TAUT FOLIATIONS
AND THE ACTIONS OF FUNDAMENTAL GROUPS
ON LEAF SPACES AND UNIVERSAL CIRCLES

YOSUKE KANO

Abstract. Let $\mathcal{F}$ be a leafwise hyperbolic taut foliation of a closed 3-manifold $M$ and let $L$ be the leaf space of the pullback of $\mathcal{F}$ to the universal cover of $M$. We show that if $\mathcal{F}$ has branching, then the natural action of $\pi_1(M)$ on $L$ is faithful. We also show that if $\mathcal{F}$ has a finite branch locus $B$ whose stabilizer $\text{Stab}(B)$ acts on $B$ nontrivially, then $\text{Stab}(B)$ is an infinite cyclic group generated by an indivisible element of $\pi_1(M)$.

1. Introduction

Unless otherwise specified, we assume throughout this article that $M$ is a closed oriented 3-manifold and $\mathcal{F}$ a codimension one transversely oriented, leafwise hyperbolic, taut foliation of $M$. Here we say that $\mathcal{F}$ is leafwise hyperbolic if there is a Riemannian metric on $M$ such that the induced metrics on leaves have constant curvature $-1$, and that $\mathcal{F}$ is taut if there is a loop in $M$ which intersects every leaf of $\mathcal{F}$ transversely. Note that by Candel [4], if $M$ is irreducible and atoroidal, then every taut foliation of $M$ is leafwise hyperbolic. Leafwise hyperbolic taut foliations have been extensively investigated by many people in connection with the theory of 3-manifolds (See e.g. Calegari’s book [2]). One of the most powerful methods of analyzing the structure of such foliations is to consider canonical actions of $\pi_1(M)$ on 1-manifolds naturally associated with $\mathcal{F}$. Two kinds of such 1-manifolds are known. The first one, denoted $L$, is the leaf space of $\tilde{\mathcal{F}}$, where $\tilde{\mathcal{F}}$ is the pullback of $\mathcal{F}$ to the universal cover $\tilde{M}$ of $M$. The action of $\pi_1(M)$ on $\tilde{M}$ induces an action of $\pi_1(M)$ on $L$. In the sequel we refer to it as the natural action. The second one is a universal circle. Given $\mathcal{F}$, by unifying circles at infinity of all the leaves of $\tilde{\mathcal{F}}$, Thurston [10] (See also Calegari-Dunfield [3]) constructs a universal circle with a canonical $\pi_1(M)$ action.

We say that $\mathcal{F}$ has branching if $L$ is non-Hausdorff. The first result of this article is the following:

**Theorem 3.2.** If $\mathcal{F}$ has branching, then the natural action on $L$ is faithful.

---

This work is partially supported by AGSST support program for young researchers, 2011, Chiba University, Japan.
This result is obtained from an investigation of both actions of $\pi_1(M)$ on the leaf space and on the universal circle (see §3). Notice that the hypothesis that $\mathcal{F}$ has branching is indispensable. In fact, just consider a surface bundle over $S^1$ foliated by fibers. Notice also that Calegari and Dunfield [3, Theorem 7.10] have already shown that for any taut foliation one can modify it by suitable Denjoy-like insertions so that the natural action associated with the resulting foliation becomes faithful. In the case where the foliation is leafwise hyperbolic and has branching, our result is stronger than theirs in that we assure faithfulness without performing any modifications.

Next we consider the stabilizer of a branch locus of $\mathcal{F}$. We call a subset $B$ of $L$ a positive (resp. negative) branch locus if $B$ contains at least two points and if it can be expressed in the form $B = \lim_{t \to 0} \nu_t$ for some interval $\{\nu_t \in L \mid 0 < t < \epsilon\}$ embedded in $L$ in such a way that the parameter $t$ is incompatible (resp. compatible) with the orientation of $L$ (Note that $L$ has a natural orientation induced from the transverse orientation of $\tilde{\mathcal{F}}$). Branch loci have been studied e.g. in [5], [9]. For a branch locus $B$ we define the stabilizer of $B$ by $\text{Stab}(B) = \{\alpha \in \pi_1(M) \mid \alpha(B) = B\}.$

In the case where a branch locus $B$ is finite, we obtain the following results about the action of $\text{Stab}(B)$ on $B$ (see §5 for details).

**Theorem 5.2.** Let $B$ be a finite branch locus of $L$. If an element of $\text{Stab}(B)$ fixes some point of $B$ then it fixes all the points of $B$.

We remark that for Anosov foliations Fenley [5, Theorem D] proves related results to this theorem.

Let $\pi : \tilde{M} \to M$ be the covering projection. For a leaf $\lambda$ of $\tilde{\mathcal{F}}$, we denote by $\lambda$ the projected leaf $\pi(\lambda)$ of $\mathcal{F}$.

**Theorem 5.3.** Let $B$ be a finite branch locus of $L$. Then, for any $\lambda \in B$,

1. if $\text{Stab}(B)$ is trivial, $\lambda$ is diffeomorphic to a plane, and
2. if $\text{Stab}(B)$ is nontrivial, $\lambda$ is diffeomorphic to a cylinder.

**Theorem 5.6.** Let $B$ be a finite branch locus of $L$ with nontrivial stabilizer. Then the stabilizer $\text{Stab}(B)$ is isomorphic to $\mathbb{Z}$.

We say that $\alpha \in \pi_1(M)$ is divisible if there is some $\beta \in \pi_1(M)$ and an integer $k \geq 2$ such that $\alpha = \beta^k$. Otherwise we say $\alpha$ is indivisible.

**Theorem 5.7.** Let $B$ be a finite branch locus of $L$ such that $\text{Stab}(B)$ acts on $B$ nontrivially. Then a generator of $\text{Stab}(B)$ ($\cong \mathbb{Z}$) is indivisible.

For an oriented loop $\gamma$ in $M$, we say that $\gamma$ is tangential if $\gamma$ is freely homotopic to a leaf loop (a loop contained in a single leaf) of $\mathcal{F}$, and that $\gamma$ is positively (resp. negatively) transversal if $\gamma$ is freely
homotopic to a loop positively (resp. negatively) transverse to $\mathcal{F}$. As a final topic of this article, we study relations between the infiniteness of branch loci and the existence of a non-transversable leaf loop in $M$ (see §6). One of the results we obtain is the following:

**Theorem 6.5.** Suppose $\mathcal{F}$ has branching. If there is a non-contractible leaf loop in $M$ which is not freely homotopic to a loop transverse to $\mathcal{F}$, then $\mathcal{F}$ has an infinite branch locus.

This article is organized as follows. In Section 2, we briefly review the Calegari-Dunfield construction of a universal circle. Using their construction, we prove the faithfulness of the natural action of $\pi_1(M)$ on $L$ in Section 3. In Section 4, we introduce a notion of comparable sets and give several basic properties of such sets, which are applied in Section 5 to the investigation of the structure of finite branch loci and their stabilizers. In Section 6, we study how the non-transversability of leaf loops in $M$ is related to the infiniteness of branch loci in $L$.

2. Universal circle

In this section we briefly recall the definition of a universal circle after [3].

Let $\tilde{M}$, $\mathcal{F}$ and $L$ be as in the introduction. For $\lambda, \mu \in L$ we write $\lambda < \mu$ if there is an oriented path in $\tilde{M}$ from $\lambda$ to $\mu$ which is positively transverse to $\mathcal{F}$. We say that $\lambda$ and $\mu$ are comparable if either $\lambda \leq \mu$ or $\lambda \geq \mu$. For a leaf $\lambda$ of $\mathcal{F}$, the endpoint map $e : T_p\lambda - \{0\} \to S^1_\infty(\lambda)$ from the tangent space of $\lambda$ at $p$ to the ideal boundary of $\lambda$ takes a vector $v$ to the endpoint at infinity of the geodesic ray $\gamma$ with $\gamma(0) = p$ and $\gamma'(0) = v$. The circle bundle at infinity is the disjoint union $E_\infty = \bigcup_{\lambda \in L} S^1_\infty(\lambda)$ with the finest topology such that the endpoint map $e : T\mathcal{F} - \text{(zero section)} \to E_\infty$ is continuous. A continuous map $\phi : X \to Y$ between oriented 1-manifolds homeomorphic to $S^1$ is monotone if it is of mapping degree one and if the preimage of any point of $Y$ is contractible. A gap of $\phi$ is the interior in $X$ of such a preimage. The core of $\phi$ is the complement of the union of gaps.

**Definition 2.1.** A universal circle $S^1_{\text{univ}}$ for $\mathcal{F}$ is a circle together with a homomorphism $\rho_{\text{univ}} : \pi_1(M) \to \text{Homeo}^+(S^1_{\text{univ}})$ and a family of monotone maps $\phi_\lambda : S^1_{\text{univ}} \to S^1_\infty(\lambda)$, $\lambda \in \mathcal{F}$, satisfying the following conditions:

1. For every $\alpha \in \pi_1(M)$, the following diagram commutes:

$$
\begin{array}{ccc}
S^1_{\text{univ}} & \xrightarrow{\rho_{\text{univ}}(\alpha)} & S^1_{\text{univ}} \\
\phi_\lambda \downarrow & & \phi_{\alpha(\lambda)} \downarrow \\
S^1_\infty(\lambda) & \xrightarrow{\alpha} & S^1_\infty(\alpha(\lambda))
\end{array}
$$

3
2. If \( \lambda \) and \( \mu \) are incomparable then the core of \( \phi_\lambda \) is contained in the closure of a single gap of \( \phi_\mu \) and vice versa.

Calegari-Dunfield’s way of construction of a universal circle is as follows. Let \( I = [0, 1] \) be the unit interval. A marker for \( F \) is a continuous map \( m : I \times \mathbb{R}^+ \to \tilde{M} \) with the following properties: (1) For each \( s \in I \), the image \( m(s \times \mathbb{R}^+) \) is a geodesic ray in a leaf of \( \tilde{F} \). We call these rays the horizontal rays of \( m \)\footnote{By abuse of notation we refer to such an interval as a marker.}.

(2) For each \( t \in \mathbb{R}^+ \), the image \( m(I \times t) \) is transverse to \( \tilde{F} \) and of length smaller than some constant depending only on \( \tilde{F} \).

We use the interval notation \([\lambda, \mu]\) to represent the oriented image of an injective continuous map \( c : I \to L \) such that \( c(0) = \lambda \) and \( c(1) = \mu \), and refer it to the interval from \( \lambda \) to \( \mu \). Here, notice that the orientation of such an interval is induced from that of \( I \) (not from that of \( L \)).

Let \( J = [\lambda, \mu] \) be an interval in \( L \) and let \( m \) be a marker which intersects only leaves of \( \tilde{F}|_J \). Then the endpoints of the horizontal rays of \( m \) form an interval in \( E_\infty|_J \) which is transverse to the circle fibers. By abuse of notation we refer to such an interval as a marker.

For each \( \nu \in J \), the intersection of \( S^1_\infty(\nu) \) with the union of all markers are dense in \( S^1_\infty(\nu) \). If two markers \( m_1, m_2 \) in \( E_\infty|_J \) are not disjoint, their union \( m_1 \cup m_2 \) is also an interval transverse to the circle fibers. It follows that a maximal such union of markers is still an interval. Again by abuse of notation we call such an interval a marker.

A continuous section \( \tau : J \to E_\infty|_J \) is admissible if the image of \( \tau \) does not cross (but might run into) any marker. The leftmost section \( \tau(p, J) : J \to E_\infty|_J \) starting at \( p \in S^1_\infty(\lambda) \) is an admissible section which is anticlockwisemost among all such sections if the order of \( J \) is compatible with that of \( L \), and clockwise otherwise. For any \( p \) the leftmost section starting at \( p \) exists.

Let \( B = \lim_{t \to 0} \nu_t \) be a branch locus and let \( \mu_1, \mu_2 \in B \). For each \( t > 0 \), let \( \alpha_t = [\mu_1, \nu_t] \) and \( \beta_t = [\nu_t, \mu_2] \). Then, we can define a map \( r_t : S^1_\infty(\mu_1) \to S^1_\infty(\mu_2) \) by \( r_t(p) = \tau(\tau(\tau(p, \alpha_t))(\nu_t), \beta_t)(\mu_t) \). As \( t \) tends to 0, \( r_t \) converges to a constant map. We denote the image of the constant map by \( r(\mu_1, \mu_2) \in S^1_\infty(\mu_2) \) and call it the turning point from \( \mu_1 \) to \( \mu_2 \).

Given an incomparable pair \( \lambda, \mu \in L \), we define a “path” from \( \lambda \) to \( \mu \) to be a disjoint union of finitely many intervals \([\hat{\nu}_{i-1}, \hat{\nu}_i] \), \( 1 \leq i \leq n \), in \( L \) (some of them may be degenerated to singletons), with the following properties: (1) \( \hat{\nu}_0 = \lambda \) and \( \hat{\nu}_n = \mu \), (2) \( \hat{\nu}_i \) and \( \hat{\nu}_i \) belong to a common branch locus for each \( 1 \leq i \leq n - 1 \), and (3) \( n \) is minimal under the conditions (1) and (2). Note that a path connecting any two points in \( L \) exists and is unique.

For a point \( p \) in \( S^1_\infty(\lambda) \), the special section \( \sigma_p : L \to E_\infty \) at \( p \) is defined as follows. First, set \( \sigma_p(\lambda) = p \). Next, pick any point \( \mu \in L \). We define \( \sigma_p(\mu) \) as follows: When \( \mu \) is comparable with \( \lambda \), then \( \sigma_p \) is
defined on $[\lambda, \mu]$ to be the leftmost section starting at $p$. When $\mu$ is
incomparable with $\lambda$, let $\prod_{i=1}^{n}[\tilde{\nu}_{i-1}, \tilde{\nu}_i] \,(n > 1)$ be the path from $\lambda$ to $\mu$. We then put $r = r(\tilde{\nu}_{n-1}, \tilde{\nu}_{n-1}) \in S^{1}_{\infty}(\tilde{\nu}_{n-1})$ and define $\sigma_p$ on the
interval $[\tilde{\nu}_{n-1}, \tilde{\nu}_n]$ by $\sigma_p = \sigma_r$. This completes the definition of $\sigma_p$.

Let $\mathcal{S}$ be the union of the special sections $\sigma_p$ as $p$ varies over all
points in all circles $S^{1}_{\infty}(\lambda)$ of points $\lambda$ in $L$. By [3] Lemma 6.25], the
set $\mathcal{S}$ admits a natural circular order. The universal circle $S^{1}_{\text{univ}}$ will be
derived from $\mathcal{S}$ as a quotient of the order completion of $\mathcal{S}$ with respect
to the circular order. Then one can observe that any element of $S^{1}_{\text{univ}}$ is a section $L \to E_{\infty}$.

3. Faithfullness of the Action

In this section, we show that if $\mathcal{F}$ has branching, then the natural
action of $\pi_1(M)$ on the leaf space $L$ is faithful.

As explained in §2, every element $\sigma$ of $S^{1}_{\text{univ}}$ is a section $\sigma : L \to E_{\infty} = \bigcup_{\lambda \in L} S^{1}_{\lambda}(\lambda)$ and that the maps $\phi_{\lambda} : S^{1}_{\text{univ}} \to S^{1}_{\lambda}(\lambda)$ are defined
by $\phi_{\lambda}(\sigma) = \sigma(\lambda)$. For a point $x$ in $S^{1}_{\lambda}(\lambda)$, we define a (possibly degenerate)
closed interval $I_x$ in $S^{1}_{\text{univ}}$ by $I_x = \{\sigma \in S^{1}_{\text{univ}} \mid \sigma(\lambda) = x\}$. Then,
for any $x$ the interval $I_x$ is nonempty because the special section $\sigma_x$ at
$x$ belongs to $I_x$.

Let $\lambda \in L$ and $\alpha \in \pi_1(M)$ be such that $\alpha(\lambda) = \lambda$. Then, $\alpha$, as the
restriction of a covering transformation of $\tilde{M}$ to $\lambda$, induces an isometry
of the hyperbolic plane $\lambda$, (hence also a projective transformation of
$S^{1}_{\lambda}(\lambda)$). We notice that this isometry is a hyperbolic element (meaning
that its trace is greater than 2). In fact, since it has no fixed points in $\lambda$, it is not elliptic. If it were parabolic, then it would yield in $M$
a non-contractible loop whose length can be made arbitrarily small,
contradicting the compactness of $M$.

The following is a key lemma.

**Lemma 3.1.** Let $B = \lim_{t \to 0} \nu_t$ be a branch locus of the leaf space of
$L$. If $\alpha \in \pi_1(M)$ fixes two distinct points $\mu_1$ and $\mu_2$ in $B$ and also fixes
the interval $\{\nu_t \mid 0 < t < \epsilon\}$ pointwise, then $\alpha$ is trivial in $\pi_1(M)$.

**Proof.** Suppose $\alpha$ is nontrivial. Let $p_1, q_1 \in S^{1}_{\infty}(\mu_1)$ and $p_2, q_2 \in S^{1}_{\infty}(\mu_2)$
be the fixed points of $\alpha$, and let $r_1 \in S^{1}_{\infty}(\mu_1)$ be the turning point from $\mu_2$ to $\mu_1$. Without loss of generality, we assume that $p_1 \neq r_1$. Note
that by construction of the universal circle, the special sections $\sigma_p$ and
$\sigma_{q_i}$ in $S^{1}_{\text{univ}}$ are fixed by $\rho_{\text{univ}}(\alpha)$ for $i = 1, 2$, and therefore the images
$\phi_{\nu_t}(\sigma_{p_1})$ and $\phi_{\nu_t}(\sigma_{q_i})$ are fixed by $\alpha$ for any $t \in (0, \epsilon)$.

We claim that if $t$ is sufficiently close to 0, then $\phi_{\nu_t}(\sigma_{p_1})$ and $\phi_{\nu_t}(I_{r_1})$
are disjoint in $S^{1}_{\infty}(\nu_t)$. Take two distinct points $x$ and $y$ in $S^{1}_{\infty}(\mu_1) - \{p_1, r_1\}$ so that the 4-tuple $(p_1, x, r_1, y)$ lies in circular order. Then,
for sufficiently small $t > 0$, the 4-tuple $(\sigma_{p_1}(\nu_t), \sigma_x(\nu_t), \sigma_{r_1}(\nu_t), \sigma_y(\nu_t))$
lies in $S^{1}_{\infty}(\nu_t)$ also in circular order. Let $K$ be the closed interval in
$S^{1}_{\infty}(\nu_t)$ with boundary points $\sigma_x(\nu_t)$ and $\sigma_y(\nu_t)$ which contains $\sigma_{r_1}(\nu_t)$.
Since $I_{r_1}$ contains $\sigma_{r_1}$ but does not contain $\sigma_{p_1}, \sigma_x$ and $\sigma_y$, it follows that $\phi_{\nu_1}(I_r)$ is contained in $K$. In particular, $\phi_{\nu_1}(\sigma_{p_1})$ and $\phi_{\nu_1}(I_{r_1})$ are disjoint. This shows the claim.

For $t$ sufficiently close to 0, the two points $\sigma_{p_2}(\nu_t)$ and $\sigma_{q_2}(\nu_t)$ are distinct. Since $\sigma_{p_2}$ and $\sigma_{q_2}$ are contained in $I_{r_1}$, the 3 points $\sigma_{p_1}(\nu_t), \sigma_{p_2}(\nu_t)$ and $\sigma_{q_2}(\nu_t)$ are also mutually distinct. Thus, we find at least 3 fixed points of $\alpha$ in $S^1_{\nu_t}(\nu_t)$, contradicting to the fact that $\alpha$ is a nontrivial orientation preserving isometry of the hyperbolic plane $\nu_t$. □

Now, the first main result of this article is the following:

**Theorem 3.2.** Let $M$ be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of $M$. If $\mathcal{F}$ has branching, then the natural action of $\pi_1(M)$ on the leaf space of $\mathcal{F}$ is faithful.

**Proof.** This is a direct consequence of Lemma 3.1. □

### 4. Comparable sets

Contents of this section are unrelated to leafwise hyperbolicity. For $\alpha \in \pi_1(M)$, we define the comparable set $C_\alpha$ for $\alpha$ to be the subset of $L$ consisting of points $\lambda$ such that $\lambda$ and $\alpha(\lambda)$ are comparable. Below we collect some basic properties of comparable sets.

Obviously, $\alpha(C_\alpha) = C_\alpha$, $C_\alpha = C_{\alpha^{-1}}$ and $C_\alpha \subset C_{\alpha^k}$ for every $k > 0$.

We say that $\mathcal{F}$ has one-sided branching in the positive (resp. negative) direction if $L$ has positive (resp. negative) branch loci but has no negative (resp. positive) ones. If $L$ has both positive loci and negative loci, then we say $\mathcal{F}$ has two-sided branching.

**Lemma 4.1.** Let $\mathcal{F}$ have one-sided branching in the positive direction, and let $\alpha \in \pi_1(M)$. Suppose $\lambda$ and $\mu$ are points in $L$ such that $\lambda$ is a common lower bound of $\mu$ and $\alpha(\mu)$. Then $\lambda \in C_\alpha$.

**Proof.** Since the natural action preserves the order of $L$, the inequality $\lambda < \mu$ implies $\alpha(\lambda) < \alpha(\mu)$. Thus, by the hypothesis, $\alpha(\mu)$ is a common upper bound of $\lambda$ and $\alpha(\lambda)$. Since $\mathcal{F}$ has no branching in the negative direction, it follows that $\lambda$ and $\alpha(\lambda)$ are comparable. □

From this lemma we see the following fact: Let $\mathcal{F}$ and $\alpha$ be as above. Then, there is $\lambda \in L$ such that $\{\mu \in L \mid \mu < \lambda\} \subset C_\alpha$.

**Lemma 4.2.** Let $\alpha \in \pi_1(M)$ and let $\lambda, \mu \in C_\alpha$. Then the path $\gamma$ from $\lambda$ to $\mu$ is entirely contained in $C_\alpha$. Furthermore, if $\gamma$ is represented as $\gamma = \prod_{i=1}^{n} [\tilde{\nu}_{i-1}, \tilde{\nu}_i]$ ($\tilde{\nu}_0 = \lambda, \tilde{\nu}_n = \mu$) by using a union of intervals, then $\tilde{\nu}_i, \tilde{\nu}_i$ are fixed by $\alpha$ for each $1 \leq i \leq n - 1$.

**Proof.** Let $p : L \to K$ be the Hausdorffification of $L$ (namely, $K$ is the space obtained from $L$ by identifying each of the branch loci to a single point). Put $\delta = p(\gamma)$. We see that $\delta$ is a “real path” zigzaggedly embedded in $K$. Let $a$ and $b$ be the embedded intervals in $K$ joining
Proposition 4.4. Let \( l \) be any point in \( C \). If \( \alpha(\lambda) \neq \lambda \) then the open interval \( \alpha^{-1}(\lambda) \) and \( \alpha(\lambda) \) is contained in \( C \) and contains \( \lambda \). Thus, \( \lambda \) is an interior point of \( C \). Next, we consider the case where \( \alpha(\lambda) = \lambda \). Take any point \( \mu \in L \) with \( \lambda < \mu \). Then the interval \( [\lambda, \mu] \) is mapped by \( \alpha \) orientation preservingly onto the interval \( [\lambda, \alpha(\mu)] \). Since \( L \) is an oriented 1-manifold, there must exist \( \nu \in (\lambda, \mu] \) such that \( [\lambda, \nu] \) is contained in \( C \). Similarly, we can find \( \eta < \lambda \) such that \( (\eta, \lambda] \) is contained in \( C \). Consequently, we have \( \lambda \in (\eta, \nu) \subset C \), which means \( \lambda \) is an interior point of \( C \). This proves the proposition.

For a path \( \gamma = \prod_{i=1}^{n} [\hat{\nu}_{i-1}, \hat{\nu}_{i}] \), we call \( n \) the length of \( \gamma \) and denote it by \( l(\gamma) \).

Proposition 4.3. For any \( \alpha \in \pi_1(M) \), \( C_\alpha \) is connected and open.

Proof. Connectedness is already shown in Lemma 12. We will prove openness. Let \( \lambda \) be any point in \( C \). If \( \alpha(\lambda) \neq \lambda \) then the open interval bounded by \( \alpha^{-1}(\lambda) \) and \( \alpha(\lambda) \) is contained in \( C \) and contains \( \lambda \). Thus, \( \lambda \) is an interior point of \( C \). Next, we consider the case where \( \alpha(\lambda) = \lambda \). Take any point \( \mu \in L \) with \( \lambda < \mu \). Then the interval \( [\lambda, \mu] \) is mapped by \( \alpha \) orientation preservingly onto the interval \( [\lambda, \alpha(\mu)] \). Since \( L \) is an oriented 1-manifold, there must exist \( \nu \in (\lambda, \mu] \) such that \( [\lambda, \nu] \) is contained in \( C \). Similarly, we can find \( \eta < \lambda \) such that \( (\eta, \lambda] \) is contained in \( C \). Consequently, we have \( \lambda \in (\eta, \nu) \subset C \), which means \( \lambda \) is an interior point of \( C \). This proves the proposition.

For a path \( \gamma = \prod_{i=1}^{n} [\hat{\nu}_{i-1}, \hat{\nu}_{i}] \), we call \( n \) the length of \( \gamma \) and denote it by \( l(\gamma) \).

Proposition 4.4. Let \( \alpha \in \pi_1(M) \) and \( \lambda \in L \) be such that \( \lambda \notin C_\alpha \). If the path from \( \lambda \) to \( \alpha(\lambda) \) has odd length, then \( C_{\alpha^k} = \emptyset \) for any \( k > 0 \).

Proof. Let \( \gamma \) be the path joining \( \lambda \) to \( \alpha(\lambda) \). Since \( l(\gamma) \) is odd, there occur no overlappings in composing \( k \) paths \( \gamma, \alpha(\gamma), \ldots, \alpha^{k-1}(\gamma) \) successively, and the result \( \gamma * \alpha(\gamma) * \cdots * \alpha^{k-1}(\gamma) \) is the path from \( \lambda \) to \( \alpha^k(\lambda) \).
Figure 1. \( \alpha(\delta_0) \) is shown as a broken line in the case \( \hat{\nu}_m \in C_\alpha \), and as a dotted line in the case \( \alpha(\hat{\nu}_m) \in (\hat{\nu}_m, \check{\nu}_{m+1}] \).

It follows that \( \lambda \) and \( \alpha^k(\lambda) \) are incomparable. Put \( A = \coprod_{i \in \mathbb{Z}} \alpha^i(\gamma) \). Notice that if \( \mu \in A \) then \( \mu \notin C_\alpha \), so we assume \( \mu \notin A \). For two points \( \nu \) and \( \eta \) in \( L \) we denote by \( \gamma^\nu_\eta \) the path from \( \nu \) to \( \eta \). Let \( \delta = \bigcap_{\nu \in A} \gamma^\nu_\eta \). Then, \( \delta \) is disjoint from \( A \). But, there exists a branch locus \( B \) intersecting \( A \) such that either \( B \cap \delta \neq \emptyset \), or one of the endpoints of \( \delta \) converges to \( B \). Thus, we have an estimate

\[
\min\{l(\gamma^\nu_\eta) \mid \nu \in B, \eta \in \alpha^k(B), k > 0\} \geq 2
\]

because \( l(\gamma) \geq 3 \). Since the path \( \gamma^\mu_{\alpha^k(\mu)} \) must pass through the branch loci \( B \) and \( \alpha^k(B) \), we have \( l(\gamma^\mu_{\alpha^k(\mu)}) \geq 2 \). Namely, \( \mu \) and \( \alpha^k(\mu) \) are incomparable.

Lemma 4.5. Let \( \alpha \in \pi_1(M) \) and \( \lambda \in L \) be such that \( \lambda \notin C_\alpha \) and that \( \lambda \in C_{\alpha^k} \) for some \( k > 1 \). Let \( \gamma = \coprod_{i=1}^n [\hat{\nu}_{i-1}, \hat{\nu}_i] \) \( (\hat{\nu}_0 = \lambda, \hat{\nu}_n = \alpha(\lambda)) \) be the path from \( \lambda \) to \( \alpha(\lambda) \). Then \( \alpha(\hat{\nu}_m) = \hat{\nu}_m \) and \( \alpha^k(\hat{\nu}_m) = \check{\nu}_m \) where \( m = l(\gamma)/2 \) (which is an integer by the above proposition).

Proof. Let \( \gamma_j \) be the path from \( \lambda \) to \( \alpha^j(\lambda) \), and let \( \delta_0 \) and \( \delta_1 \) be the paths from \( \lambda \) to \( \hat{\nu}_m \), and from \( \hat{\nu}_m \) to \( \alpha(\lambda) \), respectively. By reversing the transverse orientation of \( F \) if necessary, we can assume that \( \hat{\nu}_m \) and \( \hat{\nu}_m \) belong to a common positive branch locus.

First, we show that \( \hat{\nu}_m \notin C_\alpha \). Suppose on the contrary that \( \hat{\nu}_m \in C_\alpha \). Note that the length of the path \( \alpha(\delta_0) \) joining \( \alpha(\lambda) \) to \( \alpha(\hat{\nu}_m) \) is \( l(\gamma)/2 \). So if \( \hat{\nu}_m \) and \( \alpha(\hat{\nu}_m) \) are comparable, the intersection \( \gamma \cap \alpha(\delta_0) \) must coincide with \( \delta_1 \) as a set. Therefore \( \hat{\nu}_m \geq \alpha(\hat{\nu}_m) \). See Fig. 1. Then \( \hat{\nu}_m \geq \alpha^{k-1}(\hat{\nu}_m) \), and we have \( \gamma_k = \delta_0 * [\hat{\nu}_m, \alpha^{k-1}(\hat{\nu}_m)] * \alpha^{k-1}(\delta_1) \). Since
\(\gamma_k\) passes through \(\alpha^{k-1}(\hat{\nu}_m)\) and \(\alpha^{k-1}(\check{\nu}_m)\), it follows that \(\lambda\) and \(\alpha^k(\lambda)\) are incomparable, which contradicts the choice of \(\lambda\).

Next, we show that \(\alpha(\check{\nu}_m) \notin (\check{\nu}_m, \check{\nu}_{m+1}]\). Suppose not. Then the branch locus obtained from the embedded interval \((\check{\nu}_m, \alpha(\check{\nu}_m))\) contains \(\alpha(\check{\nu}_m)\), that is, \(\check{\nu}_m\) and \(\alpha(\check{\nu}_m)\) are comparable. See Fig. [1] Since we are assuming that \(\hat{\nu}_m\) and \(\check{\nu}_m\) belong to a common positive branch locus, we have \(\hat{\nu}_m < \alpha(\check{\nu}_m)\). Then \(\hat{\nu}_m < \alpha^{k-1}(\check{\nu}_m)\), and therefore \(\gamma_k = \delta_0 * [\hat{\nu}_m, \alpha^{k-1}(\check{\nu}_m)] * \alpha^{k-1}(\delta_1)\). Since \(\gamma_k\) passes through \(\check{\nu}_m\) and \(\hat{\nu}_m\), it follows that \(\lambda\) and \(\alpha^k(\lambda)\) are incomparable, which is a contradiction.

Finally, we consider other cases. If \(\alpha(\check{\nu}_m) \neq \hat{\nu}_m\), we have \(l(\alpha^{j+1}(\gamma) - \alpha^j(\gamma)) > l(\gamma)/2\) for all \(0 \leq j < k\). Therefore, we have

\[
1 < l(\gamma_1) < l(\gamma_2) < \cdots < l(\gamma_k) = 1
\]

This contradiction shows that \(\alpha(\check{\nu}_m) = \hat{\nu}_m\). We also have that \(\alpha^k(\check{\nu}_m) = \hat{\nu}_m\). Otherwise, \(\gamma_k = \delta_0 * \alpha^k(\delta_0)^{-1}\), and therefore \(\gamma_k\) passes through \(\hat{\nu}_m\) and \(\alpha^k(\check{\nu}_m)\) which belong the common branch locus. It follows that \(\lambda \notin C^*\), which is a contradiction. \(\square\)

**Lemma 4.6.** Let \(\alpha \in \pi_1(M)\) and let \(B\) be an \(\alpha\)-invariant branch locus. If \(\{\nu_t\}_{0 < t < \varepsilon}\) be an embedded interval such that \(B = \lim_{t \to 0} \nu_t\), then there exists \(0 < \varepsilon' < \varepsilon\) such that \(\nu_t\) is in \(C_\alpha\) for any \(t \in (0, \varepsilon')\).

**Proof.** Let \(\{\nu_t\}_{0 < t < \varepsilon}\) be as in the hypothesis of the lemma. Then, \(\alpha(\nu_t) = \lim_{t \to 0} \alpha(\nu_t)\). Since \(B = \alpha(\nu_t)\), two intervals \(\{\nu_t\}_{0 < t < \varepsilon}\) and \(\{\alpha(\nu_t)\}_{0 < t < \varepsilon}\) are both asymptotic to \(B\) from the same direction as \(t\) tends to 0. This with the fact that \(L\) is a 1-manifold implies that the two intervals coincide near \(B\). Thus, the conclusion of the lemma follows. \(\square\)

## 5. Branch Loci and Their Stabilizers

In this section we focus on a branch locus of the leaf space \(L\). We consider the case where a branch locus is a finite set and clarify the structure of the stabilizer of such a locus.

**Lemma 5.1.** Let \(B\) be a finite branch locus and let \(\alpha \in \text{Stab}(B)\). If \(\rho_{\text{univ}}(\alpha)\) has a fixed point in \(S^1_{\text{univ}}\), then \(\alpha\) fixes \(B\) pointwise.

**Proof.** Let \(\alpha \in \text{Stab}(B)\) be a nontrivial element satisfying the hypothesis of the lemma, and let \(\lambda\) be any point of \(B\). Then, since \(B\) is finite, there exists some \(k \in \mathbb{N}\) such that \(\alpha^k(\lambda) = \lambda\). Notice here that \(\alpha^k\) is nontrivial in \(\pi_1(M)\), because by tautness of \(\mathcal{F}\) and by Novikov’s theorem ([S]) our manifold \(M\) is aspherical and hence has no torsion in \(\pi_1(M)\) ([1] Corollary 9.9]). Now, let us suppose by contradiction that \(\alpha(\lambda) \neq \lambda\). Let \(r \in S^1_{\text{univ}}(\lambda)\) be the turning point from \(\alpha(\lambda)\) to \(\lambda\) and let \(p \in S^1_{\text{univ}}(\alpha\lambda)\) be one of the two fixed points of \(\alpha^k\) which is different from \(r\). Then the special section \(\sigma_p\) in \(S^1_{\text{univ}}\) is fixed by \(\rho_{\text{univ}}(\alpha^k)\). This with the hypothesis that \(\rho_{\text{univ}}(\alpha)\) has a fixed point implies that \(\sigma_p\) must be
fixed by \( \rho_{\text{univ}}(\alpha) \) itself. So we have \( \rho_{\text{univ}}(\alpha)(\sigma_p) \in I_p \). On the other hand, since \( \alpha(p) \in \alpha(\lambda) \), it follows from the definition of turning point that \( \rho_{\text{univ}}(\alpha)(\sigma_p) = \sigma_{\alpha(p)} \in I_r \). This is a contradiction because \( I_p \) and \( I_r \) are disjoint. 

**Theorem 5.2.** Let \( M \) be a closed oriented 3-manifold, and \( \mathcal{F} \) a transversely oriented leafwise hyperbolic taut foliation of \( M \). Suppose \( \mathcal{F} \) has a finite branch locus \( B \). If an element of \( \text{Stab}(B) \) fixes some point of \( B \) then it fixes all the points of \( B \).

**Proof.** Let \( \lambda \) be the \( \alpha \)-fixed point in \( B \), and let \( p, q \in S^1_{\infty}(\lambda) \) be the fixed points of \( \alpha \). Then \( \sigma_p, \sigma_q \in S^1_{\text{univ}} \) are fixed by \( \rho_{\text{univ}}(\alpha) \). The proof follows by Lemma 5.1. \( \square \)

The next result gives information on topological types of leaves in a finite branch locus.

**Theorem 5.3.** Let \( M \) and \( \mathcal{F} \) be as in Theorem 5.2, and let \( B \) be a finite branch locus of \( \mathcal{L} \). Then, for any \( \lambda \in B \),

1. if \( \text{Stab}(B) \) is trivial, \( \lambda \) is diffeomorphic to a plane, and
2. if \( \text{Stab}(B) \) is nontrivial, \( \lambda \) is diffeomorphic to a cylinder.

**Proof.** Let \( \lambda \in B \). Since \( \mathcal{F} \) is taut, by Novikov’s theorem ([8]) the inclusion map of each leaf of \( \mathcal{F} \) into \( M \) is \( \pi_1 \)-injective. So, if \( \lambda \) is not a plane, there exists a nontrivial element \( \alpha \in \pi_1(M) \) such that \( \alpha(\lambda) = \lambda \). This \( \alpha \) must belong to \( \text{Stab}(B) \), showing the first statement of the theorem.

To prove the second statement, suppose that \( \text{Stab}(B) \) is nontrivial. Then, we can first observe that \( \lambda \) is not a plane. In fact, let \( \gamma \) be any nontrivial element of \( \text{Stab}(B) \). Since \( B \) is finite, \( \gamma^n(\lambda) = \lambda \) for some \( n \in \mathbb{N} \). By the same argument as in the proof of Lemma 5.1, we see that \( \gamma^n \) nontrivial in \( \pi_1(M) \). This shows the observation. Now, by way of contradiction, let us assume \( \lambda \) is not a cylinder, either. Then, again by \( \pi_1 \)-injectivity of the inclusion \( \lambda \to M \), we can find elements \( \alpha, \beta \in \pi_1(M) \) generating a free subgroup of rank 2 such that \( \alpha(\lambda) = \beta(\lambda) = \lambda \). These two elements are hyperbolic as isometries of \( \lambda \) and having no common fixed point on \( S^1_{\infty}(\lambda) \). Let \( \mu \) be another leaf in \( B \), and let \( r \in S^1_{\infty}(\lambda) \) be the turning point from \( \mu \) to \( \lambda \). By exchanging \( \alpha \) and \( \beta \) if necessary, we may assume \( \alpha(\mu) \neq r \). Then, \( \alpha^k(r) \neq \alpha^{l}(r) \) for any \( k \neq l \in \mathbb{Z} \). Pick a point \( s \in S^1_{\infty}(\mu) \) and consider the special section \( \sigma_s \) at \( s \). Then, \( \rho_{\text{univ}}(\alpha^k)(\sigma_s) = \sigma_{\alpha^k(s)} \) is the special section at \( \alpha^k(s) \). Since \( \sigma_{\alpha^k(s)}(\lambda) = \phi_\lambda \circ \rho_{\text{univ}}(\alpha^k)(\sigma_s) = \alpha^k \circ \phi_\lambda(\sigma_s) = \alpha^k(r) \), it follows that \( \alpha^k(r) \) is the turning point from \( \alpha^k(\mu) \) to \( \lambda \). In particular, \( \alpha^k(\mu) \neq \alpha^l(\mu) \) for \( k \neq l \), hence, \( B \) contains infinitely many elements \( \alpha^k(\mu), k \in \mathbb{Z} \), contradicting the finiteness of \( B \). \( \square \)

**Remark 5.4.** The author does not know whether or not there exists a finite branch locus which has a trivial stabilizer.
Proposition 5.5. Let $B = \{\lambda_1, \ldots, \lambda_n\}$ be a finite branch locus which has a nontrivial stabilizer and let $r_i^j \in S^1_{\infty}(\lambda_i)$ be the turning point from $\lambda_i$ to $\lambda_i$. Then there exists $1 \leq k \leq n$ such that the set of turning points $\{r_k^j \mid j \neq k\}$ is a single point in $S^1_{\infty}(\lambda_k)$.

Proof. By Theorem 5.3 each $\lambda_i$ is a cylindrical leaf. Let $\gamma$ be an element in $\text{Stab}(B)$ such that $\gamma(\lambda_1) = \lambda_1$ and that a loop in $M$ representing $\gamma$ is freely homotopic to a loop in $\lambda_1$ generating $\pi_1(\lambda_1)$. By Theorem 5.2 $\gamma$ fixes all points in $B$. Let $p_i, q_i \in S^1_{\infty}(\lambda_i)$ be the fixed points of $\gamma$. Note that $r_i^j \in \{p_i, q_i\}$ for any $i, j$. Otherwise, $B$ cannot be finite by the same argument as in the proof of Theorem 5.3. Now, we suppose that $\{r_i^j \mid j \neq 1\} = \{p_1, q_1\}$. After renumbering the indices if necessary, we can assume that $r_i^j = p_1$ for $2 \leq j < n_1$ and $r_i^j = q_1$ for $n_1 \leq j < n$, where $3 \leq n_1 \leq n$. Then, we claim that $r_i^{n_1} = r_i^{1}$ for $1 \leq j < n_1$. In fact, let $2 \leq j < n_1$, and take 4 points $x, y, z, w$ as follows: $x, y$ are in $S^1_{\infty}(\lambda_1) - \{p_1, q_1\}$ such that the 4-tuple $(p_1, x, y, q_1)$ is circularly ordered, $z \in S^1_{\infty}(\lambda_j)$ and $w \in S^1_{\infty}(\lambda_{n_1}) - \{r_i^{n_1}\}$. Then, $\sigma_z \in I_{p_1}, \sigma_w \in I_{q_1}$ and the 4-tuple $(I_{p_1}, \sigma_x, I_{q_1}, \sigma_y)$ is circularly ordered in $S^1_{\text{univ}}$. Furthermore, $\sigma_x, \sigma_y \in I_{r_i^{n_1}}$, and $\sigma_w \notin I_{r_i^{n_1}}$. It follows that $\sigma_z \in I_{r_i^{n_1}}$, that is, $r_i^{n_1}$ is the turning point from $\lambda_j$ to $\lambda_{n_1}$. This proves the claim.

Now, if $\{r_i^{m_1} \mid j \neq n_1\} = \{r_i^{n_1}\}$ we can put $k = n_1$. Otherwise, by renumbering the indices again, we can assume that $r_i^{m_1} = r_i^{n_1} = p_1$ for $1 \leq j < n_2$ ($j \neq n_1$), and $r_i^{m_1} = q_1$ for $n_2 \leq j < n$, where $n_1 < n_2 \leq n$. Similarly, we have $r_i^{m_2} = r_i^{n_2}$ for $1 \leq j < n_2$. Since $B$ is finite, we can find a desired $k$ after repeating this process finitely many times. \square

Theorem 5.6. Let $M$ and $F$ be as in Theorem 5.2. Let $B$ be a finite branch locus of $L$ with nontrivial stabilizer. Then the stabilizer $\text{Stab}(B)$ is isomorphic to $\mathbb{Z}$.

Proof. Let $B = \{\lambda_1, \ldots, \lambda_n\}$, and let $\gamma$ be as in the proof of the above proposition. If $\text{Stab}(B)$ acts on $B$ trivially, then for any $\alpha \in \text{Stab}(B)$ there exists an integer $k$ such that $\alpha = \gamma^k$. Thus $\gamma$ is a generator of $\text{Stab}(B)$. So we assume that $\text{Stab}(B)$ acts on $B$ nontrivially. Let $p_i, q_i \in S^1_{\infty}(\lambda_i)$ be the fixed points of $\gamma$, and let $r_i^j \in S^1_{\infty}(\lambda_i)$ be the turning point from $\lambda_j$ to $\lambda_i$ for $i \neq j$. By Proposition 5.5, without loss of generality we can assume that $\{r_i^j \mid j \neq 1\}$ is a single point. Put $\text{Stab}(B)(\lambda_1) = \{\alpha(\lambda_1) \mid \alpha \in \text{Stab}(B)\} = \{\lambda_1, \ldots, \lambda_m\}$ where $1 < m \leq n$. Since the natural action preserves the set of turning points, $\{r_i^j \mid 1 \leq j \leq n, j \neq i\}$ is also a single point for any $i \leq m$. Let us denote this single point by $p_i$. It follows that the subset $\{\sigma_{p_i} \mid 1 \leq i \leq m\}$ of $S^1_{\text{univ}}$ is kept invariant by homeomorphisms $\rho_{\text{univ}}(\alpha)$ for $\alpha \in \text{Stab}(B)$. After renumbering indices if necessary, we can assume that the $m$-tuple $(\sigma_{p_1}, \ldots, \sigma_{p_m})$ is circularly ordered in $S^1_{\text{univ}}$. Let $\beta \in \text{Stab}(B)$ be such that $\rho_{\text{univ}}(\beta)(\sigma_{p_1}) = \sigma_{p_2}$, that is, $\beta(\lambda_1) = \lambda_2$. Then we have $\beta(\lambda_i) = \lambda_{i+1}$ where the indices $i$ are taken
modulo \( m \). Since \( \beta \gamma \beta^{-1}(\lambda_k) = \lambda_1 \), we have \( \beta \gamma \beta^{-1} = \gamma k \) for some \( k \neq 0 \). Moreover, there is \( l \neq 0 \) such that \( \beta^l = \gamma^l \). It follows that \( \beta^{km} = \gamma^{kl} = \beta^l \beta^{-l} = \beta^m \), that is, \( \beta^{(k-1)m} \) is trivial. If \( k \neq 1 \), \( \beta \) is a torsion element in \( \pi_1(M) \), which is a contradiction. Therefore \( k = 1 \) and we have that \( \gamma \) and \( \beta \) commute. Since \( \pi_1(M) \) is torsion-free, the subgroup \( \langle \gamma, \beta \mid \gamma^l \beta^{-m} \rangle \) must be isomorphic to \( \mathbb{Z} \). It follows that there is \( \delta \in \pi_1(M) \) such that \( \gamma = \delta^i \) and \( \beta = \delta^j \) where \( i \neq 0 \) and \( j \neq 0 \). Let \( \alpha \) be any element in \( \text{Stab}(B) \). Then \( \alpha(\lambda_1) = \lambda_i \) for some \( 1 \leq i \leq m \). By the choice of \( \gamma \) and \( \beta \), we have that \( \alpha \) can be represented as a word in \( \gamma, \beta \), hence, in \( \delta \). It follows that \( \text{Stab}(B) \) is isomorphic to \( \mathbb{Z} \). \( \square \)

We say that \( \alpha \in \pi_1(M) \) is infinitely divisible if for any integer \( \ell \), there are \( k > \ell \) and \( \beta \in \pi_1(M) \) such that \( \alpha = \beta^k \).

**Theorem 5.7.** Let \( M \) and \( F \) be as in Theorem 5.2 and let \( B \) be a finite branch locus of \( L \) such that \( \text{Stab}(B) \) acts on \( B \) nontrivially. Then a generator of \( \text{Stab}(B) \) (\( \cong \mathbb{Z} \)) is indivisible.

**Proof.** Let \( B = \{ \lambda_1, \ldots, \lambda_n \} \). By Theorem 5.6 \( \text{Stab}(B) \) is generated by some single element \( \alpha \). We assume by contradiction that \( \alpha \) is divisible. Since \( M \) is aspherical (as was noted in the proof of Lemma 5.1), \( \pi_1(M) \) has no infinitely divisible elements (see [3, Theorem 4.1]). Hence, there exists an indivisible element \( \beta \in \pi_1(M) \) such that \( \alpha = \beta^k \) for some \( k > 1 \). Note that since \( \beta \notin \text{Stab}(B) \), the points \( \lambda_i \in B \) and \( \beta(\lambda_i) \in \beta(B) \) are distinct for any \( i \). Moreover, we see that they are incomparable for any \( i \). In fact, if \( \lambda_i \) and \( \beta(\lambda_i) \) were comparable, say \( \lambda_i \prec \beta(\lambda_i) \), then \( \lambda_i \prec \beta^k(\lambda_i) = \alpha(\lambda_i) \), contradicting the assumption that \( \alpha \in \text{Stab}(B) \). Let \( \{\nu_t\}_{0 < t < \epsilon} \) be an embedded interval such that \( B = \lim_{t \to 0} \nu_t \). Since \( B \) is \( \alpha \)-invariant, it follows from Lemma 4.6 that there is \( \nu \in \{\nu_t\}_{0 < t < \epsilon} \) such that \( \nu \in C_\alpha = C_{\beta^k} \). We can (and do) take such \( \nu \) so that \( \nu \) also satisfies that \( \nu \notin C_\beta \). Let \( \bigcup_{l=1}^l [\tilde{\nu}_{l-1}, \tilde{\nu}_l] \) \( (l > 1) \) be the path joining \( \nu \) to \( \beta(\nu) \). By the choice of \( \nu \) and by Lemma 4.5 we have \( \beta^k(\tilde{\nu}_m) = \tilde{\nu}_m \) where \( m = l/2 \). It follows that \( \rho_{\text{min}}(\beta^k) \) has a fixed point in \( S_{\text{min}}^1 \). By Lemma 5.7 \( \beta^k = \alpha \) fixes all points in \( B \), which is a contradiction. \( \square \)

**Remark 5.8.** The author does not know whether or not there is a finite branch locus \( B \) such that \( \text{Stab}(B) \) acts on \( B \) trivially and is generated by a divisible element.

We will give an example of a tautly foliated compact 3-manifold admitting a finite branch locus whose stabilizer acts on the locus nontrivially. We remark that a recipe how to construct such a locus has already been provided in [3, Example 3.7], and our construction follows it.

**Example 5.9.** Let \( P = D^2 - (E_1 \cup E_2) \) be the unit disk in \( \mathbb{R}^2 \) with two open disks removed, where \( E_1, E_2 \) are disks centered \( (-\frac{1}{2}, 0), (\frac{1}{2}, 0) \) with
radius $\frac{1}{4}$ respectively. On $P$ we consider a standard singular foliation $\mathcal{G}$ (see Fig. 2) satisfying the following properties: (1) $\mathcal{G}$ has $(0, 0)$ as its unique singular point, which is of saddle type, (2) $\mathcal{G}$ is transverse to $\partial P$, (3) all leaves of $\mathcal{G}$ (except the 4 separatrices) are compact, and (4) $\mathcal{G}$ is symmetric with respect to both the $x$-axis and $y$-axis.

Put $S_0 = \partial D^2$, $S_1 = \partial E_1$ and $S_2 = \partial E_2$. Let $(P', \mathcal{G}')$ be a copy of $(P, \mathcal{G})$, and let $c : P' \to P$ be the identification. We construct a double $\Sigma = P \cup P'$ using diffeomorphisms $g_i : S_i \to c(S_i)$ ($i = 0, 1, 2$) to glue $S_i$ to $c(S_i)$, where $c^{-1} \circ g_i$ is a rotation by an irrational angle for $i = 0$ and by angle $\pi$ for $i = 1, 2$. Then $\mathcal{G}$ and $\mathcal{G}'$ induce a singular foliation $\mathcal{G}''$ of $\Sigma$ with two saddle singularities and without any saddle connection.

By construction, there exists a $\mathcal{G}''$-preserving homeomorphism $\rho$ of $\Sigma$ which fixes the singular points and which is the rotation by $\pi$ near the singular points. Fix a hyperbolic structure on $\Sigma$. Then each leaf of $\mathcal{G}''$ except the singular points and the separatrices is isotopic to a unique embedded geodesic, and the closure of the union of these geodesics constitutes a geodesic lamination, say $\lambda$, on $\Sigma$. Note that the two complementary regions $Q_1$ and $Q_2$ to $\lambda$ are ideal open squares. There exists a $\lambda$-preserving homeomorphism $\psi$ of $\Sigma$ isotopic to $\rho$. Let $M$ be the mapping torus of $\psi$, that is, $M = \Sigma \times [0, 1]/(s, 1) \sim (\psi(s), 0)$. Then $\lambda$ induces a surface lamination $\Lambda$ of $M$ whose complementary regions $R_i$ are $Q_i$-bundles over $S^1$ for $i = 1, 2$. Denote by $p_i : R_i \to S^1$ the bundle projection. Now we extend $\Lambda$ to a foliation $\mathcal{F}$ of $M$ by filling $R_i$ ($i = 1, 2$) with leaves diffeomorphic to $Q_i$ as follows. Denote the boundary components of $R_i$ by $C_{i1}$ and $C_{i2}$, which are open cylinders. Let $\gamma_i$ be an oriented loop in $R_i$ such that $p_i|\gamma_i$ is a diffeomorphism onto $S^1$. Then the composition $\gamma_i^2 = \gamma_i \ast \gamma_i$ is isotopic to a leaf loop $\gamma_{ij}$ of $C_{ij}$ which is a generator of $\pi_1(C_{ij})$. We foliate $R_i$ as a product by leaves isotopic to the fibers $Q_i$ so that the holonomy along $\gamma_{i1}$ is contracting and the holonomy along $\gamma_{i2}$ is expanding. Then the resulting foliation $\mathcal{F}$ is taut and has two-sided branching, and each end of a lift of $\gamma_i$ to $\tilde{M}$ gives a branch locus consisting of two points. Let $\alpha_i$ be an element

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{A singular foliation $\mathcal{G}$ of $P$.}
\end{figure}
in \( \pi_1(M) \) whose conjugacy class corresponds with the free homotopy class of \( \gamma_i \). Then, \( \alpha_i \) belongs to the stabilizer of some branch locus and acts on the locus nontrivially, as desired.

6. LOOPS AND ACTIONS

Given a loop in a tautly foliated manifold \((M, \mathcal{F})\), it is natural to ask whether it is transversable, or tangentiabile, to \( \mathcal{F} \). In this section, we observe that these properties of loops are expressed completely in the language of the natural action. Furthermore, we consider relations between such properties and the branching phenomenon of \( \mathcal{F} \).

We do not need to assume leafwise hyperbolicity in the first two propositions below.

**Proposition 6.1.** Let \( \gamma \) be a loop in \( M \), and \( \alpha \) an element in \( \pi_1(M,p) \) whose conjugacy class corresponds with the free homotopy class of \( \gamma \). Then, \( \gamma \) is tangentiabile if and only if the action of \( \alpha \) on \( L \) has a fixed point. Similarly, \( \gamma \) is positively (resp. negatively) transversable if and only if there is a point \( \lambda \) in \( L \) such that \( \alpha(\lambda) > \lambda \) (resp. \( \alpha(\lambda) < \lambda \)).

**Proof.** Let \( \lambda \) be a leaf of \( \mathcal{F} \) and suppose that the deck transformation \( \alpha \) leaves \( \lambda \) invariant. Take any point \( x \) in \( \lambda \) and join \( x \) to \( \alpha(x) \) by a path in \( \lambda \). Then it projects down to a leaf loop in \( M \) freely homotopic to \( \alpha \). Conversely, suppose \( \gamma \) is a leaf loop in \( M \). Join the base point \( p \) to a point of \( \gamma \) by a path \( c \). Then, the loop \( c \ast \gamma \ast c^{-1} \) represents an element of \( \pi_1(M,p) \) conjugate to \( \alpha \). Obviously it has a fixed point, hence so does \( \alpha \). The claim on transversability is also shown easily. \( \square \)

According to the proposition, we can define the tangentiability and transversability for elements in \( \pi_1(M) \) in terms of the natural action: An element \( \alpha \) in \( \pi_1(M) \) is **tangentiabile** if there is a leaf \( \lambda \) in \( L \) such that \( \alpha(\lambda) = \lambda \). It is positively (resp. negatively) **transversable** if there is a leaf \( \lambda \) in \( L \) such that \( \alpha(\lambda) > \lambda \) (resp. \( \alpha(\lambda) < \lambda \)).

We remark here that \( \pi_1(M) \) can have an element which is neither tangentiabile nor transversable. Such an element exists if and only if \( \mathcal{F} \) has two-sided branching (Calegari [1, Lemma 3.2.2]).

**Proposition 6.2.** Let \( \alpha \in \pi_1(M) \). Suppose there are points \( \lambda, \mu \in L \) such that \( \alpha(\lambda) > \lambda \) and \( \alpha(\mu) < \mu \). Then there exists a point \( \nu \in L \) such that \( \alpha(\nu) = \nu \). Moreover, if \( \lambda \) and \( \mu \) are incomparable, then such \( \nu \) can be found in some branch locus.

**Proof.** If \( \lambda \) and \( \mu \) are comparable, then the conclusion follows immediately from the intermediate value theorem. If \( \lambda \) and \( \mu \) are incomparable, then the conclusion follows from Lemma 4.2. \( \square \)

This proposition means that if a loop in \( M \) is both positively and negatively transversable to \( \mathcal{F} \), then it is tangentiabile to \( \mathcal{F} \). Such a statement might also follow from the Poincaré-Bendixson theorem. But,
here we gave a proof purely in terms of the natural action of $\pi_1(M)$ on $L$.

In the following we assume leafwise hyperbolicity and observe that tangentiability and/or transversability of loops in $M$ and the infinite-ness of branch loci are closely related.

**Theorem 6.3.** Let $M$ be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of $M$ with one-sided branching. Suppose that there is non-contractible leaf loop $\gamma$ in $M$ which is not transversable. Then every branch locus of $L$ is an infinite set.

**Proof.** Suppose that there exists a finite, say positive, branch locus $B = \{\lambda_1, \cdots, \lambda_n\}$. Let $\alpha$ be an element in $\pi_1(M)$ whose conjugacy class corresponds with the free homotopy class of $\gamma$. By Proposition 6.1, $\alpha$ has a fixed leaf in $L$, and for each $\mu \in L$ if $\mu$ is not fixed by $\alpha$ then $\mu \notin C_\alpha$. Let $\nu$ be a fixed point of $\alpha$. By Lemma 4.1 for every $\eta$ with $\eta \leq \nu$, we have $\eta \in C_\alpha$, and therefore $\alpha(\eta) = \eta$. By replacing $\mathcal{F}$ with $\beta(\mathcal{F})$ for some $\beta \in \pi_1(M)$ if necessary, we can assume that $\lambda_1 \leq \nu$ and therefore $\alpha(\lambda_1) = \lambda_1$. This implies in particular that $B$ is $\alpha$-invariant. Since $B$ is finite, by Theorem 5.2 we have $\alpha(\lambda_i) = \lambda_i$ for any $1 \leq i \leq n$. By Lemma 3.1 $\alpha$ must be trivial, which is a contradiction. □

**Corollary 6.4.** Let $M$ be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of $M$ with every leaf dense. Suppose that there is non-contractible leaf loop $\gamma$ in $M$ which is not transversable. Then every branch locus of $L$ is an infinite set.

**Proof.** Suppose there is a finite branch locus $B$. Let $\alpha \in \pi_1(M)$ be as in the proof of the preceding theorem. By Proposition 6.1 $\alpha$ has a fixed leaf in $L$, and for each $\mu \in L$ if $\mu$ is not fixed by $\alpha$ then $\mu \notin C_\alpha$. Let $\nu$ be a fixed point of $\alpha$. By Lemma 4.1 for every $\eta$ with $\eta \leq \nu$, we have $\eta \in C_\alpha$, and therefore $\alpha(\eta) = \eta$. By replacing $\mathcal{F}$ with $\beta(\mathcal{F})$ for some $\beta \in \pi_1(M)$ if necessary, we can assume that $\lambda_1 \leq \nu$ and therefore $\alpha(\lambda_1) = \lambda_1$. This implies in particular that $B$ is $\alpha$-invariant. Since $B$ is finite, by Theorem 5.2 we have $\alpha(\lambda_i) = \lambda_i$ for any $1 \leq i \leq n$. By Lemma 3.1 $\alpha$ must be trivial, which is a contradiction. □

**Theorem 6.5.** Let $M$ be a closed oriented 3-manifold, and $\mathcal{F}$ a transversely oriented leafwise hyperbolic taut foliation of $M$ with branching. Suppose that there is a non-contractible leaf loop $\gamma$ in $M$ which is not transversable. Then $L$ has an infinite branch locus.

**Proof.** Let $\alpha$ be as in Theorem 6.3. Then $\alpha$ has a fixed point $\nu \in L$, and for each $\mu \in L$ if $\mu$ is not fixed by $\alpha$ then $\mu \notin C_\alpha$. Without loss of generality, we assume that $\mathcal{F}$ has a positive branch locus. We claim that there exist some $\nu' > \nu$ such that $\nu'$ and $\alpha(\nu')$ are incomparable. Put $L' = \{\mu \mid \mu > \nu\}$. Then we can observe that $L'$ is a submanifold of $L$ with one-sided branching in the positive direction and contains at least one branch locus. If $\alpha$ fixes all leaves in $L'$, then by applying Lemma 3.1 to a branch locus in $L'$ we obtain that $\alpha$ is trivial in $\pi_1(M)$, which contradicts the hypothesis that $\alpha$ is represented by a non-contractible loop. Therefore, there exists some $\nu' \in L$ which is not fixed by $\alpha$. Since
such $\nu'$ does not belong to $C_\alpha$, the claim is shown. Since $\nu < \nu'$ and $\alpha(\nu) = \nu$, it follows that $\nu$ is a common lower bound for $\nu'$ and $\alpha(\nu')$. Thus, the fact that $\nu'$ and $\alpha(\nu')$ are incomparable implies that there is a unique $\lambda \in [\nu, \nu']$ such that $\mu \in [\nu, \nu']$ is fixed by $\alpha$ if and only if $\mu \in [\nu, \lambda)$. Evidently, $\lambda$ belongs to some $\alpha$-invariant branch locus, say $B$. Also note that $\rho_{\text{univ}}(\alpha)$ has a fixed point because $\alpha$ fixes a point in $L$. We now show $B$ is infinite. Suppose not. Then, by Lemma 5.1 all leaves in $B$ are $\alpha$-fixed, and, by Lemma 3.1 $\alpha$ must be trivial, which is a contradiction. □

References

[1] D. Calegari, The Gromov norm and foliations, *Geom. Funct. Anal.* 10 (2000), 1423-1447.
[2] D. Calegari, Foliations and the Geometry of 3-Manifolds, Oxford Math. Monographs, Oxford Univ. Press, Oxford, 2007.
[3] D. Calegari and N. Dunfield, Laminations and groups of homeomorphisms of the circle, *Invent. Math.* 152 (2003), 149-204.
[4] A. Candel, Uniformization of surface laminations, *Ann. Sci. Éc. Norm. Sup.* 26 (1993), 489-516.
[5] S. Fenley, The structure of branching in Anosov flows of 3-manifolds, *Comment. Math. Helv.* 73 (1998), 259-297.
[6] S. Friedl, Centralizers in 3-manifold groups. *RIMS Kokyuroku* 1747 (2011), 23-34.
[7] J. Hempel, 3-manifolds, Ann. of Math. Studies, 86, Princeton Univ. Press, Princeton, N.J., 1976.
[8] S. P. Novikov, Topology of foliations, *Trudy Moskov. Mat. Obsh.* 14 (1965), 248-278.
[9] S. Shields, Branching in the universal cover of taut foliations, *Pacific J. Math.* 203 (2002), 235-256.
[10] W. P. Thurston, Three-manifolds, Foliations and Circles, II. unfinished manuscript, 1998.

Department of Mathematics and Informatics, Graduate School of Science, Chiba University, Chiba 263-8522, Japan
E-mail address: ykano@g.math.s.chiba-u.ac.jp