EXACT REPRESENTATION OF TRUNCATED VARIATION OF BROWNIAN MOTION

PIOTR MIŁOŚ

Abstract. In the recent papers \[8, 9, 10\] the truncated variation has been introduced, characterized and studied in various stochastic settings. In this note we uncover an intimate link to the Skorokhod problem. Further, we exploit it to give an explicit representation of the truncated variation of a Brownian motion. More precisely, we prove that the inverse of this process is, up to a minor time shift, a Lévy subordinator with the exponent \(\sqrt{2q}\tanh(c\sqrt{q/2})\).

This also gives a representation of a solution of the two-sided Skorokhod problem for a Brownian motion.

1. Representation of the truncated variation

Given \(f : [a; b] \mapsto \mathbb{R}\), a càdlàg function, its truncated variation on the interval \([a; b]\) is defined by

\[
TV^c(f, [a; b]) := \sup_n \sup_{a \leq t_1 < t_2 < \ldots < t_n \leq b} \sum_{i=1}^{n-1} \phi_c(f(t_{i+1}) - f(t_i)), \quad c \geq 0,
\]

\(\phi_c(x) = \max\{|x| - c, 0\}\) (note that \(TV^0(f, [a; b])\) is the total variation). Closely related notions of the upward truncated variation and the downward truncated variation denoted by \(UTV^c\), respectively \(DTV^c\) are defined by putting \(\phi_c(x) = \max\{x - c, 0\}\), respectively \(\phi_c(x) = \max\{-x - c, 0\}\) in (1.1). The truncated variation has two interesting variational characterizations - [9, (1.2) and (2.2)]. We recall one of them:

\[
TV^c(f, [a; b]) = \inf \left\{ TV^0(g, [a; b]) : g \text{ such that } \|g - f\|_\infty \leq \frac{1}{2}c \right\},
\]

where \(\|g\|_\infty := \sup\{|g(x)| : x \in [a; b]\}\). Moreover, the infimum is attained for some function \(g^c\), which can be effectively characterized. We skip the precise description at the moment (instead we refer the reader to [9, Section 2.1] and the proof of Proposition 2 below), yet the basic idea is simple. Following [7, Remark 2.4] we notice that “\(g^c\) is the most lazy function, which changes its value only if it is necessary to stay in the tube defined by \(\|g^c - f\|_\infty \leq c/2\).” This can be seen on the following picture:

![An example of function (in red) and \(g^c\) (in various colors).](image-url)

This observation, explored from a different perspective, will lead us to a new characterization of \(g^c\) in the language of the Skorokhod problem. First we recall that

**Lemma 1.** Let \(f \in \mathcal{C}([a; b], \mathbb{R}), c > 0\). For any \(x \in [f(a) - c/2, f(a) + c/2]\) there exist unique functions \(f_1, f_2 \in \mathcal{C}([a; b], \mathbb{R})\) such that

\[\text{Date: 11.11.2013.}\]
Proposition 2. For this special choice of $x$ the solution of the Skorokhod problem given by Lemma \ref{lem:skorokhod} fulfills
\[ g^c(t) = x + f_1(s) - f_2(s). \]
Consequently for any $t \in [a; b]$ we have
\[ TV^c(f, [a; t]) = f_1(t) + f_2(t). \]
Moreover, one easily observes that $f_1(s) = f_2(s) = 0$ if $s \in [a; \tau_U \land \tau_D].$

Remark 3. The choice of $x$ for the starting point might seem little mysterious. Again this is a sign of the “laziness”. One chooses $x$ such that $g^c$ can remain constant as long as it possible (which can be seen on the picture above, in which case we have $\tau_U < \tau_D$).

**Proof.** This proposition is an easy consequence of known results thus we present it in a sketchy way. We are going to check that $f_1$ is the same as UTV$^c$ and $f_2$ is DTV$^c$. By \cite[(2.4)]{9}, we know that the unique minimizer, $g^c$, \cite[(1.2)]{1} is given by
\[ g^c(s) := x + UTV^c(f, [a; s]) - DTV^c(f, [a; s]). \]
By Lemma \ref{lem:skorokhod} it is enough to check conditions 1-4. Condition 1 is trivial, condition 2 holds by \cite[(2.1)]{9}. Concerning condition 3, it is easy to check that $f_1$ is continuous and increasing. We now recall \cite[Section 2.2]{7}, it studies function $f^c$, which by \cite[Theorem 2.1]{7} is the same as our $g^c$. We see this function grows if and only if the running maximum $M_k^c(s)$ grows and $s \in [T^c_{U,k}; T^c_{D,k})$. It is straightforward to check that these are precisely $s$ for which $h(s) = -c/2$. \hfill \Box

2. Truncated variation of Brownian Motion

We will apply the results of the previous section to study the truncated variation of a Brownian motion. Theorem \ref{thm:truncated_variation} and Theorem \ref{thm:truncated_variation2} are the main results of our paper. Let us recall definitions of $\tau_U$ and $\tau_D$ stated before Proposition \ref{prop:truncated_variation} and that $g^c$ is the unique minimizer in the problem \cite[(1.2)]{1}. These quantities will be used as a functions of Brownian paths. We define the processes $\{X^c_t\}_{t \in \mathbb{R}}$ and $\{Y^c_t\}_{t \in \mathbb{R}}$ by putting $X^c_t = Y^c_t = 0$ for $t \leq 0$ and
\[ X^c_t := \sum_{k=-\infty}^{+\infty} L^c_{tk} - \sum_{k=-\infty}^{+\infty} L^c_{tk+1/c}, \]
\[ Y^c_t := \sum_{k=-\infty}^{+\infty} L^c_{tk}, \]
where $L^c$ is the local time of a Brownian motion at the level $a \in \mathbb{R}$.

**Theorem 4.** Let $B$ be a Brownian motion independent of $X^c, Y^c$. Then
\[ B^c_t = \begin{cases} \begin{align*} (B_{\tau_U} - c/2) + X^c_{t-\tau_U}, & \text{if } \tau_U < \tau_D, \end{align*} \end{cases} \]
\[ (B_{\tau_D} + c/2) - X^c_{t-\tau_D}, & \text{if } \tau_U > \tau_D. \]
and
\[ TV^c(B, [0; t]) = TV(B^c)[0, t] = d Y^c_{t-(\tau_U \land \tau_D)}. \]
In both cases above $= d$, the equality in distribution, is understood on the process level.

**Remark 5.** It is easy to prove that $\tau_U \land \tau_D$ has exponentially decaying tails. Further, we note that $\tau_U \land \tau_D$ is the same as $\theta(c)$ in \cite{4}. Thus its density is given by \cite[(2.3)]{4}. The Laplace transform is also provided therein.
The process $Y^c$ is non-decreasing and can be given a straightforward description in terms of Lévy processes. We define its generalized inverse $\{S^c_t\}_{t \geq 0}$ by

$$S^c_t := \inf \{ s \geq 0 : Y^c_s = t \}.$$  

**Theorem 6.** The process $S^c$ is a Lévy process with the exponent $\Phi^c(q) := (-\log \mathbb{E}e^{-qs^c})/t$, given by

$$\Phi^c(q) = \sqrt{2q} \tanh \left( c\sqrt{q}/2 \right), \quad q \geq 0.$$  

**Remark 7.** Potentially this result can be used to repro [9] Theorem 5 in case of $X$ being a Brownian motion. Indeed, the process $\{c^{-1}S^c_t\}_{t \geq 0}$ is inverse of the process $cTV(X,t)$. By Theorem 6 it is a Lévy process with the exponent $q \mapsto \Phi^c(c^{-1}q)$.

**Remark 8.** Without loss of generality we may choose $c = 1/2$ then the (2.2) can be regarded as the local time of a Brownian motion on circle at point 1/2. Processes of this kind were studied e.g. in [11, 1].

**Remark 9.** The theorem gives also a representation for the two-sided Skorokhod problem for Brownian motion, like the one studied in [6].

**Proof.** (of Theorem 6) By Proposition 2 for $t \leq \tau_U \wedge \tau_D$ the process $B^c$ is constant. For the rest of the proof we assume, without loss of generality, that $\tau_U < \tau_D$ (the other case follow by taking $-B$). We have $x = B_{\tau_u} - c/2$ and we want to identify the law joint low of $f_1$ and $f_2$ (the path-wise analogues of $f_1$ and $f_2$ in Proposition 2). We have $f_1(t) = f_2(t) = 0$ for $t \in [0, \tau_U]$. At the terminal point of this interval we have

$$B_{\tau_u} - (x + f_1(\tau_u) + f_2(\tau_u)) = c/2.$$  

By the unicity of solutions of the Skorokhod problem the task boils down to ensuring that $[0, +\infty) \ni t \mapsto f_1(\tau_u + t) =: \tilde{f}_1(t)$ and $[0, +\infty) \ni t \mapsto f_2(\tau_u + t) =: \tilde{f}_2(t)$ fulfills conditions of Lemma 1 with for $x = -c/2$ and the function $t \mapsto B_{\tau_u + t} - B_{\tau_u}$. By the strong Markov property this process is a Brownian motion independent of its evolution up to time $\tau_U$.

We are going to find the joint law of $\tilde{f}_1$ and $\tilde{f}_2$. To this end we consider the function $F_c$ given by

$$F_c(x) := \sum_{k = -\infty}^{+\infty} |x - 2kc| 1_{x \in (2k-1)c,(2k+1)c} - c/2.$$  

One easily calculates its left derivative

$$\partial^- F_c(x) = \sum_{k = -\infty}^{+\infty} 1_{x \in (2kc,(2k+1)c)} - \sum_{k = -\infty}^{+\infty} 1_{x \in ((2k-1)c,2kc]},$$

and the weak second derivative

$$\partial^2 F_c = \sum_{k = -\infty}^{+\infty} \delta_{2kc} - \delta_{(2k+1)c}.$$  

Let $B$ be a Brownian motion. Let us consider $F_c(B)$. By the Itô-Tanaka formula [12] Theorem VI.1.5 (the formula as stated applies only to convex functions but it is easy to represent $F_c = F^1_c - F^2_c$ where both $F^1_c$ and $F^2_c$ are piecewise linear and convex) we have

$$(2.4) \quad F_c(B_t) = \int_0^t \partial^- F_c(B_s) dB_s + \sum_{k = -\infty}^{+\infty} L^{2kc}_t - \sum_{k = -\infty}^{+\infty} L^{(2k+1)c}_t.$$  

Let us notice that by the Lévy theorem $\beta_t := \int_0^t \partial^- F_c(B_s) dB_s$ defines another Brownian motion. A path-wise application of Lemma 1 reveals that $x = c/2$, $h(t) = F_c(B_t)$, $f(t) = \beta_t$, $\tilde{f}_1(t) = \sum_{k = -\infty}^{+\infty} L^{2kc}_t$, $\tilde{f}_2(t) = \sum_{k = -\infty}^{+\infty} L^{(2k+1)c}_t$ is the solution of the Skorokhod problem. Putting the pieces together we obtain the representation (2.4). □

**Proof.** (of Theorem 6) First, we are going to prove that $S^c$ is a Lévy process. We will check conditions of [13] Definition 1.6. Points 2 and 5 are straightforward. Let us prove 1 for the case $0 \leq t_0 < t_1$ (the case of general $n$ follows by induction). $S^c_{t_1}$ is clearly a stopping time with respect to the filtration of the underlying Brownian motion. We have $S^c_{t_2} - S^c_{t_1} = \inf \{ s \geq 0 : Y^c_{s+t_1} - Y^c_{s+t_2} \geq t_2 - t_1 \} = \inf \{ s \geq 0 : \theta S^c_s \circ Y^c_s \geq t_2 - t_1 \}$, where $\theta$ is the shift operator. By the strong Markov property we conclude that $S^c_{t_2} - S^c_{t_1}$ is independent of $S^c_{t_1}$ concluding the proof of 1. A very similar argument covers 3. Finally 4, can be proved by the fact that $\mathbb{P} (L^0_t > 0) = 1$ for any $t > 0$. 

$\square$
Further, the proof will use many notions of the theory of Brownian motion. As they are standard, instead of introducing them formally (which would be very lengthy), we refer the reader to \cite{112}.

Process $S^c$ is clearly non-decreasing (i.e. it is a subordinator). Our aim is to calculate its Lévy exponent $\Phi^c$. To this end let us denote by $n$ the Itô excursion measure (for details we refer to \cite{112} Chapter XIII). Let $\delta$ denote the length of the excursion and $\rho = \inf \{s \geq 0 : |w_s| = c\}$ (by $w$ we denote the excursion itself); by convention we put $\rho = +\infty$ if the defining set is empty. By the Itô decomposition, \cite{112} Theorem XII.2.4 for any $q \geq 0$ we have

\begin{equation}
\Phi^c(q) = \int_{(0,\infty)} (1 - e^{-qy})(n(\rho \in dy; \rho < +\infty) + n(\zeta \in dy; \rho = +\infty)) =: I_1 + I_2.
\end{equation}

Let us now denote the stopping time $\tau_a := \inf \{s \geq 0 : B_s = a\}$ for $a \in \mathbb{R}$ and $\mathbb{P}_x$ the measure under which the Brownian motion $B$ starts from $x$. \cite{13} Proposition 2 suggests that for some constant $k > 0$ we have

\begin{equation}
I_1 = \lim_{x \searrow 0} I_1^I(x), \quad I_1^I(x) := kx^{-1} \int_{(0,\infty)} (1 - e^{-qy})\mathbb{P}_x(\tau_c \in dy; \tau_c < \tau_0),
\end{equation}

and

\begin{equation}
I_2 = \lim_{x \searrow 0} I_2^I(x), \quad I_2^I(x) := kx^{-1} \int_{(0,\infty)} (1 - e^{-qy})\mathbb{P}_x(\tau_0 \in dy; \tau_c > \tau_0).
\end{equation}

Proving these relations is surprisingly lengthy so we postpone it until later. Now we are going to show how they imply our result. By \cite{13} (8.8) we have

\begin{equation}
I_1 = k \lim_{x \searrow 0} x^{-1} \mathbb{E}_x(1 - e^{-q\tau_c}; \tau_c < \tau_0) = k \lim_{x \searrow 0} x^{-1} \left( \frac{W^{(0)}(x)}{W^{(0)}(c)} - \frac{W^{(q)}(x)}{W^{(q)}(c)} \right),
\end{equation}

where $W^{(q)}(x) = \sqrt{2/q} \sinh(\sqrt{2qx})$ and $W^{(0)}(x) = 2x$ (as indicated in \cite{13} Exercise 8.2 p.233)). Simple calculations reveal that

\begin{equation}
I_1 = k \left( \frac{1}{c} - \frac{\sqrt{2q}}{\sinh(\sqrt{2qc})} \right).
\end{equation}

Similarly, by \cite{13} (8.9), we have

\begin{equation}
I_2 = k \lim_{x \searrow 0} x^{-1} \mathbb{E}_x(1 - e^{-q\tau_0}; \tau_0 < \tau_c) = k \lim_{x \searrow 0} x^{-1} \left( \frac{Z^{(0)}(x) - Z^{(0)}(c)}{W^{(0)}(c)} - \frac{Z^{(q)}(x) + Z^{(q)}(c)}{W^{(q)}(c)} \right),
\end{equation}

where $Z^{(0)}(x) = \cosh(\sqrt{2qx})$ and $Z^{(0)}(x) = 1$. Performing some standard calculations we get

\begin{equation}
I_2 = k \left( \sqrt{2q} \coth(\sqrt{2qc}) - \frac{1}{c} \right).
\end{equation}

We thus have $I_1 + I_2 = k\sqrt{2q}\tanh(c\sqrt{q}/2)$. The constant $k$ have yet to be determined. This, in principle could be done using \cite{13} but we shall do this by comparing to results of \cite{9} Theorem 1 in the case of a Brownian motion. As indicated in Remark \cite{7} we are to study a Lévy process with the exponent $q \rightarrow \Phi^c(c^{-1}q)$. One easily checks that $\lim_{c \searrow 0} \Phi^c(c^{-1}q) = kq$ which describes a drift process with speed $2k$. By \cite{9} Theorem 1 we conclude that $k = 1$. To avoid unnecessary notation we will omit writing $k$ in the further part of the proof.

The last step of this proof is justifying (2.6) and (2.7). In both the cases we will introduce two additional quantities $I^m_1$ and $I^m_2(\alpha)$, where $\alpha \in \{1, 2\}$. The parameter $m$ controls some discretisation which is required to apply \cite{13} Proposition 2. We will show that for $\alpha \in \{1, 2\}$ we have

\begin{equation}
\lim_{m \rightarrow +\infty} I^m_1 = I_1,
\end{equation}

\begin{equation}
\lim_{m \rightarrow +\infty} \sup_{x \in (0,1/2)} |I^m_1(x) - I_1(x)| = 0,
\end{equation}

\begin{equation}
\forall m \lim_{x \searrow 0} I^m_1(x) = I^m_1.
\end{equation}

This will be enough to show (2.6) and (2.7). Indeed, we know already that $\lim_{x \searrow 0} I_1^I(x)$ exists, it is enough to find a sequence $\{x_n\}_{n \geq 0}$ such that $x_n \rightarrow 0$ and $I_1^I(x_n) \rightarrow I_1$. To this end we fix $\epsilon > 0$ and choose $m$ such that $|I^m_1 - I_1| \leq \epsilon$ and $\sup_{x \in (0,1/2)} |I^m_1(x) - I_1(x)| \leq \epsilon$, finally we can find $x \in (0,1/2)$ such that $|I^m_1(x) - I_1^m| \leq \epsilon$. For this $x$ we have $|I_1 - I_1(x)| \leq 3\epsilon$. Further reasoning is standard.
Let us start with (2.7). We define \( \{a^m_k\}_{k \in \{-\infty, \ldots, +\infty\}} \) where \( m \in \mathbb{N} \) by

\[
(2.11) \quad a^m_k := \begin{cases} (1 - \frac{k}{m})^{-2} & \text{for } k \leq 0, \\ 1 + \frac{k}{m} & \text{for } k > 0. \end{cases}
\]

One easily verifies that for any \( m \) we have \( a^m_k \leq a^m_{k+1} \). By the fact that function \( x \mapsto 1 - e^{-qx} \) is increasing we have

\[
J^m := \sum_{k = -\infty}^{+\infty} \left(1 - e^{-qa^m_{k+1}}\right) n(\zeta \in [a^m_{k-1}, a^m_k]; \rho = +\infty) \leq I_2 \leq \sum_{k = -\infty}^{+\infty} \left(1 - e^{-qa^m_k}\right) n(\zeta \in [a^m_{k-1}, a^m_k]; \rho = +\infty) =: K^m.
\]

We are going to prove that \( \lim_{m \to +\infty} K^m - J^m = 0 \) and consequently

\[
(2.12) \quad \lim_{m \to +\infty} K^m = I_2.
\]

We have\(^1\)

\[
(2.13) \quad 0 \leq K^m - J^m \leq \sum_{k = -\infty}^{+\infty} \left(e^{-qa^m_{k+1}} - e^{-qa^m_k}\right) n(\zeta \in [a^m_{k-1}, a^m_k]) \
\leq 0 \sum_{k = -\infty}^{0} (a^m_k - a^m_{k-1}) n(\zeta \in [a^m_{k-1}, a^m_k]) + \frac{1}{m} \sum_{k = 1}^{+\infty} e^{-qa^m_k-1} n(\zeta \in [a^m_{k-1}, a^m_k]),
\]

where we used \( (e^{-qa^m_{k+1}} - e^{-qa^m_k}) \leq e^{-qa^m_{k+1}}(a^m_k - a^m_{k-1}) \). By the special choice of sequence \( a^m \) and [12] Proposition XII.2.8 for some \( C > 0 \) we have \( n(\zeta \in (a^m_{k-1}, a^m_k]) \leq C/m \) which holds for \( k \leq 0 \). The first sum can be upper-bounded by

\[
\frac{1}{m} \sum_{k = -\infty}^{0} (a^m_k - a^m_{k-1}) = \frac{1}{m} a^m_0 \to 0.
\]

The second term of (2.13) is bounded from above by the following integral

\[
\frac{1}{m} \int_1^{+\infty} e^{-qx} n(\zeta \in dy) = 0.
\]

We have thus proven (2.12). Now we define the aforementioned \( I^m_2 \), namely we put

\[
I^m_2 := \sum_{k = -\infty}^{+\infty} \left(1 - e^{-qa^m_k}\right) n(\zeta \in [a^m_{k-1}, a^m_k]; a^m_{k-1} \leq \rho).
\]

Obviously, we have \( K^m \leq I^m_2 \) and further we can estimate

\[
0 \leq I^m_2 - K^m \leq \sum_{k = -\infty}^{+\infty} \left(1 - e^{-qa^m_k}\right) n(\zeta \in [a^m_{k-1}, a^m_k]; \rho \in [a^m_{k-1}, a^m_k])
\leq \int \left(1 - e^{-q\zeta(w)}\right) \sum_{k = -\infty}^{+\infty} 1_{\zeta(w) \in [a^m_{k-1}, a^m_k]} 1_{\rho(w) \in [a^m_{k-1}, a^m_k]} n(dw).
\]

In the last expression we integrate over the space of excursions (we refer the reader to [12] Section XII] for details). By [12] Proposition XII.2.8 one checks that \( \int (1 - e^{-q\zeta(w)}) n(dw) < +\infty \). The expression

\[
\sum_{k = -\infty}^{+\infty} 1_{\zeta(w) \in [a^m_{k-1}, a^m_k]} 1_{\rho(w) \in [a^m_{k-1}, a^m_k]} \]

is bounded by 1 and converges point-wise to 0 (for any \( w \)). Thus by Lebesgue’s dominated convergence we get that \( \lim_{m \to +\infty} I^m_2 - K^m \). This and (2.12) yields (2.8).

Now we define the aforementioned \( I^m_2(x) \) by

\[
(2.14) \quad I^m_2(x) := \sum_{k = -\infty}^{+\infty} \left(1 - e^{-qa^m_k}\right) \frac{1}{x} \mathbb{P}_x(\tau_0 \in [a^m_{k-1}, a^m_k]; a^m_{k-1} \leq \tau_e).
\]

\(^1\)From now on notation \( x \leq y \) denotes a situation when there exists a constant \( C > 0 \) such that \( x \leq Cy \) and \( C \) is irrelevant for calculations.
We note that by [3 Proposition 2] every summand converge to the corresponding summand of \( I^m_2 \). Let us consider terms with \( k \geq 1 \). We have

\[
\frac{1}{x} \mathbb{P}_x (\tau_0 \in [a_{k-1}^m, a_k^m]; a_{k-1}^m \leq \tau_c) \leq \frac{\mathbb{P}_x (\tau_0 \geq 1)}{x} \mathbb{P}_x (a_{k-1}^m \leq \tau_c; a_{k-1}^m \leq \tau_0 | \tau_0 \geq 1) \lesssim \exp (-C a_{k-1}^m),
\]

for some constant \( C > 0 \). Indeed, \( \mathbb{P}_x (\tau_0 \geq 1) = \mathbb{P}_0 (\inf_{t \in [0,1]} W_t \geq -x) \approx (2/\pi)^{1/2} x \) (see [12 Section III.3.7]). Secondly, by the strong Markov property \( \mathbb{P}_x (a_{k-1}^m \leq \tau_c; a_{k-1}^m \leq \tau_0 | \tau_0 \geq 1) \leq \mathbb{P}_{c/2}(\forall t \in [0,a_{k-1}^m-1] W_t \in (0,c)) \lesssim \exp (-C(a_{k-1}^m - 1)) \). The last term is clearly summable, thus Lebesgue’s dominated theorem implies that

\[
\lim_{x \to 0^+} \sum_{k=1}^{+\infty} \frac{1}{x} \mathbb{P}_x (\tau_0 \in [a_{k-1}^m, a_k^m]; a_{k-1}^m \leq \tau_c) = \sum_{k=1}^{+\infty} \left( 1 - e^{-q\tau_c} \right) n(\zeta \in [a_{k-1}^m, a_k^m]; a_{k-1}^m \leq \rho).
\]

Next, we treat the case of \( k \leq 0 \). Using [12 Remark 1 after Proposition III.3.8] we estimate

\[
(1 - e^{-q\tau_c}) \frac{1}{x} \mathbb{P}_x (\tau_0 \in [a_{k-1}^m, a_k^m]; a_{k-1}^m \leq \tau_c) \leq \left( 1 - e^{-q\tau_c} \right) \frac{1}{x} \int_{a_{k-1}^m}^{a_k^m} \frac{x}{(2\pi y^2)^{1/2}} \exp \left( -\frac{x^2}{2y} \right) dy \leq \left( 1 - e^{-q\tau_c} \right) \frac{a_k^m - a_{k-1}^m}{(a_k^m - a_{k-1}^m)^{3/2}} \leq \frac{a_k^m - a_{k-1}^m}{(a_k^m - a_{k-1}^m)^{1/2} + a_k^m - a_{k-1}^m}.
\]

One checks that \( a_k^m - a_{k-1}^m \lesssim (1 - k/m)^{-3} \), this implies that the last expression is a sequence summable in \( k \). Analogously as before we have convergence for \( \sum_{k=-\infty}^{0} \ldots \). Put together they give (2.10).

We notice that \( I^m_2(x) \) defined in (2.14) can be expressed as

\[
I^m_2(x) = \int_{(0, +\infty)} \left( 1 - e^{-qf(m)(y)} \right) \frac{1}{x} \mathbb{P}_x (\tau_0 \in dy; g^m(y) \leq \tau_c),
\]

where \( g^m(y) = a_{k-1}^m \) and \( f^m(y) = a_k^m \) whenever \( y \in [a_{k-1}^m, a_k^m] \). We recall \( I_2(x) \) defined in (2.7) and consider

\[
|I^m_2(x) - I_2(x)| \leq \int_{(0, +\infty)} \left( e^{-qy} - e^{-qf(m)(y)} \right) \frac{1}{x} \mathbb{P}_x (\tau_0 \in dy; g^m(y) \leq \tau_c)
+ \int_{(0, +\infty)} \left( 1 - e^{-qy} \right) \frac{1}{x} \mathbb{P}_x (\tau_0 \in dy; \tau_c \in [g^m(y), y]) =: J^m(x) + K^m(x).
\]

We estimate

\[
J^m(x) \lesssim \int_{(0,1)} \frac{e^{-qy} - e^{-qf(m)(y)}}{y} y \mathbb{P}_x (\tau_0 \in dy) + \int_{[1, +\infty)} \left( e^{-qy} - e^{-qf(m)(y)} \right) \mathbb{P}_x (\tau_c \geq g^m(y)|\tau_0 \geq 1) dy.
\]

We denote \( l^m(y) := e^{-qy} - e^{-qf(m)(y)} \). One checks that \( \forall \varphi \in (0,1) \) we have \( l^m(y) \leq 2 \) and \( \lim_{y \to +\infty} l^m(y) = 0 \). For \( y \geq 1 \) we have \( e^{-qy} - e^{-qf(m)(y)} \lesssim 1/m \) and \( \mathbb{P}_x (\tau_c \geq g^m(y)|\tau_0 \geq 1) \lesssim e^{-Cy} \). We recall also [12 Remark 1 after Proposition III.3.8] to get

\[
J^m(x) \lesssim \int_{(0,1)} l^m(y) \frac{1}{(2\pi y^2)^{1/2}} \exp \left( -\frac{x^2}{2y} \right) dy + \frac{1}{m} \int_{[1, +\infty)} e^{-Cy} dy \lesssim \int_{(0,1)} \frac{l^m(y)}{(2\pi y^2)^{1/2}} dy + \frac{1}{m}.
\]

We can see that the last expression does not depend on \( x \) and converges to 0 when \( m \to +\infty \). We proceed to \( K^m \).

We recall that \( \mathbb{P}_x (\tau_c \leq \tau_0) \lesssim \tau \) and thus

\[
K^m(x) \lesssim \mathbb{P}_x (\tau_c \in (g^m(\tau_0), \tau_0)|\tau_c \leq \tau_0).
\]

We observe that the strong Markov property yields \( \mathbb{P}_x (\tau_c \in (g^m(\tau_0), \tau_0)|\tau_c \leq \tau_0) \leq \mathbb{P}_x (\tau_0 \leq 1/m) \to 0 \). This is enough to conclude that \( \sup_{x \in (0,1)} K^m(x) \to 0 \) when \( m \to +\infty \). And thus also (2.9). Having checked that (2.8), (2.9) and (2.10) hold for \( i = 2 \) we conclude (2.7).

Now we turn to (2.6). We use \( \{a_k^m\}_{k \in (-\infty, +\infty)} \) given by (2.11) and define

\[
I_1^m := \sum_{l \geq 1} \sum_{k \geq 1} \left( 1 - e^{-q\tau_0} \right) n(\rho \in [a_l^m, a_{l+1}^m]; \zeta \in [a_k^m, a_{k+1}^m]).
\]
Obviously we have $I_m^0 \leq I_1$ and further

$$I_1 - I_m^0 \leq \sum_{l} \left( e^{-qn_m} - e^{-qn_{m+1}} \right) n(\rho \in [a_l^m, a_{l+1}^m]; \rho < +\infty)$$

$$+ \sum_k \left( 1 - e^{-qn_k} \right) n(\rho \in [a_k^m, a_{k+1}^m]; \zeta \in [a_k^m, a_{k+1}^m]) =: K_m^0 + J_m^0.$$  

We consider $K_m^0$. We have

$$K_m^0 = \int l_m(y)(1 - e^{-qy})n(\rho < +\infty),$$

where $l_m(y) := (e^{-qf_m(y)} - e^{-qg_m(y)})/(1 - e^{-qy})$. We have

$$(2.17) \quad \int_{0, +\infty}(1 - e^{-qy})n(\rho \in dy; \rho < +\infty) = \int_{0, +\infty} qe^{-qy}n(\rho \geq y; \rho < +\infty)dy < +\infty,$$

where the last estimate follows by $n(\rho \geq y; \rho < +\infty) \leq n(\zeta > y) \lesssim y^{-1/2}$, which is asserted in [12, Proposition XII.2.8]. It is also easy to verify that $\sup_{0 \leq \rho \leq l_m(y)} \leq C$ for some $C > 0$ and that $l_m \to 0$ point-wise. Dominated Lebesgue’s theorem implies $K_m^0 \to 0$. To deal with $J_m^0$ one checks that $1 - e^{-qg_\rho(y)} \lesssim 1 - e^{-qy}$

$$J_m^0 \lesssim \int_{(0, +\infty)} (1 - e^{-qy}) n(\rho \in dy; \zeta \in [g_m(y), f_m(y)])dy$$

$$\leq \int_{(0, +\infty)} (1 - e^{-qy}) n(\rho \in dy) + \int_{(\epsilon, +\infty)} n(\rho \in dy; \zeta \in [g_m(y), f_m(y)])dy.$$

The measure $n$ is finite on the set $\rho \geq c$ hence the second term converges by the Lebesgue dominated theorem (as the conditions converge to 0). Further, by (2.17), the first integral is equal to some $C(\epsilon) > 0$, which $\lim_{\epsilon \to 0} C(\epsilon) = 0$. The above facts are enough to conclude that $J_m^0 \to 0$. In this way we have established (2.8).

Now we define the approximation sequence $I_m^0(x)$ by

$$I_m^0(x) := \sum_{l} \sum_{k > l} \left( 1 - e^{-qn_k} \right) \frac{1}{x} \mathbb{P}_x (\tau_c \in [a_l^m, a_{l+1}^m]; \tau_0 \in [a_k^m, a_{k+1}^m]).$$

We are going to show (2.10). To this end we define

$$I_m^0(x; h) = \sum_{l = -h}^{h} \sum_{k = l+1}^{h} \left( 1 - e^{-qn_k} \right) \frac{1}{x} \mathbb{P}_x (\tau_c \in [a_l^m, a_{l+1}^m]; \tau_0 \in [a_k^m, a_{k+1}^m]), \quad h \in \{2, 3, \ldots\}.$$  

Each of $I_m^0(x; h)$ contains only finite number of terms so by [3, Proposition 2] we have

$$\lim_{h \to 0} I_m^0(x; h) = \sum_{l = -h}^{h} \sum_{k = l+1}^{h} \left( 1 - e^{-qn_k} \right) n(\tau_c \in [a_l^m, a_{l+1}^m]; \tau_0 \in [a_k^m, a_{k+1}^m]).$$

To conclude (2.10) we are going to show that for any fixed $m$ we have

$$(2.19) \quad \lim_{h \to +\infty} \sup_{x} (I_m^0(x) - I_m^0(x; h)) = 0.$$  

Firstly, we notice that

$$\sum_{k = h}^{+\infty} \sum_{l = -h}^{l \leq k} \left( 1 - e^{-qn_k} \right) \frac{1}{x} \mathbb{P}_x (\tau_c \in [a_l^m, a_{l+1}^m]; \tau_0 \in [a_k^m, a_{k+1}^m]) \leq \sum_{k = h}^{+\infty} \sum_{l = -h}^{l \leq k} \frac{1}{x} \mathbb{P}_x (\tau_0 \in [a_k^m, a_{k+1}^m]) \leq \frac{1}{x} \mathbb{P}_x (\tau_0 \geq a_k^m).$$

The last term converges to 0 with $h \to +\infty$ uniformly in $x$. Next, we treat

$$\sum_{l = -h}^{l \leq k} \sum_{k = l+1}^{h} \left( 1 - e^{-qn_k} \right) \frac{1}{x} \mathbb{P}_x (\tau_c \in [a_l^m, a_{l+1}^m]; \tau_0 \in [a_k^m, a_{k+1}^m]) \leq \sum_{l = -h}^{l \leq h} \frac{1}{x} \mathbb{P}_x (\tau_c \in [a_l^m, a_{l+1}^m]; \tau_0 \geq \tau_c)$$

$$\leq \frac{1}{x} \mathbb{P}_x (\tau_c \leq a_{l+1}^m; \tau_0 \geq \tau_c).$$

We are now to analyze the last expression. By the strong Markov property we have

$$\frac{1}{x} \mathbb{P}_x (\tau_c \leq a_{l+1}^m; \tau_0 \geq \tau_c) \leq \frac{1}{x} \mathbb{P}_x (\tau_c \leq a_{l+1}^m; \tau_0 \geq \tau_{c/2}) = \frac{\mathbb{P}_x (\tau_0 \geq \tau_{c/2})}{x} \mathbb{P}_x (\tau_c \leq a_{l+1}^m; \tau_0 \geq \tau_{c/2})$$

$$= \frac{1}{x} \mathbb{P}_x (\tau_0 \geq \tau_{c/2}) \mathbb{P}_{c/2} (\tau_c \leq a_{l+1}^m) \leq \mathbb{P}_{c/2} (\tau_c \leq a_{l+1}^m) \to 0.$$
We notice that the last expression does not involve \( x \), thus we again we have obtained convergence uniform in \( x \) and finish the proof of (2.10) and consequently (2.11).

Our final task is (2.9). We recall (2.18) and we decompose

\[
I_1(x) = \sum_{l} \sum_{k>l} \int_{[a^m_k, a^m_{k+1})} \left( 1 - e^{-qy} \right) \frac{1}{x} P_x (\tau_c \in dy; \tau_0 \in [a^m_k, a^m_{k+1}])
\]

\[+ \sum_{k} \int_{[a^m_k, a^m_{k+1})} \left( 1 - e^{-qy} \right) \frac{1}{x} P_x (\tau_c \in dy; \tau_c \leq \tau_0; \tau_0 \in [a^m_k, a^m_{k+1}]) \cdot
\]

We recall (2.18) and consider

\[
|I_1^m(x) - I_1(x)| \lesssim \sum_{l} \sum_{k>l} \int_{[a^m_k, a^m_{k+1})} \left( e^{-qa^m_k} - e^{-qa^m_{k+1}} \right) \frac{1}{x} P_x (\tau_c \in dy; \tau_0 \in [a^m_k, a^m_{k+1}])
\]

\[+ \sum_{k} \left( 1 - e^{-qa^m_{k+1}} \right) \frac{1}{x} P_x (\tau_c \in [a^m_k, a^m_{k+1}); \tau_c \leq \tau_0; \tau_0 \in [a^m_k, a^m_{k+1}]) =: J^m(x) + K^m(x).\]

We deal with the first term as follows

\[
J^m(x) \lesssim \sum_{l} \sum_{k>l} \int_{[a^m_k, a^m_{k+1})} (a^m_{k+1} - a^m_l) \frac{1}{x} P_x (\tau_c \in dy; \tau_0 \in (a^m_k, a^m_{k+1}); \tau_c/2 \leq \tau_0)
\]

\[\lesssim \sum_{l} \sum_{k>l} \int_{[a^m_k, a^m_{k+1})} (a^m_{k+1} - a^m_l) P_x (\tau_c - \tau_c/2 \in dy; \tau_0 + \tau_c/2 \in (a^m_k, a^m_{k+1}]; \tau_c/2 \leq \tau_0)
\]

\[\lesssim \max(a^m_{k+1} - a^m_l).\]

Obviously the last expression independent of \( x \) and convergences to 0. Similarly, one can estimate

\[
K^m(x) \lesssim \sum_{k} P_{c/2} (\tau_c \in (a^m_k, a^m_{k+1}); \tau_0 \in (a^m_k, a^m_{k+1})).
\]

It is easy to see that this converge to 0. In this way we have shown (2.9). This completes the proof.

**Acknowledgments.** The author wish to thank prof. Andreas Kyprianou for useful discussions and in particular for the ideas which lead to the proof of Theorem 6. Further, the author is grateful to dr. R. Łochowski for useful comments and suggestions.

The research was partially supported by MNiSW grant N N201 397537.

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