Coin flipping is a fundamental cryptographic task where spatially separated Alice and Bob wish to generate a fair coin flip over a communication channel. It is known that ideal coin flipping is impossible in both classical and quantum theory. In this work, we give a short proof that it is also impossible in generalized probabilistic theories under the generalized no-restriction hypothesis. Our proof relies crucially on a formulation of cheating strategies as semi-infinite programs, i.e., cone programs with infinitely many constraints. This introduces a formalism which may be of independent interest to the quantum community.

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I. INTRODUCTION

In this paper, we consider the possibility of cryptography in theories more general than quantum or classical theory. One may ask why this is a worthwhile endeavour, and for this we give several reasons. The first reason is to future-proof current results, which is important in the context of cryptography. While developing quantum cryptography and computation, the community quickly came to realize that classical cryptography results need to be re-evaluated for the new quantum era. Since results in quantum cryptography typically rely on the validity of quantum mechanics being a faithful description of nature, these too all have to re-evaluated if quantum theory is one day superseded by a new theory, regardless of how minor or radical the departure from quantum mechanics is. Another reason is to gain a better understanding of results in quantum theory. For instance, it is insightful to sit back and think about what parts of quantum theory were needed to prove a result. Did we require entanglement? Were we just assuming these states are in superposition? Can we reprove this only assuming the no-signaling principle? By answering such questions, we gain a better understanding of quantum mechanics itself as well as the resources necessary for performing particular tasks.

In this and many other works in cryptography, optimization theory is a key ingredient in the analysis. On a high level, we want to maximize how much someone can “cheat” a protocol, whereby it is understood that the inability to cheat translates into security, and vice versa. The goal is often to design protocols which minimize cheating. We, however, take the opposite approach in this work and prove a limitation on designing any protocol for a particular task, namely coin flipping, discussed below.

II. COIN FLIPPING

Coin flipping is the cryptographic task where Alice and Bob generate a random bit $b$ over a communication channel such that when Alice and Bob are honest, both output the same bit $b$ and this bit is uniformly random [1]. Coin flipping is a primitive that is used mainly for building larger, more sophisticated cryptographic protocols in the two-party setting, and hence an understanding of its properties, along with its security limitations, is important. For example, coin flipping has been used in the creation of optimal oblivious transfer protocols [2], is related to bit commitment (see, for example, Refs. [3–5]), and variants have been studied such as weak coin flipping [6], unbalanced coin flipping [7], and die rolling [8]. Moreover, since secure oblivious transfer implies secure bit commitment [9] which in turn implies secure coin flipping [1] proving the insecurity of coin flipping in a generalized probabilistic theory (GPT) setting (as we do in this paper) automatically implies the insecurity of these other tasks, as well as any others that imply secure coin flipping.

More formally, the coin flipping task is as follows. Suppose Alice has a set of strategies (basically, a description of how she interacts with Bob) given by the set $\mathcal{A}$ and Bob has a set of strategies given by the set $\mathcal{B}$. We do not just consider deterministic strategies but also those that occur as the result of some measurement procedure. We denote the probability of a pair of strategies occurring as $\text{Prob}(A, B)$ which is between 0 and 1 for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

A coin-flipping protocol consists of the following:
(1) A triple of strategies for Alice \((A_0, A_1, A_{\text{abort}})\) which correspond to the measurement outcomes of some deterministic strategy \(A_{\text{det}}\) and

(2) A triple of strategies for Bob \((B_0, B_1, B_{\text{abort}})\) which correspond to the measurement outcomes of some deterministic strategy \(B_{\text{det}}\), satisfying

\[
\text{Prob}(A_b, B_b) = 1/2 \quad \text{for} \quad b \in \{0, 1\}. \tag{1}
\]

The conditions above ensure that the protocol behaves as expected, that the bit \(b\) is uniform and shared between Alice and Bob. Ideally, we wish that neither Alice nor Bob can cheat by digressing from protocol and disturbing the conditions given by (1). However, this may not be the case, and as such, we need to measure this disturbance. The security measure in coin flipping is given by the amount a dishonest Alice or a dishonest Bob can bias the output distribution away from uniform. To make this formal, we define the following symbols:

- \(P_{A,b}^\ast\): The maximum probability that dishonest Alice can force honest Bob to accept the outcome \(b\).
- \(P_{B,b}^\ast\): The maximum probability that dishonest Bob can force honest Alice to accept the outcome \(b\).
- \(\epsilon\): The bias of the coin-flipping protocol defined as

\[
\epsilon := \max \{P_{A,0}^\ast, P_{A,1}^\ast, P_{B,0}^\ast, P_{B,1}^\ast\} - 1/2. \tag{2}
\]

We wish to design protocols such as to minimize \(\epsilon\), with a perfect protocol having \(\epsilon = 0\). In classical and quantum theory, this is known to be impossible \([10,11]\). In this work, we show that under some assumptions on \(\mathcal{A}\) and \(\mathcal{B}\), \(\epsilon\) can be lower bounded by a positive constant, thus showing near-perfect coin flipping is impossible in any theory satisfying those assumptions.

To study the range of possible \(\epsilon\), we need to study the four quantities \(P_{A,0}^\ast, P_{A,1}^\ast, P_{B,0}^\ast, P_{B,1}^\ast\). Let us first consider \(P_{B,0}^\ast\). We can write this succinctly by the rudimentary optimization problem:

\[
P_{B,0}^\ast = \sup_{B \in \mathcal{B}} \{\text{Prob}(A_0, B)\}. \tag{3}
\]

This optimization problem exactly captures how much Bob can force Alice to output 0 maximized over all physical strategies he can perform. Before studying this problem using optimization theory, we require a mathematical structure on the quantities involved. We now discuss such a structure which is given by the study of generalized probabilistic theories.

### III. Generalized Probabilistic Theories (GPTs)

To study (3) more generally than quantum and classical theory, we require a more general setting for physical theories. Here we work in the framework of generalized probabilistic theories which formalizes any physical theory with an operational description. There have been many approaches to GPTs; see, for example, Refs. \([12–21]\) for introductions to these frameworks. GPTs have been successfully used for studying cryptography \([13,22–27]\) and computation \([28–36]\) in theories more general than quantum theory. We, however, do not actually need to introduce the full framework of GPTs for the purposes of this work. Instead, we just consider the structure that any such theory would impose on the sets of strategies for Alice and Bob.

As mentioned above, we do not just want to consider the strategies which occur deterministically, but those which may correspond to obtaining a particular outcome in some experiment. That is, given a strategy \(A \in \mathcal{A}\) for Alice and a strategy \(B \in \mathcal{B}\) for Bob, we obtain a probability \(\text{Prob}(A, B)\) that these two strategies jointly occur. In particular, there is always a “zero strategy” \(0 \in \mathcal{A}\) such that \(\text{Prob}(0, B) = 0\) for all \(B \in \mathcal{B}\). Conceptually, one can think of this as Alice aborting the protocol or simply not taking part in the first place.

First, we assume that these spaces of strategies are convex where we interpret convex combinations as probabilistic mixtures. That is, we assume that

\[
pA_1 + (1 - p)A_2 \tag{4}
\]

is in the set \(\mathcal{A}\) and represents the strategy where with probability \(p\) Alice uses strategy \(A_1\) and with probability \(1 - p\) Alice uses strategy \(A_2\). Given this understanding of the convex structure, the calculated probabilities must satisfy

\[
\text{Prob} \left( \sum_i p_i A_i, B \right) = \sum_i p_i \text{Prob}(A_i, B) \tag{5}
\]

where the set \(\{p_i\}\) form a probability distribution. An equivalent equation holds for convex combinations of Bob’s strategies. This means that a strategy for Alice induces a linear functional on the space of strategies for Bob (and vice versa).

Rather than working directly with the spaces of strategies \(\mathcal{A}\) and \(\mathcal{B}\), we work with operational equivalence classes of strategies. We say that two strategies \(A_1\) and \(A_2\) are operationally equivalent if

\[
\text{Prob}(A_1, B) = \text{Prob}(A_2, B), \quad \forall B \in \mathcal{B} \tag{6}
\]

and similarly for Bob’s strategies. We denote these equivalence classes as \(\tilde{A}\) and \(\tilde{B}\). (While we are working with the equivalence classes of strategies, our main result applies to the original strategies themselves; the equivalence classes just provide a convenient tool for our proof. We elaborate on this in the Appendix B.)

Note that our earlier assumptions imply that \(\tilde{A}\) and \(\tilde{B}\) are both convex sets in some vector space \(V\) which are bounded and have nonempty interior. (For completeness, we prove that the sets are bounded in Appendix B.) Moreover, we assume that the vector space \(V\) is finite-dimensional. This assumption is typically made in the study of GPTs for technical convenience. It can, however, be motivated by the idea that in a tomographic characterization of the strategies of Alice, one can only, in practice, perform a finite number of different experiments and therefore we must characterize the strategies by a finite number of probabilities.

Employing the Riesz representation theorem \([37]\) in the case of linear functionals on finite-dimensional vector spaces, one can show that we can always compute the probabilities as

\[
\text{Prob}(A, B) = \langle \tilde{A}, \tilde{B} \rangle \tag{7}
\]

From now on we take \(\tilde{A}\) as the set of Alice’s strategies (similarly \(\tilde{B}\) as the set of Bob’s strategies) and hence drop the
that are cal theory and restrict both Alice and Bob to a set of strategies prove anything meaningful. For example, consider any physically feasible strategy for Bob can be physically possible up to some small error. To avoid GPTs with these unequally realistic assumptions, we make the assumption that any mathematically possible strategy for Bob can be physically realized.

To formally define this lack of restriction for Bob, we start with defining important quantities studied in convex analysis. The polar set of the set $C$ is given as

$$C^o := \{W : (W, Z) \leq 1, \forall Z \in C\}$$

and its dual cone is given as

$$C^* := \{W : (W, Z) \geq 0, \forall Z \in C\}.$$  \hspace{1cm} (10)

Notice we have $B \subseteq A^* \cap A^o$ and $A \subseteq B^* \cap B^o$ because every choice of strategies for Alice and Bob yields a proper probability.

We can now define our physical assumption.

**Definition 1.** The generalized no-restriction hypothesis for Bob states that $B = A^* \cap A^o$.

To support this assumption, one can argue that if Alice knows that her set of strategies is given as $A$ then to be able to guarantee security against Bob she should not make any assumptions about what Bob can do. In other words, we also maximize over all physical theories, which in this case translates to allowing Bob to have the largest set of strategies as possible.

This is closely related to the (standard) no-restriction hypothesis [19], which is a commonly used assumption in the study of GPTs that can be expressed as the idea that all mathematically possible measurements are physically allowed. Here, we generalize this idea to the level of arbitrary strategies.

One could equally well consider Bob’s perspective and assume the generalized no-restriction hypothesis for Alice, i.e., $A = B^* \cap B^o$. Surprisingly these two assumptions are not equivalent; see Fig. 1 for an example of this fact. However, for the purposes of this work, we need to only assume it for one party. We henceforth assume it for Bob, but by symmetry the following arguments can be adapted to the case where it is assumed instead for Alice.

**IV. A PHYSICAL ASSUMPTION**

Clearly some assumption on the sets $A$ and $B$ is required to prove anything meaningful. For example, consider any physical theory and restrict both Alice and Bob to a set of strategies that are $\epsilon$-close to their honest strategies. This allows us to define a (rather boring) GPT in which ideal coin flipping is possible up to some small error. To avoid GPTs with these unnecessary restrictions, we make the assumption that any mathematically feasible strategy for Bob can be physically realized.

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Proof. We first show that the feasible region of (13) is bounded. To this end, we define the function
\[ f(Y) = \max_{X \in A \delta} |\langle X, Y \rangle|, \]
which is finite since \( A \delta \) is finite. It can be easily checked that this is a norm (since \( A \delta \) contains a basis) and is bounded for all \( B \) satisfying the constraints of (13). Since all norms are equivalent in finite-dimensional vector spaces, we know there exists a \( \tau > 0 \) such that \( \|B\|_2 \leq \tau \) for all \( B \) feasible in (13).

Fix \( B \) feasible in (13) and \( A \in A \delta \). We now wish to scale \( B \) by some constant \( c > 0 \) to ensure \( \langle A, cB \rangle \leq 1 \) [and thus \( cB \) is feasible in (12)]. Then, for \( X \in A \delta \), \( \delta \)-close to \( A \), we have
\[ \langle B, A \rangle = \langle B, X \rangle + \langle B, A - X \rangle \]
\[ \leq \langle B, X \rangle + \|B\|_2 \|A - X\|_2 \]
\[ \leq 1 + \tau \delta. \]
Thus, \( B + \tau \delta \) is feasible in (12). This implies that
\[ P_{\text{Bob},0}^B \leq P_{\text{Bob},0}^c \leq (1 + \tau \delta) P_{\text{Bob},0}. \]
Taking limits finishes the proof.

VI. DISCUSSION

We now prove a lower bound on the product of Alice’s cheating probability and the relaxation of Bob’s cheating probability. This is the key step in proving our main result, which takes advantage of the simplified structure of the relaxed problem.

Lemma 3. \( P_{\text{Alice},0}^A \cdot P_{\text{Bob},0}^B \geq 1/2 \), for all \( \delta > 0 \).

Proof. Let \( B \in \text{int}(B) = \text{int}(A^* \cap A^* \cap X) \) in which exists since \( B \) has nonempty interior by construction. Then \( B' := B/2 \) satisfies \( B' \in \text{int}(A^*) \) and \( \langle B', X \rangle < 1 \) for all \( X \in A \delta \). This is known as a strictly feasible solution. Since \( P_{\text{Bob},0}^B \) is bounded from above by Eq. (18), the strong duality theorem for cone programming (see, for example, Ref. [46]) states that \( P_{\text{Bob},0}^B \) is equal to
\[ \min_{y_X \geq 0} \left\{ \sum_{X \in A \delta} y_X : \sum_{X \in A \delta} y_X X = A_0 \in (A^*)^0 \right\} \]
and this problem attains \(^2\) an optimal solution \( y_X^* \). Thus, we have \( P_{\text{Bob},0}^B \) is equal to
\[ A := \frac{1}{P_{\text{Bob},0}^B} \sum_{X \in A \delta} y_X^* X = \sum_{X \in A \delta} \left( \frac{y_X^*}{\sum_{X \in A \delta} y_X^*} \right) X. \]
Notice that \( A \in A \delta \) by convexity and \( A - \frac{1}{P_{\text{Bob},0}^B} A_0 \in (A^*)^* \) by the constraints in (19). Suppose Alice uses \( A \) as her strategy to force Bob to accept outcome 0. Then we have
\[ P_{\text{Alice},0}^A \cdot \langle A, B_0 \rangle \geq \frac{1}{2 P_{\text{Bob},0}^B} \langle A_0, B_0 \rangle = \frac{1}{2 P_{\text{Bob},0}^B}, \]
since \( B_0 \in B \subseteq A^* \) and \( \langle A_0, B_0 \rangle = 1/2 \) from Eq. (1).

\(^2\)Note that attainment of an optimal dual solution is not always stated explicitly in the proofs of strong duality, but it is indeed the case.

By combining the two lemmas, we have that \( P_{\text{Alice},0}^* \cdot P_{\text{Bob},0}^* \geq 1/2 \), and therefore the maximum of the two probabilities is at least \( 1/\sqrt{2} \). This gives the same lower bound on the bias Kitaev gave for the case of quantum theory [11], which was later reproved by Gutoski and Watrous using a representation of quantum strategies [47].

Theorem 4. Any coin-flipping protocol in a GPT satisfying the generalized no-restriction hypothesis for Bob (and/or Alice) satisfies \( \epsilon \geq 1/\sqrt{2} - 1/2 \approx 0.207 \). In particular, either Alice or Bob can force an outcome with probability at least \( 1/\sqrt{2} \).

Since quantum theory satisfies the generalized no-restriction hypothesis for both Alice and Bob [47], we have another proof that coin flipping is impossible in quantum theory.
time-independent point games. This, in a nutshell, strips away all the “time-dependent” information of the protocol. Our framework and proof, on the other hand, completely strips away all notion of time as it does not explicitly rely on the round-to-round strategy descriptions, and thus might make this point game reduction simpler or even trivial. It might even expose a GPT in which perfectly secure weak coin flipping is attainable in a finite number of rounds of communication. It would be interesting to find such a GPT, if one exists, and compare it to quantum theory where it is known that perfectly secure weak coin flipping only exists in the limit of infinite rounds of communication [6,50].

In short, if one were to develop GPT weak coin-flipping protocols with small bias, then the lower bound presented in this work might be achievable by imitating the quantum protocol. It would be interesting to see which GPTs allow for secure weak coin flipping, whether it is proved using point games, semi-infinite programming, or another yet-to-be-discovered method.

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APPENDIX A: PROOF THAT OUR MAIN RESULT APPLIES TO THE ORIGINAL STRATEGY CONTEXT (BEFORE MODDING OUT BY THE EQUIVALENCE RELATIONS)

Recall that the set of strategies for Alice (resp. Bob) is denoted by $A$ (resp. $B$) and we use $\bar{A}$ (resp. $\bar{B}$) to denote the same set after modding out by the equivalence relation:

$$A_1 \sim A_2 \iff \text{Prob}(A_1, B) = \text{Prob}(A_2, B) \quad \text{for all } B \in B$$

(A1)

(and the analogous equivalence relation for Bob). Recall that we have an inner product such that $\text{Prob}(A, B) = \langle A, B \rangle$ for all $A \in \bar{A}$ and $B \in \bar{B}$.

A protocol is defined by a triple $(A_0, A, A_{\text{abort}}) \subset A$ for Alice and a triple $(B_0, B, B_{\text{abort}}) \subset B$ for Bob satisfying certain properties that are not needed for the following discussion. Let $(\bar{A}_0, \bar{A}, \bar{A}_{\text{abort}})$ and $(\bar{B}_0, \bar{B}, \bar{B}_{\text{abort}})$ be the triples of equivalence classes for Alice and Bob, respectively.

We have shown in the paper that, under the generalized no-restriction hypothesis on $\bar{A}$ or $\bar{B}$, we have at least one of the two following conditions holding:

1. There exists $\bar{B} \in \bar{B}$ such that $\langle \bar{A}_0, \bar{B} \rangle \geq 1/\sqrt{2}$; or
2. There exists $\bar{A} \in \bar{A}$ such that $\langle \bar{B}_0, \bar{A} \rangle \geq 1/\sqrt{2}$.

Suppose the first condition holds. Then take any $B \in B$ in the same equivalence class as $\bar{B}$. Then we have

$$\text{Prob}(A_0, B) = \langle \bar{A}_0, \bar{B} \rangle \geq 1/\sqrt{2}. \quad (A2)$$

A similar argument exists if the second condition holds. Thus, our main result follows in the original context of the strategies as well, namely,

1. There exists $B \in B$ such that $\text{Prob}(A_0, B) \geq 1/\sqrt{2}$; or
2. There exists $A \in \bar{A}$ such that $\text{Prob}(B_0, A) \geq 1/\sqrt{2}$.

APPENDIX B: PROOF THAT ALICE AND BOB’S STRATEGIES ARE BOUNDED IF THEY HAVE NONEMPTY INTERIORS

Suppose that $\bar{A}$ and $\bar{B}$ are the convex sets representing Alice and Bob’s strategies, respectively. [Note that we are using the set of strategies after we have modded out by the equivalence relation (A1).] We now show that they are each bounded if they have nonempty interiors. To this end, it suffices to prove this for any norm since we are working in a finite-dimensional real vector space. Since $\bar{B}$ has nonempty interior, we can define a basis

$$\mathcal{I} = \{\bar{B}_1, \ldots, \bar{B}_n\},$$

where $\bar{B}_1, \ldots, \bar{B}_n \in \bar{B}$. Define the function

$$f(X) = \sup_{\bar{B} \in \mathcal{I}} |\langle X, \bar{B} \rangle|,$$

which is clearly non-negative and finite for all $X$.

Moreover, since $\langle \bar{A}, \bar{B} \rangle = \text{Prob}(A, B)$ for all $\bar{A} \in \bar{A}$ and $\bar{B} \in \bar{B}$, we have $f(\bar{A}) \leq 1$ for all $\bar{A} \in \bar{A}$. Thus, all that remains is to prove that $f$ is a valid norm.

We now show the triangle inequality. For any $X$ and $Y$, we have

$$f(X + Y) = \sup_{\bar{B} \in \mathcal{I}} |\langle X + Y, \bar{B} \rangle|$$

$$\leq \sup_{\bar{B} \in \mathcal{I}} (|\langle X, \bar{B} \rangle| + |\langle Y, \bar{B} \rangle|)$$

$$\leq \sup_{\bar{B} \in \mathcal{I}} |\langle X, \bar{B} \rangle| + \sup_{\bar{B} \in \mathcal{I}} |\langle Y, \bar{B} \rangle|$$

$$= f(X) + f(Y). \quad (B6)$$

Thus, the triangle inequality holds. For any scalar $\alpha \in \mathbb{R}$, we have

$$f(\alpha X) = \sup_{\bar{B} \in \mathcal{I}} |\langle \alpha X, \bar{B} \rangle|$$

$$= \sup_{\bar{B} \in \mathcal{I}} (|\langle \alpha \rangle| |\langle X, \bar{B} \rangle|)$$

$$= |\alpha| \sup_{\bar{B} \in \mathcal{I}} |\langle X, \bar{B} \rangle|$$

$$= |\alpha| f(X). \quad (B10)$$

The last property to show is when $f(X) = 0$, we must have $X = 0$. Let $X$ be such that $f(X) = 0$. Then clearly we must have $\langle X, \bar{B} \rangle = 0$ for all $\bar{B} \in \mathcal{I}$. Since $\mathcal{I}$ is a basis we have $\langle X, Y \rangle = 0$ for all vectors $Y$. By setting $Y = X$, we have that $\|X\| = 0$ implying that $X = 0$, as desired.
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