Holographic entanglement entropy of the near horizon 1/4 BPS F-D$p$ bound states

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Abstract

It was shown in \cite{1} that the near horizon limit of the 1/4 BPS threshold F-D$p$ (for $0 \leq p \leq 5$, $p \neq 4$) bound state solutions of type II string theories give rise to space-time metrics endowed with Lifshitz scaling along with hyperscaling violation. Here we compute the holographic entanglement entropy of this system for all $p \neq 4$ (for $p = 4$ the space-time has AdS\textsubscript{2} structure). For $p = 3, 5$, we get the expected area law behavior of the entanglement entropy. For $p = 0, 1$, the entanglement entropy has new area law violations and has the behavior which is in between the linear and logarithmic behaviors. For $p = 2$, we get a logarithmic violation of the area law. We also compute the entanglement entropy at finite temperature and show that as the temperature rises, the entanglement entropy makes a crossover to the thermal entropy of the system. We thus obtain the string theoretic realization of holographic EE and various of its aspects noted earlier for generic metric with hyperscaling violation.

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1 Introduction

An entanglement entropy (EE) is inherently a quantum mechanical concept and is defined as the von Neumann entropy of the reduced density matrix of a subsystem $A$ of the full quantum system obtained by taking the trace over the degrees of freedom of the subsystem $B$ complement to $A$. This measures how the two subsystems $A$ and $B$ are entangled with each other (see, for example, [2, 3, 4, 5, 6, 7, 8], including some reviews). The concept has found many applications in various branches of physics, particularly, in condensed matter physics [9, 10, 11, 12, 13]. It has been realized in the study of condensed matter systems that to describe quantum phases of matter and their transitions which occur at nearly zero temperature, the relevant quantity to characterize these phases are their patterns of entanglement and not the conventional order parameters [9, 12, 13]. Therefore, EE is potentially useful to study systems at or near quantum criticality.

For low dimensional ($d < 2$) quantum field theory or quantum many body systems it is known [4, 5] how to compute the EE, however, for higher dimensional ($d \geq 2$) quantum theory the computation is not easy for a generic subsystem $A$ even for free theories. Motivated by this and also by looking at the similarity with the Bekenstein-Hawking black hole entropy, Ryu and Takayanagi [14, 15] gave a prescription to obtain the EE in any dimensions using the holography and AdS/CFT [16, 17]. The idea is that the EE of a region $A$ (with boundary $\partial A$) in a time slice of $(d + 1)$ dimensional CFT can be calculated by the minimal area of the manifold $\gamma_A$ embedded in the time slice of the bulk AdS$_{d+2}$ spacetime, such that $\partial \gamma_A = \partial A$, and is given by the relation [14]

$$S_A = \frac{\text{Area}(\gamma_A^\text{min})}{4G_{N}^{d+2}}$$  \hspace{1cm} (1)

In (1) $\gamma_A^{\text{min}}$ denotes the manifold with minimal area whose boundary is $\partial A$. Also $G_N^{d+2}$ is the $(d + 2)$ dimensional Newton’s constant. The expression in (1) is known as the holographic entanglement entropy and in lower dimensions it has been checked to match with the known results of a quantum system [14]. In higher dimensions also $S_A$ calculated from the gravity side can be seen to give correct qualitative behavior expected from field theory results.

Although originally the holographic EE was given using AdS/CFT, but it is believed to hold even for the non-AdS/non-conformal correspondence or in general for gauge/gravity duality [18, 19, 20, 21, 22, 23]. Holographic EE has been shown to give the correct behavior even for the gravity theories having Lifshitz scaling symmetry as well as theories with hyperscaling violation [24, 25, 26, 27]. As some condensed matter systems near quantum critical point show such scaling behavior [28, 29], these gravity theories may describe the dual of those condensed matter systems and therefore studying their holographic EE can help us to understand various quantum phases of these systems. This has been addressed in some recent papers [26, 30, 31, 27].
In this paper we compute\(^3\) the holographic EE for the geometries obtained by taking the near horizon limit of 1/4 BPS, threshold F-Dp bound state solutions of type II string theories. In an earlier paper \([1]\) we have shown that these geometries show a Lifshitz scaling symmetry along with a hyperscaling violation and may be dual to some condensed matter systems near critical point. We find that for \(p = 3, 5\), the holographic EE has the usual area law behavior \([2, 3]\). However, for \(p = 0, 1, 2\), the area law is violated indicating new phases in dual field theory. For \(p = 0, 1\), we find that the holographic EE has a behavior which is in between linear and logarithmic behaviors shown earlier \([26]\) for a system with hyperscaling violations. Here we find a string theory realizations of that. For \(p = 2\), we get a logarithmic violation of the area law indicating that the corresponding dual system represents compressible metallic states with hidden Fermi surface \([30, 31]\). We also compute the holographic EE at finite temperature by extending our results to the non-extremal F-Dp bound state solutions. We study both the low temperature and the high temperature behavior of this EE and find that at high temperature the EE makes a crossover to the thermal entropy of the system \([32]\).

This paper is organized as follows. In section 2, we give the computation of holographic EE for the dimensionally reduced, near-horizon F-Dp geometry and discuss their behaviors in various cases. In section 3, we discuss the finite temperature extension of the holographic EE and discuss the low and high temperature behaviors. We conclude in section 4.

## 2 Entanglement entropy of F-Dp system

In \([1]\) we constructed the 1/4 BPS, threshold F-Dp (for \(0 \leq p \leq 5\)) bound state solutions of type II string theories. In the near horizon limit the string frame metric, the dilaton and the form-fields of these solutions in a suitable coordinate take the form

\[
d s^2 = Q_2^2 u^{2-p} \left[ -\frac{d t^2}{Q_1 Q_2 u^{4(5-p)}} + \sum_{i=1}^{p} \frac{(d x_i)^2}{Q_2 u^{2}} + \frac{(d x^{p+1})^2}{Q_1 u^{2}} + \frac{4}{(4 - p)^2} \frac{d u^2}{u^2} + d \Omega_{7-p}^2 \right]
\]

\[
e^{2 \phi} = \frac{Q_2^{3-p}}{Q_1} u^{1 - (6-p)(1-p)/(4-p)}
\]

\[
B_{[2]} = -\frac{1}{Q_1 u^{2(6-p)/(4-p)}} d t \wedge d x^{p+1}, \quad A_{[p+1]} = -\frac{1}{Q_2 u^{(6-p)/(4-p)}} d t \wedge d x^1 \wedge \cdots \wedge d x^p
\]  

(2)

Here \(Q_1, Q_2\) are the F-string and Dp-brane charges respectively. \(B_{[2]}\) is the NSNS 2-form which couples to F-string and \(A_{[p+1]}\) is the RR (\(p+1\))-form which couples to Dp-brane. Note

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\(^3\)We would like to emphasize that the holographic EE has been computed for the generic Lifshitz metric in \([24]\) (the special case of D3-D7 scaling solution is also discussed there) and for a generic metric with hyperscaling violation in \([26]\). However, it is not clear in \([26]\) how such a space-time, particularly, a Lifshitz metric with hyperscaling violation can be obtained from any fundamental theory. We here use the solutions of the type F-Dp, as mentioned below, of string theory and show how the various properties of the EE observed in \([26]\) can be realized for different values of \(p\).
that in (2) $p \neq 4$, since for $p = 4$, the metric has AdS$_2$ structure. Also in the above while for $p < 4$, $u \to 0$ corresponds to going to the UV and $u \to \infty$ corresponds to going to the IR, for $p > 4$, $u \to 0$ corresponds to going to the IR and $u \to \infty$ corresponds to going to the UV. It is clear from (2) that the part of the metric in square bracket is invariant under the scale transformations

$$t \to \lambda^{\frac{2(5-p)}{4-p}} t \equiv \lambda^z t, \quad x^{1,2,...,(p+1)} \to \lambda x^{1,2,...,(p+1)}, \quad u \to \lambda u$$

where $\lambda$ is a scale parameter and $z$ is the dynamical scaling exponent and has a value $2(5-p)/(4-p)$. However, the full metric is not scale invariant and therefore it has a hyperscaling violation. Now since the dilaton in this case is not constant, in order to find the hyperscaling violation exponent we have to compactify the metric on $S^{7-p}$ and then express the $(p+3)$-dimensional reduced metric in Einstein frame. It has the form,

$$ds^2_{p+3} = Q_1^{p+1} Q_2 u \frac{2^{2(p+1)} u^{(4-p)} (p-2)}{Q_1 Q_2 u^{(4-p)} (p+1)} \left[ -\frac{dt^2}{Q_1 Q_2 u^{(4-p)} (p+1)} + \frac{\sum_{i=1}^{p} (dx^i)^2}{Q_2 u^2} + \frac{(dx^{p+1})^2}{Q_1 u^2} + \frac{4}{(4-p)^2} \frac{du^2}{u^2} \right]$$

Now it is clear that the metric transforms under (3) as,

$$ds_{p+3} \to \lambda^{\frac{p(4-p)-(p-2)}{(4-p)(p+1)}} ds_{p+3} \equiv \lambda^{\theta/d} ds_{p+3}$$

where $\theta$ is the hyperscaling violation exponent which has the value $p - (p - 2)/(4 - p)$ in this case and $d = p + 1$ is the spatial dimension of the boundary theory.

In this section we compute the EE on a strip embedded on the boundary of the fixed time slice of the $(p+3)$-dimensional bulk geometry given in (4) using the proposal of Ryu and Takayanagi [14, 15]. We note that the supergravity solution (2) is valid as long as the effective string coupling $e^\phi \ll 1$ and the curvature of the metric $R \ll \ell_s^{-1}$, where $\ell_s$ is the fundamental string length. This gives restrictions on the radial parameter $u$ (which is related to the energy parameter in the boundary theory) in terms of the charges of the F-string and D$p$-branes and we will discuss them later.

To find the holographic EE, we first calculate the minimal area of the manifold embedded in the time slice of the background (4) bounded by the edge of $A$, i.e., $\partial A$ which is a strip given by $-\ell \leq x^{p+1} \leq \ell$ and $0 \leq x^i \leq L$, with $i = 1, 2, \ldots, p$. The area of the embedded manifold is,

$$\text{Area}(\gamma_A) = \int d^{p+1}x \sqrt{g}$$

where $g$ is the determinant of the metric induced on $\gamma_A$. For the strip just mentioned, the above area reduces to

$$\text{Area}(\gamma_A) = (Q_1 Q_2)^{\frac{1}{2}} L^p \int_{-\ell}^{\ell} dx^{p+1} u^{-\frac{2}{4-p}} \sqrt{1 + \frac{4 Q_1 u^2}{(4-p)^2}}$$
where \( \dot{u} = \partial u / \partial x^{p+1} \) and \( u(x^{p+1}) \) gives the embedding of the edge of the strip into the time slice of the bulk. Note that here the boundary is located at \( u = 0 \) for \( p < 4 \) and it is at \( u \rightarrow \infty \) for \( p > 4 \). Now to minimize the area \((\text{I})\) we use the equation of motion which has the form,

\[
\dot{u} = \frac{4 - p}{2\sqrt{Q_1}} \sqrt{\left(\frac{u_*}{u}\right)^{\frac{4}{4-p}} - 1}
\]

where \( u_* \) is the constant of motion. The solution of this equation has the form that \( u \) starts from 0 or \( \infty \) (for \( p < 4 \) or \( p > 4 \)) i.e., at the boundary and then comes all the way up to \( u = u_* \), where there is a turning point \( (\dot{u} = 0) \) and goes back again to \( u = 0 \) or \( \infty \). The equation \((\text{S})\) can be easily integrated to obtain the unknown constant of motion \( u_* \) in terms of the width of the strip \( \ell \) as,

\[
u_* = \frac{4\ell}{\sqrt{Q_1}\pi} \left( \frac{8-p}{4} \right)^{\frac{1}{4}} \Gamma \left( \frac{6-p}{4} \right)
\]

Now substituting \((\text{S})\) into \((\text{I})\) and replacing \( u \) by a dimensionless variable \( x = u/u_* \), we obtain the area as,

\[
\text{Area}(\gamma_A) = 2Q_1 Q_2^\frac{3}{2} L^p u_*^\frac{2-p}{4} \int_{u_{\min} u_*}^{u_{\max} u_*} dx \frac{x^2}{4 - p} \sqrt{1 - x^{4-p}}
\]

We would like to remark that the above area \((\text{I})\) is actually divergent near the boundary \( u = 0 \) for \( p < 4 \) and \( u \rightarrow \infty \) for \( p > 4 \). So, we put a cut-off \( u_{\min}, u_{\max} \) for \( p < 4 \) and \( p > 4 \) respectively. So, putting the proper integration limit, the holographic EE can be obtained from \((\text{I})\) as,

\[
S_A = \frac{1}{4G_N^{p+3}} \frac{2Q_1 Q_2^\frac{3}{2} L^p}{4 - p} u_*^\frac{2-p}{4} \int_{u_{\min} u_*}^{u_{\max} u_*} dx \frac{x^2}{4 - p} \sqrt{1 - x^{4-p}}
\]

where the lower limits \( u_{\min}/u_* \) and \( u_{\max}/u_* \) refer to \( p < 4 \) and \( p > 4 \) cases respectively. The integral in \((\text{I})\) can be easily evaluated for \( p \neq 2 \). Actually, for \( p \neq 2 \), we write the results in two parts, one away from the boundary which is in general finite and the other near the boundary when we take \( u_{\min} \rightarrow 0 \) (for \( p < 4 \)) or \( u_{\max} \rightarrow \infty \) (for \( p > 4 \)). To extract the part near the boundary we expand the square root in the integrand for \( x \) nearly zero for \( p < 4 \) and \( 1/x \) nearly zero for \( p > 4 \). The leading contribution of the integral near the boundary has the form,

\[
\int_{u_{\min} u_*}^{u_{\max} u_*} dx \frac{x^2}{4-p} \sqrt{1 - x^{4-p}} \approx \frac{4 - p}{2 - p} \left( \frac{u_{\min, \max}}{u_*} \right)^{\frac{2-p}{4-p}}
\]

The part away from the boundary is the regularized integral and has finite contribution given
by,
\[
\int_{0, \infty}^{1} \frac{dx}{x^{2/p} \sqrt{1 - x^{4/p}}} = \frac{4 - p}{2 - p} \left( \frac{u_{\min, \max}}{u_*} \right)^{2-p/4}
\]
\[
= \frac{4 - p}{4} \sqrt{\pi} \frac{\Gamma \left( \frac{2-p}{4} \right)}{\Gamma \left( \frac{4-p}{4} \right)}
\]
(13)

The lower limits \((0, \infty)\) in (13) refer to \(p < 4\) and \(p > 4\) cases. Therefore the holographic EE for the strip on the boundary of F-Dp solution has the form,
\[
S_A = \frac{Q_1 Q_2^{\frac{1}{4}}}{2G_N^{p+3}} L^p \left( \frac{u_{\min, \max}}{2 - p} \right)^{2-p/4} + \frac{\pi^2}{4} \frac{\Gamma \left( \frac{2-p}{4} \right)}{Q_1^{2/p} \Gamma \left( \frac{4-p}{4} \right)} \left( \frac{\Gamma \left( \frac{8-p}{4} \right)}{\Gamma \left( \frac{6-p}{4} \right)} \right)^{2-p/4} \ell^{2-p/4}
\]
(14)

In the above \(u_{\min, \max}\) refers to \(p < 4\) and \(p > 4\) cases. The holographic EE given in (14) is valid for \(p = 0, 1, 3,\) and 5. As we will discuss, for \(p = 2\), the holographic EE has a logarithmic violation of the area law [30, 31]. From (14) we see that for \(p = 0, 1\), the first term actually vanishes as we take \(u_{\min} \to 0\) and \(S_A\) is finite for these cases. For \(p = 0\), \(S_A \sim \ell^2\) and for \(p = 1\), \(S_A \sim L \ell^2\). In these cases the EE has the behavior which is in between the logarithmic and linear behaviors. In fact these two cases are where the hyperscaling violation exponent \(\theta = p - (p - 2)/(4 - p)\) lies between \(d = p + 1\) and \(d - 1 = p\), i.e. \(d - 1 < \theta < d\) and therefore as noted in [26], there are new violations of area law indicating that there are new phases for these kind of systems. On the other hand for \(p = 3\), we find \(S_A = \alpha_3 L^p/u_{\min}^{\theta-\phi} + \beta_3 L^p/\ell^{\theta-\phi}\), where \(\alpha_3\) and \(\beta_3\) are known constants (as given in (14)) and \(\theta = 2\). For \(p = 5\), we have \(S_A = \alpha_5 L^p/u_{\max}^{\theta-\phi} + \beta_5 L^p/\ell^{\theta-\phi}\), where \(\alpha_5, \beta_5\) are known constants (as given in (14)) and \(\theta = 8\). Thus for \(p = 3, 5\) we get the usual area law of the holographic EE [2, 2] for a system having Lifshitz scaling with hyperscaling violation.

For \(p = 2\), the relation (9) is still valid and from there we get
\[
u_* = \frac{2\ell}{\sqrt{Q_1}}
\]
(15)

However, the integration in (11) gives a logarithmically divergent contribution. Using (15), we find the holographic EE for \(p = 2\) case as,
\[
S_A = \frac{Q_1 Q_2^{\frac{1}{4}}}{4G_N^{3}} L^2 \log \frac{4\ell}{\sqrt{Q_1 u_{\min}}}
\]
(16)

We thus find that in this case the holographic EE gives a logarithmic violation of the area law. This was interpreted in the boundary theory as the presence of hidden Fermi surface [30, 31].

Note that the holographic EE (14), (16) have been calculated here using the supergravity configuration [2] which is valid under certain range of parameter \(u\) as mentioned earlier. These ranges have been discussed in [1] and here we discuss them in the context of EE.
Let us first consider the case of odd $p$. For $p = 1$, the gravity description (2) is valid when $Q_2 \ll Q_1$ and $u \gg 1/Q_2^{3/2} \gg 1/Q_1^{3/2}$ and so the EE (14) is valid also in this range. However, when $Q_2 \geq Q_1$, we have to go to the S-dual frame and the gravity description is then valid for $u \gg 1/Q_1^{3/2} \gg 1/Q_2^{3/2}$. We calculated the EE also in the S-dual frame and found that they have exactly the same form (14) as calculated from the original description except that the range of validity is different. Also the cut-off parameter $u_{\text{min}}$ can be taken close to the boundary only if we have both $Q_1, Q_2 \gg 1$. For $p = 3$, the supergravity description (2) as well as the holographic EE (14) are valid in the range $1/Q_1^{1/6} \ll u \ll Q_2^{1/2}$. Here $u_{\text{min}}$ can be taken close to the boundary if we assume $Q_1 \gg 1$. When $u \leq 1/Q_1^{1/6}$, we have to go to the S-dual description and again, as in the case of $p = 1$, we find that the holographic EE has exactly the same form (14) as in the original description. In the S-dual frame $u_{\text{min}}$ can be taken as close to the boundary as possible without any restriction to the charges. The same thing happens for $p = 5$ as well. Here the supergravity description (2) and the holographic EE (14) are valid in the range $Q_2^{1/6} \ll u \ll (Q_1 Q_2)^{1/4}$. In this case $u_{\text{max}}$ can be taken close to the boundary if $Q_1 Q_2 \gg 1$. However, when $u \geq (Q_1 Q_2)^{1/4}$, we have to go to the S-dual description. But the holographic EE has exactly the same form (14) as in the original description. In the S-dual description $u_{\text{max}}$ can be taken close to the boundary without any restriction to the charges.

For even $p$, the situation is slightly different. So, for example, for $p = 0$, the supergravity description (2) as well as the holographic EE (14) are valid in the range $1/Q_2 \ll u \ll Q_1^{2/3}/Q_2$ and the cut-off parameter $u_{\text{min}}$ can be taken close to the boundary if $Q_2 \gg 1$. However, when $u \geq Q_1^{2/3}/Q_2$, the dilaton is large and we have to uplift the solution to M-theory. Since here the boundary theory has one dimension higher, we find that the holographic EE in this case has exactly the same form (14) except that the overall factor $L^0/G_N^3$ is replaced by $L^1/G_N^1$. The same is true for $p = 2$ case as well. Here, the supergravity description (2) as well as the holographic EE (14) are valid in the range $u \gg Q_2^{1/4}/Q_1^{1/2}$ along with $Q_2 \gg 1$. $u_{\text{min}}$ in this case can be taken close to the boundary if $Q_1 \gg 1$ such that $Q_2^{1/4}/Q_1^{1/2} \ll 1$. However, when $u \leq Q_2^{1/4}/Q_1^{1/2}$ we have to uplift the solution to M-theory. Again the holographic EE can be found to have exactly the same form as (14) except that the overall factor $L^2/G_N^5$ is replaced by $L^3/G_N^6$.

Before we conclude this section, we here make a comparison of our results on holographic EE and those obtained in [26]. The metric used in [26] has a quite generic form and is not obtained from any fundamental theory. In particular, it is not known whether the space-time is in general stable or not. Whereas our metrics are obtained from the near horizon limit of some known string theory solutions [1] which preserve at least a quarter of the space-time susy and are therefore stable. Although the forms of the holographic EE given in eqs.(4.24) and (4.26) of [26] are similar to the forms we obtained in this paper in (14) and (16), the details are quite different. So, for example, the holographic EE in eqs.(4.24) and (4.26) of
depend on the AdS radius $R$ and a cross-over scale $r_F$, whereas, the holographic EE in this paper depends on two charge parameters $Q_1$ and $Q_2$ (see (14) and (16)) of the F-strings and D$p$-branes respectively. By comparing them we find that these parameters are related as $R^d/r_F^d = Q_1 Q_2^{d/2}$. However, to understand the precise relations between them, i.e., how $R$ and $r_F$ are separately related to the charges $Q_1$ and $Q_2$, we need to obtain the near horizon F-D$p$ solution as a deformation of a relativistic solution. This at present is not known. The boundary theory in [26] always lives at $r \to 0$ (in their notation), whereas in this paper the boundary theory lives at $u \to 0$ for $p < 4$, but lives at $u \to \infty$ for $p > 4$. Also note that the various holographic properties of the EE observed in [26] for generic metric are concretely realized in our string theory solutions for various values of $p$. Thus, for example, for $p = 3, 5$ ($p = 5$ case is not included in [26]), the EE has the usual area law, for $p = 2$, the EE has logarithmic violation of the area law and for $p = 0, 1$, the EE has new area law violations which is in between linear and logarithmic behaviors. The string theory realizations discussed in [26] are for $z = 1$, i.e., for the relativistic cases whereas the string theory realizations we have discussed are of Lifshitz-like and non-relativistic. Finally, as the solutions in [26] are not obtained from string theory, there is no restriction as such on the radial parameter. Whereas the solutions discussed in this paper are valid as long as the effective string coupling (the dilaton) and the curvature of the metric (in units of $\alpha'$) remain small. These give restrictions on the radial parameter $u$ as we have discussed. We find that even in the strongly coupled region the holographic EE have very similar structures as those of the original solutions.

3 Holographic EE at finite temperature

In this section we compute the holographic EE at finite temperature for the strip embedded on fixed time slice of the boundary of the F-D$p$ system. For this purpose we start from the non-extremal F-D$p$ bound state solutions and then take the near horizon limit. In a suitable coordinate the near horizon metric in the string frame and the dilaton of this solution take the form,

$$ds^2 = Q_2^2 Q_1^{-\frac{2}{1-p}} \left[ -\frac{f(u)dt^2}{Q_1 Q_2 u^\frac{4(5-p)}{4-p}} + \sum_{i=1}^{p} (dx_i)^2 + \frac{(dx_{p+1})^2}{Q_1 u^2} + \frac{4}{(4-p)^2} \frac{du^2}{f(u)u^2} + d\Omega_{7-p}^2 \right]$$

$$e^{2\phi} = Q_2^\frac{2-p}{2} Q_1^{-\frac{4-p}{4-p}} u^\frac{(6-p)(1-p)}{(4-p)}$$

(17)

where $f(u) = 1 - (u/u_h)^{2(6-p)/(4-p)}$ and $u = u_h$ is the radius of the event horizon. As before compactifying the above metric on $S^{7-p}$ and writing the resultant metric in Einstein frame
we get,
\[
 ds^2_{p+3} = Q_1^{p+1} Q_2 u^{2(4-p)-(p-2)} \left[ - \frac{f(u)dt^2}{Q_1 Q_2 u^{(4-p)}(4-p)} + \sum_{i=1}^{p-1} (dx^i)^2 + \frac{(dx^{p+1})^2}{Q_1 u^{2}} + \frac{4}{(4-p)^2} f(u) u^2 \right]
\]

From this black hole metric we can calculate its temperature and it has the form
\[
 T = \frac{6 - p}{4\pi (Q_1 Q_2)^{\frac{1}{2}} u_h^{1-p}}
\]

The thermal entropy or Bekenstein-Hawking entropy which is proportional to the area of the black hole can, therefore, be written as,
\[
 S_T = \frac{1}{4 G_N^{p+3}} (Q_1 Q_2)^{\frac{1}{2}} V \left( \frac{4\pi (Q_1 Q_2)^{\frac{1}{2}}}{6 - p} \right)^{\frac{1}{4-p}} T^{-\frac{1}{4-p}}
\]

Note that the expression (20) for the entropy holds good for $p < 5$. For $p = 5$, as we see from (19), the temperature does not depend on $u_h$ and therefore the thermal entropy in that case is independent of $T$ which implies that the specific heat vanishes. The specific heat is positive only for $p < 5$ cases and therefore we will restrict our discussion only for those cases.

We now calculate the holographic EE at finite temperature, for the same strip as in the previous section, from the near horizon non-extremal F-Dp configuration (18). The area expression (7) in this case will be modified as,
\[
 \text{Area}(\gamma_A) = (Q_1 Q_2)^{\frac{1}{2}} L^p \int_{-\ell}^{\ell} dx^{p+1} u^{-\frac{2}{4-p}} \sqrt{1 + \frac{4Q_1 \dot{u}^2}{(4-p)^2 f(u)}}
\]

The equation of motion (8), would therefore be modified as,
\[
 \dot{u} = \frac{4 - p}{2\sqrt{Q_1}} \sqrt{f(u)} \sqrt{\left( \frac{u_*}{u} \right)^{\frac{2}{4-p}} - 1}
\]

As before, introducing a dimensionless variable $x = u/u_*$, the above equation can be integrated to obtain,
\[
 \ell = \frac{\sqrt{Q_1}}{4 - p} u_* I \left( \frac{u_*}{u_h} \right)
\]

where,
\[
 I \left( \frac{u_*}{u_h} \right) = \int_0^1 dx \frac{x^{\frac{2}{4-p}}}{\sqrt{1 - \left( \frac{x u_*}{u_h} \right)^{2\frac{(4-p)}{4-p}} \sqrt{1 - x^{\frac{2}{4-p}}}}}
\]

Now writing $dx^{p+1} = du/\dot{u}$ in the area expression (21) and using $\dot{u}$ from (22), the area expression reduces to,
\[
 \text{Area}(\gamma_A) = \frac{2Q_1 Q_2^{\frac{1}{2}} L^p u_*^{\frac{2}{4-p}}}{4 - p} \tilde{I} \left( \frac{u_*}{u_h} \right)
\]
where

\[ \tilde{I} \left( \frac{u_s}{u_h} \right) = \int_{u_{\text{min}}/u_s}^{1} dx \frac{x^{-\frac{1}{4-p}}}{\sqrt{1 - \left( \frac{x u_s}{u_h} \right)^{2(6-p)/4-p}}} \sqrt{1 - x^{\frac{4}{4-p}}} \]  \hspace{1cm} (26)

Note that even though the integral (24) is convergent near the boundary \((u = 0)\), the integral (20) is divergent there and therefore we have put a cut-off \(u_{\text{min}}\) in \(\tilde{I}\). The finite temperature EE therefore takes the form,

\[ S_A^{\text{finite}} = \frac{Q_1 Q_2^\frac{3}{2} L^p u_s^{\frac{2-p}{p}}}{2 G_N^{p+3} (4 - p)} \tilde{I} \left( \frac{u_s}{u_h} \right) \]  \hspace{1cm} (27)

Now in order to express the finite temperature EE in terms of the temperature and \(\ell\), as in the case of thermal entropy (20), we have to evaluate \(\tilde{I}\). However, this integral can not be performed analytically. It can be evaluated only numerically. For that we have to first numerically integrate \(\tilde{I}\) in (24) to obtain \(u_s\) in terms of \(\ell\) and \(u_h\) (which in turn is related to the temperature \(T\) by the relation (19)) and then use that to numerically obtain \(S_A^{\text{finite}}\) from (27). Here we discuss the low and the high temperature behaviors of \(S_A^{\text{finite}}\) as done in [26].

From (19) we find that \(T \sim u_h^{-2(5-p)/(4-p)}\) and so for \(p < 4\), as \(u_h \to 0\), i.e. as \(u_h\) goes to the boundary, \(T \to \infty\) and as \(u_h \to \infty\), i.e. in the extremal limit, \(T \to 0\) as expected. It is, therefore, clear that in the small temperature regime, \(u_s/\ell \to 0\) as \(T \to 0\). Now in this approximation the integral (24) can be evaluated to the leading order to give \(u_s\) in terms of \(\ell\) which has the same form as given in eq. (19). Once we have this, the integral (26) can be evaluated by expanding the factor in the denominator \([1 - \left( \frac{x u_s}{u_h} \right)^{2(6-p)/4-p}]^{1/2}\) for small \(u_s/\ell\) which in turn gives \(S_A^{\text{finite}}\) in the form,

\[ S_A^{\text{finite}} \approx \frac{Q_1 Q_2^\frac{3}{2} L^p}{2 G_N^{p+3}} \left[ \frac{(u_{\text{min}})^{\frac{2-p}{4}}}{2 - p} + \frac{\pi^{-\frac{1}{4-p}}}{4 \pi^2 Q_1} \frac{(2-p)^{\frac{2-p}{4}}}{\Gamma \left( \frac{8-p}{4} \right)} \frac{(4\pi Q_2^{-\frac{3}{4}})}{\Gamma \left( \frac{4-p}{4} \right)} \frac{(8-p)^{\frac{2-p}{4}}}{\Gamma \left( \frac{8-p}{4} \right)} \right]^{\frac{2-p}{4-p}} \ell^{\frac{2-p}{4-p}} + \frac{\sqrt{\pi} \Gamma \left( \frac{14-3p}{4} \right)}{8 \Gamma \left( \frac{10-3p}{4} \right)} \left( \frac{4\pi Q_1 \Gamma \left( \frac{8-p}{4} \right)}{\sqrt{Q_1} \pi \Gamma \left( \frac{6-p}{4} \right)} \right)^{\frac{14-3p}{4}} \left( \frac{4\pi Q_2^6}{6 - p} \right)^{\frac{6-p}{4}} \ell^{\frac{2-p}{4-p}} \left( \ell T \right)^{\frac{2(6-p)}{4-p}} + \cdots \]  \hspace{1cm} (28)

The first two terms in (28) is precisely the zero temperature expression of the holographic EE given earlier in (14) and the last term is the finite temperature correction for small temperature. Note that here our result is valid only for \(p < 4\) and that is why in the first term we have only \(u_{\text{min}}\). We point out that the finite temperature correction we have obtained here has the same structure as given in [26]. Also note that for \(p = 2\), the zero temperature form of the holographic EE has a logarithmic violation of area law and is given in (16) but the correction term due to small temperature is still given by the third term in (28) with \(p = 2\).
On the other hand when the temperature is large, \( u_h \sim u_* \). In this case both the integrals \( I \) given in (24) and \( \tilde{I} \) given in (26) are dominated by the pole at \( u = 1 \) and therefore have the same values, i.e., \( I \approx \tilde{I} \). So, substituting \( I \) from (23) as \( I = (4 - p)\ell / (\sqrt{Q_1 u_*}) \) into the holographic EE at finite temperature in (27) and then expressing \( u_h \sim u_* \) in terms of temperature from (19) we obtain,

\[
S_A^{\text{finite}} \approx \frac{1}{4G_N^{p+3}} (Q_1 Q_2)^{\frac{1}{2}} (L^p 2\ell) \left( \frac{4\pi (Q_1 Q_2)^{\frac{1}{2}}}{6 - p} \right)^{\frac{1}{6 - p}} T^{\frac{1}{3 - p}} \tag{29}
\]

This has precisely the same form as the Bekenstein-Hawking entropy given in (20) if we identify \( (L^p 2\ell) = V \). We thus find that the holographic EE at finite temperature indeed makes a cross over to the thermal entropy as the temperature is increased [26, 32].

We remark that the finite temperature extensions of the holographic EE for the generic hyperscaling violating geometries have also been discussed in [26], we here give a concrete string theoretic realizations of those from the non-extremal F-Dp solutions. We have given the exact expressions of the low temperature (28) and the high temperature (29) behavior of the holographic EE, whereas, in [26], an approximate expressions are given showing the temperature dependence of the holographic EE at finite temperature. As before our expressions depend on the charges of the F-strings, \( Q_1 \) and Dp-branes, \( Q_2 \), but in [26], the corresponding expressions depend on the AdS radius \( R \) and the cross-over scale \( r_F \).

### 4 Conclusion

To conclude, in this paper we have computed the holographic entanglement entropy of the near horizon geometry of the threshold, 1/4 BPS F-Dp \((p \neq 4)\) bound state solutions of type II string theories using the prescription of Ryu and Takayanagi [14, 15]. The geometry was shown earlier to have a Lifshitz scaling with hyperscaling violation and the corresponding boundary theory may describe certain condensed matter systems near quantum critical point. We have computed the holographic EE for these systems for both zero and finite temperature. For \( p = 0, 1 \) we have found that the holographic EE are finite and have new area law violations having behaviors in between the linear and logarithmic behaviors. As indicated in [26], this may imply new phases in the dual theory and we have a string theory realizations of that. For \( p = 2 \), we have a logarithmic violations of area law and this indicates that the dual theory describes compressible metallic states with hidden Fermi surface. For \( p = 3, 5 \) we have the usual area law. We have extended our results for the non-extremal F-Dp system and obtained the Hawking temperature and the thermal or Bekenstein-Hawking entropy of these systems. We found that only for \( p < 4 \), the specific heat of the system is positive and so we calculated the holographic EE for F-Dp system only for \( p < 4 \) at finite temperature. It has been noted that in general the finite temperature holographic EE can not be expressed
in a closed analytic form. Some of the integrals in the expression can only be computed numerically. However, we have discussed the low and the high temperature behaviors of the Holographic EE expressions at finite temperature. We found that as the temperature of the system is increased the Holographic EE makes a cross over to the thermal entropy of the system indicating the existence of a cross over function interpolating between the entanglement and thermal entropy.

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