Quantum-classical correspondence for supersymmetric Gaudin magnets with boundary

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Abstract

We extend duality between the quantum integrable Gaudin models with boundary and the classical Calogero-Moser systems associated with root systems of classical Lie algebras \( B_N, C_N, D_N \) to the case of supersymmetric \( gl(m|n) \) Gaudin models with \( m + n = 2 \). Namely, we show that the spectra of quantum Hamiltonians for all such magnets being identified with the classical particles velocities provide the zero level of the classical action variables.

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1 Introduction: an overview

**KZ equations and many-body systems.** In this paper we study the quantum-classical duality appeared previously in a number of different contexts [7, 8, 16, 18]. In the general case it is a certain relation between classical integrable many-body systems and quantum spin chains or Gaudin models. In its simplest form the duality relation follows from the quasiclassical limit of the Matsuo-Cherednik projection [12]. Namely, consider the $\mathfrak{gl}(2)$ Knizhnik-Zamolodchikov equations

$$
\kappa \partial_{z_i} \Psi = H^G_i \Psi, \quad \Psi \in \mathcal{H}, \quad i = 1, \ldots, N, \hspace{1cm} (1.1)
$$

where the operators $H^G_i$ are the Gaudin Hamiltonians [6]

$$
H^G_i = w^{(i)} + \hbar \sum_{k \neq i} \frac{P_{ik}}{z_i - z_k}, \quad i = 1, \ldots, N \hspace{1cm} (1.2)
$$

acting on the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$, $P_{ik}$ are permutation operators exchanging $i$-th and $k$-th tensor components of $\mathcal{H}$, $\hbar$ and $\kappa$ are constant parameters (in general complex), and $w^{(i)}$ acts as constant diagonal (twist) matrix $\text{diag}(\omega, -\omega)$ in the $i$-th component of $\mathcal{H}$.

The Matsuo-Cherednik construction provides a symmetrized projection $\langle \Omega | \Psi \rangle$ of the solution $\Psi$ (1.1) to a solution of the eigenvalue problem for the quantum Calogero-Moser $N$-body system [2]:

$$
\left( -\frac{\kappa^2}{2} \sum_{i=1}^{N} \partial_{z_i}^2 + (\hbar - \kappa) \hbar \sum_{i<j}^{N} \frac{1}{(z_i - z_j)^2} \right) \langle \Omega | \Psi \rangle = E \langle \Omega | \Psi \rangle, \hspace{1cm} (1.3)
$$

where the eigenvalue $E$ is a function of the twist parameter $\omega$. The dual vector $\langle \Omega | \in \mathcal{H}^*$ is invariant with respect to the action of permutation operators. Details and generalizations can be found in [5].

The quasiclassical limit $\kappa \to 0$ of the Knizhnik-Zamolodchikov equations (1.1), with $\Psi$ expanded as $\Psi = (\Psi_0 + \kappa \Psi_1 + \ldots)e^{S/\kappa}$, with some function $S = S(z_1, \ldots, z_N)$ leads to the eigenvalue problems

$$
H^G_i \psi = H^G_i \Psi, \quad H^G_i = \partial_{z_i} S, \quad \psi = \Psi_0 \in \mathcal{H}, \quad i = 1, \ldots, N \hspace{1cm} (1.4)
$$
for the commuting Hamiltonians of the Gaudin model. At the same time the quasiclassical limit of the spectral problem (1.3) provides some value $H^{CM} = E_0(\omega)$ of the classical Calogero-Moser Hamiltonian

$$H^{CM} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \sum_{i<j}^{N} \frac{g^2}{(q_i - q_j)^2}, \quad p_i = \dot{q}_i$$

(1.5)

with the following identification of variables. The positions of classical particles $q_i$ are identified with the marked points $z_i$ of the Gaudin model (as in the Schrödinger equation (1.3)), the coupling constant $g$ is equal to the Planck constant $\hbar$, and the classical velocities $\dot{q}_i$ are identified with the eigenvalues $H^G_i$ of the quantum Gaudin Hamiltonians:

$$q_j = z_j, \quad g = \hbar \quad \text{and} \quad \dot{q}_j = H^G_j, \quad j = 1, \ldots, N.$$

(1.6)

Similar fixation holds true for all higher Hamiltonians in involution of the Calogero-Moser model:

$$H^{CM}_k = E_k(\omega),$$

(1.7)

so that we finally obtain all action variables be fixed. Equations (1.7) define some Lagrangian submanifold in the classical $2N$-dimensional phase space. Its definition depends on the data of the initial KZ equations (1.1).

**Lax matrix and Bethe ansatz.** In order to clarify the duality relation between the Gaudin model (1.4) and the Lagrangian submanifolds the classical Lax matrix of the Calogero-Moser system should be used. For the model (1.5) it is of the following form:

$$L^{CM}_{ij}(\{\dot{q}_i\}, \{q_i\}, g) = \delta_{ij} \dot{q}_i + g \frac{1 - \delta_{ij}}{q_i - q_j}, \quad i, j = 1, \ldots, N.$$  

(1.8)

$$M^{CM}_{ij} = \delta_{ij} \sum_{k \neq i} \frac{g}{(q_i - q_k)^2} - (1 - \delta_{ij}) \frac{g}{(q_i - q_j)^2}, \quad i, j = 1, \ldots, N,$$

(1.9)

so that the $N \times N$ matrix Lax equation $\dot{L} = [L, M]$ is equivalent to the equations of motion

$$\dot{p}_i = \ddot{q}_i = -\sum_{k \neq i} \frac{2g^2}{(q_i - q_k)^3}, \quad i = 1, \ldots, N.$$  

(1.10)

Recall that the eigenvalues of the Lax matrix $\text{Spec}(L^{CM}) = \{I_1, \ldots, I_N\}$ are the action variables for the model (1.5) since $H^{CM}_k = \frac{1}{k} \text{tr} (L^{CM})^k = \frac{1}{k} \sum_{i=1}^{N} I_i^k$.

To find the level of Hamiltonians $E_k(\omega)$ (1.7) (or equivalently, the level of the action variables $I_k$), one can use the algebraic Bethe ansatz for the Gaudin model. The solution of the eigenvalue problems (1.4) is as follows:

$$H^G_i = \omega + \sum_{k \neq i}^{N} \frac{\hbar}{z_i - z_k} + \sum_{\gamma \neq 1}^{M} \frac{\hbar}{\mu_\gamma - z_i}, \quad i = 1, \ldots, N,$$

(1.11)

where the parameters $\{\mu_\alpha, \alpha = 1, \ldots, M\}$ are the Bethe roots satisfying the system of $M$ Bethe equations (BE)

$$2\omega + \hbar \sum_{k=1}^{N} \frac{1}{\mu_\alpha - z_k} = 2\hbar \sum_{\gamma \neq \alpha}^{M} \frac{1}{\mu_\alpha - \mu_\gamma}, \quad \alpha = 1, \ldots, M.$$  

(1.12)
The positive integer parameter $M$ is the number of overturned spins in the Gaudin eigenvector $\psi \in \mathcal{H}$ (1.4). In what follows we assume that $M \leq [N/2]$. Using the identification (1.6) we can substitute $\tilde{q}_j = H^C_i$ into the Lax matrix (1.8) and compute the eigenvalues $I_k$ (the action variables). It appears that on shell, i.e., when the Bethe equations (1.12) are satisfied, these eigenvalues take the form \[ \text{Spec } L^\text{CM} \left( \{H^C_i \}, \{z_j \}, \hbar \right)|_{BE} = \left\{ \omega, \ldots, \omega, \omega, \omega, \ldots, \omega \right\}. \] (1.13)

That is the action variables of the classical model are twist parameters $\omega, -\omega$ with multiplicities given by the occupation numbers $N - M, M$ (the numbers of spins looking up and down in the state $\psi$). The identification of variables (1.6) can be viewed as initial conditions for the Calogero-Moser model (1.8)–(1.10), i.e. the quantum-classical duality provides some specific initial conditions for the classical model given by intersection of two Lagrangian submanifolds. The first one is the $N$-dimensional level set of $N$ classical Hamiltonians in involution defined by (1.13) and the second one is the $N$-dimensional hyperplane $q_j = z_j$. (These are initial coordinates of the particles.) In particular, if the twist is absent ($\omega = 0$), then the first Lagrangian submanifold is the zero set of the higher classical Hamiltonians.

**Determinant identities and factorization of the Lax matrix.** The derivation of (1.13) is quite tricky (see [8]). It uses some non-trivial determinant identities for the matrices of the form $L^\text{CM} \left( \{H^C_i (z_k, \mu_k, h, \omega) \}, \{z_j \}, \hbar \right)$. For the example discussed above the identity looks as follows:

\[ \det_{N \times N} \left( \mathcal{L} - \lambda I \right) = (\omega - \lambda)^{N - M} \det_{M \times M} \left( \tilde{\mathcal{L}} - \lambda I \right), \] (1.14)

where $I$ is the identity matrix and

\[
\mathcal{L}_{ij} = \delta_{ij} \left( \omega + \sum_{k \neq i} \frac{h}{q_i - q_k} + \sum_{\gamma = 1}^{M} \frac{h}{\mu_\gamma - q_i} \right) + (1 - \delta_{ij}) \frac{h}{q_i - q_j}, \quad i, j = 1, \ldots, N
\] (1.15)

\[
\tilde{\mathcal{L}}_{\alpha\beta} = \delta_{\alpha\beta} \left( \omega - \sum_{\gamma \neq \alpha} \frac{h}{\mu_\alpha - \mu_\gamma} - \sum_{k = 1}^{N} \frac{h}{q_k - \mu_\alpha} \right) + (1 - \delta_{\alpha\beta}) \frac{h}{\mu_\alpha - \mu_\beta}, \quad \alpha, \beta = 1, \ldots, M.
\] (1.16)

Identities of such type appear in studies of scalar products of Bethe vectors in quantum integrable models [1]. The proof of these identities is based on another non-trivial phenomenon -- factorization of the Lax matrices [9, 17]. For example, consider the matrix $\mathcal{L}$ (1.15) for $M = 0$. It can be represented in the following form:\[\mathcal{L} = \omega I + h(D^0)^{-1} V C_0 V^{-1} D^0 = (D^0)^{-1} V (\omega I + hC_0) V^{-1} D^0, \quad V_{ij}(q) = q_i^{j-1}, \] (1.17)

\[
(D^0)_{ij} = \delta_{ij} \prod_{k \neq i}^{N} (q_i - q_k), \quad (C_0)_{ij} = i \delta_{i+1,j},
\] (1.18)

where $V$ is the Vandermonde matrix, $D^0$ is diagonal and $C_0$ is the upper-triangular matrix. In factorization formulas for Lax matrices for Calogero-Moser models associated with the root systems of types $B, C, D$ the upper-triangular matrix

\[
\tilde{C}_{ij} = \frac{1 + (-1)^j}{2} \delta_{i+1,j}
\] (1.19)

\[\text{In this paper the notations for } V, C_0 \text{ differ from those in [8] by transposition.}\]
is also used, see (A.7), (A.10). From (1.17) it immediately follows that such matrix \( L \) has all eigenvalues equal to \( \omega \). It is the statement (1.14) for \( M = 0 \). Below we use some modifications of the above determinant identities and factorization formulae.

**From duality to correspondence: supersymmetric generalization.** In the supersymmetric case the main statement of the duality relation (1.13) is that it holds true for \( \text{gl}(1|1) \) and \( \text{gl}(0|2) \) supersymmetric Gaudin models [13, 3, 11] as well as for \( \text{gl}(2|0) \). In the general case one should take into consideration all \( m + n + 1 \) models associated with the superalgebras \( \text{gl}(m|n) \) with \( m + n \) fixed [16]. In this respect a single classical system corresponds to a number of quantum models.

Notice that duality (1.13) could be used (in principle) for a direct solution of the quantum spectral problem (1.4) without using the Bethe ansatz equations (1.12). Indeed, one may write down a system of algebraic equations for the Lax matrix (1.8) to have the eigenvalues (1.13). Solving this systems with respect to velocities one finds the spectrum (1.4) as \( H^G_{ij} = \dot{q}_j(\{q_k\}, \omega, \hbar) \). However, it follows from the above statement that a given solution for velocities may correspond to one or another Gaudin model among the \( m + n + 1 \) models, and it is not clear in general to which one. To clarify the underlying combinatorics is an interesting open problem.

**Calogero-Moser systems and Gaudin models with boundary.** In our previous paper [18] we studied the duality for Calogero-Moser models associated with the classical root systems of simple Lie algebras, i.e. to the root systems of \( B_N, C_N \) and \( D_N \) types. These models were introduced in [14]. In the rational case the Hamiltonian is of the following form:

\[
H = \frac{1}{2} \sum_{a=1}^{N} p_a^2 - g_2^2 \sum_{a<b}^{N} \left( \frac{1}{(q_a - q_b)^2} + \frac{1}{(q_a + q_b)^2} \right) - g_4^2 \sum_{a=1}^{N} \frac{1}{(2q_a)^2} - g_1^2 \sum_{a=1}^{N} \frac{1}{q_a^2}. \tag{1.20}
\]

It depends on three free parameters, the coupling constants \( g_2, g_4 \) and \( g_1 \). Two of them \( (g_4 \text{ and } g_1) \) can be unified in (1.20) into a single combination \( g_4^2/4 + g_1^2 \). We do not do that since the two last terms are associated with roots of different types, and in the trigonometric (and elliptic) extensions these terms are different. The root systems \( B_N, C_N \) and \( D_N \) are distinguished by values of the coupling constants, see (A.4).

The size of the matrices participating in the Lax pair for the model (1.20) is \( 2N \times 2N \) \((C, D)\) or \((2N + 1) \times (2N + 1) \) \((B)\). The corresponding factorization formulas of the type (1.17)–(1.18) were derived in [17], see (A.7), (A.10). Based on this knowledge we showed in [18] that (1.20) is quantum-classically dual to the boundary Gaudin magnet – the Gaudin limit of the XXX quantum spin chain with (some special) boundary conditions introduced by Sklyanin [15]. Using the algebraic Bethe ansatz for the boundary Gaudin models [10] we derived the underlying matrix identities (A.17) and (A.23) of type (1.14) and computed the levels of the classical Hamiltonians. In contrast to \( A_{N-1} \) type Calogero-Moser model [15], all eigenvalues of \( BCD \)-type Lax matrix are equal to zero:

\[
\det \left( L^\text{CM} \left( \{H_j^G\}, \{z_j\}, \hbar \right) \right)_{BE} - \lambda I = (-\lambda)^r, \tag{1.21}
\]

where \( r = 2N \) for \( C_N, D_N \) root systems and \( r = 2N + 1 \) for \( B_N \).
**Purpose of the paper** is to extend the quantum-classical duality between the quantum-boundary Gaudin magnet and the classical Calogero-Moser model of types $BCD$ to the correspondence. Namely, a single classical model (associated to the root system of type $B$, $C$ or $D$) is related to three supersymmetric Gaudin magnets associated with superalgebras $\mathfrak{gl}(2|0)$, $\mathfrak{gl}(1|1)$ and $\mathfrak{gl}(0|2)$. The eigenvalues and Bethe equations are described in the next section. It should be stressed that our previous construction for $\mathfrak{gl}(2|0)$ \cite{[18]} can not be straightforwardly extended to the supersymmetric case. Some additional computational tricks are required for the proof. Some details of the proof are given in Section 3. The details of the Lax representation for the Calogero-Moser models of types $BCD$, including factorization formulae and determinant identities, can be found in the appendix. In the Conclusion we summarize the obtained results.

## 2 Supersymmetric Gaudin model with boundary

We consider $\mathfrak{gl}(m|n)$ Gaudin magnets with open boundary conditions and $m+n=2$, which arise by applying Gaudin limit to the $\mathbb{Z}_2$-graded quantum spin chains. The rational models are defined through the graded permutation operator

$$ P = \sum_{i,j=1}^{m+n} (-1)^{p(j)} E_{ij} \otimes E_{ji}, \quad P(x \otimes y) = (-1)^{p(x)p(y)} (y \otimes x), \quad x, y \in \mathbb{C}^{m|n}. \quad (2.1) $$

See the notation in (A.24)–(A.28). The supersymmetric Yang’s $R$-matrix is of the form

$$ R(u) = I + \frac{\eta}{u} P. \quad (2.2) $$

It satisfies the graded Yang-Baxter equation

$$ R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2). \quad (2.3) $$

For the construction of integrable spin chains with open boundary conditions one needs the reflection equations for boundary $K$-matrices $K^\pm(u)$:

$$ R_{12}(u_1 - u_2)K^-(u_1)R_{21}(u_1 + u_2)K^+(u_2) = K^+(u_2)R_{12}(u_1 + u_2)K^-(u_1)R_{21}(u_1 - u_2), \quad (2.4) $$

$$ R_{12}(u_2 - u_1)(K_1^+(u_1))^{t_1} R_{21}(-u_1 - u_2 - (m-n)\eta)(K_2^+(u_2))^{t_2} = (K_2^+(u_2))^{t_2} R_{12}(-u_1 - u_2 - (m-n)\eta)(K_1^+(u_1))^{t_1} R_{21}(u_2 - u_1), $$

where $t_i$ means super transposition in the $i$-th tensor component (A.27).

In this paper we consider the following diagonal solution\footnote{General $K$-matrices presumably correspond to the general Lax matrix of the Calogero-Moser model \cite{[4]}, which has no restriction \cite{[A.3]} for the coupling constants. However the factorization properties and determinant identities are yet unknown for the general Lax pair.} of the reflection equations (2.4):

$$ K^-(u) = \begin{pmatrix} 1 + \frac{\alpha \eta}{u} & 0 \\ 0 & -1 + \frac{\beta \eta}{u} \end{pmatrix}, \quad K^+(u) = \begin{pmatrix} 1 - \frac{\beta \eta}{u + \frac{m-n}{2} \eta} & 0 \\ 0 & -1 - \frac{\beta \eta}{u + \frac{m-n}{2} \eta} \end{pmatrix}. \quad (2.5) $$
Relations (2.3) (2.4) guarantee that the transfer matrices

\[ T(u) = \text{str}_0 \left( K_0^+(u)R_{01}(u-z_1) \ldots R_{0N}(u-z_N)K_0^-(u)R_{0N}(u+z_N) \ldots R_{01}(u+z_1) \right) \]

(2.6)

commute for different values of the spectral parameter: \([T(u), T(v)] = 0\) for all \(u, v\). The Gaudin model appears in the limit \(\varepsilon \to 0\) after the substitution \(\eta = \varepsilon \hbar\):

\[ T(u) = (-1)^{p(1)} + (-1)^{p(2)} + \varepsilon \hbar \gamma(u) + \varepsilon^2 \hbar^2 T^G(u) + O(\varepsilon^3), \]

(2.7)

where \(\gamma(u)\) is a scalar function. The notation \(p(i)\) is defined in (A.24). The expression

\[ T^G(u) = -\frac{2\alpha \beta}{u^2} + \frac{1}{\hbar} \sum_{i=1}^{N} \left( \frac{H_i^G}{u - z_i} - \frac{H_i^G}{u + z_i} \right) \]

(2.8)

is the Gaudin transfer matrix, and the operators \(H_i^G\) are commuting Gaudin Hamiltonians:

\[ \frac{1}{\hbar} H_i^G = \frac{(2\xi + (-1)^{p(1)} - (-1)^{p(2)})\sigma_3^{(i)}}{2z_i} + \sum_{k \neq i} \left( \frac{P_{ik}}{z_i - z_k} + \frac{\sigma_3^{(i)} P_{ik} \sigma_3^{(i)}}{z_i + z_k} \right), \quad \xi = \alpha - \beta, \]

(2.9)

where \(P_{ik}\) are the graded permutation operators (2.1) exchanging \(i\)-th and \(k\)-th tensor components of the Hilbert space \((\mathbb{C}^{m|n})^\otimes N\). The Gaudin spectral problems are

\[ H_i^G \psi = H_i^G \psi. \]

(2.10)

The eigenvalues \(H_i^G\) of these operators are obtained via the algebraic Bethe ansatz technique [3].

The Gaudin model (2.8), (2.9) with \(2N\) marked points \(z_1, \ldots, z_N, -z_1, \ldots, -z_N\) will be shown to be dual to the Calogero-Moser models of types \(C_N\) and \(D_N\). Besides this case, we also need the one related to the \(B_N\) root system. It comes from the transfer matrix (2.8) by substituting \(N \to N + 1\) and \(z_{N+1} = 0\). Then we have \(2N + 1\) marked points \(z_1, \ldots, z_N, 0, -z_1, \ldots, -z_N\). In this case we fix the parameter \(\xi = p(1) - p(2)\). Consider the solutions of the eigenvalue problem (2.10) in all the cases [3].

**2N marked points.** The eigenvalues of the Gaudin Hamiltonians (2.9) are of the form

\[ \frac{1}{\hbar} H_i^G = \frac{\xi - p(1) + p(2)}{z_i} + \sum_{k \neq i} \left( \frac{(-1)^{p(1)}}{z_i - z_k} + \frac{(-1)^{p(1)}}{z_i + z_k} \right) - \sum_{l=1}^{M} \left( \frac{(-1)^{p(1)}}{z_i - \mu_l} + \frac{(-1)^{p(1)}}{z_i + \mu_l} \right), \]

(2.11)

where \(\xi = \alpha - \beta\). The set of Bethe roots \(\{\mu\}_M = \{\mu_1, \ldots, \mu_M\}\) satisfy the system of \(M\) Bethe equations \((l = 1, \ldots, M)\)

\[ \frac{2\xi}{\mu_l} + (-1)^{p(1)} \sum_{k=1}^{N} \left( \frac{1}{\mu_l - z_k} + \frac{1}{\mu_l + z_k} \right) = \]

\[ = \left( (-1)^{p(1)} + (-1)^{p(2)} \right) \left( \frac{1}{\mu_l} + \sum_{k \neq l} \left( \frac{1}{\mu_l - \mu_k} + \frac{1}{\mu_l + \mu_k} \right) \right). \]

(2.12)

Let us write down (2.11) and (2.12) explicitly for different superalgebras:
Here we consider the Gaudin model (2.7)–(2.8) with
\[ H = \sum_{i=1}^{N} \left( 1 \over z_{i} - z_{k} + 1 \over z_{i} + z_{k} \right) - \sum_{l=1}^{M} \left( 1 \over z_{i} - \mu_{l} + 1 \over z_{i} + \mu_{l} \right), \quad (2.13) \]

\[ 2 \xi \mu_{l} + \sum_{k=1}^{N} \left( 1 \over \mu_{l} - z_{k} + 1 \over \mu_{l} + z_{k} \right) = 2 \left( 1 \over \mu_{l} + \sum_{k \neq l} \left( 1 \over \mu_{l} - \mu_{k} + 1 \over \mu_{l} + \mu_{k} \right) \right). \quad (2.14) \]

**gl(2|0) case:**
\[ \frac{1}{\hbar} H_{i}^{G(2|0)}(\{ z \}_{N}, \{ \mu \}_{M}, \xi) = \frac{\xi}{z_{i}} + \sum_{k \neq i}^{N} \left( 1 \over z_{i} - z_{k} + 1 \over z_{i} + z_{k} \right) - \sum_{l=1}^{M} \left( 1 \over z_{i} - \mu_{l} + 1 \over z_{i} + \mu_{l} \right), \quad (2.15) \]
\[ -2 \xi \mu_{l} + \sum_{k=1}^{N} \left( 1 \over \mu_{l} - z_{k} + 1 \over \mu_{l} + z_{k} \right) = 0. \quad (2.16) \]

**gl(1|1) case:**
\[ \frac{1}{\hbar} H_{i}^{G(1|1)}(\{ z \}_{N}, \{ \mu \}_{M}, \xi) = \frac{\xi}{z_{i}} + \sum_{k \neq i}^{N} \left( 1 \over z_{i} - z_{k} + 1 \over z_{i} + z_{k} \right) - \sum_{l=1}^{M} \left( 1 \over z_{i} - \mu_{l} + 1 \over z_{i} + \mu_{l} \right), \quad (2.17) \]
\[ -2 \xi \mu_{l} + \sum_{k=1}^{N} \left( 1 \over \mu_{l} - z_{k} + 1 \over \mu_{l} + z_{k} \right) = 2 \left( 1 \over \mu_{l} + \sum_{k \neq l} \left( 1 \over \mu_{l} - \mu_{k} + 1 \over \mu_{l} + \mu_{k} \right) \right). \quad (2.18) \]

In all these cases \( H_{i}^{G_{\text{min}}} = H_{i}^{G_{\text{min}}} (\{ z \}_{N}, \{ \mu \}_{M}, \xi) \), i.e. the eigenvalues depend on \( N \) marked points \( z_{i}, M \) Bethe roots \( \mu_{j} \) and the parameter \( \xi \).

**2N + 1 marked points.** Here we consider the Gaudin model (2.7)–(2.8) with \( N + 1 \) spins and \( \xi = p(2) - p(1), z_{N+1} = 0 \). Then (2.11)–(2.12) acquire the form
\[ \frac{1}{\hbar} \tilde{H}_{i}^{G}(\{ z \}_{N}, \{ \mu \}_{M}) \]
\[ = (-1)^{p(1)} \left( 2 \over z_{i} + \sum_{k \neq i}^{N} \left( 1 \over z_{i} - z_{k} + 1 \over z_{i} + z_{k} \right) - \sum_{l=1}^{M} \left( 1 \over z_{i} - \mu_{l} + 1 \over z_{i} + \mu_{l} \right) \right), \quad (2.19) \]
\[ (-1)^{p(1)} \sum_{k=1}^{N} \left( 1 \over \mu_{l} - q_{k} + 1 \over \mu_{l} + q_{k} \right) = ((-1)^{p(1)} + (-1)^{p(2)}) \sum_{k \neq l}^{M} \left( 1 \over \mu_{l} - \mu_{k} + 1 \over \mu_{l} + \mu_{k} \right). \quad (2.20) \]

For each superalgebra (2.19)–(2.20) we have:

**gl(2|0) case:**
\[ \frac{1}{\hbar} \tilde{H}_{i}^{G(2|0)}(\{ z \}_{N}, \{ \mu \}_{M}) = 2 \over z_{i} + \sum_{k \neq i}^{N} \left( 1 \over z_{i} - z_{k} + 1 \over z_{i} + z_{k} \right) - \sum_{l=1}^{M} \left( 1 \over z_{i} - \mu_{l} + 1 \over z_{i} + \mu_{l} \right), \quad (2.21) \]
\[
\sum_{k=1}^{N} \left( \frac{1}{\mu_l - q_k} + \frac{1}{\mu_l + q_k} \right) = 2 \sum_{k \neq l}^{M} \left( \frac{1}{\mu_l - \mu_k} + \frac{1}{\mu_l + \mu_k} \right). \tag{2.22}
\]

\[
\frac{1}{\hbar} \tilde{H}^{G(1|1)}_i (\{ z \}_N, \{ \mu \}_M) = \frac{2}{z_i} + \sum_{k \neq i}^{N} \left( \frac{1}{z_i - z_k} + \frac{1}{z_i + z_k} \right) - \sum_{l}^{M} \left( \frac{1}{z_i - \mu_l} + \frac{1}{z_i + \mu_l} \right), \tag{2.23}
\]

\[
\sum_{k=1}^{N} \left( \frac{1}{\mu_l - q_k} + \frac{1}{\mu_l + q_k} \right) = 0. \tag{2.24}
\]

\[
\frac{1}{\hbar} \tilde{H}^{G(0|2)}_i (\{ z \}_N, \{ \mu \}_M) = -\frac{2}{z_i} - \sum_{k \neq i}^{N} \left( \frac{1}{z_i - z_k} + \frac{1}{z_i + z_k} \right) + \sum_{l}^{M} \left( \frac{1}{z_i - \mu_l} + \frac{1}{z_i + \mu_l} \right), \tag{2.25}
\]

\[
\sum_{k=1}^{N} \left( \frac{1}{\mu_l - q_k} + \frac{1}{\mu_l + q_k} \right) = 2 \sum_{k \neq l}^{M} \left( \frac{1}{\mu_l - \mu_k} + \frac{1}{\mu_l + \mu_k} \right). \tag{2.26}
\]

### 3 Proof of the correspondence

The statement of the correspondence is that relation (1.21) holds true for the Lax matrices (A.1) of BCD types, where velocities of the Calogero-Moser particles are identified with the eigenvalues of \( \text{gl}(2|0) \) or \( \text{gl}(1|1) \) or \( \text{gl}(0|2) \) Gaudin model Hamiltonians given in (2.11) or (2.19) for \( C, D \) and \( B \) root systems respectively. More precisely, make the following identifications:

\[
z_j = q_j, \quad j = 1, \ldots, N, \tag{3.1}
\]

\[
\dot{q}_j = H^{G(m|n)}_j (\{ q \}_N, \{ \mu \}_M, \xi) \quad \text{or} \quad \dot{q}_j = \tilde{H}^{G(m|n)}_j (\{ q \}_N, \{ \mu \}_M). \quad j = 1, \ldots, N. \tag{3.2}
\]

Next, consider the Lax matrix

\[
L (\{ \dot{q}_j \}_N, \{ q_j \}_N \mid g_1, g_2, g_4)
\]

from (A.1)–(A.4). The size of the Lax matrix is equal to \( r = 2N \) for \( C_N, D_N \) root systems and \( r = 2N + 1 \) for \( B_N \). Then we are going to prove the following statement:

\[
\det_{2N \times 2N} \left( L \left( \left\{ H^{G(m|n)}_j (\{ q \}_N, \{ \mu \}_M, \xi) \right\}_N, \{ q_j \}_N \mid g_1, g_2, g_4 \right) \right)_{BE \ (2.12)} - \lambda I = \lambda^{2N}, \tag{3.3}
\]

where

\[
\text{for } C_N: \quad g_1 = 0, g_2 = \hbar, g_4 = \sqrt{2\hbar (\xi - p(1) + p(2))}, \tag{3.4}
\]

\[
\text{for } D_N: \quad \xi = p(1) - p(2), g_1 = 0, g_2 = \hbar, g_4 = 0
\]

and

\[
\det_{(2N+1) \times (2N+1)} \left( L \left( \left\{ \tilde{H}^{G(m|n)}_j (\{ q \}_N, \{ \mu \}_M) \right\}_N, \{ q_j \}_N \mid \sqrt{2\hbar}, \hbar, 0 \right) \right)_{BE \ (2.20)} - \lambda I = -\lambda^{2N+1}. \tag{3.5}
\]

The proof for \( \text{gl}(2|0) \) with parameters (A.5) or (A.8) was given in [18]. Here we prove the cases of the superalgebras \( \text{gl}(1|1) \) and \( \text{gl}(0|2) \). We will see that the latter can be reduced to \( \text{gl}(2|0) \) using special properties of the Lax matrices. The proof for \( \text{gl}(1|1) \) requires additional computational trick based on the usage of the Frobenius matrix and matching of the parameters.
3.1 \( C_N \) and \( D_N \) root systems for \( \mathfrak{gl}(1|1) \) superalgebra

Here \( m = n = 1 \). We begin with the \( C_N \) root system since \( D_N \) comes as a particular case of it. It follows from (3.4) and (A.24) that \( p(1) = 0, p(2) = 1 \) and \( g_1 = 0, g_2 = h, g_4 = \sqrt{2}h (\xi + 1) \).

Introduce the short-hand notation for the Lax matrix of size \( 2N \times 2N \) entering (3.3):

\[
\mathcal{L}^{[1]} = L \left( \{ H_j^{G(1|1)}(\{ q \}_N, \{ \mu \}_M, \xi), \{ q \}_N | 0, h, \sqrt{2h} (\xi + 1) \} \right).
\]

(3.6)

By comparing (2.13) and (2.15) we conclude that

\[
H_j^{G(1|1)}(\{ q \}_N, \{ \mu \}_M, \xi) = H_j^{G(2|0)}(\{ q \}_N, \{ \mu \}_M, \xi + 1).
\]

(3.7)

Therefore,

\[
\mathcal{L}^{[1]} = L \left( \{ H_j^{G(2|0)}(\{ q \}_N, \{ \mu \}_M, \xi + 1), \{ q \}_N | 0, h, \sqrt{2h} (\xi + 1) \} \right),
\]

(3.8)

that is the Lax matrix has the form (A.13) written for \( \mathfrak{gl}(2|0) \) case but with \( \xi \) replaced by \( \xi + 1 \). However, we cannot use the duality statement for \( \mathfrak{gl}(2|0) \) case here since the Bethe equations in \( \mathfrak{gl}(1|1) \) case (2.16) differ from those (2.14) for \( \mathfrak{gl}(2|0) \). We can use this argument when \( M = 0 \) only (then the set of Bethe equations is empty). For \( M = 0 \) we get \( \mathcal{L}^{[1]} = L'(\xi \to \xi + 1) \) for \( L' \) (A.7). The duality relation (3.3) then follows immediately from the explicit form of matrices \( C_0 \) and \( \tilde{C} \) (1.19) because they are upper-triangular.

Suppose \( M \geq 1 \). Let us apply the determinant identity (A.17) to (3.8). We get:

\[
\det_{2N \times 2N} (\mathcal{L}^{[1]} - \lambda I) = \lambda^{2N - 2M} \det_{2M \times 2M} (\tilde{\mathcal{L}}^{[1]} - \lambda I), \quad (3.9)
\]

where \( 2M \times 2M \) dual matrix is of the form

\[
\tilde{\mathcal{L}}^{[1]} = L' \left( \{ H_j^{G(2|0)}(\{ \mu \}_M, \{ q \}_N, -\xi), \{ \mu \}_M | 0, h, -\sqrt{2h} \xi \} \right).
\]

(3.10)

Next, let us impose the Bethe equations (2.16) in (3.10) (take it “on-shell”). This yields

\[
\mathcal{L}^{[1]} \bigg|_{BE(2.16)} = L \left( \{ H_j^{G(2|0)}(\{ \mu \}_M, \{ 0 \}_N, \xi), \{ \mu \}_M | 0, h, -\sqrt{2h} \xi \} \right).
\]

(3.11)

Using explicit form of (2.13), we have:

\[
H_j^{G(2|0)}(\{ \mu \}_M, \{ q \}_N, \xi) = H_j^{G(2|0)}(\{ \mu \}_M, \{ q \}_N, -\xi) + \frac{2\xi}{\mu_j}, \quad (3.12)
\]

so that

\[
\mathcal{L}^{[1]} \bigg|_{BE(2.16)} = L_{ij} \left( \{ H_k^{G(2|0)}(\{ \mu \}_M, \{ 0 \}_N, -\xi), \{ \mu \}_M | 0, h, -\sqrt{2h} \xi \} \right) \pm \delta_{ij} \frac{2\xi}{\mu_j}, \quad (3.13)
\]

(for \( i, j = 1, \ldots, 2M \)), where the sign + is for \( 1 \leq i \leq M \) and the sign − for \( M + 1 \leq i \leq 2M \). The first term is transformed using the factorization formula (A.7):

\[
L \left( \{ H_j^{G(2|0)}(\{ \mu \}_M, \{ 0 \}_N, -\xi), \{ \mu \}_M | 0, h, -\sqrt{2h} \xi \} \right) = \hbar(D^0)^{-1} V \left( C_0 - (1 + 2\xi)\tilde{C} \right) V^{-1} D^0.
\]

(3.14)
The second term in (3.13) is a diagonal matrix, which can represented through the Frobenius companion matrix (A.14) as follows:

$$2\xi \hbar (D^0)^{-1} V (J^{-1}) V^{-1} D^0,$$

where the set of variables \( \{x_k\} \) defining \( J \) is \( (\mu_1, \ldots, \mu_M, -\mu_1, \ldots, -\mu_M) \). Finally, we obtain

$$\tilde{L}^{[1]}_{\text{BE}} = h(D^0)^{-1} V \left( C_0 - (1 + 2\xi)\tilde{C} + 2\xi J^{-1} \right) V^{-1} D^0,$$

where the matrices \( C_0 \) and \( \tilde{C} \) are as in (1.18), (1.19) (their size is \( 2M \times 2M \)), and the matrix \( J^{-1} \) (A.11) is of the form

$$J^{-1} = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
-\varepsilon_{2M-2}(\mu_1^{-1}, \ldots, \mu_M^{-1}, -\mu_1^{-1}, \ldots, -\mu_M^{-1}) & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\varepsilon_0(\mu_1^{-1}, \ldots, \mu_M^{-1}, -\mu_1^{-1}, \ldots, -\mu_M^{-1}) & 0 & \cdots & 0 \\
\end{pmatrix}.$$  \hspace{1cm} (3.17)

Here \( \varepsilon_k \) are elementary symmetric polynomials of the indicated variables. Then we obtain:

$$\det_{2M \times 2M} \left( \tilde{L}^{[1]}_{\text{BE}} - \lambda I \right) = \det_{2M \times 2M} \left( C_0 - (1 + 2\xi)\tilde{C} + 2\xi J^{-1} - \lambda I \right).$$  \hspace{1cm} (3.18)

Our aim is to show that the latter determinant equals \( \lambda^{2M} \). Using the explicit form (1.13), (1.19), (3.17) of all entering matrices, one can verify that the first row of the matrix \( C_0 - (1 + 2\xi)\tilde{C} + 2\xi J^{-1} \) consists of zeros. Also, \( (2M - 1) \times (2M - 1) \) matrix obtained from \( C_0 - (1 + 2\xi)\tilde{C} + 2\xi J^{-1} \) by removing the first column and the first row is upper-triangular. Therefore,

$$\det_{2M \times 2M} \left( C_0 - (1 + 2\xi)\tilde{C} + 2\xi J^{-1} - \lambda I \right) = \lambda^{2M}. \hspace{1cm} (3.19)$$

Together with (3.9) this completes the proof for the \( C_N \) root system and yields

$$\det_{2N \times 2N} \left( \tilde{L}^{[1]}_{\text{BE}} - \lambda I \right) = \lambda^{2N}. \hspace{1cm} (3.20)$$

The proof for the \( D_N \) root system follows from the above at \( \xi = -1 \).

### 3.2 \( C_N \) and \( D_N \) root systems for \( \mathfrak{gl}(0\vert 2) \) superalgebra

Here \( m = 0, n = 2, p(1) = 1, p(2) = 1 \) and \( g_1 = 0, g_2 = \hbar, g_4 = \sqrt{2}\hbar \xi \) for \( C_N \) (and \( \xi = 0 \) for \( D_N \)).

We are going to use the following statement:

**Lemma** Consider \( 2N \times 2N \) matrix (A.11) for \( C_N \) (and \( D_N \)) root system, i.e. the Lax matrix \( L(\{q_1\}_N, \{q_2\}_N, g_1, g_2, g_4) \)

$$L = \begin{pmatrix} P + A & B \\ -B & -P - A \end{pmatrix}, \quad A, B \in \text{Mat}(N, \mathbb{C}),$$  \hspace{1cm} (3.21)
where $P, A, B$ are given in [A.2]. Then $\det(L - \lambda I)$ is an even function of the parameter $g_4$ entering as common coefficient in the diagonal part of the matrix $B$.

The proof is given in the appendix.

Consider the Lax matrix

$$L^{0|2} = L \left( \{H_j^{G(0|2)}(\{q\}_N, \{\mu\}_M, \xi)\}, \{q\}_N | 0, h, \sqrt{2}h\xi \right) ,$$

(3.22)

where $H_j^{G(0|2)}(\{q\}_N, \{\mu\}_M, \xi)$ are given in (2.17). From the above Lemma we have

$$\det_{2N \times 2N} (L^{0|2} - \lambda I) = \det_{2N \times 2N} \left( L \left( \{H_j^{G(0|2)}(\{q\}_N, \{\mu\}_M, \xi)\}, \{q\}_N | 0, h, -\sqrt{2}h\xi \right) - \lambda I \right) .$$

(3.23)

Also notice that

$$H_j^{G(0|2)}(\{z\}_N, \{\mu\}_M, \xi) = -H_j^{G(2|0)}(\{z\}_N, \{\mu\}_M, -\xi).$$

(3.24)

Therefore,

$$\det_{2N \times 2N} (L^{0|2} - \lambda I) = \det_{2N \times 2N} \left( L \left( \{H_j^{G(0|2)}(\{q\}_N, \{\mu\}_M, \xi)\}, \{q\}_N | 0, h, -\sqrt{2}h\xi \right) - \lambda I \right)$$

$$= \det_{2N \times 2N} \left( -L^T \left( \{H_j^{G(0|2)}(\{q\}_N, \{\mu\}_M, \xi)\}, \{q\}_N | 0, h, -\sqrt{2}h\xi \right) + \lambda I \right)$$

$$\overset{3.24}{=} \det_{2N \times 2N} \left( L \left( \{H_j^{G(0|2)}(\{q\}_N, \{\mu\}_M, -\xi)\}, \{q\}_N | 0, h, -\sqrt{2}h\xi \right) + \lambda I \right) .$$

Finally, the Bethe equations (2.18) for $gl(0|2)$ are exactly the same as in the $gl(2|0)$ case (2.14) but with $\xi \rightarrow -\xi$. Thus the proof of the duality relation in this case follows from the one for $gl(2|0)$ (with sign of $\xi$ changed):

$$\det_{2N \times 2N} \left( L^{0|2}_{BE\ 2.18} - \lambda I \right) = \lambda^{2N} .$$

(3.26)

### 3.3 $B_N$ root system for $gl(1|1)$ superalgebra

Here $m = n = 1$, $p(1) = 0$, $p(2) = 1$ and $g_1 = \sqrt{2}h$, $g_2 = h$, $g_4 = 0$. The size of the Lax matrix is $(2N + 1) \times (2N + 1)$.

Introduce the matrix

$$L^{1|1}_B = L \left( \{\tilde{H}_j^{G(1|1)}(\{q\}_N, \{\mu\}_M)\}, \{q\}_N | \sqrt{2}h, 0 \right) ,$$

(3.27)

where $\tilde{H}_j^{G(1|1)}(\{q\}_N, \{\mu\}_M)$ are from (2.28). Notice that these eigenvalues coincide with those for $gl(2|0)$ case (2.21), so that

$$L^{1|1}_B = L \left( \{\tilde{H}_j^{G(2|0)}(\{q\}_N, \{\mu\}_M)\}, \{q\}_N | \sqrt{2}h, 0 \right) .$$

(3.28)

Then we use the determinant identity for the $B_N$ case [A.23]

$$\det_{(2N+1) \times (2N+1)} (L^{1|1}_B - \lambda I) = -\lambda^{2N-2M+1} \det_{2M \times 2M} (L^{1|1} - \lambda I) ,$$

(3.29)
where
\[ \tilde{\mathcal{L}}^{1|1} = L \left( \{-H_j^{G(2,0)}(\{\mu\}_M, \{q\}_N, \xi = -1)\}; \{\mu\}_M | 0, h, \sqrt{2h} \right) \] (3.30)
is a $2M \times 2M$ matrix and eigenvalues $H_j^{G(2,0)}(\{\mu\}_M, \{q\}_N, \xi = -1)$ are from (2.13) for $C_N$ case with $\xi = -1$ and $\{q\}_N, \{\mu\}_M$ interchanged:
\[ H_j^{G(2,0)}(\{\mu\}_M, \{q\}_N, \xi = -1) \]
\[ = -\frac{h}{\mu_i} + \sum_{k \neq i}^M \left( \frac{h}{\mu_i - \mu_k} + \frac{h}{\mu_i + \mu_k} \right) - \sum_{l=1}^N \left( \frac{h}{\mu_i - q_l} + \frac{h}{\mu_i + q_l} \right). \] (3.31)

Also, making the transposition we have:
\[ \det_{2M \times 2M} \left( \tilde{\mathcal{L}}^{1|1} - \lambda I \right) = \det_{2M \times 2M} \left( -(\tilde{\mathcal{L}}^{1|1})^T + \lambda I \right) \]
\[ = \det_{2M \times 2M} \left( L \left( \{H_j^{G(2,0)}(\{\mu\}_M, \{q\}_N, \xi = -1)\}; \{\mu\}_M | 0, h, \sqrt{2h} \right) + \lambda I \right). \] (3.32)

The Bethe equations (2.23) imply that the last sum in (3.31) vanishes. Then
\[ \det_{2M \times 2M} \left( \mathcal{L}^{1|1} \big|_{BE2.23} - \lambda I \right) \]
\[ = \det_{2M \times 2M} \left( L \left( \{H_j^{G(2,0)}(\{\mu\}_M, \{q\}_N, \xi = -1)\}; \{\mu\}_M | 0, h, \sqrt{2h} \right) + \lambda I \right). \] (3.33)

In this way we come to the matrix $L \left( \{H_j^{G(2,0)}(\{\mu\}_M, \{q\}_N, \xi = -1)\}; \{\mu\}_M | 0, h, \sqrt{2h} \right)$. It is exactly the case (3.11), which was previously discussed in detail (3.12)–(3.19) for generic $\xi$, and here we deal with $\xi = -1$. Therefore,
\[ \det_{2M \times 2M} \left( \tilde{\mathcal{L}}^{1|1} \big|_{BE2.23} - \lambda I \right) = \lambda^{2M}. \] (3.34)

Thus, plugging this into (3.29), we get
\[ \det_{2N \times 2N} \left( \mathcal{L}^{1|1} \big|_{BE2.23} - \lambda I \right) = -\lambda^{2N-2M+1} \lambda^{2M} = -\lambda^{2N+1}. \] (3.35)

### 3.4 $B_N$ root system for $\mathfrak{gl}(0|2)$ superalgebra

Here $m = 0, n = 2, p(1) = 1, p(2) = 1$ and $g_1 = \sqrt{2h}, g_2 = h, g_4 = 0$. The size of the Lax matrix is $(2N + 1) \times (2N + 1)$.

Introduce the Lax matrix
\[ \mathcal{L}^{0|2}_B = L \left( \{\tilde{H}_j^{G(0,2)}(\{q\}_N, \{\mu\}_M)\}; \{q\}_N | \sqrt{2h}, h, 0 \right), \] (3.36)
where the eigenvalues $\tilde{H}_j^{G(0,2)}(\{q\}_N, \{\mu\}_M)$ are from (2.25). Notice that
\[ \tilde{H}_j^{G(0,2)}(\{q\}_N, \{\mu\}_M) = \tilde{H}_j^{G(2,0)}(\{q\}_N, \{\mu\}_M) \] (3.37)
(see \((2.21)\)). Therefore,
\[
\det_{(2N+1)\times(2N+1)}(\mathcal{L}_B^{0/2} - \lambda I) = -\det_{(2N+1)\times(2N+1)}\left(-\left(\mathcal{L}_B^{0/2}\right)^T + \lambda I\right)
\]
\[
= -\det_{(2N+1)\times(2N+1)}\left(L \left(\{\tilde{H}_G^{(2)}\}^0, \{\mu\}_M\}, \{\mu\}_N|\sqrt{2}\hbar, \hbar, 0\right) + \lambda I\right).
\]

The Bethe equations for \(\text{gl}(0|2)\) case \((2.26)\) are exactly the same as for the \(\text{gl}(2|0)\) case \((2.22)\). Thus, the desired statement follows from the one for \(\text{gl}(2|0)\):
\[
\det_{(2N+1)\times(2N+1)}\left(\left.\mathcal{L}_B^{0/2}\right|_{BE(2.26)} - \lambda I\right) = -\lambda^{2N+1}.
\]

4 Conclusion

To summarize, let us formulate the final statement of the paper. Consider the set of supersymmetric Gaudin models with boundary based on the superalgebras \(\text{gl}(2|0), \text{gl}(1|1)\) and \(\text{gl}(0|2)\). For each of the Gaudin model make the following substitutions into the data of the classical Calogero-Moser models of type \(BCD\):
\[
z_j = q_j, \quad j = 1, \ldots, N
\]
and
\[
\dot{q}_j = H_j^G \quad \text{or} \quad \dot{q}_j = \tilde{H}_j^G, \quad j = 1, \ldots, N
\]
in the Lax matrix \((A.1)\), which we denote as \(L(\{\dot{q}_j\}, \{q_j\}|g_1, g_2, g_4)\). Here \(H_j^G\) and \(\tilde{H}_j^G\) are eigenvalues of the Gaudin Hamiltonians \((2.11)\) and \((2.19)\) respectively. For the classical root systems the set of the coupling constants and the Lax matrix size \(r\) are as follows:
\[
B_N : \tilde{H}_j^G (2.19), \quad g_1 = \sqrt{2}\hbar, \quad g_2 = \hbar, \quad g_4 = 0, \quad r = 2N + 1;
\]
\[
C_N : H_j^G (2.11), \quad g_1 = 0, \quad g_2 = \hbar, \quad g_4 = \sqrt{2}\hbar (\xi + p(2) - p(1)), \quad r = 2N;
\]
\[
D_N : H_j^G (2.11) \quad \text{with} \quad \xi = p(1) - p(2), \quad g_1 = 0, \quad g_2 = \hbar, \quad g_4 = 0, \quad r = 2N.
\]
If the Bethe roots \(\{\mu_k\}\) satisfy the Bethe equations (more precisely, \((2.20)\) for the \(B_N\) case, \((2.12)\) for the \(C_N\) case and \((2.12)\) with \(\xi = p(1) - p(2)\) for the \(D_N\) case), i.e., \(H_j^G\) or \(\tilde{H}_j^G\) belong to the spectrum of the Gaudin model, then all eigenvalues of the Lax matrix \(L(\{H_j^G\}, \{q_j\}|g_1, g_2, g_4)\) or \(L(\{\tilde{H}_j^G\}, \{q_j\}|g_1, g_2, g_4)\) and, therefore, all the integrals of motion, are equal to zero (see \((1.21)\)).

5 Appendix

5.1 Lax pairs and identities

The Lax matrices for the Calogero-Moser models \((1.20)\) are of sizes \((2N + 1)\times(2N + 1)\) (but they have effective size \(2N \times 2N\) when \(g_1 = 0\) \([14]\):
\[
L = \begin{pmatrix}
P + A & B & C \\
-B & -P - A & -C \\
-C^T & C^T & 0
\end{pmatrix}
\]
where $P, A, B$ are matrices of size $N \times N$ and $C$ is a column of length $N$:

$$P_{ab} = \dot{q}_a \delta_{ab}, \quad A_{ab} = \frac{g_2 (1 - \delta_{ab})}{q_a - q_b}, \quad B_{ab} = \frac{g_2 (1 - \delta_{ab})}{q_a + q_b} + \frac{g_4 \sqrt{2} \delta_{ab}}{2q_a}, \quad (C)_a = \frac{g_1}{q_a},$$

(A.2)

$a, b = 1, \ldots, N$. The Lax matrix provides the Hamiltonians through $H_k = \frac{1}{2\pi} \text{tr} L^k$. For $k = 2$ this yields (1.20). In fact, the matrix (A.1) becomes the Lax matrix of the model (1.20) if the coupling constants $g_2, g_4$ and $g_1$ satisfy the condition

$$g_1 (g_1^2 - 2g_2^2 + \sqrt{2}g_2g_4) = 0.$$ 

(A.3)

The classical root systems (of $BCD$ types) arise as follows:

- $B_N$ ($\text{so}_{2N+1}$): $g_4 = 0$, $g_1^2 = 2g_2^2$; $r = 2N + 1$, $x_{2N+1} = (q_1, \ldots, q_N, -q_1, \ldots, -q_N, 0)$;
- $C_N$ ($\text{sp}_{2N}$): $g_1 = 0$ and $r = 2N$, $x_{2N} = (q_1, \ldots, q_N, -q_1, \ldots, -q_N)$;
- $D_N$ ($\text{so}_{2N}$): $g_1 = 0$, $g_4 = 0$ and $r = 2N$, $x_{2N} = (q_1, \ldots, q_N, -q_1, \ldots, -q_N)$,

where $r$ is equal to the size of the Lax matrix (it is a dimension of the fundamental representation) and $x_r = \{x_1, \ldots, x_r\}$ is the set of coordinates on the Cartan subalgebra of the corresponding Lie algebra.

Factorization formulae. Let us write down the factorized form of the Lax matrices (A.1)–(A.4) [17]. For this purpose we use the Vandermonde matrix $V$ and the $D^0$ from (1.17)–(1.18). Both matrices are uniquely defined by a set of $r = N$ variables $x_N = \{q_1, \ldots, q_N\}$. We assume the following rule for $BCD$ cases: the matrices $V, D^0, C_0, \tilde{C}$ (1.17)–(1.18) are of size $r \times r$ constructed by means of sets of variables $x_r$ from (A.4). For $B_N$ root system $D^0$ obtained in this way should be also multiplied by diag$(I_N, I_N, \sqrt{2})$. With these definitions, we have the following factorization formulae:

- **$C_N$** and **$D_N$**: set

$$g_1 = 0, \quad g_2 = h, \quad g_4 = \sqrt{2}h\xi$$

(A.5)

and make the substitutions (these are some canonical transformations in the Hamiltonian approach)

$$\dot{q}_i \rightarrow \frac{\xi h}{q_i} + \sum_{k \neq i}^N \left( \frac{h}{q_i - q_k} + \frac{h}{q_i + q_k} \right), \quad i = 1, \ldots, N.$$ 

(A.6)

Then the matrix $L \rightarrow L'$ obtained in this way takes the form

$$L' = h(D^0)^{-1} V (C_0 - (1 - 2\xi)\tilde{C}) V^{-1} D^0.$$ 

(A.7)

This is true for the $C_N$ case, and $\xi = 0$ in (A.7) yields the $D_N$ case.

- **$B_N$**: set

$$g_1 = \sqrt{2}h, \quad g_2 = h, \quad g_4 = 0$$

(A.8)

---

The model (1.20) is integrable for arbitrary constants but this Lax representation requires the constraint (A.3). We use it since the factorization formulae and determinant identities are available for this type of the Lax representation only. Alternative Lax pairs can be found in [4].
and make the substitutions
\[ \dot{q}_i \rightarrow \frac{2\hbar}{q_i} + \sum_{k \neq i}^N \left( \frac{\hbar}{q_i - q_k} + \frac{\hbar}{q_i + q_k} \right), \quad i = 1, \ldots, N. \] (A.9)

The matrix \( L \rightarrow L'' \) obtained in this way is represented in the form
\[ L'' = \hbar(D^0)^{-1}V(C_0 + \tilde{C})V^{-1}D^0, \] (A.10)
where \( C_0 \) and \( \tilde{C} \) are the matrices defined in (1.18), (1.19) but of the size \((2N + 1) \times (2N + 1)\).

**Frobenius companion matrix** is constructed by means of coefficients of characteristic polynomial \( p(z) = \det(zI - J) = z^r + c_{r-1}z^{r-1} + c_{r-2}z^{r-2} + \ldots + c_1z + c_0 \). The matrix \( J \) and its inverse are as follows:
\[
J = \begin{pmatrix}
0 & 0 & \ldots & 0 & -c_0 \\
1 & 0 & \ldots & 0 & -c_1 \\
0 & 1 & \ldots & 0 & -c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -c_{r-1}
\end{pmatrix}, \quad J^{-1} = \begin{pmatrix}
-c_1/c_0 & 1 & 0 & \ldots & 0 \\
-c_2/c_0 & 0 & 1 & \ldots & 0 \\
-c_3/c_0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1/c_0 & 0 & 0 & \ldots & 0
\end{pmatrix}, \quad (A.11)
\]
where we assume generic case, so that \( c_0 \neq 0 \). The zeros of \( p(z) \) are eigenvalues \((x_1, \ldots, x_r)\) of \( r \times r \) matrix \( J \), and \((-1)^i c_{r-i} = c_i(x_1, \ldots, x_r)\) are the elementary symmetric polynomials defined by \( p(z) = \prod_{k=1}^r (z - x_k) = \sum_{k=0}^r (-1)^r e_{r-k}(x_1, \ldots, x_r)z^k \). The ratios of coefficients entering \( J^{-1} \) can be represented in the following way:
\[
-\frac{c_k}{c_0} = (-1)^{k+1} \frac{e_{r-k}(x_1, \ldots, x_r)}{e_r(x_1, \ldots, x_r)} = (-1)^{k+1} e_k(x_1^{-1}, \ldots, x_r^{-1}). \] (A.12)

The Vandermonde matrix \( V_f(x) = x_i^{j-1} \) brings \( J \) to the diagonal form:
\[ \text{diag}(x_1, x_2, \ldots, x_r) = VJV^{-1} \] (A.13)
or, equivalently,
\[ \text{diag}(x_1^{-1}, x_2^{-1}, \ldots, x_r^{-1}) = VJ^{-1}V^{-1}. \] (A.14)

**The determinant identities.** For the \( C_N \) and \( D_N \) root systems, following [17, 18], consider the \( 2N \times 2N \) matrix
\[
\mathcal{L} = L \left( \{H_j^{G(2|0)}\}_N(\{q\}_N, \{\mu\}_M, \xi), \{q\}_N | 0, \hbar, \sqrt{2} \hbar \xi \right). \] (A.15)
It is the Lax matrix \([A.1] L(\{q_j\} | g_1, g_2, g_4)\) of type \( C_N \), where velocities are replaced by eigenvalues of \( \text{gl}(2|0) \) Gaudin Hamiltonians \([2,13]\), and the set of constants is chosen according to \([A.3]\). Define also the dual matrix \( \tilde{\mathcal{L}} \) of size \( 2M \times 2M \):
\[
\tilde{\mathcal{L}} = L \left( \{H_j^{G(2|0)}\}_M(\{\mu\}_M, \{q\}_N, 1 - \xi), \{\mu\}_M | 0, \hbar, \sqrt{2} \hbar (1 - \xi) \right). \] (A.16)
Then the determinant identity for the matrices (A.15), (A.16) is as follows:
\[
\det_{2N \times 2N} \left( \mathcal{L} - \lambda I \right) = \lambda^{2N - 2M} \det_{2M \times 2M} \left( \tilde{\mathcal{L}} - \lambda I \right).
\] (A.17)

Similarly, for the $B_N$ root system define the matrix
\[
\mathcal{L} = L \left( \{ \tilde{H}^{G(2|0)}_j \}_N, \{ q_j \}_N \mid \sqrt{2h}, h, 0 \right)
\] (A.18)
of size $(2N + 1) \times (2N + 1)$ with the set of coupling constants (A.8) and $\tilde{H}^G_j$ are from (2.21). The dual matrix is of size $2M \times 2M$:
\[
\tilde{\mathcal{L}} = L \left( \{ -H^{G(2|0)}_j \}_M, \{ q_j \}_N, \xi = -1 \right), \quad \tilde{\mathcal{L}} \in \text{Mat}(M, \mathbb{C}),
\] (A.19)

where
\[
\tilde{A}_{ij} = \delta_{ij} \left( \frac{h}{\mu_i} + \sum_{k=1}^{N} \left( \frac{h}{\mu_i - q_k} + \frac{h}{\mu_i + q_k} \right) - \sum_{l \neq i}^{M} \left( \frac{h}{\mu_i - \mu_l} + \frac{h}{\mu_i + \mu_l} \right) \right) + h(1 - \delta_{ij})
\] (A.21)
and
\[
\tilde{B}_{ij} = \delta_{ij} \frac{h}{\mu_i} + (1 - \delta_{ij}) \frac{h}{\mu_i + \mu_j}.
\] (A.22)

Then the determinant identity reads as follows:
\[
\det_{(2N+1) \times (2N+1)} \left( \mathcal{L} - \lambda I \right) = -\lambda^{2N - 2M + 1} \det_{2M \times 2M} \left( \tilde{\mathcal{L}} - \lambda I \right).
\] (A.23)

5.2 The notation for $\text{gl}(n|m)$ matrices

We use the fundamental (defining) representation of the $\text{gl}(n|m)$ superalgebra. Elements of $\text{gl}(n|m)$ are endomorphisms of $\mathbb{Z}_2$-graded vector space $V = \mathbb{C}^{m|n}$. The parity of the basis elements of $V$ is defined through $\mathbb{Z}_2$-valued parameter
\[
p(i) = \begin{cases} 
0, & \text{for } 1 \leq i \leq m, \\
1, & \text{for } m + 1 \leq i \leq n + m.
\end{cases}
\] (A.24)

The parity of the matrix units
\[
p(E_{ij}) = p(i) + p(j) \mod 2
\] (A.25)
provides the rule for the tensor product of operators (matrices):
\[
(A \otimes B)(C \otimes D) = (-1)^{p(B)p(C)}(AC) \otimes (BD).
\] (A.26)

The super-transposition for operator-valued matrices is defined as
\[
A = \sum_{i,j=1}^{m+n} E_{ij} \otimes a_{ij} \quad \rightarrow \quad A^t = \sum_{i,j=1}^{m+n} (-1)^{p(j) + p(i)} E_{ji} \otimes a_{ij}.
\] (A.27)

The super trace is
\[
\text{str}A = \sum_{i=1}^{m} a_{ii} - \sum_{i=m+1}^{m+n} a_{ii}.
\] (A.28)
5.3 Proof of Lemma (3.21)

Let us consider the characteristic polynomial of the matrix $L$ \textcolor{red}{(3.21)}:

\[
\det_{2N \times 2N} (L - \lambda I) = \det_{N \times N} \left( (B - A - P)(B + A + P) + \lambda^2 I \right). \tag{A.29}
\]

The explicit form of these matrices is as follows:

\[
(P + A + B)_{ij} = \delta_{ij} \left( \dot{q}_i + \frac{g_4}{\sqrt{2q_i}} \right) + (1 - \delta_{ij}) \frac{2g_2q_i}{q_i^2 - q_j^2}, \tag{A.30}
\]

\[
(B - A - P)_{ij} = \delta_{ij} \left( \frac{g_4}{\sqrt{2q_i}} - \dot{q}_i \right) - (1 - \delta_{ij}) \frac{2g_2q_j}{q_i^2 - q_j^2}.
\]

Next, compute the product of the matrices (A.30):

\[
\sum_{\alpha=1}^{N} (B - A - P)_{i\alpha} (B + A + P)_{\alpha j} = \left( \frac{g_4^2}{2q_i^2} - q_i^2 \right) + \sum_{\alpha \neq i} \frac{4g_2^2q_\alpha^2}{q_i^2 - q_\alpha^2}.
\]

\[
\sum_{\alpha=1}^{N} (B - A - P)_{i\alpha} (B + A + P)_{\alpha j} = \left( \frac{g_4}{\sqrt{2q_i}} - \dot{q}_i \right) \frac{2g_2q_i}{q_i^2 - q_j^2} - \left( \frac{g_4}{\sqrt{2q_j}} + \dot{q}_j \right) \frac{2g_2q_j}{q_i^2 - q_j^2} \tag{A.31}
\]

\[
- \sum_{\alpha \neq i,j} \frac{4g_2^2q_\alpha^2}{(q_i^2 - q_\alpha^2)(q_\alpha^2 - q_j^2)} = - \frac{2q_i\dot{q}_i}{q_i^2 - q_j^2} - \frac{2q_j\dot{q}_j}{q_i^2 - q_j^2} - \sum_{\alpha \neq i,j} \frac{4g_2^2q_\alpha^2}{(q_i^2 - q_\alpha^2)(q_\alpha^2 - q_j^2)}, \quad i \neq j.
\]

Thereby, we found that every matrix element of $N \times N$ matrix (A.29) is an even function of $g_4$. Thus the characteristic polynomial (A.29) is an even function of $g_4$.

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