Study of a Scalar Field on the Maximally Extended Schwarzschild Spacetime

C Buss¹, M Casals¹,²

¹Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150 - Urca Rio de Janeiro, Brasil
²School of Mathematics & Statistics and Complex & Adaptive Systems Laboratory, University College Dublin, Belfield, Dublin 4, Ireland
E-mail: cbuss@cbpf.br

Abstract. In this paper we present preliminary results about the investigation of scalar perturbations both inside and outside a Schwarzschild black hole. We express the wave equation in Ingoing Eddington-Finkelstein coordinates (explicitly showing its separability), write local series expansions for mode solutions and use them to then numerically calculate scalar perturbations inside and outside the future event horizon. Our main motivation is to calculate, in the future, the two-point quantum correlator in Schwarzschild spacetime, with points on opposite sides of the horizon.

1. Introduction

The study of linear field perturbations of black hole spacetimes is important for understanding the properties of black holes, both classical and quantum. Most studies of black hole spacetimes have concentrated on the exterior region of the black hole (see, e.g., [1–6]). Indeed, standard coordinate systems usually only cover the exterior region. There exist, however, coordinate systems which cover both the exterior and the interior regions of the black hole. Such systems are naturally suited for studying field perturbations propagating across the event horizon of the black hole. Investigating field perturbations inside the horizon is useful for various purposes. For example, classically, it has been shown that field perturbations across the event horizon lead to instabilities in the interior of certain black holes (see, e.g., [7–9]). Quantum-mechanically, the study of objects such as the two-point quantum correlator, with one point inside the horizon and the other point outside, may yield an insight into properties of Hawking radiation. This is precisely the main motivation on the base of our work.

Some studies of the quantum correlator have already been carried out in toy-model black hole spacetimes (e.g., [10,11]). Our aim is the calculation of the quantum correlator for a scalar field in a non-rotating (Schwarzschild) black hole spacetime, with points on opposite sides of the horizon. With this aim in mind, in this paper we present preliminary results for the calculation of scalar field perturbations outside as well as inside a Schwarzschild black hole. For this purpose, we have chosen to work with Ingoing Eddington-Finkelstein (IEF) coordinates as opposed to the more widely used Schwarzschild (Schw) coordinates. The reason being that IEF coordinates not only render the wave field equation separable - as in Schwarzschild coordinates - but also they are regular across the (future) event horizon - as opposed to Schwarzschild coordinates.
Separability is a desirable property of the field equation as it facilitates the practical calculation of its solutions.

This paper is organized as follows. In Sec. 2 we introduce the maximal extension of Schwarzschild spacetime, Schwarzschild and IEF coordinate systems, the wave equation for a massless scalar field, and define various mode solutions. In Sec. 3 we present analytical results: we explicitly write out and separate the wave equation in IEF coordinates and we give series expansions for mode solutions. In Sec. 4 we present a numerical calculation of mode solutions both outside and inside the future event horizon of a Schwarzschild black hole.

2. Theoretical Background

Schwarzschild spacetime describes the (unique) spherically symmetric, asymptotically flat vacuum black hole solution of Einstein equations. These properties are manifest in the metric written in Schwarzschild coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$  \hspace{1cm} \text{(1)}$$

Here, $M$ is the mass of the black hole; $r$, outside the event horizon, is the radial coordinate associated with the circumference; $t$ is the time as measured by a stationary observer at radial infinity; and $\theta$ and $\phi$ are, respectively, polar and azimuthal angles. The event horizon is located at $r = 2M$; we note that we use geometrized units ($G = 1$ and $c = 1$). These coordinates manifestly become irregular at the horizon. It is known, however, that this spacetime can be maximally extended, that is, geodesics in the maximally extended spacetime either start/end at a curvature singularity or are “continuable infinitely with respect to its ‘natural length’” [13]. We show the maximal extension of Schwarzschild spacetime in the Penrose-Carter diagram of Fig. 1.

![Figure 1. Penrose-Carter diagram showing the maximal extension of the Schwarzschild manifold. Region II is the black hole interior, II’ is the white hole interior, while I and I’ are the external regions. $\mathcal{H}^+$ and $\mathcal{H}^-$ are the future and past event horizons, respectively. $\mathcal{J}^+$ and $\mathcal{J}^-$, future and past null infinities. $I^+$ and $I^-$ are the future and past timelike infinities, while $I^0$ are the spacelike infinities. Finally, $r = 0$ represent the curvature singularities. A massive particle in the neighborhood of an astrophysical Schwarzschild black hole would typically start its trajectory at region $I^-$ and either end up at $I^+$ without entering the black hole or else cross, perhaps, the future event horizon $\mathcal{H}^+$ entering region II. See [15].](image)

In this work, we will use IEF coordinates. In this coordinate system, the metric is written as

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$  \hspace{1cm} \text{(2)}$$

where $t$, $\theta$ and $\phi$ are as in Schwarzschild coordinates, while $v$ is an advanced-time coordinate, and is related to Schwarzschild coordinates by $v = t + r_*\Delta t$, with $r_* = r + \log\left|\frac{r-2M}{2M}\right| \in (-\infty, +\infty)$.

We note that Schwarzschild coordinates are irregular at $r = 2M$, i.e., both at $\mathcal{H}^+$ and at $\mathcal{H}^-$. On the other hand, the IEF coordinates are regular on the future event horizon $\mathcal{H}^+$, which
corresponds to the horizon of an astrophysical black hole, as well as throughout regions I and II of the Penrose-Carter diagram. In fact, it is apparent from the equations for radial null geodesics, \( dv/dr = 0 \) and \( dv/dr = 2(1 - 2M/r)^{-1} \), that this coordinate system describes well the infall of massless particles across \( H^+ \).

In this work, we consider a massless scalar field perturbation of Schwarzschild spacetime. This corresponds to a fixed background metric supporting a scalar field \( \varphi \) obeying the following (Klein-Gordon) wave equation:

\[
\nabla_\mu \nabla^\mu \varphi = 0. \tag{3}
\]

In Schwarzschild coordinates, this wave equation is separable, allowing for the decomposition of the field into mode solutions, i.e., solutions depending on the separation constants as

\[
\varphi(t, r, \theta, \phi) = \frac{e^{-i\omega t}}{r} R_{\omega l}(r) Y_{lm}(\theta, \phi), \tag{4}
\]

where \( \omega, l \) and \( m \) are the aforementioned separation constants, \( Y_{lm} \) is a spherical harmonic and \( R_{\omega l} \) satisfies a linear, second order (radial) ordinary differential equation.

One can find a complete basis of mode solutions of the wave equation by choosing any two radial null geodesics, \( dv/dr = 0 \) and \( dv/dr = 2(2M)4M \omega \), that this coordinate system describes well the infall of massless particles across \( H^+ \). This is given in Table 2.

### Table 1. Modes and Boundary Conditions - Region I.

| Mode       | Boundary Conditions           | Mode       | Boundary Conditions           |
|------------|------------------------------|------------|------------------------------|
| \( R_{\omega l}^m(r) \sim e^{-i\omega r} \) | \( R_{\omega l}^m(r) \sim B_{\text{out}}e^{i\omega r} + B_{\text{in}}e^{-i\omega r} \), \( r_s \to -\infty \) | \( R_{\omega l}^m(r) \sim e^{i\omega r} \), \( r_s \to \infty \) | \( R_{\omega l}^m(r) \sim B_{\text{out}}e^{i\omega r} + B_{\text{in}}e^{-i\omega r} \), \( r_s \to -\infty \) |
| \( A_{\text{out}}e^{i\omega r} + A_{\text{in}}e^{-i\omega r} \), \( r_s \to -\infty \) |                                                                 |                                                                 |                                                                 |

### Table 2. Modes and Boundary Conditions - Regions I and II.

| Mode       | Boundary Conditions           | Mode       | Boundary Conditions           |
|------------|------------------------------|------------|------------------------------|
| \( R_{\omega l}^m(r) \sim e^{-i\omega r} \) | \( r \to 2M^+ \) \( R_{\omega l}^m(r) \sim (r - 2M)4M \omega \) e^{-i\omega r} \), \( r \to 2M^+ \) | \( r \to 2M^- \) \( e^{4M \omega (2M - r)} \) e^{-i\omega r} \), \( r \to 2M^- \) |
| \( e^{-i\omega r} \), \( r \to 2M^- \) |                                                                 |                                                                 |                                                                 |

The quantities \( A_{\text{in/ou}} \) and \( B_{\text{in/ou}} \) in Tables 1 and 2 are complex-valued coefficients. It is worth noticing from Table 2 that the In and Out modes are continuously defined across \( r = 2M \).

As it has been mentioned, neither the Schwarzschild nor the IEF coordinate system covers the maximal extension of Schwarzschild spacetime. Indeed, in this work we only consider regions I and II of Fig. 1, which are the only regions that are expected to exist in the case of a spherical black hole created from gravitational collapse. We note, however, that it is possible to extend the modes from region I into region I’ by mirror-reflecting them with respect to the origin of the Penrose-Carter diagram (Fig. 1). See [16] for more details.

### 3. Analytical Results

In order to study the separability of Eq. (3), we use the ansatz \( \varphi(v, r, \theta, \phi) = e^{-i\omega \tau} \Psi_{\omega l}(r) \Theta_l(\theta) \Phi_m(\phi)/r \). It thus follows that \( \Psi_{\omega l}(r) = e^{-i\omega \tau} R_{\omega l}(r) \). We then obtain from Eq. (3) the following separable partial differential equation:

\[
\left(1 - \frac{2M}{r} \right) \Psi_{\omega l}'' + \left( \frac{2M}{r^2} - \frac{2i\omega}{r} \right) \Psi_{\omega l}' - \frac{2M}{r^3} + \frac{\csc^2(\theta) \Phi_m''}{r^2 \Phi_m} + \frac{\Theta_l''}{r^2 \Theta_l} + \cot(\theta) \Theta_l' = 0, \tag{5}
\]
where a prime on a function indicates derivative with respect to its argument. This yields the following radial equation:

\[
\left(1 - \frac{2M}{r}\right) \Psi''_{\omega l} + \left(\frac{2M}{r^2} - 2i\omega\right) \Psi'_{\omega l} - \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) \Psi_{\omega l} = 0.
\]  

When analysing the singular structure of this equation, we find that \( r = 0 \) and \( r = 2M \) are regular singular points and \( r = \infty \) is an irregular singular point. This means that Eq. (6) has the same singular structure as the radial differential equation in Schwarzschild coordinates.

The boundary conditions of Tables 1 and 2 are given at the singular points \( r = 2M \) and \( r = \infty \). However, we wish to numerically integrate the radial equation by imposing boundary conditions on the solution away from a singular point. For this purpose we use the following local expansions of the radial solutions.

About \( r = 2M \), we carry out a Fröbenius expansion, thus obtaining two series of the form

\[
\Psi_{\omega l}(r) = \sum_{n=0}^{\infty} a_{n\omega l}(r - 2M)^n + \alpha,
\]

where \( \alpha = 0 \) or \( 4iM\omega \), while the coefficients \( a_{n\omega l} \) can be derived by applying the ansatz (7) to Eq. (6). It is clear from Table 1 that \( \alpha = 0 \) corresponds to In modes and \( \alpha = 4iM\omega \) to Out modes.

Regarding the Up mode, its boundary conditions are set at future null infinity (\( \mathcal{J}^+ \)). Due to the irregularity of this singular point, we consider an ansatz of the form \( \Psi_{\omega l}(r) = e^{i\omega r}e^{-V(r)/r} \) and require that \( V(r) \to 0 \) as \( r \to \infty \) [18]. It is straightforward to obtain an expansion of \( V(r) \) about \( r = \infty \) up to an appropriate order. We note that, although such expansion is only asymptotic, it is nevertheless useful for setting the boundary conditions away from \( r = \infty \).

### 4. Numerical Results

The results presented in this section will be restricted to a particular mode. This mode is \{\( l = 2, \omega \approx 0.0070/M \}\}. All graphs have the same structure: dotted lines represent the real part of a function, while continuous lines represent the imaginary part.

We use the expansion (7) and the boundary conditions as given by Tables 1 and 2. Again, we give special attention to the prescription given in [17] in order to integrate numerically Eq. (6) across the future event horizon.

**Figure 2.** From left to right: In, Out and Up radial functions in the exterior region of the black hole spacetime. Dotted lines represent the real part of a function, while continuous lines represent its imaginary part. We refer the reader to Fig. 3 about the boundedness of the Up radial function.
We also calculated the Wronskian between modes in the exterior region of the black hole. Since only the Wronskian of solutions of the radial equation obtained in Schwarzschild coordinates is constant, we use the relation between $\Psi_{\omega l}(r)$ and $R_{\omega l}(r)$ to obtain the expression

$$W_{\text{Schw}}(R_{\omega l}^{1}, R_{\omega l}^{2}) = e^{-2i\omega r} \left( 1 - \frac{2M}{r} \right) W_{\text{IEF}}(\Psi_{\omega l}^{1}, \Psi_{\omega l}^{2}),$$

where $W(.,.)$ is the Wronskian. From Eq. (8) we have obtained the following values:

$$W_{\text{Schw}}(R_{\omega l}^{\text{in}}, R_{\omega l}^{\text{up}}) \approx (2.202 + i0.217)10^{6}M^{-1}$$
and

$$W_{\text{schw}}(R_{\omega l}^{\text{out}}, R_{\omega l}^{\text{up}}) \approx (2.210 + i0.112)10^{6}M^{-1}.$$

As previously said, any two types of mode solutions among In, Out and Up form a complete set of mode solutions of Eq.(6). This allows us to express $\Psi_{\omega l}^{\text{up}}$ as a linear combination of $\Psi_{\omega l}^{\text{in}}$ and $\Psi_{\omega l}^{\text{out}}$, with coefficients given by the expression

$$\Psi_{\omega l}^{\text{up}} = \frac{1}{2i\omega} W_{\text{Schw}}(R_{\omega l}^{\text{out}}, R_{\omega l}^{\text{up}}) \Psi_{\omega l}^{\text{in}} + \frac{1}{2i\omega} W_{\text{Schw}}(R_{\omega l}^{\text{in}}, R_{\omega l}^{\text{up}}) \Psi_{\omega l}^{\text{out}}. \tag{9}$$

Therefore, using Table 2 and Eq. (7), we are able to obtain In and Out modes in the interior region of the black hole, while Eq. (9) allows us to extend the Up mode across the event horizon. These results can be seen in the graphs of Fig. 3.

**Figure 3.** From left to right: In, Out and Up radial functions in the interior region of the black hole. Again, dotted lines represent the real part of a function, while continuous lines represent its imaginary part.

It is interesting to notice that, as expected from the definition of the boundary conditions in Table 2, the In radial function is smoothly defined across the event horizon, while the Out function is continuous, but has a cusp at the event horizon (see Fig. 3). Such feature is carried over to the Up mode via Eq. (9).

We also calculated the Wronskian between the pairs In-Up and Out-Up, this time across the horizon. We found a discontinuity at $r = 2M$, which was expected due to the mentioned cusp in the Out and Up modes. We have obtained the values of

$$W_{\text{Schw}}(R_{\omega l}^{\text{in}}, R_{\omega l}^{\text{up}}) \approx (2.631 + i0.237)10^{6}M^{-1}$$
and

$$W_{\text{schw}}(R_{\omega l}^{\text{out}}, R_{\omega l}^{\text{up}}) \approx (2.639 + i0.111)10^{6}M^{-1}$$ inside the horizon.

5. Conclusions

We were able to obtain various mode radial solutions and their Wronskians at high numerical precision in the exterior region of a Schwarzschild black hole. Using a prescription given by Damour and Ruffini [17], we also obtained mode solutions and their Wronskians *inside* the horizon.

Our perspective is to use this work to investigate an expected relationship between quantum modes in the internal and external regions of the Schwarzschild spacetime, as given by the two-point correlation function.
6. Acknowledgments
CB would like to thank Conselho Nacional de Desenvolvimento Científico e Tecnológico for the support given through a MSc scholarship.

Appendix A. Numerical Comparison and Tests
We have thoroughly tested the numerical solutions in order to guarantee numerical accuracy. We summarize some of these tests in Table A1.

| Object | Region | Test | Worst Relative Error |
|--------|--------|------|----------------------|
| $\Psi_{\text{in}}$ | Exterior | Series, Equation 7 | $O[10^{-19}]$ |
| $\Psi_{\text{in}}$ | Exterior | Numerical Integration similar to [20] | $O[10^{-29}]$ |
| $\Psi_{\text{in}}$ | Exterior | Jaffé Series [18] | $O[10^{-25}]$ |
| $\Psi_{\text{in}}$ | Exterior | Lhs of Equation 6 = 0 | $O[10^{-37}]$ |
| $\Psi'_{\text{in}}$ | Exterior | Derivative of series, Equation 7 | $O[10^{-17}]$ |
| $\Psi_{\text{out}}$ | Exterior | Numerical Integration similar to [20] | $O[10^{-29}]$ |
| $\Psi'_{\text{out}}$ | Exterior | Jaffé Series [18] | $O[10^{-17}]$ |
| $\Psi_{\text{in}}$ | Exterior | Lhs of Equation 6 = 0 | $O[10^{-37}]$ |
| $W_{\text{Schw}}(R_{\text{in}}, R_{\text{up}})$ | Exterior | $W_{\text{Schw}}'(R_{\text{in}}, R_{\text{up}}) = 0$ | $O[10^{-29}]$ |
| $W_{\text{Schw}}(R_{\text{out}}, R_{\text{up}})$ | Exterior | $W_{\text{Schw}}'(R_{\text{out}}, R_{\text{up}}) = 0$ | $O[10^{-30}]$ |
| $\Psi_{\text{in}}$ | Interior | Series, Equation 7 | $O[10^{-21}]$ |
| $\Psi_{\text{out}}$ | Interior | Lhs of Equation 6 = 0 | $O[10^{-34}]$ |
| $\Psi'_{\text{in}}$ | Interior | Derivative of series, Equation 7 | $O[10^{-22}]$ |
| $\Psi'_{\text{out}}$ | Interior | Numerical Integration similar to [20] | $O[10^{-19}]$ |
| $\Psi_{\text{out}}$ | Interior | Lhs of Equation 6 = 0 | $O[10^{-34}]$ |
| $W_{\text{Schw}}(R_{\text{in}}, R_{\text{up}})$ | Interior | $W_{\text{Schw}}'(R_{\text{in}}, R_{\text{up}}) = 0$ | $O[10^{-21}]$ |
| $W_{\text{Schw}}(R_{\text{out}}, R_{\text{up}})$ | Interior | $W_{\text{Schw}}'(R_{\text{out}}, R_{\text{up}}) = 0$ | $O[10^{-30}]$ |

Table A1. Numerical Tests

References
[1] Teukolsky S A 1973 Astrophys. J. 185 635
[2] Candelas P 1980 Phys. Rev. D 21 2185
[3] Leaver E W 1986 Phys. Rev. D 34 381
[4] Kay B S and Wald R M 1987 Classical and Quantum Gravity 4 893
[5] Whiting B F 1989 J. Math. Phys. 30 1301
[6] Sasaki M and Tagoshi H 2003 Living Rev. Rel. 6 6
[7] Chandrasekhar S and Hartle J B 1982 Proc. R. Soc. Lond. A 384 301
[8] Ori A 1992 Phys. Rev. Lett. 14 2117
[9] Poisson E and Israel W 1990 Phys. Rev. D 41 1796
[10] Schutzhold R and Unruh W G 2010 Phys. Rev. D 81 124033
[11] Parentani R 2010 Phys. Rev. D 82 025008
[12] Hawking S W 1974 Nature 248 (5443) 30
[13] Kruskal M D 1960 Phys. Rev. 119 1743
[14] Misner C W et al 1973 Gravitation (San Francisco: W. H. Freeman and Company)
[15] Frolov V P and Zelnikov A 2011 Introduction to black hole physics (Oxford: Oxford University Press)
[16] Frolov V P and Novikov I D 1997 Black hole physics: basic concepts and new developments, Fundamental Theories of Physics (Dordrecht: Kluwer Academic Publishers)
[17] Damour T and Ruffini R 1976 Phys. Rev. D 14 332
[18] Casals M and Ottewill A 2013 Phys. Rev. D 87 064010
[19] Philipp D and Perlick V 2015 Preprint gr-qc/1503.08101v1
[20] Wardell B et al 2014 Phys. Rev. D 89 084021