WEIGHT ZERO PART OF THE FIRST COHOMOLOGY OF COMPLEX ALGEBRAIC VARIETIES

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Abstract. We show that the weight 0 part of the first cohomology of a complex algebraic variety $X$ is a topological invariant, and give an explicit description of its dimension using a topological construction of the normalization of $X$, where $X$ can be reducible, but must be equidimensional. The first assertion is known in the $X$ compact case by A. Weber, where intersection cohomology is used. Note that the weight 1 or 2 part of the first cohomology is not a topological (or even analytic) invariant in the non-compact case by Serre’s example.

Introduction

Let $X$ be a complex algebraic variety which is assumed equidimensional and reduced in this paper. Let $\pi : \tilde{X} \to X$ be the normalization. It is known that the underlying topological space of $\tilde{X}$ can be constructed canonically from the underlying topological space of $X$, see [GM1, McC] (and (2.2) below). Set

$$F_X := \text{Coker}(\pi^* : \mathbb{Q}_X \to \pi^* \mathbb{Q}_{\tilde{X}}).$$

Then $F_X$ is a constructible sheaf supported on the non-unibranch locus of $X$, and can be rather complicated in general. There is a long exact sequence of mixed Hodge structures

$$0 \to H^0(X, \mathbb{Q}) \to H^0(\tilde{X}, \mathbb{Q}) \to H^0(X, F_X) \to H^1(X, \mathbb{Q}) \to H^1(\tilde{X}, \mathbb{Q}) \to \cdots,$$

using the mapping cone construction in [De3] or [Sa2] (where both give essentially the same, see [Sa3]). The following is known.

Proposition 1. The first cohomology $H^1(\tilde{X}, \mathbb{Q})$ has weight $\geq 1$.

Indeed, this proposition follows from the surjectivity of the morphism of fundamental groups $\sigma_* : \pi_1(X) \to \pi_1(\tilde{X})$ for a resolution of singularities $\sigma : X \to \tilde{X}$, using [FL, 0.7 (B)] or [Ko, Proposition 2.10] (see [ADH, Theorem 2.1]), since it induces the injection of mixed Hodge structures $\sigma^* : H^1(\tilde{X}, \mathbb{Q}) \hookrightarrow H^1(X, \mathbb{Q})$.

We can also deduce Proposition 1 from the following (see (1.1) below).

Proposition 2. We have the injectivity of the canonical morphism of mixed Hodge structures

$$H^1(\tilde{X}, \mathbb{Q}) \hookrightarrow \text{IH}^1(\tilde{X}, \mathbb{Q}) = \text{IH}^1(X, \mathbb{Q}).$$

Indeed, it is well known that Proposition 1 still holds by replacing $H^1(\tilde{X}, \mathbb{Q})$ with the intersection cohomology $\text{IH}^1(\tilde{X}, \mathbb{Q})$, see [Sa2, 4.5.2]. In this paper we prove the following.

Theorem 1. There are canonical isomorphisms

$$W_0 H^1(X, \mathbb{Q}) = \text{Ker}(H^1(X, \mathbb{Q}) \to \text{IH}^1(X, \mathbb{Q}))$$

$$= \text{Ker}(H^1(X, \mathbb{Q}) \to H^1(\tilde{X}, \mathbb{Q}))$$

$$= \text{Coker}(H^0(\tilde{X}, \mathbb{Q}) \to H^0(X, F_X)).$$

(The first isomorphism is shown in the $X$ compact case by A. Weber [We, Theorem 1.7].)
Theorem 2. We have the equality
\[
\dim W_0 H^1(X, \mathbb{Q}) = \dim H^0(X, \mathcal{F}_X) - b_0(\tilde{X}) + b_0(X).
\]

These assertions follow from the exact sequence (1) together with Propositions 1 and 2 if we have further the following.

Proposition 3. The cohomology group \( H^0(X, \mathcal{F}_X) \) has weight 0.

This can be shown using mixed Hodge modules, for instance, see (1.2) below. Theorem 2 is well known in the \( X \) curve case, where we have
\[
\dim \mathcal{F}_{x,x} = r_{x,x} - 1,
\]
with \( r_{x,x} \) the number of local irreducible components of \( X \) at \( x \). In the case \( \dim X \geq 2 \), it is not necessarily easy to calculate \( \dim H^0(X, \mathcal{F}_X) \) explicitly, see (2.4) below. We can calculate it if \( X \) satisfy certain strong conditions. Note that \( \dim H^0(X, \mathcal{F}_X) \) can increase strictly by shrinking \( X \), see Example (2.5) below.

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In Section 1 we prove Theorems 1 and 2 by showing Propositions 2 and 3. In Section 2 we review a topological construction of the normalization \( \tilde{X} \) of a complex analytic space \( X \), and calculate \( \dim H^0(X, \mathcal{F}_X) \) for some examples. In Appendix we review totally ramified cyclic coverings which are useful to construct examples of algebraic varieties with multi-branch points, and calculate the local monodromies of local branches.

1. Proof of the main theorems

In this section we prove Theorems 1 and 2 by showing Propositions 2 and 3.

1.1. Proof of Proposition 2. In the notation of the introduction, let \( K^* \) be the mapping cone of the canonical morphism \( \mathbb{Q} \rightarrow \text{IC}_\mathbb{X} \mathbb{Q}[−n] \) so that we have the distinguished triangle
\[
\mathbb{Q} \rightarrow \text{IC}_\mathbb{X} \mathbb{Q}[−n] \rightarrow K^* \rightarrow \text{D}^b_c(X, \mathbb{Q}),
\]
where \( n = \dim X \), and \( \text{IC}_\mathbb{X} \mathbb{Q} \) is the intersection complex, see [BBD, GM2].

Since \( \tilde{X} \) is normal, we have the following isomorphisms by an inductive construction of intersection complexes using iterated open direct images and truncations (see [GM2, 3.1]):
\[
\mathcal{H}^0(\text{IC}_\mathbb{X} \mathbb{Q}[−n]) = \mathbb{Q} \rightarrow \mathcal{H}^k(\text{IC}_\mathbb{X} \mathbb{Q}[−n]) = 0 \quad (\forall k < 0).
\]
By the long exact sequence of cohomology sheaves associated with (1.1.1), this implies
\[
\mathcal{H}^k K^* = 0 \quad (\forall k \leq 0), \quad \text{hence} \quad \mathcal{H}^0(\tilde{X}, K^*) = 0,
\]
since \( \Gamma(X, *) \) is a left exact functor. So the injectivity of the first morphism in (2) follows from the long exact sequence of cohomology groups associated with (1.1.1):
\[
\mathcal{H}^0(\tilde{X}, K^*) \rightarrow \mathcal{H}^1(\tilde{X}, \mathbb{Q}) \rightarrow \mathcal{H}^1(\tilde{X}, \mathbb{Q}),
\]
since \( \mathcal{H}^*(\tilde{X}, \mathbb{Q}) = \mathcal{H}^*(\tilde{X}, \text{IC}_\mathbb{X} \mathbb{Q}[−n]) \) by definition.

The last isomorphism in (2) follows from the commutativity of the direct image by a finite morphism (in the strong sense) with the intermediate direct image (see Remark (ii) below) in the cartesian diagram case, using the base change property as in [Sa2, 4.4.3]. Note that...
the direct image by a finite morphism $\pi$ is an exact functor of mixed Hodge modules (by the vanishing of $H^k\pi^*$ for $k \neq 0$). This finishes the proof of Proposition 2.

Remarks. (i) In this paper, $a^*_X\mathbb{Q} \in D^b\text{MHM}(X)$ with $a_X : X \to \text{pt}$ the structure morphism is denoted by $\mathbb{Q}_{h,X}$ for any complex algebraic variety $X$. Here $\mathbb{Q}$ denotes the trivial Hodge structure of type $(0,0)$ (see [De1]), and MHM$(X)$ is the category of mixed Hodge modules on $X$, see [Sa2]. We have $\mathbb{Q}_{h,X}[n] \in \text{MHM}(X)$ if $X$ is smooth and purely $n$-dimensional.

(ii) The intermediate direct image of a mixed Hodge module $\mathcal{M}$ by an open embedding $j : U \hookrightarrow X$ is defined by

$$j_*\mathcal{M} := \text{Im}(H^0 j_*\mathcal{M} \to H^0 j_*\mathcal{M}) \in \text{MHM}(X),$$

(see also [BBD]). The intersection complex $IC_X\mathbb{Q}_h$ in the category of mixed Hodge modules can be defined by $j_*(\mathbb{Q}_{h,U}[n])$ for any smooth dense open subvariety $U \subset X$ assuming that $X$ is purely $n$-dimensional.

1.2. Proof of Proposition 3. The constructible sheaf $\mathcal{F}_X$ underlies a bounded complex of mixed Hodge modules $\mathcal{M}^* \in D^b\text{MHM}(X)$ defined by

$$\mathcal{M}^* := C(\mathbb{Q}_{h,X} \to \pi_*\mathbb{Q}_{h,X}).$$

Since $\pi$ is a finite morphism (in the strong sense), we have $H^k i^*_x\mathcal{M}^* = 0 (k \neq 0)$ and $H^0 i^*_x\mathcal{M}^*$ is a Hodge structure of type $(0,0)$ for any $x \in X$, using the base change property as in [Sa2] 4.4.3, where $i_x : \{x\} \hookrightarrow X$ is the inclusion.

For a sufficiently large number of points $x_k \in X$, we have the following injective morphism of mixed Hodge structures using the functorial morphisms $\text{id} \to (i_{x_k})_!i^*_x$ (see [Sa2] 4.4.1): 

\begin{equation}
H^0(X, \mathcal{M}^*) := H^0(a_X)_!\mathcal{M}^* \hookrightarrow \bigoplus_k H^0 i^*_x \mathcal{M}^*.
\end{equation}

Here the injectivity is shown by using the finiteness of the cohomology $H^0(X, \mathcal{F}_X)$ underlying $H^0(X, \mathcal{M}^*)$. So Proposition 3 follows. (It seems also possible to apply the mapping cone construction in [De3] to the base change of $\pi : \tilde{X} \to X$ by $i_{x_k} : \{x_k\} \hookrightarrow X$. The details are left to the reader.)

1.3. Proof of Theorems 1 and 2. Restricting the exact sequence (1) to the weight 0 part and using Propositions 1 and 3, we get the exact sequence

\begin{equation}
0 \to H^0(X, \mathbb{Q}) \to H^0(\tilde{X}, \mathbb{Q}) \to H^0(X, \mathcal{F}_X) \to W_0H^1(X, \mathbb{Q}) \to 0.
\end{equation}

This implies Theorem 2 and the last equality in Theorem 1. The other equalities then follow from the exact sequence (1) and Proposition 2. This finishes the proof of Theorems 1 and 2.

1.4. Dependence of the weight filtration on the algebraic structure. The weight filtration $W$ on $H^1(X, \mathbb{Q})$ for a smooth non-compact surface is not even an analytic invariant. Indeed, $H^1(X, \mathbb{Q})$ can be pure of weight either 1 or 2 depending on the algebraic structure in the case of Serre’s example, where a 2-dimensional complex manifold $X$ has two smooth compactifications $\overline{X}_1, \overline{X}_2$ such that $\overline{X}_1$ is a ruled surface over an elliptic curve $E$ with divisor at infinity a section of the ruled surface, and $\overline{X}_2 \cong \mathbb{P}^2$ with divisor at infinity general three lines on $\mathbb{P}^2$. More precisely we have in the first case

$$H^1(X_1, \mathbb{Q}) = G_{i_1}^W H^1(X_1, \mathbb{Q}) = H^1(E, \mathbb{Q}),$$

where $X_i$ is the algebraic surface over $X$ determined by the compactification $\overline{X}_i$ ($i = 1, 2$) using GAGA, see [PS] Example 4.19, [SS] Remark 2.12, [Ha1] p. 232, [Ha2] Appendix B, Example 2.0.1.
Then $E$ is an elliptic curve. (Indeed, $(b + \alpha i)/(a + i) \notin \mathbb{R}$ for any $a, b, \in \mathbb{Q}$, since $\alpha$ is an irrational real number.) We have $G = \mathbb{C}^*$, and there is a fibration $X \to E$ whose fibers are parallel translates of $G \subset X$. This gives an algebraic structure as an semi-abelian variety on the complex Lie group $X$ (by compactifying the fibers). It corresponds to the mixed $\mathbb{Z}$-Hodge structure $H = ((H_{\mathbb{Z}}, W), (H_{\mathbb{C}}, F))$ of weights $\{-2, -1\}$, which is the dual of $H^1(X, \mathbb{Z})$, see loc. cit.

\section{2. Topological construction of the normalization.}

In this section we review a topological construction of the normalization $\tilde{X}$ of a complex analytic space $X$, and calculate $\dim H^0(X, \mathcal{F}_X)$ for some examples.

\subsection*{2.1. Algebraic and analytic normalizations.} If a reduced complex algebraic variety $X$ is normal, then so is the associated analytic space $X^{\text{an}}$; in particular, $X^{\text{an}}$ is unibranch at any point, see for instance [Mu]. This seems to be a rather nontrivial assertion (related to a version of Zariski’s Main Theorem).

\begin{remark}
Using mixed 1-motives [De3, 1.10], it is rather easy to construct examples of smooth varieties $X$ such that $H^1(X, \mathbb{Q})$ has weights 1 and 2, and the weight filtration $W_i H^1(X, \mathbb{Q})$ as a submodule of $H^1(X, \mathbb{Q})$ depends on the algebraic structure of $X$. For instance, let

\[ H_{\mathbb{Z}} = \mathbb{Z}^3, \quad H_{\mathbb{C}} = \mathbb{C}^3, \quad F^0 H_{\mathbb{C}} = \mathbb{C} v \subset H_{\mathbb{C}}, \]

where $F^{-1} H_{\mathbb{C}} = H_{\mathbb{C}}$, $F^1 H_{\mathbb{C}} = 0$, and $\alpha$ is any irrational real number.

Let $\Gamma_{0,\mathbb{Q}}$ be any 1-dimensional $\mathbb{Q}$-vector subspace of $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Set

\[ \Gamma := H_{\mathbb{Z}}, \quad \Gamma_0 := \Gamma_{0,\mathbb{Q}} \cap \Gamma, \quad \Gamma_1 := \Gamma/\Gamma_0. \]

Then $\Gamma_1$ is torsion-free, and $\Gamma_0 \cap F^0 H_{\mathbb{C}} = 0$. Put

\[ W_{-3} H_{\mathbb{Z}} = 0, \quad W_{-2} H_{\mathbb{Z}} = \Gamma_0, \quad W_{-1} H_{\mathbb{Z}} = H_{\mathbb{Z}}, \]

and

\[ V := H_{\mathbb{C}}/F^0 H_{\mathbb{C}}, \quad V_0 := \mathbb{C} \Gamma_0 \subset V, \quad V_1 := V/V_0, \quad G := V_0/\Gamma_0, \quad X := V/\Gamma, \quad E := V_1/\Gamma_1. \]

Then $E$ is an elliptic curve. (Indeed, $(b + \alpha i)/(a + i) \notin \mathbb{R}$ for any $a, b, \in \mathbb{Q}$, since $\alpha$ is an irrational real number.) We have $G = \mathbb{C}^*$, and there is a fibration $X \to E$ whose fibers are parallel translates of $G \subset X$. This gives an algebraic structure as an semi-abelian variety on the complex Lie group $X$ (by compactifying the fibers). It corresponds to the mixed $\mathbb{Z}$-Hodge structure $H = ((H_{\mathbb{Z}}, W), (H_{\mathbb{C}}, F))$ of weights $\{-2, -1\}$, which is the dual of $H^1(X, \mathbb{Z})$, see loc. cit.

\begin{remark}
The above theorem can be verified by using GAGA together with a well-known assertion that a reduced algebraic variety $X$ is normal if and only if the canonical morphism

\[ \mathcal{O}_X \to \sigma_* \mathcal{O}_X \]

is an isomorphism for a resolution of singularities $\sigma : X \to X$ (and similarly for analytic spaces). Note that the isomorphism $\mathcal{O}_{X^{\text{an}}} \xrightarrow{\sim} \sigma_* \mathcal{O}_{X^{\text{an}}}$ implies first that $X$ is unibranch at any point.

\subsection*{2.2. Local cohomology.} Let $X$ be a reduced complex analytic space of dimension $\leq n$, where $n \geq 1$. It is well known that the the number of local irreducible components of dimension $n$ at $x \in X$ is given by

\[ r_{x,x} := \dim H^{2n}_{(x)} \mathbb{Q}_X, \]

which is defined purely topologically using the local cohomology.

\begin{remark}
(i) It is known that we have the isomorphism

\[ H^{2n}_{(x)} \mathbb{Q}_X = H^{2n-1}(S_x \cap X, \mathbb{Q}), \]

where $S_x$ is a sufficiently small sphere with center $x$ in the ambient space containing $X$ locally, and is defined by using a distance function with $x$. This can be shown by using
a topological cone theorem in the local singularity theory (where the theory of tubular neighborhood system \( \text{Ma} \) is essential). This is usually used to show (2.2.1).

(ii) The assertion (2.2.1) can be proved by using only a resolution of singularities. Indeed, we have

\[
H_{2n}^2(\{x\}; \mathbb{Q}) = H_{2n}^2(j\mathbb{Q}_U),
\]

where \( j : U \hookrightarrow X \) is an open embedding such that \( U \) is smooth and \( Z := X \setminus U \subset X \) is a closed analytic subset of dimension \(< n \). This isomorphism follows by using the long exact sequence associated with the distinguished triangle

\[
j_! j^{-1} \to \text{id} \to i_* i^{-1} + 1,
\]

where \( i : Z \hookrightarrow X \) is the inclusion.

The assertion (2.2.1) is then reduced to the unibranch case, and can be proved by taking an embedded resolution of singularities

\[
\sigma : (\mathcal{X}, Y, E) \to (X, Z, x),
\]

as follows. Here we may assume that \( E := \sigma^{-1}(x) \) is a divisor on \( \mathcal{X} \) (by taking first the blow-up along \( x \)), \( Y := \sigma^{-1}(Z) \) is a divisor with normal crossings on \( \mathcal{X} \), and \( \sigma \) is an isomorphism over \( U \). (Note that \( E \) is also a divisor with normal crossings on \( \mathcal{X} \), since \( E \subset Y \).) We have

\[
\mathbb{D}\mathbb{Q}_U \cong \mathbb{Q}_U[2n], \quad j_! \circ \mathbb{D} = \mathbb{D} \circ Rj_*, \quad i^!_x \circ \mathbb{D} = \mathbb{D} \circ i_*^*, \quad (i_*)_* \circ i^!_x = R\Gamma_{\{x\}},
\]

with \( i : \{x\} \hookrightarrow X \) the inclusion. Here \( \mathbb{D} \) is the dual functor, see [Ve]. The assertion (2.2.1) in the unibranch case is then reduced to

\[
H^0 i^!* Rj_* \mathbb{Q}_U = \mathbb{Q}.
\]

Let \( \tilde{j} : U \hookrightarrow \tilde{X} \) be the inclusion so that \( \sigma \circ \tilde{j} = j \). By the base change property for the direct image by the proper morphism \( \sigma : \tilde{X} \to X \) and the pull-back by \( i_x : \{x\} \hookrightarrow X \), the assertion is further reduced to

\[
H^0(E, R\tilde{j}_* \mathbb{Q}_U|_E) = \mathbb{Q},
\]

and then follows from

\[
\tilde{j}_* \mathbb{Q}_U|_E = \mathbb{Q}_E.
\]

Here the connectivity of \( E \) is essential.

(iii) The argument in Remark (ii) above can be extended to the \( \mathbb{Z} \)-coefficient case. Here we have to show

\[
H_{2n}(X, X \setminus \{x\}; \mathbb{Z}) \cong H^0 i^!* \mathbb{D}\mathbb{Z}_X = \mathbb{Z},
\]

in the unibranch case, see [GM1 4.1]. This is not necessarily equivalent to

\[
H_{2n}(X, X \setminus \{x\}; \mathbb{Z}) \cong H_{2n}^2(\{x\}; \mathbb{Z}) = \mathbb{Z},
\]

since the \( t \)-structure on the derived category of bounded complexes of \( \mathbb{Z} \)-modules with finitely generated cohomologies is not self-dual (because of the problem of torsion), see [BBD].

2.3. Topological construction. From now on, assume \( X \) reduced and moreover purely \( n \)-dimensional (that is, \( X \) is equidimensional with dimension \( n \)). The subset of \( r \)-branch points of \( X \) can be defined by

\[
X^{(r)} := \{ x \in X \mid r_{X,x} = r \} \subset X.
\]

The local irreducible decomposition of \( X \) is obtained by using the set of unibranch points \( X^{(1)} \). (Indeed, for a sufficiently small open neighborhood \( U_x \) of \( x \in X \), it is enough to take the closures of the connected components of \( X^{(1)} \cap U_x \).)
We will denote by $I_{X,x}$ the set of local irreducible components of $(X, x)$ for $x \in X$. (More precisely, $I_{X,x}$ is defined to be the inductive limit of the set $I(U_x, x)$ consisting of the connected components of $X^{(1)} \cap U_x$ whose closure contains $x$. Here $U_x$ runs over sufficiently small open neighborhoods of $x \in X$ such that $|I(U_x, x)|$ is independent of $U_x$.)

We can define a natural topology on $$
\check{X}^{\text{top}} := \bigsqcup_{x \in X} I_{X,x}
$$ so that an open neighborhood of $\check{x} \in I_{X,x}$ is topologically identified with the normalization of the local irreducible component $\Gamma_{\check{x}}$ of $X$ at $x$ corresponding to $\check{x} \in I_{X,x}$ under the natural map $$
\pi^{\text{top}} : \check{X}^{\text{top}} \to X.
$$ Indeed, the local irreducible component $\Gamma_{\check{x}}$ of $(X, x)$ defines a multivalued (set-theoretic) section of $\pi^{\text{top}}$ over $\Gamma_{\check{x}} \subset X$. (It is univalued only over the unibranch points of $\Gamma_{\check{x}}$, for instance, if $X = \{y^2 = x^3\} \subset \mathbb{C}^2$.) We can get a fundamental neighborhood system of $\check{x} \in \check{X}^{\text{top}}$ as the image of a fundamental neighborhood system of $(\Gamma_{\check{x}}, x)$ under this multivalued section (since the normalization is a finite morphism in the strong sense, that is, proper and finite fibers). This is a topological construction of the normalization $\check{X}$ of a complex analytic space $X$, see [GM1, 4.1], [McC] for topological normalization.

2.4. Calculation of $H^0(\check{X}, \mathcal{F}_X)$. With the notation of the introduction and (2.3), we have the isomorphism

$$
(2.4.1) \quad \mathcal{F}_{X,x} = \text{Coker} (\mathbb{Q} \hookrightarrow \bigoplus_{\check{x} \in I_{X,x}} \mathbb{Q}) \quad (x \in X),
$$

and the monodromies of the local systems $\pi_* \mathcal{Q}_X|_{S_i}$, $\mathcal{F}_X|_{S_i}$ are induced by the monodromy of the set $I_{X,x}$ ($x \in S_i$), where $S_i$ is a stratum of a Whitney stratification of $X$. In particular, the monodromy group of $\pi_* \mathcal{Q}_X|_{S_i}$ is finite, and the monodromy action is semisimple.

Take $x_i \in S_i$ for each $i$. Let $\mathcal{F}_{X,x_i}^{\text{inv}} \subset \mathcal{F}_{X,x_i}$ be the monodromy invariant part by the action of $\pi_1(S_i, x_i)$. Then

$$
(2.4.2) \quad H^0(X, \mathcal{F}_X) = \bigoplus_i \mathcal{F}_{X,x_i}^{\text{inv}}, \quad \text{if all the strata } S_i \text{ are closed subsets of } X.
$$

In general we need further the information about the extensions between the local systems $\mathcal{F}_X|_{S_i}$. Choosing a path from $x_i$ to $x_j$ inside $S_j$ except for the end point, we get a morphism $$
\rho_{j,i} : \mathcal{F}_{X,x_i} \to \mathcal{F}_{X,x_j} \quad \text{if } S_i \subset \overline{S}_j.
$$

The composition $\rho_{k,j} \circ \rho_{j,i}$ coincides with $\rho_{k,i}$ up to the monodromy action of $\pi_1(S_k, x_k)$ if $S_i \subset \overline{S}_j \subset \overline{S}_k$. We have

$$
(2.4.3) \quad H^0(X, \mathcal{F}_X) = \{ (\xi_i) \in \bigoplus_i \mathcal{F}_{X,x_i}^{\text{inv}}, \quad \rho_{j,i} \xi_i = \xi_j \quad \text{if } S_i \subset \overline{S}_j \}.
$$

Remark. It is not necessarily easy to construct an example of irreducible projective variety $X$ such that $$
\dim H^0(X, \mathcal{F}_X) > 0 \quad \text{with} \quad \dim X > 1,
$$
except for the case where $X = C \times Z$ with $\dim C = 1$ or a quotient of $C \times Z$ by a finite group action. For instance it is known that, if $X \subset \mathbb{P}^N$ is a global complete intersection with $\dim X > 1$, then

$$
H^1(X, \mathbb{Q}) = 0.
$$

Here Artin’s theorem in [BBD] is used, see also [Di]. (The latter theorem can be deduced also from Cartan’s Theorem B together with the Riemann-Hilbert correspondence, see for
instance [Sa1, Lemma 2.1.18]. Note that the shifted constant sheaf $\mathbb{Q}_X[\dim X]$ (or its dual) for a locally complete intersection $X \subset Y$ corresponds to a regular holonomic $D$-module on $Y$ by the Riemann-Hilbert correspondence. This can be verified easily by using algebraic local cohomology, see for instance a remark after [RSW, 1.5.1].)

2.5. Examples. (i) For integers $a, b, c \geq 2$, set

$$X := \{y^b = x^b(x^a + z^c)\} \subset \mathbb{C}^3.$$ 

This is irreducible at 0. Indeed, we get the normalization

$$\tilde{X} := \{v^b = x^a + z^c\} \subset \mathbb{C}^3,$$

by the blowing up along the center $\{x = y = 0\} \subset \mathbb{C}^3$, where $v = y/x$ (for the normality of hypersurfaces with isolated singularities, see for instance [Ha2, II, Proposition 8.23]).

The non-unibranch locus coincides with the $b$-branch locus

$$X^{(b)} = S := \{x = y = 0\} \setminus \{0\} \subset X.$$

This is a stratum of the Whitney stratification. Note that, restricting $\tilde{X}$ to $\{x = 0\}$, we get

$$\pi^{-1}(S) = \{v^b = z^c\}.$$

Since $S$ is not closed in $X$, and $\mathcal{F}_X$ is the 0-extension of $\mathcal{F}_X|_S$, we have

$$(2.5.1) \quad H^0(X, \mathcal{F}_X) = 0.$$

In this example $H^1(X, \mathbb{Q})$ also vanishes by the exact sequence (1), since $\tilde{X}$ is contractible so that $H^1(\tilde{X}, \mathbb{Q}) = 0$.

(ii) In the above example, the action of the monodromy on $I_{X,s}$ by a generator of $\pi_1(S, s)$ $(s \in S)$ is identified with

$$\mathbb{Z}/b\mathbb{Z} \ni k \mapsto k + c \in \mathbb{Z}/b\mathbb{Z},$$

using the description of $\pi^{-1}(S)$ in (i). One may also use the description of local irreducible components of $X$ given by

$$y = x(x^a + z^c)^{1/b},$$

where $x^a$ can be neglected essentially (by restricting to $\{|x|^a \ll |z|^c\} \subset \mathbb{C}^3$), see also (A.4.1) below.

The number of orbits by the monodromy action on $I_{X,s}$ is then given by

$$e := \text{GCD}(b, c),$$

since $\mathbb{Z}/(b\mathbb{Z} + c\mathbb{Z}) = \mathbb{Z}/e\mathbb{Z}$.

Set $X' := X \setminus \{0\}$ (or $X' := X \setminus C$ with $C \subset X$ any curve containing 0 and not contained in $S$). Since the monodromy action on $\pi_*\mathbb{Q}_{\tilde{X}}|_S$ is semisimple, we then get by (2.4.1–2)

$$(2.5.2) \quad \dim H^0(X', \mathcal{F}_{X'}) = e - 1.$$

Remark. In the case $X' := X \setminus \{0\}$, we can calculate $H^1(\tilde{X}', \mathbb{Q})$ as in theory of isolated hypersurface singularities [Mi] (using the Wang sequence), and get

$$(2.5.3) \quad H^1(\tilde{X}', \mathbb{Q}) \cong H^2(F_f, \mathbb{Q})_{1}(1).$$

Here $\tilde{X}' := \tilde{X} \setminus \{0\}$ with $\tilde{X} = f^{-1}(0) \subset \mathbb{C}^3$ for $f := v^b - x^a - z^c$, and the right-hand side is the unipotent monodromy part of the Milnor fiber cohomology of $f$ up to a Tate twist. (Note that the Milnor monodromy is semisimple so that $N = 0$ in the weighted homogeneous

isolated singularity case, where \( N := (2\pi i)^{-1} \log T_u \) with \( T_u \) the unipotent part of the Milnor monodromy \( T. \) Indeed, we have the canonical isomorphism

\[ i^*_x R(j_U)_* \mathbb{Q}_U = C(N : i^*_x \psi_{j,1} \mathbb{Q}_V \to i^*_x \psi_{j,1} \mathbb{Q}_V(-1))[1], \]

where \( U := \mathbb{C}^3 \setminus f^{-1}(0) \) with the inclusion \( j_U : U \hookrightarrow V := \mathbb{C}^3, \) and \( \psi_{j,1} \) is the unipotent monodromy part of the nearby cycle functor \( \psi \) [De2]. This implies

\[ H^4 i^*_x i^*_X \mathbb{Q}_V = H^3 i^*_x R(j_U)_* \mathbb{Q}_V = H^2(F_j, \mathbb{Q}_1)(-1), \]

where \( i_X : \widetilde{X} \hookrightarrow V = \mathbb{C}^3 \) is the inclusion. Since \( \mathbb{D} \mathbb{Q}_V = \mathbb{Q}_V(3)[6], \) we have

\[ \mathbb{D}(H^4 i^*_x i^*_X \mathbb{Q}_V) = H^4 i^*_x i^*_X \mathbb{Q}_V(3)[6] = H^2(\mathbb{Q}_X(3) = H^1(\widetilde{X}', \mathbb{Q})(3), \]

where the last isomorphism follows from the distinguished triangle

\[ (i_x)_* i^*_x \to \text{id} \to R(j_{\widetilde{X}'})_* j^*_{\widetilde{X}'}, \]

with \( j_{\widetilde{X}'} : \widetilde{X}' \hookrightarrow \widetilde{X} \) the inclusion. (Here \( i_x \) denotes also the inclusion into \( \widetilde{X} \).) So (2.5.3) follows from the above two isomorphisms together with the self-duality isomorphism

\[ \mathbb{D}(H^2(F_j, \mathbb{Q}_1)) = H^2(F_j, \mathbb{Q}_1)(3), \]

which is a consequence of the duality for the unipotent monodromy part of the vanishing cycle functor \( \varphi_{j,1}, \) see for instance [Sa1, 5.2.3]. (Here the intersection form cannot be used on this unipotent monodromy part, see [St1, St2].)

The unipotent monodromy part of the vanishing cohomology \( H^2(F_j, \mathbb{Q}_1) \) has pure Hodge structure of weight 3, since \( N = 0, \) see [St1, St2] and also [Sa1, 5.1.6]. Its dimension is given by the sum of the coefficients of \( t \) and \( t^2 \) in the Steenbrink spectrum as in [St2], and the latter is expressed in this case by

\[ \text{Sp}(f) = \left( \frac{t^{1/a} - t}{1 - t^{1/a}} \right) \left( \frac{t^{1/b} - t}{1 - t^{1/b}} \right) \left( \frac{t^{1/c} - t}{1 - t^{1/c}} \right), \]

see for instance [Sa1, 1.9] and the references there. (In the case \( a = b = c, \) we may also use the Thom-Gysin sequence (see for instance [RSW, 1.3]) together with a well-known theorem of Griffiths on rational integrals for the complement of a smooth curves on \( \mathbb{P}^2. \) By the symmetry of the coefficients of the spectrum, we then get

\[ \dim H^1(\widetilde{X}', \mathbb{Q}) = 2 \cdot \# \{(i, j, k) \in \mathbb{Z}^3_{>0} \mid \frac{i}{a} + \frac{j}{b} + \frac{k}{c} = 1\}. \]

This description implies that we may have

\[ W_1 H^1(X', \mathbb{Q}) = 0, \quad \text{but} \quad H^1(X', \mathbb{Q}) = H^1(\widetilde{X}') \neq 0, \]

in the case \( e = \gcd(b, c) = 1 \) and \( \frac{a}{b}, \frac{a}{c} \in \mathbb{Z}, \) for instance, if \( (a, b, c) = (6, 3, 2). \) (Note that \( H^1(X', \mathcal{F}_{X'}) = 0, \) since \( \pi^{-1}(X'^{(b)}) \cong \mathbb{C}^* \) by \( e = 1. \)

**Appendix. Totally ramified cyclic coverings**

In this Appendix we review totally ramified cyclic coverings which are useful to construct examples of algebraic varieties with multi-branch points, and calculate the local monodromies of local branches.

**A.1. Construction.** Let \( Y \) be a complex algebraic variety, \( D \) be a locally principal effective divisor on \( Y, \) and \( \mathcal{L} \) be an invertible sheaf (that is, a locally free sheaf of rank 1) on \( Y \) such that

\[ (A.1.1) \quad \nu : \mathcal{L}^\otimes m \to \mathcal{O}_Y(-D) \subset \mathcal{O}_Y, \]
where $m \geq 2$. This isomorphism gives a structure of an $\mathcal{O}_Y$-algebra on
\[
\mathcal{A}_Y := \bigoplus_{i=0}^{m-1} \mathcal{L}^i = \mathcal{B}_Y / \mathcal{I}(D, \mathcal{L}) \quad \text{with} \quad \mathcal{B}_Y := \bigoplus_{i \geq 0} \mathcal{L}^i,
\]
where $\mathcal{I}(D, \mathcal{L}) \subset \mathcal{B}_Y$ is an ideal sheaf locally generated over $\mathcal{B}_Y$ by
\[
f_u := \otimes^m u - t(\otimes^m u) \in \mathcal{L}^m \oplus \mathcal{O}_Y \subset \mathcal{B}_Y,
\]
for a local generator $u \in \mathcal{L}$.

The \textit{totally ramified cyclic covering} of $Y$ of degree $m$ associated with $(D, \mathcal{L})$ is then defined by
\[
X(Y, D, \mathcal{L}, m) := (\text{Spec}_Y \mathcal{A}_Y)_{\text{var}} \subset L^\vee = (\text{Spec}_Y \mathcal{B}_Y)_{\text{var}}.
\]
Here $(S)_{\text{var}}$ is the algebraic variety associated with a scheme $S$ of finite type over $\mathbb{C}$ in general, and is defined by the set of closed points of $S$ together with the induced topology and structure sheaf. We will denote $X(Y, D, \mathcal{L}, m)$ also by $X$ to simplify the notation.

This construction is quite well known in algebraic geometry and Hodge theory. It was, for instance, used to generalize Kodaira’s vanishing theorem by applying C.P. Ramanujam’s idea \cite{Ram} so that the vanishing theorem is reduced to the weak Lefschetz type theorem using Hodge theory, see \cite[p. 151]{Na}, \cite{Sa2}. Proof of Proposition 2.33 and Remark 2.34(2) \cite{Vi} where the \textit{normalization} cannot be taken in our case since the multi-branch points disappear as in Example (2.5)(i)). In the literature it does not seem that the \textit{monodromies of local branches} are studied.

Since we assume that $\mathcal{L}$ is an invertible sheaf and $D$ is a locally principal divisor, the construction is quite trivial. By definition, $X$ is a locally principal divisor on the dual line bundle $L^\vee$ of the line bundle $\mathcal{L}$ corresponding to the locally free sheaf $\mathcal{L}$ (that is, the sheaf of local sections of $L$ is $\mathcal{L}$). Choosing a local trivialization of $L$ (or equivalently, of $L^\vee$), $X$ is locally defined by
\[
t^m = g_u \quad \text{in} \quad L^\vee,
\]
where $g_u := t(\otimes^m u) \in \mathcal{O}_Y$ for a local generator $u \in \mathcal{L}$ giving the local trivialization of $L^\vee$, and $t$ is the coordinate of the fiber of the line bundle $L^\vee$ associated with the local trivialization of $L^\vee$. (By definition $\mathcal{L} \subset \mathcal{B}_Y$ is identified with functions on $L^\vee$ which are \textit{linear} on each fiber of $L^\vee$, and $t$ corresponds to $u$ by this identification.) Note that $g_u$ is a local defining function of $D$ by (A.1.1), and $X$ is a ramified finite cyclic Galois covering of $Y$, where a generator of the covering transformation group acts by multiplication by $\exp(2\pi \sqrt{-1}/m)$ on the variable $t$. We say that $X$ is \textit{totally ramified}, since $\pi_Y^{-1}(D)_{\text{red}}$ is isomorphic to $D_{\text{red}}$.

\textbf{Remarks.} (i) If we fix a divisor $D$, then the invertible sheaf $\mathcal{L}$ satisfying (A.1.1) is unique up to a tensor product with an invertible sheaf $\mathcal{E}$ satisfying $\mathcal{E}^m = \mathcal{O}_Y$ (which corresponds to an $m$-torsion point of the Picard group $\text{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$). For a fixed divisor $D$, the totally ramified cyclic covering $X$ of degree $m$ associated with $(D, \mathcal{L})$ depends (as an algebraic variety over $Y$) on the choice of the invertible sheaf $\mathcal{L}$ satisfying (A.1.1) in general. This can be seen, for instance, in the case of $m = 2$ by looking at the cokernel of the canonical injection $\mathcal{O}_Y \hookrightarrow (\pi_Y)_* \mathcal{O}_X$, since the latter is isomorphic to $\mathcal{L}$ by the definition of $\mathcal{O}_X$ (that is, $(\pi_Y)_* \mathcal{O}_X = \mathcal{A}_Y$).

(ii) The construction of the totally ramified cyclic covering of degree $m$ associated with $(D, \mathcal{L})$ is \textit{compatible with base changes by dominant morphisms of $Y$}. More precisely, for
a morphism \( \phi : Y' \to Y \) such that the image of any irreducible component of \( Y' \) is not contained in \( D \), there is the canonical isomorphism

\[(A.1.3) \quad X(Y', \phi^* D, \phi^* \mathcal{L}, m) = X(Y, D, \mathcal{L}, m) \times_Y Y'.\]

**A.2. Case \( Y \) smooth.** In the notation of (A.1), assume \( Y \) connected and smooth so that any divisor on \( Y \) is locally principal. We have the irreducible decomposition

\[D = \sum_{i \in I} a_i D_i \quad (a_i \in \mathbb{Z}_{>0}),\]

For \( y \in D \), set

\[r_y(D, m) := \text{GCD}(m, a_i \: (i \in I_y)) \quad \text{with} \quad I_y := \{i \in I \mid y \in D_i\}.
\]

In the notation of (2.3), we can prove the following isomorphism as to pological spaces

\[(A.2.1) \quad \pi_Y : X^{(r)} \overset{\sim}{\to} D(m, r) := \{y \in D \mid r_y(D, m) = r\},\]

which is induced by the isomorphism as reduced algebraic varieties

\[(A.2.2) \quad \pi_Y : (X|_D)_{\text{red}} \overset{\sim}{\to} D_{\text{red}},\]

where the left-hand side means the reduced restriction of \( X \) over \( D_{\text{red}} \). Indeed, we have a factorization of a defining function of \( X \) in the line bundle

\[(A.2.3) \quad t^m - g = \prod_{q=1}^{\ell} (t^{m/e} - \theta^q \cdot \gamma) \quad \text{if} \quad g = h^e \quad \text{with} \quad m/e \in \mathbb{Z},\]

where \( \theta = \exp(2\pi \sqrt{-1}/e) \). So the isomorphism (A.2.1) is reduced to the assertion that \( X \) is unibranch at a point over \( y \in D \) if \( r_y(D, m) = 1 \).

The latter assertion is verified by taking locally a sufficiently general line \( C \) in \( Y \) which is sufficiently close to \( y \) and intersecting only smooth points of the \( D_i \: (i \in I_y) \), and then restricting the covering over \( C \). Indeed, it is well known (and is easy to show) that the local monodromies of the unramified cyclic covering over the complement of \( D \) are given around an intersection point of \( C \) and \( D_i \) by \( \gamma^{-a_i} \). Here \( X|_C \) is locally defined by \( t^m = z^{a_i} \) in \( \mathbb{C} \times C \) with \( z \) a local coordinate of \( C \), and we have the factorization

\[t^m - z^{a_i} = \prod_{q=1}^{\ell} (t - \theta^q \cdot z^{a_i/m})\]

over any simply connected open subset of \( C \setminus D \) with \( \theta' := \exp(2\pi \sqrt{-1}/m) \). Note that \( \gamma \) is identified with the generator \( \exp(2\pi \sqrt{-1}/m) \) of the covering transformation group \( \mu_m \) of the cyclic covering \( X \to Y \), where

\[(A.2.4) \quad \mu_m := \{\theta \in \mathbb{C}^* \mid \theta^m = 1\},\]

and the latter acts by the natural multiplication on the variable \( t \) in (A.1.2). (Recall that the cyclic covering is an abelian covering so that the local monodromy is independent of a path from a base point to the intersection point inside \( C \), since it is unique up to a conjugation.) So the restriction of the covering over \( C \setminus D \) is connected if \( r_y(D, m) = 1 \).

**Remark.** If \( \text{GCD}(m, a_i \: (i \in I)) = 1 \), then \( X \) is irreducible by an argument similar to the above one by taking sufficiently general curves on \( Y \), at least if \( Y \) is quasi-projective. (Note that \( Y \) is assumed connected.) Its converse does not necessarily hold as a consequence of Remark after (A.1) unless \( H_1(Y, \mathbb{Z}) = 0 \).

**A.3. Calculation of local monodromies of local branches.** With the notation and assumption of (A.2), we have a stratification of \( Y \) such that its strata (which are not assumed
smooth) are given by
\[ Y_{J,p}^\circ := Y_{J,p} \setminus \big( \bigcup_{i \notin J} D_i \big), \]
with \( Y_{J,p} (k \in K_J) \) irreducible components of \( \bigcap_{i \in J} D_i \).
These are indexed by \( J \subset I, k \in K_J \) satisfying the condition:
\[ \{ i \in I \mid D_i \supset Y_{J,p} \} = J. \]
We fix a non-empty \( J \subset I \) and \( k \in K_J \) satisfying this condition (where \( Y_{J,p} \subset D_{\text{red}} \)). We have the local system of finite sets on \( Y_{J,p}^\circ \)
\[ \{ I_{X,\pi^{-1}_Y(y)} \}_{y \in Y_{J,p}^\circ}, \]
consisting of local branches, where \( Y_{J,p}^\circ \) is identified with \( \pi^{-1}_Y Y_{J,p}^\circ \) by the isomorphism (A.2.2).
Here a local system of finite sets means a locally constant sheaf with stalks finite sets. (We will see that it is a principal \( \mu \)-bundle at the end of this section.) We can describe its local monodromies around \( Y_{J,p} \setminus Y_{J,p}^\circ \) as follows.

The stratum \( Y_{J,p}^\circ \) and its closure \( Y_{J,p} \) may have singularities in general. In the singular case, we apply Hironaka’s resolution of singularities to the variety \( Y_{J,p} \) inside \( Y \). It is given by iterating blowing-ups along smooth centers contained in the singular locus of the proper transform of \( Y_{J,p} \). We then get a projective morphism of smooth algebraic varieties
\[ \sigma_Y : \mathcal{Y} \to Y, \]
inducing an isomorphism over the complement of the singular locus of \( Y_{J,p} \) in \( Y \), and such that the proper transform \( \mathcal{Y}_{J,p} \) of \( Y_{J,p} \) in \( \mathcal{Y} \) is smooth. (This procedure is unnecessary in the \( Y_{J,p} \) smooth case.)

We further iterate blowing-ups along smooth centers contained in the non-normal-crossing locus of the intersection of the proper transform \( \mathcal{Y}_{J,p} \) with the total transforms of \( \bigcup_{i \notin J} D_i \), so that this intersection becomes a divisor with normal crossings on \( \mathcal{Y}_{J,p} \). (It is not required that the total transforms of \( \bigcup_{i \notin J} D_i \) is a divisor with normal crossings on \( \mathcal{Y} \).) This is necessary to calculate the local monodromies, since we can do it only at smooth points of the reduced intersection of \( \mathcal{Y}_{J,p} \) and the total transforms of \( \bigcup_{i \notin J} D_i \). (Here reduced intersection means that we consider the associated reduced variety. This is needed since we talk about smooth points.)

Let
\[ Z = \sum_k m_k Z_k \]
be the restriction of the total transforms of the divisor \( \sum_{i \notin J} a_i D_i \) to \( \mathcal{Y}_{J,p} \), where the \( Z_k \) are the irreducible components of the divisor with normal crossings (by the above construction). Set
\[ \mathcal{X} := X \times_Y \mathcal{Y}, \quad \text{and} \quad \mathcal{Y}_{J,p}^\circ := \mathcal{Y}_{J,p} \setminus Z. \]
Note that \( \mathcal{X} \) is a totally ramified covering of \( \mathcal{Y} \), see Remark (ii) after (A.1). Using the isomorphism (A.2.2) for \( \pi_Y : \mathcal{X} \to \mathcal{Y} \), the local monodromy of \( \{ I_{X,\pi^{-1}_Y(y)} \}_{y \in \mathcal{Y}_{J,p}^\circ} \) around a general point \( y \in Z_k \setminus \big( \bigcup_{k' \neq k} Z_{k'} \big) \) is given by
\[ (\gamma^{-m_k}), \]
where the generator \( \gamma \) of the covering transformation group \( \mu_m \) of \( \pi_Y : \mathcal{X} \to \mathcal{Y} \) can be identified with a generator of the covering transformation group of \( \mathcal{X} \vert_{\mathcal{Y}_{J,p}^\circ} \to \mathcal{Y}_{J,p}^\circ \) (with \( \mathcal{X} \) the normalization of \( \mathcal{X} \)). Indeed, the assertion can be reduced to the case \( \dim \mathcal{Y}_{J,p} = 1 \) and \( Z_k \) is a point, by restricting to a sufficiently general smooth subvariety of \( \mathcal{Y} \) which is transversal to \( Z_k \). Here we can further iterate blowing-ups along \( Z_k \) so that the proper
transform of $D_i$ for $i \not\in J$ does not contain $Z_k$, and only the exceptional divisor of the last blow-up contains $Z_k$. (Here we may also assume that the proper transforms of local branches of $D_i$ for $i \in J$, which intersect $Y_{j,p}$ without containing it completely, do not contain $Z_k$.)

Set

$$e := \text{GCD}(m, a_i (i \in J)).$$

On a neighborhood of $Z_k$, we then get the factorization of a defining function of $X$ in the line bundle

$$(A.3.2) \quad t^m - z^{m_k} \prod_{i \in J} h_i^{a_i} = e \prod_{q=1}^{n} (t^{m/e} - \theta^{q} z^{m_k/e} \prod_{i \in J} h_i^{a_i/e}).$$

Here $\theta := \exp(2\pi \sqrt{-1}/e)$, $h_i$ is the pull-back of a local defining function of $D_i$, and $z$ is an appropriate coordinate defining the exceptional divisor of the last blow-up along $Z_k$. (Note that $m/e$ and the $a_i/e (i \in J)$ are integers by the definition of $e$, but $m_k/e$ is not necessarily.)

So we get the assertion (A.3.1), since $\gamma$ acts by multiplication by $1^{1/m}$ on the variable $t$, and hence by $1^{1/e}$ on $t^{m/e}$, where $1^{1/k} := \exp(2\pi \sqrt{-1}/k)$ for $k \in \mathbb{Z}_{>0}$.

The above argument shows that the local branches form a principal $\mu_e$-bundle over $Y^n_{j,p}$, which is identified with a locally constant sheaf with stalks finite sets having a transitive free action of $\mu_e$ (where $\mu_e$ is as in (A.2.4)).

**Remark.** The global monodromies of local branches are determined by local monodromies if $H_1(Y_{j,p}, \mathbb{Z}) = 0$. Indeed, for two principal $\mu_e$-bundles $P_1$, $P_2$ over $Y^n_{j,p}$, $\text{Hom}_{\mu_e}(P_1, P_2)$ is also a principal $\mu_e$-bundle over $Y^n_{j,p}$. The latter can be extended over $Y_{j,p}$ if the local monodromies of $P_1, P_2$ coincide. Here the local monodromies are identified with morphisms

$$H_1(U_y \cap Y^n_{j,p}, \mathbb{Z}) \to \mu_e,$$

with $U_y$ an open neighborhood of $y$ in $Y_{j,p}$ (since the target is abelian). A similar assertion holds for global monodromies with $U_y \cap Y^n_{j,p}$ replaced by $Y^n_{j,p}$.

**A.4. 2-dimensional case.** With the notation and assumption of (A.3), assume further $\dim Y = 2$, and $Y_{j,p} = D_i$. In this case $Y$ is obtained by iterating point-center blowing-ups, and $Z_k$ is a point of $Y_{j,p}$ (which is a proper transform of $D_i$), and is identified with a point $y'$ of the normalization $\widetilde{D}_i$ of $D_i$ corresponding to an analytic branch $(D'_i, y) \subset (D_i, y)$. Here $e = \text{GCD}(m, a_i)$. The multiplicity $m_k$ of $Z_k$ in $Z$ (see (A.3)) can be given by

$$(A.4.1) \quad m_y' := \sum_{j \neq i} a_j (D'_i, D_j)_y,$$

with $(D'_i, D_j)_y$ the intersection number of $D'_i$ and $D_j$ at $y$, and we have the equalities

$$(A.4.2) \quad (D'_i, D_j)_y := \text{dim}_{\mathbb{C}} \mathcal{O}_{Y^{an}, y}/(g'_i, g_j) = \# (g'_i^{-1}(0) \cap g_j^{-1}(c)).$$

Here $g'_i, g_j$ are (reduced) local defining holomorphic functions of $D'_i$, $D_j$ around $y$, and $c$ is a nonzero complex number with $|c|$ sufficiently small so that $g'_i^{-1}(0)$ and $g_j^{-1}(c)$ intersect transversally at smooth points. (Here we cannot use the finite determinacy of hypersurface isolated singularities, since we have to treat two functions $g'_i, g_j$ at the same time, although $g_j$ is algebraic in the initial coordinates.) As for the middle term of (A.4.2), we have the canonical isomorphism

$$\mathcal{O}_{Y^{an}, y}/(g'_i, g_j) = \mathcal{O}_{D'_i, y} \otimes \mathcal{O}_{Y^{an}, y} \mathcal{O}_{D^{an}, y}.$$

If $D_i$ is unibranch (where $D'_i = D_i$ and $g'_i = g_i$ are algebraic), the first equality of (A.4.2) is a theorem (in the proper intersection case), see [Fu, Section 7.1] (and also [Se]).

The last term of (A.4.2) coincides with the multiplicity $m_{i,j}$ of $Z_k$ in the restriction of the total transform of $D_j$ to the proper transform of $D_i$, where the latter is identified with the normalization $\widetilde{D}_i$. Note that there is a local coordinate $\tilde{z}$ of $(\widetilde{D}_i, y')$ such that the pull-back
of $g_j$ to the proper transform $(\bar{D}_i, y')$ by the composition $\bar{D}_i \hookrightarrow Y \to Y$ coincides with $\mathcal{Z}^{m_{i,j}}$. This can be used to show (A.4.2) in the $D_i$ unibranch case, see Remark (ii) below.)

To show (A.4.2), consider the morphism

$$\rho: (\mathbb{C}^2, 0) \ni (y_1, y_2) \mapsto (z_1, z_2) = (g'_i(y_1, y_2), g_j(y_1, y_2)) \in (\mathbb{C}^2, 0),$$

where $(y_1, y_2)$ is an analytic local coordinate system of $(Y, y)$. The assertion is then shown by proving the finiteness and flatness of the morphism

(A.4.3) $$\rho|_U: U \to U' \quad \text{with} \quad U := \rho^{-1}(U'),$$

for a sufficiently small open neighborhood $U'$ of $0 \in \mathbb{C}^2$ satisfying

$$g'_i(0) \cap g_j(0) \cap U = \{0\}, \quad \text{that is,} \quad (\rho|_U)^{-1}(0) = \{0\}.$$ 

Indeed, finiteness follows from Weierstrass preparation theorem (using a graph embedding together with the factorization $\mathbb{C}^4 \to \mathbb{C}^3 \to \mathbb{C}^2$). So the direct image $\rho_*\mathcal{O}_U$ is a coherent $\mathcal{O}_{U'}$-module, and hence $R := \mathbb{C}\{y_1, y_2\}$ is a finite $R'$-module with $R' := \mathbb{C}\{z_1, z_2\}$.

For flatness, consider a short exact sequence of finite $R'$-modules

$$0 \to K \to P \to R \to 0,$$

where $P$ is free, $P/\mathfrak{m}'P \to R/\mathfrak{m}'R$ is an isomorphism with $\mathfrak{m}' := (z_1, z_2) \subset R'$ the maximal ideal, and $K$ is defined by the short exact sequence. (Note that the surjectivity of $P \to R$ follows from the isomorphism modulo $\mathfrak{m}'$ using Nakayama’s lemma.) Since $g'_i, g_j$ form a regular sequence in $R = \mathbb{C}\{y_1, y_2\}$, the Koszul complex defined by them vanishes except for the highest degree, see [Ei, Sc]. This implies that

$$\text{Tor}^1_{R'}(R'/\mathfrak{m}'R', R) = 0,$$

since the left-hand side coincides with the middle term of the Koszul complex for $g'_i, g_j$. Using the long exact sequence associated with the above short exact sequence, we then get

$$K/\mathfrak{m}'K = 0, \quad \text{hence} \quad K = 0 \quad \text{(by Nakayama’s lemma)}.$$ 

Thus $R$ is finite free over $R'$, see also [Ei, Theorem 6.8]. (Note that freeness is equivalent to flatness under the finiteness hypothesis by an argument similar to the above one, see [Ei, Sc].) So the coherent sheaf $\rho_*\mathcal{O}_U$ is a finite free $\mathcal{O}_{U'}$-module (shrinking $U'$ if necessary). We then get the last equality of (A.4.2), since both sides coincide with the rank of the free $\mathcal{O}_{U'}$-module $\rho_*\mathcal{O}_U$ counted at $(0, 0)$ and $(0, c) \in \mathbb{C}^2$ respectively.

Remarks. (i) We can show the finiteness and flatness of (A.4.3) using algebraic geometry in the $D_i$ unibranch case, although Zariski topology is not fine enough for the assertion to hold with $U'$ a Zariski open neighborhood of $0 \in \mathbb{C}^2$. By Grothendieck’s version of Zariski’s main theorem (see [Gr2, Theorem 4.4.3]), we can extend the germ of a morphism

$$(g_i, g_j): (Y, y) \to (\mathbb{C}^2, 0) \quad \text{(with } g_i, g_j \text{ algebraic)}$$

to a finite morphism $\hat{Y} \to \mathbb{C}^2$, where $\hat{Y}$ contains an open subvariety $Y^\circ$ of $Y$ as a dense open subvariety, and $\hat{Y} \setminus Y^\circ$ may contain a point over $0 \in \mathbb{C}^2$. We may assume $\hat{Y}$ normal (replacing it by the normalization if necessary) so that it is Cohen-Macaulay by Serre’s condition $S_2$, see for instance [Ha2] II, Theorem 8.22A. The finiteness and flatness of (A.4.3) then follows from this in the $D_i$ unibranch case (using [Ha2] II, Theorem 8.21A(c))). Here $U'$ is a sufficiently small neighborhood in the classical topology and $U$ is a connected component of the inverse image of $U'$. (In the non-unibranch case, etale topology would be needed, see for instance [Ray]).
(ii) In the \(D_i\) unibranch case, we can also show (A.4.2) by using a remark before (A.4.3) about the coordinate \(\tilde{z}\). Here we may assume \(Y\) smooth projective and \(D_i \cap D_j = \{y\}\) by replacing \(Y\) with a smooth projective compactification of an affine neighborhood of \(y \in Y\) and then blowing-up \(Y\) at the other intersection points of \(D_i, D_j\). Note that the intersection number of the proper transform of \(D_i\) and the total transform of \(D_j\) coincides with the intersection number \((D_i, D_j)\), since the intersection numbers of the exceptional divisors and the total transform of \(D_j\) vanish. So we get (A.4.2) in this case using the remark about \(\tilde{z}\).

A.5. Quotient variety construction. We can generalize a construction in [BR, Example 6.8] (due to J. Kollár and B. Wang) by using totally ramified cyclic coverings in (A.1) as follows.

Let \(X \to Y\) be the totally ramified cyclic covering of degree \(m \geq 2\) associated with \((D, L)\) as in (A.1), where we assume \(Y\) connected and smooth as in (A.2). Let \(e\) be an integer \(\geq 2\) dividing \(m\). Let \(Z\) be a connected complex algebraic variety having an action of \(\mu_e\) such that we have the quotient variety \(Z' := Z/\mu_e\) and the canonical morphism \(Z \to Z'\) is a finite morphism. (In loc. cit. \(Y = \mathbb{P}^1\), \(Z\) is an elliptic curve, \(Z \to Z'\) is an isogeny, \(m = e = 2\), and either \(D = 2\{0\}\) with \(L = \mathcal{O}_{\mathbb{P}^1}(-1)\) or \(D = 2\{0\} + \{1\} + \{\infty\}\) with \(L = \mathcal{O}_{\mathbb{P}^1}(-2)\).)

There is a short exact sequence of abelian groups

\[
1 \to \mu_e \to \mu_m \to \mu_{m'} \to 1,
\]

where the first morphism is the \(m'\)th power morphism with \(m' := m/e\). This gives an action of \(\mu_e\) on \(X\), and hence the diagonal action of \(\mu_e\) on \(X \times Z\). We have the quotient variety \((X \times Z)/\mu_e\), since \(X, Z\), and hence \(X \times Z\), are covered by affine open subsets which are stable by the action of \(\mu_e\). (Here we use the assertion that a finite morphism is affine.) There is a ramified cyclic covering

\[
(A.5.1) \quad \mathcal{X} := (X \times Z)/\mu_e \to (X \times Z)/(\mu_e \times \mu_e) = (X/\mu_e) \times Z',
\]

by the diagonal embedding \(\mu_e \hookrightarrow \mu_m \times \mu_e\). We can verify that \(X/\mu_e\) is the totally ramified cyclic covering of \(Y\) of degree \(m' = m/e\) associated with \((D, L^{\otimes e})\) if \(e \neq m\), and \(X/\mu_e = Y\) if \(e = m\). These imply that \(\mathcal{X}\) is projective if \(Y\) is. (Indeed, a finite morphism is projective, see [Gr1] Propositions 4.4.10 and Corollary 6.1.11.)

We have the isomorphisms

\[
(A.5.2) \quad H^1(\mathcal{X}, \mathbb{Q}) = H^1(X, \mathbb{Q})^{\mu_e} \oplus H^1(Z, \mathbb{Q})^{\mu_e} = H^1(X/\mu_e, \mathbb{Q}) \oplus H^1(Z', \mathbb{Q}).
\]

The first isomorphism follows from the K"unneth isomorphism

\[
(A.5.3) \quad H^1(X \times Z, \mathbb{Q}) = H^1(X, \mathbb{Q}) \otimes H^0(Z, \mathbb{Q}) \oplus H^1(Z, \mathbb{Q}) \otimes H^0(X, \mathbb{Q}) \otimes H^1(Z, \mathbb{Q}),
\]

since the action of \(\mu_e\) is trivial on

\[
H^0(X, \mathbb{Q}) = H^0(Z, \mathbb{Q}) = \mathbb{Q}.
\]

As a corollary of (A.5.2), we get

\[
(A.5.4) \quad W_0H^1(\mathcal{X}, \mathbb{Q}) = 0, \text{ if } Z' \text{ is smooth and } e = m,
\]

since \(X/\mu_e = Y\) is assumed smooth.

Remark. Assume the finite abelian group \(\mu_e\) acts on \(Z\) freely so that \(Z \to Z'\) is an unramified finite covering. For instance, \(Z\) is an elliptic curve \(E\), or more generally an abelian variety, and the action is given by

\[
E \ni Q \to Q + kP \in E \quad \text{for} \quad k \in \mathbb{Z}/e\mathbb{Z} \cong \mu_e,
\]

where \(Q\) is a torsion point of \(E\) of order \(e\).
In this unramified finite covering case we can get an example where a 1-dimensional stratum $S_i$ in (2.4) is not a rational curve and moreover the monodromy along $S_i$ is nontrivial. Here we have the following isomorphisms compatible with (A.5.2):

$$H^0(Z', R^1(\rho_{\mathcal{X}})_*\mathbb{Q}_X) = H^1(X, \mathbb{Q})_{\mu} = H^1(X/\mu, \mathbb{Q}),$$

where $\rho_{\mathcal{X}} : \mathcal{X} \to Z'$ is the canonical morphism. For the proof of the first isomorphism, we can verify that the monodromy action on $(R^1(\rho_{\mathcal{X}})_*\mathbb{Q}_X)_{\xi}$ by $\xi \in \pi_1(Z', z_0')$ is identified with the action of a generator $\gamma \in \mu$ on $H^1(X, \mathbb{Q})$, if $\xi$ is represented by the image of a path from the base point $z_0 \in Z$ to $\gamma(z_0) \in Z$. (Note that $\xi$ is unique up to the composition with the image of an element of $\pi_1(Z, z_0)$ in $\pi_1(Z', z_0')$.) Here $z_0'$ is the base point of $Z'$, which is the image of the base point $z_0 \in Z$.

We have the geometric monodromy coming from the product structure of $X \times Z$. Indeed, we have the parallel translation between the fibers

$$\rho_{\mathcal{X}}^{-1}(z') \cong \rho_{\mathcal{X}}^{-1}(z'') \quad \text{for sufficiently near } z', z'' \in Z',$$

since we have this for $X \times Z \to Z$ in a compatible way with the action of $\mu$. By definition the pull-back of this parallel translation to $Z$ gives the global trivialization of $X \times Z \to Z$. This implies that the geometric monodromy is independent of the choice of a path from $z_0$ to $\gamma(z_0)$ in $Z$ (using the above description of the ambiguity of $\xi \in \pi_1(Z', z_0)$).

The second isomorphism of (A.5.5) follows from the isomorphism

$$(\pi_{\mathcal{X}')}_* \mathbb{Q}_X)^{\mu} = \mathbb{Q}_{\mathcal{X}},$$

together with the complete reducibility of finite dimensional complex representations of finite groups (using the scalar extension by $\mathbb{Q} \to \mathbb{C}$), where $\pi_{\mathcal{X}'} : X \to X' := X/\mu$ is the quotient morphism. This isomorphism is a stalkwise property of the constructible sheaf $(\pi_{\mathcal{X}'})_* \mathbb{Q}_X$ with the $\mu$-action, since $\pi_{\mathcal{X}'}$ is a finite morphism. (This argument holds for any finite group action on $X$ such the quotient exists as an algebraic variety.)

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