Asymptotic dynamics of short-waves in nonlinear dispersive models

M. A. Manna and V. Merle

Physique Mathématique et Théorique, CNRS - UM2, 34095 MONTPELLIER (France)

The multiple-scale perturbation theory, well known for long-waves, is extended to the study of the far-field behaviour of short-waves, commonly called ripples. It is proved that the Benjamin-Bona-Mahony-Peregrine equation can propagate short-waves. This result contradict the Benjamin hypothesis that short-waves tends not to propagate in this model and close a part of the old controversy between Korteweg-de Vries and Benjamin-Bona-Mahony-Peregrine equations. We shown that a nonlinear (quadratic) Klein-Gordon type equation substitutes in a short-wave analysis the ubiquitous Korteweg-de Vries equation of long-wave approach. Moreover the kink solutions of $\phi^4$ and sine-Gordon equations are understood as an all orders asymptotic behaviour of short-waves. It is proved that the antikink solution of $\phi^4$ model which was never obtained perturbatively can be obtained by perturbation expansion in the wave-number $k$ in the short-wave limit.

The solution is expanded in the form of a power series in a small parameter $\epsilon$ proportional to the inverse of the wave-number $k$. The perturbative series solution is secular. It is regularized (uniform expansion), through a renormalization of the frequency. This results to the celebrated Stokes’ hypothesis on frequency-amplitude dependence in water waves [7].

The Stokes’ hypothesis is actually the second key tool of our approach. Indeed, for LW asymptotic description, the KdV [3] or MKdV [10] hierarchies occult the need of this tool, as they naturally provide the correct series expansion of the frequency.

Basic Models. Hence the problem is the asymptotic behaviour of a SW in Benjamin-Bona-Mahony-Peregrine (BBMP) [1] and in two classical relativistic nonlinear models: $\phi^4$ and sine-Gordon (SG) [2]:

$$
\begin{align*}
\text{(BBMP)} & \quad u_t + u_x - u_{xxt} = 3(u^2)_x, \\
\text{(}$\phi^4$\text{)} & \quad \phi_{xx} - \phi_{tt} = m^2\phi - \lambda\phi^3, \\
\text{(SG)} & \quad \phi_{xx} - \phi_{tt} = \frac{m^3}{\sqrt{\lambda}}\sin[(\sqrt{\lambda} m)\phi].
\end{align*}
$$

The above models have quite different intrinsic characteristics. First of all SG is an integrable models while BBMP and $\phi^4$ are not. Second the linear dispersion relation $\omega(k)$ has a finite limit as $k \to \infty$ (SW limit) for BBMP, while it is unbounded for SG and $\phi^4$. Indeed we have

$$
\begin{align*}
\omega_{(BBMP)} & = \frac{k}{1 + k^2}, \\
\omega_{(\phi^4)} & = \omega_{(SG)} = (m^2 + k^2)^{\frac{1}{2}},
\end{align*}
$$

However the phase and group velocities are all bounded in the SW limit $k \to \infty$, which is a central point in this approach as indeed this very property allows the three models to sustain short waves. Then we face the problem of the nonlinear propagation of a SW, which is the object of this work. In the following, both nonintegrable systems (BBMP and $\phi^4$) will be displayed in details, while the integrable one (SG) will be only sketched.
The BBMP Model. Let us consider a SW in (1) characterized by $k = k_0 \epsilon^{-1}$ with $k_0 \sim O(1)$ and $\epsilon \ll 1$. The plane wave solution of the linear problem $u = \exp i \{kx - \omega(k)t\}$ inspire a fast variable $\zeta = \epsilon^{-1}x$ and infinitely many slow time variables $\tau_{2n+1} = \epsilon^{2n+1}t$ ($n = 0, 1, 2, \ldots$), by expanding $\omega$ in powers of $\epsilon$.

We assume the expansion

$$u = u_0 + \epsilon^2 u_2 + \epsilon^4 u_4 + \ldots$$

and suppose the extension $u_{2n} = u_{2n}(\zeta, \tau_1, \tau_3, \cdots)$, $n = 0, 1, \cdots$. Then, the operators

$$\frac{\partial}{\partial x} = \frac{1}{\epsilon} \frac{\partial}{\partial \zeta},$$

$$\frac{\partial}{\partial t} = \epsilon \frac{\partial}{\partial \tau_1} + \epsilon^3 \frac{\partial}{\partial \tau_3} + \epsilon^5 \frac{\partial}{\partial \tau_5} + \ldots$$

allow us to study the behavior of a short-wave for large time.

BBMP gives at orders $\epsilon^{-1}, \epsilon, \epsilon^3, \ldots$, the equations (written only up $\epsilon^3$)

$$-u_{0,\zeta\tau_1} + u_0 - 3u_0^2 = 0,$$

$$\hat{L}u_2 = u_{0,\tau_1} + u_{0,\zeta\tau_3},$$

$$\hat{L}u_4 = u_{2,\tau_1} - u_{0,\tau_3} + u_{2,\zeta\tau_3} + u_{0,\zeta\zeta\tau_5} + 3(u_0^2)\zeta,$$

where $\hat{L}$ is the linear operator associated with $\epsilon^3$:

$$\hat{L}(v) = -v_{\zeta\zeta\tau_1} + v_\zeta - 6(\nu u_0)\zeta.$$  

The unique solution of $\epsilon^3$ in the form $u_0(\eta)$ with $\eta = k_0 \zeta - \omega_1 \tau_1 - \omega_3 \tau_3 - \omega_5 \tau_5, \ldots$, going to zero for $|\zeta| \to \infty$ is

$$u_0 = \frac{1}{2} \operatorname{sech}^2 \eta, \quad \omega_1 = -\frac{1}{4k_0}.$$  

The values $\omega_3, \omega_5, \cdots$, corrections to the principal frequency $\omega_1$ (Stokes’ hypothesis) are still free but will be determined later by the non-secularity requirement.

The equation $\epsilon^3$ for $u_2$ then reads:

$$\hat{L}u_2 = \{4\omega_3 k_0^2 - \omega_1 - 12 \omega_3 k_0^2 \operatorname{sech}^2 \eta\} \operatorname{sech}^2 \eta \tanh \eta,$$

and its two first right hand side terms are resonant (secular producing terms) because $\epsilon^3$:

$$\hat{L}(\operatorname{sech}^2 \eta \tanh \eta) = 0.$$ 

These secular terms are eliminated by choosing

$$\omega_3 = \frac{\omega_1}{4k_0^2} = -\frac{1}{4^2 k_0^3}.$$ 

Hence, $\epsilon^3$ yields the solution $u_2(\eta) = 4^{-1}k_0^2 u_0(\eta)$.

The equation $\epsilon^4$ for $u_4(\eta)$ contains secular producing terms originated by the first four terms in the right hand side. They can be eliminated choosing $\omega_5 = -4^{-3}k_0^{-5}$.

The solution is $u_4(\eta) = 4^{-2}k_0^{-4} u_0(\eta)$. This procedure can be repeated at any higher order $n = 0, 1, 2, \ldots$ and we obtain recursively

$$u_{2n}(\eta) = \frac{u_0(\eta)}{4^n k_0^{2n}}, \quad \omega_{2n+1} = -\frac{1}{4^n + 1} 2n + 1.$$  

Next, nicely enough, not only the perturbative series solution $\epsilon$ can be summed to give

$$u(\eta) = u_0(\eta) \sum_{n=0}^{\infty} \frac{\epsilon^{2n}}{4^n k_0^{2n}} = \frac{4k_2^2}{k_2^2 - 1} u_0(\eta).$$  

but also, by using $\omega_{2n+1}$, its argument $\eta$ in the laboratory coordinates, which results as

$$\eta = k x + \frac{1}{4k_2} \sum_{n=0}^{\infty} \frac{t}{(4k_2)^n} = k x + \frac{k t}{4k_2 - 1}.$$  

Therefore, this SW perturbation technique finally leads to the solution

$$u(x, t) = -\frac{2k^2}{1 - 4k_2^2} \operatorname{sech}^2 [k(x + t/4k_2^2 - 1)].$$  

This very expression, solution of BBMP, was obtained in $\epsilon^3$ as an asymptotic limit of a LW of small amplitude. Thus, for $t \to \infty$, the nonlinear dynamics of a SW (with an order one amplitude) and that of a LW (with small amplitude) are indistinguishable in BBMP. The equation $\epsilon^3$ is a nonlinear Klein-Gordon equation which substitutes in this SW approach the classical Korteweg-de Vries of the LW approach.

The $\phi^4$ Model. The topological antikink type solution of $\phi^4$ will be obtained by perturbation expansion starting from the constant solutions $\phi_0 = \pm m / \sqrt{\lambda}$. Hence we seek a solution $\phi(\eta)$ such that $\phi \to \mp m / \sqrt{\lambda}$ for $\eta \to \pm \infty$.

For $\eta < 0$, the function $u = \phi - m / \sqrt{\lambda}$, goes to zero for $\eta \to -\infty$ and satisfies

$$u_{xx} - u_t = -2m^2 u - 3m \sqrt{\lambda} u^2 - \lambda u^3.$$  

For unidirectional propagation the convenient fast variable $\zeta$ and the slow variables $\tau_{2n+1}$ are in this case: $\zeta = \epsilon^{-1}(x - t)$, $\tau_1 = \epsilon t$, $\tau_3 = \epsilon^3 t, \cdots$. Expanding $u$ according to $u = \epsilon^0(u_0 + \epsilon^2 u_2 + \epsilon^4 u_4 + \cdots)$, the resulting equations are (up to $\epsilon^6$),

$$\hat{L}u_0 = 0,$$

$$\hat{L}u_2 = -2u_{0,\zeta\tau_3} + u_{0,2\tau_1} - 3m \sqrt{\lambda} u_0^2,$$

$$\hat{L}u_4 = u_{2,2\tau_1} - u_{2,\zeta\tau_3} - 2u_{0,\zeta\tau_5} + 2u_{0,\tau_1 \tau_3} - 6m \sqrt{\lambda} u_0^2 u_2 - \lambda u_4.$$
with \( \hat{L} \) in this case being the linear Klein-Gordon operator

\[
\hat{L}(v) = 2v_\tau + 2m^2 v
\]  

(23)

We choose for the solution \( u_0 \) of (20) the form \( u_0 = B \exp 2\eta \) with \( k = k_0 \zeta + (4k_0)^{-1}m^2 \tau_1 + \omega_3 \tau_3 + \cdots \) and \( B \) a constant. All linear terms at the right hand side of equations for \( u_{2(n-1)} \) are secular. They can be eliminated choosing appropriately \( \omega_{2n-1} \), namely

\[
\omega_{2n-1} = -\frac{(\frac{1}{2})!}{n! (\frac{1}{2} - n)!} \frac{m^{2n}}{2^n k_0^{2n-1}}.
\]

(24)

Next the solutions read

\[
u_{2(n-1)} = B^n (\frac{\sqrt{\lambda}}{2m})^{n-1} \exp 2n\eta.
\]

(25)

With these values of \( \omega_{2n-1} \) the series for \( \eta \) can again be summed as

\[
\eta = kx - \sqrt{k^2 + \frac{m^2}{2}} t,
\]

and also the perturbative series for \( u \) only if we choose

\[
B = -\frac{2m}{\sqrt{\lambda} \kappa_0^2}.
\]

It leads for \( \phi = u + m/\sqrt{\lambda} \) to

\[
\phi = -\frac{m}{\sqrt{\lambda}} \tanh \{ kx - \sqrt{k^2 + \frac{m^2}{2}} t - \log k \}.
\]

(26)

To get this expression it is necessary to use the Fourier representation \( x < 0 \)

\[
\sum_{n=0}^{\infty} (-1)^{n+1} \delta_n \exp (2nx) = \tanh x,
\]

where \( \delta_n \) are the Neumann’s numbers \( (\delta_0 = 1, \delta_n = 2, \forall n = 1,2,3,...) \).

The above solution \( \phi \) is the antikink solution of \( \phi^4 \) (with an initial shift \( \log k/k \), which has not been obtained previously within another perturbation scheme.

The expression \( \sqrt{k^2 + \frac{m^2}{2}} \) can be interpreted as a nonlinear frequency \( \omega_{nl} \), which defines the nonlinear group velocity

\[
v = \frac{\partial \omega_{nl}}{\partial k} = \frac{k}{\sqrt{k^2 + \frac{m^2}{2}}}. \]

(27)

It is remarkable that the Lorentz invariance of (26) is precisely related to that particular velocity, indeed

\[
\phi = -\frac{m}{\sqrt{\lambda}} \tanh \{ \frac{m}{\sqrt{2}} \left( \frac{xv}{\sqrt{1 - v^2}} - \frac{t}{\sqrt{1 - v^2}} \right) - \log k \}.
\]

(28)

Note that the case \( \eta > 0 \) in the perturbative series would simply yield the solution \( \phi(-\eta) \).

The SG Model. Finally in the case of the sine-Gordon model (3), for \( \phi = \phi_0 + \epsilon^2 \phi_2 + \epsilon^4 \phi_4 + \cdots \), with \( \phi_{2n} \) functions of \( \eta = k_0 \zeta + (2k_0)^{-1}m^2 \tau_1 + \omega_3 \tau_3 + \cdots \) where \( \zeta = \epsilon^{-1}(x - t), \tau_1 = \epsilon t, \tau_3 = \epsilon^3 t, \cdots \), we obtain (up to order \( \epsilon^3 \))

\[
\hat{L}(\phi_0) = 0,
\]

(29)

\[
\hat{L}(\phi_2) = -2\phi_0, \phi_2 \phi_3 + \phi_0,2\tau_1 - \frac{\lambda}{3!} \phi_0^3,
\]

(30)

\[
\hat{L}(\phi_4) = -2\phi_2, \phi_3 + \phi_2,2\tau_1 - 2\phi_0, \phi_3 + 2\phi_0, \tau_1 \tau_3 - \frac{\lambda}{3!} \phi_0^2 \phi_2 + \frac{\lambda^2}{2!} \phi_0^4 \phi_0 + \frac{m^2}{5!} \phi_0^5.
\]

(31)

with \( \hat{L} \) being in this case the operator

\[
\hat{L}(v) = 2v_\tau + m^2 v.
\]

(32)

We choose for the solution of (32) the expression \( \phi_0 = C \exp \eta \) with \( C \) a constant. As in the previous case all the linear terms at the right hand side of equations for \( \phi_{2(n-1)} \) are secular. They can be eliminated choosing \( \omega_{2(n-1)} \) as

\[
\omega_{2n-1} = (-1)^{n+1} \frac{(\frac{1}{2})!}{n! (\frac{1}{2} - n)!} \frac{m^{2n}}{2^n k_0^{2n-1}}.
\]

(33)

Hence the solutions \( \phi_{2(n-1)} \) read

\[
\phi_{2(n-1)} = -\frac{16}{\lambda^n} C^{2n-1} \exp (2n-1) \eta
\]

(34)

The series for \( \phi \) sums for \( C = 4m/\sqrt{\lambda} \kappa_0 \) and yields

\[
\phi = \frac{4m}{\sqrt{\lambda}} \sum_{n=0}^{\infty} (-1)^{n} \frac{\exp [(2n+1)(\eta - \log k)]}{2n + 1},
\]

(35)

In this case the Lorentz invariant form of (34) appears as a function of the nonlinear phase velocity \( v = \sqrt{1 - m^2/k^2} \) as

\[
\phi = \frac{4m}{\sqrt{\lambda}} \arctan \left\{ \exp \left[ \frac{m}{\sqrt{1 - v^2}} (x - vt) - \log k \right] \right\}.
\]

(36)

Conclusion and Comments We have applied a multiple-time version of the reductive perturbation method to study the solitary-wave and the kink-wave solutions of some nonlinear dispersive models. These solutions has already been known before. The alternative way gives here to obtain them shows that they represented short-waves asymptotic dynamics \( t \rightarrow \infty \).
short-waves while KdV amplify them. Then our result answers this old controversy on the relative relevance between KdV and BBMP. Actually we proved that short-waves do propagate nonlinearly in BBMP models, and build up soliton-like solutions as $t \to \infty$.

2 - The antikink (or kink) solution of $\phi^4$ model which cannot be obtained as a perturbative solution in $\lambda$, appears as a perturbative solution in $k$ in the short-wave limit.

3 - The eq. (35) shows that the kink solution of SG is obtainable only from a short-wave dynamics, as indeed, the limit $k \to 0$ gives rise to an imaginary argument.

4 - An initial profile generically contains short-wave components which are usually neglected in favor of the long-wave components, which occurs as well through numerical discretisation of the models. As we have shown that the short-wave components asymptotically build up soliton solutions, the common understanding of a soliton as originating from long-wave is to be questioned.

5 - It is worth noting finally that, within the long-wave approach, the nonlinear character of the solution is present already at the first orders, as indeed one usually finds the Boussinesq, KdV, MKdV, etc equations. This is not the case with the short-wave approach where usually all orders (i.e. all times) are necessary to unveil the nonlinear character of the solution.

Acknowledgements The authors wish to thank J. Leon and P. Grangé for many helpful and stimulating discussion. One of us (M.A.M) is indebted to R. A. Kraenkel and J. G. Pereira for fruitfull collaboration.

[1] T. Taniuti, C. C. Weil, J. Phys. Soc. Japan 24, 941 (1968).
[2] C. S. Su, C. S. Gardner, J. Math. Phys. 10, 536 (1969).
[3] Y. Kodama, T. Taniuti, J. Phys. Soc. Japan 45, 298 (1978).
[4] A. Jeffrey, T. Kawahara, in Asymptotic Methods in Nonlinear Wave Theory (Pitman Publishing, London, 1982).
[5] R.A. Kraenkel, M. A. Manna, V. Merle, J. C. Montero, J. G. Pereira, Physical Review E 54, 2976 (1996).
[6] G. Sandri, Nuovo Cimento. B36, 67 (1965).
[7] G. B. Whitham, in Linear and Nonlinear Waves, pp 471-476 (Wiley-Interscience, New York, 1974).
[8] R. A. Kraenkel, M. A. Manna, J. G. Pereira, J. Math. Phys. 36, 307 (1995).
[9] R. A. Kraenkel, M. A. Manna, J. C. Montero, J. G. Pereira, J. Math. Phys. 36, 6882 (1995).
[10] M. A. Manna, V. Merle, Preprint Montpellier 1996. [solv-int/9703006].
[11] T. B. Benjamin, J. L. Bona, J. J. Mahony, Philos. Trans. Roy. Soc. London A 272, 47 (1972).
[12] R. Rajaraman, in Solitons and Instantons. An Introduction to Solitons and Instantons in Quantum Field Theory (North-Holland, Amsterdam, 1982).
[13] A. H. Nayfeh, in Introduction to Perturbation Techniques, (Wiley-Interscience, New York, 1993).
[14] M. D. Kruskal, in Dynamical Systems, Theory and Applications, Lecture Notes in Physics 38, ed. by J. Moser (Springer-Verlag, Berlin, 1975).