A Confront between Amati and Combo Correlations at Intermediate and Early Redshifts

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Abstract: I consider two gamma-ray burst (GRB) correlations: Amati and Combo. After calibrating them in a cosmology-independent way by employing Beziér polynomials to approximate the Observational Hubble Dataset (OHD), I perform Markov Chain Monte Carlo (MCMC) simulations within the $\Lambda$CDM and the $w$CDM models. The results from the Amati GRB dataset do not agree with the standard $\Lambda$CDM model at a confidence level $\geq 3\sigma$. For the Combo correlation, all MCMC simulations give best-fit parameters which are consistent within 1-$\sigma$ with the $\Lambda$CDM model. Pending the clarification of whether the diversity of these results is statistical, due to the difference in the dataset sizes, or astrophysical, implying the search for the most suited correlation for cosmological analyses, future investigations require larger datasets to increase the predictive power of both correlations and enable more refined analyses on the possible non-zero curvature of the Universe and the dark energy equation of state and evolution.

Keywords: gamma rays: bursts; cosmology: cosmological parameters

1. Introduction

As of today, type Ia supernovae (SNe Ia) are considered to be standard candles for high-precision distance determinations [1–4]. However, some flaws in the use of SNe Ia have been recently exposed. The first one consists of a possible luminosity evolution with the environments of SNe Ia, which plays a major role in the systematic uncertainties in their distance determination [5]. The second one is that SNe Ia are detectable at most at redshifts $z \simeq 2$ [6] and, thus, they cannot be used alone to clear the degeneracy between the standard $\Lambda$CDM cosmological model and alternative dark energy (DE) scenarios [7–10].

To overcome the above second shortcoming, distance indicators covering a wide range of $z$ have become essential for cosmological tests. In this respect, gamma-ray bursts (GRBs) have the advantage to be detectable up to $z = 9.4$ [11–14], with a peak in their redshift distribution at $z \sim 2$–2.5 [15]. Despite being affected by selection and instrumental effects [16–20], in the last two decades, various phenomenological correlations between GRB photometric and spectroscopic properties have been proposed to convert GRBs into cosmic rulers [19–30]. However, the major effect jeopardizing the standardization of GRBs is the circularity problem, which is a consequence of the fact that, due to the lack of very low-redshift GRBs, energy-spectrum correlations have to be calibrated by assuming an a priori background cosmology and fitting procedures inevitably return an overall agreement with it [31].

Recently, a model-independent calibration conceived to overcome the circularity problem [32] has been applied to the most investigated correlation involving prompt emission rest-frame peak energy $E_p$ of the GRB $\nu F_{\nu}$ spectrum and the bolometric isotropic radiated energy $E_{\text{iso}}$, the $E_p - E_{\text{iso}}$ correlation or Amati relation [19–21,23,33]. This calibration method utilizes the Observational Hubble Data (OHD)
obtained from the differential age method applied to pairs of nearby galaxies [34,35] and provides, through a model-independent fitting, measurements of the Hubble rate \( H(z) \) at arbitrary redshifts without assuming an a priori cosmological model [32,36].

In this paper I perform cosmological fits on the distance moduli obtained from the calibration of the \( E_p - E_{iso} \) correlation by means of a Markov Chain Monte Carlo (MCMC) technique by fixing the Hubble constant to the value obtained from the OHD calibration \( H_0 = (67.76 \pm 3.68) \text{ km s}^{-1} \text{ Mpc}^{-1} \) [32]. I compare the standard \( \Lambda \)CDM paradigm with its simplest DE extension, i.e., the \( \omega \)CDM model. In doing so, I here include the data from the most recent SNe Ia dataset, i.e., The Pantheon sample [37], instead of the JLA sample [38] used in a previous work [32]. I also extend this analysis focusing on another GRB correlation named Combo [29], which relates \( E_p \) with X-ray afterglow observables (such as the plateau luminosity \( L_0 \) in the rest-frame band 0.3–10 keV and its rest-frame duration \( \tau \), and The late power-law decay index \( \alpha \) ). Although for this correlation an alternative calibration has been proposed [29], I here apply the above one based on OHD [32] and perform cosmological fits on the distance moduli obtained from the Combo correlation by comparing, again, the \( \Lambda \)CDM and the \( \omega \)CDM models. Thence, the aim of this paper is to confront the results from the above cosmological fits from the Amati and Combo correlations.

In Section 2, I summarize the model-independent calibration method based on the use of OHD dataset [32]. In Section 3, I apply the above calibration technique to Amati and Combo correlations and obtain the corresponding GRB distance moduli \( \mu_h \) and \( \mu_c \), respectively. In Section 4 I perform cosmological fits to get constraints on the cosmological parameters. In particular, in Section 4.1, I show how to employ the calibrated GRB correlations and fit a) GRB data alone and b) GRB data together with Pantheon SN Ia dataset [37] to get constraints on the matter parameter \( \Omega_m \) within the \( \Lambda \)CDM model. In Section 4.2, to obtain robust bounds on the DE parameter \( w \) within the \( \omega \)CDM model, I perform a fit by using only GRB+SN dataset. In Section 4.3, I summarize the results of the MCMC simulations from the above cosmological models. In Section 5, I discuss the results, draw conclusions, and identify the perspectives of this work.

2. The OHD Model-Independent Calibration Method

The circularity problem affecting GRB correlations [17,19,22,31] enters in the definition of the generic energy/luminosity \( \mathcal{X} \) through the luminosity distance \( d_L \), i.e., \( \mathcal{X}(z,H_0,\Omega_i) \equiv 4\pi d_L^2(z,H_0,\Omega_i) \mathcal{Y}_{\text{bolo}} \), where \( \Omega_i \) are the cosmological parameters, and \( \mathcal{Y}_{\text{bolo}} \) is the rest-frame bolometric GRB fluence \( S_{\text{bolo}}(1+z)^{-1} \) for the \( E_{iso} \) definition, or The bolometric observed flux \( F_{\text{bolo}} \) for the isotropic luminosity definitions. The use of any luminosity distance definition coming from other cosmological probes may bias the GRB Hubble diagram by introducing the systematics of the selected probe itself [20,30,31,39].

Thus, the need of model-independent techniques is essential to overcome the above issues [35,40–49,49,50]. However, the model is plagued by the convergence problem [51]. Thus, I introduce a new technique described in terms of polynomials and based directly on catalogs of data. In particular, I consider the OHD datapoints. These are cosmology-independent estimates of the Hubble function \( H(z) = -(1+z)^{-1} \Delta z/\Delta t \) based on spectroscopic measurements of differential age \( \Delta t \) and redshift difference \( \Delta z \) [34]. The updated OHDs [35], shown in Figure 1, can be approximated by employing a Bézier parametric curve of degree \( n \)

\[
H_n(x) = \sum_{d=0}^{n} \beta_d h_d(x) , \quad h_d(x) = \frac{n! x^d (1-x)^{n-d}}{d!(n-d)!} , \tag{1}
\]

where \( \beta_d \) are coefficients of the linear combination of Bernstein basis polynomials \( h_d(x) \), positive in the range \( 0 \leq x \equiv z/z_m \leq 1 \), where \( z_{\text{max}} \) is the maximum \( z \) of the OHD. The only non-linear monotonic growing function up to \( z_m \) is \( H_2(z) \) obtained for \( n = 2 \) [32].
By using the above OHD interpolating function $H_2(z)$, the luminosity distance writes as

$$d_L(\Omega_k, z) = \frac{c}{H_0} \sqrt{|\Omega_k|} F_k \left( \int_0^z H_0 \sqrt{\Omega_k} \, dz' \right),$$

where $F_k(x) = \sinh(x)$ for $\Omega_k > 0$, $F_k(x) = x$ for $\Omega_k = 0$, and $F_k(x) = \sin(x)$ for $\Omega_k < 0$. Imposing a curvature parameter $\Omega_k = 0$, supported by Planck results [52], the luminosity distance becomes completely cosmology-independent

$$d_{\text{cal}}(z) = c(1 + z) \int_0^z \frac{dz'}{H_2(z')} ,$$

enabling the calibration of the energy/luminosity

$$X_{\text{cal}}(z) \equiv 4\pi d_{\text{cal}}^2(z) S_{\text{bolo}}(1 + z)^{-1} ,$$

where the errors $\sigma X_{\text{cal}}$ depend upon the GRB systematics and the statistical errors of the proposed correlation.

3. The Calibrated Amati and Combo Correlations

The $E_p - E_{\text{iso}}$ correlation is given by

$$\log \left( \frac{E_p}{\text{keV}} \right) = q + m \left[ \log \left( \frac{E_{\text{iso}}}{\text{erg}} \right) - 52 \right] .$$

By applying the procedure outlined in Section 2, from Equation (4) the calibrated isotropic energy writes as

$$E_{\text{iso}}(z) \equiv 4\pi d_{\text{cal}}^2(z) S_{\text{bolo}}(1 + z)^{-1} .$$

The fit of the calibrated correlation gives as best-fit parameters $q = 2.06 \pm 0.03$, $m = 0.50 \pm 0.02$, and extra-scatter $\sigma_A = 0.20 \pm 0.01$ dex [32] (see Figure 2, left panel).
The 193 GRB distance moduli from the calibrated Amati correlation can be computed from the standard definition $\mu_A = 25 + 5\log(d_{\text{cal}}/\text{Mpc})$. In the specific case:

$$\mu_A = 32.55 + \frac{5}{2}\left[ \frac{1}{m}\log\left(\frac{E_p}{\text{keV}}\right) - \frac{q}{m} - \log\left(\frac{4\pi S_{\text{iso}}}{\text{erg/cm}^2}\right) + \log(1+z) \right].$$  \hspace{1cm} (7)

The Hubble diagram of $\mu_A$ with $z$ and the corresponding attached errors, accounting for the GRB systematics and the statistical errors on $q$, $m$ and $\sigma_A$, are shown in Figure 2 (right panel).

![Figure 2](image1.png)

**Figure 2.** Left: the calibrated $E_p$-$E_{\text{iso}}$ correlation (black data), the best-fitting function (blue solid line) and the $1\sigma_A$ and $3\sigma_A$ limits (dark-gray and light-gray shaded regions, respectively). Right: the distribution of the GRB distance moduli $\mu_A$ as obtained from the calibrated Amati correlation. Reproduced from Ref. [32].

The Combo correlation writes as [29]

$$\log\left(\frac{L_0}{\text{erg/s}}\right) = a + b\log\left(\frac{E_p}{\text{keV}}\right) - \log\left(\frac{\tau/s}{1+a} \right),$$ \hspace{1cm} (8)

The calibration method in Equation (4) is here applied to the plateau luminosity $L_0$ as follows

$$L_0(z) \equiv 4\pi d_{\text{cal}}^2(z)F_0,$$ \hspace{1cm} (9)

where $F_0$ is the rest-frame 0.3–10 keV energy flux. The calibrated Combo correlation is shown in Figure 3 (left panel). Its best-fit parameters are $a = 50.04 \pm 0.27$, $b = 0.71 \pm 0.11$, and an extra-scatter $\sigma_C = 0.35 \pm 0.04$.

![Figure 3](image2.png)

**Figure 3.** Left: the calibrated Combo correlation (black data), the best-fit (red solid line) and the $1\sigma_C$ and $3\sigma_C$ limits (dark-orange and light-orange shaded regions, respectively). Right: the distribution of the GRB distance moduli $\mu_C$ as obtained from the calibrated Combo correlation.
The Hubble diagram of the 60 GRB distance moduli $\mu_C$ and their attached errors accounting for the systematics and the statistical errors on $a$, $b$ and $\sigma_C$, obtained from the calibrated Combo correlation, are given by [29],

$$\mu_C = -97.45 + \frac{5}{2} \left[ \log \left( \frac{a}{\text{erg/s}} \right) + b \log \left( \frac{E_p}{\text{keV}} \right) - \log \left( \frac{\tau/s}{1 + a} \right) - \log \left( \frac{4\pi F_0}{\text{erg/cm}^2/s} \right) \right]$$

(10)

and are shown in Figure 3 (right panel).

4. Results from Cosmological Fits

I here portray our theoretical scenarios that I am going to test with GRB data. I thus consider a generic version of the Hubble rate, taking pressureless matter with negligible contribution from radiation and spatial curvature, i.e., imposing $\Omega_k = 0$ [52].

I handle a DE field with the equation of state $P = w \rho$, with constant $w$. This approach involves two models of particular interest: the first, the concordance paradigm, is defined as $w \rightarrow -1$, whereas the second takes $w < 0$ with no further impositions. In particular, I write

$$H(z) = H_0 \sqrt{\Omega_m(1+z)^3 + \Omega_{DE}(1+z)^{3(1+w)}},$$

(11)

where the matter density is defined by $\Omega_m = \rho_m / \rho_c$ with $\rho_c \equiv 8\pi G / (3H_0^2)$ the critical density. The DE cosmological parameter becomes $\Omega_{DE} = 1 - \Omega_m$, being $w$ the DE parameter. In particular, Equation (11) results in the $\Lambda$CDM model for $w \equiv -1$, or else to the $w$CDM model.

To search for the best-fit parameters of the above cosmological models, I perform a MCMC numerical integration on the $\chi^2$ distribution by means of the Metropolis-Hastings algorithm implemented on a Mathematica code. This code starts by assuming priors on the fitting parameters and fixing $H_0 = 67.74$ km s$^{-1}$ Mpc$^{-1}$ as indicated in the Section 1; the search of the best-fit proceeds by using a random walk behaviour through $10^4$ steps searches for the set of model parameters minimizing the $\chi^2$.

4.1. The $\Lambda$CDM Model

To get the $\Lambda$CDM (for a different perspective, see [53]), I set $w = -1$ in Equation (11), leaving $\Omega_m$ the only parameter to be found by the MCMC simulation. At this stage I employ GRB data alone to obtain constraints on $\Omega_m$ from the minimization of the Amati/Combo relation chi square

$$\chi^2_{A/C} = \sum_{i=1}^{N_{A/C}} \left[ \frac{\mu_{A/C,i} - \mu_{GRB}^{th}(\Omega_m, z_i)}{\sigma_{A/C,i}} \right]^2,$$

(12)

where for the Amati correlation $N_A = 193$ and for the Combo correlation $N_C = 60$; $\mu_{GRB}^{th}$ is the theoretical GRB distance modulus computed from a given model. For the MCMC simulations, I assume the uniform prior $0 \leq \Omega_m \leq 1$. The fit results for the $\Lambda$CDM model are shown in Figure 4 and summarized in Table 1.
Table 1. Best-fit results and errors at 1-σ (3-σ) confidence level of the MCMC simulation for the ΛCDM model, with both GRB and GRB+SN samples, and the wCDM model, with GRB+SN sample. The $\chi^2$/DoF ratios are also indicated.

| Sample      | $w$   | $\Omega_m$ | $\chi^2$/DoF | $w$   | $\Omega_m$ | $\chi^2$/DoF |
|-------------|-------|------------|--------------|-------|------------|--------------|
| $\Lambda$CDM |       |            |              |       |            |              |
| GRB         | $-1$  | $0.43^{+0.03}_{-0.03}$ | $1126.9/192$ | $-1$  | $0.37^{+0.08}_{-0.08}$ | $49.1/59$    |
| GRB+SN      | $-1$  | $0.36^{+0.02}_{-0.01}$ | $2177.0/1240$| $-1$  | $0.30^{+0.02}_{-0.02}$ | $1084.5/1107$|
| $w$CDM      |       |            |              |       |            |              |
| GRB+SN      | $-1.15^{+0.16}_{-0.20}$ | $0.40^{+0.04}_{-0.04}$ | $2176.3/1239$| $-1.12^{+0.15}_{-0.26}$ | $0.34^{+0.06}_{-0.04}$ | $1084.2/1106$|

Figure 4. Plots of the $\chi^2_{A/C}$ distribution (red points) from the MCMC simulation for the $\Lambda$CDM model. Left: the $\chi^2_A$ of the Amati correlation; right: the $\chi^2_C$ of the Combo correlation. The blue points represent the starting point of the MCMC simulation.

Now, I perform a fit within the $\Lambda$CDM model, including the distance moduli from the SNe Ia Pantheon Sample, the largest combined sample consisting of 1048 sources Ia ranging from $0.01 < z < 2.3$ [37]. The SN Ia distance modulus is given by

$$\mu_{SN} = m_B - (M - aX_1 + \beta C - \Delta M - \Delta B) ,$$

and depends upon the $B$-band apparent $m_B$ and absolute $M$ magnitudes, the stretch $X_1$ and colour $C$ light curve factors, the luminosity-stretch $a$ and luminosity-color $\beta$ parameters, and, finally, the distance corrections $\Delta M$, based on the host galaxy mass of the SN, and $\Delta B$, based on predicted biases from simulations. $M$ does not enter the SN uncertainties, thus, the SN chi-square is

$$\chi^2_{SN} = \left(\Delta \tilde{\mu}_{SN} - \tilde{M} \tilde{1}\right)^T C^{-1} \left(\Delta \tilde{\mu}_{SN} - \tilde{M} \tilde{1}\right) ,$$

(14)
where $\Delta\mu_{SN} \equiv \mu_{SN} - \mu_{SN}^{th}(\Omega_m, z_i)$ is the vector of residuals, and the covariance matrix $C$ accounts for statistical and systematic uncertainties [54]. By analytically marginalizing over $M$ through a flat prior [55], the SN chi-square becomes independent from it, leading to

$$\chi_{SN,M}^2 = a + \log \frac{e}{2\pi} - \frac{b^2}{e},$$

(15)

in which $a \equiv \Delta\mu_{SN}^T C^{-1} \Delta\mu_{SN}$, $b \equiv \Delta\mu_{SN}^T C^{-1} 1$, and $e \equiv 1^T C^{-1} 1$. Analytical marginalizations over $\alpha$ and $\beta$ are not possible, because they enter in the SN uncertainties.

The total chi-square for the $\Lambda$CDM is thus given by

$$\chi^2(\Omega_m) = \chi_{A/C}^2(\Omega_m) + \chi_{SN,M}^2(\Omega_m),$$

(16)

and it is computed for each GRB correlation. For the MCMC simulations, I assume the same uniform priors as for the above fit involving GRB data alone. The fit results are shown in Figure 5 and summarized in Table 1.

4.2. The $w$CDM Model

Now, I perform fits using the $w$CDM model, obtained from Equation (11) by allowing the $w$ parameter free to vary. Hence, the model parameters to be found by the MCMC simulation are now $w$ and $\Omega_m$. In this case the use of GRB data alone does not provide acceptable constraints on $w$. Therefore, to obtain more robust bounds on this parameter, I make use of the combined GRB+SN sample.

The total chi-square for the $w$CDM model

$$\chi_{A/w}^2(\Omega_m, w) = \chi_{A/C}^2(\Omega_m, w) + \chi_{SN,M}^2(\Omega_m, w)$$

(17)

is computed for each GRB correlation and using $\mu_{GRB}^{th}(\Omega_m, w, z_i)$ and $\mu_{SN}^{th}(\Omega_m, w, z_i)$. For the MCMC simulations, I assume the uniform priors $0 \leq \Omega_m \leq 1$ and $-3 \leq w \leq 0$. The fit results for the $w$CDM model are shown in Figure 6 and summarized in Table 1.
4.3. Results

From the fits on the $\Lambda$CDM obtained by employing GRB datasets alone, summarized in Table 1 and portrayed in Figure 4, one immediately notices that $\Omega_m$ is higher when compared to the results from other surveys [19,20,29,30,32,56]. This discrepancy tends to be flattened for both $\Lambda$CDM and $w$CDM models, when including SN Ia dataset (see Table 1 and Figures 5 and 6).

Going into details for the Amati correlation, the results from the $\Lambda$CDM model are in tension with Planck predictions [52] at a confidence level $\geq 3\sigma$, but consistent within $1\sigma$ with previous results obtained from GRB data alone [19,20,29,30,32,56]. As stated above, this tension is slightly reduced by including SN Ia dataset. $\Lambda$CDM and $w$CDM best-fit models obtained by considering GRB+SN datasets have the same number of datapoints and just one degree of freedom (DoF) of difference (see Table 1). From their direct comparison, one can see that the $w$CDM model leads to a modest improvements in the $\chi^2$ by introducing an extra parameter. Moreover the $w$ parameter is consistent within $1\sigma$ with the $\Lambda$CDM case, i.e., $w = -1$. This represents a clear indication that, from a statistical point of view, the $\Lambda$CDM model is a better fit than the $w$CDM one to fit the SN+Amati GRB dataset.

For the Combo correlation, the results from both $\Lambda$CDM and $w$CDM models are consistent with Planck predictions [52] within $1\sigma$. Also in this case, from The direct comparison between the best-fit results, the $\Lambda$CDM model is a better fit than the $w$CDM one to fit the SN+Combo GRB dataset.

By sorting the correlations by the $\chi^2$/DoF values, one notices immediately that this ratio for Combo relation is smaller and closer to unity than the Amati one. At this stage it is not clear whether these findings are due to the fact that the Combo GRB dataset is smaller than the Amati one, leading to larger attached errors, or because the Combo correlation is indeed a more suited correlation for cosmological analyses. Future updates on both datasets may shed light on this issue and strengthen the statistical contents for both correlations.

It is worth to mention that the results listed in Table 1 and displayed in Figures 4–6 slightly differ to (though are consistent with) those obtained in Ref. [32] by employing Amati GRBs and SNe Ia. The reason for this difference has to be seeked in the different SN Ia sample used here (The Pantheon dataset) and that employed in Ref. [32] (The JLA dataset). In any case, as expected, the results are consistent at $1\sigma$ confidence level.

5. Conclusions and Discussions

I employ a new method, based on the approximation of the OHD data with Beziér polynomials [32], to calibrate in a cosmology-independent way two GRB correlations: Amati and Combo. Through this technique, GRB distance moduli have been obtained without postulating an a priori
cosmological model and just imputing the information on curvature $\Omega_k = 0$ from Planck [52]. In such a way the circularity problem affecting GRB correlations has been healed.

From the model-independently calibrated Amati and Combo correlations I performed MCMC analyses: in a first set of simulations, I tested the $\Lambda$CDM model by using (a) GRB data alone and (b) GRB data and SNe Ia from the Pantheon sample [37]; in a second set, I tested the $w$CDM model by considering only GRB+SN datasets, because GRBs alone do not provide acceptable constraints on $w$.

The results of the MCMC simulations are summarized in Table 1 and portrayed in Figures 4–6. Though consistent within 1–$\sigma$ with previous results obtained from GRB data alone [19,20,29,30,32,56], the results from the Amati correlation are in tension with Planck predictions [52] at a confidence level $\geq 3$–$\sigma$ for both $\Lambda$CDM and $w$CDM models. This findings are somewhat in line with recent claims on tensions with the $\Lambda$CDM model [57,58]. However, while the value of $\Omega_m$ is always noticeably high, within The $w$CDM model the value of the $w$ parameter is consistent within 1–$\sigma$ with the $\Lambda$CDM case, i.e., $w = −1$. The results obtained for the Amati correlation case slightly differ to (though are consistent at 1–$\sigma$ confidence level with) those obtained in Ref. [32]. This is likely due to different employed SN Ia samples, the Pantheon dataset used here and the JLA dataset employed in Ref. [32]. For the Combo correlation, all MCMC simulations give best-fit parameters which are consistent within 1–$\sigma$ with the $\Lambda$CDM model. From a statistical significance point of view, the Amati correlation has the largest dataset; however, the Combo correlation provides the smaller values of the ratio $\chi^2$/DoF. At this stage it is not clear whether the diversity of the results from the two calibrated correlations is statistical, due to the difference in the dataset size of the correlations, or astrophysical, which may help in principle in establishing the most suited correlation for cosmological analyses. However, this can be an indication that more refined analyses are needed. The increase of both datasets may shed light on the above issues and strengthen the predictive power of both correlations.

Future perspectives of this work may shed light also on the role of spatial curvature. Recent literature raises doubts about fixing the spatial curvature to zero and claims that $\Omega_k \neq 0$, though it is very small [59], or that the current constraint $\Omega_k = 0.001 \pm 0.002$ [52] is based on the pre-assumption of a flat surface in a DE analysis [60]. However, by relaxing the assumption $\Omega_k = 0$, the circularity problem is not fully healed since the luminosity distance in Equation (2) depends upon $\Omega_k$. This implies that quantities such as $E_{\text{iso}}$ and $L_0$ are now functions of $\Omega_k$. Therefore, in order to measure the curvature parameter, one may still employ the model-independent method based on Bézier polynomials to approximate the OHD data with the function $H_2(z)$, use it in the luminosity distance definition, and jointly fit $\Omega_k$ with the GRB correlation best-fit parameters [60].

Finally, the results obtained in this work are in line with $w = −1$ and a cosmological constant $\Lambda$ describing the DE evolution, as purported by the $\Lambda$CDM model. At this stage, slow and/or small DE evolution with time, cannot be excluded. On The contrary, these numerical bounds seem to rule out barotropic DE models with fast variation of $w$ at intermediate redshifts, such as all modified Chaplygin gas models, a few Cardassian universes, Braneworld cosmologies, etc. Concerning extended theories of gravity [48,51,61], provided their overall agreement with the above results, they cannot be excluded a priori. As stated above, this picture may change by exploring the possibility that there is non-zero curvature and may open new scenarios for the DE equation of state and evolution.

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