SOME RESULTS ON THE ANNIHILATORS AND ATTACHED PRIMES OF LOCAL COHOMOLOGY MODULES

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Abstract. Let \((R, \mathfrak{m})\) be a local ring and \(M\) a finitely generated \(R\)-module. It is shown that if \(M\) is relative Cohen-Macaulay with respect to an ideal \(a\) of \(R\), then \(\text{Ann}_R(H^\text{cd}_{\mathfrak{m}}(a,M)(M)) = \text{Ann}_R M/L = \text{Ann}_R M\) and \(\text{Ass}_R(R/\text{Ann}_R M) \subseteq \{p \in \text{Ass}_R M \mid \text{cd}(a, R/p) = \text{cd}(a, M)\}\), where \(L\) is the largest submodule of \(M\) such that \(\text{cd}(a, L) < \text{cd}(a, M)\). We also show that if \(H^\text{dim}_{\mathfrak{m}}(M) = 0\), then \(\text{Att}_R(H^\text{dim}_{\mathfrak{m}}-1(M)) = \{p \in \text{Supp}(M) \mid \text{cd}(a, R/p) = \text{dim} M - 1\}\), and so the attached primes of \(H^\text{dim}_{\mathfrak{m}}-1(M)\) depends only on \(\text{Supp}(M)\). Finally, we prove that if \(M\) is an arbitrary module (not necessarily finitely generated) over a Noetherian ring \(R\) with \(\text{cd}(a, M) = \text{cd}(a, R/\text{Ann}_R M)\), then \(\text{Att}_R(H^\text{dim}_{\mathfrak{m}}-1(M)) \subseteq \{p \in \text{Ass}_R M \mid \text{cd}(a, R/p) = \text{dim} M\}\).

1. Introduction

Let \(R\) be an arbitrary commutative Noetherian ring (with identity), \(a\) an ideal of \(R\) and let \(M\) be a finitely generated \(R\)-module. Recall that the \(i\)th local cohomology module of \(M\) with support in \(V(a)\) is defined by \(H^i_a(M) := \lim_{n \to \infty} \text{Ext}^i_R(R/a^n, M)\). As the structure of local cohomology modules in general seems to be quite mysterious, one tries to establish some properties providing a better understanding of these modules. Among these properties, an interesting question is determining the annihilators of local cohomology modules. This problem has been studied by several authors; see for example \([2, 3, 11, 12, 13, 16, 17, 19]\), and has led to some interesting results. A very interesting result shows that if \(R\) is a regular local ring containing a field, then \(H^i_a(R) \neq 0\), if and only if \(\text{Ann}_R(H^i_a(R)) = 0\), for all \(i \geq 0\), cf. \([11]\) (in positive characteristic) and \([13]\) (in characteristic zero). One purpose of the present paper is to establish some new results concerning of the annihilators of local cohomology modules \(H^i_a(M)\) \((i \in \mathbb{N}_0)\). As a main result in the second section, we determine the annihilators of the local cohomology module \(H^i_a(M)\) in several cases. More precisely, we shall prove the following theorem:

**Theorem 1.1.** Let \(R\) be a Noetherian ring, \(a\) an ideal of \(R\) and \(M\) a non-zero finitely generated \(R\)-module such that \(\text{cd}(a, M) = c\).

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Proposition 1.2. Let $R$ be a local (Noetherian) ring and let $x = x_1, \ldots, x_n$ be elements of $R$. Then for any finitely generated $R$-module $M$,

$$(\text{Ann}_R M)^{n+1} \subseteq (\text{Ann}_R (H^n_{x_R}(M))) \cdots (\text{Ann}_R H^n_{x_R}(M))) \subseteq \text{Ann}_R M.$$  

Another basic problems concerning local cohomology is to find the set of attached primes of $H^n_a(M)$. In Section 3, we obtain some results about the attached primes of local cohomology modules. In this section among other things, we derive the following result, which is an extension of the main theorems of [3] and [9].

Theorem 1.3. Let $R$ be an arbitrary Noetherian ring and $a$ an ideal of $R$. Let $M$ be an $R$-module (not necessarily finitely generated) with $\dim M = \dim R$. Then

$$\text{Att}_R (H^\dim M_a(M)) \subseteq \{ p \in \text{Ass}_R M \mid \text{cd}(a, R/p) = \dim M \}.$$  

The result in Theorem 1.3 is proved in Corollary 3.14. One of our tools for proving Theorem 1.3 is the following.

Proposition 1.4. Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ be an $R$-module (not necessarily finitely generated) such that $\text{cd}(a, M) = \text{cd}(a, R/\text{Ann}_R M)$. Then

$$\text{Att}_R (H^\text{cd}(a,M)_{a}(M)) \subseteq \{ p \in V(\text{Ann}_R M) \mid \text{cd}(a, R/p) = \text{cd}(a, M) \}.$$  

For an $R$-module $A$, a prime ideal $p$ of $R$ is said to be attached prime to $A$ if $p = \text{Ann}_R (A/B)$ for some submodule $B$ of $A$. We denote the set of attached primes of $A$ by $\text{Att}_R A$. This definition agrees with the usual definition of attached prime if $A$ has a secondary representation (cf. [14, Theorem 2.5]).

Another main result in Section 3 is to give a complete characterization of the attached primes of the local cohomology module $H^\dim M_a(M)$. More precisely, we shall show the following result, which is an extension of the main theorems of [3] and [9].
Theorem 1.5. Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ be a finitely generated $R$-module such that $H^d_{\mathfrak{a}}(M) = 0$. Then

$$\text{Att}_R(H^d_{\mathfrak{a}}(M)) = \{ \mathfrak{p} \in \text{Supp}(M) | \text{cd}(\mathfrak{a}, R/\mathfrak{p}) = \text{dim } M - 1 \}.$$ 

As a consequence of Theorem 1.5 we show that the set $\text{Att}_R(H^d_{\mathfrak{a}}(M))$ depends on $\text{Supp}(M)$ only, whenever $H^d_{\mathfrak{a}}(M) = 0$. More precisely, we shall show that:

Corollary 1.6. Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ and $N$ be two non-zero finitely generated $R$-modules with $\text{dim } M = d$ and $H^d_a(M) = 0$. If $\text{Supp}(M) = \text{Supp}(N)$, then $\text{Att}_R(H^d_{\mathfrak{a}}^d(\mathfrak{N})) = \text{Att}_R(H^d_{\mathfrak{a}}^d(M))$.

2. Annihilators of Top Local Cohomology Modules

Let us, firstly, recall the important concept of the cohomological dimension of an $R$-module $L$ with respect to an ideal $\mathfrak{a}$ of a commutative Noetherian ring $R$, denoted by $\text{cd}(\mathfrak{a}, L)$, is the largest integer $i$ such that $H^i_{\mathfrak{a}}(L) \neq 0$; i.e., $\text{cd}(\mathfrak{a}, L) := \sup\{ i \in \mathbb{Z} | H^i_{\mathfrak{a}}(L) \neq 0 \}$. The first main observation of this section is Theorem 2.2. The following lemma plays a key role in the proof of that theorem.

Lemma 2.1. Let $R$ be a Noetherian ring, $\mathfrak{a}$ an ideal of $R$ and let $M$ be a finitely generated $R$-module with finite dimension $c$ such that $\text{cd}(\mathfrak{a}, M) = c$. Then

$$\text{Ann}_R(H^c_{\mathfrak{a}}(M)) = \text{Ann}_R(M/H^c_{\mathfrak{a}}(M)) = \text{Ann}_R(M/\cap_{\text{cd}(\mathfrak{a}, R/\mathfrak{p}) = c} N_j).$$

Here $0 = \cap_{j=1}^n N_j$ denotes a reduced primary decomposition of zero submodule 0 in $M$ and $N_j$ is a $\mathfrak{p}_j$-primary submodule of $M$, for all $j = 1, \ldots, n$ and $\mathfrak{b} = \prod_{\text{cd}(\mathfrak{a}, R/\mathfrak{p}) = c} \mathfrak{p}_j$.

Proof. See [3, Corollary 2.7].

We are now ready to state and prove the first main result of this section.

Theorem 2.2. Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ be a non-zero finitely generated $R$-module with finite dimension $c$ such that $\text{cd}(\mathfrak{a}, M) = c$. Then

$$\text{Ann}_R(H^c_{\mathfrak{a}}(M)) = \cap_{\text{cd}(\mathfrak{a}, R/\mathfrak{p}) = c} \mathfrak{q}_i.$$ 

Here $\text{Ann}_R M = \cap_{i=1}^m \mathfrak{q}_i$, denotes a reduced primary decomposition of the ideal $\text{Ann}_R M$ and $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal of $R$, for all $i = 1, \ldots, m$. 

Proof. Let \(0 = \cap_{i=1}^{n} N_i\) denote a reduced primary decomposition of zero submodule \(0\) in \(M\) and \(N_i\) is a \(p_i\)-primary submodule of \(M\), for all \(i = 1, \ldots, n\). Since \(m\Ass_R M = m\Ass_R(R/Ann_R M)\), without loss of generality we may (and do) assume that, there is some \(t \leq \min\{m, n\}\), such that \(m\Ass_R M = \{p_1, \ldots, p_t\}\), \(m\Ass_R(R/Ann_R M) = \{p_1, \ldots, p_t\}\) and \(p_i = p_i\), for each \(i \leq t\). On the other hand, it follows from \(0 = \cap_{i=1}^{n} N_i\) that \(Ann_R M = \cap_{i=1}^{t} (N_i : R M)\) is a primary decomposition (not necessarily reduced) of \(Ann_R M\), and so \(q_i = (N_i : R M)\), for each \(i \leq t\). Now, in view of Lemma 2.1 we have

\[
Ann_R(H^c_a(M)) = Ann_R(M/ \cap_{p_i \in Ass_R M, cd(a,R/p_i')=c} N_i)
\]

\[
= \cap_{p_i \in Ass_R M, cd(a,R/p_i')=c} (N_i : R M)
\]

\[
= \cap_{i=1}^{t} Ass_R(R/Ann_R M), cd(a,R/p_i)=c q_i,
\]

as required. □

The first application of Theorem 2.2 improves the main results of [19, Corollary 2.9] and [12, Corollary 2.5].

Corollary 2.3. Let \(R\) be a Noetherian ring and \(a\) an ideal of \(R\). Let \(M\) be a non-zero finitely generated \(R\)-module with finite dimension \(c\) such that \(cd(a,M) = c\). Then the following conditions are equivalent:

(i) \(Ann_R H^c_a(M) = Ann_R M\).

(ii) \(\Ass_R(R/Ann_R M) = Att_R H^c_a(M)\).

In particular, when \(M\) is a faithful \(R\)-module we have \(Ann_R H^c_a(M) = 0\) if and only if \(\Ass_R R = Att_R H^c_a(M)\).

Proof. The assertion follows from Theorem 2.2 and [3, Corollary 3.4]. □

The following lemma will be used in Proposition 2.5 and Theorem 2.6.

Lemma 2.4. Let \(R\) be a Noetherian ring and \(a\) an ideal of \(R\). Let \(M\) and \(N\) be two non-zero finitely generated \(R\)-modules such that \(\Supp(N) \subseteq \Supp(M)\). Then \(cd(a,N) \leq cd(a,M)\).

Proof. See [3, Theorem 2.2]. □

Proposition 2.5. Let \(R\) be a Noetherian ring and let \(a, b, c\) be ideals of \(R\). Let \(M\) be a finitely generated \(R\)-module such that \(H^c_a(M/aM) \cong H^c_a(M/bM) \cong H^c_a(M)\), where \(c := cd(a,M)\) is finite. Then

\[
H^c_a(M) \cong H^c_a(M/(aM + bM)).
\]

Proof. Since

\[
H^c_a(M/aM) \cong H^c_a(M/bM) \cong H^c_a(M),
\]

it yields that \((a + b)H^c_a(M) = 0\). Now as, \(\Supp(M) = \Supp(R/Ann_R M)\), it follows from Lemma 2.4 that \(cd(a,R/Ann_R M) = c\). Now it follows from [10, Exercise 24] and
Independence theorem (cf. [5, Theorem 4.2.1]) that

$$H^c_a(M/(bM + cM)) \cong H^c_{a/R/\text{Ann}_R M}(R/\text{Ann}_R M \otimes_{R/\text{Ann}_R M} M/(bM + cM))$$

$$\cong H^c_{a/R/\text{Ann}_R M}(R/\text{Ann}_R M) \otimes_{R/\text{Ann}_R M} M/(bM + cM)$$

$$\cong H^c_{a/R/\text{Ann}_R M}(R/\text{Ann}_R M \otimes_{R/\text{Ann}_R M} M \otimes_R R/(b + c))$$

$$\cong H^c_{a/R/\text{Ann}_R M}(R/\text{Ann}_R M \otimes_{R/\text{Ann}_R M} M) \otimes_R (b + c)$$

$$\cong H^c_a(M) \otimes_R (b + c) \cong H^c_a(M)/(b + c)H^c_a(M) = H^c_a(M),$$

as required.

Theorem 2.6. Let $R$ be a Noetherian ring and $M$ a finitely generated $R$-module. Let $a$ be an ideal of $R$ such that $c := \text{cd}(a, M)$ is finite. Then the set

$$\Sigma := \{c \mid c \text{ is an ideal of } R \text{ and } H^c_a(M) \cong H^c_a(M/cM)\}$$

has a largest element with respect to inclusion, $b$ say; and $b = \text{Ann}_R(H^c_a(M))$.

Proof. Since $R$ is a Noetherian ring, the set $\Sigma$ has a maximal member, $b$ say. (Note that $\text{Ann}_R M \in \Sigma$, and so $\Sigma$ is not empty.) Since by Proposition 2.5 the sum of any two members of $\Sigma$ is again in $\Sigma$, it follows that $b$ contains every member of $\Sigma$, and so is the largest element of $\Sigma$. Now, we show that $b = \text{Ann}_R(H^c_a(M))$. To this end, as $H^c_a(M) \cong H^c_a(M/bM)$, it follows that $b \subseteq \text{Ann}_R(H^c_a(M))$. To establish the reverse inclusion, let $x \in \text{Ann}_R(H^c_a(M))$, and we show that $x \in b$. Because of $b$ is the largest element of $\Sigma$, it is enough for us to show that $H^c_a(M) \cong H^c_a(M/xM)$. As $\text{Supp}(M) = \text{Supp}(R/\text{Ann}_R M)$, it follows from Lemma 2.4 that $\text{cd}(a, R/\text{Ann}_R M) = c$, and so it follows from [10 Exercise 24] and Independence theorem (see [5, Theorem 4.2.1]) that

$$H^c_a(M/xM) \cong H^c_{a/R/\text{Ann}_R M}(R/\text{Ann}_R M \otimes_{R/\text{Ann}_R M} M/xM)$$

$$\cong H^c_{a/R/\text{Ann}_R M}(R/\text{Ann}_R M) \otimes_{R/\text{Ann}_R M} M/xM$$

$$\cong H^c_{a/R/\text{Ann}_R M}(R/\text{Ann}_R M \otimes_{R/\text{Ann}_R M} M) \otimes_{R/xR} M/xR$$

$$\cong H^c_{a/R/\text{Ann}_R M}(R/\text{Ann}_R M \otimes_{R/\text{Ann}_R M} M) \otimes_{R/xR} M = H^c_a(M) \otimes_{R/xR} M = H^c_a(M)/xH^c_a(M) = H^c_a(M),$$

as required.

The following result follows by the similar argument as in the proof of [20] Lemma 2.4.4], but we give a direct proof for the convenience of the reader.

Proposition 2.7. Let $R$ be a local (Noetherian) ring and let $x = x_1, \ldots, x_n$ be elements of $R$. Let $a$ be an ideal of $R$ such that $\text{Rad}(a) = \text{Rad}(xR)$. Then for any finitely generated $R$-module $M$,

$$(\text{Ann}_R M)^{n+1} \subseteq (\text{Ann}_R(H^0_a(M)) \cdots (\text{Ann}_R(H^n_a(M))) \subseteq \text{Ann}_R M.$$
t. Because of the cohomology modules \(H^i(x^t, M), i \in \mathbb{Z}\), of the Koszul complex \(K(\cdot)\) are annihilated by \(x^t\) for any positive integer \(t\), it follows that the support of \(H^i(x^t, M), i \in \mathbb{Z}\), is contained in \(V(\mathfrak{a})\). Now, in view of Exercise 19 Corollary 1,

\[
(\text{Ann}_R(H_{\mathfrak{a}}^0(M))) \cdots (\text{Ann}_R(H_{\mathfrak{a}}^n(M)))H^n(x^t, M) = 0,
\]

for any positive integer \(t\). Now since \(H^n(x^t, M) = M/x^tM\) we deduce that

\[
(\text{Ann}_R(H_{\mathfrak{a}}^0(M))) \cdots (\text{Ann}_R(H_{\mathfrak{a}}^n(M)))M \subseteq x^tM \subseteq \mathfrak{a}^tM,
\]

for any positive integer \(t\), and so we have

\[
(\text{Ann}_R(H_{\mathfrak{a}}^0(M))) \cdots (\text{Ann}_R(H_{\mathfrak{a}}^n(M)))M \subseteq \bigcap_{t \geq 1} \mathfrak{a}^tM.
\]

Now the result follows from the Krull’s intersection theorem. \(\square\)

Recall that the arithmetic rank of an ideal \(\mathfrak{a}\) in a Noetherian ring \(R\), denoted by \(\text{ara}(\mathfrak{a})\), is the least number of elements of \(R\) required to generate an ideal which has the same radical as \(\mathfrak{a}\), i.e.,

\[
\text{ara}(\mathfrak{a}) = \min\{n \in \mathbb{N}_0 : \exists a_1, \ldots, a_n \in R \text{ with } \text{Rad}(a_1, \ldots, a_n) = \text{Rad}(\mathfrak{a})\}.
\]

**Corollary 2.8.** Let \(R\) be a local (Noetherian) ring and \(\mathfrak{a}\) an ideal of \(R\) with \(\text{ara}(\mathfrak{a}) := n\). Let \(M\) be a finitely generated \(R\)-module. Then

\[
(\text{Ann}_R(H_{\mathfrak{a}}^0(M))) \cdots (\text{Ann}_R(H_{\mathfrak{a}}^n(M))) \subseteq \text{Ann}_R M.
\]

**Proof.** The assertion follows from Proposition 2.7 and the definition of \(\text{ara}(\mathfrak{a})\). \(\square\)

**Corollary 2.9.** Let \(R\) be a local (Noetherian) ring, \(\mathfrak{a}\) an ideal of \(R\) and let \(M\) be a non-zero finitely generated \(R\)-module such that \(\text{grade}(\mathfrak{a}, M) := g\) and \(\text{cd}(\mathfrak{a}, M) := c\). Then

\[
(\text{Ann}_R(H_{\mathfrak{a}}^0(M))) \cdots (\text{Ann}_R(H_{\mathfrak{a}}^g(M))) \subseteq \text{Ann}_R M.
\]

**Proof.** The assertion follows from Proposition 2.7 and the fact that \(\text{cd}(\mathfrak{a}, M) \leq \text{ara}(\mathfrak{a})\). \(\square\)

Before bringing the next theorem we recall the following definition.

**Definition 2.10.** Let \(R\) be a Noetherian ring, \(\mathfrak{a}\) an ideal of \(R\) and let \(M\) be a finitely generated \(R\)-module. We denote by \(T_R(\mathfrak{a}, M)\) the largest submodule of \(M\) such that \(\text{cd}(\mathfrak{a}, T_R(\mathfrak{a}, M)) < \text{cd}(\mathfrak{a}, M)\). It follows from Lemma 2.4 that

\[
T_R(\mathfrak{a}, M) = \bigcup\{N | N \leq M \text{ and } \text{cd}(\mathfrak{a}, N) < \text{cd}(\mathfrak{a}, M)\}.
\]

The following theorem improves [12, Theorem 3.3]. To this end, recall that a non-zero finitely generated \(R\)-module \(L\) is called a relative Cohen-Macaulay module with respect to an ideal \(\mathfrak{a}\) of \(R\) if there is precisely one non-vanishing local cohomology module of \(L\) with respect to \(\mathfrak{a}\); that is \(\text{grade}(\mathfrak{a}, L) = \text{Rad}(\mathfrak{a}, L)\).

**Theorem 2.11.** Let \((R, \mathfrak{m})\) be a local (Noetherian) ring and \(\mathfrak{a}\) an ideal of \(R\). Let \(M\) be a relative Cohen-Macaulay \(R\)-module with respect to \(\mathfrak{a}\) such that \(\text{cd}(\mathfrak{a}, M) := c\). Then

\[
\text{Ann}_R(H_{\mathfrak{a}}^c(M)) = \text{Ann}_R M/T_R(\mathfrak{a}, M) = \text{Ann}_R M.
\]
Proof. Since
\[ \text{Ann}_R M \subseteq \text{Ann}_R M/T_R(a, M) \subseteq \text{Ann}_R(H^c(M)), \]
it is enough for us to show that \( \text{Ann}_R(H^c(M)) \subseteq \text{Ann}_R M \). To do this, as
\[ \text{grade}(a, M) = \text{cd}(a, M), \]
the assertion follows from Corollary 2.9.

The next corollary improves [12, Corollary 3.5].

**Corollary 2.12.** Let \((R, m)\) be a local (Noetherian) ring, \(a\) an ideal of \(R\) and let \(M\) be a non-zero finitely generated \(R\)-module. Suppose that \(a\) is generated by an \(M\)-regular sequence of length \(n\). Then
\[ \text{Ann}_R(H^n_a(M)) = \text{Ann}_R(M/T_R(a, M)) = \text{Ann}_R M. \]

**Proof.** Since \(H^n(M) \neq 0\) and \(H^i_a(M) = 0\) for all \(i \neq n\), it follows that \(\text{grade}(a, M) = \text{cd}(a, M) = n\). Now, the assertion follows from Theorem 2.11.

**Remark 2.13.** Let \(R\) be a Noetherian ring and \(a\) an ideal of \(R\). Let \(M\) be a relative Cohen-Macaulay \(R\)-module with respect to \(a\) such that \(\text{cd}(a, M) := c\). Then, it is easy to see that
\[ \text{Supp}(H^c(M)) = \text{Supp}(M) \cap V(a). \]

**Corollary 2.14.** Let \((R, m)\) be a local (Noetherian) ring and \(a\) an ideal of \(R\). Let \(M\) be a non-zero finitely generated \(R\)-module such that \(\text{cd}(a, M) = 0\). Then
\[ \text{Ann}_R(H^0_a(M)) = \text{Ann}_R(M/T_R(a, M)) = \text{Ann}_R M. \]
In particular we have \(\text{Supp}(H^0_a(M)) = \text{Supp}(M) \subseteq V(a)\).

**Proof.** The assertion follows from Theorem 2.11 and Remark 2.13.

**Lemma 2.15.** Let \(R\) be a Noetherian ring and \(a\) an ideal of \(R\). Let \(M\) be a non-zero finitely generated \(R\)-module such that \(\text{cd}(a, M) = 0\). Then
\[ \text{Ann}_R(H^0_a(M)) = \text{Ann}_R(M/T_R(a, M)). \]

**Proof.** It follows from \(H^0_a(M) \cong H^0_a(M/T_R(a, M))\) that
\[ \text{Ann}_R(M/T_R(a, M)) \subseteq \text{Ann}_R(H^0_a(M)). \]
Now, let \(x \in \text{Ann}_R(H^0_a(M))\). Then from the exact sequence
\[ 0 \to (0 :_M xR) \to M \xrightarrow{x} xM \to 0, \]
we obtain the exact sequence
\[ H^0_a(M) \xrightarrow{x} H^0_a(xM) \to 0. \]
Since \(x \in \text{Ann}_R(H^0_a(M))\), we get that \(H^0_a(xM) = 0\), and so \(xM \subseteq T_R(a, M)\), as required.

\[ \square \]
Corollary 2.16. Let \((R, \mathfrak{m})\) be a local (Noetherian) ring and \(\mathfrak{a}\) an ideal of \(R\). Let \(M\) be a non-zero finitely generated \(R\)-module such that \(\text{cd}(\mathfrak{a}, M) = 1\). Then
\[
\text{Ann}_R(H^1_{\mathfrak{a}}(M)) = \text{Ann}_R(M/T_R(\mathfrak{a}, M)).
\]

Proof. Since
\[
H^1_{\mathfrak{a}}(M) \cong H^1_{\mathfrak{a}}(M/H^0_{\mathfrak{a}}(M)) \text{ and } T_R(\mathfrak{a}, M/H^0_{\mathfrak{a}}(M)) = T_R(\mathfrak{a}, M)/H^0_{\mathfrak{a}}(M),
\]
so without loss of generality we may assume that \(H^0_{\mathfrak{a}}(M) = 0\). Now, the assertion follows from Theorem 2.11. \(\square\)

Corollary 2.17. Let \((R, \mathfrak{m})\) be a local (Noetherian) ring and \(\mathfrak{a}\) an ideal of \(R\). Let \(M\) be a relative Cohen-Macaulay \(R\)-module with respect to \(\mathfrak{a}\) such that \(\text{cd}(\mathfrak{a}, M) := c\). Then
\[
\text{Ass}_R(R/\text{Ann}_R M) \subseteq \{ \mathfrak{p} \in \text{Ass}(M) \mid \text{cd}(\mathfrak{a}, R/\mathfrak{p}) = c \}.
\]

Proof. In view of Theorem 2.11 and Lemma 2.1, there is a primary decomposition (not necessarily reduced) of \(\text{Ann}_R M\),
\[
\text{Ann}_R M = \bigcap_{\mathfrak{p}_j \in \text{Ass}_R M, \text{cd}(\mathfrak{a}, R/\mathfrak{p}_j) = c} (N_j : M).
\]
Therefore, there is some reduced primary decomposition of \(\text{Ann}_R M\) as
\[
\text{Ann}_R M = \bigcap_{\mathfrak{p}_j \in \text{Ass}_R(R/\text{Ann}_R M), \text{cd}(\mathfrak{a}, R/\mathfrak{p}_j) = c} (N_j : M).
\]
Now, the assertion follows from this. \(\square\)

3. Attached primes of local cohomology modules

In this section we will investigate the attached prime ideals of local cohomology modules. As the first main result, we will give a complete characterization of the attached primes of the local cohomology module \(H^\dim M - 1_{\mathfrak{a}}(M)\), which is a generalization of the main results of [3, Theorem 3.7] and [9, Theorem 2.3]. By using this characterization we show that the set \(\text{Att}_R(H^\dim _{\mathfrak{a}} M - 1(M))\) depends on \(\text{Supp}(M)\) only, whenever \(H^\dim _{\mathfrak{a}} M(M) = 0\). We begin with:

Definition 3.1. Let \(L\) be an \(R\)-module. We say that a prime ideal \(\mathfrak{p}\) of \(R\) is an attached prime of \(L\), if there exists a submodule \(K\) of \(L\) such that \(\mathfrak{p} = \text{Ann}_R(L/K)\) or equivalently \(\mathfrak{p} = \text{Ann}_R(L/\mathfrak{p}L)\). We denote by \(\text{Att}_R L\) (resp. \(\text{mAtt}_R L\)) the set of attached primes of \(L\) (resp. the set of minimal attached primes of \(L\)).

When \(M\) is representable in the sense of [14] (e.g. Artinian or injective), our definition of \(\text{Att}_R L\) coincides with that of Macdonald [14] and Sharp [21]. The following corollary is a consequence of Theorem 2.11.

Corollary 3.2. Let \((R, \mathfrak{m})\) be a local (Noetherian) ring and \(\mathfrak{a}\) an ideal of \(R\). Let \(M\) be a relative Cohen-Macaulay \(R\)-module with respect to \(\mathfrak{a}\) such that \(\text{cd}(\mathfrak{a}, M) := c\). Then
\[
\text{mAss}_R M = \text{mAtt}_R(H^c_{\mathfrak{a}}(M)).
\]
Proof. The assertion follows from Theorem 2.11 and \[3\] Lemma 3.2. □

The following theorem is our first main result of this section which extends the main results of \[3\] Theorem 3.7 and \[9\] Theorem 2.3.

**Theorem 3.3.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ be a non-zero finitely generated $R$-module of finite dimension $d$ such that $H^d_a(M) = 0$. Then

$$\text{Att}_R H^{d-1}_a(M) = \{ p \in \text{Supp}(M) | \text{cd}(a, R/p) = d - 1 \}.$$ 

**Proof.** Since

$$H^{d-1}_a(M) \cong H^{d-1}_a(R/\text{Ann}_R M) \otimes_{R/\text{Ann}_R M} M,$$

so, by \[1\] Lemma 2.11, we have

$$\text{Att}_{R/\text{Ann}_R M} H^{d-1}_a(M) = \text{Att}_{R/\text{Ann}_R M} H^{d-1}_a(R/\text{Ann}_R M) \cap \text{Supp}_{R/\text{Ann}_R M}(M) = \text{Att}_{R/\text{Ann}_R M} H^{d-1}_a(R/\text{Ann}_R M).$$

So, in view of \[3\] Theorem 3.7, we have

$$\text{Att}_{R/\text{Ann}_R M} H^{d-1}_a(M) = \{ p/\text{Ann}_R M \in \text{Spec}(R/\text{Ann}_R M) | \text{cd}(a, R/p) = d - 1 \}.$$ 

Now, as

$$\text{Att}_R H^{d-1}_a(M) = \{ p \in \text{Supp} M | p/\text{Ann}_R M \in \text{Att}_{R/\text{Ann}_R M} H^{d-1}_a(M) \},$$

the assertion follows. □

**Corollary 3.4.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ and $N$ be two non-zero finitely generated $R$-modules with $\dim M = d$ and $H^d_a(M) = 0$. If $\text{Supp}(N) \subseteq \text{Supp}(M)$, then $\text{Att}_R(H^{d-1}_a(N)) \subseteq \text{Att}_R(H^{d-1}_a(M))$.

**Proof.** By Lemma 2.4, we can (and do) assume that $\text{cd}(a, N) = \text{cd}(a, M) = d - 1$. Now, the assertion follows from Theorem 3.3. □

The next consequence of Theorem 3.3 shows that the attached primes of the local cohomology module $H^{\dim M - 1}_a(M)$, depends only on $\text{Supp}(M)$, whenever $H^{\dim M}_a(M) = 0$.

**Corollary 3.5.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ and $N$ be two finitely generated $R$-modules with $\dim M = d$ and $H^d_a(M) = 0$. If $\text{Supp}(M) = \text{Supp}(N)$, then $\text{Att}_R(H^{d-1}_a(N)) = \text{Att}_R(H^{d-1}_a(M))$.

**Proof.** The assertion follows from Lemma 2.4 and Corollary 3.4. □

**Lemma 3.6.** Let $R$ be a Noetherian domain and $a$ an ideal of $R$ such that $\text{cd}(a, R) = 1$. Then $\text{Ann}_R H^1_a(R) = 0$.

**Proof.** Suppose in contrary that $x(\neq 0) \in \text{Ann}_R H^1_a(R)$. Then, Since $H^0_a(R) = 0$, by the exact sequence

$$0 \rightarrow R x \rightarrow R \rightarrow R/xR \rightarrow 0,$$
we get $H^0_a(R/xR) \cong H^1_a(R)$. This is a contradiction, because, $H^0_a(R/xR)$ is a finitely generated $R$-module, but $H^1_a(R)$, is not.

\[\square\]

**Proposition 3.7.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$ such that $\text{cd}(a, R) = 1$. Then

$$\text{Att}_R H^1_a(R) = \{p \in \text{Spec } R | \text{cd}(a, R/p) = 1\}.$$ 

**Proof.** In view of [3, Theorem 3.3], we have

$$\text{Att}_R H^1_a(R) \subseteq \{p \in \text{Spec } R | \text{cd}(a, R/p) = 1\}.$$ 

Now, let $p$ be a prime ideal of $R$ such that $\text{cd}(a, R/p) = 1$. Then, by Lemma 3.7, $\text{Ann}_R H^1_a(R/p) = p$. Thus by Definition 3.1, we have $p \in \text{Att}_R H^1_a(R/p)$. Now, from the exact sequence

$$0 \longrightarrow p \longrightarrow R \longrightarrow R/p \longrightarrow 0,$$

and the right exactness of $H^1_a(-)$, we deduce that $p \in \text{Att}_R H^1_a(R)$, as required. \[\square\]

The following result gives a partial answer to [4 Question (i)], in the case $\text{cd}(a, M) = 1$.

**Theorem 3.8.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ be a non-zero finitely generated $R$-module such that $\text{cd}(a, M) = 1$. Then

$$\text{Att}_R H^1_a(M) = \{p \in \text{Supp}(M) | \text{cd}(a, R/p) = 1\}.$$ 

**Proof.** The proof is similar to the proof of Theorem 3.3, by using Proposition 3.7 instead of [3, Theorem 3.7]. \[\square\]

The next corollary reproves [9, Corollary 2.4].

**Corollary 3.9.** Let $R$ be a local (Noetherian) ring and $a$ an ideal of $R$. Let $M$ be a non-zero finitely generated $R$-module such that $\text{cd}(a, M) \leq 1$. Then

$$\text{Att}_R H^1_a(M) = \text{Supp}(M) \setminus V(a).$$

In particular, if $x \in R$, then $\text{Att}_R H^1_x(M) = \text{Supp}(M) \setminus V(xR)$.

**Proof.** The assertion follows from Corollary 2.14 and Theorem 3.8. \[\square\]

**Remark 3.10.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ be a non-zero finitely generated $R$-module such that $\text{cd}(a, M) = 0$. Then it is not hard to see that $p \in V(a)$ if and only if $\text{cd}(a, R/p) = 0$, for each $p \in \text{Supp}(M)$.

**Lemma 3.11.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ be a non-zero finitely generated $R$-module such that $\text{cd}(a, M) = 0$. Then

$$\text{Att}_R H^0_a(M) = \{p \in \text{Supp}(M) | \text{cd}(a, R/p) = 0\}.$$
Proof. Since $H_0^0(M)$ is a finitely generated $R$-module, we have
\[ \text{Att}_R H_0^0(M) = \text{Supp}(H_0^0(M)) . \]
Now, the assertion follows from Remarks 2.13 and 3.10. \qed

The following theorem shows that [3, Question (i)], is true in the case $\dim M \leq 3$.

**Theorem 3.12.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ be a non-zero finitely generated $R$-module with $\dim M \leq 3$ and $c := \text{cd}(a, M)$. Then
\[ \text{Att}_R H_c^0(M) = \{ p \in \text{Supp}(M) | \text{cd}(a, R/p) = c \} . \]

**Proof.** The result follows from Lemma 3.11, Theorems 3.3, 3.8 and [3, Corollary 3.4]. \qed

It is natural to ask about the attached primes of $\text{Att}_R H_{\text{cd}(a, M)}^c(M)$ for arbitrary $R$-module $M$ (not necessarily finitely generated). The following is our second main result of this section which gives a partial answer to this question. As a consequence we extend the main results of [6] and [18].

**Theorem 3.13.** Let $R$ be a Noetherian ring and $a$ an ideal of $R$. Let $M$ be a non-zero $R$-module (not necessarily finitely generated) such that $c := \text{cd}(a, M) = \text{cd}(a, R/\text{Ann}_R M)$ is finite. Then
\[ \text{Att}_R H_{\text{cd}(a, M)}^c(M) \subseteq \{ p \in \text{Ass}_R M | \text{cd}(a, R/p) = c \} . \]

**Proof.** Let $p \in \text{Att}_R H_{\text{cd}(a, M)}^c(M)$. Then clearly we have $\text{Ann}_R M \subseteq p$ and so $\text{Supp}(R/p) \subseteq \text{Supp}(R/\text{Ann}_R M)$.

Hence by using Lemma 2.4 we see that $\text{cd}(a, R/p) \leq c$. On the other hand, it follows from $p \in \text{Att}_R H_{\text{cd}(a, M)}^c(M)$ that
\[ p/\text{Ann}_R M \in \text{Att}_{R/\text{Ann}_R M} H_{\text{cd}(a, M)}^c(M) , \]
and so
\[ H_{\text{cd}(a, M)}^c(M) \otimes_{R/\text{Ann}_R M} R/p \neq 0 . \]

Now, as
\[ H_{\text{cd}(a, M)}^c(M) \cong H_{\text{cd}(a, R/\text{Ann}_R M)}^c(R/\text{Ann}_R M) \otimes_{R/\text{Ann}_R M} M , \]
it follows that
\[ H_{\text{cd}(a, R/\text{Ann}_R M)}^c(R/\text{Ann}_R M) \otimes_{R/\text{Ann}_R M} M \otimes_{R/\text{Ann}_R M} R/p \neq 0 . \]

Consequently
\[ H_{\text{cd}(a, R/p)}^c(R/p) \otimes_{R/\text{Ann}_R M} M \neq 0 , \]
and thus $H_{\text{cd}(a, R/p)}^c(R/p) \neq 0$, as required. \qed

The final result extends the main result of [6, Theorem B].

**Corollary 3.14.** Let $R$ be an arbitrary Noetherian ring and $a$ an ideal of $R$. Let $M$ be a non-zero $R$-module (not necessarily finitely generated) such that $d := \dim M = \dim R$ is finite. Then
\[ \text{Att}_R H_d^d(M) \subseteq \{ p \in \text{Ass}_R M | \text{cd}(a, R/p) = d \} . \]
Proof. We may assume that \( \cd(a, M) = d \). Then by Theorem 3.13 it is enough to show that \( \cd(a, R/\Ann_R M) = d \). To do this, as \( \Supp(M) \subseteq \Supp(R/\Ann_R M) \), it follows from [7, Theorem 1.4] that
\[
d = \cd(a, M) \leq \cd(a, R/\Ann_R M) \leq \cd(a, R) \leq \dim R = d,
\]
as required. \( \Box \)

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