Sectorial perturbations
of self-adjoint matrices and operators

E B Davies
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Abstract

This paper considers $N \times N$ matrices of the form $A_\gamma = A + \gamma B$, where $A$ is self-adjoint, $\gamma \in \mathbb{C}$ and $B$ is a non-self-adjoint perturbation of $A$. We obtain some monodromy-type results relating the spectral behaviour of such matrices in the two asymptotic regimes $|\gamma| \to \infty$ and $|\gamma| \to 0$ under certain assumptions on $B$. We also explain some properties of the spectrum of $A_\gamma$ for intermediate sized $\gamma$ by considering the limit $N \to \infty$, concentrating on properties that have no self-adjoint analogue. A substantial number of the results extend to operators on infinite-dimensional Hilbert spaces.

AMS subject classifications:

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1 Introduction

Let $A$ be a (possibly unbounded) self-adjoint operator acting in the Hilbert space $\mathcal{H}$ and let $A_\gamma = A + \gamma B$ where $\gamma \in \mathbb{C}$ and $B$ is a bounded operator on $\mathcal{H}$. Many papers have been written about the spectral properties of the self-adjoint operators $A_\gamma$ when $B = B^*$ and $\gamma \in \mathbb{R}$, the main techniques used including variational inequalities and perturbation expansions; see [5, 6, 8] and many further references there. In this paper we concentrate on more general $B$ and assume that $\gamma$ is complex. Our main concern is to describe phenomena that have no self-adjoint analogues, an issue that has been curiously neglected. A recent paper of Rana and Wojtylak, [7], is closer to this one, but there is little technical overlap. The interplay between the asymptotic regimes $|\gamma| \to 0$ and $|\gamma| \to \infty$ is a main focus of
interest, but we also explore some spectral phenomena that arise for intermediate values of $\gamma$.

As well as being of intrinsic interest, operators of this type are relevant to non-self-adjoint Schrödinger and wave equations, for which the evolution is contractive as a function of time. In such situations every eigenvalue of $A_\gamma$ lies in an appropriate half-plane and the eigenvalue determines the energy and rate of decay of the associated eigenstate of the system. From Section 6 onwards we study rank one perturbations. As well as providing a range of phenomena that must be included in a more general theory, this is of direct relevance to the study of non-self-adjoint boundary conditions for Schrödinger operators in one dimension. The relevant perturbations of the Schrödinger operators are singular, but, if one considers instead the resolvent operators, the perturbations are rank one and bounded.

General considerations from perturbation theory imply that the set $\mathcal{R}$ of $(\gamma, \lambda) \in \mathbb{C}^2$ such that $\lambda$ is an isolated eigenvalue of $A_\gamma$ with finite algebraic multiplicity is a Riemann surface that may have branch points where the multiplicity of the eigenvalue is greater than 1; see [4]. If $B$ is relatively compact with respect to $A$ then

$$\text{Spec}(A_\gamma) = \text{Ess}(A) \cup \{\lambda : (\gamma, \lambda) \in \mathcal{R}\}$$

for every $\gamma \in \mathbb{C}$. Our goal in this paper is to understand how the geometrical structure of $\mathcal{R}$ depends upon some simple generic assumptions about $A$ and $B$.

In much of the paper we assume that $\mathcal{H}$ has finite dimension $N$. We assume that $A$ is self-adjoint and that $B$ is sectorial. The coupling constant $\gamma$ is restricted by the requirement that $\text{Im}(\langle A_\gamma f, f \rangle) \geq 0$ for all $f \in \mathcal{H}$; this is equivalent to assuming that $iA_\gamma$ is dissipative in a standard sense; see [1, Section 8.3]. Further assumptions on $B$ are made as necessary. In the particular case $B = B^* \geq 0$, which motivated our initial interest, we assume that $0 < \text{arg}(\gamma) < \pi$. Theorem 26 and Example 29 show how a substantial part of the spectrum of a large matrix may sometimes be approximated by using a carefully chosen matrix that is much smaller. Section 7 focuses on spectral properties of $A_\gamma$ that are best understood by considering the limit $N \to \infty$.

## 2 Sectorial operators

The truncation of an operator $A$ on $\mathcal{H}$ to a closed subspace $\mathcal{K}$ is defined by $A^\sharp = PAP|_\mathcal{K}$, where $P$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. We will need the following lemma.

**Lemma 1** If $A = A^*$, $aI \leq A \leq bI$ and $A^\sharp$ denotes the truncation of $A$ to $\mathcal{K}$ then $aI^\sharp \leq A^\sharp \leq bI^\sharp$. 

Proof We use variational methods. The hypotheses imply that
\[
\inf \{ \langle Af, f \rangle : f \in H \text{ and } \|f\| = 1 \} 
\leq \inf \{ \langle Af, f \rangle : f \in K \text{ and } \|f\| = 1 \}
= \inf \{ \langle A^2f, f \rangle : f \in K \text{ and } \|f\| = 1 \}.
\]
Therefore \( aI = A \). The other half of the proof is similar. □

A bounded operator \( D \) on the Hilbert space \( H \) is said to be sectorial if there exist ‘sectorial constants’ \( \sigma_1, \sigma_2 \) such that \(-\pi/2 < -\sigma_1 \leq 0 \leq \sigma_2 < \pi/2\) and
\[
\{ \langle Df, f \rangle : f \in H \} \subseteq \{ 0 \} \cup \{ z : z \neq 0 \text{ and } -\sigma_1 \leq \arg(z) \leq \sigma_2 \}. \tag{1}
\]
The theory of sectorial operators has a long history; see Sections VI.1.5 and VI.3.1 of [4]. The following lemma is adapted from [4, Theorem VI.3.2], but we include a proof for completeness.

**Lemma 2** If \( D \) is a bounded sectorial operator on \( H \) and \( f \in H \) then the following are equivalent.

(i) \( \langle Df, f \rangle = 0 \);

(ii) \( \langle (D + D^*)f, f \rangle = 0 \);

(iii) \( Df = 0 \);

(iv) \( D^*f = 0 \).

If \( K = \text{Ker}(D) \) then \( K \) and \( K^\perp \) are invariant under \( D \) and \( D^* \). Moreover \( D|_K = D^*|_K = 0 \). Both \( D|_{K^\perp} \) and \( D^*|_{K^\perp} \) are one-one with ranges that are dense in \( K^\perp \).

The truncation \( D^2 \) of \( D \) to \( K^\perp \) may be written in the form
\[
D^2 = X^{1/2}(I^2 + iE)X^{1/2} \tag{2}
\]
where \( X \) is the truncation of \( (D + D^*)/2 \) to \( K^\perp \), \( I^2 \) is the identity operator on \( K^\perp \) and \( E \) is a self-adjoint operator on \( K^\perp \) satisfying
\[
-\tan(\sigma_1)I^2 \leq E \leq \tan(\sigma_2)I^2, \tag{3}
\]
where \( \sigma_1, \sigma_2 \) are the sectorial constants of \( D \).

**Proof**

(i) implies (ii). This uses \( \langle D^*f, f \rangle = \langle Df, f \rangle \).

(ii) implies (iii) and (iv). We write \( D = D_0 + iD_1 \) where \( D_0 = D_0^* \geq 0 \) and \( D_1 = D_1^* \).

The sectorial condition is equivalent to \(-k_1D_0 \leq D_1 \leq k_2D_0 \) where \( k_r = \tan(\sigma_r) \) for
If (i) holds then $\langle D_0 f, f \rangle = 0$, so $\|D_0^{1/2} f\|^2 = \langle D_0 f, f \rangle = 0$. This implies that $D_0^{1/2} f = 0$, and hence that $D_0 f = 0$. Since $0 \leq D_1 + kD_0 \leq 2kD_0$, we also have $(\langle D_1 + kD_0 f, f \rangle = 0$, hence $(D_1 + kD_0)^{1/2} f = 0$ and then $(D_1 + kD_0) f = 0$. Therefore $D_1 f = 0$. We conclude that $Df = 0$ and $D^* f = 0$.

(iii) and (iv) separately imply (i). Both are elementary.

The property (iii) implies that $D_{|\mathcal{K}} = 0$. The property (iv) together with the general identity $\text{Ran}(D^*) = (\text{Ker}(D^*))^\perp$ implies that $\text{Ran}(D)$ is dense in $\mathcal{K}^\perp$. The corresponding statement for $D^*$ has a similar proof.

We have, finally, to prove (2) and (3). Without loss of generality we assume that $\mathcal{K} = 0$ and omit the symbol $\natural$. The operator $X = D_0$ is then one-one with dense range $\mathcal{D}$ in $\mathcal{H}$. The inequalities $-k_1 D_0 \leq D_1 \leq k_2 D_0$ are equivalent to $-k_1 I \leq E \leq k_2 I$ where $E = D_0^{-1/2} D_1 D_0^{-1/2}$ is initially defined as a quadratic form on $\mathcal{D}$. This yields (3). The bounds on the form $E$ imply that it is associated with a bounded linear operator on $\mathcal{H}$. We then have $D_1 = D_0^{1/2} E D_0^{1/2}$ and hence (2). \[\Box\]

**Corollary 3** If $D$ is sectorial and $S$ is bounded then the following are equivalent.

(i) $SDS^* = 0$;
(ii) $SD = 0$;
(iii) $SD^* = 0$.

**Proof** Assuming (i), $\langle SDS^* g, g \rangle = 0$ for all $g \in \mathcal{H}$. Therefore $\langle Df, f \rangle = 0$ for all $f \in \text{Ran}(S^*)$. Lemma 2 now implies that $Df = 0$ for all $f \in \text{Ran}(S^*)$. Hence $DS^* = 0$ and (3) holds. The proof that (i) implies (ii) is similar and the proofs that (ii) and (iii) separately imply (i) are elementary. \[\Box\]

The remainder of this section is of independent interest, but it is not used elsewhere. Given constants $\sigma_1$, $\sigma_2$ such that $-\pi/2 < -\sigma_1 \leq 0 \leq \sigma_2 < \pi/2$, the set of all bounded operators $D$ on the Hilbert space $\mathcal{H}$ such that (1) holds is a proper closed convex cone, which we denote by $C_{\sigma_1, \sigma_2}$. We say that a non-zero operator $C$ lies in $\partial C_{\sigma_1, \sigma_2}$ if $C = A + B$ and $A$, $B \in C_{\sigma_1, \sigma_2}$ imply that there exist non-negative constants $\alpha$, $\beta$ such that $A = \alpha C$ and $B = \beta C$. The set of all positive multiples of such an operator $C$ is called an extreme ray of $C_{\sigma_1, \sigma_2}$.

**Lemma 4** Let $A$, $B$, $C \in C_{\sigma_1, \sigma_2}$ and $C = A + B$. Then $\text{Ker}(C) \subseteq \text{Ker}(A)$. In particular $\text{rank}(C) = 1$ implies $A = 0$ or $\text{rank}(A) = 1$. 

4
Proof The assumptions imply that
\[ C + C^* = (A + A^*) + (B + B^*). \]
and then
\[ 0 \leq A + A^* \leq C + C^*. \]
Therefore \( \langle (C + C^*) f, f \rangle = 0 \) implies \( \langle (A + A^*) f, f \rangle = 0 \). Lemma 2 now implies that \( \text{Ker}(C) \subseteq \text{Ker}(A). \)

Theorem 5 Let \( C_{\sigma_1, \sigma_2} \) be the cone defined above. Then a non-zero operator \( A \in \partial C_{\sigma_1, \sigma_2} \) if and only if \( Af = \alpha \langle f, e \rangle e \) for all \( f \in \mathcal{H} \), where \( e \in \mathcal{H} \) satisfies \( \|e\| \neq 0 \) and \( \alpha = e^{-i\sigma_1} \) or \( \alpha = e^{i\sigma_2} \).

Proof Given \( A \in C_{\sigma_1, \sigma_2} \), let \( K_1 = \text{Ker}(A) \). If \( K_1 \) has dimension greater than 1, then by applying the spectral theorem to the self-adjoint operator \( E \) in (2), one may write \( K_1 = K_2 \oplus K_3 \) where \( K_2 \) and \( K_3 \) are non-zero orthogonal subspaces that are invariant with respect to \( E \). One then has a block decomposition
\[ I^2 + iE = \begin{pmatrix} I_2 + iE_2 & 0 \\ 0 & I_3 + iE_3 \end{pmatrix} \]
in an obvious notation. Moreover \( I_2 + iE_2 \) and \( I_3 + iE_3 \) both lie in \( C_{\sigma_1, \sigma_2} \) with respect to the relevant Hilbert spaces. It follows that \( A = A_2 + A_3 \) where \( A_2 \) and \( A_3 \) have the following block decompositions with respect to \( \mathcal{H} = K_1 \oplus K_2 \oplus K_3 \).

\[ A_2 = A_2^{1/2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 + iE_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} A_2^{1/2}, \]
\[ A_3 = A_3^{1/2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_3 + iE_3 \end{pmatrix} A_3^{1/2}. \]

The factors \( A_2^{1/2} \) do not change the sector in which the numerical range lies, so \( A_2, A_3 \in C_{\sigma_1, \sigma_2} \) and \( A \notin \partial C_{\sigma_1, \sigma_2} \).

Conversely if \( K_1 \) is one-dimensional then \( A \) has rank 1 and it is of the form \( Af = \langle f, e_1 \rangle e_2 \) for some non-zero vectors \( e_1, e_2 \) and all \( f \in \mathcal{H} \). Since
\[ \text{Ker}(A) = \{ f : \langle f, e_1 \rangle = 0 \}, \]
\[ \text{Ker}(A^*) = \{ f : \langle f, e_2 \rangle = 0 \}, \]
Lemma 2 implies that \( \text{Ker}(A) = \text{Ker}(A^*) \), from which one may deduce that \( e_2 \) is a multiple of \( e_1 \). An easy calculation using Lemma 4 shows that \( A \) is in an extreme ray if and only if the argument of \( \alpha \) has one of the two stated values. □
3 Cyclicality

This section generalizes the notion of cyclic vector to perturbations of an operator that have rank greater than 1.

**Theorem 6** Let $A$ be a (possibly unbounded) self-adjoint operator acting in the Hilbert space $\mathcal{H}$ and let $B, X$ be two bounded operators on $\mathcal{H}$. Then the following conditions are equivalent.

(i) $X e^{iAt}B = 0$ for all $t \in \mathbb{R}$;

(ii) $X(zI - A)^{-1}B = 0$ for all $z \notin \text{Spec}(A)$;

(iii) $X e^{(A+\gamma B)t}B = 0$ for some (equivalently all) $\gamma \in \mathbb{C}$ and all $t \in \mathbb{R}$.

If $A$ is bounded the above conditions are also equivalent to

(iv) $XA^nB = 0$ for all $n \geq 0$.

**Proof** We use a number of standard theorems and formulae from the theory of one-parameter semigroups; see [1, Sections 8.2, 11.4]; in finite dimensions many of these can be derived more directly. We first observe that $A = A^*$ implies that there is a one-parameter group with generator $iA$; following the usual convention we write this in the form $e^{iAt}$, where $t \in \mathbb{R}$. The boundedness of $B$ implies that there is a one parameter group, which we denote by $e^{i(A+\gamma B)t}$, whose generator is $A + \gamma B$.

(i)$\Rightarrow$(ii). This follows directly from the following formulae, the integrals being convergent in the strong operator topology. If $\text{Im}(z) < 0$ then

$$(zI - A)^{-1} = i \int_0^{\infty} e^{(-izI+iA)t} dt.$$  

If $\text{Im}(z) > 0$ then

$$(zI - A)^{-1} = -i \int_{-\infty}^0 e^{(-izI+iA)t} dt.$$  

If $z \in \mathbb{R} \setminus \text{Spec}(A)$ and $\varepsilon > 0$ then

$$(zI - A)^{-1} = \lim_{\varepsilon \to 0} (zI + i\varepsilon I - A)^{-1}.$$  

(ii)$\Rightarrow$(i). This uses the formulae

$$(sI \mp iA)^{-n-1} = \frac{(-1)^n}{n!} \frac{d^n}{ds^n} (sI \mp iA)^{-1},$$

$$e^{\pm iAt} = \lim_{n \to \infty} \left( \frac{t}{n} \right)^n \left( \frac{n}{t} I \mp iA \right)^n.$$  

6
The formulae are valid for all positive \( s, t \) and \( n \) and the limits may be taken in the strong operator topology. Both formulae may be proved by using the spectral theorem, but they are also valid at the semigroup level.

(i)⇒(iii). Assuming \( t > 0 \), this uses the formula

\[
e^{i(A+\gamma B)t} = e^{iAt} + \int_{s=0}^{t} e^{iA(t-s)}i\gamma Be^{iAs} \, ds + \int_{s=0}^{t} \int_{u=0}^{s} e^{iA(t-s)}i\gamma Be^{iA(s-u)}i\gamma Be^{iAu} \, du \, ds + \ldots,
\]

the integrals and series being convergent in the strong operator topology for all \( \gamma \in \mathbb{C} \). The proof for \( t < 0 \) is similar.

(iii)⇒(i). If (iii) holds for some \( \gamma \in \mathbb{C} \) then (i) follows by using the formula

\[
e^{iAt} = e^{i(A+\gamma B)t} - \int_{s=0}^{t} e^{i(A+\gamma B)(t-s)}i\gamma Be^{i(A+\gamma B)s} \, ds + \int_{s=0}^{t} \int_{u=0}^{s} e^{i(A+\gamma B)(t-s)}i\gamma Be^{i(A+\gamma B)(s-u)}i\gamma Be^{i(A+\gamma B)u} \, du \, ds + \ldots
\]

(i)⇔(iv). These use

\[
(iA)^n = \left. \frac{d^n}{dt^n} e^{iAt} \right|_{t=0}, \quad e^{iAt} = \sum_{n=0}^{\infty} \frac{(iAt)^n}{n!},
\]

both limits being in the operator norm.

In the context of Theorem 6, we say that the bounded operator \( B \) is cyclic for \( A \) if the conditions of the following corollary hold.

**Corollary 7** Let \( A \) be a possibly unbounded self-adjoint operator acting in the Hilbert space \( \mathcal{H} \) and let \( B \) be a bounded operator on \( \mathcal{H} \). Then the following conditions are equivalent.

(i) Whenever any of the equivalent conditions of Theorem 6 holds for some bounded operator \( X \) on \( \mathcal{H} \), it follows that \( X = 0 \).

(ii) If one defines

\[
\mathcal{L}_2 = \text{lin}\{e^{iAt}Bv : t \in \mathbb{R} \text{ and } v \in \mathcal{H}\}
\]

then \( \mathcal{L}_2 \) is dense in \( \mathcal{H} \).

(iii) If one defines

\[
\mathcal{L}_3 = \text{lin}\{(sI - A)^{-1}Bv : s \not\in \text{Spec}(A) \text{ and } v \in \mathcal{H}\}
\]

then \( \mathcal{L}_3 \) is dense in \( \mathcal{H} \).
(iv) Assuming that $A$ is bounded, if one defines

$$\mathcal{L}_4 = \operatorname{lin}\{A^n B v : n = 0, 1, 2, \ldots \text{ and } v \in \mathcal{H}\}$$

then $\mathcal{L}_4$ is dense in $\mathcal{H}$.

**Proof** (i)⇒(ii). If (ii) is false then the Hahn-Banach theorem implies that there exists a non-zero $\phi \in \mathcal{H}$ such that $\langle \phi, v \rangle = 0$ for all $v \in \mathcal{L}_2$. If one defines $X v = \langle v, \phi \rangle \phi$ for all $v \in \mathcal{H}$ then one sees that $xe^{iA} B v = 0$ for all $v \in \mathcal{H}$ but $X \neq 0$, so Theorem 6(i) is false.

(ii)⇒(i). If Theorem 6(i) is false for some non-zero $X \in \mathcal{L}(\mathcal{H})$ then $\mathcal{L}_2 \subseteq \ker(X) \neq \mathcal{H}$, so (ii) is false.

The proofs that (i)⇔(iii) and (i)⇔(iv) are very similar. □

In the following theorem and elsewhere we use the notations $C_+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $C_- = \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$. If $B$ is a sectorial operator with sectorial constants $\sigma_1$, $\sigma_2$ we define

$$S_B = \{0\} \cup \{\gamma \in \mathbb{C} : \gamma \neq 0 \text{ and } \sigma_1 < \arg(\gamma) < \pi - \sigma_2\}.$$  \hspace{1cm} (4)

The condition $\gamma \in S_B$ implies that $\gamma \langle Bf, f \rangle \in C_+ \cup \{0\}$ for all $f \in \mathcal{H}$.

**Remark 8** The conditions in Corollary 7 only depend on $B$ via the closure of its range $\mathcal{R}_0 = \{Bf : f \in \mathcal{H}\}$. In particular if $A$ and $B$ are both bounded, then $B$ is cyclic for $A$ if and only if the linear span of $\bigcup_{r \geq 0} A^r \mathcal{R}_0$ is dense in $\mathcal{H}$. Given $m \in \mathbb{N}$, let $\mathcal{R}_m$ be the orthogonal complement of $\bigcup_{r = 0}^{m-1} A^r \mathcal{R}_0$ in $\bigcup_{r = 0}^m A^r \mathcal{R}_0$. Then $\mathcal{R}_m$ are orthogonal subspaces and $B$ is cyclic for $A$ if and only if the closure of the sum of $\{\mathcal{R}_m\}_{m=0}^\infty$ is dense on $\mathcal{H}$. One may use these subspaces to represent $A$ as a self-adjoint block tridiagonal matrix. If $B$ is sectorial and $\tilde{B}_{r,s}$ is its associated block matrix, then $\tilde{B}_{0,0}$ is the truncation of $B$ to $\mathcal{R}_0$ and all other entries $\tilde{B}_{r,s}$ vanish. If $\mathcal{H}$ is finite-dimensional, one only has a finite number of non-zero subspaces. □

**Remark 9** The conditions in Corollary 7 are close to those used in the block Krylov subspace method of numerical analysis. Case 4 corresponds to the standard version of the method while Case 3 corresponds to the rational version. □

**Theorem 10** Suppose that $B$ is sectorial and that $\gamma \in S_B$. If $B$ is cyclic for $A$ and $\lambda$ is an eigenvalue of $A_\gamma$ then $\lambda \in C_+$. If $M = \text{rank}(B) < \infty$ then the geometric multiplicity of $\lambda$ is at most $M$.

**Proof** Suppose that $0 \neq f \in \text{Dom}(A_\gamma)$ and $Af + \gamma B f = \lambda f$. By calculating the imaginary part of

$$\langle Af, f \rangle + \langle \gamma B f, f \rangle = \lambda \langle f, f \rangle$$
one deduces that either \( \lambda \in C_+ \) or \( \lambda \in \mathbb{R} \) and \( \text{Im}(\langle \gamma Bf, f \rangle) = 0. \) Since \( \gamma B \) is sectorial it follows that \( \langle \gamma Bf, f \rangle = 0. \) Lemma 2 now implies that \( Bf = B^*f = 0. \) Therefore \( Af = \lambda f \) and \( e^{i\lambda t} = e^{i\lambda t} \) for all \( t \in \mathbb{R}. \) Therefore \( B^*e^{i\lambda t}f = 0 \) for all \( t \in \mathbb{R} \) and \( \langle f, e^{i\lambda t}Bv \rangle = 0 \) for all \( t \in \mathbb{R} \) and all \( v \in \mathcal{H}. \) Since \( B \) is cyclic for \( A \) it follows by Corollary 7(ii) that \( f = 0. \) The contradiction implies that \( \lambda \in C_+. \)

4 The main theorems when \( N < \infty \)

In this section we suppose that \( N = \dim(\mathcal{H}) < \infty \) and put \( M = \text{rank}(B) \) where \( B \) is sectorial. Our goal is to describe how the spectrum of \( A_\gamma = A + \gamma B \) depends on \( \gamma, \) assuming that \( \gamma \in S_B \) as defined in (4), and in particular the relationship between the spectral asymptotics for small and for large \( \gamma. \)

Under the above assumptions it is elementary that \( \text{Im}(\langle (A + t\gamma B)f, f \rangle) \) is a monotonically increasing linear function of \( t \in (0, \infty), \) as is \( \text{Im}(\text{tr}((A + t\gamma B))). \) Combining these observations with known variational results for \( B = B^* \geq 0 \) and \( \gamma > 0, \) leads to the conjecture that the imaginary part of each eigenvalue of \( A + t\gamma B \) also increases monotonically as a function of \( t. \) The following example demonstrates that this is false. It also illustrates the results in Theorem 19. Example 1.5.7 of [1], which is even simpler, provided one of the motivations for the present study.

**Example 11** Let \( A \) be the \( 5 \times 5 \) diagonal matrix with eigenvalues \( \lambda_r = r \) for \( 1 \leq r \leq 5, \) and let \( A_\gamma = A + \gamma B \) where \( B \) is the rank 2 operator

\[
Bf = \langle f, e_1 \rangle e_1 + \langle f, e_2 \rangle e_2
\]

for all \( f \in \mathbb{C}^5, \) where \( e_1 = (2, 2, 2, 2, 2) \) and \( e_2 = (3, 3, -2, -2, -2). \) Figure 1 plots the eigenvalues of \( A_\gamma \) for \( \gamma = te^{i\theta}, \) where \( 0 < t < \infty \) and \( \theta = 3\pi/8. \) The eigenvalues converge to the eigenvalues of \( A \) as \( t \to 0. \) Two of the eigenvalue curves diverge as \( t \to \infty, \) while the other three converge back to the real axis.

We shall need the following conditions. Apart from (H1), each is generic in the sense that it holds for a dense open subset of operators of the relevant type.

(H1) \( \dim(\mathcal{H}) < \infty, \ A = A^* \) and \( B \) is sectorial.

(H2) The operator \( B \) is cyclic for the operator \( A. \)
(H3) All of the eigenvalues of $A$ have algebraic multiplicity 1.

(H4) All of the non-zero eigenvalues of $B$ have algebraic multiplicity 1.

(H5) All of the eigenvalues of the truncation of $A$ to the kernel $K$ of $B$ have algebraic multiplicity 1.

**Theorem 12** Let $\gamma \in S_B$. If (H1) holds and $Z = i(A + \gamma B)$ then $\|e^{Zt}\| \leq 1$ for all $t \geq 0$. Given (H1), the condition (H2) holds if and only if there are constants $M \geq 1$ and $c > 0$ such that

$$\|e^{Zt}\| \leq Me^{-ct} \quad (5)$$

for all $t \geq 0$. Given (H1) and (H2), one can put $M = 1$ in (5) if and only if $\text{Ker}(B) = \{0\}$.

**Proof** It follows directly from (H1) that $Z$ is dissipative for every $\gamma \in S_B$ and hence that $e^{Zt}$ is a contraction semigroup for $t \geq 0$. If (H2) also holds then every eigenvalue $\lambda$ of $Z$ satisfies $\text{Re}(\lambda) < 0$, and an application of the Jordan form theorem yields (5).

**Example 13** Suppose that $A$ and $B$ satisfy (H1–5) and that every eigenvalue $\lambda$ of $A$ satisfies $\lambda > 0$. Define the operators $\tilde{A}$ and $\tilde{B}$ on $\mathcal{H} \oplus \mathcal{H}$ by $\tilde{A}(f \oplus g) =$
We assume (H1), (H2) and that \( \gamma \in S_B \) throughout the section, so that we can use Theorem [10]. We make constant use of the polynomial

\[
p(\gamma, \lambda) = \det(A + \gamma B - \lambda I).
\]

The \( \gamma \)-dependence of the spectrum of \( A_\gamma \) depends on an analysis of the algebraic surface

\[
\mathcal{R} = \{ (\gamma, \lambda) \in S_B \times \mathbb{C}_+ : p(\gamma, \lambda) = 0 \}.
\]

We will use the following classical facts.

**Proposition 14** If \( X \) is an \( N \times N \) matrix and \( q(\lambda) = \det(X - \lambda I) \) then \( q \) is a polynomial with degree \( N \) and the following are equivalent.

(i) Every eigenvalue of \( X \) has algebraic multiplicity 1;

(ii) Every root \( \lambda \) of \( q \) is simple;

(iii) There are no simultaneous solutions of \( q(\lambda) = q'(\lambda) = 0 \);

(iv) The discriminant of \( q \) is non-zero. (The discriminant of a polynomial \( q \) is a certain multiple of the square of its Vandermonde determinant, and may be written as a homogeneous polynomial with degree \( 2N - 2 \) in the coefficients of \( q \).)

Since the zeros of \( p(0, \lambda) \) all lie on the real axis, the following lemma can often be used to reduce the determination of the zeros of \( p(\gamma, \lambda) \) in \( \mathbb{C}_+^2 \) to a lower dimensional problem. See Lemma [20]. The right-hand side of (8), usually without the \( ^\natural \), is called the relative determinant of \( A_\gamma \) and \( A \).

**Lemma 15** One has

\[
\frac{p(\gamma, \lambda)}{p(0, \lambda)} = \det((I + \gamma (A - \lambda I)^{-1} B)^{\natural})
\]

where \(^\natural\) denotes the truncation of the operator to the range of \( B^* \).

**Proof** This is a combination of two identities

\[
\frac{p(\gamma, \lambda)}{p(0, \lambda)} = \det(I + \gamma (A - \lambda I)^{-1} B),
\]

\[
= \det((I + \gamma (A - \lambda I)^{-1} B)^{\natural}).
\]

11
The first equality is obtained by calculating the determinants of both sides of the identity
\[ A + \gamma B - \lambda I = (A - \lambda I)(I + \gamma(A - \lambda I)^{-1}B). \]
The second equality is proved by writing \( I + \gamma(A - \lambda I)^{-1}B \) as a 2 \( \times \) 2 block matrix using the orthogonal decomposition
\[ \mathcal{H} = \text{Ker}(B) \oplus \text{Ran}(B^*). \]

**Lemma 16** Given (H2) and (H3), there exists a finite set \( F_1 \subset S_B \), such that \( A_\gamma \) has \( N \) distinct eigenvalues, each with algebraic multiplicity 1, for every \( \gamma \in S_B \setminus F_1 \). If \( (\gamma, \lambda) \in \mathcal{R} \) and \( \gamma \notin F_1 \) then \( \frac{\partial p}{\partial \lambda}(\gamma, \lambda) \neq 0 \).

**Proof** The eigenvalues of \( A_\gamma \) are the roots of the polynomial \( q_\gamma(\lambda) = p(\gamma, \lambda) \), which is of degree \( N \) in \( \lambda \) with leading coefficient \((-1)^N\). The eigenvalues of \( A_\gamma \) lie in \( \mathbb{C}_+ \) by Theorem 10. They all have algebraic multiplicity 1 if and only if the discriminant of \( q_\gamma \) is non-zero, by Proposition 14. The coefficients of \( q_\gamma \) are polynomials in \( \gamma \), so the discriminant is also a polynomial \( r \) in \( \gamma \). The hypothesis (H3) implies that \( r(0) \neq 0 \), so \( r \) is not identically zero, and it has only a finite number of roots. The first part of the proof is completed by putting \( F_1 = \{ \gamma \in S_B : r(\gamma) = 0 \} \). The proof of the final part of the theorem uses Proposition 14 again.

**Lemma 17** Given (H2) and (H4), there exists a finite set \( F_2 \subset S_B \), such that if \( (\gamma, \lambda) \in \mathcal{R} \) and \( \gamma \notin F_2 \) then \( \frac{\partial p}{\partial \gamma}(\gamma, \lambda) \neq 0 \).

**Proof** One may evaluate \( p(\gamma, \lambda) \) by combining an orthonormal basis of \( \text{Ker}(B) \) with a set of \( M \) eigenvectors associated with the non-zero eigenvalues \( \beta_1, \ldots, \beta_M \) of \( B \). If one does so then one sees that \( q_\lambda(\gamma) = p(\gamma, \lambda) \) is a polynomial with degree (at most) \( M \) in \( \gamma \) whose leading coefficient is \( \det(A^\sharp - \lambda I^\sharp) \prod_{r=1}^{M} \beta_r \), where \( A^\sharp \) is the truncation of \( A \) to \( \text{Ker}(B) \) and \( I^\sharp \) is the identity operator on this subspace. Since \( A^\sharp \) is self-adjoint and \( \lambda \in \mathbb{C}_+ \), the determinant is non-zero and the degree of \( q_\lambda \) is \( M \).

One may see as in the proof of Lemma 16 that the roots of \( q_\lambda \) are all distinct if and only if a certain polynomial \( r(\lambda) \) is non-zero. If \( p(\gamma, \lambda) = 0 \) and \( \frac{\partial p}{\partial \gamma}(\gamma, \lambda) = 0 \) then \( r(\lambda) = 0 \). The set \( G_1 \) of roots of \( r \) is finite provided \( r \) does not vanish identically. Assuming this,
\[ F_2 = \{ \gamma \in \mathbb{C}_+ : (\gamma, \lambda) \in \mathcal{R} \text{ for some } \lambda \in G_1 \} \]

is also finite and \( \frac{\partial p}{\partial \gamma}(\gamma, \lambda) \neq 0 \) for all \( (\gamma, \lambda) \in \mathcal{R} \) such that \( \gamma \notin F_2 \).

The polynomial \( r \) is not identically zero provided the \( M \) solutions \( \gamma \) of \( \det(A + \gamma B - \lambda I) = 0 \) are distinct for all large enough \( \lambda \in \mathbb{C}_+ \). This is true if and only
if the solutions $s$ of $\lambda^{-N} \det(A + s\lambda B - \lambda I) = 0$ are distinct for all large enough $\lambda \in \mathbb{C}_+$. These solutions converge as $|\lambda| \to \infty$ to the solutions of $\det(sB - I) = 0$, which are $\beta_1^{-1}, \ldots, \beta_M^{-1}$. They are distinct by (H4). □

The following lemma will be used in the proof of Theorem 19.

**Lemma 18** Let $L$ be a bounded operator on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with block matrix

$$L = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

where the entries satisfy $\|Q\| \leq c$, $\|R\| \leq c$, $\|S^{-1}\| \leq 1/(2c)$ and $\|P^{-1}\| < \varepsilon \leq 1/(2c)$. Then $L$ is invertible and

$$\left\| L^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix} \right\| < 2\varepsilon.$$

**Proof** If one puts $X = \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}$ then $\|Y\| \leq c$ and $\|X^{-1}\| \leq 1/(2c)$. Therefore $\|YX^{-1}\| \leq 1/2$ and the perturbation expansion

$$(X + Y)^{-1} = X^{-1} \sum_{n=0}^{\infty} (-YX^{-1})^n$$

implies that $L = X + Y$ is invertible with $\|L^{-1}\| \leq 1/c$. Moreover

$$\left\| L^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & S^{-1} \end{pmatrix} \right\| \leq \|(X + Y)^{-1} - X^{-1}\| + \varepsilon$$

$$\leq \| -X^{-1}YX^{-1} + (X^{-1}YX^{-1})(YX^{-1}) - (X^{-1}YX^{-1})(YX^{-1})^2 + \ldots \| + \varepsilon$$

$$\leq 2\|X^{-1}YX^{-1}\| + \varepsilon$$

$$= 2\left\| \begin{pmatrix} 0 & P^{-1}QS^{-1} \\ S^{-1}RP^{-1} & 0 \end{pmatrix} \right\| + \varepsilon$$

$$\leq 2\varepsilon.$$ □

We use the above results to connect the spectrum of $A_\gamma$ for large and small $\gamma$. Let $\mathcal{G}$ denote the set of all continuously differentiable curves $g : [0, \infty) \to \mathbb{C}$ such that $g(0) = 0$, $g(t) \in S_B$ for every $t > 0$ and $g'(t)$ does not vanish for any $t \in [0, \infty)$. Let $F = F_1 \cup F_2$ where $F_1$ is defined as in Lemma 16 and $F_2$ is defined as in Lemma 17. Let $\mathcal{G}_0$ denote the set of all curves $g \in \mathcal{G}$ such that $g(0) = 0$, $g(t) \in S_B \setminus F$ for every $t > 0$ and $\lim_{t \to \infty} |g(t)| = \infty$. In the next theorem, one can impose stronger
conditions on \( g \) (e.g. \( C^\infty \) or real analyticity) and obtain similarly strengthened conclusions on the eigenvalue curves \( \lambda_r \).

Our main theorem below is an example of monodromy in the sense that we prove that certain one-parameter curves that avoid a finite number of singularities may have different end points even if they have the same starting point, provided they take different routes around the singularities; the difference is measured by an element of a permutation group.

**Theorem 19** Given (H2–5), let \( g \in G_0 \). Then there exist \( N \) curves \( \lambda_r \in G \) such that

\[
\text{Spec}(A_{g(t)}) = \{\lambda_1(t), \ldots, \lambda_N(t)\} \quad \text{for all } t \in [0, \infty).
\]

One can choose the ordering of these so that \( \lambda_r(0) = \alpha_r \) for all \( r \in \{1, \ldots, N\} \), where \( \alpha_r \) are the eigenvalues of \( A \) written in increasing order. Assuming this is done, there exists a \( g \)-dependent permutation \( \pi \) on \( \{1, \ldots, N\} \) such that

\[
\lim_{t \to \infty} \frac{\lambda_{\pi(r)}(t)}{g(t)} = \beta_r
\]

for \( 1 \leq r \leq M \), where \( \beta_r \) are the non-zero eigenvalues of \( B \) written in any fixed order, and

\[
\lim_{t \to \infty} \lambda_{\pi(M+r)}(t) = \delta_r
\]

for \( 1 \leq r \leq N - M \), where \( \delta_r \) are the non-zero eigenvalues of the truncation \( A^\perp \) of \( A \) to \( \text{Ker}(B) \) written in increasing order (the eigenvalues are distinct by (H5). If \( g \in G_0 \) is a real analytic curve then so are all the curves \( \lambda_r \).

**Proof** If \( t \geq 0 \) then \( g(t) \notin F \), so the eigenvalues \( \lambda_1, \ldots, \lambda_N \) of \( H_{g(t)} \) all have algebraic multiplicity 1. Perturbation theory implies that each eigenvalue of \( A_\gamma \) is an analytic function of \( \gamma \) if \( \gamma = g(t) \). Therefore each eigenvalue \( \lambda(t) \) of \( A_{g(t)} \) is a \( C^1 \) function of \( t \), or real-analytic if \( g \) is real analytic. These perturbation arguments imply all the statements of the theorem that relate to the limit \( t \to 0 \).

We next observe that \( p(g(t),\lambda(t)) = 0 \) for all \( t > 0 \). Differentiating this with respect to \( t \) yields

\[
\frac{\partial p}{\partial \gamma}(g(t),\lambda(t))g'(t) + \frac{\partial p}{\partial \lambda}(g(t),\lambda(t))\lambda'(t) = 0.
\]

By applying Lemmas 16 and 17, we deduce that \( \lambda'(t) \) is non-zero for every \( t > 0 \).

In order to prove the remainder of the theorem we need only find the asymptotic forms of the eigenvalues of \( H_\gamma \) as \( |\gamma| \to \infty \) and apply the results to \( \gamma = g(t) \) as \( t \to \infty \). The spectrum of \( A_\gamma \) is a set rather than an ordered sequence and there is no reason for any ordering of the eigenvalues of \( A_\gamma \) for large \( \gamma \) to be related to the ordering for \( \gamma = 0 \).
We start by describing the large eigenvalues of $A_\gamma$. For every $r \in \{1, \ldots, M\}$, perturbation theory and (H4) together imply that $B + \gamma^{-1}A$ has a simple eigenvalue of the form
\[
\mu_r = \beta_r + \gamma^{-1}\langle Af_r, f_r^* \rangle + O(\gamma^{-2})
\]
as $|\gamma| \to \infty$, where $f_r$ is an eigenvector of $B$ associated with the eigenvalue $\beta_r$, $f_r^*$ is an eigenvector of $B^*$ associated with the eigenvalue $\overline{\beta_r}$ and we normalize so that $\langle f_r, f_r^* \rangle = 1$. This implies that $A_\gamma$ has a simple eigenvalue of the form
\[
\lambda_r = \gamma \beta_r + \langle Af_r, f_r^* \rangle + O(\gamma^{-1})
\]
for all $r \in \{1, \ldots, M\}$.

We next use Lemma 18 to describe the small eigenvalues of $A_\gamma$. If one defines $\mathcal{H}_1 = \operatorname{Ran}(B)$ and $\mathcal{H}_2 = \operatorname{Ker}(B)$ then $\mathcal{C}^N = \mathcal{H}_1 \oplus \mathcal{H}_2$ is an orthogonal direct sum by Lemma 2. One may write
\[
A + \gamma B = \begin{pmatrix}
C + \gamma B^2 & E \\
E^* & A^2
\end{pmatrix}.
\]

where $B^2$ is the truncation of $B$ to $\mathcal{H}_1$, $C$ is the truncation of $A$ to $\mathcal{H}_1$ and $A^2$ is the truncation of $A$ to $\mathcal{H}_2$. We now add $kI$ to both sides where the constant $k$ is independent of $\gamma$ and large enough to ensure that $\|(A^2 + kI)^{-1}\| \leq 1/(2c)$, where $c = \|E\| + 1 = \|E^*\| + 1$. Using the fact that $B^2$ is invertible on $\mathcal{H}_1$, we observe that
\[
\varepsilon = \|(C + \gamma B^2 + kI)^{-1}\| = O(|\gamma|^{-1})
\]
as $|\gamma| \to \infty$. Lemma 18 now implies that $A + \gamma B + kI$ is invertible and
\[
\left\| (A + \gamma B + kI)^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & (A^2 + kI)^{-1} \end{pmatrix} \right\| = O(|\gamma|^{-1})
\]
for all large enough $|\gamma|$. Every eigenvalue $\delta_r$ of $A^2$ satisfies
\[
2 \leq 2c \leq |\delta_r + k| \leq \|A^2\| + k.
\]

Since $A^2$ is self-adjoint, a perturbation argument applied to (9) implies that there is an eigenvalue $\mu_r$ of $A + \gamma B$ such that
\[
|(\mu_r + k)^{-1} - (\delta_r + k)^{-1}| = O(|\gamma|^{-1})
\]
as $|\gamma| \to \infty$. By combining the last two equations we obtain
\[
|\mu_r - \delta_r| = O(|\gamma|^{-1})
\]
as $|\gamma| \to \infty$. Moreover, the perturbation argument proves that $\mu_r$ has the same multiplicity 1 as $\delta_r$ for all $r \in \{1, \ldots, N - M\}$.
We have now described \( N \) distinct simple eigenvalues of \( A_{\gamma} \) for all sufficiently large \( |\gamma| \). Since \( A_{\gamma} \) is an \( N \times N \) matrix there are no other eigenvalues. \( \square \)

Simple continuity arguments show that two homotopic curves \( g_1, g_2 \in \mathcal{G}_0 \) give rise to the same permutation \( \pi \). The fact that non-homotopic curves may give rise to different permutations is demonstrated in Examples 37 and 38.

## 5 Localization

In this section we describe a procedure for approximating the spectrum of \( A_{\gamma} = A + \gamma B \) in a given region of \( \mathbb{C}_+ \). We assume that \( A \) is a (possibly unbounded) self-adjoint operator on \( \mathcal{H} \), that \( \mathcal{K} \) is an auxiliary Hilbert space, that \( B = CD \) and that \( C : \mathcal{K} \to \mathcal{H}, D : \mathcal{H} \to \mathcal{K} \) are bounded operators.

We first note that the Birman-Schwinger method does not depend on self-adjointness of the perturbation. Numerically, the method is most useful when the dimension of \( \mathcal{K} \) is much smaller than that of \( \mathcal{H} \), but one need not assume that either is finite-dimensional.

**Lemma 20** If \( \lambda \notin \text{Spec}(A) \) and \( \gamma \neq 0 \) then \( \lambda \) is an eigenvalue of \( A_{\gamma} \) if and only if \( -1/\gamma \) is an eigenvalue of

\[
m(\lambda) = D(A - \lambda I)^{-1}C \in \mathcal{L}(\mathcal{K}).
\]

If \( \mathcal{K} \) is finite-dimensional then \( \lambda \notin \text{Spec}(A) \) is an eigenvalue of \( A_{\gamma} \) if and only if the jointly analytic function

\[
p(\gamma, \lambda) = \det (I + \gamma m(\lambda))
\]

vanishes.

**Proof** We start with the identity

\[
A + \gamma CD - \lambda I = (I + \gamma CD(A - \lambda I)^{-1})(A - \lambda I),
\]

both sides being regarded as linear maps from \( \text{Dom}(A) \) to \( \mathcal{H} \). Since \( A - \lambda I : \text{Dom}(A) \to \mathcal{H} \) is one-one and onto, \( \lambda \) is an eigenvalue of \( A + \gamma CD \) if and only if \(-1/\gamma \) is an eigenvalue of \( CD(A - \lambda I)^{-1} \). Both implications in the first sentence now depend on the elementary fact that if \( U, V \) are vector spaces over \( \mathbb{C} \), \( X : U \to V, Y : V \to U \) are linear operators and \( \sigma \in \mathbb{C} \) is non-zero, then \( \sigma \) is an eigenvalue of \( XY \) if and only if it is an eigenvalue of \( YX \). \( \square \)

In spite of the second statement in Lemma 20 the \( \mathcal{L}(\mathcal{K}) \)-valued function \( m \) is easier to analyze than the scalar function \( p \). One says that the analytic function \( m : \mathbb{C}_+ \to \mathcal{L}(\mathcal{K}) \) is an operator-valued Herglotz function if \( \langle m(\lambda)f, f \rangle \in \mathbb{C}_+ \) for every \( f \in \mathcal{K}\setminus\{0\} \) and \( \lambda \in \mathbb{C}_+ \); see [3].
Lemma 21 Suppose that $B = B^* \geq 0$, $C = D = B^{1/2}$, $\mathcal{K}$ is the closure of the range of $B$ and $^\natural$ is the operation of truncation to $\mathcal{K}$. Then
\[ m(\lambda) = \left( B^{1/2}(A - \lambda I)^{-1}B^{1/2} \right)^\natural \]
is a $\mathcal{L}(\mathcal{K})$-valued Herglotz function.

Proof The assumptions imply that $g = B^{1/2}f \neq 0$ and that
\[ \langle m(\lambda)f, f \rangle = \langle (A - \lambda I)^{-1}g, g \rangle, \]
which lies in $\mathbf{C}_+$ by the spectral theorem. □

Theorem 22 If $B = B^* \geq 0$ has finite rank $N$ and $\lambda \in \mathbf{C}_+$ then there are at least 1 and at most $N$ values of $\gamma$ such that $\lambda$ is an eigenvalue of $A_\gamma$; all such $\gamma$ lie in $\mathbf{C}_+$.

Proof If $f \in \text{Dom}(A)$, $f \neq 0$ and $A_\gamma f = \lambda f$ then
\[ \langle Af, f \rangle + \gamma \langle Bf, f \rangle = \lambda \langle f, f \rangle. \]
This implies that
\[ \text{Im}(\gamma) \langle Bf, f \rangle = \text{Im}(\lambda) \langle f, f \rangle. \]
Since the right hand side is positive, we deduce that $\langle Bf, f \rangle > 0$ and $\text{Im}(\gamma) > 0$.

Lemma 20 states that $\lambda$ is an eigenvalue of $A_\gamma$ if and only if $-1/\gamma$ is an eigenvalue of the $N \times N$ matrix $m(\lambda)$. The final statement of Lemma 21 implies that every eigenvalue of $m(\lambda)$ lies in $\mathbf{C}_+$. This proves that there are at least 1 and at most $N$ distinct values of $\gamma$, each of which lies in $\mathbf{C}_+$. □

The equation (11) below is a special case of the Nevanlinna-Riesz-Herglotz representation of operator-valued Herglotz functions; see [3].

Lemma 23 Under the assumptions of Lemma 21 let $P(E)$ denote the spectral projection of $A$ associated with any Borel subset $E \subseteq \mathbf{R}$. If
\[ Q(E) = (B^{1/2}P(E)B^{1/2})^\natural \]
then $Q$ is a finite, non-negative, countably additive, $\mathcal{L}(\mathcal{K})$-valued measure on $\mathbf{R}$ and
\[ m(\lambda) = \int_{\mathbf{R}} \frac{1}{s - \lambda} Q(\text{d}s) \]
for all $\lambda \in \mathbf{C}_+$. 17
The following lemma is well-known, but we include a proof for completeness.

**Lemma 24** If $f : [a, b] \to \mathbb{C}$ is bounded and measurable then

$$\| \int_a^b f(s) Q(ds) \| \leq \|f\|_\infty \|Q([a,b])\|.$$ 

**Proof** If $\phi, \psi \in \mathcal{K}$, we have

$$|\langle \int_a^b f(s) Q(ds) \phi, \psi \rangle| = |\langle \int_a^b f(s) P(ds)(B^{1/2}\phi), (B^{1/2}\psi) \rangle|$$

$$= |\langle f(A)(P([a,b])B^{1/2}\phi), (P([a,b])B^{1/2}\psi) \rangle|$$

$$\leq \|f\|_\infty \|P([a,b])B^{1/2}\phi\| \|P([a,b])B^{1/2}\psi\|$$

$$= \|f\|_\infty \langle Q([a,b])\phi, \phi \rangle^{1/2} \langle Q([a,b])\psi, \psi \rangle^{1/2}$$

$$\leq \|f\|_\infty \|Q([a,b])\| \|\phi\| \|\psi\|.$$ 

The lemma follows. \□

Our next lemma defines two operators that will be used in Theorem 26.

**Lemma 25** Let $[a,b] \subset \mathbb{R}$. Then there exist bounded, self-adjoint operators $X, Y : \mathcal{K} \to \mathcal{K}$ such that

$$X = \int_a^b Q(ds), \quad (12)$$

$$X^{1/2} Y X^{1/2} = \int_a^b sQ(ds). \quad (13)$$

Moreover $0 \leq X \leq B^3$ and $aI \leq Y \leq bI$.

**Proof** Since $X = Q([a,b])$, the inequalities for $X$ and the boundedness of $X$ follow from

$$0 \leq \langle Q([a,b])f, f \rangle \leq \langle Q(R)f, f \rangle = \langle B^2 f, f \rangle \leq \|B^2\| \|f\|^2,$$

valid for all $f \in \mathcal{K}$. If one defines

$$Z = \int_a^b sQ(ds),$$

18
then \(aX \leq Z \leq bX\). If \(X\) is invertible this immediately implies that

\[ aI \leq Y = X^{-1/2}ZX^{-1/2} \leq bI. \]

The general case follows as in Lemma 2.

Determining the eigenvalues of \(m(\lambda)\) for a given range of values of \(\lambda\) can sometimes be aided by writing

\[ m(\lambda) = m_1(\lambda) + m_2(\lambda) \]

where \(m_1\) may be computed more readily than \(m\) and \(m_2(\lambda)\) can be neglected or replaced by an appropriate approximation for the selected range of values of \(\lambda\).

Theorem 26 enables one to replace the contribution of an interval \([a, b]\) to \(m(\lambda)\) by a single operator provided \(\lambda\) is far enough away from \([a, b]\).

**Theorem 26** Let

\[ m(\lambda) = \int_{\mathbb{R}} \frac{1}{s - \lambda} Q(ds), \]

where \(\lambda \in \mathbb{C}_+\) and \(Q\) is a finite, non-negative, countably additive, \(\mathcal{L}(\mathcal{K})\)-valued measure on \(\mathbb{R}\). Given \([a, b] \subset \mathbb{R}\) and \(L > 0\), let

\[ \tilde{m}(\lambda) = \int_{s \notin [a, b]} \frac{1}{s - \lambda} Q(ds) + X^{1/2}(Y - \lambda I)^{-1}X^{1/2}, \]

where \(X\) and \(Y\) are as defined in Lemma 25. Then

\[ |m(\lambda) - \tilde{m}(\lambda)| \leq \frac{2(b - a)^2}{L^3} \|Q([a, b])\| \]

for all \(\lambda \in \mathbb{C}_+\) such that \(\text{dist}(\lambda, [a, b]) \geq L\).

**Proof** It suffices to prove that

\[ \left\| \int_{a}^{b} \frac{1}{s - \lambda} Q(ds) - X^{1/2}(Y - \lambda I)^{-1}X^{1/2} \right\| \leq \frac{2(b - a)^2}{L^3} \|Q([a, b])\| \]  

for all \(\lambda\) satisfying the stated conditions.

We first observe that

\[ \frac{1}{s - \lambda} + \frac{1}{\lambda - b} + \frac{s - b}{(\lambda - b)^2} = \frac{(s - b)^2}{(s - \lambda)(\lambda - b)^2}. \]

Integrating both sides with respect to \(Q\) over \([a, b]\) yields

\[ \int_{a}^{b} \frac{1}{s - \lambda} Q(ds) + \frac{X}{\lambda - b} + \frac{X^{1/2}YX^{1/2} - bX}{(\lambda - b)^2} = \int_{a}^{b} \frac{(s - b)^2}{(s - \lambda)(\lambda - b)^2} Q(ds), \]
and then
\[
\left\| \int_a^b \frac{1}{s - \lambda} Q(ds) + \frac{X}{\lambda - b} + X^{1/2} \frac{Y - bI}{(\lambda - b)^2} X^{1/2} \right\|
\leq \frac{(b - a)^2}{L^3} \| Q([a, b]) \|. \tag{18}
\]
by Lemma 24.

We next use the formula
\[
\frac{I}{Y - \lambda I} + \frac{I}{\lambda - b} + \frac{Y - bI}{(\lambda - b)^2} = \frac{(Y - bI)^2}{(Y - \lambda I)(\lambda - b)^2} \tag{19}
\]
to obtain
\[
\left\| X^{1/2} (Y - \lambda I)^{-1} X^{1/2} + \frac{X}{\lambda - b} + X^{1/2} \frac{Y - bI}{(\lambda - b)^2} X^{1/2} \right\|
= \left\| X^{1/2} \frac{(Y - bI)^2}{(Y - \lambda I)(\lambda - b)^2} X^{1/2} \right\|
\leq \frac{(b - a)^2}{L^3} \| Q([a, b]) \|. \tag{20}
\]
The proof of (15) is completed by combining (18) and (20).

Remark 27 One can obtain a better approximation than that in Theorem 26 if \([a, b]\) is divided into several subintervals, each of which is used to produce an extra term in the formula (14). The new \(\tilde{m}\) is, of course, more cumbersome to use numerically.

From this point onwards we assume that \(A\) is a possibly unbounded self-adjoint operator and that \(A_\gamma = A + \gamma B\) where \(Bf = \langle f, e \rangle e\) and \(e \in \mathcal{H}\) is a vector with norm 1. The Herglotz function (10) is then scalar-valued and given by the formula \(m(\lambda) = \langle (A - \lambda I)^{-1} e, e \rangle\).

If one approximates \(m\) uniformly in a chosen region by another analytic function whose zeros are more easily computed, then one can apply Rouche’s theorem to approximate the zeros of \(m\) and hence the spectrum of \(A_\gamma\). Lemma 28 is directly applicable to the polynomial \(p(\gamma, \lambda) = \det(A_\gamma - \lambda I)\), where \(\gamma\) is fixed. The connection between this and the Herglotz function \(m\) is explained in Lemma 15 and (24).

The proof of Lemma 28 can be adapted to cases in which \(p\) is not a polynomial; one needs an upper bound on the orders of its zeros, a lower bound on the distances between zeros and a lower bound on \(p\) for points that are not close to a zero.

Lemma 28 Let \(0 < \varepsilon < 1/2\), let \(U\) be a bounded open set in \(\mathbb{C}\) and let \(U_\varepsilon = \{ z \in \mathbb{C} : \text{dist}(z, U) < 2\varepsilon \}\). Let \(p\) be a monic polynomial such that \(|p(z)| > \varepsilon\) for all
\( z \in U_\varepsilon \setminus U \). Suppose that every root of \( p \) is simple and that \( |s - t| \geq 2 \) for any two distinct roots of \( p \). Let \( q \) be an analytic function on \( U_\varepsilon \) such that \( |p(z) - q(z)| < \varepsilon \) for all \( z \in U_\varepsilon \). If \( \lambda \in U_\varepsilon \) and \( p(\lambda) = 0 \) then \( \lambda \in U \) and there exists exactly one zero of \( p \) and one zero of \( q \) inside the circle \( C_{\lambda,\varepsilon} = \{ w \in U_\varepsilon : |w - \lambda| = \varepsilon \} \). If \( \lambda \in U_\varepsilon \) and \( q(\lambda) = 0 \) then \( \lambda \in U \) and there exists exactly one zero of \( p \) and one zero of \( q \) inside \( C_{\lambda,\varepsilon} \).

**Proof** The assumptions of the lemma imply immediately that neither \( p \) nor \( q \) can vanish in \( U_\varepsilon \setminus U \). Suppose that \( \lambda \in U \) and \( p(\lambda) = 0 \). Then \( |s - \lambda| \geq 2 \) for all \( s \) in the finite set \( S \) of roots of \( p \) that are not equal to \( \lambda \). If \( w \in C_{\lambda,2\varepsilon} \) then \( |w - s| > 1 \) for all \( s \in S \). Therefore

\[
|p(w)| = |(w - \lambda) \prod_{s \in S} (w - s)| > |w - \lambda| = 2\varepsilon.
\]

Since \( C_{\lambda,2\varepsilon} \) and its interior are contained in \( U_\varepsilon \), we may apply Rouché’s theorem to \( p \) and \( q \) on and inside \( C_{\lambda,2\varepsilon} \), and deduce that \( q \) has exactly one zero inside \( C_{\lambda,2\varepsilon} \). The same argument evidently applies to \( C_{\lambda,\varepsilon} \).

On the other hand if \( \lambda \in U \) and \( q(\lambda) = 0 \) then \( |p(\lambda)| < \varepsilon \). If \( T \) is the set of all roots of \( p \) then

\[
\varepsilon > |p(\lambda)| = \left| \prod_{t \in T} (\lambda - t) \right|
\]

so \( |\lambda - t| < 1 \) for at least one root of \( p \); from this point onwards we use the symbol \( t \) to refer to one such root. If \( S = T \setminus \{ t \} \) then \( |\lambda - s| > 1 \) for all \( s \in S \), so \( t \) is unique. Therefore

\[
\varepsilon > |p(\lambda)| = \left| (\lambda - t) \prod_{s \in S} (\lambda - s) \right| > |\lambda - t|.
\]

Repeating the first paragraph of the proof with \( \lambda \) replaced by \( t \), both \( p \) and \( q \) have exactly one root inside the circle \( C_{t,2\varepsilon} \). This contains the region inside \( C_{\lambda,\varepsilon} \), so both \( p \) and \( q \) have at most one root inside \( C_{\lambda,\varepsilon} \). The proof is concluded by noting that we have already observed that they have at least one root inside \( C_{\lambda,\varepsilon} \). \( \square \)

**Example 29** Given positive integers \( M_1, M_2 \) and \( L \), let \( N = M_1 + M_2 \) and let \( A \) be the diagonal \( N \times N \) matrix with entries

\[
A_{r,r} = (r - 1)/(M_1 - 1),
\]

\[
A_{M_1+s,M_1+s} = L + 1 + (s - 1)/(M_2 - 1),
\]

for \( 1 \leq r \leq M_1 \) and \( 1 \leq s \leq M_2 \), so that \( \text{Spec}(A) \subset [0,1] \cup [L+1,L+2] \). Also let \( A_\gamma = A + \gamma B \) where \( B \) is the rank one operator associated with the unit vector
$e \in \mathbb{C}^N$ defined by $e_r = (2M_1)^{-1/2}$ for $1 \leq r \leq M_1$ and $e_{M_1+s} = (2M_2)^{-1/2}$ for $1 \leq s \leq M_2$. One sees immediately that $\|e\| = 1$ in $\mathbb{C}^N$.

The continuous curves in Figure 2 show parts of six of the spectral curves of $A_\gamma$ for $\gamma = te^{i\theta}$ when $M_1 = 5$, $M_2 = 25$, $L = 4$, $0 \leq t < 20$ and $\theta = 89^\circ$. Most of the curves stay within a small distance of their starting point as $t$ increases. The curve starting at the eigenvalue 0.5 of the $30 \times 30$ matrix $A$ moves rapidly away from the real axis as $t$ increases but eventually converges to 3. There is only one curve that diverges to $\infty$ as $t \to \infty$, and a part of this appears in the top right-hand part of the figure.

The dashed curves in Figure 2 are produced in a similar manner but with $M_1 = 5$ and $M_2 = 1$, so that $\tilde{A}$ is a $6 \times 6$ matrix. Following the prescription of Theorem 26, we define $\tilde{A}_{r,r}$ as above for $1 \leq r \leq 5$, but put $\tilde{A}_{6,6} = 5.5$, so that the function $\tilde{m}$ of Theorem 26 is the Herglotz function for the pair $\tilde{A}$, $\tilde{e}$, where $\tilde{e}_r = (10)^{-1/2}$ for $1 \leq r \leq 5$ and $\tilde{e}_6 = 2^{-1/2}$. In spite of the substantial reduction in the size of the matrix, the part of the spectrum in $\{\lambda : \text{Re}(\lambda) \leq 2\}$ is almost unchanged, as predicted by Theorem 26. □

Figure 2: Spectral curves in Example 29

22
6 Rank one perturbations

In this section we obtain more detailed results of the type already considered under the assumptions that $A$ is a self-adjoint operator acting on the Hilbert space $\mathcal{H}$ and that $Bf = \langle f, e \rangle e$ for all $f \in \mathcal{H}$, where $e$ is a unit vector in $\mathcal{H}$.

We define $A_\gamma$ on $\mathcal{H}$ by

$$A_\gamma f = Af + \gamma \langle f, e \rangle e$$

(21)

where $\gamma \in C$. We summarize a few of the many results known in the case $\gamma \in \mathbb{R}$ and then consider non-real $\gamma$, for which new issues arise. We also assume that $e$ is a cyclic vector for $A$ in the sense that $\|e\| = 1$ and $\mathcal{L} = \text{lin}\{A^n e : n = 0, 1, \ldots\}$ is dense in $\mathcal{H}$. This is equivalent to $B$ being cyclic for $A$ by Corollary 7.

The four propositions below provide the general context within which our more detailed results are proved. The first is classical and may be found in [8].

**Proposition 30** Let $e$ be a cyclic vector for the bounded self-adjoint operator $A$ and let $A_\gamma$ be defined by (21). If $\gamma \in \mathbb{R}$ then every eigenvalue of $A_\gamma$ has multiplicity one. If $\alpha \in \mathbb{R}$ and $\lambda_\alpha$ is an isolated eigenvalue of $A_\alpha$, then $\lambda_\alpha$ can be analytically continued to all real $\gamma$ that are close enough to $\alpha$ and $\lambda'_\gamma > 0$ for all such $\gamma$.

**Proposition 31** If $\lambda \in \mathbb{C} \setminus \text{Spec}(A)$ and $\gamma \in \mathbb{C}$ then $\lambda$ is an eigenvalue of $A_\gamma$ if and only if

$$1 + \gamma \langle (A - \lambda I)^{-1} e, e \rangle = 0.$$  

(22)

This formula defines $\gamma$ as an analytic function of $\lambda \in \mathbb{C} \setminus \text{Spec}(A)$; one has $\gamma(\lambda) = -1/m(\lambda)$, where

$$m(\lambda) = \langle (A - \lambda I)^{-1} e, e \rangle.$$  

(23)

This is a special case of Lemma 20. In finite dimensions one may alternatively use the formula

$$\frac{\det(A + \gamma B - \lambda I)}{\det(A - \lambda I)} = \det((I + \gamma(A - \lambda I)^{-1} B)^2) = 1 + \gamma \langle (A - \lambda I)^{-1} e, e \rangle.$$  

(24)

See Lemma [15].

**Proposition 32** The function $m(\lambda)$ defined for all $\lambda \in \mathbb{C} \setminus \text{Spec}(A)$ by (23) is a Herglotz function in the sense that $m(\lambda) \in \mathbb{C}_\pm$ for all $\lambda \in \mathbb{C}_\pm$. Moreover $|m(x + iy)| < 1/|y|$ for all $x \in \mathbb{R}$ and $y \neq 0$. If $A$ is bounded then $m(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ such that $|\lambda| > \|A\|$. It follows that

$$\gamma(\lambda) = \lambda + \langle Ae, e \rangle + O(\lambda^{-1})$$

as $|\lambda| \to \infty$.  

23
Proposition 33 Suppose that $\gamma_0, \lambda_0 \in \mathbb{C}_+ \text{ satisfy (22)}$ and that $\lambda_0$ has algebraic multiplicity 1 as an eigenvalue of $A_{\gamma_0}$. Then there exists an analytic function $\lambda$ of $\gamma$ defined for all $\gamma$ in some neighbourhood of $\gamma_0$ such that $\gamma$ and $\lambda$ satisfy (22). Moreover $\lambda'(\gamma) \neq 0$ in this neighbourhood.

Proof The first statement of the proposition is a standard fact from perturbation theory for the eigenvalues of operators that depend analytically on a parameter. Given this, we differentiate (22) with respect to $\gamma$ to obtain

$$\langle (A - \lambda I)^{-1}e, e \rangle - \gamma \lambda'(\gamma) \langle (A - \lambda I)^{-2}e, e \rangle = 0.$$ 

Assuming that the neighbourhood is small enough, $\gamma \notin \mathbb{R}$ and $\langle (A - \lambda I)^{-1}e, e \rangle \neq 0$ by Proposition 32. This implies that $\lambda'(\gamma) \neq 0$. $\Box$

In the rest of this section we assume that $N = \dim(H) < \infty$, that $e \in H$ has norm one and is a cyclic vector for $A$, and that $\text{Im}(\gamma) \geq 0$. Our goal is to understand how the eigenvalues of $A_{\gamma}$ depend on $\gamma$ for very small and very large $\gamma$, and the mapping properties from the one asymptotic regime to the other. We start with the case in which $\gamma$ is real and positive.

Lemma 34 Under the assumptions of the last paragraph, let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $A$ written in increasing order and let $\delta_1, \ldots, \delta_{N-1}$ be the eigenvalues of the truncation $A^\flat$ of $A$ to $K = \{f : \langle f, e \rangle = 0\}$. Then

$$\lambda_1 < \delta_1 < \lambda_2 < \ldots < \delta_{N-1} < \lambda_N.$$ 

If one assumes that $\gamma \geq 0$ then the eigenvalues $\lambda_{r, \gamma}$ of $A_{\gamma}$ are all strictly increasing analytic functions of $\gamma$ satisfying $\lambda_{r, 0} = \lambda_r$. Moreover $\lim_{\gamma \to +\infty} \lambda_{r, \gamma} = \delta_r$ for $1 \leq r \leq N - 1$ and $\lim_{\gamma \to +\infty} \lambda_{N, \gamma} = +\infty$.

Proof This uses Proposition 30 and the variational formula for the eigenvalues of $A_{\gamma}$. $\Box$

We now turn to the study of the case $\gamma \in \mathbb{C}_+$.

Theorem 35 Given $\theta \in (0, \pi)$, define

$$S_\theta = \bigcup_{r>0} \text{Spec}(A_{te^i \theta}).$$ 

Then

$$S_\theta \cap S_\phi = \emptyset \text{ if } \theta \neq \phi$$ 

and

$$\bigcup_{\theta \in (0, \pi)} S_\theta = \mathbb{C}_+.$$ 

Moreover the limit set of each $S_\theta$ in $\mathbb{C}_+ \cup \{\infty\}$ is $\text{Spec}(A) \cup \text{Spec}(A^\flat) \cup \{\infty\}$.
Proof If $\lambda \in \mathbb{C}_+$ then (22) determines $\gamma = te^{i\theta}$ uniquely. This fact implies (26) and (27). The limit set of $S_\theta$ is the union of $\lim_{t \to 0} \Spec(A_{te^{i\theta}})$ and $\lim_{t \to +\infty} \Spec(A_{te^{i\theta}})$, both of which were determined in Theorem 19.

We now turn to the structure of the individual sets $S_\theta$. Let $\delta_1, \ldots, \delta_{N-1}$ denote the eigenvalues of $A^\flat$, written in increasing order and let $\delta_N = \infty$. As before we say that a curve $\sigma : (0, \infty) \to \mathbb{C}_+$ is simple and analytic if it is a one-one, real analytic mapping and $\sigma'(t)$ is non-zero for all $t \in (0, \infty)$.

Theorem 36 There exists a finite increasing set $T \subset (0, \pi)$ such that if $\theta \in (0, \pi) \setminus T$ then $S_\theta$ is the union of $N$ disjoint simple analytic curves. Each curve starts at some $\lambda_r \in \Spec(A)$ and ends at some $\delta_{\tau(r)}$, where $\tau$ is a permutation of $\{1, 2, \ldots, N\}$. This permutation is constant in each subinterval $J$ of $(0, \pi) \setminus T$, but it may change from one interval to another.

Proof This a corollary of Theorems 19 and 35, but some of the calculations are simpler because the polynomial $p(\gamma, \lambda)$ defined in (6) has the following explicit form. By expanding the determinant using an orthonormal basis whose first term is $e$ one obtains

\begin{align}
p(\gamma, \lambda) &= \det(A - \lambda I) + \gamma \det(A^\flat - \lambda I^\flat) \\
&= p_0(\lambda) + \gamma p_1(\lambda),
\end{align}

where $^\sharp$ denotes the truncation to $K = \{ \phi : \langle \phi, e \rangle = 0 \}$, $p_0$ is a polynomial with degree $N$ and $p_1$ is a polynomial with degree $N - 1$. The formula (29) can also be derived from (23). Let $F$ be the finite exceptional set defined just before Theorem 19. It follows from (26) and (27) that there is a finite set $T \subset (0, \pi)$ such that $\theta \in T$ if and only if $te^{i\theta} \in F$ for some $t > 0$. If $\theta \notin (0, \pi) \setminus T$ then the curve $g(t) = te^{i\theta}$ lies in $G_0$, as defined just before Theorem 19, which yields most of the statements of this theorem. $\theta \notin T$ implies that the curves are simple and non-intersecting because every $\lambda \in \mathbb{C}_+$ is associated with only one $\gamma = te^{i\theta}$ and hence with only one value of $t \in (0, \infty)$ by (22). The constancy of the permutation on each subinterval $J$ follow from the continuous dependence of the curves in $S_\theta$ on $\theta$. To prove the last statement, it is sufficient to consider the following example. □

Example 37 Let $\mathcal{H} = \mathbb{C}^2$ and let

$$A_\gamma = \begin{pmatrix} 1 + \gamma \alpha^2 & \gamma \alpha \beta \\ \gamma \alpha \beta & -1 + \gamma \beta^2 \end{pmatrix}$$

where $\gamma \in \mathbb{C}$, $\alpha > 0$, $\beta > 0$ and $\alpha^2 + \beta^2 = 1$, so that $e = (\alpha, \beta)$ has norm one and is a cyclic vector for $A = A_0$. The eigenvalues of $A_\gamma$ are

$$\lambda_{\pm, \gamma} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} + 1 - \gamma(\beta^2 - \alpha^2)}.$$
Figure 3 plots these eigenvalues for $\alpha = \sqrt{3}/2$, $\beta = 1/2$, $\gamma = te^{i\theta}$ and $0 \leq t \leq 4$. The two dashed curves correspond to the choice $\theta = 119^\circ$; the two intersecting continuous curves correspond to the choice $\theta = 120^\circ$; the two dotted curves correspond to the choice $\theta = 121^\circ$. Note that the only critical point is $\gamma_c = \{-1 + i\sqrt{3}\}$, so $T = \{120^\circ\}$. The corresponding eigenvalue of $A_{\gamma_c}$ is $\lambda_c = \{-1/2 + i\sqrt{3}/2\}$, which has algebraic multiplicity 2 but geometric multiplicity 1, by a direct computation or Theorem 10. It is clear that the permutation $\tau$ of the set $\{1, 2\}$ defined in Theorem 36 is different for $\theta < 120^\circ$ and for $\theta > 120^\circ$ and that there is no natural way of defining such a permutation for $\theta = 120^\circ$. □

![Figure 3: Spectral curves described in Example 37](image)

**Example 38** Let $A$ be the $5 \times 5$ diagonal matrix with eigenvalues $\lambda_r = r$, $r = 1, 2, 3, 4, 5$, and let $e = (1, 1, 1, 1, 1)/\sqrt{5}$. Then the limits of the eigenvalues of $A_\gamma$ as $|\gamma| \to \infty$ are given numerically by $\mu_1 = 1.35556$, $\mu_2 = 2.45608$, $\mu_3 = 3.54390$, $\mu_4 = 4.64442$ and $\mu_5 = \infty$. For each $\theta$ exactly one of the five eigenvalue curves diverges to $\infty$. The table below lists the permutations $\tau$ associated with each angle $\theta \in (0^\circ, 180^\circ)$ that is a multiple of $10^\circ$. 

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26
The first change in \( \tau \) occurs for \( \theta_1 \in (61^\circ, 62^\circ) \), while the second occurs for \( \theta_2 \in (81^\circ, 82^\circ) \). \( \square \)

7 The limit \( N \to \infty \)

The previous analysis clarifies to some extent how the spectra of a family of \( N \times N \) matrices \( A_\gamma = A + \gamma B \) depend on \( \gamma \) for very small and very large \( \gamma \). However, it does not capture the full range of phenomena that can occur for \( \gamma \) of intermediate sizes. Even if one is interested in a particular fairly large value of \( N \), one often obtains further insights by considering a family of \( N \times N \) matrices \( A_{N,\gamma} = A_N + \gamma B_N \). From this point of view the case \( N = \infty \) is regarded as an idealization that may be simpler to analyze than the original problem. Results such as Proposition 40 may then be used to estimate the difference between the two cases.

We assume throughout that \( B_N f = \langle f, e_N \rangle e_N \) for all \( f \in \mathbb{C}^N \) where \( e_N \in \mathbb{C}^N \) is a unit vector. The set of eigenvalues of \( A_{N,\gamma} \) is obtained by solving \( 1 + \gamma m_N(\lambda) = 0 \), or equivalently \( \gamma = -1/m_N(\lambda) \), where \( m_N \) are the Herglotz functions

\[
m_N(\lambda) = \langle (A_N - \lambda I_N)^{-1} e_N, e_N \rangle.
\]

See Propositions 31 and 32.

**Theorem 39** Let

\[
\mu_N(\gamma) = \max \{ \text{Im}(\lambda_{r,N,\gamma}) : 1 \leq r \leq N \}
\]

where \( \gamma \in \mathbb{C}_+ \) and \( \{ \lambda_{r,N,\gamma} \}_{r=1}^N \) are the eigenvalues of \( A_{N,\gamma} \) repeated according to their algebraic multiplicities. Then

\[
\mu_N(\gamma) \geq \text{Im}(\gamma)/N
\]

(30)
for all $N \geq 1$ and $\gamma \in \mathbb{C}_+$. Suppose that $\|A_N\| \leq c$ for all $N$ and that $m_N$ converge to $m_\infty$ locally uniformly on $\mathbb{C}_+$ as $N \to \infty$. If $\gamma \in \mathbb{C}_+$ and $-1/\gamma \notin \text{Ran}(m_\infty)$ then
\[
\lim_{N \to \infty} \mu_N(\gamma) = 0.
\]

Proof We note that $m_\infty$ is a Herglotz function, unless it is a real constant. Each function $m_N : \mathbb{C}_+ \to \mathbb{C}_+$ is surjective because $m_N(\lambda) = -1/\gamma$ has $N$ solutions $\lambda \in \mathbb{C}_+$ counting multiplicities, namely the eigenvalues of $A_{N,\gamma}$. We show in Example 41 that $m_\infty$ need not be surjective. The lower bound (30) follows from
\[
N \mu_N(\gamma) \geq \text{Im} \left( \sum_{r=1}^{N} \lambda_{r,N,\gamma} \right) = \text{Im} \left( \text{tr}(A_N + \gamma B_N) \right) = \text{Im}(\gamma).
\]

If $\gamma \in \mathbb{C}_+$ then every eigenvalue $\lambda_{r,N,\gamma}$ of $A_{N,\gamma}$ satisfies
\[
|\lambda_{r,N,\gamma}| \leq \|A_N\| + |\gamma| \|B_N\| \leq c + |\gamma|.
\]

Therefore $0 < \mu_N(\gamma) \leq c + |\gamma|$. If $\mu_N(\gamma)$ does not converge to 0 as $N \to \infty$ then there exists a subsequence $N(s)$ and a constant $c_2 > 0$ such that $\mu_{N(s)}(\gamma) \geq c_2$ for all $s$; and then a subsequence $r(s)$ such that $\text{Im}(\lambda_{r(s),N(s),\gamma}) \geq c_2$ for all $s$. Combining this with (31), there exist subsubsequences, which we again parametrize using $s$, and $\lambda \in \mathbb{C}_+$ such that $\lim_{s \to \infty} \lambda_{r(s),N(s),\gamma} = \lambda$ where $\text{Im}(\lambda) \geq c_2$ and $|\lambda| \leq c + |\gamma|$. Since $m_{N(s)}(\lambda_{r(s),N(s),\gamma}) = -1/\gamma$ for all $s$, the local uniform convergence of $m_N$ to $m_\infty$ implies that $m_\infty(\lambda) = -1/\gamma$. \hfill \Box

Theorem 39 depends on the assumption that $m_N$ converges to $m_\infty$ as $N \to \infty$. The following proposition allows one to estimate the difference between $m_N(\lambda)$ and $m_\infty(\lambda)$ for problems of the above type by putting
\[
k(s) = \frac{|g(s)|^2}{s - \lambda},
\]

where the choice of $g$ depends on the problem. It may be seen that the bound on the difference is $O(\text{Im}(\lambda)^{-2})$ as $\text{Im}(\lambda) \to 0$.

Proposition 40 Let $k$ be a continuous function on $[a,b]$ with bounded first derivative and let $N$ be a positive integer. Then
\[
\frac{b-a}{N} \sum_{r=1}^{N} k(a + r(b-a)/N) = \int_{a}^{b} k(s) \, ds + \text{rem} \quad (32)
\]

where
\[
|\text{rem}| \leq \frac{(b-a)^2}{2N} \|k'\|_\infty.
\]
Proof The left hand side of (32) is the sum of $N$ terms of the form

$$\delta k(c_r) = \int_{c_r-\delta}^{c_r} \frac{d}{ds}[(s - c_r + \delta)k(s)] ds$$

$$= \int_{c_r-\delta}^{c_r} k(s) ds + C_r$$

where $\delta = (b-a)/N$, $c_r = a + r\delta$ and

$$|C_r| = \left| \int_{c_r-\delta}^{c_r} (s - c_r + \delta)k'(s) ds \right|$$

$$\leq \|k'\|_{\infty} \int_{c_r-\delta}^{c_r} |s - c_r + \delta| ds$$

$$= \frac{\delta^2}{2}\|k'\|_{\infty}.$$ 

Summing over $r$ one obtains

$$|\text{rem}| \leq \frac{N\delta^2}{2}\|k'\|_{\infty} = \frac{(b-a)^2}{2N}\|k'\|_{\infty}.$$ 

□

Example 41 Let $A_N$ be the $N \times N$ diagonal matrix with entries $A_{N,n,n} = n/N$ for all $n$ and let $B_N$ the rank one matrix associated with the unit vector $e_{N,n} = N^{-1/2}$ for all $n$. Then $\text{Spec}(A_N) \subset [0, 1]$ and

$$m_N(\lambda) = \frac{1}{N} \sum_{r=1}^{N} \frac{1}{n/N - \lambda}.$$ 

It may be seen that $m_N$ converges locally uniformly to

$$m_\infty(\lambda) = \int_0^1 \frac{ds}{s-\lambda} = \log \left( \frac{\lambda - 1}{\lambda} \right)$$

as $N \to \infty$. It follows that the range of $m_\infty$ is $\{z \in \mathbb{C} : 0 < \text{Im}(z) < \pi\}$ and the set of $\mu \in \mathbb{C}_+$ such that $m_\infty(\lambda) = -1/\mu$ has no solution is the closed disc

$$D = \{\gamma : |\gamma - i/(2\pi)| \leq 1/(2\pi)\}.$$ 

Figure 4 provides a contour plot $\mu_N(\gamma)$ for $N = 100$, the contours corresponding to the values 0.01, 0.02, 0.03, 0.04, 0.05 of $\mu_N(\gamma)$. The circle $\partial D$ is included for comparison. □
Figure 4: Contour plot of $\mu_N(\gamma)$ for $N = 100$.

**Example 42** Let $A_N$ be the $N \times N$ diagonal matrix with entries $A_{N,n,n} = n/N$ for all $n$ and let $B_N$ the rank one matrix associated with the unit vector $e_{N,n} = N^{-1/2}g(n/N)$ where $g \in L^2(0,1)$ and $g$ is sufficiently regular. Then Spec$(A_N) \subset [0,1]$ and $m_N$ converges locally uniformly to

$$m_\infty(\lambda) = \int_0^1 \frac{f(s)ds}{s - \lambda},$$  \hspace{1cm}  \text{(34)}$$

as $N \to \infty$, where $f(s) = |g(s)|^2$. The integral in (34) is well-defined for every $\lambda \in C_+$ because $f \in L^1(0,1)$, but the form of the range of $m_\infty$ depends on whether either or both of the integrals

$$\int_0^1 \frac{f(s)}{s} \, ds, \quad \int_0^1 \frac{f(s)}{1-s} \, ds$$

is finite. In Example 41 both integrals are infinite. More generally the first integral diverges if and only if $A + \gamma B$ has a negative eigenvalue for all real negative $\gamma$, while the second integral diverges if and only if $A + \gamma B$ has a positive eigenvalue for all real positive $\gamma$.

The range of $m_\infty$ is the union of the sets $m_\infty(S_{\epsilon,r})$, where

$$S_{\epsilon,r} = \{ \lambda : \text{Im}(\lambda) \geq \epsilon \text{ and } |\lambda| \leq r \}.$$
These increase monotonically as $\varepsilon > 0$ decreases to 0 and as $r$ increases to $\infty$. The boundary $\partial S_{\varepsilon,r}$ may be parametrized as a simple closed curve $\gamma_{\varepsilon,r}$. A standard theorem in complex analysis states that each set $m_\infty(S_{\varepsilon,r})$ is the union of the range of the closed curve $\sigma_{\varepsilon,r} = m_\infty \circ \gamma_{\varepsilon,r}$ and the set of all $z$ not in this range whose winding number with respect to $\sigma_{\varepsilon,r}$ is non-zero.

The observations above allow one to compute the range of $m_\infty$ approximately by taking $\varepsilon > 0$ small enough and $r$ large enough. This is particularly easy if one can write $m_\infty$ in closed form. If $g(s) = s^{1/2}$ for all $s \in [0, 1]$ then

$$m_\infty(\lambda) = \int_0^1 \frac{s}{s-\lambda} \, ds = 1 + \lambda \log \left( \frac{\lambda - 1}{\lambda} \right).$$

Figure 5 was obtained by putting $\varepsilon = 10^{-8}$. The set of $\gamma$ for which $-1/\gamma = m_\infty(\lambda)$ is not soluble is the part of $C_\gamma$ that is inside the closed curve plotted. This curve starts at $-1$ and ends at 0. The gap observed near 0 is a numerical artifact that arises because the convergence to 0 is logarithmic.

**Theorem 43** Suppose that

$$m(\lambda) = \int_{\mathbb{R}} \frac{f(s)}{s-\lambda} \, ds$$

Figure 5: Boundary curve in Example 42
for all $\lambda \in \mathbb{C}_+$, where $f(s) \geq 0$ for all $s \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{f(s)}{1 + |s|} \, ds < \infty$$

(36)

and $0 < \|f\|_{\infty} < \infty$. Then $m(\mathbb{C}_+)$ is contained in

$$\{ z : 0 < \text{Im}(z) < \pi \|f\|_{\infty} \}.$$  

(37)

Hence $-1/\gamma = m(\lambda)$ is not soluble for any $\gamma$ in the closed disc

$$\left\{ \gamma : \left| \gamma - \frac{i}{2\pi \|f\|_{\infty}} \right| \leq \frac{1}{2\pi \|f\|_{\infty}} \right\}.$$  

(38)

**Proof** The condition (36) ensures that the integral (35) defining $m(\lambda)$ converges for all $\lambda \in \mathbb{C}_+$ and defines an analytic function of $\lambda$. The condition $f(s) \geq 0$ and $0 < \|f\|_{\infty}$ ensures that the range of $m$ is contained in $\mathbb{C}_+$.

We next observe that if $\lambda = u + iv$ where $u \in \mathbb{R}$ and $v > 0$ then

$$\text{Im}(m(\lambda)) = \int_{\mathbb{R}} \frac{v}{(u - s)^2 + v^2} f(s) \, ds.$$  

A direct estimate of this integral yields (37), and (38) follows. \qed
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References

[1] E. B. Davies, ‘Linear Operators and their Spectra’, Cambridge Studies in Advanced Mathematics, vol. 106, Camb. Univ. Press, 2007.

[2] E. B. Davies and Y. Safarov, private conversation, 2012.

[3] F. Gesztesy, N. J. Kalton, K. A. Makarov and E. Tsekanovskii, Some applications of operator-valued Herglotz functions, pp. 271–321 in ‘Operator Theory: Advances and Applications’, Vol. 123, eds. J. A. Ball et al., Birkhäuser Verlag AG, Basel, Switzerland, 2001.

[4] T. Kato, ‘Perturbation Theory for Linear Operators’, New York, Springer, 1966.

[5] C Liaw, Rank one and finite rank perturbations, preprint 2012, arXiv:1205.4376v1 [Math.SP].

[6] C. Liaw, S. Treil, Rank one perturbations and singular integral operators, Journal of Functional Analysis 257 (2009) 1947-1975.

[7] A. C. M. Rana and M. Wojtylak, Eigenvalues of rank one perturbations of unstructured matrices, Linear Algebra and its Applications 437 (2012) 589-600.

[8] B. Simon, Spectral analysis of rank one perturbations and applications, pp. 109-149 in ‘Mathematical Quantum Theory. II. Schrödinger Operators’, Vancouver, BC, 1993, CRM Proc. Lecture Notes, vol. 8, Amer. Math. Soc., Providence, RI, 1995. See also Chaps. 12-14 in B. Simon, ‘Trace Ideals and Their Applications’, 2nd ed., Amer. Math. Soc., Providence, RI, 2005.