GRADIENT ESTIMATES FOR THE DEGENERATE PARABOLIC EQUATION $u_t = \Delta F(u)$ ON MANIFOLDS AND SOME LIOUVILLE-TYPE THEOREMS

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Abstract. In this paper, we first prove a gradient estimate for the positive solutions of the ecumenic degenerate parabolic equation:

$$u_t = \Delta F(u),$$

with $F'(u) > 0$, on a complete Riemannian manifold with Ricci curvature lower bound $\text{Ric}(M) \geq -k$ with $k \geq 0$. The second part, we apply the gradient estimates to the Fast Diffusion Equations (FDE) and Porous Media Equations (PME):

$$u_t = \Delta (u^p), \quad p > 0,$$

to obtain the gradient estimates in a larger range of $p$ than the range of $p$ for Harnack inequalities and Cauchy problems in the literature, and also prove some Liouville-type theorems for positive global solutions on noncompact complete manifolds with nonnegative Ricci curvature for the Fast Diffusion Equations (FDE) and the Porous Media Equations (PME).

1. Introduction

In this paper, we consider a localized Hamilton-type gradient estimates for the positive solutions of the ecumenic degenerate parabolic equation

(1.1) $$u_t = \Delta F(u)$$

on a complete Riemannian manifold $(M^n, g)$ of dimension $n \geq 1$ with $\text{Ric}(M) \geq -k$ for some $k \geq 0$. Here $F \in C^2(0, \infty)$, $F' > 0$, and $\Delta$ is the Laplace-Beltrami operator of the metric $g$. There is a lot of literature on this kind of topics. For example, related problems such as the Fast Diffusion Equations and the Porous Media Equations have been considered by D.G. Aronson [1], G. Auchmuty and D. Bao [2], M.A. Herrero and M. Pierre [6] and S.T. Yau [10].

It is well known that, in the study of geometric analysis as well as other elliptic or parabolic equations, the gradient estimate and the Harnack inequality play a most important role. To begin with, let us review some main results on the gradient estimate and the Harnack inequality. The first one is the Harnack-type differential inequality for the heat equation by P. Li and S.T. Yau [7].

Theorem A (P. Li and S.T. Yau [7]). Let $M^n$ be a complete manifold of dimension $n \geq 2$ with $\text{Ric}(M^n) \geq -k$ for some $k \geq 0$. Suppose that $u$ is any
positive solution to the heat equation in \( B(x_0, R) \times [t_0 - T, t_0] \). Then for \( a > 1 \),

\[
\frac{|\nabla u|^2}{u^2} - a \frac{u_t}{u} \leq c_n \left( \frac{1}{R^2} + \frac{1}{T} + k \right)
\]

in \( B(x_0, R/2) \times [t_0 - T/2, t_0] \). Here \( c_n \) depends only on the dimension \( n \) and \( a \).

For the heat equation on compact manifolds without boundary, there is the another type gradient estimate by R. Hamilton [5]:

**Theorem B (R. Hamilton [5]).** Let \( M^n \) be a compact manifold without boundary and with Ricci \( (M) \geq -k \) for some \( k \geq 0 \). Let \( u \) be a smooth positive solution of the heat equation with \( u \leq M \) for all \( (x, t) \in M^n \times (0, \infty) \). Then

\[
\frac{|\nabla u|^2}{u^2} \leq \left( \frac{1}{t} + 2k \right) \ln \frac{M}{u}.
\]

The Hamilton-type gradient estimate takes up a significant position in the study of the heat equation. However, the classical Hamilton’s estimate is a global result which requires the heat equation should be posed on compact manifold without boundary. Recently, a localized Hamilton type gradient estimate was proved by P. Souplet and Q.S. Zhang [9], which can be viewed as a combination of Li-Yau’s differential Harnack inequality [7] and Hamilton’s gradient estimate [5].

**Theorem C (P. Souplet and Q.S. Zhang [9]).** Let \( M^n \) be a complete Riemannian manifold with dimension \( n \geq 1 \), Ricci \((M) \geq -k \), \( k \geq 0 \). Suppose that \( F \in C^2(0, \infty) \) with \( F' > 0 \), and \( u \) is any positive solution of the degenerate parabolic equation (1.1) in \( Q_{R, T} = B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty) \). Suppose also that \( u \leq M \in Q_{R, T} \). Then there exists a dimensional constant \( C \) such that

\[
\frac{|\nabla_x u(x, t)|}{u(x, t)} \leq C \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \right) \left( 1 + \ln \frac{M}{u} \right).
\]

for all \((x, t) \in Q_{R/2, T/2}\).

Moreover, if \( M^n \) has nonnegative Ricci curvature and \( u \) is any positive solution of the heat equation on \( M \times (0, \infty) \), then there exist dimensional constants \( C_1, C_2 \) such that

\[
\frac{|\nabla_x u(x, t)|}{u(x, t)} \leq C_1 \frac{1}{\sqrt{t}} \left( 1 + \ln \frac{u(x, 2t)}{u(x, t)} \right).
\]

for all \( x \in M^n \) and \( t > 0 \).

The first Harnack-type differential inequality dealing with the degenerate parabolic equation (1.1) was attributed to S.T. Yau [10], while the Harnack-type differential inequalities have been widely used for Porous Media Equations, see [1], [2], [6]. We state Yau’s main result as below.

**Theorem D (S.T. Yau [10]).** Let \( M^n \) be a compact Riemannian manifold without boundary, Ricci \((M) \geq 0 \). Suppose that \( F \in C^2(0, \infty) \) with \( F' > 0 \), \( c(t) \in C^1(0, \infty) \), and \( u \) is any positive solution of the degenerate parabolic equation

\[
u_t = \Delta F(u)
\]

on \( M^n \). Let \( \alpha \neq 0 \) be an arbitrary constant. Define a function \( G \) on \((0, \infty)\) by \( G'(s) = F'(s) / s \), and we abbreviate \( G = G(u) \), \( F^\kappa = F^\kappa(u), \kappa = 0, 1, 2 \).

If the conditions below are satisfied:

(A) \( |\nabla G|^2 - \alpha G_t - c(t) \leq 0 \) at \( t = 0 \);


(B). (nonlinear condition) the following quadratic inequality holds true for all $x \geq 0$
\[
0 \geq \frac{1 - \alpha}{\alpha^2} \left( \alpha u F'' - \frac{2(1 - \alpha)}{n} F' \right) x^2 + \left( \frac{4(1 - \alpha)}{n\alpha^2} - \frac{u F''}{F'} \right) c(t) x
\]
\[
- \left( \frac{2}{n} + \alpha \frac{u F''}{F'} \right) \frac{c^2(t)}{\alpha^2 F'} - c'(t)
\]
then we have for all $t > 0$ that
\[
|\nabla G|^2 - \alpha G - c \leq 0.
\]

It is natural to seek a localized Hamilton-type gradient estimate for the general parabolic equation (1.1) as P. Souplet and Q.S. Zhang [9] did for the heat equation on complete manifolds. Under some strong assumptions a localized Hamilton-type gradient estimate for the equation (1.1) was obtained by L. Ma, L. Zhao and X. Song [8]. The first main result of this paper is to generalize the result in [8] to the following Hamilton-type gradient estimate for the equation (1.1) under some weaker assumptions:

**Theorem 1.1. (Gradient Estimate).** Let $M^n$ be a complete Riemannian manifold with dimension $n \geq 1$, $Ric(M) \geq -k$, $k \geq 0$. Suppose that $F \in C^2(0, \infty)$ with $F' > 0$, and $u$ is any positive solution of the degenerate parabolic equation (1.1) in $Q_{R,T} \equiv \mathbb{B}(x_0,R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$. Define a function $G$ on $(0, \infty)$ by $G'(s) = F'(s)/s$. We denote by $U \subset (0, \infty)$ the value range of $u$. If there exist nonnegative constants $K, \alpha, \delta, \tau$ and $\gamma$ satisfying:

(A). $F'(s) \leq K, \forall s \in U$;

(B). $\alpha - G(s) \geq \delta > 0, \forall s \in U$;

(C). (Nonlinear Condition)

\[
\begin{align*}
\tau &\geq \frac{|s F''(s)|}{F'(s)}, \quad \forall s \in U \\
2 + \frac{s F''(s)}{F'(s)} \left( 2 - (n - 1) \frac{s F''(s)}{F'(s)} \frac{\alpha - G(s)}{F'(s)} \right) &\geq \gamma > 0, \quad \forall s \in U
\end{align*}
\]

then there exists a constant $C(n, \alpha, K, \tau, \gamma)$ depending only on $n$, $\delta$, $K$, $\tau$ and $\gamma$ such that $\forall (x, t) \in Q_{R/2, T/2}$, there holds

\[
|\nabla u G(u(x, t))| \leq C(n, \alpha, K, \tau, \gamma) \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \right).
\]

When $n = 1$, the Ricci curvature lower bound $k$ vanishes.

**Remark 1.1.** One might compare the above theorem with Theorem 7 in [8]. the nonlinear condition (C) here is much weaker than the one in [8]. Our Nonlinear Condition (C) can apply to a much larger class of equations. One may see it from those results on the Fast Diffusion Equations and the Porous Media Equations in the second part of the paper.
The second part of this paper, we study the Fast Diffusion Equations and the Porous Media Equations
\begin{equation}
\tag{1.3} \frac{u_t}{u} = \Delta(u^p)
\end{equation}
on a complete Riemannian manifold \( (M^n, g) \). Firstly for \( p < 1 \), we generalize the gradient estimates for (1.3) in [8], where the theorem holds only for dimension \( n = 2 \) or \( n = 3 \) with much shorter range of \( p \) (only for \( p \in (1 - 1/\sqrt{n}, 1) \) for \( n = 2, 3 \) in [8]), as the following result:

**Theorem 1.2.** Let \( M^n \) be a complete Riemannian manifold with dimension \( n \geq 1 \), \( \text{Ric}(M) \geq -k \), \( k \geq 0 \). Suppose that \( u \leq M \) is a positive solution of the Porous Media Equation (1.3) in \( Q_{R,T} = B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty) \), where
\begin{equation}
1 - \frac{4}{n + 3} < p < 1 \quad \text{for } n \geq 1.
\end{equation}

Then there exists a constant \( C(n,p) \) depending only on \( n \) and \( p \) such that
\begin{equation}
\frac{\left| \nabla u(x,t) \right|}{u(x,t)} \leq C(n,p) \left( \frac{1}{R} + \frac{M^{(1-p)/2}}{\sqrt{T}} + \sqrt{k} \right),
\end{equation}
for all \( (x,t) \) in \( Q_{R/2,T/2} \). When \( n = 1 \), the Ricci curvature lower bound \( k \) vanishes.

**Remark 1.2.** One should notice that the range of \( p \) here is \((1 - \frac{4}{n + 3}, 1)\), while previous results for the Fast Diffusion Equations on \( \mathbb{R}^n \) requires \( p \in ((1 - \frac{2}{n})^+, 1) \), for example, a survey by D.G. Aronson [1], the Cauchy problem by M.A. Herrero and M. Pierre [6], the Harnack-type inequalities by G. Auchmuty and D. Bao [2]. We can see that for \( n \geq 3 \), the range of \( p \) for our gradient estimate is larger than the range of \( p \) in previous results [1], [2], [6], [8]. Our gradient estimate will be a useful tool to study related problem for the Fast Diffusion Equations in this large range of \( p \), with which one couldn’t deal before even on \( \mathbb{R}^n \). We shall study the Cauchy problem, the Harnack-type inequalities for the Fast Diffusion Equations \((p < 1)\) in another paper.

On a noncompact manifold with nonnegative Ricci curvature, an immediate application of Theorem 1.2 is the following time-dependent Liouville theorem of the Fast Diffusion Equations, which generalizes Yau’s celebrated Liouville Theorem for positive harmonic functions.

**Theorem 1.3. (Liouville theorem)** Let \( M^n \) be a complete, noncompact manifold with nonnegative Ricci curvature. Let \( u \) be a positive ancient solution, a solution defined in all space and negative time, of the Porous Medium Equation for \( 1 - \frac{4}{n + 3} < p < 1 \), and \( L(s) \in C(\mathbb{R}) \) be any strick increasing function with \( L(s) \to \infty \) as \( s \to \infty \), such that
\[ u(x,t) = o\left( L(d(x)) + |t|^{1/(1-p)} \right) \]
near infinity. Then \( u \) is a constant.

**Remark 1.3.** One might see that the growth condition in the spatial direction in Theorem is very weak, since we might choose
\[ L(s) = \exp(\exp(\cdots(\exp(s))\cdots)) \]
with \( l \exp \) for any \( l > 0 \). Note that one might write any positive harmonic function \( v(x) \) as a positive global solution of \( \Delta(u^p) = 0 \) with \( u(x) = v(x)^{1/p} \), hence Yau’s celebrated Liouville theorem for positive harmonic functions is a special case of Theorem 1.3 for time dependent positive solutions of the Fast Diffusion Equations, while one couldn’t do this for the Heat Equations (see examples in [9]).

Secondly, we shall study the Porous Medium Equation for \( p > 1 \). We have the following gradient estimate for \( n = 1 \):

**Theorem 1.4.** Suppose that \( u \) is a positive solution of the Porous Media Equation (1.3) in \( Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty) \), with range(u) = \([m, M]\). Let \( G(u) = \frac{p}{p-1} u^{p-1} \), \( \alpha = \frac{p}{p-1} M^{p-1} (1 + \delta) \) with some constant \( \delta > 0 \). Then for any \( p > 1 \), there exists a constant \( C(p) \) depending only on \( p \) such that

\[
\frac{|\nabla \alpha G(u(x,t))|}{\alpha - G(u(x,t))} \leq C(p) \left( \frac{1 + \delta}{\delta R} + \frac{1}{\sqrt{MP^{-1} \delta T}} \right).
\]

for all \((x, t)\) in \( Q_{R/2,T/2} \).

An immediate application of this theorem is the following time-dependent Liouville theorem for the Porous Medium Equation with \( p > 1 \) and the Liouville theorem for positive solutions of \( \Delta(u^p) = 0 \) when \( n = 1 \),

**Theorem 1.5. (Liouville theorem)** Let \( u \) be a positive ancient solution, a solution defined in all space and negative time, to the Porous Medium Equation (\( p > 1 \)) on \( \mathbb{R} \), such that

\[
u(x, t) = \alpha \left( d(x)^{1/(p-1)} + |t|^{1/(p-1)} \right)
\]

near infinity. Then \( u \) is a constant.

And for \( \dim M \geq 2 \), we have the following gradient estimate:

**Theorem 1.6.** Let \( M \) be a complete Riemannian manifold with dimension \( n \geq 2 \), \( \text{Ric}(M) \geq -k \), \( k \geq 0 \). Suppose that \( u \) is a positive solution of the Porous Media Equation (1.3) in \( Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty) \), with range(u) = \([m, M]\). Let \( G(u) = \frac{p}{p-1} u^{p-1} \), \( \alpha = \frac{p}{p-1} M^{p-1} (1 + \delta) \) with some small constant \( 0 < \delta \leq \frac{4}{n-1} \). If the following pinch condition on \( m, M \) holds

\[
1 \leq \left( \frac{M}{m} \right)^{p-1} < \frac{1 + \delta}{(n-1)(p-1) + 1},
\]

then there exists a constant \( C(n,p) \) depending only on \( n \) and \( p \), and

\[
\gamma = 2p - \frac{(n-1)(p-1) M^{p-1} (1 + \delta) - m^{p-1}}{m^{p-1}} > 0,
\]

such that

\[
\frac{|\nabla \alpha G(u(x,t))|}{\alpha - G(u(x,t))} \leq C(n,p) \left( \frac{1 + \delta}{\gamma \delta R} + \frac{1}{\sqrt{\gamma \delta M^{p-1} \delta T}} + \sqrt{\frac{k}{\delta}} \right).
\]

for all \((x, t)\) in \( Q_{R/2,T/2} \).
The rest of this paper is organized as follows. In section 2, we prove Theorem 1.1 following the argument in [8], which can be regarded as a combination of the analysis in [7] and [9]. In section 3, we apply the Theorem 1.1 to study the Fast Diffusion Equation and the Porous Media Equations and prove Theorem 1.2-1.6. Here and later Ricci(M) is the Ricci curvature and a manifold is complete if every geodesic extends to infinity.

We shall use the Lemma 2.1. Let $A = (a_{ij})$ be a nonzero $n \times n$ symmetric matrix with eigenvalues $\{\lambda_k\}$, for any $a, b \in \mathbb{R}$, one has the following properties:

\[ (a) \quad |A|^2 = \sum_{i,j=1}^n a_{ij}^2 = \text{tr}(AA^T) = \sum_{k=1}^n \lambda_k^2; \]

\[ (b) \quad \max_{|v|=1} (aA + btrAI_n)(v, v) = a\lambda_i + b \sum_{k=1}^n \lambda_k, \text{ for some } 1 \leq i \leq n \]

\[ \min_{|v|=1} (aA + btrAI_n)(v, v) = a\lambda_j + b \sum_{k=1}^n \lambda_k, \text{ for some } 1 \leq j \leq n. \]

\[ (c) \quad \max_{A \in S(n)} \left| \frac{aA + btrAI_n}{|A|} (v, v) \right|^2 = (a + b)^2 + (n - 1)b^2. \]

**Proof.** (a) follows from direct computation and $A$ symmetry.

(b) follows from the facts that $\max_{|v|=1} (aA + btrAI_n)(v, v)$ and $\min_{|v|=1} (aA + btrAI_n)(v, v)$ are the maximal and minimal eigenvalue of $aA + btrAI_n$, and the eigenvalues of $aA + btrAI_n$ are $\{a\lambda_i + b \sum_{k=1}^n \lambda_k\}_{i=1}^n$.

To prove (c), apply (a) and (b), we have

\[ \max_{A \in S(n)} \left| \frac{aA + btrAI_n}{|A|} (v, v) \right|^2 = \max_{\sum_{k=1}^n \lambda_k^2 = 1} \left[ a\lambda_1 + b \sum_{k=1}^n \lambda_k \right]^2 \]

By Lagrangian multiplier method in Calculus, the extremums of $f(\lambda_1, \ldots, \lambda_n) = a\lambda_1 + b \sum_{k=1}^n \lambda_k$ under constrain $\sum_{k=1}^n \lambda_k^2 = 1$ are

\[ - (a + b)^2 + (n - 1)b^2)^{1/2} \leq a\lambda_1 + b \sum_{k=1}^n \lambda_k \leq [(a + b)^2 + (n - 1)b^2]^{1/2}. \]

The rest of this section, we shall prove Theorem 1.1. We shall define a quantity $w(x, t)$ and the starting part of the argument will be to deduce a differential inequality on $w(x, t)$, then use the well-known cut-off function of Li-Yau [7], to derive the desired bounds.
Let \( \phi = \ln u \). Since \( u \) is a solution to the equation \( u_t = \Delta (F(u)) \), simple calculation shows
\[
\phi_t = \Delta (G(u)) + \nabla G(u) \cdot \nabla \phi.
\]
Writing \( g(\phi) = G(e^\phi) \), and multiplying (2.1) by \( g'(\phi) \) and some elementary computations, we then get
\[
\begin{align*}
g_t &= g'(\Delta g + |\nabla g|^2), \quad \nabla g = g' \nabla \phi. \\
g'(\phi) &= G'(e^\phi) e^\phi = F'(u). \\
g''(\phi) &= F''(e^\phi) e^\phi = F''(u) u.
\end{align*}
\]
Set
\[
w = w(x, t) = |\nabla \ln(\alpha - g)|^2 = \frac{|\nabla g|^2}{(\alpha - g)^2},
\]
and we first derive a differential inequality for \( w \), to which we apply the maximum principle.

**Lemma 2.2.** \( w \) satisfies the following differential inequality:
\[
g' \Delta w - w_t \geq Lu^2 - 2g'kw - L_1 \nabla g \cdot \nabla w,
\]
where \( L \) and \( L_1 \) are some functions given in (2.10) and (2.11).

**Proof.** After some elementary computations in local orthonormal system as in [8], we get that
\[
w_t = 2 \frac{\nabla \cdot \nabla g_t}{(\alpha - g)^2} + 2 \frac{|\nabla g|^2 g_t}{(\alpha - g)^3}
\]
\[
= 2 \frac{\nabla g \cdot \nabla (g' \Delta g + |\nabla g|^2)}{(\alpha - g)^2} + 2 \frac{|\nabla g|^2 (g' \Delta g + |\nabla g|^2)}{(\alpha - g)^3}
\]
\[
= 2g' \frac{\nabla g \cdot \nabla \Delta g}{(\alpha - g)^2} + 2 \frac{\Delta g \nabla g \cdot \nabla g'}{(\alpha - g)^2} + 2 \frac{\nabla g \cdot |\nabla g|^2}{(\alpha - g)^2} + 2 \frac{g' |\nabla g|^2 \Delta g}{(\alpha - g)^3} + 2 \frac{|\nabla g|^4}{(\alpha - g)^3}
\]
(2.3)
\[
(2.4)
\]
\[
\Delta w = w_{jj} = 2 \left( \frac{g^2_{ij}}{(\alpha - g)^2} \right) + 2 \left( \frac{g^2_i g_j}{(\alpha - g)^3} \right)
\]
(2.5)
\[
= 2 \frac{g^2_{ij}}{(\alpha - g)^2} + 2 \frac{g_i g_{jj}}{(\alpha - g)^2} + 8 \frac{g_i g_{ij} g_j}{(\alpha - g)^3} + 2 \frac{|\nabla g|^2 \Delta g}{(\alpha - g)^3} + 6 \frac{|\nabla g|^4}{(\alpha - g)^4}
\]
By (2.3) and (2.5), we obtain that
\[
g' \Delta w - w_t = 2g' \frac{g^2_{ij}}{(\alpha - g)^2} + 2g_i \frac{g_{ijj} - g_j g_{ij}}{(\alpha - g)^2} + 8g_i \frac{g_{ij} g_j}{(\alpha - g)^3}
\]
\[
- 4 \frac{g_i g_{ij} g_j}{(\alpha - g)^3} + 6g_i \frac{|\nabla g|^4}{(\alpha - g)^4} - 2 \frac{g'' |\nabla g|^2 \Delta g}{(\alpha - g)^2} - 2 \frac{|\nabla g|^4}{(\alpha - g)^3}
\]
Bochner identity implies that
\[
g_{ij}g_{ij} - g_j g_{ij} = g_j (g_{ii} - g_{ij}) = R_{ij} g_i g_j = Ric(\nabla g, \nabla g)
\]
where $R_{ij}$ is the Ricci curvature tensor. Therefore we arrive at

$$g\Delta w - w_t = 2g' \frac{g_{ij}^2}{(\alpha - g)^2} + 2g' \frac{\text{Ric}(\nabla g, \nabla g) + 8g' g_{ij}\eta_j - 4g g_{ij}g_j}{(\alpha - g)^2} + 6g' \frac{|\nabla g|^4}{(\alpha - g)^4} - 2g' \frac{|\nabla g|^2 \Delta g}{g' (\alpha - g)^2} - 2 \frac{|\nabla g|^4}{(\alpha - g)^3}$$

(2.6)

Recalling (2.4), we have

$$\nabla g \cdot \nabla w = 2 \frac{g_{ij}g_j}{(\alpha - g)^2} + 2 \frac{|\nabla g|^4}{(\alpha - g)^3}$$

(2.7)

Adding $\left(2 - \frac{2g'}{\alpha - g} - \eta_w\right)$ to (2.7) with (2.6), where $\eta$ is a parameter function to be determined later, we conclude that

$$g'\Delta w - w_t = 2g' \frac{g_{ij}^2}{(\alpha - g)^2} + 2g' \left( \frac{2g' - \eta g''}{\alpha - g} \right) \frac{g_{ij}g_j}{(\alpha - g)^2} - 2g' \frac{|\nabla g|^2 \Delta g}{g'(\alpha - g)^2} + 4g' \frac{\text{Ric}(\nabla g, \nabla g)}{(\alpha - g)^2} - \left(2 - \frac{2g'}{\alpha - g} - \eta g'' \right) \nabla g \cdot \nabla w$$

(2.8)

Denote $f = 2g'/(\alpha - g)$, $b = g''/g'$, $A = (g_{ij})$, and $v = \nabla g/|\nabla g|$, then we have $g^2_{ij} = |A|^2$, $g g_{ij}g_j = A(v, v)|\nabla g|^2$, and $\Delta g = \text{tr}A$. From definition of $w$, the right side of (2.8) can be written as

$$2g' \frac{|A|^2}{(\alpha - g)^2} + 2(\alpha - g) \left[ 2 - 2\eta g'' \right] \frac{A(v, v)}{|A|} - b \frac{\text{tr}A}{|A|} \frac{|A|}{\alpha - g} w$$

$$+ (\alpha - g) \left( f + 2 - 2\eta b \right) w^2 + 2g' \text{Ric}(v, v) w - \left( 2 - f - \eta b \right) \nabla g \cdot \nabla w$$

$$= 2g' \left[ \frac{|A|}{(\alpha - g)} + \frac{1}{f} \left( \frac{f - \eta b}{\alpha - g} \frac{A(v, v)}{|A|} - b \frac{\text{tr}A}{|A|} \right) w \right]$$

$$+ (\alpha - g) \left( f + 2 - 2\eta b - \frac{1}{f} \left( \frac{f - \eta b}{\alpha - g} \frac{A(v, v)}{|A|} - b \frac{\text{tr}A}{|A|} \right) \right) w^2$$

$$+ 2g' \text{Ric}(v, v) w - \left( 2 - f - \eta b \right) \nabla g \cdot \nabla w$$

$$\geq \frac{\alpha - g}{f} \left( f^2 + 2 - 2\eta b \right) \left( \frac{f - \eta b}{\alpha - g} \frac{A(v, v)}{|A|} - b \frac{\text{tr}A}{|A|} \right)^2 w^2$$

(2.9) + 2g' \text{Ric}(v, v) w - \left( 2 - f - \eta b \right) \nabla g \cdot \nabla w
Apply Lemma 2.1 to (2.9), we have
\[ g' \Delta w - w_t \geq \frac{\alpha - g}{f} \left( f^2 + \left( 2 - 2\eta b \right) f - \left( f - \eta b - b \right)^2 - (n-1)b^2 \right) w^2 \]
\[ + 2g' \text{Ric}(v,v)w - \left( 2 - f - \eta b \right) \nabla g \cdot \nabla w \]
\[ = \frac{\alpha - g}{f} \left( 2(1+b)f - (n-1+[\eta+1]^2)b^2 \right) w^2 \]
\[ + 2g' \text{Ric}(v,v)w - \left( 2 - f - \eta b \right) \nabla g \cdot \nabla w \]

Next we estimate the coefficient \( L \) of \( w^2 \). Pick \( \eta = -1 \), We can bound \( L \) as the following:
\[
(2.10) \quad L = \left( \alpha - g \right) \left( 2(1+b) - (n-1)\frac{b^2}{f} \right) \geq (\alpha - g)\gamma > 0, \]
since \( 2(1+b) - (n-1)\frac{b^2}{f} \geq \gamma > 0 \) from our Nonlinear Condition (C) in Theorem 1.1. We estimate the coefficient \( L_1 \) of \( \nabla g \cdot \nabla w \) as
\[
(2.11) \quad |L_1| = \left| 2 - f + b \right| \leq 2 + \tau + f. \]

Hence from \( \text{Ric}(M) \geq -k \), we have
\[
(2.12) \quad g' \Delta w - w_t \geq Lw^2 - 2g'kw - L_1 \nabla g \cdot \nabla w \]

Now we can apply maximum principle on the differential inequality (2.12) to prove our gradient estimate (1.2). We will follow [8] and [9] to use the well-known cut-off function by Li and Yau [7] to show Theorem 1.1. We caution the reader that the calculation is not the same as that in [7] due to the difference of the first-order term.

**Proof of Theorem 1.1.** Let \( \Psi = \Psi(x,t) \) be a smooth cut-off function supported in \( Q_{R,T} \), satisfying the following properties:

1. \( \Psi = \Psi(d(x,x_0),t) \), \( \Psi = 1 \) in \( Q_{R/2,T/2} \), \( 0 \leq \Psi \leq 1 \);
2. \( \Psi \) is decreasing as a radial function in the spatial variables;
3. \( \frac{\partial \Psi}{\partial r} \leq \frac{C}{R} \), \( \frac{\partial^2 \Psi}{\partial r^2} \leq \frac{C}{R^2} \), when \( 0 < a < 1 \);
4. \( \frac{\partial^2 \Psi}{\partial r^2} \leq \frac{C}{R^2} \).

Then, from (2.12) and a straightforward calculation, one has
\[
(2.13) \quad g' \Delta (\Psi w) - (\Psi w)_t + L_1 \nabla g \cdot \nabla (\Psi w) - 2g' \frac{\nabla \Psi}{\Psi} \cdot \nabla (\Psi w) \geq L\Psi w^2 - 2kg'\Psi w + g' \Delta \Psi w - \Psi w + L_1(\nabla g \cdot \nabla \Psi)w - 2g' \frac{\left| \nabla \Psi \right|^2}{\Psi}w. \]

Hereafter we always use \( C \) to denote different constant depending on \( n \) only. We obtain the upper bounds for each term of the right-hand side of (2.13) as did in P.
Here we make use of Young’s inequality,

\begin{equation}
\left| 2kg'\Psi w \right| \leq \frac{1}{6} L \Psi w^2 + \frac{6k^2 g^2 \Psi}{L} \leq \frac{1}{6} L \Psi w^2 + C k^2 \frac{g^2}{L};
\end{equation}

\begin{equation}
\left| \Psi w \right| \leq \frac{1}{6} L \Psi w^2 + \frac{3|\Psi w|^2}{2L} \leq \frac{1}{6} L \Psi w^2 + \frac{C}{T^2} \frac{1}{L};
\end{equation}

\begin{equation}
\left| 2g' \frac{\nabla \Psi^2}{\Psi} w \right| \leq \frac{1}{6} L \Psi w^2 + \frac{6g^2 |\nabla \Psi|^4}{L} \Psi^3 \leq \frac{1}{6} L \Psi w^2 + \frac{C g^2}{R^4} \frac{1}{L};
\end{equation}

\begin{align*}
|L(\nabla g \cdot \nabla \Psi)w| &= \left| L(\nabla g \cdot \nabla \Psi) \frac{\alpha - g}{|\nabla g|} \right|^{3/2} \\
&\leq \frac{1}{6} L \Psi w^2 + \frac{C L^4(\alpha - g)^4}{L^3} |\nabla \Psi|^4 \\
&\leq \frac{1}{6} L \Psi w^2 + \frac{C L^4(\alpha - g)^4}{L^3}.
\end{align*}

Here we make use of Young’s inequality,

\[ ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall p, q > 0 \text{ with } \frac{1}{p} + \frac{1}{q} = 1. \]

Furthermore, by the properties of \( \Psi \) and the assumption on the Ricci curvature, one has

\begin{align*}
\left| -g' \Delta \Psi w \right| &\leq g' \left[ \frac{\partial^2 \Psi}{r} + (n - 1) \frac{\partial_r \Psi}{r} + \partial_r \Psi \partial_r \ln(\sqrt{g}) \right] w \\
&\leq g' \left[ \frac{\partial^2 \Psi}{r} + 2(n - 1) \frac{|\partial_r \Psi|}{r} + k |\partial_r \Psi| \right] w \\
&\leq \frac{1}{6} L \Psi w^2 + \frac{9g^2}{L} \left[ \frac{|\partial^2 \Psi|^2}{\Psi} + 4(n - 1)^2 |\partial_r \Psi|^2 \frac{R^2 \Psi}{R^2} + k \frac{|\partial_r \Psi|^2}{\Psi} \right] \\
&\leq \frac{1}{6} L \Psi w^2 + C \left[ \frac{1}{R^2} \right] \frac{1}{R^4} + \frac{k}{R^2} \frac{g^2}{L}.
\end{align*}

Inserting (2.14)-(2.18) into the right-hand side of (2.13), we deduce that

\begin{align*}
g' \Delta(\Psi w) - (\Psi w)_t + L_1 \nabla g \cdot \nabla (\Psi w) - 2g' \frac{\nabla \Psi}{\Psi} \cdot \nabla (\Psi w) \\
\geq \frac{L}{6} \Psi w^2 - C \left[ L_1^4(\alpha - g)^4 \frac{1}{R^4} + \frac{g^2}{L} \frac{1}{R^4} + \frac{1}{L \ T^2} + \frac{g^2}{L \ k^2} \right]
\end{align*}

Recalling that \( L \geq 2(\alpha - g) > 0, L_1 \leq 2 + \tau + f \) and \( f = g'/(\alpha - g) \), that is

\begin{align*}
g' \Delta(\Psi w) - (\Psi w)_t + L_1 \nabla g \cdot \nabla (\Psi w) - 2g' \frac{\nabla \Psi}{\Psi} \cdot \nabla (\Psi w) \\
\geq \frac{L}{6} \Psi w^2 - C(\gamma) \left[ \frac{(1 + f + \tau)^4}{R^4} + \frac{1}{(\alpha - g)^2} \left[ \frac{g^2}{R^4} + \frac{1}{T^2} + \frac{g^2}{L^2 k^2} \right] \right]
\end{align*}

\begin{align*}
(2.20) \geq \frac{L}{6} \Psi w^2 - C(\gamma) \left[ \frac{(1 + f + \tau)^4}{R^4} + \frac{1}{(\alpha - g)^2} \frac{1}{T^2} + f^2 k^2 \right].
\end{align*}
Suppose that the maximum of \((\Psi w)\) is reached at \((x_1, t_1) \in Q_{R,T}\). By [7], we can assume, without loss of generality, that \(x_1\) is not in the cut-locus of \(M\). Then at this point, one has \(\Delta (\Psi w) \leq 0\), \((\Psi w)_t \geq 0\) and \(\nabla (\Psi w) = 0\). Recalling that \(\alpha - g \geq \delta, 0 < g' \leq K\) and \(f \leq K/\delta\), we have

\[
\left(\Psi w^2\right)(x_1, t_1) \leq C(\gamma) \left(\frac{(\delta + K + \tau \delta)^4}{\delta^4 R^4} + \frac{1}{\delta^2} \frac{1}{T^2} + \left(\frac{K}{\delta}\right)^2 k^2\right)
\]

By assumption, the maximum of \((\Psi w)\) is reached at \((x_1, t_1) \in Q_{R,T}\), which implies that for any \((x, t) \in Q_{R,T}\)

\[
\left(\Psi w\right)^2(x, t) \leq \left(\Psi w\right)^2(x_1, t_1) \leq \left(\Psi w^2\right)(x_1, t_1)
\]

\[
\leq C(\gamma) \left(\frac{(\delta + K + \tau \delta)^4}{\delta^4 R^4} + \frac{1}{\delta^2} \frac{1}{T^2} + \left(\frac{K}{\delta}\right)^2 k^2\right)
\]

Noticing that \(\Psi(x, t) = 1\) in \(Q_{R/2, T/2}\) and \(w = |\nabla g|^2/(\alpha - g)^2\), we finally have proved

\[
\frac{|\nabla g|^2}{(\alpha - g)^2} \leq C(\gamma) \left(\frac{(\delta + K + \tau \delta)^2}{\delta^2 R^2} + \frac{1}{\delta T} + \frac{K}{\delta k}\right)
\]

which is exact the conclusion of Theorem 1.1. \(\square\)

3. Some applications

In this section, we shall study the Heat Equations, the Fast Diffusion Equations and the Porous Media Equations, and obtain some Hamilton-type gradient estimates by applying our Theorem 1.1, then prove some time-dependent Liouville Theorems on noncompact complete manifolds with nonnegative Ricci curvature. As a corollary, we obtain Yau’s celebrated Liouville theorem for positive harmonic functions: any positive harmonic function on a noncompact manifold with nonnegative Ricci curvature is a constant function.

3.1. Heat Equations: Let \(M^n\) be a complete Riemannian manifold with dimension \(n \geq 1\), \(\text{Ric}(M) \geq -k, k \geq 0\). Suppose that \(u \leq M\) is a positive solution of the heat equation

\[u_t = \Delta u\]

in \(Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)\). We may look as the case of \(F(u) = u\) in Theorem 1.1. Choose \(G(s) = \ln s\) and let \(\alpha = 1 + \ln M, K = 1, \tau = 1\) and \(\gamma = 2\) in Theorem 1.1, we obtain the Theorem C by P. Souplet and Q.S. Zhang in [9]) as a direct corollary of our Theorem 1.1.

**Corollary 3.1. (Theorem 1.1 in [9])** Let \(M^n\) be a complete Riemannian manifold with dimension \(n \geq 1\), \(\text{Ric}(M) \geq -k, k \geq 0\). Suppose that \(F \in C^2(0, \infty)\) with \(F' > 0\), and \(u\) is any positive solution of the degenerate parabolic equation (1.1) in \(Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)\). Suppose also that \(u \leq M\) in \(Q_{R,T}\). Then there exists a dimensional constant \(C\) such that

\[
\frac{|\nabla u(x,t)|}{u(x,t)} \leq C \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right) \left(1 + \ln \frac{M}{u}\right).
\]
for all \((x, t)\) in \(Q_{R/2, T/2}\).

Moreover, if \(M^n\) has nonnegative Ricci curvature and \(u\) is any positive solution of the heat equation on \(M \times (0, \infty)\), then there exist dimensional constants \(C_1, C_2\) such that

\[
\frac{\|\nabla_x u(x, t)\|}{u(x, t)} \leq C_1 \frac{1}{\sqrt{t}} \left( 1 + \ln \frac{u(x, 2t)}{u(x, t)} \right),
\]

for all \(x \in M^n\) and \(t > 0\).

Using the above gradient estimates for positive solutions of the heat equation, P. Souplet and Q.S. Zhang in [9]) proved the following time-dependent Liouville Theorem:

**Corollary 3.2.** (Theorem 1.2 in [9]) Let \(M\) be a complete, noncompact manifold with nonnegative Ricci curvature. Then the following conclusions hold.

(a) Let \(u\) be a positive ancient solution to the heat equation (that is, a solution defined in all space and negative time) such that \(u(x, t) = e^o(d(x)+\sqrt{|t|})\) near infinity. Then \(u\) is a constant.

(b) Let \(u\) be an ancient solution to the heat equation such that \(u(x, t) = o(|d(x)+\sqrt{|t|})\) near infinity. Then \(u\) is a constant.

As discussed in [9], both growth conditions of the above theorem in the spatial direction are sharp, by the same simple examples. Hence one couldn’t obtain Yau’s celebrated Liouville theorem for positive harmonic functions directly from the above time-dependent Liouville Theorem for heat equation.

### 3.2. Fast Diffusion Equations

Let \(M^n\) be a complete Riemannian manifold with dimension \(n \geq 1\), \(\text{Ric}(M) \geq -k\), \(k \geq 0\). Here we consider the Fast Diffusion Equation

\[
u_t = \Delta(u^p), \quad p < 1
\]

on \(M \times (-\infty, \infty)\). We have the following localized Hamilton-type gradient estimates for the positive solution on \(M \times (-\infty, \infty)\):

**Theorem 3.1.** Suppose that \(u \leq M\) is a positive solution of the Fast Diffusion Equation

\[
u_t = \Delta(u^p)
\]

in \(Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)\), where

\[
1 - \frac{4}{n + 3} < p < 1, \quad \text{for } n \geq 1.
\]

Then there exists a constant \(C\) depending only on \(n\) and \(p\) such that

\[
\frac{\|\nabla u(x, t)\|}{u(x, t)} \leq C \left( \frac{1}{R} + \frac{M^{(1-p)/2}}{\sqrt{T}} + \sqrt{k} \right),
\]

for all \((x, t)\) in \(Q_{R/2,T/2}\). When \(n = 1\), the Ricci curvature lower bound \(k\) vanishes.
Proof. For $0 < p < 1$, we have $F(s) = s^p$, and choose $G(s) = \frac{p}{p-1} s^{p-1}$, and let $\alpha = 0$ in Theorem 1.1. We have $F'(s) = ps^{p-1}$, $F''(s) = p(p-1)s^{p-2}$, then Condition (C) in Theorem 1.1 becomes

$\begin{cases}
\tau \geq 1 - p \\
4p - (n-1)(1-p) > 0.
\end{cases}$

which is equivalent to,

$1 - \frac{4}{n+3} < p < 1$.

Let $\gamma = \frac{(n+3)p - (n-1)}{2}$ for given $p$ in admission range (3.1). We follow the proof of Theorem 1.1, we have

$\begin{cases}
g = \frac{p}{p-1} u^{p-1}, \quad g' = pu^{p-1}, \quad g'' = p(p-1)u^{p-1} \\
f = \frac{2a'}{g} = 2(1-p), \quad b = \frac{a''}{g} = p - 1, \\
L = \gamma \frac{p}{1-p} u^{p-1}, \quad L_1 = 3 - p.
\end{cases}$

We follow the proof of Theorem 1.1 until (2.19) with the modification that the constant $C(n,p)$ here depends only on $n$ and $p$, we insert the above quantities into (2.19) to obtain

$2g' \Delta (\Psi w) - (\Psi w)_t + L_1 \nabla g \cdot \nabla (\Psi w) - 2g' \frac{\nabla \Psi}{\Psi} \cdot \nabla (\Psi w) \geq \frac{L}{6} \left[ \Psi w^2 - C(n,p) \left( \frac{1}{R^4} + \frac{1}{u^{2(p-1)}} \frac{1}{T^2} + k^2 \right) \right].$

By the same maximum argument in proof of Theorem 1.1, we have

$\Psi w^2 \leq C(n,p) \left( \frac{1}{R^4} + \frac{M^{2(1-p)}}{T^2} + k^2 \right)$

which implies

$\frac{|\nabla g|^2}{(-g)^2} \leq C(n,p) \left( \frac{1}{R^4} + \frac{M^{1-p}}{T} + k \right)$

Then the conclusion of Theorem 3.1 follows easily from the fact that

$\frac{|\nabla g|}{-g} = \frac{|\nabla G(u)|}{-G(u)} = (1-p) \frac{|\nabla u|}{u}$

An immediate application of the above gradient estimates is the following time-dependent Liouville theorem for the Fast Diffusion Equations on a noncompact manifold with nonnegative Ricci curvature:

**Theorem 3.2. (Liouville theorem)** Let $M^n$ be a complete, noncompact manifold with nonnegative Ricci curvature. Let $u$ be a positive ancient solution, a solution defined in all space and negative time, of the Fast Diffusion Equation for $1 - \frac{4}{n+3} <$
p < 1 , and \( L(s) \in C(\mathbb{R}) \) be any strict increasing function with \( L(s) \to \infty \) as \( s \to \infty \), such that
\[
u(x,t) = o\left( L(d(x)) + |t|^{1/(1-p)} \right)
\]
near infinity. Then \( u \) is a constant.

**Proof.** Since \( L(s) \) is a strict increasing function with \( L(s) \to \infty \) as \( s \to \infty \), there is inverse function \( H(s) \) of \( L(s) \) which is also a strict increasing function with \( H(s) \to \infty \) as \( s \to \infty \). Fixing \((x_0,t_0)\) in space-time and using Theorem 3.1 for \( u \) on the cube \( Q\left(\frac{1}{2}H(R^{2/(1-p)})\right.\), \( R^2) = B(x_0, \frac{1}{2}H(R^{2/(1-p)})\times [t_0 - R^2, t_0] \), and the \( M \) in Theorem 3.1 is the maximum value on the double cube \( Q(H(R^{2/(1-p)}), 2R^2) \), by our assumption on the growth condition of the function \( u \) at infinity,
\[
M_{H(R^{2/(1-p)}), 2R^2} = o\left( L(H(R^{2/(1-p)})) + R^{2/(1-p)} \right) = o(R^{2/(1-p)}).
\]
Hence by Theorem 3.1, we have
\[
\frac{\lvert \nabla u(x_0, t_0) \rvert}{u(x_0, t_0)} \leq C\left( \frac{1}{2H(R^{2/(1-p)})} + \frac{1}{R} o(R) \right).
\]
Letting \( R \to \infty \), it follows that \( \lvert \nabla u(x_0, t_0) \rvert = 0 \). Since \((x_0, t_0)\) is arbitrary, one sees that \( u = c \).

As a corollary of the above time dependent Liouville theorem for Fast Diffusion Equation, we obtain Yau’s celebrated Liouville theorem for positive harmonic functions on a complete, noncompact manifold with nonnegative Ricci curvature:

**Corollary 3.3.** (Yau’s Liouville theorem for positive harmonic functions)
any positive harmonic function on a noncompact manifold with nonnegative Ricci curvature is a constant function.

**Proof.** The proof follows immediately from the above time dependent Liouville theorem. Let \( v \) be a positive harmonic function. Choose an \( p \) with \( 1 - \frac{4}{n+3} < p < 1 \), then \( u(x) = v(x)^{1/p} \) is a positive solution of \( \Delta(u^p) = 0 \), which can be regarded as a time independent positive solution of the corresponding Fast Diffusion Equation. Define
\[
L(s) = s \max_{d(x) \leq s} u(x) + s
\]
It easy to see \( L(s) \) is a strict increasing function with \( L(s) \to \infty \) as \( s \to \infty \), and \( u(x,t) = o(L(d(x))) \) near infinity. Follow from the above Theorem, we have \( u \) must be a constant, which implies \( v \) must be a constant.

3.3. Porous Media Equations: Let \( M^n \) be a complete Riemannian manifold with dimension \( n \geq 1 \), \( Ric(M) \geq -k \), \( k \geq 0 \). Here we consider the Porous Media Equations
\[
u_t = \Delta(u^p), \quad p > 1
\]
on \( M \times (-\infty, \infty) \).

Firstly for dimension \( n = 1 \), we have the following localized Hamilton-type gradient estimates for the positive solution on \( M \times (-\infty, \infty) \):

**Theorem 3.3.** Suppose that \( u \) is a positive solution of the Porous Media Equation on \( M^n \) dimension \( n = 1 \)
\[
u_t = \Delta(u^p)
\]
in $Q_{R,T} \equiv B(x_0,R) \times [t_0 - T,t_0] \subset M \times (-\infty, \infty)$, with range($u$) = $[m,M]$. Let $G(u) = \frac{p}{p-1} u^{p-1}$, $\alpha = \frac{p}{p-1} M^{p-1}(1 + \delta)$ with some constant $\delta > 0$. Then for any $p > 1$, there exists a constant $C(p)$ depending only on $p$ such that

$$\frac{|\nabla_x G(u(x,t))|}{\alpha - G(u(x,t))} \leq C(p) \left( \frac{1 + \delta}{\delta R} + \frac{1}{\sqrt{M^{p-1} \delta T}} \right).$$

for all $(x,t)$ in $Q_{R/2,T/2}$.

**Proof.** We have $F(s) = s^p$ and choose $G(s) = \frac{p}{p-1} s^{p-1}$ in Theorem 1.1. We have $F'(s) = ps^{p-1}$, $F''(s) = p(p-1)s^{p-2}$, and $K = pM^{p-1}$, and $\alpha$, as defined in above, then Condition (A) and (B) are satisfied and Condition (C) in Theorem 1.1 becomes

$$\begin{cases} 
\tau \geq p - 1, \\
2p \geq \gamma > 0, \ \forall \ s \in [m,M]
\end{cases}$$

Condition (C) is satisfied if we choose $\gamma = 2p$ and $\tau = p - 1$. As proof of Theorem 3.1, we have

$$\begin{aligned}
g &= \frac{p}{p-1} u^{p-1}, \ g' = pu^{p-1}, \ g'' = p(p-1)u^{p-1} \\
f &= \frac{2p}{\alpha - g} = 2(p - 1) \frac{u^{p-1}}{M^{p-1}(1+\delta) - u^{p-1}} \leq \frac{2(p-1)}{\delta}
\end{aligned}$$

$$\begin{aligned}
b &= \frac{g''}{g'} = p - 1 \\
L &= \gamma(\alpha - g) = 2p^2 \frac{M^{p-1}(1+\delta) - u^{p-1}}{M^{p-1}(1+\delta) - u^{p-1}} \geq \frac{2p^2}{\delta} M^{p-1} \delta \\
L_1 &= 1 + p + f \leq 1 + p + 2(p - 1) \frac{u^{p-1}}{M^{p-1}(1+\delta) - u^{p-1}} \leq 1 + p + \frac{2(p-1)}{\delta}
\end{aligned}$$

By the same argument in proof of Theorem 3.1, we have

$$\Psi u^2 \leq C(p) \left( \frac{L_1}{R^4} + \frac{1}{L^2 T^2} \right) \leq C(p) \left( \frac{(1 + \frac{1}{\delta})^4}{R^4} + \frac{1}{(M^{p-1} \delta)^2 T^2} \right)$$

which implies

$$\frac{|\nabla g|^2}{(\alpha - g)^2} \leq C(p) \left( \frac{(1 + \frac{1}{\delta})^2}{R^2} + \frac{1}{M^{p-1} \delta T} \right)$$

Then the conclusion of Theorem 3.2 follows easily from the fact that

$$\frac{|\nabla g|}{\alpha - g} = \frac{|\nabla G(u)|}{\alpha - G(u)}$$

An immediate application of this theorem is the following time-dependent Liouville theorem for the Porous Medium Equations on $\mathbb{R}$.

**Theorem 3.4. (Liouville theorem)** Let $u$ be a positive ancient solution, a solution defined in all space and negative time, to the Porous Medium Equation $(p > 1)$ on $\mathbb{R}$, such that

$$u(x,t) = a \left( d(x)^{(p-1)} + |t|^{(p-1)} \right)$$
near infinity. Then \( u \) is a constant.

**Proof.** By our assumption the function \( u \) satisfies \( u(x,t) = o(d(x)^{1/(p-1)} + |t|^{1/(p-1)}) \) near infinity. Fixing \((x_0, t_0)\) in space-time and using Theorem 3.3 for \( u \) on the cube \( Q(R, R) = B(x_0, R) \times [t_0 - R, t_0] \), we have

\[
\frac{|\nabla u(x_0, t_0)|}{u(x_0, t_0)} \leq C(p) \left( \frac{1 + \delta}{\delta R} + \frac{1}{\sqrt{M^{p-1} \delta T}} \right) M^{p-1} \leq C(p, \delta) \left( \frac{M^{p-1}}{R} + \sqrt{\frac{M^{p-1}}{T}} \right)
\]

where \( M \) is the maximum value on the double cube \( Q(2R, 2R) \), by our assumption on the growth condition of the function \( u \) at infinity,

\[
M_{2R,2R} = o\left( R^{1/(p-1)} + R^{1/(p-1)} \right) = o\left( R^{1/(p-1)} \right).
\]

Hence we have

\[
\frac{|\nabla u(x_0, t_0)|}{u(x_0, t_0)} \leq C(p, \delta) \left( \frac{M^{p-1}}{R} + \sqrt{\frac{M^{p-1}}{R}} \right) = o(1),
\]

Letting \( R \to \infty \), it follows that \( |\nabla u(x_0, t_0)| = 0 \). Since \((x_0, t_0)\) is arbitrary, one sees that \( u = c \).

Secondly for dimension \( n \geq 2 \), we have the following localized Hamilton-type gradient estimates for the positive solution on \( M \times (-\infty, \infty) \):

**Theorem 3.5.** Let \( M \) be a complete Riemannian manifold with dimension \( n \geq 2 \), \( \text{Ric}(M) \geq -k, k \geq 0 \). Suppose that \( u \) is a positive solution of the Porous Media Equation

\[
u_t = \Delta (u^p)
\]
in \( Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty) \), with range(\( u \)) = \([m, M]\). Let \( G(u) = \frac{p}{p-1} u^{p-1} \), \( \alpha = \frac{p}{p-1} M^{p-1} (1 + \delta) \) with some small constant \( 0 < \delta \leq \frac{1}{n-1} \). If the following pinch condition on \( m, M \) holds

\[
1 \leq \left( \frac{M}{m} \right)^{p-1} < \frac{1 + \delta}{1 + \delta} \left( \frac{4p}{(n-1)(p-1)} + 1 \right),
\]

then there exists a constant \( C(n, p) \) depending only on \( n \) and \( p \), and

\[
\gamma = 2p - \frac{(n-1)(p-1) M^{p-1} (1 + \delta) - m^{p-1}}{m^{p-1}} > 0,
\]
such that

\[
\frac{|\nabla_x G(u(x,t))|}{\alpha - G(u(x,t))} \leq C(n, p) \left( \frac{\delta + 1}{\gamma^2 R} + \frac{1}{\sqrt{\gamma^2 M^{p-1} T}} + \sqrt{\frac{k}{\delta}} \right).
\]

for all \((x,t)\) in \( Q_{R/2, T/2} \).

**Proof.** We have \( F(s) = s^p \) and choose \( G(s) = \frac{s^p}{p-1} \) in Theorem 1.1. We have \( F'(s) = ps^{p-1}, F''(s) = p(p-1)s^{p-2} \), and \( \alpha = pM^{p-1} \), and \( \alpha \), as defined in above, then Condition (A) and (B) are satisfied and Condition (C) in Theorem 1.1 becomes

\[
\left\{ \begin{array}{l}
\gamma \geq p - 1, \\
2p - \frac{(n-1)(p-1) M^{p-1} (1 + \delta) - m^{p-1}}{s^{p-1}} \geq \gamma > 0, \quad \forall s \in [m, M]
\end{array} \right.
\]
which is equivalent to,
\[
\begin{cases}
\tau \geq p - 1, \\
2p - \frac{(n-1)(p-1) M^{p-1}(1+\delta)-m^{p-1}}{m^{p-1}} > 0
\end{cases}
\]
Let \( \tau = p - 1 \) and \( \gamma = 2p - \frac{(n-1)(p-1) M^{p-1}(1+\delta)-m^{p-1}}{m^{p-1}} > 0 \), the above condition as \( 0 < \delta \leq \frac{1}{n-1} \) is equivalent to our pinch condition (3.2).

As proof of Theorem 3.3, we have
\[
\begin{cases}
g = \frac{p}{p-1} u^{p-1}, \quad g' = pu^{p-1}, \quad g'' = p(p-1)u^{p-1} \\
f = \frac{2g'}{\alpha-g} = 2(p-1) \frac{u^{p-1}}{M^{p-1}(1+\delta)-u^{p-1}} \leq \frac{2(p-1)}{\delta} \\
b = \alpha - g = p - 1 \\
L = \gamma(\alpha-g) = p \frac{p}{p-1} \left[M^{p-1}(1+\delta)-u^{p-1}\right] \geq \frac{p^2}{p-1} M^{p-1} \delta \\
L_1 = 1 + p + f \leq 1 + p + 2(p-1) \frac{u^{p-1}}{M^{p-1}(1+\delta)-u^{p-1}} \leq 1 + p + \frac{2(p-1)}{\delta}
\end{cases}
\]
By the same argument in proof of Theorem 3.3, we have
\[
\Psi w^2 \leq C(n,p) \left( \frac{(\delta+1)^2}{\gamma^3 \delta^4 R^4} + \frac{1}{\gamma^2 \delta^2 M^{2(p-1)} T^2} + \frac{k^2}{\delta^2} \right)
\]
which implies
\[
\frac{|\nabla g|^2}{(\alpha-g)^2} \leq C(n,p) \left( \frac{(\delta+1)^2}{\gamma^2 \delta^2 R^2} + \frac{1}{\gamma \delta M^{p-1} T} + \frac{k}{\delta} \right)
\]
Then the conclusion of Theorem 3.5 follows easily from the fact that
\[
\frac{|\nabla g|}{\alpha-g} = \frac{|\nabla G(u)|}{\alpha-G(u)}
\]
\[\square\]

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