The adjoint trigonometric representation of displacements and a closed-form solution to the IKP of general 3C chains

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Based on the representation of rigid body displacements as adjoint matrices, the article introduces the adjoint trigonometric representation of displacements (ATRD) as a further generalization of the trigonometric representation of rotations. In comparison to the dual Rodrigues–Euler–Gauß–Gelman equation, recently reported for affine screw displacements with arbitrary, fixed pitches, the ATRD is built upon a product of a unit line and a dual angle, instead of upon a product of a unit screw and a real angle.

Due to this conceptual difference, the ATRD requires four independent parameters of a unit line instead of five when parametrizing a displacement along a unit screw. As a consequence for computational kinematics, the ATRD permits transferring the analytic solution to the inverse kinematics problem (IKP) of 3-DOF, general, spherical 3R-chains into a closed-form solution to the IKP of 6-DOF, general, affine 3C-chains.

KEYWORDS
adjoint representation, dual number functions, inverse kinematics, line geometry, Plücker vectors, principle of transference, Rodrigues formula, screw theory

1 | INTRODUCTION

According to the displacement theorem by Mozzi and Chasles, any spatial displacement of a rigid body can be expressed as a rotation about one particular line in space in combination with a translation along that same line [1]. The specific line is the affine, oriented, one-dimensional subspace of the three-dimensional Euclidean space that remains invariant with respect to the displacement under consideration: all points on the line are again mapped to points on the line. The different motion types, rotations, translations, and helical motions, are unified within screw theory [1, 2] using the concept of specific pitches associated to a screw. In the theories of dual unit quaternions [3] and Study parameters [4], the parametrization of different motion types is closely related to the tuple of a dual half angle and a unit line [5]. The concept of dual quantities is also used in the representation of spatial displacements via dual (3 × 3)-matrices [6], in the parametrization of displacements by means of dual Euler angles and Yang–Sheht–Uicker parameters [7–9], in the interpretation of Plücker vectors as dual 3-vectors [6], and in the approach of distance geometry [10].

Employing (4 × 4)-homogeneous transformation matrices for working with rigid body displacements, the relations of the matrix coordinates to the parameters of the invariant line and its associated screw involve cubical and fractional expressions for general displacements [11–13]. Recently a generalization of the Rodrigues–Euler–Gauß–Gelman (REGG) rotation formula [13–20] in trigonometric form [16, 21] has been reported that demonstrates how the relations of the matrix coordinates and the
`geometric screw parameters' simplify if $(6 \times 6)$-adjoint matrices are employed in place of $(4 \times 4)$-homogeneous matrices to represent rigid body displacements [22]. The present article extends this previous work by expressing the adjoint representation of displacements via `geometric line parameters', as a product of an invariant line with a dual angle, in coherence to the analysis reported in [5].

The main contributions are two-fold: the adjoint trigonometric representation of displacements (ATRD) is reported as a novel, principal formula together with a proof. Based on this fundamental advancement, a closed-form solution to a problem of computational kinematics is derived: Applying the principle of transference [23–26] to the solution approach for the inverse kinematics problem (IKP) of general, spherical three-revolute (3R) chains with three degrees-of-freedom (DOF) and employing the ATRD, a novel analytic method for solving the IKP of 6-DOF general, affine three-cylindric (3C) chains is achieved. The method complements former approaches in which spatial 3C chains are analyzed as subchains in spatial closed RCCC loops [7, 27–30] and closed 3CCC parallel platforms [31]. In the field of kinematic analysis, spatial kinematic chains equipped with cylindric joints are of significant importance since they appear as subchains in other mechanisms [10] but also since can be regarded as ‘relaxed’ – in the sense of ‘sub-constrained’ – versions of all chains featuring ‘simple’ (revolute, prismatic, and/or helical) joints. In order to achieve these main contributions, the article further develops technical tools to analyze the geometry of lines in space: these include the concept of a dual directed angle between two oriented lines relative to a third, the definition of novel dual number functions, as well as a dual inner product for the $(6 \times 6)$ cross-matrix representation of oriented lines.

The article is arranged according to the following structure. In Section 2, technical preparations are prepared for the remainder of the document. In Section 3, a comprehensive overview of the trigonometric representation of rotations (TRR) is provided: the section introduces an inner product for the space of skew-symmetric $(3 \times 3)$-matrices, ‘cross-matrices’ for short, that matches the usual inner product of the corresponding vectors. Finally, the trigonometric formula by Rodrigues, Euler, Gauß, and Gelman (REGG) is stated together with a proof based on an analysis of the periodicity of the cross-matrix powers. The main theoretical contributions of the article are presented in Section 4. The principle of transference is applied to the TRR, yielding the adjoint trigonometric representation of displacements (ATRD), a certain generalization of the REGG-rotation formula. In the presented shape, the generalization employs the matrix algebra of real $(6 \times 6)$-adjoint matrices and avoids the usage of matrix algebra over dual numbers. Coarsely, the proceeding follows the structure of the preceding section. In Section 5, two auxiliary functions for dual numbers are introduced: the dual bivariate inverse tangent function, for computing the two dual solutions of a dual trigonometric equation, and the dual trivariate inverse cosine function, for computing the dual relative directed angle between two lines in space with respect to a third. The main practical contribution of the article is presented in Section 6. The inverse kinematics problem of general affine 3C chains is formulated in the adjoint representation. Based on the ATRD and the two dual trigonometric functions, a novel closed-form solution is developed. The IK computations are illustrated for an example instance of a general spatial 3C chain. In Section 7, the structural geometric and algebraic properties of the TRR formula and of the ATRD formula are briefly reflected. Finally, the article is concluded in Section 8 with an outline of its contributions and potential continuations. A notation overview, further technical details, and an alternate parametrization via Cayley maps are compiled in an supplementary appendix.

## 2 PREREQUISITES

In this section, a set of definitions and conventions is compiled in preparation of the main parts of the document. The orientation of a set of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ is defined as the trivariate function

$$\text{ortn}^3(\mathbf{a}, \mathbf{b}, \mathbf{c}) := \text{sgn}(\det(\mathbf{a} \mathbf{b} \mathbf{c})).$$

(1)

A normalized vector is emphasized by using a hat. In particular, a unit vector, a vector with length one, is indicated as $\mathbf{a} = \frac{1}{\|\mathbf{a}\|} \cdot \mathbf{a}$. The operator $\ast$ expresses the combination of transposition and multiplication. In particular $\mathbf{a} \ast \mathbf{b} := \mathbf{a}^T \cdot \mathbf{b}$ denotes the inner product of two vectors and $\mathbf{A} \ast \mathbf{B} := \mathbf{A}^T \cdot \mathbf{B}$ denotes its matrix generalization.\[^1\] The operator $\otimes$ expresses the outer product, for example, as $\mathbf{a} \otimes \mathbf{b} := \mathbf{a} \cdot \mathbf{b}^T$, determining the dyadic tensor. A Plücker vector\[^2\] is a six-dimensional vector ($\mathbf{a}^6$), consisting firstly of a direction $\mathbf{n}$, and secondly of a moment $\mathbf{m}$, in the so-called ray-coordinate order [2]. An oriented line (spear) in 3D-space

\[^1\]The scalar $c = \mathbf{a} \ast \mathbf{b}$ and the matrix $\mathbf{C} = \mathbf{A} \ast \mathbf{B}$ represent cosine similarities for (matrices with columns of) normalized vectors.

\[^2\]A ‘Plücker vector’ is also called a ‘motor’ [32]: ‘Dem Kundigen mag der Hinweis genügen, daß der Motor in gewissem Sinn als der Sechsvorrichtung der reinen vierdimensionalen Liniengeometrie erscheint.’
in form of a Plücker vector is briefly denoted here as $\Lambda = (^{\star}_a)$; its moment $\mathbf{m}$ is defined by the cross product of an arbitrary, fixed point of the line, an anchor $\mathbf{a}$, with the direction of the line, as $\mathbf{m} = \mathbf{a} \times \mathbf{n}$. A unit Plücker vector is a Plücker vector that features either $\|\mathbf{n}\| = 1$, or $\|\mathbf{m}\| = 1$ and $\|\mathbf{n}\| = 0$; here indicated by a hat, as $\hat{\Lambda} = (^{\star}_a)$. The corresponding unit Plücker vector $\hat{\omega}_a$ associated to a Plücker vector $\omega_a$ is obtained by means of the ‘normalization function’ $\eta$, with the Plücker vector length $l := \{\|\mathbf{n}\| \neq 0, \|\mathbf{m}\| \neq 0\}$ [22] as

$$\eta: \begin{pmatrix} n \\ m \end{pmatrix} \mapsto \begin{pmatrix} \hat{n} \\ \hat{m} \end{pmatrix}, \quad \eta\left( \begin{pmatrix} n \\ m \end{pmatrix} \right) := \frac{1}{l} \cdot \begin{pmatrix} n \\ m \end{pmatrix}$$

In particular, the two parts of a unit screw are the unit direction $\hat{\mathbf{n}} = \frac{1}{l} \cdot \mathbf{n}$ and the normalized moment $\hat{\mathbf{m}} = \frac{1}{l} \cdot \mathbf{m}$. A screw in 3D-space in form of a unit Plücker vector is briefly denoted as $\hat{\Sigma} = (^{\star}_a)$. A Plücker vector is expressed in form of a dual 3-vector $\hat{\Sigma} = \mathbf{n} + \varepsilon \cdot \mathbf{m}$ (Appendix B), instead of the form as a real 6-vector. The $(4 \times 4)$-cross-matrix form of a unit screw $\hat{\Sigma} \cong (\hat{\mathbf{n}}, \hat{\mathbf{m}})$ reads $\hat{\Sigma}^\circ = (\hat{\mathbf{n}}^\circ \hat{\mathbf{m}}^\circ)$. The homogeneous matrix displacement $D = (R_0^t)$ is computed as the matrix exponential of the product of an angle $\phi$ and the $(4 \times 4)$-cross-matrix form $\hat{\Sigma}^\circ$ of a unit screw as

$$D = \text{dsp}(\phi; \hat{\Sigma}) = \exp(\phi \cdot \hat{\Sigma}^\circ) = \exp\left( \phi \cdot \begin{pmatrix} \hat{\mathbf{n}}^\circ \\ \hat{\mathbf{m}}^\circ \end{pmatrix} \right)$$

for $\phi \in \mathbb{R}$, $\phi \neq 0$; here indicated by a hat, as $\hat{\Sigma}$. A finite twist, representing a spatial displacement, is obtained as product of an angle with a unit screw, at $\hat{\Sigma} = \phi \cdot \hat{\Sigma}$. A Plücker vector is expressed in form of a dual 3-vector $\hat{\Sigma} = \mathbf{n} + \varepsilon \cdot \mathbf{m}$ (Appendix B), instead of the form as a real 6-vector. The $(4 \times 4)$-cross-matrix form of a unit screw $\hat{\Sigma} \cong (\hat{\mathbf{n}}, \hat{\mathbf{m}})$ reads $\hat{\Sigma}^\circ = (\hat{\mathbf{n}}^\circ \hat{\mathbf{m}}^\circ)$. The homogeneous matrix displacement $D = (R_0^t)$ is computed as the matrix exponential of the product of an angle $\phi$ and the $(4 \times 4)$-cross-matrix form $\hat{\Sigma}^\circ$ of a unit screw as

$$D = \text{dsp}(\phi; \hat{\Sigma}) = \exp(\phi \cdot \hat{\Sigma}^\circ) = \exp\left( \phi \cdot \begin{pmatrix} \hat{\mathbf{n}}^\circ \\ \hat{\mathbf{m}}^\circ \end{pmatrix} \right)$$

Finite-sum expressions, that involve cubical and fractional expressions for general displacements, are, for instance, stated in [11–13]. For a given homogeneous displacement matrix $D = (R_0^t)$, the components $R$ and $t$ are retrieved, by means of the projection operators $\Pi_R = (I_4^0)$, $\Pi_T = (0^1)$, via the matrix products

$$R = \Pi_R(D) = \Pi_R \ast D \ast \Pi_R$$
$$t = \Pi_T(D) = \Pi_T \ast D \ast \Pi_T$$

(4)

The $(6 \times 6)$-left adjoint matrix of a displacement [13, 22] is denoted in the sequel briefly as $\dot{D} = \text{Ad}(D)$. In correspondence to Equation (3), the adjoint matrix is given as

$$\dot{D} = \text{dasp}(\phi; \hat{\Sigma}) = \exp(\phi \cdot \hat{\Sigma}^\circ) = \exp\left( \phi \cdot \begin{pmatrix} \hat{\mathbf{n}}^\circ \\ \hat{\mathbf{m}}^\circ \end{pmatrix} \right)$$

where $t^\circ$ expresses the skew-symmetric matrix (Appendix C) associated to $t$, the translation vector. A simplified finite-sum expression has recently been reported in [22] for the adjoint representation. The dual inner product of two screws, given as Plücker vectors $\Sigma_A$ and $\Sigma_B$, is defined as

$$\Sigma_A \otimes \Sigma_B = \left( n_A \atop m_B \right) \otimes \left( n_A \atop m_B \right) = n_A \ast n_B + \varepsilon \cdot (n_A \ast m_B + m_A \ast n_B),$$

and is typically deduced from the interpretation of a Plücker vector as dual vector $\Sigma = n + \varepsilon \cdot m$ and from the definition of dual multiplication (compare Equation (46) in Appendix B). For unit lines $\hat{\Lambda}_A$ and $\hat{\Lambda}_B$, the dual inner product of Equation (5) matches the ‘dual cosine similarity’

$$\hat{c}_{AB} = \hat{\Lambda}_A \otimes \hat{\Lambda}_B = \left( \hat{n}_A \atop \hat{m}_B \right) \otimes \left( \hat{n}_A \atop \hat{m}_B \right) = \cos \theta_{AB} - \varepsilon \cdot d_{AB} \cdot \sin \theta_{AB},$$

3 Given a line in form of a Plücker vector $\Lambda$, the anchor $\mathbf{a}$ can generally not be retrieved. The point $\mathbf{a}^\star = (\mathbf{n} \times \mathbf{m}) / (\mathbf{n} \cdot \mathbf{n})$ is the point of $\Lambda$ that features the minimal distance to the origin/the least norm.

4 In the present article, the adjoint representation is not expressed via the ‘arguments’ of an angle $\phi$ and a unit twist $\hat{\Sigma}$ but via the ‘arguments’ of a dual angle $\hat{\phi}$ and a unit line $\hat{\Lambda}$. This reparametrization is the origin for the formulation of the adjoint trigonometric representation of displacements in Section 4 and its consequences.
the dual cosine (Appendix B) of the dual angle $\hat{\theta}_{AB} = \theta_{AB} + \epsilon \cdot d_{AB}$ which describes the rotational offset, $\theta_{AB}$, and translational offset, $d_{AB}$, of the two lines $\hat{\ell}_A$ and $\hat{\ell}_B$ in space. Applying the dual inverse cosine function (Appendix B) to the dual cosine similarity $\hat{\varphi}_{AB}$, as

$$| \hat{\varphi}_{AB} | = \hat{\mathrm{acos}}(\hat{\varphi}_{AB}) = \hat{\mathrm{acos}}(\epsilon_{AB} + \epsilon \cdot \hat{\varphi}_{AB}) = \hat{\mathrm{acos}}(\epsilon_{AB}) + \epsilon \cdot \frac{-\hat{\varphi}_{AB}}{\sin(\hat{\mathrm{acos}}(\epsilon_{AB}))},$$

the dual absolute value (Appendix B) of the dual angle $| \hat{\varphi}_{AB} | = \hat{\mathrm{acos}}(\hat{\ell}_A \oplus \hat{\ell}_B)$ between the two unit lines is obtained. The operation extends the primal inverse cosine function that determines the absolute value of the angle between two unit vectors, $| \varphi_{AB} | = \mathrm{acos}(\hat{n}_A \ast \hat{n}_B)$.\(^5\)

### 3 | TRIGONOMETRIC REPRESENTATION OF ROTATIONS

This section introduces the rotation formula by Rodrigues–Euler–Gauß in the trigonometric form, documented by Gelman [21], via three mutually reciprocal base matrices. All three matrix tensors are associated to a ‘rotation 3-vector’ and are equipped with dedicated names and operator symbols due to their geometric and algebraic significance. A supplementary overview is provided with Table 5 in the appendix.

**Orthogonal base matrices**

In accordance with [16, 21, 33], three $(3 \times 3)$-matrices, the cross-matrix $a^\otimes$, the unit-matrix $a^\odot$, and the square-matrix $a^\oslash$, associated to a vector $a \in \mathbb{R}^3$ are defined as

$$a^\otimes := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}, \quad a^\odot := -a^\otimes \cdot a^\otimes, \quad a^\oslash := (a \ast a) \cdot I - a^\odot. \quad (6)$$

The unit-matrix and the square-matrix can be characterized as ‘product matrices’ of ‘inner type’ and of ‘outer type’, respectively, as it can be observed by the identities

$$a^\odot = a^\odot \ast a^\odot, \quad a^\oslash = a \oslash a. \quad (7)$$

The product-character of the two matrices is also observed by their sum: the two matrices add up to the ‘inner product in diagonal form’:

$$I \cdot (a \ast a) = a^\otimes + a^\oslash. \quad (8)$$

Geometrically, the three matrices serve as particular operators if applied for unit vectors in matrix-vector products [22]: the projection vector, the rejection vector, and the orthogonal vector of a vector $b$ with respect to a unit vector $\hat{n}$ are obtained as

$$\pi(b; \hat{n}) = \hat{n}^\otimes \cdot b, \quad \tau(b; \hat{n}) = \hat{n}^\odot \cdot b, \quad \xi(b; \hat{n}) = \hat{n}^\oslash \cdot b. \quad (9)$$

An example for the application as geometric operators is given in Figure 1. Algebraically, the three matrices are rank-deficient for $a \neq 0$ with

$$\text{rank}(a^\otimes) = 2, \quad \text{rank}(a^\odot) = 2, \quad \text{rank}(a^\oslash) = 1.$$

**Matrix inner product**

For the scope of this article, the Lie algebra of $(3 \times 3)$-matrices is equipped with the particular inner product

$$\langle A, B \rangle_{33} := \frac{1}{2} \cdot \text{tr}(A \ast B). \quad (10)$$

\(^5\) A method for extracting the corresponding unit line (four parameters) from a given finite screw (six parameters) is described in Equation (25) and in Equation (31).

\(^6\) The directed angle between vectors and between lines is determined in Section 5 (Equation (39) and Equation (42)).
The projection vector \( \pi(b; \hat{n}) \) (red), the rejection vector \( \tau(b; \hat{n}) \) (green), and the orthogonal vector \( \chi(b; \hat{n}) \) (blue) of a vector \( b = \frac{1}{10} \cdot (6, 0, 8)^T \) (gray) with respect to the axis \( \hat{n} = (0, 0, 1)^T \). The numerical values are \( \pi(b; \hat{n}) = \frac{1}{10} \cdot (0, 0, 8)^T \), \( \tau(b; \hat{n}) = \frac{1}{10} \cdot (6, 0, 0)^T \), \( \chi(b; \hat{n}) = \frac{1}{10} \cdot (0, 6, 0)^T \).

The product \( \langle A, B \rangle_{33} \) connects the matrix algebra consistently to the typical inner product of vector algebra. In particular, the matrix inner product of two cross-matrices, \( a^\otimes \) and \( b^\otimes \), matches (based on Equation 53) the usual vector inner product: \[ \langle a^\otimes, b^\otimes \rangle_{33} = a \ast b. \] (11)

The three matrices of Equation (6) are pairwise reciprocal, as it can be read-off the equations

\[ \langle a^\otimes, a^\otimes \rangle_{33} = 0 \quad \langle a^\otimes, a^\circ \rangle_{33} = 0 \quad \langle a^\circ, a^\circ \rangle_{33} = 0. \]

The inner product of Equation (10) relates the squared norm of a cross-matrix consistently to the squared norm of a vector (via Equation 11)

\[ \|a^\otimes\|^2 = a \ast a = \|a\|^2. \] (12)

The squared norms of the three base matrices (Equation 6) for unit vectors correspond to

\[ \|\hat{n}^\otimes\|^2 = \langle \hat{n}^\otimes, \hat{n}^\otimes \rangle_{33} = 1 \quad \|\hat{n}^\circ\|^2 = \langle \hat{n}^\circ, \hat{n}^\circ \rangle_{33} = 1 \quad \|\hat{n}^\circ\|^2 = \langle \hat{n}^\circ, \hat{n}^\circ \rangle_{33} = \frac{1}{2}. \]

**Periodic powers**

For the positive powers of the unit vector’s cross matrix \( \hat{n}^\otimes \), the recursive relations

\[ (\hat{n}^\otimes)^{k \pm 4} = (\hat{n}^\otimes)^k \quad (\hat{n}^\otimes)^{k \pm 2} = -(\hat{n}^\otimes)^k \]

hold. Based on this observation, the generalized powers (for integer exponents) for a unit vector’s cross-matrix are defined as

\[ (\hat{n}^\otimes)^k := \begin{cases} + \hat{n}^\otimes & k \text{ mod } 4 = 0 \\ + \hat{n}^\circ & k \text{ mod } 4 = 1 \\ - \hat{n}^\otimes & k \text{ mod } 4 = 2 \\ - \hat{n}^\circ & k \text{ mod } 4 = 3, \end{cases} \] (13)

the augmented definition reflects the fact that the cross-matrix geometrically corresponds to a rotation of \( \frac{\pi}{2} \) within the plane \( \hat{n}^\perp \). The cyclic relation is illustrated in Figure 2. As the unit matrix \( \hat{n}^\otimes \) and the square-matrix \( \hat{n}^\circ \) represent projections, they are idempotent, i.e., the powers equal the matrices themselves

\[ (\hat{n}^\otimes)^k := \hat{n}^\otimes \quad (\hat{n}^\circ)^k := n^\circ. \] (14)

---

\( ^7 \) For the unit-matrix and the square matrix, the inner products are \( \langle a^\otimes, b^\otimes \rangle_{33} = \frac{1}{2} \cdot (3 \cdot (a \ast b) - 2 \cdot (a \ast b)^2 + (a \ast a) \cdot (b \ast b)) \) and \( \langle a^\otimes, b^\circ \rangle_{33} = \frac{1}{2} \cdot (3 \cdot (a \ast b) - 2 \cdot (a \ast b)^2) = \langle a^\otimes, b^\otimes \rangle_{33} - \frac{1}{2} \cdot (a \ast a) \cdot (b \ast b). \)
Trigonometric rotation formula

The proper orthogonal matrix \( R \) for a rotation by an angle \( \phi \) about an axis \( \mathbf{n} \), is determined as the trigonometric representation \([16, 21, 33]\) of rotations as

\[
R = \text{rot}(\phi, \mathbf{n}) = \exp(\phi \cdot \mathbf{n}^{\otimes}) = \cos(\phi) \cdot \mathbf{n}^{\otimes} + \sin(\phi) \cdot \mathbf{n}^{\otimes} + \mathbf{n}^{\otimes}.
\]  

(15)

In terms of the introduced \((3 \times 3)\) base matrices, the rotation matrix is obtained as an affine-trigonometric combination. As an alternative to the exponential map, the rotation matrix \( R \) is obtained via the Cayley map for the argument 'tan(\(\phi/2\)) \cdot \mathbf{n}^{\otimes}' (Rodrigues’ vector) in Appendix D. A formal proof for the trigonometric representation of rotation in Equation (15) is obtained by applying Taylor series expansion

\[
\exp(\phi \cdot \mathbf{n}^{\otimes}) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (\phi \cdot \mathbf{n}^{\otimes})^k
\]

\[
= I + \left( \sum_{k=1}^{\infty} \frac{1}{(2k)!} \cdot \phi^{2k} \cdot (\mathbf{n}^{\otimes})^{2k} \right) + \left( \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} \cdot \phi^{2k+1} \cdot (\mathbf{n}^{\otimes})^{2k+1} \right)
\]

\[
= (\mathbf{n}^{\otimes} + \mathbf{n}^{\otimes}) + \mathbf{n}^{\otimes} \cdot \left( \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \cdot \phi^{2k} \right) + \mathbf{n}^{\otimes} \cdot \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot \phi^{2k+1} \right)
\]

\[
= \mathbf{n}^{\otimes} + (\cos \phi) \cdot \mathbf{n}^{\otimes} + (\sin \phi) \cdot \mathbf{n}^{\otimes},
\]

and employing the periodicity of the powers of the cross-matrix (Equation 13). By means of the trigonometric representation of rotations, the invariance \( R \cdot \mathbf{n} = \mathbf{n} \) follows immediately

\[
R \cdot \mathbf{n} = \cos(\phi) \cdot \mathbf{n}^{\otimes} * \mathbf{n}^{\otimes} \cdot \mathbf{n} + \sin(\phi) \cdot \mathbf{n}^{\otimes} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{n} * \mathbf{n} = \mathbf{n},
\]  

(16)

using the cross product identity \( \mathbf{n}^{\otimes} \cdot \mathbf{n} = \mathbf{n} \times \mathbf{n} = 0 \) and the inner product identity \( \mathbf{n} * \mathbf{n} = 1 \).

4 ADJOINT TRIGONOMETRIC REPRESENTATION OF DISPLACEMENTS

The rotation formula by Rodrigues, Euler, Gauß, and Gelman is generalized in this section for spatial displacements in a particular manner. As previously, the formula is stated by means of three mutually reciprocal base matrices. Each of the three matrix tensors
is associated to a ‘screw Plücker 6-vector’ and here equipped – due to its geometric and algebraic significance – with a dedicated name and symbol. Supplementary overviews are provided with Figure 12 and Table 5 in the appendix.

**Left adjoint representation**

The left adjoint representation of a displacement, with homogeneous matrix $D = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$ of a rotation matrix $R$ and a translation vector $t$, has the shape $\hat{D} = \text{Ad}(D) = \begin{pmatrix} R & 0 \\ t^* & R \end{pmatrix}$. The $(6 \times 6)$-left adjoint matrix is defined as that operator which transforms the Plücker vector (of a screw) in correspondence to the left adjoint action of a homogeneous displacement on the homogeneous representation (of the screw), formally as

$$
\begin{pmatrix}
\begin{array}{c}
n_B \\
m_B
\end{array}
\end{pmatrix} = \hat{D} \cdot \begin{pmatrix}
\begin{array}{c}
n_A \\
m_A
\end{array}
\end{pmatrix} \Leftrightarrow \begin{pmatrix}
\begin{array}{c}
n_B^\oplus \\
m_B^\oplus
\end{array}
\end{pmatrix} = D \cdot \begin{pmatrix}
\begin{array}{c}
n_A^\oplus \\
m_A^\oplus
\end{array}
\end{pmatrix} \cdot D^{-1}
$$

The main purpose of the adjoint representation in this article is not to serve as ‘a matrix operator for the adjoint action on Plücker vectors’, but as ‘a matrix representation of rotations and translations for rigid bodies’. In this context, the adjoint matrix representation can be regarded as an alternative to the homogeneous matrix representation (see Figure 4 at the end of this section). The adjoint matrix representation can also be further seen as the ‘real manifestation’ of the displacement representation using dual (3×3)-matrices which has been employed in spatial kinematic analysis in the past [6, 8, 34, 35].

**Primal and dual unit matrices**

For emphasizing the close relationship between the adjoint matrix representation and the dual matrix representation, the ‘primal-unit-matrix’ $t$ and the ‘dual-unit-matrix’ $e$ are defined as

$$
t := \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad (17)
$$
in analogy to the definition of dual numbers (Section B). The matrices $t$ and $e$ serve as ‘primal-dual unit matrices’ to express the left adjoint matrix representation conveniently. Via $t$ and $e$, and by means of the lifting map $\Gamma(A) = \Gamma_1 \ast A \cdot \Gamma_1 + \Gamma_2 \ast A \cdot \Gamma_2$, with auxiliary matrices $\Gamma_1 = (I \ 0) \in \mathbb{R}^{3 \times 6}$ and $\Gamma_2 = (0 \ I) \in \mathbb{R}^{3 \times 6}$, the adjoint representation reads as

$$
\hat{D} = (t + e \cdot \Gamma(t^\otimes)) \cdot \Gamma(R), \quad (18)
$$
in terms of its ‘constituting’ components, the rotation matrix $R$ and the translation vector $t$. The product-shaped decomposition based on $t$ and $e$ in Equation (18) is comparable to the expression of Study parameters in terms of the rotational and the translational parameters [4].

Similarly to Equation (4), for a given adjoint displacement matrix $\hat{D}$, the components $R$ and $t$ are formally retrieved, by means of the projection operators $\hat{\Pi}_R = (\rho_6) \in \mathbb{R}^{6 \times 3}$ and $\hat{\Pi}_T = (\rho_3) \in \mathbb{R}^{6 \times 3}$, via the matrix products

$$
R = \hat{\Pi}_R(\hat{D}) = \hat{\Pi}_R \ast \hat{D} \cdot \hat{\Pi}_R,
\quad t = \hat{\Pi}_T(\hat{D}) = ((\hat{\Pi}_T \ast \hat{D} \cdot \hat{\Pi}_R) \cdot R^T)^\otimes. \quad (19)
$$
The operations for obtaining $R$ and $t$ from a homogeneous and from an adjoint displacement matrix in Equation (4) and Equation (19) are subsumed in a schematic overview within the upper part of Figure 4 in Section 4.

**Dual number matrix form**

A dual number $\tilde{x} = x + e \cdot \hat{x}$ is lifted into the form of a compatible $(6 \times 6)$-matrix by substituting the real unit by the primal unit matrix $t$ and the dual unit matrix $e$. The embedding is formalized by the operator $\overline{\cdot}$ defined as

$$
\overline{\cdot} : \mathbb{R} \rightarrow \mathbb{R}^{6 \times 6}, \quad \tilde{x} = x + e \cdot \hat{x} \mapsto \overline{(\tilde{x})} = t \cdot x + e \cdot \hat{x} = \begin{pmatrix} x \cdot I & 0 \\ \hat{x} \cdot I & x \cdot I \end{pmatrix}. \quad (20)
$$

For the remaining parts of this section, the matrix expressions

$$
\overline{\varphi} = \begin{pmatrix} \cos \varphi \cdot I & 0 \\ -s \cdot \sin \varphi \cdot I & \cos \varphi \cdot I \end{pmatrix}, \quad (\overline{\cos \varphi}) = \begin{pmatrix} \cos \varphi \cdot I & 0 \\ -s \cdot \sin \varphi \cdot I & \cos \varphi \cdot I \end{pmatrix}, \quad (\overline{\sin \varphi}) = \begin{pmatrix} \sin \varphi \cdot I & 0 \\ +s \cdot \cos \varphi \cdot I & \sin \varphi \cdot I \end{pmatrix}
$$
corresponding to a dual angle and to its dual trigonometric functions (Equation 50) are stated explicitly.
Adjoint orthogonal base matrices

As affine line generalizations to the three \((3 \times 3)\)-matrices defined in Equation (6), three \((6 \times 6)\)-matrices are stated in accordance with [22]. The cross-matrix \(\otimes\), the unit-matrix \(\boxcircle\), and the square-matrix \(\boxslash\), associated to a six-dimensional Plücker vector \(\mathbf{s} = (n \, m)\), are defined as

\[
\begin{align*}
\begin{pmatrix} n \\ m \end{pmatrix} \otimes & := \begin{pmatrix} n^\otimes & 0 \\ m^\otimes & n^\otimes \end{pmatrix} \\
\begin{pmatrix} n \\ m \end{pmatrix} \boxcircle & := - \begin{pmatrix} n \\ m \end{pmatrix} \cdot \begin{pmatrix} n \\ m \end{pmatrix} \\
\begin{pmatrix} n \\ m \end{pmatrix} \boxslash & := \mathbf{i} \cdot (n \times n) - \begin{pmatrix} n \\ m \end{pmatrix}.
\end{align*}
\]

(21)

For sake of a compact description, the matrix product \(a \otimes b\) is used as an abbreviation for the sum of two ‘twisted’ matrix products as

\[
a \otimes b := b^\otimes \ast a^\otimes + a^\otimes \ast b^\otimes.
\]

(22)

By means of this symmetric product (further properties are stated in Section C), two equalities

\[
\begin{align*}
\begin{pmatrix} n \\ m \end{pmatrix} \boxcircle = & \begin{pmatrix} n \otimes 0 \\ n \otimes m \end{pmatrix} \\
\begin{pmatrix} n \\ m \end{pmatrix} \boxslash = & \begin{pmatrix} -n \otimes m \otimes n \otimes m \end{pmatrix},
\end{align*}
\]

are stated as line-generalization of the two vector-identities in Equation (7). In terms of the primal-dual unit matrices \(\iota\) and \(\epsilon\) from Equation (17), the line cross-matrix reads

\[
\begin{pmatrix} n \\ m \end{pmatrix} \otimes = \iota \cdot \Gamma(n^\otimes) + \epsilon \cdot \Gamma(m^\otimes).
\]

(23)

The line unit-matrix and the line square-matrix are expressed in terms of the primal-dual unit matrices and of the symmetric matrix product from Equation (22) as

\[
\begin{align*}
\begin{pmatrix} n \\ m \end{pmatrix} \boxcircle = & \iota \cdot \Gamma(n^\otimes) + \epsilon \cdot \Gamma(n \otimes m) \\
\begin{pmatrix} n \\ m \end{pmatrix} \boxslash = & \iota \cdot \Gamma(n^\otimes) - \epsilon \cdot \Gamma(n \otimes m).
\end{align*}
\]

Geometrically, the three matrices, \(\otimes\), \(\boxcircle\), and \(\boxslash\), serve as geometric operators for Plücker vectors if applied in matrix-vector multiplications [2]: the line projection, the line rejection, and the line orthogonal of a line \(\Lambda_B\) with respect to a unit line \(\hat{\Lambda}_N\) are obtained as

\[
\begin{align*}
\pi(\Lambda_B; \hat{\Lambda}_N) = & \hat{\Lambda}_N^\otimes \cdot \Lambda_B \\
\tau(\Lambda_B; \hat{\Lambda}_N) = & \hat{\Lambda}_N^\boxcircle \cdot \Lambda_B \\
\chi(\Lambda_B; \hat{\Lambda}_N) = & \hat{\Lambda}_N^\boxslash \cdot \Lambda_B.
\end{align*}
\]

(24)

The obtained results from of the three line operations, generally involve non-zero pitches, indicated in compact notation as \(\pi_{\text{BN}} = \pi(\Lambda_B; \hat{\Lambda}_N)\), \(\tau_{\text{BN}} = \tau(\Lambda_B; \hat{\Lambda}_N)\), and \(\chi_{\text{BN}} = \chi(\Lambda_B; \hat{\Lambda}_N)\). For deriving the corresponding line vectors, satisfying the orthogonality of moment and direction, \(m \times n = 0\), the ‘alignment function’ \(\lambda: \mathbf{s} \mapsto \Lambda\), is defined in accordance with [22] as

\[
\lambda(\mathbf{s}) = \lambda\left(\begin{pmatrix} n \\ m \end{pmatrix}\right) = \left(n, m - \frac{n \times m \cdot n}{n \times n} \cdot n\right)^T = \Lambda
\]

(25)

8 In other terms, the line orthogonal is also obtained by means of the ‘motor product’ [32] and by means of the ‘Lie bracket’ [36].
holds as a generalization of Equation (11). For unit lines, the expression resembles the dual-cosine similarity

\[ \langle \tau(\mathbf{A}; \mathbf{\hat{N}}) \rangle \]

for the scope of this article, the Lie algebra of

\[ \mathfrak{Ad}(\mathbb{R}^3) \]

Adjoint matrix inner product

\[ \langle \mathbf{A}, \mathbf{B} \rangle_{66} := \langle \mathbf{A} \rangle_{66} \cdot \langle \mathbf{B} \rangle_{66} \]

\[ \langle \mathbf{A}, \mathbf{B} \rangle_{33} + \langle \mathbf{A}, \mathbf{B} \rangle_{33} \]

\[ \frac{1}{2} \cdot \text{tr}(\mathbf{A} \ast \mathbf{B}) + \frac{1}{2} \cdot (\text{tr}(\mathbf{A} \ast \mathbf{B}) + \text{tr}(\mathbf{A} \ast \mathbf{B})) \]

generalizing the inner product for (3 × 3)-matrices introduced in Equation (10). The adjoint matrix inner product \( \langle \mathbf{\hat{A}}, \mathbf{\hat{B}} \rangle_{66} \) is consistent to the usual inner product of two screws: for two cross-matrices, \( \mathbf{S}^\ominus_A \) and \( \mathbf{S}^\ominus_B \) of Plücker vectors \( \mathbf{S}_A \) and \( \mathbf{S}_B \), the adjoint matrix inner product evaluates via Equation (53) to

\[ \langle \mathbf{S}^\ominus_A, \mathbf{S}^\ominus_B \rangle_{66} = \mathbf{n}_A \ast \mathbf{n}_B + e \cdot (\mathbf{n}_A \ast \mathbf{m}_B + \mathbf{m}_A \ast \mathbf{n}_B) \]

identical to Equation (5). In compact form, the identity

\[ \langle \mathbf{S}^\ominus_A, \mathbf{S}^\ominus_B \rangle_{66} = \mathbf{S}_A \otimes \mathbf{S}_B \]

holds as a generalization of Equation (11). For unit lines, the expression resembles the dual-cosine similarity \( \langle \mathbf{\hat{A}}_p, \mathbf{\hat{A}}_Q \rangle_{66} \) from Equation (5), \( \cos \theta - e \cdot d \cdot \sin \theta \), of the rotational offset \( \theta \) and the translational offset \( d \) between two lines in space, \( \mathbf{\hat{A}}_p \) and \( \mathbf{\hat{A}}_Q \). By means of the adjoint matrix inner product, the pairwise reciprocity of the three base matrices is observed with

\[ \langle \mathbf{A}^\ominus, \mathbf{A}^\ominus \rangle_{66} = 0 \quad \langle \mathbf{A}^\ominus, \mathbf{A}^\ominus \rangle_{66} = 0 \quad \langle \mathbf{A}^\ominus, \mathbf{A}^\ominus \rangle_{66} = 0 \]

The squared norm of a cross-matrix is determined with Equation (28) as

\[ \| \mathbf{S}^\ominus_A \|^2 = \langle \mathbf{S}^\ominus_A, \mathbf{S}^\ominus_A \rangle_{66} = \mathbf{n}_A \ast \mathbf{n}_A + e \cdot 2 \cdot \mathbf{n}_A \ast \mathbf{m}_A \]
In consistency to the norms of Plücker vectors, the squared norm for a cross-matrix of a unit line \( \hat{\mathbf{A}} \) is evaluated to \( \| \hat{\mathbf{A}} \|^2 = 1 \). For all three base matrices of a unit line, the squared norms are determined with

\[
\| \hat{\mathbf{A}} \|^2 = \left( \hat{\mathbf{A}} \cdot \hat{\mathbf{A}} \right)_{66} = 1
\]

\[
\| \hat{\mathbf{A}} \|^2 = \left( \hat{\mathbf{A}} \cdot \hat{\mathbf{A}} \right)_{66} = 1
\]

\[
\| \hat{\mathbf{A}} \|^2 = \left( \hat{\mathbf{A}} \cdot \hat{\mathbf{A}} \right)_{66} = \frac{1}{2}
\]

in consistency to the results for unit vectors in Section 3. By means of the dual square root in Appendix B, the norm of a cross-matrix is computed as

\[
\| \mathbf{S} \|_A = \sqrt{\left( \mathbf{S}_A \cdot \mathbf{S}_A \right)_{66}} = \| n_A \| \cdot \left( 1 + \varepsilon \cdot \frac{n_A \cdot m_A}{\| n_A \|^2} \right).
\]

simplifying to \( \| \mathbf{A} \| = \| n \| \) and to \( \| \hat{\mathbf{A}} \|^2 = 1 \) for lines and unit lines. Based on the derived norm, a normalization for the ‘adjoint representation of Plücker vectors’ is given via the fraction \( \frac{1}{\| \mathbf{A} \|} \), expressed as an adjoint matrix with \( \mathbb{A} \) from Equation (20). In detail, the cross-matrix normalization is computed by the equation chain

\[
\text{nrml}(\mathbf{S}) = \left( \frac{1}{\| \mathbf{S} \|} \right) \cdot \mathbf{S} = \left( \frac{1}{\| n \|} \cdot \left( 1 - \varepsilon \cdot \frac{n \cdot m}{\| n \|^2} \right) \right) \cdot \mathbf{S} = (i \cdot n^{\mathbb{A}} + \varepsilon \cdot m^{\mathbb{A}}) \cdot \Gamma
\]

\[
= \left( i \cdot \frac{n^{\mathbb{A}}}{\| n \|} + \varepsilon \cdot \left( \frac{m^{\mathbb{A}}}{\| n \|} - \frac{n^{\mathbb{A}} \cdot n \cdot m}{\| n \|^2} \right) \cdot \Gamma \right)
\]

\[
= \left( i \cdot \hat{n}^{\mathbb{A}} + \varepsilon \cdot \left( \hat{m}^{\mathbb{A}} - \hat{n}^{\mathbb{A}} \cdot (\hat{n} \cdot \hat{m}) \right) \right) \cdot \Gamma = (\lambda(\eta(\mathbf{S})))^{\mathbb{A}} = \hat{\mathbf{A}}^{\mathbb{A}}
\]

The normalization (for the cross-matrix) of a screw yields the (cross-matrix for the) corresponding unit line, indicating the screw axis in normalized form. In the chain of Equations (31), the normalization function \( \eta \), from Equation (2), and the alignment function \( \lambda \), from Equation (25), are recovered ‘by definition’. The steps of the normalization process are illustrated in the lower part of Figure 4.

Periodic powers

For the positive powers of the unit line’s cross matrix \( \hat{\mathbf{A}}^{\mathbb{A}} \), the recursive relations

\[
(\hat{\mathbf{A}}^{\mathbb{A}})^{k+4} = (\hat{\mathbf{A}}^{\mathbb{A}})^{k} \quad (\hat{\mathbf{A}}^{\mathbb{A}})^{k+2} = - (\hat{\mathbf{A}}^{\mathbb{A}})^{k}
\]

hold. Based on this observation, the generalized powers (for integer exponents) for a unit line’s cross-matrix are defined, as a generalization of Equation (13), as

\[
(\hat{\mathbf{A}}^{\mathbb{A}})^k = \begin{cases} + \hat{\mathbf{A}}^{\mathbb{A}} & k \text{ mod } 4 = 0 \\ + \hat{\mathbf{A}}^{\mathbb{A}} & k \text{ mod } 4 = 1 \\ - \hat{\mathbf{A}}^{\mathbb{A}} & k \text{ mod } 4 = 2 \\ - \hat{\mathbf{A}}^{\mathbb{A}} & k \text{ mod } 4 = 3 \end{cases}
\]

(32)

The augmented definition reflects the fact that the \((6 \times 6)\) cross-matrix geometrically corresponds to a rotation of \( \frac{\pi}{2} \) about the affine line \( \hat{\mathbf{A}} \) in the plane \( \hat{\mathbf{A}}^\perp \). The scheme in Figure 2 illustrates the definition of Equation (32). The matrices \( \hat{\mathbf{A}}^{\mathbb{A}} \) and \( \hat{\mathbf{A}}^{\mathbb{A}} \) are idempotent projections, their powers equal

\[
(\hat{\mathbf{A}}^{\mathbb{A}})^k = \hat{\mathbf{A}}^{\mathbb{A}} \quad (\hat{\mathbf{A}}^{\mathbb{A}})^k = \hat{\mathbf{A}}^{\mathbb{A}}
\]

generalizing the corresponding Equation (14) for unit vectors.
Adjoint trigonometric formula

Based on the technical preparations of the preceding subsections, the novel, finite-sum expression of a Lie group element in terms of its line-geometric parameters is stated. The left adjoint matrix $\mathbf{\hat{D}} = \text{Ad}(\mathbf{D})$ of a displacement is determined in the adjoint trigonometric representation in terms of a ‘magnitude’ $\tilde{\phi}$ (in form of a dual angle) and a ‘direction’ $\hat{\mathbf{\lambda}}$ (in form of a unit spear) as

$$\mathbf{\hat{D}} = \text{adsp}(\tilde{\phi}; \hat{\mathbf{\lambda}}) = \exp(\tilde{\phi} \boxdot \hat{\mathbf{\lambda}}) = (\cos \tilde{\phi}) \boxdot \hat{\mathbf{\lambda}} + (\sin \tilde{\phi}) \boxdot \hat{\mathbf{\lambda}} + \hat{\mathbf{\lambda}} \boxdot.$$  

(33)

The adjoint trigonometric representation of displacements (ATRD) expresses the left adjoint matrix as an affine-trigonometric combination of the three $(6 \times 6)$-base matrices introduced in advance. As an alternative to the exponential map, the displacement matrix $\mathbf{\hat{D}}$ is obtained via the Cayley map for the argument $\tan(\tilde{\phi}/2)$ $(\text{generalized Rodrigues vector})$ in Appendix D. The structure of the formal proof for the ATRD corresponds to the spherical counterpart in Section 3 with

$$\exp(\tilde{\phi} \boxdot \hat{\mathbf{\lambda}}) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (\tilde{\phi} \boxdot \hat{\mathbf{\lambda}})^k$$

$$= I + \left( \sum_{k=1}^{\infty} \frac{(\tilde{\phi} \boxdot)^{2k}}{(2k)!} \cdot (\hat{\mathbf{\lambda}} \boxdot)^{2k} \right) + \left( \sum_{k=0}^{\infty} \frac{(\tilde{\phi} \boxdot)^{2k+1}}{(2k+1)!} \cdot (\hat{\mathbf{\lambda}} \boxdot)^{2k+1} \right)$$

$$= \hat{\mathbf{\lambda}} \boxdot + \hat{\mathbf{\lambda}} \boxdot \cdot \left( \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \cdot (\tilde{\phi} \boxdot)^{2k} \right) + \hat{\mathbf{\lambda}} \boxdot \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot (\tilde{\phi} \boxdot)^{2k+1} \right)$$

$$= \hat{\mathbf{\lambda}} \boxdot + (\cos \tilde{\phi}) \boxdot \cdot \hat{\mathbf{\lambda}} \boxdot + (\sin \tilde{\phi}) \boxdot \cdot \hat{\mathbf{\lambda}} \boxdot$$. 
employing the periodic nature of the powers in Equation (32). The invariance $\hat{D} \cdot \hat{A} = \hat{A}$, as an affine generalization to Equation (16), follows immediately by means of the adjoint trigonometric representation of displacements

$$\hat{D} \cdot \hat{A} = -(\cos \hat{\phi})^n \cdot \hat{A}^{\circ} \cdot \hat{A}^{\circ} \cdot \hat{A} + (\sin \hat{\phi})^n \cdot \hat{A}^{\circ} \cdot \hat{A} + \hat{A}^{\circ} \cdot \hat{A} = \hat{A} \quad (34)$$

using the reciprocity condition $\hat{A}^{\circ} \cdot \hat{A} = 0$ and the idempotence condition $\hat{A}^{\circ} \cdot \hat{A} = \hat{A}$. Two matrix representations of spatial displacements are compared in Figure 4. The homogeneous matrix representation, indicated on the left hand side, is opposed to the adjoint matrix representation, on the right hand side. The structural similarity between the two matrix representations and their conceptual differences can be read-off the graphics. This concerns the ‘composition’ (products) from ‘geometric parameters’ (angles, lines, screws) and the decomposition (projections) into rotational (matrix), translational (vector), and invariant (screws, lines) subcomponents.

5 | NOVEL CONCEPTS FOR DUAL QUANTITIES

In this section, two novel concepts for dual quantities are introduced. First, the dual trigonometric equation $\hat{a} \cdot \cos \hat{\phi} + \hat{b} \cdot \sin \hat{\phi} = \hat{c}$ is stated together with its closed-form solution via the novel dual bivariate inverse tangent function ‘atan2’. Second, the concept of the dual relative directed angle between two lines with respect to a third is presented and computed via the novel dual trivariate inverse cosine function $\overline{\text{acos}}3$.

**Trigonometric equations**

The two angles solving the primal trigonometric equation

$$a \cdot \cos \phi + b \cdot \sin \phi = c, \quad (35)$$

are computed, with $d = \sqrt{a^2 + b^2 - c^2}$, via the expression $\hat{\phi}^{(+)} , \hat{\phi}^{(-)} = \text{atan2}(b, a) \pm \text{atan2}(d, c)$, where

$$\text{atan2}(y, x) = \begin{cases} \text{atan}(y/x) & x > 0 \\ \text{atan}(y/x) + \text{sgn}(y) \cdot \pi & x < 0 \\ \text{sgn}(y) \cdot \pi / 2 & x = 0 \end{cases}$$

The two solutions to Equation (35) can be interpreted geometrically as the intersection points of a line and a circle [37]. An example is provided in Figure 6 together with the numerical values. The number of distinct solutions to Equation (35) can be determined, by means of the normalization term $n := \sqrt{a^2 + b^2}$ and the normalized parameter $\hat{c} := \frac{c}{n}$, via

$$\left\{ \begin{array}{ll} 0 & |\hat{c}| > 1 \\ 1 & |\hat{c}| = 1 \\ 2 & |\hat{c}| < 1 \end{array} \right.$$  

Applying the principle of transference to Equation (35), the dual trigonometric equation

$$\hat{a} \cdot \overline{\cos} \hat{\phi} + \hat{b} \cdot \overline{\sin} \hat{\phi} = \hat{c}, \quad (36)$$

is obtained. Applying the same dualization principle to the solution approach Equation (35), the two dual angles solving the dualized equation are determined, with $\hat{d} = \sqrt{\hat{a}^2 + \hat{b}^2 - \hat{c}^2}$, via the expression

$$\hat{\phi}^{(+)} , \hat{\phi}^{(-)} = \overline{\text{atan2}}(\hat{b}, \hat{a}) \pm \overline{\text{atan2}}(\hat{d}, \hat{c}) \quad (37)$$

The definitions for the square and for the square root of a dual number are provided in Appendix B. The dual bivariate inverse tangent function $\overline{\text{atan2}}$ required in Equation (37) is defined as

$$\overline{\text{atan2}}(\hat{y}, \hat{x}) = \overline{\text{atan2}}(y + e \cdot \hat{y}, x + e \cdot \hat{x}) = \overline{\text{atan2}}(y, x) + e \cdot \frac{xy - \hat{x}\hat{y}}{x^2 + y^2}, \quad (38)$$

9 The degenerate setup with $a = 0, b \neq 0, c \neq 0$ corresponds to the inversion of sine function, the degenerate setup with $a \neq 0, b = 0, c \neq 0$ corresponds to the inversion of cosine function. In both setups, the line intersecting the circle is parallel to one of the two coordinate axes.
identical to the first Taylor approximation of ‘atan2’ (in correspondence with the principle of transference, applied to real function [24], Appendix B) and generalizing the dual inverse tangent function (Equation 50). A direct geometric interpretation of the dual constraint in Equation (37) is cumbersome due to the dimension of the stated problem; the two solutions can be interpreted as the (dual) intersection points of a (dual) line with a (dual) circle, as in the primal case.10

**Dual relative directed angles**

The *dual directed angle* between two lines *relative* to a third line is introduced as a generalization to the *dual angle* between two lines (Section 2) as well as to the *directed angle* between two vectors *relative* to a third vector [33]. See Figure 5 for a graphical illustration.

As the absolute value of the *directed angle* between two vectors, \( \mathbf{a} \) and \( \mathbf{b} \), *relative* to a third vector \( \hat{\mathbf{n}} \), matches the ‘absolute angle’ between \( \mathbf{a} \) and \( \mathbf{b} \) if the third vector \( \hat{\mathbf{n}} \) is orthogonal to the vectors \( \mathbf{a} \) and \( \mathbf{b} \), the absolute value of the *dual directed angle* between two lines, \( \Lambda_A \) and \( \Lambda_B \), *relative* to a third line \( \hat{\Lambda}_N \), matches the ‘dual absolute angle’ between \( \Lambda_A \) and \( \Lambda_B \) if the third line \( \hat{\Lambda}_N \) is reciprocal to the lines \( \Lambda_A \) and \( \Lambda_B \).11

The *relative directed angle* from \( \mathbf{a} \) to \( \mathbf{b} \), measured with respect to the (directed) axis \( \hat{\mathbf{n}} \), is computed, in accordance with [33], using the trivariate inverse trigonometric function

\[
\tilde{\theta}_{AB}^{(N)} = \Delta_A(a, b) = \text{acos}^3(a, b; \hat{\mathbf{n}}) = \text{ormt}^3(a, b, \hat{\mathbf{n}}) \cdot \text{acos}(\hat{\mathbf{n}} \cdot (a) \ast (b)),
\]

compare Figure 7 for an illustration. The concept is extended from the spherical case, of intersecting lines, to the affine case, of non-intersecting lines. The *dual relative directed angle* \( \tilde{\mathbf{z}} \) from \( \Lambda_A \) to \( \Lambda_B \), measured with respect to the (directed) unit line \( \hat{\Lambda}_N \) is formally determined via the dual trivariate inverse trigonometric function ‘\( \text{acos}^3 \)’ as

\[
\tilde{\theta}_{AB}^{(N)} = \tilde{\mathbf{z}}_{\hat{\Lambda}_N}(\Lambda_A, \Lambda_B) = \text{acos}^3(\Lambda_A, \Lambda_B; \hat{\Lambda}_N).
\]

The trivariate inverse dual trigonometric function ‘\( \text{acos}^3 \)’ is developed within four steps. In the first step, the unit lines of the line rejections of \( \hat{\Lambda}_A \) and \( \hat{\Lambda}_B \) with respect to \( \hat{\Lambda}_N \) are computed

\[
\hat{\Lambda}_{A \perp N} = \lambda(\eta(\tau(\hat{\Lambda}_A; \hat{\Lambda}_N))) \quad \hat{\Lambda}_{B \perp N} = \lambda(\eta(\tau(\hat{\Lambda}_B; \hat{\Lambda}_N))).
\]

The required functions are introduced in the previous sections. See Equation (21) for the line rejection ‘\( \tau \)’, Equation (25) for the alignment function ‘\( \lambda \)’, and Equation (2) for the normalization function ‘\( \eta \)’. Alternately, the matrix normalization ‘\( \text{nrml} \)’ in Equation (31) could be applied. In the second step, the dual inner product of the lines \( \hat{\Lambda}_{A \perp N} \) and \( \hat{\Lambda}_{A \perp N} \), see Equation (5) and Equation (28), is determined:

\[
\hat{\mathbf{c}}_{AB}^{(N)} = c + e \cdot d = \hat{\Lambda}_{A \perp N} \otimes \hat{\Lambda}_{B \perp N}.
\]

The dual number \( \hat{c} \) represents the dual cosine similarity of the two lines measured within the plane \( \hat{\Lambda}^1_N \). In the third step, the dual cosine similarity is transformed into an dual absolute relative angle

\[
| \tilde{\theta}_{AB}^{(N)} | = \text{acos}(\hat{c}_{AB}^{(N)}).
\]

---

10 As an alternative to using the dual bivariate inverse tangent function, the two solutions to Equation (36) are given via the direct formulas \( \phi_{\pm} \), \( \phi_{\pm} = \text{atan2}(b, a) \pm \text{atan2}(d, c) \) for the primal part and \( s_{\pm} \), \( s_{\pm} = \frac{d \cdot \cos \phi_{\pm} \pm b \cdot \sin \phi_{\pm}}{a \cdot \sin \phi_{\pm} \pm b \cdot \cos \phi_{\pm}} \) for the dual part, such that \( | \phi_{\pm} | = | \phi_{\pm} - s_{\pm} | \).

11 The concepts of ‘relative directed’ angle and ‘absolute’ angle match, if the third vector \( \hat{\mathbf{n}} \) is aligned with \( \mathbf{a} \times \mathbf{b} \) and if the third line \( \hat{\Lambda}_N \) is aligned with \( \Lambda_A^2 \cdot \Lambda_B^2 \).
using the dual inverse cosine function (Equation 50). In the fourth step, the absolute angle

\[
\phi = \frac{\pi}{2}
\]

is defined, by means of the auxiliary matrix

\[
\mathcal{M} \triangleq \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}
\]

and the parameters

\[
\hat{a} = \frac{a}{d}, \quad \hat{b} = \frac{b}{d}, \quad \hat{c} = \frac{c}{d}, \quad \hat{d} = \frac{d}{d} = 1,
\]

the trigonometric equation is reformulated to

\[
\frac{1}{2} \cos \phi + \frac{4}{5} \sin \phi = \frac{7}{10}
\]

and the identities \(d^2 + \hat{e}^2 = \hat{e}^2 + d^2 = 1\) hold. The distance of midpoint \(O\) equals \(\hat{e} = \| - O + M \| = \frac{7}{10}\). The distances of the intersection points and the midpoint equal \(d = \sqrt{1 - \hat{e}^2} = \| - M + S^+ \| = \| - M + S^- \| = \frac{\sqrt{15}}{10}\). On the right hand side, the sketch has been rotated by \(\phi_0\), so that \(A\) is aligned vertically and \(\Lambda^+\) horizontally, instead of the standard axes using the dual inverse cosine function (Equation 50).

In the fourth step, the absolute angle \(|\hat{d}|\) is transferred into the directed angle \(\theta_{AB}^{(N)}\) using the orientation of the three lines by means of the pattern ‘signed term = orientation \(\times\) absoluteterm’. The orientation of three lines is a generalized concept of the orientation of three vectors (Equation 1). The orientation of three, \(\mathbf{A}_A, \mathbf{A}_B,\) and \(\mathbf{A}_N\) is defined, by means of the auxiliary matrix \(\mathbf{N}_{ABN} = (\mathbf{n}_A, \mathbf{n}_B, \mathbf{n}_N)\), holding the direction vectors of the lines, via the case distinction

\[
\text{ornt}_3(\mathbf{A}_A, \mathbf{A}_B, \mathbf{A}_N) \begin{cases} 
\text{sgn}(\langle \hat{r}_A(n_A) \times \hat{r}_B(n_B) \rangle * \mathbf{n}_N) & \text{det}(\mathbf{N}_{ABN}) \neq 0 \\
\text{sgn}(\langle \hat{r}_N(\mathbf{A}_B) - \hat{r}_N(\mathbf{A}_A) \rangle * \mathbf{n}_N) & \text{det}(\mathbf{N}_{ABN}) = 0.
\end{cases}
\]

In the generic case of \(\text{det}(\mathbf{N}_{ABN}) \neq 0\), the line orientation equals the orientation of the three direction vectors, \(\text{ornt}_3(\mathbf{A}_A, \mathbf{A}_B, \mathbf{A}_N) = \text{ornt}_3(\mathbf{n}_A, \mathbf{n}_B, \mathbf{n}_N)\). In the degenerate case of \(\text{det}(\mathbf{N}_{ABN}) = 0\), the orientation of three lines is obtained via the point projections \(\hat{r}_N(\mathbf{A}_A)\) and \(\hat{r}_N(\mathbf{A}_B)\) from Equation (26). For \(\text{det}(\mathbf{N}_{ABN}) = 0\), the orientation matches +1 if the point projections of \(\mathbf{A}_A\) and \(\mathbf{A}_B\) on the line \(\hat{A}_N\) are ordered in accordance to the direction \(\mathbf{n}_N\) and it matches −1 otherwise. The Table 1 provides an overview for the orientation of a line triplet, of the generic case \(\text{det}(\mathbf{N}_{ABN}) \neq 0\), and of three degenerate subcases of \(\text{det}(\mathbf{N}_{ABN}) = 0\).

**Table 1** Overview of three distinct cases of degenerate line constellations.

For sake of brevity, only the line \(\mathbf{A}_A\) is compared with \(\hat{A}_N\). The distinctions for the line \(\mathbf{A}_B\) with regard to \(\hat{A}_N\) are treated analogously

| \(\mathbf{A}_A\) \# \(\mathbf{A}_B\) | \(\mathbf{A}_A\) \(\|\) \(\mathbf{A}_B\) |
|-----------------|-----------------|
| \(\mathbf{A}_A\) \# \(\hat{A}_N\) | ‘generic geometry’ | ‘indefinite constellation’ |
| \(\mathbf{A}_A\) \(\|\) \(\hat{A}_N\) | ‘coplanar line triplet’ | ‘half turn multiplicity’ |
The finite inverse kinematics problem (IKP) of a generic spatial 3C chain can be stated as the task to find joint configuration vectors \( \mathbf{q} = (\phi_{12}, s_{12}, \phi_{23}, s_{23}, \phi_{34}, s_{34}) \) that satisfy

\[
D_{14} = \text{dsp}(\phi_{12}; \hat{s}_{12}^{(0)}) \cdot \text{dsp}(s_{12}; \hat{s}_{12}^{(\infty)}) \cdot \text{dsp}(\phi_{23}; \hat{s}_{23}^{(0)}) \cdot \text{dsp}(s_{23}; \hat{s}_{23}^{(\infty)}) \cdot \text{dsp}(\phi_{34}; \hat{s}_{34}^{(0)}) \cdot \text{dsp}(s_{34}; \hat{s}_{34}^{(\infty)}) \cdot \mathbf{Z}_{14}.
\]

In particular, the condition \( \det(N_{ABN}) = 0 \) covers the ‘constellations’ (the geometric posture of all joint axes of a mechanism, given by design) of parallel, antiparallel, or coincident lines \( \Lambda_A \) and \( \Lambda_B \):

\[
sin(\text{acos}(\tau_N(\hat{n}_A) \star \tau_N(\hat{n}_B))) = 0 \quad \Rightarrow \quad \det(N_{ABN}) = 0.
\]

In Figure 8, two illustrations of geometry of a generic and of a degenerate line constellation are provided.

By means of the introduced tools, the dual inverse cosine function \( \text{acos}3 \), for computing the dual directed angle from line \( \hat{\Lambda}_A \) to \( \hat{\Lambda}_B \) relative to line \( \hat{\Lambda}_N \), is defined by the compact expression

\[
\delta_{AB}^{(N)} = \text{acos}3(\hat{\Lambda}_A; \hat{\Lambda}_B; \hat{\Lambda}_N)
\]

\[
= \begin{cases} 
\text{ornt}3(\Lambda_A; \Lambda_B; \Lambda_N) \cdot \text{acos}(\hat{\Lambda}_{A \perp N} \odot \hat{\Lambda}_{B \perp N}) & \det(N_{ABN}) \neq 0 \\
\text{ornt}3(\Lambda_A; \Lambda_B; \Lambda_N) \cdot s \cdot \|\pi_N^V(\Lambda_B) - \pi_N^V(\Lambda_A)\| & \det(N_{ABN}) = 0.
\end{cases}
\]

In contrast to the dual absolute directed angle between two lines, given in Equation (5) in terms of dual vector algebra and in Equation (11) in terms of real matrix algebra, the dual relative directed angle, computed in Equation (42), respects a ‘third’ line. Geometrically, a dual relative directed angle \( \delta_{AB}^{(N)} = \bar{\theta} \) represents that angle \( \bar{\theta} \) which ‘rotates’ \( \Lambda_A \), by computing \( \Lambda'_A = \exp((\bar{\theta} + \epsilon \cdot 0)\odot \hat{\Lambda}_A^\perp) \odot \Lambda_A \), so ‘far’ such that the transformed direction of \( \Lambda'_A \) projects to the same vector in the plane \( \hat{\Lambda}_N \) as the direction of \( \Lambda_B \) does. The dual part of the dual angle \( \bar{\theta} \) represents that shifts \( \epsilon \) which ‘translates’ \( \Lambda_A \), by computing \( \Lambda'_A = \exp((0 + \epsilon \cdot \hat{s})\odot \hat{\Lambda}_A^\perp) \odot \Lambda_A \), so ‘far’ such that the transformed location of \( \Lambda'_A \) projects to the same point on the line \( \hat{\Lambda}_N \) as the line \( \Lambda_B \) does. In conclusion, the rotational part of \( \delta_{AB}^{(N)} \) corresponds to the directed arc length between the projections onto the unit circle within the plane \( \hat{\Lambda}_N \) and the translational part of \( \delta_{AB}^{(N)} \) corresponds to the directed vector length between the projections onto the line \( \hat{\Lambda}_N \) itself.

### 6 | ANALYTIC SOLUTION TO THE IKP OF GENERAL 3C CHAINS

This section provides the novel analytic solution to the inverse kinematics problem of general affine 3C chains in the adjoint representation. The solution is based on the ATRD, the generalized REGG formula for spatial displacements, introduced in Section 4, and on the novel dual inverse trigonometric functions ‘atan2’ and ‘acos3’, introduced in Section 5. The method is illustrated with two examples.

**Problem statements**

The finite inverse kinematics problem (IKP) of a generic spatial 3C chain can be stated as the task to find joint configuration vectors \( \mathbf{q} = (\phi_{12}, s_{12}, \phi_{23}, s_{23}, \phi_{34}, s_{34}) \) that satisfy
In contrast to the POE formulation, each joint displacement acted 'with respect to the preceding element' if an 'iterative formulation' would be stated, for example, following the convention by Sheth and Uicker [40].

The analytic solution to the inverse kinematics problem of general 3C chains form is achieved in four steps. In the first step, the problem is stated based on the ATRD from Section 4. In the second step, the second dual angle $\phi_{23}$ is computed by means of the dual bivariate inverse tangent function from Section 5. In the third and fourth step, the first and the third dual angles, $\phi_{12}$ and $\phi_{34}$, are determined based on the concept of the dual relative directed angle from Section 5.

Solution method

The analytic solution to the inverse kinematics problem of general 3C chains form is achieved in four steps. In the first step, the problem is stated based on the ATRD from Section 4. In the second step, the second dual angle $\phi_{23}$ is computed by means of the dual bivariate inverse tangent function from Section 5. In the third and fourth step, the first and the third dual angles, $\phi_{12}$ and $\phi_{34}$, are determined based on the concept of the dual relative directed angle from Section 5.

In contrast to the POE formulation, each joint displacement acted 'with respect to the preceding element' if an 'iterative formulation' would be stated, for example, following the convention by Sheth and Uicker [40].
For a fixed constellation of joint axes (by design) and a fixed target pose (by application), the abbreviations \( \hat{M}_{ij} = \text{adsp}(\hat{\phi}_{ij}; \hat{\lambda}_{ij}) = \exp(\hat{\phi}_{ij}^{[\|]} \cdot \hat{\lambda}_{ij}^{[\|]}) \) are defined for the ‘moving’ variable displacements and \( \hat{S}_{14} = D_{14} \cdot (\hat{Z}_{14})^{-1} \) for the ‘static’ reference displacement, to express the IKP statement from Equation (44) briefly as

\[
\hat{S}_{14} = M_{12} \cdot \hat{M}_{23} \cdot \hat{M}_{34}.
\]

In the first step, the ansatz [22, 41] for computing the second dual angle is given by exploiting the invariance of the first and third axis with respect to those displacements, explicitly stated in Equation (34), as

\[
\hat{\lambda}_{12} \odot \hat{S}_{14} \cdot \hat{\lambda}_{34} = \hat{\lambda}_{12} \odot M_{12} \cdot \hat{M}_{23} \cdot \hat{M}_{34} \cdot \hat{\lambda}_{34} = \hat{\lambda}_{12} \odot \hat{M}_{23} \cdot \hat{\lambda}_{34}
\]

\[
= \hat{\lambda}_{12} \odot \left( (\cos \phi_{23}^{[\|]} \cdot \hat{\lambda}_{23} + (\sin \phi_{23}^{[\|]} \cdot \hat{\lambda}_{23}^{[\|]} + \hat{\lambda}_{23}^{[\|]} \cdot \hat{\lambda}_{34} \right) \hat{\lambda}_{34}.
\]

As a second step, the trigonometric representation of displacement Equation (33) is employed

\[
(\cos \phi_{23}^{[\|]} \cdot \hat{\lambda}_{23} + (\sin \phi_{23}^{[\|]} \cdot \hat{\lambda}_{23}^{[\|]} + \hat{\lambda}_{23}^{[\|]} \cdot \hat{\lambda}_{34} = \hat{\lambda}_{12} \odot (\hat{S}_{14} - \hat{\lambda}_{23}^{[\|]} \cdot \hat{\lambda}_{34}.
\]

Thus, the matrix constraint equation is resolved to the scalar constraint equation

\[
\cos \phi \cdot \hat{\lambda}_{12} \odot \hat{\lambda}_{23} \cdot \hat{\lambda}_{34} + \sin \phi \cdot \hat{\lambda}_{12} \odot \hat{\lambda}_{23}^{[\|]} \cdot \hat{\lambda}_{34} = \hat{\lambda}_{12} \odot (\hat{S}_{14} - \hat{\lambda}_{23}^{[\|]} \cdot \hat{\lambda}_{34}
\]

in shape of the dual trigonometric equation \( \hat{a} \cdot \cos \phi + \hat{b} \cdot \sin \phi = \hat{c} \) in Section 5 (Equation 36). The four dual parameters required to solve the equation, by means of applying the dual inverse tangent function \( \text{atan2} \), are determined by

\[
\hat{a} = \hat{\lambda}_{12} \odot \hat{\lambda}_{23} \cdot \hat{\lambda}_{34} \quad \hat{c} = \hat{\lambda}_{12} \odot (\hat{S}_{14} - \hat{\lambda}_{23}^{[\|]} \cdot \hat{\lambda}_{34}
\]

\[
\hat{b} = \hat{\lambda}_{12} \odot \hat{\lambda}_{23}^{[\|]} \cdot \hat{\lambda}_{34} \quad \hat{d} = \sqrt{\hat{a}^2 + \hat{b}^2 - \hat{c}^2}.
\]

The two solutions of Equation (37) provide dual angles \( \phi_{23}^{(+)} \) and \( \phi_{23}^{(-)} \) admissible to the second joint. The values for the first and for the third joint are determined in dependence of the chosen value for the second joint, by means of the function ‘\( \text{acos3} \)’ introduced in Section 5. The two dual angles for the first joint are determined in the assignment

\[
\phi_{12}^{(+)} \cdot \phi_{12}^{(-)} = \text{acos3} \left( \hat{S}_{14} \cdot \hat{\lambda}_{34}, M_{23} \cdot \hat{\lambda}_{34}, -\hat{\lambda}_{12} \right).
\]

Similarly, the two dual angles for the third joint are obtained with

\[
\phi_{34}^{(+)} \cdot \phi_{34}^{(-)} = \text{acos3} \left( \hat{S}_{14} \cdot \hat{\lambda}_{34}, (\hat{M}_{23})^{-1} \cdot \hat{\lambda}_{12}, +\hat{\lambda}_{34} \right).
\]

Overall, two solutions to the inverse kinematics problem of 3C chains in Equation (44) are obtained. The joint configurations \( q^{(+)} = (\phi_{12}^{(+)} \cdot \phi_{23}^{(+)} \cdot \phi_{34}^{(+)} \) and \( q^{(-)} = (\phi_{12}^{(-)} \cdot \phi_{23}^{(-)} \cdot \phi_{34}^{(-)} \) are triplets of dual angles. Each of the triplets contains three angles and three shifts for the configuration of the three cylindric joints.

**Example**

The application of the closed-form solution method for the IKP of spatial 3C chains is illustrated by a kinematic chain which features joint axes that resemble the line constellation depicted in Figure 3. The axes of the three cylindric joints are given by the direction vectors, \( n_{12} = (1, 0, 0)^T, n_{23} = (-1, -4, 2)^T, \) and \( n_{34} = (0, 0, 1)^T \), and by the anchor points, \( a_{12} = (0, 0, 0)^T, \) \( a_{23} = (6, 0, 8)^T, \) and \( a_{34} = (0, 4, 8)^T \). In the zero reference posture (depicted on the left hand side of Figure 9), the effector is positioned at \( a_{eq} = (-4, +4, 10)^T \) and oriented as the global origin. The solution method is once applied for zero reference pose (in Figure 9) and once for the pose depicted in Figure 10. In the second case, the displacement \( D_{14} \) of the effector with respect to the origin corresponds to

\[
R_{14} \approx \begin{pmatrix}
0.747 & -0.591 & -0.304 \\
0.587 & 0.802 & -0.114 \\
0.311 & -0.093 & 0.945
\end{pmatrix} 
\]

\[
t_{14} \approx \begin{pmatrix}
-0.491 \\
0.009 \\
0.908
\end{pmatrix}.
\]
Figure 9  Example kinematic chain with three cylindric joints. The gray disc represents the base link of the chain, the blue disk the end-effector link. The chain is drawn in the zero reference posture and in the second configuration that features the end effector at the identical pose, with joint values \( \phi_{12} \approx -126.87^\circ, s_{12} \approx -21.6, \phi_{23} \approx -126.87^\circ, s_{23} \approx -4.09, \) and \( \phi_{34} \approx -126.87^\circ, s_{34} \approx -14.58. \) The rotation offsets \( \phi \) of the joint displacements are indicated via the (gray/blue) directional markings on the ‘motor’ elements, the translation offsets \( s \) via the (yellow) cylinder rods connecting those.

Figure 10  Illustration of the example kinematic chain in two configurations for one target pose. The gray disc represents the base link of the chain, the blue disk the end-effector link. The joint configuration on the left hand side corresponds to the numerical values \( \phi_{12} \approx 10^\circ, s_{12} = 0.5, \phi_{23} \approx 20^\circ, s_{23} = 1, \) and \( \phi_{34} \approx 30^\circ, s_{34} = 2. \) The configuration on the right hand side is given by \( \phi_{12} \approx -123.09^\circ, s_{12} = -18.17, \phi_{23} \approx 172.45^\circ, s_{23} = -5.1, \) and \( \phi_{34} \approx -161.44^\circ, s_{34} = 11.48. \)

In both figures, the links are indicated as three dimensional splines, connecting to the adjacent joints asymptotically to the directions of the respective joint axes. The cylindric joint are indicated by tuples of blue and gray cylinders. Markings on the cylinder surfaces indicate the rotational joint offsets. The yellow ‘rod’ connecting the cylinders highlight the translational joint offsets. The illustrations exemplify the consistency of the solutions that are obtained by the analytic method from Section 6. Table 2 provides an overview of the (approximate) numerical values for the inverse kinematics solutions.

| Table 2  | Overview of inverse kinematic solutions for two example poses |
|----------|--------------------------------------------------|
| Fig.     | Config.  | \( \phi_{12} \) | \( s_{12} \) | \( \phi_{23} \) | \( s_{23} \) | \( \phi_{34} \) | \( s_{34} \) |
| 9        | left     | 0°             | 0             | 0°             | 0             | 0°             | 0             |
|          | right    | \(-126.87^\circ\) | \(-21.6\) | \(-126.87^\circ\) | \(-4.09\) | \(-126.87^\circ\) | \(-14.58\) |
| 10       | left     | 10°            | 0.5           | 20°            | 1             | 30°            | 2             |
|          | right    | \(-123.09^\circ\) | \(-18.17\) | 172.45°        | \(-5.1\)     | \(-161.44^\circ\) | 11.48         |
FIGURE 11  Overview of the three reciprocal (Equation 29) base elements (red, green, blue) of the trigonometric representation of rotations (left) and of the adjoint trigonometric representation of displacements (right). For sake of readability, only the generalized scheme on the right hand side is described. The figure indicates three different spherical shapes: the outer encircles all elements of the $(6 \times 6)$-matrix representation in contrast to the line representation via Plücker vectors (top left) and dual three-vectors (top right). The spatial (partially indicated) sphere emphasizes the three orthogonal base matrices. The planar circle represents the cyclic nature of the rank-4-tensors and comprehends Figure 2. The argument of the exponential function in the adjoint trigonometric formula Equation (33) (see also Figure 4), $\tilde{\phi} \otimes \tilde{\Lambda}^\oplus$, is an element of the tangent space (thin green arrow, in ‘$\pi$-direction’ $\tilde{\Lambda}^\oplus$) at identity (the element $\tilde{\Lambda}^\oplus$). The exponential function ‘lifts’ $\tilde{\phi} \otimes \tilde{\Lambda}^\oplus$, an element of ‘flat’ Lie algebra, to $\tilde{D}$, an element of the ‘curved’ Lie group (illustrated by the gray arrow).

7 | DISCUSSION

The adjoint trigonometric formula ATRD in Equation (33) (Section 4) serves as the central tool for solving the IKP general spatial 3C chains in Section 6 by transforming matrix equations into (dual) scalar equations. Due to the properties of the $(6 \times 6)$ matrix representations employed for lines and displacements, the ATRD serves a direct generalization (according to the principle of transference) of the trigonometric formula TRR in Equation (15) (Section 3) in terms of the $(3 \times 3)$ matrix representations employed for vectors and rotations. The TRR and the ATRD formulas both obey to the pattern

$$\exp(\phi \cdot \text{crossmat}) = \cos(\phi) \cdot \text{unitmat} + \sin(\phi) \cdot \text{crossmat} + \text{squaremat}. \quad (45)$$

In order to interpret these trigonometric formulas in an ‘intuitive’ manner, Figure 11 provides two drawings that outline their common ‘geometric’ structure. The figure indicates the reciprocity of the three base matrices and comprehends the period-four cyclicity of the cross-matrix (Figure 2). It further motivates the two trigonometric formulas as particular extensions\(^{13}\) of Euler’s formula $\tilde{z} = \exp(i \cdot \phi) = \cos(\phi) + i \cdot \sin(\phi)$. For comparability with the pattern in Equation (45), Euler’s formula is also circumscribed by

$$\exp(\phi \cdot \text{imag}) = \cos(\phi) \cdot \text{one} + \sin(\phi) \cdot \text{imag} + \text{zero}.$$

The exponential assigns an element of the Lie groups’ manifold $(\mathcal{R} | \tilde{D} | \tilde{z})$ to an element $(\phi \cdot \tilde{n}^\oplus | \tilde{\phi} \otimes \tilde{\Lambda}^\oplus | \phi \cdot \tilde{\Lambda}^\oplus)$ of the group’s Lie algebra, the tangent space at identity $(\tilde{n}^\oplus | \tilde{\Lambda}^\oplus | 1)$. Using other algebraic representations for lines or displacements than the $(6 \times 6)$ matrix representations, these structural analogies would be difficult to obtain. Table 3 indicates the advantageous features of the $(6 \times 6)$ representation in comparison with three other algebraic representations, $(6 \times 1)$ Plücker vectors, $(4 \times 4)$ matrices, and $(3 \times 1)$ dual vectors, frequently used for parametrizing lines and rigid body displacements.

\(^{13}\)The powers of the $(\tilde{n}^\oplus | \tilde{\Lambda}^\oplus | \phi)$ feature cyclicity of period four.
TABLE 3  Comparison of four algebraic representations of oriented unit lines. In the first three rows, the four representation are characterized by their names, their used symbols, and their ambient space. In the fourth row it is emphasized that a \((3 \times 1)\) dual vector generally contains non-real elements. Next, its is highlighted that the set of Plücker vector does not feature a Lie algebra structure (there is no exponential map the lifts \((6 \times 1)\) Plücker vectors to a Lie group of rigid body displacements). Further, only the matrix representations permit using techniques of linear algebra (for example, the computation of a determinant or trace). The subsequent row highlights the property of the \((6 \times 6)\)-representation to allow the definition of three orthogonal, geometrically meaningful base matrices (Figure 11). The last row indicates that only the representations of \((6 \times 6)\) matrices and dual vectors preserve structure when generalizing from a linear (intersecting lines) to an affine-linear (skew lines) geometric setting.

| Name                  | Plücker vector | \((4 \times 4)\)-matrix | \((6 \times 6)\)-matrix | dual vector |
|-----------------------|----------------|-------------------------|-------------------------|-------------|
| Symbol                | \(\hat{\mathbf{A}}\) | \(\hat{\mathbf{A}}^\circ\) | \(\hat{\mathbf{A}}^\circ\) | \(\hat{\mathbf{A}}\) |
| Dimensions            | \(\mathbb{R}^{(6\times1)}\) | \(\mathbb{R}^{(4\times4)}\) | \(\mathbb{R}^{(6\times6)}\) | \(\mathbb{R}^{(3\times1)}\) |
| Real number algebra   | ✓              | ✓                       | ✓                       | ✓           |
| Lie algebra           | ✓              | ✓                       | ✓                       | ✓           |
| Linear algebra        | ✓              | ✓                       | ✓                       | ✓           |
| Three base matrices   | ✓              | ✓                       | ✓                       | ✓           |
| Transference principle| ✓              | ✓                       | ✓                       | ✓           |

8 | CONCLUSIONS

The concept of the adjoint trigonometric representation of displacements has been introduced as the major theoretical contribution of the article. The concept is obtained as a novel ‘dual-angle/affine-line generalization in shape of adjoint matrices’ of the REGG-rotation formula, the trigonometric representation of rotations. In the future, the adjoint trigonometric formula could find applications in various contexts of computing displacements in spatial rigid body systems. As the mayor practical contribution, the adjoint trigonometric representation has been employed in the adjoint formulation of the inverse kinematics problem of general affine 3C chains to obtain a novel, closed-form solution to this particular problem. The analytic solution could be used as a subroutine in solution schemes for solving kinematic problems of related mechanical designs.

In context to these major results, the article features additional contributions. The set of left adjoint \((6 \times 6)\) matrices is identified as an alternate tool for dealing with rigid body motions in a comparison to the set of homogeneous \((4 \times 4)\) matrices. It has been indicated that the left adjoint matrices represent a real-number version of the \((3 \times 3)\) matrices with dual entries. The findings are summarized in a brief discussion. With regard to line geometry and screw theory, the cross-matrix representation of an oriented line is presented as an alternative to \((6 \times 1)\) Plücker vectors and to dual \((3 \times 1)\) vectors. The algebra of cross-matrices is equipped with a dual inner product that induces those properties that are established for the Plücker vector representation of lines. For example, the inner product provides the means for obtaining the unit line of the axis associated to a screw via the induced normalization of the cross-matrix representation. With respect to dual angles, the article extends the concept of a dual angle lines to the concept of a dual directed angle between two oriented lines relative to a third. Its computation is conducted via a novel, trivariate dual inverse cosine function respecting the orientation of three lines in space accordingly. Further, a bivariate dual inverse tangent function is introduced via the transference principle that permits solving a dual trigonometric equation in closed form. The obtained tools for studying spatial lines and screws can be employed in a broad context of computational geometry and kinematics in the future.

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APPENDIX A: NOMENCLATURE

The scheme in Figure A1 indicates the interrelations of the representation of a screw as a 6-vector, as a $(4 \times 4)$-cross matrix, and as a $(6 \times 6)$-cross matrix. Table A2 gives an overview of the matrices that serve as the reciprocal base for the REGG equations in $(3 \times 3)$-shape and $(6 \times 6)$-shape. An overview of the symbols used in this article is provided in Table A1.

**TABLE A1** An overview of used symbols

| Symbols | Meanings | References |
|---------|----------|------------|
| $a \otimes b$ | dyadic vector product | Section 2 |
| $a \cdot b$ | inner product for vectors | Section 2 |
| $A \otimes B$ | inner product for vector sets (matrices) | Section 2 |
| $S_A \otimes S_B$ | dual inner product for Plücker vectors | Equation (5) |
| $(a^\oplus, b^\oplus)_{33}$ | $(3 \times 3)$-matrix inner product | Equation (10) |
| $(S_A^\oplus, S_B^\oplus)_{66}$ | $(6 \times 6)$-matrix inner product | Equation (27) |
| $a \otimes b$ | symmetric matrix product | Equation (22) |
| rot | $(3 \times 3)$ rotation matrix | Equation (15) |
| dsp | $(4 \times 4)$ homogeneous displacement matrix | Equation (3) |
| adsp | $(6 \times 6)$ adjoint displacement matrix | Equation (33) |
| $\hat{x} = x + \epsilon \cdot \hat{x}$ | dual number | Appendix B |
| $(\hat{x})^{\otimes} = \hat{t} \cdot \hat{x} + \epsilon \cdot \hat{\hat{x}}$ | dual number $(6 \times 6)$-matrix | Equation (20) |
| $n^\otimes, A^\otimes$ | vector and line cross-matrix | Equation (6) and Equation (21) |
| $n^\otimes, A^\otimes$ | vector and line unit-matrix | Equation (6) and Equation (21) |
| $n^\otimes, A^\otimes$ | vector and line square-matrix | Equation (6) and Equation (21) |
| $\pi$ | vector and line projection | Equation (9) and Equation (24) |
| $\tau$ | vector and line rejection | Equation (9) and Equation (24) |
| $\chi$ | vector and line orthogonal | Equation (9) and Equation (24) |
| $\pi^\tau$ | point projection | Equation (26) |
| $\Pi_R, \Pi_T$ | $(4 \times 4)$ matrix projections | Equation (4) |
| $\Pi_R, \Pi_T$ | $(6 \times 6)$ matrix projections | Equation (19) |
| $\Gamma$ | $(6 \times 6)$ matrix generation | Equation (18) and Equation (23) |
| $\eta$ | Plücker normalization function | Equation (2) |
| $\lambda$ | Plücker alignment function | Equation (25) |
| nrml | $(6 \times 6)$-matrix normalization function | Equation (31) |
| ornt3 | orientation of three vectors and lines | Equation (1) and Equation (41) |
| pow | dual power function | Equation (47) |
| atan2 | dual bivariate inverse tangent function | Equation (38) |
| acos3 | dual trivariate inverse cosine function | Equation (42) |

**FIGURE A1** Interrelations of algebraic representations of a finite screw: the $(6 \times 1)$-Plücker-vector representation $\$, the $(4 \times 4)$-matrix representation $\otimes$, and the $(6 \times 6)$-matrix representation $\oplus$ are interrelated via the two cross-matrix operators $\otimes$ and $\oplus$ and the left adjoint map ‘ad’
APPENDIX B: DUAL ALGEBRA

A dual number is a sum of two real numbers, denoted in this article by \( \hat{a} = u + \varepsilon \cdot \hat{u} \), with the specific property of the dual unit \( \varepsilon \) that its square equals zero, \( \varepsilon^2 = 0 \). The definitions of dual operations in the following can be deduced from this algebraic property and the general formula \([24]\)

\[
\hat{f}(\hat{x}, \hat{y}) = f(x + \varepsilon \cdot \hat{x}, y + \varepsilon \cdot \hat{y}) = f(x, y) + \varepsilon \cdot \frac{\partial}{\partial x}(f) + \varepsilon \cdot \hat{y} \cdot \frac{\partial}{\partial y}(f).
\]

Further overviews of dual algebras and dual functions are, for example, provided in \([18, 26]\). The product of two dual quantities is determined as

\[
\hat{u} \cdot \hat{v} = (u + \varepsilon \cdot \hat{u}) \cdot (v + \varepsilon \cdot \hat{v}) = u \cdot v + \varepsilon \cdot (u \cdot \hat{v} + \hat{u} \cdot v).
\]

(B1)

The dual power function \( \hat{a}^{\hat{x}} = \text{pow}(\hat{a}, \hat{x}) \) is determined, for \( a \neq 0 \) and \( x \neq 0 \), via

\[
\text{pow}(\hat{a}, \hat{x}) = \text{pow}(a + \varepsilon \cdot \hat{a}, x + \varepsilon \cdot \hat{x}) = a^x + \varepsilon \cdot (x \cdot \hat{a} \cdot |a|^{x-1} + \hat{x} \cdot \log |a| \cdot |a|^x).
\]

(B2)

In particular, the definition of the dual power function above incorporates the dual square, the dual square root, as well as the dual inverse:

\[
(a + \varepsilon \cdot \hat{a})^2 = a^2 + \varepsilon \cdot 2 \cdot \hat{a} \cdot |a| \quad \quad \sqrt{a + \varepsilon \cdot \hat{a}} = \sqrt{a} + \varepsilon \cdot \frac{\hat{a}}{2 \sqrt{a}}
\]

\[
(a + \varepsilon \cdot \hat{a})^{-1} = a^{-1} - \varepsilon \cdot \frac{\hat{a}}{|a|^2}.
\]

The absolute value of a dual number,

\[
|\hat{a}| = |a + \varepsilon \cdot \hat{a}| = \sqrt{\hat{a}^2} = |a| + \varepsilon \cdot \hat{a},
\]

is used to determine a normalized dual number via

\[
\hat{\hat{a}} = \hat{a} \cdot |\hat{a}|^{-1} = \frac{a}{|a|} + \varepsilon \cdot \left( \frac{\hat{a}}{|a|} - \frac{a}{|a|} \cdot \frac{\hat{a}}{|a|} \right) = \hat{a} + \varepsilon \cdot (\hat{\hat{a}} - \hat{a} \cdot \hat{\hat{a}}).
\]

(B4)

The equations build the skeleton for the screw normalization in cross-matrix form in Equation (31). Three basic dual trigonometric functions and their inverse counterparts are given by the equations

\[
\begin{align*}
\sin(\hat{\varphi}) &= \sin(\varphi) + \varepsilon \cdot \hat{\varphi} \cdot \cos(\varphi) \\
\sin(\hat{\varphi}) &= \sin(\varphi) + \varepsilon \cdot \hat{\varphi} \cdot \cos(\varphi) \\
\cos(\hat{\varphi}) &= \cos(\varphi) - \varepsilon \cdot \hat{\varphi} \cdot \sin(\varphi) \\
\cos(\hat{\varphi}) &= \cos(\varphi) - \varepsilon \cdot \hat{\varphi} \cdot \sin(\varphi) \\
\tan(\hat{\varphi}) &= \tan(\varphi) + \varepsilon \cdot \frac{\hat{\varphi}}{\cos^2(\varphi)} \\
\tan(\hat{\varphi}) &= \tan(\varphi) + \varepsilon \cdot \frac{\hat{\varphi}}{\cos^2(\varphi)} \\
\arctan(\hat{\varphi}) &= \arctan(\varphi) + \varepsilon \cdot \hat{\varphi} \cdot \cos^2(\arctan(\varphi)) \\
\arctan(\hat{\varphi}) &= \arctan(\varphi) + \varepsilon \cdot \hat{\varphi} \cdot \cos^2(\arctan(\varphi))
\end{align*}
\]

(B5)
APPENDIX C: MATRIX IDENTITIES

Cross-matrices and products
The matrix entries of a cross matrix $a^\otimes$ is
\[
a^\otimes = -(a^\otimes)^T = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix},
\] (C1)
and its transpose equals its negative. As an expansion of $a^\circ = (a \ast a) \cdot I - a^\otimes$ (Equation 8), the identity
\[
a^\otimes \ast a^\otimes = (a \ast a) \cdot I - a \otimes a
\] (C2)
is observed. For the product of two-cross matrices, the entries are given as
\[
a^\otimes \ast b^\otimes = -(a^\otimes \cdot b^\otimes) = -(b^\otimes \cdot a^\otimes)^T = \begin{pmatrix} a_2b_2 + a_3b_3 & -b_1a_2 & -b_1a_3 \\ -b_2a_1 & a_1b_1 + a_3b_3 & -b_2a_3 \\ -b_3a_1 & -b_3a_2 & a_1b_1 + a_2b_2 \end{pmatrix}
\]
For the cross-matrix products $a^\otimes \ast b^\otimes = -(a^\otimes \cdot b^\otimes)$, the identity
\[
a^\otimes \ast b^\otimes = (a \ast b) \cdot I - b \otimes a,
\] (C3)
relating to the inner product and the outer product, holds as a ‘dyadic/bivariate’ generalization to Equation (8) and Equation C2.

Symmetric matrix product
The concept of the symmetric matrix product, defined as
\[
a \otimes b := b^\otimes \ast a^\otimes + a^\otimes \ast b^\otimes
\]
in Equation (22), can be regarded of an extension of the identity $n^\otimes = n^\circ \ast n^\otimes$ in Equation (7). For the anti-symmetric vector cross product, the related identity
\[
(a \times b)^\otimes = b^\otimes \ast a^\otimes - a^\otimes \ast b^\otimes
\]
holds. The symmetric matrix product can be restated with Equation (53) as
\[
a \otimes b = 2 \cdot (a \ast b) \cdot I - (a \otimes b + b \otimes a).
\] (C4)
In case of a monovariate product, Equation (54) simplifies to
\[
a \otimes a = 2 \cdot ((a \ast a) \cdot I - a \otimes a).
\]
With the symmetric matrix product, unit-matrix, and square-matrix are expressed as
\[
a^\otimes_0 = \frac{1}{2} \cdot (a \otimes a) \quad a^\circ = (a \ast a) \cdot I - \frac{1}{2} \cdot (a \otimes a).
\]

APPENDIX D: CAYLEY MAPS
In 1965, Dimentberg [29] reported an extension of Rodrigues’ rotation formula, in terms of Rodrigues’ rotation vector [43] and its dualized analogue, following the principle of transference. The consistency of the REGG formula (Equation 15) and the ATRD (Equation 33) with Dimentberg’s equations follows by means of the (adjoint matrix representation of) Rodrigues’ vector
\[
\lambda \cdot \hat{n}^\otimes := \tan \left(\frac{\phi}{2}\right) \cdot \hat{n}^\otimes,
\]
and (the adjoint matrix representation of) its dualized version

\[(\tilde{\lambda})^\square \cdot \tilde{\hat{\Lambda}}^\square := \left(\frac{\tan(\frac{\phi}{2})}{\boxbar}\right)^\square \cdot \tilde{\hat{\Lambda}}^\square,\]

and by means of the Cayley map

\[\text{cay}(x) = (1 + x) \circ (1 - x)^{-1}.\] (D1)

employed [44] for rotations and displacements. With these definitions, the rotation matrix \( R \) is obtained, in terms of the tangent-half-angle parameter \( \lambda = \tan(\phi/2) \), by

\[ R = \text{cay}(\lambda \cdot \tilde{n}^\otimes) = \frac{1 - \lambda^2}{1 + \lambda^2} \cdot \tilde{R}^\otimes + \frac{2 \cdot \lambda}{1 + \lambda^2} \cdot \tilde{R}^\otimes + \tilde{R}^\otimes, \] (D2)

in coherence with the formulation by Dimentberg [29] and, via the tangent half-angle identities

\[
\cos(\phi) = \frac{1 - \lambda^2}{1 + \lambda^2}, \quad \sin(\phi) = \frac{2\lambda}{1 + \lambda^2},
\]

in coherence with the REGG formula in Equation (15). Analogously, the adjoint displacement \( \tilde{D} \) is obtained, in terms of the dual tangent-half-angle parameter \( \tilde{\lambda} = \tan(\tilde{\phi}/2) \), by

\[ \tilde{D} = \text{cay}(\tilde{\lambda} \cdot \tilde{\tilde{n}}^\otimes) = \left(\frac{1 - \tilde{\lambda}^2}{1 + \tilde{\lambda}^2}\right)^\square \cdot \tilde{\hat{\Lambda}}^\square + \left(\frac{2 \cdot \tilde{\lambda}}{1 + \tilde{\lambda}^2}\right)^\square \cdot \tilde{\hat{\Lambda}}^\square + \tilde{\hat{\Lambda}}^\square, \] (D3)

in coherence with the structure-preserving formulation by Dimentberg [29], and via the dual tangent-half angle identities

\[
\tilde{\cos}(\tilde{\phi}) = \frac{1 - \tilde{\lambda}^2}{1 + \tilde{\lambda}^2}, \quad \tilde{\sin}(\tilde{\phi}) = \frac{2\tilde{\lambda}}{1 + \tilde{\lambda}^2},
\]

in coherence with the ATRD in Equation (33). For the dual parameter \( \tilde{\lambda} \), the equalities \( \tilde{\lambda} = \lambda + e \cdot \tilde{\phi} \cdot \frac{1 + \lambda^2}{2} \) and \( \tilde{\lambda}^2 = \lambda^2 + e \cdot \tilde{\phi} \cdot (1 + \lambda^2) \) hold. In order to prove Cayley’s identity in Equation D2, Cayley’s map is stated for Rodrigues’ vector \( \lambda \cdot \tilde{n}^\otimes \) as

\[ \text{cay}(\lambda \cdot \tilde{n}^\otimes) = (I + \lambda \cdot \tilde{n}^\otimes) \cdot (I - \lambda \cdot \tilde{n}^\otimes)^{-1}. \]

Observing the generalized inverse equality

\[ (I - \lambda \cdot \tilde{n}^\otimes)^+ = (I + \lambda^2 \cdot \tilde{n}^\otimes)^{-1} \cdot (I + \lambda \cdot \tilde{n}^\otimes) \]

with \( (I - \lambda \cdot \tilde{n}^\otimes)^+ = (I - \lambda \cdot \tilde{n}^\otimes)^{-1} \), the further simplification

\[ (I + \lambda^2 \cdot \tilde{n}^\otimes)^{-1} = (\tilde{n}^\otimes + \tilde{n}^\otimes \cdot (1 + \lambda^2))^{-1} = \tilde{n}^\otimes + \frac{1}{1 + \lambda^2} \cdot \tilde{n}^\otimes, \]
A formal proof of the dual Cayley identity from Equation (D3) follows via the principle of transference in formal consistency employing the product identities from Table D1, the generalized inverse equality
\[
\left(\frac{\nu}{\eta} - \frac{\nu}{\eta} \right) = \left(\frac{\nu}{\eta} + \frac{\nu}{\eta} \right) \cdot \left(\frac{\nu}{\eta} + \frac{\nu}{\eta} \right)^{-1}
\]

and employing the product identities listed in Table D1, the Cayley identity of Equation D2 is obtained via the chain of equations
\[
\text{cay} \left( \lambda \cdot \hat{n}^\oplus \right) = \left( I + \lambda \cdot \hat{n}^\oplus \right) \cdot \left( I - \lambda \cdot \hat{n}^\oplus \right)^{-1}
\]
\[
= \left( I + \lambda \cdot \hat{n}^\oplus \right)^2 \cdot \left( I + \lambda^2 \cdot \hat{n}^\oplus \right)^{-1}
\]
\[
= \left( I + 2\lambda \cdot \hat{n}^\oplus - \lambda^2 \cdot \hat{n}^\oplus \right) \cdot \left( I + \lambda^2 \cdot \hat{n}^\oplus \right)^{-1}
\]
\[
= \left( \hat{n}^\oplus + 2\lambda \cdot \hat{n}^\oplus \right) \cdot \left( \hat{n}^\oplus + \frac{1}{1 + \lambda^2} \cdot \hat{n}^\oplus \right)
\]
\[
= \hat{n}^\oplus + \left( 2\lambda \cdot \hat{n}^\oplus \right) \cdot \left( \frac{1}{1 + \lambda^2} \cdot \hat{n}^\oplus \right)
\]
\[
= \hat{n}^\oplus + \frac{2\lambda}{1 + \lambda^2} \cdot \hat{n}^\oplus + \frac{1 - \lambda^2}{1 + \lambda^2} \cdot \hat{n}^\oplus.
\]

A formal proof of the dual Cayley identity from Equation (D3) follows via the principle of transference in formal consistency (as the proof of Equation (33) formally follows the proof of Equation (15)) via the chain of equations
\[
\text{cay} \left( \lambda \cdot \hat{\lambda}^\ominus \right) = \left( t + (\lambda) \cdot \hat{\lambda}^\ominus \right) \cdot \left( t - (\lambda) \cdot \hat{\lambda}^\ominus \right)^{-1}
\]
\[
= \left( t + (\lambda) \cdot \hat{\lambda}^\ominus \right)^2 \cdot \left( t + (\lambda^2) \cdot \hat{\lambda}^\ominus \right)^{-1}
\]
\[
= \left( t + (2\lambda) \cdot \hat{\lambda}^\ominus - (\lambda^2) \cdot \hat{\lambda}^\ominus \right) \cdot \left( t + (\lambda^2) \cdot \hat{\lambda}^\ominus \right)^{-1}
\]
\[
= \left( \hat{\lambda}^\ominus + (2\lambda) \cdot \hat{\lambda}^\ominus \right) \cdot \left( \hat{\lambda}^\ominus + \left( \frac{1}{1 + \lambda^2} \right) \cdot \hat{\lambda}^\ominus \right)
\]
\[
= \hat{\lambda}^\ominus + \left( (2\lambda) \cdot \hat{\lambda}^\ominus \right) \cdot \left( \frac{1}{1 + \lambda^2} \right) \cdot \hat{\lambda}^\ominus
\]
\[
= \hat{\lambda}^\ominus + \left( \frac{2\lambda}{1 + \lambda^2} \right) \cdot \hat{\lambda}^\ominus + \frac{1 - \lambda^2}{1 + \lambda^2} \cdot \hat{\lambda}^\ominus.
\]

employing the product identities from Table D1, the generalized inverse equality
\[
(t - (\lambda) \cdot \hat{\lambda}^\ominus)^+ = (t + (\lambda^2) \cdot \hat{\lambda}^\ominus)^{-1} \cdot (t + (\lambda) \cdot \hat{\lambda}^\ominus)
\]
with $(t - (\lambda) \cdot \hat{\lambda}^\ominus)^+ = (t - (\lambda) \cdot \hat{\lambda}^\ominus)^{-1}$, and the simplification
\[
(t + (\lambda^2) \cdot \hat{\lambda}^\ominus)^{-1} = (\hat{\lambda}^\ominus + (1 + \lambda^2) \cdot \hat{\lambda}^\ominus)^{-1} = \hat{\lambda}^\ominus + \left( \frac{1}{1 + \lambda^2} \right) \cdot \hat{\lambda}^\ominus.
\]