Computing divisorial gonality is hard

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Abstract

The gonality $\text{gon}(G)$ of a graph $G$ is the smallest degree of a divisor of positive rank in the sense of Baker-Norine. In this note we show that computing $\text{gon}(G)$ is NP-hard by a reduction from the maximum independent set problem. The construction shows that computing $\text{gon}(G)$ is moreover APX-hard.

1 Introduction and notation

In [4], Baker and Norine developed a theory of divisors on finite graphs in which they uncovered many parallels between finite graphs and Riemann surfaces. In particular, they stated and proved a graph theoretical analogue of the classical Riemann-Roch theorem. See [3, 7, 11] for background on the interplay between divisors on graphs, curves and tropical curves.

As observed in [4], there is also close connection between divisor theory and the chip-firing game of Björner-Lovász-Shor [6]. See [13] for the connection to the Abelian sandpile model and Biggs’ dollar game [5].

An important parameter associated to a connected graph $G$, in the context of divisor theory, is the (divisorial) gonality $\text{gon}(G)$ of $G$. This is the smallest degree of a positive rank divisor. In terms of the chip-firing game, the gonality can then be expressed as $\text{gon}(G) = 2|E| - |V| - \lambda_1(Q(G))$, where $\lambda_1(Q(G))$ is the maximum number of chips that can be placed on the graph $G = (V, E)$ such that adding a chip at an arbitrary node will result in a finite game.

There are only few graph classes for which the gonality is known. Examples include trees, complete multipartite graphs and $n \times m$ grid graphs, see [10]. A conjectured upper bound is $\text{gon}(G) \leq \frac{|E| - |V| + 4}{2\Delta(G)}$ matching the classical Brill-Noether bound, see [3].

In [10], it was show that the treewidth of $G$ is a lower bound on $\text{gon}(G)$, as conjectured in [9]. In [2], a lower bound $\text{gon}(G) \leq \frac{|V|\lambda_1(Q(G))}{2\Delta(G)}$ is given in terms of the smallest nonzero eigenvalue $\lambda_1(Q(G))$ of the Laplacian and the maximum degree $\Delta(G)$. For the Erdős-Rényi graph, it is known that asymptotically almost surely $\text{gon}(G(n, p)) = n - o(n)$ (provided that $np \to \infty$), see [8, 2].

In this note we show that computing the gonality of a graph is NP-hard, see Theorem 2.2.

Graphs and divisors

Let $G = (V, E)$ be a connected graph. We allow graphs to have parallel edges, but no loops. For $A, B \subseteq V$, we denote by $E(A, B)$ the set of edges with an end in $A$ and an end in $B$ and by $E[A] := E[A, A]$ the set of edges with both ends in $A$. For distinct nodes $u, v \in V$, we use the abbreviation $E(u, v) := E(\{u\}, \{v\})$ for the set of edges between $u$ and $v$. By $\deg(u)$ we denote the degree of a node $u$. The Laplacian of $G$ is the matrix $Q(G) \in \mathbb{Z}^{V \times V}$ defined by

$$Q(G)_{uv} := \begin{cases} \deg(u) & \text{if } u = v, \\ |E(u, v)| & \text{otherwise}. \end{cases} \quad (1)$$

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1 Under the assumption that $G$ has at least 3 nodes.
Proof. This follows directly from Lemma 1.1. □

Lemma 1.2. Let $D \geq 0$ be a divisor on $G$ and let $u, v \in V$. Then $u \not\sim_D v$ if and only if for some effective divisor $D' \sim D$ we can fire on a subset $U$ with $u \in U$, $v \notin U$. In particular, $u \sim_D v$ if and every $u \sim v$ cut has more than $\deg(D)$ edges.

Proof. This follows as no chips can move along a $D$-blocking edge. □

Lemma 1.3. Let $D \geq 0$ be a divisor on $G = (V,E)$. Let $F$ be the set of $D$-blocking edges and let $U$ be a component of $(V,E \setminus F)$. Then for every effective divisor $D' \sim D$ we have $\sum_u \notin U D'(u) = \sum_{u \in U} D(u)$.

Proof. This follows as no chips can move along a $D$-blocking edge. □

2 Proof of main theorem

To prove hardness of computing $\gon(G)$, we use the following construction. Let $G = (V,E)$ be a graph. Let $M := 3|V| + 2|E| + 2$. Construct a graph $\hat{G}$ in the following way. Start with a single node $T$. For every $v \in V$ add three nodes: $v, v', T_v$. For every edge $e \in E(u,v)$, make two nodes $e_u$
and $e_v$. The edges of $\hat{G}$ are as follows. For every $e \in E(u,v)$, add an edge between $e_u$ and $e_v$, add $M$ parallel edges between $u$ and $e_u$ and $M$ parallel edges between $e_v$ and $v$. For every $v \in V$, add three parallel edges between $v'$ and $T_v$, $M$ parallel edges between $v$ and $v'$ and $M$ parallel edges between $T_v$ and $T$. Observe that $M$ is equal to the number of nodes of $\hat{G}$ plus one. See Figure 1 for an example.

Figure 1: On the left a graph $G$, on the right the corresponding $\hat{G}$, where the $M$-fold parallel edges are drawn as bold edges. Here, $M = 12 + 8 + 2 = 22$.

**Lemma 2.1.** Let $G = (V,E)$ be a graph and let $\hat{G}$ be the graph as constructed above. Then the following holds

$$\text{gon}(\hat{G}) = 4|V| + |E| + 1 - \alpha(G).$$

**Proof.** We first show that $\text{gon}(\hat{G}) \geq 4|V| + |E| + 1 - \alpha(G)$. Let $D$ be an effective divisor on $\hat{G}$ with $\deg(D) = \text{gon}(\hat{G})$. Consider the equivalence relation $\sim_D$ on $V(\hat{G})$. Clearly, $\deg(D) \leq |V(\hat{G})| < M$. Hence, by Lemma 1.2, the $M$-fold edges are $D$-blocking. It follows that $T$ is equivalent to all nodes $T_v, v \in V$, and every node $v \in V$ is equivalent $v'$ to and to all nodes $e_v$. By Lemma 1.3, the numbers of chips on each of the sets $\{v\}$, $\{T\}$, $\{v', T_v\}$ and $\{e_u, e_v\}$ are constant over all effective $D' \sim D$. Hence, form the fact that $\text{rank}(D) \geq 1$ it follows that

- $D(v) \geq 1$ for all $v \in V \cup \{T\}$
- $D(v') + D(T_v) \geq 2$ if $v \sim_D T$ and $D(v') + D(T_v) \geq 3$ otherwise.
- $D(e_u) + D(e_v) \geq 2$ if $u \sim_D v$ and $D(e_u) + D(e_v) \geq 1$ otherwise.

Now let $(U_0 \cup \{T\}) \cup U_1 \cup \cdots \cup U_k$ be the partition of $V \cup \{T\}$ induced by $\sim_D$. By the above, we see that

$$\deg(D) \geq |V| + 1 + 3|V| - |U_0| - |E| - |E[U_0]|.$$  \hspace{1cm} (5)

Since $-|U| + E[U] \geq -\alpha(G)$ for any subset $U \subseteq V$, we find that $\text{gon}(G) = \deg(D) \geq 4|V| + 1 + |E| - \alpha(G)$.

Now we show that equality can be attained. For this, let $S \subseteq V$ be an independent set in $G$ of size $\alpha(G)$. Take the partition $V \cup \{T\} = U_0 \cup U_1 \cup \cdots \cup U_k$ where $U_0 = S \cup \{T\}$ and $U_1, \ldots, U_k$ are singletons. Every edge of $G$ has endpoints in two distinct sets $U_i$ and $U_j$ with $i < j$. We orient the edge from $U_i$ to $U_j$. We now define the effective divisor $D$ on $\hat{G}$ as follows.

$$D(v) = 1 \quad \text{for every } v \in V \cup \{T\}$$
$$D(v') = D(T_v) = 1 \quad \text{for every } v \in S$$
$$D(T_v) = 3 \quad \text{for every } v \in V \setminus S$$
$$D(v') = 0 \quad \text{for every } v \in V \setminus S$$
$$D(e_u) = 1 \quad \text{for every edge } e \text{ with tail } u$$
$$D(e_v) = 0 \quad \text{for every edge } e \text{ with head } v$$
It is easy to check that $\deg(D) = 4|V| + 1 + |E| - \alpha(G)$. Now for every $i = 1, \ldots, k$ let $V_i := U_i \cup \cdots \cup U_k$ and let

$$W_i := V_i \cup \{v' \mid v \in V_i\} \cup \{e_u \mid u \in V_i \text{ is end point of an edge } e\}.$$  

Then for every $i$, $D' := D + Q(\hat{G})1_{W_i}$ is effective. Furthermore, $D'(v') \geq 1$ for every $v \in U_i$ and $D'(e_u) \geq 1$ for every edge with head in $U_i$. It follows that $D$ has positive rank.

**Theorem 2.2.** Computing $\text{gon}(G)$ is NP-hard, in fact it is APX-hard even on cubic graphs.

**Proof.** Since computing $\alpha(G)$ is NP-hard, it follows from Lemma 2.1 that also computing $\text{gon}(G)$ is NP-hard. In [1] it was shown that computing $\alpha(G)$ is in fact APX-hard even on cubic graphs. Since for cubic graphs $G$ we have $\text{gon}(\hat{G}) = \frac{11}{2}|V| + 1 - \alpha(G)$, and $\frac{1}{4}|V| \leq \alpha(G) \leq |V|$, it follows that computing $\text{gon}(G)$ is APX-hard.

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