THE FIRST SIMULTANEOUS SIGN CHANGE AND NON-VANISHING OF HECKE EIGENVALUES OF NEWFORMS

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ABSTRACT. Let \( f \) and \( g \) be two distinct newforms which are normalized Hecke eigenforms of weights \( k_1, k_2 \geq 2 \) and levels \( N_1, N_2 \geq 1 \) respectively. Also let \( a_f(n) \) and \( a_g(n) \) be the \( n \)-th Fourier-coefficients of \( f \) and \( g \) respectively. In this article, we investigate the first sign change of the sequence \( \{a_f(p^\alpha)a_g(p^\beta)\} \) where \( p \) is a prime number. We further study the non-vanishing of the sequence \( \{a_f(n)a_g(n)\} \) and derive bounds for first non-vanishing term in this sequence. We also show, using ideas of Kowalski-Robert-Wu and Murty-Murty, that there exists a set of primes \( S \) of natural density one such that for any prime \( p \in S \), the sequence \( \{a_f(p^\alpha)a_g(p^\beta)\} \) has no zero elements. This improves a recent work of Kumari and Ram Murty. Finally, using \( \mathfrak{B} \)-free numbers, we investigate simultaneous non-vanishing of coefficients of \( m \)-th symmetric power \( L \)-functions of non-CM forms in short intervals.

1. INTRODUCTION

For positive integers \( k \geq 2, N \geq 1 \), let \( S_k(N) \) be the space of cusp forms of weight \( k \) for the congruence subgroup \( \Gamma_0(N) \) and \( S_k^{new}(N) \) be the subspace of \( S_k(N) \) consisting of newforms. We investigate arithmetic properties of Fourier-coefficients of \( f \in S_k^{new}(N) \) which are normalized Hecke eigenforms. This question has been studied extensively by several mathematicians. In recent works, Kowalski, Lau, Soundararajan and Wu [15] and later Matomäki [22] showed that any \( f \in S_k^{new}(N) \) which is a normalized Hecke eigenform is uniquely determined by the signs of their Hecke eigenvalues at primes. In this article, we investigate simultaneous sign change and non-vanishing of Hecke eigenvalues of such forms. More precisely, for \( z \in \mathcal{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\} \), \( q := e^{2\pi i z} \), let

\[
\begin{align*}
\begin{align*}
(f(z) = \sum_{n=1}^{\infty} a_f(n)q^n &\in S_k^{new}(N_1) \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} a_g(n)q^n \in S_k^{new}(N_2) \\
\end{align*}
\end{align*}
\]

be two newforms which are normalized Hecke eigenforms. Here we study first sign change and non-vanishing of the sequence \( \{a_f(n)a_g(n)\} \).

The question of simultaneous sign change for arbitrary cusp forms was first studied by Kohnen and Sengupta [14] under certain conditions which were later removed by the first

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author, Kohnen and Rath [9]. In the later paper, the authors prove infinitely many sign change of the sequence \( \{a_f(n)a_g(n)\}_{n \in \mathbb{N}} \). Here we prove the following theorem.

**Theorem 1.** Let \( N_1, N_2 \) be square-free, \( N := \text{lcm}[N_1, N_2] \) and \( f \in S^\text{new}_{k_1}(N_1) \), \( g \in S^\text{new}_{k_2}(N_2) \) be two distinct normalized Hecke eigenforms with Fourier expansions as in (1). Then there exists a prime power \( p^\alpha \) with \( \alpha \leq 2 \) and

\[
p^\alpha \ll \max \left\{ \exp(c \log^2(\sqrt{q(f)} + \sqrt{q(g)})), \left[ N^2 \left(1 + \frac{k_2 - k_1}{2}\right) \left(\frac{k_1 + k_2}{2}\right)\right]^{1+\epsilon} \right\}
\]

such that \( a_f(p^\alpha)a_g(p^\alpha) < 0 \). Here \( c > 0 \) is an absolute constant and \( q(f), q(g) \) are analytic conductors of Rankin-Selberg \( L \)-functions of \( f \) and \( g \) respectively.

We use Rankin-Selberg method and an idea of Iwaniec, Kohnen and Sengupta [12] to prove Theorem 1. This theorem can be thought of as a variant of Strum’s result about distinguishing two newforms by their Fourier-coefficients. This result can be compared with the results of Lau-Liu-Wu [19], Kohnen [13], Kowalski-Michel-Vanderkam [16], Ram Murty [25] and Sengupta [32].

Next we investigate sign changes of the sequence \( \{a_f(n)a_g(n^2)\}_{n \in \mathbb{N}} \) in short intervals. This question of sign change for the sequence \( \{a_f(n)a_g(n)\}_{n \in \mathbb{N}} \) in short intervals was considered by Kumari and Ram Murty (see [18, Theorem 1.6]). Here we prove the following.

**Theorem 2.** Let

\[
f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S^\text{new}_{k_1}(N_1) \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} a_g(n)q^n \in S^\text{new}_{k_2}(N_2)
\]

be two distinct normalized Hecke eigenforms. For any sufficiently large \( x \) and any \( \delta > \frac{17}{18} \), the sequence \( \{a_f(n)a_g(n^2)\}_{n \in \mathbb{N}} \) has at least one sign change in \((x, x + x^\delta)\). In particular, the number of sign changes for \( n \leq x \) is \( \gg x^{1-\delta} \).

Sign changes of Hecke eigenvalues implies non-vanishing of Hecke eigenvalues. The question of non-vanishing of Hecke eigenvalues has been studied by several mathematicians. One of the fundamental open problem in this area is a question of Lehmer which predicts that \( \tau(n) \neq 0 \) for all \( n \in \mathbb{N} \), where \( \tau(n) \) is the Ramanujan’s \( \tau \)-function defined as follows;

\[
\Delta(z) := \sum_{n=1}^{\infty} \tau(n)q^n := q \prod_{n=1}^{\infty} (1-q^n)^4.
\]

It is well known that \( \Delta(z) \in S_{12}(1) \) is the unique normalized Hecke eigenform. We now investigate non-vanishing of the sequence \( \{a_f(p^m)a_g(p^m)\}_{m \in \mathbb{N}} \) and our first theorem in this direction is the following.
Theorem 3. Let
\[ f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S^{new}_{k_1}(N_1) \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} a_g(n)q^n \in S^{new}_{k_2}(N_2) \]
be two distinct normalized Hecke eigenforms. Then for all primes \( p \) with \( (p, N_1N_2) = 1 \), the set
\[ \{ m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) \neq 0 \} \]
has positive density.

The first author along with Kohnen and Rath (see Theorem 3 of [9]) showed that for infinitely many primes \( p \), the sequence \( A_p := \{ a_f(p^m)a_g(p^m) \}_{m \in \mathbb{N}} \) has infinitely many sign changes and hence in particular, \( A_p \) has infinitely many non-zero elements. Theorem 3 shows that for all primes \( p \) with \( (p, N_1N_2) = 1 \), the non-zero elements of the sequence \( A_p \) has positive density and hence does not follow from Theorem 3 of [9]. Our next theorem strengthens Theorem 1.2 of Kumari and Ram Murty [18].

Theorem 4. Let
\[ f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S^{new}_{k_1}(N_1) \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} a_g(n)q^n \in S^{new}_{k_2}(N_2) \]
be two distinct normalized non-CM Hecke eigenforms. Then there exists a set \( S \) of primes with natural density one such that for any \( p \in S \) and integers \( m, m' \geq 1 \), we have
\[ a_f(p^m)a_g(p^{m'}) \neq 0. \]

Now we shall consider the question of the first simultaneous non-vanishing which is analogous to the question considered in Theorem 1. Our result here is as follows.

Theorem 5. Let
\[ f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_{k_1}(N_1) \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} a_g(n)q^n \in S_{k_2}(N_2) \]
be two distinct normalized Hecke eigenforms. Further assume that \( N := \text{lcm}[N_1, N_2] > 12 \). Then there exists a positive integer \( 1 < n \leq (2 \log N)^4 \) with \( (n, N) = 1 \) such that
\[ a_f(n)a_g(n) \neq 0. \]

Further, when \( N \) is odd, then there exists an integer \( 1 < n \leq 16 \) with \( (n, N) = 1 \) such that
\[ a_f(n)a_g(n) \neq 0. \]

Note that \( a_f(1)a_g(1) = 1 \neq 0 \) but we are trying to find the first natural number \( n > 1 \) with \( (n, N) = 1 \) for which \( a_f(n)a_g(n) \neq 0 \) which we call the first non-trivial simultaneous non-vanishing. Though first simultaneous sign change (see [19], also Theorem 1 above) implies first
non-trivial simultaneous non-vanishing but the bound proved in Theorem 5 is much stronger for first non-trivial simultaneous non-vanishing.

The paper is organized as follows. In the next section, we introduce notations and briefly recall some preliminaries. In sections 3 to 7, we provide proofs of theorems mentioned in the introduction. Finally, in the last section, using \( \mathbb{B} \)-free numbers, we deduce certain results about simultaneous non-vanishing of coefficients of symmetric power \( L \)-functions of non-CM forms in short intervals.

2. Notation and Preliminaries

Throughout the paper, \( p \) denotes a prime number and \( \mathcal{P} \) denotes the set of all primes. We say that a subset \( S \subset \mathcal{P} \) has natural density \( d(S) \) if the limit

\[
\lim_{x \to \infty} \frac{\# \{ p \in \mathcal{P} : p \leq x \text{ and } p \in S \}}{\# \{ p \in \mathcal{P} : p \leq x \}}
\]

exists and equal to \( d(S) \). For any non-negative real number \( x \), we denote the greatest integer \( n \leq x \) by \( \lceil x \rceil \). Let \( A \) be a subset of the set of natural numbers. Then we say the density of the set \( A \) is \( d(A) \) if the limit

\[
\lim_{x \to \infty} \frac{\# \{ n \leq x : n \in A \}}{\# \{ n \leq x \}}
\]

exists and equal to the real number \( d(A) \). For any \( n, m \in \mathbb{N} \), we shall denote the greatest common divisor of \( n \) and \( m \) by \( (n,m) \).

For a normalized Hecke eigenform \( f \in S_{k}^{new}(N) \) with Fourier expansion

\[
f(z) = \sum_{n \geq 1} a_{f}(n)q^{n},
\]

we write

\[
\lambda_{f}(n) := \frac{a_{f}(n)}{n^{(k-1)/2}}.
\]

From the theory of Hecke operators, we know

\[
\lambda_{f}(1) = 1 \quad \text{and} \quad \lambda_{f}(m)\lambda_{f}(n) = \sum_{d|(m,n),(d,N)=1} \lambda_{f}\left(\frac{mn}{d^2}\right).
\]

Also by a celebrated work of Deligne, we have

\[
|\lambda_{f}(n)| \leq d(n) \quad \text{for all } (n,N) = 1,
\]

where \( d(n) \) denotes the number of positive divisors of \( n \).

The following result of Kowalski-Robert-Wu [17, Lemma 2.3] (see also Murty-Murty [26, Lemma 2.5] ) plays an important role in the proof of Theorem 4.
Lemma 6. Let
\[ f(z) := \sum_{n=1}^{\infty} a_f(n)q^n \in S_{k}^{new}(N) \]
be a normalized non-CM Hecke eigenform. For \( \nu \geq 1 \), let
\[ P_{f,\nu} := \{ p \in \mathcal{P} \mid p \nmid N \text{ and } \lambda_f(p^\nu) = 0 \}. \]
Then for any \( \nu \geq 1 \), we have
\[ #(P_{f,\nu} \cap [1, x]) \ll_{f, \delta} x \frac{(\log x)^{1+\delta}}{x}, \]
for any \( x \geq 2 \) and \( 0 < \delta < 1/2 \). Here the implied constant depends on \( f \) and \( \delta \). Let
\[ P_f := \bigcup_{\nu \in \mathbb{N}} P_{f,\nu}. \]
Then for any \( x \geq 2 \) and \( 0 < \delta < 1/2 \), we have
\[ #(P_f \cap [1, x]) \ll_{f, \delta} x \frac{(\log x)^{1+\delta}}{x}, \]
where the implied constant depends only on \( f \) and \( \delta \).

We now recall some well known properties of Rankin-Selberg \( L \)-function associated with \( f \in S_{k_1}^{new}(N_1) \) and \( g \in S_{k_2}^{new}(N_2) \) which are normalized Hecke eigenforms. Suppose that \( k_1 \leq k_2 \). One can now define the Rankin-Selberg \( L \)-function as follows
\[ R(f, g; s) := \sum_{n \geq 1} \lambda_f(n)\lambda_g(n)n^{-s}, \]
which is absolutely convergent for \( \Re(s) > 1 \) and hence it defines a holomorphic function there. Let \( M := \gcd(N_1, N_2) \) and \( N := \text{lcm}[N_1, N_2] \) be square-free. By the work of Rankin \cite{29} (see also \cite{27}, page 304), one knows that the function \( \zeta_N(2s)R(f, g; s) \) is entire if \( f \neq g \), where \( \zeta_N(s) \) is defined by
\[ \zeta_N(s) := \prod_{p \mid N} (1 - p^{-s})^{-1} \quad \text{for } \Re(s) > 1. \]
(5)

We also have the completed Rankin-Selberg \( L \)-function
\[ R^*(f, g; s) := (2\pi)^{-2s} \Gamma(s + \frac{k_2 - k_1}{2})\Gamma(s + \frac{k_1 + k_2}{2} - 1) \prod_{p \mid M} (1 - c_p p^{-s})^{-1} \zeta_N(2s)R(f, g; s) \]
with \( c_p = \pm 1 \) depending on the forms \( f \) and \( g \). It is well known by the works of Ogg (see \cite{27} Theorem 6) and Winnie Li (see \cite{20} Theorem 2.2) that the completed Rankin-Selberg \( L \)-function satisfies the functional equation
\[ R^*(f, g; s) = N^{1-2s} R^*(f, g; 1-s). \]
(7)
3. Proof of Theorem 1

Throughout this section, we assume that \( N_1 \) and \( N_2 \) are square-free and \( f \in S_{k_1}^{\text{new}}(N_1) \), \( g \in S_{k_2}^{\text{new}}(N_2) \) are two distinct normalized Hecke eigenforms with \( 1 < k_1 \leq k_2 \). In order to prove Theorem 1, we need to prove the following Propositions.

Proposition 7. For square-free integers \( N_1, N_2 \), let \( f \in S_{k_1}^{\text{new}}(N_1) \), \( g \in S_{k_2}^{\text{new}}(N_2) \) be normalized Hecke eigenforms with \( f \neq g \) and let \( N := \text{lcm}[N_1, N_2] \) and \( M := (N_1, N_2) \). Then for any \( t \in \mathbb{R} \) and \( \epsilon > 0 \), one has

\[
\zeta_N(2 + 2\epsilon + 2it)R(f, g; 1 + \epsilon + it) \ll \epsilon \quad \text{for all} \quad \epsilon > 0,
\]

and

\[
\zeta_N(-2\epsilon + 2it)R(f, g; -\epsilon + it) \ll \epsilon \quad \text{for all} \quad \epsilon > 0,
\]

where \( \zeta_N(s) \) is defined in (5).

Proof. Since \( \zeta_N(2 + 2\epsilon + 2it) \) and \( R(f, g; 1 + \epsilon + it) \) are absolutely convergent for \( \Re(s) > 1 \), we have the first inequality. To derive the second inequality, we use functional equation. From the functional equation (7), we have

\[
(8) \quad \zeta_N(2 - 2s) \cdot R(f, g; 1 - s) = (2\pi)^{2s} \cdot N^{2s - 1} \cdot \frac{\Gamma(s + \frac{k_2 - k_1}{2})}{\Gamma(1 - s + \frac{k_2 - k_1}{2})} \cdot \frac{\Gamma(s + \frac{k_1 + k_2}{2} - 1)}{\Gamma(-s + \frac{k_1 + k_2}{2})} \cdot \prod_{p|M} \frac{1 - c_p p^{s-1}}{1 - c_p p^{-s}} \cdot \zeta_N(2s) \cdot R(f, g; s).
\]

Using Stirling’s formula (see page 57 of [10]), we have

\[
\left| \frac{\Gamma(1 + \frac{k_2 - k_1}{2} + \epsilon + it)}{\Gamma(\frac{k_2 - k_1}{2} - \epsilon + it)} \right| \ll \epsilon \left( 1 + \frac{k_2 - k_1}{2} \right)^{1+2\epsilon} |1 + it|^{1+2\epsilon}
\]

and

\[
\left| \frac{\Gamma(k_1 + k_2 + \epsilon + it)}{\Gamma(k_1 + k_2 - 1 - \epsilon + it)} \right| \ll \epsilon \left( \frac{k_1 + k_2}{2} \right)^{1+2\epsilon} |1 + it|^{1+2\epsilon}.
\]

For all \( t \in \mathbb{R} \), we also have

\[
\left| \prod_{p|M} (1 - c_p p^{-1-\epsilon-it})^{-1} \right| = \left| \prod_{p|M} \sum_{m \geq 0} (c_p p^{-1-\epsilon-it})^m \right| \leq \prod_{p|M} \sum_{m \geq 0} (p^{-1-\epsilon})^m \ll 1
\]

and

\[
\left| \prod_{p|M} (1 - c_p p^{\epsilon+it}) \right| = \prod_{p|M} |1 - c_p p^{\epsilon+it}| \leq \prod_{p|M} (1 + p^\epsilon) \leq \prod_{p|M} p^{1+\epsilon} \ll M^{1+2\epsilon}.
\]

Putting \( s = 1 + \epsilon + it \) in (8) and using the above estimates along with the first inequality, we get the second inequality. \( \square \)

The next proposition provides convexity bound for Rankin-Selberg \( L \)-function \( R(f, g; s) \).
Proposition 8. For square-free integers \(N_1, N_2\), let \(f \in S^\text{new}_{k_1}(N_1)\), \(g \in S^\text{new}_{k_2}(N_2)\) be normalized Hecke eigenforms with \(f \neq g\) and \(N := \text{lcm}[N_1, N_2]\). Then for any \(t \in \mathbb{R}, \epsilon > 0\) and \(1/2 < \sigma < 1\), one has
\[
R(f, g; \sigma + it) \ll \epsilon \, N^{2(1-\sigma+\epsilon)} \left(1 + \frac{k_2 - k_1}{2}\right)^{1-\sigma+\epsilon} \left(\frac{k_1 + k_2}{2}\right)^{1-\sigma+\epsilon} (3 + |t|)^{2(1-\sigma+\epsilon)}.
\]

To prove this proposition, we shall use the following strong convexity principle due to Rademacher.

Proposition 9 (Rademacher [28]). Let \(g(s)\) be continuous on the closed strip \(a \leq \sigma \leq b\), holomorphic and of finite order on \(a < \sigma < b\). Further suppose that
\[
|g(a + it)| \leq E|P + a + it|^\alpha, \quad |g(b + it)| \leq F|P + b + it|^\beta,
\]
where \(E, F\) are positive constants and \(P, \alpha, \beta\) are real constants that satisfy
\[
P + a > 0, \quad \alpha \geq \beta.
\]
Then for all \(a < \sigma < b\) and for all \(t \in \mathbb{R}\), we have
\[
|g(\sigma + it)| \leq (E|P + \sigma + it|^\alpha) \frac{b-a}{\beta-a} (F|P + \sigma + it|^\beta) \frac{2}{\beta-a}.
\]

We are now ready to prove Proposition 8.

Proof. We apply Proposition 9 with
\[
a = -\epsilon, \quad b = P = 1 + \epsilon, \quad F = C_2,
\]
\[
E = C_1 \, N^{2+4\epsilon} \left(1 + \frac{k_2 - k_1}{2}\right)^{1+2\epsilon} \left(\frac{k_1 + k_2}{2}\right)^{1+2\epsilon}, \quad \alpha = 2 + 4\epsilon, \quad \beta = 0,
\]
where \(C_1, C_2\) are absolute constants depending only on \(\epsilon\). Thus for any \(-\epsilon < \sigma < 1 + \epsilon\), we have
\[
\zeta_N(2\sigma + 2it)R(f, g; \sigma + it) \ll \epsilon \left[N^{\frac{2+4\epsilon}{1+2\epsilon}} \left(1 + \frac{k_2 - k_1}{2}\right) \left(\frac{k_1 + k_2}{2}\right)^{1-\sigma+\epsilon} (1 + \sigma + \epsilon + |t|)^{2(1-\sigma+\epsilon)}\right].
\]

Note that for \(1/2 < \sigma < 1 + \epsilon\), one knows
\[
|\zeta_N(2\sigma + 2it)|^{-1} \ll \epsilon \log \log(N + 2) \cdot |1 + it|^{\epsilon}.
\]
Combining all together, we get Proposition 8. \(\square\)

As an immediate corollary, we have

Corollary 10. For square-free integers \(N_1, N_2\), let \(f \in S^\text{new}_{k_1}(N_1)\), \(g \in S^\text{new}_{k_2}(N_2)\) be normalized Hecke eigenforms with \(f \neq g\) and \(N := \text{lcm}[N_1, N_2]\). Then for any \(t \in \mathbb{R}\) and any \(\epsilon > 0\), one has
\[
R(f, g; 3/4 + it) \ll \epsilon \left[N^2 \left(1 + \frac{k_2 - k_1}{2}\right) \left(\frac{k_1 + k_2}{2}\right)^{1/4+\epsilon} (3 + |t|)^{1/2+\epsilon}\right].
\]
Proposition 11. For square-free integers $N_1, N_2$, let $f \in S_{k_1}^{new}(N_1)$, $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms with $f \neq g$ and $N := \text{lcm}[N_1, N_2]$. Then for any $\epsilon > 0$, one has

$$\sum_{\substack{n \leq x, \\ (n, N) = 1, n \text{ square-free}}} \lambda_f(n)\lambda_g(n) \log^2(x/n) \ll \epsilon \left[ N^2 \left( 1 + \frac{k_2 - k_1}{2} \right) \left( \frac{k_1 + k_2}{2} \right) \right]^{1/4+\epsilon} x^{3/4}. $$

Proof. For any $\epsilon > 0$, we know by Deligne’s bound that

$$\lambda_f(n)\lambda_g(n) \ll \epsilon n^\epsilon. $$

Hence by Perron’s summation formula (see page 56 and page 67 of [24]), we have

$$\sum_{\substack{n \leq x, \\ (n, N) = 1, n \text{ square-free}}} \lambda_f(n)\lambda_g(n) \log^2(x/n) = \frac{1}{\pi i} \int_{1+\epsilon+i\infty}^{1+\epsilon-i\infty} R^b(f, g; s) \frac{x^s}{s^3} ds$$

where

$$R^b(f, g; s) = \prod_{p \mid N} \left( 1 + \frac{\lambda_f(p)\lambda_g(p)}{p^s} \right), \quad \Re(s) > 1. $$

Further

$$R(f, g; s) = R^b(f, g; s)H(s), $$

where $H(s)$ has an Euler product which converges normally for $\Re(s) > 1/2$. Now we shift the line of integration to $\Re(s) = 3/4$. Observing that there are no singularities in the vertical strip bounded by the lines with $\Re(s) = 1 + \epsilon$ and $\Re(s) = 3/4$ and using Proposition 8 along with (11), we have

$$\sum_{\substack{n \leq x, \\ (n, N) = 1, n \text{ square-free}}} \lambda_f(n)\lambda_g(n) \log^2(x/n) = \frac{1}{\pi i} \int_{3/4-\infty}^{3/4+i\infty} R^b(f, g; s) \frac{x^s}{s^3} ds.$$ 

The above observations combined with Corollary 10 then implies that

$$\sum_{\substack{n \leq x, \\ (n, N) = 1, n \text{ square-free}}} \lambda_f(n)\lambda_g(n) \log^2(x/n) \ll \epsilon N^{1/2+\epsilon} \left( 1 + \frac{k_2 - k_1}{2} \right)^{1/4+\epsilon} \left( \frac{k_1 + k_2}{2} \right)^{1/4+\epsilon} x^{3/4}. $$

This completes the proof of the proposition. \(\square\)

Our next lemma will play a key role in proving Theorem 1.

Lemma 12. For square-free integers $N_1, N_2$, let $f \in S_{k_1}^{new}(N_1)$, $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms with $f \neq g$ and $N := \text{lcm}[N_1, N_2]$. Also assume that for any $\alpha \leq 2$, $\lambda_f(p^\alpha)\lambda_g(p^\alpha) \geq 0$ for
all $p^k \leq x$. Then for $x \geq \exp(c \log^2(\sqrt{q(f)} + \sqrt{q(g)}))$, we have

$$\sum_{n \leq x, \atop (n,N)=1, n \text{ square-free}} \lambda_f(n)\lambda_g(n) \gg \frac{x}{\log^2 x}.$$

Here $q(f), q(g)$ are analytic conductors of Rankin-Selberg $L$-functions of $f$ and $g$ respectively with

(12) $q(f) \leq k_1^2 N_1^2 \log \log N_1$ and $q(g) \leq k_2^2 N_2^2 \log \log N_2$

and $c > 0$ is an absolute constant.

**Proof.** Using Hecke relation (3), for any prime $(p,N) = 1$, we know that

$$\lambda_f(p^2)\lambda_g(p^2) = [\lambda_f(p)\lambda_g(p)]^2 - \lambda_f(p)^2 - \lambda_g(p)^2 + 1.$$

By hypothesis $\lambda_f(p^2)\lambda_g(p^2) \geq 0$ for all $p \leq \sqrt{x}$. Hence for any $p \leq \sqrt{x}$ and $(p,N) = 1$, we have

$$\lambda_f(p)^2\lambda_g(p)^2 \geq \lambda_f(p)^2 + \lambda_g(p)^2 - 1.$$

This implies that

$$\sum_{p \leq \sqrt{x}, \atop (p,N)=1} \lambda_f(p)^2\lambda_g(p)^2 \geq \sum_{p \leq \sqrt{x}, \atop (p,N)=1} \lambda_f(p)^2 + \sum_{p \leq \sqrt{x}, \atop (p,N)=1} \lambda_g(p)^2 - \sum_{p \leq \sqrt{x}, \atop (p,N)=1} 1.$$

Using standard analytic techniques and prime number theorem for Rankin-Selberg $L$-functions of $f$ and $g$ respectively (see [11], pages 94-95, 110-111 for further details), we see that

$$\sum_{p \leq \sqrt{x}, \atop (p,N)=1} \lambda_f(p)^2\lambda_g(p)^2 \geq c_1 \frac{\sqrt{x}}{\log x}$$

provided $x \geq \exp(c \log^2(\sqrt{q(f)} + \sqrt{q(g)}))$, where $c, c_1 > 0$ are absolute constants and $q(f), q(g)$ are as in equation (12). Using the hypothesis

$$\lambda_f(p)\lambda_g(p) \geq 0 \quad \text{and} \quad \lambda_f(p^2)\lambda_g(p^2) \geq 0$$

for all $p, p^2 \leq x$ and assuming that $x \geq \exp(c \log^2(\sqrt{q(f)} + \sqrt{q(g)}))$, we have

$$\sum_{n \leq x, \atop (n,N)=1, n \text{ square-free}} \lambda_f(n)\lambda_g(n) \geq \frac{1}{2} \sum_{p,q \leq \sqrt{x}, \atop (pq,N)=1, \atop p \neq q} \lambda_f(pq)\lambda_g(pq)$$

$$= \frac{1}{2} \left( \sum_{p \leq \sqrt{x}, \atop (p,N)=1} \lambda_f(p)\lambda_g(p) \right)^2 - \frac{1}{2} \sum_{p \leq \sqrt{x}, \atop (p,N)=1} \lambda_f(p)^2\lambda_g(p)^2.$$
Now using Deligne’s bound, we get
\[
\sum_{\substack{n \leq x, \\ (n,N) = 1, \\ n \text{ square-free}}} \lambda_f(n)\lambda_g(n) \geq \frac{1}{2} \left( \sum_{p \leq \sqrt{x}, \\ (p,N) = 1} \lambda_f(p)\lambda_g(p) \frac{\lambda_f(p)\lambda_g(p)}{4} \right)^2 - 8 \sum_{p \leq \sqrt{x}, \\ (p,N) = 1} 1
\]
\[
= \frac{1}{32} \left( \sum_{p \leq \sqrt{x}, \\ (p,N) = 1} \lambda_f(p)^2\lambda_g(p)^2 \right)^2 + O \left( \frac{\sqrt{x}}{\log x} \right)
\]
\[
\gg \frac{x}{\log^2 x}.
\]
This completes the proof of the lemma. □

We are now in a position to complete the proof of Theorem 1.

Proof. Assume that \(\lambda_f(p^\alpha)\lambda_g(p^\alpha) \geq 0\) for all \(p^\alpha \leq x\) with \(\alpha \leq 2\). By Lemma 12, we see that
\[
\sum_{n \leq x/2, \\ (n,N) = 1, \\ n \text{ square-free}} \lambda_f(n)\lambda_g(n) \log^2(x/n) \gg \sum_{n \leq x/2, \\ (n,N) = 1, \\ n \text{ square-free}} \lambda_f(n)\lambda_g(n) \gg \frac{x}{\log^2 x}
\]
provided \(x \geq \exp(c \log^2(\sqrt{q(f)} + \sqrt{q(g)}))\), where \(c > 0, q(f), q(g)\) are as in Lemma 12. Now comparing (10) and (13), for any \(\epsilon > 0\), we have
\[
x \ll \epsilon^* \max \left\{ \exp(c \log^2(\sqrt{q(f)} + \sqrt{q(g)})), \left[ N^2 \left( 1 + \frac{k_2 - k_1}{2} \right) \left( \frac{k_1 + k_2}{2} \right) \right]^{1+\epsilon} \right\},
\]
where \(c, q(f), q(g)\) are as before. Here we have used Lemma 4 of Choie and Kohnen. This completes the proof of Theorem 1. □

4. PROOF OF THE THEOREM 2

We now state a Lemma which we shall use to prove Theorem 2.

Lemma 13. Let \(\{a_n\}_{n \in \mathbb{N}}\) and \(\{b_m\}_{m \in \mathbb{N}}\) be two sequences of real numbers such that
\[
\begin{align*}
(1) \quad a_n &= O(n^{\alpha_1}), \quad b_m = O(m^{\alpha_2}), \\
(2) \quad \sum_{n,m \leq x} a_nb_m &\ll x^\beta, \\
(3) \quad \sum_{n,m \leq x} a_n^2b_m^2 &= cx + O(x^\gamma),
\end{align*}
\]
where \(\alpha_1, \alpha_2, \beta, \gamma \geq 0\) and \(c > 0\) such that \(\max\{\alpha_1 + \alpha_2 + \beta, \gamma\} < 1\). Then for any \(r\) satisfying
\[
\max\{\alpha_1 + \alpha_2 + \beta, \gamma\} < r < 1,
\]
there exists a sign change among the elements of the sequence \(\{a_nb_m\}_{n,m \in \mathbb{N}}\) for \(n, m \in [x, x^r]\). Consequently, for sufficiently large \(x\), the number of sign changes among the elements of the sequence \(\{a_nb_m\}_{n,m \in \mathbb{N}}\) with \(n, m \leq x\) are \(\gg x^{1-r}\).
Proof. Suppose that for any \( r \) satisfying
\[
\max\{\alpha_1 + \alpha_2 + \beta, \gamma\} < r < 1,
\]
the elements of the sequence \( \{a_nb_m\}_{n,m \in \mathbb{N}} \) have same signs in \([x, x + x^r]\). This implies that
\[
x^r \ll \sum_{x \leq n, m \leq x + x^r} a_n^2 b_m^2 \ll x^{\alpha_1 + \alpha_2} \sum_{x \leq n, m \leq x + x^r} a_n b_m \ll x^{\alpha_1 + \alpha_2 + \beta},
\]
which is a contradiction. This completes the proof of the Lemma. \( \square \)

Lemma \([13]\) can be thought of as a generalization of a Lemma of Meher and Ram Murty (see \([23, \text{Theorem 1.1}]\)) when \( b_1 = 1 \) and \( b_m = 0 \) for all \( m > 1 \). We are now in a position to prove Theorem \([2]\).

Proof. In order to apply Lemma \([13]\) we need to verify the following conditions for the elements of the sequence \( \{\lambda_f(n) \lambda_g(n^2)\}_{n \in \mathbb{N}} \). Note that

1. Ramanujan-Deligne bound implies that
\[
\lambda_f(n) \lambda_g(n^2) = O(n^\epsilon)
\]
for all \( n \in \mathbb{N} \).
2. By a recent work of Lü \([21, \text{Theorem 1.2(2)}]\) (see also Kumari and Ram Murty \([18]\)), one has
\[
\sum_{n \leq x} \lambda_f(n) \lambda_g(n^2) \ll x^{5/7} (\log x)^{-\theta/2},
\]
where \( \theta = 1 - \frac{8}{3\pi} = 0.1512 \ldots \)
3. In the same paper, Lü (see \([21, \text{Lemma 2.3(ii)}]\)) as well as Kumari and Ram Murty \([18]\) also proved that
\[
\sum_{n \leq x} \lambda_f(n)^2 \lambda_g(n^2)^2 = cx + O(x^{17/15 + \epsilon}),
\]
where \( c > 0 \).

Theorem \([2]\) now follows from Lemma \([13]\) by choosing \( a_n = \lambda_f(n) \) and \( b_m := \lambda_g(m^2) \) for all \( m, n \in \mathbb{N} \) and considering the sequence \( \{a_nb_n\}_{n \in \mathbb{N}} \). \( \square \)

5. PROOF OF THE THEOREM \([3]\)

Using equation \([4]\), one can write
\[
\lambda_f(p) = 2 \cos \alpha_p \quad \text{and} \quad \lambda_g(p) = 2 \cos \beta_p
\]
with \(0 \leq \alpha_p, \beta_p \leq \pi\). Using the Hecke relation (3) for any prime \((p, N_1N_2) = 1\), one has

\[
\lambda_f(p^m) = \begin{cases} 
(14) \quad (-1)^m(m + 1) & \text{if } \alpha_p = \pi; \\
m + 1 & \text{if } \alpha_p = 0; \\
\sin((m+1)\alpha_p) \sin \alpha_p & \text{if } 0 < \alpha_p < \pi.
\end{cases}
\]

and

\[
\lambda_g(p^m) = \begin{cases} 
(15) \quad (-1)^m(m + 1) & \text{if } \beta_p = \pi; \\
m + 1 & \text{if } \beta_p = 0; \\
\sin((m+1)\beta_p) \sin \beta_p & \text{if } 0 < \beta_p < \pi.
\end{cases}
\]

Theorem 5 now follows from the following four cases.

Case (1): When \(\alpha_p = 0\) or \(\pi\) and \(\beta_p = 0\) or \(\pi\), then by the equation (14) and equation (15), we see that

\[
\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) \neq 0\} = \mathbb{N}.
\]

In this case all elements of the sequence \(\{a_f(p^m)a_g(p^m)\}_{m \in \mathbb{N}}\) are non-zero.

Case (2): Suppose that at least one of \(\alpha_p, \beta_p\), say \(\alpha_p = 0\) or \(\pi\) and \(\beta_p \in (0, \pi)\). If \(\beta_p/\pi \notin \mathbb{Q}\), there is nothing to prove. Now if \(\beta_p/\pi = r/s\) with \((r, s) = 1\), then we have

\[
\#\{m \leq x \mid a_f(p^m)a_g(p^m) \neq 0\} = \#\{m \leq x \mid a_g(p^m) \neq 0\} = \lfloor x \rfloor - \left\lfloor \frac{x}{s} \right\rfloor.
\]

Hence the set \(\{m \mid a_f(p^m)a_g(p^m) \neq 0\}\) has positive density.

Case (3): Suppose that \(\alpha_p = \beta_p \in (0, \pi)\), i.e. \(\alpha_p/\pi = \beta_p/\pi \in (0, 1)\). If \(\alpha_p/\pi \notin \mathbb{Q}\), then \(a_f(p^m)a_g(p^m) \neq 0\) for all \(m \in \mathbb{N}\) as \(\sin m\alpha_p \neq 0\) for all \(m \in \mathbb{N}\). If \(\alpha_p/\pi \in \mathbb{Q}\), say \(\alpha_p/\pi = t/s\), where \(r, s \in \mathbb{N}\) with \((r, s) = 1\), then we have \(\sin m\alpha_p = 0\) if and only if \(m\) is an integer multiple of \(s\) and hence

\[
\#\{m \leq x : a_f(p^m)a_g(p^m) \neq 0\} = \lfloor x \rfloor - \left\lfloor \frac{x}{s} \right\rfloor.
\]

Hence the set in (2) has positive density.

Case (4): Assume that \(\alpha_p, \beta_p \in (0, \pi)\) with \(\alpha_p \neq \beta_p\). If both \(\alpha_p/\pi, \beta_p/\pi \notin \mathbb{Q}\), then there is nothing to prove. Next suppose that one of them, say \(\alpha_p/\pi \in \mathbb{Q}\) with \(\alpha_p/\pi = t/s\) with \((r, s) = 1\) and \(\beta_p/\pi \notin \mathbb{Q}\). Then we have

\[
\#\{m \leq x : a_f(p^m)a_g(p^m) \neq 0\} = \#\{m \leq x : a_f(p^m) \neq 0\} = \lfloor x \rfloor - \left\lfloor \frac{x}{s} \right\rfloor.
\]

Hence the set in (2) has positive density.

Now let both \(\alpha_p/\pi, \beta_p/\pi \in \mathbb{Q}\). If \(\alpha_p/\pi = r_1/s_1\) and \(\beta_p/\pi = r_2/s_2\) with \((r_i, s_i) = 1\), for \(1 \leq i \leq 2\), then

\[
\#\{m \leq x : a_f(p^m)a_g(p^m) \neq 0\} = \#\{m \leq x : a_f(p^m) \neq 0\} \cap \{m \leq x : a_g(p^m) \neq 0\}.
\]
Note that both $s_1$ and $s_2$ can not be 2 as otherwise $\alpha_p = \beta_p$. Since
\[
\#\{m \leq x : a_f(p^m)a_g(p^m) = 0\} = \#\{m \leq x : a_f(p^m) = 0\} \cup \{m \leq x : a_g(p^m) = 0\} \\
\leq \left\lfloor \frac{x}{s_1} \right\rfloor + \left\lceil \frac{x}{s_2} \right\rceil,
\]
the set in (2) has positive density. This completes the proof of Theorem 3.

6. PROOF OF THEOREM 4

Using Lemma 6, we see that for any $x \geq 2$ and $0 < \delta < 1/2$
\[
\#\{p \leq x : a_f(p^m) = 0 \text{ for some } m \geq 1\} \ll_{f,\delta} \frac{x}{(\log x)^{1+\delta}},
\]
where the implied constant depends only on $f$ and $\delta$. We have the same estimate for the form $g$ as well. Therefore for any $x \geq 2$ and $0 < \delta < 1/2$, we have
\[
\#\{p \leq x : a_f(p^m)a_g(p^{m'}) = 0 \text{ for some } m, m' \geq 1\} \ll_{f,g,\delta} \frac{x}{(\log x)^{1+\delta}},
\]
where the implied constant depends on $f, g$ and $\delta$. Hence
\[
\#\{p \leq x : a_f(p^m)a_g(p^{m'}) \neq 0 \text{ for all } m, m' \geq 1\} \\
= \pi(x) - \#\{p \leq x : a_f(p^m)a_g(p^{m'}) = 0 \text{ for some } m, m' \geq 1\},
\]
where $\pi(x)$ denotes the number of primes up to $x$. Now using prime number theorem as well as the identity (16), we have
\[
\#\{p \leq x : a_f(p^m)a_g(p^{m'}) \neq 0 \text{ for all } m, m' \geq 1\} \sim \frac{x}{\log x}.
\]
Hence the set
\[
\{p \in \mathcal{P} : a_f(p^m)a_g(p^{m'}) \neq 0 \text{ for any integers } m, m' \geq 1\}
\]
has natural density 1.

7. PROOF OF THEOREM 5

We keep the notations in this section as in section 5. To prove Theorem 5, we start by proving the following Proposition.

**Proposition 14.** Let
\[
f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_{k_1}(N_1) \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} a_g(n)q^n \in S_{k_2}(N_2)
\]
be two distinct normalized Hecke eigenforms. Then for any prime $p$ with $(p, N_1N_2) = 1$, there exists an integer $m$ with $1 \leq m \leq 4$ such that $a_f(p^m)a_g(p^m) \neq 0$. 
Proof. Note that $a_f(p^m)a_g(p^m) \neq 0$ is equivalent to $\sin(m+1)\alpha_p \sin(m+1)\beta_p \neq 0$. If $a_f(p)a_g(p) \neq 0$, then we are done. Now suppose $a_f(p)a_g(p) = 0$, then either $a_f(p) = 0$ or $a_g(p) = 0$.

**Case (1):** If $a_f(p) = 0 = a_g(p)$, then $\alpha_p = \beta_p = \pi/2$. Hence we have

$$a_f(p^2)a_g(p^2) = p^{k_1+k_2-2} \neq 0.$$  

**Case (2):** Suppose that at least one of $a_f(p), a_g(p) \neq 0$. Without loss of generality assume that $a_f(p) = 0$ and $a_g(p) \neq 0$, then $\alpha_p = \pi/2$ and $\beta_p \neq \pi/2$. Now if $\beta_p = 0$ or $\pi$, then $a_g(p^2) = 3p^{k_2-1}$. Hence we have

$$a_f(p^2)a_g(p^2) = -3p^{k_1+k_2-2} \neq 0.$$  

If $\beta_p \notin \{0,\pi/2,\pi\}$, then this implies that $a_g(p^2) = p^{(k_2-1)}\frac{\sin 3\beta_p}{\sin \beta_p}$. Now if $a_f(p^2)a_g(p^2) = 0$, then $\beta_p \in \{\pi/3,2\pi/3\}$ as $0 < \beta_p < \pi$. Then we have

$$\frac{a_f(p^4)a_g(p^4)}{p^{2(k_1+k_2-2)}} = \frac{2}{\sqrt{3}} \sin \frac{5\pi}{2} \sin \frac{5\pi}{3} \quad \text{or} \quad \frac{a_f(p^4)a_g(p^4)}{p^{2(k_1+k_2-2)}} = \frac{2}{\sqrt{3}} \sin \frac{5\pi}{2} \sin \frac{10\pi}{3}.$$  

Since neither $\sin(5\pi/2)\sin(5\pi/3)$ nor $\sin(5\pi/2)\sin(10\pi/3)$ is equal to zero, this completes the proof of Proposition 14. \hfill \Box

*Proof.* We now complete the proof of the first part of Theorem 5 by showing the existence of a prime $p \leq 2 \log N$ with $(p, N) = 1$ and then using Proposition 14. We know by a theorem of Rosser and Schoenfeld (see [30, p. 70]) that

$$\sum_{p \leq x} \log p > 0.73x \quad \text{for all} \quad x \geq 41.$$  

Using this, one can easily check that

$$\sum_{p \leq x} \log p > \frac{x}{2} \quad \text{for all} \quad x \geq 5.$$  

Now consider the following product

$$\prod_{p \leq 2 \log N} p = \exp \left( \sum_{p \leq 2 \log N} \log p \right) > N,$$  

which confirms the existence of such a prime. Proof of the second part of Theorem 5 follows immediately by applying Proposition 14. \hfill \Box
8. **$\mathcal{B}$-FREE NUMBERS AND SIMULTANEOUS NON-VANISHING IN SHORT INTERVALS**

In this section, we first list certain properties of $\mathcal{B}$-free numbers and their distribution in short intervals to derive simultaneous non-vanishing of Hecke eigenvalues. Erdős [8] introduced the notion of $\mathcal{B}$-free numbers and showed the existence of these numbers in short intervals.

**Definition 1.** Let us assume that 

$$\mathcal{B} := \{b_1, b_2, \ldots\} \subset \mathbb{N}$$

be such that 

$$(b_i, b_j) = 1 \text{ for } i \neq j \text{ and } \sum_{i \geq 1} \frac{1}{b_i} < \infty.$$ 

One says that a number $n \in \mathbb{N}$ is $\mathcal{B}$-free if it is not divisible by any element of the set $\mathcal{B}$.

The distribution of $\mathcal{B}$-free numbers in short intervals has been studied by several mathematicians (see [3], [31], [34], [35], [37]). Balog and Ono [4] were first to use $\mathcal{B}$-free numbers to study non-vanishing of Hecke eigenvalues.

For a non-CM cusp form $f \in S_k(N)$ with Fourier coefficients $\{a_f(n)\}_{n \in \mathbb{N}}$, Serre (see [33, page 383]) defined the function 

$$i_f(n) := \max \{m \in \mathbb{N} | a_f(n + j) = 0 \text{ for all } 0 < j \leq m\}$$

which is now known as gap function. Alkan and Zaharescu [1] proved that 

$$i_{\Delta}(n) \ll_{\Delta} n^{1/4+\epsilon}$$

for Ramanujan $\Delta$-function. Kowalski, Robert and Wu [17], using distribution of $\mathcal{B}$-free numbers in short intervals showed that 

$$i_f(n) \ll_f n^{7/17+\epsilon}$$

where $f \in S_k^{new}(N)$ is any normalized Hecke eigenform. Recently, Das and Ganguly [6] showed that 

$$i_f(n) \ll_f n^{1/4+\epsilon}$$

for any $f \in S_k(1)$.

In this article, we will study simultaneous non-vanishing of Hecke eigenvalues using $\mathcal{B}$-free numbers. This question was first considered by Kumari and Ram Murty [18]. We now introduce the set of $\mathcal{B}$-free numbers as constructed by Kowalski, Robert and Wu [17]. These numbers will play an important role in our work.

Let $\mathfrak{P}$ be a subset of $\mathcal{P}$ such that 

$$(17) \quad \#(\mathfrak{P} \cap [1, x]) \ll \frac{x^\rho}{(\log x)^{\eta_x}}$$
where \( \rho \in [0, 1] \) and \( \eta_{\rho} \)'s are real numbers with \( \eta_1 > 1 \). Let us define
\[
(18) \quad \mathcal{B}_\mathcal{P} := \mathcal{P} \cup \{ p^2 \mid p \in \mathcal{P} - \mathcal{P} \}.
\]
Write \( \mathcal{B}_\mathcal{P} = \{ b_i \mid i \in \mathbb{N} \} \). Note that \( (b_i, b_j) = 1 \) for all \( b_i, b_j \in \mathcal{B}_\mathcal{P} \) with \( b_i \neq b_j \). To show \( \sum_{i \in \mathbb{N}} \frac{1}{b_i} < \infty \), it is enough to show that \( \sum_{p \in \mathcal{P}} \frac{1}{p} < \infty \). Applying equation (17) and partial summation formula, one has
\[
\sum_{p \leq x} \frac{1}{p} = \frac{1}{x} \sum_{p \leq x} 1 + \int_2^x \frac{1}{t^2} \left( \sum_{p \leq t} 1 \right) dt \ll \mathcal{P} \frac{x^{\rho - 1}}{(\log x)^{\eta_{\rho}}} + \int_2^x t^{\rho - 2} \left( \log t \right)^{\eta_{\rho}} dt \ll \mathcal{P} 1.
\]

With these notations, Kowalski, Robert and Wu (see Corollary 10 of [17]) proved the following Theorem.

**Theorem 15 (Kowalski, Robert and Wu).** For any \( \epsilon > 0 \), \( x \geq x_0(\mathcal{P}, \epsilon) \) and \( y \geq x^{\theta(\rho) + \epsilon} \), we have
\[
\# \{ x < n \leq x + y \mid n \text{ is } \mathcal{B}_\mathcal{P}-\text{free} \} \gg \mathcal{P} \epsilon y,
\]
where
\[
(19) \quad \theta(\rho) := \begin{cases} 
\frac{1}{4} & \text{if } 0 \leq \rho \leq \frac{1}{3}; \\
\frac{10\rho}{19\rho+7} & \text{if } \frac{1}{3} < \rho \leq \frac{9}{17}; \\
\frac{3\rho}{4\rho+3} & \text{if } \frac{9}{17} < \rho \leq \frac{15}{28}; \\
\frac{5}{16} & \text{if } \frac{15}{28} < \rho \leq \frac{5}{8}; \\
\frac{22\rho}{24\rho+29} & \text{if } \frac{5}{8} < \rho \leq \frac{9}{10}; \\
\frac{7\rho}{9\rho+8} & \text{if } \frac{9}{10} < \rho \leq 1. 
\end{cases}
\]

We now study simultaneous non-vanishing in short arithmetic progression using the distribution of \( \mathcal{B} \)-free numbers. The question of the distribution of \( \mathcal{B} \)-free numbers in short arithmetic progression was first considered by Alkan and Zaharescu [2]. In this direction, Wu and Zhai (see Proposition 4.1 of [36]) have the following result about distribution of \( \mathcal{B} \)-free numbers in short arithmetic progression.

**Theorem 16 (Wu and Zhai).** Let \( \mathcal{B}_\mathcal{P} \) be as in (18). For any \( \epsilon > 0 \), \( x \geq x_0(\mathcal{P}, \epsilon) \), \( y \geq x^{\psi(\rho) + \epsilon} \) and \( 1 \leq a \leq q \leq x^{\epsilon} \) with \( (a, q) = 1 \), one has
\[
\# \{ x < n \leq x + y \mid n \text{ is } \mathcal{B}_\mathcal{P}-\text{free and } n \equiv a(\text{mod } q) \} \gg \mathcal{P} \epsilon y/q,
\]
where
\[
(20) \quad \psi(\rho) := \begin{cases} 
\frac{29\rho}{46\rho+19} & \text{if } \frac{190}{323} < \rho \leq \frac{166}{173}; \\
\frac{17\rho}{26\rho+12} & \text{if } \frac{166}{173} < \rho \leq 1.
\end{cases}
\]

Using above results, we now have the following non-vanishing Theorem for certain multiplicative function.
Theorem 17. Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function and let $N \geq 1$ be a positive integer. Define
\begin{equation}
\Psi_{f,N} := \{ p \in \mathcal{P} \mid f(p) = 0 \} \cup \{ p \in \mathcal{P} \mid p | N \}.
\end{equation}
Also assume that $\Psi_{f,N}$ satisfies condition (17). Then

1. For any $\epsilon > 0$, $x \geq x_0(\Psi_{f,N}, \epsilon)$ and $y \geq x^{\theta(\rho)+\epsilon}$, we have
$$\# \{ x < n \leq x + y \mid (n,N) = 1, n \text{ square-free and } f(n) \neq 0 \} \gg \Psi_{f,N}, \epsilon, y,$$
where $\theta(\rho)$ is as in (19).

2. For any $\epsilon > 0$, $x \geq x_0(\Psi_{f,N}, \epsilon)$, $y \geq x^{\psi(\rho)+\epsilon}$ and $1 \leq a \leq q \leq x^\epsilon$ with $(a,q) = 1$, we have
$$\# \{ x < n \leq x + y : (n,N) = 1, n \text{ square-free}, n \equiv a(mod \ q) \text{ and } f(n) \neq 0 \} \gg \Psi_{f,N}, \epsilon, y/q,$$
where $\psi(\rho)$ is as in (20).

Proof. Define
$$\mathcal{B}_{\Psi_{f,N}} := \mathcal{P} \cup \{ p^2 \mid p \in \mathcal{P} - \Psi_{f,N} \}.$$

Then first part of Theorem 17 now follows from Theorem 15. Applying Theorem 16, we get the second part of Theorem 17. □

As an immediate corollary, we have

Corollary 18. Let $E_1/\mathbb{Q}$ and $E_2/\mathbb{Q}$ be two non-CM elliptic curves which have the same conductor $N$. Let
$$L(E_i,s) = \sum_{n=1}^{\infty} a_{E_i}(n)n^{-s}, \quad i = 1,2$$
be their Hasse-Weil $L$-functions. If $f_{E_i}(z) = \sum_{n=1}^{\infty} a_{E_i}(n)q^n$ for $i = 1,2$ are the associated weight two newforms, then

1. For any $\epsilon > 0$ and $y \geq x^{33/94+\epsilon}$, we have
$$\# \{ x < n \leq x + y \mid n \text{ square-free and } a_{E_1}(n)a_{E_2}(n) \neq 0 \} \gg_{E_1,E_2,\epsilon} y.$$

2. For any $\epsilon > 0$, $x \geq x_0(E_1, E_2, \epsilon)$, $y \geq x^{87/214+\epsilon}$ and $1 \leq a \leq q \leq x^\epsilon$ with $(a,q) = 1$, we have
$$\# \{ x < n \leq x + y \mid (n,N) = 1, n \text{ square-free and } n \equiv a(mod \ q) \text{ and } a_{E_1}(n)a_{E_2}(n) \neq 0 \} \gg_{E_1,E_2,\epsilon} y/q.$$

Proof. Let $\pi_E(x)$ be the number of supersingular primes up to $x$ for a non-CM elliptic curve $E/\mathbb{Q}$. By the work of Elkies [7], we have
$$\# \{ p \leq x : a_E(p) = 0 \} \ll_E x^{3/4}.$$

Considering $f(n) := a_{E_1}(n)a_{E_2}(n)$, one easily sees that $\Psi_{f,N}$ satisfies condition (17) with $\rho = 3/4$ and $\eta_\rho = 0$. Now by using Theorem 17, we get the Corollary. □
Kumari and Ram Murty have proved similar results for non-CM cusp forms which are newforms and normalized Hecke eigenforms of weight \( k > 2 \).

As a second corollary, we have the following simultaneous non-vanishing result for coefficients of symmetric power \( L \)-functions.

To state the corollary, we need to introduce few more notations. Let \( f \in S^\text{new}_k(N) \) be a normalized Hecke eigenform with Fourier coefficients \( \{a_f(n)\}_{n \in \mathbb{N}} \). Set \( \lambda_f(n) = a_f(n)/n^{(k-1)/2} \) and suppose that for \( p \nmid N \), \( \alpha_{f,p}, \beta_{f,p} \) are the Satake \( p \)-parameter of \( f \). Then the un-ramified \( m \)-th symmetric power \( L \)-function of \( f \) is defined as follows:

\[
L_{\text{unr}}(\text{sym}^m f, s) := \prod_{p \mid N} \prod_{0 \leq j \leq m} (1 - \alpha_{f,p}^j \beta_{f,p}^{m-j} p^{-s})^{-1} = \sum_{n \geq 1} \lambda_f^{(m)}(n)n^{-s}.
\]

We now have the following corollary.

**Corollary 19.** Let \( f \in S^\text{new}_k(N_1) \) and \( g \in S^\text{new}_k(N_2) \) be normalized non-CM Hecke eigenforms. Let \( N := \text{lcm}[N_1, N_2] \). Then

1. For any \( \epsilon > 0 \), \( x \geq x_0(f, g, \epsilon) \) and \( y \geq x^{7/17+\epsilon} \), we have
   \[
   \#\{x < n \leq x + y : n \text{ is square-free and } \lambda_f^{(m)}(n)\lambda_g^{(m)}(n) \neq 0\} \gg_{f,g,\epsilon} y.
   \]
2. For any \( \epsilon > 0 \), \( x \geq x_0(f, g, \epsilon) \), \( y \geq x^{17/38+\epsilon} \) and \( 1 \leq a \leq q \leq x^{\epsilon} \) with \( (a, q) = 1 \), we have
   \[
   \#\{x < n \leq x + y : (n, N) = 1, n \text{ square-free, } n \equiv a(mod \, q) \text{ and } \lambda_f^{(m)}(n)\lambda_g^{(m)}(n) \neq 0\} \gg_{f,g,\epsilon} y/q.
   \]

**Proof.** Let

\[ \mathfrak{P}_{f,g,m} := \{p \in \mathcal{P} : p \mid N \text{ or } \lambda_f^{(m)}(p)\lambda_g^{(m)}(p) = 0\} \]

Since \( \lambda_f^{(m)}(p) = \lambda_f(p^m) \), using Lemma 6 we see that \( \mathfrak{P}_{f,g,m} \) satisfies condition [17]. Note that \( f(n) := \lambda_f^{(m)}(n)\lambda_g^{(m)}(n) \) is a multiplicative function and hence we can apply Theorem [17] to complete the proof of Corollary 19. 

**Remark 8.1.** Note that Corollary [19] implies simultaneous non-vanishing of Hecke eigenvalues in sparse sequences. More precisely, let \( f \in S^\text{new}_k(N_1) \) and \( g \in S^\text{new}_k(N_2) \) be normalized non-CM Hecke eigenforms. Also let \( N := \text{lcm}[N_1, N_2] \). Then

1. For any \( \epsilon > 0 \), \( x \geq x_0(f, g, \epsilon) \) and \( y \geq x^{7/17+\epsilon} \), we have
   \[
   \#\{x < n \leq x + y : (n, N) = 1, n \text{ square-free and } \lambda_f(n^m)\lambda_g(n^m) \neq 0\} \gg_{f,g,\epsilon} y.
   \]
2. For any \( \epsilon > 0 \), \( x \geq x_0(f, g, \epsilon) \), \( y \geq x^{17/38+\epsilon} \) and \( 1 \leq a \leq q \leq x^{\epsilon} \) with \( (a, q) = 1 \), we have
   \[
   \#\{x < n \leq x + y : (n, N) = 1, n \text{ square-free, } n \equiv a(mod \, q) \text{ and } \lambda_f(n^m)\lambda_g(n^m) \neq 0\} \gg_{f,g,\epsilon} y/q.
   \]

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