The primordial curvature perturbation in the ekpyrotic Universe

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Abstract

In the ekpyrotic scenario the Universe is initially collapsing, the energy density coming from a scalar field with a negative exponential potential. On the basis of a calculation ignoring the gravitational back-reaction the authors of the scenario claim that during collapse the vacuum fluctuation creates a perturbation in the comoving curvature, which has a flat spectrum in accordance with observation. In this note the back-reaction is included, and it is found that the spectrum during collapse is strongly scale-dependent with negligible magnitude. The spectrum is continuous across the bounce if the spacetime is smooth, making it unlikely that the ekpyrotic scenario can be compatible with observation.

Introduction

In the ekpyrotic Universe [1, 2] (called in a slightly different version [3] the pyrotechnic Universe) the Big Bang originates when a brane moving in an extra dimension collides with a fixed one. Except around the time of the collision, the extra dimension can be integrated out to give a four-dimensional field theory with Einstein gravity and flat space. Before the collision, the Universe is collapsing, with the energy density provided by the field $\phi$ which defines the position of the moving brane. In this phase the Hubble parameter $H$, the energy density $\rho$ and the negative potential $V$ are related by the Friedmann equation

$$\rho = 3M_p^2H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi).$$

(1)

The Universe is supposed initially to be almost static with negligible energy density. The energy density increases as the Universe collapses, while remaining negligible compared with the potential. When the moving brane collides with the stationary one, the energy density is converted to radiation, and at the same time the Universe starts to expand.

This ekpyrotic scenario is similar to the pre-big-bang scenario [4, 5, 6]. In both cases the 4-D field theory breaks down at the bounce, whose description would in principle require a string theory calculation (though the calculation may be more tractable in the ekpyrotic case since the bounce can take place far below the string scale). The main
difference is that in the pre-big-bang scenario the scalar fields responsible for the energy density during collapse are supposed to have negligible potential.

The adiabatic density perturbation responsible for structure in the Universe is conveniently characterized by the curvature perturbation $R$ seen by comoving observers [7, 8, 9, 10]. Its spectrum $P_R$ must be practically scale-independent (flat). A flat spectrum is generated during inflation with a sufficiently flat potential, and existing calculations [1, 3] seem to indicate that it is also generated during ekpyrotic collapse provided that the potential has a string-inspired exponential form,

$$V = -V_0 \exp \left( -\sqrt{\frac{2}{p}} \frac{\phi}{M_p} \right). \tag{2}$$

(In [1, 3], the potential was actually taken to be an exponential function of a field with slightly non-canonical normalization, leading to a slightly scale-dependent spectrum, but for simplicity we focus on the case where the canonically-normalized field appears in the potential.) However, these calculations ignore gravitational back-reaction. In this note we include the back-reaction and obtain a quite different result.

The unperturbed Universe The case where the Universe is dominated by a field with an exponential potential has already been studied in the context of power-law inflation [1, 11]. We focus first on the unperturbed case. In addition to the Friedman equation one needs the field equation

$$\ddot{\phi} + 3H \dot{\phi} + V' = 0, \tag{3}$$

or equivalently the relation

$$M_p^2 \dot{H} = -\frac{1}{2} \dot{\phi}^2. \tag{4}$$

With an exponential potential there is an exact solution of Eqs. (1) and (4), corresponding to

$$a \propto |t|^p \tag{5}$$

$$H = \frac{p}{t} \tag{6}$$

$$\frac{1}{2} \dot{\phi}^2 = \frac{1}{p} M_p^2 H^2 \tag{7}$$

$$V = \frac{3p - 1}{p} M_p^2 H^2. \tag{8}$$

We will also need the conformal time defined by $d\tau = dt/a$, which can be taken as

$$\tau = \frac{p}{p - 1} \frac{1}{aH}. \tag{9}$$

Choosing $t > 0$ and $p > 1$ gives power-law inflation. Choosing $t < 0$ and $p = \frac{1}{3}$ gives the pre-big-bang scenario in which there is collapse with no potential. Choosing $t < 0$ and $p \ll \frac{1}{3}$ gives the ekpyrotic scenario in which there is collapse with a dominant potential.
As it is hardly more effort we deal with the general case \(0 < p < \frac{1}{3}\) giving collapse with a potential of arbitrary magnitude. In the ekpyrotic case,

\[
\rho \ll \dot{\phi}^2 \tag{10}
\]

\[
\frac{1}{2} \dot{\phi}^2 + V \simeq 0 \tag{11}
\]

\[
\ddot{\phi} + V' \simeq 0 \tag{12}
\]

\[
a \simeq \text{const}. \tag{13}
\]

The curvature perturbation generated during inflation Let us first summarize the way in which the vacuum fluctuation of the inflaton field \(\phi\) generates the curvature perturbation \([8, 9, 10]\). Each perturbation \(\delta \phi\) with wavenumber \(k/a\) evolves independently, and using conformal time the quantity \(u \equiv a \delta \phi\) has the same dynamics as a free field living in flat spacetime, with wavenumber \(k\) but with some time-dependent mass-squared. Well before horizon exit at \(aH = k\) the mass is negligible, and at the classical level the perturbation is then supposed to vanish (no particles). There is however a vacuum fluctuation, which after horizon exit becomes a classical gaussian perturbation \(\delta \phi\). Evaluated on spatially flat slices, it determines the curvature perturbation \(\mathcal{R}\) seen by comoving observers \([12]\),

\[
\mathcal{R} = -H \delta \phi / \dot{\phi}. \tag{14}
\]

This quantity is constant during inflation, and on the usual assumption that the pressure perturbation is adiabatic it remains constant until horizon entry. For any sufficiently flat potential the spectrum of \(\mathcal{R}\) is flat in accordance with observation.

The calculation ignoring the metric perturbation Next we summarize the calculation presented in \([1, 3]\), working therefore with the case \(p \simeq 0\) The essential assumption of that calculation is that the metric perturbation may be ignored, leading to

\[
\ddot{\delta \phi} + [(k/a)^2 + V''] \delta \phi = 0. \tag{15}
\]

This can be written

\[
\ddot{\delta \phi} + \left( \frac{k^2}{a^2} - \frac{2}{t^2} \right) \delta \phi = 0. \tag{16}
\]

We give first a rough argument \([3]\). The amplification of the quantum fluctuation takes place around the epoch \((k/a)^2 = |V''|\). Indeed, well before this epoch the quantum fluctuation is that of a massless free field in flat spacetime, which on dimensional grounds\(^1\) the authors of \([1]\) use the Guth-Pi-Olson (GPO) formalism \([13, 14]\) for cosmological perturbations, which however is easily translated to the more standard formalism used here. Indeed, the GPO variable \(S\) is \(\frac{2}{3}(k/aH)^2 \mathcal{R}\) and this quantity immediately after the bounce \([1]\) (or inflation \([14]\)) is equated with \(-\frac{2}{3}(k/aH)^2 H \delta \phi / \dot{\phi}\) evaluated just before the bounce. Comparing with Eq. \((14)\) we see that the \(\delta \phi\) of the GPO formalism is evaluated on spatially flat slices. We see also that the use of the GPO formalism requires \(\mathcal{R}\) to be continuous across the bounce, which in contrast with the case of inflation is a non-trivial assumption.

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has the spectrum \( P_\phi \sim (k/a)^2 \). For an estimate, we take this expression to be valid also at the epoch \( (k/a)^2 = |V''| \). Well after this epoch, the negative mass-squared \( V'' \) dominates, and \( \delta \phi \) increases. In fact, \( \delta \phi \sim 1/t \sim \phi \), so that

\[
P_\phi / \dot{\phi}^2 \sim |V'' / V| = 2/pM_p^2.
\]

(17)

This gives \( P_R \sim (2/p)(H/M_p^2)^2 \), which when evaluated at reheating is supposed \([1, 3]\) to give the primordial curvature perturbation for the subsequent hot big bang.

To obtain a more precise result, we go to the quantity \( u \equiv a \delta \phi \), and conformal time \( \tau \).

Since we are working in the limit \( p \simeq 0 \) where \( a \) can be taken to be constant, this change is trivial, Eq. (4) becoming

\[
\frac{d^2 u}{d\tau^2} + (k^2 - 2/\tau^2)u = 0.
\]

(18)

This is the same equation as for slow-roll inflation, and using flat spacetime field theory to define the initial condition at \( \tau^2 \gg k^2 \) one finds \([12, 3, 10]\) at \( \tau^2 \ll k^2 \) a perturbation with spectrum

\[
P_u = (2\pi \tau)^{-2}.
\]

(19)

In the case of slow-roll inflation, \( \tau = -1/(aH) \) leading to \( P_\phi = (H/2\pi)^2 \) and

\[
P_R = (H/\dot{\phi})^2(H/2\pi)^2.
\]

(20)

In our case, \( \tau \simeq t/a \) leading to \( P_\phi = (2\pi t)^{-2} \) and

\[
P_R = \frac{1}{8\pi^2 p} \left( \frac{H}{M_p} \right)^2.
\]

(21)

The calculation including the metric perturbation In the case of slow-roll inflation the metric perturbation is indeed negligible compared with the field perturbation. In general though, they are of the same order. Using the spatially flat slicing, \( u \equiv a \delta \phi \) satisfies \([12]\)

\[
\frac{d^2 u}{d\tau^2} + \left( k^2 - \frac{1}{z} \frac{d^2 z}{d\tau^2} \right) u = 0,
\]

(22)

where

\[
z = a \dot{\phi} / H
\]

(23)

For a practically static Universe, \( z \propto \dot{\phi} \), which using Eq. (12) gives Eq. (18). For slow-roll inflation, \( z \propto a \propto 1/\tau \) giving Eq. (18). In our case, \( z \propto t^p \propto \tau^{2 p} \) giving \([11]\),

\[
\frac{d^2 u}{d\tau^2} + \left( k^2 + \frac{p(1-2p)}{(1-p)^2} \tau^{-2} \right) u = 0.
\]

(24)

Power-law inflation corresponds to \( p > 1 \), giving a negative mass-squared which amplifies the quantum fluctuation \([11]\). The collapsing case we are considering corresponds to \( 0 < p < \frac{1}{3} \), giving a positive mass-squared which does not amplify the quantum fluctuation.
We could obtain an estimate by repeating the previous rough argument, but choose instead to go straight to the precise result. We are interested in the late-time era when \( k^2 \) is negligible, and the spectrum is then

\[
P_R = \frac{r(p)}{2\pi} \left[ \frac{H}{M_p} \right] \left( \frac{k}{aH} \right)^{\frac{1}{1-p}}.
\]  

(25)

In contrast with the earlier result Eq. (21), this correctly-calculated quantity is time-independent (because \( a \propto t^p \)). In particular,

\[
P_R = \frac{r(p)}{2\pi} \sqrt{\frac{p}{2}} \left( \frac{H_{\text{reh}}}{M_p} \right) \left( \frac{k}{a_{\text{reh}}H_{\text{reh}}} \right)^{\frac{1}{1-p}}.
\]  

(26)

On cosmological scales \( k/a_{\text{reh}}H_{\text{reh}} \sim e^{-N} \) with \( N \lesssim 60 \), the precise value depending on \( H_{\text{reh}} \) and the subsequent cosmology [9, 10]. The spectral index is

\[
n = 1 + \frac{2}{1 - p}.
\]  

(27)

which gives \( n = 3 \) for the ekpyrotic case and reproduces the known result \( n = 4 \) for the pre-big-bang case.

**The bounce and beyond** We shown that the curvature perturbation during collapse is unviable, but it needs to be evolved to the epoch of horizon entry before one can say that the ekpyrotic (or pre-big-bang) scenario is unviable. The first step it to go across the bounce. The continuity of \( \mathcal{R} \) is assumed in pre-big-bang discussions and we have seen that it was also assumed in the ekpyrotic scenario [1, 3]. In a separate paper [15] it is demonstrated that \( \mathcal{R} \) is indeed continuous in the approximation that the bounce occurs on a unique slice of spacetime. Also, its near-constancy is maintained just after the bounce provided that this slice is smoothly embedded in spacetime.

The validity of these approximations can be ascertained only by a calculation going beyond the 4-D framework. Such a calculation may well violate the assumptions and generate a curvature perturbation, but there is no reason to suppose that its spectrum will be flat. Accepting the validity of the assumptions \( \mathcal{R} \) will be constant until horizon entry, in contradiction with observation, unless there is either a non-adiabatic pressure perturbation or anisotropic pressure [7, 8, 9, 10]. It may be possible to generate a non-adiabatic pressure perturbation through the vacuum fluctuation of field *not* contributing significantly to the energy density during collapse, but judging experience with the pre-big-bang case [4, 3] it will be difficult to make the spectrum of the fluctuation flat. It seems fair to say that, compared with the inflationary scenario, the ekpyrotic and pre-big-bang scenarios are disfavored by the requirement of structure formation.

**Conclusion** In the ekpyrotic scenario the Universe is initially contracting under the influence of an exponential potential. We have shown that the curvature perturbation
responsible for the origin of structure is not generated while the Universe is collapsing, and have argued that it is unlikely to be generated subsequently.

As was pointed out in [3], a simple modification of the ekpyrotic scenario is to generalize the string-inspired potential Eq. (2) by adding a constant to it. One then has an inflation model [16], which can generate a viable curvature perturbation in the usual way.

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References

[1] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, hep-th/0103239.
[2] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, hep-th/0105212.
[3] R. Kallosh, L. Kofman and A. Linde, hep-th/0104073.
[4] G. Veneziano, hep-th/0002094; R. Brustein, M. Gasperini, M. Giovannini, V. F. Mukhanov and G. Veneziano, Phys. Rev. D51 (1995) 6744.
[5] J. E. Lidsey, D. Wands and E. J. Copeland, Phys. Rep. 337 (2000) 343.
[6] A. Buonanno, M. Lemoine and K. A. Olive, Phys. Rev. D62 (2000) 083513.
[7] J. M. Bardeen, Phys. Rev. D22 (1980) 1882.
[8] D. H. Lyth & A. R. Liddle, Phys. Rep. 231 (1993) 1.
[9] D. H. Lyth & A. R. Liddle, Cosmological Inflation and Large-Scale Structure, Cambridge University Press (2000).
[10] D. H. Lyth and A. Riotto, Phys. Rep. 314 (1998) 1.
[11] D. H. Lyth and E. D. Stewart, Phys. Lett. B274 (1992) 168.
[12] V. F. Mukhanov, JETP Lett. 41, 493 (1985); M. Sasaki, Prog. Theor. Phys. 76 (1986) 1036.
[13] D. W. Olson, Phys. Rev. D14 (1976) 327.
[14] A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. 49 (1982) 1110.
[15] D. H. Lyth, in preparation.
[16] G. Dvali and S.-H. H. Tye, Phys. Lett. B450 (1999) 72; G. Dvali, Q. Shafi and S. Solganik, hep-th/0105203; C. P. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh and R.-J. Zhang, hep-th/0105204; S. Alexander, hep-th/0106097; G. Shiu and S.-H. H. Tye, hep-th/0106274.