Counting homotopy classes of mappings via Dijkgraaf-Witten invariants

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Abstract

Suppose $\Gamma$ is a finite group acting freely on $S^n$ ($n \geq 3$ being odd) and $M$ is any closed oriented $n$-manifold. We show that, given an integer $k$, the set $\text{deg}^{-1}(k)$ of based homotopy classes of mappings with degree $k$ is finite and its cardinality depends only on the congruence class of $k$ modulo $\#\Gamma$; moreover, $\# \text{deg}^{-1}(k)$ can be expressed in terms of the Dijkgraaf-Witten invariants of $M$.

key words: homotopy class, degree, topological spherical space form, Dijkgraaf-Witten invariant.

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1 Introduction

The topic of nonzero degree mappings between manifolds has a history of longer than 20 years. For a 3-manifold $M$, the homotopy set $[M, \mathbb{R}P^3]$ is important in the study of Lorentz metrics on a space-time model in which $M$ corresponds to the space part (c.f. [10]), and in this context the existence of a degree-one mapping of any quotient of $S^3$ by a free action of a finite group to $\mathbb{R}P^3$ was completely determined by Shastri-Zvengrowski [15]. The existence of degree-one mappings from 3-manifolds $M$ to lens spaces was studied by Legrand-Wang-Zieschang [10] and some conditions were obtained and expressed in terms of the torsion part of $H_1(M)$. Based on their work of computing cohomology ring, Bryden-Zvengrowski [2] gave sufficient and necessary conditions for the existence of degree-one mappings from Seifert 3-manifolds to lens spaces in terms of geometric data. As for another problem, the set of self-mapping degrees of 3-manifolds in Thurston’s picture are all

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determined by Sun-Wang-Wu-Zheng [17]. In high dimension, degrees of mappings between \((n - 1)\)-connected \(2n\)-manifolds were studied in Ding-Pan [5], Duan-Wang [6] and Lee-Xu [9]. For other problems, see [14,16] and the references therein.

In this article we focus on mappings from a closed oriented \(n\)-manifold \(M\) (with \(n \geq 3\) odd) to a topological spherical space form \(S^n/\Gamma\) where \(\Gamma\) is a finite group with an orientatation-preserving free action on \(S^n\). Nowadays we have a fairly good understanding of such \(\Gamma\) in general and a complete classification of the spaces \(S^n/\Gamma\) when \(n = 3\). An important achievement of geometric topology is the characterization of finite group \(\Gamma\) which can act freely on some \(S^n\) by Madsen-Thomas-Wall [11]: a finite group \(\Gamma\) can act freely on some \(S^n\) if and only if \(\Gamma\) satisfies the \(p^2\)- and \(2p\)-conditions for all prime \(p\), i.e. all subgroups of \(\Gamma\) of order \(p^2\) or \(2p\) are cyclic. In dimension 3, a more concrete classification is known: by the work of Perelman, all 3-dimensional topological spherical space forms are spherical space forms, i.e. \(\Gamma < SO(4)\) acting on \(S^3\) by isometries; and the determination of 3-dimensional spherical space forms is a classical result due to Threlfall-Seifert and Hopf in the 1930’s ([8,18]). Besides the cyclic group case (where \(S^3/\Gamma\) is a lens space), there are four cases: the dihedral case, the tetrahedral case, the octahedral case and the icosahedral case (where the Poincaré 3-sphere is an example). For more details see [19,20].

It is a classical result of [13] that the homotopy classes of mappings \(f : M \to S^n/\Gamma\) are classified by the degree \(\deg f\) and the induced homomorphism \(\pi_1(f) : \pi_1(M) \to \Gamma\). We find that the based homotopy set \([M, S^n/\Gamma]\) has a nice structure, namely, it is an affine set modeled on \(\mathbb{Z}\). Given an integer \(k\), the subset of homotopy classes of mappings with degree \(k\) is a finite set and its cardinality can be expressed in terms of the Dijkgraaf-Witten invariants of \(M\).

Dijkgraaf-Witten theory was first proposed in [4] by the two authors naming the theory, as a 3-dimensional topological quantum field theory. Later it was generalized to any dimension by Freed [7]. Recall that for a given finite group \(\Gamma\) and a cohomology class \([\omega] \in H^n(B\Gamma; U(1))\), the Dijkgraaf-Witten invariant of a closed \(n\)-manifold \(M\) is defined as

\[
Z^{[\omega]}(M) = \frac{1}{\#\Gamma} \sum_{\phi : \pi_1(M) \to \Gamma} \langle [\omega], f(\phi)_* [M] \rangle,
\]

where \(f(\phi)_* : H_n(M; \mathbb{Z}) \to H_n(B\Gamma; \mathbb{Z})\) is induced by the associated mapping \(f(\phi) : M \to B\Gamma\) which is unique up to based homotopy. The TQFT axioms enable us to compute \(Z^{[\omega]}(M)\) by a cut-and-paste process.
2 Mappings to topological spherical space forms

We assume that all manifolds are closed, oriented and equipped with a basepoint, and all mappings and homotopies are base-point preserving.

From now on we suppose $\Gamma$ acts freely on $S^n$ and denote $m = \#\Gamma$. The classifying space $B\Gamma$ can be obtained by attaching higher dimensional cells to $S^n/\Gamma$ to kill homotopy groups. Let $\varphi : S^n/\Gamma \to B\Gamma$ denote the inclusion map, and let $p : S^n \to S^n/\Gamma$ denote the covering map.

**Lemma 2.1.** Suppose $M$ is an $n$-manifold, and $f_0, f_1 : M \to S^n/\Gamma$ are two mappings such that $\pi_1(f_0) = \pi_1(f_1) : \pi_1(M) \to \Gamma$, then $\deg f_0 \equiv \deg f_1 \pmod{m}$. If furthermore $\deg f_0 = \deg f_1$, then $f_0$ is homotopic to $f_1$.

**Proof.** The composition $\varphi \circ f_0$ is homotopic to $\varphi \circ f_1$ since $\pi_1(f_0) = \pi_1(f_1) : \pi_1(M) \to \Gamma$; choose a homotopy $h : M \times [0, 1] \to B\Gamma$. Let $K \subset M$ be the $(n-1)$-skeleton of $M$, then $M = K \cup D^n$. By cellular approximation, we may find $h' : K \times [0, 1] \to S^n/\Gamma$ such that $\varphi \circ h' \simeq h|_{K \times [0, 1]} : K \times [0, 1] \to B\Gamma$, and $h'|_{K \times \{i\}} = f_i$ for $i = 0, 1$.

From $f_0, f_1$ and $h'$ we can construct a mapping

$$g : S^n = \partial(D^n \times [0, 1]) \hookrightarrow M \times \{0, 1\} \cup K \times [0, 1] \to S^n/\Gamma.$$ 

Let $g \circ f_0$ denote the composite

$$M \cong S^n \# M \to S^n \vee M \xrightarrow{g \circ f_0} S^n/\Gamma.$$ 

Then $g \circ f_0 \simeq f_1$, hence $\deg g = \deg f_1 - \deg f_0$.

Clearly $g$ lifts to a mapping $\overline{g} : S^n \to S^n$ and $\deg g = m \deg \overline{g}$. \hfill $\square$

From the proof we see that there is a free action of $[S^n, S^n/\Gamma] \cong \mathbb{Z}$ on $[M, S^n/\Gamma]$. By cellular approximation, each mapping $M \to B\Gamma$ is homotopic to a mapping $M \to S^n/\Gamma$. Summarizing, we have a diagram

$$
\begin{array}{c}
[M, S^n/\Gamma] \\
\downarrow_{\deg}
\end{array} \xrightarrow{\pi_1} \hom(\pi_1(M), \Gamma)$$

such that

- The map $\pi_1$ is surjective, and $\deg \times \pi_1 : [M, S^n/\Gamma] \to \mathbb{Z} \times \hom(\pi_1(M), \Gamma)$ is injective.
For any $\phi \in \text{hom}(\pi_1(M), \Gamma)$, the set $\deg(\pi_1^{-1}(\phi))$ is a congruence class modulo $m$.

It is well-known that (see Section 1.6 of [1], for instance), by splicing the acyclic $Z\Gamma$ complex

$$0 \to Z \to C_n(S^n) \to \cdots \to C_0(S^n) \to Z \to 0$$

we obtain a periodic resolution of $Z$ as a trivial $Z\Gamma$-module and hence $H^*_n(B\Gamma; Z) \cong Z/mZ$ with a preferred generator coming from the fundamental class of $S^n/\Gamma$. By the Universal Coefficient Theorem,

$$H^*_n(B\Gamma; U(1)) \cong \text{hom}(H^*_n(B\Gamma; Z), U(1)) \cong Z/mZ$$

and the pairing $H^*_n(B\Gamma; U(1)) \times H_n(B\Gamma; Z) \to U(1)$ is given by

$$\langle l, k \rangle = \zeta^{kl}_m,$$

where

$$\zeta_m = \exp\left(\frac{2\pi i}{m}\right).$$

**Theorem 2.2.** Each homotopy set $\deg^{-1}(k)$ is finite. More precisely,

$$\# \deg^{-1}(k) = \sum_{l \in \mathbb{Z}/m\mathbb{Z}} Z^l(M) \cdot \zeta_{m}^{-kl}.$$

**Proof.** There is a one-to-one correspondence

$$\deg^{-1}(k) \leftrightarrow \{ \phi : \pi_1(M) \to \Gamma : f(\phi)_*[M] = \overline{k} \},$$

hence

$$\# \deg^{-1}(k) = \# \{ \phi : \pi_1(M) \to \Gamma : f(\phi)_*[M] = \overline{k} \},$$

$$Z^\overline{k}(M) = \frac{1}{m} \cdot \sum_{\overline{k} \in \mathbb{Z}/m\mathbb{Z}} \# \deg^{-1}(k) \cdot \zeta_{m}^{kl}.$$  

The formula for $\# \deg^{-1}(k)$ is obtained by Fourier transformation.

**Remark 2.3.** In [12] Murakami-Ohtsuki-Okada defined an invariant of 3-manifold which can be shown to depend only on $\beta_1(M)$ (the first Betti number of $M$) and the linking paring $\lambda$ on $\text{Tor}H_1(M)$. They proved a formula expressing the DW invariant $Z^\overline{k}(M)$ (for a cyclic group) in terms of their invariant. Combining this with our result, we see that in dimension 3 there is an indirect connection from $\beta_1(M)$ and $\lambda$ to $\# \deg^{-1}(k)$, and the result of [10] provided an evidence for the case $k = 1$. 

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Example 2.4. Let us consider the special case when $n = 3$, $M$ is the Seifert 3-manifold with orientable base $M_O(g; (a_1, b_1), \cdots, (a_r, b_r))$ and $\Gamma = \mathbb{Z}/m\mathbb{Z}$. The Dijkgraaf-Witten invariants of such manifolds are computed by the first author in [3]. By Theorem 3.3 of [3],

$$Z_l^g(M) = m^{2g-2} \sum_{h,s \in \mathbb{Z}/m\mathbb{Z}} \prod_{j=1}^r \left( \sum_{z_j \in \mathbb{Z}/m\mathbb{Z}} \zeta_{m^2}^{la_j b_j z_j^2 - (2l\tilde{h} + ms)\tilde{b}_j \tilde{z}_j} \right),$$

where $x \mapsto \tilde{x}$ denotes the obvious map $\mathbb{Z}/m\mathbb{Z} \to \{0, 1, \cdots, m-1\}$.

Thus

$$\# \text{deg}^{-1}(k) = m^{2g-2} \sum_{l,h,s \in \mathbb{Z}/m\mathbb{Z}} \zeta_{m}^{-kl} \prod_{j=1}^r \left( \sum_{z_j \in \mathbb{Z}/m\mathbb{Z}} \zeta_{m^2}^{la_j b_j z_j^2 - (2l\tilde{h} + ms)\tilde{b}_j \tilde{z}_j} \right).$$

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