as a subgroupoid $G_2 \subset U_2$ with algebroid $\mathfrak{G}_2 \subset J^2T$ in accordance with the general theory and using the formulas in Section 2, we express these submanifolds locally as the zero set of some functions. We define the complete integrability of $G_2$ and $\mathfrak{G}_2$, a concept which plays a fundamental role in this note. In Section 4 we outline a proof that the Lie’s third theorem is equivalent to the well known uniformization theorem in Riemannian geometry (Propositions 9, 10). In Section 5 we define the algebroid and groupoid curvatures $R$ and $\mathfrak{R}$ show that their vanishing is equivalent to complete integrability and also to the constant curvature condition of the metric (Propositions 11, 12). In Section 6 we take a brief look at affine structures. Since all pre-homogeneous structures are studied on the same footing, all our propositions in Sections 4, 5, 6 are identical with those in [2], except that we do not touch characteristic classes here. It has become apparent to us that we will never be able to finish the reference [22] in [2], so we decided to give here the definition of a pre-homogeneous structure and formulate some problems which we believe are fundamental for the theory, which is the content of Section 7.

## 2 Riemannian and affine vertex connections

Let $G_k(n)$ denote the $k$'th order jet group in $n$-variables. The elements of $G_k(n)$ are $k$-jets $j_k(f)^{\circ o}$ of local diffeomorphisms $f$ of $\mathbb{R}^n$ with $f(o) = o$ where $o$ is the origin. The composition $\circ$ is defined by $j_k(f)^{\circ o} \circ j_k(g)^{\circ o} \overset{def}{=} j_k(f \circ g)^{\circ o}$. The
projection $\pi$ of jets gives the exact sequence of Lie groups

$$0 \longrightarrow K_{k+1,k}(n) \longrightarrow G_{k+1}(n) \longrightarrow G_k(n) \longrightarrow 1 \tag{2}$$

where $K_{k+1,k}(n)$ is a vector group. In this note, all projections induced by the projection of jets will be denoted by $\pi$ and all splittings by $\varepsilon$. Up to Section 7, we will have $k = 1$ in (2). We refer to [30] for some basic structure theorems for $G_k(n)$ and to [22] for an explicit matrix representation of $G_k(n)$.

In the coordinates of $\mathbb{R}^n$, an element $a \in G_2(n)$ is of the form $(a^i_j, a^i_{jk})$ and the chain rule of differentiation shows that the group operation is given by

$$(a^i_j, a^i_{jk})(b^j_l, b^j_{lk}) = (a^i_j b^l_j, a^i_j b^j_{lk} + a^l_{ik} b^j_k) \tag{3}$$

We use summation convention in (3). For simplicity of notation, we denote $a = (a^i_j, a^i_{jk})$ by $(a_1, a_2)$ and write the group operation (3) as

$$ab = (a_1, a_2)(b_1, b_2) = (a_1 b_1, a_1 b_2 + a_2(b_1)^2) \tag{4}$$

Clearly, $a^i_j = \delta^i_j$, $a^i_{jk} = 0$ defines the identity which we denote by $(I, 0)$. It is easy to check that $(a^i_j, a^i_{jk})^{-1} = (\varepsilon(a^{-1})^j_i, \varepsilon(a^{-1})^j_i a^r_s (a^{-1})^r_j (a^{-1})^s_k)$ which we write as

$$(a_1, a_2)^{-1} = (a_1^{-1}, -a_1^{-1} a_2 (a_1^{-1})^2) \tag{5}$$

using our notation in (4). We have $(I, a)(I, b) = (I, a + b)$ and $(I, a)^{-1} = (I, -a)$. We can write (2) now as $(I, a_2) \longrightarrow (a_1, a_2) \longrightarrow a_1$.

We now define a splitting $\varepsilon : O(n) \rightarrow G_2(n)$. Let $G(0)$ denote the isometry group of $\mathbb{R}^n$, that is, $G(0) = O(n) \ltimes \mathbb{R}^n$ where $\ltimes$ denotes semidirect product. For $g = (\xi, a) \in G(0)$, $x \in \mathbb{R}^n$, we have $(gx)^i = \xi^i x^s + b^i$. Therefore, if $h(p) = p$ for some $h \in G(0)$ and $p \in \mathbb{R}^n$, then $[j_1(h)^{p,p}]^i_j = \xi^i$ and $[j_2(h)^{p,p}]^i_j = 0$. Therefore any $h \in G(0)$ which stabilizes $p$ (therefore any $g \in G(0)$) is determined by its 1-arrow $j_1(g)^{p,p}$ (note the crucial role of translations!) and it follows that $m = 1$ where $m$ is the geometric order of the Klein geometry $(G(0), O(n))$.

Remark 1 In the definition of the geometric order of the Klein geometry $(G, M)$ in [3], the connectedness of $G$ is used to ensure that the adjoint representation of $H_p$ on the Lie algebra of $G$ is faithful (see Lemma 5.1 in [3]). Henceforth we will always assume this latter condition so that geometric order is defined. Also, if $M$ is simply connected, then the geometric order of $(G, M)$ is equal to the infinitesimal order of $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h}$ is the Lie algebra of some stabilizer (see [3] for the geometric and infinitesimal orders and the second paragraph of Section 7 of this note).

Thus $O(n)$ injects into $G_2(n)$ as

$$\varepsilon : O(n) \rightarrow G_2(n) \tag{6}$$

$$a \rightarrow (a, 0)$$

4
Since the above derivation does not use the orthogonality of the matrix $\xi$, we may replace $O(n)$ with $G_1(n)$ and $\varepsilon$ splits also $G_1(n)$ (in fact, any subgroup of $G_1(n)$) by the same formula in (6)). Therefore $m = 1$ also for the affine group $A = G_1(n) \ltimes \mathbb{R}^n$.

Now we denote the conjugacy classes of $\varepsilon O(n)$ and $\varepsilon G_1(n)$ inside $G_2(n)$ by \{\(G(0), \mathbb{R}^n, SO(n)\) and \{\(A, \mathbb{R}^n, G_1(n)\) respectively. It is crucial to observe how the independence of $p$ and the coordinates around $p$ explained in the Introduction is “trivialized” by translations which will not be at our disposal if we replace $\mathbb{R}^n$ with some arbitrary $M$.

These two objects play a fundamental role in this note, so we make

**Definition 2** The conjugacy classes \{\(G(0), \mathbb{R}^n, O(n)\) and \{\(A, \mathbb{R}^n, G_1(n)\) are the vertex connections of the Klein geometries \((G(0), \mathbb{R}^n)\) and \((A, \mathbb{R}^n)\) respectively.

We denote the vertex connections in Definition 2 by $R$ and $A$ and call them Riemannian and affine respectively. So $\varepsilon O(n)$ and $\varepsilon G_1(n)$ are some “easy” representatives of $R$ and $A$. These choices are irrelevant from a theoretical standpoint but greatly simplify local computations.

Now (4) gives $$(a_1, a_2) = (a_1, 0)(a_1, 0)^{-1}(a_1, a_2) = (a_1, 0)(I, a_1^{-1}a_2)$$ which expresses the semidirect product structure

$$G_2(n) = G_1(n) \ltimes K_{2,1}(n)$$

(7)

In fact, $G_1(n)$ splits inside $G_k(n)$ for all $k \geq 1$, $n \geq 1$ in the same way, that is, $G_k(n) = G_1(n) \ltimes K_{k,1}$. In more abstract terms, an algebraic group is the semidirect product of its maximal reductive and maximal nilpotent subgroups and the decomposition $G_k(n) = G_1(n) \ltimes K_{k,1}$ is a special case ([30], Theorem 2.6). We believe that the splitting of (2) for all $n \geq 1$ occurs only for $k = 1$ and (2) never splits for $n \geq 2$, $k \geq 2$. We will see in Section 7 that the Schwarzian derivative arises from the splitting of (2) for $n = 1, k = 2$. All splittings of (2) for $n = 1$ and arbitrary $k$ are determined in [27] on the level of Lie algebras. We also refer to [30] for a detailed study of the solvable Lie group $G_k(1)$. Now, even though (2) splits very rarely, there exist an abundance of splittings inside (2) for arbitrarily large values of $n$, $k$ arising from Klein geometries $(G, M)$ where $n = \dim M$, $k =$ geometric order of $(G, M)$. These splittings form the backbone of the present approach as explained in the Introduction.

Now let $G(1) \overset{df}{=} O(n + 1)$, $G(-1) \overset{df}{=} O(1, n)$ so that we have

$$G(1)/O(n) = S^n, \quad G(0)/O(n) = \mathbb{R}^n, \quad G(-1)/O(n) = \mathbb{H}^n$$

(8)

The geometric order $m = 1$ for the Klein geometries in (8). More generally, if $H \neq \{e\}$ is compact and $G/H$ is simply connected, then $m = 1$. This is equivalent to the statement that the isotropy representation of $H$ is faithful. It follows that the vertex connections \{\(G(1), S^n, O(n)\), \{\(G(-1), \mathbb{H}^n, O(n)\) are defined as explained in the Introduction. We claim
\[ \{G(1), S^n, O(n)\} = \{G(0), R^n, O(n)\} = \{G(-1), H^n, O(n)\} \quad (9) \]

We will see in Section 4 that (9) is a consequence of the uniformization theorem in Riemannian geometry.

We now take a closer look at \( \{A, R^n, G_1(n)\} \).

**Proposition 3** Let \( \sigma_1, \sigma_2 : G_1(n) \to G_2(n) \) be two group homomorphisms satisfying \( \pi \circ \sigma_i = \text{id} \). Then \( \sigma_1 G_1(n) = k(\sigma_2 G_1(n)) k^{-1} \) for some \( k \in K_{2,1}(n) \triangleleft G_2(n) \).

To prove the assertion, let \( \sigma : G_1(n) \to G_2(n) \) be any such homomorphism. So \( \sigma(a) = (a, \phi(a)) \) for some function \( \phi \). Now \( \sigma(ab) = \sigma(a) \sigma(b) \) and (4) give

\[ \phi(ab) = a \phi(b) + \phi(a)b^2 \quad a, b \in G_1(n) \quad (10) \]

We successively let \( a = \lambda I \) and \( b = \lambda I \) in (10) and get

\[ \phi(\lambda b) = \lambda \phi(b) + \phi(\lambda)b^2 \quad \phi(b\lambda) = b\phi(\lambda) + \phi(b)\lambda^2 \quad b \in G_1(n) \quad (11) \]

where \( \lambda \) denotes \( \lambda I \) in (11). Since \( \lambda b = b \lambda \), (11) gives \( \lambda \phi(b) + \phi(\lambda)b^2 = b\phi(\lambda) + \phi(b)\lambda^2 \). Solving for \( \phi(b) \), we obtain

\[ (b, \phi(b)) = \left( b, \frac{1}{\lambda^2 - \lambda} (\phi(\lambda I)b^2 - b\phi(\lambda I)) \right) \quad \lambda \in R, \quad \lambda \neq 0, 1, \quad b \in G_1(n) \]

\[ = (I, \frac{\phi(\lambda I)}{\lambda^2 - \lambda})(b, 0)(I, \frac{\phi(\lambda I)}{\lambda^2 - \lambda})^{-1} \quad (12) \]

We observe that the RHS of (12) is independent of \( \lambda \). Setting \( k = \frac{\phi(\lambda I)}{\lambda^2 - \lambda} \), (12) becomes \( \sigma(b) = (b, \phi(b)) = (I, k) \varepsilon(b)(I, -k) \). Therefore any splitting \( \sigma \) is conjugate to \( \varepsilon \), which proves the statement.

Note that the above proof works if we replace \( G_1(n) \) by any subgroup \( L \subset G_1(n) \) as long as \( L \) contains some \( \lambda I \) with \( \lambda \neq 1 \). Therefore it works for \( O(n) \) and \( O(1,n) \) since \( -I \in O(n), O(1,n) \). Also, note that we do not even assume the continuity of \( \sigma \) but deduce the very strong conclusion that \( \sigma(b) \) is a quadratic polynomial in \( b \).

Now our purpose is to express the right cosets of \( \varepsilon(O(n)) \) in \( G_2(n) \) as the zero set of some functions. We have

\[ (a_1, a_2)(b_1, b_2)^{-1} \in \varepsilon(O(n)) \iff (a_1, a_2)(b_1^{-1}, -b_1^{-1}b_2(b_1^{-1})^2) \in \varepsilon(O(n)) \]
\[ \iff (a_1b_1^{-1}, -a_1b_1^{-1}b_2(b_1^{-1})^2 + a_2(b_1^{-1})^2) \in \varepsilon(O(n)) \]
\[ \iff (a_1b_1^{-1})T(a_1b_1^{-1}) = I, \quad -a_1b_1^{-1}b_2(b_1^{-1})^2 + a_2(b_1^{-1})^2 = 0 \]
\[ \iff (b_1^{-1})^{-1}a_1^{-1}b_1^{-1} = I, \quad a_1b_1^{-1}b_2 = a_2 \]
\[ \iff a_1^Tb_1 = a_2^Tb_2, \quad a_1^{-1}a_2 = b_1^{-1}b_2 \quad (13) \]
We define the functions $F_1, F_2$ by $F_1(a_1, a_2) \overset{def}{=} a_1^T a_1$, $F_2(a_1, a_2) \overset{def}{=} a_1^{-1} a_2$, set $F \overset{def}{=} (F_1, F_2)$ and rewrite (13) as

$$ab^{-1} \in \varepsilon(O(n)) \iff F(a) = F(b)$$

(14)

In more detail, $F_1$ has components $F_{jk} : G_2(n) \to \mathbb{R}, 1 \leq j, k \leq n$, defined by

$$F_{jk}(a_1, a_2) = a_j^s a_k^r = (a_1^T a_1)_{jk}$$

(15)

where $s$ is summed in (15), and $F_2$ has components $F^i_{jk} : G_2(n) \to \mathbb{R}, 1 \leq i, j, k \leq n$, defined by

$$F^i_{jk}(a_1, a_2) = (a^{-1})^i_s a^s_{jk} = (a_1^{-1} a_2)^i_{jk}$$

(16)

So $F : G_2(n) \to \mathbb{R}^s, s = \dim G_2(n)$ and $F$ has constant rank $r = \dim G_2(n) - \dim \varepsilon(O(n))$. Thus the surjective map $F : G_2(n) \to \mathbb{R}^r$ has right cosets of $\varepsilon(O(n))$ as fibers. If we replace $\varepsilon(O(n))$ by $\varepsilon(G_1(n))$, the function $F$ will be defined only by $F_2$ as the condition imposed by $F_1$ will be redundant. We will continue to use the notation $F$ also in this case. This point will be relevant in Section 6.

Recall that $G_2(n)$ is an algebraic group, $\varepsilon(O(n)), \varepsilon(G_1(n)) \subset G_2(n)$ are algebraic subgroups and $F$ is a polynomial. On the other hand, if $o$ denotes the coset of $\varepsilon(O(n))$ (or $\varepsilon(G_1(n))$), we can always express the cosets near $o$ as the zero set of some smooth functions.

Now suppose $ab = c$ in $G_2(n)$. We have $F_1(c) = (a_1 b_1)^T a_1 b_1 = b_1^T a_1^T a_1 b_1 = b_1^T F_1(a) b_1$, that is

$$c_{jk} = (a_1^T a_1)_{st} b_s^i b_t^j$$

(17)

Similarly, $F_2(c) = F_2(ab) = (a_1 b_1)^{-1} (a_1 b_2 + a_2 b_1) = b_1^{-1} b_2 + b_1^{-1} (a_1^{-1} a_2) b_1 = F_2(b) + b_1^{-1} F_2(a) b_1$, that is,

$$c^i_{jk} = (a_1^T a_1)^i_{st} b_s^i b_t^j (b^{-1})^i_r + (b^{-1})^i_r b_t^j$$

(18)

Finally, it is easy to check that (17), (18) satisfy the group law, that is, the composition $F(a) \to F(ab) \to F((ab)c)$ is the same as $F(a) \to F(a(bc))$.

### 3 Riemannian structures

Let $M$ be a smooth and connected manifold with $\dim M = n$. Let $p, q \in M$ and $U^{p,q}_k$ denote the set of all $k$-jets $j_k(f)^{p,q}$ of local diffeomorphisms with source at $p$ and target at $q$. We call $j_k(f)^{p,q}$ a $k$-arrow (from $p$ to $q$). The composition of local diffeomorphisms induces a composition $U^{p,r}_k \times U^{r,q}_k \to U^{p,q}_k$.

We define the set $U_k \overset{def}{=} \cup_{p,q \in M} U^{p,q}_k$. The smooth structure of $M$ induces a natural smooth structure on $U_k$ as follows. For two coordinate patches $(U, x^i)$, $(V, y^j)$ on $M$, any $k$-arrow $f^{p,q} \in U_k$ with $p \in U$ and $q \in V$ has the unique representation $(x_i, y_j, y_{ij}, y_{ijk}, ..., y_{ijk...j})$ where $x^i$ and $y^i$ are the coordinates
of \( p, q \) respectively. With this differentiable structure, \( \mathcal{U}_k \) becomes a transitive Lie equation in finite form which is a very special groupoid (see [28], [25] for Lie equations in finite and infinitesimal forms and [18] and the references therein for general Lie groupoids and algebroids). We call \( \mathcal{U}_k \) the universal groupoid on \( M \) of order \( k \). Since a 0-arrow is an ordered pair \((p, q)\), \( \mathcal{U}_0 \) is the pair groupoid \( M \times M \). Note that a choice of coordinates around \( p \in M \) identifies the vertex group \( \mathcal{U}_k^{p, q} \) with the jet group \( G_k(n) \) for \( k \geq 1 \) and a change of these coordinates conjugates this identification with the \( k \)-jet of the coordinate change at \( p \). The projection of jets induces a projection \( \pi : \mathcal{U}_{k+1} \to \mathcal{U}_k \) and \( \pi \) is a morphism of groupoids, that is, it preserves the composition and inversion of arrows.

Now let \( \mathcal{G}_k \subset \mathcal{U}_k \) be a transitive subgroupoid. This means that the set \( \mathcal{G}_k^{p, q} \) of \( k \)-arrows of \( \mathcal{G}_k \) is nonempty for all \( p, q \in M \), the \( k \)-arrows of \( \mathcal{G}_k \) are closed under composition and inversion, and \( \mathcal{G}_k \subset \mathcal{U}_k \) is an imbedded submanifold (see [18] for details). The Lie subgroup \( \mathcal{G}_k^{p, p} \subset \mathcal{U}_k^{p, p} \) is called the vertex group of \( \mathcal{G}_k \) at \( p \).

We now fix some \( p \in M \) and choose some coordinates \((x^i)\) around \( p \). This choice identifies \( \mathcal{U}_k^{p, q} \) with \( G_k(n) \) and therefore identifies \( \mathcal{G}_k^{p, q} \) with a subgroup \( i_x(\mathcal{G}_k^{p, q}) \subset G_k(n) \). A change of coordinates \( f : (x^i) \to (y^j) \) around \( p \) conjugates \( i_x(\mathcal{G}_k^{p, p}) \) with the \( k \)-jet of \( f \) at \( p \). Therefore \( i_x(\mathcal{G}_k^{p, p}) \) and \( i_y(\mathcal{G}_k^{p, p}) \) belong to the same conjugacy class in \( G_k(n) \). This conjugacy class does not depend also on the choice of the point \( p \). To see this, let \( q \in M \) any other point, choose any \( k \)-arrow \( f_x^{p, q} \in \mathcal{U}_k^{p, q} \) and fix any coordinate system \((U, x^i)\) around \( p \). Now there exists a local diffeomorphism \( g \) with \( g(p) = q \) such that \( j_k(g)^{p, q} = f_x^{p, q} \) by the definition of \( f_x^{p, q} \). So \( g \) defines a coordinate system \((V, y^j)\) around \( q \) and the representation of \( f_x^{p, q} \) with respect to \((U, x^i)\), \((V, y^j)\) is \((\mathcal{F}, \mathcal{F}, \delta)\) where \( p = (\mathcal{F}) \). Therefore, \((V, y^j)\) imbeds \( \mathcal{G}_k^{p, q} \) into \( G_k(n) \) in the same way as \((U, x^i)\) imbeds \( \mathcal{G}_k^{q, q} \). We denote this conjugacy class by \( \{\mathcal{G}_k\} \) which is a common property of all the vertex groups of \( \mathcal{G}_k \).

Recalling the definition of the Riemannian vertex connection \( \mathbf{R} \) in Section 2, we now make the following

**Definition 4** A Riemannian structure on \( M \) is a transitive subgroupoid \( \mathcal{G}_2 \subset \mathcal{U}_2 \) such that \( \{\mathcal{G}_2\} = \mathbf{R} \)

Note that a Riemannian structure is a second order structure according to Definition 4 in the same way as a parallelizable manifold is a first order structure in [2]. We define \( \mathcal{G}_1 \overset{\text{def}}{=} \pi \mathcal{G}_2 \) and call \( \mathcal{G}_1 \) the underlying metric structure. We may have \( \mathcal{G}_2 \neq \mathcal{G}_1 \) but \( \pi \mathcal{G}_2 = \pi \mathcal{G}_1 \) that is, two different Riemannian structures may have the same underlying metric. By the definitions of \( \mathcal{G}_2 \), \( \{\mathcal{G}_2\} \), \( \mathbf{R} \), for any \( p \in M \), there exists a coordinate system \((U, x^i)\) around \( p \) such that the vertex group \( \mathcal{G}_2^{p, p} \) imbeds into \( G_2(n) \) as \((6)\). We call \((U, x^i)\) regular at \( p \). It will become clear below that regular and geodesic coordinates agree to the first order, but not necessarily to the second order.

Now, the projection \( \pi : \mathcal{G}_2 \to \mathcal{G}_1 \) induces an isomorphism on the vertex groups by the definition of \( \mathcal{G}_2 \). This fact implies that \( \pi \) is an isomorphism of
We claim that (20) does not depend on the section $e$ around change $(U, x)$ such section, then we shortly write as $s$. Section $U, x$ looks at $G$ and $\dim G$. Thus we have the the commutative diagram

$$
\begin{array}{ccc}
\mathcal{U}_2 & \xrightarrow{\pi} & \mathcal{U}_1 \\
\cup & \cup & \cup \\
\mathcal{G}_2 & \xrightarrow{\pi} & \mathcal{G}_1
\end{array}
$$

(19)

where $\pi|_{\mathcal{G}_2}$ is an isomorphism with inverse $\varepsilon \overset{def}{=} (\pi|_{\mathcal{G}_2})^{-1}$ and as remarked above, the restriction of $\varepsilon$ to the vertex groups “looks like” (6) in regular coordinates. It is crucial to observe that a groupoid is by no means determined by its vertex groups. An extreme case occurs in [2] where the vertex groups are trivial, but it is clear that the vertex groups severely restrict Riemannian and affine structures when curvature vanishes.

The above local coordinates on $\mathcal{U}_2$ show that $\dim \mathcal{U}_2 = 2 \dim M + \dim G_2(n)$ and $\dim \mathcal{G}_2 = 2 \dim M + \dim O(n)$. Now our purpose is to express the submanifold $\mathcal{G}_2 \subset \mathcal{U}_2$ locally as the zero set of some functions so that we can take a closer look at $\mathcal{G}_2$ and make some explicit local computations. So we fix some base point $e \in M$ and some regular coordinates around $e$ once and for all. Changing these choices will conjugate our formulas by some “constant” 2-arrow which does not depend on the base variables, so that our formulas will remain essentially the same when we differentiate them (the reader may keep track of this in what follows). Now let $(U, x^i)$ be some arbitrary coordinate patch on $M$. For each $p \in U$, we choose a 2-arrow of $\mathcal{G}_2$ with source at $p$ and target at $e$. This local (smooth) section $s$ has the coordinate representation $s(x) = (x^i, e^i, s^i_j(x), s^i_{jk}(x))$ which we shortly write as $s(x) = (s^i_j(x), s^i_{jk}(x)) = (s_1(x), s_2(x))$, using our symbolic notation in Section 2. We define $g_1(x) \overset{def}{=} F_1(s_1(x), s_2(x)) = s_1(x)^T s_1(x)$ and $g_2(x) \overset{def}{=} F_2(s_1(x), s_2(x)) = s_1(x)^{-1} s_2(x)$, that is

$$
g(x) \overset{def}{=} (g_1(x), g_2(x)) \overset{def}{=} F((s_1(x), s_2(x)) = F(s(x))
$$

(20)

We claim that (20) does not depend on the section $s(x)$. Indeed, if $t(x)$ is another such section, then $t(x) \circ s(x)^{-1} \in \mathcal{G}_{2}^{e} = \varepsilon O(n)$ since the coordinate system around $e$ is regular, and (14) implies $F(s(x)) = F(t(x))$. Now a coordinate change $(U, x^i) \rightarrow (V, y^j)$ transforms the components of the section $s(x)$ as

$$
(s^i_j(y), s^i_{jk}(y)) = (s^i_j(x), s^i_{jk}(x)) \ast \left( \frac{\partial x^i}{\partial y^j}, \frac{\partial^2 x^i}{\partial y^j \partial y^k} \right)
$$

(21)

or slightly $s(y) = s(x) \ast (\frac{\partial x}{\partial y})$ where $\ast$ denotes the group operation of $G_2(n)$ defined by (3). From (17), (18) and (21) we deduce
\[ g_{ij}(y) = g_{ab}(x) \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \]  (22)

\[ g_{ik}^j(y) = g_{be}^a(x) \frac{\partial x^b}{\partial y^i} \frac{\partial x^e}{\partial y^j} + \frac{\partial y^i}{\partial x^a} \frac{\partial^2 x^a}{\partial y^j} \partial y^j \]  (23)

The transformation law (22), (23) satisfies the group law by our remark at the end of Section 2. So we defined a second order geometric object \( g \) on \( M \) with components \((g_{ij}(x), g_{ik}^j(x))\) on \((U, x^i)\) subject to the transformation laws (22), (23). Observe that \( g \) is constructed from the 2-arrows of \( \mathcal{G}_2 \) using an invariance condition, that is, \( g \) has no separate presence of its own, at least from the present standpoint. We considered right cosets in (14) and fixed the target in the definition of \( s \) in order to deal with \( g_{ij} \) rather than \( g^{ij} \), but such choices are not much relevant for the theory.

At first sight, it seems that \( g \) is made up of two separate geometric objects, a metric defined by (22) and a torsionfree affine connection defined by (23). This peculiarity is due to the splitting of \( G_1(n)_r \) inside \( G_2(n) \). However, a closer look reveals that (22) and (23) are related in a subtle way. To see this, some \( f^{p,e} \in \mathcal{G}_2^{p,e} \) defines some regular coordinates \((U, x^i)\) around \( p \) by pulling back the one at \( e \) to \( p \). If we choose our section \( s \) with \( s(p) = f^{p,e} \) and define \( g(x) \) using this particular \( s(x) \), we find

\[ g_{ij}(p) = \delta_{ij}, \quad g_{ik}^j(p) = 0 \]  (24)

Now (24) shows that (22) and (23) live together, justifying our notation \( g \), and the auxiliary object defined by (23) is not far from the Levi-Civita connection (we use the terms “Levi-Civita connection” and “Christoffel symbols” synonymously). At this point, it is natural to ask why we work with some auxiliary objects which imitate the Levi-Civita connection but not work with the Levi-Civita connection itself. Propositions 11, 12 will give a rather unexpected answer to this fair question.

Henceforth in this note a Riemannian structure means a structure as defined above (we could not find a better name!). Since Definition 4 already incorporates the metric (22), Riemannian geometry \( RG \) “includes” Riemannian geometry \( MRG \). Our purpose in this note is not to show that the inclusion \( MRG \subset RG \) is proper (meaning that \( RG \) gives new results in \( MRG \), we do not know this), but to show that \( RG \) generalizes in a straightforward way to prehomogeneous geometries in such a way that one can completely avoid torsion, covariant differentiation and the Levi-Civita connection. This generalization will be based on Lie’s theorems as we will see in Section 4.

Now let \((U, x^i), (V, y^i)\) be two coordinate patches on \( M \). Using (22), (23), it is now easy to show that some 2-arrow \((x^i, y^i, \phi^i_j(x, y), \phi^j_{ik}(x, y))\) of \( \mathcal{U}_2 \) with source in \( U \) and target in \( V \) belongs to \( \mathcal{G}_2 \) if and only if it satisfies...
\[ \begin{aligned}
g_{ij}(x) &= g_{ab}(y)\phi_a^i(x, y)\phi_j^b(x, y) \\
g_{jk}^{ij}(x)\phi_a^i(x, y) &= g_{bc}^{ij}(y)\phi_j^b(x, y)\phi_k^c(x, y) + \phi_j^k(x, y)
\end{aligned} \]  

(25) gives a set of equations which define the submanifold \( \mathcal{G}_2 \subset \mathcal{U}_2 \) locally. In short, \( \mathcal{G}_2 \) consists of all 2-arrows which preserve the geometric object \( g \).

Note that if \( (\phi^i_j(x, y), \phi^j_k(x, y)) \) and \( (\phi^i_j(x, y), \phi^j_k(x, y)) \) both solve (25), then \( \phi^j_k(x, y) = 0 \).

Even though \( \mathcal{G}_2 \) looks like a purely geometric object at first sight, it is actually a nonlinear system of PDE’s made up of initial conditions which are its 2-arrows. More precisely, let \( f^{p,q} \in \mathcal{G}_2 \), choose coordinates \((U, x^i), (V, y^i)\) around \( p, q \) and write \( f^{p,q} = (\mathbf{x}, \mathbf{y}, \phi_1(\mathbf{x}, \mathbf{y}), \phi_2(\mathbf{x}, \mathbf{y})) \). Clearly, the substitution of the components of \((\mathbf{x}^i, \mathbf{y}^i, \phi_1(\mathbf{x}, \mathbf{y}), \phi_2(\mathbf{x}, \mathbf{y}))\) into (25) gives an identity since \( f^{p,q} \in \mathcal{G}_2 \). Suppose there exists a local diffeomorphism \( f : U \to V \) which satisfies the initial condition \( f(\mathbf{x}) = \mathbf{y}, \partial f^i/\partial y^k(\mathbf{x}) = \phi_1(\mathbf{x}, \mathbf{y}), \partial^2 f^i/\partial y^i\partial y^k = \phi_2(\mathbf{x}, \mathbf{y}) \), and the substitution \( f^i(x) = y^i, \partial f^i/\partial x^k(x) = \phi^i_j(x, y), \partial^2 f^i/\partial x^i\partial x^k = \phi^j_k(x, y) \) satisfies (25) identically for all \( x \), that is, all 2-arrows of \( f \) belong to \( \mathcal{G}_2 \). In this case, we call \( f \) a local solution of \( \mathcal{G}_2 \) with the initial condition \( f^{p,q} \). Clearly, a local solution \( f \) satisfies all the initial conditions defined by its 2-arrows. A global solution of \( \mathcal{G}_2 \) is a diffeomorphism \( f \in Diff(M) \) such that \( f_{ij} \) is a local solution for all coordinate patches \((U, x^i)\) on \( M \). Now \( \mathcal{G}_2 \) admits one global solution, namely \( id \), because \( j_2(id)^{p,q} \in \mathcal{G}_2^{p,q} \) for all \( p \). However, \( \mathcal{G}_2 \) may not admit any other local solutions. The other extreme is a fundamental concept.

**Definition 5** \( \mathcal{G}_2 \) is completely integrable if

i) All 2-arrows of \( \mathcal{G}_2 \) integrate to local solutions

ii) A local solution is uniquely determined on its domain by any of its 2-arrows.

Observe that 2-arrows of a local solution are determined by its 1-arrows in view of the splitting \( \varepsilon \) in (19). Thus a local solution is determined also by any of its 1-arrows. We will see in Section 5 that i) \( \Rightarrow \) ii). The reason is that \( \mathcal{G}_2 \) has the property \( \mathcal{G}_2 \simeq \mathcal{G}_1 = \pi \mathcal{G}_2 \). The complete integrability of \( \mathcal{G}_2 \) is a local condition which can be checked on coordinate patches \((U, x^i)\), because if all “short” 2-arrows of \( \mathcal{G}_2 \) integrate to local solutions, then all 2-arrows of \( \mathcal{G}_2 \) integrate to local solutions. This fact is easily shown as in the proof of Proposition 7.5 in [2]. If \( \mathcal{G}_2 \) is completely integrable, then its local solutions form a pseudogroup on \( M \) because 2-arrows of \( \mathcal{G}_2 \) are closed under composition and inversion by the definition of a groupoid. We will denote this pseudogroup by \( \mathcal{G} \) and its restriction to some \((U, x^i)\) by \( \mathcal{G}_{ij} \).

Now we want to linearize the PDE \( \mathcal{G}_2 \), which amounts to defining its algebroid \( \mathfrak{G}_2 \). First, we recall that the algebroid of \( \mathcal{U}_2 \) is the vector bundle \( J_2T \to M \) whose fiber over \( p \in M \) consists of 2-jets of vector fields at \( p \). So a section \( X_2 \) of \( J_2T \to M \) (with an abuse of notation, we will write \( X_2 \in J_2T \)) is
of the form \((X^i(x), X^j(x), X^i_{jk}(x))\) over \((U, x^i)\). There is a bracket \([\ ,\ ]\) defined on the sections of \(J_2T \to M\), called the Spencer bracket, which turns \(J_2T \to M\) into an algebroid. To define \([\ ,\ ]\), recall the Spencer operator \(D: J_3T \to J_2T \otimes T^*\) locally given by \((X^i(x), X^j(x), X^i_{jk}(x), X^i_{mjk}(x)) \to (\partial_j X^i(x) - X^i_j(x), \partial_j X^i_k(x) - X^i_{jk}(x), \partial_m X^i_{jk}(x) - X^i_{mjk}(x))\). We have the algebraic bracket \([\ ,\ ]_p: (J_3T)_p \times (J_2T)_p \to (J_2T)_p\) whose local formula is obtained by differentiating the usual formula for the bracket of two vector fields three times at \(p\) and replacing derivatives with jet variables. Clearly, \([\ ,\ ]_p\) extends to sections of \(J_3T \to M\) which we denote by \([\ ,\ ]\). Now if \(X_2, Y_2 \in J_2T\), their Spencer bracket is defined by

\[
[X_2, Y_2] \overset{\text{def}}{=} \{X_3, Y_3\} + i_{X_0}D(Y_3) - i_{Y_0}D(X_3) \tag{26}
\]

In (26), \(X_3, Y_3\) are arbitrary lifts of \(X_2, Y_2\) to \(J_3T\), \(X_0 = \pi X_2, Y_0 = \pi Y_2\) where \(\pi: J_3T \to J_0T = T\) is the projection and \(i_Z\) denotes contraction with respect to the vector field \(Z \in J_0T\). The bracket \([\ ,\ ]\) is independent of the lifts \(X_3, Y_3\). If \(X_2 = (X^i(x), X^j(x), X^i_{jk}(x))\) and \(Y_2 = (Y^i(x), Y^j(x), Y^i_{jk}(x))\), we compute

\[
\begin{align*}
[X_2, Y_2]^i &= X^a \partial_a Y^i - Y^a \partial_a X^i \tag{27} \\
[X_2, Y_2]_j &= X^a_j Y^a_i - Y^a_j X^a_i + X^a \partial_a Y^i_j - Y^a \partial_a X^i_j \\
[X_2, Y_2]_{jk} &= X^a_{jk} Y^a_i + X^a_j Y^a_{i,j} + X^a_k Y^a_{i,k} - Y^a_{jk} X^a_i - Y^a_j X^a_{i,k} - Y^a_k X^a_{i,j} \\
&\quad + X^a \partial_a Y^i_{jk} - Y^a \partial_a X^i_{jk}
\end{align*}
\]

Now (27) shows that the projection maps \(J_2T \to J_1T \to J_0T\) preserve brackets. Sometimes we will use the same notation \([\ ,\ ]\) for all these brackets. Of course, \([\ ,\ ]\) has all the properties one expects from a bracket (see [28], [25] for further details). Also, we have the prolongation map \(j_2: J_0T \to J_2T\) defined locally by \(X^i(x) \to (X^i(x), \partial_jX^i_j(x), \frac{\partial^2X^i}{\partial x^j \partial x^k}(x))\) and (27) shows that \([\ ,\ ]\) respects prolongation, that is, \(j_2[X, Y] = [j_2X, j_2Y]\).

Now, rather than defining \(\mathcal{G}_2\) abstractly, we will take a shortcut following [25] and derive the defining equations of \(\mathcal{G}_2\) from (25). This method allows one to do explicit computations in coordinates, but leaves the fundamental relation between \(\mathcal{G}_2\), \(\mathcal{G}_2\) which we will need in Section 4 in dark, as we will see. So we substitute \(y^i = x^i + tX^i(x)\) into \(g(y) = (g_{ij}(y), g^i_{jk}(y))\) in (25), substitute \(\phi_j^i(x, y) = \delta^i_j + tX^i_j(x), \phi^i_{jk}(x, y) = tX^i_j(x)\) in (25) and differentiate the resulting equations with respect to \(t\) at \(t = 0\). The result is

\[
0 = X^a(x) \partial_a g_{ij}(x) + g_{ai}(x)X^j_a(x) + g_{aj}(x)X^i_a(x) \tag{28}
\]

Now (28) defines a bundle of vectors \(\mathcal{G}_2 \to M\) whose fiber over \(p \in (U, x^i)\) consists of those points \((X^i(p), X^j(p), X^i_{jk}(p))\) of \(J_2T\) which satisfy (28). Henceforth, we will omit the variable \(x\) in (28) and use the same notation for points
$X_2 \in J_2T$ and sections $X_2 \in J_2T$ as before. Since $\dim \mathcal{G}_2 = 2 \dim M + \dim O(n)$, we have $\dim(\mathcal{G}_2)_p = \dim M + \dim O(n) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ for all $p \in M$ where $(\mathcal{G}_2)_p$ denotes the fiber of $\mathcal{G}_2 \to M$ over $p$. So $\mathcal{G}_2 \to M$ is a vector bundle of rank $\frac{n(n+1)}{2}$. This fact can be checked also directly from (28) using regular coordinates (see below). The fundamental fact is that the sections of $\mathcal{G}_2 \to M$ are closed with respect to the Spencer bracket $[\ , \ ]$. This follows from the theory ([28], [25]), but can be checked directly using (27) and (28). Thus we obtain the algebroid $\mathcal{G}_2 \to M$ and the infinitesimal version of (25):

$$
\begin{array}{ccc}
J_2T & \xrightarrow{\pi} & J_1T \\
\cup & & \cup \\
\mathcal{G}_2 & \xrightarrow{\pi} & \mathcal{G}_1 
\end{array}
$$

where $\mathcal{G}_2 \simeq \pi_{\mathcal{G}_2} \overset{\text{def}}{=} \mathcal{G}_1$. The splitting $\varepsilon \overset{\text{def}}{=} (\pi_{\mathcal{G}_2})^{-1}$ amounts to expressing $X^i_{jk}(x)$ in (28) in terms of the lower order terms. Since $[\mathcal{G}_2, \mathcal{G}_2] \subset \mathcal{G}_2$, we have $[\varepsilon X^1, \varepsilon Y_1] = \varepsilon [X^1, Y_1]$, $X_1, X_2 \in \mathcal{G}_1$. Note that the Lie algebra of sections of $\mathcal{G}_2 \to M$ (which we denoted by $\mathcal{G}_2$) is an infinite dimensional Lie algebra.

By construction, $\mathcal{G}_2 \to M$ is a linear system of PDE’s. A local solution is a vector field $X = X^i(x)$ such that the substitutions $\frac{\partial X^i}{\partial x^j} = X^i_j$, $\frac{\partial^2 X^i}{\partial x^j \partial x^\sigma} = X^i_{jk}$ identically satisfy (28), that is, the prolongation $j_2(X)$ belongs to $\mathcal{G}_2$. So the fibers of $\mathcal{G}_2 \to M$ consist of initial conditions. The zero section is of course a local solution, but $\mathcal{G}_2 \to M$ may not admit any other local solutions. On the other extreme, we have

**Definition 6** $\mathcal{G}_2 \to M$ is completely integrable if

i) For any initial condition $\xi \in (\mathcal{G}_2)_p$, there exists a local solution around $p$ satisfying this initial condition.

ii) A local solution is uniquely determined on its domain by any of its 2-jets.

Again, a local solution is actually determined by any of its 1-jets in view of the splitting $\varepsilon$ in (29) and i) $\Rightarrow$ ii) for the same reason: $\mathcal{G}_2 \simeq \mathcal{G}_1 = \pi_{\mathcal{G}_2}$ as we will see in Section 5. Now the fundamental fact is that the local solutions around $p$ satisfying the initial conditions $(\mathcal{G}_2)_p$ are closed with respect to the bracket $[\ , \ ]_{J_2T}$, because $[\mathcal{G}_2, \mathcal{G}_2]_{J_2T} \subset \mathcal{G}_2$ and $[\ , \ ]$ respects prolongation. However, we may have to restrict the domains of the local solutions to compute their bracket. Taking the germs of solutions at $p$ as the stalk of a sheaf at $p$, we obtain a coherent sheaf of Lie algebras defined on $M$, but we will prefer to work with the more intuitive presheaf in the next section. Clearly all stalks of this sheaf are isomorphic (see below). If $\mathcal{G}_2$ is completely integrable, what possible choices do we have for this Lie algebra?

It is a fundamental fact that $\mathcal{G}_2$ is completely integrable $\iff \mathcal{G}_2$ is completely integrable. In fact, $\Rightarrow$ is Lie’s first theorem, $\Leftarrow$ is Lie’s second theorem and $\iff$ amounts to constructing the exponential map $\exp : \mathcal{G}_2|U \to \mathcal{G}_2|U$ as envisioned by Lie and will be sketched in the next section.
4 Lie’s theorems, completeness and uniformization

Our purpose in this section is to outline a Lie theoretic derivation of the well known uniformization theorem of space forms in $\text{MRG}$. We will try to avoid the fundamental concepts of $\text{MRG}$ as much as possible in order to emphasize that the present approach generalizes in a quite straightforward way to all prehomogeneous structures where the main tools of $\text{MRG}$ will not be readily available.

Suppose that $\mathfrak{G}_2$ is completely integrable. Let $\mathfrak{X}$ denote the Lie algebra of vector fields on $M$ and $\mathfrak{X}_g \subset \mathfrak{X}$ the vector space of global solutions of $\mathfrak{G}_2$. We have seen that $\mathfrak{X}_g$ is a Lie algebra, but we may have $\dim \mathfrak{X}_g = 0$. Let $\mathfrak{X}_g(U)$ denote the local solutions of $\mathfrak{G}_2$ on $(U, x^i)$. We fix some $p \in U$ and define

\[
j_2(p) : \mathfrak{X}_g(U) \longrightarrow (\mathfrak{G}_2)_p
\]

\[
X = X^i(x) \longrightarrow (j_2 X)(p) = (X^i(p), \frac{\partial X^i}{\partial x^j}(p), \frac{\partial^2 X^i}{\partial x^j \partial x^k}(p))
\]

As we remarked above, $j_2 X = \varepsilon(j_1 X)$, $X \in \mathfrak{G}_2$. Recalling the algebraic bracket $\{ , \}_p : (J_2 T)_p \times (J_2 T)_p \rightarrow (J_2 T)_p$, we have

\[
(j_2[X, Y])(p) = \{\varepsilon((j_1(X)(p)), \varepsilon((j_1(Y)(p)))\}_p \quad X, Y \in \mathfrak{X}_g
\] (31)

because the last two terms on the $\text{RHS}$ of (26) vanish on solutions by the definition of the Spencer operator and the algebraic bracket coincides with the Spencer bracket. Note that the $\text{LHS}$ of (31) needs complete integrability for its definition whereas its $\text{RHS}$ is still defined if we replace $j_1(X)(p)$, $j_1(Y)(p)$ with arbitrary $\xi_1, \eta_1 \in (\mathfrak{G}_2)_p$. Since $\{ , \}$ respects prolongation, $j_2(p)$ is a homomorphism of Lie algebras. If we choose $U$ also simply connected, then any local solution with some initial condition in $(\mathfrak{G}_2)_p$ extends uniquely to $U$ (see below) and $j_2(p)$ becomes an isomorphism of Lie algebras. Therefore $\dim \mathfrak{X}_g(U) = \frac{n(n+1)}{2}$ if $U$ is simply connected which we will assume below.

Let $\mathfrak{X}_g(U, p) \subset \mathfrak{X}_g(U)$ denote the solutions which vanish at $p$. Now $\mathfrak{X}_g(U, p)$ is a subalgebra of dimension $\frac{n(n-1)}{2} = \dim \mathcal{O}(n)$. We call $\mathfrak{X}_g(U, p)$ the stabilizer subalgebra at $p$. In coordinates, the definition of $\mathfrak{X}_g(U, p)$ amounts to setting $X^i(p) = 0$ in (28). If we choose our coordinates regular at $p$, the first formula in (28) shows that $X^i_j(p)$ is skewsymmetric and the second formula in (28) gives $X^i_k(p) = 0$ so that $\mathfrak{X}_g(U, p)$ can be identified (not canonically!) with $\mathcal{O}(n)$. The isomorphism (30) identifies $\mathfrak{X}_g(U, p)$ with its image which is, of course, the splitting in (6) on the level of Lie algebras. Thus $\mathfrak{X}_g(U, p) \simeq \mathcal{L}(\mathfrak{G}_2^{p,p})$ is the Lie algebra of the vertex group $\mathfrak{G}_2^{p,p}$. Note again that the definition of $\mathfrak{X}_g(U, p)$ needs complete integrability whereas the definition of $\mathcal{L}(\mathfrak{G}_2^{p,p})$ does not. The bracket of $\mathfrak{X}_g(U, p)$ can be seen from (27) and reduces to the bracket of two orthogonal matrices in regular coordinates. The key fact is that the components $X^i$ pair with differential terms in (27), (28) so that the substitution $X^i = 0$ turns
everything to algebra “in the vertical direction”. Clearly the choice of $U$ is arbitrary in these arguments and we have the same local scenario everywhere on $M$.

We will now sketch the construction of the exponential map. We assume that $\mathfrak{g}_2$ is completely integrable. We define $i \overset{\text{def}}{=} j_2(p)^{-1}$ and fix some 2-arrow $f^{p,q} \in \mathcal{G}_2$ where $q \in U$ is arbitrary. For $X_2 \in (\mathfrak{g}_2)_p$, let $f(iX_2)(t,p)$ denote the 1-parameter group of local diffeomorphism generated by $iX_2$ such that $f(iX_2)(0,p) = p$. We consider the equation $f(iX_2)(t,p) = q$ in the unknowns $t$ and $X_2$. Since $\mathfrak{g}_0 = J_0T = T$, for any two solutions $t, X_2 = (X^i, X^i_j, X^i_{kj})$ and $\bar{t}, \bar{X}_2 = (\bar{X}^i, \bar{X}^i_j, \bar{X}^i_{kj})$ we have $t = \bar{t}$ and $X^i = \bar{X}^i$. With these unknowns solved uniquely, we now have the freedom for $X^i_j$ to account for the 1-arrow $\pi f^{p,q} \in \mathcal{G}_1$. We should make this choice such that $X_2 = (X^i, X^i_j, X^i_{kj})$, now determined by $X_1 = (X^i, X^i_j)$, will give the desired 2-arrow $f^{p,q}$. It is a remarkable and nontrivial fact that this can be done only in one way. In fact, it turns out that the equation $[j_2 f(iX_2)(t,x)]^{p,q} = f^{p,q}$ uniquely determines $t$ and $X_2$ in such a way that all 2-arrows $[j_2 f(iX_2)(t,x)]^{p,q}$ with $r$ close to $p$ belong to $\mathcal{G}_2$, that is, $j_2 f(iX_2)(t,x)$ is a local solution of $\mathcal{G}_2$ satisfying the initial condition $f^{p,q}$. Therefore, all 2-arrows of $\mathcal{G}_2$ starting from $p$ (or ending at $p$) integrate to local solutions. It follows that all 2-arrows of $\mathcal{G}_2$ inside $U$ also integrate to local solutions because any 2-arrow inside $U$ is a composition (not uniquely) of two such 2-arrows. So we conclude that $\mathcal{G}_2$ is completely integrable. The converse follows along the same lines and amounts to the trick of deriving (28) from (25).

So we have

**Proposition 7** (Lie’s 1st and 2nd theorems) $\mathcal{G}_2$ is completely integrable if and only if $\mathfrak{g}_2$ is completely integrable

Now we want to globalize Lie’s theorems.

If $\gamma$ is any continuous path from $a$ to $b$ with $a \in U$, then we can continue $X_{\gamma}(U)$ uniquely along $\gamma$ like analytic continuation. However, we may not be able to continue indefinitely as the local solutions may not approach a definite limit as we approach some point on the path.

**Definition 8** Suppose $\mathfrak{g}_2$ is completely integrable. Then $\mathcal{G}_2$ is complete if all local solutions can be continued (necessarily uniquely) indefinitely along all paths.

Observe the weakness of Definition 8: it needs complete integrability to define completeness whereas geodesic completeness of the metric needs no assumptions on the curvature. If $M$ is compact, then $\mathfrak{g}_2$ is complete. The proof is identical to the proof of Lemma 7.3 in [2]. If $\mathfrak{g}_2$ is complete and $M$ is simply connected, then the standard monodromy argument shows that any local solution globalizes uniquely to a global solution. Even if $M$ is not simply connected, local solutions may globalize. In this case we call $\mathcal{G}_2$ globalizable. For local Lie groups globalizability is defined and studied first in [19]. This is a subtle concept which we will not touch here (see [2] for a cohomological obstruction to
First, by choosing complete integrability. There are no new ideas involved and we will be very brief.

We define the stabilizer $H$ to $G$, the source of its 2-arrow (or 1-arrow) along the path as in [2]. We call details of these arguments. Clearly $H$ is complete if indefinite continuations are possible. Now the key fact is that $H$ is complete if and only if $G$ is simply connected, then so is $\tilde{G}$. Since $\tilde{M}$ is simply connected, $\tilde{G}$ globalizes on $\tilde{M}$. We will omit the rather straightforward details of these arguments. Clearly $\tilde{G}$ is locally “the same” as $G$. We denote the Lie algebra of global solutions of $\tilde{G}$ by $\tilde{\mathfrak{x}}_g$. By the construction of $\tilde{\mathfrak{x}}_g$, any $X \in \tilde{\mathfrak{x}}_g$ is globally determined by its 2-jet (or 1-jet) at any point $p \in \tilde{M}$ and therefore $\dim \tilde{\mathfrak{x}}_g = \frac{n(n+1)}{2}$. The subalgebra $\tilde{\mathfrak{x}}_g(p) \subset \tilde{\mathfrak{x}}_g$ consisting of the vector fields vanishing at $p$ is isomorphic to $\mathcal{L}(\mathfrak{g}_2^n) \simeq \mathfrak{o}(n)$. Indeed, any statement about $\tilde{\mathfrak{x}}_g(U)$ has a global analog for $\tilde{\mathfrak{x}}_g$.

We will now repeat the above construction by replacing $G$ with $\mathcal{G}_2$ assuming complete integrability. There are no new ideas involved and we will be very brief. First, by choosing $U$ simply connected, we may assume that the local solutions of $\mathcal{G}_2$ inside $U$, that is, the elements of the pseudogroup $G|_U$ are all defined on $U$. We define the stabilizer $H_p \overset{d}{=} \{ f \in G|_U \mid f(p) = p \}$. The exponential map integrates the Lie algebra $\mathfrak{x}_g(U)$ to $G|_U$ and the stabilizer subalgebra $\mathfrak{x}_g(U,p)$ to $H_p \simeq \mathcal{G}_2^n$. Now a local solution has a unique continuation along a path by translating the source of its 2-arrow (or 1-arrow) along the path as in [2]. We call $\mathcal{G}_2$ complete if indefinite continuations are possible. Now the key fact is that $\mathcal{G}_2$ is complete if and only if $\mathcal{G}_2$ is complete. To see this, we first observe that the “short” 2-arrows continue indefinitely if and only if all 2-arrows continue indefinitely. This is shown easily by expressing an arbitrary 2-arrow as a composition of short 2-arrows. Now the statement follows from the exponential map. If $M$ is simply connected, then the pseudogroup $G$ globalizes to a transformation group $G$ on $M$ which acts transitively and effectively on $M$. Any transformation $f \in G$ is globally determined by any of its 2-arrows (or 1-arrows) so the geometric order of $(G,M)$ is one. Clearly $H_p \overset{d}{=} \{ f \in G \mid f(p) = p \}$ is isomorphic to $O(n)$. Observe that the Klein geometry $(G,M)$ determines the vertex connection $\{ G,M,H \} = \mathbb{R}$ that we started with by construction. If globalization is not possible on $M$, we lift the pseudogroup $G$ to a pseudogroup $\tilde{G}$ on $\tilde{M}$ which globalizes to a global transformation group $\tilde{G}$ which acts affectively and transitively on $\tilde{M}$. The local properties of the Klein geometry $(\tilde{G}, \tilde{M})$ are “the same” as the pseudogroup $G|_U$ (which may be viewed as a local Lie group in the classical sense if consider 2-arrows eminating from some based point as in the construction of the exponential map). Now another key fact is that the deck transformations $\text{Deck}(\tilde{M}) \simeq \pi_1(M)$ belong to $\tilde{G}$ since they commute with the projection $\tilde{M} \to M$ and they act as a discontinuous group on $\tilde{M}$.

Before we state the next proposition, we need one abstract construction which sheds further light on the above scenario. The covering map $\rho : \tilde{M} \to M$ pulls back the groupoid $\mathcal{G}_2$ to a groupoid $\rho^{-1}\mathcal{G}_2$ by pulling back its arrows. This construction does not need complete integrability and works with all Lie groupoids. In particular, the algebroid $\mathfrak{G}_2$ pulls back to $\rho^{-1}\mathfrak{G}_2$ which is the
algebroid of $\rho^{-1}G_2$. If $G_2 (\mathfrak{G}_2)$ is completely integrable, then $\rho^{-1}G_2 (\rho^{-1}\mathfrak{G}_2)$ is also completely integrable. As we remarked above, completeness of $\mathfrak{G}_2$ implies completeness of $\rho^{-1}\mathfrak{G}_2$. Conversely, $\rho^{-1}\mathfrak{G}_2$ is complete if and only if $\rho^{-1}\mathfrak{G}_2$ is globalizable (and therefore $\mathfrak{G}_2$ is also complete, compare to [14], Theorem 4.6, pg. 176). Similar statements hold for $G_2$. So we showed above that $\rho^{-1}\mathfrak{G}_2$ globalizes to $\hat{\mathfrak{X}}_g$ if and only if $\rho^{-1}G_2$ globalizes to $\hat{G}$. By the definition of the exponential map, $\hat{G}$ has $\hat{\mathfrak{X}}_g$ as its infinitesimal generators.

To summarize, we state

**Proposition 9** (Lie’s 3rd theorem) $\rho^{-1}\mathfrak{G}_2$ integrates to a Lie algebra of global vector fields $\tilde{\mathfrak{X}}_g$ on $\tilde{M}$ if and only if $\rho^{-1}G_2$ integrates to a global transformation group $\tilde{G}$ on $\tilde{M}$. In both cases, $\tilde{G}$ has $\hat{\mathfrak{X}}_g$ as its infinitesimal generators. The Klein geometry $(\tilde{G}, \tilde{M})$ defines the vertex connection $R$ and has the above stated properties.

Note also that some $f^{p,q} \in \mathfrak{G}_2^{p,q}$ induces an isomorphism $f^{p,q} : (\mathfrak{G}_1)_p \to (\mathfrak{G}_1)_q$ (this association does not need complete integrability, see [3]) which lifts to the adjoint action of $G$ on its Lie algebra $\tilde{\mathfrak{X}}_g$.

Now what possibilities do we have for the Klein geometry $(\tilde{G}, \tilde{M})$? Let $(G_1, M_1), (G_2, M_2)$ be two Klein geometries. A morphism $(G_1, M_1) \to (G_2, M_2)$ is a pair $(\phi, f)$ where $\phi : G_1 \to G_2$ is a Lie group homomorphism and $f : M_1 \to M_2$ is a smooth map satisfying $f(gx) = \phi(g)f(x)$. We define an isomorphism $(G_1, M_1) \simeq (G_2, M_2)$ in the obvious way. Now it is natural to believe that $(G, \tilde{M})$ is isomorphic to one of

$$
(G(1), \mathbb{S}^n) \quad (G(0), \mathbb{R}^n) \quad (G(-1), \mathbb{H}^n)
$$

We can deduce (32) from the uniformization theorem of $MRG$. To do this, we need to show two more facts.

1) $\mathfrak{G}_2$ (or $G_2$) is complete $\iff$ the underlying metric is geodesically complete.
2) $\mathfrak{G}_2$ (or $G_2$) is completely integrable $\iff$ the underlying metric has constant curvature.

We will prove 2) in Section 7. However, it is also possible to deduce (32) from a Lie theoretic statement which therefore implies uniformization theorem of $MRG$. To do this, let $\mathcal{A}$ denote the set of all Klein geometries $(G, G/H)$ satisfying the following properties.

i) $H = O(n)$ and $O(n) \subset G$ is a Lie subgroup.
ii) $G/O(n)$ is simply connected.
iii) $G$ acts effectively on $G/O(n)$ with geometric order $m = 1$
iv) $(G, G/H, H) = \mathbb{R}$

The requirement $m = 1$ in iii) is redundant by ii) and the compactness of $O(n)$. With these assumptions, we need to show that any $(G, G/H) \in \mathcal{A}$ is isomorphic to one of (32). Note that $(\tilde{G}, \tilde{M})$ satisfies all the requirements.

The above group theoretic statement is actually a statement about Lie algebra pairs $(\mathfrak{g}, \mathfrak{o}(n))$ in view of Remark 1 (see also Section 7). We believe that iv) is also redundant and is a consequence.
At any rate, we will state

**Proposition 10** (Lie’s 3rd theorem, refined form) The Klein geometry \((\tilde{G}, \tilde{M})\) in Proposition 9 is isomorphic to one of (32)

Propositions 9, 10 may seem somewhat surprising at first, but actually they are quite expected. Indeed, recall the classical formula for the Lie derivative of a metric:

\[
\mathcal{L}_X(g_{ij}) = X^a \partial_a g_{ij} + g_{ai} \partial X^i \partial_x j + g_{aj} \partial X^i \partial x^i + g^{a}_{ij} \partial X^a \partial_x^i.
\]

If we replace \((X^i, \partial X^i / \partial x^j)\) by the 1-jet \(X_1 = (X^i, X^i_j) \in J_1 T\), we get the RHS of the first formula in (28). Now any linear geometric object of arbitrary order can be Lie-differentiated with respect to a vector field as explained in [34]. If we compute the Lie derivative of the second order geometric object \(g\) with respect to a vector field and replace \((X^i, \partial X^i / \partial x^j, \partial^2 X^i / \partial x^i \partial x^j)\) by \(X_2 = (X^i, X^i_j, X^i_{jk}) = \epsilon(X_1) \in J_2 T\), we get the formulas on the RHS of (28). So the solutions of (28) are Killing vector fields for \(g\). Since this computation deduces Killing vector fields from Lie derivative, covariant differentiation in tensor calculus must be a special case of Lie derivative. The derivation of the formula (21) in [2] shows that this is indeed the case. We believe that covariant differentiation in tensor calculus owes its existence to the splitting of (2) in the exceptional case \(k = 1, n \geq 1\). We hope that this crucial point will become more transparent in the next section.

5 Curvature

Definitions 5, 6 have a serious deficiency: they are not effective. How do we decide complete integrability of \(G_2, \mathcal{G}_2\) from (25), (28)? Our purpose in this section is to define two curvatures \(\mathcal{R}, \mathcal{R}\) where \(\mathcal{R} = G_2\)-curvature and \(\mathcal{R} = \mathcal{G}_2\)-curvature. We will show that \(\mathcal{G}_2\) is completely integrable \(\Leftrightarrow \mathcal{R} = 0\) and \(G_2\) is completely integrable \(\Leftrightarrow \mathcal{R} = 0\). Therefore the conditions \(\mathcal{R} = 0, \mathcal{R} = 0\) may be taken as the definitions of complete integrability. The main message here is that curvature is always an obstruction to complete integrability for pre-homogeneous structures.

We start with (28). We separate the second formula in (28) from the first one and rewrite it as an equivalent first order system

\[
\begin{align*}
\partial_j X^i & = X^i_j, \\
\partial_k X^i_j & = -g^i_{ak} X^a_j - g^a_{ij} X^a_k + g^a_{jk} X^a_i - X^a \partial_a g^i_{jk}.
\end{align*}
\]

Now (33) expresses the derivatives of the unknown functions \((X^i, X^i_j)\) in terms of the functions themselves. The find the integrability conditions of (33), we differentiate the second formula with respect to \(x^r\), substitute back from (33) and alternate \(r, k\). The result is

\[
\mathcal{R}^i_{kr,j} \overset{\text{def}}{=} \left[ \hat{\mathcal{R}}^i_{kr,j} \right]_{[kr]} = 0
\]

where \([kr]\) denotes alternation of the indices \(k, r\) and
\[ R_{ki,j} \overset{def}{=} X_k^i \partial_r g_{aj} + X^b (\partial^2 r g_{jk} + g^{a}_{jb} \partial_h g_{ra} - g^a_{ab} \partial_b g_{jr}) + X_j^b (\partial_r g_{bk} - g_{ak} g_{br}) + X_j^a (\partial_b g_{jk} - g^a_{b} g_{br} + g^a_{jk} g_{ra}) - X_k^a (\partial_b g_{jk} + g^a_{jk} g_{ra}) \] (35)

Observe the occurrence of the Riemann curvature tensor twice in (35) if we replace \( g_{kj} \) with the Christoffel symbols \( \Gamma^i_{kj} \). Using (24), one can show that this Riemann curvature tensor satisfies many identities as the genuine one.

Now, by the well known existence and uniqueness theorem for first order systems of PDE’s with initial conditions (see, for instance, [17], pg.224-227), if (35) is satisfied identically for all points \( X_1 = (X^i, X^j) \in J_1T \), then we may choose an arbitrary initial condition \((X^i(p), X^j(p)) \in (J_1T)_p \) and solve (33) uniquely for a vector field \( X^i(x) \) defined around \( p \) and satisfying \((j_1X)(p) = (X^i(p), X^j(p)) \). Of course, \( j_2X \in J_2T \) but the problem is that we may not have \( j_2X \in \mathcal{G}_2 \) even if we choose \((X^i(p), X^j(p)) \in (\mathcal{G}_1)_p \). Obviously, there is a missing integrability condition which should involve differentiation of the first formula in (28). So we join the first formula of (28) to (33) and rewrite (33) as

\[ \partial_j X^i = X^j \] (36)

Clearly, (28) is equivalent to (36). In the old works, (36) is called a mixed system due to the constraint coming from the second equation. It is shown in these works that such a system reduces to a first order system without any constraint after successive prolongations (see, for instance, [9], [32]). However, such proofs implicitly assume that some rank conditions are satisfied so that the implicit function theorem is applicable. Around 1970, these rank conditions are organized by D.C.Spencer and his coworkers into a powerful technique, now called Spencer cohomology. For instance, see [26], pg 254-255 for a direct proof in coordinates which derives the constant curvature condition (49) from the formal integrability of (28) and makes heavy use of Spencer cohomology (this proof assumes \( g^i_{kj} = \Gamma^i_{kj} \) and adds the second equation of (28) to the first equation as a trick, see pg. 251).

We will now derive the missing integrability condition by an elementary method which avoids Spencer cohomology. (48) below will justify that this method recovers all the integrability conditions. So we differentiate the second equation of (36) with respect to \( x^k \), substitute \( \partial_k X^a_j \) from the third equation, and alternate \( k, i \) in \( \partial_k X^a_j \). Another straightforward computation gives

\[ R_{ki,j} \overset{def}{=} \left[ \frac{\partial}{\partial x^k} R_{ki,j} \right]_{[ki]} = 0 \] (37)
where

\[ \hat{R}_{ki,j} \overset{\text{def}}{=} X^a_k \partial_a g_{ij} + X^a_i \partial_k g_{aj} + X^a_j \partial_k g_{ai} + X^a_i \partial_k g_{aj} \quad (38) \]

\[-X^b_j g_{ai} g_{bk} - X^b_k g_{ai} g_{bj} + X^b_k g_{ai} g_{bj} - X^b_k g_{ai} g_{bj} \]

We define the horizontal (over \( M \)) 2-form \( \mathcal{R} \) by

\[ \mathcal{R}_{ij} \overset{\text{def}}{=} (\mathcal{R}_{ij,r}, \mathcal{R}_{ij,r}') \quad (39) \]

We write \( \mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2) \) and call \( \mathcal{R} \) the algebroid curvature. Now a section of the dual bundle \((J_1T)^* \rightarrow M\) is locally of the form \( \xi^i = (\xi^1, \xi^2) \) and pairs with a section \((X^i, X^j)\) of \( J_1T \rightarrow M \) linearly to the function \( \xi_0 X^a + \xi_1 X^b \). We denote this pairing by \((X_1, \xi^1)\). We believe \( \mathcal{R}(X_0, Y_0)(Z_1) \in \mathcal{G}_1^* \) and \( \mathcal{R}(X_0, Y_0)(Z_1) \in (J_1T)^*, X_0, Y_0 \in J_0T = T, Z_1 \in \mathcal{G}_1, Z_1 \in J_1T \). A direct proof of these statements in coordinates requires formidable amount of computation. Assuming this for the moment, it follows that \((W_1, \mathcal{R}(X_0, Y_0)(Z_1))\) and \((W_1, \mathcal{R}(X_0, Y_0)(Z_1))\) are functions on \( M \). In particular, the horizontal 2-form \((Z_1, \mathcal{R}(X_0, Y_0)(Z_1))\) lives in the variational complex and descends to a 2-form on the quotient \( J_1T/\mathcal{G}_1 \) if \( \mathcal{G}_2 \) is completely integrable. At any rate, \( \mathcal{R} \) is not a tensor but a second order object. We believe that the component \( \mathcal{R}_2 \) is separated from the full curvature \( \mathcal{R} \) and tamed into the Riemann curvature tensor in the same way as \( (23) \) is separated from \( (22) \) and tamed into covariant differentiation. We should recall here that Riemann writes only one formula in his foundational Habilitationsschrift and the Riemann curvature tensor is introduced later by others as a part of covariant differentiation and tensor calculus.

We now turn to \((25)\). We first single out the fraud Riemann curvature tensor.

\[ R^i_{rj,k}(x) \overset{\text{def}}{=} [\partial_r g^j_{ik}(x) - g^b_{rk}(x) g^b_{jk}(x)]_{[rj]} \]

We differentiate the second equation in \((25)\) with respect to \( x^r \) assuming that \( y = y(x) \) is a solution, substitute back \( \frac{\partial \phi^i_j(x,y)}{\partial x^s} \) from this equation and alternate \( r,j \). For simplicity of notation, we write \( \phi^i_j \) for \( \phi^i_j(x,y,y) \) and \( \phi^i_j \) for \( \phi^i_j(x,y)^{-1} \), keeping in mind that \( \phi^i_j \) and \( \phi^i_j \) depend on both source and target variables. The result is

\[ \mathcal{R}^i_{rj,k} \overset{\text{def}}{=} R^d_{ab,c}(x) \phi^d_{ik}(\hat{\phi})^r_{cj}(\hat{\phi})^b_{dk} - R^i_{rj,k}(y) = 0 \quad (40) \]

The formula \((40)\) is well known from tensor calculus (see [10] for a proof that \((40)\) implies \((49)\) when \( g^b_{jk} = \Gamma^b_{jk} \)). By the same method above, we now differentiate the first equation of \((25)\) with respect to \( x^k \), substitute \( \frac{\partial \phi^i_j(x,y)}{\partial x^k} \) from the second equation and alternate \( k,j \) in \( \frac{\partial \phi^i_j(x,y)}{\partial x^r} \). The final result is

\[ \mathcal{R}^i_{kj} \overset{\text{def}}{=} [\hat{R}^i_{kj} ]_{[kj]} = 0 \quad (41) \]
Now any \( T \) e the base point \( G \) coordinates around it. The main idea is to “factor” all 1-arrows of \( G \). We hope that the next computation will further clarify this how the concept of torsion emerges as a necessity this point, we will leave it to the reader to work out and judge for h erself/herself variables when we differentiate, which is not possible in the principal bundle. At the derivation of \( R \) as long as we do not differentiate.

Thus the principal bundle \( \cup \) This factorization is the passage from the principal bundle to the groupoid.

Clearly, \( \alpha^q.e, \beta^q.e \) are not unique in this factorization because \( \lambda^e.e = \lambda^e.e \circ \lambda^q.e \) and \( \lambda^q.e = \lambda^e.e \circ \lambda^p.e \) work too with arbitrary \( \lambda^e.e \in G_2^p.e \). This factorization is the passage from the principal bundle to the groupoid. Thus the principal bundle \( \cup \) and the groupoid \( G_2 \) are equivalent objects as they determine each other, as long as we do not differentiate. Indeed, in the derivation of \( R \) in (43), we regard both source and targets of the arrows as variables when we differentiate, which is not possible in the principal bundle. At this point, we will leave it to the reader to work out and judge for himself/herself how the concept of torsion emerges as a necessity from this deficiency of the principal bundle. We hope that the next computation will further clarify this point.

We need the above factorization only for \( G_1 \). We now have \( \phi^i_j(x, y) = \alpha^{-1}_i(y, e) \) \( \beta^i_j(x, e) \). For simplicity, we omit the base point \( e \) from our notation and write \( \phi(x, y) = \alpha(x)^{-1} \circ \beta(y) \), that is,

\[
\phi(x, y) = \alpha^{-1}_i(y, e) \beta^i_j(x, e)
\]

We now substitute (44) into (25). The new equations obtained in this way are of course equivalent to (25). We now repeat the above derivation of \( R \) using these new equations. This amounts to substituting (44) into (35) and (38). Now (40) becomes

\[
\left[ \mathcal{R}^i_{rj,k}(x, \beta(x)) \right]_{[rj]} - \left[ \mathcal{R}^i_{rj,k}(y, \alpha(y)) \right]_{[rj]} = 0
\]

where

\[
\mathcal{R}^i_{rj,k}(x, \beta(x)) \overset{def}{=} R^d_{ab,c}(x)\beta^a_j(x)\beta^{-1}(x)^b_j\beta^{-1}(x)^c_k\]

\[
\mathcal{R}^i_{rj,k}(y, \alpha(x)) \overset{def}{=} R^d_{ab,c}(y)\alpha^a_j(y)\alpha^{-1}(y)^b_j\alpha^{-1}(y)^c_k
\]
We now substitute (44) into the peculiar formula (38). A surprising computation shows that (41) becomes

\[
\begin{align*}
\tilde{R}_{ik,j}(x, \beta(x)) & - \tilde{R}_{ik,j}(y, \alpha(y))_{[ik]} = 0 \quad (46)
\end{align*}
\]

where

\[
\begin{align*}
\tilde{R}_{ik,j}(x, \beta(x)) & \overset{\text{def}}{=} \beta^{-1}(x)^a\beta^{-1}(x)^b\beta^{-1}(x)^c(\partial_a g_{bc}(x) + g_{da}(x)g_{bc}(x)) \quad (47) \\
\tilde{R}_{ik,j}(y, \alpha(y)) & \overset{\text{def}}{=} \alpha^{-1}(y)^a\alpha^{-1}(y)^b\alpha^{-1}(y)^c(\partial_a g_{bc}(y) + g_{da}(y)g_{bc}(y))
\end{align*}
\]

Therefore, \( \mathcal{R} = 0 \) if and only if

\[
\begin{align*}
\tilde{R}_{rj,k}(x, \beta(x))_{[rj]} & = c^i_{rj,k} \quad (48) \\
\tilde{R}_{rj,k}(x, \beta(x))_{[rj]} & = c_{rj,k}
\end{align*}
\]

for some constants \( c^i_{rj,k}, c_{rj,k} \). Now (48) is the \( \text{MC} \) equations for the pseudogroup \( G \) or the global transformation group \( \tilde{G} \) in Section 4 (see [23] for \( \text{MC} \) equations for more general pseudogroups than considered here). The recovery of the group shows that our elementary method gives all the integrability conditions.

Thus we have

**Proposition 11** The following conditions are equivalent

i) \( G_2 \) is completely integrable

ii) \( \mathcal{R} = 0 \) on \( G_1 \)

iii) \( G_2 \) is completely integrable

iv) \( \mathcal{R} = 0 \) on \( G_1 \)

v) \( \text{MC} \) equations (48) hold

Now the first equation of (28) is formally integrable if and only if the constant curvature condition

\[
\mathfrak{R}_{jr}^i = c(\delta^i_k g_{jr} - \delta^i_r g_{jk}) \quad (49)
\]

holds. This equivalence is well known. As we remarked above, it is proved in [25], pg. 254-255 and also in [7], Proposition 2.13. Clearly, complete integrability implies the formal integrability of the first (in fact both) equation of (28). Now the weaker concept of formal integrability is equivalent to complete integrability as shown in [13] in the case of pseudogroups of finite type. Therefore

**Proposition 12** The conditions of Proposition 11 are equivalent to

i) (28) is formally integrable

ii) (49) holds
Propositions 11, 12 have a surprising consequence. Let $G_2, \tilde{G}_2$ two Riemannian structures defining the same metric structure, that is, $\pi G_2 = \pi \tilde{G}_2$. Then $R = 0$ if and only if $\tilde{R} = 0$. Indeed, both conditions are equivalent to (49) which can be checked working with the metric only. Therefore, as far as (49) is concerned (but possibly not further!), all the auxiliary objects (23) are equal and the Levi-Civita connection has no priority, answering the fair question in Section 2. We recall here again that Definition 4 excludes nothing from $MRG$ and what it includes is topologically trivial since Riemannian structures according to Definition 4 are in 1-1 correspondence with reductions of the structure group $G_2(n)$ of the second order principal (co)frame bundle to the subgroup $\varepsilon O(n)$ which is homotopically equivalent to $O(n)$.

We conclude this section by clarifying some ambiguities in [2] which were not clear to us at the time of writing [2]. If we linearize the groupoid with defining equations (39) in [2], we arrive at (21) in [2]. The vector fields which solve (21) should be called right invariant vector fields according to the convention in [2] but infinitesimal generators according to this note. No attention is paid to the infinitesimal generators in [2] but everything is based on left invariant vector fields. On the other hand, it is the infinitesimal generators which integrate to solutions of (21) whereas the left invariant vector fields integrate to right local diffeomorphisms whose 1-arrows are computed by the formula (45) in [2]. This corresponds to the well known fact that left invariant vector fields integrate to right translations and right invariant vector fields integrate to left translations on a Lie group. Therefore, Definition 4.1, Lemma 4.2 and some related arguments are not essential for the main purpose of [2] and torsion can be avoided also in [2] as in this note.

6 Affine structures

Definition 13 An affine structure on $M$ is a subgroupoid $A_2 \subset U_2$ with $\{A_2\} = A$.

The defining equations of $A_2$ are given by the second formula in (25) where we should replace $g^i_{jk}$ by $\Gamma^i_{jk}$. So $A_2$ is nothing but a torsionfree affine connection on the first order principal (co)frame bundle of $M$. Clearly, a Riemannian structure canonically determines an affine structure. Now the algebroid $\mathfrak{A}_2$ is defined by the second formula in (28). All the constructions and propositions in Sections 3, 4, 5 carry over word by word. There is only one Klein geometry up to isomorphism in the uniformization theorem (see [14], [34] for a thorough study) which can be derived also from a Lie theoretic statement. The $R_1$ and $\mathfrak{R}_1$ components of $R$ and $\mathfrak{R}$ disappear. The $MC$ equations involve only these components. We omit further details as our main purpose in this note was to use affine structures to emphasize some facts in Riemannian geometry.
7 Pre-homogeneous structures

This note clearly scratches only the tip of an iceberg and leaves many questions unanswered even in affine geometry let alone Riemannian geometry. Still, we feel that it may be useful to formulate some open problems (admittedly by far obvious) for pre-homogeneous structures with the hope that they may activate some research.

At this stage, it is quite clear how to define a pre-homogeneous structure. However there is a technical difficulty we should clarify first. Let \( G \) be a connected Lie group, \( H \neq \{e\} \subset G \) a discrete subgroup with \( H \cap Z(G) = \{e\} \), that is, \( G \) acts effectively on \( M = G/H \). If \( N \) denotes the base manifold \( G \), then \( m = 0 \) for the Klein geometry \((G, N)\) but \( m = 1 \) for \((G, M)\). Thus the geometric order may decrease by one (but not more, see [3]) when we pass to a covering. The main point here is that geometric order is a global concept whereas the vertex connection is actually a local concept. This is clear from the construction of the local pseudogroup \( G_U \) in Section 4 which determines the vertex connection. Indeed, we need not determine all \( g \in G \) in the formula \( \mathcal{H}_m = \{g\} \) in the Introduction but only those \( g \) near the identity. In [3] we defined also the infinitesimal order \( \overline{m} \) of an effective infinitesimal Klein geometry \((\mathfrak{g}, \mathfrak{h})\). As in Remark 1, if \( M \) is simply connected in \((G, M)\), then \( m = \overline{m} \). So the concept needed in this note is \( \overline{m} \) but we based our study on \( m \) because it is more intuitive and geometric than \( \overline{m} \) in the same way as the subtle nonlinear object \( \mathfrak{g}_2 \) is easier to grasp geometrically than its linearization \( \mathfrak{g}_2 \).

Now let \((G, M)\) be a Klein geometry with geometric order \( m \) and \( M \) simply connected. As explained in the Introduction, we have the vertex connection \( \{G, M, H\} \) as a conjugacy class in \( G_{m+1} \). Recall the definition of \( \{G_{m+1}\} \) in Section 3, we make

Definition 14 A pre-homogeneous structure on \( M \) of order \( m+1 \) is a transitive Lie subgroupoid \( G_{m+1} \subset U_{m+1} \) such that \( \{G_{m+1}\} = \{G, N, H\} \) for some Klein geometry \((G, N)\) of geometric order \( m \). The vertex connection \( \{G, N, H\} \) is the model for \( G_{m+1} \).

Therefore, Riemannian and affine structures as defined in Sections 3, 4 are special pre-homogeneous structures. We now check that a parallelizable manifold as defined in [2] (see Definition 3.1 in [2]) is another special case. So let \( G \) be a any Lie transformation group which acts simply transitively on \( M \). With \( H_0 \) as defined in the Introduction, we have \( H_0 = \{g\} \). Therefore \( m = 0 \) and all stabilizers are identity. It follows that any Klein geometry \((G, M)\) determines the same vertex connection \( \{G, M, \{e\}\} \) with the representative \((o, \delta^i_j)\) where \( o \) is the origin of \( \mathbb{R}^n \) and the isomorphism classes of such simply connected Lie groups is the same as all Lie algebras. Omitting the base point from our notation, the stabilizer \( \{id\} \) injects into \( G_1(n) \) as \( j_1(id) = I \) whose conjugacy class is \( \{I\} \). Therefore, for \( m = 0 \), a pre-homogeneous structure on \( M \) is a first order groupoid \( \mathcal{H}_1 \) on \( M \) with vertex groups \( \mathcal{H}_1^{p,p} = j_1(id)^{p,p} \), which is equivalent to the Definition 3.1 in [2]. To construct the geometric object, the right
(or left) coset space $G_1(n)/I = G_1(n)$ and we define the components of $F$ by $F_j^i(a) \overset{\text{def}}{=} a^i_j$, $a \in G_1(n)$, so that $ab^{-1} = I \iff F(a) = F(b)$ as in (14). Note that $F$ is again globally defined and a polynomial map of degree one. Now we construct the first order geometric object $\omega$ in the same way as we did $g$ in Section 3: $\omega$ has components $\omega_j^i(x)$ with $\det a \neq 0$ and $\omega_j^i(p) = \delta_j^i$ in regular coordinates. Some 1-arrow of $U$ preserves $\omega$ if and only if it belongs $H_1$ and we get the defining equations (39) in [2]. Note that we can define $\omega_j^i(x)$ using (50) we check that $\omega$ is a homomorphism. As in the derivation of (7), we now have

\[ (a_1, a_2, a_3)(b_1, b_2, b_3) = (c_1, c_2, c_3) \]
\[ c_1 = a_1 b_1 \]
\[ c_2 = a_1 b_2 + a_2 (b_1)^2 \]
\[ c_3 = a_1 b_3 + 3a_2 b_1 b_2 + a_3 (b_1)^3 \]

We define the map $\varepsilon : G_2(1) \to G_3(1)$ by $\varepsilon(a_1, a_2) = (a_1, a_2, \frac{3}{2} \frac{a_2^2}{a_1})$ and using (50) we check that $\varepsilon$ is a homomorphism. As in the derivation of (7), we now have

\[ (a_1, a_2, a_3) = \begin{pmatrix} a_1, a_2, \frac{3}{2} \frac{(a_2)^2}{a_1} \end{pmatrix} \begin{pmatrix} a_1, a_2, \frac{3}{2} \frac{(a_2)^2}{a_1} \end{pmatrix}^{-1} (a_1, a_2, a_3) \]
\[ = \begin{pmatrix} a_1, a_2, \frac{3}{2} \frac{(a_2)^2}{a_1} \end{pmatrix} \begin{pmatrix} 1, 0, a_3 - \frac{3}{2} \frac{(a_2)^2}{a_1} \end{pmatrix} \]

and we observe that $a_3 - \frac{3}{2} \frac{(a_2)^2}{a_1}$ is the defining formula for the Schwarzian derivative! Thus (51) defines the semidirect product

\[ G_3(1) = G_2(1) \ltimes K_{3,1}(1) \]

where $K_{3,1}(1) = \mathbb{F}$, but with a fundamental difference: (7) holds for all $n \geq 1$ whereas (51) holds only for $n = 1$, that is, we can define an affine structure on any smooth manifold whereas to define the above structure we must have, at least for the moment, $\dim M = 1$. We can now define the groupoid $G_3$, but what will we be dealing with? So it seems reasonable to ask

**Q1 :** Do all splittings inside (2) arise from Klein geometries?

On the infinitesimal level, **Q1** asks whether all finite dimensional Lie subalgebras of the Lie algebra of formal vector fields are induced by Klein geometries $(g, h)$. There is a striking relation of (51) to Riccati equation ([1]).
Let \((G, M)\) be a Klein geometry such that \(H \subset G\) is an algebraic subgroup. We do not know whether the injection of \(H\) inside \(G_{m+1}(n)\) given by Proposition 5.5 in [3] imbeds \(H\) as an algebraic subgroup. As remarked in Section 2, we can always express \(G_{m+1}(n)/H\) locally as a zero set, but the problem is to find \(F\) explicitly.

**Q2**: Given \((G, M)\), is there an effective algorithm for deriving the defining equations of \(G_{m+1}\) in some canonical form?

Recently the moving frame method of Cartan is perfected to its final form in the remarkable papers \([11],\ [12]\) in terms of a powerful and concrete algorithm (see also \([21]\) for a simplified overview). We believe that this algorithm will play a decisive role in answering **Q2**. In fact, we believe that the generalization of the Erlangen Program that we propose is a special case and, we hope, also a geometric justification of the moving frame method in \([11],\ [12]\) in the case of transitive actions of Lie groups, that is, pseudogroups of finite type.

Once we have the defining equations of \(G_{m+1}\), the rest is as in this note, \([2]\) and \([25]\): We linearize \(G_{m+1}\) to get the algebroid \(\mathfrak{g}_{m+1}\). In these equations the top order one will express \((m+1)^{th}\) order jet in terms of lower order jets but the lower order equations may not be as innocent as in (28).

**Q3**: Is there an effective algorithm for computing the components of the curvatures \(R, \mathcal{R}\)?

We again believe that the moving frame method will be of great help here.

To our embarrassment, we are unable to prove the easy should be \(\mathcal{R} = 0 \Rightarrow \mathcal{R} = 0\) using (35), (38), (40), (42).

**Q4**: Give the invariant definitions of \(\mathcal{R}, \mathcal{R}\) and prove the Lie’s theorems \(\mathcal{R} = 0 \iff \mathcal{R} = 0\).

The uniformization theorem in its full formulation gives rise to two problems. First, when \(\mathcal{R} = 0\), the completeness of a pre-homogeneous structure is defined in the same way. The equivalence \(G_{m+1}\) complete \(\Leftrightarrow\) \(G_{m+1}\) complete is again not difficult to show. However, metric completeness in \(MRG\) suggests that completeness should not need \(R = 0\). We believe that the exponential map exists in some weak form without the assumption \(R = 0\) (see remark 3) on pg. 22 of \([2]\)) but are unable to make much progress with it. Completeness is surely one of the most subtle concepts in the theory.

**Q5**: What is the definition of completeness of a pre-homogeneous structure in the presence of curvature?

Second, it is quite clear that the number of isomorphism classes of simply connected Lie groups defining the vertex connection \(\{G, N, H\}\) is equal to the number of the possible uniformizing Klein geometries up to isomorphism. We believe that this number and the vertex connection are uniquely determined by the filtration (9) in [3]. For \(m = 0\), this number is infinite since it is equal to the number of isomorphism classes of all Lie algebras. We have seen that it is three for Riemannian structures and one for affine structures.

**Q6**: Is this number always finite for \(m \geq 1\)?

The constant curvature condition (49), though it is quite natural and geometric from the point of view of \(MRG\), is a total mystery for us and we are
unable to express the constant in (49) in terms of the structure constants in (48).

Q7: Is (49) peculiar to Riemannian geometry or a particular instance of a more general phenomenon?

Finally we come to the surely most subtle part of the theory.

Q8: Develop the theory of characteristic classes (both primary and secondary) for pre-homogeneous structures.

We believe that the variational (bi)complex ([33], [4], [31],[20]) and the more recent invariant variational (bi)complex ([4], [5],[16]) will play a fundamental role in such development.

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Ercüment Ortaçgil, Mathematics Department, Boğaziçi University, Bebek, 34342, Istanbul, Turkey
e-mail: ortacgil(}@boun.edu.tr