INVERSE SPECTRAL PROBLEM FOR ANALYTIC DOMAINS I: BALIAN-BLOCH TRACE FORMULA

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Abstract. This is the first in a series of papers [Z3, Z4] on inverse spectral/resonance problems for analytic plane domains Ω. In this paper, we present a rigorous version of the Balian-Bloch trace formula [BB1, BB2]. It is an asymptotic formula for the trace

\[ \text{Tr}_{\Omega} R_{\rho}(k + i\tau \log k) \]

as \( k \to \infty \) with \( \tau > 0 \). When the support of \( \hat{\rho} \) contains the length \( L_{\gamma} \) of precisely one periodic reflecting ray \( \gamma \), then the asymptotic expansion of \( \text{Tr}_{\Omega} R_{\rho}(k + i\tau \log k) \) is essentially the same as the wave trace expansion at \( \gamma \). The raison d’être for this approach to wave invariants is that they are explicitly computable. Applications of the trace formula will be given in the subsequent articles in this series. For instance, in [Z3, Z4] we will prove that analytic domains with one symmetry are determined by their Dirichlet (or Neumann) spectra. Although we only present details in dimension 2 for the sake of simplicity, the methods and results extend with few modifications to all dimensions.

1. Introduction

This paper is the first in a series of articles devoted to inverse spectral and resonance problems for analytic domains \( \Omega \subset \mathbb{R}^2 \) with Dirichlet (or Neumann) boundary conditions [Z3, Z4]. As in the earlier articles [Z1, Z2, ISZ], the aim is to recover the domain \( \Omega \) from spectral invariants associated to special closed orbits of the billiard flow, specifically the wave trace invariants \( B_{\gamma^r, j} \) at iterates \( \gamma^r \) of a single bouncing ball orbit \( \gamma \). Such wave invariants are polynomials in the Taylor coefficients \( f_{\pm}^{(j)}(0) \) of the defining functions \( f_{\pm} \) of \( \Omega \) at the endpoints of \( \gamma \) and our goal is to recover the coefficients \( f_{\pm}^{(j)}(0) \) from the invariants \( B_{\gamma^r, j} \) as \( r \) varies, and hence to recover the analytic domain. In this series, we will achieve this goal when \( \Omega \) satisfies one symmetry condition. Except at the very last inverting step of [Z3, Z4], we do not use any symmetry or analyticity assumptions on \( \Omega \) and our computations of wave invariants are valid on all smooth plane domains. We restrict to plane domains to simplify the exposition; the methods extend in a straightforward way to domains in \( \mathbb{R}^n \).

The path we take towards the wave invariants \( B_{\gamma^r, j} \) is the one initiated by Balian-Bloch in the classic papers [BB1, BB2]. As recalled below, these papers are concerned with the asymptotics of a (regularized) resolvent trace rather than the trace of the wave group, and are based on the Neumann series representation for the resolvent kernel in terms of the free resolvent. Better known to physicists than to mathematicians, the Balian-Bloch papers were one of the origins of the Poisson relation for manifolds with boundary (see below and [CV2]). The purpose of this first paper in the series is to give a rigorous version of the Balian-Bloch approach. In [Z3], we use it to calculate wave invariants in terms of Feynman diagrams and amplitudes and to reduce the inverse spectral problem to concrete combinatorial problems.

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In subsequent articles, we will demonstrate the usefulness of this reduction with the following applications:

- In [Z3] we prove that simply connected analytic domains with one symmetry (that reverses a bouncing ball orbit) are determined among such domains by their Dirichlet spectra;
- In [Z4] we prove that exterior domains with one mirror symmetry are determined by their resonance poles. This is a resonance analogue of our inverse spectral results;
- These results should admit extensions to domains formed by flipping the graph of an analytic function around the $x$-axis, allowing for corners at the $x$-intercept. It is easy to construct examples of this kind.

The crucial advantage of our present approach is the very explicit nature of the trace formula, which allows us to obtain relatively simple formula for lower order terms in wave trace expansions. These lower order terms allow us to remove one symmetry from the inverse results of ourself and of Iantchenko-Sjöstrand-Zworski [ISZ]. The rather concrete formulae for wave invariants (see [13]) are valid for all plane domains and the symmetry assumption is only used to reduce the amount of data required to determine the domain.

We should emphasize that we view the Balian-Bloch asymptotic expansions as primarily a computational device. Although one could give a new and self-contained proof of the Poisson relation for bounded Euclidean domains by this approach, we do not do so here. In particular, we use the Melrose-Sjostrand results on propagation of singularities to microlocalize traces to orbits. Also, we do not explain how to cancel so-called ‘ghost orbits’ of non-convex domains, i.e. closed billiard trajectories which do not stay within the domain. To explain how the Balian-Bloch approach improves on normal forms or microlocal parametrix constructions, and to explain its connection to the usual Poisson relation, we now give an informal exposition of the Balian-Bloch trace formula.

1.1. Introduction to wave trace and Balian-Bloch. Let $\Omega$ denote a bounded $C^\infty$ plane domain, and let

$$E^e_\Omega(t) = \cos t \sqrt{-\Delta_\Omega}$$

denote the even part of the wave group of the Dirichlet Laplacian $\Delta_\Omega$. As recalled in §3, the singular points $t$ of the distribution trace $Tr E_\Omega(t)$ are contained in the length spectrum $Lsp(\Omega)$ of $\Omega$, i.e. the set of lengths $t = L_\gamma$ of closed orbits of the billiard flow $\Phi^t$ of $\Omega$ (i.e. straight-line motion in $\Omega$ with the Snell law of reflection at the boundary; cf. §2). In particular, at lengths $L_\gamma$ of periodic reflecting rays $\gamma$ the trace has a complete singularity expansion; its coefficients are known as the wave trace invariants of $\gamma$.

The wave trace invariants at a periodic reflecting ray may be obtained from a dual semi-classical asymptotic expansions as $k \to \infty$ for the regularized trace of the Dirichlet resolvent

$$R_\Omega(k + i\tau) := -(-\Delta_\Omega + (k + i\tau)^2)^{-1}, \quad (k \in \mathbb{R}, \tau \in \mathbb{R}^+).$$

We emphasize that $\Delta_\Omega$ (the usual Dirichlet Laplacian) is a negative operator, so the signs of the two terms are opposite. In the classical work of Seeley and others on resolvent traces, asymptotics are taken along the vertical line (or a ray of non-zero slope) in the upper half plane $k + i\tau$ and traces are polyhomogeneous functions of $k$. Here, we are taking asymptotics along horizontal lines (or logarithmic curves) in this half-plane and obtain oscillatory asymptotics reflecting the behaviour of closed geodesics.
We fix a non-degenerate periodic reflecting ray $\gamma$ (cf. §2). For technical convenience we will assume that the reflection points of $\gamma$ are points of non-zero curvature of $\partial \Omega$. This is a minor assumption, but it simplifies one argument (Proposition 7.1) and we use it again in [Z3]. We briefly explain there how to remove this assumption. We also let $\hat{\rho} \in C_0^\infty(L_\gamma - \epsilon, L_\gamma + \epsilon)$ be a cutoff, equal to one on an interval $(L_\gamma - \epsilon/2, L_\gamma + \epsilon/2)$ which contains no other lengths in $\text{Lsp}(\Omega)$ occur in its support. We then define the smoothed (and localized) resolvent by

$$R_\rho(k + i\tau) := \int_\mathbb{R} \rho(k - \mu)(\mu + i\tau)R_\Omega(\mu + i\tau)d\mu.$$  

When $\gamma, \gamma^{-1}$ are the unique closed orbits of length $L_\gamma$, it follows from the Poisson relation for manifolds with boundary (see §3 and [GM, PS]; see also Proposition (3.1)) that the trace $\text{Tr}_1 \Omega R_\rho((k + i\tau))$ of the regularized resolvent on $L^2(\Omega)$ admits a complete asymptotic expansion of the form:

$$\text{Tr}_1 \Omega R_\rho(k + i\tau) \sim e^{(ik - \tau)L_\gamma} \sum_{j=1}^\infty (B_{\gamma,j} + B_{\gamma^{-1},j})k^{-j}, \quad k \to \infty$$

with coefficients $B_{\gamma,j}, B_{\gamma^{-1},j}$ determined by the jet of $\Omega$ at the reflection points of $\gamma$, in the sense that

$$\text{Tr}_1 \Omega R_\rho(k + i\tau) - e^{(ik - \tau)L_\gamma} \sum_{j=1}^R (B_{\gamma,j} + B_{\gamma^{-1},j})k^{-j} = O(|k|^{-R}).$$

The coefficients $B_{\gamma,j}, B_{\gamma^{-1},j}$ are thus essentially the same as the wave trace coefficients at the singularity $t = L_\gamma$. Our main goal in this paper is to give a useful algorithm for calculating them explicitly in terms of the defining function of $\partial \Omega$.

In fact, it is technically more convenient to consider asymptotics of traces along logarithmic curves $k + i\tau \log k$ in the upper half plane. We therefore modify the regularization (1) to

$$R_\rho(k + i\tau \log k) := \int_\mathbb{R} \rho(k - \mu)(\mu + i\tau \log k)R_\Omega(\mu + i\tau \log \mu)d\mu.$$  

In place of (2) we will get

$$\text{Tr}_1 \Omega R_\rho(k + i\tau \log k) \sim e^{(ik)L_\gamma k^{-\tau}L_\gamma} \sum_{j=1}^\infty (B_{\gamma,j} + B_{\gamma^{-1},j})k^{-j}, \quad k \to \infty.$$  

The additional power law decay $k^{-\tau L_\gamma}$ will not cause problems in our study of wave invariants at a fixed closed geodesic, because the the errors have the accuracy of $k^{-\infty}$.

In principle, one could obtain sufficiently explicit formulae for $B_{\gamma,j}$ by applying the method of stationary phase to a microlocal parametrix at $\gamma$ (cf. [GM] [PS]), or by constructing a Birkhoff normal form for $\Delta$ at $\gamma$ [Z1, Z2, SZ, SSZ]. However, in practice we have found the Balian-Bloch approach more effective. Its starting point is to write the Dirichlet Green’s kernel $G_\Omega(k + i\tau, x, y)$ of $R_\Omega(k + i\tau)$ as a Neumann series (called the multiple reflection expansion in [BB]) in terms of the free Green’s function $G_0(k + i\tau, x, y)$, i.e. the kernel of...
the free resolvent $R_{\Omega}(k + i\tau) = -\left(\Delta_0 + (k + i\tau)^2\right)^{-1}$ on $\mathbb{R}^2$:
\[
G_{\Omega}(k + i\tau, x, y) = G_0(k + i\tau, x, y) + \sum_{M=1}^{\infty}(-2)^M G_M(k + i\tau, x, y), \text{ where}
\]
\[
G_M(k + i\tau, x, y) = \int_{(\partial \Omega)^M} \partial_{v_y} G_0(k + i\tau, x, q_1)G_0(k + i\tau, q_M, y) \times \prod_{j=1}^{M-1} \partial_{v_y} G_0(k + i\tau, q_{j+1}, q_j)ds(q_1) \cdots ds(q_M),
\]
where $ds(q)$ denotes arclength on $\partial \Omega$ and $\partial_{v_y}$ is the interior unit normal operating in the second variable. (The only change in the case of the Neumann Green’s kernel is that the $M$th term has an additional factor of $(-1)^M$; by making this change, all our methods and results extend immediately to the Neumann case.) The terms are regularized as in (1) by setting
\[
G_{M,\rho}(k + i\tau \log k) = \int_{\mathbb{R}} \rho(k - \mu)(\mu + i\tau \log \mu)G_M(\mu + i\tau \log \mu)d\mu
\]
There exists an explicit formula for $G_0(k + i\tau)$ in terms of Hankel functions ($\S 4$), from which it appears that the traces $Tr 1_{\Omega} G_{M,\rho}(k + i\tau \log k) = \int_{\Omega} G_{M,\rho}(k + i\tau \log k, x, x)dx$ are (formally) oscillatory integrals with phases
\[
L(x, q_1, \ldots, q_M, x) = |x - q_1| + |q_1 - q_2| + \cdots + |q_M - x|,
\]
equal to the length of the polygon with vertices at points $(q_1, \ldots, q_M) \in (\partial \Omega)^M$ and $x \in \Omega$. The smooth critical points correspond to the $M$-link periodic reflecting rays of the billiard flow of $\Omega$ of length $L_\gamma$, satisfying Snell’s law at each vertex (for short, we call such polygons Snell polygons). Since the amplitudes and phases only involve the free Green’s function, they are known explicitly and the coefficients of the stationary phase expansion of $Tr 1_{\Omega} R_{\rho}((k + i\tau \log k))$ can be calculated explicitly.

Thus, our plan for determining $\Omega$ from wave invariants at a bouncing ball orbit $\gamma$ is as follows: We chose $\rho$ to localize at the length of the $r$th iterate of $\gamma$. We then localize the integrals $G_{\rho,M}(k + i\tau \log k)$ to small intervals around the endpoints $\{(0, -L/2), (0, L/2)\}$ and parametrize the two components, as above, by $y = f_-(x)$, resp. $y = f_+(x)$. (See figure ??) The integral over $(\partial \Omega)^M$ is then reduced to an integral over $(-\epsilon, \epsilon)^M$ with phase and amplitude given by canonical functions of $f_-(x), f_+(x)$. Applying the stationary phase method, we obtain coefficients which are polynomials in the data $f_+(0), f_-^j(0)$. By examining the behaviour of the coefficients under iterates $\gamma^r$ of the bouncing ball orbit, we try to determine all Taylor coefficients $f_+(0)$ from these stationary phase coefficients. In [Z3, Z4, Z5], we will see that this method succeeds at least if the domain has one symmetry (so that $f_+ = -f_-$).

A number of technical problems must be overcome in turning this idea into a proof. The problem is that the free Green’s kernel (and its normal derivative) only possesses a WKB formula away from the diagonals $q_i = q_{i+1}$, and worse, it is singular along these diagonals. Hence, in the Balian-Bloch approach (unlike the parametrix approach), the following problems arise:

- The individual terms $G_{M,\rho}$ must be regularized, i.e. converted into sums of standard oscillatory integrals. Until then, it is not even clear that the trace of each term has asymptotic an asymptotic expansion, or that it localizes at critical points;
• There are non-smooth critical points (corresponding to Snell polygons in which at least one edge collapsed), and their contribution to the stationary phase expansion must be determined;

• Since the multiple reflection expansion $\sum_{n=0}^\infty$ is an infinite series, it must be explained how the full series produces an asymptotic expansion $\mathcal{O}(1)$ of $Tr_\Omega R_\rho(k + i\tau \log k)$, and why each term in $\mathcal{O}(1)$ depends on only the trace of a finite number of terms of $\mathcal{O}(1)$. In particular, the ‘tail’ trace must be estimated.

Let us briefly outline how we deal with these difficulties. At the same time, we will justify the effort involved by explaining how the new results are obtained by this method.

1.2. Regularizing the terms of the Neumann series. We first explain how we regularize the individual terms in the Neumann series. For expository reasons, we suppress the role of the interior variable $x$ in the introduction.

As will be explained in §4, the multiple reflection expansion is derived from an exact formula of potential theory $\mathcal{O}(1)$ for the Dirichlet resolvent:

$$ R_\Omega(k + i\tau) = R_0(k + i\tau) - \mathcal{D}(k + i\tau)(I + N(k + i\tau))^{-1}r_\Omega S\mathcal{D}(k + i\tau), $$

by expanding the operator $(I + N(k + i\tau))^{-1}$ in a geometric series. Here, $\mathcal{D}(k + i\tau)$ (resp. $S\mathcal{D}(k + i\tau)$) is the double (resp. single) layer potential (see §3), $S\mathcal{D}(k + i\tau)$ is the transpose, and $N(k + i\tau)$ is the boundary integral operator on $L^2(\partial\Omega)$ induced by $\mathcal{D}(k + i\tau)$ (see §3). Also, $R_0(k + i\tau)$ is the free resolvent on $\mathbb{R}^2$, and $r_\Omega$ is the restriction to the boundary. The existence of the inverse $(I + N(k + i\tau))^{-1}$ is guaranteed by Fredholm theory.

As will be discussed in detail in §4 (see also §5), the operator $N(k + i\tau)$ is a hybrid Fourier integral operator. It has the singularity of a homogeneous pseudodifferential operator of order $-1$ on the diagonal (in fact, it is of order $-2$ in dimension $2$, see Proposition 4.1). This is the way it is normally described in potential theory §4. However, away from the diagonal, it has a WKB approximation which exhibits it as a semi-classical Fourier integral operator with phase $d_\partial\Omega(q, q') = |q - q'|$ on $\partial\Omega \times \partial\Omega$, the boundary distance function of $\Omega$. To make this precise, we introduce a cutoff $\chi(k^{1-\delta}|q - q'|)$ to the diagonal, where $\delta > 1/2$ and where $\chi \in C_0^\infty(\mathbb{R})$ is a cutoff to a neighborhood of 0. We then put

$$ N(k + i\tau) = N_0(k + i\tau) + N_1(k + i\tau), \quad \text{with} \quad \chi(k^{1-\delta}|q - q'|) = 0, $$

$$ N_0(k + i\tau, q, q') = \chi(k^{1-\delta}|q - q'|)N(k + i\tau, q, q'), $$

$$ N_1(k + i\tau, q, q') = (1 - \chi(k^{1-\delta}|q - q'|))N(k + i\tau, q, q'). $$

As will be shown in Proposition 4.3, $N_1((k + i\tau), q, q')$ is a semiclassical Fourier integral operator with phase equal to $d_\partial\Omega(q, q')$.

Roughly speaking, the boundary distance function generates the billiard map of $\partial\Omega$ (see §2.2 for the definition). This is literally correct only on convex domains, since $d_\partial\Omega(q, q')$ generates both the interior and exterior billiard map, and hence on non-convex domains its canonical relation contains ‘ghost orbits’ (orbits which may exit and re-enter the domain). Because we are microlocalizing to one periodic reflecting ray $\gamma$, ghost orbits play no essential role in this paper and we think of $N_1(k + i\tau)$ as ‘quantizing’ the billiard map.
Now consider the powers $N(k + i\tau)^M$ which arise when expanding $(I - N(k + i\tau))^{-1}$ in a geometric series. We write

$$\begin{align*}
(N_0 + N_1)^M &= \sum_{\sigma: \{1, \ldots, M\} \to \{0, 1\}} N_{\sigma(1)} \circ N_{\sigma(2)} \circ \cdots \circ N_{\sigma(M)}. 
\end{align*}$$

To regularize $N^M$ is essentially to remove all of the factors of $N_0$ from each of these terms. This is obviously not possible for the term $N_0^M$ but it is possible for the other terms. In Proposition 6.1, we show that $N_0N_1$ and $N_1N_0$ are semiclassical Fourier integral operators of the same type as $N_1$ (and with the same phase), but with an amplitude of one lower degree in $k$. Thus, the term $N_1^M$ is of the highest order in the sum. We will see that it is the only important term for the inverse spectral problem. In Proposition 7.1, we similarly break up the layer potentials in (7) and analyze compositions with these. Further we compose with a special kind of semiclassical cutoff operator $\chi(x, k^{-1}D_x)$ to a neighborhood of the orbit on both sides of (7).

1.3. The M-aspect. Since the geometric series expansion of $(I + N(k + i\tau))^{-1}$ is very slowly converging, we write it as a partial geometric series and a remainder:

$$\frac{1}{I + N(k + i\tau)} = \sum_{M=0}^{M_0} (-1)^M N(k + i\tau)^M + R_{M_0},$$

where $R_{M_0} = N(k + i\tau)^{M_0+1} (I + N(k + i\tau))^{-1}$.

The partial geometric series is regularized by the methods described in the previous section. We now explain how to estimate the tail when evaluating wave invariants at a closed orbit $\gamma$. In our applications, $\gamma$ is a bouncing ball orbit, so we will assume it is one in the remainder estimate.

We use elementary inequalities on traces to reduce the estimate of the tail trace

$$\begin{align*}
Tr_{\gamma} R_{M_0}(k + i\tau \log k)\chi(x, k^{-1}D_x)
\end{align*}$$

to an estimate of $Tr N_{M_0} \chi_{\partial\gamma}(k + i\tau \log k)\chi_{\partial\gamma}^* N_{M_0}^*$, where $\chi_{\partial\gamma}$ is a semiclassical cutoff operator on $\partial\Omega$ to the periodic orbit of the billiard map corresponding to $\gamma$. As will be verified in §8 after expanding as in §10, the operator $N_{M_0} \chi_{\partial\gamma}(k + i\tau \log k)\chi_{\partial\gamma}^* N_{M_0}^*$ can be regularized as above as a sum over $\sigma$ of semiclassical Fourier integral operators whose phases involve lengths $\mathcal{L}$ of $M_0 - |\sigma|$-link billiard trajectories, where $|\sigma|$ represents the number of $N_0$ factors in a term. The cutoff operator will force the direction of the first link to point nearly along $\gamma$ and hence will force each link of the critical $M_0 - |\sigma|$-link paths to have length $\sim L_\gamma$. We then use the fact that the spectral parameter varies over a logarithmic curve: due to the imaginary part of $\tau \log k$ of the phase, each link in the critical path gets weighted by the factor $e^{-\tau L_\gamma \log k} = k^{-\tau}$. On the other hand, each removal of a factor $M_0$ lowers the order in $k$ by one. Combining these two effects, we show in Proposition 8.3 and Lemma 8.1 that for $M_0$ sufficiently large, the remainder term will be of lower order than any prescribed power $k^{-R}$.

1.4. Main results. This regularization procedure provides a method of explicitly calculating wave invariants associated to an $M$-link periodic reflecting ray. After regularizing the initial terms of the multiple reflection expansion and discarding the remainder, we may apply stationary phase to rather simple and canonical oscillatory integrals.
Our main result may be summarized in the following theorem. The term ‘canonical’ means ‘independent of the domain $\Omega$.’

**Theorem 1.1.** Let $\gamma$ be a primitive non-degenerate $m$-link reflecting ray, whose reflection points are points of non-zero curvature of $\partial \Omega$, and let $\hat{\rho} \in C^\infty_0(\mathbb{R})$ be a cut off satisfying $\text{supp} \hat{\rho} \cap \text{Lsp}(\Omega) = \{rL_r\}$. Then there exists an effective algorithm for obtaining the wave invariants at $\gamma$ consisting of the following steps:

(i) Each term $G_{M,\rho}(k + i\tau \log k)$ of (5) defines a kernel of trace class;

(ii) The traces $\text{Tr}_{1\Omega} G_{M,\rho}(k + i\tau \log k)$ can be regularized in a canonical way as oscillatory integrals with canonical amplitudes and phases;

(iii) The stationary phase method applies to these oscillatory integrals. The coefficients $B_{\gamma^r, j}$ of a given order $j \leq R$ are obtained by summing the expansions for $\text{Tr}_{1\Omega} G_{M,\rho}(k + i\tau \log k)$ for $M \leq M_R$. There exists $M_0$ such that the tail trace (12) is $O(k^{-R})$.

(iv) The coefficients of the term $B_{\gamma^r, j}$ are universal polynomials in the $2j + 2$-jet of the defining function of $\partial \Omega$ at the reflection points.

(v) If $\gamma$ is a bouncing ball orbit, then modulo an error term $R_{2r} (j^{2j-2} f(0))$ depending only on the $(2j - 2)$-jet of $f$ at the endpoints, we obtain the formula

$$B_{\gamma^r, j-1} + B_{\gamma^{-r}, j-1} = a_{j,r,+} f^{(2j)}(0) + a_{j,r,-} f^{(2j)}(0) + b_{j,r,+} f^{(2j-1)}(0) + b_{j,r,-} f^{(2j-1)}(0),$$

where the coefficients are polynomials in the matrix elements $h^q$ of the inverse of the Hessian of the length function $L$ at $\gamma^r$ with universal coefficients.

For instance, if $\gamma$ is elliptic and invariant under an isometric involution $\sigma$ of $\Omega$, then

$$B_{\gamma^r, j-1} + B_{\gamma^{-r}, j-1} = r \{ 2(h^{11})^j f^{(2j)}(0) + 2(h^{11})^j \frac{1}{2 - 2 \cos \alpha/2} + (h^{11})^{j-2} \sum_{q=1}^{2r} (h^{q})^3 \} f^{(3)}(0) f^{(2j-1)}(0) \}.$$  

Here, $e^{\pm 2\pi ic}$ are the eigenvalues of the Poincare map $P_\gamma$ of $\gamma$. To our knowledge, no such explicit formula for the wave invariants has been available before.

The most important aspect of Theorem 1.1 is the algorithm for computing the coefficients of $B_{\gamma^r, j}$ in terms of the defining functions of $\partial \Omega$, as illustrated in (v) and by the explicit formula (13). That is truly what distinguishes the result from previous approaches to wave invariants and the Poisson relation. For instance, recovering a $\mathbb{Z}_2$-invariant $\partial \Omega$ is tantamount to sifting the Taylor coefficients of $f$ at 0 from the expression (13), and we see that this amounts to a fine study of sums of powers in the matrix coefficients $h^3$ of the inverse Hessian of $L$. We needed the complete symbol of the wave trace to obtain this result, and no such details have previously been obtained by any other method. The gain in effectiveness is borne out in the subsequent articles [Z3, Z4]. Although we will not do so here, the proof extends with no essential changes bounded smooth domains in $\mathbb{R}^n$.

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## 2. Billiards and the length functional

We collect here some notation and background results on plane billiards which we will need below, mainly following the reference Kozlov-Trechev [KT].

Let $\Omega$ denote a simply connected analytic plane domain with smooth boundary $\partial \Omega$ of length $2\pi$. We denote by $T = \mathbb{R} \setminus 2\pi \mathbb{Z}$ the unit circle and parametrize the boundary counterclockwise by arc-length starting at some point $q_0 \in \partial \Omega$:

\begin{equation}
q : T \to \partial \Omega \subset \mathbb{R}^2, \quad q(\phi) = (x(\phi), y(\phi)), \quad |\dot{q}(\phi)| = 1, \quad q(0) = q_0.
\end{equation}
We similarly identify the \( m \)-fold Cartesian product \((\partial \Omega)^m\) of \(\partial \Omega\) by \(T^m\), and denote a point of the latter by \((\phi_1, \ldots, \phi_m)\).

By an \( m \)-link **periodic reflecting ray** of \(\Omega\) we mean a billiard trajectory \(\gamma\) which intersects \(\partial \Omega\) transversally at \( m \) points \(q(\phi_1), \ldots, q(\phi_m)\) of intersection, and reflects off \(\partial \Omega\) at each point according to Snell’s law

\[
q(\phi_{j+1}) - q(\phi_j) \quad \text{and the unit inward normal at} \quad q(\phi_j) \quad \text{(cf. [AG, EP] and [Z6])}
\]

We have:

\[
\nu_{q(\phi_j)} = \frac{q(\phi_j) - q(\phi_{j-1})}{|q(\phi_j) - q(\phi_{j-1})|} \cdot \nu_{q(\phi_j)}.
\]

Here, \(\nu_{q(\phi)}\) is the inward unit normal to \(\partial \Omega\) at \(q(\phi)\). We refer to the segments \(q(\phi_j) - q(\phi_{j-1})\) as the **links** of the trajectory. An \( m \)-link periodic reflecting ray is thus the same as an \( m \)-link polygon in which the Snell law holds at each vertex. Since they will come up often, we make:

**Definition 2.1.** By \(P_{(\phi_1, \ldots, \phi_m)}\) we denote the polygon with consecutive vertices at the points \((q(\phi_1), \ldots, q(\phi_m))\) \(\in (\partial \Omega)^n\). The polygon is called:

- **Non-singular** if \(\phi_j \neq \phi_{j+1}\) for all \(j\)
- **Snell** if \(P_{(\phi_1, \ldots, \phi_m)}\) is non-singular and if \([16]\) holds for each pair of consecutive links \(\{q(\phi_j) - q(\phi_{j-1}), q(\phi_{j+1}) - q(\phi_j)\}\);
- **Singular Snell** if \(P_{(\phi_1, \ldots, \phi_m)}\) has fewer than \(n\) distinct vertices, but each non-singular pair of consecutive links satisfies Snell’s law.

We will denote the acute angle between the link \(q(\phi_{j+1}) - q(\phi_j)\) and the inward unit normal \(\nu_{q(\phi_{j+1})}\) by \(\angle(q(\phi_{j+1}) - q(\phi_j), \nu_{q(\phi_j)})\) and that between \(q(\phi_{j+1}) - q(\phi_j)\) and the inward unit normal at \(q(\phi_j)\) by \(\angle(q(\phi_{j+1}) - q(\phi_j), \nu_{q(\phi_j)})\), i.e. we put

\[
\frac{q(\phi_{j+1}) - q(\phi_j)}{|q(\phi_{j+1}) - q(\phi_j)|} \cdot \nu_{q(\phi_j)} = \cos \angle(q(\phi_{j+1}) - q(\phi_j), \nu_{q(\phi_j)}).
\]

We also use the notation \(\angle(q(\phi_{j+r}) - q(\phi_j), \nu_{q(\phi_j)})\) for the angle between the link \(q(\phi_{j+r}) - q(\phi_j)\) and the unit inward normal at \(q(\phi_j)\).

The function \(\angle(q(\phi_{j+1}) - q(\phi_j), \nu_{q(\phi_j)})\) is well-defined on \(T^m\) minus the diagonals \(\Delta_{\hat{k}, \hat{j}+1} = \{\phi_j = \phi_{j+1}\}\). It has a continuous extension across the diagonals according to the following

**Proposition 2.2.** \(\cos \angle(q(\phi) - q(\phi'), \nu_{q(\phi)}) = -\frac{1}{2} \kappa(\phi)|\phi' - \phi| + O(|\phi' - \phi|^2)\).

**Proof.** (cf. [AC, EP] and [Z6]) We have:

\[
(q(\phi')) - q(\phi) \cdot \nu_{q(\phi)} = -\frac{1}{2} (\phi - \phi')^2 \kappa(\phi) + O((\phi - \phi')^3).
\]

Now divide by \(|q(\phi) - q(\phi')|\).

\(
\square
\)

2.1. **Length functional.** We first define a length functional on \(T^M\) by:

\[
L(\phi_1, \ldots, \phi_M) = |q(\phi_1) - q(\phi_2)| + \cdots + |q(\phi_{M-1}) - q(\phi_M)|.
\]
It is clear that \( L \) is a smooth function away from the ‘large diagonals’ \( \Delta_{j,j+1} := \{ \phi_j = \phi_{j+1} \} \), where it has \(|x|\) singularities. We have:

\[
\begin{align*}
\frac{\partial}{\partial \phi_j} |q(\phi_j) - q(\phi_{j-1})| &= -\sin \angle(q(\phi_j) - q(\phi_{j-1}), \nu_{q(\phi_j)}), \\
\frac{\partial}{\partial \phi_j} |q(\phi_{j+1}) - q(\phi_j)| &= \sin \angle(q(\phi_{j+1}) - q(\phi_j), \nu_{q(\phi_{j+1})})
\end{align*}
\]

\( (18) \)

\( \implies \frac{\partial}{\partial \phi_j} \partial = \sin \angle(q(\phi_{j+1}) - q(\phi_j), \nu_{q(\phi_{j+1})}) - \sin \angle(q(\phi_j) - q(\phi_{j-1}), \nu_{q(\phi_j)}). \)

The condition that \( \frac{\partial}{\partial \phi_j} \partial = 0 \) is thus that the 2-link defined by the triplet \((q(\phi_{j-1}), q(\phi_j), q_{i+1})\) is Snell at \( \phi_j \). A smooth critical point of \( L \) on \( \mathbf{T}^M \) is thus the same as an \( M \)-link Snell polygon.

We will also be concerned with the length functional \( L : \Omega \times \mathbf{T}^M \to \mathbb{R}^+ \) defined by:

\[
L(x, \phi_1, \ldots, \phi_M) = |x - q(\phi_1)| + |q(\phi_1) - q(\phi_2)| + \cdots + |q(\phi_{M-1}) - q(\phi_M)| + |q(\phi_M) - x|,
\]

which is smooth away from the diagonals \( x = q(\phi_1), x = q(\phi_M) \) together with \( \Delta_{j,j+1} \). Its gradient in the \( x \)-variable is given by

\[
\nabla_x L = \frac{x - q(\phi_1)}{|x - q(\phi_1)|} + \frac{x - q(\phi_M)}{|x - q(\phi_M)|};
\]

so that a smooth critical point \( x \) of \( L(x, \phi_1, \ldots, \phi_M) \) corresponds to triple \((\phi_M, x, \phi_1)\) whose 2-link is straight at \( x \). We sum up in the following well-known proposition, due to Poincare. For background, see [KT].

**Proposition 2.3.** A smooth critical point \((x, \phi_1, \ldots, \phi_M)\) of \( L \) on \( \Omega \times \mathbf{T}^M \) corresponds to an \( M \)-link Snell polygon with vertices \((x, \phi_1, \ldots, \phi_M)\).

We will also need the formula for the interior normal derivative \( \frac{\partial}{\partial \nu_y} = \nu_{q(\phi_{j+1})} \cdot \nabla_y \) along the boundary of the link-lengths:

\[
(21) \quad \frac{\partial}{\partial \nu_y} |q(\phi_j) - y|_{y=q(\phi_{j+1})} = \frac{q(\phi_{j+1}) - q(\phi_j) - q(\phi_{j+1})}{|q(\phi_{j+1}) - q(\phi_{j+1})|} \cdot \nu_{q(\phi_{j+1})} = \cos \angle(q(\phi_{j+1}) - q(\phi_j), \nu_{q(\phi_j)}).
\]

2.2. **Billiard flow and length spectrum.** The (geometer’s) billiard flow \( \Phi^t \) of \( \Omega \) is the flow on \( \mathbf{T}^* \Omega \) which is partially defined by Euclidean motion in the interior and Snell’s law of reflection at the boundary. We refer to the billiard orbits as trajectories or rays, and when they have only transversal intersections with the boundary we refer to them as transversally reflecting rays. The straight line segments between intersection points are called links.

At (co-) vectors tangential to \( \partial \Omega \) this law does not uniquely define the flow unless \( \Omega \) is convex, in which case tangentially intersecting rays can only travel along the boundary. In the non-convex case, there exist rays which intersect \( \partial \Omega \) tangentially and at such points the geometr’s billiard flow is not uniquely defined. The propagation of singularities theorem for domains with boundary (cf. [AM, MS, PS]) largely resolves this ambiguity, and completely resolves it for analytic domains. It defines the billiard flow as the broken bicharacteristic flow of the wave operator, i.e. the trajectories along which singularities of solutions of the wave equation move. Roughly speaking, the trajectories are transversally reflecting rays and
limits of such rays with many small links. The limit rays intersect the boundary tangentially and then glide for some time along the boundary and then re-enter the domain. Since small links can only occur in the convex part of the boundary, the entrance and exit points to the boundary of a plane domain occur at its inflection points. There exists unique continuation of geodesics unless $\partial \Omega$ has infinite order contact with a tangent line, and of course this cannot occur for analytic domains. For background and further discussion we refer to [PS, GM, M]. Pictures of gliding rays may be found [GM, M].

In the Poisson relation for the wave equation, it is the analysts’ billiard flow (propagation of singularities) which is relevant, and henceforth we assume the billiard flow defined as the broken bicharacteristic flow for the wave equation. We then define the length spectrum $L_{sp}(\Omega)$ to be the set of lengths of periodic orbits of the billiard flow ([PS], Definition (1.2.9)).

By the billiard map $\beta$ of $\Omega$ we mean the map induced by $\Phi^t$ on $B^*\partial\Omega$: if $(q, \eta) \in B^*\partial\Omega$, we may add a multiple of the unit normal to obtain an inward pointing unit vector $v$ at $q$. We then follow the billiard trajectory of $v$ until it hits the boundary, and then define $\beta(q, \eta)$ to be its tangential projection.

In the case of strictly convex domains, periodic orbits are either periodic reflecting rays $\gamma$ or closed geodesics on $\partial \Omega$ ([PS], Ch. 7). By periodic $n$-link reflecting ray, we mean a periodic orbit of the billiard flow $\Phi^t$ on $T^*\Omega$ whose projection to $\Omega$ has only transversal intersections with $\partial \Omega$. That is, $\gamma$ is a Snell polygon with $n$ sides. (Here, and henceforth, we often do not distinguish notationally between an orbit of $\Phi^t$ and its projection to $\Omega$.)

2.2.1. Poincare map and Hessian of the length functional. The linear Poincare map $P_\gamma$ of $\gamma$ is the derivative at $\gamma(0)$ of the first return map to a transversal to $\Phi^t$ at $\gamma(0)$. By a non-degenerate periodic reflecting ray $\gamma$ we mean one whose linear Poincare map $P_\gamma$ has no eigenvalue equal to one. For the definitions and background, we refer to [PS][KT].

There is an important relation between the spectrum of the Poincare map $P_\gamma$ of a periodic $n$-link reflecting ray and the Hessian $H_n$ of the length functional at the corresponding critical point of $L : T^n \to \mathbb{R}$. For the following, see [KT] (Theorem 3).

**Proposition 2.4.** We have:

$$\det(I - P_\gamma) = -\det(H_n) \cdot (b_1 \cdots b_n)^{-1},$$

where $b_j = \frac{\partial^2|q(\phi_{j+1} - q(\phi_j))|}{\partial \phi_j \partial \phi_{j+1}}$.

3. **Wave trace and resolvent trace asymptotics**

The spectral invariants we will use in determining $\Omega$ are essentially the wave trace invariants associated to bouncing ball orbits. As will be recalled below, these invariants are coefficients of the asymptotic expansion of the trace $Tr_1 E_\Omega(t)$ of the Dirichlet wave group around in singularities at lengths of periodic billiard trajectories. Dual to the wave trace singularity expansion, at least formally, is the asymptotics as $k \to \infty$ of the trace $Tr_1 R_\Omega(k+i\tau)$ of the Dirichlet resolvent. Further, we will regularize the trace and relate the resolvent trace coefficients at a periodic reflecting ray to the corresponding wave trace invariants.

3.1. **Resolvent and Wave group.** By the Dirichlet Laplacian $\Delta_\Omega$ we mean the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ with domain $\{u \in H^1_0(\Omega) : \Delta u \in L^2\}$; thus, in our notation, $\Delta_\Omega$ is a negative
operator. We denote by \( E_\Omega(t, x, y) = \cos t \sqrt{-\Delta_\Omega(x, y)} \) the fundamental solution of the mixed wave equation with Dirichlet boundary conditions:

\[
\begin{cases}
\frac{\partial^2 E_\Omega}{\partial t^2} = \Delta E & \text{on } \mathbb{R} \times \Omega \\
E_\Omega(0, x, y) = \delta(x - y) & \frac{\partial E_\Omega}{\partial t}(0, x, y) = 0 \\
E_\Omega(t, x, y) = 0 & (t, x, y) \in \mathbb{R} \times \partial \Omega \times \Omega.
\end{cases}
\] (22)

The resolvent of the Laplacian \( \Delta_\Omega \) on \( \Omega \) with Dirichlet boundary conditions is the operator on \( L^2(\Omega) \) defined by

\[
R_\Omega(k + i\tau) = -(\Delta_\Omega + (k + i\tau)^2)^{-1}, \quad \tau > 0.
\]

The resolvent kernel, which we refer to as the Dirichlet Green’s function \( G_\Omega(k + i\tau, x, y) \) of \( \Omega \subset \mathbb{R}^2 \), is by definition the solution of the boundary problem:

\[
\begin{cases}
(\Delta_x + (k + i\tau)^2)G_\Omega(k + i\tau, x, y) = -\delta(x - y), \quad (x, y \in \Omega) \\
G_\Omega(k + i\tau, x, y) = 0, \quad x \in \partial \Omega.
\end{cases}
\] (23)

To clarify our sign conventions, let us specify them in the case of the free Laplacian \( \Delta_0 \) on \( \mathbb{R}^2 \). Our \( \Delta \) is negative, so the symbol of the free resolvent is \( (|\xi|^2 - (k + i\tau)^2)^{-1} \). The free Green’s function has the asymptotic behaviour

\[
G_0(k + i\tau, x, y) \sim \frac{e^{i(k+i\tau)|x-y|}}{[(k + i\tau)|x-y|]^{1/2}}, \quad ([|k + i\tau||x-y|] \to \infty)
\]
hence is oscillatory in \( k \) and has the exponentially decay \( e^{-\tau|x-y|} \) when \( \tau > 0 \). We will later place the spectral parameter on the logarithmic curve \( k + i\tau \log k \), where it has a power law decay in \( k \) as well as oscillatory behaviour. For further discussion of the signs, see [1] (p. 142).

The resolvent may be expressed in terms of the (even) wave operator as

\[
R_\Omega(k + i\tau) = \frac{1}{k + i\tau} \int_0^\infty e^{i(k+i\tau)t} E_\Omega(t) dt, \quad (\tau > 0)
\] (24)

which holds because

\[
\frac{1}{\lambda^2 - (k + i\tau)^2} = \frac{1}{k + i\tau} \int_0^\infty e^{i(k+i\tau)t} \cos \lambda t dt, \quad (\forall \lambda \in \mathbb{R}, \tau \in \mathbb{R}^+).
\]

Given \( \hat{\rho} \in C^\infty_0(\mathbb{R}^+) \), we have defined the smoothed resolvent \( R_\rho(k + i\tau) \) in [1]. By (24) we can rewrite it in terms of the wave kernel as:

\[
R_\rho(k + i\tau) = \int_0^\infty \int_{\mathbb{R}} \rho(k - \mu)e^{i(\mu + i\tau)t} E_\Omega(t) dt d\mu
\]

\[
= \int_0^\infty \hat{\rho}(t)e^{i(k+i\tau)t} E_\Omega(t) dt
\]

\[
= \rho(k + i\tau + \sqrt{\Delta_\Omega}) + \rho(k + i\tau - \sqrt{\Delta_\Omega})
\] (25)

This is essentially the smoothing used in the study of wave invariants in [DG].
3.2. Wave trace and resolvent trace asymptotics. We now recall the classical results about the asymptotics of \( \text{Tr} \Omega R_\rho(k + i\tau) \) (cf. [GM] [PS]). From the last formula in (25) we note that

\begin{equation}
\text{Tr} \Omega R_\rho(k + i\tau) = \sum_{j=1}^{\infty} \left[ \rho(k - \lambda_j - i\tau) + \rho(k + \lambda_j - i\tau) \right].
\end{equation}

Note that \( \rho \) is an entire function since \( \hat{\rho} \in C^\infty_0(\mathbb{R}) \). We also note that

\begin{equation}
\text{Tr} \Omega R_\rho(k + i\tau) = \sum_{j=1}^{\infty} \rho(k - \lambda_j - i\tau) + O(k^{-\infty}), \quad (k \to \infty, \tau > 0)
\end{equation}

since \( \lambda_j > 0 \) for all \( j \). This is essentially the same expression studied in [DG, GM].

Dual to the sum is the trace of the even part of the Dirichlet wave group, i.e. the distribution in \( t \) defined by

\[ \text{Tr} \Omega E_\Omega(t) := \int_\Omega E_\Omega(t, x, x) \, dx = \sum_{j=1}^{\infty} \cos t\lambda_j \]

where

\[ \Delta \phi_j = \lambda_j^2 \phi_j, \quad \phi_j|_{\partial \Omega} = 0, \quad \langle \phi_i, \phi_j \rangle = \delta_{ij}, \quad E_\Omega(t, x, y) = \sum_j \cos t\lambda_j \phi_j(x)\phi_j(y). \]

The singular support of the wave trace is contained in the set \( \text{Lsp}(\Omega) \) of lengths of generalized broken geodesics ([AM], [GM], Theorem; and [PS]): More precisely, for any bounded smooth domain, we have

\[ \text{singsupp Tr} \Omega E_\Omega(t) \subset \text{Lsp}(\Omega). \]

If \( \Omega \) belongs to a certain residual set \( \mathcal{R} \), then

\[ \text{singsupp Tr} \Omega E_\Omega(t) = \text{Lsp}(\Omega). \]

When \( L_\gamma \) is the length of a non-degenerate periodic reflecting ray \( \gamma \), and when \( L_\gamma \) is not the length of any other generalized periodic orbit, then \( \text{Tr} \Omega E_\Omega(t) \) is a Lagrangean distribution in the interval \((L_\gamma - \epsilon, L_\gamma + \epsilon)\) for sufficiently small \( \epsilon \), hence \( \text{Tr} \Omega E_\rho(k + i\tau) \) has a complete asymptotic expansion in powers of \( k^{-1} \). Let us recall the precise statement (see [GM], Theorem 1, and also page 228; see also [PS] Theorem 6.3.1).

Let \( \gamma \) be a non-degenerate billiard trajectory whose length \( L_\gamma \) is isolated and of multiplicity one in \( \text{Lsp}(\Omega) \). Then for \( t \) near \( L_\gamma \), the trace of the wave group has the singularity expansion

\[ \text{Tr} \Omega E_\Omega(t) \sim a_\gamma(t - L_\gamma + i0)^{-1} + a_{\gamma 0} \log(t - L_\gamma + i0) + \sum_{k=1}^{\infty} a_{\gamma k}(t - L_\gamma + i0)^k \log(t - L_\gamma + i0) \]

where the coefficients \( a_{\gamma k} \) (the wave trace invariants) are calculated by the stationary phase method from the Lagrangean parametrix \( \hat{E} \).

Recall that \( \hat{E}(t) \) is a microlocal parametrix in that it approximates \( E_\Omega(t) \) modulo regular kernels in a sufficiently small conic neighborhood \( \Gamma_L \) of \( \mathbb{R}^+ \gamma \).

We will need the following equivalent statement:
Corollary 3.1. Assume that $\gamma$ is a non-degenerate periodic reflecting ray, and let $\hat{\rho} \in C^\infty_0(L_\gamma - \epsilon, L_\gamma + \epsilon)$, equal to one on $(L_\gamma - \epsilon/2, L_\gamma + \epsilon/2)$ and with no other lengths in its support. Then $\text{Tr} 1_{\Omega} R_\rho(k + i\tau)$ admits a complete asymptotic expansion of the form (2). The coefficients $B_{\gamma,j}$ are canonically related to the wave invariants $a_{\gamma,j}$.

Proof. By (25), we have $R_\rho = E_\rho$ where

$$E_\rho(k + i\tau) := \int_{\mathbb{R}} e^{(ik - \gamma)t} \hat{\rho}(t) E_{\Omega}(t) dt. \tag{28}$$

The corollary thus follows immediately from the Poisson relation.

As mentioned in the introduction, we actually use a variant of this result:

Corollary 3.2. Under the same assumptions, $\text{Tr} 1_{\Omega} R_\rho(k + i\tau \log k)$ admits a complete asymptotic expansion of the form (2) with the same coefficients $B_{\gamma,j}$.

Proof. We use (28) but with $\tau \log k$ in place of $\tau$. We then substitute the microlocal parametrix $\hat{E}(t)$ and calculate

$$\text{Tr} 1_{\Omega} \int_{\mathbb{R}} e^{(ik - \gamma \log k)t} \hat{\rho}(t) \hat{E}_{\Omega}(t) dt$$

asymptotically by the stationary phase method. Since supp $\hat{\rho}$ is contained in $\mathbb{R}_+$, the factor $e^{-\tau \log kt}$ may be absorbed into the amplitude and decreases its order. The result follows exactly as in Corollary (3.1) by the stationary phase method.

3.3. Microlocal cutoff. In this section, we explain that the regularized wave trace expansion $\text{Tr} 1_{\Omega} R_\rho(k)$ at a periodic reflecting ray $\gamma$ can be microlocalized to $\gamma$, i.e. equals $\text{Tr} 1 R_\rho(k) \chi(k)$, where $\chi(k)$ is a (specially adapted) cutoff to $\gamma$. In principle this should be obvious, but we include some details since we were unable to find a suitable reference. For simplicity we assume that $\gamma$ is a bouncing ball orbit.

The cutoff consist of there terms,

$$\chi(k) = \chi_+(k) + \chi_0(k) + \chi_-(k)$$

corresponding to the top, middle and bottom of the orbit. Let $U = U_+ \cup U_0 \cup U_-$ be a small strip around $\pi(\gamma)$ in $\mathbb{R}^2$, with $U_\pm$ be a small neighborhood of $U \cap \partial \Omega_\pm$ (the top/bottom boundary component). Further, let $(\theta, r) \rightarrow q(\theta) + r \nu_q(\theta)$ denote Fermi normal coordinates along $U_+$ and we denote the dual symplectic coordinates by $p_+, p_\theta$. We use the same notation for $U_-$, anticipating that no confusion will arise. We then define semiclassical pseudodifferential cutoff operators of the form:

- $\chi_\pm(r, \theta, k^{-1}D_\theta)$ on $U_\pm$, with $\chi_\pm(r, \phi, p_\theta)$ supported in $U_+ \times \{ |p_\theta| < \epsilon \}$ and with $\chi_\pm(r, \theta, p_\theta) \equiv 1$ for all $0 \leq r < \epsilon, \phi \in (-\epsilon, \epsilon), |p_\theta| \leq \epsilon/2$.
- $\chi_0(x, k^{-1}D_x)$ is properly supported in $U_0 \times U_0$ and $\chi_0(x, \xi)$ is supported in a small neighborhood of $U_0 \times \{ (0, 10) \}$.

One can apply $\chi(k)$ to functions supported in $\overline{U \cap \Omega}$. In particular, the cutoff resolvent is defined near the boundary by

$$\chi_\pm(k) R_{\Omega, \rho}(k + i\tau)(r, \theta, r', \theta') = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ik(\theta - \theta')} p_\theta R_{\Omega}(k + i\tau, r, \theta'; r', \theta') \chi_\pm(r, \theta, p_\theta) dp_\theta d\theta'.$$
Naturally, the introduction of the cutoff does not change the trace modulo negligible terms. For the sake of completeness, we sketch the proof that the full resolvent trace is unchanged.

**Lemma 3.3.** \( \text{Tr} \Omega R_{\Omega, \rho}(k + i\tau \log k) - \text{Tr} \Omega \chi(k) R_{\Omega, \rho}(k + i\tau \log k) = O(k^{-\infty}). \)

**Proof.** It is known [AM] (see also [GM]) that
\[
\text{Tr} \Omega R_{\Omega, \rho}(k + i\tau \log k) \sim \text{Tr} \Omega \tilde{R}_{\Omega, \rho}(k + i\tau),
\]
where \( \tilde{R}_{\Omega, \rho}(k + i\tau) \) is a microlocal parametrix for the semiclassical Dirichlet resolvent in neighborhood of \( \gamma \). To be precise, Andersson-Melrose [AM] proved that there exists a microlocal parametrix for \( E_D(t) \), the even Dirichlet wave group, near any transversal reflecting ray. The Fourier-Laplace transform of this parametrix is a semiclassical resolvent parametrix. The parametrix is defined throughout \( U \).

We verify this using (25). First, we may write
\[
(29) \quad R_{\rho D}(k)\chi(k) = \int_0^{\infty} \hat{\rho}(t) E_D(t) \chi(r, y, |D_t|^{-1}D_y)e^{i(k + i\tau)t} dt. \]

Calculating the expansion in the statement of the Lemma is the same as computing the wave front set of the trace of (29) near \( rL_\gamma \). Thus, the statement is equivalent to saying that
\[
(30) \quad \text{WF}[\text{Tr} E_D(t)(I - \chi(r, y, |D_t|^{-1}D_y)) \cap (rL_\gamma - \epsilon, rL_\gamma + \epsilon)] = \emptyset. \]

To prove this, we recall that \( \text{WF}(E_D(t, x, y)) \) is the space-time graph
\[
(31) \quad \Gamma = \{(t, \tau, x, \xi, x', \xi') \in T^*(\mathbb{R} \times \Omega^c \times \Omega^c) : \tau = -|\xi|, \quad G^t(x, \xi) = (x', \xi')\},
\]
of the generalized billiard flow. Now in Fermi coordinates \( (r, \rho dr, y, \eta dy) \), \( \mathbb{R}^+ \gamma \) is ray in the direction of \( dr \). Since the space-time graph of the cotangent bundle along \( \gamma \) may be described in normal coordinates as a neighborhood of
\[
\{(t, \tau, t, \tau, 0, 0)\},
\]
a conic neighborhood may be described in these coordinates by \(|y| \leq \epsilon, |\eta/\tau| \leq \epsilon\). This is precisely the set to which \( \tilde{\chi}_\gamma(r, y, |D_t|^{-1}D_y) \) microlocalizes. Emptyness of the WF in (30) follows from the calculus of wave front sets, which implies that only diagonal points in \( \Gamma \) contribute, i.e. periodic orbits of the billiard flow, and from our assumption that \( \gamma^r \) is the only orbit with period in the given set.

The only potentially confusing issue is in the choice of cutoff operator, which must be rather special since it operators on a manifold with boundary. Let us verify in another way that this kind of cutoff operator acts as a microlocal cutoff to \( \gamma \).

According to [GM] [AM], we can calculate the trace using a microlocal parametrix
\[
\tilde{E}(t, x, y) = \int_{\mathbb{R}^2} e^{i\phi(t, x, y, \xi)} a(t, x, y, \xi) d\xi
\]
for the Dirichlet wave kernel. We take its Fourier-Laplace transform to get a semiclassical microlocal parametrix

\[ \tilde{R}_{\Omega,\rho}(k + i\tau, x, y) = \int_{\mathbb{R}^2} \int_0^\infty \hat{\rho}(t) e^{i(k + i\tau)t} e^{i\phi(t, x, y, \xi)} a(t, x, y, \xi) d\xi. \]

We now change variables \( \xi \rightarrow k\xi \) to obtain

\[ \tilde{R}_{\Omega,\rho}(k + i\tau, x, y) = k^2 \int_{\mathbb{R}^2} \int_0^\infty \hat{\rho}(t) e^{ik\phi(t, x, y, \xi)} e^{-\tau t} a(t, x, y, k\xi) d\xi, \]

where the phase is \( \Phi = t + \phi(t, x, y, \xi) \). We now apply \( \chi(k) \) to get

\[ \chi(k) \tilde{R}_{\Omega,\rho}(k + i\tau, x, y) = k^2 \int_{\mathbb{R}^2} \int_0^\infty \hat{\rho}(t) e^{ik[t + \phi(t, x, y, \xi)]} \tilde{\chi}(x, d_x\phi) e^{-\tau t} a(t, x, y, k\xi) d\xi, \]

where

\[ \tilde{\chi}(x, d_x\Phi) = \chi(x, d_x\phi) A(k, x, d_x\phi) \]

with \( A \) a symbol of order 0. The assumption on the cutoff implies that \( \tilde{\chi} \equiv 1 \) near \( \gamma \). Since the trace is computed by applying stationary phase to the trace of this oscillatory integral, it is unchanged modulo rapidly decaying errors by the cutoff.

4. Multiple Reflection Expansion of the Resolvent

The purpose of this section is to review the ‘multiple-reflection’ expansion of the Dirichlet Green’s function of a bounded plane domain. This is the term in [BB1, BB2] for the Neumann series expression for the Dirichlet Green’s function in terms of double layer potentials. The same method also works for Neumann boundary conditions, but for simplicity we only explicitly treat the Dirichlet case. We refer to [T], Chapter 5 for background in potential theory.

The method of layer potentials ([T], II, § 7. 11) seeks to solve (23) in terms of the ‘layer potentials’ \( G_0(k + i\tau, x, q), \partial_\nu G_0(k + i\tau, x, q) \in \mathcal{D}'(\Omega \times \partial\Omega) \), where \( \nu \) is the interior unit normal to \( \Omega \), where \( \partial_\nu = \nu \cdot \nabla \), and where \( G_0(k + i\tau, x, y) \) is the ‘free’ Green’s function of \( \mathbb{R}^2 \), i.e. of the kernel of the free resolvent \( -(\Delta_0 + (k + i\tau)^2)^{-1} \) of the Laplacian \( \Delta_0 \) on \( \mathbb{R}^2 \). The free Green’s function in dimension two is given by:

\[ G_0(k + i\tau, x, y) = H_0^{(1)}((k + i\tau)|x - y|) = \int_{\mathbb{R}^2} e^{i(x - y, \xi)}(|\xi|^2 - (k + i\tau)^2)^{-1} d\xi. \]

Here, \( H_0^{(1)}(z) \) is the Hankel function of index 0. In general, the Hankel function of index \( \nu \) has the integral representations ([T], Chapter 3, §6)

\[ H_\nu^{(1)}(z) = 2e^{-iz\nu} \frac{\Gamma(\nu + 1/2)}{i\nu \Gamma(\nu + 1/2)} (\frac{z}{2})^\nu \int_1^\infty e^{iz(t^2 - 1)^{\nu - 1/2}} dt \]

\[ = \left( \frac{2}{\pi^2} \right)^{1/2} e^{iz^2} \frac{1}{i\nu} \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu + 1/2)} \int_0^\infty e^{-sz} s^{-1/2} (1 - \frac{s}{2z})^{-\nu - 1/2} ds. \]

From the first, resp. second, representation we derive the asymptotics:

\[ H_\nu^{(1)}((k + i\tau)|x - y|) \sim \begin{cases} -\frac{1}{2\pi} \ln(|k + i\tau||x - y|) & \text{as } |k + i\tau||x - y| \rightarrow 0, \text{ if } \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi} \frac{2i^{\nu}}{|k + i\tau||x - y|^\nu} & \text{as } |k + i\tau||x - y| \rightarrow 0, \text{ if } \nu > 0 \\ e^{i((k + i\tau)|x - y| - \nu\pi/2 - \pi/4)} (\frac{1}{|k + i\tau||x - y|^\nu})^{1/2} & \text{as } |k + i\tau||x - y| \rightarrow \infty. \end{cases} \]
Here it is assumed that $\tau > 0$.

The single layer, respectively double layer, potentials are the operators

$$\mathcal{S}(k+i\tau)f(x) = \int_{\partial\Omega} G_0(k+i\tau,x,q)f(q)ds(q), \quad \mathcal{D}(k+i\tau)f(x) = \int_{\partial\Omega} \frac{\partial}{\partial q} G_0(k+i\tau,x,q)f(q)ds(q),$$

where $ds(q)$ is the arc-length measure on $\partial\Omega$. They induce boundary operators

$$(i) \quad S(k+i\tau)f(q) = \int_{\partial\Omega} G_0(k+i\tau,q,q')f(q')ds(q'),$$

$$(ii) \quad N(k+i\tau)f(q) = 2\int_{\partial\Omega} \frac{\partial}{\partial q} G_0(k+i\tau,q,q')f(q')ds(q')$$

which map $H^s(\partial\Omega) \rightarrow H^{s+1}(\partial\Omega)$. Furthermore one has (cf. [T II, Proposition 11.5])

$$(i) \quad S(k+i\tau), N(k+i\tau) \in \Psi^{-1}(\partial\Omega),$$

$$(ii) \quad (I + N(k+i\tau)) : H^s(\partial\Omega) \rightarrow H^s(\partial\Omega) \text{ is an isomorphism.}$$

To understand the role of these operators, it helps to recall that the Poisson integral operator

$$PI_\Omega(k+i\tau) : H^s(\partial\Omega) \rightarrow H^{s+1/2}(\Omega), \quad PI_\Omega(k+i\tau)u(x) = \int_{\partial\Omega} \partial_\nu G_\Omega(k+i\tau,x,q)u(q)ds(q)$$

may be expressed in the form

$$PI_\Omega(k+i\tau) = 2\mathcal{D}(k+i\tau)(I + N(k+i\tau))^{-1}. \tag{38}$$

We refer to [T I (see Chapter 5, Proposition 1.7; see also [LM]) for background on layer potentials.

We include some other facts about the kernel $N(k+i\tau)$ which will be used to estimate remainders in the multiple reflection expansion (see Proposition [12] and Lemma [32]). Further and more precise estimates will be given in [Z5].

**Proposition 4.1.** Suppose that $\partial\Omega$ is $C^1$. Then:

(i) $N(k+i\tau,q(\phi_1),q(\phi_2)) \in C^{1-\epsilon}(T \times T)$.

(ii) $N(k+i\tau)$ is a Hilbert-Schmidt operator on $L^2(\partial\Omega)$, with $\|N(k+i\tau)\|_{HS} \leq C|k|^{1/2}$.

(iii) $N(k+i\tau) \in \Psi^{-2}(\partial\Omega)$, hence it is a trace-class operator on $L^2(\partial\Omega)$.

**Proof.** (i) By definition,

$$N(k+i\tau,q(\phi_1),q(\phi_2)) = \partial_{\nu_\nu} G_0(k+i\tau,q(\phi_1),y)|_{y=q(\phi_2)}$$

$$= -(k+i\tau)H_1^{(1)}(|k+i\tau||q(\phi_1)-q(\phi_2)||) \cos \angle(q(\phi_2)-q(\phi_1),\nu_q(\phi_2)).$$

Now at $r \sim 0$, we have (see [32], also [AS, 9.1] and [EP]),

$$H_1^{(1)}(r) = \frac{2i}{\pi r} + O(r(1 + \log r)). \tag{39}$$

We correspondingly define

$$N(k+i\tau) = N_{\text{sing}} + N_{\text{reg}}(k+i\tau), \quad \text{with}$$

$$N_{\text{sing}}(\phi,\phi') = \frac{1}{2\pi|q(\phi)-q(\phi')|^2} \nu_q(\phi') \cdot \frac{q(\phi)-q(\phi')}{|q(\phi)-q(\phi')|} = -\frac{1}{4\pi} \kappa(\phi) + O((\phi - \phi')).$$
In fact, the vanishing of the numerator to order 2 implies that \( N_{\text{sing}} \) is a \( C^\infty \) kernel. Hence the smoothness of \( N(k + i\tau, q(\phi_1), q(\phi_2)) \) equals that of \( N_{\text{reg}}(k + i\tau, q(\phi_1), q(\phi_2)) \). It follows that the kernel has the regularity of \( x \log x \), so \( N(k + i\tau, q(\phi_1), q(\phi_2)) \) is Lipschitz continuous of any exponent \( \alpha < 1 \).

(ii) Obviously the kernel is Hilbert-Schmidt. We need to estimate

\[
(40) \quad \int_{\partial \Omega \times \partial \Omega} |N(k + i\tau, q_1, q_2)|^2 ds(q_1)ds(q_2).
\]

For the sake of brevity, we only sketch the norm estimate, referring to [25] for further details. We break up the domain of integration into the three regions

\[
|q - q'| \leq |k + i\tau|^{-1}, \quad |k + i\tau|^{-1} \leq |q - q'| \leq |k + i\tau|^{-2/3}, \quad |q - q'| \geq |k + i\tau|^{-2/3}.
\]

In the first (near diagonal) region, the kernel is bounded above by a constant independent of \( k + i\tau \). Indeed, both the singularity and the factor of \( (k + i\tau) \) cancel in the most singular term, as can be seen from (39) or (33). The smoother terms also cancel the factor of \( (k + i\tau) \) in region (i).

In region (ii), the Hankel factor has a uniform upper bound. The cosine factor puts in \( |q - q'| \). Hence, in any region \( |k + i\tau|^{-1} \leq |q - q'| \leq |k + i\tau|^{-r} \) the integral is dominated by

\[
|k + i\tau|^2 \int_0^{|k + i\tau|^{-r}} x^2 dx = O(|k + i\tau|^{2-3r}).
\]

So it is uniformly bounded if \( r \geq 2/3 \).

In region (iii), we can (and will) use the WKB expansion of the Hankel function. The square of the Hankel function contributes the factor \( |(k + i\tau)|^{-1} \), cancelling one power in (41). Taking the square root of the result gives the stated estimate. \( \square \)

(iii) It is standard that \( N(k + i\tau) \in \Psi^{-1}(\partial \Omega) \). The smoothness of \( N_{\text{sing}} \) noted in (i) shows that the symbol of \( N \) of order \(-1\) vanishes; hence, it is actually of order \(-2\). Therefore, it is of trace class.

An alternative proof is by the Hille-Tamarkin theorem (cf. [30]) theorem, which states that a Hilbert-Schmidt kernel \( K(x, y) \) on an interval (or circle) is trace class if it is \( \text{Lip}_\alpha \) in one of its variables with \( \alpha > 1/2 \).

This completes the proof of the Proposition. \( \square \)

In [25], we prove that the operator norm of \( N(k + i\tau) \) on \( L^2(\Omega) \) is uniformly bounded in \( k \) for \( \tau > 0 \). On the other hand, it appears that \( N(k + i\tau) \) is not a contraction for any fixed \( \tau > 0 \).

4.1. **Multiple-reflection expansion: Dirichlet boundary conditions.** To solve (23), one puts \( \Gamma(k + i\tau, x, y) = G_\Omega(k + i\tau, x, y) - G_0(k + i\tau, x, y) \) and solves the boundary value problem

\[
(41) \quad \begin{cases}
(\Delta_x - (k + i\tau)^2)\Gamma(k + i\tau, x, y) = 0, \quad x, y \in \Omega \\
r_\Omega \Gamma(k + i\tau, q, y) = -r_\Omega G_0(k + i\tau, q, y), \quad q \in \partial \Omega.
\end{cases}
\]

Here,

\[
r_\Omega u = u|_{\partial \Omega}
\]
is the restriction operator acting in the first variable.

We try to solve (41) with \( \Gamma(k + i\tau, \cdot, y) \) of the form: \( \Gamma(k + i\tau, x, y) = D\ell(k + i\tau)\mu \). If we denote by

\[
    f_+(q) = \lim_{x \to q, x \in \Omega} f(x) \in C(\Omega),
\]

then one has (cf. \( \text{[II]} \) II, Proposition 11.1)

\[
    [D\ell(k + i\tau)f]_+(x) = 1/2f(x) + 1/2N(k + i\tau)f(x).
\]

Hence the equation for \( \mu(k + i\tau, q, y) \) is given by:

\[
    (I + N(k + i\tau))\mu(k + i\tau, q, y) = -r_\Omega G_0(k + i\tau, q, y), \quad (q, y) \in \partial\Omega \times \Omega.
\]

By (36) there exists a unique solution,

\[
    \mu(k + i\tau, q, y) = -(I + N(k + i\tau))^{-1}r_\Omega G_0(k + i\tau, q, y), \quad (q, y) \in \partial\Omega \times \Omega,
\]

where by (35), \( (I + N(k + i\tau))^{-1} \in \Psi^0(\partial\Omega) \). It follows that

\[
    R_\Omega(k + i\tau) = R_\Omega(0) + D\ell(k + i\tau)(I + N(k + i\tau))^{-1}r_\Omega R_\Omega(0 + i\tau)
\]

More precisely, the kernels are related by the Neumann series

\[
    G_\Omega(k + i\tau, x, y) = \sum_{M=0}^{\infty}(-2)^M G_M(k + i\tau, x, y), \quad \text{where, for } M \geq 1
\]

\[
    G_M(k + i\tau, x, y) = \int_{\partial\Omega} \frac{\partial}{\partial y} G_0(k + i\tau, x, q_1) \Pi_{j=1}^{M-1} \frac{\partial}{\partial y} G_0(k + i\tau, q_j, q_{j+1}) G_0(k + i\tau, q_M, y) \Pi_{j=1}^{M} ds(q_j).
\]

Here, \( \frac{\partial}{\partial y} \) is short for \( \frac{\partial}{\partial \nu_y} \). This expansion of \( G_\Omega(k + i\tau, x, y) \) of \( R_\Omega(k + i\tau) \) is referred to in \( \text{[BB1]} \) as the multiple-reflection expansion. We regularize \( G_M \) to \( G_{M,\rho} \) as in (45). By substituting in the multiple reflection expansion (38), we obtain a multiple reflection expansion for \( R_\rho \).

In the following proposition, we assume the boundary is smooth since we are dealing with analytic boundaries. Some additional work is needed if the boundary is only assumed piecewise smooth, as in [4].

**Proposition 4.2.** Suppose that \( \partial\Omega \) is \( C^\infty \). Then, for \( \tau \geq 0 \), \( G_{M,\rho}(k + i\tau, x, y) \) defines for each \( M \) a trace class operator on \( L^2(\Omega) \).

**Proof.** First, the \( M = 0 \) term \( \int_{\mathbb{R}} \rho(k - \mu)1_\Omega(\mu + i\tau)R_\rho(\mu + i\tau)1_\Omega d\mu \) is easily seen to be trace class since it is the restriction of a smoothing operator to \( \Omega \).

Next, we observe that for \( M \geq 1 \), \( G_M(k + i\tau, x, y) \) is the Schwartz kernel of \( D\ell(k + i\tau)N(k + i\tau)^{M-1}r_\Omega R_\Omega(0 + i\tau)1_\Omega \). Let us consider each factor. Using (38), it follows that

\[
    D\ell(k + i\tau) : H^s(\partial\Omega) \to H^{s+1/2}(\Omega) \text{ continuously.}
\]

Further (see [II], Chapter 4, Proposition 4.5), the restriction operator satisfies \( r_\Omega : H^s(\Omega) \to H^{s-1/2}(\partial\Omega) \), for \( s > 0 \). (See also [LM]). And \( 1_\Omega R_\Omega(0 + i\tau)1_\Omega : H^s(\Omega) \to H^{s+2}(\Omega) \).

It follows that for \( M \geq 2 \),

\[
    D\ell(k + i\tau)N(k + i\tau)^{M-1}r_\Omega R_\Omega(0 + i\tau)1_\Omega : H^s(\Omega) \to H^{s+2(M-1)+2}(\Omega).
\]

Hence, this operator is certainly of trace class on \( L^2(\partial\Omega) \) if \( M \geq 2 \).
The case \( M = 1 \) is different from the others, so it seems worth considering it separately. The operator \( D(\mu)Y_\Omega R_0(\mu)\Omega_1 \) has Schwartz kernel
\[
\int_{\mathbb{R}} \int_{y_\Omega} \rho(k - \mu) (\mu + i\tau) \partial_{\nu_\Omega} G_0(\mu, x, q(\phi)) G_0(\mu, \phi, y) d\phi.
\]
The double layer potential has kernel
\[
\partial_{\nu_\Omega} G_0(\mu, x, q(\phi)) = - (\mu + i\tau) H^{(1)}_1((\mu + i\tau)|q(\phi_1) - x|) \cos \angle(q(\phi) - x, \nu_\phi).
\]
and as for \( N(k + i\tau) \) above, we write
\[
\partial_{\nu_\Omega} G_0(k + i\tau, x, q(\phi)) = K_{reg}(k + i\tau, x, q(\phi)) + K_{sing}(k + i\tau, x, q(\phi))
\]
where
\[
K_{sing}(k + i\tau, x, q(\phi)) = \frac{1}{2\pi} \frac{1}{x - q(\phi)} \cdot \nu_\phi.
\]
It is clear that
\[
\int_{\mathbb{R}} K_{reg}(\mu + i\tau, x, q(\phi)) G_0(\mu + i\tau, q(\phi), y) d\phi
\]
is a composition of a Hilbert-Schmidt kernel from \( L^2(\Omega) \to L^2(\partial\Omega) \) and one from \( L^2(\partial\Omega) \to L^2(\Omega) \). Hence, this term is of trace class. The \( K_{sing} \) term has kernel
\[
\int_{\mathbb{R}} \frac{1}{|x - q(\phi)|} \cdot \nu_\phi \{ \int_{\mathbb{R}} \rho(k - \mu)(\mu + i\tau) G_0(\mu + i\tau, q(\phi), y) d\mu \} d\phi
\]
is of trace class. However, the bracketed operator is a smoothing operator from \( L^2(\Omega) \to L^2(\partial\Omega) \). Hence, the composition is of trace class.

\[\square\]

4.2. The operators \( N_0 \) and \( N_1 \). We now describe \( N_1(k + i\tau) \) of \( \mathbb{R} - \Omega \) as a semiclassical Fourier integral operator. For related results, see [HZ]. We denote by \( \chi \) a smooth bump function which is supported on \([-1, 1]\) and equals 1 on \([-1/2, 1/2]\). We also now put in \( h = k^{-1+\delta} \).

We denote by \( S^p_\delta(T^m) \) the class of symbols \( a(k, \phi_1, \ldots, \phi_m) \) satisfying:
\[
(|k^{-1}D_\phi|^\alpha a(k, \phi)| \leq C_\alpha |k|^{-p\delta|\alpha|}, \quad (|k| \geq 1).
\]

**Proposition 4.3.** \( N_1(k + i\tau) \) is a semiclassical Fourier integral operator of order 0 associated to the billiard map. More precisely,
\[
N_1(k + i\tau, q(\phi_1), q(\phi_2)) = (1 - \chi(k^{1-\delta}(\phi_1 - \phi_2)))
\]
\[
(1 + i\tau)^\frac{1}{2} a_1(k + i\tau, q(\phi_1), q(\phi_2)) e^{i(k + i\tau)|q(\phi_1) - q(\phi_2)|}
\]
with \( a_1(k + i\tau, q(\phi_1), q(\phi_2)) \in S_\delta^0(T^2) \).

**Proof.** We begin by analyzing the amplitudes of the Hankel functions.

**Lemma 4.4.** There exist amplitudes \( a_0, a_1 \) such that:

(i) \( H_0^{(1)}((k + i\tau)z) = e^{i(k + i\tau)z} a_0((k + i\tau)z), \) where \( (1 - \chi(k^{1-\delta}z)) a_0((k + i\tau)z) \in S_\delta^{-1/2}(\mathbb{R}) \);

(ii) \( (k + i\tau) H_1^{(1)}((k + i\tau)z) = (k + i\tau)^\frac{1}{2} e^{i(k + i\tau)z} a_1((k + i\tau)z), \) with \( (1 - \chi(k^{1-\delta}z)) a_1((k + i\tau)z) \in S_\delta^0(\mathbb{R}) \).
Proof. By the explicit formula (35),

\[ a_0((k + i\tau)z) = \left(\frac{2}{\pi(k + i\tau)z}\right)^{1/2} \int_0^\infty e^{-s}s^{-1/2}(1 - \frac{s}{2i(k + i\tau)z})^{-1/2}ds. \]

We claim that the integral defines a polyhomogeneous symbol of order 0 in \(|(k + i\tau)z| \to \infty\). Indeed, applying the binomial theorem to the factor \((1 - \frac{s}{2i(k + i\tau)z})^{-1/2}\) gives

\[ a_0((k + i\tau)z) = \left(\frac{2}{\pi(k + i\tau)z}\right)^{1/2} \sum_{j=0}^{N} C(-1/2, j)(\frac{s}{2i(k + i\tau)z})^j ds + R_N((k + i\tau)z) \]

where \(C(-1/2, j)\) are binomial coefficients, where \(c_j\) are the resulting constants, and where \(R_N(s, (k + i\tau)z) = \int_0^\infty e^{-s}s^{-1/2}R_N((k + i\tau)z, s)ds\) with \(R_N((k + i\tau)z, s)\) the \(N\)th order remainder in the Taylor series expansion of \((1 - \frac{s}{2i(k + i\tau)z})^{-1/2}\). It is evident that \(R_N(s, (k + i\tau)z) = O(|(k + i\tau)z|^{-(N+1/2)})\). Differentiating with \(k^{-1}D_z\) similarly gives

\[ (k^{-1}D_z)^\alpha a_0((k + i\tau)z) = \sum_{j=0}^{N} c_j((k + i\tau)z)^{-j-1/2-|\alpha|} + (k^{-1}D_z)^\alpha R_N(kz) \]

\[ = O(|kz|^{-1/2+|\alpha|}). \]

Now \(1 - \chi(k^{1-\delta}z)\) clearly belongs to \(S_0^0(\mathbb{R})\). Since \(|kz| \geq k\delta\) on supp \(1 - \chi(k^{1-\delta}z)\), we conclude that

\[ (k^{-1}D_z)^\alpha(1 - \chi(k^{1-\delta}z))a((k + i\tau)z) = O(k^{-1/2+|\alpha|\delta}), \]

proving (i).

By definition, \(H_1^{(1)}(z) = -\frac{d}{dz}H_0^{(1)}(z)\), so (ii) follows immediately from (i).

We now complete the proof of the Proposition. The amplitude of \(N_1\) is then

\[ a_1(k + i\tau, q(\phi_1), q(\phi_2)) := a_1((k + i\tau)|q(\phi_1) - q(\phi_2)|) \cos \vartheta_{1,2}, \]

where \(\vartheta_{1,2} = \angle(q(\phi_2) - q(\phi_1), \nu(q(\phi_2)))\).

Since \(q : T \to \partial\Omega\) is smooth, the metrics \(|\phi_1 - \phi_2|_T\) and \(|q(\phi_1) - q(\phi_2)|_{\mathbb{R}^2}\) are equivalent, and we use the former. Thus, we need to check that

\[ |(k^{-1}D_\phi)^n a_1((k + i\tau)|q(\phi_1) - q(\phi_2)|) \cos \vartheta_{1,2}| \leq Ck^{-n\delta}. \]

By repeatedly differentiating (18) away from the diagonal, we find

\[ |D_\phi^a(q(\phi_1) - q(\phi_2))| \leq C_\alpha|\phi_1 - \phi_2|^{1-|\alpha|}. \]

It follows from (18) and by the chain rule that

\[ (k^{-1}D_\phi)^a a_1((k + i\tau)|q(\phi_1) - q(\phi_2)|) = \sum_{j=0}^{N} (k^{-1}D_z)^j a_1((k + i\tau)z)|_{z=|q(\phi_1) - q(\phi_2)|} \times \sum_{\gamma_1,...,\gamma_j, |\gamma| = |\alpha| - j} C_{\alpha, \gamma_j} \Pi_{l=1}^{j} (k^{-1}D_\phi)^{\gamma_l} \sin \vartheta_{1,2}. \]

By Lemma (18)(ii),

\[ |(1 - \chi(k^{1-\delta}(\phi_1 - \phi_2)))(k^{-1}D_z)^j a_1((k + i\tau)z)|_{z=|q(\phi_1) - q(\phi_2)|} \leq C_j k^{-j\delta}. \]
4.3. **Layer potentials as semi-classical Fourier integral operators.** In this section, we give a description of the layer potentials $S\ell, D\ell$ as Fourier integral operators from $L^2(\partial\Omega) \to L^2(\Omega)$ which parallels that of Proposition 4.3. These operators also appear in the trace formula, and we will be needing the results of this section in §7. First, we must introduce suitable cutoff operators away from the diagonal.

To define the cutoff operator, we need to introduce coordinates. We separate $\Omega$ into two zones depending on the distance $r(x, \partial\Omega)$ to $\partial\Omega$. We set

$$
\begin{cases}
(\partial\Omega)_\epsilon &= \{ x \in \Omega : r(x, \partial\Omega) < \epsilon \} \\
\Omega_\epsilon &= \Omega \setminus (\partial\Omega)_\epsilon.
\end{cases}
$$

Here, $\epsilon$ is sufficiently small so that the exponential map

$$
\exp : N^\perp(\partial\Omega) \to \Omega, \quad \exp_{q(\phi_0)} r\nu_{q(\phi_0)} = q(\phi_0) + r\nu_{q(\phi_0)}
$$

from the inward normal bundle along the boundary to the interior is a diffeomorphism from vectors of length $< \epsilon$ to $(\partial\Omega)_\epsilon$. Also, as above, $\nu_{q(\phi_0)}$ is the interior unit normal at $q(\phi_0)$. The exponential map induces Fermi normal coordinates $(\phi, r)$ on the annulus $(\partial\Omega)_\epsilon$, with $r = r(x, \partial\Omega)$ the distance from $x$ to $\partial\Omega$. We denote the Jacobian of the exponential map along the boundary by $J(\phi, r)$. We introduce the corresponding cutoff

$$
\chi_{\partial\Omega}(x) := \chi(1 - \delta^{-1} r),
$$

where it is understood that the cutoff is supported in a tube around the boundary where the Fermi normal coordinates are defined. We further introduce an angular cutoff as for the boundary integral operators. Near the boundary we may use Fermi normal coordinates $(r, \theta)$ and we introduce $\chi(1 - \delta^{-1} (\theta - \phi))$ where $q = q(\phi)$ in arclength coordinates.

We then break up each layer potential into several pieces using the cutoffs. For instance, we first make a radial cutoff of the single layer:

$$
\chi_{\partial\Omega}(x)S\ell + (1 - \chi_{\partial\Omega}(x))S\ell.
$$

We then further break up $\chi_{\partial\Omega}(x)S\ell$ into

$$
\chi_{\partial\Omega}(x)S\ell(k + i\tau, (r, \theta), q(\phi)) = \chi_{\partial\Omega}(x)\chi(1 - \delta^{-1} (\theta - \phi))S\ell(k + i\tau, (r, \theta), q(\phi))
$$

$$
+ \chi_{\partial\Omega}(x)(1 - \chi(1 - \delta^{-1} (\theta - \phi)))S\ell(k + i\tau, (r, \theta), q(\phi)).
$$

We do likewise with the double layer potential.

We now show that, when suitably cutoff away from the diagonal singularities, the operators $S\ell, D\ell$ are semiclassical Fourier integral operators with phase function $|x - q| : \Omega \times \partial\Omega \to \mathbb{R}$. 

Since $|\phi_1 - \phi_2| \geq k^{-1+\delta}$ on supp $(1 - \chi(k^{-1-\delta}(\phi_1 - \phi_2)))$ it follows from (53) that

$$
|1 - \chi(k^{-1-\delta}(\phi_1 - \phi_2))(1 + D_{\phi})\gamma_{\ell}\sin \theta | \leq C|\gamma|\delta^{-1}.
$$

The same kind of estimate is correct for the factor $\cos \theta_{12}$. The proof follows from the combination of (51), (52), (53) with the fact that $(1 - \chi(k^{-1-\delta}(\phi_1 - \phi_2))) \in S^0(T^2)$. 

$\square$
The phase parametrizes the canonical relation
\[(61) \Gamma = \{(x, \frac{x-q}{|x-q|}, q, -\frac{x-q}{|x-q|} \cdot T_q) \} \subset T^*\Omega \times T^*\partial\Omega\]
which is the graph of the interior-to-boundary billiard map \(\beta\), which takes an interior vector \((x, \frac{x-q}{|x-q|})\) to the tangential component of the tangent vector(s) to the billiard ray it generates at its intersection point(s) with the boundary. The natural projections \(p : \Gamma \to T^*\Omega\), resp. \(q : \Gamma \to T^*\partial\Omega\) have singularities at points where the billiard ray intersects the boundary tangentially. Away from these singular points, \(p\) resp. \(q\) are submersions with discrete, resp. 1 dimensional fibers. Phases of this type arose in the work of Carles on-Sjolin (though not in relation to boundary value problems) and are discussed in detail in Sogge [So]. Since we are cutting off the diagonal we have eliminated grazing rays. In the case of convex domains, the cutoff layer potentials are non-degenerate Fourier integral operators, while in the non-convex they have degeneracies at points \((x, \frac{x-q}{|x-q|}, q, 0) \in \Gamma\). These too will be eliminated when we microlocalize to the orbit \(\gamma\).

We have:

**Proposition 4.5.** The operators
\[\begin{align*}
(1 - \chi_{\partial\Omega}^{k-1+\delta}(x))& \mathcal{D}(k+i\tau) \quad (\text{resp. } \mathcal{S}(k+i\tau)); \\
\chi_{\partial\Omega}^{k-1+\delta}(x)(1 - \chi(k^{1-\delta}(\theta - \phi)))& \mathcal{D}((k + i\tau, (r, \theta), q(\phi))) \quad (\text{resp. } \mathcal{S})
\end{align*}\]
are semiclassical Fourier integral operator of order \(-1/4\) (\(\mathcal{D}\)), resp. \(-5/4\) (\(\mathcal{S}\)), associated to the canonical relation \(\Gamma\). More precisely, there exist amplitudes such that
\[\begin{align*}
(i) \quad (1 - \chi_{\partial\Omega}^{k-1+\delta}(x))& \mathcal{D}(k+i\tau, x, q(\phi)) \\
& = (1 - \chi_{\partial\Omega}^{k-1+\delta}(x))(k+i\tau)^{\frac{3}{2}}A_1((k + i\tau), x, q(\phi))e^{i(k+i\tau)|x-q(\phi)|}, \\
(ii) \quad \chi_{\partial\Omega}^{k-1+\delta}(r)(1 - \chi(k^{1-\delta}(\phi - \theta)))& \mathcal{D}((k + i\tau, x, q(\phi))) \\
& = (k+i\tau)^{\frac{3}{2}}A_2(k+i\tau, r, \theta, \phi)e^{i(k+i\tau)q(\theta)+\nu_q(\phi)-q(\phi)}
\end{align*}\]
with \(A_j(k+i\tau, r, \theta, \phi) \in S^0_\ell(T^2)\). The analogous statements are true for \(\mathcal{S}\).

**Proof.** The proof is similar to that of Proposition 4.3. Due to the cutoffs, the kernel is in its semiclassical regime where the WKB approximation is valid and the orders of the amplitudes can be read off from Lemma 4.3. After substituting \(z = |x - q(\phi)|\) we get the amplitudes:

\[\begin{align*}
& a_0((k+i\tau)|x - q(\phi)|) \text{ for } \mathcal{S}\; \text{;} \\
& (k+i\tau)a_1((k+i\tau)|x - q(\phi)|) \cos \angle(x - q(\phi), \nu_{q(\phi)}) \text{ for } \mathcal{D}.
\end{align*}\]

We note that the order convention on Fourier integral operators operating between spaces of unequal dimensions \(n_1, n_2\) is that \(k^{n_1/4+n_2/4}\) times an amplitude of order 0 on the stationary phase set defines an operator of order 0. Thus, the cutoff \(\mathcal{D}\) has an order of 1/4 less than \(N_1\). Since the normal derivative introduces a factor of \((k+i\tau)\) in \(\mathcal{D}\) but not in \(\mathcal{S}\), the order of \(\mathcal{D}\) is one higher than \(\mathcal{S}\).

To complete the proof, we need to check, that differentiations in \(k^{-1}D_x, k^{-1}D_\phi\) in case (i), resp. \(k^{-1}D_\theta, k^{-1}D_r, k^{-1}D_\phi\) in case (ii), lower the order of the amplitudes by \(k^{-\delta}\). The only difference to Proposition 4.3 is that \(\angle q(\phi_2) - q(\phi_1), \nu_{q(\phi_2)}\) there is replaced by \(\angle(x -
$q(\phi), \nu_{q(\phi)} = \angle(q(\theta) + r\nu_{q(\theta)} - q(\phi), \nu_{q(\phi)})$. As before, each derivative at most increases the number of factors of $|x - q(\phi)|^{-1}$ by one. On the support of the cutoff, each such factor counts $k^{1-\delta}$. Due to the accompanying factor of $k^{-1}$ we see that each derivative $k^{-1}D$ decreases the order of the amplitude by $k^{-\delta}$.

4.4. Integral formulas. In §8, we are going to need an asymptotic formula and remainder estimate for some integrals involving Hankel functions. The reasons for the assumptions on the parameters $a, b$ will be clarified at that time. The constraint $1 > \delta > 1/2$ on $h = k^{-1+\delta}$ originates in the following.

**Proposition 4.6.** Let $a \in (-1, 1)$, let $\Re b \geq 1$, $\Im b > 0$ and let $R \in \mathbb{N}$. Also, let $\chi \in C_0^\infty(\mathbb{R})$ be an even cutoff function, equal to one near 0. Then

$$
\int_0^\infty \chi(k^{-\delta}x) \cos(ax)H_0^{(1)}(bx) dx = \frac{1}{\sqrt{b^2 - a^2}} + O(k^{-R\delta}|a - b|^R) + O(k^{-R\delta}),
$$

where the $O$-symbol is uniform. The equation may be differentiated any number of times in $a$ with the same remainder estimate.

**Proof.** The result is suggested by the standard formula

$$
\int_0^\infty \cos(ax)H_0^{(1)}(bx) dx = \frac{1}{\sqrt{b^2 - a^2}}
$$

for the Fourier transform of a Hankel function \cite{O, AG}. To deal with the cutoff we proceed as follows.

Since $1_{[1,\infty]}(t)(t^2 - 1)^{-1/2}$ is a tempered distribution on $\mathbb{R}$, and since $\chi(k^{-\delta}x) \cos(ax)$ is a Schwartz function, we may replace the Hankel function by its Fourier integral formula \cite{B2} to obtain

(62)

$$
\int_0^\infty \chi(k^{-\delta}x) \cos(ax)H_0^{(1)}(bx) dx = \int_1^\infty \left\{ \int_0^\infty \chi(k^{-\delta}x) \cos(ax)e^{ibt} dx \right\}(t^2 - 1)^{-1/2} dt
$$

$$
= k^\delta \sum \pm \int_0^\infty \int_0^\infty 1_{[1,\infty]}(t)(t^2 - 1)^{-1/2} \chi(x)e^{ikx(bt \pm a)x} dx dt.
$$

The phase has a (non-degenerate) critical point at $x = 0$, $t = \pm a/b$ (since $b \neq 0$). However, with our assumptions on $(a, b)$, the critical point lies (slightly) outside of the support $[1, \infty]$ of the $dt$ integral, hence the operator

$$
L = k^{-\delta} \frac{1}{bt \pm a} D_x
$$

is well-defined on the support of the integral. We cannot integrate by parts in the $dt$ integral due to singularities in the amplitude, but we can (and will) integrate by parts repeatedly in the $dx$ integral with $L$, which reproduces the phase.

The first time we integrate by parts with $L$, we obtain

(63)

$$
\sum \pm \int_0^\infty \int_0^\infty 1_{[1,\infty]}(t)(t^2 - 1)^{-1/2} \chi(x)\frac{1}{bt \pm a} D_x e^{ikt(bt \pm a)x} dx dt
$$

$$
= \int_0^\infty ((bt \pm a)^{-1}1_{[1,\infty]}(t)(t^2 - 1)^{-1/2} dt
$$

$$
+ \sum \pm \int_0^\infty \int_0^\infty 1_{[1,\infty]}(t)(t^2 - 1)^{-1/2} (D_x \chi(x))\frac{1}{bt \pm a} e^{ikt(bt \pm a)x} dx dt.
$$
We then integrate the second term by parts repeatedly with \( L \) and observe that no further boundary terms are picked up since \( (D_x\chi(x)) \equiv 0 \) near \( x = 0 \). We then break up the \( dt \) integral into \( \int_1^2 \{ \cdots \} dt + \int_2^\infty \{ \cdots \} dt \). On the \( dt \)-integral over \([2, \infty]\), the factors of \( \frac{1}{\delta x^a} \) obtained from the \( R \) partial integrations render the integral absolutely convergent and the factors of \( k^{-\delta} \) show that it is of order \( O(|k|^{-R\delta}) \) with coefficients independent of \((a, b)\). In the integral over \([1, 2]\) we only have \( |bt \pm a| \geq |b - a| \). After \( R \) partial integrations we therefore obtain an estimate of \( k^{-R \delta} |b - a|^{-R} \), for the integral. The same kind of argument applies if we first differentiate the integral \( n \) times in \( a \), without changing the remainder estimate.

To complete the proof, we note (with [AG]) that

\[
\int_1^\infty \sum_\pm (bt \pm a)^{-1}(t^2 - 1)^{-1/2}dt = \frac{1}{\sqrt{b^2 - a^2}}.
\]

\[\square\]

**Corollary 4.7.** With the same assumptions as above, assume further that \( |a - b| \geq C k^{-1 + \delta} \). Then

\[
\int_0^\infty \chi(k^{-\delta}x) \cos(ax) H_0^{(1)}(bx)dx = \frac{1}{\sqrt{b^2 - a^2}} + O(k^{R(1 - 2\delta)}),
\]

where the \( O \)-symbol is uniform. Thus, for any \( \delta \) with \( 1 > \delta > 1/2 \), the remainder is rapidly decaying. The equation may be differentiated any number of times in \( a \) with the same remainder estimate.

We will also need a slight extension of this result:

**Proposition 4.8.** With the same notation and assumptions as above, we have

\[
\int_0^\infty \chi(k^{-\delta}x) \cos(ax) H_0^{(1)}(b\sqrt{r^2 + u^2}) \cos(au)dudx = -i e^{-ir\sqrt{b^2 - a^2}} \sqrt{b^2 - a^2} + O(k^{R(1 - 2\delta)}),
\]

where the \( O \)-symbol is uniform for \( a \) in the interval above.

**Proof.** The proof proceeds much as before except that we now use the identity: ([O], (14.16)-(14.18))

\[
\int_0^\infty H_0^{(1)}(b\sqrt{r^2 + u^2}) \cos(au)du = -i e^{-ir\sqrt{b^2 - a^2}} \sqrt{b^2 - a^2}, \quad \text{valid if} \ b^2 - a^2 > 0.
\]

Details are left to the reader. \[\square\]

### 4.5. The case of the unit disc.

The only computable example is the unit disc, and the reader might wish to use it to check the calculations and estimates. Due to the \( S^1 \) symmetry, \( N(k) \) is a convolution operator and there are simple commutation relations with the layer potentials. We have:

\[
\begin{align*}
N(k, \theta, \phi) &= k \sum_{n \in \mathbb{Z}} H_n^{(1)}(k) J_n(k) e^{in(\theta - \phi)} \\
S\ell(k, (r, \theta), \phi) &= \sum_{n \in \mathbb{Z}} H_n^{(1)}(k) J_n(kr) \cos n(\phi - \theta) \\
D\ell(k, (r, \theta), \phi) &= k \sum_{n \in \mathbb{Z}} H_n^{(1)}(k) J_n'(kr) \cos n(\phi - \theta)
\end{align*}
\]  
(65)
These layer potential identities follow from the so-called Graf addition theorem:

\[(66) \quad G_0(k, x, y) = \sum_{n \in \mathbb{Z}} H_n^{(1)}(k \max\{|x|, |y|\}) H_n(k \min\{|x|, |y|\}) \cos n \angle(x, y),\]

which is valid for general domains.

5. Microlocalizing the trace

As discussed in the introduction, we are going to use (67) to calculate the wave invariants at \( \gamma \). In Lemma 3.3, we discussed the standard microlocalization of the trace to \( \gamma \). We now prove that we can use a microlocal cutoff operator on the boundary as well as on the domain. This obvious sounding statement is not actually obvious in the present approach since the layer potentials are not standard semiclassical Fourier integral operators.

Let us first review Lemma 3.3 in the language of layer potentials. It asserts that

\[(67) \quad \rho \ast \text{Tr}[D(k + i\tau)(I + N(k))^{-1tr}S(k + i\tau)^{tr} \circ (1 - \chi) \sim 0,\]

where \( \chi(k) \) is a semiclassical cutoff in \( \Omega \) to a microlocal neighborhood of \( \gamma \) which has the form \( \chi(r, y, k^{-1}D_y) \) near the boundary. Here, \( (r, y) \) are Fermi normal coordinates in \( \Omega^c \) near the endpoints of \( \gamma \), with \( r \) the distance to the boundary. Thus, differentiations are only in tangential directions.

We now verify that the analogue of (67) remains correct if we use a microlocal cutoff on the boundary. The billiard map \( \beta \) is a cross section of the billiard flow, and in this cross section a bouncing ball \( \gamma \) corresponds to a periodic orbit which we denote \( \partial \gamma \in B^*\partial \Omega \) of period 2. We choose the boundary parametrization so that \( \phi = 0 \) is one of the endpoints of the segment (which we also denote \( \partial \gamma \)) in \( \Omega \). To microlocalize the boundary operators to the periodic point, we introduce a semiclassical pseudodifferential cutoff operator \( \chi_{\partial \gamma}(\phi, k^{-1}D_\phi) \) with complete symbol \( \chi_{\partial \gamma}(\phi, \eta) \) supported in \( V_\varepsilon := \{ (\phi, \eta) : |\phi|, |\eta| \leq \varepsilon \} \).

**Proposition 5.1.** Suppose as above that \( \text{supp} \rho \cap Lsp(\Omega) = \{ rL_\gamma \} \), and let \( \chi_{\partial \gamma}(\phi, k^{-1}D_\phi) \) be a semiclassical microlocal cutoff to the billiard-map orbit corresponding to \( \gamma \). Then:

\[
\rho \ast \text{Tr}[D(k + i\tau)(I + N(k))^{-1tr}S(k + i\tau)^{tr} \circ \chi_{\partial \gamma}(k) \circ S(k + i\tau)^{tr}.
\]

**Proof.** By (25) and by (37)-(38), we have:

\[
\begin{cases}
(i) \quad S(k + i\tau)^{tr} = \gamma G_0(k + i\tau) = \int_0^\infty e^{it(k+i\tau)} r_{\Omega 2} E_0(t) dt, \\
(ii) \quad 1_\Omega (D(k + i\tau) \circ (I + N(k + i\tau))^{-1}(x, q) = \int_0^\infty e^{it(k+i\tau)} r_{\Omega 2} E_D(t) dt,
\end{cases}
\]

where \( r_{\Omega 2} = \partial_u |_{\partial \Omega} \). We have subcribed the restriction operators to clarify which variables they operate on. The composition \( D(k + i\tau) \circ (I + N(k))^{-1} \circ (1 - \chi_{\partial \gamma}(k) \circ S(k + i\tau)^{tr} \) may be written (with the relevant value of \( k \)) as

\[(68) \quad \int_0^\infty e^{it(k+i\tau)} \{ \int_0^t r_{\Omega 2} E_D (t-s) \circ (1 - \chi_{\partial \gamma})(y, |D_i|^{-1}D_y) r_{\Omega 1} \circ E_0(s) ds \} dt.\]
We therefore have:
\[
\rho * \mathcal{D}\ell \circ (I + N(k))^{-1} \circ (1 - \chi_{\partial\gamma}(k)) \circ \mathcal{S}^{\text{tr}}
\]
(69)
\[
= \int_0^\infty \hat{\rho}(t)[\int_0^t r_{\Omega_2} E_D(t-s) \circ (I - \chi_{\partial\gamma})(y, |D_t|^{-1}D_y) \circ r_{\Omega_1} E_0(s)ds]e^{(k+i\tau)t}dt.
\]
The statement of the Proposition is equivalent to:
(70) \(WF[Tr \int_0^t r_{\Omega_2} E_D(t-s) \circ (I - \chi_{\partial\gamma})(y, |D_t|^{-1}D_y) \circ r_{\Omega_1} E_0(s)ds] \cap [rL_\gamma - \epsilon, rL_\gamma + \epsilon] = \emptyset.\)

We now claim that the integrand
\[
V(s, t) = Tr r_{\Omega_2} E_D(t-s) \circ (I - \chi_{\partial\gamma})(y, |D_t|^{-1}D_y) \circ r_{\Omega_1} E_0(s)
\]
is a smooth function for \(t \in (rL_\gamma - \epsilon, rL_\gamma + \epsilon)\) and for \(s \in (0, rL_\gamma + \epsilon)\). Indeed, the singular support consists of \((s, t)\) such that there exists a closed billiard orbit of length \(t\) outside a phase space neighborhood of \(\gamma^r\) which consists of a straight line segment of length \(s\) from a point \(x \in \Omega\) to a boundary point \(q\), followed by a generalized billiard orbit of length \(t - s\) from \(q\) back to \(x\). By our assumption on the length spectrum, the only possible orbit with length \(t \in (rL_\gamma - \epsilon, rL_\gamma + \epsilon)\) is \(\gamma^r\) of length \(r\); but the cutoff has removed this orbit. Since the integrand is smooth, the integral over \(s \in [0, t]\) determines a smooth function of \(t \in L_\gamma - \epsilon, rL_\gamma + \epsilon\).

\[\square\]

**Remark 5.2.** The Proposition is obvious in the case of the unit disc, although in this case it is only natural to cutoff in the frequency variable since all radial geodesics are bouncing ball orbits. Lemma 3.1 and Proposition 5.1 are then equivalent, since the cutoff to the right of \(\mathcal{S}^{\text{tr}}\) has the form \(\chi(k^{-1}D_\phi)\) and commutes with \(\mathcal{S}^{\text{tr}}\).

### 6. Regularizing the boundary integrals

The purpose of this section is to analyze the compositions in (10) with the semiclassical cutoff \(\chi_{\partial\gamma}\) on the boundary as semiclassical Fourier integral operators. Since the role of the imaginary part of the spectral parameter is not important here, we write it simply as \(\tau\) and only substitute \(\tau \log k\) when it is needed in (8).

For ease of notation, we write the terms of (10) as
(71) \(N_\sigma := N_{\sigma(1)} \circ N_{\sigma(2)} \circ \cdots \circ N_{\sigma(M)}\)
and we put
(72) \(|\sigma| = \#\sigma^{-1}(0) = \) the number of \(N_0\) factors occurring in \(N_\sigma\).

**Proposition 6.1. (A)** Suppose that \(N_\sigma\) is not of the form \(N_0^M\). Then for any integer \(R > 0\), \(N_\sigma \circ \chi_{\partial\gamma}(k + i\tau)\) may be expressed as the sum
\[
N_\sigma = F_\sigma(k, \phi_1, \phi_2) + K_R,
\]
where \(F_\sigma\) is a semiclassical Fourier integral kernel of order \(-|\sigma|\) associated to \(\beta_\sigma^{M-|\sigma|}\) and a remainder \(K_R\), which is a bounded kernel which is uniformly of order \(k^{-R}\).

**B** \(N_0^M \circ \chi_{\partial\gamma} \sim N_{0M} \circ \chi_{\partial\gamma}\), where \(N_{0M}\) is a semiclassical pseudodifferential operator of order \(-M\).
The proof will be broken up into a sequence of Lemmas. First we will consider the compositions $N_0 \circ N_1$, $N_1 \circ N_0$ without the cutoff $\chi_{\partial \gamma}$. Then we consider iterated compositions with the cutoff. Finally we discuss the special term $N_0^M \circ \chi_{\partial \gamma}$.

6.1. **The compositions** $N_0 \circ N_1$. Here we characterize the composition $N_0 \circ N_1$. Essentially the same result holds for $N_1 \circ N_0$.

The composed kernel equals

\[
N_0 \circ N_1(k + i\tau, \phi_1, \phi_2) := (k + i\tau) \int_{T} \chi(k^{-1+\delta}(\phi_1 - \phi_3))(1 - \chi(k^{-1+\delta}(\phi_2 - \phi_3)))
\]

(73)

and

\[
H_1^{(1)}((k\mu + i\tau)|q(\phi_3) - q(\phi_1)|) \cos \angle(q(\phi_3) - q(\phi_1), \nu_{q(\phi_3)})N_1((k\mu + i\tau, q(\phi_2), q(\phi_3)).
\]

The somewhat technical nature of the following lemma is due in part to the lack of a cutoff to $\gamma$.

**Lemma 6.2.** For any $R \in \mathbb{N}$, there exists an amplitude $A_{01}(k + i\tau, \phi_1, \phi_2)$ such that

- (i) $N_0 \circ N_1(k + i\tau, \phi_1, \phi_2) = (1 - \chi(k^{-1-\delta}(\phi_1 - \phi_2))k^{(\frac{5}{2} - 3\delta)}e^{ik|q(\phi_1) - q(\phi_2)|}A_{01}(k + i\tau, \phi_1, \phi_2) + K_R(k + i\tau, \phi_1, \phi_2)$;

- (ii) $A_{01}(k + i\tau, \phi_1, \phi_2) \in S^0_\delta(T^2 \times \mathbb{R})$.

- (iii) $K_R$ is a bounded kernel which is uniformly of order $k^{-R}$.

**Proof.** Using Lemma 4.4 and Proposition 4.3, we rewrite $N_1$ in terms of phases and amplitudes. The proof is then based on a change of variables and on use of the explicit cosine transform of the Hankel function given in Proposition 4.6 to evaluate integrals involving the ‘difficult factor’ $N_0(k + i\tau)$, i.e. $H_1^{(1)}((k\mu + i\tau)|q(\phi_1) - q(\phi_3)|$.

It is convenient to first change variables

$\phi_3 \to \vartheta = \phi_1 - \phi_3$,

and then to change variables $\vartheta \to u$, with:

\[
u := \begin{cases}
|q(\phi_3) - q(\phi_1)|, & \phi_1 \geq \phi_3 \\
-|q(\phi_3) - q(\phi_1)|, & \phi_1 \leq \phi_3
\end{cases} = \begin{cases}
|q(\phi_1 - \vartheta) - q(\phi_1)|, & \vartheta \geq 0 \\
-|q(\phi_1 - \vartheta) - q(\phi_1)|, & \vartheta \leq 0
\end{cases}
\]

(74)

In other words, we change from the intrinsic distance along $\partial \Omega$ to chordal distance. Due to the cutoff, the variable $u$ may be taken to range over $(-k^{-1+\delta}, k^{-1+\delta})$, so the change of variables is well-defined and smooth on the support of the integrand. The purpose of this change of variables is to simplify the difficult factor in (73):

\[
H_1^{(1)}((k\mu + i\tau)|q(\phi_3) - q(\phi_1)|) \to H_1^{(1)}((k\mu + i\tau)|u|).
\]

Now we consider the other factors. After the change of variables, we have:
identities (cf. \[EP\]):

\[ a = \sin \vartheta_{1,2}, \]  

with \( \vartheta_{1,2} = \angle(q(\phi_2) - q(\phi_1), \nu_{q(\phi_2)}) \).

These statements follow in a routine way from Proposition 4.3 and from the following identities (cf. \[EP\]):

\[
\begin{align*}
(i) \ q(\phi) - q(\phi') &= (\phi - \phi')T(\phi') - \frac{1}{2}\kappa(q(\phi))(\phi - \phi')^2\nu_{q(\phi')} + O((\phi - \phi')^3) \\
(ii) \ |q(\phi) - q(\phi')|^2 &= (\phi - \phi')^2 + O((\phi - \phi')^4) \\
(iii) \ |q(\phi) - q(\phi')| &= |\phi - \phi'| + O(\phi - \phi')^3 \\
(iv) \ \langle(q(\phi) - q(\phi'), \nu_{q(\phi')}) &= \frac{-1}{2}(\phi - \phi')^2\kappa(\phi') + O((\phi - \phi')^3).
\end{align*}
\]

Here, \( T(\phi) \) denotes the unit tangent vector at \( q(\phi) \). It follows that \( K(u) = -1/2\kappa(q(j_0)) + O(|u|^2) \). For further details on the claims above, we refer to the Appendix \[Z6\].

Taking into account the factor \((k + i\tau)\) in front of the integral, it follows that the composed kernel \[73\] can be expressed in the form of Proposition (6.2) (i), with

\[
A(k + i\tau, \phi_1, \phi_2) = \int_{-\infty}^{\infty} \tilde{\chi}(k^{1-\delta}u)(1 - \chi(k^{1-\delta}(\phi_2 - \phi_1 - u)))|u|e^{ikau}h_1^{(1)}((k + i\tau)|u|)G((k + i\tau), u, \phi_1, \phi_2)du,
\]

where \( G((k + i\tau), u, \phi_1, \phi_2) \) is a symbol in \( k \) of order 3/2 and smooth in \( u \), and where \( \tilde{\chi}(k^{1-\delta}u) = \chi(k^{1-\delta}(\phi_1 - \phi_0)) \). The cutoff has been changed slightly under the change of variables, but for notational simplicity we retain the old notation for it.

We now change variables again, \( u' = ku \) (and then drop the prime), to get

\[
A(k + i\tau, \phi_1, \phi_2) = k^{-2}\int_{-\infty}^{\infty} \tilde{\chi}(k^{1-\delta}u)(1 - \chi(k^{1-\delta}(\phi_2 - \phi_1 - k^{-1}u)))|u|e^{i\tau u}h_1^{(1)}((k + i\tau)|u|)G((k + i\tau, \frac{u}{k}, \phi_1, \phi_2)du,
\]

with \( b = 1 + i(\tau/k) \).

**Remark 6.3.** We note that the entire amplitude now has order \(-1/2\) at least formally. Since we have not yet microlocalized to \( \gamma \), we will temporarily obtain a worse estimate, but in the final step we will show that this is the correct order.

Since \( |u| \leq k^\delta \) on the support of the cutoff, we have \( \frac{|u|}{k} \leq k^{-1+\delta} \), and then the Taylor expansion of \( G((k + i\tau, u, \phi_1, \phi_2) \) at \( u = 0 \) produces an asymptotic series

\[
k^{-3/2} G((k + i\tau, \frac{u}{k}, \phi_1, \phi_2) = \sum_{n=0}^{p} k^{-n}u^nG_n((k + i\tau, \phi_1, \phi_2) + k^{-p}u^pR_p(k, \frac{u}{k}, \phi_1, \phi_2),
\]
where \( G_n(k+i\tau, \phi_1, \phi_2) \) is a symbol of order 0, and where \( R_p \) is the remainder,
\[
R_p(k, \frac{u}{k}, \phi_1, \phi_2) = \frac{1}{p!} \int_0^1 (1-s)^{p-1} G^{(p)}(k+i\tau, s\frac{u}{k}, \phi_1, \phi_2) ds.
\]
Since \( G(k+i\tau) \) is a symbol of order 0, \( G^{(p)}(k+i\tau, s, \phi_1, \phi_2) \) is uniformly bounded for \( |s| \leq 1 \) and hence
\[
|R_p(k, \frac{u}{k}, \phi_1, \phi_2)| \leq C_p \quad \text{for} \quad |u| \leq k^\delta.
\]
Second, we Taylor expand the cutoff around \( u = 0 \) to one order:
\[
(1 - \chi(k^{1-\delta}(\phi_2 - \phi_1 - k^{-1}u)) = (1 - \chi(k^{1-\delta}(\phi_2 - \phi_1)) + k^{-\delta} u S_1(k, \frac{u}{k}, \phi_1, \phi_2),
\]
with
\[
S_1(k, \frac{u}{k}, \phi_1, \phi_2) = \int_0^k \chi'(k^{1-\delta}(\phi_1 - \phi_2 - s\frac{u}{k})) ds.
\]
We then write:
\[
G(1 - \chi) = (G_p + R_p)((1 - \chi) + S_p) = G_p(1 - \chi) + G_p S_p + [R_p((1 - \chi) + S_p)],
\]
where \( G_p \) is the \( p \)th Taylor polynomial of \( G \). We claim that the first terms is a Fourier integral kernel of the type described in (i)-(iii); that the second kernel has this form but multiplied by cutoffs \( \chi^1(k^{1-\delta}(\phi_1 - \phi_2)) \) which vanish except in a small band \( Ck^{-1+\delta} \leq |\phi_1 - \phi_2| \leq 10Ck^{-1+\delta} \); and that the remaining two terms define an error of the type (iv).

Consider the first term:
\[
\sum_{n=1}^{R} G_n(k+i\tau, \phi_1, \phi_2) (1 - \chi(k^{1-\delta}(\phi_1 - \phi_2))) k^{-n-2} \int_{-\infty}^{\infty} \tilde{\chi}(k^{-\delta}u)|u|u^n e^{iau} H_1^{(1)}(b|u|) du.
\]
The integral may be expressed in the form
\[
\frac{\partial}{\partial b} \frac{\partial}{\partial a^n} \int_{-\infty}^{\infty} \tilde{\chi}(k^{-\delta}u) e^{iau} H_0^{(1)}(b|u|) du|_{a=\sin \vartheta_{1,2}, b=(1+i\tau/k)}.
\]
Since the terms in the finite part of the Taylor expansion of \((1 - \chi(k^{1-\delta}(\phi_2 - \phi_1 - k^{-1}u))\) vanish if \( |\phi_2 - \phi_1| \leq k^{-1+\delta} \), we have \( a = \sin \vartheta_{1,2} \in (-1 + k^{-1+\delta}, 1 - k^{-1+\delta}) \). So we may apply Proposition \[\text{[1.6]}\] - Corollary \[\text{[1.7]}\] to evaluate \[\text{[81]}\] asymptotically as
\[
i^{-n} \frac{\partial}{\partial a^n} \int_0^{\infty} \tilde{\chi}(k^{-\delta}x) \cos(ax) H_0^{(1)}(bx) dx|_{a=\sin \vartheta_{1,2}, b=(1+i\tau/k)}
\]
\[
= i^{-n} \frac{\partial}{\partial a^n} (1 - a^2)^{-3/2} + O(k^{-\delta}|a - 1|) + i\tau|^{-(3+2n)}
\]
\[
= P_n(a) (1 - a^2)^{-3/2} + O(k^{-\delta}|a - 1|) + i\tau|^{-(3+2n)}
\]
\[
= (\cos \vartheta_{1,2})^{-3/2} + O(k^{-\delta} k^{(3+2n)(1-\delta)}).
\]
Here, \( P_n \) as an \( n \)th degree polynomial which we will not need to evaluate.

Let us analyze the order in \( k \) of this part of the amplitude. We now put in the factor of \( k^{-n-2} \) in front of \[\text{[81]}\]. We observe that \( \cos \vartheta_{1,2} \) can be as small as \( k^{-1+\delta} \) on the support.
of the cutoffs in the integral (which do not include a cutoff to $\gamma$). In view of the factor $(\cos \vartheta_{1,2})^{-(3+2n)}$, we estimate the contribution of the $n$th term to the amplitude as

$$k^{-n-2}(\cos \vartheta_{1,2})^{-(3+2n)} \leq k^{-n-2}k_{(3+2n)(1-\delta)} = k^{n(1-2\delta)+(1-3\delta)}.$$  

Each further derivative in $k^{-1}D$ gives a further factor of $k^{-\delta}$. Since $\delta > 1/2$ the terms decrease in order with $n$, and since there are only a finite number of terms, we conclude that $k^{-3/2}A(k+i\tau, \phi_1, \phi_2)$ lies in $S_{(1-3\delta)}$. This proves (i) - (ii) modulo the remainder estimate.

**Remark 6.4.** In Lemma 6.5 we will make use of the cutoff $\chi_{\phi_\gamma}$ at the end to eliminate these factors of $\cos \vartheta_{1,2}$.

We next turn to the term $G_p S_p$, which is the most difficult of the remainder terms. As in (81) of the previous step, the key point is to analyze the integrals

$$i^{-n}k^{-\frac{1}{2}}\frac{\partial}{\partial \vartheta_{1,2}} \int_0^1 \int_{\infty}^t \tilde{\chi}(k^{-\delta} u) \chi'(k^{-1-\delta}(\phi_1 - \phi_2) - s u) e^{i\vartheta_{1,2}} H_0^{(1)}(b|u|) duds |_{a = \sin \vartheta_{1,2}, b = (1+i\tau/k)}.$$  

To deal with this integral, we need to adapt the method of proof of Proposition (4.6) - Corollary (4.7) to take into account the additional cutoff.

As before, we have:

$$\int_0^1 \int_{\infty}^t \tilde{\chi}(k^{-\delta} x) \chi'(k^{-1-\delta}(\phi_1 - \phi_2) - s x) e^{i\vartheta_{1,2}} H_0^{(1)}(b|u|) dx ds$$

$$= k^{-\delta} \sum_{\pm} \int_0^1 \int_{\infty}^t 1_{[1,\infty]}(t) (t^2 - 1)^{-1/2} \chi(x) \chi'(k^{-1-\delta}(\phi_1 - \phi_2) - sx) e^{i\vartheta_{1,2}} (bt \pm a x) dx ds dt.$$  

The phase is the same as in Proposition (4.6), and we again integrate by parts in $dx$ with $L = k^{-\delta} \frac{1}{b t \pm a} D_x$.

The first time we integrate by parts with $L$, we obtain

$$\chi'(k^{-1-\delta}(\phi_1 - \phi_2)) \int_0^t ((bt \pm a)^{-1} 1_{[1,\infty]}(t) (t^2 - 1)^{-1/2} dt$$

$$+ \sum_{\pm} \int_0^1 \int_{\infty}^t 1_{[1,\infty]}(t) ((t^2 - 1)^{-1/2} (D_x \chi(x) \chi'(k^{-1-\delta}(\phi_1 - \phi_2) - sx)) |_{bt \pm a} e^{i\vartheta_{1,2}} (bt \pm a x) dx ds dt.$$  

We then continue to integrate the second term by parts repeatedly with $L$. But unlike the case of Proposition (4.6), we do pick up boundary terms each time when $D_x$ falls on the second cutoff factor. For instance, the next iterate produces the boundary term:

$$k^{-\delta} \chi''(k^{-1-\delta}(\phi_1 - \phi_2)) \int_0^t ((bt \pm a)^{-2} 1_{[1,\infty]}(t) (t^2 - 1)^{-1/2} dt$$

$$+ k^{-\delta} \sum_{\pm} \int_0^1 \int_{\infty}^t 1_{[1,\infty]}(t) ((t^2 - 1)^{-1/2} (D_x^2 \chi(x) \chi'(k^{-1-\delta}(\phi_1 - \phi_2) - sx)) |_{bt \pm a} e^{i\vartheta_{1,2}} (bt \pm a x) dx ds dt.$$  

The boundary terms are similar to the kind in the $G_p \chi$ terms, except multiplied by $\chi'(k^{-1-\delta}$ and higher derivatives. This establishes the claimed form for the first two terms.

As for the integral, we break up the $dt$ integral into $\int_1^2 \{ \cdots \} dt + \int_2^\infty \{ \cdots \} dt$ as we did in Proposition (4.6). The same observations show that after $R$ partial integrations, the remainder term is bounded by $k^{-R\delta} |b-a|^{-R}$. For the integral. The boundary terms have the form of $\frac{1}{\sqrt{b^2 - a^2}}$ times the series $\sum_{j=1}^R k^{-\delta j} \chi^{(j)}(k^{-1-\delta}(\phi_1 - \phi_2))$.  

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We now sketch the estimate of the last two remainder terms. There are two kinds of terms, those from \( G \) which descend in steps of \( k^{-1} \) and those of \( \chi \) which descend in steps of \( k^{-\delta} \). If we retain the principal term of \( \chi \) and the first \( n \) terms of \( G \), then we obtain a remainder of

\[
(87) \quad k^{-n-2} \int_{-\infty}^{\infty} \chi(k^{-\delta}u)|u|^n R_n(u/k, \phi_1, \phi_2) e^{iu} H_1^{(1)}(|u|) du
\]

of the \( n \)th order remainder in the Taylor expansion of the amplitude. By (90), we can bound the \( R_n \) and exponential factors by a uniform constant \( C \). We can also bound the Hankel factor by \(|u|^{-1/2} \) for \(|u| \geq 1 \); the singularity at \( u = 0 \) is cancelled by the \( u^n \). Thus, the integral is bounded by \( k^{-n-2} \int_0^1 x^{n+1/2} dx \sim k^{-n-2} k^{(n+3/2)} = k^{-1-\delta-1/2} \). So for \( n \) sufficiently large we have an arbitrarily small remainder. We can choose \( n \) to obtain the order stated in (iv).

\[ \square \]

6.2. **Proof of (A).** By iterating the Lemma and composing with \( \chi_{\partial \gamma} \), we can improve Lemma (10) to the statement in (A) of the Proposition. First, some more notation. Any term \( N_\sigma \) of (10) can be expressed as a product

\[
(88) \quad N_0^{s_0} \circ N_1^{s_1} \cdots \circ N_r^{s_r} N_0^{t_1},
\]

of blocks of \( N_0 \) and \( N_1 \), with \( \sum_{j=1}^r s_j = |\sigma| \) and \( \sum_{j=1}^r (s_j + t_j) = M \). The number \( r = r(\sigma) \) counts the number of blocks of \( N_0 \). We now compose with \( \chi_{\partial \gamma} \), and successively eliminate the blocks \( N_0^{s_1} \) from right to left using one factor of \( N_1 \) to the right. There is a slight notational problem since it could happen that \( t_1 = 0 \), in which case one should instead use the factor of \( N_1 \) to the left in eliminating a block of \( N_0 \). Since the process is analogous in that case, we will assume for simplicity of notation that \( t_1 \neq 0 \).

**Lemma 6.5.** For any term \( N_\sigma \) of (10) except for \( N_0^M \), and for any \( R \in \mathbb{N} \), there exist \((s_1, \ldots, s_r)\) as above and amplitudes \( A_j(k + i\tau, \phi_1, \phi_2) \), (with \( j = 1, \ldots, r \)) of order \(-s_j\) such that:

- (i) \( N_\sigma \circ \chi_{\partial \gamma} = N_1^{t_1-1} \circ M_1 \circ N_1^{t_2-1} \circ \cdots \circ N_r^{t_r-1} \circ M_1 \circ N_1^{t_1-1} \circ \chi_{\partial \gamma} + K_R \), where each \( M_j \) is a semiclassical Fourier integral kernel of the form
  \[
  M_j(k + i\tau, \phi_1, \phi_2) = k^{1/2} \left( (1 - \chi(k^{-\delta}(\phi_2 - \phi_2))) e^{i(k|q(\phi_1) - q(\phi_2)|)} A_j(k + i\tau, \phi_1, \phi_2) \right),
  \]
  with \( A_j(k + i\tau, \phi_1, \phi_2) \in S^{-s_j}_\delta(T^2 \times \mathbb{R}) \).

- (iv) \( K_R \) is a bounded kernel which is uniformly of order \( k^{-R} \).

**Proof.** We work from right to left using the argument of Lemma (6.2) repeatedly to remove all of the \( N_0 \) factors in each block. This can be done because we only used knowledge of the phase and of the order of the amplitude to obtain (i) - (iii) of the Lemma. In each such \( M_j \) we may choose \( R \) large enough so that the remainder for this block, when composed with the remaining factors of \( N_0, N_1 \) satisfies (iii). We then define

\[
(89) \quad F_\sigma = N_r^{t_r-1} \circ M_r \circ N_1^{t_1-1} \circ \cdots \circ N_1^{t_2-1} \circ M_1 \circ N_1^{t_1-1} \circ \chi_{\partial \gamma}.
\]

It is a semiclassical Fourier integral kernel with phase

\[
(90) \quad L_\sigma(\phi_1, \ldots, \phi_{M-|\sigma|}) = |q(\phi_1) - q(\phi_2)| + \cdots + |q(\phi_{M-|\sigma|}) - q(\phi_{M-|\sigma|-1})|.
\]

Now let us use the composition with \( \chi_{\partial \gamma} \). Since \( N_r^{t_r-1} \circ M_r \circ N_1^{t_1-1} \circ \cdots \circ N_1^{t_2-1} \circ M_1 \circ N_1^{t_{1}-1} \) is a semiclassical Fourier integral operator, its composition with \( \chi_{\partial \gamma} \) microlocalizes the kernel
to the periodic orbit of \(\beta\) corresponding to \(\gamma\). That is, on the support of the cutoff \(\chi_{\partial\gamma}\), critical points correspond to Snell paths in which each link points roughly in the direction of \(\vartheta\) bounded above. As remarked in the proof of Lemma 6.2, each removal of 6.5 with the cutoff. The change of variables eliminated two powers of \(\varepsilon\). However, upon microlocalizing to the periodic orbit, there is a uniform lower bound for \(\cos\vartheta_{1,2}\), and hence these factors are bounded above. As remarked in the proof of Lemma 6.2, each removal of \(N_0\) decreases the order by 1 on the set where \(\cos\vartheta_{1,2}\) is bounded uniformly below. Thus, the order of \(F_\sigma\) is \(-|\sigma|\). This improves the order estimate to the statement in (A) and completes the proof of this part of the Proposition.

6.3. The term \(N_0^M \circ \chi_{\partial\gamma}\). Our first step in proving (B) of the Proposition is:

**Lemma 6.6.** \(N_0 \circ \chi_{\partial\gamma}\) is a semiclassical pseudodifferential operator of order \(-1\).

**Proof.** The composed kernel equals

\[
N_0 \circ \chi_{\partial\gamma}(k + i\tau, \phi_1, \phi_2) := (k + i\tau) \int T \chi(k^{-1+\delta}(\phi_1 - \phi_2))(1 - \chi(k^{-1+\delta}(\phi_2 - \phi_3)))
\]

(91)

\[
H_1^{(1)}((k\mu + i\tau)|q(\phi_3) - q(\phi_1)|) \cos \angle(q(\phi_3) - q(\phi_1), \nu_q(\phi_1)) \chi_{\partial\gamma}(k, q(\phi_2), q(\phi_3)) d\phi_3,
\]

where

\[
\chi_{\partial\gamma}(k, q(\phi_2), q(\phi_3)) = k \int \mathbb{R} e^{i(k(\varphi - \varphi_3) - \eta)} \chi_{\partial\gamma}(\varphi_2, \eta) d\eta
\]

where \(\chi_{\partial\gamma}(\varphi, \eta)\) now denotes (with a slight abuse of notation) the symbol of \(\chi_{\partial\gamma}\), a smooth phase space cutoff to \((0, 0), \beta(0, 0), \) the coordinates of the periodic point of period 2.

We make the same change of variables as in Lemma 6.2, which again takes

\[
\begin{cases}
(i) H_1^{(1)}((k\mu + i\tau)|q(\phi_3) - q(\phi_1)|) \rightarrow H_1^{(1)}((k\mu + i\tau)|u|);

(ii) \cos \angle(q(\phi_2) - q(\phi_2 + \vartheta), \nu_q(\phi_1)) \rightarrow |u|K(\phi_2, u), \quad \text{with } K \text{ smooth in } u;

(iii) e^{ik(\varphi_2 - \varphi_3) - \eta} \chi_{\partial\gamma}(\varphi_2, \eta) \rightarrow e^{ik(\varphi_2 - \varphi_1)} e^{ik\vartheta} e^{i\eta \cdot \eta} = \chi_{\partial\gamma}(\varphi_2, \eta) e^{ik(\varphi_2 - \varphi_1)} e^{iku \cdot \eta} A(k, u, \eta) \chi_{\partial\gamma}(\varphi_2, \eta),
\end{cases}
\]

where \(A(k, u, \eta)\) is an amplitude of order 0.

Then (91) becomes

\[
(k + i\tau)(1 - \chi(k^{-1+\delta}(\varphi_2 - \varphi_1))) \int \mathbb{R} e^{ik(\varphi_2 - \varphi_1)} a(k + i\tau, \phi_1, \phi_2, \eta) \chi_{\partial\gamma}(\varphi_2, \eta) d\eta,
\]

(92)

with \(a(k + i\tau, \phi_1, \phi_2, \eta) = \int T \chi(k^{-1+\delta}u) H_1^{(1)}((k\mu + i\tau)|u|) |u| e^{iku \cdot \eta} A_1(k, u, \eta) dud\eta\),

for another amplitude of order 0. This is the same kind of integral we analyzed in (87) and (81) of Lemma 6.2, and we obtain the same description of \(a(k, \phi_1, \phi_2)\) as for \(A(k + i\tau, \phi_1, \phi_2)\) except that the parameter \(a\) in (81) is now \(\eta\). Thus, we obtain powers of \((1 - \eta^2)^{-1}\), which blow up on the unit sphere bundle of \(\partial\Omega\). These points of course correspond to tangential (or grazing) rays, and on the support of the cutoff \(\chi_{\partial\gamma}(1 - \eta^2)^{-1}\) is uniformly bounded. Thus, we obtain the statement of the Lemma precisely as in Lemma 6.2 and its improvement Lemma 6.5 with the cutoff. The change of variables eliminated two powers of \(k\), leaving an amplitude
of order 0; an amplitude of order 0 defines a semiclassical pseudodifferential operator of order $-1$.

To complete the proof of $B$ it suffices to iterate Lemma 6.6. On each application, we have a new cutoff operator, but there is no essential change in the argument. This completes the proof of Proposition 6.2.

### 7. Regularizing the interior integral

We further need regularize the integrals involving the outer factors of $D\ell(k+i\tau)$ and of $S\ell(k+i\tau)^{\text{tr}}$. Since we are taking a trace, we can (and will) cycle the factor of $D\ell(k+i\tau)$ to the right of $S\ell^{\text{tr}}(k+i\tau)\chi(k)$ to obtain an operator

\[(93)\]

\[S\ell\chi(k)D\ell : L^2(\partial\Omega) \to L^2(\partial\Omega), \quad S\ell^{\text{tr}}\chi(k)D\ell(q, q') = \int_{\Omega} G_0(\lambda, q, x)\chi(k^{-1}D_x, x)\partial_\nu G_0(\lambda, x, q')dx.\]

on $\partial\Omega$ (or $S^1$, after parametrizing it). Here, $\chi(k)$ is a semiclassical cutoff to a neighborhood of a periodic orbit $\gamma$. For simplicity we will assume that $\gamma$ is a bouncing ball orbit, although the same method would apply to a general periodic reflecting ray.

We note that use of the inside/outside duality as in [Z5] would in effect make this section unnecessary, at the expense of forcing a stronger hypothesis on the simplicity of the length spectrum. Indeed, the integrals over the inside/outside would add up to the integral in (93) but over $\mathbb{R}^2$ instead of $\Omega$. This integral is easily evaluated to be $N'(k+i\tau)$. We will now show that the integral over $\Omega$ (with the cutoff in place) produces a similar kind of semi-classical Fourier integral operator.

Before stating the precise results, we give some heuristics on the composition (93). In Proposition 4.5, we described the layer kernels away from their diagonal singularities. We are now including the latter singularities. In addition to $\Gamma$ of (61), the wave front description now also includes the relation

\[(94)\]

\[\Delta_s = \{(q, \xi, q, \eta) : \xi|_{\partial\Omega} = \eta\} \subset T^*\Omega \times T^*\partial\Omega,\]

which carries the singularities of the kernel. Intuitively, $S\ell, D\ell$ are singular Fourier integral kernels associated to the union (which we write as a sum) of the two canonical relations $\Delta_s + \Gamma$, and hence the composition $S\ell^t \circ \chi(k) \circ D\ell$ should be associated to the composition

\[(\Delta_s + \Gamma)^t \circ (\Delta_s + \Gamma) = \Delta_s^t \circ \Delta_s + \Delta_s^t \circ \Gamma + \Gamma^t \circ \Delta_s + \Gamma^t \circ \Gamma.\]

It is clear that

\[\Gamma^t \circ \Gamma = \Delta_s^t \circ \Gamma = \Gamma^t \circ \Delta_s = \Gamma_\beta,\]

the graph of the billiard map. Also,

\[\Delta_s^t \circ \Delta_s = \Delta_\beta,\]

the diagonal of $T^*\partial\Omega$. We therefore expect the composition to contain these two components, precisely as $N'(k+i\tau)$ does in the combined inside/outside case. The following proposition confirms this. It also produces a cutoff function, which confirms Proposition 5.1.
Proposition 7.1. \( S\ell\chi(k)D\ell \sim \chi\partial_\gamma(kD_\theta, \theta)[D_0 + D_1] \), where \( D_1 \) is a semiclassical Fourier integral operator of order \(-1\) associated to the billiard map, where \( D_0 \) is a semiclassical pseudodifferential operator of order \(-3\), and where \( \chi\partial_\gamma(kD_\theta, \theta) \) is a microlocal cutoff to the \( \beta \)-orbit of \((0,0)\).

Proof. We break up each term as in \((69)\) and then \((70)\) and analyze each one separately.

7.0.1. The most regular terms. The regular terms are those of the form

\[
\int_\Omega (1 - \chi^{k^{-1+\delta}}(x)))^2 G_0(\lambda, q, x) \chi(k^{-1}D_x, x) \partial_\gamma G_0(\lambda, x, q') dx.
\]

or where the cutoff has the form \((1 - \chi^{k^{-1+\delta}}(x)))\chi^{k^{-1+\delta}}(x))\).

Lemma 7.2. The regular terms of the form \((65)\) define Fourier integral kernels of the form

\[
k^{1/2} e^{i(k+irq)(q,q')} |\chi\partial_\gamma(q, q', q - q')| A(k, \tau, \phi, \phi') \]

where \( A \) is a semiclassical amplitude of order \(-1\), and where \( \chi\partial_\gamma \) is a cutoff to \( \gamma \). Thus, \((65)\) defines a semiclassical Fourier integral operator of order \(-1\) associated to \( \beta \).

Proof. Each factor of the product is described by Proposition 4.5. Further, we may take to be the product of a frequency cutoff \( \chi(k^{-1}D_\theta) \) and a spatial cutoff \( \psi_\gamma(x) \) to a strip around \( \gamma \). Thus, the integral has the form:

\[
(k + ir) \int_\Omega (1 - \chi^{k^{-1+\delta}}(x)))^2 e^{i((k+ir)|x-q'| |x-q'|)} A(k, x, q, q') dx
\]

where \( A \) is a semiclassical smooth amplitude of order \(-1\). The stationary phase set is defined by

\[
\{ x : d_x|x-q| = -d_x|x-q'| \iff \frac{x-q}{|x-q|} = -\frac{x-q'}{|x-q'|} \}.
\]

First, we see that \( x \in \overline{qq'} \) (the line segment between \( q, q' \)). The ray \( \overline{qq'} \) is constrained by the cutoff to point in the nearly vertical direction in the strip containing \( \gamma \), hence \(|q-q'| \geq C > 0\) on the support of the cutoff. (Note that the phase is the sum, not the difference, of the distances since we are taking the transpose, not the adjoint, of \( S\ell(k + ir) \). For the adjoint, the critical point equation would force \( q = q' \).

The ray \( \overline{qq'} \) is thus a critical manifold of the phase, and the phase equals \(|q - q'| \) along it. To prove Lemma 7.2, we show that this critical manifold is non-degenerate and determine the amplitude by stationary phase. We choose rectangular coordinates \((s, t)\) oriented so that the \( t \) axis is the ray \( \overline{qq'} \) and so that the \( s \)-axis is orthogonal to it. Then \( q = (0, b), q' = (0, b') \) for some \( b, b' \in \mathbb{R} \) and the phase may be written \( \Psi = (s^2 + (t - b)^2)^{1/2} + (s^2 + (t - b')^2)^{1/2} \). Hence, \( \Psi_s = s((s^2 + (t - b)^2)^{-1/2} + (s^2 + (t - b')^2)^{-1/2}) \) and on the stationary phase set \( s = 0 \) the second derivative is simply \( \Psi_{ss}(0, t) = \frac{|(t - b)^2|^{-1/2} + ((t - b')^2)^{-1/2}}{2} \). It is clear that \( \Psi_{ss}(0, t) \) is uniformly bounded below and since \( |b - b'| \geq C \) on the support of the cutoff it is also uniformly bounded above. Stationary phase introduces a factor of \( k^{-1/2} \) so the resulting amplitude is a product of \( k^{1/2} \) with an amplitude of order \(-1\) and hence the composition defines a semiclassical Fourier integral operator of order \(-1\).
7.0.2. **Singular terms.** Therefore we only need to consider the integral

\[(97) \int_{\Omega} [\lambda_{\alpha\Omega}(x)]^2 \lambda(\lambda^{-1}D_x, x) \partial_{\nu} G_0(\lambda, x, q') dx.\]

With no essential loss of generality, we redefined the cutoff to remove the square. We write \(x = (r, \theta)\) and then break up the integral into the sum of four terms corresponding to the cutoffs:

- (i) \(\lambda_{\alpha\Omega}(r)^{1+\delta}(\theta - \phi)\chi(k^{1-\delta}(\theta - \phi'));
- (ii) \(\lambda_{\alpha\Omega}(r)^{1+\delta}(1 - \chi(k^{1-\delta}(\theta - \phi)))\chi(k^{1-\delta}(\theta - \phi'));
- (iii) \(\lambda_{\alpha\Omega}(r)^{1+\delta}(\theta - \phi)(1 - \chi(k^{1-\delta}(\theta - \phi')));
- (iv) \(\lambda_{\alpha\Omega}(r)^{1+\delta}(1 - \chi(k^{1-\delta}(\theta - \phi))(1 - \chi(k^{1-\delta}(\theta - \phi')));

7.0.3. **Codimension zero case.** This refers to case (iv). There are no singularities in the integrand due to the cutoff. We can use the WKB approximation in each Green’s function and apply the cutoff operator in the smooth variables \(\phi, \phi'\)

\[7.0.4. \text{Lemma (7.2)}\]

\[7.3. \text{Lemma (7.3)}\]

\[7.4. \text{Lemma (7.4)}\]

7.0.4. **Codimension one case.** This applies to cases (ii) - (iii), which are quite similar although not identical. We do case (ii); case (iii) is similar.

We have:

\[(98) \int_0^e \int_0^{2\pi} \chi_{\alpha\Omega}(r)^{1+\delta}(1 - \chi(k^{1-\delta}(\theta - \phi))(1 - \chi(k^{1-\delta}(\theta - \phi')))\chi|\nabla_x|_{x(r, \theta) - q'}e^{i(k + i\tau)|x(r, \theta) - q'| + q'-x(r, \theta)|} A(k, r, \theta, q, q') r dr d\theta.\]

There is no essential difference to the regular terms in (95). We therefore have the same result.

**Lemma 7.3.** Integral (98) defines a Fourier integral kernel of order \(-1\) of the same form as Lemma (7.2).

**Lemma 7.4.** Integrals (98) define Fourier integral kernels of the type

\[k^{1/2}\partial_{\nu}(q, q', \frac{q - q'}{|q - q'|})e^{i(k + i\tau)|q| - q|q'|} A(k, \tau, q, q'),\]

with \(A\) an amplitude of order \(-2\), i.e. they are semiclassical Fourier integral operators of order \(-2\) associated to \(\beta\).

**Proof.** The proof is reminiscent of that of Lemma 6.2 but is somewhat more complicated. For the sake of brevity, we concentrate on the new details and do not discuss the error estimate, which is similar to that in the proof of Lemma 6.2.

We substitute the WKB approximation for \(\partial_{\nu} G_0(\lambda, r, \theta, q(\phi'))\) (see Proposition 4.5) but not for \(G_0(\lambda, q(\phi), r, \theta)\) and apply the cutoff operator to the WKB expression. The integral
of concern is thus:

\[ H_0^{(1)}((k + i\tau)|q(\phi) + rv_q(\phi) - q(\phi)|)\chi(\frac{\gamma(q(\phi) + rv_q(\phi) - q(\phi))}{|q(\phi) + rv_q(\phi) - q(\phi)|} \cdot T_\theta) \]

\[ A(k + i\tau, \theta, r, \phi') e^{i(k + i\tau)|q(\theta) + rv_q(\phi) - q(\phi')|}(1 - \chi(k^{1-\delta}|q(\theta) - q(\phi')|)r d\theta dr, \]

where \( A \in S_{-1/2}^1 \).

We cannot use the WKB expression for the factor \( H_0^{(1)}((k + i\tau)|q(\theta) + r\nu_q(\theta) - q(\phi)|) \), so we deal with it by changing variables and explicitly integrating. The change of variables is given by \((r, \phi) \rightarrow (r, u)\), with:

\[
u(\theta) := \begin{cases} 
\frac{|q(\theta) + r\nu_q(\theta) - q(\phi)|^2 - r^2}{2(1 - r)(1 - \cos(\phi - \theta))}, & \phi \geq \theta \\
-(|q(\theta) + r\nu_q(\theta) - q(\phi)|^2 - r^2)^{1/2}, & \theta \geq \phi
\end{cases}
\]

Here, \( \phi - \theta \) and \( u \) range only over \((-k^{-1+\delta}, k^{-1+\delta})\). We claim that \( u(\phi) \) is smooth and invertible with uniform bounds on derivatives as \( r \) varies on \([0, \epsilon_0]\).

For the sake of brevity, we only verify this in the basic case of a circle of radius \( a \) and refer to \cite{Z GU} for the routine extension to the general case, which only requires showing that the quadratic approximation is sufficient to determine the smoothness of the change of variables. We recall here that we have microlocalized the integral to \( \gamma \) and that \( \partial \Omega \) has non-vanishing curvature at the reflection points of \( \gamma \). We then have:

\[ (|q(\theta) + r\nu_q(\theta) - q(\phi)|^2 - r^2) = 2(1 - r)(1 - \cos(\phi - \theta)). \]

For \(|\theta - \phi| \leq -k^{-1+\delta}, (1 - \cos(\theta - \phi)) \) has a smooth square root, given by the stated formula.

We now consider the effect of this change of variables on the remaining factors. For the exponential factor, we have

\[
\{ e^{i(k + i\tau)|q(\theta) + rv_q(\phi) - q(\phi')|} \rightarrow e^{i(k + i\tau)|q(\phi) - q(\phi')|}, A(k, \phi, \phi', u, r),
\]

where \( a = \sin(\angle(q(\phi) - q(\phi'))) \) and where \( A_1(k, \mu, \phi, \phi', u, r) \) is a polyhomogeneous symbol of order 0 in \( k \). We note that \( F(0, 0) = 1, G(0, 0) = -1/2k(q(\phi_1)) \). Also, the cutoff transforms as:

\[ \chi_{\partial\gamma}(\frac{q(\theta) + r\nu_q(\phi) - q(\phi')}{|q(\theta) + r\nu_q(\phi) - q(\phi')|} \cdot T_\theta) \rightarrow \chi_{\partial\gamma}(\frac{q(\phi) - q(\phi')}{|q(\phi) - q(\phi')|} \cdot T_\theta) K(k, r, q, \phi, \phi', \theta), \]

where \( K(k, r, q, \phi, \phi', \theta) \) is a smooth semiclassical amplitude whose leading order term equals 1.

After multiplying together these amplitudes, and rescaling the variables \( u \rightarrow ku, r \rightarrow kr \). In the regime \( \phi - \theta = O(k^{-1+\delta}) \), we obtain we obtain an amplitude \( A(k, \phi, \phi') \) such that:

\[ \chi_{\partial\gamma}(q, q', \frac{q - q'}{|q - q'|} e^{i(k + i\tau)|q(\phi) - q(\phi')|} A(k, \phi, \phi'); \]

\[ A(k, \phi, \phi') = k^{-1} \int_0^\infty \int_{-\infty}^{\infty} \chi(k^{1-\delta}u) \chi(k^{-\delta}r) \frac{e^{i\mu(u)} \sqrt{u^2 + r^2}}{u^2 + r^2} (b\sqrt{u^2 + r^2}) A_1((k + i\tau), u, \phi, \phi', r, \mu)) du dr, \]

\[ H_0^{(1)}((k + i\tau)|q(\phi) + rv_q(\phi) - q(\phi')|) \chi(\frac{q(\theta) + rv_q(\phi) - q(\phi')}{|q(\theta) + rv_q(\phi) - q(\phi')|} \cdot T_\theta) \]

\[ A(k + i\tau, \theta, r, \phi') e^{i(k + i\tau)|q(\theta) + rv_q(\phi) - q(\phi')|}(1 - \chi(k^{1-\delta}|q(\theta) - q(\phi')|)r d\theta dr, \]
where $b = \mu(1 + i\tau/k)$ and where $A_1$ is an amplitude of order $-1/2$. As a check on the order, we note that the normal derivative contributed a factor of $(k + i\tau)$ and the change of variables put in $k^{-2}$, leading to the stated power of $k$.

To complete the proof, we need to show that $A \in S_{\delta}^{-1/2}(T \times \mathbb{R})$. We analyse this integral (102), working by induction on the Taylor expansions of $A_1$ in the integral

$$\int_0^\infty \int_{-\infty}^\infty \chi(k^{-\delta} u) \chi(k^{-\delta} r) e^{i(au + ru)} H_0^{(1)}(b\sqrt{u^2 + r^2}) dudr.$$

We recall from Proposition (4.8) that:

$$J(r; a, b) := \int_0^\infty H_0^{(1)}(b\sqrt{u^2 + r^2}) \cos(au) du = -ie^{-ir\sqrt{b^2 - a^2}} \sqrt{b^2 - a^2}, \text{ valid if } b^2 - a^2 > 0.$$

Taylor expanding the $\cos \angle q(\theta) + r\nu_q(\theta) - q(\phi), \nu_q(\phi))$ leads to the following integrals:

$$\left\{ \begin{array}{ll}
\int_0^\infty e^{-iau} H_0^{(1)}(b\sqrt{u^2 + r^2}) \frac{r}{\sqrt{u^2 + r^2}} du = -\frac{\partial}{\partial \theta} J_k(r; a, b) \\
\int_0^\infty e^{-iau} H_1^{(1)}(b\sqrt{u^2 + r^2}) \frac{u}{\sqrt{u^2 + r^2}} du = \left[ \frac{\partial}{\partial \theta} - r \frac{\partial}{\partial \theta} \right] J_k(r; a, b)
\end{array} \right.$$

As in Proposition (4.6), the cutoff factor gives lower order terms in $k$.

We then integrate in $dr$. The basic integrals are:

$$\left\{ \begin{array}{ll}
\int_0^\infty e^{ia2r} \chi(k^{-\delta} r) \frac{\partial}{\partial \theta} e^{-ir\sqrt{b^2 - a^2}} dr &= -\frac{i}{b} \int_0^\infty e^{ia2r} \chi(k^{-\delta} r) e^{-ir\sqrt{b^2 - a^2}} dr \\
\int_0^\infty e^{ia2r} \chi(k^{-\delta} r) \left[ \frac{\partial}{\partial \theta} - r \frac{\partial}{\partial \theta} \right] e^{-ir\sqrt{b^2 - a^2}} dr &= \frac{\partial}{\partial \theta} \int_0^\infty e^{ia2r} \chi(k^{-\delta} r) e^{-ir\sqrt{b^2 - a^2}} dr \\
&\quad + \frac{1}{b} \int_0^\infty e^{ia2r} \chi(k^{-\delta} r) re^{-ir\sqrt{b^2 - a^2}} dr
\end{array} \right.$$

To leading order in $k$ the exponentials $e^{ia2r}$ and $e^{-ir\sqrt{b^2 - a^2}}$ cancel, and we find the integrals grow at the rates $k^{-2+\delta} \cos \theta^{-1}_{M_1}$. By differentiating in $(a_1, a_2)$, can obtain any term in the Taylor expansion of $F$ with remainder estimate. Putting in the higher order terms in the Taylor expansions of $F$ just adds lower order terms in $k^{-1}$, producing a symbol expansion as in the previous cases. Due to the cutoff factor $\chi_{\partial^1 \theta} \left( \frac{q(\theta) - q(\phi)}{|q(\theta) - q(\phi)|} \cdot T_{\theta} \right)$, the factors of $\cos \theta^{-1}_{M_1}$ are bounded above. The details are now similar to those in the proof of Lemma 6.2.

7.0.5. Codimension two. The final integral we must consider is

$$\int_0^\infty \int_{S^1} \chi_{\partial \Omega}^{k^{-1+\delta}}(x) \chi(k^{-1+\delta}(\theta - \phi)) \chi(k^{-1+\delta}(\theta - \phi')) G_0(\lambda, q, x) \chi(k^{-1} D_x, x) \partial_x G_0(\lambda, x, q) rdrd\theta.$$

In this case, we cannot use the WKB expansion for either Green’s function and must use integral formulas for products of Hankel functions. This is the most complicated case, and it is the one producing the pseudodifferential term in Proposition 7.1.

□
Lemma 7.5. Integral (104) defines a semiclassical pseudodifferential operator of order $-3$ with kernel of the form $a(\phi, D_\phi) \circ \chi_{\partial_\phi}$, i.e. with kernel of the form

$$
k \int_{\mathbb{R}} \chi_{\partial_\phi}(\phi, \eta)) e^{ik\eta(\phi - \phi')} A_k(\phi, \phi', \eta) d\eta,$$

where $A$ is a semiclassical amplitude of order $-3$.

Proof. The integral we are considering is:

(105) $$(k + i\tau) \int_{0}^{\varepsilon} \int_{\mathbb{T}} \int_{\mathbb{R}} \chi(k^{1-\delta}r) \chi(k^{1-\delta}(q(\phi) + rv_q(\theta) - q(\phi)))$$

$$\chi(k^{1-\delta}(q(\theta) + rv_q(\theta) - q(\phi'))) H_0^{(1)}((k + i\tau)|q(\theta) + rv_q(\theta) - q(\phi)|)$$

$$\chi(k^{-1}D_{\theta}, \theta, r) H_1^{(1)}((k + i\tau)|q(\theta) + rv_q(\theta) - q(\phi')|) \angle q(\theta) + rv_q(\theta) - q(\phi'), \nu_q(\theta)) rd\theta dr.$$ We substitute the Fourier integral formula for $\chi(k^{-1}D_{\theta}, \theta, r)$ to obtain:

(106) $$(k + i\tau) \int_{0}^{\varepsilon} \int_{\mathbb{T}} \int_{\mathbb{R}} \chi(k^{1-\delta}r) \chi(k^{1-\delta}(q(\phi) + rv_q(\theta) - q(\phi)))$$

$$\chi(k^{1-\delta}(q(\theta) + rv_q(\theta) - q(\phi'))) H_0^{(1)}((k + i\tau)|q(\theta) + rv_q(\theta) - q(\phi)|)$$

$$\chi(p_\theta, \theta, r)e^{ikpv(\theta - \theta')} H_1^{(1)}((k + i\tau)|q(\theta') + rv_q(\theta') - q(\phi')|)$$

$$\angle q(\theta') + rv_q(\theta') - q(\phi'), \nu_q(\theta')) rd\theta dr dp_\theta.$$ We then make the change of variables $(\theta, \theta', r) \rightarrow (u, u', r)$ defined in (107) with respect to the pairs $(\theta; \phi, r) \rightarrow (u, r)$ and $(\theta'; \phi', r) \rightarrow (u', r)$. Thus,

(107) $$u := \begin{cases} 
|q(\theta) + rv_q(\theta) - q(\phi)|^2 - r^2)^{1/2}, & \phi \geq \theta \\
-(|q(\theta) + rv_q(\theta) - q(\phi)|^2 - r^2)^{1/2}, & \theta \geq \phi
\end{cases}$$

while $u'$ is defined similarly with $(\theta', \phi')$ replacing $(\theta, \phi)$. We note that $u \sim (\theta - \phi)$ since

$$u^2 = |q(\theta) - q(\phi)|^2 - 2r\kappa(\phi)|q(\phi)|(|\theta - \phi| \cdot \sin \frac{|q(\theta) - q(\phi)|}{|q(\theta) - q(\phi)|} \cdot \nu_q(\theta)) + \cdots$$

This gives

(106) $$\int_{0}^{\varepsilon} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi(k^{1-\delta}u') \chi(k^{1-\delta}u)$$

$$\chi(p_\theta, u(\theta, r, \phi, r)) e^{ikpv(\theta(\theta, r) - \theta'(u', \phi', r))}$$

$$H_1^{(1)}((k + i\tau) \sqrt{u^2 + r^2}) H_0^{(1)}((k + i\tau) \sqrt{(u')^2 + r^2})$$

$$\frac{r F(u', r) + (u')^2 G(u', r)}{\sqrt{(u')^2 + r^2}} B_k(u, u', r, \phi, \phi') dudu'dp_\theta.$$ Here, $B_k$ is a smooth, polyhomogeneous amplitude. We recall that $\chi(p_\theta, u(\theta, r, \phi, r))$ is compactly supported in $p_\theta$ so that there is no problem of convergence of the $dp_\theta$ integral.
We then change to scaled variables \( k^{-1}u, k^{-1}u', k^{-1}r \) and expand
\[
\theta(k^{-1}u, k^{-1}r, \phi) = k^{-1}u + \phi + \cdots
\]
to obtain
\[
\tag{109} \chi(p_\theta, 0, \phi) = e^{ikp_\theta(\phi - \phi')} A_k(\phi, \phi', p_\theta),
\]
where
\[
A_k(\phi, \phi', p_\theta) = k^{-3} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \chi(k^{-3} \rho) \chi(k^{-3} u) \chi(k^{-3} u') \]
\[
\chi'(p_\theta, \theta(k^{-1}u, k^{-1}r, \phi), k^{-1}r) e^{ip_\theta(u - u')}
\]
\[
H_1^{(1)}(b_1 \sqrt{(u')^2 + r^2})(\frac{r}{\sqrt{(u')^2 + r^2}}) H_0^{(1)}(b_0 \sqrt{u^2 + r^2})(\frac{rF(u', r) + (u')^2G(u', r)}{\sqrt{(u')^2 + r^2}})
\]
\[
B_k(k^{-1}u, k^{-1}r, \phi, \phi', k^{-1}u') du'dr.
\]

We would like to prove that \( A_k \) is a symbol. As above, we work by induction on the Taylor expansion of \( F, G, B_k \). Polynomials in \( (u, u', r) \) may be expressed as sums of derivatives with respect to parameters \( (a_0, a_1, b_0, b_1) \) at \( a_0 = a_1 = \) of the integrals
\[
k^{-3} \int_0^\infty \int_{-\infty}^\infty \chi(k^{-3} \rho) \chi(k^{-3} u) \chi(k^{-3} u') \]
\[
\chi(p_\theta, \theta(k^{-1}u, k^{-1}r, \phi), k^{-1}r) e^{ip_\theta(u - u')}
\]
\[
H_1^{(1)}(b_1 \sqrt{(u')^2 + r^2})(\frac{r}{\sqrt{(u')^2 + r^2}}) H_0^{(1)}(b_0 \sqrt{u^2 + r^2}) du'dr,
\]
or the same with \( (\frac{r}{\sqrt{(u')^2 + r^2}}) \) replacing \( (\frac{r}{\sqrt{(u')^2 + r^2}}) \).

We now do the \( du_1 \) and \( du_0 \) integrals first as in the \( \phi_0 \sim \phi_1 \) case. The zeroth order terms in the usual Taylor expansions produce \( dr \) integrals of a form similar to \( \tag{103} \):
\[
\left\{
\begin{array}{l}
\frac{1}{\sqrt{b_0^2 - a_0^2}} b_1 \int_0^\infty e^{ia_2 r} \chi(k^{-3} r) e^{-i r \sqrt{b_0^2 - a_0^2}} dr \\
\frac{-i}{\sqrt{b_0^2 - a_0^2}} b_1 \frac{\partial}{\partial b_1} \int_0^\infty e^{ia_2 r} \chi(k^{-3} r) e^{-i r \sqrt{b_0^2 - a_0^2}} dr \\
\frac{1}{\sqrt{b_0^2 - a_0^2}} b \int_0^\infty e^{ia_2 r} \chi(k^{-3} r) r e^{-i r \sqrt{b_0^2 - a_0^2}} dr
\end{array}\right.
\]
\[
\tag{111}
\left\{
\begin{array}{l}
\frac{1}{\sqrt{b_0^2 - a_0^2}} b_1 \int_0^\infty e^{ia_2 r} \chi(k^{-3} r) e^{-i r \sqrt{b_0^2 - a_0^2}} dr \\
\frac{-i}{\sqrt{b_0^2 - a_0^2}} b_1 \frac{\partial}{\partial b_1} \int_0^\infty e^{ia_2 r} \chi(k^{-3} r) e^{-i r \sqrt{b_0^2 - a_0^2}} dr \\
\frac{1}{\sqrt{b_0^2 - a_0^2}} b \int_0^\infty e^{ia_2 r} \chi(k^{-3} r) r e^{-i r \sqrt{b_0^2 - a_0^2}} dr
\end{array}\right.
\]
The rest proceeds as in the proof of Lemma \( \tag{6.2} \). We conclude that
\[ \Box \]

8. Tail estimate

In this section we provide the remainder estimate. As discussed in the Introduction, the remainder comes from the Neumann series:
\[
(I + N(k + i\tau))^{-1} = \sum_{M=0}^{M_0} (-1)^M N(k + i\tau)^M + \mathcal{R}_{M_0}, \quad \mathcal{R}_{M_0} = N(k + i\tau)^{M_0 + 1} (I + N(k + i\tau))^{-1}.
\]
It is now important to take the imaginary part to be of the form \( \tau \log k \). We further specify \( \rho \) to have the following properties: \( \hat{\rho}(t) = \rho_0(t - L) \) where \( \rho_0 \in C_0^\infty(\mathbb{R}) \) is non-negative and supported in an \( \epsilon \)-interval around 0.

**Lemma 8.1.** For any \( R \), there exists \( M_0 = M_0(R) \) and

\[
Tr \rho * D(1 + i \tau \log k)R_{M_0} (1 + i \tau \log k)D(1 + i \tau \log k)A^{\nu} = O(k^{-R}).
\]

*Proof.* By Proposition 5.1 we can insert the cutoff \( \chi_{\partial \gamma} \) into the trace to obtain:

\[
(113) \quad \rho \circ (1 + i \tau \log k)S(1 + i \tau \log k)\chi(k)
\]

where \( S(k, q, x) = \mu_0G_0(k, x, y) \) is the transpose of the single layer potential. The cutoffs may be inserted at various points, and for simplicity of notation we often write only one in (the other can then also be inserted).

We regard the trace as a Hilbert-Schmidt inner product for Hilbert-Schmidt operators from \( L^2(\partial \Omega) \) to \( L^2(\Omega) \). For ease of notation, we the remainder at step \( M_0 + 1 \) and the trace is then

\[
\rho * (D(1 + i \tau \log k)(1 + N(1 + i \tau \log k))^{-1}N(1 + i \tau \log k), [N(1 + i \tau \log k)]^{\nu}HS).
\]

Since \( |\rho| = \rho_0 \) is a probability measure, we can estimate the convolution integral by Cauchy-Schwarz inequality as

\[
\rho_0 |(D(1 + i \tau \log k)(1 + N(1 + i \tau \log k))^{-1}N(1 + i \tau \log k), [N(1 + i \tau \log k)]^{\nu}HS) |^2.
\]

We further apply Schwartz’ inequality to the inner product to obtain the upper bound

\[
\rho_0 \cdot ||D(1 + i \tau \log k)(1 + N(1 + i \tau \log k))^{-1}N(1 + i \tau \log k)||^2 \text{HS}
\]

(116)

\[
\cdot ||N(1 + i \tau \log k)\chi_{\partial \gamma}S(1 + i \tau \log k)\chi(k)||^2 \text{HS}.
\]

We now separately estimate each factor. The first estimate is crude but sufficient for our purposes.

**Lemma 8.2.** \( ||D(1 + i \tau \log k)(1 + N(1 + i \tau \log k))^{-1}N||_{\text{HS}} = O(k^{1/2}) \).

*Proof.* We first use the inequality \( ||AB||_{\text{HS}} \leq ||A|| ||B||_{\text{HS}} \) where \( ||A|| \) is the operator norm to bound

\[
||D(1 + i \tau \log k)(1 + N(1 + i \tau \log k))^{-1}N||_{\text{HS}} \leq ||D(1 + i \tau \log k)(1 + N(1 + i \tau \log k))^{-1}|| ||N||_{\text{HS}}.
\]

Here, \( || \cdot || \) denotes the \( L^2(\partial \Omega) \to L^2(\Omega) \) operator norm.

By Proposition 4.1 (ii), we have that

\[
||N^2(1 + i \tau \log k)||_{\text{HS}} = O(k^{1/2}).
\]

By (37)-(38), the norm \( ||D(1 + i \tau \log k)(1 + N(1 + i \tau \log k))^{-1}|| \) is the norm of the Poisson operator. We claim that

\[
(117) \quad ||PI(1 + i \tau)||_{L^2(\partial \Omega) \to L^2(\Omega)} \leq \tau^{-1/2 + \epsilon}.
\]

This estimate is probably crude but it is sufficient for our purposes. We are not aware of any prior estimates on mapping norms of the Poisson kernel in the \( k \) aspect.
We first rewrite the Poisson integral of (37) as
\[ PI(k, x, q) = r_\Omega X_\nu G_\Omega(k, x, y) \]
where \( X_\nu \) is any smooth vector filed on \( \Omega \) which agrees with \( \partial_\nu \) on \( \partial\Omega \). The operator \( r_\Omega X_\nu \) is (roughly) of order \( 3/2 \), so we rewrite the composition as
\[ PI(k + i\tau, x, q) = r_\Omega X_\nu \Delta_\Omega^{3/4 - \epsilon} \Delta_\Omega^{3/4 + \epsilon} G_\Omega(k + i\tau, x, y). \]
Here, \( \Delta_\Omega \) is the Dirichlet Laplacian and the fractional powers are defined by the method of Seeley \[S1\]. Both \( X_\nu \) and \( r_\Omega \) operate in the \( y \) variable.

To prove (117), we use the (well known) fact that \( \Delta_\Omega^{3/4 - \epsilon} \) is a bounded operator on \( L^2(\Omega) \rightarrow H^{3/2 + \epsilon}(\Omega) \) for any \( \epsilon > 0 \). (See \[LM\], Theorem 9.4 for proof of the continuity for \( \epsilon > 0 \) and Theorem 9.5 for proof of lack of continuity if \( \epsilon = 0 \)). So the first factor is bounded independently of \( k + i\tau \).

For the second, we use that \( \Delta_\Omega^{3/4 + \epsilon} G_\Omega(k + i\tau, x, y) \) is the kernel of \( \Delta_\Omega^{3/4 + \epsilon} (\Delta_\Omega + (k + i\tau)^2)^{-1} \). This is clearly a bounded normal operator on \( L^2(\Omega) \), so its \( L^2(\partial\Omega) \rightarrow L^2(\partial\Omega) \) operator norm is given by
\[ \| \Delta_\Omega^{3/4 + \epsilon} (\Delta_\Omega + (k + i\tau)^2)^{-1} \|_{L^2(\Omega) \rightarrow L^2(\Omega)} = \max_j \frac{\lambda_j^{3/2 + \epsilon}}{|\lambda_j^2 + (k + i\tau)^2|}. \]
It is elementary to maximise this function, and one finds that the order of magnitude of the maximum occurs when \( \lambda_j \sim k \), and it then has the form
\[ \frac{k^{3/2 + \epsilon}}{|\tau k|} \sim \frac{k^{1/2 + \epsilon}}{\tau}. \]
This completes the proof of the Claim and hence of the Lemma. \( \square \)

We now give the crucial estimate. It explains why we did not need sharp estimates in the previous step.

**Proposition 8.3.** For any \( R \), there exists \( M_0 \) such that:
\[ \| \rho * N^{M_0}(k + i\tau \log k)\chi_{\partial\gamma} S^\ell(k + i\tau \log k)\chi(k) \|_{HS}^2 = O(k^{-R}). \]

**Proof.** \( \square \)

We estimate \( \| S^\ell(k + i\tau \log k)\chi(k^{-1}D_x) \| \) by a power of \( k \) as above, and thus reduce the proposition to estimating
\[ \| \rho * N(k + i\tau \log k)^{M_0} \chi_{\partial\gamma} \|_{HS}^2 \]
where the \( HS \) norm is now on Hilbert-Schmidt operators on \( L^2(\partial\Omega) \). We write out the Hilbert-Schmidt norm square as the trace:
\[ Tr \rho * N(k + i\tau \log k)^{M_0} \chi_{\partial\gamma} \chi_{\partial\gamma}^* N(k + i\tau \log k)^{M_0} \]
where the trace is on $L^2(\partial \Omega)$.

By Proposition 6.1

$$\tag{123} \sum_{\sigma_1, \sigma_2 : Z_{M_0} \to \{0,1\}} Tr \rho \ast F_{\sigma_1} \chi_{\partial \gamma} \chi_{\partial \gamma}^* F_{\sigma_2}^*$$

plus errors which may be assumed to be $O(k^{-R})$. We recall that $F_{\sigma}$ is a Fourier integral operator of order $-|\sigma|$ associated to $\beta^{M_0-|\sigma|}$. In addition, the phase of $F_{\sigma}$ has the form

$$\tag{125} ik(\mathcal{L}_{\sigma_1} - \mathcal{L}_{\sigma_2}) - \tau \log k(\mathcal{L}_{\sigma_1} + \mathcal{L}_{\sigma_2}),$$

where

$$\mathcal{L}_{\sigma}(q_1, \ldots, q_{M_0-|\sigma|}) = |q_1 - q_2| + \cdots + |q_{M_0-|\sigma|-1} - q_{M_0-|\sigma|}|$$

is the length of an $M_0 - |\sigma|$-link. The sign difference in the first term reflects the composition of $N^{M_0}$ with its adjoint, and since the second term comes from the real part of the phase there is no sign change.

We now estimate the traces by applying apply stationary phase. We are only interested in the order of the trace and not in the coefficients, so we argue qualitatively. The terms of the stationary phase expansion of $Tr \rho \ast F_{\sigma_1} \chi_{\partial \gamma} \chi_{\partial \gamma}^* F_{\sigma_2}^*$ correspond to the fixed points of the canonical transformation $\beta^{M_0-|\sigma|} \circ \beta^{-M_0-|\sigma|} = \beta^{-|\sigma_1|+|\sigma_2|}$ which lie in the support of the cutoffs $\chi_{\partial \gamma}$. By assumption, the only periodic point of $\beta$ in the support of the cutoff is the period 2 orbit corresponding to $\gamma$. Thus, other critical points (which we will call general critical points) can only occur when $|\sigma_1| = |\sigma_2|$, which we henceforth write as $|\sigma|$. Equivalently, the general critical points correspond to a closed path obtained by first following any $M_0 - |\sigma|$-link Snell path from a variable point $q_1 \in \partial \Omega$ to an endpoint $q_0$, and then reversing along the same path back to $q_1$. The general critical points thus form a non-degenerate critical manifold parametrized by $(q_1, q_0) \in \partial \Omega \times \partial \Omega$.

Now consider critical points in the support of $\chi_{\partial \gamma}$. It vanishes unless the first link points roughly in the direction of the first link of $\gamma$. Since critical paths are Snell, this forces all links to point roughly in the directions of links of $\gamma$. It follows that all links of critical paths (including $\gamma^*$ and the general ones) have lengths $\sim CL_\gamma$ for some absolute constant $C > 0$. Note that $C = 1/2$ for $\gamma$, so this is approximately correct for all links. Thus, the imaginary part of the phase introduces the damping factors

$$e^{-2C\tau \log k(M_0 - |\sigma|) L_\gamma}$$

into the stationary phase expansion. It follows that

$$\tag{127} Tr \rho \ast F_{\sigma_1} \chi_{\partial \gamma} \chi_{\partial \gamma}^* F_{\sigma_2}^* \sim k^{-2C\tau(M_0 - |\sigma|)L_\gamma} k^{-2|\sigma|} k^{-(M_0 - |\sigma|)+1} \int_{\partial \Omega \times \partial \Omega} \alpha$$

for some smooth density $\alpha$. We arrived at this order due to:

- the order $-|\sigma|$ of $F_{\sigma}$ (Proposition 6.1);
- The damping factor (126);
- Application of stationary phase to an $2M_0 - (|\sigma_1| + |\sigma_2|)$-fold integral with a 2-dimensional non-degenerate stationary phase manifold.
Given $R$, we need to choose $(M_0, \tau)$ so that
\[-2C\tau(M_0 - |\sigma|)L_\gamma - 2|\sigma| - (M_0 - |\sigma|) + 1 \leq -R\]
\[
\iff (M_0 - |\sigma|)(2C\tau L_\gamma + 1) + 2|\sigma| \geq R + 1.
\]
for every $\sigma$. For the case $|\sigma| = M_0$, it suffices to have $M_0 \geq (R+1)/2$. Otherwise, $M_0 - |\sigma| \geq 1$ and it suffices to pick $\tau \geq \frac{R+1}{C L_\gamma}$. With these choices of $(M_0, \tau)$ the inequality is true for all $\sigma$. 
\[
\square
\]
\[
\square
\]

9. Completion of the proof of Theorem 1.1

To complete the proof it suffices to determine the trace asymptotics of the regularized finite sums
\[
\sum_{M=0}^{M_0} \sum_{\sigma: \{1, \ldots, M\} \rightarrow \{0,1\}} Tr \rho \ast N_{\sigma(1)} \circ N_{\sigma(2)} \circ \cdots \circ N_{\sigma(M)} \circ \chi_\partial \gamma 
\]
\[
\circ S^{\ell r}(k + i\tau \log k) \circ \chi(k) \circ D\ell(k + i\tau \log k).
\]

We recall (see [1]) that $\rho \ast A(k)$ is short for
\[
\int_{\mathbb{R}} \rho(k - \mu)(\mu + i\tau)A(\mu)d\mu.
\]

**Proposition 9.1.** Let $\gamma$ be a periodic $m$-link reflecting ray, and let $\gamma^r$ be its $r$th iterate. Let $\hat{\rho} \in C_0^\infty(rL_\gamma - \epsilon, rL_\gamma + \epsilon)$ be a smooth cutoff, equal to one near $rL_\gamma$ and containing no other lengths in its support. Then there exist coefficients $a_{\gamma^r, \sigma, j}$ such that
\[
Tr \rho \ast N_{\sigma(1)} \circ N_{\sigma(2)} \circ \cdots \circ N_{\sigma(M)} \chi_\partial \gamma S^{\ell r}(k + i\tau \log k) \circ \chi(k) \circ D\ell(k + i\tau)
\]
\[
\sim \begin{cases} 
 e^{(i(k-r\log k)rL_\gamma)k^{-|\sigma|(\sum_{j=1}^{R}(a_{\gamma^r, \sigma, j} + a_{\gamma^{r-1}, \sigma, j}))k^{-j} + O(k^{-R})}}, & (M \geq mr), \\
 O_R(k^{-R}), & M < mr.
\end{cases}
\]
Here, $|\sigma| = \{j : \sigma(j) = 0\}$. In the special case where $M = mr$ and $\sigma(j) = 1$ for all $j$, we write $a_{\gamma^{r+1}, \sigma, j} = b_{\gamma^{r+1}, j}$.

The proof consists of a sequence of Lemmas. We begin by collecting the results of Propositions [6.1 and [7.1]

**Lemma 9.2.** $Tr \rho \ast N_{\sigma(1)} \circ N_{\sigma(2)} \circ \cdots \circ N_{\sigma(M)} \chi_\partial \gamma S^{\ell r}(k + i\tau \log k) \circ \chi(k) \circ (k)D\ell(k + i\tau)$ is a finite sum of oscillatory integrals of the form
\[
k^{(M-|\sigma|+3)/2} \int_{\mathbb{R}} \int_{\mathbb{T}^{M-1-|\sigma|}} e^{ik[(1-\mu)t+\mu L_\sigma(q(1), \ldots, q(\phi_{M-|\sigma|}))]} e^{-\tau \log k L(q(1), \ldots, q(\phi_{M-|\sigma|}))} 
\]
\[
\chi(q(\phi_1) - q(\phi_2), \phi_1) A(k\mu, \phi_1, \ldots, \phi_{M-|\sigma|}) \hat{\rho}(t) dt d\mu d\phi_1 \cdots d\phi_{M-|\sigma|},
\]
where
\[
\begin{cases}
 L_\sigma(q(1), \ldots, q(\phi_{M-|\sigma|})) = |q(\phi_1) - q(\phi_2)| + \cdots + |q(\phi_{M-|\sigma|}) - q(\phi_1)|\n 
\chi(q(\phi_1) - q(\phi_2), \phi_1) = \end{cases}
\]
and where \( A(k, \phi_1, \ldots, \phi_{M-|\sigma|}) \in S^{-|\sigma|}_0 \).

**Proof.** There are two somewhat different cases, namely the case where \(|\sigma| < M\) and the case where \(|\sigma| = M\).

9.0.6. **Case (i):** \(|\sigma| < M\). By Proposition 6.1

\[
N_{\sigma(1)} \circ N_{\sigma(2)} \circ \cdots \circ N_{\sigma(M)}(k, \phi_1, \phi_2) = F_{\sigma}(k, \phi_1, \phi_2),
\]

where \( F_{\sigma} \) is a semiclassical Fourier integral kernel of the form

\[
F_{\sigma}(k, \phi_1, \phi_2) = e^{i(k+i\tau)|q(\phi_1)-q(\phi_2)|} \chi(k^{1-\delta}(\phi_1 - \phi_2))A_{\sigma}(k, \phi_1, \phi_2),
\]

with \( A_{\sigma} \in I^{-|\sigma|}(T^2) \). By Proposition 6.1 the full composition

\[
N_{\sigma(1)} \circ N_{\sigma(2)} \circ \cdots \circ N_{\sigma(M)}S^{\ell \tau}(k + i\tau)\chi_{\partial \gamma} \circ \chi(k) \circ D\ell(k + i\tau)
\]

has the form

\[
F_{\sigma}(k) \circ \chi_{\partial \gamma} \circ [D_0 + D_1],
\]

where the operators \( D_0, D_1 \) are from Proposition 7.1, \( D_1 \) puts in an extra factor of \( T \) and an amplitude of order \(-1\). All of these operators are semiclassical Fourier integral operators with symbols in \( S^0_0 \) and therefore they can be composed in the standard way. Therefore we have:

\[
F_{\sigma}(k) \circ \chi_{\partial \gamma}S^{\ell \tau}(k + i\tau \log k) \circ \chi(k) \circ D\ell(k + i\tau \log k) \in I^{-|\sigma|-1}_0(T^2).
\]

We then unravel \( \rho^* \) and recall that the factor \((k - \mu)\) raises the order by one. Finally we change variables \( \mu \to k\mu \) which again raises the order by one.

9.0.7. **Case (ii):** \(|\sigma| = M\). By Proposition 6.1 \( N_0^M \circ \chi_{\partial \gamma} \) is a \(-M\)th order semiclassical pseudodifferential operator, so it suffices to consider composition of the form \( A_{-M} \circ \chi_{\partial \gamma}[D_0 + D_1] \). The statement is clear in this case.

The next step is to show that the stationary phase method applies to oscillatory integrals in \( I^{-r}_\delta(Y, \Phi) \). This is almost obvious, but for the sake of completeness we include the proof.

**Lemma 9.3.** Let \( I_k(a, \Phi) \in I^{-r}_\delta(Y, \Phi) \), and let \( C_\Phi \) denote the set of critical points of \( \Phi \), and assume that \( C_\Phi \) is a non-degenerate critical manifold of codimension \( q \). Suppose that \( 1/2 < \delta < 1 \). Then:

\[
I_k(a, \Phi) \sim \begin{cases} 
  k^{-r-q/2} \sum_{j=0}^{R} a_j k^{-j} + \text{Rem}_{r,R}(a, \Phi, R) k^{-R-1}, & C_\Phi \neq \emptyset \\
  \text{Rem}_{M,R}(a, \Phi, R) k^{-R}, & C_\Phi = \emptyset
\end{cases}
\]

where \( a_j \) is a polynomial in the jet of the amplitude \( a \) and phase \( \Phi \) at \( C_\Phi \), and where the remainder \( \text{Rem}_{R}(a, \Phi, k) \leq ||a||_{C^{3R}} + ||\Phi||_{C^{3R}} \).

**Proof.** We consider an oscillatory integral

\[ I_k(a, \Phi) = \int_Y e^{ik\Phi} A(k, y) dy, \]
with \( \Phi \in S^0_\delta \), with \( A(k, y) \in S^0_\delta(Y) \). By assumption, the critical set \( C_\Phi = \{ y : \nabla_y \Phi = 0 \} \) of the phase is a non-degenerate critical manifold. We choose a cutoff \( \psi \) supported near \( C_\Phi \) and write

\[
I(a, \Phi) = I(a\psi, \Phi) + I(a(1 - \psi), \Phi).
\]

We now show that if \( a \in S^0_\delta \), for some \( r \), then \( I(a(1 - \psi), \Phi) = O(k^{-R}) \) for any \( R > 0 \). The implicit constant is of linear growth in \( r \).

In the usual way, we integrate by parts repeatedly with the operator:

\[
L_y = \frac{1}{k|\nabla_y \Phi|^2} \nabla_y \Phi \cdot \nabla_y,
\]

that is, we apply the transpose

\[
L_y^t = L_y + \frac{1}{k} \nabla \cdot \left( \frac{\nabla_y \Phi}{|\nabla \Phi|^2} \right)
\]

to the amplitude. The second term is a scalar multiplication.

We first observe that the coefficients of \( L \) belong to \( S^0_\delta(Y \times \mathbb{R} \times \mathbb{R}) \). Indeed, by assumption \( L \in S^0_\delta(Y) \). In the expression

\[
\nabla \cdot \left( \frac{\nabla \Phi}{|\nabla \Phi|^2} \right) = \frac{\Delta \Phi}{|\nabla \Phi|^2} + \frac{\nabla^2 \Phi(\nabla \Phi, \nabla \Phi)}{|\nabla \Phi|^4} = \frac{\Delta_y L}{|\nabla \Phi|^2} + \frac{\nabla^2_y L(\nabla L, \nabla L)}{|\nabla \Phi|^4},
\]

it is then obvious that the numerator belongs to \( S^0_\delta(Y) \). Since the denominator \( |\nabla \Phi|^2 = |\nabla_y L|^2 \) is bounded below on \( \text{supp}(1 - \psi) \), it follows that the coefficients belong to \( S^0_\delta(Y) \).

We now verify that each partial integration lowers the symbol order by one unit of \( k^\delta \), i.e. that \( (k^{-1}L)^R A(k, y) \in S^0_\delta^{-R\delta}(Y) \). We prove this by induction on \( R \). As \( R \to R + 1 \), we apply one of two terms of \( L^t \). We know that each differentiation improves the symbol order by one unit of \( k^\delta \). But each coefficient multiplies by an element of \( S^0_\delta(Y) \), hence preserves symbol order.

The remainder has the form

\[
|\text{Rem}_{M,R}(a, \Phi, R)| \leq \sup_{(t, \mu, x, \phi)} \max_{\alpha: \alpha \leq 2R} |D^\alpha \rho(t)(1 - \psi_T)(y)\chi(1 - \mu)A_k((t, \mu, x, \phi)|.
\]

Since \( \rho(t)(1 - \psi_T)(y)\chi(1 - \mu)A_k((t, \mu, x, \phi) \in S^0_\delta(\mathbb{R} \times \mathbb{R} \times Y) \), the right side is \( O(k^{(1-\delta)2R}) \).

If we choose \( \delta \) satisfying \( 1 > \delta > 1/2 \), then \( k^{-R}k^{(1-\delta)2R} = k^{-(2\delta-1)R} \) gives a negative power of \( k \). This is sufficient for the proof of the remainder estimate.

This completes the proof.

\[
\square
\]

Combining Lemmas 9.2-9.3, we obtain

\[
k^{(M-|\sigma|)/2} \int_{\mathbb{R}} \int_{\mathbb{T}^M_{-|\sigma|+1}} e^{ik[(1-\mu)t+\mu L(q(\phi_1),\ldots,q(\phi_{M-|\sigma|}))]} e^{-\tau \log kE(q(\phi_1),\ldots,q(\phi_{M-|\sigma|+1}))} \\
\chi(q(\phi_1)-q(\phi_2),\phi_1)A(k\mu, \phi_1, \ldots, \phi_{M-|\sigma|})\hat{\rho}(t)dt d\mu d\phi_1 \cdots d\phi_{M-|\sigma|+1}
\]

\[
\sim \left\{ \begin{array}{ll}
e^{ik - \tau \log kL} k^{-|\sigma|}\{\sum_{j=1}^R (a_{\gamma,\sigma,j} + a_{\gamma-r,\sigma,j})k^{-j} + O(k^{-R})\}, & (M \geq mr), \\
O_R(k^{-R}), & M < mr
\end{array} \right.
\]

\[
(136)
\]
Indeed, the critical point set of the phase \( \Phi = t(1 - \mu) + \mu \mathcal{L} \) is given by:

\[
C_\Phi = \{(t, \mu, \phi_1, \ldots, \phi_{M-|\sigma|}) : \mu = 1, t = \mathcal{L}(\phi_1, \ldots, \phi_{M-|\sigma|}), q(\phi_1), \ldots, q(\phi_{M-|\sigma|}) = \gamma^r \}.
\]

Clearly, the stationary phase set is empty if \( M < mr \). When \( M = rm \) the phase is non-degenerate and we obtain the result stated in Proposition 9.1 by stationary phase. This completes the proof of Theorem 1.1.

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