GLOBAL SOLVABILITY AND GENERAL DECAY OF A TRANSMISSION PROBLEM FOR KIRCHHOFF-TYPE WAVE EQUATIONS WITH NONLINEAR DAMPING AND DELAY TERM

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Abstract. A transmission problem for Kirchhoff-type wave equations with nonlinear damping and delay term in the internal feedback is considered under a memory condition on one part of the boundary. By virtue of multiplier method, Faedo-Galerkin approximation and energy perturbation technique, we establish the appropriate conditions to guarantee the existence of global solution, and derive a general decay estimate of the energy, which includes exponential, algebraic and logarithmic decay etc.

1. Introduction. We consider a transmission problem for Kirchhoff-type wave equations with nonlinear damping and delay term

\begin{align*}
  u_{tt}(x,t) - (1 + \|\nabla u(x,t)\|_{L^2(\Omega_1)}^2)\Delta u(x,t) \\
  + \mu_1 \beta(u_t(x,t)) + \mu_2 \varphi(u_t(x,t - \tau)) = 0, \quad (x,t) \in \Omega_1 \times (0, +\infty), \\
  v_{tt}(x,t) - (1 + \|\nabla v(x,t)\|_{L^2(\Omega_2)}^2)\Delta v(x,t) = 0, \quad (x,t) \in \Omega_2 \times (0, +\infty),
\end{align*}

subject to a memory condition on a part of the boundary and transmission conditions

\begin{align*}
  u(x,t) + \int_0^t g(t-s)(1 + \|\nabla u(x,s)\|_{L^2(\Omega_1)}^2) \frac{\partial u(x,s)}{\partial \nu} ds \\
  = 0, \quad (x,t) \in \Gamma_2 \times (0, +\infty), \\
  v(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0, +\infty), \\
  (1 + \|\nabla u(x,t)\|_{L^2(\Omega_1)}^2) \frac{\partial u(x,t)}{\partial \nu} = (1 + \|\nabla v(x,t)\|_{L^2(\Omega_2)}^2) \frac{\partial v(x,t)}{\partial \nu}, \\
  u(x,t) = v(x,t), \quad (x,t) \in \Gamma_1 \times (0, +\infty),
\end{align*}

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and initial conditions
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad (x, t) \in \Omega_1, \quad (1.6) \]
\[ u_t(x, t) = f_0(x, t), \quad (x, t) \in \Omega_1 \times (-\tau, 0), \quad (1.7) \]
\[ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad (x, t) \in \Omega_2, \quad (1.8) \]
where \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is a bounded domain with a smooth boundary \( \partial \Omega = \Gamma_0 \cup \Gamma_2, \; \Gamma_0 \cap \Gamma_2 = \emptyset \). \( \Gamma_0 \) is a boundary of small ball \( B(x_0) \) containing \( x_0 \) in \( \Omega \), \( \Omega_2 \subset \Omega \) is a subdomain outside of \( B(x_0) \) with smooth boundary \( \Gamma_0 \cup \Gamma_1 \), and \( \Omega_1 = \Omega \setminus (\Omega_2 \cup B(x_0)) \) is a subdomain with smooth boundary \( \Gamma_2 \cup \Gamma_1 \). \( \nu \) denotes the unit normal vector pointing towards the exterior of \( \Omega_1 \) and there exists a constant \( \delta > 0 \) such that \( m \cdot \nu \geq \delta > 0 \) on \( \Gamma_2 \), where \( m := m(x) = x - x_0 \) (as shown in Figure 1). \( \mu_1 \) and \( \mu_2 \) are real parameters with \( \mu_1 > 0 \) and \( \mu_2 \neq 0 \), \( \tau > 0 \) is the delay, \( g \) is a positive function and \( f_0 \) is given history belonging to a suitable space. For simplicity, \( u(x, t) \) and \( u(x, t - \tau) \) are hereinafter referred to as \( u \) and \( u(t - \tau) \) respectively, where there is no confusion.

![Figure 1. An example of \( \Omega \).](image-url)

Our transmission model \((1.1)-(1.8)\) arises in several applications in physics and biology. For example, the model of the transverse vibration of a membrane composed by two different materials in \( \Omega_1 \) and \( \Omega_2 \).

In the past decades, there have been many authors investigated wave equations and systems with damping and showed that the dissipation produced by internal or boundary damping can lead to decay of the solution, see \([5, 11, 13-15, 38, 41, 45]\) and the references therein. For example, Cavalcanti et al. \([13]\) studied the mixed boundary value problem of linear degenerate wave equations with nonlinear boundary feedback and boundary memory source, and established the existence and exponential decay of the global solution. Later, Park and Bae \([38]\) considered a Kirchhoff-type wave equation
\[ u_{tt} - (1 + \| \nabla u \|_2^2) \Delta u - \Delta u_t = 0, \quad (x, t) \in \Omega \times (0, +\infty), \]
and obtained the result that the energy decays exponentially or polynomially under a memory condition on a part of the boundary. Santos et al. \([41]\) investigated the following Kirchhoff-type wave equation
\[ u_{tt} - M(\| \nabla u \|_2^2) \Delta u - \Delta u_t + f(u) = 0, \quad (x, t) \in \Omega \times (0, +\infty), \]
where \( M(s) \in C^1(0, +\infty) \) with \( M(s) \geq m_0 > 0 \) and \( M(s)s \geq \int_0^s M(r)dr, \; \forall s \geq 0 \). They proved that the energy decays exponentially or polynomially under the same conditions in \([38]\). Moreover, we referred to \([11, 14, 15]\) for the decay estimate of degenerate wave equations with localized damping and viscoelastic damping, and system of linear equations with boundary memory dissipations. The general decay
of a thermoelastic laminated beam with past history was researched in [16, 30] and the general and optimal decay result for a Moore-Gibson-Thompson equation with memory was established in [28].

It is well known that delay effect, which arises in many practical problems, may be the source of instability. Hence, the control of PDEs with time delay has become an active area of research in recent years. For example, it was proved in [17, 18, 36, 37, 44] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable, unless additional conditions or control terms were used. For the influence of linear delay term on the stability, we see [1, 22, 23, 27, 31]. A boundary stabilization problem for the wave equation with interior delay was studied in [1]. The authors derived an exponential stability result under some Lions geometric condition. Kirane and Said-Houari [23] considered the viscoelastic wave equation with a delay

\[ u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_1 u_t + \mu_2 u_t(x, t - \tau) = 0, \quad (x, t) \in \Omega \times (0, +\infty), \]

where \(\mu_1\) and \(\mu_2\) are positive constants. Under the hypothesis of \(0 \leq \mu_1 \leq \mu_2\), they established a general decay estimate of the energy. Later, Liu [31] improved this result by considering the equation with a time-varying delay whose coefficient \(\mu_2\) is not necessarily positive, and Feng [22] studied the plate equation. Furthermore, we refer to [27] for the decay of a weak viscoelastic equation with time-varying delay.

However, there are few results on the effect of nonlinear delay, which can be seen in [8, 9, 20, 25]. Benaissa and Louhibi [9] considered the following wave equation with nonlinear delay

\[ u_{tt} - \Delta u + \mu_1 \beta(u_t) + \mu_2 \beta(u_t(x, t - \tau)) = 0, \quad (x, t) \in \Omega \times (0, +\infty), \]

where \(\beta : \mathbb{R} \to \mathbb{R}\) is a nondecreasing continuous function satisfying some appropriate nonlinear conditions. Under Dirichlet boundary condition, they derived the global existence and asymptotic behavior of the solution. In [8], Benaissa et al. obtained the uniform decay estimate of the Dirichlet initial boundary value problem for the following viscoelastic equation with nonlinear delay in the internal feedback

\[ u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_1 \beta(u_t) + \mu_2 \varphi(u_t(x, t - \tau)) = 0, \quad (x, t) \in \Omega \times (0, +\infty). \]

Recently, Li and Chai [25] studied the Euler-Bernoulli plate equation with localized nonlinear internal feedback

\[ u_{tt} - \mathcal{A}^2 u + a(x)[\mu_1 \beta(u_t) + \mu_2 \varphi(u_t(x, t - \tau))] = 0, \quad (x, t) \in \Omega \times (0, +\infty). \]

They established the energy decay estimate by Riemannian geometric method. Djilali [20] considered the nonlinear Timoshenko beam system with nonlinear delay in one-dimensional space

\[ \rho_1 u_{tt} - K(u_x + v)_x = 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \]
\[ \rho_2 v_{tt} - b v_{xx} + K(u_x + v)_x + \int_0^t g(t-s)v_{xx}(x,s)ds + \mu_1 \beta(v_t) + \mu_2 \varphi(v_t(x, t - \tau)) = 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \]

and deduced the global existence and uniform decay rate of the energy.
For the transmission problem, [2, 7, 12, 19, 26, 29, 33-35] studied the existence, regularity, controllability and decay estimate of solution for the transmission problems with Laplacian operators. For example, Marzocchi [33] proved that the solution for a semilinear transmission problem in one-dimensional space between elastic and thermoelastic materials decays exponentially. This result was extended to $N$-dimensional space by Marzocchi and Naso [34]. Most recently, Liu et.al [29] established the general decay of the solution for a transmission problem in memory-type thermoelasticity with second sound. Moreover, Bastos and Raposo [7] and Cavalcanti et.al [12] investigated the exponential stability of a transmission problem with frictional damping and a transmission problem of viscoelastic waves with hereditary memory, respectively.

Recently, there are many new results for transmission problems with operators of Kirchhoff-type, see [6, 10, 24, 35, 39, 40, 43]. Bae [6] investigated the transmission problem for wave equations given by

$$u_{tt} - \|\nabla u\|_{\Omega_1}^2 \Delta u + |u|^\alpha u = 0, \quad (x, t) \in \Omega_1 \times (0, +\infty),$$

$$v_{tt} - \|\nabla v\|_{\Omega_2}^2 \Delta v + |v|^\beta v = 0, \quad (x, t) \in \Omega_2 \times (0, +\infty).$$

Under a memory condition on a part of the boundary, he studied the global existence of the solution and showed that the energy of the problem have the same decay rate with the relaxation function, which decays exponentially or polynomially. Later, Park [39, 40] considered the uniform decay rate of the transmission problem of the Kirchhoff-type wave equations.

On the other hand, for the transmission problem with delay, Benseghir [10] studied a linear transmission problem with a delay term in one-dimensional space

$$u_{tt} - au_{xx} + \mu_1 u_t + \mu_2 u_t(x, t - \tau) = 0, \quad (x, t) \in L \times (0, +\infty),$$

$$v_{tt} - bv_{xx} = 0, \quad (x, t) \in (L_1, L_2) \times (0, +\infty),$$

where $L = (0, L_1) \cup (L_2, L_3)$. Under the assumption that the effect of delay is weaker than damping ($\mu_2 < \mu_1$), he showed the exponential stability of the solution by introducing a suitable Lyapunov functional. Li et al. [24] studied the following linear transmission system with long-time memory and delay term

$$u_{tt} - au_{xx} + \int_0^{+\infty} g(t - s)u_{xx}(x, s)ds + \mu_1 u_t + \mu_2 u_t(x, t - \tau) = 0, \quad (x, t) \in L \times (0, +\infty),$$

$$v_{tt} - bv_{xx} = 0, \quad (x, t) \in (L_1, L_2) \times (0, +\infty),$$

By assuming $\mu_2 \leq \mu_1$, they proved the well-posedness result by using semigroup theory and Hille-Yosida theorem. Furthermore, they established a general decay result, which the exponential and polynomial decay are only special cases. Moreover, we refer to [43] for the similar transmission problem with short-time memory.

In view of the works mentioned above, it is clear that research on the global well-posedness and general uniform energy decay for Kirchhoff-type transmission problem (1.1)-(1.8) has not been started yet. The main difficulties lie in finding the competitive relationship between Kirchhoff-type operators, nonlinear delay, nonlinear damping, and boundary memory term. Motivated by these observations, we investigate the global existence of solution to problem (1.1)-(1.8) and its uniform decay by using multiplier method, Faedo-Galerkin approximation and energy perturbation technique, which includes exponential, algebraic and logarithmic decay etc.
The remainder of this paper is organized as follows. In Sect. 2, we present some preliminaries and state the main results. In Sect. 3, we prove a global solvability of problem (1.1)-(1.8). A general decay estimate of energy is derived in Sect. 4.

2. Preliminaries and main results. In this section, we introduce some materials needed in the proof of our results and state the main results.

Throughout this paper, positive constant $C$ is not necessarily the same at each occurrence and we define

$$H^1_0(\Omega_2) := \{ v \in H^1(\Omega_2) : v = 0 \text{ on } \Gamma_0 \},$$

$$V := \{(u,v) \in H^1(\Omega_1) \times H^1_0(\Omega_2) : u = v, \, (1 + \| \nabla u \|_{\Omega_1}^2) \frac{\partial u}{\partial v} = (1 + \| \nabla v \|_{\Omega_2}^2) \frac{\partial v}{\partial v} \},$$

$$(u,v)_{\Omega_i} := \int_{\Omega_i} u(x)v(x)dx, \, i = 1,2, \, (u,v)_{\Gamma_j} := \int_{\Gamma_j} u(x)v(x)dx, \, j = 1,2.$$ For a Banach space $X$, $\| \cdot \|_X$ represents the norm of $X$. It's convenient to denote $\| \cdot \|_{L^2(\Omega_1)}$ and $\| \cdot \|_{L^2(\Gamma_j)}$ by $\| \cdot \|_{\Omega_1}$ and $\| \cdot \|_{\Gamma_j}$, respectively. Moreover, we give some simple notations.

$$h \ast u(t) := \int_0^t h(t-s)u(s)ds,$$

$$h \circ u(t) := \int_0^t h(t-s)[u(t) - u(s)]ds,$$

$$h \circ u(t) := \int_0^t h(t-s)[u(t) - u(s)]^2ds.$$ A direct calculation derives

$$\langle h \ast u, u \rangle_{\Gamma_2} = -\frac{1}{2} \frac{d}{dt} \left[ \int_{\Gamma_2} (h \circ u) d\Gamma - \left( \int_0^t h(s)ds \right) \| u \|_{\Gamma_2}^2 \right]$$

$$- \frac{1}{2} h(t)\| u \|_{\Gamma_2}^2 + \frac{1}{2} \int_{\Gamma_2} (h' \circ u) d\Gamma,$$ \hspace{1cm} (2.1)

and

$$\| h \circ u \|_{\Gamma_2}^2 \leq \left( \int_0^t |h(s)|ds \right) \int_{\Gamma_2} (|h| \circ u) d\Gamma.$$ \hspace{1cm} (2.2)

Differentiating (1.3), we arrive at the following Volterra equation

$$(1 + \| \nabla u \|_{\Omega_1}^2) \frac{\partial u}{\partial v} + \frac{1}{g(0)} g' \ast (1 + \| \nabla u \|_{\Omega_1}^2) \frac{\partial u}{\partial v} = -\frac{1}{g(0)} u_t.$$ By using of the Volterra's inverse operator, we get

$$(1 + \| \nabla u \|_{\Omega_1}^2) \frac{\partial u}{\partial v} = -\frac{1}{g(0)} (u_t + k \ast u_t),$$

where the resolvent kernel satisfies $k(t) + \frac{1}{g(0)} (g' \ast k)(t) = -\frac{1}{g(0)} g'(t)$. Denoting $r = \frac{1}{g(0)}$, then the aforementioned equality can be written as

$$(1 + \| \nabla u \|_{\Omega_1}^2) \frac{\partial u}{\partial v} = -ru_t + k(0)u - k(t)u_0 + k' \ast u.$$ \hspace{1cm} (2.3)

The identity (2.3) implies (1.3).

The following assumptions are made to state the corresponding results. We begin with some assumptions on nonlinear functions $\beta$ and $\varphi$. 


The hypothesis (2.6) is only used to prove global existence of the solution. Noting that $\Phi$ is an odd increasing function of $C^1$ and there exist constants $\varphi_0$, $\varphi_1 > 0$ such that

\[ |\varphi'(s)| \leq \varphi_0, \]  

and

\[ \varphi(s)s \leq \varphi_1 \beta(s)s. \]  

**Remark 1.** Noting that $\varphi$ is an odd and increasing function, we find

\[ 0 \leq \Phi(s) \leq \varphi(s)s \leq \varphi_1 \beta(s)s, \]

where $\Phi(s) = \int_0^s \varphi(r) dr$.

**Remark 2.** The hypothesis (2.6) is only used to prove global existence of the solution to problem (1.1)-(1.8). For the energy estimate, we may remove this restriction of linear growth order on function $\varphi$.

For the resolvent kernel $k$, as in [39], we assume that:

**H3:** $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a function of $C^2$ such that

\[ k(0) > 0, \lim_{t \to +\infty} k(t) = 0, k'(t) \leq 0, \]

and there exists a nonincreasing continuous function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

\[ k''(t) \geq -\xi(t)k'(t), \ \forall t \geq 0, \text{ and } \int_0^{+\infty} \xi(s) ds = +\infty. \]

Next, we present the definition of weak solution to our problem.

**Definition 1.** For given initial data $(u_0, v_0) \in \left( H^2(\Omega_1) \times H^2(\Omega_2) \right) \cap V$, $(u_1, v_1) \in V$, and $f_0 \in L^2(\Omega_1 \times (-\tau, 0))$. Functions $(u, v) \in C(0, T; V)$ are called the weak solutions of problem (1.1)-(1.8), if $(u, v)$ satisfies initial conditions $(u(0), v(0)) = (u_0, v_0)$, $(u_t(0), v_t(0)) = (u_1, v_1)$, $u_t(x, t) = f_0(x, t)$, $\forall t \in (-\tau, 0)$, and

\[
\begin{align*}
\int_0^T \int_{\Omega_1} u_t \phi dx dt + \int_0^T \left( 1 + \|\nabla u\|_{L_1}^2 \right) \int_{\Omega_1} \nabla u \cdot \nabla \phi dx dt & \\
+ \mu_1 \int_0^T \int_{\Omega_1} \beta(u_1) \phi dx dt + \mu_2 \int_0^T \int_{\Omega_1} \varphi(u_1(t-\tau)) \phi dx dt & \\
+ \int_0^T \int_{\Omega_2} v_t \psi dx dt + \int_0^T \left( 1 + \|\nabla v\|_{L_2}^2 \right) \int_{\Omega_2} \nabla v \cdot \nabla \psi dx dt & \\
= -r \int_0^T \int_{\Gamma_2} |u_t + k(0)u + k'u - k(t)u(0)| \phi dx dt,
\end{align*}
\]

$\forall (\phi, \psi) \in V$ and $T > 0$.

As for the global solvability for problem (1.1)-(1.8) in time, we get the following result.
Theorem 1. Suppose that $|\mu_2| \leq \frac{\mu_1}{2\varphi_1}$ and (H1)-(H3) hold. Then for $(u_0, v_0) \in (H^2(\Omega_1) \times H^2(\Omega_2)) \cap V$, $(u_1, v_1) \in V$, $f_0 \in L^2(\Omega_1 \times (-T, 0))$ satisfying the compatibility conditions

$$(1 + \|\nabla u_0\|_{\Omega_1}^2) \frac{\partial u_0}{\partial \nu} + ru_1 = 0, \text{ on } \Gamma_2, \quad v_0 = 0, \text{ on } \Gamma_0,$$

$$u_0 = v_0, \quad (1 + \|\nabla v_0\|_{\Omega_1}^2) \frac{\partial v_0}{\partial \nu} = (1 + \|\nabla v_0\|_{\Omega_1}^2) \frac{\partial v_0}{\partial \nu}, \text{ on } \Gamma_1,$$

there exists a unique weak solution $(u, v)$ of problem (1.1)-(1.8) such that

$$(u, v) \in C((0, +\infty); V) \cap C^1((0, +\infty); L^2(\Omega_1) \times L^2(\Omega_2)).$$

In order to state the asymptotic behavior of the energy, we define the energy functional as

$$E(t) = \frac{1}{2} \|u\|_{\Omega_1}^2 + \frac{1}{2} \|v\|_{\Omega_2}^2 + \frac{1}{2} \|\nabla u\|_{\Omega_1}^2 + \frac{1}{2} \|\nabla v\|_{\Omega_2}^2 + \frac{1}{4} \|\nabla u\|_{\Omega_1}^4$$

$$+ \frac{1}{4} \|\nabla v\|_{\Omega_2}^4 + \frac{r}{2} \int_{\Omega_1} (k't + u)^2 - \frac{r}{2} \int_{\Gamma_2} (k' \circ u) d\Gamma$$

$$+ \zeta \int_{t-T}^t \int_{\Omega_1} \beta(u_\rho(x, \rho)) u_\rho(x, \rho) dx d\rho,$$

where $\zeta$ is a positive constant such that

$$|\mu_2| \varphi_1 < \zeta < \mu_1 - |\mu_2| \varphi_1.$$  

(2.12)

Then, we have the following general decay for (1.1)-(1.8).

Theorem 2. Let $(u, v)$ be the solution of (1.1)-(1.8). Assuming $|\mu_2| < \frac{\mu_1}{2\varphi_1}$ and (H1)-(H3) hold. Then for $t_0 > 0$ large enough, there exist constants $C_0 > 0$ and $\varpi > 0$ such that

(i): $E(t) \leq C_0 E(0) e^{-\varpi \int_0^t \xi(s) ds}$, for all $t \geq t_0$, if $u_0 = 0$ on $\Gamma_2$,

(ii): otherwise, $E(t) \leq C_0 \left[ E(0) + \|u_0\|_{\Omega_1}^2 \int_0^t k^2(s) e^{-\varpi \int_0^s \xi(r) dr} ds \right] e^{-\varpi \int_0^t \xi(s) ds}$, for all $t \geq t_0$.

Remark 3. The exponential decay and polynomial decay in previous literatures are special cases of the result in Theorem 2. In fact, if we take

$$k(t) = e^{-\sigma t}, \quad \sigma > 0, \quad \xi(t) = \sigma;$$

$$k(t) = \frac{1}{(1+t)^{\sigma}}, \quad \sigma > 0, \quad \xi(t) = \frac{1+\sigma}{1+t};$$

$$k(t) = \frac{1}{\ln(\ln(3+t))}, \quad \xi(t) = \frac{1}{(3+t)^2 \ln(3+t)},$$

then by Theorem 2, the energy may decay exponentially, polynomially, and logarithmically, respectively.
3. Global solvability. In this section, by using Feado-Galerkin approximation technique and multiplier method, we prove Theorem 1.

The proof of Theorem 1: We divide the proof into four steps.

Step 1. Feado-Galerkin approximation.

Let \( \{ (\phi_j, \psi_j) \} \) be a basis in \( V \), which is orthogonal in \( L^2(\Omega_1) \times L^2(\Omega_2) \). For \( \forall n \geq 1 \), denoting \( V_n = \text{span}\{ (\phi_1, \psi_1), (\phi_2, \psi_2), \ldots, (\phi_n, \psi_n) \} \).

We define the approximations

\[
(u^{(n)}(x, t), v^{(n)}(x, t)) := \sum_{j=1}^{n} b_{jn}(t) (\phi_j(x), \psi_j(x)),
\]

where \( (u^{(n)}, v^{(n)}) \) is the solution of the following finite dimensional Cauchy problem

\[
\int_{\Omega_1} u^{(n)}_{tt} \phi_j \, dx + (1 + \| \nabla u^{(n)} \|_{L^2}^2) \int_{\Omega_1} \nabla u^{(n)} \cdot \nabla \phi_j \, dx \\
+ \mu_1 \int_{\Omega_1} \beta (u^{(n)}_t) \phi_j \, dx + \mu_2 \int_{\Omega_1} \varphi (u^{(n)}_t(t - \tau)) \phi_j \, dx \\
+ \int_{\Omega_2} v^{(n)}_{tt} \psi_j \, dx + (1 + \| \nabla v^{(n)} \|_{L^2}^2) \int_{\Omega_2} \nabla v^{(n)} \cdot \nabla \psi_j \, dx \\
= -r \int_{\Gamma_2} [u^{(n)}(t) + k(0)u^{(n)} + k' * u^{(n)} - k(t)u_n] \phi_j \, dx,
\]

(3.1)

and

\[
(u_{0n}, v_{0n}) = \left( (u^{(n)}(0), v^{(n)}(0)) \right) \rightarrow (u_0, v_0), \text{ in } (H^2(\Omega_1) \times H^2(\Omega_2)) \cap V,
\]

(3.2)

\[
(u_{1n}, v_{1n}) = \left( (u^{(n)}_t(0), v^{(n)}_t(0)) \right) \rightarrow (u_1, v_1), \text{ in } V,
\]

(3.3)

\[
f_{0n} = u^{(n)}_t(x, t) \rightarrow f_0(x, t), \text{ in } L^2(\Omega_1 \times (-\tau, 0)).
\]

(3.4)

According to the standard theory of ordinary differential equations, the finite dimensional problem (3.1)- (3.4) has a unique solution \( (b_{jn}(t))_{j=1, \ldots, n} \) defined on \([0, T_n)\), \( T_n > 0 \). The extension of these solutions to the whole interval \([0, T]\), for all \( T > 0 \), is a consequence of the first estimate which we are going to prove below.

Step 2. Energy estimates.

Estimate I: Multiplying (3.1) by \( b'_{jn}(t) \) and summing on \( j \), then using (2.1) we have

\[
\frac{d}{dt} \left\{ \frac{1}{2} \left\| u^{(n)}_t \right\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \left\| v^{(n)}_t \right\|_{L^2(\Omega_2)}^2 + \frac{1}{2} \left\| \nabla u^{(n)} \right\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \left\| \nabla v^{(n)} \right\|_{L^2(\Omega_2)}^2 \right\} \\
+ \frac{1}{4} \left\| \nabla v^{(n)} \right\|_{L^2(\Omega_2)}^4 + \frac{r}{2} k(t) \left\| u^{(n)} \right\|_{L^2(\Omega_2)}^2 - \frac{r}{2} \int_{\Gamma_2} \left( k' \circ u^{(n)} \right) \, d\Gamma \\
= -\mu_1 \int_{\Omega_1} \beta (u^{(n)}_t) u^{(n)}_t \, dx - \mu_2 \int_{\Omega_1} \varphi (u^{(n)}_t(t - \tau)) u^{(n)}_t \, dx \\
+ r k(t) \int_{\Gamma_2} u_{0n} u_t^{(n)} \, dx - \frac{r}{2} \int_{\Gamma_2} \left( k'' \circ u^{(n)} \right) \, d\Gamma + \frac{r}{2} k'(t) \left\| u^{(n)} \right\|_{L^2(\Omega_2)}^2.
\]

(3.5)
Noting that
\[
\zeta \frac{d}{dt} \int_{t-\tau}^{t} \int_{\Omega_i} \beta \left( u_{\rho}^{(n)}(x, \rho) \right) u_{\rho}^{(n)}(x, \rho) dx d\rho
\]
\[= \zeta \int_{\Omega_i} \beta \left( u_{t}^{(n)}(x) \right) u_{t}^{(n)}(x) dx - \zeta \int_{\Omega_i} \beta \left( u_{t}^{(n)}(t - \tau) \right) u_{t}^{(n)}(t - \tau) dx,
\]
where \( \zeta \) is a positive constant such that
\[|\mu_2| \varphi_1 \leq \zeta \leq \mu_1 - |\mu_2| \varphi_1.\]
Combining (3.5) and (3.6), we can derive
\[
\frac{d}{dt} E^{(n)}(t) + r \left\| u_{t}^{(n)} \right\|_{\Gamma_2}^{2} + \frac{r}{2} \int_{\Gamma_2} \left( k'' \circ u^{(n)} \right) d\Gamma - \frac{r}{2} k'(t) \left\| u^{(n)} \right\|_{\Gamma_2}^{2}
\]
\[= - (\mu_1 - \zeta) \int_{\Omega_i} \beta \left( u_{t}^{(n)}(x) \right) u_{t}^{(n)}(x) dx - \mu_2 \int_{\Omega_i} \varphi \left( u_{t}^{(n)}(t - \tau) \right) u_{t}^{(n)}(x) dx
\]
\[\quad - \zeta \int_{\Omega_i} \beta \left( u_{t}^{(n)}(t - \tau) \right) u_{t}^{(n)}(t - \tau) dx + r k(t) \int_{\Gamma_2} u_{0n} u_{t}^{(n)} d\Gamma,
\]
where
\[
E^{(n)}(t) = \frac{1}{2} \left\| u_{t}^{(n)} \right\|_{\Omega_i}^{2} + \frac{1}{2} \left\| u_{t}^{(n)} \right\|_{\Omega_2}^{2} + \frac{1}{2} \left\| \nabla u^{(n)} \right\|_{\Omega_1}^{2} + \frac{1}{2} \left\| \nabla u^{(n)} \right\|_{\Omega_2}^{2}
\]
\[+ \frac{1}{4} \left\| \nabla u^{(n)} \right\|_{\Omega_1}^{4} + \frac{1}{4} \left\| \nabla u^{(n)} \right\|_{\Omega_2}^{4} - \frac{r}{2} \int_{\Gamma_2} \left( k'' \circ u^{(n)} \right) d\Gamma
\]
\[+ \frac{r}{2} k(t) \left\| u^{(n)} \right\|_{\Gamma_2}^{2} + \zeta \int_{t-\tau}^{t} \int_{\Omega_i} \beta \left( u_{\rho}^{(n)}(x, \rho) \right) u_{\rho}^{(n)}(x, \rho) dx d\rho.
\]

Let’s denote \( \Phi^* \) to be the conjugate function of the convex function \( \Phi \), i.e.,
\[\Phi^* = \sup_{t \in \mathbb{R}+} (st - \Phi(t)).\] Then \( \Phi^* \) is the Legendre transform of \( \Phi \) which is given by (see Arnold [4, pp.61-62])
\[\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi(\Phi')^{-1}(s), \quad \forall s \geq 0,
\]
and satisfies the inequality
\[s t \leq \Phi^*(s) + \Phi(t), \quad \forall s, t \geq 0.
\]
Taking the definition of \( \Phi \) into account, we get
\[\Phi^*(s) = s \varphi^{-1}(s) - \Phi(\varphi^{-1}(s)), \quad \forall s \geq 0.
\]
Noting that \( \varphi \) is an odd function, from (H2), (3.9), and (3.10), we obtain
\[
\varphi \left( u_{t}^{(n)}(t - \tau) \right) u_{t}^{(n)} \leq \varphi \left( u_{t}^{(n)}(t - \tau) \right) u_{t}^{(n)}(t - \tau) - \Phi \left( u_{t}^{(n)}(t - \tau) \right) + \Phi \left( u_{t}^{(n)} \right)
\]
\[\leq \varphi_1 \beta \left( u_{t}^{(n)}(t - \tau) \right) u_{t}^{(n)}(t - \tau) + \varphi_1 \beta \left( u_{t}^{(n)} \right) u_{t}^{(n)}.
\]
Moreover, it follows from Young’s inequality that
\[
r k(t) \int_{\Gamma_2} u_{0n} u_{t}^{(n)} d\Gamma \leq \frac{r}{2} \left\| u_{t}^{(n)} \right\|_{\Gamma_2}^{2} + \frac{r k^2(t)}{2} \left\| u_{0n} \right\|_{\Gamma_2}^{2}.
\]
Then by (3.2)-(3.4) we can deduce
\[
\frac{d}{dt} E^{(n)}(t) + \frac{r}{2} \left[ \left\| u_t^{(n)} \right\|^2_{\Gamma^2} + \int_{\Gamma^2} \left( k'' \circ u^{(n)} \right) d\Gamma - k'(t) \left\| u^{(n)} \right\|^2_{\Gamma^2} \right] \\
\leq \frac{r k^2(t)}{2} \| u_{tt}^{(n)} \|^2_{\Omega^2} - (\mu_1 - \zeta - |\mu_2| \varphi_1) \int_{\Omega^1} \beta\left(u_t^{(n)}\right) u_t^{(n)} dx \\
- (\zeta - |\mu_2| \varphi_1) \int_{\Omega^1} \beta\left(u_t^{(n)}(t - \tau)\right) u_t^{(n)}(t - \tau) dx.
\]
(3.13)

Integrating (3.13) over \((0, t), 0 < t \leq T\), and then using Gronwall’s inequality and (3.2)-(3.4), we obtain the first estimate
\[
\left\| u_t^{(n)} \right\|^2_{\Omega^1} + \left\| v_t^{(n)} \right\|^2_{\Omega^2} + \left\| u^{(n)} \right\|^2_{\Omega^1} + \left\| v^{(n)} \right\|^2_{\Omega^2} + \left\| u_t^{(n)} \right\|^4_{\Omega^1} + \left\| v_t^{(n)} \right\|^4_{\Omega^2} \\
+ \left\| u_t^{(n)} \right\|^2_{\Omega^2} + \int_{t-\tau}^t \int_{\Omega^1} \beta\left(u_t^{(n)}(x, \rho)\right) u_t^{(n)}(x, \rho) dx d\rho + \int_0^t \left\| u_t^{(n)}(s) \right\|^2_{\Omega^2} ds \\
\leq L_1,
\]
(3.14)
where \(L_1 > 0\) is a constant independent of \(n\).

**Estimate II:** First of all, we are going to estimate the initial data \(\left\| u_t^{(n)}(0) \right\|^2_{\Omega^1}\) and \(\left\| v_t^{(n)}(0) \right\|^2_{\Omega^2}\). Multiplying (3.1) by \(b_{jn}^{(n)}(t)\), and summing on \(j\), we can obtain
\[
\left\| u_t^{(n)} \right\|^2_{\Omega^1} + \left(1 + \left\| u^{(n)} \right\|^2_{\Omega^1}\right) \int_{\Omega^1} \nabla u^{(n)} \cdot \nabla u_t^{(n)} dx \\
+ \mu_1 \int_{\Omega^1} \beta(u_t^{(n)}) u_t^{(n)} dx + \mu_2 \int_{\Omega^1} \varphi(u_t^{(n)}(t - \tau)) u_t^{(n)} dx \\
+ \left\| v_t^{(n)} \right\|^2_{\Omega^2} + \left(1 + \left\| v^{(n)} \right\|^2_{\Omega^2}\right) \int_{\Omega^2} \nabla v^{(n)} \cdot \nabla v_t^{(n)} dx \\
= - r \int_{\Gamma^2} \left[ u_t^{(n)} + k(0) u^{(n)} + k' u^{(n)} - k(t) u_t^{(n)}(0) \right] u_t^{(n)} dx.
\]
(3.15)

Considering \(t = 0\) in (3.15), we have
\[
\left\| u_t^{(n)}(0) \right\|^2_{\Omega^1} + \left\| v_t^{(n)}(0) \right\|^2_{\Omega^2} \\
\leq \left\{ \left(1 + \left\| u^{(n)} \right\|^2_{\Omega^1}\right) \left\| \Delta u_{tt}^{(n)} \right\|_{\Omega_1} + \mu_1 \left\| \beta(u_t^{(n)}) \right\|_{\Omega_1} + \mu_2 \left\| \varphi(f_{tt}^{(n)}(-\tau)) \right\|_{\Omega_1} \right\} \left\| u_t^{(n)}(0) \right\|^2_{\Omega_1} \\
+ \left(1 + \left\| v^{(n)} \right\|^2_{\Omega^2}\right) \left\| \Delta v_{tt}^{(n)} \right\|_{\Omega_2} \left\| v_t^{(n)}(0) \right\|^2_{\Omega_2}.
\]
(3.16)

Noting that \(\beta(u_t^{(n)})\) and \(\varphi(f_{tt}^{(n)}(-\tau))\) are bounded in \(L^2(\Omega_1)\) by (H1), (H2) and (3.4). Then by (3.2)-(3.4) we can deduce
\[
\left\| u_t^{(n)}(0) \right\|^2_{\Omega_1} + \left\| v_t^{(n)}(0) \right\|^2_{\Omega_2} \leq C, \forall n \in \mathbb{N}.
\]
(3.17)
Now, differentiating (3.1) with respect to $t$, multiplying it by $b''_{fn}(t)$, and summing on $j$, we have

$$
\begin{align*}
\frac{d}{dt} \left\{ \frac{1}{2} \left\| u''_{tt}(t) \right\|_{\Omega}^2 + \frac{1}{2} \left\| v''_{tt}(t) \right\|_{\Omega}^2 + \frac{1}{2} \left\| \nabla u''(t) \right\|_{\Omega_1}^2 + \frac{1}{2} \left\| \nabla v''(t) \right\|_{\Omega_2}^2 \\
+ \frac{1}{2} \left\| \nabla u''(t) \right\|_{\Omega_1}^2 \left\| \nabla u''_{tt}(t) \right\|_{\Omega_1}^2 + \frac{1}{2} \left\| \nabla v''(t) \right\|_{\Omega_2}^2 \left\| \nabla v''_{tt}(t) \right\|_{\Omega_2}^2 \\
+ \left( \int_{\Omega_1} \nabla u''(t) \cdot \nabla u''_{tt}(t) \, dx \right)^2 + \left( \int_{\Omega_2} \nabla v''(t) \cdot \nabla v''_{tt}(t) \, dx \right)^2 + \frac{rk(0)}{2} \left\| u''_{tt}(t) \right\|_{\Gamma_2}^2 \right\}
\end{align*}
$$

$$
= 3 \left\| u''_{tt}(t) \right\|_{\Omega_1}^2 \left\| \nabla u''(t) \right\|_{\Omega_1} \left\| \nabla u''_{tt}(t) \right\|_{\Omega_1} + 3 \left\| \nabla u''_{tt}(t) \right\|_{\Omega_2}^2 \left\| \nabla v''(t) \right\|_{\Omega_2} \left\| \nabla v''_{tt}(t) \right\|_{\Omega_2} \\
- \mu_1 \int_{\Omega_1} \beta' \left( u''(t) \right) \left\| u''_{tt}(t) \right\|_{\Omega_1}^2 \, dx - \mu_2 \int_{\Omega_2} \varphi' \left( u''(t) \right) \left\| u''_{tt}(t) \right\|_{\Omega_2}^2 \, dx \\
- r \left\| u''_{tt}(t) \right\|_{\Gamma_2}^2 + rk(t) \int_{\Gamma_2} \left( k'' \ast u''(t) \right) u''_{tt}(t) \, d\Gamma \\
- rk'(0) \int_{\Gamma_2} u''(t) u''_{tt}(t) \, d\Gamma. \tag{3.18}
\end{align*}
$$

In addition, we have

$$
\frac{d}{dt} \int_{t-\tau}^{t} \int_{\Omega_1} \left| u''_{tt}(t, \rho) \right|^2 \, dx \, d\rho = \left\| u''_{tt}(t) \right\|_{\Omega_1}^2 - \left\| u''_{tt}(t-\tau) \right\|_{\Omega_1}^2. \tag{3.19}
$$

Combining (3.18) and (3.19), we can derive

$$
\begin{align*}
\frac{d}{dt} E_1''(t) + \mu_1 \int_{\Omega_1} \beta' \left( u''(t) \right) \left\| u''_{tt}(t) \right\|_{\Omega_1}^2 \, dx + \frac{1}{2} \left\| u''_{tt}(t) \right\|_{\Omega_1}^2 \left( r \right)_{\Gamma_2}^2 \\
= 3 \left\| \nabla u''(t) \right\|_{\Omega_1}^2 \left\| \nabla u''_{tt}(t) \right\|_{\Omega_1} + 3 \left\| \nabla u''_{tt}(t) \right\|_{\Omega_2}^2 \left\| \nabla v''(t) \right\|_{\Omega_2} \\
- \mu_2 \int_{\Omega_2} \varphi' \left( u''(t) \right) \left\| u''_{tt}(t) \right\|_{\Omega_2}^2 \, dx + \frac{1}{2} \left\| u''_{tt}(t) \right\|_{\Omega_1}^2 \left( r \right)_{\Gamma_2}^2 \\
+ rk(t) \int_{\Gamma_2} \left( k'' \ast u''(t) \right) u''_{tt}(t) \, d\Gamma \\
- rk'(0) \int_{\Gamma_2} u''(t) u''_{tt}(t) \, d\Gamma. \tag{3.20}
\end{align*}
$$

where

$$
E_1''(t) = \frac{1}{2} \left\| u''(t) \right\|_{\Omega_1}^2 + \frac{1}{2} \left\| v''(t) \right\|_{\Omega_2}^2 + \frac{1}{2} \left\| \nabla u''(t) \right\|_{\Omega_1}^2 + \frac{1}{2} \left\| \nabla v''(t) \right\|_{\Omega_2}^2 \\
+ \frac{1}{2} \left\| \nabla u''(t) \right\|_{\Omega_1}^2 \left\| \nabla u''_{tt}(t) \right\|_{\Omega_1}^2 + \frac{1}{2} \left\| \nabla v''(t) \right\|_{\Omega_2}^2 \left\| \nabla v''_{tt}(t) \right\|_{\Omega_2}^2 \\
+ \left( \int_{\Omega_1} \nabla u''(t) \cdot \nabla u''_{tt}(t) \, dx \right)^2 + \left( \int_{\Omega_2} \nabla v''(t) \cdot \nabla v''_{tt}(t) \, dx \right)^2 \\
+ \frac{rk(0)}{2} \left\| u''_{tt}(t) \right\|_{\Gamma_2}^2 + \frac{1}{2} \int_{t-\tau}^{t} \left| u''_{tt}(x, \rho) \right|^2 \, dx \, d\rho. \tag{3.21}
$$
By virtue of Young’s inequality, we obtain

\[
-\mu_2 \int_{\Omega_1} \varphi' \left( u^{(n)}_t (t) \right) u^{(n)}_tt \, dx \\
\leq \varepsilon_1 \left\| u^{(n)}_tt (t) \right\|^2_{\Omega_1} + \frac{\varphi_0^2}{4 \varepsilon_1} \left\| u^{(n)}_t \right\|^2_{\Omega_1},
\]

(3.22)

\[
-r k'(t) \int_{\Gamma_2} u_0n u^{(n)}_t \, d\Gamma \leq \varepsilon_2r \left\| u^{(n)}_t \right\|^2_{\Gamma_2} + \frac{r}{4 \varepsilon_2} (k'(t))^2 \left\| u_0n \right\|^2_{\Gamma_2},
\]

(3.23)

\[
-\rho \int_{\Gamma_2} (k'' + u^{(n)}) u^{(n)}_t \, d\Gamma \leq \varepsilon_2r \left\| u^{(n)}_t \right\|^2_{\Gamma_2} + \frac{r}{4 \varepsilon_2} (k'(0))^2 \left\| u^{(n)} \right\|^2_{\Gamma_2},
\]

(3.24)

and

\[
-\rho k'(0) \int_{\Gamma_2} u^{(n)} u^{(n)}_t \, d\Gamma \leq \varepsilon_2r \left\| u^{(n)}_t \right\|^2_{\Gamma_2} + \frac{r}{4 \varepsilon_2} (k'(0))^2 \left\| u^{(n)} \right\|^2_{\Gamma_2},
\]

(3.25)

where \( 0 < \varepsilon_1 < \frac{1}{2} \) and \( 0 < \varepsilon_2 < \frac{1}{3} \). Substituting (3.22)-(3.25) into (3.20), we can derive

\[
\frac{d}{dt} E_1^{(n)}(t) + \mu_1 \int_{\Omega_1} \beta' \left( u^{(n)}_t \right) \left\| u^{(n)}_t \right\|^2 \, dx \\
+ \left( \frac{1}{2} - \varepsilon_1 \right) \left\| u^{(n)}_tt (t) \right\|^2_{\Omega_1} + r(1 - 3 \varepsilon_2) \left\| u^{(n)}_t \right\|^2_{\Gamma_2} \\
\leq 3 \left\| \nabla u^{(n)}_t \right\|^3_{\Omega_1} \left\| \nabla u^{(n)}_t \right\|^3_{\Omega_1} + 3 \left\| \nabla v^{(n)}_t \right\|^3_{\Omega_2} \left\| \nabla v^{(n)}_t \right\|^3_{\Omega_2} + \left( \frac{1}{2} + \frac{\varphi_0^2}{4 \varepsilon_1} \right) \left\| u^{(n)}_t \right\|^2_{\Omega_1} \\
+ \frac{r}{4 \varepsilon_2} \left\| k''(t-s) \right\|_{L^1(0,+,\infty)} \left\| u^{(n)}(s) \right\|^2_{\Gamma_2} \\
+ \frac{r}{4 \varepsilon_2} (k'(0))^2 \left\| u_0n \right\|^2_{\Gamma_2} + \frac{r}{4 \varepsilon_2} (k'(0))^2 \left\| u^{(n)} \right\|^2_{\Gamma_2}.
\]

(3.26)

By combining (3.14), (3.17) and (3.26) we have

\[
\frac{d}{dt} E_1^{(n)}(t) + \mu_1 \int_{\Omega_1} \beta' \left( u^{(n)}_t \right) \left\| u^{(n)}_t \right\|^2 \, dx \\
+ \left( \frac{1}{2} - \varepsilon_1 \right) \left\| u^{(n)}_tt (t) \right\|^2_{\Omega_1} + r(1 - 3 \varepsilon_2) \left\| u^{(n)}_t \right\|^2_{\Gamma_2} \\
\leq 3 \sqrt{L_1} \left( \left\| \nabla u^{(n)}_t \right\|^3_{\Omega_1} + \left\| \nabla v^{(n)}_t \right\|^3_{\Omega_2} \right) + \left( \frac{1}{2} + \frac{\varphi_0^2}{4 \varepsilon_1} \right) \left\| u^{(n)}_t \right\|^2_{\Omega_1} \\
- \frac{rL_1}{4 \varepsilon_2} k'(0) \left\| k''(t-s) \right\|_{L^1(0,+,\infty)} + \frac{C_r}{4 \varepsilon_2} (k'(t))^2 + \frac{rL_1}{4 \varepsilon_2} (k'(0))^2.
\]

(3.27)
where $L_2 > 0$ is a constant independent of $n$.

**Step 3. Pass to the limit.**

It follows from the first prior estimate (3.14) and second prior estimate (3.27) that

\[
\begin{align*}
\{ u^{(n)} \} & \text{ is bounded in } L^\infty (0, T; H^1(\Omega_1)), \\
\{ u_t^{(n)} \} & \text{ is bounded in } L^\infty (0, T; H^1(\Omega_1)), \\
\{ u_{tt}^{(n)} \} & \text{ is bounded in } L^\infty (0, T; L^2(\Omega_1)), \\
\beta \left( u_t^{(n)} \right) u_t^{(n)} & \text{ is bounded in } L^1 (\Omega_1 \times (0, T)), \\
\varphi \left( u_t^{(n)} (t - \tau) \right) u_t^{(n)} (t - \tau) & \text{ is bounded in } L^1 (\Omega_1 \times (0, T)), \\
\{ u^{(n)} \} & \text{ is bounded in } L^\infty (0, T; L^2(\Gamma_2)), \\
\{ u_t^{(n)} \} & \text{ is bounded in } L^\infty (0, T; L^2(\Gamma_2)), \\
\{ u_{tt}^{(n)} \} & \text{ is bounded in } L^2 (0, T; L^2(\Gamma_2)), \\
\{ v^{(n)} \} & \text{ is bounded in } L^\infty (0, T; H^1(\Omega_2)), \\
\{ v_t^{(n)} \} & \text{ is bounded in } L^\infty (0, T; H^1(\Omega_2)), \\
\{ v_{tt}^{(n)} \} & \text{ is bounded in } L^\infty (0, T; L^2(\Omega_2)), \\
\end{align*}
\]

for all $T \geq 0$. Noting (H1) and (H2), from (3.28), we also obtain the following estimates

\[
\begin{align*}
\left\| \beta \left( u_t^{(n)} \right) \right\|_{L^2(\Omega_1 \times (0, T))} & \leq \hat{C}, \\
\left\| \beta \left( u_t^{(n)} (t - \tau) \right) \right\|_{L^2(\Omega_1 \times (0, T))} & \leq \hat{C},
\end{align*}
\]

where $\hat{C}$ is a positive constant independent of $n$ and $t$. Therefore, (3.28) and (3.29) permit us to obtain subsequences of $\{ u^{(n)} \}$, $\{ v^{(n)} \}$ (we still denote the subsequences by $\{ u^{(n)} \}$, $\{ v^{(n)} \}$ for convenience) such that

\[
\begin{align*}
\{ u^{(n)} \}, v^{(n)} \rightarrow (u, v) \text{ weakly star in } L^\infty (0, T; V), \\
\{ u_t^{(n)} \}, v_t^{(n)} \rightarrow (u_t, v_t) \text{ weakly star in } L^\infty (0, T; V), \\
\{ u_{tt}^{(n)} \}, v_{tt}^{(n)} \rightarrow (u_{tt}, v_{tt}) \text{ weakly star in } L^\infty (0, T; L^2(\Omega_1) \times L^2(\Omega_2)), \\
\beta \left( u_t^{(n)} \right) \rightarrow \chi_1 \text{ weakly in } L^2(\Omega_1 \times (0, T)), \\
\varphi \left( u_t^{(n)} (t - \tau) \right) \rightarrow \chi_2 \text{ weakly in } L^2(\Omega_1 \times (0, T)), \\
u_t^{(n)} \rightarrow u_t \text{ strongly in } L^2 (0, T; L^2(\Gamma_2)).
\end{align*}
\]
Also, noting that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, thanks to Aubin-Lions Theorem [42], we deduce that
\[
\left( u^{(n)}, v^{(n)} \right) \rightarrow (u, v) \text{ strongly in } C(0, T; V),
\]
\[
\left( u^{(n)}_t, v^{(n)}_t \right) \rightarrow (u_t, v_t) \text{ strongly in } C(0, T; L^2(\Omega_1) \times L^2(\Omega_2)),
\]
and
\[
u^{(n)}_t \rightarrow u_t, \text{ a.e. in } \Omega_1 \times (0, T),
\]
\[
u^{(n)}_t(t - \tau) \rightarrow u_t(t - \tau), \text{ a.e. in } \Omega_1 \times (0, T),
\]
(3.30)
(3.31)

Arguing as Lemma 3.1, 3.2 in [9], from (3.28) and (3.30), we have
\[
\beta \left( u^{(n)}_t \right) \rightarrow \beta(u_t) \text{ in } L^1(\Omega_1 \times (0, T)),
\]
and therefore
\[
\beta \left( u^{(n)}_t \right) \rightarrow \beta(u_t) \text{ weakly in } L^2(\Omega_1 \times (0, T)).
\]
(3.32)

Similarly, from (3.28), (3.30) and (H2), we also deduce
\[
\varphi \left( u^{(n)}_t(t - \tau) \right) \rightarrow \varphi(u_t(t - \tau)) \text{ weakly in } L^2(\Omega_1 \times (0, T)).
\]
(3.33)

Taking the first estimate and the continuity of trace operator $\gamma_0 : H^1(\Omega_1) \rightarrow H^{\frac{1}{2}}(\Gamma_2)$ into account, we have
\[
\{ u^{(n)} \} \text{ is bounded in } L^2(0, T; H^2(\Gamma_2)),
\]
\[
\{ u^{(n)}_t \} \text{ is bounded in } L^2(0, T; H^2(\Gamma_2)),
\]
\[
\{ u^{(n)}_{tt} \} \text{ is bounded in } L^2(0, T; L^2(\Omega_2)).
\]

Now the second estimate (3.27) implies
\[
(1 + \| \nabla u^{(n)} \|^2_{\Omega_1}) u^{(n)} \rightarrow (1 + \| \nabla u \|^2_{\Omega_1}) u \text{ strongly in } C(0, T; H^1_0(\Omega_1)),
\]
\[
(1 + \| \nabla v^{(n)} \|^2_{\Omega_2}) v^{(n)} \rightarrow (1 + \| \nabla v \|^2_{\Omega_2}) v \text{ strongly in } C(0, T; H^1_0(\Omega_2)).
\]

Thus we can pass to the limit in (3.1), (3.2)-(3.4) to obtain
\[
\int_0^T \int_{\Omega_1} u_{tt} \phi dx dt + \int_0^T \left( 1 + \| \nabla u \|^2_{\Omega_1} \right) \int_{\Omega_1} \nabla u \cdot \nabla \phi dx dt
\]
\[
+ \mu_1 \int_0^T \int_{\Omega_1} \beta(u_t) \phi dx dt + \mu_2 \int_0^T \int_{\Omega_1} \varphi(u_t(t - \tau)) \phi dx dt
\]
\[
+ \int_0^T \int_{\Omega_2} v_{tt} \psi dx dt + \int_0^T \left( 1 + \| \nabla v \|^2_{\Omega_2} \right) \int_{\Omega_2} \nabla v \cdot \nabla \psi dx dt
\]
\[
= - r \int_0^T \int_{\Omega_2} [u_t + k(0)u + k(t)u(t)] \phi dx dt,
\]
for $\forall (\phi, \psi) \in L^2(0, T; V)$ and $T > 0$, and the limit function $(u, v)$ satisfies the initial conditions and history value, i.e.
\[
(u(0), v(0)) = (u_0, v_0), \quad (u_t(0), v_t(0)) = (u_1, v_1),
\]
\[
u_t(x, t) = f_0(x, t), \quad \forall t \in (-\tau, 0).
\]

Therefore, the limit function $(u, v)$ is a weak solution of problem (1.1)-(1.8).
Step 4. Uniqueness.

Let \((\overline{u}, \overline{v})\) and \((\hat{u}, \hat{v})\) be two solutions of problem (1.1)-(1.8). Then \((\hat{u}, \hat{v}) = (\overline{u} - \hat{u}, \overline{v} - \hat{v})\) verifies

\[
\hat{u}_{tt} - [(1 + \|\nabla \overline{u}\|_{\Omega_1}^2)\Delta \overline{u} - (1 + \|\nabla \hat{u}\|_{\Omega_1}^2)\Delta \hat{u}] + \mu_1 [\beta (\overline{u}_t) - \beta (\hat{u}_t)] \\
+ \mu_2 [\varphi (\overline{u}_t(t-\tau)) - \varphi (\hat{u}_t(t-\tau))] = 0, \ (x, t) \in \Omega_1 \times (0, +\infty),
\]

\[(3.34)\]

\[
\hat{v}_{tt} - [(1 + \|\nabla \overline{v}\|_{\Omega_2}^2)\Delta \overline{v} - (1 + \|\nabla \hat{v}\|_{\Omega_2}^2)\Delta \hat{v}] = 0, \ (x, t) \in \Omega_2 \times (0, +\infty),
\]

\[(3.35)\]

\[
[\|(1 + \|\nabla \overline{u}\|_{\Omega_1}^2)\|_{\Omega_1} \frac{\partial \overline{u}}{\partial v} - (1 + \|\nabla \hat{u}\|_{\Omega_1}^2)\frac{\partial \hat{u}}{\partial v}]
\]

\[
= -r[\hat{u} + k(0)\hat{u} + k' \hat{u}], \ (x, t) \in \Gamma_2 \times (0, +\infty),
\]

\[(3.36)\]

\[
v = 0, \ (x, t) \in \Gamma_0 \times (0, +\infty),
\]

\[(3.37)\]

\[
((1 + \|\nabla \overline{v}\|_{\Omega_2}^2)\|_{\Omega_2} \frac{\partial \overline{v}}{\partial v} - (1 + \|\nabla \hat{v}\|_{\Omega_2}^2)\frac{\partial \hat{v}}{\partial v}]
\]

\[
= (1 + \|\nabla \overline{u}\|_{\Omega_2}^2)\|_{\Omega_2} \frac{\partial \overline{u}}{\partial v} - (1 + \|\nabla \hat{v}\|_{\Omega_2}^2)\frac{\partial \hat{v}}{\partial v},
\]

\[(3.38)\]

\[
\hat{u} = \hat{v}, \ (x, t) \in \Gamma_1 \times (0, +\infty),
\]

Multiplying (3.34) and (3.35) by \(\hat{u}_t\) and \(\hat{v}_t\), and integrating over \(\Omega_1\) and \(\Omega_2\), respectively, using (3.36)-(3.38) and (2.1), we get the identity

\[
\frac{d}{dt} \left\{ \frac{1}{2} \|\hat{u}_t\|_{\Omega_1}^2 + \frac{1}{2} \|\hat{v}_t\|_{\Omega_1}^2 + \frac{1}{2} \|\nabla \hat{u}\|_{\Omega_1}^2 + \frac{1}{2} \|\nabla \hat{v}\|_{\Omega_1}^2 + \frac{1}{2} \|\nabla \overline{u}\|_{\Omega_1}^2 \|\nabla \overline{v}\|_{\Omega_1},
\]

\[
+ \frac{1}{2} \|\nabla \overline{v}\|_{\Omega_2}^2 \|\nabla \hat{v}\|_{\Omega_2}^2 + \frac{r}{2} k(t) \|\hat{u}\|_{\Gamma_2}^2 - \frac{r}{2} \int_{\Gamma_2} (k' \circ \hat{u}) d\Gamma \right\}
\]

\[
= \|\nabla \hat{u}\|_{\Omega_1}^2 \int_{\Omega_1} \nabla \overline{u} \cdot \nabla \hat{u}_t dx - \int_{\Omega_1} \nabla \hat{u} \cdot (\nabla \overline{u} + \nabla \hat{u}) dx \int_{\Omega_1} \nabla \hat{u} \cdot \nabla \hat{u}_t dx
\]

\[
+ \|\nabla \hat{v}\|_{\Omega_2}^2 \int_{\Omega_2} \nabla \overline{v} \cdot \nabla \hat{v}_t dx - \int_{\Omega_2} \nabla \hat{v} \cdot (\nabla \overline{v} + \nabla \hat{v}) dx \int_{\Omega_2} \nabla \hat{v} \cdot \nabla \hat{v}_t dx
\]

\[- \mu_1 \int_{\Omega_1} [\beta (\overline{u}_t) - \beta (\hat{u}_t)] \hat{u}_t dx - \mu_2 \int_{\Omega_1} [\varphi (\overline{u}_t(t-\tau)) - \varphi (\hat{u}_t(t-\tau))] \hat{u}_t dx
\]

\[- r \|\hat{u}_t\|_{\Gamma_2}^2 - \frac{r}{2} \int_{\Gamma_2} (k'' \circ \hat{u}) d\Gamma + \frac{r}{2} k'(t) \|\hat{u}\|_{\Omega_1}^2.
\]

\[(3.42)\]

In addition, we have

\[
\frac{d}{dt} \int_{t-\tau}^t \int_{\Omega_1} |\hat{u}_\rho(x, \rho)|^2 dx d\rho = \|\hat{u}_t\|_{\Omega_1}^2 - \|\hat{u}_t(t-\tau)\|_{\Omega_1}^2.
\]

\[(3.43)\]
Combining (3.42) and (3.43), observing the monotonicity of \( \beta \) and using Young’s inequality, we can derive

\[
\frac{1}{2} \frac{d}{dt} \left\{ \| \tilde{u}_t \|_{\Omega_1}^2 + \| \tilde{v}_t \|_{\Omega_2}^2 + \| \nabla \tilde{u} \|_{\Omega_1}^2 + \| \nabla \tilde{v} \|_{\Omega_2}^2 + \| \nabla \bar{\pi} \|_{\Omega_1}^2 \| \nabla \tilde{u} \|_{\Omega_1}^2 + \| \nabla \bar{\pi} \|_{\Omega_2}^2 \| \nabla \tilde{v} \|_{\Omega_2}^2 \right\}
\]

\[
- r \int_{\Gamma_2} (k' \circ \tilde{u}) \, d\Gamma + rk(t) \| \tilde{u} \|_{\Gamma_2}^2 + \int_{t}^{t_{t-r}} \int_{\Omega_1} |\tilde{u}_\rho(x, \rho)|^2 \, dx \, d\rho
\]

\[
\leq C \left( \| \nabla \tilde{u} \|_{\Omega_1}^2 + \| \nabla \tilde{v} \|_{\Omega_2}^2 \right),
\]

from estimates (3.14) and (3.27). Integrating (3.44) over \((0, t)\), \(0 < t \leq T\) and then using Gronwall’s lemma, we get

\[
\| \tilde{u}_t \|_{\Omega_1}^2 + \| \tilde{v}_t \|_{\Omega_2}^2 + \| \nabla \tilde{u} \|_{\Omega_1}^2 + \| \nabla \tilde{v} \|_{\Omega_2}^2 + \int_{t_{t-r}}^{t} \int_{\Omega_1} |\tilde{u}_\rho(x, \rho)|^2 \, dx \, d\rho = 0.
\]

Hence, uniqueness follows.

With the four steps above, we get the global well-posedness of solution for the problem (1.1)-(1.8).

4. General decay estimate. In this section, by virtue of the energy perturbation technique, we give the proof of our main result Theorem 2 in detail.

Firstly, we introduce the following lemmas which play a key observation in the proof.

**Lemma 1.** Let \((u, v)\) be the solution of (1.1)-(1.8) and assume (H2) holds, then we have

\[
\frac{d}{dt} E(t) \leq - \frac{r}{2} \| u_t \|_{\Omega_2}^2 - (\mu_1 - \zeta - |\mu_2| \varphi_1) \int_{\Omega_1} \beta (u_t) u_t \, dx
\]

\[
- (\zeta - |\mu_2| \varphi_1) \int_{\Omega_1} \beta (u_t(t - \tau)) u_t(t - \tau) \, dx
\]

\[
- \frac{r}{2} \int_{\Gamma_2} (k'' \circ u) \, d\Gamma + \frac{r}{2} k'(t) \| u \|_{\Gamma_2}^2 + \frac{r}{2} k^2(t) \| u_0 \|_{\Gamma_2}^2.
\]

**Proof.** Multiplying (1.1) and (1.2) by \(u_t\) and \(v_t\), integrating over \(\Omega_1\) and \(\Omega_2\), respectively, using (1.3)-(1.5) and (2.1), we get the identity

\[
\frac{d}{dt} \left\{ \frac{1}{2} \| u_t \|_{\Omega_1}^2 + \frac{1}{2} \| v_t \|_{\Omega_2}^2 + \frac{1}{2} \| \nabla u \|_{\Omega_1}^2 + \frac{1}{2} \| \nabla v \|_{\Omega_2}^2 + \frac{1}{4} \| \nabla u \|_{\Omega_1}^4 + \frac{1}{4} \| \nabla v \|_{\Omega_2}^4 \right\}
\]

\[
= - \mu_1 \int_{\Omega_1} \beta (u_t) u_t \, dx - \mu_2 \int_{\Omega_1} \varphi (u_t(t - \tau)) u_t \, dx - r \| u_t \|_{\Gamma_2}^2
\]

\[
+ rk(t) \int_{\Gamma_2} u_t \, dx - \frac{r}{2} \int_{\Gamma_2} (k'' \circ u) \, d\Gamma + \frac{r}{2} k'(t) \| u \|_{\Gamma_2}^2
\]

\[
+ \frac{d}{dt} \left[ \frac{r}{2} \int_{\Gamma_2} (k' \circ u) \, d\Gamma - \frac{r}{2} k(t) \| u \|_{\Gamma_2}^2 \right].
\]

Noting that

\[
\zeta \frac{d}{dt} \int_{t-\tau}^{t} \int_{\Omega_1} \beta (u_\rho(x, \rho)) \, u_\rho(x, \rho) \, dx \, d\rho
\]

\[
= \zeta \int_{\Omega_1} \beta (u_t) u_t \, dx - \zeta \int_{\Omega_1} \beta (u_t(t - \tau)) u_t(t - \tau) \, dx.
\]

(4.3)
Similar to the derivation of (3.11), from (H2), we can deduce
\[ \varphi(u(t - \tau)) u_t \leq \varphi_1 \beta(u(t - \tau)) u_t(t - \tau) + \varphi_1 \beta(u_t) u_t. \] (4.4)
Combining (4.2)-(4.4) and (2.11), we can derive
\[ \frac{d}{dt} E(t) \leq - (\mu_1 - \zeta - |\mu_2| \phi_1) \int_{\Omega_1} \beta(u_t) u_t dx \\
- (\zeta - |\mu_2| \phi_1) \int_{\Omega_1} \beta(u_t(t - \tau)) u_t(t - \tau) dx \\
- r \|u_t\|_{L_2}^2 - \frac{r}{2} \int_{\Gamma_2} (k'' \circ u) d\Gamma + \frac{r}{2} k'(t) \|u_t\|_{L_2}^2 \\
+ r k(t) \int_{\Gamma_2} u_0 u_t dx. \] (4.5)
By virtue of Young’s inequality, we obtain
\[ r k(t) \int_{\Gamma_2} u_0 u_t dx \leq \frac{r}{2} \|u_t\|_{L_2}^2 + \frac{r}{2} k^2(t) \|u_0\|_{L_2}^2. \] (4.6)
Substituting (4.6) into (4.5), then we can derive the result of Lemma 1. \( \square \)

**Remark 4.** From the range of \( \zeta \), we can see that \( \mu_1 - \zeta - |\mu_2| \phi_1 > 0 \) and \( \zeta - |\mu_2| \phi_1 > 0 \). However, since
\[ \frac{r}{2} k^2(t) \|u_0\|_{L_2}^2 \geq 0, \]
\( E(t) \) may be not nonincreasing.

Now we define the functional
\[ \Lambda_1(t) := \int_{\Omega_1} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] u_t dx + \int_{\Omega_2} \left[ (m \cdot \nabla v) + \left( \frac{N}{2} - \theta \right) v \right] v_t dx, \]
where \( 0 < \theta < 1 \) is a constant which will be determined later.

**Lemma 2.** Let \( (u, v) \) be a solution of problem (1.1)-(1.8) and suppose (H1)-(H3) hold, then for \( t_0 > 0 \) large enough, there exist \( \alpha_1 > 0 \) such that
\[ \frac{d}{dt} \Lambda_1(t) \leq - \theta \|u_t\|_{H_1}^2 - (1 - \theta) \|\nabla u\|_{H_1}^4 - \alpha_1 \|\nabla u\|_{H_1}^2 + \left( \frac{R}{2} + \frac{r^2}{\eta_3} \right) \|u_t\|_{L_2}^2 \\
+ \frac{\beta_1 + \beta_3}{4\eta_1} \int_{\Omega_1} \beta(u_t) u_t dx + \frac{(\beta_1 + \beta_3) \phi_1^2}{4\eta_2} \int_{\Omega_1} \beta(u_t(t - \tau)) u_t(t - \tau) dx \\
+ \frac{r^2}{\eta_3} k^2(t) \|u_0\|_{L_2}^2 - \frac{r^2}{\eta_3} k(0) \int_{\Gamma_2} (k' \circ u) d\Gamma - \theta \|v_t\|_{L_2}^2 \\
- (1 - \theta)(1 + \|\nabla v\|_{L_2}^2) \|\nabla v\|_{L_2}^2, \] (4.7)
where \( R = \max\{|x - x_0| : x \in \Omega\} \), \( \eta_i (i = 1, 2, 3) \) are sufficiently small positive constants and
\[ \alpha_1 = \alpha_1(\eta_1, \eta_2, \eta_3) \\
= \frac{1}{2} - 2 \left( \frac{R^2 + \left( \frac{N}{2} - \theta \right)^2 \lambda^2}{\lambda^2} \right) (\mu_1^2 \eta_1 + \mu_2^2 \eta_2) \\
- 2 \left( \frac{N}{2} - \theta \right)^2 \lambda^2 \eta_3 > 0. \]
Proof. By (1.1) and Green formula, we can derive

\[
\frac{d}{dt} \int_{\Omega_t} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] u_t \, dx \\
= \int_{\Omega_t} (m \cdot \nabla u_t) u_t \, dx - (1 + \|\nabla u\|_{\Omega_t}^2) \int_{\Omega_t} \nabla (m \cdot \nabla u) \cdot \nabla u \, dx \\
+ \left( \frac{N}{2} - \theta \right) \|u_t\|_{\Omega_t}^2 - \left( \frac{N}{2} - \theta \right) (1 + \|\nabla u\|_{\Omega_t}^2) \|\nabla u\|_{\Omega_t}^2 \\
+ \int_{\partial \Omega_t} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] (1 + \|\nabla u\|_{\Omega_t}^2) \frac{\partial u}{\partial \nu} \, d\Gamma \\
- \mu_1 \int_{\Omega_t} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \beta(u_t) \, dx \\
- \mu_2 \int_{\Omega_t} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \varphi(u_t(t - \tau)) \, dx. \tag{4.8}
\]

Noting that

\[
\int_{\Omega_t} (m \cdot \nabla u_t) u_t \, dx = -\frac{N}{2} \|u_t\|_{\Omega_t}^2 + \frac{1}{2} \int_{\partial \Omega_t} (m \cdot \nu) |u_t|^2 \, d\Gamma, \tag{4.9}
\]

and

\[
- \int_{\Omega_t} \nabla (m \cdot \nabla u) \cdot \nabla u \, dx \\
= - \int_{\Omega_t} \sum_{i,j=1}^{N} \left[ \frac{\partial}{\partial x_i} \left( m_j \frac{\partial u}{\partial x_j} \right) \frac{\partial u}{\partial x_i} \right] \, dx \\
= - \int_{\Omega_t} \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial m_j}{\partial x_j} \, dx - \frac{1}{2} \int_{\Omega_t} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_i} \right)^2 m_j \, dx \\
= \left( \frac{N}{2} - 1 \right) \|\nabla u\|_{\Omega_t}^2 - \frac{1}{2} \int_{\partial \Omega_t} (m \cdot \nu) |\nabla u|^2 \, d\Gamma. \tag{4.10}
\]

Substituting (4.9), (4.10) into (4.8), then we have

\[
\frac{d}{dt} \int_{\Omega_t} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] u_t \, dx \\
= - \theta \|u_t\|_{\Omega_t}^2 - (1 - \theta)(1 + \|\nabla u\|_{\Omega_t}^2)\|\nabla u\|_{\Omega_t}^2 + \frac{1}{2} \int_{\partial \Omega_t} (m \cdot \nu) |u_t|^2 \, d\Gamma \\
- \frac{1}{2} \int_{\partial \Omega_t} (m \cdot \nabla u) \cdot \nabla u \, d\Gamma \\
+ \int_{\partial \Omega_t} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] (1 + \|\nabla u\|_{\Omega_t}^2) \frac{\partial u}{\partial \nu} \, d\Gamma \\
- \mu_1 \int_{\Omega_t} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \beta(u_t) \, dx \\
- \mu_2 \int_{\Omega_t} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \varphi(u_t(t - \tau)) \, dx. \tag{4.11}
\]
Similarly, using (1.2) and Green formula, we can derive

\[
\frac{d}{dt} \int_{\Omega_2} \left[ (m \cdot \nabla v) + \left( \frac{N}{2} - \theta \right) v \right] v_t \, dx
\]

\[
= -\theta\|v_t\|_{\Omega_2}^2 - (1 - \theta)(1 + \|\nabla v\|_{\Omega_2}^2)\|\nabla v\|_{\Omega_2}^2 + \frac{1}{2} \int_{\partial\Omega_2} (m \cdot \nabla v)|v|^2 \, d\Gamma \\
- \frac{1}{2}(1 + \|\nabla v\|_{\Omega_2}^2) \int_{\partial\Omega_2} (m \cdot \nabla v)|v|^2 \, d\Gamma \\
+ \int_{\partial\Omega_2} \left[ (m \cdot \nabla v) + \left( \frac{N}{2} - \theta \right) v \right] (1 + \|\nabla v\|_{\Omega_2}^2) \frac{\partial v}{\partial \nu} \, d\Gamma \tag{4.12}
\]

where \( \vec{\nu} \) denotes the normal vector pointing towards the exterior of \( \Omega_2 \). Adding (4.11) to (4.12) and using transmission conditions (1.5), we can get

\[
\frac{d}{dt} \Lambda_1(t) = -\theta\|u_t\|_{\Omega_1}^2 - (1 - \theta)(1 + \|\nabla u\|_{\Omega_1}^2)\|\nabla u\|_{\Omega_1}^2 \\
+ \frac{1}{2} \int_{\Omega_2} (m \cdot \nabla u)|u_t|^2 \, d\Gamma - \frac{1}{2}(1 + \|\nabla u\|_{\Omega_1}^2) \int_{\Omega_2} (m \cdot \nabla u)|u|^2 \, d\Gamma \\
- \mu_1 \int_{\Omega_1} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \beta(u) \, dx \\
- \mu_2 \int_{\Omega_1} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \varphi(u(t - \tau)) \, dx \\
- \theta\|v_t\|_{\Omega_2}^2 - (1 - \theta)(1 + \|\nabla v\|_{\Omega_2}^2)\|\nabla v\|_{\Omega_2}^2 \\
+ \frac{1}{2} \int_{\Gamma_0} (m \cdot \nabla v)|v_t|^2 \, d\Gamma - \frac{1}{2}(1 + \|\nabla v\|_{\Omega_2}^2) \int_{\Gamma_0} (m \cdot \nabla v)|v|^2 \, d\Gamma \\
+ \int_{\Gamma_0} \left[ (m \cdot \nabla v) + \left( \frac{N}{2} - \theta \right) v \right] (1 + \|\nabla v\|_{\Omega_2}^2) \frac{\partial v}{\partial \nu} \, d\Gamma. \tag{4.13}
\]

Since \( \frac{\partial v}{\partial x_i} = \vec{\nu} \frac{\partial v}{\partial \nu}, i = 1, ..., N \) and \( m \cdot \vec{\nu} \leq 0 \) on \( \Gamma_0 \), we have

\[
- \frac{1}{2}(1 + \|\nabla v\|_{\Omega_2}^2) \int_{\Gamma_0} (m \cdot \nabla v)|v|^2 \, d\Gamma + (1 + \|\nabla v\|_{\Omega_2}^2) \int_{\Gamma_0} (m \cdot \nabla v) \frac{\partial v}{\partial \nu} \, d\Gamma = 0.
\]

Moreover, since \( v = 0 \) on \( \Gamma_0 \), (4.13) can be rewritten as

\[
\frac{d}{dt} \Lambda_1(t) \leq -\theta\|u_t\|_{\Omega_1}^2 - (1 - \theta)(1 + \|\nabla u\|_{\Omega_1}^2)\|\nabla u\|_{\Omega_1}^2 \\
+ \frac{1}{2} \int_{\Omega_2} (m \cdot \nabla u)|u_t|^2 \, d\Gamma - \frac{1}{2}(1 + \|\nabla u\|_{\Omega_1}^2) \int_{\Omega_2} (m \cdot \nabla u)|u|^2 \, d\Gamma \\
- \mu_1 \int_{\Omega_1} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \beta(u) \, dx \\
+ \int_{\Omega_1} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \varphi(u(t - \tau)) \, dx \\
- \mu_2 \int_{\Omega_1} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \beta(u) \, dx \\
- \theta\|v_t\|_{\Omega_2}^2 - (1 - \theta)(1 + \|\nabla v\|_{\Omega_2}^2)\|\nabla v\|_{\Omega_2}^2 \\
+ \frac{1}{2} \int_{\Gamma_0} (m \cdot \nabla v)|v_t|^2 \, d\Gamma - \frac{1}{2}(1 + \|\nabla v\|_{\Omega_2}^2) \int_{\Gamma_0} (m \cdot \nabla v)|v|^2 \, d\Gamma \\
+ \int_{\Gamma_0} \left[ (m \cdot \nabla v) + \left( \frac{N}{2} - \theta \right) v \right] (1 + \|\nabla v\|_{\Omega_2}^2) \frac{\partial v}{\partial \nu} \, d\Gamma.
\]
\[-\mu_2 \int_{\Omega_1} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \varphi(u(t - \tau)) dx \]
\[-\theta \|v_t\|_{\Omega_2}^2 - (1 - \theta) (1 + \|\nabla v\|_{\Omega_2}^2) \|\nabla v\|_{\Omega_2}^2. \tag{4.14}\]

Using Young’s inequality to the fifth to seventh terms in the right side of (4.14), respectively, we have

\[
\left| -\mu_1 \int_{\Omega_1} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \beta(u_t) dx \right|
\leq 2 \left[ R^2 + \left( \frac{N}{2} - \theta \right) \lambda_1^2 \right] \mu_1^2 \eta_1 \|\nabla u\|_{\Omega_1}^2 + \frac{1}{4\eta_1} \int_{\Omega_1} |\beta(u_t)|^2 dx, \tag{4.15}\]

\[
\left| -\mu_2 \int_{\Omega_1} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] \varphi(u(t - \tau)) dx \right|
\leq 2 \left[ R^2 + \left( \frac{N}{2} - \theta \right) \lambda_2^2 \right] \mu_2^2 \eta_2 \|\nabla u\|_{\Omega_1}^2 + \frac{1}{4\eta_2} \int_{\Omega_1} |\varphi(u(t - \tau))|^2 dx, \tag{4.16}\]

and

\[
\left| \int_{\Gamma_2} \left[ (m \cdot \nabla u) + \left( \frac{N}{2} - \theta \right) u \right] (1 + \|\nabla u\|_{\Omega_2}^2) \frac{\partial u}{\partial \nu} d\Gamma \right|
\leq 2 R^2 \eta_3 \|\nabla u\|_{F_2}^2 + 2 \left( \frac{N}{2} - \theta \right) \lambda_2^2 \eta_3 \|\nabla u\|_{\Omega_1}^2 + \frac{1}{4\eta_3} \left| (1 + \|\nabla u\|_{\Omega_1}^2) \frac{\partial u}{\partial \nu} \right|_{\Gamma_2}^2
\leq 2 R^2 \eta_3 \|\nabla u\|_{F_2}^2 + 2 \left( \frac{N}{2} - \theta \right) \lambda_2^2 \eta_3 \|\nabla u\|_{\Omega_1}^2
+ \frac{r^2}{4\eta_3} \|u_t + k(0)u - k(t)u_0 + k' \ast u\|_{\Gamma_2}^2
\leq 2 R^2 \eta_3 \|\nabla u\|_{F_2}^2 + 2 \left( \frac{N}{2} - \theta \right) \lambda_2^2 \eta_3 \|\nabla u\|_{\Omega_1}^2 + \frac{r^2}{\eta_3} \|u_t\|_{F_1}^2 + \frac{r^2}{\eta_3} k^2(0)\|u_0\|_{F_2}^2
+ \frac{r^2}{\eta_3} k^2(t)\|u\|_{F_2}^2 - \frac{r^2}{\eta_3} k(0) \int_{\Gamma_2} (k' \circ u) d\Gamma, \tag{4.17}\]

where \(\eta_i (i = 1, 2, 3)\) are sufficiently small positive constants and we have used inequality (2.2) and the following identity

\[(1 + \|\nabla u\|_{\Omega_2}^2) \frac{\partial u}{\partial \nu} = -r [u_t + k(0)u - k(t)u_0 + k' \ast u]
- r [u_t + k(t)u - k(t)u_0 - k' \circ u].\]

Besides, \(\lambda\) and \(\lambda_1\) are the optimal constants of trace inequality and the first eigenvalue of \(-\Delta\) with Dirichlet boundary condition, respectively, i.e., \(\|u\|_{F_2}^2 \leq \lambda \|\nabla u\|_{\Omega_1}^2\) and \(\|u\|_{\Omega_1}^2 \leq \lambda_1 \|\nabla u\|_{\Omega_1}^2\).
Substituting (4.15)-(4.17) into (4.14), we can derive
\[
\frac{d}{dt} A_1(t) \leq - \theta \|u_t\|^2_{\Omega_1} - (1 - \theta) \|\nabla u\|^4_{\Omega_1} - \alpha_1(\eta_1, \eta_2, \eta_3) \|\nabla u\|^2_{\Omega_1} \\
+ \left( \frac{R}{2} + \frac{\nu^2}{\eta_3} \right) \|u_t\|^2_{\Omega_2} + \frac{1}{4\eta_1} \int_{\Omega_1} |\beta(u_t)|^2 dx \\
+ \frac{1}{4\eta_2} \int_{\Omega_1} |\varphi(u_t(t - \tau))|^2 dx + \frac{\nu^2}{\eta_3} |u_0|^2_{\Omega_2} \\
- \frac{\nu^2}{\eta_3} k(0) \int_{\Gamma_2} (k' \circ u) d\Gamma - \left( \frac{\delta}{2} - 2R^2 \eta_3 \right) \|\nabla u\|^2_{\Omega_2} \\
- \left[ \frac{1 - \theta}{2\lambda^2 k(t)} - \frac{\nu^2}{\eta_3} \right] k(t) ||u||^2_{\Omega_2} \\
- \theta \|\nu_t\|^2_{\Omega_2} - (1 - \theta)(1 + \|\nabla v\|^2_{\Omega_2}) \|\nabla v\|^2_{\Omega_1},
\]
(4.18)

Denote
\[
\Omega_A^1 = \{ x \in \Omega_1 : |u_t| \geq \epsilon' \}, \quad \Omega_B^1 = \{ x \in \Omega_1 : |u_t(t - \tau)| \geq \epsilon' \}.
\]

Then it follows from (H1) that
\[
\int_{\Omega_A^1} |\beta(u_t)|^2 dx \leq \beta_1 \int_{\Omega_A^1} \beta(u_t) u_t dx \leq \beta_1 \int_{\Omega_1} \beta(u_t) u_t dx,
\]
(4.19)

and
\[
\int_{\Omega_1 \setminus \Omega_A^1} |\beta(u_t)|^2 dx \leq \beta_3 \int_{\Omega_1 \setminus \Omega_A^1} \beta(u_t) u_t dx \leq \beta_3 \int_{\Omega_1} \beta(u_t) u_t dx.
\]
(4.20)

Similarly, from (H2), we have
\[
\int_{\Omega_B^1} |\varphi(u_t(t - \tau))|^2 dx \leq \varphi_1^2 \beta_1 \int_{\Omega_B^1} \beta(u_t(t - \tau)) u_t(t - \tau) dx \\
\leq \varphi_1^2 \beta_1 \int_{\Omega_1} \beta(u_t(t - \tau)) u_t(t - \tau) dx,
\]
(4.21)

and
\[
\int_{\Omega_1 \setminus \Omega_B^1} |\varphi(u_t(t - \tau))|^2 dx \leq \varphi_2^2 \beta_3 \int_{\Omega_1 \setminus \Omega_B^1} \beta(u_t(t - \tau)) u_t(t - \tau) dx \\
\leq \varphi_2^2 \beta_3 \int_{\Omega_1} \beta(u_t(t - \tau)) u_t(t - \tau) dx.
\]
(4.22)

Substituting (4.19)-(4.22) into (4.18), we obtain
\[
\frac{d}{dt} A_1(t) \leq - \theta \|u_t\|^2_{\Omega_1} - (1 - \theta) \|\nabla u\|^4_{\Omega_1} - \alpha_1(\eta_1, \eta_2, \eta_3) \|\nabla u\|^2_{\Omega_1} \\
+ \left( \frac{R}{2} + \frac{\nu^2}{\eta_3} \right) \|u_t\|^2_{\Omega_2} + \frac{1}{4\eta_1} \int_{\Omega_1} |\beta(u_t)|^2 dx \\
+ \frac{(\beta_1 + \beta_3)\varphi_1^2}{4\eta_2} \int_{\Omega_1} \beta(u_t(t - \tau)) u_t(t - \tau) dx \\
+ \frac{\nu^2}{\eta_3} k^2(t) ||u_0||^2_{\Omega_2} + \frac{\nu^2}{\eta_3} k(0) \int_{\Gamma_2} (k' \circ u) d\Gamma
\]
\[
- \left( \frac{\delta}{2} - 2R^2\eta_3 \right) \| \nabla u \|^2_{\Gamma_2} - \left[ \frac{1 - \theta}{2\lambda^2k(t)} - \frac{r^2}{\eta_3} \right] k(t) \| u \|^2_{\Gamma_2} \\
- \theta \| v_t \|^2_{\Omega_2} - (1 - \theta)(1 + \| \nabla v \|^2_{\Omega_2}) \| \nabla v \|^2_{\Omega_2}.
\]  
(4.23)

Choosing \( \eta_i (i = 1, 2, 3) \) small enough such that

\[
\alpha_1 = \alpha_1(\eta_1, \eta_2, \eta_3)
\]

\[
= \frac{1 - \theta}{2} - 2 \left[ R^2 + \left( \frac{N}{2} - \theta \right)^2 \lambda_1^2 \right] \left( \mu_1^2 \eta_1 + \mu_2^2 \eta_2 \right)
\]

\[
- 2 \left( \frac{N}{2} - \theta \right)^2 \lambda^2 \eta_3 > 0,
\]

and

\[
\frac{\delta}{2} - 2R^2\eta_3 > 0.
\]

Then it follows from \( \lim_{t \to \infty} k(t) = 0 \) in (H3) that our result holds for \( t_0 > 0 \) large enough.

Next, we define the functional

\[
\Lambda_2(t) = \int_{t-\tau}^t \int_{\Omega_1} e^{\rho-t} \beta(u_\rho(x, \rho))u_\rho(x, \rho)dxd\rho.
\]

Then we have the following lemma.

**Lemma 3.** The functional \( \Lambda_2 \) satisfies

\[
\frac{d}{dt} \Lambda_2(t) \leq - C(\tau) \left[ \int_{t-\tau}^t \int_{\Omega_1} \beta(u_\rho(x, \rho))u_\rho(x, \rho)dxd\rho \\
+ \int_{\Omega_1} \beta(u_t(t-\tau))u_t(t-\tau)d\rho \right] + \int_{\Omega_1} \beta(u_t)u_t d\rho,
\]

where \( C(\tau) \) is a positive constant only depending on \( \tau \).

**Proof.** We use the method introduced by [3] to prove this lemma. Taking the derivative of \( \Lambda_2(t) \) directly, we have

\[
\frac{d}{dt} \Lambda_2(t) = - \int_{t-\tau}^t \int_{\Omega_1} e^{\rho-t} \beta(u_\rho(x, \rho))u_\rho(x, \rho)dxd\rho
\]

\[
- e^{-\tau} \int_{\Omega_1} \beta(u_t(t-\tau))u_t(t-\tau)d\rho + \int_{\Omega_1} \beta(u_t)u_t d\rho
\]

\[
\leq - C(\tau) \left[ \int_{t-\tau}^t \int_{\Omega_1} \beta(u_\rho(x, \rho))u_\rho(x, \rho)dxd\rho \\
+ \int_{\Omega_1} \beta(u_t(t-\tau))u_t(t-\tau)d\rho \right] + \int_{\Omega_1} \beta(u_t)u_t d\rho,
\]

where \( C(\tau) \) is a positive constant depending only on \( \tau \).

Now, we give the proof of Theorem 2 in detail.

**The proof of Theorem 2:** Define the Lyapunov functional as

\[
L(t) := M_1 E(t) + M_2 \Lambda_1(t) + M_3 \Lambda_2(t),
\]

where \( M_i (i = 1, 2, 3) \) are positive constants which will be determined later.
Differentiating $L(t)$ directly and making use of Lemma 1-Lemma 3, we can derive

$$\frac{d}{dt} L(t) \leq -M_2 \theta \|u_t\|_{\Omega_1}^2 - (1 - \theta) M_2 \|\nabla u\|_{\Omega_1}^4 - M_2 \alpha_1 \|\nabla u\|_{\Omega_1}^2,$$

$$- C(\tau) M_3 \int_{t-\tau}^t \int_{\Omega_1} \beta(u_\rho(x, \rho)) u_\rho(x, \rho) dx d\rho$$

$$- \left[ (\mu_1 - \zeta - |\mu_2| \varphi_1) M_1 - \frac{(\beta_1 + \beta_3) M_2}{4\eta_1} - M_3 \right] \int_{\Omega_1} \beta(u_t) u_t dx$$

$$- \left[ (\zeta - |\mu_2| \varphi_1) M_1 - \frac{(\beta_1 + \beta_3) \varphi_1^2 M_2}{4\eta_2} + C(\tau) M_3 \right]$$

$$\times \int_{\Omega_1} \beta(u_t(t - \tau)) u_t(t - \tau) dx$$

$$- \left[ \frac{M_1 r}{2} - \left( \frac{r^2}{\eta_3} + \frac{R}{2} \right) M_2 \right] \|u_t\|_{\Omega_1}^2 + \left( \frac{M_1 r}{2} + \frac{M_2 r^2}{\eta_3} \right) k^2(t) \|u_0\|_{\Omega_2}^2$$

$$- \left[ \frac{M_1 r}{2} + \frac{M_2 r^2}{\eta_3} \right] k(0) \int_{\Gamma_2} (k' \circ u) d\Gamma$$

$$- M_2 \theta \|u_1\|_{\Omega_2}^2 - (1 - \theta) M_2 (1 + \|\nabla v\|_{\Omega_2}) \|\nabla v\|_{\Omega_2}^2,$$

(4.25)

Choosing $M_i$, $i = 1, 2, 3 > 0$ appropriately such that

$$\mu_1 - \zeta - |\mu_2| \varphi_1) M_1 - \frac{(\beta_1 + \beta_3) M_2}{4\eta_1} - M_3 > 0,$$

$$\zeta - |\mu_2| \varphi_1) M_1 - \frac{(\beta_1 + \beta_3) \varphi_1^2 M_2}{4\eta_2} + C(\tau) M_3 > 0,$$

and

$$\frac{M_1 r}{2} - \left( \frac{r^2}{\eta_3} + \frac{R}{2} \right) M_2 > 0,$$

and since $E(t)$ is equivalent to

$$\|u_t\|_{\Omega_1}^2 + \|v_t\|_{\Omega_2}^2 + \|\nabla u\|_{\Omega_1}^2 + \|\nabla v\|_{\Omega_2}^2 + \frac{1}{4} \|\nabla u\|_{\Omega_1}^4 + \frac{1}{4} \|\nabla v\|_{\Omega_2}^4$$

$$+ k(t) \|u_0\|_{\Omega_2}^2 - \int_{\Gamma_2} (k' \circ u) d\Gamma + \int_{t-\tau}^t \int_{\Omega_1} \beta(u_\rho(x, \rho)) u_\rho(x, \rho) dx d\rho,$$

we know that there exist positive constants $\kappa_1$, $\kappa_2$, $\kappa_3$, such that

$$\frac{d}{dt} L(t) \leq -\kappa_1 E(t) + \kappa_2 k^2(t) \|u_0\|_{\Gamma_2}^2 - \kappa_3 \int_{\Gamma_2} (k' \circ u) d\Gamma,$$

(4.26)

for all $t \geq t_0$.

Multiplying (4.26) by $\xi(t)$ and using (H3) and (4.1), we can obtain

$$\xi(t) \frac{d}{dt} L(t) \leq -\kappa_1 \xi(t) E(t) + \kappa_2 \xi(t) k^2(t) \|u_0\|_{\Gamma_2}^2 - \kappa_3 \xi(t) \int_{\Gamma_2} (k' \circ u) d\Gamma$$

$$\leq -\kappa_1 \xi(t) E(t) + \kappa_2 \xi(t) k^2(t) \|u_0\|_{\Gamma_2}^2 + \kappa_3 \int_{\Gamma_2} (k' \circ u) d\Gamma$$

$$\leq -\kappa_1 \xi(t) E(t) + \kappa_2 \xi(t) k^2(t) \|u_0\|_{\Gamma_2}^2$$

$$+ \kappa_3 \left[ -\frac{2}{r} \frac{d}{dt} E(t) + k^2(t) \|u_0\|_{\Gamma_2}^2 \right].$$

(4.27)
Integrating this inequality from 0, then we have
\[
\frac{d}{dt} \left[ \xi(t)L(t) + \frac{2\kappa_3}{r} E(t) \right] \leq -\kappa_1 \xi(t)E(t) + [\kappa_2 \xi(0) + \kappa_3] k^2(t)\|u_0\|_{L_2}^2.
\]  
(4.28)

Now, define the functional
\[
\mathcal{L}(t) = \xi(t)L(t) + \frac{2\kappa_3}{r} E(t),
\]
for all \( t \geq t_0 \). Then it is easy to verify that \( \mathcal{L}(t) \) is equivalent to \( E(t) \) and there exist constants \( \gamma_1, \gamma_2 > 0 \) such that
\[
\frac{d}{dt} \mathcal{L}(t) \leq -\gamma_1 \xi(t)\mathcal{L}(t) + \gamma_2 k^2(t)\|u_0\|_{L_2}^2,
\]  
(4.29)

for all \( t \geq t_0 \).

Case (i): If \( u_0 = 0 \) on \( \Gamma_2 \), inequality (4.29) becomes
\[
\frac{d}{dt} \mathcal{L}(t) \leq -\gamma_1 \xi(t)\mathcal{L}(t).
\]
Integrating this inequality from 0 to \( t \), we have
\[
\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\gamma_1 \int_0^t \xi(s)ds},
\]  
(4.30)

for all \( t \geq t_0 \). It follows from the equivalence relation between \( \mathcal{L}(t) \) and \( E(t) \) that there exists a constant \( C > 0 \) such that
\[
E(t) \leq CE(0)e^{-\gamma_1 \int_0^t \xi(s)ds},
\]
for all \( t \geq t_0 \).

Case (ii): If \( u_0 \neq 0 \) on \( \Gamma_2 \), setting
\[
\mathcal{F}(t) = \mathcal{L}(t) - \gamma_2 \|u_0\|_{L_2}^2 e^{-\gamma_1 \int_0^t \xi(s)ds} \int_0^t k^2(s)e^{\gamma_1 \int_0^r \xi(r)dr}ds,
\]  
(4.31)

for all \( t \geq t_0 \). Then by calculating directly, and using (4.29) we get
\[
\frac{d}{dt} \mathcal{F}(t) \leq -\gamma_1 \xi(t)\mathcal{F}(t),
\]
for all \( t \geq t_0 \). Integrating this inequality over \( (0, t) \), we have
\[
\mathcal{F}(t) \leq \mathcal{F}(0)e^{-\gamma_1 \int_0^t \xi(s)ds},
\]  
(4.32)

for all \( t \geq t_0 \). Combining (4.31) and (4.32), we can derive
\[
\mathcal{L}(t) \leq \left[ \mathcal{L}(0) + \gamma_2 \|u_0\|_{L_2}^2 \int_0^t k^2(s)e^{\gamma_1 \int_0^r \xi(r)dr}ds \right] e^{-\gamma_1 \int_0^t \xi(s)ds},
\]  
(4.33)

for all \( t \geq t_0 \). It follows from the equivalence relation between \( \mathcal{L}(t) \) and \( E(t) \) that there exists a constant \( C > 0 \) such that
\[
E(t) \leq C \left[ E(0) + \gamma_2 \|u_0\|_{L_2}^2 \int_0^t k^2(s)e^{\gamma_1 \int_0^r \xi(r)dr}ds \right] e^{-\gamma_1 \int_0^t \xi(s)ds},
\]
for all \( t \geq t_0 \). Consequently, the proof of Theorem 2 is completed.

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