On the discrete Wigner function for SU(N)

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Abstract. We present a self-consistent theoretical framework for finite-dimensional discrete phase spaces that leads us to establish a well-grounded mapping scheme between Schwinger unitary operators and generators of the special unitary group SU(N). This general mathematical construction provides a sound pathway to the formulation of a genuinely discrete Wigner function for arbitrary quantum systems described by finite-dimensional state vector spaces. To illustrate our results, we obtain a general discrete Wigner function for the group SU(3) and apply this to the study of a particular three-level system. Moreover, we also discuss possible extensions to the discrete Husimi and Glauber-Sudarshan functions, as well as future investigations on multipartite quantum states.

Keywords: Discrete Wigner Function, Group SU(N), Finite-Dimensional Discrete Phase Spaces, Schwinger Unitary Operators

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1. Introduction

Recent advances in theoretical and experimental investigations consolidate the Wigner function as a fundamental mathematical tool of wide-ranging use in several branches of physics, chemistry, and engineering [1]. Originally formulated eighty-seven years ago by Eugene Paul Wigner [2], this function represents a historical landmark in the continuous phase-space representations of quantum mechanics [3,4]. Despite this enormous success, it is important to emphasize that its counterpart in the finite-dimensional discrete phase-space representations has been the subject of sound theoretical approaches dating back a few decades [5]. In this connection, Tilma et al [6] recently proposed a general approach for constructing Wigner functions associated with continuous representations (Euler angles) that allows us to describe arbitrary spin systems. In particular, this kind of construction makes explicit use of symmetries related to the special unitary group SU(N), although it produces Wigner functions that are difficult to visualize for multipartite systems. This apparent disadvantage can be circumvented by means of a theoretical formalism that encompasses the generators of SU(N) and their representatives in the finite-dimensional discrete phase spaces, producing, as a by-product, discrete Wigner functions that describe the multipartite quantum states characterized by finite state vector spaces.

Nowadays, it is noticeable the efforts developed by some authors in the past to establish a quantum-algebraic framework for finite-dimensional discrete phase spaces that allows, within other things, to properly describe the quasiprobability distribution functions in complete analogy with their continuous counterparts, as well as in different scenarios of admissible applications in quantum mechanics [7–37]. The main goal of this paper is to fulfil the aforementioned gap through a self-consistent theoretical framework for finite-dimensional discrete phase spaces that permits to implement a well-succeeded mapping scheme between generators of the special unitary group and Schwinger unitary operators. This one-to-one correspondence represents a real twofold-gain for our purposes, since it leads us to establish the discrete representatives of these generators in such phase spaces (by means of the \( \text{mod}(N) \)-invariant unitary operator basis [11]) and to obtain a general expression for the discrete Wigner function – here related to SU(N) – which describes arbitrary finite quantum systems. As a consequence, the first standard examples that emerge from this general formalism take into account the groups SU(2) and SU(3), with special emphasis on the respective discrete Wigner functions. The study of a particular three-level system serves, in this case, as a toy model employed to illustrate the discrete Wigner function for SU(3) and its possible implications in physics. To conclude, we also discuss some relevant points associated with possible extensions to the discrete Husimi and Glauber-Sudarshan quasiprobability distribution functions, arbitrary spin systems, and multipartite quantum states.

This paper is structured as follows. In Section 2 we fix a preliminary mathematical background on finite-dimensional discrete phase spaces that leads us to establish a first definition of a discrete Wigner function via \( \text{mod}(N) \)-invariant unitary operator
basis, and also general expressions for mean values. In Section 2, we briefly review the basic aspects of the group SU(N) with focus on the complete orthonormal operator basis formed by its $N^2 - 1$ generators. Next, we show the formal connection with the finite-dimensional discrete phase spaces through a well-grounded mapping scheme that allows to construct straight relations between generators of the special unitary group and Schwinger unitary operators, as well as to determine the discrete representatives of these generators in such phase spaces. This section culminates by presenting an elegant mathematical expression for the discrete Wigner function – now associated with the group SU(N) – which allows us to investigate arbitrary quantum systems described by finite-dimensional state vector spaces. Section 4 is dedicated to illustrate our results through the group SU(3), where the corresponding discrete Wigner function is used to investigate a particular three-level system. To conclude, Section 5 contains an interesting discussion on possible extensions to the discrete Husimi and Glauber-Sudarshan functions, as well as future applications on multipartite quantum states. Two mathematical appendices concerned with important topics and calculational details of certain expressions used in the previous sections were added: Appendix A shows, in particular, how the parity operator is connected with the $\text{mod}(N)$-invariant operator basis in the continuum limit; while Appendix B exhibits the relations between Gell-Mann and Schwinger unitary operators, and their corresponding mapped expressions in the finite-dimensional discrete phase spaces.

2. Finite-dimensional discrete phase spaces

Let us begin with establishing the mathematical prerequisites related to the Schwinger unitary operators and its corresponding symmetrized basis [8], whose operational aspects allow to build an effective self-consistent theoretical framework for the $\text{mod}(N)$-invariant unitary operator basis [11], and subsequently, for the discrete Wigner function [13]. The sentence “finite-dimensional discrete phase space” means, henceforth, a finite mesh with $N^2$ points labelled by discrete variables.

Definition 1 (Schwinger). Let $\hat{U}$ and $\hat{V}$ describe a pair of unitary operators defined in a $N$-dimensional state vector space, as well as $\{ |u_\alpha\rangle, |v_\beta\rangle \}$ denote their respective orthonormal eigenvectors related by the symmetrical finite Fourier kernel (and/or inner product) $\langle u_\alpha | v_\beta \rangle = \frac{1}{\sqrt{N}} \omega^{\alpha \beta}$ with $\omega := \exp\left( \frac{2 \pi i}{N} \right)$. The general properties\footnote{Further results and properties associated with $\hat{U}$ and $\hat{V}$ can be found in Ref. [13].}

\begin{align*}
\hat{U}^\eta |u_\alpha\rangle &= \omega^{\alpha \eta} |u_\alpha\rangle, & \hat{V}^\xi |v_\beta\rangle &= \omega^{\beta \xi} |v_\beta\rangle, \\
\hat{U}^\eta |v_\beta\rangle &= |v_{\beta + \eta}\rangle, & \hat{V}^\xi |u_\alpha\rangle &= |u_{\alpha - \xi}\rangle, \\
\hat{U}^N &= \hat{1}, & \hat{V}^N &= \hat{1}, & \hat{V}^\xi \hat{U}^\eta &= \omega^{\eta \xi} \hat{U}^\eta \hat{V}^\xi,
\end{align*}

where $\hat{1}$ corresponds to the identity operator, consist of fundamental basic mathematical rules that characterize the aforementioned unitary operators. Besides, the discrete labels $\{ \alpha, \beta, \eta, \xi \}$ obey the arithmetic modulo $N$.\footnote{Further results and properties associated with $\hat{U}$ and $\hat{V}$ can be found in Ref. [13].}
Remark. The commutation relation \([\hat{U}, \hat{V}] = (1 - \omega)\hat{U}\hat{V} \) for Schwinger unitary operators and its complementary property \(\hat{V}\hat{U} = \omega\hat{U}\hat{V}\), lead us to establish some related additional results as follow:

\[
\begin{align*}
[\hat{U}, [\hat{U}, \ldots [\hat{U}, \hat{V}] \ldots]] &= (1 - \omega)^p \hat{U}^p \hat{V}, \\
[\hat{V}, [\hat{V}, \ldots [\hat{V}, \hat{U}] \ldots]] &= (1 - \omega^* )^q \hat{V}^q \hat{U}, \\
(\hat{U}\hat{V})^q &= \omega^{\frac{1}{2}(q-1)}\hat{U}^q \hat{V}^q,
\end{align*}
\]

with \(p, q \in \mathbb{N}^*\). On the other hand, such results permit to infer that exists an uncertainty relation underlying the operators \(\hat{U}\) and \(\hat{V}\). In the recent past \([39]\), it was introduced a quantum-algebraic framework embracing a new uncertainty principle for these unitary operators that generalizes and strengthens the Massar-Spindel inequality \([40]\) – this inequality determines, in particular, a new set of restrictions upon the selective process of signals and wavelet bases. Within this scope, it is worth mentioning that Bagchi and Pati \([41]\) also derived new uncertainty relations for arbitrary unitary operators acting on finite-dimensional state vector spaces, whose respective tighter bounds were obtained for different situations. Recently, it was shown that minimum-uncertainty states saturate the Massar-Spindel inequality \([42]\).

Definition 2 (Schwinger). Let us introduce the set of \(N^2\) operators

\[
\hat{S}_\xi(\eta, \xi) = \frac{1}{\sqrt{N}} \omega^{\frac{1}{2}\eta\xi} \hat{U}^\eta \hat{V}^\xi \quad (\eta, \xi = 0, \ldots, N - 1),
\]

which represents a symmetrized version of the unitary operator basis \(\hat{S}(\eta, \xi) = \frac{1}{\sqrt{N}} \hat{U}^\eta \hat{V}^\xi\) originally proposed by Schwinger \([38]\). The labels \(\eta\) and \(\xi\) are associated with the discrete dual variables of an \(N^2\)-dimensional phase space; besides, it can be also verified by direct inspection that \(\hat{S}(\eta, \xi)\) is invariant under the changes \(\hat{U} \to \hat{V}\) and \(\hat{V} \to \hat{U}^{-1}\), followed by \(\eta \to \xi\) and \(\xi \to -\eta\) (this result depicts the pre-symplectic character of the symmetrized operator basis \([8, 10]\)). Additionally, note that both the inverse element

\[
\hat{S}_\xi^{-1}(\eta, \xi) = \hat{S}_\xi^\dagger(\eta, \xi) = \hat{S}_\xi(\eta, -\xi)
\]

and the similarity transformation

\[
\left[\sqrt{N} \hat{S}_\xi(\alpha, -\beta)\right] \hat{S}_\xi(\eta, \xi) \left[\sqrt{N} \hat{S}_\xi^\dagger(\alpha, -\beta)\right] = \omega^{-\beta n + \alpha \xi} \hat{S}_\xi(\eta, \xi)
\]

represent good examples of relevant mathematical properties associated with \(\hat{S}_\xi(\eta, \xi)\) – see Ref. \([13]\) for complementary results. Summarising, \(\{\hat{S}_\xi(\eta, \xi)\}_{\eta, \xi = 0, \ldots, N - 1}\) constitutes a complete orthonormal operator basis that allows us to determine all possible dynamical quantities belonging to the physical system under investigation; consequently, any linear operator \(\hat{O}\) can be decomposed in this basis as

\[
\hat{O} = \sum_{\eta, \xi = 0}^{N - 1} \mathcal{O}(\eta, \xi) \hat{S}_\xi(\eta, \xi),
\]

\(\mathcal{O}(\eta, \xi)\) being a function of \(\eta\) and \(\xi\). It is worth noting that \(\text{Tr}[\hat{S}_\xi^\dagger(\eta', \xi')\hat{S}_\xi(\eta, \xi)] = \delta_{\eta, \eta'} \delta_{\xi, \xi'} \delta_{\eta', \xi'} \) establishes a fundamental property since it ensures that such a decomposition is unique. The superscript \([\mathcal{N}]\) on the Kronecker deltas denotes that this function is different from zero when its labels are \(\text{mod}(N)\)-congruent.
where the coefficients $O(\eta, \xi)$ are given by $\text{Tr}[\hat{S}_\eta^\dagger(\eta, \xi)\hat{O}]$. When $\hat{O} \equiv \hat{\rho}$, these coefficients describe the discrete Wigner characteristic function $\chi_W(\eta, \xi) := \text{Tr}[\hat{S}_\eta^\dagger(\eta, \xi)\hat{\rho}]$.

**Definition 3** (Galetti–de Toledo Piza). The mod($N$)-invariant unitary operator basis was originally defined as the discrete Fourier transform of the symmetrized basis [11]

$$\hat{G}(\mu, \nu) = \frac{1}{\sqrt{N}} \sum_{\eta, \xi = 0}^{N-1} \omega^{-(\mu\eta + \nu\xi)} \omega^{-\frac{1}{2}N\Phi(\eta, \xi; N)} \hat{S}_\eta(\eta, \xi),$$

being the additional phase $\Phi(\eta, \xi; N) := NI^N_\eta I^N_\xi - \eta I^N_\xi - \xi I^N_\eta$ responsible for the mod($N$)-invariance of this new operator basis [11], where $I^N_\xi = \left[ \frac{\xi}{N} \right]$ corresponds to the integer part of $\xi$ with respect to $N$. Such an additional phase is irrelevant in describing single operators, however, it may be meaningful for products of operators and/or composition laws [13] By analogy with Equation (3), the decomposition of any linear operator $\hat{O} = \frac{1}{N} \sum_{\mu, \nu = 0}^{N-1} O(\mu, \nu)\hat{G}(\mu, \nu)$

(5)

can also be verified in such a case, with $O(\mu, \nu) = \text{Tr}[\hat{G}^\dagger(\mu, \nu)\hat{O}]$ exhibiting a one-to-one correspondence between operators and functions belonging to an $N^2$-dimensional phase space characterized by the discrete labels $\mu$ and $\nu$. In particular, these coefficients allow us to define, for $\hat{O} \equiv \hat{\rho}$, the discrete Wigner function $W(\mu, \nu) := \text{Tr}[\hat{G}^\dagger(\mu, \nu)\hat{\rho}]$.

Remark. There is an immediate link between both the coefficients $O(\mu, \nu)$ and $O(\eta, \xi)$, given by the discrete Fourier transform

$$O(\mu, \nu) = \frac{1}{\sqrt{N}} \sum_{\eta, \xi = 0}^{N-1} \omega^{\mu\eta + \nu\xi} \omega^{-\frac{1}{2}N\Phi(\eta, \xi; N)} O(\eta, \xi).$$

(6)

Equation (6) exhibits essentially the mathematical role of the discrete Fourier transform in connecting the $N^2$-dimensional discrete phase spaces described by the pair $(\mu, \nu)$ and its respective dual pair $(\eta, \xi)$; then, it becomes quite easy to show the connection between discrete Wigner function and its respective characteristic function, that is

$$W(\mu, \nu) = \frac{1}{\sqrt{N}} \sum_{\eta, \xi = 0}^{N-1} \omega^{\mu\eta + \nu\xi} \omega^{-\frac{1}{2}N\Phi(\eta, \xi; N)} \chi_W(\eta, \xi),$$

(7)

in complete analogy to the continuous case [13]. Summarizing, the functions $O(\mu, \nu)$ and $W(\mu, \nu)$ correspond, in this context, to a well-established one-to-one mapping between operators and functions embedded in a finite phase space characterized by the discrete variables $\mu$ and $\nu$. Note that mappings of Hermitian operators in the $N^2$-dimensional discrete phase space lead us to obtain real functions.

\* In fact, the term $\omega^{-\frac{1}{2}N\Phi(\eta, \xi; N)}$ can be suppressed when one deals with discrete labels obeying the arithmetic modulo $N$.

\*\* In particular, when one deals with discrete momentum and coordinate-like variables, these variables assume integer values in the symmetric interval $[-\ell, \ell]$ for $\ell = \frac{N-1}{2}$ fixed and a given odd $N$ – see, for instance, Ref. [13] for description of discrete coherent states.
Next, we present an important property related to the trace of the product of two bounded operators $\hat{A}$ and $\hat{B}$,

$$\text{Tr}[\hat{A}\hat{B}] = \frac{1}{N} \sum_{\mu,\nu=0}^{N-1} A(\mu, \nu)B(\mu, \nu) = \sum_{\eta,\xi=0}^{N-1} \mathcal{A}(\eta, \xi)\mathcal{B}(-\eta, -\xi),$$  \hspace{1cm} (8)

which corresponds to the overlap of their respective mappings in the $G$ or $S_S$ bases; in addition, if one considers the density operators $\hat{\rho}_1$ and $\hat{\rho}_2$, it turns immediate to establish that

$$\text{Tr}[\hat{\rho}_1\hat{\rho}_2] = \frac{1}{N} \sum_{\mu,\nu=0}^{N-1} W_1(\mu, \nu)W_2(\mu, \nu) = \sum_{\eta,\xi=0}^{N-1} \chi_{W, 1}(\eta, \xi)\chi_{W, 2}(-\eta, -\xi).$$  \hspace{1cm} (9)

Similarly, an expression for the mean value $\langle \hat{O} \rangle \equiv \text{Tr}[\hat{O}\hat{\rho}]$ can also be obtained from this property,

$$\langle \hat{O} \rangle = \frac{1}{N} \sum_{\mu,\nu=0}^{N-1} O(\mu, \nu)W(\mu, \nu) = \sum_{\eta,\xi=0}^{N-1} O(\eta, \xi)\chi_{W}(-\eta, -\xi).$$  \hspace{1cm} (10)

So, all the relevant quantities needed to describe the kinematical/dynamical content of physical systems with finite state spaces have, in this description of quantum mechanics, their representatives in the finite-dimensional discrete phase space. A few years ago, by means of the extended Cahill-Glauber formalism for finite-dimensional spaces [19, 21], the aforementioned results were generalized in order to include the discrete Husimi and Glauber-Sudarshan functions.

Now, let us say a few words about $\hat{G}(\mu, \nu)$ and $\hat{S}_S(\eta, \xi)$, since both the discrete bases are connected through the Equation [4]. There are other proposals of discrete bases for finite-dimensional phase spaces in literature, with convenient inherent mathematical properties, which can also be applied in analogous quantum systems [5]. In particular, Klimov and co-workers [22,25] proposed equivalent mathematical expressions of discrete bases for finite state spaces, where they basically showed that discrete Wigner functions depend on the specific phase choice for such bases. So, in Appendix A we present certain interesting aspects of those aforementioned discrete bases and their consequences on the different definitions of discrete Wigner functions.

3. Mappings in SU(N)

After establishing a self-consistent theoretical framework for finite-dimensional discrete phase spaces, let us now implement, at this moment, an important set of mathematical and physical results involving the elements of the special unitary group SU(N) and the Schwinger unitary operators. In particular, we will show how the generators of the Lie algebra $su(N)$ can be mapped upon $N^2$-dimensional discrete phase-space representatives through the use of $\text{mod}(N)$-invariant unitary operator basis; consequently, as a relevant by-product of this biunivocal mapping, a discrete Wigner function for $N$-level systems associated with the Hilbert space $\mathcal{H}_N$ can be properly obtained.
Definition 4. Let \( \hat{g}_i \) with \( i = 1, \ldots, N^2 - 1 \) denote the generators of the Lie algebra \( su(N) \) characterized by \( N \times N \) skew-Hermitian matrices obeying the relations \( \hat{g}_i^\dagger = \hat{g}_i \), \( \text{Tr}[\hat{g}_i] = 0 \), and \( \text{Tr}[\hat{g}_i \hat{g}_j] = 2\delta_{ij} \). In addition, the structure constants \( \mathcal{F}_{ijk} = -\frac{i}{4} \text{Tr}[\hat{g}_i \hat{g}_j \hat{g}_k] \) (antisymmetric tensor) of this algebra are associated with the commutation relation

\[
[\hat{g}_i, \hat{g}_j] = 2i \sum_{k=1}^{N^2-1} \mathcal{F}_{ijk} \hat{g}_k,
\]

whereas the anti-structure constants

\[
\mathcal{D}_{ijk} = \frac{1}{4} \text{Tr}[\{\hat{g}_i, \hat{g}_j\} \hat{g}_k] \quad \text{(symmetric tensor)}
\]

are related to the anticommutation relation

\[
\{\hat{g}_i, \hat{g}_j\} = \frac{4}{N} \delta_{ij} \hat{I}_N + 2 \sum_{k=1}^{N^2-1} \mathcal{D}_{ijk} \hat{g}_k,
\]

where \( \hat{I}_N \) represents the \( N \)-dimensional unit matrix. Note that both the constants \( \mathcal{F}_{ijk} \) and \( \mathcal{D}_{ijk} \) can be found in tabulated form in the literature for different values of \( N \) \([45,46]\).

Remark. To construct systematically the generators \( \{\hat{g}_i\}_{i=1,\ldots,N^2-1} \) of the group SU(N), let us initially define two Hermitian operators \( \hat{U}_{\alpha,\beta} := \hat{P}_{\alpha,\beta} + \hat{P}_{\beta,\alpha} \) and \( \hat{V}_{\alpha,\beta} := -i(\hat{P}_{\alpha,\beta} - \hat{P}_{\beta,\alpha}) \) \((0 \leq \alpha < \beta \leq N - 1)\)

which are specific combinations of the transition operators \( \hat{P}_{\alpha,\beta} = |\alpha\rangle \langle \beta| \); in both cases, \( |\alpha\rangle \) and \( |\beta\rangle \) denote complete orthonormal bases in \( \mathcal{H}_N \). In addition, let us also introduce the Hermitian operator

\[
\hat{\mathcal{W}}_{\gamma} = \sqrt{\frac{2}{(\gamma + 1)(\gamma + 2)}} \left[ \sum_{\sigma=0}^{\gamma} \hat{\mathcal{P}}_{\sigma,\sigma} - (\gamma + 1) \hat{\mathcal{P}}_{\gamma+1,\gamma+1} \right] \quad (0 \leq \gamma \leq N - 2),
\]

where \( \hat{\mathcal{P}}_{\sigma,\sigma} = |\sigma\rangle \langle \sigma| \) corresponds to the projection operators. In this way, the set formed by the \( N^2 - 1 \) orthogonal operators

\[
\{\hat{g}\} = \{\hat{W}_{0,1}, \hat{W}_{0,2}, \ldots, \hat{W}_{1,2}, \ldots, \hat{V}_{0,1}, \hat{V}_{0,2}, \ldots, \hat{V}_{1,2}, \ldots, \hat{W}_0, \hat{W}_1, \ldots, \hat{W}_{N-2}\}
\]

completely characterizes the generators of SU(N) – indeed, it corresponds to a complete orthonormal operator basis \([47]\).

In this general algebraic-theoretical approach, the decomposition of any linear operator \( \hat{O} \) is expressed as follows:

\[
\hat{O} = \frac{1}{N} \text{Tr}[\hat{O}] \hat{I}_N + \frac{1}{2} \sum_{i=1}^{N^2-1} O_i \hat{g}_i,
\]  

(11)
where the coefficients $\mathcal{O}_i$ are given by $\text{Tr}[\hat{g}_i \hat{O}]$. So, for $N$-level quantum systems related to the Hilbert space $\mathcal{H}_N$, the associated density operator $\hat{\rho}$ can be promptly determined from their $N^2 - 1$ mean values $\langle \hat{g}_i \rangle = \text{Tr}[\hat{g}_i \hat{\rho}]$, namely,

$$\hat{\rho} = \frac{1}{N} \mathbb{I}_N + \frac{1}{2} \sum_{i=1}^{N^2-1} \langle \hat{g}_i \rangle \hat{g}_i.$$  \hfill (12)

Now, from the experimental point of view, it is sufficient to measure the components of the Bloch vector $\mathbf{g} = (\langle \hat{g}_1 \rangle, \ldots, \langle \hat{g}_{N^2-1} \rangle) \in \mathbb{R}^{N^2-1}$ to obtain an acceptable description of such states [48, 49]. Next, let us combine Equations (11) and (12) in order to achieve a compact expression for the mean value

$$\langle \hat{O} \rangle = \frac{1}{N} \text{Tr}[\hat{O}] + \frac{1}{2} \sum_{i=1}^{N^2-1} \mathcal{O}_i \langle \hat{g}_i \rangle.$$ \hfill (13)

An immediate extension of this result refers to the product of two operators $\hat{A}$ and $\hat{B}$, that is

$$\langle \hat{A} \hat{B} \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle + \frac{1}{4} \sum_{i,j=1}^{N^2-1} A_i B_j \left( \langle \hat{g}_i \hat{g}_j \rangle - \langle \hat{g}_i \rangle \langle \hat{g}_j \rangle \right),$$ \hfill (14)

being the second term of the right-hand side responsible for correlations associated with generators and quantum states.

### 3.1. Connections with finite-dimensional discrete phase spaces

To begin with, let $\mathcal{H}_N$ describe the $N$-dimensional state vector space [50] previously stated in Section 2. Therefore, it seems quite reasonable to assume that $|\alpha\rangle \equiv |u_\alpha\rangle$, which implies that both the transition and projection operators can be also written as specific combinations of the Schwinger unitary operators, this important connection being the necessary link to characterize the generators of SU($N$) in finite-dimensional discrete phase spaces.*

Let us start with the mappings of $\hat{\mathcal{P}}_{\alpha,\beta}$ in the $N^2$-dimensional discrete phase space, namely,

$$\hat{\mathcal{P}}_{\alpha,\beta} = |u_\alpha\rangle \langle u_\beta| = \frac{1}{N} \sum_{\eta=0}^{N-1} \omega^{-\eta \alpha} \hat{U}^\eta \hat{V}^{\beta-\alpha} = \frac{1}{\sqrt{N}} \sum_{\eta=0}^{N-1} \omega^{-\frac{\eta}{2}(\alpha+\beta)} \hat{S}_S(\eta, \beta - \alpha),$$

as well as the projection operators

$$\hat{\mathcal{P}}_{\sigma,\sigma} = |u_\sigma\rangle \langle u_\sigma| = \frac{1}{N} \sum_{\eta=0}^{N-1} \omega^{-\eta \sigma} \hat{U}^\eta = \frac{1}{\sqrt{N}} \sum_{\eta=0}^{N-1} \omega^{-\eta} \hat{S}_S(\eta, 0).$$

* In fact, such a connection is justified through Equation (5) for $\hat{O} \equiv \hat{g}_i$ and it corresponds to a change of basis,

$$\hat{g}_i = \frac{1}{N} \sum_{\mu,\nu=0}^{N-1} \text{Tr}[\hat{G}^i(\mu, \nu) \hat{g}_i] \hat{G}(\mu, \nu) \quad (i = 1, \ldots, N^2 - 1).$$
These particular results enable us to obtain the Hermitian operators
\begin{align}
\hat{U}_{\alpha,\beta} &= \frac{1}{N} \sum_{\eta=0}^{N-1} \hat{U}^\eta \left( \omega^{-\eta\alpha} \hat{V}^{\beta-\alpha} + \omega^{-\eta\beta} \hat{V}^{N-(\beta-\alpha)} \right), \\
\hat{V}_{\alpha,\beta} &= -\frac{i}{N} \sum_{\eta=0}^{N-1} \hat{U}^\eta \left( \omega^{-\eta\alpha} \hat{V}^{\beta-\alpha} - \omega^{-\eta\beta} \hat{V}^{N-(\beta-\alpha)} \right),
\end{align}
and
\begin{equation}
\hat{W}_\gamma = \sqrt{\frac{2}{(\gamma + 1)(\gamma + 2)}} \frac{1}{N} \sum_{\eta=0}^{N-1} \left[ \sum_{\sigma=0}^{\gamma} \omega^{-\eta\sigma} - (\gamma + 1)\omega^{-\eta(\gamma+1)} \right] \hat{U}^\eta
\end{equation}
as functions of the Schwinger unitary operators, obeying the restrictions imposed on the discrete labels \(\alpha, \beta, \gamma\). Since \(\hat{U}_{\alpha,\beta}, \hat{V}_{\alpha,\beta},\) and \(\hat{W}_\gamma\) are responsible for the generators of the group SU(N), their respective mappings represent a sound mathematical framework that leads us to an alternative description of physical systems in finite-dimensional discrete phase spaces.

For example, let us now consider the well-known group SU(2) and its corresponding generators \(\{\hat{g}\} = \{\hat{g}_1, \hat{g}_2, \hat{g}_3\} = \{\hat{U}_{0,1}, \hat{V}_{0,1}, \hat{W}_0\}\) written in terms of the unitary operators \(\hat{U}\) and \(\hat{V}\): \(\hat{g}_1 = \hat{V}, \hat{g}_2 = -i\hat{U}\hat{V},\) and \(\hat{g}_3 = \hat{U}\). The matrix representation of these results in the basis \(\{|u_0\}, |u_1\}\) reproduces exactly the Pauli matrices \(\{\hat{\sigma}_i\}_{i=x,y,z}\) in a one-to-one correspondence, namely, \(\hat{\sigma}_i = \hat{V}\delta_{ix} - i\hat{U}\hat{V}\delta_{iy} + \hat{U}\delta_{iz}\); consequently, the mod(2)-invariant unitary operator basis \(\{\hat{u}_{\ell}\}_{\ell=1,2,3}\) achieves the simple form
\begin{equation}
\hat{G}(\mu, \nu) = \frac{1}{2} \left[ \hat{I}_2 + \langle -1\rangle^\mu \hat{\sigma}_x + \langle -1\rangle^{\mu+\nu+1} \hat{\sigma}_y + \langle -1\rangle^\mu \hat{\sigma}_z \right].
\end{equation}
Next, we briefly define the density-matrix vectorial space for \(N\)-level systems.

### 3.2. The density matrix

Now, let us characterize the density-matrix (or equivalently, density-operator) space
\begin{equation}
\mathcal{L}_{+,1}(\mathcal{H}_N) = \{ \hat{\rho} \in \mathcal{L}(\mathcal{H}_N) \mid \text{Tr}[\hat{\rho}] = 1, \ \hat{\rho} = \hat{\rho}^\dagger, \ \rho_\ell \geq 0 \ (\ell = 1, \ldots, N) \}
\end{equation}
for \(N\)-level systems associated with the Hilbert space \(\mathcal{H}_N\), which exhibits three important basic premises for \(\hat{\rho}\): (i) \(\text{Tr}[\hat{\rho}] = 1\) (the normalization condition is preserved), (ii) \(\hat{\rho} = \hat{\rho}^\dagger\) (by definition, \(\hat{\rho}\) consists of a Hermitian matrix), and (iii) \(\rho_\ell \in \mathbb{R}_+\) (the eigenvalues are positive). The notation \(\mathcal{L}(\mathcal{H}_N)\) corresponds to the set of linear operators on \(\mathcal{H}_N\) \([15]\). Note that \(\text{Tr}[\hat{\rho}^2] \leq 1\) can be considered as a further property, but not necessary under certain circumstances, being the equality reached in this situation only for pure states. In fact, this specific property permits us to connect both the discrete Wigner function \([17]\) and Bloch vector, since the relation
\begin{equation}
\text{Tr}[\hat{\rho}^2] = \frac{1}{N} \sum_{\mu,\nu=0}^{N-1} W^2(\mu, \nu) = \frac{1}{N} + \frac{1}{2} |\mathbf{g}|^2 \leq 1
\end{equation}
is always verified. With respect to the set of eigenvalues \( \{ \rho_\ell \}_{\ell=1,\ldots,N} \), they are basically determined from the polynomial equation
\[
P(\rho) := \det(\hat{\rho} - \rho \mathbb{I}_N) = \rho^N - S_1 \rho^{N-1} + S_2 \rho^{N-2} + \ldots + (-1)^N S_N = 0,
\]
whose coefficients satisfy the recurrence relation \( [49] \)
\[
S_r = \frac{1}{r} \sum_{s=1}^{r} (-1)^{s-1} \text{Tr}[\hat{\rho}^s] S_{r-s} \quad (r \geq 2)
\]
such that \( S_0 := 1 \) and \( S_1 = 1 \). This mathematical recipe presents some disadvantages associated with the nontrivial calculations of terms like \((\text{Tr}[\hat{\rho}^m])^n\) – see Refs. \([48,49]\) for technical details.

3.3. The discrete Wigner function

To determine the discrete Wigner function for \( N \)-level systems associated with \( \mathcal{H}_N \), only two results are needed: the first one corresponds to its definition in the \( N^2 \)-dimensional discrete phase space, \( W(\mu, \nu) := \text{Tr}[\mathcal{G}^\dagger(\mu, \nu) \hat{\rho}] \), as stated in Section 2; while the second one refers to the expansion \([12]\) of the density operator \( \hat{\rho} \), which deals with the generators \( \{ \hat{g}_i \}_{i=1,\ldots,N^2-1} \) of the group \( \text{SU}(N) \). This mixing of remarkable results leads to establish a solid theoretical background that allows us, among other things, to determine a compact expression for the discrete Wigner function, that is \( \# \)
\[
W(\mu, \nu) = \frac{1}{N} + \frac{1}{2} \sum_{i=1}^{N^2-1} \langle \hat{g}_i \rangle (\hat{g}_i)(\mu, \nu) \quad (0 \leq \mu, \nu \leq N - 1)
\]
where \( \langle \hat{g}_i \rangle \) denotes the components of the Bloch vector and \( (\hat{g}_i)(\mu, \nu) \) corresponds to its respective mapped expressions in the \( \text{mod}(N) \)-invariant operator basis. Therefore, the protagonism from these generators turns completely evident in Equation \((20)\). For this reason, the general mapped expressions of the Hermitian operators \( \hat{\mathcal{W}}_{\alpha,\beta} \), \( \hat{\mathcal{Y}}_{\alpha,\beta} \), and \( \hat{\mathcal{W}}_\gamma \) assume, in this context, the following forms for \( N > 2 \) (see Appendix B):
\[
(\hat{\mathcal{W}}_{\alpha,\beta})(\mu, \nu) = 2\delta^{[N]}_{\mu,\varepsilon} \cos \left[ \frac{2\pi \nu}{N} (\beta - \alpha) \right] \quad \text{if} \quad \alpha + \beta = 2\varepsilon \quad \text{and} \quad \varepsilon \in \mathbb{N}^*, \quad \text{with} \quad \alpha \neq \beta,
\]
\[
(\hat{\mathcal{Y}}_{\alpha,\beta})(\mu, \nu) = 2\delta^{[N]}_{\mu,\varepsilon} \sin \left[ \frac{2\pi \nu}{N} (\beta - \alpha) \right] \quad \text{if} \quad \alpha + \beta = 2\varepsilon \quad \text{and} \quad \varepsilon \in \mathbb{N}^*, \quad \text{with} \quad \alpha \neq \beta,
\]
\[
\hat{\chi}_\nu(\eta, \xi) = \frac{1}{\sqrt{N}} \delta^{[N]}_{\eta,0} \delta^{[N]}_{\xi,0} + \frac{1}{2} \sum_{i=1}^{N^2-1} \langle \hat{g}_i \rangle (\eta, \xi) \quad (0 \leq \eta, \xi \leq N - 1),
\]
with \( \langle \hat{g}_i \rangle (\eta, \xi) \) referring to the mapped expressions of the generators in the symmetrized unitary operator basis \( S^\dagger \) – see Equation \((7)\) for connection with discrete Wigner function.
On the discrete Wigner function for SU(N)

\[
(\hat{W}_\gamma)(\mu, \nu) = \sqrt{\frac{2}{(\gamma + 1)(\gamma + 2)}} \left[ \sum_{a=0}^{\gamma} \delta^{[N]}_{\mu,a} - (\gamma + 1)\delta^{[N]}_{\mu,\gamma+1} \right]. \tag{23}
\]

Therefore, given a physical system with \(\hat{\rho}\) described by SU(N), the terms \{\langle \hat{g}_i \rangle\}_{i=1, \ldots, N^2-1} can be promptly determined in this case, and consequently, the discrete Wigner function \(W(\mu, \nu; t)\) is introduced via \(\hat{\rho}(t)\).

To illustrate our results let us consider the group SU(2) once again. So, the discrete Wigner function for two-level systems \[51\] is given in this context by

\[
W(\mu, \nu) = \frac{1}{2} \left[ 1 + (-1)^\nu P_x + (-1)^{\mu+\nu+1} P_y + (-1)^\mu P_z \right] \quad (0 \leq \mu, \nu \leq 1),
\]

where \(P_i = \text{Tr}[\hat{\rho}\hat{\sigma}_i] \in [-1, 1]\) for \(i = x, y, z\) represents the components of the polarization vector \(P\) with \(P_x^2 + P_y^2 + P_z^2 \leq 1\) (the saturation occurs for pure states). Moreover, this function can be measured since the components of \(P\) are experimentally obtained \[52,53\]. However, it is worth to emphasize that different SU(2) Wigner function approaches using continuous variables have emerged from the discussion on atomic coherent states in the recent past \[54–59\]. Notwithstanding their inherent technical difficulties, the approaches were applied with broad success on a collection of two-level systems \[56,57\] or even in the tomographic reconstruction of a spin-squeezed state of a Bose-Einstein condensate \[59\]. Finally, let us say some few words about the SU(2) Wigner function here exhibited: it corresponds to a genuinely discrete quasiprobability distribution function and obeys the criterion ‘easy-to-handle’ – see also Ref. \[34\].

With respect to the discrete SU(N) Wigner function \(W(\mu, \nu; t)\), its elegance and simplicity hide possible difficulties inherent to the calculation of generators for high values of \(N\), as well as the determination of the elements \(\langle \hat{g}_i \rangle\) and \(\langle \hat{g}_i \rangle(\mu, \nu)\) for all \(i = 1, \ldots, N^2-1\). This apparent disadvantage can be circumvented through the symmetries and dynamical characteristics of the physical system under investigation \[24\]. Tilma and coworkers \[60\] showed, a few years ago, that generalized SU(N)-symmetric coherent states can be used to construct the continuous quasiprobability distribution functions (namely, the Wigner, Husimi, and Glauber-Sudarshan functions \[43\]); subsequently, the authors proposed an alternative framework for computing Wigner functions which describe physical systems with arbitrary dimensions \[6\]. Therefore, Equation \(\langle \hat{g}_i \rangle\) can be interpreted as a discrete version of their results.

4. Application: the group SU(3)

Up to now the first example treated throughout the text was the group SU(2), namely, a single two-level quantum system where a general qubit state can be promptly represented by means of its discrete Wigner function. The next natural step is to obtain the discrete Wigner function for three-level quantum systems related to the group SU(3) and mainly described by

\[
\hat{\rho} = \\
\begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} \\
\rho_{12}^* & \rho_{22} & \rho_{23} \\
\rho_{13}^* & \rho_{23}^* & \rho_{33}
\end{pmatrix} \in \mathcal{L}_{+1}(\mathcal{H}_3). \tag{24}
\]
For such a task, let us initially consider the results established for the SU(3) generators in Appendix B as well as their respective mapped expressions \{\{(\hat{\lambda}_1)_{(\mu, \nu)}\}\}_{1=1,...,8} in the finite-dimensional discrete phase space. These results allow, in particular, to determine compact expressions for the components \((\langle \hat{\lambda}_1 \rangle, \ldots, \langle \hat{\lambda}_8 \rangle)\) of the Bloch vector \(g\),

\[
\langle \hat{\lambda}_1 \rangle = 2\text{Re}(\rho_{12}), \quad \langle \hat{\lambda}_2 \rangle = -2\text{Im}(\rho_{12}), \quad \langle \hat{\lambda}_3 \rangle = \rho_{11} - \rho_{22}, \\
\langle \hat{\lambda}_4 \rangle = 2\text{Re}(\rho_{13}), \quad \langle \hat{\lambda}_5 \rangle = -2\text{Im}(\rho_{13}), \quad \langle \hat{\lambda}_6 \rangle = 2\text{Re}(\rho_{23}), \\
\langle \hat{\lambda}_7 \rangle = -2\text{Im}(\rho_{23}), \quad \langle \hat{\lambda}_8 \rangle = \frac{\sqrt{3}}{3} (\rho_{11} + \rho_{22} - 2\rho_{33}).
\]  

(25)

In this way, the discrete SU(3) Wigner function is properly established with the help of Equation (20) for \(N = 3\),

\[
W(\mu, \nu) = \frac{1}{3} + \frac{1}{3} \left(2\delta^{[3]}_{\mu,0} - \delta^{[3]}_{\mu,1} - \delta^{[3]}_{\mu,2}\right)\rho_{11} - \frac{1}{3} \left(\delta^{[3]}_{\mu,0} - 2\delta^{[3]}_{\mu,1} + \delta^{[3]}_{\mu,2}\right)\rho_{22} \\
- \frac{1}{3} \left(\delta^{[3]}_{\mu,0} + \delta^{[3]}_{\mu,1} - 2\delta^{[3]}_{\mu,2}\right)\rho_{33} + 2\delta^{[3]}_{\mu,1} \left[ \cos \left(\frac{4\pi\nu}{3}\right) \text{Re}(\rho_{13}) - \sin \left(\frac{4\pi\nu}{3}\right) \text{Im}(\rho_{13}) \right] \\
+ \frac{2}{3} \sin \left[ \left(\mu - \frac{1}{2}\right) \frac{\pi}{3} \right] \left[ \cos \left(\frac{2\pi\nu}{3}\right) \text{Re}(\rho_{12}) - \sin \left(\frac{2\pi\nu}{3}\right) \text{Im}(\rho_{12}) \right] \\
+ \frac{2}{3} \sin \left[ \left(\mu - \frac{1}{2}\right) \frac{\pi}{3} \right] \left[ \cos \left(\frac{2\pi\nu}{3}\right) \text{Re}(\rho_{23}) - \sin \left(\frac{2\pi\nu}{3}\right) \text{Im}(\rho_{23}) \right].
\]  

(26)

This expression is completely general and can be particularly applied to the description of a qutrit [61,63], and its three-dimensional visualization represents a complementary result to that discussed in Ref. [64] – see Table 1 for \(0 \leq \mu, \nu \leq 2\).

To illustrate this result, we consider a specific three-level system described by

\[
\hat{\rho} = \begin{pmatrix}
\frac{1}{3} & \varphi_1 & \varphi_2 \\
\varphi_1 & \frac{1}{3} & \varphi_3 \\
\varphi_2 & \varphi_3 & \frac{1}{3}
\end{pmatrix}, \quad (\varphi_1, \varphi_2, \varphi_3 \in \mathbb{R}^+),
\]  

(27)

where \(\varphi_1, \varphi_2, \) and \(\varphi_3\) are associated with the respective transition rates between the states \(0 \rightleftharpoons 1, 0 \rightleftharpoons 2,\) and \(1 \rightleftharpoons 2,\) these states being equally populated in the ratio of \(\frac{1}{3}\).
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Figure 1. Three-dimensional plots of the discrete Wigner function $W(\mu, \nu)$ described by Equation (28) versus $(\mu, \nu) \in [0, 2]$ with $N = 3$ fixed, and different transition rates: (a) $\varphi_1 = \varphi_2 = \varphi_3 = \frac{1}{3}$, (b) $\varphi_1 = 0$ and $\varphi_2 = \varphi_3 = \frac{1}{3}$, (c) $\varphi_1 = \varphi_3 = \frac{1}{3}$ and $\varphi_2 = 0$, (d) $\varphi_1 = \varphi_2 = \frac{1}{3}$ and $\varphi_3 = 0$, (e) $\varphi_1 = \varphi_3 = 0$ with $\varphi_2 = \frac{1}{3}$; and finally, (f) $\varphi_1 = \varphi_2 = 0$ and $\varphi_3 = \frac{1}{3}$. In such a case, (a) represents a pure state with $W(1, 1) = W(1, 2) \approx -0.44$ and $W(1, 0) \approx 1.89$ reflecting its respective minimum and maximum values. Moreover, (e) and (f) depict situations where the discrete Wigner function is positive.

In this toy model, the condition $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \leq \frac{1}{3}$ asserts that $\rho_\ell \in \mathbb{R}_+$ for $\ell = 1, 2, 3$ (namely, the eigenvalues assume real and positive values), the saturation being reached in this case only for pure states. Following, the discrete Wigner function (26) can be expressed in such a case as follows:

$$ W(\mu, \nu) = \frac{1}{3} + \frac{2}{3} \sin \left[ (\mu - \frac{1}{2}) \frac{\pi}{3} \right] \cos \left( \frac{2\pi\nu}{3} \right) \varphi_1 + 2\delta^{[3]}_{\mu,1} \cos \left( \frac{4\pi\nu}{3} \right) \varphi_2 $$

$$ + \frac{2}{3} \sin \left[ (\mu - \frac{2}{2}) \frac{\pi}{3} \right] \cos \left( \frac{2\pi\nu}{3} \right) \varphi_3. $$

Figure 1 shows the 3D plots of this expression as a function of $(\mu, \nu)$ for different values of $\varphi_1$, $\varphi_2$, and $\varphi_3$. For instance, the negative values appearing in (a-d) at the points $(1, 1)$ and $(1, 2)$ represent a quantum signature of the nonclassical effects associated with the particular three-level system under investigation (in these cases, at least two transitions are allowed). However, (e-f) exhibit only positive values related to the transitions $0 \leftrightarrow 2$ or $1 \leftrightarrow 2$ – see Table 1. Note that this specific toy model can be promptly generalized in order to incorporate more realistic effects such as different population rates and complex transition rates.

Recently, Martins and coworkers [65] exhibited a set of results on the $*$-product for SU(3) Wigner functions over SU(3)/U(2). In particular, these functions were defined in a symplectic manifold which corresponds to a classical phase space that is, by its turn,
related to a specific set of orbit-type coherent states. It is worth stressing that Equation (26) describes the SU(3) Wigner function defined upon a finite-dimensional phase space labelled by genuine discrete variables associated with spin representations, which differs from that aforementioned result.

5. Concluding remarks

In this paper, we have established a self-contained theoretical framework for the discrete Wigner function associated with a wide class of arbitrary quantum systems characterized by a finite space of states. Indeed the connection between SU(N) generators and Schwinger unitary operators (and vice versa) via the mod(N)-invariant unitary operator basis paves the way to introduce a finite-dimensional phase space which is genuinely discrete. Thus, the discrete Wigner function obtained from this guideline is completely general since it allows, within other possibilities, to describe arbitrary spin systems and also to provide a theoretical background for nonrelativistic studies on particle physics models. Next, let us focus our attention on effective gains and future perspectives derived from this manuscript which deserve to be properly discussed.

- The mathematical framework exposed here can be promptly generalized in order to include the extended Cahill-Glauber formalism for finite-dimensional spaces [19][21]. For this task, it is enough to substitute the mod(N)-invariant operator basis $\hat{G}(\mu, \nu)$ by its extended version

$$
\hat{T}^{(s)}(\mu, \nu) = \frac{1}{\sqrt{N}} \sum_{\eta_\xi = -\ell}^{\ell} \omega^{-(\mu_\eta + \nu_\xi)} \omega^\frac{1}{N} \Phi(\eta_\xi; N) [\mathcal{K}(\eta, \xi)]^{-s} \hat{S}_S(\eta, \xi),
$$

where the additional term $\mathcal{K}(\eta, \xi)$ is expressed through a nontrivial sum of products of Jacobi theta functions for integer arguments with $|s| \leq 1$ [19]. So, Equation (20) assumes the general form

$$
F^{(s)}(\mu, \nu) := \text{Tr}[\hat{T}^{(s)}(\mu, \nu) \hat{\rho}] = \frac{1}{N} + \frac{1}{2} \sum_{i=1}^{N^2-1} \langle \hat{g}_i \rangle \langle \hat{g}_i \rangle^{(s)}(\mu, \nu),
$$

such that for $s = -1, 0, +1$ the parametrized function $F^{(s)}(\mu, \nu)$ recovers the discrete Husimi, Wigner, and Glauber-Sudarshan functions, respectively.

- A first application of this quantum-algebraic framework is associated with the study of bipartite systems. For example, let $\mathcal{H}_{N_1}$ and $\mathcal{H}_{N_2}$ denote both the Hilbert spaces of the parts 1 and 2, as well as $\mathcal{H}_N = \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}$ characterize the Hilbert space of the total system for $N = N_1 N_2$. In addition, let $\{\hat{g}_{1,i}\}_{i=1, \ldots, N_1^2-1}$ and $\{\hat{g}_{2,j}\}_{j=1, \ldots, N_2^2-1}$ be the respective generators of the groups SU($N_1$) and SU($N_2$) used to describe each part, with SU($N$) = SU($N_1$) \otimes SU($N_2$) being responsible for bipartite system. From the kinematical point of view, there are two distinct approaches to deal with such a system via discrete Wigner functions: the first one corresponds to write the density matrix $\hat{\rho}$ of the total system in terms of the aforementioned generators $\hat{g}_{1,i}$ and $\hat{g}_{2,j}$...
related to the parts 1 and 2 through the techniques developed in Refs. \[66,67\], and then obtain the corresponding discrete Wigner function; the second one consists of using Equation (20) as a guideline, where now the generators \(\{\hat{g}_k\}_{k=1,\ldots,N^2-1}\) of the total system entered the scene. Following, two-qubits X-states \[68\] and qubit-qutrit systems \[69\] represent two typical examples where these approaches can be applied to the study on entanglement effects of bipartite systems, and also in the study of hybrid systems \[70\].

To conclude, let us discuss some pertinent points associated with more general spin systems and multipartite quantum states. It is worth of mention that \(\hat{G}(\mu, \nu)\) was already used in the description of spin squeezing effects in straight connection with entanglement properties of the modified Lipkin-Meshkov-Glick model via discrete Wigner function for high values of \(N\) \[30\] – see Ref. \[24\] for a discussion on spin-tunneling processes involving an original version of the aforementioned model, where now the discrete Husimi function has been extensively used. Moreover, the active and fruitful field of research on N-qubit X-states \[71\] represents, nowadays, an interesting scenario of possible applications for discrete Wigner function where the study on maximally genuine multipartite entangled mixed states will take place \[72\].

### Appendix A. The parity operator

The discussion on the equivalence between large \(N\) limits of quantum theories and classical limits certainly reveals a heated historical debate that has dragged on for many decades – for instance, see Refs. \[73–75\]. In this appendix we will show, in particular, how the continuum limit of a genuinely discrete quantum approach takes place, emphasizing, by its turn, the continuum limit \(N \rightarrow \infty\) of \(\hat{G}(0,0)\) in connection with the continuous parity operator. Furthermore, we will also present an alternative form of the \(\text{mod}(N)\)-invariant unitary operator basis written in terms of the discrete parity operator.

**Definition 5.** Let \(\hat{F}\) denote the finite-dimensional discrete Fourier operator defined in terms of the set \(\{|u_\beta\rangle, |v_\beta\rangle\}_{\beta=0,\ldots,N-1}\) of orthonormal eigenvectors,

\[
\hat{F} := \sum_{\beta=0}^{N-1} |u_\beta\rangle \langle u_\beta| = \frac{1}{\sqrt{N}} \sum_{\beta,\beta'=0}^{N-1} \omega^{\beta\beta'} |u_{\beta'}\rangle \langle u_\beta| \Rightarrow \hat{F}^\dagger \hat{F} = \hat{F} \hat{F}^\dagger = \hat{I}.
\]

The additional properties \[76,77\]

\[
\hat{F}^2 = \sum_{\beta=0}^{N-1} |u_{-\beta}\rangle \langle u_\beta|, \quad \hat{F}^3 = \sum_{\beta=0}^{N-1} |v_{-\beta}\rangle \langle u_\beta|, \quad \text{and} \quad \hat{F}^4 = \hat{I},
\]

lead us to find out that \(\hat{F}\) is a periodic operator with 4-period. Furthermore, the property associated with \(\hat{F}^2\) also permits to establish the discrete parity operator \(\hat{P}\) by means of the relation \(\hat{P} := \hat{F}^2\).
Remark. Now, let us clarify an important point related to the \( \text{mod}(N) \)-invariant unitary operator basis \((4)\). Initially, we should observe that

\[
[\sqrt{N} \hat{S}_S(\nu, -\mu)] \hat{S}_S(\eta, \xi) [\sqrt{N} \hat{S}_S^\dagger(\nu, -\mu)] = \omega^{-(\mu \eta + \nu \xi)} \hat{S}_S(\eta, \xi)
\]

depicts a similarity transformation, this particular result being responsible for rewriting \( \hat{G}(\mu, \nu) \) as follows:

\[
\hat{G}(\mu, \nu) = [\sqrt{N} \hat{S}_S(\nu, -\mu)] \hat{G}(0, 0) [\sqrt{N} \hat{S}_S^\dagger(\nu, -\mu)],
\]

where

\[
[\sqrt{N} \hat{S}_S(\nu, -\mu)][\sqrt{N} \hat{S}_S^\dagger(\nu, -\mu)] = [\sqrt{N} \hat{S}_S^\dagger(\nu, -\mu)][\sqrt{N} \hat{S}_S(\nu, -\mu)] = \hat{I}.
\]

Therefore, Equation (A.3) not only represents an unitary transformation of \( \hat{G}(0, 0) \), but also establishes a displacement of a given initial point \((0, 0)\) to the final point \((\mu, \nu)\) on the associated operator space – for a discussion on construction and properties of the discrete coherent states, see Ref. [13]. Furthermore, note that \( \hat{G}(0, 0) \) apparently does not exhibit any connection with the previously defined discrete parity operator.

An interesting question then emerges from our considerations on Schwinger unitary operators and \( \text{mod}(N) \)-invariant unitary operator basis \( \hat{G}(\mu, \nu) \): “Can \( \hat{G}(0, 0) \) describe the parity operator in the continuum limit \( N \to \infty \)? To answer this particular question, let us initially adopt the mathematical prescription for the continuum limit established in Refs. [15, 16] through the following steps:

- Let \( \hat{Q} \) and \( \hat{P} \) denote, respectively, the discrete coordinate and momentum operators defined in a \( N \)-dimensional state vector space, which obey the eigenvalue equations

\[
\hat{Q}|q_\alpha\rangle = q_\alpha|q_\alpha\rangle \quad \text{and} \quad \hat{P}|p_\beta\rangle = p_\beta|p_\beta\rangle,
\]

where \( q_\alpha := \varepsilon q_0 \alpha \) and \( p_\beta := \varepsilon p_0 \beta \) represent two distinct sets of discrete eigenvalues labeled by \( \alpha, \beta \in [-\ell, \ell] \) for \( \ell = \frac{N-1}{2} \) and \( N \) odd, with \( \varepsilon = \sqrt{\frac{2\pi}{N}} \) fixed. For simplicity, the distances between successive eigenvalues of \( \hat{Q} \) and \( \hat{P} \) are maintained constants, that is, \( D_q = \varepsilon q_0 \) and \( D_p = \varepsilon p_0 \). Besides, the real parameters \( q_0 \) (coordinate unity) and \( p_0 \) (momentum unity) satisfy the relation \( q_0 p_0 = \hbar \); consequently, the discrete eigenvalues can be written as \( q_\alpha = D_q \alpha \) and \( p_\beta = D_p \beta \).

- The connection between \( (\hat{Q}, \hat{P}) \) and \( (\hat{U}, \hat{V}) \) follows the mathematical prescription

\[
\hat{U} = \exp\left(\frac{i}{\hbar} D_q \hat{Q}\right) \quad \text{and} \quad \hat{V} = \exp\left(\frac{i}{\hbar} D_p \hat{P}\right),
\]

which implies that \( \omega^{\frac{1}{2} \nu \xi} \hat{U} \hat{V} \) present in \( \hat{G}(0, 0) \) leads to the expression

\[
\omega^{\frac{1}{2} \nu \xi} \hat{U} \hat{V} = \exp\left(-\frac{i}{2\hbar} p_\eta q_\xi\right) \exp\left(\frac{i}{\hbar} p_\eta \hat{Q}\right) \exp\left(-\frac{i}{\hbar} q_\xi \hat{P}\right)
\]

with \( p_\eta = D_p \eta \) and \( q_\xi = -D_q \xi \). Therefore, \( \hat{G}(0, 0) \) assumes the form

\[
\hat{G}(0, 0) = \sum_{p_\eta = -D_p}^{R_p} \sum_{q_\xi = -D_q}^{R_q} \frac{D_p D_q}{2\pi \hbar} \exp\left(-\frac{i}{2\hbar} p_\eta q_\xi\right) \exp\left(\frac{i}{\hbar} p_\eta \hat{Q}\right) \exp\left(-\frac{i}{\hbar} q_\xi \hat{P}\right).
\]
where the limits \( R_p = \ell D_p \) and \( R_q = \ell D_q \) are related to the maximum range of each discrete spectrum \( \{ p_\eta \} \) and \( \{ q_\xi \} \).

- The continuum limit \( N \to \infty \) of \( \hat{\mathcal{G}}(0, 0) \) reobtains the Weyl-Wigner mapping kernel
  \[
  \hat{\Delta}(0, 0) = \int_{\mathbb{R}^2} \frac{dpdq}{2\pi\hbar} \exp\left(-\frac{i}{2\hbar}pq\right) \exp\left(\frac{i}{\hbar}p\hat{Q}\right) \exp\left(-\frac{i}{\hbar}q\hat{P}\right),
  \]
  or alternatively,
  \[
  \hat{\Delta}(0, 0) = 2\int_{\mathbb{R}} dx \langle -x | x \rangle = 2\hat{\mathcal{P}}. \tag{A.4}
  \]

In other words, the link with continuous parity operator \( \hat{\mathcal{P}} \) becomes evident in such a case. Summarizing, the answer of that first initial question is yes and particularly given by (A.4).

For sake of completeness, it should be stressed that similar quantum representations of finite-dimensional discrete phase spaces can also be constructed from this context \[5\] and worked out to describe the discrete Wigner function. Exemplifying, let us introduce the unitary operator basis \[14, 17\]
\[
\hat{D}(\eta, \xi) := \omega^{-\{2^{-1}\eta\xi\}} \hat{U}_\eta \hat{V}_\xi \quad (\eta, \xi \in [-\ell, \ell]) \tag{A.5}
\]
which differs from \( \hat{S}(\eta, \xi) \) through the presence, among other things, of a specific phase here depicted by \( \omega^{-\{2^{-1}\eta\xi\}} \). The argument showed in this phase obeys, in particular, the following rule: \( 2\{2^{-1}\eta\xi\} = \eta\xi + kN (\forall k \in \mathbb{Z}) \), namely, \( 2^{-1} \) represents the multiplicative inverse of \( 2 \) in \( \mathbb{Z}_N \) for \( N \) odd or prime. Besides, the property
\[
\hat{\mathcal{P}} = \frac{1}{N} \sum_{\eta, \xi = -\ell}^\ell \hat{D}(\eta, \xi)
\]
exhibits a straightforward connection with the discrete parity operator, which allows us to establish the \( \text{mod}(N) \)-invariant unitary operator basis \( \{ \hat{\Delta}(\mu, \nu) \}_{\mu, \nu = -\ell, \ldots, \ell} \) as being an unitary transformation on \( \hat{\mathcal{P}} \), that is,
\[
\hat{\Delta}(\mu, \nu) := \hat{D}(\mu, \nu) \hat{\mathcal{P}} \hat{D}_\dagger(\mu, \nu) = \hat{D}(\mu, \nu) \hat{\Delta}(0, 0) \hat{D}_\dagger(\mu, \nu). \tag{A.6}
\]
Equation (A.6) can be comprehended as a further discrete counterpart of the continuous case, such that \( W(\mu, \nu) := \text{Tr}[\hat{\Delta}(\mu, \nu) \hat{\rho}] \) defines its respective Wigner function \[28,37\] ††

Appendix B. Prolegomenon for SU(3)

The construction method of the SU(3) generators follows the mathematical prescription exposed in Section 3 for \( N = 3 \). Firstly, let us mention that SU(3) is basically constituted by the eight generators
\[
\{ \hat{g} \} = \{ \hat{\mathcal{U}}_{0,1}, \hat{\mathcal{V}}_{0,1}, \hat{\mathcal{W}}_0, \hat{\mathcal{U}}_{0,2}, \hat{\mathcal{V}}_{0,2}, \hat{\mathcal{W}}_{1,2}, \hat{\mathcal{V}}_{1,2}, \hat{\mathcal{W}}_1 \} \tag{B.1}
\]
†† A pertinent discussion, from the algebraic point of view, involving the effects of \( \hat{S}(\eta, \xi) \) and \( \hat{D}(\eta, \xi) \) on the respective definitions of discrete Wigner function can be found in Ref. \[22\], where, in particular, both the discrete Wigner functions were properly calculated for the discrete vacuum coherent state with \( N = 23 \) and illustrated by means of tridimensional plots.
whose expressions, explicitly written as a function of the Schwinger unitary operators, are listed in Table B1. Furthermore, their respective matrix representations in the basis \{ |u_0 \rangle, |u_1 \rangle, |u_2 \rangle \} provide exactly the well-known Gell-Mann matrices \( \lambda \)'s, namely,

\[
\hat{\lambda}_1 \equiv \hat{\Psi}_{0,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_2 \equiv \hat{\Psi}_{0,1} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\hat{\lambda}_3 \equiv \hat{\Psi}_{0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_4 \equiv \hat{\Psi}_{0,2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

\[
\hat{\lambda}_5 \equiv \hat{\Psi}_{0,2} = \begin{pmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\lambda}_6 \equiv \hat{\Psi}_{1,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
\hat{\lambda}_7 \equiv \hat{\Psi}_{1,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{\lambda}_8 \equiv \hat{\Psi}_{1} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}.
\]

As a further result, note that the product of two Gell-Mann matrices also satisfies the relation

\[
\hat{\lambda}_i \hat{\lambda}_j = \frac{2}{3} \delta_{ij} \hat{\lambda}_3 + \sum_{k=1}^{8} (\hat{D}_{ijk} + i \hat{F}_{ijk}) \hat{\lambda}_k \quad (i, j = 1, \ldots, 8)
\]

with \( \text{Tr}[\hat{\lambda}_i \hat{\lambda}_j] = 2 \delta_{ij} \), for which the non-null structure constants are given by \[48\]

\[
\begin{align*}
\hat{F}_{123} &= 1, \\
\hat{F}_{458} &= \hat{F}_{678} = \frac{\sqrt{3}}{2}, \\
\hat{F}_{147} &= \hat{F}_{246} = \hat{F}_{257} = \hat{F}_{345} = -\hat{F}_{156} = -\hat{F}_{367} = \frac{1}{2}, \\
\hat{D}_{118} &= \hat{D}_{228} = \hat{D}_{338} = -\hat{D}_{888} = \frac{\sqrt{3}}{3}, \\
\hat{D}_{448} &= \hat{D}_{558} = \hat{D}_{668} = \hat{D}_{778} = -\frac{\sqrt{3}}{6}, \\
\hat{D}_{146} &= \hat{D}_{157} = \hat{D}_{256} = \hat{D}_{344} = \hat{D}_{355} = -\hat{D}_{247} = -\hat{D}_{366} = -\hat{D}_{377} = \frac{1}{2}.
\end{align*}
\]

An interesting discussion about Lie algebra \( su(3) \) and possible applications in particle physics, emphasizing strictly its algebraic structure, can be found in Refs. \[78\] \[81\].

According to the previous results established in Section 3 for the generators of SU(N), the mappings of \( \{ \hat{g}_i \}_{i=1, \ldots, N^2-1} \) in the \( N^2 \)-dimensional dual discrete phase space – here labelled by the pair \(( \eta, \xi )\) – can be properly written as

\[
(\hat{\Psi}_{\alpha,\beta})(\eta, \xi) = \frac{1}{\sqrt{N}} \left[ \omega^{-\eta \alpha} \delta_{\xi,\beta-\alpha}^{[N]} + \omega^{-\eta \beta} \delta_{\xi,N-(\beta-\alpha)}^{[N]} \right] \omega^{-\frac{1}{2} \eta \xi},
\]

\[
(\hat{\Psi}_{\alpha,\beta})(\eta, \xi) = -\frac{1}{\sqrt{N}} \left[ \omega^{-\eta \alpha} \delta_{\xi,\beta-\alpha}^{[N]} - \omega^{-\eta \beta} \delta_{\xi,N-(\beta-\alpha)}^{[N]} \right] \omega^{-\frac{1}{2} \eta \xi},
\]
Table B1. Explicit constructions of the SU(3) generators as specific combinations of the unitary operators $\hat{U}$ and $\hat{V}$ – see Equations (15)-(17) for $N = 3$ and $\omega = \exp (\frac{2\pi i}{3})$.

| Part 1. Relations among Gell-Mann and Schwinger unitary operators |
|---------------------------------------------------------------|
| $\lambda_1 = \frac{1}{2} (\hat{V} + \hat{V}^2 + \hat{U}\hat{V} + \omega^*\hat{U}^2\hat{V} + \omega\hat{U}^2\hat{V}^2)$ |
| $\lambda_2 = -\frac{1}{2} (\hat{V} - \hat{V}^2 + \hat{U}\hat{V} + \omega^*\hat{U}^2\hat{V} - \omega\hat{U}^2\hat{V}^2)$ |
| $\lambda_3 = \frac{1}{2} [(1 - \omega^*)\hat{U} + (1 - \omega)\hat{U}^2]$ |
| $\lambda_4 = \frac{1}{2} (\hat{V} + \hat{V}^2 + \omega^*\hat{U}\hat{V} + \hat{U}\hat{V}^2 + \hat{U}^2\hat{V}^2)$ |
| $\lambda_5 = \frac{1}{2} (\hat{V} - \hat{V}^2 + \omega^*\hat{U}\hat{V} + \hat{U}\hat{V}^2 - \hat{U}^2\hat{V}^2)$ |
| $\lambda_6 = \frac{1}{2} (\hat{V} + \hat{V}^2 + \omega^*\hat{U}^2\hat{V} + \omega\hat{U}\hat{V}^2 + \omega^*\hat{U}^2\hat{V}^2)$ |
| $\lambda_7 = -\frac{1}{2} (\hat{V} - \hat{V}^2 + \omega^*\hat{U}^2\hat{V} + \omega\hat{U}\hat{V}^2 - \omega^*\hat{U}^2\hat{V}^2)$ |
| $\lambda_8 = -\sqrt{3} (\omega\hat{U} + \omega^*\hat{U}^2)$ |

| Part 2. Inverse relations for some combinations of unitary operators |
|---------------------------------------------------------------|
| $\hat{U} = \frac{1}{2} [(1 - \omega)\lambda_3 - \sqrt{3} \omega^*\lambda_8]$ |
| $\hat{V} = \frac{1}{2} [\lambda_1 + \lambda_4 + \lambda_6 + i(\hat{\lambda}_2 - \hat{\lambda}_5 + \hat{\lambda}_7)]$ |
| $\hat{U}^2 = \frac{1}{2} [(1 - \omega^*)\lambda_3 - \sqrt{3} \omega \lambda_8]$ |
| $\hat{V}^2 = \frac{1}{2} [\lambda_1 + \lambda_4 + \lambda_6 - i(\hat{\lambda}_2 - \hat{\lambda}_5 + \hat{\lambda}_7)]$ |
| $\hat{U}\hat{V} = \frac{1}{2} [\lambda_1 + \omega^*\lambda_4 + \omega\lambda_6 + i(\hat{\lambda}_2 - \omega^*\hat{\lambda}_5 + \omega\hat{\lambda}_7)]$ |
| $\hat{U}^2\hat{V} = \frac{1}{2} [\lambda_1 + \omega\lambda_4 + \omega^*\lambda_6 + i(\hat{\lambda}_2 - \omega\hat{\lambda}_5 + \omega^*\hat{\lambda}_7)]$ |
| $\hat{U}\hat{V}^2 = \frac{1}{2} [\omega\hat{\lambda}_1 + \hat{\lambda}_4 + \omega^*\hat{\lambda}_6 - i(\omega^*\hat{\lambda}_2 - \omega\hat{\lambda}_5 + \omega^*\hat{\lambda}_7)]$ |
| $\hat{U}^2\hat{V}^2 = \frac{1}{2} [\omega^*\hat{\lambda}_1 + \hat{\lambda}_4 + \omega\hat{\lambda}_6 - i(\omega\hat{\lambda}_2 - \hat{\lambda}_5 + \omega\hat{\lambda}_7)]$ |

\[
(\mathcal{W}_\gamma)(\eta, \xi) = \sqrt{\frac{2}{(\gamma + 1)(\gamma + 2)}} \frac{1}{\sqrt{N}} \sum_{\sigma=0}^{\gamma} \omega^{-\eta\sigma} - (\gamma + 1)\omega^{-\eta(\gamma + 1)} \omega^{-\frac{1}{2}N\xi} \delta_{\xi,0}^{[N]}.
\]

In particular, these expressions represent the respective dual counterparts of Equations (21)-(23), since both the results are connected through the discrete Fourier transform

\[
(\hat{g}_i)(\mu, \nu) = \frac{1}{\sqrt{N}} \sum_{\eta,\xi=0}^{N-1} \omega^{\mu\eta + \nu\xi} \omega^{-\frac{1}{2}N\xi(\eta,\xi;N)} (\hat{g}_i)(\eta, \xi).
\]  

(B.4)

For instance, if one considers the group SU(3), the dual representatives $(\hat{\lambda}_i)(\eta, \xi)$ assume
the following explicit forms:

\[
\begin{align*}
\hat{\lambda}_1(\eta, \xi) &= \frac{\sqrt{3}}{3} \left( \delta_{\xi,1}^{[3]} + \omega^{-\eta} \delta_{\xi,2}^{[3]} \right) \omega^{-\frac{1}{2} \eta \xi}, \\
\hat{\lambda}_2(\eta, \xi) &= -i \frac{\sqrt{3}}{3} \left( \delta_{\xi,1}^{[3]} - \omega^{-\eta} \delta_{\xi,2}^{[3]} \right) \omega^{-\frac{1}{2} \eta \xi}, \\
\hat{\lambda}_3(\eta, \xi) &= \frac{\sqrt{3}}{3} \left( 1 - \omega^{-\eta} \right) \omega^{-\frac{1}{2} \eta \xi} \delta_{\xi,0}^{[3]}, \\
\hat{\lambda}_4(\eta, \xi) &= \frac{\sqrt{3}}{3} \left( \delta_{\xi,2}^{[3]} + \omega^{-2\eta} \delta_{\xi,1}^{[3]} \right) \omega^{-\frac{1}{2} \eta \xi}, \\
\hat{\lambda}_5(\eta, \xi) &= -i \frac{\sqrt{3}}{3} \left( \delta_{\xi,1}^{[3]} - \omega^{-2\eta} \delta_{\xi,2}^{[3]} \right) \omega^{-\frac{1}{2} \eta \xi}, \\
\hat{\lambda}_6(\eta, \xi) &= \frac{\sqrt{3}}{3} \left( \omega^{-\eta} \delta_{\xi,1}^{[3]} + \omega^{-2\eta} \delta_{\xi,2}^{[3]} \right) \omega^{-\frac{1}{2} \eta \xi}, \\
\hat{\lambda}_7(\eta, \xi) &= -i \frac{\sqrt{3}}{3} \left( \omega^{-\eta} \delta_{\xi,1}^{[3]} - \omega^{-2\eta} \delta_{\xi,2}^{[3]} \right) \omega^{-\frac{1}{2} \eta \xi}, \\
\hat{\lambda}_8(\eta, \xi) &= \frac{1}{3} \left( 1 + \omega^{-\eta} - 2\omega^{-2\eta} \right) \omega^{-\frac{1}{2} \eta \xi} \delta_{\xi,0}^{[3]}.
\end{align*}
\]

The desired results are a consequence of taking the discrete Fourier transform for each expression separately, that is,

\[
\begin{align*}
\hat{\lambda}_1(\mu, \nu) &= \frac{2}{3} \sin \left( \left( \frac{\mu - \frac{1}{2}}{2} \right) \frac{\pi}{3} \right) \cos \left( \frac{2\pi \nu}{3} \right), \\
\hat{\lambda}_2(\mu, \nu) &= \frac{2}{3} \sin \left( \left( \frac{\mu - \frac{1}{2}}{2} \right) \frac{\pi}{3} \right) \sin \left( \frac{2\pi \nu}{3} \right), \\
\hat{\lambda}_3(\mu, \nu) &= \delta_{\mu,0}^{[3]} - \delta_{\mu,1}^{[3]}, \\
\hat{\lambda}_4(\mu, \nu) &= 2\delta_{\mu,1}^{[3]} \cos \left( \frac{4\pi \nu}{3} \right), \\
\hat{\lambda}_5(\mu, \nu) &= 2\delta_{\mu,1}^{[3]} \sin \left( \frac{4\pi \nu}{3} \right), \\
\hat{\lambda}_6(\mu, \nu) &= \frac{2}{3} \sin \left( \left( \frac{\mu - \frac{3}{2}}{2} \right) \frac{\pi}{3} \right) \cos \left( \frac{2\pi \nu}{3} \right), \\
\hat{\lambda}_7(\mu, \nu) &= \frac{2}{3} \sin \left( \left( \frac{\mu - \frac{3}{2}}{2} \right) \frac{\pi}{3} \right) \sin \left( \frac{2\pi \nu}{3} \right), \\
\hat{\lambda}_8(\mu, \nu) &= \frac{\sqrt{3}}{3} \left( \delta_{\mu,0}^{[3]} + \delta_{\mu,1}^{[3]} - 2\delta_{\mu,2}^{[3]} \right). \\
\end{align*}
\]

These mapped expressions represent important parts for the evaluation of discrete SU(3) Wigner function.

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