Dynamical spectral structure of density fluctuation near the QCD critical point

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Abstract The expression for the dynamical spectral structure of the density fluctuation near the QCD critical point has been derived using linear response theory within the purview of Israel–Stewart relativistic viscous hydrodynamics. The change in the spectral structure of the system as it approaches the critical point has been studied. The effects of the critical point have been introduced in the system through a realistic equation of state and the scaling behaviour of various transport coefficients and thermodynamic response functions. We have found that the Rayleigh and Brillouin peaks are distinctly visible when the system is away from the critical point but the peaks tend to merge near the critical point. The sensitivity of the structure of the spectral function on wave vector ($k$) of the sound wave has been demonstrated. It has been shown that the Brillouin peaks get merged with the Rayleigh peak because of the absorption of sound waves in the vicinity of the critical point.

1 Introduction

Relativistic Heavy Ion Collisions experiments (RHIC-E) are aimed at exploring the properties of high temperature \cite{1} and high density \cite{2} state of strongly interacting matter. The degrees of freedom of hadronic matter under extreme conditions of temperature and densities are deconfined quarks and gluons and their interaction is governed by Quantum Chromodynamics (QCD). The study of the deconfined state of quarks and gluons is relevant for understanding the evolution of the micro-second old universe, the composition of the core of the compact astrophysical objects (e.g. neutron star) and properties of non-abelian gauge theory in medium \cite{3–5}. The RHIC-E provide opportunities to verify different theoretical predictions on the thermal nature of QCD matter \cite{6}. Results of these experiments indicate that after collision of two heavy ions at relativistic energies, a strongly interacting perfect fluid medium consisting of quarks and gluons—called quark-gluon plasma (QGP) \cite{7}—is formed \cite{8–11}. The fluid with high internal pressure expands hydrodynamically, cools and revert to the hadronic phase. The thermal properties of the QGP can be described by two independent variables, the temperature ($T$) and baryonic chemical potential ($\mu$) associated with the conservation of net baryon number in the system. The values of $T$ and $\mu$ of the system depend on the energy of collisions of the nuclei. Numerical simulation of QCD in the high $T$ and low $\mu(\rightarrow 0)$ region predict that the quark-hadron transition is a cross-over \cite{12–16}. However, calculations based on several QCD inspired models indicate that at high $\mu$ region the transition is first-order \cite{15,17}. Therefore, it is expected that between the cross-over and the first-order transitions there exists a region in the $T – \mu$ plane where the first order transition ends and crossover begins \cite{18}. This point is called the Critical End Point (CEP) in the QCD phase diagram. At present, precise lattice QCD results for high $\mu$ are not available due to the well-known sign problem for spin 1/2 particles. As a result, the prediction of the precise position of the CEP is not possible from first principle calculations \cite{19}. Due to the lack of first-principle calculations there have been large numbers of effective field theory-based studies on the QCD phase at non-zero baryon density \cite{20–25}. Different QCD-
based effective models have predicted diverse locations of CEP in the QCD phase diagram. The holographic model for five-dimensional black holes indicated the critical coordinate as \((\mu_c, T_c) = (783 \text{ MeV, } 143 \text{ MeV})\) \cite{26}. These model studies indicate the existence of CEP in the phase diagram. However, the position of the CEP is still ambiguous as its location depends on the parameters of the models used. It is argued in Refs. \cite{27–29} that it is unlikely that \(\mu < 2T\) at the CEP. In the present work, we chose \((\mu_c, T_c) = (367, 154)\) MeV. However, the results and conclusion drawn from such a selection of \(\mu_c\) and \(T_c\) will remain valid for other values of \(\mu_c\) and \(T_c\) too.

Extensive efforts have been given to search for the CEP in the Beam Energy Scan (BES) program in Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory (BNL) to create fireball at different \(\mu\) and \(T\) by varying the centre of mass energies \((\sqrt{s})\) \cite{30}. One of the most promising signatures of the CEP is a non-monotonic behaviour of beam energy dependence of higher-order cumulants of the baryon fluctuations reflected through the net proton production. The net proton yield has been calculated by using both QCD based models \cite{31–34} and gauge/gravity correspondence \cite{35}. It is found in Ref. \cite{36} that the path to the critical point is influenced by far from equilibrium initial conditions leading to dramatically different \((T, \mu)\) due to viscous effects. When the CEP is approached, the fluctuations become very large, which is related to the diverging nature of the correlation length \((\xi)\). However, due to the critical slowing down, correlation length does not grow as much and limits within 2–3 fm at most \cite{37}. The effect of critical slowing down has been taken into account with the slow hydrodynamic modes, recently developed by Stephanov et al \cite{38}. Recently it has been shown \cite{39} that in the presence of the CEP the QGP will suppress all the waves having finite wavelengths.

It is well-known that the density–density correlation length diverges at the critical point. The effects of the divergence on the baryon number fluctuations, particle correlations and correlations of density fluctuations will have a better chance to be detected provided the fluctuation survives the evolution of the hadronic phase. It is important to understand the correlations of these fluctuations theoretically for identifying signatures of the CEP in data. The fluid dynamical descriptions of the system are valid when the ratio, \(\frac{\xi}{\lambda} \ll 1\). However, at the CEP \(\xi\) diverges resulting in the break down of the fluid dynamics. However, it is possible to identify a region near the CEP where the fluid dynamics remain valid and can be used to understand the properties of the system \cite{40}. The effects of CEP on the evolution of matter go as input through the equation of state (EoS) and via the critical behaviour of the transport coefficients and the thermodynamic response functions of the fluid.

The correlation of density fluctuations can be investigated through the spectral structure \((S_{nn}(k, \omega))\) in Fourier space near the CEP containing the Rayleigh (R) and Brillouin (B) peaks corresponding to thermal and mechanical fluctuations, respectively \cite{40,41}. The spectral function has been studied experimentally in the condensed matter physics laboratories to estimate the speed of sound \((c_s)\) by using scattering of light. The position of B-peaks in the \(S_{nn}(k, \omega)\) is connected to \(c_s\).

The integrated intensities under the R and B peaks and their widths are connected to various transport coefficients like thermal conductivity \((\kappa)\), shear \((\eta)\), and bulk \((\xi)\) viscosities and response functions like specific heats at constant volume \((C_V)\) and pressure \((C_P)\). Sadly, such external probes are not available to examine the properties of QCD matter near the CEP.

The dynamical spectral structure, \(S_{nn}(k, \omega)\) has been estimated in Ref. \cite{42} but the effect of EoS containing the critical point was ignored there. It has been shown in this work that the EoS plays a vital role in determining the behaviour of \(S_{nn}(k, \omega)\), especially its strength at the R peak changes by several orders of magnitude when the effects of the CEP in EoS is incorporated. The EoS has strong effects on B-peaks too because it determines \(c_s\) and hence the location of the B peaks. It is also very important to understand whether all the hydrodynamic modes travel at the same speed or not. The R-peak and the B-peaks will be closer for slower modes even at points away from the CEP. Therefore, the structure of \(S_{nn}(k, \omega)\) will shed light on the speed of the perturbation propagating as a sound wave.

The paper is organized as follows. The EoS containing the effects of CEP is discussed in the next section. The expression for \(S_{nn}(k, \omega)\) has been derived in Sect. 3 within the scope of the Israel–Stewart relativistic hydrodynamics \cite{43} in Eckart frame of reference. Section 4 is devoted to discuss the critical behaviour of transport coefficients and response functions. Results are presented in Sects. 5 and 6 is dedicated to summary and discussions.

2 Equation of state (EoS)

One of the main objectives of the present work is to investigate the role of the CEP on density fluctuation. In order to understand this role, it is required to construct an EoS which contains the effects of the CEP. The details of the construction of the EoS can be found in Refs. \cite{44–46} (for a pedagogic approach see \cite{47}). Therefore, we do not wish to discuss the details about the construction of the EoS here but only quote the main results to make the article self-contained and refer to \cite{44–47} for details.

The universality hypothesis suggests that the three-dimensional (3D) Ising model and the QCD belongs to the same universality class. Therefore, the results of the 3D Ising model can be mapped onto the QCD phase diagram. It is well-known that magnetization, \(M(r, \mathcal{H})\) of Ising model is
a function of magnetic field \( (\mathcal{H}) \) and reduced temperature \( (r = (T - T_c) / T_c, \ T_c \) is the critical temperature) is analogous to a critical entropy density, \( s_c(T, \mu) \) in QCD. We can express \( M \) at any temperature \( (T) \) as a function of \( r \) and \( \mathcal{H} \) with the location of the CEP at \( (r, \mathcal{H}) = (0, 0) \). Thus, \( r < 0 \) represents the first-order phase transition and \( r > 0 \) indicates the crossover transition regions respectively. The linear mapping between the CEP of the Ising model and the CEP of the QCD phase diagram is performed by assuming the following relation \[44\],

\[
\mathcal{H} = \frac{T - T_c}{\Delta T_c}, \quad r = \frac{\mu - \mu_c}{\Delta \mu_c}
\]

where \((\mu_c, T_c)\) represents the location of the CEP in the QCD phase diagram with \( \mu_c \) and \( T_c \) as the critical values of the baryonic chemical potential and temperature, respectively. The critical region can be introduced by the values of \( \Delta T_c \) and \( \Delta \mu_c \) chosen as the extensions along \( T \) and \( \mu \) axis, respectively. The QCD critical entropy density can be expressed by using the mapping as:

\[
s_c = \frac{M(r, \mathcal{H})}{\Delta T_c} = M \left( \frac{\mu - \mu_c}{\Delta \mu_c}, \frac{T - T_c}{\Delta T_c} \right) \frac{1}{\Delta T_c}
\]

where the form of \( M \) (magnetization) of the 3D Ising model is given by the parametric representation in terms of the two variables \( R \) and \( \theta \) as \[44-46\]:

\[
M = M_0 R^{\beta \theta}, \quad \mathcal{H} = \mathcal{H}_0 R^{\delta \theta} (\theta - 0.76201 \theta^3 + 0.00804 \theta^5), \quad r = R (1 - \theta^2), \quad (R \geq 0, -1.154 \leq \theta \leq +1.154)
\]

where \( \beta' = 0.326 \) and \( \delta = 4.8 \) are the critical exponents and \( M_0, \mathcal{H}_0 \) are normalisation constants which can be evaluated by imposing the following condition,

\[
M(r = -1, \mathcal{H} = 0^+) = 1, \quad M(r = 0, \mathcal{H} = 1) = 1, \quad M(r = 0, \mathcal{H} = -1) = sgn[\mathcal{H}] \eta \eta^{1/\delta}
\]

The parametrization of \( M \) in terms of \( (R, \theta) \) mapped to \( (r, h) \) and then ultimately the entropy density for QCD is calculated by mapping \( (r, h) \) to \( (\mu, T) \) as clearly indicated in \[45\]. In order to construct the EoS, we first construct a dimensionless entropy density as

\[
\tilde{S} = A(\Delta T_c, \Delta \mu_c) s_c(T, \mu)
\]

where \( A \) is defined as

\[
A(\Delta T_c, \Delta \mu_c) = B \sqrt{\Delta T_c^2 + \Delta \mu_c^2}
\]

Here \( B \) is also a dimensionless number which represents the extension of the critical region. In this work, we use \((T_c, \mu_c) = (154 \text{MeV}, 367 \text{MeV})\) with \((\Delta T_c, \Delta \mu_c, B) = (0.1 \text{GeV}, -0.2 \text{GeV}, 2)\). The construction of the entropy density is done by appropriately connecting entropy densities of the QGP \((s_Q)\) and the hadronic \((s_H)\) phases with use of \( \tilde{S} \) as a switching function. The final result reads as:

\[
s(T, \mu) = \frac{1}{2} [1 - \tanh \tilde{S}(T, \mu)] s_Q(T, \mu) \quad + \frac{1}{2} [1 + \tanh \tilde{S}(T, \mu)] s_H(T, \mu)
\]

We calculate \( s_Q \) by using the following expression \[48\]

\[
s_Q(T, \mu) = \frac{32 + 21 N_f}{45} \pi^2 T^3 + \frac{N_f}{9} \mu^2 T
\]

with \( N_f \) being the number of quark flavors. In Eq. (8) massless quarks and gluons are considered. However, the effects of thermal mass can be taken into account through the effective degeneracy of quarks and gluons \[49\] (see Ref. [50]). To estimate the effective degeneracy with thermal quark mass we consider the energy density of quarks,

\[
e_q(T, \mu, m_q) = \frac{g_q^\text{eff}}{(2\pi)^3} \int dp^3 p^3 [\exp(E_i - \mu/m_q) - 1]
\]

The effective degeneracy is given by

\[
g_q^\text{eff} = \frac{\epsilon_q(T, \mu, m_q)}{g_q}
\]

Similarly for gluons the effective degeneracy can be estimated by using the relation

\[
g_g^\text{eff} = \frac{\epsilon_g(T, m_g)}{g_g}
\]

The \( g_q^\text{eff} = 5.76 \) with thermal mass at \((\mu, T) = (367, 158)\) MeV, whereas for massless quark the degeneracy is 6. In case of gluon with thermal mass \( g_g^\text{eff} = 14.2 \) which is 16 for massless gluon. For \((\mu, T) = (367, 158)\) MeV the thermal momenta of quarks and gluons are 1064 MeV and 497 MeV respectively where as the respective thermal masses are 102 MeV and 208 MeV, respectively. This indicates that the effect of thermal mass at the critical point will not be significant.

The hadronic entropy density \((s_H)\) can be estimated from the following expression \[51,52\],

\[
s_H(T, \mu) = \pm \sum_i \frac{g_i}{2 \pi^2} \int_0^\infty dp^2 \int_0^\infty dE_i E_i - \mu_i
\]

where the sum extends over all hadrons with mass up to 2.5 GeV \[52\], \( g_i \) represents the statistical degeneracy factor and \( E_i = \sqrt{p_i^2 + m_i^2} \) is the energy of the \( i \)th hadron of mass \( m_i \) and momentum, \( p_i \). Once we know the entropy density, the thermodynamic quantities e.g. baryon number density,
pressure and energy density can be evaluated as follows. The net baryon number density \( n \) is given by
\[
n(T, \mu) = \int_0^T \frac{\partial s(T', \mu)}{\partial \mu} dT' + n(0, \mu)
\] (13)
and the pressure can be estimated as
\[
p(T, \mu) = \int_0^T s(T', \mu)dT' + p(0, \mu)
\] (14)
where \( n(0, T) \) and \( p(0, \mu) \) are the net baryon density and pressure respectively at \( T = 0 \). Finally, the energy density is given by,
\[
e(T, \mu) = Ts(T, \mu) - p(T, \mu) + \mu n
\] (15)

To get the first order phase boundary, the discontinuity in the entropy density along the transition line also needs to be considered. We add the following term to Eq. (13) to take into account this possibility (for \( T > T_c \))
\[
\left| \frac{\partial T_c(\mu)}{\partial \mu} \right| \left[ s(T_c(\mu) + \Delta, \mu) - s(T_c(\mu) - \Delta, \mu) \right]
\] (16)
where \( \frac{\partial T_c}{\partial \mu} = tan\theta_c \), is the tangent at the \( T_c \) and \( \Delta(\rightarrow 0) \) is a small deviation in \( T \) from \( T_c \). The value of \( T_c \) for the first order transition depends on \( \mu \) as indicated by Eq. (16).

In this context we contrast the entropy density, velocity of sound and the baryon susceptibility \( \chi_B \) obtained in the present work with the corresponding lattice QCD results [53, 54] in the \( \mu \to 0 \) limit.

3 Brief review on the relativistic viscous fluid dynamics: Israel–Stewart (IS) theory in Eckart frame

In Navier–Stokes (NS) hydrodynamics the dissipative flux is assumed to be proportional to the first-order gradient of the hydrodynamic field which leads to acausal and unstable solutions. However, there are some recent developments to overcome the problems of causality and stability in first-order theory too. It has been shown in Refs. [55–58] that the first-order hydrodynamics can be made hyperbolic to obtain the stable and causal solutions in the presence of shear viscosity, bulk viscosity and thermal conductivity.

In this work, we use the IS relativistic hydrodynamic theory [43] in the Eckart frame of reference [59] to investigate the properties of the fluid near the CEP. It was developed earlier by Muller in the context of non-relativistic fluids [60]. The IS method generalizes the standard model of the entropy current for out of equilibrium systems and then enforces the second law of thermodynamics in the simplest possible way. The second-order theories contain several relaxation time scales and if these time scales are too short then the second-order theory may also be acausal. The equivalence and correspondence between stable first-order hydrodynamics and IS hydrodynamics can be found in Refs. [61–63]. The baryonic chemical potential, \( \mu \), is introduced in the system through the conservation of the net baryon number, \( n \), for the description of the fluid near the CEP. In general, there could be many choices for the frame of references (a frame can be attached to each conserved charge of the system). However, the most widely used choices are Landau–Lifshitz (LL) [64] and Eckart [59] frames of references. The LL frame represents a local rest frame where the energy dissipation is zero but the net-number dissipation (diffusion) is non-zero, whereas the Eckart frame represents a local rest frame where the net-charge dissipation is vanishing but the energy dissipation is non-vanishing. We adopt the Eckart frame of reference where the particle current \( (N^\mu) \) can be written as:
\[
N^\mu = nu^\mu
\] (17)
where \( u^\mu \) is the fluid four velocity with \( u^\mu u_\mu = -1 \).

The conserved baryon number density is given by \( n = -u_\mu N^\mu \) and the energy momentum tensor is expressed as,
\[
\Pi^{\mu\nu} = \epsilon u^\mu u^\nu + (p + \Pi)\Delta^{\mu\nu} + g^{\mu\nu} q^\mu q^\nu + \Pi^{\mu\nu}
\] (18)
where, \( g^{\mu\nu} = (-1, 1, 1, 1) \) is the metric tensor, \( \epsilon \) is energy density, \( p \) is the pressure, \( \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \) is the projection operator. It has the properties \( \Delta^{\mu\nu} u_\nu = \Delta^{\mu\nu} \Delta^\nu = 0 \). In the above equation \( \Pi \) and \( \Pi^{\mu\nu} \) represent the trace part (bulk pressure) and the symmetric transverse traceless part (shear pressure) of the symmetric viscosity tensor respectively and \( q^\mu \) is the heat flow vector. They describe the out of equilibrium properties of the fluid and satisfy the constraints,
\[
u_\mu q^\mu = u_\mu \Pi^{\mu\nu} = \pi^{\nu\mu} = \pi^{\mu\nu} = \pi^{\mu\nu} = 0
\] (19)
As the Eckart frame is characterized by no (baryonic) charge flow, the four velocity is expressed as \( u^\mu = N^\mu / \sqrt{1 - N^\alpha N_\alpha} \).

The Muller–Israel–Stewart theory takes into account of general relativity with the method of Grad’s 14-moment approximation and successfully restores the causality condition (as far as the relaxation time scales are not too small) for the relativistic viscous fluids by considering second order gradients of hydrodynamic variables [65–67]. The general forms of \( \Pi, \Pi^{\mu\nu}, q^\mu \) which contain additional coefficients arising due to inclusion of second order gradients are [43,65,68,69]:
\[
\Pi = -\xi [\partial_\mu u^\mu + \beta_0 D\Pi - \bar{a}_0 \partial_\mu q^\mu]
\]
\[
\pi^{\lambda\mu} = -\kappa T \Delta^{\lambda\mu} \left[ \partial_\mu u_\rho + \beta_2 D\pi_{\alpha\beta} - \bar{a}_1 \partial_\mu q_\rho \right]
\]
\[
q^{\lambda} = -\kappa T \Delta^{\lambda\mu} \left[ \frac{1}{T} \partial_\mu T + Du_\mu + \bar{b}_1 Dq_\mu - \bar{a}_0 \partial_\mu \Pi \right]
\] (20)
where, $D \equiv u^\mu \partial_\mu$, is known as co-moving derivative and in the local rest frame (LRF) $D \Pi = \Pi$ represents the time derivative. Here $\eta$, $\zeta$, $\kappa$ are the coefficient of shear viscosity, bulk viscosity and thermal conductivity respectively. The double symmetric traceless projection operator is defined by $A^{\mu \nu \beta} = \frac{1}{2} [A^{\mu \nu} A^{\beta} + A^{\mu \beta} A^{\nu} - \frac{2}{3} A^{\mu \nu} A^{\beta}]$. The relaxation times for the bulk pressure ($\tau_\Pi$), the heat flux ($\tau_\epsilon$) and the shear tensor ($\tau_\eta$) are defined as [70]

$$\tau_\Pi = \zeta \beta_0, \quad \tau_\epsilon = T \beta_1, \quad \tau_\eta = 2 \eta \beta_2$$

(21)

The relaxation lengths which couple heat flux and bulk pressure ($l_{\Pi q}$, $l_{\eta q}$), the heat flux and shear tensor ($l_{\eta \pi}$, $l_{\eta q}$) are defined as follows:

$$l_{\Pi q} = \zeta \alpha_0, \quad l_{\eta q} = k_B T \alpha_0, \quad l_{\eta \pi} = k_B T \alpha_1, \quad l_{\eta q} = 2 \eta \alpha_1$$

(22)

The quantities, $\beta_0$, $\beta_1$, $\beta_2$ are called relaxation coefficients, and $\alpha_0$, $\alpha_1$ are coupling coefficients. The $\alpha_0, \alpha_1, \beta_1$ appearing in Eq. (20) are related to relaxation and coupling coefficients as

$$\alpha_0 - \alpha_0 = \alpha_1 - \alpha_1 = - (\beta_1 - \beta_1) = - \left( (\epsilon + p) \right)^{-1}$$

(23)

The relaxation and coupling coefficients are given by [43]

$$\alpha_0 = (D_{41} D_{20} - D_{31} D_{30}) \Delta \phi \Delta \Omega J_{21} J_{31}$$

$$\alpha_1 = (J_{41} J_{21} - J_{31} J_{32}) \Delta \phi \Delta \Omega J_{21} J_{31}$$

$$\beta_0 = \frac{3 \beta}{\phi^2 \Omega^2} [J_{52} - \frac{3}{D_{20}} (J_{31} J_{30} - J_{41} J_{20})]$$

$$\beta_1 = \frac{D_{41}}{\Lambda^{2 n m} J_{21} J_{31}}$$

(24)

$$\beta_2 = \frac{\beta J_{52}}{2 \phi^2}$$

where, $\beta = m / T$, and $m$ is the mass of the particle

$$D_{ij} = D_{j+1,i} D_{j-1,i} - J_{ij}^2, \quad \phi = \frac{e + p}{\beta}, \quad \psi = \frac{e + p}{n m}, \quad \Lambda = 1 + \frac{5 \phi}{n m} - \psi^2, \quad \Omega = \frac{2 \delta n \delta \phi}{\delta n \delta m} - 5$$

(25)

where $i, j$ are integers. The quantities $\phi, \psi$ and $\Lambda$ are calculated by using their relations with $e, p$ and $n$. $J_{ij}$ is defined as,

$$J_{ij} = \frac{A_0}{(2 j + 1)!} \int_0^\infty N \Delta \sinh^{2(i+1)} R \cosh^{-2j} R dR$$

(26)

and

$$N = \frac{1}{\exp[\beta \cosh R - \alpha] - \epsilon}$$

(27)

is the Synge’s distribution function, $\alpha = \mu / T$ and

$$\Delta = 1 + \epsilon N, \quad A_0 = 4 \pi m^3$$

(28)

We have taken $\epsilon' = 0.01$. In the ultra-relativistic limit, $(\beta \rightarrow 0)$, one gets [43],

$$\alpha_0 = 6 \beta^{-2} p^{-1}, \quad \alpha_1 = - \frac{1}{4} p^{-1}, \quad \beta_0 = 216 \beta^{-4} p^{-1}, \quad \beta_1 = \frac{5}{4} p^{-1}, \quad \beta_2 = \frac{3}{4} p^{-1}$$

(29)

We have evaluated the spectral structure in the ultra-relativistic limit by using Eq. (29) as well with the relaxation and coupling coefficients given by Eq. (24) and found negligible difference in the final results. The thermal masses of quarks ($m_q$) and gluons ($m_g$) [5] are taken as:

$$m_q^2 = \pi \alpha_s \frac{N_c^2 - 1}{4 N_c} (T^2 + \frac{\mu^2}{\pi^2})$$

(30)

and

$$m_g^2 = (C_A + N_f / 2) \frac{4 \pi \alpha_s T^2}{6}$$

(31)

where $C_A = 3, N_c$ is the number of color and $N_f$ is the number of flavors and $\alpha_s$ is the strong coupling. We have taken $\alpha_s = 0.2$.

The relativistic viscous fluid are described by the following two equations corresponding to the conservations of energy-momentum and the net baryon number:

$$\partial_\mu T^{\mu \nu} = 0$$

$$\partial_\mu N^{\mu \nu} = 0$$

(32)

### 3.1 Linearized Hydrodynamic Equations

The hydrodynamic Eq. (32) are non-linear, partial differential equations which are difficult (if not impossible) to solve analytically in general. Presently our goal is to obtain the $S_{nm}(k, \omega)$ of the dynamical density fluctuations in wave vector-frequency space. The hydrodynamical equations mentioned above can be linearized to describe small perturbations in thermodynamical variables (small deviations from the equilibrium values of the variable). These linearized equations can be solved to obtain the density fluctuations. Let $Q_0 (Q)$ denote a thermodynamic quantity in (away from) equilibrium. For small perturbation $\delta Q$ can be written as:

$$Q = Q_0 + \delta Q$$

where $Q$ can be any of the quantities among $n, e, u^\mu, q^\mu, s, \Pi, \pi^{a \mu}$, etc ($v_0$ has been taken as zero here).

The linearized hydrodynamic equations around the equilibrium [39,42,71–73] become:

$$0 = \frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta v$$

$$0 = h_0 \frac{\partial \delta \nu}{\partial t} + \nabla (\delta P + \delta \Pi) + \frac{\partial \delta q}{\partial t} + \nabla \cdot \delta \pi$$

$$0 = \delta \Pi + \zeta [\nabla \cdot \delta v + \beta_0 \frac{\partial \delta \Pi}{\partial t} - \alpha_0 \nabla \cdot \delta q]$$

$$0 = \delta \pi^{ij} + \eta [\partial^j \delta v^i + \partial^i \delta v^j]$$
where, $h_0 = \epsilon_0 + p_0$ is the equilibrium enthalpy density. Now we decompose the fluid four velocity along the directions parallel and perpendicular to the direction of wave vector, $k$ and call them $\delta v_{||}$ and $\delta v_{\perp}$ respectively, i.e. $k \cdot \delta v_{\perp} = 0$.

The hydrodynamic equations can be solved for a given set of initial condition, $n(0), v_i(0), T(0), q(0), \Pi(0)$ and $\pi(0)$, by using the Fourier-Laplace transformation defined by:

$$\delta Q(k, \omega) = \int_{-\infty}^{\infty} d^3 r \int_0^\infty dt e^{-i(k \cdot r - \omega t)} \delta Q(r, t)$$

The $\delta p$ and $\delta s$ can be written in terms of the independent variables $n$ and $T$ as follows by using the thermodynamic relations:

$$\begin{align*}
\delta p &= \left( \frac{\partial p}{\partial n} \right)_T \delta n + \left( \frac{\partial p}{\partial T} \right)_n \delta T \\
\delta s &= \left( \frac{\partial s}{\partial n} \right)_T \delta n + \left( \frac{\partial s}{\partial T} \right)_n \delta T
\end{align*}$$

We use Eqs. (34) and (35) to write down the longitudinal linearized hydrodynamic equation as:

$$\delta Q(k, \omega) = \mathbb{M} \delta Q(k, 0)$$

where,

$$\begin{align*}
\mathbb{M} &= \begin{bmatrix}
\delta n(k, \omega) \\
\delta v_{||}(k, \omega) \\
\delta \Pi(k, \omega) \\
\delta v_{\perp}(k, \omega) \\
\delta \pi(k, \omega) \\
\delta T(k, \omega)
\end{bmatrix} \\
\delta Q(k, 0) &= \begin{bmatrix}
\delta n(k, 0) \\
\delta v_{||}(k, 0) + \frac{1}{n_0} \delta q_{||}(k, 0) \\
i \omega \beta_0 k_0 \delta \Pi(k, 0) - 2 \beta_2 \eta \delta \pi_{||}(k, 0) - 2 \beta_1 \delta v_{\perp}(k, 0) - n_0 \left( \frac{\partial s}{\partial n} \right)_T \delta T(k, 0) + n_0 \left( \frac{\partial s}{\partial T} \right)_n \delta T(k, 0)
\end{bmatrix}
\end{align*}$$

We are concerned here about the density fluctuation which is given by,

$$\delta n(k, \omega) = \left[ \mathbb{M}^{-1}_{11} - n_0 \left( \frac{\partial s}{\partial n} \right)_T \mathbb{M}^{-1}_{16} \right] \delta n(k, 0) + 2 \beta_2 \eta \delta \pi_{||}(k, 0) + 2 \beta_1 \delta v_{\perp}(k, 0) + (1 + 2 i \omega \beta_2 \eta)$$

We define $S'_{nn}(k, \omega)$ by the following expression:

$$S'_{nn}(k, \omega) = \left\{ \delta n(k, \omega) \delta n(k, 0) \right\}$$

The correlation between two independent thermodynamic variables, say, $Q_i$ and $Q_j$ vanishes i.e.

$$\left\{ \delta Q_i(k, \omega) \delta Q_j(k, 0) \right\} = 0, \quad i \neq j$$

The required correlator, $S'_{nn}(k, \omega)$ is obtained as:

$$S'_{nn}(k, \omega) = \left[ \mathbb{M}^{-1}_{11} - n_0 \left( \frac{\partial s}{\partial n} \right)_T \mathbb{M}^{-1}_{16} \right] \delta n(k, 0) \delta n(k, 0)$$

Finally, $S_{nn}(k, \omega)$ is defined as:

$$S_{nn}(k, \omega) = \left\{ \frac{S'_{nn}(k, \omega)}{\delta n(k, 0) \delta n(k, 0)} \right\}$$

The variation of $S_{nn}(k, \omega)$ with $k$ and $\omega$ for given values of transport coefficients and other relevant thermodynamic variables are studied below. The full expression for $S_{nn}(k, \omega)$
has been given in Appendix A. In the small $k$ limit with $\alpha_0 \to 0$, $\alpha_1 \to 0$, $\beta_0 \to 0$, $\beta_1 \to 0$, $\beta_2 \to 0$ the results for Navier–Stokes hydrodynamics can be recovered from the full expression given in Appendix A. We will see below that the variation of $S_{nn}(k, \omega)$ with $\omega$ admits three peaks positioned at $\omega = 0$ and $\omega = \pm \omega_B$. The $\omega_B$ is a function of $k$, velocity of sound and other thermodynamic variables. The peak at $\omega = 0$ is called the R-peak and the doublet symmetrically situated at $\pm \omega_B$ are called B-peaks. The quantities $\left( \frac{\partial \rho}{\partial n} \right)_T$, $\left( \frac{\partial \rho}{\partial T} \right)_n$, $\left( \frac{\partial s}{\partial n} \right)_T$, $\left( \frac{\partial s}{\partial T} \right)_n$ appearing in $S_{nn}(k, \omega)$ can be evaluated in terms of relevant thermodynamic variables (see Appendix). In condensed matter physics the $S_{nn}(k, \omega)$ is measured by utilizing the relation between intensity of light scattering with density–density correlation [40]. However, any such direct measurement of the corresponding critical opalescence in QCD is not possible. The possibility of the measurement of QCD opalescence by measuring jet quenching has been proposed in Ref. [74].

4 Behaviour of $S_{nn}(k, \omega)$ near the CEP

The dynamical structure function, $S_{nn}(k, \omega)$ defined in Eq. (43) depends on the transport coefficients, $\eta$, $\kappa$ and $\gamma$ as well as on the four partial derivatives, $\left( \frac{\partial \rho}{\partial n} \right)_T$, \left( \frac{\partial \rho}{\partial T} \right)_n$, $\left( \frac{\partial s}{\partial n} \right)_T$, $\left( \frac{\partial s}{\partial T} \right)_n$. It also depends on the coupling coefficients ($\alpha_0$, $\alpha_1$) and the relaxation coefficients ($\beta_0$, $\beta_1$, $\beta_2$) which infiltrate into $S_{nn}(k, \omega)$ through the IS hydrodynamics. The partial derivatives appearing in $S_{nn}(k, \omega)$ contain the effects of the CEP through the EoS. The behaviour of the transport coefficients and the response functions near the CEP is characterized by the critical exponents. As the CEP is approached some of these quantities start to diverge. A thermodynamic variable, $f(r)$ near the CEP can be written as [41]:

$$f(r) = Ae^\lambda(1 + Be^\gamma + \cdots)$$ (44)

where, $\gamma > 0$, $r = (T - T_c)/T_c$ is the reduced temperature. The critical exponent $\lambda$ can be defined as:

$$\lambda = \lim_{r \to 0} \frac{\ln f(r)}{\ln(r)}$$ (45)

$\lambda$ can be either +ve or -ve correspondingly $f(r)$ will vanish or diverge at the CEP.

Now the partial derivatives appearing in the expression for $S_{nn}(k, \omega)$ can be expressed as (see Appendix):

$$\left( \frac{\partial \rho}{\partial n} \right)_T = \frac{1}{n_0\kappa_T}, \quad \left( \frac{\partial \rho}{\partial T} \right)_n = \mu n_0 c_s^2 \frac{C_V}{C_p}$$

$$\left( \frac{\partial s}{\partial n} \right)_T = \frac{h_0 c_s^2 \alpha_p}{n_0 \gamma}, \quad \left( \frac{\partial s}{\partial T} \right)_n = \frac{C_V}{T}$$ (46)

where, $\kappa_T = \frac{1}{n_0} \left( \frac{\partial n}{\partial P} \right)_T$, is the isothermal compressibility, $\alpha_p = -\frac{1}{n_0} \left( \frac{\partial n}{\partial P} \right)_p$ is the volume expansivity coefficient, $C_p$ and $C_V$ are the specific heats at constant pressure and volume respectively with their ratio, $C_p / C_V = \gamma$. The critical behaviour of various transport coefficients and response functions plays crucial roles to determine the strength of the signal in the presence of the CEP. We take the following $r$ dependence near the CEP for the present purpose [46, 75, 76].

$$\kappa_T = \kappa_T^0 |r|^{-\gamma'}, \quad C_V = C_0 |r|^{-\alpha}, \quad C_p = \frac{\kappa_0 T_0}{n_0} \left( \frac{\partial P}{\partial T} \right)_n |r|^{-\gamma'}$$

$$C_s^2 = \frac{T_0}{n_0 h_0 C_0} \left( \frac{\partial P}{\partial T} \right)_n |r|^{\alpha'}, \quad \alpha_p = \frac{\partial P}{\partial T} \n_0 |r|^{-\gamma'}$$

$$\eta = \eta_0 |r|^{1+4\alpha/2-\gamma'}, \quad \zeta = \zeta_0 |r|^{-\alpha'}, \quad \kappa = \kappa_0 |r|^{-\alpha}$$ (47)

where, $\alpha$, $\gamma'$, $\alpha'$, $\alpha_c$ are the critical exponents. Here $\alpha = 0.11$ and $\gamma' = 1.2$. For liquid-gas critical point, $\alpha_c = 4d - \alpha = 1.78$ (here, $d = 3$, $v = 0.63$), and $\alpha_c = 0.63$. For consistency the partial derivative, $\left( \frac{\partial n}{\partial T} \right)_T$ has been evaluated using the the EoS. The critical behaviours of the second order coupling and relaxation coefficients enter into the calculation through the thermodynamic variables which contain the effects of the CEP (Sect. 2) as evident from Eqs. (24), (25) and (29).

5 Results and discussion

Now we discuss the effects of the CEP on the EoS and subsequently on the $S_{nn}(k, \omega)$. We have evaluated the $S_{nn}(k, \omega)$ by including the effects of the CEP through: (i) the EoS and (ii) the critical behaviour of various transport coefficients as discussed above. As mentioned in Sect. 2 we assume that the CEP is located at $(\mu, T) = (367, 154)$ MeV in the QCD phase diagram. This means that the entropy density (first order derivative of free energy) changes continuously for $\mu < 367$ MeV and $T > 154$ MeV, reflecting the change to crossover from first-order transition (characterized by a discontinuity in entropy density) for $\mu > 367$ MeV and $T < 154$ MeV [39]. The variation of $s/T^2$ with $T$ as obtained from Eq. (7) in the limit of $\mu \to 0$ is depicted in Fig. 1 and contrasted with the result from lattice QCD [53] at vanishing $\mu$. The baryon density susceptibility ($\chi_2^B$) and the velocity of sound have been plotted in Fig. 2a, b respectively to substantiate the performance of the EoS used in the present work.

Figure 3a shows the comparison of $S_{nn}(k, \omega)$ for first-order (NS) and second-order (IS) hydrodynamics. The results show a visible difference in the $S_{nn}(k, \omega)$ estimated by using NS and IS hydrodynamics. Figure 3b displays the variation of $S_{nn}(k, \omega)$ with $\omega$ when the system is away from CEP represented by $r = 0.2$. To investigate the effects of EoS we keep the transport coefficients finite with values, $\eta/s, \zeta/s, \kappa T/s = 1/4\pi$. The relevant coupling and relax-
The variation of $s/T^3$ with $T$ is plotted and compared our result with Lattice result [53]. We observe that for (i) there are three distinct peaks (red line). The central one is bigger and called the R-peak which originates from the entropy fluctuation at constant pressure i.e. due to the thermal fluctuation. The symmetric doublet about the R-peak are the B-peaks. The B-peaks originate from pressure fluctuation at constant entropy which is connected to sound waves. We find that the B-peaks are weaker than the R-peaks. (ii) We exclude the effects of EoS to investigate the effects of transport coefficients only near the CEP via their critical exponents in evaluating $S_{nn}(k, \omega)$. The dotted green curve in Fig. 3b depicts the $S_{nn}(k, \omega)$ obtained by using the scaling laws i.e when the parametric forms of the transport coefficients and response functions mentioned in Eq. (47) are used. The B-peaks in scenario (ii) are sub-dominant, almost like a broad shoulder.

![Graph 1](image1)

**Fig. 1** The variation of $s/T^3$ with $T$ is plotted and compared our result with Lattice result [53]

![Graph 2](image2)

**Fig. 2** a The $\chi_B^2 / T^2$ obtained from the parametrization of the present work has been displayed as a function of $T/T_c$. The result is compared with lattice QCD results [54]. b The variation of $c_s^2$ with $T/T_c$ from lattice QCD result [53] and present work have been depicted

![Graph 3](image3)

**Fig. 3** a Variation of $S_{nn}(k, \omega)$ with $\omega$ for ($r = 0.2$) and $k = 0.1$ fm$^{-1}$ are plotted when the system is away from CEP for first-order (NS) and second-order (IS) hydrodynamics. b Variation of $S_{nn}(k, \omega)$ with $\omega$ for ($r = 0.2$) and $k = 0.1$ fm$^{-1}$ in second order hydrodynamics. The results obtained with $\eta/s = \xi/s = \kappa T/s = 1/4\pi$ is shown by red line. The green line is obtained when parametric form of transport coefficients and response functions are used
Fig. 4 $S_{nn}(k, \omega)$ near the critical point for ($r = 0.01$) and $k = 0.1 \text{ fm}^{-1}$. The red curve represents the effects of EoS and $\eta/s = \zeta/s = \kappa T/s = 1/4\pi$. The green line is obtained by using the scaling hypothesis of thermodynamic variables near CEP and both effects are shown in blue curve. Inset plot is for the broader range in $\omega$ ($0.2 \leq \omega \leq 0.2$).

Closer to the CEP, however, the scenario changes drastically. The $S_{nn}(k, \omega)$ near the CEP ($r = 0.01$) shows only the R-peak with height increased by more than an order of magnitude and the B-peaks vanish due to the absorption of sound waves near the CEP (Fig. 4). The vanishing of B-peaks can be understood from the fact that in the leading order of Brillouin frequencies, $\pm \omega_B \sim c_s k$ where $c_s$ is the velocity of sound. The effects of EoS are seen to reduce the height of the R-peak (red line). The velocity of sound is determined by EoS, therefore, the position as well as the height of the B-peaks will depend on the EoS.

The $S_{nn}(k, \omega)$ as a function of $\omega$ for smaller $k$ ($k = 0.02 \text{ fm}^{-1}$) is plotted in Fig. 4 when only the effects of the EoS is considered. The red (blue) line corresponds to results close to the CEP with $r = 0.01$ (away from CEP with $r = 0.2$, B-peaks are not visible because of the scale chosen along $\omega$ axis). We observe that at smaller $k$ the value of R-peak gets bigger as well as sharper. A comparison with results shown in Fig. 5 reveals that near the CEP, the B-peaks vanish and the R-peak gets sharper and bigger.

Motivated by the above results shown in Figs. 4 and 5 i.e. by observing the sensitivity of the results on the $k$ values we investigate the behaviour of spectral function for different hydrodynamic modes (k-modes). We consider the variation of $S_{nn}(k, \omega)$ with $k$ and $\omega$ near the CEP in Fig. 6. We observe that for all values of $k$, the B-peaks merge with the R-peak.

The height of the peak is maximum in the neighbourhood of $k \rightarrow 0$. In Fig. 7, the $S_{nn}(k, \omega)$ is plotted against $\omega$ and $k$ when the system is away from the CEP. We observe that the B-peaks shift away from R-peak with increase in $k$. This is better reflected in Fig. 8, where the Brillouin frequency is plotted against $k$. A linear variation of $\omega$ with $k$ is obtained for given values of $T$ and $\mu$. The slope ($c_s$) of the line is found to be $c_s = 0.532$ (i.e $c_s^2 = 0.283$) close to the speed of sound $c_s^2 = 0.25$ obtained from Eq. (64) at the same value of temperature and chemical potential.

Therefore, we find that the B-peaks move toward the R-peak as the system approach the CEP, and ultimately merge with R-peak at the CEP. This suggests that the speed of propagation of all the hydrodynamic modes vanishes at the CEP. This is consistent with the finding that the speed of sound reaches the minimum at the QCD critical point. The positions of the B-peak depend on $k$ when the system is away from the
CEP which indicates that the different k-modes travel with different speeds in the medium.

Partons produced in RHIC-E with relatively low \( p_T \), on subsequent scattering, produce a locally thermalized hot and dense medium of QGP, whereas, the high \( p_T \) partons do not contribute in the medium formation, however, they pass through the medium as jets with associated radiated partons. The supersonic partons can produce perturbations in the medium. The response of the medium to such perturbations, are reflected on the spectra of the produced hadrons at the freeze-out hyper-surface. Specifically, the appearance of two maxima at \( \Delta \phi = \pi \pm 1.2 \) radian in the quenched away side jet or the double-hump in the correlation function of the jet, the structure is explained as the effect of the Mach cone produced due to hydrodynamic response to the perturbation created by jets [77].

The Mach cone appears as double hump in the two particle correlation in the low momentum domain of associated particles. The CEP can suppress the double hump in the two particle correlation contrary to the other mechanisms (e.g. (i) deflection of away side jets due to strong asymmetric flow, (ii) Cherenkov radiation, (iii) radiation of gluons at large angle) which can create the double hump. These mechanisms have the ability to obscure the suppression due to the CEP by producing double humps which makes the detection of the CEP hard. Keeping these issues in mind, however, we note the following points. (i) The double hump of the away side jets may originate due to the deflection by the strong asymmetric flow in non-central collisions and third flow harmonics \( (v_3) \) due to initial state fluctuations [78,79]. However, if the system passes through the CEP, the flow will be highly suppressed and hence, the deflection too will be strongly reduced and hence the ability to create double hump by deflection will also weaken. (ii) The Cherenkov radiation of gluons by the away side jet propagating through the medium [80,81] may also give rise to double hump. However, Cherenkov radiation is unlikely to be responsible for the double hump because of the lack of observed momentum dependence of the location of the double peaks of associated particles. (iii) Gluon radiation at large angle by the away side jet propagating through the medium and consequently its change of direction can be responsible for double hump creation. The quantitative prediction of the Mach cone positions studied through three particle correlation [82–84], and the momentum independence of the location of the double hump indicate that the observed double humps originate from Mach cone effects. The vanishing of the Mach cone will, therefore, strongly indicate the existence of the CEP.

In non-relativistic limit, it has been shown that the width of the R-peak is determined by \( \kappa \), the widths of the B-peaks can be expressed in terms of \( \kappa, \eta, \zeta \) and the ratio \( C_P/C_V \) [40]. The integrated intensity under the R and B peaks are determined by the ratio \( C_V/C_P \) which is connected with the ratio of isothermal to adiabatic compressibilities. Therefore, the spectral function contains enormous information about the thermodynamic state of the system. However, the spectral function of QGP calculated as a function of \( \omega \) and \( k \) which are not directly accessible through experiment i.e. the lack of external probes (as the case of light scattering in condensed matter physics) makes the construction of \( S_{nn}(k, \omega) \) difficult experimentally. In such a situation, one has to depend on the indirect construction \( S_{nn}(k, \omega) \). Since the width of the R and B peaks and the area under these peaks are related to the transport coefficients and thermodynamic response functions, therefore, the determination of these coefficients by some other experimental measurements will help in constructing the spectral functions.

Some kind of mapping between \( k \) and some experimentally measurable variables is required to gain insight on \( S_{nn}(k, \omega) \). The spectral function can be measured by using the bin by bin density fluctuations. Then \( k \)'s can be connected to the inverse angular separation of the bins as \( \delta \phi \sim \frac{1}{kR} \), where \( R \) is the radius of the freeze-out surface (analogous to the analysis of temperature fluctuation of cosmic microwave background radiation [85]). Therefore, the study of the bin by bin density correlation in \( \delta \phi \sim (kR)^{-1} \) in transverse plane for different beam energies will carry the effects of CEP. However, for \( k \sim 0.1 \text{fm}^{-1} \), \( \delta \phi \) is large for typical freeze-out radius in RHIC-E, (\sim 5 \text{ fm}). This implies that the correlation can be observed for large angular separation. That is the pattern corresponding to these \( k \) values will show up over the angular size \( \delta \phi \).

6 Summary and discussions

The correlation of density fluctuation near the QCD critical point has been studied by using linearized perturbative equations obtained from IS hydrodynamics. The spectral structure, \( S_{nn}(k, \omega) \) of the density fluctuation has been derived
rigorously by keeping all the relevant transport coefficients and response functions non-zero. The effects of the EoS and the critical behaviour of various transport coefficients on $S_{\eta \eta}(k, \omega)$ have been investigated. While interpreting the suppression of Mach cone structure as a signature of the CEP, the mode dependence of propagation speed should be taken into consideration. These findings suggest that beam energy dependence of the correlation of number density fluctuation of produced particles in RHIC-E has the potential to carry the signature of the critical point.

In condensed matter physics, the effects of the critical point have been investigated by measuring the intensity of the light scattered from the system. In contrast to this, no such external probe is available for detecting the CEP in QCD. However, several possibilities for the detection of the CEP in QCD have been discussed in the literature. In a condensed matter system, the phenomenon of critical opalescence at the CEP is considered as a signal of large density fluctuations [40]. The possibility of detecting the phenomenon of QCD opalescence by measuring the suppression of hadronic spectra in RHIC-E ($R_{AA}$) has been indicated in Ref. [74]. The $R_{AA}$ can be used to estimate the opacity factor, $\mathcal{K}$ as:

$$\mathcal{K} = -\frac{\ln(R_{AA})}{R_{HBT}} \quad (48)$$

where $R_{HBT}$ is the Hanbury–Brown–Twiss (HBT) radius of the system.

The Fourier coefficients of the azimuthal distributions of particles can be used to understand various properties of the matter produced in RHIC-E. The coefficient of $\cos 3\phi$ (triangular flow) sheds light on the initial fluctuations, similarly, the coefficients of $\cos 2\phi$ (elliptic flow) can be used to discern the EoS of the system. Near the CEP, the order of the harmonics vary as $\sim 1/\lambda_c$ where $\lambda_c$ is the the wavelength of the pressure perturbation (sound) which diverges at the CEP and hence all the harmonics will vanish [39]. However, the experimentally measured spectra contain contribution from all the space-time points i.e. from all possible values of $T$ and $\mu$ not only from the CEP. Therefore, even if the system passes through the CEP the Fourier coefficients may not vanish, but the CEP may suppress them.

In Ref. [75] the mode-mode coupling theory has been used to study the existence and detection of the CEP. It is comprehensively shown that the thermal conductivity at the CEP diverges which induces a sharp change in the two particle correlation of fluctuations in rapidity space. Detection of such modifications in the correlation function may confirm the existence of CEP. The suppression of fluctuations in temperature ($\Delta T$) and baryonic chemical potential ($\Delta \mu$) due to the divergence of thermodynamic response functions at the CEP can signal the presence of the CEP [31,32]. The suppression in $\Delta T$ and $\Delta \mu$ will be reflected through the transverse momentum ($p_T$) spectra of hadrons and proton to pion ratio respectively.

In reality, the possibility of the trajectories passing through the critical point may be infrequent. This limits the fluctuations near the CEP. These fluctuations will remain out of equilibrium due to critical slowing down [86]. The creation of defects due to the CEP in QCD system like cosmology [87] and condensed matter [88] systems and their detection will be extremely exciting. The appearance of Kibble-Zurek length scale and its connection with spatial correlations have been studied in Ref. [89]. The measurement of enhancement of non-flow correlations in presence of the CEP as a function of $\sqrt{\xi}$ can be used to detect the CEP [89].

Rigorously speaking fluid dynamics works in the region where the condition, $k << q$ is satisfied where $q$ is the inverse of correlation length which can be expressed as $q = q_o r^\nu$, $q_o$ is a constant and $\nu$ is the critical index with a numerical value $\nu = 0.73 \pm 0.02$ [76]. This fundamental assumption becomes inoperative since $\xi$ diverges ($q \to 0$) at the CEP. However, there will be a region in the neighbourhood of the CEP where the predictions of fluid dynamics may be useful. Following a procedure similar to condensed matter system [40] we can write, $k << q_o r^\nu$ for the validity of the hydrodynamics, which implies,

$$T > T_c \left[ 1 + \left( \frac{k}{q_o} \right)^{1/\nu} \right] \quad (49)$$

Since hydrodynamics is an effective theory for soft physics (large wavelength or small wave vector, $k$), fluid dynamics can be applied in the neighbourhood of the CEP but becomes invalid at CEP because of the divergence of the correlation length. The response of the trajectories in the neighbourhood of the CEP in the $T-\mu$ plane to the initial conditions away from equilibrium has been investigated in Ref. [90].

The physics of hadronic matter under extreme conditions of temperatures or densities and the QCD phase transition has been considered as the condensed matter physics of elementary particles [91] where the relevant microscopic interaction is controlled by non-abelian gauge theory in contrast to the condensed matter physics governed by abelian gauge theory. In condensed matter physics, the theoretical results on dynamical spectral structure can be compared with the experimentally measured intensity of scattered light. It has been shown that experimentally measured widths of R and B-peaks have been connected to the Landau-Placzek ratio, $C_F/C_V - 1 = \gamma - 1$. Identification of appropriate probes analogous to light in the abelian system will go a long way to reveal the physics of QCD matter near the CEP. It has been shown [92] that the dynamical universality class
of the QCD critical point belong to model H [93] with a linear combination of the chiral condensate and the baryon density as hydrodynamic mode. The evaluation of baryon fluctuation by using the correlation length obtained in [92] will be an interesting problem to address, however, this is beyond the scope of the present work.

The electromagnetic (EM) probes of QGP (see [49] for review) i.e. real photons and lepton pairs can be used to study the evolution of the system from the pristine partonic stage to the final hadronic stage through an intermediary phase transition or cross-over. The photons and lepton can bring the information of the thermodynamic state of their production points [49,94] efficiently as their mean free paths are larger than the size of the fireball created in RHIC-E. This fact should be contrasted with hadrons which are subjected to rescattering in the medium and consequently loose information of their production point. Therefore, in principle, the photons can bring the information of the CEP very efficiently. In Ref. [95] the photon spectra has been evaluated by using second-order dissipative hydrodynamics and the most updated rate of photon productions from QGP and hadronic phases. However, the effects of CEP on EoS, various transport coefficients and response functions are required to be included to get the imprints of CEP on the photon spectra which is beyond the scope of the present work.

The detection of the CEP of the extremely transient QGP state is a huge challenge. Theoretical evaluation of various transport coefficients (η, ζ and κ) and thermodynamic response functions (C\textsubscript{V}, κ\textsubscript{T}, etc.) from the first principle appears to be a distant possibility at the moment. The measurement of these coefficients unambiguously experimentally looks remote too. But these coefficients are required to construct the \( S_{\eta \kappa}(k, \omega) \). Therefore, all the possibilities to calculate or measure these coefficients must be explored. In view of this fact, several possibilities for the detection of the CEP have been discussed. The Mach cone is one of the most important signal for the detection of the CEP. The other useful signal is the electromagnetic probes (photons and dileptons). Work in this direction is under progress [96].

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Appendix A

In this appendix, the expression for the dynamical spectral structure, \( S_{\eta \kappa}(k, \omega) \) derived by considering contributions up to second order in transport coefficients (i.e. \( \eta^2, \zeta^2, \kappa^2, \eta \zeta, \eta \kappa, \kappa \zeta \)) has been provided. The coupling and relaxation coefficients (\( \tilde{\alpha}_0, \tilde{\alpha}_1, \rho_0, \rho_1, \beta_2 \)) have been taken non-zero in obtaining the results displayed in the text, but have been taken as zero in the following to avoid a more lengthy and complex expressions.

\[
\begin{align*}
S'_{\eta \kappa}(k, \omega) &= k^2 n_0 \left[ \int \frac{d^3 p}{(2\pi)^3} \right] \frac{1}{\omega^2 n_0 T^2 \left( \frac{\partial s}{\partial T} \right)_n} \left[ h_0 k T \omega + \left( \frac{\partial \rho}{\partial T} \right)_n k^2 T \omega \left( \frac{\partial \rho}{\partial T} \right)_n \right] \\
&\times \left( \frac{\partial s}{\partial T} \right)_n + k^2 T \kappa (\zeta + \frac{4}{3} \eta) \right] \\
&\times \left( \frac{\partial s}{\partial n} \right)_n \left( \frac{\partial \rho}{\partial T} \right)_n k (\zeta + \frac{4}{3} \eta) + T^3 \kappa \omega^2 \right]
\end{align*}
\]
where
\[ S_{nn}(k, \omega) = \frac{S'_{nn}(k, \omega)}{\delta n(k, 0) \delta n(k, 0)} \]  
(51)

The expression for \( S_{nn}(k, \omega) \) contain derivatives of several thermodynamics quantities. In this appendix we recast these derivatives in terms of response functions like: isothermal and adiabatic compressibilities (\( \kappa_T \) and \( \kappa_s \)), specific heats (\( C_P \) and \( C_V \)), baryon number susceptibility (\( \chi_B \)) and velocity of sound (\( c_s \)), etc. The baryon number density \( n \) and the entropy density \( s \) can be written as:
\[ n = \left( \frac{\partial p}{\partial \mu} \right)_T; \quad s = \left( \frac{\partial p}{\partial T} \right)_\mu \]  
(52)

Baryon number susceptibility, isothermal compressibility and adiabatic compressibility are given by,
\[ \chi_B = \left( \frac{\partial n}{\partial \mu} \right)_T; \quad \kappa_T = \frac{1}{n_0} \left( \frac{\partial n}{\partial p} \right)_T; \quad \kappa_s = \frac{1}{n_0} \left( \frac{\partial n}{\partial \rho} \right)_s \]  
(53)

Specific heats can be expressed as:
\[ C_P = T \left( \frac{\partial s}{\partial T} \right)_p; \quad C_V = T \left( \frac{\partial s}{\partial T} \right)_V = T \left( \frac{\partial s}{\partial T} \right)_n \]  
(54)

Now we write down the expression for partial derivatives, \( \left( \frac{\partial p}{\partial T} \right)_n \), \( \left( \frac{\partial n}{\partial T} \right)_T \), \( \left( \frac{\partial s}{\partial T} \right)_T \) and \( \left( \frac{\partial s}{\partial n} \right)_T \) below. \( \left( \frac{\partial p}{\partial T} \right)_n \) can be evaluated as:
\[ \left( \frac{\partial p}{\partial T} \right)_n = \frac{\partial (p, n)}{\partial (T, n)} = \frac{\partial (p, n)}{\partial (T, p)} \frac{\partial (T, p)}{\partial (s, p)} \frac{\partial (s, p)}{\partial (s, \epsilon)} \frac{\partial (s, \epsilon)}{\partial (s, n)} \frac{\partial (s, n)}{\partial (T, n)} \]  
(55)

Using the relation,
\[ d\epsilon = T \, ds + \mu \, dn \quad \text{and} \quad \mu = \left( \frac{\partial \epsilon}{\partial n} \right)_s \]  
(56)

we write:
\[ \left( \frac{\partial p}{\partial T} \right)_n = \mu n C_V \left( \frac{C_V}{C_P} \right)_n \]  
(57)

Next we consider \( \left( \frac{\partial p}{\partial n} \right)_T \):
\[ \left( \frac{\partial p}{\partial n} \right)_T = \frac{\partial (p, T)}{\partial (n, T)} = \left( \frac{\partial (p, T)}{\partial (n, s)} \right) \frac{\partial (n, s)}{\partial (n, T)} \]  
(58)

\[ \left( \frac{\partial p}{\partial n} \right)_T = \frac{\partial (p, T)}{\partial (n, T)} = \frac{\partial (p, T)}{\partial (n, s)} \frac{\partial (n, s)}{\partial (n, T)} \]  
(59)

\[ = \frac{\partial \epsilon}{\partial n}\left( \frac{\partial p}{\partial s}_{s/n} \right) \frac{\partial s}{\partial \epsilon}_s \frac{\partial s}{\partial T}_n \]  
(60)

The factor, \( \left( \frac{\partial s}{\partial T} \right)_T \), can be written as:
\[ \left( \frac{\partial s}{\partial T} \right)_n = \frac{1}{T} \left( \frac{\partial s}{\partial T} \right)_T = \frac{C_V}{T} \]  
(61)

For fixed net baryon number, \( c_n \) can be written as \( c_n = C_V \). Therefore,
\[ \left( \frac{\partial s}{\partial T} \right)_n = \frac{C_V}{T} \]  
(62)

We evaluate the derivative \( \left( \frac{\partial s}{\partial n} \right)_T \) as:
\[ \left( \frac{\partial s}{\partial n} \right)_T = -\left( \frac{\partial s}{\partial n} \right)_n \frac{\partial T}{\partial n} = -\frac{1}{T} \left( \frac{\partial s}{\partial T} \right)_n \frac{\partial T}{\partial n} \]  
(63)

The velocity of sound is given by:
\[ c_s^2 = \left( \frac{\partial p}{\partial s} \right)_s/n = \frac{n d\mu + s dT}{\mu d\mu + n dT} = \frac{n F dT + s dT}{\mu d\mu + n dT} = \frac{n F dT + s dT}{\mu d\mu + n F dT + \mu (\frac{dn}{dT})_T dT + T (\frac{ds}{dT})_T dT + T (\frac{ds}{dT})_T dT + T (\frac{ds}{dT})_T dT + T (\frac{ds}{dT})_T dT} \]  
(64)

where,
\[ F = \left( \frac{\partial s}{\partial T} \right)_n = \frac{\partial n}{\partial T} \]  
(65)

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