SPHALERONS AT FINITE TEMPERATURE

Sylvie Braibant, Yves Brihaye
Faculté des Sciences, Université de Mons-Hainaut
B-7000 Mons, Belgium

Jutta Kunz
Instituut voor Theoretische Fysica, Rijksuniversiteit te Utrecht
NL-3508 TA Utrecht, The Netherlands
and
FB Physik, Universität Oldenburg, Postfach 2503
D-2900 Oldenburg, Germany

Abstract

We construct the sphaleron for several temperature dependent effective potentials. We determine the sphaleron energy as a function of temperature and demonstrate that the sphaleron energy at a given temperature \( T \) is well approximated by the sphaleron energy at temperature zero scaled by the ratio of the vacuum expectation values of the Higgs field at temperatures \( T \) and zero. We address the cosmological upper bound on the Higgs mass.

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Introduction

The observation [1] that the baryon asymmetry of the universe (BAU) might possibly be explained in the framework of the standard model attracted a lot of attention [2-7]. While in previous scenarios the BAU was produced at high temperature, e.g. in GUTs, Kuzmin et al. [1] realized, that the BAU might be produced during the electroweak phase transition.

When, on the one hand, one considers a scenario for baryogenesis, where the baryon asymmetry is produced by some mechanism at high temperature one has to require that the presently observed BAU survives the electroweak phase transition. When, on the other hand, one considers a scenario for baryogenesis, where the BAU is generated during the electroweak phase transition, one must require that the baryon number violating transitions have a large enough rate. In both cases one needs to evaluate the rate of baryon number violation in the broken phase of the electroweak model. (See [6,7] for reviews.)

When the temperature cools down, the universe goes through the electroweak phase transition, where the symmetry is broken by the Higgs potential. In the broken phase, the structure of the vacuum becomes non-trivial. Topologically distinct vacua are separated by finite energy barriers, whose height is determined by the sphaleron energy, $E_{sp}$. The sphaleron [8], an unstable solution of the electroweak model, plays a central role in the generation of the baryon asymmetry, since the rate of baryon number violating transitions is largely determined by the Boltzmann factor

$$\Gamma \sim \exp \left( -\frac{E_{sp}(T)}{T} \right).$$

The existence of the BAU might even yield cosmological constraints for the parameters of the standard model. Requiring for instance that the BAU, generated during a first order electroweak phase transition, does survive till the present time, means that the baryon number violating transitions have to be out of thermal equilibrium after the phase transition. This constraints the value of the energy barrier, i.e. the sphaleron energy $E_{sp}(T)$, in the region of temperature where the phase transition occurs. Shaposhnikov [9] has derived the model-independent
relation
\[ \frac{E_{\text{sp}}(T_t)}{T_t} > 45 , \tag{2} \]
where \( T_t \) denotes the transition temperature. Relation (2) can be used to obtain an upper bound on the Higgs mass [2,7,9].

At the moment a major source of uncertainty in computing the rate of baryon number violating transitions and in extracting the cosmological upper bound for the Higgs mass lies in the inclusion of finite temperature effects in the electroweak model. Lacking a satisfactory alternative method, the technique of effective potentials, computed perturbatively by resumming the dominant Feynman diagrams, is used to describe the interactions of the standard model in the neighbourhood of the critical temperature [10]. The simplest temperature dependent effective potential yields a second order phase transition [6]. Recently several “improved” effective potentials have been proposed [9,11-15], which contain cubic terms in the Higgs field, providing a first order electroweak phase transition. For these temperature dependent effective potentials one must then reevaluate the sphaleron energy \( E_{\text{sp}}(T) \) and the transition rate (1) [16].

The purpose of this note is to determine the sphaleron energy in the region of the phase transition for three increasingly sophisticated temperature dependent effective potentials. Besides computing the sphaleron energy numerically we employ for the sphaleron energy the simple formula
\[ E(\lambda, T) = E(\lambda, T = 0) \frac{\langle \phi(T) \rangle}{\langle \phi(0) \rangle} , \tag{3} \]
where \( \langle \phi(T) \rangle \) is the vacuum expectation value of the Higgs field at temperature \( T \). Comparison of both energies then yields the quality of the approximation (3) for the more sophisticated temperature dependent effective potentials. We also address the model-independent bound (2) for the sphaleron energy and the implications for the cosmological upper bound on the Higgs mass for the potentials considered.

**Sphalerons at zero temperature**

Let us first consider the classical lagrangian for the electroweak interactions. It is sufficient to treat the mixing angle \( \theta_w \) perturbatively, as demonstrated recently
In leading order, we therefore consistently set the $U(1)$ gauge field equal to zero. The bosonic sector of the electroweak model then reduces to

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\phi), \quad (4a)$$

with

$$V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2, \quad \phi \equiv \sqrt{2}(\Phi^\dagger \Phi)^{1/2}. \quad (4b)$$

Sphalerons are saddle points of the classical energy functional [8]. In order to construct sphalerons we choose the gauge $A_0 = 0$, and we employ the static spherically symmetric ansatz for the fields

$$\phi(\vec{r}) = vL(r), \quad A_i^a(\vec{r}) = \frac{G(r)}{gr^2} \epsilon_{iba} r_b. \quad (5)$$

The classical energy functional then reads

$$E = \frac{2\pi M_W}{g^2} \int dx \left[ \frac{G^2(G-2)^2}{x^2} + 2\frac{dG}{dx}^2 + 2G^2L^2 \right.\left. + 4x^2 \frac{dL}{dx}^2 + \frac{8\lambda}{g^2} x^2(L^2 - 1)^2 \right], \quad (6)$$

where $x$ is the dimensionless coordinate

$$x = M_W r$$

and $M_W$ and $M_H$ are the masses of the gauge and Higgs bosons

$$M_W = \frac{g v}{2}, \quad M_H = v\sqrt{2\lambda}.$$ 

In the calculations we use the values $v \approx 246$ GeV and $g \approx 0.65$, corresponding to $M_W = 80$ GeV.

In order to have non-trivial regular, finite energy solutions, the radial functions $G(x)$ and $L(x)$ must obey the boundary conditions

$$G(0) = 2, \quad G(\infty) = 0, \quad L(0) = 0, \quad L(\infty) = 1. \quad (7)$$

For the energy functional (6) with boundary conditions (7) one saddle point solution is known, the sphaleron [8], whose energy increases from 7 TeV for $M_H = 0$ to
13 TeV for $M_H = \infty$. For $M_H < 12M_W$ the sphaleron has precisely one direction of instability. For $M_W > 12M_W$ new directions of instability of the sphaleron appear, which are associated with new solutions of the electroweak model. (These are the bisphalerons [18,19], which are based on the general spherically symmetric ansatz for the fields involving three functions for the gauge fields and two functions for the Higgs field.)

**Sphalerons at finite temperature**

In order to introduce finite temperature effects into the electroweak model and to describe the electroweak phase transition, one has to replace the Mexican hat potential (4b) by a temperature dependent effective potential [6]. The expression for the energy functional and the equations of motion are then modified accordingly. We now discuss the effects of three temperature dependent effective potentials on the sphaleron solution.

**Case I.**

The simplest approximation for the temperature dependent effective potential consists of supplementing the tree level potential by the leading term of the high-temperature expansion [6]. Neglecting logarithmic terms, this effective potential reads [6]

$$V(\phi, T) = \frac{\lambda}{4} \phi^4 - \frac{\lambda}{2} v^2 \phi^2 + \frac{\gamma T^2}{2} \phi^2, \quad \gamma = \frac{2M_W^2 + M_Z^2 + 2M_t^2}{4v^2}.$$ \hspace{1cm} (8)

This potential has a transition between the broken phase for $T < T_c$ and the unbroken phase for $T > T_c$ at

$$T_c^2 = \frac{\lambda v^2}{\gamma}.$$ \hspace{1cm} (9)

The vacuum expectation value of the Higgs field $\langle \phi(T) \rangle$ is a continuous function of temperature

$$\langle \phi(T) \rangle = 0 \quad \text{for} \ T > T_c,$$

$$\langle \phi(T) \rangle = v\left(1 - \frac{\gamma T^2}{\lambda v^2}\right)^{1/2} \quad \text{for} \ T < T_c.$$ \hspace{1cm} (10)

The corresponding sphaleron solutions possess a nice property. They can be constructed from their zero temperature counterpart by a suitable scaling of $x$ and
of $L(x)$
\[
\tilde{x} = x/a , \quad \tilde{L}(x) = aL(x) , \quad a^2 = \left(1 - \frac{\gamma T^2}{\lambda v^2}\right) .
\] (11)
In this case the sphaleron energy $E(\lambda, T)$ is exactly given by formula (3)
\[
E(\lambda, T) = E(\lambda, T = 0)\langle \phi(T) \rangle \langle \phi(0) \rangle ,
\]
where $\langle \phi(0) \rangle \equiv v = 246$ GeV.

Since the phase transition is of second order, this effective potential leads to a restoration of the baryon-antibaryon symmetry in the broken phase shortly after the phase transition. It can hardly be reconciled with the observed BAU.

Case II. $\theta_w = 0$

Considering higher orders in the perturbative calculation one obtains corrections to the temperature dependent effective potential. In the next order one finds the so called “one loop improved” potential [7]
\[
V(\phi, T) = \frac{\lambda}{4} \phi^4 - \frac{\lambda}{2} v^2 \phi^2 + \frac{\gamma T^2}{2} \phi^2 - \delta T \phi^3 , \quad \delta = \frac{2M_W^3 + M_Z^3}{4\pi v^2} .
\] (12)
The new term, cubic in $\phi$, now renders the phase transition first order.

For this effective potential there are three relevant critical temperatures. For low temperatures, the minimum of the potential is attained at some $\langle \phi(T) \rangle \neq 0$ and $\phi = 0$ corresponds to a local maximum of $V$. At $T = T_c$ (defined in eq. (9)) the nature of the extremum at the origin changes. $\phi = 0$ turns into a local minimum, which is separated from the absolute minimum by a small potential barrier. The two minima become degenerate at a temperature $T = T_b$, and $\phi = 0$ becomes the absolute minimum for $T > T_b$. Thus at high temperatures the symmetry is restored. The local minimum then disappears at $T = T_a$. To summarize, the expectation value of the Higgs field is given by the absolute minimum of the potential
\[
\langle \phi(T) \rangle = 0 \quad \text{for} \quad T > T_b ,
\]
\[
\langle \phi(T) \rangle > 0 \quad \text{for} \quad T < T_b ,
\] (10')
and one observes a first order phase transition. Choosing the masses $M_W = M_Z = 80$ GeV, $M_H = 45$ GeV and $M_t = 120$ GeV the three critical temperatures correspond to
\[
T_c = 0.2902v , \quad T_b = 0.2962v , \quad T_a = 0.2965v
\]
in units of $v = 246$ GeV.

For the “one loop improved” potential (12) the last term in the energy functional (6) must be replaced by

$$\frac{8\lambda}{g^2} x^2 \left[ L^4 - 2L^2 + \frac{\gamma T^2}{2\lambda v^2} L^2 - \frac{4\delta T}{\lambda v} L^3 + C \right],$$

where the constant $C$ must be adjusted such that the potential approaches zero asymptotically. The boundary conditions for the function $L(x)$ become

$$L(0) = 0, \quad L(\infty) = \frac{\langle \phi(T) \rangle}{v},$$

where $\langle \phi(T) \rangle$ denotes the non-trivial minimum of the potential $V(\phi, T)$.

We have analysed the sphaleron for this effective potential numerically. The sphaleron is physically meaningful only for $T < T_b$, but the solutions can be constructed up to $T < T_a$. The profiles of $G(x)$ and $L(x)$ are illustrated in Figs. 1 and 2 for the temperatures $T = 0$ and $T = T_b$, using the parameters $M_W = M_Z = 80$ GeV, $M_H = 45$ GeV and $M_t = 120$ GeV. The energy of the sphaleron is shown in Fig.3 (solid line) as a function of temperature for the same parameters. The sphaleron energy obtained for this potential with the approximation formula (3) is also shown in Fig.3 (dashed line). Obviously, eq. (3) constitutes a rather good approximation, since its values for $E_{sp}(T)$ exceed the exact values for the potential (12) only slightly, typically by 4% at $T = T_c$ up to 8% at $T = T_b$.

In Fig.4 we present the ratio $E_{sp}(T)/T$ for the parameters $M_W = M_Z = 80$ GeV, $M_H = 45$ GeV and $M_t = 120$ GeV. For convenience, we have represented the temperature in Fig.4 via the variable $\xi$

$$\xi = \frac{\lambda^2}{\delta^2} \left( 1 - \left( \frac{T_c}{T} \right)^2 \right)$$

defined [20] such that the critical temperatures $T_c, T_b, T_a$ correspond to $\xi = 0, \xi = 2, \xi = 9/4$, respectively. Variation of the top quark mass $M_t$ within the experimental bounds hardly effects the ratio $E_{sp}(T)/T$. This observation is easily understood at the critical temperature $T_c$ by applying the approximation formula (3)

$$\frac{E_{sp}(T_c)}{T_c} = \frac{E_{sp}(0)}{v} \frac{3\delta}{\lambda}.$$
which is independent of the top quark mass $M_t$.

The model-independent relation (2) provides a cosmological upper bound on the Higgs mass, depending on the effective temperature dependent potential. In Fig. 5 we present the ratio $E_{sp}(T)/T$ at the critical temperature $T_c$ as a function of the Higgs mass for the “one loop improved” potential for the parameters $M_W = M_Z = 80$ GeV and $M_t = 120$ GeV. To satisfy relation (2) the Higgs mass must be smaller than 46 GeV (practically independent of $M_t$), which is considerably below the experimental lower bound on the Higgs mass, $M_H > 60$ GeV.

Case II. $\theta_w \neq 0$

So far we have considered the sphaleron in the limit of vanishing mixing angle (i.e. $\theta_w = 0$). At zero temperature this constitutes an excellent approximation to the physical case (i.e. $\theta_w \approx 30^\circ$). The energy of the sphaleron at the physical mixing angle is only lower by 1%. Computing the first order correction in $\theta_w$ to the sphaleron energy at finite temperature, we observe that it is also 1%, if $M_W = M_Z$ is chosen in the effective potential. Including the mixing angle dependence of the parameters $\gamma$ and $\delta$ in the effective potential, however, leads to a considerable effect for the ratio $E_{sp}(T)/T$. For comparison with the $\theta_w = 0$ case we show in Fig. 4 the ratio $E_{sp}(T)/T$ for the masses $M_W = 80$ GeV, $M_Z = 92$ GeV, $M_H = 45$ GeV and $M_t = 120$ GeV. By including the mixing angle dependence of the gauge boson masses explicitly, the cosmological bound on the Higgs mass is slightly improved. Here it increases to $M_H < 50$ GeV.

In Ref. [8] it was observed that the sphaleron carries a large magnetic dipole moment. In the leading order approximation in $\theta_w$ the magnetic dipole moment is given by

$$
\mu = \lim_{x \to \infty} 2x^3 p(x) \frac{e}{\alpha_w M_W},
$$

where $p(x)$ is determined by the equation

$$
x^2 p'' + 4xp' = \frac{L^2 G}{2},
$$

and satisfies the boundary conditions

$$
p'(0) = 0, \quad p(x \gg 1) \approx \frac{\mu}{x^3}.
$$
Our numerical analysis indicates that the magnetic dipole moment depends slightly on temperature, when expressed in units of $e/\alpha_w M_W(T)$. Employing in the effective potential the parameter set $M_W = 80$ GeV, $M_H = 45$ GeV, $M_t = 120$ GeV and $M_Z = 92$ GeV, we find for the magnetic moment in these units

\[
\mu(T = 0) = 1.90 , \quad \mu(T = T_c) = 1.98(4.0) , \quad \mu(T = T_b) = 2.02(6.1) ,
\] (17)

(where the values in brackets are in units of $e/\alpha_w M_W(0)$).

**Case III.**

Finally let us briefly discuss the sphaleron when Debye screening effects are taken into account in the effective potential. Assuming $M_W = M_Z$, the effective potential with Debye screening effects reads [9,21]

\[
V_{sc}(\phi, T) = \frac{\lambda}{4} \phi^4 - \frac{\lambda}{2} \frac{v^2}{\phi^2} + \frac{T^2}{8} \phi^2 \left( \frac{3M_W^2 + 2M_t^2}{v^2} \right) - \frac{T}{4\pi} \left( \phi^3 \frac{2M_W^3}{v^3} + \left( \frac{11}{6} g^2 T^2 + \frac{M_W^2}{v^2} \phi^2 \right)^{3/2} \right) .
\] (18)

The effective potentials (12) and (18) differ in their cubic pieces. Potentials like (18) have also been considered by Khoze [15] in an attempt to find a potential suitable to describe the barrier between the vacua and not only the vacua themselves.

The sphaleron energies obtained for the potential (18) are shown in Fig.3 (solid curve) along with the energies obtained for the potential (12). The same parameters have been used for both potentials. Also for this potential formula (3) for the sphaleron energy (dashed curve) represents a good approximation.

At a fixed temperature the sphaleron obtained for the potential with Debye screening effects is heavier than the one corresponding to the “one loop improved” potential. But on the other hand the phase transition for the potential with Debye screening effects occurs at a higher temperature. Considering the model-independent relation (2) we find for the effective potential with Debye screening

\[
E_{sp}(T)/T \approx 35 \quad \text{for} \quad T = T_c \approx 0.3075 ,
\]
\[
E_{sp}(T)/T \approx 25 \quad \text{for} \quad T = T_b \approx 0.311 .
\] (19)
The ratio $E_{sp}(T)/T$ is shown in Fig. 5 for the critical temperature $T_c$ as a function of the Higgs mass for the parameters $M_W = M_Z = 80$ GeV and $M_t = 120$ GeV. The curve obtained with Debye screening effects included is distinctly below the curve for the “one loop improved” potential. While initially there was hope that potentials incorporating the Debye screening effects would allow for a higher cosmological bound on the Higgs mass [9], these results indicate, that taking Debye screening effects into account is not favourable for the bound on the Higgs mass. Indeed, we find from the model-independent relation (2) for the potential (18) a cosmological upper bound on the Higgs mass $M_H < 40$ GeV as compared to $M_H < 46$ GeV for the “one loop improved” potential (12).

Conclusions

In this paper we have presented the sphaleron energy as a function of temperature for three effective temperature dependent potentials. We have demonstrated that the simple scaling formula (3) represents a good approximation for the sphaleron energy for the “one loop improved” potential and for the effective potential with Debye screening effects included. We conjecture, that the formula will also be good for other effective potentials, i.e. that it is sufficient to know the minimum of the respective effective potential to obtain a good estimate of its sphaleron energy.

The model-independent relation (2) provides a cosmological upper bound on the Higgs mass, depending on the respective effective potential considered. For $\theta_w = 0$ we have obtained the bounds $M_H < 46$ GeV for the “one loop improved” potential and the even lower value $M_H < 40$ GeV for the effective potential with Debye screening effects included. We therefore conclude that Debye screening effects are not favourable for the cosmological upper bound on the Higgs mass. In contrast, considering the mixing angle dependence of the effective potential, when calculating the sphaleron energy, does shift the upper bound on the Higgs mass to a higher value, $M_H < 50$ for the “one loop improved” potential.

A bound of $M_H < 50$ GeV (resp. $M_H < 46$ GeV) is inconsistent with the present limit from the LEP experiments, $M_H > 60$ GeV. However, the bound on the Higgs mass is sensitive to the effective potential. Thus employment of a more sophisticated temperature dependent potential (than the ones considered here)
might reconcile the bound (2) with the experimental limit.

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Figure captions

Figure 1
The gauge field function $G(x)$ is shown for the sphaleron obtained with the potential (12) with parameters $M_H = 45$ GeV, $M_W = M_Z = 80$ GeV, $M_t = 120$ GeV. The solid (dashed) line represents the solution for $T = 0$ ($T = T_b$).

Figure 2
The Higgs field function $L(x)$ is shown for the sphaleron obtained with the potential (12) with parameters $M_H = 45$ GeV, $M_W = M_Z = 80$ GeV, $M_t = 120$ GeV. The solid (dashed) line represents the solution for $T = 0$ ($T = T_b$).

Figure 3
The sphaleron energy $E_{sp}$ (in TeV) is plotted as a function of temperature $T$ (in units of $v$), for the potentials (12) and (18). The solid lines represent the values obtained by numerically solving the equations of motion, the dashed lines represent the values obtained from the simple approximation (3).

Figure 4
The ratio $E_{sp}(T)/T$ is plotted as a function of the variable $\xi$ (see eq. (14)) for the potential (12) for the parameters $M_H = 45$ GeV, $M_W = 80$ GeV, $M_Z = 80$ GeV (resp. $M_Z = 92$ GeV) and $M_t = 120$ GeV.

Figure 5
The ratio $E_{sp}(T_c)/T_c$ is plotted as a function of the Higgs mass $M_H$ (in units of GeV) for the potentials (12) and (18) (solid lines) for the parameters $M_W = M_Z = 80$ GeV and $M_t = 120$ GeV. The dashed line represents the model-independent relation (2).