Orthogonal Impulse Response Analysis in Presence of Time-Varying Covariance

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Abstract

In this paper the orthogonal impulse response functions (OIRF) are studied in the non-standard, though quite common, case where the covariance of the error vector is not constant in time. The usual approach for taking into account such behavior of the covariance consists in applying the standard tools to (rolling) sub-periods of the whole sample. We underline that such a practice may lead to severe upward bias. We propose a new approach intended to give what we argue to be a more accurate resume of the time-varying OIRF. This consists in averaging the Cholesky decomposition of nonparametric covariance estimators. In addition an index is developed to evaluate the heteroscedasticity effect on the OIRF analysis. The asymptotic behavior of the different estimators considered in the paper is investigated. The theoretical results are illustrated by Monte Carlo experiments. The analysis of U.S. inflation data shows the relevance of the tools proposed herein for an appropriate analysis of economic variables.

Keywords: Impulse response analysis; Kernel smoothing; Time-varying covariance; VAR models

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1 Introduction

In time series econometrics it is common to investigate sub-samples of a full time series in order to capture changes in the data. Reference can be made to Dees and Saint-Guilhem (2011) or Diebold and Yilmaz (2014) who considered rolling windows. In order to accommodate possible regime switches, Bernanke and Mihov (1998a) constitute different sub-periods for measuring monetary policy. Strongin (1995) split the data considered for the study according to the Federal Reserve operating procedures. Nazlioglu, Soytas and Gupta (2015) propose a pre-crisis, in-crisis and post-crisis split type to carry out a volatility spillover analysis, while Strohsal, Proaño and Wolters (2019) consider pre and post 1985 financial liberalization samples. Blanchard and Simon (2001), Stock and Watson (2005) and Alter and Beyer (2014) use both rolling windows and static periods to describe non constant dynamics in the series they study.

In this paper we consider the analysis of the orthogonal impulse response functions (OIRF) in the case of vector autoregressive (VAR) models with constant autoregressive parameters but with time-varying covariance structure. In the literature, it is often admitted that the conditional mean is constant while the variance is time-varying (see Bernanke and Mihov (1998b), Sims (1999), Stock and Watson (2002), Kew and Harris (2009) or Patilea and Raïssi (2012) among others). In addition it is widely known that non constant variance is common for economic variables. For instance Sensier and van Dijk (2004) found that more than 80% of the 214 U.S. economic variables they studied have a non constant variance. In our multivariate context the test proposed by Aue, Hörmann, Horváth and Reimherr (2009) can be used for break detection in the covariance structure. Our main message, focused on the OIRF analysis, is that if one wishes to work with (rolling or fixed) sub-samples, it is advisable one to carry out a pointwise estimation, and then resume it using averages according to the periods of interest to obtain an accurate picture of the non constant dynamics. This idea leads us to introduce in the following what will be called the averaged OIRF. Let us point out that applying the standard tools to sub-samples can, in some sense, lead to bias distortions in resuming the time-varying dynamics of a series. In the following we present an example which illustrates our point.
1.1 Univariate example: Korean Won/USD exchange rate

Let us consider the log differences of the monthly exchange rate of South Korean Won to one U.S. Dollar from May 1981 to June 2018. From the left panel of Figure 1 we can observe that the post Asian crisis variance is clearly greater than that of pre crisis period. In particular sharp increases can be noted during the Asian and the 2007-2008 financial crises. Using the adaptive approach introduced by Xu and Phillips (2008), the conditional mean is filtered by fitting an AR model.

Using this simple framework we illustrate the ways of resuming the time-varying impulse response functions i periods after a rescaled impulse hits the variable of an univariate series. Let us define by $\sigma_t^2 = g^2(t/T)$ the (unobserved) innovations variance at time $1 \leq t \leq T$, where $g(.)$ is a function fulfilling some regularity conditions. As the (unobserved) moving average coefficients $\phi_i$ are constant in our case, it suffices to focus on the changes in the variance to capture the evolutions of the rescaled impulse response functions (IRF) $\phi_i \sigma_{t-i}$. The usual way to resume the time varying IRF over a given period would be to estimate the standard tool that assume a constant variance, using for instance rolling windows. This would lead to estimate $\phi_i \int g^2 (.)^{0.5}$ (which will be called the approximate IRF), whereas $\phi_i \int g$ (which will be called the averaged IRF) is more sound to resume the IRF. Here, the integrals account for the averaging over the time window. Indeed, if the purpose is to find a local approximation of the IRF, averaging over the values in a time window around the time of interest seems more reasonable than considering a kind of norm. Clearly the averaged and approximated IRF are in general different, as long as the variance structure is non constant. More precisely, more the variance varies over time, larger the discrepancy between the averaged and the approximated IRF is. Therefore, in the following we also propose to build an indicator of the variability of the variance based on the discrepancy between the averaged and approximated IRFs. It is important to underline that the robustness/stability studies often rely on the simple (graphical) examination of the different OIRF. As this way of proceeding is subjective, our indicator is intended to quantify such a kind of analyses.

In order to have an idea about the differences between the two quantities, the corresponding estimators (defined below in the paper) are displayed in Figure 1 for $i = 0$. As expected the approximated IRF estimates are noticeably greater than the averaged

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1The term rescaled is taken from Lütkepohl (2005,p53).
IRF estimates as the changes are marked. For example we found that the approximated IRF can be greater than the averaged IRF by up to 120% for the Won/dollar exchange rate. In particular it can be seen that large differences between the resuming quantities are consequences of fast changes in the series. For instance the initial shock due to the Asian crisis for the Won/dollar exchange rate in December 1997 can be viewed as not representative of the post Asian crisis regime. Also the few extreme values due to the 2008 crisis do not really reflects the post Asian crisis period. However the analysis based on the approximated IRF seems more impacted by these short crisis periods than the one based on the averaged IRF. This can be explained by the fact that the approximated IRF uses the squared observations (or residuals), while the averaged IRF considers estimates of the variance structure.

The structure of this study is as follows. In section 2 the vector autoregressive model with unconditionally heteroscedastic innovations is presented. Next, different possible concepts of OIRF that could be considered in our framework are discussed. Moreover, we introduce a scalar variance variability index that measures the departure from the standard constant variance VAR setup. The Section 3 is dedicated to the estimators and their asymptotic properties. The time-varying OIRF estimator is introduced and its nonparametric rate of convergence is derived. In Section 3.2 and Section 3.3 the estimators of the approximated and averaged OIRF are defined. Their asymptotic behavior is also studied. In Section 3.4 we introduce the estimator of our variance variability index and derive its asymptotic properties. In Section 4 Monte Carlo experiments are conducted to compare the finite sample properties of the different estimators of the OIRF. U.S. energy and transportation inflation data are studied to underline the usefulness of the proposed tools. The proofs are relegated to the Appendix.

2 Time-varying orthogonal impulse response functions

Following the usual approach for impulse response analysis between variables, consider a vector autoregressive (VAR) model for the series $X_t \in \mathbb{R}^d$:

$$X_t = A_{01}X_{t-1} + \cdots + A_{0p}X_{t-p} + u_t$$ (1)
where \( u_t \) is the error term and the \( A_{0i} \)'s are the autoregressive parameters matrices, supposed to be such that \( \det A(z) \neq 0 \) for all \( |z| \leq 1 \), with \( A(z) = I_d - \sum_{i=1}^{p} A_{0i} z^i \). Here, the covariance of the system is allowed to vary in time. More precisely, the covariance of the process \( (u_t) \) is denoted by \( \Sigma_t := G(t/T)G(t/T)' \), where \( r \mapsto G(r) \), \( r \in (0,1] \), is a \( d \times d \)-matrix valued function. With the rescaling device used by Dahlhaus (1997), the process \( (X_t) \) should be formally written in a triangular form. Herein, the double subscript is suppressed for notational simplicity.

The specification we consider allows for commonly observed features as cycles, smooth or abrupt changes for the covariance, and is widely used in the literature (see e.g. Xu and Phillips (2008) and references therein). In practice the lag length \( p \) in (1) is unknown but can be fixed using the tools proposed in Patilea and Raïssi (2013) and Raïssi (2015) under our assumptions.

In the sequel, the model (1) is considered re-written as follows:

\[
X_t = (\tilde{X}_t' \otimes I_d)\vartheta_0 + u_t
\]
\[
u_t = H_t\epsilon_t,
\]
where \( (\epsilon_t) \) is an iid centered process with \( E(\epsilon_t\epsilon'_t) = I_d \) and

\[
\vartheta_0 = \text{vec}(A_{01}, \ldots, A_{0p})
\]
is the vector of parameters. Herein the vec(\( \cdot \)) operator consists in stacking the columns of a matrix into a vector. The matrix \( H_t \) is the lower triangular matrix of the Cholesky decomposition of the errors’ covariance, that is \( \Sigma_t = H_tH'_t \). The matrix \( I_d \) is the \( d \times d \)-identity matrix. The usual Kronecker product is denoted by \( \otimes \) and \( \tilde{X}_t = (X'_t, \ldots, X'_{t-p+1})' \). We also define

\[
\Phi_0 = I_d, \quad \Phi_i = \sum_{j=1}^{i} \Phi_{i-j}A_{0j}, \quad (2)
\]

\( i = 1, \ldots \), with \( A_{0j} = 0 \) for \( j > p \). The \( \Phi_i \)'s correspond to the coefficients matrices of the infinite moving average representation of \( (X_t) \). Under our assumptions the components of the \( \Phi_i \)'s decrease exponentially fast to zero.

If the errors’ covariance \( \Sigma \) is assumed constant, then we can define \( d \times d \) standard OIRF

\[
\theta(i) := \Phi_i H, \quad i = 1, 2, \ldots
\]
where here $H$ is the lower triangular matrix of the Cholesky decomposition of $\Sigma$. See Lütkepohl (2005, p59). Let us denote by $\hat{\vartheta}_{OLS}$ the ordinary least squares (OLS) estimator of the autoregressive parameters and define $\hat{\Sigma}$ the OLS estimator of the constant errors covariance matrix. Using $\hat{\vartheta}_{OLS}$ and $\hat{\Sigma}$ it is easy to see that an estimator of $\theta(i)$ can be built. Under standard assumptions it can be shown that such estimators are consistent, $\sqrt{T}$-asymptotically Gaussian. See Lütkepohl (2005, p110). However it clearly appears that the classical OIRF cannot take into account for the time-varying instantaneous effects properly, and may be misleading in our non standard but quite realistic framework.

2.1 tv-OIRF

In the framework of the model (1), a common alternative to the classical OIRF is the time-varying OIRF (tv-OIRF hereafter)

$$\theta_r(i) := \Phi_i H(r), \quad i \geq 1,$$

for each $r \in (0, 1]$, and where $H(r)$ is the lower triangular matrix of the Cholesky decomposition of $\Sigma(r) = G(r)G(r)^\prime$. The parameter $r$ gives the time where the impulse response analysis is conducted. In other words, the counterpart of the usual OIRF in the case of time-varying variance is the two arguments function

$$(r, i) \mapsto \theta_r(i), \quad (r, i) \in (0, 1] \times \{1, 2, \ldots\}.$$  

The form (4) implicitly arises when models with constant autoregressive parameters but time varying variance are used to analyse the data (see Bernanke and Mihov (1998a), Stock and Watson (2002) or Xu and Phillips (2008) among others for this kind of models). When the covariance of the errors is constant, for each $i$ the map $r \mapsto \theta_r(i)$ is constant, and thus we retrieve the standard case. Nevertheless, in general these maps are not constant and are typically estimated at nonparametric rates, as it will be shown in the following.

Some resume of the tv-OIRF through time could be sometimes more convenient to take into account for time-varying dynamics in the series. In the sequel, we consider two approaches for resuming the tv-OIRF over a given sub-period (rolling or static). First, we replace the matrix $H(r)$ in equation (1) by the lower triangular matrix of the Cholesky decomposition of the realized variance, that is the average of the variance,
over a given period around \( r \). This will yield to what we shall call \textit{approximated OIRF}. Typically this corresponds to the usual practice which consists in applying the standard method to periods (see e.g. Stock and Watson (2005) or Beetsma and Giuliodori (2012)). Second, keeping in mind that we are looking for a resume that locally approximates the tv-OIRF, which is tantamount to looking for a resume of \( H(r) \) appearing in equation [4] we introduce the \textit{averaged OIRF} that is obtained by replacing the matrix \( H(r) \) with the average of the lower triangular matrix of the Cholesky decomposition of \( \Sigma(\cdot) \) over a given period around \( r \). Both resumes we consider could be estimated at parametric rates and, considering static or rolling periods, could be used for an analysis of the series. However, as argued in the Introduction, the averaged OIRF should be preferred. Before presenting the approximate and averaged approaches, let us point out that, as usual, resuming the OIRF does not makes the shocks orthogonal pointwise. Note however that such property is approximately reached when the period or rolling window is sufficiently small.

### 2.2 Approximated OIRF

The usual way to resume the OIRF in presence of a non constant covariance in our framework is to consider the following quantities

\[
\tilde{\theta}_q^r(i) = \Phi_i \tilde{H}(r), \quad i \geq 1, \tag{5}
\]

where \( \tilde{H}(r) \) is the lower triangular matrix of the Cholesky decomposition of the positive definite matrix \( q^{-1} \int_{r-q/2}^{r+q/2} \Sigma(v) dv \) with \( 0 < r - q/2 < r + q/2 < 1 \). Again the standard case is retrieved if the covariance structure is assumed constant. If \( r \) does not corresponds to a covariance break, we have \( \tilde{\theta}_q^r(i) \approx \theta_r(i) \) for small enough \( q \). Since the Cholesky of the integrated variance structure does not really reflects the evolutions of \( H(\cdot) \), we will refer to [5] as approximate OIRF in the sequel. For fixed \( r \) and \( q \), the quantities \( \tilde{\theta}_q^r(i) \) could be estimated at parametric rates. As stated above, this tool is usually considered using rolling windows to capture general time varying patterns. The approximated OIRF are also computed to contrast between static periods.
2.3 Averaged OIRF

By definition, the approximated OIRF could be quite different from $\theta_r(i)$. Given the definition of $\theta_r(i)$, a more natural way to approach it would be to average the lower triangular matrix of the Cholesky decomposition over a window around $r$. We propose a new alternative way to resume the tv-OIRF (4) based on the quantities

$$\bar{\theta}_q^r(i) := \Phi_i \bar{H}(r)$$

where

$$\bar{H}(r) := \frac{1}{q} \int_{r-q/2}^{r+q/2} H(v) dv, \quad i \geq 1,$$

$0 < q < 1$ is fixed by the practitioner, and $r$ is such that $0 < r - q/2 < r + q/2 < 1$. The standard case is retrieved if the errors covariance is assumed constant. On the other hand if $r$ does not correspond to an abrupt break of the covariance structure, we clearly have $\bar{\theta}_q^r(i) \approx \theta_r(i)$ when $q$ is small.

2.4 Variance variability indices

In this section we propose an index, that is a scalar, to measure the departure from a constant covariance matrix situation in a VAR model. We could write

$$\bar{\theta}_q^r(i) = \bar{\theta}_q^r(i) I_{r,q}$$

with

$$I_{r,q} = \bar{H}(r)^{-1} \bar{H}(r).$$

Let us define

$$i_{r,q} = \|I_{r,q}\|_2^2,$$

where here $\| \cdot \|_2$ denotes the spectral norm of a matrix. In this case, $i_{r,q}$ is equal to the square of the largest eigenvalue of $I_{r,q}$, which has only real, positive eigenvalues. By elementary matrix algebra properties, we also have

$$i_{r,q} = \max_{a \in \mathbb{R}^d, a \neq 0} \frac{\text{Var}(a'X_{app})}{\text{Var}(a'X_{avg})},$$

where $X_{avg}$ and $X_{app}$ are $d$-dimensional random vectors with variances $\bar{H}(r)\bar{H}(r)'$ and $\bar{H}(r)\bar{H}(r')$, respectively. In the statistical literature, a quantity like $i_{r,q}$ is usually called the first relative eigenvalue of one matrix (here $\bar{H}(r)\bar{H}(r)'$) with respect to the other matrix (here $\bar{H}(r)\bar{H}(r')$). See, for instance, Flury (1985). By construction, in our context, the eigenvalues of the matrix $\bar{H}(r)^{-1} \bar{H}(r)$ are real numbers larger than or equal to 1, as shown in the following.
The index \( i_{r,q} \) is inspired by the OIRF analysis. It is designed to provide a measure of variability through the contrast between two possible definitions of OIRF that coincide in the case of a covariance \( \Sigma \) constant over time. Another simple index could be defined as

\[
j_{r,q} = \left\| \frac{1}{q} \int_{r-q/2}^{r+q/2} \Sigma(v)dv - \bar{H}(r)\bar{H}(r)' \right\|_2^2.
\]  

By elementary properties of the spectral norm,

\[
j_{r,q} = \max_{a \in \mathbb{R}^d, \|a\|=1} \{ \text{Var}(a'X_{\text{app}}) - \text{Var}(a'X_{\text{avg}}) \}.
\]

The technical assumptions of the paper may be found in Section 5.2.

**Lemma 2.1.** Under the Assumption \( A1 \),

1. \( i_{r,q} \geq 1 \) and \( i_{r,q} = 1 \) if and only if \( v \mapsto H(v) \) is constant on \( (r - q/2, r + q/2) \);
2. \( j_{r,q} \geq 0 \) and \( j_{r,q} = 0 \) if and only if \( v \mapsto H(v) \) is constant on \( (r - q/2, r + q/2) \)

In our context, for any \( 0 < q < 1 \), a map \( r \mapsto i_{r,q} \) (resp. \( r \mapsto j_{r,q} \)) constant equal to 1 (resp. 0) means the covariance of \( X_t \) is constant in time. For simplicity, in the sequel we will focus on index \( i_{r,q} \) which is invariant to multiplication of the errors’ covariance matrix by a positive constant. Large values of \( i_{r,q} \) indicates a large variability in the variance of the vector series around the time \( rT \).

## 3 OIRF estimates when the variance is varying

Let us first briefly recall the estimation methodology for heteroscedastic VAR models of Patilea and Raïssi (2012), Section 4. First, we consider the OLS estimator of the autoregressive parameters

\[
\tilde{\vartheta}_{\text{OLS}} = \left( \sum_{t=1}^{T} \tilde{X}_{t-1} \tilde{X}_{t-1}' \otimes I_d \right)^{-1} \text{vec} \left( \sum_{t=1}^{T} X_t \tilde{X}_{t-1}' \right).
\]

Patilea and Raïssi (2012) showed that

\[
\sqrt{T}(\tilde{\vartheta}_{\text{OLS}} - \vartheta_0) \Rightarrow \mathcal{N}(0, \Lambda_3^{-1}\Lambda_2\Lambda_3^{-1}),
\]

where

\[
\Lambda_2 = \int_0^1 \sum_{i=0}^{\infty} \left\{ \tilde{\Phi}_i (1_{p \times p} \otimes \Sigma(r)) \tilde{\Phi}_i' \right\} \otimes \Sigma(r)dr, \quad \Lambda_3 = \int_0^1 \sum_{i=0}^{\infty} \left\{ \tilde{\Phi}_i (1_{p \times p} \otimes \Sigma(r)) \tilde{\Phi}_i' \right\} \otimes I_d dr,
\]

\[
(3)
\]
with $1_{p \times p}$ the $p \times p$ matrix with components equal to one, and $\tilde{\Phi}_i$ is a block diagonal matrix $\tilde{\Phi}_i := \text{diag}(\Phi_i, \Phi_{i-1}, \ldots, \Phi_{i-p+1})$. The matrices $\Phi_i$, $i \geq 0$, are defined in equation [2], and $\Phi_i = 0$ for $i < 0$.

Next, let us consider kernel estimators of the time-varying covariance matrix. Denote by $A \odot B$ the Hadamard (entrywise) product of two matrices of same dimension $A$ and $B$. For $t = 1, \ldots, T$, define the symmetric matrices

$$\hat{\Sigma}_t = \sum_{j=1}^{T} w_{tj} \odot \hat{u}_j \hat{u}_j' ,$$

(4)

where the $\hat{u}_t = X_t - (\tilde{X}'_{t-1} \otimes I_d) \hat{\vartheta}_{OLS}$ are the OLS residuals. The $(k,l)$--element, $k \leq l$, of the $d \times d$ matrix of weights $w_{tj}$ is given by

$$w_{tj}(b_{kl}) = (Tb_{kl})^{-1}K((t - j)/(Tb_{kl})),$$

with $b_{kl}$ the bandwidth and $K(\cdot)$ a nonnegative kernel function. For any $r \in (0,1]$, the value $\Sigma(r)$ of the covariance function could be estimated by $\hat{\Sigma}_{[rT]}$. (Here and in the following, for a number $a$, we denote by $[a]$ the integer part of $a$, that is the largest integer number smaller or equal to $a$.) For all $1 \leq k \leq l \leq d$ the bandwidth $b_{kl}$ belongs to a range $B_T = [c_{\min}b_T, c_{\max}b_T]$ with $c_{\min}, c_{\max} > 0$ some constants and $b_T \downarrow 0$ at a suitable rate specified below. In practice the bandwidths $b_{kl}$ can be chosen by minimization of a cross-validation criterion. This estimator is a version of the Nadaraya-Watson estimator considered by Patilea and Raïssi (2012). Here, we replace the denominator by the target density, that is the uniform density on the unit interval which is constant equal to 1. A regularization term may be needed to ensure that the matrices $\hat{\Sigma}_t$ are positive definite (see Patilea and Raïssi (2012)). Another simple way to circumvent the problem is to select a unique bandwidth $b = b_{kl}$, for all $1 \leq k, l \leq d$.

With at hand an estimator of $\Sigma(r)$, we could define $\hat{H}_{[rT]}$, the lower triangular matrix of the Cholesky decomposition of $\hat{\Sigma}_{[rT]}$, as the estimator of $H(r)$. Below, we establish the rates of convergence of these nonparametric estimators. For $r \in (0,1)$, let $\Sigma(r-) = \lim_{r \uparrow r} \Sigma(\tilde{r})$ and $\Sigma(r+) = \lim_{r \downarrow r} \Sigma(\tilde{r})$. Moreover, by definition let $\Sigma(1+) = 0$. Let $H(r-)$ and $H(r+)$ be defined similarly. In the following, $\| \cdot \|_F$ is the Frobenius norm, while $\sup_{B_T}$ denotes the supremum with respect the bandwidths $b_{kl}$ in $B_T$.

**Proposition 3.1.** Assume that Assumptions A0-A2 in the Appendix hold true. Then,
for any $r \in (0, 1]$,

$$\sup_{B_T} \left\| \hat{\Sigma}_{[T,r]} - \frac{1}{2} \{ \Sigma(r-) + \Sigma(r+) \} \right\|_F = O_P \left( b_T + \sqrt{\log(T)/Tb_T} \right)$$

and

$$\sup_{B_T} \left\| \hat{H}_{[T,r]} - \frac{1}{2} H_{\pm}(r) \right\|_F = O_P \left( b_T + \sqrt{\log(T)/Tb_T} \right),$$

where $H_{\pm}(r)$ is the lower triangular matrix of the Cholesky decomposition of $\Sigma(r-) + \Sigma(r+)$. 

The rate of convergence of $\hat{\Sigma}_{[T,r]}$ and $\hat{H}_{[T,r]}$ is given by a bias term, with the standard rate one could obtain when estimating Lipschitz continuous functions nonparametrically, and a variance term which has is multiplied by a logarithm factor, the price to pay for the uniformity with respect to the bandwidth.

The above estimation of the non constant covariance structure could be used to define the adaptive least squares (ALS) estimator

$$\hat{\vartheta}_{ALS} = \tilde{\Sigma}^{-1} \text{vec} \left( \tilde{\Sigma} \tilde{X} \right),$$

where

$$\tilde{\Sigma}_X = T^{-1} \sum_{t=1}^{T} \tilde{X}_{t-1} \tilde{X}_{t-1}' \otimes \tilde{\Sigma}_t^{-1} \quad \text{and} \quad \tilde{\Sigma}_\tilde{X} = T^{-1} \sum_{t=1}^{T} \tilde{\Sigma}_t^{-1} X_t \tilde{X}_t'.$$

By minor adaptation of the proofs in Patilea and Raïssi (2012), in order to take into account the simplified change in the definition of the weights $w_{ij}$, it could be shown that, uniformly with respect to $b \in B_T$, $\hat{\vartheta}_{ALS}$ is consistent in probability and

$$\sqrt{T}(\hat{\vartheta}_{ALS} - \vartheta_0) \Rightarrow \mathcal{N}(0, \Lambda_1^{-1}),$$

where

$$\Lambda_1 = \int_0^1 \sum_{i=0}^{\infty} \left\{ \Phi_i(1_{p \times p} \otimes \Sigma(r)) \tilde{\Phi}_i' \right\} \otimes \Sigma(r)^{-1} dr.$$ 

Patilea and Raïssi (2012) showed that $\Lambda_3^{-1} \Lambda_2 \Lambda_3^{-1} - \Lambda_1$ is a positive semi-definite matrix.

### 3.1 The tv-OIRF nonparametric estimator

The natural way to build estimates of the time-varying OIRF defined in equation (4) is to plugin estimates of the $\Phi_i$ and $H(r)$. For estimating $\Phi_i$ we use $\tilde{\Phi}_i^{als}$ which are
obtained as in (2), but considering the ALS estimator of the $A_0i$’s. By the arguments used in the proof of Proposition 3.3 below, this estimator has the $O_p(1/\sqrt{T})$ rate of convergence. Using the nonparametric estimator of $H(r)$ we introduced above, we obtain what we will call the \textit{ALS estimator of $\theta_r(i)$}, that is

$$\hat{\theta}_r(i) := \hat{\Phi}^{als}_i \hat{H}_{[rT]}, \quad r \in (0, 1].$$  

(7)

Even if $\hat{\Phi}^{als}_i$ has an improved variance compared to the estimator one would obtain using the OLS estimator of the $A_0i$’s, the estimator $\hat{\theta}_r(i)$ still inherits the nonparametric rate of convergence of $\hat{H}_{[rT]}$ described in Proposition 3.1. Hence, analyzing the variations of the estimated curves $r \mapsto \hat{\theta}_r(i)$, for various $i$, suffers from lower, nonparametric convergence rates. In section 3.3 we propose to use instead of $\hat{\theta}_r(i)$ averages over the values in a neighborhood of $r$, that is a window containing $r$. In particular, this allows to recover parametric rates of convergence of the estimators.

### 3.2 Approximated orthogonal impulse response function estimates

The results of this part are only stated as they are direct consequences of arguments in Patilea and Raïssi (2012) and standard techniques (see Lütkepohl (2005)). Let the usual estimator of (5),

$$\hat{\theta}_r(i) := \hat{\Phi}^{ols}_i \hat{\theta}^{ols}_r H(r),$$  

(8)

where $\hat{H}(r)$ is the lower triangular matrices of the Cholesky decomposition of

$$\hat{S}_T(r) = \frac{1}{[qT]} + \sum_{k=-[qT/2]}^{[qT/2]} \hat{u}_{[rT]-k} \hat{u}'_{[rT]-k},$$  

(9)

with $\hat{u}_{[rT]-k}$ the OLS residuals and $\hat{\Phi}^{ols}_i$ are the estimators of the MA coefficients obtained from the OLS estimators of the autoregressive parameters. Recall that (8) is used to evaluate the OIRF in the standard homoscedastic case (see Lütkepohl (2005) Section 3.7), but is also commonly considered to evaluate tv-OIRF (static periods or rolling windows). The expression (8) is suitable at least asymptotically, since by the
Proof Lemma 3.2 below

\[
\frac{1}{[qT] + 1} \sum_{k=-[qT]/2}^{[qT]/2} \hat{u}_{[rT]-k} \hat{u}_{[rT]-k}^\prime = \frac{1}{[qT] + 1} \sum_{k=-[qT]/2}^{[qT]/2} u_{[rT]-k} u_{[rT]-k}^\prime + o_p(1/\sqrt{T})
\]

\[= \frac{1}{q} \int_{r-q/2}^{r+q/2} \Sigma(v)dv + O_p(1/\sqrt{T}). \tag{10}\]

In order to specify the asymptotic behavior of \(\hat{\theta}_r(i)\), we first state a result which can be proved using similar arguments to those of Lemma 7.4 of Patilea and Raïssi (2010). Let \(\hat{\zeta}_t := \text{vech}(\hat{u}_t \hat{u}_t^\prime)\), \(\zeta_t := \text{vech}(u_t u_t^\prime)\) and \(\Gamma_t := \text{vech}(\Sigma(t/T)) = \text{vech}(\Sigma_t)\), where the vech operator consists in stacking the elements on and below the main diagonal of a square matrix. Define \(\Gamma(r) := \text{vech} \left( q^{-1} \int_{r-q/2}^{r+q/2} \Sigma(v)dv \right) \) and \(\hat{\Gamma}(r) := ([qT] + 1)^{-1} \sum_{k=-[qT]/2}^{[qT]/2} \hat{\zeta}_{[rT]-k} \) for \(r < 1\). Introduce also the functions \(\Gamma(\cdot)\) and \(\Delta(\cdot)\) given by \(\Gamma(\cdot) = \text{vech}(\Sigma(\cdot))\) and \(\Delta(t/T) = E(\zeta_t \zeta_t^\prime)\).

**Lemma 3.2.** Under the assumptions A0-A3 in the Appendix, we have

\[
\sqrt{T} \left( \hat{\varrho}_{OLS} - \varrho_0 \right) \Rightarrow N \left( 0, \begin{pmatrix} \Lambda_3^{-1} \Lambda_2 \Lambda_3^{-1} & 0 \\ 0 & \Omega(r) \end{pmatrix} \right), \tag{11}\]

with \(\hat{\varrho}_{OLS}\) defined in (1), \(\Lambda_2\), \(\Lambda_3\) defined in (3) and

\[
\Omega(r) = \frac{1}{q} \int_{r-q/2}^{r+q/2} \{\Delta(v) - \Gamma(v)\Gamma(v)^\prime}\}dv.
\]

Now define the commutation matrix \(K_d\) such that \(K_d \text{vec}(G) = \text{vec}(G')\), and the elimination matrix \(L_d\) such that \(\text{vech}(G) = L_d \text{vec}(G)\) for any square matrix \(G\) of dimension \(d \times d\). Introduce the \(pd \times pd\) matrix

\[
A = \begin{pmatrix} A_{01} & \cdots & \cdots & A_{0p} \\
I_d & 0 & \cdots & 0 \\
0 & \ddots & 0 & \vdots \\
0 & 0 & I_d & 0 \\
\end{pmatrix} \tag{12}
\]

and the \(d \times pd\)-dimensional matrix \(J = (I_d, 0, \ldots, 0)\). We are in position to state the asymptotic behavior of the classical approximated OIRF estimator. Note that this result can be obtained using the same arguments of Lütkepohl (2005) Proposition 3.6, together with (11).
Proposition 3.3. Under the Assumptions A0-A1 in the Appendix, we have for all $r \in (q/2, 1 - q/2)$ and as $T \to \infty$

\[
\sqrt{T} \text{vec}\left( \frac{z_q^d}{\theta_r(i) - \bar{\theta}_r^q(i)} \right) \Rightarrow \mathcal{N}\left(0, C_i(r)\Lambda^{-1}_2\Lambda^{-1}_3 C_i(r)' + D_i(r)\Omega(r)D_i(r)\right), \ i = 0, 1, 2, ...
\]  

(13)

where $C_0 = 0$, $C_i(r) = \left( \bar{H}(r) \otimes I_d \right) \left( \sum_{m=0}^{i-1} J(A^r)^{i-1-m} \otimes \Phi_m \right)$, $i = 1, 2, ...$, $\bar{H}(r)$ is given in (5), and

\[D_i(r) = (I_d \otimes \Phi_i) \Xi(r), \ i = 0, 1, 2, ...
\]

with

\[\Xi(r) = L_d' \left[ L_d (I_d^2 + K_d) \left( \bar{H}(r) \otimes I_d \right) L_d \right]^{-1}.
\]

We propose an alternative approximated OIRF estimator based on the more efficient estimator $\hat{\theta}_{ALS}$ defined in equation (5) and the estimators $\hat{\Phi}^{als}_i$ of the coefficients $\hat{\Phi}_i$ of the infinite moving average representation of $(X_t)$. More precisely,

\[\frac{z_q^{q,als}}{\theta_r(i) - \bar{\theta}_r^q(i)} := \hat{\Phi}^{als}_i \hat{\theta}_r(i),
\]  

(14)

a new approximated OIRF estimator. Below we state its asymptotic distribution.

Proposition 3.4. Let the conditions of Proposition 3.3 and the Assumption A2 in the Appendix hold true. With the notation defined in Proposition 3.3 we have for all $r \in (q/2, 1 - q/2)$ and as $T \to \infty$

\[
\sqrt{T} \text{vec}\left( \frac{z_q^{q,als}}{\theta_r(i) - \bar{\theta}_r^q(i)} \right) \Rightarrow \mathcal{N}\left(0, C_i(r)\Lambda^{-1}_2\Lambda^{-1}_3 C_i(r)' + D_i(r)\Omega(r)D_i(r)\right), \ i = 0, 1, 2, ...
\]

(15)

Moreover, the difference between the asymptotic variance of vec\left( \frac{z_q^d}{\theta_r(i) - \bar{\theta}_r^q(i)} \right) given in equation (13) and the asymptotic variance of vec\left( \frac{z_q^{q,als}}{\theta_r(i) - \bar{\theta}_r^q(i)} \right) is a positive semi-definite matrix.

The proof of Proposition 3.4 is omitted since it follows the steps of the proof of Proposition 3.3 and use the results of Patilea and Raïssi (2012) on the convergence in law of $\hat{\theta}_{ALS}$. In particular, they proved that $\Lambda^{-1}_2\Lambda^{-1}_3 - \Lambda^{-1}_1$ is a positive semi-definite matrix and this implies that $\theta_r(i)$ is a lower variance estimator of $\bar{\theta}_r^q(i)$.

Although the standard $\theta_r(i)$, or the more efficient estimator $\frac{z_q^{q,als}}{\theta_r(i) - \bar{\theta}_r^q(i)}$ are easily to compute, for the reasons we detailed above, we believe that their limit it is not the
appropriate tool to resume the evolution of the tv-OIRF \(^4\). Instead we propose to use an estimator of the averaged OIRF. To build such an estimate of the averaged OIRF with negligible bias, we need a slightly modified kernel estimator of \(\Sigma(\cdot)\) that we introduce in the next section.

### 3.3 New OIRF estimators with time-varying variance

In this section, we propose an alternative estimator for the approximated OIRF and an estimator for the averaged OIRF we introduced in section \(2.3\). To guarantee \(\sqrt{T}\)-asymptotic normality for these estimators, we implicitly need suitable estimators of integral functionals under the form

\[
\frac{1}{q} \int_{r-q/2}^{r+q/2} A(v) \Sigma(v) dv
\]

with \(A(\cdot)\) some given matrix-valued function. The estimator of such integral obtained by plugging in the nonparametric estimator of the covariance structure introduced in equation \(24\) would not be appropriate as it suffers from boundary effects. More details on this problem are provided in section \(5.1\) in the Appendix. Therefore, in the sequel, we construct alternative, bias corrected estimators for such integral functionals.

For \(-[(q+h)T/2] \leq k \leq [(q+h)T/2]\), we define

\[
\hat{V}_{[rT]-k} = \frac{1}{T} \sum_{j=[(r-(q-h)/2)T]+1}^{[r+(q-h)/2)T]} \frac{1}{hL} \left( \frac{[rT] - k - j}{hT} \right) \hat{u}_j \hat{u}_j'.
\]  

(16)

Hereafter, for simplicity, we use the same bandwidth \(h\) for all the \(d^2\) components of the estimated matrix-valued integrals. Note that \(\hat{V}_{[rT]-k}\) is an estimator of \(\Sigma_{[rT]-k}\).

Next, let \(\hat{H}_{[rT]-k}\) denote the lower triangular matrix of the Cholesky decomposition of \(\hat{V}_{[rT]-k}\), that is

\[
\hat{V}_{[rT]-k} = \hat{H}_{[rT]-k} \hat{H}'_{[rT]-k}.
\]  

(17)

We propose the following adaptive least squares estimators of the time-varying averaged OIRF:

\[
\hat{\theta}_i^q(i) = \hat{\Phi}_i^{als} \hat{H}(r),
\]  

(18)

where

\[
\hat{H}(r) = \frac{1}{[qT]+1} \sum_{k=-[(q+h)T/2]}^{[(q+h)T/2]} \hat{H}_{[rT]-k}.
\]  

(19)
Proposition 3.5. If assumptions A0-A2 hold true, then for all \( r \in (q/2, 1 - q/2) \) and as \( T \to \infty \)
\[
\sqrt{T} \text{vec} \left( \hat{\theta}_i^2(i) - \bar{\theta}_i^2(i) \right) \Rightarrow N(0, \overline{\mathbf{C}}_i(r)\Lambda^{-1}_i \overline{\mathbf{C}}_i(r)' + D_i(r)\Omega(r)D_i(r)'), \quad i = 0, 1, 2, ...
\]
with \( D_i(r) \) and \( \Omega(r) \) defined in Proposition 3.3 and
\[
\overline{\mathbf{C}}_i(r) = \left( \frac{1}{q} \int_{r-q/2}^{r+q/2} H(v)'dv \otimes I_d \right) \left( \sum_{m=0}^{i-1} J \text{A}'(\cdot)^{-1-m} \otimes \Phi_m \right).
\]

3.4 Estimation of the variance variability index

Finally, we build estimators for the variance variability index introduced in section 2.4. In the proof of Proposition 3.5 it is shown that the estimator \( \hat{H}(r) \) defined in equation (19) behaves \( \sqrt{T} \)-asymptotically normal centered at \( \bar{H}(r) := \frac{1}{q} \int_{r-q/2}^{r+q/2} H(v)dv \). Then, the estimator of the index \( i_{r,q} \) is
\[
\hat{i}_{r,q} = \left\| \bar{H}(r)^{-1} \hat{H}(r) \right\|_2^2, \quad 0 < r - q/2 < r + q/2 < 1, \quad (20)
\]
where \( \hat{H}(r) \) is defined in (19) and \( \hat{H}(r) \) the lower triangular matrix of the Cholesky decomposition of \( \hat{S}_T(r) \) defined as in equation (9).

Proposition 3.6. Let assumptions A0-A2 hold true. Let \( 0 < q < 1/2 \) and \( r \in (q/2, 1 - q/2) \). If \( i_{r,q} > 1 \) and all other eigenvalues of the matrix \( \bar{H}(r)\hat{H}(r)^{-1} \hat{H}(r) \) are strictly smaller than \( i_{r,q} \), then \( \sqrt{T} \left( \hat{i}_{r,q} - i_{r,q} \right) \) converges in distribution to a centered normal variable. If \( i_{r,q} = 1 \), then \( \hat{i}_{r,q} - 1 = o_P(1/\sqrt{T}) \).

The estimator \( \hat{i}_{r,q} \) has a non standard rate of convergence in the case of constant variance \( \Sigma(\cdot) \). Determining this rate and its limit in distribution remains an open problem to be studied in the future.

4 Numerical illustrations

Several papers in the literature have documented potential problems for the statistical analysis or the interpretation of the OIRF. For instance Benkwitz et al. (2000) pointed out several issues related to the building of bootstrap confidence intervals (see also Lütkepohl et al. (2015) and references therein for recent developments in this field).
Furthermore we refer to Lütkepohl (2005), Section 2.3, for a discussion on the problems of the variables ordering or the missing of relevant variables. In order to address these issues, numerous settings were proposed in the literature. Note also that the standard tools for solving some of these issues cannot be directly applied or are computationally intractable in many times. For instance this can explain why confidence intervals are in general not displayed when time-varying covariance is taken into account in the literature. Such interesting topics deserve a complete work in our framework, and are beyond the scope of this article. Hence our numerical outputs will focus on the asymptotic behaviors of the OIRF and of the heteroscedasticity index \( i_{r,q} \) introduced above.

In particular, for the approximated OIRF approach, we will consider the estimator (14) which benefits from the more accurate ALS estimation in comparison to the classical estimator given in (8). The approximated OIRF estimator will be compared to the averaged OIRF estimator (18).

### 4.1 Monte Carlo experiments

In this part the \( \tilde{H}(r) \) will be computed using two bandwidths, \( h_1 = \frac{q}{2\sqrt{3}} T^{-1/3} \) and \( h_2 = \frac{q}{2\sqrt{3}} T^{-2/7} \), to illustrate the effect of the bandwidth choice on the OIRF analysis. The constant \( q/2\sqrt{3} \) corresponds to the standard deviation of a uniform distribution on an interval of length \( q \), while the rates \( T^{-1/3} \) and \( T^{-2/7} \) are two possible theoretical choices. In each experiment, 1000 independent trajectories of the following bivariate VAR(1) system are simulated

\[
X_t = AX_{t-1} + u_t, \quad u_t = H_t \epsilon_t, \quad \epsilon_t \sim \text{Gaussian iid}
\]

where

\[
A = \begin{pmatrix} 0.5 & -0.3 \\ 0.1 & 0.3 \end{pmatrix},
\]

and the \( \epsilon_t \)'s are standard Gaussian iid. The covariance of the errors terms \( \Sigma_t := H_t H_t' \) is driven by a matrix of functions \( \Sigma_t = \Sigma(t/T) \) with

\[
\Sigma(r) = \begin{pmatrix} \sigma_{11}^2(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}^2(r) \end{pmatrix},
\]

where \( \sigma_{11}^2(r) = 1.4 + \delta f(r) \) for a fixed non constant function \( f(\cdot) \) and \( \delta \geq 0 \). \( \sigma_{11}^2(r) \) is plotted in Figure 2. The others components of the covariance matrix are set as follows:
\[ \sigma_{22}(r) = 0.5\sigma_{11}(r) \quad \text{and} \quad \sigma_{12}(r) = \sigma_{21}(r) = \sqrt{\sigma_{11}^2(r)\sigma_{22}^2(r)} \times 0.7. \]

The patterns displayed by the covariance structure are intended to mimic business cycle behavior commonly observed for economic variables (see the Korean Won/USD exchange rate example). Note that when \( \delta = 0 \), we retrieve the homoscedastic case. Samples \( T = 100, 200, 400 \) and 800 are considered in the sequel.

In the Monte Carlo investigation, the changes through time are studied by considering the subsample \((0.5; 0.5)\), that is taking \( q = 0.5 \) and \( r = 0.5 \) (i.e. \( i_{0.5,0.5} \)). In order to avoid lengthy outputs, we only display the results for the orthogonalized response of the first variable for an impulse from its own past taking \( i = 1 \). The corresponding averaged (resp. approximated) OIRF will be denoted by \( \hat{\theta}^{i_{0.51}, 0.5} \) (resp. \( \tilde{\theta}^{i_{0.51}, 0.5} \)).

We begin with a comparison between the averaged and approximated approaches for resuming the OIRFs. All the outputs concerning the OIRF are obtained setting \( \delta = 1 \). In Figure 3 the relative differences between the averaged and approximated estimators are displayed. It appears that the approximated OIRF are in the order of 10% greater than the averaged OIRF. The ratio is even always positive for \( T = 400 \) and \( T = 800 \). Recall that the approximated approach does not rely on the adequate way to resume the Cholesky decompositions of the covariance. Hence we can conclude that the approximated approach delivers an upwards distorted picture of the OIRF when compared to the averaged approach. Now let us turn to the illustration of the asymptotic results in Proposition 3.3 and 3.5. From the Q-Q plots displayed in Figure 4 and 5, we can remark that the different OIRF estimates seems to behave as normal, even for small samples. In particular we did not notice major differences between the estimators of the averaged OIRF obtained using the bandwidths \( h_1 \) and \( h_2 \).

In this part we analyze the finite sample behavior of the index estimator defined in (20). Recall that the index is intended to capture the discrepancy between the homoscedastic and the heteroscedastic cases. Figure 6 and 7 correspond to a heteroscedasticity parameter \( \delta = 1 \). In Figure 8 various values are considered for \( \delta \), meanwhile the outputs for the homoscedastic case, \( \delta = 0 \), are displayed in Figure 9. From Figure 6, it can be seen that the normal approximation is not met for small samples. As the sample is increased, the results become better. Figure 7 and 9 show that the estimator \( \hat{i}_{r,q} \) seems to converge to the true value, whether \( i_{r,q} > 0 \) (the heteroscedastic case) or when \( i_{r,q} = 1 \) is in the border of the possible values (the homoscedastic case). All
these observations illustrate the statements of Proposition 3.6. Finally the ability of
the index to detect heteroscedastic situations is studied by allowing values from zero to
one for $\delta$. From Figure 8 it emerges that the $\hat{i}_{r,q}$ clearly take increasing values as $\delta$ is
far from zero. This suggests that the proposed index is relevant to decide whether the
approximated or averaged OIRF should be applied.

4.2 Real data analysis

We assess the discrepancy between the approximated and the averaged OIRF for the
seasonally adjusted log first differences of the monthly U.S. energy and transportation
consumer price indexes (CPI) for all urban consumers. The series taken from June 1,
1979 to May 1, 2019 ($T = 480$) are plotted in Figure 10. The effects of energy prices
shocks on other macroeconomic variables are commonly investigated in the applied
econometric literature. This can be explained by the importance of the energy sector
in world economies or finance markets. The reader is referred to papers published in
specialized journals like *Energy Economics*, *Energy Policy* or papers with JEL codes
Q43: *energy and macroeconomics* and C32: *time series models*. In general such kind
of data may exhibit fast variance changes (from the early 2000’s in our case). At first
glance this suggests that our methodology can deliver a quite different picture of the
OIRF when compared to the standard approach.

First a VAR(1) model is adjusted to the series to capture the conditional mean.
Following the ordering argument of Lütkepohl (2005, p61), the first component corre-
sponds to the energy CPI and the second one to the transportation CPI. Indeed it
is reasonable to think that there is no instantaneous effects from the transportation
CPI to the energy CPI. The model adequacy is checked using the portmanteau tests
proposed in Patilea and Raïssi (2013). The existence of second order dynamics in the
residuals is tested by considering the Monte Carlo cross validation portmanteau test
proposed in Patilea and Raïssi (2014). Our outputs not displayed here show that a
deterministic specification for the variance structure seems adequate.

Now we turn to the analysis of the time varying OIRF. A possible illustration would
consist in mapping the whole sample into 10 and 5 subsamples of same size, and apply

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The data can be downloaded from the website of the research division of the Federal Reserve Bank of
Saint Louis https://fred.stlouisfed.org/
the tools proposed in the paper. We first examine the $i_{r,q}$ indexes to evaluate how the heteroscedasticity can affect the standard approach in resuming the OIRF. From Tables 1 and 2 the $i_{r,q}$’s can be very far from one due to the fast variance changes. This suggests that the approximate OIRF should be avoided. In the sake of comparison of the different methods, the averaged and approximated OIRF are displayed in Figures 11 and 12. In view of the heteroscedasticity in the data, it appears that the approximated OIRF are dramatically biased when compared to the averaged OIRF. For instance we noted that the approximated OIRF are greater than the averaged OIRF upwards 27% for the 5 subsamples mapping. The greater discrepancies were found in period exhibiting sharp evolutions of the variance.

From our real data analysis it turns out that the standard approach, which consists in computing the approximate OIRF, can be quite misleading in presence of heteroscedasticity. Indeed considering the approximated OIRF leads to an oversized estimation of the OIRF. This would occur especially when economic crises or specific political events generate smooth fast or abrupt changes in the variance of the variables. Noting that rolling window or by periods analyses are actually performed to compare pre and post situations related to such events, it clearly appears that the averaged OIRF provide a reliable estimation of resumed OIRF.
5 Appendix

5.1 Kernel estimates of the covariance function integrals

As mentioned in section 3.3, we need suitable estimators of integral functionals

\[ \frac{1}{q} \int_{r-q/2}^{r+q/2} A(v) \Sigma(v) \, dv \]

with \( A(\cdot) \) some given matrix-valued function. The estimator of such integrals obtained by plugging in the nonparametric estimator of the covariance structure introduced in equation (4) would be asymptotically biased due to boundary effects.

To explain the rationale of the alternative nonparametric estimator we propose, we will assume for the moment that the \( d \times d \)-matrices \( u_j u'_j \) are available for all \( 1 \leq j \leq T \). Let us consider the generic real-valued random quantity

\[ S_T(r) = \int_{r-q/2}^{r+q/2} a(v) \left[ \frac{1}{|J|} \sum_{j \in J} \omega_{v,j}(h) u_j^{(k)} u_j^{(l)} \right] \, dv = \frac{1}{|J|} \sum_{j \in J} \left[ \int_{r-q/2}^{r+q/2} a(v) \omega_{v,j}(h) \, dv \right] u_j^{(k)} u_j^{(l)}, \]

where \( J = \{ j_{\text{min}}, \ldots, j_{\text{max}} \} \subset \{1, \ldots, T\} \) is a set of consecutive indices that will be specified below and \( |J| = j_{\text{max}} - j_{\text{min}} + 1 \) is the cardinal of \( J \); \( \omega_{v,j}(h) = h^{-1} L(h^{-1}(v - j/T)) \) with \( h \) a deterministic bandwidth with a rate that will be specified below, and \( L(\cdot) \) is a bounded symmetric density function with support \([-1, 1]\); \( a(\cdot) \) is a given differentiable function with Lipschitz continuous derivative; \( u_j^{(k)} \) and \( u_j^{(l)} \) are components of \( u_j \) and \( E(u_j^{(k)} u_j^{(l)}) = \Sigma^{(k,l)}(j/T) \), that is the \((k,l)\) cell of the matrix \( \Sigma(j/T) \).

By a change of variables and Taylor expansion,

\[ \int_{r-q/2}^{r+q/2} a(v) \frac{1}{h} L \left( \frac{v - j/T}{h} \right) \, dv = \int_{(r-q/2-j/T)/h}^{(r+q/2-j/T)/h} a(j/T + uh) L(u) \, du \\
= a(j/T) \int_{(r-q/2-j/T)/h}^{(r+q/2-j/T)/h} L(u) \, du + ha'(j/T) \int_{(r-q/2-j/T)/h}^{(r+q/2-j/T)/h} u L(u) \, du + O(h^2). \]

To avoid large bias, we aim at using the properties \( \int_{-1}^{1} L(u) \, du = 1 \) and \( \int_{-1}^{1} u L(u) \, du = 0 \). For this purpose, any \( j \in J \) should satisfy the conditions \((r+q/2-j/T)/h \geq 1\) and \((r-q/2-j/T)/h \leq -1\). That is, the indices set \( J \) should be defined such that

\[ \forall j \in J, \quad (r-q/2+h)T \leq j \leq (r+q/2-h)T. \]

Let us define

\[ j_{\text{min}} = [(r-q/2+h)T] + 1 \quad \text{and} \quad j_{\text{max}} = [(r+q/2-h)T]. \]
Then, uniformly with respect to \( j \in J \),
\[
\int_{r-q/2}^{r+q/2} a(v) \frac{1}{h} L \left( \frac{v-j/T}{h} \right) dv - a(j/T) = O(h^2).
\]
Note that \(|J| = [(q - 2h)T] \) and \(|J|/(q - 2h)T = 1 + O(1/T)\).

Now, we could deduce
\[
S_T = \frac{1}{|J|} \sum_{j \in J} a(j/T) \{ u_j^{(k)} u_j^{(l)} - \Sigma^{(k,l)}(j/T) \} + \frac{1}{|J|} \sum_{j \in J} a(j/T) \Sigma^{(k,l)}(j/T) + O(h^2)
\]
\[
= \Delta_T + \frac{1}{q - 2h} \int_{r-q/2+\mu}^{r+q/2-h} a(v) \Sigma^{(k,l)}(v) dv + O(T^{-1}) + O(h^2).
\]

Let us comment on these findings. To make the reminder \( O(h^2) \) negligible, we will need to impose \( Th^4 \to 0 \). For instance, we could consider a bandwidth \( h \) under the form
\[
h = c \frac{q}{2\sqrt{3}} T^{-2/7}, \quad \text{for some constant } c > 0.
\]
The factor \( q/2\sqrt{3} \) takes into account the standard deviation of a uniform design on the interval \([r - q/2, r + q/2] \). The term \( \Delta_T \) is a sum of independent centered variables and will have a Gaussian limit. Finally, let us focus on the last integral and notice that
\[
\frac{1}{q - 2h} \int_{r-q/2+\mu}^{r+q/2-h} a(v) \Sigma^{(k,l)}(v) dv = \frac{1}{q} \int_{r-q/2}^{r+q/2} a(v) \Sigma^{(k,l)}(v) dv + O(h).
\]

Thus \( S_T \) preserves a non negligible bias as an estimator of \( q^{-1} \int_{r-q/2}^{r+q/2} a(v) \Sigma^{(k,l)}(v) dv \).
The solution we will propose to remove this bias is to define estimates like \( S_T \) with modified \( q \) and thus with modified bounds \( j_{\text{min}} \) and \( j_{\text{max}} \) of the set \( J \).

### 5.2 Assumptions

**Assumption A0:** (a) The process \((\epsilon_t)\) is iid such that \( E(\epsilon_t \epsilon_t') = I_d \), with \( I_d \) the \( d \times d \) identity matrix, and \( \sup_t \| \epsilon_i,t \|_{\mu} < \infty \) for some \( \mu > 8 \) and for all \( i \in \{1, \ldots, d\} \) with \( \| . \|_{\mu} := (E \| . \|_{\mu}^\mu)^{1/\mu} \) and \( \| . \| \) being the Euclidean norm. Moreover \( E \left( \epsilon^{(i)}_t \epsilon^{(j)}_t \epsilon^{(k)}_t \right) = 0 \), \( i, j, k \in \{1, \ldots, d\} \).

(b) The matrix \( A \) given in (12) is of full rank.

The covariance of the system (11) is allowed to vary in time according to assumption A1 below.
Assumption A1: We assume that $H_t = G(t/T)$, where the matrices $G(\cdot)$ are lower triangular matrices with positive diagonal components. The components $\{g_{k,l}(r) : 1 \leq k, l \leq d\}$ of the matrices $G(r)$ are measurable deterministic functions on the interval $(0, 1]$, with $\forall 1 \leq k, l \leq d, \sup_{r \in (0, 1]} |g_{k,l}(r)| < \infty$. The functions $g_{k,l}(\cdot)$ satisfy a Lipschitz condition piecewise on a finite partition of $(0, 1]$ in sub-intervals (the partition may depend on $k, l$). The matrix $\Sigma(r) = G(r)G(r)'$ is assumed positive definite for all $r$ and $\inf_{r \in (0, 1]} \lambda_{\min}(\Sigma(r)) > 0$ where $\lambda_{\min}(\Gamma)$ denotes the smallest eigenvalue of the symmetric matrix $\Gamma$.

The OIRF estimates we investigate in the following are obtained as products between a functional of the innovation vectors $u_t$ (the estimator of $\varphi_0$) and a centered functional of matrices $u_t u_t'$ (the estimator of some square root matrix built using the covariance structure $\Sigma(\cdot)$). The $\sqrt{T}$-asymptotic normality of the OIRF estimators is then deduced from the asymptotic behavior of the two factors. The condition $E \left( \left( \epsilon_t^{(i)} \epsilon_t^{(j)} \epsilon_t^{(k)} \right) \right) = 0, \ i, j, k \in \{1, \ldots, d\}$, is a convenient condition for simplifying the asymptotic variance of our estimators, that is making it block diagonal. It is in particular fulfilled if the errors are supposed Gaussian. The asymptotic results could be also deduced if this condition fails, the asymptotic variance of the estimators would then include some additional covariance terms.

Assumption A2: (i) The kernel $K(\cdot)$ is a bounded symmetric density function defined on the real line such that $K(\cdot)$ is nondecreasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$ and $\int_{\mathbb{R}} |v| K(v) dv < \infty$. The function $K(\cdot)$ is differentiable except a finite number of points and the derivative $K'(\cdot)$ is a bounded integrable function. Moreover, the Fourier Transform $\mathcal{F}[K](\cdot)$ of $K(\cdot)$ satisfies $\int_{\mathbb{R}} |s\mathcal{F}[K](s)| ds < \infty$.

(ii) The bandwidths $b_{kl}, 1 \leq k \leq l \leq d$, are taken in the range $B_T = [c_{\min} b_T, c_{\max} b_T]$ with $0 < c_{\min} < c_{\max} < \infty$ and $b_T + 1/T b_T^{2+\gamma} \to 0$ as $T \to \infty$, for some $\gamma > 0$.

(iii) The kernel $L(\cdot)$ is a symmetric bounded Lipschitz continuous density function with support in $[-1, 1]$.

(iv) The bandwidth $h$ satisfies the condition $h^4 T + 1/T h^2 \to 0$ as $T \to \infty$.

5.3 Proofs

In the sequel, $c, c', c''$ and $C, C', C''$ are constants, possibly different from line to line.
Proof of Lemma 2.1. First, note that

$$
\tilde{H}(r)\tilde{H}(r)' - \tilde{H}(r)\tilde{H}(r)' = \frac{1}{q} \int_{r-q/2}^{r+q/2} H(v)\tilde{H}(v)' \, dv
$$

is a positive semi-definite matrix, whatever the values of $r$ and $q$ are. Moreover,

$$
\tilde{H}(r)\tilde{H}(r)' = \tilde{H}(r)\tilde{H}(r)' \quad \text{if and only if} \quad H(\cdot) \text{ is constant on } (r-q/2, r+q/2) \quad (22)
$$

Indeed, for any $a \in \mathbb{R}^d$,

$$
a' \left\{ \tilde{H}(r)\tilde{H}(r)' - \tilde{H}(r)\tilde{H}(r)' \right\} a
= \frac{1}{q} \int_{r-q/2}^{r+q/2} a' \left( H(v) - \frac{1}{q} \int_{r-q/2}^{r+q/2} H(u) \, du \right) \left( H(v) - \frac{1}{q} \int_{r-q/2}^{r+q/2} H(u) \, du \right)' \, dv
= \frac{1}{q} \int_{r-q/2}^{r+q/2} a' \left( H(v) - \frac{1}{q} \int_{r-q/2}^{r+q/2} H(u) \, du \right)^2 \, dv \geq 0.
$$

This shows that $\tilde{H}(r)\tilde{H}(r)' - \tilde{H}(r)\tilde{H}(r)'$ is positive semi-definite. Next, under our assumptions, for each $a \in \mathbb{R}^d$ the map

$$
v \mapsto \left\| a' \left( H(v) - \frac{1}{q} \int_{r-q/2}^{r+q/2} H(u) \, du \right) \right\|^2 \quad (23)
$$

is piecewise continuous on $(0,1)$. Thus, if $\tilde{H}(r)\tilde{H}(r)' = \tilde{H}(r)\tilde{H}(r)'$, then necessarily, for each $a$, the map $23$ is constant equal to zero. This implies that $H(\cdot)$ is constant on $(r-q/2, r+q/2)$. Conversely, when $H(\cdot)$ is constant, then $\tilde{H}(\cdot) = \tilde{H}(\cdot)$ and thus $\tilde{H}(r)\tilde{H}(r)' = \tilde{H}(r)\tilde{H}(r)'$. Finally, the two statements in the lemma are direct consequences of (22) and the positive semi-definiteness of $\tilde{H}(r)\tilde{H}(r)' - \tilde{H}(r)\tilde{H}(r)'$. \qed

Proof of Proposition 3.1. For the convergence of $\tilde{\Sigma}_{[r,T]}$ let us recall that $\Sigma(r) = G(r)G(r)'$ and the components $\{g_{k,l}(\cdot) : 1 \leq k, l \leq d\}$ of $G(\cdot)$ are bounded piecewise Lipschitz continuous functions. Let $U_t(\vartheta) = u_t(\vartheta)u_t(\vartheta)'$ with $u_t(\vartheta) = X_t - (\bar{X}_{t-1} \otimes I_d)\vartheta$ for some $\vartheta \in \mathbb{R}^{d^2}$. Thus $U_t(\vartheta_0) = u_tu_t'$ and $U_t(\hat{\vartheta}_{OLS}) = \hat{u}_t\hat{u}_t'$. By elementary matrix algebra,

$$
\left\| U_t(\hat{\vartheta}_{OLS}) - U_t(\vartheta_0) \right\|_F \leq 2d\sqrt{p} \|G\|_\infty \left\| \hat{\vartheta}_{OLS} - \vartheta_0 \right\| \left\| \bar{X}_{t-1} \right\| \|\epsilon_t\| + d^2p \left\| \hat{\vartheta}_{OLS} - \vartheta_0 \right\|^2 \left\| \bar{X}_{t-1} \right\|^2.
$$
Herein, $\| \cdot \|_F$, $\| \cdot \|$ and $\| \cdot \|_{\infty}$ are the Frobenius, Euclidian and uniform norms, respectively. By the triangle inequality, the monotonicity of $K(\cdot)$ and the rate of $\| \hat{\vartheta}_{\text{OLS}} - \vartheta_0 \|$, deduce

$$\sup_{B_T} \left\| \hat{\Sigma}_{[rT]} - \sum_{j=1}^{T} w_{[rT],j} \odot u_{j} u_j' \right\|_F \leq \frac{1}{c_{\min} T b_T} \sum_{j=1}^{T} K \left( \frac{[rT] - j}{c_{\max} b_T T} \right) \left\| \hat{u}_j u_j' - u_{j} u_j' \right\|_F$$

$$= O_P(1/\sqrt{T}).$$

Next, we can write

$$\sum_{j=1}^{T} w_{[rT],j} \odot u_{j} u_j' = \sum_{j=1}^{T} w_{[rT],j} \odot \left\{ u_{j} u_j' - E(u_{j} u_j') \right\} + \sum_{j=1}^{T} w_{[rT],j} \odot E(u_{j} u_j')$$

$$=: \Sigma_{1,[rT]} + \Sigma_{2,[rT]}.$$

Let $\sigma_{1,[rT]}^{(k,l)}$, $\sigma_{2,[rT]}^{(k,l)}$, $\Sigma(r)^{(k,l)}$ and $\Sigma_j^{(k,l)}$ denote the $(k,l)$ elements of the matrices $\Sigma_{1,[rT]}$, $\Sigma_{2,[rT]}$, $\Sigma(r)$ and $E(u_{j} u_j')$, respectively.

First we study the bias. For any $r \in (0,1)$, since $K(\cdot)$ is symmetric, we have

$$\sigma_{2,[rT]}^{(k,l)} = \frac{1}{T b_{kl}} \sum_{j=1}^{T} K \left( \frac{j - [rT]}{T b_{kl}} \right) \Sigma_j^{(k,l)}$$

$$= \frac{1}{b_{kl}} \int_{[1/T, (1+T)/T]} K \left( \frac{sT - [rT]}{T b_{kl}} \right) \Sigma^{(k,l)} ds$$

$$z = (s-r)/b_{kl}$$

$$= \int_{[1-Tr b_{kl}, (1+T-Tr)/b_{kl}]} K \left( \frac{(r + z b_{kl})T - [rT]}{T b_{kl}} \right) \Sigma(r + z b_{kl})^{(k,l)} dz$$

$$+ \int_{[-r/b_{kl}, 0)} K \left( \frac{(r + z b_{kl})T - [rT]}{T b_{kl}} \right) \left\{ \Sigma(r + z b_{kl})^{(k,l)} - \Sigma(0)^{(k,l)} \right\} dz$$

$$+ \int_{[0, (1-r)/b_{kl}]} K \left( \frac{(r + z b_{kl})T - [rT]}{T b_{kl}} \right) \left\{ \Sigma(r + z b_{kl})^{(k,l)} - \Sigma(r+)^{(k,l)} \right\} dz$$

$$+ \Sigma(r+)^{(k,l)} \int_{[-r/b_{kl}, 0)} K \left( \frac{(r + z b_{kl})T - [rT]}{T b_{kl}} \right) dz$$

$$+ \Sigma(r-)^{(k,l)} \int_{[0, (1-r)/b_{kl}]} K \left( \frac{(r + z b_{kl})T - [rT]}{T b_{kl}} \right) dz + O(1/Tb_T).$$

Next, on the intervals where the Lipschitz property holds true, for $z \geq 0$ we have

$$\left| \Sigma(r + z b_{kl})^{(k,l)} - \Sigma(r+)^{(k,l)} \right| \leq L_{c_{\max}} |z| b_T,$$

and for $z < 0$ we have

$$\left| \Sigma(r + z b_{kl})^{(k,l)} - \Sigma(r-)^{(k,l)} \right| \leq L_{c_{\max}} |z| b_T,$$
for some constant $L$. Meanwhile, for any $b_{kl} \in B_T$,

$$0 \leq \frac{[(r + zb_{kl})T - [rT]]}{T b_{kl}} - z \leq \frac{1}{T c_{\min} b_T},$$

so that, since $K(\cdot)$ is piecewise Lipschitz continuous, except at most a finite number of values $z$,

$$\sup_{b_{kl} \in B_T} \left| K \left( \frac{[(r + zb_{kl})T - [rT]]}{T b_{kl}} \right) - K(z) \right| \leq \frac{1}{T c_{\min} b_T}.$$  

Finally, for $r \in (0, 1)$, $r/b_{kl}$ and $(1 - r)/b_{kl}$ tend to infinity and thus

$$\inf_{b_{kl} \in B_T} \int_{[-r/b_{kl}, 0]} K \left( \frac{[(r + zb_{kl})T - [rT]]}{T b_{kl}} \right) dz \uparrow \frac{1}{2}$$

and

$$\inf_{b_{kl} \in B_T} \int_{[0,(1-r)/b_{kl})} K \left( \frac{[(r + zb_{kl})T - [rT]]}{T b_{kl}} \right) \uparrow \frac{1}{2}.$$  

The case $r = 1$ could be treated with similar arguments. Gathering facts, deduce that, for any $r \in (0, 1]$, the rate of the bias term is

$$\sup_{B_T} \left\| \hat{\Sigma}_{2, [rT]} - \frac{1}{2} \{ \Sigma(r-) + \Sigma(r+) \} \right\|_F = O(b_T + 1/T b_T).$$

For the variance term $\Sigma_{1, [rT]}$, we could use the properties of the empirical process indexed by families of functions of polynomial complexity. Here the family of functions are indexed by the constants that multiplies the rate $b_T$ to define the bandwidths for each element $(k, l)$ in the matrix. The polynomial complexity is guaranteed by the monotonicity of $K(\cdot)$ and by the fact that the polynomial complexity is preserved by finite unions. We apply Theorem 3.1 in van der Vaart and Wellner (2011) for each component $(k, l)$ with the family

$$\mathcal{F}_T = \{(r, u^{(k)}, u^{(l)}) \mapsto K(a - r/c b_T)u^{(k)}u^{(l)} : r \in (0, 1], a > 0, c_{\min} \leq c \leq c_{\max} \},$$

the envelope $F(r, u^{(k)}, u^{(l)}) = Cu^{(k)}u^{(l)}$ for some constant $C > 0$, $p = (\mu - 4)/(\mu - 8) > 1$, $\delta^2 = c' b_T$ for some constant $c' > 0$. In this case $J(\delta, F, L_2) \leq C' \delta \sqrt{\log(1/\delta)} \leq C'' \sqrt{b_T \log(T)}$, for some constants $C', C'' > 0$. Deduce that

$$\sup_{B_T} \left\| \hat{\Sigma}_{1, [rT]} \right\|_F = O_P \left( \sqrt{\log(T)/T b_T} \right).$$

For the second part of the results on the matrices $H$, it suffices to apply a perturbation bound for the Cholesky factorization, as for instance in Theorem 3.1 of Chang and Stehlé (2010), to deduce

$$\sup_{B_T} \left\| \hat{H}_{[rT]} - \frac{1}{2} H_{\pm}(r) \right\|_F \leq C \left\| \frac{1}{2} H_{\pm}(r) \right\|_F \sup_{B_T} \left\| \hat{\Sigma}_{[rT]} - \frac{1}{2} \{ \Sigma(r-) + \Sigma(r+) \} \right\|_F,$$

for some constant $C$. Now the proof in complete. \qed
Proof of Lemma 3.2. First let us write from the mean value Theorem:

\[ \text{vech}(\text{Int}_{T,r,q}(\hat{U})) = \text{vech}(\text{Int}_{T,r,q}(U)) + \frac{\partial \text{vech}(\text{Int}_{T,r,q}(U(\vartheta)))}{\partial \vartheta^*}|_{\vartheta = \vartheta^*} (\hat{\vartheta}_{\text{OLS}} - \vartheta_0), \]

where \( \hat{U}_t = \hat{u}_t \hat{u}_t' \), \( U_t = u_t u_t' \), \( U_t(\vartheta) = u_t(\vartheta)u_t(\vartheta)' \) and \( u_t(\vartheta) = X_t - (\hat{X}_t' - I_d)\vartheta \) for some \( \vartheta \in \mathbb{R}^{d^2p} \) and \( \vartheta^* \) between \( \hat{\vartheta}_{\text{OLS}} \) and \( \vartheta_0 \). Noting that \( \frac{\partial u_t(\vartheta)}{\partial \vartheta} = -((\hat{X}_t' - I_d)\vartheta) \), the consistency of \( \hat{\vartheta}_{\text{OLS}} \) and \( \text{vech}(\text{Int}_{T,r,q}(U(\vartheta))) \)

Using the \( \sqrt{T} \)-convergence of \( \hat{\vartheta}_{\text{OLS}} \) (see (2)), this implies that

\[ \sqrt{T} \text{vech}(\text{Int}_{T,r,q}(\hat{U}) - \text{Int}_{T,r,q}(\Sigma)) = \sqrt{T} \text{vech}(\text{Int}_{T,r,q}(U) - \text{Int}_{T,r,q}(\Sigma)) + o_p(1), \quad (24) \]

where we recall that \( \text{Int}_{r,q}(\Sigma) = q^{-1} \int_{r-q/2}^{r+q/2} \Sigma(u)dv. \)

Next, we investigate the joint distribution of

\[ \sqrt{T} \left[ (\hat{\vartheta}_{\text{OLS}} - \vartheta_0)', \{\text{vech}(\text{Int}_{T,r,q}(U) - \text{Int}_{T,r,q}(\Sigma))\} \right]' \]. \quad (25) \]

We write:

\[
\left( \begin{array}{c}
\hat{\vartheta}_{\text{OLS}} - \vartheta_0 \\
\text{vech}(\text{Int}_{T,r,q}(U) - \text{Int}_{T,r,q}(\Sigma))
\end{array} \right) = \left( \begin{array}{cc}
\text{vech}(\text{Int}_{T,0.5,1}(\bar{X})) \otimes I_d & -1 \\
0 & I_{d(d+1)/2}
\end{array} \right) \left( \begin{array}{c}
\Upsilon_t^1 \\
\Upsilon_t^2
\end{array} \right),
\]

where \( \bar{X}_{t-1} = \bar{X}_{t-1} \bar{X}_t' \), \( \Upsilon_t^1 = \text{vec}(\text{Int}_{T,0.5,1}(X'u)) \), with \( X'u = u_t \bar{X}_t' \) and

\[ \Upsilon_t^2 = \text{vech}(\text{Int}_{T,r,q}(U) - \text{Int}_{T,r,q}(\Sigma)). \]

The vector \( \Upsilon_t = (\Upsilon_t^1', \Upsilon_t^2')' \) is a martingale difference since the process \( (u_t) \) is independent. On the other hand we have \( T^{-1} \sum_{t=1}^{T} \text{Int}_{T,0.5,1}(\bar{X}) \otimes I_d \to \Lambda_3 \), from Patiilea and Raïssi (2012). Then from the Lindeberg CLT and the Slutsky Lemma, (25) is asymptotically normally distributed with mean zero. For the asymptotic covariance matrix in (11), the top left block is given from the asymptotic normality result (2), while the bottom right block can be obtained using the same arguments of Patiilea and Raïssi (2010), Lemma 7.1, 7.2, 7.3 and 7.4. The asymptotic covariance matrix is block diagonal since we assumed that \( E(u_{it}u_{jt}u_{kt}) = 0, i, j, k \in \{1, \ldots, d\} \) in A1, together with considering again that \( u_t \) is independent with respect to the past of \( X_t \). Hence the asymptotic matrix of (25) is given as in (11). \( \square \)
To simplify the reading, before proceeding to the next proofs, let us put the orthogonal impulse response function (OIRF) notation in a nutshell. First, let $S \mapsto C(S)$ be the operator that maps a positive definite matrix into the lower triangular matrix of the Cholesky decomposition of $S$. Next, consider a matrix-valued function $r \mapsto A(r)$, $r \in (0, 1]$, and, for any $r \in (0, 1]$, $0 < q < 1$ such that $0 < r - q/2 < r + q/2 < 1$, let

$$Int_{r,q}(A) = \frac{1}{q} \int_{r-q/2}^{r+q/2} A(v)dv \quad Int_{T,r,q}(A) = \frac{1}{[qT]} \sum_{k=-[qT/2]}^{[qT/2]} A_{[rT]-k}.$$ 

If $\sup_{r \in (0,1)} \|A(r)\|_F < \infty$ and the components of $A(\cdot)$ are piecewise Lipschitz continuous on each sub-intervals of a finite number partition of $(0,1]$, then there exists a constant $c$ such that

$$\sup_{r,q} \|Int_{r,q}(A) - Int_{T,r,q}(A)\|_F \leq cT^{-1}.$$ 

Now, we could rewrite the theoretical IRF we introduced above as follows: for any $i \geq 1$,

(approximated OIRF) \quad \tilde{\theta}^q(i) = \Phi_i \tilde{H}(r) = \Phi_i C(Int_{r,q}(\Sigma)),$$

and

(averaged OIRF) \quad \bar{\theta}^q(i) = \Phi_i \int r \mapsto A(v)dv = \Phi_i Int_{T,r,q}(C(\Sigma)).$$

Moreover, the estimators we introduced could be rewritten as follows: with the matrix-valued function $r \mapsto \hat{U}(r) = \hat{u}_{[rT]}^{[rT]}$, the usual approximated OIRF estimator is

$$\tilde{\theta}^q_r(i) = \Phi_i \hat{H}(r) = \Phi_i C(Int_{T,r,q}(\hat{U}));$$

the new approximated OIRF estimator is

$$\tilde{\theta}^q_{r,als}(i) = \Phi_i C(Int_{T,r,q}(\hat{U}));$$

and the averaged OIRF estimator is

$$\hat{\theta}^q(i) = \Phi_i \frac{q + h}{q} Int_{T,r,q+h}(C(\hat{V}));$$

with

$$\frac{q + h}{q} \int T,r,q+h(C(\hat{V})) = \tilde{H}(r)\{1 + O(1/T)\} = \frac{1 + O(1/T)}{[qT] + 1} \sum_{k=-(q+h)T/2}^{[q+h)T/2]} \hat{H}_{[rT]-k};$$
and \( \hat{H}_{[rT]−k} \) the lower triangular matrix of the Cholesky decomposition of \( \hat{V}_{[rT]−k} \), where \( \hat{V}_{[rT]−k} \) is defined in equation (16).

Next, let us recall the differentiation formula of the Cholesky operator
\[
\Delta := \frac{\partial \text{vec}(C(\Sigma))}{\partial \text{vec}(\Sigma)} = (I_d \otimes C(\Sigma))Z(C(\Sigma)^{-1} \otimes C(\Sigma)^{-1}),
\]
(26)
where \( Z \) is a diagonal matrix such that \( Z \text{vec}(A) = \text{vec}(\Phi(A)) \) for any \( d \times d \)-matrix \( A \). Here \( \Phi \) takes the lower-triangular part of a matrix and halves its diagonal:
\[
\Phi(A)_{ij} = \begin{cases} 
A_{ij} & \text{if } i > j \\
\frac{1}{2}A_{ij} & \text{if } i = j \\
0 & \text{if } i < j
\end{cases}
\]
Note that
\[
(C(\Sigma)^{-1} \otimes C(\Sigma)^{-1})\text{vec}(\Sigma) = \text{vec}(C(\Sigma)^{-1}\Sigma(C(\Sigma)')^{-1}) = \text{vec}(I_d)
\]
and thus
\[
\Delta \text{vec}(\Sigma) = (I_d \otimes C(\Sigma))Z\text{vec}(I_d) = (I_d \otimes C(\Sigma))\text{vec}(\Phi(I_d))
\]
\[
= \text{vec}(C(\Sigma)\Phi(I_d)) = \frac{1}{2}\text{vec}(C(\Sigma)).
\]

**Proof of Proposition 3.3** Using our notations we write
\[
\tilde{\theta}_r(i) - \tilde{\theta}_r^q(i) = \tilde{\Phi}_i^{\text{ols}}C(\text{Int}_{T,r,q}(\hat{U})) - \Phi_i C(\text{Int}_{r,q}(\Sigma)).
\]
From (24) and the consistency of the OLS estimator, we have
\[
\sqrt{T}(\tilde{\Phi}_i^{\text{ols}}C(\text{Int}_{T,r,q}(\hat{U})) - \Phi_i C(\text{Int}_{r,q}(\Sigma))) = \sqrt{T}(\tilde{\Phi}_i^{\text{ols}}C(\text{Int}_{T,r,q}(U)) - \Phi_i C(\text{Int}_{r,q}(\Sigma))) + o_p(1).
\]
Now let us write
\[
\sqrt{T}\text{vec }\left[ \tilde{\Phi}_i^{\text{ols}}C(\text{Int}_{T,r,q}(U) - \Phi_i C(\text{Int}_{r,q}(\Sigma)) \right] = \sqrt{T}\text{vec }\left[ (\tilde{\Phi}_i^{\text{ols}} - \Phi_i)C(\text{Int}_{r,q}(\Sigma)) \right] \\
+ \Phi_i(C(\text{Int}_{T,r,q}(U) - C(\text{Int}_{r,q}(\Sigma))) \\
+ (\tilde{\Phi}_i^{\text{ols}} - \Phi_i)(C(\text{Int}_{T,r,q}(U)) - C(\text{Int}_{r,q}(\Sigma))).
\]
(27)
For the third term in the right hand side of (27), \( \sqrt{T}\text{vec }\{\tilde{\Phi}_i^{\text{ols}} - \Phi_i\} \) is asymptotically normal as we can apply the delta method from \textbf{A0}(b), Lemma 3.2 and Rule (8)
Appendix A.13 in Lütkepohl (2005). Similarly using Rule (10) in Appendix A.13 of Lütkepohl (2005) and Lemma 3.2 again, \( \sqrt{T} \text{vec}\{(C(\text{Int}_{T,r,q}(U)) - C(\text{Int}_{r,q}(\Sigma)))\} \) is asymptotically normal. Hence we have

\[
(\hat{\Phi}^{\text{ols}}_i - \Phi_i)(C(\text{Int}_{T,r,q}(U)) - C(\text{Int}_{r,q}(\Sigma))) = O_p(T^{-1}),
\]

so that we obtain

\[
\sqrt{T} \text{vec}\left[\hat{\Phi}^{\text{ols}}_i C(\text{Int}_{T,r,q}(U)) - \Phi_i C(\text{Int}_{r,q}(\Sigma))\right] = \sqrt{T} \text{vec}\left[(\hat{\Phi}^{\text{ols}}_i - \Phi_i)C(\text{Int}_{r,q}(\Sigma))\right] + \Phi_i(C(\text{Int}_{T,r,q}(U)) - C(\text{Int}_{r,q}(\Sigma))) + o_p(1).
\]

For the last equality we used (26) and the delta method argument. The convergence asympotically normal. Hence we have

\[
\text{vec}\left\{\Phi_i(C(\text{Int}_{T,r,q}(U)) - C(\text{Int}_{r,q}(\Sigma)))\right\} = (I_d \otimes \Phi_i) \text{vec}\{C(\text{Int}_{T,r,q}(U)) - C(\text{Int}_{r,q}(\Sigma))\}\}
\]

and

\[
\text{vec}\left\{(\hat{\Phi}^{\text{ols}}_i - \Phi_i)C(\text{Int}_{r,q}(\Sigma))\right\} = (C(\text{Int}_{T,r,q}(U)) - C(\text{Int}_{r,q}(\Sigma)))\{1 + o_p(1)\}.
\]

Proof of Proposition 3.5 Let us fix \( \tilde{q} \in (0,1) \) and consider the definitions in equations (16) and (19) with the generic \( \tilde{q} \) replacing \( q \). Note that, given \( A(\cdot) \) a \( d \times d \)-matrix valued function defined on \( (0,1] \) with differentiable elements that have Lipschitz continuous derivatives, we have

\[
\text{Int}_{T,r,\tilde{q}+2h} \left(\text{vec}(A\tilde{V})\right) = \frac{1}{|q+2h|T} \sum_{k=-|\tilde{q}+2h|T/2}^{[\tilde{q}+2h]T/2} \text{vec}(A_{[rT]-k}\tilde{V}_{[rT]-k}) + O_p(1/T)h
\]

\[
= \frac{1}{\tilde{q} + 2h} \frac{1}{T} \sum_{j=[(r-\tilde{q}/2)T]+1}^{[(r+\tilde{q}/2)T]} \text{vec} \left( \int_{r-\tilde{q}/2-h}^{r+\tilde{q}/2+h} \frac{1}{h} L \left( \frac{v - j/T}{h} \right) A(v)dv \right) \tilde{u}_j \tilde{u}_j^\top + O_p(1/T)\]

\[
= \frac{1}{\tilde{q} + 2h} \frac{1}{T} \sum_{j=[(r-\tilde{q}/2)T]+1}^{[(r+\tilde{q}/2)T]} \text{vec} \left( A(j/T)\tilde{u}_j \tilde{u}_j^\top \right) + O_p(h^2 + 1/T)\]

\[
= \frac{\tilde{q}}{\tilde{q} + 2h} \frac{1}{|q|T} \sum_{j=-|\tilde{q}/2|T/2}^{[\tilde{q}/2]T/2} \text{vec} \left( A((|rT| - j)/T)\tilde{u}_j \tilde{u}_j^\top \right) + O_p(h^2 + 1/T)\]

\[
= \frac{\tilde{q}}{\tilde{q} + 2h} \text{Int}_{T,r,\tilde{q}} \left(\text{vec}(A\tilde{U})\right) + O_p(h^2 + 1/T)\right)\]  \( (28) \)
Moreover, we can write
\[
\text{vec}(C(\hat{V})) - \text{vec}(C(\Sigma)) = \Delta \left[ \text{vec}(\hat{V}) - \text{vec}(\Sigma) \right] \{1 + o_\Delta(1)\}.
\]

Gathering facts, we could now study the asymptotic equivalent of \(\text{vec}(\text{Int}_{T,r,q+h}(C(\hat{V})))\).

We have
\[
\text{vec} \left( \text{Int}_{T,r,q+h} \left( C \left( \hat{V} \right) \right) \right) = \text{Int}_{r,q+h} \left( \text{vec} \left( C \left( \hat{V} \right) \right) \right)
\]
\[
= \text{Int}_{r,q+h} \left( \text{vec} \left( C \left( \Sigma \right) \right) \right) + O_\Delta(1/Th)
\]
\[
+ \left\{ \text{Int}_{T,r,q+h}(\Delta \text{vec}(\hat{V})) - \text{Int}_{r,q+h}(\Delta \text{vec}(\Sigma)) + O_\Delta(1/Th) \right\} \{1 + o_\Delta(1)\}
\]
\[
= \frac{q}{q+h} \text{Int}_{r,q} \left( \text{vec} \left( C \left( \Sigma \right) \right) \right) + O_\Delta(1/Th)
\]
\[
+ \frac{1}{q+h} \int_{r-(q+h)/2}^{r-q/2} \text{vec}(C(\Sigma(v)))dv + \frac{1}{q+h} \int_{r+q/2}^{r+(q+h)/2} \text{vec}(C(\Sigma(v)))dv
\]
\[
+ \left\{ \frac{q-h}{q+h} \text{Int}_{T,r,q-h}(\Delta \text{vec}(\hat{U})) + O_\Delta(h^2+1/Th) - \text{Int}_{r,q+h}(\Delta \text{vec}(\Sigma)) + O_\Delta(1/Th) \right\}
\]
\[
\times \{1 + o_\Delta(1)\},
\]

where for replacing \(\text{Int}_{T,r,q+h}(\Delta \text{vec}(\hat{V}))\) we use the equation (28) with \(q - h\) instead of \(\bar{q}\). Moreover, since \(\Delta \text{vec}(\Sigma) = (1/2)\text{vec}(C(\Sigma))\), we also have
\[
\text{Int}_{r,q+h}(\Delta \text{vec}(\Sigma)) = \frac{q-h}{q+h} \text{Int}_{r,q-h}(\Delta \text{vec}(\Sigma))
\]
\[
+ \frac{1}{2} \frac{1}{q+h} \int_{r-(q+h)/2}^{r-q/2} \text{vec}(C(\Sigma(v)))dv + \frac{1}{2} \frac{1}{q+h} \int_{r-q/2}^{r-(q-h)/2} \text{vec}(C(\Sigma(v)))dv
\]
\[
+ \frac{1}{2} \frac{1}{q+h} \int_{r+q/2}^{r+(q+h)/2} \text{vec}(C(\Sigma(v)))dv + \frac{1}{2} \frac{1}{q+h} \int_{r+(q-h)/2}^{r+q/2} \text{vec}(C(\Sigma(v)))dv
\]
\[
= \frac{q-h}{q+h} \text{Int}_{r,q-h}(\Delta \text{vec}(\Sigma))
\]
\[
+ \frac{1}{q+h} \int_{r-(q-h)/2}^{r-q/2} \text{vec}(C(\Sigma(v)))dv + \frac{1}{q+h} \int_{r+q/2}^{r+(q+h)/2} \text{vec}(C(\Sigma(v)))dv + O(h^2),
\]

where for the last equality we use the change of variables \(v \to v-h/2\) (resp. \(v \to v+h/2\)) in the integral on the interval \([r-q/2, r-(q-h)/2]\) (resp. \([r+q/2, r+(q+h)/2]\)) and the Lipschitz property of the elements on \(\Sigma(\cdot)\). Thus, we could write
\[
\text{vec} \left( \text{Int}_{T,r,q+h} \left( C \left( \hat{V} \right) \right) \right) = \frac{q}{q+h} \text{Int}_{r,q} \left( \text{vec} \left( C \left( \Sigma \right) \right) \right)
\]
\[
+ \frac{q-h}{q+h} \left\{ \text{Int}_{T,r,q-h}(\Delta \text{vec}(\hat{U})) - \text{Int}_{r,q-h}(\Delta \text{vec}(\Sigma)) \right\} + O_\Delta(h^2 + 1/Th).
\]

(29)
That means
\[
\sqrt{T} \left( \frac{q + h}{q} \text{vec} \left( \text{Int}_{T,r,q+h} \left( C \left( \hat{V} \right) \right) \right) - \text{vec} \left( \text{Int}_{r,q} \left( C \left( \Sigma \right) \right) \right) \right)
\]
\[
= \left\{ 1 - \frac{h}{q} \right\} \sqrt{T} \left\{ \text{Int}_{T,r,q-h} \left( \Delta \text{vec} \left( \bar{U} \right) \right) - \text{Int}_{r,q-h} \left( \Delta \text{vec} \left( \Sigma \right) \right) \right\} + O_{p} \left( \sqrt{Th^4} + 1/\sqrt{Th^2} \right)
\]
\[
= \sqrt{T} \left\{ \text{Int}_{T,r,q-h} \left( \Delta \text{vec} \left( \bar{U} \right) \right) - \text{Int}_{r,q-h} \left( \Delta \text{vec} \left( \Sigma \right) \right) \right\} + O_{p} \left( h + \sqrt{Th^4} + 1/\sqrt{Th^2} \right).
\]

It also means that
\[
\frac{q + h}{q} \text{vec} \left( \text{Int}_{T,r,q+h} \left( C \left( \hat{V} \right) \right) \right) = \text{vec} \left( \text{Int}_{r,q} \left( C \left( \Sigma \right) \right) \right) + O_{p} \left( 1/\sqrt{T} \right).
\]

Now, we have the ingredients to derive the asymptotic normality of our averaged OIRF estimator
\[
\tilde{\theta}_{r}^{q}(i) = \tilde{\Phi}_{i}^{als} \frac{q + h}{q} \text{Int}_{T,r,q+h} \left( C \left( \hat{V} \right) \right),
\]
of the averaged OIRF \( \bar{\theta}_{r}^{q}(i) = \Phi_{i} \text{Int}_{r,q} \left( C \left( \Sigma \right) \right) \). First, note that by (31) and the \( \sqrt{T} \)-convergence of \( \text{vec}(\tilde{\Phi}_{i}^{als}) \)
\[
\sqrt{T} \text{vec} \left( \tilde{\theta}_{r}^{q}(i) - \bar{\theta}_{r}^{q}(i) \right) = \text{vec} \left[ \sqrt{T} \left( \tilde{\Phi}_{i}^{als} - \Phi_{i} \right) \text{Int}_{r,q} \left( C \left( \Sigma \right) \right) \right] + o_{p} \left( 1 \right).
\]

By (30), the \( \sqrt{T} \)-asymptotic normality of
\[
(I_d \otimes \Phi_{i}) \left\{ \frac{q + h}{q} \text{vec} \left( \text{Int}_{T,r,q+h} \left( C \left( \hat{V} \right) \right) \right) - \text{vec} \left( \text{Int}_{r,q} \left( C \left( \Sigma \right) \right) \right) \right\}
\]
follows from the CLT applied to
\[
\sqrt{T} \left( I_d \otimes \Phi_{i} \right) \left\{ \text{Int}_{T,r,q-h} \left( \Delta \text{vec} \left( \bar{U} \right) \right) - \text{Int}_{r,q-h} \left( \Delta \text{vec} \left( \Sigma \right) \right) \right\}
\]
\[
= \sqrt{T} \left( I_d \otimes \Phi_{i} \right) \Delta \left\{ \text{vec} \left( \text{Int}_{T,r,q-h} \left( \bar{U} \right) \right) - \text{vec} \left( \text{Int}_{r,q-h} \left( \Sigma \right) \right) \right\} \left\{ 1 + o_{p} \left( 1 \right) \right\}
\]
\[
= \sqrt{T} \left( I_d \otimes \Phi_{i} \right) \Delta \left\{ \text{vec} \left( \text{Int}_{T,r,q} \left( U \right) \right) - \text{vec} \left( \text{Int}_{r,q} \left( \Sigma \right) \right) \right\} \left\{ 1 + o_{p} \left( 1 \right) \right\}.
\]

The result follows from the \( \sqrt{T} \)-asymptotic normality of \( \text{vec}(\tilde{\Phi}_{i}^{als}) \) and the zero-mean condition for the product of any three components of the error vector, see Assumption A1. □
Let us note that, taking \( A(\cdot) \) equal to the identity matrix \( I_d \) in (28) we can deduce that new approximated OIRF estimator could be equivalently defined, with \( \bar{q} = q \), as equal to
\[
\sqrt{\frac{q + 2h}{q}} \tilde{\Phi}_i^{als} C(\text{Int}_{\tilde{T},r,q+2h}(\tilde{V})),
\]
where here \( \tilde{V} \) is defined in (16). The difference between the two definitions is asymptotically negligible. More precisely,
\[
\frac{z_{q,als}}{\theta_r} (i) = \tilde{\Phi}_i^{als} C \left( \text{Int}_{\tilde{T},r,q} \left( \tilde{U} \right) \right) = \sqrt{\frac{q + 2h}{q}} \tilde{\Phi}_i^{als} C(\text{Int}_{\tilde{T},r,q+2h}(\tilde{V})) + O_p(h^2 + 1/Th)
\]
provided \( Th^4 + 1/Th^2 \to 0 \).

Proof of Proposition 3.6 In the sequel, when we use the \( o_p(\cdot) \) and \( O_p(\cdot) \) symbols for a vector or a matrix, it should be understood as used for their norms. Recall that
\[
i_{r,q} = \left\| \bar{H}(r)^{-1} \bar{H}(r) \right\|_2^2,
\]
where
\[
\bar{H}(r) = \text{Int}_{r,q}(C(\Sigma)) \quad \text{and} \quad \bar{H}(r) = C(\text{Int}_{r,q}(\Sigma)).
\]
The estimator we propose is
\[
\tilde{i}_{r,q} = \left\| \tilde{H}(r)^{-1} \tilde{H}(r) \right\|_2^2
\]
where
\[
\tilde{H}(r) = \frac{1}{[qT] + 1} \sum_{k=-[(q+h)T/2]}^{[(q+h)T/2]} \tilde{H}_{\lfloor r+q/2 \rfloor - k} = \frac{q + h}{q} \text{Int}_{r,q+h}(C(\tilde{V}))
\]
and
\[
\tilde{H}(r) = C(\tilde{S}_T(r))
\]
with \( \tilde{S}_T(r) \) some estimator of \( q^{-1} \int_{r-q/2}^{r+q/2} \Sigma(v)dv \).
By (29)

\[
\text{vec} \left( \tilde{H}(r) \right) = \text{Int}_{r,q} \left( \text{vec} \left( C \left( \Sigma \right) \right) \right) \\
+ \frac{q - h}{q} \left\{ \text{Int}_{T,r,q-h} \left( \Delta \text{vec}(U) \right) - \text{Int}_{r,q-h} \left( \Delta \text{vec}(\Sigma) \right) \right\} + O_{\mathbb{P}}(h^2 + 1/Th) \\
= \text{Int}_{r,q} \left( \text{vec} \left( C \left( \Sigma \right) \right) \right) + \frac{q - h}{q} \left\{ \Delta \text{vec}(\text{Int}_{T,r,q-h}(U) - \text{Int}_{r,q-h}(\Sigma)) \right\} + o_{\mathbb{P}}(1/\sqrt{T}) \\
= \text{vec} \left( \tilde{H}(r) \right) + \frac{q - h}{q} G_{T,r,q-h} + o_{\mathbb{P}}(1/\sqrt{T}),
\]

with \( \Delta \) defined in (26). If we consider

\[
\tilde{S}_T(r) = \text{Int}_{T,r,q}(\tilde{U}) = \text{Int}_{T,r,q}(U) + o_{\mathbb{P}}(1/\sqrt{T})
\]

and use the identity

\[
\text{vec}(C(\tilde{S}_T(r))) - \text{vec}(C(\text{Int}_{r,q}(\Sigma))) = \Delta \left[ \text{vec}(\tilde{S}_T(r)) - \text{vec}(\text{Int}_{r,q}(\Sigma)) \right] \{ 1 + o_{\mathbb{P}}(1) \}
\]

\[
= \Delta \left[ \text{vec}(\text{Int}_{T,r,q}(U) - \text{Int}_{r,q}(\Sigma)) \right] \{ 1 + o_{\mathbb{P}}(1) \},
\]

we deduce

\[
\text{vec} \left( \tilde{H}(r) \right) = \text{vec} \left( \tilde{H}(r) \right) + G_{T,r,q} + o_{\mathbb{P}}(1/\sqrt{T}).
\]

Note that

\[
\frac{q - h}{q} G_{T,r,q-h} - G_{T,r,q} = O_{\mathbb{P}}(h/\sqrt{T}).
\]

We deduce from above

\[
\tilde{H}(r)^{-1} \tilde{H}(r) = \left[ I_d + \tilde{H}(r)^{-1} \left\{ \tilde{H}(r) - \tilde{H}(r) \right\} \right]^{-1} \tilde{H}(r)^{-1} \left[ \tilde{H}(r) + \left\{ \tilde{H}(r) - \tilde{H}(r) \right\} \right]
\]

\[
= \left[ I_d - \tilde{H}(r)^{-1} \left\{ \tilde{H}(r) - \tilde{H}(r) \right\} + O_{\mathbb{P}}(1/T) \right] \tilde{H}(r)^{-1} \left[ \tilde{H}(r) + \left\{ \tilde{H}(r) - \tilde{H}(r) \right\} \right]
\]

\[
= \tilde{H}(r)^{-1} \tilde{H}(r) + \tilde{H}(r)^{-1} \left\{ \tilde{H}(r) - \tilde{H}(r) \right\} - \tilde{H}(r)^{-1} \left\{ \tilde{H}(r) - \tilde{H}(r) \right\} \tilde{H}(r)^{-1} \tilde{H}(r)
\]

\[
+ O_{\mathbb{P}}(1/T)
\]

\[
= \tilde{H}(r)^{-1} \tilde{H}(r) + \tilde{H}(r)^{-1} \left\{ \text{ivec}(G_{T,r,q}) \right\} - \frac{q - h}{q} \tilde{H}(r)^{-1} \left\{ \text{ivec}(G_{T,r,q-h}) \right\} \tilde{H}(r)^{-1} \tilde{H}(r)
\]

\[
+ O_{\mathbb{P}}(h^2 + 1/Th + 1/T),
\]

where \( \text{ivec}() \) denotes the inverse of the \( \text{vec}() \) operator: for any matrix \( A \), \( \text{ivec}(\text{vec}(A)) = A \). In particular, we deduce that in the case where \( \Sigma() \) is constant on the interval \([r - q/2, r + q/2]\), and thus \( i_{r,q} = 1 \), we have

\[
\left\| \tilde{H}(r)^{-1} \tilde{H}(r) - I_d \right\|_2 = o_{\mathbb{P}}(1/\sqrt{T}).
\]
As a consequence,
\[
\left\| \tilde{H}(r)^{-1} \tilde{H}(r) \right\|_2 - 1 = \left\| \tilde{H}(r)^{-1} \tilde{H}(r) \right\|_2 - \| I_d \|_2 \\
\leq \left\| \tilde{H}(r)^{-1} \tilde{H}(r) - I_d \right\|_2 = o_P(1/\sqrt{T}),
\]
and thus
\[
\hat{i}_{r,q} - 1 = o_P(1/\sqrt{T}).
\]

In the case where \( \Sigma(\cdot) \) is not constant on the interval \([r - q/2, r + q/2]\), and thus \( i_{r,q} > 1 \), let us note that \( i_{r,q} \) is also the largest eigenvalue of the symmetric matrix
\[
\tilde{H}(r)'\tilde{H}(r)^{-1}\tilde{H}(r)^{-1}\tilde{H}(r).
\]

By the decomposition of \( \tilde{H}(r)^{-1} \tilde{H}(r) \) we have
\[
\tilde{H}(r)'\tilde{H}(r)^{-1}\tilde{H}(r)^{-1}\tilde{H}(r) = \left\{ \tilde{H}(r)^{-1} \tilde{H}(r) + M_{T,r,q} + o_P(1/\sqrt{T}) \right\} \times \left\{ \tilde{H}(r)^{-1} \tilde{H}(r) + M_{T,r,q} + o_P(1/\sqrt{T}) \right\}
\]
\[
= \tilde{H}(r)'\tilde{H}(r)^{-1}\tilde{H}(r)^{-1}\tilde{H}(r) + \mathcal{H}_{T,r,q} + o_P(1/\sqrt{T}),
\]
where
\[
\mathcal{H}_{T,r,q} = M_{T,r,q}'\tilde{H}(r)^{-1} \tilde{H}(r) + \tilde{H}(r)'\tilde{H}(r)^{-1} M_{T,r,q},
\]
\[
M_{T,r,q} = \tilde{H}(r)^{-1} \{ \text{ivec}(G_{T,r,q}) \} - \tilde{H}(r)^{-1} \{ \text{ivec}(G_{T,r,q}) \} \tilde{H}(r)^{-1} \tilde{H}(r)
\]
and, recall, \( G_{T,r,q} = \Delta \text{vec}(\text{Int}_{T,r,q-h}(U) - \text{Int}_{r,q-h}(\Sigma)) \). By the delta-method and the differential of the first eigenvalue of a symmetric matrix, see Theorem 7, section 8, Magnus and Neudecker (1988),
\[
\sqrt{T} \left( \hat{i}_{r,q} - i_{r,q} \right) = v_1' \sqrt{T} \mathcal{H}_{T,r,q} v_1 + o_P(1) = (v_1' \otimes v_1') \text{vec}(\sqrt{T} \mathcal{H}_{T,r,q}) + o_P(1),
\]
with \( v_1 \) a normalized eigenvector associated to the largest eigenvalue \( i_{r,q} \) of the matrix \( \mathcal{H}_{T,r,q} \). Finally, CLT guarantees that \( \text{vec}(\sqrt{T} \mathcal{H}_{T,r,q}) \) convergences in distribution to a Gaussian limit. The result follows.
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Tables and Figures

Table 1: The $\hat{i}_{r,q}$’s and $\hat{j}_{r,q}$’s for the CPI data using a 5 subsamples mapping.

| periods | 1   | 2   | 3   | 4   | 5   |
|---------|-----|-----|-----|-----|-----|
| $\hat{i}_{r,q}$ | 1.33 | 1.23 | 1.32 | 1.70 | 1.34 |

Table 2: The same as in Table 1 but for a mapping with 10 subsamples.

| periods | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\hat{i}_{r,q}$ | 1.28 | 1.66 | 1.27 | 1.25 | 1.33 | 1.59 | 1.68 | 1.92 | 1.94 | 1.58 |

Figure 1: The log differences of the Korean Won/USD exchange rate ($FX_t$) on the left panel ($100\times\log(FX_t/FX_{t-1})$) from May 1981 to June 2018, with sample size $T = 446$. The data are available from the website of the Federal Reserve Bank of St. Louis (www.fred.stlouisfed.org), the identification being EXKOUS. On the right panel the averaged (full line) and approximated (dotted line) estimations of the IRF are displayed.
Figure 2: The variance structure $\sigma_{11}^2(r)$ of the first innovations component of the simulated process (21).

Figure 3: The relative differences between the approximated and averaged OIRFs: $100 \times \left( \hat{\theta}_{0.5,11} \left( \frac{1}{\hat{\theta}_{0.5,11}^{(1)}} - 1 \right) \right)$, (see equations (14) and (19)). The results corresponding to a bandwidth with $T^{-1/3}$ (resp. $T^{-2/7}$) decreasing rate is displayed on the left (resp. on the right).
Figure 4: The normal Q-Q plot of the approximated OIRFs of order one, that is $\sqrt{T} \cdot (\hat{\theta}_{0.5}^{0.5,11} - \hat{\theta}_{0.5}^{0.5,11}(1))'$, over the $N = 1000$ iterations.

Figure 5: The normal Q-Q plot of the averaged OIRFs of order one $\sqrt{T} \cdot (\bar{\theta}_{0.5}^{0.5,11} - \bar{\theta}_{0.5}^{0.5,11}(1))'$s. The results corresponding to a bandwidth with a $T^{-1/3}$ (resp. $T^{-2/7}$) decreasing rate are displayed on the top (resp. on the bottom) panels.
Figure 6: The normal Q-Q plot of the $\sqrt{T}(\hat{i}_{r,q} - i_{r,q})$'s. The results corresponding to a bandwidth with a $T^{-1/3}$ (resp. $T^{-2/7}$) decreasing rate are displayed on the top (resp. on the bottom) panels.

Figure 7: The box-plots of the $\hat{i}_{r,q}$'s for different sample sizes. The horizontal line corresponds to the true value. The results corresponding to a bandwidth with a $T^{-1/3}$ (resp. $T^{-2/7}$) decreasing rate are displayed on the left (resp. on the right) panels.
Figure 8: The box-plots of the $\hat{i}_{r,q}$'s for different values for the heteroscedasticity parameter $\delta$. As $\delta$ is far from zero, the heteroscedasticity is more marked. The results corresponding to a bandwidth with a $T^{-1/3}$ (resp. $T^{-2/7}$) decreasing rate are displayed on the left (resp. on the right) panel.

Figure 9: The box-plots of the $\hat{i}_{r,q}$'s for different sample sizes in the homoscedastic case (the true value is equal to one). The results corresponding to a bandwidth with a $T^{-1/3}$ (resp. $T^{-2/7}$) decreasing rate are displayed on the left (resp. on the right) panel.
Figure 10: The log first differences of the US energy CPI on the right and transportation CPI on the left from June 1, 1979 to May 1, 2019.

Figure 11: The approximated OIRF given in (19) in the left panel and the averaged OIRF given in (14) and in the right panel. The sample is mapped considering 10 subsamples.
Figure 12: The same as in Figure 11 but with 5 subsamples mapping the whole sample.