Exponentiated Weibull-Geometric Distribution and its Applications

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Abstract

In this paper a new lifetime distribution, which is called the exponentiated Weibull-geometric (EWG) distribution, is introduced. This new distribution obtained by compounding the exponentiated Weibull and geometric distributions. The EWG distribution includes as special cases the generalized exponential-geometric (GEG), complementary Weibull-geometric (CWG), complementary exponential-geometric (CEG), exponentiated Rayleigh-geometric (ERG) and Rayleigh-geometric (RG) distributions.

The hazard function of the EWG distribution can be decreasing, increasing, bathtub-shaped and unimodal among others. Several properties of the EWG distribution such as quantiles and moments, maximum likelihood estimation procedure via an EM-algorithm, Rényi and Shannon entropies, moments of order statistics, residual life function and probability weighted moments are studied in this paper. In the end, we give two applications with real data sets to show the flexibility of the new distribution.

Keywords: EM-algorithm, Exponentiated Weibull distribution, Order statistics, Residual life function.

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1. Introduction

The Weibull and exponentiated Weibull (EW) distributions in spite of their simplicity in solving many problems in lifetime and reliability studies, do not provide a reasonable parametric fit to some practical applications.

Recently, attempts have been made to define new families of probability distributions that extend well-known families of distributions and at the same time provide great flexibility in

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modeling data in practice. One such class of distributions generated by compounding the well-known lifetime distributions such as exponential, Weibull, generalized exponential, exponentiated Weibull and etc with some discrete distributions such as binomial, geometric, zero-truncated Poisson, logarithmic and the power series distributions in general. The non-negative random variable $Y$ denoting the lifetime of such a system is defined by $Y = \min_{1 \leq i \leq N} X_i$ or $Y = \max_{1 \leq i \leq N} X_i$, where the distribution of $X_i$ belongs to one of the lifetime distributions and the random variable $N$ can have some discrete distributions, mentioned above.

This new class of distributions has been received considerable attention over the last years. The exponential-geometric (EG), exponential-Poisson (EP), exponential-logarithmic (EL), exponential power series (EPS), Weibull-geometric (WG), Weibull-power series (WPS), exponentiated exponential Poisson (EEP), complementary exponential-geometric (CEG), two-parameter Poisson-exponential, generalized exponential power series (GEPS), exponentiated Weibull-Poisson (EWP) and generalized inverse Weibull-Poisson (GIWP) distributions were introduced and studied by Adamidis and Loukas [2], Kus [13], Tahmasbi and Rezaei [25], Chahkandi and Ganjali [7], Barreto-Souza et al. [4], Morais and Barreto-Souza [19], Barreto-Souza and Cribari-Neto [3], Louzada-Neto et al. [14], Cancho et al. [6], Mahmoudi and Jafari [15], Mahmoudi and Sepahdar [16] and Mahmoudi and Torki [17].

In this article, we propose a new four-parameter distribution, referred to as the EWG distribution which contains as special sub-models the generalized exponential-geometric (GEG), complementary Weibull-geometric (CWG), complementary exponential-geometric (CEG), exponentiated Rayleigh-geometric (ERG) and Rayleigh-geometric (RG) distributions. The hazard function of the EWG distribution can be decreasing, increasing, bathtub-shaped and unimodal. Several properties of the EWG distribution such as quantiles and moments, maximum likelihood estimation procedure via an EM-algorithm, Rényi and Shannon entropies, moments of order statistics, residual life function and probability weighted moments are studied in this paper.

The paper is organized as follows. In Section 2, we review the EW distribution and its properties. In Section 3, we define the EWG distribution. The density, survival and hazard rate functions and some of their properties are given in this section. Section 4 provides a general expansion for the quantiles and moments of the EWG distribution. Its moment generating function is derived in this section. Rényi and Shannon entropies of the EWG distribution are
given in Section 5. Section 6 provides the moments of order statistics of the EWG distribution. Residual life function of the EWG distribution is discussed in Section 7. In Section 8 we explain probability weighted moments. Mean deviations from the mean and median are derived in Section 9. Section 10 is devoted to the Bonferroni and Lorenz curves of the EWG distribution. Estimation of the parameters by maximum likelihood via an EM-algorithm and inference for large sample are presented in Section 11. In Section 12, we studied some special sub-models of the EWG distribution. Applications to real data sets are given in Section 13 and conclusions are provided in Section 14.

2. Exponentiated Weibull distribution: A brief review

Mudholkar and Srivastava [20] introduced the EW family as extension of the Weibull family, which contains distributions with bathtub-shaped and unimodal failure rates besides a broader class of monotone failure rates. One can see Mudholkar et al. [21], Mudholkar and Huston [22], Gupta and Kundu [12], Nassar and Eissa [23] and Choudhury [9] for applications of the EW distribution in reliability and survival studies.

The random variable $X$ has an EW distribution if its cumulative distribution function (cdf) takes the form

$$F_X(x) = \left(1 - e^{-(\beta x)\gamma}\right)^\alpha, \quad x > 0,$$

where $\gamma > 0$, $\alpha > 0$ and $\beta > 0$, which is denoted by $\text{EW}(\alpha, \beta, \gamma)$. The corresponding probability density function (pdf) is

$$f_X(x) = \alpha \gamma \beta x^{\gamma-1} e^{-(\beta x)\gamma} \left(1 - e^{-(\beta x)\gamma}\right)^{\alpha-1}.$$

The survival and hazard rate functions of the EW distribution are

$$S(x) = 1 - \left(1 - e^{-(\beta x)\gamma}\right)^\alpha,$$

and

$$h(x) = \alpha \gamma \beta x^{\gamma-1} e^{-(\beta x)\gamma} \left(1 - e^{-(\beta x)\gamma}\right)^{\alpha-1} \left[1 - \left(1 - e^{-(\beta x)\gamma}\right)^\alpha\right]^{-1},$$

respectively. The $k$th moment about zero of the EW distribution is given by

$$E(X^k) = \alpha \beta^{-k} \Gamma\left(\frac{k}{\gamma} + 1\right) \sum_{j=0}^{\infty} (-1)^j \left(\frac{\alpha - 1}{j}\right) (j + 1)^{-\left(\frac{k}{\gamma} + 1\right)}.$$
Note that for positive integer values of $\alpha$, the index $j$ in previous sum stops at $\alpha - 1$, and the above expression takes the closed form

$$E(X^k) = \alpha \beta^{-k} \Gamma\left(\frac{k}{\gamma} + 1\right) A_k(\gamma),$$

(4)

where

$$A_k(\gamma) = 1 + \sum_{j=1}^{\alpha-1} (-1)^j \binom{\alpha-1}{j} (j+1)^{-\left(\frac{k}{\gamma}+1\right)}, \quad k = 1, 2, 3, \cdots,$$

(5)

in which $\Gamma(.)$ denotes the gamma function (see, Nassar and Eissa [23] for more details).

3. The EWG distribution

Consider the random variable $X$ having the EW distribution where its cdf and pdf are given in (1) and (2).

Given $N$, let $X_1, \cdots, X_N$ be independent and identically distributed (iid) random variables from EW distribution. Let the random variable $N$ is distributed according to the geometric distribution with pdf

$$P(N = n) = (1 - \theta)\theta^{n-1}, \quad n = 1, 2, \cdots, \quad 0 \leq \theta < 1.$$

Let $Y = \max(X_1, \cdots, X_N)$, then the conditional cdf of $Y|N = n$ is given by

$$F_{Y|N=n}(y) = \left(1 - e^{-\beta y}\right)^{\alpha n},$$

(6)

which is the EW distribution with parameters $n\alpha$, $\beta$, $\gamma$, and denoted by EW($n\alpha, \beta, \gamma$). The exponentiated Weibull-geometric (EWG) distribution, denoted by EWG($\alpha, \beta, \gamma, \theta$), is defined by the marginal cdf of $Y$, i.e.,

$$F_Y(y) = \frac{(1 - \theta) \left(1 - e^{-\beta y}\right)^\alpha}{1 - \theta \left(1 - e^{-\beta y}\right)^\alpha}.$$  

(7)

This new distribution includes some sub-models such as the complementary exponential-geometric (CEG), generalized exponential-geometric (GEG), complementary Weibull-geometric (CWG), exponentiated Rayleigh-geometric (ERG) and Rayleigh-geometric (RG) as special cases. The pdf of the EWG distribution is given by

$$f_Y(y) = \frac{(1 - \theta) \alpha \gamma \beta \gamma^{-1} e^{-\beta y} \left(1 - e^{-\beta y}\right)\alpha^{-1}}{\left[1 - \theta \left(1 - e^{-\beta y}\right)\alpha\right]^2},$$

(8)
where $\alpha, \beta, \gamma > 0$ and $0 \leq \theta < 1$.

The survival function and hazard rate function of the EWG distribution, are given respectively by

$$S(y) = \frac{1 - (1 - e^{-(\beta y)^\gamma})^\alpha}{1 - \theta (1 - e^{-(\beta y)^\gamma})^\alpha}, \quad (9)$$

and

$$h(y) = \frac{(1 - \theta)\alpha\gamma\beta y^{\gamma-1}e^{-(\beta y)^\gamma}(1 - e^{-(\beta y)^\gamma})^{\alpha-1}}{[1 - \theta (1 - e^{-(\beta y)^\gamma})^\alpha][1 - (1 - e^{-(\beta y)^\gamma})^\alpha]}.$$  

**Proposition 1.** For $|\theta| < 1$, it is easy to prove that the density of EWG distribution can be written as an infinite mixture of EW distributions. If $|z| < 1$ and $k > 0$, we have the series representation

$$(1 - z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k + j)}{\Gamma(k)j!} z^j. \quad (10)$$

If $|\theta| < 1$, expanding $[1 - \theta (1 - e^{-(\beta y)^\gamma})^\alpha]^{-2}$ as in Eq. (10), the density function (8) can be demonstrated by

$$f_Y(y) = (1 - \theta)\alpha\gamma\beta y^{\gamma-1}e^{-(\beta y)^\gamma} \sum_{j=0}^{\infty} (j + 1)\theta^j (1 - e^{-(\beta y)^\gamma})^{\alpha(j+1)-1}.$$  

Using the EW density (2), we obtain

$$f_{EWG}(y; \alpha, \beta, \gamma, \theta) = (1 - \theta) \sum_{j=0}^{\infty} \theta^j f_{EW}(y; \alpha(j + 1), \beta, \gamma). \quad (11)$$

Various mathematical properties of the EWG distribution for $|\theta| < 1$, can be obtained from Eq. (11) and the corresponding properties of the EW distribution.

**Proposition 2.** The density of EWG distribution can be expressed as infinite linear combination of density of the biggest order statistic of $X_1, \cdots, X_n$, where $X_i \sim EW(\alpha, \beta, \gamma)$ for $i = 1, 2, \cdots, n$. we have

$$f_{EWG}(y) = \sum_{n=1}^{\infty} (G(y))^n P(N = n) = \sum_{n=1}^{\infty} g_{X(n)}(y) P(N = n),$$

in which $g_{X(n)}(y)$ is the pdf of $X(n) = \max(X_1, \cdots, X_n)$. 


4. Quantiles and moments of the EWG distribution

The \( p \)th quantile of the EWG distribution is given by

\[
x_p = \beta^{-1}\left\{ \left[ -\log \left( 1 - \left( \frac{p}{1-\theta(1-p)} \right)^{\frac{1}{\alpha}} \right) \right]^{\frac{1}{\gamma}} \right\},
\]

which is used for data generation from the EWG distribution. In particular, the median of the EWG distribution is given by

\[
x_{0.5} = \beta^{-1}\left\{ \left[ -\log \left( 1 - \left( \frac{1}{2-\theta} \right)^{\frac{1}{\alpha}} \right) \right]^{\frac{1}{\gamma}} \right\}.
\]

Suppose that \( Y \sim \text{EWG}(\alpha, \beta, \gamma, \theta) \), and \( X(n) = \max(X_1, \cdots, X_n) \), where \( X_i \sim \text{EW}(\alpha, \beta, \gamma) \) for \( i = 1, 2, \cdots, n \), then the \( k \)th moment of \( Y \) is given by

\[
E(Y^k) = \sum_{n=1}^{\infty} P(N = n)E(Y^k|n) = \sum_{n=1}^{\infty} P(N = n)E(X^k_{(n)})
\]

\[
= \sum_{n=1}^{\infty} P(N = n)\alpha \beta^{-k} \Gamma \left( \frac{k}{\gamma} + 1 \right) \sum_{j=0}^{\infty} (-1)^j \binom{n-1}{j}(j+1)^{-(\frac{k}{\gamma}+1)}
\]

\[
= (1-\theta)\alpha \beta^{-k} \Gamma \left( \frac{k}{\gamma} + 1 \right) \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n\theta^{n-1}(-1)^j \binom{n-1}{j}(j+1)^{-(\frac{k}{\gamma}+1)}.
\]

For positive integer values of \( \alpha \), the index \( j \) in above expression stops at \( \alpha - 1 \).

Using Eq. (12), the moment generating function of the EWG distribution is given by

\[
M_Y(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(Y^i)
\]

\[
= \sum_{i=0}^{\infty} \frac{t^i}{i!} \left[ (1-\theta)\alpha \beta^{-i} \Gamma \left( \frac{i}{\gamma} + 1 \right) \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n\theta^{n-1}(-1)^j \binom{n-1}{j}(j+1)^{-(\frac{i}{\gamma}+1)} \right]
\]

\[
= \alpha(1-\theta) \sum_{i=0}^{\infty} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n\theta^{n-1}(-1)^j \binom{n-1}{j}(j+1)^{\frac{i(\beta)}{\beta}} \Gamma \left( \frac{i}{\gamma} + 1 \right) \frac{(t/\beta)^i}{i!}(j+1)^{-(\frac{i}{\gamma}+1)}.
\]

(13)

According to Eq. (12), the mean and variance of the EWG distribution are given respectively by

\[
E(Y) = (1-\theta)\alpha \beta^{-1} \Gamma \left( \frac{1}{\gamma} + 1 \right) \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n\theta^{n-1}(-1)^j \binom{n-1}{j}(j+1)^{-(\frac{1}{\gamma}+1)},
\]

(14)

and

\[
Var(Y) = (1-\theta)\alpha \beta^{-2} \Gamma \left( \frac{2}{\gamma} + 1 \right) \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n\theta^{n-1}(-1)^j \binom{n-1}{j}(j+1)^{-(\frac{2}{\gamma}+1)} - E^2(Y),
\]

(15)

Where \( E(Y) \) is given in Eq. (14)
5. Rényi and Shannon entropies

For a random variable with pdf \( f \), the Rényi entropy is defined by \( I_R(r) = \frac{1}{1-r} \log \left\{ \int f^r(y) dy \right\} \), for \( r > 0 \) and \( r \neq 1 \). For the EWG distribution, the power series expansion gives

\[
\int_0^\infty f^r(y) dy = [\alpha \gamma^r (1 - \theta)]^r \sum_{j=0}^\infty \frac{\beta^j \Gamma(2r+j)}{\Gamma(2r) j!} \int_0^\infty y^{(\gamma-1)r} e^{-r(\beta y)^\gamma} (1 - e^{-(\beta y)^\gamma})^{(j+r)-r} dy
\]

\[
= [\alpha \gamma^r (1 - \theta)]^r \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^k \frac{\beta^j \Gamma(2r+j)}{\Gamma(2r) j!} \frac{\Gamma(r-\frac{j}{\gamma})}{(r-\frac{j}{\gamma})} \theta^j.
\]

Substituting from (16), the Rényi entropy is given by

\[
I_R(r) = \frac{1}{1-r} \log \left\{ [\alpha (1 - \theta)]^r (\beta \gamma)^{-1} \sum_{j=0}^\infty \sum_{k=0}^\infty (-1)^k \frac{\alpha (j+r)-r}{k} \frac{\Gamma(2r+j)}{\Gamma(2r) j!} \frac{\Gamma(r-\frac{j}{\gamma})}{(r-\frac{j}{\gamma})} \theta^j \right\}.
\]

The Shannon entropy which is defined by \( E[-\log(f(Y))] \), is derived from \( \lim_{r \to 1} I_R(r) \).

6. Moments of order statistics

Let the random variable \( Y_{r:n} \) denotes the \( r \)th order statistic \( (Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}) \) in a sample of size \( n \) from the EWG distribution. The pdf of \( Y_{r:n} \) for \( r = 1, \cdots, n \), is given by

\[
f_{r:n}(y) = \frac{1}{B(r, n-r+1)} f(y) F(y)^{r-1} [1 - F(y)]^{n-r}, \quad y > 0.
\]

where \( F(y) \) and \( f(y) \) are the cdf and pdf of the random variable \( Y \). Substituting from (7) and (8) into (18), gives

\[
f_{r:n}(y) = \frac{\alpha \gamma^r (1 - \theta)^r}{B(r, n-r+1)} \gamma^{-1} e^{-(\beta y)^\gamma} \frac{(1-e^{-(\beta y)^\gamma})^{\alpha r-1} [1 - (1-e^{-(\beta y)^\gamma})^\alpha]^{n-r}}{[1-\theta(1-e^{-(\beta y)^\gamma})^\alpha]^{n+r}}, \quad y > 0.
\]

Also the cdf of \( Y_{r:n} \) is given by

\[
F_{r:n}(y) = \sum_{k=r}^n \binom{n}{k} [F(y)]^k [1 - F(y)]^{n-k}
\]

\[
= \sum_{k=r}^n \frac{(1-\theta)^k (1-e^{-(\beta y)^\gamma})^{\alpha k} (1-(1-e^{-(\beta y)^\gamma})^\alpha)^{n-k}}{(1-\theta(1-e^{-(\beta y)^\gamma})^\alpha)^n}.
\]

Using the binomial expansion, series expansion \( (10) \) and after some calculations, the \( k \)th moment of the \( r \)th order statistic \( Y_{r:n} \) is given by

\[
E(Y_{r:n}^k) = \frac{\alpha \beta^{-k} (1 - \theta)^r}{B(r, n-r+1)} \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} (-1)^j \theta^i \binom{n-i}{i} \binom{n-r}{j} \frac{\Gamma(k+1)}{\Gamma(s+1)} \frac{\Gamma(k+i+1)}{(s+i+1)}
\]
The pdf, cdf and $k$th moment of the smallest and biggest order statistics, i.e., $Y_{1:n}$ and $Y_{n:n}$, can be obtained by setting $r = 1$ and $n$ in Eqs. (19)-(21).

7. Residual life function of the EWG distribution

Given that a component survives up to time $t \geq 0$, the residual life is the period beyond $t$ until the time of failure and defined by expectation of the conditional random variable $X|X > t$. In reliability, it is well known that the mean residual life function and ratio of two consecutive moments of residual life, determine the distribution uniquely (Gupta and Gupta, [11]). Therefore, we obtain the $r$th order moment of the residual life via the general formula

$$m_r(t) = E[(Y - t)^r|Y > t] = \frac{1}{S(t)} \int_t^\infty (y - t)^r f(y)dy,$$

(22)

where $S(t) = 1 - F(t)$, is the survival function.

In what seen this onwards, we use the expressions

$$\int_t^\infty x^{\gamma+s-1}e^{-(k+1)(\beta x)}dx = \frac{1}{\gamma \beta^{\gamma+s}}(k+1)^{-\left(1+\frac{s}{\gamma}\right)}\Gamma(k+1)(\beta)^{\gamma}(1 + \frac{s}{\gamma}),$$

and

$$\int_0^t x^{\gamma+s-1}e^{-(k+1)(\beta x)}dx = \frac{1}{\gamma \beta^{\gamma+s}}(k+1)^{-\left(1+\frac{s}{\gamma}\right)}\Gamma_t(k+1)(\beta)^{\gamma}(1 + \frac{s}{\gamma}),$$

where $\Gamma_t(s) = \int_t^\infty x^{s-1}e^{-x}dx$ is the upper incomplete gamma function and $\Gamma_t(s) = \int_0^t x^{s-1}e^{-x}dx$ is the lower incomplete gamma function.

Applying series expansion (10), the binomial expansion for $(y - t)^r$ and substituting $S(y)$ given by (9) into (22), the $r$th moment of the residual life of the EWG is given by

$$m_r(t) = \frac{\alpha(1 - \theta)}{S(t)} \sum_{i=0}^{r} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+k}(j+1)t^j\theta^i}{(k+1)^{1+\gamma} \beta^{r-i}} \binom{r}{i} \left( \frac{\alpha(j+1) - 1}{k} \right) \Gamma(k+1)(\beta)^{\gamma} \left( 1 + \frac{r-i}{\gamma} \right).$$

(23)

Another important representation for the EWG is the mean Residual life (MRL) function obtain by setting $r = 1$ in Eq. (23). The importance of the MRL function is due to its uniquely determination of the lifetime distribution as well as the failure rate (FR) function. Lifetimes can exhibit IMRL (increasing MRL) or DMRL (decreasing MRL). MRL functions that first decreases (increases) and then increases (decreases) are usually called bathtub (upside-down bathtub) shaped, BMRL (UMRL). Many authors such as Ghitany [10], Mi [18], Park [24] and Tang et al. [26] have been studied the relationship between the behaviors of the MLR and FR.
functions of a distribution.

The following theorem gives the MRL function of the EWG distribution.

**Theorem 1.** The MRL function of the EWG distribution is given by

\[
m_1(t) = \left[ \frac{\alpha(1 - \theta)}{\beta S(t)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha(i + 1) - 1}{j} \theta^j(i + 1)(j + 1)^{-\left(1 + \frac{j}{\gamma}\right)} \Gamma(j + 1)(\beta t)^\gamma \right] - t.
\]

Setting \( r = 2 \) in (23), the variance of residual life function of the EWG distribution can be obtained using \( m_1(t) \) and \( m_2(t) \).

8. Estimation and inference

In this section, we study the estimation of the parameters of the EWG distribution. Let \( Y_1, Y_2, \cdots, Y_n \) be a random sample with observed values \( y_1, y_2, \cdots, y_n \) from EWG distribution with parameters \( \alpha, \beta, \gamma \) and \( \theta \). Let \( \Theta = (\alpha, \beta, \gamma, \theta)^T \) be the parameter vector. The total log-likelihood function is given by

\[
l_n \equiv l_n(y; \Theta) = n[\log \alpha + \log \gamma + \gamma \log \beta + \log(1 - \theta)] + (\gamma - 1) \sum_{i=1}^{n} \log y_i - \sum_{i=1}^{n} (\beta y_i)^\gamma
\]

\[+ (\alpha - 1) \sum_{i=1}^{n} \log[1 - e^{-(\beta y_i)^\gamma}] - 2 \sum_{i=1}^{n} \log[1 - \theta(1 - e^{-(\beta y_i)^\gamma})^\alpha].\]

The associated score function is given by

\[
U_n(\Theta) = (\partial l_n/\partial \alpha, \partial l_n/\partial \beta, \partial l_n/\partial \gamma, \partial l_n/\partial \theta)^T,
\]

where

\[
\frac{\partial l_n}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1 - e^{-(\beta y_i)^\gamma}) + 2\theta \sum_{i=1}^{n} \log(1 - e^{-(\beta y_i)^\gamma}) \frac{(1 - e^{-(\beta y_i)^\gamma})^\alpha}{1 - \theta(1 - e^{-(\beta y_i)^\gamma})^\alpha},
\]

\[
\frac{\partial l_n}{\partial \beta} = \frac{n}{\beta} - \gamma \beta^{\gamma - 1} \sum_{i=1}^{n} y_i^\gamma + (\alpha - 1) \gamma \beta^{\gamma - 1} \sum_{i=1}^{n} \frac{y_i^\gamma e^{-(\beta y_i)^\gamma}}{1 - \theta(1 - e^{-(\beta y_i)^\gamma})^\alpha}
\]

\[+ 2\theta \alpha \gamma \beta^{\gamma - 1} \sum_{i=1}^{n} \frac{y_i^\gamma e^{-(\beta y_i)^\gamma} (1 - e^{-(\beta y_i)^\gamma})^\alpha}{1 - \theta(1 - e^{-(\beta y_i)^\gamma})^\alpha},
\]

\[
\frac{\partial l_n}{\partial \gamma} = \frac{n}{\gamma} + n \log \beta + \sum_{i=1}^{n} \log y_i - \sum_{i=1}^{n} \log(\beta y_i) \gamma y_i
\]

\[+ (\alpha - 1) \sum_{i=1}^{n} \frac{\log(\beta y_i) (\beta y_i)^\gamma e^{-(\beta y_i)^\gamma}}{1 - e^{-(\beta y_i)^\gamma}} + 2\theta \alpha \sum_{i=1}^{n} \frac{\log(\beta y_i) (\beta y_i)^\gamma e^{-(\beta y_i)^\gamma} (1 - e^{-(\beta y_i)^\gamma})^\alpha}{1 - \theta(1 - e^{-(\beta y_i)^\gamma})^\alpha} - 1,
\]

\[
\frac{\partial l_n}{\partial \theta} = -\frac{n}{\theta} + 2 \sum_{i=1}^{n} \frac{(1 - e^{-(\beta y_i)^\gamma})^\alpha}{1 - \theta(1 - e^{-(\beta y_i)^\gamma})^\alpha}.
\]

The maximum likelihood estimation (MLE) of \( \Theta \), say \( \hat{\Theta} \), is obtained by solving the nonlinear system \( U_n(\Theta) = 0 \). The solution of this nonlinear system of equation has not a closed form. For interval estimation and hypothesis tests on the model parameters, we require the information
matrix. The $4 \times 4$ observed information matrix is

$$I_n(\Theta) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\gamma} & I_{\alpha\theta} \\ I_{\alpha\beta} & I_{\beta\beta} & I_{\beta\gamma} & I_{\beta\theta} \\ I_{\alpha\gamma} & I_{\beta\gamma} & I_{\gamma\gamma} & I_{\gamma\theta} \\ I_{\alpha\theta} & I_{\beta\theta} & I_{\gamma\theta} & I_{\theta\theta} \end{bmatrix},$$

whose elements are given in Appendix.

Applying the usual large sample approximation, MLE of $\Theta$ i.e. $\hat{\Theta}$ can be treated as being approximately $N_d(\Theta, J_n(\Theta)^{-1})$, where $J_n(\Theta) = E[I_n(\Theta)]$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\Theta} - \Theta)$ is $N_d(0, J(\Theta)^{-1})$ where $J(\Theta) = \lim_{n \to \infty} n^{-1}I_n(\Theta)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\Theta)$ is replaced by the average sample information matrix evaluated at $\hat{\Theta}$, say $n^{-1}I_n(\hat{\Theta})$. The estimated asymptotic multivariate normal $N_d(\Theta, I_n(\hat{\Theta})^{-1})$ distribution of $\hat{\Theta}$ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. An 100$(1 - \gamma)$ asymptotic confidence interval for each parameter $\Theta_r$ is given by

$$ACI_r = (\hat{\Theta}_r - Z_{\gamma/2}\sqrt{\hat{I}rr}, \hat{\Theta}_r + Z_{\gamma/2}\sqrt{\hat{I}rr}),$$

where $\hat{I}rr$ is the $(r, r)$ diagonal element of $I_n(\hat{\Theta})^{-1}$ for $r = 1, 2, 3, 4$, and $Z_{\gamma/2}$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

9. Submodels of the EWG distribution

The EWG distribution contains some sub-models for the special values of $\alpha$, $\beta$ and $\gamma$. Some of these distributions are discussed here in details.

9.1. Complementary Weibull-geometric distribution

The complementary Weibull-geometric (CWG) distribution is a special case of the EWG distribution for $\alpha = 1$. Our approach here is complementary to that of Barreto-Souza et al. in introducing the Weibull-geometric (WG) distribution. The pdf, cdf and hazard rate function of the CWG distribution, are given respectively by

$$f(y) = \frac{(1 - \theta)\gamma\beta^\gamma y^{\gamma - 1}e^{-\gamma(\beta y)^\gamma}}{[1 - \theta (1 - e^{-\gamma(\beta y)^\gamma})]^2}, \quad (24)$$
\[ F(y) = \frac{(1 - \theta)(1 - e^{-(\beta y)^\gamma})}{1 - \theta (1 - e^{-(\beta y)^\gamma})}, \]  

(25)

and

\[ h(y) = \frac{(1 - \theta)^\beta y^\gamma - (1 - e^{-(\beta y)^\gamma})}{[1 - \theta (1 - e^{-(\beta y)^\gamma})]\left[1 - (1 - e^{-(\beta y)^\gamma})\right]}. \]  

(26)

According to Eqs. (14) and (15), the mean and variance of the CWG distribution are given by

\[ E(Y) = \frac{(1 - \theta)(\Gamma(1 + \frac{1}{\gamma}) - 1)}{\Gamma(1 + \frac{1}{\gamma})} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n\theta^{n-1}(-1)^j \binom{n-1}{j} (j + 1)^{-\left(\frac{1}{2} + \frac{1}{\gamma}\right)}, \]  

(27)

and

\[ Var(Y) = \frac{2(1 - \theta)(\Gamma(1 + \frac{2}{\gamma}) - 1)}{\Gamma(1 + \frac{2}{\gamma})} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n\theta^{n-1}(-1)^j \binom{n-1}{j} (j + 1)^{-\left(\frac{3}{2} + \frac{1}{\gamma}\right)} - E^2(Y), \]  

(28)

where \( E(Y) \) is given in Eq. (27).

9.2. Generalized exponential-geometric distribution

The generalized exponential-geometric (GEG) distribution is a special case of the EWG distribution for \( \gamma = 1 \). This distribution is introduced and analyzed in details by Mahmoudi and Jafari [15]. The pdf, cdf and hazard rate function of the GEG distribution, are given respectively by

\[ f(y) = \frac{(1 - \theta)^\alpha e^{-(\beta y)} (1 - e^{-(\beta y)})^{\alpha - 1}}{1 - \theta (1 - e^{-(\beta y)})^{\alpha}}, \]  

and

\[ F(y) = \frac{(1 - \theta)(1 - e^{-(\beta y)})^\alpha}{1 - \theta (1 - e^{-(\beta y)})^{\alpha}}, \]

and

\[ h(y) = \frac{(1 - \theta)^\alpha e^{-(\beta y)} (1 - e^{-(\beta y)})^{\alpha - 1}}{1 - \theta (1 - e^{-(\beta y)})^{\alpha}} \left[1 - (1 - e^{-(\beta y)})^{\alpha}\right]. \]

According to Eqs. (14) and (15), the mean and variance of the GEG distribution are

\[ E(Y) = \frac{\alpha(1 - \theta)(\Gamma(1 + \frac{1}{\gamma}) - 1)}{\Gamma(1 + \frac{1}{\gamma})} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n\theta^{n-1}(-1)^j \binom{n\alpha - 1}{j} (j + 1)^{-2}, \]  

(29)

and

\[ Var(Y) = \frac{2\alpha(1 - \theta)(\Gamma(1 + \frac{2}{\gamma}) - 1)}{\Gamma(1 + \frac{2}{\gamma})} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n\theta^{n-1}(-1)^j \binom{n\alpha - 1}{j} (j + 1)^{-3} - E^2(Y), \]  

where \( E(Y) \) is given in Eq. (29).
9.3. Complementary exponential-geometric distribution

The complementary exponential-geometric (CEG) distribution is a special case of the EWG distribution for $\alpha = 1$ and $\gamma = 1$. The pdf, cdf and hazard rate function of the CEG distribution are given respectively by

$$f(y) = \frac{(1 - \theta) e^{-(\beta y)}}{[1 - \theta (1 - e^{-(\beta y)})]^2},$$

$$F(y) = \frac{(1 - \theta) (1 - e^{-(\beta y)})}{1 - \theta (1 - e^{-(\beta y)})},$$

and

$$h(y) = \frac{(1 - \theta) e^{-(\beta y)}}{[1 - \theta (1 - e^{-(\beta y)})] [1 - (1 - e^{-(\beta y)})^\alpha]}. $$

According to Eqs. (14) and (15), the mean and variance of the CEG distribution, are given respectively by

$$E(Y) = (1 - \theta) \beta^{-1} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n \theta^{n-1} (-1)^j \binom{n}{j} / (j+1)^2,$$

and

$$Var(Y) = (1 - \theta) \beta^{-2} \Gamma(3) \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n \theta^{n-1} (-1)^j \binom{n}{j} / (j+1)^3 - E^2(Y),$$

where $E(Y)$ is given in Eq. (30).

9.4. Exponentiated Rayleigh-geometric distribution

The exponentiated Rayleigh-geometric distribution (ERG) distribution is a special case of the EWG distribution for $\gamma = 2$. The pdf, cdf and hazard rate function of the ERG distribution are given respectively by

$$f_Y(y) = \frac{2(1 - \theta) \alpha \beta^2 y e^{-(\beta y)^2} \left(1 - e^{-(\beta y)^2}\right)^{\alpha-1}}{[1 - \theta (1 - e^{-(\beta y)^2})^\alpha]^2},$$

$$F_Y(y) = \frac{(1 - \theta) \left(1 - e^{-(\beta y)^2}\right)^\alpha}{1 - \theta (1 - e^{-(\beta y)^2})^\alpha},$$

and

$$h(y) = \frac{2(1 - \theta) \alpha \beta^2 y e^{-(\beta y)^2} \left(1 - e^{-(\beta y)^2}\right)^{\alpha-1}}{[1 - \theta (1 - e^{-(\beta y)^2})^\alpha] [1 - (1 - e^{-(\beta y)^2})^\alpha].}$$

The mean and variance of the ERG distribution, are given respectively by

$$E(Y) = (1 - \theta) \alpha \beta^{-1} \Gamma\left(\frac{1}{2} + 1\right) \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n \theta^{n-1} (-1)^j \binom{n\alpha - 1}{j} (j+1)^{-\left(\frac{1}{2}+1\right)},$$

(31)
and

\[ Var(Y) = (1 - \theta)\alpha \beta^{-2} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n^\theta n^{-1} (-1)^j \binom{n-1}{j} (j+1)^{-2} - E^2(Y), \]

where \( E(Y) \) is given in Eq. (31).

9.5. Rayleigh-geometric distribution

The Rayleigh-geometric distribution (RG) distribution is a special case of the CWG distribution, obtained by choosing \( \gamma = 2 \) in CWG distribution. Setting \( \gamma = 2 \) in Eqs. (24)-(26) gives the pdf, cdf and hazard rate function of the RG distribution as

\[ f(y) = \frac{2(1 - \theta)\beta^2 y e^{-\theta(\beta y)^2}}{[1 - \theta (1 - e^{-\theta(\beta y)^2})]^2}, \]

\[ F(y) = \frac{(1 - \theta) \left( 1 - e^{-\theta(\beta y)^2} \right)}{1 - \theta (1 - e^{-\theta(\beta y)^2})}, \]

and

\[ h(y) = \frac{2(1 - \theta)\beta^2 y e^{-\theta(\beta y)^2}}{e^{-\theta(\beta y)^2} \left[ 1 - \theta (1 - e^{-\theta(\beta y)^2}) \right]}, \]

According to Eqs. (27) and (28), the mean and variance of the RG distribution are given by

\[ E(Y) = (1 - \theta)\beta^{-1} \Gamma \left( \frac{1}{2} + 1 \right) \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n^\theta n^{-1} (-1)^j \binom{n-1}{j} / (j+1)^{-\left(\frac{1}{2}+1\right)}, \quad (32) \]

and

\[ Var(Y) = (1 - \theta)\beta^{-2} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} n^\theta n^{-1} (-1)^j \binom{n-1}{j} / (j+1)^{-2} - E^2(Y), \]

where \( E(Y) \) is given in Eq. (32).

10. Conclusion

We propose a new four-parameter distribution, referred to as the EWG distribution which contains as special sub-models the generalized exponential-geometric (GEG), complementary Weibull-geometric (CWG), complementary exponential-geometric (CEG), exponentiated Rayleigh-geometric (ERG) and Rayleigh-geometric (RG) distributions. The hazard function of the EWG distribution can be decreasing, increasing, bathtub-shaped and unimodal. Several properties of the EWG distribution such as quantiles and moments, maximum likelihood estimation procedure via an EM-algorithm, Rényi and Shannon entropies, moments of order statistics, residual life function and probability weighted moments are studied. Finally, we fitted EWG model to two real data sets to show the potential of the new proposed distribution.
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