Effective construction of a class of positive operators in Hilbert space, which do not admit triangular factorization

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Abstract

A class of non-factorable positive operators is constructed. As a result, pure existence theorems in the well-known problems by Ringrose, Kadison and Singer are substituted by concrete examples.

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1 Introduction

To introduce the main notions of the triangular factorization (see [3, 5, 7, 13, 14, 19]) consider a Hilbert space $L^2(a, b)$ ($-\infty \leq a < b \leq \infty$). The orthogonal projectors $P_\xi$ in $L^2(a, b)$ are defined by the relations

$$(P_\xi f)(x) = f(x) \text{ for } a < x < \xi, \quad (P_\xi f)(x) = 0 \text{ for } \xi < x < b \ (f \in L^2(a, b)).$$

Denote the identity operator by $I$.

Definition 1.1 A bounded operator $S_-$ on $L^2(a, b)$ is called lower triangular if for every $\xi$ the relations

$$S_- Q_\xi = Q_\xi S_- Q_\xi,$$

where $Q_\xi = I - P_\xi$, are true. The operator $S_-^*$ is called upper triangular.
Definition 1.2 A bounded, positive definite and invertible operator \( S \) on \( L^2(a,b) \) is said to admit a left (right) triangular factorization if it can be represented in the form

\[
S = S_- S_+ \quad (S = S_+ S_-),
\]

(1.2)

where \( S_- \) and \( S_-^{-1} \) are bounded and lower triangular operators.

Further, we often write factorization meaning a left triangular factorization.

In paper [19] (see p. 291) we formulated necessary and sufficient conditions under which the positive definite operator \( S \) admits a triangular factorization. The factorizing operator \( S_-^{-1} \) was constructed in the explicit form. We proved that a wide class of operators admits a triangular factorization [19].

D. Larson proved [7] the existence of positive definite and invertible but non-factorable operators. In the present article we construct concrete examples of such operators. In particular, the following operator

\[
Sf = f(x) - \mu \int_0^\infty \frac{\sin \pi (x-t)}{\pi (x-t)} f(t) dt, \quad f(x) \in L^2(0, \infty), \quad 0 < \mu < 1 \quad (1.3)
\]

is positive definite and invertible but non-factorable. Using positive definite and invertible but non-factorable operators we have managed to substitute pure existence theorems [7] by concrete examples in the well-known problems posed by J.R. Ringrose [12], R.V. Kadison and I.M. Singer [5]. We note that Kadison-Singer problem was posed independently by I. Gohberg and M.G. Krein [4].

The non-factorable operator \( S \), which is defined by formula (1.3), is used in a number of theoretical and applied problems (in optics [21], random matrices [23], generalized stationary processes [10, 11], and Bose gas theory [9]). The results obtained in the paper are interesting from this point of view too.

2 A special class of operators and corresponding differential systems

In this section we consider operators \( S \) of the form

\[
Sf = f(x) - \mu \int_0^\infty h(x-t)f(t) dt, \quad f(x) \in L^2(0, \infty), \quad (2.1)
\]
where $\mu = \overline{\mu}$ and $h(x)$ admits representation

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\lambda} \rho(\lambda) d\lambda. \quad (2.2)$$

We suppose that the function $\rho(\lambda)$ satisfies the following conditions

1. The function $\rho(\lambda)$ is real and bounded

   $$|\rho(\lambda)| \leq U^2, \quad U > 0 \quad (-\infty < \lambda < \infty). \quad (2.3)$$

2. $\rho(\lambda) = \rho(-\lambda) \in L(-\infty, \infty)$. Hence, the function $h(x)$ $(-\infty < x < \infty)$ is continuous and real. The corresponding operator

   $$Hf = \int_{0}^{\infty} h(x-t)f(t)dt \quad (2.4)$$

is self-adjoint and bounded, where $||H|| \leq U$. We introduce the operators

$$S_\xi f = f(x) - \mu \int_{0}^{\xi} h(x-t)f(t)dt, \quad f(x) \in L^2(0, \xi), \quad 0 < \xi < \infty. \quad (2.5)$$

The following statement is true.

**Proposition 2.1** If $-1/U < \mu < 1/U$, then the operator $S_\xi$, which is defined by formula (2.5), is positive definite, bounded and invertible.

Hence, we have

$$S_\xi^{-1} f = f(x) + \int_{0}^{\xi} R_\xi(x, t, \mu) f(t)dt. \quad (2.6)$$

The function $R_\xi(x, t, \mu)$ is jointly continuous in $x, t, \xi, \mu$. M.G. Krein (see [4], Ch. IV, Section 7) proved that

$$S_b^{-1} = (I + V_+)(I + V_-), \quad 0 < b < \infty, \quad (2.7)$$

where the operators $V_+$ and $V_-$ are defined in $L^2(0, b)$ by the relations

$$(V_+ f)(x) = (V_- f)(x) = \int_{0}^{x} R_\xi(x, t, \mu) f(t)dt. \quad (2.8)$$
The Krein’s formula (2.7) is true for the Fredholm class of operators. The operator $S_\text{b}$ belongs to this class. The kernel of the operator $V_\text{-}$ does not depend of $b$. Hence, if the operator $S$ admits the factorization, then formula (2.7) holds for the case $b = \infty$ too, i.e.

$$S^{-1} = (I + V_+)(I + V_-). \quad (2.9)$$

**Remark 2.1** Relation (2.9) also follows from Theorem 2.1 in the paper [19].

Let us introduce the function

$$q_1(x) = 1 + \int_0^x R_x(x, t, \mu)dt. \quad (2.10)$$

Using the relation $R_x(x, t, \mu) = R_x(x - t, 0, \mu)$ (see [4], formula (8.12)), we obtain

$$q_1(x) = 1 + \int_0^x R_x(u, 0, \mu)du. \quad (2.11)$$

According to the well-known Krein’s formula ( [4], Ch. IV, formulas (8.3) and (8.14)) we have

$$q_1(x) = \exp \left\{ \int_0^x R_t(t, 0, \mu)dt \right\}. \quad (2.12)$$

Together with $q_1(x)$ we shall consider the function

$$q_2(x) = M(x) + \int_0^x M(t)R_x(x, t, \mu)dt, \quad (2.13)$$

where

$$M(x) = \frac{1}{2} - \mu \int_0^x h(s)ds. \quad (2.14)$$

The functions $q_1(x)$ and $q_2(x)$ generate the 2×2 differential system

$$\frac{dW}{dx} = i z J H(x) W, \quad W(0, z) = I_2. \quad (2.15)$$

Here $W(x, z)$ and $H(x)$ are 2×2 matrix functions and $J$ is a 2×2 matrix:

$$H(x) = \begin{bmatrix} q_2^2(x) & 1/2 \\ 1/2 & q_1^2(x) \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (2.16)$$
Note that according to [18] (see formulas (53) and (56) therein) we have:

\[ q_1(x)q_2(x) = 1/2. \]  

(2.17)

It is easy to see that

\[ JH(x) = T(x)PT^{-1}(x), \]  

(2.18)

where

\[ T(x) = \begin{bmatrix} q_1(x) & -q_1(x) \\ q_2(x) & q_2(x) \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \]  

(2.19)

Consider the matrix function

\[ V(x, z) = e^{-izx/2}T^{-1}(x)W(x, z)T(0). \]  

(2.20)

Due to (2.15)-(2.20) we get

\[ \frac{dV}{dx} = (iz/2)jV + \Gamma(x)V, \quad V(0) = I_2, \]  

(2.21)

where

\[ \Gamma(x) = \begin{bmatrix} 0 & B(x) \\ B(x) & 0 \end{bmatrix}, \quad j = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \]  

\[ B(x) = \frac{q_1'(x)}{q_1(x)} = R_x(x, 0, \mu). \]  

(2.22)

(2.23)

Let us introduce the functions

\[ \Phi_n(x, z) = v_{1n}(x, z) + v_{2n}(x, z) \quad (n = 1, 2), \]  

(2.24)

\[ \Psi_n(x, z) = i[v_{1n}(x, z) - v_{2n}(x, z)] \quad (n = 1, 2), \]  

(2.25)

where \( v_{in}(x, z) \) are elements of the matrix function \( V(x, z) \). It follows from (2.21) that

\[ \frac{d\Phi_n}{dx} = (z/2)\Psi_n - B(x)\Phi_n, \quad \Phi_1(0, z) = \Phi_2(0, z) = 1, \]  

(2.26)

\[ \frac{d\Psi_n}{dx} = -(z/2)\Phi_n + B(x)\Psi_n, \quad \Psi_1(0, z) = -\Psi_2(0, z) = i. \]  

(2.27)

Consider again the differential system (2.15) and the solution \( W(x, z) \) of this system. The element \( w_{1,2}(\xi, z) \) of the matrix function \( W(\xi, z) \) can be represented in the form (see [16], p. 54, formula (2.6))

\[ w_{1,2}(\xi, z) = iz \left( (I - zA)^{-1}1, S^{-1}\xi^{-1}\right)_{\xi}, \]  

(2.28)

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where the operator $A$ has the form

$$Af = i \int_0^x f(t)dt.$$  \hfill (2.29)

It is well-known that

$$(I - zA)^{-1}1 = e^{izx}.$$  \hfill (2.30)

We can obtain a representation of $W(\xi, z)$ without using the operator $S^{-1}_\xi$. Indeed, it follows from $(2.20)$, $(2.21)$, and $(2.25)$ that

$$W(x, z) = (1/2)e^{ixz/2}T(x) \begin{bmatrix} \Phi_1 - i\Psi_1 & \Phi_2 - i\Psi_2 \\ \Phi_1 + i\Psi_1 & \Phi_2 + i\Psi_2 \end{bmatrix} T^{-1}(0).$$  \hfill (2.31)

According to equality (2.11) we have $q_1(0) = 1$. Due to (2.19) we infer

$$T(0) = \begin{bmatrix} 1 & -1 \\ 1/2 & -1/2 \end{bmatrix}, \quad T^{-1}(0) = \begin{bmatrix} 1/2 & 1 \\ -1/2 & 1 \end{bmatrix}.$$  \hfill (2.32)

Further we plan to use a Krein’s result from [6]. For that purpose we introduce the functions

$$P(x, z) = e^{ixz/2}[\Phi(x, z) - i\Psi(x, z)]/2,$$  \hfill (2.33)

$$P^*(x, z) = e^{ixz/2}[\Phi(x, z) + i\Psi(x, z)]/2,$$  \hfill (2.34)

where

$$\Phi(x, z) = \Phi_1(x, z) + \Phi_2(x, z), \quad \Psi(x, z) = \Psi_1(x, z) + \Psi_2(x, z).$$  \hfill (2.35)

Using (2.26), (2.27) and (2.33), (2.34) we see that the pair $P(x, z)$ and $P^*(x, z)$ is a solution of the following Krein system

$$\frac{dP}{dx} = izP - B(x)P^*, \quad \frac{dP^*}{dx} = -B(x)P,$$  \hfill (2.36)

where

$$P(0, z) = P^*(0, z) = 1.$$  \hfill (2.37)

It follows from (2.33) and (2.34) that

$$P(x, z) - P^*(x, z) = -ie^{ixz/2}\Psi(x, z).$$  \hfill (2.38)
3 Non-factorable positive definite operators, a sufficient condition

We assume that the following relation is true:
\[ M(x) = (1 - \mu)/2 + q(x), \quad q(x) \in L^2(0, \infty), \quad (3.1) \]
where the function \( M(x) \) is defined by (2.14). Condition (3.1) can be rewritten in an equivalent form:
\[ \int_0^\infty h(x)dx = 1/2, \quad \int_x^\infty h(x)dx \in L^2(0, \infty). \quad (3.2) \]

Now, we need the relations (see [15], Ch. 1, formulas (1.37) and (1.44)):
\[ S_\xi 1 = M(x) + M(\xi - x), \quad S_\xi = U_\xi S_\xi U_\xi, \quad (3.3) \]
where \( U_\xi f(x) = f(\xi - x), \quad 0 \leq x \leq \xi \). It follows from (3.1) and (3.3) that
\[ S_\xi 1 = 1 - \mu + q(x) + U_\xi q(x). \quad (3.4) \]

Hence the relation
\[ S_\xi^{-1} 1 = \frac{1}{1 - \mu} [1 - r_\xi(x) - U_\xi r_\xi(x)] \quad (3.5) \]
is true. Here \( r_\xi(x) = S_\xi^{-1} q(x) \). Using formulas (2.28), (3.1), and (3.5), we obtain the following representation of \( w_{1,2}(\xi, z) \).

**Lemma 3.1** The function \( w_{1,2}(\xi, z) \) has the form
\[ w_{1,2}(\xi, z) = e^{iz\xi} G(\xi, z) - \overline{G(\xi, \bar{z})}, \quad (3.6) \]
where
\[ G(\xi, z) = \frac{1}{1 - \mu} \left[ 1 - iz \int_0^\xi e^{-ixx} r_\xi(x)dx \right]. \quad (3.7) \]

Note that the operator \( S \) is positive definite, bounded and invertible. According to (2.7) we have
\[ Q(x) = (I + V_-)q(x) \in L^2(0, \infty). \quad (3.8) \]

Hence, there exists a sequence \( x_n \) such that
\[ Q(x_n) \to 0, \quad x_n \to \infty. \quad (3.9) \]

Now, we prove the following statement.
Lemma 3.2 Let relation (3.9) be true. Then we have
\[ \lim_{x_n \to \infty} q_1(x_n) = \frac{1}{\sqrt{1-\mu}}. \] (3.10)

Proof. In view of (2.10), (2.13), and (3.1) we get
\[ q_2(x) = q_1(x)(1-\mu)/2 + Q(x). \] (3.11)
Taking into account the relation \( q_1(x)q_2(x) = 1/2 \) (see [18], formulas (53) and (56)), we obtain the equality
\[ 1/2 = q_1^2(x)(1-\mu)/2 + q_1(x)Q(x). \] (3.12)
Formula (3.10) follows directly from (3.9), (3.12), and inequality \( q_1(x) > 0 \).
\[ \Box \]

It follows from (2.19) and (3.10) that
\[ T(x_n) \to \begin{bmatrix} C & -C \\ 1/2C & 1/2C \end{bmatrix}, \quad x_n \to \infty, \quad C = 1/\sqrt{1-\mu}. \] (3.13)
Hence, in view of (2.32), (2.33), (2.35), and (3.13) the following assertion is true.

Lemma 3.3 Let \( x_n \) tend to \( \infty \). Then, \( w_{1,2} \) has the following asymptotics
\[ w_{1,2}(x_n, z) = -iCe^{ix_nz/2} \Psi(x_n, z) \left( 1 + o(1) \right). \] (3.14)

Lemma 3.4 Suppose that the operator \( S \) admits a factorization. Then we have
\[ \lim_{\xi \to \infty} e^{-iz\xi}w_{1,2}(\xi, z) = G(z), \quad \Im z < 0, \] (3.15)
\[ \lim_{\xi \to \infty} w_{1,2}(\xi, z) = -\overline{G(z)}, \quad \Im z > 0. \] (3.16)
where
\[ G(z) = \frac{1}{1-\mu} \int_0^\infty e^{-izx}r(x)dx, \quad r(x) = S^{-1}q(x). \] (3.17)
Proof. According to (2.9) we have $S^{-1} = I + V_-$, where $V_-$ is defined by (2.8). Hence, the operator function $S^{-1}_\xi$ strongly converges to the operator $S^{-1}$ when $\xi \to \infty$. Then the function $r_\xi(x) = S^{-1}_\xi q(x)$ strongly converges to $r(x) = S^{-1}q(x)$, when $\xi \to \infty$, and $r(x) \in L^2(0, \infty)$. Using (3.6) and (3.7) we obtain relations (3.15) and (3.16). The lemma is proved. □

From Lemma 3.4 we derive the following important assertion.

**Proposition 3.1** If at least one of the equalities (3.15) and (3.16) is not true, then the corresponding operator $S$ does not admit factorization.

Note that a new approach to the notion of the limit of a function was used in Lemma 3.2. Namely, we introduce a continuous function $F(x)$, which belongs to $L(0, \infty)$, and consider sequences $x_n \to \infty$, such that

$$F(x_n) \to 0.$$  \hfill (3.18)

**Definition 3.1** We say that the function $f(x)$ tends to $A$ almost sure (a.s.) if relation (3.18) implies

$$f(x_n) \to A, \quad x_n \to \infty.$$  \hfill (3.19)

Equality (3.10) can be written in the form

$$\lim_{x \to \infty} q_1(x) = \frac{1}{\sqrt{1 - \mu}}, \quad \text{a.s.}$$  \hfill (3.20)

**Remark 3.1** From heuristic point of view ”almost all” sequences $x_n \to \infty$ satisfy relation (3.18). This is the reason of using the probabilistic term ”almost sure”.

4 A class of non-factorable positive definite operators

Introduce a partition

$$0 = a_0 < a_1 < \ldots < a_n = a,$$  \hfill (4.1)
and consider the function $\rho(\lambda) = \rho(-\lambda)$ such that

$$\rho(\lambda) = \begin{cases} 
0, & a \leq \lambda, \\
b_{k-1}, & a_{k-1} \leq \lambda < a_k,
\end{cases} \quad (4.2)$$

where

$$b_0 = 1; \quad -1 \leq b_k \leq 1 \quad (0 < k \leq n - 1). \quad (4.3)$$

In the case of $\rho$ given by (4.2) and (4.3) we can put $U = 1$ in (2.3). Further we investigate the operators $S$, which are defined by formulas (2.1), (2.2), and (4.2). The spectral function $\sigma(\lambda)$ of the corresponding system (2.36) is absolutely continuous and such that (see [6]):

$$\sigma'(\lambda) = \left[1 - \mu \rho(\lambda)\right]/(2\pi). \quad (4.4)$$

**Remark 4.1** The operators $S$, which are defined by formulas (2.1), (2.2), and (4.2), appear in the theory of generalized stationary processes of white noise type (see [10,11]). If $n = 1$ and $a_1 = \pi$, then the corresponding operator $S$ has the form (1.3).

It follows from (2.2) and (4.2) that

$$h(x) = \frac{1}{\pi} \sum_{k=1}^{n} b_k \frac{\sin a_k x - \sin a_{k-1} x}{x}. \quad (4.5)$$

According to (4.4) we have

$$\int_{-\infty}^{\infty} \frac{\log\sigma'(u)}{1 + u^2} du < \infty. \quad (4.6)$$

It follows from (4.7) (see [6]) that

$$\int_{0}^{\infty} |P(x, z_0)|^2 dx < \infty, \quad \Im z_0 > 0. \quad (4.7)$$

Hence, there exists a sequence $x_n$ such that

$$|P(x_n, z_0)|^2 \to 0, \quad x_n \to \infty. \quad (4.8)$$

Now, we use the corrected form of Krein’s theorem (see [6,20]):
Proposition 4.1 1) There exists the limit
\[ \Pi(z) = \lim_{x_n \to \infty} P_*(x_n, z), \] (4.9)
where the convergence is uniform at any bounded closed set of the upper half-plane \( \Im z > 0 \).

2) The function \( \Pi(z) \) can be represented in the form
\[ \Pi(z) = \frac{1}{\sqrt{2\pi}} \exp \left\{ \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{1 + tz}{(z - t)(1 + t^2)} \log \sigma'(t) dt + i\alpha \right\}, \] (4.10)
where \( \alpha = \bar{\alpha} \). Here \( \sigma \) is the spectral function of system (2.36), which corresponds to \( \rho \) given by (4.2) and (4.3), that is, this \( \sigma \) is defined by (4.4).

Remark 4.2 The function \( |Q(x)|^2 + |P(x, z_0)|^2 \) belongs to the space \( L(0,\infty) \). Hence, there exists a sequence \( x_n \) such that relations (3.9) and (4.8) are true simultaneously.

If (4.5) holds, then the following conditions are fulfilled:
\[ 0 < \delta \leq ||S|| \leq \Delta < \infty, \quad \int_0^{\infty} |h(x)|^2 dx < \infty. \] (4.11)
Therefore, in formula (4.10) we have (see [18], Proposition 1):
\[ \alpha = 0. \] (4.12)

One can easily see that
\[ -\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{1 + tz}{(z - t)(1 + t^2)} \log(2\pi) dt = \frac{1}{2} \log(2\pi). \] (4.13)
It follows from (4.10), (4.12), and (4.13) that \( \Pi(z) \) has the form
\[ \Pi(z) = \prod_{k=0}^{n-1} \left[ \left( \frac{a_{k+1} + z}{a_{k+1} - z} \right) \left( \frac{a_k - z}{a_k + z} \right)^{\log(1-b_k\mu)/2i\pi} \right], \quad \Im z > 0. \] (4.14)

Next, we prove the main result of this paper.

Theorem 4.1 The bounded positive definite and invertible operator \( S \), which is defined by formulas (2.1) and (4.5), does not admit a left triangular factorization.
Proof. Taking into account Lemma 3.3 and relations (2.38), (4.8), and (4.9) we have

\[ \lim_{x_n \to \infty} w_{1,2}(x_n, z) = -C \Pi(z), \quad \exists z > 0, \quad C = 1/\sqrt{(1 - \mu)}. \] (4.15)

Now, we use the following relations

\[ \lim_{y \to +0} \left( \frac{a_{k+1} - iy}{a_{k+1} + iy} \right) \left( \frac{a_k + iy}{a_k - iy} \right) = 1, \quad k > 0, \] (4.16)

\[ \lim_{y \to +0} \left( \frac{a_{k+1} - iy}{a_{k+1} + iy} \right) \left( \frac{a_k + iy}{a_k - iy} \right) = -1, \quad k = 0. \] (4.17)

Formulas (4.14), (4.16), and (4.17) imply that

\[ \lim_{y \to +0} \Pi(iy) = \sqrt{(1 - \mu)}. \] (4.18)

Suppose that the operator \( S \) admits a factorization. It follows from the asymptotics of sinus integral (see [2], Ch. 9, formulas (2) and (10)), that the kernel \( h(x) \), defined by formula (4.5), satisfies conditions (3.2). Hence, the conditions of Lemma 3.4 are fulfilled. Comparing formulas (3.16) and (4.15), we see that

\[ -\lim_{y \to +0} G(-iy) = -1/(1 - \mu) \neq -C \lim_{y \to +0} \Pi(iy) = -1. \] (4.19)

Hence, the relation (3.16) is not true. According to Proposition 3.1 the operator \( S \) does not admit a factorization. The theorem is proved. \( \square \)

5 Examples instead of existence theorems

Let the nest \( N \) be the family of subspaces \( Q_{\xi} L^2(0, \infty) \). The corresponding nest algebra \( Alg(N) \) is the algebra of all linear bounded operators in the space \( L^2(0, \infty) \) for which every subspace from \( N \) is an invariant subspace. Put \( D_N = Alg(N) \cap Alg(N)^* \). The set \( N \) has multiplicity one if the diagonal \( D_N \) is abelian, that is, \( D_N \) is a commutative algebra. We can see that the lower triangular operators \( S_- \) form the algebra \( Alg(N) \), the corresponding diagonal \( D_N \) is abelian, and it consists of the commutative operators

\[ T_\varphi f = \varphi(x)f, \quad f \in L^2(0, \infty), \] (5.1)
where $\phi(x)$ is bounded. Hence, the introduced nest $N$ has the multiplicity 1.

**Ringrose Problem.** Let $N$ be a multiplicity one nest and $T$ be a bounded invertible operator. Is $TN$ necessarily multiplicity one nest?

We obtain a concrete counterexample to Ringrose’s hypothesis.

**Proposition 5.1** Let the positive definite, invertible operator $S$ be defined by the relations (2.1) and (4.5). The set $S^{1/2}N$ fails to have multiplicity 1.

**Proof.** We use the well-known result (see [3], p. 169):
The following assertions are equivalent:
1. The positive definite, invertible operator $T$ admits factorization.
2. $T^{1/2}$ preserves the multiplicity of $N$.
We stress that in our case the set $N = Q_\xi L^2(0, \infty)$ is fixed.) The operator $S$ does not admit the factorization. Therefore, the set $S^{1/2}N$ fails to have multiplicity 1. The proposition is proved. □

Next, consider the operator

$$Vf = \int_0^x e^{-(x+y)}f(y)dy, \quad f(x) \in L^2(0, \infty). \quad (5.2)$$

An operator is said to be hyperintransitive if its lattice of invariant subspaces contains a multiplicity one nest. Note that the lattice of invariant subspaces of the operator $V$ coincides with $N$, see [8] and [22] (Ch. 11, Theorem 150). Hence we deduce the answer to Kadison-Singer [5] and to Gohberg-Krein [4] question.

**Corollary 5.1** The operator $W = S^{1/2}VS^{-1/2}$ is a non-hyperintransitive compact operator.

Indeed, the lattice of the invariant subspaces of the operator $W$ coincides with $S^{1/2}N$.

**Remark 5.1** The existence parts of Theorem 4.1, Proposition 5.1, and Corollary 5.1 are proved by D.R. Larson [7].
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