On Kinetic Equations Modeling Evolution of Systems in Mathematical Biology

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Abstract. We develop a rigorous formalism for the description of the kinetic evolution of interacting entities modeling systems in mathematical biology within the framework of the evolution of marginal observables. For this purpose we construct the mean field asymptotic behavior of a solution of the Cauchy problem of the dual BBGKY hierarchy (Bogolyubov-Born-Green-Kirkwood-Yvon) for marginal observables of the dynamical systems based on the Markov jump processes, exhibiting the intrinsic properties of the living entities. The constructed scaling limit is governed by the set of recurrence evolution equations, namely by the dual Vlasov-type hierarchy. Moreover, the relationships of the dual Vlasov hierarchy for the limit marginal observables with the Vlasov-type kinetic equation is established.

Key words: kinetic equation, dual BBGKY hierarchy, Markov jump process, scaling limit, soft active matter.

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1 Introduction

Recently the considerable advance in solving the problem of rigorous modeling of the kinetic evolution of systems with a large number of constituents (entities) of mathematical biology, in particular, systems of large number of cells, is observed [1]-[9] (and see references cited therein).

The many-entity biological systems [2], [10] are dynamical systems displaying a collective behavior which differs from the statistical behavior of usual gases [11], [12]. In the first place their own distinctive features is connected with the fact that entities show the ability to retain various complexity features [2]-[8]. To specify such nature of entities we consider the dynamical system suggested in [10], [13] which is based on the Markov jump processes that must represent the intrinsic properties of living creatures.

In the work we develop a new approach to the description of the kinetic evolution of large number of interacting entities within the framework of the evolution of marginal observables. To this end we describe the evolution of interacting entities by the dual BBGKY hierarchy for marginal observables of such system and construct the mean field scaling asymptotics of its solution. For obvious reasons the description of collective behavior in terms of observables of living constituents of biological systems is more suitable in comparison with the formalism of states.

We outline the structure of the paper and the main results. In introductory section 2 we adduce the basic facts on the description of the evolution of systems of finitely many entities


of various subpopulations of mathematical biology introduced in paper [10]. In section 3 we develop one more approach to the description of the evolution of many-entity systems in terms of the hierarchies of evolution equations for marginal observables and marginal distribution functions which underlie of kinetic models. In particular, a nonperturbative solution of the Cauchy problem of the dual BBGKY hierarchy for the marginal observables is constructed. In section 4 we prove the main result concerning to the description of the kinetic evolution of interacting entities within the framework of the evolution of marginal observables. The mean field asymptotics of a nonperturbative solution of the Cauchy problem of the dual BBGKY hierarchy for entities is constructed. Furthermore, the relationships of the dual Vlasov hierarchy for the limit marginal observables with the Vlasov-type kinetic equation is established. Finally in section 5 we conclude with some observations and perspectives for future research.

2 Preliminary facts

A description of many-constituent systems is formulated in terms of two sets of objects: observables and states. The functional of the mean value of observables defines a duality between observables and states and as a consequence there exist two approaches to the description of the evolution of such systems, namely in terms of the evolution equations for observables and for states. In this section we adduce some preliminary facts about dynamics of finitely many entities of various subpopulations described within the framework of nonequilibrium grand canonical ensemble [12].

2.1 The stochastic dynamics of many-entity systems

We consider a system of entities of various \( M \) subpopulations introduced in paper [10] in case of non-fixed, i.e. arbitrary, but finite average number of entities. Every \( i \)-th entity is characterized by: \( u_i = (j_i, u_i) \in \mathcal{J} \times \mathcal{U} \), where \( j_i \in \mathcal{J} \equiv (1, \ldots, M) \) is a number of its subpopulation, and \( u_i \in \mathcal{U} \subset \mathbb{R}^d \) is its microscopic state [10]. The stochastic dynamics of entities of various subpopulations is described by the semigroup \( e^{t \Lambda} = \oplus_{n=0}^{\infty} e^{t \Lambda_n} \) of the Markov jump process defined on the space \( C_{\gamma} \) of sequences \( b = (b_0, b_1, \ldots, b_n, \ldots) \) of measurable bounded functions \( b_n(u_1, \ldots, u_n) \) that are symmetric with respect to permutations of the arguments \( u_1, \ldots, u_n \) and equipped with the norm:

\[
\|b\|_{C_{\gamma}} = \max_{n \geq 1} \frac{\gamma^n}{n!} \|b_n\|_{C_n} = \max_{n \geq 1} \frac{\gamma^n}{n!} \max_{j_1, \ldots, j_n} \max_{u_{i_1}, \ldots, u_{i_m}} \left| b_n(u_{i_1}, \ldots, u_{i_m}) \right|,
\]

where \( \gamma < 1 \) is a parameter. The operator \( \Lambda_n \) (the Liouville operator of \( n \) entities) is defined on the subspace \( C_n \) of the space \( C_{\gamma} \) and it has the following structure [10]:

\[
(\Lambda_n b_n)(u_1, \ldots, u_n) = \sum_{m=1}^{M} \varepsilon^{m-1} \sum_{i_1 \neq \ldots \neq i_m=1}^{n} (A^{[m]}(i_1, \ldots, i_m)b_n)(u_1, \ldots, u_n) =
\]

\[
= \sum_{m=1}^{M} \varepsilon^{m-1} \sum_{i_1 \neq \ldots \neq i_m=1}^{n} a^{[m]}(u_{i_1}, \ldots, u_{i_m}) \left( \int_{\mathcal{J} \times \mathcal{U}} A^{[m]}(v; u_{i_1}, \ldots, u_{i_m}) \times \right. \\
\left. \times b_n(u_1, \ldots, u_{i_{i-1}}, v, u_{i_{i+1}}, \ldots, u_n) dv - b_n(u_1, \ldots, u_n) \right),
\]

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\]

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= \sum_{m=1}^{M} \varepsilon^{m-1} \sum_{i_1 \neq \ldots \neq i_m=1}^{n} a^{[m]}(u_{i_1}, \ldots, u_{i_m}) \left( \int_{\mathcal{J} \times \mathcal{U}} A^{[m]}(v; u_{i_1}, \ldots, u_{i_m}) \times \right. \\
\left. \times b_n(u_1, \ldots, u_{i_{i-1}}, v, u_{i_{i+1}}, \ldots, u_n) dv - b_n(u_1, \ldots, u_n) \right),
\]
where \( \varepsilon > 0 \) is a scaling parameter \([14]\), the functions \( a^m(u_1, \ldots, u_m) \), \( m \geq 1 \), characterize the interaction between entities, in particular, in case of \( m = 1 \) it is the interaction of entities with an external environment. These functions are measurable positive bounded functions on \((J \times U)^n\) such that:

\[
0 \leq a^m(u_1, \ldots, u_m) \leq a^*_m,
\]

where \( a^*_m \) is some constant. The functions \( A^m(v; u_1, \ldots, u_m) \), \( m \geq 1 \), are measurable positive integrable functions which describe the probability of the transition of the \( i_1 \) entity in the microscopic state \( u_i \) to the state \( v \) as a result of the interaction with entities in the states \( u_{i_2}, \ldots, u_{i_m} \) (in case of \( m = 1 \) it is the interaction with an external environment). The functions \( A^m(v; u_1, \ldots, u_m) \), \( m \geq 1 \), satisfy the conditions:

\[
\int_{J \times U} A^m(v; u_1, \ldots, u_m)dv = 1, \quad m \geq 1.
\]

We refer to paper \([10]\), where examples of the functions \( a^m \) and \( A^m \) are given in the context of biological systems.

In case of \( m = 1 \) generator \([1]\) has the form \( \sum^{n}_{i_1=1} \Lambda^n_{i_1}(i_1) \) and it describes the free stochastic evolution of entities. The case of \( m \geq 2 \) corresponds to a system with the \( m \)-body interaction of entities in the sense accepted in kinetic theory \([15]\). The \( m \)-body interaction of entities is the distinctive property of biological systems in comparison with many-particle systems, for example, gases of atoms with a pair interaction potential.

On the space \( C_n \) the one-parameter mapping \( e^{tA_n} \) is a bounded \(*\)-weak continuous semigroup of operators \([16]\).

### 2.2 The evolution equations for observables and states

The observables of a system of a non-fixed number of entities of various subpopulations are the sequences \( O = (O_0, O_1(u_1), \ldots, O_n(u_1, \ldots, u_n), \ldots) \) of functions \( O_n(u_1, \ldots, u_n) \) defined on \((J \times U)^n\) and \( O_0 \) is a real number. The evolution of observables is described by the sequences \( O(t) = (O_0, O_1(t, u_1), \ldots, O_n(t, u_1, \ldots, u_n), \ldots) \) of the functions

\[
O_n(t) = e^{tA_n}O^0_n, \quad n \geq 1,
\]

i.e. they are the corresponding solution of the Cauchy problem of the Liouville equation (or the Kolmogorov forward equation) with initial data \( O^0_n \).

The average values of observables (mean values of observables) are determined by the following positive continuous linear functional defined on the space \( C_n \):

\[
\langle O \rangle(t) = (I, D(0))^{-1}(O(t), D(0)) \doteq (I, D(0))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(J \times U)^n} du_1 \ldots du_n O_n(t) D^0_n, \quad (2)
\]

where \( D(0) = (1, D^0_1, \ldots, D^0_n, \ldots) \) is a sequence of nonnegative functions \( D^0_n \) defined on \((J \times U)^n\) that describes the states of a system of a non-fixed number of entities of various subpopulations.
at initial time and \((I, D(0)) = \sum_{n=0}^{\infty} \frac{1}{n!} \int (\mathcal{J} \times \mathcal{U})^n \, du_1 \ldots du_n \, D^n_0 \) is a normalizing factor (the grand canonical partition function).

Let \(L^\alpha_1 = \oplus_{n=0}^{\infty} \alpha^n L^1_n \) be the space of sequences \(f = (f_0, f_1, \ldots, f_n, \ldots)\) of the integrable functions \(f_n(u_1, \ldots, u_n)\) defined on \((\mathcal{J} \times \mathcal{U})^n\), that are symmetric with respect to permutations of the arguments \(u_1, \ldots, u_n\), and equipped with the norm:

\[
\|f\|_{L^\alpha_1} = \sum_{n=0}^{\infty} \alpha^n \|f_n\|_{L^1} = \sum_{n=0}^{\infty} \alpha^n \sum_{j_1 \in \mathcal{J}} \ldots \sum_{j_n \in \mathcal{J}} \int_{\mathcal{U}^n} du_1 \ldots du_n \|f_n(u_1, \ldots, u_n)\|
\]

where \(\alpha > 1\) is a parameter. Then for \(D(0) \in L^1\) and \(O(t) \in C_\gamma\) average value functional (2) exists and it determines a duality between observables and states.

As a consequence of the validity for functional (2) of the following equality:

\[
(I, D(0))^{-1}(O(t), D(0)) = (I, D(0))^{-1}(e^{tA} O(0)) D(0)) = (I, e^{tA^*_n} D(0))^{-1}(O(0)) e^{tA^*_n} D(0)) \equiv (I, D(t))^{-1}(O(0), D(t)),
\]

where \(e^{tA^*_n} = \oplus_{n=0}^{\infty} e^{tA^*_n}\) is the adjoint semigroup of operators with respect to the semigroup \(e^{tA} = \oplus_{n=0}^{\infty} e^{tA_n}\), it is possible to describe the evolution within the framework of the evolution of states. Indeed, the evolution of all possible states, i.e. the sequence \(D(t) = (1, D_1(t, u_1), \ldots, D_n(t, u_1, \ldots, u_n), \ldots) \in L^1\) of the distribution function \(D_n(t), n \geq 1\), is determined by the formula:

\[
D_n(t) = e^{tA^*_n} D^0_n, \quad n \geq 1,
\]

where the operator \(A^*_n\) is the adjoint operator to operator (1) and on \(L^1_n\) it is defined as follows:

\[
(A^*_n f_n)(u_1, \ldots, u_n) = \sum_{m=1}^{M} \varepsilon^{m-1} \sum_{i_1 \neq \ldots \neq i_m=1}^{n} \left( \int_{\mathcal{J} \times \mathcal{U}} A^{[m]}(u_{i_1}, v, u_{i_2}, \ldots, u_{i_m}) a^{[m]}(v, u_{i_2}, \ldots, u_{i_m}) f_n(u_1, \ldots, u_n) dv - a^{[m]}(u_{i_1}, \ldots, u_{i_m}) f_n(u_1, \ldots, u_n) \right),
\]

where the functions \(A^{[m]}\) and \(a^{[m]}\) are defined as above in (1).

The function \(D_n(t)\) is a solution of the Cauchy problem of the dual Liouville equation (or the Kolmogorov backward equation).

On the space \(L^1_n\) the one-parameter mapping \(e^{tA^*_n}\) is a bounded strong continuous semigroup of operators (16).

### 3 Hierarchies of evolution equations for entities of various subpopulations

For the description of microscopic behavior of many-entity systems we also introduce the hierarchies of evolution equations for marginal observables and marginal distribution functions known as the dual BBGKY hierarchy and the BBGKY hierarchy, respectively (12). These hierarchies are constructed as the evolution equations for one more method of the description of observables and states of finitely many entities.
3.1 Marginal observables and states of many-entity systems

An equivalent approach to the description of observables and states of many-entity systems is given in terms of marginal observables $B(t) = (B_0, B_1(t, u_1), \ldots, B_s(t, u_1, \ldots, u_s), \ldots)$ and marginal distribution functions $F(0) = (1, F_1^0(u_1), \ldots, F_s^0(u_1, \ldots, u_s), \ldots)$. Considering formula (2), marginal observables and marginal distribution functions are introduced according to the equality:

$$\langle O \rangle(t) = (I, D(0))^{-1}(O(t), D(0)) = (B(t), F(0)),$$

where $(I, D(0))$ is a normalizing factor defined as above. If $F(0) \in L^1_\alpha$ and $B(0) \in C_\gamma$, then at $t \in \mathbb{R}$ the functional $(B(t), F(0))$ exists under the condition that: $\gamma > \alpha^{-1}$.

Thus, the relationship of marginal distribution functions $F(0) = (1, F_1^0, \ldots, F_s^0, \ldots)$ and the distribution functions $D(0) = (1, D_1^0, \ldots, D_s^0, \ldots)$ is determined by the formula:

$$F_s^0(u_1, \ldots, u_s) = (I, D(0))^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathcal{J} \times \mathbb{R})^n} du_{s+1} \cdots du_{s+n} D_s^0(u_1, \ldots, u_{s+n}), \quad s \geq 1,$$

and, respectively, the marginal observables are determined in terms of the observables as follows:

$$B_s(t, u_1, \ldots, u_s) = \sum_{n=0}^{s} \frac{(-1)^n}{n!} \sum_{j_1 \neq \cdots \neq j_n = 1} O_{s-n}(t, (u_1, \ldots, u_s) \setminus (u_{j_1}, \ldots, u_{j_n})), \quad s \geq 1.$$

Two equivalent approaches to the description of the evolution of many interacting entities are the consequence of the validity of the following equality for the functional of mean values of marginal observables:

$$(B(t), F(0)) = (B(0), F(t)),$$

where $B(0) = (1, B_0^0, \ldots, B_s^0, \ldots)$ is a sequence of marginal observables at initial moment.

We remark that the evolution of many-entity systems is usually described within the framework of the evolution of states by the sequence $F(t) = (1, F_1^0(t, u_1), \ldots, F_s^0(t, u_1, \ldots, u_s), \ldots)$ of the marginal distribution functions $F_s(t, u_1, \ldots, u_s)$ governed by the BBGKY hierarchy for entities [10]:

$$\frac{\partial}{\partial t} F_s(t, u_1, \ldots, u_s) = \Lambda^s_s F_s(t, u_1, \ldots, u_s) + \sum_{k=1}^{s} \frac{1}{k!} \sum_{i_1 \neq \cdots \neq i_k = 1}^{s} \sum_{n=1}^{M-k} \frac{M-k-n-1}{n!} \times$$

$$\int_{(\mathcal{J} \times \mathbb{R})^n} du_{s+1} \cdots du_{s+n} \sum_{j_1 \neq \cdots \neq j_{k+n} \in (i_1, \ldots, i_k, s+1, \ldots, s+n)} \Lambda^{s+k+n}(j_1, \ldots, j_{k+n}) F_{s+n}(t, u_1, \ldots, u_{s+n}),$$

$s \geq 1$,

where on $L^1_n$ the adjoint Liouville operator $\Lambda^s_s$ is defined by formula (3) and we used notations accepted above.
3.2 The dual BBGKY hierarchy for entities of various subpopulations

The evolution of a non-fixed number of entities of various subpopulations within the framework of marginal observables is described by the following Cauchy problem of the dual BBGKY hierarchy [17]:

\[
\begin{align*}
\frac{d}{dt} B_s(t, u_1, \ldots, u_s) &= \Lambda_s B_s(t, u_1, \ldots, u_s) + \sum_{n=1}^{s} \frac{1}{n!} \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \times \\
&\times \sum_{j_1 \neq \ldots \neq j_k=1}^{s} \varepsilon^{k-1} \Lambda^{[k]}(j_1, \ldots, j_k) \sum_{i_1 \neq \ldots \neq i_n \in \{j_1, \ldots, j_k\}} B_{s-n}(t, (u_1, \ldots, u_s) \setminus \{u_i_1, \ldots, u_i_n\}),
\end{align*}
\]

(4)

\[
B_s(t, u_1, \ldots, u_s) |_{t=0} = B_s^{0,\varepsilon}(u_1, \ldots, u_s), \quad s \geq 1,
\]

(5)

where the operators \(\Lambda_s\) and \(\Lambda^{[k]}\) are defined by formulas (1) and the functions \(B_s^{0,\varepsilon}\), \(s \geq 1\), are a scaled initial data.

The simplest examples of recurrence evolution equations (4) have the form

\[
\begin{align*}
\frac{d}{dt} B_1(t, u_1) &= \Lambda^{[1]}(1) B_1(t, u_1), \\
\frac{d}{dt} B_2(t, u_1, u_2) &= (\sum_{i=1}^{2} \Lambda^{[1]}(i) + \varepsilon \Lambda^{[2]}(1, 2)) B_2(t) + \varepsilon \Lambda^{[2]}(1, 2) B_1(t, u_1) + \Lambda^{[2]}(2, 1) B_1(t, u_2)).
\end{align*}
\]

The solution \(B(t) = (B_0, B_1(t, u_1), \ldots, B_s(t, u_1, \ldots, u_s), \ldots)\) of the Cauchy problem of recurrence evolution equations (4), (5) is given by the following expansions:

\[
B_s(t, u_1, \ldots, u_s) = \sum_{n=0}^{s} \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n=1}^{s} \mathcal{A}_{1+n}(t, \{Y \setminus Z\}, Z) B_{s-n}^{0,\varepsilon}(u_1, \ldots, u_{j_1-1}, u_{j_1+1}, \ldots, u_{j_n-1}, u_{j_n+1}, \ldots, u_s), \quad s \geq 1,
\]

(6)

where the \((1+n)th\) order cumulant of the semigroups \(\{e^{t\Lambda_k}\}_{t \in \mathbb{R}}, k \geq 1\), is determined by the formula [17]:

\[
\mathcal{A}_{1+n}(t, \{Y \setminus Z\}, Z) = \sum_{\mathbf{p} : (\{Y \setminus Z\}, Z) = \bigcup_i Z_i} \prod_{Z_i \subset \mathbf{p}} (-1)^{|\mathbf{p}| - 1} \prod_{|Z_i| = 1} e^{t\Lambda_0(\mathbf{z}_i)},
\]

(7)

the sets of indexes are denoted by \(Y \equiv (1, \ldots, s)\), \(Z \equiv (j_1, \ldots, j_n) \subset Y\), the set \(\{Y \setminus Z\}\) consists from one element \(Y \setminus Z = (1, \ldots, j_1 - 1, j_1 + 1, \ldots, j_n - 1, j_n + 1, \ldots, s)\) and the mapping \(\theta(\cdot)\) is the declustering operator defined as follows: \(\theta(\{Y \setminus Z\}, Z) = Y\).

The simplest examples of marginal observables (1) are given by the following expansions:

\[
\begin{align*}
B_1(t, u_1) &= \mathcal{A}_1(t, 1) B_1^{0,\varepsilon}(u_1), \\
B_2(t, u_1, u_1) &= \mathcal{A}_1(t, \{1, 2\}) B_2^{0,\varepsilon}(u_1, u_2) + \mathcal{A}_2(t, 1, 2)(B_1^{0,\varepsilon}(u_1) + B_1^{0,\varepsilon}(u_1)),
\end{align*}
\]
where first and second order cumulants (7) are determined by the corresponding formulas:
\[ A_1(t, \{1, 2\}) = e^{t\Lambda_2}, \]
\[ A_2(t, 1, 2) = e^{t\Lambda_2} - e^{t\Lambda^{[1]}(1)}e^{t\Lambda^{[1]}(2)}. \]

For initial data \( B(0) = (B_0, B_1^{0,e}, \ldots, B_s^{0,e}, \ldots) \in \mathcal{C} \) the sequence \( B(t) \) of marginal observables given by expansions (6) is a classical solution of the Cauchy problem of the dual BBGKY hierarchy for entities (4), (5).

4 The kinetic evolution in terms of marginal observables of entities

To consider mesoscopic properties of interacting entities we develop an approach to the description of the kinetic evolution within the framework of the evolution equations for marginal observables. For this purpose we construct the mean field asymptotics [14] of a solution of the Cauchy problem of the dual BBGKY hierarchy for entities (see also [18], [19]).

4.1 The mean field limit of a solution of the dual BBGKY hierarchy for entities

We restrict ourself by the case of \( M = 2 \) subpopulations to simplify the cumbersome formulas and consider the mean field scaling limit of nonperturbative solution (6) of the Cauchy problem of the dual BBGKY hierarchy for entities (4), (5).

Let for initial data \( B_0, B_s^{0,e} \in \mathcal{C} \) there exists the limit function \( b_0^s \in \mathcal{C} \)
\[ w^* - \lim_{\varepsilon \to 0} (\varepsilon^{-s}B_s^{0,e} - b_0^s) = 0, \]
then for arbitrary finite time interval there exists the mean field limit of solution (6) of the Cauchy problem of the dual BBGKY hierarchy for entities (4), (5) in the sense of the \(*\)-weak convergence of the space \( \mathcal{C} \)
\[ w^* - \lim_{\varepsilon \to 0} (\varepsilon^{-s}B_s(t) - b_s(t)) = 0, \] (8)
and it is determined by the expansion:
\[ b_s(t, u_1, \ldots, u_s) = \sum_{n=0}^{s-1} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n e^{(t-t_1)\Lambda^{[1]}(k_1)} \sum_{i_1 \neq j_1 = 1}^s \Lambda^{[2]}(i_1, j_1) e^{(t_1-t_2)\Lambda^{[1]}(l_1)} \ldots e^{(t_{n-1}-t_n)\Lambda^{[1]}(k_n)} \sum_{i_n \neq j_n = 1}^s \Lambda^{[2]}(i_n, j_n) e^{(t_n-t_{n+1})\Lambda^{[1]}(l_n)} \ldots e^{(t_n-t_{n+1})\Lambda^{[1]}(k_n)} b_0^{s-n}((u_1, \ldots, u_s) \setminus (u_{j_1}, \ldots, u_{j_n})). \]
The proof of this statement is based on formulas for cumulants of asymptotically perturbed semigroups of operators (7).

If \( b^0 \in C_\gamma \), then the sequence \( b(t) = (b_0, b_1(t), \ldots, b_s(t), \ldots) \) of limit marginal observables (9) is generalized global in time solution of the Cauchy problem of the dual Vlasov hierarchy for entities:

\[
\frac{\partial}{\partial t} b_s(t) = \sum_{j=1}^s \Lambda^{[1]}(j) b_s(t) + \sum_{j_1 \neq j_2 = 1}^s \Lambda^{[2]}(j_1, j_2) b_{s-1}(t, u_1, \ldots, u_{j_2-1}, u_{j_2+1}, \ldots, u_s), \quad (10)
\]

\[
b_s(t, u_1, \ldots, u_s) \mid_{t=0} = b^0_s(u_1, \ldots, u_s), \quad s \geq 1,
\]

where in recurrence evolution equations (10) the operators \( \Lambda^{[1]}(j) \) and \( \Lambda^{[2]}(j_1, j_2) \) are defined by formula (1).

### 4.2 The relationships of kinetic equations for marginal observables and states

We consider initial states of statistically independent entities specified by a one-particle marginal distribution function, namely \( f^{(c)} \equiv (1, f^0_1(u_1), \ldots, \prod_{i=1}^s f^0_i(u_i), \ldots). \) Such states are intrinsic for the kinetic description of many-entity systems [12, 14].

If \( b(t) \in C_\gamma \) and \( f^0_i \in L^1(\mathcal{J} \times U) \), then under the condition that: \( \|f^0_i\|_{L^1(\mathcal{J} \times U)} < \gamma \), there exists the mean field scaling limit of the mean value functional of marginal observables and it is determined by the following series expansion:

\[
\langle b(t) \mid f^{(c)} \rangle = \sum_{s=0}^\infty \frac{1}{s!} \int_{(\mathcal{J} \times U)^s} du_1 \ldots du_s b_s(t, u_1, \ldots, u_s) \prod_{i=1}^s f^0_i(u_i).
\]

Then for the mean value functionals of the limit additive-type marginal observables the following representation is true:

\[
\langle b^{(1)}(t) \mid f^{(c)} \rangle = \sum_{s=0}^\infty \frac{1}{s!} \int_{(\mathcal{J} \times U)^s} du_1 \ldots du_s b^{(1)}_s(t, u_1, \ldots, u_s) \prod_{i=1}^s f^0_i(u_i) = \int_{(\mathcal{J} \times U)} du_1 b^{(1)}_1(0, u_1) f_1(t, u_1).
\]

In equality (12) the function \( b^{(1)}_s(t) \) is given by a special case of expansion (9), namely

\[
b^{(1)}_s(t, u_1, \ldots, u_s) =
\]

\[
= \int_0^t dt_1 \ldots \int_{0}^{t_{s-2}} dt_{s-1} e^{(t-t_1) \sum_{k_1=1}^s \Lambda^{[1]}(k_1)} \sum_{i_1 \neq j_1 = 1}^s \Lambda^{[2]}(i_1, j_1) e^{(t_1-t_2) \sum_{i_2=1, i_1 \neq j_1}^s \Lambda^{[1]}(i_2)} \ldots \sum_{i_{s-1} \neq j_{s-1} = 1, i_{s-1} \neq j_{s-2} = 1}^s \Lambda^{[2]}(i_{s-1}, j_{s-1}) \times e^{(t_{s-2} - t_{s-1}) \sum_{k_{s-1}=1, k_{s-1} \neq (j_1, \ldots, j_{s-2})}^s \Lambda^{[1]}(k_{s-1})} \sum_{i_{s-1} \neq j_{s-1} = 1, i_{s-1} \neq j_{s-1} \neq (j_1, \ldots, j_{s-2})}^s \Lambda^{[2]}(i_{s-1}, j_{s-1}) \times e^{(t_{s-1} - t_{s-1}) \sum_{l_{s-1}=1, l_{s-1} \neq (j_1, \ldots, j_{s-1})}^s \Lambda^{[1]}(l_{s-1})} b^{(1)}_1(0, (u_1, \ldots, u_s) \setminus (u_{j_1}, \ldots, u_{j_{s-1}})), \quad s \geq 1,
\]

\[
\]
and the limit one-particle distribution function $f_1(t, u_1)$ is represented by the series expansion:

$$f_1(t, u_1) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \int_{(J \times U)^n} du_2 \ldots du_{n+1} e^{(t-t_1)A^*[1](1)} \Lambda^*[2](1, 2) \prod_{j_1=1}^2 e^{(t_1-t_2)A^*[1](j_1)} \ldots \prod_{j_n=1}^n e^{(t_{n-1}-t_n)A^*[1](j_n)} \sum_{i=1}^n \Lambda^*[2](i, n+1) \prod_{i=1}^{n+1} e^{t_i A^*[1](j_i)} \prod_{i=1}^{n+1} f_1^0(u_i),$$

where the operator $\Lambda^*[2]$ is defined as a particular case of the operators $\Lambda^*[m]$, $m \geq 1$:

$$\Lambda^*[m](i_1, \ldots, i_m)f_n(u_1, \ldots, u_n) = \left( \int_{J \times U} A^*[m](u_1, v, u_2, \ldots, u_{i_m}) a^*[m](v, u_2, \ldots, u_{i_m}) \times f_n(u_1, u_{i_1-1}, v, u_{i_1+1}, \ldots, u_n) dv \right).$$

For initial data $f_1^0 \in L^1(J \times U)$ limit marginal distribution function (13) is a strong solution of the Cauchy problem of the Vlasov kinetic equation for entities [10]:

$$\frac{\partial}{\partial t} f_1(t, u_1) = \Lambda^*[1](1)f_1(t, u_1) + \int_{J \times U} du_2 \Lambda^*[2](1, 2)f_1(t, u_1)f_1(t, u_2),$$

$$f_1(t, u_1)|_{t=0} = f_1^0(u_1),$$

or in case of general interactions of entities (11) the Vlasov kinetic equation takes the form

$$\frac{\partial}{\partial t} f_1(t, u_1) = \Lambda^*[1](1)f_1(t, u_1) + \sum_{n=1}^{M-1} \frac{1}{n!} \int_{(J \times U)^n} du_2 \ldots du_{n+1} \sum_{j_1 \neq \ldots \neq j_{n+1} \in \{1, \ldots, n+1\}} \Lambda^*[n+1](j_1, \ldots, j_{n+1}) \prod_{i=1}^{n+1} f_1(t, u_i).$$

Correspondingly, in the Heisenberg picture of the evolution of entities a chaos property fulfills. It follows from the equality for the mean value functionals of the limit $k$-ary marginal observables, i.e. the sequences $b^{(k)}(0) = (0, \ldots, 0, b^{(k)}_k(u_1, \ldots, u_k), 0, \ldots)$,

$$\langle b^{(k)}(t) | f^{(c)} \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{(J \times U)^s} du_1 \ldots du_s b^{(k)}_s(t, u_1, \ldots, u_s) \prod_{i=1}^s f_1^0(u_i) = \frac{1}{k!} \int_{(J \times U)^k} du_1 \ldots du_k b^{(k)}_k(0, u_1, \ldots, u_k) \prod_{i=1}^k f_1(t, u_i), \quad k \geq 2,$$

where the limit one-particle marginal distribution function $f_1(t, u_1)$ is determined by series expansion (13).

Thus, an equivalent approach to the description of the kinetic evolution of entities in terms of the Vlasov-type kinetic equation (14) is given by the dual Vlasov hierarchy (10) for the additive-type marginal observables. In case of the $k$-ary marginal observables the evolution governed by the dual Vlasov hierarchy (11) means the propagation of initial chaos (15) in terms of the $k$-particle marginal distribution functions.
5 Conclusion and outlook

We developed a new approach to the description of kinetic evolution of large number of interacting constituents (entities) of mathematical biology within the framework of the evolution of marginal observables of these systems. Such representation of the kinetic evolution seems in fact the direct mathematically fully consistent formulation modeling collective behavior of biological systems, since the notion of state is more subtle and implicit characteristic of living entities.

A mean field scaling asymptotics of non-perturbative solution (6) of the dual BBGKY hierarchy (4) for marginal observables is constructed. We established that the limit additive-type marginal observables governed by the dual Vlasov hierarchy (10) gives an equivalent approach to the description of the kinetic evolution of many entities in terms of a one-particle distribution function governed by the Vlasov kinetic equation (14). Moreover, the kinetic evolution of the \( k \)-ary marginal observables governed by the dual Vlasov hierarchy means the property of the propagation of initial chaos (15) for the \( k \)-particle marginal distribution functions within the framework of the evolution of states.

One of the advantages of suggested approach in comparison with the conventional approach of the kinetic theory [11], [12], [20] is the possibility to construct kinetic equations in scaling limits in the presence of correlations at initial time which can characterize the analogs of condensed states of systems of statistical mechanics for interacting entities or soft active matter [21].

We note that the developed approach is also related to the problem of a rigorous derivation of the non-Markovian kinetic-type equations from underlying many-entity dynamics [19] which make it possible to describe the memory effects of collective dynamics of entities modeling systems in mathematical biology.

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