Reconstruction of $SU(5)$ Grand Unified Model In Noncommutative Geometry Approach

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Abstract

Based on the generalized gauge theory on $M^4 \times Z_2 \times Z_3$, we reconstructed the realistic $SU(5)$ Grand Unified model by a suitable assignment of fermion fields. The action of group elements $Z_2$ on fermion fields is the charge conjugation while the action of $Z_3$ elements represent generation translation. We find that to fit the spontaneous symmetry breaking and gauge hierarchy of $SU(5)$ model a linear term of curvature has to be introduced. A new mass relation is obtained in our reconstructed model.

PACS number(s): 02.40. -k, 11.15. -q, 12.10. Dm

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1 Introduction

In recent years, it is believed that Non-commutative geometry extends the basic geometry framework of physics\cite{1, 2}. The most remarkable results are that in Standard Model the Higgs fields may be considered as a kind of gauge field by the same footing as Yang-Mills fields and the Yukawa couplings can be introduced as a kind of gauge coupling. These topics have been studied by many works\cite{3}—\cite{10}. It is also interesting to quest whether the same description stands when we go from Standard Model to Grand Unification theories (e.g. $SU(5)$ GUT\cite{11}), in which Higgs fields are introduced as input data in model building. By enlarging the discrete points model first proposed by A.Connes\cite{1, 3}, A. Chamsedine et al \cite{5} provided a generalized formula, which gave a clue to study more extensive model beyond the Standard Model, such as $SU(5)$ and $SO(10)$ Grand unified models. But there are lots of details need to be further studied.

In our previous works\cite{7, 8}, we constructed generalized gauge theory on discrete group $Z_2$. In this approach, we enlarged space-time to five dimensions with the 5-th “coordinate” containing only two points of $Z_2$, assigned left and right handed Fermion fields according to the discrete group “coordinate” and wrote down a Lagrangian of fermion fields, which is not only the function of the space–time coordinates but also of the discrete group ”coordinate”. The most important point of this approach was that the derivatives on discrete group were included in the Lagrangian. Similar to the case of the ordinary Yang–Mills gauge theory, when we require the Lagrangian be invariant under the gauge group which is a function of space time and of discrete group, then Higgs appear in the covariant derivative and Yukawa coupling is naturally introduced by the gauge coupling. Furthermore,we constructed the Weinberg-Salam model and the Electroweak-strong interaction model and tried to endow discrete group with some physical meaning.

In this paper, we first develop our previous approach to the case of $M^4 \times Z_2 \times Z_3$ and reconstruct the realistic $SU(5)$ Grand Unified model of three generation fermions with generalized gauge theory on $M^4 \times Z_2 \times Z_3$. A similar generalized gauge theory on $M^4 \times Z_2 \times Z_3$ has also been discussed in a $CP$ violation toy model\cite{11}. We distinguish the left and right hand parts of fermions by two elements of the discrete group $Z_2$, differentiate three families by three elements of the discrete group $Z_3$ and connect fermions by charge conjugation transformation on discrete points of $Z_2$ and by generation translation on discrete points of $Z_3$. Since there are two mass scales in the $SU(5)$ model characterizing the spontaneous symmetry breaking of $SU(5)$ to $SU(3) \times SU(2) \times U(1)$ and then to $SU(3) \times U(1)$, if we want to get this gauge hierarchy, we need to add the linear term of curvature $F$, which is first proposed by Sitarz\cite{6}.

The plan of this paper is as follows. In section 2, we give the basic notion of gauge theory on $M^4 \times Z_2 \times Z_3$. In section 3, we build $SU(5)$ model using the generalized gauge theory on $M^4 \times Z_2 \times Z_3$. In section 4, we discuss the symmetry broken phenomenon.

2 Notation of gauge theory on $M^4 \times Z_2 \times Z_3$

In this section we shall give the basic notion of construction gauge theory on $M^4 \times Z_2 \times Z_3$. More detailed account of construction can be found in \cite{3, 4}.
Let \( x^\mu \) denote the coordinate on \( M^4 \) and \( g \) label the points of discrete group \( Z_2 \times Z_3 \). The differentiation of an arbitrary function on product space \( M^4 \times Z_2 \times Z_3 \) has the following form,

\[
df = \partial_\mu f dx^\mu + \partial_g f \chi^g, \quad g \in Z_2 \times Z_3,
\]

where \( dx^\mu \) and \( \chi^g \) are basis of one forms on \( M^4 \) and \( Z_2 \times Z_3 \) respectively. The partial derivative \( \partial_g \) is defined as follows:

\[
\partial_g f(x,h) = (f(x,h) - R_g f(x,h)) = (f(x,h) - f(x,h \cdot g)).
\]

From the defination we can obtain a lot of ralations: ones for the product of one-forms

\[
dx^\mu \hat{\otimes} dx^\nu = -dx^\nu \hat{\otimes} dx^\mu
\]

ones for the multiplication of one-form by functions

\[
f(x,h)dx^\mu = dx^\mu f(x,h), \quad \chi^g f(x,h) = R_g f(x,h) \chi^g,
\]

and ones by acting derivative operator on one forms

\[
\frac{ddx^\mu}{dt} = 0, \quad d\chi^g = -C^g_{p,h} \chi^p \hat{\otimes} \chi^h,
\]

where structure constants \( C^g_{p,h} = \delta^g_p + \delta^g_h - \delta^g_{ph}(\delta^e_{ph} - 1) \). The general gauge potential \( A \) on \( M^4 \times Z_2 \times Z_3 \) can be written as:

\[
A = A_\mu dx^\mu + \sum_{g \in Z_2 \times Z_3} \phi_g \chi^g
\]

The unitarity of gauge group enforces that \( A^* = -A \). Thus, since \((dx^\mu)^* = dx^\mu\) and \((\chi^g)^* = -\chi^{g^{-1}}\), we obtain

\[
(A_\mu)^\dagger = -A_\mu, \quad \phi_g^\dagger = R_g \phi_{g^{-1}}.
\]

The curvature two form \( F = dA + A \hat{\otimes} A \) splits into terms,

\[
F = \frac{1}{2} F_{\mu\nu} dx^\mu \hat{\otimes} dx^\nu + F_{\mu g} dx^\mu \hat{\otimes} \chi^g + F_{gh} \chi^g \hat{\otimes} \chi^h,
\]

where

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],
\]

\[
F_{\mu g} = \partial_\mu \Phi_g + A_\mu \Phi_g - \Phi_g R_g (A_\mu),
\]

\[
F_{gh} = \partial_g \phi_h + \phi_g R_g \phi_h - C^k_{gh} \phi_k
\]

with \( \Phi_g = 1 - \phi_g \).
To construct the Yang-Mills action, we need to define the metric

\[
\begin{align*}
    &<dx^\mu, dx^\nu> = g^{\mu\nu}, \quad <\chi^g, \chi^h> = \eta^{gh}, \\
    &<dx^\mu \wedge dx^\nu, dx^\sigma \wedge dx^\rho> = \frac{1}{2} (g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}), \\
    &<dx^\mu \otimes \chi^g, dx^\nu \otimes \chi^h> = g^{\mu\nu} \eta^{gh}, \\
    &<\chi^g \otimes \chi^h, \chi^{g'} \otimes \chi^{h'}> = \eta^{gg'} \eta^{hh'}. 
\end{align*}
\]

(2.8)

where \( \eta^{gh} = \eta_g \delta^{gh} \). After taking such a form of the metric, the Yang-Mills Lagrangian becomes

\[
L_N = -\frac{1}{N} \int_G <F, \bar{F}> = \frac{1}{N} \int_G \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \eta_g F_{\mu g} F^{\mu g} - \eta_g \eta_h F_{gh} F^{\cdot gh} \right). 
\]

(2.9)

It was found [6] that there exist a possibility of adding an extra gauge invariant term to the Yang-Mills action, which is linear in the curvature \(<F>\),

\[
L_L = -\frac{1}{N} \int_G <F> = -\frac{1}{N} \int_G F_{gh} \eta^{gh} = -\frac{1}{N} \int_G \eta_g F_{gg}^{-1}. 
\]

(2.10)

Let us add this term to Yang-Mills action with an arbitrary scaling parameter \( \alpha \). We obtain the bosonic sector Lagrangian

\[
L = L_N + \alpha L_L. 
\]

(2.11)

In the next section, we will find that this Lagrangian is needed in the construction of the \( SU(5) \) model.

3 Generalized \( SU(5) \) gauge theory on \( M^4 \times Z_2 \times Z_3 \)

In this section, we build the \( SU(5) \) gauge theory on \( M^4 \times Z_2 \times Z_3 \) by using generalized gauge theory on discrete group [7, 8]. According to the basic knowledge of \( SU(5) \) model, we first set fermion fields on discrete group, then write down gauge fields in terms of gauge potential, at last give Lagrangian of gauge fields by noncommutative differential geometry approach.

3.1 Fields on \( M^4 \times Z_2 \times Z_3 \)

From the basic knowledge of \( SU(5) \) model [4], we know that one family of left(or right) handed fermions can be accommodated in an \( SU(5) \) reducible representation of \( 5^* + 10 \).
According to representation of $SU(5)$, we write down the first family fermions as following:

$$5^* : \psi_L = \begin{bmatrix} d_1^C \\ d_2^C \\ d_3^C \\ e^- \\ -\nu_e \end{bmatrix}_L, \quad 5 : \psi_R^C = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ -\nu_e^C \end{bmatrix}_R$$

$$10 : \chi_L = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & u_3^C & -u_2^C & u_1 & d_1 \\ -u_3^C & 0 & u_1^C & u_2 & d_2 \\ u_2^C & -u_1^C & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{bmatrix}_L$$

$$10^* : \chi_R^c = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & u_3 & -u_2 & u_1^C & d_1^C \\ -u_3 & 0 & u_1 & u_2^C & d_2^C \\ u_2 & -u_1 & 0 & u_3^C & d_3^C \\ -u_1^C & -u_2^C & -u_3^C & 0 & e^- \\ -d_1^C & -d_2^C & -d_3^C & -e^- & 0 \end{bmatrix}_R$$

The other two families can be written similarly by replacing $u, d, e, \nu_e$ by $c, s, \mu, \nu_\mu$ and $t, b, \tau, \nu_\tau$.

From observation of three family fermions and their left-right hand parts, $5^* + 10$ and $5 + 10^*$, we find it is possible to assign them with respect to elements of discrete group $Z_2 \times Z_3$, which is product of discrete groups $Z_2$ and $Z_3$. Discrete group $Z_2$ has two elements $Z_2 = \{ e, Z | Z^2 = e \}$, discrete group $Z_3$ has three elements $Z_3 = \{ e, r, r^2 | r^3 = e \}$. So the direct product group $Z_2 \times Z_3$ has six elements

$$Z_2 \times Z_3 = \{ e, r, r^2, Zr, Zr^2 | Z^2 = e, r^3 = e, Zr = rZ \}.$$

In this paper, we use two elements of $Z_2$ to distinguish left-right hand fermions and use three elements of $Z_3$ to distinguish three families. So our working manifold should be $M^4 \times Z_2 \times Z_3$.

According to discrete group $Z_2 \times Z_3$, we arrange Fermions as following:

$$\psi(x,e) = \begin{bmatrix} \psi^C \\ \chi^C \end{bmatrix}^1_R, \quad \psi(x,r) = \begin{bmatrix} \psi^C \\ \chi^C \end{bmatrix}^2_R, \quad \psi(x,r^2) = \begin{bmatrix} \psi^C \\ \chi^C \end{bmatrix}^3_R,$$

$$\psi(x,Z) = \begin{bmatrix} \psi \\ \chi \end{bmatrix}^1_L, \quad \psi(x,rZ) = \begin{bmatrix} \psi \\ \chi \end{bmatrix}^2_L, \quad \psi(x,r^2Z) = \begin{bmatrix} \psi \\ \chi \end{bmatrix}^3_L,$$

where $[i]$ represents the $i$-th generation of Fermions. It is important to note that the actions $R_g, g \in Z_2 \times Z_3$ on fermions have definite physical meaning. We find that the action $R_Z$ is nothing but the charge conjugation transformation, which inter-changes left-right hand fermions in $5^*+10$ and $5+10^*$ and the action $R_{r^i}, i = 1, 2, 3$ is the translation between different generations,

$$R_{r^i}\psi^j = \psi^{[i+j] \mod 3}, \quad i = 1, 2; \quad j = 1, 2, 3$$
As we did in [7, 8], to build gauge theory on space $M^4 \times Z_2 \times Z_3$, we should introduce free fermion Lagrangian first,

$$L(x, g) = \overline{\psi}(g) \left[ i\gamma^\mu (\mathring{\partial}_\mu - \partial_\mu) - U(\partial_Z + \partial_{Zr} + \partial_{Zr^2}) - U_1(\partial_r + \partial_{r^2}) \right] \psi(g),$$

(3.3)

where $U, U_1$ are parameters with mass dimension. Here we just choose two free parameters in front of partial derivatives of discrete group, a special case that depend on the model to be building, since these parameters are directly related to the mass of Higgs particles and there are only two mass scales of Higgs fields in minimum $SU(5)$ model. In fact, $U$ and $U_1$ are parameters relate to the distance among discrete points in non-commutative geometry approach.

Similar to the reason that leads to the introduction of Yang-Mills fields, it is reasonable to require that the Lagrangian (3.3) be invariant under gauge transformations $H(x, g), g \in Z_2 \times Z_3$, where $H$ are functions depending not only on $M^4$ but also on discrete group. So one should introduce covariant derivative in Lagrangian (3.3).

Gauge invariant Lagrangian under $SU(5)$ group should be written as follows:

$$L_F(x, g) = \overline{\psi}(g) \left[ i\gamma^\mu (\mathring{D}_\mu - \partial_\mu) - U(D_Z + D_{Zr} + D_{Zr^2}) - U_1(D_r + D_{r^2}) \right] \psi(g),$$

(3.4)

$$D_\mu = \partial_\mu + ig A_\mu, \quad D_g = \partial_g + \phi_g R_g, \quad g \in Z_2 \times Z_3,$$

where

$$A(e) = \begin{pmatrix} (A_{k,l}) \\ (A_{mn,pq}^*) \end{pmatrix}, \quad A(Z) = \begin{pmatrix} (A_{k,l}^*) \\ (A_{mn,pq}) \end{pmatrix},$$

(3.5)

$(A_{k,l})$ is a $5 \times 5$ matrix valued on 24 generators of $SU(5)$ group and the corresponding matrix elements are $A_{k,l}; (A_{mn,pq})$ is a $25 \times 25$ matrix with $mn, pq$ denoting the row and column indices of the matrix and the matrix element are

$$A_{mn,pq} = A_{m,p} \delta_{n,q} + A_{n,q} \delta_{m,p}.$$

Because the gauge transformations are independent of generations, we should set Yang-Mills potentials to be the same in different generations. This means $A(e) = A(r) = A(r^2)$ and $A(Z) = A(rZ) = A(r^2 Z)$.

In minimum $SU(5)$ model, there are two Higgs multiplets which belong to the adjoint and the vector representations respectively. Only the vector Higgs field appears in Yukawa coupling. In Yukawa terms of Lagrangian (3.4), it is easy to find that $\phi_Z, \phi_{rZ}, \phi_{r^2Z}$ connect left-right hand fermions and $\phi_r, \phi_{r^2}$ connect fermions with the same chirality. So only $\phi_Z, \phi_{rZ}, \phi_{r^2Z}$ fields appear in Yukawa terms while $\phi_r, \phi_{r^2}$ fields do not. To get the minimum $SU(5)$ model, we arrange vector representation in $\phi_Z, \phi_{rZ}, \phi_{r^2Z}$ and adjoint representation in $\phi_r, \phi_{r^2}$. Thus we write down the
Higgs fields as following,

\[ g = e \quad g = r \quad g = r^2 \]

\[
\begin{align*}
\phi_Z(g) &= \left[ \begin{array}{cc}
0 & f_{11}(H^*_{i,mn}) \\
f_{11}(H^*_{pq,z}) & e_{11}(H_{pq,mn})
\end{array} \right] ;
\phi_{rZ}(g) &= \left[ \begin{array}{cc}
0 & f_{22}(H^*_{i,mn}) \\
f_{22}(H^*_{pq,z}) & e_{22}(H_{pq,mn})
\end{array} \right] ;
\phi_{r^2Z}(g) &= \left[ \begin{array}{cc}
0 & f_{33}(H^*_{i,mn}) \\
f_{33}(H^*_{pq,z}) & e_{33}(H_{pq,mn})
\end{array} \right]
\]

where \((H_{i,mn})\) is a 5 \times 25 matrix, \((H^*_{pq,j})\) is a 25 \times 5 matrix, \(H_{pq,mn}\) is a 25 \times 25 matrix and the elements are

\[
H_{i,mn} = H_m\delta_{i,n} - H_n\delta_{i,m},
\]

\[
H_{pq,j} = H_p\delta_{q,j} - H_q\delta_{p,j},
\]

\[
H_{pq,mn} = \epsilon_{pqmn}kH_k
\]

Higgs fields on discrete points \(Z, rZ, r^2Z\) may be defined by Hermitian condition \(\phi^{\dagger}_g = R_g\phi_{g^{-1}}\), which are

\[
\phi^{\dagger}_Z = R_Z\phi_Z, \phi^{\dagger}_{rZ} = R_{rZ}\phi^{\dagger}_{rZ}, \phi^{\dagger}_{r^2Z} = R_{r^2Z}\phi_{r^2Z}.
\]

The other two components of Higgs fields \(\phi_r, \phi_{r^2}\) are set as,

\[
\begin{align*}
\phi_r(g) &= I\left[ \begin{array}{cc}
t_1(\Sigma_{i,j}) & s_1(\Sigma_{pq,mn}) \\
s_1(\Sigma_{pq,mn}) & e_1(\Sigma_{pq,mn})
\end{array} \right];
\phi_{rZ}(g) &= I\left[ \begin{array}{cc}
t_2(\Sigma_{i,j}) & s_2(\Sigma_{pq,mn}) \\
s_2(\Sigma_{pq,mn}) & e_2(\Sigma_{pq,mn})
\end{array} \right];
\phi_{r^2Z}(g) &= I\left[ \begin{array}{cc}
t_3(\Sigma_{i,j}) & s_3(\Sigma_{pq,mn}) \\
s_3(\Sigma_{pq,mn}) & e_3(\Sigma_{pq,mn})
\end{array} \right],
\end{align*}
\]

where \(I = \sqrt{-1}\) and \(t_i s_i\) are real parameters,

\[
\Sigma_{pq,mn} = \Sigma_{p,q}\delta_{q,n} + \Sigma_{q,n}\delta_{p,m}
\]

(\(\Sigma_{i,j}\)) is a 5 \times 5 traceless Hermitian matrix, i.e \((\Sigma_{i,j}) = (\Sigma_{i,j})^\dagger\) and \(Tr\Sigma = 0\).

The Hermitian condition \(\phi^{\dagger}_{r^2} = R_{r^2}\phi_r\) gives the values of \(\phi_{r^2}\) on discrete points as,

\[
\begin{align*}
\phi_r(g) &= -I\left[ \begin{array}{cc}
t_3(\Sigma_{i,j}) & s_3(\Sigma_{pq,mn}) \\
s_3(\Sigma_{pq,mn}) & e_3(\Sigma_{pq,mn})
\end{array} \right];
\phi_{rZ}(g) &= -I\left[ \begin{array}{cc}
t_1(\Sigma_{i,j}) & s_1(\Sigma_{pq,mn}) \\
s_1(\Sigma_{pq,mn}) & e_1(\Sigma_{pq,mn})
\end{array} \right];
\phi_{r^2Z}(g) &= -I\left[ \begin{array}{cc}
t_2(\Sigma_{i,j}) & s_2(\Sigma_{pq,mn}) \\
s_2(\Sigma_{pq,mn}) & e_2(\Sigma_{pq,mn})
\end{array} \right],
\end{align*}
\]

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Actually, we impose a symmetry \( R_z \phi = -\phi^* \), in the assignments of the fields \( \phi_r, \phi_r^2 \). It is interesting to find that this constraint corresponds to the discrete symmetry which was introduced in standard \( SU(5) \) grand unification model. In \( SU(5) \) model, we require the Higgs potential terms is invariant under transformation \( H \rightarrow -H \), \( \sum \rightarrow -\sum \), which can remove unwanted terms in the potential.

### 3.2 Lagrangian of Model

After taking the assignments of Yang-Mills fields and Higgs fields, now we are ready to write down the Lagrangian of fermionic sector form (3.4), which include couplings of gauge fields.

\[
\mathcal{L}_F = \sum_{A,k} \overline{\psi}_{k,A} i\gamma^\mu D_\mu \psi_{k,A} + \sum_{A,k,l} \overline{\chi}_{kl,A} i\gamma^\mu D_\mu \chi_{kl,A} + 2 \sum_{A,B} \sum_{pqklm} M_{1A,B} \chi^C_{pq,kl,A} \epsilon_{pqklm} H^*_m + h.c.
\]

(3.6)

where \( A, B \) are generation indices, the other indices are that of \( SU(5) \) group and \( M_{1A,B} \) \( M_{2A,B} \) are elements of matrix

\[
M_1 = \begin{bmatrix}
    f_{11} & f_{12} & f_{13} \\
    f_{21} & f_{22} & f_{23} \\
    f_{31} & f_{32} & f_{33}
\end{bmatrix}
\]

\[
M_2 = \begin{bmatrix}
    e_{11} & e_{12} & e_{13} \\
    e_{21} & e_{22} & e_{23} \\
    e_{31} & e_{32} & e_{33}
\end{bmatrix}
\]

It is easy to show that

\[
\overline{\chi}^C_{ij,A} \chi_{kl,B} \epsilon_{ijklm} H^*_m = \overline{\chi}^C_{kl,B} \chi_{ij,A} \epsilon_{ijklm} H^*_m
\]

(3.7)

so we set \( M_2 \) to be symmetric matrix, i.e \( e_{AB} = e_{BA} \), or

\[
M_2 = \begin{bmatrix}
    e_{11} & e_{12} & e_{13} \\
    e_{12} & e_{22} & e_{23} \\
    e_{13} & e_{23} & e_{33}
\end{bmatrix}
\]

The Lagrangian of bosonic sector may be derived from the generalized differential calculation on \( M^4 \times Z_2 \times Z_3 \). Using those assignments of fields on discrete groups and the basic knowledge of non-commutative geometry, we can obtain the Lagrangian of gauge fields. In the calculation, for simplicity, we set \( \eta_z = \eta_{rZ} = G \) and note \( \eta_r = G_1 \). Because the calculation is fairly cumbersome, we only write the result here.
\[ \mathcal{L}_G = -\frac{1}{N} < F, \bar{F} > \]
\[ = -\frac{g^2}{4N} 66 F_{\mu \nu} F^{\mu \nu} + \frac{16\beta}{N} G D_\mu H^\dagger D^\mu H \]
\[ + \frac{4\alpha}{N} G_1 T r(D_\mu \Sigma^\dagger D^\mu \Sigma) - [V(H, \Sigma) + V(\Sigma) + V(H)] \]
\[ \text{(3.8)} \]

where
\[ \alpha = t_1^2 + t_2^2 + t_3^2 + 10(s_1^2 + s_2^2 + s_3^2), \]
\[ \beta = T r(2M_1 M_1^\dagger + 3M_2 M_2^\dagger), \]

and
\[ D_\mu H = (\partial_\mu + ig A_\mu) H \]
\[ D_\mu \Sigma = \partial_\mu \Sigma + ig(A_\mu \Sigma - \Sigma A_\mu), \]

which show that Higgs fields \( H \) and \( \Sigma \) are vector and adjoint representations of \( SU(5) \) group. Here we wrote gauge bosons, Higgs fields \( \Sigma \) and \( H \) in their matrix forms\[14\] as

\[ A = \frac{1}{\sqrt{2}} \begin{bmatrix}
 [G -2B/ \sqrt{30}]_{\alpha \beta} & X_1 & Y_1 \\
 X_2 & Y_2 \\
 X_3 & Y_3 \\
 X_1^\dagger & X_2^\dagger & X_3^\dagger & W^3/\sqrt{2} + 3B/\sqrt{30} & W^\dagger \\
 Y_1^\dagger & Y_2^\dagger & Y_3^\dagger \\
 \end{bmatrix} \]
\[ \text{(3.9)} \]

\[ \Sigma = \begin{bmatrix}
 [\Sigma_8]^\alpha_{\beta} & - 2\Sigma_0/\sqrt{30} \\
 \Sigma X_1 & \Sigma Y_1 \\
 \Sigma X_2 & \Sigma Y_2 \\
 \Sigma X_3 & \Sigma Y_3 \\
 \Sigma X_1^\dagger & \Sigma X_2^\dagger & \Sigma X_3^\dagger \\
 \Sigma X_1^\dagger & \Sigma X_2^\dagger & \Sigma X_3^\dagger \\
 \end{bmatrix}, \]
\[ \text{(3.10)} \]

\[ H = \begin{bmatrix}
 H_{t_1} \\
 H_{t_2} \\
 H_{t_3} \\
 H_{d_1} \\
 H_{d_2} \\
 \end{bmatrix} \]
\[ \text{(3.11)} \]

Before giving the expression of potential, we normalize the coefficient of dynamics terms in above Lagrangian, so we can take values of normalization constant \( N \) and metrics \( G, G_1 \) as follows:

\[ N = 66g^2 = 16\beta \frac{G}{U^2} = 4\alpha \frac{G_1}{U_1^2} \]
\[ G = \frac{33}{8} g^2 U_2 \]
\[ G_1 = \frac{33}{2} g^2 U_1^2. \]

Then the Lagrangian of gauge fields becomes:
\[
\mathcal{L}_G = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu H'^\dagger D_\mu H + Tr(D_\mu \Sigma'^\dagger D^\mu \Sigma)
- [V(H, \Sigma) + V(\Sigma) + V(H)],
\]
(3.12)

and the potential is given as following,
\[
V(\Sigma) = -m_2^2 Tr\Sigma^2 + \lambda_1 (Tr\Sigma^2)^2 + \lambda_2 Tr\Sigma^4
\]
\[
V(H) = -m_2^2 H'^\dagger H + \lambda_3 (H'^\dagger H)^2
\]
\[
V(H, \Sigma) = \lambda_4 (Tr\Sigma^2) H'^\dagger H + \lambda_5 H'^\dagger \Sigma^2 H
\]

where
\[
m_1^2 = g^2 U_1^2 (\frac{33}{2\alpha} - \frac{99}{32} \frac{\alpha U^4}{\beta^2 U^2_2})
\]
\[
m_2^2 = g^2 U_2^2 (\frac{33}{4\beta} - 66 \frac{1}{\alpha^2 U^2_1})
\]
(3.13)

\[ \lambda_1 = \frac{99}{2} g^2 Tr(SS'^\dagger) \]
\[ \lambda_2 = \frac{33 g^2}{4} Tr(TT'^\dagger + 10TrSS'^\dagger) \]
\[ \lambda_3 = \frac{33 g^2}{4} Tr\{[Diag(M^1_1 M^1_1)]^2 + [Diag(M^1_2 M^2_1)]^2 + 2[Diag(M^1_2 M^2_2)]^2\} \]
\[ \lambda_4 = \frac{33 g^2}{8} Tr[Diag(M^1_1 M^1_1)T + Diag(M^1_1 M^1_1)S + 4Diag(M^1_2 M^2_2)S] \]
\[ \lambda_5 = \frac{33 g^2}{8} Tr[Diag(M^1_1 M^1_1)S - 2Diag(M^1_2 M^2_1)S - Diag(M^1_2 M^1_1)T]. \]

In the above expressions, we used the following notations,
\[
T = \frac{1}{\alpha} \begin{bmatrix}
  t_1^2 + t_2^2 & t_1^2 + t_2^2 & t_1^2 + t_2^2 \\
  t_1^2 + t_2^2 & t_1^2 + t_2^2 & t_1^2 + t_2^2 \\
  t_1^2 + t_2^2 & t_1^2 + t_2^2 & t_1^2 + t_2^2 \\
\end{bmatrix}
\]
\[
S = \frac{1}{\alpha} \begin{bmatrix}
  s_1^2 + s_2^2 & s_1^2 + s_2^2 & s_1^2 + s_2^2 \\
  s_1^2 + s_2^2 & s_1^2 + s_2^2 & s_1^2 + s_2^2 \\
  s_1^2 + s_2^2 & s_1^2 + s_2^2 & s_1^2 + s_2^2 \\
\end{bmatrix}
\]

It is easy to show that
\[ Tr(T + 10S) = 2; \]
For a 3 \times 3 matrix M, we define Diag(M) as the diagonal part of M
\[
Diag(M) = \begin{bmatrix}
  M_{11} & 0 & 0 \\
  0 & M_{22} & 0 \\
  0 & 0 & M_{33} \\
\end{bmatrix}
\]
To express above formulas in a simple form, we redefine parameters by absorbing some constants in free parameter $U$ and $U_1$,

\[ \mu = \frac{33g^2U}{\beta}, \quad \mu_1 = \frac{33g^2U_1}{\alpha}, \]

and

\[ \hat{s}_1 = \frac{s_1^2 + s_2}{\alpha}, \quad \hat{t}_1 = \frac{t_1^2 + t_3}{\alpha}, \]
\[ \hat{s}_2 = \frac{s_1^2 + s_3}{\alpha}, \quad \hat{t}_2 = \frac{t_1^2 + t_2}{\alpha}, \]
\[ \hat{s}_3 = \frac{s_2^2 + s_3}{\alpha}, \quad \hat{t}_3 = \frac{t_2^2 + t_3}{\alpha} \]

where $\hat{s}_i, \hat{t}_i (i = 1, 2, 3)$, is positive real numbers.

So we can write $T$, $S$ and $m_1^2$ and $m_2^2$ in a simple form as,

\[ T = \begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{bmatrix}, \quad S = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{bmatrix} \]

\[ m_1^2 = \mu_1^2\left(\frac{1}{2} - \frac{3}{32}\mu_1^4\right), \quad m_2^2 = \mu^2\left(1 - 2\frac{\mu_1^2}{\mu_2}\right). \]

In the previous calculation, we only take into account the term $\langle F, F \rangle$. It can be shown that in this case the Higgs potential can’t give correct symmetry breaking mechanism of $SU(5)$ group. Fortunately, if the term $\langle F \rangle$ is introduced in the Lagrangian, we can get correct results. It is easy to show that,

\[ \langle F \rangle = \frac{16}{U^2}G\beta H \dagger H + 4\alpha \frac{G_1}{U^2}Tr(\Sigma^2) \]

We should introduce Lagrangian as follows

\[ \mathcal{L} = -\frac{1}{N}(\langle F, F \rangle + q' \langle F \rangle). \]

One finds that

\[ -\frac{q'}{N} < F >= -q'H \dagger H - q' Tr(\Sigma^2). \]

We set $q' = q^2\mu_1^2$ and recalculate the Lagrangian of gauge fields and find that only coefficients $m_1^2, m_2^2$ are modified as,

\[ m_1^2 = \mu_1^2\left(\frac{1}{2} + q - \frac{3}{32}\mu_1^2\right) \]
\[ m_2^2 = \mu^2\left(\frac{1}{4} + (q - 2)\frac{\mu_1^2}{\mu_2}\right). \]
4 The Realistic $SU(5)$ Model and Higgs Mechanism

In the last section, we have completed the model building of generalized gauge theory on $M^4 \times Z_2 \times Z_3$, where the potential of Higgs fields is derived directly from the calculation of non-commutative geometry. But some of crucial points need to be studied further, such as, does the potential provide the desired mechanism of gauge symmetries breaking, (i.e $SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)$) and if the results suit to describe the physical phenomenon?

4.1 Realistic $SU(5)$ Model

It is known that there are two mass scales in $SU(5)$ model: masses of the $X,Y$ and of $W$ gauge boson masses. There exits a vast hierarchy of gauge symmetries, $M_X$ larger than $M_W$ by something like 12 order of magnitude. In this section, we will show that the model we built in last section may give rise the desired symmetries broken and gauge hierarchy, if we impose the following conditions among parameters,

$$\mu_1 \ll \mu$$

$$F = \frac{30\lambda_4 + 9\lambda_5}{60\lambda_1 + 14\lambda_2} < 1$$

(4.1)

$$q = \frac{4 + F}{2(1 - F)}.$$  

Now if the conditions (4.1) is under consideration, we may write down the Bosonic part Lagrangian of the model as

$$L_G = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu H^\dagger D_\mu H + Tr(D_\mu \Sigma^\dagger D^\mu \Sigma)$$

$$+ m_1^2 Tr \Sigma^2 - \lambda_1 (Tr \Sigma^2)^2 - \lambda_2 Tr \Sigma^4$$

$$+ m_2^2 H^\dagger H - \lambda_3 (H^\dagger H)^2 - \lambda_4 Tr \Sigma^2 H^\dagger H - \lambda_5 H^\dagger \Sigma^2,$$  

(4.2)

where

$$m_1^2 = \frac{5}{2(1 - F)} \mu_1^2, \quad m_2^2 = \frac{1}{4} \mu^2 + F m_1^2,$$

$$\lambda_1 = \frac{99}{20} g^2 Tr(SS^\dagger)$$

$$\lambda_2 = \frac{33}{4} g^2 Tr(TT^\dagger + 10 Tr SS^\dagger)$$

$$\lambda_3 = \frac{33}{8} g^2 Tr\{[Diag(M_1 M_1^\dagger)]^2 + [Diag(M_1^\dagger M_1)]^2 + 2[Diag(M_2 M_2^\dagger)]^2$$

$$+ 2[Diag(M_1 M_2)]^2\}$$

(4.3)

$$\lambda_4 = \frac{33}{8} g^2 Tr[Diag(M_1 M_1^\dagger)T + Diag(M_1 M_1^\dagger)S + 4 Diag(M_2 M_2^\dagger)S]$$

$$\lambda_5 = \frac{33}{8} g^2 Tr[Diag(M_1 M_1^\dagger)S - 2 Diag(M_2 M_2^\dagger)S - Diag(M_1 M_1^\dagger)T],$$

In those expressions, matrices $M_1$, $M_2$, $T$ and $S$ are defined as:
\[ M_1 = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}, \quad M_2 = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \]

\[ T = \begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{bmatrix}, \quad S = \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{bmatrix}, \]

where \( \hat{s}_i, \hat{t}_i \) are positive real numbers and satisfy the condition \( \frac{1}{3} \text{Tr}(T + 10S) = 1 \).

So far we have constructed a realistic \( SU(5) \) model. Our next task is to research whether it gives us the desired physical results.

### 4.2 Symmetry Breaking

Since for parameters \( \lambda_1 \) and \( \lambda_2 \) in (4.3), the following conditions are true, \( \lambda_2 > 0 \) and \( \lambda_1 > -\frac{7}{30} \lambda_2 \), potential \( V(\Sigma) \) reaches its minimum at

\[ \Sigma_0 = V_1 \begin{bmatrix} 2 \\ 2 \\ 2 \\ -3 \\ -3 \end{bmatrix} \]

where \( V_1^2 = \frac{m_1^2}{60 \lambda_1 + 14 \lambda_2} \), which was derived by Li[13]. For the first stage, \( SU(5) \) gauge symmetry is spontaneous broken down to \( SU(3) \times SU(2) \times U(1) \) as the scalar \( \Sigma \) develops VEV, \( < \Sigma > = \Sigma_0 \). Because \( \Sigma \) is a scalar in the adjoint representation of \( SU(5) \), mass terms for the \( G_3, W_r, B \) fields remain to be zero, while the \( X \) and \( Y \) bosons acquire their masses

\[ M_X = M_Y = \sqrt{\frac{25}{2}} g V_1 \]

For the second stage, gauge symmetries \( SU(3) \times SU(2) \times U(1) \) are broken to \( SU(3) \times U(1) \) as scalar field \( H \) takes its VEV as

\[ < H > = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_2 \end{bmatrix}, \]

where \( V_2^2 = \frac{m_2^2}{\lambda_3} \),

\[ m_d^2 = m_2^2 - (30 \lambda_4 + 9 \lambda_5) V_1^2 = \frac{1}{4} \mu^2. \]
Then bosons $W$ and $B$ obtain masses,

$$M_W = \frac{1}{2} g V_2, \quad M_B = \sqrt{\frac{2}{5}} g V_2.$$ 

Meanwhile Higgs fields also obtain their masses in this model, their values are listed in the following table,

| Scalar fields | $[\text{mass}]^2$ |
|---------------|------------------|
| $[\Sigma_8]_{\alpha \beta}$ | $20 \lambda_2 V_1^2$ |
| $[\Sigma_3]_{\alpha \beta}$ | $80 \lambda_2 V^2_1$ |
| $\Sigma_\theta$ | $4m_1^2$ |
| $H_{\alpha \alpha}$ | $\lambda_3 V_2^2 + 5\lambda_5 V_1^2$ |
| $H_{\alpha r}$ | $\lambda_3 V_2^2$ |

(4.4)

It is interesting to note that

$$\frac{m_d^2}{m_W^2} = \frac{33}{\beta^2} Tr\{[\text{Diag}(M_1 M_1^\dagger)]^2 + [\text{Diag}(M_1^\dagger M_1)]^2 + 2[\text{Diag}(M_2 M_2^\dagger)]^2 + 2[\text{Diag}(M_2^\dagger M_2)]^2\}$$

is a quantity which depends on fermionic mass matrix. This relation does not exist in the original $SU(5)$ Grand Unified Model.

Because parameters $\mu$ and $\mu_1$ were chosen to be $\mu \ll \mu_1$ in conditions (4.1), it is easy to find $V_2 \ll V_1$ in VEV, which means that we may realize the masses of gauge bosons $X$ and $Y$ to be as heavy as 12 order of that of gauge bosons $W$ and $B$. Therefore the gauge hierarchy problem is fitting here. In fact, to realize $\mu \ll \mu_1$, we should take $U \ll U_1$ in the fermion lagrangian (3.4). From the point view of non-commutative geometry approach, $U$ is a paramter labeling the distant between two discrete points of $Z_2$ and $U_1$ is that labeling the distant among three discrete points of $Z_3$, These two geometry quantities control the mass scales of symmetries broken in our model.

## 5 Concluding remarks

We have first constructed a $SU(5)$ model with generalized gauge theory on $M^4 \times Z_2 \times Z_3$. We have shown that the Higgs mechanism is automatically included in the generalized gauge theory by introducing the Higgs fields as a kind of gauge fields with respect to the discrete groups and the Yukawa couplings automatically given by the generalized gauge coupling principle. Then we arrange the parameters appropriately and obtain the minimum $SU(5)$ grand unified model.

In the model, the Higgs potential can lead to the spontaneous symmetry broken mechanism of $SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)$, and they took place in two different gauge hierarchy scalars. There are also two scalars $H$ and $\Sigma$, the vector and adjoint representations of
SU(5) group to break down gauge symmetry and enable the particles massive. In construction of the model, we arrange $H$ and $\sum$ in the connection matrices $\phi_{Z}$, $\phi_{Zr}$, $\phi_{Zr^2}$, $\phi_{r}$ and $\phi_{r^2}$. I want to emphasize that this assignments is unique in general, if they are set in a “wrong” place, their transformation properties under SU(5) group will not be satisfied. It is worthy while to point out that the hierarchy scalars depends on two geometry quantities, i.e.the distant of two discrete points in $Z_2$ and that of three discrete points in $Z_3$. One of the interesting starting point of this approach is to understand the discrete groups $Z_2$ and $Z_3$ as charge conjugation transformation and generation translation in the free fermion lagrangian, although they are broken after the arrangements of gauge fields. This is completely different from previous work.

There exist some differences between the parameters of the reconstructed model and the standard SU(5) grand unified model. In Standard SU(5) Model, there are following free parameters:

- $g$ — SU(5) coupling constant,
- $M_1, M_2$ — mass matrices,
- $m_1^2, m_2^2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ — parameters in potential.

In our reconstructed model, coupling constant $g$, mass matrices $M_1, M_2$ , and $m_1, m_2$ are also free parameters, instead of parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, we introduced two matrices $S$ and $T$ and they satisfy condition $\frac{1}{2}Tr(T+10S) = 1$. On apparent observation, the number of parameters is the same in these two models, but now parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are functions of $M_1, M_2, S, T$ in the reconstructed model, so they are not as free as in the standard SU(5) model. One result of this property is that the ratio of $M_{Hd}/M_W$ is a function of mass matrices which means there exists a complex relation among the masses of particles at tree level. Therefore, it needs to be studied further whether there are more relations. This approach is also available to study more extensive models such as the left-right symmetry model, the SO(10) grand unified model and the supersymmetry model etc. We will study these issues elsewhere.

Acknowledgement

This work is supported in part by the National Science Foundation and Chinese Post Doctoral Foundation. The authors would like to thank Prof. H-Y Guo, K. Wu and Z.Y. Zhao for helpful discussions and Dr. C. Liu for useful comments.

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