We analyse the effects of thermal conduction in a relativistic fluid, just after its departure from hydrostatic equilibrium, on a time scale of the order of thermal relaxation time. It is obtained that the resulting evolution will critically depend on a parameter defined in terms of thermodynamic variables, which is constrained by causality requirements.
1 Introduction.

Most part of the life of stars (at any stage of evolution), may be described on the basis of the quasi-static approximation (slowly evolving regime). This is so, because most relevant processes in star interiors take place on time scales that are usually, much larger than the hydrostatic time scale \[1,2\]. However, during their evolution, self-gravitating objects may pass through phases of intense dynamical activity for which the quasi-static approximation is clearly not reliable (e.g., the quick collapse phase preceding neutron star formation). All these phases of star evolution (“slow” and “quick”) are generally accompanied by intense dissipative processes, usually described in the diffusion approximation. This assumption, in its turn, is justified by the fact that frequently, the mean free path of particles responsible for the propagation of energy in stellar interiors is very small as compared with the typical length of the star.

In this work we shall study the influence of thermal conduction on the evolution of a self-gravitating system out of hydrostatic equilibrium (in the “quick” phase).

However, instead of following its evolution long time after its departure from equilibrium, we shall evaluate the system immediately after such departure. Here “immediately” means on a time scale of the order of thermal relaxation time, before the establishment of the steady state resistive flow.

Doing so we shall avoid the introduction of numerical procedures which might lead to model dependent conclusions.

On the other hand, however, we shall obtain only indications about the tendency of the object and not a complete description of its evolution.

As we shall see, there appears a local parameter formed by a specific combination of thermal relaxation time, thermal conductivity, proper energy density and pressure, which critically affects the evolution of the object and which is constrained by causality requirements.

The paper is organized as follows.

In the next section the field equations, the conventions and other useful formulae are introduced. In section 3 we briefly present the equation for the heat conduction. The central problem is analysed in section 4 and a discussion of results is given in the last section.
2 Field Equations and Conventions.

We consider spherically symmetric distributions of collapsing fluid, which for sake of completeness we assume to be anisotropic, undergoing dissipation in the form of heat flow, bounded by a spherical surface Σ.

The line element is given in Schwarzschild-like coordinates by

\[
\text{d} s^2 = e^\nu \text{d} t^2 - e^\lambda \text{d} r^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)
\]  

(1)

where \( \nu(t, r) \) and \( \lambda(t, r) \) are functions of their arguments. We number the coordinates: \( x^0 = t; \ x^1 = r; \ x^2 = \theta; \ x^3 = \phi. \)

The metric (1) has to satisfy Einstein field equations

\[
G^\mu_\nu = -8\pi T^\mu_\nu
\]  

(2)

which in our case read (3):

\[
-8\pi T^0_0 = -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right)
\]  

(3)

\[
-8\pi T^1_1 = -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right)
\]  

(4)

\[
-8\pi T^2_2 = -8\pi T^3_3 = - \frac{e^{-\nu}}{4} \left( 2\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu}) \right)
\]

\[
+ \frac{e^{-\lambda}}{4} \left( 2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r} \right)
\]  

(5)

\[
-8\pi T^0_1 = -\frac{\dot{\lambda}}{r}
\]  

(6)

where dots and primes stand for partial differentiation with respect to \( t \) and \( r \) respectively.

In order to give physical significance to the \( T^\mu_\nu \) components we apply the Bondi approach (3).

Thus, following Bondi, let us introduce purely locally Minkowski coordinates \((\tau, x, y, z)\).
\[ d\tau = e^{\nu/2} dt \quad dx = e^{\lambda/2} dr \quad dy = r d\theta \quad dz = r \sin \theta d\phi \]

Then, denoting the Minkowski components of the energy tensor by a bar, we have

\[ \bar{T}_0^0 = T_0^0 \quad \bar{T}_1^1 = T_1^1 \quad \bar{T}_2^2 = T_2^2 \quad \bar{T}_3^3 = T_3^3 \quad \bar{T}_{01} = e^{-(\nu+\lambda)/2} T_{01} \]

Next, we suppose that when viewed by an observer moving relative to these coordinates with velocity \( \omega \) in the radial direction, the physical content of space consists of an anisotropic fluid of energy density \( \rho \), radial pressure \( P_r \), tangential pressure \( P_\perp \) and radial heat flux \( \hat{q} \). Thus, when viewed by this moving observer the covariant tensor in Minkowski coordinates is

\[
\begin{pmatrix}
\rho & -\hat{q} & 0 & 0 \\
-\hat{q} & P_r & 0 & 0 \\
0 & 0 & P_\perp & 0 \\
0 & 0 & 0 & P_\perp
\end{pmatrix}
\]

Then a Lorentz transformation readily shows that

\[ T_0^0 = \bar{T}_0^0 = \frac{\rho + P_r \omega^2}{1 - \omega^2} + \frac{2Q \omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} \quad (7) \]

\[ T_1^1 = \bar{T}_1^1 = -\frac{P_r + \rho \omega^2}{1 - \omega^2} - \frac{2Q \omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} \quad (8) \]

\[ T_2^2 = T_3^3 = \bar{T}_2^2 = \bar{T}_3^3 = -P_\perp \quad (9) \]

\[ T_{01} = e^{(\nu+\lambda)/2} \bar{T}_{01} = -\frac{(\rho + P_r) \omega e^{(\nu+\lambda)/2}}{1 - \omega^2} - \frac{Q e^{\nu/2} e^\lambda}{(1 - \omega^2)^{1/2}}(1 + \omega^2) \quad (10) \]

with

\[ Q \equiv \frac{\hat{q} e^{-\lambda/2}}{(1 - \omega^2)^{1/2}} \quad (11) \]
Note that the velocity in the \((t, r, \theta, \phi)\) system, \(dr/dt\), is related to \(\omega\) by

\[
\omega = \frac{dr}{dt} e^{(\lambda - \nu)/2} \quad (12)
\]

At the outside of the fluid distribution, the spacetime is that of Vaidya, given by

\[
ds^2 = \left(1 - \frac{2M(u)}{R}\right) du^2 + 2dudR - R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (13)
\]

where \(u\) is a time-like coordinate such that \(u = \text{constant}\) is (asymptotically) a null cone open to the future and \(R\) is a null coordinate \((g_{RR} = 0)\). It should be remarked, however, that strictly speaking, the radiation can be considered in radial free streaming only at radial infinity.

The two coordinate systems \((t, r, \theta, \phi)\) and \((u, R, \theta, \phi)\) are related at the boundary surface and outside it by

\[
u = t - r - \frac{2M}{R} \ln \left(\frac{r^2}{2M} - 1\right) \quad (14)
\]

\[R = r \quad (15)\]

In order to match smoothly the two metrics above on the boundary surface \(r = r_\Sigma(t)\), we have to require the continuity of the first fundamental form across that surface. As result of this matching we obtain

\[
[P_r]_\Sigma = \left[Q e^{\lambda/2} \left(1 - \omega^2\right)^{1/2}\right]_\Sigma = [\hat{q}]_\Sigma \quad (16)
\]

expressing the discontinuity of the radial pressure in the presence of heat flow, which is a well known result [4].

Next, it will be useful to calculate the radial components of the conservation law

\[
T^\mu_{\nu;\mu} = 0 \quad (17)
\]

After tedious but simple calculations we get

\[
(-8\pi T^1_1)' = \frac{16\pi}{r} (T^1_1 - T^2_2) + 4\pi\nu' (T^1_1 - T^0_0) + \frac{e^{-\nu}}{r} \left(\dot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\nu}}{2} \right) \quad (18)
\]
which in the static case becomes
\[ P_r' = -\frac{\nu'}{2}(\rho + P_r) + \frac{2(P_\perp - P_r)}{r} \] (19)
representing the generalization of the Tolman-Oppenheimer-Volkoff equation for anisotropic fluids [5].

3 Heat Conduction Equation.

As we mentioned in the introduction, in the study of star interiors it is usually assumed that the energy flux of radiation (and thermal conduction) is proportional to the gradient of temperature (Maxwell-Fourier law or Eckart-Landau in general relativity).

However it is well known that the Maxwell-Fourier law for the radiation flux leads to a parabolic equation (diffusion equation) which predicts propagation of perturbation with infinite speed (see [6]–[8] and references therein). This simple fact is at the origin of the pathologies [9] found in the approaches of Eckart [10] and Landau [11] for relativistic dissipative processes.

To overcome such difficulties, different relativistic theories with non-vanishing relaxation times have been proposed in the past [12]–[15]. The important point is that all these theories provide a heat transport equation which is not of Maxwell-Fourier type but of Cattaneo type [18], leading thereby to a hyperbolic equation for the propagation of thermal perturbation.

Accordingly we shall describe the heat transport by means of a relativistic Israel-Stewart equation [8], which reads
\[ \tau \frac{Dq^\alpha}{Ds} + q^\alpha = \kappa P^{\alpha\beta} (T_{\beta} - Ta_{\beta}) - \tau u^\alpha q_{\beta} a^\beta - \frac{1}{2}\kappa T^2 \left( \frac{\tau}{\kappa T^2} u^\beta \right)_;\beta q^\alpha \] (20)
where \( \kappa, \tau, T, q^\beta \) and \( a^\beta \) denote thermal conductivity, thermal relaxation time, temperature, the heat flow vector and the components of the four acceleration, respectively. Also, \( P^{\alpha\beta} \) is the projector onto the hypersurface orthogonal to the four velocity \( u^\alpha \).

In our case this equation has only two independent components, which read, for \( \alpha = 0 \)
\[ \tau e^{(\lambda - \nu)/2} \left( Q \dot{\omega} + \dot{Q} \omega + Q \omega \dot{\lambda} \right) + \tau \left( Q' \omega^2 + Q \omega' + \frac{Q \omega^2 \lambda'}{2} \right) \]
\[ + \frac{\tau Q^2}{r} + Q \omega e^{\lambda/2} \left( 1 - \omega^2 \right)^{1/2} = -\frac{\kappa \omega^2 \dot{T} e^{-\nu/2}}{(1 - \omega^2)^{1/2}} - \frac{\kappa \omega T' e^{-\lambda/2}}{(1 - \omega^2)^{1/2}} \]
\[ - \frac{\nu'}{2} \kappa T \omega e^{-\lambda/2} - \frac{1}{2} Q \omega \left( e^{(\lambda - \nu)/2} \dot{T} + \omega T' \right) \]
\[ - \frac{1}{2} \tau Q \omega \left[ e^{(\lambda - \nu)/2} \left( \frac{\omega \dot{\omega}}{1 - \omega^2} + \frac{\dot{\lambda}}{2} \right) + \left( \frac{\omega'}{1 - \omega^2} + \frac{\nu' \omega}{2} \right) \right] \]
\[ + \frac{1}{2} \tau Q \omega \left[ \frac{1}{\kappa} \left( e^{(\lambda - \nu)/2} \dot{\kappa} + \omega \kappa' \right) + \frac{2}{T} \left( e^{(\lambda - \nu)/2} \dot{T} + \omega T' \right) \right] \]
\[ + \left( \tau e^{(\lambda - \nu)/2} - \frac{\kappa T \omega e^{-\nu/2}}{(1 - \omega^2)^{1/2}} \right) \times \left( \frac{\omega \dot{\lambda}}{2} + \frac{\dot{\omega}}{1 - \omega^2} \right) \]
\[ + \left( \tau Q - \frac{\kappa T \omega e^{-\lambda/2}}{(1 - \omega^2)^{1/2}} \right) \times \frac{\omega \omega'}{1 - \omega^2} \]

(21)

and for \( \alpha = 1 \)

\[ \tau e^{(\lambda - \nu)/2} \left( \dot{Q} + \frac{Q \dot{\lambda}}{2} + \frac{Q \omega^2 \dot{\lambda}}{2} \right) + \tau \omega \left( Q' + \frac{Q \lambda'}{2} \right) \]
\[ + \frac{\tau Q \omega}{r} + Q e^{\lambda/2} \left( 1 - \omega^2 \right)^{1/2} = -\frac{\kappa \omega \dot{T} e^{-\nu/2}}{(1 - \omega^2)^{1/2}} - \frac{\kappa T' e^{-\lambda/2}}{(1 - \omega^2)^{1/2}} \]
\[ - \frac{\nu'}{2} \kappa T e^{-\lambda/2} - \frac{1}{2} Q \omega \left( e^{(\lambda - \nu)/2} \dot{T} + \omega T' \right) \]
\[ - \frac{1}{2} \tau Q \omega \left[ e^{(\lambda - \nu)/2} \left( \frac{\omega \dot{\omega}}{1 - \omega^2} + \frac{\dot{\lambda}}{2} \right) + \left( \frac{\omega'}{1 - \omega^2} + \frac{\nu' \omega}{2} \right) \right] \]
\[ + \frac{1}{2} \tau Q \omega \left[ \frac{1}{\kappa} \left( e^{(\lambda - \nu)/2} \dot{\kappa} + \omega \kappa' \right) + \frac{2}{T} \left( e^{(\lambda - \nu)/2} \dot{T} + \omega T' \right) \right] \]
\[ + \left( \tau Q \omega e^{(\lambda - \nu)/2} - \frac{\kappa T e^{-\nu/2}}{(1 - \omega^2)^{1/2}} \right) \times \left( \frac{\omega \dot{\lambda}}{2} + \frac{\dot{\omega}}{1 - \omega^2} \right) 
\]
\[ + \left( \tau Q \omega - \frac{\kappa T e^{-\lambda/2}}{(1 - \omega^2)^{1/2}} \right) \times \frac{\omega \omega'}{1 - \omega^2} \]  
(22)

where the expressions

\[ u^\mu = \left( \frac{e^{-\nu/2}}{(1 - \omega^2)^{1/2}}, \frac{\omega e^{-\lambda/2}}{(1 - \omega^2)^{1/2}}, 0, 0 \right) \]
(23)

\[ q^\mu = Q \left( \omega e^{(\lambda - \nu)/2}, 1, 0, 0 \right) \]  
(24)

have been used.

We are now ready to get into the central problem of this work.

### 4 Thermal Conduction and Departure from Hydrostatic Equilibrium.

Let us now consider a spherically symmetric fluid distribution which initially may be in either hydrostatic and thermal equilibrium (i.e. $\omega = Q = 0$), or slowly evolving and dissipating energy through a radial heat flow vector.

Before proceeding further with the treatment of our problem, let us clearly specify the meaning of “slowly evolving”. That means that our sphere changes on a time scale which is very large as compared to the typical time in which it reacts on a slight perturbation of hydrostatic equilibrium. This typical time is called hydrostatic time scale. Thus a slowly evolving system is always in hydrostatic equilibrium (very close to), and its evolution may be regarded as a sequence of static models linked by (6).

As we mentioned before, this assumption is very sensible, since the hydrostatic time scale is usually very small.

Thus, it is of the order of 27 minutes for the sun, 4.5 seconds for a white dwarf and $10^{-4}$ seconds for a neutron star of one solar mass and 10 Km radius [2].

In terms of $\omega$ and metric functions, slow evolution means that the radial velocity $\omega$ measured by the Minkowski observer, as well as time derivatives are so small that their products and second order time derivatives may be
neglected (an invariant characterization of slow evolution may be found in [14]).
Thus [17]

\[ \ddot{\nu} \approx \ddot{\lambda} \approx \dot{\lambda} \dot{\nu} \approx \dot{\nu}^2 \approx \omega^2 \approx \dot{\omega} = 0 \]  

(25)

As it follows from (6) and (10), \( Q \) is of the order \( O(\omega) \). Thus in the slowly evolving regime, relaxation terms may be neglected and (20) becomes the usual Landau-Eckart transport equation [17].

Then, using (25) and (18) we obtain (19), which as mentioned before is the equation of hydrostatic equilibrium for an anisotropic fluid. This is in agreement with what was mentioned above, in the sense that a slowly evolving system is in hydrostatic equilibrium.

Let us now return to our problem. Before perturbation, the two possible initial states of our system are characterized by:

1. Static

\[ \dot{\omega} = \dot{Q} = \omega = Q = 0 \]  

(26)

2. Slowly evolving

\[ \dot{\omega} = \dot{Q} = 0 \]  

(27)

\[ Q \approx O(\omega) \neq 0 \quad (small) \]  

(28)

where the meaning of “small” is given by (25).

Let us now assume that our system is submitted to perturbations which force it to depart from hydrostatic equilibrium but keeping the spherical symmetry. We shall study the perturbed system on a time scale which is small as compared to the thermal adjustment time.

Then, immediately after perturbation (“immediately” understood in the sense above), we have for the first initial condition (static)

\[ \omega = Q = 0 \]  

(29)

\[ \dot{\omega} \approx \dot{Q} \neq 0 \quad (small) \]  

(30)

whereas for the second initial condition (slowly evolving)

\[ Q \approx O(\omega) \neq 0 \quad (small) \]  

(31)
\[
\dot{Q} \approx \dot{\omega} \neq 0 \quad \text{(small)} 
\] (32)

As we shall see below, both initial conditions lead to the same final equations.

Let us now write explicitly eq. (18). With the help of (7)–(10), we find after long but trivial calculations

\[
\begin{align*}
\frac{P_r'}{1 - \omega^2} + \frac{\rho'\omega^2}{1 - \omega^2} + \frac{2\omega'\rho}{1 - \omega^2} + \frac{2\omega'P_r}{(1 - \omega^2)^2} \\
+ \frac{2\omega^3\omega'\rho}{(1 - \omega^2)^2} + \frac{2Q'\omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} + \frac{2Q\omega'e^{\lambda/2}}{(1 - \omega^2)^{1/2}} + \frac{2Q\omega^2\omega'e^{\lambda/2}}{(1 - \omega^2)^{3/2}} \\
+ \frac{2}{r} \left[ \frac{4\pi r^3}{r - 2m} \left( \rho + P_r\omega^2 \right) - \frac{Q\omega e^{\lambda/2}}{(1 - \omega^2)^{3/2}} \right] \frac{12\pi r^3}{r - 2m} \left( \frac{Q\omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} \right)^2 \\
+ (\rho + P_r) \frac{\omega^2}{1 - \omega^2} + (P_r - P_\perp) + \frac{2Q\omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} + \frac{(\rho + P_r) 1 + \omega^2}{2} \frac{m}{1 - \omega^2} \frac{m}{r - 2m} \\
+ \frac{Q\omega e^{\lambda/2}}{(1 - \omega^2)^{1/2}} \frac{m}{r - 2m} + \frac{2\pi r^3}{r - 2m} \left( P_r + \rho\omega^2 \right) (P_r + \rho \omega^2) (\rho + P_r) \frac{1 + \omega^2}{(1 - \omega^2)^2} \\
+ \frac{8\pi r^3}{r - 2m} \left( P_r + \rho\omega^2 \right) \frac{Q\omega e^{\lambda/2}}{(1 - \omega^2)^{3/2}} + \frac{4\pi r^3}{r - 2m} Q\omega e^{\lambda/2} (\rho + P_r) \frac{1 + \omega^2}{(1 - \omega^2)^{3/2}} \\
= \frac{e^{-\nu}}{8\pi r} \left( \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} - \frac{\dot{\lambda}^2}{2} \right)
\end{align*}
\] (33)

which, when evaluated immediately after perturbation, reduces to

\[
P_r' + \frac{(\rho + P_r) m}{r^2 (1 - 2m/r)} + \frac{4\pi r}{(1 - 2m/r)} \left( P_r \rho + P_r^2 \right) + \frac{2 (P_r - P_\perp)}{r} = \frac{e^{-\nu}}{8\pi r} \ddot{\lambda}
\] (34)

for both initial states.

On the other hand, an expression for \( \ddot{\lambda} \) may be obtained by taking the time derivative of (E)

\[
\dot{\lambda} = -8\pi r e^{(\nu + \lambda)/2} \left( \rho + P_r \right) \frac{\omega}{1 - \omega^2} \frac{\dot{\nu}}{2} + Q e^{\lambda/2} \frac{1 + \omega^2}{(1 - \omega^2)^{1/2}} \frac{\dot{\nu}}{2}
\]
\[
\begin{align*}
+ \frac{(\rho + P_r) \omega \dot{\lambda}}{1 - \omega^2} & + Qe^{\lambda/2} \frac{1 + \omega^2}{(1 - \omega^2)^{1/2}} \dot{\lambda} + (\ddot{\rho} + \dot{P}_r) \frac{\omega}{1 - \omega^2} \\
+ (\rho + P_r) \dot{\omega} & \frac{1 + \omega^2}{(1 - \omega^2)^{1/2}} + \dot{Q}e^{\lambda/2} \frac{1 + \omega^2}{(1 - \omega^2)^{1/2}} \\
+ Qe^{\lambda/2} \omega \dot{\omega} \frac{3 - \omega^2}{(1 - \omega^2)^{3/2}}
\end{align*}
\] (35)

which, in its turn, when evaluated after perturbation, reads

\[
\ddot{\lambda} = -8 \pi r e^{(\nu + \lambda)/2} \left[ (\rho + P_r) \dot{\omega} + \dot{Q}e^{\lambda/2} \right]
\] (36)

replacing \(\ddot{\lambda}\) by (36) en (34), we obtain

\[-e^{(\nu - \lambda)/2} R = (\rho + P_r) \dot{\omega} + \dot{Q}e^{\lambda/2}\] (37)

where \(R\) denotes the left-hand side of the TOV equation, i.e.

\[
R \equiv \frac{dP_r}{dr} + \frac{4\pi r P_r^2}{1 - 2m/r} + \frac{P_r m}{r^2 (1 - 2m/r)} + \frac{4\pi r \rho P_r}{1 - 2m/r} + \\
+ \frac{\rho m}{r^2 (1 - 2m/r)} - \frac{2 (P_{\perp} - P_r)}{r} \\
= P'_r + \frac{\nu'}{2} (\rho + P_r) - \frac{2}{r} (P_{\perp} - P_r)
\] (38)

The physical meaning of \(R\) is clearly inferred from (38). It represents the total force (gravitational + pressure gradient + anisotropic term) acting on a given fluid element. Obviously, \(R > 0/R < 0\) means that the total force is directed inward/outward of the sphere.

Let us now turn back to thermal conduction equation (20). Evaluating its \(t\)-component (given by Eq.(21)) immediately after perturbation, we obtain for the first initial configuration (static), an identity. Whereas the second case (slowly evolving) leads to

\[
\omega \left( T' + \frac{T'^{\nu'}}{2} \right) = 0
\] (39)

which is to be expected, since before perturbation, in the slowly evolving regime, we have according to Eckart-Landau (valid in this regime)
\[ Q = -\kappa e^{-\lambda} \left( T' + \frac{T'\nu'}{2} \right) \]  

(40)

Therefore, the quantity in bracket is of order \( Q \). Then immediately after perturbation this quantity is still of order \( O(\omega) \), which implies (39). The corresponding evolution of the \( r \)-component of the equation (20) yields, for the initially static configuration

\[ \tau \dot{Q} e^{\lambda/2} = -\kappa T \dot{\omega} \]  

(41)

where the fact has been used that after perturbation

\[ Q = 0 \quad \Rightarrow \quad T' = -\frac{T'\nu'}{2} \]  

(42)

For the second case, the \( r \)-component of heat transport equation yields also (11), since after perturbation the value of \( Q \) is still given by (40), up to \( O(\omega) \) terms.

Finally, combining (37) and (41)

\[ \dot{\omega} = -\frac{e^{(\nu-\lambda)/2} R}{(\rho + P_r)} \times \frac{1}{1 - \frac{\kappa T}{\tau(\rho + P_r)}} \]  

(43)

or, defining the parameter \( \alpha \) by

\[ \alpha \equiv \frac{\kappa T}{\tau(\rho + P_r)} \]  

(44)

\[ -e^{(\nu-\lambda)/2} R = (\rho + P_r) \dot{\omega} (1 - \alpha) \]  

(45)

Let us first consider the \( \alpha = 0 \) case. Then, last expression has the obvious “Newtonian” form

\[ \text{Force} = \text{mass } \times \text{acceleration} \]

since, as it is well known, \((\rho + P_r)\) represents the inertial mass density and by “acceleration” we mean the time derivative of \( \omega \) and not \((a_\mu a^\mu)^{1/2}\). In this case \((\alpha = 0)\), an outward/inward acceleration \((\dot{\omega} > 0/\dot{\omega} < 0)\) is associated with an outwardly/inwardly \((R < 0/R > 0)\) directed total force (as one expects!).

However, in the general case \((\alpha \neq 0)\) the situation becomes quite different. Indeed, the inertial mass term is now multiplied by \((1 - \alpha)\), so that if \(\alpha = 1\),
we obtain that $\dot{\omega} \neq 0$ even though $R = 0$. Still worse, if $\alpha > 1$, then an outward/inward acceleration is associated with an inwardly/outwardly directed total force. However as we shall see in next section, causality requirements constrain $\alpha$ to be less than 1.

The last term in (20) is frequently omitted (the so-called “truncated” theory) [19]. In the context of this work both components of this term vanish and therefore all results found above are independent of the adopted theory (Israel-Stewart or truncated).

5 Discussion.

Restrictions based on stability and causality were derived in [9] for Israel-Stewart thermodynamics. According to equation (134) of [9], it follows that we must have $\alpha < 1$ in order to guarantee that thermal pulses are subluminal. In fact there is a similar parameter in the case of bulk viscous perturbations [8], which due to causality and stability limits should be smaller than one.

Before concluding we would like to make the following remarks:

1. Observe the formal similarity between the critical point and the equation of state for an inflationary scenario ($\rho = -P_r$).

2. It should be clear that in the context of the perturbation scheme used here, we get information only about the tendency of the system. To find out the real influence of the critical point on the evolution of the object, the full integration of the equations is required. Calculations involving such integrations have been performed in the past [26]–[28], however in neither one of the examples examined there, the system reaches the critical point. Furthermore, our configurations are initially in global thermal equilibrium, which is a highly idealized situation. In this sense, it should be stressed that our aim here is not to model a real star but to study some specific aspects of relativistic diffusion.

3. It should be clear that the analysis presented here depends strictly on the validity of the diffusion approximation, which in turn depends on the assumption of local thermodynamical equilibrium (LTE). Therefore, only small deviations from LTE can be considered in the context of this work.
4. For the sake of completeness we have considered an anisotropic fluid (instead of an isotropic one), leaving the origin of such anisotropy completely unspecified. As it is apparent, anisotropy does not affect the most important result obtained here (Eq.(45)). However, should anisotropy be related to viscosity, then for consistency the anisotropic pressure tensor should be subjected to the Israel-Stewart causal evolution equation for shear viscosity.

Acknowledgments.

This work has been partially supported by the Spanish Ministry of Education under grant No. PB94-0718.

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