ASYMPTOTIC TOPOLOGICAL REGULARITY OF CAT(0) SPACES

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Abstract. We study asymptotic topological regularity of CAT(0) spaces. We prove that if a purely n-dimensional, proper, geodesically complete CAT(0) space has small volume growth, then it is homeomorphic to the n-dimensional Euclidean space. We also discuss asymptotic geometry of proper, geodesically complete CAT(0) spaces of small volume growth.

1. Introduction

A CAT(0) space is defined as a geodesic metric space that is globally non-positively curved in the sense of Alexandrov. Since Gromov [20] formulated the definition, CAT(0) spaces have played central roles in geometry of metric spaces of non-positive curvature (see e.g., [8]).

A Hadamard manifold is by definition a simply connected, complete Riemannian manifold of non-positive sectional curvature. The Cartan–Hadamard theorem in Riemannian geometry tells us that every n-dimensional Hadamard manifold is diffeomorphic to the n-dimensional Euclidean space $\mathbb{R}^n$. A connected, complete Riemannian manifold is CAT(0) if and only if it is a Hadamard manifold.

Gromov [19] asked the question whether there exists a convex geodesic metric space that is a topological n-manifold differ from $\mathbb{R}^n$. We notice that every convex geodesic metric space is contractible. Every CAT(0) space is convex. For the case of $n \geq 5$, Davis–Januszkiewicz [15, Theorem 5b.1] gave an affirmative answer, in fact, showed that for each $n \in \mathbb{N}$ with $n \geq 5$, there exists a piecewise flat CAT(0) polyhedron that is a topological n-manifold not homeomorphic to $\mathbb{R}^n$ (see also [4]).

In the case of $n \leq 3$, if a convex geodesic metric space is a topological n-manifold, then it is homeomorphic to $\mathbb{R}^n$ ([9], [36]). Thurston [40, Theorem 1.6] proved that if a CAT(0) topological 4-manifold possesses a tame point, then it is homeomorphic to $\mathbb{R}^4$. As far as the author knows, the question of Gromov [19] for CAT(0) spaces remained open for the case of $n = 4$. (see e.g., [16, Section 5]). Lytchak, Stadler, and
the author [30] recently prove that every CAT(0) topological 4-manifold is homeomorphic to $\mathbb{R}^4$.

In the present paper, from a viewpoint of asymptotic geometry, we study problems of when a CAT(0) space is homeomorphic to $\mathbb{R}^n$. In global Riemannian geometry, many problems on the structure of open Riemannian manifolds of non-negative Ricci curvature has been studied in terms of their volume growths. Cheeger–Colding [14, Theorem A.1.11] concluded that if an $n$-dimensional open Riemannian manifold of non-negative Ricci curvature has sufficiently large volume growth, then it is diffeomorphic to $\mathbb{R}^n$. In this paper, we describe conditions for CAT(0) spaces of small volume growth to be homeomorphic to $\mathbb{R}^n$.

1.1. Main results. We denote by $\mathcal{H}^n$ the $n$-dimensional Hausdorff measure. We denote by $\omega^0_n(r)$ the $n$-dimensional Hausdorff measure of a metric ball in $\mathbb{R}^n$ of radius $r$. For a point $p$ in a metric space, we denote by $U_r(p)$ the open metric ball of radius $r$ around $p$. For a metric space $X$, we define a non-negative value $G_0^n(X) \in [0, \infty]$ by

$$G_0^n(X) := \limsup_{t \to \infty} \frac{\mathcal{H}^n(U_t(p))}{\omega^0_n(t)}$$

for some base point $p$ in $X$. We remark that $G_0^n(X)$ does not depend on the choice of base points. We call $G_0^n(X)$ the (upper) $n$-dimensional Euclidean volume growth of $X$.

A metric space is said to be proper if every closed bounded subset is compact. A geodesic metric space is geodesically complete if every geodesic can be extended to a local geodesic defined on $\mathbb{R}$. We notice that if a CAT(0) space is geodesically complete, then every geodesic can be extended to a geodesic line defined on $\mathbb{R}$.

Let $X$ be a proper, geodesically complete CAT(0) space. We denote by $X^n$ the $n$-dimensional part of $X$ determined as the set of all points in $X$ at which all sufficiently small open metric balls have topological dimension $n$. A relative volume comparison of Bishop–Gromov type (Proposition 2.7) for CAT(0) spaces tells us that if $X^n$ is non-empty, then the following hold: (1) $G_0^n(X) \geq 1$; (2) if $G_0^n(X)$ is finite, then for any $p \in X^n$ the limit superior in (1.1) turns out to be the limit.

We say that a separable metric space is purely $n$-dimensional if every non-empty open subset has topological dimension $n$. A proper, geodesically complete CAT(0) space $X$ is purely $n$-dimensional if and only if $X = X^n$ ([28, Theorem 1.2]).

We prove the following asymptotic topological regularity:

**Theorem 1.1.** For every $\epsilon \in (0, \infty)$, and for every $n \in \mathbb{N}$, there exists $\delta \in (0, \infty)$ satisfying the following property: If a purely $n$-dimensional, proper, geodesically complete CAT(0) space $X$ satisfies $G_0^n(X) < 1 + \delta$, then $X$ is $(1 + \epsilon)$-bi-Lipschitz homeomorphic to $\mathbb{R}^n$. 
In Theorem 1.1, the pureness on the dimension is essential since we can construct counterexamples possessing lower dimensional subsets. For instance, for each \( n \in \mathbb{N} \) with \( n \geq 2 \), the one-point union \( \mathbb{R}^n \vee \mathbb{R} \) of \( \mathbb{R}^n \) and \( \mathbb{R} \) equipped with the gluing metric is an \( n \)-dimensional, proper, geodesically complete CAT(0) space with \( G^n_0(\mathbb{R}^n \vee \mathbb{R}) = 1 \).

We describe the following optimal condition of small volume growth for purely \( n \)-dimensional CAT(0) spaces to be homeomorphic to \( \mathbb{R}^n \):

**Theorem 1.2.** If a purely \( n \)-dimensional, proper, geodesically complete CAT(0) space \( X \) satisfies \( G^n_0(X) < \frac{3}{2} \), then it is homeomorphic to \( \mathbb{R}^n \).

We denote by \( T \) the discrete metric space consisting of three points with pairwise distance \( \pi \). The condition of \( G^n_0(X) \) in Theorem 1.2 is optimal since for the \( l^2 \)-product metric space \( \mathbb{R}^{n-1} \times C_0(T) \) of \( \mathbb{R}^{n-1} \) and the Euclidean cone \( C_0(T) \) over \( T \) we have \( G^n_0(\mathbb{R}^{n-1} \times C_0(T)) = \frac{3}{2} \).

We obtain the following characterization as the critical case:

**Theorem 1.3.** If a purely \( n \)-dimensional, proper, geodesically complete CAT(0) space \( X \) satisfies \( G^n_0(X) = \frac{3}{2} \), then \( X \) is either homeomorphic to \( \mathbb{R}^n \) or isometric to the \( l^2 \)-product metric space \( \mathbb{R}^{n-1} \times C_0(T) \), where \( C_0(T) \) is the Euclidean cone over \( T \).

Once we know Theorem 1.3, for homology manifolds we can hope to relax the condition on \( G^n_0 \) in Theorem 1.2. We notice that every complete CAT(0) homology manifold is proper and geodesically complete (\cite[Corollary I.3.8]{8}, \cite[Lemma 4.1]{28}, \cite[Lemma 3.1]{29}).

We prove the following regularity of CAT(0) homology manifolds:

**Theorem 1.4.** For every \( n \in \mathbb{N} \), there exists \( \delta \in (0, \infty) \) satisfying the following property: If a complete CAT(0) homology \( n \)-manifold \( X \) satisfies \( G^n_0(X) < \frac{3}{2} + \delta \), then it is homeomorphic to \( \mathbb{R}^n \).

Thurston \cite[Theorem 3.3]{10} showed that every homology 3-manifold with an upper curvature bound is a topological 3-manifold. Lytchak and the author \cite[Theorem 1.2]{29} proved that for every homology \( n \)-manifold \( M \) with an upper curvature bound there exists a locally finite subset \( E \) of \( M \) such that \( M - E \) is a topological \( n \)-manifold.

### 1.2. Relations with asymptotic geometry.

The Tits boundaries of CAT(0) spaces have been utilized in asymptotic geometry concerning flat subspaces. For instance, Kleiner–Leeb \cite{23} employed the Tits boundaries in their studies of rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. Leeb \cite{25} described metric characterizations of symmetric spaces and Euclidean buildings in terms of their Tits boundaries. Related subsequent studies on metric characterizations can be seen in \cite{7, 21, 27, 31, 37, 38}, and so on. We notice that every complete CAT(0) space \( X \) admitting a geodesic ray has the Tits boundary \( \partial_T X \) that is a complete CAT(1) space.
Let $X$ be a proper, geodesically complete CAT(0) space. It seems to be well-known that if $\partial_T X$ is isometric to the $(n-1)$-dimensional standard unit sphere $\mathbb{S}^{n-1}$, then $X$ is isometric to $\mathbb{R}^n$. This rigidity follows from an observation of Leeb [25, Proposition 2.1], obtained as a generalization of Schroeder’s work in [13, Appendix 4] for Hadamard manifolds. Indeed, Leeb [25, Proposition 2.1] showed that for an arbitrary proper CAT(0) space, if we find a subspace $\Sigma$ of the Tits boundary such that $\Sigma$ is isometric to $\mathbb{S}^{n-1}$ and does not bound a unit hemisphere, then there exists an $n$-dimensional flat subspace $\Pi$ with $\partial_T \Pi = \Sigma$.

As a result of asymptotic geometric regularity, we prove that if $\partial_T X$ is sufficiently close to $\mathbb{S}^{n-1}$ with respect to the Gromov–Hausdorff distance, then $X$ is bi-Lipschitz homeomorphic to $\mathbb{R}^n$ (see Theorems 5.11 and 5.12). A desired bi-Lipschitz homeomorphism is given by a map on $X$ with Busemann function coordinates. When we analyze such a regular map on $X$ with Busemann function coordinates, we utilize the ideas of the theory of strainer maps on GCBA spaces with distance function coordinates developed by Lytchak and the author [28].

We emphasize that $\partial_T X$ is not necessarily compact, and it is not necessarily geodesically complete, although $X$ is proper and geodesically complete. We show that if $X$ has the Gromov–Hausdorff asymptotic cone $C_\infty X$, then $C_\infty X$ is isometric to the Euclidean cone over $\partial_T X$; in particular, $\partial_T X$ is compact and geodesically complete (Proposition 3.4). We also prove that a purely $n$-dimensional, proper, geodesically complete CAT(0) space is doubling if and only if it has the Gromov–Hausdorff asymptotic cone, and if and only if it has finite $n$-dimensional Euclidean volume growth (Proposition 3.7). These observations enable us to prove our main results of asymptotic topological regularity.

1.3. Outlines of the proofs of the main results. Let $X$ be a purely $n$-dimensional, proper, geodesically complete CAT(0) space. Suppose that $X$ has finite $n$-dimensional Euclidean volume growth. Then $X$ has the Gromov–Hausdorff asymptotic cone isometric to the Euclidean cone $C_0(\partial_T X)$ over the Tits boundary $\partial_T X$; in particular, $\partial_T X$ is compact, geodesically complete, and purely $(n-1)$-dimensional; moreover,

$$G_0^0(X) = \frac{\mathcal{H}^{n-1}(\partial_T X)}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}$$

(Propositions 3.4 and 3.7). According to the upper bounds for $G_0^0(X)$ in Theorems 1.1–1.4, the Tits boundary $\partial_T X$ is characterized as follows:

1. If we have $G_0^0(X) < 1 + \delta$ for sufficiently small $\delta$, then a volume rigidity result of the author [32, Theorem 1.10] for CAT(1) spaces implies that $\partial_T X$ is bi-Lipschitz homeomorphic to $\mathbb{S}^{n-1}$.

2. If we have $G_0^0(X) < 3/2$, then a volume sphere theorem of Lytchak and the author [29, Theorem 8.3] for CAT(1) spaces implies that $\partial_T X$ is homeomorphic to $\mathbb{S}^{n-1}$.
(3) If we have $G^0_n(X) = 3/2$, then a characterization of the author [34, Theorem 1.1] for CAT(1) spaces implies that $\partial_T X$ is either homeomorphic to $S^{n-1}$ or the spherical suspension $S^{n-2} * T$.

(4) If we have $G^0_n(X) < 3/2 + \delta$ for sufficiently small $\delta$, and if $X$ is a homology $n$-manifold, then $\partial_T X$ is a homology $(n - 1)$-manifold, and hence a volume sphere theorem of the author [34, Theorem 1.2] for CAT(1) homology manifolds implies that $\partial_T X$ is homeomorphic to $S^{n-1}$.

From properties (1)–(4) listed above, we can derive Theorems 1.1–1.4. In the proofs of Theorems 1.2, 1.3, and 1.4 when we prove that $X$ is a topological $n$-manifold, we use the local topological regularity theorem of Lytchak and the author [29, Theorem 1.1]. In the proof of Theorem 1.3 in order to determine the geometric structure of $X$, we describe a volume regularity condition for CAT(1) spaces to be almost isometric to a compact spherical building (Proposition 6.4).

1.4. Organization. In Section 2, we discuss basic concepts in the geometry of metric spaces with curvature bounded above.

In Section 3, we study relations between Gromov–Hausdorff asymptotic cones of CAT(0) spaces and their Tits boundaries, and we show the observations mentioned in Subsection 1.2.

In Section 4, we recall the basic properties of strainer maps discussed by Lytchak and the author [28]. Using the homotopic stability theorem [28, Theorem 13.1] of fibers of strainer maps, we prove that if a proper, geodesically complete CAT(0) space $X$ has the Gromov–Hausdorff asymptotic cone, then any sufficiently large metric sphere in $X$ is homotopy equivalent to $\partial_T X$ (Theorem 4.6); in particular, if in addition $X$ is a topological $n$-manifold, and if $\partial_T X$ is simply connected, then it is homeomorphic to $\mathbb{R}^n$ (Theorem 4.7).

In Section 5, for a proper, geodesically complete CAT(0) space $X$, we study a map $F : X \to \mathbb{R}^m$ with Busemann function coordinates satisfying a regular property, called a Busemann strainer map. We prove that if $X$ is $n$-dimensional, then a Busemann strainer map $F : X \to \mathbb{R}^n$ becomes a bi-Lipschitz homeomorphism (Propositions 5.9 and 5.10). We also show the result of asymptotic geometric regularity (Theorems 5.11 and 5.12) mentioned in Subsection 1.2.

In Section 6, we discuss the proofs of Theorems 1.1–1.4.

1.5. Problem. As a natural question beyond Theorem 1.4, we pose the following asymptotic regularity problem for CAT(0) spaces:

**Problem 1.1.** Let $n \in \mathbb{N}$ satisfy $n \geq 4$. Let $c_n$ be the supremum of $c \in (3/2, \infty)$ for which every complete CAT(0) homology $n$-manifold $X$ with $G^0_n(X) \leq c$ is homeomorphic to $\mathbb{R}^n$.

(1) Find the concrete value $c_n$. 


(2) Describe all complete CAT(0) homology $n$-manifolds $X$ satisfying $G^0_n(X) = c_n$ in the maximal critical case.

This problem is closely related to a volume pinching problem posed in [34, Problem 1.1] for CAT(1) spaces (cf. [12]).

Acknowledgments. The author would like to express his gratitude to Alexander Lytchak for valuable discussions and helpful comments in private communications. The author would like to thank Takashi Shioya for his interest in this work. The author would also like to thank the referees for carefully reading the manuscript and for giving helpful comments.

2. Preliminaries

We refer the readers to [1], [2], [3], [5], [8], [10], [11] for the basic facts on metric spaces with an upper curvature bound.

2.1. Metric spaces. Let $r \in (0, \infty)$. For a point $p$ in a metric space, we denote by $U_r(p)$, $B_r(p)$, and $S_r(p)$ the open metric ball of radius $r$ around $p$, the closed one, and the metric sphere, respectively.

Let $X$ be a metric space. A subset $A$ of $X$ is called an $r$-net of $X$ if $\bigcup_{p \in A} U_r(p)$ coincides with $X$. A subset $A$ of $X$ is said to be $r$-separated if every pair of points in $A$ has distance at least $r$. Due to Zorn’s lemma, for every subset $W$ of $X$, for every $s \in (0, \infty)$ there exists a maximal $s$-separated set $A$ in $W$; in this case, $A$ is an $s$-net of $W$.

For $N \in \mathbb{N}$, a metric space $X$ is said to be $N$-doubling if every open metric ball of radius $r$ in $X$ can be covered by at most $N$ open metric balls of radius $r/2$ in $X$. A metric space $X$ is doubling if $X$ is $N$-doubling for some $N$. If a metric space $X$ is $N$-doubling, then so is every metric subspace of $X$. A metric space $X$ is doubling if and only if there exists some $N \in \mathbb{N}$ such that for each $s \in (0, \infty)$, and for each $x \in X$, every $s$-separated set in $U_{2s}(x)$ has at most $N$ elements.

For a metric space $X$ with metric $d_X$, and for a positive number $\lambda \in (0, \infty)$, we denote by $\lambda X$ the rescaled metric space defined as $(X, \lambda d_X)$. If a metric space $X$ is $N$-doubling, then for every $\lambda \in (0, \infty)$ the rescaled metric space $\lambda X$ is also $N$-doubling.

Let $X$ be a metric space with metric $d_X$. Let $d_X \wedge \pi$ be the $\pi$-truncated metric on $X$ defined by $d_X \wedge \pi := \min\{d_X, \pi\}$. The Euclidean cone $C_0(X)$ over $X$ is defined as the cone $[0, \infty) \times X / \{0\} \times X$ over $X$ equipped with the Euclidean metric $d_{C_0(X)}$ given by

$$d_{C_0(X)}([(t_1, x_1)], [(t_2, x_2)])^2 := t_1^2 + t_2^2 - 2t_1 t_2 \cos \left( (d_X \wedge \pi)(x_1, x_2) \right).$$

For simplicity, we write an element $[(t, x)]$ in $C_0(X)$ as $tx$, and denote by 0 the vertex of $C_0(X)$. For metric spaces $Y$ and $Z$, we denote by $Y \ast Z$ the spherical join of $Y$ and $Z$. Note that $C_0(Y \ast Z)$ is isometric to the $\ell^2$-direct product metric space $C_0(Y) \times C_0(Z)$ of $C_0(Y)$ and $C_0(Z)$. 

2.2. Maps between metric spaces. Let $c \in (0, \infty)$. Let $X$ be a metric space with metric $d_X$, and $Y$ a metric space with metric $d_Y$. A map $f : X \to Y$ is said to be $c$-Lipschitz if $d_Y(f(x_1), f(x_2)) \leq cd_X(x_1, x_2)$ for all $x_1, x_2 \in X$. A map $f : X \to Y$ is said to be $c$-bi-Lipschitz if $f$ is $c$-Lipschitz, and if $d_Y(x_1, x_2) \leq cd_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$ (consequently, $c \in [1, \infty)$). A 1-bi-Lipschitz homeomorphism is nothing but an isometry, and a 1-bi-Lipschitz embedding is an isometric embedding. A map $f : X \to Y$ is $c$-open if for any $r \in (0, \infty)$ and $x \in X$ such that $B_{cr}(x)$ is complete, the ball $U_r(f(x))$ in $Y$ is contained in the image $f(U_r(x))$ of the ball $U_r(x)$ in $X$. In the case where $X$ is complete, if a map $f : X \to Y$ is $c$-open, then $f$ is surjective; indeed, for every $x \in X$, and for every $y \in Y$ with $y \neq f(x)$, by setting $t := d_Y(f(x), y)$, we find $x_0 \in U_{2ct}(x)$ with $y = f(x_0)$ since the ball $B_{2ct}(x)$ in $X$ is complete, and hence the ball $U_{2t}(f(x))$ in $Y$ is contained in the image $f(U_{2t}(x))$. Moreover, in the case where $X$ is complete, a $c$-Lipschitz map $f : X \to Y$ is a $c$-bi-Lipschitz homeomorphism for some $c \in [1, \infty)$ if and only if $f$ is an injective $c$-open map.

A map $\varphi : X \to Y$ is said to be a $c$-approximation between $X$ and $Y$ if $\varphi(X)$ is a $c$-net of $Y$, and if for all $x_1, x_2 \in X$ we have

$$|d_Y(\varphi(x_1), \varphi(x_2)) - d_X(x_1, x_2)| < c.$$  

If there exists a $c$-approximation $\varphi : X \to Y$, then there exists a $2c$-approximation $\psi : Y \to X$ such that for all $x \in X$ and $y \in Y$ we have $d_X((\psi \circ \varphi)(x), x) < 2c$ and $d_Y((\varphi \circ \psi)(y), y) < 2c$.

2.3. Geodesic metric spaces. Let $X$ be a metric space. A geodesic $\gamma : I \to X$ means an isometric embedding from an interval $I$. For a pair of points $p, q$ in $X$, a geodesic $pq$ in $X$ from $p$ to $q$ means the image of an isometric embedding $\gamma : [a, b] \to X$ from a bounded closed interval $[a, b]$ with $\gamma(a) = p$ and $\gamma(b) = q$. A geodesic $\gamma : I \to X$ is called a ray if $I = [0, \infty)$, and $\gamma$ is called a line if $I = \mathbb{R}$.

For $r \in (0, \infty]$, a metric space $X$ is said to be $r$-geodesic if every pair of points in $X$ with distance smaller than $r$ can be joined by a geodesic in $X$. A metric space is geodesic if it is $\infty$-geodesic. A geodesic metric space is proper if and only if it is complete and locally compact.

For $r \in (0, \infty]$, a subset $C$ of a metric space is said to be $r$-convex if $C$ itself is $r$-geodesic as a metric subspace, and if every geodesic joining two points in $C$ is contained in $C$. A subset $C$ of a metric space is convex if $C$ is $\infty$-convex.

2.4. Gromov–Hausdorff topology. We denote by $d_{GH}$ the Gromov–Hausdorff distance between metric spaces. If for $c \in (0, \infty)$ two metric spaces $X$ and $Y$ satisfy $d_{GH}(X, Y) < c$, then there exists a $2c$-approximation between $X$ and $Y$. We say that a sequence $(X_i)$ of metric spaces converges to a metric space $X$ in the Gromov–Hausdorff topology if $\lim_{i \to \infty} d_{GH}(X_i, X) = 0$. Due to the Gromov precompactness...
theorem, for fixed \( r \in (0, \infty) \) and \( N \in \mathbb{N} \), every sequence of \( N \)-doubling compact metric spaces of diameter at most \( r \) has a Gromov–Hausdorff convergent subsequence whose limit is \( N \)-doubling.

We say that a sequence \( (X_i, p_i) \) of pointed geodesic metric spaces converges to a pointed metric space \( (X, p) \) in the pointed Gromov–Hausdorff topology if for every \( r \in (0, \infty) \) there exists a sequence \( (\epsilon_i) \) in \( (0, \infty) \) with \( \lim_{i \to \infty} \epsilon_i = 0 \) such that for each \( i \) there exists a \( \epsilon_i \)-approximation \( \varphi_i: B_r(p) \to B_r(p_i) \) with \( \varphi_i(p) = p_i \); in this case, we write \( (X, p) = \lim_{i \to \infty} (X_i, p_i) \). If a sequence \( (X_i, p_i) \) of pointed proper geodesic metric spaces converges to a metric space \( (X, p) \) in the pointed Gromov–Hausdorff topology, then \( X \) is proper and geodesic.

2.5. CAT(\( \kappa \)) spaces. For \( \kappa \in \mathbb{R} \), we denote by \( M^n_\kappa \) the simply connected, complete Riemannian \( n \)-manifold of constant curvature \( \kappa \), and denote by \( D_\kappa \) the diameter of \( M^n_\kappa \). A metric space \( X \) is said to be CAT(\( \kappa \)) if \( X \) is \( D_\kappa \)-geodesic, and if every geodesic triangle in \( X \) with perimeter smaller than \( 2D_\kappa \) is not thicker than the comparison triangle with the same side lengths in \( M^n_\kappa \).

Let \( X \) be a CAT(\( \kappa \)) space. Every pair of points in \( X \) with distance smaller than \( D_\kappa \) can be uniquely joined by a geodesic. Let \( p \in X \). For every \( r \in (0, D_\kappa/2] \), the balls \( U_r(p) \) and \( B_r(p) \) are convex. Along the geodesics emanating from \( p \), for every \( r \in (0, D_\kappa) \) the balls \( U_r(p) \) and \( B_r(p) \) are contractible inside themselves. Every open subset of \( X \) is an ANR \( [35, 24] \). For \( x, y \in U_{D_\kappa}(p) - \{ p \} \), we denote by \( \angle_p(x, y) \) the angle at \( p \) between \( px \) and \( py \). Put \( \Sigma^\prime_pX := \{ px \mid x \in U_{D_\kappa}(p) - \{ p \} \} \). The angle \( \angle_p \) at \( p \) is a pseudo-metric on \( \Sigma^\prime_pX \). The space of directions \( \Sigma^\prime_pX \) at \( p \) is defined as the \( \angle_p \)-completion of the quotient metric space \( \Sigma^\prime_pX/\angle_p = 0 \). For \( x \in U_{D_\kappa}(p) - \{ p \} \), we denote by \( x'_p \in \Sigma^\prime_pX \) the starting direction of \( px \) at \( p \). The tangent space \( T_pX \) at \( p \) is defined as \( C_0(\Sigma^\prime_pX) \). The space \( \Sigma^\prime_pX \) is CAT(1), and the space \( T_pX \) is CAT(0).

In fact, for a metric space \( \Sigma \), the Euclidean cone \( C_0(\Sigma) \) is CAT(0) if and only if \( \Sigma \) is CAT(1). For two metric spaces \( Y \) and \( Z \), the spherical join \( Y * Z \) is CAT(1) if and only if \( Y \) and \( Z \) are CAT(1).

2.6. Ideal boundaries of CAT(0) spaces. Let \( X \) be a metric space with metric \( d_X \). Two rays \( \gamma_1, \gamma_2: [0, \infty) \to X \) are said to be asymptotic if \( \sup_{t \in [0, \infty)} d_X(\gamma_1(t), \gamma_2(t)) \) is finite. The asymptotic relation gives an equivalence relation on the set of all rays in \( X \). The ideal boundary \( \partial_\infty X \) of \( X \) is defined as the set of all asymptotic equivalence classes of rays in \( X \). For a ray \( \gamma \) in \( X \), we denote by \( \gamma(\infty) \) the asymptotic equivalent class of \( \gamma \) in \( \partial_\infty X \).

Let \( X \) be a complete CAT(0) space. For every \( p \in X \), and for every \( \xi \in \partial_\infty X \), there exists a unique ray \( \gamma: [0, \infty) \to X \) with \( \gamma(0) = p \) and \( \gamma(\infty) = \xi \). For \( p \in X \) and \( \xi \in \partial_\infty X \), we denote by \( \gamma_{p\xi} \) the unique ray emanating from \( p \) to \( \xi \), by \( p\xi \) the image of \( \gamma_{p\xi} \), and by \( \xi'_p \in \Sigma^\prime_pX \) the starting direction of \( p\xi \) at \( p \). For \( p \in X \), and \( \xi, \eta \in \partial_\infty X \), we denote
by \( \angle_p(\xi, \eta) \) the angle at \( p \) between \( p\xi \) and \( p\eta \). The angle metric \( \angle \) on \( \partial_\infty X \) is defined by \( \angle(\xi, \eta) := \sup_{p \in X} \angle_p(\xi, \eta) \). The Tits metric \( d_T \) on \( \partial_\infty X \) is defined as the length metric on \( \partial_\infty X \) induced from \( \angle \). Notice that \( \angle = \min\{d_T, \pi\} \), and \( d_T \) possibly takes the value \( \infty \). We denote by \( \partial_T X \) the ideal boundary \( \partial_\infty X \) equipped with the Tits metric \( d_T \), and call it the Tits boundary of \( X \). Then \( \partial_T X \) is a complete CAT(1) space. The Euclidean cone \( C_0(\partial_T X) \) is isometric to the Euclidean cone \( C_0(\partial_\infty X) \) over the ideal boundary \( \partial_\infty X \) with the angle metric \( \angle \).

We recall the following basic asymptotic property (see e.g., [8, Proposition II.9.8, Corollary II.9.10]):

**Lemma 2.1.** Let \( X \) be a complete CAT(0) space with metric \( d_X \). Let \( p \in X \). Then for all \( \xi_1, \xi_2 \in \partial_T X \), and for all \( a_1, a_2 \in (0, \infty) \), we have

\[
d_{C_0(\partial_T X)}(a_1\xi_1, a_2\xi_2) = \lim_{t \to \infty} \frac{d_X(\gamma_{p\xi_1}(a_1t), \gamma_{p\xi_2}(a_2t))}{t},
\]

as the monotone non-decreasing limit. Moreover, for every \( \lambda \in (0, \infty) \) the map \( \Phi^\lambda_p : C_0(\partial_T X) \to \lambda X \) defined by \( \Phi^\lambda_p(a\xi) := \gamma_{p\xi}(a/\lambda) \) is 1-Lipschitz, where \( \gamma_{p\xi} : [0, \infty) \to X \) is the ray in \( X \) from \( p \) to \( \xi \).

We recall the following splitting theorem for CAT(0) spaces ([8, Theorem II.9.24], and [3, Appendix 4] in the Riemannian setting).

**Proposition 2.2.** ([8]) Let \( X \) be a geodesically complete, complete CAT(0) space. If \( \partial_T X \) isometrically splits as a spherical join \( \Sigma_1 \ast \Sigma_2 \), then \( X \) is isometric to an \( \ell^2 \)-direct product metric space \( X_1 \times X_2 \) of geodesically complete, complete CAT(0) spaces \( X_1 \) and \( X_2 \) such that for each \( j \in \{1, 2\} \) the space \( \partial_T X_j \) is isometric to \( \Sigma_j \).

2.7. Geodesically complete CAT(\( \kappa \)) spaces. We refer the readers to [28] for the basic properties of GCBA spaces, that is, locally compact, separable, locally geodesically complete metric spaces with an upper curvature bound. Recall that a CAT(\( \kappa \)) space is said to be \textit{locally geodesically complete} (or has \textit{geodesic extension property}) if every geodesic defined on a compact interval can be extended to a local geodesic beyond endpoints. A CAT(\( \kappa \)) space is \textit{geodesically complete} if every geodesic can be extended to a local geodesic defined on \( \mathbb{R} \). Every locally geodesically complete, complete CAT(\( \kappa \)) space is geodesically complete. The geodesical completeness for compact (resp. proper) CAT(\( \kappa \)) spaces is preserved under the (resp. pointed) Gromov–Hausdorff limit.

Let \( X \) be a proper, geodesically complete CAT(\( \kappa \)) space. For every \( p \in X \), the space \( \Sigma_p X \) is compact and geodesically complete, and \( T_p X \) is proper and geodesically complete. In fact, for a CAT(1) space \( \Sigma \), the Euclidean cone \( C_0(\Sigma) \) is geodesically complete if and only if \( \Sigma \) is geodesically complete and not a singleton. For two CAT(1) spaces \( Y \) and \( Z \), the spherical join \( Y \ast Z \) is geodesically complete if and only if \( Y \) and \( Z \) are geodesically complete and not a singleton.
2.8. Dimension of CAT(κ) spaces. Let X be a separable CAT(κ) space. The covering (topological) dimension \( \dim X \) satisfies
\[
\dim X = 1 + \sup_{p \in X} \dim \Sigma_p X = \sup_{p \in X} \dim T_p X
\]
(22). Assume in addition that X is proper and geodesically complete. Every relatively compact open subset of X has finite covering dimension (see [28, Subsection 5.3]). The covering dimension \( \dim X \) is equal to the Hausdorff dimension of X, and equal to the supremum of \( m \) such that X has an open subset \( U \) homeomorphic to \( \mathbb{R}^m \) ([28, Theorem 1.1]).

From the studies in [28, Subsection 11.3] on the stability of dimension, we can derive the following (see [34, Lemmas 2.1 and 2.3]):

**Lemma 2.3.** Let \((X_i, p_i)\) be a sequence of pointed proper geodesically complete CAT(κ) spaces. Assume that \((X_i, p_i)\) converges to a pointed metric space \((X, p)\) in the pointed Gromov–Hausdorff topology. Then we have
\[
\dim X \leq \liminf_{i \to \infty} \dim X_i.
\]
If in addition each \( X_i \) is purely \( n \)-dimensional, then so is \( X \).

On the Gromov–Hausdorff topology, we have the following continuity (see [34, Lemma 2.2]):

**Lemma 2.4.** Let \((X_i)\) be a sequence of compact geodesically complete CAT(κ) spaces. Assume that \((X_i)\) converges to a metric space \( X \) in the Gromov–Hausdorff topology. Then we have
\[
\lim_{i \to \infty} \dim X_i = \dim X.
\]

We say that a separable metric space is pure-dimensional if it is purely \( n \)-dimensional for some \( n \).

We have the following characterization ([29, Proposition 8.1]):

**Proposition 2.5.** ([29]) Let \( X \) be a proper, geodesically complete, geodesic CAT(κ) space. Let \( W \) be a connected open subset of \( X \). Then the following are equivalent:

1. \( W \) is pure-dimensional;
2. for every \( p \in W \) the space \( \Sigma_p X \) is pure-dimensional;
3. for every \( p \in W \) the space \( T_p X \) is pure-dimensional.

2.9. On the volumes of CAT(κ) spaces. For \( \kappa \in \mathbb{R} \) and \( r \in (0, D_\kappa) \), we denote by \( \omega_\kappa^n(r) \) the \( n \)-dimensional Hausdorff measure of any metric ball in \( M^\kappa_\kappa \) of radius \( r \) if \( n \geq 2 \), and by \( \omega_\kappa^1(r) \) the 1-dimensional Hausdorff measure of \([-r, r]\).

We recall that for every proper, geodesically complete CAT(κ) space \( X \), the \( n \)-dimensional part \( X^n \) of \( X \) coincides with the set of all points \( p \in X \) with \( \dim \Sigma_p X = n - 1 \); if in addition \( \dim X = n \), then \( X^n \) also coincides with the support of \( \mathcal{H}^n \) ([28, Theorem 1.2]). For CAT(κ)
spaces, we have the following absolute volume comparison of Bishop–Günther type ([32, Proposition 6.1], [34, Proposition 3.1]):

**Proposition 2.6.** ([32]) Let $X$ be a proper, geodesically complete CAT($\kappa$) space, and let $p \in X$ be a point with $p \in X^n$. Then for every $r \in (0, D_\kappa)$ we have

$$\mathcal{H}^n(U_r(p)) \geq \omega^n_\kappa(r).$$

Moreover, if in addition $X$ is purely $n$-dimensional, then the equality holds if and only if the pair $(U_r(p), p)$ is isometric to $(U_r(\tilde{p}), \tilde{p})$ for any point $\tilde{p} \in \mathcal{M}_n^\kappa$.

Furthermore, we have the following relative volume comparison of Bishop–Gromov type ([32, Proposition 6.3], [34, Proposition 3.2]):

**Proposition 2.7.** ([32]) Let $X$ be a proper, geodesically complete CAT($\kappa$) space, and let $p \in X$ be a point with $p \in X^n$. Then the function $f_p: (0, D_\kappa) \to [1, \infty[$ defined by

$$f_p(t) := \frac{\mathcal{H}^n(U_t(p))}{\omega^n_\kappa(t)}$$

is monotone non-decreasing.

Let $(X_i)$ be a sequence of compact geodesically complete CAT($\kappa$) spaces of dim $X_i = n$ converging to a metric space $X$ in the Gromov–Hausdorff topology. By Lemma 2.4 the compact, geodesically complete CAT($\kappa$) space $X$ satisfies dim $X = n$. Let $(X_i, p_i)$ be a sequence of pointed, proper geodesically complete CAT($\kappa$) spaces of dim $X_i = n$ converging to a pointed metric space $(X, p)$ in the pointed Gromov–Hausdorff topology. By Lemma 2.3 the proper, geodesically complete CAT($\kappa$) space $X$ satisfies dim $X \leq n$.

We quote the volume convergence theorem for CAT($\kappa$) spaces in [32, Theorem 1.1] in the following form:

**Theorem 2.8.** ([32]) If a sequence $(X_i)$ of $n$-dimensional, compact geodesically complete CAT($\kappa$) spaces converges to some metric space $X$ in the Gromov–Hausdorff topology, then we have

$$\mathcal{H}^n(X) = \lim_{i \to \infty} \mathcal{H}^n(X_i).$$

If a sequence $(X_i, p_i)$ of pointed, $n$-dimensional, proper geodesically complete CAT($\kappa$) spaces converges to some pointed metric space $(X, p)$ in the pointed Gromov–Hausdorff topology, then for every $r \in (0, \infty)$ with dim $U_r(p) = n$ we have

$$\mathcal{H}^n(U_r(p)) = \lim_{i \to \infty} \mathcal{H}^n(U_r(p_i)).$$

The second half of Theorem 2.8 on the pointed Gromov–Hausdorff convergence, not shown explicitly in [32, Theorem 1.1], can be proved by a similar argument to that discussed in [32, Section 4].
Remark 2.1. Lytchak and the author [28] introduced a positive Radon measure on an arbitrary GCBA space, called the canonical measure, whose restriction to the \(m\)-dimensional part coincides with \(H^m\) (see [28, Theorem 1.4]). As a generalization of Theorem 2.8, [28] proved the continuity of the canonical measure with respect to the Gromov–Hausdorff topology for a sequence of compact GCBA spaces of dimension, curvature and diameter bounded from above and injectivity radius bounded below by some constants [28, Theorem 1.5], and general local statements [28, Section 12]. Cavallucci and Sambusetti [13] show the upper semi-continuity of the canonical measure of balls with respect to the pointed Gromov–Hausdorff topology for a sequence of proper GCBA spaces [13, Lemma 2.7], and formulate the continuity under a uniform measure doubling condition [13, Corollary 5.7].

We next recall the following volume regularity ([32, Theorem 1.10]):

**Theorem 2.9.** ([32]) For every \(\epsilon \in (0, \infty)\), and for every \(m \in \mathbb{N}\), there exists \(\delta \in (0, \infty)\) satisfying the following property: If a purely \(m\)-dimensional, compact, geodesically complete CAT(1) space \(\Sigma\) satisfies
\[
H^m(\Sigma) < H^m(S^m) + \delta,
\]
then \(\Sigma\) is \((1 + \epsilon)\)-bi-Lipschitz homeomorphic to \(S^m\).

3. Gromov–Hausdorff asymptotic cones

In this section, we denote by \(O_\infty\) the set of all sequences \((\lambda_i)\) in \((0, \infty)\) with \(\lim_{i \to \infty} \lambda_i = 0\). We discuss basic properties of Gromov–Hausdorff asymptotic cones of CAT(0) spaces.

3.1. Asymptotic cones and Tits boundaries. Let \(X\) be a proper geodesic metric space. Let \(p \in X\). For a sequence \((\lambda_i) \in O_\infty\), the Gromov–Hausdorff asymptotic cone \(C^{(\lambda_i)}_{\infty} X\) of \(X\) with scale \((\lambda_i)\) is defined by
\[
(C^{(\lambda_i)}_{\infty} X, p_\infty) := \lim_{i \to \infty} (\lambda_i X, p),
\]
if the pointed Gromov–Hausdorff limit \(\lim_{i \to \infty} (\lambda_i X, p)\) exists, where \(p_\infty\) is called the limit base point of \(p\). Notice that \(C^{(\lambda_i)}_{\infty} X\) does not depend on the choice of the base point \(p\), and \(C^{(\lambda_i)}_{\infty} X\) is proper.

The following lemma seems to be well-known as a consequence of the Gromov precompactness theorem:

**Lemma 3.1.** If a proper geodesic metric space \(X\) is \(N\)-doubling, then for some \((\lambda_i) \in O_\infty\) there exists the Gromov–Hausdorff asymptotic cone \(C^{(\lambda_i)}_{\infty} X\) of \(X\) with scale \((\lambda_i)\) such that \(C^{(\lambda_i)}_{\infty} X\) is \(N\)-doubling.

**Proof.** Take \(p \in X\) and \(r \in (0, \infty)\). Let \((\lambda_i) \in O_\infty\). Then the compact ball \(B_r(p)\) in \(\lambda_i X\) is \(N\)-doubling for all \(i\). From the Gromov precompactness theorem we deduce that the pointed sequence \((\lambda_i X, p)\) has a Gromov–Hausdorff convergent subsequence whose limit is \(N\)-doubling.
Discussing diagonal arguments adequately, we see that $X$ has an $N$-doubling Gromov–Hausdorff asymptotic cone with some scale. □

For a proper CAT(0) space, the existence of a Gromov–Hausdorff asymptotic cone with some scale leads to the compactness of the Tits boundary. Namely, we have the following:

**Lemma 3.2.** Let $X$ be a proper CAT(0) space whose Tits boundary $\partial_T X$ is non-empty. If $X$ has a Gromov–Hausdorff asymptotic cone $C^{(\lambda_i)}_\infty X$ of $X$ with some scale $(\lambda_i)$, then $\partial_T X$ is compact.

**Proof.** Assume that there exists a Gromov–Hausdorff asymptotic cone $C^{(\lambda_i)}_\infty X$ of $X$ with some scale $(\lambda_i)$. Then for a fixed base point $p \in X$ the pointed sequence $(\lambda_i X, p)$ converges to $(C^{(\lambda_i)}_\infty X, p_\infty)$ in the pointed Gromov–Hausdorff topology, where $p_\infty$ is the limit base point of $p$.

It suffices to show that $\partial_T X$ is totally bounded. Let $r \in (0, \pi)$, and let $Z_r$ be a maximal $r$-separated subset of $\partial_T X$. Now the sequence of compact metric balls $B_1(p; \lambda_i X)$ of radius $1$ around $p$ in $\lambda_i X$ converges to the compact ball $B_1(p_\infty)$ in $C^{(\lambda_i)}_\infty X$. For some sequence $(\epsilon_i) \in O_\infty$, for each $i$ we can take an $\epsilon_i$-approximation $\varphi_i : B_1(p; \lambda_i X) \to B_1(p_\infty)$. By the compactness of $B_1(p_\infty)$, we find a constant $N_0 \in \mathbb{N}$ such that every $\sin(r/2)$-separated subset of $B_1(p_\infty)$ has at most $N_0$ elements.

Suppose that there exists an $r$-separated subset $\{\xi_1, \ldots, \xi_{N_0+1}\}$ of $Z_r$. For each $j \in \{1, \ldots, N_0+1\}$, we put $x_j^i := \gamma_{p \xi_j} (1/\lambda_i)$, where $\gamma_{p \xi_j}$ is the ray in $X$ from $p$ to $\xi_j$. By selecting a subsequence if necessary, we may assume that for each $j \in \{1, \ldots, N_0+1\}$ the sequence $(\varphi_i(x_j^i))$ converges to a point $x_j^\infty$ in $B_1(p_\infty)$. Let $d_X$ be the metric on $X$, and $d_\infty$ the metric on $C^{(\lambda_i)}_\infty X$. From Lemma 2.11 it follows that for every $\epsilon \in (0, \infty)$, for all distinct $j, k \in \{1, \ldots, N_0+1\}$ we have

$$d_\infty (x_j^\infty, x_k^\infty) > d_\infty (\varphi_i (x_j^i), \varphi_i (x_k^i)) - \epsilon > (\lambda_i d_X) (x_j^i, x_k^i) - \epsilon_i + \epsilon$$

$$> d_{C_0(\partial_T X)} (\xi_j, \xi_k) - \epsilon_i - 2\epsilon = 2 \sin \frac{d_T (\xi_j, \xi_k)}{2} - \epsilon_i - 2\epsilon$$

$$\geq 2 \sin \frac{r}{2} - \epsilon_i - 2\epsilon \geq \sin \frac{r}{2} - 2\epsilon$$

for all sufficiently large $i$. Hence the set $\{x_1^\infty, \ldots, x_{N_0+1}^\infty\}$ is $\sin(r/2)$-separated in $B_1(p_\infty)$. This contradicts the choice of the constant $N_0$.

Thus $Z_r$ has at most $N_0$ elements, and hence it is a finite $(r/2)$-net. Therefore $\partial_T X$ is totally bounded. □

### 3.2. Asymptotic cones and Euclidean cones

A pointed metric space $(X, p)$ is said to be isometric to a pointed metric space $(Y, q)$ if there exists an isometry $f : X \to Y$ with $f(p) = q$.

Let $X$ be a proper geodesic metric space. The Gromov–Hausdorff asymptotic cone $C_\infty X$ of $X$ is defined by

$$(C_\infty X, p_\infty) := \lim_{\lambda \to 0} (\lambda X, p),$$
Let \( \epsilon \) if the limit exists; more precisely, if for every sequence \((\lambda_i) \in O_\infty\) there exists the Gromov–Hausdorff asymptotic cone \( C^{(\lambda_i)}_\infty X \) of \( X \) with scale \((\lambda_i)\), and if for all \((\lambda_i), (\lambda_i') \in O_\infty\) the Gromov–Hausdorff asymptotic cones \( C^{(\lambda_i)}_\infty X, p_\infty \) and \( C^{(\lambda_i')}_\infty X, p'_\infty \) are isometric to each other.

Next we prove the following:

**Lemma 3.3.** Let \( X \) be a proper, geodesically complete CAT(0) space. Assume that \( \partial_T X \) is compact. Let \( p \in X \) be a point. Then \( X \) has the Gromov–Hausdorff asymptotic cone \( C^\infty X \) such that \( (C^\infty X, p_\infty) \) is isometric to \( (C_0(\partial_T X), 0) \), where \( p_\infty \) is the limit base point of \( p \). More precisely, for every \( r \in (0, \infty) \), and for every \( \epsilon \in (0, \infty) \), there exists \( \lambda_0 \in (0, \infty) \) such that for each \( \lambda \in (0, \lambda_0) \) the map \( \Phi^\lambda_p : B_r(0) \to B_r(p) \) from the ball \( B_r(0) \) in \( C_0(\partial_T X) \) to the ball \( B_r(p) \) in \( \lambda X \) defined by

\[
\Phi^\lambda_p(a\xi) := \gamma_{p\xi} (a/\lambda)
\]

is a surjective 1-Lipschitz \( \epsilon \)-approximation.

**Proof.** Take \( r \in (0, \infty) \). Define a function \( \theta_r : (0, 2r) \to (0, \infty) \) by

\[
\theta_r(t) := 2 \sin^{-1} \frac{t}{2r}.
\]

Let \( \epsilon \in (0, 2r) \). From Lemma 2.1 we derive that if \( \xi, \eta \in \partial_T X \) satisfy \( d_T(\xi, \eta) < \theta_r(\epsilon) \), then for every \( \lambda \in (0, \infty) \) we have

\[
(\lambda d_X)(\gamma_{p\xi}(r/\lambda), \gamma_{p\eta}(r/\lambda)) < \epsilon.
\]

Since \( \partial_T X \) is compact, we can find a finite \( \theta_r(\epsilon) \)-net \( \{\xi_1, \ldots, \xi_m\} \) of \( \partial_T X \). By Lemma 2.1 there exists \( \lambda_0 \in (0, \infty) \) such that for every \( \lambda \in (0, \lambda_0) \) we have

\[
|d_{C_0(\partial_T X)}((l_1\xi)\zeta_m, (l_2\xi)\zeta_m) - (\lambda d_X)(\gamma_{p\xi_1}(l_1\epsilon/\lambda), \gamma_{p\xi_2}(l_2\epsilon/\lambda))| < \epsilon
\]

for all \( l_1, l_2 \in \{0, 1, \ldots, \lfloor r/\epsilon \rfloor\} \) and for all \( m_1, m_2 \in \{1, \ldots, m\} \).

Fix \( \lambda \in (0, \lambda_0) \). As shown in Lemma 2.1, the map \( \Phi^\lambda_p \) is 1-Lipschitz. From the geodesical completeness of \( X \) it follows that \( \Phi^\lambda_p \) is surjective. To verify that \( \Phi^\lambda_p \) is a \( 9\epsilon \)-approximation, we pick \( \xi_1, \xi_2 \in \partial_T X \), and \( a_1, a_2 \in (0, r) \). For some \( m_1, m_2 \in \{1, \ldots, m\} \), we have \( \xi_1 \in U_{\theta_r(\epsilon)}(\zeta_m) \) and \( \xi_2 \in U_{\theta_r(\epsilon)}(\zeta_{m_2}) \). In addition, for some \( l_1, l_2 \in \{1, \ldots, \lfloor r/\epsilon \rfloor\} \), we have \( |l_1\epsilon - a_1| < \epsilon \) and \( |l_2\epsilon - a_2| < \epsilon \). Then

\[
d_{C_0(\partial_T X)}(a_1\xi_1, a_2\xi_2) < d_{C_0(\partial_T X)}((l_1\epsilon)\zeta_m, (l_2\epsilon)\zeta_{m_2}) + 4\epsilon
\]

\[
< (\lambda d_X)(\gamma_{p\xi_1}(l_1\epsilon/\lambda), \gamma_{p\xi_2}(l_2\epsilon/\lambda)) + 5\epsilon
\]

This implies that \( \Phi^\lambda_p \) is a \( 9\epsilon \)-approximation. Thus \( X \) has the Gromov–Hausdorff asymptotic cone \( C^\infty_\lambda X \) isometric to \( C_0(\partial_T X) \).

In summary, we conclude the following:
Proposition 3.4. Let $X$ be a proper, geodesically complete CAT(0) space. Then the following are equivalent:

1. $X$ has the Gromov–Hausdorff asymptotic cone $C_\infty X$;
2. $X$ has a Gromov–Hausdorff asymptotic cone $C^{(\lambda)} X$ with some scale $(\lambda_i)$;
3. $\partial_T X$ is compact.

In this case, for every $p \in X$ the pointed limit $(C_\infty X, p_\infty)$ is isometric to $(C_0(\partial_T X), 0)$, where $p_\infty$ is the limit base point of $p$. In particular, the following hold:

1. $\partial_T X$ is geodesically complete, and not a singleton; moreover, if $X$ is doubling, then $\partial_T X$ is doubling;
2. if $\dim X = n$, then $\dim \partial_T X = n - 1$;
3. if $X$ is pure-dimensional, then $\partial_T X$ is also pure-dimensional.

Proof. In Lemma 3.2 and Lemma 3.3, we already show the implications $(2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$. As shown in Lemma 3.3, we see that $X$ has the Gromov–Hausdorff asymptotic cone $C_\infty X$ such that $(C_\infty X, p_\infty)$ is isometric to $(C_0(\partial_T X), 0)$ for the limit base point $p_\infty$.

1. Since $C_0(\partial_T X)$ is isometric to $C_\infty X$, it is geodesically complete, and hence $\partial_T X$ is geodesically complete and not a singleton; moreover, if $X$ is $N$-doubling, then so is $C_\infty X$. Hence $\partial_T X$ is doubling.

2. Assume that $\dim X = n$. Applying Lemma 2.3 to the Gromov–Hausdorff asymptotic cone $C_\infty X$, we see $\dim C_\infty X \leq n$. On the other hand, since there exists a point $x \in X$ with $\dim \Sigma_x X = n - 1$, we have $\dim \Sigma_{p_\infty} (C_\infty X) \geq n - 1$ ([28, Lemma 11.5]); in particular, we see $\dim C_\infty X \geq n$. Hence $\dim C_0(\partial_T X) = n$, and $\dim \partial_T X = n - 1$.

3. Assume that $X$ is purely $n$-dimensional. From Lemma 2.3 it follows that $C_\infty X$ is purely $n$-dimensional. Hence $C_0(\partial_T X)$ is purely $n$-dimensional. Therefore $\partial_T X$ is purely $(n - 1)$-dimensional. \qed

Remark 3.1. Let $X$ be a complete CAT(0) space. If $X$ has telescopic dimension $\leq n$ in the sense of Caprace–Lytchak [12], then the Tits boundary $\partial_T X$ has geometric dimension $\leq n - 1$ in the sense of Kleiner [22] ([12, Proposition 2.1]). If $X$ is proper and $\dim X \leq n$, then $\dim_C \partial_T X \leq n - 1$, where $\dim_C \partial_T X$ is the supremum of the topological dimensions of compact subsets of $\partial_T X$ ([18, Proposition 1.8]).

3.3. Asymptotic cones and volume growths. We first show the following volume convergence:

Proposition 3.5. Let $X$ be a proper, geodesically complete CAT(0) space of $\dim X = n$. If $X$ has the Gromov–Hausdorff asymptotic cone $C_\infty X$, then for every $p \in X$ we have

\[
\lim_{\lambda \to 0} \frac{\mathcal{H}^n (U_{1/\lambda}(p))}{\omega_0^n (1/\lambda)} = \frac{\mathcal{H}^{n-1} (\partial_T X)}{\mathcal{H}^{n-1} (\mathbb{S}^{n-1})}.
\]
Proof. Let \((\lambda_i) \in O_\infty\). By Proposition [34], the sequence \((\lambda_i, p)\) of the pointed proper metric spaces converges to the pointed proper metric space \((C_0(\partial_T X), 0)\) in the pointed Gromov–Hausdorff topology. For each \(i\), let \(U_1(p; \lambda_i X)\) be the open metric ball of radius 1 around \(p\) in \(\lambda_i X\). By Proposition [34] (2), for the open ball \(U_1(0)\) in \(C_0(\partial_T X)\) we see \(\dim U_1(0) = n\). Therefore from Theorem [28] we derive

\[
\lim_{i \to \infty} \mathcal{H}^n (U_1(p; \lambda_i X)) = \mathcal{H}^n (U_1(0)).
\]

For the open ball \(U_{1/\lambda_i}(p)\) in \(X\), we have

\[
\mathcal{H}^n (U_1(p; \lambda_i X)) = \lambda_i^n \mathcal{H}^n (U_{1/\lambda_i}(p))
\]

for all \(i\). Combining (3.2) and (3.3) leads to

\[
\lim_{i \to \infty} \frac{\mathcal{H}^n (U_{1/\lambda_i}(p))}{\omega^n_0 (1/\lambda_i)} = \frac{\mathcal{H}^n (U_1(0))}{\omega^n_0 (1)} = \frac{\mathcal{H}^{n-1} (\partial_T X)}{\mathcal{H}^{n-1} (S^{n-1})}.
\]

The last equality follows from [32] Lemma 6.4 and the fact that both \(C_0(\partial_T X)\) and \(\mathbb{R}^n\) are Euclidean cones. Thus we obtain the desired equality [31].

We next prove the following lemma:

Lemma 3.6. For every \(n \in \mathbb{N}\), and for every \(c \in [1, \infty)\), there exists a constant \(N_0 \in \mathbb{N}\) satisfying the following property: Let \(X\) be a purely \(n\)-dimensional, proper, geodesically complete CAT(0) space. If \(X\) satisfies \(\mathcal{G}^n_0(X) \leq c\), then \(X\) is \(N_0\)-doubling.

Proof. It suffices to find a constant \(N_0 \in \mathbb{N}\) depending only on \(n\) and \(c\) such that for every \(r \in (0, \infty)\) and for every \(x \in X\), any \(r\)-separated set contained in \(U_{2r}(x)\) has cardinality \(\leq N_0\). Take a maximal \(r\)-separated set \(\{x_1, \ldots, x_N\}\) in \(B_{2r}(x)\), so that \(B_{2r}(x)\) is contained in \(\bigcup_{j=1}^N U_j(x_j)\). Observe that \(U_{r/2}(x_j) \cap U_{r/2}(x_k)\) is empty for all distinct \(j, k \in \{1, \ldots, N\}\). Since \(X\) is purely \(n\)-dimensional, for each \(j \in \{1, \ldots, N\}\) the value \(\mathcal{H}^n(U_{r/2}(x_j))\) is positive and finite. Choose a number \(k_0 \in \{1, \ldots, N\}\) such that \(\mathcal{H}^n(U_{r/2}(x_{k_0}))\) is minimal among \(\mathcal{H}^n(U_{r/2}(x_1)), \ldots, \mathcal{H}^n(U_{r/2}(x_N))\). Notice that for each \(j \in \{1, \ldots, N\}\) the ball \(U_{r/2}(x_j)\) is contained in \(U_{5r/2}(x_{k_0})\). Hence we have

\[
N \mathcal{H}^n (U_{r/2}(x_{k_0})) \leq \mathcal{H}^n \left( \bigcup_{j=1}^N U_{r/2}(x_j) \right) \leq \mathcal{H}^n (U_{5r/2}(x_{k_0})).
\]

From Propositions [26] and [27], and from \(\mathcal{G}^n_0(X) \leq c\), we derive

\[
N \leq \frac{\mathcal{H}^n (U_{5r/2}(x_{k_0}))}{\mathcal{H}^n (U_{r/2}(x_{k_0}))} \leq \frac{\mathcal{H}^n (U_{5r/2}(x_{k_0}))}{\omega^n_0 (r/2)} = 5^n \frac{\mathcal{H}^n (U_{5r/2}(x_{k_0}))}{\omega^n_0 (5r/2)}
\]

\[
\leq 5^n \mathcal{G}^n_0(X) \leq 5^n c.
\]

Letting \(N_0 := \lceil 5^n c \rceil\) completes the proof. \(\square\)
Summing up, we conclude the following:

**Proposition 3.7.** Let $X$ be a purely $n$-dimensional, proper, geodesically complete CAT(0) space. Then the following are equivalent:

1. $X$ is doubling;
2. $X$ has the Gromov–Hausdorff asymptotic cone $C_{\infty} X$;
3. For some $c \in [1, \infty)$ we have $G^n_0(X) \leq c$.

In this case, for every $x \in X$ we have

$$
\frac{\mathcal{H}^{n-1}(\Sigma_x X)}{\mathcal{H}^{n-1}(S^{n-1})} \leq \frac{\mathcal{H}^{n-1}(\partial_T X)}{\mathcal{H}^{n-1}(S^{n-1})} = G^n_0(X).
$$

**Proof.** In Proposition 3.5 and Lemma 3.6 we already verify $(2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$, respectively. The implication $(1) \Rightarrow (2)$ follows from Lemma 3.1 and Proposition 3.4.

Now we show (3.4). For every $x \in X$, there exists a 1-Lipschitz map $f_x : \partial_T X \to \Sigma_x X$ defined by $f_x(\xi) := \xi_x'$, where $\xi_x' \in \Sigma_x X$ is the starting direction of $x \xi$ at $x$. Since $X$ is geodesically complete, the map $f_x$ is surjective. Hence we see the inequality in (3.4). From Proposition 3.5 we derive the equality in (3.4). This finishes the proof.

**Remark 3.2.** Cavallucci and Sambusetti [13] prove the compactness results for various classes consisting of GCBA spaces with respect to the Gromov–Hausdorff topology, and to the pointed one, which are closely related to the contents in this section.

### 4. Homotopy at infinity

The goal of this section is to prove that if a proper, geodesically complete CAT(0) space has the Gromov–Hausdorff asymptotic cone, then any sufficiently large metric sphere is homotopy equivalent to the Tits ideal boundary. In the proof, we use the homotopic stability of fibers of strainer maps discussed by Lytchak and the author [28].

#### 4.1. Almost spherical points

Let $\delta \in (0, \infty)$. Let $\Sigma$ be a compact, geodesically complete CAT(1) space with metric $d_{\Sigma}$ of diameter $\pi$. A point $\xi \in \Sigma$ is said to be $\delta$-spherical if there exists $\eta \in \Sigma$ such that for every $\zeta \in \Sigma$ we have

$$
d_{\Sigma}(\xi, \zeta) + d_{\Sigma}(\zeta, \eta) < \pi + \delta;
$$

in this case, the pair of $\xi$ and $\eta$ are said to be opposite. For a point $\xi \in \Sigma$, a point $\eta \in \Sigma$ is an antipode of $\xi$ if $d_{\Sigma}(\xi, \eta) = \pi$, and the set of all antipodes of $\xi$ is denote by $\text{Ant}(\xi)$.

From the triangle inequality and the extendability of geodesics to length $\pi$, we derive the following ([28, Lemma 6.3]):
Lemma 4.1. ([28]) Let $\Sigma$ be a compact geodesically complete CAT(1) space with metric $d_\Sigma$ of diameter $\pi$. Then $\xi, \eta \in \Sigma$ are opposite $\delta$-spherical points if and only if $d_\Sigma(\eta, \zeta) < \delta$ for any $\zeta \in \text{Ant}(\xi)$; in this case, $d_\Sigma(\xi, \eta) > \pi - \delta$, and $\text{Ant}(\xi)$ has diameter $< 2\delta$; moreover, for every $\zeta \in \text{Ant}(\xi)$ the pair of $\xi$ and $\zeta$ are opposite $2\delta$-spherical points.

An $m$-tuple $(\xi_1, \ldots, \xi_m)$ of points in $\Sigma$ is said to be $\delta$-spherical if there exists another $m$-tuple $(\eta_1, \ldots, \eta_m)$ of points in $\Sigma$ such that

1. $\xi_j$ and $\eta_j$ are opposite $\delta$-spherical points for all $j \in \{1, \ldots, m\}$;
2. $d_\Sigma(\xi_j, \xi_k), d_\Sigma(\eta_j, \eta_k)$, and $d_\Sigma(\xi_j, \eta_k)$ are smaller than $\pi/2 + \delta$ for all distinct $j, k \in \{1, \ldots, m\}$;

in this case, the pair of $(\xi_1, \ldots, \xi_m)$ and $(\eta_1, \ldots, \eta_m)$ are opposite.

By Lemma 4.1 we have the following ([28, Corollary 6.1]):

Lemma 4.2. ([28]) Let $\Sigma$ be a compact geodesically complete CAT(1) space with metric $d_\Sigma$ of diameter $\pi$, and let $(\xi_1, \ldots, \xi_m)$ be an $m$-tuple of $\delta$-spherical points in $\Sigma$. Then the following hold:

1. if $(\xi_1, \ldots, \xi_m)$ is a $\delta$-spherical $m$-tuple, then
   $$\frac{\pi}{2} - 2\delta < d_\Sigma(\xi_j, \xi_k) < \frac{\pi}{2} + \delta$$
   for all distinct $j, k \in \{1, \ldots, m\}$;
2. if for all distinct $j, k \in \{1, \ldots, m\}$ we have
   $$\frac{\pi}{2} - \delta < d_\Sigma(\xi_j, \xi_k) < \frac{\pi}{2} + \delta,$$
   then for each $\eta_j \in \Sigma$ with $d_\Sigma(\xi_j, \eta_j) = \pi$, $j \in \{1, \ldots, m\}$, the $m$-tuples $(\xi_1, \ldots, \xi_m)$ and $(\eta_1, \ldots, \eta_m)$ are opposite $2\delta$-spherical.

4.2. Strainers. Based on [28, Section 7], we discuss the notions of strainers for proper geodesically complete CAT($\kappa$) spaces.

Let $X$ be a proper, geodesically complete CAT($\kappa$) space. A point $x \in X$ is said to be $(m, \delta)$-strained if $\Sigma_x X$ admits a $\delta$-spherical $m$-tuple.

We say that an open metric ball in $X$ is tiny if the radius is smaller than $\min\{1, \Delta_0/100\}$. An open metric ball $U_{r_0}(x_0)$ in $X$ has capacity bounded by $N$ if $B_{5r_0}(x_0)$ is $N$-doubling. Notice that every tiny ball has capacity bounded by $N$ for some $N$ ([28, Proposition 5.1]).

Let $U_{r_0}(x_0)$ be a tiny ball in $X$. For a point $x \in U_{r_0}(x_0)$, we say that an $m$-tuple $(p_1, \ldots, p_m)$ of points in $B_{5r_0}(x_0) - \{x\}$ is an $(m, \delta)$-strainer at $x$ if the $m$-tuple $(p_1', \ldots, p_m')$ of the starting directions at $x$ is $\delta$-spherical in $\Sigma_x X$. For a subset $W$ of $U_{r_0}(x_0)$, we say that an $m$-tuple $(p_1, \ldots, p_m)$ of points in $B_{5r_0}(x_0) - W$ is an $(m, \delta)$-strainer at $W$ if for every $x \in W$ the $m$-tuple $(p_1, \ldots, p_m)$ is an $(m, \delta)$-strainer at $x$.

We review the following basic observation ([28, Proposition 7.3]):

Proposition 4.3. ([28]) Let $\delta \in (0, \infty)$. Let $X$ be a proper, geodesically complete CAT($\kappa$) space. Then for every $p \in X$ there exists $r \in (0, \Delta_0)$ such that the point $p$ is a $(1, \delta)$-strainer at $U_r(p) - \{p\}$. 

Theorem 4.5. Let $\delta \in (0, \infty)$. Let $X$ be a proper, geodesically complete CAT($\kappa$) space. If $4m\delta < 1$, then for every $(m, \delta)$-strained point $x \in X$ we have $\dim T_x X \geq m$.

Notice that if a tiny ball $U_{r_0}(x_0)$ in $X$ satisfies $\dim U_{r_0}(x_0) = n$, then $n$ is the largest number such that there exists a point $x \in U_{r_0}(x_0)$ at which $\Sigma_x X$ is isometric to $\mathbb{R}^n$. In addition, $n$ is the largest number such that there exists an $(n, 1/4n)$-strained point in $U_{r_0}(x_0)$ (see [28, Proposition 11.1]).

4.3. Strainer maps. As well as [28, Section 8], we discuss the notions of strainer maps for proper geodesically complete CAT($\kappa$) spaces.

Let $X$ be a proper, geodesically complete CAT($\kappa$) space with metric $d_X$. For a point $p \in X$, we denote by $d_p$ the distance function from $p$ defined by $d_p(x) := d_X(p, x)$. We say that a map $f: U \to \mathbb{R}^m$ from an open subset $U$ in a tiny ball in $X$ is an $(m, \delta)$-strainer map if there exists an $(m, \delta)$-strainer $(p_1, \ldots, p_m)$ at $U$ with $f = (d_{p_1}, \ldots, d_{p_m})$. Every $(m, \delta)$-strainer map is $2\sqrt{m}$-Lipschitz. If $4m\delta \leq 1$, then every $(m, \delta)$-strainer map $f: U \to \mathbb{R}^m$ is $2\sqrt{m}$-Lipschitz and $2\sqrt{m}$-open; in particular, the Hausdorff dimension of $U$ is at least $m$ ([28, Lemma 8.2]). Moreover, the Lipschitz constant and the openness constant of strainer maps can be chosen close to 1 in the following sense: For a given constant $N \in \mathbb{N}$, for every $\epsilon \in (0, 1)$, and for every $m \in \mathbb{N}$, there exists $\delta \in (0, \infty)$ such that every $(m, \delta)$-strainer map with domain contained in a tiny ball of capacity bounded by $N$ is $(1 + \epsilon)$-Lipschitz and $(1 + \epsilon)$-open ([28, Corollary 8.4]).

4.4. Homotopic stability of fibers of strainer maps. We now recall the following homotopic stability theorem of [28] concerning fibers of strainer maps on proper geodesically complete CAT($\kappa$) spaces. (see a more general statement [28, Theorem 13.1]):

Theorem 4.5. ([28]) Let $(X_i)$ be a sequence of proper geodesically complete CAT($\kappa$) spaces, and let $X$ be a proper, geodesically complete CAT($\kappa$) space. Assume that a sequence $(U_{r_0}(x_i))$ of tiny balls with radius $r_0$ in $X_k$ has capacity uniformly bounded by $N_0$, and a sequence $(B_{10r_0}(x_i), x_i)$ of pointed compact metric balls in $X_i$ converges to a pointed compact metric ball $(B_{10r_0}(x), x)$ in $X$ in the pointed Gromov–Hausdorff topology. For an open subset $U$ contained in $U_{r_0}(x)$, and for an $(m, \delta)$-strainer $(p_1, \ldots, p_m)$ at $U$ with $20m\delta \leq 1$, let $f: U \to \mathbb{R}^m$ be an $(m, \delta)$-strainer map with $f = (d_{p_1}, \ldots, d_{p_m})$. Assume that for some $c \in \mathbb{R}^m$ the fiber $f^{-1}\{c\}$ is compact. Let $(\Pi_i)$ be a sequence of compact subsets contained in $U_{r_0}(x_i)$ such that $(\Pi_i)$ converges to $f^{-1}\{c\}$ in the Gromov–Hausdorff topology. Let $(p_{i,j}, \ldots, p_{m,j})$ be a sequence of $m$-tuples of points in $B_{5r_0}(x_i)$ such that $(p_{i,j})$ converges to $p_j$ for all $j \in \{1, \ldots, m\}$. Then for every sequence $(c_i)$ in $\mathbb{R}^m$ with $\lim_{i \to \infty} c_i = c$
there exists a positive number \( r \in (0, \infty) \) such that for the sequence \((f_i)\) of the maps \( f_i : U_r(\Pi_i) \to \mathbb{R}^m \) given by \( f_i = (d_{p_1}, \ldots, d_{p_m}) \) the following hold for all sufficiently large \( i \):

1. each \( f_i \) is an \((m, \delta)\)-strainer map;
2. the fiber \( f_i^{-1}(\{c\}) \) is compact, and the sequence \((f_i^{-1}(\{c\}))\) converges to \( f^{-1}(\{c\}) \) in the Gromov–Hausdorff topology.
3. the fiber \( f_i^{-1}(\{c\}) \) is homotopy equivalent to the fiber \( f^{-1}(\{c\}) \).

### 4.5. Homotopic stability at infinity

We deduce the following:

**Theorem 4.6.** Let \( X \) be a proper, geodesically complete CAT(0) space. If \( X \) has the Gromov–Hausdorff asymptotic cone \( C_\infty X \), then for every \( p \in X \) there exists a sufficiently large \( t_0 \in (0, \infty) \) such that for an arbitrary \( t \in [t_0, \infty) \) the metric sphere \( S_t(x) \) is homotopy equivalent to the Tits boundary \( \partial_T X \).

**Proof.** Let \( p \in X \) be arbitrary. Take a sequence \((\lambda_i)\) in \((0, \infty)\) with \( \lim_{i \to \infty} \lambda_i = 0 \). Since \( X \) has the Gromov–Hausdorff asymptotic cone \( C_\infty X \), as seen in Proposition 3.4, the pointed Gromov–Hausdorff limit \( \lim_{i \to \infty}(\lambda_i X, p) \) exists, and it is isometric to \((C_0(\partial_T X), 0)\); moreover, \( C_0(\partial_T X) \) is proper and geodesically complete.

Take a unit open metric ball \( U_1(0) \) in \( C_0(\partial_T X) \) as a tiny ball. Since the sequence \((\lambda_i X, p)\) converges to \((C_0(\partial_T X), 0)\), the sequence \((U_1(p))\) of tiny balls in \( \lambda_i X \) has capacity uniformly bounded. Furthermore, the sequence \((B_{10}(p))\) of pointed compact metric balls in \( \lambda_i X \) converges to the pointed compact metric ball \((B_{10}(0), 0)\) in \( C_0(\partial_T X) \).

Choose \( \delta \in (0, 1) \) with \( 2\delta \leq 1 \). Let \( f : U_1(0) - \{0\} \to \mathbb{R} \) denote the distance function from 0 defined by \( f(t\xi) := t \). Since \( C_0(\partial_T X) \) has the structure of a Euclidean cone, we see that \( f \) is a \((1, \delta)\)-strainer map. Take \( c \in (0, 1) \). The fiber \( f^{-1}(\{c\}) \) is the compact metric sphere \( S_c(0) \) in \( C_0(\partial_T X) \), and it is homeomorphic to \( \partial_T X \).

For each \( i \), let \( f_i : U_1(p) - \{p\} \to \mathbb{R} \) be the distance function from \( p \) defined by \( f_i(x) := d_{\lambda_i X}(p, x) \), where \( U_1(p) \) is the open metric ball in \( \lambda_i X \) and \( d_{\lambda_i X} \) is the metric on \( \lambda_i X \). Then the fiber \( f_i^{-1}(\{c\}) \) is the compact metric sphere \( S_c(p) \) in \( \lambda_i X \). Observe that the sequence \((f_i^{-1}(\{c\}))\) converges to \( f^{-1}(\{c\}) \) in the Gromov–Hausdorff topology. From Theorem 4.5 it follows that each fiber \( f_i^{-1}(\{c\}) \) is homotopy equivalent to the fiber \( f^{-1}(\{c\}) \) for all sufficiently large \( i \). Thus the metric sphere \( S_{c/\lambda_i}(p) \) in \( X \) is homotopy equivalent to \( \partial_T X \) for all sufficiently large \( i \). This finishes the proof. \( \square \)

### 4.6. Simply connectedness at infinity

A topological \( n \)-manifold \( M \) is said to be *simply connected at infinity* if there exists a sequence \((K_i)\) of compact subsets of \( M \) with \( M = \bigcup_{i=1}^{\infty} K_i \) satisfying the following properties for all \( i \):

1. \( K_i \subset K_{i+1} \);
2. every loop in \( M - K_{i+1} \) is
contractible in \( M - K_i \). A contractible topological \( n \)-manifold is homeomorphic to \( \mathbb{R}^n \) if and only if it is simply connected at infinity ([9, 36] for the case of \( n = 3 \), [17] for \( n = 4 \), and [39] for \( n \geq 5 \)).

From Theorem 4.6 we derive the following:

**Theorem 4.7.** Let \( X \) be a proper, geodesically complete CAT(0) space. Assume that \( X \) has the Gromov–Hausdorff asymptotic cone \( C_\infty X \). If \( X \) is a topological \( n \)-manifold, and if \( \partial_T X \) is simply connected, then \( X \) is homeomorphic to \( \mathbb{R}^n \).

**Proof.** We prove that \( X \) is simply connected at infinity. Let \( p \in X \) be arbitrary. Let \( t_0 \in (0, \infty) \) be sufficiently large as in Theorem 4.6 so that for every \( t \in [t_0, \infty) \) the metric sphere \( S_t(p) \) is homotopy equivalent to \( \partial_T X \). Choose a monotone decreasing sequence \( (\lambda_i) \) in \( (0, 1/t_0) \) with \( \lim_{i \to \infty} \lambda_i = 0 \). Then the sequence \( (B_{1/\lambda_i}(p)) \) of compact subsets of \( X \) satisfies \( X = \bigcup_{i=1}^{\infty} B_{1/\lambda_i}(p) \) and \( B_{1/\lambda_i}(p) \subseteq B_{1/\lambda_{i+1}}(p) \) for each \( i \).

Now it suffices to show that every loop in \( X - B_{1/\lambda_{i+1}}(p) \) is contractible in \( X - B_{1/\lambda_i}(p) \). Let \( \sigma : [0, 1] \to X - B_{1/\lambda_i}(p) \) be an arbitrary loop. Set \( t_i := 1/\lambda_{i+1} \). Let \( \varphi_i : X - B_{t_i}(p) \to S_{t_i}(p) \) be the geodesic contraction map defined by \( \varphi_i(x) := \gamma_{px}(t_i) \), where \( \gamma_{px} : [0, d_X(p, x)] \to X \) is the unit-speed geodesic in \( X \) from \( p \) to \( x \), and \( d_X \) is the metric on \( X \). Since \( X \) is CAT(0), the map \( \varphi_i \) is continuous. Then the loops \( \varphi_i \circ \sigma \) can be joined by a homotopy along the geodesics emanating from \( p \); indeed, the map \( h_i : S^1 \times [0, 1] \to X - B_{1/\lambda_i}(p) \) defined by

\[
h_i(s, t) := \gamma_{px}(s)((1 - t)d_X(p, \sigma(s)) + tt_i)
\]

is a homotopy in \( X - B_{1/\lambda_i}(p) \) from \( \sigma \) to \( \varphi_i \circ \sigma \). Since \( \partial_T X \) is simply connected, so is \( S_{t_i}(p) \). Hence \( \varphi_i \circ \sigma \) is contractible in \( S_{t_i}(p) \), and hence \( \sigma \) is contractible in \( X - B_{1/\lambda_i}(p) \). This completes the proof. \( \square \)

5. **Asymptotic geometric regularity**

5.1. **Busemann functions on CAT(0) spaces.** Let \( X \) be a complete CAT(0) space with metric \( d_X \). Let \( \gamma : [0, \infty) \to X \) be a ray in \( X \). The **Busemann function** \( b_\gamma : X \to \mathbb{R} \) along \( \gamma \) is defined by

\[
b_\gamma(x) := \lim_{t \to \infty} (d_X(x, \gamma(t)) - t).
\]

The Busemann function \( b_\gamma \) is 1-Lipschitz and convex. For \( r \in \mathbb{R} \), we denote by \( B_r(\gamma) \) the closed \((-r)\)-horoball \( b_\gamma^{-1}((-\infty, -r]) \). Note that for every \( x \in X \), and for every \( r \in (0, \infty) \) with \( b_\gamma(x) > -r \), we have

\[
d_X(x, B_r(\gamma)) = b_\gamma(x) + r.
\]

For every \( x \in X \), and for every \( y \in X - \{x\} \), by letting \( \xi := \gamma(\infty) \), we have the first variation formula

\[
(b_\gamma \circ \gamma_{xy})'(a) = -\cos \angle_x(\xi, y),
\]

where \( \gamma_{xy} : [a, b] \to X \) is the geodesic from \( x \) to \( y \), and \( (b_\gamma \circ \gamma_{xy})'(a) \) is the right derivative of \( b_\gamma \circ \gamma_{xy} \) at \( a \) (see e.g., [18] Lemma 3.3)).
We first show the following basic property:

**Lemma 5.1.** Let $X$ be a complete CAT(0) space. Let $\gamma : [0, \infty) \to X$ be a ray in $X$. Let $\xi := \gamma(\infty)$. If for distinct $x, y \in X$ we have $b_\gamma(x) = b_\gamma(y)$, then $\angle_x(\xi, y) \leq \pi/2$ and $\angle_y(\xi, x) \leq \pi/2$.

**Proof.** Take distinct $x, y \in X$ with $b_\gamma(x) = b_\gamma(y)$. Let $\gamma_{xy} : [a, b] \to X$ be the geodesic from $x$ to $y$. Since $b_\gamma$ is convex along $\gamma_{xy}$, for every $t \in [a, b]$ we have $b_\gamma(\gamma_{xy}(t)) \leq b_\gamma(x)$. From the first variation formula (5.2) we derive

$$-\cos \angle_x(\xi, y) = (b_\gamma \circ \gamma_{xy})'(a) \leq 0,$$

and hence $\angle_x(\xi, y) \leq \pi/2$. Similarly, we see $\angle_y(\xi, x) \leq \pi/2$. □

### 5.2. Strainers at infinity

We now introduce the following:

**Definition 5.1.** Let $\delta \in (0, \infty)$. Let $X$ be a proper, geodesically complete CAT(0) space. We say that a point $\xi \in \partial_\infty X$ is $\delta$-spherical if there exists some $\eta \in \partial_\infty X$ such that for every $\zeta \in \partial_\infty X$ we have

$$\angle(\xi, \zeta) + \angle(\zeta, \eta) < \pi + \delta;$$

in this case, the pair of $\xi$ and $\eta$ are said to be opposite. We say that an $m$-tuple $(\xi_1, \ldots, \xi_m)$ of points in $\partial_\infty X$ is an $(m, \delta)$-strainer at infinity if there exists another $m$-tuple $(\eta_1, \ldots, \eta_m)$ of points in $\partial_\infty X$ such that

1. $\xi_j$ and $\eta_j$ are opposite $\delta$-spherical points for all $j \in \{1, \ldots, m\}$;
2. $\angle(\xi_j, \xi_k), \angle(\eta_j, \eta_k)$, and $\angle(\xi_j, \eta_k)$ are smaller than $\pi/2 + \delta$ for all distinct $j, k \in \{1, \ldots, m\}$;

in this case, the pair of $(\xi_1, \ldots, \xi_m)$ and $(\eta_1, \ldots, \eta_m)$ are opposite.

By the definition of the angle metric $\angle$, we have:

**Lemma 5.2.** Let $X$ be a proper, geodesically complete CAT(0) space, and let $(\xi_1, \ldots, \xi_m)$ be an $(m, \delta)$-strainer at infinity. Then for every $x \in X$ the $m$-tuple $((\xi_1)'_x, \ldots, (\xi_m)'_x)$ of the directions in $\Sigma_x X$ forms a $\delta$-spherical $m$-tuple. Moreover, if $(\xi_1, \ldots, \xi_m)$ and $(\eta_1, \ldots, \eta_m)$ are opposite $(m, \delta)$-strainers at infinity, then the $m$-tuples $((\xi_1)'_x, \ldots, (\xi_m)'_x)$ and $((\eta_1)'_x, \ldots, (\eta_m)'_x)$ are opposite $\delta$-spherical points in $\Sigma_x X$.

If there exists an $(m, \delta)$-strainer at infinity, then via Lemma 5.2 we can apply the local statements obtained in [28].

Similarly to [28, Lemma 7.6], we verify that the existence of strainers at infinity guarantees the existence of almost flat ideal triangles:

**Lemma 5.3.** Let $X$ be a proper, geodesically complete CAT(0) space. Let $\xi \in \partial_\infty X$ be a $(1, \delta)$-strainer at infinity. Then for every pair of distinct points $x, y \in X$ the following hold:

1. $\pi - 2\delta < \angle_x(\xi, y) + \angle_y(\xi, x) \leq \pi$;
2. If $b_\gamma(x) = b_\gamma(y)$ for a ray $\gamma$ in $X$ with $\xi = \gamma(\infty)$, then

$$\frac{\pi}{2} - 2\delta < \angle_x(\xi, y) \leq \frac{\pi}{2}.$$
Proof. Since $X$ is CAT(0), we know $\angle_x(\xi, y) + \angle_y(\xi, x) \leq \pi$ (see e.g., [8 Proposition II.9.3]). By Lemma 5.1, if $b_\gamma(x) = b_\gamma(y)$ for a ray $\gamma$ in $X$ with $\xi = \gamma(\infty)$, then $\angle_x(\xi, y) \leq \pi/2$ and $\angle_y(\xi, x) \leq \pi/2$.

Take a point $\eta \in \partial_x X$ for which $\xi$ and $\eta$ are opposite $(1, \delta)$-strainers at infinity. Similarly to the case of $\xi$, we know $\angle_x(\eta, y) + \angle_y(\eta, x) \leq \pi$. By Lemmas 4.1 and 5.2, we have
\[
\angle_x(\xi, y) + \angle_x(\eta, y) \geq \angle_x(\xi, \eta) > \pi - \delta,
\angle_y(\xi, x) + \angle_y(\eta, x) \geq \angle_y(\xi, \eta) > \pi - \delta.
\]
Therefore $\angle_x(\xi, y) + \angle_y(\xi, x) > \pi - 2\delta$. Moreover, if $b_\gamma(x) = b_\gamma(y)$, then we obtain $\angle_x(\xi, y) > \pi/2 - 2\delta$. This proves the lemma. \qed

5.3. Strainer maps at infinity. Let $X$ be a proper, geodesically complete CAT(0) space. For an arbitrary $m$-tuple $(\gamma_1, \ldots, \gamma_m)$ of rays in $X$, by letting $f_j := b_{\gamma_j}$ for $j \in \{1, \ldots, m\}$, we obtain the map $F = (f_1, \ldots, f_m): X \to \mathbb{R}^m$ with Busemann function coordinates. Put $\xi_j := \gamma_j(\infty)$ for $j \in \{1, \ldots, m\}$. By the first variation formula (5.2) for Busemann functions, the map $F$ is differentiable at all $x \in X$ with differential $D_x F: T_x X \to \mathbb{R}^m$ determined as
\[
(D_x F)(rv) = -r \left( \cos \angle_x((\xi_1)_x', v), \ldots, \cos \angle_x((\xi_m)_x', v) \right).
\]

Similarly to [28, Lemma 8.1], we see the following:

Lemma 5.4. Set $c := 1/4m$. Let $X$ be a proper, geodesically complete CAT(0) space. Let $(\gamma_1, \ldots, \gamma_m)$ be an $m$-tuple of rays in $X$, and put $f_j := b_{\gamma_j}$ for $j \in \{1, \ldots, m\}$. Assume that for every $x \in X$ there exist $m$-tuples $(v_1^\pm, \ldots, v_m^\pm)$ of directions in $\Sigma_x X$ satisfying the following:

1. $\pm (D_x f_j)(v_j^\pm) > 1 - c$ for all $j \in \{1, \ldots, m\}$;
2. $|(D_x f_k)(v_j^\pm)| < 2c$ for all distinct $j, k \in \{1, \ldots, m\}$.

Then the map $F = (f_1, \ldots, f_m): X \to \mathbb{R}^m$ with Busemann function coordinates is 2-open, where $\mathbb{R}^m$ denotes the $m$-dimensional real vector space with $\ell_1$-norm.

Definition 5.2. Let $X$ be a proper, geodesically complete CAT(0) space. For an $m$-tuple $(\gamma_1, \ldots, \gamma_m)$ of rays in $X$, let $f_j := b_{\gamma_j}$ for the Busemann function along $\gamma_j$ for $j \in \{1, \ldots, m\}$. We say that the map $F = (f_1, \ldots, f_m): X \to \mathbb{R}^m$ is a Busemann $(m, \delta)$-strainer map if the $m$-tuple $(\gamma_1(\infty), \ldots, \gamma_m(\infty))$ in $\partial_\infty X$ is an $(m, \delta)$-stainer at infinity.

By Lemmas 5.2 and 5.4 and by applying the same idea as [28, Lemma 8.2] to our setting, we have:

Lemma 5.5. Let $X$ be a proper, geodesically complete CAT(0) space. If $4m\delta < 1$, then every Busemann $(m, \delta)$-strainer map $F: X \to \mathbb{R}^m$ is $2\sqrt{m}$-Lipschitz and $2\sqrt{m}$-open; in particular, if $X$ admits a Busemann $(m, \delta)$-strainer map, then the Hausdorff dimension of $X$ is at least $m$.\[\]

\[\]
Mimicking the limiting arguments in [28] Subsection 8.3] together with Lemmas 5.2 and 5.3 similarly to [28] Lemma 8.3, we can obtain:

**Lemma 5.6.** For a given constant $N \in \mathbb{N}$, for every $\epsilon \in (0, 1)$, and for every $m \in \mathbb{N}$, there exists $\delta \in (0, \infty)$ satisfying the following: Let $X$ be a proper, geodesically complete $\text{CAT}(0)$ space, and let $F: X \to \mathbb{R}^m$ be a Busemann $(m, \delta)$-strainer map. If an open convex subset $U$ of $X$ is $N$-doubling, then for every $x \in U$ the differential $D_x F: T_x X \to \mathbb{R}^m$ satisfies the following:

1. $|(D_x F)(v)| < 1 + \epsilon$ for all $v \in \Sigma_x X$;
2. For every $u \in S^{m-1}$ in $\mathbb{R}^m$ there exists an element $rv \in T_x X$ with $(D_x F)(rv) = u$ and $r < 1 + \epsilon$.

Due to the Lyuchak open map theorem [26] Theorem 1.2 for metric spaces, we conclude the following as well as [28] Corollary 8.4:

**Proposition 5.7.** For a given constant $N \in \mathbb{N}$, for every $\epsilon \in (0, 1)$, and for every $m \in \mathbb{N}$, there exists $\delta \in (0, \infty)$ satisfying the following property: Let $X$ be a proper, geodesically complete $\text{CAT}(0)$ space, and let $F: X \to \mathbb{R}^m$ be a Busemann $(m, \delta)$-strainer map. If an open convex subset $U$ of $X$ is $N$-doubling, then the restriction $F|U$ of $F$ to $U$ is $(1 + \epsilon)$-Lipschitz and $(1 + \epsilon)$-open. In particular, if $X$ is $N$-doubling, then $F$ is $(1 + \epsilon)$-Lipschitz and $(1 + \epsilon)$-open.

### 5.4. Differentials of strainer maps at infinity

Similarly to [28] Proposition 8.5, we see the following:

**Proposition 5.8.** Let $X$ be a proper, geodesically complete $\text{CAT}(0)$ space, and let $F: X \to \mathbb{R}^m$ be a Busemann $(m, \delta)$-strainer map with $4m\delta < 1$. Let $\gamma: [a, b] \to X$ be a geodesic in $X$. Then for all $s, t \in [a, b)$

$$|(F \circ \gamma)'_+(s) - (F \circ \gamma)'_+(t)| < 4\delta \sqrt{m},$$

where $(F \circ \gamma)'_+(t)$ is the right derivative of $F \circ \gamma$ at $t$. If in addition $\gamma$ contains at least two distinct points on a single fiber of $F$, then for all $t \in [a, b)$ we have

$$|(F \circ \gamma)'_+(t)| < 6\delta \sqrt{m}.$$
To show the second inequality, we assume that $F(\gamma(r)) = F(\gamma(s))$ holds for $r, s \in [a, b]$ with $r < s$. By Lemma \ref{lemma:2.1} (2) and \ref{lemma:2.1},
\[
| (b_{n_j} \circ \gamma)'_+ (r) | = | \cos \alpha_j (r) | < 2 \delta.
\]
In particular, we have $| (F \circ \gamma)'_+ (r) | < 2 \delta \sqrt{m}$. This together with the first inequality leads to the second one for all $t \in [a, b]$.

5.5. Fully strained CAT(0) spaces. We recall that a $c$-open map from a complete metric space is surjective; moreover, a $c$-Lipschitz map from a complete metric space for some $c \in [1, \infty)$ is a $c$-bi-Lipschitz homeomorphism if and only if it is an injective $c$-open map (as mentioned in Subsection 2.2).

Now we prove the following:

**Proposition 5.9.** Let $X$ be a proper, geodesically complete CAT(0) space of dim $X = n$, and let $F: X \to \mathbb{R}^n$ be a Busemann $(n, \delta)$-strainer map with $100n\delta < 1$. Then $F$ is a $2\sqrt{n}$-bi-Lipschitz homeomorphism.

**Proof.** Let $(\gamma_1, \ldots, \gamma_n)$ be the $n$-tuples of the rays in $X$ such that for each $j \in \{1, \ldots, n\}$ the Busemann function $b_{\gamma_j}$ is the $j$-th coordinate of $F$. Let $\xi_j := \gamma_j(\infty)$. As seen in Lemma \ref{lemma:5.2}, for every $x \in X$ the $n$-tuple $((\xi_1)_x', \ldots, (\xi_n)_x')$ of the directions in $\Sigma_x X$ forms a $\delta$-spherical $n$-tuple. By Lemma \ref{lemma:5.3} it suffices to prove that $F$ is injective.

Suppose that for some distinct $y, z \in X$ we have $F(y) = F(z)$. Let $\gamma: [a, b] \to X$ be a geodesic from $y$ to $z$. By Proposition \ref{prop:4.3}, we can choose $t \in [a, b]$ sufficiently close to $b$ such that the point $z$ is a $(1, \delta)$-strainer at $\gamma(t)$; in particular, the direction $z'_{\gamma(t)} \in \Sigma_{\gamma(t)} X$ is $\delta$-spherical in $\Sigma_{\gamma(t)} X$. From Proposition \ref{prop:5.8} it follows that $| (F \circ \gamma)'_+ (t) | < 6 \delta \sqrt{n}$; more precisely, for all $j \in \{1, \ldots, n\}$ we have $| \cos \angle_{\gamma(t)} (\xi_j, z) | < 6 \delta$. By Lemma \ref{lemma:4.2}, the $(n + 1)$-tuple $((\xi_1)'_{\gamma(t)}, \ldots, (\xi_n)'_{\gamma(t)}, z'_{\gamma(t)})$ of $\delta$-spherical directions in $\Sigma_{\gamma(t)} X$ is $12\delta$-spherical; in other words, the point $\gamma(t)$ is $(n + 1, 12\delta)$-strained. This property together with Lemma \ref{lemma:4.4} implies $\dim T_{\gamma(t)} X \geq n + 1$. Hence $\dim X \geq n + 1$. This is a contradiction.

Combining Propositions \ref{prop:5.7} and \ref{prop:5.9}, we conclude the following:

**Proposition 5.10.** For a given constant $N \in \mathbb{N}$, for every $\epsilon \in (0, 1)$, and for every $n \in \mathbb{N}$, there exists $\delta \in (0, \infty)$ satisfying the following property: Let $X$ be an $N$-doubling, proper, geodesically complete CAT(0) space of dim $X = n$, and let $F: X \to \mathbb{R}^n$ be a Busemann $(n, \delta)$-strainer map. Then $F$ is a $(1 + \epsilon)$-bi-Lipschitz homeomorphism.

5.6. Asymptotic regularity. We denote by $\Delta_1^{n-1}$ the standard spherical $(n - 1)$-simplex in $\mathbb{S}^{n-1}$ defined by
\[
\Delta_1^{n-1} := \{ (u_1, \ldots, u_n) \in \mathbb{S}^{n-1} | u_1 \geq 0, \ldots, u_n \geq 0 \}.
\]
We also denote by \( \text{rad} \Delta_1^{n-1} \) the radius of \( \Delta_1^{n-1} \) defined by

\[
\text{rad} \Delta_1^{n-1} := \inf_{u \in \Delta_1^{n-1}} \sup_{v \in \Delta_1^{n-1}} d_{S^{n-1}}(u, v),
\]

where \( d_{S^{n-1}} \) is the metric on \( S^{n-1} \). We notice that \( \text{rad} \Delta_1^{n-1} < \pi/2 \).

We now prove the following asymptotic geometric regularity:

**Theorem 5.11.** For every \( n \in \mathbb{N} \), there exists \( \delta \in (0, \infty) \) depending only on \( n \) satisfying the following property: Let \( X \) be a proper, geodesically complete \( \text{CAT}(0) \) space. If \( \partial_T X \) satisfies \( d_{\text{GH}} (\partial_T X, S^{n-1}) < \delta \), then \( X \) is \( 2\sqrt{n} \)-bi-Lipschitz homeomorphic to \( \mathbb{R}^n \).

**Proof.** Let \( \delta \in (0, \infty) \) be small enough. By the assumption, there exists a \( 2\delta \)-approximation \( \varphi : S^{n-1} \to \partial_T X \). Let \((e_1, \ldots, e_n)\) be an orthonormal \( n \)-tuple of points in \( S^{n-1} \), and let \((-e_1, \ldots, -e_n)\) be the antipodal one. For \( j \in \{1, \ldots, n\} \), we put \( \xi_j := \varphi(e_j) \) and \( \eta_j := \varphi(-e_j) \). Then the \( n \)-tuples \((\xi_1, \ldots, \xi_n)\) and \((\eta_1, \ldots, \eta_n)\) are opposite \((n, 10\delta)\)-strainers at infinity. By Lemma 5.2, for every \( x \in X \), the \( m \)-tuples \(( (\xi_1)_x', \ldots, (\xi_n)_x' ) \) and \(( (\eta_1)_x', \ldots, (\eta_n)_x' ) \) are opposite \( 10\delta \)-spherical points in \( \Sigma_x X \).

For each \( j \in \{1, \ldots, n\} \), we take a ray \( \gamma_j \) in \( X \) with \( \xi_j = \gamma_j(\infty) \). For the \( n \)-tuple \((\gamma_1, \ldots, \gamma_n)\) of the rays in \( X \), we obtain a Busemann \((n, 10\delta)\)-strainer map \( F : X \to \mathbb{R}^n \) with coordinate \((b_{\gamma_1}, \ldots, b_{\gamma_n})\). By Lemma 5.5, the map \( F \) is \( 2\sqrt{n} \)-Lipschitz and \( 2\sqrt{n} \)-open.

We are going to prove that \( F \) is injective. Suppose that for some distinct \( y, z \in X \) we have \( F(y) = F(z) \). Let \( \gamma : [a, b] \to X \) be a geodesic from \( y \) to \( z \). By the same way as that discussed in the proof of Proposition 5.3, we see that for \( t \in [a, b] \) sufficiently close to \( b \), the \((n + 1)\)-tuple \(( (\xi_1)'_{\gamma(t)}, \ldots, (\xi_n)'_{\gamma(t)}, z'_{\gamma(t)} \) \) of \( 10\delta \)-spherical directions in \( \Sigma_{\gamma(t)} X \) is \( 120\delta \)-spherical; in other words, the point \( \gamma(t) \) is \((n + 1, 120\delta)\)-strained. Since \( X \) is geodesically complete, we can find \( \zeta \in \partial_T X \) such that the geodesic \( yz \) parametrized by \( \gamma \) is contained in the ray \( y\zeta \) from \( y \) to \( \zeta \). By Lemma 4.2 (1), for all \( j \in \{1, \ldots, n\} \) we have

\[
d_T(\xi_j, \zeta) \geq \angle(\xi_j, \zeta) \geq \angle(\xi_j)'_{\gamma(t)} \left( (\xi_j)'_{\gamma(t)}, z'_{\gamma(t)} \right) > \frac{\pi}{2} - 240\delta.
\]

The \( m \)-tuples \(( (\xi_1)'_{\gamma(t)}, \ldots, (\xi_n)'_{\gamma(t)} \) and \(( (\eta_1)'_{\gamma(t)}, \ldots, (\eta_n)'_{\gamma(t)} \) are opposite \( 10\delta \)-spherical points in \( \Sigma_{\gamma(t)} X \). By Lemmas 4.3 and 4.2 (1),

\[
d_T(\eta_j, \zeta) \geq \angle(\eta_j, \zeta) \geq \angle(\eta_j)'_{\gamma(t)} \left( (\eta_j)'_{\gamma(t)}, z'_{\gamma(t)} \right)
\]

\[
\geq \angle(\gamma(t) \left( (\xi_j)'_{\gamma(t)}, (\eta_j)'_{\gamma(t)} \right) - \angle(\gamma(t) \left( (\xi_j)'_{\gamma(t)}, z'_{\gamma(t)} \right) > \frac{\pi}{2} - 130\delta.
\]

Since the map \( \varphi \) is a \( 2\delta \)-approximation, we can choose a point \( u_0 \in S^{n-1} \) with \( d_T(\varphi(u_0), \zeta) < 4\delta \). Then for the metric \( d_{S^{n-1}} \) on \( S^{n-1} \) we have

\[
d_{S^{n-1}}(e_j, u_0) > \frac{\pi}{2} - 250\delta, \quad d_{S^{n-1}}(-e_j, u_0) > \frac{\pi}{2} - 140\delta.
\]
On the other hand, for every \( u \in S^{n-1} \) we find \( k \in \{1, \ldots, n\} \) such that \( d_{S^{n-1}}(e_k, u) \leq \text{rad} \Delta_1^{n-1} \) or \( d_{S^{n-1}}(-e_k, u) \leq \text{rad} \Delta_1^{n-1} \) for the radius \( \text{rad} \Delta_1^{n-1} \) of the standard spherical \((n - 1)\)-simplex \( \Delta_1^{n-1} \) in \( S^{n-1} \). This yields a contradiction, provided \( \delta \) is sufficiently small, since we have \( \text{rad} \Delta_1^{n-1} < \pi/2 \). Therefore we see that \( F \) is injective.

This finishes the proof of Theorem 5.11.

From Proposition 5.10 we derive the following:

**Theorem 5.12.** For a given constant \( N \in \mathbb{N} \), for every \( \epsilon \in (0,1) \), and for every \( n \in \mathbb{N} \), there exists \( \delta \in (0, \infty) \) satisfying the following property: Let \( X \) be an \( N \)-doubling, proper, geodesically complete CAT(0) space. If \( \partial T X \) satisfies \( d_{GH}(\partial T X, S^{n-1}) < \delta \), then \( X \) is \((1 + \epsilon)\)-bi-Lipschitz homeomorphic to \( \mathbb{R}^n \).

**Proof.** Let \( \delta \in (0, \infty) \) be sufficiently small. From the assumption on \( d_{GH} \), as discussed in the proof of Theorem 5.11, we can find a Busemann \((n, 10\delta)\)-strainer map \( F: X \to \mathbb{R}^n \) that is a \( 2\sqrt{n} \)-bi-Lipschitz homeomorphism. In particular, we have \( \dim X = n \). Proposition 5.10 leads to the conclusion.

### 6. Asymptotic Topological Regularity

In this section, we prove Theorems 1.1, 1.2, 1.3, and 1.4.

#### 6.1. Proof of Theorem 1.1

Let \( \epsilon \in (0, \infty) \) and \( n \in \mathbb{N} \) be arbitrary. Let \( \delta \in (0,1) \) be sufficiently small. Let \( X \) be a purely \( n \)-dimensional, proper, geodesically complete CAT(0) space. Assume that we have \( G_\delta^n(X) < 1 + \delta \). By Lemma 3.6 for some constant \( N_0 \in \mathbb{N} \) depending only on \( n \), the space \( X \) is \( N_0 \)-doubling. From Proposition 3.7 it follows that \( X \) has the Gromov–Hausdorff tangent cone \( C_\infty X \); moreover,

\[
\frac{\mathcal{H}^{n-1}(\partial_T X)}{\mathcal{H}^{n-1}(S^{n-1})} < 1 + \delta.
\]

Proposition 3.4 implies that \( \partial_T X \) is a purely \((n - 1)\)-dimensional, compact, geodesically complete CAT(1) space. Theorem 2.9 leads to that for some function \( \vartheta_n: [0, \infty) \to (0, \infty) \) depending only on \( n \) with \( \lim_{t \to 0} \vartheta_n(t) = 0 \), the Tits boundary \( \partial_T X \) is \((1 + \vartheta_n(\delta))\)-bi-Lipschitz homeomorphic to \( S^{n-1} \), provided \( \delta \) is small enough. Therefore we have \( d_{GH}(\partial_T X, S^{n-1}) < \vartheta_n(\delta) \). From Theorem 5.12 we deduce that \( X \) is \((1 + \epsilon)\)-bi-Lipschitz homeomorphic to \( \mathbb{R}^n \). This completes the proof.

#### 6.2. Manifold recognitions and sphere theorems

We quote the local topological regularity theorem for CAT(\( \kappa \)) spaces established by Lytchak and the author [29, Theorem 1.1] in the following form:

**Theorem 6.1.** (29) Let \( W \) be an open subset of a proper CAT(\( \kappa \)) space \( X \). Then the following are equivalent:

1. \( W \) is a topological \( n \)-manifold;
(2) for every \( x \in W \) the space \( \Sigma_x X \) is homotopy equivalent to \( \mathbb{S}^{n-1} \);
(3) for every \( x \in W \) the space \( T_x X \) is homeomorphic to \( \mathbb{R}^n \).

We say that a triple of points in a CAT(1) space is a \textit{tripod} if the three points have pairwise distance at least \( \pi \). Lytchak and the author [29] proved a \textit{capacity sphere theorem} for CAT(1) spaces stating that if a compact, geodesically complete CAT(1) space admits no tripod, then it is homeomorphic to a sphere ([29 Theorem 1.5]). As its application, [29] showed the following volume sphere theorem for CAT(1) spaces ([29 Theorem 8.3], and [33] for the case of \( m = 2 \)):

**Theorem 6.2.** ([29]) If a purely \( m \)-dimensional, compact, geodesically complete CAT(1) space \( \Sigma \) satisfies

\[
\mathcal{H}^m (\Sigma) < \frac{3}{2} \mathcal{H}^m (\mathbb{S}^m),
\]

then \( \Sigma \) is homeomorphic to \( \mathbb{S}^m \).

The assumption of \( \mathcal{H}^m \) in Theorem 6.2 is optimal since the spherical join \( \mathbb{S}^{m-1} \ast T \) satisfies \( \mathcal{H}^m (\mathbb{S}^{m-1} \ast T) = (3/2) \mathcal{H}^m (\mathbb{S}^m) \). We can construct a CAT(1) \( m \)-sphere admitting a tripod whose \( m \)-dimensional Hausdorff measure is equal to \( (3/2) \mathcal{H}^m (\mathbb{S}^m) \).

**Example 6.1.** ([34]) The spherical join \( \mathbb{S}^{m-2} \ast T \) can be represented by the quotient metric space \( \bigsqcup_{j=1,2,3} \mathbb{S}_{+,j}^{m-1} / \sim \) obtained by gluing three closed unit \((m - 1)\)-hemispheres \( \mathbb{S}_{+,j}^{m-1} \) along their boundaries \( \partial \mathbb{S}_{+,j}^{m-1} = \partial \mathbb{S}_{+,k}^{m-1} \). For \( j = 1, 2, 3, 3 + 1 = 1 \), let \( \Sigma^{m-1}_j \) be the isometrically embedded unit \((m - 1)\)-spheres \( \mathbb{S}_{+,j}^{m-1} \cup \mathbb{S}_{+,j+1}^{m-1} / \sim \) in \( \mathbb{S}^{m-2} \ast T \) obtained by the relation \( \partial \mathbb{S}_{+,j}^{m-1} = \partial \mathbb{S}_{+,j+1}^{m-1} \). We take three copies of closed unit \( m \)-hemispheres \( \mathbb{S}_{+,j}^m \) for \( j \in \{1, 2, 3\} \). Let \( \Sigma \) be the quotient metric space obtained as

\[
\Sigma := (\mathbb{S}^{m-2} \ast T) \sqcup \left( \bigsqcup_{j=1,2,3} \mathbb{S}_{+,j}^m \right) / \sim
\]

by attaching \( \mathbb{S}_{+,j}^m \) to \( \mathbb{S}^{m-2} \ast T \) along \( \Sigma^{m-1}_j = \partial \mathbb{S}_{+,j}^m \) for each \( j \in \{1, 2, 3\} \). We call \( \Sigma \) the \textit{m-triplex}. The \( m \)-triplex \( \Sigma \) is a purely \( m \)-dimensional, compact, geodesically complete CAT(1) space that is homeomorphic to \( \mathbb{S}^m \). This space has a tripod and satisfies \( \mathcal{H}^m (\Sigma) = (3/2) \mathcal{H}^m (\mathbb{S}^m) \). We notice that the 1-triplex is by definition a circle of length \( 3\pi \).

The author obtained the following characterization ([34 Theorem 1.1], and [33] for the case of \( m = 2 \)) of CAT(1) spaces of small volume:

**Theorem 6.3.** ([34]) If a purely \( m \)-dimensional, compact, geodesically complete CAT(1) space \( \Sigma \) satisfies

\[
\mathcal{H}^m (\Sigma) = \frac{3}{2} \mathcal{H}^m (\mathbb{S}^m),
\]

then \( \Sigma \) is homeomorphic to \( \mathbb{S}^m \).
then $X$ is either homeomorphic to $\mathbb{S}^m$ or isometric to $\mathbb{S}^{m-1} \ast T$. If in addition $X$ has a tripod, then $X$ is isometric to either the $m$-triplex or $\mathbb{S}^{m-1} \ast T$.

6.3. **Proof of Theorem 1.2.** Take a purely $n$-dimensional, proper, geodesically complete $\text{CAT}(0)$ space $X$. Assume that $\mathcal{G}_0^n(X) < \frac{3}{2}$. By Proposition 3.7, the space $X$ is doubling, and it has the Gromov–Hausdorff asymptotic cone $C_\infty X$; moreover, for every $x \in X$ we have

$$\frac{\mathcal{H}^{n-1}(\Sigma_x X)}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \leq \frac{\mathcal{H}^{n-1}(\partial_T X)}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} = \mathcal{G}_0^n(X) < \frac{3}{2}.$$ 

By Proposition 2.5, the space $\Sigma_x X$ is purely $(n-1)$-dimensional. The volume sphere theorem 6.2 implies that $\Sigma_x X$ is homeomorphic to $\mathbb{S}^{n-1}$. Due to the local topological regularity theorem 6.1, we conclude that $X$ is a topological $n$-manifold. We may assume $n \geq 3$. From Proposition 3.4 we deduce that $\partial_T X$ is a purely $(n-1)$-dimensional, compact, geodesically complete $\text{CAT}(1)$ space. The volume sphere theorem 6.2 implies that $\partial_T X$ is homeomorphic to $\mathbb{S}^{n-1}$. From Theorem 1.7 we conclude that $X$ is homeomorphic to $\mathbb{R}^n$. This finishes the proof of Theorem 1.2.

6.4. **Asymptotic volume regularity.** Lytchak [27, Corollary 1.4] showed that if there exists a surjective 1-Lipschitz map from a compact spherical building of dimension $m$ onto a geodesically complete $\text{CAT}(1)$ space $Y$, then $Y$ is a spherical building of dimension $m$. Notice that every spherical building is a geodesically complete $\text{CAT}(1)$ space. We refer the readers to [8, Chapter II.10 Appendix] for basics on spherical (and Euclidean) buildings.

We show the following volume regularity of $\text{CAT}(1)$ spaces, which is a generalization of [32, Proposition 7.1] (see also Theorem 2.9):

**Proposition 6.4.** For some function $\vartheta_m : [0, \infty) \to (0, \infty)$ depending only on $m$ with $\lim_{t \to 0} \vartheta_m(t) = 0$, the following holds: Let $Y$ be a purely $m$-dimensional, proper, geodesically complete $\text{CAT}(1)$ space. Let $Z$ be a compact spherical building of dimension $m$. Let $f : Y \to Z$ be a surjective 1-Lipschitz map. If for $\epsilon \in (0, \infty)$ we have

$$(6.1) \quad \mathcal{H}^m(Y) < \mathcal{H}^m(Z) + \epsilon,$$

then $f$ is a $\vartheta_m(\epsilon)$-approximation. In particular, if $\mathcal{H}^m(Y) \leq \mathcal{H}^m(Z)$, then $Y$ is isometric to $Z$.

**Proof.** Suppose that for $\epsilon \in (0, \infty)$ we have (6.1). Let $y_1, y_2 \in Y$ satisfy $d_Z(f(y_1), f(y_2)) < d_Y(y_1, y_2)$, where $d_Y$ and $d_Z$ are the metrics on $Y$ and on $Z$, respectively. Set

$$s_0 := \frac{d_Z(f(y_1), f(y_2))}{2}, \quad t_0 := \frac{d_Y(y_1, y_2)}{2}.$$ 

It suffices to show $t_0 - s_0 < \vartheta_m(\epsilon)$ for some $\vartheta_m(\epsilon)$. 

Since $Z$ is a spherical building of dimension $m$, there exists a closed $\pi$-convex subset $\Sigma$ of $Z$ containing $f(y_1), f(y_2)$ such that $\Sigma$ is isometric to $S^m$. Take a geodesic $f(y_1)f(y_2)$ in $\Sigma$, and the midpoint $z \in f(y_1)f(y_2)$ between $f(y_1)$ and $f(y_2)$. For each $j \in \{1, 2\}$, we put
\[
U_j := U_{t_0}(y_j) \cap f^{-1}(\Sigma), \quad \bar{U}_j := U_{t_0}(f(y_j)) \cap \Sigma.
\]
From Proposition 2.5, it follows that $\Sigma_{y_j}Y$ is purely $(m-1)$-dimensional. We can take a surjective 1-Lipschitz map $\varphi_{y_j} : \Sigma_{y_j}Y \to \Sigma_{f(y_j)}\Sigma$ onto $\Sigma_{f(y_j)}\Sigma$ that is isometric to $S^{m-1}$ (Proposition 11.3). We define a map $\Phi_{y_j} : U_j \to \bar{U}_j$ by $\Phi_{y_j}(y) := \exp_{f(y_j)} d\nu(y, y) \varphi_{y_j}(y_{y_j})$, where $\exp_{f(y_j)}$ is the exponential map from $T_{f(y_j)}\Sigma$. Then $\Phi_{y_j}$ is surjective. Since $Y$ is CAT(1), the map $\Phi_{y_j}$ is 1-Lipschitz. Hence we have
\[
\mathcal{H}^m(U_j) \leq \mathcal{H}^m(U_j).
\]
From the choices of $U_j$ and $\bar{U}_j$ it follows that $\Sigma - (\bar{U}_1 \cup \bar{U}_2)$ is contained in $f(f^{-1}(\Sigma) - (U_1 \cup U_2))$. Since $f$ is 1-Lipschitz, we have
\[
\mathcal{H}^m(\Sigma - (\bar{U}_1 \cup \bar{U}_2)) \leq \mathcal{H}^m(f^{-1}(\Sigma) - (U_1 \cup U_2)) = \mathcal{H}^m(f^{-1}(\Sigma)) - \mathcal{H}^m(U),
\]
and hence we obtain
\[
\mathcal{H}^m(Z) = \mathcal{H}^m(\Sigma) + \mathcal{H}^m(Z - \Sigma) \leq \mathcal{H}^m(f^{-1}(\Sigma)) - \mathcal{H}^m(U) + \mathcal{H}^m(f^{-1}(Z - \Sigma)) = \mathcal{H}^m(Y) - \mathcal{H}^m(U).
\]
Thus by (6.1) we have $\mathcal{H}^m(\bar{U}) < \epsilon$, so $\mathcal{H}^m(U_{t_0-s_0}(z) \cap \Sigma) < \epsilon$. This implies that $t_0 - s_0 < \vartheta_m(\epsilon)$ holds for some $\vartheta_m(\epsilon)$. \qed

For $k \in \mathbb{N}$ with $k \geq 2$, we denote by $T_k$ the discrete metric space consisting of $k$ points with pairwise distance $\pi$.

As an application of Proposition 6.3 we show the following asymptotic volume regularity:

**Proposition 6.5.** For some function $\vartheta_n : [0, \infty) \to (0, \infty)$ depending only on $n$ with $\lim_{t \to 0} \vartheta_n(t) = 0$, the following holds: Let $X$ be a purely $n$-dimensional, doubling, proper, geodesically complete CAT(0) space. Assume that for some $p \in X$ the space $\Sigma_pX$ of directions at $p$ is isometric to $S^{n-2} \ast T_k$ for some $k$. If for $\epsilon \in (0, \infty)$ we have
\[
\mathcal{H}^{n-1}(\partial T X) < \mathcal{H}^{n-1}(S^{n-2} \ast T_k) + \epsilon,
\]
then we have \( d_{\text{GH}}(\partial_T X, S^{n-2} \ast T_k) < \vartheta_n(\epsilon) \). If in addition we have \( \mathcal{H}^{n-1}(\partial_T X) \leq \mathcal{H}^{n-1}(S^{n-2} \ast T_k) \), then \( \partial_T X \) is isometric to \( S^{n-2} \ast T_k \); in particular, \( X \) is isometric to \( \mathbb{R}^{n-1} \times C_0(T_k) \).

**Proof.** Proposition 6.4 implies that \( \partial_T X \) is a purely \((n-1)\)-dimensional, compact, geodesically complete CAT(1) space. Since \( \Sigma_x \) is isometric to \( S^{n-2} \ast T_k \), it is a compact spherical building of dimension \( n-1 \). Take the 1-Lipschitz map \( f_p: \partial_T X \to \Sigma_x \) defined by \( f_p(\xi) := \xi'_p \). By the completeness of \( X \), the map \( f_p \) is surjective. Since we have (6.4) for \( \epsilon \), Proposition 6.4 implies that \( f_p \) is a \( \vartheta_n(\epsilon) \)-approximation for some \( \vartheta_n(\epsilon) \), and hence \( d_{\text{GH}}(\partial_T X, S^{n-2} \ast T_k) < \vartheta_n(\epsilon) \).

Assume in addition that we have \( \mathcal{H}^{n-1}(\partial_T X) \leq \mathcal{H}^{n-1}(S^{n-2} \ast T_k) \). Proposition 6.4 implies that \( \partial_T X \) is isometric to \( S^{n-2} \ast T_k \). From Proposition 2.2 we deduce that \( X \) is isometric to \( X_1 \times X_2 \) for some proper, geodesically complete CAT(0) spaces \( X_1 \) and \( X_2 \) such that \( \partial_T X_1 \) is isometric to \( S^{n-2} \) and \( \partial_T X_2 \) consists of \( k \) points. Then \( X_1 \) is isometric to \( \mathbb{R}^{n-1} \). From the existence of the point \( p \) at which \( \Sigma_x \) is isometric to \( S^{n-2} \ast T_k \), we see that \( X_2 \) is isometric to \( C_0(T_k) \). Thus \( X \) is isometric to \( \mathbb{R}^{n-1} \times C_0(T_k) \).

**6.5. Proof of Theorem 1.3.** Let us consider a purely \( n \)-dimensional, proper, geodesically complete CAT(0) space \( X \). Assume that we have \( G_0^n(X) = 3/2 \). By Proposition 3.7, the space \( X \) is doubling, and it has the Gromov–Hausdorff asymptotic cone \( C_\infty X \); moreover, for every \( x \in X \) we have

\[
\frac{\mathcal{H}^{n-1}(\Sigma_x X)}{\mathcal{H}^{n-1}(S^{n-1})} \leq \frac{\mathcal{H}^{n-1}(\partial_T X)}{\mathcal{H}^{n-1}(S^{n-1})} = G_0^n(X) = \frac{3}{2}.
\]

From Proposition 2.5 it follows that \( \Sigma_x X \) is purely \((n-1)\)-dimensional. Theorems 6.2 and 6.3 imply that \( \Sigma_x X \) is either homeomorphic to \( S^n \) or isometric to \( S^{n-2} \ast T \); if in addition \( \Sigma_x X \) has a tripod, then \( \Sigma_x X \) is isometric to either the \( n \)-triplex or \( S^{n-2} \ast T \). From Proposition 6.4 we deduce that \( \partial_T X \) is a purely \((n-1)\)-dimensional, compact, geodesically complete CAT(1) space. Theorem 6.3 implies that \( \partial_T X \) is either homeomorphic to \( S^n \) or isometric to \( S^{n-2} \ast T \); if in addition \( \partial_T X \) has a tripod, then \( \partial_T X \) is isometric to either the \( n \)-triplex or \( S^{n-2} \ast T \).

Suppose first that \( \partial_T X \) is isometric to \( S^{n-2} \ast T \). Then from Proposition 2.2 we conclude that \( X \) is isometric to \( \mathbb{R}^{n-1} \times C_0(T) \).

Suppose next that \( \partial_T X \) is homeomorphic to \( S^n \), and \( \partial_T X \) has no tripod. In this case, for every \( x \in X \), the space \( \Sigma_x X \) has no tripod too because of the existence of a surjective 1-Lipschitz map from \( \partial_T X \) onto \( \Sigma_x X \). In particular, we have \( \mathcal{H}^{n-1}(\Sigma_x X) < (3/2) \mathcal{H}^{n-1}(S^{n-1}) \). Due to Theorem 6.2 we conclude that \( \Sigma_x X \) is homeomorphic to \( S^{n-1} \).

From the local topological regularity theorem 6.1 it follows that \( X \) is a topological \( n \)-manifold. Then \( X \) is homeomorphic to \( \mathbb{R}^n \); indeed, in the case of \( n \geq 3 \), from Theorem 4.7 we derive the conclusion.
Suppose next that $\partial_T X$ is homeomorphic to $S^n$, and $\partial_T X$ has a tripod. In this case, $\partial_T X$ is isometric to the $n$-triplex. Then $X$ admits no point at which the space of directions is isometric to $S^{n-1} \ast T$; indeed, if we would find a point $p \in X$ at which $\Sigma_p X$ is isometric to $S^{n-1} \ast T$, then Proposition 6.5 implies that $\partial_T X$ would be isometric to $S^{n-1} \ast T$ too. Hence for every $x \in X$ the space $\Sigma_x X$ is homeomorphic to $S^{n-1}$. From Theorem 6.1 we see that $X$ is a topological $n$-manifold. Using Theorem 4.7 we conclude that $X$ is homeomorphic to $\mathbb{R}^n$.

Thus we have completed the proof of Theorem 1.3. \quad \square

6.6. CAT($\kappa$) homology manifolds. Let $H_*$ denote the singular homology with $\mathbb{Z}$-coefficients. A locally compact, separable metric space $M$ is said to be a homology $n$-manifold if for every $p \in M$ the local homology $H_*(M, M - \{p\})$ at $p$ is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\})$, where 0 is the origin of $\mathbb{R}^n$. A homology $n$-manifold $M$ is a generalized $n$-manifold if $M$ is an ANR of dim $M < \infty$. Every generalized $n$-manifold has dimension $n$. Due to the theorem of Moore (see [41, Chapter IV]), for each $n \in \{1, 2\}$, every generalized $n$-manifold is a topological $n$-manifold.

Every CAT($\kappa$) homology $n$-manifold is a geodesically complete generalized $n$-manifold. We refer the readers to [29] for advanced studies of homology manifolds with an upper curvature bound.

We recall the following stability ([29, Lemma 3.3]):

**Lemma 6.6.** ([29]) Assume that a sequence $(X_i, p_i)$ of pointed proper geodesically complete CAT($\kappa$) spaces converges to some proper geodesically complete CAT($\kappa$) space $(X, p)$ in the pointed Gromov–Hausdorff topology. If each $X_i$ is a homology $n$-manifold, then so is $X$.

We now show the following:

**Proposition 6.7.** Let $X$ be a doubling, proper, geodesically complete CAT(0) space. If $X$ is a homology $n$-manifold, then $\partial_T X$ is a compact homology $(n - 1)$-manifold and $C_0(\partial_T X)$ is a homology $n$-manifold.

**Proof.** From Propositions 3.4 and 3.7 it follows that for every $p \in X$ the space $X$ has the Gromov–Hausdorff asymptotic cone $(C_\infty X, p_\infty)$ isometric to $(C_0(\partial_T X), 0)$, where $p_\infty$ is the limit base point of $p$. Assume that $X$ is a homology $n$-manifold. By Lemma 6.6 the cone $C_0(\partial_T X)$ is a homology $n$-manifold. Since $C_0(\partial_T X) - \{0\}$ is homeomorphic to $\partial_T X \times \mathbb{R}$, we see that $\partial_T X$ is a homology $(n - 1)$-manifold. \quad \square

For CAT(1) homology manifolds, the author proved the following volume sphere theorem ([34, Theorem 1.2]):

**Theorem 6.8.** ([34]) For every $m \in \mathbb{N}$, there exists $\delta \in (0, \infty)$ depending only on $m$ such that if a compact CAT(1) homology $m$-manifold $\Sigma$ satisfies

$$\mathcal{H}^m(\Sigma) < \frac{3}{2} \mathcal{H}^m(S^m) + \delta,$$

then...
then $\Sigma$ is homeomorphic to $S^n$.

6.7. **Proof of Theorem 1.4.** Let $\delta \in (0, 1)$ be sufficiently small. Let $X$ be a complete CAT(0) homology $n$-manifold. Assume that we have $G^n_0(X) < 3/2 + \delta$. By Proposition 3.7 the space $X$ is doubling, and it has the Gromov–Hausdorff asymptotic cone $C_\infty X$; moreover, for every $x \in X$ we have

$$\frac{H^{n-1}(\Sigma_x X)}{H^{n-1}(S^{n-1})} \leq \frac{H^{n-1}(\partial_T X)}{H^{n-1}(S^{n-1})} = G^n_0(X) < \frac{3}{2} + \delta.$$

By Proposition 2.5, the space $\Sigma_x X$ is purely $(n-1)$-dimensional. The volume sphere theorem 6.8 for homology manifolds implies that $\Sigma_x X$ is homeomorphic to $S^{n-1}$. From the local topological regularity theorem 6.1 we see that $X$ is a topological $n$-manifold.

We may assume $n \geq 3$. From Proposition 6.7 we deduce that $\partial_T X$ is a compact CAT(1) homology $(n-1)$-manifold. The volume sphere theorem 6.8 for homology manifolds implies that $\partial_T X$ is homeomorphic to $S^{n-1}$. From Theorem 4.7 we conclude that $X$ is homeomorphic to $\mathbb{R}^n$. This finishes the proof of Theorem 1.4. □

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