Family of multipartite separability criteria based on a correlation tensor

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A family of separability criteria based on correlation matrix (tensor) is provided. Interestingly, it unifies several criteria known before like, e.g., computable cross-norm or realignment criterion (CCNR), de Vicente criterion, and derived recently separability criterion based on symmetric informationally complete positive operator valued measures (SIC POVMs). It should be stressed that, unlike the well-known correlation matrix criterion or criterion based on local uncertainty relations, our criteria are linear in the density operator and hence one may find unexplored classes of entanglement witnesses and positive maps. Interestingly, there is a natural generalization to multipartite scenario using multipartite correlation matrix. We illustrate the detection power of the above criteria on several well-known examples of quantum states.

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I. INTRODUCTION

Quantum entanglement is one of key features of quantum theory and provides a crucial resource for modern quantum technologies like quantum communication, quantum cryptography, and quantum calculations [1,2], One of the tasks of the theory of quantum entanglement is to derive criteria which enables one to distinguish separable and entangled states [1,3]

Recall, that a state of a bipartite system living in $\mathcal{H}_A \otimes \mathcal{H}_B$ represented by a density matrix $\rho$ is separable if [4]

$$\rho = \sum_k p_k \rho_A^k \otimes \rho_B^k,$$

where $p_k$ is a probability distribution and $\rho_A^k (\rho_B^k)$ are density operators of subsystem $A$ ($B$). For low-dimensional bipartite systems 2 $\otimes$ 2 (qubit-qubit) and 2 $\otimes$ 3 (qubit-qutrit), this problem is completely solved by the celebrated Peres-Horodecki criterion: A state is separable if and only if it is positive partial transpose (PPT), that is, $\rho^T := (\text{id} \otimes T)\rho \geq 0$ [5,6]. However, for higher dimensional systems and systems composed of more than two parties, the problem is notoriously difficult (actually, it belongs to the class of so-called NP-hard problems [7], that is, the problem is not solvable by polynomial in time algorithm).

There are several separability criteria developed in the past 20 years of activity (see the reviews [1,3]). Any entangled state $\rho$ of a bipartite system can be detected a suitable entanglement witness, that is, a Hermitian operator $W$ acting in $\mathcal{H}_A \otimes \mathcal{H}_B$ such that for all separable states $\text{Tr}(W\rho_{\text{sep}}) \geq 0$ but $\text{Tr}(W\rho) < 0$ [1,3,8,9]. The well-known criterion based on positive maps states that $I_A \otimes \Phi(\rho) \geq 0$ for all positive maps $\Phi$ (it recovers PPT criterion if one takes $\Phi = T$). These two criteria are necessary, sufficient, and related to each other via Choi-Jamiolkowski isomorphism. While classification of entanglement witnesses (equivalently: positive maps) is not known (except the lowest dimensional cases), there is a number of other criteria [1,3] which are not universal, i.e., do not allow us to detect all entangled states, but are easily applicable and in particular allow us to detect many PPT entangled states. The prominent example is realignment or computable cross-norm (CCNR) criterion [10–12]. There are also separability criteria which are nonlinear in the state of the system, like, for example, criteria based on local uncertainty relations (LURs) [13], extensions of realignment criterion [14], or covariance matrix criterion (CMC) [15–17] (see also Ref. [16] for the unifying approach). Separability criteria based on correlation tensor were also analyzed in Ref. [18].

In this paper, we propose a unification of several bipartite separability criteria based on correlation matrix (or correlation tensor). In this category, apart from CCNR, one finds, e.g., de Vicente criterion (dV) [19], separability criterion derived in Ref. [20], and recent criterion based on SIC POMVs (ESIC) [21]. The criterion presented here in general is not stronger that CMC but we provide an example of a PPT state which is not detected by filtered CMC [15,16] (LFCMC) but is detected by this one. Our result is then generalized to multipartite scenario. We stress that the criteria presented here are linear in the density operator and hence may be used to construct other classes of entanglement witness and positive maps.

II. BIPARTITE SYSTEMS

Consider a bipartite system living in $\mathcal{H}_A \otimes \mathcal{H}_B$ with dimensions $d_A$ and $d_B$, respectively (in what follows, we assume $d_A \leq d_B$). Let $G_A^\rho$ and $G_B^\rho$ denote arbitrary orthonormal basis in $B(\mathcal{H}_A)$ and $B(\mathcal{H}_B)$, that is, $(G_A^\mu | G_A^\nu)_{HS} = \delta_{\mu\nu}$, and the same for $G_B^\rho$ (where $(X^Y)_B := \text{Tr}(X^Y)$ is a Hilbert-Schmidt inner product). Now, given a bipartite state $\rho$, one defines the following correlation matrix:

$$C_{\alpha\beta} = (G_A^\alpha \otimes G_B^\beta)_{HS} = \text{Tr}(\rho G_A^\alpha \otimes G_B^\beta).$$

If $\rho$ is separable, then the CCNR criterion gives the following bound for the trace norm of $C$:

$$\|C\|_{tr} \equiv \text{Tr}\sqrt{CC^T} \leq 1.$$
The norm $\|C\|_\infty$ does not depend upon the particular orthonormal basis $G^\alpha_A$ and $G^\beta_B$. Let us take a particular basis consisting of Hermitian operators such that $G^\alpha_0 = \mathbb{I}/\sqrt{d_A}$ and $G^\beta_0 = \mathbb{I}/\sqrt{d_B}$ (we call it the canonical basis). It is clear that $G^\alpha_0$ and $G^\beta_0$ are traceless for $\alpha, \beta > 0$. The canonical basis gives rise to the following generalized Bloch representation:

$$\rho = \frac{\mathbb{I}}{d_A} \otimes \frac{\mathbb{I}}{d_B} + \sum_{i>0} r^A_i G^A_i \otimes \frac{\mathbb{I}}{d_B} + \sum_{j>0} r^B_j \frac{\mathbb{I}}{d_A} \otimes G^B_j$$

$$+ \sum_{i,j>0} t_{ij} G^A_i \otimes G^B_j = \sum_{i=0}^{d_A-1} \sum_{j=0}^{d_B-1} c_{i,j} \rho_{i,j} \otimes G^A_i \otimes G^B_j,$$

where $r_i^A$ and $r_j^B$ are generalized Bloch vectors corresponding to reduced states $\rho_A$ and $\rho_B$, respectively, and $t_{ij}$ is usually called a correlation tensor, that is, one finds for the reduced states

$$\rho_A = \text{Tr}_B \rho = \frac{\mathbb{I}}{d_A} + \sum_{i>0} r^A_i G^A_i$$

and

$$\rho_B = \text{Tr}_A \rho = \frac{\mathbb{I}}{d_B} + \sum_{j>0} r^B_j G^B_j.$$

We denote $C_{i,j}$ defined by the canonical basis by $C_{i,j}$. Clearly $\|C_{\text{can}}\|_\infty = \|C\|_\infty$. Let us introduce two square diagonal matrices:

$$D^A_x = \text{diag}\{x, 1, \ldots, 1\}, \quad D^B_y = \text{diag}\{y, 1, \ldots, 1\},$$

where $D^A_x$ is $d_A \times d_A$ and $D^B_y$ is $d_B \times d_B$, and the real parameters $x, y \geq 0$. Now comes the main result:

**Theorem 1.** If $\rho$ is separable, then

$$\|D^A_x C_{\text{can}} D^B_y\|_\infty \leq N_A(x) N_B(y),$$

where

$$N_A(x) = \sqrt{\frac{d_A - 1 + x^2}{d_A}} \quad N_B(y) = \sqrt{\frac{d_B - 1 + y^2}{d_B}},$$

for arbitrary $x, y \geq 0$.

**Proof.** Separability implies that $\rho$ is a convex combination of product states and hence (due to the triangle inequality for the norm) it is enough to check (5) for a product state $\rho_A \otimes \rho_B$. One finds for the correlation matrix

$$(C_{\text{can}})_{\alpha\beta} = R^A_{\alpha} R^B_{\beta},$$

where $R^A_0 = 1/\sqrt{d_A}$, $R^A_1 = 1$ ($i \geq 1$), and similarly for $R^B_\beta$. It implies $\|C_{\text{can}}\|_\infty = \|R_A\| \|R_B\|$, where $\|R_A\|^2 = \frac{1}{d_A} + \|r^A\|^2$ (and the same for $R_B$). Let us observe that

$$(D^A_x C_{\text{can}} D^B_y)_{\alpha\beta} = (R^A_\alpha R^B_\beta),$$

with $R^A_\alpha = (x/\sqrt{d_A}, r^A)$ and $R^B_\alpha = (y/\sqrt{d_B}, r^B)$. It implies

$$\|D^A_x C_{\text{can}} D^B_y\|_\infty = \sqrt{\frac{x^2}{d_A} + \|r^A\|^2} \sqrt{\frac{y^2}{d_B} + \|r^B\|^2}.$$

Finally, positivity of $\rho_A$ and $\rho_B$ requires that

$$\text{Tr} \rho_A^2 \leq 1, \quad \text{Tr} \rho_B^2 \leq 1,$$

which imply that the corresponding Bloch vectors $r^A$ and $r^B$ satisfy

$$\|r^A\|^2 \leq \frac{d_A - 1}{d_A}, \quad \|r^B\|^2 \leq \frac{d_B - 1}{d_B},$$

and hence formula (5) easily follows.

Note that using well-known inequality [22]

$$\|D^A_x C_{\text{can}} D^B_y\|_\infty \leq \|D^A_x\|_\infty \|C_{\text{can}}\|_\infty \|D^B_y\|_\infty,$$

where $\|X\|_\infty = \sigma_{\text{max}}(X)$ (maximal singular value of $X$), one finds for separable state $\|D^A_x C_{\text{can}} D^B_y\|_\infty \leq \|D^A_x\| \|D^B_y\|$ and hence if $x, y > 1$ it implies $\|D^A_x C_{\text{can}} D^B_y\|_\infty \leq xy$. Note, however, that this condition is much weaker than (5).

**III. RELATION TO OTHER SEPARABILITY CRITERIA**

Clearly $(x, y) = (1, 1)$ reproduces CCNR criterion. Interestingly, $(x, y) = (0, 0)$ reproduces separability criterion derived by de Vicente [19]. If $d_A = d_B$, then CCNR criterion is stronger than the dV criterion. However, for bipartite states $\rho$ such that $\rho_A = \mathbb{I}_A/d_A$ and $\rho_B = \mathbb{I}_B/d_B$, the dV criterion is stronger than CCNR if $d_A \neq d_B$, and they are equivalent if $d_A = d_B$ [19]. Interestingly, we found another example of such a criterion in Ref. [20]. After suitable renormalization, the result of Ref. [20] corresponds to $(x, y) = (\sqrt{2}/d_A, \sqrt{2}/d_B)$.

In a recent paper [21], authors proposed an interesting separability criterion based on symmetric informationally complete positive operator valued measure (SIC POVM). Recall that a family of $d^2-1$ rank-1 operators $\Omega_1 = \frac{1}{d^2} |\psi_1\rangle\langle\psi_1|$ in $d$-dimensional Hilbert space defines SIC POVM if and only if (iff)

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{d \delta_{ij} + 1}{d + 1} - \frac{d^2}{d + 1} \sum_{i=1}^{d^2} \Omega_i = \mathbb{I}_d.$$

There is an old conjecture by Zauner that SIC POVM exists for any $d$ [23] (see also Ref. [24]). So far these objects have been found for several dimensions (see Refs. [25] and [26] for the recent progress). It is, therefore, clear that the result of Ref. [21] was restricted to specific dimensions only. Here we show that this criterion is universal (valid for any $d_A$ and $d_B$). Moreover, it belongs to our class (5) with $(x, y) = (\sqrt{d_A + 1}, \sqrt{d_B + 1})$. The separability criterion (so called ESIC criterion) derived in Ref. [21] states that if $\rho$ is separable, then

$$\|P\|_\infty \leq \frac{2}{\sqrt{d_A(d_A + 1) d_B(d_B + 1)}},$$

where $P_{\alpha\beta} = \langle \Pi^0_\alpha \otimes \Pi^0_\beta \rangle$ and $\Pi^0_\alpha$ and $\Pi^0_\beta$ are elements of SIC POVMs in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. It was conjectured in Ref. [21] that the ESIC criterion is stronger than the CCNR criterion. This conjecture is supported by several examples and numerical analysis (cf. Ref. [21]).

Let us observe that if $\Pi_d$ define SIC POVM in $d$-dimensional Hilbert space, then

$$G^{(\text{SIC})}_{\alpha\beta} := \sqrt{d(d + 1)} \Pi_d - \frac{\sqrt{d + 1} + 1}{\sqrt{d^3}} \mathbb{I}_d,$$

defines on orthonormal basis in $\mathbb{B} \mathcal{(H)}$, that is, $\langle G^{(\text{SIC})}_{\alpha\beta} | G^{(\text{SIC})}_{\gamma\delta} \rangle_{\text{HS}} = \delta_{\alpha\beta \gamma \delta}$. Note that this is not a canonical
basis. Indeed, $G_{\alpha}^{(\pm)}$ is not proportional to $\mathbb{I}_d$. However, it enjoys the following properties:

$$ \text{Tr} \, G_{\alpha}^{(\pm)} = \pm \frac{1}{\sqrt{d}}, \quad \sum_{\alpha} G_{\alpha}^{(\pm)} = \pm \sqrt{d} \mathbb{I}_d. \quad (10) $$

In what follows, we take $G_\alpha := G_{\alpha}^{(-)}$ (but the final result applies for $G_\alpha^{(+)}$ as well). Direct calculation shows

$$ \sqrt{d_A(d_A+1)}d_B(d_B+1)P = A CB, \quad (11) $$

where $C_{\alpha\beta} = (G_A^{\alpha} \otimes G_B^{\beta})$ is a correlation matrix defined in terms of $G_\alpha = G_{\alpha}^{(-)}$ and

$$ A = \mathbb{I}_A \otimes \mathbb{I}_B + a \mathcal{J}_A \otimes \mathcal{J}_A, $$

$$ B = \mathbb{I}_A \otimes \mathbb{I}_B + b \mathcal{J}_B \otimes \mathcal{J}_B, $$

where $\mathcal{J}_A$ is $d_A \times d_A$ matrix such that $[\mathcal{J}_A]_{ij} = 1$ (and similarly for $\mathcal{J}_B$). Finally,

$$ a = \frac{\sqrt{d_A+1} - 1}{d_A^2}, \quad b = \frac{\sqrt{d_B+1} - 1}{d_B^2}. \quad (12) $$

This way, we reformulated ESIC criterion (8) in terms of the correlation matrix $C_{\alpha\beta}$ as follows: If $\rho$ is separable, then

$$ \|ACB\|_\tau \leq 2. \quad (13) $$

It should be stressed that here $C_{\alpha\beta}$ is not a canonical matrix and hence (13) cannot be immediately related to (5). Note, however, that because the trace norm is unitarily invariant, one has

$$ \|ACB\|_\tau = \|UAU^\dagger(UV^\dagger)^{\tau}V^{\tau}\|_\tau, $$

for arbitrary unitary matrices $U$ and $V$. Taking $U$ and $V$ such that they diagonalize $A$ and $B$, respectively, one obtains

$$ \|ACB\|_\tau = \|D_A^{\text{can}}D_B^{\text{can}}\|_\tau, $$

with $(x, y) = (\sqrt{d_A+1}, \sqrt{d_B+1})$. It proves that the original assumption about the existence of two SIC POVMs $\{\Pi_{\alpha}^{A}\}$ and $\{\Pi_{\beta}^{B}\}$ is not essential and the ESIC criterion universally holds for arbitrary $d_A$ and $d_B$.

Finally, the covariance matrix criterion (CMC) [15,16] supplemented by the procedure of local filtering (LFCMC) turned out to be very powerful criterion. Interestingly, for $d_A \leq d_B$ (but $d_B - d_A$ is not too big, cf. Ref. [16]), this criterion is equivalent to (supplemented by a local filtering) dV criterion [19]. Now, in our case if $r_1 = 0$ and $r_2 = 0$, one finds

$$ \|D_A^{\text{can}}D_B^{\text{can}}\|_\tau = \frac{xy}{\sqrt{d_Ad_B}} + \|D_0^{\text{can}}D_0^{\text{can}}\|_\tau, $$

and hence one may wonder whether it is possible to obtain a stronger result than dV criterion. One easily finds that the function $N_\alpha(x)N_\beta(y) - \frac{xy}{\sqrt{d_Ad_B}}$ realizes minimum for $x = \sqrt{d_B/T} = y = \sqrt{d_A/T}$ which reproduces dV [19]. Hence, it proves that within a class of states with maximally mixed marginals (and $d_B - d_A$ is not too big) dV condition is the strongest one.

IV. CLASS OF ENTANGLEMENT WITNESS

Now, we show that this separability criterion gives rise to the whole class of entanglement witnesses. Let us recall that for any $m \times n$ matrix $X$ its trace norm is given by the following formula:

$$ \|X\|_\tau = \max_{O \in \mathcal{O}(m,n)} \langle O(X)\rangle_{HS}, \quad (14) $$

where the maximum is performed over all isometry $m \times n$ matrices $O$. Let $\rho$ be a separable state in $\mathcal{H}_A \otimes \mathcal{H}_B$. One has therefore for any fixed $(x, y)$

$$ \|D_A^{\text{can}}D_B^{\text{can}}\|_\tau \leq N_\alpha(x)N_\beta(y) $$

and hence

$$ 0 \leq N_\alpha(x)N_\beta(y) - \|D_A^{\text{can}}D_B^{\text{can}}\|_\tau $$

$$ = N_\alpha(x)N_\beta(y)\text{Tr}(\rho \mathbb{I}_A \otimes \mathbb{I}_B) $$

$$ - \max_{O \in \mathcal{O}(d_A,d_B)} \langle O(D_A^{\text{can}}D_B^{\text{can}})\rangle_{HS} $$

$$ = N_\alpha(x)N_\beta(y)\text{Tr}(\rho \mathbb{I}_A \otimes \mathbb{I}_B) $$

$$ + \min_{O \in \mathcal{O}(d_A,d_B)} \langle O(D_A^{\text{can}}D_B^{\text{can}})\rangle_{HS}. \quad (15) $$

Therefore, for an arbitrary isometry $O$

$$ \text{Tr}(W^{\tau}_x \rho) \geq 0, \quad (16) $$

where

$$ W^{\tau}_x = N_\alpha(x)N_\beta(y) \mathbb{I}_A \otimes \mathbb{I}_B + \sum_{\alpha, \beta} \tilde{O}_{\alpha\beta} G_\alpha^{\alpha} \otimes G_\beta^{\beta} \quad (17) $$

and the “deformed” isometry $\tilde{O}^{\alpha\beta}$ reads

$$ \tilde{O}^{\alpha\beta} = (D_A^{\text{can}})^{\alpha\beta}O^{\beta\text{can}}(D_B^{\text{can}})^{\beta\alpha}, $$

Finally, $W^{\tau}_x$ has the following structure:

$$ W^{\tau}_x = \sum_{\alpha, \beta} w^{\alpha\beta} G_\alpha^{\alpha} \otimes G_\beta^{\beta} \quad (18) $$

with

$$ w^{00} = \sqrt{(d_A + 1 + x^2)(d_B - 1 + y^2)} + xyO^{00}, $$

and

$$ w^{0\beta} = \frac{x}{\sqrt{d_A}} O^{0\beta}, \quad w^{\alpha0} = \frac{y}{\sqrt{d_B}} O^{\alpha0}, \quad w^{\alpha\beta} = O^{\alpha\beta} $$

for $\alpha, \beta > 0$. This way, one obtains a big class of witnesses parameterized by $d_A \times d_B$ isometry $O$ and two non-negative parameters $x, y$.

V. MULTIPARTITE CRITERION

Our separability criterion (5) may be generalized for the multipartite scenario: Consider $N$ partite system living in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, and let $G^{(k)}_\eta$ denotes an orthonormal basis in $\mathcal{B}(\mathcal{H}_k)$. Given a state $\rho$, define a correlation (hyper)matrix

$$ C_{\alpha_1 \cdots \alpha_N} \equiv \{G^{(k)}_{\alpha_1} \otimes \cdots \otimes G^{(N)}_{\alpha_N}\}_\rho. $$

In order to derive generalization of (5), let us reformulate the definition of the trace norm (14)

$$ \|X\|_\tau = \sup_M \frac{|\langle M(X)\rangle_{HS}|}{\|M\|_{\infty}}, \quad (19) $$

where the supremum is taken over all matrices of appropriate size. It is well known that the supremum is always realized.
by some isometry [as in (14)]. Now, we generalize (19) to an arbitrary $N$ tensor $X_{i_1,\ldots,i_N}$, where
\[
(M^N | X^N)_{HS} = \sum_{i_1,\ldots,i_N} M^N_{i_1,\ldots,i_N} X^N_{i_1,\ldots,i_N},
\]
and the spectral (operator) norm is defined as follows:
\[
\|M^N\|_{\infty} := \sup_{x_{i_1} \cdots x_{i_N}} \left| \sum_{i_1,\ldots,i_N} M^N_{i_1,\ldots,i_N} x_{i_1} \cdots x_{i_N} \right|.
\]
The $N$-partite CCNR criterion reads as follows:

**Proposition 1.** If $N$-partite state is fully separable, then $\|C^N\|_\infty \leq 1$.

**Proof.** Again, it is enough to check it for a product state $\rho^1 \otimes \cdots \otimes \rho^N$. Since the trace norm does not depend upon the basis, let us take the canonical one. One finds for the correlation hypermatrix
\[
C^N_{i_1,\ldots,i_N} = R^1_{i_1} \cdots R^N_{i_N},
\]
where the vector $R^k \in \mathbb{R}^{d_k}$ reads
\[
R^k_{i_k} = (G^k_{i_k} x^k)_{pq} = \text{Tr}(\rho^k G^k_{i_k} x^k) = (1/\sqrt{d_k}) x^k
\]
and $x^k$ is a Bloch vector of $\rho^k$. One has
\[
(M^N | C^N)_{HS} \leq \|M^N\|_\infty \|R^1_{i_1}\| \cdots \|R^N_{i_N}\| \leq \|M^N\|_\infty
\]
and hence $\|C^N\|_\infty \leq 1$.

To generalize (5), let us define $N$ diagonal $d_k^2 \times d_k^2$ matrices
\[
D^k_{i_k} = \text{diag}(x_k, 1, \ldots, 1),
\]
and $C^N_{x_1,\ldots,x_N}$ defined as follows:
\[
C^N_{i_1,\ldots,i_N} = C^N_{x_1,\ldots,x_N} (D^1_{i_1})_{i_1,i_1} \cdots (D^N_{x_N})_{i_N,i_N}.
\]
One proves the following:

**Theorem 2.** If $\rho$ is fully separable, then
\[
\|C^N(x_1,\ldots,x_N)\|_\infty \leq N^k (x_1) \cdots N^k (x_N),
\]
where
\[
N^k (x_k) = \sqrt{d_k - 1 + x_k^2} / d_k.
\]
for $k = 1, \ldots, N$.

The proof is similar to that of Theorem 1. Indeed, taking again a product state $\rho^1 \otimes \cdots \otimes \rho^N$, one finds
\[
(M^N | C^N(x_1,\ldots,x_N))_{HS} \leq \|M^N\|_\infty |R^1_{i_1}| \cdots |R^N_{x_N}|,
\]
where $|R^k_{i_k}| = (x_k/\sqrt{d_k}) x^k$, and hence
\[
\|C^N(x_1,\ldots,x_N)\|_\infty \leq |R^1_{i_1}| \cdots |R^N_{x_N}|.
\]
Finally, note that $|R^k_{i_k}|^2 = x_k^2/d_k + |x_k|^2 \leq N^k (x_k)$ due to $|x_k|^2 \leq (d_k - 1)/d_k$, which ends the proof.

Actually, the trace-norm (or more generally Ky-Fan norm) was generalized for $N$ tensors using a procedure of so-called unfoldings [27]: Given an $X^N \in C^{d_1} \otimes \cdots \otimes C^{d_N}$, one defines an $n$-unfolding (or an $n$-mode matricization of $X^N$) $X_{i_1}$ which is a $d_k \times d_n$ matrix with $X_{i_1} = (d_1 d_2 \ldots d_k) / d_n$ (see Ref. [27] for a precise definition). Now, the Ky-Fan norm of $X^N$ is defined as follows:
\[
\|X^N\|_n := \max_n \|X^N_{i_1}\|_n.
\]
Using the same arguments, one easily derives the following:

**Proposition 2.** If $\rho$ is fully separable, then
\[
\|C^N(x_1,\ldots,x_N)\|_\infty \leq N^k (x_1) \cdots N^k (x_N).
\]

Note, however, that due to $\|X^N\|_1 \leq \|X^N\|_\infty$ the separability criterion based on (22) is weaker than (20). The procedure of unfolding gives rise to a family of matrices, each of which only controls bipartite entanglement in $d_k \times d_n$ system. Interestingly, criterion (22) for $x_k = 0 (k = 1, \ldots, N)$ was already derived in Ref. [28], and for $x_k = \sqrt{2}/d_k$ it was derived in Ref. [20]. It should be clear that if each $\mathcal{H}_k$ allows for the existence of SIC POVM, then for $x_k = \sqrt{d_k + 1}$ one obtains a multipartite generalization of ESIC criterion from Ref. [21]. However, as we already observed, the existence of SICs is not essential.

**VI. DETECTION POWER**

In Ref. [12], Rudolph constructed an example of a two qubit state which is entangled (and hence not positive under partial transpose(PPT)) but it is not detected by CCNR criterion. It turns out that such state is always detected by our criterion for sufficiently big $\alpha$ and $\beta$ (cf. Appendix A for details). However, contrary to CMC, it does not detect all NPT qubit-qubit states.

Let us consider two one-parameter families of two-qutrit states constructed from unextendable product basis (UPB) [29,30]. The first family contains states of the form $\rho^{PP} = p \rho^{PP} + (1 - p) \mathbb{I}_2 \otimes \mathbb{I}_2/9$, where $\rho^{PP}$ is a bound entangled state constructed by use of the pentagon pyramid (PP) construction. The second family contains states of the form $\rho^{T} = p \rho^{T} + (1 - p) \mathbb{I}_3 \otimes \mathbb{I}_2/9$, where $\rho^{T}$ is a bound entangled state constructed by use of the Tiles (T) construction. We compare detection thresholds in these families with regard to $dV$, CCNR, ESIC, and LFCMC criteria:

|        | dV   | CCNR | ESIC | LFCMC |
|--------|------|------|------|-------|
| PP     | .9371| .8785| .8739| .8639 |
| Ti     | .9493| .8897| .8845| .8722 |

whereas our criterion detects entanglement in the PP family for $p \geq 0.8721$ ($x = 0.4059,7$) and in Ti family for $p \geq 0.8822$ ($x = 2442.1$). Our criterion detects more than linear criteria ($dV$, CCNR, and ESIC) but less than nonlinear LFCMC.

Now, we provide an example of a qutrit-qutrit state which is detected neither by CCNR nor by ESIC but it is detected by (5). Consider a chessboard state [31] defined in terms of four orthogonal vectors in $\mathbb{C}^3 \otimes \mathbb{C}^3$:

\[
|V_1\rangle = \left| m, 0, 0; s, 0, 0; n, 0, 0 \right>,
\]
\[
|V_2\rangle = \left| 0, a, 0; b, 0, c; 0, 0, 0 \right>,
\]
\[
|V_3\rangle = \left| n^*, 0, 0; 0, -m^*, 0; s, 0, 0 \right>,
\]
\[
|V_4\rangle = \left| 0, -b^*, 0; a^*, 0, 0; d, 0, 0 \right>,
\]
giving rise to $\rho = \mathcal{N} \sum_i |V_i\rangle \langle V_i|$, with $\mathcal{N}$ being a normalization factor. Let us consider the mixture with white noise $\rho_p = p \rho + (1 - p) \mathbb{I}_6 \otimes \mathbb{I}_2/9$. It is shown in Appendix B that by taking a suitable parameters we may construct a PPT state.
is a natural generalization to a multipartite scenario using multi-
entanglement witnesses and positive maps. Interestingly, there
known CMC or LUR, these criteria are linear in the density
operator space. It should be stressed that, unlike the well-
known criteria: (0,0), dV; (1,1), CCNR; and (2,2), ESIC.

These criteria are based on the universal object—correlation
CCNR or realignment criterion, de Vicente criterion, and de-

Interestingly, it unifies several criteria known before like, e.g.,
CMC [21], but is detected by (5) for \( |t| > 0 \). One
finds for the correlation matrix

\[
C^\text{can} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & r \\
0 & t & 0 & 0 \\
0 & 0 & -t & 0 \\
s & 0 & 0 & 1 + r - s
\end{pmatrix},
\]

and hence \( \|C^\text{can}\|_\infty = |t| + g(r,s) \) [12], and in general
\( \|C^\text{can}\|_\infty \leq 1 \) even if \( |t| > 0 \).

Now, using our criterion, one finds

\[
D_x^y C^\text{can} D_y^x = \frac{1}{2} \begin{pmatrix}
xy & 0 & 0 & xr \\
0 & t & 0 & 0 \\
0 & 0 & -t & 0 \\
sy & 0 & 0 & 1 + r - s
\end{pmatrix},
\]

and hence for a separable (PPT) state

\[
\|D_x^y C^\text{can} D_y^x\|_\infty = |t| + f(x,y,r,s),
\]

\[
f(x,y,r,s) = \sqrt{\lambda_+(x,y,r,s)} + \sqrt{\lambda_-(x,y,r,s)},
\]

\[
\lambda_\pm(x,y,r,s) = \frac{1}{8} \left((1 + r - s)^2 + s^2 x^2 + r^2 y^2 + x^2 y^2 \pm \sqrt{[(1 + r - s)^2 + s^2 x^2 + r^2 y^2 + x^2 y^2]^2 - 4(1 + r)^2(1 - s)^2 x^2 y^2}\right).
\]

Note that in the limit \( x, y \to \infty \)

\[
\lambda_+(x,y,r,s) \to \frac{x^2 y^2}{4}, \quad \lambda_-(x,y,r,s) \to 0, \quad f(x,y,r,s) \to \frac{xy}{2},
\]

and hence for PPT (separable) state our criterion

\[
|t| + f(x,y,r,s) \leq \sqrt{\frac{1 + x^2}{2}} \sqrt{\frac{1 + y^2}{2}}
\]

gives in the limit \( x, y \to \infty \) the condition \( |t| \leq 0 \) which recovers PPT condition for (A1).
APPENDIX B: CHESSBOARD STATE [31]

Taking the following parameters,
\[ a = 0.3346, \quad b = -0.1090, \quad c = -0.6456, \]
\[ d = 0.8560, \quad m = 0.4690, \quad n = -0.3161, \]
\[ s = -1.0178, \quad t = -0.6085, \quad p = 0.8062, \]
one obtains the following PPT density matrix (whose entanglement is not detected by realignment, CMC criterion, and ESIC criterion):
\[
\rho_p = \begin{bmatrix}
0.0964 & 0 & -0.1118 & 0 & 0 & 0 & 0.0450 & 0 & 0 \\
0 & 0.0505 & 0 & 0 & 0 & 0 & 0 & -0.0218 & 0 \\
-0.1118 & 0 & 0.2641 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0505 & 0 & 0 & 0 & 0.0165 & 0 \\
0 & 0 & 0.0753 & 0 & 0.0664 & 0 & 0.0668 & 0 & 0 \\
0 & -0.0506 & 0 & 0.1655 & 0 & 0.1191 & 0 & 0 & 0 \\
0.0450 & 0 & 0 & 0.0668 & 0 & 0 & 0.1082 & 0 & 0 \\
0 & -0.0218 & 0 & -0.0671 & 0 & 0 & 0 & 0.1931 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0215 \\
\end{bmatrix}.
\] (B1)

We calculate the quantity
\[ \sqrt{\frac{2 + x^2}{3}} \sqrt{\frac{2 + y^2}{3}} - \| D_A^\dagger C^{\text{can}} D_B^\dagger \|_w, \]
for \((x, y) = (5.8, 5.9)\). It should be non-negative for separable states. We get \(\approx -5.45 \times 10^{-3}\) and hence we detect entanglement in the state. On the other hand, performing a local filtering of \(\rho_p\)
\[ \rho_{LF} = \frac{A \otimes B \rho_B A^\dagger \otimes B^\dagger}{\text{Tr}(A \otimes B \rho_B A^\dagger \otimes B^\dagger)} \]
with operators
\[
A = \begin{bmatrix}
1.2970 & 0 & -0.0770 \\
0 & 1.4374 & 0 \\
-0.0892 & 0 & 1.2698 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0.9171 & 0 & 0.1126 \\
0 & 0.7412 & 0 \\
0.1126 & 0 & 0.6961 \\
\end{bmatrix}.
\] (B4)

one obtains a state \(\rho_{LF}\) with maximally mixed partial traces:
\[
\rho_{LF} = \begin{bmatrix}
0.0962 & 0 & -0.0717 & 0 & 0.0067 & 0 & 0.0466 & 0 & 0.0113 \\
0 & 0.0497 & 0 & -0.0028 & 0 & -0.0480 & 0 & -0.0033 & 0 \\
-0.0717 & 0 & 0.1878 & 0 & 0.0718 & 0 & 0.0113 & 0 & -0.0131 \\
0 & -0.0028 & 0 & 0.0980 & 0 & 0.0522 & 0 & -0.0827 & 0 \\
0.0067 & 0 & 0.0718 & 0 & 0.1095 & 0 & 0.0821 & 0 & 0.0052 \\
0 & -0.0480 & 0 & 0.0522 & 0 & 0.1259 & 0 & -0.0069 & 0 \\
0.0466 & 0 & 0.0113 & 0 & 0.0821 & 0 & 0.1391 & 0 & 0.0194 \\
0 & -0.0034 & 0 & -0.0827 & 0 & -0.0069 & 0 & 0.1740 & 0 \\
0.0113 & 0 & -0.0131 & 0 & 0.0052 & 0 & 0.0194 & 0 & 0.0198 \\
\end{bmatrix}.
\] (B5)

Calculating quantity
\[ \sqrt{\frac{2}{3}} \sqrt{\frac{2}{3}} - \| D_A^\dagger C^{\text{can}} D_B^\dagger \|_w, \]
one gets \(\approx 5.41 \times 10^{-3}\), and hence the state is not detected by the covariance matrix criterion after local filtering makes its partial traces maximally mixed.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[2] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, UK, 2000).
[3] O. Gühne and G. Tóth, Phys. Rep. 474, 1 (2009).
[4] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[5] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[6] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).
[7] L. Gurvits, in Proceedings of the 35th ACM Symposium on Theory of Computing (ACM Press, New York, 2003), pp. 10–19.

[8] B. M. Terhal, Phys. Lett. A 271, 319 (2000).

[9] D. Chruściński and G. Sarbicki, J. Phys. A 47, 483001 (2014).

[10] K. Chen and L.-A. Wu, Quantum Inf. Comput. 3, 193 (2003).

[11] K. Chen and L.-A. Wu, Phys. Rev. A 69, 022312 (2004).

[12] O. Rudolph, Quant. Info. Proc. 4, 219 (2005).

[13] H. F. Hofmann and S. Takeuchi, Phys. Rev. A 68, 032103 (2003).

[14] C.-J. Zhang, Y.-S. Zhang, S. Zhang, and G.-C. Guo, Phys. Rev. A 77, 060301(R) (2008).

[15] O. Gühne, P. Hyllus, O. Gittsovich, and J. Eisert, Phys. Rev. Lett. 99, 130504 (2007).

[16] O. Gittsovich and O. Gühne, Phys. Rev. A 81, 032333 (2010).

[17] M. Li, S.-M. Fei, and Z.-X. Wang, J. Phys. A 41, 202002 (2008).

[18] P. Badziag, C. Brukner, W. Laskowski, T. Paterek, and M. Żukowski, Phys. Rev. Lett. 100, 140403 (2008); W. Laskowski, M. Markiewicz, T. Paterek, and M. Żukowski, Phys. Rev. A 84, 062305 (2011).

[19] J. D. Vicente, Quant. Inf. Comput. 7, 624 (2007).

[20] M. Li, J. Wang, S.-M. Fei, and X. Li-Jost, Phys. Rev. A 89, 022325 (2014).

[21] J. Shang, A. Asadian, H. Zhu, and O. Gühne, Phys. Rev. A 98, 022309 (2018).

[22] R. A. Horn and C. R. Johnson, Matrix Analysis (Cambridge University Press, Cambridge, UK, 2013).

[23] G. Zauner, Quantendesigns - Grundzüge einer nichtkommutativen Designtheorie, Ph.D. thesis, University of Vienna, Vienna, Austria, 1999 (see also G. Zauner, Int. J. Quantum Inf. 9, 445 (2011) for the English translation).

[24] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, J. Math. Phys. 45, 2171 (2004).

[25] C. A. Fuchs, M. C. Hoang, and B. C. Stacey, Axioms 6, 21 (2017).

[26] M. Appleby, T.-Y. Chien, S. Flammia, and S. Waldron, J. Phys. A 51, 165302 (2018).

[27] L. De Lathauwer, B. De Moor, and J. Vandewalle, SIAM J. Matrix Anal. A 21, 1253 (2002).

[28] A. S. M. Hassan and P. S. Joag, Quantum Inf. Comput. 8, 773 (2007).

[29] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. 82, 5385 (1999).

[30] D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Commun. Math. Phys. 238, 379 (2003).

[31] D. Bruss and A. Peres, Phys. Rev. A 61, 030301(R) (2000).

[32] J. I. de Vicente and M. Huber, Phys. Rev. A 84, 062306 (2011).

[33] K. Chen and L. Wu, Phys. Lett. A 306, 14 (2002).

[34] B. C. Hiesmayr and M. Huber, Phys. Rev. A 78, 012342 (2008).

[35] B. C. Hiesmayr, M. Huber, and P. Krammer, Phys. Rev. A 79, 062308 (2009).

[36] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A 62, 052310 (2000).