Using Commensurabilities and Orbit Structure to Understand Barred Galaxy Evolution

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1 INTRODUCTION

The disc of a galaxy, as described by an axisymmetric radial profile, contains only orbits that appear in the $x-y$ plane to be rosettes of varying eccentricities determined by the radial energy. Such regular orbits have constant phase-space coordinates and are considered integrable, and can be represented by a combination of three fundamental frequencies. As stated by the Jeans theorem (Jeans 1915) for an axisymmetric system, the distribution function is a function of the classical integrals of motion, the energy $E$ and the angular momentum $L_z$. This principle has been used in a number of analytic studies over the past century (see Binney & Tremaine 2008), including recent advancements (Binney & McMillan 2016).

Additionally, the Jeans equations (Jeans 1922) have been used to perform assessments of the orbital structure of real galaxies under the assumption that galaxies are semi-isotropic, i.e. they can be described by the classical integrals of motion (Cappellari 2008).

Unfortunately, the assumption that galaxies are semi-isotropic rapidly breaks down for realistic galaxies and dynamical models. While typical rosette orbits in an axisymmetric system fill an area of the disc after many orbital periods, non-axisymmetries in the system create new families of commensurate (or resonant) orbits. Commensurate orbits are governed by the equation

\[ m\Omega_p = l_1\Omega_r + l_2\Omega_\phi + l_3\Omega_z \]

where $\Omega_p$, $\Omega_r$, $\Omega_\phi$, and $\Omega_z$ are the polar coordinate frequencies of a given orbit and $l_1$, $l_2$, and $l_3$ are some pattern frequencies, e.g. the frequency of a bar or spiral arms. Commensurate orbits are closed curves and have formally zero volume. They are the sites where the classical integrals of motion can change, leading to secular evolution.

Even in the case of relatively simple potentials, such as an exponential stellar disc embedded in a spherical dark matter halo, finding the distribution function, fundamental frequencies, and/or commensurate orbital structure analytically can rapidly become intractable. Few axisymmetric potentials that resemble real galaxies can be described via separable potentials that allow for analytic characterization (de Zeeuw & Lynden-Bell 1985). Further, the inclusion of non-axisymmetric features, such as a bar, can render the potential calculation virtually impossible\(^1\). Simply changing the halo model from a central cusp to a central core is known to alter the families of bar orbits present near the center of the galaxy (Merritt & Valluri 1999). Thus, it is difficult to constrain the orbital structure of realistic, evolving galaxies. The lack of techniques in the literature for determining orbital families in evolving potentials applicable to realistic galaxies (e.g. non-axisymmetries) motivates finding new methodologies that determine the orbital content of a galaxy simulation.

\(^1\) For simple analytic bar potential expressions, extensions of analytic studies are able to make some progress (e.g. Binney 2018).
disc galaxy. We present techniques suitable for studying orbits in evolving potentials in this paper.

Analytic and idealized numerical studies of potentials representing barred galaxies show a basic resonant structure that underpins the bar represented by the commensurate $x_1$ orbit, which arises from the inner Lindblad resonance (ILR, where $2\Omega_r = -\Omega_\phi + 2\Omega_\phi$; Contopoulos & Papayannopoulos 1980; Contopoulos & Grosbol 1989; Skokos et al. 2002). However, small adjustments to the mass of the model bar can admit new commensurate subfamilies of $x_1$ orbits, necessitating a model-by-model (or galaxy-by-galaxy) orbital census. For simulated galaxies, by taking the measure of the orbital families present means being able to (nearly) instantaneously identify the family of a given orbit, as the orbit may be changing over a handful of dynamical times. Worse, the orbits may only be near a closed orbit in phase-space, blurring the morphological features of commensurate orbits. Bifurcations of prominent orbit families result from alterations to the potential shape, giving rise to families such as the 1/1 (sometimes stylized 1:1) orbit, a family which results from the bifurcation of the $x_1$ family (Contopoulos 1983; Papayannopoulos & Petrou 1983; Martinet 1984; Petrou & Papayannopoulos 1986). For consistency, we refer to this orbit throughout this work as an $x_{1b}$ orbit, denoting that the family is a bifurcation of the standard $x_1$ orbit.

Finding resonant features has proven particularly difficult in an analytic framework (Binney & Tremaine 2008). Even more difficult is identifying the act of ‘trapping’, or capture into resonant orbits, which by definition truly fixed potential models cannot inform. Recently, techniques to describe orbits observed in self-consistent simulations such as those drawn from $N$-body simulations, i.e., those that are allowed to evolve with gravitational responses, have been used to find the rate at which orbits change families and join different structures (Petersen et al. 2016, hereafter PWK16).

Despite the effort placed into studying both analytic potentials and self-consistent simulations, a vast gulf of understanding exists between analytic, fixed, and self-consistent potentials. Linear or weakly non-linear galaxy dynamics can only be extended so far: at some point, the distortions become so strong that it is insufficient to consider perturbations to the system and one must treat the entire system in a self-consistent manner. However, fully self-consistent simulations are encumbered by the many parameters necessary to describe a galaxy, all of which are difficult to control when designing self-consistent model galaxies. Designing model galaxies that match observations of real galaxies, including the Milky Way (MW), is a challenging process. Cosmological simulations circumvent this problem by simulating many galaxies, some of which resemble the Milky Way (Garrison-Kimmel et al. 2018; Nelson et al. 2018), but they cannot reach the resolution required to track an ensemble of individual orbits. Hence, the need for a bridge between analytic and self-consistent work: fixed potential orbital analysis.

Within this framework, many previous studies have used frequency analysis to characterize the properties of orbits, describing orbits by their frequencies in various independent dimensions. The result is a partitioning of orbits into families that reside on integer relations between frequencies, using equation (1). Attempting frequency analysis on an ensemble of real orbits is a natural extension, via either spectral methods (Binney & Spergel 1982; Binney & Spergel 1984) or frequency mapping (Laskar 1993; Valluri et al. 2012, 2016). In some cases, one may compute accurate frequencies and orbit families can be identified on the fly. However, this condition is only realised when the evolution of the system is slow, or the evolution is artificially frozen. Therefore, we develop a new methodology based on a simple and robust geometric algorithm that permits the unambiguous determination of orbital features from frequency analysis while operating on short orbital time series so as to be physically relevant for evolving simulations. We emphasize that the utility of this method extends beyond the proof-of-concept presented here. An advantage to this orbit atlas analysis is its ability to move beyond the standard methods of locating resonances based on frequencies. We are able to empirically determine the location of the closed orbits in both physical and conserved-quantity space.

In this paper, we apply this simple methodology and characterize orbital structure, with a particular emphasis on commensurate orbits, to understand barred galaxy evolution. This work presents significant upgrades to one orbit analysis tool previously published (PWK16), as well as an entirely new algorithm. The goal of this project is to compare orbits between fixed potential simulations and fully self-consistent simulations to discern the evolution of different structures in the self-consistent simulation. Along the way, we demonstrate that (1) we can efficiently dissect bar orbits into dynamically relevant populations, (2) commensurate orbit families can be efficiently found and tracked through time across different fixed-potential realizations, (3) commensurate orbits provide a useful method to analyze self-consistent simulations, and (4) one may infer the dynamical status of barred galaxies from this methodology.

The organization of the paper is as follows. We describe the models we studied in the course of this work and present new techniques in Section 2. Results from different fixed potential models are presented in Section 3. We then discuss the implications of the findings for interpreting other models in Section 3.3. We discuss the implications of the results for observational studies in Section 4. We then use the lessons from our fixed potential analysis to interpret the evolution in the self-consistent simulations in Section 5. We conclude and propose future steps in Section 6.

## 2 METHODS

We first present the initialization and execution of self-consistent disc and halo simulations in Section 2.1. An overview of the improved $k$-means orbit classifier for closed orbit identification presented in PWK16 is discussed in Section 2.2. The extracted potentials we use for detailed study of fixed-potential integration are described Section 2.3 and the determination of the bar position and pattern speed in Sections 2.3.2 and 2.3.3, respectively. In Section 2.4, we describe the creation of an orbit atlas for each model, including the initial condition population (Section 2.4.1), and integration method (Section 2.4.2).

### 2.1 Simulations

We employ two galaxy simulations in this work. The simulations used here are updated slightly from the simulations presented in PWK16, including a modestly more concentrated halo and significantly longer time integration. We justify both changes at the end of this section.

#### 2.1.1 Initial Conditions

Both simulations feature an initially spherically-symmetric Navarro-Frank-White (NFW) dark matter halo radial profile (Navarro et al. 1997), which we generalize to include a core where the density $\rho_0(r)$ becomes constant with radius:
Table 1. Potential models used in the detailed fixed potential study.

| Potential Number | Simulation Name, Time | Potential Name | Scalelength $R_{\text{vir}}$ [kpc] | Disc Mass $M_{\text{halo}}$ [M$_{\odot}$] | $r_{\text{M}_h=2M_d}$ [R$_{\text{vir}}$] | Pattern Speed $\Omega_p$ [rad/T$_{\text{vir}}$] |
|------------------|----------------------|----------------|-----------------------------|-----------------------------|-------------------|-------------------------|
| I                | Cusp Simulation $T=0$ | Exponential Cusp | 0.01                        | 0.025                       | 0.0050            | 0.0167                   | 90                      |
| II               | Cusp Simulation $T=2$ | Barred Cusp     | 0.01                        | 0.025                       | 0.0050            | 0.0172                   | 37.5                    |
| III              | Core Simulation $T=0$ | Exponential Core | 0.01                        | 0.025                       | 0.0022            | 0.0317                   | 70                      |
| IV               | Core Simulation $T=2$ | Barred Core     | 0.01                        | 0.025                       | 0.0024            | 0.0282                   | 55.6                    |

Figure 1. Circular velocity curves as a function of radius, computed for the cusp and core simulations at $T = 0$ and $T = 2$. The left panel shows the two exponential disc models ($T = 0$, the initial conditions of each simulation), while the right panel shows the two barred models ($T = 2$, after moderate evolution in each simulation). Both panels are color coded as shown above the panels. The solid lines are the circular velocity at each radius computed from the monopole for the total system. The dashed (dotted) lines are the monopole-calculated circular velocity for the disc (halo) component only.

\[ \rho_0(r) = \frac{\rho_0 r_s^3}{(r + r_s)(r + a)} \]  

where $\rho_0$ is a normalization set by the chosen mass, $r_s = 0.04 R_{\text{vir}}$ is the scale radius, $R_{\text{vir}}$ is the virial radius, and $r_c$ is a radius that sets the size of the core. $r_s$ is related to the concentration, $c$, of a halo by $r_s = R_{\text{vir}}/c$. The halo has $c = 25$, consistent with a normal distribution of halo concentrations from recent cosmological simulations (Fitts et al. 2018; Lovell et al. 2018). The normalization of the halo is set by the choice of virial units for the simulation, such that $R_{\text{vir}} = M_{\text{halo}}/v_{\text{vir}} = T_{\text{vir}} = 1$. Scalings for the MW suggest that $R_{\text{vir}} = 1 = 300$ kpc, $M_{\text{halo}} = 1 = 1.4 \times 10^{12}$ M$_{\odot}$, $v_{\text{vir}} = 1 = 140$ km s$^{-1}$, and $T_{\text{vir}} = 1 = 2$ Gyr. The motivation behind generalizing the NFW profile to include a core lies in the ambiguity of the central density of dark matter halos in observed galaxies, including the MW (McMillan 2017). The pure NFW profile extracted from a dark matter-only simulations is cuspy. The first two potentials have $r_s = 0$ and, therefore, we refer to these as ‘cusp’ potentials, extracted from the ‘cusp simulation’ (Table 1).

We embed an exponential disc in the halo, where the three-dimensional structure of the disc is given as an exponential in radius and an isothermal sech$^2$ distribution in the vertical dimension:

\[ \rho_d(z, r) = \frac{M_d}{8\pi a^2} e^{-r/a} \text{sech}^2\left(\frac{z}{z_0}\right) \]  

where $M_d = 0.025 M_{\text{vir}}$ is the disc mass (in line with estimates for the present-day MW; Bland-Hawthorn & Gerhard 2016), $a = 0.01 R_{\text{vir}}$ is the disc scale length, and $z_0 = 0.001 R_{\text{vir}}$ is the disc scale height, which is constant across the disc.

We set $r_s = 0.02 R_{\text{vir}} (= 2a)$ for the second simulation, and we refer to the simulation as the ‘core simulation’. We tailor $\rho_0$ for the cored simulation initial condition such that the virial masses are equal to that of the cusp simulation, i.e. $M_{\text{vir, cusp}} = M_{\text{vir, core}} = 1$. We again embed a 0.025 $M_{\text{vir}}$ initially exponential disc in this halo (Table 1).

Both simulations presented here have $N_{\text{disc}} = 10^6$ and $N_{\text{halo}} = 10^7$, the number of particles in the disc and halo component, respectively. The disc particles have equal mass. We employ a ‘multimass’ scheme for the halo to increase the number of particles in the vicinity of the disc. The halo particles have a number density $n_{\text{halo}} \propto r^{-\alpha}$ with $\alpha = 2.5$. The resolution of the inner halo, $r < 0.05 R_{\text{vir}} (= 5a)$, is improved by roughly a factor of 100, making the mass of the average halo particle in the vicinity of the disc equal to that of the disc particles. This is equivalent to using $10^{10}$ halo particles.

As in PWK16, the halo velocities are realised from the distribution function produced by an Eddington inversion of the density profile (see Binney & Tremaine 2008). Eddington inversion provides an isotropic distribution, roughly consistent with the observed distributions in dark matter-only ΛCDM simulations. The disc velocities are chosen by solving the Jeans’ equations in cylindrical coordinates in the combined disc–halo potential, also as in PWK16. We employ standard techniques such as those found in Binney & Tremaine (2008). The radial velocity dispersion is set by the choice of the Toomre $Q$ parameter such that

\[ \sigma_r^2(r) = \frac{3.36 \Sigma(r) Q}{\Omega_c(r)} \]  

where $\Sigma(r)$ is the disc surface density, and the radial frequency, $\Omega_c$, is given by

\[ \Omega_c^2(r) = r \frac{d \Omega_p^2}{dr} + 4 \Omega_p^2, \]  

where $\Omega_p$ is the azimuthal frequency. Our choice of $Q = 0.9$ is motivated by our desire to form a bar in a short time period.

As discussed in PWK16, the maximum contribution to the total circular velocity by the disc, $f_D \equiv V_{c, d}/V_{c, tot}$, for typical disc galaxies is $f_D = 0.4 \sim 0.7$, with $\langle f_D \rangle = 0.57$ (Martinsson et al. 2013). Our cusp simulation has $f_D = 0.65$ and our core simulation has $f_D = 0.75$. With the new simulations, we evolve until $T_{\text{vir}} = 4.5$. For a MW-like galaxy, this is equivalent to 9 Gyr. We acknowledge that it is unrealistic to expect that a galaxy will evolve in a purely secular fashion for half the age of the universe, without interactions or mass accretion. However, integrating the simulations for a substantial time allows for a full range of evolutionary states to develop as discussed below, which help to probe the dynamical mechanisms behind bar evolution in the real universe.

2.1.2 N-body Simulation

To integrate orbits, we require a description of the potential at all points in physical space. We accomplish this using a bi-
orthogonal basis set of density-potential pairs. We generate density-potential pairs using the basis function expansion (BFE) algorithm implementation EXP (Weinberg 1999). In the BFE method (Clutton-Brock 1972, 1973; Hernquist & Ostriker 1992), a system of bi-orthogonal potential-density pairs are calculated and used to approximate the potential and force fields in the system. The functions are calculated by numerically solving the Sturm-Liouville equation for eigenfunctions of the Laplacian. The description and study of the eigenfunctions that describe the potential and density is the focus of a companion paper (Petersen et al. 2019b).

For the halo, which is a nearly spherical system in our model, we use spherical harmonics given by \( Y_l^m \) where \( m \leq l \), with the radial functions determined from the corresponding model NFW potential such that the radial function corresponding to the lowest order \( Y_0^0 \) spherical harmonic matches the potential and density of the input radial NFW profile exactly. To capture evolution, the halo is described by \( \ell_{\text{halo}} (\ell_{\text{halo}} + 2) + 1 \times n_{\text{halo}} \) terms, where \( \ell_{\text{halo}} \) is the maximum order of spherical harmonics retained and \( n_{\text{halo}} \) is the maximum order of radial terms kept per \( \ell \) order.

A cylindrical basis represents the disc, as described in Weinberg (1999). The cylindrical basis is expanded into \( n_{\text{disc}} \) azimuthal harmonics with \( n_{\text{disc}} \) radial subspaces. Each subspace has a potential function with corresponding force and density functions. The lowest-order disc pair matches the initial equilibrium profile of the analytic functional form given in equation (3). As in all eigenfunction solutions, the full series is orthogonal and complete. Successive terms probe finer spatial structure. We truncate the series to follow structure formation over a physically interesting range of scales, which has the added benefit of reducing small-scale noise including two body scattering. We may select different features by excluding functions where structural variations are not of interest.

The potential at any point in the simulation is represented by \( \left(n_{\text{disc}} + 1\right) \times n_{\text{halo}} \) coefficients for the corresponding orthogonal functions. The disc basis functions are identical between the cusp and core models. The halo basis functions are necessarily different to capture the initial density profile in the lowest-order term. We retain azimuthal and radial terms \( m_{\text{disc}} = l_{\text{halo}} = n_{\text{halo}} \leq 6, n_{\text{disc}} \leq 12, n_{\text{halo}} \leq 20 \) chosen for both the disc and halo depending upon the simulation goals. We discuss the effect on our results owing to the inclusion or exclusion of higher-order harmonic subspaces \( (m = 3, 4, 5, 6) \) in detail in Section 3.1.3. The halo has a larger number of radial \( n \) terms to probe similar scales in the disc vicinity. The disc basis is truncated at \( r = 0.2 R_{\text{vir}} \), outside of which we calculate its contribution using the monopole term only.

As will become important below, EXP allows for an easy calculation of the potential from both the initial galaxy mass distribution as well as the evolved galaxy mass distribution. Owing to the functional representation of the basis, we can create an extremely high-accuracy force field from nearly any distribution of particles. The key limitation of the BFE method lies in the loss of flexibility in the truncated bases; large deviations from the equilibrium disc or halo will not be well represented. However, this limitation can also be a great asset in gaining physical insight; if we restrict the evolution to physically important functions, one can make valuable comparisons with other methods such as matrix methods or standard perturbation theory.

2.2 Computing Trapping

We have developed several improvements to the \( k \)-means method of PWK16 that enables one to determine the membership in different orbital subpopulations beyond the bar-supporting orbits during the simulations. No conclusions from PWK16 change as a result of this upgrade; the classification is simply more detailed with the new scheme described here. The more sophisticated algorithm builds upon the same \( k \)-means technique, but uses additional diagnostics related to the distribution of apsides within the \( k \) clusters to determine membership in orbit families. The details of the classification procedure are described in Appendix A. In this section we give a qualitative overview and discuss the theoretical motivation behind our cluster-based orbit classification.

The orbits that make up a galaxy model are both a reflection of, and support, the potential of the galaxy. The pioneering work of Contopoulos & Papayannopoulos (1980) presented a census of bar-supporting orbits, including the principal \( x_1 \) family. Called the ‘backbone’ of the bar, \( x_1 \) orbits exist at various energies set by the shape of the potential. However, determining family membership in self-consistent models has remained elusive. The concept of the trapping of orbits into reinforcing structures in the potential is a dynamically complex, but straightforward, process under idealized conditions. In the case of perturbation theory, one may compute a capture criterion or trapping rate (e.g. Contopoulos 1978; Henrard 1982; Binney & Tremaine 2008; Daniel & Wyse 2015), i.e., the probability that an orbit joins a particular resonance parented by some closed commensurate orbit for which the potential may be specified.

In a self-consistent evolving galaxy, the process and probability of being captured into a resonance—and even the location resonance itself—is difficult to ascertain. Several techniques have focused on the use of ‘frozen’ potentials. First, a model is evolved self-consistently up to some time. Then the potential is frozen and orbits are then integrated in the fixed potential to determine the orbital structure. We use a hybrid approach where we simultaneously analyze frozen potentials and self-consistent simulations. With input from analytic orbit family descriptions, we hope to dissect our models using the \( k \)-means methodology at every timestep to determine the constituent orbits while the simulation undergoes self-consistent evolution. We call the identification of orbit families during self-consistent evolution ‘in vivo’ classification. In practice, this means selecting some finite time window of the orbit’s evolu-

Figure 2. Three primary self-consistent bar orbit families classified from the cusp simulation near \( T = 2 \). The upper panels are the trajectories, while the lower panels are the time-integrated densities, or relative occupation (i.e. showing where the trajectory moves faster or slower such that an orbit resides at a position for longer). The orbits are organized from largest radial extent to smallest, with the red bars indicating 0.5\( r \) in each panel. From left to right: (a) A standard \( x_1 \) orbit. (b) A bifurcated \( x_{13} \) orbit. (c) An ‘other’ bar orbit, in this case, a nearly 4:2 orbit. All orbits are plotted in the frame rotating with the bar.
tion with in which we determine membership in an orbital family. The \(k\)-means classifier is largely insensitive to variations in family membership on time scales smaller than half the rotation period of the bar.

The mass that supports the bar feature is a fundamental quantity in a barred galaxy model. However, determining the trapped mass is not an easy task, as we must empirically find parameters for determining trapped orbits in vivo, and both systematic and random errors cause uncertainty. Despite this, our \(k\)-means technique efficiently locates and identifies orbits that are members of the bar, sacrificing only minimal time resolution. The required time resolution is on the order of a handful of turning points per orbit. Other classifiers rely on an instantaneous spatial or kinematic determination of the disc galaxy structure. The strength of our \(k\)-means method is that it depends only on the positions of the turning points relative to the bar angle. This makes the methodology (a) fast and (b) independent of detailed simulation processing. The closest analog to our procedure found in the literature is that of Molloy et al. (2015), who used rotating frames to more accurately calculate the epicyclic frequency. However, this procedure is only robust for orbits that are not changing their family over multiple dynamical times. Our method is robust to orbits that are only trapped for one or two dynamical times.

We classify three primary types of bar orbit, with prototypical orbits for each shown in Figure 2:

(i) \(x_1\) orbits, the standard bar-supporting orbit (panel a of Figure 2),
(ii) \(x_{1b}\) orbits, a subfamily resulting from a bifurcation of the \(x_1\) family that are often referred to in the literature as 1/1 orbits (panel b of Figure 2)\(^2\),
(iii) ‘Other’ bar-supporting orbits that are coherently aligned with the bar potential but are not part of the \(x_1\) family, generally demonstrating higher-order behavior (panel c of Figure 2).

The orbits in Figure 2 are drawn from the cusp simulation as having been trapped into their respective (sub)families at \(T = 2\). Each orbit has the time series from the cusp simulation \(T = 1.8 - 2.2\) plotted in the upper row, with the time-averaged orbit density shown in the bottom row. In these examples, as in most cases drawn from self-consistent simulations, the true nature of the orbit is difficult to determine from the trajectory, but becomes apparent from the time-integrated location, motivating our inclusion of the time-integrated location, or ‘relative occupation’ in space, throughout this work.

2.3 Fixed Potentials

An in-depth study where one investigates the potential at every timestep is computationally intractable. Therefore, we select four example potentials where we fully decompose and describe the orbit structure, and apply the general results to the evolution of barred systems in later sections.

2.3.1 Potential Selection

From each of the cusp and core simulations, we compute the fixed potential at two times, \(T_{\text{vir}} = 0\) and \(T_{\text{vir}} = 2\), in which we will characterize the orbit structure. As discussed in Section 2.1.2, to extract the potential structure at any given time in the simulation, we compute the coefficients for the basis functions used for integration by EXP. Each particle’s contribution can be calculated by projecting the particle onto the tabulated basis functions. We calculate the potential for the entire ensemble by accumulating the contribution from all particles in the system, resulting in coefficients that serve as the weights for the different functions. The coefficients for the exponential models are calculated from the initial distribution (\(T_{\text{vir}} = 0\)), and are dominated by the lowest order term by design. The coefficients for the barred models (\(T_{\text{vir}} = 2\)) are calculated from the self-consistent evolution of the systems. The first potential is the initial exponential disc embedded in the spherical NFW cusp halo, Potential I (Exponential Cusp). We also self-consistently evolve the exponential disc to a time after a bar has formed to \(T_{\text{vir}} = 2\), Potential II (Barred Cusp). Similarly, we choose the analogous time points for the cored simulation, Potential III (Exponential Core), and Potential IV (Barred Core). The bar in the core self-consistent model is also still slowing and evolving, including active lengthening at the time we selected. The evolution of the core simulation is discussed in Section 5.2.

In Figure 1 we show the circular velocity calculated from the monopole contribution (i.e. the enclosed mass) as a function of radius (solid lines). The four potentials are color coded as indicated in the figure. In both panels, we decompose the total circular velocity into contributions from the halo (dotted lines) and disc (dashed lines). As the initial discs are the same between the cusp and core simulations, differences in total circular velocity are caused by the halo. The halo models remain largely unchanged between the Exponential and Barred version of the models, with modest (<10 per cent) changes to the enclosed mass between the initial and barred states within a scalelength. Both models become more concentrated with time. Further, the monopole of the disc models, which are identical in the Exponential Cusp and Exponential Core models, are remarkably similar in the Barred states. In the Barred Cusp model, the disc contribution is nearly identical outside of two disc scalelengths, but appreciably different inside of two disc scalelengths. The Barred Core model deviates significantly from the initial distribution out to four scalelengths, the result of a rapidly growing instability during the bar formation epoch that rearranges the entire disc distribution.

2.3.2 Bar Parameters

A crucial ingredient in the algorithm that identifies trapped orbits is the phase angle of the bar. Previously we had employed ellipse fitting (PWK16), the most traditional bar determination metric, or stellar surface density Fourier analysis. Both of these methods are subject to large-scale contributions that may not be related to the actual bar feature, such as spiral structure. The harmonic basis itself provides a filter of the key length scales for the \(m = 2\) power. A harmonic method for determining the gross rotational properties of the bar is more robust than an ellipse fitting method, which is biased by the selection of bar metrics, such as the chosen ellipticity where the bar ends.

Our choice of the \(n = 2\) radial order to determine the bar phase angle is a balance between the undesired power of the \(m = 2\) in spiral arms in the outer disc and the largest scale of desired
power in the vicinity of a disc scalelength. We have verified that the \( m = n = 2 \) function produces the best characterization of the bar pattern speed for all the models studied in this work. A comparison with the \( m = 2, n = 1 \) harmonic shows that the position angle varies only modestly from the \( m = n = 2 \) harmonic. However, a periodicity from the resolution of spiral arm structure appears, hence our choice to use \( m = 2, n = 2 \).

### 2.3.3 Figure Rotation

The dynamics and orbital structure are set by the pattern speed of the bar, \( \Omega_p \). The rotation of the model introduces the Coriolis and centrifugal forces in the bar frame, which depend on \( \Omega_p \). For the barred potential models, we determine \( \Omega_p \) by calculating finite differences in the rate of change of the coefficient phase in a finite window of the time series of coefficients from the self-consistent simulation. We calculate the instantaneous uncertainty to be 5 per cent. Fortunately, we find that variations of 5 per cent to the pattern speed make little difference to the resultant orbital structure. For the exponential potentials, Potentials I and III, we test two pattern speeds: \( \Omega_p = 0 \), which reveals the unperturbed structure of the disc and halo system, and an estimated \( \Omega_p \) from the self-consistent simulation. We estimate \( \Omega_p \) as \( T \to 0 \) using the coefficient phases as above for the earliest possible time, \( T \approx 0.2 \). For the exponential cusp we use \( \Omega_p = 90 \) and for the exponential core we use \( \Omega_p = 70 \). We apply these pattern speeds to the \( T = 0 \) potential models below. As we shall see, the introduction of figure rotation, and thus Coriolis and centrifugal forces, reveals that orbital structure varies with \( \Omega_p \).

In Figure 3, we show the classical analysis of resonance radii computed from the potential and extracted pattern speed. The left panels plot the exponential potentials (Potentials I and III) and we see that lowering the assumed pattern speed moves the calculated corotation radius outward. We also observe that the ILR does not exist at all in the exponential core model for all realistic values of \( \Omega_p \). The outer Lindblad resonance (OLR) exists for all values of \( \Omega_p \). However, in the barred cusp, the radius of the OLR occurs at such large radii (and thus low stellar density) so that it would have little influence on the structure of the disc. In the right panels, we plot the barred potentials (Potentials II and IV) and compute the frequency along the axis of the bar. Selecting other azimuths would result in the computed locations of the resonances moving outward in radius. In both the right panels, we assume the calculated pattern speed to estimate the location of the key resonances. The presence of the bar, despite the increased concentration of mass (obvious in the changed circular velocity curve at \( r < 2\alpha \), cf. Figure 1), results in the location of the key resonances occurring at larger radii than in their exponential counterparts. The bar perturbation and the mass rearrangement resulting from secular evolution create an ILR in the barred core potential (Potential IV) where none existed in the exponential core case (Potential III), as well as creating a second ILR at a larger radius in the barred cusp potential (Potential II).

#### 2.4 Orbit Atlas Construction

We detail the construction of an atlas of orbits for the four studied fixed galaxy potentials described above. The atlas consists of a time-series of orbits for each model with a range of initial conditions. In section 2.4.1, we describe the initial conditions for these orbits. In section 2.4.2 we describe both the principles of orbit integration and the details of our implementation.

##### 2.4.1 Initial Condition Selection

In PWK16 we describe phase space using energy and angular momentum. Angular momentum was expressed as a fraction of the angular momentum of a circular orbit at the same radius, i.e. \( \kappa \equiv L_{z,\text{orbit}}/L_{z,\text{circular}} \) to create a roughly rectangular grid that extends from radial to circular orbits. For this work, we choose a more observationally-motivated set of dimensions: apocenter radius (\( R_{\text{apo}} \)) and apocenter tangential velocity (\( V_{\text{apo}} \)). In these coordinates a radial orbit, i.e. \( \kappa = 0 \), corresponds to \( V_{\text{apo}} = 0 \) and a circular orbit, i.e. \( \kappa = 1 \), corresponds to \( V_{\text{apo}} = v_c(R_{\text{apo}}) \). We define the orbital apocenters along the major axis of the bar potential. We have investigated other release angles, but find that the bar axis is the most illustrative of the dynamics. In certain cases, it is necessary to use off-axis release angles to find orbits that are known to be relevant (described below), but we do not perform an exhaustive search of parameter space. We reserve a detailed study of the off-axis release angles for future work. While the dimensions do not fully sample phase space, this ‘pseudo-phase-space’ gives a intuitive understanding of the system, and can be directly applied to observations.

For this study, we also restrict orbits to the plane. The inclusion of non-planar motion would be straightforward, although the phase-space is complex to explore. We will investigate vertical commensurabilities in a future work. We choose to uniformly sample the \( R_{\text{apo}} - V_{\text{apo}} \) plane, despite large regions of this space being irrelevant for physical systems (e.g. highly radial orbits at large radii in the disc) because the space that is physically inaccessible for a regular galaxy is still of intrinsic interest to a wholistic study.
Figure 4. Relative orbit area plots for the four models, including two values of $\Omega_p$ for each of the exponential models. In each panel, we highlight and label key commensurabilities identified with the geometric algorithm in white. We plot and label the locations of corotation and the outer Lindblad resonance, as computed numerically from the monopole where possible, in cyan. The commensurabilities are discussed in detail in Section 3.1 for the cusp model and Section 3.2 for the core model. The gray region at $0.0 < a < 0.2$ was not integrated, owing to the limits of numerical resolution in this study. In the lowest two panels, we show the circular velocity along the bar major axis, computed from the potential, as a red dashed line.

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of galaxy evolution and will be discussed further in future works. We choose $R_{apo} \in (0.2a, 5a)$ and $V_{apo} \in (-0.2, 1.6)$. The lower limit on $R_{apo}$ is one of practicality; the technique for understanding orbit structure discussed below does not apply to very small radius orbits owing to the requirement that orbits are regular, an assumption that is not guaranteed at $r < 0.2a$. We will return to the question of central orbits in future work. The choice of maximum $R_{apo}$ is driven by a desire to study the structure of the stellar disc; outside of this radius, where the stellar density has significantly diminished, we do not observe any strong commensurabilities.

We wish to explore the entirety of the relevant phase space and from an initial study of the simulations we see that retrograde orbits play some role in the dynamics of the disc at small radii. Hence, we truncate $V_{apo}$ at $V_{apo} = -0.2$ to study the relevant retrograde phase space. Similarly, at large $V_{apo}$, we include values beyond the circular velocity since some orbits may occasionally be driven to larger velocities than the circular velocity at a given radius by a non-axisymmetric potential.

2.4.2 Integration

The integration of the time-series of orbits to assemble the atlas could be accomplished using a variety of integrators, so long as the integrator has high accuracy over tens of dynamical times. In the rest of this section, we describe specific details of to our integration scheme, which is based on the leapfrog integrator used in EXP. The integrator we use here also includes the following features beyond the standard leapfrog integrator: (1) an implemenation of adaptive timesteps from EXP, described in PWK16, with minimum timestep thresholds; (2) completion criteria set by either total integrated time or the number of apsides reached. We match the timestep to the minimum timestep in the self-consistent simulations, $\Delta t_{\text{EXP}} = 3.2 \times 10^{-4}$. We truncate the evolution after 50 radial periods have been completed or a maximum of $\Delta T_{\text{vir}} = 0.64$.

As noted in Section 2.1.2, each component is defined by a unique set of basis functions. Therefore, in addition to filtering the spatial scales to those of interest, the integration can either include the whole potential as extracted from the self-consistent simulation, or one can pick and choose elements for computational efficiency and accuracy. By excluding higher-order terms that do not influence the integration of individual orbits, we can achieve $\approx n_{\text{aps}}/1 - \Omega_p^2$ per speedups, where $n (l)$ is the total number of radial (azimuthal) halo functions and $n' (l')$ is the number of retained radial (azimuthal) halo functions. After inspecting the signal-to-noise ratio in the coefficients, we choose not retain higher order halo azimuthal terms with $l > 2$, resulting in an 88 per cent speedup of the halo calculation, without any important differences in the results.

Our integration is left flexible in the following ways: (1) the number of azimuthal harmonics in the disc may be specified at runtime, which allows for our primary test cases of restricting to only the monopole potential and of eliminating odd harmonics; (2) the range of radial basis functions, which allows for noise-based experiments; and (3) the pattern speed of the bar is also a runtime variable. We do not apply odd multiplicity azimuthal harmonics, which are empirically determined in the self-consistent simulations to have a different pattern speed than the even multiplicity azimuthal harmonics. In principle, we could use different values of $\Omega_p$ for individual harmonic orders, e.g. $\Omega_p, m = 1$ and $\Omega_p, m = 2$, allowing for an investigation of the dipole’s influence separately from that of the quadruple. We aim to study this phenomena in future work.

| Model                     | $x_{14.1}$ | $x_{18}$ | CR | $3n$ |
|---------------------------|------------|----------|----|------|
| Exponential Cusp (I), $\Omega_p = 0$ | ×          | ×        | ×  | ×    |
| Exponential Cusp (I), $\Omega_p = 90$ | ×          | ×        | ×  | ×    |
| Barred Cusp (II)          | ✓          | ✓        | ✓  | ✓    |
| Exponential Core (III), $\Omega_p = 0$ | ×          | ×        | ×  | ×    |
| Exponential Core (III), $\Omega_p = 70$ | ×          | ×        | ×  | ×    |
| Barred Core (IV)          | ✓          | ✓        | ✓  | ✓    |

Table 2. Comparison of different orbital families present in the potential models.

2.5 Geometric Algorithm

From a time series of discrete $(x, y, z)$ points, we use Delaunay triangulation (DT) to compute the physical volume that an individual orbit occupies, transforming a discrete time-series of points to a volume. As we are restricting our analysis to the disc plane in this study, the problem is simplified to two-dimensional DT and we can calculate an area. We have tested three-dimensional DT, and will make vertical commensurabilities that are revealed in three dimensions the focus of future work. The full procedure to compute the individual orbit filling areas is described in Appendix B.

We refer to the time-series of integrated orbits associated with each model as the orbit atlas. When processed with the geometric algorithm to compute orbit areas and placed on the $R_{apo} - V_{apo}$ plane, we refer to this as the commensurability map. The loci of $A \approx 0$ defines orbit families, which we refer to as valleys. Valleys may be strong (wide valleys with large regions of $A \approx 0$) or weak (narrow valleys with only a small path satisfying $A \approx 0$). The valleys provide a skeleton of the orbits in a given potential, tracing the commensurate orbits that support the structure of the galaxy model. We therefore refer to the figures that show the orbit area at each point in the $R_{apo} - V_{apo}$ plane as orbital skeletons.

The identification of commensurate orbits provides an important theoretical link between a perturbation theory interpretation and fully self-consistent simulations (Contopoulos & Papayannopoulos 1980; Tremaine & Weinberg 1984; Weinberg & Katz 2007a,b) to find and describe trapped orbits.

3 FIXED POTENTIAL STUDY RESULTS

We apply the tools described above to the potentials described in Section 2.3 with the goal of locating and identifying the commensurate orbit families in each. Commensurability maps for the six orbit atlases calculated from the four potential models are shown in Figure 4. The maps are in $R_{apo} - V_{apo}$ space. The color map shows the values of the area $A$, as sampled by the initial conditions listed above. The color scheme is uniform throughout the paper; colors in Figure 4 may be compared to the two orbits in Figure B1 for intuition on the colormap. The white lines in Figure 4 are the identified valleys. We do not plot all the valleys identified, but rather restrict ourselves to those with dynamical import to avoid confusion. Further, where possible, we use the monopole-calculated frequency to calculate the location of CR and OLR.

With the valleys mapped, we can see that commensurabilities follow tracks through physically adjoining regions of the galaxy model by inspecting the morphology of orbits. In many cases, the valleys intersect. At these points, we expect to find weakly chaotic behavior in a self-consistent simulation. Where we can identify commensurabilities, important regions of the galaxy
model can be queried for other physically important quantities, such as angular momentum transfer, which we discuss below. Overall, we find similar orbit families to classic analytic studies (e.g. Contopoulos & Grosbol 1989; Athanassoula 1992; Sellwood & Wilkinson 1993). Figure 4 bears some resemblance to the so-called ‘characteristic diagram’ found in the analytic literature, although the advantage of the commensurability map is to additionally demonstrate the area over which orbits may resemble the parent orbit. Table 2.5 summarizes the observed orbit families for each of the galaxy models.

We first discuss the orbit families in the cusp models before turning to the cored models. For both sets of models, we begin with the non-rotating axisymmetric models to understand the unperturbed commensurabilities (Sections 3.1.1 and 3.2.1). We then impose a pattern speed upon the axisymmetric models (Sections 3.1.2 and 3.2.2), followed by the bar-like non-axisymmetric models in Sections 3.1.3 and 3.2.3. In Section 3.3, we present the results of applying the geometric algorithm to orbits extracted from the self-consistent simulation. We compare the differences between the fixed potential models and the self-consistent simulations in Section 3.4.

3.1 Cusp Models

3.1.1 Exponential Cusp (I), $\Omega_p = 0$

The zero pattern speed exponential cusp potential (Potential I) reveals the inherent commensurate families that arise when a disc is embedded in a dark matter halo. We present the commensurability map in the upper left panel of Figure 4. We overlay the orbital skeleton as determined via our geometric algorithm.

The circular orbit curve (labeled) appears as the most clearly defined valley. Crossing the circular orbit valley are several $mn$ commensurabilities, where $n$ is the radial order and $m$ is the azimuthal order, satisfying Equation (1). The 3:2 commensurability (labelled) is the strongest commensurability crossing the circular orbit curve. As we shall see below, the $mn$ where $m = 3$ families exist in barred potentials as well. This family include the 3:1 family, which has been previously studied in the literature (Athanassoula 1992), and is considered a bifurcation of the $x_1$ family. Here, we treat the 3:$n$ families as a separate resonance that often overlaps in phase-space with the $x_1$ family. Even radial orders do not appear in this potential. The values of $m$ increase toward smaller radius, such that the next strongest commensurability curve is 5:2, then 7:2, and so on (not labelled). These high-$m$ and $n$ resonances are not expected to be important for the evolution of the system. We will show that this expectation is realised in Section 5. A physically uninteresting radial orbit commensurability valley also appears at $V_{apo} = 0$.

Additionally, while we show the parameter space above the circular velocity curve ($V_{apo} > v_c (R_{apo})$), we do not expect to see many orbits populating the region of phase space that is super-circular in a real galaxy.$^3$ For a barred model, this is not explicitly true as orbits that are perpendicular to the bar at some radius may appear faster than circular when parallel to the bar axis. In the barred models, launching at velocities faster than circular will lead to orbits that are commensurate but oriented perpendicular to the bar, to which we are not sensitive. Thus, we will tend to restrict our discussion of features to those related to regions of phase space that have a lower energy than that of a circular orbit at the same radius.

3.1.2 Exponential Cusp (Potential I), $\Omega_p = 90$°

The rotating exponential cusp, also with the underlying Potential I, reveals new structure not present in the non-rotating version of the potential when we impose a pattern speed of $\Omega_p = 90$, an estimate for the initial formation pattern speed of the bar. We present its commensurability structure in the middle left panel of Figure 4. Owing to the axisymmetric nature of the potential, the circular orbit valley is unchanged from the same potential model with $\Omega_p = 0$. However, the radial orbit commensurability seen at $V_{apo} = 0$ in the non-rotating case is not well defined in the rotating model, occupying a negligible region of phase space that is below the resolution of the commensurability map.

$^3$ In fact, this must be true for an axisymmetric model.
A rotating model admits strong low-order resonances, including the inner Lindblad resonance (ILR), corotation (CR), and the outer Lindblad resonance (OLR). In this axisymmetric model, we can use the lines shown in the upper left panel of Figure 3 to guide our interpretation. We expect to see ILR at $r = 0.2a$, CR at $r = 1.3a$, and OLR at $r = 2.8a$ along the circular orbit track. We can compute the location of CR and OLR using the monopole component of the potential, following the description in Appendix B. With the monopole-derived commensurabilities placed on Figure 4 in cyan, we see an extended patch of low relative orbit area intersecting the circular orbit track at $r = 1.3a$ coincident with the cyan CR line. At lower tangential velocities than circular orbits, and thus higher eccentricities, corotation appears as a thin track that descends in velocity with a mild dependence on radius ($1.4a < R_{apo} < 1.8a$), deviating from the cyan track for high eccentricities ($V_{apo} \rightarrow 0$). Additionally, one can see OLR crossing the circular rotation track at $r = 2.8a$, and continuing to lower $V_{apo}$. However, we do not see an obvious ILR in the diagram, as one expects given the minimum radius of the orbit map and the estimate from Figure 3.

Many higher-order resonances are clearly seen as low-area (dark) loci. These features correspond to the higher order resonances discussed above for the nonrotating model. They are unlikely to be important in a time-varying potential where the pattern speed and underlying potential changes faster than the orbital time for the high-order closed orbit.

3.1.3 Barred Cusp (Potential II)

The barred cusp model (Potential II) admits different classes of orbital families from the exponential cusp in either the non-rotating or rotating cases (Potential I), owing to the strong non-axisymmetric disturbance. We classify three types or subfamilies of $x_{1b}$ orbits:

(i) $x_{1b}$ orbits, which may be symmetric or asymmetric about the axis perpendicular to the bar. A symmetric ‘infinity’ orbit is shown in panel a of Figure 5, and an asymmetric ‘smile’ $x_{1b}$ orbit is shown in panel b (the asymmetric orbits are an example of an orbit which is more readily identified from an off-axis release).

(ii) $x_{1b}$ orbits, short-period bifurcated standard $x_1$ orbits (with ‘ears’, panel ‘c’ of Figure 5).

(iii) $x_{1b}$ orbits, long-period elongated $x_1$ orbits.

We identified the orbit families through visual inspection of the orbit atlas. While many $m:n$ orbits with $m > 1$ are clearly observed in Figure 4, we choose to mark only the strongest (lowest-order) $m$ family, namely, where $m = 3$. As several low-order even $n$ orders comprise the $m = 3$ feature, and are co-located, this is labeled as $3:n$ in Figure 4. In panel ‘c’ of Figure 5, we plot an example 3:2 orbit. All orbits that are asymmetric across the $x$-axis in Figure 5 have corresponding mirror image orbits, where an orbit with one symmetry leads the bar pattern and an orbit with the other symmetry trails it. With a fine enough grid, we find arbitrarily high order commensurabilities (see, e.g. the unidentfied structure in Figure 4 from the 3:n position to CR and beyond). In this work, we restrict our analysis to the low-order strong commensurabilities that form the persistent orbital structure of the barred galaxy.

In Figure 6, we show an example CR orbit in the barred cusp potential (Potential II), which has a strong CR feature. CR is the lowest-order resonance present in the model, with wide-ranging dynamical effects for secular evolution discussed extensively in the literature (see Sellwood 2014 for a review). CR orbits are particularly easy to recover using the geometric algorithm owing to the minimal area spanned by their trajectory, evident in Figure 6. For Figure 4, we use the monopole component of the potential to compute the location of CR, as in the above section, following the procedure in Appendix B. The monopole-calculated commensurability is largely consistent with an area-based commensurability valley in Figure 4. Owing to the long radial periods near CR in this model, the skeleton–tracing algorithm described in Appendix B identifies large regions with $A < 0.1$ and shallow slopes of $\delta A$, making tracing valleys ambiguous. We, therefore, opt to include only the monopole-derived commensurabilities at radii outside of the bar radius.

The inclusion of higher-order harmonic subspaces, $m > 2$, plays a large role in the orbital structure, allowing new families to appear owing to relatively small but important changes in the potential. In particular, the exclusion of the $m > 2$ azimuthal subspaces from the barred cusp potential model (II) results in an appreciably different orbital structure, including the disappearance of the $x_{1b}$ family. Inspection of all orbits that are part of the $x_{1b}$ family when $m \leq 6$ reveals that the $x_{1b}$ track no longer exists when we restrict

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4 While non-intuitive, ‘symmetric’ in the case of $x_{1b}$ orbits refers to symmetry across the axis perpendicular to the bar in papers on $x_{1b}$ orbits (Contopoulos & Grosbol 1989, e.g.). Thus panel a of Figure 5 is a ‘symmetric’ $x_{1b}$ orbit, owing to the $y$-axis symmetry, and panel b is asymmetric. Without any figure rotation, the symmetric $x_{1b}$ orbit looks exactly like an infinity sign, that is, the crossing point is centered rather than off-center, the ‘antibanana’ of Miralda-Escude & Schwarzschild (1989).
the potential to \( m \leq 2 \). However, this should not be interpreted as evidence that \( 2 < m \leq 6 \) causes new resonant structure into which the \( x_{1b} \) orbits are trapped, rather that \( 2 < m \leq 6 \) distorts the potential shape allowed by the quadrupole only into a potential that admits \( x_{1a} \) orbits. In Section 5, we will see that \( x_{1b} \) orbits are important for growing the bar in length and mass. In Figure 7, we integrate the same orbits as in panels a and b of Figure 5, except that we limit the harmonic subspaces included in the potential to \( m \leq 2 \). The orbits are no longer \( x_{1a} \) orbits. The infinity morphology \( x_{1a} \) orbit is now a part of the less dynamically complex \( x_1 \) family. The smile morphology \( x_{1b} \) orbit has become a ‘boxlet’ orbit.

This finding also suggests that \( m = 2 \) parameterizations of the MW bar\(^5\), such as those derived from the potential of Dehnen (2000), Antoja et al. (e.g. 2014); Monari et al. (e.g. 2016, 2017); Hunt et al. (e.g. 2018), may entirely miss important families of orbits, even if the orbits do not appear to exhibit four-fold symmetry. Other recent models for the MW have suggested the importance of the \( m = 4 \) component of the bar for reproducing the observed velocities near the Sun (Hunt & Bovy 2018).

In the barred cusp potential (Potential II), CR intersects the circular orbit track at \( R_{apo} = 3.1a \). Owing to the relatively slow pattern speed, \( \Omega_p = 37.5 \), CR is at a fairly large radius in this potential model; if we assume that the radial terminus of the \( x_{1a} \) family is the length of the bar, then the ratio of corotation to the bar length in this model is \( R = \frac{\Omega_0}{\Omega_p} = 1.47 \), well within the ‘slow’ regime for bars (Athanassoula 1992). That the pattern speed has slowed so greatly since formation has implications for the observed fast bar-slow bar tension (see e.g. PWK16), suggesting that an observational test that can assist in the determination of the fast bar-slow bar tension (see e.g. PWK16), suggesting that an ob-

\( \text{computed for the trajectories. Analytic work (Contopoulos 1988) suggests that the UHR can arise as a continuation of the } x_{1a} \text{ or-

3.2 Core Model

3.2.1 Exponential Core (Potential III), \( \Omega_p = 0 \)

The commensurability map for the nonrotating exponential cored model shares many similarities with the nonrotating cusp model (Potential I). A comparison of the cusp (I) and cored (III) nonro-

3.2.2 Exponential Core (Potential III), \( \Omega_p = 70 \)

We plot the commensurability map for the rotating exponential cored model in the middle right panel of Figure 4. While we ap-

\(^5\) Some orbits do not show any apparent structure in the inertial frame, filling in an entire circle, but appear to be rectangular ‘boxes’ in the rotating frame owing to the inner quadrupole of the bar (\( \Phi_{bar} \propto x^2 \)) approximating a harmonic potential. These orbits have been called boxlets (Miralda-Escude & Schwarzschild 1989; Lees & Schwarzschild 1992; Schwarzschild 1993). The maximum radial extent of boxlets informs the structure of the potential, but is reserved for future work studying the innermost regions of the potential models.

\(^6\) Ellipsoid-derived bar models such as the Ferrers bar (Binney & Tremaine 2008) will naturally admit \( m = 4 \) power, depending on the axis ratio, such that an increase in axis ratio will increase the \( m = 4 \) power relative to \( m = 2 \). Additionally, the density profile of the bar will contribute to the \( (m = 4)/(m = 2) \) ratio, with an increase in central density leading to a lower \( (m = 4)/(m = 2) \) ratio.
core model (III) to that of the rotating exponential cusp (I). For \( v < v_c(r) \), corotation reaches larger values of \( R_{apo} \) than those reached in the rotating exponential cusp model, reaching a maximum at \( R_{apo} = 2.1a, V_{apo} = 0.4 \). We observe a stronger 3:2 commensurability at larger values of \( R_{apo} \) compared to the rotating exponential cusp model (I). This is expected owing to the combination of a lower pattern speed as well as a shallower potential (and thus shallower energy gradient) for the cored model. CR and OLR computed from the monopole intersect the circular velocity curve at positions more or less in agreement with the estimate from Figure 3.

### 3.2.3 Barred Core (Potential IV)

The barred core model (IV), in contrast to the non-rotating and rotating exponential core models (III), demonstrates clear differences from that of the barred cusp (II). While many of the major commensurabilities remain intact, albeit at different locations in \( R_{apo} - V_{apo} \) space, we do not observe the \( x_{1b} \) family. As in the barred cusp model, we see both \( x_{1a} \) and \( x_{1l} \) orbits, where the \( x_{1l} \) orbits again are spatially co-existent with the \( m = 3 \) series of commensurabilities\(^7\). Despite the models having approximately identical disc monopoles (cf. Figure 1), the difference in the halo model and the larger pattern speed of the bar, \( \Omega_p = 55.6 \), results in CR being located at a significantly smaller radius than in the barred cusp model (II). CR intersects the circular orbit curve at \( R_{apo} = 1.8a \), compared to \( R_{apo} = 3.1a \) for the barred cusp. Similarly, OLR appears at a smaller radius when compared to the barred cusp model (II). Comparing CR with the maximum radial extent of the \( x_{1a} \) orbits at \( R_{apo} = 1.4a \), the ratio of CR to bar length ends up as \( R = 1.28 \), within the fast bar regime, versus \( R = 1.47 \) for the barred cusp (II).

### 3.3 Fixed-Potential vis-a-vis Self-Consistent Simulations

As discussed in Section 2.3, both of the barred potentials are drawn from larger self-consistent simulations. In this section, we place the single snapshots back into the self-consistent context, using outputs drawn from the self-consistent simulations. To retain similarity with the chosen fixed potential models (extracted at \( T = 2.0 \)), we use phase-space outputs satisfying \( T \in [1.8, 2.2] \), with \( dT = 0.002 \), for a total of 200 outputs.

In Figure 8, we use the phase-space outputs to generate self-consistent orbit area maps. We do this by first transforming the 200 outputs to a frame co-rotating with the bar, then feed the sequences for each orbit to the geometric algorithm. An accurate orbit area requires a series of \( \geq 2 \) orbital periods. Orbits with \( R_{apo} \leq 5a \) generally satisfy this criterion, and thus have interpretable results from the geometric algorithm. For each orbit, we then calculate its apocenter in the \( \Delta T = 0.4 \) window and the corresponding tangential velocity at apocenter. The orbits are put into rectangular bins in the \( R_{apo} - V_{apo} \) diagram with \( dr = 0.1a \) and \( dV = 0.05 \). For each bin, we calculate the lowest decile (10\(^{th} \) percentile) relative area from the distribution of relative areas found by the geometric algorithm. We tested alternate particle selection criteria per bin, including mean, median, and minimum, and find that the lowest decile value provided an appropriate balance between feature extraction and overemphasis of outliers that appear in given bins owing to errors in determining \( R_{apo} \) and \( V_{apo} \).

The left panel of Figure 8 shows the results for self-consistent orbits drawn from the barred cusp model. Many regions in the \( R_{apo} - V_{apo} \) plane are not populated in the self-consistent simulation (gray regions). Regions of low relative orbit area correspond to the commensurability tracks from Figure 4 superposed in white. The maximum \( V_{apo} \) for a given \( R_{apo} \) is set by the circular velocity curve, as expected. The region at \( R_{apo} < 2a \) is dominated by nearly commensurate bar orbits close to the \( x_1 \) commensurability valley. However, the distribution is not symmetric in \( V_{apo} \) around the \( x_1 \) valley, but is biased to larger \( V_{apo} \). A close inspection of the self-consistent simulation supports the existence of this bias, which exhibits orbits that resemble \( x_{1b} \) orbits primarily leading the bar, expected for larger \( V_{apo} \) orbits. A larger \( V_{apo} \) at fixed \( R_{apo} \) relative to the \( x_1 \) commensurability valley implies that the orbit has a larger angular momentum than the pattern of the bar itself, and the system is still evolving. At \( R_{apo} = 2a \) and \( V_{apo} = 1.2 \), a valley appears that is not prominently seen in the tracks from Figure 4. An artificially drawn extension of the \( 3:n \) series commensurabilities following an isoenergy line would approximately account for this valley, which suggests that the fixed potential integrations may be missing some key ingredient that affects the self-consistent evolution. We discuss some possible explanations in Section 5.3.

The right panel of Figure 8 shows the results for orbits drawn from the self-consistent barred core model. We again see that the phase space is limited by the circular velocity curve. Once again, orbits with low relative orbit area gather near the \( x_{1a} \) track. A second low relative orbit area feature, at similar \( V_{apo} \) to the \( x_{1a} \) track but at \( R_{apo} \sim 3a \), is also apparent. Comparison with fixed potential orbits reveals that this feature is probably an extension of the \( 3:n \) orbits to lower energies. CR creates a clear valley in the self-consistent barred core model, which extends along an isoenergy track to significantly lower values of \( V_{apo} \), more than the commensurability tracks from Figure 4 would suggest. The relative prominence of features at \( V_{apo}(R_{apo}) < v_c(R_{apo}) \) in the self-consistent orbits when compared to the fixed potential orbits suggests that a missing dynamical degree of freedom in the fixed potential integration, as we also found in the barred cusp model comparison above. Additionally, we find boxlets in the barred core model from inspection of the self-consistent simulation that are below the minimum radius of the orbit atlas (\( R_{apo} < 0.2a \)) owing to the nearly harmonic potential resulting from the inner halo density profile.

### 3.4 Summary

The primary goal of identifying orbit families is to use the unique features of the orbital families present in each model to both understand the dynamical mechanisms and infer the underlying potential to discriminate between halo models. Comparison between the fixed potential integration and the self-consistent simulations yields the following results:

(i) We empirically locate CR and OLR, finding that the barred cusp potential (II) CR location is at a substantially larger radius than the barred core potential (IV) owing to the pattern speed rate of change and the size of the bar.

(ii) We find the presence of the \( x_{1b} \) orbit family in the barred cusp potential, but not in the barred cored potential (or either of the axisymmetric models).

\(^7\) While not a formal phase-space, as discussed elsewhere in this work, residing near the same location in the orbit atlas (\( R_{apo}, V_{apo} \)) implies that orbits and families must be adjacent in phase-space as well, as the mapping to \( (E, \kappa) \) is unique.
(iii) The $x_{11}$ and $3:n$ commensurabilities are co-located at the end of the bar.

(iv) Orbit families observed in the self-consistent simulation cannot be recovered without the $m = 4$ harmonic included in the potential.

The most obvious difference between the barred cusp (II) and core (IV) potentials is the presence of the $x_{1b}$ track, which affects the structure of the orbits in the barred cusp model. Additionally, the $3:n$ valleys lie in populated regions of phase-space in the barred cusp model. Inspection of the orbits in the self-consistent model shows that a common channel to add additional orbits the bar is the transition from $3:n$, where $n = 1$ or $n = 2$, to $x_{1s}$ orbits over short timescales. This fueling of the bar is the result of resonance passage that imparts a change in orbital actions, as per standard perturbation theory. Additionally, regions with commensurability valley intersections (formally a heteroclinic connection) may lead to orbit family switching—the ‘weak chaos’ discussed in Weinberg (2015a,b). In this case the $3:n$ commensurability serves as a conduit by which orbits can join the $x_{1s}$ family, which in turn can trade orbits with the $x_{1s}$ family\(^8\). No such channel exists in the barred core case as the phase-space region where the $3:n$ and $x_{1s}$ orbit valleys intersect is not populated (cf. Figure 8). We will see in Section 5 that the lack of such a channel affects the evolutionary state of a barred galaxy.

4 APPLICATION TO OBSERVATIONS

We describe the tools presented here in the context of observations and observational interpretation. The fixed potential analysis is useful for making direct inferences about the presence of different commensurabilities in observed galaxies. In a future application, we can train orbit finding algorithms to detect complex, sparsely sampled members of the orbit library.

The fixed potential integrations presented here form a bridge between analytic potential study and fully self-consistent simulations. Canonical works on the dynamics of a disc and halo system have relied on the use of potentials with separability, such that the actions can be directly calculated (see, for example, Binney & Tremaine (2008)). Unfortunately, this means that we, as dynamists, are reduced to studying careful simulations, having to convince ourselves of their validity and the validity of the inferences that we make. Dynamical models that one fits to galaxies assume axisymmetry, which will automatically disagree with the findings presented here. Both classic and modern MW potentials are largely assumed to be axisymmetric, in stark contrast to a multitude of observations indicating that the MW is strongly barred (e.g. Bland-Hawthorn & Gerhard 2016). The fixed potential clearly has an incredible utility for determining orbital structure and provides a complete picture of the orbital structure in the system.

We provide an instantaneous description of the phase space in $R_{apo} - V_{apo}$ coordinates to directly parallel observations made with integral field units (IFUs). In Figure 9, we show the stellar mass as a function of instantaneous tangential velocity and instantaneous radius computed from the phase space distribution of the particles in the self-consistent simulation. To normalize the density map, we find the maximum mass in $R_{apo} - V_{apo}$ space, set the value to be equal to 1 and scale all the other masses accordingly. In contrast to the discussion in Section 3.3, here we use only the instantaneous information from the phase space distribution. As orbits spend a larger fraction of their time near apocenter, the signal is not as diluted as one might fear, and hence we undertake a direct comparison of instantaneous quantities and apocenter quantities.

The upper panels of Figure 9 show the cusp potentials, with the exponential cusp potential (I) on the left and the barred cusp

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\(^8\) This mechanism is distinct from ‘strong chaos’, as in Chirikov (1979). For an example of a similar mechanism at work as ‘radial migration’, a rapid process resulting from transient perturbations, see Sellwood & Binney (2002).
Figure 9. Instantaneous tangential velocity, calculated as \( v_T = (\dot{x}y - \dot{y}x)/R \) versus instantaneous radius, drawn from the corresponding self-consistent simulation. The color map is the mass relative to the maximum mass at the same radius \( R \) in the \((v_T, R)\) plane, for the four potential models. Overlaid on each are the traced commensurabilities from Figure 4, including assumed values of \( \Omega_p \) for the exponential models. The circular velocity along the bar major axis, computed from the potential, is shown as a dashed red line in the right two panels.

The barred cusp potential (II) on the right. Overlaid on each of the panels in white are the corresponding commensurability traces from Figure 4. The rotating exponential cusp model (I) places corotation at a smaller radius, \( r_{\text{CR}} = 1.3a \), than the majority of the disc mass\(^9\). Both the rotating and non-rotating exponential cusp potentials admit a zero-order 3:2 frequency at \( r_{\text{CR}} = 2.6a \), as the 3:2 frequency valley results from the disc and halo mass distributions, independent of the bar.

The barred cusp potential (II, the upper right panel of Figure 9) shows a number of features in the mass distribution that correlate with the commensurability traces. In particular, the mass associated with the bar resides within the maximum radius of the \( x_1b \) track, with the material at the end of the bar (\( r = 1.8a \)) spatially coincident with the \( x_1b \) and \( 3:n \) orbits. The location of corotation is exterior to the majority of the disc mass. Local minima in mass density along the circular orbit track as one moves outward in radius appear to correlate with higher-order commensurabilities, suggesting that the orbit families in these regions are unstable.

For the exponential core (III, the lower left panel of Figure 9), like the exponential cusp, the bulk of the mass distribution lies along the empirically-determined circular orbit track (shown in white). The spread in the measured tangential velocity values (at a fixed radius), reflects non-circular motions in the self-consistent simulation. As with the exponential cusp, corotation is interior to the majority of the disc mass, at roughly the same radius, \( r = 1.5a \).

The barred core model (IV) is shown in the lower right panel of Figure 9. Here, the contrast with the barred cusp model is striking. The mass distribution is more continuous between the eccentric bar orbits (\( r < 1.4a \)) and the nearly circular orbits (\( r > 1.4a \)). Without the presence of an \( x_1b \) commensurability track, the bar is limited by the \( x_1s \) track. Additionally, the \( x_1l \) and \( 3:n \) commensurabilities, despite residing in nearly the same physical region of phase space, do not appear to control the structure of the mass density. Rather, that role is ceded to corotation, which meets the circular orbit track at \( r = 2.0 \), where a pile-up of orbits occurs. We again see that higher-order resonances (in this case part of the \( 5:n \) series) cause a disruption in the circular orbit track at \( r = 3a \). Orbits are largely too eccentric to take part in the OLR commensurability track. Taken together, the barred cusp (II) and barred core (IV) models provide examples of which commensurabilities are responsible for structure in barred galaxies. In both models, the \( x_1 \) family dominates the structure of the bar itself, with the mass distribution apparently correlating with commensurabilities. The differences between the cusp and core models are significant enough to discern with an IFU and \( \delta v \approx 10 \) km s\(^{-1}\) velocity resolution targeting galaxies close enough to achieve \( \delta r \approx 0.5 \) kpc resolution.

The \( R_{\text{apo}} - V_{\text{apo}} \) diagram may be constructed using IFU data for a range of inclinations (\( 20^\circ < i < 70^\circ \)) using a simple pro-
cess: (i) transform the line-of-sight velocity distribution to \(x\) and \(y\) velocities, with the \(x\) axis aligned with the bar major axis, (ii) bin the position and velocity data by radius and velocity, (iii) plot the binned data in the \(R_{apo} - V_{apo}\) plane. As shown in Figure 9, the features associated with commensurabilities, e.g. the end of the bar in \(R_{apo} - V_{apo}\) space, are discernable as a local minima in the \(R_{apo} - V_{apo}\) distribution even without knowing the true apocenter values. The orbits spread proportionally more time at apocenter than at any other point in the orbit, leading to an effective weighted average. prominently reflect the closed orbits at resonances.

The \(R_{apo} - V_{apo}\) diagram suggests that the role of CR is appreciably different between the barred cusp and core models. CR plays a larger role in the barred core model, where the resonance is in a well-populated region of phase space, but has little effect in the barred cusp model, where CR is located at a larger radius than the bulk of the disc material. Indeed, the large radius of CR in the barred cusp model suggests that it plays a minimal role in the evolution after the assembly of the bar at \(T > 1\) (\(\approx 2\) Gyr scaled to the MW). The clearest diagnostic is the near-discontinuity in the mass distribution at the end of bar in the barred cusp model, while the barred core model maintains a track connecting bar and non-bar orbits.

Many of the orbits in the self-consistent simulation reside at considerable phase-space distance from true commensurabilities; even in an apparently slowly evolving or steady-state evolutionary phase, orbits can be distant from the true parent orbit. Models that rely on the construction of mass distributions from closed orbits, such as Schwarzschild orbit superposition (Schwarzschild 1979; van der Marel et al. 1998) and made-to-measure (Syer & Tremaine 1996; Dehnen 2009; Morganti & Gerhard 2012; Portail et al. 2015) techniques could be biased against important orbital families that are short-lived, yet crucial for the dynamics, such as the \(x_{1b}\) family. Therefore, we caution that this limitation (and others discussed in Section 5.3) must be considered when interpreting the evolution from models that are constructed using closed orbit libraries. This may be particularly true for the MW, which shows evidence for continued dynamical evolution in the disc (Antoja et al. 2018).

5 SELF-CONSISTENT SIMULATION INTERPRETATION

Now we interpret the time-dependent simulations through the lens of the fixed potential orbital structure in Sections 5.1 and 5.2, and also discuss the limitations of the methodology (Section 5.3). Owing to the caveats discussed elsewhere in this work, we do not expect this to be a complete assessment of the processes at work in the simulation, but rather it illuminates the effect of the potential on orbital structure and the relationship between the orbital structure and the resultant evolution. The lessons learned from the fixed potential reveals some mechanisms of bar formation and evolution in the self-consistent simulations. In particular, we are able to identify the appearance of orbit families and correlate these events with distinct evolutionary phases. The cusp and core simulations evolve differently in time, but evolve similarly for a similar orbit structure. To see this, we present the evolution of the cusp simulation in Section 5.1, then the core simulation in Section 5.2. We draw comparisons and contrasts between the two in Section 5.4. In each of the first two sections, we describe the overall evolution of the simulation before looking at the underlying orbital structure.

In each simulation, we identify three phases for the bar: (i) assembly, (ii) growth, and (iii) steady-state. The assembly phase begins at roughly the same time for both models, after which the evolution rapidly diverges. For the cusp model the assembly continues at a slower pace, before transitioning smoothly into the growth phase, then finally reaches a steady state at late times. The core simulation rapidly assembles, then hits a steady-state plateau. At late times, as the bar slowly transfers angular momentum to the halo and the mass distribution rearranges, the bar begins to grow again.

5.1 Cusp Simulation Evolution

In Figure 10, we illustrate the power of geometric commensurability finding to reveal different mechanisms at work in the cusp simulation. The upper panel shows the trapped fraction versus time for the two populations described in Section 2.2 using our \(k\)-means orbit classifier. We also include the subfamily of \(x_{1b}\) orbits as a dotted black line. The first population to appear are the ‘other’ bar-supporting orbits (blue), which exhibit clear \(k = 2\) power but modest coherence in apsis locations. This is consistent with a standard picture of orbit apsis precession building the bar. With enough other bar-supporting orbits in place, the \(x_{1}\) family appears (the black line in Figure 10) at \(T = 0.4\). The two populations grow in tandem until \(T = 1\), when the \(x_{1}\) family begins to dominate. Eventually, the rapid assembly of the bar draws to a close at \(T = 1.4\), and the \(x_{1}\) orbits grow at a slower rate until \(T = 2.4\), during which time some ‘other’ bar-supporting orbits are converted into \(x_{1}\) orbits. Near \(T = 3\), the \(x_{1}\) orbits experience an oscillation before the bar stops growing at \(T = 4\). The dynamics of this oscillation is addressed in a companion work, Petersen et al. (2019b).

The gray line in the upper panel of Figure 10 depicts the total bar mass, here the sum of the \(x_{1}\) and ‘other’ bar supporting orbits. We highlight three phases of bar evolution: bar assembly, bar growth, and steady-state. After the assembly and growth phases, the bar is 30 per cent of the entire disc population, in line with estimates for the bar-to-disc ratio in the MW (Bland-Hawthorn & Gerhard 2016).

The assembly phase of the bar is marked by nearly equal contributions from the \(x_{1}\) family and ‘other’ bar orbits. With the additional information provided by the commensurability map (the lower left of Figure 10), we see that the relative weakness of the \(x_{1}\) family is expected, as the family is both truncated at \(R_{apo} \approx 1a\) and features a prominent break in the \(x_{1}\) track, particularly as compared to the later growth and steady-state panels. CR, while present at \(T = 0.6\), has begun to migrate outward substantially from its initial position \((r_{CR}, T=0.0, a=1.4a, t_{CR}, T=0.6, a=2.0a)\). CR has not yet migrated outward enough to be at larger radii than the majority of the disc mass distribution. The outer disc, \(r > 3a\), appears nearly unevolved at this time.

The middle panels of Figure 10 are the results of extracting the potential at the times labelled in the upper panel of Figure 10 and integrating the standard initial grid described in Section 2.4.1, constructing an orbit atlas. As in Figure 4, the color in the middle panels indicates \(A\), the fraction of the area an orbit fills relative to a circle with the same radius as \(R_{apo}\), as calculated in Appendix B. The bottom panels are similar to those in Figure 9 but at the labelled times. When present, we mark in the lower panels of Figure 10 the bar-parenting family \(x_{1}\), the bifurcation of the bar-parenting family \(x_{1b}\), and the location of corotation orbits, CR, following the procedure in Appendix B. Many other weak higher-order \(m:n\) resonances are also present, in particular during the bar growth phase, indicative of a rich resonant structure. The differences in the lower panels reveal the mechanisms behind the three distinct phases. During bar assembly, the location of the families in \(R_{apo} - V_{apo}\) space changes rapidly, resulting in discontinuities.
Figure 10. Upper panel: Disc trapped fraction versus time for the cusp simulation, in system units. The black line is the bar-parenting $x_1$ family, the blue line is a collection of other bar-supporting orbits that are not formally members of the $x_1$ family, and the thin dotted black line are orbits in the $x_{1b}$ subfamily (a subset of the black line). The gray line in the upper panel is the sum of the black and blue lines, which is the total mass trapped in and supporting the bar. Three distinct epochs are highlighted: bar assembly, bar growth, and steady-state. A dashed line indicates the central time for each epoch, when we extract the corresponding potential from the self-consistent cusp simulation and construct an orbit atlas. Middle panels: Computation from the geometric algorithm at each of the times indicated in the upper panel as a function of $R_{apo}$ and $V_{apo}$. The color indicates $A$, the fraction of the area an orbit fills relative to a circle with the same radius as $R_{apo}$. Strong commensurabilities are marked and the evolutionary implications are described in the text. White lines correspond to geometric algorithm-calculated commensurabilities and cyan lines correspond to monopole-calculated commensurabilities. The circular velocity along the bar major axis at the corresponding time is shown as a dashed red line in each panel. Lower panels: Instantaneous tangential velocity, calculated as $v_T = (x\dot{y} - y\dot{x})/R$, versus instantaneous radius, drawn from the corresponding central times of each epoch (assembly, growth, steady-state) during the simulation. The colormap is normalized relative to the locus of mass for the disc in $R_{apo} - V_{apo}$ space. Overlaid on each are the traced commensurabilities as in the middle panels, as well as the calculated circular velocity along the bar major axis.

and resonances that appear as narrow valleys in $R_{apo} - V_{apo}$ space. While the bar is in the growth phase, prominent $x_1$ and $x_{1b}$ families are present, and a large density of valleys (resonances) at the end of the bar ($R_{apo} > 2a$ and $V_{apo} > 1$) and near corotation continue to feed the growth as orbits pass through resonances and lose angular momentum. During the growth period, the fraction of $x_{1b}/x_1$ orbits consistently increases, to a maximum fraction of 40 per cent. When the bar has reached a steady state, the resonant valleys have become more well defined but fewer in number, and the bar orbits have settled into a lower-energy $x_1$ valley as the $x_{1b}$ valley has disappeared. Fewer resonances at the end of the bar and beyond causes the bar to no longer grow. As the structure in the barred cusp potential (II) shown in Figure 4 is for $T = 2$, we see similarities with the bar growth phase in Figure 10.

The correlation between bar growth and the presence of the $x_{1b}$ orbits suggests another possible mechanism beyond standard secular evolution by resonance passage: co-located resonances in phase space could result in family switching. In this scenario, an orbit is trapped into one family but owing to the intersection of two resonances in phase space, it may switch orbit families at the separatrix crossing as discussed above. If the $x_{1b}$ family can be the result of both mechanisms, it may be expected that when the...
mechanisms are available, a larger fraction of orbits will end up as $x_{1b}$ orbits, growing the bar. Inspection of the left panel of Figure 8 suggests that the $x_{1b}$ family may also cut through a more populated region of phase space, increasing the number of orbits that may be exposed to the trapping mechanisms. We study the trapping mechanism via a torque-based analysis in a companion paper, Petersen et al. (2019a).

The loss of the $x_{1b}$ family for the $m \leq 2$ potential (Section 3.1.3) prompts us to develop another line of inquiry to investigate the role of $m > 2$ harmonics in the disc, their effect on the $x_{1b}$ population, and the subsequent evolution of the model. To this end, we perform a numerical experiment where we restrict the azimuthal indices in the disc to $m \leq 2$, but at a point after the bar has already formed. EXP allows for an easy manipulation of different basis functions to investigate the role of individual harmonics on the overall evolution. Using the cusp simulation, we duplicate the simulation at $T = 0.9$. Allowing the simulation to proceed, we suppress the $m > 2$ terms in the disc by applying an error function prefactor. The error function is centered at $T_{\text{off}} = 1.2$ with a width of $\delta T_{\text{off}} = 0.12$, which corresponds to roughly two bar periods. The $m > 2$ coefficients are fully suppressed by $T = 1.5$. Construction of commensurability maps at $T = 1.2$ and $T = 1.4$, when the prefactor on the $m > 2$ terms is 0.5 and 0.0 respectively, conclusively demonstrates that the $x_{1b}$ family is not present when the model is restricted to the dipole and quadrupole terms. Further, the bar rapidly evolves to a new, shorter configuration rather than continuing to grow as in the unmodified cusp simulation. The probable interpretation of this experiment is that the $m = 4$ component of the potential is necessary for $x_{1b}$ orbits to exist, and that these orbits are part of the main backbone of the long bar, even though the $x_{1b}$ orbits provide the principal observed length.

Identifying orbit families in self-consistent simulations takes on importance in the context of a MW study. Binney et al. (1991) interpreted observations of gas dynamics toward the center of the MW to be the result of $x_1$ orbits and the $x_{2b}$ family, eccentric orbits elongated perpendicular to the bar. While the non-bifurcated $x_1$ family becomes more eccentric as one moves inward, as noted by Binney et al. (1991), the $x_{1b}$ family remains highly elongated even to the end of the bar, a point which may have observational implications for the MW. Additionally, although we have not discussed the $x_2$ orbits in this paper, our method to compute trapping and the geometric algorithm are both suitable for the identification and classification of $x_2$ orbits.

5.2 Core Simulation Evolution

Some elements of the core simulation are similar to that of the cusp simulation, including the observations of multiple distinct phases of bar evolution. However, we find differences within the distinct phases. In particular, a transient steady-state phase precedes the growth phase in this model. As in Figure 10, the gray line in the upper panel of Figure 11 measures the total bar mass. After the assembly and growth phases, the bar is 28 per cent of the entire disc population, still in line with observational results. However, for a substantial fraction of time during the simulation, the bar is < 25 per cent of the total disc population, in contrast to the more massive bar, > 30 per cent in the cusp simulation.

The bar assembly phase begins at roughly the same time as in the cusp simulation, $T = 0.4$. However, the $x_1$ family, shown in black, has already begun to dominate the bar mass fraction by $T = 0.6$, when we make the first commensurability map. Despite the relative mass equality between $x_1$ orbits and other bar orbits, the commensurability map does not show a prominent, contiguous $x_1$ valley from $R_{\text{apo}} = 0$ to $R_{\text{apo}} = a$. The $x_1$ valley is instead broken by the presence of a second commensurability valley at $r = 0.8a$ which is a result of the rapidly changing potential during the assembly phase.

After $T = 0.6$, instead of the ‘other’ bar orbits (blue line) gradually becoming $x_1$ orbits as in the cusp simulation, the two populations remain distinct until $T = 2.4$. Having reached a steady state at $T = 1.4$, one would be excused for believing the evolution of the system were complete. However one entire time unit later at $T = 2.4$, the $x_1$ family begins growing, and captures some of the ‘other’ bar orbits into $x_1$ members (the steady-state label in the upper panel of Figure 11; the orbit atlas processed with the geometric algorithm is shown in the bottom middle panel). From $2.4 < T < 3.8$ the bar grows, in much the same manner as the bar growth phase in the cusp simulation. We label this phase bar growth, and calculate the commensurability map from the orbit atlas in the bottom right panel of Figure 11. During this time period, the fraction of $x_{1b}/x_1$ orbits begins increasing, from 15 per cent of the $x_1$ family to a maximum fraction of 30 per cent.

The commensurability map for the transient steady-state phase in the lower center of Figure 11 resembles the steady-state evolution of the cusp simulation (the rightmost column in Figure 10), particularly in the number of observed commensurate valleys. The closed orbit parenting the $x_1$ family and corotation is clear, as well as several other higher-order resonances. By comparison to the right panel of Figure 8, which is near to panel b in simulation time, we see that the phase space near the resonances is not populated. However, in the bar growth panels, the lower right of Figure 11, we see a rich resonant structure, akin to that during the bar growth phase in the cusp simulation. The higher-order resonances have swept outward into the bulk of the phase-space density in the simulation, which is not particularly different in phase-space distribution from Figure 8. Additionally, the $x_{1b}$ family has appeared within the bar radius.

5.3 Limitations of Fixed Potential Analysis

We have presented many orbital snapshots of evolving barred galaxies, however, the orbits themselves evolve with time. Thus, our results describe the times we have chosen to study in depth and the dynamics they illustrate. Future work will attempt a temporal analysis of orbital structure, using the same technique.

A comparison between Figures 4, 8, and 9 suggests that many of the higher order features in Figure 4 may not truly exist in the self-consistent simulation owing to system evolution. When the period of the closed orbit is larger than the secular evolution time scale for the evolving barred galaxy, the fixed-potential approximation is invalid. Further, we noted subtle but important differences between the fixed potential and self-consistent orbits in Section 3.3, indicating that the fixed potential orbits are missing some dynamical degree of freedom. This will limit the applicability of fixed-potential analyses to both self-consistent simulations and observations. The culprit is likely a time-dependent feature that was excluded in order to make the fixed potential integration stable: (1) odd azimuthal harmonics, (2) harmonic interaction and/or multiple pattern speeds, and (3) a frozen noise spectrum.

Inspection of orbits drawn from the self-consistent time series suggest that $x_1$ orbits are robust against the inclusion of odd harmonics in the self-consistent simulation, so it is likely not simply the inclusion of the odd azimuthal harmonics that will create the observed behavior. Based on a reconstruction of the perturbing...
Figure 11. Upper panel: Disc trapped fraction versus time for the core simulation, in system units. The colors are as in Figure 10. Three distinct epochs are highlighted: bar assembly, steady-state, and bar growth. In this simulation, the bar reaches an unstable steady state before growing further. A dashed line indicates the central time for each epoch, when we extract the corresponding potential from the self-consistent cusp simulation and construct an orbit atlas. Middle panels: Computation from the geometric algorithm at each of the times indicated in the upper panel as a function of $R_{apo}$ and $V_{apo}$ in white, with monopole-calculated commensurabilities marked in cyan. The circular velocity along the bar major axis at the corresponding time, is shown as a dashed red line in each panel. Colors and indicators are again as in Figure 10. Lower panels: Instantaneous velocity versus instantaneous radius corresponding to the central time for each evolutionary epoch. The commensurability tracks and circular velocity curves from the middle panels are overlaid.

$m = 1$ disturbance, the peak of the response is at $r < a$, and thus unlikely to change the orbital structure in the outer disc more than in the inner disc. To test the role of modal interplay, we have integrated orbits in potential models where the azimuthal series only included the monopole $m = 0$ and the quadrupole $m = 2$ as a representation of the bar. These models disagree with the structure observed in the simulation. One example is the presence of the $x_{1b}$ orbits, which do not exist if the bar is represented by a quadrupole only.

All azimuthal harmonics $m > 0$ have the same pattern speed imposed. We have checked that this assumption is consistent with the simulation for even harmonic orders. A future investigation will allow for variable pattern speed by azimuthal order. Lastly, it is possible that the choice of any single snapshot may freeze unwanted small-scale noise into the potential. While we believe that the self-consistent field technique will largely smooth out such aphysical fluctuations in the potential, such as small-scale Poisson noise, it is not guaranteed that our implementation of the potential is completely free of aphysical noise on small scales.

For all three concerns, the agreement in identified orbits between the $k$-means classifier of self-consistent orbits (Figure 2) and orbits integrated in the corresponding fixed potential (Figure 5) suggests that the sources of uncertainty discussed in this section are subdominant.

5.4 Summary

The differences in the evolution between the cusp and core models are easy to describe and difficult to explain. Despite this, one can draw several simple conclusions from the comparison of the evo-
volutionary status in the self-consistent simulation and the features in the commensurability maps:

(i) Bar assembly is a multi-feature event: it produces transients and multiple patterns at small and large scales compared to the bar scale. Attempts to construct the commensurability map reveal complex structure in the inner disc ($r < a$) that cannot be simply explained by the barred potential models examined in Section 3.

(ii) Despite the apparent differences in evolution of the bar, the bar growth phase in both the cusp and core simulations includes the presence of the $x_{1b}$ family. While the two simulations do not comprise an exhaustive study of parameter space, the similarities in the dynamical mechanism (e.g. the appearance of $x_{1b}$ orbits) present an interesting explanation for an avenue of bar growth.

(iii) A steady-state phase may either follow (in the case of the cusp simulation) or precede (in the case of the core simulation) the growth phase.

(iv) Despite all that can be gleaned from the commensurability map, it must be used in tandem with other diagnostics to fully interpret simulations.

6 CONCLUSION

In this paper, we present a new geometric algorithm for identifying orbit families in arbitrary potentials. The geometric algorithm generalizes fixed potential studies to evolving potentials. We apply the algorithm to two self-consistent simulations (the cusp and the core simulation), from which we select four potentials (at $T = 0$ and $T = 2$ for each simulation) to learn about the commensurability structure in MW-like models. We fully characterize the orbit structure of the models and completely identify the closed-orbit structure.

Our main finding from the fixed potential analysis is that the resulting ‘commensurability maps’ characterize the orbit families admitted by different potential models. The allowed orbit families are sensitive to the shape of the halo profile, allowing for a differentiation between the underlying potential by observing orbits in self-consistent simulations. We also rediscover the $x_{1b}$ family of orbits, referred to as 1/1 orbits in the early literature, but largely excluded from recent potential studies. We demonstrate that the $x_{1b}$ orbits are harbingers of bar growth. With the geometric algorithm, we are able to draw connections with self-consistent simulations and analytic works. In particular, we interpret previously identified epochs of bar evolution (assembly, growth, and steady state) using commensurability maps. The distinctions between different commensurability maps, such as the presence or absence of known key orbit families correlates with distinct evolutionary phases of the barred galaxy, and may be used to assess its dynamical state.

We propose a simple new method to interpret IFU data by using a pseudo phase space, the $R_{apo} - V_{apo}$ (or $r - v$) plane. For external galaxies, different commensurability map models may be compared in the $R_{apo} - V_{apo}$ plane to ascertain whether the barred galaxy is in a steady-state or growing phase, based on the location of breaks and features in the $R_{apo} - V_{apo}$ plane (cf. Figure 9).

This methodology can be connected to observations of real orbits in the MW. Observations in the near future (e.g. Gaia, SDSS V) will reveal more about the orbit structure of the inner MW. We predict that if the MW has either a cusped dark matter profile or an old bar, that $x_{1b}$ orbits will be present. If observed, these orbits would be an indicator of long-term stability in the bar, as they comprise an extremely stable family in the self-consistent simulations. This work drastically improves upon previous studies of possible orbit structure in a MW-like barred galaxy. In particular, popularly used potentials for the MW, such as MWPotential14 from galpy (Bovy 2015), are known simplifications that meet only the most rudimentary requirements for matching the potential of the MW. Our initial conditions were chosen to resemble the MW in disc-to-halo mass ratio, disc scale length to scale height ratio, and general rotation curve shape, but we make little attempt to match the data for the MW beyond scaling the system to match the virial units of the MW. Rather, our aim was a description of dynamical mechanisms that we expect to be phenomena prominent in observed barred galaxies.

Additional applications for this methodology include an extension to other non-separable realistic potentials and studying the rate at which orbits transition between families through coupling to self-consistent simulations. These rates could be connected to simple chemical models to attempt to explain chemically-distinct components of galaxies. In the future, we plan to extend the method to three dimensions and develop fit potentials for a range of realistic galaxies.

Finally, analyses such as those presented here are just one way of studying the dynamics of barred galaxies. One can also look at the torques as in Petersen et al. (2019a) or harmonics as in Petersen et al. (2019b) to gain further insights.

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APPENDIX A: ORBIT CLASSIFICATION

For an orbit conserving $L_z$ and $E$, the location of the radial turning points or apsides are set by the orbital frequencies. As we are most interested in the classification of orbits related to the bar, we evaluate the coherence of the apsis locations relative to the minimum of the bar potential, i.e. the bar position angle. We use this coherence to classify orbits into different families with a clustering algorithm. We use k-means (Lloyd 1982) to partition apsides for a given orbit into families that show similar morphologies. We then connect to known classical orbit families. Briefly, the k-means algorithm iteratively separates a collection of points into $k$ clusters by minimizing the sum of the distance between each point and the center of a determined cluster. The distance metric used in this work is standard Cartesian space.

The steps to prepare an orbit for k-means analysis are as follows:

(i) Extract $x$, $y$, and $z$ time series for a given orbit.
(ii) Determine the apsides.
(iii) Transform the $x$, $y$ positions of the apsides into a frame where the bar position angle is aligned with the $x$ axis.
(iv) For each apsis: determine the 19 other nearest apsides in time and run the k-means algorithm on the $x_{bar} - y_{bar}$ positions of the apsides.

The output of the k-means algorithm is two-fold: (1) the partitioned $k$ clusters of apsides, and (2) the location of the spatial center (equivalently listed in $(R_{\text{cluster}}, \theta_{\text{cluster}})$ or $(R_{\text{cluster}}, \phi_{\text{cluster}})$ for each cluster. Therefore, to determine membership in the bar, a set of metrics relating to the apsides is calculated for each orbit:

(i) $(\delta \theta_{\text{bar}}) \equiv \max (\langle \theta_{\text{bar}} \rangle_N)_{\text{apsides}}$, the trapping metric from PKW16 that assesses the average angular separation in radians from the bar axis, $\theta_{\text{bar}}$, for $N$ apsides in $k$ clusters. $(\delta \theta_{\bar{\text{bar}}})$ is the maximum angular separation from the bar for the $k$ clusters. $N$ is a parameter set based on the dynamical time of the bar, which we set to $N = 20$ for all analyses in this work.
(ii) $(R_{\text{cluster}})$, the radius of each cluster center, averaged over $k$ clusters.
(iii) $(\vec{R}_{\text{cluster}})$, the variance in radius for all apsides in a cluster, averaged over $k$ clusters. A larger value of $(\sigma_{R_{\text{cluster}}})$ relative to $(R_{\text{cluster}})$ implies no trapping. A threshold on this ratio effectively removes false positive detections.
(iv) $(\sigma_{\theta_{\text{cluster}}})$, the variance in position angle for all apsides in a cluster, averaged over $k$ clusters. Variation in this quantity is the product of both uncertainty in the bar angle as well as being possibly indicative of a family that would be better fit by an increase in the number of clusters $k$.
(v) $\dot{\Omega}$, the instantaneous radial frequency, computed as the finite difference in time between the central apsis and the next nearest apsis in time. This quantity is used to calculate orbits that fall...
below the Nyquist frequency for time sampling as well as orbits whose radial frequency makes it impossible for the orbit to be a bar member.

Some combination of these five quantities will describe different orbit families to within an acceptable contamination tolerance. The values for each family are presented in Table A. We estimate our contamination rate at 1 per cent from visual inspection of classified orbits. Choosing the correct metrics for family classification is largely model-independent. However, the dynamical time of a given model can affect the time resolution; determinations are noisier in models with rapidly changing pattern speeds.

To calibrate the thresholds for trapped orbits given the five metrics, we tabulate the quantities listed above for all orbits at some late time, when secular evolution is relatively slow. Significantly, we tabulate the quantities listed above for all orbits at time sampling as well as orbits of classified orbits. Choosing the correct metrics for family classification between different orbit families to be consistent members of a single family will show small values of \( \langle \sigma_{\delta x} \rangle_k \) for unrelated apsides. We empirically find the best descriminator between two families to be \( \langle \sigma_{\theta x} \rangle_k \). We assume that orbits with \( \Omega_x > \Omega_{\bar{y}} \) are not a part of the bar.

Once the process verifying the chosen criteria is complete, we proceed with a full analysis of the simulation from the beginning. The closest apsis in time is chosen as representative of the orbit’s current status. We attempt no interpolation to increase the effective time resolution and use single apsides. Therefore, the time resolution for studying coherence depends on the radial period of individual orbits, though we find that time resolution is a small part of the already small overall uncertainty. We analyze both the cusp and core simulations in their entirety using this methodology in Section 5.

We identify four major improvements over PKW16:

(i) Implementation of the ‘k-means++’ technique of Arthur & Vassilvitskii (2007) when the standard (Lloyd’s) k-means technique (Lloyd 1982) fails (approximately 0.6 per cent of orbits in the cusp model, the primary calibrator for the methodology).

(ii) Use the closest \( N \) apsides in time to the indexed time, rather than enforcing \( N \) apsides on either side of the target time.

(iii) Set a threshold, \( T_{\text{thresh}} \), that is some multiple of the bar period \( T_{\bar{y}} \) in which the \( N \) apsides must reside. This guards against choosing unrelated apsides. The threshold is exceeded for approximately 15 per cent of the fiducial model orbits, at which point the orbit is not analyzed at that timestep.

(iv) The inclusion of \( \langle \sigma_{\theta x} \rangle_k \) allows subdivision into 2:1, 4:1, and higher families even while using \( k = 2 \).

Table A lists the empirically-determined classification criteria for two families of orbits in our cusp and core simulations. Table A also lists an ‘Undetermined’ classifier for orbits with \( \Omega_x \) above the Nyquist frequency of the time-series sampling. These are nearly always orbits that are close to the center. Additionally, while \( x_{2,3,4} \) orbits exist in small quantities in our models, these orbits play little if any role in the dynamics described in this work, and so we do not focus on their classification here.

We also have interest in empirically locating the bifurcated members of the \( x_1 \) family, the \( x_{1b} \) orbits, which we refer to as a subfamily. We employ a secondary classification scheme on orbits that we determine to be part of the larger \( x_1 \) family. Unfortunately, the coherence metrics for the apsides for \( x_{1b} \) and \( x_{1a,1} \) orbits are indistinguishable. However, a clear morphological difference in the bifurcated \( x_{1b} \) orbits and the short- or long-period \( x_1 \) orbits is the presence of varying numbers of \( x_{\bar{y} a} \) and \( y_{\bar{y} a} \) local maxima. Equivalently, these are points where the velocity in one dimension relative to the bar goes to zero. Therefore, a classification scheme that separately identifies local maxima in the time-series of \( x_{\bar{y} a} \) and \( y_{\bar{y} a} \) distinguishes between the subfamilies. Such a classification scheme is computationally expensive, requiring tracking the entire time-series for a given orbit after it has been identified as an \( x_1 \) orbit. As shown in this work, \( x_1 \) orbits may be 30 per cent of all disc orbits in the simulation. The classification requires a precision transformation to an occasionally ill-defined bar frame, and the transition between the parent orbits of \( x_1 \) subfamilies may be rapid, increasing the classification uncertainty.

Despite this uncertainty, we make estimates of membership in the \( x_{1b} \) family from the \( x_{\bar{y} a} \) and \( y_{\bar{y} a} \) frequencies, \( \Omega_{x_{\bar{y} a}} \) and \( \Omega_{y_{\bar{y} a}} \), as determined by the local maxima of the \( x_{\bar{y} a} \) and \( y_{\bar{y} a} \) time series. The \( x_{1b} \) orbits trace two morphologies: infinity symbol-like orbits, and smile- or frown-like orbits. Infinity symbol orbits have \( \Omega_{x_{\bar{y} a}} / \Omega_{y_{\bar{y} a}} = 1.5 \). Smile and frown orbits have \( \Omega_{x_{\bar{y} a}} / \Omega_{y_{\bar{y} a}} = 2 \) because the strongest smiles and frowns actually counter-rotate in the bar frame. The subclassification into \( x_{1b} \) orbits benefits from the distinction with standard \( x_1 \) orbits, which have \( \Omega_{x_{\bar{y} a}} / \Omega_{y_{\bar{y} a}} = 1 \) or \( \Omega_{x_{\bar{y} a}} / \Omega_{y_{\bar{y} a}} = 3 \) (in the case of \( x_1 \) orbits with so-called ‘ears’, see Figure B1). Far away from the closed orbit, the classification between \( x_1 \) and \( x_{1b} \) becomes subjective. Therefore, we offer only a coarse estimate of the membership, assuming that orbits with \( 1.5 \leq \Omega_{x_{\bar{y} a}} / \Omega_{y_{\bar{y} a}} \leq 2 \) are \( x_{1b} \) orbits, which we can then classify into infinity or smile/frown orbits by the presence or absence of counter-rotation in the rotating frame. The broad classification by frequency is a necessity as orbits do not spend large fractions of time as closed-orbit members of the subfamilies, with small integer combinations of \( \Omega_{x_{\bar{y} a}} \) and \( \Omega_{y_{\bar{y} a}} \), but rather exhibit modest resemblance to the parent orbit as secular evolution proceeds.

We estimate the membership in the bifurcated families in this work. Owing to the uncertainty for any given orbit at a particular time, we consider orbits only over large windows of time during our analysis, reducing the uncertainty, but limiting the time resolution of our estimates for the fraction of \( x_{1b} \) orbits to \( dT = 0.1 \). This coarse time resolution is sufficient to track global trends in the \( x_{1b} \) subfamily relative to the overall \( x_1 \) membership.

Lastly, the sign of the maxima can also help determine the preferred orientation of bifurcated orbits, which are asymmetric with respect to either the bar major axis (in the case of ‘symmetric’ \( x_{1b} \) orbits) or the bar minor axis (in the case of ‘asymmetric’ \( x_{1b} \) orbits). That is, we may determine whether the crossing point in the infinity orbits is preferentially located toward one end of the bar, or whether the counter-rotating portion of the smile/frown orbits, such as the example in panel ‘b’ of Figure 5, is toward one direction along the bar minor axis. Such an asymmetry of crossing points or counter-rotating directions is responsible for the \( m = 1 \) mode, discussed in Petersen et al. (2019b).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Family & \( \langle \theta_{x \bar{y}} \rangle_{20} \) & \( \Omega_x \) & \( \langle \sigma_{\delta x} \rangle_k \) & \( \langle \sigma_{\theta x} \rangle_k \) \\
\hline
x_1 & \([0, \frac{\pi}{6}]\) & \(\frac{1}{237}\) & \([0, 0.1a]\) & \([0, \frac{\pi}{16}]\) \\
Other Bar & \([0, \frac{\pi}{4}]\) & \(\frac{1}{237}\) & \([0, 0.1a]\) & \(-\) \\
Undetermined & - & \(\frac{1}{237}\) & - & - \\
\hline
\end{tabular}
\caption{Table A1. Membership definitions for being classified into families. ‘-’ indicates that no constraint was placed on the parameter.}
\end{table}
APPENDIX B: THE GEOMETRIC ALGORITHM

B1 Orbit Area Measurement

Given a series of samples at discrete times for an orbit, we wish to approximate the area that an orbit would sample in the limit where \( dt \to 0 \) and \( T \to \infty \). To measure the area of an annulus or volume of a sphere that a discrete set of orbit time samples would eventually fill as \( T \to \infty \), we require a tessellation technique that transforms a discrete time series of points into an integrable area \( dt \to 0 \). One such computational technique is Delaunay triangulation. We construct a procedure that uses Delaunay triangulation (DT), taking an input that is a set of two-dimensional points in the \( x-y \) plane and returning a single value that is the (normalized) computed orbit area from the sum of individual tessellated triangles.10

The steps to calculate the area of a given orbit from the time series of discrete points are as follows:

(i) Integrate the orbit in a given rotating potential using discrete timestep \( dt \) as in Section 2.4.2 to obtain a set of two-dimensional points in the bar frame.
(ii) Transform the orbit points to a frame co-rotating with the imposed bar pattern.
(iii) Compute the triangulation of the transformed points by applying DT to the \((x, y)\) orbit points.
(iv) Prune the triangulation by eliminating triangles with axis ratios above some threshold.
(v) Compute the area of each remaining triangle and sum to obtain \( A \), the area of the orbit.
(vi) (optional) Normalize the area of the orbit by the area of a circle with radius \( r_{\text{max}} \), the maximum distance from the inertial center in the time series.

We use the computational geometry library CGAL (CGAL Project 2018) to perform the DT. From the input of a time-series of vertices comes a list of triangles with length-ordered sides \( a, b, c \), where \( a \) is the length of the longest side. The low density simplicies (and therefore least important for computing the area) are removed by ignoring triangles with extreme axis ratios, which we choose to be \( \frac{b}{a} > 10 \). Adjusting this threshold to 5 or 15 does not produce qualitatively different results.

Examples of the technique are shown in Figure B1. The upper panel shows the integration of two orbits over the entire time window, \( T = 0.64 \), which is 2000 steps at \( dt = 3.2 \times 10^{-5} \). Both orbits are shown in the frame co-rotating with the bar. While the black orbit has sampled the entire phase-space trajectory, the purple orbit has not. It is clear that the purple orbit will fill an entire torus in physical space given enough time. In the lower panel, we show the triangulation for each orbit. The black orbit, nearly closed, features vanishingly small triangles, while the purple orbit, which previously only sampled a fraction of the torus, is now filled with triangles. We can now evaluate the area in physical space that each orbit occupies.

As described in the optional final step above, to compare the area of orbits with hugely different energies, we normalize by the area of a circle with radius \( r_{\text{max}} \), the maximum distance from the inertial center in the time series.

As shown in Figure B1, a commensurate orbit will occupy a smaller area in physical space than a non-commensurate orbit. We exploit this to find commensurabilities in the potential. True commensurabilities occupy a vanishingly small volume in phase space, suggesting that tracing valleys in orbit area will follow commensurate orbit tracks. The procedure we use to trace commensurabilities in the \( R_{\text{apo}} - V_{\text{apo}} \) plane is as follows:

(i) Identify all orbits below a certain threshold in normalized area. We use \( A < 0.02 \), which balances the finite measure-

10 An equivalent procedure may be followed to generalize the orbit measurement to a volume, i.e. by using all three dimensions for the orbit. In such a procedure, triangles become tetrahedrons from which a volume can be computed.

\[ A_{\text{norm}} = \frac{\sum_k T_k / (\pi R_{\text{apo}}^2)}{A} \]

Figure B1. Two orbits in the barred cusp potential, Potential II. The black orbit starts from \((R_{\text{apo}}, v_{\text{apo}}) = (0.02, 0.45)\), and the purple orbit starts from \((R_{\text{apo}}, v_{\text{apo}}) = (0.02, 1.05)\). The upper panel shows the integration of the orbits in the potential for \( \Delta T = 0.64 \), with the orbits presented in a frame co-rotating with the bar. The lower panel shows the same orbits, with the fraction of a circle that the orbit fills in (area) computed using the geometric algorithm. Some residual triangles not successfully trimmed by the simplex rules are seen in the black orbit, limiting the absolute precision of the technique.

B2 Skeleton Tracing

As shown in Figure B1, a commensurate orbit will occupy a smaller area in physical space than a non-commensurate orbit. We exploit this to find commensurabilities in the potential. True commensurabilities occupy a vanishingly small volume in phase space, suggesting that tracing valleys in orbit area will follow commensurate orbit tracks. The procedure we use to trace commensurabilities in the \( R_{\text{apo}} - V_{\text{apo}} \) plane is as follows:

(i) Identify all orbits below a certain threshold in normalized area. We use \( A < 0.02 \), which balances the finite measure-
ment accuracy from the triangulation while still excluding non-
commensurate orbits.

(ii) Connect contiguous areas using a standard marching-
squares algorithm that checks adjacent orbits in the \( R_{\text{apo}} - V_{\text{apo}} \)
grid to determine which adjacent orbits meet the threshold criteria
and subsequently connect the points, which we call the threshold
map.

(iii) Perform valley-finding on the threshold map using the algo-
rithm of Steger (1998). The algorithm calculates the Hessian ma-
trix of the threshold map by convolving the threshold map with
derivatives of a Gaussian smoothing kernel and then determining
the vanishing point of the gradient, i.e. a valley.

(iv) Inspect individual orbits in the atlas near the commensu-
rability for connection to known families of orbits and frequency
ratios.

B3 Monopole-calculated Commensurabilities

One may numerically compute the location of resonances in an ax-
issymmetric potential by determining the table of frequencies for
each energy \( E \) and \( \kappa \equiv L_{z,\text{orbit}}/L_{z,\text{circular}} \) in a grid and solving
Equation (1) for a given combination of \((m, l_1, l_2, l_3)\). For our ex-
ansion of the potentials in harmonic orders, simply selecting the
monopole component of the potential is sufficient to reduce the po-
tential to an axisymmetric case. For orbits outside of the bar radius,
at evolutionary stages after the bar has formed, the monopole is
a reasonable approximation for the potential in the plane. For our
analysis restricted to the \((x, y)\) plane, \( l_3 = 0 \).

We calculate the \((E, \kappa)\) locations of CR \((m, l_1, l_2) = (2, 0, 2)\)
and OLR \((m, l_1, l_2) = (2, 1, 2)\) using the monopole approxima-
tion. In general, the resonances have little dependence on \( E \), exist-
ating at a single value of \( E \) for all \( \kappa \). To place the \((E, \kappa)\)-calcuated
locations of resonances on the \((R_{\text{apo}}, V_{\text{apo}})\)-based figures in this
work, we compute the transformation between \((R_{\text{apo}}, V_{\text{apo}})\) and
\((E, \kappa)\). In the axisymmetric case, the mapping is monotonic. The
location of ILR, \((m, l_1, l_2) = (2, -1, 2)\), is not possible to approx-
imate using this method owing to the strongly non-axisymmetric
potential at those radii. In this case, the geometric algorithm and
skeleton tracing are preferred.