Sequential Systems of Reflected Backward Stochastic Differential Equations with Application to Impulse Control

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Abstract
We consider a system of finite horizon, sequentially interconnected, obliquely reflected backward stochastic differential equations (RBSDEs) with stochastic Lipschitz coefficients. We show existence of solutions to our system of RBSDEs by applying a Picard iteration approach. Uniqueness then follows by relating the limit to an auxiliary impulse control problem. Moreover, we show that the solution to our system of RBSDEs is connected to weak solutions of a stochastic differential game where one player implements an impulse control while the opponent plays a continuous control that enters the drift term. As all our arguments are probabilistic and hence hold in a non-markovian framework, we are able to consider the setting where the underlying uncertainty in the game stems from an impulsively and continuously controlled path-dependent stochastic differential equation driven by Brownian motion.

1 Introduction
Backward stochastic differential equations (BSDEs) has been a topic of rapid development during the last decades. Non-linear BSDEs were independently introduced in [24] and [7] and has since found numerous applications. El Karoui et. al. introduced the notion of reflected backward stochastic differential equations (RBSDEs) and demonstrated a link between RBSDEs and optimal stopping in [12]. This was later exploited in a series of articles [15, 18, 20] as a means of finding solutions to optimal switching problems. In [18] existence and uniqueness of solutions to an interconnected systems of reflected BSDEs were shown. Furthermore, it was shown that the solutions are related to optimal switching problems under Knightian uncertainty. Important contributions from the perspective of the present work are also
that consider BSDEs where the Lipschitz coefficient on the $z$-variable of the driver is a stochastic process and the more recent work presented in [10] where a RBSDE with stochastic Lipschitz coefficient is solved.

Although the literature on discretely indexed systems of RBSDEs related to switching problems has grown considerably in the last decade, there is this far no work that deals with systems of RBSDEs related to general impulse control problems. In the present work we aim to add to the literature on RBSDEs by considering a sequentially arranged system of RBSDEs, namely (the notation will be explained later)

\[
\begin{aligned}
Y^v_t &= \xi^v + \int_t^T f^v(s, Y^v_s, Z^v_s)ds - \int_t^T Z^v_s dW_s + K^v_T - K^v_t, \quad \forall t \in [0, T], \\
Y^v_t &\geq \sup_{b \in U} \{Y^v_{\psi(t,b)} - c^v(t, b)\}, \quad \forall t \in [0, T], \\
\int_0^T (Y^v_t - \sup_{b \in U} \{Y^v_{\psi(t,b)} - c^v(t, b)\})dK^v_t &= 0.
\end{aligned}
\tag{1.1}
\]

As opposed to the setting in previous works our family of RBSDEs is continuously parameterized (the parameter $v$ is an impulse control). Moreover, to make our results more applicable, we allow the driver $f^v$ to satisfy a Lipschitz condition on the $z$-variable that is formulated in terms of a stochastic process. We rely on a Picard iteration approach and the main obstacle we face is showing continuity of the map $(t, b) \mapsto Y^v_{\psi(t,b)}$ that appears in the barrier to (1.1). In particular, we cannot use a “no free loop” property (see e.g. Step 5 in the proof of Theorem 3.2 in [18]). Instead we rely on a uniform convergence argument that requires some intricate analysis.

A strong motivation for the introduction of non-linear BSDEs was their close connection to various types of stochastic control problems (see e.g. [13, 25, 30]) and stochastic differential games [16, 17, 19]. Analogously, our main motivation for studying (1.1) is its relation to stochastic differential games (SDGs) of impulse versus continuous control.

In impulse control the control-law takes the form $u = (\tau_1, \ldots, \tau_N; \beta_1, \ldots, \beta_N)$, where $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_N$ is a sequence of times when the operator intervenes on the system and $\beta_j$ is the impulse that the operator affects the system with at time $\tau_j$.

We restrict our attention to the case of a Brownian filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}$ and assume that the $\tau_j$ are $\mathbb{F}$-stopping times and that $\beta_j$ is $\mathcal{F}_{\tau_j}$-measurable and take values in a compact subset $U$ of $\mathbb{R}^d$.

We extend the results in [22] to the two-player, zero-sum game setting by considering a weak formulation of the problem of maximizing the reward functional

\[
J(u, \alpha) := \mathbb{E}\left[ \int_0^T \phi(t, X^{u,\alpha}_t, \alpha_t)dt + \psi(X^{u,\alpha}_T) - \sum_{j=1}^N \ell(\tau_j, X^{u,\alpha}_{\tau_j-1} , \beta_j) \right] \tag{1.2}
\]

over impulse controls, $u$, when simultaneously a minimization is performed over continuous controls $\alpha := (\alpha_s)_{0 \leq s \leq T}$, taking values in a compact subset $A$ of $\mathbb{R}^d$. Here, $[u]_j := (\tau_1, \ldots, \tau_{N \wedge j}; \beta_1, \ldots, \beta_{N \wedge j})$ and $X^{u,\alpha}$ solves the impulsively controlled path-dependent SDE......
for $j = 1, \ldots, N$, with $\tau_{N+1} := \infty$. By considering systems of reflected BSDEs with drivers that satisfy a stochastic Lipschitz condition we are able to relax the common assumption that $|\sigma^{-1}(t, x)a(t, x, \alpha)|$ is bounded and instead assume a linear growth, i.e. that $|\sigma^{-1}(t, x)a(t, x, \alpha)| \leq k_L (1 + \sup_{s \in [0, t]} |x_s|)$, for some constant $k_L > 0$.

The remainder of the article is organised as follows. In the next section we set the notation and specify what we mean by a solution to (1.1). Moreover, in this section we relating solutions to (1.1) to weak formulations of the SDG at hand. This is followed by some concluding remarks in Section 5.

### 2 Preliminaries

We let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space, where $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ is the augmented natural filtration of a $d$-dimensional Brownian motion $W$ and $\mathcal{F} := \mathcal{F}_T$, where $T \in (0, \infty)$ is the horizon.

Throughout, we will use the following notation:

- We let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$ and for any other probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, we denote by $\mathbb{E}^\mathbb{Q}$ expectation with respect to $\mathbb{Q}$.
- $\mathcal{P}_\mathbb{F}$ is the $\sigma$-algebra of $\mathbb{F}$-progressively measurable subsets of $[0, T] \times \Omega$.
- For $p \geq 1$, we let $\mathcal{S}_p^\mathbb{F}$ be the set of all $\mathbb{R}$-valued, $\mathcal{P}_\mathbb{F}$-measurable, continuous processes $(Z_t : t \in [0, T])$ such that $\|Z\|_{\mathcal{S}_p^\mathbb{F}} := \mathbb{E}\left[\sup_{t \in [0, T]} |Z_t|^p\right]^{1/p} < \infty$.
- For $p \geq 1$, we let $\mathcal{S}_p^p$ be the set of all $\mathbb{R}$-valued, $\mathcal{P}_\mathbb{F}$-measurable, càglàd processes $(Z_t : t \in [0, T])$ such that $\|Z\|_{\mathcal{S}_p^p} < \infty$.
- We let $\mathcal{H}_p^\mathbb{F}$ denote the set of all $\mathbb{R}^d$-valued $\mathcal{P}_\mathbb{F}$-measurable processes $(Z_t : t \in [0, T])$ such that $\|Z\|_{\mathcal{H}_p^\mathbb{F}} := \mathbb{E}\left[\left(\int_0^T |Z_t|^2 dt\right)^{p/2}\right]^{1/p} < \infty$.
- For any probability measure $\mathbb{Q}$, we let $\mathcal{S}_p^\mathbb{Q}$ and $\mathcal{H}_p^\mathbb{Q}$ be defined as $\mathcal{S}_p$ and $\mathcal{H}_p$, respectively, with the exception that the norm is defined with expectation taken with respect to $\mathbb{Q}$, i.e. $\|Z\|_{\mathcal{S}_p^\mathbb{Q}} := \mathbb{E}^\mathbb{Q}\left[\sup_{t \in [0, T]} |Z_t|^p\right]$ and $\|Z\|_{\mathcal{H}_p^\mathbb{Q}} := \mathbb{E}^\mathbb{Q}\left[\left(\int_0^T |Z_t|^2 dt\right)^{p/2}\right] < \infty$.  

\[ X_t^{u,\alpha} = x_0 + \int_0^t a(s, (X_r^{u,\alpha})_{r \leq s}, \alpha_s)ds + \int_0^t \sigma(s, (X_r^{u,\alpha})_{r \leq s})dW_s, \quad \forall t \in [0, \tau_1) \]

\[ X_t^{u,\alpha} = \Gamma(t_j, X_{t_j}^{[u]j-1,\alpha}, \beta_j) + \int_{t_j}^t a(s, (X_r^{u,\alpha})_{r \leq s}, \alpha_s)ds + \int_{t_j}^t \sigma(s, (X_r^{u,\alpha})_{r \leq s})dW_s, \quad \forall t \in [t_j, \tau_{j+1}) \]
• We let $T$ be the set of all $\mathbb{P}$-stopping times and for each $\eta \in T$ we let $T_\eta$ be the corresponding subsets of stopping times $\tau$ such that $\tau \geq \eta$, $\mathbb{P}$-a.s.

• For each $\tau \in T$, we let $\mathcal{I}(\tau)$ be the set of all $\mathcal{F}_\tau$-measurable random variables taking values in $U$, so that $\mathcal{I}(\tau)$ is the set of all admissible interventions at time $\tau$.

• We let $\mathcal{U}$ be the set of all $u = (\tau_1, \ldots, \tau_N; \beta_1, \ldots, \beta_N)$, where $(\tau_j)_{j=1}^\infty$ is a non-decreasing sequence of $\mathbb{P}$-stopping times taking values in $[0, T]$, $\beta_j \in \mathcal{I}(\tau_j)$ and $N$ is an $\mathcal{F}_T$-measurable, integer valued random variable such that $\{N \geq j\}$ on $\{\tau_j < T\}$. Throughout, we also set $\tau_0 := 0$.

• We let $\mathcal{U}^f$ denote the subset of $u \in \mathcal{U}$ for which $N$ is $\mathbb{P}$-a.s. finite (i.e. $\mathcal{U}^f := \{u \in \mathcal{U} : \mathbb{P}[\{\omega \in \Omega : N > k, \forall k > 0\}] = 0\}$) and for all $k \geq 0$ we let $\mathcal{U}^k := \{u \in \mathcal{U} : N \leq k, \mathbb{P} - \text{a.s.}\}$.

• For $\eta \in T$ we let $\mathcal{U}_\eta$ (and $\mathcal{U}_\eta^f$ resp., $\mathcal{U}_\eta^k$) be the subset of $\mathcal{U}$ (and $\mathcal{U}^f$ resp. $\mathcal{U}^k$) with $\tau_1 \geq \eta$, $\mathbb{P}$-a.s.

• We let $\mathcal{A}$ be the set of all $\mathcal{P}_\mathcal{F}$-measurable processes $(\alpha_t)_{0 \leq t \leq T}$ taking values in $A$ and for each $t \in [0, T]$ we let $\mathcal{A}_t$ be the set of all $\mathcal{P}_\mathcal{F}$-measurable processes $(\alpha_s)_{t \leq s \leq T}$ taking values in $A$.

• We denote by $\mathcal{D}$ the set of all double sequences $(t_1, \ldots; b_1, \ldots)$ where $(t_j)_{j \geq 1}$ is a non-decreasing sequence in $[0, T]$ and $b_j \in U$ for $j \geq 1$.

• We let $\mathcal{D}^f$ be the subset of $\mathcal{D}$ with all finite sequences and for $k \geq 0$ we let $\mathcal{D}^k$ be the subset of sequences with precisely $k$ interventions, i.e. sequences of the type $(t_1, \ldots, t_k; b_1, \ldots, b_k)$.

• Throughout, we let $v = (t, b)$, with $t := (t_1, \ldots, t_n)$ and $b := (b_1, \ldots, b_n)$, where $n$ is possibly infinite, denote a generic element of $\mathcal{D}$.

• For $v = (t, b) \in \mathcal{D}^f$ and $v' = (t', b') \in \mathcal{D}$ we introduce the concatenation, denoted by $\circ$, defined as $v \circ v' := (t_1, \ldots, t_n, t'_1 \vee t_n, \ldots, t'_n \vee t_n, b_1, \ldots, b_n, b'_1, \ldots, b'_n)$. Furthermore, for $v \in \mathcal{D}$, we define the truncation to $k \geq 0$ interventions as $[v]_k := (t_1, \ldots, t_{k \wedge n}; b_1, \ldots, b_{k \wedge n})$.

• We extend $\circ$ to a map from $\mathcal{U}^f \times \mathcal{U}$ to $\mathcal{U}$ by letting $u \circ u' := (\tau_1, \ldots, \tau_N, \tau'_1 \vee \ldots \vee \tau'_n \vee \ldots \vee \tau_N; \beta_1, \ldots, \beta_N, \beta'_1, \ldots, \beta'_n)$ where $\pi(u)$ is the smallest $\mathcal{F}$-stopping time such that $\tau_N \leq \pi(u)$, $\mathbb{P}$-a.s.

• For each $u \in \mathcal{U}^f$ we let $u(t) := [u]_{N(t)}$ with $N(t) := \max\{j \geq 0 : \tau_j \leq t\}$.

• We introduce the norm $\|v\|_{\mathcal{D}^f} := \sum_{j=1}^n (|t_j| + |b_j|)$ on $\mathcal{D}^f$ and let $\|v - v'\|_{\mathcal{D}^f} := \infty$ whenever $n \neq n'$.

• We let $\ast$ denote stochastic integration and set $(X \ast W)_{t,s} = \int_s^t X_r dW_r$.

• We let $\mathcal{E}$ denote the Doléans-Dade exponential and use the notation

$$\mathcal{E}(X \ast W)_{t,s} = e^{\int_s^t X_r dW_r - \frac{1}{2} \int_s^t |X_r|^2 dr}.$$ 

Also, we write $\mathcal{E}(X \ast W)_t := \mathcal{E}(X \ast W)_{0,t}$.

• For any $\mathcal{P}_\mathcal{F}$-measurable process $\zeta$ such that $\mathbb{E}[\mathcal{E}(\zeta \ast W)_T] = 1$, we define $Q^\zeta$ to be the probability measure equivalent to $\mathbb{P}$, such that $dQ^\zeta = \mathcal{E}(\zeta \ast W)_T d\mathbb{P}$.

• For any non-negative, $\mathcal{P}_\mathcal{F}$-measurable càdlàg process $L$ we let $Q^L$ denote the set of all probability measures $Q$ on $(\Omega, \mathcal{F})$ such that $dQ = \mathcal{E}(\zeta \ast W)_T d\mathbb{P}$, for some $\mathcal{P}_\mathcal{F}$-measurable process $\zeta$, with $|\zeta_t| \leq L_t$ for all $t \in [0, T]$ (outside of a $\mathbb{P}$-null set).
Throughout, we generally suppress dependence on \( P \) hold in the \( P \)-a.s. sense.

Furthermore, we define the following set:

**Definition 2.1** We let \( \mathcal{O}_c \) be the set of all \( \mathcal{P}_\mathcal{F} \otimes \mathcal{B}(U) \)-measurable maps\(^1\) \( h : \Omega \times [0, T] \times U \to \mathbb{R} \) such that for each \( \tau \in \mathcal{T} \) and \( \beta \in \mathcal{I}(\tau) \) we have \( h(\tau, \beta) \in L^p(\Omega, \mathcal{F}, \mathbb{P}) \) for all \( p \geq 0 \) and (outside of a \( \mathbb{P} \)-null set) the map \((t, b) \mapsto h(t, b)\) is jointly continuous.

**Definition 2.2** We refer to a family of processes \( ((X^v_t)_{0 \leq t \leq T} : v \in U^f) \) as being **consistent** if for each \( u \in U^f \), the map \( h : [0, T] \times \Omega \times U \to \mathbb{R} \) given by \( h(t, b) = X^u_{t\cap(t,b)} \) is \( \mathcal{P}_\mathcal{F} \otimes \mathcal{B}(U) \)-measurable and for each \( \tau \in \mathcal{T} \) and each \( \beta \in \mathcal{I}(\tau) \) we have \( X^\mu_{\tau\cap(\tau, \beta)} = h(\tau, \beta) \), \( \mathbb{P} \)-a.s.

One of the main objectives of the present work is to show that (1.1) admits a unique solution. We, therefore, need to define what we mean by a solution to (1.1).

**Definition 2.3** A solution to (1.1) is a family \((Y^v, Z^v, K^v)_{v \in U^f} \), where

(i) the family \((Y^v)_{v \in U^f} \) is consistent and for each \( v \in U^f \), we have \( Y^v \in S^2 \) with a norm that is uniformly bounded in \( v \) (i.e. \( \sup_{v \in U^f} \|Y^v\|_{S^2} < \infty \)) and \((t, b) \mapsto Y^v_{t\cap(t,b)} \in \mathcal{O}_c \),

(ii) \( Z^v \in \mathcal{H}^2 \) for each \( v \in U^f \); and

(iii) \( K^v \in S^2 \) is non-decreasing with \( K^v_0 = 0 \) for each \( v \in U^f \).

### 2.1 RBSDEs with Stochastic Lipschitz Coefficient

Our approach will rely heavily on the available theory of reflected backward SDEs. In particular, we have the following result (a proof of which can be found in Appendix A):

**Proposition 2.4** Assume that

(i) **There is a** \( \mathbb{P} \)-a.s. non-negative, \( \mathcal{P}_\mathcal{F} \)-measurable, continuous process \((L_t : t \in [0, T]) \) (our stochastic Lipschitz coefficient) **such that for all** \( \mathcal{P}_\mathcal{F} \)-measurable processes \((\xi_t : t \in [0, T]) \) with \( |\xi_t| \leq L_t \) for all \( t \in [0, T] \) (outside of a \( \mathbb{P} \)-null set) we have \( \mathbb{E}[|\mathcal{E}(\xi \ast W)|_T q^T] < \infty \) and \( \mathbb{E}^{Q^\xi}[|\mathcal{E}(-\xi \ast W^\xi)|_T q^T] < \infty \).

(ii) **The terminal value** \( \xi \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}) \).

(iii) **The driver** \( (t, \omega, y, z) \mapsto f(t, y, z) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is \( \mathcal{P}_\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable. Furthermore,

(a) **For each** \( p \geq 1 \), we have

\[
\mathbb{E}\left[ \int_0^T |f(s, 0, 0)|^p ds \right] < \infty.
\] (2.1)

(b) **There is a constant** \( k_f > 0 \) **such that**

\[
|f(t, y', z') - f(t, y, z)| \leq k_f |y' - y| + L_t |z' - z|.
\] (2.2)

\(^1\) Throughout, we generally suppress dependence on \( \omega \) and refer to \( h \in \mathcal{O}_c \) as a map \((t, b) \mapsto h(t, b)\).
In addition, \( Y \) can be interpreted as the Snell envelope in the following way

\[
\begin{align*}
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s + K_T - K_t, \\
Y_t \geq S_t, \ \forall t \in [0, T] \text{ and } \int_0^T (Y_t - S_t) dK_t = 0.
\end{align*}
\]  

(2.3)

Furthermore\(^2\),

\[
\|Y\|_{S^p}^p + \|Z\|_{\mathcal{H}^p}^p + \|K\|_{S^p}^p \leq C \left( \|S^+\|_{S^2_p}^p + \mathbb{E} \left[ \left| \xi \right|^2 p + \left( \int_0^T \left| f(s, 0, 0) \right| ds \right)^{q^2 p} \right]^{1/q^2} \right)
\]

and if \((\tilde{Y}, \tilde{Z}, \tilde{K})\) is a solution to (2.3) with parameters \((\tilde{f}, \tilde{\xi}, \tilde{S})\) then

\[
\begin{align*}
\|\tilde{Y} - Y\|_{S^p}^p + \|\tilde{Z} - Z\|_{\mathcal{H}^p}^p + \|\tilde{K} - K\|_{S^p}^p \\
\leq C \left( \|\tilde{S} - S\|_{S^2_p}^{q^2 p} \Psi_T^{1/q^2 p} + \mathbb{E} \left[ |\tilde{\xi} - \xi|^2 p + \left( \int_0^T |\tilde{f}(s, Y_s, Z_s) - f(s, Y_s, Z_s)| ds \right)^{q^2 p} \right]^{1/q^2} \right).
\end{align*}
\]  

(2.5)

where

\[
\Psi_T := \mathbb{E} \left[ \left| \tilde{\xi} \right|^q p + |\xi|^q p + \left( \int_0^T |\tilde{f}(s, 0, 0)| + |f(s, 0, 0)| ds \right)^{q^4 p} \right]^{1/q^4} + \sup_{t \in [0, T]} |(\tilde{S}_t)^+ + (S_t)^+|^{q^4 p}.
\]

In addition, \( Y \) can be interpreted as the Snell envelope in the following way

\[
Y_t = \operatorname{ess} \sup_{\tau \in T_t} \mathbb{E} \left[ \int_t^\tau f(s, Y_s, Z_s) ds + S_{\tau} \mathbbm{1}_{[\tau < T]} + \xi \mathbbm{1}_{[\tau = T]} \right] \mathcal{F}_t.
\]  

(2.6)

In particular, with \( D_t := \inf \{ r \geq t : Y_r = S_r \} \wedge T \) we have the representation

\[
Y_t = \mathbb{E} \left[ \int_t^{D_t} f(s, Y_s, Z_s) ds + S_{D_t} \mathbbm{1}_{[D_t < T]} + \xi \mathbbm{1}_{[D_t = T]} \right] \mathcal{F}_t
\]

(2.7)

and \( K_{D_t} - K_t = 0, \mathbb{P}\text{-a.s.} \)

\(^2\) Throughout, \( C \) will denote a generic positive constant that may change value from line to line.

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3 Sequential Systems of Reflected BSDEs in Finite Horizon

In this section we move on to the sequential system of reflected BSDEs in (1.1). To be able to use our results for impulse control we must allow the stochastic Lipschitz coefficient in (1.1) to depend on the control parameter $v \in \mathcal{U}^f$, rendering us a family of stochastic Lipschitz coefficients $(L^v : v \in \mathcal{U}^f)$ with $L^v \in \mathcal{S}^p$ for all $v \in \mathcal{U}^f$ and $p \geq 0$.

3.1 Assumptions

We introduce the following sets of probability measures on $(\Omega, \mathcal{F})$.

**Definition 3.1** We let $\mathcal{P}^v := \mathcal{P}^{L^v}$ and define $\mathcal{K}^v$ to be the set of all $\mathcal{P}_\mathcal{F}$-measurable processes $\zeta$ with $|\zeta_t| \leq L^v_t$ for all $t \in [0, T]$ (outside of a $\mathbb{P}$-null set). Moreover, for all $t \in [0, T]$, we let $\mathcal{P}^v_t := \bigcup_{u \in \mathcal{U}^f_t} \mathcal{P}^{u(t)}_{\mathcal{F}_t}$ and $\mathcal{K}^v_t := \bigcup_{u \in \mathcal{U}^f_t} \mathcal{K}^{u(t)}_{\mathcal{F}_t}$. We also use the shorthands $\mathcal{P}_0 := \mathcal{P}^v_0$ and $\mathcal{K}_0 := \mathcal{K}^v_0$.

To streamline presentation we will formulate our assumptions on the coefficients in terms of the existence of a family of bounding processes:

**Definition 3.2** We say that a family of processes $(L^v, \Lambda^{v,v',v}, \tilde{K}^{v,v',k,p} : (v, v') \in \mathbb{U}^k \times \mathbb{D}^k, v \in \mathcal{U}^f, k \geq 0, p \geq 1)$ is a bounding family if for each $k \geq 0$ and $p, \kappa \geq 1$, there is a $C > 0$ and a $p' \geq 1$ such that for all $v \in \mathcal{U}^f$ and $v, v' \in \mathcal{D}^k$ and some $q' > 1$, we have:

(i) $L^v \in \mathcal{S}^p$ is non-decreasing, the map $v \mapsto L^v$ is continuous from $\mathcal{D}^f$ to $\mathcal{H}^2$.

Moreover, for all $\zeta \in \mathcal{K}^v$ and $Q \in \mathcal{P}^v$ we have $\mathbb{E}^Q[|\mathcal{E}(R * W^Q)\mathcal{F}_T|^{q'}] \leq C$ (where $W^Q$ is a Brownian motion under $Q$). Moreover, $\mathbb{E}[\mathcal{E}^{q'}(\int_0^T |L^v_s|^{dQ} ds)] \leq C$.

(ii) $\Lambda^{v,v',v} \in \mathcal{H}^p$ is a càdlàg process and the map $(v, v') \mapsto \int_0^T |\Lambda^{v,v',v}|^2 ds$ is $\mathbb{P}$-a.s. continuous with $\int_0^T |\Lambda^{v,v',v}|^2 ds = 0$.

(iii) $\tilde{K}^{v,p} \in \mathcal{S}^2$ with $\|\tilde{K}^{v,p}\|_{\mathcal{S}^2} \leq C$ and $(\tilde{K}^{v,p})^r \leq (\tilde{K}^{v,p})^r$ for $r \geq 1$.

(iv) $\tilde{K}^{v,v',k,p} \in \mathcal{S}^1$ with $\|\tilde{K}^{v,v',k,p}\|_{\mathcal{S}^1} \leq C \|v' - v\|_{\mathcal{D}^f}$.

Moreover, for each $r > 1$, there is a $C > 0$ such that

$$\text{ess sup}_{u \in \mathcal{U}^f_t} \mathbb{E}\left[|L^{v\circ u}_t|^p |\mathcal{F}_t\right] \leq \tilde{K}_t^{v,p},$$

$$\text{ess sup}_{u \in \mathcal{U}^f_t} \mathbb{E}\left[\left(\int_0^T |\Lambda^{v,v',u}_s|^2 ds\right)^{p/2} |\mathcal{F}_t\right] \leq \tilde{K}_t^{v,v',k,p},$$

$$\text{ess sup}_{u \in \mathcal{U}^f_t} \mathbb{E}\left[\sup_{s \in \mathcal{T}_u} |\tilde{K}^{v\circ u(s),p}_s| |\mathcal{F}_t\right] \leq C \tilde{K}_t^{v,p},$$

for all $Q \in \mathcal{P}^v_t$.

We will make the following assumptions on the involved coefficients:
Assumption 3.3  There is a bounding family \( \{L^v, \Lambda^v, v' : (v, v') \in \bigcup_{k \geq 1} D^k \times D^k, \ v \in \mathcal{U}^f \}, k \geq 0, p \geq 1 \) such that for each \( k \geq 0, p, \kappa \geq 1, v \in \mathcal{U}^f \) and \( v, v' \in D^k \) we have:

(i) The map \( (\omega, v) \mapsto \xi^v : \Omega \times D^f \to \mathbb{R} \) is \( \mathcal{F}_T \otimes \mathcal{B}(D^f) \)-measurable, \( \mathbb{P} \)-a.s. continuous in \( v \) and satisfies

\[
\mathrm{ess\ sup}_{u \in \mathcal{U}^f} \mathbb{E}\left[|\xi^v_{\text{out}}|^p | \mathcal{F}_t \right] \leq \tilde{K}^v_{t, p}
\]  

(3.4)

and

\[
\mathrm{ess\ sup}_{u \in \mathcal{U}^f} \mathbb{E}\left[|\xi^v_{\text{out}} - \xi^v_{\text{in}}|^p | \mathcal{F}_t \right] \leq \tilde{K}^v_{t, k, p}.
\]  

(3.5)

(ii) The intervention cost \( c^v \) is such that \( (t, b) \mapsto -c^v(t, b) \in \mathcal{O}_c \) and satisfies

\[
\inf_{(t, b) \in [0, T] \times \mathcal{U}} c^v(t, b) \geq \delta,
\]  

(3.6)

for some \( \delta > 0 \). Moreover,

\[
\mathrm{ess\ sup}_{u \in \mathcal{U}^f} \mathbb{E}\left[|c^{v|u}|N^{-1}(\tau_N, \beta_N) - c^{v|u}|N^{-1}(\tau_N, \beta_N)|^p | \mathcal{F}_t \right] \leq \tilde{K}^{v, k, p}.
\]  

(3.7)

(iii) We have \( \xi^v \geq \sup_{b \in U}\{\xi^{v|u}(T, b) - c^v(T, b)\} \), \( \mathbb{P} \)-a.s.

(iv) The map \( (v, t, \omega, y, z) \mapsto f^v(t, y, z) : D^f \times [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is \( \mathcal{B}(D^f) \otimes \mathcal{P}_\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable and for each \((y, z) \in \mathbb{R}^{1+d}\) the map \( v \mapsto f^v(\cdot, y, z) \) is a continuous map from \( D^f \) to \( \mathcal{H}^2 \). Furthermore, we have

(a) the bound

\[
\mathrm{ess\ sup}_{u \in \mathcal{U}^f} \mathbb{E}\left[\int_{t}^{T} |f^v_{\text{out}}(s, 0, 0)|^p ds | \mathcal{F}_t \right] \leq \tilde{K}^{v, p}.
\]  

(3.8)

(b) the Lipschitz condition

\[
|f^{v'|u}(t, y', z') - f^{v|u}(t, y, z)| \leq k_f |y' - y| + (L^y_{t|u} \vee L^{y'}_{t|u})|z' - z|
\]

\[
+ (1 + |z| + |z'|) \Lambda^{v, v, u}_t,
\]  

(3.9)

for all \((t, v, v', y, y', z, z') \in [0, T] \times \bigcup_{k \geq 1}(D^k \times D^k) \times \mathbb{R}^{2(1+d)} \), \( \mathbb{P} \)-a.s., for all \( u \in \mathcal{U}^f \); and

(c) for \( u := (\tau_1, \ldots, \tau_N; \beta_1, \ldots, \beta_N) \in \mathcal{U}^f \) we have the causality property

\[
f^u(t, y, z) = \sum_{j=0}^{N} \mathbb{I}_{[\tau_j, \tau_{j+1})}(t) f^{[u]}(t, y, z),
\]
where \( \tau_0 := 0 \) and \( \tau_{N+1} = \infty \).

Before moving on to show existence and uniqueness of solutions to (1.1) under Assumption 3.3 we give the following auxiliary result:

**Lemma 3.4** For each \( p \geq 1 \) there is a \( r' > 1 \) such that \( R^v \) defined as

\[
R^v_t := \text{ess sup}_{\zeta \in K^v_t} \mathbb{E}\left[ |\mathcal{E}(\zeta * W)_{t,T}|^{r'} |\mathcal{F}_t\right],
\]

and \( \tilde{R}^v \) defined as

\[
\tilde{R}^v_t := \text{ess sup}_{\zeta \in K^v_t} \mathbb{E}^{Q^v}\left[ |\mathcal{E}(-\zeta * W^\zeta)_{t,T}|^{r'} |\mathcal{F}_t\right],
\]

where \( W^\zeta \) is a \( Q^v \)-Brownian motion, are both \( \mathcal{F}_t \)-measurable, càdlàg processes and \( \|R^v\|_{\mathcal{S}^p} \) and \( \|\tilde{R}^v\|_{\mathcal{S}^p} \) are uniformly bounded in \( v \in U^f \).

**Proof** By continuity of the map \( v \mapsto L^v : D^f \to \mathcal{H}^2 \) it follows that \( R^u \) is continuous on \([\tau_j, \tau_{j+1})\) for \( j = 0, \ldots, N + 1 \). For \( x \geq 0 \), we let \( \tau^x := \inf\{ s \geq 0 : R^u_s \geq x \} \wedge T \) and note that for \( \theta > 1 \), we have by right-continuity that

\[
\mathbb{P}\left[ \sup_{t \in [0,T]} |R^u_t|^{\theta} \geq x^\theta \right] = \mathbb{P}\left[ |R^u_{\tau^x}|^{\theta} \geq x^\theta \right] \leq \frac{\mathbb{E}\left[ |R^u_{\tau^x}|^{\theta} \right]}{x^\theta}.
\]

However, for each \( \epsilon > 0 \), there is a \( u^\epsilon \in U^f_{\tau^x} \) and a \( \zeta^\epsilon \), with \( |\zeta^\epsilon| \leq \|1_{[\tau^x, T]}(t)L^v(t)\|_{\text{ou}} \), such that \( R^u_{\tau^x} < \mathbb{E}\left[ |\mathcal{E}(\zeta^\epsilon * W)_{\tau^x,T}|^{r'} |\mathcal{F}_{\tau^x}\right] + \epsilon \). In particular, it follows that

\[
|R^u_{\tau^x}|^{\theta} \leq 2^{\theta-1}\left( \mathbb{E}\left[ \mathcal{E}(r' \zeta^\epsilon * W)_{\tau^x,T} e^{\frac{(r')^2 - r'}{2} \int_{\tau^x}^T |\zeta^\epsilon|^2 ds} |\mathcal{F}_{\tau^x}\right]^{\theta} + \epsilon^{\theta} \right)
\]

which implies that

\[
\mathbb{E}\left[ |R^u_{\tau^x}|^{\theta} \right] \leq 2^{\theta-1}\left( \mathbb{E}\left[ \mathcal{E}(r' \zeta^\epsilon * W)_{\tau^x,T} e^{\frac{(r')^2 - r'}{2} \int_{\tau^x}^T |\zeta^\epsilon|^2 ds} |\mathcal{F}_{\tau^x}\right] + \epsilon^{\theta} \right)
\]

Since \( \epsilon > 0 \) was arbitrary we find that

\[
\mathbb{P}\left[ \sup_{t \in [0,T]} |R^u_t|^{\theta} \geq x^\theta \right] \leq 2^{\theta-1}\sup_{\zeta \in K^0_{\theta}} \mathbb{E}^{Q^v}\left[ e^{\frac{(r')^2 - r'}{2} \int_{0}^T |\zeta(t)|^2 ds} |\mathcal{F}_{\tau^x}\right].
\]
Now, with \( q' = q'/r' \) and \( \bar{q} = \bar{q}'/(\bar{q}' - 1) = q'/(q' - r') \) we have
\[
\mathbf{E}^{Q^{'\xi}} \left[ e^{\rho \xi (\xi - \bar{r} - \bar{r}') \int_{0}^{T} |\xi_{s}|^{2}ds} \right] \leq \mathbf{E} \left[ \left| \mathcal{E}(r' \xi \ast W) \right|^{\bar{q}'} \right]^{1/\bar{q}'} \mathbf{E} \left[ e^{\rho \bar{q}' (\xi - \bar{r} - \bar{r}') \int_{0}^{T} |\xi_{s}|^{2}ds} \right]^{1/\bar{q}'}.
\]
For any \( \theta > 1 \) and arbitrary \( p \geq 1 \) the coefficient \( \rho \bar{q}' (r' \bar{r} - r') \bar{r}' \) can be made arbitrarily small by choosing \( r' > 1 \) sufficiently small and since there is a \( C \geq 0 \) such that
\[
\mathbf{E} \left[ e^{q' \int_{0}^{T} |\xi_{s}|^{2}ds} \right] \leq \sup_{u \in \mathcal{U}^f} \mathbf{E} \left[ e^{q' \int_{0}^{T} |L_{u,s}|^{2}ds} \right] \leq C
\]
by Definition 3.2. (i) we conclude that
\[
\mathbb{P} \left[ \sup_{t \in [0,T]} |R_{v}^{f}|^{p} \geq x \right] \leq \frac{C}{x^{\theta}}.
\]
In particular, using integration by parts, we find that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |R_{v}^{f}|^{p} \right] = \int_{0}^{\infty} \left( \frac{C}{x^{\theta}} \wedge 1 \right) \, dx < \infty,
\]
showing that \( R_{v}^{f} \in \mathcal{S}^{p} \) with norm uniformly bounded in \( v \). The result for \( \bar{R}_{v}^{f} \) follows similarly.

**Lemma 3.5** For each \( p \geq 1 \) there is a \( r' > 1 \) such that \( \bar{R}_{v}^{f} \) defined as
\[
\bar{R}_{v}^{f} := \text{ess sup}_{u \in \mathcal{U}^f} \mathbb{E} \left[ \sup_{s \in [0,T]} \text{ess sup}_{s \in [0,T]} \mathbb{E} \left[ |\mathcal{E}(\xi \ast W)_{s,T}|^{p} \right| \mathcal{F}_{s} \right]\right]_{\mathcal{F}_{f}}
\]
is a \( \mathcal{P}_{T} \)-measurable, càdlàg process and \( \| \bar{R}_{v}^{f} \|_{\mathcal{S}^{p}} \) is uniformly bounded in \( v \in \mathcal{U}^f \).

**Proof** Let \( r' > 1 \) be such that \( \| R_{v}^{f} \|_{\mathcal{S}^{p}} \) is uniformly bounded in \( v \in \mathcal{U}^f \) for some \( \theta > 1 \). For \( x \geq 0 \), we let \( \tau^{x} := \text{inf} \{ s \geq 0 : \bar{R}_{v}^{f} \geq x \} \wedge T \) and note that for each \( \epsilon > 0 \), there is a \( u^{\epsilon} \in \mathcal{U}^{f} \) such that \( \bar{R}_{v}^{f} < \mathbb{E} \left[ \sup_{s \in [\tau^{x},T]} R_{s}^{v,u^{\epsilon}} |\mathcal{F}_{\tau^{x}} \right] + \epsilon \). Jensen’s inequality now gives
\[
\mathbb{E} \left[ |\bar{R}_{v}^{f}|^{p\theta} \right] \leq 2^{p\theta-1} \left( \mathbb{E} \left[ \sup_{s \in [\tau^{x},T]} |R_{s}^{v,u^{\epsilon}}|^{p\theta} \right] + \epsilon^{p\theta} \right).
\]
Since \( \epsilon > 0 \) was arbitrary and the first term is bounded by \( 2^{p\theta-1} \sup_{u \in \mathcal{U}^f} \| R_{v} \|_{\mathcal{S}^{p\theta}} \), the result follows by repeating the last steps in the proof of Lemma 3.4.

Throughout this section, we assume that \( r' > 1 \) is small enough that \( \| R_{v} \|_{\mathcal{S}^{3}}, \| \bar{R}_{v}^{f} \|_{\mathcal{S}^{3}} \) and \( \| \bar{R}_{v}^{f} \|_{\mathcal{S}^{3}} \) are bounded uniformly in \( v \in \mathcal{U}^f \) and let \( r \) be such that \( \frac{1}{r} + \frac{1}{r} = 1. \)
3.2 An Approximating Sequence

In this section we outline a Picard type approximation scheme, that will ultimately lead us to the conclusion that (1.1) has a solution under Assumption 3.3. We note that for all \( v \in \mathcal{U}^f \),

\[
Y^{v,0}_t = \xi^v + \int_t^T f^v(s, Y^{v,0}_{s}, Z^{v,0}_{s}) ds - \int_t^T Z^{v,0}_s dW_s \tag{3.10}
\]

admits a unique solution by Proposition 2.4 (note that (3.10) can be seen as a reflected BSDE with barrier \( S \equiv -\infty \)). We, thus, consider the following sequence of families of reflected BSDEs

\[
\begin{aligned}
Y^{v,k}_t = & \xi^v + \int_t^T f^v(s, Y^{v,k}_{s}, Z^{v,k}_{s}) ds - \int_t^T Z^{v,k}_s dW_s + K^{v,k}_T - K^{v,k}_t \\
Y^{v,k}_t & \geq \sup_{b \in \mathcal{U}} \{ Y^{v\circ(t,b),k-1}_t - c^v(t,b) \} \\
\int_0^T (Y^{v,k}_t - \sup_{b \in \mathcal{U}} \{ Y^{v\circ(t,b),k-1}_t - c^v(t,b) \}) dK^{v,k}_t &= 0.
\end{aligned}
\tag{3.11}
\]

for \( k \geq 1 \). Hypothesis We will make use of the following induction hypothesis:

**Hypothesis (RBSDE. l)** There is a family of pairs \( (Y^{v,0}, Z^{v,0}) : v \in \mathcal{U}^f \) and a sequence of families of triples \( (Y^{v,k}, Z^{v,k}, K^{v,k}) : v \in \mathcal{U}^f \) \( 1 \leq k \leq l \) such that:

i) For each \( v \in \mathcal{U}^f \), the pair \( (Y^{v,0}, Z^{v,0}) \in \mathcal{S}^2 \times \mathcal{H}^2 \) solves (3.10) and the triple \( (Y^{v,k}, Z^{v,k}, K^{v,k}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{S}^2 \) solves (3.11) for \( k = 1, \ldots, l \).

ii) For all \( k \in \{0, \ldots, l\} \) and each \( j \geq 0 \), the map \( h^{k,j} : [0, T] \times \mathcal{D}^j \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}) \) \((t, v) \mapsto Y^{v,k}_t\) is jointly continuous (outside of a \( \mathbb{P}\)-null set) and, moreover, for each \( v \in \mathcal{U}^f \) we have \( Y^{v,k}_t = \sum_{j=0}^\infty \mathbb{1}_{[N=j]} h^{k,j}(t, v) \) for all \( t \in [0, T] \) outside of a \( \mathbb{P}\)-null set.

Recall Definition 2.3 specifying what we mean by a solution to (2.3). We extend this definition to incorporate solutions to (3.10) and (3.11) as well. Through this definition the first condition in Hypothesis RBSDE.l implies that \( \sup_{v \in \mathcal{U}^f} \| Y^{v,k} \| < \infty \) and also dictates the regularity of the map \( (t, b) \mapsto Y^{v\circ(t,b)}_t \). The second statement, on the other hand, is a stronger version of the consistency property for families of processes introduced in Definition 2.2. To simplify presentation we will refer to the second property as strong consistency.

**Proposition 3.6** Hypothesis RBSDE.0 holds.

**Proof** Existence of solutions to (3.10) with \( Y^v \in \mathcal{S}^2 \) and \( Z^v \in \mathcal{H}^2 \) and uniform boundedness (specified in (i) of Definition 2.3) follows from Proposition 2.4 with barrier \( S \equiv -\infty \).
For $\kappa \geq 1$ and $\mathbf{v}, \mathbf{v}' \in \mathcal{D}^\kappa$ we have by Proposition 2.4, Assumption 3.3 and Definition 3.2.(iv) that for each $p \geq 1$, there is a $p' \geq 1$ such that

$$
\| Y^{\mathbf{v},0} - Y^{\mathbf{v}',0} \|_{S^{p'}} + \| Z^{\mathbf{v},0} - Z^{\mathbf{v}',0} \|_{T^{p'}} \\
\leq C \mathbb{E} \left[ |\xi^{\mathbf{v}} - \xi^{\mathbf{v}'}|^2 p' + \left( \int_0^T |f^{\mathbf{v}}(s, Y_s^{\mathbf{v},0}, Z_s^{\mathbf{v},0}) - f^{\mathbf{v}'}(s, Y_s^{\mathbf{v}',0}, Z_s^{\mathbf{v}',0})| ds \right)^2 p'^2 \right]^{1/2} \\
\leq C \mathbb{E} \left[ |\xi^{\mathbf{v}} - \xi^{\mathbf{v}'}|^2 p' + \left( \int_0^T \left( 1 + |Z_s^{\mathbf{v},0}| \right) \Lambda_s^{\mathbf{v},\mathbf{v}',0} ds \right)^2 p'^2 \right]^{1/2} \\
\leq C \mathbb{E} \left[ |\xi^{\mathbf{v}} - \xi^{\mathbf{v}'}|^2 p' + \left( \int_0^T \left( 1 + |Z_s^{\mathbf{v},0}|^2 \right) ds \int_0^T |\Lambda_s^{\mathbf{v},\mathbf{v}',0}|^2 ds \right) p'^2 / 2 \right]^{1/2} \\
\leq C \mathbb{E} \left[ |\xi^{\mathbf{v}} - \xi^{\mathbf{v}'}|^2 p' \right]^{1/2} \\
+ \left( 1 + \mathbb{E} \left[ \left( \int_0^T |Z_s^{\mathbf{v},0}|^2 ds \right)^{2/p'} \right]^{1/2} \right) \mathbb{E} \left[ \left( \int_0^T |\Lambda_s^{\mathbf{v},\mathbf{v}',0}|^2 ds \right)^{p'/2} \right]^{1/2} \\
\leq C \| \mathbf{v}' - \mathbf{v} \|_{\mathcal{D}^{p'}}
$$

where $C > 0$ does not depend on $\mathbf{v}, \mathbf{v}'$. By picking $p = (m + 1)\kappa + 1$, Kolmogorov’s continuity theorem (see e.g. Theorem 72 in Chapter IV of [28]) guarantees the existence of a family of processes $(\hat{Y}^\mathbf{v}, \hat{Z}^\mathbf{v} : \mathbf{v} \in \mathcal{D}^\kappa)$, where for each $\mathbf{v} \in \mathcal{D}^\kappa$, $\hat{Y}^\mathbf{v} \in \mathcal{S}^2$ and $\hat{Z}^\mathbf{v} \in \mathcal{H}^2$, such that, outside of a $\mathbb{P}$-null set, $(s, \mathbf{v}) \mapsto \hat{Y}^\mathbf{v}_s$ is continuous in $\mathbf{v}$ uniformly in $s$ and $\mathbf{v} \mapsto \hat{Z}^\mathbf{v}_s$ is $L^2([0, T])$-continuous (and in particular that $(s, \mathbf{v}) \mapsto \int_s^T Z^\mathbf{v}_r dW_r$ is continuous in $\mathbf{v}$ uniformly in $s$) and moreover $(\hat{Y}^\mathbf{v}_s, \hat{Z}^\mathbf{v}_s) = (Y^\mathbf{v}_s, Z^\mathbf{v}_s), \mathbb{P}$-a.s.

Now, since this holds for all $\kappa \geq 0$ it is clear that for each $\mathbf{v} \in \mathcal{D}^f$, the pair $(\hat{Y}^\mathbf{v}_s, \hat{Z}^\mathbf{v}_s : s \in [0, T])$ solves the corresponding BSDE (3.10) and that the map $\mathbf{v} \mapsto (\hat{Y}^\mathbf{v}, \hat{Z}^\mathbf{v}) : \mathcal{D}^f \to \mathcal{S}^2 \times \mathcal{H}^2$ is continuous.

To establish the strong consistency in Hypothesis RBSDE.0(ii), we need to show that for any $\mathbf{v} \in \mathcal{U}^f$, the pair $(\hat{Y}^\mathbf{v}, \hat{Z}^\mathbf{v}) \in \mathcal{S}^2 \times \mathcal{H}^2$ solves the BSDE corresponding to the control $\mathbf{v}$. We let $(\mathbf{v}_j)_{j \geq 0}$ be an approximating sequence in $\mathcal{U}^f$ (i.e. $\mathbf{v}_j \in \mathcal{U}^f$ and $\mathbf{v}_j \to \mathbf{v}, \mathbb{P}$-a.s. as $j \to \infty$) taking values in a countable subset of $\mathcal{D}^f$. By continuity we have

$$
\sup_{s \in [0, T]} |\hat{Y}^{\mathbf{v}_j,0}_s - \hat{Y}^{\mathbf{v},0}_s| \to 0,
$$

$\mathbb{P}$-a.s., as $j \to \infty$. Moreover, due to continuity of the map $\mathbf{v} \mapsto \xi^\mathbf{v}$ we have

$$
\xi^{\mathbf{v}_j} \to \xi^\mathbf{v}
$$

and by Assumption 3.3 we get
\[
\int_0^T |f^{v_j}(s, \hat{Y}_{s}^{v_j,0}, \hat{Z}_{s}^{v_j,0}) - f^v(s, \hat{Y}_{s}^{v,0}, \hat{Z}_{s}^{v,0})|\,ds \\
\leq \int_0^T |f^{v_j}(s, \hat{Y}_{s}^{v_j,0}, \hat{Z}_{s}^{v_j,0}) - f^v(s, \hat{Y}_{s}^{v,0}, \hat{Z}_{s}^{v,0})|\,ds \\
+ \int_0^T |f^v(s, \hat{Y}_{s}^{v,0}, \hat{Z}_{s}^{v,0}) - f^v(s, \hat{Y}_{s}^{v,0}, \hat{Z}_{s}^{v,0})|\,ds \\
\leq \int_0^T (1 + 2|\hat{Z}_{s}^{v,0}|) \Lambda_s^{v,0} \,ds \\
+ \int_0^T (k_j|\hat{Y}_{s}^{v_j,0} - \hat{Y}_{s}^{v,0}| + L_s^{v_j}|\hat{Z}_{s}^{v_j,0} - \hat{Z}_{s}^{v,0}|)\,ds
\]
which tends to 0, \(\mathbb{P}\)-a.s., as \(j \to \infty\). Moreover, for each \(j \geq 0\), the pair \((\hat{Y}^{v_j}, \hat{Z}^{v_j})\) solves (3.10) with control \(v_j\) and we conclude that
\[
\hat{Y}_{v,0} = \lim_{j \to \infty} \left\{ \xi^{v_j} + \int_\eta^T f^{v_j}(s, \hat{Y}_{s}^{v_j,0}, \hat{Z}_{s}^{v_j,0})\,ds - \int_\eta^T \hat{Z}_{s}^{v_j,0} \,dW_s \right\} \\
= \xi^v + \int_\eta^T f^v(s, \hat{Y}_{s}^{v,0}, \hat{Z}_{s}^{v,0})\,ds - \int_\eta^T \hat{Z}_{s}^{v,0} \,dW_s,
\]
for each \(\eta \in \mathcal{T}\) and strong consistency follows. \(\square\)

We now turn to the reflected BSDEs (3.11). To obtain estimates for the triple \((Y^{v,k}, Z^{v,k}, K^{v,k})\) we rely on Proposition 2.4 to reduce the system of reflected BSDEs to a single non-reflected BSDE with jumps. We, thus, introduce the following BSDE:

**Definition 3.7** For \(v, u \in \mathcal{U}^\ell\), let the pair \((U^{v,u}, V^{v,u}) \in S^2_\mathcal{T} \times \mathcal{H}^2\) (recall that \(S^2_\mathcal{T}\) is the set of \(\mathcal{F}_{\mathcal{T}}\)-measurable càglàd processes with finite \(S^2\)-norm) be the unique solution to the BSDE

\[
U_{v,u} = \xi^{v,u} + \int_t^T f^{v,u}(s, U_s^{v,u}, V_s^{v,u})\,ds - \int_t^T V_s^{v,u} \,dW_s \\
- \sum_{j=1}^{N} \mathbb{I}_{[\tau_j \geq t]} c^{v,u}_{j-1}(\tau_j, \beta_j), \tag{3.12}
\]

whenever a unique solution exists and let \(U^{v,u} \equiv -\infty\), otherwise.

**Proposition 3.8** For each \(k \geq 0\), \(v \in \mathcal{U}^\ell\) and \(u \in \mathcal{U}^l\) the BSDE (3.12) admits a unique solution and \(V^{v,u} \in \mathcal{H}^2_\mathcal{Q}\) for all \(\mathcal{Q} \in \mathfrak{Q}^v\).

**Proof** Existence of a unique solution to (3.12) follows from repeated use of Proposition 2.4 since the intervention costs belong to \(L^p(\Omega, \mathcal{F}, \mathbb{P})\) for all \(p \geq 1\). Moreover, a similar argument gives that
\[ \| V^v,u \|_{H^p} \leq C E \left[ | \xi^v |^2 + \int_0^T \| f^{v,u}(s, 0, 0) \|^{2p} \, ds \right] + C E \left[ | \rho^v |_{L^q}^{1/q} \right] \leq C. \]

Now,
\[ \| V^u,v \|_{H^2}^2 = E \left[ E(\xi \ast W)_T \int_0^T | V^u,v_s |^2 \, ds \right] \leq E \left[ | E(\xi \ast W)_T |^{q'/q} \right]^{1/q'} \| V^u,v \|_{H^q}^2 \]
and the assertion follows. \qed

In addition, we introduce the following notation:

**Definition 3.9** For \( v, u \in U \) such that (3.12) admits a unique solution, we define
\[ \gamma^v,u_s := \frac{f^{v,u}(s, U^v_s, U^v_s) - f^{v,u}(s, 0, V^v,u)}{U^v_s} \mathbb{1}_{[U^v_s \neq 0]} \]
and for \( 0 \leq s \leq t \leq T \), we set \( e^v,u_s := e^t_s \gamma^v,u \, dr \) and \( e^v,u_t := e^v,u_0 \). Moreover, we define
\[ \xi^v,u := \frac{f^{v,u}(s, 0, V^v,u) - f^{v,u}(s, 0, 0)}{| V^v,u_s |^2} (V^v,u_s)^\top \mathbb{1}_{[V^v,u_s \neq 0]} \]
and let \( \mathbb{Q}^v,u := \mathbb{Q}^\xi^v,u \), the probability measure, equivalent to \( \mathbb{P} \), under which \( W^v,u := W - \int^\cdot_0 \xi^v,u_s \, ds \) is a Brownian motion.

Before we move on to show that Hypothesis **RBSDE.l** holds for all \( l \geq 0 \) we give three helpful lemmas. **Lemma 3.10** Assume that Hypothesis **RBSDE.l** holds for some \( l \geq 0 \), then for each \( p \geq 1 \), there is a \( C > 0 \) (that does not depend on \( l \)) such that
\[ \| (\sup_{b \in U} | Y^v_o(t,b) | : t \in [0, T]) \|_{S^p} \leq C. \] (3.13)
Proof We let \( \tilde{v} := v \circ (t, b) \) and set \( \tau_i^* := \inf \{ s \geq t : Y_s^\tilde{v},l = \sup_{b \in U} \{ Y_s^{\tilde{v} \circ (s, b),l-1} - c^\tilde{v}(s, b) \} \} \wedge T \) and have by Proposition 2.4 and consistency that

\[
Y_t^\tilde{v},l = 1_{[\tau_i^* < T]} \sup_{b' \in U} \{ Y_t^{\tilde{v} \circ (\tau_i^*, b'),l-1} - c^\tilde{v}(\tau_i^*, b') \} + 1_{[\tau_i^* = T]} \xi^\tilde{v} \]

\[
= \int_t^{\tau_i^*} f^\tilde{v}(s, Y_s^\tilde{v},l, Z_s^\tilde{v},l) ds - \int_t^{\tau_i^*} Z_s^\tilde{v},l dW_s
\]

where \( \beta_i^* \) can be chosen to be \( \mathcal{F}_{\tau_i^*} \)-measurable by continuity of the map \( b' \mapsto Y_t^{\tilde{v} \circ (\tau_i^*, b'),l-1} - c^\tilde{v}(\tau_i^*, b') \) and the measurable selection theorem (see e.g. Chapter 7 in [3] or [11]).

Now, we can continue and inductively define \( \tau_j^* := \inf \{ s \geq \tau_{j-1}^* : Y_s^{\tilde{v} \circ (\tau_1^*, ..., \tau_{j-1}^*, \beta_1^*, ..., \beta_{j-1}^*),l-j} \geq 1 - c^\tilde{v}(\tau_1^*, ..., \tau_{j-1}^*, \beta_1^*, ..., \beta_{j-1}^*) \circ (s, b),l-j \} \), where \( \beta_j^* \) can be chosen to be \( \mathcal{F}_{\tau_j^*} \)-measurable by continuity of the map \( b' \mapsto Y_t^{\tilde{v} \circ (\tau_1^*, ..., \tau_{j-1}^*, \beta_1^*, ..., \beta_{j-1}^*),l-j} - c^\tilde{v}(\tau_1^*, ..., \tau_{j-1}^*, \beta_1^*, ..., \beta_{j-1}^*) \circ (s, b) \} \wedge T \) for \( j = 1, \ldots, l \), and take \( \beta_N^* \) to be the corresponding \( \mathcal{F}_{\tau_N^*} \)-measurable maximizer. By induction we get that

\[
Y_t^\tilde{v},l = \xi^{v_{out}^*} + \int_t^T \sum_{j=0}^{N^*} 1_{[\tau_j^*,\tau_{j+1}^*)}(s) f^{v_{out}^*[u^*]}(s, Y_s^{\tilde{v} \circ [u^*],l-j}, Z_s^{\tilde{v} \circ [u^*],l-j}) ds - \int_t^T \sum_{j=0}^{N^*} 1_{[\tau_j^*,\tau_{j+1}^*)}(s) Z_s^{\tilde{v} \circ [u^*],l-j} dW_s - \sum_{j=1}^{N^*} c^{v_{out}^*[u^*]}(\tau_j^*, \beta_j^*),
\]

(3.14)

where \( u^* := (\tau_1^*, ..., \tau_{N^*}^*; \beta_1^*, ..., \beta_{N^*}^*) \) with \( N^* := \max \{ j \in \{0, \ldots, l \} : \tau_j^* < T \} \) and using the convention that \( \tau_0^* = 0 \) and \( \tau_{N^*+1}^* = T \).

In particular, (3.14) implies by comparison and positivity of the intervention cost that \( U_t^{v_{out},\emptyset} \leq Y_t^\tilde{v},l \leq \sup_{u \in \mathcal{U}} U_t^{v_{out} \circ [u],\emptyset} \), \( \mathbb{P} \)-a.s., and we find that \( |Y_t^\tilde{v},l| \leq \sup_{u \in \mathcal{U}} |Y_t^{v_{out} \circ [u],\emptyset}| \). Furthermore, by continuity of the map \( b \mapsto Y_t^{v_{out} \circ (t,b)} \) we can find a \( \beta^* \in \mathcal{I}(t) \) such that \( \sup_{b \in U} |Y_t^{v_{out} \circ (t,b)}| = |Y_t^{v_{out} \circ (t,\beta^*)}| \), \( \mathbb{P} \)-a.s., and we conclude that \( \sup_{b \in U} |Y_t^{v_{out} \circ (t,b)}| \leq \sup_{u \in \mathcal{U}^{t+1}} |U_t^{v_{out} \circ [u],\emptyset}| \). Now, for arbitrary \( u \in \mathcal{U}^T \) we have that

\[
U_t^{v_{out} \circ [u],\emptyset} = e_t^{v_{out} \circ [u],\emptyset} + \int_t^T e_{t,s}^{v_{out} \circ [u],\emptyset} f^{v_{out} \circ [u]}(s, 0, 0) ds - \int_t^T e_{t,s}^{v_{out} \circ [u],\emptyset} V_s^{v_{out} \circ [u],\emptyset} dW_s^{v_{out} \circ [u],\emptyset}.
\]
This gives, since $V^{v_0, \emptyset} \in \mathcal{H}_{Q^{v_0, \emptyset}}^2$ (see Proposition 3.8), that
\[
|U_t^{v_0, \emptyset}|^p \leq C E^{Q^{v_0, \emptyset}} \left[ |\xi^{v_0}|^p + \int_t^T |f^{v_0}(s, 0, 0)|^p \, ds \right] \mathcal{F}_t \\
\leq C E \left[ |\mathcal{E}(\xi^{v_0, \emptyset} \ast W)_{t, T}|^{r'} |\mathcal{F}_t| \right]^{1/r'} E \left[ |\xi^{v_0}|^p \right]^{1/r} \\
+ \int_t^T |f^{v_0}(s, 0, 0)|^p \, ds |\mathcal{F}_t|^{1/r} \leq C (R_t^v)^{1/r'} (\tilde{K}_t^{v, p^r})^{1/r}, \quad (3.15)
\]
where the last inequality follows by Assumption 3.3 since $|\xi^{v_0, \emptyset}| \leq L^{v_0}$. In particular, continuity of $(\sup_{b \in U} Y_t^{v_0(t, b), l} : t \in [0, T])$ implies that, outside of a $\mathbb{P}$-null set, we have
\[
\sup_{b \in U} |Y_t^{v_0(t, b), l}|^p \leq C (R_t^v)^{1/r'} (\tilde{K}_t^{v, p^r})^{1/r} \quad \text{for all } t \in [0, T].
\]
We can thus apply Hölder’s inequality to find that
\[
\|(\sup_{b \in U} Y_t^{v_0(t, b), l} : t \in [0, T])\|_{S^p} \leq C \|(R_t^v)^{1/r'} (\tilde{K}_t^{v, p^r})^{1/r}\|_{S^1} \\
\leq C R_t^v \|\tilde{K}_t^{v, p^r}\|_{S^1}^{1/r} \leq C
\]
by Lemma 3.4 and since $\tilde{K}_t^{v, p} \in S^1$ for all $p \geq 1$. \hfill \Box

**Lemma 3.11** Assume that Hypothesis RBSDE.l holds for some $l \geq 0$, then for each $p \geq 1$ there is a $C > 0$, that does not depend on $l$, such that
\[
\|Y^{v, l+1}\|_{S^p} + \|Z^{v, l+1}\|_{H^p} + \|K^{v, l+1}\|_{S^p} \leq C,
\]
for all $v \in \mathcal{U}^l$.

**Proof** This is immediate from Proposition 2.4 and Lemma 3.10. \hfill \Box

**Lemma 3.12** Assume that Hypothesis RBSDE.l holds for some $l \geq 0$, then for each $\kappa \geq 0$ and $p \geq 1$, there is a $C > 0$ and a $p' \geq 1$ such that for any $v, v' \in \mathcal{D}\kappa$, we have
\[
\|(\sup_{b \in U} |Y_t^{v_0(t, b), l} - Y_t^{v_0(t, b), l}| : t \in [0, T])\|_{S^{p'}} \leq C \|v' - v\|_{\mathcal{D}^{p'\kappa}}.
\]

**Proof** We let $u^*$ and $u^t := (\tau^t_1, \ldots, \tau^t_N, \beta_1^t, \ldots, \beta^t_N)$ be the controls obtained by repeating the construction in the proof of Lemma 3.11, starting from $Y_t^{v_0(t, b), l}$ and $Y_t^{v_0(t, b), l}$, respectively, instead of $Y_t^{v_0(t, b), l}$. By Proposition 3.8 it follows that for each
\( v \in \mathcal{U}^I \) and \( u \in \mathcal{U}^{I+1} \), there is a unique pair \((U^{v,u}, V^{v,u}) \in S^2 \times \mathcal{H}^2 \) that solves (3.12), i.e.

\[
U^v_{t,u} = \xi^{v,u} + \int_t^T f^{v,u}(s, U^{v,u}_s, V^{v,u}_s)ds - \int_t^T V^{v,u}_s dW_s - \sum_{j=1}^N \mathbb{1}_{[\tau_j \geq t]} c^{v[u]}(\tau_j, \beta_j).
\]

By a trivial argument we find that \( Y^{v_{o(t,b),l}}_t = U^{v_{o(t,b),u}}_t = \text{ess sup}_{u \in \mathcal{U}^{I+1}} U^{v_{o(t,b),u}}_t \) and conclude that

\[
|Y^{v_{o(t,b),l}}_t - Y^{v_{o(t,b),l}}_t| \leq |U^{v_{o(t,b),u}}_t - U^{v_{o(t,b),u}}_t| + |U^{v_{o(t,b),u}}_t - U^{v_{o(t,b),u}}_t| \\
\leq 2 \text{ess sup}_{u \in \mathcal{U}^{I+1}} |U^{v,u}_t - U^{v,u}_t|
\]

Letting \( \delta U^u := U^{v,u}_t - U^{v,u}_t \) and \( \delta V^u := V^{v,u}_t - V^{v,u}_t \) and writing \( \delta \Box^u := \Box^{v,u} - \Box^{v,u} \) (for \( \Box = \xi, f \) and \( c \)) gives

\[
e_t \delta U^u_t = e_T \delta \xi^u_t + \int_t^T e_s \delta f^u(s, U^{v,u}_s, V^{v,u}_s)ds - \int_t^T e_s \delta V^u_s dW_s - \sum_{j=1}^N e_{\tau_j} \delta c^{[u]}(\tau_j, \beta_j), \quad (3.18)
\]

where \( e_t := e_t^{I_{\mathcal{U}^I}} \gamma^{ds} \) with

\[
\gamma_s := \frac{f^{v,u}(s, U^{v,u}_s, V^{v,u}_s) - f^{v,u}(s, U^{v,u}_s, V^{v,u}_s)}{U^{v,u}_s - U^{v,u}_s} \mathbb{1}_{[U^{v,u}_s \neq U^{v,u}_s]}
\]

and \( W^\xi := W - \int_0^T \zeta ds, \) with

\[
\zeta_s := \frac{f^{v,u}(s, U^{v,u}_s, V^{v,u}_s) - f^{v,u}(s, U^{v,u}_s, V^{v,u}_s)}{|V^{v,u}_s - V^{v,u}_s|^2} (V^{v,u}_s - V^{v,u}_s)^\top \mathbb{1}_{[V^{v,u}_s \neq V^{v,u}_s]}.
\]

is a Brownian motion under the measure \( \mathbb{Q}^\xi \in \mathcal{P}^\xi_t \) given by \( d\mathbb{Q}^\xi = \mathcal{E}(\xi \ast W)_T d\mathbb{P} \).
For \( u \in U_t^{+1} \), by again appealing to Proposition 3.8, we have that \( \delta V^u \in \mathcal{H}_{t\mathbb{Q}^u}^2 \) and since \( e^{-k_fT} \leq e_t \leq e^{k_fT} \), taking conditional expectation in (3.18) gives

\[
|\delta U_t^u| \leq C\mathbb{E}_{t\mathbb{Q}^u}^{|\delta \xi^u| + \int_0^T |\delta f^u(s, U_s^{Y,u}, V_s^{Y,u})|ds + \sum_{j=1}^N |\delta e^{[u]_{j-1}}(\tau_j, \beta_j)| |F_t|}
\leq C\mathbb{E}_{t\mathbb{Q}^u}^{|\delta \xi^u| + \left( \int_t^T |\Lambda_s^{Y,Y,u}|^2 ds \right)^{1/2} \left( 1 + \int_t^T |V_s^{Y,u}|^2 ds \right)^{1/2} + \sum_{j=1}^N |\delta e^{[u]_{j-1}}(\tau_j, \beta_j)| |F_t|},
\]

where we have used the Lipschitz condition on \( f \) to arrive at the last inequality. In particular, since \( N \leq l+1, \mathbb{P}\)-a.s., this gives that

\[
|\delta U_t^u|^p \leq C\mathbb{E}_{t\mathbb{Q}^u}^{|\delta \xi^u|^p + \left( \int_t^T |\Lambda_s^{Y,Y,u}|^2 ds \right)^{p/2} \left( 1 + \left( \int_t^T |V_s^{Y,u}|^2 ds \right)^{p/2} \right) + \sum_{j=1}^N |\delta e^{[u]_{j-1}}(\tau_j, \beta_j)|^p |F_t|}
\leq C (R_t^{Y})^{1/r'} \left( (\tilde{K}_t^{Y,Y,l+1,pr})^{1/r} + (\tilde{K}_t^{Y,Y,l+1,2pr})^{1/2r} \right) \left( 1 + \mathbb{E}\left[ \left( \int_t^T |V_s^{Y,u}|^2 ds \right)^{pr} |F_t| \right]^{1/2r} \right),
\]

as \( |\xi_t| \leq L_t^{Y,ou} \). Applying the standard change of measure approach it follows that

\[
\mathbb{E}_{t\mathbb{Q}^u}^{|\int_t^T |V_s^{Y,u}|^2 ds |F_t|} \leq C\mathbb{E}_{t\mathbb{Q}^u}^{2p} \left[ |\xi^{Y,ou}|^{2p} + \left( \int_t^T |f^{Y,ou}(s, 0, 0)|ds \right) \right] + \sum_{j=1}^N |e^{[u]_{j-1}}(\tau_j, \beta_j)|^{2p} |F_t|.
\]

Changing back to \( \mathbb{P}\)-expectation we get by the Girsanov theorem that

\[
\mathbb{E}\left[ \left( \int_t^T |V_s^{Y,u}|^2 ds \right)^{pr} |F_t| \right] = \mathbb{E}_{t\mathbb{Q}^u}^{|\int_t^T |V_s^{Y,u}|^2 ds |F_t|} \leq C (R_t^{Y})^{1/r'} \mathbb{E}_{t\mathbb{Q}^u}^{|\xi^{Y,ou}|^{2pr^2} + \left( \int_t^T |f^{Y,ou}(s, 0, 0)|ds \right)^{2pr^2} + \sum_{j=1}^N |e^{[u]_{j-1}}(\tau_j, \beta_j)|^{2pr^2} |F_t|^{1/r} \right) \leq C (R_t^{Y})^{1/r'} (\tilde{K}_t^{Y,Y,2pr^3})^{1/r^2}.
\]
Combined, this gives that
\[
|\delta U_t| \leq C (R_t^{1\prime})^{1\prime} ((\tilde{K}_t^{\prime, l+1, pr})^{1\prime} + (\tilde{K}_t^{\prime})^{1/2r'}) (R_t^{1/2rr'})^{(R_t^{1/2rr'})^1/2r^3})
\]

Hence, repeated application of Hölder’s inequality gives, with \(\delta Y_t := \sup_{b \in U} |Y_t^{\nu_o(t,b), l} - Y_t^{\nu(t,b), l}|\), that
\[
\|\delta Y\|_{S^p}^p \leq C \left( \frac{\|R^{1\prime}\|_{S^l}^{1\prime} \|\tilde{K}^{\prime, l+1, pr}\|_{S^l}^{1\prime} + \|\tilde{K}^{\prime, l+1, 2pr}\|_{S^l}^{1/2}}{1 + \|\tilde{K}^{\prime, 2pr^3}\|_{S^l}^{1/2r^3}} \right)
\]

Hence, as \(\|\tilde{K}^{\prime, 2pr^3}\|_{S^l}^1 \leq \|\tilde{K}^{\prime, 2pr^3}\|_{S^2} \leq C\) it follows that
\[
\|\delta Y\|_{S^p}^p \leq C \left( \frac{\|\tilde{K}^{\prime, l+1, pr}\|_{S^l}^{1\prime} + \|\tilde{K}^{\prime, l+1, 2pr}\|_{S^l}^{1/2}}{1 + \|\tilde{K}^{\prime, 2pr^3}\|_{S^l}^{1/2r^3}} \right)
\]

and the result follows by Definition 3.2.iv).

**Proposition 3.13** Hypothesis RBSDE.\(1\) holds for all \(l \geq 0\).

**Proof** We note that the triple \((Y^{v,l+1}, Z^{v,l+1}, K^{v,l+1})\) solves a reflected BSDE with barrier \((\sup_{b \in U} \{-c^v(t,b) + Y_t^{v_o(t,b), l} : t \in [0,T]\})\). By Lemma 3.10 we find that \((\sup_{b \in U} \{-c^v(t,b) + Y_t^{v_o(t,b), l} : t \in [0,T]\}) \in S^p\) for all \(p \geq 1\) uniformly in \(v\). Whenever the statement in Hypothesis RBSDE.\(l\) holds for some \(l \geq 0\), then Proposition 2.4 guarantees the existence of a unique triple \((Y^{v,l+1}, Z^{v,l+1}, K^{v,l+1})\) solving (3.11) with \(k = l + 1\). Moreover, we have
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |\sup_{b \in U} \{-c^v(t,b) + Y_t^{v_o(t,b), l} - Y_t^{v_o(t,b), l}\}|^p \right] 
\]
\[
\leq C \mathbb{E} \left[ \sup_{t \in [0,T]} \sup_{b \in U} |c^v(t,b) - c^v(t,b)|^p + \sup_{t \in [0,T]} |Y_t^{v_o(t,b), l} - Y_t^{v_o(t,b), l}|^p \right].
\]

By (2.5) of Proposition 2.4, Assumption 3.3 and Lemma 3.12 it, thus, follows by repeating the argument in the proof of Proposition 3.6 that for each \(p \geq 1\), there is a \(p' \geq 1\) such that

\[\square\]
\[ \|Y^{\nu,l+1} - Y^{\nu,l+1}\|_{S''} + \|Z^{\nu,l+1} - Z^{\nu,l+1}\|_{H'} + \|K^{\nu,l+1} - K^{\nu,l+1}\|_{S''} \leq C\|\nu' - \nu\|_{D'}. \]

For \( \nu', \nu \in D^x \) we let \( p = (m+1)\kappa + 1 \) and Kolmogorov’s continuity theorem implies the existence of a family of processes \((\hat{Y}^v, \hat{Z}^v, \hat{K}^v : v \in D^x)\), with \((\hat{Y}^v, \hat{Z}^v, \hat{K}^v) \in S^2 \times H^2 \times S^2\), such that (outside of a \( \mathbb{P}\)-null set) \( v \mapsto \hat{Y}^v \) and \( v \mapsto \hat{Z}^v \) are uniformly continuous, \( L^2([0, T]) \) continuous (and that \( v \mapsto \int_T^s Y_t^vdW_t \) is continuous in \( v \) uniformly in \( s \)) and uniformly continuous, respectively, and moreover \((\hat{Y}^v_s, \hat{Z}^v_s, \hat{K}^v_s) = (Y^v_s, Z^v_s, K^v_s), \mathbb{P}\)-a.s. for all \( v \in D^x \) and \( s \in [0, T] \).

Now, taking countable unions this extends to \( D^f \), and for all \( v \in D^f \), the triple \((\hat{Y}^v, \hat{Z}^v, \hat{K}^v)\) solves the BSDE (3.11) for \( Y^{\nu,l+1} \).

Furthermore, for any \( v \in U^f \) and any approximating sequence \((v_j)_{j \geq 0}\) taking values in a countable dense subset of \( D^f \) with \( v_j \to v, \mathbb{P}\)-a.s., and \( v_j \in U^f \), we have by repeating the argument in the proof of Proposition 3.6 that

\[
\hat{Y}_\eta^v = \lim_{j \to \infty} \left\{ \xi^{v_j} + \int_\eta^T f^{v_j}(s, \hat{Y}_s^{v_j}, \hat{Z}_s^{v_j})ds - \int_\eta^T \hat{Z}_s^{v_j}dW_s + \hat{K}_T^{v_j} - \hat{K}_\eta^{v_j} \right\} = \xi^v + \int_\eta^T f^v(s, \hat{Y}_s^v, \hat{Z}_s^v)ds - \int_\eta^T \hat{Z}_s^v dW_s + \hat{K}_T^v - \hat{K}_\eta^v,
\]

for all \( \eta \in T \). Finally, by Helly’s convergence theorem (see e.g. [23], p. 370) we have

\[
\lim_{j \to \infty} \int_0^T \left( \hat{Y}_s^v - \sup_{b \in U} \{-c^v(s, b) + \hat{Y}_s^{v_0(s,b),l}\} \right) d\hat{K}_s^{v_j} = \int_0^T \left( \hat{Y}_s^v - \sup_{b \in U} \{-c^v(s, b) + \hat{Y}_s^{v_0(s,b),l}\} \right) d\hat{K}_s^v
\]

and since

\[
\int_0^T \left| \hat{Y}_s^v - \sup_{b \in U} \{-c^v(s, b) + \hat{Y}_s^{v_0(s,b),l}\} - \left( \hat{Y}_s^{v_j} - \sup_{b \in U} \{-c^{v_j}(s, b) + \hat{Y}_s^{v_j_0(s,b),l}\} \right) \right| d\hat{K}_s^{v_j} \leq \hat{K}_T^{v_j} \left( \sup_{x \in [0,T] \times U} |\hat{Y}_s^v - \hat{Y}_s^{v_j}| + \sup_{(s,b) \in [0,T] \times U} |c^v(s, b) - c^{v_j}(s, b)| \right)
\]

\[
+ \sup_{(s,b) \in [0,T] \times U} |\hat{Y}_s^{v_0(s,b),l} - \hat{Y}_s^{v_j_0(s,b),l}|.
\]
which tends to zero, \( \mathbb{P} \)-a.s., as \( j \to \infty \) we get that
\[
\int_0^T \left( \hat{Y}_s^v - \sup_{b \in U} \{-c^v(s, b) + \hat{Y}_s^{v_0(s, b), l}\} \right) d\hat{K}_s^v
= \lim_{j \to \infty} \int_0^T \left( \hat{Y}_s^{v_j} - \sup_{b \in U} \{-c^{v_j}(s, b) + \hat{Y}_s^{v_j(s, b), l}\} \right) d\hat{K}_s^{v_j}
= 0
\]
and we conclude that Hypothesis \textbf{RBSDE} \( l + 1 \) holds as well. The statement of the proposition now follows by an induction argument.

\[\square\]

### 3.3 Convergence of the Scheme

We now show that there exists a limit family of triples \((\tilde{Y}^v, \tilde{Z}^v, \tilde{K}^v : v \in \mathcal{U}^\ell) := \lim_{k \to \infty}(Y^{v, k}, Z^{v, k}, K^{v, k} : v \in \mathcal{U}^\ell)\) that solves the sequential system of reflected BSDEs (1.1). This result relies heavily upon the following two lemmas and their corollaries.

**Lemma 3.14** For \( v \in \mathcal{U}^\ell \) and \( k \geq 0 \), assume that \( u^* \in \mathcal{U}^k_t \) is such that \( U_t^{v, u^*} = \text{ess sup}_{u \in \mathcal{U}^k_t} U_t^{v, u} \). Then, for each \( p \geq 1 \), there is a \( C > 0 \), that does not depend on \( v \) or \( k \), such that
\[
\mathbb{E}\left[ \sup_{s \in [t, T]} |U_s^{v, u^*}|^p + \left( \int_t^T |V_s^{v, u^*}|^2 ds \right)^{p/2} \left| F_t \right| \right] \leq C(\tilde{R}_t^{v} R_t^{v})^{1/r'} (1 + \tilde{R}_t^{v, 2pr^3})^{1/r^3}.
\]

**Proof** For the bound on \( U^{v, u^*} \) we note that by (3.15) we have
\[
|U_s^{v, u^*}|^p = |Y_s^{v_0(u^*(s^-)), k-N*(s^-)}|^p \leq C(R_s^{v, u^*(s^-)})^{1/r'} (\tilde{K}_s^{v_0(u^*(s^-)), pr})^{1/r}
\]
and since by definition we have \( R_s^{v, u^*(s)} = R_s^{v, u^*} \), we find that
\[
\mathbb{E}\left[ \sup_{s \in [t, T]} |U_s^{v, u^*}|^p \left| F_t \right| \right] \leq \mathbb{E}\left[ \sup_{s \in [t, T]} (R_s^{v, u^*})^{1/r'} (\tilde{K}_s^{v, u^*(s^-), pr})^{1/r} \left| F_t \right| \right]
\]
\[\leq \text{ess sup}_{u \in \mathcal{U}^k_t} \mathbb{E}\left[ \sup_{s \in [t, T]} R_s^{v, u^*} \left| F_t \right| \right]^{1/r'} \mathbb{E}\left[ \sup_{s \in [t, T]} \tilde{K}_s^{v_0(u^*(s^-)), pr} \left| F_t \right| \right]^{1/r}
\]
\[\leq C(\tilde{R}_t^{v})^{1/r'} (1 + \tilde{R}_t^{v, 2pr^3})^{1/2r} \quad (3.19)
\]
where we have used Jensen’s inequality and (3.3) to reach the last inequality. The first bound then follows by Jensen’s inequality since \( \tilde{R}_t^{v} R_t^{v} \geq 1, \mathbb{P}\)-a.s.
We apply Ito’s formula to \((U^{v,u*})^2\) and get that

\[
|U_t^{v,u*}|^2 + \int_t^T |V_s^{v,u*}|^2 \, ds \\
= |\xi_{vou*}|^2 + 2 \int_t^T U_s^{v,u*} f^{vou*}(s, U_s^{v,u*}, V_s^{v,u*}) \, ds - 2 \int_t^T U_s^{v,u*} V_s^{v,u*} \, dW_s \\
+ \sum_{j=1}^{N^*} \left( -2U_{\tau_j}^{v,u*} c^{vou*[u*]j-1}(\tau_j^*, \beta_j^*) + |c^{vou*[u*]j-1}(\tau_j^*, \beta_j^*)|^2 \right) \\
\leq |\xi_{vou*}|^2 + 2 \int_t^T \left( \gamma_s^{v,u*} |U_s^{v,u*}|^2 + U_s^{v,u*} f^{vou*}(s, 0, 0) \, ds \\
- 2 \int_t^T U_s^{v,u*} V_s^{v,u*} \, dW_s^{v,u*} + 4 \sup_{s \in [t,T]} |U_s^{v,u*}| \sum_{j=1}^{N^*} |c^{vou*[u*]j-1}(\tau_j^*, \beta_j^*)|,
\]

where the last term appears after applying the relation \(|c^{vou*[u*]j-1}(\tau_j^*, \beta_j^*)| \leq 2 \sup_{s \in [t,T]} |U_s^{v,u*}|. Using the relation \(ab \leq \frac{1}{2} (\kappa a^2 + \frac{1}{\kappa} b^2)\) for \(\kappa > 0\) we get

\[
|U_t^{v,u*}|^2 + \int_t^T |V_s^{v,u*}|^2 \, ds \leq |\xi_{vou*}|^2 + (C + 2\kappa) \sup_{s \in [t,T]} |U_s^{v,u*}|^2 + \int_t^T |f^{vou*}(s, 0, 0)|^2 \, ds \\
- 2 \int_t^T U_s^{v,u*} V_s^{v,u*} \, dW_s^{v,u*} + \frac{2}{\kappa} \left( \sum_{j=1}^{N^*} c^{vou*[u*]j-1}(\tau_j^*, \beta_j^*) \right)^2.
\]

On the other hand, applying the usual manipulations to (3.12) we get

\[
U_t^{v,u*} = e_{t,T}^{v,u*} \xi_{vou*}^{v,u*} + \int_t^T e_{t,s}^{v,u*} f^{vou*}(s, 0, 0) \, ds - \int_t^T e_{t,s}^{v,u*} V_s^{v,u*} \, dW_s^{v,u*} \\
- \sum_{j=1}^{N^*} e_{t,\tau_j}^{v,u*} c^{vou*[u*]j-1}(\tau_j^*, \beta_j^*). \\
\]

Rearranging terms now gives us (with \(e_{t,.} = e_{t,.}^{v,u*}\))

\[
\sum_{j=1}^{N} e_{t,\tau_j} c^{vou*[u*]j-1}(\tau_j^*, \beta_j^*) = e_{t,T} \xi_{vou*}^{v,u*} + \int_t^T e_{t,s} f^{vou*}(s, 0, 0) \, ds \\
- \int_t^T e_{t,s} V_s^{v,u*} \, dW_s^{v,u*} - U_t^{v,u*}
\]
From (3.20) we have that

\[
\left( \sum_{j=1}^{N^*} c^{u_0[u^*]_j-1}(\tau_j^*, \beta_j^*) \right)^2 \leq C(|\xi^{v_0u^*}|^2 + \int_t^T |f^{v_0u^*}(s, 0, 0)|^2 ds \\
+ \left| \int_t^T e_{t,s} V_s^{v,u^*} dW_s^{v,u^*} \right|^2 + |U_t^{v,u^*}|^2).
\]

Putting this together gives

\[
|U_t^{v,u^*}|^2 + \int_t^T |V_s^{v,u^*}|^2 ds \leq C \left( 1 + \kappa + \frac{1}{\kappa} \right) \left( |\xi^{v_0u^*}|^2 + \int_t^T |f^{v_0u^*}(s, 0, 0)|^2 ds + \sup_{s \in [t,T]} |U_s^{v,u^*}|^2 \right) \\
- 2 \int_t^T U_s^{v,u^*} V_s^{v,u^*} dW_s^{v,u^*} + \frac{C}{\kappa} \left| \int_t^T e_{t,s} V_s^{v,u^*} dW_s^{v,u^*} \right|^2.
\]

Raising both sides to \(p/2\) and taking the conditional expectation we find that

\[
\mathbb{E}^{Q^{v,u^*}} \left[ \left( \int_t^T |V_s^{v,u^*}|^2 ds \right)^{p/2} \bigg| F_t \right] \\
\leq C \mathbb{E}^{Q^{v,u^*}} \left[ \left( 1 + \kappa \frac{p}{2} + \frac{1}{\kappa p/2} \right) \left( |\xi^{v_0u^*}|^p + \int_t^T |f^{v_0u^*}(s, 0, 0)|^p ds \right) \\
+ \sup_{s \in [t,T]} |U_s^{v,u^*}|^p \right] + \left( \int_t^T |U_s^{v,u^*} V_s^{v,u^*}|^2 ds \right)^{p/4} + \frac{1}{\kappa p/2} \left( \int_t^T |V_s^{v,u^*}|^2 ds \right)^{p/2} \bigg| F_t \right]
\]

and since

\[
\mathbb{E}^{Q^{v,u^*}} \left[ \left( \int_t^T |V_s^{v,u^*}|^2 ds \right)^{p/4} \bigg| F_t \right] \\
\leq \frac{1}{2} \mathbb{E}^{Q^{v,u^*}} \left[ \kappa \sup_{s \in [t,T]} |U_s^{v,u^*}|^p + \frac{1}{\kappa} \left( \int_t^T |V_s^{v,u^*}|^2 ds \right)^{p/2} \bigg| F_t \right]
\]

we arrive at the inequality

\[
\mathbb{E}^{Q^{v,u^*}} \left[ \left( \int_t^T |V_s^{v,u^*}|^2 ds \right)^{p/2} \bigg| F_t \right] \\
\leq C \mathbb{E}^{Q^{v,u^*}} \left[ |\xi^{v_0u^*}|^p + \int_t^T |f^{v_0u^*}(s, 0, 0)|^p ds + \sup_{s \in [t,T]} |U_s^{v,u^*}|^p \bigg| F_t \right] (3.21)
\]
by choosing $\kappa > 0$ sufficiently large. Under $\mathbb{P}$ this rewrites as

$$
\mathbb{E}
\left[
\left(\int_t^T |V_{v,u^*}^*|^2 ds\right)^{p/2} \bigg| F_t
\right]
\leq C \left(\tilde{R}_t^v R_t^v \right)^{1/r'} \mathbb{E}
\left[
|\xi_{v,u^*}|^{pr^2} \bigg| F_t
\right]^{1/r'}
+ \int_t^T |f_{v,u^*}^*|^{pr^2} ds
+ \sup_{s \in [t,T]} |U_{v,u^*}^*|^{pr^2} \left|F_t\right|^{1/r}.
$$

(3.22)

The desired result now follows by setting $p \leftarrow pr^2$ in (3.19) and using Jensen’s inequality while noting that $\tilde{R}_t^v \geq 1$, $\mathbb{P}$-a.s.

\[ \square \]

**Corollary 3.15** For $v \in \mathcal{U}^f$, $t \in [0, T]$, $\beta \in \mathcal{I}(t)$ and $k \geq 0$, assume that $u^* \in \mathcal{U}_t^k$ is such that $U_{v, u^*}^* = \text{ess sup}_{u \in \mathcal{U}_t^k} U_{v, u}^*$. Then, for each $p \geq 1$, there is a $C > 0$ that does not depend on $v$ or $k$, such that

$$
\mathbb{E}
\left[
\sup_{s \in [t,T]} |U_{v, u^*}^*|^p + \left(\int_t^T |V_{v, u^*}^*|^2 ds\right)^{p/2} \bigg| F_t
\right]
\leq C \left(\tilde{R}_t^v R_t^v \right)^{1/r'} \left(1 + \tilde{K}_t^v 2^{pr^3}\right)^{1/r^3}.
$$

**Proof** This follows immediately by making suitable manipulations, i.e. setting $v \leftarrow v \circ (t, \beta)$, in the proof of Lemma 3.14.

\[ \square \]

**Lemma 3.16** For $v \in \mathcal{U}^f$ and $k \geq 0$, assume that $u^* \in \mathcal{U}_t^k$ is such that $U_{v, u^*}^* = \text{ess sup}_{u \in \mathcal{U}_t^k} U_{v, u}^*$. Then, for each $p \geq 1$, there is a $C > 0$ that does not depend on $v$ or $k$, such that

$$
\mathbb{E}
\left[
(N^*)^p \bigg| F_t
\right]
\leq C \mathbb{E}
\left[
\left(\sum_{j=1}^N c_j^{v, u^*}_{j-1} (\tau_j^*, \beta_j^*)\right)^p \bigg| F_t
\right]
\leq C \left(\tilde{R}_t^v R_t^v \right)^{1/r'} \left(1 + \tilde{K}_t^v 2^{pr^3}\right)^{1/r^3}.
$$

**Proof** Since the intervention costs are bounded from below by $\delta > 0$, we have

$$
\delta N^* \leq \sum_{j=1}^N c_j^{v, u^*}_{j-1} (\tau_j^*, \beta_j^*).$$
from which the first inequality follows. Now, from (3.20) we have

\[
\mathbb{E}^{Q_v,u^*} \left[ \left( \sum_{j=1}^{N} C_v^{(u^*)}_{j-1} (\tau_j^*, \beta_j^*) \right)^p \left| \mathcal{F}_t \right. \right]
\]

\[
\leq C \mathbb{E}^{Q_v,u^*} \left[ |\xi_v^{(u^*)}|^p + \int_t^T |f^{v,u^*}(s, 0, 0)|^p ds + \left( \int_t^T |V_s^{v,u^*}|^2 ds \right)^{p/2} + |U_t^{v,u^*}|^p \left| \mathcal{F}_t \right. \right]
\]

\[
\leq C \mathbb{E}^{Q_v,u^*} \left[ |\xi_v^{(u^*)}|^p + \int_t^T |f^{v,u^*}(s, 0, 0)|^p ds + \sup_{s \in [t,T]} |U_s^{v,u^*}|^p \left| \mathcal{F}_t \right. \right]
\]

where we have used (3.21) to get the last inequality. Now the result is immediate from the last part in the proof of Lemma 3.14.

\[\Box\]

**Corollary 3.17** For \( v \in \mathcal{U}^f \), \( t \in [0, T] \), \( \beta \in \mathcal{I}(t) \) and \( k \geq 0 \), assume that \( u^* \in \mathcal{U}^f_k \) is such that \( U_t^{v,(t,\beta),u^*} = \text{ess sup}_{u \in \mathcal{U}^f_k} U_t^{v,(t,\beta),u} \). Then, for each \( p \geq 1 \), there is a \( C > 0 \) that does not depend on \( v \) or \( k \), such that

\[
\mathbb{E} \left[ (N^*)^p \left| \mathcal{F}_t \right. \right] \leq C \mathbb{E} \left[ \left( \sum_{j=1}^{N} C_v^{(t,\beta)}(u^*)_{j-1} (\tau_j^*, \beta_j^*) \right)^p \left| \mathcal{F}_t \right. \right]
\]

\[
\leq C (R_t^{v} R_t^{v} R_t^{v})^{1/r'} (1 + K_t^{v,2pr^3})^{1/r^3}.
\]

**Proof** This follows by repeating the argument in the proof of Lemma 3.16 after making the swap \( v \leftarrow v \circ (t, \beta) \).

\[\Box\]

We are now ready to tackle the convergence of the sequence \( Y^{v,k} \), this is done in the following proposition

**Proposition 3.18** There exists a limit family \( (\tilde{Y}^v : v \in \mathcal{U}^f) \) such that for all \( v \in \mathcal{U}^f \) (outside of a \( \mathbb{P} \)-null set) and each \( p \geq 2 \), we have

(i) \( Y^{v,(\cdot),k} \nearrow \tilde{Y}^{v,(\cdot)} \) pointwisely, and

(ii) \( \| \sup_{b \in \mathcal{U}} |\tilde{Y}^{v,(\cdot),b} - Y^{v,(\cdot),b},k| \|_{S^p} \to 0 \) as \( k \to \infty \).

**Proof** The sequence \( (Y^{v,(\cdot),k})_{k \geq 0} \) is non-decreasing and \( \mathbb{P} \)-a.s. bounded by Lemma 3.10. Thus it converges pointwisely, \( \mathbb{P} \)-a.s., and i) follows.

We now turn our focus to the second claim and note that for each \( t \in [0, T] \), continuity and measurable selection implies that there is a \( \beta \in \mathcal{I}(t) \) such that

\[
\sup_{b \in \mathcal{U}} |Y_{t}^{v,(t,b),k} - Y_{t}^{v,(t,b),k'}| = |Y_{t}^{v,(t,b),k} - Y_{t}^{v,(t,b),k'}|
\]

To simplify notation we set \( \tilde{v} \leftarrow v \circ (t, \beta) \) and have by Corollary 3.17 that if \( u^* = (\tau^*_1, \ldots, \tau^*_N; \beta^*_1, \ldots, \beta^*_N) \in \mathcal{U}^f_k \) is such that \( U_t^{\tilde{v},u^*} = Y_{t}^{v,k} \), then \( \mathbb{E}[N^* | \mathcal{F}_t] \leq C (R_t^{v} R_t^{v} R_t^{v})^{1/r'} (1 + K_t^{v,2pr^3})^{1/r^3} \) and, in particular, we find that \( \mathbb{E}[1_{[N^*>k]} | \mathcal{F}_t] \leq C (R_t^{v} R_t^{v} R_t^{v})^{1/r'} (1 + K_t^{v,2pr^3})^{1/r^3} / k' \) for all \( k' \geq 1 \).
For any $k'$ with $0 \leq k' \leq k$, the truncation $[u^*]_{k'}$ belongs to $\overline{U}^k$. We, thus, have

$$U_t[\bar{\nu}^*]_{k'} \leq Y_t[\bar{\nu}^{k'}] \leq Y_t[\bar{\nu}^k].$$

Since $Y_t[\bar{\nu}^k] = U_t[\bar{\nu}^*]$, this gives

$$|Y_t[\bar{\nu}^{k'}] - Y_t[\bar{\nu}^k]| \leq U_t[\bar{\nu}^*] - U_t[\bar{\nu}^k]$$

Moreover, since the intervention costs are positive, we have that

$$U_t[\bar{\nu}^k] - U_t[\bar{\nu}^*] \leq \xi f\tilde{\nu}^k - \xi f\tilde{\nu}^* + \int_t^T (f\tilde{\nu}u^* (s, U_s[\bar{\nu}^k], V_s[\bar{\nu}^k])) ds$$

$$- \int_t^T (V_s[\bar{\nu}^*] - V_s[\bar{\nu}^k]) dW_s.$$
Taking the conditional expectation, using that $V^{\bar{v},u^*}$, $V^{\bar{v},[u^*]_k'} \in \mathbb{H}^2_{\mathcal{L}^2}$ by Proposition 3.8 and noting that the right-hand side is non-zero only when $N^* > k'$ gives

$$U_t^{\bar{v},u^*} - U_t^{\bar{v},[u^*]_{k'}} \leq C \mathbb{E}^{\mathbb{Q}^t}[\mathbb{I}_{[N^* > k']} \left| \mathcal{F}_T \right|] + \int_T^T \left| \left( f^{\bar{v},u^*} - f^{\bar{v},[u^*]_{k'}} \right) (s, U_s^{\bar{v},u^*}, V_s^{\bar{v},u^*}) \right| ds \left| \mathcal{F}_T \right|.$$

By Hölder’s inequality we find that

$$|Y_t^{\bar{v},k} - Y_t^{\bar{v},k'}|^p \leq C \mathbb{E}^{\mathbb{Q}^t}[\mathbb{I}_{[N^* > k']} \left| \mathcal{F}_T \right|]^{1/2} \mathbb{E}^{\mathbb{Q}^t} \left[ \left| \xi^{\bar{v},u^*} - \xi^{\bar{v},[u^*]_{k'}} \right|^2 |2p + |L_T^{\bar{v},u^*}|^4 + |L_T^{\bar{v},[u^*]_k'}|^4 + \left( \int_T^T \left| V_s^{\bar{v},u^*} \right|^2 ds \right)^2 \right]^{1/2} \mathbb{E}^{\mathbb{Q}^t} \left[ \left| \xi^{\bar{v},u^*} - \xi^{\bar{v},[u^*]_{k'}} \right|^2 |2p + \sup_{s \in [t,T]} \left| U_s^{\bar{v},u^*} \right|^2 + |L_T^{\bar{v},u^*}|^4 + |L_T^{\bar{v},[u^*]_k'}|^4 + \left( \int_T^T \left| V_s^{\bar{v},u^*} \right|^2 ds \right)^2 \right]^{1/2} \mathbb{E}^{\mathbb{Q}^t} \left[ \left| \xi^{\bar{v},u^*} - \xi^{\bar{v},[u^*]_{k'}} \right|^2 |2p + \sup_{s \in [t,T]} \left| U_s^{\bar{v},u^*} \right|^2 + |L_T^{\bar{v},u^*}|^4 + |L_T^{\bar{v},[u^*]_k'}|^4 + \left( \int_T^T \left| V_s^{\bar{v},u^*} \right|^2 ds \right)^2 \right]^{1/2} \leq C (C_T^v)^{1/2} (R_t^{\bar{v}} R_t^{\bar{v}} R_t^{\bar{v}})^{1/12} (1 + K_t^{v,2p^3})^{1/12} (1 + K_t^{v,8p^4})^{1/12} / (k')^{1/2},$$

where we have used (3.1) and Corollary 3.15 to arrive at the last inequality. Since $\sup_{b \in U} |\tilde{Y}^{\nu, (\cdot, b), k'} - Y^{\nu, (\cdot, b), k}|$ is continuous and the right hand side of the above equation is a càdlàg process this extends to all $t \in [0, T]$ (outside of a $\mathbb{P}$-null set) and we can take the $S^1$-norm followed by Hölder’s inequality to get that

$$\| \sup_{b \in U} \left| \tilde{Y}^{\nu, (\cdot, b), k'} - Y^{\nu, (\cdot, b), k} \right| \|_{S^p}$$

$$\leq C \|(R_t^{\nu})^{1/12} (R_t^{\nu} R_t^{\nu} R_t^{\nu})^{1/12} (1 + K_t^{v,2p^3})^{1/12} (1 + K_t^{v,8p^4})^{1/12} / (k')^{1/2}.$$

$$\leq C \| R_t^{\nu} \| S^1 \| (R_t^{\nu} R_t^{\nu} R_t^{\nu})^{1/12} (1 + K_t^{v,2p^3})^{1/12} \| S^1 \| / (k')^{1/2}$$

$$\leq C \| R_t^{\nu} \| S^1 \| R_t^{\nu} R_t^{\nu} \| S^1 \| (1 + K_t^{v,8p^4})^{1/2} \| S^1 \|^2 / (k')^{1/2}$$

$$\leq C / (k')^{1/2}.$$
where \( C > 0 \) is independent of \( k, k' \). The last inequality holds since there is a \( C > 0 \) such that

\[
\| \vec{R}^v R^v \vec{R}^v \|_{S^3} \leq \| \vec{R}^v \|_{S^3} \| R^v \| \| \vec{R}^v \|_{S^3} \leq C,
\]

for all \( v \in \mathcal{U}^f \). Finally, taking the limit as \( k \to \infty \), \((i)\) and Fatou’s lemma gives that

\[
\| \sup_{b \in \mathcal{U}} Y^v_{t_0} + f^v \|_{S^p} = C/(k')^{1/2}.
\]

**Proposition 3.19** There is a family \((\tilde{Y}^v, \tilde{K}^v : v \in \mathcal{U}^f)\) such that \((\tilde{Y}^v, \tilde{Z}^v, \tilde{K}^v : v \in \mathcal{U}^f)\) is a solution to (1.1).

**Proof** Having established that \( \| \sup_{b \in \mathcal{U}} Y^v_{t_0} + f^v \|_{S^p} \to 0 \) as \( k, k' \to \infty \) in the previous proposition it follows by Proposition 2.4 that \( \| Z^v_{t_0} - Z^v_{k'} \|_{S^2} \to 0 \) as \( k, k' \to \infty \). In particular, \((Z^v_{t_0})_{k \geq 0}\) is a Cauchy sequence in the Hilbert space \( \mathcal{H}^2 \) and we conclude that there is a \( \tilde{Z}^v \in \mathcal{H}^2 \) such that \( Z^v_{t_0} \to \tilde{Z}^v \) in \( \mathcal{H}^2 \).

Now, letting \( \tilde{K}^v \) be defined by \( \tilde{K}^v = 0 \) and

\[
\tilde{K}^v_T - \tilde{K}^v_t = \sup_{r \in [t, T]} \left( \xi^v + \int_r^T f^v(s, \tilde{Y}^v_s, \tilde{Z}^v_s)ds - \int_r^T \tilde{Z}^v_s dW_s - \sup_{b \in \mathcal{U}} \{ \tilde{Y}^v_{t_0}(r, b) - c^v(r, b) \} \right)\]

we note that \( \| (\tilde{K}^v - K^v_{t_0}) \|_{S^2} \to 0 \) as \( k \to \infty \) where \( \eta_t := \inf\{s \geq 0 : L^v_s \geq l\} \land T \) and by Lemma 3.11 we have that \( \tilde{K}^v \in S^2 \). Since \( L^v \) is continuous, and thus has \( \mathbb{P}\)-a.s. bounded trajectories, we find that

\[
\begin{aligned}
\tilde{Y}^v_t &= \xi^v + \int_t^T f^v(s, \tilde{Y}^v_s, \tilde{Z}^v_s)ds - \int_t^T \tilde{Z}^v_s dW_s + \tilde{K}^v_T - \tilde{K}^v_t, \quad \forall t \in [0, T], \\
\tilde{Y}^v_t &\geq \sup_{b \in \mathcal{U}} \{ \tilde{Y}^v_{t_0}(t, b) - c^v(t, b) \}, \quad \forall t \in [0, T], \\
\int_0^T (\tilde{Y}^v_t - \sup_{b \in \mathcal{U}} \{ \tilde{Y}^v_{t_0}(t, b) - c^v(t, b) \})dK^v_t &= 0.
\end{aligned}
\]

Finally, the map \( (t, b) \mapsto \tilde{Y}^v_{t_0}(t, b) \in \mathcal{C} \) by uniform convergence. \( \square \)

### 3.4 Uniqueness by a Verification Argument

**Theorem 3.20** The finite horizon sequential system of reflected BSDEs (1.1) admits a unique solution \((Y^v, Z^v, K^v : v \in \mathcal{U}^f)\) and \( Y^v_{t_0} = \sup_{u \in \mathcal{U}_t} U^v_{t_0} \) with \( u^* = (\tau^v_1, \ldots, \tau^v_{n_N} ; \beta_1, \ldots, \beta_{n_N}) \in \mathcal{U}_t \) defined as:

- \( \tau^v_j := \inf \left\{ s \geq \tau^v_{j-1} : Y^v_{t_0}[u^*]_{j-1} = \sup_{b \in \mathcal{U}} \{ Y^v_{t_0}[u^*]_{j-1}(s, b) - c^v[u^*]_{j-1}(s, b) \} \right\} \land T \),
- \( \beta^v_j = \arg \max_{b \in \mathcal{U}} \{ Y^v_{t_0}[u^*]_{j-1}(\tau^v_j, b) - c^v[u^*]_{j-1}(\tau^v_j, b) \} \)

and \( N^* = \sup \{ j : \tau^v_j < T \} \), with \( \tau^v_0 := T \).

\( \square \) Springer
Proof Assume that \((Y^v, Z^v, K^v : v \in \mathcal{U}^f)\) is a solution to (1.1) (i.e. \((Y^v : v \in \mathcal{U})\) is a consistent family such that \((t, b) \mapsto Y^v_{t,v(t,b)}\) is continuous and it satisfies equation (1.1)). Using Proposition 2.4 together with consistency \((\tilde{1})\)), using Proposition 2.4 together with consistency \(\tilde{N}^k\), we get

\[
Y^v_t = Y^v_{\tau^*_k} + \int_{\tau^*_k}^{\tau^*_k-1} f^v_{\tau^*_k}(s, Y^v_{\tau^*_k}, Z^v_{\tau^*_k}) ds - \int_{\tau^*_k}^{\tau^*_k-1} c^v_{\tau^*_k}(Y^v_{\tau^*_k}, Z^v_{\tau^*_k}) ds
\]

and uniqueness follows.

Concerning optimality let \(\tilde{u} = (\tilde{\tau}_1, \ldots, \tilde{\tau}_N; \tilde{\beta}_1, \ldots, \tilde{\beta}_N) \in \mathcal{U}^f\) and note that if \(\tilde{N} \geq 1\), then

\[
Y^v_t \geq U^v_{\tilde{\tau}_1} + \int_{\tilde{\tau}_1}^{\tilde{\tau}_1} f^v_{\tilde{\tau}_1}(s, U^v_{\tilde{\tau}_1}, V^v_{\tilde{\tau}_1}) ds - \int_{\tilde{\tau}_1}^{\tilde{\tau}_1} c^v_{\tilde{\tau}_1}(U^v_{\tilde{\tau}_1}, V^v_{\tilde{\tau}_1}) ds
\]

where \([u]_2 := (\tilde{\tau}_2, \ldots, \tilde{\tau}_N; \tilde{\beta}_2, \ldots, \tilde{\beta}_N)\). Successively repeating this process while considering the fact that \(\tilde{u} \in \mathcal{U}^f\) eventually leads us to the conclusion that \(Y^v_t \geq U^v_{\tilde{\tau}_1}\).

\[\square\]

4 Application to SDGs Involving Impulse Control

We now apply the above results to find solutions to stochastic differential games of impulse versus continuous control under weak formulation. In particular, we are interested in finding a saddle point for the game, i.e. a pair \((u^*, \alpha^*) \in \mathcal{U}^f \times \mathcal{A}\) such that

\[
J(u, \alpha) = J(u^*, \alpha^*) \leq J(u^*, \alpha)
\]

for all \((u, \alpha) \in \mathcal{U}^f \times \mathcal{A}\). Since we consider a weak formulation, each pair \((u, \alpha) \in \mathcal{U}^f \times \mathcal{A}\) gives rise to a probability measure \(\mathbb{Q}\) and a corresponding Brownian motion \(W^\mathbb{Q}\) such that (1.3) and (1.4) admits a solution on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q}, W^\mathbb{Q})\) and \(J(u, \alpha)\) is
to be interpreted as the corresponding cost functional under the expectation induced by $Q$.

Throughout, we assume the following forms on the drift and volatility terms in the forward SDE (1.3) and (1.4),

$$a(t, x, \alpha) = \begin{bmatrix} a_1(t, x) \\ a_2(t, x, \alpha) \end{bmatrix} \quad \text{and} \quad \sigma(t, x) = \begin{bmatrix} \sigma_{1,1}(t, x) & 0 \\ \sigma_{2,1}(t, x) & \sigma_{2,2}(t, x) \end{bmatrix},$$

where $a$ is of at most linear growth in the data $x$ and $\sigma$ is uniformly bounded. The drift is split into two terms $a_1 : [0, T] \times D \to \mathbb{R}^d_1$ (we let $D$ denote the set of all càdlàg functions $x : [0, T] \to \mathbb{R}^d$) and $a_2 : [0, T] \times D \times A \to \mathbb{R}^d_2$ with $d = d_1 + d_2$ the total dimension. The diffusion coefficient has a component $\sigma_{2,2} : [0, T] \times D \to \mathbb{R}^{d_2 \times d_2}$ that has an inverse, $\sigma_{2,2}^{-1}$, which is uniformly bounded on $[0, T] \times D$.

For the purpose of solving (4.1) let $f_u$ be given by

$$f_u(t, \omega, y, z) := \inf_{\alpha \in A} H^u(t, \omega, z, \alpha) =: H^{*,u}(t, \omega, z), \quad (4.2)$$

where

$$H^u(t, \omega, z, \alpha) := z \tilde{a}(t, (X^{u}_s)_{s \leq t}, \alpha) + \phi(t, X^{u}_t, \alpha),$$

with

$$\tilde{a}(t, x, \alpha) := \begin{bmatrix} 0 \\ \sigma_{2,2}^{-1}(t, x)a_2(t, x, \alpha) \end{bmatrix},$$

and $X^{u}$ is the unique solution to the impulsively controlled forward SDE

$$X^{u}_t = x_0 + \int_0^t \tilde{a}(s, (X^{u}_r)_{r \leq s}) ds + \int_0^t \sigma(s, (X^{u}_r)_{r \leq s}) dW_s, \quad \text{for } t \in [0, \tau_1], \quad (4.3)$$

$$X^{u}_t = \Gamma(t, j, X^{[\alpha]}_{\tau_j} - \beta_j) + \int_{\tau_j}^t \tilde{a}(s, (X^{u}_r)_{r \leq s}) ds$$

$$+ \int_0^t \sigma(s, (X^{u}_r)_{r \leq s}) dW_s, \quad \text{for } t \in [\tau_j, \tau_{j+1}], \quad (4.4)$$

with

$$\tilde{a}(t, x) := \begin{bmatrix} a_1(t, x) \\ 0 \end{bmatrix}.$$

Our approach to solving the above optimization problem is to define a measure $Q^{u,\alpha}$ under which $W^{u,\alpha}_t = W_t - \int_0^t \tilde{a}(s, (X^{u}_r)_{r \leq s}, \alpha^*_s) ds$ is a Brownian motion, where $\alpha^*$ is a measurable selection of a minimizer in (4.2). In particular, we note that for any

\[3\] We use the notation $(x_s)_{s \leq t}$ in arguments to emphasise that a function, for example, $\phi : [0, T] \times D \times A \to \mathbb{R}$ at time $t$ only depend on the trajectory of $x$ on $[0, t]$. Springer
Assumption 4.1
For any given by (4.2) attains a unique solution.

Before we move on to show optimality of the above scheme, we give assumptions on \( a, \sigma \) and \( \Gamma \) and \( \phi, \psi \) and \( \ell \) under which the sequential system (1.1) with driver given by (4.2) attains a unique solution.

(i) The function \( \Gamma : [0, T] \times \mathbb{D} \times U \to \mathbb{R}^d \) satisfies the Lipschitz condition

\[
|\Gamma(t, (x_s)_{s \leq t}, b) - \Gamma(t', (x'_s)_{s \leq t'}, b')| \\
\leq C \left( \int_0^{t\wedge t'} |x'_s - x_s| ds + |x'_{t'} - x_t| \\
+ (|t' - t| + |b' - b|) \left( 1 + \sup_{s \leq t} |x_s| + \sup_{s \leq t'} |x'_s| \right) \right)
\]

and the growth condition

\[
|\Gamma(t, (x_s)_{s \leq t}, b)| \leq K_\Gamma \vee |x_t|.
\]

for some constant \( K_\Gamma > 0 \).

(ii) The coefficients \( a : [0, T] \times \mathbb{D} \times A \to \mathbb{R}^d \) and \( \sigma : [0, T] \times \mathbb{D} \to \mathbb{R}^{d \times d} \) are continuous in \( t \) (and \( \alpha \) when applicable) and satisfy the growth conditions

\[
|a(t, (x_s)_{s \leq t}, \alpha)| \leq C (1 + \sup_{s \leq t} |x_s|), \\
|\sigma(t, (x_s)_{s \leq t})| \leq C
\]

and the Lipschitz continuity

\[
|\tilde{a}(t, (x_s)_{s \leq t}) - \tilde{a}(t, (x'_s)_{s \leq t})| + |\sigma(t, (x_s)_{s \leq t}) - \sigma(t, (x'_s)_{s \leq t})| \leq C \sup_{s \leq t} |x'_s - x_s|,
\]

\[
\left| \int_0^t [\tilde{a}(s, (x_r)_{r \leq s}) - \tilde{a}(s, (x'_r)_{r \leq s})] ds \right| \leq C \int_0^t |x'_s - x_s| ds,
\]

\[
\left| \int_0^t [\sigma(s, (x_r)_{r \leq s}) - \sigma(s, (x'_r)_{r \leq s})] ds \right| \leq C \int_0^t |x'_s - x_s|^2 ds.
\]

Moreover, for each \((t, x) \in [0, T] \times \mathbb{D}\), the matrix \( \sigma_{2,2}(t, (x_s)_{s \leq t}) \) has an inverse, \( \sigma_{2,2}^{-1}(t, (x_s)_{s \leq t}) \), that is uniformly bounded on \([0, T] \times \mathbb{D}\) and

\[
\left| \int_0^t [\tilde{a}(s, (x_r)_{r \leq s}, \hat{\alpha}(s)) - \tilde{a}(s, (x'_r)_{r \leq s}, \hat{\alpha}(s))] ds \right| \leq C \int_0^t |x'_s - x_s|^2 ds,
\]

for all measurable functions \( \hat{\alpha} : [0, T] \to A \).
(iii) The running reward \( \phi : [0, T] \times \mathbb{R}^d \times A \to \mathbb{R} \) is \( B([0, T] \times \mathbb{R}^d \times A) \)-measurable, continuous in \( \alpha \) and the terminal reward \( \psi : \mathbb{R}^d \to \mathbb{R} \) is \( B(\mathbb{R}^d) \)-measurable. Moreover, we have the growth condition

\[
|\phi(t, \xi, \alpha)| + |\psi(\xi)| \leq C^g (1 + |\xi|^p)
\]

for some \( C^g > 0 \) and all \( \xi \in \mathbb{R}^d \), and there is a nondecreasing function \( C^L : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for each \( K > 0 \),

\[
|\phi(t, \xi, \alpha) - \phi(t, \xi', \alpha)| + |\psi(\xi) - \psi(\xi')| \leq C^L(K)|\xi - \xi'|,
\]

whenever \( |\xi| \vee |\xi'| \leq K \).

(iv) The intervention cost \( \ell : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}_+ \) is jointly continuous in \( (t, \xi, b) \), bounded from below, i.e.

\[
\ell(t, \xi, b) \geq \delta > 0,
\]

and locally Lipschitz in \( \xi \) and Hölder continuous in \( t \), i.e. there is a nondecreasing function \( C^L_\ell : \mathbb{R}_+ \to \mathbb{R}_+ \) such that for each \( K > 0 \),

\[
|\ell(t, \xi, b) - \ell(t', \xi', b)| \leq C^L_\ell(K)|\xi - \xi'| + C^H_\ell|t' - t|^{\varsigma},
\]

whenever \( |\xi| \vee |\xi'| \leq K \) for some \( C^H_\ell, \varsigma > 0 \).

Under these assumptions, we note that \( f^u \) as defined in (4.2) is stochastic Lipschitz with Lipschitz coefficient

\[
L^v_t := \sup_{s \in [0, t]} \sup_{\alpha \in A} |\bar{a}(\cdot, (X^u_s)_{s \leq \cdot}, \alpha)| \vee C,
\]

where \( C > 0 \) is chosen to eliminate jumps. Moreover, for some \( k_L > 0 \), we have

\[
L^v_t \leq k_L (1 + \sup_{s \in [0, t]} |X^u_s|).
\]

### 4.1 Some Preliminary Estimates

We now present some preliminary estimates of moments and stability of solutions to (4.3) and (4.4). Towards the end of the section we will prove that any necessary changes of measure are feasible.

**Proposition 4.2** Under Assumption 4.1, the path-dependent SDE (4.3) and (4.4) admits a unique solution for each \( u \in \mathcal{U} \). Furthermore, the solution has moments of all orders, in particular we have for \( p \geq 0 \), that

\[
\sup_{u \in \mathcal{U}} \mathbb{E} \left[ \sup_{t \in [0, T]} |X^u_t|^p \right] \leq C,
\]
where \( C = C(p) \) and

\[
\sup_{v \in \mathcal{U}^f} \mathbb{E} \left[ \sup_{t \in [0,T]} \text{ess sup}_{u \in \mathcal{U}^f_t} \left| \mathbb{E} \left[ \sup_{s \in [t,T]} |X_s^{\psi(t)ou}|^p \right] \right|^{\frac{p}{2}} \right] \leq C, \tag{4.7}
\]

where \( C = C(\rho, p) \).

**Proof** See Proposition 5.4 in [26]. In particular, both moment estimates follows by noting that

\[
\mathbb{E} \left[ \sup_{s \in [t,T]} |X_s^v|^p \right] \leq C \left( 1 + \sup_{s \in [0,t]} |X_s^v|^p \right), \tag{4.8}
\]

for all \( v \in \mathcal{U}^f \) and all \( v, v' \in \mathcal{D}^k \).

**Lemma 4.3** For each \( k, \kappa \geq 0 \) and \( p \geq 1 \), there is a \( C \geq 0 \) such that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \text{ess sup}_{u \in \mathcal{U}^f_t} \mathcal{E} \left[ \sup_{s \in [0,T]} \mathcal{E} \left[ \sup_{s \in [t,T]} |X_s^{v\psi}/\psi - X_s^{v\psi}/\psi |^2 \right] \left| \mathcal{F}_t \right| \right] \right] \leq C \|v' - v\|_{\mathcal{D}^f}^p,
\]

for all \( v \in \mathcal{U}^f \) and all \( v, v' \in \mathcal{D}^k \).

**Proof** See the proof of Lemma 5.5 in [26].

A fundamental assumption in Sect. 3 is the existence of a \( q' > 1 \) and a \( C > 0 \) such that \( \sup_{v \in \mathcal{V}} \mathbb{E} \left[ |\mathcal{E}(\xi * W^Q)|^q \right] \leq C \) for all \( Q \in \mathcal{F}_0 \). In the following two lemmas we show that since \( L^q \leq k_L (1 + \sup_{s \in [0,1]} |X_s^v|) \), this statement is true.

**Lemma 4.4** For \( v \in \mathcal{U}^f \) and \( \xi > 1 \), let \( \mathcal{Y}^{v, \xi} \) be the set of all \( \mathcal{F} \)-measurable processes \( \xi \) with \( |\xi| \leq L^v_q \) for all \( t \in [0, T] \) (outside of a \( \mathcal{F} - \)null set) such that \( \mathbb{E} \left[ \mathcal{Y}^{r \xi, W} \right] = 1 \) for all \( r \in [1, \xi] \). Then, there is a \( q' > 1 \) such that \( \sup_{u \in \mathcal{U}^f} \sup_{\xi \in \mathcal{Y}^{v, \xi}} \mathbb{E} \left[ |\mathcal{E}(\xi * W)|^q \right] < \infty \).

**Proof** For \( q' \in [1, \xi] \) we have

\[
\mathbb{E} \left[ |\mathcal{E}(\xi * W)|^q \right] = \mathbb{E} \left[ \mathcal{E}(q' \xi * W) \right] e^{-\frac{(q')^2-q'}{2} f_0^T |\xi|^2 ds}
\]

\[
= \mathbb{E} \left[ Q^{q' \xi} \left[ e^{-\frac{(q')^2-q'}{2} f_0^T |\xi|^2 ds} \right] \right].
\]

Now, since \( |\xi| \leq k_L (1 + \sup_{s \in [0,1]} |X_s^v|) \) we have (see Lemma 1 in [2])

\[
\mathbb{E} \left[ Q^{q' \xi} \left[ e^{-\frac{(q')^2-q'}{2} f_0^T |\xi|^2 ds} \right] \right] \leq \mathbb{E} \left[ e^{-\frac{(q')^2-q'}{2} h(q')(1 + \sup_{s \in [0,1]} |W_s|)} \right],
\]

where \( h : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is bounded on compacts and we conclude that the left hand side is finite for \( q' > 1 \) sufficiently small. \( \square \)
Lemma 4.5 Let $\zeta$ be a $\mathcal{P}_{\mathbb{F}}$-measurable process with trajectories in $\mathbb{D}$ such that for some $v \in \mathcal{U}^f$, we have $|\zeta_t| \leq k_L(1 + \sup_{s \in [0,t]}|X^v_s|)$ for all $t \in [0, T]$, then $\mathbb{E}[\mathcal{E}(\zeta * W)_T] = 1$.

Proof We will reach the result by adapting the proof of Lemma 7 in [14] to solutions of impulsively controlled path-dependent SDEs (see also Lemma 0 of Sect. 5 in [2]). Since $(\mathcal{E}(\zeta * W)_t : 0 \leq t \leq T)$ is a $\mathbb{P}$-a.s. non-negative local martingale it is a supermartingale and we only need to show that $\mathbb{E}[\mathcal{E}(\zeta * W)_T] \geq 1$. For $M \geq 0$ and $t \in [0, T]$, we define the sets

$$C_M(t) := \left\{ x \in \mathbb{D} : k_L \left( 1 + \sup_{s \in [0,t]} |x_s| \right) < M \right\}$$

Then for each $M \geq 0$, $(C_M(t))_{t \in [0, T]}$ is a non-increasing collection of open subsets of $\mathbb{D}$ and:

(a) If for some $x \in \mathbb{D}$ we have $x \in C_M(s)$ and $x \not\in C_M(t)$ for some $0 \leq s < t \leq T$, then there is a $t' \in (s, t]$ such that $x \in C_M(r)$ for all $r \in [0, t')$ and $x \not\in C_M(t')$.

(b) For each $\epsilon > 0$, there is a $M \geq 0$ such that $\mathbb{P}[X^v \in C_M(T)] > 1 - \epsilon$ for all $v \in \mathcal{U}^f$.

Here, the second property follows from Proposition 4.2. Moreover, let $D_M(t) := \{ \omega : X^v,\zeta \in C_M(t) \} \in \mathcal{F}_t$, where $X^{u,\zeta}$ solves (1.3) and (1.4) with drift $a = \tilde{a} + \sigma \zeta$.

We first restrict our attention to the situation when $v \in \mathcal{U}^k$ for some $k \geq 0$, and note that (by arguing as in the proof of Lemma 4.3) we have

$$|X^{v,\zeta}_t - X^v_t| \leq C \left( \int_0^t \left( |\tilde{a}(s, (X^{v,\zeta}_r)_{r \leq s}) - \tilde{a}(s, (X^v_r)_{r \leq s})| + |\sigma(s, (X^{v,\zeta}_r)_{r \leq s}) - \sigma(s, (X^v_r)_{r \leq s})| ds \right. 
+ \left. \int_0^t \left( |\sigma(s, (X^{v,\zeta}_r)_{r \leq s}) - \sigma(s, (X^v_r)_{r \leq s})| dW_s \right) \right) + \left| \int_0^t (\sigma(s, (X^{v,\zeta}_r)_{r \leq s}) - \sigma(s, (X^v_r)_{r \leq s})) dW_s \right| \right)$$

$$\leq C \left( \int_0^t \left( |X^{v,\zeta}_s - X^v_s| + |\zeta_s| \right) ds + \Xi^{v,\zeta}_t \right),$$

where

$$\Xi^{v,\zeta}_t := \sup_{t' \in [0,t]} \left| \int_0^{t'} \left( \sigma(s, (X^{v,\zeta}_r)_{r \leq s}) - \sigma(s, (X^v_r)_{r \leq s})) dW_s \right) \right|.$$
Applying Grönwall’s inequality together with the fact that $|ζ_t| ≤ kL (1 + sup_{s \in [0,t]} |X^v_s|)$ gives that for any $T' \in [0, T]$,

$$\sup_{t \in [0,T']} |X^v_{t,ζ} - X^v_t| \leq C (1 + \int_0^{T'} \sup_{r \in [0,s]} |X^v_r| ds + E^v_{T'}),$$

Now, for all $ω \in D_M(T)$ we have $\sup_{t \in [0,T]} |X^v_{t,ζ}| < M$ and we can apply Grönwall’s inequality once more to obtain

$$\sup_{t \in [0,T]} |X^v_t| \leq C (1 + M + E^v_{T}),$$

Letting $E_M(t) := \{ ω \in D_M(t) : E^v_{t,ζ} \leq M \} \in F_t$ we note that

(c) For $ω \in E_M(t)$ we have $ζ_t ≤ C (1 + M)$, where $C$ does not depend on $t$ or $M$.

Now, set

$$ζ^M_t := 1_{E_M(t)} ζ_t$$

and let $X^{v,ζ,M} := X^{v,ζ,M}$. Since $|ζ^M_t| ≤ C (1 + M)$, the Novikov condition holds for any constant multiple of $ζ^M$. In particular, we conclude that the $Q_M$ defined by $dQ_M := E(ζ^M * W)_T dP$ is a probability measure. Moreover, for some $q' > 1$ we have

$$\mathbb{E}^{Q_M} \left[ (E^v_{T})^2 \right] ≤ C \mathbb{E} \left[ |E(ζ^M * W)_T| q' \right]^{1/q'} \mathbb{E} \left[ \left( \int_0^T (|σ(s, (X^v_{r,ζ})_{r \leq s})|^2 + |σ(s, (X^v_r)_{r \leq s})|^2) ds \right)^{q/2} \right]^{1/q} ≤ C$$

by Assumption 4.1.ii, where $\frac{1}{q'} + \frac{1}{q} = 1$ and, by Lemma 4.4, $C > 0$ can be chosen independently from $M$. This gives that

(d) For each $ε > 0$, there is a $M ≥ 0$ such that $Q_M' (\{ ω : E^v_{T} ≤ M' \}) > 1 - ε$ for all $M' ≥ M$.

Making use of b) and d) we find that for each $ε > 0$, there is a $M ≥ 0$ such that

$$1 - ε < P (\{ ω : X^v ∈ C_M(T) \}) = Q_M (\{ ω : X^{v,ζ,M} ∈ C_M(T) \})$$

and

$$Q_M (\{ ω : E^v_{T} > M \}) < ε.$$
Combining these gives that
\[ 1 - \epsilon < Q_M((\omega : X^{v, \zeta, M} \in C_M(T)) \cap \{ \omega : \mathbb{E}^{v, \zeta}_T \leq M \}) + Q_M((\omega : \mathbb{E}^{v, \zeta}_T > M)) \]
\[ \leq Q_M((\omega : X^{v, \zeta, M} \in C_M(T)) \cap \{ \omega : \mathbb{E}^{v, \zeta}_T \leq M \}) + \epsilon \]

Moreover, by property (a) above and right-continuity we have that
\[ \{ \omega : X^{v, \zeta, M} \in C_M(t) \} \cap \{ \omega : \mathbb{E}^{v, \zeta}_t \leq M \} = EM(t) \]
so that
\[ 1 - 2\epsilon < Q_M(EM(T)) = \mathbb{E}\left[ |\mathbb{E}(\zeta^M W)_T| \mathbb{1}_{EM(T)} \right] \leq \mathbb{E}\left[ |\mathbb{E}(\zeta * W)_T| \right]. \]

Since \( \epsilon > 0 \) was arbitrary, this proves the assertion whenever \( v \in U^k \) for some finite \( k \geq 0 \). To get the result for arbitrary \( v := (\tau_1, \ldots, \tau_N; \beta_1, \ldots, \beta_N) \in U^f \), we define the sets
\[ F_k(t) := \left\{ \omega : |\zeta_t| \leq kL \left( 1 + \sup_{s \in [0, t]} |X_s^{v,k}| \right) \right\} \supset \{ \omega : N \leq k \} \]
for all \( k \geq 0 \) and let
\[ \zeta_t^{[v]k} = \mathbb{1}_{F_k(t)} \zeta_t. \]

Now, for any \( v \in U^f \) we have by definition that \( \mathbb{P}(\{ \omega : N > k, \forall k \geq 0 \}) = 0 \) and so we can by again appealing to Lemma 4.4 find a \( k \geq 0 \) such that
\[ Q^{[v]k}(N > k) := \mathbb{E}\left[ |\mathbb{E}(\zeta^{[v]k} W)_T| \mathbb{1}_{N > k} \right] \leq \mathbb{E}\left[ |\mathbb{E}(\zeta^{[v]k} W)_T|^{q'} \right] < \epsilon \]
implying that
\[ 1 - \epsilon < Q^{[v]k}(N \leq k) = \mathbb{E}\left[ |\mathbb{E}(\zeta^{[v]k} W)_T| \mathbb{1}_{N \leq k} \right] \leq \mathbb{E}\left[ |\mathbb{E}(\zeta^{v} W)_T| \mathbb{1}_{N \leq k} \right] \leq \mathbb{E}\left[ |\mathbb{E}(\zeta^{v} W)_T| \right] \]
and the assertion follows as \( \epsilon > 0 \) was arbitrary.

\[ \square \]

**Corollary 4.6** There is a \( q' > 1 \) such that \( \sup_{\zeta \in \mathbb{K}_0} \mathbb{E}\left[ |\mathbb{E}(\zeta * W)_T|^{q'} \right] < \infty. \)

**Proof** Lemma 4.5 shows that for each \( \zeta > 1 \) the \( T^{v, \zeta} \) in Lemma 4.4 is in fact all \( \mathcal{P}_\mathbb{P} \)-measurable processes \( \zeta \) with \( |\zeta_t| \leq k_L \left( 1 + \sup_{s \in [0, t]} |X_s^v| \right) \) for all \( t \in [0, T] \) (outside of a \( \mathbb{P} \)-null set).

The above corollary gives the following:
Proposition 4.7  Under Assumption 4.1, the path-dependent SDE (1.3) and (1.4) admits a weak solution for each \((u, \alpha) \in \mathbb{U} \times \mathbb{A}\). Furthermore, the solution has moments of all orders, in particular we have for \(p \geq 0\), that

\[
\sup_{u \in \mathbb{U}^f, \mathbb{Q} \in \mathbb{Q}^u} \mathbb{E}^\mathbb{Q}\left[ \sup_{t \in [0, T]} |X^u_t|^p \right] \leq C, \tag{4.9}
\]

where \(C = C(p)\) and

\[
\sup_{v \in \mathbb{U}^f} \mathbb{E}\left[ \sup_{t \in [0, T]} \text{ess sup}_{s \in [t, T]} |\mathbb{E}^\mathbb{Q}\left[ \sup_{s \in [t, T]} |X^{v(t) \circ u}_{s}|^p \big| \mathcal{F}_t \right]|^p \right] \leq C, \tag{4.10}
\]

where \(C = C(\rho, p)\).

Moreover, there is a \(q' > 1\) and a \(C > 0\) such that for each \(v \in \mathbb{U}^f\) and all \(\zeta \in \mathbb{K}^v\) and \(\mathbb{Q} \in \mathbb{Q}^v\) we have \(\mathbb{E}^\mathbb{Q}[\mathbb{E}(\zeta * W^\mathbb{Q})_{T}^{[q']} \leq C\) (where \(W^\mathbb{Q}\) is a Brownian motion under \(\mathbb{Q}\)).

**Proof**  Existence of a weak solution to (1.3) and (1.4) follows by taking \(\zeta_t = \tilde{a}(t, (X^v_s)_{s \leq t}, \alpha_t)\) and using Lemma 4.5 to conclude that under \(\mathbb{Q}^{u, \alpha}\), the process \(W^{u, \alpha} := W - \int_0^T \tilde{a}(t, (X_s)_s)_{s \leq t}, \alpha_t)dt\) is a Brownian motion.

The moment estimates (4.9) and (4.10) now follow by repeating the steps in the proof of Proposition 5.4 in [26] and the last assertion follows by repeating the steps in Lemmas 4.4 and 4.5 while referring to the bound (4.9) rather than (4.6).

\[\top\]

4.2 The Corresponding Sequential System of Reflected BSDEs

In the present section we show that there is a unique family \((Y^v, Z^v, K^v)\) that solves the sequential system of reflected BSDEs

\[
\begin{align*}
Y^v_t &= \psi(X^v_T) + \int_t^T H^{a,v}(s, Z^v_s)ds - \int_t^T Z^v_s dW_s + K^v_t - K^v_t, \quad \forall t \in [0, T], \\
Y^v_t &\geq \sup_{b \in \mathbb{U}} \{ Y^{v(t,b)}_t \} - \ell^v(t, X^v_t, b)), \quad \forall t \in [0, T], \\
\int_0^T (Y^v_t - \sup_{b \in \mathbb{U}} \{ Y^{v(t,b)}_t \} - \ell^v(t, X^v_t, b)))dK^v_t = 0,
\end{align*}
\]

(4.11)

where \(\ell^u(t, X^u_t, b) := \infty \mathbb{I}_{[0, \tau^u)}(t) + \ell(t, X^u_t, b)\) making (4.11) a non-reflected BSDE on \([0, \pi(u))\). In the remainder of the article we will drop the superscript \(v\) in \(\ell^v\) but remind ourselves that no reflection can occur before the time of the last intervention in \(v\). Then, we will leverage the result in Theorem 3.20 to find a weak solution to the SDG in finite horizon.

Letting

\[
\tilde{K}^{v,p} := 2^{p-1} (((1 + T)(C^g)^p + (k_L)^p) \\
\left( 1 + \text{ess sup}_{u \in \mathbb{U}^f} \mathbb{E}\left[ \sup_{s \in [0, T]} |X^{v(t) \circ u}_{s}|^p \big| \mathcal{F}_t \right] \right) : 0 \leq t \leq T),
\]

\[\subseteq\] Springer
we note that
\[
\text{ess sup}_{u \in U_f} \mathbb{E} \left[ | L_{T}^{v,0}|^p + | \psi(X_{T}^{v,0})|^p + \int_{t}^{T} | H^{*,v,0}(s,0) |^p \, ds \right] \leq \tilde{K}^{v,p}_t.
\]

Moreover, by Proposition 4.2 we have that \( \sup_{v \in U_f} \| \tilde{K}^{v,p} \|_{\mathcal{S}^p} < \infty \), for all \( p \geq 1 \) and by (4.8) it follows that \( (\tilde{K}^{v,p} : v \in U_f, p \geq 1) \) satisfies the relation in (3.3).

On the other hand \( H^{*,v}(t, z', \cdot) - H^{*,v}(t, z) \) contains the term \( \phi(t, X_t^v) - \phi(t, X_t^v) \) and so \( H^{*,v} \) generally fails to satisfy the conditions in Assumption 3.3 since \( \phi \) is only locally Lipschitz in \( x \). The same thing applies to \( \psi \) and \( \ell \) and we will rely on a localization argument leading us to introduce
\[
H^{*,u,m,n}(t, \omega, z) := \inf_{\alpha \in A} H^{u,m,n}(t, \omega, z, \alpha),
\]

where
\[
H^{u,m,n}(t, \omega, z, \alpha) := z \bar{a}(t, (X_s^u)_{s \leq t}, \alpha) + \phi^{m,n}(t, X_t^u, \alpha),
\]

with \( \phi^{m,n} : [0, T] \times \mathbb{R}^d \times A \rightarrow \mathbb{R} \) given by \( \phi^{m,n}(t, x, \alpha) := \phi^{+,m}(t, x, \alpha) - \phi^{-,n}(t, x, \alpha) \), where \( (\phi^{+,m})_{m \geq 0} \) and \( (\phi^{-,n})_{n \geq 0} \) are both non-decreasing sequences of Borel-measurable, non-negative functions that are Lipschitz continuous in \( x \) and continuous in \( \alpha \) such that \( \phi^{+,m} = \phi^+(x) \) on \( |x| \leq m \) and \( \phi^{-,n}(x) = \phi^-(x) \) on \( |x| \leq n \).

Similarly, for \( m, n \geq 0 \), we let \( \psi^{m,n} : \mathbb{R}^d \rightarrow \mathbb{R} \) be given by \( \psi^{m,n}(x) := \psi^{+,m}(x) - \psi^{-,n}(x) \), where \( (\psi^{+,m})_{m \geq 0} \) and \( (\psi^{-,n})_{n \geq 0} \) are both non-decreasing sequences of non-negative Lipschitz continuous functions such that \( \psi^{+,m} = \psi^+(x) \) on \( |x| \leq m \) and \( \psi^{-,n}(x) = \psi^-(x) \) on \( |x| \leq n \) and let \( \ell^n : [0, T] \times \mathbb{R}^d \times U \rightarrow [\delta, \infty) \) be a non-decreasing sequence of jointly continuous functions that are Lipschitz continuous in \( x \) and Hölder continuous in \( t \) (uniformly in the other variables) and satisfy \( \ell^n(t, x, b) = \ell(t, x, b) \) on \( |x| \leq n \) and \( \ell^n(t, x, b) \geq \delta \) for all \( t, x, b \in [0, T] \times \mathbb{R}^d \times U \).

We now consider the following localized form of (4.11)

\[
\begin{cases}
Y_t^{v,m,n} = \psi^{m,n}(X_t^v) + \int_{t}^{T} H^{*,v,m,n}(s, Z_s^{v,m,n}) \, ds \\
- \int_{t}^{T} Z_s^{v,m,n} \, dW_s + K_T^{v,m,n} - K_t^{v,m,n}, \quad \forall t \in [0, T],
\end{cases}
\]

(4.12)

\[
Y_t^{v,m,n} \geq \sup_{b \in U} \{ Y_t^{v,0(t,b),m,n} - \ell^n(t, X_t^v, b) \}, \quad \forall t \in [0, T],
\]

\[
\int_{0}^{T} (Y_t^{v,m,n} - \sup_{b \in U} \{ Y_t^{v,0(t,b),m,n} - \ell^n(t, X_t^v, b) \}) \, dK_t^{v,m,n} = 0.
\]

Since
\[
| H^{*,v,m,n}(t, z') - H^{*,v,m,n}(t, z) | \leq C ((1 + |L_t^{v'}| \lor |L_t^{v}|) |z' - z| \\
+ (|z| + |z'|) \sup_{\alpha \in A} |\bar{a}(t, (X_s^{v'})_{s \leq t}, \alpha) \\
- \bar{a}(t, (X_s^{v})_{s \leq t}, \alpha) |) \\
+ |\phi^{m,n}(t, X_t^{v'}) - \phi^{m,n}(t, X_t^{v})|,
\]

\( \square \) Springer
Lemma 4.3 and Assumption 4.1 implies the existence of a family $(A_{t}^{v,v',u} : (v, v') \in \cup_{k \geq 1} \mathcal{D}^{k} \times \mathcal{D}^{k}, u \in \mathcal{U}^{I})$ and a family $(K_{t}^{v,v',k,p} : (v, v') \in \cup_{k \geq 1} \mathcal{D}^{k} \times \mathcal{D}^{k}, k, p \geq 0)$ satisfying the conditions in Definition 3.2 and Assumption 3.3 and it follows by Proposition 3.19 and Theorem 3.20 that there is a unique family $(Y_{t}^{v,m,n}, Z_{t}^{v,m,n}, K_{t}^{v,m,n})$ that solves the sequential system of reflected BSDEs in (4.12).

We have,

Lemma 4.8 For $v, u \in \mathcal{U}^{I}$, let $(U_{t}^{v,u,m,n}, V_{t}^{v,u,m,n}) \in S_{t}^{2} \times \mathcal{H}^{2}$ solve

$$U_{t}^{v,u,m,n} = \psi_{m,n}(X_{T}^{v,u}) + \int_{t}^{T} H_{s}^{v,u,m,n}(s, V_{s}^{v,u,m,n})ds - \int_{t}^{T} V_{s}^{v,u,m,n}dW_{s}$$

$$- \sum_{j=1}^{N} \mathbb{1}_{[t \leq \tau_{j}]}\ell_{n}(\tau_{j}, X_{\tau_{j}}^{v,u}[\omega]_{j-1}, \beta_{j}),$$

(4.13)

whenever it has a unique solution and set $U^{v,u,m,n} = -\infty$, otherwise. Then, there is a $C > 0$ that does not depend on $m, n$ such that whenever $u^{*} \in \mathcal{U}^{I}$ is such that $\text{ess sup}_{u \in \mathcal{U}^{I}} U_{t}^{v,u,m,n} = U_{t}^{v,u^{*},m,n}$, we have

$$|U_{t}^{v,u^{*},m,n}|^{2} + \mathbb{E}_{Q}^{t}\left[\int_{t}^{T} |V_{s}^{v,u^{*},m,n}|^{2}ds + (N^{*})^{2}\mathcal{F}_{T}\right]$$

$$\leq C(1 + \text{ess sup}_{u \in \mathcal{U}^{I}} \mathbb{E}_{Q}^{t}\left[\sup_{s \in [t,T]} |X_{s}^{v,u} |^{2}\mathcal{F}_{T}\right]),$$

(4.14)

for all $Q \in \mathcal{P}^{v,u}$.\[\square\]

Proof First note that whenever $u^{*} \in \mathcal{U}^{I}$ is a maximizer then (4.13) admits a unique solution. The bounds on $|U_{t}^{v,u,m,n}|^{2}$ and $\mathbb{E}_{Q}^{t}\left[\int_{t}^{T} |V_{s}^{v,u,m,n}|^{2}ds \mathcal{F}_{T}\right]$ now follow by repeating the argument in the proof of Lemma 3.14 while noting that

$$\mathbb{E}_{Q}^{t}\left[|\psi_{m,n}(X_{T}^{v,u})|^{p} + \int_{t}^{T} |H_{s}^{v,u,m,n}(s, 0)|^{p}ds \mathcal{F}_{T}\right]$$

$$\leq C(1 + \mathbb{E}_{Q}^{t}\left[\sup_{s \in [0,T]} |X_{s}^{v,u} |^{p}\mathcal{F}_{T}\right]).$$

From this, the bound on $N^{*}$ is immediate from (3.20).\[\square\]

The statement of Lemma 4.8 holds for all $Q \in \mathcal{P}^{v,u}$. Here it is notable that, since for any $m, n, m', n > 0$ the drivers $H_{t}^{v,m,n}$ and $H_{t}^{v,m',n'}$ have the same stochastic Lipschitz coefficients, the set $\mathcal{P}^{v,u}$ is not parameterized by $m, n$. This is a key property when deriving the following stability result:

Lemma 4.9 For each $m \geq 0$ and $p \geq 1$ we have

$$\| \sup_{b \in U} |Y_{.,b}^{v^{0}(.,),m,n'} - Y_{.,b}^{v^{0}(.,),m,n} | \|_{S^{p}} \to 0$$

(4.15)
as \( n, n' \to \infty \).

**Proof** We have,

\[
|Y_t^{v(t,b),m,n} - Y_t^{v(t,b),m,n'}| \leq |U_t^{v(t,b),u^*,m,n} - U_t^{v(t,b),u^*,m,n'}| + |U_t^{v(t,b),u^*,m,n} - U_t^{v(t,b),u^*,m,n'}|, 
\]

where \( u^* \) and \( u' \) are elements of \( \mathcal{U}_t \) such that \( \sup_{u \in \mathcal{U}_t} U_t^{v(t,b),u,m,n} = U_t^{v(t,b),u^*,m,n} \) and \( \sup_{u \in \mathcal{U}_t} U_t^{v(t,b),u,m,n'} = U_t^{v(t,b),u^*,m,n'} \). We now consider the first term and suppress the references to the control strategies (i.e. \( v \circ (t, b) \) and \( u^* \)) in the superscript, we have

\[
U_t^{m,n} - U_t^{m,n'} = (\psi^{m,n}(X_T) - \psi^{m,n'}(X_T)) + \int_t^T (H^{*,m,n}(s, V_s^{m,n}) - H^{*,m,n'}(s, V_s^{m,n'}))\,ds 
- \int_t^T (V_s^{m,n} - V_s^{m,n'})\,dW_s 
- \sum_{j=1}^{N^*} (\ell^n(\tau_j, X_{\tau_j}, \beta_j) - \ell^{n'}(\tau_j, X_{\tau_j}, \beta_j)). 
\]

Taking the conditional expectation under the measure \( Q^{n,n'} \) where \( W_t - \int_0^t s_{s}^{n,n'}\,ds \) is a martingale, with

\[
s_{s}^{n,n'} = \frac{H^{*,m,n}(s, V_s^{m,n}) - H^{*,m,n'}(s, V_s^{m,n'})}{|V_s^{m,n} - V_s^{m,n'}|^2} (V_s^{m,n} - V_s^{m,n'})^\top \mathbf{1}_{[V_s^{m,n} \neq V_s^{m,n'}]}, 
\]

gives

\[
|U_t^{m,n} - U_t^{m,n'}| 
\leq C \mathbb{E}_{Q^{n,n'}}[|\psi^{m,n}(X_T) - \psi^{m,n'}(X_T)| + \int_t^T |H^{*,m,n}(s, V_s^{m,n}) 
- H^{*,m,n'}(s, V_s^{m,n})|\,ds + \sum_{j=1}^{N^*} |\ell^n(\tau_j, X_{\tau_j}, \beta_j) 
- \ell^{n'}(\tau_j, X_{\tau_j}, \beta_j)| \big| \mathcal{F}_t] 
\leq C \mathbb{E}_{Q^{n,n'}}[\mathbb{1}_{|X_T| \geq n}(1 + |X_T|^\rho) + \int_t^T \mathbb{1}_{|X_s| \geq n}(1 + |X_s|^\rho)\,ds 
+ \sum_{j=1}^{N^*} (1 + |X_{\tau_j}|^\rho) \mathbb{1}_{|X_{\tau_j}| \geq n} \big| \mathcal{F}_t], 
\]

\( \mathcal{F}_t \) being the filtration generated by \( W_t \) and \( \mathbf{1}_{|X_{\tau_j}| \geq n} \).
where \( n := n \land n' \). As \( H^{*,m,n} \) and \( H^{*,m,n'} \) have the same \( z \)-coefficient and thus also the same stochastic Lipschitz coefficient we find that \( Q^{n,n'} \in \mathbb{P}^\mathbb{V} \). In particular, we have

\[
|U^{m,n}_t - U^{m,n'}_t| \leq C \mathbb{E}^{Q^{n,n'}} \left[ \mathbb{I}_{[\sup_{s \in [t,T]} |X_s| > 2]} (1 + \sup_{s \in [t,T]} |X_s|^\rho) (1 + N^*) \right]_{\mathcal{F}_t}^{1/2} \\
\leq C \mathbb{E}^{Q^{n,n'}} \left[ \mathbb{I}_{[\sup_{s \in [t,T]} |X_s| > 2]} (1 + \sup_{s \in [t,T]} |X_s|^\rho) \right]_{\mathcal{F}_t}^{1/2} \\
\leq C \text{ ess sup}_{u \in U_t^f, Q \in \mathbb{P}^\mathbb{V}} \mathbb{E}^{Q} \left[ \mathbb{I}_{[\sup_{s \in [t,T]} |X^\mathbb{V}_s| > 2]} \right]_{\mathcal{F}_t}^{1/4} \\
\left( 1 + \text{ ess sup}_{u \in U_t^f, Q \in \mathbb{P}^\mathbb{V}} \mathbb{E}^{Q} \left[ \sup_{s \in [t,T]} |X^\mathbb{V}_s|^4 \right] \right). 
\]

The result now follows by Proposition 4.7.

Now, as clearly \( \| \sup_{b \in U} \| \ell^p (\cdot, X^v, b) - \ell^p (\cdot, X^v, b) \|_{\mathcal{S}^p} \rightarrow 0 \) as \( n, n' \rightarrow \infty \) for all \( p \geq 1 \), Lemma 4.9 and Proposition 2.4 implies that

\[
\lim_{n,n' \rightarrow \infty} \{ \| Y^{v,m,n'}_\cdot - Y^{v,m,n}_\cdot \|_{\mathcal{S}^2} + \| Z^{v,m,n'} - Z^{v,m,n} \|_{\mathcal{H}^2} + \| K^{v,m,n'} - K^{v,m,n} \|_{\mathcal{S}^2} \} = 0. 
\]

For each \( m \geq 0 \) we note that \( (Y^{v,m,n})_{n \geq 0} \) is a non-increasing sequence of continuous processes that is \( \mathbb{P} \)-a.s. bounded and we have that \( Y^{v,m,n} \) converges pointwisely to a progressively measurable process \( Y^{v,m} := \lim_{n \rightarrow \infty} Y^{v,m,n} \). Furthermore, by Lemma 4.9 we find that \( Y^{v,m} \) is continuous and thus belongs to \( \mathcal{S}^2 \).

**Proposition 4.10** For each \( m \geq 0 \), there is a family of pairs \( (Z^{v,m}, K^{v,m} : v \in U^f) \) such that \( (Y^{v,m}, Z^{v,m}, K^{v,m} : v \in U^f) \) is the unique solution to

\[
\begin{align*}
Y^{v,m}_t &= \psi^m(X^v_T) + \int_t^T H^{*,v,m}(s, Z^{v,m}_s) ds - \int_t^T Z^{v,m}_s dW_s + K^{v,m}_T, \quad \forall t \in [0, T], \\
Y^{v,m}_t &= \sup_{b \in U} \{ Y^{v,\psi(b),m}_t - \ell(t, X^{v}_t, b) \}, \quad \forall t \in [0, T], \\
\int_0^T (Y^{v,m}_t - \sup_{b \in U} \{ Y^{v,\psi(b),m}_t - \ell(t, X^{v}_t, b) \}) dK^{v,m}_t = 0, \\
\end{align*}
\]

where \( H^{*,u,m}(t, \omega, z) := \inf_{\alpha \in A} H^{u,m}(t, \omega, z, \alpha) \), with

\[
H^{u,m}(t, \omega, z, \alpha) := \bar{z} \hat{a}(t, (X^u_s)_{s \leq t}, \alpha) + \phi^+,m(t, X^u_t, \alpha) - \phi^-(t, X^u_t, \alpha),
\]

and \( \psi^m(x) := \psi^+,m(x) - \psi^-(x) \).

\[ \text{ Springer} \]
\begin{proof}
Let $\eta_k := \inf\{s \geq 0 : |X_s^\nu| \geq k\} \land T$. Then, for all $n \geq k$ it follows that $(Y^v,m,n, Z^v,m,n, K^v,m,n)$ solves

\begin{align*}
\left\{ \begin{array}{l}
Y^v,m_n = Y^v,m_{\eta_k} + \int_{\eta_k}^n H^{s,v,m}(s, Z^v,m_s)ds - \int_{\eta_k}^n Z^v,m_s dW_s + K^v,m_{\eta_k}, \\
Y^v,m \geq \sup_{b \in U}\{Y^v_0 - \ell(t, X_t^\nu, b)\}, \quad \forall t \in [0, \eta_k], \\
\int_{0}^{\eta_k} (Y^v,m_n - \sup_{b \in U}\{Y^v_0 - \ell(t, X_t^\nu, b)\})dK^v,m = 0.
\end{array} \right.
\end{align*}

Moreover, by Proposition 3.19 and Theorem 3.20 there is a unique family of triples $(\hat{Y}^v,m, \hat{Z}^v,m, \hat{K}^v,m)$ that solves

\begin{align*}
\left\{ \begin{array}{l}
\hat{Y}^v,m = Y^v,m_n + \int_{\eta_k}^n H^{s,v,m}(s, \hat{Z}^v,m_s)ds - \int_{\eta_k}^n \hat{Z}^v,m_s dW_s + \hat{K}^v,m - \hat{K}^v,m, \\
\hat{Y}^v,m \geq \sup_{b \in U}\{\hat{Y}^v_0 - \ell(t, X_t^\nu, b)\}, \quad \forall t \in [0, \eta_k], \\
\int_{0}^{\eta_k} (\hat{Y}^v,m_n - \sup_{b \in U}\{\hat{Y}^v_0 - \ell(t, X_t^\nu, b)\})d\hat{K}^v,m = 0.
\end{array} \right.
\end{align*}

Letting, $n \to \infty$ we have by Lemma 4.9 and Proposition 2.4 that

\begin{align*}
\| (\hat{Y}^v,m - Y^v,m_n) \|_{[0, \eta_k]} \leq 2 + \| (\hat{Z}^v,m - Z^v,m_n) \|_{[0, \eta_k]} \leq 2 + \| (\hat{K}^v,m - K^v,m,n) \|_{[0, \eta_k]} \to 0
\end{align*}

and we find that there is a family of pairs $(Z^v,m, K^v,m : v \in \mathcal{U}^f)$ such that for each $k \geq 0$,

\begin{align*}
\left\{ \begin{array}{l}
Y^v,m = Y^v,m_{\eta_k} + \int_{\eta_k}^n H^{s,v,m}(s, Z^v,m_s)ds - \int_{\eta_k}^n Z^v,m_s dW_s + K^v,m_{\eta_k} - K^v,m, \\
Y^v,m \geq \sup_{b \in U}\{Y^v_0 - \ell(t, X_t^\nu, b)\}, \quad \forall t \in [0, \eta_k], \\
\int_{0}^{\eta_k} (Y^v,m_n - \sup_{b \in U}\{Y^v_0 - \ell(t, X_t^\nu, b)\})dK^v,m = 0.
\end{array} \right.
\end{align*}

Now, since there is a $\mathbb{P}$-a.s. finite $k_0(\omega)$ such that $\eta_k = T$ for all $k \geq k_0$, existence of a solution to (4.16) follows.

Uniqueness is established by repeating steps in the proof of Theorem 3.20. \qed

By an identical argument to that used in Lemma 4.9 we find that

\begin{align*}
\lim_{m,m' \to \infty} \sup_{b \in U} \| Y^v(.,b),m' - Y^v(.,b),m \|_{\mathcal{S}^p} = 0
\end{align*}

for all $p \geq 1$ and we conclude that:

\textbf{Proposition 4.11} \textit{The sequential system of reflected BSDEs (4.11) has a unique solution.}

\textbf{Proof} The result follows by letting $m \to \infty$ and using an identical argument to that of Proposition 4.10. \qed

\textbf{Remark 4.12} We note that the verification result of Theorem 3.20 holds in the present setting as well.

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4.3 A Solution to the Stochastic Differential Game

We are now ready to solve the SDG by relating optimal controls to solutions of the sequential system of reflected BSDEs (4.11). However, before proceeding we need to narrow down the set of admissible impulse controls that we search over in order to guarantee that (4.13) admits a unique solution.

**Definition 4.13** We let $U^m$ be the set subset of $U^l$ with all $u = (\tau_1, \ldots, \tau_N; \beta_1, \ldots, \beta_N)$ such that $N$ has moments of all orders, i.e. for each $k \geq 0$ we have $\mathbb{E}[(N)^k] < \infty$.

**Lemma 4.14** For each $u \in U^l$, there is a $\hat{u} \in U^m$ such that

$$J(u, \alpha) \leq J(\hat{u}, \alpha)$$

for all $\alpha \in A$.

**Proof** For $(u, \alpha) \in U^l \times A$ we let $(B^{u, \alpha}, E^{u, \alpha})$ solve

$$B^{u, \alpha}_t = \psi(X_T^{u, \alpha}) + \int_t^T \phi(s, X_s^{u, \alpha}, \alpha_s)ds - \int_t^T E_s^{u, \alpha} dW_s$$

whenever a solution exists in $S^2 \times \mathcal{H}^2$. We define the set of sensible impulse controls, $U^l$, as the subset of $u \in U^l$ such that for each $t \in [0, T]$ and $\alpha \in A$,

$$\mathbb{E}\left[\psi(X_T^{u, \alpha}) + \int_t^T \phi(s, X_s^{u, \alpha}, \alpha_s)ds - \sum_{j \leq j} \ell(\tau_j, X^{[u]}_{\tau_j-1, \alpha}, \beta_j) | F_t\right]$$

$$\geq -C^q (1 + T) (1 + \text{ess sup}_{u' \in U^l, \alpha' \in A} \mathbb{E}\left[\sup_{s \in [T]} |X_s^{(u')0 \alpha'} \alpha(t) \alpha' | F_t\right])$$

(4.18)

where $u(t) = (\tau_1, \ldots, \tau_N(t-); \beta_1, \ldots, \beta_N(t-))$ with $N(t-) := \max\{j \geq 0 : \tau_j < t\}$ and $\alpha(t-) \circ \alpha' := 1_{[0, t]} \alpha + 1_{[t, T]} \alpha'$. Then for each $u \in U^l$ we obtain a $u' \in U^l$ by removing all future interventions whenever (4.18) does not hold. Moreover, $u'$ dominates $u$ in the sense that $J(u', \alpha) > J(u, \alpha)$ for all $\alpha \in A$.

Now, whenever $u \in U^l$ there is a $(B^{u, \alpha}, E^{u, \alpha}) \in S^2 \times \mathcal{H}^2$ that solves (4.17). We will build on the argument in Lemma 3.14 to show that $U^l \subset U^m$. We thus assume that $u \in U^l$. Rearranging the terms in (4.17) gives

$$\sum_{j=1}^N \ell(\tau_j, X^{[u]}_{\tau_j-1, \alpha}, \beta_j) = \psi(X_T^{u, \alpha}) - B^{u, \alpha}_0 + \int_0^T \phi(s, X_s^{u, \alpha}, \alpha_s)ds$$

$$- \int_0^T E_s^{u, \alpha} dW_s,$$

(4.19)
where we know that all terms on the right hand side, except for the last (martingale) term, have moments of all orders. By Ito’s formula we have

\[|B_0^{u,\alpha}|^2 + \int_0^T |E_s^{u,\alpha}|^2 ds = |\psi(X_T^{u,\alpha})|^2 + 2 \int_0^T B_s^{u,\alpha} \phi(s, X_s^{u,\alpha}, \alpha_s) ds - 2 \int_0^T B_s^{u,\alpha} E_s^{u,\alpha} dW_s + \sum_{j=1}^N (-2B_{\tau_j}^{u,\alpha} \ell(\tau_j, X_{\tau_j}^{[u]}, \beta_j) + |\ell(\tau_j, X_{\tau_j}^{[u]}, \beta_j)|^2) \]

\[\leq |\psi(X_T^{u,\alpha})|^2 + (1 + 2\kappa) \sup_{t \in [0,T]} |B_t^{u,\alpha}|^2 + \int_0^T |\phi(s, X_s^{u,\alpha}, \alpha_s)|^2 ds - 2 \int_0^T B_s^{u,\alpha} E_s^{u,\alpha} dW_s + \frac{2}{\kappa} \left( \sum_{j=1}^N \ell(\tau_j, X_{\tau_j}^{[u],\alpha}, \beta_j) \right)^2,\]

for \(\kappa > 0\) (where we have used that \(\ell(\tau_j, X_{\tau_j}^{[u],\alpha}, \beta_j) \leq 2 \sup_{t \in [0,T]} |B_t^{u,\alpha}|\)). Using (4.19), the growth conditions on \(\phi\) and \(\psi\) and the fact that \(u \in U\) together with (4.8) gives that

\[\int_0^T |E_s^{u,\alpha}|^2 ds \leq C \left( 1 + \kappa + \frac{1}{\kappa} \right) \left( 1 + \sup_{t \in [0,T]} |X_t^{u,\alpha}|^2 \right) - 2 \int_0^T B_s^{u,\alpha} E_s^{u,\alpha} dW_s + \frac{4}{\kappa} \left( \int_0^T E_s^{u,\alpha} dW_s \right)^2.\]

Raising both sides to \(p \geq 1\) followed by taking the expectation and applying the Burkholder-Davis-Gundy inequality gives

\[E\left[ \left( \int_0^T |E_s^{u,\alpha}|^2 ds \right)^p \right] \leq C \left( 1 + \kappa + \frac{1}{\kappa} \right) + C E\left[ \left( \int_0^T B_s^{u,\alpha} E_s^{u,\alpha} |2 ds\right)^{p/2} + \frac{1}{\kappa} \left( \int_0^T |E_s^{u,\alpha}|^2 \right)^p \right]\]

\[\leq C \left( 1 + \kappa + \frac{1}{\kappa} \right) + \frac{C}{\kappa} \left( \int_0^T |E_s^{u,\alpha}|^2 \right)^p\]

where we have used the relation \(2ab \leq \kappa a^2 + \frac{1}{\kappa} b^2\) together with the bound on \(E\left[ \sup_{t \in [0,T]} |B_t^{u,\alpha}|^2 \right]\) resulting from the fact that \(u \in U^f\) to reach the last inequality. Now, choosing \(\kappa\) sufficiently large it follows that the left hand side is finite. Finally, as the left hand side of (4.19) is greater than \(\delta N\) we conclude that \(u \in U^m\). \(\square\)

By Benes’ selection Theorem ([2], Lemma 5, pp. 460), there exists, for each \(v \in U^f\), a \(\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(A)\)-measurable function \(\alpha^v(t, \omega, z)\) such that for any given \((t, \omega, z) \in \mathbb{R}^d \times \Omega \times A\), and any given \(s \in [0,T]\), \(s \neq t\), we see that

\[u(t, \omega, z) = E_s^{u,\alpha^v} = \frac{1}{\alpha^v(t, \omega, z)} \int_s^T \phi(r, X_r^{u,\alpha^v}, \alpha_r^{u,\alpha^v}) dr + \frac{1}{\alpha^v(t, \omega, z)} \int_s^T E_r^{u,\alpha^v} dW_r.\]
[0, T] \times \Omega \times \mathbb{R}^d$, we have

\[ H^u(t, \omega, z, \alpha^u(t, \omega, z)) = \inf_{\alpha \in \mathcal{A}} H^u(t, \omega, z, \alpha), \]

\(P\)-a.s.

The following theorem shows that we can extract the optimal pair \((u^*, \alpha^*)\) from the family of maps \((\alpha^v(t, \omega, z) : v \in \mathcal{U}^f)\) and the solution to (4.11).

**Theorem 4.15** Let the family \((Y^v, Z^v, K^v : v \in \mathcal{U}^f)\) be a solution to (4.11). Then the pair \((u^*, \alpha^*)\) is optimal in the sense of (4.1) and \(Y^0_0 = J(u^*, \alpha^*)\).

**Proof** For \(u \in \mathcal{U}^u\) we let \((U^u, V^u) \in S_I \times \mathcal{H}^2\) be the unique solution to

\[ U^u_t = \psi(X^u_t) + \int_t^T H^{*,u}(s, V^u_s) ds - \int_t^T V^u_s dW_s - \sum_{j=1}^N \mathbb{1}_{[\tau^*_j, \tau^*_{j+1})}(t) \alpha^*[u^*]_j(t, Z^u_{j} [u^*]_j), \]

with \(\tau^*_{N+1} := \infty\) is optimal in the sense of (4.1) and \(Y^0_0 = J(u^*, \alpha^*)\).

Then \(\|V^u\|_{\mathcal{H}^p} < \infty\) for all \(p \geq 1\) implying that \(V^u \in \mathcal{H}^2_Q\) for all \(Q \in \mathcal{P}_0\) and, since \(U^u_0\) is \(\mathcal{F}_0\)-measurable and \(\mathcal{F}_0\) is trivial, we have

\[ U^u_0 = \mathbb{E}^{Q^u, \alpha^*} \left[ \psi(X^u_0) + \int_0^T H^{*,u}(s, V^u_s) ds - \int_0^T V^u_s dW_s - \sum_{j=1}^N \mathbb{1}_{[\tau^*_j, \tau^*_{j+1})}(t) \ell(t, X^u_{\tau^*_j} [u^*]_{\tau^*_{j+1}}), \beta_j) \right] \]

\[ = \mathbb{E}^{Q^u, \alpha^*} \left[ \psi(X^u_0) + \int_0^T \phi(s, X^u_s, \alpha^*_s) ds - \int_0^T V^u_s dW^u_s, \alpha^* - \sum_{j=1}^N \mathbb{1}_{[\tau^*_j, \tau^*_{j+1})}(t) \ell(t, X^u_{\tau^*_j} [u^*]_{\tau^*_{j+1}}), \beta_j) \right] \]

\[ = J(u, \alpha^*) \]

where now \(Q^{u, \alpha}\) is the measure, equivalent to \(\mathbb{P}\), under which \(W^{u, \alpha} := W - \int_0^- \tilde{a}(s, (X^u_s)_{r \leq s}, \alpha_s) ds\) is a martingale. Moreover, a straightforward extension of Lemma 3.16 to impulse controls with an unbounded number of interventions shows that \(u^* \in \mathcal{U}^u\) and we conclude by Theorem 3.20 (see Remark 4.12) that

\[ Y^0_0 = U^u_0 = \sup_{u \in \mathcal{U}^u} U^u_0 = \sup_{u \in \mathcal{U}^u} U^u_0. \]
Combined with Lemma 4.14, the above gives

\[ Y_0^\alpha = J(u^*, \alpha^*) = \sup_{u \in \mathcal{U}} U_0^u = \sup_{u \in \mathcal{U}} J(u, \alpha^*) = \sup_{u \in \mathcal{U}} J(u, \alpha^*). \]

To show that \( \alpha^* \) is an optimal response it is enough to show that \( \alpha^* \) is a minimizer of \( \alpha \mapsto J(u^*, \alpha) \). However, for any \( (u, \alpha) \in \mathcal{U}^n \times \mathcal{A} \), we have

\[
U_0^u = \mathbb{E}^{Q,u,\alpha}[\psi(X_T^u) + \int_0^T H^u(s, V_s^u) \, ds - \int_0^T V_s^u \, dW_s - \sum_{j=1}^N \ell(\tau_j, X_{\tau_j}^{[u]j-1}, \beta_j)]
\]

\[
= \mathbb{E}^{Q,u,\alpha}[\psi(X_T^u) + \int_0^T \phi(s, X_s^u, \alpha_s) \, ds - \int_0^T V_s^u \, dW_s^\alpha - \sum_{j=1}^N \ell(\tau_j, X_{\tau_j}^{[u]j-1}, \beta_j)]
+ \mathbb{E}^{Q,u,\alpha}[\int_0^T (H^u(s, V_s^u) - H^u(s, V_s^u, \alpha_s)) \, ds]
\]

\[ \leq J(u, \alpha) \]

and we conclude that \((u^*, \alpha^*)\) is a saddle-point for the game. \( \square \)

## 5 Conclusions

The present work considers a sequentially arranged system of reflected BSDEs parameterized by impulse controls. Using only probabilistic arguments, we show existence and uniqueness of solutions under a stochastic Lipschitz condition on the driver. Moreover, we relate the solution to our system of BSDEs to a weakly formulated stochastic differential game where one player implements an impulse control while the adversary player plays a continuous control that does not enter the diffusion coefficient. Since all arguments are probabilistic, we do not need to rely on any Markov property of solutions to the underlying controlled SDE, enabling us to handle path-dependence in the drift and diffusion coefficient as well as in the impulse-to-jump map, \( \Gamma \), of the impulse controller.

A practical application of the result lies within robust control where the impulse controller is ambiguous about the model for the drift term in the controlled SDE. A cautious operator could incorporate this ambiguity into the model by considering the worst case, thus rendering a zero-sum stochastic differential game where the adversary (nature) controls the drift term. In this regard, a number of questions are left open. An issue of clear interest is when the adversary player is allowed to control the diffusion coefficient as well. In the robust control framework this setting corresponds to ambiguity about both the drift and the diffusion coefficient. A recent development in this direction is [27] where path-dependent PDEs [8, 9] and second order BSDEs [29] (abbreviated 2BSDEs) are used to represent solutions to path-dependent zero-sum stochastic differential games of continuous versus continuous control in a weak formulation. A natural next step would thus be to develop the present work to incorporate 2BSDEs.
Our game is one of impulse versus continuous control. Alternatively, one could consider the case where both players play impulse controls. This problem was approached in the Markovian framework in [6] but the extension to the path-dependent setting remains an open problem.

Finally, we leave the important issue of tractable numerical approximation algorithms as a topic of future research.

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### A Proof of Proposition 2.4

In this section we give a proof of Proposition 2.4. Since the result can be useful in other contexts as well, it is written as a stand-alone piece.

#### A.1 Classical Result Under Deterministic Lipschitz Condition

Our approach will rely heavily on the available theory of reflected backward SDEs and we, therefore, recall the following important result:

**Theorem A.1** (El Karoui et. al. [12]) Assume that

(a) $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

(b) The barrier $S$ is real-valued, $\mathcal{P}_F$-measurable and continuous with $S^+ \in \mathcal{S}^2$ and $S_T \leq \xi$.

(c) $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is such that $f(\cdot, y, z) \in \mathcal{H}^2$ for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and for some $k_f > 0$ and all $(y, y', z, z') \in \mathbb{R}^{2(1+d)}$ we have

$$|f(t, y', z') - f(t, y, z)| \leq k_f(|y' - y| + |z' - z|).$$

Then there exists a unique triple $(Y, Z, K) := (Y_t, Z_t, K_t)_{0 \leq t \leq T}$ with $Y, K \in \mathcal{S}^2$ and $Z \in \mathcal{H}^2$, where $K$ is non-decreasing with $K_0 = 0$, such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t,$$

$$Y_t \geq S_t, \forall t \in [0, T] \text{ and } \int_0^T (Y_t - S_t) dK_t = 0. \quad (A.1)$$
Furthermore,

\[ \|Y\|_{S^2}^2 + \|Z\|_{H^2}^2 + \|K\|_{S^2}^2 \leq C \mathbb{E}\left[ |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds + \sup_{t \in [0, T]} |(S_t)^+|^2 \right]. \]

(A.2)

In addition, \( Y \) can be interpreted as the Snell envelope in the following way

\[ Y_t = \text{ess sup}_{\tau \in T_t} \mathbb{E}\left[ \int_\tau^T f(s, Y_s, Z_s) ds + S_{\tau} 1_{[\tau < T]} + \xi 1_{[\tau = T]} |F| \right] \]

and with \( D_t := \inf \{ r \geq t : Y_r = S_r \} \wedge T \) we have the representation

\[ Y_t = \mathbb{E}\left[ \int_{D_t} f(s, Y_s, Z_s) ds + S_{D_t} 1_{[D_t < T]} + \xi 1_{[D_t = T]} |F| \right] \]

and \( K_{D_t} - K_t = 0, \mathbb{P}\text{-a.s.} \)

Moreover, if \((\tilde{Y}, \tilde{Z}, \tilde{K})\) is the solution to the reflected BSDE with parameters \((\tilde{\xi}, \tilde{f}, \tilde{S})\), then

\[ \|\tilde{Y} - Y\|_{S^2}^2 + \|\tilde{Z} - Z\|_{H^2}^2 + \|\tilde{K} - K\|_{S^2}^2 \]

\[ \leq C \left( \|\tilde{S} - S\|_{S^2} \Psi_T^{1/2} + \mathbb{E}\left[ |\tilde{\xi} - \xi|^2 + \int_0^T |\tilde{f}(s, Y_s, Z_s) - f(s, Y_s, Z_s)|^2 ds \right] \right). \]

(A.3)

where

\[ \Psi_T := \mathbb{E}\left[ |\tilde{\xi}|^2 + |\xi|^2 + \int_0^T (|\tilde{f}(s, 0, 0)|^2 + |f(s, 0, 0)|^2) ds + \sup_{t \in [0, T]} |(\tilde{S}_t)^+ + (S_t)^+|^2 \right]. \]

Existence and uniqueness of solutions to (A.1) under the assumption that the Lipschitz coefficient on the \( z \)-variable is a stochastic process was recently shown in [10]. In the next section we show that the moment and stability estimates in Theorem A.1 translates to the case with stochastic Lipschitz coefficients and exponents \( p \neq 2 \) as well.

**A.2 Extension to Stochastic Lipschitz Coefficient**

We consider the non-reflected BSDE with parameters \((f, \xi)\),

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad \forall t \in [0, T] \]  

(A.4)
and the reflected BSDE with parameters \((f, \xi, S)\) given in (2.3) that we recall is
\[
\begin{aligned}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \\
Y_t &\geq S_t, \forall t \in [0, T] \text{ and } \int_0^T (Y_t - S_t) dK_t = 0.
\end{aligned}
\] (A.5)

Throughout this section we will assume that for some \(q' > 1\) and all \(p \geq 1\) the following holds:

**Assumption A.2**

(i) There is a \(\mathbb{P}\)-a.s. non-negative, \(\mathcal{F}_t\)-measurable, continuous process \((L_t : t \in [0, T])\) (our stochastic Lipschitz coefficient) such that for all \(\mathcal{F}_t\)-measurable processes \((\xi_t : t \in [0, T])\) with \(|\xi_t| \leq L_t\) for all \(t \in [0, T]\) (outside of a \(\mathbb{P}\)-null set) we have \(\mathbb{E}[|\xi W_T|^{q'}] < \infty\) and \(\mathbb{E}_Q[|\xi - W_T|^{q'}] < \infty\).

(ii) The terminal value \(\xi \in L^p(\Omega, \mathcal{F}_T, \mathbb{P})\).

(iii) The driver \((t, \omega, y, z) \mapsto f(t, \omega, y, z) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}\) is \(\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)\)-measurable. Furthermore, we have

(a) The bound
\[
\mathbb{E}\left[ \int_0^T |f(s, 0, 0)|^p ds \right] < \infty. \tag{A.6}
\]

(b) There is a constant \(k_f > 0\) such that
\[
|f(t, y', z') - f(t, y, z)| \leq k_f |y' - y| + L_t |z' - z| \tag{A.7}
\]
for all \((t, y, y', z, z') \in [0, T] \times \mathbb{R}^{1+d}, \mathbb{P}\)-a.s.

(c) The barrier \(S\) is real-valued, \(\mathcal{F}_t\)-measurable and continuous with \(S^+ \in S^p\) and \(S_T \leq \xi, \mathbb{P}\)-a.s.

Under this set of assumptions, letting \(q > 1\) be such that \(\frac{1}{q'} + \frac{1}{q^2} = 1\), we have the following:

**Theorem A.3** The BSDE (A.4) has a unique solution \((Y, Z) \in S^p \times \mathcal{H}^p\), with
\[
\|Y\|_{S^p}^p + \|Z\|_{\mathcal{H}^p}^p \leq \mathbb{C} \mathbb{E}\left[ \left| \xi \right|^{q^2} + \left( \int_0^T |f(s, 0, 0)| ds \right)^{q^2} \right]^{1/q^2}.
\]
Furthermore, if \((\tilde{Y}, \tilde{Z})\) is a solution to (A.4) with parameters \((\tilde{f}, \tilde{\xi})\), then
\[
\|\tilde{Y} - Y\|_{S^p}^p + \|\tilde{Z} - Z\|_{\mathcal{H}^p}^p \leq \mathbb{C} \mathbb{E}\left[ \left| \tilde{\xi} - \xi \right|^{q^2} + \left( \int_0^T |\tilde{f}(s, Y_s, Z_s) - f(s, Y_s, Z_s)| ds \right)^{q^2} \right]^{1/q^2}.
\]

**Proof** This follows by repeating the arguments in the proofs of Lemma 3.1 and Lemma 3.2 in [21]. □
As a step towards obtaining moment estimates for solutions to (A.5) we follow the convention in [10] and let \((Y^{m, n}, Z^{m, n}, K^{m, n}) \in \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{S}^2\) be the unique solution to (A.5) with parameters \((\mathbb{1}_{[L \leq m]} f^+, \mathbb{1}_{[L \leq n]} f^-, \xi, S)\), i.e.

\[
\begin{align*}
Y^{m, n}_t &= \xi + \int_t^T f^{m, n}(s, Y^{m, n}_s, Z^{m, n}_s) \, ds + \int_t^T Z^{m, n}_s \, dW_s + K^{m, n}_T - K^{m, n}_t, \\
Y^{m, n}_t &\geq S_t, \ \forall t \in [0, T] \text{ and } \int_0^T (Y^{m, n}_t - S_t) \, dK^{m, n}_t = 0.
\end{align*}
\]

where \(f^{m, n}(t, y, z) := \mathbb{1}_{[L \leq m]} f^+(t, y, z) - \mathbb{1}_{[L \leq n]} f^-(t, y, z)\) is a standard Lipschitz driver.

**Lemma A.4** For all \(m, n \geq 0\) we have \(Z^{m, n} \in \mathcal{H}^2_Q\) for all \(Q \in \mathcal{P}^{L \wedge (m \vee n)}\).

**Proof** For each \(Q \in \mathcal{P}^{L \wedge (m \vee n)}\), there is a \(\mathcal{F}_t\)-measurable process, \(\zeta\), such that \(|\zeta_t| \leq L_t \wedge (m \vee n)\), for all \(t \in [0, T]\) (outside of a \(\mathbb{P}\)-null set) and \(dQ := \mathcal{E}(\zeta \cdot W) d\mathbb{P}\). By the Girsanov theorem (see e.g. Chapter 15 in [5]), it follows that under the probability measure \(Q\), the process \(W^\zeta := W - \int_0^T \zeta_s ds\) is a Brownian motion. Moreover, the triple \((Y^{m, n}, Z^{m, n}, K^{m, n})\) satisfies

\[
\begin{align*}
Y^{m, n}_t &= \xi + \int_t^T (f^{m, n}(s, Y^{m, n}_s, Z^{m, n}_s) - \xi_s Z^{m, n}_s) \, ds - \int_t^T Z^{m, n}_s dW^\zeta_s + K^{m, n}_T - K^{m, n}_t, \\
Y^{m, n}_t &\geq S_t, \ \forall t \in [0, T] \text{ and } \int_0^T (Y^{m, n}_t - S_t) \, dK^{m, n}_t = 0.
\end{align*}
\]

Since this is a reflected BSDE with standard Lipschitz driver we can apply (A.2) to get that

\[
\begin{align*}
\mathbb{E}^Q\left( \int_0^T |Z^{m, n}_s|^2 \, ds \right) \\
&\leq C \mathbb{E}^Q \left( |\xi|^2 + \int_0^T |f^{m, n}(s, 0, 0)|^2 \, ds + \sup_{t \in [0, T]} |(S_t)^+|^2 \right) \\
&\leq C \mathbb{E} \left[ \mathcal{E}(\zeta \cdot W)_T \left( |\xi|^2 + \int_0^T |f^{m, n}(s, 0, 0)|^2 \, ds + \sup_{t \in [0, T]} |(S_t)^+|^2 \right) \right] \\
&\leq C \mathbb{E} \left[ \mathcal{E}(\zeta \cdot W)_T q^{q'} 1^{1/q'} \mathbb{E} \left[ |\xi|^{2q} + \int_0^T |f^{m, n}(s, 0, 0)|^{2q} \, ds + \sup_{t \in [0, T]} |(S_t)^+|^{2q} \right]^{1/q} \right]^{1/q}
\end{align*}
\]

and the result follows by Assumption A.2.

**Lemma A.5** There is a \(C > 0\) such that

\[
\|Y^{m, n}\|_{\mathbb{S}^p}^p + \|Z^{m, n}\|_{\mathcal{H}^p}^p + \|K^{m, n}\|_{\mathbb{S}^p}^p \\
\leq C \mathbb{E} \left[ \mathbb{E} \left[ |\xi|^{2q} p + \left( \int_0^T |f^{m, n}(s, 0, 0)| ds \right)^{2q} p + \sup_{t \in [0, T]} |(S_t)^+|^{2q} p \right]^{1/q} \right]^{1/q^2}.
\] (A.8)

for all \(m, n \geq 0\).
Proof We define
\[ \gamma_s := \frac{f^{m,n}(s, Y^{m,n}_s, Z^{m,n}_s) - f^{m,n}(s, 0, Z^{m,n}_s)}{Y^{m,n}_s} \mathbb{1}_{\{Y^{m,n}_s \neq 0\}} \]

and set \( e_t := e^{\int_0^t \gamma_s \, ds} \). By Assumption A.2 we have that \( e^{-k_f T} \leq e_t \leq e^{k_f T} \). Applying Ito’s formula to \( e_t Y^{m,n}_t \) gives that for any \( \tau \in \mathcal{T}_r \) we have
\[
eq e_t Y^{m,n}_\tau + \int_t^\tau e_s f^{m,n}(s, 0, 0) + \zeta_s Z^{m,n}_s \, ds - \int_t^\tau e_s Z^{m,n}_s \, dW_s + \int_t^\tau e_s dK^{m,n}_s
\]

where
\[
\zeta_s := \frac{f^{m,n}(s, 0, Z^{m,n}_s) - f^{m,n}(s, 0, 0)}{|Z^{m,n}_s|^2} (Z^{m,n}_s)\mathbb{1}_{\{Z^{m,n}_s \neq 0\}}.
\]

By the Girsanov theorem, it follows that under the probability measure \( \mathbb{Q}^\xi \) defined as \( d\mathbb{Q}^\xi := \mathcal{E}(\xi \ast W)_T \, d\mathbb{P} \) the process \( W^\xi_t := W_t - \int_0^t \zeta_s \, ds \) is a Brownian motion. Furthermore, we have
\[
eq e_t Y^{m,n}_t = e_t Y^{m,n}_\tau + \int_t^\tau e_s f^{m,n}(s, 0, 0) \, ds - \int_t^\tau e_s Z^{m,n}_s \, dW^\xi_s + \int_t^\tau e_s dK^{m,n}_s.
\]

Taking the conditional expectation while picking \( \tau = D_t := \inf\{t \geq 0 : Y^{m,n}_t = S_t\} \wedge T \) and appealing to Lemma A.4 which implies that the stochastic integral is a \( \mathbb{Q}^\xi \)-martingale gives
\[
eq e_t Y^{m,n}_t = \mathbb{E}^{\mathbb{Q}^\xi} \left[ \mathbb{1}_{[D_t < T]} e^\gamma S_{D_t} + \mathbb{1}_{[D_t = T]} e^\gamma \right] + \int_0^{D_t} e_s f^{m,n}(s, 0, 0) \, ds \mathcal{F}_t.
\]

Doob’s maximal inequality together with the bounds on \( e_t \) gives that
\[
\mathbb{E}^{\mathbb{Q}^\xi} \left[ \sup_{t \in [0,T]} |Y^{m,n}_t|^p \right] \leq C \mathbb{E}^{\mathbb{Q}^\xi} \left[ \left( \int_0^T |f^{m,n}(s, 0, 0)| \, ds \right)^p \sup_{t \in [0,T]} |(S_t)^+|^p + |\xi|^p \right].
\]
Changing back to our original probability measure \( \mathbb{P} \) on the right hand side gives

\[
\mathbb{E}^{Q_T} \left[ \sup_{t \in [0,T]} |Y_t^{m,n}|^p \right] \\
\leq C \mathbb{E} \left[ \mathcal{E}(\zeta \ast W)_T \left( \left( \int_{0}^{T} |f^{m,n}(s, 0, 0)|ds \right)^p + \sup_{t \in [0,T]} |(S_t)^+ + |\xi|^p | \right) \right] \\
\leq C \mathbb{E} \mathcal{E}(\zeta \ast W)^{q'} T^{1/q'} \mathbb{E} \left[ \left( \int_{0}^{T} |f^{m,n}(s, 0, 0)|ds \right)^{qp} + \sup_{t \in [0,T]} |(S_t)^+|^q + |\xi|^q \right]^{1/q}.
\]

Moreover, as

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t^{m,n}|^p \right] = \mathbb{E}^{Q_T} \left[ \mathcal{E}(\zeta \ast W)^{q'} T^{1/q'} \left( \int_{0}^{T} |f^{m,n}(s, 0, 0)|ds \right)^{qp} + \sup_{t \in [0,T]} |(S_t)^+|^{qp} + |\xi|^q \right]^{1/q},
\]

the estimate for \( Y^{m,n} \) follows.

To arrive at the estimate for \( Z^{m,n} \) we apply Ito’s formula to \( (Y^{m,n})^2 \) and get that

\[
|Y_t^{m,n}|^2 + \int_{t}^{T} |Z_{s}^{m,n}|^2 ds = |\xi|^2 + 2 \int_{t}^{T} Y_{s}^{m,n} f^{m,n}(s, Y_{s}^{m,n}, Z_{s}^{m,n}) ds \\
- 2 \int_{t}^{T} Y_{s}^{m,n} Z_{s}^{m,n} dW_s + 2 \int_{t}^{T} Y_{s}^{m,n} dK_{s}^{m,n} \\
= |\xi|^2 + 2 \int_{t}^{T} (\gamma_s |Y_{s}^{m,n}|^2 + Y_{s}^{m,n} f^{m,n}(s, 0, 0)) ds \\
- 2 \int_{t}^{T} Y_{s}^{m,n} Z_{s}^{m,n} dW_s + 2 \int_{t}^{T} Y_{s}^{m,n} dK_{s}^{m,n}. \tag{A.10}
\]

Furthermore, applying the relation \( ab \leq \frac{1}{2} \kappa a^2 + \frac{1}{2 \kappa} b^2 \) with \( \kappa > 0 \) and the Skorokhod condition \( \int_{0}^{T} (Y_{s}^{m,n} - S_s) dK_{s}^{m,n} = 0 \) yields

\[
\int_{t}^{T} Y_{s}^{m,n} dK_{s}^{m,n} \leq \int_{t}^{T} (S_s)^+ dK_{s}^{m,n} \leq \frac{\kappa}{2} \sup_{r \in [t,T]} |(S_r)^+|^2 + \frac{1}{2 \kappa} |K_T^{m,n}|^2.
\]

Now, Proposition 2.2 in [12] gives that

\[
K_T^{m,n} - K_t^{m,n} = \sup_{r \in [t,T]} \left( \xi + \int_{r}^{T} f^{m,n}(s, Y_{s}^{m,n}, Z_{s}^{m,n}) ds - \int_{r}^{T} Z_{s}^{m,n} dW_s - S_r \right)^{-} \\
= \sup_{r \in [t,T]} \left( \xi + \int_{r}^{T} (\gamma_s Y_{s}^{m,n} + f^{m,n}(s, 0, 0)) ds - \int_{r}^{T} Z_{s}^{m,n} dW_s - S_r \right)^{-}
\]

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and, in particular, we have that

\[
K_T^{m,n} \leq |\xi| + \int_0^T \left( k_f |Y_s^{m,n}| + |f^{m,n}(s, 0, 0)| \right) ds
+ 2 \sup_{r \in [0,T]} \left| \int_0^r Z_s^{m,n} dW_s^\xi \right| + \sup_{r \in [0,T]} |(S_r)^+| \tag{A.11}
\]

Inserted into (A.10) this gives (with \(\kappa \geq 1\))

\[
\int_0^T |Z_s^{m,n}|^2 ds \leq \frac{C}{\kappa} \sup_{r \in [0,T]} \left| \int_0^r Z_s^{m,n} dW_s^\xi \right|^2 - 2 \int_0^T Y_s^{m,n} Z_s^{m,n} dW_s^\xi
+ C(1 + \kappa)(|\xi|^2 + \sup_{r \in [0,T]} |Y_r^{m,n}|^2 + \sup_{r \in [0,T]} |(S_r)^+|^2
+ \left( \int_0^T |f^{m,n}(s, 0, 0)| ds \right)^2)
\]

or

\[
\mathbb{E}^Q \left[ \left( \int_0^T |Z_s^{m,n}|^2 ds \right)^{p/2} \right]
- \frac{C}{\kappa^{p/2}} \sup_{r \in [0,T]} \left| \int_0^r Z_s^{m,n} dW_s^\xi \right|^p - C \left| \int_0^T Y_s^{m,n} Z_s^{m,n} dW_s^\xi \right|^{p/2}
\leq C(1 + \kappa^{p/2}) \mathbb{E}^Q \left[ |\xi|^p + \sup_{r \in [0,T]} |Y_r^{m,n}|^p + \sup_{r \in [0,T]} |(S_r)^+|^p + \left( \int_0^T |f^{m,n}(s, 0, 0)| ds \right)^p \right].
\]

On the other hand, applying the Burkholder-Davis-Gundy inequality gives that

\[
\mathbb{E}^Q \left[ \left( \int_0^T |Z_s^{m,n}|^2 ds \right)^{p/2} \right] - \frac{C}{\kappa^{p/2}} \sup_{r \in [0,T]} \left| \int_0^r Z_s^{m,n} dW_s^\xi \right|^p - C \left| \int_0^T Y_s^{m,n} Z_s^{m,n} dW_s^\xi \right|^{p/2}
\geq \left( 1 - \frac{C}{\kappa^{p/2}} \right) \mathbb{E}^Q \left[ \left( \int_0^T |Z_s^{m,n}|^2 ds \right)^{p/2} \right] - C \mathbb{E}^Q \left[ \left( \int_0^T |Y_s^{m,n} Z_s^{m,n}|^2 ds \right)^{p/4} \right]
\]

By again using that \(ab \leq \frac{1}{2}ka^2 + \frac{1}{2k}b^2\) we get

\[
\mathbb{E}^Q \left[ \left( \int_0^T |Y_s^{m,n} Z_s^{m,n}|^2 ds \right)^{p/4} \right] \leq C \mathbb{E}^Q \left[ \kappa \sup_{s \in [0,T]} |Y_s^{m,n}|^p + \frac{1}{\kappa^p} \left( \int_0^T |Z_s^{m,n}|^2 ds \right)^{p/2} \right]
\]

and the estimate follows by choosing \(\kappa > 0\) sufficiently large and repeating the steps above to change back to the original measure \(\mathbb{P}\). Finally, the bound for \(K^{m,n}\) is immediate from (A.11) the above.

\[\square\]

**Lemma A.6** There is a constant \(C > 0\) such that, whenever \((Y^{m,n}, Z^{m,n}, K^{m,n})\) and \((\tilde{Y}^{m,n}, \tilde{Z}^{m,n}, \tilde{K}^{m,n})\) are solutions to (A.5) with parameters \((f^{m,n}, \xi, S)\) and
\( \tilde{\theta}_{\tilde{m}, \tilde{n}}, \tilde{\xi}, \tilde{S} \), respectively, then

\[
\| \tilde{Y}_{\tilde{m}, \tilde{n}} - Y_{m,n} \|_{SP} + \| \tilde{Z}_{\tilde{m}, \tilde{n}} - Z_{m,n} \|_{HF} + \| \tilde{K}_{\tilde{m}, \tilde{n}} - K_{m,n} \|_{SP} \\
\leq C \left( \| \tilde{S} - S \|_{SP}^{p/2} \Psi_T^{1/2} \\
+ \mathbb{E} \left[ |\tilde{\xi} - \xi|^q + \left( \int_0^T |\tilde{f}_{\tilde{m}, \tilde{n}}(s, Y_s^{m,n}, Z_s^{m,n}) - f_{m,n}(s, Y_s^{m,n}, Z_s^{m,n})|ds \right)^{q^2} \right]^{1/q^2} \right),
\]

where

\[
\Psi_T := \mathbb{E} \left[ |\tilde{\xi}|^{q^2} + |\xi|^{q^2} + \left( \int_0^T (|\tilde{f}_{\tilde{m}, \tilde{n}}(s, 0, 0)| + |f_{m,n}(s, 0, 0)|)ds \right)^{q^4} \right. \\
+ \sup_{t \in [0,T]} |(\tilde{S}_t) + (S_t)|^{q^4} \right]^{1/q^4}.
\]

**Proof** For ease of notation we omit the superscripts \( m, n \) and \( \tilde{m}, \tilde{n} \) and have

\[
\tilde{Y}_t - Y_t = \tilde{\xi} - \xi + \int_t^T (\tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s) - f(s, Y_s, Z_s))ds \\
- \int_t^T (\tilde{Z}_s - Z_s) dW_s + \tilde{K}_T - \tilde{K}_t - (K_T - K_t)
\]

Now, let

\[
y_s := \frac{\tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s) - f(s, Y_s, Z_s)}{\tilde{Y}_s - Y_s} \mathbb{1}_{[\tilde{Y}_s \neq Y_s]}
\]

and

\[
\zeta_s := \frac{\tilde{f}(s, Y_s, Z_s) - f(s, Y_s, Z_s)}{|\tilde{Z}_s - Z_s|^2}(\tilde{Z}_s - Z_s)^\top \mathbb{1}_{[\tilde{Z}_s \neq Z_s]}
\]

and set \( e_t := e_t^{\nu_t} ds \). Letting \( \delta Y := \tilde{Y} - Y, \delta \xi := \tilde{\xi} - \xi, \delta Z := \tilde{Z} - Z \) and \( \delta f := \tilde{f} - f \) we get that for any \( \tau \in T_t \) we have

\[
e_t \delta Y_t = e_t \delta Y_t + \int_t^\tau e_s (\delta f(s, Y_s, Z_s) + \zeta_s \delta Z_s)ds - \int_t^\tau e_s \delta Z_s dW_s + \int_t^\tau e_s d(\delta K)_s
\]

\[
= e_t \delta Y_t + \int_t^\tau e_s \delta f(s, Y_s, Z_s)ds - \int_t^\tau e_s \delta Z_s dW_s + \int_t^\tau e_s d(\delta K)_s
\]

\( \square \)
where \( W^\xi_t = W_t - \int_0^t \zeta_s ds \). Now set \( \tau = D_t (= \inf \{ r \geq t : Y_r = S_r \} \wedge T ) \) and we find that

\[
e_t \delta Y_t = e_t \delta Y_{D_t} + \int_t^{D_t} e_s \delta f (s, Y_s, Z_s) ds - \int_t^{D_t} e_s \delta Z_s dW^\xi_s + \int_t^{D_t} e_s (d \tilde{K}_s - d K_s)
\]

\[
\geq \mathbb{1}_{\{D_t < T\}} e_{D_t} \delta S_{D_t} + \mathbb{1}_{\{D_t = T\}} e_T \delta \xi + \int_t^{D_t} e_s \delta f (s, Y_s, Z_s) ds - \int_t^{D_t} e_s \delta Z_s dW^\xi_s.
\]

On the other hand, by picking \( \tau = \tilde{D}_t := \inf \{ r \geq t : \tilde{Y}_r = \tilde{S}_r \} \wedge T \) we get

\[
e_t \delta Y_t \leq \mathbb{1}_{\{\tilde{D}_t < T\}} e_{\tilde{D}_t} \delta S_{\tilde{D}_t} + e_T \mathbb{1}_{\{\tilde{D}_t = T\}} \delta \xi + \int_t^{\tilde{D}_t} e_s \delta f (s, Y_s, Z_s) ds - \int_t^{\tilde{D}_t} e_s \delta Z_s dW^\xi_s.
\]

Taking the conditional expectation with respect to the measure \( Q^\xi \), with \( dQ^\xi = Ex(W)^T dP \), we find that

\[
e_t |\delta Y_t| \leq \mathbb{E}^{Q^\xi} \left[ \sup_{r \in [0, T]} e_r |\delta S_r| + e_T |\delta \xi| + \int_0^T e_s |\delta f (s, Y_s, Z_s)| ds \bigg| \mathcal{F}_t \right]
\]

and the bound for \( \| \tilde{Y}_\tilde{m}, \tilde{n} - Y_{m,n} \|_{S^p} \) follows by appealing to the argument of Lemma A.5 while noting that \( |\gamma_s| \leq k_f \) and \( |\zeta_s| \leq L_s \).

Applying Ito's formula to \( \delta Y^2 \) gives (similarly to (A.10) that)

\[
|\delta Y_t|^2 + \int_t^T |\delta Z_s|^2 ds = |\xi|^2 + 2 \int_t^T (\gamma_s |\delta Y_s|^2 + \delta Y_s \delta f (s, Y_s, Z_s)) ds
\]

\[- 2 \int_t^T \delta Y_s \delta Z_s dW^\xi_s + 2 \int_t^T \delta Y_s d(\delta K)_s
\]

\[
\leq |\xi|^2 + 2 \int_t^T (k_f |\delta Y_s|^2 + \delta Y_s \delta f (s, Y_s, Z_s)) ds
\]

\[- 2 \int_t^T \delta Y_s \delta Z_s dW^\xi_s + 2 \int_t^T \delta S_s d(\delta K)_s,
\]

(A.12)

where the last inequality follows from the fact that \( \int_0^T \delta Y_s d(\delta K)_s \leq \int_0^T \delta S_s d(\delta K)_s \).

Now,

\[
\int_0^T \delta S_s d(\delta K)_s \leq \sup_{r \in [0, T]} |\delta S_r| (\tilde{K}_T + K_T)
\]
which gives that

$$
\mathbb{E}^Q \left[ \left( \int_0^T |\delta Z_s|^2 ds \right)^{p/2} \right] 
\leq C \mathbb{E}^Q \left[ |\xi|^q + \sup_{r \in [0,T]} |\delta Y_s|^q + \left( \int_0^T |\delta f(s, Y_s, Z_s)| ds \right)^q \right]^{2p/q} 
+ \int_0^T |\delta Y_s \delta Z_s^m \, dW_s|^{p/2} + \sup_{r \in [0,T]} |\delta S_r|^{p/2} \left( (\tilde{K}_T)^{q/2} + (K_T)^{q/2} \right) 
$$

and repeating the last steps in the proof of Lemma A.5 we have that

$$
\mathbb{E} \left[ \left( \int_0^T |\delta Z_s|^2 ds \right)^{p/2} \right] \leq C \mathbb{E} \left[ |\xi|^q + \sup_{r \in [0,T]} |\delta Y_s|^q + \left( \int_0^T |\delta f(s, Y_s, Z_s)| ds \right)^q \right]^{2p/q} 
\sup_{r \in [0,T]} |\delta S_r|^{q/2} \left( (\tilde{K}_T)^{q/2} + (K_T)^{q/2} \right) \right]^{1/q^2} 
$$

Finally,

$$
\mathbb{E} \left[ \sup_{r \in [0,T]} |\delta S_r|^{q/2} \left( (\tilde{K}_T)^{q/2} + (K_T)^{q/2} \right) \right]^{1/q^2} 
\leq \sqrt{2} \mathbb{E} \left[ \sup_{r \in [0,T]} |\delta S_r|^{q/2} \right]^{1/2} \left[ (\tilde{K}_T)^{q/2} + (K_T)^{q/2} \right]^{1/2} 
$$

and the result follows by applying Lemma A.5 to the last term. \(\square\)

**Proposition A.7** The reflected BSDE (A.5) admits a unique solution \((Y, Z, K) \in S^2 \times \mathcal{H}^2 \times S^2\), with \(K\) non-decreasing and \(K_0 = 0\). Furthermore, we have

$$
\|Y\|_{S^p}^p + \|Z\|_{\mathcal{H}^p}^p + \|K\|_{S^p}^p \leq C \left( \|S^+\|_{S^{q/2}_p}^p + \mathbb{E} \left[ |\xi|^q + \left( \int_0^T |f(s, 0, 0)| ds \right)^q \right]^{p/q^2} \right) 
$$

(A.13)

and if \((\tilde{Y}, \tilde{Z}, \tilde{K})\) is a solution to (A.5) with parameters \((\tilde{\xi}, \tilde{\xi}, \tilde{S})\) then

$$
\|\tilde{Y} - Y\|_{S^p}^p + \|\tilde{Z} - Z\|_{\mathcal{H}^p}^p + \|\tilde{K} - K\|_{S^p}^p 
\leq C \left( \|\tilde{S} - S\|_{S^{q/2}_p}^{p/2} \Psi_T^{1/2} + \mathbb{E} \left[ |\tilde{\xi} - \xi|^q \right]^{p/q^2} + \left( \int_0^T |\tilde{f}(s, Y_s, Z_s) - f(s, Y_s, Z_s)| ds \right)^q \right)^{1/q^2} 
$$

(A.14)
where

\[ \Psi_T := \mathbb{E}\left[ |\hat{\xi}|^{q^4 p} + |\xi|^{q^4 p} + \left( \int_0^T (|\tilde{f}(s, 0, 0)| + |f(s, 0, 0)|) ds \right)^{q^4 p} \right] \]

\[ + \sup_{t \in [0, T]} |(\hat{S}_t)^+ + (S_t)^+|^{q^4 p} \]

\[ \left(1/4^4\right). \]

**Proof** The first part of the proof largely follows that of Lemma 2.3 in [10] and is included for the sake of completeness. Taking the limit on the right-hand side of (A.8) and appealing to dominated convergence we find that

\[ \|Y^{m,n}\|_{\mathcal{S}^p}^p + \|Z^{m,n}\|_{\mathcal{H}^p}^p + \|K^{m,n}\|_{\mathcal{S}^p}^p \]

\[ \leq C \left( \|S^+\|_{\mathcal{S}^{2,p}}^p + \mathbb{E}\left[ |\xi|^{q^2 p} + \left( \int_0^T |f(s, 0, 0)| ds \right)^{q^2 p} \right]^{1/2} \right) \]

\[ \leq C, \]

by Assumption A.2 for all \( m, n \geq 0 \) and \( p \geq 1 \). In particular, since for fixed \( m \) comparison implies that the sequence \( (Y^{m,n})_{n \geq 0} \) of continuous processes is non-increasing, the above bound implies that \( Y^m := \lim_{n \to \infty} Y^{m,n} \exists, \mathbb{P}\text{-a.s.}, \) and by Fatou’s lemma it satisfies \( \|Y^m\|_{\mathcal{S}^p} \leq C. \) Furthermore, we have by Lemma A.6 that

\[ \|Y^{m,n} - Y^{m,n'}\|_{\mathcal{S}^2} \to 0, \]

as \( n, n' \to \infty \) implying that \( Y^m \) is a continuous process. Moreover, \( (Z^{m,n})_{m,n \geq 0} \) is a uniformly bounded double-sequence in \( \mathcal{H}^2 \) and by again appealing to Lemma A.6 we have that

\[ \|Z^{m,n} - Z^{m,n'}\|_{\mathcal{H}^2} \to 0, \]

as \( n, n' \to \infty \). We conclude that there is a \( Z^m \) and consequently also a \( K^m \) such that \( Z^{m,n} \to Z^m \) in \( \mathcal{H}^2 \) and \( K^{m,n} \to K^m \), in \( \mathcal{S}^2 \) as \( n \to \infty \). Now, let \( \eta_l := \inf\{s > 0 : L_s \geq l\} \land T \) and note that by Theorem A.1 there is a unique triple \( (\hat{Y}^m_t, \hat{Z}^m_t, \hat{K}^m_t)_{0 \leq t \leq \eta_l} \) such that

\[ \begin{cases} \hat{Y}^m_t = Y^m_{\eta_l} + \int_{t}^{\eta_l} f^m(s, \hat{Y}^m_s, \hat{Z}^m_s) ds - \int_{t}^{\eta_l} \hat{Z}^m_s dW_s + \hat{K}^m_{\eta_l} - \hat{K}^m_t, & \forall t \in [0, \eta_l], \\ \hat{Z}^m_t \geq S^+_t, & \forall t \in [0, \eta_l] \text{ and } \int_{t}^{\eta_l} (\hat{Y}^m_s - S^+_s) d\hat{K}^m_t = 0, \end{cases} \]

with \( f^m := \mathbb{1}_{(L_s \leq m]} f^+ - f^- \). A trivial modification of the stability result in (A.3) to random terminal times gives that

\[ \|(Y^{m,n} - \hat{Y}^m)_{[0, \eta_l]}\|_{\mathcal{S}^2}^2 + \|(Z^{m,n} - \hat{Z}^m)_{[0, \eta_l]}\|_{\mathcal{H}^2}^2 + \|(K^{m,n} - \hat{K}^m)_{[0, \eta_l]}\|_{\mathcal{S}^2}^2 \to 0, \]

as \( n \to \infty \) and by uniqueness of limits it follows that \( (Y^m, Z^m, K^m)_{[0, \eta_l]} = (\hat{Y}^m, \hat{Z}^m, \hat{K}^m)_{[0, \eta_l]} \) in \( \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{S}^2 \). Since \( \eta_l = T \) outside of a \( \mathbb{P}\)-null set for
l ( = l(ω)) sufficiently large, we conclude that \((Y^m, Z^m, K^m)\) is the unique solution to
\[
\begin{aligned}
Y^m_t &= \xi + \int_t^T f^m(s, Y^m_s, Z^m_s) \, ds - \int_t^T Z^m_s \, dW_s + K^m_T - K^m_t, \quad \forall t \in [0, T], \\
Y^m_t &\geq S^m_t, \quad \forall t \in [0, T] \text{ and } \int_0^T (Y^m_t - S^m_t) \, dK^m_t = 0.
\end{aligned}
\]

From the above, the bounds (A.13) and (A.14) with \(f = f^m\) are obtained through Lemma A.5 and Lemma A.6, respectively, by Fatou’s lemma and dominated convergence.

Letting \(m \to \infty\) and repeating the above argument the result follows. \(\Box\)

**Corollary A.8** If \((Y, Z, K)\) solves (A.5), then \(Y\) can be interpreted as the Snell envelope in the following way
\[
Y_t = \text{ess sup}_{\tau \in T_t} \mathbb{E} \left[ \int_{\tau}^{T} f(s, Y_s, Z_s) \, ds + S_\tau \mathbbm{1}_{[\tau < T]} + \xi \mathbbm{1}_{[\tau = T]} \middle| \mathcal{F}_t \right]. \quad (A.15)
\]

In particular, with \(D_t := \inf \{ r \geq t : Y_r = S_r \} \land T\) we have the representation
\[
Y_t = \mathbb{E} \left[ \int_{t}^{D_t} f(s, Y_s, Z_s) \, ds + S_{D_t} \mathbbm{1}_{[D_t < T]} + \xi \mathbbm{1}_{[D_t = T]} \middle| \mathcal{F}_t \right] \quad (A.16)
\]
and \(K_{D_t} - K_t = 0\), \(\mathbb{P}\)-a.s.

**Proof** Let \(\eta_l := \inf \{ s > 0 : L_s \geq l \} \land T\), then Theorem A.1 gives that for all \(l \geq 0\) we have
\[
Y_t = \mathbb{E} \left[ \int_{t}^{D_t \land \eta_l} f(s, Y_s, Z_s) \, ds + S_{D_t \land \eta_l} \mathbbm{1}_{[D_t < T]} + \mathbbm{1}_{[D_t \geq \eta_l]} (\mathbbm{1}_{[\eta_l < T]} Y_{\eta_l} + \mathbbm{1}_{[\eta_l = T]} \xi) \middle| \mathcal{F}_t \right]
\]
and (A.16) follows by letting \(l \to \infty\) and using dominated convergence. Optimality of \(D_t\) follows similarly, implying that (A.15) holds. \(\Box\)

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