K-THEORY OF JONES POLYNOMIALS

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Abstract. We recover the Jones polynomials of knots and links from the K-theory of a cluster \(^{C^*}\)-algebra of the sphere with two cusps. In particular, an interplay between the Chebyshev and Jones polynomials is studied.

1. Introduction

Cluster algebras are a class of commutative rings introduced by [Fomin & Zelevinsky 2002] [1]. The cluster algebra of rank \(n\) is a subring \(A(x, B)\) of the field of rational functions in \(n\) variables depending on variables \(x = (x_1, \ldots, x_n)\) and a skew-symmetric matrix \(B = (b_{ij}) \in M_n(\mathbb{Z})\). The pair \((x, B)\) is called a seed. A new cluster \(x' = (x_1, \ldots, x'_k, \ldots, x_n)\) and a new skew-symmetric matrix \(B' = (b_{ij}')\) is obtained from \((x, B)\) by the exchange relations [Williams 2014] [9, Definition 2.22]:

\[
x_k x'_k = \prod_{i=1}^{n} x_i^\max(b_{ik}, 0) + \prod_{i=1}^{n} x_i^\max(-b_{ik}, 0),
\]

\[
b'_{ij} = \begin{cases} 
- b_{ij}, & \text{if } i = k \text{ or } j = k \\
 b_{ij} + |b_{ik}b_{kj} + b_{ik}b_{kj}|/2, & \text{otherwise.}
\end{cases}
\]

The seed \((x', B')\) is said to be a mutation of \((x, B)\) in direction \(k\), where \(1 \leq k \leq n\). The algebra \(A(x, B)\) is generated by the cluster variables \(\{x_i\}_{i=1}^{\infty}\) obtained from the initial seed \((x, B)\) by the iteration of mutations in all possible directions \(k\).

The Laurent phenomenon proved by [Fomin & Zelevinsky 2002] [1] says that \(A(x, B) \subset \mathbb{Z}[x^{\pm 1}]\), where \(\mathbb{Z}[x^{\pm 1}]\) is the ring of the Laurent polynomials in variables \(x = (x_1, \ldots, x_n)\). In particular, each generator \(x_i\) of the algebra \(A(x, B)\) can be written as a Laurent polynomial in \(n\) variables with the integer coefficients. The cluster algebra \(A(x, B)\) has the structure of an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. In other words, the \(A(x, B)\) is a dimension group [7, Definition 3.5.2]. We define the cluster \(^{C^*}\)-algebra \(\mathbb{A}(x, B)\) as an AF-algebra, such that \(K_0(\mathbb{A}(x, B)) \cong A(x, B)\) [7, Section 4.4].

Denote by \(S_{g,n}\) the Riemann surface of genus \(g \geq 0\) with the \(n \geq 0\) cusps. Let \(\mathcal{A}(x, S_{g,n})\) be the cluster algebra coming from a triangulation of the surface \(S_{g,n}\) [Fomin, Shapiro & Thurston 2008] [2] and let \(\mathbb{A}(x, S_{g,n})\) be the corresponding cluster \(^{C^*}\)-algebra. In what follows, we focus on the special case \(g = 0\) and \(n = 2\), i.e. when the surface \(S_{0,2}\) is a sphere with two cusps. The \(S_{0,2}\) is homotopy equivalent to an annulus \(\{z = u + iv \in \mathbb{C} \mid r < |z| \leq R\}\). The AF-algebra \(\mathbb{A}(x, S_{0,2})\) has the

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Bratteli diagram shown in Figure 1 and the surface $S_{0,2}$ has an ideal triangulation with one marked point on each boundary component [Fomin, Shapiro & Thurston 2008, Example 4.4] [2] given by the matrix:

$$B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$  (1.2)

The aim of our note is an interplay between the Laurent phenomenon in the cluster algebra $A(x, S_{0,2}) \cong K_0(A(x, S_{0,2}))$ and the Jones polynomials $V_L(t)$ [Jones 1985] [4]. The existence of such a link follows from a representation $\rho$ of the braid group $B_k$ given by the formulas in [7, Theorem 4.4.1]:

$$\begin{cases} B_{2g+1} \to A(x, S_{g,1}) \\ B_{2g+2} \to A(x, S_{g,2}) \end{cases}.$$  (1.3)

To formalize our results, denote by $\mathcal{P} \subset \mathbb{Z}[t^{\pm 1}]$ a ring generated by the set $\{V_L(t) \mid L \text{ runs all links}\}$. Denote by $x_1$ and $x_2$ the independent variables of the rank 2 cluster algebra $A(x, S_{0,2})$. Let $\mathcal{P} \subset A(x, S_{0,2})$ be an AF-algebra defined by the truncated Pascal’s diagram shown in Figure 2 [Jones 1991] [5, pp. 36, 50]. Our main results are as follows.

**Theorem 1.1.** There exists an inclusion of the rings $\mathcal{J} \subset A(x, S_{0,2})$ and an isomorphism of the dimension groups $\mathcal{J} \cong K_0(P)$ induced by the substitution:

$$t^2 = \frac{2(x_1^2 + x_2^2 + 1)}{x_1 x_2}.$$  (1.4)

**Remark 1.2.** A combinatorial approach to the Laurent phenomenon and the Jones polynomials which is based on the continued fractions and the snake graphs was studied recently by [Lee & Schiffler 2019] [6].

The paper is organized as follows. Section 2 contains notation and definitions necessary for the proof of theorem 1.1. The proof of theorem 1.1 is given in Section 3. In Section 4 we calculate the Jones polynomials of two unlinked unknotted, the Hopf link and the trefoil knot using theorem 1.1. A discussion of related results can be found in Section 5.
2. Preliminaries

We shall briefly review the Jones polynomials, skein relation and cluster algebras of rank 2. We refer the reader to [Jones 1985] [4], [Jones 1991] [5] and [Sherman & Zelevinsky 2004] [8] for a detailed account.

2.1. Jones polynomials. By $A_n$ we denote an $n$-dimensional von Neumann algebra generated by the identity and the Jones projections $e_1, e_2, \ldots, e_n$, see [Jones 1985] [4] for the definition of $e_i$. Such projections are known to satisfy the relations $e_i e_{i\pm 1} e_i = \frac{1}{[M:N]} e_i, e_i e_j = e_j e_i, \text{ if } |i - j| \geq 2,$ and the trace formula: 

$$tr(e_n x) = \frac{1}{[M:N]} tr(x), \quad \forall x \in A_n.$$ 

The reader can verify that the relations for the Jones projections $e_i$ coincide with such for the generators $\sigma_i$ of the braid group after an adjustment of the notation $\sigma_i \mapsto \sqrt{t}(t+1)e_i - 1, [M:N] = 2 + t + \frac{1}{t}$. One gets a family $\rho_t$ of representations of the braid group $B_n$ into the Jones algebra $A_n$. To get a topological invariant of the closed braid $\hat{b}$ of $b \in B_n$ coming from the trace (a character) of the representation, one needs to choose a representation whose trace is invariant under the first and the second Markov moves of the braid $b$.

The trace is invariant of the first Markov move, because two similar matrices have the same trace for any representation from the family $\rho_t$. For the second Markov move the trace formula which (after obvious substitutions) takes the form 

$$tr(b \sigma_n) = -\frac{1}{[M:N]} tr(b) tr(b^{-1}) = -\frac{1}{[M:N]} tr(b).$$

In general $tr(b \sigma_n^{-1}) \neq tr(b)$, but one can always re-scale the trace to get the equality. Indeed, the second Markov move takes the braid from $B_n$ and replaces it by a braid from $B_{n-1}$; there is a finite number of such replacements because the algorithm stops for $B_1$. Therefore a finite number of re-scalings by the constants $-\frac{1}{[M:N]}$ will give a quantity invariant under the second Markov move; the quantity is known as the Jones polynomial of the closed braid $\hat{b}$. Let $b \in B_n$ be a braid and $exp(b)$ be the sum of all powers of generators $\sigma_i$ and $\sigma_i^{-1}$ in the word presentation of $b$ and let $L := \hat{b}$ be the closure of $b$. Thus an isotopy invariant of the link $L$ is given by the quantity:

$$V_L(t) := \left(-\frac{1}{\sqrt{t}}\right)^{n-1} (\sqrt{t})^{exp(b)} tr(b). \quad (2.1)$$

2.2. Skein relation. If the links differ from each other only in a small region, the trace invariant (2.1) can be calculated recursively. Namely, it is known that $V_K(t) = 1$, where $K$ is the unknot. Recall that any link $L$ can be obtained from $K$ by a finite number of local operations of adding an overpass or underpass to the
diagram of the link $L$. Denote by $L^+$ ($L^-$, resp.) a link obtained by adding an
overpass (underpass, resp.) to the link $L$.

**Theorem 2.1.** ([4, Theorem 12]) Each $V_L(t)$ can be obtained from the surgery of
$K$ using the skein relation:

$$\frac{1}{t}V_L - tV_L^+ = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_L. \tag{2.2}$$

2.3. **Cluster algebras of rank 2.** For a pair of positive integers $b$ and $c$, we define
the matrix:

$$B = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}. \tag{2.3}$$

The reader can verify, that the exchange relations (1.1) for $B$ take the form:

$$x_{i-1}x_{i+1} = \begin{cases} 1 + x_i^b & \text{if } i \text{ odd}, \\ 1 + x_i^c & \text{if } i \text{ even}. \end{cases} \tag{2.4}$$

Let us consider the field of rational functions in two commuting independent
variables $x_1$ and $x_2$ with the rational coefficients. We shall write $\mathcal{A}(b, c)$ for a
cluster algebra of rank 2 generated by the variables $x_i$ [Sherman & Zelevinsky
2004] [8, Section 2]. Denote by $\mathcal{B}$ a basis of the algebra $\mathcal{A}(b, c)$.

**Theorem 2.2.** ([8, Theorem 2.8]) Suppose that $b = c = 2$ or $b = 1$ and $c = 4$.
Then $\mathcal{B} = \{x_p^p x_{i+1}^q \mid p, q \geq 0\} \cup \{T_n(x_1 x_4 - x_2 x_3) \mid n \geq 1\}$, where $T_n(x)$ are the
Chebyshev polynomials of the first kind.

3. **PROOF OF THEOREM 1.1**

Let us outline the main idea. Recall that each cluster variable $x_i$ in $\mathcal{A}(x, S_{0,2})$ can
be obtained from the initial cluster $(x_1, x_2)$ by a sequence of mutations described
by the exchange relations (1.1). Likewise, each Jones polynomial $V_L(t)$ can be
obtained from the $V_K(t)$ of the unknot $K$ by a local surgery of the knot diagram
described by the skein relation (2.2). Roughly speaking, it will be shown that (1.1)
and (2.2) are equivalent relations modulo the substitution $t^2 = \frac{2(x_1^2 + x_2^2 + 1)}{x_1 x_2}$. We
spilt the proof in a series of lemmas.

**Lemma 3.1.** The substitution

$$\begin{cases} W_{L^+} = \left(\frac{t+1}{\sqrt{t}}\right) V_{L^+} \\ W_{L^-} = \left(\frac{t+1}{\sqrt{t}}\right) V_{L^-} \\ W_L = (t^2 - 1)V_L \end{cases} \tag{3.1}$$

brings the skein relation (2.2) to the form:

$$W_L = t^2 W_{L^+} - W_{L^-}. \tag{3.2}$$

**Proof.** Indeed, let us multiply both sides of the equation (2.2) by $\frac{(t^2+1)\sqrt{t}}{t^2-1}$. After
obvious algebraic operations, the skein relation (2.2) can be written in the form:

$$(t^2 - 1)V_L = \frac{t+1}{\sqrt{t}} \left[V_{L^-} - t^2 V_{L^+}\right]. \tag{3.3}$$
Using the substitution (3.1), one can rewrite (3.3) in the form:

$$W_L = t^2 W_{L^+} - W_{L^-}.$$  \hspace{1cm} (3.4)

Thus the skein relation (2.2) is equivalent to the relation (3.4). Lemma 3.1 is proved. \hfill \Box

Lemma 3.2. The exchange relations (1.1) corresponding to the matrix $B$ given by formula (1.2) imply an exchange relation:

$$T_{n+1}(x) = 2(x_1 x_4 - x_2 x_3) T_n(x) - T_{n-1}(x),$$  \hspace{1cm} (3.5)

where $T_n(x)$ are the Chebyshev polynomials of the first kind.

Proof. In view of Theorem 2.2, the basis $\mathcal{B}$ of the cluster algebra $A(x, S_{0,2})$ has the form

$$\mathcal{B} = \{x^p x_{i+1}^q \mid p, q \geq 0\} \cup \{T_n(x_1 x_4 - x_2 x_3) \mid n \geq 1\}. \hspace{1cm} (3.6)$$

On the other hand, it is well known that the Chebyshev polynomials of the first kind satisfy the recurrence relation:

$$\begin{cases}
T_0(x) = 1 \\
T_1(x) = x \\
T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x).
\end{cases} \hspace{1cm} (3.7)$$

Since the $A(x, S_{0,2})$ is a cluster algebra of rank 2, one can express via the exchange relation any cluster variable $x_i$ as a rational function of the initial seed $x = (x_1, x_2)$ [Williams 2014] [9]. We let $x = (T_{n-1}(x), T_n(x))$, where $x = x_1 x_4 - x_2 x_3$. It follows from (3.7) that the cluster variable $T_{n+1}(x)$ is a rational function of the $T_{n-1}(x)$ and $T_n(x)$ of the form:

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x). \hspace{1cm} (3.8)$$

Since such a rational function is unique, we conclude that (3.8) is an exchange relation in the cluster algebra $A(x, S_{0,2})$. Lemma 3.2 is proved. \hfill \Box

Remark 3.3. Apart from (3.8) there are other exchange relations in the cluster algebra $A(x, S_{0,2})$ coming from the elements $x^p x_{i+1}^q$ of the basis $\mathcal{B}$. It is not hard to see, that it is the reason why Pascal’s triangle is Figure 2 is truncated [Jones 1991] [5, p. 36, 50] and one gets an inclusion $\mathcal{B} \subset A(x, S_{0,2})$ rather than an isomorphism.

Corollary 3.4. The skein relation (3.2) is equivalent to the exchange relation (3.5) modulo the equation

$$t^2 = 2(x_1 x_4 - x_2 x_3). \hspace{1cm} (3.9)$$

Proof. The proof is comparison of equations (3.2) and (3.5), where

$$\begin{cases}
W_{L}(t) = T_{n+1}(x) \\
W_{L^+}(t) = T_n(x) \\
W_{L^-}(t) = T_{n-1}(x).
\end{cases} \hspace{1cm} (3.10)$$

\hfill \Box
Lemma 3.5. The equation (3.9) is equivalent to the equation
\[ t^2 = 2 \left( \frac{x_1^2 + x_2^2 + 1}{x_1x_2} \right). \] (3.11)

Proof. Since the \( \mathcal{A}(x, S_{0,2}) \) is a cluster algebra of rank 2, the cluster variables \( x_3 \) and \( x_4 \) in (3.9) must be rational functions of the cluster variables \( x_1 \) and \( x_2 \). To find an explicit formula, we shall use the exchange relations (2.4). In our case \( b = c = 2 \) and the exchange relations (2.4) take the form:
\[ x_i - x_{i+1} = x_{2i+1} + 1. \] (3.12)

From (3.12) one gets the following equations:
\[ \begin{cases} 
  x_3 = \frac{x_2^2 + 1}{x_1} \\
  x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^2 + 1}{x_1^2} \end{cases} \] (3.13)

Using equations (3.13) we conclude that:
\[ x_1x_4 - x_2x_3 = \frac{x_1^2 + x_2^2 + 1}{x_1x_2}. \] (3.14)

Lemma 3.5 follows from (3.9) and (3.14). \( \square \)

Lemma 3.6. \( \mathcal{J} \subset \mathcal{A}(x, S_{0,2}) \).

Proof. (i) Recall that the ring \( \mathcal{J} \subset \mathbb{Z}[t^{\pm \frac{1}{2}}] \) is generated by all Jones polynomials \( V_L(t) \). The map \( u \mapsto t^2 \) defines an embedding \( \mathcal{J} \subset \mathbb{Z}[t^{\pm 1}] \).

(ii) On the other hand, each \( V_L(u) \) can be obtained from the Jones polynomial \( V_K(u) \) of the trivial link using the skein relations (2.2). By lemma 3.1, corollary 3.4 and lemma 3.5, one gets an inclusion:
\[ \mathcal{J} \subset \mathcal{A}(x, S_{0,2}), \] (3.15)
where \( \mathcal{J} \) is generated by the Chebyshev polynomials \( T_n \left[ 2 \left( \frac{x_1^2 + x_2^2 + 1}{x_1x_2} \right) \right] \). We refer the reader to remark 3.3 for the extra details. Lemma 3.6 is proved. \( \square \)

Lemma 3.7. \( \mathcal{J} \cong K_0(\mathcal{P}) \).

Proof. (i) Recall that the cluster algebra \( \mathcal{A}(x, S_{0,2}) \) has the structure of a dimension group [7, Section 4.4]. Lemma 3.6 says that \( \mathcal{J} \subset \mathcal{A}(x, S_{0,2}) \) is a cluster sub-algebra, hence a dimension group with the order structure inherited from the \( \mathcal{A}(x, S_{0,2}) \)

(ii) On the other hand, comparing the Bratteli diagrams in Figures 1 and 2, we conclude that \( \mathcal{P} \subset \mathcal{A}(x, S_{0,2}) \) is an inclusion of the AF-algebras. Thus one gets a commutative diagram shown in Figure 3.

(iii) In view of item (i), we obtain an isomorphism \( K_0(\mathcal{P}) \cong \mathcal{J} \) from the diagram in Figure 3. Lemma 3.7 is proved. \( \square \)

Theorem 1.1 follows from lemmas 3.5-3.7.
Theorem 1.1 by calculating the $V_L(t)$ of two unlinked unknots, the Hopf link and the trefoil knot. We refer the reader to the remarkable paper [Lee & Schiffer 2019] [6] in which the cluster algebras are used to calculate the Jones polynomials of the 2-bridge knots.

**Example 4.1.** (Two unknots) Denote by $S^1 \cup S^1$ two unlinked unknots. To calculate the Jones polynomial $V_{S^1 \cup S^1}(t)$ using theorem 1.1, we rewrite (3.8) in the form:

$$T_{n-1}(x) = 2x T_n(x) - T_{n+1}(x).$$

We let $n = 1$ and

$$
\begin{align*}
T_0(x) &= W_{S^1 \cup S^1}(t) = (t^2 - 1)V_{S^1 \cup S^1}(t) \\
T_1(x) &= W_{(S^1 \cup S^1)^+}(t) = \left(\frac{t+1}{\sqrt{t}}\right) V_{(S^1 \cup S^1)^+}(t) \\
T_2(x) &= W_{(S^1 \cup S^1)^-}(t) = \left(\frac{t+1}{\sqrt{t}}\right) V_{(S^1 \cup S^1)^-}(t),
\end{align*}
$$

compare with (3.1). From (3.2) one gets:

$$(t^2 - 1)V_{S^1 \cup S^1}(t) = t^2 \left(\frac{t+1}{\sqrt{t}}\right) V_{(S^1 \cup S^1)^+}(t) - \left(\frac{t+1}{\sqrt{t}}\right) V_{(S^1 \cup S^1)^-}(t).$$

Since $(S^1 \cup S^1)^+ \equiv (S^1 \cup S^1)^- \not\equiv K$ is the unknot, we have

$$V_{(S^1 \cup S^1)^+}(t) = V_{(S^1 \cup S^1)^-}(t) = 1.$$ 

Using (4.4) we calculate from equation (4.3):

$$(t^2 - 1)V_{S^1 \cup S^1}(t) = t^2 \left(\frac{t+1}{\sqrt{t}}\right) - \left(\frac{t+1}{\sqrt{t}}\right) = (t^2 - 1) \left(-\frac{t+1}{\sqrt{t}}\right).$$

Thus from (4.5) one gets the Jones polynomial of two unlinked unknots:

$$V_{S^1 \cup S^1}(t) = \left(-\frac{t+1}{\sqrt{t}}\right) = -t^{-\frac{1}{2}} - t^{\frac{1}{2}}.$$ 

**Example 4.2.** (Hopf link) Denote by $H$ the Hopf link. We let $n = 2$ and the substitution (4.2) brings (4.1) to the form:

$$(t^2 - 1)V_{H}(t) = t^2 \left(\frac{t+1}{\sqrt{t}}\right) V_{S^1 \cup S^1}(t) - \left(\frac{t+1}{\sqrt{t}}\right) V_{H}(t).$$
Figure 4. Bratteli diagram of the algebra $A(x, S_{1,1})$.

The $V_K(t) = 1$ for the unknot $K$ and $V_{S_1 \cup S_1}(t) = -t^{-\frac{1}{2}} - t^{\frac{1}{2}}$ for the unlinked unknots $S^1 \cup S^1$, see formula (4.6). The substitution of this data and subsequent reduction of equation (4.7) gives us the following equation:

$$
\left(\frac{t+1}{\sqrt{t}}\right) V_H(t) = -t^3 - t^2 - t - 1 = (t+1)(-t^2 - 1).
$$

(4.8)

One gets easily from (4.8) the Jones polynomial of the Hopf link:

$$
V_H(t) = -t^\frac{3}{2} - t^\frac{1}{2}.
$$

(4.9)

Example 4.3. (Trefoil knot) Denote by $T$ the trefoil knot. We let $n = 3$ and the substitution (4.2) brings (4.1) to the form:

$$(t^2 - 1)V_H(t) = t^2 \left(-\frac{t+1}{\sqrt{t}}\right) V_K(t) - \left(-\frac{t+1}{\sqrt{t}}\right) V_T(t).$$

(4.10)

The $V_K(t) = 1$ for the unknot $K$ and $V_H(t) = -t^\frac{3}{2} - t^\frac{1}{2}$ for the Hopf link $H$, see formula (4.9). The substitution of this data and subsequent reduction of equation (4.10) gives us the following equation:

$$
\left(\frac{t+1}{\sqrt{t}}\right) V_T(t) = -t^\frac{3}{2} + t^{\frac{1}{2}} + t^{\frac{3}{2}} + t^{\frac{1}{2}} = \left(\frac{t+1}{\sqrt{t}}\right) (-t^4 + t^3 + t).
$$

(4.11)

One gets from (4.11) the Jones polynomial of the trefoil knot:

$$
V_T(t) = -t^4 + t^3 + t.
$$

(4.12)

5. Remarks

An analog of theorem 1.1 for the HOMFLY polynomials [Freyd, Yetter, Hoste, Lickorish, Millet & Ocneanu 1985] [3] is proved in [7, Section 4.4.6.2]. In this case $g = n = 1$ and $S_{1,1}$ is a torus with a cusp. The matrix $B$ associated to an ideal triangulation of the Riemann surface $S_{1,1}$ has the form [Fomin, Shapiro & Thurston 2008, Example 4.6] [2]:

$$
B = \begin{pmatrix}
0 & 2 & -2 \\
-2 & 0 & 2 \\
2 & -2 & 0
\end{pmatrix}.
$$

(5.1)

It follows from the exchange relations (1.1) that the cluster $C^*$-algebra $A(x, S_{1,1})$ has the Bratteli diagram shown in Figure 4.
Remark 5.1. It is well known that the Jones polynomials can be obtained as a specialization of the HOMFLY polynomials. This fact follows from an observation that the graph in Figure 1 is a sub-graph of the graph in Figure 4. Hence we get an inclusion of the cluster C*-algebras:

\[ A(x, S_{0,2}) \subset A(x, S_{1,1}). \]  

Likewise, if one takes a double cover of the sphere by the torus ramified at four points taken for vertices of an ideal triangulation, then the triangulation of \( S_{0,2} \) is a sub-triangulation of the triangulation of \( S_{1,1} \) [Fomin, Shapiro & Thurston 2008] [2]. Equivalently, the matrix (1.2) can be obtained from a cancellation of the last row and column in the matrix (5.1). Unlike the case of the cluster algebras of rank 2 [Sherman & Zelevinsky 2004] [8, Section 2], the canonical bases for the cluster algebras of rank 3 are unknown. Therefore there is no immediate connection between the Chebyshev and HOMFLY polynomials. This fact can be viewed as a justification of the separate analysis of the Jones case.

Remark 5.2. A general result for the multivariable Laurent polynomials has been proved in [7, Theorem 4.4.1]. However, an explicit construction of such polynomials based on the exchange relations (1.1) is unclear at the moment.

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