\textit{k}-fold sums from a set with few products

Dedicated to the memory of György Elekes

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1 Introduction

Before we state our main theorems, we begin with some notation: given a finite subset \( A \) of some commutative ring, we let \( A + A \) denote the set of sums \( a + b \), where \( a, b \in A \); and, we let \( A.A \) denote the set of products \( ab \), \( a, b \in A \). When three or more sums or products are used, we let \( kA \) denote the \( k \)-fold sumset \( A + A + \cdots + A \), and let \( A^{(k)} \) denote the \( k \)-fold product set \( A.A...A \). Lastly, by \( d \ast A \) we mean the set \( da \), \( a \in A \).

The theory of sum-product inequalities has an interesting history, and began with the theorem of Erdős and Szemerédi \cite{erdos}, which says that for some \( \varepsilon > 0 \) and \( n > n_0(\varepsilon) \), we have that for any set \( A \) of \( n \) real numbers, either the sumset \( A + A \) or the product set \( A.A \) has at least \( n^{1+\varepsilon} \) elements. Further improvements to this result were achieved by Nathanson \cite{nathanson}, Ford \cite{ford}, Elekes \cite{elekes}, and finally Solymosi \cite{solymosi} and \cite{solymosi2}.

Another type of theorem that one can prove regarding sums and products is to assume that either the sumset \( A + A \) is near to being as small as possible (near to \( n \)), and then to show that \( A.A \) must be near to \( n^2 \); or, one can suppose that the product set \( A.A \) is small, and show that the sumset \( A + A \) is large. Furthermore, one can consider \( k \)-fold sums and products here. Some quite interesting results along these lines were produced by Chang \cite{chang}, \cite{chang2}, Elekes-Ruzsa \cite{elekes-ruzsa}, Elekes-Nathanson-Ruzsa \cite{elekes-nathanson-ruzsa}, and Jones-Rudnev \cite{jones-rudnev}.
There are also some related analogues in finite fields similar to these just mentioned. For example, \[3\], \[13\], \[14\], \[15\] and \[21\].

Continuing with the characteristic 0 case, Chang and Bourgain proved the following results on \(k\)-fold sums and products: Chang \[4\] showed that if \(A\) is a set of \(n\) integers, and \(|A.A| < n^{1+\varepsilon}\), then the sumset \(|kA| \gg_{\varepsilon,k} n^{k-\delta}\), where \(\delta \to 0\) as \(\varepsilon \to 0\). And then Chang and Bourgain \[2\] showed that for any \(b \geq 1\), there exists \(k \geq 1\) such that if \(A\) is a set of \(n\) integers,

\[|kA| \cdot |A^{(k)}| \gg n^b.\]

In both of these results, we would like to have that they hold for the real numbers (or even the complex numbers), instead of just the integers. Unfortunately, this appears to be out of reach at the moment.

The purpose of the present paper is to present some results towards this end. Specifically, we will prove the following two theorems.

**Theorem 1** For all \(h \geq 2\) and \(0 < \varepsilon < \varepsilon_0(h)\) we have that the following holds for all \(n > n_0(h, \varepsilon)\): if \(A\) is a set of \(n\) real numbers and

\[|A.A| \leq n^{1+\varepsilon},\]

then

\[|hA| \geq n^{\log(h/2)/2 \log 2 + 1/2 - f_h(\varepsilon)},\]

where \(f_h(\varepsilon) \to 0\) as \(\varepsilon \to 0\).

If instead of showing that \(kA\) is large, we just want to show that \(k(A.A)\) is large, we can prove a much stronger theorem:

**Theorem 2** Under the same hypotheses on \(A\) as in the theorem above, we have that

\[|h(A.A)| = |A.A + A.A + \cdots + A.A| > n^{\Theta(h/\log h)^{1/3}}.\]

### 1.1 Some remarks

While the methods in the present paper will need substantial modification to come anywhere near to proving an analogue of \[4\] for the real numbers, we feel that it might be possible to achieve bounds as good in Theorem 1 above
as we have in Theorem 2. Although this too will require a lot of work, we feel that we have a few good ideas on how to actually achieve it.

It is also worth remarking that we have several different approaches to proving a theorem of the quality of Theorem 1. In particular, it is possible to use an iterative argument involving the Szemerédi-Trotter theorem, a Szemerédi cube lemma similar to Lemma 3 below, and some “energy arguments”, to achieve similar such bounds. However, it is not as easy to see how one might go about modifying such an “incidence proof” of Theorem 1 to achieve bounds as good as in Theorem 2.

2 Preliminary lemmas and results

First, in the proofs of both theorems, we will assume that all the elements of \( A \) are positive. The reason we can assume this is that either at least \( (n - 1)/2 \) elements of \( A \) are all positive, or at least \( (n - 1)/2 \) are all negative. If we are in the negative case here, we just let \( A' \) be the negative of these negative elements; and otherwise, we just let \( A' \) be these \( (n - 1)/2 \) positive elements. Then, we simply prove our theorem using \( A' \) in place of \( A \). The effect of the lost factor of 2 will be negligible.

The proof of Theorem 2 will require the following result of Wooley [22] (see also Borwein-Erdélyi-Kós [1] for some related results).

**Theorem 3** For every \( k \geq 1 \), there exist two distinct sets

\[ \{x_1, ..., x_s\}, \{y_1, ..., y_s\} \subseteq \mathbb{Z} \]

of

\[ s < (k^2/2)(\log k + \log \log k + O(1)) \]

(when \( k = 1 \) we just delete the \( \log \log k \) term) distinct integers in each, such that

\[ \sum_{i=1}^{s} x_i^j = \sum_{i=1}^{s} y_i^j, \text{ for all } j = 1, ..., k, \]

but that

\[ \sum_{i=1}^{s} x_i^{k+1} \neq \sum_{i=1}^{s} y_i^{k+1}. \]
For the purposes of our paper, we actually require the following corollary of this theorem.

**Corollary 1** For all integers $j \geq 1$, there exists a monic polynomial $f(x)$, having only the coefficients $0, 1,$ and $-1$, having at most

$$(j^2/2)(\log j + \log \log j + O(1))$$

(again, if $j = 1$ we just delete the log log $j$ term) non-zero terms, such that $f(x)$ vanishes at $x = 1$ to order $j$, but not to order $j + 1$.

Since the number of terms of this polynomial depends only on $j$, it follows that if one performs a Taylor expansion of this polynomial about $x = 1$, one will find that

$$f(x) = \sum_{i=j}^d c_i(x - 1)^i, \quad d = \deg(f),$$

where $c_j \neq 0$, and each $c_i$ in turn is either $0$ or its absolute value can be bounded from below by some function of $j$ alone (the degree $d$ depends on $j$).

**Proof of the Corollary.** Basically, we just use the well-known fact that a polynomial

$$f(x) = \sum_{i=1}^s x^{x_i} - \sum_{i=1}^s x^{y_i}.$$

vanishes to order $j$ at $x = 1$ if and only if $f$ and its first $j - 1$ derivatives vanish at $x = 1$, where the $\ell$th derivative evaluated at $1$ looks like

$$\sum_{i=1}^s x_i(x_i - 1) \cdots (x_i - \ell + 1) - \sum_{i=1}^s y_i(y_i - 1) \cdots (y_i - \ell + 1).$$

Clearly, having all these be $0$, for $\ell = 0, 1, \ldots, j - 1$, is equivalent to having a solution to the “Tarry-Escott Problem” considered by Wooley in Theorem 3 above.

One small remaining point to consider is the fact that some of the $x_i$’s and $y_i$’s could be negative (meaning that the $f$ above is a Laurent polynomial, not a polynomial). That is easily fixed by multiplying $f$ by an appropriate power of $x$, which does not affect the vanishing properties at $x = 1$. ■

Another major theorem that we will require is the Ruzsa-Plunnecke inequality [18].
Theorem 4 Suppose that $A$ is a finite subset of an additive abelian group. Then, if

$$|A + A| \leq K|A|,$$

we will have that

$$|kA - \ell A| = |A + A + \cdots + A - A - \cdots - A| \leq K^{k+\ell}|A|.$$

We will also require the following basic lemmas.

Lemma 1 Suppose that $A$ is a set of $m^2$ positive real numbers, say they are

$$0 < a_1 < \cdots < a_{m^2},$$

such that no dyadic interval $[x, 2x]$ contains $m$ or more of the $a_j$'s. Then,

$$|kA| \gg_k m^k.$$

Proof of the lemma. Let

$$B := \{a_1, a_{2m+1}, a_{4m+1}, \ldots, a_{m^2-m+1}\}.$$

We claim that all the sums

$$b_1 + \cdots + b_k, \ b_i \in B, \ b_1 < \cdots < b_k,$$

are distinct, which would prove the lemma.

To see this, suppose we had

$$b_1 + \cdots + b_k = b'_1 + \cdots + b'_k,$$

and suppose without loss that $b_k \leq b'_k$. If $b_k < b'_k$, then $b_k < b'_k/2$, and we have

$$b'_k = b_1 + \cdots + b_k - b'_1 - \cdots - b'_{k-1} \leq b_1 + \cdots + b_k < 2b_k < b'_k,$$

contradiction. So, we can delete $b_k$ and $b'_k$ from both sides; and then, repeating the argument, we get $b_i = b'_i$, $i = 1, 2, \ldots, k$, and we are done. $\blacksquare$

The following lemma is basically a generalization of a result in [7].
Lemma 2 For every \( r, \ell \geq 1 \), \( 0 < c_1 < c_1(r, \ell) \), \( 0 \leq c_2 < c_2(r, \ell, c_1) \) and \( 0 < \varepsilon < \varepsilon(r, \ell, c_1, c_2) \), the following holds for all \( n \) sufficiently large: Suppose \( A \) is a set of \( n \) real numbers satisfying
\[
|A.A| < n^{1+\varepsilon},
\]
and suppose that
\[
A' \subseteq A, \quad B \subseteq A^{(r)}/A^{(r)},
\]
satisfy
\[
|B| > n^{c_1}, \text{ and } |A'| > n^{1-c_2}.
\]
Then, there are
\[
n^{2-O(c_2\ell+r\ell^2\varepsilon)} \text{ pairs } (a_1, a_2) \in A' \times A',
\]
such that if we let
\[
x = a_1 a_2,
\]
then there exist
\[
n^{-O(c_2\ell+r\ell^2\varepsilon)}|B|^2
\]
pairs
\[
(b_1, b_2) \in B \times B, \quad b_1 > b_2,
\]
such that if we let
\[
d = b_1/b_2,
\]
then
\[
x, \ dx, \ d^2 x, \ldots, \ d^{\ell} x \in A'.A'.
\]

Proof of the lemma. First, by Ruzsa-Plunnecke, we have that
\[
|A'/A'| \leq |A/A| < n^{1+2\varepsilon}.
\]
And then, by a simple pigeonhole argument, we have that there exists \( t_1, t_2 \in A' \) such that if
\[
A'' := (A'/t_1) \cap (t_2/A'),
\]
then
\[
|A''| > n^{1-O(c_2)},
\]
for $c_2 > 0$ small enough in terms of $\varepsilon$. Of course, this means that if we let $t = t_1 t_2$ then

$$A'' := A' \cap (t/A')$$ satisfies $|A''| = |A'| > n^{1-O(c_2)}$. Note that

$$A''/A'' \subseteq t^{-1} \ast (A'.A').$$

What we will show is that for lots of pairs

$$(e_1, e_2) \in A'' \times A'', \text{ and } y = e_1/e_2,$$

the set $A''/A''$ contains “lots” of progressions

$$y, dy, d^2 y, ..., d^\ell y, \text{ where } d \in B/B.$$ (1)

If we can do this, then

$$x, dx, d^2 x, ..., d^\ell x \in A'.A', \text{ where } x = ty.$$

Showing that $A''/A''$ contains such a progression amounts to showing that the following system has non-trivial solutions:

$$\frac{b_1 e_1}{b_2 e_2} = \frac{e_3}{e_4}, \quad \frac{b_1^2 e_1}{b_2^2 e_2} = \frac{e_5}{e_6}, \quad ..., \quad \frac{b_1^\ell e_1}{b_2^\ell e_2} = \frac{e_{2\ell+1}}{e_{2\ell+2}}.$$ (2)

Another way to write this is

$$b_1 e_1 e_4 = b_2 e_2 e_3$$
$$b_1^2 e_1 e_6 = b_2^2 e_2 e_5$$
$$...$$
$$b_1^\ell e_1 e_{2\ell+2} = b_2^\ell e_2 e_{2\ell+1}.$$

Note that both sides of the equations belong to

$$t^{-2} \ast (B^{(\ell)}(A')^{(4)}) \subseteq t^{-2} \ast (A')^{(r\ell+4)},$$

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which has at most

\[ n^{1+O(r\ell \varepsilon)} \]

elements, by Ruzsa-Plunnecke. So, there are

\[ \leq n^{\ell+O(r\ell^2 \varepsilon)} \]

values that the \( \ell \) expressions on the left-hand-side in this system, in total, can take on, and the same goes for the right-hand-side.

Now, since there are at least

\[ n^{\ell+1-O(c_2 \ell)} |B| \]

choices for

\[ e_1, e_4, e_6, e_8, \ldots, e_{2\ell+2}, b_1, \]

that make up the left-hand-sides – similarly, the right-hand-sides – it is clear that we get at least

\[ n^{\ell+2-O(c_2 \ell + r\ell^2 \varepsilon)} |B|^2 \]

solutions. For \( c_1, c_2 \) small enough in terms of \( r, \ell \) and for \( \varepsilon > 0 \) small enough in terms of \( r, \ell, c_1, c_2 \), this exceeds the number of trivial solutions, which is

\[ |B| \sum_{e \in A''/A'''} r(e) \leq |B| \cdot |A''/A'''| \cdot |A'''|^\ell = n^{\ell+1+O(\ell c_2)} |B|, \]

where \( r(e) \) is the number of representations \( e = e'/e'' \), where \( e', e'' \in A''' \).

So, there are at least

\[ n^{\ell+2-O(c_2 \ell + r\ell^2 \varepsilon)} |B|^2 \]

non-trivial solutions, when \( c_1, c_2 \) and \( \varepsilon > 0 \) are small enough. So, for the average tuple

\[ (e_4, e_6, \ldots, e_{2\ell+2}) \in (A'')^\ell, \]

we have that there are at least

\[ n^{2-O(c_2 \ell + r\ell^2 \varepsilon)} |B|^2 \]

four-tuples

\[ (e_1, e_2, b_1, b_2) \in A'' \times A'' \times B \times B, \]
such that the system has a solution (which must be unique, since the remaining $c_i$’s are determined exactly). Clearly this proves the lemma. Note that guaranteeing that we can work with $b_1 > b_2$ can be guaranteed simply by taking reciprocals in (2).

And now we state two more general-purpose lemmas, the first of which is perhaps better known as a the “Szemeredi cube lemma”, and is used in the proof of Theorem 1 only, while the other lemma is used in the proofs of both theorems.

**Lemma 3** The following holds for all $k \geq 2$, $0 < c < c_0(k)$, $0 < \varepsilon < \varepsilon_0(k, c)$ and $n > n_0(k, c, \varepsilon)$: suppose that $A$ is a set of $n$ real numbers such that

$$|A.A| < n^{1+\varepsilon},$$

and suppose that $B \subseteq A$ satisfies

$$|B| \geq n^{c}.$$

Then, there exists

$$\theta_1, \ldots, \theta_k \in B/B,$$

each $\theta_i > 1,$

such that for at least

$$n^{1-O_k(c)}$$

values $d \in A$ we have that all the numbers

$$\theta_1^{\gamma_1} \cdots \theta_k^{\gamma_k} d,$$

where each $\gamma_i \in \{0, 1\}$,

belong to $A$. (Note that for each such $d$, this means that $2^k$ different elements belong to the set $A$.)

**Proof of the lemma.** The proof is inductive: we will construct

$$D_0 := A,$$

such that

$$D_i = D_{i-1} \cap (\theta_i^{-1} \ast D_{i-1}),$$

where $\theta_i \in B/B$, $\theta_i > 1$, is chosen greedily to maximize $D_i$, given $D_{i-1}$. 

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Suppose that we have already shown that
\[ |D_{i-1}| = n^{1-O_k(c)}. \]
Then, consider the product set
\[ BD_{i-1} = \{bd : b \in B, d \in D_{i-1}\} \subseteq A.A. \]
Since \(|A.A| < n^{1+\varepsilon}\), we have that
\[ |BD_{i-1}| \leq n^{1+\varepsilon}, \]
and therefore we easily see that there exists
\[ s, t \in B, s \neq t, s/t \neq \theta_1, \ldots, \theta_{i-1}, \ s > t, \]
such that
\[ |(s \ast D_{i-1}) \cap (t \ast D_{i-1})| > n^{1-O_k(c)}, \]
for \(\varepsilon > 0\) sufficiently small in terms of \(k, c\).
So, letting \(\theta_i = s/t > 1\), we are done, because
\[ |D_{i-1} \cap (\theta_i^{-1} \ast D_{i-1})| = |(s \ast D_{i-1}) \cap (t \ast D_{i-1})| > n^{1-O_k(c)}, \]
as claimed. \(\blacksquare\)

**Lemma 4** Suppose that \(C\) is a set of real numbers, and
\[ 1 = \delta_0 > \delta_1 > \delta_2 > \cdots > \delta_{k-1} > 0 \]
are positive real numbers such that if we define the ratios
\[ \alpha_i := \delta_i/\delta_{i-1}, \ i = 1, 2, \ldots, k-1, \]
them for all pairs \(c, d \in C, c > d\),
we have
\[ c/d - 1 > 2k\alpha_2, \ldots, 2k\alpha_{k-1}. \]
Next, partition \(C\) into any disjoint sets
\[ C = C_1 \cup C_2 \cdots \cup C_k, \]
where for \( i < j \) we have that every element of \( C_i \) is greater than every element of \( C_j \). Let us express this as

\[ C_i > C_j, \text{ for } i < j. \]

Then, we have that all sums

\[ c_1 + c_2\delta_1 + \cdots + c_k\delta_{k-1}, \ c_1, \ldots, c_k \in C, \]

are distinct.

**Proof of the lemma.** Suppose that, on the contrary, two of these sums are equal. Then, it would mean that

\[ c_1 + c_2\delta_1 + \cdots + c_k\delta_{k-1} = c'_1 + c'_2\delta_1 + \cdots + c'_k\delta_{k-1}. \tag{3} \]

Suppose without loss that \( c_1 \geq c'_1 \). Now let us suppose that, in fact, \( c_1 > c'_1 \). Then, we have that

\[ \frac{c_1}{c'_1} - 1 = \sum_{i=2}^{k} \left( \frac{c'_i}{c'_1} - \frac{c_i}{c'_1} \right) \delta_{i-1}. \]

From the fact that \( C_1 > C_2, \ldots, C_k \), we have that the right-hand-side here is bounded from above in absolute value by

\[ \sum_{i=2}^{k} 2\delta_i \leq 2k\delta_2 \leq 2k\alpha_2 < \frac{c_1}{c'_1} - 1, \]

which is impossible. We conclude that \( c_1 = c'_1 \).

Now suppose for proof by induction we have shown that

\[ c_i = c'_i, \ i = 1, 2, 3, \ldots, j, \text{ where } j \leq k - 1. \]

We now show that

\[ c_{j+1} = c'_{j+1}, \]

which would clearly prove the lemma (we get \( c_k = c'_k \) for free once the other \( c_i = c'_i \) are established).

We begin by deleting the terms \( c_i\delta_{i-1} \) and \( c'_i\delta_{i-1} \) from both sides of (3), for \( i = 1, 2, \ldots, j \). So, we are left with

\[ \sum_{i=j+1}^{k} c_i\delta_{i-1} = \sum_{i=j+1}^{k} c'_i\delta_{i-1}, \]
which can be rewritten as
\[(c_{j+1}/c'_{j+1} - 1) = \sum_{i=j+2}^{k} (c'_i/c'_{j+1} - c_i/c'_{j+1})\delta'_i, \tag{4}\]
where
\[\delta'_i := \delta_i/\delta_j \leq \alpha_i, \ i = j + 1, \ldots, k - 1.\]

We assume without loss that \(c_{j+1} \geq c'_{j+1}\). If, in fact, \(c_{j+1} > c'_{j+1}\), then the absolute value of the right-hand-side of \(4\) is clearly bounded from above by
\[2k\alpha_{j+1} < c_{j+1}/c'_{j+1} - 1,\]
which is a contradiction. We conclude that \(c_{j+1} = c'_{j+1}\), and therefore the induction step is proved, as is the lemma. \[\square\]

3 Proof of Theorem 1

We suppose that the elements of \(A\) are
\[a_1 < a_2 < \cdots < a_n,\]
which we assume are all positive by the remarks at the beginning of section 2.

Let \(0 < \delta < 1/2\) be some parameter that we will choose as small as needed later, and let \(k \geq 2\) be some parameter that we will let depend on \(h\) later. Let
\[s := \lfloor n^\delta \rfloor,\]
and set
\[B := \{a_j, a_{j+1}, \ldots, a_{j+s}\},\]
where \(j\) is chosen so that
\[a_{j+s}/a_j \text{ is minimal.} \tag{5}\]
We may assume that \(B\) lies in some interval
\[B \subseteq [x, 2x],\]
since otherwise each consecutive block of \( s + 1 \) elements of \( A \) lies in its own interval of this form, disjoint from those corresponding to other blocks of elements of \( A \); and therefore, by Lemma 1 we could conclude that

\[
|kA| \geq n^{\Omega(\sqrt{k})},
\]

which would prove our theorem.

Having \( B \) lie in a dyadic interval implies that all ratios \( \theta = b_2/b_1, b_1, b_2 \in B \), satisfy

\[
\theta - 1 \in [0, 1).
\]

Next, we let \( c = \delta \) and apply Lemma 3 and let \( \theta_1, ..., \theta_{k-1} \in B/B \) denote the numbers that result from this lemma (using \( k - 1 \) in place of \( k \)); and, let \( C_0 \) denote the set of all \( n^{1-O_k(\delta)} \) elements \( d \in A \) that the lemma produces, and write

\[
C_0 := \{c_1, ..., c_{n'}\}, \quad c_1 < c_2 < \cdots < c_{n'}, \quad n' > n^{1-O_k(\delta)}.
\]

Then, let

\[
C := \{c_1, c_{1+sk2k}, c_{1+2sk2k}, c_{1+3sk2k}, \ldots\}
\]

be the set of every \( sk2^k \)th element of \( C_0 \). Note that since the elements \( \theta_i \in B/B \), and \( B \) satisfies (5), we have that

\[
\frac{\theta_{2^j}}{\theta_{j-1}} < c_2/c_1, \quad c_1, c_2 \in C, \quad c_2 > c_1,
\]

and therefore

\[
2k(\theta_j - 1) < \frac{\theta_{2^j}}{\theta_{j-1}} < c_2/c_1 - 1.
\]

It follows that if we let

\[
\delta_i := (\theta_1 - 1)(\theta_2 - 1)\cdots(\theta_i - 1), \quad i = 1, 2, ..., k - 1,
\]

then we have for \( i = 1, 2, ..., k - 2 \) that

\[
\delta_{i+1}/\delta_i = \theta_{i+1} - 1 < (c_2/c_1 - 1)/2k.
\]

We almost are ready to apply Lemma 4—all we have to do is partition \( C \), which we do simply by letting

\[
C = C_1 \cup \cdots \cup C_k,
\]

\[13\]
where $C_1$ consists of the largest $\left\lceil |C|/k \right\rceil$ elements of $C$, $C_2$ consists of the next largest $\left\lceil |C|/k \right\rceil$ elements of $C$, and so on.

Lemma 4 now tells us that all the sums

$$c_1 + c_2\delta_1 + \cdots + c_k\delta_{k-1}, \ c_i \in C_i,$$

are distinct. This then results in

$$\gg (|C|/k)^k \gg_k n^{k(1-O_k(\delta))}.$$ distinct sums.

These sums, in turn, can be re-written as just sums and differences of elements from $A$ as follows: by expressing the $\delta_{i-1}$ back in terms of the $\theta_j$'s, we find that

$$c_i\delta_{i-1} = c_i(\theta_1 - 1)\cdots(\theta_{i-1} - 1) = (-1)^{i-1}c_i + (-1)^{i-2}\theta_1c_i + \cdots$$

Each term here looks like

$$\pm c_i\theta_1^{\gamma_1}\cdots\theta_{i-1}^{\gamma_i}, \ \text{where} \ \gamma_j \in \{0,1\},$$

and we know from our use of Lemma 3 that all such numbers belong to $\pm A$.

It is easy to see, then, that all the sums (6) can be re-expressed as subsets of $KA - LA$, where

$$K, L < 2^{k-2} + 2^{k-3} + \cdots + 1 < 2^{k-1}.$$ So,

$$|2^{k-1}A|^2 > |KA| \cdot |LA| \geq |KA - LA| > n^{k(1-O_k(\delta))},$$

from which it follows that upon letting $h = 2^{k-1},$

$$|hA| \geq n^{\log(h)/2}\log^{2+1/2}\log 2 + 1/2 - g_{h}(\delta),$$

where $g_h(\delta) \to 0$ as $\delta \to 0$. Of course, this only works for when $h$ is a power of 2; by bounding general $h$ between two consecutive powers of 2, we can conclude that

$$|hA| \geq n^{\log(h/2)/2}\log^{2+1/2}\log 2 + 1/2 - g_{h}(\delta).$$

This completes the proof of our theorem, by choosing $\delta > 0$ small enough, and then choosing $\epsilon > 0$ even smaller as needed.
4 Proof of Theorem 2

Write out the elements of $A$ in increasing order as

$$a_1 < a_2 < \cdots < a_n,$$

which we assume are all positive by the remarks at the beginning of section 2.

Let $\delta > 0$ be some parameter that we will choose later as function of $h$ alone, and let $s$ and $B$ be as in the beginning of the proof of Theorem 1. In fact, we may assume that

$$B \subseteq [x, (1 + 1/\gamma(h))x],$$

for any function $\gamma(h) > 0$ we please. The reason is that if this minimal set $B$ lies in this interval, and if $a_j$ is the smallest element of $B$, then

$$[a_j, a_j + Ls] \supseteq [x, (1 + \gamma(h))^L x] \supseteq [x, 2x],$$

for $L$ large enough. And so, we may again deduce (as in the proof of Theorem 1), using Lemma 1, that

$$|hA| \gg n^{\Omega(h^{1/2})},$$

which would prove our lemma.

As in the proof of Theorem 1 we may assume that the elements of $B$ lie in some dyadic interval $[x, 2x]$, upon applying Lemma 1.

Let $k \geq 2$ be some parameter that is to depend on $h$, that we will choose later.

We now apply Lemma 2 using $\ell = M$, $r = 1$, $c_1 = \delta$, $A' = A$ (so $c_2 = 0$) and $\varepsilon > 0$ as small as needed in terms of $M$ and $\delta$, where the precise value of $M$ will be determined below, and will depend only on $h$. So, there exists

$$\theta \in B/B, \ \theta > 1,$$

such that for at least

$$|A|^{2 - O(\ell^2 \delta)}$$

pairs

$$(a_1, a_2) \in A,$$

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we have that if we let \( y = a_1a_2 \), then
\[
y, \ y\theta, \ldots, \ y\theta^\ell \in A.A.
\] (7)

So, there exists \( a_1 \in A \) and \( \theta \in B/B, \theta > 1 \), such that there are
\[
|A|^{1-O(\ell^2\delta)}
\] (8)

values \( a_2 \in A \) such that for \( y = a_1a_2 \) we have that [7] holds.

If we let the special elements \( a_2 \in A \) be
\[
\{a_{h_1}, \ldots, a_{h_{n'}}\}, \ n' > n^{1-O(\ell^2\delta)},
\]
then we define
\[
C := \{a_{h_1}, a_{h_{\lceil \sqrt{n} \rceil + 1}}, a_{h_{2\lceil \sqrt{n} \rceil + 1}}, \ldots\} \subseteq \{a_{h_1}, \ldots, a_{h_{n'}}\}. \tag{9}
\]

Note that \( C \) is basically a “well-separated” subset of those special elements \( a_2 \in A \); and, in fact, if \( \delta < 1/2 \), so that \( s > n^{1/2} \), they are so well-separated that that all ratios \( c_2/c_1, c_1, c_2 \in C, c_2 > c_1 \), have the property that
\[
c_2/c_1 > \theta. \tag{10}
\]

Later, we will prove and make use of an even stronger such inequality.

Now we apply Corollary [11] letting
\[
f_1(x), \ldots, f_{k-1}(x)
\]
be polynomials having at most \((j^2/2)(\log j + \log \log j + O(1)), j = 1, \ldots, k-1,\) terms each, each with coefficients only 0, 1, or \(-1\), that vanish at \( x = 1 \) to the orders \( 1, 2, 3, \ldots, k-1, \) respectively. Then we let
\[
M = \max(\deg(f_1), \ldots, \deg(f_{k-1})),
\]
which is a parameter that came up earlier in the proof of the present theorem (Theorem [2]). Note that \( M \) does not depend on \( n \) – it depends on \( k \), and therefore on \( h \).

Next, we set
\[
\delta_i := f_i(\theta), i = 1, 2, \ldots, k-1.
\]
Since \( f_{i+1}(x)/f_i(x) \) vanishes at \( x = 1 \) to order 1, and since \( \theta \in B/B \) and \( \theta \in [1, 2) \), we have that if we set

\[
\alpha_i := \delta_i/\delta_{i-1}, \quad i = 2, 3, \ldots, k - 1,
\]
then for every \( c_1, c_2 \in C, \ c_2 > c_1 \), we have from (10) that

\[
c_2/c_1 - 1 > \theta - 1 \gg_k \alpha_i > 0.
\]

(Note that the implied constant here depends on the sizes of the coefficients \( c_i \) in Corollary 1 and we know that these coefficients are rational numbers that depend on \( k \).) But, in fact, if \( \delta > 0 \) is small enough, then for \( c_1, c_2 \in C, \ c_2 > c_1 \), we can assume that for any function \( \gamma(k) \) of \( k \),

\[
\theta^{\gamma(k)} < c_2/c_1;
\]
and so, we may assume

\[
c_2/c_1 - 1 > 2k\alpha_i > 0.
\]

So, if we let \( C_1 \) be the largest \( |C|/k \) elements of \( C \), \( C_2 \) be the second largest \( |C|/k \) elements of \( C \), and so on, down to \( C_k \), then upon applying Lemma 4 we have that all sums

\[
a_1 + a_2 f_1(\theta) + \cdots + a_k f_k(\theta), \quad a_i \in C_i,
\]
are distinct. Since each \( C_i \) satisfies

\[
|C_i| \gg_k n^{1/3},
\]
for \( \delta > 0 \) small enough (and \( \varepsilon > 0 \) small enough in terms of \( \delta \) and \( h \)), we deduce that this produces \( n^{\Omega(k)} \) distinct sums. Now, because

\[
a_i, a_i \theta, \ldots, a_i \theta^M \in A.A,
\]
by design, we have that upon expanding out these polynomials \( f_i \) in (11) into powers of \( \theta \), we find that these \( n^{\Omega(k)} \) sums are, in fact, subsets of \( KA - LA \), where

\[
K, L \leq (1^2 + 2^2 + \cdots + (k - 1)^2/2)(\log k + \log \log k + O(1)) \ll k^3 \log k.
\]

It follows that

\[
|ck^3 \log k|A|^2 \geq |KA - LA| \geq n^{\Omega(k)}.
\]

This clearly proves the theorem upon letting \( k \gg (h/\log h)^{1/3} \).
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