Abstract
Stream is re-interpreted in terms of a Lazy monad. Future is substituted for Lazy in the obtained construct, resulting in possible parallelization of any algorithm expressible as a Stream computation. The principle is tested against two example algorithms. Performance is evaluated, and a way to improve it briefly discussed.

Categories and Subject Descriptors D.3.3 [Language Constructs and Features]: Data types; G.4 [Mathematical Software]: Algorithm design and analysis

Keywords streaming algorithm, polynomial multiplication, lazy monad

1. Introduction
Scala parallel collections [8] exploit data or SIMD parallelism, whereby a unique operation is applied in parallel to several data, independently. There are problems however, where sub-parts are not independent. In such cases, some sequence must be re-introduced, to allow certain tasks to operate only after some others have ended. This is called task parallelism or pipe-line. To achieve it, if we do not want to descend to thread level, one alternate option is to use a message passing scheme, such as the one implemented in Scala in the form of Future [3]. We seek to assemble futures in a way that allows us to obtain parallelization of some suitable algorithms. Let us take the Stream concept of a lazily evaluated List as a model. List is implemented as a chain of elementary cells:

```
class Cons(hd: A, tl: List[A])
``` extends List[A]

In Stream, tail is evaluated lazily, using a by-name parameter:

```
class Cons(hd: A, tl: => Stream[A])
``` extends Stream[A]

If, instead of waiting for the moment when it is requested, tail starts to compute itself asynchronously on a new thread, we obtain a parallel computation. The elementary cell must be modified as:

```
class Cons(hd: A, tl: Future[Stream[A]])
``` extends Stream[A]

This is illustrated in Figure 1. This idea should allow us to parallelize any algorithm that can be expressed functionally and recursively as a Stream.

2. Outline
The paper is organized as follows: in Section 3 we will introduce a Lazy monad that is semantically equivalent to the pass-by-name parameter used in Stream elementary cells. In Section 4 we will explain how to rewrite Stream in terms of this monad, or any other one for that matter, namely the Future monad. In Section 5 and 6 we will test our parallelization scheme against two example algorithms. Lastly, in Section 7 we will discuss our results and suggest some directions of improvement.

3. Lazy monad
We examine a construct that behaves like => A and at the same time obeys the monad rules. As an illustration, we take the example of the Traversable.filter method. In List, it is implemented in an imperative, iterative way:
def filter(p: A => Boolean): List[A] = {
  val b = new ListBuffer[A]
  for (x <- this) if (p(x)) b += x
  b.result
}

Functionally, this would have to be expressed recursively:

def filter(p: A => Boolean): List[A] = {
  var rest = this
  while (!rest.isEmpty && !p(rest.head))
    rest = rest.tail
  rest match {
    case head::tail => head::tail.filter(p)
    case Nil => Nil
  }
}

But it requires as many stack frames as elements in the List, resulting in stack overflows (tail call optimization is not applicable here because filter is not the last operation in the method body). In Stream, this is avoided by the pass-by-name nature of the second parameter to cons, allowing filter not to be called again immediately, and the number of stack frames to stay below reasonable level. As a result, the computation is not performed immediately but on an on-demand basis.

def filter(p: A => Boolean): Stream[A] = {
  ... if (!rest.isEmpty)
    cons(rest.head, rest.tail filter p)
  else Empty
}

To achieve the same behavior with a monad, we use again an extractor as in the List case, but we suppose that its second member is not forced, i.e. it is of type => A. Then we suppose that we can transform this type => A through a (for now putative) method map.

def filter(p: A => Boolean): Stream[A] = {
  ... rest match {
    case head::tail => head::tail.map(_ filter p)
    case Empty => Empty
  }
}

Likewise, we require the second parameter of the constructor #: to be by-name, as the laziness is to be forwarded by map. Let us now sketch the form of our Lazy monad. In order to ease later substitutions with Future, let us name it the same.

trait Future[+A] extends (() => A) {
  def flatMap[B](f: A => Future[B]) = f(apply())
}

object Future {
  def apply[A](value: => A) = new Future[A] {
    lazy val apply = value
  }
}

We find that our construct has type () => A and a method map, both as expected. We lastly endow a method to force its value, in a similar fashion as Future, for the reason given above.

object Await {
  def result[A](future: Future[A],
    duration: Duration) = future()
}

4. Stream re-interpretation

Every method of Stream can be rewritten in the same spirit. Let us skip the details, and concentrate on the implementation of elementary Cons cells. In List, the constructor’s second parameter is a normal, “flat” type. The extractor provided by the case class gives us back this value as-is.

case class ::[A](hd: A, tl: List[A]) extends List[A] {
  override def isEmpty: Boolean = false
  override def head: A = hd
  override def tail: List[A] = tl
}

Conversely, in Stream it is a by-name parameter. Since case classes disallow such a parameter type, the Cons cell must be a normal class, and no extractor is provided. Calls to tail force the value, which is memoized.

object Stream {
  class Cons[+A](hd: A, tl: Stream[A]) extends Stream[A] {
    override def isEmpty: Boolean = false
    override def head: A = hd
    override def tail: Stream[A] = tl
  }
}

Our monad-based implementation is as follows. The second parameter of the constructor is a monad. Calls to tail force the value as above. Extractions however do not, and give us back the genuine monad, thus preserving the laziness. When forced, memoization of the value occurs internally and needs not be done again in the Cons cell.
object Stream {
  case class Cons[+A](hd: A, tl: Future[Stream[A]]) extends Stream[A] {
    private[this] var defined: Boolean = _
    ...
    def tailDefined = defined
    override def tail: Stream[A] = {
      defined = true
      Await.result(tl, Duration.Inf)
    }
  }
}

Below we give the methods to get our modified Stream back and forth from/to the original Scala Stream. These are implemented recursively like the other modified Stream methods. Notice the call to future made in apply in order to wrap tails into their monadic containers. The reverse operation in unapply is simply done by forcing the value through calling tail.

object Stream {
  def apply[A](s: scala.Stream[A]): Stream[A] = {
    if (s.isEmpty) Empty
    else cons(s.head, future(apply(s.tail)))
  }
  def unapply[A](s: Stream[A]): Option[scala.Stream[A]] = Some(
    if (s.isEmpty) scala.Stream.Empty
    else scala.Stream.cons(s.head, unapply(s.tail).get))
}

5. Example: prime sieve

To evaluate our parallel algorithm, we have first tested it against a prime sieve [7]. It is not the most efficient, as it scans every divisor of a number up to the number itself instead of just its square root, but it turns out to be parallelizable according to our technique. First we give the original algorithm with the normal Stream implementation.

val n = 20000
def primes = sieve(Stream.from(2))
def sieve(s: Stream[Int]): Stream[Int] = {
  Stream.cons(s.head, 
  sieve(s.tail.filter { _ % s.head != 0 }))
} primes.force

The modified implementation of Stream entails the following modifications to the example code: use an extractor to obtain head and (wrapped) tail; call map on tail to express further operations.

val primes = sieve(Stream.range(2, n, 1))
def sieve(s: Stream[Int]): Stream[Int] = {
  s match {
    case head#::tail => head#::tail.map(s =>
      sieve(s.filter { _ % head != 0 }))
    case Empty => Empty
  }
}

6. Example: polynomial multiplication

The second example that we have tested our scheme against, is a computer algebraic algorithm of sparse polynomial multiplication. Other researches and applications of streaming algorithms for such kind of computations can be found in [5, 6]. We use multivariate polynomials, in distributive representation:

$$x = x_0 + x_1 + ... + x_n$$

$$x_i = c_i m_i$$

The test case, detailed in [2], simply consists in computing the product of two such big polynomials:

$$xy$$

Decomposing polynomial multiplication into a sequence of multiply-by-a-term-and-add operations, it is possible to express the algorithm in terms of a stream computation.

type T = Stream[(Array[N], C)]
def times(x: T, y: T) = (zero /: y) {
  (l, r) =>
  val (a, b) = r
  l + multiply(x, a, b)
}

Multiply-by-a-term is expressed functionally/recursively as follows.

def multiply(x: T, m: Array[N], c: C) = {
  x match {
    case (s, a)#::tail => {
      val (sm, ac) = (s * m, a * c)
      val result = (sm, ac)#::tail.map(multiply(_, m, c))
      if (!ac.isZero) result
      else result.tail
    }
    case Empty => Empty
  }
}

Polynomial addition is also implemented recursively. Note that the tail has to be forced in the case when one term cancels, which results in a call to Await.result. This is
Figure 2. Streaming multiply and add operations

not considered good in a regular use of Futures, but we have not been able to avoid it (and it does not occur all the time).

Figure 2 illustrates the process.

def plus(x: T, y: T) = x match {
  case (s, a)#::tailx => y match {
    case (t, b)#::taily => {
      if (s > t)
        (s, a)#::tailx.map(plus(_, y))
      else if (s < t)
        (t, b)#::taily.map(plus(x, _))
      else {
        val c = a + b
        val result =
          (s, c)#::(for (sx <- tailx;
                       sy <- taily)
                     yield plus(sx, sy))
        if (!c.isZero) result
        else result.tail
      }
    } case Empty => x
  } case Empty => y
}

7. Evaluation

To evaluate our method, we have run the examples both in sequential and parallel mode (using Lazy and Future respectively). Computations were performed on a single core Intel Atom D410 with hyperthreading and 2GB memory, under Linux version 2.6.32-5-amd64 (Debian 6.0) with java version “1.7.0_17”, OpenJDK 64-Bit Server VM (mixed mode) and scala-2.11.0-M2. The primes example was run in two versions, primes and primes_x3, until number 20000 and 60000 respectively. The polynomial multiplication example was also run in two versions, stream and stream_big, the latter using polynomials with bigger coefficients (of a factor 10000000001), in order to increase the footprint of elementary operations. According for instance to [1], the expected speedup with hyperthreading should be on the order on 1.20.

|          | seq | par(1) | par(2) |
|----------|-----|--------|--------|
| primes   | 3.4 | 5.9    |        |
| primes_x3| 15.7| 20.2   |        |
| stream   | 14  | 35.1   | 37.7   |
| stream_big| 48 | 67.5   | 49.5   |
| list     | 8.2 | 5.7    |        |
| list_big | 38.6| 22.7   |        |

This is what we obtain with a control computation, list (and list_big), which uses a more classical parallelization technique, based on parallel collections [4]. Our results are presented in Table 1 and Figure 3 and 4. On the vertical axis, seq means sequential execution, par(1) means parallel execution with available processors set to 1, and par(2) means normal parallel execution on our platform.

We make the following observations:

1. scaling does not occur in the primes example, probably due to too fine-grained elementary operations
2. the polynomial multiplication example does not scale either in the small coefficient version
3. the streaming approach, at least in the polynomial example, seems to be sound, and perform reasonably well when no parallelization is involved (stream is not worse than half as fast than list, which is a well optimized classical iterative/imperative implementation)
4. the overhead incurred by parallelization, well visible when available processors is set to 1, is compensated when the footprint of coefficients is big enough, as in
stream, stream_big, and performance increases consistently with what we can expect of hyperthreading.

As a way to improve our technique, since the minimum size of elementary computations seems to be a key factor, we suppose that grouping these in bigger chunks may provide better efficiency. This will have to be tested in forthcoming research.

8. Conclusion
We have presented a technique for parallelizing algorithms expressible as stream computations. Stream was rewritten in terms of a Lazy monad, which was then replaced by Future, enabling parallel execution of computation subparts. Two applications were proposed, for prime numbers computation and polynomial multiplication, respectively. Evaluation showed that this method has an overhead, but that it can scale nonetheless if elementary computations are big enough, even on a limited platform such as a hyperthreaded mono-processor.

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