A PRODUCT OF GAMMA FUNCTION VALUES AT FRACTIONS WITH THE SAME
DENOMINATOR

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ABSTRACT. We give an exact formula for the product of the values of Euler’s Gamma function evaluated at all rational numbers between 0 and 1 with the same denominator in lowest terms; the answer depends on whether or not that denominator is a prime power. A consequence is a surprisingly nice formula for the product of value of the Gamma function evaluated at the points of a Farey sequence.

Note: Since writing this note, I have been informed that Theorem 1 was already proved by Sándor and Tóth [5].

The purpose of this note is to establish the following classical-seeming theorem concerning Euler’s Γ-function evaluated at fractions that have the same denominator in lowest terms. The statement of the theorem uses (coincidentally) Euler’s function \( \phi(n) \), the number of integers between 1 and \( n \) that are relatively prime to \( n \), as well as von Mangoldt’s function \( \Lambda(n) \), defined to be \( \log p \) if \( n = p^r \) is a prime or a power of a prime and 0 otherwise.

**Theorem 1.** For \( n \geq 2 \),

\[
\prod_{k=1}^{n} \Gamma \left( \frac{k}{n} \right) = \frac{(2\pi)^{\phi(n)/2}}{\exp(\Lambda(n)/2)} = \begin{cases} 
(2\pi)^{\phi(n)/2} / \sqrt{p} & \text{if } n = p^r \text{ is a prime power}, \\ 
(2\pi)^{\phi(n)/2} & \text{otherwise}.
\end{cases}
\]

A few special cases of this theorem have been noted before (\( n = 2, 3, 4, 6 \) for example), and it follows for prime \( n \) from equation (1) below. Nijenhuis [4, page 4] established, by a more indirect method, the special case of the theorem where \( n \equiv 2 \pmod{4} \).

**Proof.** Gauss’s multiplication formula [11, equation (3.10)] says that

\[
\prod_{k=0}^{n-1} \Gamma \left( \frac{z+k}{n} \right) = (2\pi)^{(n-1)/2} n^{1/2-z} \Gamma(z)
\]

for any complex number \( z \) for which both sides are defined; taking \( z = 1 \) yields

\[
\prod_{k=1}^{n} \Gamma \left( \frac{k}{n} \right) = (2\pi)^{(n-1)/2} n^{-1/2}.
\]

Define the two functions

\[
F(n) = \sum_{k=1}^{n} \log \Gamma \left( \frac{k}{n} \right) \quad \text{and} \quad R(n) = \sum_{k=1}^{n} \log \Gamma \left( \frac{k}{n} \right),
\]

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It is immediate from these definitions that \( F(n) = \sum_{d|n} R(d) \); hence Möbius inversion \([3, \text{second displayed equation after equation (2.10)}]\) yields
\[
R(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).
\]
From equation (1) we see that \( F(n) = \log \left(\frac{(2\pi)^{(n-1)/2}n^{-1/2}}{2^{n-1}/2}\right) \), and so
\[
R(n) = \frac{1}{2} \log 2 \pi \sum_{d|n} \mu(d) n^{\frac{1}{2}} - \frac{1}{2} \log 2 \pi \sum_{d|n} \mu(d) - \frac{1}{2} \sum_{d|n} \mu(d) \log \frac{n}{d}.
\]
Each of these three divisor sums is standard in number theory (see \([3]\), where they appear as the first displayed equation in the proof of Theorem 2.1, equation (1.20), and the displayed equation before equation (2.10), respectively): as long as \( n \geq 2 \), we have
\[
R(n) = \left(\frac{1}{2} \log 2 \pi\right) \phi(n) - 0 - \frac{1}{2} \Lambda(n).
\]
Taking exponentials of both sides establishes the theorem.

It was known in the nineteenth century that the geometric mean of the \( \Gamma \) function on the interval \((0, 1]\) is \( \sqrt{2\pi} \), in the sense that
\[
\int_0^1 \log \Gamma(x) \, dx = \frac{1}{2} \log 2 \pi.
\]
(One can deduce this, for example, by integrating the Weierstrass formula \([1, \text{equation (2.9)}]\)
\[
\log \Gamma(z) = -\gamma z - \log z + \sum_{j=1}^{\infty} \left(\frac{1}{j} - \log \left(1 + \frac{1}{j}\right)\right)
\]
term by term; another proof uses the reflection formula \( \Gamma(z)\Gamma(1-z) = \pi \csc \pi z \) together with a known evaluation of the integral \( \int_0^{1/2} \log(\sin \pi x) \, dx \). Therefore if we multiply together \( n \) values of the \( \Gamma \) function on points in this interval, we would expect the product to be comparable to \( (2\pi)^{n/2} \). We can deduce from first principles that the product will be less than \( (2\pi)^{n/2} \) if we sample the \( \Gamma \) function at \( \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}, \) since \( \Gamma \) is decreasing on that interval; in fact, equation (1) tells us that the product will be less by a factor of precisely \( 1/\sqrt{2\pi n} \). Applying equation (1) twice, at \( 2n \) and \( n \), and dividing shows that we do better to sample at the midpoints, rather than the right-hand endpoints, of \( n \) intervals of equal length:
\[
\prod_{k=1}^{n} \Gamma\left(\frac{2k-1}{2n}\right) = \frac{(2\pi)^{n/2}}{\sqrt{2}}.
\] (2)

Theorem [1] tells us that sampling at the \( \phi(n) \) points \( \{\frac{k}{n} : 1 \leq k \leq n, (k, n) = 1\} \) curiously gives us exactly the default expectation \( (2\pi)^{\phi(n)/2} \), unless \( n \) is a prime power.

Finally, we comment that the \( \Lambda \)-function satisfies the identity \([3, \text{Section 2.2.1, exercise 1(a)}]\)
\[
\sum_{n=1}^{N} \Lambda(n) = \log \left(\text{lcm}[1, 2, \ldots, N]\right).
\]
This allows us to compute the product of the \( \Gamma \)-function sampled over points in a Farey sequence. Let \( F_N \) denote the set of all rational numbers in the open interval \((0, 1)\) whose denominator in

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Footnotes:

[3]: Reference to a specific text or source.

[1]: Reference to another text or source.

[2]: Footnote or reference marker added for citation purpose.
lowest terms is at most \( N \) (note that usually one includes the fractions \( \frac{0}{1} \) and \( \frac{1}{1} \) in this Farey sequence, but here we do not). Applying Theorem \( \Pi \) to \( n = 2, 3, \ldots, N \) and multiplying the identities together yields the formula

\[
\prod_{r \in F_N} \frac{\Gamma(r)}{\sqrt{2\pi}} = \left( \text{lcm}[1, 2, \ldots, N] \right)^{-1/2}.
\]  

(3)

It was noted by Luschny and Wehmeier [2] that this last equation is equivalent, via the reflection formula \( \Gamma(z)\Gamma(1 - z) = \pi \csc \pi z \), to the identity

\[
\text{lcm}[1, 2, \ldots, N] = \frac{1}{2} \left( \prod_{r \in F_N, r \leq 1/2} 2 \sin \pi r \right)^2;
\]

in fact they found an alternate proof using cyclotomic polynomials.

REFERENCES

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