A C++ INCARNATION OF ZERNIKE CIRCLE FUNCTIONS

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Abstract. An explicit C++ library is provided which deals with Zernike Functions over the unit circle as the main subject. The implementation includes basic means to evaluate the functions at points inside the unit circle and to convert the radial and azimuthal parameters to Noll's index and vice versa. Advanced methods allow to expand products of Zernike Functions into sums of Zernike Functions, and to convert Zernike Functions to polynomials over the two Cartesian coordinates and vice versa.

1. Basis Functions

The Zernike circle functions \( Z_{n}^{(m)}(r, \varphi) \) are products of a radial polynomial \( R_{n}^{(m)}(r) \) and an azimuthal sine or cosine function \( A_{m}(\varphi) \):

\[
Z_{n}^{(m)}(r, \varphi) = R_{n}^{(m)}(r)A_{m}(\varphi),
\]

where

\[
r = \sqrt{x^2 + y^2}, \quad 0 \leq r \leq 1
\]

is the distance to the origin of coordinates and

\[
x = r \cos \varphi; \quad y = r \sin \varphi
\]

define the azimuth angle \( \varphi \). \( n \) is the degree of \( R \), a non-negative integer. \( m \) is one of \(-n, -n+2, \ldots, n-2, n\), such that \( n - m \) is an even integer number. The normalization chosen here is [4]

\[
\int_{0}^{1} r R_{n}^{(m)}(r) R_{n'}^{(m)}(r) dr = \delta_{n,n'};
\]

\[
\int_{0}^{2\pi} A_{m}(\varphi) A_{m'}(\varphi) d\varphi = \delta_{m,m'},
\]

such that

\[
R_{n}^{(m)}(r) \equiv \sqrt{2n + 2(-1)^{n-|m|}/2} \left( \frac{n + |m|}{n - |m|} \right)^{1/2} r^{|m|/2} 2F1\left( -\frac{n - |m|}{2}, 1 + \frac{n + |m|}{2}; 1 + |m|; r^2 \right),
\]

and

\[
A_{m}(\varphi) = \begin{cases} 
\cos(m\varphi)/\sqrt{\pi m}, & m \geq 0; \\
\sin(|m|\varphi)/\sqrt{\pi |m|}, & m < 0.
\end{cases}
\]

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where

\[ \epsilon_m \equiv \begin{cases} 2, & m = 0; \\ 1, & m \neq 0. \end{cases} \]

**Remark 1.** This is slightly different from the (more common) notation in my earlier representation \([2, 3]\); therefore some of the equations are reproduced here where factors need to be replaced.

The product expansions of the azimuth functions are \([2, \S II.E]\):

\[ A_m A_{m'} = \frac{1}{2\sqrt{\epsilon_m \epsilon_{m'}}} \left\{ \begin{array}{ll} \sqrt{\epsilon_{m-m'}} A_{m-m'} + \sqrt{\epsilon_{m+m'}} A_{m+m'}, & m \geq 0, m' \geq 0; \\ A_{-|m+m'|} - \text{sgn}(m - |m'|) A_{-|m-m'|}, & m \geq 0, m' < 0; \\ \sqrt{\epsilon_{m-m'}} A_{m-m'} - A_{m+m'}, & m < 0, m' < 0. \end{array} \right. \]

The product expansions of the radial functions are:

\[ R_n^m(r) R_{n'}^{m'}(r) = \sum_{n''=m''}^{n+n'} g_{n,m,n',m'',m''} R_{n''}^{m''}(r), \]

with projections \([2, \S II.E]\)

\[ g_{n_1,m_1,n_2,m_2,n_3,m_3} = \int_0^1 r \prod_{j=1}^3 R_{n_j}^{m_j}(r) dr = \sqrt{8} \prod_{j=1}^3 (n_j + 1) \sum_{s_1=0}^{n_1} \sum_{s_2=0}^{n_2} \sum_{s_3=0}^{n_3} \frac{1}{2 + n_1 + n_2 + n_3 - 2(s_1 + s_2 + s_3)} \times \prod_{j=1}^3 (-)^{s_j} \binom{n_j - s_j}{s_j} \binom{n_j - 2s_j}{s_j} \binom{n_j - |m_j|/2}{s_j} \binom{n_j - |m_j|/2 - s_j}{s_j}. \]

The class functions which are described in the next section answer the following questions:

1. Given the numerical coefficients \(c_{n,m}\) and \(c_{n',m'}\), what are the values of the linearization coefficients \(c_{n',m'}\) in \(\sum_{n,m} c_{n,m} Z_n^{(m)} \sum_{n',m'} c_{n',m'} Z_{n'}^{(m')}\)?
2. Given the numerical coefficients \(c_{p,q}\), what are the values of the coefficients \(c_{n,m}\) in \(\sum_{p,q} c_{p,q} x^p y^q = \sum_{n,m} c_{n,m} Z_n^{(m)}\)?
3. Vice versa, given the numerical coefficients \(c_{n,m}\), what are the values of the coefficients \(c_{p,q}\) in \(\sum_{n,m} c_{n,m} Z_n^{(m)} = \sum_{p,q} c_{p,q} x^p y^q\)?

2. Implementation

2.1. Points, Locations. Points in the unit circle and in the unit sphere are represented by their 2 to 3 Cartesian coordinates and implemented in the classes **Point2D** and **Point3D**. The constructors expect the 2 or 3 Cartesian coordinates that fix the point in space. The **Point2D** object has a trivial method to extract the radial distance \(r\) and the azimuth angle \(\varphi\) of the circular coordinates from the \(x\) and \(y\) components.
2.2. Polynomials. A single term $c x^j$ of a univariate polynomial is represented by an object of the class Monomial1D. A single term $c x^p y^q$ of a bivariate polynomial is represented by an object of the class Monomial2D. A single term $c x^p y^q z^r$ of a trivariate polynomial is represented by an object of the class Monomial3D. The constructors accept the exponents $j$, the exponents $p$ and $q$ and the exponents $p$, $q$ and $r$ respectively. The coefficient $c$ is optional and set to unity if missing in the constructor. The useful operations within these classes are multiplication with or division through a constant value (which scales the coefficient), and multiplication with a single term of the same type (which essentially adds the exponents of the two factors). These operations are implemented by overloaded multiplication and division operators. There is a common method $\text{at}()$ which evaluates the term given a value $x$ on the line or a position specified by a $\text{Point2D}$ or $\text{Point3D}$ in the plane or in three-dimensional space.

Univariate polynomials $\sum_j c_j x^j$, bivariate polynomials $\sum c_{p,q} x^p y^q$ and trivariate polynomials $\sum c_{p,q,r} x^p y^q z^r$ are represented as vectors of the monomial objects in the classes Polynomial1D, Polynomial2D and Polynomial3D. Addition, subtraction and multiplications within each of them are closed operations and implemented by overloaded addition, subtraction and multiplication operators. Adding new terms is supported with overloaded $+=$ operators that accept a single monomial object or another polynomial object of the same dimension.

There is a common method $\text{at}()$ which evaluates these polynomials given a value $x$ on the line or a position specified by a $\text{Point2D}$ or $\text{Point3D}$. The function sums up the single-term components.

A special variant of the univariate polynomials are the terminating Gaussian Hypergeometric Functions $\, _2\! F_1(a, b; c; z)$ with integer parameter $a \leq 0$, two auxiliary parameters $b$ and $c$, as a function of the variable $z$. These are implemented as a class Hypergeom21 derived from Polynomial1D.

2.3. Zernike Radial Function. A radial function $R_{m,n}^m(r)$ is special case of the Hypergeometric Function constructed as in Equation (6) given the parameter $n \geq 0$ and the parameter $m$ (the latter being referred to only as $|m|$). It is represented by an object of the class ZernikeRadi, a subclass of Hypergeom21. A method in the class computes the $g$-factors of Equation (11) to support expansion of products of $R$-functions in other $R$-functions.

The implementation is wider than actually needed here: an additional argument (which defaults to 2) supports use of the radial function in $D \geq 2$ dimensions, where $D$ appears on the right hand sides of Eqs. (6) and (11) [3].

2.4. Zernike Azimuthal Function. A sine or cosine term of the form $c A_m$ is represented by an object of the class ZernikeCircAzi, which is constructed given the signed integer parameter $m$ and an optional prefactor $c$—which is set to unity of missing. Scaling of the prefactor by a constant is implemented by overloaded multiplication and division. Evaluation at some explicit angle $\varphi$ happens by calling the $\text{at}()$ member function with an argument $\varphi$ in units of radians.

A collection (arithmetic sum) of the form $\sum_m c_m A_m(\varphi)$ is represented by an object of the class ZernikeCircAziVec, where each component is stored as an element of the ZernikeCircAzi type. Adding new terms is achieved by using the
overloaded `+` operator. Scaling all terms (prefactors) at the same time is supported by overloaded `*=` operation with a constant. An overloaded multiplication implements the arithmetic multiplication by means of Equation (9).

2.5. Zernike Circle Function. A Zernike Circle Function \( cZ_n^m \) is represented by an object of the class `ZernikeCirc` which is initialized by the radial parameter \( n \), the azimuthal parameter \( m \) and an optional prefactor \( c \)—which is set to unity if missing. Violation of the two constraints on \( m \) (parity and range) are silently caught by setting the prefactor to \( c = 0 \). Instead of the two parameters \( n \) and \( m \), Noll’s index \( j \geq 1 \) can also be used to define an object of `ZernikeCirc`.

Scaling of the prefactor \( c \) is supported by overloaded `*=` and `/=` operators.

Evaluation of the function at some point in the unit circle is done by calling the `at()` member function with an argument/location specified by a `Point2D` object. The implementation constructs the factors \( R_m^n \) and \( A_m^n \) with objects of the `ZernikeRadi` and `ZernikeCircAzi` types and multiplies their values. A design decision is to set the values to zero outside the unit circle, \( r > 1 (!) \).

An arithmetic sum of the form \( \sum c_{n,m} Z_n^m \) is represented by an object of the `ZernikeCircVec` class, which represents the components as members of a vector of `ZernikeCirc`'s. Adding terms to the sum is supported by overloaded `+` operators for additional `ZernikeCirc` or `ZernikeCircVec` terms. Scaling all the coefficients \( c_{n,m} \) is achieved by overloaded `*=` and `/=` operations with constant arguments.

The most valuable functions of the implementation are:

1. The arithmetic product of two Zernike expansions is implemented by the overloaded operator `*=` which works with two factors of the `ZernikeCirc` or `ZernikeCircVec` type and produces a `ZernikeCircVec` object. This is an incarnation of [2, §II.E]; it starts with the product expansion of the \( A \)-terms provided the `ZernikeCircAziVec` class and collects the product expansion of the \( R \)-terms by calling the `g` member functions in the `ZernikeRadi` class.

2. The conversion of a polynomial \( \sum c x^p y^q \) form into a Zernike expansion is supported by constructors in the `ZernikeCircVec` class that accept arguments of the `Monomial2D` or `Polynomial2D` type. This implements [2, §II.C].

3. The conversion of a Zernike expansion \( \sum c_{n,m} Z_n^m \) into a polynomial \( \sum c x^p y^q \) is supported by constructors in the `Polynomial2D` class that accept arguments of the `ZernikeCirc` or `ZernikeCircVec` type. This implements [2, §II.D].

2.6. Polynomial Fit. If the GNU Scientific Library is available [1], an additional class `PowFit2D` is introduced, which is fed with (constructed from) a list of Cartesian \( x \) and \( y \) coordinates and function values \( f(x,y) \) at these points. There are two simple formats of entering the data into the constructor, one that reads the \( x, y, f \) triples from an ASCII file, the other providing them as a list of `Point3D` points (interpreting the third coordinate as \( f \)).

The major member function is the `fit()` function which constructs the ordinary least squares fit through the \( f \) values up to some total order of the fitting polynomial \( \sum c x^p y^q \). The limiting order \( p+q \) is an argument of the `fit()` function. This is implemented as another constructor for the `Polynomial2D` class that admits a `PowFit2D` object as its argument. Fitting of scattered data over the unit circle to Zernike Circle
Polynomials for some upper limit of the index $n$ is then a matter of converting the fitting $\text{Polynomial2D}$ to a $\text{ZernikeCircVec}$ object in a final step.

Note that this is a demonstration of library usage, but the strategy is inefficient. In practise, the Zernike coefficients would be obtained directly by solving the linear algebra in the $(r, \varphi)$ coordinates.

**References**

1. GNU scientific library, 2015, http://www.gnu.org/software/gsl/.
2. Richard J. Mathar, *Zernike basis to cartesian transformations*, arXiv:0809.2368 [physics/optics] (2008).
3. ———, *Zernike basis to cartesian transformations*, Serb. Astr. J. 179 (2009), 107–120.
4. Robert J. Noll, *Zernike polynomials and atmospheric turbulence*, J. Opt. Soc. Am. 66 (1976), no. 3, 207–211.
5. GNU operating system, GNU automake, 2015, http://www.gnu.org/software/automake/.

**Appendix A. Installation**

**A.1. Compilation.** The source code of roughly 4300 lines is available in the `anc` directory, licensed under the GNU General Public License.

It is compiled with the GNU autotools [5] via

```
autoreconf --i
./configure --prefix=$HOME
make
make install
```

This bundles the classes in a library `libZernikeCirc.a` and compiles the test routine `tstZernikeCirc`. It also searches for the `gsl` and `gslcblas` libraries and includes `PowFit2D` if these are found.

If the autotools are not available, compilation in the conventional style of

```
g++ -c -O2 [A-Z]*.cxx
ld -i -o libZernikeCirc.a *.o
g++ -o tstZernikeCirc tstZernikeCirc.cxx -L. -l ZernikeCirc
```

is an alternative. Definition of the preprocessor variables `HAVE_GSL_GSL_SF_H` and `HAVE_GSL_GSL_LINALG_H` and adding the flags `-lgsl -lgslcblas` must be done manually then, if applicable.

If doxygen is available, the API documentation can be constructed in the `html` directory with

```
make doc
doc -o html/index.html
```

**A.2. Numerical Tests.** The test program can be run with `tstZernikeCirc`

The test suite contains

1. a table of mappings of the two parameters $(n, m)$ onto Noll’s index $j$ to test the `nollIdx` member function of `$\text{ZernikeCirc}$`;
2. a table of mappings of Noll’s index $j$ to the two parameters $(n, m)$ to test basically the inverse functionality in the constructor of `$\text{ZernikeCirc}$`;
3. a double loop over pairs of objects of `$\text{ZernikeCirc}$` to test that their products `$\text{ZernikeCircVec}$` have the same value at some `$\text{Point2D}$` as expected from the product of the individual values;
(4) a loop over various terms of the \(cx^py^q\) format to test that the conversion of a \texttt{Monomial2D} object into a \texttt{ZernikeCircVec} object keeps its value for some points \texttt{Point2D} scattered in the unit circle;

(5) a loop over various terms of the \(cZ_n^{(m)}\) format to test that the conversion of a \texttt{ZernikeCirc} object into a \texttt{Polynomial2D} object keeps its value for some points \texttt{Point2D} scattered in the unit circle.

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