Towards the Classical Communication Complexity of Entanglement Distillation Protocols with Incomplete Information

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Abstract

Quantum entanglement distillation protocols are LOCC protocols between Alice and Bob that convert imperfect EPR pairs, or, in general, partially entangled bipartite states into perfect or near-perfect EPR pairs. The classical communication complexity of these protocols is the minimal amount of classical communication needed for the conversion. In this paper, we focus on the communication complexity of protocols that operate with incomplete information, i.e., where the inputs are mixed states and/or prepared adversarially.

We study 3 models of imperfect EPR pairs. In the measure-r model, r out of n EPR pairs are measured by an adversary; in the depolarization model, Bob’s share of qubits underwent a depolarization channel; in the fidelity model, the only information Alice and Bob possess is the fidelity of the shared state.

For the measure-r model and the depolarization model, we prove tight and almost-tight bounds on the outcome of LOCC protocols that don’t use communication. For the fidelity model, we prove a lower bound on the communication complexity that matches the upper bound given by Ambainis, Smith, and Yang [ASY02].

1 Introduction

1.1 Entanglement Distillation Protocols

Quantum entanglement plays a central role in quantum information theory. The phenomenon of having entangled states separated by space, is one of the quintessential features in quantum mechanics. In fact, one of the most fundamental problems in quantum information theory is to understand entanglement. In particular, a very important question is how to quantify entanglement: how do we measure the amount the entanglement of a general bipartite state?

Not only is quantum entanglement conceptually interesting, it is very useful “in practice”. If Alice and Bob share EPR pairs [EPR35], then they can perform teleportation [BBC+93]. Alice can transmit an unknown qubit to Bob by simply sending 2 classical bits. In this sense, shared EPR pairs (paired with a classical communication channel) are equivalent to a quantum channel. Furthermore, EPR pairs make “superdense coding” [BW92] possible, where Alice can transmit 2 classical bits to Bob by only sending one qubit, provided that Alice and Bob share an EPR pair a priori. However, qubits are prone to errors, and EPR pairs may decohere and become imperfect. Can Alice and Bob perform reliable teleportation and superdense coding if they share imperfect EPR pairs?

Entanglement Distillation Protocols (EDPs) provide answers to both questions mentioned above. Informally, EDPs are two-party protocols that take imperfect EPR pairs (or general entangled states) as input, and output bipartite states that are near-perfect EPR pairs. During the execution of the protocol, both parties (denoted by Alice and Bob) can perform local quantum operations (unitary transformations and measurements) on their share of qubits, and communicate classical information. Alice and Bob are not allowed to send qubits to each other. Protocols of this type are called “LOCC protocols”, standing for “Local Operation Classical Communication”. With EDPs, one can derive a quantity, namely the “distillable entanglement”, for any bipartite state. The distillable entanglement of a state is the maximum number of EPR pairs Alice and Bob can output using the optimal EDP, which proved to be a very important quantity in measuring the amount of entanglement for bipartite states. This answers the first question we mentioned above. For the second question, Alice and Bob can engage in an EDP to “distill” near perfect EPR pairs from imperfect ones, and then use the distilled EPR pairs to perform teleportation and superdense coding reliably.

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There have been numerous research efforts on entanglement distillation protocols. We only list some of the most relevant work here.

To our knowledge, Bennett, Bernstein, Popescu, and Schumacher are the first to consider the problem of producing EPR pairs from “less entangled” states. In their seminal paper [BBP+96a], they gave a protocol that converts many identical copies of pure state \( |\phi\rangle = (\cos \theta |0\rangle + \sin \theta |1\rangle) \) to perfect EPR pairs. They call this process “entanglement concentration”. In the same year, Bennett, Brassard, Popescu, Schumacher, Smolin, and Wootters [BBP+96b] studied the problem of “extracting” near-perfect EPR pairs from identical copies of mixed entangled states. This is the first time that the notion “entanglement purification protocols” was presented, which were renamed to “entanglement distillation protocols” later [B01]. They also pointed out that EDPs can be used to send quantum information through a noisy channel. Later, Bennett, DiVincenzo, Smolin and Wootters [BDS+96] improved the efficiency of the protocols in [BBP+96b] and proved a result that closely related EDPs to quantum error correcting codes, which is an alternative means to transmit quantum information reliably through a noisy channel.

Horodecki, Horodecki, and Horodecki [HHH96, HHH96] and Rains [R98a, R98b, R00] gave various asymptotic bounds on distillable entanglement for arbitrary entangled states. They considered the situation where \( n \) identical copies of a state are given as input to an LOCC protocol, which then outputs \( m \) EPR pairs. They studied the asymptotic behavior of \( m/n \) as \( n \) approaches infinity. Researchers also studied EDPs for a single copy of an arbitrary pure state, see Vidal [V99], Jonathan and Plenio [JP99], Hardy [H99], and Vidal, Jonathan, and Nielsen [VJN00]. Much of the work was built on the result of majorization by Nielsen [N99a], who is the first one that studied conditions under which one pure state can be transformed into another one by LOCC. The work above assumes that Alice and Bob know the explicit description of the state they share, and so they can act optimally.

From another direction, researchers have studied EDPs with incomplete information, where Alice and Bob don’t know the exact state they share. The state is in a mixed state, or is prepared adversarially. In this case we cannot hope that Alice and Bob would act optimally. However, there still exist protocols that do reasonably well. Bennett et. al [BBF+96b, BDS+96] studied the model where Bob’s share in the EPR pairs underwent a noisy channel, resulting in a mixed state. They showed that their protocol would “distill” near-perfect EPR pairs even when Alice and Bob don’t have the complete knowledge of the shared state. Under another circumstance, “purity-testing protocols” were studied implicitly by Lo and Chau [LC99], Shor and Preskill [SP00], and later explicitly by Barnum, Crépeau, Gottesman, Smith, and Tapp [BCG+02]. Purity-testing protocols are LOCC protocols that approximately distinguish the state of perfect EPR pairs from the rest states. Ambainis, Smith, and Yang [ASY02] pointed out that purity-testing protocols are indeed EDPs where Alice and Bob only know the fidelity of the state they share. Using constructions from [BCG+02], Ambainis, Smith and Yang constructed a “Random Hash” protocol that produces \((n-s)\) EPR pairs of conditional fidelity at least \(1 - 2^{-s/(1-\epsilon)}\) on any \(n\) qubit-pair input state of fidelity \(1-\epsilon\). Their protocol would fail with probability \(\epsilon\), and the conditional fidelity of its output is the fidelity conditioned on that the protocol doesn’t fail.

1.2 Communication Complexity

Classical communication complexity studies the minimal number of (classical) bits needed to be transmitted between multiple parties in order to collectively perform certain computation. Pioneered by Yao [Y79], it is now a very rich field in theoretical computer science, and the readers are referred to [KN97] for more information.

Quantum communication complexity, on the other hand, mostly studies the minimal number of qubits needed to be exchanged in order to perform (quantum) computation. This field is also first studied by Yao [Y93], and now it is becoming one of the central topics in quantum information theory. Numerous results have emerged, and we refer the readers to [B01] for a nice survey, and [BBC+98, K01a, K01b, R02] for some important techniques and results.

However, another class of problem, namely the classical communication complexity for quantum protocols, has being largely ignored, until very recently. This class of problem is concerned with the minimal number of classical bits needed to be communicated to perform certain quantum computation. An example is the classical communication complexity for EDPs: one may ask “how many bits do Alice and Bob need to exchange in order to distill \(n\) EPR pairs?” One reason that not many researchers pay too much attention to this problem might be the conception that classical communication is “cheap” compared to quantum communication, and thus can be ignored. However, as pointed by Lo and Popescu [LP99] there are situations where classical communication isn’t “that” cheap that can be justifiably ignored. One example is the super-dense coding [BW92]. Alice and Bob can use \(n\) qubits to transmit \(2n\) bits of classical information, if they share \(n\) EPR pairs. Nevertheless, if it takes more than \(n\) bits of classical communication to distill the \(n\) EPR pairs, it would totally destroy the purpose of super-dense coding. Furthermore, in the study of LOCC protocols over quantum states, no quantum communication
takes place, and it is interesting to study the classical communication complexity of these (quantum) protocols. The history of classical communication complexity for quantum protocols can probably trace back to the seminal paper by Bennett and Wiener [BW92], which discussed teleportation and constructed a protocol that uses $2n$ classical bits to transmit $n$ qubits. However, this topic was largely overlooked until by Lo and Popescu [LP99] and Lo [L99]. Lo and Popescu [LP99] discussed the classical communication complexity of various protocols by Bennett et. al. [BBP+96a]. They observed that the “entanglement concentration protocol” in [BBP+96a] doesn’t require any classical communication, while the “entanglement dilution protocol” requires $O(n)$ bits of classical communication for producing $n$ copies of the “diluted” state. Lo and Popescu [LP99] constructed a new dilution protocol that only uses $O(\sqrt{n})$ bits of communication. This protocol was proven to be asymptotically optimal by Hayden and Winter [HW02], and Harrow and Lo [HL02], who proved matching lower bounds for general entanglement dilution protocols. Lo [L99] studied the communication complexity for Alice and Bob to jointly prepare many copies of arbitrary (known) pure states, and proved an non-trivial upper bound. All the results above focus on a relatively simple situation, where the input are $n$ copies of a known pure state, and only the asymptotic results are known, i.e., the ratio of the amount of communication to $n$ as $n$ approaches infinity.

### 1.3 Our Contribution

In this paper, we study the classical communication complexity of EDPs with incomplete information. In this setting, Alice and Bob don’t have the complete knowledge about the input state they share. Rather, the input state is a mixed state, or is adversarially prepared. This is a natural extension to the simple model, where Alice and Bob share a pure state. In fact, we argue that this is a more “realistic setup”: it is very hard, if not impossible, to know precisely which pure state a quantum system is in. Some quite natural and commonly studied models of “noise” in quantum state are probabilistic in nature, and necessarily result in a mixed state. An example is the depolarization channel. Furthermore, EDPs that work with adversarially prepared states have the inherent worst-case behavior guarantee, and it more robust than EDPs designed only specifically for some known pure states. It is, therefore, very desirable to understand the communication complexity of EDPs that work in this setting.

We also study the precise communication complexity of EDPs, rather than their asymptotic behavior. In fact, we try to answer questions of the following fashion: “On this particular input state class, how many bits of classical communication are needed in order to just output a single EPR pair with certain quality?” We feel that it is important to understand the communication complexity in this case, where the requirement seems to be minimal. Interestingly, as we shall see later, the answer to this minimal question already yields a lot of insights into the more general problem, where Alice and Bob wish to generate EPR pairs of not only high quality, but also of large quantity.

To the best of our knowledge, this is the first paper that studies classical communication complexity of EDPs with incomplete information, and also the first paper to address the precise communication cost, rather than the asymptotic behavior. In fact, the only prior result that studied classical communication complexity of EDPs we are aware of is for the specific “entanglement concentration protocol” by Bennett et. al. [BBP+96a]. As pointed by Lo and Popescu [LP99], this protocol doesn’t need any classical communication. Notice that this particular protocol is the first EDP that appears in literature, and works in perhaps the simplest possible setting, where the input is a large copy of identical pure states. For all the related work on classical communication complexity we are aware of, they all work with a relative simple model. In this model, Alice and Bob try to convert many copies of some pure state $|\phi\rangle$ into many copies of some other state $|\psi\rangle$. The fact that only many copies of identical pure states are considered (and only asymptotic results are needed) makes a lot of techniques available, for example the Law of Large Numbers, the Central Limit Theorem, and the conversion of multiple-round protocols into single-round protocols [BBP+96a, N99a, NC00, LP99, HL02, HW02]. These techniques no longer work when we move to mixed input states and ask for precise communication complexity.

As another motivation, we point out that, as EDPs are closely related to Quantum Error Correcting Codes (QECCs), the communication complexity of EDPs is closely related to the efficiency of QECCs. Quantum error correcting codes are schemes to encode quantum states redundantly, such that if part of the states are corrupted, one can still recover the original encoded state. With QECC, Alice is able to transmit quantum states reliably through a noisy quantum channel to Bob. The readers are referred to [S95, S96, S97, NC00, P00] for more discussions on QECCs. One of the central issues concerning QECCs is to design QECCs that are efficient (i.e., has low redundancy) and robust (i.e., tolerate a wide range of noise). As pointed by Bennett et. al. [BBP+96b, BDS+96], entanglement distillation protocols can also be use to transmit quantum states reliably through a noisy channel. Alice produces EPR pairs and sends Bob’s share through the noisy channel. Then Alice and Bob engage in an EDP to “distill” near-perfect EPR pairs. Finally Alice and Bob use the shared near-perfect EPR pairs to perform teleportation and transmit the quantum states reliably. From this point of view, entanglement distillation protocols can be thought as “interactive error correcting protocols”. In fact, Bennett et. al. [BDS+96] proved a relationship connecting QECCs and EDPs; they proved that QECCs and 1-way EDPs
(where only Alice sends information to Bob and Bob doesn’t send anything back) are essentially equivalent. From any 1-way EDP, one can derive a QECC with the same parameter, and vice versa. They also showed that 2-way EDPs are more powerful than QECCs in that there exists a noisy channel for which no QECC is possible, but there exists 2-way EDPs that can transmit information through this channel. The communication complexity of EDPs somewhat corresponds to the redundancy of QECCs. As in the case of QECCs, it is therefore very desirable to construct EDPs of low communication complexity that tolerate a high level of noise. In this setting, the noise model is often adversarial or probabilistic, and both precise and asymptotic results on communication complexity are important.

We study EDPs in 3 different settings, corresponding to 3 different models of “imperfect” EPR pairs. The first model is called the measure-\(r\) model. In this model, Alice and Bob originally share \(n\) perfect EPR pairs, and then \(r\) out of these \(n\) pairs are measured in the computational basis. Each measured pair ends in a mixed state \(\frac{1}{2}(\ket{00}\bra{00} + \ket{11}\bra{11})\), and becomes disentangled. Alice and Bob have no information about which pairs are measured and which are not, but they know \(r\). In fact, we assume that the \(r\) measured pairs are adversarially chosen. This model is similar to the model used in error correcting codes (both classical and quantum). The second model is called the depolarization model. In this model, \(n\) perfect EPR pairs were produced by Alice, and then she sends Bob’s share of \(n\) qubits to Bob through depolarization channel of parameter \(p\). In other words, each of Bob’s qubits is left unchanged independently with probability \(1-p\) and is replaced by a completely mixed state with probability \(p\). It is a typical model for “noisy channels”, and in particular was studied by Bennett et. al. [BBP+96b, BDS+96]. The third model is called the fidelity model. Here, Alice and Bob only know that the fidelity of their shared state and perfect EPR pairs is \(1-\epsilon\). Alice and Bob don’t have any other information about the state. This is the model considered by Ambainis et. al. [ASY02], where they called it the “general error” model.

We obtain the following results: For the measure-\(r\) model, we obtain a tight upper bound on the fidelity of the output of protocols that don’t use communication. More precisely, we prove that in the measure-\(r\) model, the maximal fidelity of a protocol is at most \(1-r/2n\), if no communication is involved. Here we define the fidelity of a protocol to be the worse cast fidelity of the output of this protocol and the perfect EPR pairs. This bound is tight in that we also present a (very simple) protocol that achieves a fidelity of \(1-r/2n\). Interestingly, the proof seems quite non-trivial for this seemingly simple statement (and the trivial protocol that matches the bound). For the depolarization model, we obtain an almost-tight, similar bound. We prove that in the depolarization model, the maximum fidelity of a protocol is \(1-p/2\), if no communication is involved. This upper bound is almost tight, in that we also give a (very simple) protocol that achieves \(1-3p/4\). Both these upper bounds are for protocols that are only required to output 1 qubit-pair, which seems to be the minimal requirement for a “useful” EDP. For the fidelity model, we give almost tight (up to an additive constant) bounds on communication complexity of EDPs. More precisely, we prove that the maximal conditional fidelity of an EDP of \(t\) bits of communication is at most \(1-\epsilon \cdot p/2^{t+1}\), even if the EDP is only required to output 1 qubit pair. Here \(\epsilon\) is the fidelity of the input state, and \(p\) is the “ideal success probability”, which is the probability that the EDP succeeds with perfect EPR pairs (having fidelity 1) as input. Therefore, to achieve a fidelity of \(1-\delta\) on the output, \(\log(1/\delta) + \log(\epsilon \cdot p) - 1\) bits of classical communication is needed. Comparing the result from [ASY02], which constructed a protocol that uses \(\log(1/\delta) + \log(1-\epsilon)\) bits, our lower bound is tight up to an additive constant. Here we assume that both \(\epsilon\) and \(p\) are constant, which seems to be the reasonable assumption. One interesting observation is that our lower bound was proven for protocols what only output 1 qubit pair, while the matching upper bound is from a protocol that outputs many qubits (in fact, in the usual setting, the protocol outputs all but logarithmically many bit of input qubit pairs). This seems to indicate that the communication complexity is oblivious of the yield of the EDPs with respect the fidelity model. This fact is quite surprising, since it is definitely not the case for QECCs.

All the proofs in our paper are from first principles and don’t involve very complex analysis. Some techniques used in this paper would be interesting by themselves: in fact, as we pointed out earlier, the old techniques don’t work any more in our setting, when mixed states and studied and we are interested in the precise communication complexity. Therefore, we need to use new techniques, among which are an alternative definition on fidelity, which proved very useful in proving the first 2 bounds, and an observation on the “splitting” of mixed states during communication, which is useful to prove the lower bound for the fidelity model.

1.4 Outline of the Paper

In Section 2 we present some notations and definitions to be used in the rest of the paper. We prove a lower bound for the measure-\(r\) model in Section 3. We prove a lower bound for the fidelity model in Section 4. In Section 5 we prove the lower bound for the fidelity model. We conclude the paper in Section 6. Some proofs are postponed to the Appendix.
2 Notations and Definitions

All logarithms are base-2. We identify an integer with the 0-1 vector obtained from its binary representation. For a vector $v$, we write $v[j]$ to denote its $j$-th entry. For 0-1 vector $x$, we denote its Hamming weight by $|x|$, which is the number of 1’s in $x$. We define $\mathcal{B} = \{0, 1\}$, and naturally $\mathcal{B}^n = \{0, 1\}^n$. For binary strings $x$ and $y$, we use $x: y$ to denote the concatenation of these 2 strings.

Throughout the paper we are interested in finite, bipartite, symmetric quantum systems shared between Alice and Bob. We identify a “ket” $| \phi \rangle$ with a unit column vector. We assume there exists a canonical computational basis for any finite Hilbert space of dimension $N$, and we denote it by $\{ |0\rangle, |1\rangle, ..., |N-1\rangle \}$. We use superscripts to indicate which “side” a qubit or an operation belongs to. For example, a general bipartite state $| \phi \rangle$ can written as $| \phi \rangle = \sum_{i,j} \alpha_{ij} | i \rangle | j \rangle$.

There are 4 Bell states for a pair of qubits shared between Alice and Bob, and we denote them as follows:

$$
\Phi^+ = \frac{1}{\sqrt{2}} (|0\rangle^A | 0\rangle^B + |1\rangle^A | 1\rangle^B) \\
\Phi^- = \frac{1}{\sqrt{2}} (|0\rangle^A | 0\rangle^B - |1\rangle^A | 1\rangle^B) \\
\Psi^+ = \frac{1}{\sqrt{2}} (|0\rangle^A | 1\rangle^B + |1\rangle^A | 0\rangle^B) \\
\Psi^- = \frac{1}{\sqrt{2}} (|0\rangle^A | 1\rangle^B - |1\rangle^A | 0\rangle^B)
$$

We denote the state $(\Phi^+)^{\otimes n}$, which represents $n$ perfect EPR pairs, by $\Psi_n$. We also abuse the notation to use $\Psi_n$ to denote both the vector $\Psi_n$ and its density matrix $\rho(\Psi_n) | \Psi_n \rangle$ when there is no danger of confusion.

A quantum state is disentangled if it is of the form $| \psi \rangle^A \otimes | \psi' \rangle^B$. Any other pure state is entangled. A mixed state $\rho$ is disentangled if and only if it is equivalent to a state that is a mixture of disentangled pure states. Any other mixed state is entangled.

The Pauli Matrices $X$, $Y$, and $Z$ are unitary operations over a single qubit defined as

$$
X(\alpha | 0 \rangle + \beta | 1 \rangle) = \beta | 0 \rangle + \alpha | 1 \rangle \\
Y(\alpha | 0 \rangle + \beta | 1 \rangle) = i\beta | 0 \rangle - i\alpha | 1 \rangle \\
Z(\alpha | 0 \rangle + \beta | 1 \rangle) = \alpha | 0 \rangle - \beta | 1 \rangle
$$

We use $I$ to denote the identity operator.

For a unitary operator $U$, we can write it in a matrix form under the computational basis. Then we define its conjugate, $U^*$, to the entry-wise conjugate of $U$. Clearly $U^*$ is still a unitary operation.

An error model is simply a set of bipartite (mixed) states, and is often denoted by $\mathcal{M}$. We say a state $\rho$ is consistent with $\mathcal{M}$, if $\rho \in \mathcal{M}$.

2.1 Fidelity

For two (mixed) states $\rho$ and $\sigma$ in the same Hilbert space their fidelity is defined as

$$
F(\rho, \sigma) = \text{Tr}^2(\sqrt{\rho^{1/2} \sigma \rho^{1/2}}).
$$

Notice we are using a different definition as in [NC00], where the square root of (5) is used.

If $\sigma = | \varphi \rangle \langle \varphi |$ is a pure state, the definition simplifies to

$$
F(\rho, | \varphi \rangle \langle \varphi |) = \langle \varphi | \rho \varphi \rangle
$$

A special case for the fidelity is when $| \varphi \rangle = \Psi_n$ for some $n$, such that $\rho$ and $\Psi_n$ have the same dimension. In this case, we call the fidelity of $\rho$ and $| \varphi \rangle$ the fidelity of state $\rho$, and the definition simplifies to:

$$
F(\rho) = (\Psi_n | \rho \Psi_n)
$$

We are often interested in the fidelity of 2 states of unequal dimensions. In particular, we are interested in the fidelity of a general bipartite state $\rho$, and the Bell state $\Phi^+$. This coincides with the definition of fidelity when $\rho$ has dimension 2. When $\rho$ has a higher dimension, we define its base fidelity to be the fidelity of the state obtained by tracing out all but the first qubit pair of $\rho$. We denote the base fidelity of $\rho$ by $\tilde{F}(\rho)$.

It is easy to verify that the fidelity is linear with respect to ensembles, so long as one of the inputs is a pure state, as in the following claim.
Claim 1 If $\rho$ is the density matrix for a mixed state that is an ensemble $\{|p_i, \phi_i\}\}$, and $\sigma$ is the density matrix of a pure state, then we have $F(\rho, \sigma) = \sum_i p_i F(|\phi_i\rangle\langle\phi_i|, \sigma)$. 

The fidelity is also monotone with respect to trace-preserving operations [NC00].

Claim 2 For any states $\rho$ and $\sigma$ and any trace-preserving operator $E$, we have $F(E(\rho), E(\sigma)) \geq F(\rho, \sigma)$. 

One useful fact about fidelity is that any completely disentangled state has base fidelity at most $1/2$.

Lemma 1 If $\rho$ is a completely disentangled state, then $F(\rho) \leq 1/2$.

Proof: By the definition of base fidelity, we may assume that $\rho$ has dimension $2$. By Claim 1, we only need to consider the case that $\rho$ is a pure state $|\phi\rangle\langle\phi|$. Since $|\phi\rangle$ is disentangled, we may write it as

$$|\phi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle)$$

Then a direction calculation reveals that

$$\bar{F}(|\phi\rangle\langle\phi|) = \frac{1}{2} |\alpha_0\beta_0 + \alpha_1\beta_1|^2$$

$$= \frac{1}{2} (|\alpha_0|^2|\beta_0|^2 + |\alpha_1|^2|\beta_1|^2 + \alpha_0\beta_0\alpha_1^*\beta_1^* + \alpha_0^*\beta_0^*\alpha_1\beta_1)$$

$$\leq \frac{1}{2} (|\alpha_0|^2|\beta_0|^2 + |\alpha_1|^2|\beta_1|^2 + |\alpha_0\beta_0|^2 + |\alpha_1\beta_1|^2)$$

$$= \frac{1}{2} (|\alpha_0|^2 + |\alpha_1|^2)(|\beta_0|^2 + |\beta_1|^2)$$

$$= \frac{1}{2}$$

\[\Box\]

2.2 Entanglement Distillation Protocols

We give a detailed description on entanglement distillation protocols discussed in our paper. We often denote an entanglement distillation protocol by $P$. The protocol starts with a mixed state $\rho$ shared between Alice and Bob. Alice and Bob can have their private ancillary qubits, originally initialized to $|0\rangle$. A protocol is either deterministic or probabilistic. For deterministic protocols, Alice and Bob don’t share any initial random bits; for probabilistic protocols, Alice and Bob share a (classical) random string. We say a protocol $P$ is a $t$-bit protocol, if there are $t$ bits of (classical) communication during the protocol. We don’t allow protocols to have any initial entanglement as auxiliary inputs, and neither do we allow quantum channels between Alice and Bob.

An the end of a protocol, both parties output $m$ qubits, which form the output of the protocol. In addition, Alice also outputs a special symbol (either a SUCC or a FAIL). The success probability of a protocol $P$ over an input state $\rho$ is the probability that Alice outputs SUCC at the end of the protocol, and we write this as $P_{\text{SUCC}}(\rho)$. The ideal success probability of a protocol $P$ is its success probability over input $\Psi_n$. We say a protocol is ideal, if its ideal success probability is $1$. If $\sigma$ is the density matrix of the output of protocol $P$ on input $\rho$, we write it as $P(\rho) = \sigma$. If $r$ is the density matrix of the output of protocol $P$ on input $\rho$, conditioned on that Alice outputs SUCC, then we call $r$ the conditional output of protocol $P$, and write this as $P_{\text{SUCC}}^r(\rho) = r$.

For an entanglement distillation protocol $P$, we define its fidelity with respect to an error model $M$, denoted by $F_M(P)$, to be the minimal fidelity of its output over all input state consistent with $M$. In other words,

$$F_M(P) = \min_{\rho \in M} F(P(\rho)) \quad (8)$$

Similarly, we define the conditional fidelity to be the minimal fidelity of its conditional output, denoted by $F_M^c(P)$:

$$F_M^c(P) = \min_{\rho \in M} F(P^c(\rho)) \quad (9)$$

When the error model $M$ is clear from the context, it is often omitted.

3 The Measure-$r$ Model

We prove an upper bound on the fidelity of $0$-bit EDPs with respect to the measure-$r$ error model.
3.1 Notations and Definitions

We start with more notations and definitions.

A binary indicator vector, often denoted by \( \mathbf{v} \), is an \( n \)-dimensional vector, whose each entry is an element from \( \{0, 1, \ast\} \). The degree of a binary indicator vector \( \mathbf{v} \) is the number of entries that are not \( \ast \), and we write this as \( \text{deg}(\mathbf{v}) \). There are \( 2^n \cdot \binom{n}{r} \) binary indicator vectors of degree \( r \). Each binary indicator vector \( \mathbf{v} \) corresponds to a unique bipartite quantum state \( \ket{\mathbf{v}} \) in \( \mathcal{H}^2^n \) in the following way:

\[
\ket{\mathbf{v}} = \bigotimes_{j=0}^{n-1} \ket{\phi_j}, \quad \text{where} \quad \ket{\phi_j} = \begin{cases} \ket{0}^A \ket{0}^B & \text{if } v[j] = 0 \\ \ket{1}^A \ket{1}^B & \text{if } v[j] = 1 \\ \Phi^+ & \text{if } v[j] = \ast \end{cases}
\]

The state \( \ket{\mathbf{v}} \) is called an error state, where \( \mathbf{v} \) is called its error indicator vector. The degree of state \( \ket{\mathbf{v}} \) is the degree of its indicator vector. The error model for the measure-\( r \) model, denoted by \( \mathcal{M}_{n,r}^m \), is defined to be:

\[
\mathcal{M}_{n,r}^m = \{ \ket{\mathbf{v}} \mid \mathbf{v} \text{ is an } n \text{-dimensional binary indicator such that } \text{deg}(\mathbf{v}) = r \} \tag{10}
\]

An \( n \)-dimensional 0-1 vector \( x \) is consistent with a binary indicator vector \( \mathbf{v} \), if \( x[j] = v[j] \) for all \( j \) such that \( v[j] \neq \ast \). We write this as \( x \sqsubseteq \mathbf{v} \). For any \( \mathbf{v} \) of degree \( r \), there are \( 2^{n-r} \) 0-1 vectors \( x \) consistent with \( \mathbf{v} \). It is not hard to verify that

\[
\ket{\mathbf{v}} = \frac{1}{2^{(n-r)/2}} \sum_{x \sqsubseteq \mathbf{v}} \ket{x}^A \ket{x}^B \tag{11}
\]

3.2 Two Useful Lemmas

We prove 2 lemmas that would be useful for the proofs in this paper. Both lemmas are about how much “deviation” a quantum state undergoes when applied various unitary operations.

First, we consider the “deviation” of an arbitrary pure state under the operations \( \{I, X, Y, Z\} \) over its first qubit. We have the following lemma:

**Lemma 2** Let \( \ket{\phi} \) and \( \ket{\psi} \) be two pure states of the same dimension, not necessarily bipartite. Let \( I, X, Y, \) and \( Z \) be the unitary operations over the first qubit of \( \ket{\phi} \). Then we have

\[
\sum_{U \in \{I, X, Y, Z\}} |\langle\phi|U|\psi\rangle|^2 \leq 2
\]

**Proof:** We write \( \ket{\phi} = \alpha_0 \ket{0} \ket{\phi_0} + \alpha_1 \ket{1} \ket{\phi_1} \) and \( \ket{\psi} = \beta_0 \ket{0} \ket{\psi_0} + \beta_1 \ket{1} \ket{\psi_1} \)

Then we have

\[
\begin{align*}
\langle\phi|I|\psi\rangle &= \alpha_0^* \beta_0 \langle\phi_0|\psi_0\rangle + \alpha_1^* \beta_1 \langle\phi_1|\psi_1\rangle \\
\langle\phi|X|\psi\rangle &= \alpha_1^* \beta_0 \langle\phi_1|\psi_0\rangle + \alpha_0^* \beta_1 \langle\phi_0|\psi_1\rangle \\
\langle\phi|Y|\psi\rangle &= -i \alpha_1 \beta_0^* \langle\phi_1|\psi_0\rangle + i \alpha_0 \beta_1^* \langle\phi_0|\psi_1\rangle \\
\langle\phi|Z|\psi\rangle &= \alpha_0^* \beta_0 \langle\phi_0|\psi_0\rangle - \alpha_1^* \beta_1 \langle\phi_1|\psi_1\rangle
\end{align*}
\]

Therefore

\[
\sum_{U \in \{I, X, Y, Z\}} |\langle\phi|U|\psi\rangle|^2 = 2|\alpha_0 \beta_0|^2 |\langle\phi_0|\psi_0\rangle|^2 + 2|\alpha_1 \beta_1|^2 |\langle\phi_1|\psi_1\rangle|^2 + 2|\alpha_0 \beta_1|^2 |\langle\phi_0|\psi_1\rangle|^2 + 2|\alpha_1 \beta_0|^2 |\langle\phi_1|\psi_0\rangle|^2 \\
\leq 2|\alpha_0|^2 |\beta_0|^2 + 2|\alpha_1|^2 |\beta_1|^2 + 2|\alpha_0|^2 |\beta_1|^2 + 2|\alpha_1|^2 |\beta_0|^2 \\
= 2(|\alpha_0|^2 + |\alpha_1|^2)(|\beta_0|^2 + |\beta_1|^2) \\
= 2
\]

An immediate corollary is

**Corollary 1** Let \( \ket{\phi} \) be a pure state. We have \( \sum_{U \in \{I, X, Y, Z\}} |\langle\phi|U|\phi\rangle|^2 \leq 2 \).


Next, we consider quantum states and operations over bipartite systems. In particular, we study the “deviation” of a general bipartite state under unitary operations of the form $U \otimes U^*$. We interpret $U \otimes U^*$ as Alice applies $U$ to her first qubit and Bob applies $U^*$ to his first qubit. Again, we consider $U \in \{I, X, Y, Z\}$.

We have the following lemma.

**Lemma 3** Let $|\phi\rangle$ be a pure state in a bipartite system shared between Alice and Bob. Let $I$, $X \otimes X^*$, $Y \otimes Y^*$, and $Z \otimes Z^*$ be the unitary operations over the first qubit. Then we have

$$\langle \phi | \phi \rangle + \langle \phi | (X \otimes X^*) | \phi \rangle + \langle \phi | (Y \otimes Y^*) | \phi \rangle + \langle \phi | (Z \otimes Z^*) | \phi \rangle = 4 \tilde{F}(|\phi\rangle)$$

**Proof:** We first consider how the Bell states behave under these unitary operations. It is easy to verify the fidelity of $0$-bit EDPs for the measure-$r$ error model is at most $1 - r/2n$, even if the protocols are only required to output one qubit-pair. Notice that fidelity is monotone. Therefore if no protocol can output a single qubit pair of fidelity at least $1 - r/2n$, then no protocol can output multiple qubit pairs of fidelity at least $1 - r/2n$.

Notice that there exists a very simple probabilistic 0-bit protocol that has fidelity $1 - \frac{r}{2n}$. Alice and Bob use their shared random string to uniformly pick an EPR pair and output it. If this pair is measured, (which happens with probability $r/n$), the fidelity is $1/2$, and otherwise it is $1$. So the overall fidelity is exactly $1 - r/2n$. So our upper bound is tight.

| state | $\Phi^+$ | $\Phi^-$ | $\Psi^+$ | $\Psi^-$ |
|-------|---------|---------|---------|---------|
| $I \otimes I^*$ | $\Phi^+$ | $\Phi^-$ | $\Psi^+$ | $\Psi^-$ |
| $X \otimes X^*$ | $\Phi^+$ | $-\Phi^-$ | $\Psi^+$ | $-\Psi^-$ |
| $Y \otimes Y^*$ | $\Phi^+$ | $-\Phi^-$ | $-\Psi^+$ | $\Psi^-$ |
| $Z \otimes Z^*$ | $\Phi^+$ | $\Phi^-$ | $-\Psi^+$ | $-\Psi^-$ |

It is easy to see that the state $\Phi^+$ is invariant under any of the 4 operations, while other Bell states will change their signs under some operations.

---

**3.3 A Tight Bound for the No-Communication Case**

We prove that the fidelity of 0-bit EDPs for the measure-$r$ error model is at most $1 - r/2n$, even if the protocols are only required to output one qubit-pair. Notice that fidelity is monotone. Therefore if no protocol can output a single qubit pair of fidelity at least $1 - r/2n$, then no protocol can output multiple qubit pairs of fidelity at least $1 - r/2n$.

**Theorem 1** For any probabilistic 0-bit protocol $P$ that outputs one qubit pair, we have $F(P) \leq 1 - \frac{r}{2n}$ with respect to the measure-$r$ model.

Notice that there exists a very simple probabilistic 0-bit protocol that has fidelity $1 - \frac{r}{2n}$: Alice and Bob use their shared random string to uniformly pick an EPR pair and output it. If this pair is measured, (which happens with probability $r/n$), the fidelity is $1/2$, and otherwise it is $1$. So the overall fidelity is exactly $1 - r/2n$. So our upper bound is tight.
Proof: We consider a slightly different error model, where a random \( r \) out of \( n \) EPR pairs are measured. This corresponds to the density matrix

\[
\rho = \frac{1}{2^n(n/\binom{n}{r})} \sum_{\deg V = r} |\phi_V\rangle\langle\phi_V|
\]

Notice that this is the “average case” version of the measure-\( r \) model. Thus if we prove an upper bound on the fidelity of \( \mathcal{P} \) over \( \rho \), then it is also an upper bound with respect to the measure-\( r \) model.

We shall prove that no deterministic 0-bit protocol can have a fidelity higher than \( 1 - r/2n \) if \( \rho \) is the input. Then, we conclude that no probabilistic protocol can have a fidelity higher than \( 1 - r/2n \), too, since fidelity is linear.

Notice \( \mathcal{P} \) doesn’t involve any communication, we can model it as Alice and Bob both applying a unitary operation to their share of qubits, outputs the first qubit and discard the rest.

Suppose the unitary operators of Alice and Bob are \( U_A \) and \( U_B \). We denote the states under these operations by

\[
U_A |x\rangle \rightarrow |\phi_x\rangle
\]

\[
U_B |x\rangle \rightarrow |\psi_x\rangle
\]

Notice that we use “\( \rightarrow \)” instead of “\( = \)” since we allow Alice and Bob to use ancillary bits. Clearly, the vectors \( \{|\phi_x\rangle\}_x \) are orthonormal, and so are the vectors \( \{|\psi_x\rangle\}_x \).

We shall prove that

\[
\frac{1}{2^n(n/\binom{n}{r})} \sum_{\deg V = r} \left[ |\bar{F}((U_A \otimes U_B)|\phi_V\rangle\langle\phi_V|(U_A \otimes U_B))| \right] \leq 1 - \frac{r}{2n}, \tag{14}
\]

which shall imply our lemma. By Lemma 3, (14) is equivalent to

\[
\frac{1}{2^n(n/\binom{n}{r})} \sum_{\deg V = r} \sum_{U \in \{I,X,Y,Z\}} \langle \phi_V |(U_A \otimes U_B)^\dagger(U \otimes U^*)(U_A \otimes U_B)| \phi_V \rangle \leq 4(1 - \frac{r}{2n}) \tag{15}
\]

We expand the left hand side: Notice that

\[
(U_A \otimes U_B)| \phi_V \rangle = \frac{1}{2^{(n-r)/2}} \sum_{x \in V} |\phi_x\rangle |\psi_x\rangle
\]

and so we have

\[
\langle \phi_V |(U_A \otimes U_B)^\dagger(U \otimes U^*)(U_A \otimes U_B)| \phi_V \rangle = \frac{1}{2^{n-r}} \sum_{x \in V} \sum_{y \in V} \sum_{U \in \{I,X,Y,Z\}} \langle \phi_x |U| \phi_y \rangle \cdot \langle \psi_x |U^*| \psi_y \rangle
\]

for any unitary operation \( U \). So, (15) is equivalent to

\[
\frac{1}{2^n(n/\binom{n}{r})} \sum_{\deg V = r} \sum_{x \in V} \sum_{y \in V} \sum_{U \in \{I,X,Y,Z\}} \langle \phi_x |U| \phi_y \rangle \cdot \langle \psi_x |U^*| \psi_y \rangle \leq 4(1 - \frac{r}{2n}) \tag{16}
\]

However, by Cauchy-Schwartz, we have

\[
\sum_{\deg V = r} \sum_{x \in V} \sum_{y \in V} \sum_{U \in \{I,X,Y,Z\}} \langle \phi_x |U| \phi_y \rangle \cdot \langle \psi_x |U^*| \psi_y \rangle \leq \left( \sum_{\deg V = r} \sum_{x \in V} \sum_{y \in V} \sum_{U \in \{I,X,Y,Z\}} |\langle \phi_x |U| \phi_y \rangle|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{\deg V = r} \sum_{x \in V} \sum_{y \in V} \sum_{U \in \{I,X,Y,Z\}} |\langle \psi_x |U^*| \psi_y \rangle|^2 \right)^{\frac{1}{2}}
\]

Next, we estimate the terms on the right hand side:

\[
\sum_{\deg V = r} \sum_{x \in V} \sum_{y \in V} \sum_{U \in \{I,X,Y,Z\}} |\langle \phi_x |U| \phi_y \rangle|^2 = \sum_{x} \sum_{y} \sum_{U \in \{I,X,Y,Z\}} |\langle \phi_x |U| \phi_y \rangle|^2 \sum_{\deg V = r : x_1 \in V \land x_2 \in V} 1
\]

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Notice that since \( |\phi_x\rangle\)’s are all orthonormal, we have \( \sum_y |\langle \phi_x | U | \phi_y \rangle|^2 \leq 1 \) for all \( x \)’s. Thus

\[
\sum_x \sum_y \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_x | U | \phi_y \rangle|^2 \leq 2^{n+2}
\]

For any \( x \) and \( y \), we have

\[
\sum_{\deg V = r : x \subseteq V \wedge y \subseteq V} 1 = \left( \frac{n - |x \oplus y|}{n - r - |x \oplus y|} \right)
\]

The reason is simple: the only freedom for \( v \) is where to put the \((n - r)\)'s. But for every position \( k \) such that \( x[k] \neq y[k] \), we have to have \( v[k] = \ast \). Then we still have \((n - r - |x \oplus y|)\)'s we can put anywhere. So if \( x \neq y \),

\[
\sum_{\deg V = r : x \subseteq V \wedge y \subseteq V} 1 \leq \left( \frac{n - 1}{n - r - 1} \right)
\]

Also notice that by Lemma 2, we have \( \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_x | U | \phi_x \rangle|^2 \leq 2 \) for any \( x \).

Putting things together, we have

\[
\sum_{\deg V = r : x \subseteq V \wedge y \subseteq V} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_x | U | \phi_y \rangle|^2 \leq \left( \binom{n}{r} \right) \cdot \sum_{x} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_x | U | \phi_y \rangle|^2 + \left( \binom{n - 1}{r - 1} \right) \cdot \sum_{x \neq y} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_x | U | \phi_y \rangle|^2
\]

\[
= \left( \binom{n}{r} - \binom{n - 1}{r - 1} \right) \cdot \sum_{x} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_x | U | \phi_y \rangle|^2 + \binom{n - 1}{r - 1} \cdot \sum_{x} \sum_{y} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_x | U | \phi_y \rangle|^2
\]

\[
= \left( \binom{n}{r} - \binom{n - 1}{r - 1} \right) \cdot 2^{n+1} + \binom{n - 1}{r - 1} \cdot 2^{n+2}
\]

\[
= 2^{n+2} \left( \frac{n}{r} \right) \left( 1 - \frac{r}{2n} \right)
\]

Similarly, we have

\[
\sum_{\deg V = r : x \subseteq V \wedge y \subseteq V} \sum_{U \in \{I, X, Y, Z\}} |\langle \psi_x | U^* | \psi_y \rangle|^2 \leq 2^{n+2} \left( \frac{n}{r} \right) \left( 1 - \frac{r}{2n} \right)
\]

too.

Thus we have

\[
\sum_{\deg V = r : x \subseteq V \wedge y \subseteq V} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_x | U | \phi_y \rangle \cdot \langle \psi_x | U^* | \psi_y \rangle| \leq 2^{n+2} \left( \frac{n}{r} \right) \left( 1 - \frac{r}{2n} \right)
\]

which proves (16).

\[\blacksquare\]

4 The Depolarization Model

We prove an upper bound on the fidelity of 0-bit EDPs with respect to the depolarization model.

4.1 Notations and Definitions

We give notations and definitions used in this section.

We first describe the depolarization channel. A depolarization channel \( \mathcal{D} \) of parameter \( p \) is a super-operator defined as \( \text{NC00} \)

\[
\mathcal{D}(\rho) = (1 - p) \cdot \rho + p \cdot \frac{I}{2}
\]

In other words, this channel behaves in the following manner: with probability \((1 - p)\), it keeps the state untouched, and with probability \( p \), it replaces that with the completely mixed state.
It is not hard to verify that after passing the second qubit through this channel, the state $\Phi^+$ becomes a mixed state

$$\rho_p = (1 - \frac{3p}{4})|\Phi^+\rangle\langle \Phi^+| + \frac{p}{4} (|\Phi^-\rangle\langle \Phi^-| + |\Psi^+\rangle\langle \Psi^+| + |\Psi^-\rangle\langle \Psi^-|)$$

The depolarization error model of $n$ qubit pairs and parameter $n$, denoted as $\mathcal{M}_{d,n,p}$, consists of a single state: $\mathcal{M}_{d,n,p}^d = \{\rho_p^{\otimes n}\}$.

4.2 An Almost-Tight Bound for the No-Communication Case

We prove that the maximal fidelity of 0-bit EDPs for the depolarization error model is $1 - p/2$, even if the protocols are only required to output one qubit-pair.

**Theorem 2** For any probabilistic 0-bit protocol $\mathcal{P}$ that outputs one qubit pair, we have $F(\mathcal{P}) \leq 1 - \frac{p}{2}$ with respect to the depolarization model.

There exists a very simple deterministic 0-bit protocol that has fidelity $1 - \frac{3p}{4}$: Alice and Bob simply output the first qubit pair. It is very easy to verify that the fidelity of this protocol is $1 - \frac{3p}{4}$. Therefore the bound in the theorem is almost-tight (by a constant factor).

The proof to Theorem 2 is very similar to that to Theorem 1, except that it is more technical. We postpone the proof to Appendix A.

5 The Fidelity Model

We study the communication complexity of EDPs with respect to the fidelity error model.

First, we give the definition of the fidelity error model. For a bipartite system of $n$ qubit pairs, we define the fidelity error model of parameter $\epsilon$ to be the set of all bipartite systems of fidelity at least $1 - \epsilon$. We denote the error model by

$$\mathcal{M}_{f,n,\epsilon} = \{\rho \mid \rho \text{ has dimension } 2^{2n} \text{ and } F(\rho) \geq 1 - \epsilon\}$$ (17)

Notice that this error model is very different from the two previous models we studied, since it provides much less information than the previous one. As a comparison, notice that in the measure-$r$ model, all the error states have fidelity $1/2^r$, and in the depolarization model, the fidelity of the input is $(1 - 3p/4)^n$, both are very small. However, Alice and Bob have the additional information about the structure of the input states, and are able to use the information to do very well.

5.1 Two Useful Facts About Positive Operators

We present two useful facts about positive operators used in the rest of the paper.

For two positive operators $A$ and $B$, we say $A$ dominates $B$, if $A - B$ is still a positive operator, and we write this as $A \succeq B$, or equivalently, $B \preceq A$.

**Claim 3** For any positive super-operator $\mathcal{E}$ and any positive operators $A$ and $B$, if $A \succeq B$, then $\mathcal{E}(A) \succeq \mathcal{E}(B)$.

This directly follows the fact that $\mathcal{E}$ is linear and preserves the positivity of operators: If $A - B$ is a positive operator, then $\mathcal{E}(A) - \mathcal{E}(B) = \mathcal{E}(A - B)$ is also a positive operator.

**Claim 4** Let $\rho$ and $\sigma$ be density matrices such that $\rho \succeq a \cdot \sigma$, for some positive number $a$. For any POVM $\{E_m\}$, let $p_m = \text{Tr}(\rho E_m)$ and and $q_m = \text{Tr}(\sigma E_m)$ be the probabilities the measurement result being $m$ for $\rho$ and $\sigma$, respectively. Then we have $p_m \geq a \cdot q_m$.

This is obvious, since we have $p_m - a \cdot q_m = \text{Tr}((\rho - a \cdot \sigma)E_m) \geq 0$. 

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5.2 Upper and Lower Bounds for the Fidelity Model

Ambainis, Smith, and Yang proved that in the fidelity error model of parameter $\epsilon$ (which they called the “general error model”), the maximal fidelity of a protocol is $1 - \frac{2^n - 2^s - 2^n - \epsilon}{2^m - 2^n - 1 - \epsilon}$. If the protocol has $n$ qubit pairs as input, $k$ perfect EPR pairs as auxiliary input, and outputs $m$ qubit pairs. In a special case where $k = 0$ (no auxiliary input) and $m = 1$ (only one pair is output), the maximal fidelity is $1 - \frac{2^n - 2^s - \epsilon}{2^m - 1 - \epsilon} < 1 - \epsilon/2$. In other words, no “interesting” entanglement distillation protocols exist for the fidelity error model. Their result is tight, in that they also constructed a protocol, namely the “Random Permutation Protocol”, which achieves a fidelity of $1 - \frac{2^n - 2^s - 2^n - \epsilon}{2^m - 2^n - 1 - \epsilon}$. One can slightly modify this protocol to completely eliminate communication, and still maintain a high fidelity. In the original construction of the random permutation protocol, communication is used in 2 places. First, Alice and Bob communicate to agree on a common random permutation. This part of communication is not needed for a probabilistic protocol. Second, Alice and Bob communicate to check if their measurements agree. We can modify the protocol by having Alice and Bob always “pretend” that they measurements agree. A careful analysis shows that this modification won’t change the fidelity of the protocol by much. In fact, we have the following theorem:

**Theorem 3.** There exists a probabilistic 0-bit entanglement distillation protocol of fidelity $1 - \frac{2^n - 2^s - \epsilon}{2^m - 2^n - 1 - \epsilon} < 1 - \epsilon/2$ with respect to the fidelity model of parameter $\epsilon$.

The situation for conditional fidelity is very different. In fact, Ambainis et. al. proved that good protocols exist with high conditional fidelity. In particular, the following result can be easily derived from [ASY02]:

**Theorem 4.** (ASY02) For every $n$ and $s < n$, there exists probabilistic $s$-bit entanglement distillation protocols of conditional fidelity $1 - 2^{-s}/(1 - \epsilon)$ with respect to the fidelity model of parameter $\epsilon$.

**Proof’s sketch:** Consider the “Simple Random Hash” protocol in [ASY02]. The original construction for this protocol in [ASY02] has $(2n + 2)$ bits of 2-way communication. But a close examination reveals that 1 bit of 1-way communication suffices. In the original construction, Alice sends 2n bits to Bob to establish a common random string, which are not needed for a probabilistic protocol. In the original protocol, Bob also sends 1 bit of his measurement result back to Alice. This bit can also be eliminated in our model, since we allow one player (normally Alice) to output a special symbol at the end of the protocol. We then repeat the simplified 1-bit protocol for $s$ rounds sequentially, and obtain an $s$-bit protocol of conditional fidelity $1 - 2^{-s}/(1 - \epsilon)$.

Furthermore, the “Simple Random Hash” protocol only consists of 1-way communication. Also notice that this protocol is ideal, in that if the input is the perfect EPR pairs $\Psi_n$, then the protocol always succeeds. Therefore, to achieve a conditional fidelity of $1 - \delta$, only $\log(1 - \delta) - \log(1 - \epsilon)$ bits of communication is needed in the fidelity error model. Next, we shall prove a lower bound on the communication complexity.

**Theorem 5.** For any probabilistic $s$-bit protocol of ideal success probability $p$, its conditional fidelity is at most $1 - cp/2^{s+1}$ with respect to the fidelity model of parameter $\epsilon$.

Immediately from the theorem, we obtain a $\log(1 - \epsilon') - \log(1 - \epsilon') - 1$ lower bound on the communication complexity for ideal protocols of conditional fidelity $1 - \delta$. In the usual setting where $\epsilon$ is a constant, our lower bound matches the upper bound from Theorem 4 up to an additive constant. Interestingly, the theorem is proven for protocols that only output 1 qubit pair. However, this lower bound is good enough in that it matches the upper bound of the Simple Random Hash protocol, which in fact outputs many qubit pairs. In this sense, the communication complexity is “oblivious” of the yield of the EDPs. This is quite counter-intuitive.

**Proof:** WLOG we assume the protocol only outputs one qubit pair. Consider a particular input state

$$\rho_0 = (1 - \epsilon')\Psi_n + \epsilon'\frac{I}{2^{2n}}$$  \hspace{1cm} (18)

It is a mixture of the perfect EPR pairs $\Psi_n$ (with probability $1 - \epsilon'$) and the completely mixed state $\frac{I}{2^{2n}}$ (with probability $\epsilon'$). Notice that $F(\frac{I}{2^{2n}}) = \frac{1}{2^{2n}}$. So if we set $\epsilon' = \frac{2^n - \epsilon}{2^{m-1}}$, then we have $F(\rho) = 1 - \epsilon$. We shall prove that no deterministic, $s$-bit protocol has fidelity more than $1 - 2^{-s+1}cp$ over state $\rho_0$, which will imply that no probabilistic protocol can have fidelity more than $1 - 2^{-s+1}cp$, too.

We fix a deterministic protocol $P$. WLOG, we assume it proceeds in rounds: in each round, one of the two parties (Alice or Bob) applies a super-operator $E$ to his or her share of qubits, and then sends one (classical) bit to the other party. The protocol consists of $s$ rounds: one bit is sent in each round. Finally, Alice outputs the special symbol, determining if the protocol succeeds or fails.
To analyze the behavior of the protocol $\mathcal{P}$ over the input $\rho_0$, we consider how $\mathcal{P}$ behaves over state $\Psi_n$ and state $\frac{1}{2^n}$, respectively. We use $p$ (resp. $q$) to denote the probabilities that $\mathcal{P}$ succeeds over state $\Psi_n$ (resp. $\frac{1}{2^n}$). Notice $p$ is in fact the ideal success probability of protocol $\mathcal{P}$. Then it is easy to see that

$$F^c(\mathcal{P}(\rho_0)) = \frac{(1 - \epsilon')p \cdot F^c(\mathcal{P}(\Psi_n)) + \epsilon'q \cdot F^c(\mathcal{P}(\frac{1}{2^n}))}{(1 - \epsilon')p + \epsilon'q}$$  \hspace{1cm} (19)

Notice that we always have $F^c(\mathcal{P}(\Psi_n)) \leq 1$. Since $\frac{1}{2^n}$ is a disentangled state, $\mathcal{P}(\frac{1}{2^n})$ is also disentangled. Therefore we have $F^c(\mathcal{P}(\frac{1}{2^n})) \leq 1/2$ by Lemma [1]. We shall prove that

$$q \geq p^2/2^s,$$  \hspace{1cm} (20)

which will imply that

$$F(\mathcal{P}(\rho_0)) \leq \frac{(1 - \epsilon') + \epsilon'p/2^{s+1}}{(1 - \epsilon') + \epsilon'p/2^s} = 1 - \frac{\epsilon'p}{2^s + 1} \leq 1 - \epsilon p/2^{s+1}$$  \hspace{1cm} (21)

Now we prove that $q \geq p^2/2^s$. We analyse 2 cases separately: in case I, the state $\Psi_n$ is the input to the protocol; in case II, the state $\frac{1}{2^n}$ is the input to the protocol. For each case, we keep track of the local density matrices of Alice and Bob. In case I, we use $\tau_k^{I,A}$ and $\tau_k^{I,B}$ to denote the local density matrices of Alice and Bob after the $k$-th round; in case II, we use $\tau_k^{II,A}$ and $\tau_k^{II,B}$, respectively. For $k = 0$, we define the $\tau^{I,A}_0$, $\tau^{I,A}_1$, $\tau^{II,A}_0$, and $\tau^{II,A}_1$ to be the density matrices at the moment that protocol starts.

We give more definitions: after the $k$-th round, there are $2^k$ possibilities depending on the first $k$ bits communicated. For any binary string $t \in B^k$, we use $\sigma_t^{I,A}$ (resp. $\sigma_t^{I,B}$) to denote the local density matrix of Alice (resp. Bob) after the $k$-th round in case I, conditioned on that the first $k$ bits communicated so far are $t[0], t[1], ..., t[k-1]$. We use $p^I_t$ to denote the probability that this happens (that the first $k$ bits are $t[0], t[1], ..., t[k-1]$). Obviously we have $p^I_t = p^I_{t,0} + p^I_{t,1}$ for any $t \in B^k$. Furthermore, we have the following equalities

$$\sum_{t \in B^k} p^I_t = 1$$  \hspace{1cm} (22)

$$\sum_{t \in B^k} p^I_t \cdot \sigma_t^{I,A} = \tau_k^{I,A}$$  \hspace{1cm} (23)

$$\sum_{t \in B^k} p^I_t \cdot \sigma_t^{I,B} = \tau_k^{I,B}$$  \hspace{1cm} (24)

We define $\sigma_t^{II,A}$, $\sigma_t^{II,B}$, and $p^II_t$ for case II, similarly.

We use $\xi$ to denote the empty string. So we have $p^I_\xi = p^II_\xi = 1$.

One important observation is that when the protocol starts, the local density matrices for Alice and Bob are identical in both cases:

$$\sigma_\xi^{I,A} = \sigma_\xi^{I,B} = \sigma_\xi^{II,A} = \sigma_\xi^{II,B} = \frac{I}{2^n}$$  \hspace{1cm} (25)

When the protocol proceeds, the local density matrices in two cases will become different, since the state $\Psi_\alpha$ is an entangled state, while $\frac{1}{2^n}$ is not. However, they cannot differ “too far”, as we shall prove in the following lemma:

**Lemma 4** For all $k = 0, 1, ..., s - 1$ and all $t \in B^k$, we have $p^I_t \cdot \sigma_t^{I,A} \preceq \sigma_t^{II,A}$ and $p^I_t \cdot \sigma_t^{I,B} \preceq \sigma_t^{II,B}$.

**Proof:** By induction. The base case is obvious. Now the inductive case. Consider the situation at the end of the $k$-th round. Suppose the first $k$ bits sent are $t[0], t[1], ..., t[k-1]$. WLOG we assume that in the $(k + 1)$-th round, Alice applies a super-operator $\mathcal{E}$ to her share of qubits, and send one bit $\alpha$ to Bob.

First we consider the density matrix for Alice. Notice that in general, $\alpha$ is the result of the measurement from $\mathcal{E}$. Therefore, we can “split” $\mathcal{E}$ into two positive super-operators $\mathcal{E}_0$ and $\mathcal{E}_1$, such that

$$\mathcal{E}_0(\sigma_t^{I,A}) = \frac{p^I_{t,0}}{p^I_t} \cdot \sigma_t^{I,A}$$  \hspace{1cm} (26)
\[ E_1(\sigma^{I,A}_t) = \frac{p_{t,1}}{p_t} \cdot \sigma^{I,A}_t \] (27)
\[ E_0(\sigma^{II,A}_t) = \frac{p_{t,0}}{p_t} \cdot \sigma^{II,A}_t \] (28)
\[ E_1(\sigma^{II,A}_t) = \frac{p_{t,1}}{p_t} \cdot \sigma^{II,A}_t \] (29)

Intuitively, \( E_0 \) corresponds to the case that \( a = 0 \) is sent, and \( E_1 \) corresponds to the case that \( a = 1 \) is sent.

By inductive hypothesis, we have
\[ p_t \cdot \sigma^{I,A}_t \preceq \sigma^{I,A}_t \] (30)

Combining (30), (26) and (28) with Claim 3 yields that
\[ p_{t,0} \cdot \sigma^{I,A}_{t,0} = E_0(p_t \cdot \sigma^{I,A}_t) \preceq E_0(\sigma^{II,A}_t) = \frac{p_{t,0}}{p_t} \cdot \sigma^{II,A}_{t,0} \preceq \sigma^{II,A}_{t,0} \] (31)

Combining (30), (27) and (29) with Claim 3 yields that
\[ p_{t,1} \cdot \sigma^{I,A}_{t,1} = E_1(p_t \cdot \sigma^{I,A}_t) \preceq E_1(\sigma^{II,A}_t) = \frac{p_{t,1}}{p_t} \cdot \sigma^{II,A}_{t,1} \preceq \sigma^{II,A}_{t,1} \] (32)

Now we consider the local density matrix for Bob. In case I, the qubits between Alice and Bob are entangled. Therefore, the bit Alice sends to Bob carries some information about his state. In terms of the density matrix, Bob’s local density matrix will “split” from \( \sigma^{I,B}_{t,0} \) to \( \sigma^{I,B}_{t,1} \) and \( \sigma^{I,B}_{t,1} \). Notice that Bob doesn’t perform any operation to his qubits, and thus we have
\[ \sigma^{I,B}_t = \frac{p_{t,0}}{p_t} \cdot \sigma^{I,B}_{t,0} + \frac{p_{t,1}}{p_t} \cdot \sigma^{I,B}_{t,1} \] (33)

In case II, the qubits between Alice and Bob are disentangled. Therefore, the bit sent by Alice carries no information about Bob’s own state. Thus Bob’s local density matrix remains unchanged. Thus we have
\[ \sigma^{II,B}_t = \sigma^{II,B}_{t,0} = \sigma^{II,B}_{t,1} \] (34)

By inductive hypothesis, we have
\[ p_t \cdot \sigma^{I,B}_t \preceq \sigma^{II,B}_t \] (35)

Combining (33), (34), and (35), we have
\[ p_{t,0} \cdot \sigma^{I,B}_{t,0} \preceq p_t \cdot \sigma^{I,B}_t \preceq p_{t,1} \cdot \sigma^{II,B}_t = \sigma^{II,B}_{t,0} \] (36)
\[ p_{t,1} \cdot \sigma^{I,B}_{t,1} \preceq p_t \cdot \sigma^{I,B}_t \preceq p_{t,1} \cdot \sigma^{II,B}_t = \sigma^{II,B}_{t,1} \] (37)

So the inductive case is proved.

Now we are ready to prove (20). After \( s \) bits are send, Alice will decide whether to succeed or fail. In case I, we use \( r_t \) to denote the probability that Alice choose to succeed conditioned on that the bits communicated are \( t[0], t[1], ..., t[s-1] \). Notice we have \( p_t \cdot \sigma^{I,A}_t \preceq \sigma^{II,A}_t \), and thus by Lemma 3 we know that in case II, the success probability is at least \( p_{t}^{I} \cdot r_t \).

Therefore, we have
\[ p = \sum_{t \in B^s} r_t \cdot p_{t}^{I} \] (38)
\[ q \geq \sum_{t \in B^s} r_t \cdot p_{t}^{I} \] (39)
which implies that

\[ q \geq \sum_{t \in B} r_t \cdot (p_I^t)^2 \geq 1 \geq \left( \sum_{t \in B} r_t \cdot (p_I^t)^2 \right)^2 \geq \frac{p^2}{2^t} \]

This proves the theorem.

6 Conclusions and Future Work

In this paper, we studied the classical communication complexity of entanglement distillation protocols in the setting of incomplete information, where the input states are mixed states or prepared adversarially. We study on the precise communication complexity of the protocols, as opposed to the asymptotic results. We also focus on the communication complexity of EDPs of the minimal requirement on yield, i.e., only 1 qubit pair is required as output. To the best of our knowledge, this is the first paper that studies classical communication complexity in the incomplete information setting, and also the first one to study the precise communication complexity. In our setting, many techniques don’t work any more, e.g., the Law of Large Numbers, the Central Limit Theorem (both only works in the aggregated setting, where one has many copies of the identical object), and the conversion from multi-round protocols to a single-round protocol (it requires that the input state is pure, and Alice and Bob have the complete information about it).

We considered 3 error models of the input state, and proved 3 corresponding results. The first 2 results are the “base cases” for the measure-\( r \) and the depolarization models. The result upper-bound the maximum possible fidelity of 0-bit EDPs (i.e., EDPs that don’t employ any communication). Interestingly, In this case, the trivial protocols that outputs a random pair are already optimal (or near-optimal, in the depolarization model). Despite of their simple statement, these results seems non-trivial to prove. A technique in the proof is an alternative definition of the fidelity of pure states. The technique may have its independent interest. The third result is an almost tight lower bound on the communication complexity of EDPs with respect to the fidelity model. Interestingly, although the lower bound is proven for protocols of minimal yield, it matches the upper bound given by a specific protocol that has very high yield. In this sense, the communication complexity seems to be oblivious to the yield of EDPs. This observation is somewhat surprising, since this is not the case for QECCs.

We view our paper as a first step toward the much greater project of understanding the communication complexity of EDPs in general. We feel that this paper opens much more open problems than the ones it solved. We list some of the open problems that we feel interesting:

1. More Lower Bounds
   Our first 2 results on the measure-\( r \) models and the depolarization model are indeed the “base-case” result, in that they only solved the problem where there is no communication at all. What happens when there is communication? In particular, in the measure-\( r \) model, if \( r = 1 \) and \( n \geq 3 \), then there exists deterministic EDPs of fidelity 1. This contrasts with the results that the maximum fidelity of 0-bit probabilistic EDPs is \( 1 - \frac{1}{2^n} \). What about 1-bit EDPs?

2. Tighter Lower Bounds
   Our result on EDPs with respect to the depolarization channel is not tight: we managed to prove an \( 1 - p/2 \) upper bound on the fidelity of 0-bit EDPs, but the lower bound given by the trivial protocol is \( 1 - 3p/4 \). We conjecture that \( 1 - 3p/4 \) is the right upper bound but was unable to prove it.

3. EDPs with Initial Entanglement
   Our paper didn’t consider EDPs where Alice and Bob share some initial entanglement (possible in the form of EPR pairs). How would the initial entanglement affect the communication complexity?
4. Deterministic vs. Probabilistic EDPs

All the results in our paper are proven against probabilistic EDPs, where Alice and Bob share a classical random tape. Can one prove stronger results against deterministic EDPs? Is there a trade-off between the amount of shared randomness used and the amount of classical communication?

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A Proofs to the Results for the Depolarization Model

Proof: [to Theorem 2]

Notice that by changing the basis, we can write the density matrix, $\rho_p$, in another form:

$$\rho_p = (1 - p) \cdot |\Phi^+\rangle \langle \Phi^+| + \frac{p}{4} \cdot (|00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11|)$$

which gives another interpretation of the depolarization model: each EPR pair, is kept intact with probability $(1 - p)$, and is replaced by a completely mixed state with probability $p$.

This observation leads us to consider a related error model, namely the “random-corrupt” model. In a random-corrupt model of parameter $r$, $r$ EPR pairs are randomly chosen from the $n$ pairs and are “corrupted” — meaning being replaced by the completely mixed state $\frac{1}{2}(|00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11|).

It is easy to see that a depolarization error model of parameter $p$ is simply a mixture of the random-corrupt models, with probability $\binom{n}{r} p^r (1 - p)^{n-r}$ being of parameter $r$.

We shall prove that the maximal fidelity of any 0-bit protocol over the random-corrupt model of parameter $r$ is $1 - r/2n$. This will imply our theorem, since we have

$$\sum_{r=0}^{n} \binom{n}{r} p^r (1 - p)^{n-r} (1 - \frac{r}{2n}) = 1 - \frac{p}{2}$$

As in the proof to Theorem 1, we only consider deterministic protocols.

We present more notations and definitions. As extended indicator vector, often denoted by $\mathbf{u}$, is an $n$-dimensional vector, whose each entry is an element from $\{00, 01, 10, 11, \ast\}$. Its degree is the number of entries that are not $\ast$. There are $4^r \binom{n}{r}$ extended indicator vectors of degree $r$. Each extended indicator vector $\mathbf{u}$ corresponds to a unique bipartite state $|\psi_{\mathbf{u}}\rangle$ in the following way:

$$|\psi_{\mathbf{u}}\rangle = \bigotimes_{j=0}^{n-1} |\phi_j\rangle,$$

where $|\phi_j\rangle = \begin{cases} |0\rangle_A |0\rangle_B & \text{if } v[j] = 00 \\ |0\rangle_A |1\rangle_B & \text{if } v[j] = 11 \\ |1\rangle_A |0\rangle_B & \text{if } v[j] = 10 \\ |1\rangle_A |1\rangle_B & \text{if } v[j] = 11 \\ \Phi^+ & \text{if } v[j] = \ast \end{cases}$

We call such an $|\psi_{\mathbf{u}}\rangle$ an extended error state.

An $2n$-dimensional 0-1 vector $x$ is consistent with an extended indicator vector $\mathbf{u}$, if $x[j]; x[n + j] = u[j]$ for all $j$ such that $v[j] \neq \ast$, and $x[j] = x[n + j]$ for all $j$ such that $v[j] = \ast$. We write this as $x \subseteq \mathbf{u}$. There are $2^{n-r}$ 0-1 vectors $x$ consistent with an indicator vector of degree $r$. We view $x$ as the concatenation of $2$ $n$-dimensional vectors: $x = l;r$, and we write them as $l = LT(x)$ and $r = RT(x)$.

With the notations, we can write the extended error states as

$$|\psi_{\mathbf{u}}\rangle = \frac{1}{2^{(n-r)/2}} \sum_{x \subseteq \mathbf{u}} |LT(x)\rangle^A |RT(x)\rangle^B \quad (44)$$

We define the discrepancy of $x$ to be $\text{DIS}(x) = LT(s) \oplus RT(s)$, where $\oplus$ stands for bit-wise XOR. The degree of discrepancy of $x$ is $|\text{DIS}(x)|$, the Hamming weight of $\text{DIS}(x)$. Clearly, there are $\binom{n}{d}2^n$ 0-1 vectors of dimension $2n$ having degree of discrepancy $d$. Furthermore, if $x$ has degree of discrepancy $d$, then the number of degree-$r$ extended indicator vectors $\mathbf{u}$ such that $x \subseteq \mathbf{u}$ is $\binom{n-d}{r-d}$. This is because for every $j$ such that $x[j] \neq x[n + j]$, we must have $u[j] = x[j]; x[n + j]$ in order to have $x \subseteq \mathbf{u}$. So the only freedom for $\mathbf{u}$ is to put $(n-r)$ $\ast$’s in the $n-d$ places where $x[j] = x[n + j]$.

Now we consider an arbitrary 0-bit protocol. We model it as Alice and Bob both applying a unitary operation to their share of qubits, outputs the first qubit and discard the rest. Suppose the unitary operators of Alice and Bob are $U_A$ and $U_B$. We denote the states under these operations by

$$U_A |x\rangle \rightarrow |\phi_x\rangle$$
$$U_B |x\rangle \rightarrow |\psi_x\rangle$$

Then as in the proof to Theorem 1, we shall prove that

$$\frac{1}{4^r \binom{n}{r}} \sum_{\text{deg} \mathbf{u} = r} \left[ \sum_{U \in \{I,X,Y,Z\}} \langle \psi_{\mathbf{u}} \rangle |(U_A \otimes U_B)^4(U \otimes U^*)(U_A \otimes U_B)| \psi_{\mathbf{u}}\rangle \right] \leq 4(1 - \frac{r}{2n}) \quad (45)$$
which will imply our theorem.

Notice that
\[(U_A \otimes U_B)|\psi_u\rangle = \frac{1}{2^{(n-r)/2}} \sum_{x \in u} |\phi_{LT(x)}\rangle |\psi_{RT(x)}\rangle\]
and so we have
\[\langle \psi_u |(U_A \otimes U_B)^{\dagger}(U \otimes U^*) (U_A \otimes U_B) | \psi_u \rangle = \frac{1}{2^{n-r}} \sum_{x \in u} \sum_{y \in u} \langle \phi_{LT(x)} | U | \phi_{LT(y)} \rangle \cdot \langle \psi_{RT(x)} | U^* | \psi_{RT(y)} \rangle\]

So we only need to prove that
\[\frac{1}{2^{n+r}(r)} \sum_{\deg u=r} \sum_{x \in u} \sum_{y \in u} \sum_{U \in \{I,X,Y,Z\}} \langle \phi_{LT(x)} | U | \phi_{LT(y)} \rangle \cdot \langle \psi_{RT(x)} | U^* | \psi_{RT(y)} \rangle \leq 4(1 - \frac{r}{2n}) \quad (46)\]

By Cauchy-Schwartz, we have
\[\left( \sum_{\deg u=r} \sum_{x \in u} \sum_{y \in u} \sum_{U \in \{I,X,Y,Z\}} \langle \phi_{LT(x)} | U | \phi_{LT(y)} \rangle \cdot \langle \psi_{RT(x)} | U^* | \psi_{RT(y)} \rangle \right)^{\frac{1}{2}} \leq \left( \sum_{\deg u=r} \sum_{x \in u} \sum_{y \in u} \sum_{U \in \{I,X,Y,Z\}} |\langle \phi_{LT(x)} | U | \phi_{LT(y)} \rangle|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{\deg u=r} \sum_{x \in u} \sum_{y \in u} \sum_{U \in \{I,X,Y,Z\}} |\langle \psi_{RT(x)} | U^* | \psi_{RT(y)} \rangle|^2 \right)^{\frac{1}{2}}\]

Now we estimate
\[\left\sum_{\deg u=r} \sum_{x \in u} \sum_{y \in u} \sum_{U \in \{I,X,Y,Z\}} |\langle \phi_{LT(x)} | U | \phi_{LT(y)} \rangle|^2 \right\]

Notice we can write \(x = LT(x); (LT(x) \oplus DIS(x))\) and \(y = LT(y); (LT(y) \oplus DIS(y))\). If there exists an extended indicator vector \(u\) such that \(x \subseteq u\) and \(y \subseteq u\), we must have \(DIS(x) = DIS(y)\). This is because that for every \(j\) such that \(DIS(x)[j] = 1\), \(x[j]\) and \(x[n+j]\) differ. Thus we must have \(v[j] = x[j]; x[n+j]\), which implies that \(v[j] = y[j] = y[n+j]\), and \(DIS(y)[j] = 1\). In fact, for every \(j\) such that \(DIS(x)[j] = 1\), we have \(x[j] = y[j]\) and \(x[n+j] = y[n+j]\).

So we have
\[\sum_{\deg u=r} \sum_{x \in u} \sum_{y \in u} \sum_{U \in \{I,X,Y,Z\}} |\langle \phi_a | U | \phi_b \rangle|^2 \sum_{c \in \mathbb{B}^n} \deg u = r: ([a:(a \oplus c)] \subseteq u \land [(b:(b \oplus c)) \subseteq u] 1 \]

by a substituting \(a\) for \(LT(x)\), \(b\) for \(LT(y)\), and \(c\) for \(DIS(x)\).

Now we fix \(a\) and \(b\), and compute
\[\sum_{c \in \mathbb{B}^n} \deg u = r: ([a:(a \oplus c)] \subseteq u \land [(b:(b \oplus c)) \subseteq u] 1 \]

We define \(k = |a \oplus b|\). For every \(j\) where \(a[j] \neq b[j]\), we must have \(c[j] = 0\) and \(u[j] = \ast\). For every \(j\) where \(a[j] = b[j]\), if we have \(c[j] = 1\), then we must \(u[j] = a[j]; (a[j] \oplus 1)\); if we have \(c[j] = 0\), then \(u\) can be either \(a[j]; a[j] \oplus \ast\) or \(\ast\). Therefore, of \(n-k\) positions where \(a[j] = b[j]\), \(r\) would be chosen where \(vcu\) has a non\-\(\ast\) entry. Of these \(r\) places, one has the freedom to choose \(c[j] = 0\) or \(c[j] = 1\). For all other places, \(c[j] = 0\) and \(u = \ast\). So we have
\[\sum_{c \in \mathbb{B}^n} \deg u = r: ([a:(a \oplus c)] \subseteq u \land [(b:(b \oplus c)) \subseteq u] 1 = 2^r \cdot \binom{n-k}{r}\]
In other words,

\[
\sum_{\deg \mathbf{u} = r} \sum_{x \in \mathbf{u}} \sum_{y \in \mathbf{u}} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_{LT(x)} | U | \phi_{LT(y)} \rangle|^2 = \sum_{a \in B^n} \sum_{b \in B^n} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_a | U | \phi_b \rangle|^2 \cdot 2^r \cdot \left( \frac{n - |a \oplus b|}{r} \right) \quad (47)
\]

Since $|\phi_a \rangle$'s are orthogonal, we have

\[
\sum_a \sum_b \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_a | U | \phi_b \rangle|^2 \leq 2^{n+2}
\]

Also by Lemma 2, we have

\[
\sum_a |\langle \phi_a | U | \phi_a \rangle|^2 \leq 2^{n+1}
\]

Therefore

\[
\sum_{a \in B^n} \sum_{b \in B^n} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_a | U | \phi_b \rangle|^2 \cdot 2^r \cdot \left( \frac{n - |a \oplus b|}{r} \right)
\]

\[
\leq \sum_a |\langle \phi_a | U | \phi_a \rangle|^2 \cdot 2^r \left[ \binom{n}{r} - \binom{n-1}{r} \right] + 2^r \binom{n-1}{r} \sum_{a \in B^n} \sum_{b \in B^n} \sum_{U \in \{I, X, Y, Z\}} |\langle \phi_a | U | \phi_b \rangle|^2
\]

\[
\leq 2^{n+r+1} \left[ \binom{n}{r} - \binom{n-1}{r} \right] + 2^{n+r+2} \binom{n-1}{r}
\]

\[
= 2^{n+r+2} \binom{n}{r} \left( 1 - \frac{r}{2n} \right)
\]

which implies (46), which implies the theorem.