ON CERTAIN ZETA FUNCTIONS ASSOCIATED WITH BEATTY SEQUENCES

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Abstract. Let \( \alpha > 1 \) be an irrational number of finite type \( \tau \). In this paper, we introduce and study a zeta function \( Z^{\#}_{\alpha}(r, q; s) \) that is closely related to the Lipschitz-Lerch zeta function and is naturally associated with the Beatty sequence \( \mathcal{B}(\alpha) := (\lfloor \alpha m \rfloor)_{m \in \mathbb{N}} \). If \( r \) is an element of the lattice \( \mathbb{Z} + \mathbb{Z} \alpha^{-1} \), then \( Z^{\#}_{\alpha}(r, q; s) \) continues analytically to the half-plane \( \{ \sigma > -1/\tau \} \) with its only singularity being a simple pole at \( s = 1 \). If \( r \not\in \mathbb{Z} + \mathbb{Z} \alpha^{-1} \), then \( Z^{\#}_{\alpha}(r, q; s) \) extends analytically to the half-plane \( \{ \sigma > 1 - 1/(2\tau^2) \} \) and has no singularity in that region.

1. Introduction and statement of results

The Lipschitz-Lerch zeta function is defined in the half-plane \( \{ \sigma := \Re s > 1 \} \) by an absolutely convergent series

\[
\zeta(z, q; s) := \sum_{n=0}^{\infty} \frac{e(zn)}{(n+q)^s},
\]

where \( e(t) := e^{2\pi it} \) for all \( t \in \mathbb{R} \), and by analytic continuation it extends to meromorphic function on the whole \( s \)-plane. The function \( \zeta(z, q; s) \) was introduced by Lipschitz [20] for real \( z \) and \( q > 0 \); see also Lipschitz [21]. It also bears the name of Lerch [19], who showed that for \( \Im z > 0 \) and \( q \in (0, 1) \) the functional equation

\[
\zeta(z, q; 1-s) = (2\pi)^{-s} \Gamma(s) \left( e(\frac{1}{2}s - zq)\zeta(-q, z; s) + e(-\frac{1}{2}s + zq)\zeta(q, 1-z; s) \right)
\]

holds; this is called Lerch’s transformation formula. For an interesting account of the analytic properties of the Lipschitz-Lerch zeta function and related functions, we refer the reader to the work of Lagarius and Li [15–18]; see also Apostol [2].

If \( z \in \mathbb{Z} \), then \( \zeta(z, q; s) = \zeta(0, q; s) \) is the Hurwitz zeta function; in this case, \( \zeta(z, q; s) \) has a simple pole at \( s = 1 \) but no other singularities in the \( s \)-plane. On the other hand, if \( z \in \mathbb{R} \setminus \mathbb{Z} \) or \( \Im z > 0 \), then \( \zeta(z, q; s) \) is an entire function of \( s \).

For a given real number \( \alpha > 0 \), the homogeneous Beatty sequence associated with \( \alpha \) is the sequence of natural numbers defined by

\[
\mathcal{B}(\alpha) := ([\alpha m])_{m \in \mathbb{N}},
\]

where \( [\cdot] \) denotes the floor function: \( [t] \) is the greatest integer \( \leq t \) for any \( t \in \mathbb{R} \). Beatty sequences appear in a wide variety of unrelated mathematical settings, and their arithmetic properties have been extensively explored in the literature; see, for example, [1, 3–6, 8, 9, 11, 12, 22, 23, 28] and the references therein.
In this paper, we introduce and study a variant of the Lipschitz-Lerch zeta function that is naturally associated with the Beatty sequence $B(\alpha)$. Specifically, let us denote
\[ Z_\alpha(r, q; s) := \sum_{n \in B(\alpha)} \frac{e(rn)}{(n + q)^s}, \]  
where $r \in \mathbb{R}$ and $q \in (0, 1)$. For technical reasons, the work in this paper is focused on properties of the function
\[ Z_\alpha^+(r, q; s) := e^{\pi ir} Z_\alpha(r, q; s) + e^{-\pi ir} Z_\alpha(-r, 1 - q; s). \]
Further, we assume that $\alpha > 1$ is irrational and of finite type (see §2.2). Note that for rational $\alpha$, the Beatty sequence $B(\alpha)$ is a finite union of arithmetic progressions, and therefore the analytic properties of $Z_\alpha(r, q; s)$ and $Z_\alpha^+(r, q; s)$ can be gleaned from well known properties of the Lipschitz-Lerch zeta function.

The series (1.1) converges absolutely the half-plane $\{\sigma > 1\}$, uniformly on compact regions, hence $Z_\alpha(r, q; s)$ is analytic there; this implies that $Z_\alpha^+(r, q; s)$ is analytic in $\{\sigma > 1\}$ as well.

Since $B(\alpha)$ is a set of density $\alpha^{-1}$ in the set of natural numbers, it is reasonable to expect that $Z_\alpha(r, q; s)$ is closely related to the function $\alpha^{-1}\zeta(r, q; s)$. This belief is strengthened by the fact that if $\alpha^{-1} + \beta^{-1} = 1$, then the set of natural numbers can be split as the disjoint union of $B(\alpha)$ and $B(\beta)$, so we have
\[ Z_\alpha(r, q; s) + Z_\beta(r, q; s) + q^{-s} = \alpha^{-1}\zeta(r, q; s) + \beta^{-1}\zeta(r, q; s). \]
Naturally, one might also expect that $Z_\alpha^+(r, q; s)$ is closely related to the function $\alpha^{-1}\zeta^+(r, q; s)$, where
\[ \zeta^+(r, q; s) := e^{\pi ir}\zeta(r, q; s) + e^{-\pi ir}\zeta(-r, 1 - q; s). \]
As it turns out, such expectations are erroneous. Our first theorem establishes that $Z_\alpha^+(r, q; s)$ has a simple pole at $s = 1$ whenever $r$ is an element of the lattice $\mathbb{Z} + \mathbb{Z}\alpha^{-1}$; this lattice is a dense subset of $\mathbb{R}$. By contrast, the function $\alpha^{-1}\zeta^+(r, q; s)$, being a linear combination of Lipschitz-Lerch zeta functions, can only have a pole when $r$ is an integer.

**Theorem 1.1.** Let $\alpha > 1$ be an irrational number of finite type $\tau$. Suppose that $r = k\alpha^{-1} + \ell$ for some integers $k$ and $\ell$, and let $q \in (0, 1)$. Then the function
\[ Z_\alpha^+(r, q; s) - \alpha^{-1}\zeta^+(r, q; s) \]  
continues analytically to the half-plane $\{\sigma > -1/\tau\}$ with a simple pole at $s = 1$ and no other singularities. The residue at $s = 1$ is $2(-1)^{\ell}\alpha^{-1}$ when $k = 0$, and it is $2(-1)^{\ell}(\sin(\pi k \alpha^{-1}))/((\pi k))$ for $k \neq 0$.

In particular, taking $r := 0$ and $q := \frac{1}{2}$ above, we have
\[ Z_\alpha^+(0, \frac{1}{2}; s) := 2Z_\alpha(0, \frac{1}{2}; s) = \sum_{n \in B(\alpha)} \frac{2}{(n + \frac{1}{2})^s} \]
and
\[ \zeta^+(0, \frac{1}{2}; s) := 2\zeta(0, \frac{1}{2}; s) = \sum_{n=0}^{\infty} \frac{2}{(n + \frac{1}{2})^s} = (2^{s+1} - 2)\zeta(s), \]
where $\zeta(s)$ is the Riemann zeta function studied by Riemann [24] in 1859. Since $\zeta(s)$ has a simple pole at $s = 1$ with residue one, from Theorem 1.1 we deduce the following corollary.

**Corollary 1.2.** Let $\alpha > 1$ be irrational of finite type $\tau$. The Dirichlet series

$$Z_\alpha(0, \frac{1}{2}; s) := \sum_{n \in \mathbb{B}(\alpha)} \frac{1}{(n + \frac{1}{2})^s}$$

converges absolutely in $\{\sigma > 1\}$ and extends analytically to $\{\sigma > -1/\tau\}$, where it has a simple pole at $s = 1$ with residue $2\alpha^{-1}$ and no other singularities.

When $r := 0$ and $q := \frac{1}{2}$ our proof of Theorem 1.1 shows that the function

$$F_\alpha^2(0, \frac{1}{2}; s) = \pi^{-s/2} \Gamma(s/2) \left(2Z_\alpha(0, \frac{1}{2}; s) - \alpha^{-1}(2^{s+1} - 2)\zeta(s) + 2^s\right)$$

is analytic at $s = 0$. Since $\Gamma(s/2)$ has a pole at $s = 0$, one sees that $Z_\alpha(0, \frac{1}{2}; s)$ takes the same value at $s = 0$ regardless of the choice of $\alpha$.

**Corollary 1.3.** For every irrational $\alpha > 1$ of finite type, $Z_\alpha(0, \frac{1}{2}; 0) = -\frac{1}{2}$.

Our second theorem is complementary to Theorem 1.1; it establishes that $Z_\alpha^2(r, q; s)$ does not have a pole at $s = 1$ whenever the real number $r$ is not contained in the lattice $\mathbb{Z} + \mathbb{Z}\alpha^{-1}$.

**Theorem 1.4.** Let $\alpha > 1$ be an irrational number of finite type $\tau$. Let $q \in (0, 1)$, and suppose that $r$ is a real number not of the form $k\alpha^{-1} + \ell$ with $k, \ell \in \mathbb{Z}$. Then the function (1.2) continues analytically to the half-plane $\{\sigma > 1 - 1/(2\pi^2)\}$ with no singularities in that region.

2. Preliminaries

2.1. **General notation.** Throughout the paper, we fix an irrational number $\alpha > 1$ of finite type $\tau = \tau(\alpha)$ (see §2.2), and we set $\gamma := \alpha^{-1}$. Note that $\gamma$ has the same type $\tau$.

As stated earlier, we write $e(t) := e^{2\pi it}$ for all $t \in \mathbb{R}$. We use $|t|$ and $\{t\}$ to denote the greatest integer not exceeding $t$ and the fractional part of $t$, respectively. The notation $\langle\langle t\rangle\rangle$ is used to represent the distance from the real number $t$ to the nearest integer; in other words,

$$\langle\langle t\rangle\rangle := \min_{n \in \mathbb{Z}} |t - n| = \min \left\{\{t\}, 1 - \{t\}\right\} \quad (t \in \mathbb{R}). \quad (2.1)$$

For every real number $t$, we denote by $[t]$ the integer that lies closest to $t$ if $t \notin \frac{1}{2} + \mathbb{Z}$, and we put $\lfloor t \rfloor := |t|$ if $t \in \frac{1}{2} + \mathbb{Z}$. Then

$$t = \begin{cases} 
\lfloor t \rfloor + \langle\langle t\rangle\rangle & \text{if } \{t\} \leq \frac{1}{2}, \\
\lfloor t \rfloor - \langle\langle t\rangle\rangle & \text{if } \{t\} > \frac{1}{2}.
\end{cases} \quad (2.2)$$

In what follows, any implied constants in the symbols $O$, $\ll$ and $\gg$ may depend on the parameters $\alpha, r, q, \varepsilon$ but are independent of other variables unless indicated otherwise. For given functions $F$ and $G$, the notations $F \ll G, G \gg F$
and \( F = O(G) \) are all equivalent to the statement that the inequality \( |F| \leq c|G| \) holds with some constant \( c > 0 \).

## 2.2. Discrepancy and type.

The **discrepancy** \( D(M) \) of a sequence of (not necessarily distinct) real numbers \( (a_m)_{m=1}^M \) contained in \([0, 1)\) is given by

\[
D(M) := \sup_{I \subseteq [0, 1)} \left| \frac{V(I, M)}{M} - |I| \right|
\]

where the supremum is taken over all intervals \( I \) in \([0, 1)\), \( V(I, M) \) is the number of positive integers \( m \leq M \) such that \( a_m \in I \), and \(|I|\) is the length of \( I \).

The **type** \( \tau = \tau(\gamma) \) of a given irrational number \( \gamma \) is defined by

\[
\tau := \sup \{ t \in \mathbb{R} : \liminf_{n \to \infty} n^t \langle \langle \gamma n \rangle \rangle = 0 \}.
\]

Using Dirichlet’s approximation theorem, one sees that \( \tau \geq 1 \) for every irrational number \( \gamma \). The theorems of Khinchin \([10]\) and of Roth \([25, 26]\) assert that \( \tau = 1 \) for almost all real numbers (in the sense of the Lebesgue measure) and all irrational algebraic numbers \( \gamma \), respectively; see also \([7, 27]\).

Given an irrational number \( \gamma \), the sequence of fractional parts \( \{n\gamma\}_{n=1}^\infty \) is known to be uniformly distributed in \([0, 1)\) (see \([14, \text{Example 2.1, Chapter 1}]\)). In the case that \( \gamma \) is of finite type, the following more precise statement holds (see \([14, \text{Theorem 3.2, Chapter 2}]\)).

**Lemma 2.1.** Let \( \gamma \) be a fixed irrational number of finite type \( \tau \). Then, for every \( \delta \in \mathbb{R} \) the discrepancy \( D_{\gamma, \delta}(M) \) of the sequence \( \{\gamma m + \delta\}_{m=1}^M \) satisfies the bound

\[
D_{\gamma, \delta}(M) \ll M^{-1/(\tau+\varepsilon)},
\]

where the implied constant depends only on \( \gamma \) and \( \varepsilon \).

## 2.3. Functional equations of theta functions.

**Lemma 2.2.** For any real numbers \( v, w \) let

\[
\Theta_{v,w}(u) := e\left(\frac{u}{2}w^2\right) \sum_{n \in \mathbb{Z}} e^{-\pi(n+v)^2u}e(wn) \quad (u > 0).
\]

Then

\[
\Theta_{v,w}(u) = u^{-1/2} \Theta_{w,-v}(u^{-1}).
\]

**Proof.** Let \( u > 0 \) be fixed, and put

\[
f(x) := e^{-\pi(x+v)^2u}e(wx) \quad (x \in \mathbb{R}).
\]

The Fourier transform of \( f \) is given by

\[
\hat{f}(x) := \int_{-\infty}^{\infty} f(y)e(xy)dy = \int_{-\infty}^{\infty} e^{g(x,y)}dy,
\]

where

\[
g(x, y) := -\pi(y + v)^2u + 2\pi i(x + w)y.
\]

Since

\[
g(x, y) = -\pi(y + v - i(x + w)u^{-1})^2u - 2\pi iv(x + w) - \pi(x + w)^2u^{-1},
\]

it follows that
it follows that
\[ \hat{f}(x) = e^{-\pi(x+w)^2u^{-1}} e(-v(x+w)) \int_{-\infty}^{\infty} e^{-\pi(y+iyu^{-1}+ixu^{-1})^2u} dy. \] (2.5)

Using Cauchy’s theorem to shift the line of integration vertically, we see that the integral in (2.5) is equal to
\[ \int_{-\infty}^{\infty} e^{-\pi y^2 u} dy = u^{-1/2} \int_{-\infty}^{\infty} e^{-\pi y^2} dy = u^{-1/2}. \]
Consequently,
\[ \hat{f}(x) = u^{-1/2} e^{-\pi(x+w)^2u^{-1}} e(-v(x+w)) \quad (x \in \mathbb{R}). \]

Applying the Poisson Summation Formula
\[ \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n), \]
we immediately deduce the functional equation (2.4).

2.4. The pulse wave. Let \( 1_\alpha \) denote the indicator function of \( B(\alpha) \); that is,
\[ 1_\alpha(n) := \begin{cases} 1 & \text{if } n = \lfloor \alpha m \rfloor \text{ for some } m \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases} \] (2.6)

In this notation we have
\[ Z_\alpha(r, q; s) = \sum_{n=1}^{\infty} \frac{1_\alpha(n)e(rn)}{(n+q)^s}. \]

Let \( X_\gamma \) be the periodic function defined by
\[ X_\gamma(t) := \begin{cases} 1 & \text{if } \{t\} \in (0, \gamma), \\ \frac{1}{2} & \text{if } t \in \mathbb{Z} \text{ or } t \in \gamma + \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases} \] (2.7)

Since \( X_\gamma \) is periodic of bounded variation, and \( X_\gamma(t) = \frac{1}{2}(X_\gamma(t^+) + X_\gamma(t^-)) \) for all \( t \in \mathbb{R} \), its Fourier series converges everywhere, and we have
\[ X_\gamma(t) = \lim_{K \to \infty} \sum_{|k| \leq K} \hat{X}_\gamma(k)e(kt), \]
where the Fourier coefficients are given by
\[ \hat{X}_\gamma(0) := \gamma \quad \text{and} \quad \hat{X}_\gamma(k) := \frac{1 - e(-k\gamma)}{2\pi ik} \quad (k \neq 0). \]

For any irrational \( \alpha > 1 \), it is easy to see that a natural number \( n \) lies in the Beatty sequence \( B(\alpha) \) if and only if \( \{-n\gamma\} \in (0, \gamma) \). Using this characterization, the indicator function \( 1_\alpha \) given by (2.6) satisfies
\[ 1_\alpha(n) = X_\gamma(-n\gamma) = \lim_{K \to \infty} \sum_{|k| \leq K} \hat{X}_\gamma(k)e(-kn\gamma). \] (2.8)

From now on, we regard \( 1_\alpha \) as a function on all of \( \mathbb{Z} \) by defining the value \( 1_\alpha(n) \) at an arbitrary integer \( n \) via the relation (2.8).
Using our hypothesis that \( \alpha \) is of finite type, the relation (2.8) can be made more explicit; namely, for any positive real number \( K \), we have the estimate
\[
\mathbf{1}_\alpha(n) = \sum_{|k| \leq K} \tilde{X}_\gamma(k)e(-kn\gamma) + O\left( \max\{1, |n|^{r+\varepsilon}\}K^{-1} \right) \quad (n \in \mathbb{Z}) \tag{2.9}
\]
for any given \( \varepsilon > 0 \). Indeed, for each nonzero integer \( n \) let
\[
S_K(n; u) := \sum_{k \leq u} (1 - e(-k\gamma))e(-kn\gamma) \quad (u > 0).
\]
Using standard estimates for exponential sums (see, e.g., Korobov \[13\]) we have
\[
S_K(n; u) \ll \langle \langle n\gamma \rangle \rangle^{-1} + \langle \langle n\gamma + \gamma \rangle \rangle^{-1}
\]
Since \( \gamma \) is of type \( \tau \), this implies that the bound
\[
S_K(n; u) \ll |n|^{r+\varepsilon}
\]
holds, and therefore
\[
\sum_{k > K} \tilde{X}_\gamma(k)e(-kn\gamma) = \int_K^\infty \frac{dS_K(n; u)}{2\pi iu} \ll \frac{|n|^{r+\varepsilon}}{K}.
\]
Bounding \( \sum_{k > K} \tilde{X}_\gamma(-k)e(kn\gamma) \) in a similar manner, we deduce (2.9) in the case that \( n \neq 0 \). When \( n = 0 \), we have by (2.8):
\[
\mathbf{1}_\alpha(0) = \gamma + \lim_{K \to \infty} \sum_{0 < |k| \leq K} e(k\gamma) e(\gamma) \frac{e(k\gamma)}{2\pi i k}.
\]
Writing
\[
S_K(0; u) := \sum_{k \leq u} e(k\gamma) \quad (u > 0),
\]
we have \( S_K(0; u) \ll \langle \gamma \rangle^{-1} = \alpha \ll 1 \), and therefore
\[
\sum_{k > K} \tilde{X}_\gamma(k) = \int_K^\infty \frac{dS_K(0; u)}{2\pi iu} \ll K^{-1}.
\]
This yields (2.9) in the case that \( n = 0 \).

3. The proofs

For all \( u > 0 \) we denote
\[
\Psi^+(r, q; u) := \sum_{n=0}^{\infty} e^{-\pi(n+q)^2u} e(rn),
\]
\[
\Psi_\alpha^+(r, q; u) := \sum_{n=0}^{\infty} e^{-\pi(n+q)^2u} \mathbf{1}_\alpha(n) e(rn).
\]
and we also put
\[
\Psi(r, q; u) := \sum_{n \in \mathbb{Z}} e^{-\pi(n+q)^2u} e(rn),
\]
\[
\Psi_\alpha(r, q; u) := \sum_{n \in \mathbb{Z}} e^{-\pi(n+q)^2u} \mathbf{1}_\alpha(n) e(rn).
\]
It is easy to see that
\[
\max \left\{ |\Psi(r, q; u)|, |\Psi_\alpha(r, q; u)| \right\} \ll e^{-\pi q u} \tag{3.1}
\]
holds for \(u > 0\), and the relations
\[
\Psi(r, q; u) = \Psi^+(r, q; u) + e(-r)\Psi^+(-r, 1 - q; u), \tag{3.2}
\]
\[
\Psi_\alpha(r, q; u) = \Psi^+_\alpha(r, q; u) + e(-r)\Psi^+_\alpha(-r, 1 - q; u), \tag{3.3}
\]
are immediate. Indeed, (3.2) follows from the fact that the polynomial \((n + \frac{1}{2})^2\) is invariant under the map \(n \mapsto -n - 1\). To prove (3.3), we note that (2.8) implies
\[
\mathbf{1}_\alpha(n) = \gamma + \lim_{K \to \infty} \sum_{0 < |k| \leq K} \frac{\sin(\pi k \gamma)}{\pi k} \cos(2\pi k \gamma (n + \frac{1}{2})) ,
\]
hence \(\mathbf{1}_\alpha(n) = \mathbf{1}_\alpha(-n - 1)\) for all \(n \in \mathbb{Z}\).

Next, recall that
\[
\pi^{-s/2} \Gamma(s/2) \mu^{-s} = \int_0^\infty e^{-\pi \mu^2 u} u^{s/2 - 1} du \quad (\sigma > 0)
\]
for every positive real number \(\mu\). Taking into account that \(\mathbf{1}_\alpha(0) = \frac{1}{2}\) in view of (2.7) and (2.8), the function
\[
F^\sharp_\alpha(r, q; s) := \pi^{-s/2} \Gamma(s/2) (Z_\alpha(r, q; s) - \gamma \zeta(r, q; s) + \frac{1}{2} q^{-s})
\]
satisfies the relation
\[
F^\sharp_\alpha(r, q; s) = \int_0^\infty (\Psi^+_\alpha(r, q; u) - \gamma \Psi^+(r, q; u)) u^{s/2 - 1} du \quad (\sigma > 1).
\]
Therefore, using (3.2) and (3.3) it follows that
\[
F^\sharp_\alpha(r, q; s) := e^{\pi ir} F^+_\alpha(r, q; s) + e^{-\pi ir} F^+_\alpha(-r, 1 - q; s)
\]
\[
= \pi^{-s/2} \Gamma(s/2) (Z^\sharp_\alpha(r, q; s) - \gamma \zeta^2(r, q; s) + \frac{1}{2} e^{\pi ir} q^{-s} + \frac{1}{2} e^{-\pi ir} (1 - q)^{-s})
\]
satisfies the relation
\[
F^\sharp_\alpha(r, q; s) = e^{\pi ir} \int_0^\infty (\Psi_\alpha(r, q; u) - \gamma \Psi(r, q; u)) u^{s/2 - 1} du \quad (\sigma > 1).
\]
To prove Theorems 1.1 and 1.4, it suffices to show that \(F^\sharp_\alpha(r, q; s)\) continues analytically in an appropriate manner according to whether or not \(r\) lies in the lattice \(\mathbb{Z} + \mathbb{Z} \alpha^{-1}\).

To simplify the notation, we put
\[
F(s) := F^\sharp_\alpha(r, q; s) \quad \text{and} \quad \Phi(u) := \Psi_\alpha(r, q; u) - \gamma \Psi(r, q; u),
\]
and write
\[
F(s) = e^{\pi ir} \left( F_0(s) + F_\infty(s) \right),
\]
where
\[
F_0(s) := \int_0^1 \Phi(u) u^{s/2 - 1} du \quad \text{and} \quad F_\infty(s) := \int_1^\infty \Phi(u) u^{s/2 - 1} du.
\]
In view of (3.1) it is clear that the integral \(F_\infty(s)\) converges absolutely and uniformly on compact regions of \(\mathbb{C}\), hence \(F_\infty(s)\) continues to an entire function of \(s \in \mathbb{C}\). Thus, the analytic continuation of \(F(s)\) reduces to that of \(F_0(s)\).
Let $K$ be a real-valued function such that $K(u) \geq 2$ for all $u > 0$. For the moment, let $u > 0$ be fixed. Using (2.9) along with the bound
\[
\sum_{n \in \mathbb{Z}} e^{-\pi(n+q)^2u} \max\{1, |n|^{\tau+\varepsilon}\} \ll 1,
\]
we have
\[
\Psi_0(r, q; u) = \sum_{n \in \mathbb{Z}} e^{-\pi(n+q)^2u} \mathcal{E}(rn) \sum_{|k| \leq K(u)} \tilde{X}_\gamma(k) \mathcal{E}(-kn\gamma) + O(K(u)^{-1}).
\]
Reversing the order of summation and recalling (2.3), we see that
\[
\Psi_0(r, q; u) = \sum_{|k| \leq K(u)} \tilde{X}_\gamma(k) \mathcal{E}(\frac{1}{2}q(k\gamma - r)) \Theta_{q, r - k\gamma}(u) + O(K(u)^{-1}).
\]
Since $\tilde{X}_\gamma(0) = \gamma$, and $\Psi(r, q; u) = e(-\frac{1}{2}qr) \Theta_q(u)$ by (2.3), we derive the estimate
\[
\Phi(u) = \Phi_K(u) + O(K(u)^{-1}),
\]
where
\[
\Phi_K(u) := \sum_{0 < |k| \leq K(u)} \tilde{X}_\gamma(k) \mathcal{E}(\frac{1}{2}q(k\gamma - r)) \Theta_{q, r - k\gamma}(u). \tag{3.4}
\]
In particular,
\[
F_0(s) = \int_1^\infty \Phi_K(u^{-1})u^{-s/2-1}du + O\left(\int_0^1 K(u)^{-1}u^{\sigma/2-1}du\right). \tag{3.5}
\]
By the functional equation (2.4) and the definition (2.3),
\[
\Theta_{q, r - k\gamma}(u) = u^{-1/2}\Theta_{r - k\gamma, -q}(u^{-1}) = u^{-1/2}e\left(\frac{1}{2}q(k\gamma - r)\right) \sum_{n \in \mathbb{Z}} e^{-\pi(n+r-k\gamma)^2u^{-1}} \mathcal{E}(-qn).
\]
Combining this expression with (3.4) we have
\[
\Phi_K(u^{-1}) = u^{1/2} \sum_{0 < |k| \leq K(u^{-1})} \tilde{X}_\gamma(k) \mathcal{E}(q(k\gamma - r)) \sum_{n \in \mathbb{Z}} e^{-\pi(n+r-k\gamma)^2u} \mathcal{E}(-qn). \tag{3.6}
\]
Now, we make the specific choice
\[
K(u) = K_L(u) := \max\{2, u^{-\varepsilon}L\} \quad \text{with} \quad \theta := \frac{1 - \varepsilon}{2(\tau + \varepsilon)},
\]
where $L$ is a large positive real number, and $\varepsilon > 0$ is fixed (and small). In particular, for all large $u$ we have
\[
\Phi_{K_L}(u^{-1}) = u^{1/2} \sum_{0 < |k| \leq u^{\theta}L} \tilde{X}_\gamma(k) \mathcal{E}(q(k\gamma - r)) \sum_{n \in \mathbb{Z}} e^{-\pi(n+r-k\gamma)^2u} \mathcal{E}(-qn).
\]
Note that (3.5) takes the form
\[
F_0(s) = \int_1^\infty \Phi_{K_L}(u^{-1})u^{-s/2-1}du + O\left(L^{-1}(\sigma + 2\theta)^{-1}\right) \tag{3.7}
\]
provided that $\sigma > -2\theta$.

To proceed further, for each integer $k$ we write
\[
r - k\gamma = \left\lfloor r - k\gamma \right\rfloor + \nu_k\langle r - k\gamma \rangle
\]
with \( \nu_k \in \{\pm 1\} \) as in (2.2). Making the change of variables \( n \mapsto n - \lfloor r - k\gamma \rfloor \) in the inner summation of (3.6), it follows that
\[
\Phi_{K_L}(u^{-1}) = u^{1/2} \sum_{0 < |k| \leq u^\theta L} \tilde{X}_\gamma(k) e(-q\nu_k \langle r - k\gamma \rangle) \sum_{n \in \mathbb{Z}} e^{-\pi(n + \nu_k \langle r - k\gamma \rangle)^2 u} e(-qn).
\]
We introduce the notation
\[
\delta_u := \min \{ \langle r - k\gamma \rangle : |k| \leq u^\theta L \}, \tag{3.8}
\]
and let \( \kappa_u \) denote an integer for which
\[
|\kappa_u| \leq u^\theta L \quad \text{and} \quad \langle r - \kappa_u\gamma \rangle = \delta_u.
\]
Then we have
\[
\Phi_{K_L}(u^{-1}) = u^{1/2} (\Upsilon_L^{(1)}(u) + \Upsilon_L^{(2)}(u) + \Upsilon_L^{(3)}(u)),
\]
where
\[
\Upsilon_L^{(1)}(u) := \sum_{0 < |k| \leq u^\theta L} \tilde{X}_\gamma(k) e(-q\nu_k \langle r - k\gamma \rangle) \sum_{n \in \mathbb{Z}} e^{-\pi(n + \nu_k \langle r - k\gamma \rangle)^2 u} e(-qn),
\]
\[
\Upsilon_L^{(2)}(u) := \sum_{0 < |k| \leq u^\theta L} \tilde{X}_\gamma(k) e(-q\nu_k \langle r - k\gamma \rangle) e^{-\pi \langle r - k\gamma \rangle^2 u},
\]
\[
\Upsilon_L^{(3)}(u) := \tilde{X}_\gamma(\kappa_u) e(-q\nu_{\kappa_u} \delta_u) e^{-\pi \delta_u^2}.\]
By (3.7) it follows that the estimate
\[
F_0(s) = G_L^{(1)}(s) + G_L^{(2)}(s) + G_L^{(3)}(s) + O(L^{-1}(\sigma + 2\theta)^{-1})
\]
holds provided that \( \sigma > -2\theta \), where
\[
G_L^{(j)}(s) := \int_1^\infty \Upsilon_L^{(j)}(u) u^{-s/2 - 1/2} du \quad (j = 1, 2, 3).
\]
Thus, the analytic continuation of \( F_0(s) \) rests on the analytic properties of the integrals \( G_L^{(j)}(s) \).
Since \( \tilde{X}_\gamma(k) \ll |k|^{-1} \) and \( \langle r - k\gamma \rangle \leq 1/2 \) for every \( k \neq 0 \), the bound
\[
\Upsilon_L^{(1)}(u) \ll u^{1/2} e^{-\pi u/4} \log(\max\{2, u^\theta L\})
\]
is obvious. For fixed \( L \), this implies that the integral \( G_L^{(1)}(s) \) converges absolutely for all \( s \in \mathbb{C} \), uniformly on compact regions, and hence \( G_L^{(1)}(s) \) is an entire function of \( s \).
To determine the analytic behavior of \( G_L^{(2)}(s) \), we consider two distinct cases according to the size of \( \delta_u \).

**Case 1:** \( \delta_u \geq u^{-1/2} \log u \). By the definition of \( \delta_u \) (see (3.8)) it follows that \( \langle r - k\gamma \rangle \geq u^{-1/2} \log u \) for all \( k \) such that \( |k| \leq u^\theta L \). Since \( \tilde{X}_\gamma(k) \ll |k|^{-1} \) for every \( k \neq 0 \), we have
\[
\left| \Upsilon_L^{(2)}(u) \right| \ll \sum_{0 < |k| \leq u^\theta L} |k|^{-1} e^{-\pi \langle r - k\gamma \rangle^2 u} \ll \log(u^\theta L) u^{-\pi \log u}. \tag{3.9}
\]
Note that the bound
\[
\left| \Upsilon_L^{(3)}(u) \right| \ll u^{-\pi \log u} \tag{3.10}
\]
also holds in this case.

**Case 2:** \( \delta_u \leq u^{-1/2} \log u \). Let \( k \) be such that \( 0 < |k| \leq u^\theta L \) and \( k \neq \kappa_u \). Then

\[
\begin{align*}
r - k\gamma &= \lfloor r - k\gamma \rfloor + \nu_k \langle r - k\gamma \rangle, \\
r - \kappa_u \gamma &= \lfloor r - \kappa_u \gamma \rfloor + \nu_{\kappa_u} \delta_u, \\
\kappa_u \gamma - k\gamma &= [\kappa_u \gamma - k\gamma] + \nu \langle \kappa_u \gamma - k\gamma \rangle,
\end{align*}
\]

where \( \nu \in \{\pm 1\} \) as in (2.2), and therefore

\[
\langle r - k\gamma \rangle \equiv \nu_k \nu_{\kappa_u} \delta_u + \nu \nu_{\kappa_u} \langle \kappa_u \gamma - k\gamma \rangle \pmod{1} \tag{3.11}
\]

Noting that \( |\kappa_u - k| \leq 2u^\theta L \), and using the fact that \( \gamma \) is of type \( \tau \), we also have

\[
\langle \kappa_u \gamma - k\gamma \rangle \gg |k - \kappa_u|^{-\tau - \varepsilon} \gg (u^\theta L)^{-\tau - \varepsilon} = L^{-\tau - \varepsilon} u^{-1/2 + \varepsilon/2}. \tag{3.12}
\]

As \( \delta_u = o(u^{-1/2 + \varepsilon/2}) \) as \( u \to \infty \), from (3.11) and (3.12) we derive the lower bound

\[
\langle r - k\gamma \rangle \gg L^{-\tau - \varepsilon} u^{-1/2 + \varepsilon/2} \quad (0 < |k| \leq u^\theta L, \ k \neq \kappa_u).
\]

Arguing as in Case 1, this implies that

\[
|\Upsilon_L^{(2)}(u)| \ll \log(u^\theta L) \exp(-\pi L^{-2\varepsilon - 2\varepsilon u^\varepsilon}). \tag{3.13}
\]

For fixed \( L \), the bounds (3.9) and (3.13) together imply that the integral \( G_L^{(2)}(s) \) converges absolutely for all \( s \in \mathbb{C} \), uniformly on compact regions, and therefore \( G_L^{(2)}(s) \) is an entire function of \( s \).

Turning now to the analytic behavior of \( G_L^{(3)}(s) \), we consider two distinct cases according to whether or not \( \delta_u \) vanishes on the interval \((1, \infty)\).

First, suppose that \( \delta_u = 0 \) for some \( u > 1 \). In view of the definition (3.8), this condition is equivalent to the statement that \( r = k\gamma + \ell \) for some (uniquely determined) integers \( k \) and \( \ell \). In this case, one has \( \delta_u = 0 \) and \( \kappa_u = k \) for all sufficiently large \( u \). In particular, for some sufficiently large real number \( U_L \), one sees that \( \Upsilon_L^{(3)}(u) = \widetilde{X}_\gamma(k) \) once \( u \geq U_L \). Consequently, if we denote

\[
G_L^{(4)}(s) := G_L^{(3)}(s) - \frac{2\widetilde{X}_\gamma(k)}{s - 1} = \int_1^{U_L} (\Upsilon_L^{(3)}(u) - \widetilde{X}_\gamma(k)) u^{-s/2 - 1/2} du,
\]

then clearly \( G_L^{(4)}(s) \) converges absolutely for all \( s \in \mathbb{C} \), uniformly on compact regions, and so \( G_L^{(4)}(s) \) is an entire function of \( s \). Putting everything together, we have therefore shown that

\[
F^\varepsilon_\alpha(r, q; s) := F^\varepsilon_\alpha(r, q; s) - \frac{2e^{\pi i r} \widetilde{X}_\gamma(k)}{s - 1} = H_L(s) + O(L^{-1}(\sigma + 2\theta)^{-1}),
\]

where

\[
H_L(s) := e^{\pi i r} (F_\infty(s) + G_L^{(1)}(s) + G_L^{(2)}(s) + G_L^{(4)}(s))
\]

is an entire function of \( s \in \mathbb{C} \) for any fixed \( L \). The sequence \( (H_L(s))_{L \geq 1} \) converges uniformly to \( F^\varepsilon_\alpha(r, q; s) \) on every compact subset of the half-plane \( \{\sigma > -2\theta\} \), hence \( F^\varepsilon_\alpha(r, q; s) \) is analytic in the same region. Since \( \theta \to 1/(2\tau) \) as \( \varepsilon \to 0^+ \), Theorem 1.1 follows.

Next, we suppose that \( \delta_u \neq 0 \) for all \( u > 1 \). Observe that the map \( u \to \delta_u \) is a positive nonincreasing step function which tends to zero as \( u \to \infty \). Let
Moreover, (3.10) immediately yields (3.14) in the case that \( j \in \mathbb{N} \) above), it follows that every nonnegative integer \( j \) for every \( u \in (1, \infty) \), independent of the index \( j \), for all \( u \in (1, \infty) \), let \( I_j \) denote the open interval \((u_j, u_{j+1})\). To prove Theorem 1.4, it suffices to establish the upper bound

\[
|\Upsilon_L^{(3)}(u)| \ll L^{-292} u^{-292} (u \in I_j)
\]

for every \( j \geq 0 \), where the implied constant in (3.14) may depend on \( L \) but is independent of the index \( j \). Indeed, since

\[
G_L^{(3)}(s) = \int_1^\infty \Upsilon_L^{(3)}(u) u^{-s/2-1/2} du = \sum_{j \geq 0} \int_{I_j} \Upsilon_L^{(3)}(u) u^{-s/2-1/2} du,
\]

the bound (3.14) implies that the integral \( G_L^{(3)}(s) \) converges absolutely throughout the half-plane \( \{ \sigma > 1-2\theta^2 \} \), uniformly on compact regions, and thus \( G_L^{(3)}(s) \) is analytic in that region. Then

\[
F^{(3)}_{\alpha}(r, q; s) = H_L(s) + O\left( L^{-1}(\sigma+2\theta)^{-1} \right),
\]

where

\[
H_L(s) := e^{\pi i r} \left( F_{\infty}(s) + G_L^{(1)}(s) + G_L^{(2)}(s) + G_L^{(3)}(s) \right)
\]

is analytic in the half-plane \( \{ \sigma > 1-2\theta^2 \} \). The sequence \( (H_L(s))_{L \geq 1} \) converges uniformly to \( F^{(3)}_{\alpha}(r, q; s) \) on every compact subset of \( \{ \sigma > 1-2\theta^2 \} \), hence \( F^{(3)}_{\alpha}(r, q; s) \) is analytic in that half-plane. Since \( \theta \to 1/(2\tau) \) as \( \varepsilon \to 0^+ \), Theorem 1.4 follows.

It remains to establish (3.14). To this end, put

\[
\Omega^+ := \{ j \geq 0 : \delta_u > u^{-1/2} \log u \quad \text{for all} \quad u \in I_j \}.
\]

\[
\Omega^- := \{ j \geq 0 : \delta_u < u^{-1/2} \log u \quad \text{for all} \quad u \in I_j \}.
\]

By the manner in which the sequence \((u_j)_{j \geq 0}\) is constructed (especially, see \((ii)\) above), it follows that every nonnegative integer \( j \) lies either in \( \Omega^+ \) or in \( \Omega^- \). Moreover, (3.10) immediately yields (3.14) in the case that \( j \in \Omega^+ \). Therefore, it remains to show that (3.14) holds for integers \( j \in \Omega^- \).

Let \( j \in \Omega^- \) be fixed. For all \( u \in I_j \) we have \( \delta_u < u^{-1/2} \log u \) (by \((ii)\) above) and \( \kappa_u \neq 0 \) (since \( \delta_u \neq 0 \)). Using the estimates

\[
\hat{\mathcal{K}}_\gamma(\kappa_u) = \frac{1 - e(-\kappa_u \gamma)}{2\pi i \kappa_u} = \frac{1 - e(-r)}{2\pi i \kappa_u} \left( 1 + O\left( u^{-1/2} \log u \right) \right)
\]

and

\[
e(-q \nu_{\kappa_u} \delta_u) = 1 + O\left( u^{-1/2} \log u \right),
\]

we derive that

\[
\Upsilon_L^{(3)}(u) = \frac{1 - e(-r)}{2\pi i \kappa_u} e^{-\pi \delta_u^2} \left( 1 + O\left( u^{-1/2} \log u \right) \right) \ll \kappa_u^{-1}.
\]

Consequently, to prove (3.14) it is enough to establish the lower bound

\[
\kappa_u \gg_L u^{292} (u \in I_j).
\]

(3.15)
Let \( \Delta_j := \delta_{u_j} \). Using \((i)\) and \((ii)\) above, and taking into account that \( \delta_u \) is a right-continuous function of \( u \) by (3.8), we see that \( \delta_u = \Delta_j \) and \( \delta_u < u^{-1/2} \log u \) for all \( u \in \mathcal{I}_j \). Put \( k_j := \kappa_{u_j} \), so that \( \langle r - k_j \gamma \rangle = \Delta_j \), and note that \( |k_j| \leq u^\theta_j L \).

On the other hand, for any integer \( k \) with \( |k| < u^\theta_j L \) write \( |k| = u^\theta_j L \) with some real number \( u < u_j \); then

\[
\langle\langle r - k \gamma \rangle \rangle \geq \delta_u > \Delta_j.
\]

This argument shows that

\[
|k_j| = u^\theta_j L. \tag{3.16}
\]

Since \( j \in \Omega^- \), the argument given in Case 2 implies that there is precisely one integer \( k \) that satisfies both inequalities

\[
|k| < u^\theta_{j+1} L \quad \text{and} \quad \langle\langle r - k \gamma \rangle \rangle \leq u_j^{-1/2} \log u_j \tag{3.17}
\]

(namely, the integer \( k = \kappa_j \)). On the other hand, using (2.1) in combination with the definition of discrepancy and Lemma 2.1, one sees that the number of integers \( k \) satisfying (3.17) is

\[
2u_j^{-1/2} \log u_j \cdot u^\theta_{j+1} L + O \left( (u^\theta_{j+1} L)^{1-1/(\tau+\varepsilon)} \right).
\]

For large \( j \), this leads to a contradiction unless both bounds

\[
2u_j^{-1/2} \log u_j \cdot u^\theta_{j+1} L \ll 1
\]

and

\[
2u_j^{-1/2} \log u_j \cdot u^\theta_{j+1} L \ll (u^\theta_{j+1} L)^{1-1/(\tau+\varepsilon)}
\]

satisfied. We deduce that

\[
u^\theta_{j+1} \ll_L u_j. \tag{3.18}
\]

Now let \( u \in \mathcal{I}_j \). Since

\[
|\kappa_u| \leq u^\theta L \quad \text{and} \quad \langle\langle r - \kappa_u \gamma \rangle \rangle = \delta_u = \Delta_j,
\]

the integer \( k = \kappa_u \) satisfies both inequalities in (3.17). Consequently, \( \kappa_u = k_j \), and using (3.16) and (3.18) we have

\[
|\kappa_u| = |k_j| = u^\theta_j L \gg_L u^\theta_{j+1} L^2 > u^\theta L^2.
\]

This is the required bound (3.15), and our proof of Theorem 1.4 is complete.

REFERENCES

[1] A. Abercrombie, “Beatty sequences and multiplicative number theory.” Acta Arith. 70 (1995), 195–207.
[2] T. Apostol, “On the Lerch zeta function.” Pacific J. Math. 1 (1951), 161–167.
[3] W. Banks and I. Shparlinski, ‘Prime numbers with Beatty sequences,’ Colloq. Math. 115 (2009), no. 2, 147–157.
[4] W. Banks and I. Shparlinski, “Non-residues and primitive roots in Beatty sequences.” Bull. Austral. Math. Soc. 73 (2006), 433–443.
[5] W. Banks and I. Shparlinski, “Short character sums with Beatty sequences.” Math. Res. Lett. 13 (2006), 539–547.
[6] A. Begunts, “An analogue of the Dirichlet divisor problem.” Moscow Univ. Math. Bull. 59 (2004), no. 6, 37–41.
[7] Y. Bugeaud, Approximation by algebraic numbers. Cambridge Tracts in Mathematics, 160. Cambridge University Press, Cambridge, 2004.
A. Fraenkel and R. Holzman, “Gap problems for integer part and fractional part sequences.” *J. Number Theory* 50 (1995), 66–86.

V. Guo, “Piatetski-Shapiro primes in a Beatty sequence.” *J. Number Theory* 156 (2015), 317–330.

A. Khinchin, “Zur metrischen Theorie der diophantischen Approximationen.” *Math. Z.* 24 (1926), no. 4, 706–714.

T. Komatsu, “A certain power series associated with a Beatty sequence.” *Acta Arith.* 76 (1996), 109–129.

T. Komatsu, “The fractional part of $n \vartheta + \varphi$ and Beatty sequences.” *J. Théor. Nombres Bordeaux* 7 (1995), 387–406.

N. Korobov, *Exponential sums and their applications.* Translated from the 1989 Russian original by Yu. N. Shakhov. Mathematics and its Applications (Soviet Series), 80. Kluwer Academic Publishers Group, Dordrecht, 1992.

L. Kuipers and H. Niederreiter, *Uniform distribution of sequences.* Pure and Applied Mathematics. Wiley-Interscience, New York-London-Sydney, 1974.

J. Lagarias and W.-C. Li, “The Lerch zeta function I. Zeta integrals.” *Forum Math.* 24 (2012), no. 1, 1–48.

J. Lagarias and W.-C. Li, “The Lerch zeta function II. Analytic continuation.” *Forum Math.* 24 (2012), no. 1, 49–84.

J. Lagarias and W.-C. Li, “The Lerch zeta function III. Polylogarithms and special values.” *Res. Math. Sci.* 3 (2016), Paper No. 2, 54 pp.

J. Lagarias and W.-C. Li, “The Lerch zeta function IV. Hecke operators.” *Res. Math. Sci.* 3 (2016), Paper No. 33, 39 pp.

M. Lerch, “Note sur la fonction $\mathfrak{R}(w, x, s) = \sum_{k=0}^{\infty} \frac{e^{2k\pi i x}}{(w+k)^s}$.” *Acta Math.* 11 (1887), no. 1-4, 19–24.

R. Lipschitz, “Untersuchung einer aus vier Elementen gebildeten Reihe.” *J. Reine Angew. Math.* 54 (1857), 313–328.

R. Lipschitz, “Untersuchung der Eigenschaften einer Gattung von unendlichen Reihen.” *J. Reine Angew. Math.* 105 (1889), 127–156.

G. Li and W. Zhai, “The divisor problem for the Beatty sequences.” *Acta Math. Sinica* 47 (2004), 1213–1216 (in Chinese).

K. O’Bryant, “A generating function technique for Beatty sequences and other step sequences.” *J. Number Theory* 94 (2002), 299–319.

B. Riemann, “Ueber die Anzahl der Primzahlen unter einer gegebenen Größe.” *Monatsberichte der Berliner999, 1859.

K. Roth, “Rational approximations to algebraic numbers.” *Mathematika* 2 (1955), 1–20.

K. Roth, “Corrigendum to ‘Rational approximations to algebraic numbers’.” *Mathematika* 2 (1955), 168.

W. Schmidt, *Diophantine approximation.* Lecture Notes in Mathematics, 785. Springer, Berlin, 1980.

R. Tijdeman, “Exact covers of balanced sequences and Fraenkel’s conjecture.” *Algebraic number theory and Diophantine analysis (Graz, 1998)*, 467–483, de Gruyter, Berlin, 2000.

I. Vinogradov, *The method of trigonometrical sums in the theory of numbers.* Dover Publications, Inc., Mineola, NY, 2004.

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