Incremental Stability and Performance Analysis of Discrete-Time Nonlinear Systems using the LPV Framework

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Abstract: The dissipativity framework is widely used to analyze stability and performance of nonlinear systems. By embedding nonlinear systems in an LPV representation, the convex tools of the LPV framework can be applied to nonlinear systems for convex dissipativity based analysis and controller synthesis. However, as has been shown recently in literature, naive application of these tools to nonlinear systems for analysis and controller synthesis can fail to provide the desired guarantees. Namely, only performance and stability with respect to the origin is guaranteed. In this paper, inspired by the results for continuous-time nonlinear systems, the notion of incremental dissipativity for discrete-time nonlinear systems is proposed, whereby stability and performance analysis is done between trajectories. Furthermore, it is shown how, through the use of the LPV framework, convex conditions can be obtained for incremental dissipativity analysis of discrete-time nonlinear systems. The developed concepts and tools are demonstrated by analyzing incremental dissipativity of a controlled unbalanced disk system.

Keywords: Nonlinear Systems, Stability and Stabilization, Incremental Dissipativity, Discrete-Time Systems

1. INTRODUCTION

Stability and performance analysis are important tools to analyze quantitative properties of the behavior of a system and for the formulation of control synthesis algorithms. Many of these tools that are currently used in industry still rely on the systematic results of the Linear Time-Invariant (LTI) framework. Most notably, the dissipativity framework introduced in Willems (1972) allows for the simultaneous analysis of stability and performance of dynamical systems. These results form the cornerstone for many of the powerful and computationally efficient Linear Matrix Inequality (LMI) based analysis and synthesis procedures that exists for LTI systems, e.g. $\mathcal{H}_\infty$ and $\mathcal{H}_2$ based analysis and control, see Scherer and Weiland (2015) for an overview. However, as performance demands and system complexity are ever increasing in many application fields, the ability for LTI methods to cope with these systems is getting increasingly more difficult. Hence, the use of nonlinear analysis and control methods has become of increasing interest over the last decades. Nevertheless, many of the existing nonlinear control methods only focus on ensuring stability of the closed-loop system and hence have no systematic way to incorporate performance shaping, as available in the LTI case. While some dissipativity based results for $\mathcal{L}_2$ performance and passivity analysis of nonlinear systems exist (Van der Schaft (2017)), they are often cumbersome to use, requiring expert knowledge. The Linear Parameter-Varying (LPV) framework (Shamma (1988)) sought to overcome some of these issues by extending the results from the LTI framework to be used with LPV models, see Hoffmann and Werner (2015) for an overview. By embedding the behavior of a nonlinear system in an LPV representation (Tóth (2010)), and in turn trading complexity of the problem for conservativeness of the results, the convex analysis and synthesis results to ensure stability and performance of the LPV framework could easily and systemically be applied to nonlinear systems.

However, in recent research it has been pointed out that in some cases the results of the LPV framework fail to provide the desired guarantees in order to analyze or synthesize controllers for nonlinear systems (Scorletti et al. (2015); Koelewijn et al. (2020)). Namely, the LPV framework is only able to guarantee asymptotic stability for the origin of the nonlinear system, hence, e.g. in the case of disturbance rejection and/or reference tracking this is violated. The core issue of this is the use of the classical dissipativity framework, which expresses stability of only the origin of the system. For LTI systems, such classical dissipativity also implies stability of other forced equilibria, while for nonlinear systems this is not the case. Hence, in order to have a general stability and performance analysis framework for nonlinear systems an equilibrium independent notion of stability and dissipativity needs to be adopted.

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Incremental stability (Angeli (2002)), convergence (Pavlov et al. (2006)) and contraction (Lohmiller and Slotine (1998)) are such equilibrium independent stability notions, whereby stability of the differences between trajectories or of the variation along trajectories is considered. Incremental and differential (based on contraction) notions of dissipativity have also been considered which can be thought of as modeling the energy storage between or along trajectories analogous to the standard dissipativity framework modeling the energy storage with respect to single point of neutral storage. For Continuous-Time (CT) nonlinear systems these results are discussed in Verhoek et al. (2020). These methods have also been developed into convex LPV based control methods, and have successfully been applied to reference tracking and disturbance rejection of nonlinear systems (Scorletti et al. (2015); Koelwijn et al. (2019)).

The aforementioned results on equilibrium independent stability and dissipativity analysis offer great potential to provide convex tools for nonlinear controller synthesis but are currently limited to CT nonlinear systems. Nevertheless, most control algorithms are implemented digitally, hence, analysis and control of Discrete-Time (DT) systems plays an important role. Moreover, the recent resurgence in data-based methods for analysis and control of nonlinear systems also rely on DT systems analysis. While incremental and contraction based stability results have been extended to DT domain, see e.g. Tran et al. (2018), similar mental and contraction based stability results have been systems also rely on DT systems analysis. While incre-

Consider a nonlinear system of the form

\[ x(k + 1) = f(x(k), w(k)); \]
\[ z(k) = h(x(k), w(k)); \]
\[ x(0) = x_0; \]

where \( x(k) \in \mathcal{X} \subseteq \mathbb{R}^n \) is the state with initial condition \( x_0 \in \mathcal{X} \), \( x(k) \in \mathcal{W} \subseteq \mathbb{R}^m \) is the generalized disturbance, \( z(k) \in \mathcal{Z} \subseteq \mathbb{R}^m \) the generalized performance and \( k \in \mathbb{N} \) is the discrete-time instant. The sets \( \mathcal{X}, \mathcal{W} \) and \( \mathcal{Z} \) are open and convex, containing the origin. The solutions of (1) satisfy (1) in the ordinary sense and are restricted to \( k \in \mathbb{N} \). The functions \( f : \mathcal{X} \times \mathcal{W} \rightarrow \mathcal{X} \) and \( h : \mathcal{X} \times \mathcal{W} \rightarrow \mathcal{Z} \) are assumed to be Lipschitz continuous, such that \( f(0, 0) = 0 \) and \( h(0, 0) = 0 \), and such that for all initial conditions \( x_0 \in \mathcal{X} \) there is a unique solution \( (x, w, z) \in (\mathcal{X} \times \mathcal{W} \times \mathcal{Z})^\mathbb{N} \). We define the set of solutions of (1) as

\[ \mathfrak{B} := \{ (x, w, z) \in (\mathcal{X} \times \mathcal{W} \times \mathcal{Z})^\mathbb{N} \mid (x, w, z) \text{satisfies (1)} \}. \]

Furthermore we define the state transition map \( \phi_k : \mathcal{N} \times \mathcal{X} \times \mathcal{W} \rightarrow \mathcal{X} \), such that

\[ x(k) = \phi_k(x_0, x_0, w), \]

which is the state \( x(k) \in \mathcal{X} \) at discrete-time instant \( k \in \mathbb{N} \), with \( k > k_0 \), when the system is driven from \( x_0 \in \mathcal{X} \) at time instant \( k_0 \in \mathbb{N} \) by input signal \( w \in \mathcal{W} \).

In order to simultaneously analyze performance and stability of nonlinear systems, dissipativity theory is widely used, which has its roots in Willems (1972) for continuous-time systems and has also been extended to DT systems, see Byrnes and Lin (1994).

**Definition 1.** (Dissipativity) Byrnes and Lin (1994)).

A system of the form (1) is dissipative with respect to the supply function \( s : \mathcal{X} \times \mathcal{Z} \rightarrow \mathbb{R} \) if there exists a positive definite storage function \( V : \mathcal{X} \rightarrow \mathbb{R}^+ \) with \( V(0) = 0 \) such that for all \( k \in \mathbb{N} \) and \( (x, w, z) \in \mathfrak{B} \)

\[ V(x(k + 1)) - V(x(k)) \leq s(w(k), z(k)), \]

or equivalently, for all \( k \in \mathbb{N} \), \( (x, w, z) \in \mathfrak{B} \) and \( x_0 \in \mathcal{X} \)

\[ V(x(k + 1)) - V(x_0) \leq \sum_{j=0}^{k} s(w(j), z(j)). \]

Performance notions such as the induced \( \ell_2 \)-gain and passivity of DT nonlinear systems can be analyzed by specific choices of the supply function \( s \) (Van der Schaft (2017); Scherer and Welland (2015)). Furthermore, under some restriction of the supply function, dissipativity implies stability of the uncontrollable system.

**Theorem 2.** (Stability). If a system of the form (1) is dissipative, according to Definition 1, with continuous positive definite storage function \( V \) and the supply function \( s \) satisfies that \( s(0, z) \leq 0 \), \( \forall z \in \mathcal{Z} \) (negative semi-definite), then, the origin, i.e. \( x = 0 \), is a stable equilibrium point of (1). In case \( s \) satisfies that \( s(0, z) < 0 \), \( \forall z \in \mathcal{Z} \setminus \{0\} \) (negative
Theorem 5. Consider a nonlinear system with a continuous storage function $V$ and $s(0, z) \leq 0, \forall z \in Z$ if it holds from (4) that
$$V(x(k+1)) - V(x(k)) \leq 0.$$  
(6)
Hence, the system satisfies the condition for stability, see Kalman and Bertram (1960), and $V$ is a Lyapunov function. Asymptotic stability can be proven similarly. □

Remark 3. The supply functions corresponding to e.g. $\ell_2$-gain, $s(w, z) = \gamma^2 \|w(k)\|_2^2 - \|z(k)\|_2^2$, and passivity, $s(w, z) = z(k)^T w(k) + w(k)^T z(k)$, satisfy the assumptions on the supply function taken in Theorem 2.

As mentioned in the introduction, the standard dissipativity framework only analyzes the internal energy of the system with respect to a single storage (equilibrium) point, often taken as the origin of the state-space associated with the nonlinear representation. However, it is often of interest to analyze a set of equilibrium points/trajectories, e.g. in the case of reference tracking or disturbance rejection, which is cumbersome to be performed with the standard dissipativity results. Equilibrium independent dissipativity notions such as incremental dissipativity allow to efficiently handle these cases. Incremental dissipativity is an extension of the dissipativity results which takes into account multiple trajectories of a system and can be thought of as analyzing the energy flow between trajectories. The corresponding theory for CT nonlinear systems has been developed in Verhoek et al. (2020); Van der Schaft (2017). Next, we propose analogous results for incremental dissipativity of DT nonlinear systems.

3. INCREMENTAL STABILITY AND PERFORMANCE ANALYSIS

3.1 Incremental Dissipativity

Similar to the incremental dissipativity definition for CT systems in Verhoek et al. (2020) we define incremental dissipativity of DT nonlinear systems as follows:

Definition 4. (Incremental Dissipativity) A system of the form (1) is incrementally dissipative with respect to the supply function $s : \mathcal{W} \times \mathcal{W} \times Z \times Z \rightarrow \mathbb{R}$ if there exists a storage function $V : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+$ with $V(x, x) = 0$ such that for all $k \in \mathbb{N}$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathcal{B}$ and $x_0, \tilde{x}_0 \in \mathcal{X}$
$$V(x(k+1), \tilde{x}(k+1)) - V(x(k), \tilde{x}(k)) \leq s(w(k), \tilde{w}(k), z(k), \tilde{z}(k)), \quad (7)$$
or equivalently, for all $k \in \mathbb{N}$, $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathcal{B}$ and $x_0, \tilde{x}_0 \in \mathcal{X}$
$$V(x(k+1), \tilde{x}(k+1)) - V(x_0, \tilde{x}_0) \leq \sum_{j=0}^{k} s(w(j), \tilde{w}(j), z(j), \tilde{z}(j)). \quad (8)$$

Similar to standard dissipativity, incremental dissipativity also implies stability of the nonlinear system under some restrictions of the supply function.

Theorem 5. (Incremental stability) If a system of the form (1) is incrementally dissipative according to Definition 4 with a continuous storage function $V$ and the supply function $s$ satisfies that $s(w, w, z, z) < 0, \forall w \in \mathcal{W}$ and $\forall z, \tilde{z} \in \mathcal{Z}, z \neq \tilde{z}$ (negative definite) and $s(w, w, w, z) = 0, \forall w \in \mathcal{W}, z \in \mathcal{Z}$, then, the system is incrementally asymptotically stable.

Proof. If $s(w, w, z, z) < 0, \forall w \in \mathcal{W}$ and $\forall z, \tilde{z} \in \mathcal{Z}, z \neq \tilde{z}$ and $s(w, w, z, z) \leq 0, \forall w \in \mathcal{W}, z \in \mathcal{Z}$ it holds from (7) that for all $k \in \mathbb{N}$ and $x, \tilde{x} \in \mathcal{B}, x \neq \tilde{x}$,
$$V(x(k+1), \tilde{x}(k+1)) - V(x(k), \tilde{x}(k)) < 0. \quad (9)$$

Hence, the systems satisfies the conditions for incremental asymptotic stability, see Tran et al. (2018), and $V$ is an incremental stability Lyapunov function. Similar results implying (non-asymptotic) stability can be formulated for the case that $s(w, w, z, z) \leq 0, \forall w \in \mathcal{W}, z \in \mathcal{Z}$, $z \neq \tilde{z}$ (negative semi-definite), see Van der Schaft (2017).

In this work we will focus on supply functions of the form
$$s(w, \tilde{w}, z, \tilde{z}) = \begin{bmatrix} w - \tilde{w} \\ z - \tilde{z} \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} w - \tilde{w} \\ z - \tilde{z} \end{bmatrix}, \quad (10)$$
with $Q \succ 0$, such that for all $(x, w) \in \mathcal{X} \times \mathcal{W}$
$$V(x, \tilde{x}) = (x - \tilde{x})^T P(x - \tilde{x}), \quad (11)$$
with $P > 0$, such that for all $(x, w) \in \mathcal{X} \times \mathcal{W}$
$$\begin{bmatrix} I & 0 \\ A_\delta(x, w) & B_\delta(x, w) \end{bmatrix}^T \begin{bmatrix} -P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\delta(x, w) & B_\delta(x, w) \end{bmatrix} \leq 0, \quad (12)$$
where
$$A_\delta(x, w) = \frac{\partial f}{\partial x}(x, w), \quad B_\delta(x, w) = \frac{\partial f}{\partial w}(x, w), \quad C_\delta(x, w) = \frac{\partial g}{\partial x}(x, w), \quad D_\delta(x, w) = \frac{\partial g}{\partial w}(x, w). \quad (13)$$

Proof. According to Definition 4, the system (1) is dissipative with respect to a supply function $s$ if (7) holds for all $k \in \mathbb{N}$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathcal{B}$. Hence, (1) is incrementally (Q,S,R)-dissipative if for all $k \in \mathbb{N}$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathcal{B}$ it holds that, omitting dependence on time for brevity,
$$\Delta_k \left[ (x - \tilde{x})^T P(x - \tilde{x}) - (w - \tilde{w})^T Q(w - \tilde{w}) - 2(w - \tilde{w})^T S(z - \tilde{z})^T R(z - \tilde{z}) \right] \leq 0, \quad (14)$$
where $\Delta_k$ is the discrete-time difference operator, defined as $\Delta_k w(k) = v(k+1) - v(k)$. For $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathcal{B}$, define the initial conditions as $(x(0) := x_0$ and $(\tilde{x}(0) := \tilde{x}_0$ respectively, such that $x(k) = \phi_k(k, 0, x_0, w)$ and $\tilde{x}(k) = \phi_k(k, 0, \tilde{x}_0, \tilde{w})$. Then, define
$$x_0(\lambda) := x_0 + \lambda (x_0 - \tilde{x}_0), \quad (15)$$
$$\tilde{w}(\lambda) := \tilde{w}(k) + \lambda (w(k) - \tilde{w}(k)), \quad (16)$$
with $\lambda \in [0, 1]$ and
$$\hat{x}(k, \lambda) := \phi_k(k, 0, \lambda x_0(\lambda), \tilde{w}(\lambda)), \quad (17)$$
Combining the results of (21), (22), (23) and (25), we then given by
\[\dot{x}(k + 1, \lambda) = f(\dot{x}(k, \lambda), \dot{w}(k, \lambda));\] \[\dot{z}(k, \lambda) = h(\dot{x}(k, \lambda), \dot{w}(k, \lambda)).\] (18a, 18b)

The first term on the left-hand side of inequality (14) can then be expressed as
\[\Delta_k \left[(\dot{x}(k, 1) - \dot{x}(k, 0))^\top P(\dot{x}(k, 1) - \dot{x}(k, 0))\right].\] \[(19)\]
Using the Fundamental Theorem of Calculus, (19) can be expressed as
\[\Delta_k \left[\int_0^1 \dot{x}(k, \lambda) d\lambda\right]^\top P \left[\int_0^1 \dot{x}(k, \lambda) d\lambda\right] = \int_0^1 \Delta_k \left[\dot{x}(k, \lambda)^\top P \dot{x}(k, \lambda)\right] d\lambda.\] \[(20)\]

The second term on the left-hand-side of inequality (14) can be expressed, using (16), as
\[-(\dot{w}(k, 1) - \dot{w}(k, 0))^\top Q(\dot{w}(k, 1) - \dot{w}(k, 0)) = -\int_0^1 (\dot{w}(k, 1) - \dot{w}(k, 0))^\top Q(\dot{w}(k, 1) - \dot{w}(k, 0)) d\lambda = -\int_0^1 \delta w(k, \lambda)^\top Q \delta w(k, \lambda) d\lambda,\] \[(22)\]
where \(\delta w(k, \lambda) = \frac{\partial}{\partial \lambda} \dot{w}(k, \lambda) = w(k) - \dot{w}(k)\) (by definition (16)). The third term in (14) can similarly be expressed as
\[-2(\dot{w}(k, 1) - \dot{w}(k, 0))^\top \int_0^1 \delta z(k, \lambda) d\lambda = -2 \int_0^1 \delta w(k, \lambda)^\top \delta z(k, \lambda) d\lambda,\] \[(23)\]
where \(\delta z(k, \lambda) = \frac{\partial}{\partial \lambda} \dot{z}(k, \lambda).\) Finally, the fourth term in (14) can be expressed as
\[-(\dot{z}(k, 1) - \dot{z}(k, 0))^\top R(\dot{z}(k, 1) - \dot{z}(k, 0)) = \left[\int_0^1 \delta z(k, \lambda) d\lambda\right]^\top (-R) \left[\int_0^1 \delta z(k, \lambda) d\lambda\right].\] \[(24)\]

Assuming that \(R < 0\) or \(R = 0\), hence, \(-R > 0\) or \(-R = 0\), by Lemma 16 it holds that
\[\left[\int_0^1 \delta z(k, \lambda) d\lambda\right]^\top (-R) \left[\int_0^1 \delta z(k, \lambda) d\lambda\right] \leq \int_0^1 \delta z(k, \lambda)^\top (-R) \delta z(k, \lambda) d\lambda.\] \[(25)\]

Combining the results of (21), (22), (23) and (25), we obtain that, omitting dependence on time for brevity,
\[\Delta_k \left[(x - \hat{x})^\top P(x - \hat{x}) - (w - \hat{w})^\top Q(w - \hat{w}) - 2\delta w^\top S(z - \hat{z}) - \delta z^\top R(z - \hat{z}) \leq \int_0^1 \Delta_k \left[\delta x(k, \lambda)^\top P \delta x(k, \lambda) - \delta w(k, \lambda)^\top Q \delta w(k, \lambda) - 2\delta w^\top S \delta z(k, \lambda) - \delta z(k, \lambda) R \delta z(k, \lambda)\right] d\lambda.\] \[(26)\]

Hence, if it holds that
\[\int_0^1 \Delta_k \left[\delta x(k, \lambda)^\top P \delta x(k, \lambda) - \delta w(k, \lambda)^\top Q \delta w(k, \lambda) - 2\delta w^\top S \delta z(k, \lambda) - \delta z(k, \lambda) R \delta z(k, \lambda)\right] d\lambda \leq 0,\] \[(27)\]
then, condition (14) holds, meaning the system is incrementally \((Q,S,R)\)-dissipative. Furthermore, (27) holds if
\[\Delta_k \left[\delta x(k, \lambda)^\top P \delta x(k, \lambda) - \delta w(k, \lambda)^\top Q \delta w(k, \lambda) - 2\delta w^\top S \delta z(k, \lambda) - \delta z(k, \lambda) R \delta z(k, \lambda)\right] \leq 0.\] \[(28)\]
As \(f, h \in C_1\), taking the derivative w.r.t. \(\lambda\) for (18) results in
\[\delta x(k + 1, \lambda) = A_k(\dot{x}(k, \lambda), \dot{w}(k, \lambda)) \delta z(k, \lambda) + B_{\delta}(\dot{x}(k, \lambda), \dot{w}(k, \lambda)) \delta w(k, \lambda);\] \[(29a)\]
\[\delta z(k, \lambda) = C_k(\dot{x}(k, \lambda), \dot{w}(k, \lambda)) \delta z(k, \lambda) + D_{\delta}(\dot{x}(k, \lambda), \dot{w}(k, \lambda)) \delta w(k, \lambda).\] \[(29b)\]

Hence, (28) can be written, omitting dependence on time for brevity, as
\[(\star)^\top P \left[\begin{array}{ccc} A_k(x, w) & B_{\delta}(x, w) & P \delta x - \delta z^\top P \delta x - \delta w^\top Q \delta w - 2\delta w^\top S \delta z + D_{\delta}(x, w) \delta w\end{array}\right] \begin{array}{c} \delta x \\ \delta z \\ \delta w \end{array} \leq 0.\] \[(30)\]
which should hold for all \(k \in \mathbb{N}\) and \((\hat{x}, \hat{w}, \hat{z}) \in \mathcal{X}\). By Willems (1972), condition (30) can equivalently be checked by verifying (30) on the value set, hence, checking (30) for all \(\delta x \in \mathbb{R}^{n_x}\), \(\delta w \in \mathbb{R}^{n_w}\), \(\hat{x} \in \mathcal{X}\); \(x, w \in \mathcal{W}\) implies that (30) holds for all \(k \in \mathbb{N}\) and \((\hat{x}, \hat{w}, \hat{z}) \in \mathcal{X}\). Consequently, (30) holds if
\[(\star)^\top P \left[\begin{array}{ccc} A_k(x, w) & B_{\delta}(x, w) & P \delta x - \delta z^\top P \delta x - \delta w^\top Q \delta w - 2\delta w^\top S \delta z + D_{\delta}(x, w) \delta w\end{array}\right] \begin{array}{c} \delta x \\ \delta z \\ \delta w \end{array} \leq 0.\] \[(31)\]
holds for all \(\delta x \in \mathbb{R}^{n_x}\), \(\delta w \in \mathbb{R}^{n_w}\), \(x \in \mathcal{X}\), and \(w \in \mathcal{W}\). Hence, equivalently, (31) holds if for all \(x, w \in \mathcal{X} \times \mathcal{W}\) condition (12) holds. Consequently, if condition (12) holds, condition (14) holds, which in turn implies that the system is incrementally \((Q,S,R)\)-dissipative. \(\square\)

**Remark 7.** Like in the CT case in Verhoeff et al. (2020), the DT incremental dissipativity condition derived in Theorem 6 can be related to differential dissipativity and contraction analysis as we will show. Namely, based on the original nonlinear system (1), which we will refer to as the primal form of the system, with \(f, h \in C_1\), we formulate the system
\[\left[\begin{array}{c} \delta x(k + 1) \\ \delta z(k) \end{array}\right] \leq \left[\begin{array}{ccc} A_k(x(k), w(k)) & B_{\delta}(x(k), w(k)) & \delta x(k) \\ C_k(x(k), w(k)) & D_{\delta}(x(k), w(k)) & \delta z(k) \end{array}\right],\] \[(32)\]
where \((x, w, z) \in \mathcal{X}\), \(\delta x(k) \in \mathbb{R}^{n_x}\), \(\delta w \in \mathbb{R}^{n_w}\) and \(\delta z(k) \in \mathbb{R}^{n_z}\), often referred to as the differential form of the system, see Verhoeff et al. (2020), or variational dynamics, see Crouch and Van der Schaft (1987). It is straightforward to derive that “standard dissipativity”, see Definition 1, of the differential form (32), referred to as differential dissipativity, is equivalent with verifying condition (12) in Theorem 6. This is exploited in the next sections to arrive at computationally efficient checks for incremental dissipativity. See also Tran et al. (2018) and references therein for more information on differential stability and contraction analysis of DT systems.
3.2 Nonlinear Performance

Using standard (Q,S,R)-dissipativity, many useful performance notions can be retrieved such as $\ell_2$-gain performance and passivity. As we will show, incremental versions of these performance notions can be introduced and analyzed using the results of Section 3.

Incremental $\ell_2$-gain

Definition 8. ($\ell_2$-gain). A nonlinear system of the form (1) is said to have a finite incremental $\ell_2$-gain, denoted as $\ell_2$-gain, if for all $w, \bar{w} \in \ell_2$ and $x_0, \tilde{x}_0 \in \mathcal{X}$, with $(x,w,z), (\bar{x}, \bar{w}, \bar{z}) \in \mathcal{B}$, there is a finite $\gamma \geq 0$ and a function $\zeta(x, \tilde{x}) \geq 0$ with $\zeta(x, \tilde{x}) \geq 0$ such that
\[
\|z - \bar{z}\|_2 \leq \gamma \|w - \bar{w}\|_2 + \zeta(x, \bar{x}).
\] (33)

The induced $\ell_2$-gain of the system is the infimum of $\gamma$ such that (33) still holds.

Next we will show how the $\ell_2$-gain of a NL system (1) can be analyzed using the results of Theorem 6.

Lemma 9. ($\ell_2$-gain through incremental dissipativity). A nonlinear system of the form (1) has a finite $\ell_2$-gain of $\gamma$ if it is incrementally (Q,S,R)-dissipative with $\gamma = \gamma^2 I$, $S = 0$ and $R = -I$.

Proof. If a nonlinear system of the form (1) is incrementally (Q,S,R)-dissipative with $\gamma = \gamma^2 I$, $S = 0$ and $R = -I$, it holds that there exists a positive definite storage function $V : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ with $V(x, x) = 0$ such that for all $k \in \mathbb{N}$, $(x,x,z), (\bar{x}, \bar{w}, \bar{z}) \in \mathcal{B}$ and $x_0, \tilde{x}_0 \in \mathcal{X}$
\[
V(x(k+1), \tilde{x}(k+1)) = V(x_0, \tilde{x}_0) \leq \sum_{j=0}^{\infty} \gamma^2 (w(j) - \bar{w}(j))^\top (w(j) - \bar{w}(j))
- (z(j) - \bar{z}(j))^\top (z(j) - \bar{z}(j)).
\] (34)

If the system is incrementally (Q,S,R)-dissipative with $\gamma = \gamma^2 I$, $S = 0$ and $R = -I$, it is also incrementally stable, as $R < 0$ which implies that $s(w, w, z, \bar{z})$ is negative definite, see Theorem 5. Hence, $\lim_{k \to \infty} \|x(k) - \bar{x}(k)\| = 0$. Therefore, $\lim_{k \to \infty} V(x(k+1), \tilde{x}(k+1)) = 0$, as $V(x, x) = 0$ and (7) becomes
\[
V(x_0, \tilde{x}_0) \leq \sum_{j=0}^{\infty} \gamma^2 (w(j) - \bar{w}(j))^\top (w(j) - \bar{w}(j))
- (z(j) - \bar{z}(j))^\top (z(j) - \bar{z}(j)).
\] (35)

which can be written as
\[
\|z - \bar{z}\|_2^2 \leq \gamma^2 \|w - \bar{w}\|_2^2 + V(x_0, \tilde{x}_0).
\] (36)

Hence, this implies that there exist a $\zeta(x, \tilde{x}) \geq 0$ with $\zeta(x, x) = 0$ such that (33) holds (Van der Schaft (2017)).

Theorem 10. ($\ell_2$-gain analysis). A nonlinear system of the form (1) with $f, h \in \mathcal{L}_1$ has a finite $\ell_2$-gain of $\gamma$ if there exists a $P > 0$ such that for all $x \in \mathcal{X}$ and $w \in \mathcal{W}$
\[
\begin{bmatrix}
P & A_\delta(x, w) & B_\delta(x, w) & 0 \\
PA_\delta^\top(x, w) & 0 & PC_\delta^\top(x, w) & 0 \\
P & 0 & C_\delta^\top(x, w) & \gamma I \\
0 & 0 & \gamma I & \gamma I
\end{bmatrix} \geq 0.
\] (37)

Proof. Based on Lemma 9, a nonlinear system of the form (1) has a finite $\ell_2$-gain if it is incrementally (Q,S,R)-dissipative with $\gamma = \gamma^2 I$, $S = 0$ and $R = -I$. Furthermore, based on Theorem 6 a nonlinear system of the form (1) with $f, h \in \mathcal{L}_1$ is incrementally (Q,S,R)-dissipative, where $R < 0$ or $R = 0$, with a storage function of the form (11) if (12) holds. For $\ell_2$-gain analysis, $R = -I < 0$, hence, we can use Theorem 6. Combining these results gives us that in order for (1) to have a finite $\ell_2$-gain the following condition needs to be satisfied: there exists a $P > 0$ such that for all $(x, w) \in \mathcal{X} \times \mathcal{W}$
\[
\begin{bmatrix}
I & 0 & A_\delta(x, w) & B_\delta(x, w) \\
-A_\delta^\top(x, w) & 0 & 0 & P \\
A_\delta(x, w) & B_\delta(x, w) & 0 & I \\
0 & I & C_\delta(x, w) & D_\delta(x, w)
\end{bmatrix} \leq 0.
\] (38)

This condition can simply be rewritten into (37) by defining $P = \gamma P^{-1}$, taking a Schur complement and applying a congruence transformation.

Incremental passivity

Similar to the definitions in Van der Schaft (2017); Verhoek et al. (2020) we define (DT) incremental passivity as follows:

Definition 11. (Incremental passivity). A nonlinear system of the form (1) is said to be incrementally passive if it is incrementally dissipative, see Definition 4, with respect to the supply function
\[
s(w, \bar{w}, z, \bar{z}) = (w - \bar{w})^\top (z - \bar{z}) + (z - \bar{z})^\top (w - \bar{w}).
\] (39)

Theorem 12. (Incremental passivity analysis). A nonlinear system of the form (1) with $f, h \in \mathcal{L}_1$ is incrementally passive if there exists a $P > 0$ such that for all $x \in \mathcal{X}$ and $w \in \mathcal{W}$
\[
\begin{bmatrix}
P & A_\delta^\top(x, w) & 0 & C_\delta^\top(x, w) & \gamma I \\
PA_\delta(x, w) & 0 & PC_\delta(x, w) & 0 & \gamma I \\
P & 0 & C_\delta(x, w) & 0 & \gamma I \\
0 & 0 & D_\delta(x, w) & 0 & \gamma I \\
0 & 0 & 0 & \gamma I & \gamma I
\end{bmatrix} \geq 0.
\] (40)

Proof. According to Definition 11, a system of the form (1) is incrementally passive if it is incrementally dissipative with respect to the supply function $s$ by (39). This supply function can also be written in (Q,S,R) form, see (10), by taking $Q = 0$, $S = -I$ and $R = 0$. By using the results of Theorem 6, and filling in $Q = 0$, $S = -I$ and $R = 0$ in condition (12), it can simply be rewritten into (40) by taking a Schur complement and congruence transformation.

Remark 13. Note that the obtained conditions for $\ell_2$-gain and incremental passivity analysis result in checking positive semi-definiteness of a matrix, while in literature these, or similar conditions, are often found as positive definiteness checks. The positive definite versions of the conditions can simply be retrieved by making the incremental dissipativity check strict, i.e. changing $\leq$ to $<$ in (7), which then imply the strict versions of the conditions found in this paper.

4. CONVEX ANALYSIS USING THE LPV FRAMEWORK

As shown in Section 3, the condition for incremental dissipativity can be written in terms of LMIs which needs to be checked for infinitely many values of a scheduling-variable $\rho \in \mathcal{P} \subset \mathbb{R}^n$. Hence, we make use of the developed LPV
approaches to make the proposed incremental dissipativity conditions computationally feasible.

As we have shown in Section 3 the resulting incremental dissipativity conditions for a system (1) are related to standard dissipativity of its differential form (32). Hence, we embed the differential form of the nonlinear system in an LPV model.

Definition 14. (LPV embedding). Assume we have a nonlinear system of the form (1) with \( f, h \in \mathcal{C}_1 \) and with differential form given by (32). The LPV state-space model given by

\[
\begin{bmatrix}
\mathbf{\dot{x}}(k+1) \\
\mathbf{\delta z}(k)
\end{bmatrix} =
\begin{bmatrix}
A(\rho(k)) & B(\rho(k)) \\
C(\rho(k)) & D(\rho(k))
\end{bmatrix}
\begin{bmatrix}
\mathbf{\delta x}(k) \\
\mathbf{b}(k)
\end{bmatrix},
\]

where \( \rho(k) \in \mathcal{P} \subset \mathbb{R}^{n_\rho} \) is the scheduling-variable is an LPV embedding on the compact convex region \( \mathcal{X} \times \mathcal{W} \) of the differential form (32) if there exists a function, called the scheduling-map, \( \psi : \mathbb{R}^{n_\rho} \times \mathbb{R}^{n_\psi} \rightarrow \mathbb{R}^{n_\rho} \) such that under a given choice of function class for \( A, \ldots, D \), e.g. affine, polynomial, etc., \( A(\psi(x,w)) = A_\psi(x,w), \ldots, D(\psi(x,w)) = D_\psi(x,w) \) for all \( x \in \mathcal{X}, w \in \mathcal{W} \) and \( \psi(\mathcal{X} \times \mathcal{W}) \subseteq \mathcal{P} \) where \( \mathcal{P} \) is a (minimal) convex hull with \( n \) vertices. By specific choice of the embedding region \( \mathcal{X} \times \mathcal{W} \), either the full state-space can be embedded of the original NL model (1) in which case \( \mathcal{X} \times \mathcal{W} \supseteq \mathcal{A} \times \mathcal{W} \) or part of the state-space can be embedded, in which case \( \mathcal{X} \times \mathcal{W} \subseteq \mathcal{X} \times \mathcal{W} \).

Theorem 15. (Incremental Dissipativity LPV Analysis). Assume a system of the form (1) with \( f, h \in \mathcal{C}_1 \) and with an LPV embedding on the compact region \( \mathcal{X} \times \mathcal{W} \) of its differential form given by (41), see Definition 14, with scheduling-variable \( \rho \), scheduling-map \( \psi \) and such that \( \psi(\mathcal{X} \times \mathcal{W}) \subseteq \mathcal{P} \). The system (1) is incrementally (Q,S,R)-dissipative, on the region \( \mathcal{X} \times \mathcal{W} \) with respect to the supply function \( s \), given by (10) with \( R < 0 \) or \( R = 0 \), and with storage function \( V \) given by (11) with \( P > 0 \), if for all \( \rho \in \mathcal{P} \)

\[
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A(\rho) & B(\rho) \\
C(\rho) & D(\rho)
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & P
\end{bmatrix}
\begin{bmatrix}
-I & 0 \\
0 & -I
\end{bmatrix}
\begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
0 & I
\end{bmatrix}
\geq 0.
\]

Proof. A system of the form (1) with \( f, h \in \mathcal{C}_1 \) is incrementally (Q,S,R)-dissipative on the region \( \mathcal{X} \times \mathcal{W} \), there exists a \( P > 0 \) such that, for all \( (x,w) \in \mathcal{X} \times \mathcal{W}, \) condition (12) holds. As \( \rho \in \mathcal{P} \) and \( \psi(\mathcal{X} \times \mathcal{W}) \subseteq \mathcal{P} \), checking whether there exists a \( P > 0 \) such that for all \( \rho \in \mathcal{P} \) condition (42) holds implies that for all \( (x,w) \in \mathcal{X} \times \mathcal{W} \) condition (12) holds.

The resulting condition that needs to be checked for incremental (Q,S,R)-dissipativity using the LPV framework in Theorem 15 is similar to the condition that needs to be checked for standard (Q,S,R)-dissipativity of DT LPV systems, see e.g. the \( \ell^2 \)-gain results in Apkarian et al. (1995); De Oliveira et al. (2002). Note however the proposed incremental dissipativity analysis uses an LPV embedding of the differential form (32), while standard dissipativity analysis uses an LPV embedding of the primal form (1). As the proposed analysis results for incremental (Q,S,R)-dissipativity can be casted a standard (Q,S,R)-dissipativity analysis problem of an LPV system, all of the techniques to reduce the evaluation of an infinite set of LMIs to only checking a finite set of LMIs from the LPV framework can be used. Often for this, \( A, \ldots, D \) are needed to be restricted to an affine function in the embedding (41). The most common techniques are polytopic, multiplier or gridding-based approaches, see Hoffmann and Werner (2015) for an overview. Although the same tools from the LPV framework can be used for checking incremental dissipativity and ‘standard’ dissipativity of nonlinear systems, we would like to stress that the underlying dissipativity and stability concepts are very different. Namely, using the incremental dissipativity tools developed in this paper, global stability and performance guarantees can be given for the nonlinear system, while standard dissipativity tools can only provide performance and stability analysis with respect to single equilibrium point, often the origin of the state-space representation of the nonlinear system.

5. EXAMPLE

In this section, we apply the results from Section 3 in order to analyze incremental dissipativity of a controlled unbalanced disk. The CT dynamics of the unbalanced disk system, see Fig. 1, can be expressed in nonlinear state space form by neglecting the fast electrical dynamics:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t); \\
\dot{x}_2(t) &= \frac{Mgl}{\tau} \sin(x_1(t)) - \frac{1}{\tau} x_2(t) + \frac{K_m}{\tau} u(t); \\
\end{align*}
\]

where \( M \) is the mass attached to the disk and \( x_1 \) [rad] its angular position, \( x_2 \) [rad/s] its angular velocity, \( u \) [V] is the control input voltage, \( g \) is the gravitational acceleration, \( \ell \) the length of the pendulum, \( J \) the inertia of the disk and \( K_m \) and \( \tau \) are the motor constant and friction coefficient respectively. The values of the physical parameters of the system are given in Table 1.

We discretize equation (43) using a fourth order Runge-Kutta (RK4) method, where the control input is assumed to be constant over the sampling period. More specifically, assuming the CT dynamics are \( \dot{x}(t) = f(x(t), u(t)) \), we have the RK4 discretized dynamics given by

\[
x(k+1) = x(k) + \frac{T_s}{6}(\varphi_1(k) + 2\varphi_2(k) + 2\varphi_3(k) + \varphi_4(k)),
\]

where

\[
\begin{align*}
\varphi_1(k) &= f_\epsilon(x(k), u(k)), \\
\varphi_2(k) &= f_\epsilon(x(k) + \frac{\tau}{2}\varphi_1(k), u(k)), \\
\varphi_3(k) &= f_\epsilon(x(k) + \frac{\tau}{2}\varphi_2(k), u(k)), \\
\varphi_4(k) &= f_\epsilon(x(k) + \tau\varphi_3(k), u(k)),
\end{align*}
\]

and where \( T_s \) is the sample time. Applying this method to the CT dynamics of the unbalanced disk (43), with a sample time \( T_s = \frac{1}{f_s} \) second, results in a DT nonlinear state-space representation of the form

\[
x(k+1) = f(x(k), u(k)),
\]

where \( x(k) = \text{col}(x_1(k), x_2(k)) \). For the discretized version of the unbalanced disk (46), a DT LTI controller is heuristically designed in order to achieve reference tracking. This controller is given by

\[
\begin{align*}
x_c(k+1) &= x_c(k) + B_x u_c(k); \\
y_c(k) &= C_x x_c(k) + D_c u_c(k);
\end{align*}
\]
where \( x_c \) is the state, \( u_c \) is the input and \( y_c \) is the output of the controller. For the LTI controller, \( B_c = [1 \ 0] \), \( C_c = -0.5 \) and \( D_c = [-10 \ -1] \) are chosen, corresponding to a PID controller. The closed-loop interconnection of plant and controller is given in Fig. 2, where \( K \) is the DT LTI controller (47), \( G \) is the discretized unbalanced disk dynamics (46), \( w \) is the input disturbance and \( z \) the angle of the disk. The controller \( K \) in this configuration can be thought of as a PID controller for regulation of the disk angle at zero and rejection of constant input disturbances. The closed-loop interconnection results in a system of the form (1).

Using Definition 14, the differential form of the closed-loop dynamics of the DT LTI controller and discretized unbalanced disk dynamics is computed and is embedded in an LPV representation on the compact region \( \{x_1(k) \in [-\pi, \pi], x_2(k) \in [-10, 10] \text{ and } u(k) \in [-10, 10], \text{ with scheduling-variable } \rho = \text{col}(\rho_1, \rho_2, \rho_3) = \text{col}(x_1, x_2, u) \} \). Next, an upper-bound for the induced \( \ell_2 \)-gain of the closed-loop interconnection on the compact region is computed using the results of Theorem 10 and Theorem 15. To reduce the infinite set of LMI to a finite set of LMI, a gridding-based method is used, due to the complexity of the discretized plant, whereby the compact region of the LPV embedding is equidistantly gridded with 11 grid-points in each dimension resulting in a total of 1331 grid-points. Solving the optimization problem results in an upper-bound for the induced \( \ell_2 \)-gain of \( \gamma = 0.220 \) for the closed-loop interconnection on the compact region. In order to compute the closed-loop \( \ell_2 \)-gain of the closed-loop interconnection, the DT primal form in the plant (46) is embedded in a grid-based LPV model using the technique described in Koelewijn and Tóth (2021) on the aforementioned equidistant grid. The closed-loop interconnection of the LTI controller and primal form of the plant obtains an upper-bound for the \( \ell_2 \)-gain \(^2\) of \( \gamma_{cl} = 0.219 \)

1. One can also restrict the controller state \( x_c \) and the generalized disturbance \( w \) to compact sets such that \( u(k) \in [-10, 10] \), although these are not explicitly given.

2. Note that the \( \ell_2 \)-gain is smaller than the \( \ell_2 \)-gain, as the \( \ell_2 \)-gain is a stronger notion.

### Table 1. Parameters of the unbalanced disk.

|    | \( M \) | \( g \) | \( l \) | \( J \) | \( K_m \) | \( \tau \) |
|----|--------|--------|--------|--------|--------|--------|
|    | 0.076  | 9.8    | 0.041  | 2.4 \( 10^{-4} \) | 11     | 0.40   |

Fig. 1. Unbalanced disk setup.

Fig. 2. Closed-loop interconnection of DT controller \( K \) and discretized dynamics of the unbalanced disk \( G \).

For comparison, an LPV version of the controller is also heuristically designed, where \( B_c \) is taken the same as for the LTI controller (47), but \( C_c \) and \( D_c \) are made parameter-varying by taking \( C_c(\rho) = -0.5 - \frac{1}{20} \sin(\rho_1) \) and \( D_c(\rho) = [-10 - 2 \cos(\rho_1) -1] \) (hence, they only vary in \( \rho_1 = x_1 \)). For the closed-loop interconnection of the LPV controller and the primal form of the plant an upper-bound for the \( \ell_2 \)-gain is computed using a standard grid-based LPV method, resulting in \( \gamma_{cl} = 0.179 \), which is better than the closed-loop interconnection with the LTI controller. However, unlike the closed-loop with the LTI controller, the closed-loop with LPV controller does not have a bounded \( \ell_2 \)-gain.

### 6. CONCLUSION

In this paper extensions of the CT incremental dissipativity framework to DT nonlinear systems have been

\(^3\) Note that during simulation all the scheduling-variables stayed within the compact-set.
proposed, along with convex conditions to analyze it. The proposed analysis condition use the LPV framework for efficient computation of the various incremental performance notions. The DT incremental (Q,S,R)-dissipativity results, analogous to the CT results, show that incremental (Q,S,R)-dissipativity of DT systems can be evaluated by evaluating ‘standard’ dissipativity of their differential form, i.e. the dynamics of the variations along the systems trajectory. Moreover, using the LPV framework, this problem can then be casted as a standard dissipativity check of an LPV model, which allows for the many computational techniques of the LPV framework to be used to efficiently solve nonlinear performance analysis problems using convex optimization. These results pave the way for development of efficient synthesis techniques to ensure incremental dissipativity of DT nonlinear systems. For future research, we aim to develop such synthesis techniques and extend the analysis results to allow for a state dependent quadratic matrix of the storage function in order to reduce conservativeness.

REFERENCES

Angeli, D. (2002). A Lyapunov Approach to Incremental Stability Properties. *IEEE Transaction on Automatic Control*, 47(3), 410–421.

Apkarian, P., Gahinet, P., and Becker, G. (1995). Self-scheduled $\mathcal{H}_\infty$ Control of Linear Parameter-varying Systems: a Design Example. *Automatica*, 31(9), 1251–1261.

Byrnes, C.I. and Isidori, A. (1989). Geometric Control Theory for Nonlinear Systems. In *Mathematical Theory of Control*. Springer-Verlag.

Crouch, P.E. and Van der Schaft, A.J. (1989). *Variational and Hamiltonian Control Systems*. Springer.

De Oliveira, M.C., Geromel, J.C., and Bernussou, J. (2002). Extended $H_2$ and $H_\infty$ norm characterizations and controller parameterizations for discrete-time systems. *International Journal of Control*, 75(9), 666–679.

Hoffmann, C. and Werner, H. (2015). A Survey of Linear Parameter-Varying Control Applications Validated by Experiments or High-Fidelity Simulations. *IEEE Transactions on Control Systems Technology*, 23(2), 416–433.

Kalman, R.E. and Bertram, J.E. (1960). Control System Analysis and Design Via the “Second Method” of Lyapunov - II Discrete-Time Systems. *Transactions of the ASME*, 82(2), 394–400.

Koelewijn, P.J.W., Sales Mazzoccante, G., Tóth, R., and Weiland, S. (2020). Pitfalls of Guaranteeing Asymptotic Stability in LPV Control of Nonlinear Systems. In *Proc. of the 2020 European Control Conference*, 1573–1578.

Koelewijn, P.J.W. and Tóth, R. (2021). Automatic Grid-based LPV Embedding of Nonlinear Systems. Technical report, Eindhoven University of Technology. URL https://research.tue.nl/en/publications/automatic-grid-based-lpv-embedding-of-nonlinear-systems.

Koelewijn, P.J.W., Tóth, R., and Nijmeijer, H. (2019). Linear Parameter-Varying Control of Nonlinear Systems based on Incremental Stability. In *Proc. of the 3rd IFAC Workshop on Linear Parameter Varying Systems*, volume 52, 38–43.

Lohmiller, W. and Slotine, J.J.E. (1998). On Contraction Analysis for Non-linear Systems. *Automatica*, 34(6), 683–696.

Pavlov, A., van de Wouw, N., and Nijmeijer, H. (2006). *Uniform Output Regulation of Nonlinear Systems*. Birkhäuser Boston.

Rudin, W. (1976). *Principles of Mathematical Analysis*. McGraw-Hill, 3rd edition.

Scherer, C.W. and Weiland, S. (2015). Linear Matrix Inequalities in Control. URL https://www.imng.uni-stuttgart.de/mst/files/LectureNotes.pdf.

Scorletti, G., Formion, V., and De Hillerin, S. (2015). Toward nonlinear tracking and rejection using LPV control. In *Proc. of the 1st IFAC Workshop on Linear Parameter Varying Systems*, volume 48, 13–18.

Shamma, J.S. (1988). Analysis and Design of Gain Scheduled Control Systems. Ph.D. thesis, Massachusetts Institute of Technology.

Tran, D.N., Rüffer, B.S., and Kellett, C.M. (2018). Convergence Properties for Discrete-time Nonlinear Systems. *IEEE Transactions on Automatic Control*, 64(8), 3415–3422.

Tóth, R. (2010). Modeling and Identification of Linear Parameter-Varying Systems. Springer-Verlag.

Van der Schaft, A. (2017). *L_2-Gain and Passivity Techniques in Nonlinear Control*. Springer International Publishing AG, 3rd edition.

Verhoek, C., Koelewijn, P.J.W., and Tóth, R. (2020). Convex Incremental Dissipativity Analysis of Nonlinear Systems. *arXiv preprint arXiv:2006.14201*.

Willems, J.C. (1972). Dissipative Dynamical Systems Part I: General Theory. *Archive for Rational Mechanics and Analysis*, 45(5), 321–351.

Appendix A. NORM INTEGRAL INEQUALITY

Lemma 16. Given a positive definite $M \in \mathbb{R}^{n \times n}$, i.e. $M > 0$, and a continuous function $\phi : [0, 1] \rightarrow \mathbb{R}^n$, then

$$
\left( \int_0^1 \phi(t) \, dt \right)^T M \left( \int_0^1 \phi(t) \, dt \right) \leq \int_0^1 \phi(t)^T M \phi(t) \, dt
$$

(A.1)

Proof. As $M$ is positive, we can define the Euclidean vector space with

$$
\|v\| := \sqrt{v^T M v},
$$

(A.2)

where $v \in \mathbb{R}^n$. By the Cauchy-Schwarz inequality, for a continuous function $\phi : [0, 1] \rightarrow \mathbb{R}^n$

$$
\left\| \int_0^1 \phi(t) \, dt \right\| \leq \int_0^1 \|\phi(t)\| \, dt,
$$

(A.3)

see Rudin (1976). Furthermore, it also holds that for a function $\psi : [0, 1] \rightarrow \mathbb{R}$

$$
\left\| \int_0^1 \psi(t) \, dt \right\|^2 \leq \left( \int_0^1 1 \, dt \right) \left( \int_0^1 |\psi(t)|^2 \, dt \right) = \left( \int_0^1 |\psi(t)|^2 \, dt \right).
$$

(A.4)

Hence, using (A.3) and (A.4), with $\psi(t) = \|\phi(t)\|$, we get

$$
\left\| \int_0^1 \phi(t) \, dt \right\|^2 \leq \left( \int_0^1 \|\phi(t)\| \, dt \right)^2 \leq \int_0^1 \|\phi(t)\|^2 \, dt.
$$

(A.5)

Using the norm definition (A.2), this results in (A.1).