Constraint Satisfaction over Generalized Staircase Constraints

Shubhadip Mitra
smitr@iitk.ac.in
Dept. of Computer Science and Engineering,
Indian Institute of Technology, Kanpur,
India

Partha Dutta
parthdut@in.ibm.com
IBM Research Lab,
Bangalore,
India

Arnab Bhattacharya
arnabb@iitk.ac.in
Dept. of Computer Science and Engineering,
Indian Institute of Technology, Kanpur,
India

Abstract

One of the key research interests in the area of Constraint Satisfaction Problem (CSP) is to identify tractable classes of constraints and develop efficient solutions for them. In this paper, we introduce generalized staircase (GS) constraints which is an important generalization of one such tractable class found in the literature, namely, staircase constraints. GS constraints are of two kinds, down staircase (DS) and up staircase (US). We first examine several properties of GS constraints, and then show that arc consistency is sufficient to determine a solution to a CSP over DS constraints. Further, we propose an optimal $O(cd)$ time and space algorithm to compute arc consistency for GS constraints where $c$ is the number of constraints and $d$ is the size of the largest domain. Next, observing that arc consistency is not necessary for solving a DSCSP, we propose a more efficient algorithm for solving it. With regard to US constraints, arc consistency is not known to be sufficient to determine a solution, and therefore, methods such as path consistency or variable elimination are required. Since arc consistency acts as a subroutine for these existing methods, replacing it by our optimal $O(cd)$ arc consistency algorithm produces a more efficient method for solving a USCSP.

Keywords: Constraint Satisfaction Problem (CSP), Staircase Constraint, Monotone Constraint, CRC Constraint, Generalized Staircase Constraint

1 Introduction

Constraint Satisfaction Problem (CSP) is one of the key techniques in artificial intelligence that offers a simple formal framework to represent and solve several problems in temporal reasoning, job scheduling, pattern matching and natural language processing [28]. Interestingly, in many such problems, the underlying constraints are of a specific type. For this reason, one of the key research interests in the area of CSP is to identify tractable classes of constraints and develop efficient solutions for them. In this paper, we introduce generalized staircase (GS) constraints which is an important generalization of one such tractable class introduced in [11], namely, staircase constraints.

Figure 1 shows the relationships of GS constraints with other constraint classes. The class of GS constraints is the union of two constraint classes, namely, down staircase (DS) and up staircase (US). While GS constraints strictly generalize the class of staircase constraints (S, shown as a pie-shape), they are a strict subclass of connected row convex (CRC) constraints [11]. Although related constraint classes such as CRC, max-closed, min-closed and monotone constraints have been well studied in the literature, to the best of our knowledge, the specific class of GS constraints have not been investigated earlier. GS constraints are interesting to study as many temporal constraints involving bounded intervals that arise in temporal reasoning and temporal databases [13, 19, 31, 32] are often GS constraints. Due to their monotonic and convex structure, they admit significantly simpler and faster solutions than CRC constraints.
A GS constraint can be either a down staircase (DS) or an up staircase (US) constraint. A DS (respectively, US) constraint is a CRC constraint \([11]\) such that if \([p, q]\) and \([p', q']\) are images of consecutive rows in its constraint matrix, then \(p \leq p' \wedge q \leq q'\) (respectively, \(p \geq p' \wedge q \geq q'\)). Referring to the constraint matrices shown in Figure 2, a DS (respectively, US) constraint has images of consecutive rows shifted towards right (respectively, left) as we move down the matrix.

1.1 Motivation

Consider the following simplified problem of identifying cyclones at a given location from weather records. Suppose a cyclonic instance is characterized by the following set of events: hourly rainfall exceeding \(\alpha\) units (event \(A\)), hourly mean wind speed exceeding \(\beta\) units (event \(B\)), and clockwise wind direction (event \(C\)). The goal is to determine tuples of the event pattern is represented through a constraint network in Figure 3. Considering the constraint network in Figure 3, a DS (respectively, US) constraint has images of consecutive rows shifted towards right (respectively, left) as we move down the matrix.

The weather database contains a list of instances of events \(A, B, C\) sorted on \(x\) where \(x\) denotes an instance of event \(X\). The goal is to determine tuples of the form \((a, b, c)\) where \(a, b, c\) are instances of events \(A, B, C\) respectively, such that all the above constraints are satisfied. Modeling it as a CSP, we consider variables \(X_A, X_B, X_C\) for events \(A, B, C\) respectively, each of which have domains as the time points of respective event instances, and the constraints as shown in the constraint network in Figure 3. Considering the constraint matrix for the first constraint, if \(a_i, a_i' \in A\) have images \([b_j, b_k]\) and \([b_j', b_k']\) respectively in \(B\) (i.e., \(b_j\) to \(b_k\) satisfy the constraint for \(a_i\), etc.), and \(i \leq i'\), then \(j \leq j'\) and \(k \leq k'\) (the subscripts denote the indices of the instances in the sorted order). This is the down staircase property as shown in Figure 2(a). Similarly, the other constraints (ii) and (iii) are also DS constraints.

Problems such as the above are common in complex event processing applications \([1, 4, 5, 6, 7, 8, 19, 23, 24, 26, 27, 29, 31, 32, 33, 34, 35, 37]\) where the goal is to efficiently detect occurrences of complex events which are usually represented as patterns of events sharing some temporal relationship with each other. Examples of complex events include state changes in business and industrial processes, problems in enterprise systems and state changes in the environment. However, in most of the existing literature, due to the simplicity of the patterns, they are represented using constructs similar to regular expressions and the detection of complex events is realized by using finite state automata \([1, 5, 6, 7, 8, 23, 24, 26, 31, 37]\) where the goal is to recognize arbitrary patterns of temporal events sharing temporal relationships between pairs of events. Hence, we consider a CSP based approach where the event pattern is represented through a constraint network. In addition, we observe that the constraints involved in such constraint networks are usually DS constraints, as in the above example.

Another application of DSCSP is scheduling (or timetabling) of resources (or facilities) based on their availability. There is a variable for each resource whose domain consists of the times of availability of the given resource. The constraints involved are usually unary or binary. The unary constraints are typically simple to process, and thus, domain values that do not satisfy those are easy to eliminate. The binary constraints involve bounded temporal intervals, which can be expressed as DS constraints. Solving this constraint network would offer a feasible schedule for the resources. Such constraint-based timetabling have been considered in \([9, 10, 17]\).

1.2 Contributions

In this paper, we make the following contributions:

1. We examine the structure and properties of GS con-
constraints, besides studying its relationships with the existing constraint classes.

2. We show that, for DS constraints, arc consistency is sufficient to determine a solution. Further, we present an optimal algorithm, called ACiDS, for computing arc consistency for GS constraints in \( O(cd) \) time and space, where there are \( n \) variables, \( c \) constraints and the largest domain size is \( d \).

3. Observing that arc consistency is not necessary for solving a DSCSP, we propose a more efficient algorithm, DSCSP Solver, for the same.

4. With regard to US constraints, arc consistency is not sufficient to determine a solution, and therefore, methods such as path consistency or variable elimination are required. Since arc consistency acts as a subroutine for these existing methods, replacing any known arc consistency algorithm by our optimal \( O(cd) \) algorithm produces a more efficient method for solving a USCSP.

1.3 Organization

The rest of the paper is organized as follows. Section 2 covers the preliminary concepts of constraint satisfaction and formally defines the class of GS constraints. Section 3 outlines the related work in this area. We next discuss the relationships of GS constraints with the existing constraint classes and its closure properties in Section 4. In Section 5, we first show that arc consistency is sufficient to solve a DSCSP and later present an efficient arc consistency based algorithm, ACiDS, to solve a DSCSP. We also discuss how the ACiDS algorithm is useful to solve a USCSP, although arc consistency is not sufficient to solve it. In Section 6, we present a more efficient algorithm, DSCSP Solver, to solve a DSCSP. Finally, we conclude in Section 7.

2 Preliminaries

2.1 CSP over GS Constraints

A finite binary constraint satisfaction problem (CSP) \( \mathcal{P} = (\mathcal{X}, \mathcal{D}, \mathcal{C}) \) is defined as a set of \( n \) variables \( \mathcal{X} = \{X_1, X_2, \ldots, X_n\} \), a set of domains \( \mathcal{D} = \{D_1, D_2, \ldots, D_n\} \) where \( D_i \) is the finite set of possible values that can be assigned to \( X_i \), and a set of \( c \) constraints \( \mathcal{C} = \{C_{ij}\} \) where \( C_{ij} \) is a binary constraint involving \( X_i \) and \( X_j \). We assume that the domain of each variable consists of at most \( d \) distinct integers sorted in an increasing order. A constraint \( C_{ij} \) on the ordered set of variables \( (X_i, X_j) \) specifies the admissible combinations of values for \( X_i \) and \( X_j \). If a pair of values \( (v_i, v_j) \) \((v_i \in D_i, v_j \in D_j)\) satisfy the constraint \( C_{ij} \), then we denote it as \((v_i, v_j) \in C_{ij}\). For the sake of simplicity, we assume that there exists at most one constraint between any pair of variables. Given a constraint \( C_{ij} \), the constraint \( C_{ji} \) refers to a transposition of \( C_{ij} \), i.e., if \((v_i, v_j) \in C_{ij}\), then \((v_j, v_i) \in C_{ji}\). Further we assume that all constraints are binary, i.e., they involve only two variables. It was stated in [16] that any \( r \)-ary CSP for any fixed \( r \geq 1 \) can be converted to an equivalent binary CSP [25]. Therefore, it is reasonable to consider CSPs with only binary constraints. A binary constraint can always be represented by a constraint matrix [21], showing the admissible combinations of values.

A CSP \( \mathcal{P} \) asks whether there exists an assignment of the variables \( \mathcal{X} \) such that all the constraints are satisfied. If an assignment \( X_i = v_i, \forall i \) is a solution, then \( v_i \) is said to be a member of a solution.

Consider the set of vertices \( V_\mathcal{P} = \{V_1, V_2, \ldots, V_n\} \) where \( V_i \) corresponds to variable \( X_i \). A pair of variables \( (X_i, X_j) \) that has a constraint \( C_{ij} \) is referred to by the arc \((i, j)\) which represents an edge between \( V_i \) and \( V_j \). The set of all arcs is denoted by \( arc(\mathcal{P}) \). The graph induced by the vertex set \( V_\mathcal{P} \) and the edge set \( arc(\mathcal{P}) \) is referred to as the constraint network of \( \mathcal{P} \) and is denoted by \( \mathcal{N}_\mathcal{P} \).

Following [11], a constraint \( C_{ij} \) is row-convex if and only if in each row of the matrix representation of \( C_{ij} \), all the ones are consecutive, i.e., no two ones within a single row are separated by a zero in that same row. The reduced form of a constraint \( C_{ij} \), denoted by \( C_{ij}^* \), is obtained by removing all the empty rows and columns in its matrix representation. The domain of \( X_i \) through the constraint \( C_{ij} \), denoted by \( D_i(C_{ij}) \), is the set \{v \in D_i | \exists v_j : (v, v_j) \in C_{ij}\}. Suppose \( C_{ij} \) is a row-convex constraint and \( v_i \in D_i(C_{ij}) \). The image of \( v_i \) in \( C_{ij} \) is the set \{v_j \in D_j(v_i) \in C_{ij}\}. Due to row convexity of \( C_{ij} \), this set can be represented as an interval \([w_1, w_m] \) (over the domain \( D_j(C_{ji}) \)) and we denote \( w_1 \) and \( w_m \) by \( \min(C_{ij}, v_j) \) and \( \max(C_{ij}, v_j) \) respectively. We denote by \( \text{succ}(v_j, D_j(C_{ji})) \) and \( \text{pred}(v_j, D_j(C_{ji})) \) the successor and the predecessor of \( v_j \) in \( D_j(C_{ji}) \) respectively. For ease of notation, we simply use \( \min(v_j), \max(v_j), \text{succ}(v_j), \text{pred}(v_j) \) when there is no ambiguity on the underlying domain.

Referring to the example in Figure 2(a), \( \min(a_3) = b_4, \max(a_3) = b_4, \text{succ}(b_4) = b_4 \) and \( \text{pred}(b_4) = b_2 \). A row-convex constraint \( C_{ij} \) is connected row-convex (CRC) if and only if the images \([a, b]\) and \([a', b']\) of two consecutive rows in \( C_{ij}^* \) are such that \( a' \leq \text{succ}(b) \land b' \geq \text{pred}(b) \).
A constraint \( C_{ij} \) is said to be monotone if and only if \((v, w) \in C_{ij}\) implies \((v', w') \in C_{ij}\) for all values \(v_i, v'_i \in D_i, v_j, v'_j \in D_j\) such that \(v'_i \geq v_i\) and \(v'_j \leq v_j\), i.e., referring to the constraint matrix of \( C_{ij}\), if \((v_i, v_j) \in C_{ij}\), then all cells to the left of \((v_i, v_j)\) on the same row are in \( C_{ij}\) and all cells below \((v_i, v_j)\) on the same column are in \( C_{ij}\). However, a complementary definition of monotone constraints was proposed in \cite{14}, where a constraint \( C_{ij} \) is said to be monotone if and only if \((v_i, v_j) \in C_{ij}\) implies \((v'_i, v'_j) \in C_{ij}\) for all values \(v_i, v'_i \in D_i, v_j, v'_j \in D_j\) such that \(v'_i \leq v_i\) and \(v'_j \geq v_j\), i.e., referring to the constraint matrix of \( C_{ij}\), if \((v_i, v_j) \in C_{ij}\), then all cells to the right of \((v_i, v_j)\) on the same row are in \( C_{ij}\) and all cells above \((v_i, v_j)\) on the same column are in \( C_{ij}\). An example of each of them is shown in Figure 4 (a) and Figure 4 (b). Now we consider a generalization of monotone constraints, introduced in \cite{11}, as following.

**Definition 1.** Let \( \leq \) and \( \geq \) be total orderings on \( D_i\) and \( D_j\), respectively. A constraint \( C_{ij} \) is \((\leq, \geq)\)-monotone if and only if \((v_i, v_j) \in C_{ij}\) implies \((v'_i, v'_j) \in C_{ij}\) for all values \(v_i, v'_i \in D_i, v_j, v'_j \in D_j\) such that \(v'_i \leq v_i\) and \(v'_j \geq v_j\). Staircase constraints are \((\alpha, \beta)\)-monotone constraints where \(\alpha, \beta \in \{\leq, \geq\}\).

Now we consider a generalization of staircase constraints, as a generalization of staircase constraints. GS constraints are of two types, down staircase (DS) and up staircase (US), defined as below.

**Definition 2.** A constraint \( C_{ij} \) is a down staircase (DS) constraint if and only if it is row convex and for any \(u, v \in D_i\) such that \(v = succ(u)\), the following conditions (DS property) hold: \(min(C_{ij}, u) \leq min(C_{ij}, v)\) and \(max(C_{ij}, u) \leq max(C_{ij}, v)\).

**Definition 3.** A constraint \( C_{ij} \) is an up staircase (US) constraint if and only if it is row convex and for any \(u, v \in D_i\) such that \(v = succ(u)\), the following conditions (US property) hold: \(min(C_{ij}, u) \geq min(C_{ij}, v)\) and \(max(C_{ij}, u) \geq max(C_{ij}, v)\).

Figure 4 shows examples of both the constraints.

Following the above definitions, it is clear that this class of GS constraints is a strict subclass of CRC constraints. Further, \((\leq, \geq)\)-monotone and \((\geq, \leq)\)-monotone constraints are strict subclasses of DS constraints while \((\leq, \leq)\)-monotone and \((\geq, \geq)\)-monotone are strict subclasses of US constraints (see Figure 1). A binary CSP with all its constraints as DS (respectively US) constraints is referred to as a DSCSP (respectively USCSP).

### 2.2 Arc Consistency and Path Consistency

Following \cite{14}, an arc \((i, j)\) is arc consistent if and only if \(\forall v_i \in D_i, \exists v_j \in D_j\) such that \((v_i, v_j) \in C_{ij}\) (\(v_i\) is said to support \(v_j\)). For arc consistency, we assume that constraint \( C_{ij} \in C\) if and only if \( C_{ji} \in C\), i.e., arc \((i, j)\) and arc \((j, i)\) are treated distinct, and that both are in \(arc(P)\). If each \(arc(i, j) \in arc(P)\) is arc consistent, then the constraint network \(N_P\) is said to be arc consistent.

Following \cite{11}, a constraint network \(N_P\) is path consistent if and only if for every triple \((X_i, X_k, X_j)\) of variables such that \(arc(i, k), arc(k, j), arc(i, j) \in arc(P)\), for every \(v_i \in D_i\) and \(v_j \in D_j\) such that \((v_i, v_j) \in C_{ij}\), there exists \(v_k \in D_k\) such that \((v_i, v_k) \in C_{ik}\) and \((v_k, v_j) \in C_{kj}\).

Two constraint networks are equivalent if they have the same set of solutions. Given a constraint network, the task of an arc consistency (respectively path consistency) algorithm is to generate an equivalent constraint network that is arc consistent (respectively path consistent).

### 3 Related Work

Although CSPs are NP-hard, several tractable classes of CSPs have been identified \cite{22}. Among the notable tractable classes of constraints are functional, anti-functional, monotone, min-closed, max-closed and connected row convex (CRC) constraints. In this paper, we are considering another important tractable class of generalized staircase (GS) constraints.

In this work, we propose an algorithm named ACiDS, for computing arc consistency over DS constraints. In
this context, we note that there are several existing arc consistency algorithms for general arbitrary constraints. The most prominent ones include AC3 [18], AC4 [20], AC5 [14], AC6 [3] and AC7 [12]. Although AC3 has non-optimal worst case time complexity ($O(n^3d^2)$), it is still considered as the basic constraint propagation algorithm owing to its simplicity. Its successors AC4, AC5, AC6 and AC7 have optimal worst case time complexity of $O(cd^2)$. AC5 is a generic arc consistency algorithm that offers a framework to compute arc consistency for any constraint network and can be optimized for certain classes of constraints. A refinement of AC3 in the form of AC2001 was proposed in [2] which has an optimal worst case time complexity of $O(cd^2)$. None of these algorithms assume knowledge of the semantics of the underlying constraints. Although they agree on the worst case time complexity (besides AC3), their performance varies for different constraint classes. In addition, the tightness and slackness of the constraints also affect the performance for any given constraint class. Since arc consistency is sufficient to solve a DSCSP, as shown in Theorem 2 any DSCSP can be solved by any of these arc consistency algorithms (AC4, AC6, AC7 or AC2001) with a worst case time complexity of $O(cd^2)$. The arc consistency algorithm presented in this paper, namely ACIDS, has an optimal worst case time complexity of $O(cd)$ for the class of DS constraints, thus improving the $O(cd^2)$ bound for solving DS constraints using the existing arc consistency algorithms. This algorithm is a specialization of the AC5 algorithm. Similar specializations for several classes of arithmetic constraints such as functional, anti-functional and monotone constraints were stated in [14]. For each of these classes, AC5 determines a solution in $O(cd)$ optimal time.

GS constraints are a subclass of CRC constraints. While path consistency is sufficient to solve a CSP over CRC constraints, we show that arc consistency is sufficient to solve a CSP over DS constraints (please refer to Theorem 2). However, it is not known that whether US constraints are solvable using arc consistency. For the class of CRC constraints, a path consistency based algorithm was proposed in [11] that runs in $O(n^3d^2)$ time. Based on variable-elimination, [36] stated an $O(n^d\sigma^2 + cd^2)$ time algorithm for CRC constraints where $\sigma \leq n$ is the degree of elimination of the triangulated graph of the underlying constraint network. Both these algorithms for CRC constraints use arc consistency as a subroutine which takes $O(cd^2)$ time. US constraints being a subclass of CRC constraints can be solved by either of these techniques mentioned above. Our arc consistency algorithm for GS constraints that runs in $O(cd)$ time, acting as a subroutine, thus improves the time complexity of the algorithm by [36] to $O(n^d\sigma^2 + cd)$ (which is linear in $d$), when applied to US constraints.

Referring to the classes of max-closed and min-closed constraints, [15] showed that their solutions can be computed in $O(e^2d^2\Delta^2)$ time where $\Delta \leq d$ is the maximum number of supports that a value in a domain can have. In this paper, we show that a constraint is a DS if and only if it is both max-closed and min-closed. Our DS algorithms have significantly lower time complexities for this restricted subclass.

4 Properties of GS Constraints

4.1 Relationships with other Constraint Classes

In this section, we discuss the relationship of GS constraints with the existing constraint classes. Figure 1 illustrates the space of GS constraints. Since some of the relationships of GS constraints shown in this figure have already been covered so far, we next explain those that have not yet been discussed.

Following [15], a constraint $C$ is max-closed if and only if $\forall (u,v), (u',v') \in C$, implies $(\max\{u,u'\}, \max\{v,v'\}) \in C$. Similarly, a constraint $C$ is min-closed if and only if $\forall (u,v), (u',v') \in C$, implies $(\min\{u,u'\}, \min\{v,v'\}) \in C$. The authors in [15] claimed that if a constraint is both max-closed and min-closed, then it is row convex. If a constraint is either a $(\leq, \geq)$-monotone or $(\geq, \leq)$-monotone, then it is both max-closed and min-closed. We integrate these claims in the following result.

Theorem 1. A constraint is down staircase if and only if it is both max-closed and min-closed.

Proof. Suppose $C_{ij}$ is a constraint that is both max-closed and min-closed, and therefore, row convex [15]. For any $u, w \in D_i$ such that $w = \text{succ}(u)$, the images of $u, w$ in $D_j$ are intervals, say $[u, v]$ and $[v', w']$. Since $u < w$, we claim that $v \leq v'$ and $w \leq w'$. If $v' < v$, then using the min-closed property $(u,v') \in C_{ij}$. This contradicts the assumption that the image of $u$ in $D_j$ is $[v, w]$. If $w' < w$, then using the max-closed property $(u', w') \in C_{ij}$, thus contradicting the assumption that $u'$ has the image $[v', w']$ in $D_j$. Hence, $C_{ij}$ must be a down staircase constraint.

Now, suppose $C_{ij}$ is a down staircase constraint. From the down staircase property, for any $u, w \in D_i$ such that $u < w$, if the images of $u, w$ in $D_j$ are $[u, w]$ and $[v', w']$, then $v \leq v'$ and $w \leq w'$. If $w < w'$, both max-closed and min-closed properties are trivially satisfied.
Next, consider the case $v \leq v' < w \leq w'$, and any $y, y' \in D_j$ such that $v \leq y \leq w$ and $v' \leq y' \leq w$. If $y \leq y'$, then the min-closed and max-closed properties are trivially satisfied for any pair $(u, y), (u', y') \in C_{ij}$. Hence, assume $y > y'$. However, such $y'$ exists only for $\text{succ}(v') \leq y \leq w$. Observe that $(u, y') \in C_{ij}$ for $v' \leq y' < y$. Also $(u', y) \in C_{ij}$ for any $v' \leq y' < y \leq w$. Hence, both min-closed and max-closed properties are satisfied for any pair $(u, y), (u', y') \in C_{ij}$.

Figure 5 shows the intersection space of max-closed and min-closed constraints as DS constraints. Now we consider some of the other relationships shown in the figure. We note that there are CRC constraints that are not GS, but min-closed (Figure 5(a)), or max-closed (Figure 5(b)). From Figure 5(c) and Figure 5(d), we also see the existence of constraints that are either min-closed or max-closed, but not CRC. Figure 5(e) and Figure 5(f) illustrate existence of GS constraints that are both DS and US. While Figure 5(e) is not a staircase constraint, Figure 5(f) is staircase. Figure 4 shows that any staircase constraint is either max-closed, or min-closed, or both. This implies that any US constraint that is also a staircase constraint, is either max-closed or min-closed, but not both. We also claim that if a US constraint is either a max-closed or a min-closed constraint, it must be a staircase constraint. However, we note that there are US constraints that are neither max-closed nor min-closed. Figure 2(b) illustrates this fact.

4.2 Closure Properties of GS Constraints

Following [11], we know that CRC constraints are closed under the following operations: transposition, intersection and composition. Transposition of the constraint matrix implies that if $(v_i, v_j) \in C_{ij}$, then $(v_j, v_i) \in C_{ji}$. Intersection of two binary constraints $C_{ij}$ and $C'_{ij}$ implies that $(v_i, v_j) \in C_{ij} \cap C'_{ij}$ if and only if $(v_i, v_j) \in C_{ij} \land (v_i, v_j) \in C'_{ij}$. Composition refers to multiplication of two constraint matrices, i.e., given two binary constraints $C_{ij}$ and $C_{jk}$, the constraint $C_{ik} = C_{ij} \times C_{jk}$ is such that $(v_i, v_k) \in C_{ik}$ if and only if $\exists v_j (v_i, v_j) \in C_{ij} \land (v_j, v_k) \in C_{jk}$.

These closure properties are necessary for path consistency which ensures tractability of CRC constraints. In this section, we discuss the closure properties for the subclasses of GS constraints, namely DS and US constraints. We show that DS constraints are closed under transposition, intersection and composition. US constraints are, however, closed under transposition and intersection, but not under composition. In fact composition of two US constraints result in a DS constraint.

Lemma 1. DS constraints are closed under transposition.

Proof. Suppose $C_{ij}$ is a DS constraint, and $C_{ji}$ is its transpose.

From Theorem 1 we know that $C_{ij}$ is both min-closed and max-closed. Hence, for any two tuples $(a, b), (c, d) \in C_{ij}$, $(\min\{a, c\}, \min\{b, d\}) \in C_{ij}$ and $(\max\{a, c\}, \max\{b, d\}) \in C_{ij}$. From the definition of transposition, for every $(u, v) \in C_{ij}$, $(v, u) \in C'_{ji}$. Therefore, for any two tuples $(b, a), (d, c) \in C_{ji}$, $(\min\{b, d\}, \min\{a, c\}) \in C_{ji}$ and $(\max\{b, d\}, \max\{a, c\}) \in C_{ji}$. Hence, $C_{ji}$ is both min-closed and max-closed. From Theorem 1 we conclude that $C_{ji}$ is a DS constraint.

Lemma 2. DS constraints are closed under intersection.

Proof. Suppose $C_{ij}, C'_{ij}$ are two DS constraints on the same domains $D_i$ and $D_j$. Assume that $A_{ij} = C_{ij} \cap C'_{ij}$ denote the intersection of $C_{ij}$ and $C'_{ij}$. Suppose $A_{ij}$ is not a DS constraint.

Consider any two non-empty rows of $A_{ij}$, $u, v \in D_i$ such that $u < v$. Suppose $u, v$ have the respective images $[a, b], [c, d]$ in $C_{ij}$. Similarly, let $u, v$ have images $[a', b'], [c', d']$ respectively in $C'_{ij}$. Note that $\min(A_{ij}, u) = \max\{a, a'\}$ and $\max(A_{ij}, u) = \min\{b, b'\}$. Similarly, $\min(A_{ij}, v) = \max\{c, c'\}$ and $\max(A_{ij}, v) = \min\{d, d'\}$.

From the closure properties of CRC constraints in [11], $A_{ij}$ must be a CRC constraint. For $C_{ij}$, not to be a DS constraint, either of the following two cases must hold: (1) $\min(A_{ij}, u) > \max(A_{ij}, v)$. From the DS property of $C_{ij}$, $a \leq c$. Similarly, $a' \leq c'$. This, however, contradicts $\min(A_{ij}, u) = \max\{a, a'\} > \max\{c, c'\} =
\( \min(A_{ij}, v) \).

(2) \( \max(A_{ij}, u) > \max(A_{ij}, v) \). From the DS property of \( C_{ij}, b \leq d \) and \( b' \leq d' \). This, however, contradicts \( \max(A_{ij}, u) = \min(b, b') > \min(d, d') = \max(A_{ij}, v) \).

\[ \blacksquare \]

**Lemma 3.** **US constraints are closed under transposition.**

**Proof.** Suppose \( C_{ij} \) is a US constraint, and \( C_{ji} \) is its transpose. Suppose \( C_{ji} \) is not a US constraint.

Consider any two rows of matrix representation of \( C_{ji} \), say \( u, v \in D_j \) such that \( u < v \). Suppose the images of \( u, v \) in \( C_{ji} \) are \([a, b]\) and \([c, d]\) respectively where \( a, b, c, d \in D_i \). From the closure properties of CRC constraints in \([11]\), \( C_{ji} \) must be a CRC constraint. For \( C_{ji} \) not to be a US constraint, either of the following two cases must hold:

1. \( a < c \). From the definition of \( \max \) values, \( v \leq \max(c) \). Since \( a < c = \min(v) \) and \( C_{ij} \) is row convex, \( \max(a) < v \). From the above relations, it follows that \( \max(a) < \max(c) \). This, however, contradicts the US property of \( C_{ij} \).

2. \( b < d \). From the definition of \( \min \) values, \( \min(b) \leq u \). Since \( b = \max(u) < d \) and \( C_{ij} \) is row convex, \( u < \min(d) \). From the above relations, it follows that \( \min(b) < \min(d) \). This, however, contradicts the US property of \( C_{ij} \).

\[ \blacksquare \]

**Lemma 4.** **US constraints are closed under intersection.**

**Proof.** Suppose \( C_{ij}, C'_{ij} \) are two US constraints on the same domains \( D_i \) and \( D_j \). Assume that \( A_{ij} = C_{ij} \cap C'_{ij} \) denote the intersection of \( C_{ij} \) and \( C'_{ij} \). Suppose \( A_{ij} \) is not a US constraint.

Consider any two non-empty rows of \( A_{ij}, u, v \in D_i \) such that \( u < v \). Suppose \( u, v \) have images \([a, b]\), \([c, d]\) respectively in \( C_{ij} \) and \([a', b']\), \([c', d']\) respectively in \( C'_{ij} \). Note that \( \min(A_{ij}, u) = \min(a, a') \) and \( \max(A_{ij}, u) = \min(b, b') \). Similarly, \( \min(A_{ij}, v) = \max(c, c') \) and \( \max(A_{ij}, v) = \min(d, d') \).

From the closure properties of CRC constraints in \([11]\), \( A_{ij} \) must be a CRC constraint. For \( C_{ij} \) not to be a US constraint, either of the following two cases must hold:

1. \( \min(A_{ij}, u) < \min(A_{ij}, v) \). From the US property of \( C_{ij}, a \geq c \). Similarly, \( a' \geq c' \). This, however, contradicts \( \min(A_{ij}, u) = \max(a, a') < \min(c, c') = \min(A_{ij}, v) \).

2. \( \max(A_{ij}, u) < \max(A_{ij}, v) \). From the US property of \( C_{ij}, b \geq d \) and \( b' \geq d' \). This, however, contradicts \( \max(A_{ij}, u) = \min(b, b') < \min(d, d') = \max(A_{ij}, v) \).

\[ \blacksquare \]
min(C_{ik}, v) and max(C_{ik}, u) \leq \max(C_{ik}, v). Hence, $C_{ik}$ is a DS constraint.

Owing to monotonicity and row convexity, GS constraints admit much simpler and efficient algorithms to compute the above operations, as opposed to those for CRC constraints. From the arguments given in the above proofs, it is easy to see that each of these operations can be performed in $O(d)$ space and time for both DS and US constraints. Since DS constraints are solvable using arc consistency (as described in Section 5), we do not need these closure properties. However, for US constraints, if we use either of the existing approaches for CRC constraints (mentioned in Section 5), we require these properties, and hence, lies the advantage of efficient algorithms to compute these operations.

5 Arc Consistency for GS Constraints

In this section, we first show that arc consistency is sufficient to determine a solution to a DSCSP, following which we present an algorithm, ACiDS (Arc Consistency for Down Staircase constraints), to compute arc consistency for a DS CSP. We also discuss how the same algorithm can be used for computing arc consistency for a US CSP.

5.1 Sufficiency of Arc Consistency for Solving DSCSP

Theorem 2. Arc consistency is sufficient to determine a solution to a DSCSP.

Proof. Given a DSCSP $P$, suppose $\forall i, D_i, D'_i$ respectively denote the domain of $X_i$ before and after achieving arc consistency, and $\forall i, f_i, l_i$ denote the first and last values of $D'_i$ respectively ($f_i \leq l_i$). We claim that the sets $S_f = \{f_1, f_2, \ldots, f_n\}$ and $S_l = \{l_1, l_2, \ldots, l_n\}$ are solutions to $P$.

Suppose $S_f$ is not a solution. Then there must exist a constraint $C_{ij}$ such that $(f_i, f_j) \notin C_{ij}$. Since $f_i \in D'_i$, there exists a smallest available support say $v_j \in D'_j$ such that $(f_i, v_j) \in C_{ij}$. Similarly, since $f_j \in D'_j$, there exists a smallest available support say $v_i \in D'_i$ such that $(v_i, f_j) \in C_{ij}$. From the definition of $f_i, f_j$, it follows that since $(f_i, f_j) \notin C_{ij}$, hence $f_i < v_i$ and $f_j < v_j$. Since $C_{ij}$ is min-closed, $(\min\{f_i, v_i\}, \min\{f_j, v_j\}) \in C_{ij}$, which is a contradiction.

Therefore, $S_f$ is a solution to $P$. Similarly, we can show that $S_l$ is also a solution to $P$. □

5.2 ACiDS Algorithm

5.2.1 Algorithm Overview

We now present the algorithm ACiDS (pseudocode in Algorithm 2) that computes arc consistency for a DSCSP, following which we use either of the existing approaches for CRC constraints. From the arguments given in the above 5.1 Sufficiency of Arc Consistency for Solving DSCSP, we observe that since $C_{ij}$ is a DS constraint, a solution to $C_{ij}$ can be obtained by using the AC algorithm for a DS constraint. However, for US constraints, if $C_{ij}$ is a US constraint, then the tuple $(k, j, v_j)$ is added into a queue $Q$ which essentially holds values due to constraint propagation while pruning. After dequeuing the tuple $(k, j, v_j)$ at a later stage, it checks whether $v_j$ was the last available support for any $v_k \in D_k$, and puts all such values into $Q$. The process terminates when $Q$ becomes empty.

5.2.2 The MIN Data Structure

For efficiently computing the set of values in $D_i$ for which $v_j$ is the only support, we introduce a new data structure $MIN$. We define $MIN(v_j, i)$ as the set of values in $D_i$ for which $v_j$ is the smallest available support in $D_j$ over the constraint $C_{ij}$. Initially, we set $MIN(v_j, i) = \{v_i | v_j = \min(C_{ij}, v_i)\}$. If $\hat{v}_i \in D_i$ such that $v_j = \min(C_{ij}, v_i)$, then $MIN(v_j, i) = \phi$. In Figure 5(a), $MIN(b_1, i) = [a_1, a_2], MIN(b_3, i) = [a_3, a_4], MIN(b_5, i) = [a_5]$, and $MIN(b_2, i) = MIN(b_3, i) = \phi$. Due to the DS property of $C_{ij}$, the set $MIN(v_j, i)$ can be represented as an interval. To illustrate this, suppose $MIN(v_j, i) = \{v_1, v_3, v_6\}$ where $v_i \in D_i$ for $1 \leq i \leq 6$. Since $C_{ij}$ is a DS constraint and $\min(v_1) = \min(v_3) = \min(v_6)$, it implies that $v_2, v_4$ and $v_5$ does not have any support in $D_j$ over the constraint $C_{ij}$. Hence, we can also denote $MIN$ in an interval form as $MIN(v_j, i) = [v_1, v_6]$ which essentially marks the boundaries of $MIN(v_j, i)$.
Algorithm 1 ACiDS

1: procedure MAIN($P = (X, D, C)$)  
2: Initialize()  
3: for each $C_{ij} \in \mathcal{C}$ do  
4:    ArcCons(i, j, $\Delta$)  
5:  Enqueue(i, $\Delta$, $Q$)  
6:  Remove($\Delta$, $D_i$)  
7: while $Q \neq \emptyset$ do  
8:    Dequeue(i, j, v, $Q$)  
9:  LocalArcCons(i, j, v, $\Delta$)  
10:  Enqueue(i, $\Delta$, $Q$)  
11:  Remove($\Delta$, $D_i$)

1: procedure INITIALIZE  
2: $Q \leftarrow \{\}$  
3: $\forall v_j \in D_j$, $\forall C_{ij} \in \mathcal{C}$, $MIN(v_j, i) \leftarrow \phi$  
4: for each $C_{ij} \in \mathcal{C}$ do  
5:    $\delta_{SCAN} \leftarrow \text{next value of } D_i$, $\delta_{SCAN}$ respectively  
6:    while $\delta_{SCAN} \neq \phi \land \delta_{SCAN} \neq \phi$ do  
7:    while $\min(\delta_{SCAN}) > \delta_{SCAN}$ do  
8:      $\delta_{SCAN} \leftarrow \min(\delta_{SCAN})$  
9:      $MIN(\delta_{SCAN}, i) \leftarrow MIN(\delta_{SCAN}, i) \cup \{\delta_{SCAN}\}$  
10:      scan_i \leftarrow $\delta_{SCAN}$  
11: procedure ENQUEUE(i, $\Delta$, $Q$)  
12: for each $v_i \in D_i$ do  
13:  $Q \leftarrow Q \cup \{(j, i, v_i) \mid \forall (j, i) \in \mathcal{P}\}$  
14: procedure DEQUEUE(i, j, v, $Q$)  
15: if $\phi \neq \phi$ then  
16:  $Q \leftarrow Q \setminus \{(i, j, v)\}$  
17: procedure REMOVE(v, $D_i$)  
18: for each $C_{ij} \in \mathcal{C}$ do  
19:  if $v \in D_i(C_{ij})$ then  
20:    $\text{pred}(\text{succ}(v, D_i(C_{ij}))) \leftarrow \text{pred}(\text{succ}(v, D_i(C_{ij})))$  
21:  $\text{succ}(\text{pred}(v, D_i(C_{ij}))) \leftarrow \text{succ}(\text{pred}(v, D_i(C_{ij})))$  
22:  $D_i(C_{ij}) \leftarrow D_i(C_{ij}) \setminus \{v\}$  
23:  $D_i \leftarrow D_i \setminus \{v\}$

Algorithm 2 ArcCons and LocalArcCons

1: procedure ArcCons($i, j, \Delta$)  
2:  $\Delta \leftarrow \{v_i \in D_i \mid [v_j \in D_j : (v_j, v_i) \in C_{ij}]\}$  
3: procedure LocalArcCons($i, j, v_j, \Delta$)  
4:  if $MIN(v_j, i) = \phi$ then  
5:  return  
6:  $\Delta \leftarrow \phi$  
7:  $MIN(v_j, i) = [v_{i_1}, v_{i_2}]$ for some $v_{i_1}, v_{i_2} \in D_i$  
8:  $v'_{ij} \leftarrow \text{succ}(v_j)$  
9:  $\delta_{SCAN} \leftarrow v_{i_1}$  
10: while $(v'_{ij} = \phi \lor \max(C_{ij}, \delta_{SCAN}) < v'_{ij}) \land (\delta_{SCAN} \leq v_{i_2})$ do  
11:  $\text{pre}(\text{succ}(\text{succ}(\text{pre}(v_{ij}))) \leftarrow \text{pre}(\text{succ}(\text{pre}(v_{ij}))))$  
12:  $MIN(v'_{ij}, i) \leftarrow MIN(v'_{ij}, i) \cup [\delta_{SCAN}, v_{i_2}]$

The algorithm begins with initialization of data structures: queue $Q$ to an empty queue and $MIN$ as per the definition stated above. Following this, $ArcCons(i, j, \Delta)$ is called for each constraint $C_{ij} \in \mathcal{C}$. $ArcCons(i, j, \Delta)$ identifies those values $v_i \in D_i$ such that $v_j \notin D_i(C_{ij})$ and adds them to the set $\Delta$. Further, $Enqueue(i, j, \Delta, Q)$ queues tuples of the form $(j, i, v_j)$ into $Q$ for all $(arc(i, j, \Delta)) \in \mathcal{P}$ and all $v_j \in D_j$. Then, the call to $Remove(\Delta, D_i)$ removes all $v_i \in D_i$ from $D_i$ and $D_i(C_{ij})$ for each constraint $C_{ij}$. In addition, it resets the $\text{pred}$ and $\text{succ}$ values for the preceding and succeeding values of $v_i$ in $D_i$.

The algorithm continues until $Q$ becomes empty. Within the while loop (lines 10-11), a tuple $(i, j, v_j)$ is dequeued from $Q$ using $Dequeue(i, j, v_j, Q)$ and then the subroutine $LocalArcCons(i, j, v_j, \Delta)$ is invoked.

The heart of the ACiDS algorithm lies in the subroutine $LocalArcCons(i, j, v_j, \Delta)$. Firstly, it determines the set $\Delta = \{v_i \in MIN(v_j, i)\mid \text{succ}(v_j) = \phi \lor \max(v_i) < \text{succ}(v_j)\}$ as the set of values $v_i \in D_i$ for which $v_j \notin D_j$ was the only available support. Since the constraint is a DS, these values of $\Delta$ can be identified by sequentially accessing the values in the $MIN$ set using the $\text{succ}$ values, without visiting any $v \notin \Delta$. Hence, this step requires at most $O(\Delta)$ operations. Secondly, if $v_{ij}' = \text{succ}(v_j) \neq \phi$, it adds the set $\{MIN(v_j, i) - \Delta\}$ to the set $MIN(v_{ij}')$. This is to ensure that those values $v_i \in D_i$ for which $v_j$ was the smallest available support in $D_j$ but not the only remaining support, will henceforth have $v_{ij}'$ as the smallest available support. Since $MIN$ sets are stored as disjoint intervals, this step can be completed in $O(1)$ time. It is interesting and important to note here that if $v_j$ is not the smallest support for any $v_i$, i.e., $MIN(v_j, i) = \phi$, then no data structure needs to be updated and $LocalArcCons()$ returns immediately.

exploits this disjoint interval property of $MIN$ sets. During the course of the algorithm, the $MIN$ sets and the $\text{pred}$, $\text{succ}$ values get updated, while $\min$ and $\max$ values remain unchanged.

5.2.3 Re-visiting ArcCons() and LocalArcCons()

We re-design the generic subroutines $ArcCons()$ and $LocalArcCons()$, stated in ACS5, specifically for DSCSP in Algorithm 2.

If $v_i \notin D_i(C_{ij})$, i.e., $v_i$ has no support in $D_j$ over the constraint $C_{ij}$, then we define $\min(C_{ij}, v_i) = \max(C_{ij}, v_i) = \phi$. We use pointer $\delta_{SCAN}$ to scan the values of $D_i$. When $\delta_{SCAN}$ points to $v_i$, we say $\delta_{SCAN} = v_i$. Further, $\delta_{SCAN} = \phi$ if and only if $\delta_{SCAN}$ reaches the end of domain $D_i$. Similarly, we assume that $\text{succ}(v_i) = \phi$ if and only if $v_i$ is the largest value in $D_i$. 


5.2.4 Correctness of ACiDS Algorithm

In order to establish the correctness of the ACiDS algorithm which is a specialization of AC5, we essentially need to show the correctness of LocalArcCons(). In order to do so, we first show that \( MIN(v_{ij}, i) \) always retains the set of values in \( D_i \) for which \( v_i \) is the smallest available support in \( D_j \) over the constraint \( C_{ij} \).

**Lemma 7.** At the end of any iteration of \( LocalArcCons(i, j, v_j, \triangle) \) over a given ordered pair \((i, j)\), for any \( v_i \in D_i \setminus \triangle \) and \( v_j \in D_j \), \( v_i \in MIN(v_j, i) \) if and only if \( v_j \) is the smallest available support for \( v_i \) over the constraint \( C_{ij} \).

**Proof.** We prove this lemma by applying induction on iteration \( r \).

Consider any \( v_i \in D_i \). Suppose \( min(C_{ij}, v_i) = v_j \). Note that during initialization, \( v_i \in MIN(v_j, i) \). Now, consider the call to \( LocalArcCons(i, j, v_j, \triangle) \) for the first time \((r = 1)\) over the pair \((i, j)\). If \( v_{ij} \neq v_j \), \( v_i \) continues to be in \( MIN(v_j, i) \) and \( v_j \) continues to be the smallest available support for \( v_i \). If, however, \( v_{ij} = v_j \), then the following two cases arise:

1. If \( succ(v_j) = \phi \) or \( max(C_{ij}, v_i) < succ(v_j) \), then using the DS property, it implies that \( v_i \) lost its only support \( v_j \) and is, thus, inserted to \( \triangle \). In this case, the hypothesis is no longer applicable as \( v_i \notin D_i \setminus \triangle \).

2. If \( max(C_{ij}, v_i) \geq succ(v_j) \) such that \( succ(v_j) \neq \phi \), \( v_j \) is added to \( MIN(v_j', i) \) (line 12) where \( v_j' = succ(v_j) \). By definition of \( succ() \), \( v_j' \) is the next greatest value after \( v_j \) in \( D_j(C_{ij}) \). Since \( v_j = min(C_{ij}, v_j) \) is already pruned from \( D_j \), \( v_j' \) is the smallest available support for \( v_i \) over the constraint \( C_{ij} \).

Hence, the hypothesis holds for \( r = 1 \).

Suppose the hypothesis holds for \( r = p \). Now consider the call to \( LocalArcCons(i, j, v_j, \triangle) \) for the \( r = p+1 \) iteration over the pair \((i, j)\). If \( v_{ip} \neq v_j \), then following the induction hypothesis, \( v_i \) continues to be in \( MIN(v_j, i) \) and \( v_j \) continues to be the smallest available support for \( v_i \). If, however, \( v_{ip} = v_j \), then the following two cases arise:

1. If \( succ(v_j) = \phi \) or \( max(C_{ij}, v_i) < succ(v_j) \), then using the DS property, it implies that \( v_i \) lost its only support \( v_j \) and is, thus, inserted to \( \triangle \). In this case, the hypothesis is no longer applicable as \( v_i \notin D_i \setminus \triangle \).

2. If \( max(C_{ij}, v_i) \geq succ(v_j) \) such that \( succ(v_j) \neq \phi \), \( v_j \) is added to \( MIN(v_j', i) \) (line 12) where \( v_j' = succ(v_j) \). By definition of \( succ() \), \( v_j' \) is the next greater value after \( v_j \) in \( D_j(C_{ij}) \). Following the induction hypothesis, for \( r = p+1 \), since \( v_j \) was the smallest available support for \( v_i \) over the constraint \( C_{ij} \) and now \( v_j \) is already pruned from \( D_i \), \( v_j' \) is the smallest available support for \( v_i \) at the end of the \( r = p^{th} \) induction step.

Thus, the induction hypothesis follows.

The next lemma proves the correctness of the subroutine \( LocalArcCons() \).

**Lemma 8.** \( LocalArcCons(i, j, v_j, \triangle) \) correctly computes the set: \( \triangle = \{ v_i \in D_i | (v_i, v_j) \in C_{ij}, \forall v_j' \in D_j, (v_i, v_j') \notin C_{ij} \} \).

**Proof.** From Lemma 7, \( MIN(v_j, i) = \phi \), if and only if \( v_j \) is not the smallest available support for any \( v_i \) in \( D_i \). Hence, pruning \( v_j \) would not make any \( v_i \) inconsistent. Therefore, the algorithm returns immediately.

Now consider the case \( MIN(v_j, i) \neq \phi \). Note that any \( v_i \) that is supported by \( v_j \) has \( v_j \) as the only available support in \( D_j \) that can potentially support any \( v_i \).

Hence each \( v_i \in MIN(v_j, i) \) is added to \( \triangle \).

(2) If \( succ(v_j') \neq \phi \), then suppose \( MIN(v_j, i) = [v_{i1}, \ldots, v_{i2}, v_{i3}, \ldots, v_{ik}] \), such that \( max(v_{i2}) < succ(v_j') \) and \( max(v_{i3}) \geq succ(v_j') \). Consider the set of values \( v_i \in [v_{i2}, v_{i3}] \). From Lemma 7, \( v_j \) is the smallest available support for each of these \( v_i \). Further, since \( C_{ij} \) is a DS, \( max(v_i) < succ(v_j') \). Hence after pruning \( v_j \), \( v_j \) has no support in \( D_j \) and is thus added to \( \triangle \). Also observe that since \( max(v_{i3}) \geq succ(v_j') \), any \( v'_j \in [v_{i2}, v_{i3}] \) has \( succ(v_j') \) as a valid support after pruning \( v_j \). Hence, \( v'_j \) is added to \( \triangle \). 



5.2.5 Complexity of ACiDS Algorithm

We first analyze the time complexity of ACiDS. The subroutine \( Initialize() \) runs in \( O(cd) \) time. For each constraint \( C_{ij} \in C \), this procedure simultaneously scans the domains \( D_i \) and \( D_j \) without revisiting any value. Next, we note that \( ArcCons(i, j, \triangle) \) runs in \( O(d) \) time. From the description of \( LocalArcCons() \) in Section 5.2 we argue that it requires \( O(\triangle) \) time.

It was stated in [14] that if the time complexity of \( ArcCons() \) is \( O(d) \) and that of \( LocalArcCons() \) is \( O(\triangle) \), then the algorithm AC5 runs in \( O(cd) \) time. Since the \( ArcCons() \) and \( LocalArcCons() \) in ACiDS take \( O(d) \) and \( O(\triangle) \) time, we conclude that ACiDS also runs in \( O(cd) \) time.

Analyzing the space complexity, we note that since each DS constraint can be stored as a list of intervals, the total input space requirement for a DSCSP is \( O(cd) \). During the execution, since any \( v_i \in D_i \) is enqueued at most
once for each arc \((j, i) \in \mathcal{A}(P)\), the queue length of \(Q\) is at most \(O(cd)\). By a similar argument, the \(MIN\) data structure takes at most \(O(cd)\) space. Since at any point, \(\Delta\) holds values of any one domain, \(\Delta\) takes no more than \(O(d)\) space. Hence, the ACiDS algorithm uses at most \(O(cd)\) space.

The \(O(cd)\) space and time complexity is optimal for any CSP with \(c\) constraints, since each constraint needs \(\Omega(d)\) space, and at least one scan of any domain requires \(\Omega(d)\) time. The ACiDS algorithm improves the complexity bound of \(O(cd^2)\) in [36] for the subclass DS of CRC constraints.

5.3 Arc Consistency for USCSP

In this section, we show that with minor modifications in the way we scan the domains in the subroutines, \(INITIALIZE()\) and \(LOCALARCCONS()\), the ACiDS algorithm can compute arc consistency for a USCSP. The modified subroutines are presented in Algorithm 3. In addition to the assumptions stated in the ACiDS algorithm, we assume \(scan_i = \phi\) if and only if \(scan_i\) reaches the end of the domain on either side, and \(pred(v_i) = \phi\) if and only if \(v_i\) is the smallest value of the domain. It is easy to follow that arc consistency for a USCSP can also be achieved with the same time and space complexity of \(O(cd)\) as in the case of DSCSP. However, since arc consistency is not known to be sufficient to determine a solution to a USCSP, techniques such as path consistency or variable elimination are required. Since arc consistency is a basic subroutine in these techniques, replacing that by our modified ACiDS algorithm produces a faster solution. It improves the running time of \(O(\sigma^2nd + cd^2)\) (where \(\sigma \leq n\) is a problem specific parameter) of the best known algorithm [36] to \(O(\sigma^2nd + cd)\), which is linear in \(d\).

6 A Faster Solver for DSCSP

This section presents a more efficient algorithm DSCSP SOLVER (Algorithm 4) to solve a DSCSP than ACiDS. We observe that a solution to a DSCSP can be found even without achieving arc consistency. Exploiting the DS property of the DS constraints and the total ordering of the variable domains, the algorithm incrementally scans the variable domains in a manner such that no value is revisited. At each point, we check whether the values being scanned form a solution. Only when it is confirmed that the current value cannot form a solution is the next value scanned.
The algorithm maintains a queue \( Q \) of variable indices. A variable index \( i \in Q \) if and only if it is needed to check whether \( (\text{scan}_i, \text{scan}_j) \in C_{ij} \) for all the constraints \( C_{ij} \) imposed on \( X_i \). The algorithm also uses pointers \( \text{scan}_i \) and boolean variables \( \text{flag}_i \) for each variable \( X_i \in X \).

The variable \( \text{sum} \) maintains the sum of all the \( \text{flag} \) variables. As stated earlier, we assume that for each pair of variables \((X_i, X_j)\), there is at most one constraint \( C_{ij} \), and we need not consider its transpose \( C_{ji} \) separately. (In the AGiDS algorithm, we considered both \( C_{ij}, C_{ji} \in C \).)

DSCSP Solver first calls \( \text{Initialize}() \) for initializing the \( \text{scan}_i \) pointer to point to the first value of \( D_i \) and \( \text{flag}_i \) to 0 for each \( X_i \in X \). The queue \( Q \) is initialized to contain all the variable indices.

Then, the algorithm proceeds in iterations. In each iteration, we dequeue a variable index \( j \) and consider all the constraints \( C_{jk} \in C \). We keep incrementing \( \text{scan}_j \) (and \( \text{scan}_k \)) until it reaches a value which has a support in \( D_k (D_j \), respectively). If \( \text{scan}_j < \text{min}(\text{scan}_k) \), we set \( \text{scan}_j = \text{min}(\text{scan}_k) \) and reconsider all the constraints imposed on \( X_j \). Also, if \( \text{scan}_k < \text{min}(\text{scan}_j) \), we set \( \text{scan}_k = \text{min}(\text{scan}_j) \). Further, if \( \text{flag}_j = 1 \), we reset it to 0 and add \( k \) to \( Q \). Since \( \text{sum} \) maintains \( \sum_{i=1}^{n} \text{flag}_i \), \( \text{sum} \) is decremented by 1 as a consequence. At the end of the for loop (lines 5-18), if \( \text{scan}_j, \text{scan}_k \neq \phi \), then for \( \text{scan}_j \), every constraint \( C_{jk} \) imposed on \( X_j \) is satisfied by \( \text{scan}_k \). As a result, \( \text{flag}_j \) moves from 0 to 1 and, therefore, \( \text{sum} \) is incremented by 1. Whenever \( \text{sum} = n \), it implies \( \forall j \ \text{flag}_j = 1 \), which indicates that \( (\text{scan}_j, \text{scan}_k) \in C_{jk} \) for all constraints \( C_{jk} \in C \). This in return signifies that at this point, the assignment \( X_i = \text{scan}_i \) is a solution to the DSCSP.

### 6.1 Correctness of DSCSP Solver

To prove the correctness of the algorithm, we first show that the answer reported by the algorithm is indeed a solution. The following simple claims lead to this fact.

**Lemma 9.** At the end of the for loop in lines 5-18, \( \text{flag}_j = 1 \ \forall j \) if and only if \( j \notin Q \).

**Proof.** Consider a variable index \( j \in Q \). After \( j \) is dequeued from \( Q \), the for loop in lines 5-18 is processed. At the end of the for loop, \( \text{flag}_j \) is set to 1. On the other hand, before any variable index \( k \) is added to \( Q \) in line 8, \( \text{flag}_k \) is decremented to 0 in line 16.

**Lemma 10.** At any stage of the algorithm, \( \text{sum} = \sum_{i=1}^{n} \text{flag}_i \).

**Proof.** From the previous lemma, it is clear that when any variable index \( j \) is dequeued from \( Q \) in line 19, \( \text{flag}_j = 1 \). Later, \( \text{flag}_j \) is set to 1 in line 19. And immediately, \( \text{sum} \) is also incremented. On the other hand, when for any variable index \( i \), whenever \( \text{flag}_k \) is decremented from 1 to 0 in line 16, \( \text{sum} \) is also decremented by 1.

**Lemma 11.** When \( \text{flag}_j \) moves from 0 to 1 (in line 19), then \( (\text{scan}_j, \text{scan}_k) \in C_{jk} \) for each constraint \( C_{jk} \) on \( X_j \).

**Proof.** At a high level, the proof follows from the observation that before \( \text{flag}_j \) gets incremented in line 19 every constraint on \( X_j \) is verified to be true for the current value of the \( \text{scan} \) pointers.

Consider the for loop in lines 5-18. Consider any constraint \( C_{jk} \in C \). Note that during the execution of the algorithm, any \( \text{scan} \) pointer is never decremented. Following the down staircase property of \( C_{jk} \), it is clear that if \((\text{scan}_j, \text{scan}_k) \notin C_{jk} \) such that \( \text{scan}_j = D_j(C_{jk}) \) and \( \text{scan}_k = D_k(C_{kj}) \), either of the following two cases are possible but not both: (i) \( \text{scan}_j < \min(C_{jk}, \text{scan}_k) \), or (ii) \( \text{scan}_k < \min(C_{jk}, \text{scan}_j) \).

From the algorithm, in case (i), \( \text{scan}_j \) is incremented to \( \min(C_{kj}, \text{scan}_k) \). This ensures that \((\text{scan}_j, \text{scan}_k) \in C_{jk} \). However, since \( \text{scan}_j \) has been modified, we need to reconsider all the constraints \( C_{jk} \) on \( X_j \). Arguing similarly for case (ii), it is clear that at the end of the for loop in lines 5-18, when \( \text{flag}_j \) is set to 1, \((\text{scan}_j, \text{scan}_k) \in C_{jk} \) for each constraint \( C_{jk} \) on \( X_j \).

**Corollary 1.** At any stage during the execution of the algorithm, (i) for any constraint \( C_{jk} \), if both \( \text{flag}_j = 1 \) and \( \text{flag}_k = 1 \), then \((\text{scan}_j, \text{scan}_k) \in C_{jk} \), and (ii) if \( \text{sum} = n \), then there is a solution, namely, \( \forall i X_i = \text{scan}_i \).

Now we show that the algorithm would never miss a solution, if it exists. The key point to note here is that \( \text{scan}_i \) is not incremented until it is confirmed that the current value (being pointed by \( \text{scan}_i \)) cannot be a member of any solution to the DSCSP. The following lemma formally proves this claim.

**Lemma 12.** If \( v_i \in D_i \) is a member of a solution, then at every stage during the execution of the algorithm, \( \text{scan}_i \leq v_i \).

**Proof.** Suppose \( v_i \) is the first value which got crossed by \( \text{scan}_i \) during the execution of the algorithm even when it is a member of a solution \( S \). Suppose the solution is \( S = \{v_k, X_k = v_k \} \).

Consider the time when \( \text{scan}_i \) is about to be incremented from \( v_i \). Observe that at this time, \( \text{scan}_i = v_i \) and \( \text{scan}_j \leq v_j \) for all \( j \neq i \). The value \( v_i \) can get crossed if and only if for some constraint \( C_{ij} \), either \( v_i \notin D_i(C_{ij}) \) or \( v_i < \min(\text{scan}_j) \). If \( v_i \notin D_i(C_{ij}) \),
then $v_i$ cannot be a member of a solution, and hence, this is a contradiction. Considering the latter case, suppose $\text{scan}_j = v'_j < v_j$. Since $(v_j, v_i) \in C_{ji}$, $\min(v_j) \leq v_i$. However, $\min(v'_j) > v_i \geq \min(v_j)$ contradicts the DS property of $C_{ji}$.

Hence, we have shown that the solution reported by the DSCSP Solver is valid and that it would never miss to detect a solution, if one exists. This concludes the correctness analysis.

6.2 Complexity of DSCSP Solver

From the above results, it follows that if $\forall i, v_i \in D_i$ is the smallest value in $D_i$ which is a member of a solution, then the algorithm would report the solution $\forall i, X_i = v_i$. The complexity analysis leverages this fact.

Theorem 3. If $\delta_j$ is the degree of node $V_j$ (corresponding to variable $X_j$) in the constraint network, and $v_{s_j}$ is the smallest value in $D_j$ that is a member of a solution, then the algorithm DSCSP Solver takes time $O\left(\sum_{j=1}^{n} \delta_js_j\right) = O(cs)$ where $s = \max_j \{s_j\}$.

Proof. For a given variable $X_j$ that is dequeued from $Q$, the maximum number of iterations of the for loop in lines 5:15 is $s_j$ as the number of increments of $\text{scan}_j$ is at most $s_j - 1$. Now consider any given iteration of this for loop for the variable $X_j$. Suppose $\text{scan}_j = v_j$. Before $\text{scan}_j$ gets incremented during this iteration, $v_j$ is compared with at most $\text{scan}_k$ for all $k$ such that $C_{jk} \in C$. This implies that in this iteration, $v_j$ is compared with at most $\delta_j$ values. Hence, the result follows.

In the absence of any solution, the algorithm takes $O(cd)$ time because at least one of the $\text{scan}$ pointers must scan the entire domain. Analyzing the space complexity, besides the input which takes $O(cd)$ space, this algorithm requires only $O(n)$ additional space for storing the $\text{scan}$ pointers.

Comparing the DSCSP Solver with the ACiDS algorithm, the key advantages are the following: (i) DSCSP Solver is much simpler to implement, (ii) it need not scan the entire domain if there is a solution, and hence, offers an improved time complexity of $O\left(\sum_{j=1}^{n} \delta_js_j\right)$ as opposed to $O(cd)$ time in presence of a solution, and (iii) its additional space requirement is $O(n)$ as opposed to $O(cd)$ in the ACiDS algorithm.

7 Conclusions

This paper investigated the class of generalized staircase (GS) constraints, which generalizes the class of staircase constraints, while being a subclass of CRC constraints. We explored the properties of the two subclasses of GS, namely, down staircase (DS) and up staircase (US) constraints, besides studying their relationships with the existing constraint classes.

We proposed two algorithms for solving DSCSP, namely, ACiDS, based on arc consistency, and DSCSP Solver, based on incremental scan of domains. We showed that DSCSP Solver is more efficient as compared to ACiDS. For solving USCSP, we require either path consistency or variable elimination technique. Our optimal ACiDS algorithm acting as a subroutine in these techniques, offer a faster solution to USCSP.

In the future, more efficient arc consistency scheme for CRC constraints may be investigated. In addition, interesting generalizations of GS constraints and applications to real life problems will be explored.

References

[1] J. Agrawal, Y. Diao, D. Gyllstrom, and N. Immerman. Efficient pattern matching over event streams. In SIGMOD, pages 147–160, 2008.
[2] C. Bessi`ere. Refining the basic constraint propagation algorithm. In IJCAI, pages 309–315, 2001.
[3] C. Bessi`ere and M.-O. Cordier. Arc-consistency and arc-consistency again. In AAAI, pages 108–113, 1993.
[4] R. Bhargavi, V. Vaidehi, P. T. V. Bhuvaneswari, P. Balamuralidhar, and M. G. Chandra. Complex event processing for object tracking and intrusion detection in wireless sensor networks. In ICARCV, pages 848–853, 2010.
[5] L. Brenna, J. Gehrke, M. Hong, and D. Johansen. Distributed event stream processing with non-deterministic finite automata. In DEBS, pages 3:1–3:12, 2009.
[6] Q. Chen, Z. Li, and H. Liu. Optimizing complex event processing over rfid data streams. In ICDE, pages 1442–1444, 2008.
[7] X. Chuanfei, L. Shukuan, W. Lei, and Q. Jianzhong. Complex event detection in probabilistic stream. In APWEB, pages 361–363, 2010.
[8] G. Cugola and A. Margara. TESLA: a formally defined event specification language. In DEBS, pages 50–61, 2010.
[9] S. Deris, S. Omatu, and H. Ohta. Timetable planning using the constraint-based reasoning. *Computers & Operations Research*, 27(9):819–840, 2000.

[10] S. B. Deris, S. Omatu, H. Ohta, and P. A. B. D. Samat. University timetabling by constraint-based reasoning: A case study. *J. Operational Research Society*, 48(12):1178–1190, 1997.

[11] Y. Deville, O. Barette, and P. Van Hentenryck. Constraint satisfaction over connected row convex constraints. In *IJCAI*, volume 15, pages 405–411, 1997.

[12] C. B. Ere and E. C. Freuder. Using inference to reduce arc consistency computation. In *IJCAI*, pages 592–598, 1995.

[13] M. Hadjieleftheriou, N. Mamoulis, and Y. Tao. Continuous constraint query evaluation for spatiotemporal streams. In *STD*, pages 348–365, 2007.

[14] P. V. Hentenryck, Y. Deville, and C.-M. Teng. A generic arc-consistency algorithm and its specializations. *Artif. Intell.*, 57(2-3):291–321, 1992.

[15] P. Jeavons and M. C. Cooper. Tractable constraints on ordered domains. *Artif. Intell.*, 79(2):327–339, 1995.

[16] V. Kumar. Algorithms for constraint satisfaction problems: A survey. *AI Magazine*, 13(1):32–44, 1992.

[17] W. Legierski. Constraint-based reasoning for timetabling. *Artificial Intelligence Method: AI-METH*, Gliwice, Poland, 2002.

[18] A. Mackworth. Consistency in networks of relations. *Artificial Intelligence*, 8(1):99–118, 1977.

[19] S. Mitra, P. Dutta, S. Kalyanaraman, and P. Pradhan. Spatio-temporal patterns for problem determination in it services. In *SCC*, pages 49–56, 2009.

[20] R. Mohr and T. C. Henderson. Arc and path consistency revisited. *Artif. Intell.*, 28(2):225–233, 1986.

[21] U. Montanari. Networks of constraints: Fundamental properties and applications to picture processing. *Information Sciences*, 7(0):95 – 132, 1974.

[22] J. Pearson and P. Jeavons. A survey of tractable constraint satisfaction problems. Technical Report CSD-TR-97-15, Royal Holloway Univ. of London, 1997.

[23] S. Peng, Z. Li, L. Chen, Y. Nie, and Q. Chen. Complex event processing over multi-granularity rfid data streams. In *ICCSIT*, pages 235–239. IEEE, 2009.

[24] P. R. Pietzuch, B. Shand, and J. Bacon. Composite event detection as a generic middleware extension. *IEEE Network*, 18(1):44–55, 2004.

[25] F. Rossi, C. Petrie, and V. Dhar. On the equivalence of constraint satisfaction problems. In *ECAI*, pages 550–556, 1990.

[26] N. P. Schultz-Møller, M. Migliavacca, and P. Pietzuch. Distributed complex event processing with query rewriting. In *DEBS*, pages 4:1–4:12, 2009.

[27] K. Teymourian, O. Streibel, A. Paschke, R. Alnemr, and C. Meinel. Towards semantic event-driven systems. In *NTMS*, pages 1–6, 2009.

[28] E. P. K. Tsang. *Foundations of constraint satisfaction*. Academic Press, 1993.

[29] C. Vairo, G. Amato, S. Chessa, and P. Valleri. Modeling detection and tracking of complex events in wireless sensor networks. In *SMC*, pages 235–242, 2010.

[30] P. van Beek and R. Dechter. On the minimality and decomposability of row-convex constraint networks. *J. ACM*, 42(3):543–561, 1995.

[31] K. Walzer, T. Breddin, and M. Groch. Relative temporal constraints in the Rete algorithm for complex event detection. In *DEBS*, pages 147–155, 2008.

[32] F. Wang, S. Liu, P. Liu, and Y. Bai. Bridging physical and virtual worlds: complex event processing for rfid data streams. In *EDBT*, pages 588–607, 2006.

[33] T. Wang, M. Srivatsa, D. Agrawal, and L. Liu. Spatio-temporal patterns in network events. In *CoNEXT*, pages 3:1–3:12, 2010.

[34] Y. Wang, J. Cai, and S. Yang. Plan based parallel complex event detection over rfid streams. In *ICISE*, pages 315–319, 2009.

[35] J. Xingyi, L. Xiaodong, K. Ning, and Y. Baoping. Efficient complex event processing over rfid data stream. In *ICIS*, pages 75–81, 2008.

[36] Y. Zhang. Fast algorithm for connected row convex constraints. In *IJCAI*, pages 192–197, 2007.

[37] J. Zhu, Y. Huang, and H. Wang. A formal descriptive language and an automated detection method for complex events in RFID. In *COMPSAC*, pages 543–552, 2009.