Algebra of Higher Antibrackets

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Abstract

We present a simplified description of higher antibrackets, generalizations of the conventional antibracket of the Batalin-Vilkovisky formalism. We show that these higher antibrackets satisfy relations that are identical to those of higher string products in non-polynomial closed string field theory. Generalization to the case of $Sp(2)$-symmetry is also formulated.
1 Introduction

Lagrangian BRST quantization gets its most succinct formulation in the antibracket formalism of Batalin-Vilkovisky [1]. The basic objects of that approach, the antibracket itself and a so-called \( \Delta \)-operator (to be reviewed below), turn out to belong to a general algebraic structure that has attracted considerable attention recently, in particular in connection with a geometric interpretation and covariant generalizations [2].

The conventional antibracket of the Batalin-Vilkovisky formalism can be viewed as being based on a 2nd-order odd differential operator \( \Delta \) satisfying \( \Delta^2 = 0 \). In (super) Darboux coordinates it takes the simple form [1]

\[
\Delta = (-1)^{\epsilon_A+1} \frac{\delta^r}{\delta \phi^A} \frac{\delta^r}{\delta \phi^*_A} , \tag{1.1}
\]

where to each field \( \phi^A \) one has a matching “antifield” \( \phi^*_A \) of Grassmann parity \( \epsilon(\phi^*_A) = \epsilon(\phi^A) + 1 \). The antifields are conventional antighosts of the Abelian shift symmetry that for flat functional measures leads to the most general Schwinger-Dyson equations [3].

Given \( \Delta \) as above, one can define an odd (statistics-changing) antibracket \( (F,G) \) from the failure of \( \Delta \) to act like a derivation:

\[
\Delta(FG) = F(\Delta G) + (-1)^{\epsilon_G}(\Delta F)G + (-1)^{\epsilon_G}(F,G) . \tag{1.2}
\]

The antibracket so defined automatically satisfies the following relations. First, it has an exchange symmetry of the kind

\[
(F,G) = (-1)^{\epsilon_F\epsilon_G+\epsilon_F+\epsilon_G}(G,F) . \tag{1.3}
\]

It also acts like a derivation in the sense of a generalized Leibniz rule:

\[
\begin{align*}
(F,GH) &= (F,G)H + (-1)^{\epsilon_G(\epsilon_F+1)}G(F,H) \\
(FG,H) &= F(G,H) + (-1)^{\epsilon_G(\epsilon_H+1)}(F,H)G ,
\end{align*} \tag{1.4}
\]

and it satisfies a Jacobi identity,

\[
\sum_{\text{cycl.}} (-1)^{\epsilon_F+1}(\epsilon_H+1)(F,(G,H)) = 0 . \tag{1.5}
\]

In addition, there is a useful relation between the \( \Delta \)-operator and its associated antibracket:

\[
\Delta(F,G) = (F,\Delta G) - (-1)^{\epsilon_G}(\Delta F,G) . \tag{1.6}
\]

Recently, two of the present authors [4] showed that the antibracket formalism is open to a natural generalization. In a path-integral formulation, this generalization can be derived by considering general field transformations \( \phi^A \to g^A(\phi',a) \), where \( a^i \) represent certain collective fields [5]. The idea is to impose on the Lagrangian path integral the condition that certain Ward identities are preserved throughout the quantization procedure. If one imposes the most general set of Ward identities possible – the Schwinger-Dyson equations – through an unbroken Schwinger-Dyson BRST symmetry [6], one can recover the antibracket formalism of Batalin and Vilkovisky by integrating out certain ghosts \( c^A \) (the antifields \( \phi^*_A \) being simply the antighosts corresponding to \( c^A \)). For flat functional measures this corresponds to local shift transformations of the fields \( \phi^A \). If the measure is not flat, or if one wishes to impose a more restricted set of Ward identities through the BRST symmetry, the \( \Delta \)-operator and the associated antibracket will differ from those of the conventional Batalin-Vilkovisky formalism. In ref. [3] it was
shown how the Batalin-Vilkovisky $\Delta$-operator (1.1) can be viewed as an Abelian operator corresponding to the Abelian shift transformation $\phi^A \rightarrow \phi^A - a^A$. The analogous non-Abelian $\Delta$-operator for general transformations $\phi^A \rightarrow g^A(\phi', a)$ was derived in ref. [5]:

$$\Delta G \equiv (-1)^{\epsilon_i} \left[ \frac{\delta^r}{\delta \phi^i} \frac{\delta^r}{\delta \phi^*_i} G \right] u^A_i + \frac{1}{2} (-1)^{\epsilon_i+1} \left[ \frac{\delta^r}{\delta \phi^*_{ij}} \frac{\delta^r}{\delta \phi^*_{ij}} G \right] \phi^*_i U^k_{ji},$$  \hspace{1cm} (1.7)

where the $U^k_{ij}$ are the structure coefficients for the supergroup of transformations. They are related to the field transformations $g^A(\phi', a)$ by the relation

$$\frac{\delta^r}{\delta a^i} u^A_i (\phi) = \frac{\delta^r}{\delta a^i} g^A(\phi', a) \bigg|_{a=0}.$$  \hspace{1cm} (1.8)

The $\Delta$-operator of eq. (1.7) can be shown to be nilpotent [5], and it gives rise to a new non-Abelian antibracket by use of the relation (1.2). Explicitly, this antibracket takes the form [5]

$$\langle F, G \rangle \equiv (-1)^{\epsilon_i (\epsilon_{A+1})} \frac{\delta^r F}{\delta \phi^i} u^A_i \frac{\delta^r G}{\delta \phi^A} - \frac{\delta^r F}{\delta \phi^A} u^A_i \frac{\delta^r G}{\delta \phi^*_i} + \frac{\delta^r F}{\delta \phi^*_i} \phi^*_i U^k_{ij} \frac{\delta^r G}{\delta \phi^*_{ij}}.$$  \hspace{1cm} (1.10)

In ref. [5] this non-Abelian antibracket was derived directly in the path integral (by integrating out the ghosts $c^A$), but it can readily be checked that it is related to the associated $\Delta$-operator (1.7) in the manner expected from (1.2). Because this particular non-Abelian $\Delta$-operator is of 2nd order, the corresponding antibracket automatically satisfies all the properties (3-6).

As shown in ref. [4], even this non-Abelian antibracket is open to generalizations. One first notices that the non-Abelian $\Delta$ is nothing but the Hamiltonian BRST operator $\Omega$ of a certain constraint algebra in an unusual representation, that of Hamiltonian ghost momentum. Taking the most general non-Abelian BRST operator $\Omega$ of an arbitrary non-Abelian open algebra, one can then construct the corresponding general $\Delta$-operator by going to the ghost momentum representation [4]. This leads naturally to the concept of higher (non-Abelian) antibrackets. Interestingly, much of the appropriate mathematical machinery for such a formalism already exists in the mathematics literature [7, 8]. There is also a surprising connection between the algebra of these higher antibrackets and that of so-called strongly homotopy Lie algebras (for a very readable account, written for physicists, see ref. [9]), which appear in string field theory [10].

Interest in general Batalin-Vilkovisky algebras has recently arisen also in the context of two-dimensional topological field theory and string theory [11]. One should expect the higher antibrackets to play a rôle there as well [8].

From the point of view of quantization of field theories, perhaps the most important reason for studying the algebraic structure behind higher antibrackets comes from the expectation that even the conventional Batalin-Vilkovisky $\Delta$-operator will be modified by higher-order quantum corrections originating from operator-ordering ambiguities in the Hamiltonian framework. This obviously makes it important to study the Master Equation for arbitrary higher-order $\Delta$-operators, and to understand their associated BRST structure.

\[1\] Taking for convenience that the supergroup is semi-simple, with $(-1)^{\epsilon_i} U^k_{ij} = 0$.

\[2\] We are grateful to I.A. Batalin for explaining this aspect to us.
The purpose of the present paper is partly to present a simplified construction of the higher antibrackets introduced in ref. [4], partly to show how they can be generalized in a natural manner to a situation in which one has simultaneous BRST and anti-BRST symmetry. In fact, these two symmetries can, not surprisingly, be combined into an $Sp(2)$-symmetry. The mathematical analogue of this is an $Sp(2)$-covariant strongly homotopy Lie algebra. While this algebra may be of interest in its own right, it also points towards the existence of an $Sp(2)$ BRST–anti-BRST symmetric version of closed string field theory, as we shall show towards the end of our paper. This will then provide a comprehensive setting for the possible generalizations of the usual Batalin-Vilkovisky quantization formalism, and its $Sp(2)$ extensions.

We start in section 2 with a brief review of how higher antibrackets naturally arise if one generalizes the Batalin-Vilkovisky formalism from shift symmetries (which generate the usual Batalin-Vilkovisky $\Delta$-operator) to more general transformations. This is only to set the stage for what follows, because we are in this paper interested in the study of the higher antibrackets independently of such considerations. We then proceed to a discussion of the Koszul construction of higher brackets and antibrackets based on general differential operators $\Delta$ (section 2.1). Some useful mathematical background is introduced in section 2.2, and we show how to reformulate this construction in a simple fashion. In section 2.3 we discuss the precise connection to strongly homotopy Lie algebras, and prove a useful lemma related to the algebra of two sets of higher brackets. As an explicit realization in terms of chosen coordinates, we describe the algebra by means of a suitable vector field in section 2.4. The analogue of the strongly homotopy Lie algebra structure associated with our generalized higher brackets is discussed in section 2.6. Section 2.5 is our first return to physics applications: we discuss the definition of a generalized Master Equation, first introduced in ref. [4]. This leads us to the subject of BRST symmetry in this higher-antibracket framework. When formulated as the possibility of deforming a given solution of the Master Equation by the addition of BRST-exact terms, it is of interest to find the associated symmetry algebra. While the most simple choice of symmetry transformations corresponds to an algebra that is open, we show how in a simple manner one can add “equation of motion terms” to the transformations in order to make the algebra close. We also discuss finite symmetry transformations. In section 3 we turn our attention to some intriguing parallels between higher antibrackets and the so-called “string products” in closed string field theory [18, 10, 19], when as $\Delta$-operator one takes the BRST charge $Q$. Section 4 is devoted to the construction of an $Sp(2)$-symmetric analogue of the higher-antibracket BRST symmetry. Section 5 contains our conclusions. Finally, in two appendices we propose some generalizations which lie slightly outside the main line of the paper. In the first (Appendix A), we show how one can introduce yet higher levels of generalizations of the higher antibrackets discussed in the main text. While their rôle in physics applications is totally obscure, we nevertheless find it interesting that such a further generalization is possible. In Appendix B we discuss generalizations of the so-called “main identities”, valid already at the level of the normal higher antibrackets. These new identities contain new information in cases where, for example, $\Delta$ is no longer nilpotent, or, as discussed in section 4, when one imposes an $Sp(2)$ symmetry as well.

2 Higher Antibrackets

As explained in ref. [4], one can introduce obvious generalizations of the Batalin-Vilkovisky $\Delta$-operator by considering the most general Hamiltonian BRST operator $\Omega$ in the ghost momentum representation. Start with a representation of first class constraints

$$[\hat{G}_i, \hat{G}_j] = i \hat{G}_k U^k_{ij}$$ (2.1)
of the form\[13\]
\[ G_i \equiv -i\frac{\delta^r}{\delta \phi_i} u_i^A, \tag{2.2} \]
which involves a right-derivative acting to the left. Because the constraints in this representation act to the left, one must choose a representation of the Hamiltonian ghost (super) Heisenberg algebra
\[ [\eta^i, \mathcal{P}_j] = \eta^i \mathcal{P}_j - (-1)^{(\epsilon_i+1)(\epsilon_j+1)} \mathcal{P}_j \eta^i = i \delta^i_j \tag{2.3} \]
which also involves operators acting to the left. In the ghost momentum representation, this is
\[ \eta^j_i = i(-1)^{\epsilon_i} \frac{\delta}{\delta \mathcal{P}_j} \eta^i. \tag{2.4} \]

One of the observations in ref. \[4\] is that to pass to the Lagrangian $\Delta$-operator, one identifies the Hamiltonian ghost $\mathcal{P}_j$ with the Lagrangian antighost ("antifield") $\phi^*_j$. The most general Hamiltonian BRST operator $\Omega$ \[12\], in this representation takes the form \[4\]
\[ \Omega = (-1)^i \frac{\delta^r}{\delta \phi^*_i} u_i^A \frac{\delta^r}{\delta \phi_i^*} + \sum_{n=1}^{\infty} \phi^*_{i_n} \cdots \phi^*_{i_1} U^i_{j_1 \cdots j_n+1} \frac{\delta^r}{\delta \phi^*_{j_{n+1}}} \cdots \frac{\delta^r}{\delta \phi^*_{j_1}}, \tag{2.5} \]
where
\[ U^{i_1 \cdots i_n}_{j_1 \cdots j_{n+1}} = \frac{(-1)^{i_1 \cdots i_{n-1}}}{(n+1)!} (-1)^{\epsilon_{j_1} + \cdots + \epsilon_{j_{n+1}}} U^{i_1 \cdots i_n}_{j_1 \cdots j_{n+1}} \frac{\delta^r}{\delta \phi^*_{j_{n+1}}} \cdots \frac{\delta^r}{\delta \phi^*_{j_1}}. \tag{2.6} \]
The functions $U^{i_1 \cdots i_n}_{j_1 \cdots j_{n+1}}$ are generalized structure "constants" of the possibly open algebra. The infinite sum in eq. (2.5) may terminate at finite order. For example, for ordinary super Lie algebras where the structure coefficients $U^{i}_{j}^{k}$ are just constant supernumbers, the series terminates at the first term.

The $\Delta$-operator is now defined through
\[ \Delta F \equiv F \Omega. \tag{2.7} \]
One immediate consequence of the fact that the quantized Hamiltonian BRST operator satisfies $[\Omega, \Omega] = 2\Omega^2 = 0$, is that $\Delta$ also is nilpotent. One sees that in the case of an ordinary non-Abelian Lie algebra the general definitions (2.5) and (2.7) reproduce the $\Delta$-operator of eq. (1.7). The ordinary Batalin-Vilkovisky formalism corresponds to Abelian shift transformations
\[ G_A = -i \frac{\delta^r}{\delta \phi^A}, \tag{2.8} \]
for which the general definitions (2.3) and (1.7) lead to the usual Batalin-Vilkovisky $\Delta$-operator (1.4).

These preliminary remarks only serve as to motivate the study of higher-order $\Delta$-operators, and their associated antibrackets. They show that such higher-order $\Delta$-operators exist in the field theory context, and can be defined by a natural generalization of the Batalin-Vilkovisky $\Delta$-operator. But in what follows we shall neither make explicit use of the form (2.5), nor of the precise manner in which it gives rise to new higher-order $\Delta$-operators.
2.1 The Koszul Construction

In this subsection, let $\Delta$ denote a Grassmann-odd differential operator with the properties

$$\Delta^2 = 0 \ , \ \Delta(1) = 0 \ .$$

Motivated by the previous examples, we assume that $\Delta$ differentiates from the right. In physics, one will normally not need the case where $\Delta(1) \neq 0$, but exceptions exist, and these cases can be treated with equal ease (see below). One can also relax the condition of nilpotency without encountering difficulties.

Following Koszul [7], one can define a unique antibracket $(F, G)$, even when $\Delta$ is not of 2nd order. This is the content of eq. (1.2), which holds in all generality. The antibracket so defined is a measure of the failure of $\Delta$ to act like a graded derivation. This antibracket will automatically satisfy the exchange relation (1.3). The relation (1.6) also holds in all generality. But in general both the Leibniz rule (1.4) and the Jacobi identity (1.5) will be violated.

Koszul suggests that the antibracket derived from eq. (1.2) be used to define a “three-bracket”, which measures the failure of the antibracket $(F, G)$ to act like a derivation. This construction can proceed in an iterative way to define higher and higher antibrackets. We use the notation of ref. [7], and introduce objects $\Phi^n_\Delta$ which are directly related to the higher antibrackets. The lowest antibracket, the “one-bracket” is essentially identified with the $\Delta$-operator itself, while the higher antibrackets can be derived from it. In detail,

$$\begin{align*}
\Phi^1_\Delta(A) &= (-1)^{\epsilon A} \Delta(A) \\
\Phi^2_\Delta(A, B) &= (-1)^{\epsilon A + \epsilon B} \Delta(AB) - (-1)^{\epsilon A} \Delta(A)B - (-1)^{\epsilon A + \epsilon B} A\Delta(B) \\
\Phi^3_\Delta(A, B, C) &= (-1)^{\epsilon A + \epsilon B + \epsilon C} \Delta(ABC) - (-1)^{\epsilon A + \epsilon B + \epsilon C} A\Delta(BC) - (-1)^{\epsilon A + \epsilon B} \Delta(AB)C \\
&\quad + (-1)^{\epsilon A + \epsilon B} A\Delta(B)C - (-1)^{\epsilon A + \epsilon B + \epsilon C + \epsilon A \epsilon B} B\Delta(AC) + (-1)^{\epsilon B(\epsilon A + 1) + \epsilon A} B\Delta(A)C \\
&\quad + (-1)^{\epsilon A + \epsilon B + \epsilon C} AB\Delta(C) \\
&\vdots \quad \vdots
\end{align*}$$

All higher antibrackets are Grassmann-odd in the sense that

$$\epsilon\{\Phi^n_\Delta(A_1, \ldots, A_n)\} = \sum_{i=1}^n \epsilon A_i + 1 \ ,$$

and they satisfy a simple exchange relation:

$$\Phi^n_\Delta(A_i, \ldots, A_{i-1}, A_i, \ldots, A_n) = (-1)^{\epsilon A_{i-1} \epsilon A_i} \Phi^n_\Delta(A_i, \ldots, A_i, A_{i-1}, \ldots, A_n) \ .$$

This latter relation suggests that it is more natural to view the comma in $\Phi^n_\Delta$ as a graded (supercommutative) and associatiave product. We use this product notation in the next sections.

The usual antibracket of the Batalin-Vilkovisky formalism, the “two-bracket”, is defined by

$$(A, B) \equiv (-1)^{\epsilon A} \Phi^2_\Delta(A, B) \ .$$

Note that when the usual antibracket acts like a graded derivation, the “three-bracket” defined through $\Phi^3_\Delta$ vanishes identically.

An extra sign factor appears because our $\Delta$-operator is based on right-derivatives. To facilitate a comparison with the definitions of Koszul [7], we choose to compensate explicitly for the fact that our $\Delta$ operator is based on right derivatives. This causes some additional sign factors in the subsequent equation.
Akman [8] has organized the above definition of higher antibrackets in a very convenient iterative sequence:

\[
\begin{align*}
\Phi_\Delta^1(A) & = (-1)^{\epsilon_A} \Delta(A) \\
\Phi_\Delta^2(A, B) & = \Phi_\Delta^1(AB) - \Phi_\Delta^1(A)B - (-1)^{\epsilon_A} A \Phi_\Delta^1(B) \\
\Phi_\Delta^3(A, B, C) & = \Phi_\Delta^2(ABC) - \Phi_\Delta^2(AB)C - (-1)^{\epsilon_B} B \Phi_\Delta^2(A, C) \\
\vdots \\
\Phi_\Delta^{n+1}(A_1, \ldots, A_{n+1}) & = \Phi_\Delta^n(A_1, \ldots, A_n A_{n+1}) - \Phi_\Delta^n(A_1, \ldots, A_n) A_{n+1} \\
& \quad - (-1)^{\epsilon_{A_n}(\epsilon_{A_1} + \cdots + \epsilon_{A_{n-1}} + 1)} A_n \Phi_\Delta^n(A_1, \ldots, A_{n-1}, A_{n+1}) .
\end{align*}
\]

(2.14)

If \( \Phi_\Delta^k \) acts like a derivation, \( \Phi_\Delta^{k+1} \) vanishes identically, and the iteration terminates.

When \( \Phi_\Delta^2 \) fails to act like a derivation of the kind (1.4), it also fails to fulfill the Jacobi identity (1.3). Instead, one finds

\[
\sum_{\text{cycl.}} (-1)^{(\epsilon_A + 1)(\epsilon_C + 1)} (A, (B, C)) = (-1)^{\epsilon_A \epsilon_C + 1} \Phi_\Delta^1(\Phi_\Delta^3(A, B, C)) \\
+ \sum_{\text{cycl.}} (-1)^{\epsilon_A \epsilon_C + 1} \Phi_\Delta^3(\Phi_\Delta^1(A), B, C) .
\]

(2.15)

So \( \Phi_\Delta^1 \) equivalently measures the failure of the Jacobi identity for the usual antibracket. In terms of the \( \Phi_\Delta^n \)'s themselves, the (broken) Jacobi identity takes the form

\[
\sum_{\text{cycl.}} (-1)^{\epsilon_A \epsilon_C + 1} \Phi_\Delta^2(A, \Phi_\Delta^2(B, C)) = (-1)^{\epsilon_A \epsilon_C + 1} \Phi_\Delta^1(\Phi_\Delta^3(A, B, C)) \\
+ \sum_{\text{cycl.}} (-1)^{\epsilon_A \epsilon_C + 1} \Phi_\Delta^3(\Phi_\Delta^1(A), B, C) .
\]

(2.16)

The above construction shows explicitly that \( \Phi_\Delta^n \) can be defined directly in terms of the lowest bracket \( \Phi_\Delta^1 \). However, the defining equations are highly cumbersome when \( n \) is large, and it is therefore useful to have a more compact formulation. In order to be more precise, we will introduce some mathematical notation that turns out to be very convenient. Because we wish to compare directly with Koszul [7], we will give up the condition that the \( \Delta \)-operator is based on right-derivatives (as is more natural from the BRST-charge definition, and the Batalin-Vilkovisky formalism) and allow it to act as a higher-order left-derivative (as is more natural from the mathematical point of view). The translation between the two conventions is of course trivial. To avoid confusion, the analogous \( \Delta \)-operators will in the following be denoted by capital roman letters \( S, T, \) etc.

### 2.2 An Algebraic Definition

Let \( \mathcal{A} \) be a supercommutative algebra with unit \( 1 \) over the complex field \( C \). Furthermore, let \( T\mathcal{A} \) denote the tensor algebra of \( \mathcal{A} \):

\[
T\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}^{\otimes n} = C + \mathcal{A} + \mathcal{A} \otimes \mathcal{A} + \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} + \ldots .
\]

(2.17)

We distinguish between the unit element in the algebra \( 1 \in \mathcal{A} \) and the unit element in the field \( 1 \in C \) by using boldface type for the algebra unit. Note in particular that \( 1 \otimes A = 1 \cdot A = A \in \mathcal{A} \), but \( 1 \otimes A \in \mathcal{A} \otimes \mathcal{A} \) for an element \( A \in \mathcal{A} \).

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4 This result has a well-known analogy in the theory of even (Poisson) brackets.
The quotient algebra $S \mathcal{A} = T \mathcal{A} / I$ is the (super)symmetrized tensor algebra of $\mathcal{A}$, where $I$ denotes the two-sided ideal generated by the (super)commutator, i.e. elements of the form:

$$[A \otimes B] \equiv A \otimes B - (-1)^{\epsilon A \epsilon B} B \otimes A , \ A, B \in \mathcal{A} .$$

(2.18)

We will mainly work in the (super)symmetrized tensor algebra $S \mathcal{A}$, which by construction is an associative and supercommutative algebra with respect to the tensor product $\otimes$:

$$A \otimes B = (-1)^{\epsilon A \epsilon B} B \otimes A .$$

(2.19)

It would actually be interesting to do the construction for an associative but non-commutative algebra $\mathcal{A}$, and without super-symmetrizing with respect to the tensor product. But for the sake of clarity we will for the moment assume graded commutativity, and we will also (super)symmetrize the tensor product.

Besides, without guidance from physics it is not obvious which of the many ways of generalizing to the non-commutative case we should choose. Akman [8] has provided a most natural definition, which turns out to coincide with a certain expression in terms of supercommutators which we will provide below.

Define a multiplication map $\sim : S \mathcal{A} \to \mathcal{A}$, which takes tensor product $\otimes$ into the product “·” of the algebra $\mathcal{A}$:

$$\tilde{1} = 1 , \quad \tilde{A} = A ,$$

$$\sim (A_1 \otimes \ldots \otimes A_n) = A_1 \cdot \ldots \cdot A_n .$$

(2.20)

For each linear operator $T : \mathcal{A} \to \mathcal{A}$ the composed map $T \circ \sim : S \mathcal{A} \to \mathcal{A}$ is also, in a slight abuse of notation, denoted by $T$. In particular, we point out that with this definition $T(1) = T(1)$.

At this stage define a co-multiplication (cf. [7]) $\lambda : S \mathcal{A} \to S \mathcal{A} \times S \mathcal{A}$

$$\lambda(1) = (1,1) ,$$

$$\lambda(A_1 \otimes \ldots \otimes A_n) = ((A_1,1) - (1,A_1)) \otimes \ldots \otimes ((A_n,1) - (1,A_n)) .$$

(2.21)

Here $S \mathcal{A} \times S \mathcal{A}$ is equipped with a graded product $\otimes$:

$$(A,B) \otimes (C,D) = (-1)^{\epsilon B \epsilon C} (A \otimes C, B \otimes D) .$$

(2.22)

We can understand the curious sign-factor as originating from permuting $B$ and $C$. $S \mathcal{A} \times S \mathcal{A} \cong S \mathcal{A} \otimes S \mathcal{A}$ has a canonical map onto $S \mathcal{A}$, where the cross product $\times$ is substituted with the tensor product $\otimes$.

We now define a map $\Phi_T : S \mathcal{A} \to S \mathcal{A}$ for a linear operator $T$ as

$$\Phi_T \equiv (T \times \text{Id}_{S \mathcal{A}}) \circ \lambda .$$

(2.23)

In this way $\Phi_T(1) = T(1)$, while $\Phi_T(1) = T(1) - T(1) \otimes 1$. The operator $T$ only operates on the first copy of $S \mathcal{A}$ in $S \mathcal{A} \times S \mathcal{A}$ while leaving the second copy untouched. We can invoke this action for practical calculations with the help of an omit operator $\wedge_T : S \mathcal{A} \to S \mathcal{A}$

$$\tilde{1}^T = 1 ,$$

$$(A \otimes B)^{\wedge_T} = \tilde{A}^T \otimes \tilde{B}^T ,$$

$$T(A \otimes \tilde{B}^T) = (T(A), B) \approx T(A) \otimes B .$$

(2.24)
So whenever an argument of $T$ is decorated with the omit-operator, the argument should be removed from the argument-list of $T$, and appear outside to the right (or left) instead. We emphasize that the omit-operation in general involves a sign factor. For instance,

$$T(\hat{A}^T \otimes B) = (-1)^{\epsilon_{AB}} T(B) \otimes A . \quad (2.25)$$

With this definition we can write

$$\Phi_T(A_1 \otimes \ldots \otimes A_n) = T((A_1 - \hat{A}_1^T) \otimes \ldots \otimes (A_n - \hat{A}_n^T))$$

$$= \sum_{i_1, \ldots, i_n = 0}^1 (-1)^{\sum_{j > k} \epsilon_{A_j} \epsilon_{A_k} (1 - i_k)} T(A_{i_1}^i \otimes \ldots \otimes A_{i_n}^n) \otimes (-A_1)^{1 - i_1} \otimes \ldots \otimes (-A_n)^{1 - i_n} . \quad (2.26)$$

We have here employed the obvious conventions $A^0 \equiv 1$ and $A^1 \equiv A$. A useful way of writing this is

$$\Phi_T(A_1 \otimes \ldots \otimes A_n) = \left[ \ldots [\hat{T} \otimes A_1] \otimes \ldots \otimes \hat{A}_n \right] 1 , \quad (2.27)$$

where $\hat{T}$ operates on every argument to the right.

At the present stage the connection between the map $\Phi_T$ and the corresponding higher antibrackets $\Phi^n_T$ may not yet be obvious. Roughly, the commas used to separate the entries in the higher antibrackets in the previous subsection have been replaced by the tensor products here. This is of course only a matter of notation, and clearly immaterial. (And we shall freely alternate between the two ways of writing it).

To see that we are really very close to having defined the higher antibrackets $\Phi^n_T$, let us evaluate the lowest cases of $\Phi_T$:

$$\Phi_T(1) = T(1)$$

$$\Phi_T(A) = \left[ \hat{T} \otimes A \right] 1 = T(A) - T(1) \otimes A$$

$$\Phi_T(A \otimes B) = \left[ \hat{T} \otimes A \right] \otimes B \otimes 1$$

$$= T(A \otimes B) - T(A) \otimes B - (-1)^{\epsilon_{AB}} T(B) \otimes A + T(1) \otimes A \otimes B$$

$$\Phi_T(A \otimes B \otimes C) = \left[ \left[ \hat{T} \otimes A \right] \otimes B \right] \otimes C \otimes 1$$

$$= T(A \otimes B \otimes C) - T(A \otimes B) \otimes C - (-1)^{\epsilon_{AB}} T(B) \otimes C \otimes A$$

$$- (-1)^{\epsilon_{AB}} T(C) \otimes A \otimes B + T(A) \otimes B \otimes C + (-1)^{\epsilon_{AB}} T(B) \otimes C \otimes A$$

$$+ (-1)^{\epsilon_{AB}} T(C) \otimes A \otimes B - T(1) \otimes A \otimes B \otimes C . \quad (2.28)$$

The higher antibracket $\Phi^n_T : S^n A \to A$ of order $n$ is now finally defined by

$$\Phi^n_T = \Phi_T \bigg|_{S^n A} \equiv \Phi_T \circ \pi_{S^n A} . \quad (2.29)$$

This means that

$$\Phi^n_T(A_1 \otimes \ldots \otimes A_n) = \left( T((A_1 - \hat{A}_1^T) \otimes \ldots \otimes (A_n - \hat{A}_n^T)) \right)^n$$

$$= \sum_{i_1, \ldots, i_n = 0}^1 (-1)^{\sum_{j > k} \epsilon_{A_j} \epsilon_{A_k} (1 - i_k)} T(A_{i_1}^{i_1} \ldots A_{i_n}^{i_n})(-A_1)^{1 - i_1} \ldots (-A_n)^{1 - i_n} . (2.30)$$

We emphasize a particular useful representation of $\Phi^n_T$:

$$\Phi^n_T(A_1 \otimes \ldots \otimes A_n) = \left[ \ldots \left[ \hat{T} \otimes A_1 \right], \ldots, A_n \right] 1 . \quad (2.31)$$
This immediately leads to the following recursion relation:

\[ \Phi_T^{n+1}(A_1 \otimes \ldots \otimes A_{n+1}) = \Phi_T^n(A_1 \otimes \ldots \otimes A_n A_{n+1}) - \Phi_T^n(A_1 \otimes \ldots \otimes A_n) A_{n+1} - (-1)^{\epsilon A_n} \epsilon A_{n+1} \Phi_T^n(A_1 \otimes \ldots \otimes A_{n-1} \otimes A_{n+1}) A_n. \]  

(2.32)

which agrees with that of eq. (2.14).

Finally, let us evaluate some of the lowest cases:

\[
\begin{align*}
\Phi_T^0(1) &= T1 = T(1) \\
\Phi_T^1(A) &= [T, A]1 = T(A) - T(1)A \\
\Phi_T^2(A \otimes B) &= [[T, A], B]1 = T(AB) - T(A)B - (-1)^{\epsilon A\epsilon B} T(B)A + T(1)AB.
\end{align*}
\]

(2.33)

Specializing to the case of \( T(1) = T(1) = 0 \), this definition is seen to agree with the one of eq. (2.14), once translated into an operator \( T \) differentiating from the left. The more general definition with \( T(1) \) not necessarily vanishing can of course (since the above considerations are based on Koszul’s construction) be found in ref. [7] as well.

Normally, \( T \) is a differential operator. Note that if \( T \) is a (left) multiplication operator, then all brackets vanish identically, except for the zero bracket.

It may also be of interest to note that it is possible to invert the relation between the operator \( T \) and \( \Phi_T \). One way is to project \( \Phi_T \) into the algebra \( A \) itself: \( \pi_A \circ \Phi_T = T \). The following relations hold in the tensor algebra as well:

\[
\Phi_T((A_1 + \hat{A}_1) \Phi_T) \otimes \ldots \otimes (A_n + \hat{A}_n \Phi_T)) = T(A_1 \otimes \ldots \otimes A_n) = \tilde{T}(A_1 \otimes \ldots \otimes A_n) = \left( \Phi_T((A_1 + \hat{A}_1 \Phi_T) \otimes \ldots \otimes (A_n + \hat{A}_n \Phi_T)) \right) \sim .
\]

(2.34)

### 2.3 The Strongly Homotopy Lie Algebra

There is an intriguing connection between the algebra of higher antibrackets based on Grassmann odd and nilpotent operators, and strongly homotopy Lie algebras [8][9].

**LEMMA:** Let \( S, T \in \text{Hom}_C(A, A) \) and assume \( A \) is an algebra (and hence with a product). Then

\[ \Phi_{ST} = \Phi_S \circ b_{\Phi_T}^{-1} + \left| \Phi_S, \Phi_T \right| \]  

(2.35)

and (by operating with tilde on both sides)

\[ \tilde{\Phi}_{ST} = \tilde{\Phi}_S \circ b_{\Phi_T}^{-1} + \left| \tilde{\Phi}_S, \tilde{\Phi}_T \right| \]  

(2.36)

Here the co-derivation \( b_{\Phi_T}^{-1} \) is defined as

\[ b_{\Phi_T}^{-1}(A_1 \otimes \ldots \otimes A_n) = \sum_{i_1, \ldots, i_n=0}^1 (-1)^{\epsilon A_j^i (1-i_k)} \tilde{\Phi}_T(A_i^i \otimes \ldots \otimes A_n^i) \otimes A_1^{1-i_1} \otimes \ldots \otimes A_n^{1-i_n}. \]  

(2.37)

The Lemma also contain the first example of a bracket-brackets \( \left| \Phi_S, \Phi_T \right| \):

\[ \left| \Phi_S, \Phi_T \right| A_1 \otimes \ldots \otimes A_n \]  

\[ = \sum_{r=0}^n \frac{1}{r!(n-r)!} \sum_{\pi \in S_n} (-1)^{\epsilon \sigma + \epsilon T(\epsilon A_{\pi(1)} + \ldots + \epsilon A_{\pi(r)})} \Phi_S(A_{\pi(1)} \otimes \ldots \otimes A_{\pi(r)}) \]
\[ \otimes \Phi_T^{n-r}(A_{\pi(r+1)} \otimes \ldots \otimes A_{\pi(n)}) \, . \tag{2.38} \]

This is the simplest of an infinite tower of bracket-brackets. One can associate a tilded pendant
\[ \{ \tilde{\Phi}_S, \tilde{\Phi}_T \} = |\Phi_S, \Phi_T\rangle \, . \tag{2.39} \]

We refer to appendix A and B for a throughout presentation of co-derivation and bracket-brackets. Here we will merely note that the second term in (2.35) with these generalizations can take the following disguises:
\[ \{ \tilde{\Phi}_S, \tilde{\Phi}_T \} = \{ \Phi_S, \Phi_T \} = \left\{ \begin{array}{ccc} S & \text{Id}_{S_A} & \mid \sum & T & \text{Id}_{S_A} & \mid & S & \text{Id}_{S_A} & T & \text{Id}_{S_A} \end{array} \right\} = \left\{ \begin{array}{ccc} S & 1 & 1 \end{array} \right\} . \tag{2.40} \]

Let us insert arguments \( A_1, \ldots, A_n \). The lemma can then be stated as
\[ \Phi_{ST}(A_1 \otimes \ldots \otimes A_n) = \sum_{r=0}^{n} \frac{1}{r!(n-r)!} \sum_{\pi \in S_n} (-1)^{\epsilon_S} \left( \Phi_T(A_{\pi(1)} \otimes \ldots A_{\pi(n)}) \otimes A_{\pi(r+1)} \otimes \ldots \otimes A_{\pi(n)} \right) \]
\[ + \sum_{r=0}^{n} \frac{1}{r!(n-r)!} \sum_{\pi \in S_n} (-1)^{\epsilon_S+\epsilon_r(\epsilon_{\pi(1)}+\ldots+\epsilon_{\pi(r)})} \Phi_S(A_{\pi(1)} \otimes \ldots A_{\pi(r)}) \otimes \Phi_T^{n-r}(A_{\pi(r+1)} \ldots \otimes A_{\pi(n)}) \, . \tag{2.41} \]

\( \epsilon_\pi \) is the Grassmann parity originating from permuting Grassmann graded quantities:
\[ A_{\pi(1)} \ldots A_{\pi(n)} = (-1)^{\epsilon_\pi} A_1 \ldots A_n \tag{2.42} \]

**Proof of lemma:** It is clearly enough to prove the lemma for bosonic arguments \( A_1, \ldots, A_n \). The first term on the righthand side is:
\[ \sum_{r=0}^{n} \frac{1}{r!(n-r)!} \sum_{\pi \in S_n} \Phi_S \left( \Phi_T(A_{\pi(1)} \otimes \ldots A_{\pi(r)}) \otimes A_{\pi(r+1)} \otimes \ldots \otimes A_{\pi(n)} \right) \]
\[ = \sum_{i_1, \ldots, i_n = 0} \Phi_S \left( \Phi_T(A_{\pi(1)} \otimes \ldots A_{\pi(n)}) \otimes A_{\pi(1)}^{i_1} \otimes \ldots \otimes A_{\pi(n)}^{i_n} \right) \]
\[ = \sum_{i_1, \ldots, i_n = 0} \sum_{j_1 = 0}^{i_1} \ldots \sum_{j_n = 0}^{i_n} \Phi_S \left( T(A_{\pi(1)} \otimes \ldots A_{\pi(n)}) (-A_1)^{i_1} \ldots A_{n}^{i_n} \right) \]
\[ = \sum_{i_1, \ldots, i_n = 0} \sum_{j_1 = 0}^{i_1} \ldots \sum_{j_n = 0}^{i_n} \sum_{k_1 = 0}^{k_1} \ldots \sum_{k_n = 0}^{k_n} \left( T(A_{\pi(1)} \otimes \ldots A_{\pi(n)}) (-A_1)^{i_1} \ldots A_{n}^{i_n} \right) \]
\[ \times (-A_1)^{j_1} \ldots (-A_n)^{j_n} \otimes A_1^{i_1} \otimes \ldots \otimes A_n^{i_n} \]
\[ = (k_0 = 0)\text{-terms} + (k_0 = 1)\text{-terms} . \tag{2.43} \]

It is straight forward to see that the \((k_0 = 1)\text{-terms}\) are the left hand side of the lemma:
\[ (k_0 = 1)\text{-terms} = \sum_{i_1, \ldots, i_n = 0} \sum_{j_1 = 0}^{i_1} \ldots \sum_{j_n = 0}^{i_n} \sum_{k_1 = 0}^{k_1} \ldots \sum_{k_n = 0}^{k_n} \left( T(A_{\pi(1)} \otimes \ldots A_{\pi(n)}) \right) \]
\[ \times (-A_1)^{j_1} \ldots (-A_n)^{j_n} \otimes A_1^{i_1} \otimes \ldots \otimes A_n^{i_n} \]
\[ \otimes (-A_1)^{j_1} \otimes \ldots \otimes (-A_n)^{j_n} . \]
\[ S_n = \begin{cases} 1; & n = 2; \\ 0; & \text{otherwise.} \end{cases} \]

due to a cancellation between terms in which \( S \) is not operating directly on \( T \). Note that in case of \( k_0 = 0 \) the \( S \)- and \( T \)-expressions are always multiplied. The \( (k_0 = 0) \)-terms are minus the second term on the righthand side in the lemma:

\[ \begin{align*}
- (k_0 = 0) & \text{-terms} = \sum_{\ell_1, \ldots, \ell_n = 0}^{1} \sum_{i_1 = 0}^{i_1} \cdots \sum_{j_n = 0}^{j_n} \sum_{k_0 = 0}^{k_0} S(A_1^{(1-i_1)(1-k_1)} \otimes \cdots \otimes A_n^{(1-i_n)(1-k_n)}) \\
& \quad \otimes T(A_1^{i_1j_1} \cdots A_n^{i_nj_n}(-A_1)^{i_1(1-j_1)} \cdots (-A_n)^{i_n(1-j_n)}) \\
& \quad \otimes (-A_1)^{(1-i_1)k_1} \otimes \cdots \otimes (-A_n)^{(1-i_n)k_n} \\
& = \sum_{\ell_1, \ldots, \ell_n = 0}^{1} \Phi_S(A_1^{\ell_1} \otimes \cdots \otimes A_n^{\ell_n}) \otimes \Phi_T(A_1^{1-\ell_1} \otimes \cdots \otimes A_n^{1-\ell_n}) \\
& = \Phi_\Phi(S \otimes T) A_1 \otimes \cdots \otimes A_n \}.
\end{align*} \]

An anti-supersymmetrization in \( S \) and \( T \) of the tilded version of the lemma cause the second terms to drop out:

\[ \tilde{\Phi}_{[S,T]} = \tilde{\Phi}_{[S \circ b_{[T]}}, \]

or equivalently, with arguments \( A_1, \ldots, A_n \) inserted:

\[ \Phi_{[S,T]}^n(A_1 \otimes \cdots \otimes A_n) = \sum_{r=0}^{n} \frac{1}{r!(n-r)!} \sum_{\pi \in \mathcal{S}_n} (-1)^{\epsilon_r} \Phi_{[S,T]}^{n-r+1}(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(r)} \otimes A_{\pi(r+1)} \otimes \cdots \otimes A_{\pi(n)}) \] .

This contains the main identities for strongly homotopy Lie algebras. (We borrow the terminology “main identity” from closed string field theory \[10\], where analogous expressions play an important rôle; see section 3). Let us write out the first few identities.

\[ n = 0: \]

\[ \Phi_{[S,T]}^0 = \Phi_{[S \circ b_{[T]}}}^0. \]

\[ n = 1: \]

\[ \Phi_{[S,T]}^1(A) = \Phi_{[S,T]}^0(\Phi_{[T]}(A) \otimes A) + \Phi_{[S,T]}^1(A). \]

\[ n = 2: \text{ Leibnitz rule for a (not necessarily odd) Laplacian and associated (anti)bracket} \]

\[ \Phi_{[S,T]}^2(A_1 \otimes A_2) = \Phi_{[S,T]}^3(\Phi_{[T]}^0(A_1 \otimes A_2) \\
+ \sum_{\pi \in \mathcal{S}_2} (-1)^{\epsilon_r} \Phi_{[S,T]}^1(A_{\pi(1)} \otimes A_{\pi(2)}) \\
+ \Phi_{[S,T]}^2(\Phi_{[T]}^0(A_1 \otimes A_2)) \\
= \Phi_{[S,T]}^3(\Phi_{[T]}^0 \otimes A_1 \otimes A_2) \]

\[ ^5 \text{Here, and throughout our paper, } [A,B] \text{ denotes the graded commutator: } [A,B] \equiv AB - (-1)^{\epsilon_A \epsilon_B} BA. \]
\[ + \Phi_{[S}^2 \left( \Phi_{T]}^1 (A_1) \otimes A_2 \right) \]
\[ + ( -1 )^{\epsilon_{A_1} \epsilon_{A_2}} \Phi_{[S}^2 \left( \Phi_{T]}^1 (A_2) \otimes A_1 \right) \]
\[ + \Phi_{[S}^1 \left( \Phi_{T]}^2 (A_1 \otimes A_2) \right) . \]  

(2.50)

\[ n = 3: \text{ Jacobi identity} \]
\[ \Phi_{[S,T]}^3 (A_1 \otimes A_2 \otimes A_3) = \Phi_{[S}^3 \left( \Phi_{T]}^0 \otimes A_1 \otimes A_2 \otimes A_3 \right) \]
\[ + \frac{1}{2} \sum_{\pi \in S_3} ( -1 )^{\epsilon_{\pi}} \Phi_{[S}^1 \left( \Phi_{T]}^1 (A_{\pi(1)}) \otimes A_{\pi(2)} \otimes A_{\pi(3)} \right) \]
\[ + \frac{1}{2} \sum_{\pi \in S_3} ( -1 )^{\epsilon_{\pi}} \Phi_{[S}^2 \left( \Phi_{T]}^2 (A_{\pi(1)}) \otimes A_{\pi(2)} \otimes A_{\pi(3)} \right) \]
\[ + \Phi_{[S}^1 \left( \Phi_{T]}^3 (A_1 \otimes A_2 \otimes A_3) \right) \]
\[ = \Phi_{[S}^4 \left( \Phi_{T]}^0 \otimes A_1 \otimes A_2 \otimes A_3 \right) \]
\[ + \Phi_{[S}^3 \left( \Phi_{T]}^1 (A_1) \otimes A_2 \otimes A_3 \right) \]
\[ + ( -1 )^{\epsilon_{A_1} (\epsilon_{A_2} + \epsilon_{A_3})} \Phi_{[S}^3 \left( \Phi_{T]}^1 (A_2) \otimes A_3 \otimes A_1 \right) \]
\[ + ( -1 )^{(\epsilon_{A_1} + \epsilon_{A_2}) \epsilon_{A_3}} \Phi_{[S}^3 \left( \Phi_{T]}^1 (A_3) \otimes A_1 \otimes A_2 \right) \]
\[ + \Phi_{[S}^2 \left( \Phi_{T]}^2 (A_1 \otimes A_2) \otimes A_3 \right) \]
\[ + ( -1 )^{\epsilon_{A_1} (\epsilon_{A_2} + \epsilon_{A_3})} \Phi_{[S}^2 \left( \Phi_{T]}^2 (A_2 \otimes A_3) \otimes A_1 \right) \]
\[ + ( -1 )^{(\epsilon_{A_1} + \epsilon_{A_2}) \epsilon_{A_3}} \Phi_{[S}^2 \left( \Phi_{T]}^2 (A_3 \otimes A_1) \otimes A_2 \right) \]
\[ + \Phi_{[S}^1 \left( \Phi_{T]}^3 (A_1 \otimes A_2 \otimes A_3) \right) . \]  

(2.51)

It is quite amazing that the main identities for strongly homotopy Lie algebras, which in closed string field theory rely on non-trivial geometric properties in moduli space [10], here can be derived as a purely algebraic result due to an assumed existence of a product (so that \( \mathcal{A} \) is an algebra, and not just a vector space). If one does not assume the existence of this product, one can reformulate the right hand side of the main identity (2.44) in terms of nilpotency of co-derivations \( b_{\Phi_T} \):

\[ \tilde{\Phi}_{[S} \circ b_{\Phi_T]} = 0 \iff b_{\tilde{\Phi}_{[S} \circ b_{\Phi_T]} = 0 \iff b_{\tilde{\Phi}_{[S} \circ b_{\Phi_T]} = 0 \]  

(2.52)

This follows quite easily from (A.16) and (A.17).

2.4 Coordinate Representation

We will now translate the above construction into a description with explicitly chosen coordinates. Let \( \{ e_a | a \in I \} \) denote a vector basis for \( \mathcal{A} \), and \( \{ \eta^a | a \in I \} \) the dual basis in \( \mathcal{A}^* \), so that

\[ \eta^a (e_b) = \delta_b^a \]  

(2.53)

Without loss of generality we can take the coordinates \( A^a \) of a general element \( A = \sum_a A^a e_a \) to be bosonic, i.e. the basis vectors are supposed to carry the Grassmann grading. Purchasing further the vector space structure of \( \mathcal{A} \), one can identify the space \( \text{Hom}_C(S^n \mathcal{A}, S^m \mathcal{A}) \) of linear operators : \( S^n \mathcal{A} \rightarrow S^m \mathcal{A} \), with \( S^m \mathcal{A} \otimes S^n(\mathcal{A}^*) \), the set of \( S^m \mathcal{A} \)-valued homogeneous polynomials in \( \mathcal{A} \) of degree \( n \):

\[ T^{(m,n)} = e_{a_1} \ldots e_{a_m} T_{b_1 \ldots b_n}^{a_1 \ldots a_m} \eta^{b_1} \ldots \eta^{b_n} . \]  

(2.54)
Here

\[ e_{a_1} \ldots e_{a_m} \equiv e_{a_1} \otimes \ldots \otimes e_{a_m} \in S^m \mathcal{A} \]

\( \eta^{b_1} \ldots \eta^{b_n} \in S^n(\mathcal{A})^* \cong S^n(\mathcal{A}^*) \)

\[ \eta^{b_1} \ldots \eta^{b_n} (e_{a_1} \ldots e_{a_m}) \equiv \begin{cases} (-1)^{\epsilon_b} \sum_{\pi \in S_m} (-1)^{\epsilon_a} \delta_{a_{x(1)}}^{b_1} \ldots \delta_{a_{x(m)}}^{b_n} & \text{for } n = m \\ 0 & \text{otherwise} \end{cases} \] (2.55)

and \( \epsilon_{\pi} \) is the Grassmann parity originating from permuting the Grassmann-graded quantities:

\[ e_{a_{(1)}} \ldots e_{a_{(n)}} = (-1)^{\epsilon_a} e_{a_1} \ldots e_{a_n}. \] (2.56)

\[ \epsilon_a = \sum_{i>j} \epsilon_{a_i} \epsilon_{a_j} \pmod{2}. \] (2.57)

To avoid the sign-factor \( \epsilon_b \) appearing in (2.55), it is convenient to define a contraction symbol which first organizes all basis vectors \( e_a \) to the right and all dual vectors \( \eta^a \) to the left, and then contracts:

\[ [\eta^{a_1} e_{b_1} \ldots \eta^{a_n} e_{b_n}] \equiv (-1)^{\epsilon_a} \eta^{a_1} \ldots \eta^{a_n} (e_{b_1} \ldots e_{b_n}) = \sum_{\pi \in S_n} (-1)^{\epsilon_\pi} \delta_{b_{x(1)}}^{a_1} \ldots \delta_{b_{x(n)}}^{a_n}. \] (2.58)

In other words, the objects (super)commute freely under this contraction symbol. For fixed set of vectors \( e_{a_1}, \ldots, e_{a_m} \), note that the norm of a contraction is (no sum over \( a_1, \ldots, a_m \)):

\[ [\eta^{a_1} e_{a_1} \ldots \eta^{a_m} e_{a_m}] = \begin{cases} m_1! \ldots m_r! \\ 0 \end{cases} \] (2.59)

where \( m_1, \ldots, m_r \) are the multiplicities of the vectors in the set \( \{e_{a_1}, \ldots, e_{a_m}\} \). \( (m_1 + \ldots + m_r = m) \).

The second alternative in eq. (2.59) simply occurs when Grassmann odd vectors have multiplicity \( > 1 \). If all vectors are odd the norm is therefore either 0 or 1.

Next define a “symmetrizer projection operator” by

\[ P_{b_1 \ldots b_n}^{a_1 \ldots a_n} = \frac{1}{n!} \sum_{\pi \in S_n} (-1)^{\epsilon_\pi} \delta_{a_{x(1)}}^{b_1} \ldots \delta_{a_{x(n)}}^{b_n} = \frac{1}{n!} [\eta^{a_1} e_{b_1} \ldots \eta^{a_n} e_{b_n}] . \] (2.60)

\[ P_{b_1 \ldots b_n}^{a_1 \ldots a_n} P_{c_1 \ldots c_n}^{b_1 \ldots b_n} = P_{c_1 \ldots c_n}^{a_1 \ldots a_n} , \quad (-1)^{\epsilon_a} P_{b_1 \ldots b_n}^{a_1 \ldots a_n} = (-1)^{\epsilon_b} P_{b_1 \ldots b_n}^{a_1 \ldots a_n} , \] (2.61)

\[ P_{b_1 \ldots b_n}^{a_1 \ldots a_n} e_{a_1} \ldots e_{a_1} = e_{b_1} \ldots e_{b_n} , \quad P_{b_1 \ldots b_n}^{a_1 \ldots a_n} \eta_{b_1} \ldots \eta_{b_n} = \eta_{a_1} \ldots \eta_{a_n} . \]

Define the (super)symmetrized coefficients of an operator \( T \) by

\[ (T^{\text{sym}})_{b_1 \ldots b_n}^{a_1 \ldots a_m} \equiv P_{c_1 \ldots c_m}^{a_1 \ldots a_m} T_{d_1 \ldots d_n}^{c_1 \ldots c_m} P_{b_1 \ldots b_n}^{d_1 \ldots d_n} = \frac{(-1)^{\epsilon_a + \epsilon_b}}{n! \cdot m!} \eta^{a_1} \ldots \eta^{a_m} (T(e_{b_1} \ldots e_{b_n})) . \] (2.62)

In case of symmetric coefficients this yields an inversion of eq. (2.54):

\[ (T^{\text{sym}})_{b_1 \ldots b_n}^{a_1 \ldots a_m} = T_{b_1 \ldots b_n}^{a_1 \ldots a_m} . \] (2.63)
The composition of two operators $S, T \in \text{Hom}_C(SA, SA)$ is then

$$S \circ T = \sum_{k,\ell,m,n=0}^{\infty} e_{a_1} \ldots e_{a_k} S^{a_1 \ldots a_k}_{b_1 \ldots b_\ell} \left( \eta^{b_1} \ldots \eta^{b_\ell} (e_{c_1} \ldots e_{c_m}) \right) T^{c_1 \ldots c_m}_{d_1 \ldots d_n} \eta^{d_1} \ldots \eta^{d_n},$$

or, in terms of coefficients,

$$((S \circ T)^\text{sym})^{a_1 \ldots a_m}_{b_1 \ldots b_n} = \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{\epsilon}{\ell!} \left( S^{\text{sym}} \right)^{a_1 \ldots a_m}_{c_1 \ldots c_\ell} (T^{\text{sym}})^{c_1 \ldots c_\ell}_{b_1 \ldots b_n}.$$  

(2.64)

Let us now define a normal ordering in which all basis vectors $e_a$ are moved to the left and all dual vectors $\eta^a$ are moved to the right, while respecting the Grassmann grading:

$$:e_a \eta^b : \equiv e_a \eta^b, \quad :\eta^a e_b : \equiv (-1)^{\epsilon_a \epsilon_b} e_b \eta^a.$$  

(2.65)

We can then write

$$\text{Id}_{SA} = : \exp(e_a \eta^a) : = \sum_{k=0}^{\infty} \frac{1}{k!} : e_{a_1} \eta^{a_1} \ldots e_{a_k} \eta^{a_k} : = \sum_{k=0}^{\infty} (-1)^{\epsilon_a} \frac{\epsilon}{k!} e_{a_1} \ldots e_{a_k} \eta^{a_1} \ldots \eta^{a_k},$$

and co-derivation (cf. eq. (A.14-A.15))

$$b_T = : T \exp(e_a \eta^a) : = \sum_{k,n=0}^{\infty} \frac{1}{k!} : e_a T^{a_1 \ldots a_n}_{a_1 \ldots a_n} \eta^{a_1} \ldots \eta^{a_n} e_{b_1} \eta^{b_1} \ldots e_{b_k} \eta^{b_k} : .$$

(2.66)

Note that the particular bracket $|T_1, \ldots, T_k|$ defined in eq. (A.13) is just the normal-ordered product:

$$|T_1, \ldots, T_k| = : T_1 \ldots T_k : .$$

(2.67)

We can represent the dual basis vectors $\eta^a$ by a left derivative acting to the right:

$$\eta^a = \frac{\delta^l}{\delta e_a}.$$  

(2.68)

or analogously represent the basis vectors $e_a$ by a right derivative acting to the left:

$$e_a = \frac{\delta^r}{\delta \eta^a}.$$  

(2.69)

Then the contraction (2.55) can be written

$$\eta^{b_1} \ldots \eta^{b_n} (e_{a_1} \ldots e_{a_m}) = \left[ \begin{array}{c} \delta^l \vdots \delta^l \\ \delta e_{b_1} \ldots \delta e_{b_n}, e_{a_1} \ldots e_{a_m} \end{array} \right]_{\epsilon=0} = \left[ \begin{array}{c} \eta^{b_1} \ldots \eta^{b_n}, \delta^r \vdots \delta^r \\ \delta \eta^{a_1} \ldots \delta \eta^{a_m} \end{array} \right]_{\eta=0}. $$

(2.70)

The conditions $\epsilon = 0$ resp. $\eta = 0$ simply ensure that the contraction is non-zero only when $n = m$. Let us at this point mention a handy representation of the symmetrizer projection operator:

$$P^{a_1 \ldots a_m}_{b_1 \ldots b_n} = \frac{1}{n!} \frac{\delta^r}{\delta \eta^{b_1}} \ldots \frac{\delta^r}{\delta \eta^{b_n}} (\eta^{a_1} \ldots \eta^{a_m})_{\eta=0}.$$  

(2.71)
An operator \( T \in \text{Hom}_C(SA,A) \) with precisely one outgoing slot/entry can be represented by a vector field operating to the left:

\[
\vec{T} = \sum_{n=0}^{\infty} \delta^n T^a_{a\ldots a} \eta^a \ldots \eta^a.
\]  

(2.74)

Note also that the action of \( \circ b_T \) can be described by the vector field without letting \( \eta = 0 \):

\[
S \circ b_T = \left[ S, \vec{T} \right] = \sum_{k,\ell,n=0}^{\infty} \frac{\delta^r}{\delta \eta^{b_k}} \cdots \frac{\delta^r}{\delta \eta^{b_{\ell}}} S^{b_1 \ldots b_k}_{c_1 \ldots c_\ell} \left[ \eta^{c_1} \ldots \eta^{c_\ell}, \frac{\delta^r}{\delta \eta^{a_n}} \right] T^a_{a\ldots a} \eta^a \ldots \eta^a.
\]

(2.75)

Or, in terms of coordinates,

\[
(S \circ b_T)^{a_1 \ldots a_n}_{b_1 \ldots b_n} = \sum_{r=1}^{n} (-1)^{(e_T + e_c + e_{b_r} + \ldots + e_{b_n}) \langle e_{b_r+1} + \ldots + e_{b_{r-1}} \rangle} \sum_{\pi \in S_n} (-1)^{(e_T + e_c + e_{b_{\pi(n)}} + \ldots + e_{b_{\pi(r-1)}})} n!
\]

\[
(S \circ b_T)^{a_1 \ldots a_m}_{b_1 \ldots b_n} = \sum_{r=1}^{n} (-1)^{(e_T + e_c + e_{b_{\pi(n)}} + \ldots + e_{b_{\pi(r-1)}})} n!
\]

(2.76)

\[
((S \circ b_T)^{\text{sym}})^{a_1 \ldots a_m}_{b_1 \ldots b_n} = \sum_{r=1}^{n} \frac{1}{n!} \sum_{\pi \in S_n} (-1)^{(e_T + e_c + e_{b_{\pi(n)}} + \ldots + e_{b_{\pi(r-1)}})} n!
\]

(2.77)

This has as one important implication that (the generalized version of) the main identity (2.46) for a strongly homotopy Lie algebra can be formulated as a contraction between vector fields:

\[
\tilde{\Phi}_{[S,T]} = \tilde{\Phi}_{[S \circ b_T]} = \left[ \tilde{\Phi}_{[S]}, \tilde{\Phi}_{T} \right].
\]

(2.78)

In the last expression the larger outer square brackets denote a contraction i.e. action of the last vector field on the former, and the smaller inner square brackets means anti(super)symmetrization in \( S \) and \( T \).

The vector field is

\[
\tilde{\Phi}_T = \sum_{n=0}^{\infty} \delta^n T^a_{a\ldots a} \eta^a \ldots \eta^a.
\]

(2.79)

and \( \Phi^2_{T_{ab}} \) are usual Lie algebra structure constants. In particular, when \( S = T \) and \( T^2 = 0 \), the whole main identity of strongly homotopy Lie algebras can then be expressed as the nilpotency condition of this new vector field. A description of strongly homotopy Lie algebras in similar terms has been discussed in ref. [15]. Stasheff [14] expresses the main identity of strongly homotopy Lie algebras in an analogous way, but without going to particular coordinates.

Notice that the main identity takes the following form in terms of symmetrized components:

\[
\left( \tilde{\Phi}_{[S,T]}^{\text{sym}} \right)^a_{b_1 \ldots b_n} = \sum_{r=1}^{n} \frac{1}{n!} \sum_{\pi \in S_n} (-1)^{(e_T + e_c + e_{b_{\pi(n)}} + \ldots + e_{b_{\pi(r-1)}})} n!
\]

(2.79)
When written in this form, one also sees that the notion of strongly homotopy Lie algebras is open to a very natural generalization.

2.5 A Master Equation and the BRST Symmetry

So far all properties of the higher brackets have been derived in a general frame without any particular applications in mind. Clearly, for the usual Batalin-Vilkovisky Lagrangian quantization program, only one-brackets and two-brackets are required. This is because the BRST Ward Identities one wishes to impose on the Lagrangian path integral are Schwinger-Dyson equations. The BRST operator of Schwinger-Dyson equations can, for flat functional measures, be chosen to be Abelian [6], and the associated ∆-operator is then, as explained in section 2, of 2nd order in the appropriate representation of fields and antifields. But even in the conventional Lagrangian path integral one may wish to impose other BRST Ward Identities (subsets of the full set of Schwinger-Dyson equations), and the associated ∆-operator may then be of higher order [4, 5]. Interestingly, this imposes the formalism of higher antibrackets as the natural generalization of the Batalin-Vilkovisky scheme. Both the (quantum) Master Equation and the (quantum) BRST operator of the Batalin-Vilkovisky antifield quantization are then seen as very special cases in a much more general framework. We begin the discussion of this with a few useful relations.

We have already seen how the higher brackets can be given a nice formulation in terms of commutators (see eq. (2.33)). Let us for later convenience define a modified operator $X_{T;B_1,...,B_k}$ associated with the operator $T$:

$$\tilde{X}_{T;B_1,...,B_k}(A) = \left[\ldots \left[\tilde{T}, B_1\right], \ldots, B_k\right] A,$$

(2.81)

where $B_1,\ldots,B_k \in A$ are fixed elements. It then follows immediately that

$$X_{T;A_1,...,A_n}(1) = \Phi^n_T(A_1 \otimes \ldots \otimes A_n)$$

$$X_{X_{T;A_1,...,A_n},B_1,...,B_k} = X_{T;A_1,...,A_n,B_1,...,B_k}.$$  

(2.82)

Notice that this last relation tells us how we can generate higher and higher brackets by composition!

Consider the formal exponential function

$$e^\otimes A = \sum_{n=0}^{\infty} \frac{1}{n!} A^\otimes n = 1 + A + \frac{1}{2} A \otimes A + \frac{1}{6} A \otimes A \otimes A + \ldots \in SA.$$  

(2.83)

Using this notation, we can write down a very useful formula

$$\Phi_T(e^\otimes A \otimes B_1 \otimes \ldots \otimes B_k) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^n_T(A^\otimes n \otimes B_1 \otimes \ldots \otimes B_k)$$

$$= e^{-A} \left[\ldots \left[\tilde{T}, B_1\right], \ldots, B_k\right] e^A$$

$$= e^{-A} \tilde{X}_{T;B_1,...,B_k} e^A.$$  

(2.84)
The case \( k = 0 \) is just what we would call the quantum Master Equation

\[ \Delta \exp \left( \frac{i}{\hbar} S \right) = 0 \]  

(2.85)

associated with the operator \( T \)

\[ \Phi_T(e^\otimes A) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^n_T(A^\otimes n) = e^{-A} \rightarrow T e^A . \]  

(2.86)

Here \( A = \frac{i}{\hbar} S \) is identified with the action\(^6\) and \( T \) with the nilpotent Grassmann odd Laplacian:

\[ T(F) = (-1)^{\epsilon_F} \Delta(F) . \]  

(2.87)

Using the formalism described above, this equation is easily rewritten in terms of the higher antibrackets

\[ 0 = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i}{\hbar} \right)^n \Phi^n_T(S \otimes \ldots \otimes S) = e^{-\frac{i}{\hbar} S} \rightarrow T e^{\frac{i}{\hbar} S} \equiv M(S) . \]  

(2.88)

The name quantum Master Equation is justified by the fact that in the special case of the Abelian (and 2nd order) Schwinger-Dyson BRST operator \( \Delta \) it reduces to the Batalin-Vilkovisky quantum Master Equation. Moreover, for more general nilpotent \( \Delta \)'s it corresponds to the quantum Master Equation when requiring given subsets of this full set of equations (see ref. [3]).

For given \( B_1, \ldots, B_k \in A \), the bracket \( \Phi^n_T \) (with \( n \geq k \)) automatically generates an \( (n-k) \)-bracket:

\[ \Phi^{n-k}(A_1 \otimes \ldots \otimes A_{n-k}) = \Phi^n_T(B_1 \otimes \ldots \otimes B_k \otimes A_1 \otimes \ldots \otimes A_{n-k}) = \Phi^n_{X_{T;B_1,\ldots,B_k}}(A_1 \otimes \ldots \otimes A_{n-k}) . \]  

(2.89)

In particular, a conventional “two-antibracket” \( (A, B) \) can always be generated from the higher antibrackets. Also, the Master Equation (2.85) can in this terminology be seen as the sum of “zero-antibrackets” generated by the action \( S \) itself:

\[ \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i}{\hbar} \right)^k \Phi^0_{X_{T;S,\ldots,S}} = 0 . \]  

(2.90)

Suppose the Master Equation terminates after a finite order of terms, as happens when \( \Delta \) is of finite order:

\[ \sum_{k=0}^{N} \frac{1}{k!} \left( \frac{i}{\hbar} \right)^k \Phi^k_T(S \otimes \ldots \otimes S) = 0 . \]  

(2.91)

From the physics perspective it is more natural to view this as an expansion in \( \hbar \):

\[ \sum_{k=0}^{N} \left( \frac{\hbar}{i} \right)^k \frac{N!}{(N-k)!} \Phi^{(N-k)}_T(S \otimes \ldots \otimes S) = 0 . \]  

(2.92)

This also suggests a solution \( S \) expressed as an \( \hbar \)-expansion, beginning with the “classical action” \( S_0 \):

\[ S = S_0 + \sum_{n=1}^{\infty} \hbar^n S_n . \]  

(2.93)

\(^6\)Of course taken to be Grassmann-even.
To leading order in the expansion, this leads to the $N$-th order “classical Master Equation”,

$$
\Phi_T^N(S_0 \otimes \ldots \otimes S_0) = 0 ,
$$

while to next order in $\hbar$ we get

$$
\Phi_T^{N-1}(S_0 \otimes \ldots \otimes S_0) + i\Phi_T^N(S_1 \otimes S_0 \otimes \ldots \otimes S_0) = 0 ,
$$

and so on.

It is curious to note that when the $\Delta$-operator is of infinite order, and the full Master Equation therefore does not truncate, this solution in terms of an $\hbar$-expansion loses its meaning. The “classical” antibracket is then pushed to infinity, and the analysis must start with the lowest antibracket $\Delta$ instead.

In conventional Batalin-Vilkovisky quantization, the BRST operator is composed of two pieces, a classical part and a “quantum correction” (see, e.g., ref. [16]):

$$
\sigma^r F = (F,S) - i\hbar \Delta F .
$$

We have given $\sigma$ the superscript “$r$” to indicate that it acts with right-derivatives in our conventions (due to $\Delta$). The most obvious generalisation to the case where the three-brackets (and perhaps higher brackets as well) do not vanish, would be (rescaling with a factor $\frac{i}{\hbar}$, and converting to left derivatives):

$$
\sigma(F) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i}{\hbar} \right)^n \Phi_T^{n+1}(F \otimes S^\otimes n) = \Phi_T (F \otimes e^{\otimes \bar{\hbar}S}) = e^{-\bar{\hbar}S} \left[ T, F \right] e^{\bar{\hbar}S} .
$$

$\sigma$ can be given a meaning purely in terms of higher antibrackets. The nilpotency of $\sigma$ depends on the right to use the Master Equation $M(S) = 0$ before all differentiations are carried out (recall that the brackets in general contains differential operators):

$$
\sigma(\sigma(\epsilon)) = \Phi_T \left( \Phi_T (\epsilon \otimes e^{\otimes \bar{\hbar}S}) \otimes e^{\otimes \bar{\hbar}S} \right) = -\Phi_T \left( M(S) \otimes \epsilon \otimes e^{\otimes \bar{\hbar}S} \right) = 0 .
$$

The last equality is a consequence of the main identity (2.46):

$$
0 = \Phi_T \circ h \bar{\hbar} \Phi_T \left( \epsilon \otimes e^{\otimes \bar{\hbar}S} \right) = \Phi_T \left( \Phi_T (\epsilon \otimes e^{\otimes \bar{\hbar}S}) \otimes e^{\otimes \bar{\hbar}S} \right) + \Phi_T \left( \Phi_T (e^{\otimes \bar{\hbar}S}) \otimes \epsilon \otimes e^{\otimes \bar{\hbar}S} \right) .
$$

Let us test this generalization $\sigma$ by searching for variations $\delta S$ of the action $S$ that preserve the Master Equation. Variation $\frac{i}{\hbar} \delta S = \sigma(\epsilon)$ of the form (2.97) only preserve the Master Equation (2.85) “on-shell” (where we apply the Master Equation $M(S) = 0$ before all differentiations has been done; this terminology becomes particularly obvious in the case of string field theory – see later):

$$
\delta \left( T e^{\bar{\hbar}S} \right) = \frac{i}{\hbar} T \left( e^{\bar{\hbar}S} \delta S \right)
$$

It may look as if one can have an “off-shell” invariance of the Master Equation with respect to a slightly different type of variations:

$$
\frac{i}{\hbar} \delta S = \sigma(\epsilon) \equiv e^{-\bar{\hbar}S} T \left( \epsilon e^{\bar{\hbar}S} \right)
$$

This is however not quite true. For instance, if one varies the equivalent form (2.88) of the master equation, one gets:

$$
\delta M(S) = \frac{i}{\hbar} \left( -M(S) \delta S + e^{-\bar{\hbar}S} T e^{\bar{\hbar}S} \delta S \right) .
$$
Here variation \( \tilde{\sigma}(\epsilon) \) of the form (2.101) only preserve (2.88) “on-shell”. However the alternative \( \tilde{\sigma} \) does have the nice property that nilpotency, \( \tilde{\sigma}^2 = 0 \), is a direct consequence of \( T \) being nilpotent.

The reason why the meaning of “on-shell” and “off-shell” here becomes somewhat obscured, can be traced back to the fact that neither \( \bar{\sigma} \) nor \( \sigma \) are derivations, i.e. do not fulfill the Leibnitz rule.

Finally, let us mention that in the case of \( \sigma \), the invariance of the master equation (2.88) can be directly related to the nilpotency of \( \sigma \):

\[
\delta_S M(S) = \frac{i}{\hbar} \bar{\Phi}_T \left( \delta_S \otimes e^{\partial \Phi S} \right) = \sigma(S) \ . \tag{2.103}
\]

Both \( \sigma \) and \( \tilde{\sigma} \) can obviously be viewed as BRST symmetry operators, and, since \( S \) in the BRST context is taken to satisfy the quantum Master Equation, in fact coincide. From the BRST viewpoint the fact that deformations \( S \rightarrow S + \delta S \) of a solution \( S \) to the Master Equation still satisfy this Master Equation is seen as the possibility of adding BRST-exact terms \( \sigma(\epsilon) \) (or \( \tilde{\sigma}(\epsilon) \)) to the action.

### 2.6 The Transformation Algebra

When the BRST transformations alternatively are viewed as transformations of the action \( S \), one would like to find the possible algebra of such transformations. This has already been done in the framework of the conventional Batalin-Vilkovisky formalism by Hata and Zwiebach [2] (note that their odd Laplacian consists of left derivatives, so we denote it by \( T \), to be consistent). Letting

\[
\frac{i}{\hbar} \delta_S S = \sigma(\epsilon) = \frac{\bar{\epsilon}}{\hbar} + \frac{i}{\hbar} (S, \epsilon) \ , \tag{2.104}
\]

they find

\[
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] F(S) = \delta_{\epsilon_1} \delta_{\epsilon_2} F(S) \ , \tag{2.105}
\]

i.e., the algebra of transformations on \( S \) is just the algebra of the conventional antibracket. Here \( F \) is a general expression in \( S \). Let us consider the analogous transformation algebra in the general case. The algebra corresponding to \( \sigma \) does not close in general, but yields instead an algebra it is natural to call “open” (again a terminology motivated by closed string field theory; see ref. 10, eqs. (4.60-4.61)):

\[
[\delta_{\epsilon_2}, \delta_{\epsilon_1}] F(S) = \delta_{\epsilon_3} F(S) + \bar{\Phi}_T \left( M(S) \otimes \epsilon_1 \otimes \epsilon_2 \otimes e^{\partial \Phi S} \right) F'(S) \ , \tag{2.106}
\]

with

\[
\epsilon_3 = \bar{\Phi}_T \left( \epsilon_1 \otimes \epsilon_2 \otimes e^{\partial \Phi S} \right) \ . \tag{2.107}
\]

The additional terms on the right hand side of (2.106) are here to be understood as “equation of motion” terms, and the gauge algebra is then of the usual open kind. In the conventional case of vanishing three-bracket, the “equation of motion” term in (2.106) drops out, and (2.107) boils down to \( \epsilon_3 = \Phi_T^2(\epsilon_1 \otimes \epsilon_2) \), thereby reproducing (2.105).

The easiest way to derive eq. (2.106) is by using the main identity (2.46),

\[
0 = \bar{\Phi}_T \circ b_{\bar{\Phi}_T} \left( \epsilon_1 \otimes \epsilon_2 \otimes e^{\partial \Phi S} \right) = \bar{\Phi}_T \left( \bar{\Phi}_T \left( \epsilon_1 \otimes e^{\partial \Phi S} \right) \otimes \epsilon_2 \otimes e^{\partial \Phi S} \right) - \bar{\Phi}_T \left( \Phi_T \left( e_2 \otimes e^{\partial \Phi S} \right) \otimes \epsilon_1 \otimes e^{\partial \Phi S} \right)
\]  
\[
+ \bar{\Phi}_T \left( \Phi_T \left( e_{\epsilon_1} \otimes e^{\partial \Phi S} \right) \otimes e^{\partial \Phi S} \right) + \bar{\Phi}_T \left( \Phi_T \left( e^{\partial \Phi S} \right) \otimes \epsilon_1 \otimes e_{\epsilon_2} \otimes e^{\partial \Phi S} \right) \ , \tag{2.108}
\]
and noting that
\[ \frac{i}{\hbar} \delta_{\epsilon_2} \delta_{\epsilon_1} S = \frac{i}{\hbar} \tilde{\Phi}_T \left( \delta_{\epsilon_2} S \otimes \epsilon_1 \otimes e^{\otimes \frac{i}{\hbar} S} \right) = \tilde{\Phi}_T \left( \epsilon_2 \otimes e^{\otimes \frac{i}{\hbar} S} \right) \otimes \epsilon_1 \otimes e^{\otimes \frac{i}{\hbar} S} \right). \] (2.109)

Interestingly, the algebra can be made to close by choosing the transformations $\bar{\sigma}$ instead. As we have emphasized before, the two transformations $\bar{\sigma}$ and $\sigma$ are equal "on-shell":
\[ \bar{\sigma}(\epsilon) = \sigma(\epsilon) + M(S)\epsilon. \] (2.110)

The closed algebra corresponding to $\bar{\sigma}$ is:
\[ [\bar{\delta}_{\epsilon_2}, \bar{\delta}_{\epsilon_1}] F(S) = \bar{\delta}_{\epsilon_3} F(S), \] (2.111)
with
\[ \epsilon_3 = \frac{i}{\hbar} \epsilon_1 \bar{\delta}_{\epsilon_2} S - \frac{i}{\hbar} \epsilon_2 \bar{\delta}_{\epsilon_1} S = \epsilon_1 \bar{\sigma}(\epsilon_2) - \epsilon_2 \bar{\sigma}(\epsilon_1). \] (2.112)

This holds even without assuming nilpotency of $T$.

Having found new nilpotent operators $\sigma$ and $\bar{\sigma}$ generated by nilpotent $T$-operators, it is natural to consider the higher antibrackets generated by $\sigma$ or $\bar{\sigma}$.

\[ \Phi_n^\sigma(A_1 \otimes \ldots \otimes A_n) = \bar{\Phi}_T \left( e^{\otimes \frac{i}{\hbar} S} \otimes A_1 \otimes \ldots \otimes A_n \right). \] (2.113)

This is the natural generalisation of the BRST operator to more entries.

Apart from the zero-bracket, the two sets of higher brackets $\Phi_n^\sigma, \Phi_n^{\bar{\sigma}}$ are equal:
\[ \Phi_n^{\bar{\sigma}} = \Phi_n^\sigma, \quad n \neq 0, \] (2.114)
because the difference $\sigma - \bar{\sigma}$ is a (left) multiplication operator(cf. (2.110)). Note that
\[ \sigma = \Phi_1^\sigma = \Phi_1^{\bar{\sigma}}. \] (2.115)

The careful reader will have noticed that each time $\sigma$ was treated in the past two sections, we chose, whenever possible, arguments that did not involve the assumption of a product for the algebra $\mathcal{A}$. For instance (2.108) could be derived easier with the help of (2.113) and (2.50).

To summarize, the benefits of $\sigma$ are chiefly that it can be written purely in terms of higher brackets, i.e. without the use of a product\(^7\), while $\bar{\sigma}$ have the nicest properties with respect to nilpotency, invariance of master equation and closure of the transformation algebra.

### 2.7 Finite Transformations

If we keep the perspective that $\sigma$ and $\bar{\sigma}$ can be seen as valid deformations $\delta S$ of a solution $S$ to the Master Equation $M(S) = 0$, it is natural to ask for the analogous finite deformations of $S$. In the case of the conventional Batalin-Vilkovisky formalism, this has also been considered by Hata and Zwiebach.

\(^7\)A point crucial for understanding why it so far has been $\sigma$ only which has surfaced in closed string field theory. Interestingly, $\sigma$ is not seen as a BRST transformation in closed string field theory, but rather as a gauge transformation. We will return to this point in section 3.
Note that one only has to apply the fixed-point integral equation
\[ \delta_r A = \sigma(\epsilon) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi_T^{n+1}(\epsilon \otimes A^\otimes n) = \tilde{\Phi}_T(\epsilon \otimes e^{\otimes A}). \quad (2.116) \]

Here \( \epsilon \) and \( T \) are supposed to have the same Grassmann parity and \( A \) is bosonic. The above transformations correspond, as mentioned previously, to gauge transformations in closed string field theory \([10]\) (there with \( A = \kappa \Psi \) being a string field). The transformation parameter \( \epsilon \equiv \epsilon_0 \) can be split into a finite constant \( \epsilon_0 \) of same Grassmann parity as \( \epsilon \) and a bosonic infinitesimal parameter \( dt \). We want to integrate up this expression to finite transformations. It follows that we have a 1. order initial value problem:

\[ \frac{d}{dt}A(t) = \tilde{\Phi}_T(\epsilon_0 \otimes e^{\otimes A(t)}) \]
\[ A(t = 0) = A_0. \quad (2.117) \]

This can be rewritten as an integral equation

\[ A(t) = A_0 + \int_0^t ds \tilde{\Phi}_T(\epsilon_0 \otimes e^{\otimes A(s)}). \quad (2.118) \]

Let us define \( a(t) \equiv e^{\otimes A(t)} \). Exponentiating the integral equation yields:

\[ a(t) = a_0 \otimes \exp \int_0^t ds \tilde{\Phi}_T(\epsilon_0 \otimes a(s)). \quad (2.119) \]

Iterating this “fixed-point integral equation” infinitely many times gives:

\[ a(t_1) = a_0 \otimes \exp \int_0^{t_1} dt_2 \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \exp \int_0^{t_2} dt_3 \tilde{\Phi}_T \left( \epsilon_0 \otimes \ldots \right. \right. \]
\[ \ldots \otimes a_0 \otimes \exp \int_0^{t_n} \right. dt_{n+1} \tilde{\Phi}_T \left( \epsilon_0 \otimes \ldots \right) \right) \quad (2.120) \]

Projecting \( A(t_1) = \pi_\mathcal{A} a(t_1) \) to the original algebra \( \mathcal{A} \) results in

\[ A(t_1) = A_0 + \int_0^{t_1} dt_2 \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \exp \int_0^{t_2} dt_3 \tilde{\Phi}_T \left( \epsilon_0 \otimes \ldots \right. \right. \]
\[ \ldots \otimes a_0 \otimes \exp \int_0^{t_n} \right. dt_{n+1} \tilde{\Phi}_T \left( \epsilon_0 \otimes \ldots \right) \right) \quad (2.121) \]

Note that one only has to apply the fixed-point integral equation \( n \) times, to get the \( n \)'th order contribution with respect to the transformation parameter \( \epsilon_0 \). The first few orders in the parameter \( \epsilon_0 \) are:

\[ A(\epsilon_0) = A_0 + \tilde{\Phi}_T(\epsilon_0 \otimes a_0) + \frac{1}{2} \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \tilde{\Phi}_T(\epsilon_0 \otimes a_0) \right) \]
\[ + \frac{1}{6} \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \tilde{\Phi}_T(\epsilon_0 \otimes a_0) \right. \otimes \tilde{\Phi}_T(\epsilon_0 \otimes a_0) \right) \]
\[ + \frac{1}{6} \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \tilde{\Phi}_T(\epsilon_0 \otimes a_0) \right) \quad (2.122) \]
\[ + \frac{1}{24} \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \tilde{\Phi}_T(\epsilon_0 \otimes a_0) \otimes 3 \right) \]
\[ + \frac{1}{8} \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \tilde{\Phi}_T(\epsilon_0 \otimes a_0) \otimes \tilde{\Phi}_T(\epsilon_0 \otimes a_0) \right) \]
\[ + \frac{1}{24} \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \tilde{\Phi}_T(\epsilon_0 \otimes a_0) \right) \right) \]
\[ + \frac{1}{24} \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \tilde{\Phi}_T \left( \epsilon_0 \otimes a_0 \otimes \tilde{\Phi}_T(\epsilon_0 \otimes a_0) \right) \right) + \mathcal{O} \left( (\epsilon_0)^5 \right). \quad (2.122) \]

Although eq. (2.121) gives the finite transformation in closed form by taking the limit \( n \to \infty \), it is clearly not very useful beyond the expansion in \( \epsilon_0 \) (illustrated to \( \mathcal{O}(\epsilon_0^5) \) above). It is therefore of more interest to consider the order-by-order expansion. Let us first comment on the type of terms that can arise. Besides the zeroth-order term \( A_0 \), all terms begin (and end) with a bracket \( \tilde{\Phi}_T \), i.e. two brackets are never multiplied at the lowest level of nesting. Note that the symmetry factor \( \frac{1}{8} \) in the above expression breaks the otherwise apparent factorial pattern of the first orders, so the rule for giving the coefficients is clearly not that simple. In general, the symmetry factor for a term can be deduced according to two simple rules found empirically in ref. [17], and which easily can be read off from formula (2.121). The rule is the following. For each bracket \( \tilde{\Phi}_T \) appearing in the considered term, do the following:

- If \( k \) entries are equal, divide by \( \frac{1}{k!} \).
- Divide by the total number \( N \) of \( \epsilon_0 \)'s appearing somewhere inside the bracket (i.e. also the \( \epsilon_0 \)'s in further nested brackets).

These two simple rules suffice in determining the whole expansion. Of course, one can as easily simply expand eq. (2.121).

### 3 Connection to Closed String Field Theory

Non-polynomial closed string field theory [18, 10] is based on a so-called “string product” which shares a number of properties with higher antibrackets. This is particularly obvious in the conventions of Zwiebach [10], which we will follow here. For an arbitrary genus \( g \), the \( n \)th string product is denoted by \([A_1, \ldots, A_n]_g\). It has \( n \) entries of states (string fields) \( A_i \), and it maps these states into a new state.\(^8\)

This string product is supercommutative,

\[ [A_1, \ldots, A_{i-1}, A_i, \ldots, A_n]_g = (-1)^{\epsilon A_i - 1} [A_1, \ldots, A_i, A_{i-1}, \ldots, A_n]_g, \quad (3.1) \]

and Grassmann-odd:

\[ \epsilon([A_1, \ldots, A_n]_g) = \sum_{i=1}^{n} \epsilon A_i + 1. \quad (3.2) \]

The string product also carries ghost number (the same for any genus \( g \)), but this notion is not of importance for what follows. In addition to the string product, an important rôle is played by the BRST operator \( Q \).

\(^8\)In closed string field theory, these states are assumed to be annihilated by certain operators \( b^-_g \) and \( L^-_g \) (a property the string product inherits), but this assumption is not required in the following considerations, when restricted to properties of the string products alone.
In classical closed string field theory, corresponding to genus zero, the string product satisfies a so-called “main identity” of the form \[ Q[A_1, \ldots, A_n]_0 + \sum_{i=1}^{n} (-1)^{\epsilon_{A_1}+\ldots+\epsilon_{A_{i-1}}} [A_1, \ldots, QA_i, \ldots, A_n]_0 + \sum_{\{i_l, j_k\}} \sigma(i_l, j_k) [A_{i_1}, \ldots, A_{i_l}, [A_{j_1}, \ldots, A_{j_k}]_0]_0, \] (3.3)

where the last sum is restricted to \( l \geq 1, k \geq 2, \) and \( l + k = n. \) The sign factor \( \sigma(i_l, j_k) \) is what is picked up by the prescribed reordering of terms, using the fact the string product is supercommutative.

The BRST operator \( Q \) is defined on a given conformal background, and the whole string field theory is then also defined on such a background. At genus zero, this means that the “zero-product” corresponding to \( n = 0 \) must be taken to vanish:

\[ [\cdot]_0 = 0. \] (3.4)

(The corresponding definition away from a conformal background will be discussed later.) The first non-trivial string product is thus the “one-product”, a linear map that takes one string state into another. It is given by

\[ [A]_0 \equiv QA. \] (3.5)

The classical non-polynomial closed string field theory action can then be written \[ S(\Psi) = \frac{1}{\kappa^2} \sum_{n=2}^{\infty} \kappa^n/n! \{\Psi, \ldots, \Psi\}_0, \] (3.6)

where \( \{A, B_1, \ldots, B_n\}_0, \) a Grassmann-even \((n+1)\)-bracket, is defined by an inner product,

\[ \{A, B_1, \ldots, B_n\}_0 \equiv \langle A, [B_1, \ldots, B_n]_0 \rangle, \] (3.7)

with the following exchange relation:

\[ \langle A, B \rangle = (-1)^{(\epsilon_A+1)(\epsilon_B+1)} \langle B, A \rangle. \] (3.8)

A more familiar expression for the closed string field theory action is obtained by using \( \{\Psi, \Psi\}_0 \equiv \langle \Psi, [\Psi]_0 \rangle = \langle \Psi, Q\Psi \rangle \) \[ S(\Psi) = \frac{1}{2} \langle \Psi, Q\Psi \rangle + \frac{1}{\kappa^2} \sum_{n=3}^{\infty} \kappa^n/n! \{\Psi, \ldots, \Psi\}_0, \] (3.9)

where the last bracket has \( n \) entries. The classical equations of motion then take the form

\[ Q\Psi + \frac{1}{\kappa} \sum_{n=2}^{\infty} \kappa^n/n! [\Psi, \ldots, \Psi]_0 = 0. \] (3.10)

Finally, the closed string field theory action (3.6) is left invariant by the following gauge transformations:

\[ \delta_\epsilon \Psi = Q\epsilon + \sum_{n=1}^{\infty} \kappa^n/n! [\Psi, \ldots, \Psi, \epsilon]_0 = \sum_{n=0}^{\infty} \kappa^n/n! [\Psi, \ldots, \Psi, \epsilon]_0. \] (3.11)
If we compare these string field theory expressions with the identities among higher antibrackets we derived in the previous sections, it is tempting to identify the $n$th string product at genus zero with the $n$th antibracket generated by an odd operator $T$:

$$[A_1, \ldots, A_n]_0 = \Phi^T_n(A_1, \ldots, A_n).$$  \hspace{1cm} (3.12)

The obvious obstruction to such an identification is the lack of a simple product “ $\cdot$ ” of, using the notation of section 2, the algebra $A$. Still, let us consider the similarities. We have already listed the pertinent properties of the string products. All of these properties are shared with the higher antibrackets: They are both Grassmann-odd, graded commutative under exchange of entries, and the crucial "main identity" of the string products is recognized as being identical to the identity (2.47) of higher antibrackets.

Consider now the equations of motion (3.10), which in the previous section played the rôle of the full Master Equation (the action $S$ replacing the string field $\Psi$). We can view this equation, with its infinite sum of higher brackets (or string products), from two points of view. Either as a clever way of representing the particular combination of exponential functions without reference to the algebra by means of which these exponential functions could be defined, or as a very complicated way of writing the simple formula

$$e^{-\kappa \Psi} \overrightarrow{T} e^{\kappa \Psi} = 0$$  \hspace{1cm} (3.13)

through its power series expansion. Of course, to give meaning to $\exp(\kappa \Psi)$, we would have to assume that it is possible to redefine ghost number assignments so that $\kappa \Psi$ becomes of ghost number zero. Closed string field theory is tied to the formulation in terms of a power series expansion.

Similarly, the gauge symmetry of string field theory (3.11) can be understood as the infinite-series expansion of the simple expression

$$\delta_\epsilon \Psi = e^{-\kappa \Psi} \overrightarrow{T} \epsilon e^{\kappa \Psi}.$$  \hspace{1cm} (3.14)

We have also already seen the usefulness of the higher-antibracket formalism when deriving what in closed string field theory is viewed as the analogous finite gauge transformations (in section 2.9). In all of these cases, we can use the algebra $A$ to derive results with far greater ease, and whenever these results are expressible in terms of higher antibrackets alone (without using the new product) we find that the expressions coincide with those of closed string field theory.

It is also of interest to see the results of subsection 2.8 from the point of view of closed string field theory. It was noted by Ghoshal and Sen \cite{20} that there is an apparent clash between the gauge symmetry of closed string field theory being open off-shell, while the gauge transformations of the low-energy effective field theory derived from this theory form an algebra which closes off-shell. By analyzing special cases, they found that the usual gauge transformations of closed string field theory combine with "trivial" gauge transformations (proportional to the equations of motion) to give the proper transformations (which close) of the low-energy theory. Ghoshal and Sen in fact conjecture that all gauge transformations of closed string field theory can be organized in such a manner (by adding suitable "equation of motion terms") that the algebra eventually closes off-shell. Our symmetry operator $\bar{\sigma}$ is precisely of this kind, but it cannot as it stands be given an interpretation in closed string field theory, since it – in contrast to $\sigma$ – involves the product of string fields discussed above.

### 3.1 Beyond Conformal Backgrounds

An interesting place for considering the analogy between string products and higher antibrackets is that of closed string field theory in a background that is not conformal. Zwiebach \cite{10} has analyzed
the fate of the string product algebra in this situation.

So far the analogy has been based on the assumption that the “zero-product” (3.4) is vanishing. Away from a conformal background this zero-product will no longer vanish. Zwiebach calls it \( F \), and distinguishes the new string products by a prime \([10]\):

\[
\left[ \cdot \right]'_0 = F . 
\] (3.15)

Denoting, accordingly, also the new BRST-like operator by \( Q' \), some of the first few identities that generalize the “main identity” of eq. (3.3) read \([10]\):

\[
Q'F = 0 \quad (3.16)
\]
\[
Q^2A + [F, A]'_0 = 0 \quad (3.17)
\]

This last equation (3.17) gives the violation of \( Q' \)-nilpotency away from a conformal background. The analogue of \( Q \) differentiating the two-product (1.6) becomes

\[
Q'[A_1, A_2]'_0 + [Q'A_1, A_2]'_0 + \cdots = 0 ,
\] (3.18)

and the higher-order identities can also be worked out.

The appearance of a non-trivial zero-product \( F \) has a completely natural explanation in terms of higher antibrackets: it corresponds to the inclusion of a non-trivial zero-bracket \( \Phi^0_T \).

In closed string field theory, one studies the behavior away from a conformal background by shifting the string field:

\[
\Psi \to \Psi_0 + \Psi ,
\] (3.19)

where \( \Psi_0 \) does not solve the classical equation of motion. The precise connection between such a shift and the emergence of a new algebra of string brackets that now involves the zero-product is easily understood if one accepts the formulation in terms of higher antibrackets and the new, assumed, string product. Consider the equation of motion for the unshifted field. We can write it as

\[
0 = e^{-\kappa \Psi} \overrightarrow{T} e^{\kappa \Psi} = e^{-\kappa(\Psi - \Psi_0)} \overrightarrow{T} e^{\kappa(\Psi_0 + \kappa(\Psi - \Psi_0))}
\]
\[
= e^{-\kappa(\Psi - \Psi_0)} \overrightarrow{T}' e^{\kappa(\Psi - \Psi_0)} ,
\] (3.20)

that is, the equation of motion for the shifted string field \( \Psi - \Psi_0 \), with respect to a new nilpotent BRST operator,

\[
\overrightarrow{T}' \equiv e^{-\kappa \Psi_0} \overrightarrow{T} e^{\kappa \Psi_0}(\cdot) ,
\] (3.21)

a conjugate version of \( \overrightarrow{T} \). Recall that \( \overrightarrow{T} \) here acts on everything to its right.

Because \( \Psi_0 \) is assumed not to solve the classical equation of motion, it follows immediately that the new string product algebra will have a non-trivial zero-product:

\[
\overrightarrow{T}'1 = e^{-\kappa \Psi_0} \overrightarrow{T} e^{\kappa \Psi_0}
\]
\[
= \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} [\Psi_0, \ldots, \Psi_0]_0 \neq 0 ,
\] (3.22)

\( i.e., \) precisely (\( \kappa \) times) the left hand side of the equation of motion for \( \Psi_0 \) (which by assumption is non-vanishing). We identify it as \( F = \left[ \cdot \right]'_0 \) above. Note, incidentally, that

\[
\overrightarrow{T}'F = \overrightarrow{T}' \overrightarrow{T}'(1) = 0 ,
\] (3.23)

\( ^9 \)The analogous study of shifts with a \( \Psi_0 \) that still solves the equations of motion (corresponding to a new, but still conformal, background), was first considered by Sen \([19]\). In this case the zero-product must still be taken to vanish.

25
but this identity is *not* the same as eq. (3.17). The “BRST-like” operator $Q'$ is the one-bracket $\Phi_1^\prime T'$, associated with $T'$, not the BRST operator $T'$ itself.

The whole sequence of main identities can of course now be rewritten in terms of $\vec{T}'$, rather than $\vec{T}$. The only new feature compared with the usual main identities of closed string field theory is that the 0-bracket $\Phi^0$ is non-vanishing. In particular, one sees immediately that the first identities (3.17) and (3.17) are trivially included in eq. (2.47), and similarly for the higher identities. They are all contained in eq. (2.47).

Note that $\vec{T}'$ is nilpotent simply as a consequence of $\vec{T}$ being nilpotent. It can be viewed as a genuine BRST operator corresponding to the shifted background. The “BRST-like” operator $Q'$ of closed string field theory [10] is in the present context rather seen as the one-bracket; it is not nilpotent when $F \neq 0$.

We have thus shown that when shifting the string field $\Psi$ by $\Psi_0$, almost all of the formalism remains intact, and in particular almost everything can eventually be expressed in terms of string products. It is therefore not surprising that these results can also be derived directly on the basis of string products alone [10]. They just appear with far more ease in the present picture. There are also interesting exceptions, such as the new nilpotent BRST operator $\vec{T}'$. This is the appropriate BRST operator for shifted backgrounds, but it cannot be expressed solely in terms of antibrackets (or string products), and therefore has no obvious analogue in closed string field theory.

### 4 An $Sp(2)$-Symmetric Formulation

As discussed in section 2, the higher antibrackets give rise to a BRST symmetry which is a generalization of the BRST symmetry of Batalin and Vilkovisky. An obvious question to ask is whether one analogously can find a formulation that includes both BRST symmetry and anti-BRST symmetry. There have been various suggestions for Lagrangian BRST formulations à la Batalin and Vilkovisky which includes the extended BRST–anti-BRST symmetry. All these have from the outset included the BRST–anti-BRST symmetries in an $Sp(2)$ symmetry. The original approach is due to Batalin, Lavrov and Tyutin [21], and it has recently been suggested that this formulation be rephrased in terms of what has been called “triplectic quantization” [22].

The main new ingredient of an $Sp(2)$-symmetric formulation of conventional Lagrangian quantization is a Grassmann-odd vector field $V$, which satisfies $V^2 = 0$, and which must be added to the $\Delta$-operator. We take $V$ to be a differential operator based on a right-derivative. In the original formulation of ref. [21], the following relations are assumed:

\[ V^{(a} V^{b)} = 0 \]
\[ \Delta^{(a} \Delta^{b)} = 0 \]
\[ \Delta^{(a} V^{b)} + V^{(a} \Delta^{b)} = 0. \]

Here $a, b, \ldots$ denote indices in $Sp(2)$, the invariant tensor of which is $\epsilon_{ab}$. Symmetrization in $Sp(2)$ indices is defined by

\[ F^{(a} G^{b)} \equiv F^a G^b + F^b G^a, \]

\[ F^{(a} G^{b)} = 0, \quad \Delta^{(a} V^{b)} + V^{(a} \Delta^{b)} = 0. \]

---

10 For alternative viewpoints, suggestions for alternative but physically equivalent schemes, and a derivation of the relation between the two different schemes, see ref. [23].

11 In the first formulation of the $Sp(2)$-symmetric triplectic quantization [22], the condition (4.3) was required to hold even before antisymmetrization in the $Sp(2)$-indices. However, as was noted in ref. [24], the above more general condition is all that is required. See also ref. [25].
and these indices are raised and lowered by the $\epsilon$-tensor.

In refs. [22, 24, 25] the $\Delta^a$-operators are assumed to be of purely 2nd order, while the $V^a$-operators are assumed to be of purely 1st order. However, in actual applications it is usually the combinations

$$\Delta^a_{\pm} \equiv \Delta^a \pm \frac{i}{\hbar} V^a$$

(4.5)

which appear. This suggests that we should simply view $\Delta^a_{\pm}$ as more general 2nd-order odd differential operators (still excluding a constant term). It follows from (4.1), (4.2) and (4.3) that

$$\Delta_{\pm} \{ \Delta_{\pm} \} = 0 .$$

(4.6)

Since by definition $V^a$ is of first order, the antibrackets defined by use of either $\Delta^a$ or $\Delta^a_{\pm}$ will coincide. These antibrackets are born with an $Sp(2)$-index:

$$(F, G)^a \equiv (-1)^{\epsilon_G} \Delta^a(FG) - \Delta^a(F)G - (-1)^{\epsilon_G} F \Delta^a(G) .$$

(4.7)

The above antibrackets satisfy the usual exchange relation (1.3), and the same graded Leibniz rules (1.4). The analogue of the graded Jacobi identity (1.5) reads

$$\sum_{\text{cycl.}} (-1)^{\epsilon_F + 1}(\epsilon_H + 1) (F, (G, H)^{\{a\}}_{b}) = 0 ,$$

(4.8)

and the $Sp(2)$-covariant version of the relation (1.6) is

$$\Delta^{\{a} (F, G)^{b\}} = (F, \Delta^{\{a} G^{b\}}) - (-1)^{\epsilon_G} (\Delta^{\{a} F^{b\}} G^{a\}} .$$

(4.9)

Furthermore, it follows from the above definitions that the vector fields $V^a$ differentiate the antibrackets according to

$$V^{\{a} (F, G)^{b\}} = (F, V^{\{a} G^{b\}}) - (-1)^{\epsilon_G} (V^{\{a} F^{b\}} G^{a\}} .$$

(4.10)

This implies that also the relation (4.9) remains valid if we replace $\Delta^a$ by $\Delta^a_{\pm}$.

Our task is now to generalize the above construction to the case of higher antibrackets. The obvious starting point is to introduce two higher-order $\Delta^a$-operators, and proceed as in section 2, using the $Sp(2)$-algebra (4.2). The analogous $V^a$-operator, taken by definition to be always of 1st order, can be introduced trivially by letting $\Delta^a \rightarrow \Delta^a_{\pm} \equiv \Delta^a \pm (i/\hbar) V^a$, where $V^a$ simply equals the 1st-order part of $\Delta^a$. The main ingredient is therefore the existence of two odd differential operators of arbitrary order, and with the algebra of $\Delta^a$ as in (1.2). As in the previous section, we can include the case of a possibly non-vanishing constant pieces in these differential operators as well, corresponding to $\Delta^a(1)$ not necessarily being zero.

### 4.1 $Sp(2)$-Covariant Higher Antibrackets

All necessary ingredients for the extension of the above $Sp(2)$-symmetric formulation to the higher-antibracket BRST symmetry have been given in section 2. In particular, we refer to section 2.4, where we gave the algebra of higher antibrackets generated by two nilpotent operators $S, T$. In accordance with the above formalism, we shall here denote these operators by $\Delta^a$ and $\Delta^b$. All the subsequent manipulations remain valid if we replace these by $\Delta_{\pm}$ through the definition (4.5). Because of the
proliferation of indices, we drop the subscript $\Delta$ on the higher antibrackets, and just indicate the relevant $\Delta$-operator by its $Sp(2)$-index $a$. For simplicity, we take
\[ \Delta^a(1) = 0 , \tag{4.11} \]
so that there are no zero-brackets (they can of course trivially be included).

With the operators $\Delta^a$ being Grassmann-odd, the algebra of $Sp(2)$-symmetric higher antibrackets can then be written:
\[ \sum_{r=1}^{n} \frac{1}{r!(n-r)!} \sum_{\pi \in S_n} (-1)^{\epsilon_a} \Phi_{\{a\}}^{n-r+1} \left( \Phi_{\{b\}}^{1}(A_{\pi(1)} \otimes \ldots \otimes A_{\pi(r)}) \otimes A_{\pi(r+1)} \otimes \ldots \otimes A_{\pi(n)} \right) = 0 . \tag{4.12} \]
This algebra contains all the usual identities of $Sp(2)$-symmetric quantization as outlined above, and the appropriate generalization if higher antibrackets are included. In detail, the first identity is nothing but $Sp(2)$-nilpotency of the operators $\Delta^a$ (cf. eq. (4.6)):
\[ \Phi_{\{a\}}^1(\Phi_{\{b\}}^1(A)) = 0 , \tag{4.13} \]
while the 2nd identity gives the $Sp(2)$-covariant rule for how $\Delta^a$ differentiates the “two-antibracket” (as in eq. (4.13)):
\[ \Phi_{\{a\}}^1(\Phi_{\{b\}}^2(A_1 \otimes A_2)) = -\Phi_{\{a\}}^2(\Phi_{\{b\}}^1(A_1) \otimes A_2) - (-1)^{\epsilon_a} \Phi_{\{a\}}^2(\Phi_{\{b\}}^1(A_2) \otimes A_1) . \tag{4.14} \]
(Note that this identity is not altered by the presence of higher order operators in the $\Delta^a$’s). The next identity is the $Sp(2)$-covariant analogue of the Jacobi identity (4.3), including its possible breaking when the $\Delta^a$’s are of order 3 or higher:
\[ \Phi_{\{a\}}^2(\Phi_{\{b\}}^2(A_1 \otimes A_2) \otimes A_3) + (-1)^{\epsilon_a} \Phi_{\{a\}}^2(\Phi_{\{b\}}^2(A_2 \otimes A_3) \otimes A_1) \\
+(-1)^{\epsilon_a} \Phi_{\{a\}}^2(A_3 \otimes A_1) \otimes A_2 = -\Phi_{\{a\}}^3(\Phi_{\{b\}}^1(A_1) \otimes A_2 \otimes A_3) \\
-(-1)^{\epsilon_a} \Phi_{\{a\}}^3(\Phi_{\{b\}}^1(A_2) \otimes A_3 \otimes A_1) - (-1)^{\epsilon_a} \Phi_{\{a\}}^3(A_3 \otimes A_1 \otimes A_2) \\
-\Phi_{\{a\}}^3(\Phi_{\{b\}}^1(A_1 \otimes A_2 \otimes A_3)) . \tag{4.15} \]
The subsequent identities are of course completely new, involving higher and higher order of antibrackets. They can be read off directly from eq. (2.17). Also the higher main identity (B.8) generates a series of new identities. We quote the first few:
\[ 0 = \Phi_{\{a\}}^1\left( \Phi_{\{b\}}^1\left( \Phi_{\{c\}}^1(A \otimes A_2) \right) \right) . \tag{4.16} \]
This identity is rather trivial, but it turns out to be very convenient for proving that no other independent $Sp(2)$ main identities exist. The next reads
\[ 0 = \Phi_{\{a\}}^1\left( \Phi_{\{b\}}^1\left( \Phi_{\{c\}}^1(A_1 \otimes A_2) \right) \right) + \Phi_{\{a\}}^1\left( \Phi_{\{b\}}^1\left( \Phi_{\{c\}}^1(A_1) \otimes A_2 \right) \right) \\
+(-1)^{\epsilon_a} \Phi_{\{a\}}^2\left( \Phi_{\{b\}}^1\left( \Phi_{\{c\}}^1(A_2) \otimes A_1 \right) \right) + \Phi_{\{a\}}^2\left( \Phi_{\{b\}}^1\left( \Phi_{\{c\}}^1(A_1) \otimes A_2 \right) \right) \\
+(-1)^{\epsilon_a} \Phi_{\{a\}}^2\left( \Phi_{\{b\}}^1\left( \Phi_{\{c\}}^1(A_2) \otimes A_1 \right) \right) + \Phi_{\{a\}}^2\left( \Phi_{\{b\}}^1\left( \Phi_{\{c\}}^1(A_1) \otimes A_2 \right) \right) \\
+(-1)^{\epsilon_a} \Phi_{\{a\}}^{1+1}\Phi_{\{b\}}^{1+1}\Phi_{\{c\}}^1(A_2 \otimes \Phi_{\{c\}}^1(A_1)) . \tag{4.17} \]
The interesting point about these new identities (the first few of which of course are valid also in conventional $Sp(2)$ BRST quantization) is that they do not involve symmetrizations in the $Sp(2)$ indices. The higher identities can be read off from eq. (B.8) in Appendix B.
These higher main identities are qualified guesses for what will arise in an $Sp(2)$ symmetric formulation of genus zero closed string field theory.

We next turn to the question of the corresponding BRST operators. In the conventional $Sp(2)$-covariant scheme of ref. [21], one can show [23] – as expected – that the two symmetries are generated by the two antibrackets and the solution to the Master Equations

$$\Delta^a \exp \left[ \frac{i}{\hbar} S \right] = 0 .$$

(4.18)

More interestingly, also in this context one derives a “quantum BRST operator” (see the 2nd reference of [23]), which reads

$$\sigma^a \epsilon = (\epsilon, S)^a + V^a \epsilon - (i\hbar) \Delta^a \epsilon ,$$

(4.19)

where the first-order contribution $V^a$ to $\Delta^a$ explicitly separates out.

Consider now the corresponding BRST operators in the generalized situation in which one has higher antibrackets. Repeating the exercise of the analogous situation without $Sp(2)$ symmetry, letting

$$T^a(F) = (-1)^{\ell F} \Delta^a(F) .$$

(4.20)

one finds immediately that the appropriate generalization is (rescaling by a factor of $(i/\hbar)$, converting to left-derivatives, and lowering the $Sp(2)$ index):

$$\sigma^a \epsilon = \frac{\Phi_a}{\infty} (\epsilon \otimes e^{\frac{i}{\hbar} S})$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i}{\hbar} \right)^n \Phi^{n+1}_a \left( \epsilon \otimes S^{\otimes n} \right)$$

$$= e^{-\frac{i}{\hbar} S} [T_a, \epsilon] e^{\frac{i}{\hbar} S} ,$$

(4.21)

and similarly for the associated BRST operator $\bar{\sigma}^a$, which one can define completely analogous to the case without $Sp(2)$ symmetry. When expanded as a possibly infinite sum, the first three terms of eq. (4.21) agree with the corresponding $Sp(2)$ quantum BRST operator of ref. [23]. The new terms involve higher and higher antibrackets precisely as anticipated. By construction,

$$\sigma_a \sigma_b = 0 ,$$

(4.22)

to all orders.

5 Conclusions

Higher antibrackets provide us with a rich mathematical background for studying various quantization problems in physics. They give the obvious generalization of the Batalin-Vilkovisky formalism to situations in which the $\Delta$-operator is of order 3 or higher. When viewed from this more general perspective even the original Batalin-Vilkovisky formalism is seen in a completely new light. Many of the ingredients of the Lagrangian BRST formalism suddenly become very natural. For example, in the conventional Batalin-Vilkovisky formalism the quantum Master Equation involves both the conventional antibracket (the two-antibracket from the present perspective) and the $\Delta$-operator. Usually, the need for the quantum correction in the form of this $\Delta$-operator is viewed as a kind of coincidence, the result of a particular correction from the path integral measure to the classical BRST transformation of the action. Similarly, the “quantum correction” to the classical BRST transformation due to this $\Delta$-operator is seen as a (slightly annoying) modification of the otherwise fully “anticanonical” formalism.
that only involves the use of a two-antibracket: a Grassmann-odd analogue of the Poisson bracket. What we have seen here, is that the \( \Delta \)-operator is in no way mysteriously present in the formalism. It plays two rôles: First, it is the operator by which higher antibrackets are formed, and second, it really is to be viewed as a “one-antibracket”, completely on par with the conventional antibracket. If \( \Delta(1) \) would not vanish, this identification would no longer hold. The quantum Master Equation is based on \( \Delta \), and it holds in all generality that this equation can be expressed solely in terms of the higher antibrackets generated by \( \Delta \).

The fact that an almost-canonical formulation\(^{12}\) of the Lagrangian quantization program exists, is thus in many respects coincidental, and not fundamental. It is due to the fact that in the conventional representation of fields and antifields the BRST operator of Schwinger-Dyson BRST symmetry (and hence \( \Delta \)) is of 2nd order. In general, a Master Equation of the form

\[
\Delta \exp \left( \frac{i}{\hbar} S \right) = 0 \quad (5.1)
\]

will contain an infinite series of arbitrarily high antibrackets. The canonical considerations are of course limited to the two-antibracket.

From the Lagrangian BRST quantization point of view it is interesting that the appearance of the \( \Delta \)-operator can be traced to a totally different origin: that of integrating out ghosts while keeping the antighosts \(^{3}\). Also from this point of view the \( \Delta \)-operator immediately appears on an equal footing with the two-antibracket: the same ghost integration that introduces the conventional antibracket in the BRST operator also simultaneously introduces the \( \Delta \)-operator. It is nevertheless astonishing that the whole mathematical framework of higher antibrackets can be derived by simple ghost-field integrations in the Lagrangian path integral \(^{5, 4}\). The fact that there is an analogous construction from the ghost momentum representation of Hamiltonian BRST quantization \(^{4}\) hints at new and unexpected relations between the Hamiltonian and Lagrangian BRST schemes.

In this paper we have focused on some of the more mathematical aspects of the theory of higher brackets. The formulation has been greatly simplified, thereby providing a much cleaner setting for the field theory aspects. Of course, one most interesting result is the close correspondence between higher antibrackets and the so-called string products of closed string field theory \(^{3}\). We have argued that there are many hints at the existence of a new product of string fields by means of which non-polynomial closed string field theory could have at its origin a formulation based on exponentials (defined within this product). This remains speculation at the present stage, but even if it should turn out not to be possible to realize such a product in closed string field theory, our formalism may still be of use in this context. Namely, one may conjecture that at least all those results which can be expressed solely with the help of brackets (or, here, string products) may still be valid in closed string field theory. Then the product may be used only in intermediate steps, to simplify the calculations.

The BRST symmetry associated with higher antibrackets is part of a more general BRST–anti-BRST symmetry, and we have shown how they both can be included in a manifestly \( Sp(2) \)-covariant formulation. As an amusing by-product of this, we can also write down \( Sp(2) \)-covariant analogues of the closed string field theory equations of motions, and the corresponding \( Sp(2) \)-extended gauge symmetries. For the path integral of conventional quantum field theory, the associated \( Sp(2) \)-covariant BRST symmetry is required when one imposes certain identities as \( Sp(2) \)-BRST Ward Identities in the path integral, as discusses in the analogous case without \( Sp(2) \) symmetry in ref. \(^{5}\). It is interesting that this \( Sp(2) \)-covariant formulation in a most natural manner arises from the mathematical structure of strongly homotopy Lie algebras.

\(^{12}\)With respect to an odd Poisson-like bracket, the usual antibracket.
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A Generalizations

In section 2.2 we discussed the construction of higher antibrackets based on the co-multiplication $\lambda$ of eq. (2.21). Interestingly, this lends itself to a natural generalization which we will outline in this appendix. It is based on a map of degree $k$ called $\lambda^{\epsilon_1,\ldots,\epsilon_k}_{t_1,\ldots,t_k} : SA \to \prod_{i=1}^k SA$. Define it as follows:

$$
\lambda^{\epsilon_1,\ldots,\epsilon_k}_{t_1,\ldots,t_k}(1) = (1,1,\ldots,1)
$$

$$
\lambda^{\epsilon_1,\ldots,\epsilon_k}_{t_1,\ldots,t_k}(A_1 \otimes \ldots \otimes A_n)
= \left( t_1(A_1,1,\ldots,1) + (-1)^{\epsilon_{A_1}} \epsilon_{t_2} t_2(1,A_1,1,\ldots,1) + \ldots + (-1)^{\epsilon_{A_1}(\epsilon_{t_{n-1}} + \epsilon_{t_n})} t_k(1,\ldots,1,A_1) \right)
\otimes \ldots \otimes 
\left( t_1(A_n,1,\ldots,1) + (-1)^{\epsilon_{A_n}} \epsilon_{t_2} t_2(1,A_n,1,\ldots,1) + \ldots + (-1)^{\epsilon_{A_n}(\epsilon_{t_{n-1}} + \epsilon_{t_n})} t_k(1,\ldots,1,A_n) \right)
$$

(A.1)

Here $\epsilon_1, \ldots, \epsilon_k$ are $\pm 1$, and $t_1, \ldots, t_k$ complex numbers. Even though $\lambda^{\epsilon_1,\ldots,\epsilon_k}_{t_1,\ldots,t_k}$ does not depend on $\epsilon_1$, it is natural to introduce an $\epsilon_1$. Here $\prod_{i=1}^k SA$ is equipped with a graded product $\otimes$:

$$
(A_1, \ldots, A_k) \otimes (B_1, \ldots, B_k) = (-1)^{\sum_{i>j} \epsilon_{A_i} \epsilon_{B_j}} (A_1 \otimes B_1, \ldots, A_k \otimes B_k).
$$

(A.2)

It turns out to be more convenient allowing for tensor valued operator $T$, i.e. linear maps: $SA \to SA$ instead of just working with ordinary linear operators: $A \to A$. We can now define generalized higher brackets for $T_1, \ldots, T_k \in \text{Hom}_C(SA,SA)$ and complex numbers $t_1, \ldots, t_k$

$$
\begin{vmatrix}
T_1 & \cdots & T_k \\
\hline
t_1 & \cdots & t_k
\end{vmatrix}
\equiv (T_1 \times \ldots \times T_k) \circ \lambda^{\epsilon_1,\ldots,\epsilon_k}_{t_1,\ldots,t_k} \in \text{Hom}_C(SA,SA).
$$

(A.3)

respectively

$$
\begin{vmatrix}
T_1 & \cdots & T_k \\
\hline
t_1 & \cdots & t_k
\end{vmatrix}
\equiv \begin{vmatrix}
T_1 & \cdots & T_k \\
\hline
t_1 & \cdots & t_k
\end{vmatrix} \sim \in \text{Hom}_C(SA,A).
$$

(A.4)

With the above generalization, we can write (cf. (2.21), (2.23), (2.30) resp. (2.31). ) :

$$
\lambda = \lambda_{1,-1}^{\epsilon_T}.
$$

$$
\Phi_T = \begin{vmatrix}
T & \text{Id}_{SA} \\
1 & -1
\end{vmatrix}
$$

$$
\bar{\Phi}_T = \begin{vmatrix}
T & \text{Id}_{SA} \\
1 & -1
\end{vmatrix}
$$

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\[
T = \begin{pmatrix} \Phi_T & \text{Id}_{S\mathcal{A}} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \Phi_T & \text{Id}_{S\mathcal{A}} \\ 1 & 1 \end{pmatrix}.
\] (A.5)

The first few brackets can be rewritten as
\[
\begin{align*}
\left| T \atop t \right| A_1 \otimes \ldots \otimes A_n & = t^n T(A_1 \otimes \ldots \otimes A_n) \\
\left| S \atop s \atop t \right| A_1 \otimes \ldots \otimes A_n & = (S \otimes T) \left( (sA_1^T + tA_1^S) \otimes \ldots \otimes (sA_n^T + tA_n^S) \right) \\
& = \sum_{\ell_1, \ldots, \ell_n = 0}^1 (-1)^{\ell_T} \sum_j \epsilon_{A_j} \ell_j + \sum_j \epsilon_{A_j} (1 - |\ell_j|) \sum_{j > k} \epsilon_{A_k} \epsilon_{A_h} \ell_j (1 - |\ell_k|) (1 - |\ell_j|) \ell_k + \ell_j \ell_k \\
& \otimes T((tA_1)^{(1 - \ell_1) \otimes \ldots \otimes (tA_n)^{(1 - \ell_n)}) \\
& \otimes U((uA_1)^{(\ell_1) \otimes \ldots \otimes (uA_n)^{\ell_n})}.
\end{align*}
\] (A.6)

Here \( \ell^\pm = \frac{1}{2}(|\ell| + \ell) \) is just the positive (resp. negative) part of the real number \( \ell \). This is clearly not a systematic way of describing the generalized brackets for more than three operator entries. In order to proceed into higher numbers of operator entries, we use characteristic functions. Let \( \chi_s \) be the characteristic function associated with the statement \( s \). \( \chi_s = 1 \) if \( s \) is true, and \( \chi_s = 0 \) if \( s \) is false. Then
\[
\begin{align*}
\left| T_1 \ldots \ldots \ldots T_k \atop t_1 \ldots \ldots \ldots t_k \right| A_1 \otimes \ldots \otimes A_n & = (T_1 \otimes \ldots \otimes T_k) \left( (t_1 A_1^T_{T_2 \ldots T_k} + t_1 A_1 T_3 \ldots T_k + \ldots + t_1 A_1 T_{1 \ldots T_{k-1}}) \\
& \otimes \ldots \otimes (t_n A_n^T_{T_2 \ldots T_k} + t_n A_n T_3 \ldots T_k + \ldots + t_n A_n T_{1 \ldots T_{k-1}}) \right) \\
& = \sum_{\ell_1, \ldots, \ell_n = 1}^k (-1)^{\ell_T} \sum s_{\ell_T} \epsilon_{A_1} \chi_{\ell_1 = s} + \sum_{p > q} \epsilon_{A_p} \epsilon_{A_q} \sum_{\ell_T} \chi_{\ell_p = \ell_q} \chi_{\ell_q = s} \\
& T_1((t_1 A_1)^{\chi_{\ell_1 = 1} \otimes \ldots \otimes (t_n A_n)^{\chi_{\ell_n = 1}}) \\
& \otimes \ldots \otimes T_k((t_1 A_1)^{\chi_{\ell_1 = k} \otimes \ldots \otimes (t_n A_n)^{\chi_{\ell_n = k}}),
\end{align*}
\] (A.7)

The generalized higher antibrackets are all graded symmetric:
\[
\begin{pmatrix} T_1 \ldots \ldots \ldots T_k \atop t_1 \ldots \ldots \ldots t_k \end{pmatrix} = (-1)^{\ell_T} \begin{pmatrix} T_{\ell(1)} \ldots \ldots \ldots T_{\ell(k)} \atop t_{\ell(1)} \ldots \ldots \ldots t_{\ell(k)} \end{pmatrix},
\] (A.8)

where \((-1)^{\ell_T}\) is the sign factor originating from permuting Grassmann graded quantities:
\[
(T_1, \ldots, T_k) \leftrightarrow (T_{\ell(1)}, \ldots, T_{\ell(k)}).
\] (A.9)

They are restricted linear and enjoy simple composition properties:
\[
\begin{pmatrix} T_1 \ldots \ldots \ldots T_i + T_i'' \ldots \ldots \ldots T_k \atop t_1 \ldots \ldots \ldots t_i \end{pmatrix} = \begin{pmatrix} T_1 \ldots \ldots \ldots T_i' \ldots \ldots \ldots T_k \atop t_1 \ldots \ldots \ldots t_i \end{pmatrix} + \begin{pmatrix} T_1 \ldots \ldots \ldots T_i'' \ldots \ldots \ldots T_k \atop t_1 \ldots \ldots \ldots t_i \end{pmatrix}
\] (A.10)
When all the coefficients $t_1, \ldots, t_k = 1$ are equal to 1, one can say a lot more. First of all let us simplify the notation in this special case:

$$|T_1, \ldots, T_k| \equiv \begin{vmatrix} T_1 & \cdots & T_k \\ t_1 & \cdots & t_k \end{vmatrix}.$$  \hfill (A.13)

Following Zwiebach ([10], eq. (4.100)), we define a co-derivation $b_T$ for an operator $T \in \text{Hom}_C(SA, SA)$.

$$b_T \equiv |T, \text{Id}_{SA}| \equiv \begin{vmatrix} T & \text{Id}_{SA} \\ 1 & 1 \end{vmatrix}$$  \hfill (A.14)

$$b_T(A_1 \cdots \otimes A_n) = T((A_1 + \tilde{A}_1^T) \otimes \ldots \otimes (A_n + \tilde{A}_n^T)) = \sum_{i_1, \ldots, i_n=0}^1 (-1)^{\epsilon_{A_1^i} \epsilon_{A_n^i}(1-i_k)} T(A_1^{i_1} \otimes \ldots \otimes A_n^{i_n}) \otimes A_1^{1-i_1} \otimes \ldots \otimes A_n^{1-i_n}. \hfill (A.15)$$

We also note the following simple relations

$$b_{S+T} = b_S + b_T,$$

$$\pi_A \circ b_T = T,$$

$$b_{\Phi_T} = T = \tilde{b}_{\tilde{\Phi}_T}.$$  \hfill (A.16)

The two last statements only holds for operator $T$, which is not tensor valued, i.e. $T \in \text{Hom}_C(SA, A)$. Less obvious are the following identities:

$$b_S \circ b_T = b_{S \circ b_T} + b_{[S, T]}$$

$$|S_1, \ldots, S_k| \circ b_T = \sum_{i=1}^k (-1)^{\epsilon_{S_i+1} \cdots \epsilon_{S_k}} |S_1, \ldots, S_i \circ b_T, \ldots, S_k|,$$  \hfill (A.17)

where $T \in \text{Hom}_C(SA, A)$. The first identity in (A.17) is an important special case of the second identity. Many of the above (and coming) constructions can actually be carried out in a vector space frame just as well, i.e. not assuming a dot product for the algebra $A$. For instance the $\Phi_T$ and $b_T$ construction works without a dot, if $T \in \text{Hom}_C(SA, SA)$. The most notable exceptions are the tilde operation, the higher brackets $\Phi^n_T$, and in particular the recursion relation (2.32). However, one can impose the existence of the higher brackets $\Phi^n_T$ (and their so-called “main identity”: see below) as a principle. For instance in closed string field theory the higher brackets can be built up from a geometric consideration on moduli space $\Sigma$.

**B Higher Main identities**

The purpose of this appendix is to show that by applying the lemma (2.33) and (A.17) several times, one can derive higher order versions of the same lemma. Unfortunately, there is no closed expression
for $\Phi_{T_1T_2\ldots T_k}$ in terms of higher brackets $\Phi_{T_1}, \Phi_{T_2}, \ldots, \Phi_{T_k}$ alone, but there is a fairly simple graphical representation, which we now sketch.

We will argue that $\Phi_{T_1T_2\ldots T_k}$ can be understood as a restricted sum over oriented and connected tree diagrams with $k$ 1-, 2- and 3-vertices.

First of all, we take every line in the tree to run between vertices. In particular: every external leg is assumed decorated with an external point, a “1-vertex”. All other vertices but a root-vertex are supposed to have at least one in-going line. Because there are at most three lines connected to each vertex, one can draw all oriented lines in the tree horizontally downwards, and vertically to the right.

- Each vertex corresponds to a higher bracket $\tilde{\Phi}_{T_i}$.
- A horizontally connected collection of $r-1$ oriented lines $r = 1, 2, 3, \ldots$, corresponds to a $r$-bracket-bracket $\left| \Phi_{T_{i_1}}, \ldots, \Phi_{T_{i_r}} \right|$, where $i_1 < i_2 < \ldots < i_r$ (cf. definition (A.13)). Of course one can skip the horizontal orientation $i_1 < i_2 < \ldots < i_r$ inside a bracket-bracket, at the cost of introducing a symmetry factor $\frac{1}{r!}$ for each bracket-bracket.
- A downward line corresponds to the action of the co-derivation $(\cdot) \circ b_{(-)}$

$$\tilde{\Phi}_{T_i} \circ b_{\left| \Phi_{T_{i_1}}, \ldots, \Phi_{T_{i_r}} \right|} \right),$$

with $i < i_1, \ldots, i < i_r$. (A conventional higher bracket $\Phi_{T_i} = \left| \Phi_{T_i} \right|$ is also considered to be a 1-bracket-bracket.) An incoming downward lines actual attachment position to a bracket-bracket is immaterial, and tree diagram with different incoming attachment position are considered equal, and should only be counted as one.

- Each tree is given a sign, because of the permutation of Grassmann-graded brackets within it. The easiest way to specify this sign is to enumerate the vertices, which is basically the same as specifying a permutation $\tau \in S_k$ that takes the enumeration of the operators $T_1, T_2, \ldots, T_k$ into this enumeration of the vertices. The sign is then computed as the sign originating from simply permuting Grassmann graded quantities:

$$\left( T_1, \ldots, T_k \right) \mapsto \left( T_{\tau(1)}, \ldots, T_{\tau(k)} \right).$$

The vertex enumeration goes as follows: Start at the left-uppermost vertex, proceed downwards if possible, else to the right. When entering a bracket-bracket, start with the left entry. When hitting an end-bracket-bracket, go back to the last furcation point (that is, the next-to-last bracket-bracket), then go to the right, etc.

**Proof:** (sketched here only for the bosonic case). We use induction in the total number $k$ of 1-,2- and 3-vertices. From the lemma

$$\Phi_{T_1T_2\ldots T_{k+1}} = \Phi_{T_1T_2\ldots T_k} \circ b_{\Phi_{T_{k+1}}} + \left| \Phi_{T_1T_2\ldots T_k}, \Phi_{T_{k+1}} \right|$$

Now each tree with $k+1$ vertices of the above type can be grown from a tree with $k$ vertices by attaching either an extra $\Phi_{T_{k+1}}$-entry to the right (which gives a horizontal growth) in a bracket-bracket, or a downward growth $\circ b_{\Phi_{T_{k+1}}}$, from a vertex, if there is not already an outgoing downward line there. It is easy to see from (A.17) that the action of $\circ b_{\Phi_{T_{k+1}}}$ on all the trees with $k$ vertices $\Phi_{T_1T_2\ldots T_k}$ yields all the trees with $k + 1$ vertices exactly once, except the diagram where the root-bracket-bracket is enlarged by an entry to the right. This tree is then built via the second term on the right hand side of (B.3).
It is clear that these generalized (higher) main identities quickly become totally unwieldy when written out in full. In the special case of just one Grassmann-odd operator $T$, the higher main identities actually give no genuinely new information when $T^2 = 0$. This nilpotent case can be seen using the graphical representation, where the main identity (2.33) states that one vertical line (with a bracket at each end) is equal to zero. This means that an end-bracket-bracket that contains precisely one bracket causes the tree to vanish. Any end-bracket-bracket containing more brackets causes the tree to vanish, because the brackets are Grassmann odd. However, already in the simple case of just one odd operator $T$ which is not nilpotent, the above generalized main identities relate brackets based on $T^n$ (up to as many powers possible while still having $T^n \neq 0$) to those based on $T$. Koszul [7] has given one particular example of these identities, but no general prescription for finding them.

The case of two anticommuting odd operators $T_1$ and $T_2$ is even more interesting. Let us restate the main identity (2.40) as

$$0 = \Phi_{T_{(a)}} \circ b_{\Phi_{T_{(b)}}}$$

(B.7)

At the next level, one new main identity arises:

$$0 = \Phi_{T_{(a)}} \circ b_{\Phi_{T_{(b)}}} \circ b_{\Phi_{T_{(c)}}} + \Phi_{T_{(a)}} \circ b_{\Phi_{T_{(b)}} \circ \Phi_{T_{(c)}}}$$

(B.8)
Shown in more details, the content of this new main identity involving two operators $T_1$ and $T_2$ is:

\[
0 = \sum_{r,s=0}^{r+s \leq n} \frac{1}{r!(n-r)!} \sum_{\pi \in S_n} (-1)^{\epsilon_{\pi}} \Phi^{n-r-s+1}_{T_a} \left( \Phi^{r}_{T_b} (A_{\pi(1)} \otimes \ldots \otimes A_{\pi(r)}) \right) \\
\otimes \Phi_{T_c}^{s} (A_{\pi(r+1)} \otimes \ldots \otimes A_{\pi(r+s)}) \otimes \Phi_{T_c}^{n-r-s+1} (A_{\pi(r+s+1)} \otimes \ldots \otimes A_{\pi(n)})
\]

\[
+ \sum_{r,s=0}^{r+s \leq n} \frac{1}{r!(n-r)!} \sum_{\pi \in S_n} (-1)^{\epsilon_{\pi} + \epsilon_{A_{\pi(1)}} + \ldots + \epsilon_{A_{\pi(r)}}} \Phi^{n-r-s+2}_{T_a} \left( \Phi^{r}_{T_b} (A_{\pi(1)} \otimes \ldots \otimes A_{\pi(r)}) \right) \\
\otimes \Phi_{T_c}^{s} (A_{\pi(r+1)} \otimes \ldots \otimes A_{\pi(r+s)}) \otimes A_{\pi(r+s+1)} \otimes \ldots \otimes A_{\pi(n)}
\]

(B.9)

Remarkably, this identity holds without any kind of symmetrisation in the indices $a, b, c = 1, 2$. Only the case $b \neq c$ is truly a new identity. For other combinations of $a, b, c = 1, 2$, eq. (B.8) can be deduced from the original main identity and symmetry arguments. One can prove that all nilpotent $Sp(2)$-symmetric higher main identities can be derived from these two main identities. This follows from:

\[
0 = \left| \Phi_{T_a}, \Phi_{T_b}, \Phi_{T_c}, \ldots \right|
\]

(B.10)

\[
0 = \sum_{\text{cycl. } a,b,c} \left| \Phi_{T_a} \circ b_{\Phi_{T_b}}, \Phi_{T_c}, \ldots \right|
\]

(B.11)

The analogous $Sp(2)$-symmetric formulations have been discussed in section 4.
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