Proxy Voting for Better Outcomes

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Abstract

We consider a social choice problem where only a small number of people out of a large population are sufficiently available or motivated to vote. A common solution to increase participation is to allow voters use a proxy, that is, transfer their voting rights to another voter. Considering social choice problems on metric spaces, we compare voting with and without the use of proxies to see which mechanism better approximates the optimal outcome, and characterize the regimes in which proxy voting is beneficial.

When voters’ opinions are located on an interval, both the median mechanism and the mean mechanism are substantially improved by proxy voting. When voters vote on many binary issues, proxy voting is better when the sample of active voters is too small to provide a good outcome. Our theoretical results extend to situations where available voters choose strategically whether to participate. We support our theoretical findings with empirical results showing substantial benefits of proxy voting on simulated and real preference data.

1 Introduction

In his 1969 paper, James Miller envisioned a world where technology enables people to vote from their homes [18]. With the rise of participatory democracies, the formation of many overlapping online communities, and the increasing use of polls by companies and service providers, this vision is turning into reality.

New online voting apps provide an easy way for people to report and aggregate their preferences, from simple direct polls (such as those used by Facebook and Doodle), through encrypted large-scale applications (e.g. electionbuddy.com), to sophisticated tools that use AI to guide group selection, such as robovote.org. As a result, each of us is prompted to vote in various formats multiple times a day: we vote for our union members and approve their decisions, on meeting times, and even on the temperature in our office.1

Is direct democracy coming back? Can it replace representative democracy and parliaments? As it turns out, many online voting instances and polls have low participation rates [5, 15], presumably since most people consider them insignificant, low-priority,

1 http://design-milk.com/comfy-app-brings-group-voting-workplace-thermostat
or simply a burden. The actual decisions in many of these polls are often taken by a small group of dedicated and active voters, with little or no involvement from most people who could have voted. The outcome in such cases may be completely unrepresentative for the entire population, e.g., if the motivation of the active voters depends on their position or other factors. Even if the set of active voters is selected at random and is thus representative in expectation, there may be too few voters for a reliable outcome. For example, Mueller et al. [20] argue that to function well, such a “random democracy” would require over 1000 representatives.

Proxy voting lets voters who are unable or uninterested to vote themselves transfer their voting rights to another person—a proxy. Proxy voting is common in politics and in corporates [23], and plays an important role in existing and planned systems for e-voting and participatory democracies [21]. Yet there is only a handful of theoretical models dealing with proxy voting, and our understanding of its effects are limited (see Discussion).

In this paper, we model voters’ positions as points in a metric space aggregated by some function $g$ (specifically, Median, Mean, or Majority). For example, a voter’s position may be her preferred pension policy in the union’s negotiation with management (say, how much to save on a scale of 0 to 10). The optimal policy is an aggregate over the preferences of all employees. Since actively participating in union’s meeting costs time and effort, we consider a subset of active voters selected from the population (either at random or by strategic self-selection), and ask whether the accuracy can be improved by allowing inactive voters to use a proxy at no cost. Following Tullock [27], we weigh the few active voters (who are used as proxies) according to their number of followers, and assume that inactive voters select the “nearest” active voter as a proxy. For example, a person who is unable to attend the next union meeting could use an online app to select a colleague with similar preferences as her proxy, thereby increasing his weight and influencing the outcome in her direction.

The intuition for why proxy voting should increase accuracy is straight-forward: opinions that are more “central” or “representative” would attract followers and gain weight, whereas the weight of “outliers” that distort the outcome will be demoted. However as we will see, this reasoning does not always work in practice. Thus it is important to understand the conditions in which proxy voting is expected to improve accuracy, especially when voters behave strategically.

1.1 Contribution and Structure

We dedicate one section to each common mechanism, and show via theorems and empirical results that proxy voting usually has a significant positive effect on accuracy, and hence welfare. For the Median mechanism on a line (Section 3), proxy voting may only increase the accuracy, often substantially. For the Mean mechanism on a line (Section 4), we show improvement in expectation if active voters are sampled from the population at random. The last domain contains multiple independent binary issues, where a Majority vote is applied to each issue (Section 5). Here we show that proxy voting essentially leads to a “dictatorship of the best expert,” which increases accuracy when the sample is small and/or when voters have high disagreements. Interestingly,
results on real preference data are even more positive, and we analyze the reasons in the text. We further characterize equilibria outcomes when voters strategically choose whether to become active (i.e., use as proxies), and show that most of our results extend this strategic setting. Results are summarized in Table [1].

2 Preliminaries

\(\mathcal{X}\) is the space, or set of possible voter’s preferences, or types. In this paper \(\mathcal{X} \subseteq \mathbb{R}^k\) for some \(k \geq 1\) dimensions, thus each type can be thought of as a position in space. We use the \(\ell_p\) distance metric on \(\mathcal{X}\). In particular, we will consider two spaces: an unknown interval \(\mathcal{X} = [a, b]\) for some \(a, b \in \mathbb{R} \cup \{\pm \infty\}\), and multiple binary issues \(\mathcal{X} = \{0, 1\}^k\). Note that this means that all \(\ell_p\) norms coincide (not true e.g. for \(\mathcal{X} = \mathbb{R}^2\)).

We assume an infinite population of voters, that is given by a distribution \(\mathcal{X} = \{a, b\}\) for all \(x \in \mathcal{X}\). We say that \(f\) over the interval \([a, b]\) is symmetric if there is a point \(c\) s.t. \(f(c - x) = f(c + x)\) for all \(x\). We say that \(f\) over the interval \([a, b]\) is \([weakly]\) single-peaked if there is a point \(c \in \mathcal{X}\) s.t. \(f\) is \([weakly]\) increasing in \([\min a, z]\) and \([weakly]\) decreasing in \([z, b]\). \(f\) is \(single-dipped\) if the function \(-f\) is single-peaked. For example, (truncated) Normal distributions are single-peaked, and Uniform distributions are weakly single peaked. We denote the cumulative distribution function corresponding to \(f\) by \(F(X) = P_{z \sim f}(z < x)\).

Mechanisms A mechanism \(g : \mathcal{X}^n \rightarrow \mathcal{X}\) (also called a voting rule) is a function that maps any profile (set of positions) to a winning position.

Two particular mechanisms we will consider for the interval setting are the Mean mechanism, \(\text{mm}(S) = \frac{1}{|S|} \sum_{s_i \in S} s_i\), and the Median mechanism, \(\text{md}(S) = \min\{s_i \in S \text{ s.t. } |\{j : s_j \leq s_i\}| \geq |\{j : s_j > s_i\}|\) (see Fig. [1]).

For the binary issues we will focus on a simple Majority mechanism that aggregates each issue independently according to the majority of votes. That is, \((\text{mj}(S))^{(j)} = 1\) if \(|\{i : s_i^{(j)} = 1\}| > |\{i : s_i^{(j)} = 0\}|\) and 0 otherwise, where \(s^{(j)}\) is the \(j\)’th entry of position vector \(s\). In all mechanisms we break ties lexicographically towards the lower outcome.

All of our three mechanisms naturally extend to such infinite populations, as the Median, Mean, and Majority of \(f\) (in their respective domains) are well defined. The mechanisms also extend to weighted finite populations. E.g. for \(n\) agents with positions \(S\) and weights \(w = \{w_1, \ldots, w_n\}\), the weighted mean is defined as \(\text{mm}(S, w) = \frac{1}{\sum_{i \leq n} w_i} \sum_{i \leq n} w_is_i\), and similarly for the Median and Majority.

In our model, a finite subset \(N\) of \(n\) agents are selected out of the whole population, and only these agents can vote. We follow \([20]\) in assuming that positions \(S_N = \{s_1, \ldots, s_n\}\) are sampled i.i.d. from \(f\). We can think of these as voters who happen to be available at the time of voting, or voters for which this voting is important enough to consider participation.

In our basic setup, the unavailable voters abstain, while all agents vote. The result is \(g(S_N)\). Yet two problems may prevent us from getting a good outcome. First, \(N\) may be too small for \(g(S_N)\), the decision made by the agents, to be a good estimation
Figure 1: The top figure shows the preferences of 4 agents on an interval, as well as the outcomes of the median and mean mechanisms. In the middle figure we see the weight of each agent under proxy selection, assuming $f$ is a uniform distribution on the whole interval, as well as the modified outcomes. The bottom figure shows the outcome under proxy voting if agent 2 becomes inactive, and $M = \{1, 3, 4\}$. The dotted line marks $\text{mn}(f) = \text{md}(f) = 5$.

Proxies and weights Our main focus in this paper is characterizing the regime in which voting by proxy is beneficial. In this setup each inactive voter specifies one of the active agents as a proxy to vote on her behalf. Given a set $M$ of active agents, the decisions of inactive voters are specified by a mapping $J_M : \mathcal{X} \rightarrow M$, where $J_M(x) \in M$ is the proxy of any voter located at $x \in \mathcal{X}$. We label the Proxy setup as $P$, in contrast to the Basic setup denoted as $B$. We highlight that all voters select a proxy, whether they are part of $N$ or not.

Without further constraints, we will assume that the proxy of a voter at $x$ is always its nearest active agent, i.e. the agent whose position (or preferences) are most similar to $x$. Thus for every set $M$, we get a partition (a Voronoi tessellation) of $\mathcal{X}$ and can compute the weight of each active agent $j$ by integrating $f$ over the corresponding cell. Formally, $J_M(x) = \arg\min_{j \in M} \|x - s_j\|$ and $w_j = \int_{x \in \mathcal{X}, J_M(x) = j} f(x) dx$. The outcome of each mechanism $g$ for agents $N$ is then defined as $g^B(S_N) = g(S_N)$ in the Basic scenario, and $g^P(S_N) = g(S_N, w_N)$ in the Proxy scenario, where $w_N$ is computed according to $f$ as above (see Fig. 1). The distribution $f$ should be inferred from the context.

Equilibrium under strategic participation In our strategic scenarios the agents $N$ are players in a complete information game, whose (ordinal) utility exactly matches their preferences as voters. I.e., they prefer an outcome that is as close as possible to their own position. Each agent has two actions: active and inactive. In addition, a voter who is otherwise indifferent between the two possible outcomes (i.e. he is not pivotal)
will prefer to remain inactive, a behavior known as lazy-bias \[10\]. We refer to these strategic/lazy-bias scenarios by adding $+L$ to either $B$ or $P$. Agents may not misreport their position.

When there are no proxies (scenario $B+L$) this strategic decision is very simple, since each agent has a single vote which may or may not be pivotal (and when it is pivotal it always helps the agent). On the other hand, if voting by proxy is allowed (scenario $P+L$), any change in the set of active agents changes the proxy selection and thus the weights of all remaining agents. Recall that $w_M$ denotes the weights we get under proxy selection with active set $M$. Then for all $i \notin M$, agent $i$ prefers to join set $M$ iff $\|g(S_{M \cup \{i\}}, w_{M \cup \{i\}}) - s_i\| < \|g(S_M, w_M) - s_i\|$.

For example, if agent 2 in Fig. 1 (bottom) becomes inactive, we get no change in the Median outcome $\text{md}(S_M, w_M)$, and thus agent 2 prefers to become inactive (it is also possible that an agent strictly loses when becoming active).

A pure Nash equilibrium, or equilibrium for short, is a subset $M \subseteq N$ s.t. no agent in $M$ prefers to be inactive, and no agent in $N \setminus M$ prefers to be active. While it is possible that there are multiple equilibria (or none at all), this will turn out not to be a problem in most cases we consider. We thus define $g^{B+L}(S_N) = g(S_M)$ and $g^{P+L}(S_N) = g(S_M, w_M)$, where $M \subseteq N$ is the set of active agents in equilibrium.

To recap, an instance is defined by a population distribution $f$, a scenario $Q \in \{B, P, B+L, P+L\}$, a mechanism $g \in \{\text{md}, \text{mm}, \text{mj}\}$ and a sample size $n$. We sample a finite profile of $n$ agents i.i.d. from $f$, whose locations are $S_N$. Then, according to the scenario, either all of $N$ are active, or we get a subset $M$ of active agents. The votes of all active agents are aggregated according to $g$, with or without being weighted by $w_M$, the number of their inactive followers. Finally, the outcome of mechanism $g^Q(S_N)$ depends on a subset of these parameters, according to the scenario $Q$.

**Evaluation** We do not consider here the reasons for using one mechanism over another, and simply assume that $g(f)$ reflects the best possible outcome to the society or to the designer. We want to measure how close is $g^Q(S_N)$ to the optimal outcome $g(f)$. We define the error as the distance between $g^Q(S_N)$ and $g(f)$, i.e., $\|g^Q(S_N) - g(f)\|$.

The loss of a mechanism $g$ is calculated according to its expected error—the expected squared distance from the optimum—over all samples of $m$ available voters.

$$L^Q(n) = \mathbb{E}_{S_N \sim f^n} \left[ \|g^Q(S_N) - g(f)\|^2 \right],$$

where the mechanism $g$ and the distribution $f$ can be inferred from the context, and the expectation is over all subsets of $n$ positions sampled i.i.d. from distribution $f$ (sometimes omitted from the subscript).

We note that the loss is the sum of two components \[28\]: the (squared) bias $\mathbb{E}[g^Q(S_N) - g(f)]^2$ and the variance $\mathbb{V}[g^Q(S_N)]$. A mechanism $g$ is unbiased for $(Q, f)$ if $\mathbb{E}[g^Q(S_N)] = g(f)$. For example in the Basic scenario, mechanisms mm and mj are unbiased for $(B, f)$ regardless of $f$, and md is unbiased for $(B, f)$ if $f$ is symmetric, but not for other (skewed) distributions.

Our primary goal is to characterize the conditions under which proxy voting improves the outcome, i.e. $L^{P[+L]}(n) < L^{B[+L]}(n)$. 

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3 Median Voting on an Interval

The Median mechanism is popular for two primary reasons. First, it finds the point that minimizes the sum of distances to all reported positions, i.e. \( \text{md}(S) \in \arg\min_{x \in X} \sum_{s_i \in S} |s_i - x| \). Second, in strategic settings where agents might misreport their positions, it is known that the Median mechanism is group strategyproof [19], meaning that no subset of agents can gain by misreporting.

3.1 Random participation

Suppose all \( n \) agents sampled from \( f \) are active. Let \( j^* \in N \) be the proxy closest to \( x^* = \text{md}(f) \), and \( s^* = s_{j^*} \).

Lemma 1. \( \text{md}^P(S_N) = s^* \) for any distribution \( f \).

Proof. Recall that \( \text{md}^P(S_N) = \text{md}(S_N, w) \) where \( w_j \) is the weight of voters using \( j \in N \) as a proxy. All voters \( x \geq \text{md}(f) \) are mapped to one of the proxies \( j^*, j^* + 1, \ldots, n \), thus \( \sum_{j=j^*}^n \geq 1/2 \) and \( \text{md}(S_M, w) \geq s^* \). Similarly, all voters \( x \leq \text{md}(f) \) are mapped to one of the proxies \( 1, 2, \ldots, j^* \), thus \( \sum_{j=1}^{j^*} \geq 1/2 \) and \( \text{md}(S_N, w) \leq s^* \). Thus \( \text{md}^P(S_N, w) = s^* \).

Thus \( \text{md}^P(S_N) \) always returns the proxy closest to \( x^* = \text{md}(f) \), whereas \( \text{md}^B(S_N) \) returns some \( j \in N \), meaning that the error is never higher with proxy voting. i.e., \( |\text{md}^P(S) - x^*| \leq |\text{md}^B(S) - x^*| \) for any \( S \). In particular, the loss (expected error) is weakly better.

Corollary 2. For the Median mechanism, \( \mathcal{L}^P(n) \leq \mathcal{L}^B(n) \) for any distribution \( f \) and sample size \( n \).

Proof.

\[
\mathcal{L}^P(n) = E[(\text{md}^P(S_N) - x^*)^2] \leq E[(\text{md}^B(S_N) - x^*)^2] = \mathcal{L}^B(n).
\]

Note that for symmetric distributions, both of \( \text{md}^B(S_N) = \text{md}(S_N) \) and \( \text{md}^P(S_N) = \text{md}(S_N, w_N) \) are unbiased from symmetry arguments. Therefore to compute or bound the loss we just need to compute the variance of \( \text{md}^Q(S_N) \). For the unweighted median, this problem was solved by Laplace (see [25] for details): Let \( x^* = \text{md}(f) \) be the median of symmetric distribution \( f \) s.t. \( f(x^*) > 0 \). The variance of \( \text{md}(S_N) \) is given by (approximately) \( \frac{1}{4n f(x^*)^2} = \Theta(1/n) \). Since for any distribution the loss (or MSE) is lower-bounded by the variance, we get that for the Median mechanism, \( \mathcal{L}^B(n) = \Omega(\frac{1}{n}) \).

We argue that the loss decreases quadratically faster with the number of agents once proxy voting is allowed.

\[2\text{We will assume in this section that } f(x) > 0 \text{ in some environment of } x^*, \text{ which is a very weak assumption.} \]
Conjecture 3. For the Median mechanism, $L^P(n) = O(\frac{1}{n^2})$ for any distribution $f$.

The rest of this section is dedicated to supporting this conjecture. In particular, we prove it for symmetric distributions, and show empirically that it holds for other distributions as well. Further, for Uniform and single-peaked distributions, we can upper-bound the constant in the expression.

Theorem 4. For the Median mechanism, $L^P(n) = O(\frac{1}{n^2})$, for any symmetric distribution $f$.

Proof. W.l.o.g. we can assume $x^* = 0$, and that the support of $f$ is the interval $[-1, 1]$. What is the expected distance between $s^*$ and $\text{md}(f) = x^* = 0$? We can translate each proxy $x_i$ to $y_i = |x_i - x^*| = |x_i|$. Note that $y_i$ come from some distribution $f'$ on $[0, 1]$. By our assumption, $f(x)$ is strictly positive in some $\epsilon$ environment of $x^* = 0$, i.e. $f(x) > \alpha$ for all $x \in [-\epsilon, \epsilon]$, for some $\alpha, \epsilon > 0$. We thus have that $f'(z) > \alpha$ for all $z \leq \epsilon$. Then for the cumulative distribution $F'(z)$, we have that $F'(z) > za$ for all $z \leq \epsilon$, and $F'(z) > \epsilon a$ for all $z > \epsilon$.

Recall that by Lemma 1, the error is exactly $|s^* - x^*| = |s^*|$. The random variable $s^* = \min y_i$ is the minimum of $n$ variables sampled i.i.d. from $f'[0, 1]$. The distribution of the minimum is well known and in particular for all $z \in [0, 1]$,

$$Pr(s^* > z) = (Pr_{Z \sim f'[0,1]}(Z > z))^n = (1 - F'(z))^n.$$  

For $z = \epsilon$, we get $Pr(s^* > \epsilon) = (1 - F'(\epsilon))^n \leq (1 - \epsilon a)^n$.

There is some $n_\epsilon$ s.t. for all $n > n_\epsilon$, $Pr(s^* > \epsilon) < \frac{1}{n^3}$ since the left terms drops exponentially fast. Thus assume $n > n_\epsilon$. Let $T_n = \left\lfloor 2n \cdot \epsilon \right\rfloor$. 

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\[ \mathcal{L}^p(n) = \text{VAR}_f(s^*) = E_f[(s^*)^2] \leq \sum_{t=1}^{2n} \Pr(s^* \in \left[ \frac{t-1}{2n}, \frac{t}{2n} \right]) \left( \frac{t}{2n} \right)^2 \]

(bound by steps)

\[ = \sum_{t=1}^{T_n} \Pr(s^* \in \left[ \frac{t-1}{2n}, \frac{t}{2n} \right]) \left( \frac{t}{2n} \right)^2 + \sum_{t=T_n+1}^{2n} \Pr(s^* \in \left[ \frac{t-1}{2n}, \frac{t}{2n} \right]) \left( \frac{t}{2n} \right)^2 \]

\[ \leq \sum_{t=1}^{T_n} \Pr(s^* \in \left[ \frac{t-1}{2n}, \frac{t}{2n} \right]) \left( \frac{t}{2n} \right)^2 + \Pr(s^* > \epsilon) \]

\[ \leq \sum_{t=1}^{T_n} \Pr(s^* \in \left[ \frac{t-1}{2n}, \frac{t}{2n} \right]) \left( \frac{t}{2n} \right)^2 + \frac{1}{n^3} \]

\[ = \sum_{t=1}^{T_n} \left( \Pr(s^* > \frac{t-1}{2n}) - \Pr(s^* > \frac{t}{2n}) \right) \left( \frac{t}{2n} \right)^2 + \frac{1}{n^3} \]

\[ \leq \sum_{t=1}^{T_n} \Pr(s^* > \frac{t-1}{2n}) \left( \frac{t}{2n} \right)^2 + \frac{1}{n^3} \]

\[ \leq \frac{1}{4n^2} \sum_{t=1}^{T_n} \Pr(s^* > \frac{t-1}{2n}) t^2 + \frac{1}{n^3} \leq \frac{1}{4n^2} \sum_{t=1}^{T_n} (1 - F'(\frac{t-1}{2n})) t^2 + \frac{1}{n^3} \]

\[ \leq \frac{1}{4n^2} \sum_{t=1}^{T_n} \left( 1 - \alpha \frac{t-1}{2n} \right) t^2 + \frac{1}{n^3} \leq \frac{1}{4n^2} \sum_{t=1}^{T_n} e^{-\Theta(t)} t^2 + \frac{1}{n^3} \]

\[ = \frac{1}{4n^2} \sum_{t=1}^{T_n} e^{-\Theta(t)} t^2 + \frac{1}{n^3} \leq \frac{1}{4n^2} C + \frac{1}{n^3} = O(\frac{1}{n^2}), \]

where \( C \) is some constant.

Further, for Uniform \( f = U[-1, 1] \) we can derive a tighter bound on the sum of the series and show \( \mathcal{L}^p(n) < \frac{4}{n^2} \). In fact for any single-peaked \( f \) on \([-1, 1]\) we have \( \mathcal{L}^p(n) < \frac{7}{n^2} \).

We simulated the effect of proxy voting on the Median mechanism in Figure 2. We can see that the (log of the) loss for each distribution closely resembles \( \log(\frac{c}{n^2}) = c' - 2 \log(n) \), where the constant \( c' \) depends on the distribution. This also holds for the asymmetric distributions, which supports our Conjecture 3. In particular, this means that the loss under proxy voting drops much faster than the loss in the Basic scenario, which is roughly \( \frac{1}{n} \).
Figure 2: The top figure shows $L_Q(n)$ (in log scale) as a function of $n$ for various distributions. SP/SD/symm stands for Single-peaked / Single-dipped/ Symmetric distributions. Each point is based on 1000 samples of size $n$.

3.2 Strategic participation

We show that when participation is strategic the outcome of proxy voting is not affected, whereas the unweighted sample median becomes unboundedly worse. Suppose all voters in $N$ are indexed in increasing order by their location, so that $\text{argmin } S_N = 1$.

**Proposition 5.** In the Basic scenario, for any distribution $f$ and any set of agents $N$, there is a unique equilibrium of $\text{md}^{B+L}$ where $M = \{1\}$ (i.e., the lowest agent). Further, the game is weakly acyclic, i.e. there is a sequence of best replies from any initial state to this equilibrium.

Clearly this means that $L^{B+L}(n) < L^B(n)$, and only gets worse as we increase the sample size $n$.

The intuition is that due to tie-breaking, either all agents below current median, or all agents above it, are non-pivotal.

**Proof.** Note first that if $M = \{1\}$ then the single active agent is pivotal by definition. Any other agent $i > 1$ is non-pivotal since $\text{md}(\{s_1, s_i\}) = s_1$ by our tie-breaking assumption, thus $M = \{1\}$ is an equilibrium.

Consider any subset of active agents $M \subseteq N$ s.t. $|M| > 1$. If $m$ is even, then all agents above the median $g(M)$ are non-pivotal. If $m$ is odd then all agents below the median are non-pivotal. Thus there is at least one agent in $M$ who prefers to become inactive. This continues until $|M| = 1$.

Finally, if $M = \{i\}$ for some $i > 1$, we have the following sequence of best-replies: any agent $j < i$ is pivotal, and in particular $j = 1$. Thus agent 1 will become active. Now agent $i$ is no longer pivotal so becomes inactive.

On the other hand, while lazy bias decreases participation in the Proxy scenario, this does not increase the loss.
**Theorem 6.** In the Proxy scenario, for any distribution \( f \) and any set of agents \( N \), there is a unique equilibrium of \( \text{md}^{P+L} \) where \( M = \{ j^* \} \) (the agent closest to \( x^* \)). Further, the game is weakly acyclic, i.e. there is a sequence of best replies from any initial state to this equilibrium.

In particular, \( \mathcal{L}^{P+L}(n) = \mathcal{L}^P(n) \) for any distribution \( f \).

**Proof.** If \( j^* \notin M \) is inactive, then for \( M \cup \{ j^* \} \) the outcome becomes \( s_j \), rather than \( s_k \) (where \( k = J_M(x^*) \)), which \( j^* \) prefers. If \( j^* \) is active, and \( j \neq j^* \) quits, then all votes above \( s_j \), are still mapped to \( j^* \) or higher (and similarly for votes below \( s_j \)). Thus the outcome remains the same which means \( j \) is not pivotal. \( \square \)

### 4 Mean Voting on an Interval

The Mean mechanism is perhaps the simplest and most common way to aggregate positions. For positions \( S_N \) on the interval the outcome is \( \text{mn}(S_N) = \frac{1}{n} \sum_{i \in N} s_i \), which is known to minimize the sum of square distances to all agents.

#### 4.1 Random Participation

Assume that \( f \) is a symmetric distribution, so that \( \text{mn}^Q(S_N) \) is unbiased under all scenarios. When we apply the Mean mechanism, the loss in the basic scenario is simply the sample variance.

**Proposition 7.** Let \( f \) be a symmetric, weakly single-peaked distribution, and suppose \(|N| = 2\). Then, for any \( S_N \), \( \| \text{mn}^B(S_N) - x^* \| \leq \| \text{mn}^B(S_N) - x^* \| \). That is, for any pair of agents the proxy-weighted mean is weakly better than the unweighted mean.

**Proof.** Suppose w.l.o.g. that the support of \( f \) is \([-1,1]\), that \( f \) is symmetric around \( x^* = 0 \), that \( s_1 < s_2 \), and that \( x = \text{mn}(S_N) = \frac{2s_1 + s_2}{2} \geq 0 \). Then for the basic (unweighted) scenario,

\[
\| \text{mn}^B(S_N) - x^* \| = |\text{mn}^B(S_N)| = |\text{mn}(S_N)| = |x| = x.
\]

Since \( f(\cdot) \) in single-peaked, the CDF \( F(\cdot) \) is convex in \([-1,0]\) and concave in \([0,1]\), thus for all \( z \geq 0 \), \( F(z) \geq \frac{z+1}{2} \). In particular \( F(x) \geq \frac{x+1}{2} \).

In the proxy (weighted) scenario, agent 1 gets all voters below point \( x \), i.e. \( w_1 = F(x) \), whereas \( w_2 = 1 - F(x) \). Thus

\[
\text{mn}(S_N, w) = w_1 s_1 + w_2 s_2 = F(\hat{x}) s_1 + (1 - F(\hat{x})) s_2
\]

\[
= s_2 + F(\hat{x})(s_1 - s_2) \leq s_2 + \hat{x} + \frac{1}{2}(s_1 - s_2) \quad \text{as } s_1 - s_2 < 0
\]

\[
= \frac{s_1 + s_2}{2} + \hat{x} \frac{s_1 - s_2}{2} = \hat{x} + \hat{x} \frac{s_1 - s_2}{2}
\]

\[
= \hat{x}(1 + \frac{s_1 - s_2}{2}) \in [-\hat{x}, \hat{x}] \quad \Rightarrow \quad \text{as } s_1 - s_2 < 0
\]

\[
\| \text{mn}^P(S_N) - x^* \| = |\text{mn}(S_N, w)| \leq |\hat{x}| = |\text{mn}^B(S_N) - x^*|,
\]

as required. \( \square \)
We now turn to evaluate this term. Consider 3 agents on \( X = [0, 1] \), located at \( S_N = \{ \frac{1}{3}, \frac{2}{3}, 1 \} \). For a Uniform distribution \( f \), the optimal outcome is \( x^* = \text{mm}(f) = \frac{1}{3} \). In the Basic scenario, \( \text{mm}^B(S_N) = \frac{1}{3} \) while with proxies, \( \text{mm}^P(S_N) = \frac{2}{3} \) and \( \text{mm}^P(S_N, w_N = \{ \frac{1}{3}, 0, \frac{1}{3} \}) = \frac{1}{3} + \frac{4}{3} = \frac{17}{9} \).

The question is under which distributions \( f \) the loss is improved on average by weighing the samples. We show analytically that this holds for uniform distributions and provide similar simulation results for other distributions.

**Proof sketch.** Suppose w.l.o.g. that the support of \( f \) is \([-1, 1]\), that \( f \) is symmetric around \( x^* = 0 \), that \( s_1 < s_2 \), and that \( \hat{x} = \text{mm}(S_N) = \frac{s_2 + s_3}{2} \geq 0 \). Then for the basic (unweighted) scenario, \( ||\text{mm}^B(S_N) - x^*|| = ||\text{mm}^B(S_N)|| = ||\text{mm}(S_N)|| = |\hat{x}| = \hat{x} \).

Since \( f \) in single-peaked, \( F \) is convex in \([-1, 0]\) and concave in \([0, 1]\), thus for all \( z \geq 0 \), \( F(z) \geq \frac{z+1}{2} \). In particular \( F(\hat{x}) \geq \frac{\hat{x} + 1}{2} \).

In the proxy (weighted) scenario, agent 1 gets all voters below point \( \hat{x} \), i.e. \( w_1 = F(\hat{x}) \), whereas \( w_2 = 1 - F(\hat{x}) \). We can compute the weights and show that \( \text{mm}(S_N, w) = (1 + \frac{s_2 - s_3}{3}) \in [-x, x] \).

This means that \( ||\text{mm}^P(S_N) - x^*|| = ||\text{mm}(S_N, w)|| \leq |\hat{x}| \), i.e. weakly better than \( \text{mm}^B(S_N) \).

For larger sets of agents this is not true in general. Even for the Uniform distribution there are examples with more agents where proxy voting leads to a less accurate outcome:

\[
\begin{array}{c|c|c}
0 & \text{mm}(S_N) = \frac{1}{3} & \text{mm}(S_N, w_N) = \frac{17}{32} \\
\hline
\text{mn}(S_N) = \frac{1}{3} & \frac{4}{3} & \frac{5}{3} \\
\end{array}
\]

Consider 3 agents on \( X = [0, 1] \) with \( S_N = \{ \frac{1}{3}, \frac{2}{3}, 1 \} \). For a Uniform distribution \( f \), the optimal outcome is \( x^* = \text{mm}(f) = \frac{1}{3} \). In the Basic scenario, \( \text{mm}^B(S_N) = \frac{1}{3} \) while with proxies, \( \text{mm}^P(S_N) = \frac{2}{3} \) and \( \text{mm}^P(S_N, w_N = \{ \frac{1}{3}, 0, \frac{1}{3} \}) = \frac{1}{3} + \frac{4}{3} = \frac{17}{9} \).

Consider the uniform distribution over the interval \([-1, 1]\) (w.l.o.g., as we can always rescale). In the Basic scenario, we know from \([3]\) that \( L^B(n) = \mathbb{V}[\text{mm}(S_N)] = \frac{1}{5n} \). The next proposition indicates that the loss under proxy voting decreases quadratically faster than without proxies (as with the median mechanism).

**Proposition 8.** For the Mean mechanism, when \( f = U[-1, 1] \), \( L^P(S_N) = \frac{2}{n^2} (1 + O(\frac{1}{n})) \).

**Proof.** We first note that the weighted mean \( \text{mm}(S_N, w_N) \), is an unbiased estimator of the distribution mean from symmetry argument, and therefore \( L^P(S_N) = \mathbb{V}[\text{mm}(S_N, w_N)] \).

We now turn to evaluate this term. \( \mathbb{V}[\text{mm}(S_N)] = \mathbb{E} \left[ (\text{mm}(S_N))^2 \right] \).

\[
\text{mm}(S_N, w_N) = \frac{1}{n} \sum_{j=1}^{n} w_j s_j
\]

Here \( w_j \) is the number of voters that elect representative \( j \) as their proxy. Since the number of vote \( n \) is large, \( \alpha \) is the corresponding share of the probability distribution,
In the Uniform distribution $U[-1, 1]$ we can compute the weights:

$$w_j = F \left( \frac{s_{j+1} + s_j}{2} \right) - F \left( \frac{s_{j-1} + s_j}{2} \right)$$

where we set $s_0 = -2 - s_1$, $s_{n+1} = 2 - s_n$ for convenience. Therefore $mn(S_N, w_N)$ can be written as

$$\sum_{i=1}^{n} w_j s_j = \frac{1}{4} \sum_{i=1}^{n} s_j (s_{j+1} - s_{j-1}) = \frac{s_n + s_1}{2} + \frac{s_1^2 - s_n^2}{4}$$

by telescopic cancellation. Here $s_n$ and $s_1$ are the two extremes representatives. Now, since the joint distribution of $(s_1, s_n)$ is explicitly known [3],

$$\Pr(s_1 = x, s_n = y) = n \cdot (n - 1) \cdot \left( \frac{1}{2} \right)^2 \cdot \left( \frac{y - x}{2} \right)^{n-2}$$

it is possible to evaluate it precisely by integration. We get

$$\mathbb{E} \left[ mn(S_N, w)^2 \right] = \frac{2((n - 5)n + 14)n \cdot (n - 1)}{\prod_{i=1}^{n}(n + t)}$$

$$= \frac{2}{n^2} \left( 1 + O(n^{-1}) \right) \cdot \left( \frac{1}{2} \right)^2 \cdot \left( \frac{y - x}{2} \right)^{n-2}$$

We should note that the estimator $\frac{s_1 + s_n}{2}$ is known to minimize the MSE for the uniform distribution. It is interesting that the estimator obtained by proxy voting is so similar. 

Recall that $\frac{s_1 + s_n}{2}$ is the maximum likelihood estimator of $\mathbb{E}[f] = mn(f)$ for the uniform distribution, and $\mathbb{V}[s_1 + s_n] = \frac{2}{\pi^2} \left( 1 + O(\frac{1}{n}) \right)$, which means that $\mathcal{L}^P / \mathcal{L}^B \rightarrow 4$.

While proxy voting may have adverse effect on the mean in specific samples, our proof shows that on average, proxy voting leads to a substantial gain under the Uniform distribution. Other common distributions displayed the same effect. Fig. 3 shows proxy voting leads to a substantial improvement over the unweighted mean of active voters for various distributions.

### 4.2 Strategic participation

In the basic (non-proxy) scenario, it is easy to see that every voter is always pivotal with any active set unless $s_i = mn(M)$. Thus in every equilibrium $M \subseteq N$, $mn(S_M) = mn(S_N)$, and for any distribution $f$, and $\mathcal{L}^{B+L}(n) = \mathcal{L}^{B}(n)$. 

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In the proxy setting things get more involved. The following lemma analyzes the best response of agents in cases where the voter’s population is monotonic is some region.

**Lemma 9.** (A) It is a dominant strategy for both argmin_i \{S_N\} and argmax_i \{S_N\} to be active;

(B) Consider three agents, s_1 < s_2 < s_3 s.t. \{1,3\} \subseteq M. Suppose f is strictly decreasing in [s_1, s_3]. Agent 2 prefers to be active if \text{mn}(S_{M\cup\{s_2\}}) < s_2, and prefers to be inactive if \text{mn}(S_{M\backslash\{s_2\}}) \geq s_2. The reverse condition applies for increasing f. If f is constant, agent 2 always prefer to be inactive.

**Proof.** (A) is obvious. For (B), consider a set of active agents M^- such that \{s_1, s_3\} \subseteq M^- and s_2 \notin M^- . Define M^+ = M^- \cup \{s_2\}. The population decision boundary points are the intermediate points between the different agents \alpha = (s_1 + s_2)/2, \gamma = (s_2 + s_3)/2 and \beta = (s_1 + s_3)/2. The result of the decision mechanism is denoted as g_{M^-}, or correspondingly, g_{M^+}. We note that \text{mn}(S_{M^+}) - \text{mn}(S_{M^-}) equals (s_2 - s_1) \int_{\alpha}^{\beta} f(x)\,d(x) + (s_2 - s_3) \int_{\beta}^{\gamma} f(x)\,d(x).

By the intermediate value theorem, there exists a point x_1 \in [\alpha, \beta] such that \int_{\alpha}^{\beta} f(x)\,d(x) = (\beta - \alpha) f(x_1) = \frac{(s_3 - s_2)f(x_1)}{2}.

Similarly, there is x_2 \in [\beta, \gamma] such that \int_{\beta}^{\gamma} f(x)\,d(x) = \frac{(s_2 - s_1)f(x_2)}{2}. Therefore,

\[
\text{mn}(S_{M^+}) - \text{mn}(S_{M^-}) = \frac{(s_2 - s_1)(s_3 - s_2)(f(x_1) - f(x_2))}{2}
\]  

(3)

This expression is positive if f(x_1) > f(x_2). If f is monotonic decreasing in [s_1, s_3] this holds, while if f is monotonic increasing we have \text{mn}(S_{M^+}) < \text{mn}(S_{M^-}). If \text{mn}(S_{M^-}) < s_2 and f is increasing, it is not beneficial for s_2 to become active. Likewise, if \text{mn}(S_{M^-}) > s_2 and f is monotonic decreasing s_2 will not be active.
Finally, if $f$ is constant, then $f(x_1) = f(x_2)$ and agent $s_2$ does not affect the result and will be inactive.

Before considering general probability distributions, we apply the previous lemma for the particular case of the uniform distribution. We show that even when the voters are strategic, the result equilibrium is the optimal configuration.

**Proposition 10.** In the Proxy scenario, for the Uniform distribution and any set of agents $N$, there is a unique equilibrium of $\text{mn}^{P+L}$ where $M = \{\argmin S_N, \argmax S_N\}$ (i.e., the two extreme agents). Further, the game is weakly acyclic, i.e. there is a sequence of best replies from any initial state to this equilibrium.

**Proof.** Lemma 9(A) says it is beneficial that the two most extreme agents be active. Due to Part (B), all other agents will quit.

Our last result for uniform distributions shows that strategic behavior, despite lowering the number of active agents, leads to a more accurate outcome than in the non-strategic case. In fact, it can be shown that no other estimator outperforms $\text{mn}^{P+L}$ for the Uniform distribution.

**Corollary 11.** For the Mean mechanism, for any sample $S_N$, under the unique equilibrium of $\text{mn}^{P+L}$ for Uniform $f$, $\text{mn}^{P+L}(S_N) = \text{mn}^P(S_N)$. In particular, $\mathcal{L}^{P+L}(n) = \mathcal{L}^P(n)$.

**Proof.** w.l.o.g. $f = U[-1, 1]$. For any sample $S_N$, let $S_M = \{s_1, s_n\}$ contain the two extreme samples. Let $w_M = (w_1^*, w_n^*)$ denote the weights of these samples under proxy voting, when there are no other agents. We have that

$$
\text{mn}^{P+L}(S_N) = \text{mn}(S_M, w_M) = \frac{1}{2}(s_1 w_1^* + s_n w_n^*)
= \frac{1}{2} \left( s_1 \left( \frac{s_1 + s_n}{2} + 1 \right) + s_n \left( 1 - \frac{s_1 + s_n}{2} \right) \right)
= \frac{s_1 + s_n}{2} + \frac{s_1^2 - s_n^2}{4}
= \text{mn}(S_N, w_N) = \text{mn}^P(S_N),
$$

(by Eq. (2))

as required.

We now turn to analyze more general distributions, and we first focus on the single peak case. Denote the peak location as $x$, the smallest agent in $[x, \infty]$ as $s_+$ and the largest agent in $[-\infty, x]$ as $s_-$. Set $\alpha = (s_+ + s_-)/2$ as the intermediate point between $s_+$ and $s_-$. Assume, w.l.o.g. that $f(s_-) \geq f(s_+)$. We call a given set of agents $M$ an equitable partition if $\text{mn}^P(S_M) \in [s_-, s_+]$ and $f(A) \geq f(s_+)$ (Fig. 4a).

**Proposition 12.** Consider a single peaked distribution $f$ and a profile $S_N$. If $S_N$ is an equitable partition, then there is an equilibrium of $\text{mn}^{P+L}$ where all agents are active $M = N$. In particular, the error is the same as in $\text{mn}^P$. 

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Figure 4: a) An equitable partition of a single peaked distribution. b) A feasible equilibrium in a single dip scenario. Note that in both cases the distribution is not necessarily symmetric.

Proof. Consider the set \( M = N \). Following Eq. 3, \( s_+ \) will not quit from the active set if \((s_2 - s_1)(s_3 - s_2)(f(x_1) - f(x_2)) > 0\).

We shall now show that \( f(x_1) > f(s_+) \). Assume \( A < x \). As \( f(A) \leq f(s_+) \), for every \( y \in [A,x] \) we have \( f(y) > f(A) \geq f(s_+) \) as \( f \) is increasing in \([A,x]\). Likewise, \( f(y) > f(s_+) \) as \( f \) is decreasing in \([x,s_+]\) therefore \( f(x_1) \), the mean value of \( f \) in \([A,s_+]\) satisfies \( f(x_1) > f(s_+) \). Now, \( f(s_+) > f(x_2) \) as \( f \) is monotonic decreasing in \([x,\infty]\). Therefore, the former expression is positive, and \( s_+ \) will stay in the equilibrium set.

Now, Lemma 9(A) shows that the most extreme agents \( s_1, s_n \) will be active, while Lemma 9(B) shows every agent between \([s_+, s_n]\) and \([s_1, s_-]\) are also active. Namely, all the agents are active.

This shows that proxy voting may achieve maximal participation in a single peak setup. Next, we address the single dip setting.

Proposition 13. Consider a single dipped distribution where the dip location is \( x \). Consider any equilibrium \( M \subseteq N \), and assume w.l.o.g that \( \min(S_M) \leq x \). Then, \( M \) contains at most two agents in \([\min(S_N), \min(S_M)]\) and at most two agents in \([\max(S_M), \max(S_N)]\).

Proof. Lemma 9(A) shows that the two most extreme agents \( s_1, s_n \) are always active. Denote the dip location as \( x \). Consider some active agents set \( M \), and assume \( \min(S_M) \leq x \). Lemma 9(B) shows that there can not be more than two agent in \([x, s_n]\) and \([s_1, \min(S_M)]\). Consider an equilibrium set that contain active agents in \( \mathcal{A} = [\min(S_M), x] \). Denote the maximal active agent in \( \mathcal{A} \) as \( y \). Then Lemma 9(B) indicates that all agents in \([\min(S_M), y]\) are active, while there is only one active agent in \([s_1, \min(S_M)]\), which is \( s_1 \).

If there are no active agent in \( \mathcal{A} \), then Lemma 9(B) show that are at most two agents in \([s_1, \min(S_M)]\) and in \([s, s_n]\).

We see the possible emergence of four active agents, or parties, at the center-right, center-left, extreme right and extreme left. If the distribution is heavily skewed, we
expect some parties to emerge between the dip location and the decision rule, balancing the result.

5 Binary Issues

In this section \( \mathcal{X} = \{0, 1\}^k \) and \( \mathbf{m}_j(S) \) outputs a binary vector according to the majority on each issue. In the most general case, \( f \) can be an arbitrary distribution over \( \{0, 1\}^k \). However, we assume that issues are conditionally independent in the following way: first a number \( P \) is drawn from a distribution \( h \) over \([0, 1]\), and then the position on each issue is ‘1’ w.p. \( P \). That is, the position of a voter on all issues is \((s^{(1)}, \ldots, s^{(k)})\), where \( s^{(j)} \) are random variables sampled i.i.d from a Bernoulli distribution \( Ber(P) \), and \( P \) is a random variable sampled from \( h \). Since \( h \) induces \( f \) we sometimes use them interchangeably.

Evaluation W.l.o.g. denote the majority opinion on each issue as 0, meaning that \( x^* = \mathbf{m}_j(f) = (0, 0, \ldots, 0) \). The expected rate of ‘1’ opinions is \( \mu \equiv \mathbb{E}_{P \sim h}[P] < 0.5 \).

5.1 Random participation

Suppose that each agent is wrong w.p. exactly \( \mu < 0.5 \), i.e. \( t_i \sim Ber(\mu) \) is the opinion of agent \( i \) on a particular issue. Then the probability that the majority is wrong on this issue is \( \Pr(\sum_i t_i > n/2) \), as stated by the Condorcet Jury Theorem. In our case, \( t_i \sim Ber(P_i) \), where \( P_i \) differs among agents, and this case of independent heterogeneous variables was covered in [13], which showed:

\[
\Pr(\sum_i t_i > n/2) = \Pr(Z_{\mu,n} > n/2),
\]

where \( Z_{\mu,n} \sim Bin(\mu, n) \) and \( \mu = \frac{1}{n} \sum_i P_i \). Since the loss is additive along issues, \( \mathcal{L}_B(n) = k \Pr(Z_{\mu,n} > n/2) \).

We now turn to analyze the Proxy scenario. Assume w.l.o.g. that \( P_1, P_2, \ldots, P_n \) are sorted in increasing order. As \( k \) increases, \( P_i \) provides a good prediction of how many 1’s and 0’s will be in \( s_i \). This enables us to predict how inactive agents will select their proxies: an agent with parameter \( P_i < 0.5 \) will almost always select agent 1 and an agent with \( P_i > 0.5 \) will select agent \( n \) w.h.p.
Lemma 14. For every position $z < 0.5$, \( \Pr(\exists j \in N \text{ s.t. } \|s_j - z\| < \|s_1 - z\|) < n \cdot e^{-bk} \) for some constant $b$. The same holds for $z > 0.5$ and $s_n$.

Proof. Note that $P_1 < P_j$ for all $j > 1$. In addition, we denote a topic disagreement indicator $I_{i,j}^{(l)} = \left[s_i^{(l)} \neq s_j^{(l)}\right]$. For each agent $i$ with $P_i < 0.5$, $\forall j > 1$,

$$\Pr(\|s_i - s_j\| < \|s_i - s_1\|) = \Pr(\sum_{l=1}^{k} I_{i,j}^{(l)} < \sum_{l=1}^{k} I_{i,1}^{(l)})$$

$$= \Pr(\sum_{l=1}^{k} I_{i,j}^{(l)} - \sum_{l=1}^{k} I_{i,1}^{(l)} < 0)$$

define

$$q_1 = P_i(1 - P_1) + (1 - P_i)P_1$$
$$q_2 = P_i(1 - P_j) + (1 - P_i)P_j$$

Since $P_i < 0.5$, $P_1 > P_j \Rightarrow q_1 > q_2$

$$X_1 = \sum_{l=1}^{k} I_{i,1}^{(l)} \sim \text{Binomial}(k, q_1)$$
$$X_2 = \sum_{l=1}^{k} I_{i,j}^{(l)} \sim \text{Binomial}(k, q_2)$$

$$\Pr(\sum_{l=1}^{k} I_{i,1}^{(l)} < \sum_{l=1}^{k} I_{i,j}^{(l)}) = \Pr(X_1 - X_2 < 0)$$

Since $k \to \infty$ and $q_1, q_2$ are constants, a normal approximation to binomial distribution will be sufficiently accurate for our purpose.

$$X_1 \approx Z_1 \sim N(kq_1, kq_1(1 - q_1))$$
$$X_2 \approx Z_2 \sim N(kq_2, kq_2(1 - q_2))$$
$$(Z_1 - Z_2) \sim N(k(q_1 - q_2), k(q_1(1 - q_1) + q_2(1 - q_2)))$$
$$\Pr(X_1 - X_2 < 0) \approx \Pr(Z_1 - Z_2 < 0)$$

$$= \Phi(-\frac{k(q_1 - q_2)}{\sqrt{k(q_1(1 - q_1) + q_2(1 - q_2))}})$$

$$= \Phi(-\frac{\sqrt{k}(q_1 - q_2)}{q_1(1 - q_1) + q_2(1 - q_2)}) = \Phi(-a \cdot \sqrt{k})$$

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for some positive constant $a$. Note that for $x < -1$, $\Phi(x) < O(e^{-x^2})$, thus $\Pr(X_1 > X_2) < e^{-bk}$ for some constant $b > 0$. By the union bound, $\Pr (\exists j \in M \text{ s.t. } \|s_j - z\| < \|s_1 - z\|) \leq (m - 1) \Pr(X_1 > X_2) < me^{-bk}$. 

This means that when there are many issues, all voters with $z < 0.5$ will cast their votes to agent 1, thus $w_1 = \Pr_{z \sim h}(z < 0.5)$, $w_n = \Pr_{z \sim h}(z > 0.5)$. Hence one of the agents $\{1, n\}$ is effectively a dictator, depending on whether the median of $h$ is below or above 0.5. From now on we will assume that agent 1 is the dictator, as this occurs with high probability as $k \to \infty$ under most distributions with $\mu < 0.5$. Thus (for sufficiently large $k$),

$$\|m^P(S_N) - x^*\| = k \min_{i \in N} (P_i) = kP_1. \quad (5)$$

To recap, under scenario $B$ the majority mechanism is equivalent to unweighted majority of a size $n$ committee, while under scenario $P$, the mechanism is equivalent to a dictatorship of the best expert (i.e., the most conformist agent).

Given a particular distribution $h$, we can calculate $L^P(n)$ analytically or numerically. E.g. when $h = U[0, a]$ (note $a = 2\mu$),

$$L^P(n) = k\mathbb{E} P_{n \sim U(0,a)} \left[ \min_{i \in N} P_i \right] = \frac{ka}{n+1} = \frac{2\mu k}{n+1}.$$  

We can infer from Eqs. (4) and (5) that the proxy voting is beneficial in cases where the best expert out-performs the majority decision on average. Specifically, when the sample is small and/or the signal of agents is weak ($\mu$ is close to 0.5). See Fig. 5.

### 5.2 Strategic participation

In general there may be multiple equilibria that are difficult to characterize, and whose outcomes $m_j(S_M)$ may be very different from $x^*$. However we can show that for a sufficiently high $k$, there is (w.h.p) only one equilibrium outcome in each of the mechanisms $m^B+L$, $m^P+L$.

Intuitively, the reason is as follows. For every agent $i \in N$ there is w.h.p an issue for which she is pivotal, and thus the only equilibrium in scenario $B+L$ will be $M = N$. 

Figure 5: The loss $L^Q(n)$ (in log scale), for distributions $h = U[0, 2\mu = 0.66]$ (left); $h = N(\mu = 0.33, \sigma = 0.3)$ (right).
(w.h.p). In scenario $P+L$, the entire weight is distributed between the active agents with the lowest and highest $P$. This means that the best agent is always pivotal and thus active. Regardless of which other agents become active, we get that $mj^{P+L}(S_N) = mj(S_M, w_M) = s_1 = mj^P(S_N)$. The probability that any other equilibrium exists and affects the loss goes to zero.

**Basic setting** For any $M \subseteq N$, denote by $Y_M$ the event that set $M$ is an equilibrium in the game $mj^{B+L}(S_N)$. We bound the probability that $N$ is not the unique equilibrium.

**Lemma 15.** $\Pr(\neg Y_N \lor (\exists M \subseteq N, Y_M)) < e^{2n-k}$. Note that for $k \gg n \cdot 2^{n+1}$ the bound tends to 0.

**Proof.** For a binary vector $q \in \{0, 1\}^n$, we denote by $Z_q$ the event that for some issue $j \leq k$, $q_i = s_i^{(j)}$ for all $i \in N$. We also denote $Z^* = \bigcup_{q \in \{0, 1\}^n} Z_q$.

We first argue that $Z^*$ entails both $Y_N$ and $\neg Y_M$ for any $M \subseteq N$. Consider first the set $N$, and voter $i \in N$. If $n$ is odd consider some vector $q$ where $q_i = 1$ and all other voters split evenly between 0 and 1. Since $Z^*$ holds, there is an issue $j$ s.t. $q_{i'} = s_{i'}^{(j)}$ for all $i' \in N$. We get that $mj^j(N)^{(j)} = 1$ but $mj^j(N \setminus \{i\})^{(j)} = 0$, i.e. $i$ is pivotal and will thus not quit. If $n$ is even we proceed in a similar way except $q_i = 0$ and all of $N$ split evenly between 0 and 1.

For any smaller set $M$, consider some $i \in N \setminus M$, where $|M| = m$. If $m$ is even we consider a vector $q$ where $q_i = 1$ and all voters in $M$ split evenly between 0 and 1. We get that there is an issue $j$ where $mj^j(M)^{(j)} = 0$ but $mj^j(N \cup \{i\})^{(j)} = 1$, i.e. $i$ is pivotal and will join ($M$ is not stable). If $m$ is odd we proceed in a similar way except $q_i = 0$ and all of $M \cup \{i\}$ split evenly between 0 and 1.

It is left to bound $Pr(\neg Z^*)$. Indeed, for any $q$ and $j \leq k$, the probability that $q = s^{(j)}$ is exactly $2^{-n}$, and thus

$$Pr(\neg Z^*) \leq \sum_q Pr(\neg Z_q) = \sum_{q \in \{0, 1\}^n} \prod_{j \leq k} Pr(s^{(j)} \neq Z_q)$$

$$= \sum_{q \in \{0, 1\}^n} \prod_{j \leq k} (1 - 2^{-n}) = \sum_{q \in \{0, 1\}^n} (1 - 2^{-n})^k = 2^n (1 - 2^{-n})^k$$

$$\leq 2^n e^{-k/2^n} < e^{2n-k}.$$  

Any other equilibrium occurs with negligible probability, and has a bounded effect on the loss.

**Corollary 16.** As $k \to \infty$, the probability that $N$ is the unique equilibrium of $mj^{B+L}(S_N)$ tends to 1. In particular, $|L^{B+L}(n) - L^B(n)| \to 0$. 

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Proxy voting From Lemma 14, we know that for every set $M \subseteq N$, the most extreme voter $j = 1$ gets the votes of all inactive voters with $P_i < 0.5$, and in particular is pivotal (w.h.p., as $k$ is large enough). Thus voter 1 is active in any equilibrium, and is in fact a dictator as in the non-strategic scenario.

Finally, since we assume that the median of $h$ is less than 0.5, $j = 1$ is a dictator. As no other voter in $N$ is pivotal on any issue, they all become inactive. Thus under the same assumptions of Lemma 14

**Corollary 17.** As $k \to 0$, the probability that $M = \{1\}$ is the unique equilibrium of $m_j^S(S_N)$ tends to 1. In particular, $|L^P + L(n) - L^P(n)| \xrightarrow{k \to \infty} 0$.

### 5.3 Empirical Evaluation

We evaluate proxy voting on real data to avoid two unrealistic assumptions in our theoretical model: that the number of issues $k$ is very large, and that $i$’s votes on all issues are i.i.d.

We examine several data sets from PrefLib [17]: The first few datasets are Approval ballots of French presidential 2002 elections over 16 candidates in several regions (ED-26). We treat each candidate as an “issue” and each voter can either agree with the issue (approve this candidate) or disagree. $P_i$ is the fraction of issues on which voter $i$ disagrees with the majority.

We also considered two datasets of ordinal preferences: sushi preferences (ED-14) and AGH course selection (AD-9). The translation to a binary matrix is by checking for each pair of alternatives $(\alpha, \beta)$ whether $\alpha$ is preferred over $\beta$. This leaves us with 45 and 36 binary issues in the sushi and AGH datasets, respectively. A subset of $k = 15$ issues were sampled at each iteration in order to get results that are more robust (we thus get a “sushi distribution” and “AGH distribution” instead of a single dataset).

We first consider the weight distribution among agents (Fig. 6). The weight of agents is decreasing in $P_i$, meaning that agents with higher agreement with the majority opinion get more followers, with the best agents getting a significantly higher weight. This is related to the theoretical result that the best expert get $> 0.5$ weight, but is much less extreme. Also there is no weight concentration on the worst agent (this can be explained by the ‘Anna Karenina principle’ as each bad agent errs on different issues). In other words, allowing proxies does not result in a dictatorship of the best active agent, but in meritocracy of the better active agents.

This leads us to expect better performance than the theoretical prediction when comparing Proxy voting to the Basic setting. Indeed, Fig. 6(right) and Fig. 7 show that in all datasets $L^P(n) < L^B(n)$ except for very small samples in the French election datasets. This gap increases quickly with the sample size.
Figure 6: On the left, the average weight of each agent, in increasing order of $P_i$ (best agent on the left). On the right, the loss with (red) and without (blue) proxies. Results for the other datasets were similar.

Figure 7: The ratio of $L_B(n)$ and $L_P(n)$ (in log scale) for all datasets. Each point based on 1000 samples.

6 Discussion and Related Work

Our results, summarized in Table 1, provide a strong support for proxy voting when agents' positions are placed on a line, especially when the Median mechanism is in use. In contrast, when positions are (binary) multi-dimensional, proxy voting might concentrate too much power in the hands of a single proxy, and increase the error. However we also showed that on actual data this rarely happens and analyzed the reasons. These findings corroborate our hypothesis that proxy voting can improve representation across several domains. We are looking forward to study the effect of proxy voting in other domains, including common voting functions that use voters’ rankings.

Proxy voting, and our model in particular, are tightly related to the proportional representation problem, dealing with how to select representatives from a large population. A recent paper by Skowron [24] considers the selection of representatives who then use voting to decide on issues that affect the society. In our case, selection is

3Note that Hamming distance between agents’ positions equals the Kendal-Tau distance between their ordinal preferences.

4“Happy families are all alike; every unhappy family is unhappy in its own way” [26].
Table 1: A summary of our results. The first three lines show the effect of proxy voting when all agents are active. Results marked by (*) are obtained by simulations. The bottom lines summarize the effect of strategic voting with lazy bias.

| Rule:                      | Median                              | Mean                  | Majority ($k \to \infty$) with many issues |
|----------------------------|-------------------------------------|-----------------------|---------------------------------------------|
| Proxy better for any $S_N$| Yes                                 | $f$ SP + symmetric $n = 2$ | No                                          |
| $L^P < L^B$                | Always                              | $f$ Uniform           | depends on the best                        |
| $L^P \ll L^B$              | $f$ symmetric (* most $f$)          | $f$ Uniform (* most $f$) | agent (* real data)                        |

unique equilibrium of $g^{P+L}$ always $f$ Uniform always always

$L^B+L \geq L^B$ always always always

$L^{P+L} \leq L^P$ always $f$ Uniform (* some SP $f$) always

random as suggested in [20], and representatives are weighted proportionally to the number of voters that pick them as proxies, as originally suggested by Tullock [27]. It is interesting to note that political systems where public representatives are selected at random (“sortition”) have been applied in practice [8]. Our results suggest that such systems could be improved by weighting the representatives after their selection. Setting the weight proportionally to the number of followers seems natural, but it is an open question whether there are even better ways to set these weights.

Closest to our work is a model by Green-Armytage [12], where voters select proxies and use the Median rule to decide on each of several continuous issues. Decisions are evaluated based on their square distance from the “optimal” one. However even if the entire population votes, the outcome may be suboptimal, as Green-Armytage assumes people perceive their own position (as well as others’ position) with some error. He then focuses on how various options for delegating one’s vote may contribute to reducing her expressive loss, i.e. the distance from her true opinion to her ballot. In contrast, expressive losses do not play a roll in our model, where the sources of inaccuracy are small samples and/or strategic behavior.

Alger [1] considers a model with a fixed set of political representatives on an interval (as in our model), but focuses mainly on the ideological considerations of the voters and the political implications rather than on mathematical analysis. Our very positive results on the use of proxies in the Median mechanism support Alger’s conclusions, albeit under a somewhat different model of voters incentives. Alger also points out that proxy voting significantly reduces the amount of communication involved in collecting ballots on many issues.

Other models allow chains of voters who use each other as proxies [11, 6], or social influence that effectively increases the weight of some voters [2].

Indeed, we believe that a realistic model of proxy voting would have to take into account such topological and social factors in addition to statistics and incentives. E.g., [2] shows the benefits of a bounded degree, which in our model may allow a way to
bound excessive weights. Social networks may also be a good way to capture correlations in voters’ preferences [22], and can thus be used to extend our results beyond independent voters.

**Strategic behavior** We showed that most of our results hold when participation is strategic. What if voters (either active or inactive) could mis-report their position? Note that inactive voters have no reason to lie under the Median and the Majority mechanisms, due to standard strategyproofness properties. However active agents may be able to affect the outcome by changing the partition of followers. We can also consider more nuanced strategic behavior, for example where an agent also cares about her number of followers regardless of the outcome. More generally, strategic considerations under proxy voting combine challenges from strategic voting with those of strategic candidacy [14, 9], and would require a careful review of the assumptions of each model.

Other open questions include the effect of proxy voting on diversity, fairness, and participation. It is argued that diverse representatives often reach better outcomes [16], and fairness attracts much attention in the analysis of voting and other multiagent systems [4, 29, 7]. The effect on participation and engagement may also be quite involved, since allowing voters to use a proxy may increase the participation level of some who would otherwise not be represented, but on the other hand may lower the incentive to vote actively, thereby reducing overall engagement of the society.

Finally, the future of proxy voting depends on the development and penetration of novel online voting tools and social apps, such as those mentioned in the Introduction. We hope that sharing of data and insights will promote research on the topic, and set new challenges for mechanism design.

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