Nonlinear pseudo-bosons versus hidden Hermiticity: II. The case of unbounded operators

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Abstract
A close parallelism between the notions of nonlinear pseudobosons and those of an apparent non-Hermiticity of observables as shown in part I (2011 J. Phys. A: Math. Theor. 44 415305) is demonstrated to survive the transition to the quantum models using an unbounded metric in the so-called standard Hilbert space of states.

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1. Introduction
The core of the \textit{difference} between the current bosonic and fermionic quantum states is reflected by the respective number operators with eigenvalues which may be any non-negative integer for bosons and just 0 or 1 for fermions. The most natural \textit{unification} of these states is being achieved under the notion of supersymmetry [1]. The latter concept finds a further generalization in the models exhibiting the so-called nonlinear supersymmetry (NLSUSY, meaning, in essence [2], that the anticommutator of the so-called charges becomes equal to a nonlinear polynomial function of the Hamiltonian) or, alternatively, in the models composed of the so-called nonlinear pseudobosons (NLPB) [3].

There exists [4] a close relationship between the abstract NLSUSY algebras and their representations in terms of certain manifestly non-Hermitian operators (or, more explicitly [5], cryptohermitian operators) of quantum observables with real spectra. Remarkably enough, the latter observables may very traditionally be selected as ordinary differential linear Hamiltonians. In a different context, their large subclass (called, conveniently, \textit{PT}-symmetric Hamiltonians and sampled by Bessis’ and Zinn–Justin’s [6] $H = -\partial_x^2 + ix^3$) has recently been made extremely popular by Carl Bender and coauthors [7, 8].

In our preceding paper I [9], we demonstrated that there also exists a similarly close connection between the \textit{same} class of the cryptohermitian Hamiltonian (of Hamiltonian-like) operators $H \neq H^\dagger$ and the class of the generalized, NLPB number operators $M \neq M^\dagger$. At
the same time, we felt it rather unfortunate that the rigorous formulation of the expected third possible connection between the NLPB systems and NLSUSY algebras was still missing.

We saw one of the reasons in the emergence of a number of subtle technical difficulties attributed to the unbounded-operator nature of the Hamiltonians \( H \neq H^\dagger \) which are needed in the NLSUSY model building [10]. As a consequence, our formulation of the equivalence between the notions of the cryptohermiticity and NLPB characteristics of quantum systems in paper I relied heavily on the assumptions of the boundedness of the operators entering the scene.

In particular, the latter constraint has been applied to the so-called metric operator \( \Theta \) which enters the definition of the inner product in the so-called standard physical Hilbert space of states \( \mathcal{H}^{(S)} \) (this notation was introduced in [5]). Under such a constraint we followed the notation conventions introduced in the series of recent papers by one of us (FB) and spoke about the ‘regular’ NLPB systems in paper I.

In this context, our present paper II will start from an appropriate weakening of the assumptions. This will enable us to formulate, rigorously, the third, ‘missing’ connection between the NLPB systems and NLSUSY algebras.

Our constructions will start from a systematic clarification of the appropriate definitions. Firstly, the notion of the cryptohermitian Hamiltonians will be left reserved for the class of bounded operators \( H \neq H^\dagger \). The phenomenologically inspired consistency of the use of such a severely restricted class has been advocated by Scholtz et al [11] who imagined that the related simplification of the mathematics proves vital, in their case of interest, for the practical feasibility of the interacting-boson-model-inspired variational calculations of the spectra of the heavy nuclei.

In the present context, motivated by the needs of supersymmetry, the overall situation is much less easy. First of all, one can no longer restrict one’s attention to the bounded (i.e. in our notation, cryptohermitian) Hamiltonians \( H \neq H^\dagger \). In order to reflect such an important change of perspective, we will rechristen the ‘unbounded cryptohermitian’ Hamiltonians \( H \neq H^\dagger \) as ‘quasi-Hermitian’ Hamiltonians. Such a terminological aspect of the problem was also discussed in the introductory part of our preceding paper I. In this paper, such a terminological convention may find an independent and very sound historical support in the introduction of ‘quasi-Hermiticity’, by Dieudonné [12], as early as in 1964.

Within the broadened perspective, the present usage of the name of quasi-Hermitian Hamiltonians will be mostly accompanied by the concrete selection of an ordinary differential linear Hamiltonian, like the \( \mathcal{P}\mathcal{T} \)-symmetric Hamiltonians cited above. Let us remind readers that we have shown in paper I that the notions of regular NLPB and cryptohermiticity are, under certain sound assumptions, equivalent. One of the assumptions used throughout that paper is related to the fact that the intertwining operator is bounded with a bounded inverse or, equivalently, that the two sets of eigenstates of \( M \) and \( M^\dagger \) are Riesz bases. However, in the above-mentioned physical applications (and many other ones), this is not ensured at all. In these cases, the role of the unbounded operators becomes crucial.

In this paper, we will show that many of our previous results can still be extended when the unbounded operators are involved. The paper is organized as follows: in section 2, we will return to the notion of ‘hidden’ Hermiticity [5] and distinguish, for our present purposes at least, between its form called cryptohermiticity (in which one assumes that the operators are bounded) and its generalized, unbounded-operator form which will be called here, for the sake of definiteness, quasi-Hermiticity. Subsequently we return to the definition of NLPB and focus on the case in which these cease to be regular. In such a setting, we will outline parallels as well as differences between the results of paper I. Section 3 is then devoted to examples, while our conclusions are given in section 4.
2. Observables and metrics: bounded versus unbounded

2.1. Cryptohermiticity versus quasi-Hermiticity

Let us once more return to the above-mentioned unification of bosons with fermions and recall the popular idea of their arrangement into the so-called supersymmetric multiplets. This idea found a wide acceptance by theoretical particle physicists, although, up to now, it does not seem to be supported by any persuasive experimental evidence. This is the main ‘hidden’ reason why the formalism has been thoroughly tested via the toy-model formalism of the so-called supersymmetric quantum mechanics (SUSYQM [13]). The simplification proved suitable for the purpose. For the sake of brevity, attention is restricted just to a system composed of a single linear fermion in combination with an arbitrarily large \( n \)-plet of linear bosons [14].

Fortunately, the subsequent study of SUSYQM found an independent and fruitful motivation in its own, mostly purely formal byproducts. *Pars pro toto* we might mention the development of the concept of the shape invariance of solvable two-particle potentials, etc.

One of the other useful byproducts of the study of SUSYQM may be seen, paradoxically, in its incompleteness as noted by Jewicki and Rodrigues [15]. On an abstract level this point may be characterized as a sort of incompatibility between the analytic implementation and the algebraic essence of the formalism. Indeed, in the latter context, one reveals that a different angular-momentum-like parameter \( \ell \) enters, in principle, the two partner Hamiltonian-like operators via the centrifugal-like interaction term \( \sim \ell (\ell + 1)/r^2 \). In the former context, as a consequence, one must very carefully discuss the boundary conditions in the origin.

Fortunately, in the traditional SUSYQM found in textbooks, it is quite easy to satisfy these \( \ell \)-dependent boundary conditions (and to ignore the whole ‘algebraic’ shortcoming) simply using a brute-force suppression of the ‘dangerous’ \( \ell \)-dependence of the Hamiltonians in question. Roughly speaking, one simply decides to restrict attention just to the special cases in which \( \ell (\ell + 1) = 0 \) [1].

An unexpectedly successful alternative recipe of the extension of the theory to all of the ‘reasonable’ real \( \ell > -1/2 \) (performing, in effect, its regularization) has been found in the small-circle complexification of the coordinate \( r \) near the origin [16]. Such an origin-avoiding regularization of the Schrödinger equation breaks, naturally, the manifest Hermiticity of the Hamiltonian and/or partner sub-Hamiltonians in question. For this reason, one must be rather careful—in our present paper, we will return to the domain covered by textbooks using the recipes as summarized rather briefly in [5] or in our preceding paper I.

At this point, it is important to emphasize that in the latter two papers (as well as in their ‘fathers-founders’ predecessor [11]) the formalism of the so-called cryptohermitian quantum mechanics is built upon the mathematics-simplifying assumption that all of the operators entering the game are bounded. We are now interested in discussing the mathematically more sophisticated version of the formalism where the emphasis is being shifted to the differential versions of the operators, with a number of illustrative differential-equation examples as reviewed, say, in long papers [8, 17].

For an incorporation of the related necessary weakening of the assumptions let us first introduce the following.

**Definition 1.** Let us consider two operators \( H \) and \( \Theta \) acting on the Hilbert space \( \mathcal{H} \), with \( \Theta \) self-adjoint, positive and invertible. Let us call \( H^\dagger \) the adjoint of \( H \) in \( \mathcal{H} \) with respect to its scalar product and introduce the conjugate operator \( H^\perp = \Theta^{-1}H^\dagger \Theta \), whenever it exists. We will say that \( H \) is quasi-Hermitian with respect to \( \Theta \) (QH w.r.t. \( \Theta \)) if \( H = H^\perp \).
2.2. Quasi-Hermiticity versus the NLPB properties

It is worth reminding the readers that we are interested in the case in which $\Theta$ and $\Theta^{-1}$ are unbounded. Using standard facts in the functional calculus it is obvious that in the assumptions of definition 1 the operators $\Theta^{\pm1/2}$ are well defined. Hence, we can introduce an operator $h := \Theta^{1/2} H \Theta^{-1/2}$, at least if the domains of the operators allow us to do so. More explicitly, $h$ is well defined if, taken $f \in D(\Theta^{-1/2})$, $\Theta^{-1/2} f \in D(H)$ and, moreover, if $H \Theta^{-1/2} f \in D(\Theta^{1/2})$.

Of course, the latter requirements are surely satisfied if $H$ and $\Theta^{\pm1/2}$ are bounded. This option was considered in paper I. Otherwise, due care is required, forcing us to introduce the following, slightly modified terminology.

**Definition 2.** Assume that $H$ is $\text{QH}$ w.r.t. $\Theta$ for $H$ and $\Theta$ as above. $H$ is well behaved w.r.t. $\Theta$ if (i) $h = \Theta^{1/2} H \Theta^{-1/2}$ exists and is self-adjoint, $h = h^\dagger$; (ii) $h$ has only discrete eigenvalues $\epsilon_n$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, with eigenvectors $e_n$: $he_n = \epsilon_n e_n$, $n \in \mathbb{N}_0$, and (iii) if $\mathcal{E} := \{e_n\}$ is an o.n. basis on $\mathcal{H}$.

This definition is slightly different from that considered in paper I, and it is more convenient in the present context where $\Theta$ is assumed to be unbounded. Similarly, the general notion of NLPB should also incorporate the cases which are not regular.

**Definition 3.** Given two operators $a$ and $b$ acting on the Hilbert space $\mathcal{H}$, we will say that the triple $(a, b, [\epsilon_n])$ such that $\epsilon_0 = 0 < \epsilon_1 < \cdots < \epsilon_n < \cdots$ is a family of NLPB if the following four properties hold:

1. **$P1$:** a nonzero vector $\Phi_0$ exists in $\mathcal{H}$ such that $a \Phi_0 = 0$ and $\Phi_0 \in D^\infty(b)$.
2. **$P2$:** a nonzero vector $\eta_0$ exists in $\mathcal{H}$ such that $b^\dagger \eta_0 = 0$ and $\eta_0 \in D^\infty(a^\dagger)$.
3. **$P3$:** calling
   \[
   \Phi_n := \frac{1}{\sqrt{\epsilon_n}} b^n \Phi_0, \quad \eta_n := \frac{1}{\sqrt{\epsilon_n}} a^n \eta_0, \tag{2.1}
   \]
   we have, for all $n \geq 0$, $\Phi_n \in D(a)$, $\eta_n \in D(b^\dagger)$ and
   \[
   a \Phi_n = \sqrt{\epsilon_n} \Phi_{n-1}, \quad b^\dagger \eta_n = \sqrt{\epsilon_n} \eta_{n-1}. \tag{2.2}
   \]
4. **$P4$:** the sets $\mathcal{F}_\Phi = \{\Phi_n, \ n \geq 0\}$ and $\mathcal{F}_\eta = \{\eta_n, \ n \geq 0\}$ are bases of $\mathcal{H}$.

The definitions in (2.1) are well posed in the sense that because of $P1$ and $P2$, the vectors $\Phi_n$ and $\eta_n$ are well-defined vectors of $\mathcal{H}$ for all $n \geq 0$ [18]. In paper I, we further assumed that $\mathcal{F}_\Phi$ and $\mathcal{F}_\eta$ are Riesz bases of $\mathcal{H}$. Under such a constraint, we called our NLPB regular (NLRPB). Now, we will consider the fully general case in which the latter condition is not satisfied. For the sake of brevity of our discussion, we will, at the same time, skip the not too interesting possibility of having the multiplicity $m(\epsilon_n)$ of some eigenvalues $\epsilon_n$ greater than 1.

Definition 2 above will then imply that the set $\mathcal{E}$ produces a resolution of the identity which we write in the bra-ket language as

\[
\sum_{n=0}^{\infty} |\epsilon_n\rangle \langle \epsilon_n| = 1.
\]

Proceeding further in a close parallel with paper I, let us now introduce the manifestly not self-adjoint operators

\[
M = ba, \quad M^\dagger = a^\dagger b^\dagger. \tag{2.3}
\]
We can check that
\[ \Phi_n \in D(M) \cap D(b) \cap D(a), \quad \eta_n \in D(\mathfrak{M}) \cap D(a^\dagger) \cap D(b^\dagger), \]
as well as
\[ b\Phi_n = \sqrt{\epsilon_{n+1}} \Phi_{n+1}, \quad a^\dagger \eta_n = \sqrt{\epsilon_{n+1}} \eta_{n+1}, \quad (2.4) \]
\[ M\Phi_n = \epsilon_n \Phi_n, \quad \mathfrak{M}\eta_n = \epsilon_n \eta_n \quad (2.5) \]
which follow from definitions (2.1) and (2.2). Incidentally, this does not automatically imply that, for instance, \( D(a) \) is exactly the linear span of the \( \Phi_n, D_\Phi \), but only that \( D(a) \supseteq D_\Phi \). The eigenvalue equations themselves imply that the vectors in \( \mathcal{F}_\Phi \) and \( \mathcal{F}_\eta \) are mutually orthogonal,
\[ \langle \Phi_n, \eta_m \rangle = \delta_{n,m}, \quad (2.6) \]
having fixed the normalization of \( \Phi_0 \) and \( \eta_0 \) in such a way that \( \langle \Phi_0, \eta_0 \rangle = 1 \). Recalling [18], we remind readers that conditions \{P1, P2, P3, P4\} are equivalent to \{P1, P2, P3', P4\}, where \( P3' \): the vectors \( \Phi_n \) and \( \eta_n \) defined in (2.1) satisfy (2.6).

Let us now complement \( M \) and \( \mathfrak{M} \) by a pair of further operators
\[ N := ab, \quad \mathfrak{N} := N^\dagger = b^\dagger a^\dagger. \quad (2.7) \]
It is easy to check that \( \Phi_n \in D(N) \), \( \eta_n \in D(\mathfrak{N}) \), and that \( N\Phi_n = \epsilon_{n+1} \Phi_n \) and \( \mathfrak{N}\eta_n = \epsilon_{n+1} \eta_n \) for all \( n \geq 0 \). If the sequence \{\( \epsilon_n \)\} diverges for diverging \( n \), it is clear that the operators involved here, \( a, b, M, N \) and so on, are unbounded. Moreover, as already discussed in the introduction, the intertwining operator between \( M \) and \( M^\dagger \), see below, will also turn out to be unbounded, in contrast to what happens in paper I. For this reason, we will pay particular attention to this aspect of our construction.

To begin with, let us define an operator \( X \) on a certain dense domain \( D(X) \) as follows:
\[ D(X) \ni f \to Xf := \sum_{k=0}^\infty \langle \eta_k, f \rangle \Phi_n. \]
The set \( D(X) \) contains, for instance, all the vectors of \( \mathcal{F}_\Phi \), whose linear span is dense in \( \mathcal{H} \); hence, the norm closure of \( D(X) \) is all of \( \mathcal{H} \), so that \( X \) is well defined. Now, for all \( f \in D(X) \) and for all \( m \geq 0 \), we see that \( \langle \eta_m, (Xf - f) \rangle = 0 \). Therefore, since \( \mathcal{F}_\eta \) is complete in \( \mathcal{H}, Xf = f \). In other words, \( X \) is the identity operator on \( D(x) \) and it can be extended to all \( \mathcal{H} \). Then, using Dirac’s bra-ket notation, we can write
\[ \sum_n | \Phi_n \rangle \langle \eta_n | = \sum_n | \eta_n \rangle \langle \Phi_n | = 1. \quad (2.8) \]
Let us now define two more operators, \( S_\Phi \) and \( S_\eta \), on their domains \( D(S_\Phi) \) and \( D(S_\eta) \), by letting \( h = \sum_n \langle \Phi_n, h \rangle \eta_n \) be in \( D(S_\Phi) \) and setting
\[ S_\Phi h = \sum_n \langle \Phi_n, h \rangle \Phi_n. \quad (2.9) \]
Analogously, let \( f = \sum_n \langle \eta_n, f \rangle \Phi_n \) be in \( D(S_\eta) \). Then, we define
\[ S_\eta f = \sum_n \langle \eta_n, f \rangle \eta_n. \quad (2.10) \]
In Dirac’s notation, this means that \( S_\Phi := \sum_n | \Phi_n \rangle \langle \Phi_n | \) and \( S_\eta := \sum_n | \eta_n \rangle \langle \eta_n | \). It is clear that both these operators are densely defined. Indeed, calling as before \( D_\Phi \) and \( D_\eta \), respectively, the linear spans of \( \mathcal{F}_\Phi \) and \( \mathcal{F}_\eta \), we see that \( D_\Phi \subseteq D(S_\Phi) \) and \( D_\eta \subseteq D(S_\eta) \). In particular,
\[ S_\Phi \Phi_n = \eta_n, \quad S_\eta \eta_n = \Phi_n \quad (2.11) \]
for all $n \geq 0$. The last equations have an interesting consequence: since $\mathcal{F}_\beta$ and $\mathcal{F}_\alpha$ are not Riesz bases\(^3\), $S_\eta$ and $S_\Phi$ are necessarily unbounded operators. This means that they cannot be considered, or called \textit{frame operators}, as in our previous papers, since in the standard frame theory, the frame operator is necessarily bounded with a bounded inverse. It is also easy to check that they are both positive definite, $(h, S_\Phi h) > 0$, $(f, S_\eta f) > 0$, for all nonzero $h \in D(S_\Phi)$ and $f \in D(S_\eta)$, and that they are the inverse of the other:

$$S_\eta = S_\Phi^{-1}.$$  

(2.12)

For that we have to prove that if $f \in D(S_\eta)$, then $S_\eta f \in D(S_\Phi)$ and $S_\Phi(S_\eta f) = f$, and that if $h \in D(S_\Phi)$, then $S_\Phi h \in D(S_\eta)$ and $S_\eta(S_\Phi h) = h$.

Let $f \in D(S_\eta)$ be a norm-limit $f = \| \cdot \|_{\mathcal{H}} \lim_{N \to \infty} f_N$ of $f_N = \sum_{k=0}^{N} (\eta_k, f)\Phi_k$, with $S_\eta f_N$ converging uniformly in $\mathcal{H}$ to what we call $(S_\eta f)$. In other words, both $\{f_N\}$ and $\{S_\eta f_N\}$ are $\| \cdot \|$-Cauchy sequences. To check that $(S_\eta f)$ belongs to $D(S_\Phi)$, it is enough to check that $\{S_\Phi(S_\eta f_N)\}$ is a $\| \cdot \|$-Cauchy sequence as well. This is true since $S_\Phi(S_\eta f_N) = f_N$ for all $N$, which is a $\| \cdot \|$-Cauchy sequence by assumption, converging to $f$. This concludes half of what we had to prove. The proof of the other half is similar.

A direct computation finally shows that $D(S_\eta N) = D(S_\eta)$, $D(S_\Phi N) = D(S_\Phi)$ and that $S_\eta = S_\eta^0$ and $S_\Phi = S_\Phi^0$.

\textbf{Remark.} An apparently simpler definition of $S_\eta$ and $S_\Phi$ would consist in fixing their domains to be exactly $D_\Phi$ and $D_\eta$, respectively. This is equivalent to a restriction of the operators considered so far. However, this choice is not appropriate for us since, in particular, it is not clear if for instance $D(S_\Phi) = D(S_\Phi^0)$ [12]. Nevertheless, similar restrictions will be quite useful in the next section.

2.3. Relating $M$ and $M'$ for non-regular NLPB

We are now interested in deducing a relation between $M$ and $M'$ using the operators $S_\Phi$ and $S_\eta$. The starting point is the eigenvalue equation $M\Phi_n = \epsilon_n \Phi_n$, together with the equality $\eta_n = S_\eta \Phi_n$ obtained before. Hence, $M\Phi_n \in D(S_\eta)$ and we have that $S_\eta(M\Phi_n) = \epsilon_n \eta_n$ for all $n \geq 0$. This equation also implies that $\eta_n \in D(S_\eta MS_\Phi)$, and that for all $n \geq 0$,

$$(S_\eta MS_\Phi - \mathfrak{M})\eta_n = 0.$$  

(2.13)

This equation, by itself, is not enough to ensure that $S_\eta M\Phi_n = \mathfrak{M}$. We know (see [19], problem 50) that for an unbounded operator $A$, the validity of equation $A\epsilon_n = 0$ for all vectors $\epsilon_n$ of a basis still does not imply, in general, that $A = 0$.\(^4\) In other words, even if it is rather reasonable to imagine that $(S_\eta M\Phi_n - \mathfrak{M})\eta_n = 0$ implies that $S_\eta M\Phi_n = \mathfrak{M}$, this is not guaranteed at all. For this reason, as already anticipated in the previous remark, we define the following restrictions:

$$M_0 = M \mid_{D_\Phi}, \quad N_0 = N \mid_{D_\Phi}, \quad \mathfrak{M}_0 = \mathfrak{M} \mid_{D_\Phi}, \quad \mathfrak{N}_0 = \mathfrak{N} \mid_{D_\Phi}.$$  

(2.14)

For these operators, we can prove that

$$S_\eta M_0 S_\Phi = \mathfrak{M}_0, \quad M_0' = \mathfrak{M}_0,$$  

(2.15)

as well as

$$S_\eta N_0 S_\Phi = \mathfrak{N}_0, \quad N_0' = \mathfrak{N}_0.$$  

(2.16)

\(^3\) Recall that this is the situation we are interested in here. The case in which these are Riesz bases was already considered in paper I.

\(^4\) In order to be so, we should have $A\epsilon_n = 0$ for all bases!
Indeed we can check that, for instance, \( D(\mathfrak{B}_0) = D(S_\eta M_0 S_\eta) = D_\eta \) and that the operators \( \mathfrak{B}_0 \) and \( S_\eta M_0 S_\eta \) coincide on \( D_\eta \). Therefore, for these restrictions, formulas analogous to those found in paper I are recovered.

The following theorem, which extends to nonregular NLPB an analogous result proven in paper I, can now be deduced.

**Theorem 4.** Let \( H \) be well behaved w.r.t. \( \Theta \), with \( \Theta = \Theta^\dagger \) unbounded, positive and invertible. Then, it is possible to introduce two operators \( a \) and \( b \) on \( \mathcal{H} \), and a sequence of real numbers \( \{\epsilon_n, n \in \mathbb{N}_0\} \), such that the triple \((a, b, [\epsilon_n])\) is a family of nonregular NLPB.

Vice versa, if \((a, b, [\epsilon_n])\) is a family of nonregular NLPB, two operators can be introduced, \( H \) and \( \Theta \), such that \( \Theta = \Theta^\dagger \) is unbounded, positive and invertible, and \( H \) is well behaved w.r.t. \( \Theta \).

**Proof.** The proof is slightly different from that given for the bounded operator, so that we will give it here.

First, we assume that \( H \) is well behaved w.r.t. \( \Theta \), where \( \Theta = \Theta^\dagger \) is an unbounded, positive and invertible operator. Of course, our hypotheses imply that (i) \( H^\dagger := \Theta^{-1/2}H\Theta^{-1/2} \) is well defined and coincides with \( H \); (ii) that \( h = \Theta^{1/2}H\Theta^{-1/2} \) is also well defined, and self-adjoint; (iii) that \( \mathcal{E} \) is an o.n. basis of eigenvectors of \( h \), with eigenvalues \( \{\epsilon_n\} \), of \( H \): \( \epsilon_n = \epsilon_n\epsilon_n \) for all \( n \geq 0 \).

Therefore, \( \Theta^{1/2}H\Theta^{-1/2}\epsilon_n = \epsilon_n\epsilon_n\epsilon_n \in D(\Theta^{-1/2}) \); consequently, \( H(\Theta^{-1/2}\epsilon_n) = \epsilon_n(\Theta^{-1/2}\epsilon_n) \). This suggests to define the vectors \( \Phi_n := \Theta^{-1/2}\epsilon_n \), which belong to \( D(H) \) and satisfy the eigenvalue equation \( H\Phi_n = \epsilon_n\Phi_n \). Since \( h = h^\dagger \), we can repeat the same considerations starting from \( h^\dagger \). Hence, defining \( \eta_n := \Theta^{1/2}\epsilon_n \), we deduce that \( \eta_n \in D(H^\dagger) \) and that \( H^\dagger\eta_n = \epsilon_n\eta_n \). The sets \( \mathcal{F}_\Phi := \{\Phi_n, n \geq 0\} \) and \( \mathcal{F}_\eta := \{\eta_n, n \geq 0\} \) can be proven to be bases of \( \mathcal{H} \). Indeed, let us take a vector \( f \in D(\Theta^{1/2}) \), such that \( f \) is orthogonal to all the vectors in \( \mathcal{F}_\eta \). Therefore, we have, for all \( n \geq 0 \),

\[
0 = \langle f, \eta_n \rangle = \langle f, \Theta^{1/2}\epsilon_n \rangle = \langle \Theta^{1/2}f, \epsilon_n \rangle,
\]

which implies that \( \Theta^{1/2}f = 0 \), so that \( f = 0 \) as well. Using standard results, see [20] for instance, we conclude that all the elements of \( \mathcal{H} \) can be expanded in terms of \( \mathcal{F}_\eta \), which is therefore a basis of all of \( \mathcal{H} \). Analogously, we can check that \( \mathcal{F}_\Phi \) is a basis of \( \mathcal{H} \). However, due to the fact that \( \Theta^{1/2}f \) are unbounded, \( \mathcal{F}_\eta \) and \( \mathcal{F}_\Phi \) are not Riesz bases.

Let us now define two operators \( a \) and \( b \) on \( D(a) = D(b) := D_\eta \) as follows: let \( f = \sum_{k=0}^N c_k\Phi_k \) be a generic vector in \( D_\Phi \). Then,

\[
a f := \sum_{k=0}^N c_k\sqrt{\epsilon_k}\Phi_{k-1}, \quad b f := \sum_{k=0}^N c_k\sqrt{\epsilon_{k+1}}\Phi_{k+1}. \tag{2.17}
\]

In particular, these imply that \( a \Phi_n = \sqrt{\epsilon_n} \Phi_{n-1} \) and that \( b \Phi_n = \sqrt{\epsilon_{n+1}} \Phi_{n+1} \) for all \( n \geq 0 \).

Now, recalling that \( \epsilon_0 = 0 \), we deduce that \( a\Phi_0 = 0 \). Also iterating the raising equation above, we find that \( \Phi_n := \frac{1}{\sqrt{\epsilon_n}} b^n \Phi_0 \), which implies, in particular, that \( \Phi_0 \in D^\infty(b) \). Hence, condition P1 of definition 3 is satisfied.

To check condition P2, we first have to compute \( a^\dagger \) and \( b^\dagger \). It is possible to check that for all \( n \geq 0 \), \( \eta_n \in D(a^\dagger) \cap D(b^\dagger) \), and that

\[
a^\dagger\eta_n = \sqrt{\epsilon_{n+1}}\eta_{n+1}, \quad b^\dagger\eta_n = \sqrt{\epsilon_n}\eta_{n-1}. \tag{2.18}
\]

so that, clearly, \( b^\dagger\eta_0 = 0 \) and, again acting iteratively, \( \eta_n \in D^\infty(a^\dagger) \). In fact, we find that \( \eta_n := \frac{1}{\sqrt{\epsilon_n}} a^\dagger\eta_0 \). Condition P3 is clearly true, while condition P4 was already proved.

Let us now prove the converse implication, that is, let us see how NLPB produce two operators, \( H \) and \( \Theta \), satisfying definition 2.
This is a consequence of equation (2.15), \( S_B M_0 S_B = \Theta M_0 \), which we can rewrite it as \( M_0 = S^{-1}_B M_0 S_B \). Hence, the operators \( H \) and \( \Theta \) in definition 1 are easily identified: \( H \) is \( M_0 \), while \( \Theta \) is \( S_B \), and \( M_0 = QH \) w.r.t. \( S_B \). With this in mind, the operator \( h \) becomes \( h = S^{1/2}_B M_0 S^{-1/2}_B \). First of all, we need to understand if \( h \) is well defined. For that, recalling the properties of \( S_B \) and using the spectral theorem, we deduce that \( S_B^{\pm 1/2} \) are well defined.

Let us now observe that if \( f \in D(S_B) \), then \( f \in D(S^{1/2}_B) \). This follows from the equality \( \langle f, S_B f \rangle = \| S^{1/2}_B f \|^2 \). Analogously, if \( h \in D(S^{1/2}_B) \), then \( h \in D(S^{-1/2}_B) \). Therefore, since \( \Phi_n \in D(S_B) \), \( \Phi_n \in D(S^{1/2}_B) \) as well, so that we can define new vectors of \( \mathcal{H} \) as \( e_n := S^{1/2}_B \Phi_n \), \( n \geq 0 \). Note that \( e_n \in D(S^{1/2}_B) \cap D(S^{-1/2}_B) \). In fact, we have \( S^{1/2}_B e_n = S_B \Phi_n = \eta_n \) and \( S^{-1/2}_B e_n = \Phi_n \). It follows that \( e_n \in D(h) \) and that \( he_n = e_n \). Standard arguments, [20], finally show that the linear span of \( \mathcal{E} := \{ e_n \} \) is dense in \( \mathcal{H} \), showing in this way that \( h \) is well defined. Finally, we can also check from the definition that \( \langle e_n, e_m \rangle = \delta_{n,m} \): \( \mathcal{E} \) is an o.n. basis of \( \mathcal{H} \). It is now clear that \( h = h^\dagger \).

We want to briefly consider few consequences of this theorem, which are very similar to those found in paper I.

1. Dirac’s representations of the operators introduced so far can again be easily deduced. Thus, we have

\[
a = \sum_{n=0}^{\infty} \sqrt{\epsilon_n} |\Phi_{n-1}\rangle \langle \eta_n|, \quad b = \sum_{n=0}^{\infty} \sqrt{\epsilon_{n+1}} |\Phi_{n+1}\rangle \langle \eta_n|.
\]

(2.19)

We can also deduce the similar expansions for \( a^\dagger \) and \( b^\dagger \) and for

\[
h = \sum_{n=0}^{\infty} \epsilon_n |e_n\rangle \langle e_n|, \quad H = \sum_{n=0}^{\infty} \epsilon_n |\Phi_n\rangle \langle \eta_n| \quad \text{and} \quad H^\dagger = \sum_{n=0}^{\infty} \epsilon_n |\eta_n\rangle \langle \Phi_n|.
\]

2. As in paper I, operators \( S_B \) and \( S_B \), and their square roots, behave as intertwining operators.

This is exactly the same kind of result we have deduced for regular pseudo-bosons, where biorthogonal Riesz bases and intertwining operators are recovered. For instance, equation (2.15) produces the following intertwining relation: \( S_B M_0 = \Theta M_0 S_B \).

3. Even if \( h \) is not required to be factorizable, it turns out that it can still be written as \( h = b_{\Theta} a_{\Theta} \), where \( a_{\Theta} = \Theta^{1/2}_B a \Theta^{-1/2}_B \) and \( b_{\Theta} = \Theta^{1/2}_B b \Theta^{-1/2}_B \). We can write \( [a_{\Theta}, b_{\Theta}] = \Theta^{1/2}_B [a, b] \Theta^{-1/2}_B \neq [a, b] \), but only if \( [a, b], \Theta^{1/2}_B = 0 \), which is the case for linear pseudo-bosons. Thus, the Hamiltonian \( h \) can be written in a factorized form at a formal level at least.

3. Non-regular NLPB in differential-operator realizations

This section will be divided into two parts, with the first one offering a physical motivation and background of what will be discussed in the second subsection.

3.1. Nonlinear supersymmetries

3.1.1. Antilinear operators. In a historical perspective and in the context of physics and quantum theory, the emergence of the pair of non-selfadjoint factorized operators (2.3) may be traced back to [10]. In this paper, the usual form of SUSYQM (in which one traditionally assumes that \( M = M' \) [14]) has been generalized. In our present language, the idea of [10] (cf also its presentation in a broader context in [4]) may be characterized as lying in the use of nonlinear regular pseudo-bosons. Indeed, in the approach of [10] using \( M \neq M' \), the
supersymmetry connecting bosons with fermions has been realized in the representation space spanned by states defined by equation (2.5).

The quantum system presented in [10] may be recalled here as our first illustration of the immediate applicability of the general NLRPB formalism in the very concrete physical and phenomenologically oriented situations. Firstly, following the notation of [10], we have to define the pair of the factorized sub-Hamiltonian operators

\[ M = M^{(±)} = B^{(±)} A^{(±)} \]

where

\[ A^{(±)} = \frac{d}{dx} + W^{(±)}(x), \quad B^{(±)} = -\frac{d}{dx} + W^{(±)}(x). \]

(3.1)

Once we fix a real constant \( \varepsilon > 0 \) and select, for the sake of definiteness,

\[ W^{(±)}(x) = ± \left[ \frac{1}{x ± i\varepsilon} - i(x ± i\varepsilon)^2 \right] \]

(3.2)

the main result is the validity of the refactorization

\[ M^{(+) T} A^{(-)} B^{(-) T} T \]

(3.3)

The complex-conjugation antilinear operator \( T \) can be interpreted as mimicking the time-reversal operation performed over the system.

The readers are recommended to find more details (e.g. the relevant older references and/or a generalization of the ansatz (3.2) in [10]). It is worth adding that the transition to non-Hermitian interactions makes the model truly inspiring. Its structure may be perceived as an immediate predecessor of the introduction of the abstract concept of pseudo-bosons in [21] where also an immediate follow-up preprint [22] has been cited.

A few years later, a further so-called tobogganic generalization of the whole formalism has been proposed and summarized, say, in the recent compact review paper [23]. The core of the generalization lied in the Riemann-surface-adapted generalization of the operator \( T \neq T^{-1} \). Due to the circumstances, one must set

\[ A = -T \frac{d}{dx} + TW^{(-)}(x), \quad B = \frac{d}{dx} T^{-1} + W^{(-)}(x) T^{-1} \]

i.e. one must redefine further the creation- and annihilation-like operators of equation (3.1).

3.1.2. Regularizations by complexifications. Among the illustrative textbook quantum systems of SUSYQM, a special role is played by the one-dimensional harmonic-oscillator Schrödinger equation

\[ \left( -\frac{d^2}{dr^2} + r^2 \right) \psi(r) = E \psi(r), \quad \psi(r) \in L_2(-\infty, \infty). \]

(3.4)

In [24], this example found a natural PT-symmetric two-parametric generalization in the so-called Kratzer harmonic oscillator

\[ \left( -\frac{d^2}{dr^2} + \frac{G}{(r - ic)^2} + r^2 - 2icx - c^2 \right) \psi(x) = E \psi(x), \quad \psi(x) \in L_2(-\infty, \infty). \]

(3.5)

Here the real constant \( c \neq 0 \) regularizes the centrifugal-like spike at any coupling strength \( G = \alpha^2 - 1/4 \) so that the wavefunctions may be defined as living on the whole real line. The parameter \( \alpha \) should be chosen positive and, in the simpler, non-degenerate case, non-integer. This implies [24] that the complete set of normalizable eigenfunctions may be numbered by the quasi-parity \( q = ± 1 \) and by the excitation \( n = 0, 1, 2, \ldots \). At the respective \( c \)-independent bound-state energies

\[ E = E_{qn} = 4n + 2 + 2q\alpha \]

(3.6)
wavefunctions become defined, in closed form, in terms of Laguerre polynomials $L_n^{(\gamma)}(x)$,
\[
\varphi(x) = \text{const.}(x - ic)^{-\gamma + 1/2} e^{-(x-ic)^2/2} L_n^{(-\gamma)}((x - ic)^2). \tag{3.7}
\]
Naturally, the new spectrum of energies is not equidistant, though it is still real and composed of the two equidistant subspectra.

Although the reality of the energies (3.6) of the states (2.1) themselves (possessing, in addition, the so-called unbroken PT symmetry [7]) seemed to be in contrast with the manifest non-Hermiticity of the underlying operators $M$, the puzzle has been clarified in [25]. We were able to show there that our apparently non-Hermitian model (3.5) generating the real spectrum of energies may be re-interpreted as self-adjoint. For this purpose, we showed, in [25], that the inner product may be modified in such a way that the induced norm remains positive definite. We also showed that in spite of the immanent ambiguity of such ‘hidden-Hermiticity-mediating’ changes of the inner product, one of the most natural definitions of a unique inner product may be based on the use of ‘quasi-parity’ [25] (which is now better known as ‘charge’ [8]).

3.1.3. The implementation of supersymmetry. In the ultimate stage of development of the SUSYQM construction as presented in [26], we were able to describe one of the most natural deformations of the structure of the creation- and annihilation-operator algebra. Its detailed form followed from the $c \neq 0$ regularization of the singular harmonic oscillator of equation (3.5) where the regularized Hamiltonian will be denoted by the superscripted bracket symbol $H^{[\alpha]}$ in what follows.

Our construction just paralleled the standard supersymmetrization of the current, regular harmonic oscillator (3.4) (cf [14] for details). Firstly, we replaced the above-proposed cubic-oscillator toy-model superpotential of equation (3.2) by its harmonic-oscillator alternative
\[
W^{(\gamma)}(x) = x - ic - \frac{\gamma + 1/2}{x - ic}, \quad c > 0 \tag{3.8}
\]
with any real $\gamma$. Secondly, in the manner compatible with the supersymmetric recipe yielding the two (namely ‘left’ and ‘right’, $\gamma$-numbered) families of quantum Hamiltonians
\[
H^{(\gamma)}_{(L)} = BA = \hat{p}^2 + W^2 - W', \quad H^{(\gamma)}_{(R)} = AB = \hat{p}^2 + W^2 + W', \tag{3.9}
\]
we verified that
\[
H^{(\gamma)}_{(L)} = H^{[\alpha]} - 2\gamma - 2, \quad H^{(\gamma)}_{(R)} = H^{[\beta]} - 2\gamma, \tag{3.10}
\]
where $\alpha = |\gamma|$ and $\beta = |\gamma + 1|$, respectively. Further details may be found in [26] and [27].

3.2. Non-regular pseudo-bosons

3.2.1. The hidden Lie-algebraic structures. In [26], we revealed the existence of the second-order differential operators
\[
A^{(-\gamma-1)}A^{(\gamma)} = A^{(\gamma-1)}A^{(-\gamma)} = A(\alpha) \\
B^{(-\gamma)}B^{(\gamma-1)} = B^{(\gamma)}B^{(-\gamma-1)} = B(\alpha)
\]
which acted as the true respective annihilation and creation operators in our spiked and complex harmonic-oscillator model
\[
A(\alpha)\mathcal{L}^{(\gamma)}_{N+1} = c_5(N, \gamma)\mathcal{L}^{(\gamma)}_N, \quad B(\alpha)\mathcal{L}^{(\gamma)}_N = c_5(N, \gamma)\mathcal{L}^{(\gamma)}_{N+1},
\]
where $c_5(N, \gamma) = -4\sqrt{(N + 1)(N + \gamma + 1)}$. The corresponding generalization of the pseudobosonic version of the Heisenberg algebra has been shown, in [28], for the Lie algebra
sl(2, IR) with the renormalized generators $A(α)/\sqrt{32}$, $B(α)/\sqrt{32}$ and $H^{(α)}/4$ and with the commutators

$$A(α)B(α) - B(α)A(α) = 8H^{(α)}$$

and

$$A(α)H^{(α)} - H^{(α)}A(α) = 4A(α), \quad H^{(α)}B(α) - B(α)H^{(α)} = 2B(α).$$

3.2.2. Reinterpretation. The operators $A(α)$ and $B(α)$, and the functions $L^{(γ)}_N$, allow us to construct a nontrivial example of NLPB satisfying p1–p4 of definition 3. For that we begin to define two operators $a := -A(α)$ and $b = -B(α)$, a countable family of vectors $Φ_n := L^{(γ)}_N$ and the following sequence of non-negative numbers: $ε_n = c_5(n + 1, γ)^2$. Then, $aΦ_n = e_n Φ_{n−1}$ and $bΦ_n = e_n Φ_{n+1}$. Let now $H$ be the closure of the linear span of the vectors $Φ_n$s, which, in general, is a proper subset of $H$. $H$ is a Hilbert space, in which a unique biorthogonal basis $F_0 = \{η_n\}$, $⟨η_n, Φ_m⟩ = δ_{n,m}$, can be introduced [29]. The first vector of this biorthogonal set, $η_0$, satisfies condition P2 of definition 3. Indeed we have

$$⟨b^iη_0, Φ_k⟩ = ⟨η_0, bΦ_k⟩ = √ε_{k+1}⟨η_0, Φ_{k+1}⟩ = 0$$

for all $k ≥ 0$. Hence, being $F_0$ complete, $b^iη_0 = 0$. To check now that $η_0$ belongs to $D^∞(a^1)$, we consider the following scalar product: $⟨a^iΦ_n, η_0⟩$, which is zero whenever $k > n$ due to the lowering property of $a$ on the set $F_0$. On the other hand, if $k ≤ n$, we deduce that $⟨a^iΦ_n, η_0⟩ = √ε_{k}δ_{n,k}$. Therefore, since $δ_{n,k} = ⟨Φ_n, η_k⟩$, we deduce that $⟨Φ_n, (\frac{1}{∞}_k)(a^1)^kη_0 − η_k⟩ = 0$ for all $n ≥ 0$. Hence, using once more the completeness of $F_0$, we deduce that $η_k = √ε_{k+1}(a^1)^kη_0$ for all $k$: this shows that $η_0 ∈ D^∞(a^1)$ and that the various vectors of the unique biorthogonal basis $F_0$ introduced above are related as in equation (2.1). To fulfill all the requirements of definition 3, we finally have to prove that $b^iη_n = √ε_nη_{n−1}$. The proof goes like this: for all $k ≥ 0$,

$$⟨b^iη_n, Φ_k⟩ = ⟨η_n, bΦ_k⟩ = √ε_{k}δ_{n−1,k} = √ε_n⟨η_{n−1}, Φ_k⟩,$$

so that $⟨b^iη_n − √ε_nη_{n−1}, Φ_k⟩ = 0$ for all $k$. Since $F_0$ is complete, this proves that $b$ is a lowering operator for $F_0$, as required. We conclude that NLPB can be constructed out of this model, but they are not, in general, regular, due to the fact that the operator $S_h$ mapping $F_0$ to $F_3$ is, in general, unbounded. Incidentally, we also deduce that $Φ_n$ and $η_n$ are respectively eigenstates of $H^{(α)}$ and $H^{(α)∗}$ with the same eigenvalue, $\frac{1}{2}(ε_{n+1} − ε_n)$. This is connected to the fact that these operators are related to the operators $M = ba$ and $M^\dagger = a^1b^1$ (and to their specular counterparts $N = ab$ and $N^\dagger = b^1a^1$) introduced in section 2.

4. Conclusions

The key motivation of our present NLPB-related studies I and II was twofold. Firstly, in a series of papers [30–36] one of us (FB) considered the canonical commutation relations $[a, b] = 1$ in a generalized version in which $b$ was not necessarily equal to $a^1$. In parallel, in another series of papers (cf, e.g., their most recent samples [37–41]), the second one of us (MZ) studied the possibility of the weakening of the Hermiticity of the observables in a few quantum systems of an immediate phenomenological appeal and/or methodical interest.

In paper I, we announced the possibility of connecting these two alternative points of view. In particular, we addressed the problem while simplifying its technical aspects by the acceptance of the operator-boundedness assumptions as currently made in the physics literature.
This enabled us to clarify the role of the metric (specifying the inner products in the correct Hilbert space of states) from the NLPB point of view and vice versa. We also endorsed the message of [11] and [5] by re-recommending the practical use of the factorizations of the metrics \( \Theta \) into the individual Dyson-map factors \( \Omega \).

Later on we consulted several less accessible mathematics-oriented references (e.g. [42]) and imagined that there exist many situations in physics (with some of them being cited above) in which the picture provided by the bounded operators appears insufficient. For this reason, we returned to the subject of paper I. In its present continuation, we incorporated the above-mentioned new knowledge and perspective into a necessary weakening of the underlying mathematical assumptions.

In paper II, we revealed, first of all, that in the territory of unbounded operators the functional structure obtained from \( a \) and \( b \), and from the so-called pseudo-bosons related to these, may be much richer than the one described in paper I. Still, many ideas of paper I survived the generalization. In particular, in the presently specified quasi-Hermitian case, we still succeeded in the clarification of the conditions under which one can still work with the NLPB formalism where the two biorthogonal bases remain obtainable as eigenstates of the two number-like operators, \( M \) and \( M^\dagger \), with eigenvalues which are not equal to integers in general.

In a certain synthesis of our originally separate starting positions, we showed that even in the quasi-Hermitian context with unbounded operators, the doublet of the generalized number operators \( M \) and \( M^\dagger \) (and of \( N \) and \( N^\dagger \) as well) may still be perceived as interconnected by an intertwining operator. The latter intertwinner has been shown specified using the two sets of eigenstates. At this point, we made an ample use of the extended notion of pseudo-bosons in which their essential characteristics play the role. In this sense, we believe that the role of the generalized number operators might acquire more and more relevance in the future applications of the formalism where the boundedness of the operators of the observables cannot be guaranteed and where, in addition, the Hermiticity of these operators is ‘hidden’.

We dare to believe that our present results might encourage a further growth of interest in the practical use of quasi-Hermitian operators of observables in applied quantum theory. Keeping in mind the presence of many obstacles and mathematical puzzles in this field, we expect that the present clarification of at least some of them might re-encourage the mathematically sufficiently rigorous further search for the representations of quantum systems in which the inner product in the standard Hilbert space is nontrivial. In particular, our present results might encourage a return to all of those constructions where the physical inner products proved formally represented by unbounded metric operators \( \Omega \) while their formal factorization \( \Theta = \Omega^\dagger \Omega \) into ‘microscopic’ Dyson maps would still be difficult to perform.

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