A SOBOLEV ESTIMATE FOR THE ADJOINT RESTRICTION OPERATOR

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Abstract. In this note we consider the adjoint restriction estimate for hypersurface under additional regularity assumption. We obtain the optimal $H^s$-$L^q$ estimate and its mixed norm generalization. As applications we prove some weighted Strichartz estimates for the propagator $e^{it(-\Delta)^{\alpha/2}}\varphi$, $\alpha > 0$.

1. Introduction

The Fourier extension operator (the adjoint of restriction operator) $R^*$ for the sphere is defined by

$$R^*f(x) = \int_{S^d} e^{ix\cdot\xi} f(\xi) d\sigma(\xi), \quad x \in \mathbb{R}^{d+1}.$$ 

Here $S^d$ denotes the unit sphere in $\mathbb{R}^{d+1}$ and $d\sigma$ is the induced Lebesgue measure on $S^d$. The problem which is known as restriction problem for the sphere is to determine the range of $p, q$ for which

$$(1.1) \quad \|R^*f\|_{L^q(\mathbb{R}^{d+1})} \leq C\|f\|_{L^p(S^d)}$$

holds. As it can be easily seen by Knapp’s example and the asymptotic expansion of $\hat{d\sigma}$, (1.1) holds only if $q > \frac{2(d+1)}{d}$ and

$$(d + 2)/q \leq d(1 - 1/p).$$

When $d = 1$, (1.1) on the optimal range was obtained by Zygmund [28] (see [7] for earlier result due to Fefferman and Stein [7]). It has been conjectured that the necessary condition is sufficient for (1.1) in higher dimensions but it still remains open. When $p = 2$, the sharp boundedness is due to Tomas [23] and Stein [16]. Result beyond Tomas-Stein’s range was first obtained by Bourgain when $d = 2$, and further progresses were made by works of Wolff [26], Tao, Vargas and Vega [22], Tao and Vargas [21], and Tao [20]. (Also see [27, 12, 25, 13] for results regarding different types of hypersurfaces.) Most recent improvement is due to Bourgain-Guth [4], which relies on the multilinear restriction estimate in Bennett, Carbery and Tao [2]. Especially, when $p = q$ the estimate is known to be true for $p \in (56/17, \infty)$ in $\mathbb{R}^{d+1}$ (see [11] p.1265) and for $p \in (p_0(d), \infty)$ in $\mathbb{R}^{d+1}$, $d \geq 3$ where $p_0(d) = 2 + 12/(4d + 1 - k)$ if $d + 1 \equiv k \pmod{3}$, $k = -1, 0, 1$.

In this note, we consider the estimate (1.1) by taking a different perspective. Let us denote by $\mathcal{H}^s$ the $L^2$ Sobolev space of order $s$ on the sphere. The main purpose of this paper is to study the restriction estimate

$$(1.2) \quad \|R^*f\|_q \leq C\|f\|_{\mathcal{H}^s}$$

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and to find the optimal range of $s, q$ for which (1.2) holds. When $s = 0$, by the necessary condition and Tomas-Stein theorem (1.2) holds if and only if $q \geq 2(d + 2)/d$. It is natural to expect that the range of $q$ gets wider if $f$ is assumed to have an additional regularity, to say $s > 0$. However, for $q \leq 2(d + 1)/d$ this estimate is not allowed because $\hat{a} \sigma \not \in L^q$. Hence, the estimate (1.2) is of interest for $q$ satisfying $2(d + 1)/d < q < 2(d + 2)/d$. By Knapp type example again, it can be shown (see the paragraph below Theorem 1.2) that (1.2) is possible only if

$$s \geq s_q = s_q(d) =: \frac{d + 2}{q} - \frac{d}{2}.$$  

(1.3)

The estimate (1.2) for $s = s_q$ can be deduced from the sharp restriction estimate (1.1) with $(d + 2)/q = d(1 - 1/p)$ by making use of the embedding $L^p(S^d) \subset H^\alpha(S^d)$ with $\alpha \geq d/2 - d/q$ and $p \neq \infty$. Since we have the sharp restriction estimate for $d = 1$, we get (1.2) for $4 < q < 6$ with the optimal regularity. In higher dimensions it seems natural to expect that the estimate (1.2) holds whenever $2(d + 1)/d < q < 2(d + 2)/d$ and (1.3) is satisfied. As it turns out, this is indeed the case and the problem is much easier than the restriction estimate (1.1). This is mainly due to the fact that the inequality (1.2) is $L^2$-based.

The following is our first result.

**Theorem 1.1.** Let $d \geq 1$ and $q < 2(d + 2)/d$. Then (1.2) holds if and only if $s \geq s_q$ and $q > 2(d + 1)/d$.

**Mixed norm generalization.** In what follows we consider a more general class of operator. Let $\phi$ be a smooth function. Then we define an extension operator $E$ by

$$Ef(x,t) = E_\phi f(x,t) = \int e^{i(x,\xi + t\phi(\xi))} a(\xi) f(\xi) d\xi, \ (x,t) \in \mathbb{R}^d \times \mathbb{R},$$

where $a$ is a smooth function supported in $B(0, 1)$. We denote by $B(z, r)$ the ball of radius $r$ which is centered at $z$. Generalizing (1.2) we consider the estimate

$$\|Ef\|_{L^q_x(L^r_t(\mathbb{R}^d))} \leq C\|f\|_{H^s}.$$  

(1.4)

Here $H^s$ is the usual inhomogeneous Sobolev space of order $s$. If the hessian matrix $H\phi$ of $\phi$ is nonsingular on the support of $a$, then by the well known Strichartz estimate (1.4) holds for $s = 0$ and $q, r$ satisfying $2 \leq q, r \leq \infty$, $d/r + 2/q \leq d/2$, $(q, r, d) \neq (2, \infty, 2)$ (see (1.1)). Let us set

$$s_c = s_c(r, q, d) = \frac{d}{r} + \frac{2}{q} - \frac{d}{2}.$$  

(1.5)

In what follows we obtain a mixed norm generalization of (1.2).

**Theorem 1.2.** Suppose $\det H\phi \neq 0$ on the support of $a$. Then if $2 \leq q, r < \infty$, $d/r + 2/q > d/2$ and $d/r + 1/q < d/2$, then (1.4) holds whenever $s \geq s_c$. If $d/r + 1/q = d/2$ and $q \neq 2$, then the weak type estimate $\|Ef\|_{L^q_x(L^{\infty}_t(\mathbb{R}^d))} \leq C\|f\|_{H^s}$ holds.

Theorem 1.1 is an obvious consequence of Theorem 1.2 as it can be shown by making use of proper coordinates patches (for example see [19, Sec. 3, Ch. 4]). It is easy to see (1.4) is no longer true if $d/r + 1/q \geq d/2$ because $E_\phi(1) \not \in L^q_xL^r_t$ as it can be shown easily by making use of Lemma 2.1. And the regularity condition is also optimal since (1.4) fails if $s < s_c$. Indeed, let us take $f(\xi) = \lambda^{s_c} \eta(\lambda \xi)$ with...
a smooth compactly supported $\eta$ so that $\|f\|_{H^s} \lesssim \lambda^s$. Then it is easy to see that $|Ef(x,t)| \gtrsim \lambda^{-\frac{d}{2}}$ if $|t| \leq c\lambda^2$ and $|x + \nabla\phi(0)t| \leq c\lambda$ for a small $c > 0$. Hence (1.4) implies
\[ \lambda^{-\frac{d}{2}} \lambda^{\frac{d}{q} + \frac{2}{q}} \lesssim \lambda^s. \]
Letting $\lambda \to \infty$ we see the condition (1.3).

Remark 1. Since $E$ is a localized operator, in Theorem 1.2 we may replace $H^s$ by the homogeneous Sobolev space $\dot{H}^s$ provided that $0 < s < d/2$. The same is also true for Theorem 1.1. Indeed, it is easy to see that for $0 < s < d/2$
\[ \|f\beta\|_{H^s} \lesssim \|f\|_{\dot{H}^s}, \]
where $\beta \in C_0^\infty(\mathbb{R}^d)$.

Application to Strichartz estimates. The adjoint restriction estimates are closely related to the Strichartz estimates. Especially, various space time estimates for the Schrödinger operator can be deduced from restriction estimate for the paraboloid. (See [14] for more detailed discussion.) As applications of Theorem 1.2 we obtain some weighted Strichartz estimates on the range where the usual Strichartz estimates (See [14] for more detailed discussion.) As applications of Theorem 1.2 we obtain some weighted Strichartz estimates on the range where the usual Strichartz estimates are not allowed.

Let $\alpha > 0$ and $\alpha \neq 1$. We first consider the weighted Strichartz estimates for the fractional Schrödinger operator
\[ (\ref{eq:1.6}) \]
\[ \|e^{i((t-x)^2/2)}\varphi\|_{L^2_t(L^\alpha_x)} \lesssim \|x|^{-\mu} \|(-\Delta)^{\nu/2}\varphi\|_2. \]

Corollary 1.3. Let $d \geq 1$, $\alpha > 0$, $\alpha \neq 1$ and $2 \leq q, r < \infty$. If $d/r + 2/q > d/2$ and $d/r + 1/q < d/2$, then (1.6) holds provided that $\mu = s_c$ and $\nu = \frac{2-\alpha}{q}$.

We now consider the case $\alpha = 1$, to say the wave operator. Let us set $s_c^w := s_c(q, r, d - 1)$.

Corollary 1.4. Let $d \geq 2$, $\alpha = 1$ and $2 \leq q, r < \infty$. If $(d-1)/r + 2/q > (d-1)/2$ and $(d-1)/r + 1/q < (d-1)/2$, then (1.6) holds provided that $\mu = s_c^w$ and $\nu = \frac{1}{2} + \frac{1}{q} - \frac{1}{r}$.

Let $\Delta_\omega$ be the Laplace-Beltrami operator on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ which is given by $\Delta_\omega = \sum_{1 \leq i < j \leq d} \Omega_{i,j}^2$, $\Omega_{i,j} = \omega_i \partial_j - \omega_j \partial_i$. Then define a Sobolev norm $\|f\|_{H_{sph}^\nu}$ by setting
\[ \|f\|_{H_{sph}^\nu}^2 = \int_0^\infty \int_{S^{d-1}} |(1 - \Delta_\omega)^{\nu/2} f(r\omega)|^2 d\sigma_\omega r^{d-1} dr. \]
We now consider the estimate
\[ (\ref{eq:1.7}) \]
\[ \|(-\Delta)^{\gamma/2} e^{it\sqrt{-\Delta}} \varphi\|_{L^2_t(L^\alpha_x)} \lesssim C \|\varphi\|_{H_{sph}^\nu}. \]
This type of inequality was studied by Sterbenz [18] to extend the range of admissible $q, r$ by making use of angular regularity (also see [6, 5, 9] for related results and references therein). It is known (see [18]) that this estimate holds only if
\[ \nu \geq s_c^w, \quad \frac{d-1}{r} + \frac{1}{q} < \frac{d-1}{2}. \]
Sterbenz [18] showed that (1.7) holds provided $\nu > s_c^w$. Our result enables us to obtain the endpoint regularity estimate when $q \geq r$. 

\[ \text{Corollary 1.3.} \]

\[ \|f\beta\|_{H^s} \lesssim \|f\|_{\dot{H}^s}, \]

\[ \text{Corollary 1.4.} \]

\[ \|e^{i((t-x)^2/2)}\varphi\|_{L^2_t(L^\alpha_x)} \lesssim \|x|^{-\mu} \|(-\Delta)^{\nu/2}\varphi\|_2. \]

\[ \text{Remark 1.} \]

\[ \text{Application to Strichartz estimates.} \]

\[ \text{Let $\Delta_\omega$ be the Laplace-Beltrami operator on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ which is given by $\Delta_\omega = \sum_{1 \leq i < j \leq d} \Omega_{i,j}^2$, $\Omega_{i,j} = \omega_i \partial_j - \omega_j \partial_i$. Then define a Sobolev norm $\|f\|_{H_{sph}^\nu}$ by setting} \]

\[ \|f\|_{H_{sph}^\nu}^2 = \int_0^\infty \int_{S^{d-1}} |(1 - \Delta_\omega)^{\nu/2} f(r\omega)|^2 d\sigma_\omega r^{d-1} dr. \]

\[ \text{We now consider the estimate} \]

\[ \|(-\Delta)^{\gamma/2} e^{it\sqrt{-\Delta}} \varphi\|_{L^2_t(L^\alpha_x)} \lesssim C \|\varphi\|_{H_{sph}^\nu}. \]

\[ \text{This type of inequality was studied by Sterbenz [18] to extend the range of admissible $q, r$ by making use of angular regularity (also see [6, 5, 9] for related results and references therein). It is known (see [18]) that this estimate holds only if} \]

\[ \nu \geq s_c^w, \quad \frac{d-1}{r} + \frac{1}{q} < \frac{d-1}{2}. \]

\[ \text{Sterbenz [18] showed that (1.7) holds provided $\nu > s_c^w$. Our result enables us to obtain the endpoint regularity estimate when $q \geq r$.} \]
Corollary 1.5. Let \( d \geq 2 \) and \( 2 \leq q, r < \infty \). If \( q \geq r, (d-1)/r + 2/q > (d-1)/2 \) and \( (d-1)/r + 1/q < (d-1)/2 \), then (1.7) holds provided that \( \nu \geq s_c^u \).

In addition to \( \wedge \) and \( \vee \), we use \( \mathcal{F}, \mathcal{F}^{-1} \) to denote Fourier, inverse Fourier transforms, respectively. The paper is organized as follows. In section 2 we provide a few preliminaries for the proof of Theorem 1.2. In section 3 we prove Theorem 1.2 and the proofs of Corollaries 1.3–1.5 are given in Section 4.

2. Preliminaries

For the proof of the estimate (1.4) we may assume that \( \phi \) is close to a quadratic form. More precisely, let \( \phi \) be a smooth function satisfying that \( \det H\phi \) is nonsingular. Then we may assume that for a sufficiently small \( \epsilon_0 > 0 \)

\[
(\epsilon_1)
\phi(\xi) = \frac{1}{2} \xi^t M \xi + E(\xi)
\]

and \( \|E\|_{C^\ell(B(0,2))} \leq C\epsilon_0 \) for a sufficiently large positive integer \( \ell \) where \( M \) is the diagonal matrix with nonzero entries \( \pm 1 \).

Parabolic rescaling. Indeed, let \( \xi_0 \) be a point in \( B(0,1) \). By decomposing \( a \) into finite number of smooth functions which are supported in small balls we only need to consider individually the localized operator

\[
\int e^{i(x\xi + t\phi(\xi))} a_{\xi_0, \epsilon_0}(\xi) f(\xi) d\xi,
\]

where \( a_{\xi_0, \epsilon_0} \) is smooth function supported in \( B(\xi_0, \epsilon_0) \). By Taylor expansion we have

\[
\phi(\xi) = \phi(\xi_0) + \nabla \phi(\xi_0) \cdot (\xi - \xi_0) + \frac{1}{2}(\xi - \xi_0)^t H\phi(\xi_0)(\xi - \xi_0) + O(|\xi - \xi_0|^3).
\]

By discarding harmless factors, translation \( \xi \to \xi + \xi_0 \) and the linear change of variables \( (x, t) \to (x + t\nabla \phi(\xi_0), t) \) we may assume

\[
\phi(\xi) = \frac{1}{2} \xi^t H\phi(\xi_0) \xi + O(|\xi|^3)
\]

and then making a linear change of variables for both \( x \) and \( \xi \) we may further simplify \( \frac{1}{2} \xi^t H\phi(\xi_0) \xi \) to the form \( \frac{1}{2}(\xi_1^2 \pm \xi_2^2 \pm \cdots \pm \xi_d^2) = \frac{1}{2} \xi^t M \xi \) (diagonalization and rescaling). These operations essentially don’t effect the estimate (1.4) except changing the constant \( C \). Now one can make the effect of error term small by further scaling

\[
\xi \to \epsilon_0 \xi, \quad (x, t) \to (\epsilon_0^{-1} x, \epsilon_0^{-2} t)
\]

which changes \( x \cdot \xi + t(\frac{1}{2} \xi^t M \xi + O(|\xi|^3)) \) to \( x \cdot \xi + t(\frac{1}{2} \xi^t M \xi + O(\epsilon_0 |\xi|^3)) \). Hence we get (2.1).

Asymptotic of oscillatory integral. From the above assumption (2.1) \( \nabla \phi(\xi) \) is close to \( M \xi \). Hence, with sufficiently small \( \epsilon_0 \) we may assume that \( \xi \to \nabla \phi(\xi) \) is a diffeomorphism on \( B(0,2) \) such that there is a unique smooth function \( \eta : B(0,15/8) \to \mathbb{R}^d \) such that

\[
\nabla \phi(\eta(x)) = -x.
\]

Then we define

\[
(2.2) \quad \psi(x) = x \cdot \eta(x) + \phi(\eta(x)).
\]
By Fourier inversion we write
\[ F = \beta \parallel \text{EP} \parallel \]
\[ (3.1) \]
\[ d/r \]
It is easy to handle \[ (3.2) \]
\[ Ef \]
Since \[ |\nabla_\xi(\xi \cdot x + t\phi(\xi))| \geq |x| \text{ if } |x| \geq 5t/4 \] (here we are assuming \( 0 < \epsilon_0 \ll 1 \), by routine integration by parts we see that for any \( M > 0 \)
\[ (2.3) \]
\[ \int e^{i(x-\xi + t\phi(\xi))}a(\xi)d\xi \lesssim (1 + |x|)^{-M}(1 + |t|)^{-M}. \]
For the other case we need the following which can be shown by the stationary phase method. It is a special case of Theorem 7.7.6 in Hörmander [10] (see p.222).

**Lemma 2.1.** Suppose that \( \phi \) is given by \[ (2.1) \] and \( \text{supp } a \subset B(0, 1) \). Then if \( |t| \geq 1 \) and \( |x| \lesssim t \), for every positive integer \( N \)
\[ \int e^{i(x-\xi + t\phi(\xi))}a(\xi)d\xi = \sum_{l=0}^{N} t^{-\frac{d}{2}-l}e^{it\psi(\xi)}a_l\left(\frac{T}{t}\right) + O(|t|^{-N-\frac{d}{2}-1}), \]
where \( a_l \) is a bounded smooth function with compact support.

### 3. Proof of Theorem 1.2

To begin with we assume that the operator \( E \) is defined by \( \phi \) which satisfies \[ (2.1) \] with a small \( \epsilon_0 > 0 \) and a large \( \ell \). By time reversal symmetry it is sufficient to show
\[ \|E(f)\|_{L^q_t((0,\infty), L^r(x))} \lesssim \|f\|_{H^s}. \]
From the Strichartz estimate and Plancherel’s theorem we recall the estimate
\[ \|E(f)\|_{L^q_t((0,T), L^r(x))} \lesssim \|f\|_2 \]
which holds for \( T > 0 \) and \( q, r \) which satisfy \( d/r+2/q = d/2 \), \((r,q,d) \neq (\infty,2,2)\), and by Plancherel’s theorem and Hölder’s inequality we also have \( \|Ef\|_{L^q_t((0,T), L^r(x))} \lesssim T^{\frac{1}{q}}\|f\|_2 \) for \( q \geq 1 \). Interpolation between these two estimates gives
\[ (3.1) \]
\[ \|E(f)\|_{L^q_t((0,T), L^r(x))} \lesssim T^{\frac{1}{q}}\left(\frac{q}{2} + \frac{q}{2} - 2\right)\|f\|_2 \]
for \( q, r \geq 2 \) satisfying \( d/r+2/q \geq d/2 \). Note that \( s_c > 0 \) if \( d/r + 2/q > d/2 \) and \( d/r + 1/q < d/2 \). Hence, we obviously need only to show that
\[ \|E(f)\|_{L^q_t(1,\infty), L^r(x)} \lesssim \|f\|_{H^s}. \]
From now on we assume that \( t \geq 1 \).

Let \( \beta \in C_c^\infty(1/2,2) \) such that \( \sum_{k=0}^{\infty} \beta(2^{-k}t) = 1 \) for \( t > 0 \) and let us denote by \( P_k \)
the Littlewood-Paley projection operator which is given by \( F(P_k f) = \beta(2^{-k} \cdot )f \).
And we also set \( \beta_0 = 1 - \sum_{1}^{\infty} \beta(2^{-k}t) \) and define \( P_{\leq 0} \) by \( F(P_{\leq 0} f) = \beta_0(|\cdot |)f \).

Using the projection operators we decompose \( Ef \) so that
\[ (3.2) \]
\[ Ef = EP_{\leq 0} f + \sum_{k=1}^{\infty} EP_k f. \]
It is easy to handle \( EP_{\leq 0} f \). Let us set
\[ K(x,t) = \int e^{i(x-\xi + t\phi(\xi))}a(\xi)d\xi. \]
By Fourier inversion we write
\[ EP_{\leq 0} f(x,t) = \int K(x - y, t)\beta_0(|y|) f^\vee(y)dy. \]
Then by \((2.3)\) and Lemma 2.1, \(|K(x, t)| \leq |t|^{-\frac{d}{2}} \chi_{B(0, \frac{\tau}{2})}(\frac{x}{t}) + (1 + |x|)^{-M}|t|^{-M}\) for \(N > 0\). Hence it follows that
\[
|EP_0 f(x, t)| \leq C \int \left(|t|^{-\frac{d}{2}} \chi_{B(0, \frac{\tau}{2})}(\frac{x - y}{t}) + (1 + |x - y|)^{-M}|t|^{-M}\right) |\beta_0(|y|)| f'(y) |dy.
\]
Since \(\beta_0(|\cdot|)\) is supported in \(B(0, 2)\), by Cauchy-Schwarz inequality and Plancherel’s theorem we see that
\[
|EP_{\leq 0} f(x, t)| \leq C |t|^{-\frac{d}{2}} (\chi_{\{|x| \leq \frac{\tau}{2}|t| + 2\}} + (1 + |x|)^{-M}) \|\beta_0 f'|_1
\]
\[
\leq C |t|^{-\frac{d}{2}} (\chi_{\{|x| \leq \frac{\tau}{2}|t| + 2\}} + (1 + |x|)^{-M}) \|f\|_2.
\]
So by taking integration it follows that
\[
\|EP_{\leq 0} f\|_{L^q_t(L^r_x)} \leq C |t|^{-\frac{d}{2} + \frac{d}{q} - \frac{d}{2}} \|f\|_{L^q_t(L^r_x)} \|f\|_2 \leq C \|f\|_2
\]
because \(d/r + 1/q < d/2\).

For \(k \geq 1\) and a large constant \(C > 0\), we set
\[
\chi^\circ_k(t) = \chi_{[1, C2^k]}(t), \quad \chi^c_k(t) = \chi_{[C2^k, \infty)}(t).
\]
We break the sum in \((3.2)\) such that
\[
\sum_{k=1}^\infty EP_k f = \sum_{k=1}^\infty \chi^\circ_k EP_k f + \sum_{k=1}^\infty \chi^c_k EP_k f.
\]
The contribution from the first summation is easy to handle. In fact, since \(\frac{d}{r} + \frac{d}{q} - \frac{d}{2} > 0\), using \((3.1)\)
\[
\left\| \sum_{k=1}^\infty \chi^\circ_k EP_k f \right\|_{L^q_t L^r_x} \leq \sum_{k=1}^\infty \left\| \chi^\circ_k EP_k f \right\|_{L^q_t L^r_x} \leq C \sum_{k=1}^\infty 2^{k\left(\frac{d}{r} + \frac{d}{q} - \frac{d}{2}\right)} \|P_k f\|_{L^2}
\]
\[
\leq C \left( \sum_{k=1}^\infty 2^{-k\left(\frac{d}{r} + \frac{d}{q} - \frac{d}{2}\right)} \right) \left( \sum_{k=1}^\infty 2^{k\left(\frac{d}{r} + \frac{d}{q} - \frac{d}{2}\right)} \|P_k f\|^2_{L^2} \right) \leq C \|f\|_{H^\infty}.
\]
Hence we are reduced to showing
\[
(3.4) \quad \left\| \sum_{k=1}^\infty \chi^c_k EP_k f \right\|_{L^q_t L^r_x} \leq C \|f\|_{H^\infty}.
\]

We now use the asymptotic expansion in Lemma 2.1. Let \(A\) be a smooth function supported in \([-3/2, 3/2]\) such that \(A = 1\) on \([-5/4, 5/4]\). Then we use Lemma 2.1 for \(K(x, t)A(x/t)\) and \((2.3)\) for \(K(x, t)(1 - A(x/t))\) to get
\[
K(x, t) = \sum_{l=0}^N t^{-\frac{d}{2} - l} e^{i\psi(l)} A_l(x/t) + e(x, t)
\]
where \(e(x, t) = O((1 + |x| + |t|)^{-N - \frac{d}{2} - 1})\) and \(A_l\) is a smooth function supported in \(B(0, 3/2)\). For simplicity we set
\[
A_l(x, t) = \sum_{l=0}^N t^{-l} A_l(x/t).
\]
We now define
\[ \widetilde{E}f(x, t) = t^{-\frac{d}{2}} \int e^{it\psi(x/t)} \widetilde{A}(\frac{x - y}{t}) f^\vee(y)dy, \quad \mathcal{R}f(x, t) = \int e(x - y, t) f^\vee(y)dy. \]

Since \( E f = \int K(x - y, t)f^\vee(y)dy \), clearly \( E f = \widetilde{E}f + \mathcal{R}f \). Hence, the left hand side of (3.5) is bounded by
\[ \left\| \sum_{k=1}^{\infty} \chi_k^\vee \widetilde{E}P_k f \right\|_{L^1_x L^\infty_y} + \left\| \sum_{k=1}^{\infty} \chi_k^\vee \mathcal{R}P_k f \right\|_{L^1_x L^\infty_y}. \]

The contribution from \( \sum_{k=1}^{\infty} \chi_k^\vee \mathcal{R}P_k f \) is easy to control. In fact, since \( q \geq 2 \), with sufficiently large \( N \) (using \( c(x, t) = O((1 + |x| + |t|-N^{-1}) \)) and Young’s inequality we see that for \( s > 0 \)
\[ \left\| \sum_{k=1}^{\infty} \chi_k^\vee \mathcal{R}P_k f \right\|_{L^1_x L^\infty_y} \leq C \sum_{k=1}^{\infty} \left\| \chi_k^\vee \mathcal{R}P_k f \right\|_{L^1_x L^\infty_y} \leq C \sum_{k=1}^{\infty} \| \beta(2^{-k}|\cdot|) f^\vee \|_{L^1_x L^\infty_y} \leq C \| f \|_{H^s}. \]

For the last inequality we use Cauchy-Schwarz inequality and Plancherel’s theorem. Hence, to get the desired bounds, by multiplying harmless factor \( e^{-it\psi(x/t)} \) it is sufficient to consider the operator which is defined by
\[ \widetilde{E}_\psi P_k f(x, t) = e^{-it\psi(x/t)} \widetilde{E}P_k f(x, t). \]

Now, for (3.4) it is sufficient to show that
\[ \left\| \sum_{k=1}^{\infty} \chi_k^\vee(t) \widetilde{E}_\psi P_k f \right\|_{L^1_x L^\infty_y} \leq C \| f \|_{H^s}. \]

Let us set
\[ m(k, y, \xi) = 2^{kd} \int e^{i(t\psi(x - \frac{2^k}{t} - \psi(x) - x)\xi)} \widetilde{A}(x - \frac{2^k y}{t})dx. \]

Then by scaling \( y \to 2^k y \) we get
\[ \mathcal{F}_x(\widetilde{E}_\psi P_k f(t, \cdot))(\xi) = t^{-\frac{d}{2}} \int m(k, y, \xi) \beta(y) f^\vee(2^k y)dy. \]

Here \( \mathcal{F}_x \) denotes the Fourier transform in \( x \). In order to get (3.5) we need the following lemma which shows if \( t \gg 2^k \) the Fourier transform of \( \widetilde{E}_\psi P_k f(t, t) \) is essentially supported in the set \( \{ \xi : |\xi| \sim 2^k \} \).

**Lemma 3.1.** Let \( 1/2 \leq |y| \leq 2 \). If \( |\xi| \geq B2^k \) or \( |\xi| \leq B^{-1/2} \) for some large \( B > 0 \), then for any \( M > 0 \)
\[ |\partial^2_\xi m(k, y, \xi)| \leq C(\max\{2^k, |\xi|\})^{-M} \]
with \( C \) independent of \( k, y \).
Proof. To see this, we consider the phase function of the integral in (3.6)

$$ t\psi(x - \frac{2ky}{t}) - t\psi(x) - x \cdot \xi. $$

From (2.2) and (2.1), we have $\nabla \psi(x) = \eta(x) = Mx + E(x)$ where $\|E\|_{C^1(B(0,2))} \lesssim \epsilon_0$. Hence the Hessian matrix of $\psi$ is close to the matrix $M$. Since $|y| \sim 1$ and $\frac{2k}{t} \ll 1$, it is easy to see

$$ |\nabla \psi(x - \frac{2ky}{t}) - \nabla \psi(x)| = \left| M\frac{2ky}{t} + O(\epsilon_0\frac{2ky}{t}) \right| \sim \frac{2k}{t}. $$

Since $|\xi| \geq B2^k$ or $|\xi| \leq B^{-1}2^k$ for some large $B > 0$, we get

$$ |\nabla_x(t\psi(x - \frac{2ky}{t}) - t\psi(x) - x \cdot \xi)| \gtrsim \max(2^k, |\xi|). $$

Note that $\tilde{A}(\cdot - \frac{2ky}{t})$ is supported in $B(0, 7/4)$ with large enough $B$, by integration by parts we get desired inequality. \hfill \square

Now we break

$$ \tilde{E}_\psi P_k f(t\cdot, t) = (I - \tilde{P}_k)\tilde{E}_\psi P_k f(t\cdot, t) + \tilde{P}_k\tilde{E}_\psi P_k f(t\cdot, t), $$

where $\tilde{P}_k$ is a projection operator defined by $\mathcal{F}(\tilde{P}_k f) = \tilde{\beta}(2^{-k} |\xi|)\tilde{f}(\xi)$ with $\tilde{\beta} \in C_0^\infty(1/2B, 2B)$ satisfying $\tilde{\beta} = 1$ on $(B^{-1}, B)$. By integration by parts with (3.3), it follows that if $1/2 \leq |y| \leq 2$

$$ |\mathcal{F}^{-1}\left((1 - \tilde{\beta}(2^{-k} \cdot |\eta|)m(k, y, \cdot)\right)| \leq C2^{-Mk}(1 + |x|)^{-M}. $$

Hence, by (3.5) we get

$$ |(I - \tilde{P}_k)\tilde{E}_\psi P_k f(tx, t)| \leq Ct^{-\frac{d}{2}}2^{-Mk}(1 + |x|)^{-M} \int |\beta(y)f^\vee(2ky)|dy. $$

Then, by Schwarz’s inequality and Plancherel’s theorem, $|(I - \tilde{P}_k)\tilde{E}_\psi P_k f(tx, t)| \leq Ct^{-\frac{d}{2}}2^{-Mk}(1 + |x|)^{-M}\|P_k f\|_2$. Hence

$$ \|\chi_k^c(t)(I - \tilde{P}_k)\tilde{E}_\psi P_k f(t\cdot, t)\|_{L_x^s} \leq Ct^{-\frac{d}{2}}2^{-Mk}\|P_k f\|_2. $$

Using this and Littlewood-Paley inequality, we now see that

$$ \left\| \sum_{k=1}^{\infty} \chi_k^c(t)\tilde{E}_\psi P_k f(t\cdot, t) \right\|_{L_x^s} \leq t^\frac{d}{2} \left\| \sum_{k=1}^{\infty} \chi_k^c P_k f(t\cdot, t) \right\|_{L_x^s} + \frac{t^d}{2} \left\| \sum_{k=1}^{\infty} \chi_k^c(I - \tilde{P}_k)\tilde{E}_\psi P_k f(t\cdot, t) \right\|_{L_x^s} $$

$$ \leq Ct^\frac{d}{2} \chi_k^c(t) \left( \sum_{k=1}^{\infty} \|\tilde{E}_\psi P_k f(t\cdot, t)\|_{L_x^s}^2 \right)^{\frac{1}{2}} + Ct^\frac{d}{2} \sum_{k=1}^{\infty} 2^{-Mk}\|f\|_2.$$
Since \( q \geq 2 \), taking integration in \( t \), by Minkowski’s inequality we arrive at
\[
\left\| \sum_{k=1}^{\infty} \chi_k^c \tilde{E}_\psi P_k f(\cdot, t) \right\|_{L^q L^r_x} \leq C \left( \sum_{k=1}^{\infty} \left\| t^\beta \chi_k^c \tilde{E}_\psi P_k f(\cdot, t) \right\|_{L^q_x}^2 \right)^{\frac{1}{2}} + \sum_{k=1}^{\infty} 2^{-Mk} \| P_k f \|_2 \| t^{\frac{d}{2} - \frac{d}{q}} \|_q.
\]

The second term in the right hand side is clearly bounded by \( C \| f \|_{H^s} \) if \( d/r + 1/q < d/2 \). Therefore we are reduced to showing that
\[
\sum_{k=1}^{\infty} \left\| t^\beta \chi_k^c \tilde{E}_\psi P_k f(\cdot, t) \right\|_{L^q_x}^2 \leq C \| f \|_{H^s}.
\]

For this it is sufficient to show the following.

**Lemma 3.2.** If \( |t| \geq B 2^k \) for some large \( B > 0 \), then for \( 1 \leq r \leq \infty \)
\[
\| \tilde{E}_\psi P_k f(\cdot, t) \|_{L^r_x} \leq C t^{-\frac{d}{4}} \| P_k f \|_r.
\]

**Proof.** Since \( (P_k f)^\vee \) is supported in \( \{ y : 2^{k-1} \leq |y| \leq 2^{k+1} \} \), we may put in a harmless smooth function \( \beta_0 \) so that
\[
\tilde{E}_\psi P_k f(x, t) = e^{-it\psi(x)} t^{-\frac{d}{4}} \int e^{it\psi(x-y)} \tilde{A}(x-y) \beta_0(y) (2^{-k}|y|)(P_k f)^\vee(y) dy,
\]
where \( \beta = \beta_0 \beta \) and \( \beta_0 \) is supported in \( [2^{-2}, 2^2] \). By rescaling we have
\[
\tilde{E}_\psi P_k f(tx, t) = t^{-\frac{d}{4}} \int \mathcal{K}(x, z, k) P_k f(z) dz,
\]
where
\[
\mathcal{K}(x, z, k) = 2^k e^{-it\psi(x)} \int e^{it\psi(x-2^k \frac{y}{t})} e^{i2^k z y} \tilde{A}(x - 2^k \frac{y}{t}) \beta_0(y) dy.
\]

Since \( |y| \sim 1 \) and \( t \geq B 2^k \), considering the phase part of this integral we see that
\[
\nabla_y \left( t \psi(x - 2^k \frac{y}{t}) + 2^k z y \right) = 2^k (z - \nabla \psi(x)) + O(1). \]
Therefore, by integration by parts we get
\[
|\mathcal{K}(x, z, k)| \leq C 2^{kd} (1 + 2^k |z - \nabla \psi(x)|)^{-N}.
\]

Since \( \tilde{A} \) is supported in \( B(0, 3/2) \), it follows that \( \text{supp} \mathcal{K}(\cdot, z, k) \subset B(0, 7/4) \) if \( B \) is sufficiently large. From \( (2.2) \) \( x \to \nabla \psi(x) \) is a diffeomorphism on \( B(0, 15/8) \). Hence it is easy to see \( \int |\mathcal{K}(x, z, k)| dx < C \). Clearly, \( \int |\mathcal{K}(x, z, k)| dz \leq C \). Then (3.11) follows from Young’s inequality.

We now return to the proof of (3.10). We break \( \chi_k^c \) to \( \chi_{[C 2^k, C 2^k]} + \chi_{[C 2^{2k}, \infty)} \) so that
\[
\left( \sum_{k=1}^{\infty} \left\| t^\beta \chi_k^c \tilde{E}_\psi P_k f(\cdot, t) \right\|_{L^q_x}^2 \right)^{\frac{1}{2}} \leq \left( \sum_{k=1}^{\infty} \left\| t^\beta \chi_{[C 2^k, C 2^k]}(t) \tilde{E}_\psi P_k f(\cdot, t) \right\|_{L^q_x}^2 \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} \left\| t^\beta \chi_{[C 2^{2k}, \infty)} \tilde{E}_\psi P_k f(\cdot, t) \right\|_{L^q_x}^2 \right)^{\frac{1}{2}}.
\]
By rescaling we note that the first of right hand side equals
\[\left( \sum_{k=1}^{\infty} \| \widetilde{E}_k P_k f \|_{L^q_t([C2^k, C2^k], L^q_x(R^d))}^2 \right)^{\frac{1}{2}}.\]

By (3.1) it is bounded by \(C \left( \sum_{k=1}^{\infty} 2^{k(d+\frac{d}{q} - \frac{d}{2})} \| P_k f \|_{L^q_t}^2 \right)^{\frac{1}{2}} \lesssim \| f \|_{H^{s_c}}. \)

So, we only need to consider the second term. By making use of (3.11) we see
\[\left( \sum_{k=1}^{\infty} \| \widetilde{E}_k P_k f(t, t) \|_{L_x^2}^2 \right)^{\frac{1}{2}} \leq C \left( \sum_{k=1}^{\infty} \| \tilde{\psi} P_k f(t, t) \|_{L_x^2}^2 \right)^{\frac{1}{2}} \leq C \| f \|_{H^{s_c}}.\]

For the last inequality we use Bernstein’s inequality \(\| P_k f \|_q \leq C 2^{k(d - \frac{d}{q})} \| P_k f \|_2.\) This completes the proof of (3.10).

**Proof of weak type endpoint estimate.** Let us fix \(q, r\) such that \(2 \leq q, r < \infty\) and \(d/r + 1/q = d/2.\) The proof here is a minor modification of that of Theorem 1.2. So we shall be brief.

As before it suffices to show \(\| E(f) \|_{L^q_t(0, \infty), L_x^r(R^d)} \lesssim \| f \|_{H^{s_c}}.\) Because of (3.1) it is enough to prove that
\[\| E(f) \|_{L^q_t(0, \infty), L_x^r(R^d)} \lesssim \| f \|_{H^{s_c}}.\]

By (3.3) it follows that
\[\| E_{P_k} f \|_{L^q_t(0, \infty), L_x^r(R^d)} \leq C \| \tilde{\psi} \|_{L^q_t(0, \infty)} \| f \|_2 \leq C \| f \|_2\]
because \(d/r + 1/q = d/2.\) Breaking \(\sum_{k=1}^{\infty} E_{P_k} f = \sum_{k=1}^{\infty} \chi_k^c E_{P_k} f + \sum_{k=1}^{\infty} \chi_k^c R_{P_k} f,\)
for the first sum we get the desired bound by the same argument as before because \(L^q \subset L^q_{t,x}.\) Hence, we reduce to show
\[(3.12) \quad \left\| \sum_{k=1}^{\infty} \chi_k^c E_{P_k} f \right\|_{L^q_t(0, \infty), L_x^r(R^d)} \leq C \| f \|_{H^{s_c}}.\]

Then, decomposing further \(\chi_k^c E_{P_k} f = \chi_k^c \widetilde{E}_{P_k} f + \chi_k^c R_{P_k} f,\) the contribution from \(\chi_k^c R_{P_k} f\) is controlled by the bound obtained previously. So it is sufficient to show that
\[\left\| \sum_{k=1}^{\infty} \chi_k^c \tilde{E}_{P_k} f \right\|_{L^q_t(0, \infty), L_x^r(R^d)} \leq C \| f \|_{H^{s_c}}.\]

Since \(q > 2\) \(L^{q/2, \infty}\) is a Banach space, using Minkowski’s inequality and triangle inequality, from (3.9) we get
\[\left\| \sum_{k=1}^{\infty} \chi_k^c \tilde{E}_{P_k} f(\cdot, t) \right\|_{L^q_t(0, \infty), L_x^r(R^d)} \leq C \left( \sum_{k=1}^{\infty} \| t^{\frac{d}{2}} \chi_k \tilde{E}_{P_k} f(t, t) \|_{L_x^r}^2 \right)^{\frac{1}{2}} \]
\[+ \sum_{k=1}^{\infty} 2^{-NK} \| P_k f \|_2 \left\| t^{\frac{d}{2} - \frac{d}{q}} \right\|_{L^{q/2, \infty}}.\]
The second one on the right hand side is bounded by $\|f\|_{H^{\infty}}$ because $\frac{d}{2} - \frac{d}{2} \in L^{q,\infty}(0, \infty)$. So, we only need to show that

$$\left\| t^{\frac{d}{2}} \chi_{[C^{2k}, \infty]} \left| \mathcal{E}_{0} P_{k} f(t \cdot, t) \right|_{L^{q}} \right\|_{L^{q,\infty}} \leq C 2^{k \left( \frac{d}{4} + \frac{2}{2} - \frac{d}{2} \right)} \| P_{k} f \|_{2}.$$  

This follows from (3.11) because $\| t^{\frac{d}{2}} t^{-\frac{d}{2}} \chi_{[C^{2k}, \infty]} \|_{L^{q,\infty}} \leq C$. This completes the proof.

4. Strichartz estimates: Proofs of Corollaries

In this section we prove Corollary 1.3, 1.5.

**Proof of Corollary 1.3.** Since $q, r \geq 2$, using Littlewood-Paley theory and Minkowski’s inequality we have

$$\| e^{it(-\Delta)^{\alpha/2}} \varphi \|_{L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{d}))} \lesssim \sum_{k \in \mathbb{Z}} \| e^{it(-\Delta)^{\alpha/2}} P_{k} \varphi \|_{L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{d}))}^{2}.$$  

We now observe that $e^{it(-\Delta)^{\alpha/2}} P_{k} g(x) = 2^{dk} e^{it2^{\alpha k}(-\Delta)^{\alpha/2}} P_{0} g_{2^{k}}(x)$, where we denote $g_{\lambda}(x) = \lambda^{-d} g(x/\lambda)$. Since $0 < s_{c} < 1/2$, recalling Remark 1 from rescaling and Theorem 1.2 we get

$$\| e^{it(-\Delta)^{\alpha/2}} \varphi \|_{L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{d}))}^{2} \lesssim \sum_{k \in \mathbb{Z}} 2^{dk(d - \frac{d}{2} - \frac{\alpha}{4})} \| e^{it(-\Delta)^{\alpha/2}} P_{0} \varphi_{2^{k}} \|_{L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{d}))}^{2} \lesssim \sum_{k \in \mathbb{Z}} 2^{dk(d - \frac{d}{2} - \frac{\alpha}{4})} \| \beta \widehat{\varphi}(2^{k}) \|_{H^{s_{c}}}^{2}.$$  

By Plancherel’s theorem and rescaling (note that $P_{0} \varphi_{2^{k}} = 2^{-kd} (P_{k} \varphi)(2^{-k} x)$) it follows that

$$\| e^{it(-\Delta)^{\alpha/2}} \varphi \|_{L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{d}))}^{2} \lesssim \sum_{k \in \mathbb{Z}} 2^{2k(d - \frac{d}{2} - \frac{\alpha}{4})} \| \beta \varphi(2^{k}) \|_{H^{s_{c}}}^{2}.$$  

We define $P_{k}$ by $\mathcal{F}(P_{k} f) = \beta_{c}(2^{-k} \cdot | \cdot) \widehat{f}$ so that $P_{k} P_{k} = P_{k}$. (Here $\beta_{c}$ is a smooth function supported in $[2^{-2}, 2^{2}]$ such that $\beta_{c} \beta = \beta$.) Also we set $\tilde{P}_{k} = 2^{k \frac{2 - \alpha}{2} - \frac{d}{2} - \frac{\alpha}{4} \delta} (-\Delta)^{\frac{\alpha - 2}{2\gamma}} P_{k}$. Hence we get

$$\| e^{it(-\Delta)^{\alpha/2}} \varphi \|_{L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{d}))}^{2} \lesssim \int \left( \sum_{k \in \mathbb{Z}} | \tilde{P}_{k} P_{k} (-\Delta)^{\frac{\alpha - 2}{2\gamma}} \varphi |^{2} \right) | x |^{2s_{c}} \, dx.$$  

Since $0 < s_{c} < 1/2$, $| x |^{2s_{c}}$ is an $A_{2}$-weight (see [17, p.219]). Thus by vector valued inequality for $A_{p}$ weights (e.g. [3], Remarks 6.5, p. 521) it follows that

$$\| e^{it(-\Delta)^{\alpha/2}} \varphi \|_{L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{d}))}^{2} \lesssim \int \left( \sum_{k \in \mathbb{Z}} | P_{k} (-\Delta)^{\frac{\alpha - 2}{2\gamma}} \varphi |^{2} \right) | x |^{2s_{c}} \, dx.$$  

By the well known Littlewood-Paley theory (e.g. [15, p. 275], [21]) in weighted $L^{p}$ spaces the right hand side is bounded by

$$C \int | (-\Delta)^{\frac{\alpha - 2}{2\gamma}} \varphi |^{2} | x |^{2s_{c}} \, dx.$$
Therefore we get the desired inequality.

**Proof of Corollary 1.4** In order to prove Corollary 1.4 it is sufficient to show that
\[\|e^{it\sqrt{-\Delta}}\varphi\|_{L^q_t(L^r_x(\mathbb{R}^d))} \leq C\|x^{s\varphi}\varphi\|_2\]
whenever \(\widehat{\varphi}\) is supported in \(\{\xi : 1/2 < |\xi| < 2\}\). Once it is established, the rest of proof is identical with that of Corollary 1.3. By finite decompositions and rotation we may assume that \(\hat{f}\) is supported in \(\Gamma = \{\xi = (\bar{\xi}, \xi_d) : |\xi| < \xi_d/100, 1/2 < \xi_d < 2\}\).

Let us set
\[\mathcal{T}\varphi(x,t) = \int_{\Gamma} e^{ix\cdot\xi + it(|\xi| - \xi_d)}\hat{\varphi}(\xi)d\xi = \int_{\Gamma} e^{i\bar{x}\cdot\xi + it\xi_d + it\xi_d\theta(\xi/\xi_d)}\hat{\varphi}(\xi)d\xi\]
with \(\theta(\eta) = \sqrt{1 + |\eta|^2} - 1\). Then by a simple change of variables \(x_d \to x_d - t\) it is enough to show that
\[(4.1) \quad \|\mathcal{T}\varphi\|_{L^q_t(L^r_x(\mathbb{R}^d))} \leq C\|x^{s\varphi}\varphi\|_2\]
provided that \(\text{supp} \hat{\varphi} \subset \Gamma\). By Hausdorff-Young’s inequality in \(x_d\) and Minkowski’s inequality, the left hand side is bounded by
\[C\int_{1/2}^{2}\int_{\{\xi : \frac{1}{\alpha} \leq \xi \leq C\}} e^{i\bar{x}\cdot\xi + it\xi_d\theta(\xi/\xi_d)}\hat{\varphi}(\xi,\xi_d)d\xi d\xi_d.\]

Freezing \(\xi_d \in (1/2, 2)\), the Hessian matrix of \(\theta(\cdot/\xi_d)\) is non-singular. So, we apply Theorem 1.2 to the extension operator defined by \(\theta(\cdot/\xi_d)\). In fact, since \(\theta(\eta)\) is close to \(\frac{1}{2}|\eta|^2\) and \(1/2 \leq \xi_d \leq 2\), it is easy to see that there is a uniform bound \(C\) independent of \(\xi_d\) so that
\[\|\int_{\{\xi : \frac{1}{\alpha} \leq \xi \leq C\}} e^{i\bar{x}\cdot\xi + it\xi_d\theta(\xi/\xi_d)}g(\xi)d\xi\|_{L^q_t(L^r_x(\mathbb{R}^d-1))] \leq C\|x(1 - \Delta)\frac{s\varphi}{2}g\|_2.\]

Therefore, recalling Remark 1 and taking integration in \(\xi_d\), we get
\[\|\mathcal{T}\varphi\|_{L^q_t(L^r_x(\mathbb{R}^d))} \leq C\int_{1/2}^{2}\|(-\Delta)^{s\varphi/2}\hat{\varphi}(\xi,\xi_d)\|_{L^2_x}d\xi_d.\]

Then (4.1) follows by Plancherel’s theorem and Cauchy-Schwarz inequality.

**Proof of Corollary 1.5** For the proof it suffices to show the case \(q = r\). The other cases follow from interpolation with the estimate \(\|(-\Delta)^{\gamma(\infty, 2, d)/2}e^{it\sqrt{-\Delta}}\varphi\|_{L^q_t(L^r_x(\mathbb{R}^d))} \lesssim C\|\varphi\|_{H^0_{sph}}\). By Littlewood-Paley theory it is enough to show that
\[\|e^{it\sqrt{-\Delta}}\varphi\|_{L^q_t(L^r_x(\mathbb{R}^d))} \lesssim C\|\varphi\|_{H^0_{sph}}\]
for \(\varphi\) of which Fourier transform is supported in \(\{\xi : 1/2 \leq |\xi| \leq 2\}\). For this we write
\[e^{it\sqrt{-\Delta}}\varphi(x) = \frac{1}{(2\pi)^d}\int_{\frac{1}{2}}^{2}\int_{S^{d-1}} e^{irx\cdot\omega + itr}\hat{\varphi}(r\omega)dr d\omega d^{d-1}dr.\]

By Hausdorff-Young’s inequality in \(t\) and Minkowski’s inequality we get
\[\|e^{it\sqrt{-\Delta}}\varphi\|_{L^q_t(L^r_x(\mathbb{R}^d))} \lesssim \|R^0(\hat{\varphi}(\cdot))(r\cdot)\|_{L^q_t((1/2, 2):L^r_x(\mathbb{R}^d))}.\]
Theorem 1.1 and Hölder’s inequality imply
\[ \| e^{it\sqrt{-\Delta}} \varphi \|_{L^q_x(R^d)} \lesssim \| (1 - \Delta_\omega)^{\nu/2} \hat{\varphi}(r\omega) \|_{L^2_{r}(1/2,2;L^2(\mathbb{S}^{d-1}))}. \]
Now recalling \((1 - \Delta_\omega)^{\nu/2} \hat{\varphi} = F((1 - \Delta_\omega)^{\nu/2} g)\), we get the desired inequality by Plancherel’s theorem. \(\square\)

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