EMBEDDING COMPACTA INTO PRODUCTS OF CURVES

AKIRA KOYAMA(∗), JÓZEF KRASINKIEWICZ AND STANISŁAW SPIEŻ

Abstract. We present some results on $n$-dimensional compacta embeddable into $n$-dimensional Cartesian products of compacta. We pay special attention to compacta embeddable into products of 1-dimensional compacta. Most of our basic results are proven under the assumption that the compacta $X$ admit essential maps into the $n$-sphere $S^n$ (equivalently, the Čech cohomology $H^n(X) \neq 0$). Our investigations have been inspired by some results in this direction established by Borsuk, Cauty, Dydak, Koyama and Kuperberg. The results of the present paper may be viewed as an extension of the theory developed so far by these authors.

First, we prove that if $X$ is an $n$-dimensional compactum with $H^n(X) \neq 0$ that embeds in a product of $n$ curves then there exists an algebraically essential map $X \to \mathbb{T}^n$ into the $n$-torus. Then we show that the same is true if $X$ embeds in the $n$th symmetric product of a curve. The existence of such a mapping implies the existence of elements $a_1, \ldots, a_n \in H^1(X)$ whose cup product $a_1 \cup \cdots \cup a_n$ is non-zero. Consequently, rank $H^1(X) \geq n$ and cat $X > n$. In particular, $S^n$, $n \geq 2$, is not embeddable in the $n$th symmetric product of any curve. Next, we introduce some new classes of $n$-dimensional continua and show that embeddability of locally connected quasi $n$-manifolds into products of $n$ curves also implies rank $H^1(X) \geq n$. It follows that some 2-dimensional contractible polyhedra are not embeddable in products of two curves. On the other hand, we show that any collapsible 2-dimensional polyhedron can be embedded in a product of two trees. We answer a question posed by Cauty proving that closed surfaces embeddable in products of two curves can be also embedded in products of two graphs. We prove that no closed surface $\neq \mathbb{T}^2$ lying in a product of two curves is a retract of that product.

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1. Introduction

Throughout this paper we use the following standard terminology and notation. All *spaces* discussed in this paper are metrizable and all *mappings* (also called *maps*) are continuous. By a *compactum* we mean a compact (metric) space, by a *continuum* we mean a connected compactum, and by a *curve* we mean a 1-dimensional continuum. Sometimes we write $X \approx Y$ to indicate that $X$ is homeomorphic to $Y$.

By $\mathbb{B}^n$, $n \geq 1$, we denote the closed unit $n$-ball in the Euclidean $n$-space $\mathbb{R}^n$; and $\mathbb{B}^0 = \mathbb{R}^0$ stands for the one-point set $\{0\}$. A space homeomorphic to $\mathbb{B}^n$ is called a (closed) $n$-disc. By $\mathbb{S}^{n-1}$ we denote the unit $n$-sphere in $\mathbb{R}^n$ - the boundary of $\mathbb{B}^n$; and $\mathbb{S}^{-1}$ stands for the empty set. A space homeomorphic to $\mathbb{S}^n$ is called a *topological $n$-sphere*; a space homeomorphic to $\mathbb{S}^1$ is called a (topological) *circle* (or a simple closed curve). A space homeomorphic to the open unit $n$-ball $\mathbb{B}^n = \mathbb{B}^n \setminus \mathbb{S}^{n-1}$ is called an open $n$-disc. As usual, by the $n$-torus $\mathbb{T}^n$, $n \geq 1$, we mean the $n$-fold product $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. In particular, $\mathbb{T}^1 = \mathbb{S}^1$. By $\mathbb{T}^0$ we mean a one-point space. A space homeomorphic to $\mathbb{T}^n$ is called a *topological $n$-torus*. By a *dendrite* we mean a non-degenerate locally connected continuum containing no simple closed curve. A non-degenerate continuum is said to be a local dendrite if every point has a neighborhood which is a dendrite. It is known that dendrites coincide with 1-dimensional compact absolute retracts, and local dendrites – with 1-dimensional compact absolute neighborhood retracts (cf. [Kur]).

By a *complex* we mean a standard CW complex. Terminology concerning some special complexes introduced in this work will be presented in appropriate places. Throughout the paper by a *polyhedron* we mean the underlying space $|K|$ of a finite regular CW complex $K$. Recall that a CW complex $K$ is said to be regular if each cell $\sigma \in K$ admits a characteristic map $\varphi_\sigma : \mathbb{B}^n \to \sigma$, where $n = \dim \sigma$, which is a homeomorphism. Any simplicial complex $K$ is regular, such $K$ is called a triangulation of $|K|$.

In 1958 J. Nagata [Na1] discovered the following remarkable theorem.

**Theorem 1.1 (Nagata).** Every $n$-dimensional space, $n \geq 2$, can be embedded in the topological product $X_1 \times \cdots \times X_{n+1}$ of 1-dimensional spaces. \(\square\)

(This result has been subsequently discussed and refined by many authors, see e.g. [Bw], [I-M], [L], [Mi], [Ol], [St], [T].)

A few years later, in his book devoted to dimension theory, he also asked the following question: "It is an open problem whether every $n$-dimensional metric space can be topologically imbedded in the topological product of $n$ 1-dimensional metric spaces?" ([Na2],p.163). This question was answered in the negative by K. Borsuk [Bo3] in 1975. Actually, Borsuk proved the following interesting result.

**Theorem 1.2 (Borsuk).** The 2-sphere $\mathbb{S}^2$ is not embeddable in any product of two curves. (Analogous result holds for all spheres $\mathbb{S}^n$, $n \geq 3$.) \(\square\)

In the present paper we shall show that some 2-dimensional contractible (so, acyclic) polyhedra have this property as well.
The above two results justify the following notion. An $n$-dimensional space, $n \geq 2$, is said to be *ordinary* if it can be embedded in a product of $n$ 1-dimensional spaces; otherwise we call it *exceptional*. Hence $S^2$ and some other spaces are exceptional.

Thus every class of $n$-dimensional, $n \geq 2$, metric spaces splits in a natural way into two complementary subclasses - the ordinary and exceptional spaces. Generally speaking, in our paper we make an attempt at better understanding this splitting. We focus our attention on some special ordinary compacta. The fundamental problem of characterization of ordinary $n$-dimensional compacta (even for $n = 2$) seems unattainable at the moment. Using this new terminology one could have expressed our results in a more concise form.

Any 1-dimensional compactum can be embedded in the Menger curve $\mu$. It follows that an $n$-dimensional compactum is ordinary if and only if it can be embedded in the $n$-fold product $\mu^n$.

Let us recall that a mapping $f : X \to Y$, where $X$ is metric, is said to be an $\varepsilon$-*mapping* if $\text{diam} f^{-1}(y) < \varepsilon$ for each $y \in f(X)$. It is well known that any $n$-dimensional compactum $X$ admits surjective $\varepsilon$-mappings onto $n$-dimensional (compact) polyhedra for each $\varepsilon > 0$. In particular, each curve admits $\varepsilon$-mappings onto graphs for each $\varepsilon > 0$. (By a *graph* we mean a 1-dimensional connected polyhedron.) It follows that if $X$ is $n$-dimensional and ordinary then it admits $\varepsilon$-mappings into products of $n$ graphs for each $\varepsilon > 0$ (equivalently, $X$ admits $\varepsilon$-mappings into $\mu^n$). In Problem 1.4 located at the end of this chapter we inquire about the reverse implication. If $X$ admits $\varepsilon$-mappings into $Y$ for each $\varepsilon > 0$ then $X$ is said to be *quasi-embeddable* into $Y$.

In this context we mention an important and quite large class of extremely complicated ordinary continua. Namely, R. Pol [P] proved a surprising result that every $n$-dimensional, $n \geq 2$, hereditarily indecomposable continuum is ordinary. Moreover, he showed that such a continuum can be embedded in a product of $n$ hereditarily indecomposable curves.

All *manifolds* discussed in this paper are assumed to be compact and connected (possibly with non-empty boundary), unless opposite is explicitly stated. The interior of $M$ will be often denoted by $\mathring{M}$. A manifold $M$ is *closed* if its boundary is empty, $\partial M = \emptyset$. A manifold with non-empty boundary will be called *bordered manifold*. A mapping $f : X \to M$, where $M$ is a closed manifold, is said to be *essential* if every mapping $g : X \to M$ homotopic to $f$ is surjective.

The symbol $H^\ast(\cdot; G)$ is used to denote the Čech cohomology functor with coefficients in an Abelian group $G$. In some cases where no confusion is likely to occur we shall write $f^\ast$ instead of $H^\ast(f; G)$, where $f$ is a mapping. If $H^n(f; G) \neq 0$ then $f$ is said to be *non-trivial with respect to $H^n(\cdot; G)$*. A mapping $f : X \to M$, where $M$ is a closed $n$-manifold, is said to be *algebraically essential* if $H^n(f; G) \neq 0$ for some group $G$ (that is, $f$ is non-trivial with respect to $H^n(\cdot; G)$). Any algebraically essential mapping is essential. (In fact, this readily follows from the following result of M. Brown [Br, p. 94]:

*If $M$ is a bordered manifold then there is a surjective map $H : (\partial M) \times I \to M$*
such that \( H(x,0) = x \) for each \( x \in \partial M \), \( H((\partial M) \times [0,1]) = (\partial M) \times [0,1] \to M \) is an embedding, \( H^{-1}(H((\partial M) \times \{1\})) = (\partial M) \times \{1\}, \) and \( \dim H((\partial M) \times \{1\}) \leq \dim M - 1. \)

It follows that the set \( Y = H((\partial M) \times \{1\}) \) has dimension \( < \dim M \) and it is a strong deformation retract of \( M \), because \( M \) is homeomorphic to the mapping cylinder of the mapping \( \varphi : \partial M \to Y \) given by \( \varphi(x) = H(x,1). \)

The cohomology functor \( H^*(\cdot, \mathbb{Z}) \) with integer coefficients \( \mathbb{Z} \) will be abbreviated to \( H^*(\cdot) \). Thus the groups \( H^*(X, \mathbb{Z}) \) and the homomorphisms \( H^*(f, \mathbb{Z}) \) will be written briefly \( H^*(X) \) and \( H^*(f) \), respectively. By the Hopf Classification Theorem, cf. [Sp, p. 431], for any \( n \)-dimensional space \( X \), the group \( H^n(X) \) is in one-to-one correspondence with the set of homotopy classes of maps \( X \to S^n \). Non-zero elements correspond to homotopy classes of essential maps. As usual, \( H_*(X) \) denotes the singular homology functor with integer coefficients.

Let \( g_1, \ldots, g_k \) be elements of an Abelian group \( G \). They are said to be linearly independent (over \( \mathbb{Z} \)) if the equality \( n_1 g_1 + \cdots + n_k g_k = 0, \ n_i \in \mathbb{Z}, \) implies \( n_1 = \cdots = n_k = 0 \). By the rank of \( G \), denoted \( \text{rank } G \), we mean the maximal number of linearly independent elements in \( G \) (over \( \mathbb{Z} \)). We write \( G \cong H \) to denote that a group \( G \) is isomorphic to a group \( H \).

The original proof of the Borsuk theorem was not elementary. Two simpler proofs have been subsequently supplied by J. van Mill and R. Pol [M-P]. Here we supply several other proofs (and one, perhaps the simplest possible, in article 6A). Actually, we shall see that Borsuk’s theorem readily follows from each of several results proved in our paper.

Generalizing the Borsuk theorem W. Kuperberg [Ku] showed in 1978 that no closed \( n \)-manifold, \( n \geq 2, \) with finite fundamental group, can be embedded in any product of an \((n-1)\)-dimensional compactum and a curve. (Here we obtain this result as a corollary to Theorem 2B.1, see Corollary 2B.5.) In particular, the projective plane is not embeddable in any product of two curves. He also noted that

**Theorem 1.3 (Kuperberg).** Any orientable closed surface different from the 2-sphere can be embedded in a product of two graphs. \( \square \)

Using this result one can show that certain class of pretty complicated 2-dimensional continua consists of ordinary continua. Such continua have been constructed by Dranishnikov [Dr2] (called by him fractal Riemann surfaces). (Any such a continuum contains no open non-void subset embeddable in the plane and admits mappings as small as we please onto orientable surfaces of arbitrary high genus).

A common feature of the continua studied by Dranishnikov and Pol is their infinite rank of the first Čech cohomology.

Kuperberg also asked if there is a non-orientable closed surface embeddable in a product of two curves. The question was answered in 1984 by R. Cauty [C1], who proved the following remarkable result.
Theorem 1.4 (Cauty). Any non-orientable closed connected surface \( M \) can be embedded in a product of two graphs if and only if genus of \( M \) is \( \geq 6 \). □

(The latter condition is equivalent to either \( \text{rank } H^1(M) \geq 5 \), or Euler characteristic of \( M \) is \( \leq -4 \).) We take the opportunity to recall some relations between three classic numerical invariants - \( g(M), \chi(M), r(M) \) - associated with any closed surface \( M \). By \( g(M) \) - the genus of \( M \) - we mean maximal number of mutually disjoint simple closed curves on \( M \) whose union does not separate \( M \). As usual, \( \chi(M) \) stands for the Euler characteristic of \( M \). And we put \( r(M) = \text{rank } H_1(M) \). From the universal coefficient theorem for cohomology we have the following short exact sequence:

\[
0 \to \text{Ext}(H_0(M), \mathbb{Z}) \to H^1(M) \to \text{Hom}(H_1(M), \mathbb{Z}) \to 0.
\]

It follows that \( r(M) = \text{rank } H^1(M) \) as well (see [Sp, p. 244, 5.5.4]). If \( M \) is orientable then \( g(M) \) is equal to the so called handle number of \( M \) (that is, the number of handles attached to \( S^2 \) in a canonical presentation of \( M \)); for \( M \) non-orientable it is the so called crosscap number of \( M \) (that is, the number of the Möbius strips attached to \( S^2 \) in a canonical presentation) of \( M \). Moreover, we have the following equalities:

\[
\bullet \ g(M) = 1 - \frac{1}{2} \chi(M) = \frac{1}{2} r(M) \text{ for } M \text{ orientable, and}
\]
\[
\sim \bullet \ g(M) = 2 - \chi(M) = 1 + r(M) \text{ for } M \text{ non-orientable.}
\]

The present paper has been inspired by the above results and by the following theorem (which also implies the Borsuk theorem) due to Dydak and Koyama [D-K] and proved in 2000. This theorem readily follows from our Corollary 3A.4.

Theorem 1.5 (Dydak, Koyama). Let \( X \) be a compact subset of the product of \( n \) curves, \( n > 1 \), and let \( G \) be an Abelian group such \( H^n(X;G) \neq 0 \). Then \( H^1(X;G) \neq 0 \). □

* * *

The remaining part of this introduction is basically devoted to a brief summary of the main results of our paper. Some extra remarks are also included.

In Chapters 2 and 3 we first study properties of compacta admitting algebraically non-trivial maps into products of compacta. Then we investigate the cohomology ring of certain compacta embeddable into products of curves. Some consequences concerning the category of the compacta are derived. The following theorem is a simplified version of some basic results. Actually, its conclusion also holds under weaker stipulation. It is enough to require that there is a mapping \( f : X \to Y_1 \times \cdots \times Y_n \) which is non-trivial with respect to \( H^n(\cdot) \).

Theorem 1.6 (cf. Corollary 2A.5, Corollary 2B.3, and Corollary 3A.1). Let \( X \) be a compact subset of the product \( Y_1 \times \cdots \times Y_n \) of \( n \) curves with \( H^n(X) \neq 0 \). Then there is an algebraically essential map \( X \to \mathbb{T}^n \). Consequently, there exist
elements \( a_1, \ldots, a_n \in H^1(X) \) whose cup product \( a_1 \smile \cdots \smile a_n \in H^n(X) \) is non-zero. Such elements are linearly independent, hence \( \text{rank } H^1(X) \geq n \). Moreover, \( \text{cat } X > n \). □

As an application we see that the Klein bottle \( \mathbb{K} \) cannot be embedded in any product of two curves (because \( H^1(\mathbb{K}) \cong \mathbb{Z} \)). This conclusion can also be derived from the above result of Cauty. But it does not follow from the Dydak-Koyama theorem (because \( H^1(\mathbb{K}; G) = 0 \) if and only if \( G = 0 \), as \( G \) is a direct summand of \( H^1(\mathbb{K}; G) \)).

An analogous theorem holds for symmetric products, see Theorem 4G.1. It follows that no \( \mathbb{S}^n, n \geq 2 \), can be embedded in the \( n \)th symmetric product of a curve (see Corollary 4G.3). This is an analogue of the Borsuk Theorem 1.2. And it answers in the negative the following question posed by Illanes and Nadler [I-N, Question 83-14]: Is the 2-sphere embeddable in the second symmetric product of a curve? The theorem also implies that neither the projective plane \( \mathbb{P}^2 \) nor the Klein bottle \( \mathbb{K} \) can be embedded in the symmetric product of a curve (see Corollary 6G.2).

An \( n \)-dimensional continuum \( X, n \geq 1 \), is said to be a quasi \( n \)-manifold if for every point \( x \in X \) there is an open neighborhood \( V \) of \( x \) such that every closed subset of \( X \) which separates \( X \) between \( x \) and \( X \setminus V \) and has dimension \( \leq n - 1 \) admits an essential map into \( \mathbb{S}^{n-1} \) (see Section 5A for details). This and some other classes of \( n \)-dimensional continua have been defined in Chapter 5. Each class comprises all \( n \)-manifolds, and we have the following results.

**Theorem 1.7** (cf. Theorem 5B.1). Let \( X \) be a locally connected quasi \( n \)-manifold, \( n \geq 2 \), with \( H^1(X) \) of finite rank. If \( X \) embeds in a product of \( n \) curves then there exists an embedding \( g = (g_1, \ldots, g_n) : X \to P_1 \times \cdots \times P_n \) such that

1. each \( g_i \) is a monotone surjection,
2. each \( P_i \) is a graph with no endpoint. □

**Corollary 1.8** (see Corollary 5B.3). If a closed \( n \)-manifold is embeddable in a product of \( n \) curves then it is also embeddable in a product of \( n \) graphs. □

It follows that if a closed surface can be embedded in a product of two curves then it can be also embedded in a product of two graphs. This answers a question posed by Cauty. A harder variant of this question where the word ”surface” is replaced by ”2-dimensional polyhedron” is still open [C1].

We also prove a strong version of the following

**Theorem 1.9** (cf. Theorem 5D.5). Let \( X \) be a locally connected quasi \( n \)-manifold lying in a product of \( n \) curves. Then \( \text{rank } H^1(X) \geq n \). □

An edge of a 2-dimensional regular CW complex \( K \) is said to be free if it is incident with exactly one 2-cell of \( K \). Such an edge is also said to be free in \( |K| \).

**Corollary 1.10.** No contractible 2-dimensional polyhedron \( |K| \) with no free edge can be embedded in a product of two curves. □
There are two well known polyhedra satisfying the hypotheses of this corollary: the Borsuk example [Bo1] (which occurred in [Z] under the name "dunce hat"), and the "Bing house", cf. [R-S]. Hence neither can be embedded in a product of two curves.

We also prove some results on 2-dimensional polyhedra. Here we quote three of them.

**Theorem 1.11** (cf. Theorem 5E.1). Let $X$ be a 2-dimensional connected polyhedron. If $X$ can be embedded in a product of two curves and $\text{rank } H_1(X) \leq 2$, then $X$ collapses to either a point, or a graph, or a torus. In particular, $X$ is collapsible if $\text{rank } H_1(X) = 0$. □

**Theorem 1.12** (cf. Theorem 5E.3). Let $X$ be a 2-dimensional polyhedron. If $X$ is collapsible then $X$ can be embedded in a product of two trees. □

(By a tree we mean a connected graph containing no circle.)

**Theorem 1.13** (see Corollary 5E.7). The cone over an $n$-dimensional polyhedron can be embedded in a product of $n + 1$ copies of an $m$-od. □

(By an $m$-od we mean the cone over an $m$-element set.)

In Chapter 6 we present a detailed discussion of the problem of embeddability of surfaces into products of two curves. The results of Kuperberg (Theorem 1.3) and Cauty (Theorem 1.4) will be given new proofs. In section 6C the Kuperberg theorem is improved by showing that each orientable surface different from $S^2$ can be embedded in $\Theta_n \times \Theta_n$, where $n = g(M) + 1$ and $\Theta_n$ denotes the canonical $\theta_n$-curve (see 6B for definition). In section 6D we prove the following refinement of the Cauty result.

**Theorem 1.14** (cf. Theorem 6D.1). Let $M$ be a closed 2-manifold in the product $Y_1 \times Y_2$ of two curves. Then

(i) $\text{rank } H_1(M) \leq 3$ implies $M = P_1 \times P_2$, where each $P_i$ is a circle in $Y_i$;

(ii) $\text{rank } H_1(M) = 4$ implies $M \subset P_1 \times P_2$, where each $P_i$ is a $\theta$-curve in $Y_i$. □

In section 6E we show that the torus is the only surface which, when embedded in a product of two curves, is a retract of that product (see Theorem 6E.1). In section 6F we show that any bordered surface can be embedded in "the three-page book" (Theorem 6F.1). Section 6G contains some results on non-embeddability in the second symmetric product of a curve, which have been mentioned above.

One easily sees that if a space embeds in the product of two curves $X$ and $Y$ then it also embeds in the second symmetric product of a curve $Z$ (because $X \times Y$ embeds into the second symmetric product of $Z = X \vee Y$ - the one-point union (or, the bouquet) of $X$ and $Y$, here $X, Y$ are pointed spaces). We shall show that the reverse implications is not true. The former statement implies that any orientable surface different from $S^2$ and any non-orientable surface of genus $\geq 6$ can be embedded in the second symmetric product of a graph.

We complete this chapter with some remarks on $\varepsilon$-mappings. Let us begin with a general result.
**Theorem 1.15.** Let $X$ and $Y$ be $n$-dimensional compacta, $n \geq 1$. If $H^n(X) \neq 0$ then there is an $\varepsilon > 0$ such that every $\varepsilon$-mapping $f : X \to Y$ is non-trivial with respect to $H^n(\cdot)$.

*Proof.* By the hypothesis $H^n(X) \neq 0$ there is an essential map $u : X \to S^n$. By a result of Eilenberg [E] there is an $\varepsilon > 0$ such that for every surjective $\varepsilon$-mapping $v : X \to Z$ there is a mapping $w : Z \to S^n$ such that $\text{dist}(u, w \circ v) < 2$. Consequently, $u \simeq w \circ v$. Now, consider any $\varepsilon$-mapping $f : X \to Y$, and let $v_0 : X \to f(X)$ be determined by $f$. Then by the Eilenberg theorem there is a mapping $w_0 : f(X) \to S^n$ such that $u \simeq w_0 \circ v_0$. Since $\dim Y \leq n$ and $f(X)$ is a closed subset of $Y$, there is an extension $g : Y \to S^n$ of $w_0$. It follows that $u \simeq g \circ f$. Hence $H^n(f) \neq 0$ because $H^n(u) \neq 0$. This completes the proof. □

**Corollary 1.16.** (a) Let $X$ be a compactum quasi-embeddable in the product $Y_1 \times \cdots \times Y_k$ of finite dimensional compacta. If $H^n(X) \neq 0$, where $n = n_1 + \cdots + n_k$ and $n_i = \dim Y_i \geq 1$, then $X$ admits a mapping into $S^{n_1} \times \cdots \times S^{n_k}$ which is non-trivial with respect to $H^n(\cdot)$.

(b) Let $X$ be a compactum quasi-embeddable in the symmetric product $SP^n(Y)$, where $Y$ is a curve. If $H^n(X) \neq 0$ then $X$ admits a mapping into $T^n$ which is non-trivial with respect to $H^n(\cdot)$.

*Proof.* (a) By Theorem 1.15 there is a mapping $f : X \to Y_1 \times \cdots \times Y_k$ which is non-trivial with respect to $H^n(\cdot)$. Hence the conclusion follows from Corollary 2A.3 (which is independent of Theorem 1.15).

(b) As in case (a), there is a mapping $f : X \to SP^n(Y)$ which is non-trivial with respect to $H^n(\cdot)$. The conclusion follows from Theorem 4G.1. □

**Corollary 1.17.** No continuum $X$ with $H^1(X) = 0$ and $H^n(X) \neq 0$, where $n \geq 2$, is quasi-embeddable in either a product of $n$ curves or the $n$th symmetric product of a curve. □

**Note.** This corollary improves the Borsuk Theorem 1.2. Its particular case, for $X = S^n$ and the Cartesian product of $n$ curves, was observed in [M-P].

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**Problems to Chapter 1**

The following problem is of fundamental importance for the theory which has been developed in this paper and is certainly very hard.

**Problem 1.1.** Characterize $n$-dimensional ordinary compacta for each $n \geq 2$.

Any $n$-dimensional compactum can be regarded as a subset of the cube $I^{2n+1}$. Hence in an attempt of solving the above problem it may be helpful to start with studying ordinary $n$-dimensional compacta lying in lower dimensional cubes. Clearly, all $n$-dimensional compacta lying in $I^n$ are ordinary by definition. The following is the first, very special and already highly nontrivial case of the problem.
Problem 1.2. Characterize 2-dimensional ordinary compacta lying in $I^3$. (In general: Characterize $n$-dimensional ordinary compacta lying in $I^{n+1}$.)

Problem 1.3. Can $\mu^2$ be embedded in $I^4$? (In general: Can $\mu^n$, $n \geq 2$, be embedded in $I^{2n}$?)

Our next problem is of great interest, we believe it has affirmative solution.

Problem 1.4. Suppose $X$ is a compactum which admits $\varepsilon$-mappings into $\mu^n$ for each $\varepsilon > 0$ (i.e. $X$ is quasi-embeddable in $\mu^n$). Can $X$ be embedded in $\mu^n$?

In connection with Theorems 1.6 and 1.9 our next question is also of great interest.

Problem 1.5. Let $X$ be a locally connected quasi $n$-manifold lying in a product of $n$ curves. Does $X$ admit an essential map $X \rightarrow S^n$?

For a compactum $X$ and a natural $n \geq 2$ by $F_n(X)$ we denote the hyperspace of all non-void subset of $X$ composed of at most $n$ points, with the Hausdorff metric (or the Vietoris topology). We shall call it the hyperspace of $X$ of at most $n$ points. (In [Bo2] it is called the $n$th potency of $X$.) One easily sees that $F_2(S^1)$ is homeomorphic to the Möbius strip. An interesting result of Bott (see [Bo], cf. [Bo2]) asserts that $F_3(S^1)$ is homeomorphic to $S^3$. In 1947 Wu gave complete description of $H^*(F_n(S^1))$ for each $n \geq 1$ (see [Wu]). In particular, he showed that $H^{2n}(F_{2n}(S^1)) = 0$. This implies that $S^{2n}$ is not embeddable in $F_{2n}(S^1)$ because $\dim F_{2n}(S^1) = 2n$. In a recent paper by Chinen and Koyama [Ch-K] the latter result has been extended to all odd $n > 3$. Thus, no $S^n$, $n \geq 4$, can be embedded in $F_n(S^1)$. But the following harder problem is still open.

Problem 1.6. Can the $n$-sphere $S^n$, $n \geq 4$, be embedded in $F_n(X)$, where $X$ is a curve?

2. Embeddability into products of curves and the first cohomology

This chapter splits into sections 2A and 2B. In section 2A we consider mappings from and into $n$-dimensional products of compacta $Y_1, \cdots, Y_k$, where $n = \dim Y_1 + \cdots + \dim Y_k$, which induce non-trivial homomorphisms of cohomology in dimension $n$. Theorem 2A.1 states one of principal observations of this paper. It says that there is no proper subgroup of $H^n(Y_1 \times \cdots \times Y_k; G)$ which contains all images $\text{im} H^n(g)$, where $g$ is a mapping from $Y_1 \times \cdots \times Y_k$ to the product of spheres with corresponding dimensions. This implies Corollary 2A.2 on the existence of algebraically essential mappings. Next we obtain an important Corollary 2A.3 which provides a relation between algebraically essential mappings into $Y_1 \times \cdots \times Y_k$ and analogous mappings into corresponding products of spheres. In section 2B we study analogous problems in which the products are more restricted (they involve 1-dimensional factors). In that case we get a strong additional information on the first cohomology of the domain space of a mapping, see Theorem 2B.1. This enables us to give an alternative argument for the Kuperberg Theorem 1.3.
2A. Mapping products of compacta into corresponding products of spheres

To present our results we first recall some standard notation and auxiliary results.

For the $n$-sphere $S^n$, $n \geq 1$, by $\gamma_n$ we denote a generator of $H^n(S^n)$, and by $\gamma_{n,*}$ - the generator of $H^n(S^n, \ast)$ corresponding to $\gamma_n$ under the inclusion mapping $j : S^n \to (S^n, \ast)$, that is $H^n(j)(\gamma_{n,*}) = \gamma_n$.

To simplify notation we often use the letter $\nu$ to denote the homomorphisms $H^m(X) \otimes G \to H^m(X; G)$ from the universal coefficient formula (for various $X$ and $G$); and the letter $\mu$ to denote the homomorphism $H^{n_1}(X_1) \otimes \cdots \otimes H^{n_k}(X_k) \to H^n(X_1 \times \cdots \times X_k)$ from the Künneth formula [Sp, p. 237], where $n = n_1 + \cdots + n_k$.

The composition

$$
\lambda = \nu \circ (\mu \otimes 1_G) : H^{n_1}(X_1) \otimes \cdots \otimes H^{n_k}(X_k) \otimes G \to H^n(X_1 \times \cdots \times X_k; G),
$$

where $\nu : H^n(X_1 \times \cdots \times X_k) \otimes G \to H^n(X_1 \times \cdots \times X_k; G)$ in this case, will be called splitting homomorphism for $H^n(X_1 \times \cdots \times X_k; G)$. This homomorphism is natural as both $\mu$ and $\nu$ are known to be natural.

In case where $X_1, \ldots, X_k$ are finite dimensional compacta and $n_i = \dim X_i$ for each $i$, $\lambda$ is an isomorphism. In fact, the Künneth formula for the product $X_1 \times \cdots \times X_k$ and integer coefficients is the following short exact sequence

$$
0 \to [H^*(X_1) \otimes \cdots \otimes H^*(X_k)]^n \xrightarrow{\mu} H^n(X_1 \times \cdots \times X_k) \to [H^*(X_1) \ast \cdots \ast H^*(X_k)]^{n+1} \to 0.
$$

By our assumption $H^l(X_i) = 0$ for $l > \dim X_i$. So the formula takes on the form

$$
0 \to H^{n_1}(X_1) \otimes \cdots \otimes H^{n_k}(X_k) \xrightarrow{\mu} H^n(X_1 \times \cdots \times X_k) \to 0 \to 0.
$$

Thus $\mu$ is an isomorphism. From the universal coefficient formula we have the following short exact sequence

$$
0 \to H^n(X_1 \times \cdots \times X_k) \otimes G \xrightarrow{\nu} H^n(X_1 \times \cdots \times X_k; G) \to H^{n+1}(X_1 \times \cdots \times X_k) \ast G.
$$

By our assumption $\dim(X_1 \times \cdots \times X_k) \leq n$, so $H^{n+1}(X_1 \times \cdots \times X_k) = 0$. It follows that $\nu$ is an isomorphism. Consequently $\lambda$ is an isomorphism.

**Theorem 2A.1.** Let $Y_1, \ldots, Y_k$ be finite dimensional compacta, let $G$ be an Abelian group, and let $H$ be a proper subgroup of $H^n(Y_1 \times \cdots \times Y_k; G)$, where $n = n_1 + \cdots + n_k$ and $n_i = \dim Y_i \geq 1$. Then there exist mappings $\varphi_1 : Y_1 \to S^{n_1}, \ldots, \varphi_k : Y_k \to S^{n_k}$ such that the image $\text{im} H^n(\varphi_1 \times \cdots \times \varphi_k; G) \notin H$.

**First proof.** Under our assumptions the splitting homomorphism $\lambda$ is an epimorphism (in fact – an isomorphism), $\lambda^{-1}(H)$ is a proper subgroup of $H^{n_1}(Y_1) \otimes \cdots \otimes H_{n_k}(Y_k) \otimes G$. As elements of the form $a_1 \otimes \cdots \otimes a_k \otimes g$, where $a_1 \in H^{n_1}(Y_1), \ldots, a_k \in H^{n_k}(Y_k)$ and $g \in G$, generate $H^{n_1}(Y_1) \otimes \cdots \otimes H^{n_k}(Y_k) \otimes G$, one of them – say $a_1 \otimes \cdots \otimes a_k \otimes g$ – is not in $\lambda^{-1}(H)$. Hence

(i) $\lambda(a_1 \otimes \cdots \otimes a_k \otimes g) \notin H$. 


Since \( \dim Y_i = n_i \), by the Hopf-Whitney Classification Theorem, cf. [Sp, p. 431], there exist mappings \( \varphi_1 : Y_1 \to S^{n_1}, \ldots, \varphi_k : Y_k \to S^{n_k} \) such that

(ii) \( a_1 = \varphi^*_1(\gamma_{n_1}), \ldots, a_k = \varphi^*_k(\gamma_{n_k}) \).

By naturality of the splitting isomorphism, the following diagram commutes

\[
\begin{array}{ccc}
H^n(Y_1 \times \cdots \times Y_k; G) & \xrightarrow{H^n(\varphi_1 \times \cdots \times \varphi_k; G)} & H^n(S^{n_1} \times \cdots \times S^{n_k}; G) \\
\lambda & & \chi \\
H^{n_1}(Y_1) \otimes \cdots \otimes H^{n_k}(Y_k) \otimes G & \xleftarrow{\varphi_1^* \otimes \cdots \otimes \varphi_k^* \otimes 1_G} & H^{n_1}(S^{n_1}) \otimes \cdots \otimes H^{n_k}(S^{n_k}) \otimes G. 
\end{array}
\]

Hence, by (ii), we infer that

\[
\lambda(a_1 \otimes \cdots \otimes a_k \otimes g) = \lambda(\varphi^*_1(\gamma_{n_1}) \otimes \cdots \otimes \varphi^*_k(\gamma_{n_k}) \otimes g) = \\
\lambda((\varphi^*_1 \otimes \cdots \otimes \varphi^*_k \otimes 1_G)(\gamma_{n_1} \otimes \cdots \otimes \gamma_{n_k} \otimes g)) = H^n(\varphi_1 \times \cdots \times \varphi_k; G)(\lambda(\gamma_{n_1} \otimes \cdots \otimes \gamma_{n_k} \otimes g)).
\]

Thus, by (i), this completes the proof. \( \square \)

We are going to supply yet another argument for Theorem 2A.1 using (instead of the Hopf-Whitney theorem applied in the above proof) the following standard fact from cohomology theory:

For any \( n \)-dimensional polyhedron \( P \) (with a fixed triangulation) the group \( H^n(P) \) is generated by elements \( \sigma^* = H^n(v_\sigma)(\gamma_n) \), where \( \sigma \) runs over all \( n \)-simplices of \( P \), and \( v_\sigma : P \to S^n \) is a mapping transforming the interior of \( \sigma \) homeomorphically onto \( S^n \setminus \{\ast\} \) and carrying each other point to \( \ast \).

To clarify the claim let us make the following observations. We begin letting \( j : P \to (P, P^{(n-1)}) \) and \( j' : S^n \to (S^n, \ast) \) denote the inclusions. By \( \tilde{v}_\sigma : (P, P^{(n-1)}) \to (S^n, \ast) \) we denote the mapping determined by \( v_\sigma \), i.e. \( \tilde{v}_\sigma(x) = v_\sigma(x) \) for each \( x \in P \). Since \( j' \circ v_\sigma = \tilde{v}_\sigma \circ j \), we have the equalities:

\[
\sigma^* = H^n(v_\sigma)(\gamma_n) = H^n(v_\sigma)(H^n(j')(\gamma_{n,*})) = H^n(j' \circ v_\sigma)(\gamma_{n,*}) = \\
H^n(\tilde{v}_\sigma \circ j)(\gamma_{n,*}) = H^n(j)(H^n(\tilde{v}_\sigma)(\gamma_{n,*})).
\]

It is known that the group \( H^n(P, P^{(n-1)}) \) is freely generated by the elements \( H^n(\tilde{v}_\sigma)(\gamma_{n,*}) \). Since the inclusion \( j \) induces an epimorphism \( H^n(j) : H^n(P, P^{(n-1)}) \to H^n(P) \), the claim follows.

\textit{Second proof.} By the Freudenthal theorem each \( Y_i \) is the limit of an inverse sequence \( Y_{i,1} \leftarrow Y_{i,2} \leftarrow \cdots \) of \( n_i \)-dimensional polyhedra. (Each \( Y_{i,m} \) is taken with a fixed triangulation.) Let \( p_{i,m} : Y_i \to Y_{i,m} \) denote the projections. Then \( Y_1 \times \cdots \times Y_k \) is the limit of the sequence

\[
Y_{1,1} \times \cdots \times Y_{k,1} \leftarrow Y_{1,2} \times \cdots \times Y_{k,2} \leftarrow \cdots
\]
and \(p_{1,m} \times \cdots \times p_{k,m} : Y_1 \times \cdots \times Y_k \to Y_{1,m} \times \cdots \times Y_{k,m}\) are the projections. By the continuity of the Čech cohomology there exist an index \(m\) such that

\[
H' = \left( H^n(p_{1,m} \times \cdots \times p_{k,m}; G) \right)^{-1}(H)
\]

is a proper subgroup of \(H^n(Y_{1,m} \times \cdots \times Y_{k,m}; G)\). It follows that \((\lambda)^{-1}(H')\) is a proper subgroup of \(H^{n_1}(Y_{1,m}) \otimes \cdots \otimes H^{n_k}(Y_{k,m}) \otimes G\), where \(\lambda\) is the splitting isomorphism for \(H^n(Y_{1,m} \times \cdots \times Y_{k,m}; G)\). Referring to the claim preceding the proof we infer that \(H^{n_1}(Y_{1,m}) \otimes \cdots \otimes H^{n_k}(Y_{k,m}) \otimes G\) is generated by elements of the form \(\sigma^* \otimes \cdots \otimes \sigma^*_k \otimes g\), where \(\sigma_i\) is an \(n_i\)-simplex of \(Y_{i,m}\) and \(g \in G\). It follows that there exists an element \(\sigma^* \otimes \cdots \otimes \sigma^*_k \otimes g \notin (\lambda)^{-1}(H')\), where

\[
(i') \quad \sigma^*_i = H^n(v_i)((\gamma_{n_i})
\]

and \(v_i : Y_{i,m} \to S^{n_i}\) is the mapping corresponding to \(\sigma_i\). Thus

\[
(ii') \quad \lambda(\sigma^*_1 \otimes \cdots \otimes \sigma^*_k \otimes g) \notin H'.
\]

On the other hand, by an argument similar to that used in the first proof, we get the equality

\[
\lambda(\sigma^*_1 \otimes \cdots \otimes \sigma^*_k \otimes g) = H^n(v_1 \times \cdots \times v_k; G)(\lambda'(\gamma_{n_1} \otimes \cdots \otimes \gamma_{n_k} \otimes g)).
\]

Setting \(\varphi_i = v_i \circ p_{i,m} : Y_i \to S^{n_i}\) for \(i = 1, \cdots, k\), by (i'), we easily infer the conclusion. \(\square\)

Theorem 2A.1 readily implies the following corollary on existence of algebraically essential mappings.

**Corollary 2A.2.** Let \(Y_1, \cdots, Y_k\) be finite dimensional compacta such that \(H^n(Y_1 \times \cdots \times Y_k; G) \neq 0\), where \(n = n_1 + \cdots + n_k\) and \(n_i = \dim Y_i \geq 1\). Then there exist mappings \(\varphi_1 : Y_1 \to S^{n_1}, \cdots, \varphi_k : Y_k \to S^{n_k}\) such that \(H^n(\varphi_1 \times \cdots \times \varphi_k; G) \neq 0\). In particular, \(\varphi_1 \times \cdots \times \varphi_k : Y_1 \times \cdots \times Y_k \to S^{n_1} \times \cdots \times S^{n_k}\) is algebraically essential.

**Proof.** By our hypothesis the trivial group \(H = 0\) is a proper subgroup of \(H^n(Y_1 \times \cdots \times Y_k; G)\). Hence the conclusion follows from Theorem 2A.1. \(\square\)

Our next corollary points out an important relation between algebraically non-trivial mappings into products of finite dimensional compacta and algebraically essential mappings into products of spheres.

**Corollary 2A.3.** Let \(f : X \to Y_1 \times \cdots \times Y_k\) be a mapping of a compactum \(X\) to the product of finite dimensional compacta \(Y_1, \cdots, Y_k\). If \(f\) is non-trivial with respect to \(H^n(\cdot; G)\), where \(n = n_1 + \cdots + n_k\) and \(n_i = \dim Y_i \geq 1\), then there exist mappings \(\varphi_1 : Y_1 \to S^{n_1}, \cdots, \varphi_k : Y_k \to S^{n_k}\) such that

\[
(\varphi_1 \times \cdots \times \varphi_k) \circ f : X \to S^{n_1} \times \cdots \times S^{n_k}
\]

is non-trivial with respect to \(H^n(\cdot; G)\) as well.
Proof. It follows from our hypothesis that the group \( H = \ker f^* \) is a proper subgroup of \( H^n(Y_1 \times \cdots \times Y_k; G) \). Hence by Theorem 2A.1 there exist mappings \( \varphi_1 : Y_1 \to S^{n_1}, \cdots, \varphi_k : Y_k \to S^{n_k} \) such that \( \text{im} H^n(\varphi_1 \times \cdots \times \varphi_k; G) \not\subseteq H \). One easily sees that these mappings satisfy the conclusion of our corollary. \( \square \)

Next corollary immediately follows from the above result (applied to an inclusion) taking into account the following simple lemma which has been observed in [D-K].

Lemma 2A.4. Let \( X \) be a closed subset of an \( n \)-dimensional compactum \( Y \), and let \( i : X \hookrightarrow Y \) denote the inclusion mapping. Then \( H^n(i; G) : H^n(Y; G) \to H^n(X; G) \) is an epimorphism. In particular, \( H^n(i; G) \neq 0 \) if \( H^n(X; G) \neq 0 \).

Proof. Let us consider the following portion of the cohomology exact sequence of the pair \((Y,X)\):

\[
H^n(Y; G) \xrightarrow{i^*} H^n(X; G) \to H^{n+1}(Y,X; G).
\]

Note that \( H^{n+1}(Y,X; G) = 0 \) as \( \dim Y = n \). It follows that \( H^n(i; G) \) is an epimorphism. \( \square \)

Corollary 2A.5. Let \( X \) be a compactum lying in the product \( Y_1 \times \cdots \times Y_k \) of finite dimensional compacta. If \( H^n(X; G) \neq 0 \), where \( n = n_1 + \cdots + n_k \) and \( n_i = \dim Y_i \geq 1 \), then there exist mappings \( \varphi_1 : Y_1 \to S^{n_1}, \cdots, \varphi_k : Y_k \to S^{n_k} \) such that \( H^n(\varphi_1 \times \cdots \times \varphi_k)|X; G) \neq 0 \). In particular, \( X \) admits algebraically essential mappings into \( S^{n_1} \times \cdots \times S^{n_k} \).

\( \square \)

2B. Mappings into products involving 1-dimensional factors

In case of mappings into products of curves Corollary 2A.3, combined with some other facts, gives an important information on the first cohomology of the domain space.

Theorem 2B.1. Let \( f : X \to Y_1 \times \cdots \times Y_k \times Y \) be a map of a compactum \( X \) into the product of 1-dimensional compacta \( Y_1, \cdots, Y_k, \) and an \( l \)-dimensional compactum \( Y \). If \( H^{k+1}(f) \neq 0 \) then \( \text{rank } H^1(X) \geq k \).

The actual proof will be given after Lemmas 2B.2 and 2B.3 below. In the next lemma we use the following well-known fact (which follows from a result of M. Brown [Br, p. 94], see Chapter 1):

If \( X \) is a compactum and \( M \) is a closed manifold, then any mapping \( f : X \to M \) which is not surjective is homotopic to a mapping \( g \) such that \( \dim g(X) \leq \dim M - 1 \).

Lemma 2B.2. Let \( X, Y \) be compacta, let \( M \) be a closed manifold, and let \( n = \dim M + \dim Y \). If \((f,g) : X \to M \times Y \) is a mapping such that \( H^n((f,g); G) \) is non-trivial, for an Abelian group \( G \), then \( f \) is essential.
Proof. Otherwise, by the result preceding this lemma, \((f, g)\) is homotopic to a map \(h : X \to M \times Y\) such that \(\dim h(X) \leq n - 1\). Then \(H^n((f, g); G) = H^n(h; G) = 0\), a contradiction. □

Let \(A\) be a subset of a space \(X\) and let \(i : A \hookrightarrow X\) be the inclusion. For \(a \in H^n(X)\), we write briefly \(a|A\) for \(i^*(a)\). Note that for a mapping \(f : X \to Y\) and \(b \in H^n(Y)\) we have \(f^*(b)|A = (f|A)^*(b)\).

Lemma 2B.3. Let \(X\) be a compactum, and let \(f_i : X \to \mathbb{S}^1\), \(i = 1 \cdots , k\), be mappings such that \((f_1, \cdots , f_k) : X \to \mathbb{S}^1 \times \cdots \times \mathbb{S}^1\) is essential. Then elements

\[f_1^*(\gamma_1), \cdots , f_k^*(\gamma_1) \in H^1(X)\]

are linearly independent.

Proof. Let \(e = (1, 0)\) be the unit point of \(\mathbb{S}^1\). Put

\[X_i = \{x \in X : f_j(x) = e \text{ for } j \neq i\}\ .\]

We have a natural homomorphism

\[\eta : H^1(X) \to H^1(X_1) \oplus \cdots \oplus H^1(X_k)\]

defined by the formula \(\eta(a) = (a|X_1, \cdots , a|X_k)\) for \(a \in H^1(X)\). Since \(f_i(X_j) \subset \{e\}\) for \(i \neq j\), and \((f_i^*(\gamma_1)|X_j) = (f_i|X_j)^*(\gamma_1)\), we infer that

(i) \(\eta(f_i^*(\gamma_1)) = (0, \cdots , 0, (f_i|X_i)^*(\gamma_1), 0, \cdots , 0)\),

where \((f_i|X_i)^*(\gamma_1)\) stands in the \(i\)th position.

The conclusion of our lemma will follow once we show the images \(\eta(f_1^*(\gamma_1)), \cdots , \eta(f_k^*(\gamma_1))\) are linearly independent. To this end, by (i), it is enough to show that each \((f_i|X_i)^*(\gamma_1)\) is non-zero (here we refer to \(H^1(Y)\) being torsion free for any space \(Y\)). But \((f_i|X_i)^*(\gamma_1) \neq 0\) if and only if \(f_i|X_i : X_i \to \mathbb{S}^1\) is essential (this follows from the Bruschlinsky theorem). Hence we have to prove that \(f_i|X_i\) is essential. Suppose it is not true.

Then there is a homotopy \(F : X_i \times I \to \mathbb{S}^1\) such that \(F_0 = f_i|X_i\) and \(e \notin F_1(X_i)\). As \(f_i|X_i\) extends to \(f_i : X \to \mathbb{S}^1\) there is an extension \(F' : X \times I \to \mathbb{S}^1\) of \(F\) (by the homotopy extension theorem). Now we define a homotopy \(G : X \times I \to \mathbb{S}^1 \times \cdots \times \mathbb{S}^1\) by the formula

\[G(x, t) = (f_1(x), \cdots , f_{i-1}(x), F'(x, t), f_{i+1}(x), \cdots , f_k(x))\ .\]

Obviously, \(G\) is continuous and \(G_0 = f\). To complete the proof it suffices to show that

(ii) \((e, \cdots , e) \notin G_1(X)\).

(In fact, this contradicts the essentiality of \(f\).)
So fix a point $x \in X$. We have to show that $G_1(x) \neq (e, \cdots, e)$. This clearly holds if $f_j(x) \neq e$ for some $j \neq i$. Then consider the case $f_j(x) = e$ for all $j \neq i$. In such a case $x \in X_i$. Therefore

$$G_1(x) = (e, \cdots, e, F'(x, 1), e, \cdots, e) \neq (e, \cdots, e),$$

because $F'(x, 1) = F(x, 1) \neq e$. This proves (ii), and completes the proof of our lemma. □

**Proof of Theorem 2B.1**

By Corollary 2A.3 there is a mapping $(\psi_1, \cdots, \psi_k, \psi) : X \to S^1 \times \cdots \times S^1$ such that $H^{k+l}((\psi_1, \cdots, \psi_k, \psi)) \neq 0$. By Lemma 2B.2 we infer that the mapping $(\psi_1, \cdots, \psi_k) : X \to S^1 \times \cdots \times S^1$ is essential. Applying Lemma 2B.3 we conclude that $\psi_i(\gamma_1), \cdots, \psi_k(\gamma_1)$ are linearly independent in $H^1(X)$. This ends the proof. □

**Corollary 2B.4.** Let $M$ be a closed $(k+l)$-manifold lying in the product $Y_1 \times \cdots \times Y_k \times Y$ of 1-dimensional compacta $Y_1, \cdots, Y_k$, and an $l$-dimensional compactum $Y$. Then $\text{rank } H^1(M) \geq k$.

**Proof.** We are going to apply Theorem 2B.1. Notice that $\dim(Y_1 \times \cdots \times Y_k \times Y) = k+l$. Thus, by Lemma 2A.3, $H^{k+l}(i)$ is not trivial, where $i : M \to Y_1 \times \cdots \times Y_k \times Y$ denotes the inclusion. Applying Theorem 2B.1, we get the conclusion. □

Now we can give an alternative proof of Kuperberg’s theorem [Kup].

**Corollary 2B.5 (Kuperberg).** Let $M$ be a closed $(l+1)$-manifold with finite fundamental group $\pi_1(M)$. Then $M$ is not embeddable in any product $Y_1 \times Y$, where $Y_1$ is a 1-dimensional compactum and $Y$ is an $l$-dimensional compactum.

**Proof.** Suppose $M$ embeds in $Y_1 \times Y$, where $Y_1$ and $Y$ are as above. By Corollary 2B.4, it follows that $H^1(M)$ is non-trivial. On the other hand, it follows from our assumption that $H_1(M)$ is finite (being abelisation of the finite group $\pi_1(M)$, cf. [Sp, pp. 398, 391]). By the theorem on universal coefficients for cohomology we have the following exact sequence:

$$0 \to \text{Ext}(H_0(M), \mathbb{Z}) \to H^1(M) \to \text{Hom}(H_1(M), \mathbb{Z}) \to 0.$$

As $H_1(M)$ is finite, the last term is trivial. Since $H_0(M) \approx \mathbb{Z}$ is free, $\text{Ext}(H_0(M), \mathbb{Z})$ is trivial as well (cf. [Sp, 5.5.1, p. 241]). Hence $H^1(M) = 0$, a contradiction. □

3. **Embeddability into products and the cohomology ring**

This chapter has been divided in sections 3A and 3B. In section 3A we show that algebraically non-trivial mappings into products are strictly related to certain non-zero cup products in the domain space. The main result of the entire chapter is Theorem 3A.1. Corollary 3A.4 generalizes the Dydak-Koyama Theorem 1.5. In section 3B we apply those results to a study of the categories of spaces.
3A. Algebraically non-trivial mappings into products of compacta and non-zero cup products of spherical elements

In this section we show that mappings into products $Y_1 \times \cdots \times Y_k$ are non-trivial with respect to $H^n(\cdot; \bigotimes_{i=1}^k G_i)$, where $n = \dim Y_1 + \cdots + \dim Y_k$, if and only if they induce certain non-zero cup products in the domain space. We need the following notion. An element $\alpha \in H^n(Y; G)$ is called spherical if $\alpha = \nu(\varphi^*(\gamma_m) \otimes g)$ for a mapping $\varphi : Y \to S^m$ and an element $g \in G$.

**Theorem 3A.1.** Let $f = (f_1, \ldots, f_k) : X \to Y_1 \times \cdots \times Y_k$ be a mapping of a compactum $X$ in the product of finite dimensional compacta. Then $H^n(f; \bigotimes_{i=1}^k G_i) \neq 0$, where $n = n_1 + \cdots + n_k$ and $n_i = \dim Y_i \geq 1$, if and only if there exist spherical elements $\alpha_1 \in H^{n_1}(Y_1; G_1)$, $\ldots$, $\alpha_k \in H^{n_k}(Y_k; G_k)$ such that the cup product

$$f_1^*(\alpha_1) \cup \cdots \cup f_k^*(\alpha_k) \in H^n(X; \bigotimes_{i=1}^k G_i)$$

is not zero.

In particular, if $X \subset Y_1 \times \cdots \times Y_k$ and $H^n(X; \bigotimes_{i=1}^k G_i) \neq 0$ then there exist elements $a_1 \in H^{n_1}(X; G_1)$, $\ldots$, $a_k \in H^{n_k}(X; G_k)$ such that the cup product $a_1 \cup \cdots \cup a_k \in H^n(X; \bigotimes_{i=1}^k G_i)$ is not zero. (So, $H^n(X; G_i) \neq 0$ for each $i$.)

**Remark.** A simpler variant of this theorem is Corollary 3A.6, where one assumes that each $G_i = G$ and that $H^n(f; G) \neq 0$ for some $G$.

We are going to show that Theorem 3A.1 follows from Corollary 2A.4 and Lemma 3A.3 below. A short proof will be given after the proof of Lemma 3A.3. In the proof of that lemma we use Lemma 3A.2 which will be proved first. The general Künneth theorem for the Čech cohomology of finite products of compact spaces imposes some limitations on the coefficient groups. In the following lemma we point out that those limitations are immaterial in case of additional dimensional restrictions.

**Lemma 3A.2.** Let $X_1, \ldots, X_k$ be nonempty compacta with $\dim X_i = n_i$ and let $n = n_1 + \cdots + n_k$. Then the homomorphism

$$\mu' : H^{n_1}(X_1; G_1) \otimes \cdots \otimes H^{n_k}(X_k; G_k) \to H^n(X_1 \times \cdots \times X_k; \bigotimes_{i=1}^k G_i)$$

is an isomorphism.

**Proof.** The introductory remarks preceding Theorem 2A.1 show that

$$\mu : H^{n_1}(X_1) \otimes \cdots \otimes H^{n_k}(X_k) \to H^n(X_1 \times \cdots \times X_k)$$

is an isomorphism.

From the universal coefficient formula we have the exact sequence

$$0 \to H^{n_i}(X_i) \otimes G_i \xrightarrow{\nu_i} H^{n_i}(X_i; G_i) \to H^{n_i+1}(X_i) \ast G_i.$$

As $H^{n_i + 1}(X_i) = 0$, each $\nu_i$ is an isomorphism. Hence
\[(H^{n_1}(X_1) \otimes G_1) \otimes \cdots \otimes (H^{n_k}(X_k) \otimes G_k) \overset{\nu_1 \otimes \cdots \otimes \nu_k}{\longrightarrow} H^{n_1}(X_1; G_1) \otimes \cdots \otimes H^{n_k}(X_k; G_k)\]
is an isomorphism as well. Let $\varphi$ be a composition of the canonical isomorphism
\[(H^{n_1}(X_1) \otimes \cdots \otimes H^{n_k}(X_k)) \otimes \bigotimes_{i=1}^{k} G_i \rightarrow (H^{n_1}(X_1) \otimes G_1) \otimes \cdots \otimes (H^{n_k}(X_k) \otimes G_k)\]
and $\nu_1 \otimes \cdots \otimes \nu_k$. Let
\[\nu : H^n(X_1 \times \cdots \times X_k) \otimes (G_1 \otimes \cdots \otimes G_k) \rightarrow H^n(X_1 \times \cdots \times X_k; G_1 \otimes \cdots \otimes G_k)\]
be the homomorphism from the universal coefficient formula. Since $\dim(X_1 \times \cdots \times X_k) = n$, we have $H^{n+1}(X_1 \times \cdots \times X_k) = 0$. It follows, as before, that $\nu$ is an isomorphism.

Let us consider the following diagram:
\[
\begin{array}{ccc}
(H^{n_1}(X_1) \otimes \cdots \otimes H^{n_k}(X_k)) \otimes \bigotimes_{i=1}^{k} G_i & \overset{\mu \otimes 1}{\longrightarrow} & H^n(X_1 \times \cdots \times X_k) \otimes \bigotimes_{i=1}^{k} G_i \\
\varphi \downarrow & & \nu \downarrow \\
H^{n_1}(X_1; G_1) \otimes \cdots \otimes H^{n_k}(X_k; G_k) & \overset{\mu'}{\longrightarrow} & H^n(X_1 \times \cdots \times X_k; \bigotimes_{i=1}^{k} G_i),
\end{array}
\]
where $1 : \bigotimes_{i=1}^{k} G_i \rightarrow \bigotimes_{i=1}^{k} G_i$ is the identity isomorphism. By the above considerations we infer that all the upper horizontal and the vertical homomorphisms are isomorphisms. To complete the proof it is enough to observe that the diagram commutes.

\[\square\]

**Lemma 3A.3.** Let $X$ be a compactum, let $G_1, \ldots, G_k$ be a sequence of Abelian groups, and let $n_1, \ldots, n_k, n_i \geq 1$, be natural numbers. Then any mapping $f = (f_1, \ldots, f_k) : X \rightarrow S^{n_1} \times \cdots \times S^{n_k}$ is non-trivial with respect to $H^n(\cdot; \bigotimes_{i=1}^{k} G_i)$, where $n = n_1 + \cdots + n_k$, if and only if there exist elements $\alpha_1 \in H^{n_1}(S^{n_1}; G_1), \ldots, \alpha_k \in H^{n_k}(S^{n_k}; G_k)$ such that the cup product
\[f_1^*(\alpha_1) \cup \cdots \cup f_k^*(\alpha_k) \in H^n(X; \bigotimes_{i=1}^{k} G_i)\]
is non-zero. In the "only if" part we can take $\alpha_i = \nu(\gamma_{n_i} \otimes g_i)$ for some $g_i \in G_i$, and consequently, $f_i^*(\alpha_i) = \nu(f_i^*(\gamma_{n_i}) \otimes g_i)$; moreover, if $G_i = \mathbb{Z}$, then we can take $\alpha_i = \gamma_i$ (and $g_i = 1$).

**Proof.** We start with the "only if" implication. Since $\dim(S^{n_1} \times \cdots \times S^{n_k}) = n$, the homomorphism
\[\mu : H^{n_1}(S^{n_1}; G_1) \otimes \cdots \otimes H^{n_k}(S^{n_k}; G_k) \rightarrow H^n(S^{n_1} \times \cdots \times S^{n_k}; \bigotimes_{i=1}^{k} G_i)\]
(from the Künneth formula) is an epimorphism, see Lemma 3.2. As \( f^* (= H^n(f; \boxtimes_{i=1}^k G_i)) \) is not trivial, it follows that \( f^* \circ \mu \) is neither. Hence there exist elements \( \alpha_1 \in H^{n_1}(S^{n_1}; G_1), \ldots, \alpha_k \in H^{n_k}(S^{n_k}; G_k) \), such that

(i) \( (f^* \circ \mu)(\alpha_1 \otimes \cdots \otimes \alpha_k) \neq 0 \).

(Moreover we can take \( \alpha_i = \gamma_i \) if \( G_i = \mathbb{Z} \).)

By definition, \( \mu(\alpha_1 \otimes \cdots \otimes \alpha_k) = \alpha_1 \times \cdots \times \alpha_k \). Thus referring to [Sp, Corollary 5.6.14, p. 253] we get \( \alpha_1 \times \cdots \times \alpha_k = p_1^*(\alpha_1) \cdots p_k^*(\alpha_k) \), where \( p_i : S^{n_1} \times \cdots \times S^{n_k} \to S^{n_i} \) is the projection for each \( i \). Combining these equalities and [Sp, Property 5.6.8, p. 251], we get

(ii) \( (f^* \circ \mu)(\alpha_1 \otimes \cdots \otimes \alpha_k) = f^*(p_1^*(\alpha_1) \cdots p_k^*(\alpha_k)) = f^*(p_1^*(\alpha_1)) \cdots f^*(p_k^*(\alpha_k)) = f_1^*(\alpha_1) \cdots f_k^*(\alpha_k) \).

This combined with (i) completes the proof of this implication. Now let us prove the additional claim.

Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^{n_i}(S^{n_i}) \otimes G_i & \xrightarrow{f_i^* \otimes 1} & H^{n_i}(X) \otimes G_i \\
\downarrow \nu & & \downarrow \nu \\
H^{n_i}(S^{n_i}; G_i) & \xrightarrow{f_i^*} & H^{n_i}(X; G_i).
\end{array}
\]

(In the top arrow \( f_i^* = H^n(f_i) \), while the bottom arrow is \( f_i^* = H^n(f_i; G_i) \).)

As each element of the group \( H^{n_i}(S^{n_i}) \otimes G_i (= < \gamma_n \otimes G_i >) \) can be uniquely written in the form \( \gamma_n \otimes g \) (for \( g \in G_i \)) and \( \nu : H^{n_i}(S^{n_i}) \otimes G_i \to H^{n_i}(S^{n_i}; G_i) \) is an epimorphism, there exists \( g_i \in G_i \) such that \( \alpha_i = \nu(\gamma_n(\otimes g_i) \). (For \( G_i = \mathbb{Z} \), it follows from the definition of \( \nu(\gamma_n(\otimes 1) = \gamma_n(\).) By the commutativity we get

\[ f_i^*(\alpha_i) = \nu(f_i^*(\alpha_i) \otimes g_i), \]

which ends the proof of the claim.

The other implication follows from the equality \( f^* \mu(\alpha_1 \otimes \cdots \otimes \alpha_k) = f_1^*(\alpha_1) \cdots f_k^*(\alpha_k) \), see (ii). \( \square \)

Proof of Theorem 3A.1

The proof has been divided into two parts.

Part "if". By our assumption and Corollary 2A.4 there exist mappings \( \varphi_1 : Y_1 \to S^{n_1}, \cdots, \varphi_k : Y_k \to S^{n_k} \) such that \( H^n((\varphi_1 \times \cdots \times \varphi_k) \circ f; G) \neq 0 \). Since \( (\varphi_1 \times \cdots \times \varphi_k) \circ f = (\varphi_1 \circ f_1, \cdots, \varphi_k \circ f_k) \), by Lemma 3A.3 there exist elements \( \beta_1 \in H^{n_1}(S^{n_1}; G_1), \cdots, \beta_k \in H^{n_k}(S^{n_k}; G_k) \) such that the cup product

\[
(\varphi_1 \circ f_1)^*(\beta_1) \cdots (\varphi_k \circ f_k)^*(\beta_k) \in H^n(X; \boxtimes G_i)
\]

\( i=1 \)
is non-zero. Then each $\alpha_i = (\varphi_i)^*(\beta_i)$ is a spherical element of $H^{n_i}(Y_i; G_i)$, and we have

$$(f_1)^*(\alpha_1) \cup \cdots \cup (f_k)^*(\alpha_k) = (\varphi_1 \circ f_1)^*(\beta_1) \cup \cdots \cup (\varphi_k \circ f_k)^*(\beta_k),$$

which proves this part.

Part "only if". By our assumption there exist elements $\alpha_1 \in H^{n_1}(Y_1; G_1), \ldots, \alpha_k \in H^{n_k}(Y_k; G_k)$ such that the cup product

$$(f_1)^*(\alpha_1) \cup \cdots \cup (f_k)^*(\alpha_k) \in H^n(X; \bigotimes_{i=1}^k G_i)$$

is non-zero. Let $p_i : Y_1 \times \cdots \times Y_k \to Y_i$ denote the projection. Then we have

$$(f_1)^*(\alpha_1) \cup \cdots \cup (f_k)^*(\alpha_k) = (p_1 \circ f)^*(\alpha_1) \cup \cdots \cup (p_k \circ f)^*(\alpha_k) = f^*((p_1)^*(\alpha_1) \cup \cdots \cup (p_k)^*(\alpha_k)),$$

which completes the proof. □

Theorem 3A.1 implies the following generalization of the Dydak-Koyama theorem.

**Corollary 3A.4.** Let $f : X \to Y_1 \times \cdots \times Y_k$ be a mapping of a compactum $X$ into the product of finite dimensional compacta, and let $n = n_1 + \cdots + n_k$, where $n_i = \dim Y_i \geq 1$. If $H^n(f; G) \neq 0$, then $H^{n_i}(X; G) \neq 0$ for each $i$.

In particular, if $X \subset Y_1 \times \cdots \times Y_k$ and $H^n(X; G) \neq 0$, then $H^{n_i}(X; G) \neq 0$ for each $i$.

**Proof.** Without loss of generality we can assume $i = 1$. Thus we have to show that $H^{n_1}(X; G) \neq 0$. To this end apply Theorem 3A.1 with $G_1 = G$ and $G_j = \Z$ for $j \neq 1$. Thus we infer that there exist elements $a_1 \in H^{n_1}(X; G), a_2 \in H^{n_2}(X), \ldots, a_k \in H^{n_k}(X)$ such that $a_1 \cup \cdots \cup a_k \neq 0$. Consequently, $a_1 \neq 0$, which proves the first statement of our corollary.

To prove the second statement, we repeat the argument used in the proof of Corollary 2A.4. □

The final result of this section - Corollary 3A.6 - represents a simplified version of Theorem 3A.1 in case where all groups $G_1, \ldots, G_k$ are equal and somehow restricted. In the proof we refer to Lemma 3A.5 which makes the simplification possible. The lemma shows that the hypothesis $H^n(f; \bigotimes^k G) \neq 0$ from Theorem 3A.1 can be replaced by $H^n(f; G) \neq 0$ (where $\bigotimes^k G$ denotes $G \otimes \cdots \otimes G$). Lemma 3A.5 is a consequence of Theorem A1 from the Appendix.
Lemma 3A.5. Let $f : X \to Y$ be a mapping between compacta and let $\dim Y = n$. Suppose $G$ is either a direct sum of cyclic groups or a non-torsion Abelian group. Then $H^n(f; G) \neq 0$ implies $H^n(f; \bigotimes^k G) \neq 0$.

Proof. By the universal coefficient theorem the diagram

\[
\begin{array}{ccccccc}
H^n(Y) \otimes G' & \xrightarrow{f^* \otimes 1_{G'}} & H^n(X) \otimes G' \\
\downarrow & & \downarrow \\
H^n(Y; G') & \xrightarrow{H^n(f; G')} & H^n(X; G')
\end{array}
\]

commutes for any group $G'$. Since $H^n(f; G) \neq 0$ and the left vertical homomorphism is an epimorphism (because $\dim Y \leq n$), the homomorphism

\[ f^* \otimes 1_G : H^n(Y) \otimes G \to H^n(X) \otimes G \]

is non-trivial (in fact, take $G' = G$ in the diagram). Therefore, since $G$ is either a direct sum of cyclic groups or a non-torsion Abelian group, by Theorem A1 from the Appendix the homomorphism

\[ f^* \otimes 1_{\otimes^k G} : H^n(Y) \otimes \bigotimes^k G \to H^n(X) \otimes \bigotimes^k G \]

is non-trivial as well. Consequently, since the right vertical homomorphism is a monomorphism, (take $G' = \bigotimes^k G$ in the diagram) the homomorphism $H^n(f; \bigotimes^k G)$ is non-trivial (in fact, take $G' = \bigotimes^k G$ in the diagram), which completes the proof.

Corollary 3A.6. Let $f : X \to Y_1 \times \cdots \times Y_k$ be a mapping of a compactum $X$ into the product of finite dimensional compacta, and let $n = n_1 + \cdots + n_k$, where $n_i = \dim Y_i \geq 1$. Suppose $G$ is either a direct sum of cyclic groups or a non-torsion Abelian group. Then $H^n(f; G) \neq 0$ implies that there exist elements $a_1 \in H^{n_1}(X; G), \ldots, a_k \in H^{n_k}(X; G)$ such that the cup product $a_1 \smile \cdots \smile a_k \in H^n(X; \bigotimes^k G)$ is not zero.

Proof. This corollary directly follows from Theorem 3A.1 and Lemma 3A.5. □

3B. Categories of spaces

In this section we are going to show that for any compactum its embeddability into products is related to its category.

Let us recall the classic definition of the category (cf. [Sp]): A space $X$ is said to have category $\leq k$ (written: $\text{cat } X \leq k$) if there exists a closed covering $\{F_1, \ldots, F_k\}$ of $X$ such that each $F_i$ is contractible in $X$. We shall deal with a modification of this notion.
Let $G_1, \ldots, G_k$ be Abelian groups and let $n_1, \ldots, n_k$ be natural numbers. A space $X$ is said to have category $(G_1, \ldots, G_k; n_1, \ldots, n_k)$ (written: $X \in \text{cat}(G_1, \ldots, G_k; n_1, \ldots, n_k)$) if there exists a closed covering $\{F_1, \ldots, F_k\}$ of $X$ such that each homomorphism

$$H^{n_i}(F_i \hookrightarrow X; G_i)$$

is trivial. In such a case, it is sometimes said that $X$ has trivial decomposition relatively $(G_1, \ldots, G_k; n_1, \ldots, n_k)$. To shorten notation we write $(G; n_1, \ldots, n_k)$ for $(G, \ldots, G; n_1, \ldots, n_k)$.

Let us recall some basic properties of the cup product which will be used in the proof of our next lemma.

Let $f : X \to Y$ be a mapping and suppose $f(A_1) \subset B_1, \ldots, f(A_k) \subset B_k$ for some subsets $A_1, \ldots, A_k$ of $X$ and $B_1, \ldots, B_k$ of $Y$. Let $f_i : (X, A_i) \to (Y, B_i)$, $i = 1, \ldots, k$, be defined by $f$. Then for any sequence $b_1 \in H^{n_1}(Y, B_1; G_1), \ldots, b_k \in H^{n_k}(Y, B_k; G_k)$ we have

1. $b_1 \cup \cdots \cup b_k \in H^n(Y, B_1 \cup \cdots \cup B_k; \bigotimes_{i=1}^k G_i)$, where $n = n_1 + \cdots + n_k$, and
2. $f_i^*(b_1 \cup \cdots \cup b_k) = f_i^*(b_1) \cup \cdots \cup f_i^*(b_k)$.

**Lemma 3B.1.** Let $X$ be a compactum with category $(G_1, \ldots, G_k; n_1, \ldots, n_k)$. Then for every sequence $a_1 \in H^{n_1}(X; G_1), \ldots, a_k \in H^{n_k}(X; G_k)$ the product $a_1 \cup \cdots \cup a_k \in H^n(X; \bigotimes_{i=1}^k G_i)$ is equal to 0.

**Proof.** From the hypothesis we infer that there exist a closed covering $\{F_1, \ldots, F_k\}$ of $X$ such that each homomorphism $H^{n_i}(F_i \hookrightarrow X; G_i)$ is trivial. Consider the following portion of the cohomology exact sequence of $(X, F_i)$:

$$H^{n_i}(X, F_i; G_i) \xrightarrow{j_i^*} H^{n_i}(X; G_i) \to H^{n_i}(F_i; G_i)$$

where $j_i : X \to (X, F_i)$ is defined by $i_X$. Hence the right homomorphism is trivial. So, as $a_i \in H^{n_i}(X; G_i)$ there is an element $b_i \in H^{n_i}(X, F_i; G_i)$ such that $j_i^*(b_i) = a_i$. Setting $Y = X$, $f = i_X$, $A_i = \emptyset$ and $B_i = F_i$, we have $i_X^* : X \to (X, F_1 \cup \cdots \cup F_k)$ and $(i_X)^i = j_i$. It follows from (2) that

$$\overline{i_X^*}(b_1 \cup \cdots \cup b_k) = j_1^*(b_1) \cup \cdots \cup j_k^*(b_k) = a_1 \cup \cdots \cup a_k .$$

Since $X = F_1 \cup \cdots \cup F_k$, we have $H^n(X, F_1 \cup \cdots \cup F_k; \bigotimes_{i=1}^k G_i) = 0$. Thus $b_1 \cup \cdots \cup b_k = 0$, hence $a_1 \cup \cdots \cup a_k = 0$, which completes the proof. □

**Theorem 3B.2.** Let $X$ be an $n$-dimensional compactum and let $Y_1, \ldots, Y_k$ be compacta such that $n = n_1 + \cdots + n_k$, where $n_i = \dim Y_i \geq 1$. If there is a mapping $f : X \to Y_1 \times \cdots \times Y_k$ such that $H^n(f : \bigotimes_{i=1}^k G_i) \neq 0$ then $X$ is not of category $(G_1, \ldots, G_k; n_1, \ldots, n_k)$.

**Proof.** Suppose, on the contrary, that $X \in \text{cat}(G_1, \ldots, G_k; n_1, \ldots, n_k)$. Then by Theorem 3A.1 there exist elements $a_1 \in H^{n_1}(X; G_1), \ldots, a_k \in H^{n_k}(X; G_k)$ such that $a_1 \cup \cdots \cup a_k \in H^n(X; \bigotimes_{i=1}^k G_i)$ is not 0. This contradicts Lemma 3B.1, and completes the proof. □
Corollary 3B.3. Let $X \subset Y_1 \times \cdots \times Y_k$ be a compactum lying in the product of compacta, and let $G_1, \cdots, G_k$ be Abelian groups. If $H^n(X; \bigotimes_{i=1}^k G_i) \neq 0$, where $n = n_1 + \cdots + n_k$ and $n_i = \dim Y_i \geq 1$, then $X$ is not of category $(G_1, \cdots, G_k; n_1, \cdots, n_k)$.

Proof. This corollary follows from Theorem 3B.2 because $H^n(X \hookrightarrow Y_1 \times \cdots \times Y_k; \bigotimes_{i=1}^k G_i) \neq 0$ (cf. the proof of the second assertion of Corollary 3A.4).

Corollary 3B.4. Let $X$ be a compactum embeddable in the product $Y_1 \times \cdots \times Y_n$ of $n$ curves. If $H^n(X) \neq 0$ then $\text{cat } X > n$.

4. Embedding compacta into symmetric products of curves

In this chapter we establish a basic result on symmetric products of curves, see Theorem 4E.1. Its principal assertion is analogous to Corollary 2A.3 and reads: any compactum which admits algebraically non-trivial mapping into the $n$th symmetric product of a curve admits an algebraically non-trivial mapping into the $n$-torus $\mathbb{T}^n$. In order to prove this result we need several auxiliary observations which will be presented in a series of sections. The most important result applied in the proof is the result of B.W. Ong [On] on the symmetric products of a bouquet of circles, we recall it in section 4D. The remaining assertions of Theorem 4E.1 follow then from Lemma 2B.3 and Theorem 3A.1. (The results of this chapter formulated for compacta hold for compact Hausdorff spaces as well.)

4A. Symmetric products as functors

As usual, $S_n$, $n \geq 1$, denotes the $n$th symmetric group, i.e. the permutation group of the set $\{1, \cdots, n\}$. There is a standard (left) action of $S_n$ on $X^n$ (permuting coordinates of points) for any non-void set $X$. Points of the diagonal $\Delta^n(X) = \{(x, \cdots, x) : x \in X\}$ are the only fixed points of this action. Hence any orbit $S_n \cdot (x, \cdots, x)$ is a one-point set $\{(x, \cdots, x)\}$.

First, recall the definition of the symmetric product. Given a space $X \neq \emptyset$ and $n \geq 1$ the $n$th symmetric product of $X$, denoted $SP^n(X)$, is the space of all orbits $S_n \cdot x$, where $x \in X^n$, of the standard (continuous) action of $S_n$ on $X^n$. We identify $SP^1(X)$ with $X$ under the assignment $\{x\} \to x$. Let $q : X^n \to SP^n(X)$ denote the quotient mapping taking each point to the orbit of that point, i.e. $q(x) = S_n \cdot x$. In other words, for any $x, y \in X^n$ we have

1. $q(x) = q(y)$ if and only if $y = \alpha \cdot x$ for some $\alpha \in S_n$.

It follows that

2. $q$ is open and closed.

Notice that $SP^n$ is a functor from the category of topological spaces to itself. In fact, if $f : X \to Y$ then $f^n : X^n \to Y^n$ is an equivariant mapping in the following sense:
Hence for compacta we have more:

the symmetric product because the Nagata Theorem 1.1 implies that $Y$ is embedded in the symmetric product $X$.

Thus orbits in $\{x\}$ are regarded as pointed with $(\ast)\cdots (\ast)$. Let $X$ and $Y$ are 1-dimensional then so is $Y$. This combined with the Nagata Theorem 1.1 implies that any $n$-dimensional metrizable space can be embedded in the symmetric product $SP^{n+1}(Y)$, where $Y$ is metric 1-dimensional.

For compacta we have more: any $n$-dimensional compactum can be embedded in the symmetric product $SP^{n+1}(\mu)$, where $\mu$ stands for the Menger curve. This holds because $SP^n(\mu)$ contains a copy of the product $\mu^n$ (as $\mu$ is locally homogeneous).

Moreover, $SP^n$ is a homotopy functor, i.e. if $f \simeq g$ then $SP^n(f) \simeq SP^n(g)$. Hence $SP^n$ preserves the homotopy type of spaces. This property is important for our discussion. Another important property of that functor is continuity: if the projections $\{p_i : X \to X_i\}$ represent the limit of the inverse sequence $\{X_1 \xleftarrow{p_{1,2}} X_2 \xleftarrow{p_{2,3}} \cdots\}$ then the projections $\{SP^n(p_i) : SP^n(X) \to SP^n(X_i)\}$ represent the limit of the inverse sequence $\{SP^n(X_1) \xleftarrow{SP^n(p_{1,2})} SP^n(X_2) \xleftarrow{SP^n(p_{2,3})} \cdots\}$.

4B. Symmetric products of a pointed space

For $1 \leq k \leq l$ there is a canonical monomorphism $s_{k,l} : S_k \to S_l$ defined as follows: $s_{k,l}(\beta)$, for $\beta \in S_k$, is a permutation on $\{1, \cdots, l\}$ which acts as $\beta$ on $\{1, \cdots, k\}$ and keeps each element $> k$ fixed.

Now consider a pointed space $X$ with a base point $\ast$. Then each space $X^k, k \geq 1$, is regarded as pointed with $(\ast, \cdots, \ast)$ as the base point. Put $X^0 = \{\ast\}$ and regard it as pointed as well. Let $u_{k,l} : X^k \to X^l$ be canonical embedding given by: $(x_1, \cdots, x_k) \to (x_1, \cdots, x_k, \ast, \cdots, \ast)$. This embedding preserves the base points and is equivariant in the following sense:

(4) $u_{k,l}(\beta \cdot x) = s_{k,l}(\beta) \cdot u_{k,l}(x)$ for each $x \in X^k$.

Thus orbits in $X^k$ go under $u_{k,l}$ into orbits in $X^l$. Moreover, we have the following property:

(5) If $u_{k,l}(y) = \alpha \cdot u_{k,l}(x)$ for $x, y \in X^k$ and $\alpha \in S_l$, then $y = \beta \cdot x$ for some $\beta \in S_k$. 

It follows that

\[(6)\ (S_k \cdot u_{k,l}(x)) \cap u_{k,l}(X^k) = u(S_k \cdot x) \text{ for each } x \in X^k.\]

Thus \(u_{k,l}\) induces a mapping \(v_{k,l} : SP^k(X) \to SP^l(X), \) \(S_k \cdot x \to S_l \cdot u_{k,l}(x),\) which makes the following diagram commutative:

\[
\begin{array}{ccc}
X^k & \xrightarrow{u_{k,l}} & X^l \\
q \downarrow & & \downarrow q \\
SP^k(X) & \xrightarrow{v_{k,l}} & SP^l(X).
\end{array}
\]

One can define analogous commutative diagram for \(k = 0\) as well, setting \(SP^0(X) = \{\ast\},\) \(u_{0,l}(\ast) = (\ast, \ldots, \ast)\) and \(v_{0,l}(\ast) = ((\ast, \ldots, \ast))\). Notice that \(v_{k,k} = id_{SP^k(X)}\) and \(v_l \circ v_{k,l} = v_{l,m} \circ v_{k,l} = v_{l,m} \text{ for } k \leq l \leq m.\) By (5), \(v_{k,l}\) is injective, hence \(v_{k,l}\) is an embedding if \(X\) is compact - we call it canonical embedding. Therefore, the image \(SP^{k,l}(X) = v_{k,l}(SP^k(X))\) is a copy of \(SP^k(X)\) in \(SP^l(X)\) in that case. In this way one obtains a canonical direct sequence of embeddings of symmetric products of \(X:\)

\[
SP^0(X) \xrightarrow{v_{0,1}} SP^1(X) \xrightarrow{v_{1,2}} SP^2(X) \xrightarrow{v_{2,3}} \cdots.
\]

All the mappings belong to the category of pointed spaces.

A pointed space \(X\) with a base point \(\ast\) is said to be a bouquet of pointed subspaces \((X_j, \ast)\), for \(j \in J\), if each \(X_j\) is a closed subset of \(X\), \(X_j \neq \{\ast\}\), \(X = \bigcup_{j \in J} X_j\), and \(X_j \cap X_{j'} = \{\ast\} \text{ for } j \neq j'\). The spaces \(X_j\) are called leaves.

4C. Symmetric products of a bouquet of \(k\) circles and skeleta of torus \(T^k\)

If \(K_1, \ldots, K_n\) are \(CW\) complexes then we put

\[
K_1 \Box \cdots \Box K_n = \{\sigma_1 \times \cdots \times \sigma_n : \sigma_1 \in K_1, \ldots, \sigma_n \in K_n\}.
\]

Notice that \(K_1 \Box \cdots \Box K_n\) is a \(CW\) complex, it is called product cell complex of the complexes. Given characteristic maps of cells of \(K_i\)'s the cells of \(K_1 \Box \cdots \Box K_n\) have natural characteristic maps obtained by taking products of the corresponding characteristic maps of factors.

The circle \(S^1\) will be regarded as a \(CW\) complex with cell structure \(S = \{\sigma^0, \sigma^1\}\) composed of a 0-cell \(\sigma^0 = \{1\}\) and a 1-cell \(\sigma^1 = S^1\). Consequently, for any \(k \geq 1\), the \(k\)-torus \(T^k = S^1 \times \cdots \times S^1, k\) times, will be regarded as a \(CW\) complex with the cell structure \(S \Box \cdots \Box S, k\) times. The cell structure will be called canonical \(CW\) structure on \(T^k\). The space \(|(S \Box \cdots \Box S)^{(n)}|\) of the \(n\)th skeleton of \(S \Box \cdots \Box S\), where \(n \geq 0,\) carries the \(CW\) complex structure \((S \Box \cdots \Box S)^{(n)};\) with that structure it will be denoted by \((T^k)^{(n)}\) and called the \(n\)th skeleton of the torus \(T^k\). For each \(J \subset \{1, \ldots, k\}\) by \(\sigma_J\) we denote the cell \(\sigma_1 \times \cdots \times \sigma_k \in (S \Box \cdots \Box S),\) where \(\sigma_j = \sigma^1\) if \(j \in J,\) and \(\sigma_j = \sigma^0\) otherwise. It is a cell of dimension \(|J|, \dim \sigma_J = |J|\). Topologically, it is a torus of the same dimension. The cell \(\sigma_0\) is the unique 0-cell of \(S \Box \cdots \Box S,\) hence lies in every cell. One easily sees that
(1) $\sigma_J \cap \sigma_{J'} = \sigma_{J \cap J'}$.

Each cell $\sigma_J$ is a retract $\mathbb{T}^k$ by a retraction $\mathbb{T}^k \to \sigma_J$ which takes each point $(x_1, \ldots, x_k)$ to $(y_1, \ldots, y_k)$, where $y_i = x_i$ if $i \in J$, and $y_i = 1$ otherwise. This retraction will be called canonical.

Any two $m$-cells of the canonical CW structure on $\mathbb{T}^k$ can be joined by a chain of $m$-cells so that any two consecutive $m$-cells meet in an $(m-1)$-cell. If $n \leq k$ then any cell of the $n$th skeleton lies in an $n$-cell. Thus, we see that, using terminology of 5A, the $n$th skeleton (for $k \geq 2$) is close to be a simple ramified $n$-manifold complex (it is not such a complex because it is not regular).

Now we recall the main result concerning the bouquets of circles (see [On, Theorem 3.1]).

**Theorem 4C.1 (Ong).** Let $\bigcup_{i=1}^k S_i$, $k \geq 1$, be a bouquet of circles. Then, for any $n \geq 1$, there is a homotopy equivalence $\psi: SP^n(\bigcup_{i=1}^k S_i) \to (\mathbb{T}^k)(n)$. $\square$

**Note.** Moreover, one can construct $\psi$ to be a surjective mapping with fibers being absolute retracts.

### 4D. Algebraic properties of skeleta of a torus

Assume $n \leq k$. Then put $J_n = \{J : J \subset \{1, \ldots, k\}, |J| = n\}$. For each $J \in J_n$ let

$$r_J : (\mathbb{T}^k)(n) \to \sigma_J$$

denote the retraction induced by the canonical retraction $\mathbb{T}^k \to \sigma_J$ (under our assumption $\sigma_J \subset (\mathbb{T}^k)(n)$).

**Lemma 4D.1.** For $n \leq k$ the homomorphism

$$h : \bigoplus_{J \in J_n} H^n(\sigma_J) \to H^n((\mathbb{T}^k)(n)), \quad (a_J)_{J \in J_n} \to \sum_{J \in J_n} H^n(r_J)(a_J),$$

is an isomorphism.

**Proof.** We may assume that $n \geq 1$. (For $n = 0$ the homomorphism $h$ is the identity isomorphism on $H^0(\sigma_0)$.) Let $\rho : (\mathbb{T}^k)(n) \to (\mathbb{T}^k)(n)/(\mathbb{T}^k)(n-1)$ denote the quotient mapping. For each $J \in J_n$ the image $\rho(\sigma_J)$ is an $n$-sphere which we denote by $S_J$. Since $(\mathbb{T}^k)(n) = \bigcup_{J \in J_n} \sigma_J$ and $\sigma_J \cap \sigma_{J'} \subset (\mathbb{T}^k)(n-1)$ for $J, J' \in J_n$, $J' \neq J$, we have $(\mathbb{T}^k)(n)/(\mathbb{T}^k)(n-1) = \bigvee_{J \in J_n} S_J$ (a bouquet of $n$-spheres).

For each $J \in J_n$, let $r_J' : \bigvee_{J \in J_n} S_J \to S_J$ be the retraction which takes $S_{J'}$, $J' \neq J$, to the base point. Then the following diagram

$$
\begin{array}{ccc}
(\mathbb{T}^k)(n) & \xrightarrow{r_J} & \sigma_J \\
\rho \downarrow & & \rho \downarrow \\
\bigvee_{J \in J_n} S_J & \xrightarrow{r_J'} & S_J
\end{array}
$$
commutes. Consequently, the diagram

\[
\begin{array}{c}
H^n((\mathbb{T}^k)^{(n)}) & \xleftarrow{h} & \bigoplus_{J \in J_n} H^n(\sigma_J) \\
H^n(\rho) & \uparrow & \bigoplus H^n(\rho|\sigma_J) \\
H^n(\bigvee_{J \in J_n} S_J) & \xleftarrow{h'} & \bigoplus_{J \in J_n} H^n(S_J)
\end{array}
\]

commutes, where \(h'\) is a homomorphism analogous to \(h\).

One easily sees that \(H^n(\rho)\) is an epimorphism. (Indeed, this follows from examination of a portion of the homomorphism \(CS(\bigvee_{J \in J_n} S_J, *) \rightarrow CS((\mathbb{T}^k)^{(n)}, (\mathbb{T}^k)^{(n-1)})\) between the cohomology exact sequences induced by the relative homeomorphism \(\rho : ((\mathbb{T}^k)^{(n)}, (\mathbb{T}^k)^{(n-1)}) \rightarrow (\bigvee_{J \in J_n} S_J, *)\).) The homomorphism \(\bigoplus H^n(\rho|\sigma_J)\) is an isomorphism since each \(H^n(\rho|\sigma_J)\) is an isomorphism. It is well known that \(h'\) is an isomorphism. Thus, by commutativity of the above diagram, \(h\) is an epimorphism.

For each \(J \in J_n\), let \(i_J : \sigma_J \rightarrow (\mathbb{T}^k)^{(n)}\) denote the inclusion. If \(J, J' \in J_n\) and \(J' \neq J\), then \((r_{J'} \circ i_J)(\sigma_J) = \sigma_{J \cap J'} \subset (\sigma_J)^{(n-1)}\), hence \(H^n(r_{J'} \circ i_J) = 0\). And \(H^n(r_J \circ i_J) = id_{H^n(\sigma_J)}\), since \(r_J \circ i_J = id_{\sigma_J}\). It follows that the composition

\[
\bigoplus_{J \in J_n} H^n(\sigma_J) \xrightarrow{(H^n(\rho|\sigma_J))} H^n((\mathbb{T}^k)^{(n)}) \xrightarrow{h} \bigoplus_{J \in J_n} H^n(\sigma_J)
\]

is the identity homomorphism. Thus \(h\) is a monomorphism. Consequently, \(h\) is an isomorphism. This ends the proof of the lemma. \(\square\)

4E. The Main Theorem

Now we have at disposal all necessary tools needed for proving the following main theorem of this chapter.

**Theorem 4E.1.** Let \(f : X \rightarrow SP^n(Y)\) be a map from a compact space \(X\) to the \(n\)th symmetric product of a curve \(Y\) such that the induced homomorphism \(f^* : H^n(SP^n(Y)) \rightarrow H^n(X)\) is non-trivial. Then there is a map \(g : SP^n(Y) \rightarrow \mathbb{T}^n\) such that \(H^n(g \circ f) : H^n(\mathbb{T}^n) \rightarrow H^n(X)\) is non-zero. Consequently, \(\text{rank } H^1(X) \geq n\) and there exist elements \(a_1, \ldots, a_n \in H^1(X)\) such that \(a_1 \sim \cdots \sim a_n \neq 0\).

**Note.** An analogous theorem holds if we take any Abelian group as the coefficient group for cohomology in place of \(\mathbb{Z}\).

**Proof.** We can present \(Y\) as the limit \(Y = \text{invlim}\{P_1 \leftarrow P_2 \leftarrow \cdots\}\), where \(P_i\) are graphs. Then \(SP^n(Y) = \text{invlim}\{SP^n(P_1) \leftarrow SP^n(P_2) \leftarrow \cdots\}\). Since \(f^* : H^n(SP^n(Y)) \rightarrow H^n(X)\) is non-trivial, by the continuity of Čech cohomology, there is an index \(l \geq 1\) such that

\[(pr_1 \circ f)^* : H^n(SP^n(P_l)) \rightarrow H^n(X)\]

is nontrivial, where \(pr_1 : SP^n(Y) \rightarrow SP^n(P_l)\) is the projection.
Since functor $SP^n$ preserves homotopy type, we may assume that $P_l$ is a bouquet of circles, $P_l = \bigvee_{i=1}^k S_i$, where $k \geq 1$. By Theorem 4C.1 there is a homotopy equivalence $\psi : SP^n(\bigvee_{i=1}^k S_i) \to (\mathbb{T}^k)^{(n)}$. Therefore, the homomorphism

$$(\psi \circ pr_l \circ f)^* : H^n((\mathbb{T}^k)^{(n)}) \to H^n(X)$$

is nontrivial. It follows that $k \geq n$. By Theorem 4F.1 there is $J \in J_n$ such that the homomorphism

$$(r_J \circ \psi \circ pr_l \circ f)^* : H^n(\sigma_J) \to H^n(X)$$

is nontrivial. As $\sigma_J$ is homeomorphic to $\mathbb{T}^n$, this concludes the proof of the first assertion of the theorem. To obtain the remaining assertions we apply Lemma 2B.3 and Theorem 3A.1.

**Corollary 4E.2.** Let $X$ be a compactum embeddable in the $n$th symmetric product of a curve. If $H^n(X) \neq 0$ then rank $H^1(X) \geq n$ and there exist elements $a_1, \cdots, a_n \in H^1(X)$ such that $a_1 \sim \cdots \sim a_n \neq 0$.

**Note.** In his recent paper R. Cauty [C2] formulates the following related result: *under the above assumptions, if $H^n(X) \neq 0$ then $H^1(X) \neq 0$. His proof resorts to more sophisticated techniques.*

**Proof.** Let $f : X \to SP^n(Y)$ be an embedding, where $Y$ is a curve. Since dim $SP^n(Y) = n$, combining the hypothesis $H^n(X) \neq 0$ with the Hopf Extension Theorem, we infer that there is an element $\alpha \in H^n(\sigma_J)$ such that $H^n(f)(\alpha) \neq 0$. Then the conclusion follows from the above theorem.

Next corollary directly follows from the above corollary because $H^n(M) \neq 0$ for any closed manifold $M$.

**Corollary 4E.3.** If a closed $n$-manifold $M$, $n \geq 2$, can be embedded in the $n$th symmetric product of a curve then rank $H^1(M) \geq n$. In particular, $\mathbb{S}^n$ is not embeddable in any $n$th symmetric product of a curve. The same holds for any $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$, where $n_1 + \cdots + n_k = n$ and $n_i \geq 2$.

**Note.** The second assertion is an analog of the Borsuk Theorem. It follows from [C2] as well.

### 5. Locally connected generalized manifolds in products of curves

This chapter splits into sections 5A-5E. In section 5A we introduce broad classes of continua (each including in particular all closed manifolds) and call them quasi manifolds, pseudo manifolds, para manifolds and ramified manifolds, and establish some basic results. In Theorem 5B.1 we prove that embeddings of locally connected
quasi $n$-manifolds in products of $n$ curves can be factored through special embeddings in products of $n$ graphs with no endpoint. Then we construct an example of a closed surface lying in a product of two curves whose image under either projection is not a graph (Example 5B.2). Theorem 5B.1 has noteworthy consequences. For example, it follows that no locally connected and unicoherent quasi $n$-manifold can be embedded in any product of $n$ curves. In section 5C we present a list of basic properties of ramified pseudo $n$-manifolds lying in products of $n$ graphs. To obtain these properties we carefully study the "fibers" of the projections restricted to the pseudo-manifold. In section 5D we prove a fundamental Theorem 5D.5 on algebraic structure of locally connected quasi $n$-manifolds lying in products of $n$ curves. In particular, this implies that there exist contractible 2-dimensional polyhedra not embeddable in products of two curves. Thus we reveal acyclic polyhedra which have the same property as $S^2$ in the Borsuk theorem. In section 5E we prove that any 2-dimensional collapsible polyhedron (in particular, any cone over a graph) can be embedded in a product of two trees.

5A. Definitions and general properties of certain generalized manifolds

Let $K$ be a CW complex. Then the open cells of $K$ (that is, the interiors $\hat{\sigma}$ of the cells $\sigma \in K$) form a partition of $|K|$, i.e. they are mutually disjoint and cover $|K|$. It follows from the definition of a CW complex that for each skeleton $K^{(n)}$ the space $|K^{(n)}|$ is the union of open cells with dimension $\leq n$. If $\dim \sigma = n$ then $\partial \sigma \subset |K^{(n-1)}|$ and $\hat{\sigma} \cap |K^{(n-1)}| = \emptyset$. A cell $\sigma \in K$ is said to be proper if it is a union of open cells of $K$. If each cell of $K$ is proper then $K$ is said to have proper cells.

Proposition 5A.1. A CW complex $K$ has proper cells if and only if for each two cells $\sigma, \tau \in K$ the condition $\hat{\sigma} \cap \tau \neq \emptyset$ implies $\sigma \subset \tau$ (that is, $\sigma$ is a face of $\tau$).

Corollary 5A.2. If $K_1, \ldots, K_n$ are CW complexes with proper cells then $K_1 \sqcap \cdots \sqcap K_n$ has proper cells as well.

Notice that there exist finite CW complexes with proper cells which are not regular. (The canonical CW structure on the torus $\mathbb{T}^k$ has this property.) But the converse is true:

Lemma 5A.3. Any regular CW complex $K$ has proper cells.

Proof. We must show that any cell $\tau \in K$ is the union of some open cells. By induction we may assume that this holds for all cells with dimension $< n+1 = \dim \tau$, $n \geq 0$. Note that $\partial \tau$ is a subset of the union of the $n$-cells $\sigma_1, \ldots, \sigma_k$ of $K$ such that $\hat{\sigma}_i \cap \tau \neq \emptyset$. It remains to show that $\sigma_i \subset \tau$ for each $i$. To this end fix $i$. As $\partial \tau \cap \hat{\sigma}_i$ is a non-void closed subset of $\hat{\sigma}_i$, to get the conclusion, it is enough to prove that $\partial \tau \cap \hat{\sigma}_i$ is open in $\hat{\sigma}_i$. Since $\partial \tau$ is an $n$-sphere in $|K^{(n)}|$ and $\hat{\sigma}_i$ is open in $|K^{(n)}|$, for each point $x \in \partial \tau \cap \hat{\sigma}_i$ there is an open $n$-cell containing $x$ and wholly lying in this intersection. By the Brouwer Domain Invariance Theorem such a ball is a neighborhood of $x$ in $\hat{\sigma}_i$. Thus $\partial \tau \cap \hat{\sigma}_i$ is open in $\hat{\sigma}_i$, which completes the proof.
From the classic Borsuk Separation Theorem relating closed sets separating $S^n$ to their essential mappings into $S^{n-1}$ (cf. [E-S, p. 302]) we infer the following fact.

Lemma 5A.4. For any $n$-manifold $M$ and a point $x_0 \in M$ there is an open neighborhood $V$ of $x_0$ in $M$ such that every closed subset $F$ of $M$ separating $M$ between $x_0$ and $M \setminus V$ admits an essential map into $S^{n-1}$. (In fact, this holds for every neighborhood $V$ which is an open $n$-disc.) $\square$

The revealed property can be used to define, for each natural $n \geq 1$, a new class of $n$-dimensional continua (comprising in particular all connected $n$-manifolds) playing an important role in our investigations. Namely, an $n$-dimensional continuum $X$ is said to be a \textit{quasi $n$-manifold at a point} $x \in X$ if there is an open neighborhood $V$ of $x$ in $X$ such that every closed subset $F$ of $X$ with $\dim F \leq n-1$ separating $X$ between $x$ and $X \setminus V$ admits an essential map into $S^{n-1}$. (Recall that a closed set $F \subset X$ is said to separate $X$ between subsets $A$ and $B$ if there exist disjoint open sets $U$ and $V$ such that $X \setminus F = U \cup V$, $A \subset U$ and $B \subset V$.)

Notice that $X$ is a quasi $1$-manifold at $x \in X$ if and only if $x$ is not an endpoint of $X$. (A point of a space is said to be its \textit{endpoint} if that point admits arbitrarily small open neighborhoods whose boundaries are one-point sets.) If $X$ is a quasi $n$-manifold at $x$ then every neighborhood of $x$ is $n$-dimensional. If $V$ is as in the first definition then any other open neighborhood $W$ of $x$ contained in $V$ has the separation property as well. Also note that if a closed set $F \subset V$ separates $V$ between $x$ and $\partial V$ then $F$ separates $X$ between $x$ and $X \setminus V$. If $X$ is a quasi $n$-manifold at every point of $X$ then it is called a \textit{quasi $n$-manifold}. Notice that an $n$-dimensional continuum which is a union of quasi $n$-manifolds is a quasi $n$-manifold as well.

An $n$-dimensional continuum $X$ is said to be a \textit{para $n$-manifold} if each point of $X$ belongs to an open $n$-disc lying in $X$ (not necessarily open in $X$). In other words, $X$ is a union of open $n$-discs.

If $X$ is an $n$-dimensional continuum and a point $x \in X$ is an element of an open $n$-disc lying in $X$ then $X$ is a quasi $n$-manifold at $x$. (This follows from the observation that if $E$ is an open disc in $X$ then $E$ is open in $\overline{E}$.) Hence any para $n$-manifold is a quasi $n$-manifold. This simple criterion can be used to detect many interesting quasi $n$-manifolds which are not $n$-manifolds. For instance, both the “Bing house” and the “dunce hat” are para $2$-manifolds, so they are also quasi $2$-manifolds. Those examples are widely known primarily for being $2$-dimensional contractible and not collapsible polyhedra.

And it is convenient to introduce other generalizations of $n$-manifolds. First, for an $n$-dimensional continuum $X$ define the following subsets:

$$P(X) = \{x \in X : x \text{ is an element of an open } n\text{-cell lying in } X \text{ and open in } X\};$$

$$R(X) = \{x \in X : x \text{ is an element of an open } n\text{-cell lying in } X\}.$$  

Thus $P(X) = X$ if and only if $X$ is an $n$-manifold, and $R(X) = X$ if and only if $X$ is a para $n$-manifold.
Now we define the generalizations. An $n$-dimensional continuum $X$ is said to be a pseudo $n$-manifold (ramified $n$-manifold, respectively) if $P(X)$ ($R(X)$, respectively) is dense in $X$ and $\dim[X \setminus P(X)] \leq n - 2$ ($\dim[X \setminus R(X)] \leq n - 2$, respectively). If, in addition, $P(X)$ ($R(X)$, respectively) is connected then $X$ is said to be a simple pseudo $n$-manifold (simple ramified $n$-manifold, respectively).

The set of $n$-cells of a regular CW complex $K$, $n \geq 1$, is said to be chain connected if every two $n$-cells of $K$ can be joined by a finite sequence of $n$-cells of $K$ such that every two successive cells meet along an $(n-1)$-cell. If each $(n-1)$-cell of $K$ is a face of an $n$-cell, the above property holds if and only if the space $|K^{(n)}| \setminus |K^{(n-2)}|$ is connected.

Let $K$ be an $n$-dimensional finite regular CW complex. Then $K$ is said to be a para $n$-manifold (resp., quasi $n$-manifold, pseudo $n$-manifold, ramified $n$-manifold) complex if the polyhedron $|K|$ is a para $n$-manifold (resp., quasi $n$-manifold, pseudo $n$-manifold, ramified $n$-manifold). To name the complexes from the classes defined above for which the set of $n$-cells is chain connected, we add the adjective simple to the basic names, e.g. a simple para $n$-manifold complex, etc.

Notice that any ramified $n$-manifold complex $L$ lying in a simple pseudo $n$-manifold complex $K$, coincides with $K$. In particular, this holds if $|K|$ is an $n$-manifold (as $K$ is a simple pseudo $n$-manifold complex in this case).

(In the literature, the term pseudo $n$-manifold is often used in a more restrictive sense to mean the space $|K|$, where $K$ is a simple pseudo $n$-manifold complex.)

**5A.5 Proposition.** Let $K$ be an $n$-dimensional finite regular CW complex. Then

(i) $K$ is a pseudo $n$-manifold (simple pseudo $n$-manifold, respectively) complex if and only if each cell of $K$ is a face of an $n$-cell, each $(n-1)$-cell of $K$ is incident with exactly two $n$-cells (and the set of $n$-cells of $K$ is chain connected, respectively);

(ii) $K$ is a ramified $n$-manifold (simple ramified $n$-manifold, respectively) complex if and only if each cell of $K$ is a face of an $n$-cell, each $(n-1)$-cell of $K$ is incident with at least two $n$-cells (and the set of $n$-cells of $K$ is chain connected, respectively). □

Our main observation in this subsection is Theorem 5A.8 below which describes a basic property of quasi $n$-manifolds and ramified $n$-manifolds lying in $n$-dimensional polyhedra. In the proof Lemma 5A.7 is needed. In the proof of Lemma 5A.7 we need in turn Lemma 5A.6 below; it is relatively simple but not trivial. Lemma 5A.6 is certainly known to many topologists, and can be proved using various arguments. We supply possibly the shortest one.

**Lemma 5A.6.** No proper closed subset of $\mathbb{S}^n$ admits an essential map into $\mathbb{S}^n$.

**Proof.** We may assume that $n \geq 1$. Consider a proper closed subset $F$ of $\mathbb{S}^n$. Since $\mathbb{S}^n$ is a compact subset of $\mathbb{R}^{n+1}$, the set $F$ can be regarded as a compact subset of $\mathbb{R}^{n+1}$. Then note that $\mathbb{R}^{n+1} \setminus F$ is connected. So, by the Borsuk Separation Theorem [E-S, p. 302], $F$ admits no essential map into $\mathbb{S}^n$. □
Lemma 5A.7. (a) Let $X$ be a quasi $n$-manifold at a point $x$. If $U$ is a neighborhood of $x$ in $X$ and $h : U \to \mathbb{R}^n$ is an embedding, then $h(U)$ is a neighborhood of $h(x)$ in $\mathbb{R}^n$.

(b) Let $X$ be a ramified $n$-manifold and let $U$ be a non-void open subset of $X$. If $h : U \to \mathbb{R}^n$ is an embedding such that $h(U)$ is closed, then $h(U) = \mathbb{R}^n$.

Proof of (a). Suppose, to the contrary, that $h(x) \in \partial h(U)$. Let $V$ be an open neighborhood of $x$ in $X$ with $\overline{V} \subset U$ such that any closed subset $F$ of $X$ separating $x$ and $X \setminus V$ admits an essential map into $S^{n-1}$. Since $h(U \setminus V)$ is a closed subset of $h(U)$ not containing $h(x)$ there is an open ball $B(h(x), \varepsilon_0)$ in $\mathbb{R}^n$ such that $B(h(x), \varepsilon_0) \cap h(U \setminus V) = \emptyset$. Also, there is a sphere $S = \partial B(h(x), \varepsilon)$, $0 < \varepsilon < \varepsilon_0$, such that $S \not\subset h(U)$. Since $S$ separates $\mathbb{R}^n$ between $x$ and $h(U \setminus V)$, the intersection $E = S \cap h(U)$ separates $h(U)$ between $h(x)$ and $h(U \setminus V)$. Note that $E$ is a proper subset of $S$. As $E = S \cap h(V) = S \cap h(\overline{V})$, the set $E$ is a compact subset of $h(V)$ which separates $h(\overline{V})$ between $h(x)$ and $h(\partial V)$. It follows that $F = h^{-1}(E)$ is a compact subset of $V$ which separates $\overline{V}$ between $x$ and $\partial V$. Hence $F(\subset V)$ is a closed subset of $X$ which separates $X$ between $x$ and $X \setminus V$. By our choice of $V$, there is an essential map $F \to S^{n-1}$. But $F(\approx E)$ is homeomorphic to a proper closed subset of $S^{n-1}$, which contradicts Lemma 5A.6.

Proof of (b). It is enough to show that $h(U)$ is dense in $\mathbb{R}^n$. Let $V = \text{int } h(U)$. First we show that $V$ is dense in $h(U)$. In fact, since $R(X)$ is dense $X$ and $U$ is open, the image $h(R(X) \cap U)$ is dense in $h(U)$. On the other hand, $h(R(X) \cap U)$ is open in $\mathbb{R}^n$, by the Brouwer Invariance of Domain Theorem. Hence $h(R(X) \cap U) \subset V$, so $V$ is dense in $h(U)$. Therefore, $\partial V = h(U) \setminus V$. Consequently, $\partial V \subset h(R(X))$, hence $\dim \partial V \leq n - 2$, by the definition of a ramified $n$-manifold. Therefore, $\mathbb{R}^n \setminus \partial V$ is connected (cf. [E, Theorem 1.8.13, p. 77]). It follows that $V$ is dense in $\mathbb{R}^n$, for otherwise $\partial V$ separates $\mathbb{R}^n$. Hence $h(U) = \mathbb{R}^n$, which completes the proof. □

An easy modification of the above argument proves the following

Corollary 5A.8. If $f : X \to Y$ is an embedding of a ramified $n$-manifold into a simple pseudo $n$-manifold, then $h(x) = Y$. □

Theorem 5A.9. Let $X$ be either a quasi $n$-manifold or a ramified $n$-manifold. If $f : X \to |K|$ is an embedding, where $|K|$ is an $n$-dimensional polyhedron, then $h(X) = |L|$, where $L$ is a ramified $n$-manifold subcomplex of $K$. Moreover, if $X$ is a pseudo $n$-manifold, then $L$ is a pseudo $n$-manifold complex.

Proof. First we prove that $f(X)$ is a union of $n$-cells of $K$. To this end, it is enough to show that each point $y \in f(X)$ is an element of an $n$-cell of $K$ which lies in $f(X)$. Notice that there is an open neighborhood $V$ of $y$ in $f(X)$ such that, for each cell $\sigma \in K$, the condition $V \cap \sigma \neq \emptyset$ implies $y \in \sigma$. Since $\dim V = n$, there is an $n$-cell $\sigma_0 \in K$ such that $V \cap \sigma_0 \neq \emptyset$. Hence $y \in \sigma_0$. It remains to show that $\sigma_0 \subset f(X)$. As $f(X)$ is closed, it is enough to show that $\sigma_0 \subset f(X)$. Notice that $f(X) \cap \sigma_0$ is a non-void closed subset of $\sigma_0$. On the other hand, this set is open in $f(X)$. It follows that $U = f^{-1}(\sigma_0)$ is non-void and open in $X$, and $f(U) = f(X) \cap \sigma_0$. Since
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By Lemma 5A.7 we infer that \( f(X) \cap \hat{\sigma}_0 \) is also open in \( \hat{\sigma}_0 \), in both cases under discussion. Consequently, \( f(X) \cap \hat{\sigma}_0 = \hat{\sigma}_0 \). Hence \( \hat{\sigma}_0 \subset f(X) \), as desired. The remaining assertions simply follow from the Borsuk Separation Theorem, cf. [E-S, p. 302]. □

By the above theorem, for any polyhedron \(|K|\) which is either a quasi \( n \)-manifold or a ramified \( n \)-manifold, \( K \) is a ramified \( n \)-manifold complex. Moreover, the converse holds for \( n = 1, 2 \). In general, the converse fails: the suspension of the "dunce hat" (or the "Bing house") can be presented as a polyhedron \(|K|\) such that \( K \) is a ramified 3-manifold complex but \(|K|\) is no quasi 3-manifold.

Let \( K \) be a ramified \( n \)-manifold complex. By its combinatorial component we mean any maximal simple ramified \( n \)-manifold subcomplex of \( K \). One easily sees that \( K \) is the union of combinatorial components, and any two different combinatorial components meet in a subcomplex of dimension \( \leq n - 2 \).

We have the following diagram of inclusions in the class of connected polyhedra:

\[
\{\text{para } n\text{-manifolds}\} \supset \{\text{special para } n\text{-manifolds}\} \\
\cap \\
\{\text{quasi } n\text{-manifolds}\} \supset \{\text{special quasi } n\text{-manifolds}\} \\
\cap \\
\{\text{ramified } n\text{-manifolds}\} \supset \{\text{special ramified } n\text{-manifolds}\} \\
\cup \\
\{\text{pseudo } n\text{-manifolds}\} \supset \{\text{special pseudo } n\text{-manifolds}\}.
\]

5B. On locally connected quasi manifolds.

From embeddings into products of curves to embeddings into products of graphs

Here we prove a useful theorem on factorization of embeddings of quasi manifolds into products of curves through embeddings into product of graphs.

Theorem 5B.1. Let \( X \) be a locally connected quasi \( n \)-manifold such that \( H^1(X) \) has finite rank. If \( f = (f_1, \ldots, f_n) : X \to Y_1 \times \cdots \times Y_n \) is an embedding of \( X \) in the product of \( n \) curves, then there exist mappings \( g = (g_1, \ldots, g_n) : X \to P_1 \times \cdots \times P_n \) and \( h = h_1 \times \cdots \times h_n : P_1 \times \cdots \times P_n \to Y_1 \times \cdots \times Y_n \) such that \( f_i = h_i \circ g_i \) for each \( i = 1, \ldots, n \) (hence \( f = h \circ g \)), where \( g_i : X \to P_i \) is a monotone surjection, \( P_i \) is a graph with no endpoint (that is, \( P_i \) is a quasi 1-manifold), and \( h_i : P_i \to Y_i \) is \( 0 \)-dimensional.

In particular, if \( X \) is embeddable in a product of \( n \) curves, then there exists an embedding \( (g_1, \ldots, g_n) : X \to P_1 \times \cdots \times P_n \), where each \( g_i : X \to P_i \) is a monotone surjection, \( P_i \) is a graph with no endpoint, and rank \( H^1(P_i) \leq \text{rank } H^1(X) \).

Note. It follows that if \( f_i : X \to Y_i \) is monotone, then \( f_i(X) \) is a graph. In fact, in this case \( f_i(X) = h_i(P_i) \) and \( h_i : P_i \to Y_i \) is an embedding. If \( f_i \) is not
monotone then \( f_i(X) \) need not be a graph, see Example 5B.3. The proof given below shows that if \( X \) is a non-degenerate connected polyhedron (or any non-degenerate locally connected continuum whose \( H^1(X) \) has finite rank) then \( f_i(X) \) is a local dendrite.

\[ \square \]

**Proof.** By the Whyburn factorization theorem, there is a factorization \( f_i = h_i \circ g_i \),

\[ X \xrightarrow{g_i} P_i \xrightarrow{h_i} Y_i, \]

where \( g_i \) is a monotone surjection, and \( h_i \) is 0-dimensional. Since \( Y_i \) is 1-dimensional and \( h_i \) is 0-dimensional we infer that \( \dim P_i \leq 1 \) (by a theorem of Hurewicz). Clearly, \( g = (g_1, \ldots, g_n) : X \to P_1 \times \cdots \times P_n \) is an embedding. Since \( \dim X = n \), it follows that \( \dim P_i > 0 \) for each \( i \). Therefore, \( P_i \) is a locally connected curve, as \( g_i \) is a surjection. Since \( g_i \) is a monotone surjection and \( H^1(X) \) has finite rank, \( P_i \) is actually a local dendrite (see [Kr, Lemma 3.1]). Hence each point of \( P_i \) has a closed neighborhood which is a dendrite. First we show that

\[ (*) \text{ P}_i \text{ has no endpoint.} \]

For suppose \( P_i \) has an endpoint \( z_0 \). Since \( g_i(X) = P_i \), there is a point \( x_0 \in X \) such that \( g_i(x_0) = z_0 \). Since \( X \) is a quasi \( n \)-manifold at \( x_0 \) there is an open neighborhood \( V \) of \( x_0 \) in \( X \) such that

1. any closed \((n-1)\)-dimensional subset of \( X \) separating \( X \) between \( x_0 \) and \( X \setminus V \) admits an essential map to \( S^{n-1} \).

Now we shall show that there is an open neighborhood \( U \) of \( g(x_0) \) in \( P_1 \times \cdots \times P_n \) such that

2. \( \overline{U} \cap g(X \setminus V) = \emptyset \),
3. \( \dim \partial U = n - 1 \),
4. \( \partial U \) is contractible.

To construct \( U \) we assume, without loss of generality, that \( i = 1 \). Then \( g(x_0) = (y_1, y_2, \ldots, y_n) \), where \( y_1 = z_0 \). Note that \( g(V) \) is a neighborhood of \( g(x_0) \) in \( g(X) \), hence any small enough \( U \) satisfies (2). Since \( z_0 \) is an endpoint of \( P_1 \), and each \( P_j \) is a local dendrite, there exist sets \( U_1, \ldots, U_n \), as small as we please, such that each \( U_j \) is an open and connected neighborhood of \( y_j \) in \( P_j \) with \( \partial U_j \) finite, each \( \overline{U}_j \) is a dendrite, and \( \partial U_1 \) is a one-point set. Then the set \( U = U_1 \times \cdots \times U_n \) has the desired properties. In fact, as \( U_j \)'s are small, \( U \) satisfies (2). Then note that

\[ \partial U = \bigcup_{j=1}^{n} (\overline{U}_1 \times \cdots \times \overline{U}_{j-1} \times \partial U_j \times \overline{U}_{j+1} \times \cdots \times \overline{U}_n). \]

Hence (3) follows. To prove (4), note that \( (\partial U_1) \times \overline{U}_2 \times \cdots \times \overline{U}_n \) is a strong deformation retract of \( \partial U \) (because \( \partial U_1 \), as a one-point set, is a strong deformation retract of \( \overline{U}_1 \)). Hence (4) follows from the fact that \( (\partial U_1) \times \overline{U}_2 \times \cdots \times \overline{U}_n \) is contractible.
Now consider the set \( F = \partial_{g(X)}(U \cap g(X)) \). Observe that it is a closed subset of \( g(X) \) such that

(5) \( F \subset \partial U \),
(6) \( F \) separates \( g(X) \) between \( g(x_0) \) and \( g(X \setminus V) \).

It follows that

(7) \( g^{-1}(F) \) is closed, \((n-1)\)-dimensional, and separates \( X \) between \( x_0 \) and \( X \setminus V \).

Now we are ready to complete the proof of (*). Note that by (1) and (7) there is an essential map \( \varphi : F \to S^{n-1} \). By (3) and (5) there is a continuous extension \( \varphi^* : \partial U \to S^{n-1} \) of \( \varphi \). However, by (4), \( \varphi^* \) is null-homotopic, hence so is \( \varphi \), a contradiction. This proves (*).

Next we show that

(**) \( P_i \) is a graph.

To prove (**) recall that \( P_i \) is a local dendrite. Since \( P_i \) has no endpoint, it contains a circle. (Otherwise it is a dendrite, hence contains an endpoint.) It follows that the union of all simple closed curves in \( P_i \) is a (not necessarily connected) graph. Enlarging this set by the union of a finite collection of arcs (e.g., adding arcs in \( P_i \) irreducibly connecting different components of the union), we get a connected graph \( Q_i(\subset P_i) \) such that for each component \( C \) of \( P_i \setminus Q_i \) we have

(8) \( C \) is a dendrite and \( \partial C \) consists of a single point.

To prove (**) it suffices to show that \( Q_i = P_i \).

Suppose, on the contrary, that \( P_i \setminus Q_i \neq \emptyset \). Then consider a component \( C \) of \( P_i \setminus Q_i \). It is an open set in \( P_i \). By (1), \( C \) is a dendrite and \( \partial C \) is a one-point set. There is a point \( z_0 \in C \) which is an endpoint of the dendrite \( C \). Since \( C \) is open, \( z_0 \) is an endpoint of \( P_i \) as well. This contradicts (**) and ends the proof of (**) otherwise.

As \( g_i \) is a monotone surjection, the induced homomorphism \( H^1(g_i) : H^1(P_i) \to H^1(X) \) is a monomorphism by the Vietoris-Begle Theorem for \( n = 1 \) (see e.g. [Sp, 6.9, Theorem 15]). Therefore, \( \text{rank } H^1(P_i) \leq \text{rank } H^1(X) \). This completes the proof. \( \square \)

It is well known that for any closed \( n \)-manifold \( M \) the group \( H^1(M) \) has finite rank. Consequently, Theorem 5B.1 implies the following

**Corollary 5B.2.** If a closed \( n \)-manifold is embeddable in a product of \( n \) curves, then it is also embeddable in a product of \( n \) graphs. \( \square \)

**Note.** This corollary answers a question posed (for surfaces) by R. Cauty. A harder variant of this question (for \( n \)-dimensional polyhedra) is still open [C1]. \( \square \)

**Example 5B.3.** There exist a curve \( X \) which is not a graph, and a closed orientable surface \( M \) in the product \( X \times X \) such that both projections \( pr_1 : X \times X \to X \) map \( M \) onto \( X \). Moreover, \( M \) is invariant under the canonical involution on \( X \times X \) which interchanges the coordinates.
Proof. First we construct a closed orientable surface \( N \) in the product \( Y_1 \times Y_2 \) of two curves such that

1. \( Y_1 \) is not a graph,
2. the projections \( q_i : Y_1 \times Y_2 \to Y_i \) map \( N \) onto \( Y_i \).

Define \( Y_1 \) to be the union \( Y_1 = \alpha_0 \cup \alpha_1 \cup \beta_1 \cup \beta_2 \) of four arcs with common endpoints \( a, b \) such that \( \alpha_0 \cup \alpha_1 \cup \beta_1 \) is a \( \theta \)-curve, \( \beta_2 \cap (\alpha_0 \cup \alpha_1) = \emptyset \), and \( \beta_1 \cap \beta_2 \) is a compact set with infinitely many components. Then \( Y_1 \) satisfies (1). Define \( Y_2 \) to be a graph given by the formula: \( Y_2 = T_0 \cup T_1 \), where \( T_0, T_1 \) are two oriented circles whose intersection \( T_0 \cap T_1 = L_0 \cup L_1 \), where \( L_0, L_1 \) are disjoint oriented arcs coherently oriented with each \( T_i \). In such a case, each \( T_i \) can be presented as the union of four arcs with disjoint interiors, \( T_i = A_i \cup B_i \cup L_0 \cup L_1 \), such that \( S_1 = A_0 \cup A_1 \) and \( S_2 = B_0 \cup B_1 \) are disjoint circles. We define the surface \( N \) in \( Y_1 \times Y_2 \) by the formula

\[
N = \alpha_0 \times T_0 \cup \alpha_1 \times T_1 \cup \beta_1 \times S_1 \cup \beta_2 \times S_2.
\]

One easily verifies that \( N \) is an orientable surface satisfying (2).

To construct the promised example we proceed as follows. Choose a homeomorphism \( h : \alpha_0 \to A_0 \). Then define \( X \) to be the quotient space \( X = (Y_1 \cup Y_2) / x \sim h(x) \) for each \( x \in \alpha_0 \). By (1) we infer that \( X \) is a curve but not a graph. Let \( X_i = h_i(Y_i) \), where \( h_i : Y_i \to X \) are canonical embeddings. Clearly, \( X = X_1 \cup X_2 \) and \( X_1 \cap X_2 = A \), where \( A = h_1(\alpha_0) = h_2(A_0) \) is an arc. Let \( t : Y_1 \times Y_2 \to Y_2 \times Y_1 \) denote the map given by \( t(y, z) = (z, y) \). Then we define \( M(\subset X \times X) \) as follows:

\[
M = [(h_1 \times h_2)(N) \cup (h_2 \times h_1)(t(N))] \setminus (\hat{A} \times \hat{A}).
\]

One easily verifies that \( M \) is invariant under canonical involution on \( X \times X \). Since \( \alpha_0 \times A_0 \subset N \subset Y_1 \times Y_2 \), \( A_0 \times \alpha_0 \subset t(N) \subset Y_2 \times Y_1 \). Hence \( A \times A = (h_1 \times h_2)(\alpha_0 \times A_0) \subset (h_1 \times h_2)(N) \subset (h_1 \times h_2)(Y_1 \times Y_2) = X_1 \times X_2 \). Likewise, \( A \times A \subset (h_2 \times h_1)(t(N)) \subset X_2 \times X_1 \). Hence we have

\[
(h_1 \times h_2)(N) \cap (h_2 \times h_1)(t(N)) = A \times A.
\]

Thus \( M \) is the connected sum of orientable surfaces \((h_1 \times h_2)(N)\) and \((h_2 \times h_1)(t(N))\), hence it is an orientable surface. Applying (2), we easily see that both projections \( pr_i : X \times X \to X \) map \( M \) onto \( X \). \( \Box \)

In connection with the above proof let us notice the following fact.

Note. Let \( M \) be any compactum lying in the product \( P_1 \times P_2 \) of two graphs. If \( A \) is an arc in \( P_1 \) with \( p_1^{-1}(A) = A \times (S_1 \cup \cdots \cup S_k), k \geq 2 \), where \( S_1, \ldots, S_k \) are disjoint circles in \( P_2 \), then \( M \) can be embedded in the product \( P_1' \times P_2 \) in such a way that \( P_1' \) is a curve and the image of \( M \) under the projection \( P_1' \times P_2 \to P_1' \) is not a graph. (In fact, \( P_1' \) can be obtained from \( P_1 \) by adding an arc \( B \) with the same endpoints as that of \( A \) in such a way that \( A \cup B \) is not a graph.) \( \Box \)
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The surface $M$ constructed in Example 5B.3 meets the diagonal of $X \times X$. Below we present another example of a surface in the product $P \times P$, where $P$ is a graph, which is disjoint from the diagonal of the product and is invariant under the canonical involution on $P \times P$.

**Example 5B.4.** There exist a graph $P$ and a closed orientable surface $M$ in $P \times P$ such that: $M$ is disjoint with the diagonal of $P \times P$, both projections $pr_i : P \times P \rightarrow P$ map $M$ onto $P$, and $M$ is invariant under the canonical involution on $P \times P$.

**Proof.** Fix any number $n \geq 4$. The graph $P$ is defined to be a subset of $S^1 \times I$ given by

$$P = (S^1 \times \{0, 1\}) \cup \{z_0, \ldots, z_{n-1}\} \times I,$$

where $z_j = \exp\left(\frac{2\pi i j}{n}\right)$, $j = 0, \ldots, n-1$. Then define arcs $A_j \times \{0\}$, $A_j \times \{1\}$, $I_j$ and circles $S_j$ in $P$ as follows:

$$A_j = \{\exp(2\pi it) : t \in \left[\frac{j}{n}, \frac{j+1}{n}\right]\}, \quad I_j = \{z_j\} \times I, \quad S_j = I_j \cup (A_j \times \{0, 1\}) \cup I_{j+1}.$$

(All indices in this construction are reduced modulo $n$.) Finally, define tori $T_j$ to be the subsets of $P \times P$ given by

$$T_j = S_j \times S_{j+2}.$$

Notice that the intersection

$$D_j = T_i \cap T_{j+1} = I_{j+1} \times I_{j+3}$$

is a disc. Now we are ready to define the surface $M$, put

$$M = (T_0 \cup \cdots \cup T_{n-1}) \setminus (\hat{D}_0 \cup \cdots \cup \hat{D}_{n-1}).$$

One easily verifies that $M$ has all the desired properties. □

**5C. Ramified manifolds in products of graphs**

Here we establish some properties of ramified $n$-manifolds lying in products of $n$ graphs. These properties will find essential applications in the next section and in Chapter 6.

*Throughout this section we consider fixed graphs $P_1 = |K_1|$, $\ldots$, $P_n = |K_n|$, $n \geq 2$, where each $K_i$ is a regular 1-dimensional CW complex.*

(So, each 1-cell of $K_i$, i.e. an edge, is an arc; its endpoints are called vertices.) Let us recall that by $K_1 \sqcap \cdots \sqcap K_n$ we denote the cell structure on $P_1 \times \cdots \times P_n$ defined by

$$K_1 \sqcap \cdots \sqcap K_n = \{\sigma_1 \times \cdots \times \sigma_n : \sigma \in K_1, \ldots, \sigma_n \in K_n\}.$$

Also, we consider a fixed ramified $n$-manifold $M = |K(M)|$ lying in $P_1 \times \cdots \times P_n$, where $K(M)$ is a subcomplex of $K_1 \sqcap \cdots \sqcap K_n$.

By Theorem 5A.9 we have
Property (a). $K(M)$ is a ramified $n$-manifold complex; in particular, $M$ is the union of $n$-cells of $K(M)$. □

We adopt the following notation. For a non-void subset $J$ of the index set $\{1, \ldots, n\}$ let:

- $P_J = \prod_{j \in J} P_j$,
- $K_J$ denote the cell structure on $P_J$ induced by $\{K_j : j \in J\}$,
- $p_J$ denote the restriction to $M$ of the projection $pr_J : P_1 \times \cdots \times P_n \to P_J$ (in particular, $p_{\{1, \ldots, n\}} : M \to P_1 \times \cdots \times P_n$ is the inclusion mapping,
- $n_J = |J|$,
- $J^c = \{1, \ldots, n\} \setminus J$ (therefore, $n_J = n - n_J$). Notice that

For any cell $\sigma = \sigma_1 \times \cdots \times \sigma_n \in K$, $\sigma_j \in K_j$, the restriction $p_J|\sigma$ is the projection onto $\sigma_J = \prod_{j \in J} \sigma_j$. In this sense we say that $p_J$ preserves the cell structures $K$ and $K_J$. It follows from Property (a) that

Property (b). $p_J(M) = |K'_J|$, where $K'_J$ is a subcomplex of $K_J$. Moreover, $|K'_J|$ is a ramified $n_J$-manifold; if $M$ is a (simple) pseudo $n$-manifold, then $p_J(M)$ is a (simple) pseudo $n_J$-manifold. (In the sequel we abbreviate $K'_J$ to $K'_J$.) □

From this point on to the end of this section we assume that $J$ is a proper non-void subset of $\{1, \ldots, n\}$.

For every cell $\tau \in K'_{J^c}$, we define $P_J(\tau)$ to be the union of all $n_J$-cells $\sigma \in K_J$ such that $\sigma \times \tau \subset M$. From Property (a) we infer

Property (c). $M = \bigcup \{P_J(\tau) \times \tau : \tau \in K'_{J^c} \text{ is an } n_{J^c}-\text{cell}\}$. □

Property (d). If $\tau$ is a face of a cell $\tau' \in K'_{J^c}$ then $P_J(\tau) \supset P_J(\tau')$. □

Moreover, for any point $y \in p_{J^c}(M)$ let $P_J(y) = \{x \in P_J : (x, y) \in M\}$. Thus, $P_J(y) \times \{y\} = p_J^{-1}(y)$, and $P_J(y) \subset p_J(M)$.

Property (e). For any cell $\tau \in K'_{J^c}$ and any point $y \in \hat{\tau}$ we have $P_J(y) = P_J(\tau) = \bigcup \{P_J(\tau') : \tau' \in K'_{J^c} \text{ is an } n_{J^c}-\text{cell with face } \tau\}$.

Proof. Note that $P_J(y) \supset P_J(\tau) \supset P_J(\tau')$ for each cell $\tau' \in K'_{J^c}$ with face $\tau$. So, it remains to justify the inclusion

$$P_J(y) \subset \bigcup \{P_J(\tau') : \tau' \in K'_{J^c} \text{ is an } n_{J^c}-\text{cell with face } \tau\}.$$ 

To this end, consider a point $x \in P_J(y)$. Then $(x, y) \in M$. By Property (a), $(x, y)$ belongs to an $n$-cell $\sigma \times \tau' \subset M$, where $\sigma$ is an $n_J$-cell in $K_J$ and $\tau'$ is an $n_{J^c}$-cell in $K'_{J^c}$. As $y \in \hat{\tau}$, $\tau$ is a face of $\tau'$. It follows that $x \in \bigcup \{P_J(\tau') : \tau' \in K'_{J^c} \text{ is an } n_{J^c}-\text{cell with face } \tau\}$, which ends the proof. □
Property (f). For any cell $\tau \in K'_{je}$ the set $P_J(\tau)$ is a finite disjoint union of ramified $n_J$-manifolds in $P_J = |K_J|$. Moreover, if $M$ is a pseudo $n$-manifold and $\tau$ is an $n_{je}$-cell then $P_J(\tau)$ is a finite union of disjoint pseudo $n_J$-manifolds.

Proof. If $\tau$ is an $n_{je}$-cell then both assertions follow from the fact that each $(n-1)$-cell $\sigma \times \tau \in K(M)$, where $\sigma \in K_J$ is an $(n_J-1)$-cell, is a face of at least two (exactly two if $M$ is a pseudo $n$-manifold) $n$-cells $\sigma_1 \times \tau$, $\sigma_2 \times \tau \in K(M)$. Consequently, the first assertion for arbitrary $\tau \in K'_{je}$ follows from Property (e).

Property (g). $p_J(M) = \bigcup \{P_J(\tau) : \tau \in K'_{je} \text{ is a } k-\text{cell} \}$ for each $k = 0, \ldots, n_{je}$.

Proof. For $k = n_{je}$ this follows from Property (c). Applying Property (e) we obtain the general case.

Property (h). If $P_J(w)$ is an $n_j$-manifold for each vertex $w \in K'_{je}$, then $p_J(M) = P_J(w_0)$ for any vertex $w_0$ of $K'_{je}$.

Proof. By Property (g) (with $k = 0$), it suffices to prove that

(*) $P_J(w_1) = P_J(w_2)$ for any two vertices $w_1, w_2 \in K'_{je}$.

To this end, consider a 1-cell $\tau \in K'_{je}$ with vertices $w$ and $w'$. Then, by Properties (f) and (d), $P_J(\tau)$ is a finite union of ramified $n_J$-manifolds contained in both $n_J$-manifolds $P_J(w)$ and $P_J(w')$. It follows that $P_J(w) = P_J(\tau) = P_J(w')$, as no proper ramified $n_J$-manifold is contained in an $n_J$-manifold. Thus the condition (*) is a consequence of the connectivity of the complex $K'_{je}$.

Property (i). If $p_J(M)$ is a simple pseudo $n_J$-manifold then $M = p_J(M) \times p_{J'}(M)$.

Proof. For every $\tau \in K'_{je}$ the set $P_J(\tau)$ is a ramified $n_J$-manifold contained in the simple pseudo $n_J$-manifold $p_J(M)$, so $P_J(\tau) = p_J(M)$. Hence the assertion follows from Property (c).

Property (j). Let $j \in \{1, \ldots, n\}$. Then $p_j(M)$ is a circle if and only if $P_j(v)$ is a circle for each vertex $v \in K'_{\{j\}e}$.

Proof. This follows from Property (h) combined with Property (i).

Let us define yet another symbol:

$$J(M) = \{j \in \{1, \ldots, n\} : p_j(M) \text{ is a circle} \}.$$ 

If $J(M)$ is not empty then $M$ is said to have projections onto a circle.

Property (k). Suppose $J(M)$ is a proper non-void subset of $\{1, \ldots, n\}$ then $M = p_{J(M)}(M) \times p_{J(M)'}(M)$, where $p_{J(M)}(M) = \prod_{j \in J(M)} P_j(M)$ is a $|J(M)|$-dimensional torus, and $p_{J(M)'}(M)$ ($\subset \prod_{j \in J(M)'} P_j$) is a ramified $|J(M)'|$-dimensional manifold with no projection onto a circle. If $J(M) = \{1, \ldots, n\}$ then $M = p_1(M) \times \cdots \times p_n(M)$ is an $n$-dimensional torus.
Proof. First notice that $p_{J(M)}(M)$ is a $|J(M)|$-dimensional torus $\prod_{j \in J(M)} p_j(M)$. That follows from the fact that $p_{J(M)}(M)$ is a ramified $|J(M)|$-dimensional manifold (see Property (b)) lying in the torus $\prod_{j \in J(M)} p_j(M)$ of the same dimension. Next notice that $M = p_{J(M)}(M) \times p_{J(M)^c}(M)$ by Property (i). Analogous arguments can be used to prove the remaining assertions. □

5D. Product structure of generalized manifolds lying in products of graphs

To prove the main result of this section we need the following

Lemma 5D.1. Let $p_i : X \to P_i$ and $q_i : S_i \to X$, $i = 1, \ldots, n$, be any mappings such that

$$H_1(p_i \circ q_j) : H_1(S_j) \to H_1(P_i)$$

is a monomorphism for each $i = j$, and the zero homomorphism for $i < j$. Then the homomorphism

$$\varphi : H_1(S_1) \oplus \cdots \oplus H_1(S_n) \to H_1(X),$$

defined by $\varphi(x_1, \ldots, x_n) = H_1(q_1)(x_1) + \cdots + H_1(q_n)(x_n)$, is a monomorphism. Consequently, $\text{rank } H_1(X) \geq \text{rank } H_1(S_1) + \cdots + \text{rank } H_1(S_n)$.

Proof. To prove the lemma consider an element $\alpha = (\alpha_1, \ldots, \alpha_n) \in H_1(S_1) \oplus \cdots \oplus H_1(S_n)$ such that $\varphi(\alpha) = 0$. It is enough to show that $\alpha = 0$. We prove this by finite induction. So, assume that for some $k$, $1 \leq k \leq n$, $\alpha_i = 0$ if $i \leq k - 1$. (For $k = 1$ this condition is empty.) It remains to show that $\alpha_k = 0$. To this end note that

$$H_1(p_k)(\varphi(\alpha)) = H_1(p_k)(H_1(q_k)(\alpha_k) + \cdots + H_1(q_n)(\alpha_n)) = H_1(p_k \circ q_k)(\alpha_k).$$

Thus $H_1(p_k \circ q_k)(\alpha_k) = 0$, consequently, $\alpha_k = 0$, which ends the proof. □

In the following lemma we keep the notation and assumptions of section 5C.

Lemma 5D.2. Let $M$ be a ramified $n$-manifold lying in the product $P_1 \times \cdots \times P_n$ of graphs. Then $\text{rank } H_1(M) \geq n$. If $\text{rank } H_1(M) = n + k$ and $k < n$, then $|J(M)| \geq n - k$. Therefore, for $0 < k < n$, we have $M = p_{J(M)}(M) \times p_{J(M)^c}(M)$, where $p_{J(M)}(M) = \prod_{j \in J(M)} p_j(M)$ is a $|J(M)|$-dimensional torus and $p_{J(M)^c}(M)$ is a ramified $|J(M)^c|$-dimensional manifold with no projection onto a circle. For $k = 0$ the set $M$ coincides with the $n$-torus $p_1(M) \times \cdots \times p_n(M)$.

Proof. Let $v_j$, $j = 1, \ldots, n$, denote a vertex of $K_{\{j\}}'$. By Property (f), $P_j(v_j)$ is a finite union of ramified 1-manifolds. To continue the proof we apply Lemma 5D.1 as follows.

Let $q_j : P_j(v_j) \to M$ be the map such that $p_j \circ q_j$ is the inclusion $P_j(v_j) \hookrightarrow P_j$ and $(p_{\{j\}}^c \circ q_j)(P_j(v_j)) = \{v_j\}$. Since $p_j \circ q_j$ is an inclusion into a graph, $H_1(p_j \circ q_j)$ is a monomorphism. If $i \neq j$ then $H_1(p_i \circ q_j)$ is the zero homomorphism as the image of $p_i \circ q_j$ is a point. Thus, by Lemma 5D.1, we obtain
(*) \( \text{rank } H_1(M) \geq \text{rank } H_1(P_1(v_1)) + \cdots + H_1(P_n(v_n)). \)

Notice that \( \text{rank } H_1(P_j(v_j)) \geq 1. \) Hence \( \text{rank } H_1(M) \geq n, \) which proves the first assertion.

Now we prove the second one. To this end, pick the vertices \( v_j \) so that \( \text{rank } H_1(P_j(v_j)) \geq \text{rank } H_1(P_j(w_j)) \) for each vertex \( w_j \in K_j' \). Let

\[ J_0 = \{ j \in \{1, \cdots, n\} : \text{rank } H_1(P_j(v_j)) = 1 \}. \]

Since \( \text{rank } H_1(M) = n + k \), by (*) we infer that \( \text{rank } H_1(P_j(v_j)) \geq 2 \) for at most \( k \) indices \( j \). Hence \( J_0 \) consists of at least \( n - k \) indices. If \( j \in J_0 \) then \( \text{rank } H_1(P_j(w_j)) = 1 \) for each vertex \( w_j \in K_j'. \) Hence \( P_j(w_j) \) is a circle. By Property (h) we infer that \( p_j(M) \) is a circle for each \( j \in J_0 \). It follows that \( J_0 \subset J(M) \) (actually, \( J_0 = J(M) \)). Hence \( |J(M)| \geq n - k \). The remaining assertions follow directly from Property (k). \( \square \)

**Lemma 5D.3.** Let \( X \) be a compactum and let \( A \) be a closed subset of \( X \). If \( \dim X \leq m \) and \( H^m(X) = 0 \) then \( H^n(A) = 0 \) for each \( n \geq m \).

**Proof.** Since \( H^n(X) = 0 \) and \( H^{n+1}(X, A) = 0 \), the conclusion follows from the cohomology exact sequence of the pair \( (X, A) \). \( \square \)

**Lemma 5D.4.** Let \( X_i, i = 1, 2, \) be non-degenerate continua such that each point of \( X_i \) admits a closed neighborhood with trivial \( n_i \)-dimensional cohomology, where \( n_i = \dim X_i \). If \( X_1 \times X_2 \) is a quasi \((n_1 + n_2)\)-manifold then each \( X_i \) is a quasi \( n_i \)-manifold.

**Note.** Every polyhedron \( P \) satisfies the condition from this lemma: each point of \( P \) admits a closed neighborhood which is contractible. \( \square \)

**Proof.** Let \( x_1 \in X_1 \) and \( x_2 \in X_2 \) be arbitrary points. We must construct open neighborhoods \( V_i \) of \( x_i \) in \( X_i \) satisfying the definition of a quasi \( n_i \)-manifold. Let \( n = n_1 + n_2 \). Since \( X_1 \times X_2 \) is a quasi \( n \)-manifold, there is an open neighborhood \( V \) of the point \( (x_1, x_2) \) in \( X_1 \times X_2 \) such that for every closed \((n-1)\)-dimensional set \( F \) separating \( X_1 \times X_2 \) between \( (x_1, x_2) \) and \( (X_1 \times X_2) \setminus V \) we have \( H^{n-1}(F) \neq 0 \). Since the same condition holds for any open neighborhood of \((x_1, x_2)\) contained in \( V \), by Lemma 5D.3 and our hypothesis about \( X_i \), we may assume that \( V = V_1 \times V_2 \), where each \( V_i \) is an open neighborhood of \( x_i \) in \( X_i \) such that \( H^{n_i}(\overline{V_i}) = 0 \), where \( \overline{V_i} \) stands for the closure of \( V_i \) in \( X_i \). We shall show that these \( V_i \)'s are the desired neighborhoods.

By Lemma 5D.3 again, it follows that

\((*)\) for any closed subset \( A \) of \( X_i \) contained in \( \overline{V_i} \) and any \( k \geq n_i \) we have \( H^k(A) = 0 \).

Now consider a closed \((n_i-1)\)-dimensional subset \( F_i \) of \( X_i \) separating \( X_i \) between \( x_i \) and \( X_i \setminus V_i \). Then \( X_i \setminus F_i = U_i \cup W_i \), where \( U_i, W_i \) are disjoint open sets in \( X_i \) such that \( x_i \in U_i \) and \( X_i \setminus V_i \subseteq W_i \). Then \( \overline{U_i} \subseteq V_i \), and the boundary \( \partial U_i \) (\( \subset F_i \)
separates $X_i$ between $x_i$ and $X_i \setminus V_i$. It follows that $\partial(U_1 \times U_2)$ separates $X_1 \times X_2$ between $(x_1, x_2)$ and $(X_1 \times X_2) \setminus V$. Thus $H^{n-1}(\partial(U_1 \times U_2)) \neq 0$.

Then consider the following portion of the Mayer-Vietoris cohomology exact sequence of the couple $\{\partial U_1 \times \overline{U}_2, \overline{U}_1 \times \partial U_2\}$:

$$H^{n-2}(\partial U_1 \times \partial U_2) \xrightarrow{\delta^*} H^{n-1}(\partial(U_1 \times U_2)) \to H^{n-1}(\partial U_1 \times \overline{U}_2) \oplus H^{n-1}(\overline{U}_1 \times \partial U_2).$$

(The sequence takes this form because $\partial U_1 \times \partial U_2 = (\partial U_1 \times \overline{U}_2) \cap (\overline{U}_1 \times \partial U_2)$ and $\partial(U_1 \times U_2) = (\partial U_1 \times \overline{U}_2) \cup (\overline{U}_1 \times \partial U_2)$.) By the Künneth formula and $(*)$, $H^{n-1}(\partial U_1 \times \overline{U}_2) = 0$ and $H^{n-1}(\overline{U}_1 \times \partial U_2) = 0$. It follows that $\delta^*$ is an epimorphism. Thus $H^{n-2}(\partial U_1 \times \partial U_2) \neq 0$ since $H^{n-1}(\partial(U_1 \times U_2)) \neq 0$. Again, by the Künneth formula and $(*)$, $H^{n-2}(\partial U_1 \times \partial U_2) = H^{n-1}(\partial U_1) \otimes H^{n-2}(\partial U_2)$. It follows that both $H^{n-1}(\partial U_1)$ and $H^{n-2}(\partial U_2)$ are not trivial.

Note that $H^{n-1}(F_i, \partial U_i) = 0$ since $\dim F_i = n_i - 1$. Thus, from the cohomology exact sequence of the couple $(F_i, \partial U_i)$ and $(*)$, it follows that the homomorphism $H^{n-1}(F_i) \to H^{n-1}(\partial U_i)$ induced by the inclusion $\partial U_i \to F_i$ is an epimorphism. Consequently, each $H^{n-1}(F_i)$ is not trivial, which concludes the proof of our lemma. □

**Theorem 5.5.** Let $X$ be a locally connected quasi $n$-manifold, $n \geq 1$, lying in the product $Y_1 \times \cdots \times Y_n$ of $n$ curves. Then rank $H_1(X) \geq n$. Moreover, if rank $H_1(X) = n + k$ and $k < n$ then the set

$$J(X) = \{ j \in \{1, \ldots, n\} : pr_j(X) is a circle\},$$

where $pr_j : Y_1 \times \cdots \times Y_n \to Y_j$ stand for the projections, contains at least $n - k$ elements and $X = \bigcap_{j \in J(X)} pr_j(X) \times X'$, where $X'$ is a quasi $|J(X)^c|$-manifold in $Y_{j^c} = \prod_{i \in J^c} Y_i$. (For $k = 0$ we have $X = pr_1(X) \times \cdots \times pr_n(X)$, hence $X$ is homeomorphic to the torus $\mathbb{T}^n$ in this case.)

**Proof.** Let $f = (f_1, \ldots, f_n) : X \to Y_1 \times \cdots \times Y_n$ be the inclusion of $X$ in the product of $n$ curves, i.e., each $f_i : X \to Y_i$ is the restriction of the projection $pr_i : Y_1 \times \cdots \times Y_n \to Y_i$. Clearly, we can assume that rank $H_1(X)$ is finite. By Theorem 5B.1, there exist mappings $g = (g_1, \ldots, g_n) : X \to P_1 \times \cdots \times P_n$ and $h = h_1 \times \cdots \times h_n : P_1 \times \cdots \times P_n \to Y_1 \times \cdots \times Y_n$ such that $f_i = h_i \circ g_i$ for each $i = 1, \ldots, n$ (so $f = h \circ g$), where $g_i : X \to P_i$ is a monotone surjection, and $P_i$ is a graph.

By Theorem 5A.8, $M = g(X)$, is a ramified $n$-manifold lying in the product $P_1 \times \cdots \times P_n$. Notice that rank $H_1(M) = rank H_1(X)$.

By Lemma 5D.2, rank $H_1(X) \geq n$. Moreover, if rank $H_1(X) = n + k$, where $k < n$, then $J(X)$ contains at least $n - k$ indices. And $M = pr_{J(X)}(M) \times pr_{J(X)^c}(M)$, where $pr_{J(X)}(M) = \prod_{j \in J(X)} pr_j(M)$. (If $k = 0$ then $M = pr_1(M) \times \cdots \times pr_n(M)$.)

Thus, we have $pr_j(M) = P_j$ for each $j \in J(X)$ as each $g_j$ is surjective.

Since $f = h \circ g$ is an inclusion it follows that $h|M$ is an embedding of $M = \prod_{j \in J(X)} P_j \times pr_{J(X)^c}(M)$ in the product $Y_1 \times \cdots \times Y_n$. It follows that $h_j : P_j \to Y_j$
is an embedding for each \( j \in J(X) \) and also \( h_{J(X)}|p_{J(X)}(M) \) is an embedding of \( p_{J(X)}(M) \) into \( Y_{J(X)}c \), where \( h_{J(X)} = \prod_{i \in J(X)c} h_i \). Observe that \( h_j(p_j) = f_j(X) \) for each \( j \in J(X) \) and \( h_j(p_j)(M) = f_j(X)(M) \), where \( f_j(X) = \prod_{i \in J(X)c} f_i \). It follows that \( X = f_1(X) \times \cdots \times f_n(X) \) if \( k = 0 \) and that \( X = (\prod_{j \in J(X)} f_j(X)) \times f_j(X)(X) \) if \( 0 < k < n \). Also note that \( f_j(X) \subset Y_j \) is a circle for each \( j \in J(X) \) and that \( f_j(X)(X) \subset Y_{J(X)c} \).

If \( 0 < k < n \), by Lemma 5D.4 and Property (b), \( p_{J(X)c}(M) \) is a quasi \((n - |J(X)|)-\)manifold. It follows that \( f_{J(X)c}(M) \) is a quasi \((n - |J(X)|)-\)manifold too, as the image of \( p_{J(X)c}(M) \) by the embedding \( h_{J(X)c}|p_{J(X)c}(M) \). This ends the proof of the theorem. □

Applying the Künneth formula we infer the following

**Corollary 5D.6.** No product \( T^{n-k} \times M^k \), where \( 2 \leq k \leq n \) and \( M^k \) is a closed \( k \)-manifold with \( H^1(M^k) = 0 \), can be embedded in a product of \( n \) curves. □

**Remark.** This corollary implies the Borsuk theorem [B3].

The "Bing house" and the "dunce hat" are contractible quasi 2-manifolds. Hence the first cohomology group of both examples is trivial. Thus the theorem implies that

**Corollary 5D.7.** Both the "Bing house" and the "dunce hat" are 2-dimensional compact contractible polyhedra, and neither can be embedded in a product of two curves. □

**Note.** This corollary shows that the number of factors in the Nagata embedding theorem cannot be lessened to \( n \), even for contractible \( n \)-dimensional polyhedra. □

**Corollary 5D.8.** Let \( X \) be an \( n \)-manifold (in general, let \( X \) be a locally connected quasi \( n \)-manifold which is also a pseudo \( n \)-manifold), \( n \geq 1 \), lying in the product \( Y_1 \times \cdots \times Y_n \) of \( n \) curves. If rank \( H_1(X) \leq n + 1 \) then \( X = S_1 \times \cdots \times S_n \), where each \( S_j \) is a circle in \( Y_j \).

**Proof.** By Theorem 5D.5, there is a set \( J \subset \{1, \cdots, n\} \) composed of \( n-1 \) elements such that

\[
(*) \quad X = (\prod_{j \in J} S_j) \times X',
\]

where each \( S_j \) is a circle in \( Y_j \) and \( X' \) is a quasi 1-manifold in \( Y_i \), where \( i \) is the element of the set \( \{1, \cdots, n\} \setminus J \). Since \( X' \) is locally connected and rank \( H_1(X') \leq 2 \) it follows that \( X' \) is a graph with no endpoint. As \( X \) is a pseudo \( n \)-manifold, by \((*)\) it follows that \( X' \) contains no triod, hence it is a circle. This completes the proof.

5E. Contractible 2-dimensional polyhedra in products of two graphs

In this section we prove a result which in a particular case gives a noteworthy property of 2-dimensional polyhedra acyclic in dimension 1 and embeddable in products of two curves. As neither the "Bing house" nor the "dunce hat" have
this property, we get another argument for non-embeddability of those examples in products of two curves.

**Theorem 5E.1.** Let $|K|$ be a 2-dimensional connected polyhedron, where $K$ is a regular CW complex. If $|K|$ can be embedded in a product of two curves and rank $H_1(|K|) \leq 2$, then $K$ collapses to either a point, or a quasi 1-manifold, or a torus. In particular, $K$ is collapsible if rank $H_1(|K|) = 0$.

**Remark.** Also this theorem implies the Borsuk theorem [B3].

**Proof.** Let $\kappa = \{K = K_0 \setminus K_1 \setminus \cdots \setminus K_n\}$ be a maximal sequence of subcomplexes of $K$ such that each successive complex is obtained from the preceding one by an elementary collapsing. It follows that

\[ (*) \quad K_n \text{ is connected and } H_1(|K_n|) = H_1(|K|). \]

Suppose $|K_n|$ is 1-dimensional. Then, since $\kappa$ is maximal, $K_n$ has no endpoint. Hence $|K_n|$ is a quasi 1-manifold.

Now suppose $|K_n|$ is 2-dimensional. By $X$ we denote the union of all 2-cells of $K_n$. Then $X = |K'_n|$, where $K'_n$ is a subcomplex of $K_n$. Since $\kappa$ is maximal, each 1-cell of $K'_n$ is a face of at least two different 2-cells of $K'_n$. Thus each component of $X$ is a ramified pseudo 2-manifold. Since $|K_n| \setminus X$ is 1-dimensional, by exactness of the homology sequence of $(|K_n|, X)$, it follows that $H_1(X \hookrightarrow |K_n|)$ is a monomorphism. Consequently, rank $H_1(X) \leq 2$. Thus, by Theorem 5D.5, $X$ is homeomorphic to a torus. To complete the proof it suffices to show that $X = |K_n|$. Suppose, to the contrary, that $X$ is a proper subset of $|K_n|$. Let $C$ be a component of the closure (in $|K_n|$) of $|K_n| \setminus X$. Note that $C$ is a 1-dimensional connected subpolyhedron of $|K_n|$ intersecting $X$ in a finite (nonzero) number of points. Since $\kappa$ is maximal each endpoint of $C$ belongs to $X$. Observe that if $C$ has exactly one endpoint then it contains a circle $S$, and if $C$ has at least two endpoints then it contains an arc $L$ with end points in $X$. Note that, in the first case rank $H_1(X \cup S) = 3$ and in the second case rank $H_1(X \cup L) = 3$. It follows that rank $H_1(|K_n|) \geq 3$, contrary to our assumption. Thus in this case $|K_n|$ is homeomorphic to a torus.

Note that rank $H_1(|K_n|) \geq 1$ if $|K_n|$ is a quasi 1-manifold and rank $H_1(|K_n|) = 2$ if $|K_n|$ is homeomorphic to a torus. It follows that $|K_n|$ is a point if rank $H_1(|K_n|) = 0$. This ends the proof. \( \square \)

Theorem 5E.3 below is a partial inverse of Theorem 5E.1. In the proof of 5E.3 we need the following

**Lemma 5E.2.** Let $K_1$ and $K_2$ be regular 1-dimensional CW complexes, and let $A$ be an oriented arc in $|K_1| \times |K_2|$ which is a union of 1-cells of $K_1 \sqcap K_2$. Then there exist regular 1-dimensional CW complexes $K'_1 \supset K_1$ and $K'_2 \supset K_2$, and a disc $D \subset |K'_1| \times |K'_2|$, such that

1. Each component of $|K'_1| \setminus |K_1|$ is a 1-cell of $K'_1$ with one endpoint removed,
2. $D$ is a union of 2-cells of $K'_1 \sqcap K'_2$ and $D \cap (|K_1| \times |K_2|) = A$.  

Proof. Let $p$ denote the initial point of $A$. Without loss of generality we may assume that $A$ is the union of arcs $A_1, \ldots , A_n$, $n \geq 1$, such that: $p \in A_1$, $A_j \subset pr_1^{-1}(v_j)$ for $j$ odd, where $v_j$ is a vertex of $K_1$, $A_j \subset pr_2^{-1}(w_j)$ for $j$ even, where $w_j$ is a vertex of $K_2$, $A_j$ meets $A_{j+1}$ in a common endpoint for each $j$, and no other arcs intersect. To get $K_1'$ we enlarge $K_1$ by 1-cells $v_j'v_j$ for $j$ odd with all $v_j' \notin K_1$ different. Similarly, to get $K_2'$ we enlarge $K_2$ by 1-cells $w_j'w_j$ for $j$ even with all $w_j' \notin K_2$ different. Hence (i) holds. Define $D$ to be the union of the following discs:

- $v_j'v_j \times pr_2(A_j)$ for $j$ odd,
- $pr_1(A_j) \times w_j'w_j$ for $j$ even,
- $v_j'v_j \times w_{j+1}'w_{j+1}$ for $j$ odd, and
- $v_{j+1}'v_{j+1} \times w_j'w_j$ for $j$ even.

One can easily verify that property (ii) holds as well. □

**Theorem 5E.3.** Let $K$ be a regular 2-dimensional CW complex. If $K$ is collapsible then $|K|$ is embeddable in a product of two trees.

**Proof.** We shall prove a stronger version of this theorem:

(0) there exist trees $|K_1|$, $|K_2|$ and an embedding $h : |K| \rightarrow |K_1| \times |K_2|$ such that $h(\sigma)$ is a union of cells of $K_1 \sqcup K_2$ for each cell $\sigma \in K$.

Let

$$|K| = |L_n| \setminus \cdots \setminus |L_0| = \{\ast\}$$

be a sequence of elementary collapses of $|K|$ to a point $\ast$. The proof of (0) will be completed ones we show that (0) holds for each $L_m$, $m = 0, \ldots , n$, in place of $K$. This in turn will be proved by induction on $m$. Evidently, if $m = 0$ the assertion (0) is true. Now assume (0) holds for $m-1 \geq 0$. We prove it for $m$. By our assumption (0) holds for $L_{m-1}$. Hence there exist an embedding $h' : |L_{m-1}| \rightarrow |K_1| \times |K_2|$ as in (0). Since $|L_{m}| \setminus |L_{m-1}|$ is an elementary collapsing, $|L_{m}|$ is a union of $|L_{m-1}|$ and $\tau$, where $\tau$ is either a 1-cell or a 2-cell of $L_{m}$. If $\tau$ is a 1-cell then $|L_{m-1}| \cap \tau$ is a vertex $u_0 \in L_{m-1}$. Then $h'(u_0) = (v_1, v_2)$, where $v_i$ is a vertex of $K_i$. If $\tau$ is a 2-cell then $|L_{m-1}| \cap \tau$ is an arc $A'$ which is a union of 1-cells of $L_{m-1}$, so the arc $A = h'(A')$ is a union of 1-cells of $K_1 \sqcup K_2$. One easily sees that in the first case $|L_{m}|$ embeds in $|K_1| \times |K_2|$ as in (0), where $K_2' = (K_2, v_2) \vee (I, 0)$ (the one-point union). In the second case condition (0) follows from Lemma 5E.2. This ends the proof. □

Final results of this section are devoted to embeddability of cones over polyhedra into products.

**Theorem 5E.4.** Let $P$ be a $(k+l+1)$-dimensional polyhedron, where $k, l \geq 0$. Then there exist polyhedra $P'$ and $P''$ with dim $P' = k$ and dim $P'' = l$ such that the cone over $P$ can be embedded in the product of cones over $P'$ and $P''$.

The above proposition is a consequence of the following two lemmas.
Lemma 5E.5. Let $P$, $k$ and $l$ be as 5E.4. Then there exist polyhedra $P'$ and $P''$ with $\text{dim} P' = k$ and $\text{dim} P'' = l$ such that $P$ can be embedded in the join $P' \ast P''$.

Proof. Let $P = |K|$, where $K$ is a simplicial complex. Put $P' = |K^{(k)}|$, where $K^{(k)}$ is the $k$-skeleton of $K$. Define $P''$ to be the dual to $P'$ in $P$, i.e. $P''$ is the union of all simplices of the barycentric subdivision of $K$ which are disjoint with $P'$. Then $\text{dim} P' = k$ and $\text{dim} P'' = l$. Observe that $P$ is pl isomorphic to a subpolyhedron of the join $P' \ast P''$. This follows from the fact that for any simplex $\sigma \in K$ with $\text{dim} \sigma \geq k$ we have $\sigma = \sigma' \ast \sigma''$, where $\sigma'$ is the $k$-skeleton of $\sigma$ (with respect to the standard simplicial structure on $\sigma$) and $\sigma''$ is the dual to $\sigma'$ in $\sigma$. □

Lemma 5E.6. Let $P'$ and $P''$ be polyhedra. Then the product of cones over $P'$ and $P''$ is homeomorphic to the cone over the join $P' \ast P''$.

Proof. For a polyhedron $Q$ let $aQ$ denote the cone with vertex $a$ and base $Q$. According to the definition of link (cf. [R-S, p. 2]), $Q$ may be considered as a link of the vertex $a$ in $aQ$. The conclusion of our lemma is the following special case of a known formula (see [R-S, p. 24, Ex. (3)]) for the link of a point in the product of two polyhedra: $\text{lk}((a',a''),a'P' \times a''P'') \cong P' \ast P''$. □

Corollary 5E.7. The cone over any $n$-dimensional polyhedron can be embedded in the product of $n + 1$ copies of an $m$-od.

Proof. We can prove this result by induction on the dimension $n$ of the polyhedron. We start the induction with 0-dimensional polyhedra, in which case the proof is obvious. The inductive step is proven applying Theorem 5E.4 for $k = 0$ and $l = n - 1$. □

Note that neither Lemma 5E.5 nor Corollary 5E.7 (and thus Theorem 5E.4) can be extended to a more general case of continua in place of polyhedra. For example the Menger curve can not be embedded in the join of two 0-dimensional compacta. Neither the cone over the Menger curve is embeddable in the product of two cones over 0-dimensional compacta.

Problems to Chapter 5

Problem 5A.1. Is it possible to characterize a polyhedron $|K|$ which is a quasi $n$-manifold in terms of the complex $K$ itself?

Problem 5B.1. Let $X$ be a locally connected quasi $n$-manifold, $n \geq 2$, with $H^1(X)$ of finite rank, and let $(f_1, \cdots, f_n) : X \rightarrow Y_1 \times \cdots \times Y_n$ be an embedding in a product of curves. Is it possible to approximate mappings $f_i$ by mappings $f_i' : X \rightarrow Y_i$ so that $(f_1', \cdots, f_n')$ is still an embedding and each $f_i'(X)$ is a graph?

Problem 5D.1. Characterize quasi $n$-manifolds embeddable in products of $n$ graphs.

Problem 5D.2. Let $X$ be a locally connected pseudo $n$-manifold lying in a product of $n$ curves. Is $X$ a locally connected quasi $n$-manifold?

Problem 5D.3. Characterize ordinary closed 3-manifolds. Must such a manifold be a product of non-degenerate factors?
6. Embedding surfaces into products of two curves

This chapter splits into sections 6A - 6G. In section 6A we present a short and elementary proof of the Borsuk theorem on non-embeddability of $S^2$ in products of two curves. In section 6B we establish some general properties of closed surfaces lying in products of two graphs. In section 6C we show that any orientable surface of genus $g$ can be monotonically embedded in the product of two $\theta_{g+1}$-curves (see section 6B for definitions). The main objective of the entire chapter is discussed in section 6D devoted to certain results of R. Cauty [C] concerning embeddability of non-orientable surfaces in products of curves. In section 6E we prove that among closed surfaces lying in a product of two curves the torus is the only one which is a retract of that product. In section 6F we show that any surface with non-empty boundary can be embedded in the "three-page book". In section 6G we consider embeddability in the second symmetric product of curves.

6A. Another proof of the Borsuk theorem

Suppose there is an embedding $f : S^2 \to Y_1 \times Y_2$, where $Y_1$ and $Y_2$ are curves. Then $f = (f_1, f_2)$, where $f_i : S^2 \to Y_i, i = 1, 2$. By the Whyburn factorization theorem, there exist a continuous monotone surjection $g_i : S^2 \to X_i$ and a 0-dimensional mapping $h_i : X_i \to Y_i$ such that $f_i = h_i \circ g_i$. As $g_i$ is continuous and monotone, $X_i$ is a locally connected continuum contractible with respect to $S^1$ (equivalently: $H^1(X_i) = 0$) [Kur, §57, I, Theorem 2, p. 434]. As $h_i$ is 0-dimensional, $X_i$ is 1-dimensional by the Hurewicz theorem [Kur, §45, VI, Theorem 1, p. 114]. It follows that $X_i$ is an absolute retract (a dendrite) (see e.g. [Kur, §57, III, Corollary 8, p. 442 and §53, IV, Theorem 16, p. 344]). Then we can continue in two different ways:

(a) Since $\dim(Y_1 \times Y_2) = 2$, there is an extension $\varphi : Y_1 \times Y_2 \to S^2$ of the inverse $f^{-1} : f(S^2) \to S^2$ (see e.g. [Kur, §53, VI, Theorem 1, p. 354]). Note that $f$ is homotopic to a constant mapping, because each $f_i$ is (as $X_i$ is contractible). It follows that $\varphi \circ f : S^2 \to S^2$ is homotopic to a constant. But $\varphi \circ f = 1_{S^2}$, a contradiction.

(b) Note that $X_1 \times X_2$ is a 2-dimensional absolute retract. On the other hand, one easily sees that $(g_1, g_2) : S^2 \to X_1 \times X_2$ is an embedding. Thus $S = (g_1, g_2)(S^2)$ is a topological 2-sphere in $X_1 \times X_2$. It follows that $S$ is a retract of $X_1 \times X_2$ (see e.g. [Kur, §53, VI, Theorem 1, p. 354]), hence $S$ is an absolute retract as well, a contradiction. $\square$

6B. On surfaces lying in products of two graphs

This section is devoted to general properties of closed surfaces lying in products of two graphs. We apply the notation and conventions of Section 5C for $n = 2$. Below we rephrase the general Properties (a)-(j) from section 5C for this special dimension. Therefore, no proof of the obtained properties will be given. (The restated properties will be designated by adding primes to the letters designating corresponding properties from Section 5C.)
Throughout this section we consider a special pseudo 2-manifold $M \subset P_1 \times P_2$, where $P_i = |K_i|$ are graphs and $K_i$ are regular 1-dimensional CW complexes.

From Theorem 5A.4 we infer that $M = |K(M)|$, where $K(M)$ is a subcomplex of $K_1 \sqcup K_2$. Moreover, $K(M)$ is a special pseudo 2-manifold complex, hence any ramified 2-manifold subcomplex of $K(M)$ coincides with $K(M)$. Let us recall that $p_i = pr_i|M$.

**Property (a’).** $M$ is the union of 2-cells of $K(M)$. □

**Property (b’).** $p_i(M) = |K'_i|$, where $K'_i$ is a subcomplex of $K_i$ with no endpoint. □

For any cell $\tau \in K'_2$ the set $P_1(\tau)$ is the union of all 1-cells $\sigma \in K_1$ such that $\sigma \times \tau \subset M$. (The set $P_2(\sigma)$ for $\sigma \in K_1$ has analogous description.) For any point $y \in p_2(M)$ the set $P_1(y) = \{x \in P_1 : (x, y) \in M\}$. Thus, $P_1(y) \times \{y\} = p_2^{-1}(y)$, and $P_1(y) \subset p_1(M)$. (The set $P_2(x)$ for $x \in p_1(M)$ has analogous description.) We list certain properties of the sets $P_1(\tau)$ and $P_1(y)$; analogous properties are valid for the other pair of sets.

**Property (c’).** $M = \bigcup\{P_1(\tau) \times \tau : \tau \in K'_2 \text{ is a 1-cell}\}$. □

**Property (d’).** If $w$ is a vertex of a 1-cell $\tau \in K'_2$ then $P_1(w) \supset P_1(\tau)$. □

**Property (e’).** For any cell $\tau \in K'_2$ and any point $y \in \tau$ we have $P_1(y) = P_1(\tau)$. If $\tau = \{v\}$ is a vertex then $P_1(\tau) = \bigcup\{P_1(\tau') : \tau' \in K'_2 \text{ is a 1-cell with vertex } v\}$. □

**Property (f’).** For any 1-cell $\tau \in K'_2$ the set $P_1(\tau)$ is a finite union of disjoint circles. □

**Property (g’).** $p_1(M) = \bigcup\{P_1(\tau) : \tau \in K'_2 \text{ is a } k\text{-cell}\}$ for $k = 0, 1$. □

**Property (h’).** If $P_1(w)$ is a circle for each vertex $w \in K'_2$, then $p_1(M) = P_1(w_0)$ for any vertex $w_0$ of $K'_2$. □

**Property (i’).** If $p_1(M)$ is a circle then $M = p_1(M) \times p_2(M)$. □

**Property (j’).** $p_1(M)$ is a circle if and only if $P_1(v)$ is a circle for each vertex $v \in K'_2$.

If both $p_i : M \to P_i$ are surjections (i.e. $p_i(M) = P_i$, or $K'_i = K_i$), then $M$ is said to be surjectively embedded in $P_1 \times P_2$.

For any two mappings $h_i : P_i \to Y_i$, where $Y_i$ is a curve, we have the following

**Property (l).** Suppose $M$ is surjectively embedded in $P_1 \times P_2$. If $h_1 \times h_2 : P_1 \times P_2 \to Y_1 \times Y_2$ is injective on $M$ and every two $P_2(\sigma), \sigma \in K_1$, intersect, then $h_1 : P_1 \to Y_1$ is an embedding.

**Proof.** For suppose $h_1(x) = h_1(x')$ for some $x, x' \in P_1$; we have to show that $x = x'$. Let $x \in \hat{\sigma}$ and $x' \in \hat{\sigma'}$ for some $\sigma, \sigma' \in K_1$. Then $P_2(x) = P_2(\sigma)$
and \( P_2(x') = P_2(\sigma') \) by Property (e'). By our assumption, there is a point \( y \in P_2(\sigma) \cap P_2(\sigma') = P_2(x) \cap P_2(x') \). Then points \((x, y), (x', y)\) are different, both belong to \( M \) and are sent to \((h_1(x), h_2(y))\) by \( h_1 \times h_2\), a contradiction. This proves Property (I). \( \square \)

Let \( \Theta_n, n \geq 1 \), denote the graph in \( S^2 \) composed of \( n \) meridians \( \mu_1^n, ..., \mu_n^n \) such that each \( \mu_j^n \) passes through \( z_j = (\cos 2\pi \frac{j}{n}, \sin 2\pi \frac{j}{n}, 0) \). (So, the meridians are equally and cyclicly spaced on \( S^2 \); if no confusion is likely to occur, we abbreviate \( \mu_j^n \) to \( \mu_j \).) The poles \( p = (0, 0, 1) \) and \(-p = (0, 0, -1)\) are common endpoints of the meridians. Hence \( \Theta_n = |K(n)| \), where \( K(n) \) denotes the regular 1-dimensional CW-complex with the meridians as 1-cells (edges) and with the poles as 0-cells (vertices). Any space homeomorphic to \( \Theta_n \) will be called a \( \theta_n \)-curve, its points corresponding to the poles of \( \Theta_n \) will be called poles of the curve. Note that a \( \theta_3 \)-curve is commonly known as the \( \theta \)-curve.

**Property (m).** Suppose \( M \) is surjectively embedded in \( P_1 \times P_2 \), where \( P_2 = \Theta_{n+1}, n \geq 1 \). Then, for any 1-cell \( \sigma \in K_1 \), the set \( P_2(\sigma) \) is a circle (composed of two meridians). Analogous property holds in case \( P_1 = \Theta_{m+1} \). \( \square \)

In case where both \( p_1 : M \to P_1 \) and \( p_2 : M \to P_2 \) are monotone, \( M \) is said to be **monotonically embedded** in \( P_1 \times P_2 \).

**6C. Surfaces in products of \( \theta_n \)-curves**

We keep notation of section 6B. First we prove that any orientable surface can be monotonically embedded in the product of two \( \theta_n \)-curves. Applying some results of the previous article we show that any two copies of an orientable surface lying in the product of two \( \theta_n \)-curves are equivalent up to a special automorphism of the entire product.

For each natural \( m \geq 1 \) define \( M^0_{2m} \) to be the subset of \( \Theta_{m+1} \times \Theta_{m+1} \) given by

\[
M^0_{2m} = \bigcup_{j=0}^{m} \mu_j^{m+1} \times (\mu_j^{m+1} \cup \mu_{j+1}^{m+1}).
\]

Here \( \mu_{m+1}^{m+1} = \mu_0^{m+1} \). One easily sees that this space can be also expressed in the "dual" form:

\[
M^0_{2m} = \bigcup_{j=0}^{m} (\mu_{j-1}^{m+1} \cup \mu_j^{m+1}) \times \mu_j^{m+1}.
\]

Here \( \mu_{m+1}^{m+1} = \mu_m^{m+1} \). Let \( p_i : M^0_{2m} \to \Theta_{m+1} \) denote the restriction to \( M^0_{2m} \) of the projection \( p_i : \Theta_{m+1} \times \Theta_{m+1} \to \Theta_{m+1} \). The mappings \( q_1 : M^0_{2m} \to \Theta_{m+1} \times \{-p\} \) and \( q_2 : M^0_{2m} \to \{-p\} \times \Theta_{m+1} \) defined by \( q_1(z_1, z_2) = (z_1, -p) \) and \( q_2(z_1, z_2) = (-p, z_2) \), for all \((z_1, z_2) \in M^0_{2m}\), are monotone retractions (equivalent to the corresponding mappings \( p_i \)) onto \( \theta_{m+1} \)-curves. And their diagonal mapping

\[
(q_1, q_2) : M^0_{2m} \to (\Theta_{m+1} \times \{-p\}) \times (\{-p\} \times \Theta_{m+1}) \approx (\Theta_{m+1} \times \Theta_{m+1})
\]
is an embedding (equivalent to the inclusion \((p_1, p_2) : M^0_{2m} \to \Theta_{m+1} \times \Theta_{m+1}\)). The properties of \(q'_i\)'s follow from corresponding properties of \(p'_i\)'s (see the proof below).

**Lemma 6C.1.** The set \(M^0_{2m}\) is a closed orientable surface of genus \(m\) monotonically embedded in \(\Theta_{m+1} \times \Theta_{m+1}\).

**Proof.** By definition it follows that both \(p_i\) are surjective. Hence the monotonicity of the embedding is a consequence of the equalities: 
\[ p^{-1}_1(x) = \{x\} \times (\mu^{m+1}_j \cup \mu^{m+1}_{j+1}) \] 
for any \(x \in \mu^{m+1}_j\), 
\[ p^{-1}_2(y) = (\mu^{m+1}_{j-1} \cup \mu^{m+1}_j) \times \{y\} \] 
for any \(y \in \mu^{m+1}_j\); 
\[ p^{-1}(v) = \{v\} \times \Theta_{m+1} \] 
for \(v, w \in \{p, -p\}\). So, to complete the proof, it remains to show that:

\[(*) M^0_{2m} \text{ is a closed orientable surface of genus } m.\]

We give two proofs of this assertion.

**First proof of \((*)\).** This proof is by induction on \(m\). Clearly, \((*)_1\) holds because \(M^0_2 = \mu^2_0 \times (\mu^2_0 \cup \mu^2_1) \cup \mu^2_1 \times (\mu^2_1 \cup \mu^2_0) = \Theta_2 \times \Theta_2\) is a torus. Then suppose \((*)_m\), \(m \geq 1\), has been proved; it remains to show that \((*)_{m+1}\) holds too. To this end, observe that \(M^0_{m+2} = \mu^{m+2}_0 \times (\mu^{m+2}_0 \cup \mu^{m+2}_1) \cup \mu^{m+2}_1 \times (\mu^{m+2}_1 \cup \mu^{m+2}_0) \cup \cdots \cup \mu^{m+2}_m \cup \mu^{m+2}_{m+1}\) is a topological copy of \(M^0_{2m}\) in \(\Theta_{m+2} \times \Theta_{m+2}\). Hence, by the inductive assumption, \(M^0\) is a closed orientable surface of genus \(m\). Let \(T\) denote the torus \(\mu^{m+2}_m \cup \mu^{m+2}_{m+1}\) \(\mu^{m+2}_0 \cup \mu^{m+2}_1\). Then note that the intersection \(M^0 \cap T\) is the disc \(\mu^{m+2}_m \times \mu^{m+2}_0\), and \(M^0_{2(m+1)} = (M^0 \cup T) \setminus \mu^{m+2}_m \times \mu^{m+2}_0\). It follows that \(M^0_{2(m+1)} = M^0 \#T\), so \(M^0_{2(m+1)}\) is an orientable surface of genus \(m+1\), which proves \((*)_{m+1}\).

**Second proof of \((*)\).** This argument appeals to the following obvious fact: If \(N\) is a 2-disc with \(m\) holes then the doubling \(M = 2N\) is an orientable closed surface of genus \(m\). Let us recall that by definition the doubling \(2N\) is the union of two copies of \(N\) glued along \(\partial N\). In other words, \(2N\) is the subset of the product \(N \times I\) defined by
\[ 2N = \partial (N \times I) = (\partial N) \times I \cup N \times \partial I.\]

So, it suffices to show that \(M^0_{2m} \simeq 2N\). Actually, we shall prove more: \(M^0_{2m}\) can be obtained as the image of a special embedding \(\varphi : 2N \to \Theta_{m+1} \times \Theta_{m+1}\).

To this end we take \(N\) quite specific. Notice that \(S^2 = D_0 \cup \cdots \cup D_m\), where each \(D_j\) stands for the 2-disc in \(S^2\) from \(\mu^{m+1}_{j-1}\) to \(\mu^{m+1}_j\) (that is, \(D_j\) consists of the points which lie on meridians passing through points \((\cos 2\pi t, \sin 2\pi t, 0)\) with \(t \in [\frac{j-1}{m+1}, \frac{j}{m+1}]\)). (To make the notation less complicated we abbreviate \(\mu^{m+1}_{j+1}\) to \(\mu_j\).) For each \(j\) choose a 2-disc \(E_j\) lying in the interior of \(D_j\), and put \(C_j = D_j \setminus \hat{E}_j\). Then each triple \((C_j, \partial D_j, \partial E_j)\) is homeomorphic to the cylinder triple \((S^1 \times \hat{I}, S^1 \times \{0\}, S^1 \times \{1\})\). Now, define \(N\) to be the set
\[ N = S^2 \setminus (\hat{E}_0 \cup \cdots \cup \hat{E}_m).\]

One easily sees that \(N = C_0 \cup \cdots \cup C_m\) and \(C_j \cap C_{j+1} = \mu_j\) for each \(j\).
On the other hand we have
\[ 2N = \bigcup_{j=0}^{m} (C_j \times \{0,1\}) \cup (\partial E_j \times I). \]

Notice that, for each \( j \), both sets \((C_j \times \{0,1\}) \cup (\partial E_j \times I)\) and \((\mu_{j-1} \cup \mu_j) \times \mu_j\) are homeomorphic to the cylinder \(S^1 \times I\). Moreover, there exist homeomorphisms \(\varphi_j : (C_j \times \{0,1\}) \cup (\partial E_j \times I) \rightarrow (\mu_{j-1} \cup \mu_j) \times \mu_j\) such that \(\varphi_j(z,0) = (z,-p)\) and \(\varphi_j(z,1) = (z,p)\) for each \( z \in \mu_{j-1} \cup \mu_j \). Combining these homeomorphisms we get the desired embedding \(\varphi\). (See Figure 6C.1 where the doubling \(2N\), with \( m = 2 \) and a special embedding in \(\mathbb{R}^3\), is depicted.) \(\square\)

Any sequence \(\nu_0, \ldots, \nu_k\) of different meridians of \(\Theta_n\) induces a sequence
\[ S_0, \ldots, S_k \]
of topological circles in \(\Theta_n\), where \(S_j = \nu_j \cup \nu_{j+1}\) for \( j = 0, \ldots, k-1 \) and \(S_k = \nu_k \cup \nu_0\). This sequence will be called a cycle of circles induced by \(\nu_0, \ldots, \nu_k\). The following assertion immediately follows from this definition and Lemma 6C.1.

**Theorem 6C.2.** Let \( M \) be any subset of the product \(\Theta_n \times \Theta_{n'}\), \( n, n' \geq 2 \), which can be expressed in the form
\[ M = \sigma_0 \times S_0 \cup \cdots \cup \sigma_m \times S_m, \]
where \(\sigma_0, \ldots, \sigma_m\) are different meridians in \(\Theta_n\), and \(S_0, \ldots, S_m\) is a cycle of circles in \(\Theta_{n'}\). Then \( M \) is a closed orientable surface of genus \( m \).

Moreover, for any embeddings \(h : \Theta_{m+1} \rightarrow \Theta_n\) and \(h' : \Theta_{m+1} \rightarrow \Theta_{n'}\) (preserving the poles) such that \(h(\mu_j^{m+1}) = \sigma_j\) and \(h'(\mu_j^{m+1}) = \tau_j\), for each \( j = 1, \ldots, m \), where \(\tau_0, \ldots, \tau_m\) induces the cycle \(S_0, \ldots, S_m\), the embedding \(h \times h' : \Theta_{m+1} \times \Theta_{m+1} \rightarrow \Theta_n \times \Theta_{n'}\) takes \(M_{2m}^0\) onto \( M \). \(\square\)

**Corollary 6C.3.** For each \( m \geq 1 \) the product \(\Theta_{m+1} \times \Theta_{m+1}\) contains a closed orientable surface \( M \) of genus \( m \) in such a position that it is invariant under the canonical involution on that product. Moreover, \( M \) meets the diagonal of \(\Theta_{m+1} \times \Theta_{m+1}\) along a circle which does not separate \( M \).

**Proof.** We shall show that the subset \( M \) of \(\Theta_{m+1} \times \Theta_{m+1}\) given by the formula (here we write \(\mu_j\) instead of \(\mu_j^{m+1}\))
\[ M = (\mu_0 \times \mu_0) \cup (\bigcup_{i=0}^{m-1} (\mu_i \times \mu_{i+1} \cup \mu_{i+1} \times \mu_i)) \cup (\mu_m \times \mu_m), \]
has the desired properties. Obviously, \( M \) is symmetric under the canonical involution on \(\Theta_{m+1} \times \Theta_{m+1}\). Moreover, \( M \) meets the diagonal of \(\Theta_{m+1} \times \Theta_{m+1}\) along
the circle $S = D_0 \cup D_m$, where $D_j$ denotes the diagonal of $\mu_j \times \mu_j$. One can easily verify that $M \setminus S$ is connected (the vertices $(-p, p)$ and $(p, -p)$ can be connected by an arc lying in $\mu_0 \times \mu_1 \setminus S$ and any other point of $M \setminus S$ can be connected to one of these points by an arc lying in $M \setminus S$). Thus, by Theorem 6C.2, it remains to show that $M$ can be expressed in the form

$$M = \sigma_0 \times (\tau_0 \cup \tau_1) \cup \cdots \cup \sigma_m \times (\tau_m \cup \tau_0),$$

where $\sigma_0, \cdots, \sigma_m$ and $\tau_0, \cdots, \tau_m$ are sequences of different meridians in $\Theta_{m+1}$. The constructions of the sequences $(\sigma)$ and $(\tau)$ depend on parity of $m$.

If $m$ is even then we put (instead of $\mu_{i+1}$'s we write just their subscripts)

$$(\sigma) = (0, 1, 3, \cdots, m - 1, m, m - 2, m - 4, \cdots, 4, 2),$$

$$(\tau) = (1, 0, 2, 4, \cdots, m - 2, m, m - 1, m - 3, \cdots, 5, 3).$$

If $m$ is odd then we put

$$(\sigma) = (0, 1, 3, \cdots, m - 2, m, m - 1, m - 3, \cdots, 4, 2),$$

$$(\tau) = (1, 0, 2, 4, \cdots, m - 1, m - 2, m - 4, \cdots, 5, 3).$$

One can verify that these sequences have the desired properties. $\square$

**Theorem 6C.4.** Let $M$ be a special pseudo 2-manifold with rank $H_1(M) = 2m$ lying in the product $\Theta_n \times \Theta_{n'}$, $n, n' \geq 2$. Then $M$ can be expressed in the form

$$M = \sigma_0 \times P_2(\sigma_0) \cup \cdots \cup \sigma_m \times P_2(\sigma_m),$$

where $\sigma_0, \cdots, \sigma_m$ is a sequence of different meridians in $\Theta_n$, and the sets $P_2(\sigma_0), \cdots, P_2(\sigma_m)$ form a cycle of circles in $\Theta_{n'}$. In particular, $M$ is an orientable closed surface of genus $m$.

**Proof.** Consider a sequence $\sigma_0 \cdots \sigma_k$ of length $k + 1$, with $k \geq 0$, such that

1. $\sigma_0 \cdots \sigma_k$ are different 1-cells of $K'(n)$,
2. $P_2(\sigma_j) = \tau_j \cup \tau_{j+1}$ for each $j = 0, \cdots, k$,
3. $\tau_0 \cdots \tau_k$ are different 1-cells of $K'(n')$.

(Such sequences exist: the one-element sequence $\sigma_0$, for any 1-cell $\sigma_0$ of $K'(n)$, has these properties. In fact, by Property (f) the set $P_2(\sigma_0)$ is a circle. Hence $P_2(\sigma_0) = \tau_0 \cup \tau_1$ and $\tau_0 \neq \tau_1$.) We investigate the union

$$M[k] = \bigcup_{j=0}^k \sigma_j \times (\tau_j \cup \tau_{j+1}).$$

(The 2-cells of $K(M)$ lying in $M[k]$ make the sequence

$$\sigma_0 \times \tau_0, \sigma_0 \times \tau_1, \sigma_1 \times \tau_1, \cdots, \sigma_k \times \tau_k, \sigma_k \times \tau_{k+1}$$
such that each cell borders upon the next one along in turn
\[ \sigma_0 \times \{-p, p\}, \{-p, p\} \times \tau_1, \ldots, \sigma_k \times \{-p, p\}. \]

In subsequent discussion we take for \( k \) the maximal number for which the conditions (1)-(3) hold. Then we have
\[ (4) \; \tau_{k+1} = \tau_0. \]

For suppose \( \tau_{k+1} \neq \tau_0 \). Then \( \tau_{k+1} \neq \tau_j \) for each \( j = 0, \ldots, k \). In fact, else \( \tau_{k+1} = \tau_j \) for some \( 0 < j < k \). Thus \( P_1(\tau_j) = \sigma_{j-1} \cup \sigma_j \cup \sigma_k \). So, by (1), \( P_1(\tau_j) \) is not a circle, contrary to Property(m). Next, let \( P_1(\tau_{k+1}) = \sigma_k \cup \sigma_{k+1} \). Then \( \sigma_{k+1} \neq \sigma_j \) for each \( j \). Else \( \sigma_{k+1} = \sigma_j \) for some \( 0 \leq j < k \). Consequently, \( P_2(\sigma_j) = \tau_j \cup \tau_{j+1} \cup \tau_{k+1} \) is not a circle, because \( \tau_0, \ldots, \tau_{k+1} \) are all different. This again contradicts Property (m). Combining these properties, one sees that \( \sigma_0, \ldots, \sigma_{k+1} \) is a sequence of length \( k + 2 \) satisfying (1)-(3), contrary to maximality of \( k \).

It follows from (1)-(4) that
\[ M[k] = \sigma_0 \times P_2(\sigma_0) \cup \cdots \cup \sigma_k \times P_2(\sigma_k) \]
is a subset of \( \Theta_n \times \Theta_{n'} \) of the form which has been discussed in Theorem 6C.2. Consequently, \( M[k] \) is a closed orientable surface of genus \( k \) contained in \( M \). This implies \( M[k] = M \). In addition, \( k = m \) because \( 2m = \text{rank} \; H_1(M) = \text{rank} \; H_1(M[k]) = 2k \), which completes the proof. \( \square \)

The following corollary is a special case of the above theorem.

**Corollary 6C.5.** Let \( M \) be a closed surface of genus \( m \) lying in the product \( \Theta_n \times \Theta_{n'}, n, n' \geq 2 \). Then \( M \) can be written in the form
\[ M = \sigma_0 \times P_2(\sigma_0) \cup \cdots \cup \sigma_m \times P_2(\sigma_m), \]
where \( \sigma_0, \ldots, \sigma_m \) is a sequence of different meridians in \( \Theta_n \), and \( P_2(\sigma_0), \ldots, P_2(\sigma_m) \) is a cycle of circles in \( \Theta_{n'} \). In particular, \( M \) is orientable. \( \square \)

Two final assertions directly follow from the above discussion.

**Corollary 6C.6.** Let \( M \) be a closed surface of genus \( m \) lying in the product \( \Theta_n \times \Theta_{n'}, n, n' \geq 2 \). Then both \( p_1(M) \) and \( p_2(M) \) are \( \Theta_{m+1} \)-curves and \( M \) is monotonically embedded in \( p_1(M) \times p_2(M) \). \( \square \)

**Corollary 6C.7.** Let \( M_1 \) and \( M_2 \) be two copies of a closed surface lying in the product \( \Theta_n \times \Theta_{n'}, n, n' \geq 2 \). Then there are homeomorphisms \( h : \Theta_n \to \Theta_n \), and \( h' : \Theta_{n'} \to \Theta_{n'} \) such that \( M_2 = (h \times h')(M_1) \). \( \square \)

6D. On Cauty’s results about embeddability of non-orientable surfaces into products of graphs

In this section we mostly comment on some beautiful results of Cauty [C1] concerning embeddability as in the title. In particular, we give slightly different and hopefully simpler descriptions of the embeddings occurring in the Cauty proof. The main results are essentially due to Cauty.
Theorem 6D.1. (cf. [C1]) Let $M$ be a special pseudo 2-manifold in the product $Y_1 \times Y_2$ of two curves. Then

(i) $\text{rank } H_1(M) \leq 3$ implies $M = P_1 \times P_2$, where each $P_i$ is a circle in $Y_i$;

(ii) $\text{rank } H_1(M) = 4$ implies $M \subset P_1 \times P_2$, where each $P_i$ is a $\theta$-curve in $Y_i$.

(Hence $M \approx \mathbb{T}^2 \# \mathbb{T}^2$ and $M$ is monotonically embedded in $P_1 \times P_2$).

Remark. Note that (i) implies the Borsuk theorem [Bo3].

In order to prove Theorem 6D.1 we need the following three lemmas. The first one will be used in the proof of Lemma 6D.4.

Lemma 6D.2. Let $S_j$ be a closed subset of a graph $P_j$, $j = 1, 2$, and let $S_0$ be either the empty set or a circle in $P_1$ such that $S_0 \setminus S_1 \neq \emptyset$. Suppose $p_j : M \to P_j$ and $q_i : S_i \to M$, $i = 0, 1, 2$, are mappings satisfying the conditions:

(i) $H_1(p_1 \circ q_2) = 0$,

(ii) $p_j \circ q_j : S_j \to P_j$, and $p_1 \circ q_0 : S_0 \to P_1$ are the inclusions.

Then there is a monomorphism

$$\varphi : H_1(S_0) \oplus H_1(S_1) \oplus H_1(S_2) \to H_1(M).$$

Proof. We will prove that the homomorphism $\varphi$ defined by

$$\varphi(x_0, x_1, x_2) = H_1(q_0)(x_0) + H_1(q_1)(x_1) + H_1(q_2)(x_2)$$

is a monomorphism.

Since $H_2(P_j, S_j) = 0$, by (ii) and the exactness of the homology sequence of $(P_j, S_j)$ we have

(1) $H_1(p_j \circ q_j)$ is a monomorphism.

First, assume that $S_0 = \emptyset$. Note that $H_1(S_0)$ is trivial. Thus, by (i) and (1), Lemma 4D.1 implies that $\varphi$ is a monomorphism.

Now, assume $S_0$ is a circle in $P_1$ such that $S_0 \setminus S_1 \neq \emptyset$. Then there is a retraction $r : P_1 \to S_0$ such that $r(S_1)$ is a proper subset of $S_0$. Define $p_0 = r \circ p_1$. Note that

(1') $H_1(p_0 \circ q_0)$ is a monomorphism.

Observe that $p_0 \circ q_1 = r \circ p_1 \circ q_1 : S_1 \to S_0$ is homotopic to a constant map as $(p_1 \circ q_1)(S_1) = S_1$ and $r(S_1) \subset S_0$. Thus $H_1(p_0 \circ q_1) = 0$. Also, $H_1(p_0 \circ q_2) = H_1(r \circ p_1 \circ q_2) = 0$, by (i). Thus we have

(2) $H_1(p_i \circ q_j) = 0$ if $0 \leq i < j \leq 2$.

Hence, by (1), (1') and (2), Lemma 4D.1 implies that $\varphi$ is a monomorphism in this case as well. \qed

In the following two lemmas 6D.3 and 6D.4 we consider two graphs $P_1 = |K_1|$ and $P_2 = |K_2|$, and a special pseudo 2-manifold $M \subset P_1 \times P_2$ as in 6B. We keep the notation of section 6B.
Lemma 6D.3. Suppose $M$ is monotonically embedded in $P_1 \times P_2$. If either $P_1$ or $P_2$ is the one-point union of finitely many circles, then both $P_1$ and $P_2$ are circles and $M = P_1 \times P_2$.

Proof. It is enough to consider the case where $P_1$ is the one-point union of circles $S_1, \ldots, S_k$, where $k \geq 1$. There is a 1-cell $\sigma \in K_1$ lying in $S_1$. Then for any 1-cell $\tau \in K_2$ laying in $P_2(\sigma)$ (this set is well defined because $p_1$ is surjective) we have $P_1(\tau) = S_1$. (In fact, as $p_2$ is monotone, by Properties (e') and (f') in 6B, we infer that $P_1(\tau)$ is a circle. Hence $P_1(\tau) = S_1$, because $\sigma \subset P_1(\tau)$.)

Therefore $S_1 \times P_2(\sigma) \subset M$. Again, by an argument as above, $P_2(\sigma)$ is a circle. Hence $M = S_1 \times P_2(\sigma)$. Consequently, $P_1 = S_1$, $P_2 = P_2(\sigma)$ and $M = P_1 \times P_2$, which completes the proof. $\square$

Lemma 6D.4. Suppose $M$ is monotonically embedded in $P_1 \times P_2$. Then we have:

(i) If $\text{rank } H_1(M) \leq 3$ then both $P_1$, $P_2$ are circles and $M = P_1 \times P_2$.

(ii) If $\text{rank } H_1(M) = 4$ then both $P_1$, $P_2$ are $\theta$-curves.

Proof. Let $v$ be a vertex of $K_1(= K_1')$ such that $\text{rank } H_1(P_2(v)) \geq \text{rank } H_1(P_2(v'))$ for each vertex $v' \in K_1$, and let $w$ be a vertex of $K_2(= K_2')$ such that $\text{rank } H_1(P_1(w)) \geq \text{rank } H_1(P_1(w'))$ for each vertex $w' \in K_2$. Put $S_1 = P_1(w)$ and $S_2 = P_2(v)$.

First, we shall show that

(1) if either $\text{rank } H_1(S_1) = 1$ or $\text{rank } H_1(S_2) = 1$ then both $P_1$ and $P_2$ are circles and $M = P_1 \times P_2$.

Suppose $\text{rank } H_1(S_1) = 1$. Then $\text{rank } H_1(P_1(w')) = 1$, so $P_1(w')$ is a circle, for each vertex $w' \in K_2'$. By Property (h') in 6B, $P_1 = p_1(M)$ is the circle $S^1$. Thus, by Lemma 6D.3, the consequent of implication (1) holds. By a similar argument, the consequent holds as well if $\text{rank } H_1(S_2) = 1$.

Now, we will shall show that

(2) if $\text{rank } H_1(S_1) \geq 2$ and $\text{rank } H_1(S_2) \geq 2$ then $\text{rank } H_1(M) \geq 4$.

We intend to apply Lemma 6D.2. Define $q_i : S_i \to M$ by $q_1(x) = (x, w)$, $q_2(y) = (v, y)$. With this notation, and for $S = \emptyset$, the hypotheses of Lemma 6D.2 are fulfilled. Thus we infer that

(3) $\text{rank } H_1(M) \geq \text{rank } H_1(S_1) + \text{rank } H_1(S_2)$.

This implies (2). Observe that (2) and (1) imply (i).

Now assume $\text{rank } H_1(M) = 4$. Then, by (1), $\text{rank } H_1(S_i) \geq 2$ for each $i$. Consequently, by (3), $\text{rank } H_1(S_1) = \text{rank } H_1(S_2) = 2$. It follows that each $S_i$ is a union of two different circles which intersect in a connected set. We shall show that $P_1 = S_1$.

By Property (g') in 6B, it suffices to show that $P_1(\tau) \subset S_1$ for each 1-cell $\tau \in K_2$. Suppose, to the contrary, $P_1(\tau) \not\subset S_1$ for a 1-cell $\tau$. By our assumptions and Property (f'), $S_0 = P_1(\tau)$ is a circle. Define $q_0 : S_0 \to M$ by $q(x) = (x, y_0)$, where $y_0 \in \tau$ is any point. With these new data, the hypotheses of Lemma 6D.2 are
fulfilled again. Hence, by this lemma, rank $H_1(M) \geq 5$, contrary to rank $H_1(M) = 4$. So, indeed $P_1 = S_1$.

Thus $P_1$ is a union of two circles which intersect in a connected set. As $P_1$ is connected, the intersection is an arc or a point. The later case is excluded by Lemma 6D.3. So $P_1$ is a $\theta$-curve.

By similar argument $P_2$ is a $\theta$-curve as well. This ends the proof of (ii), which ends the proof of our lemma. □

Proof of Theorem 6D.1

Suppose a special pseudo 2-manifold $M$ is a subset of the product $Y_1 \times Y_2$ of two curves. By Theorem 4B.1 we can assume that $M$ is also monotonically embedded in the product $P_1 \times P_2$ of two graphs. Thus, the assumptions about $P_1$, $P_2$ and $M$ from subsection 6B are fulfilled, and the hypothesis of 6D.4 is fulfilled as well.

From the algebraic assumptions about $H_1(M)$ and Lemma 6D.4, it follows that both $P_i$ are either circles or $\theta$-curves. Therefore, by Property (f’), any two sets $P_2(\sigma), \sigma \in K_1$, and any two sets $P_1(\tau), \tau \in K_2$, intersect. By the same Theorem 4B.1, there exist two mappings $h_1 : P_1 \rightarrow Y_1$ and $h_2 : P_2 \rightarrow Y_2$ such that their product mapping $h_1 \times h_2 : P_1 \times P_2 \rightarrow Y_1 \times Y_2$ coincides with the identity on $M$. Thus, by Property (l) from 6B, we infer that both mappings $h_1 : P_1 \rightarrow Y_1$ and $h_2 : P_2 \rightarrow Y_2$ are embeddings. Therefore we can think of $P_i$ as a subset of $Y_i$, which completes the proof. □

Theorem 6D.5. (R. Cauty) Any closed surface whose rank of $H_1$ is \(\geq 5\) can be embedded in a product of two graphs.

Note 1. Combining Theorems 6D.1 and 6D.5 with other known results we can summarize the final status of the problem of embeddability of 2-manifolds into products of two curves in the following way.

For closed surfaces we have: Any surface which can be embedded in the product of two curves can be also embedded in the product of two graphs (see Corollary 5B.2). Among orientable surfaces the sphere $S^2$ is the only one not embeddable in any product of two graphs (see [Bo3] and [Ku], cf. 6A, the Remarks following 5D.6, 6D.1, and Theorem 6C.1). Any non-orientable surface, except the following five: the projective plane $\mathbb{P}^2$, the Klein bottle $K = \mathbb{P}^2 \# \mathbb{P}^2$, $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$, $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$ and $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$, can be embedded in a product of two graphs (see [C1]). (The above sequence of surfaces could have been written down in a more concise form using the well known equivalence $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \approx \mathbb{T}^2 \# \mathbb{P}^2$.) The rank of the 1st homology of these non-orientable surfaces is 0, 1, 2, 3, 4, respectively. Therefore, there are exactly six closed surfaces not embeddable in any product of two curves: one orientable - $S^2$ - with genus 0, and five non-orientable with genus 1, 2, 3, 4 and 5. Thus we have exactly six exceptional closed surfaces.

For surfaces with non-empty boundary we show in section 6F that each of them can be embedded in the product of the simple triod $T$ and the unit interval $I$ (the
product is often called the "three-page book". (This result is probably known to a number of topologists.)

**Note 2.** For each $n \geq 3$ there exist exceptional closed orientable $n$-manifolds different from $S^n$ (see Corollary 5D.6). □

We are going to prove Theorem 6D.5 by an argument somewhat different from the original one devised by R. Cauty. Let $M$ be a closed 2-manifold with rank $H_1(M) \geq 5$, we have to show that $M$ can be embedded in the product of two graphs.

The original proof is based on the following idea: if $M_1$ and $M_2$ are two closed surfaces and each can be embedded in a product of two graphs, then the same is true of the connected sum $M_1 \# M_2$ ([C1], [Ku]). Moreover, rank $H_1(M \# \mathbb{T}^2) = \text{rank } H_1(M) + 2$. Since any torus lies in a product of two graphs, and any closed orientable surface different from $S^2$ can be obtained from a torus by repeated applications of the operation of connected sum with $\mathbb{T}^2$, any closed orientable surface can be embedded in a product of two graphs ([Ku]). On the other hand, any closed non-orientable surface whose rank of $H_1$ is $\geq 5$ can be obtained by the same construction starting from a closed non-orientable surface whose rank of $H_1$ is either 5 or 6. So, the proof will be done once we show that either of the two surfaces embeds in a product of two graphs. The desired embeddings have been constructed by R. Cauty [C1].

Our approach is slightly different: Theorem 6D.5 is obtained as a direct consequence of Theorem 6C.1 (about embeddability of orientable surfaces) and the following two lemmas (about embeddedness of non-orientable surfaces).

**Lemma 6D.6.** Any closed non-orientable surface $M$ whose rank of $H_1$ is odd and $\geq 5$ can be embedded in a product of two graphs.

**Proof.** Suppose rank $H_1(M) = 2k + 1$, where $k \geq 2$. We shall show that $M$ can be embedded in the product $P_1 \times P_2$ of two graphs. Define $P_1 = S_1 \cup S'_1$, where $S_1$ and $S'_1$ are oriented circles such that their intersection $S_1 \cap S'_1$ is a union of $k$ disjoint arcs $A_1, \ldots, A_k$, and the orientations induce the same orientation on $A_1$ and opposite orientations on $A_2$. And define $P_2 = S_2 \cup S'_2$ to be a $\theta$-curve, with $S_2$ and $S'_2$ being oriented circles such that the intersection $S_2 \cap S'_2$ is an arc $A$, and the orientations induce opposite orientations on $A$. It remains to construct a non-orientable surface $\tilde{M}$ in $P_1 \times P_2$ homeomorphic to $M$.

There are 1-dimensional CW complexes $K_1$ and $K_2$ such that: $P_i = |K_i|$, each $A_j$ is the carrier of a 1-cell $\sigma_j$ of $K_1$ oriented coherently with $S_1$, and $A$ is the carrier of a 1-cell $\tau$ of $K_2$ oriented coherently with $S_2$. Suppose $\partial \tau = w_1 - w_0$, where $w_i$ are vertices of $\tau$.

Consider the tori $T = S_1 \times S_2$, $T' = S'_1 \times S'_2$. Note that their intersection

$$T \cap T' = (S_1 \cap S'_1) \times (S_2 \cap S'_2) = (A_1 \cup \cdots \cup A_k) \times A$$

is the union of $k$ disjoint 2-cells $\sigma_1 \times \tau$, $\cdots$, $\sigma_k \times \tau$ of $K_1 \sqcup K_2$. Put $\tilde{M} = N \cup N'$, where $N = T \setminus (\sigma_1 \times \tau \cup \cdots \cup \sigma_k \times \tau)$ and $N' = T' \setminus (\sigma_1 \times \tau \cup \cdots \cup \sigma_k \times \tau)$. One easily sees that $\tilde{M}$ is a (connected) surface. We shall show that $\tilde{M}$ is not orientable.
For suppose $\widetilde{M}$ is orientable. Then $H_2(\widetilde{M}) \approx \mathbb{Z}$ and each 2-cell of $K_1 \sqcup K_2$ lying in $\widetilde{M}$ can be assigned an orientation such that the 2-chain $z$ which is the sum of oriented 2-cells with coefficient 1 is a 2-cycle (representing a generator of $H_2(\widetilde{M}))$. Then $z = c + c'$, where $c$ (respectively $c'$) is the sum of the oriented 2-cells lying in $N$ (respectively in $N'$). Since $N$ is orientable we may assume that the orientations of the 2-cells lying in $N$ are induced by the orientations of $S_1$ and $S_2$. As $0 = \partial z = \partial c + \partial c'$ we have $\partial c = -\partial c'$. Since the oriented 1-cell $\sigma_1 \times w_1$ enters $\partial c$ with coefficient 1, it must enter $\partial c'$ with coefficient -1. Let $\tau_1 \in K_2$ be the 1-cell with carrier $S_2 \setminus \hat{A}$ oriented coherently with $S_2$. Let $\tau'_1 \in K_2$ be the 1-cell with carrier $S'_2 \setminus \hat{A}$ oriented coherently with $S'_2$. It follows that the 2-cell $\sigma_1 \times \tau'_1$ lying in $N'$ has been assigned the orientation $\sigma_1 \otimes \tau'_1$. Consequently, the orientations of the remaining 2-cells lying in $N'$ are also induced by the orientations of $S'_1$ and $S'_2$. But this leads to a contradiction: the oriented 1-cell $\sigma_2 \otimes w_1$ enters $\partial c$ with coefficient 1, and $\partial c'$ with coefficient 1 as well.

Now we prove that $\text{rank } H_1(\widetilde{M}) = 2k + 1$. Note that $\widetilde{M}$ is the union of two tori with the interiors of $k$ disjoint discs removed, hence the Euler characteristic $\chi(\widetilde{M}) = \chi(N) + \chi(N') - \chi(N \cap N') = -(k) + (k) - 0 = -2k$. So, $\text{rank } H_1(\widetilde{M}) = 1 - \chi(\widetilde{M}) = 2k + 1$. It follows that $\widetilde{M} \approx M$. □

The ideas used in the above proof can be easily generalized to give a proof of the following much more general result.

**Proposition 6D.7.** Let $M$ be an $n$-manifold, $n \geq 2$, and let $h : M \to M$ be an involution. Suppose $M = M_0 \cup M_1$ is a union of two bordered $n$-manifolds such that $h(M_i) = M_{1-i}$, where $i = 0, 1$, and $M_0 \cap M_1 = N_0 \sqcup N_1$ is the disjoint union of orientable $(n - 1)$-manifolds such that $h(N_i) = N_i$. If $h$ preserves an orientation of $N_0$ and reverses an orientation of $N_1$, then $M$ is not orientable. □

In the next lemma we use some graphs $P_n$, $n \geq 3$. To describe $P_n$ we first pick some points in the plane $\mathbb{R}^2$. Denote $o = (0, 0)$, and let $v_i = (\cos \frac{2\pi i}{n}, \sin \frac{2\pi i}{n})$ for $i = 0, \ldots, n - 1$. Then define

$$P_n = \bigcup_{i=0}^{n-1} (ov_i \cup v_iv_{i+1}),$$

where $v_n = v_0 = (1, 0)$. Thus $P_n = |K_n|$, where $|K_n|$ is the (simplicial) complex composed of segments and their endpoints. Let $\check{P}_n$ denote the union $\check{P}_n = P_n \cup S$, where $S$ is a circle such that $S \cap P_n = v_0v_1$.

**Lemma 6D.8.** Any closed non-orientable surface whose rank of $H_1$ is equal to $2n$ with $2n \geq 6$, can be embedded in a product of two graphs. Moreover, if $n$ is odd it can be embedded in $P_n \times P_n$, if $n$ is even it can be embedded in $\check{P}_{n-1} \times \check{P}_{n-1}$.

**Proof.** We shall prove the stronger version. To this end, we have to consider two cases.
Case 1: \( n \) is odd. To complete the proof in this case it is enough to construct a non-orientable surface \( M \) in \( P_n \times P_n \) whose rank of \( H_1 \) is equal to \( 2n \). To this end, for each \( i = 0, \ldots, n-1 \), consider:

- the oriented circles \( S_i = ov_i \cup v_i v_{i+1} \cup v_{i+1} o \) in \( P_n \),
- the tori \( T_i = S_i \times S_i \) in \( P_n \times P_n \), and
- the 2-cells \( D_i = ov_i \times ov_i \) in \( T_i \).

(Here \( v_n = v_0 \).) Then \( T_i \cap T_{i+1} = D_{i+1} \), where \( T_n = T_0 \) and \( D_n = D_0 \). Put

\[
M = M_0 \cup \cdots \cup M_{n-1},
\]

where \( M_i = T_i \setminus (\tilde{D}_i \cup \tilde{D}_{i+1}) \). First we shall show that \( M \) is a closed (connected) surface. Note that \( M \) is connected because each \( M_i \) is connected and contains \((o, o)\). One easily sees that \( M \) is locally planar at each point different from \((o, o)\). Thus, it remains to show that \( M \) is locally planar at \((o, o)\) too. To this end, note that \( K \) is a pseudo-manifold, and that all the 2-cells of \( K \) containing \((o, o)\) can be arranged into a sequence

\[
\begin{align*}
&ov_0 \times ov_1 \rightarrow ov_2 \times ov_1 \rightarrow ov_2 \times ov_3 \rightarrow \cdots \rightarrow ov_{n-1} \times ov_0 \rightarrow \\
&ov_1 \times ov_0 \rightarrow ov_1 \times ov_2 \rightarrow ov_3 \times ov_2 \rightarrow \cdots \rightarrow ov_0 \times ov_{n-1}.
\end{align*}
\]

(Each cell \( ov_i \times ov_{i'} \) is followed by \( ov_{i'+1} \times ov_{i+1} \), where indices are reduced modulo \( n \). Note also that both cells of the \( i \)th column lie in \( M_i \) and have only the point \((o, o)\) in common.) Each cell borders upon the next one along in turn:

\[
\{o\} \times ov_1, ov_2 \times \{o\}, \ldots, ov_{n-1} \times \{o\}, \{o\} \times ov_0,
\]

\[
ove \times \{o\}, \{o\} \times ov_2, \ldots, \{o\} \times ov_{n-1}, ov_0 \times \{o\}.
\]

This shows that the union of the 2-cells is a 2-disc whose interior in \( M \) contains point \((o, o)\). Thus, \( M \) is locally planar at that point. Hence \( M \) is a closed surface.

Now we shall show that \( \chi(M) = 1 - 2n \). First, one can prove by induction that the set \( N = (T_0 \cup \cdots \cup T_{n-2}) \setminus (\tilde{D}_1 \cup \cdots \cup \tilde{D}_{n-2}) \) is the connected sum \( T_0 \# \cdots \# T_{n-2} \). Then we note that \( M = N' \cup M_{n-1} \), where \( N' = N \setminus (\tilde{D}_0 \cup \tilde{D}_{n-1}) \). Also note that \( N' \cap M_{n-1} = \partial D_0 \cup \partial D_{n-1} \) is the one-point union of two circles. Consequently,

\[
\chi(M) = \chi(N') + \chi(M_{n-1}) - \chi(N' \cap M_{n-1}) = [(n-2)(-2)+(-2)]+(-2)-(-1) = -2n+1.
\]

It follows that \( M \) is not orientable (as the Euler characteristic of an orientable surface is even). Finally, rank \( H_1(M) = 1 - \chi(M) = 2n \), which concludes the proof in Case 1.

---

1 There is an action of \( \mathbb{Z}_{2n} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_n \) on \( M \), with \((o, o)\) as a unique fixed point, such that the sequence represents a cyclic orbit of the 2-cell \( ov_0 \times ov_1 \), and each cell is followed by its image under a generating homeomorphism. (The generating homeomorphism is the restriction to \( M \) of the homeomorphism \( h : P_n \times P_n \rightarrow P_n \times P_n \) defined by \( h(x, y) = (\rho(y), \rho(x)) \), where \( \rho : P_n \rightarrow P_n \) is the rotation around \( o \) through the angle \( \frac{2\pi}{n} \).)
Case 2: $n$ is even. As in Case 1, it is enough to construct a non-orientable surface $M'$ in $\tilde{P}_{n-1} \times \tilde{P}_{n-1}$ with rank of $H_1$ equal to $2n$. Since $n - 1$ is odd and $\geq 3$, by Case 1 there is a closed non-orientable surface $M$ in $P_{n-1} \times P_{n-1}$ with rank of $H_1$ equal to $2(n - 1)$. Note that $A = \text{cl}(S \setminus P_{n-1})$ is an arc. It follows that $M' = (M \cup (S \times S)) \setminus (\tilde{A} \times \tilde{A})$ is homeomorphic to $M \# T^2$. Thus, $M'$ is a closed non-orientable surface in $\tilde{P}_{n-1} \times \tilde{P}_{n-1}$ with $\chi(M') = -2(n - 1) - 1$. Consequently, rank $H_1(M') = 1 + 2(n - 1) + 1 = 2n$, which concludes the proof in Case 2. □

6E. Retracting products onto surfaces

In connection with the proof in section 6A, let us make some observations concerning the existence of retractions from $Y_1 \times Y_2$ onto closed surfaces $M \subset Y_1 \times Y_2$, where $Y_1$ and $Y_2$ are curves. We prove that the only surface which is such a retract is the torus.

**Theorem 6E.1.** Let $M$ be a closed surface in the product $Y_1 \times Y_2$ of two curves.

(A) If $M$ is a torus then $M = P_1 \times P_2$, where each $P_i$ is a circle in $Y_i$. Consequently, $M$ is a retract of $Y_1 \times Y_2$.

(B) If $M$ is not a torus then $M$ is not a retract of $Y_1 \times Y_2$.

**Remark.** Part (B) implies the Borsuk theorem [Bo3] (because any topological 2-sphere in a 2-dimensional space is a retract of that space, see e.g., [Kur, §53, VI, Theorem 1, p.354]). □

Part (A) directly follows from condition (i) of Theorem 6D.1, and the fact that any circle in a curve $Y$ is a retract of $Y$ (see e.g., [Kur, §53, VI, Theorem 1, p.354]). To prove Part (B) we need the following fact. With no doubt, it is known to many (all?) topologists. As we are unable to indicate an appropriate reference, a proof is supplied for completeness.

**Lemma 6E.2.** Let $M$ be a closed surface and let $S$ be a circle in $M$. If $S$ is contractible in $M$ then $S$ bounds a disc $D \subset M$, i.e. $S = \partial D$.

**Proof.** Let $p : \tilde{M} \to M$ be the universal covering projection. Then $\tilde{M}$ is homeomorphic to either $S^2$ or $\mathbb{R}^2$. By the homotopy lifting property of $p$, there is a circle $\tilde{S} \subset \tilde{M}$ such that $p$ maps $\tilde{S}$ bijectively onto $S$. We shall show that there is a disc $\tilde{D} \subset \tilde{M}$ bounded by $\tilde{S}$ such that for every covering transformation $\tau : \tilde{M} \to \tilde{M}$ different from the identity we have

1. $\tau(\tilde{D}) \cap \tilde{D} = \emptyset$.

First consider any covering transformation $\tau : \tilde{M} \to \tilde{M}$ different from the identity. If $\tilde{M} \approx S^2$ then $\tau$ is unique and is an involution. Note that $\tau(\tilde{S}) \cap \tilde{S} = \emptyset$.
In fact, \( \tau(\tilde{S}) \not\subseteq \tilde{S} \), otherwise \( p \) would not be injective on \( \tilde{S} \) as \( \tau \) has no fixed point. Then suppose (2) is false. Therefore, by the above observation, we get a simple triod \( T = oa \cup ob \cup oc \) such that \( oa \cup ob \subset \tilde{S} \) and \( oc \subset \tau(\tilde{S}) \). But this is impossible because \( p \) is a local homeomorphism and \( p(T) \subset S \). This proves (2).

Next choose the disc \( \tilde{D} \) in \( \tilde{M} \) bounded by \( \tilde{S} \). If \( \tilde{M} \approx \mathbb{R}^2 \) then such a disc is uniquely determined (by the Jordan theorem); if \( \tilde{M} \approx S^2 \) then take \( \tilde{D} \) disjoint with \( \tau(\tilde{S}) \) (which can be done by (2)). Observe that

\[
(3) \quad \tau(\tilde{S}) \cap \tilde{D} = \emptyset.
\]

According to the choice of the disc, it is enough to prove (3) only for \( \tilde{M} \approx \mathbb{R}^2 \).

Suppose, to the contrary, that (3) is false. Then \( \tau(\tilde{S}) \subset \tilde{D} \) by (2). Consequently, \( \tau(\tilde{D}) \subset \tilde{D} \) because \( \tau(\tilde{D}) \) is a disc bonded by \( \tau(\tilde{S}) \) (as \( \tau \) is a homeomorphism of the plane). Thus, by the Brouwer Fixed Point Theorem, \( \tau \) has a fixed point, a contradiction. This proves (3). Then note that

\[
(4) \quad \tilde{D} \not\subseteq \tau(\tilde{D}).
\]

Else \( \tau^{-1}(\tilde{D}) \subset \tilde{D} \), a contradiction (as \( \tau^{-1} \) is another covering transformation different from the identity). Now we are ready to prove (1).

Suppose \( \tau(\tilde{D}) \cap \tilde{D} \neq \emptyset \). By (4) there is an arc \( ab \subset \tilde{D} \) meeting \( \tau(\tilde{D}) \) only at \( b \). It follows from (2) that \( ab \cup \tau(\tilde{D}) \) is a 2-umbrella in \( \tilde{M} \), a contradiction. This proves (1).

It follows from (1) that \( p \) maps \( \tilde{D} \) homeomorphically onto \( p(\tilde{D}) \). Therefore, \( D = p(\tilde{D}) \) is the desired disc in \( M \) bounded by \( S \), which completes the proof.

**Proof of Part (B) of Theorem 6E.1**

Let \( M \) be a closed surface different from the torus and lying in the product \( Y_1 \times Y_2 \) of two curves.

First we discuss the special case where \( Y_1 = P_1 \) and \( Y_2 = P_2 \) are graphs and \( M \) is monotonically embedded in \( P_1 \times P_2 \). Let \( K_i \) denote a regular complex such that \( P_i = \partial K_i \). Then we adopt the notation of Subsection 6B.

By Property (b) in 6B, there are 1-cells \( \sigma_0 \in K_1 \) and \( \tau_0 \in K_2 \) such that \( \sigma_0 \times \tau_0 \subset M \). By Property (f') in 6D the sets \( P_1(\tau_0) = S_1 \) and \( P_2(\sigma_0) = S_2 \) are circles. Let

\[
M_1 = S_1 \times \tau_0 \cup \sigma_0 \times S_2, \quad M_2 = M \setminus (S_1 \times \tau_0 \cup \sigma_0 \times S_2),
\]

and let \( D_0 = (S_1 \setminus \sigma_0) \times (S_2 \setminus \tau_0) \). Note that \( D_0 \) is a disc, \( M_1 \) and \( M_2 \) are surfaces with boundary, \( M = M_1 \cup M_2 \) and \( M_1 \cap M_2 = \partial M_1 = \partial M_2 = \partial D_0 \).

Now suppose \( M \) is a retract of \( P_1 \times P_2 \). Then \( S = \partial D_0 \) is contractible in \( M \), because \( S \) is contractible in \( D_0 \) and \( D_0 \subset P_1 \times P_2 \). Thus, by Lemma 6E.2 there is a disc \( D \subset M \) bounded by \( S \). It follows that either \( D = M_1 \) or \( D = M_2 \). However, \( M_1 = (S_1 \times S_2) \setminus D_0 \) is not a disc, hence \( D = M_2 \). Therefore, \( M = M_1 \cup D \) is a torus. This contradicts our assumption, and ends the proof in the special case.
In the general case, by Theorem 4.3, there exist two graphs $P_1, P_2$, an embedding $g : M \to P_1 \times P_2$, and a map $h : P_1 \times P_2 \to Y_1 \times Y_2$ such that

(i) $g(M)$ is monotonically embedded in $P_1 \times P_2$, and

(ii) $h \circ g : M \to Y_1 \times Y_2$ is the inclusion.

Since Part (B) holds in the special case, it follows from (i) that $g(M)$ is not a retract of $P_1 \times P_2$. Suppose $M$ is a retract of $Y_1 \times Y_2$ and let $r : Y_1 \times Y_2 \to M$ be a retraction. Then, by (ii), $g \circ r \circ h(x) = x$ for each $x \in g(M)$. Thus $g \circ r \circ h : P_1 \times P_2 \to g(M)$ is a retraction, a contradiction. This contradiction completes the proof of Part (B) in the general case. □

6F. Embedding bordered surfaces in the "three-page book"

By a bordered surface we mean a connected 2-manifold with non-empty boundary. The "three-page book" stands for the product $I \times T$, where $T = a_0v \cup a_1v \cup a_2v$ is a simple triod (i.e. a union of three segments $a_iv$ mutually disjoint except a common endpoint $v$). In this section we prove the following theorem.

**Theorem 6F.1.** Any bordered surface can be embedded in the "three-page book".

**Proof.** To this end, consider a bordered surface $M$ and let $k (\geq 1)$ be the number of components of $\partial M$. It is known that

(1) $\chi(M) \leq 2 - k$.

In order to show that $M$ can be embedded in the "three-page book" we recall a convenient classic model of $M$ (cf. [Ma, p. 43]). One gets the model by repeated application of the following procedure: given a bordered surface $N$ we take a long rectangular strip and pass both ends of the strip to the boundary of $N$ so that the ends do not overlap on the boundary of $N$. This produces a bordered surface whose Euler characteristic is equal to $\chi(N) - 1$. Attaching such a strip to a closed disc we get either an annulus or the Möbius band. In the first case the strip is called "annular" and in the other "twisted".

We begin with a closed disc $D$. Let $S_k$ denote the bordered surface obtained from $D$ by attaching $k - 1$ annular strips as shown in Figure 6F.1.

(FIGURE 6F.1)

The bordered surface which results is a model of the 2-sphere $S^2$ with the interiors of $k$ disjoint discs removed. Therefore,

(2) $\chi(S_k) = 2 - k$.

Consequently,

(3) $\chi(M) \leq \chi(S_k)$.

To get the final model of $M$ we shall modify $S_k$ so as to keep the number of boundary components the same and reduce the Euler characteristic to $\chi(M)$. It is necessary to distinguish two essentially different cases.

**Case I:** $M$ is orientable. Then $\chi(M) = l - k$, where $l$ is even. By (2) and (3) we infer that $\chi(M) = \chi(S_k) - 2m_0$, where $m_0 \geq 0$. The desired model of $M$ is
obtained by attaching \( m_0 \) pairs of "crossed" annular strips to the boundary of \( D \) as shown in Figure 6F.2.

**Case II**: \( M \) is non-orientable. Then \( \chi(M) = \chi(S_k) - m_1 \), where \( m_1 \geq 1 \). To get the desired model of \( M \) we attach \( m_1 \) "twisted" strips to \( S_k \) as shown in Figure 6F.3.

To complete the proof of our theorem it is enough to embed the above models in the product \( I \times T \), where \( T \) is the simple triod. If \( M \) is orientable then one can easily detect a copy of the model in \( I \times T \) as shown in Figure 6F.4. (For the disc corresponding to \( D \) we take the rectangle \( I \times a_2v \). The desired copy is the union of the rectangle, \( k - 1 \) annular strips lying in \( I \times a_0v \), and \( m_0 \) pairs of "crossed" annular strips lying in \( I \times (a_0v \cup a_1v) \).)

Now assume \( M \) is non-orientable. Then we define a copy of the model of \( M \) in \( I \times T \) as shown in Figure 6F.5. (This time for the disc corresponding to \( D \) we take the rectangle \( I \times a_2v \) with \( m_1 \) "gates". The desired copy is the union of the rectangle, \( k - 1 \) annular strips lying in \( I \times a_0v \), and \( m_1 \) "twisted" strips lying in \( I \times (a_0v \cup a_1v) \).)

**6G. Embedding surfaces in the second symmetric product of a curve**

As we have noted in Chapter 4, any 2-dimensional compactum can be embedded in the symmetric product \( SP^3(\mu) \), where \( \mu \) stands for the Menger universal curve. Illanes and Nadler have asked about embeddability of \( S^2 \) in the second symmetric product of the Menger curve (thus enquiring about possible extension of the Borsuk Theorem 1.2 to embeddings into symmetric products of curves). We shall show in a moment that the answer is negative, but first we note a positive result.

Notice that the Cartesian product \( X \times Y \) naturally embeds in the second symmetric product \( SP^2(X \lor Y) \) (where \( X \) and \( Y \) are pointed spaces). This combined with Theorems 6C.1 and 6D.5 implies the following corollary.

**Theorem 6G.1.** Any closed surface embeddable in the product of two graphs can be also embedded in the second symmetric product of a graph. In particular, any orientable closed surface different from \( S^2 \), and any non-orientable surface \( M \) with rank \( H_1(M) \geq 5 \) can be embedded in the second symmetric product of a graph. \( \square \)

To get additional information about possible embeddability of the six exceptional surfaces in the second symmetric product of a curve, we first formulate a particular case of Corollary 4G.3.
Corollary 6G.2. Neither the 2-sphere nor the projective plane, nor the Klein bottle can be embedded (even up to shape) in the second symmetric product of a curve. □

This corollary answers in the negative the Illanes and Nadler question.

The next lemma is needed in the proof of our final theorem.

Lemma 6G.3. Let $X$ be the union of three different topological $n$-tori $T_1, T_2, T_3$, $n \geq 2$, such that the intersection of any two different tori is an $(n-1)$-torus, and the intersection of all is non-void and not an $(n-1)$-torus. Then $X$ is not embeddable in any product of $n$ curves.

Proof. Suppose $X$ can be embedded in the product $Y_1 \times \cdots \times Y_n$ of $n$ curves. Without loss of generality we can assume that $X \subset Y_1 \times \cdots \times Y_n$. Let $\star = (\star_1, \ldots, \star_n)$ be a point of $T_1 \cap T_2 \cap T_3$. By Theorem 5D.5, for each $i = 1, 2, 3$, there exist topological circles $S_{i,1} \subset Y_1, \ldots, S_{i,n} \subset Y_n$ such that

$$T_i = S_{i,1} \times \cdots \times S_{i,n}.$$

Call $S_{i,j}$ the $j$th factor of $T_i$. The $j$th factors of two tori either coincide or meet at $\{\star_j\}$ only. In the latter case all the remaining factors coincide. Indeed, by our hypothesis, for any two different elements $i_1, i_2$, the intersection

$$T_{i_1} \cap T_{i_2} = (S_{i_1,1} \cap S_{i_2,1}) \times \cdots \times (S_{i_1,n} \cap S_{i_2,n})$$

is an $(n-1)$-torus. It follows that $S_{i_1,j} \cap S_{i_2,j}$ is a one-point of the form $\{\star_j\}$ for exactly one $j$, and $S_{i_1,k} = S_{i_2,k}$ for $k \neq j$. Then we say that $T_{i_1}$ and $T_{i_2}$ differ at index $j$. Without loss of generality we can assume that

(1) $T_1$ and $T_2$ differ at index 1.

Let us notice that

(2) $T_1$ and $T_3$ differ at index $j > 1$.

Indeed, otherwise $S_{3,1} \cap S_{1,1} = \{\star_1\}$ and $S_{3,j} = S_{1,j}$ for $j > 1$. If $S_{3,1} = S_{2,1}$ then $T_3 = T_2$, a contradiction. Thus $S_{3,1} \neq S_{2,1}$, hence $S_{3,1} \cap S_{2,1} = \{\star_1\}$ and $S_{3,j} = S_{2,j}$ for $j > 1$. Therefore, the intersection $T_1 \cap T_2 \cap T_3$ is an $(n-1)$-torus, a contradiction. Now, by (1) and (2), we infer that

$$T_2 \cap T_3 = \{\star_1\} \times S_{1,2} \times \cdots \times S_{1,j-1} \times \{\star_j\} \times S_{1,j+1} \times \cdots \times S_{1,n}$$

is an $(n-2)$-torus, a contradiction. This ends the proof. □

In our final observation we point out the fact that the symmetric product produces spaces more complex than the corresponding Cartesian product does. Already the second symmetric product may transform a relatively simple graph into a 2-dimensional polyhedron which is not embeddable in any Cartesian product of two curves. In fact, we have the following
Theorem 6G.4. The nth symmetric product, \( n \geq 2 \), of a bouquet of \( n + 1 \) topological circles cannot be embedded in the product of \( n \) curves.

Proof. Let \( \bigvee_{i=1}^{n+1} S_i \) be a bouquet of \( n + 1 \) topological circles with base point \(*\). We have to show that \( SP^n(\bigvee_{i=1}^{n+1} S_i) \) cannot be embedded in the product of \( n \) curves. To this end, it is enough to point out a subset \( Q \) of \( SP^n(\bigvee_{i=1}^{n+1} S_i) \) that is not embeddable in any product of \( n \) curves. The space \( Q \) will be defined as a subspace of an \( n \)-dimensional CW complex. The cells \( e_J \) of the complex correspond to proper sets \( J \subset \{1, \ldots, n+1\} \). If \( J = \{n_1, \ldots, n_k\} \), where \( n_1 < \cdots < n_k \), then \( e_J \) is defined as follows:

\[
e_J = q(\{(x_1, \ldots, x_n) \in (\bigvee_{i=1}^{n+1} S_i)^n : x_j \in S_{n_j} \text{ for } j = 1, \ldots, k; x_j = * \text{ for } j > k\}),
\]

where \( q : (\bigvee_{i=1}^{n+1} S_i)^n \to SP^n(\bigvee_{i=1}^{n+1} S_i) \) is the quotient map. We have exactly one 0-cell \( e_\emptyset = \{q(*, \ldots, *)\} \). One easily sees that \( \dim e_J = |J| \) and \( e_J \cap e_{J'} = e_{J \cap J'} \).

Now we are ready to define \( Q \), put \( Q = T_1 \cup T_2 \cup T_3 \), where \( T_1 = e_{\{2, \ldots, n+1\}} \), \( T_2 = e_{\{1,3,\ldots,n+1\}} \), \( T_3 = e_{\{1,2,4,\ldots,n+1\}} \). It remains to prove that \( Q \) is not embeddable in any product of \( n \) curves. Notice the following properties of the cells: each \( T_j \) is an \( n \)-torus, the intersection of any two tori \( T_j \) is an \( (n-1) \)-torus, each of the three tori contains the 0-cell, and the intersection of all three tori is the \( (n-2) \)-torus \( e_{\{4,\ldots,n+1\}} \). Thus the hypotheses of Lemma 6G.3 are fulfilled. It follows from this lemma that \( Q \) has the desired property, which ends the proof. \( \square \)

6H. Embedding surface-like continua in products of two curves

Here we apply Cauty’s Theorem 6D.5 to derive a result on embeddability of some special surface-like continua into products of two curves.

Let \( p \) be a prime number. We denote the mapping cylinder of a \( p \)-to-1 covering map \( f = f_p : S^1 \to S^1 \) by \( M(p) \). Let \( \partial M(p) \) denote the circle in \( M(p) \) corresponding to the domain of \( f \). Now we construct an inverse sequence of 2-dimensional polyhedra \( P_i \). Denote by \( P_1 \) the 2-sphere with a triangulation \( K_1 \), and assume a 2-dimensional polyhedron \( P_i \) with a triangulation \( K_i \) has been defined. Replace each 2-simplex \( \sigma \) of \( K_i \) by \( M(p)\sigma \), where \( M(p)\sigma \) is obtained from \( M(p) \) by identifying \( \partial M(p) \) with the boundary \( \partial \sigma \). In this way we obtain a polyhedron \( P_{i+1} \). Define \( \varphi_i : P_{i+1} \to P_i \) to be any mapping satisfying the conditions:

1. \( \varphi_i(x) = x \) for each \( x \in |K_{i+1}^{(1)}| \), and \( \varphi_i|M(p)\sigma \) is a relative homeomorphism of \( (M(p)\sigma, \partial M(p)\sigma) \) onto \( (\sigma, b(\sigma)) \), where \( b(\sigma) \) is the barycenter of \( \sigma \).

Let \( K_{i+1} \) be a triangulation of \( P_{i+1} \) such that \( \text{diam}[\varphi_j \circ \cdots \circ \varphi_i(\sigma)] < 1/2^{i-j} \) for each 2-simplex \( \sigma \in K_{i+1} \). Then we define the \textit{Pontrjagin surface mod } \( p \) as the inverse limit:

\[
\Pi_p = \varprojlim \{P_i, \varphi_i\}.
\]

Then using Theorem 6D.5 and a technique analogous to that in [Dr2, Lemma 2], one can prove the following:
Theorem 6H.1. The Pontrjagin surface mod 2 can be embedded in a product of two curves. □

Problems to Chapter 6

A graph $P$ is said to be a quasi-factor of a closed surface $M$ if $M$ can be surjectively embedded in $P \times Q$, where $Q$ is another graph. If $P$ contains no proper quasi-factor of $M$ then it is called minimal. If $M$ is monotonically embedded in $P \times Q$ then $P$ is said to be a monotone quasi-factor of $M$. As in the case of quasi-factors, we can define minimal monotone quasi-factors of $M$. The results from subsection 6C show that the graph $\Theta_{m+1}$ is a minimal (monotone) quasi-factor of any orientable surface of genus $m$.

Problem 6C.1. Determine all quasi-factors (surjective monotone quasi-factors) of a closed surface.

Problem 6C.2. Determine all minimal quasi-factors (minimal monotone quasi-factors) of a closed surface.

Problem 6D.1. Suppose a closed surface $M$ can be embedded in the second symmetric product of a curve. Can $M$ be embedded in the second symmetric product of a graph? (Same question for a polyhedron in place of $M$.)

Problem 6G.1. Is it possible to embed the remaining three exceptional surfaces (that is, the non-orientable surfaces with genus 3, 4 and 5) in the second symmetric product of a graph?

Problem 6H.1. Can the Pontrjagin surface mod $p$, $p \neq 2$, be embedded in a product of two curves?

Problem 6H.2. Is each Pontrjagin surface a quasi 2-manifold?

Problem 6H.3. Is it possible to embed any 3-dimensional product of two Pontrjagin surfaces in a product of three curves?

APPENDIX

The main result of this Appendix is Theorem A1 below which expresses certain property of tensor product. All groups under discussion are Abelian.

Theorem A1. Let $f : A \rightarrow B$ be a homomorphism of groups. If $G$ is a group which is either non-torsion or a direct sum of cyclic groups, and

$$f \otimes 1_G : A \otimes G \rightarrow B \otimes G$$

is non-trivial then, for each $k \geq 1$, the homomorphism

$$f \otimes 1_{\bigotimes^k G} : A \otimes \bigotimes^k G \rightarrow B \otimes \bigotimes^k G,$$
is non-trivial as well.

To prove this result we need the following Lemmas A2, A3 and A4. Lemma A2 shows that Theorem A1 can be derived from its special case where \( f \) is the inclusion homomorphism.

**Lemma A2.** Let \( f : A \to B \) be a homomorphism and let \( i : f(A) \to B \) denote the inclusion homomorphism. Then \( f \otimes 1_G : A \otimes G \to B \otimes G \) is non-trivial if and only if \( i \otimes 1_G : f(A) \otimes G \to B \otimes G \) is non-trivial, for any Abelian group \( G \).

**Proof.** Let \( f' : A \to f(A) \) be defined by \( f'(a) = f(a) \), for all \( a \in A \). Then \( f = i \circ f' \), so \( f \otimes 1_G = (i \otimes 1_G) \circ (f' \otimes 1_G) \). As \( f' \) is an epimorphism, \( f' \otimes 1_G \) is an epimorphism as well, the conclusion follows. \( \square \)

**Lemma A3.** Let \( i : A \to B \) be the inclusion of groups, where \( A \) is a \( p \)-group for some prime \( p \). If \( G_1 \) and \( G_2 \) are groups such that both homomorphisms

\[
i \otimes 1_{G_k} : A \otimes G_k \to B \otimes G_k,
\]

for \( k = 1, 2 \), are non-trivial then

\[
i \otimes 1_{G_1 \otimes G_2} : A \otimes (G_1 \otimes G_2) \to B \otimes (G_1 \otimes G_2)
\]

is non-trivial as well.

**Proof.** First we prove this lemma under additional assumption of \( B \) being a \( p \)-group. Let \( G_k^{(p)} \) be a \( p \)-basic subgroup of \( G_k \) (cf. [F, p. 136]), and let \( i_k \) denote the inclusion \( G_k^{(p)} \to G_k \). By [F, Theorem 61.1, p.261], \( 1_C \otimes i_k : C \otimes G_k^{(p)} \to C \otimes G_k \) is an isomorphism for any \( p \)-group \( C \). Consequently, since \( i \otimes 1_{G_k} \) is non-trivial and both \( A \) and \( B \) are \( p \)-groups, the homomorphism

\[
i \otimes 1_{G_k^{(p)}} : A \otimes G_k^{(p)} \to B \otimes G_k^{(p)}
\]

is non-trivial. Since \( G_k^{(p)} \) is a direct sum of cyclic \( p \) groups and infinite cyclic groups, and since tensor product commutes with direct sums, it follows that there exists a cyclic (infinite or a \( p \)-group) subgroup \( H_k \) of \( G_k^{(p)} \) (a direct summand of \( G_k^{(p)} \)), such that

(i) the homomorphism \( i \otimes 1 : A \otimes H_k \to B \otimes H_k \) is non-trivial.

Since \( \mathbb{Z} \otimes C \cong C \otimes \mathbb{Z} \cong C \) for any group \( C \), and \( \mathbb{Z}(p^r) \otimes \mathbb{Z}(p^s) \cong \mathbb{Z}(p^t) \), where \( t = \min(r,s) \) (cf. [F, p. 255]), it follows that \( H_1 \otimes H_2 \) is isomorphic to either \( H_1 \) or \( H_2 \). Consequently, by (i), the homomorphism

\[
i \otimes 1 : A \otimes (H_1 \otimes H_2) \to B \otimes (H_1 \otimes H_2)
\]

is non-trivial. Since \( H_1 \otimes H_2 \) is a direct summand of \( G_1^{(p)} \otimes G_2^{(p)} \), it follows that
(ii) the homomorphism $i \otimes 1 : A \otimes (G_1^{(p)} \otimes G_2^{(p)}) \to B \otimes (G_1^{(p)} \otimes G_2^{(p)})$ is non-trivial.

Now, by [F, p. 255, property (C)] and [F, Theorem 61.1, p. 261], for any $p$-group $C$, we have the following sequence of (natural) isomorphisms

$$(C \otimes G_1^{(p)}) \otimes G_2^{(p)} \xrightarrow{(1 \otimes j_1) \otimes 1} (C \otimes G_1^{(p)}) \otimes G_2 \xrightarrow{(1 \otimes j_2) \otimes 1} (C \otimes G_1) \otimes G_2 ;$$

so

$$1 \otimes (j_1 \otimes j_2) : C \otimes (G_1^{(p)} \otimes G_2^{(p)}) \to C \otimes (G_1 \otimes G_2)$$

is a (natural) isomorphism. Thus, by (ii), since $A$ and $B$ are $p$-groups, the homomorphism $i \otimes 1_{G_1 \otimes G_2}$ is non-trivial, which ends the proof of the lemma in case $B$ is a $p$-group.

Now, we prove the lemma in the general case. The $p$-group $A$ is contained in the $p$-component $B_p$ of the torsion part of $B$. Let $i_1 : A \hookrightarrow B_p$ and $i_2 : B_p \hookrightarrow B$ denote the inclusions homomorphisms. Then $i = i_2 \circ i_1$. Since $i \otimes 1_{G_k}$ is non-trivial, the homomorphism

$$i_1 \otimes 1_{G_k} : A \otimes G_k \to B_p \otimes G_k$$

is also non-trivial, for $k = 1, 2$. Consequently, referring to the special case, the homomorphism

$$i_1 \otimes 1_{G_1 \otimes G_2} : A \otimes (G_1 \otimes G_2) \to B_p \otimes (G_1 \otimes G_2)$$

is non-trivial.

Since $B_p$ is a pure subgroup of $B$ (see [F, p. 114]), by [F, Theorem 60.4, p. 259] it follows that

$$i_2 \otimes 1 : B_p \otimes (G_1 \otimes G_2) \to B \otimes (G_1 \otimes G_2)$$

is a monomorphism. Consequently, $i \otimes 1_{G_1 \otimes G_2} = (i_2 \otimes 1_{G_1 \otimes G_2}) \circ (i_1 \otimes 1_{G_1 \otimes G_2})$ is non-trivial, which proves the lemma. □

**Lemma A4.** Let $i : A \to B$ be an inclusion homomorphism of groups. If both $A$ and $G$ contain an element of infinite order then the homomorphism

$$i \otimes 1_G : A \otimes G \to B \otimes G$$

is non-trivial.

**Proof.** By the assumption $A$ contains an infinite cyclic subgroup $C$. Since $G$ contains an element of infinite order, the quotient $G/T(G)$ is not trivial and torsion-free. Let us consider the following commutative diagram

$$
\begin{array}{ccc}
C \otimes G & \xrightarrow{(i|C) \otimes 1_{G}} & B \otimes G \\
1_C \otimes \pi \downarrow & & 1_B \otimes \pi \downarrow \\
C \otimes (G/T(G)) & \xrightarrow{(i|C) \otimes 1_{G/T(G)}} & B \otimes (G/T(G))
\end{array}
$$
where \( \pi \) denotes the projection \( G \to G/T(G) \).

Since \( i|C : C \hookrightarrow B \) is a monomorphism and \( G/T(G) \) is torsion-free, by [F, Theorem 60.6, p. 260], \((i|C) \otimes 1_{G/T(G)}\) is a monomorphism as well. Next \( 1_C \otimes \pi \) is not trivial, because \( \pi \) is an epimorphism and \( G/T(G) \) is not trivial. Consequently, by the commutativity of the diagram, \((i|C) \otimes 1_G\) is non-trivial. Thus \( i \otimes 1_G \) is non-trivial. □

**Proof of Theorem A1**

By Lemma A2 we may assume that \( f : A \to B \) is the inclusion homomorphism. Then we divide the proof into three parts.

(I) First, we assume that \( A \) is a torsion group (and \( G \) is an arbitrary (Abelian) group). Then \( A \) is a direct sum of its \( p \)-components \( A_p \). By the assumption the homomorphism \( f \otimes 1_G : A \otimes G \to B \otimes G \) is non-trivial. Consequently, there is a prime \( p \) such that the homomorphism \((f|A_p) \otimes 1_G : A_p \otimes G \hookrightarrow B \otimes G\) is non-trivial. By Lemma A3, the homomorphism

\[(f|A_p) \otimes 1_{\bigotimes^k G} : A_p \otimes \bigotimes^k G \hookrightarrow B \otimes \bigotimes^k G\]

is non-trivial as well. Consequently, the homomorphism \((f \otimes 1_{\bigotimes^k G}) : A \otimes G \to B \otimes G\) is non-trivial, which ends the proof in this case.

(II) Now, we assume that both \( A \) and \( G \) contain an element of infinite order. Then also \( \bigotimes^k G \) contains an element of infinite order. Thus, in this case, the theorem follows by Lemma A4.

Observe that (I) and (II) imply the conclusion of the theorem in the case \( G \) contains an element of infinite order.

(III) Finally, we assume that \( G \) is the direct sum of cyclic groups. Let \( G = \bigoplus G_s \), where each \( G_s \) is a cyclic group. Then \( \bigotimes^k G \) is also a direct sum of cyclic groups, in which an isomorphic copy of each \( G_s \) appears at least once. Consequently, \( \bigotimes^k G \) has an isomorphic copy of \( G \) as a direct summand. Thus, in this case, the required conclusion follows since the tensor product commutes with the direct sum. This ends the proof of the theorem. □

Note that the conclusion of Theorem A1 holds for arbitrary group \( G \) provided the image \( f(A) \) is a torsion group. In general, if \( G \) is a torsion group, the conclusion of Theorem A1 does not hold, even if \( G \) is a \( p \)-group such that \( G \otimes G \neq 0 \).

**Example.** Let \( G = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p) \), where \( p \) is an arbitrary prime. Then \( G \) is a \( p \)-group such that \( G \otimes G = \mathbb{Z}(p) \). Let \( f : \mathbb{Z} \to \mathbb{Z} \) be a homomorphism defined by \( f(1) = p \). One can observe that \( f \otimes 1_G \) is not trivial since \( f \otimes 1_{\mathbb{Z}(p^\infty)} : \mathbb{Z} \otimes \mathbb{Z}(p^\infty) \to \mathbb{Z} \otimes \mathbb{Z}(p^\infty) \) is an epimorphism (and \( \mathbb{Z} \otimes \mathbb{Z}(p^\infty) \) is isomorphic to \( \mathbb{Z}(p^\infty) \)). On the other hand \( f \otimes 1_G \otimes 1_G \) is trivial since \( f \otimes 1_{\mathbb{Z}(p)} \) is trivial.
References

[Bi] R. Bing, Some aspects of the topology of 3-manifolds related to Poincaré conjecture, Lectures on Modern Mathematics, vol. 2, Wiley, New York, 1964, pp. 93–128.

[Bo1] K. Borsuk, Über das Phänomen der Unzerlegbarkeit in der Polyedertopologie, Comment. Math. Helv. 8 (1935), 142–148.

[Bo2] , On the third symmetric potency of circumference, Fund. Math. 3 (1949), 236–244.

[Bo3] , Remarks on the Cartesian product of two 1-dimensional spaces, Bull. Acad. Pol. Sci. Ser. Math. 23 (1975), 971–973.

[Bt] R. Bott, On the third symmetric potency of S1, Fund. Math. 39 (1952), 264–268.

[Bw] B.W. Bowers, General position properties satisfied by finite products of dendrites, Trans. Amer. Math. Soc. 288 (1985), 739–753.

[Br] M. Brown, A mapping theorem for untriangulated manifolds, Topology of 3-manifolds, Proceedings of the University of Georgia Institute (1962), Prentice-Hall, 92–94.

[C1] R. Cauty, Sur le plongement des surfaces non orientables dans un produit de deux graphes, Bull. Acad. Pol. Sci. Ser. Math. 32 (1984), 121–128.

[C2] R. Cauty, Sur le plongement dans un produit symétrique de compacts de dimension un, (preprint).

[Ch-K] N. Chinen and A. Koyama, On the symmetric product of a circle, (preprint).

[Dr1] A.N. Dranishnikov, Homological dimension theory, Russian Math. Surveys, 43:4 (1988), 11–63.

[Dr2] , On problem of Y. Sternfeld, Glasnik Mat. 27(47) (1992), 365–368.

[D-K] J. Dydak and A. Koyama, Compacta not embeddable into Cartesian products of curves, Bull. Pol. Acad. Sci. Ser. Math. 48 (2000), 51–56.

[E] S. Eilenberg, Sur les transformations à petites tranches, Fund. Math. 30 (1938), 92–95.

[E-S] S. Eilenberg and N. Steenrod, Foundations of Algebraic Topology, Princeton University Press, Princeton, New York, 1952.

[En] R. Engelking, Dimension Theory, PWN-Polish Scientific Publishers - Warszawa; North-Holland Publishing Company - Amsterdam, Oxford, New York, 1978.

[Fu] L. Fuchs, Infinite Abelian Groups, vol. I, Academic Press, New York, 1970.

[I-N] A. Illanes and S. B. Nadler, Jr., Hyperspaces, Marcel Dekker, 1999.

[I-M] I. Ivanišić, U. Milutinović, A universal separable metric space based on triangular Sierpiński curve, Top. Appl., 120 (2002), 237–271.

[Kr] J. Krasinkiewicz, On approximation of mappings into 1-manifolds, Bull. Acad. Pol. Sci. Ser. Math. 44 (1996), 431–440.

[Ku] W. Kuperberg, On embeddings of manifolds into Cartesian products of compacta, Bull. Acad. Pol. Sci. Ser. Math. 26 (1978), 845–848.

[Kur] K. Kuratowski, Topology, vol. II, PWN-Academic Press, Warsaw-New York, 1968.

[L] S.L. Lipscomb, On embedding finite-dimensional metric spaces, Trans. Amer. Math. Soc. 211 (1975), 143–160.

[Ma] W. S. Massey, Algebraic Topology: an Introduction, Springer-Verlag, 1967.

[Mi] D. Michalik, Embeddings of n-dimensional separable metric spaces into the product of Sierpiński curves, Proc. Amer. Math. Soc., 138(8) (2007), 2661–2664.

[M-P] J. van Mill and R. Pol, Remark on products of 1-dimensional compacta, Q & A General Topology, 13 (1995), 97–98.

[Na1] J. Nagata, Note on dimension theory for metric spaces, Fund. Math. 45 (1958), 143–181.

[Na2] , Modern Dimension Theory, North-Holland, Amsterdam, 1965.

[Ol] W. Olszewski, Embeddings of finite-dimensional spaces into finite product of 1-dimensional spaces, Top. Appl., 40 (1985), 93–99.

[On] B.W. Ong, The homotopy type of the symmetric products of bouquets of circles, International Journal of Mathematics 14 (2003), 489–497.
EMBEDDING COMPACTA INTO PRODUCTS OF CURVES

[P] R. Pol, A 2-dimensional compactum in the product of two 1-dimensional compacta which does not contain any rectangle, Topology Proc. 16 (1991), 133–135.

[R-S] C. P. Rourke and B. J. Sanerson, Introduction to Piecewise-Linear Topology, Spinger-Verlag, Berlin, Heiderberg, New York, 1982.

[Sp] E. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.

[St] Y. Sternfeld, Mappings in dendrites and dimension, Houston J. Math., 19 (1993), 483–497.

[T] K. Tsuda, A note on closed embeddings of finite dimensional metric spaces II, Bull. Pol. Acad. Sci., 33 (1985), 541–546.

[Wh] J. H. C. Whitehead, Combinatorial homotopy, Bull. Amer. Math. Soc., 55 (1949), 453–496.

[Wu] W. Wu, Note sur les produits essentiel symétriques des espaces topologique I, Comptes Rendus des Seance de l’Academie des Sciences, 16 (1947), 1139–1141.

[Z] E. C. Zeeman, On the dunce hat, Topology, 2 (1964), 341–358.

Department of Mathematics, Faculty of Science, Suruga, Shizuoka University, Shizuoka, 422-8529, Japan
E-mail address: sakoyam@ipc.shizuoka.ac.jp

The Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-950, Warsaw, Poland

Institute of Mathematics and Informatics, University of Opole, ul. Oleska 48, 45-052 Opole, Poland
E-mail address: jokra@impan.gov.pl

The Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-950, Warsaw, Poland
E-mail address: spiez@impan.gov.pl