Influence of partially $\tau$-embedded subgroups of prime power order in supersolubility and p-nilpotency of finite groups

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ABSTRACT
In this paper, we introduced a new concept of $pr$-embedded subgroups which belongs to an embedded class of subgroups of finite groups. A subgroup $H$ of a group $G$ is said to be a partially $\tau$-embedded subgroup in $G$ if there exists a normal subgroup $K$ of $G$ such that $HK$ is normal in $G$ and $H \cap K \leq H_{pr}G$ where $H_{pr}G$ generated by all those subgroups of $H$ which are partially $\tau$-quasinormal in $G$. We investigate the influence of some $pr$-embedded subgroups with prime power order on the structure of a finite group $G$. Some new criteria about the $p$-nilpotency and supersolubility of a finite group were obtained. Our results also generalized some earlier ones about formations.

1. Introduction
During the twentieth century, mathematicians investigated some aspects of the theory of finite groups in great depth, especially the local theory of finite groups and the theory of solvable and nilpotent groups. As a consequence, the complete classification of finite simple groups was achieved, meaning that all those simple groups from which all finite groups can be built are now known. During the second half of the twentieth century, mathematicians such as Chevalley and Steinberg also increased our understanding of finite analogs of classical groups, and other related groups. One such family of groups is the family of general linear groups over finite fields. Finite group theory fund many applications in chemistry, physics, engineering and other areas of sciences.

In this article, $G$ denotes the finite groups. All the notations are standard, as in [1,2]. Order of $G$ denoted by $|G|$, Sylow $q$-subgroup of $G$ denoted by $G_q$ and Sylow subgroup of $G$ denoted by $Syl(G)$ or simply $Syl$. partially $\tau$-quasinormal subgroups denoted by $pr$-quasinormal and partially $\tau$-embedded subgroups simply denoted by $pr$-embedded.

Many authors worked on quasinormal subgroups and gave generalizations of normal subgroups. For example, Kegel [3] gave the extension of $S$-quasinormal. $\tau$-quasinormality is the generalization of the $S$-quasinormal subgroups of finite groups. Recently, Li et al. [2] generalized $\tau$-quasinormal subgroups to $pr$-quasinormal. $H$ is $pr$-quasinormal in $G$ if every $Syl(H)$ is a Sylow subgroup of some $\tau$-quasinormal subgroup of $G$. Many authors discussed the concept of permutable groups and gave many new concepts and generalizations of permutable groups. Now we present another extension of permutable groups and by the use of the following definitions we extend the permutable groups.

Definition 1.1: $H$ is called partially $\tau$-embedded subgroup in $G$ if $K < G$ such that $HK$ is normal in $G$ and $H \cap K \leq H_{pr}G$ where $H_{pr}G$ generated by all those subgroups of $H$ which are partially $\tau$-quasinormal in $G$. We call $H_{pr}G$ the $pr$-core of $H$ in $G$.

Clearly, all $\tau$-quasinormal and $pr$-quasinormal subgroups are $pr$-embedded subgroups.

Remark 1.2: $\tau$-quasinormality and $pr$-quasinormality implies $pr$-embeddedness.

Example 1.3: Let $G=S_4$ and $H=\langle (1 4) \rangle$. Then $H$ is $pr$-embedded but $H$ is not $\tau$-quasinormal. Since $H$ and every subgroup of $G$ with order 6 containing $H$ cannot permute with every $Syl(G)$. As $T=A_4$, $HT \leq G$ and $H \cap A_4 \leq H_{pr}G=1 \leq H_{pr}G=H$. Clearly $pr$-embedded is an extension of $\tau$-quasinormal. So $pr$-embedded is an extended form of $\tau$-quasinormal and $pr$-quasinormal.

Our contribution in this direction are following theorems about $pr$-embedded.
Theorem 1.4: Let $H \leq G$ such that $G/H$ is supersolvable. If every maximal subgroup of any Sylow of $\Phi(H)$ is $p\tau$-embedded in $G$, then $G$ is supersolvable.

Theorem 1.5: Let $Q$ be a $G_p$, where $p$ is a prime divisor of $|G|$ with $(|G|, (q - 1)(q^2 - 1) \cdots (q^n - 1)) = 1$ for $n > 1$. If every maximal subgroup of $Q$ and every $n$-maximal subgroup of $Q$ is $p\tau$-embedded in $G$, then $G$ is $q$-nilpotent.

Theorem 1.6: Suppose $M \trianglelefteq G$ and $Q$ be a Sylow $q$-subgroup of $M$ such that $(|M|, q - 1) = 1$ provided $q$ is a prime divisor of $|M|$. If every largest subgroup $Q_i$ of $Q$ is $p\tau$-embedded in $G$ such that $Q_i$ does not have a $q$-supersolvable supplement in $G$, then each chief factor of $G$ between $M$ and $O_q(M)$ is cyclic.

2. Preliminaries

This section contains some basic results that help us in proving our main results.

Lemma 2.1: Suppose that $A$ and $B$ are two subgroups of a finite group $G$, then the following statements hold:

1. If $A$ is $S$-quasinormal in $G$, then $A \cap B$ is $S$-quasinormal in $B$ [4].
2. If $A$ and $B$ are $S$-quasinormal in $G$, then $A \cap B$ is $S$-quasinormal in $G$ [3].
3. If $A$ is $S$-quasinormal in $G$, then $A/A_{eq}$ is nilpotent [4].

Lemma 2.2 ([3,4]): Suppose that $H$ be an $S$-quasinormal subgroup of the finite group $G$. Then

1. If $H \leq K \leq G$, then $H$ is $S$-quasinormal in $K$.
2. If $N$ is a normal subgroup of $G$, then $HN/N$ is $S$-quasinormal in $G/N$.
3. If $K \leq G$, then $H \cap K$ is $S$-quasinormal in $K$.
4. If $K$ is normal in $G$, then $H \cap K$ is $S$-quasinormal in $G$.

Lemma 2.3 ([5]): Let $Q$ be a $S$-quasinormal $q$-subgroup of the finite group $G$ for some prime $q$, then we have $N_{\Phi}(Q) \geq O_q^1(G)$.

Lemma 2.4: Let $M \trianglelefteq G$ and $H \leq T \leq G$. Then

1. $H_{prG} \leq H$. 
2. $H_{prG} \leq H_{prK}$. 
3. $H_{prG}M/M \leq (HM/M)_{pr(G/M)}$.

Proof: Suppose that $L$ be a partially $\tau$-quasinormal subgroup of $G$ contained in $H$ and $q$ be a prime dividing $|L|$. Further suppose that $Q$ a Sylow q-subgroup of $L$ and $E$ an $S$-quasinormal subgroup of $G$ such that $Q \in y_{\Phi}(E)$. Then the proof is given as follows:

1. Let $x \in H$, then $L^x \leq H$. If $R$ is a Sylow q-subgroup of $L^x$, then $R = Q^x_1$ for some Sylow q-subgroup $Q_1 = Q$. Obviously, $Q^x_1 \in y_{\Phi}(E^x)$ and $E^x$ is an $S$-quasinormal subgroup of $G$. Hence $L^x$ is a partially $\tau$-quasinormal subgroup of $G$. This implies that (1) holds.
2. By Lemma 2.1(1), $E \cap K$ is an $S$-quasinormal subgroup of $K$. Since $Q \leq E \cap K$, $Q$ is a Sylow q-subgroup of $E \cap K$. Hence $L \leq H_{prK}$ and so $H_{prG} \leq H_{prK}$.
3. Clearly, $LM/M \leq HM/M$ and $LM/M$ is a partially $\tau$-quasinormal subgroup of $G/M$. Hence $H_{prG}M/M \leq (HM/M)_{pr(G/M)}$.

Lemma 2.5: Let $H \leq G$. Then we have the following:

1. $H$ is $p\tau$-embedded in $G$ and $H \leq N \leq G$, then $H$ is $p\tau$-embedded in $N$.
2. Suppose that $M \trianglelefteq G$ and $M \leq H$. If $H$ is a q-group and $p\tau$-embedded in $G$, then $H/M$ is $p\tau$-embedded in $G/M$.
3. Suppose that $H$ is a q-group of $G$ and $M$ is a normal q-group of $G$. If $H$ is $p\tau$-embedded in $G$, then $HM/M$ is $p\tau$-embedded in $G/M$.
4. If $H$ is $p\tau$-embedded in $G$ and $H \leq T \leq G$, then there exists $K \leq G$ such that $HK \leq G$, $H \cap K \leq H_{prG}$ and $HK \leq T$.

Proof: By hypothesis, $T \leq G$ such that $HT \leq G$, $H \cap T \leq H_{prG}$.

1. Then by Lemma 2.4

$$H(N \cap T) = N \cap HT \leq N$$

and

$$H \cap T \leq H_{prG} \leq H_{prN}$$

Hence, $H$ be $p\tau$-embedded in $N$.

2. We know that $(H/M)(TM/M)$ is normal in $G/M$, so by Lemma 2.4, we have

$$H/M \cap TM/M = (H \cap T)/M \leq H_{prG}M/M \leq (HM/M)_{pr(G/M)}$$

$$= (H/M)_{pr(G/M)}$$

So $H/M$ is $p\tau$-embedded in $G/M$.

3. Let $H$ be $p\tau$-embedded in $G$, then $T \trianglelefteq G$ such that $HT$ be $S$-quasinormal, $H \cap T \leq H_{prG}$.

Clearly

$$TM/M \leq G$$

and

$$(H/M)(TM/M) = HTM/M$$

is $S$-quasinormal in $G/M$ by Lemma 2.1(2) of [2].

Now

$$([HM : H], [HM : M]) = 1,$$
also

\[ HM/M \cap TM/M = (HM \cap T)M/M = (H \cap T)(M \cap T)M/M \]

and

\[ = (H \cap T)M/M \leq H_{prG}/M. \]

From Lemma 2.10 (Lemma 2.8 of [10]),

\[ H_{prG}/M \leq (HM/M)H_{prG}/M. \]

Hence \( HM/M \) is \( pr \)-embedded in \( G/M \).

(4) Let \( H \) be \( pr \)-embedded in \( G \). Then \( M \) is normal in \( G \) such that \( HM \) is \( S \)-quasinormal in \( G \) and

\[ H \cap M \leq H_{prG}. \]

Let \( K = M \cap T \), then \( K \) is normal in \( G \), and

\[ HK = H(M \cap T) = HM \cap T \]

is \( S \)-quasinormal by Lemma 2.1(5) of [6],

\[ HK \leq T \]

and

\[ H \cap K = H \cap M \cap T = H \cap M \leq H_{prG}. \]

\[ \square \]

**Lemma 2.6 ([7]):** Consider a group \( G \) and prime number \( p \) such that \( q^{n+1} \) does not divides \( |G| \) for integers \( n \geq 1 \). If

\[ (|G|,(q-1)(q^2-1)\ldots(q^n-1)) = 1, \]

then \( G \) is \( q \)-nilpotent.

**Lemma 2.7 ([8]):** Consider a group \( G \). If \( A \) is subnormal in \( G \) and \( A \) is a \( q \)-subgroup, then

\[ A \leq O_q(G). \]

**Lemma 2.8 ([9]):** Suppose that \( P, R, S \leq G \). Then the following statements are equivalent:

- \( P \cap RS = (P \cap R)(P \cap S) \).
- \( PR \cap PS = P(R \cap S) \).

**Lemma 2.9:** Let \( M \leq G \) such that \( G = PM \) for some \( Q \leq G \), take \( N \leq G \) a maximal subgroup with \( M \) contained in \( N \), then \( Q \cap N \) is maximal in \( Q \).

**Lemma 2.10 (Lemma 2.8 of [10]):** Let \( q \) be a prime number which divides \( |G|, (|G|,q-1) = 1 \). Then

1. If \( M \leq G \) having \( p \) order, then \( M \leq Z(G) \).
2. If \( G \) has \( G_q \) a cyclic subgroup, then \( G \) is \( q \)-nilpotent.
3. If \( N \leq G, |G:N| = q \), then \( N \leq G \).

**Lemma 2.11:** Let \( q \) be a prime divisor of the order of \( G \) in such a way \(|G|, q-1 = 1\), then

1. \( G \) is \( q \)-nilpotent provided \( G \) is \( q \)-supersoluble.
2. \( G \) is \( q \)-nilpotent provided \( G \) has cyclic Sylow \( q \)-subgroup.
3. \( X \) is normal in \( G \) provided \(|G:X| = q \) and \( X \leq G \).
4. \( N \) lies in \( Z(G) \) provided \(|N| = q \) and \( N \) is normal in \( G \).

**Proof:** Suppose \( C/D \) is any random chief factor. If \( G \) is \( q \)-supersoluble, then there are two possibilities:

1. \(|C/D| = q \) is cyclic.
2. \(|C/D| = q^2 \)-group.

If

\[ |C/D| = q, \]

then

\[ |Aut(C/D)| = q - 1. \]

As \( G/C_G(C/D) \) is isomorphic to a subgroup of \( Aut(C/D) \), then \(|G/C_G(C/D)| \) will divide \( (|G|,q-1) \). This implies

\[ C_G(C/D) = G. \]

Hence \( G \) is \( q \)-nilpotent. Proof of (2), (3) and (4) can be seen in [11, Theorem 2.8]. \[ \square \]

**Lemma 2.12 ([12], 7.19):** Let \( Y \leq G \), then

\[ Y/\Phi(Y) \leq Z_G(G/\Phi(Y)) \]

if and only if

\[ Y \leq Z_G(G). \]

**Lemma 2.13:** Suppose \( X \leq G \) is \( q \)-subgroup. Then there exists \( A \leq G \) such that \( A \) is the largest subgroup of \( X \) and is \( pr \)-embedded.

**Proof:** If the order of \( X \) is \( q \), then the theorem holds. Let \( Y \leq X \) be a normal \( q \)-subgroup which is smallest without identity. Let \( X \neq Y \). Using Lemma 2.6(2) of [13], the theorem is still satisfied by \( Y \). So with the help of induction some largest subgroup \( L \) of \( X \) will divide \( (|G|,q-1) \). This implies

\[ C_G(C/D) = G. \]

Obviously, \( L \leq X \) and \( L \leq G \). Therefore, the hypothesis is satisfied again. Now let \( X = Y \). Assume that \( L \) be the largest subgroup of \( X \). So there will be \( E \leq G \) in such a way that \( LE \) is \( S \)-quasinormal

\[ L \cap E \leq L_{prG}. \]

Then \( LE \neq L \) and \( Y \neq 1 \). If \( X \leq LE \), then

\[ X = X \cap LE = L(X \cap E). \]
Hence $X \leq E$, which shows that

$$L = L \cap E = L_{pr,G},$$

a contradiction. Now, if

$$X \nsubseteq LE,$$

then

$$L = L(E \cap X)$$

So using 2.3 of [13], $LE \cap X$ is $S$-quasinormal, which is again a contradiction. Thus, $L = L_{pr,G}$. So using 2.3 of [13], $L$ is $S$-quasinormal. Consequently, $X \leq G$ by Lemma 2.11 of [14]. Hence the lemma is proved.

### 3. Proofs of main results

In this section, we give proofs of our main theorems.

**Proof of Theorem 1.4:** The proof of this theorem is given in the following steps:

1. In this step, we show that $\Phi(G) \leq G$ is supersolvable.
   
   Since $N = N \cap G$ and $\Phi(N) = \Phi(N) \cap N \leq \Phi(G) \cap N$, so $N$, $\Phi(N)$ satisfies the theorem, then $N$ is supersolvable.

2. Here, we prove that $\Phi(G) \nsubseteq G$.
   
   Then $\text{Syl}(G) \leq \Phi(G)$ s.t. $Q \leq G$. Suppose $Q_1$ is maximal in $\text{Syl}(G)$. By hypotheses $Q_1$ is $pr$-embedded, then there is $T \unlhd G$ s.t. $G = Q_1T$ and $K = Q_1 \cap T$ is a $pr$-quasinormal subgroup. So $Q_1 \cap T = K \leq G$, by (1) $G/T, G/Q_1$ supersolvable,

   $$G/Q_1 \cap T \cong G/Q_1 \times G/T$$

   then $G/Q_1 \cap T$ be supersolvable. Now $|Q_1/|T|| = 1$, so $|Q_1/|T|| = 1$ and $|G/| = |Q_1T/| = |Q_1/|T|/|Q_1/|T|$. So $Q_1 = G$, which contradicts the hypothesis. Thus $|Q_1/|T|| \neq 1$. Since $Q_1$ is $pr$-quasinormal in $G$, then $K \leq N$ where $N$ is quasinormal.

   Since $K \leq Q \leq O^q(G)$, using Lemma 2.7, $K$ is $S$-quasinormal. Then $\exists G_p$ and $p \neq q$ so $K \unlhd PK \leq G$ and $K \leq \text{Syl}(PK)$. By 2.2 of [15], $K$ is $r$-quasinormal in $PK$. Thus by 2.1 and 2.3 of [15], $K \unlhd PK$, $O^q(G) \leq M_{G}(K)$. So $K = Q_1 \cap T \leq G$. As $Q_1 \leq G$, $PK = P \times G$ and then for every $G_p$, $T_{G_p} = P \neq T$ so $T$ is nilpotent by 5.1.4 of [16]. As $P$ char $T$, then $P \leq G$ and so nilpotent, which contradicts.

3. Here we show that $\Phi(G) = 1$.
   
   Let $\Phi(G) \neq 1$, so by (2) $\Phi(G)/\Phi(G) = \Phi(G)/\Phi(G)$, obviously each Syl of $\Phi(G)/\Phi(G)$ is $pr$-embedded in $G/\Phi(G)$ by our supposition, so $G/\Phi(G)$ is supersolvable, which implies $G$ is supersolvable, which contradicts the hypothesis.

4. Now we prove that $J(G) = \Phi(G)$.
   
   $J(G)$ is the direct product of minimal normal in $G$ by (3) and Lemma 2.5 of [17]. Then $J(G) \leq \Phi(G)$, particularly $J(G) < G$. Consider $J(G) < \Phi(G)$. We have to show $(G/J(G))/(H/J(G)) \cong H/J(H)$ is supersolvable. From 2.2 (b) of [18], every maximal of $\text{Syl}(J(G)/J(H))$ is $pr$-embedded in $G/J(G)$ by lemma 2.2(b) of [18]. Then $G/J(G)$, $\Phi(G)/J(H)$ satisfied and $G/J(G)$ is supersolvable. By (2), $J(G)$ is a proper subgroup of a $\text{Syl}_p(\Phi(G))$, particularly $J(G)$ is maximal in $\text{Syl}(G)$, then $J(G)$ is abelian or cyclic and their order $q$, 4, respectively. Hence supersolvable, which contradicts the hypothesis.

5. Finally, we complete the prove with following lines.
   
   From (4) each maximal of $\text{Syl}(J(G))$ is $pr$-embedded, so from Theorem 4.3 of [10], $G$ is supersolvable.

This completes the proof of Theorem 1.4.

**Proof of Theorem 1.5:** The proof of this theorem is given in the following steps:

1. Using Lemma 2.6, $|Q| \geq q^{n+1}$, thus every $n$-maximal subgroup $Q_n$ of $Q$ satisfies $Q_n \neq 1$.

2. Now we prove that $G$ is not simple.
   
   By hypothesis, $Q_n$ is $pr$-embedded. Using the definition of the $pr$-embedded subgroup and $K \unlhd G$ s.t. $Q_nK \leq G$, $Q_n \cap K \leq (Q_n)_{pr,G}$. Let $G$ be simple. If $K = 1$, then $1 \neq Q_nK = Q_n \leq G$, which contradicts the hypothesis. If $K = G$, then $1 < Q_n \cap K = Q_n \leq (Q_n)_{pr,G}$. We can write

   $$(Q_n)_{pr,G} = \langle V | V \text{ is a nontrivial } pr$$

   $$(\text{quasinormal of } G \text{ in } Q_n) \rangle$$

   Let $V$ be an arbitrary nontrivial $pr$-quasinormal subgroup $\leq Q_n$. Then $T \leq G$ be $S$-quasinormal subgroup such that $V$ be $\text{Syl}(T)$. As $G$ be a simple group, we have $T_G = 1$. By Lemma 2.1, $V$ is $S$-quasinormal. From the arbitrariness of $V$ and Lemma 2.1 of [2], $Q_n$ is $S$-quasinormal, so $Q_n = 1$, in contrary to (1).

3. Now we prove that $M \leq G$, where $M$ is unique and minimal.
   
   $\Phi(G)$ is equal to 1. Since $G/M$ satisfies which shows that $QM/M$ is a $\text{Syl}(G/M)$. By Lemma 2.6, we may take $|QM/M| \geq q^{n+1}$. Let $N/M$ be $n$-maximal of $QM/M$. So $N/M = N/M = (N/M)(M) \neq Q_n/M$. Obviously, $Q_n$ is an $n$-maximal subgroup of $Q$. According to supposition, $Q_n$ is $pr$-embedded. Therefore, there is $K \leq G$ such that $Q_nK \leq G$ and $Q_n \cap K \leq (Q_n)_{pr,G}$. Furthermore, we can see that $KM/M$ is normal in $G/M$, $M/q/M$. $KM/M = Q_nM/M$. $KM/M = Q_nKMK/M \leq G/M$. If $M \cap Q_nK = 1$, then $M \cap Q_n = M \cap K = 1$, $M \cap Q_nK = (M \cap Q_nK)(M \cap K)$. If $M \cap Q_nK \neq 1$, then $M \leq Q_nK$. Since $Q_n \cap M = Q_n \cap M = Q_n \cap M = Q_n \cap M = M$
M ∩ Q_n K. By Lemma 2.8, Q_n M ∩ K M = (Q_n ∩ K) M, and thus Q_n M/M ∩ K M/M = (Q_n M ∩ K M)/M = (Q_n ∩ K)/M. Hence N_p M/M ∩ K M/M = Q_n M/M ∩ K M/M = (Q_n M M ∩ (Q_m)_{pr} M/M ≤ (Q_m M)_{pr} (G/M)) by Lemma 2.4. Thus N_p M/M is pr-embedded in G/M. Then, the factor group G/M satisfies the hypothesis. It yields that G/M is q-nilpotent. As a consequence, the uniqueness of M and Φ(G) = 1 are clear.

(4) Now we prove that Q_n(G) = 1.

If L = Q_1(G) is not equal to 1, then Q/L is a Sylq(L) group. Let K/L ≤ Q/L. Then K = Q/L for some Q ≤ L. Then Q ≤ L by Lemma 2.5 that Q_1/L is pr-embedded in G/L. Besides, M_{G/L}(Q/L) = M_{G}(Q/L) ≤ (Q_n M)_{pr} (G/M) (see [11, Lemma 3.6.10]) is q-nilpotent. As a result, G/O_q(G) satisfied. It follows that G/L is q-nilpotent and so is G, which contradicts the hypothesis.

(5) Now we prove that Q_n(G) = 1.

If Q_n(G) is not equal to 1, according to step (3) M ≤ O_q(G), there is N ≤ G s.t. G = MN, M ∩ N = 1. Since Q_n(G) ∩ N is normalized by M and N, thus M yields M = Q_n(G), Q = Q ∩ N = M(Q ∩ N). As Q ∩ N < Q, then Q_1 ≤ Q is a maximal subgroup which contains Q ∩ N, and hence Q = M Q_1. Pick a maximal Q of Q contained in Q. From hypothesis K ≤ G s.t. Q_n K ≤ G and Q_n ∩ K ≤ (Q_n Q_1)_{pr}(G). Let V be a nontrivial pr-quasinormal ≤ Q_n. Then τ-quasinormal subgroup T ≤ G s.t. V is τ_q. If τ_q ≤ 1, then M ≤ τ_q ≤ T, so M ≤ V ≤ (Q_n Q_1)_{pr}(G) ≤ Q_n ≤ Q_1. Consequently, Q = M Q_1 = Q_1, which contradicts the hypothesis. Thus we have τ_q = 1. Furthermore, using Lemma 2.2, V is τ-quasinormal. From the arbitrariness of V and Lemma 2.1 of [2], (Q_n Q_1)_{pr}(G) is τ-quasinormal. By Lemma 2.3 of this paper and Lemma 2.1 of [2], Q_n(G) ≤ M_{G}(Q_n Q_1)_{pr}(G) and (Q_n Q_1)_{pr}(G) is subnormal in G. By Lemma 2.7, we have Q_n ∩ K ≤ (Q_n Q_1)_{pr}(G) ≤ Q_n(G) = M, so Q_n ∩ K ≤ (Q_n Q_1)_{pr}(G) ≤ Q_n ∩ M. Furthermore, Q_n ∩ K ≤ (Q_n Q_1)_{pr}(G) ≤ Q_n ∩ M ≤ (Q_n Q_1)_{pr}(G) ≤ (Q_n Q_1)_{pr}(G) ≤ Q_n ∩ M ≤ M. It follows that (Q_n Q_1)_{pr}(G) = Q_n ∩ M = M or (Q_n Q_1)_{pr}(G) = 1. If (Q_n Q_1)_{pr}(G) = Q_n ∩ M = M, then M ≤ Q_1, which contradicts the hypothesis. If (Q_n Q_1)_{pr}(G) = 1, then Q_n ∩ K = 1 and so Q_n ≤ Q_1. So K is q-nilpotent from Lemma 2.6. Suppose K_qr is normal q-complement of K, then K_qr ≤ G, we get K_qr = 1 by step (4), and thus there is q-subgroup K ⊆ G and Q ≤ K ≤ Q_n K ≤ O_q(G) = M. If K ⊆ G, we get K = Q_n K = M, so Q_n ≤ K, namely, Q_n ∩ K = Q_n = 1, which contradicts the hypothesis. If K = 1, then Q_n ≤ G, so M ≤ Q_n ≤ Q_1, which contradicts the hypothesis. Now it is clear that (5) holds.

(6) Now we complete the proof with the following lines:

If M ∩ Q ≤ Φ(Q), then M is q-nilpotent by Tate’s Theorem [19, IV,4.7]. Therefore, M_qr ≤ G. So M_qr ≤ Q_n(G) = 1. Moreover, M be q-group, then M ≤ O_q(G) = 1, which contradicts the hypothesis. As a result, there is a maximal Q_1 ≤ Q, s.t. Q = (Q ∩ M)Q_1. Take Q_n ≤ Q contained in Q_1. By the hypothesis, K is normal in G s.t. Q_n K ≤ G, Q_n K ≤ (Q_m M)_{pr}. Let V be a nontrivial pr-quasinormal contained in Q_n. So τ-quasinormal T ≤ G, then V be Sylq(T). If T_q ≠ 1, then M ≤ T_q ≤ T, so V ∩ M is a Sylq(M). We know V ∩ M ≤ Q_1 ∩ M ≤ Q ∩ M, Q ∩ M be Sylq(M), so V ∩ M = Q_1 ∩ M = Q ∩ M. Consequently, Q = (Q ∩ M)Q_1 = (Q_n ∩ M)Q_1 = Q_1, which contradicts the hypothesis. Hence T_q = 1, V is τ-quasinormal from Lemma 2.1. By Lemma 2.1 of [2] and arbitrariness of V, (Q_m M)_{pr} is S-quasinormal, and (Q_m M)_{pr} is subnormal using Lemma 2.1 of [2]. By Lemma 2.7 that (Q_m M)_{pr} ≤ O_q(G) = 1, so |K_qr| ≤ q_n, therefore K is q-nilpotent. Similarly, we have K_qr = 1 and so K = 1. It deduces that Q_n ≤ G, M ≤ Q_n ≤ Q_1.

This completes the proof of Theorem 1.5.

Proof of Theorem 1.6: Here we will prove the theorem by obtaining a contradiction. The proof follows in the following steps:

(1) First, we prove that K is q-nilpotent.

Let Q_1 be the largest subgroup of Q. Q_1 has a q-supersolvable supplement X ∩ K in K provided Q_1 has a q-supersolvable supplement X. Because (|K|, q − 1) = 1, this implies X ∩ K is q-nilpotent from Lemma 2.1(1). If Q_1 is pr-embedded in K, so Q_1 is also pr-embedded in K from 2.6(1) of [13]. Also, Q_1 does not have any q-nilpotent supplement in K. So by theorem 1.5 of [13], K is q-nilpotent.

(2) Now we prove that Q = K.

Using step (1), Q_n(K) is the normal Hall q-subgroup of K.

Let Q_n(K) ≠ 1. We can check it easily that our theorem is true for (G/O_q(K, K/O_q(K)). Using mathematical induction we can see G/O_q(K) be the chief factor, between 1 and K/O_q(K) is cyclic. Following each factor between K and O_q(K) is cyclic, which implies O_q(K) = 1. Hence Q = K.

(3) In this step, we prove that Φ(Q) = 1.

First, let Φ(Q) ≠ 1. Then in the light of Lemma 2.3(2), we can check easily that our theorem holds for (G/Φ(Q), Q/Φ(Q)). Every chief factor of G/Φ(Q) under Q/Φ(Q) is cyclic by our selection of (G,K). Hence cyclic by Lemma 2.12, which contradicts the hypothesis.

(4) Here we prove that every largest subgroup of Q is pr-embedded.

Let us have some largest Q_i subgroup contained in Q in such a way that T is the q-supersolvable supplement of Q_1 in G, thus QT = G with Q ∩ T ≠ 1. Because Q ∩ T ≤ T, we may suppose that
Q ∩ T contains a smallest normal subgroup L of T. Here obviously |L| = q. Since Q is elementary abelian and G = QT, this implies L ⊆ G. Here we can check that our theorem holds yet for (G/L, Q/L).

By our selection of (G, K) we can see every chief factor of G/L under Q/L is cyclic. As a consequence, every chief factor of G under Q is cyclic, which is a contradiction, hence (4) holds.

(5) Now we find the smallest normal subgroup.

Let Q ⊆ G, so using Lemma 2.13, G contains some largest normal subgroup of Q, which can’t be true because Q is of smallest order.

(6) Let L ⊆ Q of G, then Q/L ≤ N_G(L/G) and |Y| > q. Moreover, using Lemma 2.3(2) of [13], our theorem satisfies (G/L, Q/L). Thus from our selection of (G, K) = (G, Q), every chief factor of G/L under Q/L is cyclic. If |L| = q, then cyclic, which contradiction of our supposition. Now if every Q contains two smallest normal subgroups R and L of G, then LR/R ⊆ Q/R and from the isomorphism L/R ∼= L, it follows that |L| = q, a contradiction again. Thus, step (6) is true.

(7) Finally, we prove the contradiction.

Suppose that L ⊆ Q of G and L1 the largest subgroup of L. To show L1 is S-quasinormal, we can suppose that B is a complement of L in Q, as Q is an elementary abelian q-group. Also take W = L1B. Clearly, W is a largest subgroup of G. Using step (4), W is pr-embedded in G. So using Lemma 2.6(4) of [13], there will be R ⊆ G satisfying the condition, W ∩ R ≤ W_{prG}, WR ≤ Q and WR is S-quasinormal.

From Lemma 2.3 of [13], W_{prG} is S-quasinormal. Now if R = Q, so W = W_{prG} is S-quasinormal. By 2.1(5) of [2],

\[ W ∩ L = L_1C ∩ L = L_1(K ∩ L) = Y_1 \]

is S-permutable. If R = 1, this gives W = WR is S-quasinormal. As a result, L1 is S-quasinormal. Consider 1 < R < Q. Implies L ≤ R by step (6). So by using Lemma 2.1(5) of [2],

\[ L_1 = W ∩ L = W_{prG} ∩ L \]

is S-quasinormal. This implies |L| = q, which contradicts step (6).

4. Conclusions

In this paper, we check the supersolvability and nilpotency of pr-embedded subgroups. We proved that if H ⊆ G such that G/H is supersolvable and every maximal subgroup of any Syl of \( \Phi(H) \) is pr-embedded in G, then G is supersolvable. Further we proved that if Q be a Gq, where q is a prime divisor of |G| with \( (q, \frac{|G|(q^2 - 1)}{(q - 1)(q - 2)}) = 1 \) and every maximal subgroup of Q and every n-maximal subgroup of Q is pr-embedded in G, then G is q-nilpotent. At last, we prove that if M ⊆ G and Q be a Sylow q-subgroup of M such that \( (|M|, q - 1) = 1 \) provided q is a prime divisor of |M| and every largest subgroup Q1 of Q is pr-embedded in G such that Q1 does not have a q-supersolvable supplement in G, then each chief factor of G between M and Q2(M) is cyclic. Our results are the extension of existing results.

Disclosure statement

No potential conflict of interest was reported by the authors.

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