Quantum Invariants*

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Abstract
Consider the partition function
\[ Z^Q(a, g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \text{Tr}_\mathcal{H} (\gamma U(g) a e^{-Q^2 + itda}) e^{-t^2} dt. \]
In this paper we give an elementary proof that this is an invariant. This is what we mean: assume that \( Q \) is a self-adjoint operator acting on a Hilbert space \( \mathcal{H} \), and that the operator \( Q \) is odd with respect to a \( \mathbb{Z}_2 \)-grading \( \gamma \) of \( \mathcal{H} \). Assume that \( a \) is an operator that is even with respect to \( \gamma \) and whose square equals \( I \). Suppose further that \( e^{-Q^2} \) has a finite trace, and that \( U(g) \) is a unitary group representation that commutes with \( \gamma \), with \( Q \), and with \( a \). Define the differential \( da = [Q, a] \). Then \( Z^Q(a, g) \) is an invariant in the following sense: if the operator \( Q(\lambda) \) depends differentiably on a parameter \( \lambda \), and if \( d\lambda a \) satisfies a suitable bound, (we specify these regularity conditions completely in §XI) then \( Z^Q(\lambda)(a, g) \) is independent of \( \lambda \). Once we have set up the proper framework, a short calculation in §IX shows that \( \partial Z^Q(\lambda)/\partial \lambda = 0 \). These considerations apply to non-commutative geometry, to super-symmetric quantum theory, to string theory, and to generalizations of these theories to underlying quantum spaces.

I Introduction
In an earlier paper [QHA] we have studied a class of geometric invariants that arise within the framework of differential geometry and its non-commutative generalization [C1, C2, JLO]. By pairing a cocycle \( \tau \) with an operator-valued, even, square-root of unity \( a \), we obtained a specific formula for an invariant \( Z^Q(a, g) = \langle \tau^{\text{JLO}}, a \rangle \). In case that \( \tau \) is the JLO-cochain [JLO], this invariant has the numerical value \( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \text{Tr}_\mathcal{H} (\gamma U(g) a e^{-Q^2 + itda}) e^{-t^2} dt \), see [QHA]. Here \( Q \) is a self-adjoint operator and Hilbert space \( \mathcal{H} \), and \( da = [Q, a] \). We assume that the differential \( Q \) is odd with respect to the \( \mathbb{Z}_2 \)-grading \( \gamma \), while \( a \) is even. We also assume that \( g \) is an element of a group \( G \) of symmetries of \( Q \) and of \( a \). The invariant \( Z \) is not necessarily integer, but it is an integer-valued when \( g \) equals the identity element.

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In this note we present an alternative point of view to [QHA]. Rather than relating $\mathcal{Z}$ to a general theory of invariants, to entire cyclic cohomology, and to $K$-theory, we start from the formula for $\mathcal{Z}^Q(\lambda)(a, g)$ above and ask the basic question: can one see directly, when $Q = Q(\lambda)$ depends on a parameter $\lambda$, that $\mathcal{Z}^Q(\lambda)(a, g)$ is actually independent of $\lambda$?

We answer this question affirmatively, by studying a new auxiliary Hilbert space $\hat{\mathcal{H}}$ containing $\mathcal{H}$, and defining a new representation for $\mathcal{Z}$ as an expectation $\mathfrak{Z}(\lambda, a, g)$ on $\hat{\mathcal{H}}$. We replace the operator $Q(\lambda)$ on $\mathcal{H}$ with the operator $q(\lambda, a) = Q(\lambda) + \eta a$ on $\hat{\mathcal{H}}$. We call $q$ the extended supercharge. Here $\hat{\mathcal{H}}$ differs from $\mathcal{H}$ by also containing the additional independent fermionic coordinate $\eta$ chosen so that $\eta^2 = I$ and $\eta Q(\lambda) + Q(\lambda) \eta = 0$. We may interpret $\eta a$ as a connection associated with the translation in the auxiliary direction $t$, paired with $\eta$. In §VIII we define an expectation on $\hat{\mathcal{H}}$, namely

$$\mathfrak{Z}(\lambda, a, g) = \langle \langle (Ja) \rangle \rangle ,$$

where the notation is explained in (VIII.1–2). We also show that $\mathfrak{Z}(\lambda, a, g)$ and $\mathcal{Z}^Q(\lambda)(a, g)$ agree.

Once we have defined the proper framework, we show with a short calculation in §IX that $\mathcal{Z}$ is constant. We also show that $\mathfrak{Z}(\lambda, a, g)$ and $\mathcal{Z}^Q(\lambda)(a, g)$ agree.

Though the analytic part of the argument is crucial, it also remains identical to our paper [QHA]. In that other work, we formulate precisely two regularity conditions: first the regularity of $Q(\lambda)$ with respect to $\lambda$, and secondly the regularity of $a$ with respect to $Q(\lambda)$. We call the latter the fractional differentiability properties of $a$. The conditions we give are useful because they are easy to verify in a large set of examples.

Under these regularity conditions, $\mathcal{Z}^Q(\lambda)(a, g)$ is once-differentiable in $\lambda$. Furthermore, the resulting $\lambda$-derivative of $\mathcal{Z}$ equals the expression that we would obtain by interchanging the order of differentiating and the order of taking traces or integrals in the definition of $\mathcal{Z}$. The analysis to establish these facts is lengthy, but can be taken over directly from [QHA]. For the convenience of the reader, in §XI, we summarize the analytic hypotheses used in argument.

In §X we consider a different but related case with two differentials $Q_1$ and $Q_2$, but where only $Q_1$ is invariant under the symmetry group $G$. Instead we assume that $Q_1^2 + Q_2^2$ is also invariant, and that $Q_1^2 - Q_2^2$ commutes with all relevant operators. We show in this case that an expectation (X.4) has a representation similar to (I.1) and also is an invariant with respect to $\lambda$.

II The Supercharge

Our basic framework involves an odd, self adjoint operator $Q$ on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$. This means we have a self-adjoint operator $\gamma$ on $\mathcal{H}$ for which $\gamma^2 = I$. Thus $\mathcal{H}$ splits into the direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- of eigenspaces of \gamma$. The statement that $Q$ is odd means $Q \gamma + \gamma Q = 0$. In terms of
the direct sum decomposition,

\[ Q = \begin{pmatrix} 0 & Q^*_+ \\ Q_+ & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{II.1} \]

The operator \( Q \) is the supercharge\(^1\) and its square

\[ H = Q^2 = \begin{pmatrix} Q^*_+ Q_+ & 0 \\ 0 & Q_+ Q^*_+ \end{pmatrix} \tag{II.2} \]

will be referred to as the Hamiltonian. We let \( x^\gamma = \gamma x \gamma \) denote the action of \( \gamma \) on operators. We say that the operator \( x \) is even (bosonic) if \( x = x^\gamma \) and odd (fermionic) if \( x^\gamma = -x \). We define the graded differential

\[ dx = Qx - x^\gamma Q. \tag{II.3} \]

We suppose that there is a compact Lie group \( G \) with a continuous unitary representation \( U(g) \) on \( H \) such that

\[ U(g)\gamma = \gamma U(g), \quad \text{and} \quad U(g)Q = QU(g). \tag{II.4} \]

Denote the action of \( U(g) \) on the operator \( x \) by

\[ x \to x^g = U(g)xU(g)^{-1}. \tag{II.5} \]

### III The Observables

We also consider an algebra of bounded operators \( \mathfrak{A} \) on \( H \) with the properties that each \( a \in \mathfrak{A} \) is even and invariant. In other words, each \( a \in \mathfrak{A} \) commutes with \( \gamma \) and with \( U(g) \) for all \( g \in G \). We also consider \( \text{Mat}_n(\mathfrak{A}) \), the set of \( n \times n \) matrices with matrix elements (or entries) in \( \mathfrak{A} \). If \( a, b \in \text{Mat}_n(\mathfrak{A}) \) are matrices with entries \( a_{ij}, b_{ij} \in \mathfrak{A} \), we use the shorthand \( ab \) to denote the matrix with entries \( \sum_{k=1}^n a_{ik}b_{kj} \in \mathfrak{A} \).

The differential of elements of \( \mathfrak{A} \) is

\[ da = Qa - aQ = [Q, a] \tag{III.1} \]

is always defined as a quadratic form on \( H \). We make precise the nature of the differentiability of \( \mathfrak{A} \), in §XI. The operators \( \mathfrak{A} \) and the derivative \( d \) are the fundamental building blocks of non-commutative geometry; here \( \mathfrak{A} \) generalizes the notion of functions on a manifold \( M \), and \( Q \) generalizes a Dirac operator on a bundle over \( M \).

\(^1\)We are not concerned with the basic structure of \( H \) or \( Q \), aside from the possibility to perform the construction in §V.
IV The Invariant $\mathcal{Z}^{Q(\lambda)}(a, g)$

In [QHA] we gave a simple formula for an invariant. Let $Q(\lambda)$ depend on a real parameter $\lambda$. We denote the graded commutator (II.3) of $Q(\lambda)$ with $x$ by

$$d_\lambda x = Q(\lambda)x - x^\gamma Q(\lambda) .$$  \hspace{1cm} (IV.1)

For $a \in \mathfrak{A}$ define

$$\mathcal{Z}^{Q(\lambda)}(a, g) = 1\sqrt{\pi} \int_{-\infty}^{\infty} \text{Tr}_{\mathcal{H}} \left( \gamma U(g)ae^{-Q(\lambda)^2 + itd_\lambda a} \right) e^{-t^2} dt .$$  \hspace{1cm} (IV.2)

More generally, we let $a \in \text{Mat}_n(\mathfrak{A})$. In this case

$$\mathcal{Z}^{Q(\lambda)}(a, g) = 1\sqrt{\pi} \int_{-\infty}^{\infty} \text{Tr}_{\mathcal{H} \otimes \mathbb{C}^n} \left( \gamma U(g)ae^{-Q(\lambda)^2 + itd_\lambda a} \right) e^{-t^2} dt ,$$  \hspace{1cm} (IV.3)

where we extend $\gamma, U(g), Q(\lambda)^2$ to diagonal $n \times n$ matrices.

**Theorem I.** For $a \in \mathfrak{A}$, assume $a^2 = I$. Furthermore assume that $Q = Q(\lambda)$ and $d_\lambda a = [Q(\lambda), a]$ satisfy the regularity hypotheses given in §XI. Then $\mathcal{Z}^{Q(\lambda)}(a, g)$ is independent of $\lambda$.

The main point of this paper is to present a new, elementary proof of Theorem I.

V The Extended Supercharge $q$

In order to exhibit our proof, we introduce a new Hilbert space $\hat{\mathcal{H}}$ on which the operators $Q, \gamma, \mathfrak{A}$, and $U(g)$ also act. In addition, on $\hat{\mathcal{H}}$ there are two additional self adjoint operators $\eta$ and $J$, both of which have square one,

$$\eta^2 = J^2 = I ,$$  \hspace{1cm} (V.1)

and such that

$$[\eta, x] = [J, x] = 0 \text{ for } x = \gamma, a \in \mathfrak{A}, \text{ or } U(g) ,$$  \hspace{1cm} (V.2)

and also

$$\eta J + J \eta = \eta Q + Q \eta = [J, Q] = 0 .$$  \hspace{1cm} (V.3)

Let $\Gamma = \gamma J$ denote a $\mathbb{Z}_2$-grading on $\hat{\mathcal{H}}$, and for $x$ acting on $\hat{\mathcal{H}}$ let

$$x^\Gamma = \Gamma x \Gamma .$$  \hspace{1cm} (V.4)

The operator $\eta$ is our auxiliary fermionic coordinate, and $J = (-I)^{N_\eta}$ is the corresponding $\mathbb{Z}_2$ grading.\footnote{Suppose that $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ is a tensor product of bosonic and fermionic Fock spaces, that $Q$ is linear in fermionic creation or annihilation operators, and that $\gamma = (-I)^{N_\eta}$. This would be standard in the physics of supersymmetry. Suppose in addition that $\eta = b + b^*$ denotes one fermionic degree of freedom independent of those in $\mathcal{H}_f$ and acting on the two-dimensional space $\mathcal{H}_\eta$. Then take $\hat{\mathcal{H}} = \mathcal{H}_b \otimes (\mathcal{H}_f \wedge \mathcal{H}_\eta)$ and $J = (-I)^{N_\eta}$, with $Q, \gamma, a$, and $U(g)$ acting on $\hat{\mathcal{H}}$ in the natural way. This gives a realization of (V.1–3) on $\hat{\mathcal{H}}$.}
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Given $a \in \mathfrak{A}$, define the extended supercharge $q = q(\lambda, a)$ by

$$q = q(\lambda, a) = Q(\lambda) + \eta a,$$  \hspace{1cm} (V.5)

and also let

$$h = h(\lambda, a) = q(\lambda, a)^2 = Q(\lambda)^2 + a^2 - \eta d_\lambda a.$$  \hspace{1cm} (V.6)

Note that

$$q^\Gamma = -q, \quad \text{and} \quad h^\Gamma = h.$$  \hspace{1cm} (V.7)

We use the notation $d_q$ to denote the $\Gamma$-graded commutator on $\hat{H}$,

$$d_q x = qx - x^\Gamma q.$$  \hspace{1cm} (V.8)

If we need to emphasize the dependence of $q$ on $\lambda$ or $a$, then we write $d_{q(\lambda, a)} x$. We continue to reserve $d$ or $d_\lambda$ to denote the $\gamma$-graded commutator (IV.1).

VI Heat Kernel Regularization on $\hat{H}$

Let us introduce the heat kernel regularizations $\hat{X}_n$ of $X_n$ on $\hat{H}$. Let $X_n = \{x_0, \ldots, x_n\}$ denote an ordered set of $(n + 1)$ linear operators $x_j$ acting on $\hat{H}$. We call the $x_j$ vertices and $X_n$ a set of vertices. Choose $a \in \mathfrak{A}$ and let $q(\lambda, a) = Q(\lambda) + \eta a$, and $h = h(\lambda, a) = q(\lambda, a)^2$. Define the heat kernel regularization $\hat{X}_n(\lambda, a)$ of $X_n = \{x_0, \ldots, x_n\}^\wedge(\lambda, a)$ by

$$\hat{X}_n(\lambda, a) = \int_{s_j > 0} x_0 e^{-s_0 h} x_1 e^{-s_1 h} \cdots x_n e^{-s_n h} \delta(1 - s_0 - \cdots - s_n) ds_0 \cdots ds_n.$$  \hspace{1cm} (VI.1)

Note that if $T$ is any operator on $\hat{H}$ that commutes with $h = q^2$, then

$$\{x_0, \ldots, x_j T, x_{j+1}, \ldots, x_n\}^\wedge(\lambda, a) = \{x_0, \ldots, x_j, T x_{j+1}, \ldots, x_n\}^\wedge(\lambda, a).$$  \hspace{1cm} (VI.2)

Furthermore $T = J_\eta$ anti-commutes with $q(\lambda, a)$ and commutes with $h(\lambda, a)$ for all $a$.

**Proposition II.** (Vertex Insertion) Let $X_n = \{x_0, \ldots, x_n\}$ denote a set of vertices possibly depending on $\lambda$. Then with the notation $Q = \partial Q(\lambda)/\partial \lambda$, we have

$$\frac{\partial}{\partial \lambda} \{x_0, \ldots, x_n\}^\wedge(\lambda, a) = -\sum_{j=0}^n \{x_0, \ldots, x_j, d_q \dot{Q}, x_{j+1}, \ldots, x_n\}^\wedge(\lambda, a)$$

$$+ \sum_{j=0}^n \{x_0, \ldots, \frac{\partial x_j}{\partial \lambda}, \ldots, x_n\}^\wedge(\lambda, a).$$  \hspace{1cm} (VI.3)

Here

$$d_q \dot{Q} = d_{q(\lambda, a)} \dot{Q} = d_\lambda \dot{Q} + \eta [a, \dot{Q}].$$  \hspace{1cm} (VI.4)
Proof. By differentiating $\hat{X}_n$ defined in (VI.1), we obtain two types of terms. Differentiating the $x_j$’s gives the second sum in (VI.3). (This sum is absent if the $x_j$’s are $\lambda$-independent.) The other terms arise from differentiating the heat kernels. We use the identity

$$\frac{\partial}{\partial \lambda} e^{-sh} = -\int_0^s e^{-uh} \frac{\partial h}{\partial \lambda} e^{-(s-u)h} du .$$  \hfill (VI.5)

Note that

$$\frac{\partial h}{\partial \lambda} = \frac{\partial}{\partial \lambda} (q^2) = q \frac{\partial q}{\partial \lambda} + \frac{\partial q}{\partial \lambda} q = d_q \left( \frac{\partial q}{\partial \lambda} \right) = d_q \dot{Q} .$$

Explicitly

$$d_q \dot{Q} = (Q + \eta a) \dot{Q} + \dot{Q} (Q + \eta a) = d \dot{Q} + \eta [a \dot{Q}] .$$

Inserted back into the definition of $\hat{X}_n$, we observe that the differentiation of the heat kernel between vertex $j$ and vertex $j+1$ produces one new $-d_q \dot{Q}$ vertex at position $j+1$. This completes the proof of (VI.3).

Define the action of the grading $\Gamma$ on sets of vertices $X_n$ by

$$X_n \rightarrow X_n^\Gamma = \{x_0^\Gamma, x_1^\Gamma, \ldots, x_n^\Gamma\} .$$  \hfill (VI.6)

Since $q^2 = (q^\Gamma)^2$, the regularization $X_n \rightarrow \hat{X}_n$ commutes with the action of $\Gamma$, namely

$$\left( \hat{X}_n(\lambda, a) \right)^\Gamma = \left( X_n^\Gamma \right)^\wedge (\lambda, a) .$$  \hfill (VI.7)

It is also convenient to write explicitly differential of $\hat{X}_n$,

$$d_q \hat{X}_n(\lambda, a) = q \hat{X}_n(\lambda, a) - \hat{X}_n(\lambda, a)^\Gamma q = \{qx_0, x_1, \ldots, x_n\}^\wedge (\lambda, a) - \{x_0^\Gamma, \ldots, x_n^\Gamma q\}^\wedge (\lambda, a) .$$  \hfill (VI.8)

Note that

$$d_q \hat{X}_n(\lambda, a) = \sum_{j=0}^n \{x_0^\Gamma, x_1^\Gamma, \ldots, x_j^\Gamma, d_q x_j, \ldots, x_n\}^\wedge (\lambda, a) .$$  \hfill (VI.9)

One other identity we mention is

**Proposition III.** (Combination Identity) The heat kernel regularizations satisfy

$$\{x_0, x_1, \ldots, x_n\}^\wedge (\lambda, a) = \sum_{j=0}^n \{x_0, x_1, \ldots, x_j, I, x_{j+1}, \ldots, x_n\}^\wedge (\lambda, a) .$$  \hfill (VI.10)
Proof. The $j$th term on the right side of (VI.9) is

$$\{x_0, \ldots, x_j, I, x_{j+1}, \ldots, x_n\}^\lambda(\lambda, a)$$

$$= \int_{s_j>0} x_0 e^{-s_0 h} \cdots x_j e^{-(s_j + s_{j+1}) h} \cdots x_n e^{-s_{n+1} h} \delta(1 - s_0 - \cdots - s_{n+1}) ds_0 \cdots ds_{n+1}. \quad (VI.11)$$

Change the $s$-integration variables to $s'_0 = s_0, s'_1 = s_1, \ldots, s'_j = s_j + s_{j+1}, s'_{j+1} = s_{j+2}, \ldots, s'_n = s_{n+1}$, and $s'_{n+1} = s_j$. This change has Jacobian 1, and the resulting integrand has the form of the integrand for $\{x_0, \ldots, x_n\}^\lambda$ with variables $s'_0, \ldots, s'_n$, namely

$$\int_{s'_0, s'_1, \ldots, s'_{n+1}} ds'_0 \cdots ds'_n \left( \int ds'_{n+1} x_0 e^{-s'_0 h} \cdots x_n e^{-s'_n h} \delta(1 - s'_0 - \cdots - s'_n) \right), \quad (VI.12)$$

with the integrand depending on the variable $s'_{n+1}$ only through the restriction of the range of the $s'_{n+1}$ integral. The original domain of integration restricts $s'_{n+1}$ to the range $0 \leq s'_{n+1} \leq s'_j$, so the dependence of the integrand on $s'_{n+1}$ is the characteristic function of the interval $[0, s'_j]$. Thus performing the $s'_{n+1}$ integration produces a factor $s'_j$ in the $s'_0, \ldots, s'_n$-integrand. Add the similar results for $0 \leq j \leq n$ to give the factor $s'_0 + s'_1 + \cdots s'_n$. But the delta function in (VI.12) restricts this sum to be 1, so the integral of the sum is exactly $\{x_0, \ldots, x_n\}^\lambda(\lambda, a)$.

VII   Expectations on $\hat{\mathcal{H}}$

Let $a \in \mathfrak{a}$ satisfy $a^2 = I$, and let $\hat{X}_n = \hat{X}_n(\lambda, a)$ denote the heat kernel regularization of $X_n$. We define the expectation

$$\langle \langle \hat{X}_n \rangle \rangle_{\lambda, a, g} = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \text{Tr} \hat{R}(\Gamma U(g) \hat{X}_n(\lambda, ta)) \ dt. \quad (VII.1)$$

Here we choose $a^2 = I$ to ensure that the $t^2$ term in $h$ provides a gaussian convergence factor to the $t$-integral. This integral represents averaging over $a$’s whose squares are multiples of the identity.

These expectations can be considered as $(n + 1)$-multilinear expectations on sets $X_n$ of vertices. We sometimes suppress the $\lambda$- or $a$- or $g$-dependence of the expectations, or the $n$-dependence of sets of vertices. Furthermore, where confusion does not occur we omit the $^\lambda$ that we use to distinguish a set of vertices $X$ from the heat kernel regularization of the set. Thus at various times we denote $\langle \langle \hat{X}_n \rangle \rangle_{a, g}$ by $\langle \langle X \rangle \rangle$, or when we wish to clarify the dependence on $n, a$, or $g$ with some subset of these indices, or even as one of the following:

$$\langle \langle \hat{X}_n \rangle \rangle_{\lambda, a, g} = \langle \langle X \rangle \rangle = \langle \langle X \rangle \rangle_n = \langle \langle X \rangle \rangle_{n, a} = \langle \langle X \rangle \rangle_{n, a, g}, \quad (VII.2)$$

etc.

Proposition IV. With the above notation, we have the identities

$$(\Gamma\text{-invariance}) \quad \langle \langle X \rangle \rangle_n = \langle \langle X^\Gamma \rangle \rangle_n, \quad (VII.3)$$
(differential) \[ \langle\langle d_q X \rangle\rangle_n = \sum_{j=0}^{n} \langle\langle \{ x_0^\Gamma, x_1^\Gamma, \ldots, x_{j-1}^\Gamma, d_q x_j, \ldots, x_n \} \rangle\rangle, \quad (VII.4) \]

(cyclic symmetry) \[ \langle\langle x_0, x_1, \ldots, x_n \rangle\rangle = \langle\langle x_n^{I\Gamma}, x_1, x_2, \ldots, x_{n-1} \rangle\rangle, \quad (VII.5) \]

and

(combination identity) \[ \langle\langle x_0, x_1, \ldots, x_n \rangle\rangle_n = \sum_{j=0}^{n} \langle\langle x_0, x_1, \ldots, x_j, I, x_{j+1}, \ldots, x_n \rangle\rangle_{n+1}. \quad (VII.6) \]

Also, in case \( Q = Q^g \) and \( a = a^g \), then \( q = q^g \) and we have

(infinitesimal invariance) \[ \langle\langle d_{q(\lambda, t a)} X \rangle\rangle = 0. \quad (VII.7) \]

**Proof.** The symmetry \((\text{VII.3})\) is a consequence of the fact that \( \Gamma^2 = I \), and \( \Gamma \) commutes with \( U(g) \) and with \( q^2 \). The expectation of \((\text{VII.9})\) completes the proof of \((\text{VII.4})\). The proof of \((\text{VII.5})\) involves cyclicity of the trace. The identity \((\text{VII.6})\) is the expectation of \((\text{VII.10})\). To establish \((\text{VII.7})\), note that every \( \hat{X}_n \) can be decomposed uniquely as \( \hat{X}_n = \hat{X}_n^+ + \hat{X}_n^- \), where \( (\hat{X}_n^\pm)^\Gamma = \pm \hat{X}_n^\pm \). The symmetry \((\text{VII.3})\) ensures that \( \langle\langle d_{q(\lambda, t a)} X_n^+ \rangle\rangle = 0 \). On the other hand, \( q^\Gamma = -q \), together with cyclicity of the trace and \( q^g = q \) ensures that

\[ \langle\langle d_{q(\lambda, t a)} X_n^- \rangle\rangle = \langle\langle q(\lambda, t a) X_n^- \rangle\rangle + \langle\langle X_n^- q(\lambda, t a) \rangle\rangle = \langle\langle q(\lambda, t a) X_n^- \rangle\rangle + \langle\langle q(\lambda, t a)^g X_n^- \rangle\rangle = 0. \]

Except in \((\text{VII.7})\), we have implicitly assumed that the vertices \( x_j \) in \( X_n \) are \( t \)-independent. In case that \( X_n \) has one factor linear in \( t \), the heat kernel regularizations of the following agree,

\[ \{ tx_0, x_1, \ldots, x_n \}^\wedge (\lambda, t a) = \{ x_0, x_1, \ldots, tx_j, \ldots, x_n \}^\wedge (\lambda, t a), \quad (VII.8) \]

for any \( j = 0, 1, \ldots, n \). We then obtain an interesting relation for expectations,

**Proposition V.** (Integration by parts) Let \( a^2 = I \). Then

\[ \langle\langle tx_0, x_1, \ldots, x_n \rangle\rangle_n = \sum_{j=0}^{n} \frac{1}{2} \langle\langle x_0, \ldots, x_j, \eta d_\lambda a, x_{j+1}, \ldots, x_n \rangle\rangle_{n+1}. \quad (VII.9) \]

**Proof.** In order to establish \((\text{VII.9})\), we collect together the terms \( \exp(-s_j t^2) \) that occur in \( \{ x_0, \ldots, x_n \}^\wedge (t a) \). Since the integrand for the heat kernel regularization has a \( \delta \)-function restricting the variables \( s_j \) to satisfy \( s_0 + \cdots + s_n = 1 \), we obtain the factor \( \exp(-t^2) \). Write

\[ te^{-t^2} = -\frac{1}{2} \frac{d}{dt} (e^{-t^2}) \]
and integrate by parts in $t$. The resulting derivative involves the $t$-derivative of each heat kernel
\[ \exp\left(-s^2 q(\lambda, ta)^2\right) \] with the quadratic term in $t$ removed from $q^2$. Note that
\[ e^{-st^2} \frac{d}{dt} e^{-s(q^2-t^2)} = -\int_0^s e^{-uq^2} \left( \frac{d}{dt} (q^2 - t^2) \right) e^{-(s-u)q^2} du \]
\[ = \int_0^s e^{-uq^2} \eta \lambda a e^{-(s-u)q^2} du . \]
Here we use (V.6) with $ta$ replacing $a$ and with $a^2 = I$ in order to evaluate the $t$ derivative of $q^2 - t^2$. Thus each derivative introduces a new vertex equal to $\frac{1}{4} \eta \lambda a$, and the proof of (VII.9) is complete.

### VIII The Functional $\mathfrak{J}(\lambda, a, g)$

Let us consider a single vertex and $X_0 = x_0 = Ja$, where $a \in \mathfrak{A}$, and its expectation
\[ \mathfrak{J}(\lambda, a, g) = \langle \langle Ja \rangle \rangle . \] (VIII.1)

Explicitly
\[ \mathfrak{J}(\lambda, a, g) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \text{Tr} \hat{H} \left( \gamma U(g) a e^{-q(\lambda, ta)^2} \right) dt . \] (VIII.2)

This functional allows us to recover the functional $\mathfrak{J}$.

**Theorem VI.** Let $a$ satisfy $a^2 = I$. Then
\[ \mathfrak{J}(\lambda, a, g) = \mathfrak{J}^{Q(\lambda)}(a, g) . \] (VIII.3)

**Proof.** Let $h = h_0 - t\eta da$, where $h_0 = Q(\lambda)^2 + t^2$. The Hille-Phillips perturbation theory for semi-groups can be written
\[ e^{-q(\lambda, ta)^2} = e^{-h_0 + t\eta da} \]
\[ = e^{-h_0} + \sum_{n=1}^{\infty} t^n \int_{s_j > 0} e^{-s_0 h_0} \eta \lambda a e^{-s_1 h_0} \eta \lambda a \cdots \eta \lambda a e^{-s_n h_0} \delta(1 - s_0 - s_1 - \cdots - s_n) ds_0 ds_1 \cdots ds_n . \] (VIII.4)

In the $n$th term we collect all factors of $\eta$ on the left. Note that $\eta$ commutes with $a$ and $h_0$, and it anti-commutes with $d_\lambda a$. Therefore the result of collecting the factors of $\eta$ on the left is $\eta^n (-1)^{n(n-1)/2}$. If $n$ is odd, then $\eta^n = \eta$ and $\text{Tr} \mathcal{H}_\eta(\eta) = 0$. Thus only even $n$ terms contribute to (VIII.2). For even $n$, $\eta^n (-1)^{n(n-1)/2} = (-1)^{n/2} I$ and $\text{Tr} \mathcal{H}_\eta(I) = 2$. Thus (VIII.2) becomes
\[ \mathfrak{J}(\lambda, a, g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \sum_{n=0}^{\infty} (-t^2)^n \langle \{a, d_\lambda a, \ldots, d_\lambda a\} \rangle_{2n} e^{-t^2} , \] (VIII.5)
where we use expectations $\langle \rangle_n$ on $H$ similar to $\langle\langle \rangle\rangle_n$ on $\hat{H}$ (but without the $t$-integration) and defined by

$$\langle\{x_0, \ldots, x_n\}\rangle_n = \int_{s_j > 0} \text{Tr}_H (\gamma U(g) x_0 e^{-s_0 Q(\lambda)^2} \cdots x_n e^{-s_n Q(\lambda)^2}) \delta(1-s_0-s_1-\cdots-s_n) ds_0 ds_1 \cdots ds_n .$$

(VIII.6)

But using the Hille-Phillips formula once again, (VIII.6) is just

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \text{Tr}_H (\gamma U(g) ae^{-Q(\lambda)^2 + it\lambda a}) e^{-t^2} = \mathcal{Z}(Q(\lambda))(a,g).$$

(VIII.7)

(Here we use the symmetry of (VIII.7) under $\gamma$ to justify vanishing of terms involving odd powers of $d\lambda a$.) Thus we can prove that $\mathcal{Z}(Q(\lambda))(a,g)$ is independent of $\lambda$ by showing that $\mathcal{J}(\lambda,a,g)$ is constant in $\lambda$.

**IX \ \ \mathcal{J}(\lambda,a,g) Does Not Depend on $\lambda$**

We now prove Theorem I. Calculate $\partial \mathcal{J}/\partial \lambda$ using (VI.3), in the simple case of one vertex independent of $\lambda$. Thus

$$\frac{\partial}{\partial \lambda} \mathcal{J}(\lambda,a,g) = \frac{\partial}{\partial \lambda} \langle\langle \{Ja, d_{q(\lambda,ta)} \dot{Q}\} \rangle\rangle.$$

(IX.1)

Using the identity (VII.7) in the form

$$0 = \langle\langle d_{q(\lambda,ta)} \{Ja, \dot{Q}\} \rangle\rangle = \langle\langle \{d_{q(\lambda,ta)}(Ja), \dot{Q}\}\rangle\rangle + \langle\langle \{Ja, d_{q(\lambda,ta)} \dot{Q}\}\rangle\rangle,$$

(IX.2)

we have

$$\frac{\partial}{\partial \lambda} \mathcal{J}(\lambda,a,g) = \langle\langle \{d_{q(\lambda,ta)}(Ja), \dot{Q}\}\rangle\rangle.$$

(IX.3)

It is at this point that we have used $q^g = q$, namely the invariance of both $Q$ and $a$ under $U(g)$. To evaluate (IX.3), note that

$$d_{q(\lambda,ta)}(Ja) = [q(\lambda,ta), Ja] = Jd_{\lambda}a - 2tJ\eta.$$

(IX.4)

Here we use the assumption $a^2 = I$. From Proposition V we therefore infer

$$\frac{\partial}{\partial \lambda} \mathcal{J}(\lambda,a,g) = \langle\langle \{Ja, \dot{Q}\}\rangle\rangle - 2 \langle\langle tJ\eta, \dot{Q}\rangle\rangle = \langle\langle \{Ja, \dot{Q}\}\rangle\rangle - \langle\langle J\eta, \eta d_{\lambda}a, \dot{Q}\rangle\rangle - \langle\langle J\eta, \dot{Q}, \eta d_{\lambda}a\rangle\rangle.$$

(IX.5)

Since $J\eta$ commutes with $h = q^2$, and since $J\eta \dot{Q} = -\dot{Q}J\eta$, use (V12) to establish

$$\langle\langle \{J\eta, \eta d_{\lambda}a, \dot{Q}\}\rangle\rangle + \langle\langle J\eta, \dot{Q}, \eta d_{\lambda}a\rangle\rangle = \langle\langle \{I, Jd_{\lambda}a, \dot{Q}\}\rangle\rangle - \langle\langle \{I, \dot{Q}, Jd_{\lambda}a\}\rangle\rangle = \langle\langle \{Jd_{\lambda}a, \dot{Q}, I\}\rangle\rangle + \langle\langle \{Jd_{\lambda}a, I, \dot{Q}\}\rangle\rangle.$$

(IX.6)
In the last step we also use $\dot{Q} = -\dot{Q}$ and the cyclic symmetry (VII.3). Hence we can simplify (IX.6) to $\langle \{ Jd_\lambda a, \dot{Q} \} \rangle$, by applying the combination identity (VII.6). Substituting this back into (IX.5), we end up with

$$\frac{\partial}{\partial \lambda} \mathcal{Z}(\lambda, a, g) = \langle \{ Jd_\lambda a, \dot{Q} \} \rangle - \langle \{ Jd_\lambda a, \dot{Q} \} \rangle = 0.$$ (IX.7)

Thus $\mathcal{Z}(\lambda, a, g)$ is invariant under change of $\lambda$, and the demonstration is complete.

X Independent Supercharges $Q_j(\lambda)$

Let us generalize our consideration to the case that there are two self-adjoint operators $Q_1 = Q_1(\lambda)$ and $Q_2 = Q_2(\lambda)$ on $\mathcal{H}$ such that

$$Q_1 \gamma + \gamma Q_1 = Q_2 \gamma + \gamma Q_2 = Q_1 Q_2 + Q_2 Q_1 = 0.$$ (X.1)

Thus we have two derivatives $d_j a = [Q_j, a]$. We assume that the energy operator on $\mathcal{H}$ is defined by

$$H = H(\lambda) = \frac{1}{2} (Q_1 + Q_2)^2 = \frac{1}{2} (Q_1^2 + Q_2^2)$$ (X.2)

and that the operator

$$P = \frac{1}{2} (Q_1^2 - Q_2^2)$$ (X.3)

has the properties:

i) $P$ does not depend on $\lambda$.

ii) $P$ commutes with $Q_1, Q_2$ and with each $a \in \mathfrak{A}$.

iii) $U(g)$ commutes with $Q_1$ and with $H(\lambda)$.

Assumption (i) corresponds to a common situation where $P$ can be interpreted as a “momentum” operator. Then the energy, but not the momentum is assumed to depend on $\lambda$. Assumption (ii) says that $Q_1, Q_2$ are translation invariant, and that $\mathfrak{A}$ is a “zero-momentum” or translation-invariant subalgebra. According to assumption (iii), $U(g)$ commutes with $Q_2$, but $U(g)$ may not commute with $Q_2$. Under these hypotheses, and with appropriate regularity assumptions, we showed in [QHA] that for $a = a^g$ and $a^2 = I$,

$$\mathcal{Z}^{Q_j(\lambda)}(a, g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \text{Tr} \left( \gamma U(g) a e^{-H + itd_1 a - t^2} \right) dt$$ (X.4)

is independent of $\lambda$. Here we show how the framework above can also be used to show that (X.4) is constant.
We introduce on $\hat{\mathcal{H}}$ two extended supercharges $q_1 = q_1(\lambda, a) = Q_1 + \eta a$ and $q_2 = Q_2$. Thus with $\eta$ as before, $\eta Q_1 + Q_1 \eta = \eta Q_2 + Q_2 \eta = 0$. Define
\begin{equation}
    h = h(\lambda, ta) = H(\lambda) + t^2 a^2 - t\eta d_1 a. \tag{X.5}
\end{equation}
Note that
\begin{equation}
    h = q_1(\lambda, ta)^2 - P = Q_1(\lambda)^2 + t^2 a^2 - t\eta d_1 a - P, \tag{X.6}
\end{equation}
so we can eliminate $Q_2(\lambda)$ from $h$ by introducing the operator $P$, that commutes with $a, \gamma, J, U(g), \eta$, and $Q_j(\lambda)$.

Thus $P$ commutes with all operators that we consider on $\hat{\mathcal{H}}$, so we repeat the constructions of §V–IX. However, we replace $q(\lambda, ta)^2$ in the previous construction with $h(\lambda, ta)$ defined by (X.5). Also we replace $d_q x$ with $d_{q_j} x = q_j x - x^T q_j$. We use the heat kernel $\exp(-sh)$ to define the heat kernel regularization. Then define the expectation $\langle \langle \cdot \rangle \rangle$ by the formula (VII.1) with this new $h(\lambda, ta)$. As $q_1 = q_1^g$, therefore we have
\begin{equation}
    \langle \langle d_{q_1(\lambda,ta)} X \rangle \rangle = 0. \tag{X.7}
\end{equation}
However it may not be true that $q_2 = q_2^g$, so it may not be true that $\langle \langle d_{q_2(\lambda,ta)} X \rangle \rangle$ vanishes. As before, with $a^2 = I$, define
\begin{equation}
    \mathfrak{J}(\lambda, a, g) = \langle \langle J a \rangle \rangle. \tag{X.8}
\end{equation}
In this case, we establish as in the proof of Theorem VI that
\begin{equation}
    \mathfrak{J}(\lambda, a, g) = \mathfrak{J}^{Q_j(\lambda)}(a, g). \tag{X.9}
\end{equation}
Thus the proof of Theorem I shows:

**Theorem VII.** Let $a \in \mathfrak{A}$, assume $a^2 = I$, and also assume the regularity hypotheses on $Q_j(\lambda)$ and $d_1 a = [Q_1(\lambda), a]$, stated in §XI. Then the expectation $\mathfrak{J}^{Q_j(\lambda)}(a, g)$ is independent of $\lambda$.

## XI Regularity Hypotheses

As explained in the introduction, our results depend crucially on some regularity hypotheses. In order for $\mathfrak{J}$ to exist, we assume $e^{-H(\lambda)} = e^{-Q(\lambda)^2}$ exists and is trace class on $\mathcal{H}$. We give sufficient conditions to ensure this, as well as to ensure the validity of the results claimed in §I — IX. The content of §X require only minor modification of these hypotheses. We have explored the consequences of these hypotheses in [QHA].

1. The operator $Q$ is self-adjoint operator on $\mathcal{H}$, odd with respect to $\gamma$, and $e^{-\beta Q^2}$ is trace class for all $\beta > 0$. 

2. For \( \lambda \in \Lambda \), where \( \Lambda \) is a open interval on the real line, the operator \( Q(\lambda) \) can be expressed as a perturbation of \( Q \) in the form

\[
Q(\lambda) = Q + W(\lambda).
\]  

(XI.1)

Each \( W(\lambda) \) is a symmetric operator on the domain \( D = C^\infty(Q) \).

3. Let \( \lambda \) lie in any compact subinterval \( \Lambda' \subset \Lambda \). The inequality

\[
W(\lambda)^2 \leq aQ^2 + bI,
\]  

(XI.2)

holds as an inequality for forms on \( D \times D \). The constants \( a < 1 \) and \( b < \infty \) are independent of \( \lambda \) in the compact set \( \Lambda' \subset \Lambda \).

4. Let \( R = (Q^2 + I)^{-1/2} \). The operator \( Z(\lambda) = RW(\lambda)R \) is bounded uniformly for \( \lambda \in \Lambda' \), and the difference quotient

\[
\frac{Z(\lambda) - Z(\lambda')}{\lambda - \lambda'}
\]  

(XI.3)

converges in norm to a limit as \( \lambda' \to \lambda \in \Lambda' \subset \Lambda \).

5. The bilinear form \( d_\lambda a \) satisfies the bound

\[
\| R^\alpha d_\lambda a R^\beta \| < M,
\]  

(XI.4)

with a constant \( M \) independent of \( \lambda \) for \( \lambda \in \Lambda' \). Here \( \alpha, \beta \) are non-negative constants and \( \alpha + \beta < 1 \).

In certain examples we are interested in the behavior of \( \mathfrak{J}(\lambda, a, g) \) as \( \lambda \) tends to the boundary of \( \Lambda \). In this case, we may establish the constancy of \( \mathfrak{J} \) with estimates that are weaker than (1–5) at the endpoint of \( \Lambda \), by directly proving the existence and continuity of \( \mathfrak{J} \) at the endpoint. We study one such an example in [J], though other types of endpoint singularities are also of interest (often involving a \( \lambda \to \infty \) limit).

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