SPHERICAL DENSITY OF HYPERBOLIC METRIC
AND UNIFORM PERFECTNESS

TOSHIYUKI SUGAWA

Dedicated to Professor Matti Vuorinen
on the occasion of his sixty-fifth birthday

Abstract. It is well known that a hyperbolic domain in the complex plane has uniformly perfect boundary precisely when the product of its hyperbolic density and the distance function to its boundary has a positive lower bound. We extend this characterization to a hyperbolic domain in the Riemann sphere in terms of the spherical metric.

1. Introduction and main result

Let $\Omega$ be a domain in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with at least three points in its boundary $\partial \Omega \subset \hat{\mathbb{C}}$. Then, it is well known that $\Omega$ carries the hyperbolic metric $\lambda_\Omega = \lambda_\Omega(z)|dz|$, which is a complete conformal metric of constant Gaussian curvature $-4$. Such a domain is thus called hyperbolic. For instance, the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ has the hyperbolic metric of the form

$$\lambda_D(z) = \frac{1}{1-|z|^2}.$$ 

In what follows, we consider only hyperbolic domains unless otherwise stated. The hyperbolic metric $\lambda_\Omega$ can be characterized by the relation

$$\lambda_\Omega(z) = \lambda_\Omega(p(z))|p'(z)|, \quad z \in \Omega,$$

where $p : D \rightarrow \Omega$ is an analytic universal covering projection.

As general references for the hyperbolic metric and related topics, the reader may consult [6], [1], and [2]. We remark that the hyperbolic metric often refers to $2\lambda_\Omega$, which is of constant curvature $-1$. The reader should check its definition first when referring to other papers or books on the hyperbolic metric.

2010 Mathematics Subject Classification. Primary 30F45; Secondary 30C80, 51M10.

Key words and phrases. hyperbolic metric, uniformly perfect, spherical metric.

The author was supported in part by JSPS Grant-in-Aid for Scientific Research (B) 22340025.
We denote by $d_{\Omega}(z)$ the Euclidean distance from $z \in \Omega$ to the boundary $\partial \Omega$; namely,
$$d_{\Omega}(z) = \min_{a \in \partial \Omega} |z - a|.$$ 
As is easily seen, the inequality $d_{\Omega}(z)\lambda_{\Omega}(z) \leq 1$ holds for each $z \in \Omega \setminus \{\infty\}$. Moreover, if $\Omega$ is simply connected and if $\Omega \subset \mathbb{C}$, the Koebe one-quarter theorem implies the opposite inequality $d_{\Omega}(z)\lambda_{\Omega}(z) \geq 1/4$. In general, however, $d_{\Omega}(z)\lambda_{\Omega}(z)$ can be arbitrarily small. Indeed, positivity of the quantity
$$C(\Omega) = \inf_{z \in \Omega} d_{\Omega}(z)\lambda_{\Omega}(z)$$
gives the domain $\Omega$ a strong geometric constraint.

**Theorem 1.1** (Beardon and Pommerenke [3]). Let $\Omega$ be a hyperbolic domain in $\mathbb{C}$. Then $C(\Omega) > 0$ if and only if $\partial \Omega$ is uniformly perfect.

Here, a compact subset $E$ of $\hat{\mathbb{C}}$ containing at least two points is said to be uniformly perfect if there exists a constant $k \in (0, 1)$ such that $\{z \in E : kr < |z - a| < r\} \neq \emptyset$ for every $a \in E \setminus \{\infty\}$ and $0 < r < d(E)$, where $d(E)$ denotes the Euclidean diameter of $E$. Note that $d(E) = +\infty$ whenever $\infty \in E$. There are many other characterizations of uniformly perfect sets. See [11], [12], [13] and [14] in addition to [6] and [1].

In the above theorem, the assumption $\Omega \subset \mathbb{C}$ is essential. Indeed, let us consider the domain $\Delta_R = \{z \in \hat{\mathbb{C}} : |z| > R\}$ containing $\infty$. Then, the hyperbolic metric of it is expressed by
$$\lambda_{\Delta_R}(z) = \frac{R}{|z|^2 - R^2}.$$ 
Thus,
$$d_{\Delta_R}(z)\lambda_{\Delta_R}(z) = \frac{R}{|z| + R} \to 0 \quad (z \to \infty).$$ 
This phenomenon may be explained by the fact that $\Delta_R$ and $\Delta_R \setminus \{\infty\}$ cannot be distinguished merely by the distance function $d_{\Omega}(z)$.

It is therefore desirable to have a similar characterization of the uniform perfectness which is valid for domains in $\hat{\mathbb{C}}$. To this end, it is natural to employ the spherical distance instead of the Euclidean one.

We recall that the spherical (chordal) distance is defined by
$$\sigma(z, w) = \frac{|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}$$
for $z, w \in \mathbb{C}$ and $\sigma(z, \infty) = 1/\sqrt{1 + |z|^2}$ for $z \in \mathbb{C}$. Note that $0 \leq \sigma(z, w) \leq 1$. The corresponding infinitesimal form is given by
$$\sigma(z)|dz| = \frac{|dz|}{1 + |z|^2}.$$
which is known as the spherical metric and has constant Gaussian curvature $+4$. It is also convenient to use the quantity

$$\tau(z, w) = \frac{|z - w|}{1 + z\overline{w}},$$

which can also be thought of as a spherical counterpart of the Euclidean distance, although $\tau$ is not a distance function on $\hat{\mathbb{C}}$. We then consider the distances to the boundary

$$\delta_\Omega(z) = \min_{a \in \partial \Omega} \sigma(z, a) \quad \text{and} \quad \varepsilon_\Omega(z) = \min_{a \in \partial \Omega} \tau(z, a)$$

for $z \in \Omega$.

In the context of spherical geometry, it is more natural to consider the spherical density of the hyperbolic metric defined by

$$\mu_\Omega(z) = \frac{\lambda_\Omega(z)|dz|}{\sigma(z)|dz|} = (1 + |z|^2)\lambda_\Omega(z).$$

Minda [9] studied $\mu_\Omega(z)$ in relation with $\varepsilon_\Omega(z)$ and gave several estimates for $\mu_\Omega(z)$. Among others, the following result is relevant to the present paper.

**Theorem 1.2** (Minda [9]). Let $\Omega$ be a hyperbolic domain in $\hat{\mathbb{C}}$. For each $z \in \Omega$, the inequality $\varepsilon_\Omega(z)\mu_\Omega(z) \leq 1$. Moreover, equality holds at $z$ if and only if $\Omega$ is a spherical disk with center $z$.

We define spherical counterparts to $C(\Omega)$ in the following way:

$$\tilde{C}(\Omega) = \inf_{z \in \Omega} \delta_\Omega(z)\mu_\Omega(z) \quad \text{and} \quad \hat{C}(\Omega) = \inf_{z \in \Omega} \varepsilon_\Omega(z)\mu_\Omega(z).$$

**Example 1.3.** We consider the disk $\mathbb{D}_R = \{ z \in \mathbb{C} : |z| < R \}$ for $0 < R < +\infty$. It is immediate to see that $C(\mathbb{D}_R) = 1/2$. On the other hand, we compute $\mu_{\mathbb{D}_R}(z) = R(1 + |z|^2)/(R^2 - |z|^2)$, $\varepsilon_{\mathbb{D}_R}(z) = \tau(|z|, R) = (R - |z|)/(1 + R|z|)$ and $\delta_{\mathbb{D}_R}(z) = \sigma(|z|, R) = (R - |z|)/\sqrt{(1 + R^2)(1 + |z|^2)}$.

Therefore,

$$\tilde{C}(\mathbb{D}_R) = \inf_{0 < x < R} \frac{R - x}{\sqrt{(1 + R^2)(1 + x^2)}} \cdot \frac{R(1 + x^2)}{R^2 - x^2} = \inf_{0 < x < R} \frac{R\sqrt{1 + x^2}}{(R + x)\sqrt{1 + R^2}}.$$ Since the function $\sqrt{1 + x^2}/(R + x)$ is decreasing in $0 < x < 1/R$ and increasing in $1/R < x$, we obtain

$$\tilde{C}(\mathbb{D}_R) = \begin{cases} 
1/2 & \text{if } R \leq 1, \\
R/(1 + R^2) < 1/2 & \text{if } R > 1.
\end{cases}$$

We also have

$$\varepsilon_{\mathbb{D}_R}(x)\mu_{\mathbb{D}_R}(x) = \frac{R - x}{1 + Rx} \cdot \frac{R(1 + x^2)}{R^2 - x^2} = \frac{R(1 + x^2)}{(1 + Rx)(R + x)}.$$
for $0 < x < R$. Since the function $R(1+x^2)/(1+Rx)(R+x)$ is decreasing in $0 < x < 1$, increasing in $x > 1$, and tends to $1/2$ as $x \to R$, we obtain finally

$$\hat{C}(\mathbb{D}_R) = \begin{cases} 1/2 & \text{if } R \leq 1, \\ 2R/(1 + R)^2 < 1/2 & \text{if } R > 1. \end{cases}$$

The spherical diameter, namely, the diameter with respect to the distance $\sigma$, of a set $E \subset \hat{C}$ will be denoted by $\sigma(E)$. Then we observe that

$$\sigma(\hat{C} \setminus \mathbb{D}_R) = \begin{cases} 1 & \text{if } R \leq 1, \\ \sigma(R, -R) = 2R/(1 + R^2) & \text{if } R > 1. \end{cases}$$

Therefore,

$$\frac{\hat{C}(\mathbb{D}_R)}{\sigma(\hat{C} \setminus \mathbb{D}_R)} = 1/2 \quad \text{and} \quad \frac{1}{2} \leq \frac{\hat{C}(\mathbb{D}_R)}{\sigma(\hat{C} \setminus \mathbb{D}_R)} < 1$$

for any $R > 0$. Note also that the diameter of $\hat{C} \setminus \mathbb{D}_R$ with respect to $\tau$ is $+\infty$ for $R \leq 1$ and $2R/(R^2 - 1)$ for $R > 1$.

In view of the above example, we expect more uniform estimates if we consider the modified quantities

$$\tilde{C}'(\Omega) = \frac{\tilde{C}(\Omega)}{\sigma(\hat{C} \setminus \Omega)} \quad \text{and} \quad \hat{C}'(\Omega) = \frac{\hat{C}(\Omega)}{\sigma(\hat{C} \setminus \Omega)}.$$ 

Since $\delta_\Omega, \varepsilon_\Omega, \mu_\Omega, \sigma(\hat{C} \setminus \Omega)$ are invariant under the spherical isometries (see [9]), so are the quantities $\tilde{C}(\Omega), \tilde{C}'(\Omega), \hat{C}(\Omega)$ and $\hat{C}'(\Omega)$; namely, $\tilde{C}(T(\Omega)) = \tilde{C}(\Omega), \tilde{C}'(T(\Omega)) = \tilde{C}'(\Omega), \hat{C}(T(\Omega)) = \hat{C}(\Omega)$ and $\hat{C}'(T(\Omega)) = \hat{C}'(\Omega)$ for a spherical isometry $T$.

Our main result is now stated as in the following.

**Theorem 1.4 (Main Theorem).** Let $\Omega$ be a hyperbolic domain in $\mathbb{C}$. Then,

(i) $\hat{C}(\Omega) \leq 1/2$.
(ii) $\hat{C}(\Omega) \leq \tilde{C}(\Omega)$ and $\tilde{C}'(\Omega) \leq \hat{C}'(\Omega)$.
(iii) $\tilde{C}(\Omega) \leq 2\hat{C}(\Omega)$.
(iv) $\hat{C}(\Omega) \leq 4\tilde{C}'(\Omega) = 4\tilde{C}(\Omega)/\sigma(\hat{C} \setminus \Omega)$.

As an immediate corollary of the main theorem, we obtain the following characterizations of uniform perfectness of the boundary.

**Corollary 1.5.** Let $\Omega$ be a hyperbolic domain in $\hat{C}$. Then the following conditions are equivalent:

(1) $\partial\Omega$ is uniformly perfect.
(2) $\hat{C}(\Omega) > 0$.
(3) $\hat{C}(\Omega) > 0$. 

...
Harmelin and Minda [5] showed that $C(\Omega) \leq 1/2$ for a hyperbolic domain $\Omega \subset \mathbb{C}$. The above assertion (i) (and thus $C(\Omega) \leq 1/2$) can be regarded as a spherical analog of it. In addition, Mejia and Minda [8] showed that $C(\Omega) \geq 1/2$ if and only if $\Omega$ is convex. Let us mention the following result due to Minda.

**Theorem 1.6 (Minda [10, Theorem 1]).** Let $\Omega$ be a spherically convex domain in $\hat{\mathbb{C}}$ and $z \in \Omega$. Then

$$\mu_\Omega(z) \geq \frac{1 + \varepsilon_\Omega(z)^2}{2\varepsilon_\Omega(z)},$$

where equality holds if and only if $\Omega$ is a hemisphere.

In particular, $\varepsilon_\Omega(z)\mu_\Omega(z) \geq (1 + \varepsilon_\Omega(z)^2)/2 > 1/2$ and hence,

$$\tilde{C}(\Omega) \geq 1/2$$

for a spherically convex domain $\Omega$. This gives a spherical analog to the one direction of the afore-mentioned result. We observe that $\tilde{C}(D_R) = \tilde{C}(\mathbb{D}_R) = 1/2$ for $0 < R \leq 1$ and $\tilde{C}(D_R) < \tilde{C}(\mathbb{D}_R) < 1/2$ for $R \geq 1$ in Example 1.3. Since $\mathbb{D}_R$ is spherically convex if and only if $0 < R \leq 1$, we have some hope that the conditions $\tilde{C}(\Omega) \geq 1/2$ and/or $\tilde{C}(\Omega) \geq 1/2$ would characterize spherical convexity of $\Omega$.

2. Spherical geometry

In this section, we collect necessary information about the spherical geometry to prove our main theorem.

Let Möb be the group of Möbius transformations $z \mapsto (az + b)/(cz + d)$, with $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$. This is nothing but the group of analytic automorphisms of the Riemann sphere (the complex projective line) and is canonically isomorphic to $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm I\}$. Note that the action of Möb on $\hat{\mathbb{C}}$ is not isometric with respect to the spherical metric $\sigma = |dz|/(1 + |z|^2)$. We denote by $\text{Isom}^+(\hat{\mathbb{C}})$ the subgroup of Möb consisting of spherical isometries. It is a standard fact that each isometry $T \in \text{Isom}^+(\hat{\mathbb{C}})$ has either the form

$$T(z) = e^{i\theta} \frac{z - a}{1 + \overline{a}z}$$

for a real constant $\theta$ and a complex number $a \in \mathbb{C}$, or the form $T(z) = -e^{i\theta}/z$ for a real constant $\theta$, in which case we can interpret $a = \infty$. In particular, we can see that $\text{Isom}^+(\hat{\mathbb{C}})$ acts on $\hat{\mathbb{C}}$ transitively. Note that $\tau(z, a) = |T(z)|$ for the above $T$. It is also useful to note the relations

$$\varepsilon_{T(\Omega)}(T(z))\mu_{T(\Omega)}(T(z)) = \varepsilon_\Omega(z)\mu_\Omega(z)$$

and

$$\delta_{T(\Omega)}(T(z))\mu_{T(\Omega)}(T(z)) = \delta_\Omega(z)\mu_\Omega(z),$$
Lemma 2.1. Let $T \in \text{Isom}^+(\mathbb{C})$. Likewise, we also have $\hat{C}(T(\Omega)) = \hat{C}(\Omega)$.

Recall that $0 \leq \sigma(z, w) \leq 1$ and that $z$ and $w$ are called antipodal if $\sigma(z, w) = 1$, which is equivalent to $\tau(z, w) = +\infty$. It is easy to see that $z$ and $w$ are antipodal if and only if $z = -1/\bar{w}$. We write $z^* = -1/\bar{z}$ for the antipodal point of $z$. It should be noted here that $\delta_\Omega(z) < 1$ holds always for a hyperbolic domain $\Omega$.

We have a simple relation between $\sigma$ and $\tau$. Since

$$1 + \tau(z, w)^2 = \frac{|1 + z\bar{w}|^2 + |z - w|^2}{|1 + z\bar{w}|^2} = \frac{(1 + |z|^2)(1 + |w|^2)}{|1 + z\bar{w}|^2} = \frac{\tau(z, w)^2}{\sigma(z, w)^2};$$

we have

$$\sigma(z, w) = \frac{\tau(z, w)}{\sqrt{1 + \tau(z, w)^2}} \quad \text{and} \quad \tau(z, w) = \frac{\sigma(z, w)}{\sqrt{1 - \sigma(z, w)^2}}.$$

In particular, $\sigma(z, w) \leq \tau(z, w)$. We also have the relation $\delta_\Omega(z) = \varepsilon_\Omega(z)/\sqrt{1 + \varepsilon_\Omega(z)^2}$ for a hyperbolic domain $\Omega$.

We now compare $\varepsilon_\Omega(z)$ with $d_\Omega(z)$.

**Lemma 2.1.** Let $\Omega$ be a hyperbolic domain in $\mathbb{C}$ and fix a point $z \in \Omega$. Then, $\varepsilon_\Omega(z)|z| \leq 1$ and

$$\frac{\varepsilon_\Omega(z)(1 + |z|^2)}{1 + \varepsilon_\Omega(z)|z|} \leq d_\Omega(z) \leq \frac{\varepsilon_\Omega(z)(1 + |z|^2)}{1 - \varepsilon_\Omega(z)|z|}.$$

**Proof.** For brevity, set $\varepsilon = \varepsilon_\Omega(z)$ and let $\Delta = \{w \in \mathbb{C} : \tau(w, z) < \varepsilon\}$. Then, by assumption, $\Delta \subset \Omega \subset \mathbb{C}$. Let $T(w) = (z - w)/(1 + \bar{z}w)$. Note that $T^{-1} = T$. Then $\Delta = T^{-1}(\mathbb{D}_z) = T(\mathbb{D}_z)$. Since $\Delta$ does not contain $\infty$, the function $T$ does not have a pole in $\mathbb{D}_z$, which implies $\varepsilon|z| \leq 1$. If $\varepsilon|z| = 1$, $\Delta$ is a half-plane and $T$ has a pole at $z^*$. Note that the image of the diameter $[z^*, -z^*]$ of $\mathbb{D}_z$ under $T$ is a half-line perpendicular to $\partial\Delta$. The Euclidean distance from $z$ to $\partial\Delta$ is thus

$$|T(-z^*) - T(0)| = \frac{|z - z^*|}{2} = \frac{1 + |z|^2}{2|z|} \leq \frac{\varepsilon(1 + |z|^2)}{1 + \varepsilon|z|}.$$

The assertion is now confirmed in this case. We next assume that $\varepsilon|z| < 1$. We then compute

$$\frac{|1 + \bar{z}w|^2(\tau(w, z)^2 - \varepsilon^2)}{1 - \varepsilon^2|z|^2} = \left|w - \frac{(1 + \varepsilon^2)}{1 - \varepsilon^2|z|^2} \right|^2 - \left( \frac{\varepsilon(1 + |z|^2)^2}{1 - \varepsilon^2|z|^2} \right)^2,$$
which means that $\Delta$ is the disk with center $m = (1 + \varepsilon^2)z / (1 - \varepsilon^2|z|^2)$ and radius $r = \varepsilon(1 + |z|^2) / (1 - \varepsilon^2|z|^2)$. Since a point $a$ in $\partial \Delta$ belongs to $\partial \Omega$, we have

$$d_\Omega(z) \leq |z-a| \leq r + |z-m| = \frac{\varepsilon(1 + |z|^2)}{1 - \varepsilon|z|}.$$ 

On the other hand, we obtain

$$d_\Omega(z) \geq d_\Delta(z) = r - |z-m| = \frac{\varepsilon(1 + |z|^2)}{1 + \varepsilon|z|}.$$ 

Thus the proof is complete. \hfill \Box

3. Proof of the main theorem

Before the proof of the main theorem, we prepare a couple of lemmas which will be used later. We will call a map $f : \Omega \to \mathbb{C}$ disk-convex if $f$ maps any disk in $\Omega$ conformally onto a convex domain. Note that any Möbius transformation $T$ is disk-convex on $\Omega$ whenever $T(\Omega) \subset \mathbb{C}$.

Lemma 3.1. Suppose that $f$ maps a hyperbolic domain $\Omega$ in $\mathbb{C}$ conformally onto another hyperbolic domain $\Omega'$ in $\mathbb{C}$. If $f$ is disk-convex, then for each $z \in \Omega$,

$$d_\Omega(z)|f'(z)| \leq 2d_{\Omega'}(f(z)).$$

Proof. Fix $z_0 \in \Omega$ and set $d_0 = d_\Omega(z_0)$. Since $f$ is convex on the disk $\Delta = \{ z : |z-z_0| < d_0 \}$, a covering theorem for convex functions (see [4, Theorem 2.15]) implies that $f(\Delta) \supset \{ w : |w - f(z_0)| < d_0|f'(z_0)|/2 \}$. Thus $d_{\Omega'}(f(z_0)) \geq d_{f(\Delta)}(f(z_0)) \geq d_0|f'(z_0)|/2$. \hfill \Box

Since $\lambda_{\Omega'}(f(z))|f'(z)| = \lambda_\Omega(z)$, we obtain the following.

Corollary 3.2. For a hyperbolic domain $\Omega$ in $\mathbb{C}$ and a disk-convex univalent function $f : \Omega \to \mathbb{C}$,

$$d_\Omega(z)\lambda_\Omega(z) \leq 2d_{f(\Omega)}(f(z))\lambda_{f(\Omega)}(f(z)).$$

In particular, $C(\Omega) \leq 2C(f(\Omega))$.

Remark 3.3. In [5], Harmelin and Minda proved that $C(f(\Omega)) \leq AC(\Omega)$ for a conformal map $f$ with constant $A = \sqrt{1 + 3 \coth^2(\pi/4)} = 2.8241 \ldots$ and conjectured that $A$ can be reduced to 2. Later, Ma and Minda [7] obtained a better bound: $A = \sqrt{1 + 3 \coth^2(\pi/3)} = 2.4335 \ldots$

Proof of the main theorem. We first prove assertion (i). The idea employed in the proof of Harmelin and Minda [5, Theorem 4] works. Fix a point $z_0 \in \Omega$ and set $R = \varepsilon_\Omega(z_0)$. Take a boundary point $a \in \partial \Omega$ such that $R = \tau(z_0,a)$. By a suitable spherical isometry, we may assume that $z_0 = 0$ and $a > 0$ (and hence, $a = R$). Then, $D_R \subset \Omega$ and
thus \( \mu_\Omega \leq \mu_{\mathbb{D}_R} \) on \( \mathbb{D}_R \). Note also that \( \varepsilon_\Omega(x) = \varepsilon_{\mathbb{D}_R}(x) = \sigma(x, R) \) for \( 0 < x < R \). Hence, by Example 1.3,

\[
\tilde{C}(\Omega) \leq \lim_{x \to R^-} \varepsilon_\Omega(x) \mu_\Omega(x) \leq \lim_{x \to R^-} \varepsilon_{\mathbb{D}_R}(x) \mu_{\mathbb{D}_R}(x) \leq \frac{1}{2}.
\]

Assertion (ii) is obvious because \( \delta_\Omega(z) \leq \varepsilon_\Omega(z) \).

We next show assertion (iii). By definition and Lemma 2.1 we observe

\[
d_\Omega(z) \lambda_\Omega(z) \geq \frac{\varepsilon_\Omega(z) (1 + |z|^2)}{1 + \varepsilon_\Omega(|z|)} \cdot \frac{\mu_\Omega(z)}{1 + |z|^2} \geq \frac{\varepsilon_\Omega(z) \mu_\Omega(z)}{2},
\]

from which the inequality \( C(\Omega) \geq \tilde{C}(\Omega)/2 \) follows.

Finally, we show assertion (iv). Fix a point \( z \in \Omega \) and take a point \( a \in \partial \Omega \) such that \( \delta_\Omega(z) = \sigma(z, a) \). Then take a point \( b \in \tilde{\mathbb{C}} \setminus \Omega \) so that \( \max_{w \in \tilde{\mathbb{C}} \setminus \Omega} \sigma(w, a) = \sigma(b, a) \). It is easy to see the inequality

\[
\frac{1}{2} \sigma(\tilde{\mathbb{C}} \setminus \Omega) \leq \sigma(a, b) \leq \sigma(\tilde{\mathbb{C}} \setminus \Omega),
\]

where \( \sigma(\tilde{\mathbb{C}} \setminus \Omega) \) is the spherical diameter of \( \tilde{\mathbb{C}} \setminus \Omega \). Let \( T \in \text{Isom}^+(\tilde{\mathbb{C}}) \) such that \( T(b) = \infty \). Then \( a' = T(a) \neq \infty \) and \( \sigma(a, b) = \sigma(a', \infty) = 1/\sqrt{1 + |a'|^2} \). Set \( \Omega' = T(\Omega) \) and \( z' = T(z) \). Note here that \( \delta_{\Omega'}(z') = \sigma(z', a') \). Then, by the above observations and Corollary 5.2 we have

\[
\delta_\Omega(z) \mu_\Omega(z) = \delta_{\Omega'}(z') \mu_{\Omega'}(z') = \frac{\sqrt{1 + |z'|^2}}{\sqrt{1 + |a'|^2}} \cdot z' - a' |\lambda_{\Omega'}(z')|
\geq \frac{\sigma(\tilde{\mathbb{C}} \setminus \Omega)}{2} \left( \frac{1 + |z'|^2}{2} \right) d_{\Omega'}(z') \lambda_{\Omega'}(z')
\geq \frac{\sigma(\tilde{\mathbb{C}} \setminus \Omega)}{2} \cdot \frac{d_\Omega(z) \lambda_\Omega(z)}{2}.
\]

Hence, we obtain the inequality \( \tilde{C}'(\Omega) \geq C(\Omega)/4 \). \( \square \)

Acknowledgement. The author would like to express his sincere thanks to the referee for careful reading and corrections.

References

1. F. G. Avkhadiev and K.-J. Wirths, Schwarz-Pick Type Inequalities, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2009.
2. A. F. Beardon and D. Minda, The hyperbolic metric and geometric function theory, Proceedings of the International Workshop on Quasiconformal Mappings and their Applications (IWQCMA05) (India) (S. Ponnusamy, T. Sugawa, and M. Vuorinen, eds.), Narosa Publishing House, 2007, pp. 9–56.
3. A. F. Beardon and Ch. Pommerenke, The Poincaré metric of plane domains, J. London Math. Soc. (2) 18 (1978), 475–483.
4. P. L. Duren, Univalent Functions, Springer-Verlag, 1983.
5. R. Harmelin and D. Minda, *Quasi-invariant domain constants*, Israel J. Math. **77** (1992), 115–127.
6. L. Keen and N. Lakic, *Hyperbolic Geometry from a Local Viewpoint*, Cambridge University Press, Cambridge, 2007.
7. W. Ma and D. Minda, *Behavior of domain constants under conformal mappings*, Israel J. Math. **91** (1995), 157–171.
8. D. Mejia and D. Minda, *Hyperbolic geometry in k-convex regions*, Pacific J. Math. **141** (1990), 333–354.
9. D. Minda, *Estimates for the hyperbolic metric*, Kodai Math. J. **8** (1985), 249–258.
10. ______, *The hyperbolic metric and Bloch constants for spherically convex regions*, Complex Var. **5** (1986), 127–140.
11. Ch. Pommerenke, *Uniformly perfect sets and the Poincaré metric*, Arch. Math. **32** (1979), 192–199.
12. ______, *On uniformly perfect sets and Fuchsian groups*, Analysis **4** (1984), 299–321.
13. T. Sugawa, *Various domain constants related to uniform perfectness*, Complex Variables Theory Appl. **36** (1998), 311–345.
14. ______, *Uniformly perfect sets: analytic and geometric aspects* (Japanese), Sugaku **53** (2001), 387–402, English translation in Sugaku Expo. **16** (2003), 225–242.

*E-mail address: sugawa@math.is.tohoku.ac.jp*

**Graduate School of Information Sciences**
**Tohoku University**
**Sendai 980-8579**
**JAPAN**