Abstract. The critical loci of a map $f : X \rightarrow Y$ between nonsingular varieties are the locally closed subschemes $\Sigma_i(f) \subseteq X$ where the differential of $f$ has constant rank. We prove that if $f : X \rightarrow \mathbb{A}^r$ is the general member of a suitably large linear family of maps from a nonsingular variety $X$ to affine space, then the critical loci $\Sigma_i(f)$ are smooth, except in characteristic 2 where the first critical locus $\Sigma_1(f)$ may be singular at a finite set of points. We compute for such maps $f : X \rightarrow \mathbb{A}^r$ the codimensions of their loci of Thom-Boardman singularities of order 2. In characteristics different from 2, the codimensions we find agree with those found by Levine in the context of differential topology. Finally, assuming $\dim X \geq \dim Y$, we give a local description of an arbitrary map $f : X \rightarrow Y$ at points of its first critical locus $\Sigma_1(f)$. In the case of functions and nondegenerate critical points, this description recovers the usual one from Morse theory.

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1. Introduction

Fix an algebraically closed base field \( k \). In this paper we are concerned with the critical loci of maps \( f : X \to Y \) between nonsingular varieties over \( k \). By definition, the \( i \)th critical locus of such a map is the locally closed subset

\[
\Sigma^i(f) := \{ x \in X \mid \text{rank}(df(x)) = \min(\dim X, \dim Y) - i \} \subseteq X
\]
equipped with its natural determinantal subscheme structure. Here \( df \) denotes the differential \( df : T_X \to f^*T_Y \) of \( f \).

Our main result asserts the smoothness of the critical loci of the general member of a linear family of maps from a nonsingular variety to affine space. It is an algebraic analogue of a classical result of Thom [Tho56] according to which the critical loci of a suitably generic map between smooth manifolds are themselves manifolds.

**Theorem 1.1** (Theorem 13.4). Let \( X \) be a nonsingular variety of dimension \( n \). Let \( W \subseteq \Gamma(X, \mathcal{O}_X^{\oplus r}) = \text{Hom}_k(X, \mathbb{A}^r) \) be a finite-dimensional, linear subspace. Suppose that \( W \) separates jets of order 2, that is, suppose that the natural map \( W \to (\mathcal{O}_X/m_x^{i+1})^{\oplus r} \) is surjective for each \( x \in X \). Let \( f \in W \) be a general section. Let \( i \) be a nonnegative integer. Then the critical locus \( \Sigma^i(f) \subseteq X \) is either empty or of pure codimension \( i(\lfloor n - r \rfloor + i) \) in \( X \). Moreover, \( \Sigma^i(f) \) is nonsingular, unless \( \text{char}(k) = 2, i = 1, \) and either

1. \( r \geq n \); or
2. \( r = 1 \) and \( n \) is odd,

in which case the singular locus of \( \Sigma^i(f) \) consists of a finite set of points.

Theorem 1.1 may be proved in characteristic zero by a simple argument reminiscent of the one used by Thom, see Proposition 13.3 below. It is much more interesting in positive characteristic, due to the failure of Sard’s lemma.

The only previously known case of Theorem 1.1 is the case where \( r = 1 \), which is a restatement of Kollár’s algebraic Morse lemma [Kol95, Proposition 18]. To see the connection between Theorem 1.1 and Morse theory, consider a function \( f : X \to \mathbb{A}^1 \) and a point \( x \in X \), and fix étale coordinates \( x_1, \ldots, x_n \in \mathcal{O}_{X,x} \) around \( x \) (for example, a regular system of parameters). The ideal in \( \mathcal{O}_{X,x} \) corresponding to the subscheme of critical points \( \Sigma^1(f) \subseteq X \) is generated by the partial derivatives

\[
\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \in \mathcal{O}_{X,x}.
\]

Thus, if \( x \in \Sigma^1(f) \), then the following are equivalent:

- \( x \) is a reduced, isolated point of \( \Sigma^1(f) \).
- \( \Sigma^1(f) \) is nonsingular of codimension \( n \) at \( x \).
• The differentials of the partials $\partial f/\partial x_j$ are independent at $x$.

• Hessian matrix of $f$ is invertible at $x$.

In characteristic 2, the Hessian matrix has the peculiarity of being skew-symmetric, hence of even rank. It can therefore never be invertible when the dimension $n$ of $X$ is odd, which explains exception (2) in Theorem 1.1.

The necessity of exception (1) in Theorem 1.1 is illustrated by the following simple example:

**Example 1.2.** Suppose that $X = A^2$, $r = 2$, and

$$W \subseteq \Gamma(A^2, \mathcal{O}_{A^2}^{\oplus 2}) = \text{Hom}(A^2, A^2)$$

is the linear space of cubic maps, that is, of maps $A^2 \to A^2$ whose components are inhomogeneous cubic polynomials in the coordinates of the source $A^2$. Then $W$ separates 2-jets.

Suppose $\text{char}(k) = 2$ and let $f \in W$ be a generic cubic map. By a direct computation, the critical locus $\Sigma_1(f) \subseteq A^2$ is a degree-4 curve in $A^2$ that is singular at exactly one point.

Beyond critical loci, in this paper we consider second-order singularities of maps in the sense of Thom and Boardman [Tho56, Boa67]. Following Porteous [Por71], given a nonsingular variety $Y$, a morphism $f : X \to Y$, and nonnegative integers $i$ and $j$, we define a locally closed subscheme $\Sigma^{i,j}(f) \subseteq \Sigma^i(f)$ with the property that, if $\Sigma^i(f)$ is smooth of codimension $i(|n-r| + i)$ in $X$, then

$$\Sigma^{i,j}(f) = \Sigma^j(f|_{\Sigma^i(f)}).$$

We refer to this subscheme as the *locus of second order singularities $f$ with symbol $(i,j)$*. 

Our main result about loci of second-order singularities extends Levine’s computation [Lev71, p. 55] of the codimensions of these loci, from suitably generic maps between smooth manifolds, to general members of linear families of maps from a nonsingular variety to affine space.

**Theorem 1.3 (Theorem 13.4).** Let $X$ be a nonsingular variety of dimension $n$. Let

$$W \subseteq \Gamma(X, \mathcal{O}_X^{\oplus r}) = \text{Hom}(X, A^r)$$

be a finite-dimensional, linear subspace that separates jets of order 2. Let $f \in W$ be a general section. Let $i$ and $j$ be nonnegative integers. Let $m := \min(n,r)$. Then the critical locus $\Sigma^{i,j}(f) \subseteq X$ is either empty or has pure codimension

$$i(|n-r| + i) + j(n-m + i - j)(r-m+i-1) + \frac{1}{2}j(j \pm 1)(r-m+i).$$

in $X$. Furthermore, if $k$ has characteristic zero, then $\Sigma^{i,j}(f)$ is smooth.

If $\text{char}(k) \neq 2$, then the generic codimensions of Theorem 1.3 agree with those found by Levine.
To prove Theorems 1.1 and 1.3 we work in the second-order jet scheme $J^2(X,A)$. This is a vector bundle over $X$ whose fiber over a point $x \in X$ is the vector space $(\mathcal{O}_X/m_x^3)^{\oplus r}$. Its sheaf of sections is Grothendieck’s sheaf of principal parts $\mathcal{P}^2_X(\mathcal{O}_X^{\oplus r})$.

In a separate paper we use sheaves of principal parts to construct $m$th-order jet schemes $J^m(X,Y)$ for nonsingular varieties $Y$ other than affine space. When carried out in these jet schemes, the arguments of this paper yield generalizations of Theorems 1.1 and 1.3 to families of maps from between any two nonsingular varieties.

It would be interesting to know how the definitions of jet schemes and second-order singularities adopted in this paper compare with the ones introduced by Mount and Villamayor [MV74].

A crucial tool in our proof of Theorems 1.1 and 1.3 is the intrinsic differential of $df : T_X \rightarrow f^*T_Y$ at $\Sigma$.

\[ d_{\Sigma(f)}(df) : T_X|_{\Sigma(f)} \rightarrow \mathcal{H}om_{\Sigma(f)}(\ker(df|_{\Sigma(f)}), \coker(df|_{\Sigma(f)})) \]

that generalizes the Hessian bilinear form of a function at a critical point. It has the property of being surjective if, and only if, the critical locus $\Sigma$ is nonsingular and of codimension $i(|n-r|+i)$ in $X$.

Similarly to the Hessian matrix, the second-order differential $d_{\Sigma(f)}(df)$ exhibits certain symmetries. To control its rank in the proofs of Theorems 1.1 and 1.3, we use the following result:

**Theorem 1.4** (Theorem 15.5). Let $E$ and $F$ be finite-dimensional vector spaces over $k$, and let $A \subseteq E$ be a vector subspace. Write $e := \dim_k E$, $f := \dim_k F$ and $a := \dim_k A$. Let $H$ be the vector space of linear maps $h : E \rightarrow \text{Hom}(A,F)$ such that the map $A \times A \rightarrow F$ that sends $(v,w) \mapsto h(v)(w)$ is symmetric. Let $i$ and $p$ be nonnegative integers. Let $\Delta^{i,p} \subseteq H$ be the locally closed subvariety parametrizing linear maps $h \in H$ such that

1. $h$ has rank $\min(e,af) - i$, and
2. $\dim_k(\ker(h) \cap A) = p$.

If $\Delta^{i,p}$ is nonempty (see Lemma 15.6), then $\Delta^{i,p}$ is smooth of pure codimension

\[ p(n-a+p) + f \cdot [\frac{1}{2}(-p^2 \pm p) + (e-n)a] - n(e-n) \]

in $H$, where $n := \min(e,af) - i$.

Our proof of Theorem 1.4 uses some interesting geometry on Grassmannians. Let $\Sigma \subseteq H$ be the subvariety parametrizing linear maps of rank $\min(e,af) - i$. We first note that $\Delta^{i,p}$ is isomorphic to the intersection of the Tjurina transform of $\Sigma$, which is a subvariety of the product $\text{Gr}(E) \times H$, with the special Schubert cell $(\sigma_p(A) \setminus \sigma_{p+1}(A)) \times H$. Then we show this intersection is smooth using carefully chosen charts of the Grassmannian.

The last result we state in this introduction describes the local structure of a morphism $f : X \rightarrow Y$ between nonsingular varieties with $\dim X \geq$
dim $Y$, at points in the first critical locus $\Sigma^1(f)$. For functions $f : X \to \mathbb{A}^1$ and nondegenerate critical points, that is, points of $\Sigma^{1,0}(f)$, our description recovers the usual one from Morse theory.

**Theorem 1.5** (Proposition 5.5). Let $X$ and $Y$ be nonsingular varieties of respective dimensions $n$ and $r$. Let $f : X \to Y$ be a morphism. Let $x \in X$ be a point. Let $y_1, \ldots, y_r \in \mathcal{O}_{Y,f(x)}$ be étale coordinates around $f(x) \in Y$. Let $f_l := y_l \circ f \in \mathcal{O}_{X,x}$, where $l = 1, \ldots, r$, be the components of $f$ with respect to these coordinates.

Suppose that $r \leq n$ and that $df(x) : T_X(x) \to T_Y(f(x))$ has rank $r - 1$, so that $x \in \Sigma^1(f)$. By the inverse function theorem, there exist

1. a reordering of the étale coordinates $y_1, \ldots, y_r \in \mathcal{O}_{Y,f(x)}$;
2. an isomorphism of local $k$-algebras $\hat{\mathcal{O}}_{X,x} \cong k[[x_1, \ldots, x_n]]$;
3. constants $c_1, \ldots, c_{r-1} \in k$; and
4. a power series $f \in k[[x_1, \ldots, x_n]]$

in terms of which

$$(f_1, \ldots, f_r) = (c_1 + x_1, \ldots, c_{r-1} + x_{r-1}, f).$$

Let $j$ be the unique nonnegative integer such that $x \in \Sigma^{1,j}(f)$. Then $n-r+1-j$ is nonnegative, and is moreover even if char($k$) = 2. Furthermore, the reordering, isomorphism, constants and power series above may be chosen so that $f = q + h$, where

$$q := \begin{cases} x_r^2 + \cdots + x_{n-j}^2 & \text{if char}(k) \neq 2 \\ x_r x_{r+1} + \cdots + x_{n-j-1} x_{n-j} & \text{if char}(k) = 2 \end{cases}$$

and $h \in k[[x_1, \ldots, x_{r-1}, x_{n-j+1}, x_n]]$ is a power series that does not involve the variables occurring in $q$.

Proposition 5.5 follows from Proposition 7.4 below, which is a version of “Morse’s Lemma with parameters” that holds in positive characteristics. For another version, see [GN16, Lemmas 3.9 and 3.12]. We derive Proposition 7.4 from general statements about finite determinacy and versal unfoldings of power series, namely Propositions 6.3 and 6.7 below. The latter two propositions seem to be folklore, but follow from standard arguments, as we note below. Their analogues in the context of maps between smooth manifolds are special cases of [Wal81, Theorems 1.2 and 3.4].

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2. Vector bundles and degeneracy loci

Let \( X \) be a scheme. Let \( E \) be a locally free \( \mathcal{O}_X \)-module of finite rank. The vector bundle associated to \( E \) is the \( X \)-scheme
\[
\mathbf{V}(E) := \text{Spec}_X \text{Sym}(E^\vee).
\]

There is a natural isomorphism between \( E \) and the sheaf of sections of the projection \( \mathbf{V}(E) \to X \). Indeed, given a morphism of schemes \( t : T \to X \), there are natural bijections
\[
\text{Hom}_X(T, \mathbf{V}(E)) = \text{Hom}_{\mathcal{O}_X\text{-alg.}}(\text{Sym}(E^\vee), t_* \mathcal{O}_T)
\]
\[
= \text{Hom}_{\mathcal{O}_T}(t^*E^\vee, \mathcal{O}_T)
\]
\[
= \Gamma(T, t^*E)
\]
by the universal mapping properties of the relative spectrum and of the symmetric algebra, and the adjunction between pullback and pushforward.

**Definition 2.1.** The tautological section \( \tau \in \Gamma(\mathbf{V}(E), E_{\mathbf{V}(E)}) \) is the section corresponding to the identity morphism of \( \mathbf{V}(E) \).

The map
\[
\text{Hom}_X(T, \mathbf{V}(E)) \to \Gamma(T, t^*E)
\]
given by pullback of \( \tau \) coincides with the natural bijection described above.

**Example 2.2.** Suppose that the \( \mathcal{O}_X \)-module \( E \) is free with basis \( \{ v_1, \ldots, v_e \} \subseteq \Gamma(X, E) \). Let \( A^e \) be the affine space over \( \text{Spec} \mathbb{Z} \) with coordinates \( t_1, \ldots, t_e \).

Then there exists a unique isomorphism of schemes \( \mathbf{V}(E) \cong X \times A^e \) over \( X \) with respect to which
\[
\tau = t_1 \cdot \pi^*v_1 + \cdots + t_e \cdot \pi^*v_e.
\]

**Remark 2.3.** Write \( V := \mathbf{V}(E) \). Let \( \pi : V \to X \) denote the projection. The map \( E^\vee \to \pi_*\Omega_{V/X} \) that sends \( \sigma \mapsto d(\sigma \cdot \tau) \) is linear over \( \mathcal{O}_X \). Its adjoint is an \( \mathcal{O}_V \)-linear isomorphism
\[
\pi^*E^\vee \xrightarrow{\sim} \Omega_{V/X}
\]
by Example 2.2 and the computation of the sheaf of differentials of affine space.

Let \( \alpha : E \to F \) be a map of locally free \( \mathcal{O}_X \)-modules of finite rank. Let \( e \) and \( f \) respectively denote the ranks of \( E \) and \( F \). Let \( m = \min(e, f) \). Let \( i \) be a nonnegative integer.

**Definition 2.4.** The \( i \)th degeneracy locus of \( \alpha \) is defined to be the subscheme \( \Sigma^i(\alpha) \subseteq X \) where exterior power
\[
\wedge^{m-i+1} \alpha : \wedge^{m-i+1} E \to \wedge^{m-i+1} F
\]
vanishes if \( i \leq m + 1 \), and the empty scheme otherwise.

Let \( i \) be an integer such that \( 0 \leq i \leq m \).
Remark 2.5.  
(1) A point $x \in X$ lies in $\Sigma^i(\alpha)$ if, and only if, the $k(x)$-linear map $\alpha(x)$ has rank at most $m - i$.
(2) By the Laplace expansion of the determinant, we have closed immersions
\[ \emptyset = \Sigma^{m+1}(\alpha) \subseteq \Sigma^m(\alpha) \subseteq \cdots \subseteq \Sigma^0(\alpha) = X. \]
(3) If $t : T \to X$ be a morphism of schemes, then
\[ t^{-1}\Sigma^i(\alpha) = \Sigma^i(t^*\alpha) \]
as closed subschemes of $T$.

Let $\Sigma$ denote the locally closed subscheme $\Sigma^i(\alpha) \setminus \Sigma^{i+1}(\alpha) \subseteq X$.

Proposition 2.6. A morphism of schemes $t : T \to X$ factors through $\Sigma$ if, and only if, the cokernel of $t^*\alpha : t^*E \to t^*F$ is a locally free $O_T$-module of rank $f - m + i$.

Proof. Left to the reader. The key point is to show that the cokernel of the $O_\Sigma$-linear map $\alpha|_\Sigma : E_\Sigma \to F_\Sigma$ is a locally free of rank $f - m + i$. This can be done with the help of Lemma 2.8 below. □

Corollary 2.7. The kernel, image and cokernel of $\alpha|_\Sigma : E_\Sigma \to F_\Sigma$ are locally free $O_\Sigma$-modules of respective ranks $e - m + i$, $m - i$ and $f - m + i$. If $t : T \to \Sigma$ is a map of schemes, then:
\[ \ker(t^*(\alpha|_\Sigma)) = t^*\ker(\alpha|_\Sigma) \]
\[ \operatorname{im}(t^*(\alpha|_\Sigma)) = t^*\operatorname{im}(\alpha|_\Sigma) \]
\[ \operatorname{coker}(t^*(\alpha|_\Sigma)) = t^*\operatorname{coker}(\alpha|_\Sigma) \]

Proof. This follows from Proposition 2.6 and the following familiar fact. Let $W$ be a scheme and let
\[ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \]
be a short exact sequence of $O_W$-modules. If $B$ and $C$ are locally free of finite rank, then $A$ is locally free of finite rank, and the sequence (1) remains exact after pullback along any map $t : T \to W$. □

Lemma 2.8. Let $R$ be a ring. Let $\beta : M \to N$ be a map of $R$-modules. Let $A,B \subseteq M$ be submodules such that $\beta(A) \subseteq N$ is a free direct summand of finite rank $a$. Let $q$ be a nonnegative integer. The following are equivalent:

1. The map $\wedge^{q+a}(A + B) \to \wedge^{q+a}N$ induced by $\beta$ is zero.
2. The map $\wedge^qB \to \wedge^q(N/\alpha(A))$ induced by $\beta$ is zero.

Proof. Replacing $A$ and $B$ with their images under $\beta$, it suffices to consider the case where $\beta$ is injective, which is [Kle69, Lemma 2.5]. □

Let $\pi : V \to X$ be the vector bundle corresponding to the locally free $O_X$-modules $\mathcal{H}om_X(E,F)$. In symbols,
\[ V = V(\mathcal{H}om_X(E,F)). \]
Let $h : E_V \to F_V$ be the tautological map (Definition 2.1). The following result is well known.

**Proposition 2.9.** Let $i$ be an integer such that $0 \leq i \leq m$. Let $\Sigma$ denote the locally closed degeneracy locus $\Sigma'(h) \setminus \Sigma^{i+1}(h) \subseteq V$. Then $\Sigma$ is smooth of relative dimension $(e - m + i)(f - m + i)$ over $X$. The canonical $\mathcal{O}_V$-linear isomorphism $T_{\Sigma/X} \xrightarrow{\sim} \mathcal{H}om_X(E, F)_V$ of Remark 2.3 induces an $\mathcal{O}_\Sigma$-linear isomorphism

$$N_\Sigma := (T_{V/X})_{\Sigma'/\Sigma} \xrightarrow{\sim} \mathcal{H}om_\Sigma(\ker(h|_\Sigma), \coker(h|_\Sigma)).$$

**Proof.** Omitted.

### 3. The intrinsic differential

Let $k$ be a field. Let $X$ be a scheme over $k$. Let $E$ and $F$ be locally free $\mathcal{O}_X$-modules of ranks $e$ and $f$, respectively. Let $\alpha : E \to F$ be an $\mathcal{O}_X$-linear map. Let $x : T \to X$ be a morphism of schemes. We think of $x$ as a $T$-valued point of $X$.

**Proposition 3.1.** Suppose the $\mathcal{O}_X$-modules $E$ and $F$ are free. Choose bases for $E$ and $F$. Let

$$\nabla : \mathcal{H}om(E, F) \to \Omega_X \otimes \mathcal{H}om(E, F)$$

be the $k$-linear map given by differentiation of matrix entries with respect to these bases. Let

$$d_x \alpha \in \Gamma(T, x^* \Omega_X \otimes \mathcal{H}om_T(\ker(x^* \alpha), \coker(x^* \alpha)))$$

be the image of $\nabla \alpha$ under the $\mathcal{O}_T$-linear map

$$x^* \Omega_X \otimes \mathcal{H}om_T(x^* E, x^* F) \xrightarrow{(i, q)} x^* \Omega_X \otimes \mathcal{H}om_T(\ker(x^* \alpha), \coker(x^* \alpha))$$

induced by the inclusion $i : \ker(x^* \alpha) \hookrightarrow x^* E$ and the projection $q : x^* F \twoheadrightarrow \coker(x^* \alpha)$. Then $d_x \alpha$ is independent of the bases used to define it.

**Proof.** We may assume that $E = \mathcal{O}_X^{e \times e}$ and $F = \mathcal{O}_X^{f \times f}$ and that the chosen bases on these $\mathcal{O}_X$-modules are the standard ones. Let $\varphi : E \xrightarrow{\sim} E$ and $\psi : F \xrightarrow{\sim} F$ be $\mathcal{O}_X$-linear isomorphisms. Let $\tilde{\alpha} : x^* E \to x^* F$ be the composition $\psi \circ \alpha \circ \varphi^{-1}$. Let $\tilde{i} : \ker(x^* \tilde{\alpha}) \hookrightarrow x^* E$ and $\tilde{q} : x^* F \twoheadrightarrow \coker(x^* \tilde{\alpha})$ be the natural inclusion and projection maps. Let $\tilde{\varphi} : \ker(x^* \tilde{\alpha}) \xrightarrow{\sim} \ker(x^* \tilde{\alpha})$ and $\tilde{\psi} : \coker(x^* \tilde{\alpha}) \xrightarrow{\sim} \coker(x^* \tilde{\alpha})$ are $\mathcal{O}_T$-linear isomorphisms induced by $\varphi$ and $\psi$. To prove the result, it suffices to show that

$$\tilde{q} \cdot \nabla \tilde{\alpha} \cdot \tilde{i} = \tilde{\psi} \cdot q \cdot \nabla \alpha \cdot i \cdot \tilde{\varphi}^{-1}$$

as elements of $\Gamma(T, x^* \Omega_X \otimes \mathcal{H}om_T(\ker(x^* \alpha), \coker(x^* \alpha)))$.

Identifying $\alpha$, $\varphi$ and $\psi$ with matrices with entries in $\Gamma(X, \mathcal{O}_X)$ and applying the Leibniz rule, we find that

$$d \tilde{\alpha} = d \psi \cdot \alpha \cdot \varphi^{-1} + \psi \cdot d \alpha \cdot \varphi^{-1} + \psi \cdot \alpha \cdot d(\varphi^{-1})$$
as elements of $\Gamma(X, \Omega^2_X)$). Now $\alpha \cdot \varphi^{-1} \cdot \iota = \psi^{-1} \cdot \tilde{\alpha} \cdot \iota = 0$ as maps $\ker(x^*\tilde{\alpha}) \to x^*E$, and $\tilde{q} \cdot \psi \cdot \alpha = \tilde{q} \cdot \tilde{\alpha} \cdot \varphi = 0$ as maps $x^*F \to \text{coker}(x^*\tilde{\alpha})$. Thus
\[
\tilde{q} \cdot \nabla \tilde{\alpha} \cdot \iota = \tilde{q} \cdot \psi \cdot \nabla \alpha \cdot \varphi^{-1} \cdot \iota
= \tilde{\psi} \cdot q \cdot \nabla \alpha \cdot \iota \cdot \varphi^{-1},
\]
which completes the proof. \hfill \Box

**Definition 3.2.** The intrinsic differential of $\alpha : E \to F$ at $x$ is the unique section
\[
d_x \alpha \in \Gamma(T, x^*\Omega_X \otimes \mathcal{H}om_T(\ker(x^*\alpha), \text{coker}(x^*\alpha)))
\]
such that, for each open subset $U \subseteq X$ over which $E$ and $F$ are free, the restriction of $d_x \alpha$ to $x^{-1}U \subseteq T$ coincides with the section of Proposition 3.1 applied to $\alpha|_U : E_U \to F_U$ and $x|_{x^{-1}U} : x^{-1}U \to U$. When $X$ is smooth over $k$, we will regard the intrinsic differential as an $\mathcal{O}_T$-linear map
\[
d_x \alpha : x^*T_X \to \mathcal{H}om_T(\ker(x^*\alpha), \text{coker}(x^*\alpha)).
\]

Suppose that $X$ is smooth over $k$ and that the cokernel of $x^*\alpha : x^*E \to x^*F$ is a locally free $\mathcal{O}_T$-module of constant rank. Then the intrinsic differential $d_x \alpha$ may be constructed geometrically, as follows. Let $\pi : V \to X$ be the vector bundle corresponding to the locally free $\mathcal{O}_X$-modules $\mathcal{H}om_X(E, F)$. In symbols,
\[
V = \mathcal{V}(\mathcal{H}om_X(E, F)).
\]
Let $h : E_V \to F_V$ be the tautological map (Definition 2.1). Let $\tilde{\alpha} : X \to V$ be the unique section of $\pi : V \to X$ such that $\tilde{\alpha}^*h = \alpha$. Let $i$ be the nonnegative integer defined by
\[
\text{rank}(\text{coker}(x^*\alpha)) = f - \min(e, f) + i.
\]
Let $\Sigma$ denote the locally closed degeneracy locus $\Sigma^i(h) \backslash \Sigma^{i+1}(h) \subseteq V$. Then $\tilde{\alpha} \circ x : T \to V$ factors through $\Sigma$ by Proposition 2.6.

**Proposition 3.3.** With the assumptions and notation of the preceding paragraph, the intrinsic differential $d_x \alpha$ is equal to the composition of the $\mathcal{O}_T$-linear maps
\[
x^*T_X \xrightarrow{\overline{d\tilde{\alpha}}} (\tilde{\alpha} \circ x)^*T_V \xrightarrow{q} (\tilde{\alpha} \circ x)^*N_{\Sigma} \xrightarrow{\theta} \mathcal{H}om_T(\ker(x^*\alpha), \text{coker}(x^*\alpha)),
\]
where $\overline{d\tilde{\alpha}}$ denotes the differential of $\tilde{\alpha} : X \to V$; $N_{\Sigma} := T_V|_{\Sigma}/T_{\Sigma}$ denotes the normal sheaf of $\Sigma$ in $V$; $q$ denotes the canonical projection; and $\theta$ denotes the canonical isomorphism of Proposition 2.9 and Corollary 2.7.

Definition 3.2 may be motivated by the following observation.

**Remark 3.4.** By definition of transversality, the map $\tilde{\alpha} : X \to V$ is transverse to $\Sigma$ if, and only if, the intrinsic differential
\[
d_{\Sigma^i} \alpha : (T_X)_{\Sigma^i} \to \mathcal{H}om_{\Sigma^i}(\ker(\alpha|_{\Sigma^i}), \text{coker}(\alpha|_{\Sigma^i}))
\]
is surjective. By [EGA, Proposition IV.17.13.2] these conditions hold if and only if the scheme-theoretic inverse image $\tilde{\alpha}^{-1}\Sigma$ is smooth over $k$ and of codimension in $X$ equal to the codimension of $\Sigma$ in $V$. The latter codimension is equal to $i(|e - f| + i)$ by Proposition 2.9.

4. Second-order singularities

Let $k$ be a field. Let $f : X \to Y$ be a morphism of smooth schemes over $k$. Let $i$ be a nonnegative integer.

**Definition 4.1.** The $i$th critical locus of $f$ is the locally closed subscheme

$$\Sigma^i(f) := \Sigma^i(df) \setminus \Sigma^{i+1}(df) \subseteq X,$$

where $\Sigma^j(df)$ denotes the $j$th degeneracy locus of the differential $df : T_X \to f^*T_Y$.

Let $n$ and $r$ denote the (locally constant) dimension functions of $X$ and $Y$, respectively. A point $x \in X$ is contained in $\Sigma^i(f)$ if, and only if, the $k(x)$-linear map $df(x) : T_X(x) \to T_Y(f(x))$ has rank $\min(n,r) - i$.

**Definition 4.2.** Let $T$ be a scheme and let $x : T \to X$ a morphism, which we regard as a $T$-valued point of $X$. The intrinsic differential of $df : T_X \to f^*T_Y$ at $x$ is an $O_T$-linear map

$$d_x(df) : x^*T_X \to \text{Hom}_T(\ker(x^*df), \text{coker}(x^*df)).$$

The restriction

$$d^2_x f : \ker(x^*df) \to \text{Hom}_T(\ker(x^*df), \text{coker}(x^*df))$$

of this $O_T$-linear map to $\ker(x^*df) \subseteq x^*T_X$ is called the second intrinsic differential of $f$ at $x$.

When the morphism $x : T \to X$ is understood from the context, we may write $d^2_T f$ instead of $d^2_x f$ and refer to this $O_T$-linear map as the second intrinsic differential of $f$ at $T$.

The kernel and cokernel of the restriction of the differential $df : T_X \to f^*T_Y$ to $\Sigma^i(f)$ are locally free $O_{\Sigma^i(f)}$-modules, so the second-order differentials $d_{\Sigma^i(f)}(df)$ and $d^2_{\Sigma^i(f)} f$ are maps of locally free $O_{\Sigma^i(f)}$ modules. Hence we can speak of their degeneracy loci.

Let $j$ be a nonnegative integer.

**Definition 4.3.** (1) The bad locus $B^i(f)$ is the closed subscheme of $\Sigma^i(f)$ defined as follows. If $n \geq i(|n - r| + i)$, then $B^i(f)$ is the first degeneracy locus

$$B^i(f) := \Sigma^i(d_{\Sigma^i(f)}(df)) \subseteq \Sigma^i(f)$$

of the intrinsic differential of $df : T_X \to f^*T_Y$ at $\Sigma^i(f)$. If $n < i(|n - r| + i)$, then $B^i(f) = \Sigma^i(f)$. 


The locally closed subscheme
\[ \Sigma_{i,j}(f) := \Sigma^j(d^2_{\Sigma^i(f)}f) \setminus \Sigma^{j+1}(d^2_{\Sigma^i(f)}f) \subseteq \Sigma^i(f) \]
is called the locus of second-order singularities with symbol \((i, j)\).

Definition 4.3 is motivated by the following two results.

**Proposition 4.4.** The bad locus \(B^i(f)\) is the locus where \(\Sigma^i(f)\) is either not smooth or of codimension in \(X\) different from \(i(|n - r| + i)\).

**Proof.** By Remark 3.4, the critical locus \(\Sigma^i(f)\) is smooth and of codimension \(i(|n - r| + i)\) at a point \(x \in \Sigma^i(f)\) if, and only if, the intrinsic differential
\[ d_{\Sigma^i(f)}(df) : T_X|_{\Sigma^i(f)} \to \text{Hom}_\Sigma(\text{ker}(df|_{\Sigma^i(f)}), \text{coker}(df|_{\Sigma^i(f)})) \]
is surjective at \(x\). By Remark 2.5 (1) and the observation that the target of \(d_{\Sigma^i(f)}(df)\) is a locally free \(O_{\Sigma^i(f)}\)-module of rank \(i(|n - r| + i)\), this happens if, and only if, \(x \not\in B^i(f)\).

\[ \square \]

As \(\Sigma^i(f)\) is smooth away from \(B^i(f)\), it makes sense to talk about the critical loci of the restriction
\[ f|_{\Sigma^i(f) \setminus B^i(f)} : \Sigma^i(f) \setminus B^i(f) \to Y. \]

**Proposition 4.5.** Away from \(B^i(f)\), the locus of second-order singularities \(\Sigma^{i,j}(f)\) agrees with the \(j\)th critical locus of \(f\) restricted to \(\Sigma^i(f)\). In symbols:
\[ \Sigma^{i,j}(f) \setminus B^i(f) = \Sigma^j(f|_{\Sigma^i(f) \setminus B^i(f)}) \]

**Proof.** As we won't need Proposition 4.5 in the sequel, we just give the idea of the proof. Replacing \(X\) with \(X \setminus B^i(f)\), it suffices to consider the case in which \(B^i(f)\) is empty. Let \(\Sigma := \Sigma^i(f)\), let \(K := \text{ker}(df|_{\Sigma})\) and let \(C := \text{coker}(df|_{\Sigma})\). We have a diagram of \(O_{\Sigma}\)-modules:

\[
\begin{array}{cccccc}
0 & \to & T_{\Sigma} & \to & T_X|_{\Sigma} & \xrightarrow{d_{\Sigma}(df)} & \text{Hom}_\Sigma(K, C) & \to & 0 \\
& & & \downarrow{d(f|_{\Sigma})} & \downarrow{df|_{\Sigma}} & & \downarrow{f^*T_Y|_{\Sigma}} & \\
& & & & & & & & \\
\end{array}
\]

Applying Lemma 2.8 to \(K\) and \(T_{\Sigma}\) viewed as locally free, locally split \(O_{\Sigma}\)-submodules of \((T_X|_{\Sigma})\), the result follows.

\[ \square \]

5. The second intrinsic differential, locally

Let \(k\) be a field. Let \(f : X \to Y\) be a morphism of smooth schemes over \(k\). Let \(x \in X\) be a point with residue field \(k\). Let \(y := f(x) \in Y\).

Let \(y_1, \ldots, y_r \in O_{Y,y}\) be elements whose differentials form a basis for \(\Omega_{Y,y}\) as an \(O_{Y,y}\)-module (for example, a regular system of parameters at \(y\)). We think of \(y_1, \ldots, y_r\) as coordinates on a neighborhood of \(y\) in \(Y\). Let
Proposition 5.1. After reordering the coordinates \( y_1, \ldots, y_r \), we may find a regular system of parameters \( x_1, \ldots, x_n \in \mathcal{O}_{X,x} \) and constants \( c_1, \ldots, c_{m-i} \in k \) such that

\[
(f_1, \ldots, f_r) = (c_1 + x_1, \ldots, c_{m-i} + x_{m-i}, f_{m-i+1}, \ldots, f_r)
\]

in \( \mathcal{O}^{\oplus r}_{X,x} \).

Proof. The images of the differentials \( df_1, \ldots, df_{m-i} \) span a \( k \)-linear subspace of \( \Omega_X(x) \) of dimension \( m-i \). Reordering the coordinates \( y_1, \ldots, y_r \), we may assume that the images of \( df_1, \ldots, df_{m-i} \) form a basis for this subspace. Let \( x_{m-i+1}, \ldots, x_n \in m_x \) be elements whose differentials complete the images of \( df_1, \ldots, df_{m-i} \) to a basis of \( \Omega_X(x) \). Then

\[
(f_1 - f_1(x), \ldots, f_{m-i} - f_{m-i}(x), x_{m-i+1}, \ldots, x_n) \in m_x
\]

is a regular system of parameters. \( \square \)

Fix a regular system of parameters \( x_1, \ldots, x_n \in \mathcal{O}_{X,x} \) as in Proposition 5.1. Let

\[
\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\} \subseteq T_{X,x}
\]

be the basis of \( T_{X,x} \) dual to the basis \( \{dx_1, \ldots, dx_n\} \) of \( \Omega_{X,x} \). Similarly, let

\[
\left\{ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_r} \right\} \subseteq T_{Y,y}
\]

be the basis of \( T_{Y,y} \) dual to the basis \( \{dy_1, \ldots, dy_r\} \) of \( \Omega_{Y,y} \).

We identify elements of \( T_{X,x} \) with derivations \( \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \). Given \( g \in \mathcal{O}_{X,x} \), we write

\[
\partial_a g := \frac{\partial g}{\partial x_a}
\]

for all \( a = 1, \ldots, n \).

Let \( K \) and \( C \) denote the kernel and cokernel of the differential \( df(x) : T_X(x) \to T_Y(y) \).

Proposition 5.2. The images of

\[
\frac{\partial}{\partial y_{m-i+1}}, \ldots, \frac{\partial}{\partial y_r} \in T_{Y,y}
\]

in \( C \) form a basis for \( C \) as vector space over \( k \). The images of

\[
\frac{\partial}{\partial x_{m-i+1}}, \ldots, \frac{\partial}{\partial x_n} \in T_{X,x}
\]

in \( T_X(x) \) lie in \( K \) and form a basis for \( K \) as a vector space over \( k \).
Proof. In $\text{Hom}_{\mathcal{O}_{X,x}}(T_{X,x}, (f^*T_Y)_x)$, we have
\[
df = \sum_{l=1}^{r} \sum_{a=1}^{n} (\partial_a f_l) dx_a \otimes \frac{\partial}{\partial y_l} = \sum_{l \leq m-i} dx_l \otimes \frac{\partial}{\partial y_l} + \sum_{l > m-i} \sum_{a=1}^{n} (\partial_a f_l) dx_a \otimes \frac{\partial}{\partial y_l}.
\]
Therefore in $C$ we have
\[
0 = df \left( \frac{\partial}{\partial x_b} \right) = \begin{cases} 
\frac{\partial}{\partial y_b} + \sum_{l=m-i} (\partial_b f_l) \frac{\partial}{\partial y_l} & \text{if } b \leq m - i \\
\sum_{l > m-i} (\partial_b f_l) \frac{\partial}{\partial y_l} & \text{if } b > m - i.
\end{cases}
\]
The equalities (2) for $b \leq m - i$ show that the vector fields $\frac{\partial}{\partial y_l} \in T_{Y,y}$ with $l > m - i$ generate $C$, and hence form a basis for $C$. This fact combined with the equalities (2) for $b > m - i$ shows that $(\partial_b f_l)(x) = 0 \in k$ for all $l = m - i + 1, \ldots, r$ and $b = m - i + 1, \ldots, n$. It follows that the images in $T_X(x)$ of the vector fields $\frac{\partial}{\partial x_b} \in T_{X,x}$ with $b > m - i$ lie in $K$, hence form a basis for $K$.

The following proposition shows that the second intrinsic differential of $f$ at $x$ is represented by a collection of submatrices of the Hessians of $f_{m-i+1}, \ldots, f_r$.

**Proposition 5.3.** The second intrinsic differential $d_x^2 f : K \to \text{Hom}_k(K, C)$ is the unique $k$-linear map from $K$ to $\text{Hom}_k(K, C)$ such that
\[
(d_x^2 f) \left( \frac{\partial}{\partial x_a} \right) \left( \frac{\partial}{\partial x_b} \right) = \sum_{l=m-i+1}^{r} (\partial_a \partial_b f_l)(x) \frac{\partial}{\partial y_l}
\]
for all $a, b = m - i + 1, \ldots, n$.

Proof. Let $H(f) : T_{X,x} \to \text{Hom}_X(T_X, f^*T_Y)_x$ be the unique $\mathcal{O}_{X,x}$-linear map such that
\[
H(f) \left( \frac{\partial}{\partial x_a} \right) = \sum_{l=1}^{r} \sum_{b=1}^{n} (\partial_a \partial_b f_l) dx_b \otimes \frac{\partial}{\partial y_l} = \sum_{l=m-i+1}^{r} \sum_{b=1}^{n} (\partial_a \partial_b f_l) dx_b \otimes \frac{\partial}{\partial y_l}
\]
for all $a = 1, \ldots, n$. Let
\[
\delta : \text{Hom}_k(T_X(x), T_Y(y)) \to \text{Hom}_k(K, C)
\]
be the natural map induced by the inclusion $K \hookrightarrow T_X(x)$ and the quotient map $T_Y(y) \twoheadrightarrow C$. Then
\[
\delta \circ (H(f)(x)) = d_x(df : T_X \to f^*T_Y)
\]
by definition of the intrinsic differential (Definition 3.2). The result follows.

For the remainder of this section, we assume that $n \geq r$ and $i = 1$. Then
\[
(f_1, \ldots, f_r) = (x_1 + c_1, \ldots, x_{r-1} + c_{r-1}, f_r)
\]
and the second intrinsic differential $d_x^2 f : K \to \text{Hom}_k(K, C)$ is a map between $k$-vector spaces of dimension $n - r + 1$. Let $j$ be the rank-deficiency of this map, so that $d_x^2 f$ has rank $n - r + 1 - j$ and $x \in \Sigma^{i,j}(f)$.
Proposition 5.5. Let \( \mathbb{F} \) have even rank. Then there exist a power series \( h \in k[[x_1, \ldots, x_{r-1}, x_{n-j+1}, \ldots, x_n]] \) contained in the ideal
\[
\langle x_1, \ldots, x_{r-1} \rangle + \langle x_{n-j+1}, \ldots, x_n \rangle^2,
\]
a constant \( c \in k \), and an automorphism \( \varphi \) of
\[
\hat{O}_{X,x} = k[[x_1, \ldots, x_n]]
\]
as a local \( k[[x_1, \ldots, x_{r-1}]] \)-algebra such that \( \varphi(f_r) = c + q + h \).

Proof. Write \( f_e = c + g_1 + g_2 \), where \( c \in k \) is a constant, \( g_1 \) is homogeneous of degree 1 in \( x_1, \ldots, x_n \), and \( g_2 \in \langle x_1, \ldots, x_n \rangle^2 \). Then \( g_1 \) only involves the variables \( x_1, \ldots, x_{r-1} \) by the assumption the differential \( df(x) : T_X(x) \to T_Y(x) \) has rank \( r - 1 \). Let \( g_2 := g_2(0, \ldots, 0, x_e, \ldots, x_n) \in k[[x_e, \ldots, x_n]] \). By Proposition 5.3, the Hessian matrix of \( g_2 \) has rank \( n - r + 1 - j \) at the origin. Viewing \( g_2 \) as an unfolding of \( g_2 \) over \( R := k[[x_1, \ldots, x_{e-1}]] \) and applying Morse’s Lemma with Parameters (Proposition 7.4 below), we may find an automorphism \( \varphi \) of \( k[[x_1, \ldots, x_n]] \) as a local \( k[[x_1, \ldots, x_{e-1}]] \)-algebra that sends \( g_2 \) to \( q + h' \) for some power series \( h' \in k[[x_1, \ldots, x_n]] \) that does not involve the variables \( x_e, \ldots, x_{n-j} \). Setting \( h := g_1 + h' \), the result follows. \( \square \)

6. Power series with finite Milnor number

For lack of a suitable reference covering positive characteristics, we record in this section basic facts about power series with finite Milnor number. These facts belong to the larger framework of the theory of finitely determined map germs, for which we recommend Wall’s survey [Wal81].

Let \( k \) be a field. Let \( x = (x_1, \ldots, x_n) \) be a finite set of indeterminates. Let \( f \in k[[x]] \) be a power series.

Definition 6.1. The Jacobian ideal of \( f \), denoted \( \text{jac}(f) \), is the ideal generated in the power series ring \( k[[x]] \) by the partial derivatives \( \partial f / \partial x_1, \ldots, \partial f / \partial x_n \). The quotient \( k[[x]] / \text{jac}(f) \) is called the Milnor algebra of \( f \). Its (possibly infinite) dimension as a vector space over \( k \) is called the Milnor number of \( f \) and denoted by \( \mu(f) \).

Definition 6.2. Let \( r \) be a positive integer. We say that \( f \in k[[x]] \) is \( r \)-determined if for every power series \( g \in k[[x]] \) such that \( f - g \in \langle x \rangle^{r+1} \),
there exists an automorphism of $k[[x]]$ as a local $k$-algebra that sends $g$ to $f$. We say that $f$ is finitely determined if it is $r$-determined for some $r \geq 1$.

**Proposition 6.3.** If $f \in k[[x]]$ has finite Milnor number, then $f$ is finitely determined. More precisely, let $r$ be a positive integer. If $\langle x \rangle^r \subseteq \text{jac}(f)$, then $f$ is $2r$-determined.

**Proof.** This result is a less precise version of [BGM12, Theorem 2.1]. For a simple, direct argument, see the proof of [Mil68, Lemma 10.8]. \qed

The analogue of Proposition 6.3 for germs of smooth functions on Euclidean space is a very special case of [Wal81, Theorem 1.2].

Let $C$ be the category whose objects are complete, Noetherian, local $k$-algebras with residue field $k$, and whose morphisms are maps of local $k$-algebras.

**Definition 6.4.** Let $R$ be a complete local $k$-algebra in $C$.

1. An unfolding of $f$ over $R$ is a power series $F \in R[[x]]$ that maps to $f \in k[[x]]$ under the quotient map $R \to k$.
2. Let $F, F' \in R[[x]]$ be unfoldings of $f$ over $R$. A morphism (or right-equivalence) $F \to F'$ is a local $R$-algebra map $\varphi : R[[x]] \to R[[x]]$ that lifts the identity of $k[[x]]$ and sends $F$ to $F'$.

Unfoldings of $f$ over $R$ and morphisms between them form a category (in fact, a groupoid) that we denote by $D(R)$. A map $b : R \to R'$ of complete local $k$-algebras in $C$ induces an obvious functor $b^* : D(R) \to D(R')$.

**Definition 6.5.** The functor of unfoldings of $f$ is the functor $D : C \to \text{(Sets)}$ that sends a complete local $k$-algebra $R \in C$ to the set $D(R)$ of isomorphism classes of unfoldings of $f$ over $R$, and acts on morphisms in the obvious way.

**Definition 6.6.** Let $R$ be a complete local $k$-algebra in $C$. Let $F \in R[[x]]$ be a unfolding of $f$ over $R$. We say that $F$ is right-complete (or versal) if, for every complete local $k$-algebra $A$ in $C$, the map $\text{Hom}_C(R, A) \to D(A)$

that sends $b \mapsto b_* F$ is surjective.

**Proposition 6.7.** Suppose that $f$ has finite Milnor number. Let $g_1, \ldots, g_\mu \in k[[x]]$ be power series whose images span the Milnor algebra $k[[x]]/\text{jac}(f)$ as a vector space over $k$. Let $s = (s_1, \ldots, s_\mu)$ be a set of $\mu$ indeterminates. Then

$$F := f + s_1 g_1 + \cdots + s_\mu g_\mu \in k[[s, x]]$$

is a right-complete unfolding of $f$ over $k[[s]]$.

**Proof.** Follows from the proof of [GLS07, Corollary 1.17]. \qed

The analogue of Proposition 6.3 for unfoldings of germs of smooth functions on Euclidean space is a very special case of [Wal81, Theorem 3.4].
7. Morse’s Lemma with Parameters

Let $k$ be an algebraically closed field of characteristic $p \geq 0$. Let $V$ be a finite dimensional vector space over $k$. Let $q \in \text{Sym}^2(V^\vee)$ be a quadratic form on $V$. Let $h \in (V \otimes V)^\vee$ be the bilinear form associated to $q$, so that

$$h(u \otimes v) = q(u + v) - q(u) - q(v)$$

for all $u, v \in V$.

**Lemma 7.1.** Let $r$ denote the rank of $h$. Then $r$ is even if $p = 2$. Let $x_1, \ldots, x_n$ be a $k$-linear basis of $V^\vee$. Let

$$q_0 = \begin{cases} x_1^2 + \cdots + x_r^2 & \text{if } p \neq 2 \\ x_1x_2 + \cdots + x_{r-1}x_r & \text{if } p = 2. \end{cases}$$

If $p \neq 2$, then there exists a $k$-linear automorphism $\varphi$ of $V^\vee$ such that $\text{Sym}^2(\varphi)$ sends $q_0$ to $q$. If $p = 2$, then there exists a $k$-linear automorphism $\varphi$ of $V$ such that $\text{Sym}^2(\varphi)$ sends $q_0$ to either $q$ or $q + x_{r+1}^2$.

**Proof.** We consider the case where $k$ has characteristic 2, which is less standard. By [Arf41, Satz 2 on p. 150], without using the hypothesis that $k$ is algebraically closed, we may find a basis of $V^\vee$ consisting of vectors $u_1, \ldots, u_s, v_1, \ldots, u_s, w_1, \ldots, w_{n-2s}$ such that

$$q = \sum_{i=1}^s (a_iu_i^2 + b_iu_iv_i + c_i v_i^2) + \sum_{j=1}^t d_jw_j^2.$$ 

Here $s$ and $t$ are nonnegative integers satisfying $2s + t \leq n$; $a_i, b_i, c_i$ and $d_j$ are elements of $k$ for all $i = 1, \ldots, s$ and $j = 1, \ldots, t$; and $b_i \neq 0$ for all $i = 1, \ldots, s$. Using the hypothesis on $k$, it is easy to find an automorphism of $V$ that preserves the subspaces $\langle u_i, v_i \rangle$ and $\langle w_j \rangle$ of $V^\vee$ for all $i$ and $j$, and that sends

$$a_iu_i^2 + b_iu_iv_i + c_i v_i^2 \mapsto u_iu_i \quad \text{and} \quad d_jw_j^2 \mapsto w_j^2$$

for all $i$ and $j$. This implies the result, since $\sum_{j=1}^t w_j^2 = (\sum_{j=1}^t w_j)^2$. □

Let $x = (x_1, \ldots, x_n)$ be a finite set of indeterminates. Let $f \in k[[x]]$ be a power series. Suppose that $f \in \langle x \rangle^2$ and that the Hessian matrix of $f$ has rank $r$ at the origin. Then $r$ is even if $p = 2$. Let

$$q = \begin{cases} x_1^2 + \cdots + x_r^2 & \text{if } p \neq 2 \\ x_1x_2 + \cdots + x_{r-1}x_r & \text{if } p = 2. \end{cases}$$

**Lemma 7.2.** If $p \neq 2$, then there exists a local $k$-algebra automorphism $\varphi : k[[x]] \to k[[x]]$ such that $\varphi(f) \equiv q$ modulo $\langle x \rangle^3$. If $p = 2$, then there exists a local $k$-algebra automorphism $\varphi : k[[x]] \to k[[x]]$ such that either $\varphi(f) \equiv q$ or $\varphi(f) \equiv q + x_{r+1}^2$ modulo $\langle x \rangle^3$. 

Proof. Let \( n \) denote the maximal ideal \( \langle x \rangle \subset k[[x]] \). Let \( q(f) \) denote the image of \( f \in n^2 \) inside \( n^2/n^3 = \text{Sym}^2(n/n^2) \). Then \( q(f) \) is a quadratic form whose associated bilinear form is represented by the Hessian matrix of \( f \) at the origin. Therefore, by Lemma 7.1, there exists a \( k \)-linear automorphism \( \varphi_1 \) of \( n/n^2 \) such that \( \text{Sym}^2(\varphi_1) \) sends \( q(f) \) to either \( q \) or \( q + x_{r+1}^2 \). We may take \( \varphi \) to be the linear automorphism of \( k[[x]] \) corresponding to \( \varphi_1 \), which characterized by the following property: for all \( i \), the self-map of \( n/n^{i+1} = \text{Sym}^i(n/n^2) \) induced by \( \varphi \) is equal to \( \text{Sym}^i(\varphi_1) \). \( \square \)

**Proposition 7.3** (Morse’s Lemma). If \( r = n \), then there exists an automorphism of \( k[[x]] \) as a local \( k \)-algebra that maps \( f \) to \( q \).

Proof. Because \( r = n \), we have \( \langle x \rangle \subseteq \text{jac}(q) \). It follows from Proposition 6.3 that \( q \) is 2-determined. Hence it suffices to show that there exists an automorphism of \( k[[x]] \) as local \( k \)-algebra that sends \( f \) to \( q \) modulo \( \langle x \rangle^3 \). This follows from Lemma 7.2 above. \( \square \)

**Proposition 7.4.** Let \( R \) be a complete local \( k \)-algebra with residue field \( k \). Let \( F \in R[[x]] \) be a power series with residue \( f \) in \( k[[x]] \). Then there exist a power series \( h \in R[[x_{r+1}, \ldots, x_n]] \) and an automorphism of \( R[[x]] \) as a local \( R \)-algebra that sends \( F \) to \( q + h \).

Proposition 7.4 belongs to a class of results that are usually referred to as “Morse’s Lemma with Parameters” or “the Splitting Lemma” in the literature. Note that the power series \( h \) does not involve the variables occurring in \( q \).

Proof. By Lemma 7.2, there exists a local \( k \)-algebra automorphism of \( k[[x]] \) that maps \( f \) to either \( q \) or \( q + x_{r+1}^2 \) modulo \( \langle x \rangle^2 \). After lifting such an automorphism to a local \( R \)-algebra automorphism of \( R[[x]] \), we may assume that \( f \) is congruent to either \( q \) or \( q + x_{r+1}^2 \) modulo \( \langle x \rangle^2 \).

Let \( R' \) denote the complete local \( k \)-algebra \( R[[x_{r+1}, \ldots, x_n]] \). Let \( \tilde{f} \) denote the image of \( f \) under the map \( k[[x_1, \ldots, x_n]] \to k[[x_1, \ldots, x_r]] \) that sends \( x_i \mapsto x_i \) for \( i \leq r \) and \( x_i \mapsto 0 \) for \( i > r \). After replacing \( R \) by \( R' \) and \( f \) by \( \tilde{f} \), we may assume that \( r = n \) and \( f \equiv q \) modulo \( \langle x \rangle^3 \).

By Morse’s Lemma (Proposition 7.3), there exists a local \( k \)-algebra automorphism of \( k[[x_1, \ldots, x_r]] \) that sends \( f \) to \( q \). After lifting such an automorphism to a local \( R \)-algebra automorphism of \( R[[x]] \), we may assume that \( f = q \). Put differently, we may assume that \( F \) is a unfolding of \( q \) over \( R \).

By Proposition 6.7 and the assumption that \( r = n \), the power series

\[ q + t \in k[[t, x]], \]

is a versal unfolding of \( q \) over \( k[[t]] \). We may therefore find a map of local \( k \)-algebras \( a : k[[t]] \to R \) and an isomorphism of unfoldings \( \varphi : q + a(t) \to F \) of \( q \) over \( R \). The element \( h := a(t) \in R \) and the automorphism of \( R[[x]] \) underlying \( \varphi \) satisfy the conclusions of the proposition. \( \square \)
8. SHEAVES OF PRINCIPAL PARTS

The theory of Grothendieck’s sheaves of principal parts is developed in [EGA, Chapter IV]. In this section we collect basic facts about these sheaves.

Let $X$ be a scheme over a field $k$. Let $E$ be an $\mathcal{O}_X$-module. Let $m$ be a nonnegative integer. The $m$th sheaf of principal parts associated to $E$ is a sheaf of $\mathcal{O}_X$-modules that we denote by $\mathcal{P}_X^m E$ (or by $\mathcal{P}_{X/k}^m E$ when we wish to emphasize the base field). The fiber of $\mathcal{P}_X^m E$ over a $k$-rational point $x \in X$ is canonically identified with the $k$-vector space $E/m_x^{m+1}E$. More precisely, the sheaf $\mathcal{P}_X^m E$ is equipped with a $k$-linear sheaf morphism $d^m_E : E \to \mathcal{P}_X^m E$ that induces a $k$-linear isomorphism

$$E/m_x^{m+1}E \sim \mathcal{P}_X^m E \otimes k(x)$$

for each $k$-rational point $x \in X$. The pair $(\mathcal{P}_X^m E, d^m_E)$ satisfies a universal property: $d^m_E : E \to \mathcal{P}_X^m E$ is the universal $k$-linear differential operator of order $m$ from $E$ to another $\mathcal{O}_X$-module, see [Stacks, Tags 09CK and 09CT].

Using this universal property, it is not hard to see that:

1. The sheaf $\mathcal{P}_X^m E$ is generated as an $\mathcal{O}_X$-module by the image of the universal differential operator $d^m_E : E \to \mathcal{P}_X^m E$.

We write $\mathcal{P}_X^m$ and $d^m_X$ instead of $\mathcal{P}_X^m(\mathcal{O}_X)$ and $d^m_{\mathcal{O}_X}$.

If $x \in X$ is a $k$-rational point, then the quotient $\mathcal{O}_X/m_x^{m+1}$ is not merely a $k$-vector space, but a $k$-algebra. Similarly, the quotient $E/m_x^{m+1}E$ is not merely a $k$-vector space, but a module over $\mathcal{O}_X/m_x^{m+1}$. These facts have counterparts for sheaves of principal parts:

2. The $\mathcal{O}_X$-module $\mathcal{P}_X^m$ has a natural structure of $\mathcal{O}_X$-algebra. The multiplication map $\mathcal{P}_X^m \times \mathcal{P}_X^m \to \mathcal{P}_X^m$ is the unique $\mathcal{O}_X$-bilinear map such that

$$d^m_X u \cdot d^m_X v = d^m_X (uv)$$

for all sections $u, v \in \mathcal{O}_X$ defined over a common open subset of $X$.

3. The $\mathcal{O}_X$-module $\mathcal{P}_X^m E$ has a natural structure of $\mathcal{P}_X^m$-module. The multiplication map $\mathcal{P}_X^m \times \mathcal{P}_X^m E \to \mathcal{P}_X^m E$ is the unique $\mathcal{O}_X$-bilinear map such that

$$d^m_X u \cdot d^m_X s = d^m_X (us)$$

for all sections $u \in \mathcal{O}_X$ and $s \in E$ defined over a common open subset of $X$.

We note that the universal derivation $d^m_X : \mathcal{O}_X \to \mathcal{P}_X^m$ is a map of $k$-algebras, but not of $\mathcal{O}_X$-algebras.

Sheaves of principal parts have many other properties that reflect familiar facts about their fibers. Below we’ll need the following ones:

4. There exists a unique isomorphism of $\mathcal{P}_X^m$-modules

$$\mathcal{P}_X^m \otimes_{\mathcal{P}_X^m, \mathcal{O}_X} E \sim \mathcal{P}_X^m E$$

that sends $\alpha \otimes s \mapsto \alpha \cdot d^m_E s$ for all local sections $\alpha \in \mathcal{P}_X^m$ and $s \in E$ defined over a common open subset of $X$. 

(5) Let \( q \) be an integer such that \( 0 \leq q \leq m \). There exists a unique \( \mathcal{O}_X \)-linear truncation map
\[
\varepsilon_{m,q} : \mathcal{P}_X^m E \to \mathcal{P}_X^q E
\]
such that \( \varepsilon_{m,q} \circ d_X^q = d_X^q \). The map \( d_E^0 : E \to \mathcal{P}_X^0 E \) is an \( \mathcal{O}_X \)-linear isomorphism, which we use to identify \( \varepsilon_{m,0} \) with an \( \mathcal{O}_X \)-linear map \( \mathcal{P}_X^m E \to E \).

(6) Suppose that \( m \geq 1 \). There exists a unique \( \mathcal{O}_X \)-linear map \( \iota_m : \text{Sym}^m \Omega_X \otimes E \to \mathcal{P}_X^m E \) such that
\[
\iota_m (du_1 \cdots du_m \otimes s) = (d_X^m u_1 - u_1) \cdots (d_X^m u_m - u_m) \cdot d_E^m s
\]
for all local sections \( u_1, \ldots, u_m \in \mathcal{O}_X \) and \( s \in E \) defined over a common open subset of \( X \). The sequence of \( \mathcal{O}_X \)-modules
\[
0 \longrightarrow \text{Sym}^m \Omega_X \otimes \mathcal{O}_X E \xrightarrow{\iota_m} \mathcal{P}_X^m E \xrightarrow{\varepsilon_{m,m-1}} \mathcal{P}_X^{m-1} E \longrightarrow 0
\]
is right-exact in general, and is exact if either \( m = 1 \) or \( X \) is smooth over \( k \) and \( E \) is locally free.

9. Separation of jets

Let \( k \) be a field. Let \( X \) be a scheme over \( k \). Let \( E \) be an \( \mathcal{O}_X \)-module. Let \( W \subseteq \Gamma(X,E) \) be a linear subspace. Let \( m \) be a nonnegative integer. Let \( d_E^m : E \to \mathcal{P}_X^m E \) be the universal \( k \)-linear differential operator of order \( m \).

Definition 9.1. We say that \( W \) separates jets of order \( m \) if the natural \( \mathcal{O}_X \)-linear map \( \iota_m : \text{Sym}^m \Omega_X \otimes \mathcal{O}_X E \to \mathcal{P}_X^m E \) that sends \( s \otimes f \mapsto fs \) is surjective.

Proposition 9.2. If \( k \) is algebraically closed, then \( W \subseteq \Gamma(X,E) \) separates jets of order \( m \) if, and only if, the natural \( k \)-linear map \( W \to E/m_x^{m+1}E \) is surjective for all closed points \( x \in X \).

Proof. Follows immediately from the fact that the universal differential operator \( d_E^m : E \to \mathcal{P}_X^m E \) induces a \( k \)-linear isomorphism \( E/m_x^{m+1}E \to \mathcal{P}_X^m E \otimes k(x) \) for each \( k \)-rational point \( x \in X \). \( \square \)

Proposition 9.3. Let \( k \subseteq K \) be a field extension. Then \( W \subseteq \Gamma(X,E) \) separates jets of order \( m \) if, and only if, \( W \otimes_k K \subseteq \Gamma(X \times_k K, E_X \times_k K) \) separates jets of order \( m \).

Proof. Let \( \alpha : d_E^m(W) \otimes_k \mathcal{O}_X \to \mathcal{P}_X^m E \) be the natural map. By definition, \( W \) separates jets of order \( m \) if, and only if, \( \alpha \) is surjective. Let \( X_K := X \times_k K \), let \( u : X_K \to X \) be the first projection, and let \( E_K := u^*E \). Then \( u \) is faithfully flat, so \( \alpha \) is surjective if, and only if, its pullback
\[
d_E^m(W) \otimes_k \mathcal{O}_{X_K} \xrightarrow{u^*} u^*\mathcal{P}_X^m E
\]
is surjective.
The composition

\[ E \longrightarrow u_* E_K \xrightarrow{u_*(d_{E_K}^i)} u_* \mathcal{P}^i_{X/K}(E_K), \]

is a \( k \)-linear differential operator of order \( m \), hence factors through an \( \mathcal{O}_X \)-linear map

\[ \mathcal{P}^i_{X/k} E \longrightarrow u_* \mathcal{P}^i_{X/K}(E_K), \]

which by adjunction induces an \( \mathcal{O}_X \)-linear map

\[ u^* \mathcal{P}^i_{X/k} E \xrightarrow{\beta} \mathcal{P}^i_{X/K}(E_K). \]

It is a standard fact that \( \beta \) is an isomorphism. Thus \( u^* \alpha \) is surjective if, and only if, the composition \( \beta \circ u^* \alpha \) is surjective.

Unwinding definitions, we find that \( \beta \circ u^* \alpha \) is surjective if, and only if, \( W \otimes_k K \) separates jets of order \( m \). The result follows. \( \square \)

10. Jets and the intrinsic differential

Let \( k \) be a field. Let \( X \) be a scheme over \( k \). Let \( E \) and \( F \) be locally free \( \mathcal{O}_X \)-modules of ranks \( e \) and \( f \), respectively. Let \( x : T \to X \) be a morphism of schemes. Let

\[ \alpha \in \Gamma(T, x^* \mathcal{P}^1_X \mathcal{H}om_X(E,F)) \]

be a section. Let \( \bar{\alpha} : x^* E \to x^* F \) be the image of \( \alpha \) under the \( \mathcal{O}_X \)-linear truncation map \( \mathcal{P}^1_X \mathcal{H}om_X(E,F) \to \mathcal{H}om_X(E,F) \).

**Proposition 10.1.** Suppose the \( \mathcal{O}_X \)-modules \( E \) and \( F \) are free. Choose bases for \( E \) and \( F \). Let

\[ \nabla : \mathcal{H}om_X(E,F) \to \Omega_X \otimes \mathcal{H}om_X(E,F) \]

be the \( k \)-linear map given by differentiation of matrix entries with respect to these bases. Let

\[ \nabla : \mathcal{P}^1_X \mathcal{H}om_X(E,F) \to \Omega_X \otimes \mathcal{H}om_X(E,F) \]

be the unique \( \mathcal{O}_X \)-linear map such that \( \nabla \circ d_{\mathcal{H}om}(E,F) = \nabla \). Let

\[ d\alpha \in \Gamma(T, x^* \Omega_X \otimes \mathcal{H}om_T(ker \bar{\alpha}, coker \bar{\alpha})) \]

be the image of \( \nabla \alpha \) under the \( \mathcal{O}_T \)-linear map

\[ x^* \Omega_X \otimes \mathcal{H}om_T(x^* F, x^* F) \xrightarrow{(i,q)} x^* \Omega_X \otimes \mathcal{H}om_T(ker \bar{\alpha}, coker \bar{\alpha}) \]

induced by the inclusion \( i : ker \bar{\alpha} \hookrightarrow x^* E \) and the projection \( q : x^* F \to coker \bar{\alpha} \). Then \( d\alpha \) is independent of the bases used to define it.
Proof. Let \( \{v_1, \ldots, v_r\} \subseteq \Gamma(X, E) \) and \( \{w_1, \ldots, w_f\} \subseteq \Gamma(X, F) \) be the chosen \( \mathcal{O}_X \)-linear bases for \( E \) and \( F \). Write \( H := \mathcal{H}om_X(E, F) \). Let \( \theta_1 : \mathcal{O}_X^{\oplus f \times e} \xrightarrow{\sim} H \) be the \( \mathcal{O}_X \)-linear isomorphism defined by

\[
\theta_1(b) = \sum_{i,j} b_{ij}v_j^\top \otimes w_i.
\]

Let \( \tilde{\theta}_1 : (\mathcal{P}_X^{1})^{\oplus f \times e} \xrightarrow{\sim} \mathcal{P}_X^{1}H \) be the \( \mathcal{P}_X^{1} \)-linear isomorphism defined by

\[
\tilde{\theta}_1(\beta) = \sum_{i,j} \beta_{ij}d_H^1(v_j^\top \otimes w_i).
\]

Let \( d : \mathcal{P}_X^{1} \to \Omega_X \) be the unique \( \mathcal{O}_X \)-linear map such that \( d \circ d_X^1 = d \). We claim that the following diagram commutes:

\[
\begin{array}{ccc}
(\mathcal{P}_X^{1})^{\oplus f \times e} & \xrightarrow{d} & \Omega_X \otimes \mathcal{O}_X^{\oplus f \times e} \\
\downarrow \tilde{\theta}_1 & & \downarrow \text{id} \otimes \theta_1 \\
\mathcal{P}_X^{1} \mathcal{H}om_X(E, F) & \xrightarrow{\nabla} & \Omega_X \otimes \mathcal{H}om_X(E, F)
\end{array}
\]

Indeed, \( \mathcal{P}_X^{1} \) is generated as an \( \mathcal{O}_X \)-module by the image of \( d_X^1 : \mathcal{O}_X \to \mathcal{P}_X^{1} \), and for each \( b \in \mathcal{O}_X^{\oplus f \times e} \) we have

\[
\nabla(\tilde{\theta}_1(d_X^1(b))) = \nabla(d_H^1(\theta_1(b))) = \nabla(\theta_1(b)) = (\text{id} \otimes \theta_1)(db) = (\text{id} \otimes \theta_1)(\tilde{d}(d_X^1(b))).
\]

Let \( \alpha_1 \in \Gamma(T, x^*(\mathcal{P}_X^{1})^{\oplus f \times e}) \) be the matrix of \( \alpha \in \Gamma(T, x^*\mathcal{P}_X^{1} \mathcal{H}om_X(E, F)) \) with respect to the chosen bases; in other words, let \( \alpha_1 := \tilde{\theta}_1^{-1}\alpha \). Let \( \bar{\alpha}_1 \in \Gamma(T, \mathcal{O}_T^{\oplus f \times e}) \) be the image of \( \alpha_1 \) under the truncation map \( \mathcal{P}_X^{1} \to \mathcal{O}_X \).

Then \( \bar{\alpha}_1 = \theta_1^{-1}\bar{\alpha} \). We identify \( \bar{\alpha}_1 \) with an \( \mathcal{T} \)-linear map \( \bar{\alpha}_1 : \mathcal{O}_T^{\oplus e} \to \mathcal{O}_T^{\oplus f} \). Let \( \iota_1 : \ker \bar{\alpha}_1 \hookrightarrow \mathcal{O}_T^{\oplus e} \) be the natural inclusion and let \( q_1 : \mathcal{O}_T^{\oplus f} \twoheadrightarrow \coker \bar{\alpha}_1 \) be the natural projection. Let \( \tilde{\theta}_1 \) be the unique \( \mathcal{T} \)-linear isomorphism such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{O}_T^{\oplus f \times e} & \xrightarrow{\hom} & \hom_T(\mathcal{O}_T^{\oplus e}, \mathcal{O}_T^{\oplus f}) \xrightarrow{(\iota_1, q_1)} \hom_T(\ker \bar{\alpha}_1, \coker \bar{\alpha}_1) \\
\sim & \downarrow \tilde{\theta}_1 & \sim \downarrow \theta_1 \\
x^*\hom_X(E, F) & \xrightarrow{(\iota, q)} & \hom_T(\ker \bar{\alpha}, \coker \bar{\alpha})
\end{array}
\]

Then

\[
(\iota, q)(\nabla \alpha) = \tilde{\theta}_1((\iota_1, q_1)(\tilde{d}\alpha_1))
\]

as elements of \( \Gamma(T, x^*\Omega_X \otimes \hom_T(\ker \bar{\alpha}, \coker \bar{\alpha})) \).

Now let \( \{v'_1, \ldots, v'_r\} \subseteq \Gamma(X, E) \) and \( \{w'_1, \ldots, w'_f\} \subseteq \Gamma(X, F) \) be a second pair of \( \mathcal{O}_X \)-linear bases for \( E \) and \( F \). Let \( \theta_2 : \mathcal{O}_X^{\oplus f \times e} \xrightarrow{\sim} H \) and \( \tilde{\theta}_2 :
(P^1_X)^{\otimes f \times e} \to P^1_X H$ be the isomorphisms corresponding to these bases. Thus \( \theta_2 \) is \( O_X \)-linear, while \( \bar{\theta}_2 \) is \( P^1_X \)-linear. Let \( \alpha_2 := \bar{\theta}_2 \cdot \alpha \in \Gamma(T, x^*(P^1_X)^{\otimes f \times e}) \) be the matrix of \( \alpha \) with respect to this second pair of bases. Let \( \bar{\alpha}_2 \in \Gamma(T, O_{P^1_X}^{\otimes f \times e}) \) be the image of \( \alpha_2 \) under the truncation map \( P^1_X \to O_X \). Let

\[
\begin{align*}
\nu_2 : \ker \bar{\alpha}_2 & \hookrightarrow O_{P^1_X}^{\otimes e} \\
q_2 : O_{P^1_X}^{\otimes e} & \to \ker \bar{\alpha}_2 \\
\bar{\theta}_2 : \mathcal{H}om_T(\ker \bar{\alpha}_2, \ker \bar{\alpha}_1) & \to \mathcal{H}om_T(\ker \bar{\alpha}_1, \ker \bar{\alpha}_1)
\end{align*}
\]

be the natural maps, defined as in the preceding paragraph. Then the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{H}om_T(O_{P^1_X}^{\otimes e}, O_{P^1_X}^{\otimes f}) & \xrightarrow{q_1} & \mathcal{H}om_T(\ker \bar{\alpha}_1, \ker \bar{\alpha}_1) \\
\downarrow \theta_2^{-1} \theta_1 & & \downarrow \bar{\theta}_2 \cdot \theta_1 \\
\mathcal{H}om_T(O_{P^1_X}^{\otimes e}, O_{P^1_X}^{\otimes f}) & \xrightarrow{q_2} & \mathcal{H}om_T(\ker \bar{\alpha}_2, \ker \bar{\alpha}_2)
\end{array}
\]

To prove the result, let us show that

\[
\bar{\theta}_2(\nu_2, q_2)(d\alpha_2) = \bar{\theta}_1((\nu_1, q_1)(d\alpha_1)),
\]

where the equality is between elements of \( \Gamma(T, x^*\Omega_X \otimes \mathcal{H}om_T(\ker \bar{\alpha}_1, \ker \bar{\alpha}_1)) \).

Let \( \varphi \in \Gamma(X, O_X)^{\otimes e \times e} \) and \( \psi \in \Gamma(X, O_X)^{\otimes f \times f} \) be the change-of-basis matrices defined by

\[
v_j = \sum_{i=1}^e \varphi_{ij} v'_i \quad \text{and} \quad \psi_i = \sum_{j=1}^f \varphi_{ij} w'_j
\]

for all \( j = 1, \ldots, e \) and \( i = 1, \ldots, f \). Then

\[
\bar{\alpha}_2 = \theta_2^{-1} \theta_1 \alpha_1 = \psi \cdot \bar{\alpha}_1 \cdot \varphi^{-1}
\]

as elements of \( \Gamma(T, O_{P^1_X}^{\otimes f \times e}) \). Furthermore,

\[
\alpha_2 = \bar{\theta}_2^{-1} \theta_1 \alpha_1 = d^1_X \psi \cdot \alpha_1 \cdot (d^1_X \varphi)^{-1}
\]

as elements of \( \Gamma(T, x^*(P^1_X)^{\otimes f \times e}) \). Here we use that the universal differential operator \( d^1_X: O_X \to P^1_X \) is a map of \( k \)-algebras, so that \( d^1_X \varphi \in \Gamma(T, x^*(P^1_X)^{\otimes e \times e}) \) is invertible with inverse \( d^1_X (\varphi^{-1}) \).

By the Leibniz rule (Proposition 10.3 below), we have

\[
\bar{\alpha}_2 = \bar{\alpha}_2(\psi \cdot \bar{\alpha}_1 \cdot \varphi^{-1} - \psi \cdot \bar{\alpha}_1 \cdot \varphi^{-1} + \psi \cdot \bar{\alpha}_1 \cdot \varphi^{-1} + \psi \cdot \bar{\alpha}_1 \cdot \varphi^{-1})
\]

where the equalities are between elements of \( \Gamma(T, x^*\Omega_X \otimes \mathcal{H}om_T(O_{P^1_X}^{\otimes e}, O_{P^1_X}^{\otimes f})) \).

Now \( \bar{\alpha}_1 \cdot \varphi^{-1} \cdot \nu_2 = \psi^{-1} \cdot \bar{\alpha}_2 \cdot \nu_2 = 0 \) as maps \( \ker \bar{\alpha}_2 \to O_{P^1_X}^{\otimes e} \), and \( q_2 \cdot \psi \cdot \bar{\alpha}_1 =
$q_2 \cdot \bar{\alpha}_2 \cdot \varphi = 0$ as maps $\mathcal{O}_T^{\operatorname{gr}} \to \operatorname{coker} \bar{\alpha}_2$. Thus

$$(\iota_2, q_2)(\bar{d}\alpha_2) = q_2 \cdot \bar{d}\alpha_2 \cdot \iota_2$$

$$= q_2 \cdot \psi \cdot \bar{d}\alpha_1 \cdot \varphi^{-1} \cdot \iota_2$$

$$= (q_2, \iota_2)(\theta_2^{-1}\theta_1(\bar{d}\alpha_1))$$

$$= \bar{\theta}_2^{-1}\theta_1((q_1, \iota_1)(\bar{d}\alpha_1))$$

as elements of $\Gamma(T, x^*\Omega_X \otimes \mathcal{H}om_T(\ker \bar{\alpha}_2, \operatorname{coker} \bar{\alpha}_2))$. \hfill \Box

**Definition 10.2.** The *intrinsic differential* of $\alpha \in \Gamma(T, x^*\mathcal{P}_X^1 \mathcal{H}om_X(E, F))$ is the unique section

$$d\alpha \in \Gamma(T, x^*\Omega_X \otimes \mathcal{H}om_T(\ker \bar{\alpha}, \operatorname{coker} \bar{\alpha}))$$

such that, for each open subset $U \subseteq X$ over which $E$ and $F$ are free, the restriction of $d\alpha$ to $x^{-1}U \subseteq T$ coincides with the section of Proposition 10.1 applied to $x|_{x^{-1}U} : x^{-1}U \to U$ and

$$\alpha|_{x^{-1}U} \in \Gamma(x^{-1}U, x^*\mathcal{P}_X^1 \mathcal{H}om_X(E, F)) =$$

$$= \Gamma(x^{-1}U, (x|_{x^{-1}U})^*\mathcal{P}_U^1 \mathcal{H}om_U(E_U, F_U)).$$

When $X$ is smooth over $k$, we will regard the intrinsic differential as an $\mathcal{O}_T$-linear map

$$d\alpha : x^*T_X \to \mathcal{H}om_T(\ker \bar{\alpha}, \operatorname{coker} \bar{\alpha}).$$

The following result was used in the proof Proposition 10.1.

**Proposition 10.3** (Leibniz rule). Let $\bar{d} : \mathcal{P}_X^1 \to \Omega_X$ be the $\mathcal{O}_X$-linear map induced by the universal derivation $d : \mathcal{O}_X \to \Omega_X$. Let $f, g \in \mathcal{P}_X^1$ be sections defined over a common open subset of $X$, and let $\bar{f}, \bar{g} \in \mathcal{O}_X$ be their respective images under the truncation map $\mathcal{P}_X^1 \to \mathcal{O}_X$. Then

$$(4) \quad \bar{d}(fg) = \bar{d}f \cdot \bar{g} + \bar{f} \cdot \bar{d}g$$

as sections of $\Omega_X$.

**Proof.** Both sides of (4) are $\mathcal{O}_X$-bilinear functions of $f$ and $g$. As $\mathcal{P}_X^1$ is generated as an $\mathcal{O}_X$-module by the image of the universal differential operator $d_X^1 : \mathcal{O}_X \to \mathcal{P}_X^1$, we may assume that $f = d_X^1 u$ and $g = d_X^1 v$ for some $u, v \in \mathcal{O}_X$. In this case, (4) reduces to the usual Leibniz rule satisfied by the universal derivation $d : \mathcal{O}_X \to \Omega_X$. \hfill \Box

**Remark 10.4.**

1. Let $\bar{\alpha} : E \to F$ be an $\mathcal{O}_X$-linear map. Then

$$d\bar{\alpha} = d(x^*d_X^1\bar{\alpha}).$$

2. Let $T'$ be a scheme and let $t : T' \to T$ be a morphism. Then $d(t^*\alpha)$ is equal to the image of $t^*d\alpha$ under the $\mathcal{O}_{T'}$-linear map

$$t^*x^*\Omega_X \otimes t^*\mathcal{H}om_T(\ker \bar{\alpha}, \operatorname{coker} \bar{\alpha}) \xrightarrow{\mu} t^*x^*\Omega_X \otimes \mathcal{H}om_{T'}(\ker(t^*\bar{\alpha}), \operatorname{coker}(t^*\bar{\alpha}))$$
induced by the natural $\mathcal{O}_T$-linear map $t^* \ker \alpha \to \ker(t^*\alpha)$ and isomorphism $\operatorname{coker}(t^*\alpha) = t^* \operatorname{coker}(\alpha)$. In cases where $\mu$ is an isomorphism, which by Corollary 2.7 happens for example when $\operatorname{coker}(\alpha)$ is a locally free $\mathcal{O}_T$-module, we will abuse notation and write

$$t^* d\alpha = d(t^* \alpha).$$

11. Jets and second-order differentials

Let $k$ be a field. Let $X$ be a smooth scheme over $k$. Let $r$ be a positive integer. Let $x : T \to X$ be a morphism of schemes. Let $f \in \Gamma(T, x^*(\mathcal{P}^2_X)^\oplus)$ be a section. Let $f_1 \in \Gamma(T, x^*(\mathcal{P}^1_X)^\oplus)$ be the truncation of $f$ to first order.

Let $\tilde{d} : \mathcal{P}^2_X \to \Omega_X$ be the unique $\mathcal{O}_X$-linear map such that $\tilde{d} d x = d$. Let $\tilde{d} : \mathcal{P}^2_X \to \mathcal{P}^1_X \Omega_X$ be the unique $\mathcal{O}_X$-linear map such that $\tilde{d} \circ d^2_x = d^1_\Omega \circ d$.

The following diagram, where the vertical arrows are the natural truncation maps, commutes:

$$\begin{array}{ccc}
\mathcal{P}^2_X & \xrightarrow{\tilde{d}} & \mathcal{P}^1_X \Omega_X \\
\downarrow & & \downarrow \\
\mathcal{P}^1_X & \xrightarrow{\tilde{d}} & \Omega_X
\end{array}$$

Let $K$ and $C$ respectively denote the kernel and cokernel of the differential $\tilde{d} f_1 \in \Gamma(T, x^*(\mathcal{P}^1_X \Omega_X)^\oplus)$ regarded as an $\mathcal{O}_T$-linear map $\tilde{d} f_1 : x^* T_X \to \mathcal{O}_T^\oplus$.

**Definition 11.1.** The intrinsic differential of $\tilde{d} f \in \Gamma(T, x^*(\mathcal{P}^1_X \Omega_X)^\oplus)$, regarded as element of $\Gamma(T, x^* \mathcal{P}^1_X \mathcal{Hom}_X(T_X, \mathcal{O}_X^\oplus))$, is an $\mathcal{O}_T$-linear map

$$d(\tilde{d} f) : x^* T_X \to \mathcal{Hom}_T(K, C).$$

The restriction

$$d^2 f : K \to \mathcal{Hom}_T(K, C)$$

of this $\mathcal{O}_T$-linear map to $K \subseteq x^* T_X$ is called the *second intrinsic differential* of $f$.

**Remark 11.2.**

1. Let $\mathbb{A}^r$ be the $r$-dimensional affine space over $k$. Let $\tilde{f} : X \to \mathbb{A}^r$ be a morphism over $k$, which we identify with an element of $\Gamma(X, \mathcal{O}_X^\oplus)$. Then

$$d(\tilde{d}(x^* d^2_x \tilde{f})) = d(x^* \tilde{d} d^2_x \tilde{f}) = d(x^* d^1_\Omega d \tilde{f}) = d_x (d \tilde{f}),$$

where the last equality follows from Remark 10.4 (1).

2. Let $T'$ be a scheme and let $t : T' \to T$ be a morphism. If the cokernel of $\tilde{d} f_1 : x^* T_X \to \mathcal{O}_T^\oplus$ is a locally free $\mathcal{O}_T$-module, then

$$t^* d(\tilde{d} f) = d(t^* \alpha)$$

as elements of

$$\mathcal{Hom}_{T'}(t^* x^* T_X, \mathcal{Hom}_{T'}(\operatorname{coker}(\tilde{d}(t^* f_1)), \operatorname{coker}(\tilde{d}(t^* f_1))))).$$

In general, $d(\tilde{d}(t^* f))$ equals the image of $t^* d(\tilde{d} f)$ under a natural map, see Remark 10.4 (2).
For the remainder of this section, given a $k$-scheme $S$ and an $\mathcal{O}_S$-module $M$, we will write
\[ \square^2 M = \begin{cases} \text{Sym}^2 M & \text{if char}(k) \neq 2 \\ \wedge^2 M & \text{if char}(k) = 2. \end{cases} \]

Furthermore, given a submodule $A \subseteq M$ we will denote by $A \square M$ the image of $A \otimes M$ under the quotient map $M \otimes M \to \square^2 M$.

**Remark 11.3.** Let $S$ be a $k$-scheme. Let $M$ be a locally free $\mathcal{O}_S$-module of finite rank. Let
\[ \beta : \text{Sym}^2 (M^\vee) \to M^\vee \otimes M^\vee = (M \otimes M)^\vee \]
be the $\mathcal{O}_S$-linear map that sends a quadratic form on $M$ to its associated bilinear form, so that $\beta(uv) = u \otimes v + v \otimes u$ for all $u, v \in E^\vee$. Then the image of $\beta$ is the submodule $(\square^2 M)^\vee \subseteq (M \otimes M)^\vee$.

We have a diagram of natural $\mathcal{O}_T$-linear maps:
\[
\begin{array}{ccc}
x^*(\text{Sym}^2 \Omega_X)^{\oplus r} & \xrightarrow{\beta} & x^*(\Omega_X \otimes \Omega_X)^{\oplus r} \\
\downarrow & & \downarrow \\
\text{Hom}_T(x^*(\square^2 T_X), \mathcal{O}_T^{\oplus r}) & \xrightarrow{} & \text{Hom}_T(x^*(T_X \otimes T_X), \mathcal{O}_T^{\oplus r}) \\
\downarrow & & \downarrow \\
\text{Hom}_T(x^*T_X \square K, C) & \xleftarrow{} & \text{Hom}_T(x^*T_X \otimes K, C)
\end{array}
\]

Let
\[
\theta : \text{Hom}_T(x^*T_X \otimes K, C) \xrightarrow{\sim} \text{Hom}_T(x^*T_X, \text{Hom}_T(K, C)) \\
\tilde{\theta} : \text{Hom}_T(K \otimes K, C) \xrightarrow{\sim} \text{Hom}_T(K, \text{Hom}_T(K, C))
\]
be the natural $\mathcal{O}_T$-linear isomorphisms, both of which are described by the rule $b \mapsto (v \mapsto b(v \otimes -))$.

**Proposition 11.4.** The second-order differentials $d(\tilde{d}f)$ and $d^2f$ are symmetric in characteristic different from 2 and in skew-symmetric in characteristic 2. More precisely:
\[
\theta^{-1}(d(df)) \in \text{Hom}_T(x^*T_X \square K, C) \\
\tilde{\theta}^{-1}(d^2f) \in \text{Hom}_T(\square^2 K, C)
\]

**Proof.** The question being Zariski-local on $T$, we may assume that there exist global sections $x_1, \ldots, x_n \in \Gamma(X, \mathcal{O}_X)$ whose differentials from a basis for $\Omega_X$ as an $\mathcal{O}_X$-module. Then the result follows from the combination of Lemma 11.7 and the first part of Lemma 11.8 below. \qed

Let
\[ \beta' : x^*(\text{Sym}^2 \Omega_X)^{\oplus r} \to \text{Hom}_T(x^*T_X, \text{Hom}_T(K, C)) \]
be the natural map. Let
\[ 0 \longrightarrow \text{Sym}^2 \Omega_X \overset{\iota}{\longrightarrow} P^2_X \longrightarrow P^1_X \longrightarrow 0 \]
be the canonical short exact sequence.

**Proposition 11.5.** Let \( \delta \in \Gamma(T, x^*(\text{Sym}^2 \Omega_X)^{\oplus r}) \) be a section. Then
\[ \text{d}\left(\tilde{d}(f + \iota \delta)\right) = \text{d}(\tilde{d}f) + \beta \delta \]
as elements of \( \text{Hom}_T(x^*T_X; \text{Hom}_T(K, C)) \), and
\[ \text{d}^2(f + \iota \delta) = \text{d}^2 f + (\beta \delta)|_K \]
as elements of \( \text{Hom}_T(K, \text{Hom}_T(K, C)) \).

**Proof.** The question being Zariski-local on \( T \), we may assume that there exist global sections \( x_1, \ldots, x_n \in \Gamma(X, \mathcal{O}_X) \) whose differentials from a basis for \( \Omega_X \) as an \( \mathcal{O}_X \)-module. Then the result follows from the combination of Lemma 11.7 and the second part of Lemma 11.8 below. \( \square \)

**Setup 11.6.** Suppose that there exist sections \( x_1, \ldots, x_n \in \Gamma(X, \mathcal{O}_X) \) be sections whose differentials form an \( \mathcal{O}_X \)-linear basis for \( \Omega_X \). Fix a choice of such sections. Let \( \{\partial_1, \ldots, \partial_n\} \) be the basis of \( T_X = \Omega_X^\vee \) that is dual to the basis \( \{dx_1, \ldots, dx_n\} \) of \( \Omega_X \). Let \( \text{Hess} : \mathcal{O}_X \rightarrow \Omega_X \otimes \Omega_X \) be the second-order \( k \)-linear differential operator differential operator that sends
\[ \varphi \mapsto \sum_{a,b=1}^n \partial_a \partial_b \varphi \cdot dx_a \otimes dx_b. \]

For each \( a = 1, \ldots, n \), we identify \( \partial_a \in \Gamma(X, T_X) \) with a derivation \( \partial_a : \mathcal{O}_X \rightarrow \mathcal{O}_X \), and let \( \tilde{\partial}_a : P^1_X \rightarrow \mathcal{O}_X \) and \( \tilde{\partial}_a : P^2_X \rightarrow P^1_X \) be the \( \mathcal{O}_X \)-linear maps characterized by the conditions that \( \tilde{\partial}_a \circ d_X^1 = \partial_a \) and \( \tilde{\partial}_a \circ d_X^2 = d_X^1 \circ \partial_a \). Let \( \text{Hess} : P^2_X \rightarrow \Omega_X \otimes \Omega_X \) be the unique \( \mathcal{O}_X \)-linear such that \( \text{Hess} \circ d_X^2 = \text{Hess} \).

**Lemma 11.7.** Assume Setup 11.6. Let \( \bar{\theta} : \Omega_X \otimes \Omega_X \rightarrow \text{Hom}_T(T_X, (T_X)^\vee) \) be the \( \mathcal{O}_X \)-linear map that sends \( b \mapsto (v \mapsto b(v \otimes -)) \). The intrinsic differential \( \text{d}(\tilde{d}f) \) is equal to the composition of the \( \mathcal{O}_T \)-linear maps
\[ (x^*T_X) \xrightarrow{\theta|_{\text{Hess}f}} \text{Hom}_T(x^*T_X, \mathcal{O}_X^{\oplus r}) \xrightarrow{\text{coker}(\ker(\bar{f}_1))} \text{Hom}_T(\ker(\bar{f}_1), \text{coker}(\tilde{d}f_1)), \]
where the second arrow is induced by the inclusion \( \ker(d\bar{f}_1) \hookrightarrow x^*T_X \) and the projection \( \mathcal{O}_X^{\oplus r} \rightarrow \text{coker}(d\bar{f}_1) \).

**Proof.** Write \( H := \text{Hom}_X(T_X, \mathcal{O}_X^{\oplus r}) \). Let \( \nabla : H \rightarrow \Omega_X \otimes H \) be the \( k \)-linear map given by differentiation of matrix elements with respect to the basis \( \{\partial_1, \ldots, \partial_n\} \subseteq \Gamma(X, T_X) \) of \( T_X \) and the standard basis of \( \mathcal{O}_X^{\oplus r} \). Let \( \nabla : \mathcal{O}_X \otimes H \rightarrow \Omega_X \otimes H \) be the unique \( \mathcal{O}_X \)-linear map such that \( \nabla \circ d_H^1 = \nabla \). Then \( \nabla \circ d = \text{Hess} \) as \( \mathcal{O}_X \)-linear maps \( (P^2_X)^{\oplus r} \rightarrow \Omega_X \otimes H = (\Omega_X \otimes \Omega_X)^{\oplus r} \). Indeed,
$P_X^2$ is generated as an $O_X$-module and, for each local section $\varphi \in O_X^{\otimes r}$, we have

$$\nabla \dd^2 \varphi = \nabla d_1^2 \varphi$$

$$= \nabla (d \varphi)$$

$$= \nabla (\sum_b \partial_b \varphi \cdot dx_b)$$

$$= \sum_b d (\partial_b \varphi) \otimes dx_b$$

$$= \sum_{a,b} \partial_a \partial_b \varphi \cdot dx_a \otimes dx_b$$

$$= \operatorname{Hess}(\varphi)$$

$$= \overline{\operatorname{Hess}}(d_X^2 \varphi).$$

In particular, $\nabla (\dd f) = \overline{\operatorname{Hess}}(f)$ as elements of $\Gamma(T, x^*(\Omega_X \otimes H))$. The result now follows from the definition of the intrinsic differential (Definition 10.2). $\square$

**Lemma 11.8.** Assume Setup 11.6. Let $\tilde{\beta} : \text{Sym}^2 \Omega_X \to \Omega_X \otimes \Omega_X$ be the $O_X$-linear map that sends $uv \mapsto u \otimes v + v \otimes u$. Then the $O_X$-linear map

$$\overline{\operatorname{Hess}} : P_X^2 \to \Omega_X \otimes \Omega_X$$

factors through $\text{im}(\tilde{\beta}) \subseteq \Omega_X \otimes \Omega_X$. Furthermore, the composition of $\overline{\operatorname{Hess}}$ with the canonical injection $\iota : \text{Sym}^2 \Omega_X \to P_X^2$ is equal to $\tilde{\beta}$. In symbols:

$$\overline{\operatorname{Hess}} \circ \iota = \tilde{\beta}$$

**Proof.** First, we claim that $P_X^2$ is freely generated as an $O_X$-module by the sections

$$d_X^2(x_i x_j), d_X^2 x_i, d_X^2 1 \in \Gamma(X, P_X^2),$$

where $i, j = 1, \ldots, n$. To see this, let $A^n_k$ denote the affine space over $k$ with coordinates $t_1, \ldots, t_n$, and let $\varphi : X \to A^n_k$ be the unique map of $k$-schemes such that $\varphi^# t_i = x_i$ for all $i = 1, \ldots, n$. Then $\varphi$ is étale, so induces an isomorphism of $O_X$-algebras $\varphi^* P_{A^n_k} \xrightarrow{\sim} P_X^2$. This isomorphism sends $d_{A^n_k}^2 t_i \mapsto d_X^2 x_i$ for all $i = 1, \ldots, n$. Now let $\varepsilon_1, \ldots, \varepsilon_n$ be indeterminates. It is a standard fact that the unique map of $O_{A^n}$-algebras

$$O_{A^n}[\varepsilon_1, \ldots, \varepsilon_n] \to P_{A^n}^2$$

that sends $\varepsilon_i \mapsto d_{A^n}^2 t_i - t_i$ for all $i = 1, \ldots, n$ is surjective with kernel $\langle \varepsilon_1, \ldots, \varepsilon_n \rangle^2$. The claim follows.
Next, we note that
\[ \text{Hess}(d^2_X(x_i x_j)) = \sum_{a,b} \partial_a \partial_b (x_i x_j) dx_a \otimes dx_b \]
\[ = \sum_{a,b} \partial_a (x_j \partial_b x_i + x_i \partial_b x_j) dx_a \otimes dx_b \]
\[ = \sum_a \partial_a x_j dx_a \otimes dx_i + \partial_a x_i dx_a \otimes dx_j \]
\[ = dx_j \otimes dx_i + dx_i \otimes dx_j. \]

Similarly, \( \text{Hess}(d^2_X x_i) = 0 \) for all \( i = 1, \ldots, n \), and \( \text{Hess}(d^2_X 1) = 0 \). Furthermore,
\[ \text{Hess}(\iota(dx_i \cdot dx_j)) = \text{Hess}((d^2_X x_i - x_i)(d^2_X x_j - x_j)) \]
\[ = \text{Hess}((d^2_X (x_i x_j) - x_i d^2_X x_j - x_j d^2_X x_i + x_i x_j)) \]
\[ = dx_j \otimes dx_i + dx_i \otimes dx_j. \]

The result follows. \( \square \)

12. Universal singularities

Let \( k \) be a field. Let \( X \) be a smooth scheme over \( k \). Suppose that \( X \) is of pure dimension \( n \). Let \( A^r \) be the affine space over \( k \) with coordinates \( t_1, \ldots, t_r \). For each nonnegative integer \( m \), let
\[ \pi_m : J^m(X, A^r) \to X \]
be the vector bundle corresponding to the locally free \( \mathcal{O}_X \)-module \( (P^m_X) \oplus r \). Thus the set of rational points of the fiber of \( \pi_m \) over a rational point \( x \in X(k) \) is in natural bijection with the \( k \)-vector space \( (\mathcal{O}_X/m_x^{m+1}) \oplus r \).

**Definition 12.1.** The scheme \( J^m(X, A^r) \) is called the *scheme of jets of order \( m \) with source in \( X \) and target in \( A^r \)*, or the *\( m \)th jet scheme* for short.

For each nonegative integer \( m \), let
\[ s_m \in \Gamma(J^m(X, A^r), (P^m_X)^\oplus r) \]
be the tautological section.

**Definition 12.2.** Let \( f : X \to A^r \) be a morphism of \( k \)-schemes. Let \( m \) be a nonnegative integer. The *\( m \)th jet of \( f \) at \( x \)* is the unique section \( j^m f : X \to J^m(X, A^r) \) of \( \pi_m \) such that
\[ (j^m f)^* s_m = d_X^m f \]
as elements of \( \Gamma(X, (P^m_X)^\oplus r) \), where on the right we regard \( f \) as an element of \( \Gamma(X, \mathcal{O}_X^{\oplus r}) \).

Let \( T \) be a scheme and let \( x : T \to X \) be a morphism. We call the composition \( j^m_X f := j^m f \circ x \) the *\( m \)th jet of \( f \) at \( x \)*.
We think of $x$ as a $T$-valued point of $X$ and of $j^m_x f$ as a $T$-valued point of the jet scheme $J^m(X, A^r)$.

Let $\bar{d} : \mathbb{P}^1_X \to \Omega^*_X$ be the unique $\mathcal{O}_X$-linear map such that $\bar{d} \circ d^1_X = d$.

**Definition 12.3.** Let $i$ be a nonnegative integer. The $i$th universal critical locus is the locally closed subscheme

$$\Sigma^i := \Sigma^i(\bar{d}s_1) \setminus \Sigma^{i+1}(\bar{d}s_1) \subseteq J^1(X, A^r),$$

where $\Sigma^j(\bar{d}s_1)$ denotes the $j$th degeneracy locus of the $\mathcal{O}_{J^1}$-linear map $\bar{d}s_1 : (T_X)_{j^1(X, A^r)} \to \mathcal{O}^{\oplus r}_{J^1(X, A^r)}$.

**Proposition 12.4.** If $f : X \to A^r$ is a morphism of $k$-schemes, then $(j^m f)^{-1} \Sigma^i = \Sigma^i(f)$ as subschemes of $X$.

**Proof.** We have

$$(j^1 f)^{-1} \Sigma^i = (j^1 f)^{-1}(\Sigma^i(\bar{d}s_1) \setminus \Sigma^{i+1}(\bar{d}s_1))$$

$$= \Sigma^i((j^1 f)^*\bar{d}s_1) \setminus \Sigma^{i+1}((j^1 f)^*\bar{d}s_1)$$

$$= \Sigma^i(df) \setminus \Sigma^{i+1}(df)$$

$$= \Sigma^i(f).$$

Indeed, the first equality holds by definition of $\Sigma^i$, the second by Remark 2.5 (3), the third because

$$(j^1 f)^*\bar{d}s_1 = \bar{d}((j^1 f)^*s_1) = \bar{d}d^1_X f = df,$$

and the fourth by definition of $\Sigma^i(f)$. \qed

**Proposition 12.5.** The universal critical locus $\Sigma^i \subseteq J^1(X, A^r)$ is nonempty if, and only if, $0 \leq i \leq \min(n, r)$. In this case, $\Sigma^i$ is smooth over $X$ and of pure relative codimension

$$i(|n - r| + i)$$

in $J^1(X, A^r)$ over $X$.

**Proof.** If $i > \min(n, r)$, then $\Sigma^i \subseteq X$ is empty by Remark 2.5 (2). Suppose that $i \leq \min(n, r)$.

Let $H$ be the vector bundle corresponding to the locally free $\mathcal{O}_X$-module $\mathcal{H}om_X(T_X, \mathcal{O}^{\oplus r}_X)$. In symbols:

$$H := \mathcal{V}(\mathcal{H}om_X(T_X, \mathcal{O}^{\oplus r}_X)).$$

Let $h : (T_X)_H \to \mathcal{O}^{\oplus r}_H$ be the tautological map. Then the degeneracy locus $\Sigma^i(h) \setminus \Sigma^{i+1}(h) \subseteq H$ is smooth over $X$ and of pure relative codimension $i(|n - e| + i)$ in $H$ over $X$, by either Proposition 2.9 or Proposition 14.8 below.

Let $J := J^1(X, A^r)$. Let $D : J \to H$ be the morphism of vector bundles over $X$ induced by the $\mathcal{O}_X$-linear map

$$\bar{d} : (\mathbb{P}^1_X)^{\oplus r} \to \Omega^{\oplus r}_X = \mathcal{H}om_X(T_X, \mathcal{O}^{\oplus r}_X).$$
Thus $D$ is the unique morphism of $X$-schemes such that $D^* h = \bar{d}s_1$, where $s_1 \in \Gamma(J^1, (P^1_X)_{J^1})$ is the tautological section. Moreover, $D$ is smooth and surjective, since $\bar{d} : P^1_X \to \Omega_X$ is surjective. The result follows, because for each nonnegative integer $i'$, we have

$$D^{-1} \Sigma^{i'}(h) = \Sigma^{i'}(D^* h) = \Sigma^{i'}(\bar{d}s_1) = \Sigma^{i'}.$$  

For $m = 1, 2$, let $J^m := J^m(X, A^r)$. Let $q : J^2 \to J^1$ be the morphism of vector bundles over $X$ induced by the $\mathcal{O}_X$-linear truncation map $P^2_X \to P^1_X$. Let $\tilde{d} : P^2_X \to P^1_X \Omega_X$ be the unique $\mathcal{O}_X$-linear map such that $\tilde{d} \circ d^2_X = d^1_\Omega \circ d$. Then the image of the differential

$$\bar{d}s_2 \in \Gamma(J^2, (P^1_X \Omega^r_X)_{J^2})$$

under the truncation map $P^1_X \Omega_X \to \Omega_X$ is equal to

$$q^* \bar{d}s_1 \in \Gamma(J^2, (\Omega^r_X)_{J^2}).$$

Let $K$ and $C$ respectively denote the kernel and cokernel of the restriction of $\bar{d}s_1 : (T_X)_{J^1} \to \Omega^r_{J^1}$ to $\Sigma^i$. Then $\mathcal{O}_{\Sigma^i}$-modules $K$ and $C$ are locally free and

$$q^* K = \ker(q^* \bar{d}s_1|_{q^{-1}\Sigma^i}) \quad \text{and} \quad q^* C = \text{coker}(q^* \bar{d}s_1|_{q^{-1}\Sigma^i})$$

by Corollary 2.7. It follows that the intrinsic differential $d(\bar{d}s_2|_{q^{-1}\Sigma^i})$ is an $\mathcal{O}_{q^{-1}\Sigma^i}$-linear map

$$d(\bar{d}s_2|_{q^{-1}\Sigma^i}) : (T_X)_{q^{-1}\Sigma^i} \to \mathcal{H}\text{om}_{q^{-1}\Sigma^i}(q^* K, q^* C)$$

and the second intrinsic differential $d^2(s_2|_{q^{-1}\Sigma^i})$ is an $\mathcal{O}_{q^{-1}\Sigma^i}$-linear map

$$d^2(s_2|_{q^{-1}\Sigma^i}) : q^* K \to \mathcal{H}\text{om}_{q^{-1}\Sigma^i}(q^* K, q^* C).$$

Let $j$ be a nonnegative integer.

**Definition 12.6.**

1. The universal bad locus $B^i$ is the closed subscheme of $q^{-1}\Sigma^i$ defined as follows. If $n \geq i(|n - r| + i)$, then $B^i(f)$ is the first degeneracy locus

$$B^i := \Sigma^1(d(\bar{d}s_2|_{q^{-1}\Sigma^i})) \subseteq q^{-1}\Sigma^i$$

of the intrinsic differential of $\bar{d}s_2|_{q^{-1}\Sigma^i}$. If $n < i(|n - r| + i)$, then $B^i = q^{-1}\Sigma^i$.

2. The locally closed subscheme

$$\Sigma^{i,j} := \Sigma^j(d^2(s_2|_{q^{-1}\Sigma^i})) \setminus \Sigma^{j+1}(d^2(s_2|_{q^{-1}\Sigma^i})) \subseteq q^{-1}\Sigma^i$$

is called the universal locus of second-order singularities with symbol $(i, j)$ in $J^2(X, A^r)$.

The next result relates Definitions 12.6 and 4.3.

**Proposition 12.7.** Let $f : X \to A^r$ be a morphism of $k$-schemes. Then

$$(j^2 f)^{-1} B^i = B^i(f) \quad \text{and} \quad (j^2 f)^{-1} \Sigma^{i,j} = \Sigma^{i,j}(f).$$

as subschemes of $X$. 
Proof. We have \((j^2 f)^{-1} q^{-1} \Sigma^i = \Sigma^i(f)\), because \(q \circ j^2 f = j^1 f\) and, as was noted above, \((j^1 f)^{-1} \Sigma^i = \Sigma^i(f)\).

By Remark 11.2 we have
\[
(j^2 f|_{\Sigma^i(f)})^* d(\tilde{d}s_2|_{q^{-1} \Sigma^i}) = d(d(j^2 f|_{\Sigma^i(f)})^* s_2)
\]
\[
= d(d(d_X f|_{\Sigma^i(f)}))
\]
\[
= d_{\Sigma^i(f)}(df).
\]

Thus, if \(n \geq i(|n-r|+i)\), then
\[
(j^2 f)^{-1} B^i = (j^2 f|_{\Sigma^i(f)})^{-1} \Sigma^i(d(\tilde{d}s_2|_{q^{-1} \Sigma^i}))
\]
\[
= \Sigma^i((j^2 f|_{\Sigma^i(f)})^* d(\tilde{d}s_2|_{q^{-1} \Sigma^i}))
\]
\[
= \Sigma^i(d_{\Sigma^i(f)}(df))
\]
\[
= B^i(f).
\]

If \(n < i(|n-r|+i)\), then by definition \(B^i = q^{-1} \Sigma^i\) and \(B^i(f) = \Sigma^i(f)\), so
\[
(j^2 f)^{-1} \Sigma^i = \Sigma^i(f)
\]
in this case also.

In general, we have
\[
(j^2 f|_{\Sigma^i(f)})^* d^2(s_2|_{q^{-1} \Sigma}) = d_{\Sigma^i(f)}^2 f
\]
and therefore
\[
(j^2 f)^{-1} \Sigma^i = \Sigma^i(f). \tag*{\square}
\]

Theorem 12.8. Let \(m := \min(n, r)\). The universal bad locus \(B^i \subseteq q^{-1} \Sigma^i\) is nonempty if, and only if, \(1 \leq i \leq m\).

If \(n < i(|n-r|+i)\), then by definition \(B^i = q^{-1} \Sigma^i\), and \(B^i\) is smooth over \(X\) of pure relative codimension \(i(|n-r|+i)\) in \(J^2(X, A')\).

Suppose that \(n \geq i(|n-r|+i)\). If \(\text{char}(k) \neq 2\), then \(B^i \subseteq q^{-1} \Sigma^i\) has relative codimension \(n+1\) in \(J^2(X, A')\) over \(X\). The same holds if \(\text{char}(k) = 2\), with the exception of two cases:

1. The case where \(i = 1\) and \(r \geq n\).
2. The case where \(i = 0\), \(r = 1\) and \(n\) is odd.

In cases (1) and (2), the universal bad locus \(B^i\) has relative codimension \(n\) in \(J^2(X, A')\) over \(X\).

Proof. By Proposition 12.5, the universal critical locus \(\Sigma^i \subseteq J^1\) is nonempty if, and only if, \(0 \leq i \leq m\), so we may assume these inequalities hold. Then \(\Sigma^i\) is smooth over \(X\) and of pure relative codimension \(i(|n-r|+i)\) in \(J^1\) over \(X\), again by Proposition 12.5.

Recall that \(J^m := \mathcal{V}((\mathcal{T}^m)_{\mathcal{X}})\) for \(m = 1, 2\), and that \(q: J^2 \to J^1\) the morphism of vector bundles over \(X\) induced by the truncation map. This truncation map is surjective, so \(q\) is smooth and surjective. Therefore \(q^{-1} \Sigma^i\) is smooth over \(X\), and of pure relative codimension \(i(|n-r|+i)\) in \(J^2\) over \(X\). In particular, the result holds if \(n < i(|n-r|+i)\).
Suppose that $n \geq i(|n - r| + i)$. Let

$$\Box^2 T_X := \begin{cases} \text{Sym}^2 T_X & \text{if char}(k) \neq 2 \\ \wedge^2 T_X & \text{if char}(k) = 2. \end{cases}$$

Let $(T_X)_{\Sigma^1} \Box K$ be the image of $(T_X)_{\Sigma^1} \otimes K$ under the quotient map

$$(T_X \otimes T_X)_{\Sigma^1} \rightarrow (\Box^2 T_X)_{\Sigma^1}.$$ 

By Lemma 15.1 below, $(T_X)_{\Sigma^1} \Box K$ is a locally free $O_{\Sigma^1}$-module.

Let

$$\theta : \mathcal{H}om_{\Sigma^i}((T_X)_{\Sigma^i} \otimes K, C) \cong \mathcal{H}om_{\Sigma^i}((T_X)_{\Sigma^i}, \mathcal{H}om_{\Sigma^i}(K, C))$$

be the natural $O_{\Sigma^i}$-linear isomorphism that sends $b \mapsto (v \mapsto b(v \otimes -))$. By Proposition 11.4, the inverse image $\theta^{-1}(d(\tilde{d}s_2|_{q^{-1}\Sigma^i}))$ is contained the $O_{q^{-1}\Sigma^i}$-submodule

$$q^* \mathcal{H}om_{\Sigma^i}((T_X)_{\Sigma^i} \otimes K, C) \subseteq q^* \mathcal{H}om_{\Sigma^i}((T_X)_{\Sigma^i} \otimes C).$$

Let $H \rightarrow \Sigma^i$ be the vector bundle associated with the locally free $O_{\Sigma^i}$-module $\mathcal{H}om_{\Sigma^i}((T_X)_{\Sigma^i} \otimes K, C)$. Let $h : ((T_X)_{\Sigma^i} \Box K)_H \rightarrow C_H$ be the tautological $O_H$-linear map. Let $D : q^{-1}\Sigma^i \rightarrow H$ be the unique morphism of $\Sigma^i$-schemes such that

$$D^* h = \theta^{-1}(d(\tilde{d}s_2|_{q^{-1}\Sigma^i})).$$

The argument that follows is based on the following commutative diagram with Cartesian squares:

$$
\begin{array}{ccc}
B^i & \rightarrow & \Sigma^1(\theta h) \\
\downarrow & & \downarrow \\
J^2 & \rightarrow & H \\
\downarrow q & & \downarrow D \\
J^1 & \leftarrow & \Sigma^i
\end{array}
$$

The respective ranks of $K$ and $C$ as $O_{\Sigma^i}$-modules are $n - m + i$ and $r - m + i$. If $i = 0$, then $\mathcal{H}om_{\Sigma^i}(K, C) = 0$, so $\Sigma^i(\theta h)$ and $B^i$ are empty by Remark 2.5 (2).

Suppose $i > 0$. By Proposition 15.2 below, if char$(k) \neq 2$, then the first degeneracy locus $\Sigma^1(\theta h)$ has relative codimension

$$n - (n - m + i)(r - m + i) + 1 = n - i(|n - r| + i) + 1$$

in $H$ over $\Sigma^i$. Furthermore, if char$(k) = 2$, the same holds provided that

(a) $n - m + i > 1$; and
(b) if $r - m + i = 1$, then $n - m + i = n$ and $n$ is even.

Otherwise, $\Sigma^1(\theta h)$ has relative codimension $n - i(|n - r| + i)$ in $H$ over $\Sigma^i$.

Conditions (a) and (b) may be respectively rephrased as follows:

(a') If $i = 1$, then $r < n$. 


(b') If \( i = 1 \) and \( r \leq n \), then \( r = 1 \) and \( n \) is even.
Thus (a) and (b) hold if, and only if, conditions (1), (2) and (3) from the statement of the theorem are satisfied.
To prove the result it suffices to show that the relative codimension of \( B^i \) in \( q^{-1}\Sigma^i \) over \( \Sigma^i \) is equal to the relative codimension of \( \Sigma^1(\theta h) \) in \( H \) over \( \Sigma^i \). We will do this by showing that \( D \) is smooth and surjective.

Let \( G \) denote the vector bundle
\[
G := V((\text{Sym}^2 \Omega_X)^{\oplus r})
\]
regarded as an additive group scheme over \( X \). The map \( \iota \) in the canonical short exact sequence
\[
0 \rightarrow \text{Sym}^2 \Omega_X \overset{\iota}{\rightarrow} \mathcal{P}^2_X \overset{\varepsilon}{\rightarrow} \mathcal{P}^1_X \rightarrow 0
\]
induces an action of the additive group \( J^1 \)-scheme \( G_{J^1} := G \times_X J^1 \) by translations on \( J^2 \) that gives \( J^2 \) the structure of a principal \( G_{J^1} \)-bundle over \( J^1 \). This action restricts to an action of the additive group \( \Sigma^i \)-scheme \( G_{\Sigma^i} := G \times_{J^1} \Sigma^i \) on \( q^{-1}\Sigma^i \) that gives \( q^{-1}\Sigma^i \) the structure of a principal \( G_{\Sigma^i} \)-bundle over \( \Sigma^i \). The point is that the natural \( G_{\Sigma^i} \)-bundle structure.

By the diagram following Remark 11.3, there is a natural surjection of \( \mathcal{O}_{\Sigma^i} \)-modules
\[
(\text{Sym}^2 \Omega_X)^{\oplus r}_{\Sigma^i} \rightarrow \mathcal{H}om_{\Sigma^i}((T_X)_{\Sigma^i} \otimes K, C).
\]
Let \( \varphi : G_{\Sigma^i} \rightarrow H \) be the corresponding map of vector bundles over \( \Sigma^i \), which we regard as a homomorphism of additive group \( \Sigma^i \)-schemes. Then \( D : q^{-1}\Sigma^i \rightarrow H \) is \( \varphi \)-equivariant by Proposition 11.5. This implies that \( D \) is smooth and surjective, since \( \varphi \) is smooth and surjective.

\[ \square \]

**Theorem 12.9.** The universal locus of second-order singularities \( \Sigma^{i,j} \) is nonempty if, and only if,

1. \( i \leq m \); and
2. \( j \leq n - m + i \); and
3. if \( i = 0 \), then \( j = 0 \); and
4. if \( \text{char}(k) = 2 \), \( i = 1 \) and \( r \leq n \), then \( n - m + i - j \) is even.

In this case, \( \Sigma^{i,j} \) is smooth over \( X \) and of pure relative codimension
\[
i(|n - r| + i) + j(n - m + i - j)(r - m + i - 1) + \frac{1}{2} j(j \pm 1)(r - m + i)
\]
in \( J^2(X, A^r) \) over \( X \). The symbol \( \pm \) appearing in this expression should be read as “plus” if \( \text{char}(k) \neq 2 \) and as “minus” if \( \text{char}(k) = 2 \).

**Proof.** Let
\[
\square^2 K := \begin{cases} 
\text{Sym}^2 K & \text{if } \text{char}(k) \neq 2 \\
\wedge^2 K & \text{if } \text{char}(k) = 2.
\end{cases}
\]
Let
\[
\theta : \mathcal{H}om_{\Sigma^i}(K \otimes K, C) \iso\mathcal{H}om_{\Sigma^i}(K, \mathcal{H}om_{\Sigma^i}(K, C))
\]
be the natural $O_{\Sigma i}$-linear isomorphism that sends $b \mapsto (v \mapsto b(v \otimes -))$. By Proposition 11.4, the inverse image $\theta^{-1}(d^2(s_2|_{q^{-1}\Sigma i}))$ is contained the $O_{q^{-1}\Sigma i}$-submodule
\[ q^*\text{Hom}_{\Sigma i}(\square^2 K, C) \subseteq q^*\text{Hom}_{\Sigma i}(K \otimes K, C). \]

Let $H \rightarrow \Sigma^i$ be the vector bundle associated with the locally free $O_{\Sigma i}$-module $\text{Hom}_{\Sigma i}(\square^2 K, C)$. Let $h : (\land^2 K)_H \rightarrow C_H$ be the tautological $O_H$-linear map. Let $D : q^{-1}\Sigma^i \rightarrow H$ be the unique morphism of $\Sigma^i$-schemes such that
\[ D^*h = \theta^{-1}(d(s_2|_{q^{-1}\Sigma i})). \]

Consider the following commutative diagram with Cartesian squares:
\[
\begin{array}{ccc}
\Sigma^{i,j} & \longrightarrow & \Sigma^{j}(\theta h) \setminus \Sigma^{j+1}(\theta h) \\
\downarrow & & \downarrow \\
J^2 & \leftarrow & H \\
\downarrow & & \downarrow \\
J^1 & \leftarrow & \Sigma^i
\end{array}
\]

As in the proof of Theorem 12.8, the $\Sigma^j$-scheme $q^{-1}\Sigma^i$ is smooth over $X$, is of pure relative codimension $i(|n - r| + i)$ in $J^2$ over $X$, and the morphism $D$ is smooth and surjective. Thus it suffices to show that $H$ is nonempty if, and only if, (1)–(4) hold, and smooth of relative codimension
\[ j(n - m + i - j)(r - m + i - 1) + \frac{1}{2}j(j \pm 1)(r - m + i) \]
in $H$ over $\Sigma^i$ when these conditions hold. If $i > 0$, this follows from Proposition 15.3 below and the observation that the ranks of the locally free $O_{\Sigma i}$-modules $K$ and $C$ are $n - m + i$ and $r - m + i$, respectively. If $i = 0$, then $\text{Hom}_{\Sigma i}(K, C) = 0$, so by Remark 2.5 (2) the subscheme $\Sigma^j(\theta h) \setminus \Sigma^{j+1}(\theta h) \subseteq H$ is empty if $j > 0$ and equal to $H$ if $j = 0$. \[ \square \]

13. Singularities of generic maps

Let $k$ be an infinite field. Let $X$ be a smooth algebraic scheme over $k$ of pure dimension $n$. Let $r$ be positive integer. Let $A^r$ be the $r$-dimensional affine space over $k$. Let $i$ and $j$ be nonnegative integers. Let $m := \min(n, r)$.

**Proposition 13.1.** Let $f : X \rightarrow A^r$ be a morphism of $k$-schemes.

1. A necessary condition for the critical locus $\Sigma^j(f) \subseteq X$ to be nonempty is that $i \leq m$. If $\Sigma^j(f)$ is nonempty, then its codimension in $X$ is at most $i(|n - r| + i)$.

2. A necessary condition for the locus of second-order singularities $\Sigma^{i,j}(f) \subseteq X$ to be nonempty is that
   (a) $i \leq m$; and
   (b) $j \leq n - m + i$; and
   (c) if $i = 0$, then $j = 0$; and
(d) if \( \text{char}(k) = 2, \ i = 1 \) and \( r \leq n \), then \( n - m + i - j \) is even. If \( \Sigma^{i}:j(f) \) is nonempty, then its codimension in \( X \) is at most

\[
i(|n - r| + i) + j(n - m + i - j)(r - m + i - 1) + \frac{1}{2}j(j \pm 1)(r - m + i).\]

Proof. By Propositions 12.4 and 12.7, we have

\[
\Sigma^i(f) = (j^1 f)^{-1}\Sigma^i \quad \text{and} \quad \Sigma^{i}:j(f) = (j^2 f)^{-1}\Sigma^{i}:j.
\]

Thus a necessary condition for \( \Sigma^{i}(f) \) to be nonempty is that \( \Sigma^i \) be nonempty, and if \( \Sigma^i(f) \) is nonempty, then its codimension in \( X \) is at most the codimension of \( \Sigma^i \) in \( J^1(X, A^r) \). Similarly, a necessary condition for \( \Sigma^{i}:j(f) \) to be nonempty is that \( \Sigma^{i}:j \) be nonempty, and if \( \Sigma^{i}:j(f) \) is nonempty, then its codimension in \( X \) is at most the codimension of \( \Sigma^{i}:j \) in \( J^2(X, A^r) \). Hence the result follows from Proposition 12.5 and Theorem 12.9. \( \square \)

Lemma 13.2 (Atiyah-Serre). Let \( Z \) be a pure-dimensional algebraic scheme over \( k \). Let \( E \) be a locally free sheaf of constant rank on \( Z \). Let \( \Sigma \subseteq \mathcal{V}(E) \) be a locally closed subscheme of the vector bundle over \( Z \) that corresponds to \( E \). Suppose that \( \Sigma \) has pure codimension \( c \) in \( \mathcal{V}(E) \). Let \( W \subset \Gamma(Z, E) \) be a \( k \)-linear subspace of finite dimension. Suppose that \( W \) generates \( E \) as an \( \mathcal{O}_Z \)-module. Let \( s \in W \) be a general section. Then \( s^{-1}\Sigma \) is either empty or of pure codimension \( c \) in \( Z \). Furthermore, if \( \Sigma \) is smooth and \( k \) has characteristic zero, then \( s^{-1}\Sigma \) is smooth over \( k \).

Proof. Let \( \alpha : W \otimes_k \mathcal{O}_Z \to E \) be the \( \mathcal{O}_Z \)-linear map that sends \( s \otimes f \mapsto fs \). Let \( \tilde{\alpha} : Z \times_k W \to \mathcal{V}(E) \) be the map of vector bundles over \( Z \) induced by \( \alpha \). For each section \( s \in W \), we have a commutative diagram with Cartesian squares:

\[
\begin{array}{ccc}
s^{-1}\Sigma & \longrightarrow & \tilde{\alpha}^{-1}\Sigma \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Z \times_k W \\
\downarrow & & \downarrow \tilde{\alpha} \\
\text{Spec} k & \longrightarrow & W \\
pr_2 & & \\
\end{array}
\]

There is nothing to show if the second projection \( \tilde{\alpha}^{-1}\Sigma \to W \) is not dominant. Suppose that it is. By hypothesis \( \alpha \) is surjective, so \( \tilde{\alpha} \) is smooth and surjective. Thus the inverse image \( \tilde{\alpha}^{-1}\Sigma \) has pure codimension \( c \) in \( Z \times_k W \), and is smooth over \( k \) if \( \Sigma \) is. Applying generic flatness or, in characteristic zero, generic smoothness, to the second projection \( \tilde{\alpha}^{-1}\Sigma \to W \), the result follows. \( \square \)

Let

\[
W \subseteq \Gamma(X, \mathcal{O}_X^{\oplus r}) = \text{Hom}_k(X, A^r)
\]

be a \( k \)-linear subspace of finite dimension.
Proposition 13.3. Suppose that \( W \) separates jets of order 1. Let \( f \in W \) be a general section. Then the critical locus \( \Sigma^i(f) \subseteq X \) is either empty or has pure codimension \( i(n - r) + i \) in \( X \). Furthermore, if \( k \) has characteristic zero, then \( \Sigma^i(f) \) is smooth.

Proof. Follows from Propositions 12.4 and 12.5, and Lemma 13.2 applied to the \( k \)-linear subspace
\[
d^1_X(W) \subseteq \Gamma(X, (\mathcal{P}^1_X)^{\oplus r})
\]
and the universal critical locus
\[
\Sigma^i \subseteq J^1(X, A^r) = \mathbf{V}((\mathcal{P}^1_X)^{\oplus r}).
\]

Theorem 13.4. Suppose that \( W \) separates jets of order 2. Let \( f \in W \) be a general section. Then the critical locus \( \Sigma^{i,j}(f) \subseteq X \) is either empty or has pure codimension
\[
i(n - r) + j(n - m + i - j)(r - m + i - 1) + \frac{1}{2}j(j \pm 1)(r - m + i).
\]
in \( X \). Furthermore, if \( k \) has characteristic zero, then \( \Sigma^{i,j}(f) \) is smooth.

Proof. Follows from Proposition 12.7, Theorem 12.9, and Lemma 13.2 applied to the \( k \)-linear subspace
\[
d^2_X(W) \subseteq \Gamma(X, (\mathcal{P}^2_X)^{\oplus r})
\]
and the universal locus of second-order singularities
\[
\Sigma^{i,j} \subseteq J^2(X, A^r) = \mathbf{V}((\mathcal{P}^2_X)^{\oplus r}).
\]

Theorem 13.5. Suppose that \( W \) separates jets of order 2. Let \( f \in W \) be a general section. If \( \text{char}(k) \neq 2 \), then the critical locus \( \Sigma^i(f) \subseteq X \) is smooth over \( k \). The same holds if \( \text{char}(k) = 2 \), with the exception of two cases:

1. The case where \( i = 1 \) and \( r \geq n \).
2. The case where \( i = 1 \), \( r = 1 \) and \( n \) is odd.

In cases (1) and (2), the singular locus of \( \Sigma^i(f) \) consists of a finite set of points if it is nonempty.

Proof. Follows from Propositions 4.4 and 12.7, Theorem 12.8, and Lemma 13.2 applied to the \( k \)-linear subspace
\[
d^2_X(W) \subseteq \Gamma(X, (\mathcal{P}^2_X)^{\oplus r})
\]
and the universal bad locus
\[
B^i \subseteq J^2(X, A^r) = \mathbf{V}((\mathcal{P}^2_X)^{\oplus r}).
\]

14. Geometry on Grassmannians

Let \( X \) be a scheme. Let \( E \) be a locally free \( \mathcal{O}_X \)-module of finite rank \( e \). Let \( n \) an integer such that \( 0 \leq n \leq e \). Let \( G := \text{Gr}_n(E) \) be the Grassmannian of rank-\( n \) quotients on \( E \) over \( X \). Let
\[
0 \longrightarrow K \overset{\iota}{\longrightarrow} E_G \overset{q}{\longrightarrow} E_G/K \longrightarrow 0
\]
be the tautological short exact sequence on \( G \).

In the sequel we will need a few facts about the standard affine charts on a Grassmannian:

**Remark 14.1.** Suppose that \( E = E' \oplus E'' \), where \( E' \) and \( E'' \) are free \( \mathcal{O}_X \)-modules and \( E' \) has rank \( n \). Let \( U \subseteq G \) be the largest open subscheme where \( q|_{E'} : E'_G \rightarrow E_G/K \) is an isomorphism. Let \( u \) denote the composition of \( \mathcal{O}_U \)-linear maps

\[
\begin{array}{c}
E''_U \xrightarrow{q} (E_G/K)_U \xrightarrow{(q|_{E'})^{-1}} E'_U.
\end{array}
\]

We have an isomorphism of short exact sequences of \( \mathcal{O}_U \)-modules

\[
\begin{array}{c}
0 \rightarrow E''_U \xrightarrow{i'} E_U \xrightarrow{q'} E'_U \xrightarrow{q|_{E'}} 0 \\
\sim \downarrow \quad \sim \downarrow \\
0 \rightarrow K_U \xrightarrow{i} E_U \xrightarrow{q} E_U/K_U \xrightarrow{q|_{E'}} 0,
\end{array}
\]

where

\[
i' = \begin{bmatrix} -u \\ \text{id} \end{bmatrix} \quad \text{and} \quad q' = \begin{bmatrix} \text{id} & u \end{bmatrix}
\]

relatively to the direct sum decomposition \( E = E' \oplus E'' \).

Fix bases on \( E' \) and \( E'' \), so that \( u \) may be identified with a matrix with \( n \) rows, \( e - n \) columns and entries in \( \Gamma(U, \mathcal{O}_U) \). Let \( A_X^{n(e-n)} \) be the affine space of dimension \( n(e-n) \) over \( X \). Let

\[
\psi : U \rightarrow A_X^{n(e-n)}
\]

be the unique morphism of \( X \)-schemes that pulls the coordinates on \( A_X^{n(e-n)} \) back to the \( n(e-n) \) entries of the matrix representing \( u \) (in some order). It is a standard fact that \( \psi \) is an isomorphism.

Let \( A \subseteq E \) be rank-\( a \) subbundle. Let \( p \) be a nonnegative integer.

**Definition 14.2.** The \( p \)th special Schubert cycle associated to the subbundle \( A \subseteq E \), denoted \( \sigma_p(A) \), is defined to be the closed subscheme of \( G \) where \( \mathcal{O}_G \)-linear map

\[
\wedge^{a-p+1}(q|_A) : \wedge^{a-p+1}A_G \rightarrow \wedge^{a-p+1}(E_G/K)
\]

vanishes if \( a - p \geq 0 \), and the empty scheme otherwise.

**Remark 14.3.** Special Schubert cycles are examples of degeneracy loci. Indeed, let \( j \) be the unique integer such that \( \min(a,n) - j = a - p \). If \( a - p \leq n \), then \( j \geq 0 \) and

\[
\sigma_p(A) = \Sigma^j(q|_A).
\]

If \( a - p \geq n \), then \( \wedge^{a-p+1}(E_G/K) = 0 \) and therefore \( \sigma_p(A) = \emptyset \).

Intuitively, \( \sigma_p(A) \subseteq G \) is the locus where \( K \cap A_G \) has rank at least \( p \).

Suppose that \( 0 \leq a - p \leq n \).
Proposition 14.4. The Schubert cell $\sigma_p(A) \setminus \sigma_{p+1}(A)$ is smooth of relative dimension $n(e-n) - (n - a + p)p$ over $X$.

Proof. Let $w \in \sigma_p(A) \setminus \sigma_{p+1}(A)$ be a point, so that $q|_A : A_G \to E_G/K$ has rank $a - p$ at $w$. Shrinking $X$ to a neighborhood of the image of $w$ in $X$, we may assume that $E$ is a free $O_X$-module and that $A$ is spanned by the first $a$ elements a basis of $E$. Partitioning such a basis, we may find a direct sum decomposition

$$E = A' \oplus B' \oplus A'' \oplus B'',$$

where $A'$, $B'$, $A''$ and $B''$ are free $O_X$-modules such that $A'$ has rank $a - p$, $A = A' \oplus A''$, and the natural $O_G$-linear map $(A' \oplus B')_G \to E_G/K$ is an isomorphism at $w$. Hence the result follows from Lemma 14.5 below.

Lemma 14.5. Suppose that

$$E = A' \oplus B' \oplus A'' \oplus B'',$$

where $A'$, $B'$, $A''$ and $B''$ are free $O_X$-modules such that $A'$ has rank $a - p$, $A = A' \oplus A''$, and

$$A'' := A' \oplus B'' \oplus A'',$$

$E'$ := $A' \oplus B'$ and $E'' := A'' \oplus B''$, so that $E = E' \oplus E''$. Let $U \subseteq G$ and $u : E'_U \to E''_U$ be as in Remark 14.1. Let $w_{21}$ denote the composition of the $O_U$-linear maps

$$A''_U \hookrightarrow (A'' \oplus B'')_U \longrightarrow (A' \oplus B')_U \overset{pr_2}{\longrightarrow} B'_U.$$

Then $(\sigma_p(A) \setminus \sigma_{p+1}(A)) \cap U$ is the subscheme of $U$ where $w_{21} = 0$, which is isomorphic to the affine space $A^n_{X}$ over $X$.

Proof. By Lemma 2.8, the intersection $(\sigma_p(A) \setminus \sigma_{p+1}(A)) \cap U$ is equal to the closed subscheme of $U$ where the composition

$$A''_U \overset{q}{\longrightarrow} E_U/K_U \rightarrow (E_U/K_U)/q(A'_U)$$

vanishes. In view of the commutative diagram

$$\begin{array}{ccc}
(A' \oplus B')_U & \overset{pr_2}{\longrightarrow} & B'_U \\
\sim \downarrow q & & \sim \downarrow q \\
E_U/K_U & \longrightarrow & (E_U/K_U)/q(A'_U)
\end{array}$$

this intersection may also be described as the subscheme of $U$ where $w_{21} = 0$. Recalling that $U$ is isomorphic to $A^n_{X}$ as a schemes over $X$ and that $w_{21}$ may be identified with a matrix of size $(n - a + p) \times p$ whose entries are coordinates on $U$, the result follows.
Let $F$ be a locally free $\mathcal{O}_X$-module of rank $f$. Let $m := \min(e, f)$. Let $i$ be a nonnegative integer. Suppose that $n = m - i$, so that
\[ G = \text{Gr}_{m-i}(E). \]

**Definition 14.6.** Let $\alpha : E \to F$ be an $\mathcal{O}_X$-linear map. The **Tjurina transform** of the degeneracy locus $\Sigma^i(\alpha) \subseteq X$ is the subscheme $Z \subseteq G$ where $\alpha \circ \iota = 0$.

**Proposition 14.7.** There exists a unique morphism of schemes $\rho : Z \to \Sigma^i(\alpha)$ such that the following diagram commutes:
\[
\begin{array}{ccc}
Z = \{ \alpha \circ \iota = 0 \} & \xrightarrow{i} & G \\
\downarrow^\rho & & \downarrow^\pi \\
\Sigma^i(\alpha) & \xrightarrow{} & X
\end{array}
\]
This morphism is proper, surjective and induces an isomorphism of schemes
\[ Z \setminus \rho^{-1}(\Sigma^{i+1}(\alpha)) \sim \Sigma^i(\alpha) \setminus \Sigma^{i+1}(\alpha). \]

TO DO: Prove surjectivity.

**Proof.** Uniqueness of $\rho$ is clear. Existence follows from the fact that $\alpha_Z : E_Z \to F_Z$ factors though $(E_G/K)_Z$, which has rank $m - i$, so that $\wedge^{m-i+1} \alpha_Z = 0$.

Let $x : T \to \Sigma$ be a morphism of schemes. Then there exists a unique morphism of schemes $s : T \to Z$ such that $\rho \circ s = x$. Indeed, the set of such morphisms is in bijection with the set of rank-$(e - n)$ subbundles $\tilde{K} \subseteq x^* E$ contained in $\ker(x^* \alpha)$. The latter set is a singleton, since $\ker(x^* \alpha)$ is itself a rank-$(e - n)$ subbundle of $x^* E$ by Proposition 2.7. \(\square\)

The next result was stated as part of Proposition 2.9 above. We include a proof of it here to illustrate a standard argument that is a key ingredient in our proof of Theorem 15.5 below.

Let $\pi : V \to X$ be the vector bundle corresponding to the locally free $\mathcal{O}_X$-modules ${\mathfrak{Hom}_X}(E, F)$. In symbols,
\[ V = \mathbf{V}(\mathfrak{Hom}_X(E, F)). \]
Let $h : E_H \to F_H$ be the tautological map. Let $\Sigma'$ denote the locally closed subscheme $\Sigma^i(h) \setminus \Sigma^{i+1}(h) \subseteq V$.

**Proposition 14.8.** The scheme $\Sigma' \subseteq V$ is smooth of relative dimension $(e - m + i)(f - m + i)$ over $X$.

**Proof.** The fiber product $G' := G \times_X V$ is the Grassmannian of rank-$(m - i)$ quotients of $E_V$ over $V$. Let $Z' \subseteq G'$ be the Tjurina transform of the universal degeneracy locus $\Sigma^i(h) \subseteq V$. By Proposition 14.7 it suffices to show that $Z'$ is smooth of relative dimension
\[ ef - (e - m + i)(f - m + i) \]
over $X$.

Let $w \in Z'$ be a point. Shrinking $X$ to a neighborhood of the image of $w$ in $X$, we may assume that $E$ and $F$ are free $O_X$-modules. Partitioning a basis of $E$, we may find a direct sum decomposition $E = E' \oplus E''$, where $E'$ and $E''$ are free $O_X$-modules of finite rank such that $E'$ has rank $m - i$, and the map $q|_{E'} : E'_G \to E_G/K$ is an isomorphism at $w$.

Let $U \subseteq G$, $u : E''_U \to E'_U$, and $u' : E''_U \to E'_U$ be as in Remark 14.1. Then $w \in U \times_X V \subseteq G'$. The intersection $Z' \cap (U \times_X V)$ is the closed subscheme of $U \times_X V$ where $h \circ u' = 0$. Let $h_1$ and $h_2$ denote the restrictions of $h : E_V \to F_V$ to $E'_V$ and $E''_V$, respectively. Then, relatively to the direct sum decomposition $E = E' \oplus E''$, we have

$$h = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \quad \text{and} \quad u' = \begin{bmatrix} -u \\ \text{id} \end{bmatrix},$$

so that

$$h \circ u' = -h_1 u + h_2.$$

Fix bases on $E'$, $E''$ and $F$. Then $h_1$ (resp. $h_2$) may be identified with a matrix with $f$ rows, $m - i$ columns (resp. $e - m + i$ columns), and entries in $\Gamma(V, O_V)$. There exists a unique isomorphism of $X$-schemes $V \cong A^e_X$ that identifies the $ef$ entries of these two matrices (in some order) with the coordiantes on $A^e_X$. Thus

$$U \times_X V \cong A^{(m-i)(e-m+i)}_X \times_X A^{ef}_X = A^{(m-i)(e-m+i)+ef}_X.$$

It follows from this and (5) that

$$Z' \cap (U \times_X V) \cong A^M_X$$

as schemes over $X$, where

$$M := (m - i)(e - m + i) + ef - (e - m + i)f$$

$$= ef - (e - m + i)(f - m + i).$$

The result follows. \hfill \Box

15. Degeneracy loci in bundles of bilinear maps

Let $X$ be a scheme. Let $E$ and $F$ be finite, locally free $O_X$-modules of respective ranks $e$ and $f$. Let $A \subseteq E$ be a rank-$a$ subbundle. Let $\square^2 E$ denote one of two $O_X$-modules: either $\text{Sym}^2 E$ or $\wedge^2 E$. Let $A \square E$ be the image of $A \otimes E$ under the natural map $E \otimes E \to \square^2 E$.

Throughout this section, the symbol $\pm$ should be read as “plus” if $\square^2 E = \text{Sym}^2 E$, and as “minus” if $\square^2 E = \wedge^2 E$.

**Lemma 15.1.** The $O_X$-module $A \square E$ is a locally free of rank

$$\frac{1}{2}a(a \pm 1) + a(e - a).$$
Proof. The question being local on $X$, we may assume that $E$ is free with basis $\{v_1, \ldots, v_\ell\} \subseteq \Gamma(X, E)$ and that $A$ is freely generated by $v_1, \ldots, v_a$. Then $A \square E$ is freely generated by the images of the products $v_i \otimes v_j$ where $0 \leq i \leq a$, $0 \leq j \leq \ell$, and
\[
\begin{cases} 
    i \leq j & \text{if } \square^2 E = \text{Sym}^2 E \\
    i < j & \text{if } \square^2 E = \wedge^2 E.
\end{cases}
\]

The result follows. \qed

Let $\pi : V \to X$ be the vector bundle corresponding to the locally free $\mathcal{O}_X$-module $\mathcal{H}om_X(A \square E, F)$. In symbols:

$$V := V(\mathcal{H}om_X(A \square E, F))$$

Let $\tilde{h} : (A \square E)_V \to F_V$ be the tautological map. Let

$$h : E_V \to \mathcal{H}om_X(A, F)_V$$

be the image of $h$ under the natural isomorphism $\theta : \mathcal{H}om_X(A \otimes E, F) \xrightarrow{\sim} \mathcal{H}om_X(E, \mathcal{H}om(A, F))$.

Proposition 15.2. Suppose that $af \leq \ell$. If $\square^2 E = \text{Sym}^2 E$, then the first degeneracy locus $\Sigma^1(h) \subseteq V$ has relative dimension $\ell - af + 1$ in $V$. If $\square^2 E = \wedge^2 E$, then the same holds provided that
\begin{enumerate}
    \item $a > 1$; and
    \item if $f = 1$, then $a = e$ and $e$ is even.
\end{enumerate}
Otherwise, $\Sigma^1(h)$ has relative codimension $\ell - af$ in $V$ over $X$.

Proposition 15.3 ([Bra17, Proposition 2.53]). Suppose that $a = e$, so that $A = E$. Let $i$ be a nonnegative integer. Then the constant-rank locus

$$\Sigma^i(h) \setminus \Sigma^{i+1}(h) \subseteq V = V(\mathcal{H}om_X(\square^2 E, F))$$

is nonempty if, and only if,
\begin{enumerate}
    \item $i \leq e$; and
    \item $e - i$ is even if $\square^2 E = \wedge^2 E$ and $f = 1$.
\end{enumerate}

In this case, $\Sigma^i(h) \setminus \Sigma^{i+1}(h)$ is smooth of pure relative codimension

$$i(e - i)(f - 1) + \frac{1}{2}i(i + 1)f$$

in $V$ over $X$.

For each pair of nonnegative integers $i$ and $p$, let $\Delta^{i,p}$ denote the scheme-theoretic intersection

$$\Delta^{i,p} := (\Sigma^i(h) \setminus \Sigma^{i+1}(h)) \cap (\Sigma^p(h|_A) \setminus \Sigma^{p+1}(h|_A)).$$

Here we consider degeneracy loci of the $\mathcal{O}_V$-linear map $h : E_V \to \mathcal{H}om_X(A, F)_V$ and its restriction $h|_A$ to $A_V \subseteq E_V$. Thus $\Delta^{i,p}$ is a locally closed subscheme of $V$. 
Remark 15.4. Let $T$ be scheme. By Proposition 2.6, the set of morphisms $t : T \to \Delta^{i,p}$, is in natural bijection with the set of pairs $(x, \alpha)$, where $x : T \to X$ is a morphism and $\alpha : x^*E \to \mathcal{H}om_T(x^*A, x^*F)$ is an $\mathcal{O}_T$-linear map such that

1. $\theta^{-1}\alpha \in \text{Hom}_T(x^*(A \square E), x^*F)$;
2. $\text{coker}(\alpha)$ is a locally free $\mathcal{O}_T$-module of rank $af - \min(e, af) + i$; and
3. $\text{coker}(\alpha|_A)$ is a locally free $\mathcal{O}_T$-module of rank $af - \min(e, af)$.

Theorem 15.5. Let $i$ and $p$ be nonnegative integers. Let $n := \min(e, af) - i$. The scheme $\Delta^{i,p}$ is nonempty if, and only if,

1. $i \leq \min(e, af)$;
2. $\max(a - n, 0) \leq p \leq \min(a, e - n)$; and
3. $a - p$ is even if $\square^2E = \wedge^2E$ and $f = 1$.

In this case, $\Delta^{i,p}$ is smooth of pure relative codimension

$$p(n - a + p) + f \cdot \left[\frac{1}{2}(-p^2 \pm p) + (e - n)a\right] - n(e - n)$$

in $V$ over $X$.

Proof of Proposition 15.3. Proposition 15.3 is a special case of Theorem 15.5, because $\Sigma^i(h) = \Delta^{i,i}$ when $a = e$. \hfill $\square$

Proof of Proposition 15.3. The support of the degeneracy locus $\Sigma^i(h) \subseteq V$ decomposes as the disjoint union of the supports of the subschemes $\Delta^{i,p} \subseteq V$ with $i \geq 1$ and $p \geq 0$. In symbols:

$$|\Sigma^i(h)| = \bigsqcup_{i \geq 1; p \geq 0} |\Delta^{i,p}|$$

By Theorem 15.5 and Lemma 16.1 below, the minimum of the codimensions of the subschemes $\Delta^{i,p} \subseteq V$ with $i \geq 1$ and $p \geq 0$ is either $e - af + 1$ or $e - af$, as in the statement of the proposition. \hfill $\square$

Proof of Theorem 15.5. The nonemptyness statement is proved separately as Lemma 15.6 below. Suppose that (1)–(3) hold. The argument that follows is modeled on the standard one given for Proposition 14.8.

Let $G := \text{Gr}_n(E)$ be the Grassmannian of rank-$n$ quotients of $E$ over $X$. Let

$$0 \longrightarrow K \overset{\iota}{\longrightarrow} E_G \overset{q}{\longrightarrow} E_G/K \longrightarrow 0$$

be the tautological short exact sequence on $G$.

The fiber product $G' := G \times_X V$ is the Grassmannian of rank-$n$ quotients of $E_V$ over $V$. Let $Z \subseteq G'$ be the Tjurina transform of the degeneracy locus $\Sigma^i(h) \subseteq V$. Thus $Z$ is the closed subscheme of $G'$ where $h \circ \iota = 0$. Let $W$ denote the scheme-theoretic intersection of $Z$ with the Schubert cell

$$(\sigma_p(A_V) \setminus \sigma_{p+1}(A_V)) = (\sigma_p(A) \setminus \sigma_{p+1}(A)) \times_X V \subseteq G'.$$

By Proposition 14.7, the second projection $\text{pr}_2 : G' \to V$ induces an isomorphism

$$Z \setminus \text{pr}_2^{-1}(\Sigma^{i+1}(h)) \overset{\sim}{\longrightarrow} \Sigma^i(h) \setminus \Sigma^{i+1}(h).$$
We claim this isomorphism maps $W \setminus \text{pr}_2^{-1}((\Sigma^{i+1}(h)))$ isomorphically onto $\Delta^{ip}$.

To see this, let $\tilde{Z} := Z \setminus \text{pr}_2^{-1}((\Sigma^{i+1}(h)))$. By the proof of Proposition 14.7 the natural $O_Z$-linear map $K_Z \rightarrow \ker(h_Z)$ is an isomorphism over $\tilde{Z}$. This implies that $h_{\tilde{Z}}$ factors through a locally split injection

$$(E_G/K)_{\tilde{Z}} \hookrightarrow \text{Hom}_X(A, F)_{\tilde{Z}}.$$ 

Hence the subschemes of $\tilde{Z}$ where $\wedge^{a-p'+1}(q|A) = 0$ and where $\wedge^{a-p'+1}(h|A) = 0$ coincide for all $0 \leq p' \leq a + 1$. The claim follows.

Thus it suffices to show that $W$ is smooth of pure relative dimension

$$p(n-a+p) + f \cdot [\frac{1}{2}(-p^2 + p) + (e-n)a]$$

in $G' = G \times_S V$ over $X$.

Let $w \in W$ be a point. Then $\text{pr}_1(w) \in \sigma_p(A) \setminus \sigma_{p+1}(A) \subseteq G$. As in the proof of Proposition 14.4, after shrinking $X$ to a neighborhood of the image of $w$ in $X$, we may assume that the $O_X$-module $F$ is free, and find a direct sum decomposition

$$(6) \quad E = A' \oplus B' \oplus A'' \oplus B'',$$

where $A'$, $B'$, $A''$ and $B''$ are free $O_X$-modules such that $A'$ has rank $a-p$,

$$A = A' \oplus A'',$$

and the natural $O_G$-linear map $(A' \oplus B')_G \rightarrow E_G/K$ is an isomorphism at $\text{pr}_1(w)$. Let $E' := A' \oplus B'$ and $E'' := A'' \oplus B''$, so that $E = E' \oplus E''$. Let $U \subseteq G$, $\ u : E''_U \rightarrow E'_U$ and $\ i' : E''_U \rightarrow E_U$ be as in Remark 14.1.

In terms of the direct sum decompositions of $E''$ and $E'$, we may write

$$u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix},$$

where

$$u_{11} : A''_U \rightarrow A'_U \quad u_{12} : B''_U \rightarrow A'_U$$

$$u_{21} : A''_U \rightarrow B'_U \quad u_{22} : B''_U \rightarrow B'_U$$

are $O_U$-linear maps. Thus, for example, $u_{21}$ is equal to the restriction of $u : E''_U \rightarrow E'_U$ to $A''_U$ followed by the second projection $E'_U = A'_U \oplus B'_U \rightarrow B'_U$.

The fiber product $U \times_X V$ is an open subset of $G'$ that contains $w$. By Lemma 14.5, the intersection $W \cap (U \times_X V)$ is the closed subscheme of $U \times_X V$ where

$$u_{21} = 0 \quad \text{and} \quad h \circ i' = 0.$$

Fix a basis for the free $O_X$-module $F$. Let $\mathcal{B} \subseteq \Gamma(X, F^\vee)$ be the dual basis of $F^\vee$. The subscheme of $U \times_X V$ where $h \circ i' = 0$ is the same as the subscheme where $\tau h \circ i' = 0$ for all $\tau \in \mathcal{B}$. 

Fix an element $\tau \in \mathcal{B}$, so that $\tau h \in \text{Hom}_X(E, A^\vee)$. In terms of the direct sum decompositions $E = A' \oplus B' \oplus A'' \oplus B''$ and $A = A' \oplus A''$, we may write

$$\tau h = \begin{bmatrix} \alpha_{11} & \beta_{11} & \alpha_{12} & \beta_{12} \\ \alpha_{21} & \beta_{21} & \alpha_{22} & \beta_{22} \end{bmatrix}$$

where

$$\alpha_{11} : A' \to (A')^\vee \quad \alpha_{12} : A'' \to (A')^\vee \quad \alpha_{21} : A' \to (A'')^\vee \quad \alpha_{22} : A'' \to (A'')^\vee$$

and

$$\beta_{11} : B' \to (A')^\vee \quad \beta_{12} : B'' \to (A')^\vee \quad \beta_{21} : B' \to (A'')^\vee \quad \beta_{22} : B'' \to (A'')^\vee$$

are $\mathcal{O}_X$-linear maps. Note the following symmetries:

$$\alpha_{11}^\vee = \pm \alpha_{11} \quad \alpha_{22}^\vee = \pm \alpha_{22} \quad \alpha_{12}^\vee = \pm \alpha_{21}$$

The subscheme of $U \times_X V$ where $\tau h \circ \iota' = 0$ is the same as the one cut out by the equation

$$\begin{bmatrix} \alpha_{11} & \beta_{11} & \alpha_{12} & \beta_{12} \\ \alpha_{21} & \beta_{21} & \alpha_{22} & \beta_{22} \end{bmatrix} \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$
Fix $\mathcal{O}_X$-linear bases on $A', B', A''$, and $B''$. Then we may identify the schemes $U$ and $V$ with affine spaces over $X$; the $\mathcal{O}_U$-linear map $u : E'_U \to E'_U$ with a matrix whose entries are the coordinates on $U$; and the $\mathcal{O}_V$-linear map $\tilde{h} : (E \square A)_V \to F_V$ with a matrix whose entries are the coordinates on $V$.

We may view the left-hand side of (7) as a matrix whose entries are distinct coordinates on $U$; and we may view the left-hand sides of (8-11) as matrices whose entries are coordinates on $V$. The coordinates appearing in the matrices of the left-hand sides of (7-11) do not occur in the matrices of the right-hand sides of these equations.

The coordinates of $V$ occurring in $\alpha_{12}, \beta_{12}, \beta_{22}$ and on and above the diagonal of the matrix of $\alpha_{22}$, are all distinct. The matrix corresponding to $\alpha_{22}$ is symmetric if $\Box^2 E = \text{Sym}^2 E$ and skew-symmetric with zeroes along the diagonal if $\Box^2 = \wedge^2 E$. The right-hand side of (9) has the same symmetry as $\alpha_{22}$.

The equations (8-11) depend on element $\tau \in \mathcal{B}$ of the dual basis of $F$ that was fixed. Different choices of $\tau$ lead to disjoint sets of coordinates occurring in $\alpha_{12}, \alpha_{22}, \beta_{12}, \beta_{22}$.

Recalling that $W \cap (U \times_X V)$ is the subscheme of $U \times_X V$ where $u_{21} = 0$ and $\tau' h \circ \iota' = 0$ for all $\tau' \in \mathcal{B}$, we conclude that $W \cap (U \times_X V)$ is an affine space over $X$, and therefore smooth over $X$. This implies that $W$ is smooth over $X$, as the point $w \in W$ was arbitrary.

To compute the relative codimension of $W$ in $G$, recall the ranks of $A', B', A''$ and $B''$:

\[
\begin{align*}
\text{rank}(A') &= a - p \\
\text{rank}(B') &= n - a + p \\
\text{rank}(A'') &= p \\
\text{rank}(B'') &= e - n - p
\end{align*}
\]

We make the following observations:

- $u_{21} \in \text{Hom}_U(A''_U, B'_U)$, so (7) contributes $p(n - a + p)$ to the relative codimension of $W$ in $G'$.
- $\alpha_{12} \in \text{Hom}_X(A'', (A')^\vee)$, so (8) contributes $f \cdot p(a - p)$ to the relative codimension of $W$ in $G'$.
- $\alpha_{22} \in \text{Hom}_X(A'', (A'')^\vee)$, so (9) contributes $f \cdot \frac{1}{2} p(p \pm 1)$ to the relative codimension of $W$ in $G'$.
- $\beta_{12} \in \text{Hom}_X(B'', (A')^\vee)$, so (10) contributes $f \cdot (e - n - p)(a - p)$ to the relative codimension of $W$ in $G'$.
- $\beta_{22} \in \text{Hom}_X(B'', (A'')^\vee)$, so (11) contributes $f \cdot (e - n - p)p$ to the relative codimension of $W$ in $G'$.

Thus $W$ has relative codimension

\[
p(n - a + p) + f \cdot [p(a - p) + \frac{1}{2} p(p \pm 1) + (e - n - p)a]
\]

in $G'$ over $X$, as we set out to show. \qed
Lemma 15.6. Let $n := \min(e, af) - i$. The scheme $\Delta^{i,p}$ is nonempty if, and only if,

1. $i \leq \min(e, af)$;
2. $\max(a-n, 0) \leq p \leq \min(a, e-n)$; and
3. $a - p$ is even if $\square^2 E = \Lambda^2 E$ and $f = 1$.

If $\Delta^{i,p}$ is nonempty, then the projection $\pi : \Delta^{i,p} \to X$ is surjective.

Proof. By considering geometric fibers of the projection $\pi : \Delta^{i,p} \to X$, we may assume that $X$ is the spectrum of a field.

First, suppose that $\Delta^{i,p}$ is nonempty. Then $i \leq \min(e, af)$ and $p \leq a$ by Remark 15.4. If $\alpha : E \to \text{Hom}_k(A, F)$ is the $k$-linear map corresponding to a $k$-valued point of $\Delta^{i,p}$, then $\alpha$ has rank $\min(e, af) - i$ and $\alpha|_A : A \to \text{Hom}_k(A, F)$ is symmetric or alternating and of rank $a - p$. It follows that $\ker(\alpha) \cap A$ has dimension $p$. Now

$max(0, \dim(A) - \text{rank}(\alpha)) \leq \dim(\ker(\alpha) \cap A) \leq \min(\dim(A), \dim(\ker(\alpha)))$,

and (2) holds. Furthermore, a skew-symmetric matrix (with zeroes along the diagonal) has necessarily even rank, so (3) holds.

Conversely, suppose that (1), (2) and (3) hold. By (3), one of the following alternatives holds:

• $\square^2 E = \text{Sym}^2 E$.
• $\square^2 E = \Lambda^2 E$ and $a - p$ is even.
• $\square^2 E = \Lambda^2 E$, $a - p$ is odd and $f \geq 2$.

Let us exhibit a point of $\Delta^{i,p}$ assuming the third alternative holds; the other two cases are slightly simpler and left to the reader.

Let $\{v_1, \ldots, v_e\} \subseteq E$ be a basis of $E$ such that the vectors $v_1, \ldots, v_a$ freely generate $A$. Let $\{v_1^\vee, \ldots, v_a^\vee\} \subseteq E^\vee$ be the dual basis of $E^\vee$. Let $w_1, w_2 \in F$ be linearly independent vectors and let

$$\beta_1, \ldots, \beta_{n-a+p} \in \text{Hom}_k(A, F)$$

be maps such that

$$-v_{a-p}^\vee \otimes w_2 + v_2^\vee \otimes w_1, -v_1^\vee \otimes w_1, v_4^\vee \otimes w_1, -v_3^\vee \otimes w_1,$$

$$\ldots$$

$$v_{a-p-1}^\vee \otimes w_1, -v_{a-p-2}^\vee \otimes w_1, v_1^\vee \otimes w_2, \beta_1, \ldots, \beta_{n-a+p}$$

are $n = \min(e, af) - i$ linearly independent elements of $\text{Hom}_k(A, F)$. Let

$$\alpha := v_1^\vee \wedge v_2^\vee \otimes w_1 + \ldots + v_{a-p-2}^\vee \wedge v_{a-p-1}^\vee \otimes w_1 +$$

$$+ v_{a-p}^\vee \wedge v_1^\vee \otimes w_2 + \sum_{j=1}^{n-a+p} v_{a+j}^\vee \otimes \beta_j.$$ 

Then

$$\alpha \in \text{Hom}_k(A \square E, F) \subseteq \text{Hom}_k(E, \text{Hom}_k(A, F)),$$
so $\alpha$ corresponds to a point of $V$, which is contained in $\Delta^{i,p}$ by construction and Remark 15.4.

16. THE MINIMAL CODIMENSION

Let $e, a, f$ be positive integers such that $af \leq e$. Let $R$ denote the following polygonal region of the real plane:

$$R := \{(i, p) \in \mathbb{R}^2 : 1 \leq i \leq af; \ max(a - af + i, 0) \leq p \leq \min(a, e - af + i)\}$$

![Figure 1. Sketch of $R$ in the case where $(f+1)a > e+1$ and $f > 1$.](image)

Let $C_+ : \mathbb{R}^2 \to \mathbb{R}$ and $C_- : \mathbb{R}^2 \to \mathbb{R}$ be the functions defined by

$$C_\pm(i, p) = p(af - i - a + p) + f \cdot \left[-\frac{p^2}{2} \pm \frac{p}{2} + (e - af + i)a\right] - (af - i)(e - af + i).$$

Let $\Lambda := \mathbb{Z} \times (a + 2\mathbb{Z})$.

**Lemma 16.1.**

1. If $f > 1$, then the restriction of $C_+$ to $R \cap \mathbb{Z}^2$ achieves its minimum value at $(1, 0)$.
2. If $f = 1$, then the restriction of $C_+$ to $R \cap \mathbb{Z}^2$ achieves its minimum value at $(1, 1)$.
3. If $f > 1$ and $a > 1$, then the restriction of $C_-$ to $R \cap \mathbb{Z}^2$ achieves its minimum value at $(1, 0)$.
4. If $f > 1$ and $a = 1$, then the restriction of $C_-$ to $R \cap \mathbb{Z}^2$ achieves its minimum value at $(1, 1)$.
5. If $f = 1$, $a$ is even, and $a = e$, then the restriction of $C_-$ to $R \cap \Lambda$ achieves its minimum value at $(2, 2)$.
6. If $f = 1$, $a$ is even, and $a < e$, then the restriction of $C_-$ to $R \cap \Lambda$ achieves its minimum value at $(1, 2)$.
(7) If \( f = 1 \) and \( a \) is odd, then the restriction of \( C_\pm \) to \( R \cap \Lambda \) achieves its minimum value at \( (1, 1) \).

The minimum value is \( e - af + 1 \) in cases (1), (2), (3) and (5), and \( e - af \) in cases (4), (6) and (7).

**Proof.** Let \( m := \max(a - af + 1, 0) \), so that

\[
m = \begin{cases} 1 & \text{if } f = 1 \\
0 & \text{if } f > 1,
\end{cases}
\]

and let \( M := \min(a, e - af + 1) \). Let \( L_1 \) denote the vertical line segment

\[
\{(1, p) : m \leq p \leq M\}.
\]

Note that \( L_1 \) contains a unique point, namely \((1, 1)\), if \( f = 1 \) and either \( a = 1 \) or \( e = a \). Let \( L_2 \) denote the diagonal line segment

\[
\{(i, e - af + i) | 1 \leq i \leq (f + 1)a - e\}.
\]

Note that \( L_2 \) contains a unique point, namely \((1, e - af + 1)\), if \((f + 1)a = e + 1\), and is empty if \((f + 1)a \leq e\).

For each \((i, p) \in R\), we have

\[
\frac{\partial C_\pm}{\partial i}(i, p) = e - af + 2i - p \geq i > 0.
\]

Therefore the restrictions of \( C_\pm \) to \( R \cap \mathbb{Z}^2 \) and \( R \cap \Lambda \) achieve their minima on \( L_1 \cup L_2 \).

Let us consider the restrictions of \( C_\pm \) to \( L_1 \). If \( f = 1 \), suppose that \( a > 1 \) or \( a < e \), so that \( L_1 \) consists of more than one point. Let \( C_1^+ : R \to R \) and \( C_1^- : R \to R \) be the functions defined by \( C_1^+(p) = C(1, p) \). We have

\[
\frac{dC_1^\pm}{dp} = (f - 1)a - 1 - (f - 2)p \pm \frac{f}{2},
\]

so \( C_1^\pm \) is convex if \( f = 1 \), linear or constant if \( f = 2 \), and concave if \( f \geq 2 \).

First, suppose that \( f = 1 \). Then \( C_1^\pm : R \to R \) is a convex quadratic function and achieves its global minimum at the point \( p^* = 1 \pm \frac{1}{2} \). It follows that the restriction of \( C_1^\pm \) to \([1, M] \cap \mathbb{Z} \) achieves its minimum at \( p = 1 \). Moreover, the restriction of \( C_1^\pm \) to \([1, M] \cap a + 2\mathbb{Z} \) achieves its minimum at \( p = 1 \) when \( a \) is odd, and at \( p = 2 \) when \( a \) is even.

Next, suppose that \( f = 2 \), so that \( C_1^\pm : R \to R \) is a linear function. We distinguish two cases:

- If \( a = 1 \), then \( C_1^\pm \) is strictly decreasing, \( M = 1 \), and the restriction of \( C_1^\pm \) to \([0, M] \cap \mathbb{Z} \) achieves its minimum when \( p = 1 \).
- If \( a > 1 \), then \( C_1 \) is nondecreasing, so the restriction of \( C_1^\pm \) to \([0, M] \cap \mathbb{Z} \) achieves its minimum when \( p = 0 \).
Finally, suppose that $f \geq 3$. Then $C^1_\pm : \mathbb{R} \to \mathbb{R}$ is a concave quadratic function and achieves its global maximum at the point
\[ p^* := \frac{f - 1}{f - 2}a - \frac{1 \pm f/2}{f - 2}. \]
We distinguish two cases:

- If $a = 1$, then $p^* < 1/2$ and $M = 1$, so that $p^*$ is to the left of the midpoint of the interval $[0, M]$. It follows that $C^1_\pm$ achieves its minimum on $[0, M] \cap \mathbb{Z}$ when $p = 1$.
- If $a \geq 2$, then $p^* > a/2$. In general $M \leq a$, so $p^*$ is to the right of the midpoint of the interval $[0, M]$. It follows that $C^1_\pm$ achieves its minimum on $[0, M] \cap \mathbb{Z}$ when $p = 0$.

Now let us consider the behavior of the functions $C_\pm : \mathbb{R}^2 \to \mathbb{R}$ on the line segment $L_2$. Suppose that $L_2$ is neither empty nor a single point, so that $1 \leq e - af + 1 < a$. Let $C^2_\pm : \mathbb{R} \to \mathbb{R}$ and $C^2_- : \mathbb{R} \to \mathbb{R}$ be the quadratic functions defined by $C^2_\pm(i) = (i, e - af + i)$. We have
\[ \frac{dC^2_\pm}{di} = (f^2 + f - 1)a - (f - 2)(e + i) \pm f/2, \]
so $C^2_\pm$ is convex if $f = 1$, linear or constant if $f = 2$, and concave if $f \geq 2$.

First, suppose $f = 1$. Then $C^2_\pm : \mathbb{R} \to \mathbb{R}$ is convex and achieves its global maximum at the point $i^* = -(e - a) \mp \frac{1}{2}$. Note that $e - af + i \equiv a$ mod 2 if, and only if, $e \equiv i$ mod 2. Moreover, $[1, (f + 1)a - e] = [1, 2a - e]$.

- The restriction of $C^2_\pm$ to $[1, 2a - e] \cap \mathbb{Z}$ achieves its maximum at $i = 1$.
- If $a < e$, the restriction of $C^2_\pm$ to $[1, 2a - e] \cap e + 2\mathbb{Z}$ achieves its maximum at $i = 1$.
- If $a = e$, the restriction of $C^2_- \pm$ to $[1, 2a - e] \cap e + 2\mathbb{Z}$ achieves its maximum at $i = 1$ if $e$ is odd and at $i = 2$ if $e$ is even.

Next, suppose that $f = 2$. Then $C^2_\pm : \mathbb{R} \to \mathbb{R}$ is a nondecreasing linear function, so the restriction of $C^2_- \pm$ to $[1, (f + 1)a - e] \cap \mathbb{Z}$ achieves its minimum when $i = 1$.

Finally, suppose that $f \geq 3$. Then $C^2_\pm : \mathbb{R} \to \mathbb{R}$ is a concave quadratic function and $C^2_\pm$ achieves its global maximum at
\[ i^* = \frac{f^2 - f - 1}{f - 2}a \pm \frac{f/2}{f - 2} - e. \]
Then $i^*$ lies to the right of the midpoint of the interval $[1, (f + 1)a - e]$, so $C^2_- \pm$ achieves its minimum on $[1, (f + 1)a - e] \cap \mathbb{Z}$ when $i = 1$.

Statements (1)–(7) follow from the above analysis of $C^1_\pm$ and $C^2_\pm$. 

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