Using The Newton-Raphson method with Automatic Differentiation to numerically solve Implied Volatility of stock option through Binomial Model

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Abstract

In the paper written by Klibanov et al [3], it proposes a novel method to calculate implied volatility of a European stock options as a solution to ill-posed inverse problem for the Black-Scholes equation. In addition, it proposes a trading strategy based on the difference between implied volatility of the option and the volatility of the underlying stock. In addition to the Black-Scholes equation, Binomial model is another method used to price European options. And, the implied volatility can be also calculated through this model. In this paper, we apply the Newton-Raphson method together with Automatic Differentiation to numerically approximate the implied volatility of an arbitrary stock option through this model. We provide an explanation of the mathematical model and methods, the methodology, and the results from our test using the stimulated data from the Geometric Brownian Motion Model and the Binomial Model itself, and the data from the US market data from 2018 to 2021.

Keywords: Binomial Model; Implied Volatility; Option Pricing; Black-Scholes Equation; Automatic Differentiation; Newton-Raphson Method.

1 Introduction

An option is a contract that allows the holder the right to buy or sell an underlying asset or financial instrument at a specific strike price on or before a specific date. An option to buy is known as a "call option," and an option to sell is known as a "put option." Options are divided into two types based on the type of time period of implementation. An American type of option is an option that can be exercised at or before maturity, whereas a European option is an option that can only be exercised at the maturity date. In this paper, we used the market data for a total of 30,000 European call stock options.

The Black-Scholes model, also known as the Black-Scholes-Merton model, is one of the most widely used models in modern financial theory when estimating the theoretical value of options. It was developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes and was initially introduced in [1]. In the model, it uses the current stock prices, expected dividends, the option’s strike price, expected interest rates, time to expiration, and expected volatility to calculate the theoretical value of an option contract. This equation can be written as a parabolic partial differential equation which can be explicitly solved. Despite its simplicity, solving implied volatility via the equation is difficult due to its ill-posedness characteristic [3].

Another commonly-used model for option valuation is the binomial option pricing model [2]. It was first proposed by William Sharpe in the 1978 and then formalized by Cox, Ross, and Rubinstein in 1979. It traces the evolution of an option’s key underlying
variables in discrete time by means of a binomial tree. In this paper, we will discuss the method of using the binomial option pricing model to calculate the implied volatility. We will explain the methodology behind the idea as well as some numerical results from our experiments.

In the paper [3], it proposed a new mathematical method is accurately calculate the implied volatility of a European stock option. In addition, it derives a trading strategy based on the difference between the implied volatility of an option and the volatility of its underlying stock. Specifically, let σ be the volatility of a stock, ˆσ be the volatility of the corresponding option to the stock. Let S be the price of the stock, f(S, K) be the payoff or value function of the option at the maturity time T, and u(S, t), with t, the time until the maturity will occur, be the price of the option.

By assumptions from the paper, the following equation is satisfied [3]:

\[
\frac{\partial u(S, t)}{\partial t} = \frac{\hat{\sigma}^2}{2} S^2 \frac{\partial^2 u(S, t)}{\partial S^2}, \quad S > 0
\]

\[
u(S, 0) = f(S)
\]

Let K be the strike price, then the payoff function is \( f(S) = \max(S - K, 0) \), and \( u(S, t) \) is given by the Black-Scholes formula [1]:

\[
u(S, t) = S\Phi(\Theta_+(S, t)) - e^{-rt}K\Phi(\Theta_-(S, t))
\]

where the risk-free rate \( r \) is assumed to be 0. The stochastic equation of the Geometric Brownian motion for the stock price \( S_t \) with the volatility \( \sigma \) and zero drift is expressed by

\[
dS_t = \sigma S_t dW_t
\]

Since the expectation of the Wiener process, dW, is 0 and that by substituting the above equations in the Itô formula, we get the expected value of the increment of the option price on an infinitesimal time interval is

\[
\frac{(\sigma^2 - \hat{\sigma}^2)}{2} S^2 \frac{\partial^2 u(S, t)}{\partial S^2} dt
\]

Since \( \frac{\partial^2 u(S, t)}{\partial S^2} \) is the Greek \( \Gamma \) which is proved to be non-negative, and \( S^2 \) is greater than 0; thus, the expected direction of an option price depends on \( \sigma - \hat{\sigma}\sigmama; \) that is, under the assumptions, we expect the option price in the market to increase if \( \sigma - \hat{\sigma}\sigmama > 0 \), the volatility of the underlying asset is greater than the implied volatility of the option. After deriving the equation, the paper also examines the validity of the mathematical model and generates a possible winning strategy from the model. After testing the hypothesis, it gets a precision of 55.57%.

The result from [3] strengthens the role and the importance of the implied volatility of options in financial mathematics. However, the Black-Scholes equation, on which the paper relies on, is not the only single method used to price an option. Moreover, this method works only for European options.
In this paper, we firstly present the algorithm of Binomial model and explain why it can be used to find the implied volatility of an option’s price by showing the convergence from the Black-Scholes equation to the Binomial model. We discuss the method of calculating implied volatility inversely via binomial model by explaining the conditions and parameters the model is required.

Then, we test our model by employing the binomial model to calculate the implied volatility of an option price generated from Geometric Brownian motion model with $\mu = 0$, $\sigma = 0.2$, and compare the result from the Binomial Model to the result from the Black-Scholes Model, and find out the relative difference between the precision of two models.

2 Mathematical Model

2.1 Binomial Model

The Binomial Model is based on the main assumption that the price of the underlying asset, which is the stock in this case, can change in only two possible ways: either going up or going down, in the next time step. These stock prices can be estimated using Geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $S_t$ is a stock price at time $t$, $\mu$ is a drift, $\sigma$ is an volatility of the stock, and $W_t$ is a Wiener process.

Note that in the next step $\delta t$, the the Geometric Brownian motion can be approximated into two possible values [6]:

$$S_{t+\delta t} = S_t e^{\sigma \sqrt{\delta t}} \quad \text{or} \quad S_{t+\delta t} = S_t e^{-\sigma \sqrt{\delta t}}$$

Assume that the next time step, at time $T$, is the maturity of a call option with strike price $K$, the value of the function in that step can be determined by the function:

$$V(T) = \max(0, S(T) - K)$$

where $S(T)$ is the price of the stock at time $T$ (the maturity of the call option).

Since we assume that the price of the stock can go in only two directions in the next step, we can separately calculate the value of the call option as follows:

$$C^+ = \max(0, S(0) e^{\sigma \sqrt{T}} - K) = \max(0, S_0 e^{\sigma \sqrt{T}} - K)$$

$$C^- = \max(0, S(0) e^{-\sigma \sqrt{T}} - K) = \max(0, S_0 e^{-\sigma \sqrt{T}} - K)$$

Then, A single-step Binomial Model is formulated as shown below:

$$C = e^{-r \delta t} \left( \frac{1}{2} + \frac{r \sqrt{\delta t}}{2\sigma} C^+ + \left( \frac{1}{2} - \frac{r \sqrt{\delta t}}{2\sigma} \right) C^- \right)$$
where $C$ is the value of option; $r$ is a risk-free rate; $t$ is a time-step (in year); $\sigma$ is a volatility of the underlying asset.

This model can be extended into multiple steps, which is called the Lattice Model. In this case, the values of options in any possible branch of the tree can be determined by the value functions as shown above. Yet, in the middle steps, the values of options in the steps after them are required in order to calculate the option at that middle step. This can be done by solving the option values in each step backwards. In the last step, the value of the option at the current time is determined.

### 2.2 Transformation/convergence from Black-Scholes to Binomial Model

The fundamental difference between the Black-Scholes equation and the binomial model is that Black-Scholes equation considers the continuous time, whereas the binomial model considers time as a discrete variable.

One way of deriving Black-Scholes equation is letting the number of time steps in the binomial tree to approach infinity.

Suppose a binomial model with $n$ time steps is used to value a European call option, then each step has the length $T/n$. It would have $i$ upward movements and $n-i$ downward movements. According to the binomial model, the final stock price is $S(0)u^i d^{n-i}$, where $S(0)$ is the original price, $u$ is the up factor, and $d$ is the down factor.

In order to earn a profit from the option, we make a move when $\max(0, S(0)u^i d^{n-i} - K) > 0$. In other words, $S(0)u^i d^{n-i} - K > 0$, and $S(0)u^i d^{n-i} > K$.

By substituting the equation for $u$ and $d$, we get:

$$\ln\left(\frac{S(0)}{K}\right) > n\sigma \sqrt{\delta t} - 2i\sigma \sqrt{\delta t}$$

or

$$i > \frac{n}{2} - \frac{\ln\left(\frac{S(0)}{K}\right)}{2\sigma \sqrt{\delta t}}$$

Then, the equation of $C$ can be written as:

$$C = e^{-rT} (S(0)A_1 - KA_2)$$

where

$$A_1 = \sum_{i > \frac{n}{2} - (\frac{\ln\left(\frac{S(0)}{K}\right)}{2\sigma \sqrt{\delta t}})} \frac{n!}{(n-1)!i!} p^i (1 - p)^{n-1} u^i d^{n-i}$$

$$A_2 = \sum_{i > \frac{n}{2} - (\frac{\ln\left(\frac{S(0)}{K}\right)}{2\sigma \sqrt{\delta t}})} \frac{n!}{(n-1)!i!} p^i (1 - p)^{n-1}$$
Since the binomial distribution approaches a normal distribution as the number of trials approaches infinity, and the mean is \( np \) and standard deviation is \( np(1-p) \), with \( p \) the probability of success (in this case, the probability of going up), \( n \) the number of trials.

Then

\[
A_2 = N\left( np - \left( \frac{n}{2} - (\ln \left( \frac{S(0)}{K} \right) / 2\sigma \sqrt{\delta t} \right) \right) = N\left( \frac{\ln \left( \frac{S(0)}{K} \right) + (r - \frac{\sigma^2}{2})\delta t}{\sigma \sqrt{\delta t}} \right)
\]

as \( n \) goes to infinity. Similarly for \( A_1 \), we get

\[
A_1 = e^{rt} N\left( \frac{\ln \left( \frac{S(0)}{K} \right) + (r + \frac{\sigma^2}{2})\delta t}{\sigma \sqrt{\delta t}} \right)
\]

Combining \( A_1, A_2, \) and \( C \), we get

\[
d_1 = \frac{\ln \left( \frac{S(0)}{K} \right) + (r + \frac{\sigma^2}{2})\delta t}{\sigma \sqrt{\delta t}}
\]

\[
d_2 = \frac{\ln \left( \frac{S(0)}{K} \right) - (r + \frac{\sigma^2}{2})\delta t}{\sigma \sqrt{\delta t}} = d_1 - \sigma \sqrt{T}
\]

which is the Black-Scholes equation for finding a European call option.

2.3 Numerical Methods

2.3.1 Newton-Raphson method

The Newton-Raphson method is widely used to numerically solve roots of differentiable functions [7]. The algorithm can be written as the formula shown below:

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

where \( x_n \) for \( n \in \mathbb{N} \) is an approximated root for \( n^{th} \) step, and \( x_0 \) is an initial guess, and \( f \) is a function. Under stable conditions, \( x_n \) will approach to one root of \( f \).

2.3.2 Automatic Differentiation

As shown in the previous subsection, the Newton-Raphson method requires a derivative of \( f \) in order to calculate a root for \( f \). There are plenty of methods to approximate the derivative. One of the intuitive methods is to directly calculate it using theorems from calculus, so-called symbolic differentiation. However, because some formulas, such as product rules and chain rules, greatly expand functions, this method may result in an overly complex expression. This symbolic method may be applicable for single-step
binomial models, but it is exhaustive for multi-step lattice models. Another numerical method which is widely used is the finite difference method:

\[ f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x} \]

for some small \( \Delta x > 0 \). The finite element method is also introduced to calculate the derivative for the Newton-Raphson method, and the modified method is called the Secant Method.

In this project, we apply a widely-used method for differentiation, so-called Automatic Differentiation (AD). AD calculates the derivative of a function immediately after each component of the function has been computed. Also, the derivative of that part of the function is stored together with the result of that part being stored [5].

Automatic Differentiation is largely used in machine learning algorithms such as back propagation. With the structure of the Binomial/Lattice Model, the option price is calculated as a multiple of steps of calculation through a tree. We propose to use Automatic Differentiation for calculating the derivative of the model function.

3 Methodology

In this project, we approximate the implied volatility of a European stock option by using the Binomial/Lattice model.

3.1 Calculating implied volatility inversely via binomial model

As mentioned in the previous section, in order to determine the current value of an option, some parameters such as the current stock price and risk-free rate are required. These parameters can be obtained from the exchange market. Moreover, in the same place, the price of the option is determined.

Now, we let the known parameters be constants. Then, we can simply write a formula as:

\[ C = f(\sigma) \]

Then, the \( \sigma \) satisfying the equation above is the implied volatility of a corresponding stock option. It is important to note that in this paper, the 10-layer lattice model is applied. The number of the layers is determined by computational limitations. The risk-free rate is based on the 10-year treasury yield for the real-market data. And we assume the rate to be 0 for the generated dataset. Other parameters are obtained from the real market. \( C \) is known in the exchange market. Thus, \( \sigma \) can be determined by finding the root of the equation:

\[ f(\sigma) - C = 0 \]

We denote the root of the equation above by \( \hat{\sigma} \).

To finding a root of the equation, we apply Newton-Raphson’s method with initial value,
\[ x_0 = 0.2, \text{ with the tolerant rate } = 10^{-5}. \] That is, the iteration would stop at \( n \) when \[ |f(x_n) - f(x_{n-1})| < 10^{-5}. \] The maximum number of iterations is set to 100. The derivative of \( f \) is calculated using Automatic Differentiation in each iteration.

### 3.2 Testing the numerical method

Since the model used in this project is combined with multiple step calculations, it is difficult for the numerical method mentioned above to accurately approximate the implied volatility. Therefore, to assess the efficiency of the model, we tested the algorithms with simulated data.

Similar to [3], the underlying stock is simulated using a Geometric Brownian motion model with \( \mu = 0 \) and \( \sigma = 0.2 \). And, the price of options are calculated via the same binomial model, with 90 days of maturity and with 33 different volatilities. After the simulated option price is obtained, we inversely solve for the implied volatility with the method mentioned above.

![Figure 1: Simulated stock price based on Geometric Brownian motion](image)

![Figure 2: An example of simulated option prices generated from the stock prices above and Binomial model with four different volatility, strike price of 150, and fixed maturity at day 180.](image)

### 3.3 Testing with the real world data

After testing the method with synthetic data, we assess the efficiency of the model with real-world data. In this project, we apply the algorithm to 38270 European options traded...
in the market from 2018 to 2021. The last prices of an option and its underlying asset are used in this evaluation.

4 Numerical Finding

The scatter plots show the implied volatility from the binomial model with their corresponding actual implied volatility. The red diagonal line plot shows the position when a pair of implied volatility are the same. Note that the algorithm could possibly return negative implied volatility, which is not shown in these plots due to its absurdity.

According to the plots, the algorithm frequently underestimates the actual implied volatility. To address the issues of underestimation and negative results, we show the histograms together with the scatter plot for each dataset.

4.1 Generated Data

Figure 3: (left) The scatter plot of implied volatilities calculated from the binomial model and their corresponding actual values from the dataset (Generated Data); (right) The implied volatilities distributions (Generated Data).
4.2 Stock options selected from American markets from 2017-2018

![Figure 4](image)

Figure 4: (left) The scatter plot of implied volatilities calculated from the binomial model and their corresponding actual values from the dataset (The 2017-2018 US market); (right) The implied volatilities distributions (The 2017-2018 US market).

4.3 Stock options selected from American markets from 2020-2021

![Figure 5](image)

Figure 5: (left) The scatter plot of implied volatilities calculated from the binomial model and their corresponding actual values from the dataset (The 2020-2021 US market); (right) The implied volatilities distributions (The 2020-2021 US market).

5 Discussion

The three results show this method greatly underestimates the actual implied volatility. Among the plots, the result from testing the method on generated data best shows the efficiency of the method. Yet, testing with the real-world dataset reveals some flaws in its efficiency, particularly in the 2020-2021 dataset. One of the important causes of this
is that the collection of actual implied volatilities attained in the dataset was inversely solved from Black-Scholes Models. This results in high error rates in real-world data but much less in generated data. Comparing the results from different periods of real market data, we found that the results from 2020 to 2021 attained a higher error than those of the 2017-2018 market. This might be from the effect of the pandemic on the option market. Among the experiments, the results from applying the proposed method to the generated data are the most acceptable. The absolute difference between the actual value and the result from the method is less than 0.01 for each simulated option price. However, most of the results underestimate the actual implied volatility, which is shown on the corresponding histogram. Despite the fact that it is not clear how this occurs, the distribution of the difference seemingly allows us to adjust the results from the algorithm by adding the constant average error from the dataset to gain more accurate results. Yet, the adjustment needs to be deeply studied in the future.

The result from applying the method to the real stock option market from 2017 to 2018 shows more error compared to that on the generated data. In addition, the method does not wholly underestimate the actual values; it sometimes overestimates the values. However, the distribution plot still implies that the method averagely underestimates the values.

Another interesting point is that the numerical method gives negative implied volatility, which is absurd. We provide three explanations for these issues:

1. European stock options are priced based on the Black-Scholes equation, not the Binomial Model. Despite the fact that the Binomial Model converges to the model in continuous time, the implied volatility inversely calculated from the different models may not necessarily be the same. Presumably, there is no positive or real root, or implied volatility, for the Binomial Model.

2. The Newton-Raphson method gives only one root. Even if we assign the initial value, $x_0$, to be close to the average option price, under some unstable conditions, the Newton-Raphson method may give other possible roots, which are not necessarily negative, of the function. This issue can be fixed by testing the method on the different initial values and selecting the most appropriate result.

3. Unobservable real-world factors. Unlike the generated dataset, where everything is under control, the real stock market is based on uncertainty.

The results from applying the method to the real option market dataset from 2020 to 2021 show the largest error. As shown in the scatter plot, the method greatly underestimates the implied volatility. The explanations from the previous paragraphs can account for most of the results on this dataset. Importantly, the uncertainty of the market during the pandemic may influence the results.

6 Conclusion

In this paper, we provide a method to numerically calculate the implied volatility of European stock options. The method consists of two parts: the Binomial Model and a
numerical method. The Binomial model is used to price European stock options. Given
a data about options in the exchange market, including the price of the options, we
can calculate the implied volatility by inversely solving the model to get a volatility
variable pertaining to the given option price. Newton-Raphson method is used to solve
for such volatility. Yet, the numerical method requires the derivative of the binomial
model, which is hard and tedious to analytically or numerically determine. We propose to
use Automatic Differentiation, which immediately can calculate the derivative of function
after the function parameters are passed. Finally, we test the proposed method on three
datasets: the generated data, the real-market dataset from 2017 to 2018, and that from
2020 to 2021. The results show that the method works well for the generated data, but its
efficiency deceases once tested with the real-market data; yet, the result from 2017-2018
dataset is acceptable. We additionally provide the explanations of the issues found in
testing the method with each dataset.
Practically, we can use the calculated implied volatility to make an interference in option
trading decision as provided in. In addition, this method can be modified in order to
numerically calculate some parameters for other asset pricing models. For example, this
method can be applied to calculate the implied volatility for an American option which
does not follow the Black-Scholes equation; yet, this kind of option can be calculated
using modified binomial model.
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