How much in the Universe can be explained by geometry?

José B. Almeida
Universidade do Minho, Physics Department
Braga, Portugal, email: bda@fisica.uminho.pt

The paper uses geometrical arguments to derive equations with relevance for cosmology; 5-dimensional spacetime is assumed because it has been shown in other works to provide a setting for significant unification of different areas of physics. Monogenic functions, which zero the vector derivative are shown to effectively model electrodynamics and relativistic dynamics if one allows for space curvature. Applying monogenic functions to flat space, the Hubble relation can be derived straightforwardly as a purely geometrical effect. Consideration of space curvature induced by mass density allows the derivation of flat rotation curves for galaxies without appealing for dark matter. Similarly, a small overall mass density in the Universe is shown to provide a possible explanation for recent supernovae observations, without the need for a cosmological constant.

1 Introduction

Modern science is based on theories, each with its own application domain; whenever we need to predict the outcome of an experiment or observation we must resort to the relevant theory, insert the data and derive the predictions which must then be in accord with the experimental or observational results. A modern theory is based on a set of principles or axioms, from which predictions are derived by means of universally accepted rules of logic and/or mathematics. For situations well within the boundaries of one theory’s application domain the system works well but problems usually arise when the subject under scrutiny lies somewhere near the border between theories. Furthermore, we tend to think that the Universe and everything in it must somehow be ruled by global rules, applicable everywhere at all times and this contradicts the co-existence of separate theories’ domains.
One common misconception is that one global theory for the Universe, the so-called theory of everything, would spell the end of science for, by mastering such theory, one would in principle be able to predict the outcome of any conceivable experiment or observation. Things will probably never confirm this view, since science has many overlapping layers such that high-level layers are virtually independent from the way these layers are rooted on the underlying ones. It is possible to make an analogy with computer science with its programming languages in different levels. One can fully predict the behaviour of a word processing application without ever considering how bits are manipulated at machine level. There is even a more profound meaning in this analogy, resulting from the fact that we are able to identify a particular application implemented on different platforms. The application has an identity which has become detached from the platform and we have every reason to believe a similar thing happens in our Universe, with some sciences having become detached from the underlying structure.

In this paper we will address questions of cosmology in terms that may allow a high degree of unification with other areas of physics, namely with quantum mechanics. In physics one is usually confronted with the need to opt between general relativity or quantum mechanics, one theory being applicable to the very large and the other one to the very small; both theories are expected to default to Newtonian mechanics at the human scale. The situation is uncomfortable from an intellectual standpoint but it is also rather embarrassing when one tries to model the primordial Universe. At the very early stages the Universe was very small and very dense; both quantum mechanics and general relativity break down for such an extreme situation and we must humbly recognize that a different and yet unknown physics must apply. At present times, phenomena such as are observed in active galaxy nuclei (AGN) also point to the need of some sort of reconciliation between the two reigning theories of physics.

2 Physics from geometry

Is physics ultimately the way we perceive a privileged geometry? This is a fundamental question whose final answer is unknown but in the affirmative case, if one day we are fortunate enough to find that geometry, all the foundational equations of physics will emanate from the symmetries, topology and other geometric peculiarities. In spite of the fact that we can have an immediate perception of 3-dimensional geometry, geometry at large is an abstract mathematical subject which can only be expected to produce equations of physical significance if a wise assignment is made between geometrical and physical entities, namely the assignment between geometric coordinates and the dimensions of physical space and time. It is important to stress that we must be wise in the assignment process, for an improper choice of coordinates may completely hide from us the equations we would like to derive. As an example, imagine we would like to study atmospheric currents but were unaware of the Earth’s spherical shape and rotation. Ac-
cordingly we would probably choose a fixed Cartesian frame and in no way would we be able to predict winds. A similar example is that of Ptolemy’s epicycloids to model planet’s orbits; the solar centered reference system revealed a much simpler picture than the Earth centered counterpart.

The search for a putative privileged geometry of physics cannot be done blindly amid the infinite variety of geometries at our disposal; we are to be guided in this process by the work of others and by our own previous efforts and intuition. Many hints point towards the usefulness of exploring curved 5-dimensional spacetime geometries, starting with the work by Kaluza \[1\] that showed the way to the unification of general relativity with electromagnetism. Kaluza’s work was later complemented by Klein with the postulate that one of the dimensions was to be taken as compact but we will not be following that route. The author has also proposed a unified approach to physics based on 5-dimensional spacetime; a short introductory text is Almeida \[2\]; applications to quantum mechanics can be found in Almeida \[3, 4\] and an application to cosmology anticipating the present work was published in 2006 \[5\].

The point of departure for our work is the exploitation of monogenic functions in 5-dimensional spacetime with signature \((-++++)\). Monogenic functions zero the vector derivative of geometric calculus \(^1\) and as such can be seen as foundational. In the simplest case of 1-dimensional space a monogenic function is a function with zero derivative, that is, a constant with no structure; in 2-dimensional space monogenic functions are equivalent to analytic functions and increasing the space dimensionality one finds that those functions can generate many kinds of interesting geometrical structures. To start with we assume there is an associative vector product (geometric or Clifford product) defined in this space, with a commuting part and an anti-commuting part, such that

\[
ab = a \cdot b + a \wedge b, \quad ba = a \cdot b - a \wedge b.
\]  

(2.1)

The commuting part is a scalar called the inner product and the anti-commuting part is an oriented area called the outer product. We assume also a set of five independent vectors \(\{g_\alpha\}\), with \(\alpha = 0, \ldots, 4\), used as a reference frame. The metric tensor can then be constructed by

\[
g_{\alpha\beta} = g_\alpha \cdot g_\beta.
\]  

(2.2)

The frame vectors are point dependent implying a point dependence for the metric tensor; the latter has 25 degrees of freedom, the number required for modelling general relativity and electromagnetism. The particular case of flat 5-dimensional spacetime is characterized by having a metric tensor \(g_{\alpha\beta} = \text{diag}(-1, 1, 1, 1, 1)\). The exterior product

\(^1\)Detailed explanation of geometric algebra and geometric calculus can be found in Doran and Lasenby \[6\]; we will need here only a small set of the mathematical tools offered by this discipline and we will try to give the reader enough information so that he does not need a profound knowledge of the subject.
of the 5 frame vectors produces an imaginary quantity called a pseudo-scalar
\[ g_1 \wedge g_2 \wedge g_3 \wedge g_4 \wedge g_5 = iv, \quad (2.3) \]
with \( i^2 = -1 \) and \( v \) real.

The reference frame is associated with a \textit{reciprocal frame}, denoted with upper indices, whose vectors verify the relation
\[ g^\alpha \cdot g_\beta = \delta^\alpha_\beta, \quad (2.4) \]
where \( \delta^\alpha_\beta \) is the Kronecker delta. The inner product of the reciprocal frame vectors produces a reciprocal metric tensor
\[ g^\alpha \cdot g^\beta = g^{\alpha\beta} = (g_{\alpha\beta})^{-1}. \quad (2.5) \]

We use the reciprocal frame for the definition of a derivative operator called the \textit{vector derivative}; the definition is
\[ \mathbf{D} = g^\alpha \partial_\alpha, \quad (2.6) \]
where we made use of the summation convention and the compact notation for partial derivatives \( \partial_\alpha = \partial/\partial \alpha \). The vector derivative can be applied to all geometric entities, in particular to scalars and vectors as follows:

- divergence of a scalar function, \( \mathbf{D} \psi \),
- gradient of a vector function, \( \mathbf{D} \cdot \psi \),
- exterior derivative of a vector function, \( \mathbf{D} \wedge \psi \).

The exterior derivative is an oriented area which in the special case of 3-dimensional Euclidean space can be represented by the length of a vector normal to it; this is why in this space we can use the concept of curl as a the result of applying a differential operator to a vector.

The vector derivative allows us to define a special class of functions, called \textit{monogenic functions}: those that have a null derivative. A monogenic function verifies the equation
\[ \mathbf{D} \psi = 0. \quad (2.7) \]

There is a large variety of solutions to this equation, which we cannot explore in this work. In Doran and Lasenby \cite{doran_lasenby_2003} the solutions for 3-dimensional Euclidean flat space are shown to produce spherical harmonics and in \cite{doran_lasenby_2003, almeida_2005} special solutions for 5-dimensional spacetime are used for the derivation of quantum mechanics’ equations. In this paper we will be particularly interested in the ability of monogenic functions to model dynamics; the next section is dedicated to this subject.
3 Dynamics by monogenic functions

If the vector derivative is multiplied by itself the result is a second order scalar operator; we know that it is a scalar because it results from the square of a vector. A scalar operator does not alter the geometric character of the function to which it is applied; we call this second order operator a Laplacian and its definition is quite simply

\[ D^2 \psi = D(D \psi). \]  

(3.1)

Using definition (2.6) we can write

\[ D^2 \psi = g^{\alpha} \partial_{\alpha}(g^\beta \partial_\beta \psi). \]  

(3.2)

In many practical situations the frame vectors will be slowly varying functions by comparison to function \( \psi \) and when this happens we are allowed to neglect the former’s derivatives (\( \partial_\alpha g^\beta \approx 0 \)). The Laplacian then simplifies to

\[ D^2 \psi = g^{\alpha\beta} \partial_{\alpha\beta} \psi. \]  

(3.3)

Any monogenic function is by necessity a null Laplacian function; this can be verified easily by dotting Eq. (2.7) with \( D \). On the other hand a function can have null Laplacian without being monogenic; however the second order equation resulting from a null Laplacian is usually easier to solve than its first order monogenic counterpart; in many cases one can start by imposing a null Laplacian complementing that with a verification of the further restrictions imposed on the solutions by the monogenic condition. When studying dynamics this approach is usually recommended, so let us examine solutions for Eq. (3.3). We are immediately led to try a solution of the type

\[ \psi = \psi_0 e^{i p \cdot x}, \]  

(3.4)

with \( p = g^\alpha p_\alpha \) and \( x = g_\beta x^\beta \). Inserting into Eq. (3.3) we see that

\[ g^{\alpha\beta} p_\alpha p_\beta = p^2 = 0, \]  

(3.5)

as long as the derivatives of \( p \) can be neglected with respect to the derivatives of \( \psi \). Equation (3.3) includes a constant factor \( \psi_0 \) which can be any geometric entity if only the null Laplacian condition is imposed; it is the monogenic condition that clarifies the geometric character of \( \psi_0 \). Being concerned only with dynamics we can ignore this issue which becomes relevant in quantum mechanics.

Equation (3.5) tells us that vector \( p \) is null but we need to assign physical significance to its components in order to interpret this fact physically. We propose a decomposition of vector \( p \) as

\[ p = g^0 E + \mathbf{p} + g^4 m, \]  

(3.6)
where $E$ is total energy, $p = g^1 p_1 + g^2 p_2 + g^3 p_3$ is 3-dimensional momentum and $m$ is rest mass. It is particularly interesting to examine the case of flat spacetime, when the null condition for $p$ produces

$$E^2 = p^2 + m^2,$$

a well known relation from special relativity.

If we were working in the 4-dimensional spacetime of general relativity, a function $\psi$ given by Eq. (3.4) would be interpreted as an electromagnetic wave and the metric tensor would predict light bending by gravity but here we have one extra dimension, so some work must be done in order to arrive at a physical interpretation. In order to understand how the null Laplacian generates dynamics we search for the conditions that must be met for $\psi$ to remain constant. Obviously this implies that the inner product $p \cdot x$ in the exponent remains constant or, equivalently $d(p \cdot x) = 0$. Since we are only considering situations where $p$ remains essentially invariant, we can write for constant $\psi$

$$p \cdot dx = 0. \quad (3.8)$$

The left hand side of this equation is an inner product of two vectors, which can be null in two circumstances: either the two vectors are perpendicular to each other or they are both null. If $dx$ is perpendicular to $p$ we are defining a wavefront, which here has 3 dimensions and can be called an hyperplane; accordingly $\psi$ is said to be a quasi-hyperplane wave. Since it has been established by Eq. (3.5) that $p$ is null, the other possibility for constant $\psi$ happens when $dx$ is also null, that is when

$$g_{\alpha\beta} dx^\alpha dx^\beta = 0. \quad (3.9)$$

This is all we need in order to express dynamics but we can make it more familiar if we assume that $g_{4\mu} = 0$, $\mu = 0, \ldots, 3$. The equation can then be rewritten as

$$(dx^4)^2 = \frac{g_{\mu\nu}}{g_{44}} dx^\mu dx^\nu, \quad \mu, \nu = 0, \ldots, 3. \quad (3.10)$$

The reader will recognize in this equation the quadratic form of general relativity with a metric specified by $g_{\mu\nu}/g_{44}$; this will only work if the frame vectors $g_\alpha$ are independent from coordinate $x^4$. A Lagrangian can be associated with this quadratic form and geodesic trajectories derived from it so, naturally, we have a means of modelling gravitational dynamics. For this purpose it is legitimate to associate coordinate $x^0$ to time and coordinate $x^4$ to proper time; note however that a similar association may work only for gravitational dynamics an not for electrodynamics or quantum mechanics. When convenient we will denote coordinate $x^0$ with the letter $t$ and coordinate $x^4$ with the letter $\tau$.

Equation (3.9) is a scalar equation which we can manipulate in a different way to what was done above. A useful alternative consists consist in isolating $(dx^0)^2$ in the left hand side to obtain

$$(dx^0)^2 = \frac{g_{ij}}{g_{00}} dx^i dx^j, \quad (i, j) = 1, \ldots, 4, \quad (3.11)$$
where we have assumed that $g_{0i} = 0$ and the frame vectors are independent from $x^0$. We have now obtained the quadratic form of 4-space with Euclidean signature and an equation similar to the eikonal equation we are used to seeing in geometrical optics but with 4 dimensions instead of the usual 3. We call this 4-dimensional optics for obvious reasons. There are cases when we can obtain both quadratic forms and in those cases we have the choice of performing an analysis in terms of general relativity or 4-dimensional optics; the results are coincident but the perspective is entirely different. This is the case for all static metrics in general relativity; consequently they can also be examined under a 4-dimensional optics approach. In physics, looking at a problem with different approaches usually enlarges our understanding about that problem; a 4-dimensional optics approach becomes especially revealing in quantum mechanics, because quantization can be seen as akin to propagation modes in 4-dimensional waveguides.

4 Hyperspherical symmetry

Dynamics in curved space is governed by Eq. (3.9), as we have seen. However many important conclusions can be drawn from the analysis of flat space; here we should talk about kinematics rather than dynamics, because we will be dealing only with bodies in inertial movement, modelled by hyperplane 4-dimensional waves. Obviously all hyperplane waves propagate along straight lines and there is apparently not much to say about them; we shall see that things are not as simple as they seem and a few surprises will emerge from this analysis.

Returning to Eq. (3.9) but rewriting for flat space we get

$$-(dx^0)^2 + \sum_{i=1}^{4} (dx^i)^2 = 0,$$

which is a specification for null displacements or displacements on the light hypercone. This is easier to understand if we downgrade to 3-dimensional spacetime; in 3 dimensions geometrical representation becomes possible, so it is usually useful to start with 3-dimensional analysis and progress gradually to higher dimensions. A null displacement in 3-dimensional spacetime is specified by $(dx^0)^2 - (dx^1)^2 - (dx^2)^2 = 0$. All null displacements take place on the surface of a cone with apex at the current position and this cone is usually called the light cone, because light travels along null displacement paths. If we want to restrict our attention to null displacements we don’t need 3 coordinates because we know beforehand that we are restricted to a 2-dimensional surface. Since the cone’s axis is parallel to $x^0$ we can specify any displacement on its surface by a radial displacement (distance to the axis) and a polar angle variation. We have just switched to cylindrical polar coordinates because they are the most convenient to express the symmetry implied by null displacements. If we denote the radial and angular coordinates
centered at the current position by $\tau$ and $\rho$, respectively, the null displacement condition becomes $(dx^0)^2 - (d\tau)^2 - \tau^2(d\rho)^2 = 0$.

Physical spacetime is 4-dimensional and so null displacements are performed on a 3-dimensional surface to which we can call a light hypercone; in special relativity one usually refers to the light cone, in spite of the fact that this is a 3-dimensional hypersurface, but in this paper it is convenient to make the distinction clear. Again if we are interested only in null displacements 4 coordinates are one too many and the equations will gain clarity if we adopt spherical polar coordinates. One dimension above, the situation specified by Eq. (4.1) is again one of null displacements but now in 5-dimensional spacetime. One sees immediately that all displacements are now on the surface of a 4-dimensional light hypercone and that the appropriate coordinates to bring out the implied symmetry are hyperspherical polar coordinates, comprising one radial distance and 3 polar angles; we denote those coordinates by $\{\tau, \rho, \theta, \phi\}$, respectively. In hyperspherical coordinates a null displacement is expressed by

$$-(dt)^2 + (d\tau)^2 + \tau^2[(d\rho)^2 + \sin^2 \rho(d\theta)^2 + \sin^2 \rho \sin^2 \theta(d\phi)^2] = 0.$$  (4.2)

Here we see that the association of coordinate $\tau$ with proper time implies that the 3 coordinates associated with physical space are turned into distances measured on the surface of a 3-sphere so, when we use Cartesian coordinates for physical space we are actually referring to points on the hyperplane tangent to the hypersphere and not to points of constant $\tau$. We can make an analogy to displacements on the Earth’s surface, which can be approximated with displacements on a plane locally tangent to Earth, as long as displacements remain small. What we are saying here is that Cartesian coordinates can only be used on a local scale but not for cosmological distances.

## 5 Flat space kinematics

One case of special interest considers displacements with a common origin. If displacements start at the apex of the light hypercone, they can only follow generatrices characterized by $d\rho = d\theta = d\phi = 0$ and have a very simple null condition given by

$$(dt)^2 = (d\tau)^2.$$  (5.1)

For convenience we will set $\tau = t$, that is, the coordinate’s origin is placed at the hypercone’s apex. Inserting this into the equation above and denoting derivatives with respect to $t$ by a dot, we have

$$\dot{\tau} = 1,$$  (5.2)

for displacements along a generatrix.

All generatrices diverge from the hypercone’s apex; naturally the distance between two generatrices increases with the radial coordinate $\tau$. If we wish to compute distances
to a particular point, measured over the hyperspherical directrix, we can set the origin for coordinate $\rho$ at that point and introduce a distance coordinate

$$r = \tau \rho.$$  \hfill (5.3)

A simple manipulation allows us to write

$$\dot{r} = \dot{\tau} \rho + \tau \dot{\rho} = \frac{\dot{\tau}}{\tau} r.$$  \hfill (5.4)

But we have seen above that $\dot{\tau}$ is unity, so the previous equation states that two generatrices are coming apart with a velocity $\dot{r}$ proportional to their distance $r$; this is exactly what the Hubble relation says about galaxies and we are led to define the Hubble parameter by

$$H = \frac{1}{\tau}.  \hfill (5.5)$$

This is an extremely important conclusion. A physical interpretation of our geometric argument allows us to say that the wavefunctions of cosmical objects, such as galaxies or even galaxy clusters, are monogenic functions of 5-dimensional spacetime, diverging from a common origin, the big bang. As a consequence the Universe has an overall hyperspherical symmetry and we are wrong when we choose Cartesian coordinates for its description. By choosing the appropriate hyperspherical polar coordinates we detect immediately that distances between cosmical objects must grow at a rate which is exactly proportional to those distances. The Hubble relation thus acquires a purely geometrical explanation and results from the consideration of an empty Universe; there is no need for a critical mass density to justify this flat rate expansion.

In this model cosmological objects such as galaxy clusters are still objects, evolving along generatrices of the light hypercone. This contradicts the equivalence principle, in the sense that we now have an absolute definition of motion; our privileged frame, the frame of absolute stillness, is provided by all the galaxy clusters in spite of their apparent relative motion. Still objects are characterized by the relation $\dot{\tau} = 1$, while objects in motion relative to the privileged frame are characterized by $\dot{\tau} < 1$. From Eq. (4.2) we derive

$$1 - \tau^2 [\dot{\rho}^2 + \sin^2(\rho) \dot{\theta}^2 + \sin^2(\rho) \sin^2(\theta) \dot{\phi}^2] - \dot{\tau}^2 = 0.$$  \hfill (5.6)

A physical interpretation of this equation becomes easier if we make the replacement $\tau \rho \rightarrow r$:

$$1 - \left[1 + \left(\frac{\tau}{r}\right)^2\right] \dot{r}^2 - r^2 + 2 \frac{\tau}{r} \dot{\tau} \dot{r} - r^2 [\dot{\theta}^2 + \sin^2(\theta) \dot{\phi}^2] = 0.$$  \hfill (5.7)

What we usually designate by velocity is the quantity $v$ verifying the relation

$$v^2 = \dot{r}^2 + r^2 [\dot{\theta}^2 + \sin^2(\theta) \dot{\phi}^2].$$  \hfill (5.8)
Replacing above we get

\[
1 - v^2 - \left[ 1 + \left( \frac{r}{\tau} \right)^2 \right] \dot{\tau}^2 + 2 \frac{r}{\tau} \dot{r} \dot{\tau} = 0. \tag{5.9}
\]

In laboratory experiments \( r \ll \tau \), the equation simplifies to \( \dot{\tau}^2 + v^2 = 1 \) and we are taken back to special relativity.

Equation (5.9) expresses the compatibility between relativistic physics, which applies in laboratory experiments including those of high energy physics, and the physics of cosmology, which we must start to consider when the distances to our observational point become comparable to the size of the Universe. The orbits of planets involve distances which are still small compared to the Universe and we expect relativistic dynamics to work right for them but what about the dynamics of galaxies? We shall have a brief look at this subject in the next section.

6 Rotation curves of galaxies

The subject of galaxies’ dynamics is a very complex one and we will not dare to examine it in any depth in this paper. In this section we will only have a very superficial look at some possible consequences that arise from our approach to dynamics in general via monogenic functions and 5-dimensional spacetime. First of all we must check that our approach is compatible with what is generally known about orbits under gravitational field; in particular we must show that the angular velocity of a circular orbiting body varies with \( r^{-3/2} \), \( r \) being the orbital radius.

Equation (6.1) allows us to introduce a metric such as Schwarzschild’s in order to model the gravitational interaction but we will instead Yilmaz’s metric \[7, 8\], since it mathematically more convenient and it is equivalent to Schwarzschild’s for the sort of distances we will be dealing with. In spherical coordinates the quadratic form of Yilmaz’s metric can be written as

\[
(d\tau)^2 = e^{2m/r}(dt)^2 + e^{4m/r} [(dr)^2 + r^2(d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2], \tag{6.1}
\]

where \( m \) is a spherical mass, \( r \) is the distance to the center of gravity of mass \( m \) and we are using Planck or non-dimensional units \[8\]. We could use the previous equation to derive circular orbits but it will be more convenient for the development of our argument to rewrite it in the form of Eq. (6.11); it is then

\[
(dt)^2 = e^{2m/r}(d\tau)^2 + e^{4m/r} [(dr)^2 + r^2(d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2]. \tag{6.2}
\]

 Orbital equations can now be found by searching for geodesics of this space; these result from consideration of the Lagrangian defined by \[8, 10\]

\[
1 = 2L = e^{2m/r}(\dot{\tau}^2 + \dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)); \tag{6.3}
\]
we suppressed the parenthesis around the coordinates because there are no upper indices to cause confusion. This equation is usually simplified with the consideration that orbits are flat, so we are free to set $\theta = \pi/2$. Since we are interested in circular orbits we can also set $\dot{r} = 0$; the simplified Lagrangian then verifies

$$1 = 2L = e^{2m/r} \dot{r}^2 + e^{4m/r} r^2 \dot{\phi}^2. \quad (6.4)$$

The generalized momentum associated with $r$ is constant, so the corresponding Euler Lagrange equation is $\partial_r L = 0$, which expands to

$$- \frac{m}{r^3} \dot{r}^2 + \left(1 - \frac{2m}{r}\right) e^{2m/r} \dot{\phi}^2 = 0. \quad (6.5)$$

Expanding the exponential in series, taking just two terms and simplifying we get

$$\dot{\phi}^2 = \frac{m}{r^3} \dot{r}^2, \quad (6.6)$$

which is the expected relation from Newtonian dynamics with a relativistic correction given by the $\dot{\tau}^2$ factor.

Although we have based our deductions on monogenic functions, the results obtained are the same as could have been derived with standard general relativity; we have not even considered the hyperspherical symmetry that we expect to result from the imposition of the monogenic condition in 5-dimensional spacetime. If a gravitational system as a whole verifies a single monogenic function, however, we expect it to evolve on the light hypercone with apex at the center and at the time when the system was originated; we can check how such a system should evolve and try to derive some consequences for cosmological objects. Assuming the Yilmaz metric remains valid, the null displacement condition (3.9) must be written as

$$- e^{-2m/r} (dt)^2 + (d\tau)^2 + e^{-2m/r} \tau^2 [(d\rho)^2 + \sin^2 \theta (d\phi)^2] = 0. \quad (6.7)$$

The flattness of orbits allows us to set $\theta = \pi/2$, as before. We also argue that angle $\rho$ is small, for which reason its sine can be replaced by the argument. With these simplifications we are led to consider the geodesic Lagrangian

$$1 = 2L = e^{2m/r} \dot{r}^2 + e^{4m/r} (\dot{\rho}^2 + \rho^2 \dot{\phi}^2). \quad (6.8)$$

The right hand side now includes both $\rho$ and $r$ but this can be avoided if we recall that $r = \tau \rho$; making the substitution we get

$$1 = 2L = e^{2m/r} \dot{r}^2 + e^{4m/r} \left[(\dot{r} - \frac{r}{\tau} \dot{\tau})^2 + r^2 \dot{\phi}^2\right]. \quad (6.9)$$

Notice that this equation is written for pseudo-Euclidean space with signature $(++++)$ and does not have a general relativity counterpart because the metric tensor is dependent
on coordinate $\tau$; therefore it does not verify the static metric condition that we invoked in order to be able to isolate in the left hand side either $(dx^0)^2$ or $(dx^4)^2$.

Comparing Eqs. (6.9) and (6.4), we see that the difference lies in the extra term with the parenthesis $(\ddot{r} - r\dot{\tau}/\tau)$. This parenthesis is null when $\dot{r}/r = \dot{\tau}/\tau$, that we recognize as the Hubble relation seen above. When this happens the rotational velocity verifies the very simple relation

$$r^2 \dot{\phi}^2 = 1 - e^{2m/r} \dot{\tau}^2.$$  \hspace{1cm} (6.10)$$

The left hand side is the linear rotational velocity while the right hand side is nearly constant for large values of $r$; this is exactly the behaviour of orbiting material in the periphery of most galaxies. The standard explanation for this behaviour of galaxies’ rotational curves invokes large amounts of unexplained dark matter but we see here that no dark matter at all needs to be invoked; it only needs a small expansion of the orbit, at a rate similar to the Hubble expansion. The calculations for the periphery of M 31 galaxy predict an expansion velocity of 2.43 Km s$^{-1}$ in order to obtain a flat rotation curve; this is to be compared with a rotational velocity around 300 Km s$^{-1}$ and is undetectable by present day instruments. In summary, if expansion of the order of 1% of rotational velocity is detected in galaxies with flat rotation curves, this may mean that such galaxies are governed by a single monogenic function and no dark matter at all is needed to account for the observed dynamics.

7 Cosmic evolution

In Sec. 5 we verified that a monogenic function in flat space justified the option for hyperspherical coordinates and these in turn led us to the Hubble relation; this is compatible with a model for an empty Universe governed by a single monogenic function. If this works for an empty Universe, can we make it work for a Universe with a small mass density, of the order of the mass that can effectively be detected, and do away with dark matter and dark energy? In order to answer this question we recall Eq. (6.9) and consider the case of pure expansion, so that $\ddot{\phi} = 0$.

A model for the whole Universe can be constructed if we take a thin spherical shell of mass $m_s$ and radius $r$ centered on any point of the Universe. The gravitational pull on the spherical shell is exerted by the total mass $m_i$ inside the shell, which is obviously

$$m_i = \frac{4}{3} \pi r^3 \Omega,$$  \hspace{1cm} (7.1)$$

where $\Omega$ is the mass density. Equation (6.9) allows us to write the expansion rate for the shell as

$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{e^{-4m_i/r} - e^{-2m_i/r} \dot{\tau}^2}{r^2} - \left(\frac{\dot{\tau}}{\tau}\right)^2 + \frac{2\dot{r}\dot{\tau}}{\tau r}.$$  \hspace{1cm} (7.2)$$
We need to check that this equation is compatible with the Hubble relation found earlier. In an empty Universe $m_i = 0$ and $\dot{\tau} = 1$; the equation simplifies to $\dot{\tau}/r = 1/\tau$, as expected. In the standard cosmological model the flat expansion of the Universe is attributed to a critical mass density $\Omega_c$ verifying

$$\left(\frac{\dot{\tau}}{\tau}\right)^2 = \frac{8\pi\Omega_c}{3}. \quad (7.3)$$

In our hyperspherical model the critical mass density is not mass at all, but the two approaches can be made compatible if we define

$$\Omega_c = \frac{3}{8\pi\tau^2}. \quad (7.4)$$

In the presence of a small mass density we have a perturbation of the expansion rate caused by the first term on the right hand side of Eq. (7.2). The exponents in this term include the factor $m_i/r = 4\pi r^2 \Omega_c/3$; inserting the definition (7.4) we get

$$\frac{m_i}{r} = \frac{\mu \Omega_c r^2}{2\tau^2}, \quad (7.5)$$

with $\mu = \Omega/\Omega_c$. Inserting into Eq. (7.2) and taking only the first order terms for the exponentials, the perturbation term becomes

$$- \mu \Omega_c \left(\frac{\dot{\tau}}{\tau}\right)^2. \quad (7.6)$$

Taking as an approximation that the two last terms on the right hand side of Eq. (7.2) are still given by Eq. (7.3) we get

$$\left(\frac{\dot{\tau}}{\tau}\right)^2 \approx \left[\frac{8\pi}{3} - \mu \left(\frac{\dot{\tau}}{\tau}\right)^2\right] \Omega_c. \quad (7.7)$$

The important thing to note here is that the expansion rate $\dot{\tau}/r$ is dependent on $\tau$; taking $\tau$ derivatives to both sides of the equation we get

$$\frac{d}{d\tau} \left(\frac{\dot{\tau}}{\tau}\right)^2 \approx 2\mu \Omega_c \frac{\ddot{\tau}^2}{\tau^3}. \quad (7.8)$$

We conclude that the expansion rate has a positive derivative, indicating that the Universe expands faster with the passage of time. This effect has been observed and is usually attributed to a cosmological constant, however we see here that it can be explained geometrically, if we allow for a small mass density $\Omega = \mu \Omega_c$.

Tests are needed to confirm that the recent supernovae observations are compatible with the predictions from the equation above. The latter can be simplified if we make
some further approximations. We can replace \( \dot{r}/r \) and \( 1/\tau \) by the Hubble parameter \( H \), while making \( \dot{\tau} \approx 1 \); we can also take the \( \tau \) derivative to be approximately a derivative against \( t \). With those approximations the new equation is

\[
\dot{H} \approx 2\mu \Omega_c H^2.
\] (7.9)

All parameters in this equation are known from observation, with the exception of \( \mu \), which is the mass density expressed as a fraction of \( \Omega_c \); this is expected to be around 5%.

8 Conclusion

The search for a unified formulation of physics has led the author to explore 5-dimensional spacetime endowed with curvature; this space is best described by the associated geometric algebra \( G_{4,1} \), which the author uses for the study of monogenic functions. These are functions that zero the fundamental vector derivative; as such they play a fundamental role in the geometry. Monogenic functions in 5-dimensional spacetime have two important consequences: they induce an hyperspherical symmetry and they model dynamics.

The identification of an overall hyperspherical symmetry in the Universe is all that is needed to derive the Hubble relation, which then appears as a purely geometrical effect. The mysterious critical density, assigned to dark matter in the standard model of cosmology, is here given a geometrical explanation. Monogenic functions can be applied in curved geometries and show the ability to model electrodynamics and relativistic dynamics. Assuming that a mass distribution induces space curvature, we are able to show that orbits with flat rotation curves, such as are found in most galaxies, become possible; the author contends that galaxies can be modelled by a single monogenic function, avoiding the recourse to dark matter. Assuming the whole Universe to be governed by one monogenic function, it is shown that a small mass density induces an accelerated expansion, which may avoid the embarrassing appeal to a cosmological constant.

References

[1] T. Kaluza, *On the problem of unity in physics*, Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Klasse) pp. 966–972, 1921.

[2] J. B. Almeida, *Choice of the best geometry to explain physics*, 2005, arXiv: physics/0510179.

[3] J. B. Almeida, *Hidden geometric character of relativistic quantum mechanics*, J. Math. Phys. 49, 012301, 2007, arXiv: quant-ph/0606123.
José B. Almeida  

How much in the Universe can be explained by geometry?

[4] J. B. Almeida, *A geometric algebra approach to the hydrogen atom*, 2006, arXiv: physics/0602116.

[5] J. B. Almeida, *Geometric drive of the universe’s expansion*, in *1st Crisis in Cosmology Conference, CCC–I*, edited by E. Lerner and J. B. Almeida, Universidade do Minho (American Institute of Physics, Monção, Portugal, 2006), vol. 822 of *AIP Conference Proceedings Series*, pp. 110–122, arXiv: physics/0507102.

[6] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, U.K., 2003).

[7] H. Yilmaz, *New approach to general relativity*, Phys. Rev. **111**, 1417, 1958.

[8] H. Yilmaz, *New theory of gravitation*, Phys. Rev. Lett. **27**, 1399+, 1971.

[9] R. D’Inverno, *Introducing Einstein’s Relativity* (Clarendon Press, Oxford, 1996).

[10] J. L. Martin, *General Relativity: A Guide to its Consequences for Gravity and Cosmology* (Ellis Horwood Ltd., U. K., 1988).