Potential Vector Fields in $\mathbb{R}^4$ and New Generalizations of the Cauchy-Riemann System

Dmitry Bryukhov

Abstract. This paper extends approach of recent author’s paper devoted to special classes of exact solutions of the static Maxwell system in inhomogeneous isotropic media and new generalizations of the Cauchy-Riemann system in $\mathbb{R}^3$. Two families of generalizations of the Cauchy-Riemann system with variable coefficients in $\mathbb{R}^4$ are presented in the context of Non-Euclidean geometry. Analytic models of a wide range of potential meridional vector fields in $\mathbb{R}^4$ are characterized using a family of Vekua type systems in cylindrical coordinates. The specifics of meridional fields allows us to introduce the concept of four-dimensional $\alpha$-meridional mappings of the first and second kind depending on the values of a real parameter $\alpha$. In case $\alpha = 2$ tools of the radially holomorphic potential in $\mathbb{R}^4$ are developed in the context of Generalized axially symmetric potential theory (GASPT). Analytic models of potential meridional fields in $\mathbb{R}^4$ generated by Bessel functions of the first kind of integer order and quaternionic argument are described in case $\alpha = 2$. In case $\alpha = 0$ the geometric specifics of four-dimensional harmonic meridional mappings of the second kind is demonstrated explicitly in the context of the theory of Gradient dynamical systems with harmonic potential.

Mathematics Subject Classification (2010). Primary 30G35, 35J46; Secondary 30C65, 35Q99, 34A34.

Keywords. Generalizations of the Cauchy-Riemann system, Potential meridional fields in $\mathbb{R}^3$, Four-dimensional $\alpha$-meridional mappings of the second kind, The radially holomorphic potential in $\mathbb{R}^4$, Gradient dynamical systems.
1. Introduction, Preliminaries, and Notation

1.1. Introduction

A rich variety of analytic models of potential vector fields $\vec{V} = (V_0, V_1, V_2, V_3)$ in $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3)\}$ is described by the following first-order static system including a variable $C^1$-coefficient $\phi = \phi(x_0, x_1, x_2, x_3) > 0$:

$$
\begin{aligned}
\text{div} (\phi \, \vec{V}) &= 0, \\
\frac{\partial V_0}{\partial x_1} &= \frac{\partial V_1}{\partial x_0}, \quad \frac{\partial V_2}{\partial x_0} = \frac{\partial V_3}{\partial x_2}, \quad \frac{\partial V_0}{\partial x_3} = \frac{\partial V_3}{\partial x_0}, \\
\frac{\partial V_1}{\partial x_2} &= \frac{\partial V_2}{\partial x_1}, \quad \frac{\partial V_1}{\partial x_3} = \frac{\partial V_3}{\partial x_1}, \quad \frac{\partial V_2}{\partial x_3} = \frac{\partial V_3}{\partial x_2}.
\end{aligned}
$$

(1.1)

In our setting the Euclidean space $\mathbb{R}^4$ involves the longitudinal variable $x_0$ and the cylindrical radial variable $\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}$. The scalar potential $h = h(x_0, x_1, x_2, x_3)$ in simply connected open domains $\Lambda \subset \mathbb{R}^4$, where $\vec{V} = \text{grad} h$, allows us to reduce every $C^1$-solution of the system (1.1) to a $C^2$-solution of the continuity equation

$$
\text{div} (\phi \, \text{grad} h) = 0.
$$

(1.2)

The space $\mathbb{R}^4$ and the coefficient $\phi = \phi(x_0, x_1, x_2, x_3)$ may be interpreted as the phase space $X$ (or the state space) and the Jacobi last multiplier $M = M(x_0, x_1, x_2, x_3)$, respectively, in the context of the theory of Gradient dynamical systems, where $\vec{V} = \frac{d\vec{x}}{dt}$ = grad $h$ (see, e.g., [35, 36, 46, 47, 51, 55]).

Some types of harmonic potential vector fields have been extensively investigated by Brackx, Delanghe, Sommen et al. in the context of Quaternionic analysis and the theory of Quaternion-valued monogenic functions since the 1970s (see, e.g., [5, 6, 18, 19, 45, 58]). Remarkable extensions of the basic concepts of power series, holomorphicity, elementary and special functions have been developed for the past 40 years, in particular, in the context of Quaternionic analysis and its modifications (see, e.g., [24, 34, 41, 44]), of the theory of Holomorphic functions in n-dimensional space (see, e.g., [32, 33]), of the theory of Cullen regular (also referred to as “slice regular”, “slice monogenic” or “slice hyperholomorphic” in various settings) functions (see, e.g., [13, 16, 26, 29, 57].

Essentially new tools of quaternionic power series with quaternion-valued coefficients have been developed by Leutwiler and Hempfling in the 1990s to investigate analytic properties of exact solutions of the system

$$
(H_4) \quad \left\{ \begin{array}{l}
x_3 \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} \right) + 2u_3 = 0 \\
\frac{\partial u_0}{\partial x_1} = \frac{\partial u_1}{\partial x_0}, \quad \frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0}, \quad \frac{\partial u_0}{\partial x_3} = -\frac{\partial u_3}{\partial x_0}, \\
\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_3}{\partial x_1}, \quad \frac{\partial u_2}{\partial x_3} = \frac{\partial u_3}{\partial x_2}
\end{array} \right.
$$

in the context of Modified quaternionic analysis in $\mathbb{R}^4$, where $x_3 > 0$ (see, e.g., [34, 41, 42]). This system is called the system $(H_4)$ in honor of Hodge.
General class of exact solutions of the system \((H_4)\) may be equivalently represented as general class of exact solutions of the static system

\[
\begin{aligned}
\text{div} \left( x_3^{-2} \vec{V} \right) &= 0, \\
\frac{\partial \nu_0}{\partial x_1} &= \frac{\partial \nu_1}{\partial x_0}, \quad \frac{\partial \nu_0}{\partial x_2} = \frac{\partial \nu_2}{\partial x_0}, \quad \frac{\partial \nu_0}{\partial x_3} = \frac{\partial \nu_3}{\partial x_0}, \\
\frac{\partial \nu_1}{\partial x_2} &= \frac{\partial \nu_2}{\partial x_1}, \quad \frac{\partial \nu_1}{\partial x_3} = \frac{\partial \nu_3}{\partial x_1}, \quad \frac{\partial \nu_2}{\partial x_3} = \frac{\partial \nu_3}{\partial x_2},
\end{aligned}
\tag{1.3}
\]

where \(\phi = \phi(x_3) = x_3^{-2}; \quad (V_0, V_1, V_2, V_3) = (u_0, -u_1, -u_2, -u_3)\).

Assume that \(\rho > 0\). Axially symmetric Fueter’s construction in \(\mathbb{R}^4\) (see, e.g., [25,41])

\[
F = F(x) = u_0 + iu_1 + ju_2 + ku_3 = u_0(x_0, \rho) + I u_\rho(x_0, \rho),
\]

\[
x = x_0 + I \rho, \quad I = \frac{ix_1 + jx_2 + kx_3}{\rho}, \quad i^2 = j^2 = k^2 = I^2 = -1,
\tag{1.4}
\]

was characterized by the author in 2003 [8] as joint class of analytic solutions of the system \((H_4)\) and axially symmetric generalization of the Cauchy-Riemann system

\[
\begin{aligned}
(x_1^2 + x_2^2 + x_3^2) \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} \right) + 2(x_1 u_1 + x_2 u_2 + x_3 u_3) &= 0, \\
\frac{\partial u_0}{\partial x_1} &= -\frac{\partial u_1}{\partial x_0}, \quad \frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0}, \quad \frac{\partial u_0}{\partial x_3} = -\frac{\partial u_3}{\partial x_0}, \\
\frac{\partial u_1}{\partial x_2} &= \frac{\partial u_2}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_3}{\partial x_1}, \quad \frac{\partial u_2}{\partial x_3} = \frac{\partial u_3}{\partial x_2},
\end{aligned}
\tag{1.5}
\]

under conditions of

\[
u_1 x_2 = u_2 x_1, \quad u_1 x_3 = u_3 x_1, \quad u_2 x_3 = u_3 x_2, \quad x_3 > 0.
\tag{1.6}
\]

General class of exact solutions of the static system

\[
\begin{aligned}
\text{div} \left( \rho^{-2} \vec{V} \right) &= 0, \\
\frac{\partial \nu_0}{\partial x_1} &= \frac{\partial \nu_1}{\partial x_0}, \quad \frac{\partial \nu_0}{\partial x_2} = \frac{\partial \nu_2}{\partial x_0}, \quad \frac{\partial \nu_0}{\partial x_3} = \frac{\partial \nu_3}{\partial x_0},
\end{aligned}
\tag{1.7}
\]

is equivalently represented as general class of exact solutions of the system \((H_4)\), where \(\phi = \phi(\rho) = \rho^{-2}; \quad (V_0, V_1, V_2, V_3) = (u_0, -u_1, -u_2, -u_3)\).

Surprisingly, applications of the system \((H_4)\), the system \((1.5)\) and Fueter’s construction in \(\mathbb{R}^4\) \((1.4)\) in accordance with the systems \((1.3)\), \((1.7)\) have been missed.

The main goal of this paper is to develop new tools of the theory of Potential meridional vector fields in \(\mathbb{R}^4\) by means of different generalizations of the Cauchy-Riemann system with variable coefficients.

The paper is organized as follows. In Section 2, two families of generalizations of the Cauchy-Riemann system with variable coefficients in \(\mathbb{R}^4\) are provided. Analytic properties of potential meridional fields are considered in the context of Modified quaternionic analysis in \(\mathbb{R}^4\), Hyperbolic function theory in the skew-field of quaternions and the theory of Modified harmonic functions in \(\mathbb{R}^4\). In Section 3, new concept of four-dimensional α-meridional
mappings of the first and second kind, where $\alpha \in \mathbb{R}$, is introduced. In Section 4, in case $\alpha = 2$ tools of the radially holomorphic potential in $\mathbb{R}^4$ are developed using the concept of radially holomorphic functions in $\mathbb{R}^4$ introduced by Gürlebeck, Habetha, Sprößig in the context of the theory of Holomorphic functions in $n$-dimensional space. In Section 5, new analytic models of potential meridional fields in $\mathbb{R}^4$ generated by the quaternionic Fourier-Fueter cosine and sine transforms of real-valued original functions are described in case $\alpha = 2$. As a corollary, integral representations of Bessel functions of the first kind of integer order and quaternionic argument are obtained. In Section 6, in case $\alpha = 0$ the geometric specifics of four-dimensional harmonic meridional mappings of the second kind is demonstrated explicitly in the context of the theory of Gradient dynamical systems with harmonic potential, where $\vec{V} = \frac{d\vec{x}}{dt} = \text{grad} \ h, \ \Delta \ h = 0$.

1.2. Preliminaries

New families of generalizations of the Cauchy-Riemann system with variable coefficients in $\mathbb{R}^4$ may be provided by means of the following first-order system:

\[
\begin{cases}
\phi \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} \right) + \left( \frac{\partial \phi}{\partial x_0} u_0 - \frac{\partial \phi}{\partial x_1} u_1 - \frac{\partial \phi}{\partial x_2} u_2 - \frac{\partial \phi}{\partial x_3} u_3 \right) = 0, \\
\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \quad \frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0}, \quad \frac{\partial u_0}{\partial x_3} = -\frac{\partial u_3}{\partial x_0}, \\
\frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_3}{\partial x_1}, \quad \frac{\partial u_2}{\partial x_3} = \frac{\partial u_3}{\partial x_2}.
\end{cases}
\] (1.8)

Suppose that $(V_0, V_1, V_2, V_3) = (u_0, -u_1, -u_2, -u_3)$. General class of $C^1$-solutions of the system (1.1) may be equivalently represented as general class of $C^1$-solutions of the system (1.8).

The continuity equation (1.2) in the expanded form is expressed as

\[
\phi \Delta h + \frac{\partial \phi}{\partial x_0} \frac{\partial h}{\partial x_0} + \frac{\partial \phi}{\partial x_1} \frac{\partial h}{\partial x_1} + \frac{\partial \phi}{\partial x_2} \frac{\partial h}{\partial x_2} + \frac{\partial \phi}{\partial x_3} \frac{\partial h}{\partial x_3} = 0,
\] (1.9)

where the Laplacian $\Delta := \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

The equation

\[
h(x_0, x_1, x_2, x_3) = C = \text{const}
\] (1.10)

allows us to establish important properties of the equipotential hypersurfaces in simply connected open domains $\Lambda \subset \mathbb{R}^4$. Using the total differential $dh$, Eq. (1.10) is reformulated as an exact differential equation (see, e.g., [60])

\[
dh = \frac{\partial h}{\partial x_0} dx_0 + \frac{\partial h}{\partial x_1} dx_1 + \frac{\partial h}{\partial x_2} dx_2 + \frac{\partial h}{\partial x_3} dx_3 = 0.
\]

Let $\varsigma$ be a real independent variable. Assume that homogeneous first-order partial differential equation

\[
\frac{\partial h}{\partial x_0} W_0 + \frac{\partial h}{\partial x_1} W_1 + \frac{\partial h}{\partial x_2} W_2 + \frac{\partial h}{\partial x_3} W_3 = 0
\] (1.11)
is satisfied in $\Lambda$ such that

$$W_l(x_0, x_1, x_2, x_3) = \frac{dx_l}{ds} \quad (l = 0, 1, 2, 3). \quad (1.12)$$

The system (1.12) allows us to introduce the characteristic vector field $\vec{W} = (W_0, W_1, W_2, W_3)$ for equation (1.11) in $\Lambda$ in the context of Geometrical methods of the theory of ordinary differential equations (see, e.g., [3]). The scalar potential $h = h(x_0, x_1, x_2, x_3)$ is accordingly referred to as a first integral of the characteristic vector field $\vec{W} = (W_0, W_1, W_2, W_3)$ (or of its associated system $\frac{dx_0}{W_0} = \frac{dx_1}{W_1} = \frac{dx_2}{W_2} = \frac{dx_3}{W_3}$) in $\Lambda$ if and only if equation (1.11) is satisfied in $\Lambda$ (see, e.g., [3, 4, 7, 64]).

Equation (1.11) is geometrically characterized as the orthogonality condition for vector fields $\vec{V}$ and $\vec{W}$:

$$(\vec{V}, \vec{W}) = (\text{grad } h, \vec{W}) = 0. \quad (1.13)$$

Equation (1.13) is satisfied, in particular, under condition of $\vec{V} = \text{grad } h = (u_0, -u_1, -u_2, -u_3) = 0.$

**Definition 1.1.** Let $\Lambda \subset \mathbb{R}^4$ be a simply connected open domain. Every point $x^* = (x_0^*, x_1^*, x_2^*, x_3^*) \in \Lambda$ under condition of grad $h(x^*) = 0$ is called a critical point of the scalar potential $h$ in $\Lambda$. The set of critical points is called the critical set of $h$ in $\Lambda$.

Geometric and topological properties of the critical sets of the scalar potential $h$ are of particular interest to Catastrophe theory (see, e.g., [30]).

The Hessian matrix $H(h(x))$ of the scalar potential $h$ may be considered as the Jacobian matrix $J(\vec{V}(x))$ of the vector field $\vec{V}$, where $J_{lm} = \frac{\partial V_l}{\partial x_m}$ ($l, m = 0, 1, 2, 3$):

$$
\begin{pmatrix}
\frac{\partial V_0}{\partial x_0} & \frac{\partial V_0}{\partial x_1} & \frac{\partial V_0}{\partial x_2} & \frac{\partial V_0}{\partial x_3} \\
\frac{\partial V_1}{\partial x_0} & \frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & \frac{\partial V_1}{\partial x_3} \\
\frac{\partial V_2}{\partial x_0} & \frac{\partial V_2}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial x_3} \\
\frac{\partial V_3}{\partial x_0} & \frac{\partial V_3}{\partial x_1} & \frac{\partial V_3}{\partial x_2} & \frac{\partial V_3}{\partial x_3}
\end{pmatrix}
= 
\begin{pmatrix}
\frac{\partial u_0}{\partial x_0} & \frac{\partial u_0}{\partial x_1} & \frac{\partial u_0}{\partial x_2} & \frac{\partial u_0}{\partial x_3} \\
\frac{\partial u_1}{\partial x_0} & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\
\frac{\partial u_2}{\partial x_0} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\
\frac{\partial u_3}{\partial x_0} & \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}
\end{pmatrix}
\quad (1.14)
$$

The characteristic equation of the Hessian matrix $H(h(x))$ in the general four-dimensional setting is expressed as

$$\lambda^4 - I_{J(\vec{V})}\lambda^3 + II_{J(\vec{V})}\lambda^2 - III_{J(\vec{V})}\lambda + IV_{J(\vec{V})} = 0. \quad (1.15)$$

The principal invariants $I_{J(\vec{V})}, II_{J(\vec{V})}, III_{J(\vec{V})}, IV_{J(\vec{V})}$ of the Jacobian matrix (1.14) are given by formulas.
\[
I_J(\vec{v}) = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = J_{00} + J_{11} + J_{22} + J_{33},
\]

\[
II_J(\vec{v}) = \lambda_0 \lambda_1 + \lambda_0 \lambda_2 + \lambda_0 \lambda_3 + \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = J_{00}J_{11} + J_{00}J_{22} + J_{00}J_{33} + J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33},
\]

\[
III_J(\vec{v}) = \lambda_0 \lambda_1 \lambda_2 + \lambda_0 \lambda_1 \lambda_3 + \lambda_0 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_3 = J_{00}J_{11}J_{22} + J_{00}J_{11}J_{33} + J_{00}J_{22}J_{33} + J_{11}J_{22}J_{33} + 2J_{01}J_{02}J_{12} + 2J_{01}J_{03}J_{13} + 2J_{02}J_{03}J_{23} + 2J_{12}J_{13}J_{23}
\]

\[
- J_{00}(J_{12})^2 - J_{00}(J_{13})^2 - J_{00}(J_{23})^2 - J_{11}(J_{02})^2 - J_{11}(J_{03})^2 - J_{11}(J_{23})^2 - J_{22}(J_{01})^2 - J_{22}(J_{03})^2 - J_{22}(J_{13})^2 - J_{33}(J_{01})^2 - J_{33}(J_{02})^2 - J_{33}(J_{12})^2,
\]

\[
IV_J(\vec{v}) = \lambda_0 \lambda_1 \lambda_2 \lambda_3 = J_{00}J_{11}J_{22}J_{33} + 2J_{00}J_{12}J_{13}J_{23} + 2J_{01}J_{02}J_{12}J_{33} + 2J_{01}J_{03}J_{23}J_{11} + (J_{01}J_{23})^2 + 2J_{01}J_{03}J_{13}J_{22} + (J_{02}J_{13})^2 + (J_{03}J_{12})^2 - 2J_{01}J_{03}J_{12}J_{23} - 2J_{01}J_{02}J_{13}J_{23} - 2J_{02}J_{03}J_{12}J_{13} - 60J_{11}(J_{23})^2 - J_{00}J_{22}(J_{13})^2 - J_{00}J_{33}(J_{12})^2 - J_{11}J_{22}(J_{03})^2 - J_{11}J_{33}(J_{02})^2 - J_{22}J_{33}(J_{01})^2.
\]

**Definition 1.2.** Every point \( x \in \Lambda \) under condition of \( \det J(\vec{V}(x)) = 0 \) is called a degenerate point of the Jacobian matrix \( J(\vec{V}(x)) \) in \( \Lambda \).

Meanwhile, static potential vector fields in \( \mathbb{R}^4 \) may be investigated in the context of *Non-Euclidean geometry* using the Laplace-Beltrami equation (see, e.g., [19][10][34][41])

\[
\Delta_B h := \phi^{-2} \text{div}(\phi \text{ grad } h) = 0
\]

with respect to the conformal metric

\[
ds^2 = \phi(dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2).
\]

**Euclidean geometry is provided in case** \( \phi = \text{const} \). Harmonic potential fields \( \vec{V} \) in \( \mathbb{R}^4 \) satisfy the first-order static system

\[
\begin{align*}
\frac{\partial V_0}{\partial x_0} + \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3} &= 0, \\
\frac{\partial V_0}{\partial x_1} &= \frac{\partial V_1}{\partial x_0}, \quad \frac{\partial V_0}{\partial x_2} = \frac{\partial V_2}{\partial x_0}, \quad \frac{\partial V_0}{\partial x_3} = \frac{\partial V_3}{\partial x_0}, \\
\frac{\partial V_1}{\partial x_2} &= \frac{\partial V_2}{\partial x_1}, \quad \frac{\partial V_1}{\partial x_3} = \frac{\partial V_3}{\partial x_1}, \quad \frac{\partial V_2}{\partial x_3} = \frac{\partial V_3}{\partial x_2}.
\end{align*}
\]

This system is well-known in the context of *Fourier analysis on Euclidean spaces* as the Riesz system (see, e.g., [56]). On the other hand, harmonic potential fields in \( \mathbb{R}^4 \) may be equivalently represented as general class of analytic solutions of the system

\[
\begin{align*}
\frac{\partial u_0}{\partial x_0} &= \frac{\partial u_1}{\partial x_0}, \quad \frac{\partial u_0}{\partial x_1} = -\frac{\partial u_2}{\partial x_0}, \quad \frac{\partial u_0}{\partial x_2} = -\frac{\partial u_3}{\partial x_0}, \\
\frac{\partial u_1}{\partial x_1} &= \frac{\partial u_2}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_3}{\partial x_1}, \quad \frac{\partial u_2}{\partial x_3} = \frac{\partial u_3}{\partial x_2}.
\end{align*}
\]
where \((u_0, u_1, u_2, u_3) := (V_0, -V_1, -V_2, -V_3)\). This system is called the system \((R_4)\) in honor of Riesz. Four-dimensional harmonic mappings \(u = u_0 + iu_1 + ju_2 + ju_3 : \Lambda \to \mathbb{R}^4\) in the context of \textit{Quaternionic analysis} are referred to as quaternion-valued monogenic functions (see, e.g., [6, 19, 42]).

The system \((1.8)\) in the context of \textit{Modified quaternionic analysis in} \(\mathbb{R}^4\) (see, e.g., [34, 41, 42]), \textit{Hyperbolic function theory in the skew-field of quaternions} (see, e.g., [24]) and the theory of \textit{Modified harmonic functions in} \(\mathbb{R}^4\) (see, e.g., [43, 44]) may be characterized as generalized non-Euclidean modification of the system \((R_4)\) with respect to the conformal metric \((1.10)\).

1.3. Notation

The real algebra of quaternions \(\mathbb{H}\) is a four dimensional skew algebra over the real field generated by real unity 1. Three imaginary unities \(i, j, k\) satisfy to multiplication rules

\[ i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k. \]

The independent quaternionic variable is defined as

\[ x = x_0 + ix_1 + jx_2 + kx_3. \]

The quaternion conjugation of \(x\) is defined by the following automorphism:

\[ x \mapsto \overline{x} := x_0 - ix_1 - jx_2 - kx_3. \]

In such way, we deal with the Euclidean norm in \(\mathbb{R}^4\)

\[ \|x\|^2 := x\overline{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2 := r^2, \]

and the identification

\[ x = x_0 + ix_1 + jx_2 + kx_3 \sim (x_0, x_1, x_2, x_3) \]

between \(\mathbb{H}\) and \(\mathbb{R}^4\) is valid. Moreover, for every non-zero value of \(x\) an unique inverse value exists: \(x^{-1} = \overline{x}/\|x\|^2\).

The dependent quaternionic variable is defined as

\[ u = u_0 + iu_1 + ju_2 + ju_3 \sim (u_0, u_1, u_2, u_3). \]

The quaternion conjugation of \(u\) is defined by the following automorphism:

\[ u \mapsto \overline{u} := u_0 - iu_1 - ju_2 - ku_3. \]

Assume that \(x_3 > 0\). In cylindrical coordinates in \(\mathbb{R}^4\) we obtain

\[ x = x_0 + \rho (i \cos \theta + j \sin \theta \cos \psi + k \sin \theta \sin \psi), \]

where

\[ x_1 = \rho \cos \theta, \quad x_2 = \rho \sin \theta \cos \psi, \quad x_3 = \rho \sin \theta \sin \psi, \]

\[ \varphi = \arccos \frac{x_0}{r} \quad (0 < \varphi < \pi), \quad \theta = \arccos \frac{x_1}{\rho} \quad (0 \leq \theta \leq 2\pi), \]

\[ \psi = \arccot \frac{x_2}{x_3} \quad (0 < \psi < \pi). \]

**Definition 1.3.** Let \(\Omega \subset \mathbb{R}^4\) be an open set. Every continuously differentiable mapping \(u = u_0 + iu_1 + ju_2 + ju_3 : \Omega \to \mathbb{R}^4\) is called quaternion-valued \(C^1\)-function in \(\Omega\).
2. Two Types of Potential Vector Fields in \( \mathbb{R}^4 \) and Criterions of Potential Meridional Fields

To provide the first type of potential vector fields in \( \mathbb{R}^4 \), assume that \( C^1 \)-coefficient \( \phi(x_0, x_1, x_2, x_3) \) depends only on variable \( x_3 \) so that \( \phi = \phi(x_3) > 0 \). The system (1.1) is expressed as

\[
\begin{aligned}
\phi(x_3) \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} \right) - \frac{d\phi}{dx_3} u_3 = 0, \\
\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_3} = -\frac{\partial u_3}{\partial x_0},
\end{aligned}
\]  

(2.1)

New properties of analytic models of potential vector fields may be investigated in more detail in case \( \phi(x_3) = x_3^{-\alpha} \ (x_3 > 0, \alpha \in \mathbb{R}) \). The static system (1.1) is expressed as

\[
\begin{aligned}
x_3 \text{div} \ \vec{V} - \alpha V_3 = 0, \\
\frac{\partial V_0}{\partial x_0} = -\frac{\partial V_1}{\partial x_0}, \\
\frac{\partial V_0}{\partial x_1} = \frac{\partial V_1}{\partial x_1}, \\
\frac{\partial V_0}{\partial x_2} = \frac{\partial V_2}{\partial x_2}, \\
\frac{\partial V_0}{\partial x_3} = \frac{\partial V_3}{\partial x_3},
\end{aligned}
\]  

(2.2)

and the system (2.1) is simplified:

\[
\begin{aligned}
x_3 \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} \right) + \alpha u_3 = 0, \\
\frac{\partial u_0}{\partial x_1} = -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} = -\frac{\partial u_2}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_3} = -\frac{\partial u_3}{\partial x_0},
\end{aligned}
\]  

(2.3)

The system (2.3) was first introduced by Eriksson and Orelma in 2019 in the context of the theory of Hyperbolic function theory in the skew-field of quaternions [24]. This system demonstrates explicitly a family of generalizations of the Cauchy-Riemann system in accordance with the system (2.2) for different values of the parameter \( \alpha \).

When \( \alpha > 0 \), the system (2.3) may be characterized as \( \alpha \)-hyperbolic non-Euclidean modification of the system (R) with respect to the conformal metric defined on the halfspace \( \{x_3 > 0\} \) by the formula

\[
ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}.
\]

The continuity equation (1.9) takes the form of the Weinstein equation in \( \mathbb{R}^4 \) for any value of the parameter \( \alpha \) (see, e.g., [24, 34, 41, 43, 44])

\[
x_3 \Delta h - \alpha \frac{\partial h}{\partial x_3} = 0.
\]  

(2.4)

Meanwhile, nowadays solutions of the Weinstein equation (2.4) in case \( \alpha = 2 \) in the context of Hyperbolic function theory in the skew-field of quaternions are referred to as 2-hyperbolic harmonic functions in \( \mathbb{R}^4 \) [24]. The critical sets of 2-hyperbolic harmonic functions \( h = h(x_0, x_1, x_2, x_3) \) within Fueter’s construction in \( \mathbb{R}^4 \) (1.4), where \( F = F(x) = \frac{\partial h}{\partial x_0} - i \frac{\partial h}{\partial x_1} - j \frac{\partial h}{\partial x_2} - k \frac{\partial h}{\partial x_3} \).
under conditions of \( x_1 \neq 0, x_2 \neq 0, x_3 \neq 0 \) coincide with the sets of zeros of \( F = F(x) \) (on the structure of the sets of zeros of quaternionic polynomials with real coefficients see, e.g., [52, 59]).

**Definition 2.1.** Let \( \Lambda \subset \mathbb{R}^4 \ (x_3 > 0) \) be a simply connected open domain, \( \alpha > 0 \). Every exact solution of Eq. (2.4) in \( \mathbb{R}^4 \) is called \( \alpha \)-hyperbolic harmonic potential in \( \mathbb{R}^4 \).

When \( \alpha = 0 \), the system (2.3) becomes the system (2.4), while the Weinstein equation in \( \mathbb{R}^4 \) (2.4) becomes the Laplace equation.

When \( \alpha < 0 \), solutions of Eq. (2.4) in the context of Modified harmonic functions in \( \mathbb{R}^4 \) are referred to as \( -\alpha \)-modified harmonic functions in \( \mathbb{R}^4 \) (see, e.g., [43, 44]). Properties of homogeneous polynomial solutions of Eq. (2.4) in spherical coordinates on the unit half-sphere \( S^3_+ = \{(x_0, x_1, x_2, x_3) : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1, \ x_3 > 0 \} \) in \( \mathbb{R}^4 \) have been recently studied by Leutwiler.

To provide the second type of potential vector fields in \( \mathbb{R}^4 \), assume that \( C^1 \)-coefficient \( \phi(x_0, x_1, x_2, x_3) \) depends only on the cylindrical radial variable \( \rho \) so that \( \phi = \phi(\rho) > 0 \). The system (1.8) is described as

\[
\begin{align*}
\phi(\rho) \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_2}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} \right) - \left( \frac{\partial \phi(\rho)}{\partial x_1} u_1 + \frac{\partial \phi(\rho)}{\partial x_2} u_2 + \frac{\partial \phi(\rho)}{\partial x_3} u_3 \right) &= 0,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u_0}{\partial x_1} &= -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} &= -\frac{\partial u_2}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_3} &= -\frac{\partial u_3}{\partial x_0}, \\
\frac{\partial u_1}{\partial x_2} &= \frac{\partial u_2}{\partial x_1}, \\
\frac{\partial u_1}{\partial x_3} &= \frac{\partial u_3}{\partial x_1}, \\
\frac{\partial u_2}{\partial x_3} &= \frac{\partial u_3}{\partial x_2}. \\
\end{align*}
\]

\( (2.5) \)

New properties of axially symmetric analytic models of potential vector fields may be investigated in more detail in case \( \phi(\rho) = \rho^{-\alpha} \ (\rho > 0, \ \alpha \in \mathbb{R}) \). The static system (1.1) is expressed as

\[
\begin{align*}
\left( x_1^2 + x_2^2 + x_3^2 \right) \text{div} \ V - \alpha (x_1 V_1 + x_2 V_2 + x_3 V_3) &= 0, \\
\frac{\partial V_0}{\partial x_1} &= \frac{\partial V_1}{\partial x_0}, \\
\frac{\partial V_0}{\partial x_2} &= \frac{\partial V_2}{\partial x_0}, \\
\frac{\partial V_0}{\partial x_3} &= \frac{\partial V_3}{\partial x_0}, \\
\frac{\partial V_1}{\partial x_2} &= \frac{\partial V_2}{\partial x_1}, \\
\frac{\partial V_1}{\partial x_3} &= \frac{\partial V_3}{\partial x_1}, \\
\frac{\partial V_2}{\partial x_3} &= \frac{\partial V_3}{\partial x_2}, \\
\end{align*}
\]

\( (2.6) \)

and the system (2.5) is simplified:

\[
\begin{align*}
\left( x_1^2 + x_2^2 + x_3^2 \right) \left( \frac{\partial u_0}{\partial x_1} - \frac{\partial u_1}{\partial x_0} - \frac{\partial u_2}{\partial x_1} - \frac{\partial u_3}{\partial x_2} \right) + \alpha(x_1 u_1 + x_2 u_2 + x_3 u_3) &= 0, \\
\frac{\partial u_0}{\partial x_1} &= -\frac{\partial u_1}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_2} &= -\frac{\partial u_2}{\partial x_0}, \\
\frac{\partial u_0}{\partial x_3} &= -\frac{\partial u_3}{\partial x_0}, \\
\frac{\partial u_1}{\partial x_2} &= \frac{\partial u_2}{\partial x_1}, \\
\frac{\partial u_1}{\partial x_3} &= \frac{\partial u_3}{\partial x_1}, \\
\frac{\partial u_2}{\partial x_3} &= \frac{\partial u_3}{\partial x_2}. \\
\end{align*}
\]

\( (2.7) \)

This system demonstrates explicitly a family of axially symmetric generalizations of the Cauchy-Riemann system in accordance with the system (2.6) for different values of the parameter \( \alpha \).

The continuity equation (1.9) is written as

\[
(x_1^2 + x_2^2 + x_3^2) \Delta h - \alpha \left( x_1 \frac{\partial h}{\partial x_1} + x_2 \frac{\partial h}{\partial x_2} + x_3 \frac{\partial h}{\partial x_3} \right) = 0.
\]

\( (2.8) \)
When \( \alpha > 0 \), the system (2.7) may be characterized as \( \alpha \)-axial-hyperbolic non-Euclidean modification of the system \((R_4)\) with respect to the conformal metric defined outside the axis \( x_0 \) by the formula
\[
ds^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{\rho^\alpha}.
\]

**Definition 2.2.** Let \( \Lambda \subset \mathbb{R}^4 \ (\rho > 0) \) be a simply connected open domain, \( \alpha > 0 \). Every exact solution of Eq. (2.8) in \( \mathbb{R}^4 \) is called \( \alpha \)-axial-hyperbolic harmonic potential in \( \mathbb{R}^4 \).

Let us compare properties of \( \alpha \)-hyperbolic harmonic potentials and \( \alpha \)-axial-hyperbolic harmonic potentials in \( \mathbb{R}^4 \) in Cartesian coordinates. This immediately leads to the following formulation.

**Proposition 2.3 (The first criterion).** Any \( \alpha \)-hyperbolic harmonic potential \( h = h(x_0, x_1, x_2, x_3) \) in \( \Lambda \subset \mathbb{R}^4 \ (x_3 > 0) \) represents an \( \alpha \)-axial-hyperbolic harmonic potential in \( \Lambda \) if and only if \( x_2 \frac{\partial h}{\partial x_1} = x_1 \frac{\partial h}{\partial x_2}, \quad x_3 \frac{\partial h}{\partial x_2} = x_2 \frac{\partial h}{\partial x_3} \). Each condition \( x_m = 0 \) (\( m = 1, 2, 3 \)) within joint class of \( \alpha \)-hyperbolic harmonic and \( \alpha \)-axial-hyperbolic harmonic potentials implies that the component \( u_m = \frac{\partial h}{\partial x_m} \) vanishes.

**Remark 2.4.** Necessary and sufficient conditions of joint class of \( \alpha \)-hyperbolic harmonic and \( \alpha \)-axial-hyperbolic harmonic potentials in \( \mathbb{R}^4 \) coincide with conditions (1.6) of joint class of analytic solutions of the system \((H_4)\) and the system \((1.5)\).

Let us now compare properties of \( \alpha \)-hyperbolic harmonic potentials and \( \alpha \)-axial-hyperbolic harmonic potentials in \( \mathbb{R}^4 \) in cylindrical coordinates. Eq. (2.8) in cylindrical coordinates is written as
\[
\rho^2 \left( \frac{\partial^2 h}{\partial x_0^2} + \frac{\partial^2 h}{\partial \rho^2} \right) - (\alpha - 2)\rho \frac{\partial h}{\partial \rho} + \cot \theta \frac{\partial h}{\partial \theta} + \frac{\partial^2 h}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 h}{\partial \psi^2} = 0.
\]

The Weinstein equation in \( \mathbb{R}^4 \) (2.4) in cylindrical coordinates takes the following form:
\[
\rho^2 \left( \frac{\partial^2 h}{\partial x_0^2} + \frac{\partial^2 h}{\partial \rho^2} \right) - (\alpha - 2)\rho \frac{\partial h}{\partial \rho} - (\alpha - 1) \cot \theta \frac{\partial h}{\partial \theta} + \frac{\partial^2 h}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2 h}{\partial \psi^2}
- \alpha \cot \psi \frac{\partial h}{\sin^2 \theta} \frac{\partial h}{\partial \psi} = 0.
\]

This immediately leads to the following formulation.

**Proposition 2.5 (The second criterion).** Every \( \alpha \)-hyperbolic harmonic potential \( h = h(x_0, x_1, x_2, x_3) \) in \( \Lambda \subset \mathbb{R}^4 \ (x_3 > 0) \) represents an \( \alpha \)-axial-hyperbolic harmonic potential in \( \Lambda \) if and only if in cylindrical coordinates \( \frac{\partial h}{\partial \theta} = 0, \frac{\partial h}{\partial \psi} = 0 \).

**Remark 2.6.** New approach may be efficiently developed in the context of the theory of **Hyperbolic function theory in the skew-field of quaternions** and
the theory of Modified harmonic functions in \( \mathbb{R}^4 \) under conditions of \( \frac{\partial h}{\partial \varphi} = 0 \).

It should be noted that the vector \( \vec{V} = (V_0, V_1, V_2, V_3) \) within potential meridional fields in \( \mathbb{R}^4 \) is independent of two angles \( \psi \) and \( \theta \), herewith \( V_\psi := \frac{\partial h}{\partial \psi} \equiv 0 \) and \( V_\theta := \frac{\partial h}{\partial \theta} \equiv 0 \).

As it follows from the first and second criterions, new joint class of \( \alpha \)-hyperbolic harmonic and \( \alpha \)-axial-hyperbolic harmonic potentials in simply connected open domains \( \Lambda \subset \mathbb{R}^4 \) \( (x_3 > 0) \) may be characterized as general class of potential meridional fields \( \vec{V} \), where \( \phi(\rho) = \rho^{-\alpha} \). Every scalar potential \( h \) within the joint class is independent of angles \( \psi \), \( \theta \) such that

\[
\rho \left( \frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial \rho^2} \right) - (\alpha - 2) \frac{\partial g}{\partial \rho} = 0.
\]

(2.9)

Equation (2.9), where \( k = -(\alpha-2) \), is referred to as the elliptic Euler-Poisson-Darboux type equation in cylindrical coordinates (see, e.g., [2, 10, 20]), or generalized axially symmetric potential equation (GASPE) in the context of GASPT (see, e.g., [17, 31, 37, 62, 65]). Exact solutions \( g = g(x_0, \rho) \) of Eq. (2.9) are often referred to as generalized axially symmetric potentials.

Every exact solution of Eq. (2.9) as generalized axially symmetric potential indicates the existence of the Stokes’ stream function \( \hat{\gamma} = \hat{\gamma}(x_0, \rho) \), which is defined by the generalized Stokes-Beltrami system in the meridian half-plane \( (\rho > 0) \) (see, e.g., [10, 62]):

\[
\begin{cases}
\rho^{2-\alpha} \frac{\partial \hat{g}}{\partial x_0} = \frac{\partial \hat{\gamma}}{\partial \rho},
\rho^{2-\alpha} \frac{\partial \hat{\gamma}}{\partial \rho} = -\frac{\partial \hat{g}}{\partial x_0}.
\end{cases}
\]

(2.10)

The Stokes’ stream function \( \hat{\gamma} = \hat{\gamma}(x_0, \rho) \), in contrast to generalized axially symmetric potential \( g = g(x_0, \rho) \), satisfies the following equation:

\[
\rho \left( \frac{\partial^2 \hat{g}}{\partial x_0^2} + \frac{\partial^2 \hat{g}}{\partial \rho^2} \right) + (\alpha - 2) \frac{\partial \hat{g}}{\partial \rho} = 0.
\]

Consider a special class of solutions of Eq. (2.9) under condition of separation of variables \( g(x_0, \rho) = \Xi(x_0) \Upsilon(\rho) \):

\[
\frac{1}{\Xi} \frac{d^2 \Xi}{dx_0^2} + \frac{1}{\Upsilon} \frac{d^2 \Upsilon}{d\rho^2} - \frac{(\alpha - 2)}{\Upsilon \rho} \frac{d \Upsilon}{d \rho} = 0.
\]

Relations

\[
\frac{1}{\Xi} \frac{d^2 \Xi}{dx_0^2} = \frac{1}{\Upsilon} \frac{d^2 \Upsilon}{d\rho^2} - (\alpha - 2) \frac{d \Upsilon}{d \rho} = -\tilde{\beta}^2 \quad (\tilde{\beta} = \text{const} \in \mathbb{R})
\]

are equivalent to the following system of ordinary differential equations:

\[
\begin{cases}
\frac{d^2 \Xi}{dx_0^2} - \tilde{\beta}^2 \Xi = 0,
\rho^2 \frac{d^2 \Upsilon}{d\rho^2} - (\alpha - 2) \rho \frac{d \Upsilon}{d \rho} + \tilde{\beta}^2 \rho^2 \Upsilon = 0.
\end{cases}
\]

(2.11)
The first equation of the system (2.11) may be solved using hyperbolic functions: \( \Xi_\beta(x_0) = b_\beta^1 \cosh \beta x_0 + b_\beta^2 \sinh \beta x_0 \); \( b_\beta^1, b_\beta^2 = \text{const} \in \mathbb{R} \). In particular, values \( b_\beta^1 = b_\beta^2 = 1 \) imply that \( \Xi_\beta(x_0) = e^{\beta x_0} \) (see, e.g., [9]).

The second equation of the system (2.11) may be solved using linear independent solutions

\[
Y_\beta(\rho) = \rho^{\alpha - 1} \left[ a_\beta^1 J_{\alpha - \frac{1}{2}}(\beta \rho) + a_\beta^2 Y_{\alpha - \frac{1}{2}}(\beta \rho) \right]; \quad a_\beta^1, a_\beta^2 = \text{const} \in \mathbb{R},
\]

where \( J_{\alpha - \frac{1}{2}}(\beta \rho) \) and \( Y_{\alpha - \frac{1}{2}}(\beta \rho) \) are Bessel functions of the first and second kind of order \( \alpha - \frac{1}{2} \) and real argument \( \beta \rho \) (see, e.g., [21, 53, 61]).

3. Potential Meridional Fields in \( \mathbb{R}^4 \) and
Four-Dimensional \( \alpha \)-Meridional Mappings of the Second Kind

Equation (2.29) in cylindrical coordinates within potential meridional fields in \( \mathbb{R}^4 \) leads to a family of Vekua type systems for different values of the parameter \( \alpha \) investigated by Sommen et al. in the context of the theory of Quaternion-valued monogenic functions of axial type (see, e.g., [23, 49, 50])

\[
\begin{align*}
\rho \left( \frac{\partial u}{\partial x_0} - \frac{\partial u}{\partial \rho} \right) + (\alpha - 2)u_\rho &= 0, \\
\frac{\partial u}{\partial \rho} &= -\frac{\partial u}{\partial x_0}.
\end{align*}
\]

(3.1)

We should take into account that in our setting \( u_0 = \frac{\partial a}{\partial x_0}, \ u_\rho = -\frac{\partial a}{\partial \rho} \).

The static system (2.6) is reduced to the following two-dimensional system in the meridian half-plane:

\[
\begin{align*}
\rho \left( \frac{\partial V_0}{\partial x_0} + \frac{\partial V_\rho}{\partial \rho} \right) - (\alpha - 2)V_\rho &= 0, \\
\frac{\partial V_0}{\partial \rho} &= \frac{\partial V_\rho}{\partial x_0},
\end{align*}
\]

(3.2)

where \( V_0 = u_0, \ V_\rho = -u_\rho, \) and \( V_1 = V_\rho \frac{x_2}{\rho}, \ V_2 = V_\rho \frac{x_3}{\rho}, \ V_3 = V_\rho \frac{x_4}{\rho}. \)

The principal invariants of the Jacobian matrix \( J(\vec{V}(x)) \) may be demonstrated explicitly. It should be noted that the Jacobian matrix (1.14) is substantially simplified:

\[
\begin{pmatrix}
-\frac{\partial V_0}{\partial \rho} + \frac{V_\rho}{\rho}(\alpha - 2) \\
\frac{\partial V_0}{\partial x_0} - \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_2} - \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_3} - \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_4} - \frac{V_\rho}{\rho}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial V_0}{\partial \rho} + \frac{V_\rho}{\rho}(\alpha - 2) \\
\frac{\partial V_0}{\partial x_0} + \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_2} + \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_3} + \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_4} + \frac{V_\rho}{\rho}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial V_0}{\partial \rho} - \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_0} - \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_2} - \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_3} - \frac{V_\rho}{\rho} \\
\frac{\partial V_0}{\partial x_4} - \frac{V_\rho}{\rho}
\end{pmatrix}
\]

(3.3)

Theorem 3.1. Roots of the characteristic equation (1.15) of the Jacobian matrix (3.3) are given by exact formulas:

\[ \lambda_{0,1} = \frac{V_\rho}{\rho}, \]
\[ \lambda_{2,3} = \frac{(\alpha - 2)}{2} \frac{V_\rho}{\rho} \pm \sqrt{\frac{(\alpha - 2)^2}{4} \left( \frac{V_\rho}{\rho} \right)^2 - (\alpha - 2) \frac{V_\rho}{\rho} \frac{\partial V_\rho}{\partial \rho} + \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2}. \]

Proof. The principal invariants of the Jacobian matrix \((3.3)\) are written as

\[ I_{J(\vec{V})} = \alpha \frac{V_\rho}{\rho}, \]

\[ II_{J(\vec{V})} = - \left[ \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 \right] + (\alpha - 2) \frac{V_\rho}{\rho} \frac{\partial V_\rho}{\partial \rho} + (2\alpha - 3) \left( \frac{V_\rho}{\rho} \right)^2, \]

\[ III_{J(\vec{V})} = -2 \frac{V_\rho}{\rho} \left[ \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 \right] + (\alpha - 2) \left( \frac{V_\rho}{\rho} \right) \left( 2 \frac{\partial V_\rho}{\partial \rho} + \frac{V_\rho}{\rho} \right), \]

\[ IV_{J(\vec{V})} = - \left( \frac{V_\rho}{\rho} \right)^2 \left[ \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 \right] + (\alpha - 2) \left( \frac{V_\rho}{\rho} \right)^3 \frac{\partial V_\rho}{\partial \rho}. \]

The characteristic equation \((1.15)\) into the framework of the system \((3.2)\) may be factored:

\[ \left( \lambda - \frac{V_\rho}{\rho} \right)^2 \times \left[ \lambda^2 - (\alpha - 2) \frac{V_\rho}{\rho} \lambda + (\alpha - 2) \frac{V_\rho}{\rho} \frac{\partial V_\rho}{\partial \rho} - \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 - \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 \right] = 0. \]  

□

Corollary 3.2 (On the set of degenerate points). Assume that a potential meridional field \(\vec{V} = (V_0, \frac{x_1}{\rho} V_\rho, \frac{x_2}{\rho} V_\rho, \frac{x_3}{\rho} V_\rho)\) satisfies the system \((3.2)\). The set of degenerate points of the Jacobian matrix \((3.3)\) is provided by two independent equations:

\[ V_\rho = 0, \quad \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 - (\alpha - 2) \frac{V_\rho}{\rho} \frac{\partial V_\rho}{\partial \rho} = 0. \]

Corollary 3.3 (On the zero divergence condition). Assume that a potential meridional field \(\vec{V} = (V_0, \frac{x_1}{\rho} V_\rho, \frac{x_2}{\rho} V_\rho, \frac{x_3}{\rho} V_\rho)\) satisfies the system \((3.2)\), where \(\alpha \neq 0\). Every point \(x = (x_0, x_1, x_2, x_3)\), where \(\text{div} \vec{V}(x_0, x_1, x_2, x_3) = 0\), is a degenerate point of the Jacobian matrix \((3.3)\).

Remark 3.4. The second formula of Theorem 3.1 may be simplified:

\[ \lambda_{2,3} = \frac{(\alpha - 2)}{2} \frac{V_\rho}{\rho} \pm \sqrt{\left( \frac{\alpha - 2}{2} \frac{V_\rho}{\rho} - \frac{\partial V_\rho}{\partial \rho} \right)^2 + \left( \frac{\partial V_\rho}{\partial x_0} \right)^2}. \]

Geometric properties of the the Jacobian matrix \((3.3)\) allow us to introduce the concept of four-dimensional \(\alpha\)-meridional mappings of the first and second kind.
Definition 3.5. Let \( \alpha \) be a real parameter, while \( \Lambda \subset \mathbb{R}^4 \) be a simply connected open domain, where \( x_1 \neq 0, x_2 \neq 0, x_3 \neq 0 \). Assume that an exact solution \((u_0, u_1, u_2, u_3)\) of the system \((2.4)\), where \( \alpha \neq 0 \), satisfies axially symmetric conditions \( u_1 x_2 = u_2 x_1, \ u_1 x_3 = u_3 x_1, \ u_2 x_3 = u_3 x_2 \) in \( \Lambda \). Then mapping \( u = u_0 + i u_1 + j u_2 + k u_3 : \Lambda \to \mathbb{R}^4 \) is called four-dimensional \( \alpha \)-meridional mapping of the first kind, while mapping \( \Pi = u_0 - i u_1 - j u_2 - k u_3 : \Lambda \to \mathbb{R}^4 \) is called four-dimensional \( \alpha \)-meridional mapping of the second kind.

The set of degenerate points of every four-dimensional \( \alpha \)-meridional mapping of the second kind coincides with the set of degenerate points of the Jacobian matrix \((3.3)\).

Remark 3.6. The concept of four-dimensional \( \alpha \)-meridional mappings of the first and second kind may be efficiently developed in the context of the theory of Hyperbolic function theory in the skew-field of quaternions and the theory of Modified harmonic functions in \( \mathbb{R}^4 \) under conditions of \( \frac{\partial h}{\partial \rho} = 0, \frac{\partial h}{\partial \psi} = 0 \).

Let us first look at the basic properties of potential meridional fields in the context of the theory of Modified harmonic functions in \( \mathbb{R}^4 \) in case \( \alpha = -2 \), where the systems \((3.1), (3.2)\) are expressed as

\[
\begin{align*}
\rho \left( \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} \right) - 4 u_0 = 0, \\
\rho \left( \frac{\partial u_0}{\partial x_0} + \frac{\partial u_2}{\partial x_2} \right) + 4 V_0 = 0,
\end{align*}
\]

and

\[
\begin{align*}
\rho \left( \frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial \rho^2} \right) + 4 \frac{\partial g}{\partial \rho} = 0, \quad 
\rho \left( \frac{\partial^2 \hat{g}}{\partial x_0^2} + \frac{\partial^2 \hat{g}}{\partial \rho^2} \right) - 4 \frac{\partial \hat{g}}{\partial \rho} = 0.
\end{align*}
\]

The Jacobian matrix \((3.3)\) takes the following form:

\[
\begin{pmatrix}
-\frac{\partial V_0}{\partial \rho} - 4 \frac{V_0}{\rho} \\
\frac{\partial V_0}{\partial x_0} - \frac{V_0}{\rho} \\
\frac{\partial V_0}{\partial x_2} - \frac{V_0}{\rho} \\
\frac{\partial V_0}{\partial x_1} - \frac{V_0}{\rho}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial V_0}{\partial x_0} - \frac{V_0}{\rho} \\
\frac{\partial V_0}{\partial x_1} - \frac{V_0}{\rho} \\
\frac{\partial V_0}{\partial x_2} - \frac{V_0}{\rho} \\
\frac{\partial V_0}{\partial x_3} - \frac{V_0}{\rho}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial V_0}{\partial x_0} - \frac{V_0}{\rho} \\
\frac{\partial V_0}{\partial x_1} - \frac{V_0}{\rho} \\
\frac{\partial V_0}{\partial x_2} - \frac{V_0}{\rho} \\
\frac{\partial V_0}{\partial x_3} - \frac{V_0}{\rho}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial V_0}{\partial x_0} - \frac{V_0}{\rho}
\end{pmatrix}
\]

Corollary 3.7. Roots of the characteristic equation \((1.15)\) are given by the formulas

\[
\lambda_{0,1} = \frac{V_0}{\rho}, \ \lambda_{2,3} = -2 \frac{V_0}{\rho} \pm \sqrt{\left(2 \frac{V_0}{\rho} + \frac{\partial V_0}{\partial \rho}\right)^2 + \left(\frac{\partial V_0}{\partial x_0}\right)^2}.
\]
Corollary 3.8. Suppose that $\alpha = -2$, $V_0 = V_1 = V_2 = V_3$. The set of degenerate points of the Jacobian matrix (3.4) is provided by two independent equations:

$$V_\rho = 0, \quad \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 + 4\frac{V_\rho}{\rho} \frac{\partial V_\rho}{\partial \rho} = 0.$$ 

Example. Consider a generalized axially symmetric potential in case $\alpha = -2$ using Bessel function of the first kind of order $-\frac{3}{2}$ and real argument $\beta_\rho$:

$$g(x_0, \rho) = e^{\beta x_0} \rho^{-\frac{3}{2}} J_{-\frac{3}{2}}(\beta_\rho),$$

where $\rho > 0$. We deal with analytic models of $2$-modified harmonic meridional fields $\vec{V} = (V_0, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial \rho}, \frac{\partial V}{\partial \rho}, \frac{\partial V}{\partial \rho})$, where

$$V_0 = \bar{\beta} e^{\beta x_0} \rho^{\frac{3}{2}} J_{-\frac{3}{2}}(\beta_\rho), \quad V_\rho = e^{\beta x_0} \rho^{-\frac{3}{2}} \left( J_{-\frac{3}{2}}'(\beta_\rho) - \frac{3}{2\rho} J_{-\frac{3}{2}}(\beta_\rho) \right),$$

such that

$$\frac{\partial V_\rho}{\partial x_0} = \beta e^{\beta x_0} \rho^{-\frac{3}{2}} \left( J_{-\frac{3}{2}}'(\beta_\rho) - \frac{3}{2\rho} J_{-\frac{3}{2}}(\beta_\rho) \right),$$

$$\frac{\partial V_\rho}{\partial \rho} = e^{\beta x_0} \rho^{-\frac{3}{2}} \left( J_{-\frac{3}{2}}'(\beta_\rho) - \frac{3}{\rho} J_{-\frac{3}{2}}(\beta_\rho) + \frac{15}{4\rho^2} J_{-\frac{3}{2}}(\beta_\rho) \right).$$

Roots of the characteristic equation (1.15) of the Jacobian matrix (3.4) are given by the formulas

$$\lambda_{0,1} = \frac{e^{\beta x_0}}{\rho^2 \sqrt{\rho}} \left( J_{-\frac{3}{2}}'(\beta_\rho) - \frac{3}{2\rho} J_{-\frac{3}{2}}(\beta_\rho) \right),$$

$$\lambda_{2,3} = -2 \frac{e^{\beta x_0}}{\rho \sqrt{\rho}} \left( J_{-\frac{3}{2}}'(\beta_\rho) - \frac{3}{2\rho} J_{-\frac{3}{2}}(\beta_\rho) \right) \pm \frac{e^{\beta x_0}}{\rho \sqrt{\rho}} \sqrt{\beta^2 \left( J_{-\frac{3}{2}}'(\beta_\rho) - \frac{3}{2\rho} J_{-\frac{3}{2}}(\beta_\rho) \right)^2 + \left( J_{-\frac{3}{2}}''(\beta_\rho) - \frac{1}{\rho} J_{-\frac{3}{2}}'(\beta_\rho) + \frac{3}{4\rho^2} J_{-\frac{3}{2}}(\beta_\rho) \right)^2}}.$$ 

The set of degenerate points of the Jacobian matrix (3.4) is provided by two independent equations:

$$J_{-\frac{3}{2}}'(\beta_\rho) - \frac{3}{2\rho} J_{-\frac{3}{2}}(\beta_\rho) = 0,$$

$$\left( \beta^2 - \frac{4}{\rho^2} \right) \left( J_{-\frac{3}{2}}'(\beta_\rho) - \frac{3}{2\rho} J_{-\frac{3}{2}}(\beta_\rho) \right)^2 + \left( J_{-\frac{3}{2}}''(\beta_\rho) - \frac{1}{\rho} J_{-\frac{3}{2}}'(\beta_\rho) + \frac{3}{4\rho^2} J_{-\frac{3}{2}}(\beta_\rho) \right)^2 = 0.$$ 

4. The Radially Holomorphic Potential in $\mathbb{R}^4$ and Geometric Properties of Quaternionic Möbius Transformations with Real Coefficients

When $\alpha = 2$, the system (1.5) in cylindrical coordinates within Fueter’s construction in $\mathbb{R}^4$ (1.4) is reduced to the Cauchy-Riemann type system in
the meridian half-plane \( \text{(see, e.g., [2, 9, 10, 13, 16, 32])} \)

\[
\begin{align*}
\begin{cases}
\frac{\partial u_0}{\partial x_0} - \frac{\partial u_\rho}{\partial \rho} = 0, \\
\frac{\partial u_0}{\partial \rho} = -\frac{\partial u_\rho}{\partial x_0},
\end{cases}
\end{align*}
\]

where \( u_0 = \frac{\partial g}{\partial x_0}, \ u_\rho = -\frac{\partial g}{\partial \rho} \). The system \((1.7)\) is expressed as

\[
\begin{align*}
\begin{cases}
\frac{\partial V_0}{\partial x_0} + \frac{\partial V_\rho}{\partial \rho} = 0, \\
\frac{\partial V_0}{\partial \rho} = -\frac{\partial V_\rho}{\partial x_0},
\end{cases}
\end{align*}
\]

\((4.1)\)

The generalized Stokes-Beltrami system in the meridian half-plane \((2.10)\) becomes the Cauchy-Riemann type system in the meridian half-plane concerning functions \( g = g(x_0, \rho), \ \hat{g} = \hat{g}(x_0, \rho) \):

\[
\begin{align*}
\begin{cases}
\frac{\partial g}{\partial x_0} - \frac{\partial \hat{g}}{\partial \rho} = 0, \\
\frac{\partial g}{\partial \rho} = -\frac{\partial \hat{g}}{\partial x_0},
\end{cases}
\end{align*}
\]

\((4.2)\)

Generalized axially symmetric potential \( g = g(x_0, \rho) \) and the Stokes stream function \( \hat{g} = \hat{g}(x_0, \rho) \) satisfy equations

\[
\begin{align*}
\frac{\partial^2 g}{\partial x_0^2} + \frac{\partial^2 g}{\partial \rho^2} = 0, \quad \frac{\partial^2 \hat{g}}{\partial x_0^2} + \frac{\partial^2 \hat{g}}{\partial \rho^2} = 0.
\end{align*}
\]

On the other hand, an important concept of radially holomorphic functions in \( \mathbb{R}^4 \) was introduced by Gürlebeck, Habetha, Sprößig in 2008 in the context of the theory of Holomorphic functions in n-dimensional space \([32]\).

**Definition 4.1.** The radial differential operator in \( \mathbb{R}^4 \) is defined by formula

\[
\partial_{rad} G := \frac{1}{2} \left( \frac{\partial}{\partial x_0} - I \frac{\partial}{\partial \rho} \right) G := G' \quad (G = g + \hat{I} \hat{g}).
\]

Every quaternion-valued function \( G = g + \hat{I} \hat{g} \) satisfying a Cauchy-Riemann type differential equation in \( \Lambda (\rho > 0) \)

\[
\overline{\partial_{rad} G} := \frac{1}{2} \left( \frac{\partial}{\partial x_0} + I \frac{\partial}{\partial \rho} \right) G = 0
\]

\((4.3)\)

is called a radially holomorphic in \( \Lambda \), while quaternion-conjugate function \( \overline{G} = g - \hat{I} \hat{g} \) is called a radially anti-holomorphic in \( \Lambda \).

It is easy to see that general class of exact solutions of Eq. \((4.3)\) is equivalently represented as general class of exact solutions of the Cauchy-Riemann type system in the meridian half-plane concerning functions \( g = g(x_0, \rho), \ \hat{g} = \hat{g}(x_0, \rho) \) \([4.2]\) (see, e.g., \([10]\)).

The notation \( \partial_{rad} G := G' \) has been justified in \([32]\) by some clear statements. In particular, Eq. \((4.3)\) implies that

\[
G' = \frac{\partial G}{\partial x_0}.
\]

\((4.4)\)
**Definition 4.2.** Suppose that a radially holomorphic function $G = g + Ig$ in $\Lambda$ satisfies a differential equation

$$G' = F,$$

where $F = u_0 + IU_\rho$ characterizes a radially holomorphic function in $\Lambda$. The function $G$ is called a radially holomorphic primitive of $F$ in $\Lambda$.

Elementary radially holomorphic functions in $\mathbb{R}^4$ may be introduced as elementary functions of a quaternionic variable $G = G(x) = g(x_0, \rho) + Ig_\rho(x_0, \rho)$ satisfying the following relations:

$$[x^n := r^n(\cos n\varphi + I \sin n\varphi)]' = nux^{n-1};$$
$$[e^x := e^{x_0}(\cos \rho + I \sin \rho)]' = e^x;$$
$$[\cos x := \frac{1}{2}(e^{-Ix} + e^{Ix})]' = -\sin x;$$
$$[\sin x := \frac{I}{2}(e^{-Ix} - e^{Ix})]' = \cos x;$$
$$[\ln x := \ln r + I\varphi]' = x^{-1}.$$

Applications of radially holomorphic functions in $\mathbb{R}^4$ have been missed in the next book of Gürlebeck, Habetha, Sprößig [33]. Tools of the radially holomorphic potential in $\mathbb{R}^4$ in conjunction with tools of the theory of Potential meridional vector fields in $\mathbb{R}^4$ allow us to make up for the gap.

**Definition 4.3.** Radially holomorphic primitive $G = g + Ig$ in simply connected open domains $\Lambda (\rho > 0)$ into the framework of the systems (4.11), (4.12) is called the radially holomorphic potential in $\mathbb{R}^4$.

When $\alpha = 2$, the principal invariants of the Jacobian matrix (3.3)

$$
\begin{pmatrix}
-\frac{\partial V_\rho}{\partial \rho} & \frac{\partial V_\rho}{\partial x_0} x_1 & \frac{\partial V_\rho}{\partial x_0} x_2 \\
\frac{\partial V_\rho}{\partial x_0} x_1 & \frac{\partial V_\rho}{\partial x_0} x_2 + V_\rho \frac{x_1^2 + x_2^2}{\rho^2} & \frac{\partial V_\rho}{\partial \rho} x_1 \\
\frac{\partial V_\rho}{\partial x_0} x_2 & \frac{\partial V_\rho}{\partial \rho} x_2 + V_\rho \frac{x_1^2 + x_2^2}{\rho^2} & \frac{\partial V_\rho}{\partial \rho} x_2 \\
\end{pmatrix}
\right)
$$

are written as

$$I_{J(V)} = 2\frac{V_\rho}{\rho}, \quad II_{J(V)} = -\left(\left(\frac{\partial V_\rho}{\partial x_0}\right)^2 + \left(\frac{\partial V_\rho}{\partial \rho}\right)^2\right) + \left(\frac{V_\rho}{\rho}\right)^2,$$

$$III_{J(V)} = -2\frac{V_\rho}{\rho} \left[\left(\frac{\partial V_\rho}{\partial x_0}\right)^2 + \left(\frac{\partial V_\rho}{\partial \rho}\right)^2\right],$$

$$IV_{J(V)} = -\left(\frac{V_\rho}{\rho}\right)^2 \left[\left(\frac{\partial V_\rho}{\partial x_0}\right)^2 + \left(\frac{\partial V_\rho}{\partial \rho}\right)^2\right].$$
Corollary 4.4. Roots of the characteristic equation (1.15) of the Jacobian matrix (4.5) are given by the formulas

$$\lambda_{0,1} = \frac{V_\rho}{\rho}, \quad \lambda_{2,3} = \pm \sqrt{\left(\frac{\partial V_\rho}{\partial \rho}\right)^2 + \left(\frac{\partial V_\rho}{\partial x_0}\right)^2} = \pm |F'|. \quad (4.6)$$

Exact formulas (4.6) demonstrate explicitly the geometric specifics of the Jacobian matrix (4.5).

Remarkable extensions of the basic concepts of Möbius transformations with real coefficients $a, b, c, d$ within Fueter’s construction in $\mathbb{R}^4$ (1.4) $F(x) = (ax + b)(cz + d)^{-1}$, where $ad - bc = 1$, may be characterized in the context of the theory of Potential meridional vector fields in $\mathbb{R}^4$.

Example. Consider quaternionic Möbius transformations with real coefficients, where $c = 1, \ ad - b = 1$. Radially anti-holomorphic function is expressed as $\overline{F}(x) = -(\overline{\mathbf{x}} + d)^{-1} + a$. The radially holomorphic potential in $\mathbb{R}^4$ is written as

$$G = -\ln (x + d) + ax.$$ 

We deal with analytic models of meridional fields $\overline{V} = (V_0, \frac{x_1}{\rho} V_\rho, \frac{x_2}{\rho} V_\rho, \frac{x_3}{\rho} V_\rho)$, where

$$V_0 = -\frac{x_0 + d}{(x_0 + d)^2 + \rho^2} + a, \quad V_\rho = -\frac{\rho}{(x_0 + d)^2 + \rho^2},$$

$$\frac{\partial V_\rho}{\partial x_0} = \frac{2\rho(x_0 + d)}{[(x_0 + d)^2 + \rho^2]^2}, \quad \frac{\partial V_\rho}{\partial \rho} = \frac{-(x_0 + d)^2 + \rho^2}{[(x_0 + d)^2 + \rho^2]^2}.$$ 

The Jacobian matrix (4.5) takes the following form:

$$
\begin{pmatrix}
\frac{(x_0 + d)^2 - \rho^2}{[(x_0 + d)^2 + \rho^2]^2} & \frac{2(x_0 + d)x_1}{[(x_0 + d)^2 + \rho^2]^2} & \frac{2(x_0 + d)x_2}{[(x_0 + d)^2 + \rho^2]^2} & \frac{2(x_0 + d)x_3}{[(x_0 + d)^2 + \rho^2]^2} \\
\frac{2(x_0 + d)x_1}{[(x_0 + d)^2 + \rho^2]^2} & \frac{2(x_0 + d)x_2}{[(x_0 + d)^2 + \rho^2]^2} & \frac{-2x_1x_2}{[(x_0 + d)^2 + \rho^2]^2} & \frac{2x_1x_3}{[(x_0 + d)^2 + \rho^2]^2} \\
\frac{2(x_0 + d)x_2}{[(x_0 + d)^2 + \rho^2]^2} & \frac{-2x_1x_2}{[(x_0 + d)^2 + \rho^2]^2} & \frac{2(x_0 + d)x_3}{[(x_0 + d)^2 + \rho^2]^2} & \frac{2x_2x_3}{[(x_0 + d)^2 + \rho^2]^2} \\
\frac{2(x_0 + d)x_3}{[(x_0 + d)^2 + \rho^2]^2} & \frac{2x_1x_3}{[(x_0 + d)^2 + \rho^2]^2} & \frac{2x_2x_3}{[(x_0 + d)^2 + \rho^2]^2} & \frac{-2x_1x_2}{[(x_0 + d)^2 + \rho^2]^2}
\end{pmatrix} \quad (4.7)
$$

Roots of the characteristic equation (1.15) of the Jacobian matrix (4.7) are given by the formulas

$$\lambda_{0,1} = -\frac{1}{(x_0 + d)^2 + \rho^2}, \quad \lambda_{2,3} = \pm \frac{1}{(x_0 + d)^2 + \rho^2}.$$ 

Thus, the set of degenerate points of the Jacobian matrix (4.7) is empty.

Geometric properties of elementary radially holomorphic functions in $\mathbb{R}^4$ raise new issues for consideration.
5. Analytic Models of Potential Meridional Fields in $\mathbb{R}^4$
Generated by Bessel Functions of the First Kind of Integer Order and Quaternionic Argument

First outlines of the concept of integral transforms with quaternion-valued kernels within Fueter’s construction in $\mathbb{R}^4$ were presented by the author in 2003 [8]. Now tools of the Fourier-Fueter cosine and sine transforms of real-valued original functions $\tilde{\eta} = \tilde{\eta}(\tau)$ allow us to obtain integral representations for Bessel functions of the first kind of integer order $n$ and quaternionic argument $x$.

In accordance with the basic concepts of the theory of functions of a complex variable and the one-sided Laplace transform (see, e.g., [7,22,40]), every complex-valued original function $\tilde{\eta} = \tilde{\eta}(\tau)$ of a real argument $\tau$ satisfies the following conditions:

1. the function $\tilde{\eta}(\tau)$ satisfies the Hölder’s condition for any $\tau$ except for points $\tau = \tau_1, \tau_2, \ldots$, where the function $\tilde{\eta}(\tau)$ has discontinuities of the first kind (there exists a finite number of such points for every finite interval),
2. the function $\tilde{\eta}(\tau) = 0$ for any $\tau < 0$,
3. $|\tilde{\eta}(\tau)| < Me^{s_0\tau}$ with certain constants $M > 0, s_0 \geq 0$ for any $\tau \geq 0$.

In accordance with the Hölder’s condition, there exist certain constants $A > 0, 0 < \gamma \leq 1, \delta_0 > 0$ such that $|\tilde{\eta}(\tau + \delta) - \tilde{\eta}(\tau)| \leq A|\delta|^\gamma$ for any $\delta$, where $|\delta| \leq \delta_0$. The number $s_0$ is referred to as the growth rate of the function $\tilde{\eta}(\tau)$; $s_0$ may take the value 0 in case of bounded functions $\tilde{\eta}(\tau)$.

To extend some concepts of integral transforms of real-valued original functions $\tilde{\eta} = \tilde{\eta}(\tau)$ within Fueter’s construction in $\mathbb{R}^4$ (1.4), a correspondence between two functions $\tilde{\eta}(\tau)$ and $F(x)$ in the form

$$F(x) := \int_0^\infty \tilde{\eta}(\tau)K(x, \tau)d\tau,$$

where quaternion-valued function $K(x, \tau)$ belongs to Fueter’s construction in $\mathbb{R}^4$ for any $\tau \geq 0$, shall henceforth be referred to as integral transform with quaternion-valued kernel $K(x, \tau)$.

**Definition 5.1.** Let $\tilde{\eta}(\tau)$ be a real-valued original function and $\rho > 0$. Every integral transform with quaternion-valued kernel $K(x, \tau) = e^{-x\tau}$

$$F(x) := \mathcal{LF}\{\tilde{\eta}(\tau); x\} = \int_0^\infty \tilde{\eta}(\tau)e^{-x\tau}d\tau$$

is called the one-sided Laplace-Fueter transform of $\tilde{\eta}(\tau)$.

Analytic properties of radially anti-holomorphic functions in $\mathbb{R}^4$

$$\mathcal{F}(x) = \int_0^\infty \tilde{\eta}(\tau)e^{-x\tau}d\tau = \int_0^\infty \tilde{\eta}(\tau)e^{-x_0\tau}[\cos(\rho\tau) + I\sin(\rho\tau)]d\tau$$

are of particular interest to the theory of Potential meridional vector fields in $\mathbb{R}^4$. 
Definition 5.2. Let $\tilde{\eta}(\tau)$ be a real-valued original function and $\rho > 0$. Every integral transform with quaternion-valued kernel $K(x, \tau) = \cos(x\tau)$

$$F(x) := \mathcal{FF}c\{\tilde{\eta}(\tau)\}; x\} = \int_0^\infty \tilde{\eta}(\tau) \cos(x\tau) d\tau = \frac{1}{2} \int_0^\infty \tilde{\eta}(\tau)(e^{-ix\tau} + e^{ix\tau}) d\tau$$

is called the Fourier-Fueter cosine transform of $\tilde{\eta}(\tau)$.

Remark 5.3. Consider an independent quaternionic variable of the following form: $y = Ix = -\rho + Ix_0$. The Fourier-Fueter cosine transform of $\tilde{\eta}(\tau)$ may be equivalently represented by means of the one-sided Laplace-Fueter transform of $\tilde{\eta}(\tau)$:

$$\mathcal{LF}c\{\tilde{\eta}(\tau)\}; y\} = \frac{1}{2} [\mathcal{LF}\{\tilde{\eta}(\tau)\}; y\} + \mathcal{LF}\{\tilde{\eta}(\tau); -y\}].$$

Analytic models of potential meridional fields generated by the Fourier-Fueter cosine transform of $\tilde{\eta}(\tau)$ are described as $\vec{V} = (V_0, \frac{\partial}{\partial \rho} V_\rho, \frac{\partial}{\partial x} V_\rho, \frac{\partial}{\partial \rho} V_\rho)$, where

$$V_0 = \int_0^\infty \tilde{\eta}(\tau) \cosh(\rho\tau) \cos(x_0\tau) d\tau, \quad V_\rho = \int_0^\infty \tilde{\eta}(\tau) \sinh(\rho\tau) \sin(x_0\tau) d\tau.$$  

Definition 5.4. Let $\tilde{\eta}(\tau)$ be a real-valued original function and $\rho > 0$. Every integral transform with quaternion-valued kernel $K(x, \tau) = \sin(x\tau)$

$$F(x) := \mathcal{FF}s\{\tilde{\eta}(\tau)\}; x\} = \int_0^\infty \tilde{\eta}(\tau) \sin(x\tau) d\tau = \frac{I}{2} \int_0^\infty \tilde{\eta}(\tau)(e^{-ix\tau} - e^{ix\tau}) d\tau$$

is called the Fourier-Fueter sine transform of $\tilde{\eta}(\tau)$.

Remark 5.5. The Fourier-Fueter sine transform of $\tilde{\eta}(\tau)$ may be equivalently represented by means of the one-sided Laplace-Fueter transform of $\tilde{\eta}(\tau)$:

$$\mathcal{LF}s\{\tilde{\eta}(\tau)\}; y\} = \frac{I}{2} [\mathcal{LF}\{\tilde{\eta}(\tau)\}; y\} - \mathcal{LF}\{\tilde{\eta}(\tau); -y\}].$$

Analytic models of potential meridional fields generated by the Fourier-Fueter sine transform of $\tilde{\eta}(\tau)$ are described as $\vec{V} = (V_0, \frac{\partial}{\partial \rho} V_\rho, \frac{\partial}{\partial x} V_\rho, \frac{\partial}{\partial \rho} V_\rho)$, where

$$V_0 = \int_0^\infty \tilde{\eta}(\tau) \cosh(\rho\tau) \sin(x_0\tau) d\tau, \quad V_\rho = -\int_0^\infty \tilde{\eta}(\tau) \sinh(\rho\tau) \cos(x_0\tau) d\tau.$$  

Important properties of quaternionic power series with real coefficients studied by Leutwiler and Hempfling in the context of Modified quaternionic analysis in $\mathbb{R}^4$ (see, e.g., [34, 41]) allow us to extend every Bessel function of the first kind of integer order and complex argument (see, e.g., [21, 22, 61]) within Fueter’s construction in $\mathbb{R}^4$ (1.3) from the disk of radius $r$ in the complex plane $D_r = \{(x_0, x_1) : x_0^2 + x_1^2 < r^2\}$ to the ball of radius $r$ in $\mathbb{R}^4$ $B_r^4 = \{(x_0, x_1, x_2, x_3) : x_0^2 + x_1^2 + x_2^2 + x_3^2 < r^2\}$. As a corollary, every Bessel function of the first kind of integer order $n$ and quaternionic argument is represented by a quaternionic power series with real coefficients:

$$J_n(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!(n+m)!} \left(\frac{x}{2}\right)^{n+2m}.$$
Furthermore, every elementary radially holomorphic function \((\frac{x}{2})^m\) may be expanded in a series of Bessel functions of the first kind of integer order \((m + 2n)\) and quaternionic argument:

\[
\left(\frac{x}{2}\right)^m = \sum_{n=0}^{\infty} \frac{(m + 2n)(m + n - 1)!}{n!} J_{m+2n}(x) \quad (m = 1, 2, 3, \ldots)
\]

Using the Fourier-Fueter cosine transform of \(\tilde{\eta}(\tau) = \frac{\cos(2n \arccos \tau)}{\sqrt{1 - \tau^2}}\), integral representations for Bessel functions of the first kind of even integer order and quaternionic argument may be obtained:

\[
\frac{\pi}{2} (-1)^n J_{2n}(x) = \mathcal{F}\mathcal{F}c\{\tilde{\eta}(\tau); x\} = \int_0^1 \frac{\cos(2n \arccos \tau)}{\sqrt{1 - \tau^2}} \cos(x\tau) d\tau.
\]

Using the Fourier-Fueter sine transform of \(\tilde{\eta}(\tau) = \frac{\cos((2n+1) \arccos \tau)}{\sqrt{1 - \tau^2}}\), integral representations for Bessel functions of the first kind of odd integer order and quaternionic argument may be obtained:

\[
\frac{\pi}{2} (-1)^n J_{2n+1}(x) = \mathcal{F}\mathcal{F}s\{\tilde{\eta}(\tau); x\} = \int_0^1 \frac{\cos((2n+1) \arccos \tau)}{\sqrt{1 - \tau^2}} \sin(x\tau) d\tau.
\]

**Example.** Bessel function of the first kind of order zero and quaternionic argument is represented as

\[
J_0(x) = \sum_{m=0}^{\infty} (-1)^m \frac{2^m}{(m!)^2} \tilde{\eta}(\tau).
\]

Integral representation of Bessel function of the first kind of order zero and quaternionic argument is given by the formula

\[
J_0(x) = \frac{2}{\pi} \mathcal{F}\mathcal{F}c\{\tilde{\eta}(\tau); x\} = \frac{2}{\pi} \int_0^1 \frac{1}{\sqrt{1 - \tau^2}} \cos(x\tau) d\tau.
\]

As seen, remarkable analytic model of a potential meridional field generated by Bessel function of the first kind of order zero and quaternionic argument is described as

\[
\vec{V} = (V_0, \frac{\partial}{\rho} V_\rho, \frac{\partial}{\rho} V_\rho, \frac{\partial}{\rho} V_\rho),
\]

where

\[
V_0 = \int_0^1 \frac{\cosh(\rho\tau)}{\sqrt{1 - \tau^2}} \cos(x_0\tau) d\tau, \quad V_\rho = \int_0^1 \frac{\sinh(\rho\tau)}{\sqrt{1 - \tau^2}} \sin(x_0\tau) d\tau.
\]

### 6. Four-Dimensional Harmonic Meridional Mappings of the Second Kind and Gradient Dynamical Systems with Harmonic Potential

Some remarkable properties of the Hessian matrix \(\mathbf{H}(h(x))\) of harmonic potentials \(h = h(x_0, x_1, x_2, x_3)\) in the general four-dimensional setting were described by Yanushauskas in 1980 by means of homogeneous harmonic polynomials [63].

The geometric specifics of the Hessian matrix \(\mathbf{H}(h(x))\) within harmonic potential meridional fields \(\vec{V} = \text{grad } h\) may be efficiently studied using the
Jacobian matrix (3.3) in case \( \alpha = 0 \):

\[
\begin{pmatrix}
\left(- \frac{\partial V_\rho}{\partial \rho} - 2 V_\rho \right) \\
\frac{\partial V_\rho}{\partial x_1} \\
\frac{\partial V_\rho}{\partial x_2} \\
\frac{\partial V_\rho}{\partial x_3}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial V_\rho}{\partial \rho} \\
\frac{\partial V_\rho}{\partial x_1} \\
\frac{\partial V_\rho}{\partial x_2} \\
\frac{\partial V_\rho}{\partial x_3}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

(6.1)

Let us look at the basic properties of harmonic potential meridional fields in \( \mathbb{R}^3 \), where the systems (3.1), (3.2) are expressed as

\[
\begin{cases}
\rho \left( \frac{\partial u_0}{\partial \rho} - \frac{\partial u_0}{\partial x_0} \right) - 2 u_\rho = 0, \\
\frac{\partial u_0}{\partial \rho} = \frac{\partial u_0}{\partial x_0}.
\end{cases}
\]

The characteristic equation (1.5) in case \( \alpha = 0 \) is written as incomplete equation:

\[
\lambda^4 + II_{J(\vec{v})} \lambda^2 - III_{J(\vec{v})} \lambda + IV_{J(\vec{v})} = 0,
\]

(6.2)

where

\[
II_{J(\vec{v})} = - \left[ \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 \right] - 2 \frac{V_\rho}{\rho} \frac{\partial V_\rho}{\partial \rho} - 3 \left( \frac{V_\rho}{\rho} \right)^2,
\]

\[
III_{J(\vec{v})} = -2 \frac{V_\rho}{\rho} \left[ \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 \right] - 2 \left( \frac{V_\rho}{\rho} \right)^2 \left( 2 \frac{\partial V_\rho}{\partial \rho} + \frac{V_\rho}{\rho} \right),
\]

\[
IV_{J(\vec{v})} = - \left( \frac{V_\rho}{\rho} \right)^2 \left[ \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 \right] - 2 \left( \frac{V_\rho}{\rho} \right)^3 \frac{\partial V_\rho}{\partial \rho}.
\]

**Corollary 6.1.** Roots of the characteristic equation (6.2) are given by the formulas

\[
\lambda_{0,1} = \frac{V_\rho}{\rho}, \quad \lambda_{2,3} = - \frac{V_\rho}{\rho} \pm \sqrt{\left( \frac{V_\rho}{\rho} + \frac{\partial V_\rho}{\partial \rho} \right)^2 + \left( \frac{\partial V_\rho}{\partial x_0} \right)^2}.
\]

(6.3)

The set of degenerate points of the Jacobian matrix (6.1) is provided by two independent equations:

\[
V_\rho = 0, \quad \left( \frac{\partial V_\rho}{\partial x_0} \right)^2 + \left( \frac{\partial V_\rho}{\partial \rho} \right)^2 + 2 \frac{V_\rho}{\rho} \frac{\partial V_\rho}{\partial \rho} = 0.
\]

(6.4)

Exact formulas (6.3), (6.4) demonstrate explicitly the geometric specifics of the Jacobian matrix (6.1).
**Definition 6.2.** Let $\Lambda \subset \mathbb{R}^4$ be a simply connected open domain, where $x_1 \neq 0, x_2 \neq 0, x_3 \neq 0$. Assume that an exact solution $(u_0, u_1, u_2, u_3)$ of the system $(R_4)$ satisfies axially symmetric conditions $u_1 x_2 = u_2 x_1, \quad u_1 x_3 = u_2 x_3 = u_3 x_2$ in $\Lambda$. Then mapping $u = u_0 + i u_1 + j u_2 + k u_3 : \Lambda \rightarrow \mathbb{R}^4$ is called four-dimensional harmonic meridional mapping of the first kind, while mapping $\overline{u} = u_0 - i u_1 - j u_2 - k u_3 : \Lambda \rightarrow \mathbb{R}^4$ is called four-dimensional harmonic meridional mapping of the second kind.

The set of degenerate points of every four-dimensional harmonic meridional mapping of the second kind coincides with the set of degenerate points of the Jacobian matrix $(6.1)$.

**Example.** Consider a generalized axially symmetric potential in case $\alpha = 0$ using Bessel function of the first kind of order $-\frac{1}{2}$ and real argument $\beta \rho$:

$$g(x_0, \rho) = e^{\beta x_0} \rho^{-\frac{1}{2}} J_{-\frac{1}{2}}(\beta \rho).$$

We deal with analytic models of harmonic potential meridional fields $\vec{V} = (V_0, \frac{x_1}{\rho} V_\rho, \frac{x_2}{\rho} V_\rho, \frac{x_3}{\rho} V_\rho)$, where

$$V_0 = \tilde{\beta} e^{\beta x_0} \rho^{-\frac{1}{2}} J_{-\frac{1}{2}}(\beta \rho), \quad V_{\rho} = e^{\beta x_0} \rho^{-\frac{1}{2}} \left( J'_{-\frac{1}{2}}(\beta \rho) - \frac{1}{2\rho} J_{-\frac{1}{2}}(\beta \rho) \right),$$

such that

$$\frac{\partial V_0}{\partial x_0} = \tilde{\beta} e^{\beta x_0} \rho^{-\frac{1}{2}} \left( J'_{-\frac{1}{2}}(\beta \rho) - \frac{1}{2\rho} J_{-\frac{1}{2}}(\beta \rho) \right),$$

$$\frac{\partial V_\rho}{\partial \rho} = e^{\beta x_0} \rho^{-\frac{1}{2}} \left( J''_{-\frac{1}{2}}(\beta \rho) - \frac{1}{\rho} J'_{-\frac{1}{2}}(\beta \rho) + \frac{3}{4\rho^2} J_{-\frac{1}{2}}(\beta \rho) \right).$$

Roots of the characteristic equation $(6.2)$ are given by these formulas:

$$\lambda_{0,1} = \frac{e^{\beta x_0}}{\rho \sqrt{\rho}} \left( J'_{-\frac{1}{2}}(\beta \rho) - \frac{1}{2\rho} J_{-\frac{1}{2}}(\beta \rho) \right),$$

$$\lambda_{2,3} = -\frac{e^{\beta x_0}}{\rho \sqrt{\rho}} \left( J'_{-\frac{1}{2}}(\beta \rho) - \frac{1}{2\rho} J_{-\frac{1}{2}}(\beta \rho) \right) \pm$$

$$\frac{e^{\beta x_0}}{\sqrt{\rho}} \sqrt{\tilde{\beta}^2 \left( J'_{-\frac{1}{2}}(\beta \rho) - \frac{1}{2\rho} J_{-\frac{1}{2}}(\beta \rho) \right)^2 + \left( J''_{-\frac{1}{2}}(\beta \rho) + \frac{1}{4\rho^2} J_{-\frac{1}{2}}(\beta \rho) \right)^2}. $$

The set of degenerate points of the Jacobian matrix $(6.1)$ is provided by two independent equations:

$$J'_{-\frac{1}{2}}(\beta \rho) - \frac{1}{2\rho} J_{-\frac{1}{2}}(\beta \rho) = 0,$$

$$\left( \frac{1}{\rho^2} \right) \left( J'_{-\frac{1}{2}}(\beta \rho) - \frac{1}{2\rho} J_{-\frac{1}{2}}(\beta \rho) \right)^2 + \left( J''_{-\frac{1}{2}}(\beta \rho) + \frac{1}{4\rho^2} J_{-\frac{1}{2}}(\beta \rho) \right)^2 = 0.$$

Roots of the characteristic equation $(6.2)$ may be interpreted as the Poincaré stability coefficients in the context of the theory of Gradient dynamical systems with harmonic potential, where $\vec{V} = \frac{d\vec{x}}{dt} = \text{grad} \ h, \ \Delta \ h = 0$. It should be noted that properties of the sets of positive, zero and negative
values of the roots $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are of particular interest to Stability theory (see, e.g., [12, 38, 39, 51]).

References

[1] Ahlfors, L.V.: Möbius Transformations in Several Dimensions. Ordway Lectures in Mathematics. University of Minnesota, Minneapolis (1981)

[2] Aksenov, A.V.: Linear differential relations between solutions of the class of Euler-Poisson-Darboux equations. J. Math. Sci. 130(5), 4911–4940 (2005)

[3] Arnold, V.I.: Geometrical Methods in the Theory of Ordinary Differential Equations. Grundlehren der mathematischen Wissenschaften, vol. 250, 2nd edn. Springer, New York (1988)

[4] Bisi, C., Gentili, G.: Möbius transformations and the Poincaré distance in the quaternionic setting. Indiana Univ. Math. J. 58(6), 2729–2764 (2009)

[5] Brackx, F., Delange, R., Sommen, F.: Clifford Analysis. Research Notes in Mathematics, vol. 76. Pitman, Boston (1982)

[6] Brackx, F., Delanghe, R.: On harmonic potential fields and the structure of monogenic functions, Z. Anal. Anwend. 22(2), 261–273 (2003)

[7] Bronshtein, I.N., Semendyayev, K.A., Musiol, G., Muehlig, H.: Handbook of Mathematics, 6th edn. Springer, Berlin, Heidelberg (2015)

[8] Bryukhov, D.A.: Axially symmetric generalization of the Cauchy-Riemann system and modified Clifford analysis (2003) https://arxiv.org/abs/math/0302186v1 [math.CV]

[9] Bryukhov, D., Kähler, U.: The static Maxwell system in three dimensional axially symmetric inhomogeneous media and axially symmetric generalization of the Cauchy-Riemann system. Adv. Appl. Clifford Algebras 27(2), 993–1005 (2017)

[10] Bryukhov, D.: Electrostatic fields in some special inhomogeneous media and new generalizations of the Cauchy-Riemann system. Adv. Appl. Clifford Algebras 31, 61 (2021)

[11] Cao W.: On the classification of four-dimensional Möbius transformations. Proc. Edinb. Math. Soc. 50(1), 49–62 (2007)

[12] Chetayev, N.G.: The Stability of Motion. Translated from the Russian by Morton Nadler. Pergamon Press, London (1961)

[13] Colombo, F., Sabadini, I., Struppa, D.C.: Slice monogenic functions. Israel J. Math. 171, 385–403 (2009)

[14] Colombo, F., González-Cervantes, J.O., Sabadini, I.: Comparison of the various notions of slice monogenic functions and their variations. In: Simos, T.E. et al. (eds.) Numer. Anal. Appl. Math. ICNAAM 2011. AIP Conf. Proc., vol. 1389, New York, pp. 264–267 (2011)

[15] Colombo, F., Sabadini, I., Struppa, D.C.: Noncommutative Functional Calculus: Theory and Applications of Slice Hyperholomorphic Functions. Progress in Mathematics, vol.289. Birkhäuser, Basel (2011)

[16] Colombo, F., Sabadini, I., Struppa, D.C.: Entire Slice Regular Functions. SpringerBriefs in Mathematics, Springer (2016)
[17] Colton, D.: Arthur Erdélyi 1908-1977. Bull. London Math. Soc. 11, 191–207 (1979)

[18] Delange, R., Sommen, F., Souček, V.: Clifford Algebra and Spinor-Valued Functions: A Function Theory for the Dirac Operator. Mathematics and Its Applications, vol. 53. Kluwer Academic, Dordrecht (1992)

[19] Delanghe, R.: On homogeneous polynomial solutions of the Riesz system and their harmonic potentials. Complex Var. Elliptic Equ. 52(10-11), 1047–1062 (2007)

[20] Dzhaiani, G.V.: The Euler-Poisson-Darboux Equation [in Russian]. Izd. Tbilisskogo Gos. Univ., Tbilisi (1984)

[21] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F. G.: Higher Transcendental Functions, Vol.II (Bateman Manuscript Project). McGraw-Hill, New York (1953)

[22] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F. G.: Tables of Integral Transforms, Vol.I (Bateman Manuscript Project). McGraw-Hill, New York (1954)

[23] Eriksson, S.-L., Orelma, H., Vieira N.: Two-sided hypergenic functions. Adv. Appl. Clifford Algebras 27(1), 111–123 (2017)

[24] Eriksson, S.-L., Orelma, H.: Hyperbolic function theory in the skew-field of quaternions. Adv. Appl. Clifford Algebras 29, 97 (2019)

[25] Fueter, R.: Die funktionentheorie der differentialgleichungen ∆u = 0 und ΔΔu = 0 mit vier reellen variablen. Comment. Math. Helv. 7, 307–330 (1934/1935).

[26] Gal, S.G., González-Cervantes, J.O., Sabadini, I.: On some geometric properties of slice regular functions of a quaternion variable. Complex Var. Elliptic Equ. 60(10), 1431–1455 (2015)

[27] Gentili G., Struppa, D.C.: A new approach to Cullen-regular functions of a quaternionic variable, C. R. Acad. Sci. Paris 342, 741–744 (2006)

[28] Gentili, G., Stoppato, C.: Power series and analyticity over the quaternions. Math. Ann. 352(1), 113–131 (2012)

[29] Gentili, G., Stoppato, C.: Geometric function theory over quaternionic slice domains. J. Math. Anal. Appl. 495(2), 124780 (2021)

[30] Gilmore, R.: Catastrophe Theory for Scientists and Engineers. Dover Publications, New York (1993)

[31] Gryshchuk, S.V., Plaksa, S.A.: Integral representations of generalized axially symmetric potentials in a simply connected domain. Ukr. Math. J. 61(2), 195–213 (2009)

[32] Gürlebeck, K., Habetha, K., Sprößig, W.: Holomorphic Functions in the Plane and n-Dimensional Space. Birkhäuser, Basel (2008)

[33] Gürlebeck, K., Habetha, K., Sprößig, W.: Application of Holomorphic Functions in Two and Higher Dimensions. Birkhäuser, Basel (2016)

[34] Hempfling, Th., Leutwiler, H.: Modified quaternionic analysis in \( \mathbb{R}^4 \). In: Dietrich V. et al. (eds.) Clifford Algebras and Their Application in Mathematical Physics. Fundamental Theories of Physics, vol. 94, pp. 227–237. Kluwer Acad. Publ., Dordrecht (1998)
[35] Hirsch, M.W.: The dynamical systems approach to differential equations. Bull. Am. Math. Soc. (N.S.) 11(1), 1–64 (1984)

[36] Hirsch, M.W., Smale, S., Devaney, R.L.: Differential Equations, Dynamical Systems, and an Introduction to Chaos. Pure and Applied Mathematics, vol. 60, 2nd edn. Academic Press, San Diego, CA (2004)

[37] Huber, A.: On the uniqueness of generalized axially symmetric potentials. Annals of Mathematics 60(2), 351–358 (1954)

[38] Kozlov, V.V.: On the degree of instability. J. Appl. Math. Mech. 57(5), 771–776 (1993)

[39] Kozlov, V.V., Furti, S.D.: The first Lyapunov method for strongly non-linear systems of differential equations [in Russian]. In: Matrosov V.M. et al. (eds.) Nonlinear Mechanics, pp. 89–113. Physical and Mathematical Literature Pub., Moscow (2001)

[40] Lavrentyev, M.A., Shabat, B.V.: Methods of the Theory of Functions of a Complex Variable [in Russian]. 5th edn., Nauka, Moscow (1987)

[41] Leutwiler, H.: Modified Clifford analysis. Complex Var. Theory Appl. 17(3-4), 153–171 (1992)

[42] Leutwiler, H., Zeilinger, P.: On quaternionic analysis and its modifications. Comput. Methods Funct. Theory 4(1), 159–183 (2004)

[43] Leutwiler, H.: Modified spherical harmonics in four dimensions. Adv. Appl. Clifford Algebras 28, 49 (2018)

[44] Leutwiler, H.: Contributions to modified spherical harmonics in four dimensions, Complex Anal. Oper. Theory 14, 67 (2020)

[45] Lounesto, P.: Clifford Algebras and Spinors, 2nd edn. London Math. Soc. Lecture Notes Series, vol. 286. Cambridge University Press, Cambridge (2001)

[46] Nemytskii, V.V., Stepanov, V.V.: Qualitative Theory of Differential Equations. Princeton Mathematical Series, vol. 22. Princeton University Press, Princeton, NJ (1960)

[47] Nucci, M.C., Tamizhmani, K.M.: Lagrangians for biological models. J. Nonlinear Math. Phys. 19(3), 1250021 (2012)

[48] Parker, J.R., Short, I.: Conjugacy classification of quaternionic Möbius transformations. Comput. Methods Funct. Theory 9(1), 13–25 (2009)

[49] Peña Peña, D., Sommen, F.: Vekua-type systems related to two-sided monogenic functions. Complex. Anal. Oper. Theory 6(2), 397–405 (2012)

[50] Peña Peña, D., Sabadini, I., Sommen, F.: On two-sided monogenic functions of axial type. Mosc. Math. J. 17(1), 129–143 (2017)

[51] Perko, L.: Differential Equations and Dynamical Systems. Texts in Applied Mathematics vol.7, 3rd edn. Springer, New York (2001)

[52] Pogorui, A., Shapiro, M.V.: On the structure of the set of zeros of quaternionic polynomials. Complex Var. 49(6), 379–389 (2004)

[53] Polyaniin, A.D., Zaitsev, V.F.: Handbook of Exact Solutions for Ordinary Differential Equations. 2nd edn., Chapman and Hall/CRC Press, Boca Raton/London (2003)

[54] Porter, R.M.: Quaternionic Möbius transformations and loxodromes. Complex Var. Theory Appl. 36(3), 285–300 (1998)
[55] Sinai, Ya.G.: Topics in Ergodic Theory. Princeton Mathematical Series, vol. 44. Princeton University Press, Princeton, New Jersey (1994)

[56] Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton Mathematical Series, vol. 32. Princeton University Press, Princeton, New Jersey (1971)

[57] Stoppato, C.: Regular Moebius transformations of the space of quaternions. Ann. Glob. Anal. Geom. 39(4), 387–401 (2011)

[58] Sudbery, A.: Quaternionic analysis. Math. Proc. Camb. Philos. Soc., 85, 199–225 (1979)

[59] Topuridze, N.: On the roots of polynomials over division algebras. Georgian Math. J. 10(4), 745–762 (2003)

[60] Walter W.: Ordinary Differential Equations. Graduate Texts in Mathematics, vol. 182. Springer, New York (1998)

[61] Watson, G.N.: A Treatise on the Theory of Bessel Functions, 2nd edn. Cambridge Mathematical Library, Cambridge University Press, Cambridge (1995)

[62] Weinstein, A.: Generalized axially symmetric potential theory. Bull. Am. Math. Soc. 59(1), 20–38 (1953)

[63] Yanushauskas, A.I.: Lewy’s theorem on the zeros of the Hessian of a harmonic function. Sib. Math. J. 21, 861–865 (1980)

[64] Zachmanoglou, E.C., Thoe, D.W.: Introduction to Partial Differential Equations with Applications. Dover Publications, New York (1986)

[65] Zwillinger, D.: Handbook of Differential Equations, vol. 1, 3rd edn. Academic Press, San Diego (1998)

Dmitry Bryukhov

https://orcid.org/0000-0002-8977-3282

Science City Fryazino, Russia

e-mail: bryukhov@mail.ru