A NOTE ON TONELLI LAGRANGIAN SYSTEMS ON $\mathbb{T}^2$ WITH POSITIVE TOPOLOGICAL ENTROPY ON HIGH ENERGY LEVEL

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ABSTRACT. In this work we study the dynamical behavior Tonelli Lagrangian systems defined on the tangent bundle of the torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$. We prove that the Lagrangian flow restricted to a high energy level $E^{-1}_L(c)$ (i.e $c > c_0(L)$) has positive topological entropy if the flow satisfies the Kupka-Smale property in $E^{-1}_L(c)$ (i.e, all closed orbit with energy $c$ are hyperbolic or elliptic and all heteroclinic intersections are transverse on $E^{-1}_L(c)$). The proof requires the use of well-known results in Aubry-Mather’s Theory.

1. INTRODUCTION

Let $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ endowed a Riemannian metric $\langle \cdot , \cdot \rangle$. A Tonelli Lagrangian on $\mathbb{T}^2$ is a smooth function $L : T\mathbb{T}^2 \to \mathbb{R}$ that satisfies the two conditions:

- convexity: for each fiber $T_x \mathbb{T}^2 \cong \mathbb{R}^2$, the restriction $L(x, \cdot) : T_x \mathbb{T}^2 \to \mathbb{R}$ has positive defined Hessian,
- superlinearity: $\lim_{\|v\| \to \infty} \frac{L(x,v)}{\|v\|} = \infty$, uniformly in $x \in \mathbb{T}^2$.

The action of $L$ over an absolutely continuous curve $\gamma : [a, b] \to \mathbb{T}^2$ is defined by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t)) \, dt.$$ 

The extremal curves for the action are given by solutions of the Euler-Lagrange equation which in local coordinates can be written as:

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} = 0. \tag{1}$$

The Lagrangian flow $\phi_t : T\mathbb{T}^2 \to T\mathbb{T}^2$ is defined by $\phi_t(x,v) = (\gamma(t), \dot{\gamma}(t))$ where $\gamma : \mathbb{R} \to \mathbb{T}^2$ is the solution of Euler-Lagrange equation, with the initial conditions $\gamma(0) = x$ and $\dot{\gamma}(0) = v$.

The energy function $E_L : T\mathbb{T}^2 \to \mathbb{R}$ is defined by

$$E_L(x,v) = \left\langle \frac{\partial L}{\partial v}(x,v), v \right\rangle - L(x,v). \tag{2}$$

The subsets $E^{-1}_L(c) \subset T\mathbb{T}^2$ are called energy levels and they are invariant by the Lagrangian flow of $L$. Note that the superlinearity condition implies that any non-empty energy levels are compact. Therefore the flow $\phi_t$ is defined for all $t \in \mathbb{R}$.

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The Lagrangian flow of $L$ is conjugated to a Hamiltonian flow on $T^*\mathbb{T}^2$, with the canonical symplectic structure, by the Legendre transformation $\mathcal{L} : T\mathbb{T}^2 \to T^*\mathbb{T}^2$ given by:

$$\mathcal{L}(x, v) = \left( x, \frac{\partial L}{\partial v}(x, v) \right).$$

The corresponding Hamiltonian $H : T^*\mathbb{T}^2 \to \mathbb{R}$ is

$$H(x, p) = \max_{v \in T_x\mathbb{T}^2} \{ p(v) - L(x, v) \}.$$

and we have the Fenchel inequality

$$(3) \quad p(v) \leq H(x, p) + L(x, v)$$

with equality, if only if, $(x, p) = \mathcal{L}(x, v)$ or equivalently $p = \frac{\partial L}{\partial v}(x, v) \in T^*_x\mathbb{T}^2$. Therefore

$$H \left( x, \frac{\partial L}{\partial v}(x, v) \right) = E(x, v).$$

Given a nonempty energy level $E^{-1}_L(c)$, the set $H^{-1}(c) := \mathcal{L} \left( E^{-1}_L(c) \right) \subset T^*\mathbb{T}^2$ is called Hamiltonian level.

We denote by $h_{\text{top}}(L, c)$ the topological entropy of the Lagrangian flow $\phi_t|_{E^{-1}_L(c)}$, for any nonempty energy level $E^{-1}_L(c)$. The topological entropy, is a invariant that, roughly speaking, measures its orbits structure complexity. The relevant question about the topological entropy is whether it is positive or vanishes. Namely, if $\theta \in E^{-1}_L(c)$ and $T, \delta > 0$, we define the $(\delta, T)-\text{dynamical ball}$ centered at $\theta$ as

$$B(\theta, \delta, T) = \{ v \in E^{-1}_L(c) : d(\phi_t(v), \phi_t(\theta)) < \delta \text{ for all } t \in [0, T] \},$$

where $d$ is the distance function in $E^{-1}_L(c)$. Let $N_{\delta}(T)$ be the minimal quantity of $(\delta, T)-\text{dynamical ball}$ needed to cover $E^{-1}_L(c)$. The topological entropy is the limit $\delta \to 0$ of the exponential growth rate of $N_{\delta}(T)$, that is

$$h_{\text{top}}(L, c) := \lim_{\delta \to 0} \lim_{T \to \infty} \frac{1}{T} \log N_{\delta}(T).$$

Thus, if $h_{\text{top}}(L, c) > 0$, some dynamical balls must contract exponentially at least in one direction.

For example, if $\langle \cdot, \cdot \rangle$ denotes the flat metric and $L(x, v) = 1/2 \langle v, v \rangle$, then the corresponding Lagrangian flow is the geodesic flow on the flat torus, that is given by

$$\phi_t(x, v) = (x + tv \mod \mathbb{Z}^2, v).$$

It follows from straight computations, that $h_{\text{top}}(L, c) = 0$, for all $c > 0$ In this example the corresponding Hamiltonian flow is integrable. For an integrable Hamiltonian system on a four dimensional symplectic manifold under certain regularity assumptions (see [Pat91]), the topological entropy of the Hamiltonian flow restricted to a regular compact energy level is vanishes.

In [Sch16], J. P. Schröder go toward a partial answer about the integrability reverse claim of the Paternain’s theorem[Pat91] which is false in general as one already knew the
long time ago [Kat73]. Schröder proved that if the topological entropy of the Lagrangian flow on the level above the Mañé critical value is vanishes thus, for all direction $\zeta \in \mathbb{S}^1$, there are invariant Lipschitz graphs $T_{\zeta}$ ($\zeta$ with irrational slope), $T_{\zeta}^\pm$ ($\zeta$ with rational slope) over $\mathbb{T}^2$, contained in $\{E = c\}$ whose complement of its union is a tubular neighborhood of $T_{\zeta}$ and the lifted orbits from $T_{\zeta}$, $T_{\zeta}^{\pm}$ on universal cover $\mathbb{R}^2$ going to $\infty$, i.e heteroclinic orbits. He use the gap condition to prove that if in a strip between two neighboring periodic minimizes no foliation by heteroclinic minimizes exists, then there are instability regions which imply in its turn positive entropy.

The Mañé’s strict critical value of $L$, is the real number $c_0(L)$ given by

$$c_0(L) = \inf\{k \in \mathbb{R} : A_{L+k}(\gamma) \geq 0, \text{ for all contractible closed curve on } \mathbb{T}^2\}. \quad (4)$$

Here we prove that

**Theorem 1.** Let $L : \mathbb{T}^2 \to \mathbb{R}$ a Tonelli Lagrangian and $c > c_0(L)$. Suppose that the Lagrangian flows restricted to a energy level $E^{-1}_L(c)$ satisfies:

1. all closed orbits in $E^{-1}_L(c)$ are hyperbolic or elliptic, and
2. all heteroclinic intersections in $E^{-1}_L(c)$ are transverse.

Then $h_{\text{top}}(L, c) > 0$.

Let $C^r(\mathbb{T}^2)$ be the set of potentials $u : \mathbb{T}^2 \to \mathbb{R}$ of class $C^r$ endowed whit the $C^r$-topology. We recall that a subset $\mathcal{O} \subset C^r(\mathbb{T}^2)$ is called *residual* if it contains a countable intersection of open and dense subsets. In [Oli08], E. Oliveira proved a version of the Kupka-Smale Theorem for Tonelli Hamiltonian and Lagrangian systems on any closed surfaces. More precisely, it follows from [Oli08] that, for each $c \in \mathbb{R}$, there exists a residual set $\mathcal{KS}(c) \subset C^r(\mathbb{T}^2)$ such that every Hamiltonian $H_u = H + u$, with $u \in \mathcal{KS}$, satisfies the Kupka-Smale propriety, i.e, all closed orbit with energy $c$ are hyperbolic or elliptic and all heteroclinic intersections are transverse on $E^{-1}_L(c)$. See also [RR11], where L. Rifford and R. Ruggiero proved the Kupka-Smale Theorem for Tonelli Lagrangian systems on closed manifolds of any dimension.

So, if we take the residual set $\mathcal{KS}(c) \subset C^r(\mathbb{T}^2)$ given by the Kupka-Smale Theorem and by continuity of the critical values, (cf. Lemma 2.2-1 in [CIF99b]) we obtain the following corollary.

**Corollary 2.** Given $c > c_0(L)$ there exist a smooth potential $u : \mathbb{T}^2 \to \mathbb{R}$ of $C^r$-norm arbitrarily small (for any $r \geq 2$) such that $h_{\text{top}}(L - u, c) > 0$.

2. The Mather and Aubry sets

In this section we recall the definitions of the Mather sets and Aubry sets for the case of a (autonomous) Tonelli Lagrangian on the torus. The Aubry-Mather’s theory was introduced by J. Mather in [Mat91, Mat93] for convex, superlinear and time periodic

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1On arbitrary closed manifold, this equality defines the universal critical value $c_u(L)$. The value $c_0(L)$ is the infimum value of $k \in \mathbb{R}$ such that the $L + k$ action is positive on the set of all closed curves that are homologous to zero. Of course, $c_u(L) \leq c_0(L)$, and $c_u(L) = c_0(L)$ for systems on $\mathbb{T}^2$. 
Lagrangian systems on any closed manifolds. More details and proof of most of the results of this section can be seen in the original works cited above.

Let us recall the main concepts introduced by J. Mather in [Mat91]. We denote by \(\mathcal{B}(L)\) the set of all Borel probability measures, with compact support, that are invariant by the Lagrangian flow of \(L\). By duality, given \(\mu \in \mathcal{B}(L)\), there is a unique homology class \(\rho(\mu) \in H_1(T^2, \mathbb{R})\) such that

\[
\langle \rho(\mu), [\omega] \rangle = \int_{T^2} \omega \, d\mu,
\]

for any closed 1-form \(\omega\) on \(T^2\).

Then, the Mather’s \(\beta\)-function is defined by

\[
\beta(h) = \inf \left\{ \int_{T^2} L \, d\mu : \mu \in \mathcal{B}(L) \text{ with } \rho(\mu) = h \right\}.
\]

The function \(\beta : H_1(M, \mathbb{R}) \to \mathbb{R}\) is convex and superlinear. A measure \(\mu \in \mathcal{B}(L)\) that satisfies

\[
\int_{T^2} L \, d\mu = \beta(\rho(\mu))
\]

is called a \(\rho(\mu)\)-minimizing measure.

The Mather’s \(\alpha\)-function can be defined as the convex dual (or conjugate) function of \(\beta\), i.e.

\[
\alpha([\omega]) = \sup_{h \in H_1(M, \mathbb{R})} \{ [\omega], h > -\beta(h) \} = -\inf_{\mu \in \mathcal{B}(L)} \left\{ \int_M L - \omega \, d\mu \right\}.
\]

By convex duality, we have that \(\alpha\) is also convex and superlinear, and \(\alpha^* = \beta\). Moreover, a measure \(\mu_0\) is \(\rho(\mu_0)\)-minimizing if and only if there is a closed 1-form \(\omega_0\), such that \(\int_M L - \omega_0 \, d\mu_0 = -\alpha([\omega_0])\). Such a class \([\omega_0] \in H^1(M, \mathbb{R})\) is called a subderivative of \(\beta\) at the point \(\rho(\mu_0)\).

We say that \(\mu \in \mathcal{B}(L)\) is a \([\omega]\)-minimizing measure of \(L\) if

\[
\int_{T^2} L - \omega \, d\mu = \min \left\{ \int_{T^2} L - \omega \, d\nu : \nu \in \mathcal{B}(L) \right\} = -\alpha([\omega]).
\]

Let \(\mathcal{M}_L([\omega]) \subset \mathcal{B}(L)\) be the set of all \([\omega]\)-minimizing measures (it only depends on the cohomology class \([\omega]\)). The ergodic components of a \([\omega]\)-minimizing measure are also \([\omega]\)-minimizing measures, so the set \(\mathcal{M}_L([\omega])\) is a simplex whose extremal measures are ergodic \([\omega]\)-minimizing measures. In particular \(\mathcal{M}_L([\omega])\) is a compact subset of \(\mathcal{B}(L)\) with the weak*-topology.

For each \([\omega] \in H^1(T^2, \mathbb{R})\), we define the Mather set of cohomology class \([\omega]\) as:

\[
\mathcal{N}_L([\omega]) = \bigcup_{\mu \in \mathcal{M}_L([\omega])} \text{Supp}(\mu).
\]

We set \(\pi(\mathcal{N}_L([\omega])) = \mathcal{M}_L([\omega])\), and call it the projected Mather set, where \(\pi : T^2 \to T^2\) denotes the canonical projection. The celebrated Graph Theorem proved by J. Mather in [Mat91, Theorem 2], asserts that \(\mathcal{M}([\omega])\) is non-empty, compact, invariant by the
Euler-Lagrange flow and the map $\pi|_{\mathcal{L}(\omega)} : \mathcal{M}_L(\omega) \to \mathcal{M}_L(\omega)$ is a bi-Lipschitz homeomorphism. In [Car95] M. J. Carneiro proved, that this set is contained in the energy level $E^{-1}_L(\alpha(\omega))$.

Following J. Mather in [Mat93], for $t > 0$ and $x, y \in \mathbb{T}^2$, define the action potential for the Lagrangian deformed by a closed 1-form $\omega$ and the projected Aubry set for the cohomology class $[\omega]$:

$$h_\omega(x, y, t) = \inf \left\{ \int_0^t L(\gamma(s), \dot{\gamma}(s)) - \omega(\dot{\gamma}(s)) \, ds \right\},$$

where the infimum is taken over all absolutely continuous curves $\gamma : [0, t] \to \mathbb{T}^2$ such that $\gamma(0) = x$ and $\gamma(t) = y$. The infimum is in fact a minimum by Tonelli’s Theorem.

We define the Peierls barrier for the Lagrangian $L - \omega$ as the function $h_\omega : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ given by:

$$h_\omega(x, y) = \liminf_{t \to +\infty} \{ h_\omega(x, y, t) + \alpha(\omega)t \},$$

and the projected Aubry set for the cohomology class $[\omega] \in H^1(\mathbb{T}^2, \mathbb{R})$ as

$$\mathcal{A}_L([\omega]) = \left\{ x \in \mathbb{T}^2 : h_\omega(x, x) = 0 \right\}.$$

By symmetrizing $h_\omega$, we define the semidistance $\delta_{[\omega]}$ on $\mathcal{A}_L([\omega])$:

$$\delta_{[\omega]}(x, y) = h_\omega(x, y) + h_\omega(y, x).$$

This function $\delta_{[\omega]}$ is non-negative and satisfies the triangle inequality.

Finally, we define the Aubry set (that is also called static set) as the invariant set

$$\mathcal{A}_L([\omega]) = \left\{ (x, v) \in \mathbb{T}\mathbb{T}^2 : \pi \circ \phi_t(x, v) \in \mathcal{A}_L([\omega]), \forall t \in \mathbb{R} \right\}.$$

By definitions $\pi(\mathcal{A}_L([\omega])) = \mathcal{A}_L([\omega])$. In [Mat93, Theorem 6.1], J. Mather proved that this set is compact, $\mathcal{M}_L([\omega]) \subseteq \mathcal{A}_L([\omega])$ and the extension of the graph theorem to the Aubry set, i.e., the mappings $\pi|_{\mathcal{A}_L([\omega])} : \mathcal{A}_L([\omega]) \to \mathcal{A}_L([\omega])$ is a bi-Lipschitz homeomorphism.

Let us now state some important properties and results on these two invariant sets that we going to use in the proof of the Theorem 1.

By the graph property, we can define an equivalence relation in the set $\mathcal{A}_L([\omega])$: two elements $\theta_1$ and $\theta_2 \in \mathcal{A}_L([\omega])$ are equivalent when $\delta_{[\omega]}(\pi(\theta_1), \pi(\theta_2)) = 0$. The equivalence relation breaks $\mathcal{A}_L([\omega])$ into classes that are called static classes of $L$. Let $\Lambda_L([\omega])$ be the set of all static classes. We define a partial order $\preceq$ in $\Lambda_L([\omega])$ by: (i) $\preceq$ is reflexive and transitive, (ii) if there is $\theta \in \Lambda_L([\omega])$, such that the $\alpha$-limit set $\alpha(\theta) \subset \Lambda_i$ and the $\omega$-limit set $\omega(\theta) \subset \Lambda_j$, then $\Lambda_i \preceq \Lambda_j$. The following theorem was proved by G. Contreras and G. Paternain in [CP02].

**Theorem 3.** Suppose that the number of static class is finite, then given $\Lambda_i$ and $\Lambda_j$ in $\Lambda_L([\omega])$, we have that $\Lambda_i \preceq \Lambda_j$.

Let $\Gamma \subset \mathbb{T}\mathbb{T}^2$ be an invariant subset. Given $\epsilon > 0$ and $T > 0$, we say that two points $\theta_1, \theta_2 \in \Gamma$ are $(\epsilon, T)$-connected by chain in $\Gamma$, if there is a finite sequence $\{ (\xi_i, t_i) \}_{i=1}^n \subset \Gamma \times \mathbb{R}$, such that $\xi_1 = \theta_1, \xi_n = \theta_2, T < t_i$ and dist($\phi_{t_i}(\xi_i), \xi_{i+1}$) < $\epsilon$, for $i = 1, \ldots, n - 1$. 


We say that the subset $\Gamma$ is \textit{chain transitive}, if for all $\theta_1, \theta_2 \in \Gamma$ and for all $\epsilon > 0$ and $T > 0$, the points $\theta_1$ and $\theta_2$ are $(\epsilon, T)$-connected by chain in $\Gamma$. When this condition holds for $\theta_1 = \theta_2$, we say that $\Gamma$ is \textit{chain recurrent}. The proof of the following properties can be seen in [CDI97].

\textbf{Theorem 4.} $\tilde{A}_L([\omega])$ is chain recurrent.

The following theorem was proved by Mañé in [Mañ96]. A proof can be seen also in [CDI97, theorem IV].

\textbf{Theorem 5.} Let $\mu \in \mathfrak{B}(L)$. Then $\mu \in \mathcal{M}_L([\omega])$ if only if $\text{Supp}(\mu) \subset \tilde{A}_L([\omega])$.

Finally, for a closed manifold $M$, we say that a class $h \in H_1(M, \mathbb{R})$ is a \textit{rational homology} if there is $\lambda > 0$ such that $\lambda h \in i_!H_1(M, \mathbb{Z})$, where $i : H_1(M, \mathbb{Z}) \hookrightarrow H_1(M, \mathbb{R})$ denotes the inclusion. The following proposition was proved by D. Massart in [Mas97].

\textbf{Proposition 6.} Let $M$ be a closed and oriented surface and let $L$ be a Tonelli Lagragian on $M$. Let $\mu \in \mathfrak{B}(L)$ be a measure $\rho(\mu)$-minimizing such that $\rho(\mu)$ is a rational homology. Then the support of $\mu$ is a union of closed orbits or fixed points of the Lagrangian flow.

\section{3. Proof of Theorem 1}

In this section, we prove the Theorem 1.

Let $L : \mathbb{T}^2 \to \mathbb{R}$ and $c > c_0(L)$. We assume that the restricted flow

$$\phi_t|_{E_L^{-1}(c)} : E_L^{-1}(c) \to E_L^{-1}(c)$$

satisfies the two conditions:

(c1) all closed orbits are hyperbolic or elliptic, and
(c2) all heteroclinic intersections are transverse.

\textbf{Lemma 7.} Let $c > c_0(L)$ and let $h_0 \in H_1(\mathbb{T}^2, \mathbb{R}) \approx \mathbb{R}^2$ a non-zero class. Then there are a invariant probability measure $\mu_0$ and a closed 1-form $\omega_0$ with $\alpha([\omega_0]) = c$, such that:

(i) $\rho(\mu_0) = \lambda_0 h_0$ for some $\lambda_0 \in \mathbb{R}$,
(ii) $\mu_0 \in \mathcal{M}_L([\omega_0])$ and therefore $\text{Supp}(\mu_0) \subset E_L^{-1}(c)$.

\textbf{Proof.} Since $\beta$ is superlinear, we have:

$$\lim_{\lambda \to \infty} \frac{\beta(\lambda h_0)}{|\lambda h_0|} = \infty.$$  

(6)

Let $\partial \beta : H_1(\mathbb{T}^2, \mathbb{R}) \to H_1(\mathbb{T}^2, \mathbb{R})^* = H^1(\mathbb{T}^2, \mathbb{R})$ be the multivalued function such that to each point $h \in H_1(M, \mathbb{R})$ associates all subderivatives of $\beta$ in the point $h$. This is well known that, since $\beta$ is finite, then $\partial \beta(h)$ is a non empty convex cone for all $h \in H_1(\mathbb{T}^2, \mathbb{R})$, and $\partial \beta(h)$ is a unique vector if and only if $\beta$ is differentiable in $h$ (see for example [Roc70, Section 23]). We define the subset

$$S(h_0) = \bigcup_{\lambda \in \mathbb{R}} \partial \beta(\lambda h_0).$$
By (6) we have that the subset $S(h_0) \subset H^1(M, \mathbb{R})$ is not bounded. Since $\beta$ is continuous, by the above properties of the multivalued function $\partial \beta$, we have that $S(h_0)$ is a convex subset. Observe that if $\omega \in \partial \beta(0)$, then $\alpha(\omega) = c_0(L) = \min\{\alpha(\delta) : \delta \in H^1(M, \mathbb{R})\}$ and, by superlinearity of $\alpha$, the restriction $\alpha|_{S(h_0)}$ is not bounded. Therefore, by the intermediate value theorem, for each $c \in [c_0(L), \infty)$ there is $\omega_0 \in \partial \beta(\lambda_0 h_0) \subset S(h_0)$, for some $\lambda_0 \in \mathbb{R}$, such that $\alpha(\omega_0) = c$. Therefore, if $\mu_0 \in \mathcal{M}(L)$ is a $(\lambda_0 h_0)$-minimizing measure, then $\mu_0 \in \mathcal{M}_L([\omega_0])$, and $\text{Supp}(\mu_0) \subset E^{-1}_L(c)$ (see [Car95]).

Let $i : H_1(T^2, \mathbb{Z}) \hookrightarrow H_1(T^2, \mathbb{Z})$ be the inclusion. Recall that $H^1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$ and that $H^1(T^2, \mathbb{R}) \cong \mathbb{R}^2$. Then $\{(0, 1), (1, 0)\} \subset H_1(T^2, \mathbb{Z})$ is a base of $H_1(T^2, \mathbb{Z})$. We have that if $\alpha_0$, $\alpha_1$ are two closed curves in $T^2$, with $[\alpha_0] = (0, 1)$ and $[\alpha_1] = (1, 0)$ then $\alpha_0 \cap \alpha_1 \neq \emptyset$.

We fix $h_0 = (0, 1) \in H_1(T^2, \mathbb{Z})$. By applying Lemma 7 we obtain a closed 1-form $\omega_0$ and a $[\omega_0]$-minimizing measure $\mu_0$ with support in to the level $E^{-1}_L(c)$, for which the rotational vector $\rho(\mu_0)$ is a rational homology class. Therefore, by Proposition 6, the support of $\mu_0$ is formed by the union of closed orbits.

Lemma 8. The Mather’s set $\tilde{\mathcal{M}}_L([\omega_0])$ is the union of a finite number of periodic orbits for the Lagrangian flow of $L$.

Proof. The Mather’s Graph Theorem asserts that the map $\pi|_{\tilde{\mathcal{M}}_L([\omega_0])} : \tilde{\mathcal{M}}_L([\omega_0]) \to T^2$ is injective. Hence, if $\theta_1$, $\theta_2 : \mathbb{R} \to T^2$ are two distinct closed orbits of $\phi_t$ contained in $\tilde{\mathcal{M}}_L([\omega_0])$, then $\gamma_1(t) = \pi \circ \theta_1(t)$ and $\gamma_2(t) = \pi \circ \theta_2(t)$ must be simple closed curves and $[\gamma_1] = n[\gamma_2] \in H_1(T^2, \mathbb{Z})$, because otherwise $\gamma_1 \cap \gamma_2 \neq \emptyset$.

Since $c = \alpha([\omega_0]) > c_0(L) = -\beta(0)$ and $\mathcal{M}_L([\omega_0])$ is a compact set, the continuity of the map $\rho : \mathcal{B}(L) \to H_1(T^2, \mathbb{R})$ (c.f. [Mat91]) implies that there are constants $k, l \in \mathbb{R}$ such that

$$0 < k \leq |\rho(\mu)| \leq l, \quad \text{for all } \mu \in \mathcal{M}_L([\omega_0]).$$

By definition of $\tilde{\mathcal{M}}_L([\omega_0])$, we have that $\text{Supp}(\mu_0) \subset \tilde{\mathcal{M}}_L([\omega_0])$. Let $\mu_\gamma$ be the ergodic measure supported in a closed orbit $\theta(t) = (\gamma(t), \dot{\gamma}(t)) \subset \text{Supp}(\mu_0)$. Since that $\rho(\mu_0) = \lambda_0 h_0$ and by the linearity of the map $\rho$, we have that $[\gamma] = n_0 h_0$ for some $0 \neq n_0 \in \mathbb{Z}$. It follows from the definition of $\rho$, that

$$\langle \rho(\mu_\gamma), [\omega] \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t \omega(\theta(s)) \, ds = \frac{1}{|n_0|T} \int_0^{|n_0|T} \omega(\theta(s)) \, ds = \frac{1}{|n_0|T} \int_\gamma \omega,$$

for any closed 1-form $\omega$ on $T^2$. Then

$$\rho(\mu_\gamma) = \frac{[\gamma]}{|n_0|T} = \pm \frac{h_0}{T},$$

where $T > 0$ denotes the minimal period of $\gamma$. Therefore, the period of any periodic orbit contained in $\text{Supp}(\mu_0)$ is bounded.

We fix a simple closed orbit $\gamma \in \pi(Supp(\mu_0))$. Since that $[\gamma] = n_0 h_0$ we have that the set $C_\gamma = T^2 - \{\gamma\}$ define an open cylinder. Let $\mu \neq \mu_0$ be contained in $\mathcal{M}_L([\omega_0])$. 

The graph property implies that $\text{Supp}(\mu) \cap \text{Supp}(\mu_0) = \emptyset$. Then $\pi(\text{Supp}(\mu)) \subset C_\gamma$ and $\rho(\mu) \in i_*(H_1(C_\gamma, \mathbb{R})) \subset H_1(\mathbb{T}^2, \mathbb{R})$. Therefore, we have that

$$\mathcal{M}_L([\omega_0]) \subset \{ \mu \in \mathcal{M}(L) : \rho(\mu) \in \langle h_0 \rangle_{\mathbb{R}} \}.$$

Applying the Proposition 6 in each measure of the set $\mathcal{M}_L([\omega_0])$, we conclude that the set $\mathcal{M}_L([\omega_0])$ is a union of closed orbits for the Lagrangian flow. Therefore, the same arguments used on the ergodic components of $\mu_0$ imply each ergodic measure satisfies the equality (8). Then the inequality (7) implies that the period of all periodic orbit in $\mathcal{M}_L([\omega_0])$ is uniformly bounded. Therefore, it follows from the compactly of $E_L^{-1}(c)$ and condition (c1) that there is at most a finite number of closed orbits of $\phi_t|_{E_L^{-1}(c)}$ in the Mather’s set $\mathcal{M}_L([\omega_0])$.

\[ \square \]

Remark 9. It is well known that action minimizing curves do not contain conjugate points \(^2\) and a proof of this fact can be see in [CI99a, §4]. So, by Proposition A in [CI99a], for each $\theta \in \mathcal{M}_L([\omega_0])$ there exists the Green bundle along of $\phi_t(\theta)$, i.e. there is a $\phi_t$-invariant bi-dimensional subspace $E(\phi_t(\theta)) \subset T_{\phi_t(\theta)}E^{-1}(c) \cong \mathbb{R}^3$, for all $t \in \mathbb{R}$.

Therefore the linearized Poincaré map on $\theta$ has an invariant one-dimensional subspace, so the periodic orbit $\phi_t(\theta)$ can not be elliptic.

By Lemma 8, let $\gamma_i : \mathbb{R} \to \mathbb{T}^2$, with $i = 1, \ldots, n$, be a closed curves such that $\mathcal{M}_L([\omega_0]) = \bigcup_{i=1}^n \gamma_i$.

Since $\text{Supp}(\mu_0) \subset \mathcal{M}_L([\omega_0])$, we have that $[\gamma_i] = n_0 h_0 = (0, n_0) \in H_1(\mathbb{T}^2, \mathbb{Z})$, for all $i \in \{1, \ldots, n\}$.

Let $\mathcal{A}_L([\omega_0])$ the Aubry set corresponding to the cohomology class $[\omega_0]$ and let $\mathbf{A}_L([\omega_0])$ be the set of all static classes. Recall that

$$\mathcal{M}_L([\omega_0]) \subseteq \mathcal{A}_L([\omega_0]),$$

and since each static class is a connected set (proposition 3.4 in [CP02]), for each $1 \leq i \leq n$, the orbit $(\gamma_i, \dot{\gamma}_i)$ is contained in a static class. On the other hand, by the Theorem 3, we have that each static class contains at least one of the closed orbits in the set $\mathcal{M}_L([\omega_0])$. Hence the number of static classes is less than or equal to $n$.

We will separate into 2 cases: $\mathcal{M}_L([\omega_0]) = \mathcal{A}_L([\omega_0])$ or $\mathcal{M}_L([\omega_0]) \neq \mathcal{A}_L([\omega_0])$.

If $\mathcal{M}_L([\omega_0]) = \mathcal{A}_L([\omega_0])$, then each closed orbit in $\mathcal{M}_L([\omega_0])$ is a static class. Let $\Lambda_1, \ldots, \Lambda_n = \mathbf{A}_L([\omega_0])$ be the static classes. Applying the Theorem 3, we obtain that $\Lambda_i \preceq \Lambda_j$, for $i, j \in \{1, \ldots, n\}$. In particular $\Lambda_1 \preceq \Lambda_1$. Therefore, there is a point $\theta \in E_L^{-1}(c)$ such that the $\alpha$-limit set $\alpha(\theta) \subset \Lambda_1$ and the $\omega$-limit set $\omega(\theta) \subset \Lambda_1$. Since $(\gamma_1, \dot{\gamma}_1) = \Lambda_1$ is a hyperbolic orbit of $\phi_t|_{E_L^{-1}(c)}$ and the condition (c2), we have that $\Lambda_1$ has a transversal homoclinic orbit $\phi_t(\theta)$. Then $h_{\text{top}}(L, c) > 0$.

\(^2\) Two points $\theta_1 \neq \theta_2$ are said to be conjugate if $\theta_2 = \phi_r(\theta_2)$ and $V(\theta_2) \cap d_{\theta_1} \phi_r(V(\theta_1)) \neq \{0\}$, where $V(\theta) = \ker d_{\theta} \pi$ denotes the vertical bundle.
If $\tilde{M}_L([\omega_0]) \neq \tilde{A}_L([\omega_0])$, for each $\theta \in \tilde{A}_L([\omega_0]) \backslash \tilde{M}_L([\omega_0])$, by the graph property of the Aubry set $\tilde{A}_L([\omega_0])$, the curve $\gamma_\theta := \pi \circ \phi_t(\theta) : \mathbb{R} \to \mathbb{T}^2$ has no self-intersection points and $\gamma_\theta \cap M_L([\omega_0]) = \emptyset$. Moreover, by Theorem 5, we have that the $\alpha$-limit and $\omega$-limit sets are contained in the Mather’s set $\tilde{A}_L([\omega_0])$. Since a curve on $\mathbb{T}^2$, that accumulates in positive time to more than one closed curve, must have self-intersection points, we have that $\omega(\theta)$ is a single closed orbit. By the same arguments, we have that $\alpha(\theta)$ is a single closed orbit. By the condition (c2), the orbit $\phi_t(\theta)$ is a transversal heteroclinic orbit. Certainly, if $\tilde{M}_L([\omega_0])$ is a unique closed orbit, then $\phi_t(\theta)$ is a transversal homoclinic orbit, which implies $h_{top}(L, c) > 0$. In the opposite case, i.e., $n > 1$, by recurrence property (Theorem 4), then $\theta$ is an $(\epsilon, T)$-chain connected in $\tilde{A}_L([\omega_0])$ for all $\epsilon > 0$ and $T > 0$, i.e., there is a finite sequence $\{(\zeta_i, t_i)\}_{i=1}^k \subset \tilde{A}_L([\omega_0]) \times \mathbb{R}$, such that $\zeta_1 = \zeta_k = \theta, T < t_i$ and $\text{dist}(\phi_{t_i}(\zeta_i), \zeta_{i+1}) < \epsilon$, for $i = 1, \ldots, k - 1$. Since the closed orbits in $M_L([\omega_0])$ are isolated on the torus, we have that for $\epsilon$ small enough, the set $\{\pi(\zeta_i)\}_{i=1}^k \subset \tilde{A}_L([\omega_0])$ must intersect the interior of each one of the cylinders obtained by cutting the torus along the two curves $\gamma_i, \gamma_j \in \tilde{A}_L([\omega_0])$, with $1 \leq i, j \leq n$. Therefore, choosing an orientation on $\tilde{A}_L([\omega_0])$ and reordering the indices, we obtain a cycle of transversal heteroclinic orbits. This implies that $h_{top}(L, c) > 0$.

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