FRACTIONAL-ORDER OPERATORS:
BOUNDARY PROBLEMS, HEAT EQUATIONS

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Abstract. The first half of this work gives a survey of the fractional Laplacian (and related
operators), its restricted Dirichlet realization on a bounded domain, and its nonhomogeneous
local boundary conditions, as treated by pseudodifferential methods. The second half takes up
the associated heat equation with homogeneous Dirichlet condition. Here we recall recently
shown sharp results on interior regularity and on $L^p$-estimates up to the boundary, as well as
recent Hölder estimates. This is supplied with new higher regularity estimates in $L^2$-spaces
using a technique of Lions and Magenes, and higher $L^p$-regularity estimates (with arbitrarily
high Hölder estimates in the time-parameter) based on a general result of Amann. Moreover,
it is shown that an improvement to spatial $C^\infty$-regularity at the boundary is not in general
possible.

0. Introduction

This work is partly a survey of known results for the fractional Laplacian and its generalizations,
with emphasis on pseudodifferential methods and local boundary conditions. Partly it brings new results for the associated heat equation.

There is an extensive theory for boundary value problems and evolution problems for elliptic differential operators, developed through many years and including nonlinear problems, and problems with data of low smoothness. Boundary and evolution problems for pseudodifferential operators, such as fractional powers of the Laplacian, have been studied far less, and pose severe difficulties since the operators are nonlocal.

The presentation here deals with linear questions, since this is the basic knowledge one needs in any case. Our main purpose is to explain the application of pseudodifferential methods (with the fractional Laplacian as a prominent example). The boundary value theory has been established only in recent years. Plan of the paper:

(1) Fractional-order operators.
(2) Homogeneous Dirichlet problems on a subset of $\mathbb{R}^n$.
(3) Nonhomogeneous boundary value problems.
(4) Heat equations.

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Remark. There are other strategies currently in use, such as methods for singular integral operators in probability theory and potential theory (cf. e.g. [BBC03, CD14, CS98, CT04, FK13, FR17, J02, JX15, K97, R16, RS14, RS14a, RS15, RSV17, RV18]), methods for embedding the problem in a degenerate elliptic differential operator situation ([CS07] and many subsequent studies, e.g. [CSS08]), and exact calculations in polar coordinates for the ball (cf. e.g. [AJS18, DKK17, DG17, ZG16]). Each of the methods allow different types of generalizations of the fractional Laplacian, and solve a variety of problems, primarily in low-regularity spaces. It is perhaps surprising that the methods from the calculus of pseudodifferential operators have only entered the modern studies in the field in the last few years.

The new regularity results in Section 4 on heat problems, reaching beyond the recent works [FR17, G18, RV18], are: Theorem 4.2 and its corollary giving a limitation on high spatial regularity, Theorems 4.6–8 on estimates in $L_2$-related spaces for $x$-dependent operators, and Theorems 4.14–19 on high estimates in Hölder spaces with respect to time, valued in $L_2$- or $L_p$-related spaces in $x$ (including a Hölder-related space in $x$ as a limiting case).

1. Fractional-order operators

The fractional Laplacian $P = (-\Delta)^a$ on $\mathbb{R}^n$, $0 < a < 1$, has linear and nonlinear applications in mathematical physics and differential geometry, and in probability and finance. (See e.g. Frank-Geisinger [FG16], Boulenger-Himmelsbach-Lenzmann [BHL16], Gonzales-Mazzeo-Sire [GMS12], Monard-Nickl-Paternain [MNP18], Kulczycki [K97], Chen-Song [CS98], Jakubowski [J02], Bogdan-Burdzy-Chen [BBC03], Applebaum [A04], Cont-Tankov [CT04], and their references.)

The interest in probability and finance comes from the fact that $-P$ generates a semigroup $e^{-tP}$ which is a stable Lévy process. Here $P$ is viewed as a singular integral operator:

$$(-\Delta)^a u(x) = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(x) - u(x + y)}{|y|^{n+2a}} dy;$$

more general stable Lévy processes arise from operators

$$Pu(x) = PV \int_{\mathbb{R}^n} (u(x) - u(x + y))K(y) dy, \quad K(y) = \frac{K(y/|y|)}{|y|^{n+2a}},$$

where the homogeneous kernel function $K(y)$ is locally integrable, positive, and even: $K(-y) = K(y)$, on $\mathbb{R}^n \setminus \{0\}$. (1.2) can also be generalized to nonhomogeneous kernel functions satisfying suitable estimates in terms of $|y|^{-n-2a}$. Usually, only real functions are considered in probability studies.

$(-\Delta)^a$ can instead be viewed as a pseudodifferential operator ($\psi$do) of order $2a$:

$$(-\Delta)^a u = Op(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a}\mathcal{F}u(\xi)),$$

using the Fourier transform $\mathcal{F}$, defined by $\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x) dx$ (extended from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing $C^\infty$-functions, to the temperate distributions $\mathcal{S}'(\mathbb{R}^n)$). $\psi$do’s are in general defined by

$$Pu = Op(p(x,\xi))u = \mathcal{F}_{\xi\to x}^{-1}(p(x,\xi)\mathcal{F}u(\xi));$$
note that this theory operates in a context of complex functions (and distributions). Moreover, \((1.4)\) allows \(x\)-dependence in the symbol \(p(x, \xi)\).

In \((1.2)\), if \(K \in C^\infty(\mathbb{R}^n \setminus \{0\})\), the operator it defines is the same as the operator defined by \(p(\xi) = \mathcal{F}K(y)\) in \((1.4)\); here \(p(\xi)\) is homogeneous of degree \(2a\), positive and even.

As a generalization of \((1.3)\), we consider \(x\)-dependent classical \(\psi\)do’s of order \(2a \in \mathbb{R}_+\), with certain properties. That \(P\) is classical of order \(2a\) means that there is an asymptotic expansion of the symbol \(p(x, \xi)\) in a series of terms \(p_j(x, \xi), j \in \mathbb{N}_0\), that are homogeneous in \(\xi\) of order \(2a - j\) for \(|\xi| \geq 1\); the expansion holds in the sense that
\[
|\partial_\xi^\alpha \partial_x^\beta [p(x, \xi) - \sum_{j<M} p_j(x, \xi)]| \leq C_{a,\beta,M} \langle \xi \rangle^{2a-|\alpha|-M}, \text{ all } \alpha, \beta \in \mathbb{N}_0^n, M \in \mathbb{N}_0;
\]

here \(\langle \xi \rangle\) stands for \((|\xi|^2 + 1)^{1/2}\). To the operators defined from these symbols by \((1.4)\) one adds the smoothing operators mapping \(\mathcal{E}'(\mathbb{R}^n)\) to \(C^\infty(\mathbb{R}^n)\) (also called negligible operators). We assume moreover that \(p\) is \textbf{even}, meaning that
\[
p_j(x, -\xi) = (-1)^j p_j(x, \xi), \text{ all } j,
\]
and \textbf{strongly elliptic}, meaning that for a positive constant \(c\),
\[
(1.7) \quad \Re p_0(x, \xi) \geq c|\xi|^{2a} \text{ for } |\xi| \geq 1.
\]

Then \(P = \text{Op}(p(x, \xi))\) can be shown to have some of the same features as \((-\Delta)^a\). To sum up, we are assuming, with \(a \in \mathbb{R}_+:\)

**Hypothesis 1.1.** \(P\) is a classical pseudodifferential operator of order \(2a\), even and strongly elliptic (cf. \((1.4)\), \((1.6)\), \((1.7)\)).

In part of Section 4, we moreover assume \(a < 1\). For some results we consider the subset of operators defined as in \((1.2)\)ff. with a kernel function \(K(y)\) that is smooth outside 0:

**Hypothesis 1.2.** \(P\) is as in \((1.2)\), with \(K(y)\) positive, homogeneous of degree \(-n - 2a\), even, and \(C^\infty\) on \(\mathbb{R}^{n-1} \setminus \{0\}\).

This fits into the \(\psi\)do formulation, when we write \(p(\xi) = \mathcal{F}K(y)\) as \((1 - \chi(\xi))p(\xi) + \chi(\xi)p(\xi),\) with \(\chi \in C^\infty_0(\mathbb{R}^n)\) and \(\chi(\xi) = 1\) near 0. Here the first term is a symbol as under Hypothesis 1.1, and the operator defined from the second term maps large spaces of distributions (e.g. \(\mathcal{E}'(\mathbb{R}^n)\)) into \(C^\infty\)-functions (hence is a negligible operator).

To give an example of an \(x\)-dependent operator, we can mention that \((-\Delta)^a\) will take the \(x\)-dependent form if it undergoes a smooth change of coordinates. As a more general example, \(P\) can be an operator defined as \(P = A(x, D)^a\), where \(A(x, D)\) is a second-order strongly elliptic differential operator. Here \(P\) is constructed via the resolvent (Seeley [S69]).

But of course, the symbol \(p(x, \xi)\) can be taken much more general, not tied to differential operator considerations.

A difficult aspect of such operators is that they are \textbf{nonlocal}. This is a well-known feature in the pseudodifferential theory, where one can profit from pseudo-locality (namely, \(Pu\) is \(C^\infty\) on the set where \(u\) is \(C^\infty\)). In a different approach, Caffarelli and Silvestre [CS07] showed that \((-\Delta)^a\) on \(\mathbb{R}^n\) is the Dirichlet-to-Neumann operator for a degenerate elliptic \textbf{differential} boundary value problem on \(\mathbb{R}^n \times \mathbb{R}_+\); \textbf{local} in dimension \(n + 1\). This observation was then used to obtain results by transforming problems for \((-\Delta)^a\) into problems for \textbf{local} operators in one more variable, e.g. in [CSS08]. (However, in some cases where one needs to consider \((-\Delta)^a\) over a subset \(\Omega \subset \mathbb{R}^n\), the transformation might lead to equally difficult problems in the new variables.)
2. Homogeneous Dirichlet problems on a subset of $\mathbb{R}^n$

How do we get $P$ to act over $\Omega$? There are several ways to answer this. Let us first introduce an appropriate scale of $L_p$-based Sobolev spaces.

The standard Sobolev-Slobodetskiǐ spaces $W^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$ and $s \geq 0$, have a different character according to whether $s$ is integer or not. Namely, for $s$ integer, they consist of $L_p$-functions with derivatives in $L_p$ up to order $s$, hence coincide with the Bessel-potential spaces $H^s_p(\mathbb{R}^n)$, defined for $s \in \mathbb{R}$ by

$$H^s_p(\mathbb{R}^n) = \{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u}) \in L_p(\mathbb{R}^n) \}. \quad (2.1)$$

For noninteger $s$, the $W^{s,p}$-spaces coincide with the Besov spaces, defined e.g. as follows: For $0 < s < 2$,

$$f \in B^s_p(\mathbb{R}^n) \iff \| f \|_p^p + \int_{\mathbb{R}^{2n}} \frac{|f(x) - f(y) - 2f((x+y)/2)|^p}{|x+y|^{n+ps}} \, dx \, dy < \infty; \quad (2.2)$$

and $B^{s+t}_p(\mathbb{R}^n) = (1 - \Delta)^{-t/2}B^s_p(\mathbb{R}^n)$ for all $t \in \mathbb{R}$.

The Bessel-potential spaces $H^s_p$ are important because they are most directly related to $L_p$; the Besov spaces $B^s_p$ have other convenient properties, and are needed for boundary value problems in an $H^s_p$-context, because they are the correct range spaces for trace maps (both from $H^s_p$ and $B^s_p$-spaces); see e.g. the overview in the introduction to [G90]. For $p = 2$, the two scales are identical, and $p$ is usually omitted. For $p \neq 2$ they are related by strict inclusions:

$$H^s_p \subset B^s_p \text{ when } p > 2, \quad H^s_p \supset B^s_p \text{ when } p < 2. \quad (2.3)$$

When working with operators of noninteger order, the use of the $W^{s,p}$-notation can lead to confusion since the definition depends on the integrality of $s$; moreover, this scale does not always interpolate well. In the following, we focus on the Bessel-potential scale $H^s_p$, but much of what we show is directly generalized to the Besov scale $B^s_p$, and to other scales (Besov-Triebel-Lizorkin spaces). There is a more general Besov scale $B^s_{p,q}$ (cf. e.g. Triebel [T78]), where $B^s_p$ equals the special case $B^s_{p,p}$.

There is an identification of $H^s_p(\mathbb{R}^n)$ with the dual space of $H^{-s}_{p'}(\mathbb{R}^n)$, $1/p + 1/p' = 1$, in a duality consistent with the $L_2$-duality, and there is a similar result for the Besov scale.

Let $\Omega$ be a open subset of $\mathbb{R}^n$ (we shall use it with $C^\infty$-boundary, but much of the following holds under limited smoothness assumptions). One defines the two associated scales relative to $\Omega$ (the restricted resp. supported version):

$$\overline{H}^s_p(\Omega) = r^+H^s_p(\mathbb{R}^n), \quad \dot{H}^s_p(\overline{\Omega}) = \{ u \in H^s_p(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\Omega} \}; \quad (2.4)$$

here $\text{supp } u$ denotes the support of $u$ (the complement of the largest open set where $u$ is zero). Restriction from $\mathbb{R}^n$ to $\Omega$ is denoted $r^+$, extension by zero from $\Omega$ to $\mathbb{R}^n$ is denoted $e^+$ (it is sometimes tacitly understood). Restriction from $\overline{\Omega}$ to $\partial \Omega$ is denoted $\gamma_0$.

When $s > 1/p - 1$, one can identify $H^s_p(\overline{\Omega})$ with a subspace of $\overline{H}^s_p(\Omega)$, closed if $s - 1/p \notin \mathbb{N}_0$ (equal if $1/p - 1 < s < 1/p$), and with a stronger norm if $s - 1/p \notin \mathbb{N}_0$. 


The space $\dot{H}_p^s$ is in some texts indicated with a ring, zero or twiddle, as e.g. $\dot{H}_p^s$, $H_p^s$ or $\tilde{H}_p^s$. In most current texts, $\overline{H}_p^s(\Omega)$ is denoted $H_p^s(\Omega)$ without the overline (that was introduced along with the notation $\dot{H}_p$ in [H65,H85]), but we prefer to use it, since it makes the role of the space more clear in formulas where both types occur.

Now let us present some operators associated with $(−\Delta)^a$ on $\Omega$:

(a) The **restricted** Dirichlet fractional Laplacian $P_{\text{Dir}}$. It acts like $P = (−\Delta)^a$, defined on functions $u$ that are 0 on $\mathbb{R}^n \setminus \Omega$, and followed by restriction $r^+$ to $\Omega$:

$$P_{\text{Dir}} u \text{ equals } r^+ Pu \text{ when } \text{supp} u \subset \overline{\Omega}.$$  

In $L_2(\Omega)$, it is the operator defined variationally from the sesquilinear form

$$Q_0(u,v) = \frac{1}{2}c_{n,a} \int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(\bar{v}(x) - \bar{v}(y))}{|x - y|^{n+2a}} \, dx\, dy, \quad D(Q_0) = \dot{H}_p^a(\Omega).$$  

(b) The **spectral** Dirichlet fractional Laplacian $(-\Delta_{\text{Dir}})^a$, defined e.g. via eigenfunction expansions of $-\Delta_{\text{Dir}}$. It does not act like $r^+ Pe^+$. It is not often used in probability applications. (Its regularity properties in $L_p$-Sobolev spaces are discussed in [G16], which gives many references to the literature on it.)

(c) The **regional** fractional Laplacian, defined from the sesquilinear form

$$Q_1(u,v) = \frac{1}{2}c_{n,a} \int_{\Omega \times \Omega} \frac{(u(x) - u(y))(\bar{v}(x) - \bar{v}(y))}{|x - y|^{n+2a}} \, dx\, dy, \quad D(Q_1) = \overline{H}_p^a(\Omega).$$  

It acts like $r^+ Pe^+ + w$ with a certain correction function $w$.

There are still other operators over $\Omega$ that can be defined from $P$, e.g. representing suitable Neumann problems. A local Neumann condition will be discussed below in Section 3. We refer to [G16] Sect. 6, and its references, for an overview over the various choices.

We shall now focus on (a), where the operator acts like $r^+ P$.

The homogeneous Dirichlet problem, for a smooth bounded open set $\Omega$, is

$$r^+ Pu = f \text{ in } \Omega, \quad \text{supp } u \subset \overline{\Omega}.$$  

As $P$ we take $(-\Delta)^a$, or a more general $\psi\Delta$ as in Hypothesis 1.1 or 1.2.

$P_{\text{Dir}}$ in $L_2(\Omega)$ is the realization of $r^+ P$ with domain

$$D(P_{\text{Dir}}) = \{ u \in \dot{H}_p^a(\overline{\Omega}) | r^+ Pu \in L_2(\Omega) \}.$$  

When $P$ satisfies Hypothesis 1.2 (in particular, when $P = (-\Delta)^a$), then $P_{\text{Dir}}$ is positive selfadjoint; for other $P$ it is sectorial, with discrete spectrum in a sector. What can be said about the regularity of functions in the domain?

• Vishik and Eskin showed in the 1960’s (see e.g. Eskin [E81]):

$$D(P_{\text{Dir}}) = \dot{H}_p^{2a}(\overline{\Omega}) \text{ if } a < \frac{1}{2}, \quad D(P_{\text{Dir}}) \subset \dot{H}_p^{a+\frac{1}{2}-\varepsilon}(\overline{\Omega}) \text{ if } a \geq \frac{1}{2}.$$  

Ros-Oton and Serra [RS14] showed in 2014 for $(-\Delta)^\alpha$:

(2.11) \[ f \in L_\infty(\Omega) \implies u \in d^\alpha C^\alpha(\Omega) \] for small $\alpha$;

here $d(x)$ equals $\text{dist}(x, \partial\Omega)$ near $\partial\Omega$, and $C^\alpha$ is the Hölder space. They improved this later to $\alpha < a$ ($\alpha = a$ in some cases), and to more general $P$ as in (1.2), and lifted the regularity conclusions to $f \in C^\gamma, u/d^\alpha \in C^{\alpha+\gamma}$ for small $\gamma$. For (2.11), $\Omega$ was assumed to be $C^{1,1}$.

We showed in 2015 [G15], for $1 < p < \infty$ and $\Omega$ smooth:

(2.12) \[ f \in H^s_p(\Omega) \iff u \in H^{a(s+2a)}_p(\Omega), \text{ any } s \geq 0, \]

\[ f \in C^\infty(\Omega) \iff u/d^a \in C^\infty(\Omega); \]

here $H^{a(s+2a)}_p(\Omega)$ is a space introduced by Hörmander [H65] for $p = 2$. E.g. when $s = 0$,

(2.13) \[ H^{a(2a)}_p(\Omega) = \begin{cases} \dot{H}^{2a}_p(\Omega) & \text{if } a < 1/p, \\
\subset \dot{H}^{2a-\epsilon}_p(\Omega) & \text{if } a = 1/p, \\
\subset \dot{H}^{2a}_p(\Omega) + d^a H^a_p(\Omega) & \text{if } a > 1/p; \end{cases} \]

the spaces will be further explained below. (2.12) has corollaries in Hölder spaces by Sobolev embedding.

The contribution from Hörmander, accounted for in detail in [G15], is in short the following: He defined the \textit{µ-transmission property} in his book 1985, Sect. 18.2:

**Definition 2.1.** A classical $\psi do$ $P$ of order $m$ has the $\mu$-transmission property at $\partial\Omega$, when

(2.14) \[ \partial_x^\beta \partial_\xi^\alpha p_j(x, -\nu) = e^{\pi i (m - 2\mu - j - |\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, \nu), \]

for all indices; here $x \in \partial\Omega$, and $\nu$ denotes the interior normal at $x$.

The property was formulated already in a photocopied lecture note from IAS Princeton 1965-66 on $\psi do$ boundary problems [H65], handed out to a few people through the times, including Boutet de Monvel in 1968, the present author in 1980.

For $P$ of order $2a$ and even (cf. (1.6)), it holds with $\mu = a$, for any smooth subset $\Omega$ (all normal directions are covered when (1.6) holds).

The case $\mu = 0$ is the transmission condition entering in the calculus of Boutet de Monvel, described e.g. in [B71,G96,S01,G09].

Recalling that $e^+$ denotes extension by zero, let

(2.15) \[ E_a(\Omega) = e^+ d^a C^\infty(\Omega). \]

Then by [H85], Th. 18.2.18,
Theorem 2.2. The a-transmission property at $\partial\Omega$ is necessary and sufficient in order that $r^+P$ maps $E_a(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$.

Note the importance of $d^a$.

The notation in [H85] is slightly different from that in the notes [H65], which we have adapted here. The notes moreover treated solvability questions, with $f \in C^\infty(\overline{\Omega})$ or in $H^s$-spaces. The space $H^{a(s)}(\overline{\Omega})$ was introduced. Originally it was defined as “the functions supported in $\overline{\Omega}$ that are mapped into $\mathcal{P}^{s-m}(\Omega)$ for any $P$ that is elliptic of order $m$ and has the a-transmission property”, and the whole effort was to sort this out. We shall now explain the structure and its implications, for general $H^s$-spaces with $1 < p < \infty$.

Introduce first order-reducing operators of plus/minus type. For $\Omega = \mathbb{R}^n_+$, define for $t \in \mathbb{R}$:

\begin{equation}
\Xi^t_\pm = \text{Op}((\langle \xi' \rangle \pm i\xi_n)^t) \text{ on } \mathbb{R}^n.
\end{equation}

The symbols extend analytically in $\xi_n$ to $\text{Im} \xi_n \leq 0$. Hence, by the Paley-Wiener theorem, $\Xi^t_\pm$ preserve support in $\mathbb{R}^n_{\pm}$. Then for all $s \in \mathbb{R}$,

\begin{equation}
\Xi^t_+: \mathcal{H}^s_p(\mathbb{R}^n_+) \to \mathcal{H}^{s-t}_p(\mathbb{R}^n_+), \text{ with inverse } \Xi^{-t}_+,
\end{equation}

\begin{equation}
r^+\Xi^t_+ e^+: \mathcal{H}^s_p(\mathbb{R}^n_+) \to \mathcal{H}^{s-t}_p(\mathbb{R}^n_+), \text{ with inverse } r^+\Xi^{-t}_+ e^+.
\end{equation}

Here the action of $e^+$ on spaces with $s < 0$ is understood such that the operators in the families $\Xi^t_+$ and $r^+\Xi^t_+ e^+$ are adjoints for each $t \in \mathbb{R}$:

\begin{equation}
\Xi^t_-: \mathcal{H}^s_p(\mathbb{R}^n_+) \to \mathcal{H}^{s-t}_p(\mathbb{R}^n_+) \text{ has the adjoint } r^+\Xi^t_- e^+: \mathcal{H}^{s+t}_p(\mathbb{R}^n_+) \to \mathcal{H}^{s-t}_p(\mathbb{R}^n_+),
\end{equation}

with respect to an extension of the duality $\int_{\mathbb{R}^n} u\overline{v} \, dx$ (more explanation in [G15], Rem. 1.1).

Now define the a-transmission space over $\mathbb{R}^n_+$:

\begin{equation}
H^{a(s)}_p(\mathbb{R}^n_+) = \Xi^{-a}_+ e^+ \mathcal{H}^{-a}_p(\mathbb{R}^n_+), \text{ for } s - a > -1/p'.
\end{equation}

Here $e^+ \mathcal{H}^{-a}_p(\mathbb{R}^n_+)$ has a jump at $x_n = 0$ when $s - a > 1/p$; this is mapped by $\Xi^{-a}_+$ to a singularity of the type $x_n^a$.

In fact, we can show:

\begin{equation}
H^{a(s)}_p(\mathbb{R}^n_+) = \left\{ \begin{array}{ll}
\mathcal{H}^s_p(\mathbb{R}^n_+) & \text{if } -1/p' < s - a < 1/p, \\
\mathcal{H}^s_p(\mathbb{R}^n_+) + e^+ x_n^a \mathcal{H}^{s-a}_p(\mathbb{R}^n_+) & \text{if } s - a - 1/p \in \mathbb{R}_+ \setminus \mathbb{N},
\end{array} \right.
\end{equation}

with $\mathcal{H}^s_p(\mathbb{R}^n_+)$ replaced by $\mathcal{H}^{s-a}_p(\mathbb{R}^n_+)$ if $s - a - 1/p \in \mathbb{N}$.

For example, for $1/p < s - a < 1 + 1/p$, $u \in H^{a(s)}_p(\mathbb{R}^n_+)$ has the form

\begin{equation}
u = w + e^+ x_n^a K_0 \varphi,
\end{equation}
where \( w \) and \( \varphi \) run through \( \dot{H}^a_p(\mathbb{R}^n_+) \) and \( B^{s-a-1/p}_p(\mathbb{R}^{n-1}) \), respectively, and \( K_0 \) is the Poisson operator \( K_0: \varphi \mapsto F^{-1}_{t \to x}[\varphi(t)e^{-\langle \xi' \rangle x_n}] \) solving the standard Dirichlet problem

\[
(-\Delta + 1)v = 0 \text{ in } \mathbb{R}^n_+, \quad \gamma_0 u = \varphi \text{ on } \mathbb{R}^{n-1}.
\]

The analysis hinges on the following formula for the inverse Fourier transform of \((\langle \xi' \rangle + i\xi_n)^{-a-1}\), where \( e^+r^ax_n^a \) appears:

\[
F_{t \to x}^{-1}(\langle \xi' \rangle + i\xi_n)^{-a-1} = \Gamma(a + 1)^{-1}e^+r^ax_n^ae^{-\langle \xi' \rangle x_n}.
\]

The generalization to \( \Omega \subset \mathbb{R}^n \) depends on finding suitable replacements of \( \Xi_\pm \). They are a kind of generalized \( \Psi \)do's (the symbols satify some but not all of the usual symbol estimates). It was important in [G15] that we could rely on a truly pseudodifferential version \( \Lambda^{(t)}_\pm \) found in [G90].

The choice \( P = (1 - \Delta)^a \) on \( \mathbb{R}^n \) with symbol \((1 + |\xi|^2)^a\) serves as a model case with easy explicit calculations. Here one can factorize the symbol and operator directly:

\[
(2.22) \quad (1 + |\xi|^2)^a = (\langle \xi' \rangle - i\xi_n)^a(\langle \xi' \rangle + i\xi_n)^a, \quad (1 - \Delta)^a = \Xi_-^a \Xi_+^a.
\]

Let us show how to solve the model Dirichlet problem

\[
(2.23) \quad r^+(1 - \Delta)^a u = f \text{ on } \mathbb{R}^n_+, \quad \text{supp } u \subset \mathbb{R}^n_+.
\]

Say, \( f \) is given in \( \overline{H}^t_p(\mathbb{R}^n_+) \) for some \( t \geq 0 \), and \( u \) is a priori assumed to lie in \( \dot{H}^a_p(\mathbb{R}^n_+) \).

In view of the factorization (2.22),

\[
r^+(1 - \Delta)^a u = r^+\Xi_-^a \Xi_+^a u = r^+\Xi_-^a (e^+r^+ + e^-r^-) \Xi_+^a u = r^+\Xi_-^a e^+r^+ \Xi_+^a u,
\]

since \( r^- \Xi_+^a u = 0 \). \((r^- \text{ denotes restriction from } \mathbb{R}^n \text{ to } \mathbb{R}^n_+, \text{ } e^- \text{ is extension by zero on } \mathbb{R}^n \setminus \mathbb{R}^n_+.) By\ (2.17), \) the problem (2.23) is reduced by composition with \( r^+\Xi_-^a e^+ \) to the left to the problem

\[
(2.24) \quad r^+\Xi_+^a u = g, \quad \text{supp } u \subset \mathbb{R}^t_+,
\]

where \( g = r^+\Xi_-^a e^+ f \in \overline{H}^{t+a}_p(\mathbb{R}^n_+) \). Now there is an important observation, shown in Prop. 1.7 in [G15]:

**Lemma 2.3.** Let \( s > a - 1/p' \). The mapping \( \Xi_-^a e^+ \) is a bijection from \( \overline{H}^s_p(\mathbb{R}^n_+) \) to \( H^{a(s)}_p(\mathbb{R}^n_+) \) with inverse \( r^+\Xi_+^a \).

Then clearly, (2.24) is simply solved uniquely by

\[
(2.25) \quad u = \Xi_-^a e^+ g.
\]

Inserting the definition of \( g \), we can conclude:
Proposition 2.4. The problem (2.23) with \( f \) is given in \( \overline{H}_p^t(\mathbb{R}^n_+) \) for some \( t \geq 0 \), and \( u \) sought in \( \dot{H}_p^a(\mathbb{R}^n_+) \), has the unique solution
\[
(2.26) \quad u = \Xi_+^{-a} e^+ r^+ \Xi_-^{-a} e^+ f,
\]
lying in \( \Xi_+^{-a} (e^+ \overline{H}_p^{t+a}(\mathbb{R}^n_+)) = H_p^{a(t+2a)}(\mathbb{R}^n_+) \), the \( a \)-transmission space.

It is of course more difficult to treat variable-coefficient operators on curved domains. For such cases, the following result was shown in [G15]:

Theorem 2.5. Let \( P \) be a classical strongly elliptic \( \psi \)do on \( \mathbb{R}^n \) of order \( 2a > 0 \) with even symbol (i.e., \( P \) satisfies Hypothesis 1.1), and let \( \Omega \) be a smooth bounded subset of \( \mathbb{R}^n \). Let \( s > a - 1/p' \). The homogeneous Dirichlet problem (2.8), considered for \( u \in \dot{H}_p^{a-1/p'+\varepsilon}(\Omega) \), satisfies:
\[
(2.27) \quad f \in \overline{H}_p^{-2a}(\Omega) \implies u \in H_p^{a(s)}(\overline{\Omega}), \text{ the } a \text{-transmission space.}
\]

Moreover, the mapping from \( u \) to \( f \) is a Fredholm mapping:
\[
(2.28) \quad r^+ P: H_p^{a(s)}(\overline{\Omega}) \to \overline{H}_p^{-2a}(\Omega) \text{ is Fredholm.}
\]

A corollary for \( s \to \infty \) is:
\[
(2.29) \quad r^+ P: \mathcal{E}_a(\overline{\Omega}) \to C^\infty(\overline{\Omega}) \text{ is Fredholm.}
\]

The big step forward by this theorem is that it describes the domain spaces in an exact way, and shows that they depend only on \( a, s, p \), not on the operator \( P \); and this works for all \( s > a - 1/p' \).

The argumentation involves a reduction to problems belonging to the calculus of Boutet de Monvel, which is described e.g. in [B71,G96,S01,G09]. We use techniques established more recently than [H65,H85], in particular from [G90]. The basic idea is to reduce the operator, on boundary patches, to the form
\[
(2.30) \quad P \sim \Lambda_{-}^{(a)} Q \Lambda_{+}^{(a)},
\]
where \( \Lambda_{\pm}^{(a)} \) are order-reducing pseudodifferential operators, preserving support in \( \overline{\Omega} \) resp. \( \mathbb{R}^n \setminus \Omega \), and \( Q \) is of order 0 and satisfies the 0-transmission condition, hence belongs to the Boutet de Monvel calculus. We shall not dwell on the proof here, but go on to some further developments of the theory.

Remark 2.6. The assumption that the \( \psi \)do \( P \) is even, was made for simplicity, and could everywhere be replaced by the assumption that \( P \) has the \( a \)-transmission property with respect to the chosen domain \( \Omega \).

Remark 2.7. In [G14] (written after [G15]), the results are extended to many other scales of spaces, such as Besov spaces \( B_{p,q}^s \) and Triebel-Lizorkin spaces \( F_{p,q}^s \). Of particular interest is the scale \( B_{\infty,\infty}^s \), also denoted \( C_s^\cdot \), the Hölder-Zygmund scale. Here \( C_s^\cdot \) identifies with the Hölder space \( C^s \) when \( s \in \mathbb{R}_+ \setminus \mathbb{N} \), and for positive integer \( k \) satisfies \( C^{k-\varepsilon} \supset C_s^k \supset C^{k-1,1} \supset C^k \) for small \( \varepsilon > 0 \); moreover, \( C_s^0 \supset L_\infty \). Then Theorem 2.5 holds with \( H_p^a \)-spaces replaced by \( C_s^\cdot \)-spaces.
Remark 2.8. The above applications of pseudodifferential theory require that the domain $\Omega$ has $C^\infty$-boundary; in comparison, the results of e.g. Ros-Oton and coauthors in low-order Hölder spaces allow low regularity of $\Omega$, using rather different methods. There exists a pseudodifferential theory with just Hölder-continuous $x$-dependence (see e.g. Abels [A05,A05a] and references), which may be useful to reduce the present smoothness assumptions, but non-smooth coordinate changes for $\psi$do’s have not yet (to our knowledge) been established in a sufficiently useful way. At any rate, the results obtainable by $\psi$do methods can serve as a guideline for what one can aim for on domains with lower smoothness.

3. Nonhomogeneous boundary value problems

When solutions $u$ of the homogeneous Dirichlet problem lie in $d^a$ times a Sobolev or Hölder space over $\overline{\Omega}$, there is a boundary value $\gamma_0(u/d^a)$, denoted

$$(3.1) \quad \gamma_0^a u = \gamma_0(u/d^a);$$

it is viewed as a Neumann boundary value. (We omit normalizing constants for now; they are described precisely in Remark 3.2 below.)

Ros-Oton and Serra [RS14a,RS15] showed the following integration-by-parts formula:

Theorem 3.1. When $u$ and $u'$ are solutions of the homogeneous Dirichlet problem (2.8) for $(-\Delta)^a$ on $\Omega$ with $f,f' \in L^\infty(\Omega)$, $\Omega$ being $C^{1,1}$, $a > 0$, then

$$(3.2) \quad \int_{\Omega} ((-\Delta)^a u \partial_j \bar{u}' + \partial_j u (-\Delta)^a \bar{u}') \, dx = c \int_{\partial\Omega} \nu_j(x) \gamma_0^a u \gamma_0^a \bar{u}' \, d\sigma;$$

here $\nu = (\nu_1, \ldots, \nu_n)$ is the normal vector at $\partial\Omega$.

It is equivalent to a certain Pohozaev-type formula, which has important applications to uniqueness questions in nonlinear problems for $(-\Delta)^a$. It was generalized to other related $x$-independent singular integral operators in [RSV17] (with Valdinoci), and we extended it to $x$-dependent $\psi$do’s in [G16a]. (See the survey [R18] for an introduction to fractional Pohozaev identities and their applications.)

Note that the collected order of the operators in the integral over $\Omega$ is $2a+1$; the formula generalizes a well-known formula for $a = 1$ where $\gamma_0^a u$ is replaced by the Neumann trace $\gamma_0(\partial_x u)$, and the Dirichlet trace $\gamma_0 u$ is 0.

What should a nonzero Dirichlet trace be in the context of fractional Laplacians? Look at the smoothest space:

$$(3.3) \quad \mathcal{E}_a(\mathbb{R}^n) = \{ u = e^+ x_n v \mid v \in C^\infty(\mathbb{R}^n) \}.$$ 

By a Taylor expansion of $v$,

$$(3.4) \quad u(x) = x_n^a v(x',0) + x_n^{a+1} \partial_n v(x',0) + \frac{1}{2} x_n^{a+2} \partial_n^2 v(x',0) + \ldots \quad \text{for } x_n > 0.$$ 

If $u \in \mathcal{E}_{a-1}(\mathbb{R}^n)$, i.e., $u = e^+ x_n^{a-1} w$ with $w \in C^\infty(\mathbb{R}^n)$, we have analogously:

$$(3.5) \quad u(x) = x_n^{a-1} w(x',0) + x_n^a \partial_n w(x',0) + \frac{1}{2} x_n^{a+1} \partial_n^2 w(x',0) + \ldots \quad \text{for } x_n > 0.$$
Here $x_n^{a-1}w(x',0)$ is the only structural difference between (3.4) and (3.5).

This term defines the Dirichlet trace: When $u \in \mathcal{E}_{a-1}(\mathbb{R}_+^n)$, the Dirichlet trace is

\[(3.6)\]

$$
\gamma_0^{a-1}u = \gamma_0(u/x_n^{a-1}), \text{ equal to } \gamma_0 w.
$$

(Again we omit a normalizing constant.)

It is now natural to define a Neumann trace on $\mathcal{E}_{a-1}(\mathbb{R}_+^n)$ from the second term in (3.5), by

\[(3.7)\]

$$
\gamma_1^{a-1}u = \gamma_1(u/x_n^{a-1}), \text{ equal to } \gamma_1 w = \gamma_0(\partial_n w).
$$

Note that it equals $\gamma_0^a u$ if $u \in \mathcal{E}_a(\mathbb{R}_+^n)$.

**Remark 3.2.** Also higher order traces are defined on $\mathcal{E}_{a-1}(\mathbb{R}_+^n)$, namely the functions $\partial_n^k w(x',0)$ in (3.5). With the correct normalizing constants they are:

\[(3.8)\]

$$
\gamma_k^{a-1}u = \Gamma(a + k)\gamma_0(\partial_n^k(u/x_n^{a-1})), \quad k \in \mathbb{N}_0.
$$

There are analogous definitions with $a - 1$ replaced by $a - M$, $a \in \mathbb{R}_+$ and $M \in \mathbb{N}_0$; see details in [G15], in particular Th. 5.1 showing mapping properties, and Th. 6.1 showing Fredholm solvability. For $(-\Delta)^a$ in the case where $\Omega$ is the unit ball in $\mathbb{R}^n$, related definitions are given by Abatangelo, Jarohs and Saldana in [AJS18], with explicit solution formulas.

The above definitions can be carried over to $\Omega$ (where $x_n$ is replaced by $d(x)$), and they extend to $H_p^{(a-1)(s)}(\Omega)$ spaces for sufficiently large $s$, cf. [G15].

Now consider a $P$ satisfying Hypothesis 1.1. We can define the *nonhomogeneous Dirichlet problem* for functions $u \in H_p^{(a-1)(s)}(\Omega)$ (hence supported in $\overline{\Omega}$), by

\[(3.9)\]

$$
{r^+}^* P u = f \text{ in } \Omega, \quad \gamma_0^{a-1}u = \varphi \text{ on } \partial \Omega.
$$

For this we have the solvability result ([G15,G14]):

**Theorem 3.3.** For $s > a - 1/p'$,

\[(3.10)\]

$$
\{ {r^+}^* P, \gamma_0^{a-1} \}: H_p^{(a-1)(s)}(\Omega) \to \mathcal{H}_p^{-s-2a}(\Omega) \times B_p^{s-a+1/p'}(\partial \Omega)
$$

is a Fredholm mapping.

Here $B_p^{s-a+1/p'}(\partial \Omega)$ is the Besov space that usually appears as the range space for the standard Dirichlet trace operator $\gamma_0$ applied to $\mathcal{H}_p^{s-a+1}(\Omega)$. As in (2.20) (with $a$ replaced by $a - 1$), $H_p^{(a-1)(s)}(\Omega) \subset \dot{H}_p^{s}(\Omega) + e^+d^{a-1}\mathcal{H}_p^{s-a+1}(\Omega)$, when $s - a + 1/p' \in \mathbb{R}_+ \setminus \mathbb{N}$.

When $a < 1$, the factor $d(x)^{a-1}$ is unbounded, and the solutions of the form $u = d^{a-1}v$, for a nice $v$ with nonzero boundary value, blow up at $\partial \Omega$ (a detailed analysis is given in [G14] Rem. 2.10). Such solutions are called “large solutions” in Abatangelo [A15]. Nevertheless, $u \in L_p(\Omega)$ if $1 < p < 1/(1 - a)$. 
Nonhomogeneous Dirichlet problems (also with consecutive sets of boundary data) are considered in [G15,G14,G16a,A15] and the recent [AJS18].

We can moreover consider a boundary value problem where Neumann data are prescribed:

\[(3.11)\]
\[r^+ P u = f \text{ in } \Omega, \quad \gamma^{a-1}_1 u = \psi \text{ on } \partial \Omega,\]

for \(u \in H^{(a-1)(s)}(\Omega), s > a + 1/p\). (The boundary condition here is local; there have also been defined other, nonlocal Neumann problems, see the overview in [G16] Sect. 6.) To discuss the solvability of (3.11) we can construct a Dirichlet-to-Neumann operator [G18a]:

**Theorem 3.4.** Let \(K_D\) be a parametrix of the mapping

\[(3.12)\]
\[z \mapsto \gamma^{a-1}_0 z, \text{ when } r^+ P z = 0 \text{ in } \Omega,\]

(an inverse when (3.10) is a bijection). Then the mapping

\[(3.13)\]
\[S_D = \gamma^{a-1}_1 K_D,\]

the **Dirichlet-to-Neumann operator**, is a classical pseudodifferential operator of order 1 on \(\partial \Omega\), with principal symbol \(s_{DN,0}\) derived from the principal symbol of \(P\).

In particular, \(s_{DN,0}(x', \xi')\) is proportional to \(|\xi'|\) for \(|\xi'| \geq 1\), when \(P = (-\Delta)^a\), considered in local coordinates at the boundary.

And then we have:

**Theorem 3.5.** When \(S_{DN}\) is elliptic (i.e., \(s_{DN,0}(x', \xi')\) is invertible for \(|\xi'| \geq 1\), the Neumann problem (3.11) satisfies:

\[(3.14)\]
\[\{r^+ P, \gamma^{a-1}_1\} : H^{(a-1)(s)}_p(\Omega) \to H^{-2a}_p(\Omega) \times B^{s-a-1/p}_{p}(\partial \Omega)\]

is a Fredholm mapping, for \(s > a + 1/p\).

Note that the ellipticity holds in the case where \(P = (-\Delta)^a\).

It is remarkable that both the Dirichlet and the Neumann boundary operators \(\gamma^{a-1}_0\) and \(\gamma^{a-1}_1\) are **local**, in spite of the nonlocalness of the operator \(P\).

The integration by parts formula (3.1), and the consequential Pohozaev formulas, hold on \(H^a\)-spaces, where the Dirichlet trace \(\gamma^{a-1}_0 u\) vanishes and (consequently) the Neumann trace \(\gamma^{a-1}_1 u\) identifies with \(\gamma^{a}_0 u\).

The papers [RS14a,RS15,RSV17,A15,G16a] did not establish formulas where both \(\gamma^{a-1}_0 u\) and \(\gamma^{a-1}_1 u\) can be nonvanishing. However, in comparison with the standard Laplacian \(\Delta\), it is natural to ask whether there are formulas generalizing the well-known full Green’s formula with nonzero Dirichlet and Neumann data, to these operators. This question was answered in [G18a], where we showed:
Theorem 3.6. When $u, v \in H^{(a-1)(s)}(\Omega)$, then for $s > a + \frac{1}{2}$,

\begin{equation}
\int_{\Omega} (Pu \bar{v} - u \overline{P^*v}) \, dx = c_a \int_{\partial \Omega} (s_0 \gamma_1^{a-1} u \gamma_0^{a-1} \bar{v} - s_0 \gamma_0^{a-1} u \gamma_1^{a-1} \bar{v} + B \gamma_0^{a-1} u \gamma_0^{a-1} \bar{v}) \, dx',
\end{equation}

here $s_0(x') = p_0(x', \nu(x'))$ for $x' \in \partial \Omega$, and $B$ is a first-order $\psi$do on $\partial \Omega$.

Note that the only term in the right-hand side that may not be local, is the term with $B$, nonlocal in general. A closer study (work in progress) shows that $B$ vanishes if $P = (-\Delta)^a$; we also find criteria under which $B$ is local.

We shall not begin here to describe the method of proof; it consists of delicate localized $\psi$do considerations using the order-reduction operators, and elements of the Boutet de Monvel calculus.

4. Heat equations

4.1 Anisotropic spaces, H"older estimates, counterexamples to high spatial regularity.

For a given lower semibounded operator $A$ in $x$-space it is of interest to study evolution problems with a time-parameter $t$,

\begin{equation}
Au(x,t) + \partial_t u(x,t) = f(x,t) \text{ for } t > 0, \quad u(x,0) = u_0(x).
\end{equation}

Through many years, semigroup methods, as originally presented in Hille and Phillips [HP57], have been developed along with other methods from functional analysis to give interesting results for operators $A$ acting like the Laplacian and other elliptic differential operators. Also nonlinear questions have been treated, e.g. where $f$ or $A$ are allowed to depend on $u$.

For $A$ representing the fractional Laplacian and its generalizations, on $\mathbb{R}^n$ or a domain, the studies have begun more recently. A natural approach is here: To find the appropriate general strategies from the works on differential operators, and show the appropriate properties of $A$ assuring that the methods can be applied to it.

Consider the evolution problem (heat equation) associated with an operator $P$ as studied in Sections 2 and 3, with a homogeneous Dirichlet condition:

\begin{equation}
P u + \partial_t u = f \text{ on } \Omega \times I, \quad I = ]0,T[, \\
u = 0 \text{ on } (\mathbb{R}^n \setminus \Omega) \times I, \\
u|_{t=0} = u_0.
\end{equation}

Since $P_{\text{Dir}}$ is a positive selfadjoint (or sectorial) operator in $L_2(\Omega)$, there is solvability in a framework of $L_2$-Sobolev spaces.

We are interested in the regularity of solutions.

This question has been treated recently by Leonori, Peral, Primo and Soria [LPPS15] in $L_r(I; L_q(\Omega))$-spaces, by Fernandez-Real and Ros-Oton [FR17] in anisotropic H"older spaces, and by Biccari, Warma and Zuazua [BWZ18] for $(-\Delta)^a$ in local $L_p$-Sobolev spaces over
\( \Omega \). Earlier results are shown e.g. in Felsinger and Kassmann [FK13] and Chang-Lara and Davila [CD14] (Hölder properties), and Jin and Xiong [JX15] (Schauder estimates). The references in the mentioned works give further information, also on related heat kernel estimates. Very recently (November 2017), the Hölder estimates were improved by Ros-Oton and Vivas [RV18].

We have a few contributions to this subject, that we shall describe in the following.

Let us first introduce anisotropic spaces of Sobolev or Hölder type. Let \( d \in \mathbb{R}_+ \). There are the Bessel-potential types, for \( s \in \mathbb{R} \):

\[
H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) = \{ u \in S' | \mathcal{F}^{-1}(\langle \xi \rangle^{2d} + \tau^2)^{s/2d} \hat{u}(\xi, \tau) \in L_p(\mathbb{R}^{n+1}) \},
\]

\[
\mathcal{F}_p^{(s,s/d)}(\Omega \times I) = r_{\Omega \times I} H_p^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}),
\]

and there are related definitions of Besov-type with \( H_p \) replaced by \( B_p \). There are the Hölder spaces:

\[
\mathcal{C}^{(s,r)}(\Omega \times I) = L_\infty(I; \mathcal{C}^s(\Omega)) \cap L_\infty(\Omega; \mathcal{C}^r(I)), \quad \text{for } s, r \in \mathbb{R}_+;
\]

including in particular the case \( r = s/d \). (For \( s \) equal to an integer \( k \), we denote by \( \mathcal{C}^k(\Omega) \) the space of bounded continuous functions on \( \Omega \) with bounded derivatives up to order \( k \), this includes the case \( \Omega = \mathbb{R}^n \).) The spaces occur in many works; important properties are recalled e.g. in [G95] and [G18] with further references.

The Hölder-type spaces are the primary objects in the investigations of Fernandez-Real and Ros-Oton in [FR17], Ros-Oton and Vivas in [RV18]. These authors have for the Dirichlet heat problem the following results, showing the role of \( d^a \) in Hölder estimates:

**Theorem 4.1.** Let \( P \) be an operator of the form (1.2)ff., \( 0 < a < 1 \), and consider solutions of the problem (4.2).

1° [FR17], Cor. 1.6. When \( \Omega \) is a bounded open \( C^{1,1} \) subset of \( \mathbb{R}^n \), then the unique weak solution \( u \) with \( f \in L_\infty(\Omega \times I) \) and \( u_0 \in L_2(\Omega) \) satisfies

\[
\| u \|_{\mathcal{C}^{(s-1)}(\Omega \times I')} + \| u/d^a \|_{\mathcal{C}^{(s-1/2)^{(2a)}}(\Omega \times I')} \leq C(\| f \|_{L_\infty(\Omega \times I)} + \| u_0 \|_{L_2(\Omega)}),
\]

for any small \( \varepsilon > 0 \), \( I' = [t_0, T[ \) with \( t_0 > 0 \). Moreover, if \( f \in \mathcal{C}^{(\gamma,\gamma/(2a))}(\Omega \times I) \) with \( \gamma \in ]0,a[ \) such that \( \gamma + 2a \notin \mathbb{N} \), then \( u \) has the interior regularity:

\[
\| u \|_{\mathcal{C}^{(2a+\gamma,1+\gamma/(2a))}(\Omega' \times I')} \leq C' \| f \|_{\mathcal{C}^{(\gamma,\gamma/(2a))}(\Omega \times I)},
\]

for any \( \Omega' \) with \( \Omega' \subset \Omega \).

2° [RV18], Cor. 1.2. Let \( \gamma \in ]0,a[ \), \( \gamma + a \notin \mathbb{N} \). When \( \Omega \) is a bounded open \( C^{2,\gamma} \) subset of \( \mathbb{R}^n \), then \( f \in \mathcal{C}^{(\gamma,\gamma/(2a))}(\Omega \times I) \), \( u_0 \in L_2(\Omega) \) imply:

\[
\| u \|_{\mathcal{C}^{(\gamma,1+\gamma/(2a))}(\Omega \times I')} + \| u/d^a \|_{\mathcal{C}^{(a+\gamma,1+\gamma/(2a))}(\Omega' \times I')} \leq C(\| f \|_{\mathcal{C}^{(\gamma,\gamma/(2a))}(\Omega \times I)} + \| u_0 \|_{L_2(\Omega)}),
\]
for any \( I' = [t_0, T[ \) with \( t_0 > 0 \).

In other words, if we take \( \gamma = a - \varepsilon \) for a small \( \varepsilon > 0 \), (4.7) reads, with \( \varepsilon' = \varepsilon/(2a) \),

\[
\| u \|_{C^{(a-\varepsilon,3/2-\varepsilon')}_p(\Omega \times I')} + \| u/d^a \|_{C^{(2a-\varepsilon,1-\varepsilon')}_p(\Omega \times I')}
\leq C(\| f \|_{C^{(a-\varepsilon,1/2-\varepsilon')}_p(\Omega \times I')} + \| u_0 \|_{L^2(\Omega)}).
\]

(4.8)

An interesting question is whether the regularity of \( u \) can be lifted further, when \( f \) in \( 2^o \) is replaced by a more regular function. As we shall see below, this is certainly possible with respect to the \( t \)-variable; this is also shown to some extent in [FR17]. However, there are limitations with respect to the boundary behavior in the \( x \)-variable. It is shown in [G15a] that when \( P \) is as in Hypothesis 1.1, any eigenfunction \( \varphi \) of \( \text{P}_{\text{Dir}} \) associated with a nonzero eigenvalue \( \lambda \) satisfies (more on the spaces in Theorem 4.2):

\[
\varphi \in C^{(3a)}_s(\Omega) \subset d^a C^{2a}_s(\Omega) \left\{ \begin{array}{l}
d^a C^{2a}(\Omega) \text{ if } a \neq \frac{1}{2}, \\
C^{2a-\varepsilon}_s(\Omega) \text{ if } a = \frac{1}{2}, 
\end{array} \right.
\]

(4.9)

but, in the basic case \( P = (-\Delta)^a \), \( \varphi \) is not in \( d^a C^{\infty}(\Omega) \) and not either in \( C^{\infty}(\Omega) \). The function \( u(x, t) = e^{-\lambda t}\varphi(x) \) is clearly a solution of the heat equation with \( f = 0 \) (hence \( f \in C^{\infty}(I; C^{\infty}(\Omega)) \)), but \( u \) and \( u/d^a \) \( \not\in L^\infty(I'; C^{\infty}(\Omega)) \). This shows a surprising contrast to the usual regularity rules for heat equations, and it differs radically from the stationary case, where we have (2.29).

The argument can be extended from \( (-\Delta)^a \) to more general operators, and it can be sharpened to rule out also finite higher order regularities.

**Theorem 4.2.** Let \( P \) satisfy Hypothesis 1.2 with \( 0 < a < 1 \), or let \( P \) equal \( (-\Delta)^a \) with \( a > 0 \), or the fractional Helmholtz operator \( (-\Delta + m^2)^a \) with \( m > 0 \) and \( 0 < a < 1 \). Let \( \Omega \) be \( C^{\infty} \). Then any eigenfunction \( \varphi \) of \( \text{P}_{\text{Dir}} \) associated with a nonzero eigenvalue \( \lambda \) satisfies (4.9), but is not in \( C^{(a+\delta)}(\Omega) \) nor in \( C^{(3a+\beta)}_s(\Omega) \) for any \( \delta > 0 \).

**Proof.** Recall that \( C^{a}_s \) stands for the H"older-Zygmund space, which identifies with \( C^a \) when \( s \in \mathbb{R}_+ \setminus \mathbb{N} \), cf. Remark 2.7. As shown in [G14], p. 1655 and Th. 3.2, \( C^{(2a+s)}(\Omega) \) is the solution space for the homogeneous Dirichlet problem with right-hand side in \( C^{(a)}(\Omega) \); here \( C^{(2a+s)}(\Omega) \subset d^a C^{(a+s)}(\Omega) \), but there is no equality. (One reason for the lack of equality is that the functions in \( C^{(2a+s)}(\Omega) \) are in \( C^{2a+s} \) over the interior, another is that \( C^{(2a+s)}(\Omega) \) only reaches a subspace of \( d^a C^{(a+s)}(\Omega) \) near the boundary, cf. [G15] Th. 5.4.)

Assume that \( \varphi \) satisfies \( \text{P}_{\text{Dir}} \varphi = \lambda \varphi \) (\( \lambda \neq 0 \)) and is in \( C^{(a+\delta)}(\Omega) \) for a positive \( \delta < 1-a \); then in fact \( \varphi \in \hat{C}^{a+\delta}(\Omega) \) since \( \varphi \in C^{(3a+\beta)}(\Omega) \subset d^a C^{2a+\beta}(\Omega) \) implies \( \gamma_0 \varphi = 0 \). Now \( \varphi \in \hat{C}^{a+\delta}(\Omega) \) implies \( \gamma_0(\varphi/d^a) = 0 \) since \( \delta > 0 \). It is shown in [RSV17] for operators of the form (1.2)ff. with \( 0 < a < 1 \), in [RS15] for \( (-\Delta)^a \) with \( a > 0 \), and in [G16a], Ex. 4.10 for \( (-\Delta + m^2)^a \), how it follows from Pohozaev identities that

\[
\text{P}_{\text{Dir}} v = \lambda v, \quad \gamma_0(\varphi/d^a) = 0 \implies v \equiv 0.
\]

Thus \( \varphi = 0 \) and cannot be an eigenfunction. \( \square \)

This allows us to conclude:
Corollary 4.3. Consider the problem (4.2). For the operators $P$ considered in Theorem 4.2, $C^\infty$-regularity of $f$ does not imply $C^\infty$-regularity of $u$ or $u/d^a$. In fact, there exist choices of $f(x,t) \in \overline{C}^\infty(\Omega \times I)$ with solutions $u(x,t)$ satisfying

$$u \notin L_\infty(I';\overline{C}^\infty(\Omega)), \quad u \notin L_\infty(I';d^a\overline{C}^\infty(\Omega)).$$

More precisely, there exist solutions with $f(x,t) \in \overline{C}^\infty(\Omega \times I)$ such that

$$u \notin L_\infty(I';\overline{C}^{\alpha+\delta}(\Omega)), \quad u \notin L_\infty(I';C^{\alpha(3\alpha+\delta)}_\star(\Omega)), \text{ for any } \delta > 0.$$

Proof. This follows by taking $u(x,t) = e^{-\lambda t}\varphi(x)$ with an eigenfunction $\varphi$ as in Theorem 4.2. It solves the heat problem (4.2) with $f = 0$ (thus $f \in \overline{C}^\infty(\Omega \times I)$) and $u_0 = \varphi$, and it clearly satisfies (4.10) as well as (4.11). \(\square\)

Note that the $x$-regularity obtained in (4.8) is close to the upper bound.

Corollary 4.3 gives counterexamples; one can show more systematically that $\gamma_0 a u$ being nonzero prevents the solution from being in $C^\infty$ or $d^aC^\infty$ at the boundary [G18b].

4.2 Solvability in Sobolev spaces.

Now let us consider estimates in Sobolev spaces.

In [GS90] (jointly with Solonnikov) and in [G95] the author studied evolution problems for $\psi$do’s $P$ with the 0-transmission property at $\partial\Omega$, along with trace, Poisson and singular Green operators in the Boutet de Monvel calculus (cf. [B71,G90,G96]), setting up a full calculus leading to existence, uniqueness and regularity theorems in anisotropic Bessel-potential and Besov spaces as mentioned in (4.3)ff.

These works take $P$ of integer order, and do not cover the present case. We expect that a satisfactory generalization of the full boundary value theory in those works, to heat problems for our present operators, would be quite difficult to achieve. However their point of view on the $\psi$do $P$ alone, considered on $\mathbb{R}^n$ without boundary conditions, can be extended, as follows:

For a classical strongly elliptic $\psi$do $P$ of order $d \in \mathbb{R}_+$ on $\mathbb{R}^n$ (with global symbol estimates), we can construct an anisotropic symbol calculus on $\mathbb{R}^n \times \mathbb{R}$ that includes operators $P + \partial_t$ and their parametrices. It is not quite standard, since the typical strictly homogeneous symbol $|\xi|^d + i\tau$ is not $C^\infty$ at points $(0, \tau)$ with $\tau \neq 0$. But this is a phenomenon handled in [G96] by introducing classes of symbols with finite “regularity number” $\nu$ (essentially the H"older regularity of the strictly homogeneous principal symbol at points $(0, \tau)$), and keeping track of how the value of $\nu$ behaves in compositions and parametrix constructions.

The calculus gives, on $\mathbb{R}^{n+1}$ and locally in $\Omega \times I$ [G18]:

Theorem 4.4. Let $P$ be a classical strongly elliptic $\psi$do of order $d \in \mathbb{R}_+$. Then $P + \partial_t$ maps $H^{(s,s/d)}_p(\mathbb{R}^n \times \mathbb{R})$ continuously into $H^{(s-d,s/d-1)}_p(\mathbb{R}^n \times \mathbb{R})$ for any $s \in \mathbb{R}$. Moreover:

1° If $u \in H^{(r,r/d)}_p(\mathbb{R}^n \times \mathbb{R})$ for some large negative $r$ (this holds in particular if $u \in \mathcal{E}'(\mathbb{R}^{n+1})$ or e.g. $L_p(\mathbb{R};\mathcal{E}'(\mathbb{R}^n))$, then

$$P + \partial_t)u \in H^{(s,s/d)}_p(\mathbb{R}^n \times \mathbb{R}) \implies u \in H^{(s+d,s/d+1)}_p(\mathbb{R}^n \times \mathbb{R}).$$

2° Let $\Sigma = \Omega \times I$, and let $u \in H^{(s,s/d)}_p(\mathbb{R}^n \times \mathbb{R})$. Then

$$P + \partial_t)u|_{\Sigma} \in H^{(s,s/d)}_{p,\text{loc}}(\Sigma) \implies u \in H^{(s+d,s/d+1)}_{p,\text{loc}}(\Sigma).$$
This theorem works for any strongly elliptic classical $\psi$do of positive order, not just fractional Laplacians, but for example also $-\Delta + (-\Delta)^{1/2}$ or $(-\Delta)^{1/2} + b(x) \cdot \nabla + c(x)$ with real $C^\infty$-coefficients. The result extends by standard localization methods to the case where $\mathbb{R}^n$ is replaced by a closed manifold.

Note that in $2^\circ$, the regularity of $u$ is only lifted by 1 in $t$, and the hypothesis on $u$ concerns all $x \in \mathbb{R}^n$; the necessity of this is pointed out in related situations in [CD14] and [FR17].

By use of embedding theorems, we can moreover derive from the above a local regularity result in anisotropic Hölder spaces:

**Theorem 4.5.** Let $P$ and $\Sigma$ be as in Theorem 4.3. Let $s \in \mathbb{R}_+$, and let $u \in C^{(s,s/d)}(\mathbb{R}^n \times \mathbb{R}) \cap \mathcal{E}'(\mathbb{R}^n \times \mathbb{R})$. Then

$$
(P + \partial_t)u|_\Sigma \in C^{(s,s/d)}_{loc}(\Sigma) \implies u|_\Sigma \in C^{(s+d-\varepsilon,(s-\varepsilon)/d+1)}_{loc}(\Sigma),
$$

for small $\varepsilon > 0$.

Observe the similarity with (4.6) (where $d = 2a$). On one hand we have a loss of $\varepsilon$; on the other hand we have general $x$-dependent operators $P$ just required to be strongly elliptic (albeit with smooth symbols), and no upper limitations on $s$. The $\varepsilon$ might possibly be removed by working with our operators on the Hölder-Zygmund scale (cf. e.g. [G14]).

Still other spaces could be examined. There is the work of Yamazaki [Y86] on $\psi$do’s acting in anisotropic Besov-Triebel-Lizorkin spaces (defined in his work); these include the $H_p$ and $B_p$ spaces as special cases. However, the operators in [Y86] seem to be more regular (their quasi-homogeneous symbols being smooth outside of 0) than the operators $\partial_t + P$ that we study here. There is yet another type of spaces, the so-called modulation spaces on $\mathbb{R}^n$, where there are very recent results on heat equations (with nonlinear generalizations) by Chen, Wang, Wang and Wong [CWWW18].

For the case where boundary conditions at $\partial \Omega$ are imposed, there is not (yet) a systematic boundary-$\psi$do theory as in the stationary case. But using suitable functional analysis results we can make some progress, showing how the heat equation solutions behave in terms of Sobolev-type spaces involving the factor $d^a$. We henceforth restrict the attention to the case $0 < a < 1$, and to problems with initial value 0,

$$
Pu + \partial_t u = f \text{ on } \Omega \times I, \quad I = ]0,T[, \\
\partial_{\nu} u = 0 \text{ on } (\mathbb{R}^n \setminus \Omega) \times I, \\
|_{t=0} = 0.
$$

There is a straightforward result in the $L_2$-framework:

**Theorem 4.6.** Let $P$ satisfy Hypothesis 1.1 with $a < 1$, and let $\Omega$ be a smooth bounded subset of $\mathbb{R}^n$. For $f$ given in $L_2(\Omega \times I)$, there is a unique solution $u$ of (4.15) satisfying

$$
(4.16) \quad u \in L_2(I; H^{2a/(2a)}(\Omega)) \cap H^1(I; L_2(\Omega));
$$

here $H^{2a/(2a)}(\Omega) = D(P_{Dir,2})$ equals $H^{2a}(\Omega)$ if $a < 1/2$, and is as described in (2.13)ff. (with $p = 2$) when $a \geq 1/2$. 

Moreover, \( u \in \overline{C^0(I; L_2(\Omega))} \).

**Proof.** Define the sesquilinear form

\[
Q_0(u, v) = (r^+ Pu, v)_{L_2(\Omega)},
\]

first for \( u, v \in C_0^\infty(\Omega) \) and then extended by closure in \( \dot{H}^a \)-norm to a bounded sesquilinear form with domain \( \dot{H}^a(\overline{\Omega}) \). In view of the strong ellipticity, it is coercive:

\[
\text{Re} \ Q_0(u, u) \geq c_0 \| u \|^2_{\dot{H}^a} - \xi \| u \|^2_{L_2} \quad \text{with} \quad c_0 > 0, \xi \in \mathbb{R},
\]

and hence defines via the Lax-Milgram lemma a realization of \( r^+ P \) with domain (2.9); for precision we shall denote the operator \( P_{\text{Dir}, 2} \). (The Lax-Milgram construction is described e.g. in [G09], Sect. 12.4.) The adjoint is defined similarly from \( Q_0^*(u, v) = \overline{Q_0(v, u)} \), and it follows from (4.17) that the spectrum and numerical range is contained in a set

\[
\{ z \in \mathbb{C} \mid |\text{Im} \ z| \leq C(\text{Re} \ z + \xi), \text{Re} \ z > \xi_0 \},
\]

and

\[
\| (P_{\text{Dir}, 2} - \lambda)^{-1} \|_{L(L_2)} \leq c(\lambda)^{-1} \quad \text{for} \quad \text{Re} \ \lambda \leq -\xi_0.
\]

We can then apply Lions and Magenes [LM68] Th. 4.3.2, which shows that there is a unique solution \( u \) of (4.15) in \( L_2(I; D(P_{\text{Dir}, 2})) \). Here, moreover, \( \partial_t u = f - r^+ Pu \in L_2(\Omega \times I) \), so \( u \in H^1(\overline{\Omega}) \).

The last statement follows since \( \| u \|_{H^r} \leq c \| u \|_{H^{(2a)}} \), for all \( 0 < a < 1 \). This follows for \( \Omega = \mathbb{R}^n_+ \) since \( \overline{H^a(\mathbb{R}^n_+)} = \dot{H}^a(\mathbb{R}^n_+) \) if \( a < \frac{1}{2} \), and \( \overline{H^a(\mathbb{R}^n_+)} \subset \dot{H}^{\frac{1}{2} - \varepsilon}(\mathbb{R}^n_+) \) if \( a \geq \frac{1}{2} \), so that by (2.19),

\[
H^{(2a)}(\mathbb{R}^n_+) \subset \dot{H}^r(\mathbb{R}^n_+), \quad \text{hence} \quad \| u \|_{H^r} \leq c \| u \|_{H^{(2a)}},
\]

then

\[
H^{(2a)}(\overline{\Omega}) \subset \dot{H}^r(\overline{\Omega}) \subset \overline{H^r(\Omega)},
\]

for all \( 0 < a < 1 \). This follows for \( \Omega = \mathbb{R}^n_+ \) since \( \overline{H^a(\mathbb{R}^n_+)} = \dot{H}^a(\mathbb{R}^n_+) \) if \( a < \frac{1}{2} \), and \( \overline{H^a(\mathbb{R}^n_+)} \subset \dot{H}^{\frac{1}{2} - \varepsilon}(\mathbb{R}^n_+) \) if \( a \geq \frac{1}{2} \), so that by (2.19),

\[
H^{(2a)}(\mathbb{R}^n_+) = \Xi^{-a}_+ \dot{H}^a(\mathbb{R}^n_+) \subset \Xi^{-a}_+ \dot{H}^{r-a}(\mathbb{R}^n_+) = \dot{H}^r(\mathbb{R}^n_+);
\]

there is a similar proof for general \( \Omega \). Observe that \( r \leq 2a \) in all cases.

This can be used to jack up the regularity result of Theorem 4.6 by one derivative in \( t \) and an improved \( x \)-regularity, when \( f \) is \( H^r \) in \( x \) and \( H^1 \) in \( t \). Higher \( t \)-regularity can also be obtained. To do this, we shall apply the more refined Th. 4.5.2 in [LM68], introduced there for the purpose of showing higher regularities. For the convenience of the reader, we list a slightly reformulated version:
Theorem 4.7. (From [LM68] Th. 4.5.2.) Let X and H be Hilbert spaces, with X ⊂ H, continuous injection. Let A be an unbounded linear operator in X such that A − λ is a bijection from the domain $D_X(A) = \{u \in X \mid Au \in X\}$ onto X, for all $\lambda \in \mathbb{C}$ with $\text{Re}\lambda \leq -\xi_0$.

Assume moreover that for all such $\lambda$, and for $u \in D_X(A)$,

$$\| (A - \lambda)u \|_X + \langle \lambda \rangle^\beta (A - \lambda)u \|_H \geq c (\|u\|_{D_X(A)} + \langle \lambda \rangle^{\beta+1} \|u\|_H),$$

where $\beta > 0$ and $c > 0$ are given.

Let $\beta \notin \frac{1}{2} + \mathbb{N}$. The problem $Au + \partial_t u = f$ for $t \in I$, $u(0) = 0$, with $f$ given in $L_2(I; X) \cap H^\beta(I; H)$ with $f^{(j)}(0) = 0$ for $j < \beta - \frac{1}{2}$, has a unique solution

$$u \in L_2(I; D_X(A)) \cap H^{\beta+1}(I; H).$$

This allows us to show:

Theorem 4.8. Assumptions as in Theorem 4.6.

1° If $f \in L_2(I; \overline{H}^r(\Omega)) \cap \overline{H}^1(I; L_2(\Omega))$ for some $r$ satisfying (4.19), with $f|_{t=0} = 0$, then the solution of (4.15) satisfies

$$u \in L_2(I; H^{a(2a+r)}(\Omega)) \cap \overline{H}^2(I; L_2(\Omega)).$$

2° For any integer $k \geq 2$, if $f \in L_2(I; \overline{H}^r(\Omega)) \cap \overline{H}^k(I; L_2(\Omega))$ with $\partial_t^j f|_{t=0} = 0$ for $j < k$, then

$$u \in L_2(I; H^{a(2a+r)}(\Omega)) \cap \overline{H}^{k+1}(I; L_2(\Omega)).$$

It follows in particular that

$$f \in \bigcap_k \overline{H}^k(I; \overline{H}^r(\Omega)), \partial_t^j f|_{t=0} = 0 \text{ for } j \in \mathbb{N}_0 \implies u \in \bigcap_k \overline{H}^k(I; H^{a(2a+r)}(\Omega)).$$

Proof. With A acting like $r^+P$, denote

$$D_r(A) = \{v \in H^{a(2a)}(\Omega) \mid Av \in \overline{H}^r(\Omega)\}.$$

Here $D_r(A) = H^{a(2a+r)}(\Omega)$ in view of Theorem 2.5 (since $r \geq 0$). Moreover, it equals $D_{\overline{H}^r}(A) = \{v \in \overline{H}^r(\Omega) \mid Av \in \overline{H}^r(\Omega)\}$, since $H^{a(2a)}(\Omega) \subset H^r(\Omega) \subset \overline{H}^r(\Omega)$. When $\text{Re}\lambda \leq -\xi_0$, the bijectiveness of $A - \lambda$ from $H^{a(2a)}(\Omega)$ to $L_2(\Omega)$ implies bijectiveness from $D_r(A)$ to $\overline{H}^r(\Omega)$. All this shows that $D_r(A) = D_{\overline{H}^r}(A)$ is as in the start of Theorem 4.7 with $X = \overline{H}^r(\Omega)$, $H = L_2(\Omega)$. Moreover, there is an equivalence of norms

$$\|(A + \xi_0)v\|_{\overline{H}^r} \simeq \|v\|_{D_r(A)}, \text{ for } v \in D_r(A).$$

Note also that besides the inequality (4.18), that may be written

$$\langle \lambda \rangle (A - \lambda)^{-1} g \|_{L_2} \leq c \|g\|_{L_2}, \text{ for } g \in L_2(\Omega), \text{ Re}\lambda \leq -\xi_0,$$
we have that by (4.20), (4.28) for \( r = 0 \) and (4.29),

\[
\|(A - \lambda)^{-1}g\|_{\overline{H}^r} \leq c\|(A - \lambda)^{-1}g\|_{H^{a(2a)}} \simeq \|(A + \xi_0)(A - \lambda)^{-1}g\|_{L_2} \\
\leq \|g\|_{L_2} + |\lambda + \xi_0|\|(A - \lambda)^{-1}g\|_{L_2} \leq c\|g\|_{L_2}, \text{ for } g \in L_2(\Omega),
\]

so altogether,

\[
\langle \lambda \rangle\|(A - \lambda)^{-1}g\|_{L_2} + \|(A - \lambda)^{-1}g\|_{\overline{H}^r} \leq c_1\|g\|_{L_2}, \text{ for } g \in L_2(\Omega).
\]

For \( v \in D_r(A) \), with \( (A - \lambda)v \) denoted \( g \), (4.28) implies

\[
\|v\|_{D_r(A)} \simeq \|(A + \xi_0)v\|_{\overline{H}^r} \leq c_2\|(A - \lambda)v\|_{\overline{H}^r} + c_3\|\langle \lambda \rangle \|g\|_{L_2}),
\]

where we used (4.30) in the last step. Moreover, by (4.30),

\[
\langle \lambda \rangle^2\|v\|_{L_2} \leq c_1\langle \lambda \rangle\|g\|_{L_2},
\]

so we altogether find the inequality for \( v \in D_r(A) \):

\[
\|v\|_{D_r(A)} + \langle \lambda \rangle\|v\|_{L_2} \leq c_4\|(A - \lambda)v\|_{\overline{H}^r} + \langle \lambda \rangle\|(A - \lambda)v\|_{L_2}), \quad \text{Re} \lambda \leq -\xi_0.
\]

We can now apply Theorem 4.7, with \( \beta = 1 \), \( X = \overline{H}^r(\Omega) \) and \( \mathcal{H} = L_2(\Omega) \). It follows that the solution of (4.15) with \( f(x, t) \) given in \( L_2(I; \overline{H}^r(\Omega)) \cap \overline{H}^1(I; L_2(\Omega)) \), \( f|_{t=0} = 0 \), satisfies

\[
u \in L_2(I; D_r(A)) \cap \overline{H}^2(I; L_2(\Omega)),
\]

from which (4.19) follows since \( D_r(A) = H^{a(2a+r)}(\Omega) \). This shows 1°.

Now let \( k \geq 2 \). By (4.29) and (4.31), since \( v = (A - \lambda)^{-1}g \),

\[
\|v\|_{D_r(A)} + \langle \lambda \rangle^{k+1}\|v\|_{L_2} \leq c_3\|(A - \lambda)v\|_{\overline{H}^r} + \langle \lambda \rangle\|g\|_{L_2}) + c\langle \lambda \rangle^k\|g\|_{L_2} \\
\leq c_5\|(A - \lambda)v\|_{\overline{H}^r} + \langle \lambda \rangle^k\|(A - \lambda)v\|_{L_2}),
\]

which allows an application of Theorem 4.7 with \( \beta = k \), \( X = \overline{H}^r(\Omega) \), \( \mathcal{H} = L_2(\Omega) \), giving the conclusion (4.25). (4.26) follows when \( k \to \infty \). This shows 2°. □

Note in particular that \( D_r(A) = H^{a(4a)}(\Omega) \) when \( a < \frac{1}{2} \).

From the point of view of anisotropic Sobolev spaces, the solutions in (4.24) and (4.25) satisfy, since \( H^{a(2a+r)}(\Omega) \subset H^{a(2a)}(\Omega) \subset H^r(\Omega) \subset \overline{H}^r(\Omega) \),

\[
u \in L_2(I; \overline{H}^r(\Omega)) \cap \overline{H}^2(I; L_2(\Omega)) = \overline{H}^{r,2}(\Omega \times I), \quad \text{resp.} \ u \in \overline{H}^{r,k+1}(\Omega \times I),
\]

but (4.24)–(4.26) give a more refined information.

Note that \( r < 3/2 \) in all these cases. We think that a lifting to higher values of \( r \), of the conclusion of 1° concerning regularity in \( x \) at the boundary, would demand very different methods, or may not even be possible, because of the incompatibility of \( H^{a(s)}(\Omega) \) with standard high-order Sobolev spaces when \( s \) is high. Cf. also Corollary 4.3, which excludes higher smoothness in a related situation.

Next, we turn to \( L_p \)-related Sobolev spaces with general \( p \). The following result was shown for \( x \)-independent operators in [G18]:
Theorem 4.9. Let \( P \) satisfy Hypothesis 1.2 with \( a < 1 \). Then (4.15) has for any \( f \in L_p(\Omega \times I) \) a unique solution \( u(x,t) \in \overline{C}^b(I;L_p(\Omega)) \) \((1 < p < \infty)\); it satisfies:

\[
\text{(4.34)} \quad u \in L_p(I;H^a_p(\overline{\Omega})) \cap \overline{H}^1_p(I;L_p(\Omega)).
\]

The result is sharp, since it gives the exact domain for \( u \) mapped into \( f \in L_p(\Omega \times I) \).

The proof relies on a theorem of Lamberton [L87] Th. 1. It follows from Lamberton [L87] Th. 1.4.1, Expl. 1.2.1, and a strongly continuous contraction semigroup \( L^\ast \).

Remark 4.10. Another paper [BWZ17], preparatory for [BWZ18], is devoted to a computational proof of local regularity (regularity in compact subsets of \( \Omega \)) of the solutions of the stationary Dirichlet problem (2.8) for \( P = (-\Delta)^a \) with \( f \in L_p(\Omega) \). Here the authors were apparently unfamiliar with the pseudodifferential elliptic regularity theory (mentioned in [G15]) that gave the answer many years ago, see the addendum [BWZ17a]. The addendum also corrects some mistakes connected with the definition of \( W^{s,p} \)-spaces, cf. (2.3)ff. above.

The proof of local regularity is repeated in [BWZ18].

Let us recall the proof of Theorem 4.9 from [G18]: The Dirichlet realization in \( L_p(\Omega) \), namely the operator \( \Delta_{\text{Dir},p} \), acting like \( r^+ P \) with domain

\[
\text{(4.35)} \quad D(\Delta_{\text{Dir},p}) = \{ u \in \dot{H}^a_p(\overline{\Omega}) \mid r^+ Pu \in L_p(\Omega) \},
\]

coincides on \( L_2(\Omega) \cap L_p(\Omega) \) with \( \Delta_{\text{Dir},2} \) defined variationally from the sesquilinear form

\[
Q_0(u,v) = \int_{\mathbb{R}^2n} (u(x) - u(y))(\overline{v}(x) - \overline{v}(y))K(x - y) \, dxdy \text{ on } \dot{H}^a_p(\overline{\Omega}),
\]

where \( K(y) = cF^{-1}p(\xi) \), positive and homogeneous of degree \(-2a - n\).

The form has the Markovian property: When \( u_0 \) is defined from a real function \( u \) by \( u_0 = \min\{\max\{u,0\},1\} \), then \( Q_0(u_0,u_0) \leq Q_0(u,u) \). It is a so-called Dirichlet form, as explained in Fukushima, Oshima and Takeda [FOT94], pages 4–5 and Example 1.2.1, and Davies [D89]. Then, by [FOT94] Th. 1.4.1 and [D89] Th. 1.4.1–1.4.2, \( -\Delta_{\text{Dir},p} \) generates a strongly continuous contraction semigroup \( T_p(t) \) not only in \( L_2(\Omega) \) for \( p = 2 \) but also in \( L_p(\Omega) \) for any \( 1 < p < \infty \), and \( T_p(t) \) is bounded holomorphic. Then the statements in Theorem 4.9 follow from Lamberton [L87] Th. 1. \( \square \)

Remark 4.11. An advantage of the above results is that we have a precise characterization of the Dirichlet domain, namely \( D(\Delta_{\text{Dir},p}) = H^a_p(\overline{\Omega}) \). This space can be further described, as shown in [G15], cf. Th. 5.4. When \( 0 < p < 1/a \) then \( H^a_p(\overline{\Omega}) = H^2_p(\overline{\Omega}) \). Since \( \dot{H}^a_p(\overline{\Omega}) \subset \overline{H}^a_p(\overline{\Omega}) \), the solutions \( u \) are in the anisotropic space

\[
\overline{H}^{(2a,1)}_p(\mathbb{R}^n \times I), \text{ supported for } x \in \overline{\Omega}.
\]

When \( p > 1/a \), \( H^a_p(\overline{\Omega}) \) consists, as recalled earlier in (2.21)ff., of the functions \( v = w + x_n^a K_0 \varphi \) with \( w \in \dot{H}^a_p(\mathbb{R}^n \times I), \varphi \in B_p^{a-1/p}(\mathbb{R}^n-1) \); here \( K_0 \) is the Poisson operator.
$K_0: \phi \mapsto e^{-(D')^2 + a} \phi$ solving the Dirichlet problem for $1-\Delta$ with $(1-\Delta)u = 0$ and nontrivial boundary data $\phi$. Also in the curved case, $H^{a(2a)}(\Omega)$ consists of $H^{2a} \Omega)$ plus a space of Poisson solutions in $H^{a(2a)} \Omega)$ multiplied by $d^a$.

We observe moreover that if we take $r \geq 0$ such that
\begin{equation}
\begin{cases}
  r = 2a & \text{if } 0 < a < 1/p, \\
  r = a + 1/p & \text{if } 1/p \leq a < 1,
\end{cases}
\end{equation}
(note that $r \leq 2a$ in all cases), then $H^{a(2a)}(\Omega) \subset H^{r}(\Omega)$, for all $0 < a < 1$ (in view of (2.19), cf. also (4.21)). The statement (4.34) then implies that
\begin{equation}
u \in H^{r,r/(2a)}(\Omega \times I).
\end{equation}
When $f$ has a higher regularity than $L_p(\Omega \times I)$, the interior regularity can be improved a little by use of Theorem 4.4 (for the boundary regularity, see Theorem 4.18–19):

**Theorem 4.12.** Let $u$ be as in Theorem 4.9, and let $r$ satisfy (4.36). Then $u$ satisfies (4.37), and moreover, for $0 < s \leq r$,
\begin{equation}
f \in H^{s,s/(2a)}(\Omega \times I) \implies u \in H^{s+2a,s/(2a)+1}(\Omega \times I).
\end{equation}

In particular, if $a < 1/p$ and $f \in H^{2a,1}(\Omega \times I)$, then $u \in H^{4a,2}(\Omega \times I)$.

### 4.3 Higher time-regularity.

The result of Theorem 4.9 can be considerably extended by use of the theory of Amann [A97], in the question of time-regularity. Fix $p \in ]1,\infty[$. The fact that $-P_{\text{Dir},p}$ is the generator of a bounded holomorphic semigroup $T_p(t)$ in $L_p(\Omega)$ (for $t$ in a sector around $\mathbb{R}_+$ depending on $p$, cf. [D89] Th. 1.4.2), assures that there is an obtuse sector
\begin{equation}
V_\delta = \{ \lambda \in \mathbb{C} \mid \arg \lambda \in \pi/2 - \delta, 3\pi/2 + \delta \},
\end{equation}
where the resolvent $(P_{\text{Dir},p} - \lambda)^{-1}$ exists and satisfies an inequality
\begin{equation}
|\lambda| \|(P_{\text{Dir},p} - \lambda)^{-1}\|_{\mathcal{L}(L_p)} \leq M,
\end{equation}
cf. Hille and Phillips [HP57] Th. 17.5.1, or e.g. Kato [K66] Th. IX.1.23. Since $P_{\text{Dir},p}$ has a bounded inverse, $\|(P_{\text{Dir},p} - \lambda)^{-1}\|_{\mathcal{L}(L_p)} \leq c$ also holds for $\lambda$ in a neighborhood of 0, so we can replace $|\lambda|$ by $\langle \lambda \rangle$ in (4.40) (with a larger constant $M'$). Note that furthermore,
\begin{equation}
\|P_{\text{Dir},p}(P_{\text{Dir},p} - \lambda)^{-1}P_{\text{Dir},p} - \lambda)^{-1}P_{\text{Dir},p} - \lambda)^{-1}P_{\text{Dir},p} \|_{L_p} = \|P_{\text{Dir},p} - \lambda)^{-1}P_{\text{Dir},p} - \lambda)^{-1}P_{\text{Dir},p} \|_{L_p} \leq (1 + M)\|P_{\text{Dir},p} - \lambda)^{-1}P_{\text{Dir},p} - \lambda)^{-1}P_{\text{Dir},p} \|_{L_p},
\end{equation}
so that $(P_{\text{Dir},p} - \lambda)^{-1}$ is bounded uniformly in $\lambda$ from $L_p(\Omega)$ to $D(P_{\text{Dir},p})$ with the graph-norm.

Thus, if we set
\begin{equation}
E_0 = L_p(\Omega), \quad E_1 = D(P_{\text{Dir},p}) = H^{a(2a)}(\Omega),
\end{equation}
we have that $A = P_{\text{Dir},p}$ satisfies (with $B_\varepsilon = \{ |\lambda| < \varepsilon \}$ for a small $\varepsilon > 0$
\begin{equation}
\langle \lambda \rangle \|(A - \lambda)^{-1}\|_{\mathcal{L}(E_0)} + \|(A - \lambda)^{-1}\|_{\mathcal{L}(E_0,E_1)} \leq c \quad \lambda \in V_\delta \cup B_\varepsilon.
\end{equation}
We can then apply Theorem 8.8 of Amann [A97]. It is formulated with vector-valued Besov spaces $B^{s,r}_q(\mathbb{R};X)$ (valued in a Banach space $X$, a function space in the applications), where the case $q = r = \infty$ is particularly interesting for our purposes, since $B^{s,\infty}_\infty(\mathbb{R};X)$ equals the vector-valued Hölder-Zygmund space $C^s(\mathbb{R};X)$. This coincides with the vector-valued Hölder space $C^s(\mathbb{R};X)$ when $s \in \mathbb{R}_+ \setminus \mathbb{N}$, see Remark 2.7 for further information.
**Theorem 4.13.** (From [A97] Th. 8.8.) Let $s \in \mathbb{R}$ and $q, r \in [1, \infty]$. Let $E_0$ and $E_1$ be Banach spaces, with $E_1$ continuously injected in $E_0$, and let $A$ be a linear operator in $E_0$ with domain $E_1$ and range $E_0$, satisfying (4.42).

For $f$ given in $B_{q,r}^s(\mathbb{R}; E_0) \cap L_{1,\text{loc}}(\mathbb{R}; E_0)$ and supported in $\mathbb{R}_+$, the problem

$$Au + \partial_t u = f \quad \text{for } t \in \mathbb{R}, \quad \text{supp } u \subset \mathbb{R}_+,$$

has a unique solution $u$ in the space of distributions in $B_{q,r}^{s+1}(\mathbb{R}; E_0) \cap B_{q,r}^s(\mathbb{R}; E_1)$ supported in $\mathbb{R}_+$; it is described by

$$u(t) = \int_0^t e^{-(t-\tau)A} f(\tau) \, d\tau, \quad t > 0.$$

An application of this theorem with $q = r = \infty$ and $E_0, E_1$ defined by (4.41) gives:

**Theorem 4.14.** Assumptions as in Theorem 4.9. The solution $u$ satisfies: When $f \in \dot{C}_*^{s}(\mathbb{R}_+; L_p(\Omega)) \cap L_{1,\text{loc}}(\mathbb{R}, L_p(\Omega))$ for some $s \in \mathbb{R}$, then

$$u \in \dot{C}_*^{s}(\mathbb{R}_+; H_p^{a(2a)}(\Omega)) \cap \dot{C}_*^{s+1}(\mathbb{R}_+; L_p(\Omega)).$$

In particular, for $s \in \mathbb{R}_+ \setminus \mathbb{N}$,

$$f \in \dot{C}^s(\mathbb{R}_+; L_p(\Omega)) \iff u \in \dot{C}^s(\mathbb{R}_+; H_p^{a(2a)}(\Omega)) \cap \dot{C}^{s+1}(\mathbb{R}_+; L_p(\Omega)).$$

We here use conventions as in (2.4): The set of $u(x,t) \in C_*(\mathbb{R}; X)$ that vanish for $t < 0$ is denoted $\dot{C}_*^s(\mathbb{R}_+; X)$. The conclusion $\implies$ is a special case of Theorem 4.13 with $A = P_{\text{Dir}, p}$; the converse $\impliedby$ follows immediately by application of $A + \partial_t$ to $u$.

**Remark 4.15.** Amann’s theorem can of course also be applied with other choices of $B_{q,r}^s$, e.g. Sobolev-Slobodetskii spaces $W^{s,q} = B_{q,q}^s$ when $s \in \mathbb{R}_+ \setminus \mathbb{N}$; this leads to analogous statements. (Note that integer cases $s \in \mathbb{N}_0$ are not covered, in particular not the result of Theorem 4.9 for $p \neq 2$, since the $H_p^s$-spaces are not in the scale $B_{q,r}^s$ when $p \neq 2$.)

Observe the consequences:

**Corollary 4.16.** Assumptions as in Theorem 4.9. The solution $u$ satisfies:

$$f \in \dot{C}^\infty(\mathbb{R}_+; L_p(\Omega)) \implies u \in \dot{C}^\infty(\mathbb{R}_+; H_p^{a(2a)}(\Omega)).$$

Moreover,

$$f \in \dot{C}^s(\mathbb{R}_+; L_\infty(\Omega)) \implies u \in \dot{C}^s(\mathbb{R}_+; d^aC^{0-\varepsilon}(\Omega)),
\quad f \in \dot{C}^\infty(\mathbb{R}_+; L_\infty(\Omega)) \implies u \in \dot{C}^\infty(\mathbb{R}_+; d^aC^{0-\varepsilon}(\Omega)), $$

for $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\varepsilon \in ]0, a]$. 
Proof. When \( f \in \dot{C}^\infty(\mathbb{R}_+; L_p(\Omega)) \), it is in particular in \( \dot{C}^s(\mathbb{R}_+; \dot{L}_p(\Omega)) \) for all \( s \in \mathbb{R}_+ \setminus \mathbb{N} \), where (4.44) implies that \( u \in \dot{C}^s(\mathbb{R}_+; \dot{H}_p^{a(2a)}(\Omega)) \). Taking intersections over \( s \), we find (4.45).

(4.46) follows from (4.44) resp. (4.45), since \( L_\infty(\Omega) \subset L_p(\Omega) \) for \( 1 < p < \infty \), and \( \bigcap_p H_p^{a(2a)}(\Omega) \subset d^a C^{a-\varepsilon}(\Omega) \) (as already observed in [G15] Sect. 7).

The statements in (4.46) improve the results of [FR17] and [RV18] concerning time-regularity.

Amann’s theorem can also be used in situations with a higher \( x \)-regularity of \( f \). The crucial point is to obtain an estimate (4.42) on the desired pair of Banach-spaces \( E_0, E_1 \) with \( E_1 \subset E_0 \).

Lemma 4.17. Let \( 1 < p < \infty \), and let \( A \) be an operator acting like \( P_{\text{Dir},p} \) in Theorem 4.9. When \( 2a < 1/p \), \( A \) is bijective from \( H_0^{a(4a)}(\Omega) \) to \( \dot{H}^{2a}_p(\Omega) \), and we have a resolvent inequality for \( \lambda \in V_0 \cup B_\varepsilon \) as in (4.42):

\[
\langle \lambda \rangle \| (A - \lambda)^{-1} f \|_{\dot{H}^{2a}_p} + \| (A - \lambda)^{-1} f \|_{H_p^{a(4a)}} \leq c \| f \|_{\dot{H}^{2a}_p}, \quad \text{for } f \in \dot{H}^{2a}_p(\Omega).
\]

Proof. Define \( D_{s,p}(A) \) by

\[
D_{s,p}(A) = H_p^{(2a+s)}(\Omega);
\]

it will be used for \( s = 0 \) and \( 2a \). Since \( 2a < 1/p \), \( \dot{H}^{2a}_p(\Omega) \) identifies with \( \dot{H}^{2a}_p(\Omega) \), so \( A \) has the asserted bijectiveness property in view of Theorem 2.5 plus [G14] Th. 3.5 on the invariance of kernel and cokernel.

By (4.42), we have the inequality

\[
\langle \lambda \rangle \| (A - \lambda)^{-1} f \|_{L_p} + \| (A - \lambda)^{-1} f \|_{D_{0,p}} \leq c \| f \|_{L_p}, \quad \text{for } f \in L_p(\Omega),
\]

and we want to lift it to the case where \( L_p(\Omega) \) is replaced by \( E_0 = \dot{H}^{2a}_p(\Omega) \) and \( D_{0,p}(A) \) is replaced by \( E_1 = D_{2a,p}(A) \). Let \( f \in \dot{H}^{2a}_p(\Omega) = D_{0,p}(A) \), and denote \( Af = g \); it lies in \( L_p(\Omega) \), and \( \| g \|_{L_p} \approx \| f \|_{\dot{H}^{2a}_p} \). Then

\[
\| \lambda (A - \lambda)^{-1} f \|_{\dot{H}^{2a}_p} = \| - f + A (A - \lambda)^{-1} f \|_{\dot{H}^{2a}_p} = \| - f + (A - \lambda)^{-1} Af \|_{\dot{H}^{2a}_p}
\]

\[
\leq \| f \|_{\dot{H}^{2a}_p} + \| (A - \lambda)^{-1} g \|_{\dot{H}^{2a}_p} \leq \| f \|_{\dot{H}^{2a}_p} + c \| g \|_{L_p} \leq c' \| f \|_{\dot{H}^{2a}_p},
\]

where we applied (4.49) to \( g \). Next,

\[
\| (A - \lambda)^{-1} f \|_{D_{2a,p}(A)} \approx \| A (A - \lambda)^{-1} f \|_{\dot{H}^{2a}_p} \approx \| A (A - \lambda)^{-1} f \|_{\dot{H}^{2a}_p} = \| f + \lambda (A - \lambda)^{-1} f \|_{\dot{H}^{2a}_p},
\]

using (4.50). Together with (4.50), this proves (4.47). □

This leads to a supplement to Theorem 4.14:
Theorem 4.18. Assumptions as in Theorem 4.9. If $2a < 1/p$, the solution satisfies, for $s \in \mathbb{R}_+ \setminus \mathbb{N}$:

\begin{align}
 f &\in \dot{C}^s(\mathbb{R}_+; \dot{H}_p^{2a}(\Omega)) \iff u \in \dot{C}^s(\mathbb{R}_+; H_p^{a(4a)}(\Omega)) \cap \dot{C}^{s+1}(\mathbb{R}_+; \dot{H}_p^{2a}(\Omega)), \\
 f &\in \dot{C}^\infty(\mathbb{R}_+; \dot{H}_p^{2a}(\Omega)) \iff u \in \dot{C}^\infty(\mathbb{R}_+; H_p^{a(4a)}(\Omega)).
\end{align}

Proof. We apply Theorem 4.13 with $E_0 = \dot{H}^{2a}(\Omega)$, $E_1 = H^{a(4a)}(\Omega)$, where the required resolvent inequality is shown in Lemma 4.17. □

Let us also apply Amann’s theorem to the $L_2$-operators studied in Theorem 4.6, listing just the resulting Hölder estimates. Here when $I = [0,T]$, we denote by $C^+_a(I; X)$ the space of functions in $C^\infty([−\infty, T[; X)$ with support in $[0,T]$.

Theorem 4.19. Assumptions as in Theorem 4.6. With $\xi_0$ as in (4.18), there is a set $V_0 \cup B_\varepsilon$ (cf. (4.39)) such that $A = P_{\text{Dir},2} + \xi_0$ satisfies an inequality (4.42), both for the choice $\{E_0, E_1\} = \{L_2(\Omega), H^{a(2a)}(\Omega)\}$ when $a < 1$, and for the choice $\{E_0, E_1\} = \{\dot{H}^{2a}(\Omega), H^{a(4a)}(\Omega)\}$ when $a < 1/4$.

The solution of (4.15) satisfies, for $s \in \mathbb{R}_+ \setminus \mathbb{N}, I = [0,T[$:

\begin{align}
 f &\in C^+_a(I; L_2(\Omega)) \iff u \in C^+_a(I; H^{a(2a)}(\Omega)) \cap C^{a+1}_+(I; L_2(\Omega)), \\
 f &\in C^\infty_+(I; L_2(\Omega)) \iff u \in C^\infty_+(I; H^{a(2a)}(\Omega)).
\end{align}

Moreover, if $a < 1/4$,

\begin{align}
 f &\in C^+_a(I; \dot{H}^{2a}(\Omega)) \iff u \in C^+_a(I; H^{a(4a)}(\Omega)) \cap C^{a+1}_+(I; \dot{H}^{2a}(\Omega)), \\
 f &\in C^\infty_+(I; \dot{H}^{2a}(\Omega)) \iff u \in C^\infty_+(I; H^{a(4a)}(\Omega)).
\end{align}

Proof. The estimate of $(A - \lambda)^{-1}$ with $E_0 = L_2(\Omega)$, $E_1 = H^{a(2a)}(\Omega)$, is assured by the information in the proof of Theorem 4.6 that the numerical range and spectrum of $P_{\text{Dir},2}$ is contained in an angular set $\{z \in \mathbb{C} | \; |\text{Im } z| \leq C(\text{Re } z + \xi_0), \; \text{Re } z > \xi_0\}$. (This is a standard fact in the theory of operators defined from sesquilinear forms, see e.g. Cor. 12.21 in [G09].) Then the resolvent estimate holds for $\lambda \to \infty$ on the rays in a closed sector disjoint from the sector $\{ |\text{Im } z| \leq C\text{Re } z \}$.

The estimate of $(A - \lambda)^{-1}$ with $E_0 = \dot{H}^{2a}(\Omega)$, $E_1 = H^{a(4a)}(\Omega)$ now follows exactly as in Lemma 4.17, when $a < 1/4$.

For the solvability assertions, note that $u(x,t)$ satisfies $P_{\text{Dir},2}u + \partial_t u = f$ if and only if $v(x,t) = e^{-\xi_0 t}u(x,t)$ satisfies $(P_{\text{Dir},2} + \xi_0) v + \partial_t v = e^{-\xi_0 t} f$.

An application of Theorem 4.13 to $A = P_{\text{Dir},2} + \xi_0$ leads to a solvability result like (4.44) for a solution $v$ with right-hand side $f_1$. When the problem (4.15) is considered for a given function $f \in C^+_a(I; E_0)$, we extend $f$ to a function $\tilde{f} \in \dot{C}^s(\mathbb{R}_+; E_0)$, and apply the solvability result with $f_1 = e^{-\xi_0 t}\tilde{f}$; this gives a solution $v$, and we set $u = (e^{\xi_0 t}v)|_I$, solving (4.15). It has the regularity claimed in (4.52), and conversely, application of $P_{\text{Dir},2} + \partial_t$ to such a function gives a right-hand side in $C^+_a(I; E_0)$. (4.53) follows immediately.

The statements in (4.54) follow by a similar application of Theorem 4.13 with $E_0 = \dot{H}^{2a}(\Omega)$, $E_1 = H^{a(4a)}(\Omega)$. □
It seems plausible that (4.51) and (4.54) can be extended to all $a < 1$ when $\dot{H}_p^{2a}(\Omega)$ is replaced by the possibly weaker space $\dot{H}_p^r(\Omega)$, cf. (4.36), and $H_p^{a(4a)}(\Omega)$ is replaced by $H_p^{a(2a+r)}(\Omega)$ (for $p = 2$, it is to some extent obtained in (4.26)). This might be based on an extension of Amann’s strategy. The Fourier transform used in [LM68] is in [A97] replaced by multiplier theorems in a suitable sense; it could be investigated whether something similar might be done based on resolvent inequalities in the style of (4.32), (4.33).

On the other hand, we expect that the limitations on $r$ are essential in some sense, since the domain and range for the Dirichlet problem are not compatible in higher-order spaces, as noted also earlier.

**Remark 4.20.** The results in this subsection on higher $t$-regularity based on Theorem 4.13 from [A97], are in the case $p \neq 2$ restricted to the $x$-independent case $P = \text{Op}(p(\xi))$ with homogeneous symbol. However, since the proofs rely entirely on general resolvent inequalities, there is a leeway to extend the results to perturbations $P + P'$ when $P'$ is so small that such resolvent inequalities still hold; this will allow some $x$-dependence and lower-order terms. The idea can be further developed.

It is very likely that other methods could be useful and bring out further perspectives for the heat problem. One possibility could be to try to establish an $H^\infty$-calculus for the realizations of fractional Laplacians and their generalizations. This was done for elliptic differential operators with boundary conditions in a series of works, see e.g. Denk, Hieber and Prüss [DHP03] for an explanation of the theory and many references. However, pseudodifferential problems pose additional difficulties. The only contribution treating an operator within the Boutet de Monvel calculus, that we know of, is Abels [A05a]. The present pseudodifferential operators do not even belong to the Boutet de Monvel calculus, but have a different boundary behavior.

It will be interesting to see to what extent the results can be further improved by this or other tools from the extensive literature on differential heat operator problems.

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