STOCHASTIC DIFFERENTIAL EQUATIONS
DRIVEN BY G-BROWNIAN MOTION
WITH REFLECTING BOUNDARY CONDITIONS

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Abstract. In this paper, we introduce the idea of stochastic integral with respect to an increasing process in G-framework and extend G-Itô’s formula. Moreover, we study the solvability of the scalar valued stochastic differential equations driven by G-Brownian motion with reflecting boundary conditions (RGSDEs).

1. Introduction

In classical framework, Skorokhod [17, 18] firstly introduced the diffusion processes with reflecting boundary in 1960s. From then on, reflected solutions to SDEs and BSDEs were investigated by many authors. For one-dimensional case, El Karoui [3], El Karoui and Chaleyat-Maurel [4] and Yamada [24] studied reflected SDEs on a half-line, and El Karoui et al. [5] obtained the solvability of reflected BSDEs. For multidimensional case, the existence of weak solutions to reflected SDEs on a smooth domain was proved by Stroock and Varadhan [22]. After that, Tanaka [23] solved the similar problem on a convex domain by a direct approach based on the solution to Skorokhod problem. Furthermore, Lions and Sznitman [11] extended these results to a non convex domain. On the other hand, the corresponding results for reflected BSDEs can be found in Gegout-Petit and Pardoux [7], Ramasubramanian [15] and Hu and Tang [9], etc..

Motivated by uncertainty problems, risk measures and the superhedging in finance, Peng [13, 14] introduced a framework of time consistent nonlinear expectation $E[\cdot]$, in which a new type of Brownian motion is constructed and corresponding stochastic calculus was established. Moreover, Denis et al. [2] derived that $G$-expectation $E[\cdot]$ can be viewed as an upper expectation with respect to a weakly compact family $\mathcal{P}$ of probability measures, and they naturally defined a Choquet capacity $\bar{C}(\cdot)$ on $\Omega$:

$$\bar{C}(A) = \sup_{P \in \mathcal{P}} P(A), \quad A \subset \Omega.$$  

Thus, many concepts from capacity theory can be introduced to G-framework, especially, the notion of “quasi-surely” which means that a property holds true outside a polar set, i.e., outside a set $A \subset \Omega$ satisfies $\bar{C}(A) = 0$. Using the notation of “quasi-surely”, Gao [6] and Lin and Bai [1] have already worked on the solvability to SDEs driven by G-Brownian motion.

Meanwhile, Soner et al. [19, 20, 21] have established a complete theory for 2BSDEs under a uniform Lipschitz conditions which is closely related to G-expectation. In their framework, they also issued a similar notion, but their definition of “quasi-surely” means that a property holds $P$-a.s. for every probability measure $P$ in a non-dominated class of mutually singular...
measures. We notice that this definition is a little different compared to that made by $G$-capacity $\mathcal{C}$(·) in Denis et al. [2], since by their definition the null set $\mathcal{A}^c$ under each $\mathbb{P} \in \mathcal{P}$ could be different. More recently, Matoussi et al. [12] have studied the problem of reflected BSDEs with a lower obstacle.

The aim of this paper is to study the existence and uniqueness of solutions to stochastic differential equations driven by $G$-Brownian motion with reflecting boundary conditions (RGSDEs) in the sense of “quasi-surely” defined by Denis et al. [2]. The scalar valued RGSDE we consider is defined as following:

\[
\begin{aligned}
X_t &= x + \int_0^t f_s(X_s)ds + \int_0^t h_s(X_s)d(B)_s + \int_0^t g_s(X_s)dB_s + K_t, \text{ q.s., } 0 \leq t \leq T; \\
X_t &\geq S_t; \int_0^T (X_t - S_t)dK_t = 0,
\end{aligned}
\]

where $(B)$ is the quadratic variation process of $G$-Brownian motion $B$, and $K$ is an increasing process which pushes the solution $X$ upwards to be remaining above the obstacle $S$ in a minimal way. Similarly to classical reflected SDEs, the uniqueness result is deduced from a priori estimates and a solution in $\mathcal{M}_p^2([0,T])$ to (1.1) can be constructed by fixed-point iteration. To establish the comparison theorem, we need to develop an extension of $G$-Itô formula to deal with such process $X$ involves both the stochastic integrals and an increasing process. This extended $G$-Itô’s formula can have its own interest and may be used in other situations.

This paper is organized as follows: Section 2 introduces some notations and results in $G$-framework which is necessary for what follows. Section 3 introduces the stochastic calculus in a priori estimates and a solution in $\mathcal{M}_p^2([0,T])$ to (1.1) can be constructed by fixed-point iteration. To establish the comparison theorem, we need to develop an extension of $G$-Itô formula to deal with such process $X$ involves both the stochastic integrals and an increasing process. This extended $G$-Itô’s formula can have its own interest and may be used in other situations.

This paper is organized as follows: Section 2 introduces some notations and results in $G$-framework which is necessary for what follows. Section 3 introduces the stochastic calculus with respect to an increasing process in $G$-framework. Section 4 studies the reflected $G$-Brownian motion, while Section 5 is our main results.

2. G-Brownian motion, G-capacity and G-stochastic calculus

The main purpose of this section is to recall some preliminary results in $G$-framework which are needed in the sequel. The reader interested in a more detailed description of these notions is referred to Denis et al. [2], Gao [6] and Peng [14].

2.1. G-Brownian motion. Adapting the approach in Peng [14], let $\Omega$ be the space of all $\mathbb{R}$-valued continuous paths with $\omega_0 = 0$ equipped with the distance

\[
\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N}[\max_{t \in [0,N]}|\omega^1_t - \omega^2_t|] \land 1,
\]

$B$ the canonical process and $C_{l,Lip}(\mathbb{R}^n)$ the collection of all local Lipschitz functions on $\mathbb{R}^n$. For a fixed $T \geq 0$, the space of finite dimensional cylinder random variables is defined by

\[
L^0_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, 0 \leq t_1 \leq \ldots \leq t_n \leq T, \varphi \in C_{l,Lip}(\mathbb{R}^n)\},
\]

on which $\mathbb{E}[\cdot]$ is a sublinear functional satisfying: for all $X, Y \in C_{l,Lip}(\mathbb{R}^n),$

1) Monotonicity: if $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y];$
2) Sub-additivity: $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y];$
3) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, for all $\lambda \geq 0;$
4) Constant translatability: $\mathbb{E}[X + c] = \mathbb{E}[X] + c$, for all $c \in \mathbb{R}.$

The triple $(\Omega, L^0_{ip}(\Omega_T), \mathbb{E})$ is called a sublinear expectation space. A scalar valued random variable $X \in L^0_{ip}(\Omega_T)$ is $G$-normal distributed with parameters $(0, [\sigma^2, \overline{\sigma}^2])$, i.e., $X \sim \mathcal{N}(0, [\sigma^2, \overline{\sigma}^2]),$ if for each $\varphi \in C_{l,Lip}(\mathbb{R}), u(t, x) :=$
\[ \mathbb{E}[\varphi(x + \sqrt{t}X)] \] is a viscosity solution to the following PDE on \([0, +\infty) \times \mathbb{R}:
\begin{align*}
\frac{\partial u}{\partial t} - G \left( \frac{\partial^2 u}{\partial x^2} \right) &= 0;
\end{align*}
where
\[ G(a) := \frac{1}{2}(a^+ - a^-), \quad a \in \mathbb{R}. \]

**Remark 2.2.** Without loss of generality, we always assume that \( \sigma^2 = 1 \) in what follows.

**Definition 2.3.** We call a sublinear expectation \( \mathbb{E} : L^0_{lp}(\Omega_T) \to \mathbb{R} \) a \( G \)-expectation if the canonical process \( B \) is a \( G \)-Brownian motion under \( \mathbb{E}[\cdot] \), that is, for all \( 0 \leq s \leq t \leq T \), the increment \( B_t - B_s \sim \mathcal{N}(0, [(t-s)^2, (t-s)]) \) and for all \( n \in \mathbb{N} \), \( 0 \leq t_1 \leq \ldots \leq t_n \leq t \), and \( \varphi \in C_{l, Lip}(\mathbb{R}^n) \),
\[ \mathbb{E}[\varphi(B_{t_1}, \ldots, B_{t_{n-1}}, B_{t_n} - B_{t_{n-1}})] = \mathbb{E}[\psi(x_1, \ldots, x_{n-1}, \sqrt{t_n - t_{n-1}})], \]
where \( \psi(x_1, \ldots, x_{n-1}) := \mathbb{E}[\varphi(x_1, \ldots, x_{n-1})] \), \( x \in \mathbb{R}^n \).

For \( p \geq 1 \), we denote by \( L^p_G(\Omega_T) \) the completion of \( L^p_{lp}(\Omega_T) \) under the Banach norm \( \mathbb{E}[|\cdot|^p]^\frac{1}{p} \).

**2.2. G-capacity.** Derived in Denis et al. \[2\], \( G \)-expectation \( \mathbb{E}[\cdot] \) can be viewed as an upper expectation \( \bar{\mathbb{E}}[\cdot] \) associated with a weakly compact family \( \mathcal{P} \) on \( L^1_G(\Omega_T) \), i.e.,
\[ \mathbb{E}[X] = \mathbb{E}[X] := \sup_{P \in \mathcal{P}} E^P[X], \quad X \in L^1_G(\Omega_T). \]

In this sense, the domain of \( G \)-expectation can be extended from \( L^1_G(\Omega_T) \) to the space of all \( \mathcal{B}(\Omega_T) \) measurable random variables \( L^0(\Omega_T) \) by setting
\[ \bar{\mathbb{E}}[X] := \mathbb{E}^P[X], \quad X \in L^0(\Omega_T). \]

Naturally, we can define a corresponding regular Choquet capacity on \( \Omega \):
\[ \mathcal{C}(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega), \]
with respect to which, we have the following notions:

**Definition 2.4.** A set \( A \in \mathcal{B}(\Omega) \) is called polar if \( \mathcal{C}(A) = 0 \). A property is said to hold quasi-surely (q.s.) if it holds outside a polar set.

**Definition 2.5.** A random variable \( X \) is said to be quasi-continuous (q.c.) if for arbitrarily small \( \varepsilon > 0 \), there exists an open set \( O \subset \Omega \) with \( \mathcal{C}(O) < \varepsilon \) such that \( X \) is continuous in \( \omega \) on \( O^c \).

**Definition 2.6.** We say that a random variable \( X \) has a q.c. version if there exists a q.c. random variable \( Y \) such that \( X = Y \), q.s..

In a language of \( G \)-capacity, Denis et al. \[2\] proved that for \( p \geq 1 \), the function space \( L^p_G(\Omega_T) \) has a dual representation which is much more explicit to verify:

**Theorem 2.7.**
\[ L^p_G(\Omega_T) = \{ X \in L^0(\Omega_T) : X \text{ has a q.c. version, } \lim_{N \to +\infty} \mathbb{E}[|X|^p 1_{|X| > N}] = 0 \}. \]

Unlike in classical framework, the downwards monotone convergence theorem only holds true for a sequence of random variables from a subset of \( L^0(\Omega_T) \) (cf. Theorem 31 in Denis et al. \[2\]).

**Theorem 2.8.** Let \( \{X_n\}_{n \in \mathbb{N}} \subset L^1_G(\Omega_T) \) be such that \( X_n \downarrow X \), q.s., then \( \mathbb{E}[X_n] \downarrow \mathbb{E}[X] \).

**Remark 2.9.** We note that dominated convergence theorem does not exist in \( G \)-framework, even though we assume that \( \{X_n\}_{n \in \mathbb{N}} \) is a sequence in \( L^1_G(\Omega_T) \). The lack of this theorem is one of the main difficulties we shall overcome in the following sections.
2.3. G-stochastic calculus. In Peng [14], generalized Itô integrals with respect to G-Brownian motion are established:

**Definition 2.10.** A partition of \([0, T]\) is a finite ordered subset \(\pi^N_{[0, T]} = \{t_0, t_1, \ldots, t_N\}\) such that \(0 = t_0 < t_1 < \ldots < t_N = T\). We set

\[
\mu(\pi^N_{[0, T]}) := \max_{k=0,1,\ldots,N-1} |t_{k+1} - t_k|.
\]

For \(p \geq 1\), define

\[
M^p_G([0, T]) := \left\{ \eta_t = \sum_{k=0}^{N-1} \xi_k \mathbb{I}_{[t_k, t_{k+1})}(t) : \xi_k \in L^p_G(\Omega_{t_k}) \right\},
\]

and denote by \(M^p_G([0, T])\) the completion of \(M^p_G([0, T])\) under the norm

\[
||\eta||_{M^p_G([0, T])} := \left( \frac{1}{T} \int_0^T \mathbb{E}[|\eta_t|^p] dt \right)^{\frac{1}{p}}.
\]

**Remark 2.11.** By Definition 2.10, if \(\eta\) is an element in \(M^p_G([0, T])\), then there exists a sequence \(\{\eta^n\}_{n \in \mathbb{N}}\) in \(M^p_G([0, T])\), such that \(\lim_{n \to \infty} \int_0^T \mathbb{E}[|\eta_t^n - \eta_t|^p] dt \to 0\). It is readily observed that for almost every \(t \in [0, T]\), \(\{\eta^n_t\}_{n \in \mathbb{N}} \subset L^p_G(\Omega_t)\) and \(\mathbb{E}[|\eta^n_t - \eta_t|^p] \to 0\), thus \(\eta_t\) is an element in \(L^p_G(\Omega_t)\).

**Definition 2.12.** For each \(\eta \in M^2_G([0, T])\), we define

\[
\mathcal{I}_{[0, T]}(\eta) = \int_0^T \eta_t dB_s := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}} - B_{t_k}).
\]

The mapping \(\mathcal{I}_{[0, T]} : M^2_G([0, T]) \to L^2_G(\Omega_T)\) is continuous and linear and thus can be uniquely extended to \(\mathcal{I}_{[0, T]} : M^2_G([0, T]) \to L^2_G(\Omega_T)\). Then, for each \(\eta \in M^2_G([0, T])\), the stochastic integral with respect to G-Brownian motion \(B\) is defined by \(\int_0^T \eta_t dB_s := \mathcal{I}_{[0, T]}(\eta)\).

Unlike the classical theory, the quadratic variation process of G-Brownian motion \(B\) is not always a deterministic process (unless \(\varpi = \sigma\)) and it can be formulated in \(L^2_G(\Omega_t)\) by

\[
\langle B \rangle_t := \lim_{\mu(\pi^N_{[0, t]}) \to 0} \sum_{k=0}^{N-1} (B_{t_{k+1}}^2 - B_{t_k}^2) = B_t^2 - 2 \int_0^t B_s dB_s.
\]

**Definition 2.13.** For each \(\eta \in M^2_G([0, T])\), we define

\[
\mathcal{Q}_{[0, T]}(\eta) = \int_0^T \eta_t dB_s := \sum_{k=0}^{N-1} \xi_k (\langle B \rangle_{t_{k+1}} - \langle B \rangle_{t_k}).
\]

The mapping \(\mathcal{Q}_{[0, T]} : M^2_G([0, T]) \to L^2_G(\Omega_T)\) is continuous and linear and thus can be uniquely extended to \(\mathcal{Q}_{[0, T]} : M^2_G([0, T]) \to L^2_G(\Omega_T)\). Then, for each \(\eta \in M^2_G([0, T])\), the stochastic integral with respect to the quadratic variation process \(\langle B \rangle\) is defined by \(\int_0^T \eta_t d\langle B \rangle_s := \mathcal{Q}_{[0, T]}(\eta)\).

In view of the dual formulation of G-expectation as well as the properties of the quadratic variation process \(\langle B \rangle\) in G-framework, the following BDG type inequalities are obvious.

**Lemma 2.14.** Let \(p \geq 1\), \(\eta \in M^p_G([0, T])\) and \(0 \leq s \leq t \leq T\). Then,

\[
\mathbb{E}\left[ \sup_{s \leq u \leq t} \left| \int_s^u \eta_t dB_s \right|^p \right] \leq |t - s|^{p-1} \int_s^t \mathbb{E}[|\eta_u|^p] du.
\]

**Lemma 2.15.** Let \(p \geq 2\), \(\eta \in M^p_G([0, T])\) and \(0 \leq s \leq t \leq T\). Then,

\[
\mathbb{E}\left[ \sup_{s \leq u \leq t} \left| \int_s^u \eta_t dB_s \right|^p \right] \leq C_p \mathbb{E}\left[ \int_s^t |\eta_u|^2 du \right]^{\frac{p}{2}} \leq C_p |t - s|^{p-1} \int_s^t \mathbb{E}[|\eta_u|^p] du,
\]

where \(C_p\) is a positive constant independent of \(\eta\).
3. Stochastic calculus with respect to an increasing process

In this section, we define the stochastic integrals with respect to an increasing process with continuous paths and then we extend G-Itô’s formula to the case where an increasing process appears in the dynamics. In the sequel, $C$ and $M$ denote two positive constants whose values may vary from line to line.

3.1. Stochastic integrals with respect to an increasing process.

**Definition 3.1.** We denote by $M_c([0,T])$ the collection of all q.s. continuous processes $X$ whose paths $X(\omega) : t \mapsto X_t(\omega)$ are continuous in $t$ on $[0,T]$ outside a polar set $A$.

**Remark 3.2.** For example, from the proofs to Theorem 2.1 and Theorem 2.2 in Gao [6], $(\int_0^t \eta_s dB_s)_{0 \leq t \leq T}$ and $(\int_0^t \eta_s d(B_s)_0 \leq t \leq T$ have continuous modifications in $M_c([0,T])$.

**Definition 3.3.** We denote by $M_f([0,T])$ the collection of q.s. increasing processes $K \in M_c([0,T])$ whose paths $K(\omega) : t \mapsto K_t(\omega)$ are increasing in $t$ on $[0,T]$ outside a polar set $A$.

**Remark 3.4.** Obviously, an increasing process $K$ in $M_f([0,T])$ has q.s. finite total variation on $[0,T]$, and thus its quadratic variation is q.s. $0$.

**Definition 3.5.** We define, for a fixed $X \in M_c([0,T])$, the stochastic integral with respect to a given $K \in M_f([0,T])$ by

\[
\left( \int_0^T X_t dK_t \right)(\omega) = \left\{ \begin{array}{ll} 
\int_0^T X_t(\omega)dK_t(\omega), & \omega \in A^c; \\
0, & \omega \in A,
\end{array} \right.
\]

where $A$ is a polar set, and on the complementarity of which, $X(\omega)$ is continuous and $K(\omega)$ is increasing in $t$.

**Remark 3.6.** Since for a fixed $\omega \in A^c$, the function $X(\omega)$ is continuous and the function $K(\omega)$ is of bounded variation on $[0,T]$, the Riemann-Stieltjes integral on the right side always exists (cf. Hildebrandt [2]). Thus, (3.1) is well defined. Similar definition can be made for those $X$ whose paths are q.s. piecewisely continuous and without discontinuity of the second kind, i.e., for each $\omega \in A^c$, the function $X(\omega)$ is discontinuous at a finite number of points, and these discontinuous points are removable or of the first kind.

**Remark 3.7.** Given a sequence of refining partitions $\{\pi^N_{[0,T]}\}_{N \in \mathbb{N}}$ (i.e., $\pi^N_{[0,T]} \subset \pi^{N+1}_{[0,T]}$ for all $N \in \mathbb{N}$) such that $\mu(\pi^N_{[0,T]}) \to 0$, we set a sequence of binary functions:

\[
V^N_{[0,T]}(X, K)(\omega) = \sum_{k=0}^{N-1} X_{u^N_k}(\omega)(K_{t^N_{k+1}}(\omega) - K_{t^N_k}(\omega)),
\]

where $u^N_k \in [t^N_k, t^N_{k+1})$. For a fixed $\omega \in A^c$, by the Heine-Cantor theorem, $X(\omega)$ and $K(\omega)$ are uniformly continuous in $t$ on $[0,T]$. So we can find a $M_\omega > 0$ such that $K_T(\omega) < M_\omega$, then for arbitrage small $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $|t - s| < \delta$, $|X_t(\omega) - X_s(\omega)| < \varepsilon / M_\omega$. It is sufficient to choose an $N_0 \in \mathbb{N}$ such that $\mu(\pi^N_{[0,T]}) < \delta$, then, for all $N > N_0$,

\[
\left| V^N_{[0,T]}(X, K)(\omega) - \left( \int_0^T X_t dK_t \right)(\omega) \right| < \varepsilon,
\]

from which we deduce that

\[
V^N_{[0,T]}(X, K) \rightarrow \int_0^T X_t dK_t, \text{ q.s., as } N \rightarrow +\infty.
\]

The construction of the sequence (3.2) provides a q.s. approximation to the stochastic integral $\int_0^T X_t dK_t$. We note that the convergence (3.3) depends only on the sequence of refined partitions $(\pi^N_{[0,T]})_{N \in \mathbb{N}}$ but is independent of the selection of the points of division and the representatives $X_{u^N_k}$ on $[t^N_k, t^N_{k+1})$, $k = 0, 1, \ldots, N - 1, N \in \mathbb{N}$.
The following propositions can be verified directly by Definition 3.5 and the Heine-Cantor theorem.

**Proposition 3.8.** Let $X, X^1$ and $X^2 \in M_c([0,T])$, $K, K^1$ and $K^2 \in M_I([0,T])$ and $0 \leq s \leq r \leq t \leq T$, then we have

1. \( \int_s^r X_u dK_u = \int_s^r X_u dK_u + \int_r^t X_u dK_u, \) q.s.;
2. \( \int_s^r (\alpha X_u + X^2_u) dK_u = \alpha \int_s^r X^1_u dK_u + \int_r^t X^2_u dK_u, \) q.s., where \( \alpha \in L^0(\Omega) \);
3. \( \int_s^r X_u d(K^1 + K^2)_u = \int_s^r X_u dK^1_u + \int_r^t X_u dK^2_u, \) q.s.

**Remark 3.9.** By a classical argument, a q.s. continuous and bounded variation process can be viewed as the difference of two increasing processes $K_1 - K_2$, where $K_1$ and $K_2 \in M_I([0,T])$. By Proposition 3.8, the stochastic integral with respect to $K_1 - K_2$ can be defined in the same way as Definition 3.5.

**Proposition 3.10.** Let $X \in M_c([0,T])$ and $K \in M_I([0,T])$, the integral \( \int_0^T X_u dK_u \) is q.s. continuous in $t$, i.e., \( (\int_0^r X_u dK_u)_{0 \leq r \leq T} \in M_c([0,T]) \).

As showed above, (3.1) defines a random variable \( \int_0^T X_u dK_t \) in $L^0(\Omega_T)$. A nature question comes out: if we assume that for some appropriate $p$ and $q$, $X \in M^q_p([0,T])$ and $K \in M^q_G([0,T])$, this random variable \( \int_0^T X_u dK_t \) can be verified as an element in $L^p_G(\Omega_T)$ or not. In general, the answer is negative. That is because the integrability of $X$ and $K$ can not ensure the quasi-continuity of \( \int_0^T X_u dK_t \) (cf. Definition 2.3 and Theorem 2.7). More precisely, the pathwise convergence (3.3) is not necessarily uniform in $\omega$ outside a polar set $A$, and it is hard to verify directly the convergence in the sense of $L^p_G(\Omega_T)$ due to the lack of dominated convergence theorem in $G$-framework. But in some special cases, a proper sequence \( \{V^n_{[0,T]}(X,K)\}_{n \in \mathbb{N}} \) approximating to \( \int_0^T X_u dK_t \) can be found, so that the quasi-continuity is inherited during the approximation.

**Proposition 3.11.** Let $K \in M_I([0,T]) \cap M^2_G([0,T])$, $K_T \in L^2_G(\Omega_T)$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then \( \int_0^T \phi(K_t) dK_t \) is an element in $L^2_G(\Omega_T)$.

**Proof:** Consider a sequence of refining partitions \( \{\pi^N_{[0,T]}\}_{N \in \mathbb{N}} \) mentioned in Remark 3.7 and define the sequence of approximation: for $N \in \mathbb{N}$,

\[
V_{[0,T]}^N(\phi(K),K)(\omega) = \sum_{k=0}^{N-1} \phi(K_{t_{k+1}}^N)(\omega)(K_{t_{k+1}}^N(\omega) - K_{t_k}^N(\omega)).
\]

From the explanation in Remark 2.11, we can always assume that at the points of division, $K_{t_k}^N \in L^2_G(\Omega_T), k = 0, 1, \ldots, N - 1, N \in \mathbb{N}$. Since $K$ is increasing, we have

\[
\left| V_{[0,T]}^N(\phi(K),K) - \int_0^T \phi(K_t) dK_t \right| \leq \left| \int_0^T \left( \sum_{k=0}^{N-1} |K_{t_{k+1}}^N - K_{t_k}^N| I_{[t_k,t_{k+1})}(t) \right) dK_t \right|
\]

\[
\leq \sum_{k=0}^{N-1} |K_{t_{k+1}}^N - K_{t_k}^N| \leq 0, \text{ q.s., as } N \to +\infty.
\]

On the other hand, it is easy to verify by Theorem 2.7 that for all $N \in \mathbb{N}$, $V_{[0,T]}^N(\phi(K),K)$ and $\sum_{k=0}^{N-1} |K_{t_{k+1}}^N - K_{t_k}^N|^2 \in L^1_G(\Omega_T)$. Then, by Theorem 2.8

\[
\mathbb{E} \left[ \left| V_{[0,T]}^N(\phi(K),K) - \int_0^T \phi(K_t) dK_t \right| \right] \leq \mathbb{E} \left[ \sum_{k=0}^{N-1} |K_{t_{k+1}}^N - K_{t_k}^N|^2 \right] \leq 0, \text{ as } N \to +\infty.
\]

From the completeness of $L^1_G(\Omega_T)$ under $\mathbb{E}[\cdot]$, we deduce the desired result.

**Remark 3.12.** To verify that for all $N \in \mathbb{N}$, $V_{[0,T]}^N(\phi(K),K)$ and $\sum_{k=0}^{N-1} |K_{t_{k+1}}^N - K_{t_k}^N|^2 \in L^1_G(\Omega_T)$, we should assume here that $K_T \in L^2_G(\Omega_T)$. 
Proposition 3.13. Let $X$ be a q.s. continuous G-Itô process such that

$$X_t = x + \int_0^t f_s ds + \int_0^t h_s d(B)_s + \int_0^t g_s dB_s, \ 0 \leq t \leq T,$$

where $f$, $h$ and $g$ are elements in $M^p_T([0, T])$, $p > 2$. Let $K \in M_1([0, T]) \cap M^q_T([0, T])$ and $K_T \in L^p_T(\Omega_T)$, where $1/p + 1/q = 1$. Then, $\int_0^T X_t dK_t$ is an element in $L^p_T(\Omega_T)$.

Proof: Given a sequence of refining partitions $\{\pi^N_{[0, T]}\}_{N \in \mathbb{N}}$, we construct the sequence \(3.2\).

By the definitions of stochastic integrals and the BDG type inequalities, one can verify that for each $t \in [0, T]$, $X_t \in L^p_T(\Omega_t)$. Therefore, for all $N \in \mathbb{N}$, $\mathcal{V}^N_{[0, T]}(X, K) \in L^p_T(\Omega_T)$. Applying the BDG type inequalities, we have

$$\mathbb{E}[\sup_{s \leq t} |X_u - X_s|^p] \leq C \left( \sup_{0 \leq s \leq t} |X_u - X_s|^p \right)$$

Thus,

$$\mathbb{E}[\sup_{k \in [0, N) \cap \mathbb{N}} \sup_{t_k \leq t \leq t_{k+1}} |X_t - X_{t_k}^N|^p] \leq \mathbb{E} \left( \sum_{k=0}^{N-1} \sup_{t_k \leq t \leq t_{k+1}} |X_t - X_{t_k}^N|^p \right)$$

(3.5) $\leq C \left( \sup_{k \in [0, N) \cap \mathbb{N}} \sup_{t_k \leq t \leq t_{k+1}} |X_t - X_{t_k}^N|^p \right) \leq C \left( \sup_{k \in [0, N) \cap \mathbb{N}} \sup_{t_k \leq t \leq t_{k+1}} |X_t - X_{t_k}^N|^p \right)$

From the integrability of $f$, $h$ and $g$, we have

$$\mathbb{E}[\sup_{k \in [0, N) \cap \mathbb{N}} \sup_{t_k \leq t \leq t_{k+1}} |X_t - X_{t_k}^N|^p] \leq CM(\mu(\pi^N_{[0, T]}))^{p-1} + \mu(\pi^N_{[0, T]})^{\frac{p}{q}-1}.$$

For $N \in \mathbb{N}$, we calculate

$$\mathcal{V}^N_{[0, T]}(X, K) - \int_0^T X_t dK_t \leq \sum_{k=0}^{N-1} X_{t_k}^N I_{(t_k, t_{k+1})}(t) - X_t dK_t$$

$$\leq \sup_{0 \leq t \leq T} \left| \sum_{k=0}^{N-1} X_{t_k}^N I_{(t_k, t_{k+1})}(t) - X_t \right| K_T \leq K_T \sup_{k \in [0, N) \cap \mathbb{N}} \sup_{t_k \leq t \leq t_{k+1}} |X_t - X_{t_k}^N|.$$

Consequently,

$$\mathbb{E}[\mathcal{V}^N_{[0, T]}(X, K) - \int_0^T X_t dK_t] \leq \mathbb{E}[K_T \sup_{k \in [0, N) \cap \mathbb{N}} \sup_{t_k \leq t \leq t_{k+1}} |X_t - X_{t_k}^N|]$$

$$\leq \left( \mathbb{E}\left[ \sup_{k \in [0, N) \cap \mathbb{N}} \sup_{t_k \leq t \leq t_{k+1}} |X_t - X_{t_k}^N|^p \right] \right)^{\frac{1}{p}} (\mathbb{E}[K_T^2])^{\frac{1}{2}}$$

$$\leq CM(\mu(\pi^N_{[0, T]}))^{p-1} + \mu(\pi^N_{[0, T]})^{\frac{p}{q}-1} \to 0, \text{ as } N \to +\infty.$$

The desired result follows.

3.2. An extension of G-Itô’s formula. For $0 \leq s \leq t \leq T$, consider a sum of a G-Itô process and an increasing process $K$:

$$X_t = X_s + \int_s^t f_s du + \int_s^t h_s d(B)_s + \int_s^t g_s dB_s + K_t - K_s.$$

Lemma 3.14. Let $\Phi \in \mathcal{C}^2(\mathbb{R})$ be a real function with bounded and Lipschitz derivatives. Let $f$, $h$ and $g$ be bounded processes in $M^p_T([0, T])$, and $K \in M_1([0, T]) \cap M^q_T([0, T])$ satisfies for each $t \in [0, T]$,

$$\lim_{s \to t} \mathbb{E}[|K_t - K_s|^p] = 0.$$
Then,
\[
\Phi(X_t) - \Phi(X_s) = \int_s^t \frac{d\Phi}{dx}(X_u) f_u \, du + \int_s^t \frac{d\Phi}{dx}(X_u) h_u \, dB_u \\
+ \int_s^t \frac{d\Phi}{dx}(X_u) g_u \, dB_u + \int_s^t \frac{d\Phi}{dx}(X_u) dK_u \\
+ \frac{1}{2} \int_s^t \frac{d^2\Phi}{dx^2}(X_u) g_u^2 \, d\langle B \rangle_u, \text{ q.s..}
\]  
(3.7)

The proof of this lemma is based on some previous results in Peng [13] (cf. Lemma 6.1 and Proposition 6.3 in Chapter III). To avoid redundancy, we first prove a reduced lemma when \( f = h = g \equiv 0 \) to show how the increasing process \( K \) plays a role in this dynamic, and then we give sketch to indicate some key points to combine the simple lemma with the previous results in Peng [13].

**Lemma 3.15.** Let \( \Phi \in C^2(\mathbb{R}) \) be a real function with bounded and Lipschitz derivatives, and \( K \in M_f([0,T]) \cap M^2([0,T]) \). Then,
\[
\Phi(K_t) - \Phi(K_s) = \int_s^t \frac{d\Phi}{dx}(K_u) \, dK_u, \text{ q.s..}
\]

**Proof:** Consider a sequence of refining partitions \( \{\pi^N_{[s,t]}\}_{N \in \mathbb{N}} \). For \( N \in \mathbb{N} \), from the second order Taylor expansion, we have
\[
\Phi(K_t) - \Phi(K_s) = \sum_{k=0}^{N-1} (\Phi(K^N_{t_{k+1}}) - \Phi(K^N_{t_k})) \\
= \sum_{k=0}^{N-1} \frac{d\Phi}{dx}(K^N_{t_k}) (K^N_{t_{k+1}} - K^N_{t_k}) + \frac{1}{2} \sum_{k=0}^{N-1} \frac{d^2\Phi}{dx^2}(K^N_{t_k}) (K^N_{t_{k+1}} - K^N_{t_k})^2,
\]
where \( K^N_{t_k} \) satisfies \( K^N_{t_k} \leq K^N_{t_{k+1}} \leq K^N_{t_k}, \text{ q.s..} \). For the first part, similar to that in Remark 3.7, we obtain
\[
\lim_{N \to +\infty} \left| \sum_{k=0}^{N-1} \frac{d\Phi}{dx}(K^N_{t_k}) (K^N_{t_{k+1}} - K^N_{t_k}) - \int_s^t \frac{d\Phi}{dx}(K_u) \, dK_u \right| = 0, \text{ q.s..}
\]

For the second part, since \( \frac{d^2\Phi}{dx^2} \) is bounded and the quadratic variation of \( K \) on \([0,T]\) is q.s. 0, then,
\[
\frac{1}{2} \sum_{k=0}^{N-1} \frac{d^2\Phi}{dx^2}(K^N_{t_k}) (K^N_{t_{k+1}} - K^N_{t_k})^2 \leq \frac{1}{2} M \sum_{k=0}^{N-1} (K^N_{t_{k+1}} - K^N_{t_k})^2 \to 0, \text{q.s., as } N \to +\infty.
\]

The proof is complete. \( \square \)

**Sketch of the proof of Lemma 3.14** To combine the result above with the ones in Peng [14], we decompose \( X \) into \( M^X + K \), where \( M^X \) denotes the \( G \)-Itô part of \( X \). Given a sequence of refining partitions \( \{\pi^N_{[s,t]}\}_{N \in \mathbb{N}} \) for \( N \in \mathbb{N} \),
\[
\pi^N_{[s,t]} = \{t^N_0, t^N_1, \ldots, t^N_{2N}\} = \{s, s + \delta, \ldots, s + 2N \delta = t\},
\]
we have from the second order Taylor expansion

\[
\Phi(X_t) - \Phi(X_s) = \sum_{k=0}^{2N-1} \left( \Phi(X_{i_k}^N) - \Phi(X_{i_{k+1}}^N) \right)
\]

\[
= \sum_{k=0}^{2N-1} \frac{d^2\Phi}{dx^2}(X_{i_k}^N)(M_{X}^{N})_{i_{k+1}} - (M_{X}^{N})_{i_{k}})^2 + \frac{1}{2} \sum_{k=0}^{2N-1} \frac{d^2\Phi}{dx^2}(X_{i_k}^N)(M_{X}^{N})_{i_{k+1}} - (M_{X}^{N})_{i_{k}})^2
\]

\[
+ \sum_{k=0}^{2N-1} \frac{d^2\Phi}{dx^2}(\xi_k^N)(M_{X}^{N})_{i_{k+1}} - (M_{X}^{N})_{i_{k}})(K_{\xi_k^N} - K_{\xi_k^N})^2 + \frac{1}{2} \sum_{k=0}^{2N-1} \frac{d^2\Phi}{dx^2}(\xi_k^N)](K_{\xi_k^N} - K_{\xi_k^N})^2
\]

\[
+ \frac{1}{2} \sum_{k=0}^{2N-1} \left( \frac{d\Phi}{dx}(X_{i_k}^N) - \frac{d\Phi}{dx}(X_{i_{k+1}}^N) \right)(M_{X}^{N})_{i_{k+1}} - (M_{X}^{N})_{i_{k}})^2
\]

\[
= I_1^N + I_2^N + I_3^N + I_4^N + I_5^N + I_6^N,
\]

where \(\xi_k^N\) satisfies \(X_{i_{k+1}}^N \leq \xi_k^N \leq X_{i_k}^N \) q.s.

A key point in the proof is to verify the following convergence in \(M_G([0,T])\):

\[
(3.8) \quad \sum_{k=0}^{2N-1} \frac{d\Phi}{dx}(X_{i_k}^N)I_{[i_k^N,i_{k+1}^N]}(\cdot) \to \frac{d\Phi}{dx}(X) , \text{ as } N \to +\infty;
\]

and

\[
(3.9) \quad \sum_{k=0}^{2N-1} \frac{d^2\Phi}{dx^2}(X_{i_k}^N)I_{[i_k^N,i_{k+1}^N]}(\cdot) \to \frac{d^2\Phi}{dx^2}(X) , \text{ as } N \to +\infty.
\]

For the G-Itô part \(M^X\), we deduce by the BDG type inequalities

\[
(3.10) \quad \int_s^t \mathbb{E} \left[ \sum_{k=0}^{2N-1} M_{X}^{N}I_{[i_k^N,i_{k+1}^N]}(u) - M^X \right]^2 du \leq M|t-s| \Rightarrow 0 , \text{ as } N \to +\infty.
\]

For the increasing process \(K\), thanks to assumption (3.6), for each \(u \in [s,t]\),

\[
(3.11) \quad \lim_{N \to +\infty} \mathbb{E} \left[ \sum_{k=0}^{2N-1} K_{\xi_k^N}I_{[i_k^N,i_{k+1}^N]}(u) - K_u \right]^2 = 0.
\]

Moreover,

\[
\int_s^t \mathbb{E} \left[ \sum_{k=0}^{2N-1} K_{\xi_k^N}I_{[i_k^N,i_{k+1}^N]}(u) \right]^2 du \leq \int_s^t \mathbb{E}[K_u^2] du < +\infty.
\]

By dominated convergence theorem to the integral on \([s,t]\),

\[
(3.12) \quad \lim_{N \to +\infty} \int_s^t \mathbb{E} \left[ \sum_{k=0}^{2N-1} K_{\xi_k^N}I_{[i_k^N,i_{k+1}^N]}(u) - K_u \right]^2 du = 0.
\]

Combining (3.10) and (3.12), (3.8) and (3.9) are readily obtained by Lipschitz continuity of \(\frac{d\Phi}{dx}\) and \(\frac{d^2\Phi}{dx^2}\). Then, we can proceed similarly to that in Peng [14] to treat with \(I_1^N\) and \(I_2^N\).

On the other hand, due to the boundedness of \(\frac{d^2\Phi}{dx^2}\) and the boundedness and uniformly continuity of paths \(M^X(\omega)\) and \(K(\omega)\) on \([0,T]\), for \(\omega \in A^c\), we can easily get that \(I_3^N\) and \(I_4^N\) are q.s. vanished.
For \( I^N_k \), we calculate
\[
|I^N_k| \leq \frac{C}{2} \left( \sum_{k=0}^{2^N-1} |\xi^N_k - X^N_{i_k}||M^N_{i_k} - M^N_{i+1_k}|^2 + \sum_{k=0}^{2^N-1} |(\xi^N_k)^2 - K_{i_k}||M^N_{i_k} - M^N_{i+1_k}|^2 \right),
\]
where \((\xi^N_k)^2 \leq M^N_{i_k} \land M^N_{i+1_k} \leq (\xi^N_k)^2 \leq M^N_{i_k} \lor M^N_{i+1_k} \land (\xi^N_k)^2 \leq K_{i_k} \leq (\xi^N_k)^2_1 + \ldots + (\xi^N_k)^2_{2^{N-1}}, \text{q.s.}\). The result in Peng [14] shows that the first part converges to 0 in \( M^2([0,T]) \) while the second part is vanished as a result of the uniformly continuity of paths \( K(\Omega) \) on \([0,T]\), for \( \omega \in A^c \), and the q.s. boundedness of the quadratic variation of the G-Itô part \( M^X \).

For \( I^N_k \), it converges to \( \int_s^t \frac{d\Phi}{dx}(X_u)dK_u \), q.s. by Definition 5.5.

**Remark 3.16.** In the proof of classical Itô’s formula, (3.8) and (3.9) can be verified directly by the pathwise continuity of \( X \) and the dominated convergence theorem on the product space \([s,t] \times \Omega \). But in G-framework, we are short of such a theorem. In general, given an \( X \in M^2_G([0,T]) \), the sequence of step processes \( \{\sum_{k=0}^{2^N-1} X^N_{i_k} I_{[i_k,i_{k+1}]}(\cdot)\}_{N \in \mathbb{N}} \) could not converge to \( X \) in the sense of \( M^2_G([0,T]) \). Thus, (3.6) is needed to ensure that (3.11) holds true.

In fact, the left side of (3.7), particularly the term \( \int_s^t \frac{d\Phi}{dx}(X_u)dK_u \), still belongs to \( L^2(\Omega_t) \). A sufficient condition of this result is that \( K \in L^2_G(\Omega_t) \), which can be verified by choosing a sequence such that \( t_n \to t \) and for \( n \in \mathbb{N} \), \( X_{t_n} \in L^2_G(\Omega_{t_n}) \) (Remark 2.11 ensures the existence of this sequence), and by deducing from assumption 3.4.

Similar to Theorem 6.5 of Peng [14], we can extend G-Itô’s formula in Lemma 3.14 to those \( \Phi \) whose second derivates \( \frac{d^2\Phi}{dx^2} \) has polynomial growth. Unfortunately, this extension is at a cost of more restriction on the increasing process \( K \).

**Theorem 3.17.** Let \( \Phi \in C^2(\mathbb{R}) \) be a real function such that \( \frac{d^2\Phi}{dx^2} \) satisfies polynomial growth condition. Let \( f, h \) and \( g \) be bounded processes in \( M^2_G([0,T]) \) and \( K \in M^1([0,T]) \cap M^2_G([0,T]) \) satisfying for each \( t \in [0,T] \),
\[
\lim_{s \to t} \mathbb{E}[|K_t - K_s|^2] = 0,
\]
and for any \( p > 2 \), \( \mathbb{E}[K^p_t] < +\infty \). Then,
\[
\Phi(X_t) - \Phi(X_s) = \int_s^t \frac{d\Phi}{dx}(X_u)f_u du + \int_s^t \frac{d\Phi}{dx}(X_u)h_u d(B)_u \\
+ \int_s^t \frac{d\Phi}{dx}(X_u)g_u dB_u + \int_s^t \frac{d\Phi}{dx}(X_u)K_u du \\
+ \frac{1}{2} \int_s^t \frac{d^2\Phi}{dx^2}(X_u)g_u^2 d(B)_u, \text{ q.s.}.
\]

**Proof:** By the same argument in the proof of Theorem 6.5 of Peng [14], we can choose a sequence of functions \( \Phi^N \in C^2(\mathbb{R}) \) such that for any \( x \in \mathbb{R} \),
\[
|\Phi^N(x) - \Phi(x)| + \left| \frac{d\Phi^N}{dx}(x) - \frac{d\Phi}{dx}(x) \right| + \left| \frac{d^2\Phi^N}{dx^2}(x) - \frac{d^2\Phi}{dx^2}(x) \right| \leq \frac{C}{N}(1 + |x|^k),
\]
We can proceed as in Peng [14] to show that the terms on right side of (3.15), except
\[ X \]
where
\[ \omega \]
we have
\[ N \]
Borrowing the notation in the proof of Lemma 3.14 and using the BDG type inequalities,
\[ (3.15) \]
Then, from (3.14) and (3.16), we deduce that as
\[ \omega \]
are positive constants independent of
\[ N \]
Then, from (3.14) and (3.16), we deduce that as
\[ N \rightarrow +\infty, \]
\[ \Phi^N(X_t) \rightarrow \Phi(X_t), \text{ in } L^2_0(\Omega_t); \]
\[ (3.17) \]
We can proceed as in Peng [14] to show that the terms on right side of (3.15), except
\[ f^t \frac{d\Phi^N}{dx}(X_u)dK_u, \]
converge to their corresponding terms in (3.16). To complete the proof, it is sufficient to show that for \( \omega \in A^c \),
\[ \left| \int_s^t \frac{d\Phi^N}{dx}(X_u(\omega))dK_u(\omega) - \int_s^t \frac{d\Phi}{dx}(X_u(\omega))dK_u(\omega) \right| \leq C N \int_s^t (1 + |X_u(\omega)|^k)dK_u(\omega) \leq C N (1 + M^k_\omega)K_T(\omega) \rightarrow 0, \text{ as } N \rightarrow +\infty, \]
by the continuity and boundedness of paths \( X(\omega) \) and \( K(\omega) \) on \([0, T] \).

Remark 3.18. If \( |\frac{d\Phi}{dx}(x)| \leq C(1 + |x|^k) \), for some \( k \geq 1 \), then the condition on \( K \) could be weakened to \( \mathbb{E}[|K_T|^{2(k+1)}] < +\infty \).

Remark 3.19. Following exactly the procedure above, we can have similar result when a bounded variation process \( K_1 - K_2 \) appears in the dynamic.

4. Reflected G-Brownian motion

Before moving to the main result of this paper, we first consider a reduced RGSDE, that is, taking \( f = h \equiv 0 \) and \( g \equiv 1 \), only a G-Brownian motion and an increasing process drive the dynamic on the right side of (1.1). In what follows, we establish the solvability to the RGSDE of this type, i.e., the existence and uniqueness of the reflected G-Brownian Motion.

Let \( y \) be a real valued continuous function on \([0, T] \) with \( y_0 \geq 0 \). It is well-known that, there exists a unique pair \((x, k)\) of functions on \([0, T]\) such that \( x = y + k \), where \( x \) is increasing and continuous function starting from 0, moreover, the Riemann-Stieltjes integral \( \int_0^T x_tdk_t = 0 \). The solution to this Skorokhod problem on \([0, T]\) is given by
\[ x_t = y_t + k_t; \]
\[ k_t = \sup_{s \leq t} x_s, \]
which is explicit and unique.

Theorem 4.1. For any \( p \geq 1 \), there exists a unique pair of processes \((X, K)\) in \( M^p_0([0, T]) \times (M^1([0, T]) \cap M^p_0([0, T])) \), such that
\[ X_t = B_t + K_t, \text{ q.s.}, \]
and (a) $K_0 = 0$; (b) $X$ is positive; and (c) $\int_0^T X_t dK_t = 0$, q.s..

**Proof:** With the help of (4.1), we define a pair of processes $(X, K)$ pathwisely on $[0, T]$:

\[
\begin{align*}
X_t(\omega) &= B_t(\omega) + K_t(\omega); \\
K_t(\omega) &= \sup_{s \leq t} B_s(\omega).
\end{align*}
\]

(4.3)

Obviously, $K \in M_t([0, T])$ and (a), (b) and (c) are satisfied. Therefore, to complete the proof, we only need to verify that $K \in M^p_2([0, T])$.

Since for all $1 \leq p' < p$, $M^p_2([0, T]) \subset M^p_2([0, T])$, we can assume that $p > 2$ without loss of generality. Given a sequence of partitions $\{\pi^N_{[0, T]}\}_{N \in \mathbb{N}}$, we set

\[
(B^{-}_t)^N(\omega) = \sum_{k=0}^{N-1} B^{-}_N(\omega)I_{[\frac{k}{N}, \frac{k+1}{N})}(t), \quad 0 \leq t \leq T,
\]

and

\[
\sup_{0 \leq s \leq t} (B^{-}_s)^N = \sum_{k=0}^{N-1} \max_{t \in \{0, 1, \ldots, k\}} B^{-}_N I_{[\frac{k}{N}, \frac{k+1}{N})}(t), \quad 0 \leq t \leq T.
\]

We observe that both $((B^{-}_t)^N)_{0 \leq t \leq T}$ and $(\sup_{0 \leq s \leq t} (B^{-}_s)^N)_{0 \leq t \leq T}$ are step processes in $M^p_2([0, T])$. Since

\[
\mathbb{E}[\sup_{0 \leq s \leq t} (B^{-}_s)^N - \sup_{0 \leq s \leq t} B^{-}_s | p] \leq \mathbb{E}[\sup_{0 \leq s \leq t} |(B^{-}_s)^N - B^{-}_s| | p]
\]

\[
\leq \mathbb{E}[\sup_{0 \leq s \leq t} |B^{-}_N - B^{-}_s| | p] \leq \mathbb{E}[\sup_{k \in \mathbb{N}\cap[0, N]} \sup_{t \leq t+1} |B_t - B^N_t| | p],
\]

then, letting $f = h \equiv 0$ and $g \equiv 1$ in (3.5), we obtain

\[
\mathbb{E}[\sup_{0 \leq s \leq t} (B^{-}_s)^N - \sup_{0 \leq s \leq t} B^{-}_s | p] \leq C_\mu(\pi^N_{[0, T]})^\delta \to 0, \quad \text{as } N \to +\infty,
\]

which shows that $(\sup_{0 \leq t \leq T} (B^{-}_t)^N)_{0 \leq t \leq T}$ converges to $K$ in $M^p_2([0, T])$.

On the other hand, the uniqueness of such pair $(X, K)$ is inherited from the solution to Skorokhod problem pathwisely. The proof is complete.

**Remark 4.2.** We call the process $X$ in Theorem 4.1 a $G$-reflected Brownian motion on the half line $[0, +\infty)$.

Furthermore, if the $G$-Brownian motion $B$ is replaced by some $G$-Itô process, we have the following statement similar to Theorem 4.1.

**Theorem 4.3.** For some $p > 2$, consider a q.s. continuous $G$-Itô process $Y$ defined in the form of (3.4) whose coefficients are all elements in $M^p_2([0, T])$. Then, there exists a unique pair of processes $(X, K)$ in $M^p_2([0, T]) \times (M_t([0, T]) \cap M^p_2([0, T]))$ such that

\[
X_t = Y_t + K_t, \quad \text{q.s.},
\]

and (a) $X$ is positive; (b) $K_0 = 0$; and (c) $\int_0^T X_t dK_t = 0$, q.s..

We omit the proof, since it is an analogue to the proof above and deduced mainly by the integrability of the coefficients of $Y$ and (3.5).

5. Scalar valued RGSDEs

We state our main result in this section by giving the existence and uniqueness of the solutions to the scalar valued RGSDEs with Lipschitz coefficients. Besides, a comparison theorem is given at the end of this paper.
5.1. Formulation to RGSDEs. We consider the following scalar valued RGSDE:

\[
(5.1) \quad X_t = x + \int_0^t f_s(X_s)ds + \int_0^t h_s(X_s)d(B)_s + \int_0^t g_s(X_s)dB_s + K_t, \quad 0 \leq t \leq T,
\]

where

(A1) The initial condition \( x \in \mathbb{R} \);

(A2) For some \( p > 2 \), the coefficient \( f, h \) and \( g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R} \) are given functions satisfying \( f(x), h(x), g(x) \in M^p_G([0, T]) \), for each \( x \in \mathbb{R} \);

(A3) The coefficient \( f, h \) satisfying a Lipschitz condition, i.e., for each \( t \in [0, T] \) and \( x, x' \in \mathbb{R} \), \( |f_t(x) - f_t(x')| + |h_t(x) - h_t(x')| + |g_t(x) - g_t(x')| \leq C_L|x - x'| \), q.s.;

(A4) The obstacle is a G-Itô process whose coefficients are all elements in \( M^p_G([0, T]) \), and we shall always assume that \( S_0 \leq x \), q.s..

The solution of RGSDE (5.1) is a pair of processes \((X, K)\) which take values both in \( \mathbb{R} \) and satisfy:

(A5) \( X \in M^p_G([0, T]) \) and \( X_t \geq S_t, \ 0 \leq t \leq T, \) q.s.;

(A6) \( K \in M_G([0, T]) \cap M^p_G([0, T]) \) and \( K_0 = 0, \) q.s.;

(A7) \( \int_0^T (X_t - S_t) dK_t = 0, \) q.s..

5.2. Some a priori estimates and the uniqueness result. Let \((X, K)\) be a pair of solution to (5.1). Replacing \( Y_t \) by \( x + \int_0^t f_s(X_s)ds + \int_0^t h_s(X_s)d(B)_s + \int_0^t g_s(X_s)dB_s - S_t \) and \( X_t \) by \( X_t - S_t \) in (4.13), we have the following representation of \( K \) on \([0, T]\):

\[
(5.2) \quad K_t = \sup_{0 \leq s \leq t} \left( x + \int_0^s f_u(X_u)du + \int_0^s h_u(X_u)d(B)_u + \int_0^s g_u(X_u)dB_u - S_t \right)^-, \ \text{q.s}.
\]

We now give a priori estimate on the uniform norm of the solution.

Proposition 5.1. Let \((X, K)\) be a solution to (5.1). Then, there exists a constant \( C > 0 \) such that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t|^p \right] + \mathbb{E}[K_T^p] \leq C \left( |x|^p + \int_0^T (\mathbb{E}[|f_t(0)|^p] + \mathbb{E}[|h_t(0)|^p] + \mathbb{E}[|g_t(0)|^p])dt \right) + \mathbb{E} \left[ \sup_{0 \leq t \leq T} |S_t|^p \right].
\]

Proof: As \( X \) is the solution to (5.1), we obtain

\[
(5.3) \quad \mathbb{E}[\sup_{0 \leq s \leq t} |X_s|^p] \leq \mathbb{E}[\sup_{0 \leq s \leq t} |x + \int_0^s f_u(X_u)du + \int_0^s h_u(X_u)d(B)_u + \int_0^s g_u(X_u)dB_u + K_s|^p] \leq C(|x|^p + \mathbb{E}[\sup_{0 \leq s \leq t} |\int_0^s f_u(X_u)du|^p] + \mathbb{E}[\sup_{0 \leq s \leq t} |\int_0^s h_u(X_u)d(B)_u|^p] + \mathbb{E}[\sup_{0 \leq s \leq t} |\int_0^s g_u(X_u)dB_u|^p] + \mathbb{E}[|K_t|^p])
\]

Similarly to (5.3), from the representation of \( K \) (5.2), we have

\[
(5.4) \quad \mathbb{E}[K_T^p] \leq \mathbb{E}[\sup_{0 \leq s \leq t} ((x + \int_0^s f_u(X_u)du + \int_0^s h_u(X_u)d(B)_u + \int_0^s g_u(X_u)dB_u - S_s)^-)p] \leq \mathbb{E}[\sup_{0 \leq s \leq t} (|x + \int_0^s f_u(X_u)du + \int_0^s h_u(X_u)d(B)_u + \int_0^s g_u(X_u)dB_u - S_s^-|^p)] \leq C(|x|^p + \mathbb{E}[\sup_{0 \leq s \leq t} |\int_0^s f_u(X_u)du|^p] + \mathbb{E}[\sup_{0 \leq s \leq t} |\int_0^s h_u(X_u)d(B)_u|^p] + \mathbb{E}[\sup_{0 \leq s \leq t} |\int_0^s g_u(X_u)dB_u|^p] + \mathbb{E}[\sup_{0 \leq s \leq t} |S_s^-|^p])
\]
Combining (5.3) and (5.4) and applying BDG type inequalities, we get
\[\mathbb{E}[^{\sup_{0 \leq s \leq t}} |X_s|^p] + \mathbb{E}[K^p_t] \leq C(|x|^p + \int_0^t (\mathbb{E}[|f_s(X_s)|]) ds) + \mathbb{E}[\sup_{0 \leq s \leq t} |S^+_s|].\]

By condition (A3), we deduce
\[\mathbb{E}[^{\sup_{0 \leq s \leq t}} |X_s|^p] + \mathbb{E}[K^p_t] \leq C(|x|^p + \int_0^t (\mathbb{E}[|f_s(0)| + C_{L} |X_s|^p] + \mathbb{E}[|h_s(0)| + C_{L} |X_s|^p]) ds + \mathbb{E}[\sup_{0 \leq s \leq t} |S^+_s|].\]

Applying Gronwall’s lemma to \(\bar{E}\), we set
\[\Delta E = 1,\]

Then there exists a constant \(C > 0\) such that
\[\mathbb{E}[^{\sup_{0 \leq s \leq t}} |X_s|^p] \leq C \left(|x|^p + \int_0^t (\mathbb{E}[|f_s(0)|] + \mathbb{E}[|h_s(0)|]) ds \right) + \mathbb{E}[\sup_{0 \leq t \leq T} |S^+_t|], \quad 0 \leq t \leq T.\]

Putting (5.6) into (5.5), the result follows.

In the following theorem, we estimate the variation in the solutions induced by a variation in the coefficients and the obstacle process.

**Theorem 5.2.** Let \((x^1, f^1, h^1, g^1, S^1)\) and \((x^2, f^2, h^2, g^2, S^2)\) be two sets of coefficients satisfy (A1)-(A4), and \((X^1, K^1)\) the solution to the RGSDE corresponding to \((x^1, f^1, h^1, g^1, S^1)\), \(i = 1, 2\). Define
\[\Delta x = x^1 - x^2, \quad \Delta f = f^1 - f^2, \quad \Delta h = h^1 - h^2, \quad \Delta g = g^1 - g^2;\]
\[\Delta S = S^1 - S^2, \quad \Delta X = X^1 - X^2, \quad \Delta K = K^1 - K^2.\]

Then there exists a constant \(C > 0\) such that
\[\mathbb{E}[^{\sup_{0 \leq t \leq T}} |\Delta X_t|^p] \leq C \left(|\Delta x|^p + \int_0^t (\mathbb{E}[|\Delta f_s(X^1_s)|] + \mathbb{E}[|\Delta h_s(X^1_s)|]) ds \right) + \mathbb{E}[\sup_{0 \leq t \leq T} |\Delta S_t|^p].\]

**Proof:** We set
\[\Delta M^X = (M^X)_t = x^1 + \int_0^t f_s^1(X^1_s) ds + \int_0^t h_s^1(X^1_s) dB_s + \int_0^t g_s^1(X^1_s) dB_s, \quad 0 \leq t \leq T, \quad i = 1, 2;\]
and \(\Delta M^X = (M^X)_t - (M^X)_t^2). Similarly to the proof of Proposition 5.1, we calculate
\[\mathbb{E}[\sup_{0 \leq s \leq t} |\Delta M^X_s|^p] \leq \mathbb{E}[\sup_{0 \leq s \leq t} |\Delta x + \int_0^s (f_u^1(X^1_u) - f_u^2(X^2_u)) du| + \int_0^s (h_u^1(X^1_u) - h_u^2(X^2_u)) dB_u + \int_0^s (g_u^1(X^1_u) - g_u^2(X^2_u)) dB_u] \leq \mathbb{E}[\sup_{0 \leq s \leq t} |\Delta x + \int_0^s \Delta f_u(X^1_u) du + \int_0^s \Delta h_u(X^1_u) dB_u + \int_0^s \Delta g_u(X^1_u) dB_u|] \leq C(|\Delta x|^p + \int_0^t (\mathbb{E}[|\Delta f(X^1_s)|] + \mathbb{E}[|\Delta h(X^1_s)|]) ds + \int_0^t (\mathbb{E}[|\Delta X_s|] ds + \int_0^t (\mathbb{E}[|\Delta X_s|] ds).}
Then, we have

\begin{equation}
\mathbb{E}[\sup_{0 \leq s \leq t} |\Delta K_s|^p] = \mathbb{E}[\sup_{0 \leq s \leq s} |\sup_{0 \leq u \leq s} ((M^X)_{u}^1 - S_t^1) - \sup_{0 \leq u \leq s} ((M^X)_{u}^2 - S_t^2)|^p]
\end{equation}

\begin{align*}
&= \mathbb{E}[\sup_{0 \leq s \leq s} |\sup_{0 \leq u \leq s} ((M^X)_{u}^1 - S_t^1) - ((M^X)_{u}^2 - S_t^2)|^p] \\
&= \mathbb{E}[\sup_{0 \leq s \leq s} |((M^X)_{u}^1 - S_t^1) - ((M^X)_{u}^2 - S_t^2)|^p] \\
&= \mathbb{E}[\sup_{0 \leq s \leq s} |((M^X)_{u}^1 - S_t^1) - ((M^X)_{u}^2 - S_t^2)|^p] = C(\mathbb{E}[\sup_{0 \leq s \leq s} |\Delta (M^X)_s|^p] + \mathbb{E}[\sup_{0 \leq s \leq s} |\Delta S_s|^p]).
\end{align*}

Then, we have

\begin{equation}
\mathbb{E}[\sup_{0 \leq s \leq t} |\Delta X_s|^p] \leq \mathbb{E}[\sup_{0 \leq s \leq t} |(\Delta M^X)_s + \Delta K_s|^p] \\
\leq \mathbb{E}[\sup_{0 \leq s \leq t} |(\Delta M^X)_s|^p] + \mathbb{E}[\sup_{0 \leq s \leq t} |\Delta K_s|^p] \\
\leq C(\mathbb{E}[\sup_{0 \leq s \leq t} |(\Delta M^X)_s|^p] + f_0^T(\mathbb{E}[\Delta f_t(X_t^1)]^p] + \mathbb{E}[\Delta h_t(X_t^1)]^p] \\
+ \mathbb{E}[\Delta g_t(X_t^1)]^p]ds + \mathbb{E}[\sup_{0 \leq s \leq t} |\Delta S_s|^p] + f_0^T \mathbb{E}[|\Delta X_s|^p]ds).
\end{equation}

We deduce immediately the following uniqueness result by taking $x^1 = x^2 = f_1 = f_2 = h_1 = h_2$, $g^1 = g^2$ and $S^1 = S^2$ in Theorem 5.2.

**Theorem 5.3.** Under the assumptions (A1)-(A4), there exists at most one solution in $M^p_G([0, T])$ to the RGSDE (5.7).

5.3. **Existence result.** We now turn to the following existence result for RGSDE (5.1). The proof will be based on a Picard iteration.

**Theorem 5.4.** Under the assumptions (A1)-(A4), there exists a unique solution in $M^p_G([0, T])$ to the RGSDE (5.7).

**Proof:** We set $X^0 = x$ and $K^0 = 0$. For each $n > 0$, $X^{n+1}$ is given by recurrence:

\begin{equation}
X^{n+1} = x + \int_0^t f_s(X_s^n)ds + \int_0^t h_s(X_s^n)d(B)_s + \int_0^t g_s(X_s^n)dB_s + K^n_{t+1}, \quad 0 \leq t \leq T,
\end{equation}

satisfying

- (a) $X^{n+1} \in M^p_G([0, T])$, $X^{n+1} \geq S_t$, q.s.;
- (b) $K^{n+1} \in M^p_G([0, T])$, $K^{n+1}_0 = 0$, q.s.;
- (c) $\int_0^T (X^{n+1}_t - S_t)dt < \infty$.

Substituting $X^{n+1}$ by $\tilde{X}^{n+1}$ on the left side of (5.8), we know that $(X^{n+1}, K^{n+1})$ is well defined in $M^p_G([0, T]) \times (M^p_G([0, T]) \cap M^p_G([0, T]))$ by Theorem 4.3.

Firstly, we establish a priori estimates uniform in $n$ for $\{\mathbb{E}[\sup_{0 \leq s \leq t} |X^{n+1}_s|^p]\}_{n \in \mathbb{N}}$. Similarly to (5.5), we have

\begin{equation}
\mathbb{E}[\sup_{0 \leq s \leq t} |X^{n+1}_s|^p] \leq C \left( |x|^p + \int_0^T (\mathbb{E}[|f_t(0)|^p] + \mathbb{E}[|h_t(0)|^p]) \\
+ \mathbb{E}[|g_t(0)|^p])dt + \mathbb{E}[\sup_{0 \leq s \leq t} |S_t^1|^p] + \int_0^T \mathbb{E}[\sup_{0 \leq u \leq s} |X^{n+1}_u|^p]ds \right).
\end{equation}
By recurrence, it is easily to verify that, for \( n \in \mathbb{N} \),
\[
\mathbb{E} \sup_{0 \leq s \leq t} |X^n_s| \leq p(t), \quad 0 \leq t \leq T,
\]
where \( p(\cdot) \) is the solution to the following ordinary differential equation:
\[
p(t) = C \left( |x|^p + \int_0^T (\mathbb{E}[|f_t(0)|^p] + \mathbb{E}[|h_t(0)|^p] + \mathbb{E}[|g_t(0)|^p]) dt + \mathbb{E} \sup_{0 \leq t \leq T} |S^n_t|^p + \int_0^t p(s) ds \right),
\]
and \( p(\cdot) \) is continuous and thus bounded on \([0, T]\).

Secondly, for \( n \) and \( m \in \mathbb{N} \), we define
\[
u^n_{t+1} := \mathbb{E} \sup_{0 \leq s \leq t} |X^n_s - X^{n+1}_s|, \quad 0 \leq t \leq T.
\]
Following the procedures in the proof of Theorem 5.2, we have
\[
u^n_{t+1,m} \leq C \int_0^t \nu^{n,m}_s ds.
\]
Set
\[
u^n_t := \sup_{m \in \mathbb{N}} \nu^n_{t,m}, \quad 0 \leq t \leq T,
\]
then
\[0 \leq \nu^n_{t+1,m} \leq C \sup_{m \in \mathbb{N}} \int_0^t \nu^{n,m}_s ds \leq C \int_0^t \sup_{m \in \mathbb{N}} \nu^{n,m}_s ds = C \int_0^t \nu^n_s ds.
\]
Taking supremum on the left side, we obtain
\[0 \leq \nu^n_{t+1} = \sup_{m \in \mathbb{N}} \nu^n_{t,m} \leq C \int_0^t \nu^n_s ds.
\]
Finally, we define
\[\alpha_t := \lim_{k \to +\infty} \nu^n_k, \quad 0 \leq t \leq T.
\]
It is easy to find that \( \nu^n_t \leq Cp(t) \), where \( C \) is independent of \( n \). By classical Fatou’s Lemma, we have
\[0 \leq \alpha_t \leq C \int_0^t \alpha_s ds.
\]
Gronwall’s lemma gives
\[\alpha_t = 0, \quad 0 \leq t \leq T,
\]
which implies that \( \{X^n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( M^p_G([0, T]) \). We denote the limit by \( X \) and set
\[K_t = \sup_{0 \leq u \leq t} \left( x + \int_0^u f_u(X_u) du + \int_0^u h_u(X_u) dB_u + \int_0^u g_u(X_u) dB_u - S_u \right).
\]
Obviously, the pair of processes \((X, K)\) satisfies (A5) - (A7). We notice that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (f_s(X^n_s) - f_s(X_s)) ds \right|^p \right] \leq C \int_0^T \mathbb{E}[|X^n_t - X_t|^p] dt;
\]
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (h_s(X^n_s) - h_s(X_s)) dB_s \right|^p \right] \leq C \int_0^T \mathbb{E}[|X^n_t - X_t|^p] dt;
\]
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (g_s(X^n_s) - g_s(X_s)) dB_s \right|^p \right] \leq C \int_0^T \mathbb{E}[|X^n_t - X_t|^p] dt.
\]
Then, one can verify that \( K^n \) converges to \( K \) in \( M^p_G([0, T]) \) following the steps of (5.7). We conclude that the pair of processes \((X, K)\), well defined in \( M^p_G([0, T]) \times (M_1([0, T]) \cap M^p_G([0, T])) \), is a solution to (5.1). \( \square \)
Remark 5.5. Unlike in a classical RSDE, the constraint process $S$ here is assumed to be a $G$-Itô process instead of a continuous process with $\mathbb{E}[\sup_{0 \leq t \leq T} (S^n_t)^2] \leq +\infty$ (cf. El Karoui et al. [3]). In fact, this is a sufficient condition to ensure that $K^{n+1}$ is still a $M^p_G([0, T])$ process in (5.8) by Theorem 4.3, which may be weakened to:

$$\mathbb{E}[\sup_{s \leq u \leq t} |S_u - S_s|^p] \leq C|t - s|^p.$$ 

Remark 5.6. Using the approach in Soner et al. [20], this existence result still holds from a classical argument under each $P \in \mathcal{P}$, that is, a pair of process $(X^2, K^2)$ can be found to satisfy (5.1) and (A5)-(A7) in the sense of $P$-a.s. However, it is difficult to determine a universal $(X, K)$ which is the aggregation of these solutions $(X^2, K^2), P \in \mathcal{P}$.

Remark 5.7. In contrast with the fact mentioned in Remark 3.3 of Matoussi et al. [12], our results can be directly applied to the symmetrical problem i.e., the RGSDE with an upper barrier. This conclusion is due to the facts that the proof is only based on a pathwise construction and a fixed-point iteration.

5.4. Comparison principle. In this subsection, we establish a comparison principle for RGSDEs. At first, we assume additionally a bounded condition on the coefficients $f, h$ and $g$ and the obstacle process $S$, and then we remove it in the second step.

Theorem 5.8. Given two RGSDEs satisfying the conditions (A1)-(A4), we suppose in addition the following:

1. $x^1 \leq x^2$;
2. $f^i, h^i$ and $g^i = g^2 = g$ are bounded, and $S^i$ are uniformly upper bounded, $i = 1, 2$;
3. $f^1_i(x) \leq f^2_i(x)$ and $h^1_i(x) \leq h^2_i(x)$, for $x \in \mathbb{R}$, and $S^2_1 \leq S^2_T$, $0 \leq t \leq T$, q.s.

Let $(X^i, K^i)$ be a pair of solutions to the RGSDEs with data $(f^i, h^i, g, S^i)$, $i = 1, 2$, then,

$$X^1_t \leq X^2_t, 0 \leq t \leq T, \text{ q.s.}.$$ 

Proof: Since $f^i, h^i$ and $g$ are bounded, and $S^i$ are uniformly upper bounded, $i = 1, 2$, using the BDG type inequalities to (5.2), we deduce that $K^i_T$ has the moment for arbitrage large order and for $0 \leq t \leq T$, $\lim_{s \rightarrow t} \mathbb{E}[|K^i_s - K^i_t|^2] = 0$, $i = 1, 2$.

Notice that $(x^2)^2$ is not a $C^2(\mathbb{R})$ function, we have to consider $(x^2)^3$ and apply the extended $G$-Itô’s formula to $((X^i_t - X^2_t)^+)^3$,

$$((X^i_t - X^2_t)^+)^3 = 3\int_0^t ((X^i_s - X^2_s)^+)^2(f^i_s(X^i_s) - f^2_s(X^2_s))ds$$

$$+ 3\int_0^t ((X^i_s - X^2_s)^+)^2(h^i_s(X^i_s) - h^2_s(X^2_s))dB_s$$

$$+ 3\int_0^t ((X^i_s - X^2_s)^+)^2(g_s(X^i_s) - g_s(X^2_s))dB_s$$

$$+ 3\int_0^t ((X^i_s - X^2_s)^+)^2d(K^i_s - K^2_s)$$

$$+ 3\int_0^t (X^i_s - X^2_s)^+g_s(X^i_s) - g_s(X^2_s))^2d(B)_s.$$ 

(5.9)

Since on $\{X^1_t > X^2_t\}, X^1_t > X^2_t \geq S^2_t \geq S^1_t$, we have

$$\int_0^t ((X^i_s - X^2_s)^+)^2d(K^i_s - K^2_s) = \int_0^t ((X^i_s - X^2_s)^+)^2dK^i_s - \int_0^t ((X^i_s - X^2_s)^+)^2dK^2_s$$

$$\leq \int_0^t ((X^i_s - S^2_s)^+)^2dK^i_s - \int_0^t ((X^i_s - X^2_s)^+)^2dK^2_s$$

$$\leq -\int_0^t ((X^i_s - X^2_s)^+)^2dK^2_s \leq 0, \text{ q.s.}.$$ 

(5.10)
We put \([5.10]\) into \([5.9]\). Then, by Lipschitz condition \((A3)\) and taking \(G\)-expectation on both sides of \([5.10]\), we conclude

\[
\mathbb{E}[(X_1^t - X_2^t)^3] \leq C\mathbb{E}\left[ \int_0^t ((X_1^s - X_2^s)^3) \, ds \right] \leq C \int_0^t \mathbb{E}[(X_1^s - X_2^s)^3] \, ds.
\]

Using Gronwall’s lemma, it follows that \(\mathbb{E}[(X_1^t - X_2^t)^3] = 0\), which implies the result. \(\square\)

**Theorem 5.9.** Given two RGSDEs satisfying the conditions \((A1)-(A4)\), and suppose in addition the following:

1. \(x^1 \leq x^2\) and \(g^1 = g^2 = g\);
2. \(f_i^1(x) \leq f_i^2(x)\) and \(h_i^1(x) \leq h_i^2(x)\), for \(x \in \mathbb{R}\), and \(S_i^1 \leq S_i^2\), \(0 \leq t \leq T\), q.s.

Let \((X^1, K^1)\) and \((X^2, K^2)\) are two pairs of solutions to the RGSDEs above, then,

\[X_1^t \leq X_2^t, \; 0 \leq t \leq T, \; \text{q.s.} \]

**Proof:** Firstly, we define the truncation functions for the coefficients and the obstacle process: for \(N > 0\), \(\xi^N_t(x) = (-N \vee \xi_t(x)) \wedge N\), \(x \in \mathbb{R}\), where \(\xi\) denote \(f^i\), \(h^i\) and \(g\), \(i = 1, 2\), and \(S_i^N = S_i \wedge N\), \(0 \leq t \leq T\). It is easy to verify that the truncated coefficients and obstacle processes satisfy \((A2)\) and \((A3)\). Moreover, the truncation functions keep the order of the coefficients and obstacle processes, that is,

\[(f^1)^N_t(x) \leq (f^2)^N_t(x), \; (h^1)^N_t(x) \leq (h^2)^N_t(x)\] and \((S^1)^N_t \leq (S^2)^N_t\), \(0 \leq t \leq T\), q.s.

Consider the following RGSDEs on \([0, T]\), for \(i = 1, 2\),

\[(X^i)_N^t = x + \int_0^t (f^i)^N_s((X^i)_N^s) \, ds + \int_0^t (h^i)^N_s((X^i)_N^s) \, dB_s + \int_0^t g^N_s((X^i)_N^s) \, d\mathbb{E} + (K^i)_N^t;\]

satisfies

(a) \((X^i)^N_t \in M^p([0, T]), \; (X^i)^N_t \geq (S^i)^N\), q.s.;

(b) \((K^i)^N_t \in M^\infty([0, T]) \cap M^p([0, T]), \; (K^i)^N_t = 0\), q.s.;

(c) \(\int_0^T ((X^i)^N_t - (S^i)^N)d(K^i)^N_t = 0\), q.s.

By Theorem \(5.3\) we have

\[(X^1)^N_t \leq (X^2)^N_t, \; \text{q.s.}\]

Meanwhile, by Theorem \(5.2\) we have

\[
\mathbb{E}[\sup_{0 \leq t \leq T} |(X^1)^N_t - X^N_t|^p] \leq C \left( \mathbb{E}\left[ \int_0^T ((f^i)^N(t, X^i_t) - f^i(t, X^i_t))^p \right] + \mathbb{E}\left[ |(h^i)^N(t, X^i_t) - h^i(t, X^i_t)|^p \right] \right)
\]

Applying again Gronwall’s lemma, we obtain

\[
\mathbb{E}[\sup_{0 \leq t \leq T} |(X^1)^N_t - X^N_t|^p] \leq C \left( \mathbb{E}\left[ \int_0^T ((f^i)^N(t, X^i_t) - f^i(t, X^i_t))^p \right] + \mathbb{E}\left[ |(h^i)^N(t, X^i_t) - h^i(t, X^i_t)|^p \right] \right)
\]

For any \(t \in [0, T]\), we calculate

\[
\mathbb{E}[|((f^i)^N(t, X^i_t) - f^i(t, X^i_t)|^p] \leq \mathbb{E}[|f^i(t, X^i_t)|^p I_{(f^i(t, X^i_t) > N)}]
\]

\[
\leq C(\mathbb{E}[|f^i(0)|^p I_{(f^i(0) > N)}] + C_L|X^i_t|^p I_{|X^i_t| > N})
\]

\[
\leq C(\mathbb{E}[|f^i(0)|^p I_{|f^i(0)| > N}] + \mathbb{E}[|X^i_t|^p I_{|X^i_t| > N}]).
\]
Taking into consideration that \( f(0) \) and \( X^i \in M^P_{\mathcal{F}}([0, T]) \), from the argument in Remark 2.11, we know that \( f_i(0) \) and \( X^i_1 \in L^P_{\mathcal{F}}([0, T]) \) for almost every \( t \in [0, T] \). Therefore, letting \( N \to +\infty \), we have

\[
E[(f_i)^N_t(X^i_1) - f_i(X^i_1)] \to 0.
\]

Similarly, we can obtain that

\[
E[(h_i)^N_t(X^i_1) - h_i(X^i_1)] \to 0;
\]

and

\[
E[(g_i)^N_t(X^i_1) - g_i(X^i_1)] \to 0.
\]

Using dominated convergence theorem to the integrals on \([0, T]\), it follows that

\[
\lim_{N \to +\infty} \int_0^T \left( E[(f^i)^N(t, X^i_1) - f^i(t, X^i_1)^P] + E[(h^i)^N(t, X^i_1) - h^i(t, X^i_1)^P] + E[(g^N(t, X^i_1) - g(t, X^i_1)^P)] \right) dt = 0.
\]

On the other hand,

\[
E\sup_{0 \leq t \leq T} |(S^i)^N_t - S^i_t|^P \leq E\sup_{0 \leq t \leq T} \left| |S^i|^P I_{|S^i|^>N} \right| \leq E\sup_{0 \leq t \leq T} |S^i|^P I_{|\sup_{0 \leq t \leq T} |S^i|^>N}.
\]

By the proof of Theorem 4.3, we know that \( \sup_{0 \leq t \leq T} S^i_t \) is an element in \( L^P_{\mathcal{F}}(\Omega) \). So we have

\[
E\sup_{0 \leq t \leq T} \left| |(S^i)^N_t - S^i_t|^P \right| \leq E\sup_{0 \leq t \leq T} |S^i|^P I_{|\sup_{0 \leq t \leq T} |S^i|^>N} \to 0, \text{ as } N \to +\infty.
\]

Combining (5.12) and (5.13), we obtain

\[
E\sup_{0 \leq t \leq T} \left| |(X^i)^N_t - X^i_t|^P \right| \to 0, \text{ as } N \to +\infty.
\]

Then, (5.11) and (5.14) yield the desired result.

\[ \square \]

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