Completely solvable stochastic Hamiltonian system describing a continuous-time integrated climate-economy model

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Abstract
The present paper proposes a stochastic continuous-time extension of a DICE model, a well-known dynamic integrated climate-economy model. The optimal path of the proposed model is described by a coupled forward-backward stochastic differential equation which is solvable in the sense that it is reduced to solving a system of deterministic linear equations.

Keywords stochastic Hamiltonian system, DICE model, coupled forward-backward stochastic differential equation

Research Activity Group Mathematical Finance

1. Introduction

Recently, in economics, finance and their related fields, stochastic economic growth models with optimal decisions are becoming more and more fundamental tools. Stochastic dynamical optimization problems are normally solved numerically by discrete-time approximation. In this paper, we propose an alternative approach to solve the optimal economic growth problem with climate change in a very explicit way, by exemplifying an integrated climate-economy model that can be understood as a stochastic continuous-time analogue of (a version of) DICE model appearing in [1].

The DICE (abbreviation of (Dynamic Integrated Climate-Economy) models describe the economic growth under the influence of climate changes which is optimal in some sense, and are now widely used as a tool to evaluate the social cost of the concentration of Green House Gasses (GHG for short). The models are originally proposed by William Nordhaus, who was rewarded the 2018 Nobel prize for the contributions though the DICE models.

The DICE models are basically a multi-variate extension of the classical Ramsey model, as they describe the relation between the dynamics of economic growth and that of the concentration of GHG. But most of them are in discrete-time.

The continuous-time Ramsey-type models are known to have an explicit solution under a set of parameter restrictions, but in stochastic and multi-variate extensions little has been known, to the best of the authors’ knowledge, about the possibility of an explicit solution.

In classical mechanics, such a structure that allows for an explicit solution is called integrability, and studied mostly in terms of Hamiltonian dynamics. We propose to follow this strategy, that is, to work on Hamiltonian dynamics, but of a stochastic extension. This contrast with the main stream in the stochastic control theory where the Bellman principle—Hamilton-Jacobi approach—plays a central role.

In this short note we concentrate on describing stochastic Hamiltonian system for a stochastic continuous-time analogue of a DICE model, aiming to exhibit how our new framework will work. The authors also believe that the propose model itself is a contribution to the environmental economics though in this paper we are totally lacking in the discussions on the data-analysis that really evaluate the social cost of GHG concentration.

The rest of the paper is organized as follows. In Section 2.1, we describe the model and how it is related to a DICE model is discussed in Section 2.2. Section 3 is the main part of the paper, where, first in section 3.1 a coupled forward-backward stochastic differential equation is derived using a stochastic maximum principle. Then, in Section 3.2, we show how it can be solved, using a canonical transformation. The essence of the solvability is discussed in Section 4, followed by literature review of stochastic and/or continuous-time extensions of the DICE models.

2. The Stochastic Continuous-Time Integrated Economy-Climate Model

2.1 Model Description

The capital $K$ and the concentration $M$ of GHG gas evolve according to

$$\text{d}K_t = \left( e^{-b\mu} K_t^\gamma M_t^{\gamma-1} - C_t - \delta_K K_t \right) \text{d}t + \sigma_K K_t \text{d}W_t^K$$
where \( C > 0 \) is the consumption, and \( \mu > 0 \) is meant to be the regulation. The processes \( C \) and \( \mu \) are assumed to be controllable. The other parameters are exogenous; \( \gamma \) and \( \delta \) are depreciation rates, \( \tau < 1 \), and \( \gamma \in (0,1) \) stand for elasticities, \( b > 0 \) is the effectivity of the regulation. In addition, we set \( W^K \) and \( W^M \) to be Wiener processes such that \( \langle W^K, W^M \rangle = \rho t \) for some \( \rho \in (-1,1) \), and \( \sigma^K \) and \( \sigma^M \) are positive constants.

The problem is to find the optimal path \( K^* \) and \( M^* \) that attain

\[
V(t, K_t, M_t) := \sup_{C,M} \mathbb{E} \left[ \int_t^T \frac{1}{1 - \alpha} C_s^{1-\alpha} \, ds + F(K_T, M_T) \big| F_t \right],
\]

where \( \alpha \in (0, \gamma] \),

\[
F(K, M) = \kappa_1 K^{1-\alpha} - \kappa_2 M^{1-\tau}
\]

for \( \kappa_1 > 0 \) and \( \kappa_2 > 0 \).

2.2 Relation with a DICE

The DICE model in [1] is summarized as follows. The capital \( K \) evolves according with

\[
\Delta K := K_{t+1} - K_t = Q(\mu_t, K_t, L_t, A_t, \Omega_t) - \delta K_t - C_t,
\]

where \( Q \) is the Cobb-Douglas production function

\[
Q(\mu, K, L, A, \Omega) = AK^{-\gamma}L^{1-\gamma} \Omega
\]

depending on the exogenous parameters \( A \), standing for the technology level, \( \mu \), the rate of emission reduction, \( L \), the population, while \( \Omega \), the scaling factor, is endogenously given as, with exogenous parameters \( b_1, b_2, \theta_1, \theta_2 \),

\[
\Omega_t = \frac{1 - b_1 \mu_{t+1}^2}{1 + \theta_1 T_{t+1}^{\theta_2}}.
\]

Here \( T \) is the atmosphere temperature, whose evolution is modeled as

\[
\Delta T = \frac{1}{R_1} \left[ F(t) - \lambda T_{t-1} - \frac{R_2}{\tau_{12}} (T_{t-1} - T_{t-1}^*) \right],
\]

where \( \lambda, R_1, R_2 \) and \( \tau_{12} \) are exogenous parameters, and \( T^* \) is the deep-ocean temperature, evolving as

\[
\Delta T^* = \frac{1}{\tau_{12}} (T_{t-1} - T_{t-1}^*),
\]

The remaining parameter \( F \) stands for the radiative forcing form GHG, given by

\[
F(t) = 4.1 \left( \frac{\log \left( 1 + \frac{M(t)}{590} \right)}{\log 2} \right) + O(t),
\]

where \( O \) is the forcing of exogenous GHG and \( M \) is the mass of GHG in the atmosphere (+590), evolving as

\[
\Delta M = (1 - \mu_t) \sigma C(\mu_t, K_t, L_t, A_t, \Omega_t) - \delta M_{t-1},
\]

where \( \delta \), the depreciation rate, and \( \sigma \) are exogenously given. In the original DICE model, the evolution of the capital \( K \) is endogenized by considering the maximization of the consumption utility

\[
\frac{1}{1 - \alpha} \int_0^T L_t^\alpha C_t^{1-\alpha} \, ds
\]

As we have seen, the DICE model is essentially a four dimensional dynamical system of \( (K, M, T, T^*) \). We can cross a bridge between the DICE and the model in the previous subsection by the following coarse identifications:

- discrete time \( \Rightarrow \) continuous time, which is symbolically \( \Delta X \to X \) for a path \( X \).
- \( 1 - \mu_t, 1 - b_1 \mu_{t+1}^2 \Rightarrow (e^{-\alpha t}, e^{-b_1 \mu}) \)

and reducing \( (K, M, T, T^*) \) to \( (K, M(T, T^*)) \) by firstly

\[
\frac{1}{1 + \theta_1 T_{t+1}^{\theta_2}} \Rightarrow e^{-(1-\gamma)T_{t+1}},
\]

and secondly,

\[
T_t \sim \log M_t.
\]

Furthermore, by redefining \( \sigma \) and \( A \) properly, and changing the scale, and so on, we can assume that \( L_1, \sigma_t, A_t \equiv 1 \), and that the constant in the identification by (3) equals to one.

For the randomization, we let

\[
\delta_* \Rightarrow \delta_* dt - \sigma_* dW_t^*
\]

for \( * \in \{K, M\} \). They can be therefore understood as random depreciation, which might be caused by some other factors that are not taken into account in the DICE model.

Finally, by setting

\[
a = \frac{\tau}{1 - \tau} b
\]

we get (1) with \( \gamma = \alpha / \tau \). We will see later that the restriction (4) will be understood as “integrability” condition.

3. The Results

3.1 Stochastic Maximum Principle

In this subsection, we derive a stochastic Hamiltonian system applying stochastic maximum principle of Chapter 3 in [2] to the problem (2) with (1).

Let

\[
H_1(K, M, C, \mu, p_K, p_M) = p_K (e^{-b_1 \mu K} M^{1-\alpha} - \delta K K - C) + p_M \left( e^{-\alpha C} K^{\alpha} M^{1-\alpha} - \delta M M \right) + \frac{1}{1 - \alpha} C^{1-\alpha}.
\]

Then, the first order condition

\[
\begin{cases}
\partial_C H_1(s, K, M, C^*, \mu^*, p_K, p_M) = 0 \\
\partial_\mu H_1(s, K, M, C^*, \mu^*, p_K, p_M) = 0,
\end{cases}
\]

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where $C^*$ and $\mu^*$ are optimal controls, imply

$$C^* = p_K^{-\frac{1}{\alpha}},$$

$$e^{\mu^*} = (1 - \tau)^{-\frac{1}{\alpha}} \left( -\frac{p_K}{\mu M} \right)^{1 - \frac{1}{\alpha}} K^{\gamma - \frac{\gamma}{2}} \frac{d}{d\tau} \right)^{1 - \frac{1}{\alpha}}.$$

Substituting (6) into (5), we have

$$C = \left[ \lambda_K (1 - \alpha) + \frac{1}{\lambda_K} \right] e^{\lambda_K (T - s)} - \frac{1}{\lambda_K} =: f_K(s).$$

Let also $Y(s) := M(s)(p_M(s))^{\frac{1}{2}}$. Then we obtain

$$dY(s) = \begin{cases} Y(s) \left[ - \left( 1 - \frac{1}{\tau} \right) \delta_M - \frac{1}{2\tau} \left( 1 - \frac{1}{\tau} \right) \right] + \frac{1}{\mu M} \{ \lambda_M K \} \right), \right] dW^M(s),$$

with the terminal condition

$$Y(T) = M(T) (p_M(T))^{\frac{1}{2}} = \left[ (1 - \tau) \kappa_2 \right]^{\frac{1}{2}}. \quad (10)$$

Then, as above, we have that $q_M = -\sigma_M p_M M$, and that

$$dY(s) = \begin{cases} Y(s) \left[ - \left( 1 - \frac{1}{\tau} \right) \delta_M - \frac{1}{2\tau} \left( 1 - \frac{1}{\tau} \right) \right] + \frac{1}{\mu M} \{ \lambda_M K \} \right), \right] dW^M(s),$$

with the terminal condition

$$Y(T) = M(T) (p_M(T))^{\frac{1}{2}} = \left[ (1 - \tau) \kappa_2 \right]^{\frac{1}{2}}. \quad (10)$$

Since (11) is linear with respect to $Y(s)$, we obtain

$$Y(s) = \left[ - (1 - \tau) \kappa_2 \right]^{\frac{1}{2}} e^{-\lambda_M (s - T)} =: f_M(s).$$

We define $K_1(s) := K(s)^{1 - \alpha}$. By Itô formula, we get

$$dK_1(s) = (1 - \alpha) K(s)^{-\alpha} dK(s) - \frac{1}{2} (1 - \alpha) K(s)^{-\alpha-1} \{ dK(s) \}^2$$

$$= \left\{ - (1 - \alpha) \left( f_K(s)^{-1} + \delta_K + \frac{1}{2} \alpha \sigma_K^2 \right) K_1(s) + (1 - \alpha) \left[ - (1 - \tau) \right]^{\frac{1}{\tau-1}} \right\} dW^K(s).$$

Since (12) is linear with respect to $K_1(s)$, we can obtain an explicit form of $K_1$, and hence $K$. In a similar way, we can obtain explicitly the optimal path $M$.

Thus we can state our main result of the present paper.

**Theorem 1** The stochastic Hamiltonian system (7) is completely solvable.

### 4. Remarks

The key to find the solvability of (7) is that the map

$$(K, M, p_K, p_M) \mapsto (K^{1 - \frac{1}{\alpha}}, M^{1 - \frac{1}{\alpha}}, K^{\alpha}, M^{\alpha})$$

is a canonical transformation, that is, a map that sends a Hamiltonian vector field to another Hamiltonian vector field, or in other words, preserves “symplectic structure”. This can be easily verified.
The map sends the backward part the system to a deterministic linear system, which can be understood as a stochastic version of the integrability. The full characterization of a more generic form of the stochastic Hamiltonian system arising in economics will be done in the forthcoming paper by J. Akahori and K. Suzuki.

The continuous-time and stochastic extensions of the DICE models was originally proposed in [3], but solved with the Bellman principle. There have been many other studies which use a stochastic version of DICE model. [4] and [5] analyse a climate shock into DICE model. [6] uses DSICE model which is a DSGE (Dynamic stochastic general economy) extension of DICE model to study SCC. [7] studies both a level and growth rate impact of temperatures on output. [8] studies the effect of rising temperature to current equity prices. Most of them use an approximation to calibrate to the data. In contrast, we are able to find the optimal environmental policy and consumption in a very explicit way.

Disclaimer

The views expressed in the article are the authors’ own and do not represent the official views of the institutions to which the authors belong.

References

[1] W. D. Nordhaus, Managing the Global Commons: the Economics of Climate Change, MIT Press, Cambridge, 1994.
[2] J-M. Yong and X-Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer Science, New York, 1999.
[3] J. Akahori, T. Kosugi, T. Kumazaki and K-I. Oi, On a stochastic extension of integrated models for climate changes, in:Proc. of ISCIE International Symposium on Stochastic Systems Theory and its Applications, 2010 (2010), 229–234.
[4] L. Bakkensen and L. Barrage, Climate shocks, cyclones, and economic growth: bridging the micro-macro gap, NBER Working Paper 24893, 2018.
[5] R. S. Pindyck and N. Wang, The economic and policy consequences of catastrophes, Am. Econ. J.: Economic Policy, 5 (2013), 306–339.
[6] Y-G. Cai and T. S. Lontzek, The social cost of carbon with economic and climate risks, J. Political Econ., 127 (2019), 2684–2734.
[7] C. Hamble, H. Kraft and E. Schwartz, Optimal carbon abatement in a stochastic general equilibrium model with climate change, NBER Working Paper 21044, 2015.
[8] R. Bansal, D. Kiku and M. Ochoa, Climate change risk, NBER Working Paper 23009, 2016.