ON THE CONVERGENCE OF NON-INTEGER LINEAR HOPF FLOW

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Abstract. The evolution of a rotationally symmetric surface by a linear combination of its radii of curvature equation is considered. It is known that if the coefficients form certain integer ratios the flow is smooth and can be integrated explicitly. In this paper the non-integer case is considered for certain values of the coefficients and with mild analytic restrictions on the initial surface.

We prove that if the focal points at the north and south poles on the initial surface coincide, the flow converges to a round sphere. Otherwise the flow converges to a non-round Hopf sphere. Conditions on the fall-off of the astigmatism at the poles of the initial surface are also given that ensure the convergence of the flow.

The proof uses the spectral theory of singular Sturm-Liouville operators to construct an eigenbasis for an appropriate space in which the evolution is shown to converge.

1. Introduction and Results

Given a $C^2$-smooth surface $S_0 \subset \mathbb{R}^3$, consider the curvature flow problem of finding $\bar{X} : S^2 \times [0, \infty) \to \mathbb{R}^3$ with

\[
(\frac{\partial \bar{X}}{\partial t})^\perp = (ar_1 + br_2 + c) \hat{n},
\]

with $\bar{X}(S^2, 0) = S_0$,

where $a, b, c \in \mathbb{R}$, $\bar{X}$ and $\hat{n}$ are the position and normal vectors of the evolving surface, and $r_1, r_2$ are the radii of curvature of the surface. In this paper we investigate the time evolution of a strictly convex topological 2-sphere with an axis of rotational symmetry under the above linear Hopf flow [7]. The flow is parabolic when $b > 0$.

The literature on extrinsic curvature flows primarily concerns flows whose normal speed is a symmetric function in the radii of curvature [1], for which flows by mean curvature, inverse mean curvature and powers of Gauss curvature are examples [3] [4] [11]. Symmetry is necessary for the normal speed to be well defined on general surfaces [2] as the radii of curvature may be exchanged via re-parametrisation.

The current work differs in that linear Hopf flow is not a symmetric curvature flow. Rotationally symmetric surfaces support a canonical labelling of the radii of curvature associated to the meridian and profile principal foliations of the surface, therefore are good surfaces with which to explore asymmetric curvature flows. Furthermore, rotational symmetry reduces the problem to one spatial variable. This
extra tractability allows for a wider class of curvature flows to be considered e.g. \[17\] \[18\].

In \[7\] the linear Hopf flow was completely solved when the flow slope $-a/b$ takes one of the values

\[(1.2) \quad -a/b = 2n + 3,\]

with $n \in \mathbb{N}$. It was proven that the fate of an initial smooth sphere is entirely determined by the local geometry of its isolated umbilic points, in particular the order of vanishing of the difference between the radii of curvature at the poles:

$s = r_2 - r_1$.

In what follows the focus is on non-integer linear Hopf flow, i.e. the case in which $n$ as appears in equation (1.2) is non integer. In contrast to the integer case, non-round, smooth stationary solutions to the non-integer linear Hopf flow do not exist, and so the analysis is significantly complicated.

To state the main results, let $\theta$ be the angle between the normal of $S$ at a point and the axis of rotational symmetry and denote the second order Legendre differential operator for $\mu, \nu \in \mathbb{R}$ by $L_{\mu\nu}$ (given by equation (3.6) below). Let $L^2_{\sin \theta}(0, \pi)$ be the space of square integrable functions with weight $\sin \theta$ on the interval $(0, \pi)$.

**Theorem 1.1.** Consider the linear Hopf flow (1.1) and (1.2) for $n \in (-1, 1)$. Let $S_0$ be a $C^4$-smooth strictly convex rotationally symmetric initial sphere. Assume that $S_0$ satisfies the following conditions

\[(1) \quad L^m_n \left( \frac{s_0(\theta)}{\sin^{m-n} \theta} \right) \in L^2_{\sin \theta}(0, \pi),\]

\[(2) \quad n \cdot \lim_{\theta \to 0} \left( \frac{s_0(\theta)}{\sin^2 \theta} \right) = n \cdot \lim_{\theta \to \pi} \left( \frac{s_0(\theta)}{\sin^2 \theta} \right) = 0.\]

If the focal points of $S_0$ at the north and south poles coincide, the flow converges to a round sphere of radius $\frac{\sqrt{a \pm b}}{a+b}$.

Otherwise the flow converges to a non-round Hopf sphere with astigmatism at the equator given by the signed distance between the focal points at the poles of $S_0$.

If $S_0$ has isolated umbilics points at the north or south pole, a sufficient condition to imply (2) can be given in terms of the surfaces umbilic slopes, which quantify the rate at which $S_0$ becomes umbilic (see Section 2 for details).

\[(2^*) \quad At each pole of $S_0$, the umbilic slope is greater than 3.\]

This is a corollary of Proposition 2.4. It is not immediately obvious which classes of surfaces satisfy condition (1). However a sufficient condition can be given in terms of the asymptotic fall-off of $s$ and its derivatives as $\theta \to 0, \pi$.

\[(3) \quad s_0^{(i)} \sim c_i \sin^{m-i} \theta \quad \text{at} \quad \theta = 0, \pi \quad \text{for} \quad m > n+3, \quad i = 0, 1, 2 \quad \text{with} \quad \text{possibly different constants} \quad c_i \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \pi.\]

Condition (3) is in fact enough for convergence as it implies both conditions (1) and (2). Theorem 1.1 will be a consequence of the following more technical theorem.
Theorem 1.2. Let \( S_0 \) be a \( C^2 \)-smooth strictly convex rotationally symmetric 2-sphere. If the astigmatism \( s \) of \( S_0 \) satisfies both of the following conditions

(I) \[ \frac{s}{\sin^{n+2} \theta} \in L^2_{\sin \theta}(0, \pi), \quad \frac{ds}{d\theta} \in AC_{loc}(0, \pi), \quad \mathcal{L}^n_n \left( \frac{s}{\sin^{n+2} \theta} \right) \in L^2_{\sin \theta}(0, \pi), \]

(II) \[ \lim_{\theta \to 0, \pi} \left( \frac{\sin^{2n+1} \theta}{\sin^{2n+2} \theta} \frac{d}{d\theta} \left( \frac{s}{\sin^{n+2} \theta} \right) \right) = 0, \]

then the conclusion of Theorem 1.1 holds.

Theorem 1.2 has a weaker differentiability requirement of the initial surface than Theorem 1.1. Indeed, (I) implies \( S_0 \) is \( C^3 \)-smooth everywhere apart from possibly at the north and south umbilic points, and \( C^4 \)-smooth almost everywhere. Under the additional assumption that \( S_0 \) is \( C^4 \)-smooth, conditions (I) and (II) are implied by (1) and (2) respectively and Theorem 1.1 follows.

Our method of proof involves a spectral expansion in an eigenbasis of the non-integer Legendre operator, defined on the weighted Hilbert space \( L^2_{\sin \theta}(0, \pi) \). The eigenbasis is adapted to contain the stationary solutions of the linear Hopf flow. The restriction \( n \in (-1, 1) \), is needed to guarantee such an eigenbasis expansion is possible.

Section 2 fixes the notation used to describe geometrical quantities and derives some consequences of rotationally symmetry. Section 3 discusses the geometrical properties of possible stationary solutions of the linear Hopf flow and describes the evolution of important geometric quantities under the flow. Critical to this description is the second order Legendre differential operator \( \mathcal{L}^n_n \).

Section 4 reviews the theory of singular Sturm-Liouville operators, which we require to prove the existence of the adapted eigenbasis. In particular the LC property of a singular Sturm-Liouville operator and boundary conditions required to generate possible self-adjoint domains are discussed. An application of the Spectral Theorem 4.5 for LC operators then guarantees the existence of a complete orthonormal basis of eigenfunctions of such operators.

This is then applied in Section 5 where the main results are proven. This done by showing that \( \mathcal{L}^n_n \) is LC for \( n \in (-1, 1) \) and by finding boundary conditions (namely conditions (I) and (II) of Theorem 1.2) to define a self-adjoint domain for \( \mathcal{L}^n_n \), which is appropriate for the flow.

The eigenbasis is then given explicitly in terms of Legendre functions and the various geometric quantities are similarly expressed. Finally the evolution is solved in terms of the eigenbasis as an expansion which decays exponentially in time to the stationary solution of the flow.

2. Rotationally Symmetric Surfaces

The class of surfaces we concern ourselves within this work are elements of the set

\[ \mathcal{W} = \left\{ S \subset \mathbb{R}^3 : S \text{ is an embedded } C^2 \text{-smooth topological 2-sphere which is rotationally symmetric and strictly convex.} \right\}. \]
The terminology “topological sphere” which we will abbreviate as just “sphere” is taken to mean a closed surface of genus 0.

A given \( S \in \mathcal{W} \) will be orientated with outwards pointing normal \( \hat{n} \) in a right handed coordinate system \( \vec{X} = (x^1, x^2, x^3) \). Align the \( x^3 \)-axis with the axis of rotational symmetry of \( S \). \( S \) is parametrized by pushing forward the standard polar coordinates \((\theta, \phi)\) of \( S^2 \) onto \( S \) by the inverse of the Gauss map \( N^{-1} : S^2 \to S \). Hence \( \theta \in [0, \pi] \) measures the angle made between the normal vector \( \hat{n} \) of \( S \), and the axis of rotational symmetry whereas \( \phi \in [0, 2\pi] \) measures the angle made by a clockwise rotation from the \( x^2x^3 \)-plane.

As a consequence of rotational symmetry many quantities \( f : S \to \mathbb{R} \) are independent of \( \phi \), in which case we write for short-hand \( f(\theta) \) instead of \( f(\vec{X}(\theta, \phi)) \). In particular the radii of curvature of \( S \), \( r_1 \) and \( r_2 \), associated to the meridian and profile principal foliations respectively, are functions of \( \theta \) only. Of critical importance to our study is the astigmatism of \( S \) given by

\[
s = r_2 - r_1.
\]

If \( S \) is assumed strictly convex then the radii are \( C^m \)-smooth functions of the parameter \( \theta \) whenever \( S \) is \( C^{m+2} \)-smooth. Furthermore \( s \) vanishes at, and only at, the umbilic points of \( S \). In particular \( s(0) = s(\pi) = 0 \) and \( s \equiv 0 \) if and only if \( S \) is a round sphere.

Our description of \( S \) in \( \mathbb{R}^3 \) will be facilitated by the support function \( r : S \to \mathbb{R} \) defined by \( r = \vec{X} \cdot \hat{n} \). In the present setting \( r \) is a function of \( \theta \) only and \( \vec{X}(\theta, \phi) \) can be recovered from \( r \) and its derivatives via:

\[
x^1 + ix^2 = (\sin \phi + i \cos \phi) \left( r \sin \theta + \frac{dr}{d\theta} \cos \theta \right), \quad x^3 = r \cos \theta - \frac{dr}{d\theta} \sin \theta.
\]

**Remark 2.1.** For later convenience, note that the focal points \([9]\) of \( S \) at \( \theta = 0 \) and \( \pi \) lie on the axis of rotational symmetry with \( x^3 \) coordinates

\[
f_0 = r(0) - r_1(0), \quad f_\pi = -r(\pi) + r_1(\pi).
\]

This can be deduced by the above equation for \( \vec{X} \) and the equations of the focal sheets \( \vec{F}_i = \vec{X} - r_i \hat{n}, i = 1, 2 \).

We collect together some useful relationships between the above quantities on rotationally symmetric surfaces.

**Proposition 2.2.** The following relationships hold

\[
(2.1) \quad r_1 = \frac{\cos^2 \theta}{\sin \theta} \frac{d}{d\theta} \left( \frac{r}{\cos \theta} \right), \quad r_2 = \frac{d^2 r}{d\theta^2} + r,
\]

\[
(2.2) \quad r = C_2 \cos \theta + C_1 + \int \sin \theta \left[ \int \frac{s}{\sin \theta} d\theta \right] d\theta,
\]

for constants \( C_1, C_2 \). If in addition surface is \( C^3 \), the derived Codazzi-Mainardi equation holds:

\[
(2.3) \quad \frac{dr_1}{d\theta} = (r_2 - r_1) \cot \theta.
\]

**Proof.** The derivation of equation \((2.1)\) can be found in \([6]\) or derived from the definition of the support function. Equations \((2.2)\) and \((2.3)\) are derived by integrating or differentiating equation \((2.1)\) respectively, with respect to \( \theta \). \( \square \)
Remark 2.3. Given \( S \in \mathcal{W} \) with support function \( r \), the transformation \( r \mapsto r + C_1 \) translates \( S \) at each point along its normal line by a distance \( C_1 \), i.e. \( S \) moves to a parallel surface. The transformation \( r \mapsto r + C_2 \cos \theta \) translates the entire surface a distance of \( C_2 \) along the \( x^3 \)-axis. Therefore \( s \) determines the oriented, affine normal lines of the surface. See [10] for further details.

2.1. The Slope at an Isolated Umbilic.

If \( S \) possesses isolated umbilic points at its north and south poles, i.e. at \( \theta = 0, \pi \), then we define the umbilic slopes of \( S \) as

\[
\mu_0 = \lim_{\theta \to 0} \left( \frac{r_2(\theta) - r_2(0)}{r_1(\theta) - r_1(0)} \right), \quad \mu_\pi = \lim_{\theta \to \pi} \left( \frac{r_2(\theta) - r_2(\pi)}{r_1(\theta) - r_1(\pi)} \right).
\]

If \( S \) is \( C^3 \)-smooth, the umbilic slopes are just \( \frac{dr_2}{dr_1} \) evaluated at \( \theta = 0 \) and \( \pi \) respectively.

We remark that coordinate substitution \( \theta \mapsto \pi - \theta \) corresponds to reversing the direction of the \( x^3 \) axis, transforming \( \mu_0 \) into \( \mu_\pi \) and vice-versa. Hence arguments that are made concerning the umbilic at one pole will often hold at the other pole also. When this is the case we will simply say the argument follows by reflection.

The question of what values of umbilic slope are possible on various spheres is an area of active research [12] [5]. The derived Codazzi-Mainardi equation (2.3) is a necessary integrability condition for a \( C^3 \)-smooth surface to be rotationally symmetric and has some striking consequences in this direction, two of which are the following proposition and theorem.

Proposition 2.4. Let \( S \in \mathcal{W} \) be a \( C^3 \) sphere with an isolated umbilic at \( \theta = 0 \). For any \( \alpha \in \mathbb{R} \)

(1) \( \mu_0 > \alpha + 1 \implies \lim_{\theta \to 0} \left( \frac{s}{\sin^{\alpha+1} \theta} \right) = 0 \),

(2) \( \mu_0 < \alpha + 1 \implies \lim_{\theta \to 0} \left( \frac{s}{\sin^{\alpha+1} \theta} \right) = \pm \infty \).

If \( S \) has an isolated umbilic at \( \theta = \pi \) then the value of \( \mu_\pi \) dictates the behaviour of \( s \) as \( \theta \to \pi \) in the same way.

Proof. We prove the result for \( \theta = 0 \), the \( \theta = \pi \) case follows by reflection. Since the umbilic at \( \theta = 0 \) is isolated, there exists \( \delta \in (0, \frac{\pi}{2}) \) such that \( s \neq 0 \) for \( \theta \in (0, \delta] \). Therefore by the Codazzi-Mainardi equation (2.3), \( \frac{dr_2}{d\theta} \neq 0 \) on this interval also. Furthermore since the surface is assumed strictly convex and \( C^3 \)-smooth, \( \frac{dr_2}{d\theta} \) is continuous and bounded. Therefore \( \frac{dr_2}{dr_1} \) is continuous on \( (0, \delta] \).

From the Codazzi-Mainardi equation and the definition of \( s \) one can derive the separable ODE:

\[
\frac{dr_2}{dr_1} = 1 + \frac{\tan \theta}{s} \frac{ds}{d\theta}.
\]
Integrating from $\theta$ to $\delta$ gives

$$|s| = |s(\delta)| \exp \left\{ \int_{\theta}^{\delta} \cot \theta \left( 1 - \frac{dr_2}{dr_1} \right) d\theta \right\}$$

$$= \left| \frac{s(\delta)}{|\sin^\alpha \delta|} \sin^\alpha \theta \cdot \exp \left\{ \int_{\theta}^{\delta} \cot \theta \left( \alpha + 1 - \frac{dr_2}{dr_1} \right) d\theta \right\} ,$$

for all $\theta \in (0, \delta]$. Dividing both sides by $\sin^\alpha \theta$ we derive the relationship

$$\left| \frac{s(\theta)}{\sin^\alpha \theta} \right| = \left| \frac{s(\delta)}{|\sin^\alpha \delta|} \sin^\alpha \theta \cdot \exp \left\{ \int_{\theta}^{\delta} \cot \theta \left( \alpha + 1 - \frac{dr_2}{dr_1} \right) d\theta \right\} \right| \forall \theta \in (0, \delta].$$

Now let $\theta \to 0$. If $\mu_0 > \alpha + 1$ the quantity in parenthesis diverges to $-\infty$, implying $s/\sin^\alpha \theta \to 0$. On the other hand if $\mu_0 < \alpha + 1$ the quantity in parenthesis diverges to $+\infty$ which implies $|s/\sin^\alpha \theta| \to \infty$. $\square$

**Remark 2.5.** Proposition 2.4 shows how the umbilic slopes dictate the rate of vanishing of $s$ at the north and south poles. We remark that if it is the case that $\mu_0$ or $\mu_\pi$ is equal to $\alpha + 1$, then $s/\sin^\alpha \theta$ can exhibit either of the two behaviours in Proposition 2.4 or tend to a non-zero constant, as illustrated by the three examples $s = \sin^\alpha \theta \cdot \ln (2 \csc \theta)^\epsilon$ for $\epsilon = -1, 0, 1$.

**Theorem 2.6.** If $S \in W$ is $C^\infty$ and has an isolated umbilic at the north or south pole, then the umbilic slope at that pole when it exists is finite, takes a value of an odd integer greater than or equal to 3.

If furthermore $\alpha + 1$ is the value of the umbilic slope at a given pole, the limit of $s/\sin^\alpha \theta$ as we approach the pole is finite, non-zero.

**Proof.** Let $\mu$ be the umbilic slope at $\theta = 0$ and assume it is finite. Since $S$ is strictly convex, rotationally symmetric and smooth, $s$ and all odd derivatives of $s$ vanish at $\theta = 0$, i.e. $s(0) = s^{(2m+1)}(0) = 0$ for all $m \in \mathbb{N}$.

Now assume for contradiction that all even derivatives also vanish. Then $s^{(m)}(0) = 0$ for all $m \in \mathbb{N}_0$. In particular for any $\beta \in \mathbb{N}$, by L’Hôpital’s rule

$$\lim_{\theta \to 0} \left( \frac{s}{\sin^\beta \theta} \right) = \ldots = \frac{1}{\beta!} s^{(\beta)}(0) = 0.$$

Using the contrapositive of (2) in Proposition 2.4, it follows that $\mu \geq \beta + 1$ which contradicts the assumption of $\mu$ being finite. Therefore there exists some $k \in \mathbb{N}$ such that $s^{(2k)}(0) \neq 0$, without loss of generality take $k$ to be the smallest natural number such that this holds, so $s^{(m)}(0) = 0$ for all $m < 2k$. The smoothness assumption on $S$ guarantees the existence of $s^{(2k)}(0)$ and therefore we have the following limit

$$\lim_{\theta \to 0} \left( \frac{s}{\sin^{2k} \theta} \right) = \ldots = \frac{1}{(2k)!} s^{(2k)}(0) \neq 0, \pm \infty.$$

This time using the contrapositive of both (1) and (2) in Proposition 2.4 we have $\mu \geq 2k + 1$ and $\mu \leq 2k + 1$ which proves the first claim. The second claim follows from the above limit. If the isolated umbilic is at $\theta = \pi$ we argue by reflection. $\square$
Although the umbilic slopes are not generally quantised for non-smooth spheres with isolated umbilic points, the regularity of a sphere still places restrictions on the possible values of the umbilic slope.

**Proposition 2.7.** If \( S \in \mathcal{W} \) is of regularity \( C^4 \) and has isolated umbilic points, then the umbilic slopes of \( S \) are greater or equal to 3.

**Proof.** First argue at the \( \theta = 0 \) umbilic. By L’Hopital and the assumed regularity of \( S \), we have the existence of the following limit:

\[
\lim_{\theta \to 0} \left( \frac{s}{\sin^2 \theta} \right) = \lim_{\theta \to 0} \left( \frac{\frac{ds}{d\theta}}{2\cos \theta \sin \theta} \right) = \left. \frac{1}{2} \frac{d^2 s}{d\theta^2} \right|_{\theta=0}
\]

Hence by the converse of (2) in Proposition (2.4), we have that \( \mu_0 \geq 3 \). The case at \( \theta = \pi \) follows by reflection. \( \square \)

3. **Linear Hopf Flow**

3.1. **Stationary Solutions.** The stationary solutions of the linear Hopf flow (1.1) are surfaces satisfying the curvature relationship

(3.1) \( 0 = ar_1 + br_2 + c \).

The surfaces in \( \mathcal{W} \) for which a general linear curvature relationship such as equation (3.1) holds are called *linear Hopf spheres*.

**Proposition 3.1.** The linear Hopf spheres which solve equation (3.1) for given parameter values \( a, b \) and \( c \) have astigmatisms of the form

(3.2) \( s_{\text{Hopf}} = C_0 \sin^{2n+2} \theta \),

and support functions of the form

(3.3) \( r_{\text{Hopf}} = C_2 \cos \theta + C_1 + C_0 \left[ \frac{\sin^{2n+2} \theta}{2n+2} - \cos \theta \int_0^\theta \sin^{2n+1} \theta d\theta \right] \),

where \( C_0, C_1 \) and \( C_2 \) are constants and \( -a/b = 2n + 3 \).

**Proof.** Combing the derived Codazzi-Mainardi equation (2.3) with the curvature relationship (3.1) results in a separable ODE which is solved to give equation (3.2). The support function (3.3) follows by integrating equation (3.2) by quadrature as in equation (2.2). \( \square \)

Any surface with isolated umbilic points satisfying equation (3.1) necessarily has umbilic slopes taking the common value

\( \mu_0 = \mu_\pi = -a/b \).

Furthermore, if this surface is in \( \mathcal{W} \) and smooth, Theorem 2.6 implies \( -a/b = 2n + 3 \) for some \( n \in \mathbb{N}_0 \). Therefore the flow can only have smooth, non-round stationary solutions for such values of \(-a/b\), i.e. the only smooth linear Hopf spheres are ones with odd umbilic slope.

For this reason the work undertaken in [7] investigated the linear Hopf flow with \(-a/b\) restricted to an odd integer \( \geq 3 \), on smooth initial surfaces in \( \mathcal{W} \). The central result is then:
**Theorem 3.2.** [7] Let $S_0 \in W$ be a smooth initial surface with equal umbilic slopes $\mu = \mu_0 = \mu_{\pi}$. The linear Hopf flow (1.1) and (1.2) behaves in the following manner:

1. if $2n + 3 < \mu$, the evolving sphere converges exponentially through smooth convex spheres to the round sphere of radius $\frac{c}{2(n+1)}$.

2. if $2n + 3 = \mu$, an initial non-round sphere converges exponentially through smooth convex spheres to a non-round linear Hopf sphere,

3. if $2n + 3 > \mu$, the sphere diverges exponentially.

Since in the current paper we consider $n$ which is generically non-integer, we cannot expect generically smooth, non-round stationary solutions as in Theorem 3.2.

We also remark that the flow in Theorem 3.2 fixes $b = 1$, whereas in this work we allow $b > 0$. These conventions are equivalent up to a parabolic scaling $t \mapsto bt$ of equation (1.1).

### 3.2. Time Evolution of Geometric Quantities.

The curvature flow equation (1.1) is equivalent to the following evolution equation for the support function

$$
\frac{\partial r}{\partial t} = b \frac{\partial^2 r}{\partial \theta^2} + a \cot \theta \frac{\partial r}{\partial \theta} + (a + b)r + c,
$$

as can be seen from the definition of $r$ and equations (2.1). We remark that the coordinate singularities in equation (3.4) prevent us from using regular Sturm-Liouville theory to derive an associated eigenbasis, which motivates the singular theory discussed in Section 4.

The solution $r(t, \theta)$ to equation (3.4) may be found by first considering the behaviour of the astigmatism $s(t, \theta)$. The support function $r(t, \theta)$ may then be recovered via quadrature by equation (2.2) up to two time-dependent constants determined by an initial condition and equation (3.4).

**Proposition 3.3.** Under the linear Hopf flow the astigmatism evolves as

$$
\frac{\partial}{\partial t} \left( \frac{s}{\sin^{n+2} \theta} \right) = b \cdot \mathcal{L}_n^n \left( \frac{s}{\sin^{n+2} \theta} \right),
$$

where $\theta \in (0, \pi)$, $-a/b = 2n + 3$ and

$$
\mathcal{L}_n^\mu = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + (\nu + 1)\nu - \frac{\mu^2}{\sin^2 \theta},
$$

is the Legendre operator.

**Proof.** Differentiating equation (2.2) twice gives a relationship between $s$ and the derivatives of $r$. The claim follows after inserting this relationship into equation (3.4) and performing some algebraic manipulation. □

**Remark 3.4.** As expected the astigmatism of the appropriate linear Hopf sphere is stationary under the flow since

$$
\frac{s_{\text{Hopf}}}{\sin^{n+2} \theta} = \sin^n \theta \in \text{Ker} \mathcal{L}_n^\mu.
$$
A crucial ingredient for solving equation (3.5) in the case $n \in \mathbb{N}$, is that the eigenfunctions of $L^n$ are the associated Legendre polynomials. The associated Legendre polynomials, $P^m_n(\cos \theta)$, are polynomials in sine and cosine and form an orthogonal basis of $C^0[0, \pi]$. In particular they span the higher index terms ($l \geq n$) of the following astigmatism decomposition for smooth surfaces:

\[ s = \sum_{l=0}^{\infty} (a_l + b_l \cos \theta) \sin^{2l+2} \theta. \]

In [7] this decomposition enabled the flow to be solved analytically when $n \in \mathbb{N}$ by projecting the flow equation (3.5) into each eigenbasis of $L^n$ and solving for the time dependency of the coefficients $a_l(t), b_l(t)$.

The term with coefficient $a_n$ is the astigmatism of the linear Hopf sphere with umbilic slope $2n + 3$, therefore in the case of convergence of $s$ to a linear Hopf sphere, $b_l \to 0$ for all $l \in \mathbb{N}_0$ and $a_l \to 0$ for all $l \in \mathbb{N}_0 \\{n\}$ as $t \to \infty$, leaving only the linear Hopf term. In the case of convergence to a round sphere, $a_n = 0$.

We try to emulate the above argument in this paper for $n \notin \mathbb{N}$. In the non-integer case, the eigenfunctions of $L^n$ are no longer trigonometric polynomials and are not terms in the series expansion (3.7). Furthermore it is no longer clear if the non-integer Legendre functions are orthogonal or form a basis. To address this we will write the Legendre operator in its Sturm-Liouville form

\[ L^\mu = \frac{1}{\sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \nu(\nu + 1) \sin \theta - \frac{\mu^2}{\sin \theta} \right], \]

and show using Singular Sturm-Liouville theory that we can find an orthogonal basis in which the surfaces astigmatism can be decomposed.

### 4. Singular Sturm-Liouville Operators

In this section we summarize the theory of singular Sturm-Liouville problems following [21]. In Section 5 the theory will be used to deduce the existence of an orthogonal eigenbasis for the non-integer Legendre operator associated with the linear Hopf flow.

Consider the general Sturm-Liouville operator

\[ T = -\frac{1}{w(x)} \left[ \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x) \right], \]  

where $1/p, q, w \in L_{loc}((a, b); \mathbb{R})$ are locally Lebesgue integrable real-valued functions on the interval $(a, b)$ and $w > 0$. Operators of this form are called singular Sturm-Liouville operators. The Legendre operator discussed in Section 3 is an example of such.

View $T$ as a linear operator on $L^2((a, b), w(x)dx; \mathbb{C})$, the Hilbert space of complex-valued square integrable functions with weight $w$, denoted simply by $L^2_w(a, b)$. Under the standard inner-product of the $L^2_w(a, b)$ spaces

\[ \langle f, g \rangle_w = \int_a^b f(x) \cdot \overline{g(x)} \cdot w(x)dx, \]

the operator $T$ satisfies the so-called Greens formula

\[ \langle Tf, g \rangle_w = p(x) \left( f(x)g'(x) - f'(x)g(x) \right) \bigg|_{x=a}^{x=b} + \langle f, Tg \rangle_w, \]
where the evaluation of the boundary term is to be understood as a limit. 
Greens formula allows us to investigate the symmetry of $T$ in $L^2_w(a,b)$ so long as the functions $f$ and $g$ are chosen to be such that the terms in (4.2) are well defined. For this purpose the maximal domain is introduced:

$$D_{\text{max}} = \{ f \in L^2_w(a,b) : f, pf' \in AC_{\text{loc}}(a,b), \ Tf \in L^2_w(a,b) \},$$

where $AC_{\text{loc}}(a,b)$ is the space of functions which are absolutely continuous on all compact intervals of $(a,b)$.

The requirement that $f, pf' \in AC_{\text{loc}}(a,b)$, is enough to ensure that $f, pf'$ are differentiable almost everywhere and their derivatives are Lebesgue integrable, which gives meaning to equation (4.2).

In the case that the coefficient $p$ satisfies $1/p \in AC_{\text{loc}}(a,b)$, the description of $D_{\text{max}}$ simplifies to

$$D_{\text{max}} = \{ f \in L^2_w(a,b) : f' \in AC_{\text{loc}}(a,b), \ Tf \in L^2_w(a,b) \},$$

in particular the elements of $D_{\text{max}}$ must be $C^1(a,b)$ and have second derivative a.e. Such is the case with the Legendre operator (3.8).

Sturm-Liouville operators often come supplied with boundary conditions as to make the boundary term in equation (4.2) vanish, i.e.

$$\langle Tf, g \rangle_w = \langle f, Tg \rangle_w.$$

These boundary conditions constitute part of $T$’s domain of definition, which naturally must be a subset of $D_{\text{max}}$.

In non-singular Sturm-Liouville theory the well-known boundary conditions

$$\alpha_1 f(a) + \alpha_2 p(a)f'(a) = 0, \quad \beta_1 f(b) + \beta_2 p(b)f'(b) = 0,$$

define a domain $D_{S.A.}$ for $T$ which make $T|_{D_{S.A.}}$ self adjoint [15]. It is for this reason $D_{S.A.}$ will be referred to as a self adjoint domain for $T$.

In singular problems however, the quantities $f(x)$ and $p(x)f'(x)$ may not exist as $x \to a$ or $b$, even if $f \in D_{\text{max}}$. Therefore boundary conditions such as (4.5) are not appropriate. To facilitate the description of boundary conditions for singular problems the Lagrange bracket

$$[f, g]_p(x) = p(x) \left( f(x)\overline{g'(x)} - f'(x)\overline{g(x)} \right),$$

is introduced. Unlike the terms in boundary condition (4.5), the Lagrange bracket $[f, g]_p(x)$ is finite in the limits $x \to a, b$ so long as $f, g \in D_{\text{max}}$.

In order to give appropriate boundary conditions in the singular case, we first give a definition.

**Definition 4.1.** Given a singular Sturm-Liouville operator $T$, we say $T$ is limit-circle (LC) at $x = a$ if for a given $\chi \in \mathbb{C}$ all solutions of the eigenvalue equation

$$Ty = \chi y,$$

are in $L^2_w(a,c)$ for some $c \in (a,b)$. Otherwise we say $T$ is limit-point (LP) at $a$.

Similarly we say $T$ is LC at $x = b$ if correspondingly $y \in L^2_w(c,b)$ and LP at $b$ otherwise.

$T$ is said to be LC(LP) if it is LC(LP) at both $a$ and $b$.

---

1. which requires the stronger condition $1/p, q, w$ are Lebesgue integrable over $(a,b)$
Remark 4.2. \(T\) being LC or LP is independent of \(\chi \in \mathbb{C}\) [21].

The next theorem states the parallel of boundary condition (4.5) for LC Sturm-Liouville operators.

**Theorem 4.3.** [21] Let \(T\) be a LC Sturm-Liouville operator and \(\eta, \psi\) be real valued functions in \(D_{\text{max}}\) such that \([\eta, \psi]_p(a) = 1\) and \([\eta, \psi]_p(b) = 1\). Consider the separated boundary condition

\[
\begin{align*}
\alpha_1[u, \eta]_p(a) - \alpha_2[u, \psi]_p(a) &= 0 \quad (\alpha_1, \alpha_2) \in \mathbb{R}^2\setminus\{0\} \\
\beta_1[u, \eta]_p(b) - \beta_2[u, \psi]_p(b) &= 0 \quad (\beta_1, \beta_2) \in \mathbb{R}^2\setminus\{0\},
\end{align*}
\]

where

\[
[u, f]_p(x) = \lim_{x \to a^+} [u, f]_p(x) \quad [u, f]_p(b) = \lim_{x \to b^-} [u, f]_p(x).
\]

Then given \(\alpha_i, \beta_i, i = 1, 2\) as above, the domain

\[D_{S.A.} = \{y \in D_{\text{max}} : \text{equations (4.7)-(4.8) hold}\}\]

is a self adjoint-domain for \(T\), i.e. \(T|_{D_{S.A.}}\) is a self-adjoint operator.

In addition

**Proposition 4.4.** [20] If \(T\) is a LC Sturm-Liouville operator then \(T\) has a compact resolvent.

Together Theorem 4.3 and Proposition 4.4 are enough to give us an orthonormal basis due to the following spectral theorem:

**Theorem 4.5 (Spectral Theorem).** [19] Let \(\mathcal{H}\) be a separable complex Hilbert space with inner product \(\langle \cdot, \cdot \rangle\) and let \(T : \text{Dom}(T) \subseteq \mathcal{H} \to \mathcal{H}\) be a linear, self-adjoint operator on \(\mathcal{H}\) with compact resolvent. Then, there exists a sequence \((\lambda_n)_n \subset \mathbb{R}\) and a complete orthonormal basis \((e_n)_n\) of \(\mathcal{H}\) with \(e_n \in \text{Dom}(T)\) for all \(n \in \mathbb{N}\) such that

\[
\begin{align*}
(1) & \quad Te_n = \lambda_n e_n, \\
(2) & \quad \text{Dom}(T) = \{x \in \mathcal{H} : (\lambda_n \langle x, e_n \rangle)_n \in \ell^2\}, \\
(3) & \quad Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n \text{ for all } x \in \text{Dom}(T).
\end{align*}
\]

5. Proof of Theorem 1.2

We will solve equation (3.5) in terms of an eigenbasis expansion of the operator \(\mathcal{A}_n\). Theorem 1.2 will then follow by the asymptotic behaviour of the solution as \(t \to \infty\).

The proof is organised into three parts: Firstly we show the existence of the appropriate eigenbasis using the theory in Section 4. Secondly we determine the
basis explicitly as the Legendre functions \( \{ P_{m}^{-n} (\cos \theta) \}_{m=0}^{\infty} \) and derive the corresponding expansions for \( s \), \( r_1 \) and \( r \). Finally, we use this basis to solve the time evolution problem

\[
\begin{aligned}
\partial_t u &= b \cdot \mathcal{L}_n u, \quad [0, \infty) \times [0, \pi] \\
\left. u(t, \cdot) \right|_{t=0} &= u_0, \quad \{t=0\} \times [0, \pi].
\end{aligned}
\]  

(5.1)

The function space \( D_{S.A.} \) is a self-adjoint domain for \( \mathcal{L}_n \) and plays the role of an effective a parabolic boundary condition in (5.1). If \( s \) is the astigmatism of a rotationally symmetric surface, making the substitution \( u = s/\sin^{n+2} \theta \) turns the time evolution problem (5.1) into one describing the evolution of a surface astigmatism under the linear Hopf flow, i.e. equation (3.5). The solution \( s(t, \theta) \) is then integrated for the support function \( r_1(t, \theta) \) and both are shown to exhibit the asymptotic behaviour as \( t \to \infty \)

\[
\begin{aligned}
s(t, \theta) &\sim \tilde{\gamma} \cdot s_{\text{Hopf}}, \\
\left. r_1(t, \cdot) \right|_{t=0} &= r_{\text{Sphere}} + \tilde{\gamma} \cdot r_{\text{Hopf}},
\end{aligned}
\]  

(5.2)

where \( r_{\text{Sphere}} \) is the support function of a sphere with radius \(-\frac{c}{a+b}\) and \( \tilde{\gamma} \) is the signed distance between the focal points of the initial surface at \( \theta = 0 \) and \( \pi \).

### 5.1. Existence of the Eigenbasis and The Self-Adjoint Domain.

First we remark for which values of \( n \) that \( \mathcal{L}_n \) is LC.

**Proposition 5.1.** \( \mathcal{L}_n \) is LC if and only if \( n \in (-1, 1) \).

**Proof.** To check if \( \mathcal{L}_n \) is LC, solve the eigenvalue problem \( \mathcal{L}_n u = \chi u \). Recall when checking LC/LP, we are free to choose \( \chi \) as we please. If we set \( \chi := -n(n+1) \) we must solve the problem

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{du}{d\theta} \right) - \frac{n^2}{\sin^2 \theta} u = 0.
\]

If we can show that any two linearly independent solutions are square integrable (with weight \( w = \sin \theta \)), it follows that every solution is square integrable by the triangle inequality. First assume \( n = 0 \). Then the eigenvalue problem is simply

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{du}{d\theta} \right) = 0,
\]

with two linearly independent solutions \( \ln \cot \left( \frac{\theta}{2} \right) \) and a constant function. These are square integrable. Now assume \( n \neq 0 \), then two linearly independent solutions are

\[
u_\pm = \cot^{\pm n} \left( \frac{\theta}{2} \right).
\]

We have then,

\[
|u_\pm|_{L^2_{\sin \theta}(0, \pi)}^2 = \int_0^\pi \cot^{\pm 2n} \left( \frac{\theta}{2} \right) \sin \theta d\theta = 2 \int_0^\pi \left[ \cos \left( \frac{\theta}{2} \right) \right]^{1\pm 2n} \left[ \sin \left( \frac{\theta}{2} \right) \right]^{1\pm 2n} d\theta,
\]

which is convergent if and only if \(-1 < n < 1\) and therefore \( \mathcal{L}_n \) is LC if and only if \( n \in (-1, 1) \).  

\(\square\)
For such values of $n$ we may now use Theorem 4.3 to find self-adjoint domains for $\mathcal{L}^n_0$. Furthermore, as to make the convergence obvious, we’d like the eigenbasis of $\mathcal{L}^n_0$ to contain explicitly the stationary solution to equation (3.5):

$$s_{\text{Hopf}}^N = \frac{s_{\text{Hopf}}}{\sin^{n+2} \theta}$$

it is therefore necessary that $\frac{s_{\text{Hopf}}}{\sin^{n+2} \theta} \in D_{S.A}$. Finding such a self-adjoint domain will be the content of the next proposition.

**Proposition 5.2.** The separated boundary condition

(5.3) $\lim_{\theta \to 0} \left[ \sin^{2n+1} \theta \frac{d}{d\theta} \left( \frac{u}{\sin^{n} \theta} \right) \right] = 0$, $\lim_{\theta \to \pi} \left[ \sin^{2n+1} \theta \frac{d}{d\theta} \left( \frac{u}{\sin^{n} \theta} \right) \right] = 0,$

generates a self-adjoint domain for the Legendre operator

(5.4) $D_{S.A} = D_{\text{max}} \cap \{ u \in L^2_{\sin \theta}(0, \pi) : \text{boundary condition (5.3) holds}. \}$,

where $D_{\text{max}}$ is given by equation (4.3) for $(a, b) = (0, \pi)$ and $T = \mathcal{L}^n_0$. Furthermore $s_{\text{Hopf}}/\sin^{n+2} \theta \in D_{S.A}$.

**Proof.** The expression for $D_{\text{max}}$ for the Legendre operator takes the form

$$D_{\text{max}} = \{ u \in L^2_{\sin \theta}(0, \pi) : \frac{d^2}{d\theta^2} \in AC_{\text{loc}}(0, \pi), \mathcal{L}^n_0 u \in L^2_{\sin \theta}(0, \pi) \}.$$

Theorem 4.3 tells us that all self adjoint domains of the Legendre operator generated by separated boundary conditions are given by functions $u \in D_{\text{max}}$ satisfying

$$\alpha_1 [u, \eta]_{\sin \theta}(0) + \alpha_2 [u, \psi]_{\sin \theta}(\pi) = 0 \& \beta_1 [u, \eta]_{\sin \theta}(0) + \beta_2 [u, \psi]_{\sin \theta}(\pi) = 0,$$

where $\{ \eta, \psi \} \subset D_{\text{max}}$ and $[\eta, \psi]_{\sin \theta}(0) = [\eta, \psi]_{\sin \theta}(\pi) = 1$, with $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{C}^2 \setminus \{0\}$.

Take $(\eta, \psi) = \frac{1}{\sqrt{\pi}} (\sin^n \theta, \sin^{-n} \theta)$. It is straightforward to check that this choice of $\eta$ and $\psi$ satisfy the above requirements. From Theorem 4.3, the boundary conditions with this choice of $\eta$ and $\psi$ are

(5.5) $\lim_{\theta \to 0} \left[ \alpha_1 \sin^{2n+1} \theta \frac{d}{d\theta} \left( \frac{u}{\sin^{n} \theta} \right) + \frac{\alpha_2}{\sin^{2n-1} \theta} \frac{d}{d\theta} (u \cdot \sin^{n} \theta) \right] = 0,$

(5.6) $\lim_{\theta \to \pi} \left[ \beta_1 \sin^{2n+1} \theta \frac{d}{d\theta} \left( \frac{u}{\sin^{n} \theta} \right) + \frac{\beta_2}{\sin^{2n-1} \theta} \frac{d}{d\theta} (u \cdot \sin^{n} \theta) \right] = 0.$

If we let $u_{\text{Hopf}} = s_{\text{Hopf}}/\sin^{n+2} \theta$ then

$$\sin^{2n+1} \theta \frac{d}{d\theta} \left( \frac{u_{\text{Hopf}}}{\sin^{n} \theta} \right) \equiv 0,$$

and so boundary conditions (5.5)-(5.6) are satisfied by $u_{\text{Hopf}}$ only when $\alpha_2 = \beta_2 = 0$. Therefore setting $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$, gives the self adjoint domain

$$D_{S.A} = D_{\text{max}} \cap \left\{ u \in L^2_{\sin \theta}(0, \pi) : \lim_{\theta \to 0, \pi} \left[ \sin^{2n+1} \theta \frac{d}{d\theta} \left( \frac{u}{\sin^{n} \theta} \right) \right] = 0 \right\}.$$

It is easily seen that $u_{\text{Hopf}} \in D_{\text{max}}$ and so we are done. \qed
Given the astigmatism of a surface, we may characterise when \( s/\sin^{n+2}\theta \in D_{S.A.}\) in terms of the astigmatism.

**Proposition 5.3.** If \( s \) is the astigmatism of \( S \in \mathcal{W} \), then \( s/\sin^{n+2}\theta \in D_{S.A.} \) if and only if (I) and (II) of Theorem 1.2 hold.

**Proof.** We make the substitution \( u = s/\sin^{n+2}\theta \), the maximal domain conditions (5.1) and boundary condition (5.3) become (I) and (II) respectively. \( \square \)

Under stronger assumptions on the surfaces regularity, conditions (I) and (II) can be replaced by the following sufficient conditions:

**Proposition 5.4.** If \( S \in \mathcal{W} \) is in addition \( C^4 \)-smooth, it is sufficient for \( s/\sin^{n+2}\theta \in D_{S.A.} \), that both (1) and (2) of Theorem 1.1 hold.

**Proof.** We will prove that under the assumption \( S \) is \( C^4 \), (1) implies (I) and (2) implies (II). Firstly since \( S \in \mathcal{W} \) and is \( C^4 \), we have \( \frac{ds}{d\theta} \in AC_{\text{loc}}(0,\pi) \). Also, by arguing as in Proposition (2.7), \( s \) has the following asymptotic behaviours near the boundary of \((0,\pi)\)

\[
\frac{s}{\sin^{n+2}\theta} \sim \frac{1}{2\sin^n \theta} \frac{d^2s}{d\theta^2}\bigg|_{\theta=0} \quad \text{as } \theta \to 0 \quad \text{and} \quad \frac{s}{\sin^{n+2}\theta} \sim \frac{1}{2\sin^n \theta} \frac{d^2s}{d\theta^2}\bigg|_{\theta=\pi} \quad \text{as } \theta \to \pi.
\]

We state the following easy Lemma:

**Lemma 5.5.** A continuous function \( f \) on \((0,\pi)\) satisfying the growth conditions

\[
f \sim \frac{k_0}{\sin^n \theta} \quad \text{as } \theta \to 0 \quad \text{and} \quad f \sim \frac{k_\pi}{\sin^n \theta} \quad \text{as } \theta \to \pi, \quad p < 1,
\]

for constants \( k_0 \) and \( k_\pi \) is square integrable with weight \( \sin \theta \), i.e. \( f \in L^2_{\sin \theta}(0,\pi) \).

Therefore, since \( n < 1 \), lemma 5.5 gives \( s/\sin^{n+2}\theta \in L^2_{\sin \theta}(0,\pi) \). Hence the only requirement left for \( s \) to satisfy (I) is that \( \mathcal{L}^n_{S} (s/\sin^{n+2}\theta) \in L^2_{\sin \theta}(0,\pi) \) which is (1).

Secondly, we re-write condition (II) as

\[
\lim_{\theta \to 0,\pi} \left( \frac{\frac{ds}{d\theta}}{\sin \theta} - 2(n + 1) \cos \theta \frac{s}{\sin^2 \theta} \right) = 0.
\]

Substituting the limits in Proposition 2.4 in to the above gives condition (II) as the pair of equations

\[
n \cdot \lim_{\theta \to 0} \left( \frac{s}{\sin^n \theta} \right) = 0, \quad \quad n \cdot \lim_{\theta \to \pi} \left( \frac{s}{\sin^n \theta} \right) = 0,
\]

which is (2). \( \square \)

**Proposition 5.6.** Let \( S \in \mathcal{W} \) be \( C^4 \) and satisfy (3). Then both (1) and (2) of Theorem 1.1 hold.

**Proof.** First we show that (3) implies (1). By lemma 5.5, it is enough to show that (3) implies

\[
\mathcal{L}^n_{S} \left( \frac{s}{\sin^{n+2}\theta} \right) \sim \frac{k}{\sin^n \theta},
\]
around \( \theta = 0 \) for some \( p < 1 \) and some \( k \), and likewise at \( \theta = \pi \). We have

\[
\mathcal{L}^n_n \left( \frac{s}{\sin^{n+2} \theta} \right) = \frac{1}{\sin^{n+2} \theta} \frac{d^2 s}{d \theta^2} - \frac{(2n + 3) \cos \theta}{\sin^{n+3} \theta} \frac{ds}{d \theta} + \frac{2(n + 1)(1 + \cos^2 \theta)s}{\sin^{n+2} \theta}.
\]

It is easily shown that if \( s \) and its derivatives have the asymptotic behaviour given by (3), then by Lemma 5.5 the asymptotic fall off of \( \mathcal{L}^n_n \left( s/\sin^{n+2} \theta \right) \) is sufficient for \( \mathcal{L}^n_n \left( s/\sin^{n+2} \theta \right) \in L^2_{\sin \theta}(0, \pi) \). To prove (2) we have

\[
\frac{s}{\sin^2 \theta} = \frac{s}{\sin^m \theta} \cdot \sin^{m-2} \theta \to 0 \quad \text{as} \quad \theta \to 0, \pi,
\]

since \( m > n + 3 > 2 \) for \( n \in (-1, 1) \). \( \square \)

5.2. The Eigenbasis Expansion.

**Proposition 5.7.** If \( u \in L^2_{\sin \theta}(0, \pi) \) and \( n \in (-1, 1) \), then \( u \) can be decomposed in \( L^2_{\sin \theta}(0, \pi) \) as

\[
(5.7) \quad u = \gamma_{0,n} \sin^n \theta + \sum_{m=1}^{\infty} \gamma_{m,n} \mathcal{P}_{n+m}^{-n}(\cos \theta).
\]

**Proof.** Take \( \mathcal{L}^n_n \) to be the Legendre operator with the self-adjoint domain \( D_{S.A.} \) given by Proposition 5.2. By Propositions 4.4 and 5.1 and the Spectral Theorem 4.5, \( L^2_{\sin \theta}(0, \pi) \) has a complete orthonormal basis consisting of \( \mathcal{L}^n_n \)'s eigenfunctions and kernel. The kernel and eigenspaces of \( \mathcal{L}^n_n \) are spanned by the following linearly independent pairs of basis functions:

| \( n = 0 \) | \( 0 < |n| < 1 \) |
|----------------|------------------|
| Kernel \{ \( Q_0(\cos \theta), 1 \} \} | \{ \( P_n^\pm(\cos \theta), \sin^n \theta \} \} |
| Eigenspaces \{ \( Q_0(\cos \theta), P_\nu(\cos \theta) \} \} | \{ \( P_\nu(\cos \theta), P_\nu^{-n}(\cos \theta) \} \} |

We will show that \( \{ \mathcal{P}_{n+m}^{-n}(\cos \theta) \}_{m=0}^{\infty} \) are the only functions from the above list belonging to \( D_{S.A.} \), i.e. the only functions that satisfy the boundary conditions (5.3). The function \( \sin^n \theta \) and the constant function 1 are easily seen to satisfy the boundary conditions. For the other functions, we use the derivative formula [16, p362]

\[
\sin \theta \frac{d \mathcal{P}_\nu^\mu(\cos \theta)}{d \theta} = (1 - \mu + \nu) \mathcal{P}_{\nu+1}^\mu(\cos \theta) - (\nu + 1) \cos \theta \mathcal{P}_\nu^\mu(\cos \theta), \quad \forall \mu, \nu \in \mathbb{R},
\]

to write the boundary condition (5.3) with \( u = \mathcal{P}_\nu^\mu(\cos \theta) \) as

\[
(5.8) \quad \sin^{2n+1} \theta \frac{d}{d \theta} \left( \mathcal{P}_\nu^\mu(\cos \theta) \sin^n \theta \right) = (1 - \mu + \nu) \sin^n \theta \mathcal{P}_{\nu+1}^\mu(\cos \theta) - (1 + n + \nu) \cos \theta \sin^n \theta \mathcal{P}_\nu^\mu(\cos \theta).
\]

At \( \theta = 0 \), the asymptotic behavior of \( \mathcal{P}_\nu^\mu(\cos \theta) \) is given by [16, p361]

\[
(5.9) \quad \mathcal{P}_\nu^\mu(\cos \theta) \sim \frac{1}{\Gamma(1 - \mu)} \left( \frac{2}{\sin \theta} \right)^\mu, \quad \mu \neq 1, 2, 3, \ldots
\]
It follows from equation (5.8) that near $\theta = 0$ the asymptotic behavior of the boundary term is
\[
\sin^{2n+1} \theta \frac{d}{d\theta} \left( \frac{P_{\nu}^n(\cos \theta)}{\sin^n \theta} \right) \sim -\frac{2^n(n + \mu)}{\Gamma(1 - \mu)},
\]
Hence $P_{\nu}^n(\cos \theta) \in D_{S,A}$ only if $\mu = -n$. A similar argument shows that $Q_{\nu}(\cos \theta)$ must be rejected from the basis because of its bad asymptotic behaviour around $\theta = 0$ (c.f. [16, p361]). Hence the remaining eigenfunctions are

| $n = 0$ | $0 < |n| < 1$ |
|---|---|
| Kernel | $\sin^n \theta$ |
| Eigenspaces | $P_{\nu}(\cos \theta)$ | $P_{\nu}^{-n}(\cos \theta)$ |

We now use the boundary condition at $\theta = \pi$ to determine the possible values of $\nu$. We consider the case $n \neq 0$ first. In this case using together the hyper-geometric series representation for $P_{\nu}^{-n}(\cos \theta)$ [16, p353]
\[
(5.10) \quad P_{\nu}^{-n}(\cos \theta) = \frac{1}{(1 + n)(1 - n)} \left( \frac{1 + \cos \theta}{1 - \cos \theta} \right)^{-n/2} 2F_1 \left( \nu + 1, -\nu; 1 + n; \frac{1}{2} - \frac{1}{2} \cos \theta \right),
\]
and the linear transformation [16, p390]
\[
2F_1(a, b; c, z) = \frac{\Gamma(c)\Gamma(c-a-b)\Gamma(c-b)\Gamma(c-a)}{\Gamma(a)c\Gamma(b)\Gamma(c-a-b)} 2F_1(a, b; a + b + 1 - c; 1 - z) + \frac{\Gamma(c)\Gamma(c-a-b-c)(1-z)^{c-a-b}}{\Gamma(a)c\Gamma(b)\Gamma(c-a-b-b-b)} 2F_1(c-a, c-b; 1 + c - a - b; 1 - z),
\]
one gets the asymptotic behavior of $P_{\nu}^{-n}(\cos \theta)$ around $\theta = \pi$:
\[
(5.11) \quad P_{\nu}^{-n}(\cos \theta) \sim \frac{2^n\Gamma(n)}{\Gamma(n - \nu)\Gamma(1 + n + \nu)} \Gamma(\sin \theta) + \frac{\Gamma(-n)}{2^n\Gamma(1 + \nu)\Gamma(-\nu)} \sin^n \theta.
\]
We remark that we have $1/\Gamma(z) = 0$ whenever $z$ is a negative integer. The asymptotic behavior of the boundary term around $\theta = \pi$ then follows from equations (5.8) and (5.11):
\[
\frac{\sin^{2n+1}}{d\theta} \left( \frac{P_{\nu}^{-n}(\cos \theta)}{\sin^n \theta} \right) \sim \frac{2^n(1 + \nu + n)\Gamma(n)}{\Gamma(\nu + n + 2)\Gamma(n - \nu - 1)} + \frac{2^n(1 + \nu + n)\Gamma(n)}{\Gamma(\nu + n + 1)\Gamma(n - \nu)}.
\]
To satisfy the boundary condition we require the above terms to be 0 which is only the case when $\nu = n + m$ for $m \in \mathbb{N} \setminus \{0\}$, so that $1/\Gamma(n - \nu - 1) = 1/\Gamma(n - \nu) = 0$. The $n = 0$ case is again similar, we derive the asymptotic behavior of $P_{\nu}(\cos \theta)$ around $\theta = \pi$ and find that $\nu \in \mathbb{N}$ in order for the boundary term to vanish. The family $\{P_{n+m}^{-n}(\cos \theta)\}_{m=0}^\infty$ are thus the only family of eigenfunctions in $D_{S,A}$ and form an orthogonal basis of $L_{\sin \theta}^2(0, \pi)$, therefore if $u \in L_{\sin \theta}^2(0, \pi)$
\[
u = \gamma_{0,n} \sin^n \theta + \sum_{m=1}^\infty \gamma_{m,n} P_{n+m}^{-n}(\cos \theta),
\]
for expansion coefficients $\{\gamma_{m,n}\}_{m=0}^\infty$. \hfill $\square$

For well-behaved surfaces, Proposition 5.7 gives an expansion of the surfaces astigmatism in terms of the Legendre functions.
Theorem 5.8. Let $S$ be a $C^2$-smooth rotationally symmetric, strictly convex sphere with astigmatism satisfying $s / \sin^{n+2} \theta \in L^2_w(0, \pi)$. If $n \in (-1, 1)$, the following geometric quantities associated to $S$ decompose as:

$$s = \gamma_{0,n} \sin^{2n+2} \theta + \sin^{n+2} \theta \sum_{m=1}^{\infty} \gamma_{m,n} P_{n+m}^{-(n)}(\cos \theta),$$

$$r_1 = C_1 + \frac{\gamma_{0,n}}{2(n+1)} \sin^{2n+2} \theta + \sin^{n+2} \theta \sum_{m=1}^{\infty} \gamma_{m,n} \left\{ P_{n+m}^{-(n+2)}(\cos \theta) + \cot \theta P_{n+m}^{-(n+1)}(\cos \theta) \right\},$$

$$r = C_2 \cos \theta + C_1 + \gamma_{0,n} \left[ \frac{\sin^{2n+2} \theta}{2(n+1)} - \cos \theta \int_0^\theta \sin^{2n+1} \theta d\theta \right] + \sin^{n+2} \theta \sum_{m=1}^{\infty} \gamma_{m,n} P_{n+m}^{-(n+2)}(\cos \theta),$$

where $C_1 = r_1(0)$, $C_2 = r(0) - r_1(0)$ and the first equality is to be understood as an equality in $L^2_w(0, \pi)$ where as the last two are point-wise.

Proof. The expansion for $s$ follows from Proposition 5.7 by letting $s = \sin^{n+2} \theta \cdot u$. The expansion for $r_1$ can be derived by inserting the expansion of $s$ into the integrated derived Codazzi-Mainardi equation (2.3):

$$r_1(\theta) = r_1(0) + \int_0^\theta \frac{s \cos \theta}{\sin \theta} d\theta = r_1(0) + \frac{\gamma_{0,n}}{2(n+1)} \sin^{2n+2} \theta + \sum_{m=1}^{\infty} \gamma_{m,n} \int_0^\theta \cos \theta \sin^{n+1} \theta P_{n+m}^{-(n)}(\cos \theta) d\theta.$$

We then use the standard integral [16, p368]

$$\int \sin^{\mu+1} \theta \cdot P_{\nu}^{-(\mu)}(\cos \theta) d\theta = \sin^{\mu+1} \theta \cdot P_{\nu}^{-(\mu+1)}(\cos \theta) \ \forall \mu, \nu \in \mathbb{R},$$

to evaluate the following by parts:

$$\int_0^\theta \cos \theta \sin^{n+1} \theta P_{n+m}^{-(n)}(\cos \theta) d\theta = \sin^{n+2} \theta P_{n+m}^{-(n+2)}(\cos \theta) + \cos \theta \sin^{n+1} \theta P_{n+m}^{-(n+1)}(\cos \theta),$$

where the boundary term at $\theta = 0$ vanishes because of the asymptotic behaviour of $P_{n+m}^{-(n+1)}$, see equation (5.9). This completes the derivation for $r_1$, to derive the expansion of $r$ one may either insert the astigmatism decomposition into equation (2.2) and proceed as above, or we can integrate the decomposition of $r_1$ by virtue of equation (2.1). The calculation is analogous. \hfill \Box

For a given rotationally symmetric surface $S$ with astigmatism $s$, it is intriguing to ask in what way does the expansion of $s$ given in Theorem 5.8 convey geometric information about $S$?

Corollary 5.9. Let $\gamma_{m,n}$ be the expansion coefficients as given in Theorem 5.8 for $n \in (-1, 1)$. We have the following equalities

$$\gamma_{0,n} = \frac{\Gamma(n + \frac{3}{2})}{\sqrt{\pi} \Gamma(n + 1)} \cdot (f_0 - f_\pi), \quad \gamma_{1,n} = \frac{\Gamma(2n + 4)}{2n+1 \Gamma(n + 1)} \cdot (r_1(\pi) - r_1(0)).$$

hence $\gamma_{0,n}$ characterises the distance between the focal points of $S$ while $\gamma_{1,n}$ characterises the differences between the radii of curvature at each pole.
Proof. Starting with the expansion formula for $s$ in Theorem 5.8, we divide by $\sin \theta$ and integrate to find the following expression satisfied by $\gamma_{0,n}$:

$$\int_0^\pi \frac{s}{\sin \theta} d\theta = \gamma_{0,n} \int_0^\pi \sin^{2n+1} \theta d\theta,$$

where the higher order terms in the expansion vanish via orthogonality. The integral on the left hand side can be evaluated by using sequentially, equation (2.3), integration by parts, equation (2.1) and Remark 2.1:

$$\int_0^\pi \frac{s}{\sin \theta} d\theta = f_0 - f_\pi,$$

which proves the claim for $\gamma_{0,n}$ once the remaining integral on the right hand side is calculated. The claim concerning $\gamma_{1,n}$ follows by letting $\theta = \pi$ in the expansion of $r_1$. All higher order terms are zero apart from the $m = 1$ term (by equation (5.11)) and we have

$$r_1(\pi) = r_1(0) + \gamma_{1,n} \left\{ \sin^{n+2} \theta P_{n+1}^{-(n+2)}(\cos \theta) + \sin^{n+1} \theta \cos \theta P_{n+1}^{-(n+1)}(\cos \theta) \right\}|_{\theta = \pi}.$$

The $m = 1$ term satisfies

$$\sin^{n+2} \theta P_{n+1}^{-(n+2)}(\cos \theta) + \sin^{n+1} \theta \cos \theta P_{n+1}^{-(n+1)}(\cos \theta) \to \frac{\Gamma(n+1)2^{2n+2}}{\Gamma(2n+4)},$$

as $\theta \to \pi$ (again by equation (5.11)), which completes the proof. □

5.3. Time Evolution of the Eigenbasis Expansion.

In this section we give a solution for each $n \in (-1, 1)$ to the time evolution problem (5.1) for all $t \geq 0$ and for almost every $\theta \in [0, \pi]$. Furthermore this solution is unique in the sense that it is equal point-wise to a strong solution of (5.1) whenever a strong solution exists, for all most all $\theta \in [0, \pi]$, for every $t \geq 0$.

Proposition 5.10. Fix $n \in (-1, 1)$ and let $\{e_m\}_{m=1}^\infty$ be the Legendre functions $\{P_{n+m}^{-n}(\cos \theta)\}_{m=0}^\infty$, normalised with respect to the $L^2_{\sin \theta}(0, \pi)$ inner product, i.e.

$$e_m = \frac{P_{n+m}^{-n}(\cos \theta)}{||P_{n+m}^{-n}(\cos \theta)||}.$$

Let $u_0 \in D.S.A.$ have the eigenbasis decomposition

$$u_0 = \sum_{m=0}^\infty a_m e_m.$$

and $u : [0, \infty) \to L^2_{\sin \theta}(0, \pi)$ be the mapping defined by the series

$$u(t) = \sum_{m=0}^\infty A_m(t)e_m, \quad A_m(t) = a_m \exp\{- (2n+1+m)mbt\}.$$

Then, for each $t \geq 0$, $u(t)$ converges in $L^2_{\sin \theta}(0, \pi)$, is an element of $D.S.A.$ and

$$[u(t)](\theta) = \sum_{m=0}^\infty A_m(t)e_m(\theta),$$

(5.12)
point-wise for almost all $\theta \in (0, \pi)$ (with respect to the Lebesgue measure).

Furthermore, $u(t)$ is the unique classical solution to equation (5.1) for almost all
$\theta \in (0, \pi)$, $t \in (0, \infty)$ with initial data $u_0$.

Proof. First we remark that $u(0) = u_0$. Take $t \in [0, \infty)$ and let $\lambda_m = -(2n + 1 + m)m$, which is the eigenvalue of $L_n^m$ associated to $e_m$. Since $\lambda_m \leq 0$ for all $m \in \mathbb{N}$, $n \in (-1, 1)$ and $b > 0$, we have that $A_m(t)$ is decreasing in $t$, and in particular $|A_m(t)| \leq |a_m|$. It follows that since $u_0 \in L_{\sin \theta}^2(0, \pi)$, by comparison $u(t) \in L_{\sin \theta}^2(0, \pi)$.

Furthermore, by the Spectral Theorem 4.5, $D_{S.A.}$ is characterised by those elements $x \in L_{\sin \theta}^2(0, \pi)$ such that $(\lambda_n(x, e_n))_n \in \ell^2$, therefore since $u_0 \in D_{S.A.}$,

$$
\sum_{m=0}^{\infty} |\lambda_m A_m(t)|^2 \leq \sum_{m=0}^{\infty} |\lambda_m a_m|^2 < \infty,
$$

and $u(t) \in D_{S.A.}$. Therefore $u(t)$ satisfies the boundary conditions of equation (5.1).

We now show $u(t)$ is a classical solution to equation (5.1) a.e. First we show (5.12) holds, to do so we invoke the following theorem.

**Theorem 5.11 (Rademacher-Menchov).** [14, p251]

Let $\{e_n\}_n$ be an orthonormal series with respect to $L_{\sin \theta}^2(0, \pi)$. The series

$$
\sum_{n=1}^{\infty} b_n e_n(\theta),
$$

converges for almost all $\theta \in (0, \pi)$ if the sequence $(b_n)_n$ satisfies

$$
\sum_{n=0}^{\infty} (\log_2 (n + 1)b_n)^2 < \infty.
$$

To see that the sequence $(a_m)_m$ satisfies the requirement of Rademacher-Menchov, it is enough to notice that the sequence $(\lambda_m)_m$ grows quadratically with $m$, hence

$$
\sum_{m=M}^{\infty} (\log_2 (m + 1)a_m)^2 < \sum_{m=M}^{\infty} |\lambda_m a_m|^2 < \infty
$$

for some $M \in \mathbb{N}$. Therefore by Rademacher-Menchov $\sum_{m=0}^{\infty} a_m e_m(\theta)$ converges a.e., and so must $\sum_{m=0}^{\infty} A_m(t)e_m(\theta)$ for all $t \geq 0$. Now we have proved convergence, the equality in (5.12) follows by uniqueness of limits, since the partial sums of $u(t)$ converge a.e. along some sub-sequence to $u(t)$.

Fix $\theta \in [0, \pi]$ such that the $u_0(\theta)$ is equal to the series given in equation (5.12). By Fubini’s Theorem on differentiation [13, p527], since $A_m(t)$ is monotonic in $t$ for all $m \in \mathbb{N}$ we have from term by term differentiation

$$
\partial_t ([u(t)](\theta)) = \sum_{m=0}^{\infty} \lambda_m A_m(t)e_m(\theta)
$$

a.e. $t \in [0, \infty)$ (with respect to the Lebesgue measure). However this means we can find arbitrarily small $t'$ such that the above series converges and therefore converges for all $t > t'$, i.e. we have

$$
\partial_t ([u(t)](\theta)) = \sum_{m=0}^{\infty} \lambda_m A_m(t)e_m(\theta) \quad \forall t \in (0, \infty), \text{ a.e. } \theta \in (0, \pi).
$$
However since \( u(t) \in \mathcal{D}_{S.A.} \), we have \( \mathcal{L}_n^u(t) = \sum_{m=0}^{\infty} \lambda_m A_m(t) e_m, \) and because the series \( \sum_{m=0}^{\infty} \lambda_m A_m(t) e_m(\theta) \) converges a.e. with respect to \( \theta \), we have

\[
\left[ \mathcal{L}_n^u(t) \right](\theta) = \sum_{m=0}^{\infty} \lambda_m A_m(t) e_m(\theta) \quad \forall t \in [0, \infty), \text{ a.e. } \theta \in (0, \pi),
\]

\[
= \partial_t \left( [u(t)](\theta) \right) \quad \forall t \in (0, \infty), \text{ a.e. } \theta \in (0, \pi).
\]

Hence \( u(t) \) satisfies the desired time evolution equation.

Uniqueness follows from the theory of ODEs by projecting the time evolution equation into each eigenbasis.

Corollary 5.12. Let \( S_0 \in \mathcal{H} \) be an initial surface with an astigmatism \( s_0 \) satisfying \( s_0/sin^{n+2} \theta \in \mathcal{D}_{S.A.} \). Under the linear Hopf flow (1.1) and (1.2) with \( n \in (-1, 1) \), for all \( \theta \in [0, \pi], \ t \geq 0, \) the following quantities evolve as

\[
s(t, \theta) = \gamma_{0,n}(t) \sin^{2n+2} \theta + \sum_{m=1}^{\infty} \Gamma_{n,m}(t) \sin^{n+2} \theta P_{n+m}^{-n}(\cos \theta), \quad (\text{a.e. } \theta \in [0, \pi]),
\]

\[
r_1(t, \theta) = C_1(t) + \frac{\gamma_{0,n} \sin^{2n+2} \theta}{2n+2} + \sum_{m=1}^{\infty} \Gamma_{n,m}(t) \left[ P_{n+m}^{-n-1}(\cos \theta) + \cot \theta P_{n+m}^{-n}(\cos \theta) \right],
\]

\[
r(t, \theta) = C_2(t) \cos \theta + C_1(t) + \gamma_{0,n} \left[ \frac{\sin^{2n+2} \theta}{2n+2} - \cos \theta \int_0^\theta \sin^{2n+1} \theta d\theta \right]
\]

\[
+ \sum_{m=1}^{\infty} \Gamma_{n,m}(t) P_{n+m}^{-n}(\cos \theta),
\]

for all \( \theta \in [0, \pi], \) where \( \Gamma_{m,n}(t) = \gamma_{m,n} \exp \left\{ -(2n+1+m)mb \cdot t \right\} \) and \( \gamma_{m,n} \) are the decomposition coefficients of the initial surface \( S_0 \) in the basis \( \{P_{n+m}^{-n}(\cos \theta)\}_{m=0}^{\infty}. \) Furthermore the constants \( C_1(t) \) and \( C_2(t) \) evolve as

\[
C_1(t) = C_1(0)e^{-2(n+1)bt} + \frac{c \left( 1 - e^{-2(n+1)bt} \right)}{2(n+1)b}, \quad C_2 = \text{constant}.
\]

Proof. Fix \( n \in (-1, 1) \) and let \( s = \sin^{n+2} \theta \cdot u. \) Expand \( u \) as in Proposition 5.10. The coefficients \( \Gamma_{m,n}(t) \) are related to \( A_m(t) \) by \( A_m(t) = \Gamma_{m,n}(t) \|P_{n+m}^{-n}(\cos \theta)\| \). This derives the expression for \( s(t, \theta). \) The expressions for \( r_1(t, \theta) \) and \( r(t, \theta) \) follow by integration as in the proof of Theorem 5.8.
To derive the behaviour of \( C_1(t) \) and \( C_2(t) \), we ensure they satisfy equation (3.4) for the linear Hopf flow, i.e.
\[
\frac{\partial r(t, \theta)}{\partial t} = ar_1(t, \theta) + br_2(t, \theta) + c
\]
\[
= c + b \left[ s(t, \theta) - 2(n+1)r_1(t, \theta) \right].
\]
where we have first written the right hand side of (3.4) in terms of \( r_1 \) & \( r_2 \), and then used the identity \(-a/b = 2n+3\). Substituting the expansions of \( r(t, \theta) \), \( r_1(t, \theta) \) and \( s(t, \theta) \) into this equation and collecting together terms gives us the relationship
\[
\frac{\partial t}{t} C_2(t) \cos \theta + \frac{\partial t}{t} C_1(t) = c - 2(n+1)bC_1(t) + b\sin^{n+2} \theta \sum_{m=1}^{\infty} \Gamma_{m,n}(t) \Delta_{m,n},
\]
where
\[
\Delta_{m,n} = P_{n+m}^{-n}(\cos \theta) - 2(n+1)\cot \theta P_{n+m}^{-(n+1)}(\cos \theta) - (2n+2+m)(m-1)P_{n+m}^{-(n+2)}(\cos \theta).
\]
However, by letting \( \mu = -(n+2) \) and \( v = n+m \) in the following recurrence relation between the Legendre functions [16, p362]
\[
P_{\mu+2}(x) + 2(\mu+1)x \left( 1 - x^2 \right)^{-1/2} P_{\nu+1}(x) + (\nu - \mu)(\nu + \mu + 1)P_{\nu}(x) = 0,
\]
we can see that \( \Delta_{m,n} \) is identically 0, hence we have the following evolution equation
\[
\frac{\partial t}{t} C_2(t) \cos \theta + \frac{\partial t}{t} C_1(t) = c - 2(n+1)bC_1(t).
\]
It is easy to see that \( \partial t C_2(t) = 0 \). Solving the remaining ODE gives the stated time evolution of \( C_1(t) \).

\[\square\]

**Remark 5.13.** We have the following asymptotic behaviour as \( t \to \infty \),
\[
r(t, \theta) \sim C_2 \cos \theta + \frac{c}{2(n+1)b} + \gamma_{0,n} \left[ \frac{\sin^{2n+2} \theta}{2n+2} - \cos \theta \int_{0}^{\theta} \sin^{2n+1} \theta d\theta \right].
\]
This is the astigmatism of a linear Hopf sphere. If \( \gamma_{0,n} = 0 \), i.e. if the focal points of the initial surface coincide, then the astigmatism is that of a sphere with radius
\[
\frac{c}{2(n+1)b} = -\frac{c}{a+b},
\]
as claimed.

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