Second-order matter density perturbations and skewness in scalar–tensor modified gravity models

Takayuki Tatekawa$^{1,2,3}$ and Shinji Tsujikawa$^4$

$^1$ Department of Computer Science, Kogakuin University, 1-24-2 Nishi-shinjuku, Shinjuku, Tokyo 163-8677, Japan
$^2$ Research Institute for Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan
$^3$ Department of Physics, Ochanomizu University, 2-1-1 Otsuka, Bunkyo, Tokyo 112-8610, Japan
$^4$ Department of Physics, Faculty of Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

E-mail: tatekawa@cpd.kogakuin.ac.jp and shinji@rs.kagu.tus.ac.jp

Received 13 July 2008
Accepted 27 August 2008
Published 18 September 2008

Abstract. We study second-order cosmological perturbations in scalar–tensor models of dark energy that satisfy local gravity constraints, including $f(R)$ gravity. We derive equations for matter fluctuations under a sub-horizon approximation and clarify conditions under which first-order perturbations in the scalar field can be neglected relative to second-order matter and velocity perturbations. We also compute the skewness of the matter density distribution and find that the difference from the ΛCDM (CDM: cold dark matter) model is less than a few per cent even if the growth rate of first-order perturbations is significantly different from that in the ΛCDM model. This shows that the skewness provides a model-independent test for the picture of gravitational instability from Gaussian initial perturbations including scalar–tensor modified gravity models.

Keywords: dark energy theory, cosmological perturbation theory
1. Introduction

Constantly accumulating observational data [1] continue to confirm that the Universe has entered a phase of accelerated expansion following a matter-dominated epoch. The origin of the dark energy (DE) responsible for this late-time acceleration represents one of the most serious stumbling blocks in modern cosmology [2, 3]. The first step toward understanding the nature of DE is to find a signature indicating whether it originates from some modification of gravity or comes from some exotic matter with negative pressure. If gravity is modified from Einstein’s general relativity, this leaves a number of interesting experimental and observational signatures that can be tested. In particular, local gravity experiments generally place tight bounds for the parameter space of modified gravity models.

Already, many modified gravity DE models have been proposed—ranging from $f(R)$ gravity [4] ($R$ is a Ricci scalar), through scalar–tensor theory [5, 6], to braneworld scenarios [7]. The $f(R)$ gravity is probably the simplest generalization to the $\Lambda$ cold dark matter ($\Lambda$CDM) model ($f(R) = R - \Lambda$). Nevertheless it is generally not easy to construct viable $f(R)$ models that satisfy all stability, experimental and observational constraints while at the same time showing appreciable deviations from the $\Lambda$CDM model. In order to avoid a scalar degree of freedom (scalaron) as well as a graviton becoming ghosts or tachyons we require the conditions $f_{RR} > 0$ and $f_R > 0$ [8]. These conditions are also needed for the stability of density perturbations [9]. For the existence of a matter-dominated epoch followed by a late-time acceleration, the models need to be close to the $\Lambda$CDM model ($m \equiv Rf_{RR}/f_R \approx +0$) in the region $R \gg R_0$ ($R_0$ is the present cosmological Ricci scalar) [10]. Moreover the mass of the scalaron field in the region $R \gg R_0$ is sufficiently heavy for compatibility with local gravity experiments [11]–[14].
Finally, for the presence of a stable de Sitter fixed point at $r \equiv -R f_{,R}/f = -2$, we require that $0 \leq m(r = -2) \leq 1$ [10,15]. The models proposed by Hu and Sawicki [16] and Starobinsky [8] satisfy all of these requirements. They take the asymptotic form $f(R) \simeq R - \mu R_c [1 - (R/R_c)^{-2n}]$ ($\mu > 0, R_c > 0, n > 0$) in the region $R \gg R_c$ ($R_c$ is of roughly the same order as $R_0$). See [17]–[21] for other viable $f(R)$ models.

The main reason that viable $f(R)$ models are so restrictive is that the strength of a coupling $Q$ between dark energy and non-relativistic matter (such as dark matter) is large in the Einstein frame ($Q = -1/\sqrt{6}$) [22]. In the region of high density where local gravity experiments are carried out, the scalaron field $\phi$ needs to be almost frozen [11,12] with a large mass through a chameleon mechanism [23] to avoid the field mediating a long ranged fifth force. Cosmologically, this means that the field does not approach a kinematically driven $\phi$ matter-dominated era ("$\phi$MDE" [24]) in which the evolution of the scale factor is non-standard ($a \propto t^{1/2}$ [22]). The deviation from the $\Lambda$CDM model becomes important as the field begins to evolve slowly along its potential together with the decrease of the scalaron mass. In other words, the effect of modified gravity manifests itself from the late-time matter era to the accelerated epoch [8,16]. This leaves a number of interesting observational signatures for the equation of state of DE [18,20], matter power spectra [8,9,20] and convergence spectra in weak lensing [25,26].

One can generalize the analysis in $f(R)$ gravity to theories that have arbitrary constant couplings $Q$ [27]. In fact this is equivalent to the Brans–Dicke theory [28] with a scalar-field potential $V(\phi)$. By designing the potential so that the field mass is sufficiently heavy in the region of high density, it is possible to satisfy both local gravity and cosmological constraints even when $|Q|$ is of the order of unity [27]. A representative potential of this type is given by $V(\phi) = V_0 [1 - C (1 - e^{-2Q\phi})^p]$ ($V_0 > 0, C > 0, 0 < p < 1$), which covers the $f(R)$ models of Hu and Sawicki [16] and Starobinsky [8] as special cases. In particular, when $|Q|$ is of the order of unity, these models lead to large growth of the matter density perturbations ($\delta \propto t^{(\sqrt{25+48Q^2}-1)/6}$) at a late epoch of the matter era as compared to the standard growth ($\delta \propto t^{2/3}$) at an early epoch. This gives rise to a significant change of the spectral index of the matter power spectrum relative to that in the $\Lambda$CDM model [8,20]. Moreover it was recently shown that the convergence power spectrum in weak lensing observations is subject to a large modification by the non-standard evolution of matter perturbations [25].

In this paper we shall study another test of the modified gravity DE models mentioned above by evaluating a normalized skewness, $S_3 = \langle \delta^3 \rangle / \langle \delta^2 \rangle^2$, of matter perturbations. The skewness provides a good test for the picture of gravitational instability from Gaussian initial conditions [29]. If large-scale structure grows via gravitational instability from Gaussian initial perturbations, the skewness in a Universe dominated by pressureless matter is known to be $S_3 = 34/7$ in general relativity [30]. Even when a cosmological constant is present at late times, the skewness depends weakly on the expansion history of the Universe (less than a few per cent) [31–33]. This situation hardly changes in open/closed Universes [34] and Dvali–Gabadadze–Poratti braneworld models [35]. One can see some difference for the models that are significantly different from Einstein gravity—such as Cardassian cosmologies [35,36], modified gravity models that respect Birkhoff’s law [37]. In the context of dark energy coupled with dark matter, it was shown in [38] that the skewness can be a probe of the violation of equivalence principle between dark matter and (uncoupled) baryons.
In Brans–Dicke theory with cosmological constant $\Lambda$ the skewness has been calculated in [39] under the condition that the Brans–Dicke field is massless. In this case the evolution of the scale factor during the matter-dominated epoch is given by $a(t) \propto t^{(2\omega_{BD}+2)/(3\omega_{BD}+4)}$ [39], where $t$ is a cosmic time and $\omega_{BD}$ is a Brans–Dicke parameter. If the field is massless, the Brans–Dicke parameter is constrained to be $\omega_{BD} > 40,000$ [40] from solar-system experiments. This shows that the evolution of the scale factor in the matter era is very close to the standard one: $a(t) \propto t^{2/3}$. We note that an effective gravitational ‘constant’ that appears as a coefficient of matter density perturbations is also subject to change in Brans–Dicke theory. However, it was found that the skewness in such a case is given by $S_3 = (34\omega_{BD} + 56)/(7\omega_{BD} + 12)$ [39] during the matter era, which is very close to the standard one ($S_3 = 34/7$) under the condition $\omega_{BD} > 40,000$.

The $f(R)$ gravity corresponds to theory with the Brans–Dicke parameter $\omega_{BD} = 0$ [41]. Even in this situation, if the scalar field has a potential whose mass is sufficiently large in the region of high density, the $f(R)$ models can pass local gravity constraints as in the models proposed in [8, 16]. In such cases, as compared to Brans–Dicke theory with a massless field, it is expected that the skewness may show significant deviations from that in general relativity. Since the evolution of the scale factor and matter perturbations is different from that in the massless case, we cannot employ the result for the skewness presented above.

In this paper we study second-order perturbations and the skewness for Brans–Dicke theory in the presence of a potential $V(\phi)$. This is equivalent to the scalar-field action given in equation (2) by identifying the coupling $Q$ with the Brans–Dicke parameter $\omega_{BD}$ via the relation $1/(2Q^2) = 3 + 2\omega_{BD}$. In the massless case the solar-system constraint, $\omega_{BD} > 40,000$, gives the bound $|Q| \lesssim 10^{-3}$, but it is difficult to find deviations from general relativity in such a situation. Our interest is in the case in which the coupling $Q$ is of the order of $0.1 \lesssim |Q| \lesssim 1$ with a field potential that has a sufficiently large mass in the high density region. This analysis includes viable $f(R)$ models [8, 16] recently proposed in the literature. We would like to investigate to what extent the skewness differs from that in the $\Lambda$CDM model. We also derive conditions under which the contribution coming from first-order field perturbations can be neglected relative to second-order matter and velocity perturbations by starting from fully relativistic second-order perturbation equations.

2. Modified gravity models

The action for the Brans–Dicke theory [28] in the presence of a potential $V$ is given by

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} \chi R - \frac{\omega_{BD}}{2\chi} (\nabla \chi)^2 - V(\chi) \right] + S_m(g_{\mu\nu}, \Psi_m), \quad (1)$$

where $\chi$ is a scalar field coupled to a Ricci scalar $R$, $\omega_{BD}$ is a so-called Brans–Dicke parameter and $S_m$ is a matter action that depends on the metric $g_{\mu\nu}$ and matter fields $\Psi_m$. We shall use the unit $8\pi G = 1$, but we restore the bare gravitational constant $G$ when it is required.

The action (1) is equivalent to the following scalar–tensor action with the correspondence $\chi = e^{-2Q\phi}$:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} F(\phi) R - \frac{1}{2} \omega(\phi) (\nabla \phi)^2 - V(\phi) \right] + S_m(g_{\mu\nu}, \Psi_m), \quad (2)$$
where
\[ F(\phi) = e^{-2Q\phi}, \quad \omega(\phi) = (1 - 6Q^2)F(\phi). \] (3)

As we already mentioned, the constant \( Q \) is related to \( \omega_{\text{BD}} \) via the relation \( 1/(2Q^2) = 3 + 2\omega_{\text{BD}} \). In the limit \( Q \to 0 \) (i.e., \( \omega_{\text{BD}} \to \infty \)), the action (2) reduces to that for a minimally coupled scalar field \( \phi \) with a potential \( V(\phi) \). The \( f(R) \) gravity corresponds to the coupling \( Q = -1/\sqrt{6} \), i.e., \( \omega_{\text{BD}} = 0 \).

In the absence of the potential \( V(\phi) \) the coupling \( Q \) is constrained to be \( |Q| \lesssim 10^{-3} \) from solar-system tests. We are interested in the case where the presence of the potential can make the models consistent with local gravity constraints (LGC) even for \( |Q| = \mathcal{O}(1) \).

The representative potential of this type is given by [27]
\[ V(\phi) = V_0 [1 - C(1 - e^{-2Q\phi})^p] \quad (V_0 > 0, C > 0, 0 < p < 1), \] (4)

where \( V_0 \) is of the order of the present cosmological Ricci scalar \( R_0 \) in order for it to be responsible for the acceleration of the Universe today. Note that the \( f(R) \) models proposed by Hu and Sawicki [16] and Starobinsky [8] take the form \( f(R) = R - \mu R_c[1 - (R/R_c)^{-2n}] \) \((\mu > 0, R_c > 0, n > 0)\) in the region \( R \gg R_c \). These \( f(R) \) models are covered in the action (2) with (4) by identifying the field potential to be \( V = (RF - f)/2 \) with \( F = \partial f/\partial R = e^{2\phi/\sqrt{6}} \).

The background cosmological dynamics and LGC for the potential (4) have been discussed in detail in [27]. We review how the matter-dominated era is followed by the stage of late-time acceleration. This is important when we discuss the evolution of matter density perturbations in section 4. As a matter source we take into account non-relativistic matter with an energy density \( \rho_m \). In the flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric with scale factor \( a(t) \), where \( t \) is cosmic time, the evolution equations for the action (2) are
\[ 3FH^2 = \frac{1}{2}\omega\dot{\phi}^2 + V - 3HF + \rho_m, \] (5)
\[ 2F\ddot{H} = -\omega\ddot{\phi}^2 - \ddot{F} + H\dot{F} - \rho_m, \] (6)
\[ \omega \left( \dddot{\phi} + 3H\dot{\phi} + \frac{\dot{F}}{2F}\dot{\phi} \right) + V_{,\phi} - \frac{1}{2}F_{,\phi}R = 0, \] (7)
\[ \dot{\rho}_m + 3H\rho_m = 0, \] (8)

where \( H \equiv \dot{a}/a \) is the Hubble parameter and a dot represents a derivative with respect to \( t \). Note that the Ricci scalar is given by \( R = 6(2H^2 + \dot{H}) \).

We introduce the following dimensionless quantities:
\[ x_1 \equiv \frac{\dot{\phi}}{\sqrt{6H}}, \quad x_2 \equiv \frac{1}{H\sqrt{3F}}, \] (9)
and
\[ \Omega_m \equiv \frac{\rho_m}{3FH^2} = 1 - (1 - 6Q^2)x_1^2 - x_2^2 - 2\sqrt{6}Qx_1, \] (10)
where we used equation (5). We then obtain
\[ \frac{dx_1}{dN} = \frac{\sqrt{6}}{2} (\lambda x_2^2 - \sqrt{6} x_1) + \frac{\sqrt{6}Q}{2} [(5 - 6Q^2)x_1^2 + 2\sqrt{6}Q x_1 - 3x_2^2 - 1] - x_1 \frac{\dot{H}}{H^2}, \]
\[ \frac{dx_2}{dN} = \frac{\sqrt{6}}{2} (2Q - \lambda)x_1 x_2 - x_2 \frac{\dot{H}}{H^2}, \]
where \( N \equiv \ln(a) \) and \( \lambda = -V_{,\phi}/V \). The effective equation of state is defined by
\[ w_{\text{eff}} = -1 - \frac{2}{3} \frac{\dot{H}}{H^2}, \]
where
\[ \frac{\dot{H}}{H^2} = -\frac{1 - 6Q^2}{2} [3 + 3x_1^2 - 3x_2^2 - 6Q^2x_1^2 + 2\sqrt{6}Q x_1] + 3Q(\lambda x_2^2 - 4Q). \]

When \( \lambda \) is a constant (i.e., \( V(\phi) = V_0 e^{-\lambda \phi} \)), the fixed points of the system can be derived by setting \( \frac{dx_1}{dN} = \frac{dx_2}{dN} = 0 \). Even if \( \lambda \) changes with time, as is the case for the potential (4), the fixed points can be regarded as instantaneous ones. The following points can play the role of the matter-dominated epoch:

- **(M1)** \( \phi \) matter-dominated era:
  \[ (x_1, x_2) = \left( \frac{\sqrt{6}Q}{3(2Q^2 - 1)}, 0 \right), \quad \Omega_m = \frac{3 - 2Q^2}{3(1 - 2Q^2)^2}, \quad w_{\text{eff}} = \frac{4Q^2}{3(1 - 2Q^2)}. \]

- **(M2)** ‘Instantaneous’ scaling solution:
  \[ (x_1, x_2) = \left( \frac{\sqrt{6}}{2\lambda} \left[\frac{3 + 2Q\lambda - 6Q^2}{2\lambda^2}\right]^{1/2} \right), \quad \Omega_m = 1 - \frac{3 - 12Q^2 + 7Q\lambda}{\lambda^2}, \quad w_{\text{eff}} = -\frac{2Q}{\lambda}. \]

In order to realize the matter era (\( \Omega_m \simeq 1 \) and \( w_{\text{eff}} \simeq 0 \)) via the point (M1), we require the condition \( Q^2 \ll 1 \). This point was used in the coupled quintessence scenario [24] (in the Einstein frame) where the coupling is constrained to be \( |Q| \lesssim 0.1 \) from cosmic microwave background anisotropies. In \( f(R) \) gravity (\( Q = -1/\sqrt{6} \)) we have \( \Omega_m = 2 \) and \( w_{\text{eff}} = 1/3 \) (i.e., \( a \propto t^{1/2} [22] \)), which means that the point (M1) cannot be responsible for the matter era for \( |Q| \) of the order of unity.

The matter era can be realized via the point (M2) for \( |\lambda| \gg |Q| = \mathcal{O}(1) \). The parameter \( \lambda \) for the potential (4) is given by
\[ \lambda = \frac{2CpQe^{-2Q\phi}(1 - e^{-2Q\phi})^{p-1}}{1 - C(1 - e^{-2Q\phi})p}, \]
which is much larger than 1 for \( |Q\phi| \ll 1 \) (provided that \( C \) and \( p \) are not very much smaller than 1). Since \( R \simeq \rho_m/F \) during the deep matter-dominated epoch, the field \( \phi \) is stuck at the instantaneous minima characterized by the condition \( V_{,\phi}(\phi_m) + Q\rho_m \simeq 0 \).
Second-order matter density perturbations

(see equation (7)). For the potential (4) this translates into

$$\phi_m \approx \frac{1}{2Q} \left( \frac{2V_0 pC}{\rho_m} \right)^{1/(1-p)},$$

which means that $|Q\phi_m| \ll 1$ and hence $|\lambda| \gg 1$ during the deep matter era ($\rho_m \gg V_0$). When $|Q| = \mathcal{O}(1)$, the matter era is realized via the point (M2) instead of (M1).

For the dynamical system given by equations (11) and (12) there exist the following fixed points that lead to the late-time acceleration:

• (A1) The scalar-field-dominated point:
  $$\begin{align*}
  (x_1, x_2) &= \left( \frac{\sqrt{6}(4Q - \lambda)}{6(4Q^2 - Q\lambda - 1)}, \frac{6 - \lambda^2 + 8Q\lambda - 16Q^2}{6(4Q^2 - Q\lambda - 1)^2} \right)^{1/2}, \\
  \Omega_m &= 0, \\
  w_{\text{eff}} &= - \frac{20Q^2 - 9Q\lambda - 3 + \lambda^2}{3(4Q^2 - Q\lambda - 1)}. 
  \end{align*}$$

• (A2) The de Sitter point (present for $\lambda = 4Q$):
  $$\begin{align*}
  (x_1, x_2) &= (0, 1), \\
  \Omega_m &= 0, \\
  w_{\text{eff}} &= -1. 
  \end{align*}$$

The de Sitter point (A2) appears only in the presence of the coupling $Q$ (characterized by the condition $V_{\phi\phi} + QF\dot{R} = 0$ in equation (7), i.e., $\lambda = 4Q$). This can be regarded as a special case of the accelerated point (A1). For the potential (4) the parameter $|\lambda|$ is much larger than $|Q|$ during the matter era, but it gradually becomes of the same order as $|Q|$ as the system enters the accelerated epoch. It was shown in [27] that the de Sitter point (A2) is stable for $d\lambda/d\phi < 0$. As long as $|\lambda|$ continues to decrease with the growth of $|\phi|$, the solutions are finally trapped at the stable de Sitter point (A2). If the stability condition, $d\lambda/d\phi < 0$, is not satisfied, the solutions approach another accelerated point (A1).

In the following we are mainly interested in the case where the ‘instantaneous’ matter point (M2) is followed by the de Sitter point (A2). During most stages of cosmic expansion history the field $\phi$ is trapped at instantaneous minima of an effective potential induced by the matter coupling. This means that the condition $\dot{\phi}^2 \ll H^2$ is well satisfied.

The mass squared of the field $\phi$ for the potential (4) is given by

$$M^2 \equiv V_{\phi\phi} = 4V_0 C p Q^2 e^{-2Q\phi} (1 - e^{-2Q\phi})^{p-2} (1 - pe^{-2Q\phi}),$$

which is much larger than $R_0(\sim V_0)$ in the region $R \gg R_0$. In this situation it is possible to satisfy local gravity constraints in the region of high density [11, 12, 16] through a chameleon mechanism [23]. Since the field is massive inside a spherically symmetric body with radius $r_c$, only the surface part of its mass distribution contributes to the field profile outside the body. The effective coupling $Q_{\text{eff}}$ between the field and the pressureless matter is suppressed by a thin-shell parameter $\Delta r_c/r_c$ relative to the bare coupling $Q$. For the potential (4) it was shown in [12] that constraints coming from solar-system tests as well as the violation of equivalence principle give the bounds $p > 1 - 5/(9.6 - \log_{10}(Q))$ and $p > 1 - 5/(13.8 - \log_{10}(Q))$, respectively. In $f(R)$ gravity these constraints correspond to $p > 0.50$ and $p > 0.65$, respectively.
Substituting the field value (18) for equation (21), we find that the mass squared during the matter era is given by

$$M^2 \simeq \left( \frac{3^{2-p}}{2^p p C} \right)^{1/p} (1 - p) Q^2 \left( \frac{H^2}{V_0} \right)^{1/p} H^2,$$

(22)

where we used $3H^2 \simeq \rho_m$. We then find that the inequality $M^2 \gg H^2$ holds for values of $p, C, Q$ not very much smaller than unity. In the next section we shall use this property when we derive the equation for matter perturbations approximately.

3. Second-order cosmological perturbations

In this section we consider second-order cosmological perturbations for the action (2) and derive the equation for matter perturbations approximately.

3.1. Perturbation equations

Let us start with a perturbed metric including scalar metric perturbations $\alpha, \beta, \varphi$ and $\gamma$ about the flat FLRW background [42]:

$$ds^2 = -(1 + 2\alpha)dt^2 - 2a\beta_i dt dx^i + a(t)^2 \left[ (1 + 2\varphi)\delta_{ij} + 2\gamma_{ij} \right] dx^i dx^j.$$

(23)

At second order the scalar variables are written as

$$\alpha \equiv \alpha^{(1)} + \alpha^{(2)}, \quad \beta \equiv \beta^{(1)} + \beta^{(2)}, \quad \varphi \equiv \varphi^{(1)} + \varphi^{(2)}, \quad \gamma \equiv \gamma^{(1)} + \gamma^{(2)},$$

(24)

where the subscripts represent the orders of perturbations. We introduce the following quantities:

$$\chi \equiv a(\beta + a^\gamma), \quad \kappa \equiv \delta K,$$

(25)

where $\delta K$ is the perturbation of an extrinsic curvature $K$.

We decompose the scalar field $\phi$ and the quantity $F$ into background and perturbed parts:

$$\phi = \phi_0(t) + \delta\phi(t, \mathbf{x}), \quad F = F_0(t) + \delta F(t, \mathbf{x}),$$

(26)

where $\delta\phi$ and $\delta F$ depend on $t$ and a position vector $\mathbf{x}$. In the following we omit the subscript ‘0’ from background quantities. The components of the energy–momentum tensor of pressureless matter can be decomposed as

$$T^0_0 = -(\rho_m + \delta \rho_m), \quad T^0_i = -\rho_m v_i, \equiv q_{,i},$$

(27)

where $v$ is a rotation-free velocity potential. At second order, the perturbed quantities can be explicitly written as

$$\delta\phi \equiv \delta\phi^{(1)} + \delta\phi^{(2)}, \quad \delta F \equiv \delta F^{(1)} + \delta F^{(2)}, \quad \delta \rho_m \equiv \delta \rho_m^{(1)} + \delta \rho_m^{(2)}, \quad v \equiv v^{(1)} + v^{(2)},$$

(28)
Second-order matter density perturbations

The perturbation equations for the action (2), up to second order, have been derived in [43] (see also [44]). They are given by

\[ \kappa - 3H\alpha + 3\dot{\phi} + \frac{\Delta}{a^2}\chi = -\alpha \left( \frac{9}{2}H\alpha - \frac{1}{2}a^2\beta \right) + \frac{3}{2}H\beta^i\beta_i, \]  

\[ 4\pi G\delta\rho_{\text{eff}} + H\kappa + \frac{\Delta}{a^2}\varphi = \frac{1}{6}\kappa^2 - \frac{1}{4a^2}\beta_{(ij)}\beta^{ij} + \frac{1}{12a^2}(\beta^i\beta_i)^2, \]  

\[ \kappa + \frac{\Delta}{a^2}\chi - 12\pi G\rho v - \frac{3}{2}\frac{\varphi}{F}(\varphi\dot{\phi} + \dot{\varphi} - H\delta F - \dot{F}\alpha) \]

\[ = \Delta^{-1}\nabla^i \left[ -\alpha(\kappa,_{i} + 12\pi Ga_{q,i}) + \frac{3}{4a}\gamma_j(\beta^i_{[i} + \beta_i^{j]} - \frac{1}{2}\alpha_{i}\beta_{j}^{j}] \right], \]  

\[ \dot{\kappa} + 2H\kappa - 4\pi G(\delta\rho_{\text{eff}} + 3\delta P_{\text{eff}}) + \left( 3\dot{H} + \frac{\Delta}{a^2} \right) \alpha \]

\[ = \alpha\kappa - \frac{1}{a}\kappa_{i}\beta^i + \frac{1}{3}\kappa^2 + \frac{3}{2}\dot{H}(\alpha^2 - \beta^i\beta_i) + \frac{1}{a^2}(2\alpha\dot{\alpha}_{i} + \alpha_i\alpha^i)
\]

\[ - \beta^j\beta^i_{[i} - \beta^{i}_{[i}]\beta^j_{i]} + \frac{1}{a^2}\beta_{(ij)j}^{i} - \frac{1}{3a^2}(\beta^i_{[i} \beta_i^{j]}), \]

\[ \delta\dot{\rho}_m + 3H\delta\rho_m - \rho_m \left( \kappa - 3H\alpha + \frac{1}{a}\Delta v \right) \]

\[ = -\frac{1}{a}\delta\rho_{m,i}\beta^i + \delta\rho_m(\kappa - 3H\alpha) + \rho_m \left[ \alpha\kappa + \frac{3}{2}H(\alpha^2 - \beta^i\beta_i) \right]
\]

\[ - \frac{1}{a}(\alpha q_{i} + 2q^i\alpha_i), \]

\[ \dot{\varphi} + Hv - \frac{1}{a}\alpha = \rho_m \Delta^{-1}\nabla^i \left[ q_{i}(\kappa - 3H\alpha) + \frac{1}{a}\left\{ -q_{i,j}\beta^j - q_{j}\beta^j_{i} - \delta\rho_m\alpha_i \right. \right.
\]

\[ + \rho_m(\alpha\alpha_{i} - \beta^j\beta_{j}[i]) \right] \],

where

\[ \delta\rho_{\text{eff}} = \frac{1}{8\pi GF} \left[ \delta\rho_m + \omega(\dot{\phi}\dot{\phi} - \alpha\dot{\phi}^2) + \frac{1}{2}\omega\phi\delta\phi^2 - \frac{1}{2}(F_{\phi}R - 2V_{\phi})\delta\phi - 3H\delta F \right.
\]

\[ + \left( \frac{1}{2}R + \frac{\Delta}{a^2} \right) \delta F + \left( 6H\alpha - \frac{\Delta}{a^2}\chi - 3\dot{\phi} \right) \tilde{F} \]

\[ - \frac{\delta F}{F} \left( \rho_m + \frac{1}{2}\omega\phi^2 + V - 3HF \right) \],

\[ \delta P_{\text{eff}} = \frac{1}{8\pi GF} \left[ \omega(\dot{\phi}\dot{\phi} - \alpha\dot{\phi}^2) + \frac{1}{2}\omega\phi\delta\phi^2 + \frac{1}{2}(F_{\phi}R - 2V_{\phi})\delta\phi + \delta\tilde{F} + 2H\delta \tilde{F} \right.
\]

\[ - \left( \frac{1}{2}R + \frac{2\Delta}{3a^2} \right) \delta F - 2\alpha\tilde{F} - \left( \dot{\alpha} + 4H\alpha - \frac{2\Delta}{3a^2}\chi - 2\dot{\phi} \right) \tilde{F} \]

\[ - \frac{\delta F}{F} \left( \frac{1}{2}\omega\phi^2 - V + \tilde{F} + 2H\tilde{F} \right) \].
The equations for the perturbations $\delta \phi$ and $\delta F$ are
\begin{align}
\delta \ddot{\phi} + \left(3H + \frac{\dot{\omega} \dot{\phi}}{\omega} \right) \delta \dot{\phi} + \left[ -\frac{\Delta}{a^2} + \frac{\dot{\omega} \dot{\phi}}{\omega} \right] \delta \phi + \left[ \frac{2V_{,\phi} - F_{,\phi} R}{2\omega} \right] \delta \phi - \dot{\phi} \dot{\phi}
&- \left(2\ddot{\phi} + 3H \dot{\phi} + \frac{\dot{\omega} \dot{\phi}^2}{\omega} \right) \alpha - \dot{\phi} \kappa - \frac{1}{2\omega} F_{,\phi} \delta R = N_{\delta \phi}, \tag{37}
\end{align}
\begin{align}
\delta \ddot{F} + 3H \delta \dot{F} + \left( -\frac{\Delta}{a^2} - \frac{R}{3} \right) \delta F + \frac{2}{3} \omega \dot{\phi} \delta \dot{\phi} + \frac{1}{3} (\omega \cdot \dot{\phi})^2 + 2F_{,\phi} R - 4V_{,\phi} \delta \phi
&- \frac{1}{3} \delta \rho_m - \dot{F} (\kappa + \dot{\alpha}) - \left( \frac{2}{3} \omega \dot{\phi}^2 + 2\ddot{F} + 3H \dot{F} \right) \alpha + \frac{1}{3} F \delta R = N_{\delta F}, \tag{38}
\end{align}
where $N_{\delta \phi}$ and $N_{\delta F}$ are second-order terms whose explicit expressions are given in [43].

At first order the quantity $\delta \rho^{(1)} \equiv \delta \rho_m - \dot{\rho}_m a v^{(1)}$ is known to be gauge invariant [44]. In order to construct gauge invariant variables at second order, we introduce the following quantities:
\begin{align}
\delta \rho &\equiv \delta \rho_m - \dot{\rho}_m a v + \delta \rho^{(q)}, \tag{39}
\end{align}
\begin{align}
v_{\chi} &\equiv v - \frac{1}{a} \chi + v_{\chi}^{(q)}, \tag{40}
\end{align}
where $\delta \rho^{(q)}$ and $v_{\chi}^{(q)}$ are quadratic combinations of first-order terms. It was shown in [43], by defining $\delta_m \equiv \delta \rho_m / \rho_m$, that the following quantity is gauge invariant at second order:
\begin{align}
\delta &\equiv \delta_m + 3a H v - \frac{\delta \dot{\rho}_m}{\rho_m} a v + \frac{3}{2} \rho_m \dot{H} a^2 v^2 - v^i v_{,i} - 3 \rho_m a H \Delta^{-1} \nabla^i \left( \frac{\delta \rho_m}{\rho_m} v_{,i} \right), \tag{41}
\end{align}
where $\delta \rho_v \equiv \delta \rho_m - \rho_m a v$. Note that the quantity $v_{\chi}$ can also be made gauge invariant [43].

### 3.2. Approximate second-order equations

If we take the temporal comoving gauge ($v = 0$), we have $\delta = \delta_m$ and $q_{,i} = 0$ (see equations (27) and (41)). Taking $\gamma = 0$ for the spatial gauge condition, it follows that $\beta = \chi / a$ from equation (25). From equation (34) we obtain
\begin{align}
\alpha &= -\frac{1}{2} \beta_{,i} \beta^{i}, \tag{42}
\end{align}
which means that $\alpha$ is a second-order quantity. Up to second order, equations (31) and (33) are written as
\begin{align}
\kappa &= -\frac{\Delta}{a^2} \chi + \frac{3}{2} (\omega \dot{\phi} + \delta \ddot{F} - H \delta F - \dot{F} \alpha), \tag{43}
\end{align}
\begin{align}
\dot{\delta} - \kappa \delta &= -\frac{1}{a} \delta_{,i} \beta^{i}. \tag{44}
\end{align}

In order to evaluate the terms on the rhs in equation (44), it is sufficient to consider equations (40) and (43) at first order with the gauge $v = 0$. We then have $\chi^{(1)} = -a v_{\chi}$,
\[ \beta^{(1)} = -v_\chi \quad \text{and} \quad \kappa^{(1)} = \frac{\Delta v_\chi}{a} + \frac{3}{2F}(\omega \phi \delta \phi + \delta \dot{F} - H \delta F), \]

where we omitted the order of the subscript from the rhs in these equations. Hence equation (44) can be read as

\[ \dot{\delta} - \kappa = \frac{1}{a} \nabla \cdot (\delta \nabla v_\chi) + \frac{3}{2F}(\omega \phi \delta \phi + \delta \dot{F} - H \delta F)\delta. \]

In the following we consider a situation in which the scalar-field-dependent terms on the rhs in equation (45) are neglected relative to the term \( \Delta v_\chi / a \). In this case equation (46) yields

\[ \dot{\delta} - \kappa \simeq \frac{1}{a} \nabla \cdot (\delta \nabla v_\chi). \]

Later we shall confirm the validity of this approximation.

From equation (32) with equations (29), (35) and (36) we obtain

\[ \kappa + \left( 2H + \frac{\dot{F}}{2F} \right) \kappa - \frac{1}{2F} \left[ \delta \rho + 4\omega \phi \delta \dot{\phi} + (2\omega,\phi \delta \phi^2 + F,\phi R - 2V,\phi)\delta \phi + 3\delta \dot{F} + 3H \delta \dot{F} \right. \]

\[ - \left. \left( 6H^2 + \frac{\Delta}{a^2} \right) \delta F \right] = \frac{N_0 \dot{F}}{F} + N_3 - \frac{3\dot{F}}{2F} \dot{\alpha}, \]

\[ - \left[ 3H + \frac{1}{2F}(6\dot{F} + 6H \dot{F} + 4\omega \phi^2) + \frac{\Delta}{a^2} \right] \alpha, \]

where \( N_0 \) and \( N_3 \) correspond to the second-order terms on the rhs of equations (29) and (32), respectively. Following [3, 6, 27, 45] we employ the sub-horizon approximation under which the terms containing \( \kappa, \delta \rho, \Delta \delta F / a^2 \) and \( \Delta \alpha / a^2 \) are picked up in equation (48). Note that \( |\dot{F} / HF| \ll 1 \) under the condition \( |\phi| \ll H \). Apart from the term \( \Delta \alpha / a^2 \), the terms on the rhs in equation (48) are of the order of \( H^2 \alpha \) or smaller. We then have

\[ \kappa + 2H \kappa - \frac{1}{2F} \left( \delta \rho - \frac{\Delta}{a^2} \delta F \right) \simeq \frac{1}{a^2} \left[ (\nabla v_\chi) \cdot (\nabla v_\chi)^\dagger \right] \dot{\alpha}, \]

Of course this approximation is justified when the second-order term on the rhs in equation (49) is larger than the first-order (field-dependent) terms on the lhs in equation (48) that we have neglected. Later we shall derive conditions under which this approximation is valid.

Let us estimate the field perturbation \( \delta \phi \) as well as \( \delta F \). As we explained in the previous section, the field mass \( M \) defined in equation (21) is much larger than \( H \). Using the approximation in which the terms containing \( M^2, \Delta \delta \phi / a^2, \Delta \delta F / a^2, \delta \rho \) and \( \delta R \) are dominant contributions to equations (37) and (38), we obtain

\[ \left( -\frac{\Delta}{a^2} + \frac{M^2}{\omega} \right) \delta \phi - \frac{\dot{\phi}}{a} \Delta v_\chi - \frac{1}{2\omega} F,\phi \delta R \simeq 0, \]

\[ -\frac{\Delta}{a^2} \delta F - \frac{1}{3} \delta \rho - \frac{\dot{F}}{a} \Delta v_\chi + \frac{1}{3} F \delta R \simeq 0. \]
Second-order matter density perturbations

This approximation is accurate as long as an oscillating mode of the field perturbation does not dominate over the matter-induced mode \([8,27]\). Note that we have neglected second-order terms on the rhs in equations (37) and (38). Since the field is nearly frozen at the instantaneous minimum given in equation (18), the dominant second-order term corresponds to \(V_{\phi\phi\delta}\delta^2\). This term gives rise to only a tiny correction to the growth rate of perturbations. Moreover it can be neglected relative to the second-order term on the rhs in equation (49). See the appendix for the detailed estimation of such a second-order term.

On combining equations (50) and (51), we find
\[
\left(\frac{M^2}{F} - \frac{\Delta}{a^2}\right) \delta F = 2Q^2\delta\rho + \frac{\dot{F}}{a}\Delta v_\chi. \tag{52}
\]

Note that \(\delta\rho = \rho_m\delta \simeq 3FH^2\delta\) during the matter era. At first order we also have the following relation from equation (47):
\[
\frac{1}{a}\Delta v_\chi = \kappa = \dot{\delta} = cH\delta, \quad c \equiv \dot{D}/HD, \tag{53}
\]
where \(D(t)\) is the time-dependent part of \(\delta\). Since \(D(t)\) is typically proportional to \(t^n\) with \(n\) of the order of unity \([27]\), it follows that \(c = O(1)\). Hence we get
\[
\left| \frac{(\dot{F}/a)\Delta v_\chi}{2Q^2\delta\rho} \right| \simeq \left| \frac{\dot{\phi}}{QH} \right|. \tag{54}
\]

As long as the condition
\[
|\dot{\phi}| \ll |QH| \tag{55}
\]
is satisfied, we have that \(2Q^2\delta\rho| \gg |(\dot{F}/a)\Delta v_\chi|\) and
\[
\left(\frac{M^2}{F} - \frac{\Delta}{a^2}\right) \delta F \simeq 2Q^2\rho_m\delta. \tag{56}
\]

In the previous section we showed that \(|\dot{\phi}|\) is much smaller than \(H\) for the potential (4). Hence the condition (55) holds well for the values of \(|Q|\) which are not very much smaller than 1.

Equation (56) shows that the perturbation \(\delta F\) is sourced by the matter perturbation \(\delta\). Hence equation (49) can be written as
\[
\kappa + 2HK - 4\pi\rho_mG_{\text{eff}}\delta = \frac{1}{a^2}\left[ (\nabla v_\chi) \cdot (\nabla v_\chi) \right]_{,i}, \tag{57}
\]
where
\[
G_{\text{eff}}\delta \equiv \frac{1}{8\pi F} \left( \delta - \frac{1}{\rho_m} \frac{\Delta F}{a^2} \right). \tag{58}
\]

We introduce an effective potential \(\Phi\) and a peculiar velocity \(u\) as follows:
\[
\frac{\Delta\Phi}{a^2} = 4\pi\rho_mG_{\text{eff}}\delta, \tag{59}
\]
\[
u = -\nabla v_\chi. \tag{60}
\]
If one defines an effective gravitational potential $\Psi = \varphi + a H \nu_\chi$ in equation (30), it follows that $\Delta \Psi/a^2 = 4\pi G_{\text{eff}} \rho_m \delta$ at linear order, where $G_{\text{eff}}$ is different from $G_\text{eff}$ in the sign of $\Delta F/a^2$. Taking the time derivative of equation (47) and using equation (57), we get

$$\frac{\partial^2 \delta}{\partial t^2} + 2H \frac{\partial \delta}{\partial t} = \frac{1}{a^2} \nabla \cdot \left( 1 + \delta \right) \nabla \Phi + \frac{1}{a^2} \frac{\partial^2}{\partial x^i \partial x^j} \left( u^i u^j \right) . \quad (61)$$

This is our master equation that is used to compute the skewness of matter density perturbations in the next section.

4. Skewness in modified gravity

In this section we study the skewness of matter perturbations for the action (2) with the potential (4). The skewness will be derived analytically for in the matter-dominated epoch by using equation (61).

4.1. First-order perturbations

We write the solution to equation (61) in the form $\delta = \delta^{(1)} + \delta^{(2)} + \cdots$, where the subscripts represent the orders of perturbations. The equation for the first-order perturbation $\delta^{(1)}$ is

$$\frac{\partial^2 \delta^{(1)}}{\partial t^2} + 2H \frac{\partial \delta^{(1)}}{\partial t} - 4\pi \rho_m G^{(1)}_{\text{eff}} \delta^{(1)} = 0. \quad (62)$$

We express the first-order perturbations $\delta^{(1)}$ and $\delta F^{(1)}$ in plane-wave form: $\delta^{(1)} = \int \delta^{(1)}_k(t) e^{-i k \cdot x} d^3 k$ and $\delta F^{(1)} = \int \delta F^{(1)}_k(t) e^{-i k \cdot x} d^3 k$. From equation (56) we obtain

$$\delta F^{(1)}_k(t) = \frac{2Q^2 \rho_m}{M^2/F + k^2/a^2} \delta^{(1)}_k(t) \quad (63)$$

Then the temporal part of equation (62) satisfies

$$\ddot{\delta}^{(1)}_k(t) + 2H \dot{\delta}^{(1)}_k(t) - 4\pi \rho_m G^{(1)}_{\text{eff}} \delta^{(1)}_k(t) = 0, \quad (64)$$

where

$$G^{(1)}_{\text{eff}} = \frac{1}{8\pi F} \frac{(k^2/a^2)(1 + 2Q^2) + M^2/F}{(k^2/a^2) + M^2/F}. \quad (65)$$

In the early stage of the matter era, the mass $M$ is sufficiently heavy to satisfy the condition $M^2/F \gg k^2/a^2$. In this regime we have $G^{(1)}_{\text{eff}} \simeq 1/8\pi F \simeq G$, mimicking the evolution in general relativity. At late times it happens that the perturbations enter the regime $M^2/F \ll k^2/a^2$. This case corresponds to $G^{(1)}_{\text{eff}} \simeq (1 + 2Q^2)/8\pi F$, thus showing the deviation from general relativity. The transition from the former regime to the latter regime occurs at a redshift $z_k$ given by [27]

$$z_k \simeq \left( \frac{k}{a_0 H_0 Q} \right)^2 \frac{1}{(1 - p)^{1-p}} \frac{2^p p C}{(3F_0 \Omega_0^{(0)})^{2-p} H_0^2} \left( \frac{1}{1 - p} \right)^{(1/4 - p)} - 1, \quad (66)$$

where $a_0$ and $H_0$ are the present values.
Using the derivative with respect to $N = \ln(a)$, equation (64) can be written as
\[ \delta_k^{(1)''} + \left( \frac{1}{2} - \frac{3}{2} \omega_{\text{eff}} \right) \delta_k^{(1)'} - 12\pi F \Omega_m G_{\text{eff}}^{(1)} \delta_k^{(1)} = 0, \] (67)
where a prime represents the derivative in terms of $N$. As we explained in section 2, we are considering the case in which the matter era is realized via the point (M2) with $|\lambda| \gg |Q| = \mathcal{O}(1)$. Since $\Omega_m \simeq 1$ and $w_{\text{eff}} \simeq 0$ in this case, we get the following solutions:
\[ \delta_k^{(1)}(t) \propto \begin{cases} t^{2/3}, & \text{for } z \gg z_k, \\ t \frac{1}{2}(\sqrt{25+48Q^2}-1), & \text{for } z \ll z_k. \end{cases} \] (68)

In these asymptotic regimes the growth rates of first-order perturbations are independent of the wavenumber $k$. The growth rate is constrained to be $s \equiv \delta_k^{(1)}/H\dot{\delta}_k^{(1)} \lesssim 2$ from observational data, which gives the bound $|Q| \lesssim 1$ [27].

### 4.2. Conditions for the validity of approximations for reaching equation (61)

In order to reach equation (47), we have employed the approximation that the field-dependent term on the rhs in equation (45) is neglected relative to the term $\Delta v_{\chi}/a$. We have also neglected some of the first-order terms in equation (48) relative to the second-order term $(1/a^2)[(\nabla v_{\chi}) \cdot (\nabla v_{\chi})^4]_i$ in equation (49). We derive conditions under which these approximations are justified.

We write the temporal part of the first-order perturbation $\dot{v}_{\chi}^{(1)}$ as $(v_{\chi})^{(1)}(t)$. Using equations (63) and (53) together with the relation $\rho_m \simeq 3F H^2$ that holds during the matter era, we find
\[ \frac{H\delta F_k^{(1)}}{F} \simeq \frac{Q^2 H^2}{M^2/F + k^2/a^2} H\delta_k^{(1)} \simeq \frac{Q^2 H^2}{M^2/F + k^2/a^2} \frac{1}{a} |(\Delta v_{\chi})_k^{(1)}| \] (69)

As we showed in section 2, the condition $M^2/F \gg H^2$ holds for the potential (4). Moreover we are considering sub-horizon modes deep inside the horizon, i.e., $k^2 \gg a^2 H^2$.

This leads to the relation $|H\delta F_k^{(1)}/F| \ll |(\Delta v_{\chi})_k^{(1)}/a|$ in equation (69). Since the field $\phi$ is nearly frozen at instantaneous minima of its effective potential, we have the relation $|\delta \dot{F}_k^{(1)}| \ll |H\delta F_k^{(1)}|$ and hence $|\delta \dot{F}_k^{(1)}/F| \ll |(\Delta v_{\chi})_k^{(1)}/a|$. The following inequality is also satisfied:
\[ \left| \frac{1}{F} \omega \dot{\phi} \delta \phi_k^{(1)} \right| = \left| \frac{(1 - 6Q^2) \phi \delta F_k^{(1)}}{2Q} \right| \ll \frac{1}{a} |(\Delta v_{\chi})_k^{(1)}|, \] (70)

where we used equation (55). The above discussion shows that the field-dependent terms in equation (45) are neglected relative to the term $\Delta v_{\chi}/a$, thus ensuring the validity of the approximation $\kappa^{(1)} \simeq \Delta v_{\chi}^{(1)}/a$ for the modes deep inside the Hubble radius.

Using the first-order solution (53), we find that the second-order term on the rhs in equation (49) is of the order of $H^2|\delta_k^{(1)}|^2$. Meanwhile one of the first-order terms, $H^2\delta F_k^{(1)}/F$, in equation (48) has already been estimated in equation (69). The former is larger than the latter provided that
\[ |\delta_k^{(1)}| \gg \frac{Q^2 H^2}{M^2/F + k^2/a^2}. \] (71)
The rhs of equation (71) is much smaller than unity because of the condition \( M^2/F, k^2/a^3 \gg H^2 \). One can show that, under the condition (71), other field-dependent first-order terms on the lhs in equation (48) can be negligible relative to the term \((1/a^2)[(\nabla v_\chi) \cdot (\nabla v_\chi)^t] \). In summary, the master equation (61) that we have approximately derived in the previous section is reliable under the conditions (55) and (71).

4.3. Second-order perturbations and skewness

The second-order perturbation \( \delta^{(2)} \) satisfies

\[
\frac{\partial^2 \delta^{(2)}}{\partial t^2} + 2H \frac{\partial \delta^{(2)}}{\partial t} - 4\pi \rho_m G_{\text{eff}}^{(2)} \delta^{(2)} = 4\pi G_{\text{eff}} \rho_m \left( \delta^{(1)} \right)^2 + \frac{1}{a^2} \delta^{(1)} \Phi^{(1)} + \frac{1}{a^2} [u^{(1)} u^{(1)}],_{ij}, \tag{72}
\]

where \( G_{\text{eff}}^{(2)} \delta^{(2)} = \left[ \delta^{(2)} - \Delta \delta F^{(2)}/(\rho_m a^2) \right]/(8\pi F) \). When the growth rate of perturbations is dependent on \( k \), the gravitational constant \( G_{\text{eff}}^{(2)} \) is generally different from \( G_{\text{eff}}^{(1)} \). In the following we study the regime of the massless limit \( (M \rightarrow 0) \) in which the growth rate of the first-order perturbation is independent of \( k \), i.e., \( \delta^{(1)} = D(t) \delta_1(x) \) with \( D(t) = \ell(\sqrt{25+48F^2}^{-1})/6 \). In this regime we have \( G_{\text{eff}}^{(2)} = G_{\text{eff}}^{(1)} = (1 + 2\ell^2)/8\pi F \), so we simply adopt the notation \( G_{\text{eff}} \) instead of \( G_{\text{eff}}^{(1)} \) and \( G_{\text{eff}}^{(2)} \). The general relativistic case \( (M \rightarrow \infty) \) is recovered by taking the limit \( Q \rightarrow 0 \).

The first-order solution to \( u \) can be obtained by solving equation (53), i.e., \( \nabla \cdot u^{(1)} = -a \dot{\delta}^{(1)} \). It is given by

\[
u^{(1)} = -\frac{a \dot{D}}{4\pi} \int \frac{(x - x') \delta_1(x')}{|x - x'|^3} \, d^3 x' = \frac{a \dot{D}}{4\pi} \Delta_{,i}, \tag{73}
\]

where \( \Delta_{,i} \) is a spatial derivative of the quantity

\[
\Delta(x) \equiv \int \frac{\delta_1(x')}{|x - x'|} d^3 x'. \tag{74}
\]

This satisfies the relation \( \Delta_{,ii} = -4\pi \delta_1(x) \).

The last term on the rhs in equation (72) yields

\[
\frac{1}{a^2} [u^{(1)} u^{(1)}],_{ij} = \frac{1}{a^2} \left[ u^{(1)} u^{(1)} + 2u^{(1)} u^{(1)} + u^{(1)} u^{(1)} \right] = \dot{D}^2 \left[ \delta_i^2 - \frac{1}{2\pi} \delta_{1,i} \Delta_{,j} + \frac{1}{16\pi^2} \Delta_{,ij} \Delta_{,ij} \right]. \tag{75}
\]

We write the solution of equation (72) in the form [31]

\[
\delta^{(2)} = \delta_a^{(2)} + \delta_b^{(2)}, \tag{76}
\]

where \( \delta_a^{(2)} \) and \( \delta_b^{(2)} \) satisfy

\[
\frac{\partial^2 \delta_a^{(2)}}{\partial t^2} + 2H \frac{\partial \delta_a^{(2)}}{\partial t} - 4\pi G_{\text{eff}} \rho_m \delta_a^{(2)} = 4\pi G_{\text{eff}} \rho_m D^2 \delta_i^2 + \frac{D}{a^2} \Phi^{(1)} \delta_{1,i}, \tag{77}
\]

\[
\frac{\partial^2 \delta_b^{(2)}}{\partial t^2} + 2H \frac{\partial \delta_b^{(2)}}{\partial t} - 4\pi G_{\text{eff}} \rho_m \delta_b^{(2)} = \dot{D}^2 \left[ \delta_i^2 - \frac{1}{2\pi} \delta_{1,i} \Delta_{,j} + \frac{1}{16\pi^2} \Delta_{,ij} \Delta_{,ij} \right]. \tag{78}
\]
Since $\Phi_{\alpha}^{(1)} = -G_{\text{eff}}\rho_m a^3 D \Delta$ from equation (59), the rhs of equation (77) is given by $4\pi G_{\text{eff}}\rho_m D^2[\delta_{\alpha}^2 - (1/4\pi)\Delta_{\alpha}\delta_{1,1}]$. Writing the solution of $\delta_{\alpha}^{(2)}$ as $\delta_{\alpha}^{(2)} = E_a(t)\delta_a(x)$, we obtain the following equation for the temporal part:

$$\dot{E}_a + 2H\dot{E}_a - 4\pi G_{\text{eff}}\rho_m E_a = 4\pi G_{\text{eff}}\rho_m D^2,$$

where the spatial part is given by $\delta_{\alpha}(x) = \delta_{1}^2 - (1/4\pi)\Delta_{\alpha}\delta_{1,1}$. Expressing the solution of equation (78) in the form $\delta_{\alpha}^{(2)} = E_b(t)\delta_b(x)$, we get

$$\dot{E}_b + 2H\dot{E}_b - 4\pi G_{\text{eff}}\rho_m E_b = \dot{D}^2,$$

and $\delta_b(x) = \delta_1^2 - (1/2\pi)\Delta_{\alpha}\delta_{1,1} + (1/16\pi^2)\Delta_{\alpha}\Delta_{ij,ij}$.

We then find the following solution for second-order perturbations:

$$\delta^{(2)}(t, x) = E_a(t) \left[ \delta_1^2 - \frac{1}{4\pi} \Delta_{\alpha}\delta_{1,1} \right] + E_b(t) \left[ \delta_1^2 - \frac{1}{2\pi} \Delta_{\alpha}\delta_{1,1} + \frac{1}{16\pi^2} \Delta_{\alpha}\Delta_{ij,ij} \right],$$

$$= \frac{D^2 + E_a}{2} \delta_1^2 - \frac{D^2}{4\pi} \Delta_{\alpha}\delta_{1,1} + \frac{D^2}{32\pi^2} \Delta_{\alpha}\Delta_{ij,ij}.$$ (81)

In the second line we employed the fact that $E_a$ and $E_b$ are related to each other via the relation $E_a + 2E_b = D^2$.

We assume that the initial distribution of perturbations is Gaussian so that it is described by an auto-correlation function $\xi(x)$ satisfying

$$\langle \delta(x) \rangle = 0, \quad \langle \delta(x_1)\delta(x_2) \rangle = \xi(|x_1 - x_2|), \quad \langle \delta(x_1)\delta(x_2)\delta(x_3) \rangle = 0,$$

$$\langle \delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4) \rangle = \xi(|x_1 - x_2|)\xi(|x_3 - x_4|) + \xi(|x_1 - x_3|)\xi(|x_2 - x_4|) + \xi(|x_1 - x_4|)\xi(|x_2 - x_3|).$$ (82)

Since $\langle (\delta^{(1)})^3 \rangle = 0$, the quantity $\langle \delta^3 \rangle$ is given by $\langle \delta^3 \rangle = 3\langle (\delta^{(1)})^2\delta^{(2)} \rangle$ to lowest order. We then have

$$\langle \delta^3 \rangle = \frac{3}{2} D^2(D^2 + E_a)\langle \delta_1^2 \rangle - \frac{3}{4\pi} D^4\langle \delta_1^2\delta_{1,1}\Delta_{\alpha} \rangle + \frac{3}{32\pi^2} D^2(D^2 - E_a)\langle \delta_1^2\Delta_{\alpha}\Delta_{ij,ij} \rangle.$$ (83)

Since each ensemble average in equation (83) satisfies the relations $\langle \delta_1^2 \rangle = 3\xi(0)^2$, $\langle \delta_1^2\delta_{1,1}\Delta_{\alpha} \rangle = 4\pi\xi(0)^2$ and $\langle \delta_1^2\Delta_{\alpha}\Delta_{ij,ij} \rangle = (80\pi^2/3)\xi(0)^2$ [30], we obtain the skewness

$$S_3 \equiv \frac{\langle \delta^3 \rangle}{\langle \delta^2 \rangle^2} = 4 + 2\frac{E_a}{D^2},$$ (84)

where we used $\langle \delta_1^2 \rangle = \xi(0)^2$. Hence the skewness is determined by the second-order growth rate $E_a$ relative to the square of the first-order growth rate $D$.

Equation (79) can be written as

$$E_a'' + \left( \frac{1}{2} - \frac{3}{2} w_{\text{eff}} \right) E_a' - 12\pi F G_{\text{eff}}\Omega_m E_a = 12\pi F G_{\text{eff}}\Omega_m D^2.$$ (85)

Recall that $G_{\text{eff}} = (1 + 2Q^2)/8\pi F$ in the limit $M \to 0$. During the matter era realized via the point (M2) we have $\Omega_m \simeq 0$ and $w_{\text{eff}} \simeq 0$, in which case equation (85) reduces to

$$E_a'' + \frac{1}{2} E_a' - \frac{3}{2} (1 + 2Q^2) E_a = \frac{3}{2}(1 + 2Q^2) D^2.$$ (86)

Note that the skewness was calculated in [39] when the matter era is realized via the point (M1). As we already mentioned, the point (M1) cannot be used for the matter era when the coupling $|Q|$ is of the order of unity.
Second-order matter density perturbations

Figure 1. The analytic estimation (88) of the skewness during the matter-dominated epoch in the regime of the massless limit ($M \to 0$). With increase of $|Q|$ the skewness gets smaller, compared to the value $S_3 = 34/7$ in the Einstein–de Sitter Universe. However, the difference from the Einstein–de Sitter case is less than 1.7% for $|Q| \leq 1$.

Using the first-order solution $D = e^{1/4(\sqrt{25+48Q^2}-1)N}$, we get the following special solution for equation (86):

$$E_a = \frac{6(1+2Q^2)}{19 + 36Q^2 - \sqrt{25 + 48Q^2}} e^{1/2(\sqrt{25+48Q^2}-1)N}.$$  \hspace{1cm} (87)

Hence the skewness in the regime of the massless limit is given by

$$S_3 = \frac{4[22 + 42Q^2 - \sqrt{25 + 48Q^2}]}{19 + 36Q^2 - \sqrt{25 + 48Q^2}}.$$  \hspace{1cm} (88)

The general relativistic case is recovered by taking the limit $Q \to 0$:

$$S_3 = 34/7.$$  \hspace{1cm} (89)

This agrees with the skewness in the Einstein–de Sitter Universe [30] (pressureless matter without a cosmological constant).

In figure 1 we plot the analytic value (88) as a function of $|Q|$. The skewness shows some difference compared to the Einstein–de Sitter value 34/7 for $|Q| > 0.1$. When $|Q| = 1$ we have $S_3 = 4.775$, which differs from the value 34/7 only by 1.7%. For the potential (4) the first-order perturbation $\delta_k^{(1)}$ evolves from the regime $M^2/F \gg k^2/a^2$ to the regime $M^2/F \ll k^2/a^2$ for the modes relevant to large-scale structure. Hence the skewness tends to evolves from the value 34/7 to the asymptotic value given in equation (88). The estimation (88) has been derived by neglecting the transient phase around the redshift $z_k$. Since this transition occurs quickly for the models that satisfy local gravity constraints...
Second-order matter density perturbations

\[ p > 0.7 \] \[ \text{[8, 20, 27]} \], it is unlikely that the skewness is altered significantly by the presence of this transient phase.

The estimation \((88)\) does not take into account the evolution in the late-time accelerated epoch. In the \(\Lambda\)CDM model the numerical analysis shows that the skewness increases a bit during the accelerated phase from the value \(34/7 \) \( (=4.857) \) to the present value \(4.865 \) \( (\Omega_m = 0.28) \). This corresponds to growth of only \(0.16\%\). We have checked that this situation does not change much even in the presence of the coupling \(Q\). Hence the difference of the present values of the skewness from that in the \(\Lambda\)CDM model is less than a few per cent. This shows that the skewness provides a robust prediction for the picture of gravitational instability from Gaussian initial conditions, including scalar–tensor models with large couplings \((|Q| \lesssim 1)\).

5. Conclusions

In this paper we have studied the evolution of second-order matter density perturbations in a class of modified gravity models that satisfy local gravity constraints. We have considered the scalar–tensor action \((2)\), which is equivalent to the Brans–Dicke action \((1)\) with the correspondence \(1/(2Q^2) = 3 + 2\omega_{\text{BD}}\). In the presence of a field potential it is possible to satisfy local gravity constraints (LGC) even when \(|Q|\) is of the order of unity. In fact the potential \((4)\) is designed to have a large mass in the region of high density for consistency with LGC. This covers the models proposed by Hu and Sawicki \([16]\) in the context of \(f(R)\) gravity \((Q = -1/\sqrt{6})\).

Starting from second-order relativistic equations of cosmological perturbations, we have derived the equation \((61)\) for matter density fluctuations approximately. In so doing we employed the approximation that first-order perturbations in the scalar field \(\phi\) can be neglected relative to second-order matter and velocity perturbations. This is valid under the conditions \((55)\) and \((71)\), both of which can be naturally satisfied for the values of \(Q\) that we are interested in \((0.1 \lesssim |Q| \lesssim 1)\). Comparing to the \(\Lambda\)CDM model, the effective gravitational constant \(G_{\text{eff}}\) is subject to change at the late epoch of the matter era. This leads to the larger growth rate of first-order matter perturbations \((\delta_k^{(1)} \propto t^{\sqrt{25+48Q^2-1}/6})\) as compared to the standard case \((\delta_k^{(1)} \propto t^{2/3})\).

The skewness of matter distributions is determined by the second-order growth factor \(E_a\) relative to the square of the first-order growth factor \(D\). For the ‘scalar–tensor regime’ where the effective gravitational constant is given by \(G_{\text{eff}} \simeq (1 + 2Q^2)/8\pi F\), we have derived the analytic expression \((88)\) for the skewness in the matter-dominated epoch. For the ‘general relativistic regime’ where \(G_{\text{eff}} \simeq 1/8\pi F \simeq G\), we have reproduced the standard value \(S_3 = 34/7\) in the Einstein–de Sitter Universe. In modified gravity models with \(|Q| \lesssim 1\), the analytic value \((88)\) of the skewness in the asymptotic regime of the matter era is different from the value \(34/7\) by less than a few per cent. Even if we take into account the evolution of perturbations during the accelerated phase, the difference of the skewness relative to the \(\Lambda\)CDM model one remains small. The above result comes from the fact that the ratio of the second-order growth rate to the first-order one has a weak dependence on the coupling \(Q\).

When \(|Q| = \mathcal{O}(1)\) the growth rate of first-order matter perturbations is significantly different from that in the \(\Lambda\)CDM model. This gives rise to large modifications to the matter power spectrum as well as to the convergence spectrum in weak lensing, while the
skewness is hardly distinguishable from that in the ΛCDM model. This could be useful for discriminating large coupling scalar–tensor models from many other dark energy models by means of future high precision observations.

Acknowledgment

This work was partially supported by JSPS Grant-in-Aid for Scientific Research No. 30318802 (ST).

Appendix

In this appendix, we estimate the order of the second-order terms that we have neglected in equations (50) and (51). The dominant contribution of such second-order terms comes from the third derivative of the potential, i.e., $V_{,\phi\phi\phi}\delta\phi^2$ \cite{44}. Comparing to the field–mass-dependent term $V_{,\phi\phi}\delta\phi$ on the lhs in equation (50), we have $(V_{,\phi\phi\phi}\delta\phi^2)/(V_{,\phi\phi}\delta\phi) \approx -\delta\phi/\phi$ for the potential (4) under the condition $|Q\phi| \ll 1$.

Since the field stays around the instantaneous minimum given in equation (18), the field $\phi$ can be estimated as

$$\phi \simeq 3(1 - p)Q \frac{H^2}{M^2}, \quad (A.1)$$

where we used equation (21). In deriving this, we have also employed the approximate relation $\rho_m \approx 3H^2$. Note that the order of $\rho_m$ is not different from $3H^2$ even in the present epoch. Meanwhile, from equation (63), the first-order perturbation $\delta\phi^{(1)}_k$ in the Fourier space during the matter era is given by

$$\delta\phi^{(1)}_k \simeq -\frac{3QH^2}{M^2 + k^2/a^2} \delta^{(1)}_k. \quad (A.2)$$

Hence we obtain the ratio

$$\frac{\delta\phi^{(1)}_k}{\phi} \simeq -\frac{1}{1 - \frac{M^2}{1 - p M^2 + k^2/a^2} \delta^{(1)}_k}, \quad (A.3)$$

which shows that $|\delta\phi^{(1)}_k/\phi| \ll 1$ for $|\delta^{(1)}_k| \ll 1$.

The presence of the second-order term $V_{,\phi\phi\phi}\delta\phi^2$ gives rise to a correction of the order of $M^2\delta\phi^{(1)}_k/\phi$ to the mass squared $M^2$ in equation (65). In two asymptotic regimes (i) $M^2 \gg k^2/a^2$ and (ii) $M^2 \ll k^2/a^2$, this appears only as next-order corrections to the small expansion parameters $(k^2/a^2)/M^2$ (regime (i)) and $M^2/(k^2/a^2)$ (regime (ii)).

In the regime $M^2 \gg k^2/a^2$ the correction from the term $V_{,\phi\phi\phi}\delta\phi^2$ to the effective gravitational constant $G^{(1)}_{\text{eff}}$ is estimated as $\delta G^{(1)}_{\text{eff}} \approx Q^2(k^2/a^2M^2)\delta\phi^{(1)}_k/\phi$. This gives the correction to the third term on the lhs in equation (57) in the Fourier space:

$$4\pi\rho_m \delta G^{(1)}_{\text{eff}} \delta^{(1)}_k \approx \frac{Q^2 k^2/a^2}{1 - p \frac{M^2}{M^2}} H^2 \delta^{(1)}_k^2, \quad (A.4)$$

which is much smaller than the second-order term on the rhs in equation (57) which is of the order of $H^2\delta^{(1)}_k^2$. 

Journal of Cosmology and Astroparticle Physics 09 (2008) 009 (stacks.iop.org/JCAP/2008/i=09/a=009) 19
In the regime $M^2 \ll k^2/\alpha^2$ we have $\delta G^{(1)}_{\text{eff}} \approx Q^2 M^2 (\alpha^2/k^2) \delta \phi_k^{(1)}/\phi$ and hence

$$
|4\pi \rho_m \delta G^{(1)}_{\text{eff}} \delta \phi_k^{(1)}| \approx \frac{Q^2}{1-p} \left( \frac{M^2}{k^2/\alpha^2} \right)^2 H^2 \delta \phi_k^{(1)}^2,
$$

which is again much smaller than the rhs of equation (57).

The above estimation shows that neglecting second-order terms in equations (50) and (51) is justified.

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