THE RAMSEY PROPERTIES FOR GRASSMANNIANS OVER $\mathbb{R}$, $\mathbb{C}$

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Abstract. In this note we study and obtain factorization theorems for colorings of matrices and Grassmannians over $\mathbb{R}$ and $\mathbb{C}$, which can be considered metric versions of the Dual Ramsey Theorem for Boolean matrices and of the Graham-Leeb-Rothschild Theorem for Grassmannians over a finite field.

Introduction

One of the most powerful principles in Ramsey theory is the dual Ramsey theorem of R. L. Graham and B. L. Rothschild [9]. It trivially implies the classical Ramsey theorem or the much more involved Hales-Jewett theorem. The Dual Ramsey theorem is the particular instance of the Rota’s conjecture for Grassmannians over the Boolean field $\mathbb{F}_2$, and it indeed implies the Rota’s conjecture for an arbitrary finite field, proved by Graham, Leeb and Rothschild (GLR) in [8]. These statements can be categorized as a structural Ramsey theorem, the Dual Ramsey theorem as its natural generalization to finite dimensional vector spaces over an arbitrary finite field $\mathbb{F}_p$. In this paper we study the case of the infinite fields $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ in its metric form: Suppose that we endow the $n$-dimensional vector space $\mathbb{F}^n$ with a norm $\mathfrak{m}$. We can naturally identify each $k$-dimensional subspace $V$ of $\mathbb{F}^n$ with its unit ball $\text{Ball}(V, \mathfrak{m})$, i.e., the centered section of $V$ with the $\text{Ball}(\mathbb{F}^n, \mathfrak{m}) = \{v \in \mathbb{F}^n : \mathfrak{m}(v) \leq 1\}$. Thus, we can measure the distance between $V$ and $W$ by computing the Hausdorff distance $\Lambda_{\mathfrak{m}}$ between the compact and convex sets $\text{Ball}(V, \mathfrak{m})$ and $\text{Ball}(W, \mathfrak{m})$. Instead of trying to understand only discrete colorings $c : \text{Gr}(k, \mathbb{F}^n) \to r := \{0, 1, \cdots, r - 1\}$ we can now work with 1-Lipschitz mappings, called here compact colorings, $c : (\text{Gr}(k, \mathbb{F}^n), \Lambda_{\mathfrak{m}}) \to (K, d_K)$ into a compact metric space $(K, d_K)$ and ask how the restrictions of $c$ to Grassmannians $\text{Gr}(k, V)$ that are congruent to $\text{Gr}(k, \mathbb{F}^n)$ look like.

In this context, a reasonable notion of congruence $\text{Gr}(k, V) \sim_{\mathfrak{m}} \text{Gr}(k, W)$ is that $(V, \mathfrak{m})$ and $(W, \mathfrak{m})$ are linearly isometric, or equivalently when there is an affine and symmetric bijection sending the $(V, V')$-polar of $\text{Ball}(V, \mathfrak{m})$ onto the $(W, W')$-polar of $\text{Ball}(W, \mathfrak{m})$. Notice that the set-mapping associated to a linear isometry from $V$ onto $W$ defines a $\Lambda_{\mathfrak{m}}$-isometry from $\text{Gr}(k, V)$ onto $\text{Gr}(k, W)$. The corresponding quotient $\text{Gr}(k, \mathbb{F}^n)/\sim_{\mathfrak{m}}$ is canonically identified with the class $\mathcal{B}_k(\mathbb{F}^n, \mathfrak{m})$ of isometric types of $k$-dimensional subspaces of $(\mathbb{F}^n, \mathfrak{m})$, a closed subset of the Banach-Mazur compactum $\mathcal{B}_k$. In this paper we show that for the $p$-norms $\| (a_j)_j \|_p := (\sum_j |a_j|^p)^{1/p}$, if $p \in [1, \infty \setminus (2\mathbb{N} + 4)$, and for the sup norm $\| (a_j)_j \|_{\infty} := \max_j |a_j|$ we have that on each quotient $\mathcal{B}_k(\mathbb{F}^n, \| \cdot \|_p)$ there is a compatible “Gromov-Hausdorff”-metric $\gamma_p$, called here extrinsic...
metric, such that for every $k, m \in \mathbb{N}$ every compact metric space $(K, d_K)$ and every $\varepsilon > 0$ there is a dimension $n$ such that for every compact coloring $c : (\text{Gr}(k, F^n), A \mid \|_p) \to (K, d_K)$ there is some $V \in \text{Gr}(m, \mathbb{F}^n)$ that is $\| \cdot \|_p$-congruent to $F^m$ and there is a compact coloring $\hat{c} : (\mathcal{B}_k(F^n, \| \cdot \|_p), \gamma_p) \to (K, d_K)$ such that $d_K(\hat{c}(W), c(W)) \leq \varepsilon$ for every $W \in \text{Gr}(k, V)$.

In a similar way, we study factorizations of compact colorings of matrices of two kinds: $n \times k$-full rank matrices and $n$-square matrices of rank $k$, denoted by $M^{k}_{n,k}$ and by $M^{k}_{n}$, respectively. When the field $F$ is finite, we show that for large enough $n$, for every coloring $c : M^{k}_{n,k} \to r$ there is some matrix $R \in M^{n}_{m,m}$ in reduced column echelon form and a unique $\hat{c} : \text{GL}(F^k) \to r$ such that $c(R \cdot A) = \hat{c}(\text{red}(A))$ for every $A \in M^{k}_{m,k}$, where $\text{red}(A)$ is the $k$-square invertible matrix such that $A \cdot \text{red}(A)$ is in reduced column echelon form. We prove that colorings of $M^{k}_{n}$ are factorized in a similar way by, in addition, using the full rank factorization of matrices. We then analyze the colorings of these matrices over the fields $\mathbb{R}, \mathbb{C}$, and we compute the corresponding Ramsey factors in the metric context for the $p$-norms.

The proofs for the infinite fields are based on the crucial fact that when $m$ is a norm on the vector space $F^\omega$, the space of sequences $(a_n)_n$ with finitely many non-zero entries, have an approximate Ramsey property called steady approximate Ramsey property, then there is a unique Banach space $\hat{E}$ such that $E := (F^\omega, m)$ can be linearly isometrically embedded into $\hat{E}$, $\mathcal{B}_k(E)$ is dense in $\mathcal{B}_k(\hat{E})$, and such that the group $\text{Iso}(\hat{E})$ of linear isometries of $\hat{E}$, with its strong operator topology, is extremely amenable, that is, every continuous action of $\text{Iso}(\hat{E})$ on a compact space has a fixed point. The corresponding spaces to the $p$-norms are the Lebesgue space $L_p[0,1]$ if $p < \infty$, and the Gurarij space for the sup-norm.

The use of tools from topological dynamics on a pure approximate Ramsey problem is not accidental. The recent Kechris-Pestov-Todorcevic correspondence in its discrete and metric versions characterizes the extreme amenability of the automorphism groups of Fraïssé (discrete/metric) structures in terms of the (approximate) Ramsey property of the collection of finitely generated substructures (see [6, 12, 13]).

The paper is organized as follows. We first study Ramsey properties of matrices over $F_2$ and then over an arbitrary finite field $F$. In particular, we provide in Theorem 1.7 another proof of the Rota’s conjecture as a straightforward consequence of the Dual Ramsey theorem. To do this, we use basic tools from linear algebra, mainly the reduced column echelon form, that interestingly corresponds to some surjection being rigid with respect to the antilexicographical ordering, and that determines the Ramsey property (Proposition 1.9). We finish this section by introducing in Proposition 1.14 a uniqueness principle for these Ramsey factorizations. The second section is devoted to the study of Ramsey factorizations of matrices and Grassmannians over the fields $\mathbb{R}, \mathbb{C}$. We introduce the main concepts, namely $\varepsilon$-factors, and the Ramsey factors, including the extrinsic metrics, for full rank $n \times k$-matrices, Grassmannians, and $n \times n$-matrices of rank $k$, and we present our main results in Theorem 2.7, Theorem 2.13 and Theorem 2.22, respectively. The third section is devoted to the proofs of the factorization results exposed in section two. We recall the steady approximate Ramsey property (SARP) of a family of finite dimensional normed spaces and the extreme amenability of a topological group. We explain in Corollary 3.10 when a normed space of the form $E = (F^\omega, m)$ has associated a unique Banach space $\hat{E}$ that is Fraïssé, has a group of isometries that is extremely amenable, and how that gives Ramsey factors. In Subsection 3.1 we analyze these factors and we prove that they are the ones presented in Section two (Theorem 3.11). We finish with an appendix where we analyze the special case of the sup-norm, and we give explicit definitions of extrinsic metrics.
1. The Dual Ramsey Theorem and matrices over finite fields

To keep the notation unified, let $\mathbb{F}^\infty$ be the vector space over $\mathbb{F}$ consisting of all eventually zero sequences $(a_n)_{n \in \mathbb{N}}$. Let $(u_n)_{n \in \mathbb{N}}$ be the unit basis of $\mathbb{F}^\infty$, that is, each $u_n$ is the sequence whose only non-zero entry is 1 at the $n^{th}$-coordinate. In this way we identify $\mathbb{F}^n$ with the subspace $\langle u_i \rangle_{i<n}$ of $\mathbb{F}^\infty$, and then $\mathbb{F}^\infty$ with the increasing union of all $\mathbb{F}^n$.

Given $\alpha, \beta \in \mathbb{N} \cup \{\infty\}$, let $M_{\alpha, \beta}(\mathbb{F})$ be the collection of $\alpha \times \beta$-matrices with finitely many non-zero entries. In a similar manner as before, $M_{\alpha, \beta}(\mathbb{F}) = \bigcup_{n \leq \alpha, m \leq \beta} M_{n,m}(\mathbb{F})$, increasing union.

Let $M^k_{\alpha, \beta}(\mathbb{F})$ be the set of all $\alpha \times \beta$-matrices of rank $k$ with entries in $\mathbb{F}$. To lighten the notation, when there is no possible confusion, we will write $M_{\alpha, \beta}(\mathbb{F}), M^k_{\alpha, \beta}(\mathbb{F})$.

There are several equivalent ways to present the dual Ramsey theorem (DRT) of Graham and Rothschild [9]. Among these, there is a factorization result for Boolean matrices stated below as Theorem 1.4. Motivated by this, we study Ramsey-theoretical factorization results for colorings of other classes of matrices. We begin with matrices with entries in a finite field, and then conclude, in the next section, with matrices over $\mathbb{R}$ or $\mathbb{C}$.

It is well known, for example using the Gauss-Jordan elimination method, that an $n \times m$-matrix $A$ has a unique decomposition $A = \text{red}(A) \cdot \tau(A)$ where $\text{red}(A)$ is in reduced column echelon form and $\tau(A)$ is an invertible $m \times m$-matrix. We prove that when the field is finite any finite coloring of matrices over a finite field is determined, in a precise way, by $\tau$. This can be seen as an extension of the well known result of Graham, Leeb, and Rothschild on Grassmannians over a finite field [8].

**Definition 1.1** (Factors). Let $X$ be a set and $r \in \mathbb{N}$. An $r$-coloring of $X$ is a mapping $c : X \to r = \{0, 1, \ldots, r - 1\}$. A subset $Y$ of $X$ is $c$-monochromatic if $c$ is constant on $Y$. We say that a mapping $\pi : X \to K$ is a factor of $c : X \to r$ if there is some $\tilde{c} : K \to r$ such that $c = \tilde{c} \circ \pi$.

Finally, $\pi$ is a factor of $c$ in $Y \subseteq X$ if $\pi \upharpoonright Y$ is a factor of $c \upharpoonright Y$. So, $Y$ is $c$-monochromatic when the trivial constant map $\pi : X \to \{0\} = 1$ is a factor of $c$ in $Y$.

We now recall the Dual Ramsey Theorem (DRT) of Graham and Rothschild [9] (see also [14], [19]). For convenience, we present its formulation in terms of rigid surjections between finite linear orderings. Given two linear orderings $(R, <_R)$ and $(S, <_S)$, a surjective map $f : R \to S$ is called a rigid surjection when $\text{min} f^{-1}(s_0) <_R \text{min} f^{-1}(s_1)$ for every $s_0, s_1 \in S$ such that $s_0 <_S s_1$. We let $\text{Epi}(R, S)$ be the collection of rigid surjections from $R$ to $S$.

**Theorem 1.2** (Graham–Rothschild). For every finite linear orderings $R$ and $S$ such that $\#R < \#S$ and every $r \in \mathbb{N}$ there exists an integer $n > \#S$ such that, considering $n$ naturally ordered, every $r$-coloring of $\text{Epi}(n, R)$ has a monochromatic set of the form $\text{Epi}(S, R) \circ \gamma = \{\sigma \circ \gamma : \sigma \in \text{Epi}(S, R)\}$ for some $\gamma \in \text{Epi}(n, S)$.

1.1. Ramsey properties of colorings of Boolean matrices. Perhaps the most common formulation of the dual Ramsey Theorem of Graham and Rothschild is done in terms of partitions. Given $k, m, n \in \mathbb{N}$, let $\mathcal{E}_m(n)$ be the set of all partitions of $n$ into $m$ pieces. Given $\mathcal{P} \in \mathcal{E}_m(n)$, let $\langle \mathcal{P} \rangle_k$ be the set of all partitions $\mathcal{Q}$ of $n$ with $k$ pieces that are coarser than $\mathcal{P}$, i.e., such that each piece of $\mathcal{Q}$ is a union of pieces of $\mathcal{P}$.

**Theorem** (DRT, partitions version). For every $k, m \in \mathbb{N}$ and $r \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that every $r$-coloring of $\mathcal{E}_m(n)$ has a monochromatic set of the form $\langle \mathcal{P} \rangle_k$ for some $\mathcal{P} \in \mathcal{E}_m(n)$.

The following three reformulations of the Dual Ramsey Theorem are structural Ramsey results for finite Boolean algebras.

**Theorem** (DRT, Boolean algebras). Let $\mathcal{A}$ and $\mathcal{B}$ be finite Boolean algebras, and let $r \in \mathbb{N}$. Then there exists a finite Boolean algebra $\mathcal{C}$ such that every $r$-coloring of the set $\binom{\mathcal{A}}{r}$ of isomorphic copies of $\mathcal{A}$ inside $\mathcal{C}$ admits a monochromatic set of the form $\binom{\mathcal{B}}{r}$ for some $\mathcal{B}_0 \in \binom{\mathcal{B}}{r}$. 
Let $\mathcal{A}$ be a finite Boolean algebra. Any $a \in \mathcal{A}$ is represented as

$$a = \bigvee_{x \in \Gamma_a} x,$$

for a unique set of atoms $\Gamma_a$. So, any linear ordering $<$ on the sets of atoms $\text{At}(\mathcal{A})$ of $\mathcal{A}$ extends to $\mathcal{A}$ by defining $a < b$ iff $\min_c(\Gamma_a \triangle \Gamma_b) \in \Gamma_a$. Following [12], we will say that $(\mathcal{A}, <)$ is a canonically ordered (c.o.) Boolean algebra. Given c.o. Boolean algebras $\mathcal{A}$ and $\mathcal{B}$, let $\text{Emb}_<(\mathcal{A}, \mathcal{B})$ be the collection of ordering-preserving embeddings from $\mathcal{A}$ into $\mathcal{B}$, respectively.

**Theorem 1.3** (DRT, canonically ordered Boolean algebras). Given c.o. Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ and $r \in \mathbb{N}$, there is a c.o. Boolean algebra $\mathcal{C}$ such that each $r$-coloring of $\text{Emb}_<(\mathcal{A}, \mathcal{C})$ has a monochromatic set of the form $q \circ \text{Emb}_<(\mathcal{A}, \mathcal{B})$ for some $q \in \text{Emb}_<(\mathcal{B}, \mathcal{C})$.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite Boolean algebras with $k$ and $n$ atoms, respectively. Any embedding from $\mathcal{A}$ to $\mathcal{B}$ has a corresponding representing $n \times k$ matrix with entries in $\{0, 1\}$. We call the matrices arising in this fashion Boolean matrices. The set of $n \times k$ Boolean matrices will be denoted by $M^\text{ba}_{n,k}$, i.e., the set of $n \times k$ matrices with entries in $\{0, 1\}$ whose columns (which can be identified with subsets of $n$) form a $k$-partition of $n$. We let $M^\text{oba}_{n,k}$ be the set of Boolean $n \times k$-matrices that correspond to order-preserving embeddings between c.o. Boolean algebras. These are precisely the set of Boolean matrices whose columns $(P_i)_{i \in \mathbb{N}}$ further satisfy $\min P_1 < \min P_{i+1}$ for $i < k - 1$.

In the following, we identify a permutation $\sigma$ of $k$ with the associated $k \times k$ permutation matrix. This allows one to identify the group $S_k$ of permutations of $k$ with a group of unitary matrices. Let $\pi : M^\text{ba}_{n,k} \to S_k$ be the function assigning to a matrix $A$ the unique element $\pi(A)$ of $S_k$ such that $A = A_{\circ} \cdot \pi(A)$ for some (uniquely determined) matrix $A_{\circ} \in M^\text{oba}_{n,k}$. Given an $n \times m$-matrix $A$, we let $A \cdot M^\text{ba}_{m,k} = \{A \cdot B : B \in M^\text{ba}_{m,k}\}$.

**Theorem 1.4** (DRT, Boolean matrices). For every $k, m \in \mathbb{N}$ and $r \in \mathbb{N}$ there is $n$ such that for every $c : M^\text{ba}_{n,k} \to r$ there is $R \in M^\text{ba}_{n,m}$ such that $\pi$ is a factor of $c$ in $R \cdot M^\text{ba}_{m,k}$. That is, the color of $R \cdot B$ depends only on $\pi(B) = \pi(R \cdot B)$ for every $B \in M^\text{ba}_{m,k}$.

**Proof.** Let $\mathcal{C}$ be a c.o. Boolean algebra obtained by applying the Dual Ramsey Theorem for c.o. Boolean algebras—Theorem 1.3—to the power sets $\mathcal{P}(k), \mathcal{P}(m)$ canonically ordered as above by $s < t$ if and only if $\min(s \triangle t) \in s$, and to the number of colors $r^{S_k}$. Without loss of generality we can assume that $\mathcal{C}$ is equal to $\mathcal{P}(n)$ for some $n \in \omega$, since any c.o. Boolean algebra is of this form. We claim that such an $n$ satisfies the desired conclusions. Indeed, fix a coloring $c : M^\text{ba}_{n,k} \to r$. This induces a coloring $f : \text{Emb}_<(\mathcal{P}(k), \mathcal{P}(n)) \to r^{S_k}$ as follows. Let $\gamma$ be an element of $\text{Emb}_<(\mathcal{P}(k), \mathcal{P}(n))$, and let $A_{\circ} \in M^\text{ba}_{n,k}$ be the corresponding representing matrix. Define then $f(\gamma)$ to be the element $(c(A_{\circ} \cdot \sigma))_{\sigma \in S_k}$ of $r^{S_k}$. By the choice of $\mathcal{C} = \mathcal{P}(n)$ there exists $q \in \text{Emb}_<(\mathcal{P}(m), \mathcal{P}(n))$ such that $f$ is constant on $q \circ \text{Emb}_<(\mathcal{P}(k), \mathcal{P}(m))$. Let now $\bar{c} \in r^{S_k}$ be the constant value of $f$. It is now easy to see that $c(A_q \cdot B) = \bar{c}(\pi(B))$ for every $B \in M^\text{ba}_{m,k}$. \(\square\)

1.2. Ramsey properties of colorings of matrices over a finite field. It is natural to consider Ramsey properties of other classes of matrices over a field $\mathbb{F}$. We are going to see that for $\mathbb{F}$ finite there is a factorization result similar to the DRT for Boolean matrices, that extends the well known theorem by Graham, Leeb and Rothschild on Grassmannians $\text{Gr}(k, V)$, the family of all $k$-dimensional subspaces of a vector space $V$ over $\mathbb{F}$. In the following, given a sequence $(x_i)$ in a vector space $E$, we let $\langle x_i \rangle$ be its linear span inside $E$.

**Theorem 1.5** (Graham-Leeb-Rothschild [8]). Given $k, m, r \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that every $r$-coloring of $\text{Gr}(k, \mathbb{F}^n)$ has a monochromatic set of the form $\text{Gr}(k, R)$ for some $R \in \text{Gr}(m, \mathbb{F}^n)$.
This result is a particular case of the factorization theorem for injective matrices. Recall that a \( p \times q \)-matrix \( A = (a_{ij}) \) is in reduced row echelon form (RREF) when there is \( p_0 \leq p \) and (a unique) strictly increasing sequence \((j_i)_{i<p_0}\) of integers \(< q \) such that

i) \( A \cdot u_{j_i} = u_i \) for every \( i < p_0 \) and

ii) \( \langle A \cdot u_{j_i} \rangle_{j<i} = \langle u_i \rangle_{i<\ell} \) for every \( i < p_0 \).

When \( A \) is in RREF and it has rank \( p \), we define \( I_A \) as the \( q \times p \)-matrix with entries in \( \{0,1\} \), and whose nonzero entries are in the positions \((j_i,i) \) \((i < p) \). For example for the field \( \mathbb{F}_5 \) and

\[
A = \begin{pmatrix}
1 & 2 & 0 & 3 & 0 & 1 \\
0 & 0 & 1 & 4 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 3
\end{pmatrix}
\]

we have \( I_A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \) (1)

It follows that \( I_A \) is a right inverse to \( A \), i.e., \( A \cdot I_A = \text{Id}_p \). A matrix \( A \) is in reduced column echelon form (RCEF) when its transpose \( A^t \) is in RREF. Let \( \mathcal{E}_{n,m}(\mathbb{F}) \), \( \mathcal{E}(\mathbb{F}) \) be the collection of \( n \times m \)-matrices of rank \( m \) in RCEF and of full rank matrices in RCEF, respectively.

**Definition 1.6.** Let \( \tau : M_{k,n}^k(\mathbb{F}) \rightarrow \text{GL}(\mathbb{F}^k) \) be the mapping that assigns to each \( A \in M_{k,n}^k(\mathbb{F}) \) the unique \( k \times k \)-invertible matrix \( \tau(A) \) such that \( A \cdot \tau(A) \) is in RCEF. Let also \( \text{red}_c(A) := A \cdot \tau(A) \).

**Theorem 1.7** (Factorization of colorings of full rank matrices over a finite field). Given \( k, m, r \in \mathbb{N} \) there is \( n \in \mathbb{N} \) such that for every \( c : M_{n,k}^k(\mathbb{F}) \rightarrow r \) there is \( R \in \mathcal{E}_{n,m}(\mathbb{F}) \) such that \( \tau \) is a factor of \( c \) in \( R \cdot M_{m,k}^k(\mathbb{F}) \).

This gives immediately the Graham-Leeb-Rothschild Theorem—Theorem 1.5—as every \( k \)-dimensional subspace of \( \mathbb{F}^n \) can be represented as the linear span of the columns of a matrix in RCEF. The proof of Theorem 1.7 is a direct consequence of the DRT and the next propositions. In the following, we fix an ordering \( < \) on the finite field \( \mathbb{F} \) such that \( 0 < 1 \) are the first two elements of \( \mathbb{F} \). We let \( \mathbb{F}^k \) be endowed with the corresponding antilexicographic order \( \preceq_{\text{alex}} \) and we define \( \Phi_{n,k} : \text{Epi}(n,\mathbb{F}^k) \rightarrow M_{n,k}^k(\mathbb{F}) \) as the function assigning to each rigid surjection \( f \) the matrix whose rows are \( f(j) \) for every \( j < n \).

**Lemma 1.8.** \( \Phi_{n,k}(f) \) is a full rank matrix in RCEF.

**Proof.** It is clear that \( \Phi_{n,k}(f) \) is a full rank matrix. We prove that it is in RCEF. Let \( A \) be the transpose of \( \Phi_{n,k}(f) \). For each \( i \in k \), let \( j_i := \min\{j < n : A \cdot u_{j} = u_{i}\} \). Then \((j_i)_{i<k} \) is strictly increasing, since \( f \) is a rigid surjection, and if \( j < j_i \), then \( A \cdot u_{j} \preceq_{\text{alex}} u_{i} \), by the definition of \( j_i \), and the rigidity of \( f \). Therefore \( A \cdot u_{j} \in \langle u_{i} \rangle_{i<n} \). Consequently, \( A \) is in RREF. \( \square \)

The next is the key relation between matrices in RREF and rigid surjections that will allow us to use the dual Ramsey Theorem and prove Theorem 1.7.

**Proposition 1.9.** For \( A \in M_{k,n}^k(\mathbb{F}) \) the following are equivalent.

i) \( A \) is in RREF.

ii) The linear map \( T_A : \mathbb{F}^n \rightarrow \mathbb{F}^k \) represented by \( A \) in the corresponding unit bases is a rigid surjection and for every \( i < k \) there is a column of \( A \) equal to \( u_{i} \).

In particular we have the following.

**Corollary 1.10.** Suppose that \( A \in M_{m,n}^m(\mathbb{F}) \) and \( B \in M_{m,k}^k(\mathbb{F}) \).

a) If \( A \) and \( B \) are in RCEF (resp. RREF) then \( A \cdot B \) is also in RCEF (resp. RREF).

b) If \( A \) is in RCEF then \( \tau(A \cdot B) = \tau(B) \). \( \square \)
Proof of Proposition 1.9. i)⇒ii) Suppose that A is in RREF. We will prove that the canonical linear operator \( T_A : \mathbb{F}^n \rightarrow \mathbb{F}^k \), \( T_A(u_i) := A \cdot u_i \) for \( i < n \) is a rigid surjection from \( \mathbb{F}^n \) to \( \mathbb{F}^k \) endowed with the antilexicographical order \( \prec_{\text{alex}} \) described before. Let \( (j_i)_{i<k} \) be the strictly increasing sequence in \( n \) witnessing that \( A \) is in RREF. By linearity, \( T_A(0) = 0 \). Fix now \( w \in \mathbb{F}^k \).

Claim 1.10.1. \( \min_{\prec_{\text{alex}}} (T_A)^{-1}(w) = I_A \cdot w \).

From this, since \( I_A : \mathbb{F}^k \rightarrow \mathbb{F}^n \) is \( \prec_{\text{alex}} \)-increasing, we obtain that \( T_A \) is a rigid surjection.

Proof of Claim: Applied to the example in (1) and to \( w = (1,2,3) \), it should be clear that the spread \( I_A : (1,2,3) = (1,0,2,0,3,0) \) of \((1,2,3) \) is the \( \prec_{\text{alex}} \)-least element of the preimage of \((1,2,3) \) under \( T_A \). We give a detailed proof. Suppose that \((v_j)_{j<n} = \bar{v} = \min_{\prec_{\text{alex}}} \{ v \in \mathbb{F}^n : A \cdot v = w \} \). Set \( z = (z_j)_{j} := I_A(w) \). We prove by induction on \( i < k \) that \( v_j = z_j \) for every \( j \geq j_{k-i-1} \). Suppose that \( i = 0 \). Since for every \( j > j_{k-1} \) one has that \( z_j = 0 \), we obtain that \( v_j = 0 \), by \( \prec_{\text{alex}} \)-minimality of \( \bar{v} \). Let \((A)_{k-1} \) be the \((k-1)\text{-th}\) row of \( A \). It follows that \((A)_{k-1} = u_{j_{k-1}} + y \), where \( y \in \langle u_j \rangle_{j \geq j_{k-1}} \).

\[ z_{j_{k-1}} = w_{j_{k-1}} = (A)_{k-1} \cdot \bar{v} = v_{j_{k-1}}. \]

Suppose that the conclusion holds for \( i \), that is, \( v_j = z_j \) for every \( j \geq j_{k-i-1} \). We will prove that it also holds for \( i + 1 \). Since \( v \prec_{\text{alex}} z \) and \( z_j = 0 \) for every \( j_{k-i' -1} < j < j_{k-i' -1} \) and \( v_j = 0 \) for such \( j \)’s. Then the \((k-i-2)\text{-th}\) row of \( A \) is of the form \((A)_{k-i-2} = u_{j_{k-i-2}} + y \) with \( y \) in the span of \( \{ u_j : j > j_{k-i-2}, j \neq j \text{ for all } p \} \). It follows that

\[ z_{j_{k-i-2}} = w_{k-i-2} = (A)_{k-i-2} \cdot \bar{v} = v_{j_{k-i-2}}. \]

ii)⇒i) Now suppose that \( T_A \) is a rigid surjection from \( \mathbb{F}^n \) to \( \mathbb{F}^k \) with respect to the antilexicographical orderings, and that for every \( i < k \) a column of \( A \) is \( u_i \). For each \( i < k \), let \( j_i \) be the first such column of \( A \). We prove that \((j_i)_{i<k} \) witnesses that \( A \) is in RREF, that is:

Claim 1.10.2. \( T_A \langle u_j \rangle_{j<j_i} = \langle u_j \rangle_{j<i} \) for every \( i < k \).

Proof of Claim: The proof is by induction on \( i \). If \( i = 0 \), then \( T_A \langle u_j \rangle_{j<j_0} = \{0\} \) because \( u_0 \) is the second element of \( \mathbb{F}^n \) in the antilexic order, while the first element is the zero vector. Suppose the result is true for \( i \), and let us extend it to \( i + 1 \). In particular, we know that \( j_{i+1} > j_i \), and it is clear that \( \langle u_j \rangle_{j<i} \subseteq T_A \langle u_j \rangle_{j<j_i} \subseteq \langle u_j \rangle_{j<j_{i+1}} \). Suppose towards a contradiction that there exists \( j \) such that \( j_i < j < j_{i+1} \) and \( T_A(u_j) \notin \langle u_j \rangle_{j<i} \). Denote by \( \xi \) the least such \( j \). Thus, \( u_{i+1} \prec_{\text{alex}} T_A(u_\xi) \), hence there is some \( x \prec_{\text{alex}} u_\xi \) such that \( T_A(x) = u_{i+1} \). This means, by the minimality of \( \xi \), that \( T_A(u_\xi) = y + u_{i+1} \) with \( y \in \langle u_j \rangle_{j<i} \). We know that \( y \neq 0 \) by the minimality of \( j_{i+1} \); so \( u_{i+1} \prec_{\text{alex}} y + u_{i+1} \). Hence,

\[ \min(T_A)^{-1}(u_{i+1}) \prec_{\text{alex}} \min(T_A)^{-1}(y + u_{i+1}) = u_\xi. \]

So, there is \( x \in \langle u_j \rangle_{j<\xi} \) with \( T_A(x) = u_{i+1} \), which is impossible by the minimality of \( \xi \).

Proof of Theorem 1.7. Fix all parameters. We consider \( \mathbb{F}^k \) and \( \mathbb{F}^m \) antilexicographically ordered by \( \prec_{\text{alex}} \) (as explained before). Let \( n \) be obtained from the linear orderings \( (\mathbb{F}^k, \prec_{\text{alex}}), (\mathbb{F}^m, \prec_{\text{alex}}) \) and the number of colors \( r^k \), where \( \lambda = \prod_{i=0}^{k-1} (p^k - p^i) \) is the order of the group \( \text{GL}(\mathbb{F}^k) \), by applying the Dual Ramsey Theorem for rigid surjections (Theorem 1.2). We claim that \( n \) satisfies the desired conclusions. Fix a coloring \( c : M_{n,k}(\mathbb{F}) \rightarrow r \). Let \( \sigma_0 : \text{Epi}(n, \mathbb{F}^k) \rightarrow r \cdot \text{GL}(\mathbb{F}^k) \) be the coloring defined by \( \sigma_0(\sigma) := (c(\Phi_{k,n}(\sigma \cdot \Gamma^{-1}))_{\Gamma \in \text{GL}(\mathbb{F}^k)} \) for \( \sigma \in \text{Epi}(n, \mathbb{F}^k) \). By the choice of \( n \), there exists \( \rho \in \text{Epi}(n, \mathbb{F}^m) \) such that \( \sigma_0 \) is constant on \( \text{Epi}(\mathbb{F}^m, \mathbb{F}^k) \circ \rho \) with constant value \( \bar{c} \in r \cdot \text{GL}(\mathbb{F}^k) \). Let \( R := \Phi_{n,m}(\bar{c}) \). We claim that \( R \) and \( \bar{c} \) satisfy the conclusion of the statement in the theorem. It follows from Proposition 1.8 that \( R \in \mathcal{E}_{n,m}(\mathbb{F}) \). Now let \( A \in M_{m,k}(\mathbb{F}) \). We have to prove that \( c(R \cdot A) = \bar{c}(\tau(R \cdot A)) \). First, note that \( \tau(R \cdot A) = \tau(A) \), because \( R \) is in RCEF. Let \( B \) be the transpose of \( \text{red}_c(A) \) (i.e., \( B \) is the RREF of the transpose of \( A \)), and let...
$T_B : \mathbb{F}^m \to \mathbb{F}^k$ be the linear operator defined by $B$ in the corresponding canonical bases. We know by Proposition 1.9 that $T_B \in \text{Epi}(\mathbb{F}^m, \mathbb{F}^k)$.

**Claim 1.10.3.** $\Phi_{n,k}(T_B \circ \varrho) = R \cdot \text{red}_c(A)$.

**Proof of Claim:** Fix $j < m$. Then the $j^{th}$-row $(\Phi_{n,k}(T_B \circ \varrho))_j$ of $\Phi_{n,k}(T_B \circ \varrho)$ is the row vector $T_B(\varrho(j))$. Hence,

$$(\Phi_{n,k}(T_B \circ \varrho))_j = T_B(\varrho(j)) = ((\text{red}_c(A))^t \cdot ((R_j)^t)^t = (R_j) \cdot \text{red}_c(A) = (R \cdot \text{red}_c(A)).$$

So, given $\Gamma \in \text{GL}_k(\mathbb{F})$ we have that

$$c(R \cdot A) = c(R \cdot \text{red}_cA \cdot \tau(A)^{-1}) = c_0(R \cdot \text{red}_cA)(\tau(A)) = \tilde{c}(\tau(A)) = \tilde{c}(\tau(R \cdot A)).$$

1.2.1. **Square matrices of rank $k$.** We present the Ramsey factorization for finite colorings of square matrices. Recall that every $n \times m$-matrix $A$ of rank $k$ has a full rank decomposition $A = B \cdot C$ where $B \in M^k_{n \times k}$ and $C \in M^k_{k \times m}$.

**Definition 1.11.** Given $k$ and $n$, let $\tau^{(2)} : M^k_{n \times n} \to \text{GL}(\mathbb{F}^k)$ be the mapping uniquely defined by the relation $A = A_0 \cdot \tau^{(2)}(A) \cdot A_1$ for some $A_0, A_1 \in \mathcal{E}_{n,k}(\mathbb{F})$.

It is routine to see that $\tau^{(2)}$ is well defined.

**Theorem 1.12** (Factorization of colorings of square matrices over a finite field). For every $k, m, r \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that for every $c : M^k_{n \times n}(\mathbb{F}) \to r$ there are $R_0, R_1 \in \mathcal{E}_{n,m}(\mathbb{F})$ such that $\tau^{(2)}$ is a factor of $c$ in $R_0 \cdot M^k_{n \times m}(\mathbb{F}) \cdot R_1^t$.

**Proof.** Given integers $k, m$ and $r$, let $n_{\mathbb{F}}(k, m, r)$ be the minimal number $n$ such that the factorization statement in Theorem 1.7 holds for the parameters $k, m$ and $r$, and now let $n_0 := n_{\mathbb{F}}(k, m, r_{\text{GL}(\mathbb{F}^k)})$, and let $n := n_{\mathbb{F}}(k, n_0, r_{M^k_{n_0 \times k}(\mathbb{F})})$. We claim that $n$ works. Fix any $r$-coloring $f : M^k_{n \times n}(\mathbb{F}) \to r$ and $P \in \mathcal{E}_{n,n_0}(\mathbb{F})$. We define the coloring $c : M^k_{n \times n}(\mathbb{F}) \to r_{M^k_{n_0 \times k}(\mathbb{F})}$ by

$$c(A) := (f(A \cdot B^t \cdot P^t))_{B \in M^k_{n_0 \times k}(\mathbb{F})}.$$

The coloring $c$ is well defined because $A \cdot B^t \cdot P^t$ has rank $k$. Let $R \in \mathcal{E}_{n,n_0}$ and $c_0 : \text{GL}(\mathbb{F}^k) \to r_{M^k_{n_0 \times k}(\mathbb{F})}$ be such that $c(R \cdot A) = c_0(\tau(A))$ for $A \in M^k_{n \times k}(\mathbb{F})$. Define $d : M^k_{n_0 \times k}(\mathbb{F}) \to r_{\text{GL}(\mathbb{F}^k)}$ by

$$d(B) := (c_0(\Gamma)(B))_{\Gamma \in \text{GL}(\mathbb{F}^k)}.$$

Let $S \in \mathcal{E}_{n_1,m}(\mathbb{F})$ and $d_0 : \text{GL}(\mathbb{F}^k) \to r_{\text{GL}(\mathbb{F}^k)}$ be such that $d(S \cdot B) = d_0(\tau(B))$ for every $B \in M^k_{m \times k}(\mathbb{F})$. Set $R_0 = R \cdot Q$ and $R_1 := P \cdot S$, where $Q \in \mathcal{E}_{n_0,m}(\mathbb{F})$ is arbitrary. Finally, let $g : \text{GL}(\mathbb{F}^k) \to r$ be defined by $g(\Gamma) = d_0(\Gamma_0)(\Gamma_1)$, where $\Gamma = \Gamma_0 \cdot \Gamma_1^t$ are arbitrary. Notice that if $\Gamma = \Gamma_1 \cdot \Gamma_0^t$, then it follows that

$$d_0(\Gamma_0)(\Gamma_1) = d(S \cdot P_0 \cdot \Gamma_0)(\Gamma_1) = c_0(\Gamma_1)(S \cdot P_0 \cdot \Gamma_0) = c(R \cdot P_1 \cdot \Gamma_1)(S \cdot P_0 \cdot \Gamma_0) = f(R \cdot P_1 \cdot \Gamma_1 \cdot \Gamma_0 \cdot P_0^t \cdot R_1^t) = f(R_0 \cdot A \cdot P_0^t \cdot R_1^t)$$

where $P_0 \in \mathcal{E}_{m,k}(\mathbb{F})$ and $P_1 \in \mathcal{E}_{n_0,k}(\mathbb{F})$ are arbitrary. So, $g$ does not depend on the decomposition $\Gamma = \Gamma_1 \cdot \Gamma_0$'. Similarly one proves that $g(\tau^{(2)}(A)) = f(R_0 \cdot A \cdot R_1^t)$ for all $A \in M^k_{m \times m}(\mathbb{F})$. □

1.2.2. **Uniqueness.** We see that in a natural way the factors we presented are unique. We introduce the abstract notion of Ramsey factor in this context.

**Definition 1.13.** Given $\mu : M^k_{n \times k}(\mathbb{F}) \to X$, $X$ finite, and $A \subseteq \bigcup_{n \times m} M_{n \times m}(\mathbb{F})$, we say that the couple $(\mu, A)$ is a $k$-Ramsey factor when

i) $\mu(M^k_{n \times k}(\mathbb{F})) = X$,

ii) $\mu(R \cdot A) = \mu(A)$ for every $A \in M^k_{n \times k}(\mathbb{F})$ and every $R \in A \cap M^k_{n \times n}(\mathbb{F})$. 


iii) For every $m,r \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that for every $r$-coloring $c$ of $M_{m,k}^k(\mathcal{F})$ there is $R \in A \cap M_{m,n}^n(\mathcal{F})$ such that $\mu$ is a factor of $c$ in $R \cdot M_{m,k}^k(\mathcal{F})$.

We call $X$ the set of colors of $\mu$, denoted by $X_\mu$.

It follows that $(\tau, \mathcal{E})$ is a $k$-Ramsey factor, and it is the minimal one in the following precise sense.

**Proposition 1.14.** Suppose that $(\mu, A)$, $(\nu, B)$ are $k$-Ramsey factors.

a) $\#X_\mu \geq \#GL(\mathbb{F}^k) = \prod_{j=0}^{k-1}((\#\mathbb{F})^k - (\#\mathbb{F})^j)$.

b) If $A \subseteq B$, then there is a surjection $\theta : X_\mu \to X_\nu$ such that $\mu \circ \theta = \nu$.

c) If $A = B$, then there is a bijection $\theta : X_\mu \to X_\nu$ such that $\mu \circ \theta = \nu$.

**Proof.** a): In fact, we prove that if $(\mu, A)$ satisfies c) of Definition 1.13, then $\#X \geq \#GL(\mathbb{F}^k) = \prod_{j=0}^{k-1}((\#\mathbb{F})^k - (\#\mathbb{F})^j)$. Find $n \geq k, \theta : X \to GL(\mathbb{F}^k)$ and $R \in A \cap$ such that $\tau(R \cdot A) = \theta(\mu(R \cdot A))$ for every $A \in M_{m,k}^k(\mathcal{F})$. It is easy to see that $\tau : R \cdot M_{m,k}^k(\mathcal{F}) \rightarrow GL(\mathbb{F}^k)$ is surjective, hence $\theta$ is also surjective. b): With same strategy one easily proves b). c): From b) we have that $\#X_\mu = \#X_\nu$, and $\theta$ in b) must be a bijection. \qed

2. Matrices and Grassmannians over $\mathbb{R}, \mathbb{C}$

We present factorization results of compact colorings of matrices and Grassmannians over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$. There are several such results, depending on the chosen metric on the objects we color. These factorizations are approximate, because, as we deal with infinite fields, it is easily seen that the exact ones are not true; on the other hand, they apply to arbitrary colorings given by Lipschitz mappings with values in a compact metric space. Given $\alpha, \beta \in \mathbb{N} \cup \{\infty\}$, the collection of matrices $M_{\alpha,\beta}(\mathbb{F})$ can be naturally turned into a metric space by fixing two norms $m$ and $n$ on $\mathbb{F}^\alpha$ and $\mathbb{F}^\beta$, respectively, and identifying a matrix $A \in M_{\alpha,\beta}(\mathbb{F})$ with the linear operator $T_A : \mathbb{F}^\beta \rightarrow \mathbb{F}^\alpha$, $T_A(x) := A \cdot x$, $x$ as column vector (i.e., a $\beta \times 1$-matrix). This allows to define the norm $\|A\|_{m,n} := \|T_A\|_{(\mathbb{F}^\beta,m),(\mathbb{F}^\alpha,n)}$, and the corresponding distance $d_{m,n}(A,B) := \|A - B\|_{m,n} = \|T_A - T_B\|_{(\mathbb{F}^\beta,m),(\mathbb{F}^\alpha,n)}$. Also, in this way each full rank $\alpha \times k$-matrix $A$ defines a norm $\nu(A)$ on $\mathbb{F}^k$, $\nu(A)(x) := n_k(A \cdot x)$. When $m$ is a norm on $\mathbb{F}^\infty$, by identifying each $\mathbb{F}^k$ with $\langle u_j \rangle_{j<k}$, let $m_k$ be the norm on $\mathbb{F}^k$, $m_k((a_j)_{j<k}) := m(\sum_{j<k} a_j u_j)$. When there is no possible misunderstanding, we will write $d_a$ to denote $d_{m_a,m_a}$.

Recall that in general, given two normed spaces $X = (V,m)$ and $Y = (W,n)$, $L(X,Y)$ denotes the space of continuous (equivalently bounded) linear operators from $X$ to $Y$, that is again a normed space by considering the norm $\|T\| := \sup_{x \in \text{Ball}(X)} n(T(x))$, where $\text{Ball}(X) = \{x \in X : m(x) \leq 1\}$ denotes the unit ball of $X$. Let $L^k(X,Y)$ is the set of those operators of rank $k$. Since when $V$ is finite dimensional every linear mapping from $V$ to $W$ is automatically continuous, in this case, we will use also $L(V,W)$ and $L^k(V,W)$, to denote the collection of linear mappings from $V$ to $W$, and those of rank $k$, respectively. By an isometric embedding we mean a linear mapping $T : X \rightarrow Y$ such that $n(T(x)) = m(x)$ for every $x \in X$. The space of these operators is denoted by $\text{Emb}(X,Y)$.

Of particular importance will be the $p$-norms. Recall that for every $1 \leq p \leq \infty$, $\ell^p_n$ is the normed space $(\mathbb{F}^n, \| \cdot \|_p)$, where $\|(a_j)_{j<n}\|_p := \left(\sum_{j<n} |a_j|^p\right)^{1/p}$ for $p < \infty$ and $\|(a_j)_{j<n}\|_\infty := \max_{j<n} |a_j|$. Similarly one defines the $p$-norms on $\mathbb{F}^\infty$, that we denote as $\ell^p := (\mathbb{F}^\infty, \| \cdot \|_p)$, and their completions are usually denoted by $\ell_p$, for $p < \infty$ and by $c_0$, when $p = \infty$.

Roughly speaking, our factorization theorem for full rank matrices (Theorem 2.7) states that every coloring of such matrices, endowed with the $p$-metrics for $p \in [1, +\infty) \setminus 2(\mathbb{N} + 2)$ is “approximately determined” by the corresponding $\nu$ described above.

Similarly, once a norm $m$ is fixed in $\mathbb{F}^n$, $\text{Gr}(k,\mathbb{F}^n)$ turns into a metric space by considering a Hausdorff metric, and each $k$-dimensional subspace $V$ of $\mathbb{F}^n$ determines a member of the
Banach-Mazur compactum $\mathcal{B}_k$, that is, the isometry class $\tau_n(V)$ of all $k$-dimensional normed spaces isometric to $(V,m)$. We prove that when choosing $p$-norms on each $\mathbb{F}^n$ for $n$ large enough, any coloring of the $k$-Grassmannians of $\mathbb{F}^n$ is approximately determined by $\tau_n$ on some $\text{Gr}(V,k)$. We introduce a more appropriate terminology, in particular we extend the type of colorings to work with. A metric coloring of a pseudo-metric space $M$ is a 1-Lipschitz map $c$ from $M$ to a metric space $(K,d_K)$. We will say that $c$ is a $K$-coloring. Following [13], a continuous coloring is a metric coloring whose target space is the closed unit interval $[0,1]$. A compact coloring is a metric coloring whose target space is a compact metric space. For a subset $X$ of a metric space $(K,d_K)$ and $\varepsilon > 0$, the $\varepsilon$-fattening $X_\varepsilon = \{p \in K : \text{there is some } q \in X \text{ with } d(p,q) \leq \varepsilon\}$.

The oscillation $\text{osc}(c \restriction F)$ of a compact coloring $c : M \to (K,d_K)$ on a subset $F$ of $M$ is the supremum of $d_K(c(y),c(y'))$ where $y,y'$ range within $F$. When $\text{osc}(c \restriction F) \leq \varepsilon$ we also say that $c$ $\varepsilon$-stabilizes on $F$, or that $F$ is $\varepsilon$-monochromatic for $c$. A finite coloring of $M$ is a function from $c$ from $M$ to a finite set $X$; in the particular case when the target space is a natural number $r$ (identified with the set $\{0,1,\ldots,r-1\}$ of its predecessors), we will say that $c$ is an $r$-coloring. Given a finite coloring $c : M \to X$ and $\varepsilon \geq 0$, we say that a subset $F$ of $M$ is $\varepsilon$-monochromatic for $c$, or that $c$ $\varepsilon$-stabilizes on $F$, if there exists some $x \in X$ such that $F$ is included in the $\varepsilon$-neighborhood $(c^{-1}(x))_\varepsilon$ of $c^{-1}(x)$. When $\varepsilon = 0$ we will omit the use of the prefix “0-”.

**Definition 2.1** (Approximate factors). Let $(M,d_M), (N,d_N)$ and $(P,d_P)$ be metric spaces, $\varepsilon > 0$, and $c : (M,d_M) \to (N,d_N)$ and $\pi : (M,d_M) \to (P,d_P)$ be metric colorings, i.e., 1-Lipschitz maps. We say that $\pi$ is an $\varepsilon$-approximate factor (or simply $\varepsilon$-factor) of $c$ if there is some coloring $\tilde{c} : (P,d_P) \to (N,d_N)$ such that

$$
\sup_{x \in M} d_N(c(x),\tilde{c}(\pi(x))) \leq \varepsilon.
$$

That is, “up to $\varepsilon$” $c = \tilde{c} \circ \pi$. Given $M_0 \subseteq M$ we say that $\pi$ is an $\varepsilon$-factor of $c$ in $M_0$ if $\pi \restriction_{M_0} : M_0 \to P$ is an $\varepsilon$-factor of $c \restriction_{M_0}$, i.e., there is some coloring $\tilde{c} : P \to N$ such that $\sup_{x \in M_0} d_N(c(x),\tilde{c}(\pi(x))) \leq \varepsilon$.

**2.1. The statements. Ramsey factors.** As discussed above, given norms $m,n$ on $\mathbb{F}^m$ and $\mathbb{F}^n$ respectively, we regard $M_{m,n}$ as a metric space by considering a $n \times m$-matrix $A$ as the particular representation of a linear operator $T_A$ on the unit bases of suitable normed spaces $(\mathbb{F}^m,m)$ and $(\mathbb{F}^n,n)$, and then by considering the corresponding operator norm.

2.1.1. Full rank matrices. Given a vector space $V$, let $\mathcal{N}_V$ be the set of all norms on $V$, endowed with the topology of pointwise convergence. When $\dim V < \infty$, a compatible metric on $\mathcal{N}_V$ is

$$
\omega(m,n) := \log \max\{\|\text{Id}\|_{(V,m),(V,n)},\|\text{Id}\|_{(V,n),(V,m)}\},
$$

that will be called the intrinsic metric on $\mathcal{N}_V$. It is easy to see that the metric space $(\mathcal{N}_V,\omega)$ has the Heine-Borel property, that is, every closed and bounded set of it is compact. In particular, each closed $\omega$-ball is compact. Given a normed space $E = (W,\|\cdot\|)$, let $\mathcal{N}_V(E)$ be the collection of norms $m$ on $V$ such that there exists a linear isometry $T : (V,m) \to E$. In general, $\mathcal{N}_V(E)$ is not closed in $\mathcal{N}_V$, although in some natural cases is. We will write $\mathcal{N}_\alpha$ to denote $\mathcal{N}_{\alpha_0}$.

**Definition 2.2.** Suppose that $V$ is finite dimensional, $E = (W,\|\cdot\|)$ a normed space. Let

$$
\nu_{V,E} : \mathcal{L}^{\dim V}(V,W) \to \mathcal{N}_V(E)
$$

be the mapping that assigns to a 1-1 linear mapping $T : V \to W$ the norm $\nu_{V,E}(T)$ on $V$, defined by $(\nu_{V,E}(T))(x) := \|T(x)\|$, that is, the norm on $V$ that makes $T$ an isometric embedding. With a slight abuse of notation, we also write $\nu_{k,(\mathbb{F}_0,\|\cdot\|)}$ to denote the mapping $A \in M_{a,k} \mapsto \nu_{k,(\mathbb{F}_0,\|\cdot\|)}(T_A)$ that assigns to a such matrix $A$ the norm defined for each $x \in \mathbb{F}^k$ by $(\nu_{k,E}(A))(x) := \|A \cdot x\|$. 


Given a finite dimensional normed space $X = (X, \| \cdot \|_X)$ and a normed space $E = (V, \| \cdot \|_E)$, we define on $\mathcal{N}_X(E) \times \mathcal{N}_X(E)$ the $E$-extrinsic function
\[
\partial_{X,E}(m,n) := \inf \{ \| T - U \|_{X,E} : T \in \text{Emb}((X,m), E), U \in \text{Emb}((X,n), E) \}.
\]
So, $\partial_{X,E}(m,n)$ computes the minimal distance $d_{X,E}(T,U)$ between possible representations of $m$ and $n$, $\nu_{X,E}(T) = m$, $\nu_{X,E}(U) = n$. In general $\partial_{X,E}$ is not a compatible metric. Note that $\partial_{X,E}$ is a metric when $\partial_{X,E}$ satisfies the triangle inequality. The following is easy to prove.

**Proposition 2.3.** If $\partial_{X,E}$ is compatible, $\nu_{X,E} : (\mathcal{L}^\text{dim}_X(X,E), d_{X,E}) \to (\mathcal{N}_X(E), \partial_{X,E})$ is $1$-Lipschitz. \(\square\)

Recall that given a linear operator $T : X \to Y$ between normed spaces $X$ and $Y$,
\[
\| T \| = \min \{ \lambda \geq 0 : T(\text{Ball}(X)) \subseteq \lambda \cdot \text{Ball}(Y) \},
\]
and when $X$ is finite dimensional, let
\[
\| T^{-1} \| = \min \{ \lambda \geq 0 : \text{Ball}(T(X)) \subseteq \lambda \cdot T(\text{Ball}(X)) \}.
\]
When $T$ is $1$-$1$, $\| T^{-1} \| = \| U \|$, where $U : TX \to X$ is the inverse operator of $T$. Given $\alpha, \beta \in \mathbb{N} \cup \{ \infty \}$, and a norm $m \in \mathcal{N}_E$, let $M^k_{\alpha,\beta}(m;\lambda)$ be the collection of matrices in $M^k_{\alpha,\beta}$ such that the corresponding linear operator $T_A : (\mathbb{F}^j, m) \to (\mathbb{F}^{\alpha}, m)$ satisfies $\| T_A \|, \| T_A^{-1} \| \leq \lambda$.

Notice that the boundary $M^k_{\alpha,k}(m;1)$ is the collection of matrices $A$ defining isometric embeddings $T_A : (\mathbb{F}^k, m) \to (\mathbb{F}^{\alpha}, m)$, and it will be denoted by $\mathcal{E}_{\alpha,k}(m)$, and $\mathcal{E}(m) := \bigcup_{n \geq m} \mathcal{E}_{n,m}(m)$. The following is easy to prove, and highlights the interest of this collection.

**Proposition 2.4.** Let $R \in M^{\alpha}_{a,m}$.

a) The multiplication by $R$ operator $\mu_A : (M^k_{m,k}, d_m) \to (M^k_{a,k}, d_a)$, $A \mapsto R \cdot A$ defines an isometry if and only if $R \in \mathcal{E}_{a,m}(m)$.

b) If $R \in M^{\alpha}_{a,m}(m)$, then $\nu_{k, (\mathbb{F}^m, m)} \circ \mu_R = \nu_{k, (\mathbb{F}^m, m)}$.

**Proof.** a): Suppose that $X$ is a normed space of finite dimension $k$, $Y,Z$ normed spaces and suppose that $T \in \mathcal{L}(Y,Z)$ is such that the composition operator $U \in \mathcal{L}(X,Y) \mapsto TU \in \mathcal{L}(X,Z)$ is an isometry with respect to the norm metrics. Let us prove that $T$ must be an isometry. Fix a non-zero vector $y \in Y$. Let $(x_j)_{j < k}$ be an Auerbach basis of $X$, i.e., a basis consisting of normalized vectors such that its biorthogonal sequence $(x^*_j)_{j < k}$ is also normalized (see [5, Chapter 4, Theorem 13]). Let $(y_j)_{j < k}$ be a linearly independent sequence in $Y$ with $y_0 = y$. For each $n \geq 1$, let $T_n(x) = x_0^0(x)y + (1/n) \sum_{j=1}^{n-1} x^*_j(x)y_j$. It is clear that $T_n$, $(1/n)T_n$ are $1-1$. Then, $\| U \circ T_n - U \circ (1/n) \cdot T_n \| \to 0$, and $\| U \circ T_n - U \circ (1/n) \cdot T_n \| = \| T_n - (1/n) \cdot T_n \| \to 0$, hence $\| U \circ T_n \| \to \| U \|$. It follows that $\| T_n \| \to \| x^*_0 \|_Y$ and similarly $\| U \circ T_n \| \to \| x^*_0 \|_Y \cdot \| U \|$. Since $\| x^*_0 \|_Y = 1$, we obtain $\| U \| = \| y \|$. \(\square\)

Given a normed space $E = (\mathbb{F}^{\infty}, \| \cdot \|_E)$, let $\mathcal{N}_k(E;\lambda)$ ($\mathcal{N}_k(E;\lambda)$) be the closed (resp. open) ball of $\mathcal{N}_k(E)$ with respect to the intrinsic metric $\omega$ centered on the norm $\| \cdot \|_E$ in $\mathbb{F}^k$ and with radius $\lambda$, i.e., $\mathcal{N}_k(E;\lambda) = \{ n \in \mathcal{N}_k(E) : \omega(n, \| \cdot \|_E) \leq \log \lambda \}$, similarly for $\mathcal{N}_k(E;\lambda)$.

**Proposition 2.5.** a) $\mathcal{N}_k(E;\lambda) = \nu_{k,E}(M^{\alpha}_{a,k}(\| \cdot \|_E;\lambda))$ and $\mathcal{N}_k(E;\lambda) = \nu_{k,E}(M^{\alpha}_{a,k}(\| \cdot \|_E;\lambda))$.

b) If $\partial_{X,E}$ and $\omega$ are uniformly equivalent on $\omega$-bounded subsets of $\mathcal{N}_k(E)$, then every $\omega$-bounded set is $\partial_{X,E}$-totally bounded, thus, the $\mathcal{E}(\mathbb{F}^m, m)$-completion of $\mathcal{N}_k(E;\lambda)$ and $\mathcal{N}_k(E;\lambda)$ are compact.

**Proof.** b): Suppose that $A \subseteq \mathcal{N}_k(E)$ is $\omega$-bounded. Since $\mathcal{N}_k(E)$ has the Heine-Borel property, $A$ is $\omega$-totally bounded. $\partial_{X,E}$ is uniformly equivalent to $\omega$ on $A$, so $A$ is $\partial_{X,E}$-totally bounded. \(\square\)
Definition 2.6 (Ramsey factors for full-rank matrices). Let $m$ be a metric on $\mathbb{F}^\omega$, set $E_\alpha := (\mathbb{F}^\alpha, m)$ for every $\alpha \leq \omega$. We say that $m$ produces Ramsey factors for colorings of full rank matrices, when

i) $\partial E_k, E_x$ is a compatible metric on $\mathcal{N}_k(E_x)$ uniformly equivalent to $\omega$ on $\omega$-bounded sets.

ii) Given $k, m \in \mathbb{N}$, $\varepsilon > 0$, $\lambda > 1$ and a compact metric space $(K, d_K)$ there is $n \in \mathbb{N}$ such that for every $K$-coloring $c$ of $(M^k_n(m; \lambda), d_n)$ there is $R \in \mathcal{E}_{n,m}(m)$ such that the restriction $\nu_{Rk, E_x} : M^k_{n,k}(m; \lambda) \rightarrow \mathcal{N}_K(E_x; < \lambda)$ is an $\varepsilon$-factor of $c$ in $R \cdot M^k_{n,k}(m; \lambda)$; that is, there is a coloring $\tilde{c} : (\mathcal{N}_K(E_x; < \lambda), \partial E_m, E_x) \rightarrow (K, d_K)$ such that $d_K(c(R \cdot A), \tilde{c}(\nu_R(A))) \leq \varepsilon$ for every $A \in M^k_{n,k}(m; \lambda)$.

Theorem 2.7 ($p$-Factorization of colorings of full rank matrices over $\mathbb{R}$, $\mathbb{C}$). For $1 \leq p \leq \infty$, $p \neq 2\mathbb{N} + 4$, the $p$-norm $\| \cdot \|_p \in \mathcal{N}_x$ produces Ramsey factors for colorings of full rank matrices.

2.1.2. Grassmannians. Given a normed space $E = (V, \| \cdot \|_E)$ the $k$-Grassmannian $\text{Gr}(k, V)$ of $V$ is naturally a topological space, as it can be identified with the corresponding topological quotient of $E^k$ by the relation $(x_j)_{j<k} \sim (y_j)_{j<k}$ if $(x_j)_{j<k} = (y_j)_{j<k}$. If in addition $E$ is separable, this turns $\text{Gr}(k, E) := \text{Gr}(k, V)$ into a polish space. A natural compatible metric is the gap (or opening) metric (see [11]), $\Lambda_{k,E}(U, W)$ defined as the Hausdorff distance, with respect to the norm metric in $E$, between the unit balls $\text{Ball}(U, m)$ and $\text{Ball}(W, m)$, that is,

$$\Lambda_{k,E}(U, W) := \max \{ \max \min \| u - w \|_E, \max \min \| w - u \|_E \}.$$ 

Let $\text{GL}(V) \cap \mathcal{N}_V$ be the canonical action $(\Delta \cdot m) := m(\Delta^{-1}(x))$ for every $x \in V$. Notice that the intrinsic metric $\omega$ is invariant under this action. Let $\mathcal{B}_V := \mathcal{N}_V/\text{GL}(V)$ be the quotient space. Since $\omega$ is invariant under the action and $\mathcal{N}_V$ has the Heine-Borel property, so is $\mathcal{B}_V$ with its quotient metric. Observe that given a norm $m$ on $V$ of dimension $k$ there is a linear transformation $\Delta$ such that $(\Delta(u_j))_{j<k}$ is an Auerbach basis of $(V, m)$, i.e., a normalized sequence such that $m(\sum_{j<k} a_j \Delta(u_j)) \geq \max_{j<k} |a_j|$ for every sequence of scalars $(a_j)_{j<k}$. This implies that given two norms $m, n \in \mathcal{N}_V$ there is a linear transformation $\Delta$ such that $\omega(\Delta \cdot m, n) \leq \log k$, and consequently the diameter of $\mathcal{B}_V$ is at most $\log k$, hence it is compact, called the Banach-Mazur compactum. The quotient metric corresponding to $\omega$ is $2$-Lipschitz equivalent to the well-known Banach-Mazur metric

$$d_{BM}(m, n) := \log \inf_{\Delta \in \text{GL}(V) \cap \mathcal{N}_V} \| \Delta \|(V, m) \cdot \| \Delta^{-1} \|(V, n).$$

Let $\mathcal{B}_V(E)$ denote the Banach-Mazur classes corresponding to norms in $\mathcal{N}_V(E)$, or, in other words, the isometric types of finite dimensional subspaces of $E$ of the same dimension than $V$. We write $\mathcal{B}_k, \mathcal{B}_k(E)$ and $\text{Gr}(k, E)$ to denote $\mathcal{B}_k, \mathcal{B}_k(E)$ and $\text{Gr}(k, V)$, respectively.

Definition 2.8. Let $\tau_{k,E} : \text{Gr}(k, E) \rightarrow \mathcal{B}_k(E)$ be the mapping that assigns to each $k$-dimensional normed subspace $W$ of $E$ the isometric type of $(W, \| \cdot \|_E)$.

In other words, for $W \in \text{Gr}(k, E)$, $\tau_{k,E}(W) = [\nu_{k,E}(T)]_{BM}$ for some 1-1 linear function $T : \mathbb{F}^k \rightarrow W$ such that $\text{Im} T = W$. We define the $E$-Kadets mapping $\gamma_{k,E}$ on $\mathcal{B}_k(E) \times \mathcal{B}_k(E)$ by $\gamma_{k,E}(m, n) := \inf \{ \Lambda_{k,E}(U, W) : U, W \in \text{Gr}(k, E), (U, \| \cdot \|_E) \equiv (\mathbb{F}^k, m), (W, \| \cdot \|_E) \equiv (\mathbb{F}^k, n) \}$.

Definition 2.9. $\gamma_{k,E}$ is the $E$-Kadets metric when it is a compatible metric on $\mathcal{B}_k(E)$.

In the literature the Kadets metric $\gamma$ corresponds to the metric $\gamma_{C[0,1]}$ for Grassmannians of the universal space of continuous functions on the unit interval $C[0,1]$ (see [11]).

Proposition 2.10. If $\gamma_{k,E}$ is Kadets, $\tau_{k,E} : (\text{Gr}(k, E), \Lambda_E) \rightarrow (\mathcal{B}_k(E), \gamma_{k,E})$ is 1-Lipschitz. \(\square\)
Given $E = (F^a, m)$, we write $\text{Gr}_m(k, F^a)$ to denote the set of $k$-dimensional subspaces $W$ of $F^a$ so that $(W, m)$ is isometric to $(F^k, m)$, i.e., $\tau_{k,E}(W) = [m | \langle u_j \rangle_{j<k}]$. The next explains the interest of $\text{Gr}_m(k, F^a)$ and it is proved similarly to Proposition 2.4.

**Proposition 2.11.** Fix $k \leq m$ and $W \in \text{Gr}(m, E)$.

a) An invertible linear operator $\theta : W \rightarrow F^m$ defines an isometry $\Theta : (\text{Gr}(k, W), \Lambda_{k,E}) \rightarrow (\text{Gr}(k, F^m), \Lambda_{k,E}), V \mapsto \theta(V)$, if and only if $\theta : (W, m) \rightarrow (F^m, m)$ is an isometry.

b) $\tau_{k,E}(W_m) = \tau_{k,E} \circ \text{Gr}(k, W)$.

**Definition 2.12** (Ramsey factors for Grassmannians). A norm $m$ on $F^a$, $E = (F^a, m)$, produces Ramsey factors for colorings of Grassmannians when

i) $\gamma_{k,E}$ is a compatible metric.

ii) For every $k \in N$, $n > 0$, and every compact metric $(K, d_K)$ there is $n$ such that for every $c : (\text{Gr}(k, F^a), \Lambda_{k,E}) \rightarrow (K, d_K)$ there is $V \in \text{Gr}_m(m, F^a)$ such that $\tau_{k,E}$ is an $\varepsilon$-factor of $c$ in $\text{Gr}(k, V)$.

**Theorem 2.13** (Factorization of Grassmannians over $\mathbb{R}, \mathbb{C}$). For $1 \leq p \leq \infty$, $p \neq 2N + 4$, the $p$-norm $\| \cdot \|_p \in N_{\infty}$ produces Ramsey factors for colorings of Grassmannians.

Geometrically, the previous result states that restrictions of compact colorings of Grassmannians depend on shapes of their unit balls. The following statement can be considered as a version of the Graham-Leeb-Rothschild Theorem for the fields $\mathbb{R}, \mathbb{C}$.

**Corollary 2.14** (Graham-Leeb-Rothschild for $\mathbb{R}, \mathbb{C}$). For every $1 \leq p \leq \infty$, $p \neq 2N + 4$, every $k, m \in N$, every $\varepsilon > 0$ and every compact metric space $(K, d_K)$ there is $n$ such that every coloring $c : (\text{Gr}(k, F^a), \Lambda_{k,E}) \rightarrow (K, d_K)$ $\varepsilon$-stabilizes on $\text{Gr}(k, W)$ for some $W \in \text{Gr}(m, F^a)$.

**Proof.** This is direct consequence of Theorem 2.13 and the facts that for large $r$ the space $\ell_p^r$ contains almost isometric copies of $\ell_p^m$ and that $B_k(\ell_p^m)$ consists of a point. □

**Remark 2.15.** Recall that Dvoretzky’s Theorem asserts that any finite-dimensional normed space $X$ of dimension $r$ contains almost isometric copies of $\ell_p^m$ with $m$ proportional to $\log(r)$ (see [1, Theorem 12.3.6]). This means that Corollary 2.14 remains true for every norm $m$ on $F^a$.

**2.1.3. Square matrices.** Given a vector space $V$, let $V^*$ be its (algebraic) dual, the vector space of linear functions $f : V \rightarrow F$; if in addition $X = (V, m)$ is a normed space, $X^*$ will denote the (normed) dual space $\mathcal{L}(X, (F, \| \cdot \|))$, that is, the vector space of continuous linear functionals $f : V \rightarrow F$ endowed with the dual norm $m^*(f) := \sup_{\|x\|_X \leq 1} |f(x)|$. Let $\text{GL}(V) \hookrightarrow N_V^*$ be the canonical action $(\Delta \cdot n)(f) := n(\Delta^*(f))$ for $f \in V^*$. Observe that the dual mapping $^* : (N_V, \omega) \rightarrow (N_V^*, \omega)$ is a $\text{GL}(V)$-equivariant mapping, that is, given $m \in N_V$, $(\Delta \cdot m)^* = \Delta \cdot m^*$. Let $\text{GL}(V) \hookrightarrow N_V \times N_V^*$ be the action $\Delta \cdot (m, n) := (\Delta \cdot m, \Delta \cdot n)$ and let $D_V$ be the quotient space $(N_V \times N_V^*)/\text{GL}(V)$. With the compatible metric $\omega_2((m, n), (p, q)) := \omega(m, p) + \omega(n, q)$ the product $N_V \times N_V^*$ has the Heine-Borel property, so with the corresponding quotient metric $\tilde{\omega}_2$, $D_V$ also has this property. Given a normed space $E$, let $\text{D}_V(E) := (N_V(E) \times N_V^*(E))/\text{GL}(V)$. Its orbits will be denoted by $[m] = [(m_0, m_1)]$. In the next $E := (F^a, \| \cdot \|_E)$ and $D_E$, $D_k(E)$ denote $D_{F^a}$ and $D_k(E)$, respectively.

**Definition 2.16.** Let $\nu^2_{k,E} : M^k_a \rightarrow D_k(E)$ be the function that assigns to an $\alpha$-squared matrix $A$ of rank $k$, the class $\text{GL}(F^k)$-orbit of the pair $(\nu_p(A), \nu_p(C))$ for $B, C \in M^k_{\alpha,k}$ with $A = B \cdot C^*$. The fact that $\nu^2_{k,E}$ is well defined follows from the full-rank factorization of matrices.

**Proposition 2.17.** $\{(\nu_p(E(U_0), \nu(E(U_0))), \nu_p(E(U_1))) : U_0 \circ U_1^* = T_0 \circ T_1^*\}$. **Proof.** If $U_0, U_1 : F^k \rightarrow E$ are linear operators of rank $k$ then $T_0 \circ T_1^* = U_0 \circ U_1^*$. □
We define on $D_k(E) \times D_k(E)$ the function $\varrho_{k,E}$

$$\varrho_{k,E}([m],[n]) := \inf_{T,U} \|T - U\|_{E^*,E}$$

where the infimum is over bounded linear mappings $T,U : E^* \to E$ of rank $k$ admitting decompositions $T = T_0 \circ T_1^*$ and $U = U_0 \circ U_1^*$ with $T_0, U_0 : \mathbb{F}^k \to E$ and $T_1, U_1 : \mathbb{F}^k \to E$ of rank $k$ for $j = 0,1$ and such that $(\nu_{\mathbb{F}^k,E}(T_0),\nu_{\mathbb{F}^k,E}(T_1)) \in [m]$ and $(\nu_{\mathbb{F}^k,E}(U_0),\nu_{\mathbb{F}^k,E}(U_1)) \in [n]$. In the previous, we are identifying canonically $(\mathbb{F}^k)^* \oplus (\mathbb{F}^k)^*$. The following is easy to prove.

**Proposition 2.18.** If $\varrho_{k,E}$ is compatible, $\nu_{k,E}^2 : (M_k^k,d_{E^*,E}) \to (D_k(E),\varrho_{k,E})$ is 1-Lipschitz. ☐

Given a norm $m \in \mathcal{N}_\mathbb{F}$, let $M_k^k(m;\lambda)$ be the collection of $A \in M_k^k$ such that $\|T_A\|,\|T_A^{-1}\| \leq \lambda$. The next has a similar proof to that of Proposition 2.4, so we leave the details to the reader.

**Proposition 2.19.** Let $L \in M_{\mu,m}^k$ and $R \in M_{\nu,m}^k$.

a) The multiplication by $L$ and $R$ function $\mu_{L,R} : (M_{\mu,m}^k,d_{E^*,E}) \to (M_{\nu,m}^k,d_{E^*,E})$, $A \mapsto \mu_{L,R}(A) := L \cdot A \cdot R \in M_{\nu,m}^k$ is a compact metric and only if $L,R \in \mathcal{E}_{\mu,m}(\|\cdot\|_{E^*})$.

b) If $L,R \in \mathcal{E}_{\mu,m}(\|\cdot\|_{E^*})$, then $\nu_{k,E}^2(\nu_{k,E}(L),\nu_{k,E}(R)) = \nu_{k,E}^2(L,R) = \nu_{k,E}^2(\mu_{L,R}(\|\cdot\|_{E^*}))$.

Given $\lambda \geq 1$, let $D_k(\lambda) := \{([m,n]) \in D_k : \omega(m^*,n) \leq \lambda\}$, $D_k(<\lambda) := \{([m,n]) \in D_k : \omega(m^*,n) < \lambda\}$, and let $D_k(E;\lambda) := D_k(\lambda) \cap D_k(E)$, and $D_k(E;<\lambda) := D_k(<\lambda) \cap D_k(E)$.

**Proposition 2.20.** a) $D_k(\lambda)$ and $D_k(<\lambda)$ are well defined.

b) $D_k(\lambda)$ is compact.

c) $D_k(E;\lambda) = \nu_{k,E}^2(M_k^k(\|\cdot\|_{E^*};\lambda))$ and $D_k(E;<\lambda) = \nu_{k,E}^2(M_k^k(\|\cdot\|_{E^*};<\lambda))$.

**Proof.** a) follows from the fact that given $\Delta \in \text{GL}(\mathbb{F}^k)$, $\omega(\Delta \cdot m^*,\Delta \cdot n) = \omega(m^*,n)$. b): It is clear that $D_k(\lambda)$ is closed, so by the Heine-Borel property of $(D_k,\varrho_2)$, we just have to prove that $D_k(\lambda)$ is $\varrho_2$-bounded: Fix $([m,n]),([p,q]) \in D_k(\lambda)$, let $\Delta \in \text{GL}(\mathbb{F}^k)$ be such that $\omega(\Delta \cdot \cdot m,p) \leq \lambda$. Then it follows that $\omega(\Delta \cdot n,\Delta \cdot m^*) \leq \omega(\Delta \cdot n,\Delta \cdot m^*) + \omega(\Delta \cdot m^*,p^*) + \omega(p^*,q) \leq 2\lambda + k$. c) will be proved in Lemma 3.12. ☐

**Definition 2.21.** A norm $m$ on $\mathbb{F}^k$ produces Ramsey factors for colorings of square matrices if

i) $\varrho_{k,E,f}$ is a compatible metric on $D_k(E)$ uniformly equivalent to $\varrho_2$ on $\varrho_2$-bounded sets.

ii) Given $k,m \in \mathbb{N}$, real numbers $\varepsilon > 0$, $\lambda \geq 1$, and a compact metric space $(K,d_K)$, there is a $n \in \mathbb{N}$ such that for every coloring $c : (M_k^k(m;\lambda),d_{E^*,E}) \to (K,d_K)$ there are $R_0, R_1 \in \mathcal{E}_{\mu,m}(\|\cdot\|_E)$ such that the restriction $\nu_{k,E}^2 : M_k^k(m;\lambda) \to D_k(E;\lambda)$ is an $\varepsilon$-factor of $c$ in $R_0 \cdot M_k^k(m,\lambda) \cdot R_1^*$.

**Theorem 2.22** (Factorization of colorings of square matrices over $\mathbb{R}$, $\mathbb{C}$). For $1 \leq p \leq \infty$, $p \neq 2\mathbb{N} + 4$, the $p$-norm $\|\cdot\|_p \in \mathcal{N}_\mathbb{F}$ produces Ramsey factors for colorings of square matrices.

2.1.4. **Uniqueness.** We see now how when the metric on matrices/Grassmannians is fixed there are not so many options of being a Ramsey factor. Suppose that $m \in \mathcal{N}_\mathbb{F}$, $k \in \mathbb{N}$ and $\lambda > 1$. A $(k,m,\lambda)$-Ramsey factor is a pair $(\mu,\mathcal{A})$ where $\mu : (M_k^k(m;\lambda),d_m) \to (K_\mu,d_\mu)$ is a coloring (i.e., 1-Lipschitz mapping) to a compact metric space $(K_\mu,d_\mu),\mathcal{A} \subseteq \mathcal{E}(\mathbb{F}_m)$, and

i) The image of $\mu$ is dense in $K_\mu$.

ii) $\mu(RA) = \mu(A)$ for every $R \in \mathcal{A} \cap M_{m,n}^m$ and $A \in M_k^k(m;\lambda)$.

iii) For every $m$, $\varepsilon > 0$ and every compact metric $(L,d_L)$ there is a $n \in \mathbb{N}$ such that if $c : (M_k^k(m;\lambda),d_m) \to (L,d_L)$ is a coloring then there is some $R \in \mathcal{A}$ such that the restriction $\mu : R \cdot M_k^k(m;\lambda) \to (K_\mu, d_\mu)$ is an $\varepsilon$-factor of $c$ in $R \cdot M_k^k(m;\lambda)$.

Suppose that $m$ produces Ramsey factors for full rank matrices and set $E_\alpha := (\mathbb{F}^\alpha,m)$ for $\alpha \leq \infty$. We have that $\nu_{\mathbb{F}^k,E_\alpha}(M_k^k(m;\lambda)) = \mathcal{N}_k(\varepsilon) \in \mathcal{E}_{k,E_\alpha}$-totally bounded: This is because
\( \hat{c}_{E_k,E_x} \) and \( \omega \) are, by hypothesis, uniformly equivalent to \( \omega \) on \( \mathcal{N}_k(E; < \lambda) \), and this set is \( \omega \)-totally bounded because is a \( \omega \)-bounded set of \( \mathcal{N}_k \). This implies that the completion \( \mathcal{N}_k(E; < \lambda) \) is a compact space. Then it is obvious from the definition of producing Ramsey factors that 
\[
\nu_{\hat{c}_{E_k,E_x}}: M^k_{\mathcal{E}_x,k}(\mathcal{m}; < \lambda) \to \mathcal{N}_k(E; < \lambda)
\]
is a \((k, \mathcal{m}, \lambda)\)-Ramsey factor, and in fact is the minimal one:

**Proposition 2.23.** Suppose that \( \mathcal{m} \) produces Ramsey factors for full rank matrices, and suppose that \((\mu, \mathcal{A})\) is a \((k, \mathcal{m}, \lambda)\)-Ramsey factor.

a) There is some surjective coloring \( \theta : K_\mu \to \mathcal{N}_k(E_x; < \lambda) \) such that \( \nu_{\hat{c}_{E_k,E_x}} = \theta \circ \mu \).

b) If \( \mathcal{A} = \mathcal{E}(\mathcal{m}) \), then there is a surjective isometry \( \theta : K_\mu \to \mathcal{N}_k(E_x; \lambda) \) such that \( \nu_{\hat{c}_{E_k,E_x}} = \theta \circ \mu \).

Proof. a): For each \( m \) we can find 1-Lipschitz mappings \( \theta_m : (K, d_K) \to \mathcal{N}_k(E_x; \lambda) \) and \( R_m \in \mathcal{E}_{m,m}(\mathcal{m}) \) such that \( \hat{c}_{E_k,E_x}(\theta_m(\mu(R_m \cdot A)), \nu_{\hat{c}_{E_k,E_x}}(R_m \cdot A)) \leq 1/2^m \) for every \( A \in M^k_{m,k}(\mathcal{m}; < \lambda) \). By the coherence properties of \( \mu \) and \( \nu_{\hat{c}_{E_k,E_x}} \), we obtain that

\[
\hat{c}_{E_k,E_x}(\theta_m(\mu(A)), \nu_{\hat{c}_{E_k,E_x}}(A)) \leq 1/2^m \text{ for every } A \in M^k_{m,k}(\mathcal{m}; < \lambda).
\]

(2)

Given \( A \in M^k_{\mathcal{E}_x,k}(\mathcal{m}; < \lambda) \), set \( x := \mu(A) \), and let \( n \) be such that \( A \in M^k_{\mathcal{E}_x,k,n} \). Then we know from (2) that \( (\theta_n(\mu(A)))_{m \geq n} \) is a Cauchy sequence, and let \( x = \lim_x \theta_n(\mu(A)) \) be its limit. Notice that \( \theta(\mu(A)) = \nu_{\hat{c}_{E_k,E_x}}(A) \), so, since \( \nu_{\hat{c}_{E_k,E_x}}(M^k_{\mathcal{E}_x,k}(\mathcal{m}; < \lambda)) \) is dense in \( \mathcal{N}_k(E_x; < \lambda) \), we can conclude that \( \theta \) is onto. b) is an easy consequence of a).

With the obvious definitions of Ramsey factors for Grassmannians and for square matrices, the corresponding statements on \( \tau_{E_k,E_x} \) and \( \nu_\theta^2 \) are also true.

**Remark 2.24.** For some norms \( \mathcal{m} \), for example the \( p \)-norms, the completion \( \mathcal{N}_k(E; < \lambda) \) is exactly \( \mathcal{N}_k(E; \lambda) \). A sufficient condition is that for every finite dimensional subspaces \( X \) and \( Y \) of \( E_x \) there is a finite dimensional subspace \( Z \) of \( E_x \) that has isometric copies \( X_0 \) and \( Y_0 \) of \( X \) and \( Y \), respectively, such that \( X_0 \cap Y_0 = \{0\} \).

3. THE PROOFS: APPROXIMATE RAMSEY PROPERTIES AND EXTREME AMENABILITY

In Ramsey theory, the usual strategy to prove that a list of colorings is the canonical one is, giving a coloring of a class of embeddings, use the Ramsey property for an appropriate good class of embeddings and an enlarged number of colors that take now into account the transformation necessary to make an arbitrary embedding a good one. This is exactly what we did for full rank matrices over a finite field. On the approximate case, one may follow the same direct approach and obtain similar results to the ones we presented, but now obliged to deal with several approximation arguments that make the proofs somehow unnecessarily complicated. Instead, our approach is to apply a topological principle that is equivalent to a strong version of an approximate Ramsey property, and that makes the computations much more clear. This is the extreme amenability of the group of linear isometries of appropriate Banach spaces that locally are like \( l^p, \) for \( p \neq 2N + 4 \). We introduce some relevant terminology and concepts. Recall that a Banach space is a complete normed space. Given Banach spaces \( X = (X, \| \cdot \|_X) \) and \( Y = (Y, \| \cdot \|_Y) \), and given \( \delta \geq 1 \), let \( \text{Emb}_\delta(X,Y) \) be the collection of all linear functions \( T : X \to Y \) such that \( (1 + \delta)^{-1}\|x\|_X \leq \|T(x)\|_Y \leq (1 + \delta)\|x\|_X \). Notice that when \( \dim X = k < \infty \) this definition corresponds to \( \mathcal{L}^k_{1+\delta}(X,Y) \) presented before. The following concept was introduced in [6, Definition 5.1] (see also [3],[4]).

**Definition 3.1.** A family \( \mathcal{G} \) of finite dimensional normed spaces has the Steady Approximate Ramsey Property\(^\ast\) (SARP\(^\ast\)) when for every \( k \in \mathbb{N} \) and every \( \varepsilon > 0 \) there is \( \delta := \delta(k, \varepsilon) > 0 \) such that if \( X, Y \in \mathcal{G} \) and \( \dim X = k \), then there exists \( Z \in \mathcal{G} \) such that every continuous coloring \( c \) of \( \text{Emb}_\delta(X,Z) \) \( \varepsilon \)-stabilizes on \( \gamma \circ \text{Emb}_\delta(X,Y) \) for some \( \gamma \in \text{Emb}(Y,Z) \).
The \((SARP^+)\) of the classes \(\{\ell^n_p\}_n\) can be seen as a multidimensional Borsuk-Ulam principle (see [6, §§5.1.1]). In general, for a family \(\mathcal{F}\) is a strong form of amalgamation: Recall that \(\mathcal{G}\) is an amalgamation class when \(\{0\} \in \mathcal{G}\) and for every \(\varepsilon > 0\) and \(k \in \mathbb{N}\) there is \(\delta > 0\) such that if \(X \in \mathcal{G}\) has dimension \(k\), \(Y, Z \in \mathcal{G}\), and \(\gamma \in \text{Emb}_i(X, Y), \eta \in \text{Emb}_i(X, Z)\), then there are \(V \in \mathcal{G}, i \in \text{Emb}(Y, V)\) and \(j \in \text{Emb}(Z, V)\) such that \(\|i \circ \gamma - j \circ \eta\| \leq \varepsilon\). It is not difficult to see that if \(\mathcal{F}\) has the \((SARP^+)\) then it is an amalgamation class.

To an amalgamation class \(\mathcal{G}\) it corresponds a unique separable “generic” Banach space \(E\) whose family of finite dimensional substructures, denoted by \(\text{Age}(E)\), is minimal containing \(\mathcal{G}\). This is the content of the Fraïssé correspondence on the category of Banach spaces. We write \(\mathcal{G}_E\) to denote the class of subspaces of \(E\) that are isometric to some element of \(\mathcal{G}\), and \(\overline{\mathcal{G}}^\circ\) to denote the class of subspaces of elements of \(\mathcal{G}\); we write \(X \in \mathcal{G}_\overline{\mathcal{G}}\) for some element of \(\mathcal{G}\) is isometric to \(X\), and we say that \(\mathcal{G}\) is hereditary if \(Y \in \mathcal{G}\), and \(\text{Emb}(X, Y) \neq \emptyset\), then \(X \in \mathcal{G}_\overline{\mathcal{G}}\).

Finally, \(\mathcal{G} \preceq \mathcal{H}\) means that every space in \(\mathcal{G}\) is isometric to some element of \(\mathcal{H}\), and \(\mathcal{G} \equiv \mathcal{H}\) to denote that \(\mathcal{G} \preceq \mathcal{H} \preceq \mathcal{G}\). Note that if \(\mathcal{G} \equiv \mathcal{H}\), then \(\mathcal{G}\) has the \((SARP^+)\) (is an amalgamation class) if and only if \(\mathcal{H}\) has the \((SARP^+)\) (resp. is an amalgamation class).

**Theorem 3.2** (Fraïssé correspondence; [4], [6]). Let \(\mathcal{G}\) be a class of finite dimensional normed spaces.

a) If \(\mathcal{G}\) is an amalgamation class, then there is a unique separable Banach space \(E\), called the \(\mathcal{G}\)-Fraïssé limit, and denoted by \(\text{FLim}\mathcal{G}\), such that \(\overline{\mathcal{G}}_E\) is \(\text{Lim}_E\)-dense in \(\text{Age}(E)\) and \(E\) is Fraïssé, that is for every \(\varepsilon > 0\) and \(k \in \mathbb{N}\) there is \(\delta > 0\) such that the natural action \(\text{Iso}(E) \curvearrowright \text{Emb}_i(X, E)\) is \(\varepsilon\)-transitive (given \(\gamma, \eta \in \text{Emb}_i(X, E)\) there is \(g \in \text{Iso}(E)\) such that \(\|g \circ \eta - \gamma\| \leq \varepsilon\)).

b) The following are equivalent:

i) \(\mathcal{G}\) is hereditary amalgamation class that is \(d_{BM}\)-compact, that is, for every \(k\), the collection of classes \([m]\) of norms \(m \in N_k\) such that \((\mathbb{F}^k, m) \in \mathcal{G}_\overline{\mathcal{G}}\) is a closed subset of \(B_k\).

ii) There is a unique separable Fraïssé Banach space \(E\) such that \(\text{Age}(E) \equiv \mathcal{G}\).

This can be considered as the Banach space version of the Fraïssé correspondence of first order structures, that, for example, interprets several Random graphs (Rado, Henson graphs), Boolean algebras (the countable atomless one), or metric spaces (the rational Urysohn space) as Fraïssé limits. The known examples of families having the \((SARP^+)\) are related to the \(p\)-norms:

- \(\{\ell^n_p\}_n\) for all \(1 \leq p \leq \infty\): For \(p = 2\), this is a consequence of the fact, via the Kechris-Pestov-Todorcevic (KPT) correspondence (see [4, Theorem 2.12], [6, Theorem 5.10]), that the unitary group \(\text{Iso}(\ell_2)\) is extremely amenable, proved by M. Gromov and V. Milman [10], and the fact that \(\{\ell^n_2\}_n\) is an amalgamation class (see for example [6, Example 2.4.]). The case \(1 \leq p \neq 2 < \infty\) follows from the approximate Ramsey property of this class, proved in [6] and the result of G. Schechtman in [17] stating that \(\{\ell^n_p\}_n\) are amalgamation classes. The case \(p = \infty\) is proved in [4] (see also [2]) using the dual Ramsey Theorem.

- \(\text{Age}(L_p[0,1])\) for \(p \neq 2\mathbb{N} + 4\): This is a byproduct of the extreme amenability of \(\text{Iso}(L_p[0,1])\), proved by T. Giordano and V. Pestov in [7], the (KPT) correspondence, and the fact that \(\text{Age}(L_p[0,1])\) is an amalgamation class, proved in [6]. On the other direction, when \(p \in 2\mathbb{N} + 4\), it is shown in [6, Proposition 2.10.] that \(\text{Age}(L_p[0,1])\) does not have the \((SARP^+)\) because in these spaces there are finite dimensional isometric subspaces, one well complemented and the other badly complemented.

- \(\mathcal{F} = \text{Age}(C[0,1])\): This is proved in [4] (see also [2]), directly using injective envelopes and some approximations, or as a byproduct of the \((SARP^+)\) of the family \(\{\ell^n_p\}_n\) and the Kechris-Pestov-Todorcevic correspondence for Banach spaces.

The \((SARP^+)\) characterizes norms on \(F^\infty\) that produce Ramsey factors.
Theorem 3.3. Let $\mathbf{m}$ be a norm on $\mathbb{F}^\infty$, $E := (\mathbb{F}^\infty, \mathbf{m})$.

a) If $\text{Age}(E)$ has the (SARP$^+$), then $\mathbf{m}$ produces Ramsey factors for colorings of full-rank matrices, Grassmannians and square matrices.

b) If $\mathbf{m}$ produces Ramsey factors for colorings of full rank matrices, $\text{Age}(E)$ has the (SARP$^+$).

To prove b) we will use the following.

Lemma 3.4. Let $\mathbf{m}$ be a norm on $\mathbb{F}^\infty$ that produces Ramsey factors for colorings of full rank matrices, set $E := (\mathbb{F}^\infty, \mathbf{m})$. For every $k, m, r \in \mathbb{N}$, $\varepsilon > 0$, and $\lambda \in ]1, \infty[ $ there is some $n \in \mathbb{N}$ such that for every discrete coloring $c : M_{n,k}^k(m; \lambda) \to r$ there is some $R \in \mathcal{E}_{n,m}(\mathbf{m})$ such that

$$R \cdot B \in \langle \varepsilon^{-1}(c(R \cdot A)) \rangle_{\nu_{\mathbf{m},E}(A), \nu_{\mathbf{m},E}(B) + \varepsilon} \text{ for every } A, B \in M_{n,k}^k(m; < \lambda) \quad (3)$$

Proof. Fix the parameters $k, m, r \in \mathbb{N}$, $\varepsilon$, and $\lambda$. Let $n \in \mathbb{N}$ be the outcome of property ii) in Definition 2.6 when applied to $k, m, \varepsilon/2$, and the compact metric space $K := 2A_1 \text{Ball}(\ell_\infty^r)$. We claim that $n$ works. Fix $c : M_{n,k}^k(m; \lambda) \to r$, and let $f : M_{n,k}^k(m; \lambda) \to K$, $f(A) := (d(A, c^{-1}(j)))_{j < r}$. It is clear that $f$ is $1$-Lipschitz, so there is some $R \in \mathcal{E}_{n,m}(\mathbf{m})$ and $\hat{f} : \mathcal{N}_k(E; \lambda) \to K$ such that $d_{\mathcal{N}_k}(\hat{f}(\nu_{\mathbf{m},E}(A)), f(R \cdot A)) \leq \varepsilon/2$ for every $A \in M_{n,k}^k(m; < \lambda)$. Fix $A, B \in M_{n,k}^k(m; < \lambda)$. Then, $d_{K}(f(R \cdot A), f(R \cdot B)) \leq d_{\mathcal{N}_k}(\nu_{\mathbf{m},E}(A), \nu_{\mathbf{m},E}(B)) + \varepsilon$. Thus, if $j := c(R \cdot A)$, then the $j$th-coordinate of $f(R \cdot A)$ is zero, hence, the $j$th-coordinate of $c(R \cdot B)$ must satisfy that $d(R \cdot B, c^{-1}(j)) \leq d_{\mathcal{N}_k}(\nu_{\mathbf{m},E}(A), \nu_{\mathbf{m},E}(B)) + \varepsilon$, as desired. \hfill $\Box$

Proof of b) of Theorem 3.3. The proof of a) is more involved, and it will be done in several steps later. Let $\mathcal{F}$ be the collection of normed spaces of the form $(\mathbb{F}^k, \mathbf{n})$ with $\mathbf{n} \in \mathcal{N}_k(E)$ and such that $\omega(\mathbf{n}, \| \cdot \|_1) \leq k$. Since the diameter of the Banach-Mazur compactum $B_k$ is at most $k$, it follows that $\mathcal{F} \equiv \text{Age}(E)$, so the (SARP$^+$) of $\mathcal{F}$ and of $\text{Age}(E)$ are equivalent. Moreover, we prove the following equivalent discrete version of the (SARP$^+$) (see [4], [6]):

For every $k$ and $\varepsilon > 0$ there is a $\delta > 0$ such that for every $r \in \mathbb{N}$ and $X, Y \in \mathcal{F}$ with dim $X = k$ there is $Z \in \mathcal{F}$ such that every discrete coloring $c : \text{Emb}_d(X, Z) \to r$ has an $\varepsilon$-monochromatic set of the form $R \circ \text{Emb}_d(X, Y)$ for some $R \in \text{Emb}(Y, Z)$.

Fix a dimension $k$ and $\varepsilon > 0$. Notice that the collection of spaces in $\mathcal{F}$ of dimension $k$ is a $\omega$-bounded set, so, by hypothesis, the metrics $\nu_{\mathbf{m},E}(A)$ and $\omega$ are uniformly equivalent on $\mathcal{M} := \{ \mathbf{n} \in \mathcal{N}_k(E) : (\mathbb{F}^k, \mathbf{n}) \in \mathcal{F} \}$. Let $\delta > 0$ be such that such that $\mathbf{n}, \mathbf{p} \in \mathcal{M}$ are such that $\omega(\mathbf{n}, \mathbf{p}) < \delta$, then $d_{\mathcal{N}_k}(\nu_{\mathbf{m},E}(A), \nu_{\mathbf{m},E}(B)) \leq \varepsilon/2$. We claim that $\mathcal{M}$ works. For suppose that $X = (\mathbb{F}^k, \mathbf{n})$, $Y = (\mathbb{F}^m, \mathbf{p}) \in \mathcal{F}$ are such that $\text{Emb}(X) \neq \emptyset$, and $r \in \mathbb{N}$. Let $m_0 \geq m$ and $C \in \mathcal{M}_{m_0,m}$ be such that $\mathbf{n} = \nu_{\mathbb{F}^m,\mathbb{K}^{m_0,m_0}}(C)$, and let $\lambda > 1$ be such that $T_{C} \circ \text{Emb}_d(X, Y) \subseteq \{ T_{B} : B \in M_{m_0,k}^k(m; < \lambda) \}$. We use Lemma 3.4 for the parameters $k, m_0, r + 1, \varepsilon/2$ and $\lambda$ to find a corresponding $n$; set $Z := (\mathbb{F}^m, \mathbf{m})$. Now suppose that $c : \text{Emb}_d(X, Z) \to r$. We define $\tilde{c} : M_{n,k}^k(m; \lambda) \to r + 1$ by $\tilde{c}(A) = c(T_A)$ if $T_A \in \text{Emb}_d(X, Z)$ and by $\tilde{c}(A) = r$ otherwise. Let $R \in \mathcal{E}_{n,m_0}(\mathbf{m})$ be such that (3) holds. Let $\gamma := T_{R} \circ T_{C} \in \text{Emb}(Y, Z)$. We see that $\gamma \circ \text{Emb}_d(X, Y)$ is $\varepsilon$-monochromatic for $c$: Fix $\eta \in \text{Emb}(X, Y)$, and let $A \in M_{n,k}^k(m; < \lambda)$ be such that $T_A = T_{C} \circ \eta$, and let $j := c(T_{R} \circ T_{C} \circ \eta) = \tilde{c}(R \cdot A)$. Now given $\xi \in \text{Emb}_d(X, Y)$, let $B \in M_{n,k}^k(m; < \lambda)$ be such that $T_B = T_{C} \circ \xi$. Then, $\mathbf{n} := \nu_{\mathbb{F}^k,E}(A)$, $\mathbf{p} := \nu_{\mathbb{F}^k,E}(B)$ and therefore $\omega(\nu_{\mathbb{F}^k,E}(A), \nu_{\mathbb{F}^k,E}(B)) \leq \delta$, hence, $d_{\mathcal{N}_k}(\nu_{\mathbb{F}^k,E}(A), \nu_{\mathbb{F}^k,E}(B)) \leq \varepsilon/2$. This together with (3) gives that $R \cdot B = (\tilde{c}^{-1}(j))_{\nu_{\mathbb{F}^k,E}(B)}$, so there must be $D \in M_{n,k}^k(m; \lambda)$ such that $\tilde{c}(D) = j$ and such that $\| T_D - \gamma \circ \xi \| = \| T_D - T_{R} \circ T_{C} \| = d_n(D, R \cdot B) \leq \varepsilon$; since $j < r$, $T_D \in \text{Emb}(X, Z)$, so $c(T_D) = \tilde{c}(E) = j$ and we are done. \hfill $\Box$

The proof of b) of Theorem 3.3 has two main parts. The first one (Theorem 3.8) uses the fact that if $\mathcal{F}$ has the (SARP$^+$) and it is hereditary, then the isometry group $G$ of the Fraïssé limit $\text{FLim} \mathcal{F}$ is extremely amenable with its topology of pointwise convergence. This property will be used as infinitary principles can be used to conclude, via compactness arguments, the
finitary ones (e.g. infinite vs finite Ramsey, Hindman vs Folkman theorem, etc.). The fix point property of \( G \) will naturally provide abstract Ramsey factors that are \( G \)-quotients. The second part of the argument is to see that these \( G \)-quotients are in fact the desired Ramsey factors.

Recall that a topological group \( G \) is called extremely amenable when every continuous action of \( G \) on a compact Hausdorff space has a fixed point. There is a useful characterization of extreme amenability in terms of factors through quotients that we pass to explain.

Let \( (M, d) \) be a metric space, and let \( G \curvearrowright M \) be a continuous action by isometries. We write \([p]_G\) to denote the closure of the \( G \)-orbit of \( p \in M \), and \( M//G \) to denote the space of closures of \( G \)-orbits of \( M \). Since \( G \) acts by isometries the formula
\[
\overline{d}^G([p],[q]) := \inf\{d_M(p_0,q_0) : p_0 \in [p], q_0 \in [q]\}
\]
defines the quotient pseudometric induced by the quotient map \( \pi : M \to M//G \), and as we consider closures of orbits, \( \overline{d}^G \) is a metric. It is easily seen that \( \overline{d}^G \) is complete if \( d \) is complete.

Given a compact metric space \((K,d_K)\), let \( \text{Lip}((M,d_M),(K,d_K)) \) be the collection of all \( K \)-colorings of \( M \). With the topology of pointwise convergence \( \text{Lip}((M,d_M),(K,d_K)) \) is a compact space, which is metrizable when \((M,d_M)\) is separable. The continuous action \( G \curvearrowright \text{Lip}((M,d_M),(K,d_K)) \) induces a natural continuous action \( G \curvearrowright \text{Lip}((M,d_M),(K,d_K)) \), defined by setting \((g \cdot c)(p) := c(g^{-1} \cdot p)\) for every \( c \in \text{Lip}((M,d_M),(K,d_K)) \) and \( p \in M \). This is the aforementioned characterization (see [4]).

**Proposition 3.5.** Suppose that \( G \) is a Polish group, and \( d_G \) is a left-invariant compatible metric on \( G \). The following assertions are equivalent.

i) \( G \) is extremely amenable.

ii) The left translation of \( G \) on \((G,d_G)\) is finitely oscillation stable [16, Definition 1.1.11], that is, for every continuous coloring \( c : M \to [0,1] \) and every \( F \subseteq M \) finite and \( \varepsilon > 0 \) there is some \( g \in G \) such that \( \text{Osc}(c \mid g \cdot F) \leq \varepsilon \).

iii) For every action \( G \curvearrowright M \) of \( G \) on a metric space \((M,d_M)\), and for every compact coloring \( c : (M,d_M) \to (K,d_K) \) of \((M,d_M)\), there exists a compact coloring \( \hat{c} : M//G \to K \) such that for every finite \( F \subseteq M \) and \( \varepsilon > 0 \) there is some \( g \in G \) such that \( d_K(c(p),\hat{c}([p]_G)) < \varepsilon \) for every \( p \in g \cdot F \).

iv) The same as iii) where \( F \) is compact.

*Proof.* The equivalence of i) and ii) can be found in [16, Theorem 2.1.11]. The implication iii)⇒ii) is immediate, since orbit space \( G//G \) is one point. We now establish the implication i)⇒iv): Fix a 1-Lipschitz \( c : (M,d_M) \to (K,d_K) \). Let \( L \) be the closure of the \( G \)-orbit of \( c \) in \( \text{Lip}((M,d_M),(K,d_K)) \). By the extreme amenability of \( G \), there is some \( c_x \in L \) such that \( G \cdot c_x = \{c_x\} \), so we can define the quotient \( K \)-coloring \( \hat{c}([p]_G) := c_x(p) \). Given a compact subset \( F \) of \( M \), let \( g \in G \) be such that \( \max_{p \in F} d_K(c_x(p),c(g \cdot p)) < \varepsilon \). If \( x \in F \), then \( d_K(c(g \cdot x),\hat{c}([g \cdot x]_G)) = d_K(c(g \cdot x),c_x(x)) < \varepsilon \).

We apply this characterization to groups of linear isometries of a Banach space. Given two Banach spaces \( X \) and \( Y \), recall that \( \mathcal{L}(X,Y) \) is the Banach space of all bounded linear operators \( T : X \to Y \), endowed with the operator norm \( \|T\| := \sup_{\|x\| = 1} \|Tx\|_Y \), and when \( \text{Im}(T) \) of \( T \) is finite-dimensional, \( \|T\| = \|T^{-1}\| = \min \{a \geq 0 : \text{Ball}(TX) \subseteq aT(\text{Ball}(X))\} \). This case, \( \|T^*\| = \|T\| \) and \( \|T^{-1}\| = \|T\|^{-1} \). The special case when \( T : X \to Y \) is 1-1 and \( \|T\| = \|T^{-1}\| = 1 \) corresponds to \( T \) being an isometric embedding. The collection of such maps is denoted by \( \text{Emb}(X,Y) \). Let \( \mathcal{L}_\lambda(X,Y), \mathcal{L}_{<\lambda}(X,Y) \), be the set of all \( T \in \mathcal{L}(X,Y) \) with finite dimensional image such that \( \|T\|, \|T^{-1}\| \leq \lambda \), resp. \( < \lambda \). Let \( \mathcal{L}^k(X,Y) \) be the set of all \( T \in \mathcal{L}(X,Y) \) whose image is \( k \)-dimensional, and let \( \mathcal{L}_\lambda^k(X,Y) = \mathcal{L}_\lambda(X,Y) \cap \mathcal{L}^k(X,Y), \mathcal{L}_{<\lambda}^k(X,Y) = \mathcal{L}_{<\lambda}(X,Y) \cap \mathcal{L}^k(X,Y) \). Let \( \mathcal{L}^k_{\text{w}*}(X^*,X) \) be the metric space of operators \( T \in \mathcal{L}^k(X^*,X) \) such that \( T \) admits a full rank decomposition, i.e., when \( T = T_0 \circ T_1^* \) for some \( T_0,T_1 \in \mathcal{L}^k(\mathbb{R}^k,X) \). It is
an exercise to prove that this is equivalent to saying that $T$ is a $w^\ast$-to-norm continuous linear operators from $X^*$ to $X$ of rank $k$; let $L^{k,w^\ast}_x(X^*,X) := L^{k,w^\ast}_x(X^*,X) \cap \Lambda(x,X^*,X)$.

**Definition 3.6.** Let $\text{Iso}(E) \cap \mathcal{L}(X,E)$ be the canonical action by isometries $g \cdot T := g \circ T$, $\text{ Iso}(E)^2 \cap \mathcal{L}^{k,w^\ast}(E^*,E)$ be the canonical action by isometries $(g,h) \cdot T := g \circ T \circ h^*$ for $(g,h) \in \text{ Iso}(E)^2$ and $T \in \mathcal{L}^{k,w^\ast}(E^*,E)$, and let $\text{Iso}(E) \cap \text{Gr}(k,E)$ be the canonical action by isometries $g \cdot V := g(V)$.

Note that $L^k_x(X,E)$, $L^{k}_{<\lambda}(X,E)$, and $L^{k,w^\ast}_{<\lambda}(X^*,X)$, $L^{k,w^\ast}_{<\lambda}(X^*,X)$ are $\text{Iso}(E)$-closed and $\text{Iso}(E)^2$-closed, respectively. The next readily follows from Proposition 3.5, using the fact that if $G$ is extremely amenable, then $G^2$ is also extremely amenable (see [16, Corollary 6.2.10]).

**Lemma 3.7.** Suppose that $X,E$ are Banach spaces, $X$ is finite-dimensional, and $\text{Iso}(E)$ is extremely amenable. Let $k \in \mathbb{N}$, $\varepsilon > 0$, $1 \leq \lambda$, $Y \in \text{Age}(E)_\varepsilon$, and let $(K,d_K)$ be a compact metric space.

a) For every $K$-coloring $c$ of $(L^k_{<\lambda}(X,E),d_{X,E})$ there is $R \in \text{Emb}(Y,E)$ such that the quotient map $\pi : L^k_{<\lambda}(X,E) \to L^k_{<\lambda}(X,E) \mod \text{Iso}(E)$ is an $\varepsilon$-factor of $c$ in $R \circ L^k_{<\lambda}(X,Y)$.
b) For every $K$-coloring $c$ of $(\text{Gr}(k,E),\Lambda_E)$ there exists $V \in \text{Gr}(\dim Y,E)$ with $(V,\|\cdot\|_E)$ is isometric to $Y$ such that the quotient map $\pi : \text{Gr}(k,E) \to \text{Gr}(k,E) \mod \text{Iso}(E)$ is an $\varepsilon$-factor of $c$ in $\text{Gr}(k,V)$.
c) For every $K$-coloring $c$ of $(L^{k,w^\ast}_{<\lambda}(E^*,E),d_{E^*,E})$ there are $R_0,R_1 \in \text{Emb}(Y,E)$ such that the quotient map $\pi : L^{k,w^\ast}_{<\lambda}(E^*,E) \to L^{k,w^\ast}_{<\lambda}(E^*,E) \mod \text{Iso}(E)^2$ is an $\varepsilon$-factor of $c$ in $R_0 \circ L^{k,w^\ast}_{<\lambda}(Y^*,Y) \circ R_1^\ast$.

The relationship between the $(\text{SARP}^+)$ of a class of finite dimensional normed spaces and the extreme amenability of the isometry group of its Fraïssé limit is the next mix of the Fraïssé and the Kechris-Pestov-Todorcevic correspondences, that we took from [6, Corollary 5.11].

**Theorem 3.8.** If $\mathcal{G}$ is an hereditary family with the $(\text{SARP}^+)$, then the Banach-Mazur closure of $\mathcal{G}$ also has the $(\text{SARP}^+)$ and the Fraïssé limit $\text{FLim(}\mathcal{G})$ is a Fraïssé Banach space whose isometry group is extremely amenable with its strong operator topology.

**Definition 3.9.** Given a normed space $E := (\mathbb{F}^\infty,m)$ such that $\text{Age}(E)$ is an amalgamation class, we write $\hat{E}$ to denote, the Fraïssé limit $\text{FLim(}\text{Age}(E))$.

We will use the following notation. Given a normed space $E = (\mathbb{F}^\infty,m)$ and $n \in \mathbb{N}$, we set $E_n := (\langle u_j \rangle_{j<n},m)$, and given a normed space $X$ we write $\text{Age}(X)_n$ to denote the collection of subspaces of $X$ isometric to some $E_n$. The following is the asymptotic version of Lemma 3.7.

**Corollary 3.10.** Suppose that $E = (\mathbb{F}^\infty,m)$ is such that $\text{Age}(E)$ has the $(\text{SARP}^+)$, and $\text{Age}(\hat{E})_n$ is an amalgamation class. Let $k,m \in \mathbb{N}$, $\varepsilon > 0$ and $1 < \lambda$. Given a compact metric space $(K,d_K)$ there is $X \in \text{Age}(\hat{E})_\varepsilon$ such that

1) for every $K$-coloring $c$ of $L^k_x(E_k,X)$ there is $R \in \text{Emb}(E_m,X)$ such that the quotient map $\pi : L^k_x(E_k,\hat{E}) \to L^k_x(E_k,\hat{E}) \mod \text{Iso}(\hat{E})$ is an $\varepsilon$-factor of $c$ in $R \circ L^k_x(E_k,E_m)$;
2) for every $K$-coloring $c$ of $(\text{Gr}(k,X),\Lambda_X)$ there is $V \in \text{Gr}(m,X) \cap \text{Age}(\hat{E})_\varepsilon$ such that the quotient map $\pi : \text{Gr}(k,\hat{E}) \to \text{Gr}(k,\hat{E}) \mod \text{Iso}(\hat{E})$ is an $\varepsilon$-factor of $c$ in $\text{Gr}(k,V)$;
3) for every $K$-coloring $c$ of $L^{k,w^\ast}_{<\lambda}(X^*,X)$ there are $R_0,R_1 \in \text{Emb}(E_m,X)$ such that the quotient map $\pi : L^{k,w^\ast}_{<\lambda}(\hat{E}^*,\hat{E}) \to L^{k,w^\ast}_{<\lambda}(\hat{E}^*,\hat{E}) \mod \text{Iso}(\hat{E})^2$ is an $\varepsilon$-factor of $c$ in $R_0 \circ L^{k,w^\ast}_{<\lambda}(E_m^*,E_m) \circ R_1^\ast$.

**Proof.** For each $X \in \text{Age}(\hat{E})$, and each $\varepsilon > 0$, let $A_{X,\varepsilon}$ be the collection of all $Y \in \text{Age}(\hat{E})_\varepsilon$ such that $X \subseteq \varepsilon Y$. Then $\{A_{X,\varepsilon}\}_{\varepsilon,\lambda}$ is a family of subsets of $\text{Age}(\hat{E})_\varepsilon$ with the finite intersection property. Let $U$ be a non principal ultrafilter on $\text{Age}(\hat{E})_\varepsilon$ containing all $A_{X,\varepsilon}$. Now suppose for the sake of contradiction, that, for some compact space $(K,d_K)$, there is no such
$X \in \text{Age}(\hat{E})_n$ satisfying 1), 2) or 3). Since $\mathcal{U}$ is an ultrafilter there is $j = 1, 2, 3$ such that the set $\mathcal{M}_j := \{ X \in \text{Age}(\hat{E})_n : X$ does not satisfy $j \}$ belongs to $\mathcal{U}$. Suppose that $j = 1$. For each $X \in B_1$ there exists a continuous $c_X : \mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, X) \to K$ providing a counterexample. For each $T \in \mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, \hat{E})$, let $c(T) \in K$ be defined as follows. We say that $c(T) = x \in K$ if and only if for every $\varepsilon > 0$ one has that $\{ Y \in C_{T, \varepsilon} : \text{such that } T \in (\mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, Y))_\varepsilon \}$ and $d_K((T)_\varepsilon \cap \mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, Y), x) \leq \varepsilon \}$ belongs to $\mathcal{U}$. This is well defined because $K$ is compact and the set of $Y \in B_1$ such that $T \in (\mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, Y))_\varepsilon$ belongs to $\mathcal{U}$. It is easy to see that $c$ defines a coloring, i.e., that $c$ is 1-Lipschitz. Let $\pi : \mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, \hat{E}) \to \mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, \hat{E})/\text{Iso}(\hat{E})$ be the quotient mapping. By Lemma 3.7 there exist $S \in \text{Emb}(E_m, \hat{E})$ and a coloring $\tilde{c} : (\mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, \hat{E})/\text{Iso}(\hat{E}), \tilde{d}) \to (K, d_K)$ such that $d_K(c(S \circ T), \tilde{c}(\pi(T))) \leq \varepsilon/2$ for every $T \in \mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, E_m)$. Since $\text{Age}(\hat{E})_n$ is an amalgamation class, the set $C$ of $Y \in \text{Age}(\hat{E})_n$ such that there is $S_Y \in \text{Emb}(E_m, Y)$ such that $\|S_Y - S\| \leq \varepsilon/3$ belongs to $\mathcal{U}$, and since $\mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, E_m)$ is pre-compact, the set $D$ of those $Y \in C$ such that $\max\{d_K(c(Y \circ S_Y \circ T), c(S \circ T)) : T \in \mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, E_m)\} \leq \varepsilon/2$ also belongs to $\mathcal{U}$. Then $Y \in D$, $S_Y$ and $\tilde{c}$ contradicts the assumption that $c_Y$ is a counterexample. Fix $T \in \mathcal{L}^k_{\lambda, \mathcal{U}}(E_k, E_m)$. It follows that
\[
d_K(\tilde{c}(\pi(T)), c_Y(S_Y \circ T)) = d_K(\tilde{c}(\pi(S_Y \circ T)), c_Y(S_Y \circ T)) \leq d_K(\tilde{c}(\pi(S \circ T)), c(S \circ T)) + d_K(c_Y(S_Y \circ T), c(S \circ T)) \leq \varepsilon
\]
The cases $j = 2, 3$ are proved similarly, so we leave the details to the reader. $\Box$

3.1. Orbit spaces for Fraïssé Banach spaces. We see that the orbit spaces considered in Corollary 3.10 1), 2), and 3), are homeomorphic to $\mathcal{N}_k(\hat{E})$, $\mathcal{B}_k(\hat{E})$ and $\mathcal{D}_k(\hat{E})$, respectively. We also show that the $\hat{E}$-extrinsic metrics extend the corresponding $E$-extrinsic ones, finishing the proof of Theorem 3.3 a).

**Theorem 3.11.** Suppose $E = (\mathbb{F}^\infty, \| \cdot \|_E)$ is such that $\text{Age}(E)$ is an amalgamation class. Then,

a) $\hat{\mathcal{N}}_{\lambda, \mathcal{U}}(\hat{E})$ is a compatible metric on $\mathcal{N}_X(\hat{E})$ that is uniformly equivalent to $\omega$ on $\omega$-bounded sets,

$\hat{\mathcal{N}}_{\lambda, \mathcal{U}}(\hat{E})$ is a compatible metric on $\mathcal{N}_X(\hat{E})$, and $\mathcal{N}_X(\hat{E})$ is dense in $\hat{\mathcal{N}}_{\lambda, \mathcal{U}}(\hat{E})$ for every $X$.

b) $\gamma_{k, \mathcal{U}}(\hat{E})$ is a compatible metric on $\mathcal{B}_k(\hat{E})$. $\gamma_{k, \mathcal{U}}(\hat{E}) = \gamma_{k, \mathcal{U}}$ on $\mathcal{B}_k(\hat{E})$, and $\mathcal{B}_k(\hat{E})$ is dense in $\mathcal{B}_k(\hat{E})$.

c) $\mathcal{D}_{k, \mathcal{U}}(\hat{E})$ is a compatible metric on $\mathcal{D}_k(\hat{E})$ that is uniformly equivalent to $\omega_{2}$ on $\omega_{2}$-bounded sets,

$\mathcal{D}_{k, \mathcal{U}}(\hat{E}) = \mathcal{D}_{k, \mathcal{U}}(\mathcal{D}_k(\hat{E}))$, and $\mathcal{D}_k(\hat{E})$ is dense in $\mathcal{D}_k(\hat{E})$.

**Proof.** a): Suppose that dim $X = k$. Recall that we consider $\mathcal{N}_X$ with its natural topology of pointwise convergence. The mapping $\nu_{X, \hat{E}} : \mathcal{L}^k_{\lambda, \mathcal{U}}(X, \hat{E}) \to \mathcal{N}_X(\hat{E})$ is continuous, because the convergence in norm implies pointwise converge. We see that $\nu_{X, \hat{E}}(T) = \nu_{X, \hat{E}}(U)$ if and only if $[T] = [U]$. The reverse implication is clear; now suppose that $\nu_{X, \hat{E}}(T) = \nu_{X, \hat{E}}(U)$. Let $Y := T(X)$ be endowed with the $\hat{E}$-norm, and let $\theta : Y \to X$ be the inverse isometry of $T : X \to Y$. Then $U \circ \theta \in \text{Emb}(Y, E)$; so, given $\varepsilon > 0$, there is a global isometry $\alpha$ of $E$ such that $\|U \circ \alpha - \alpha \circ Y \| \leq \varepsilon$, or equivalently, $\|U - \alpha \circ T \circ \theta - \alpha \circ \theta \| \leq \varepsilon$. Since $\varepsilon$ is arbitrary, we obtain that $U \circ \theta \in [T]$. We show that $\nu_{X, \hat{E}}$ is a homeomorphism. Suppose that $(m_j)_j$ is a converging sequence in $\mathcal{N}_X(\hat{E})$ with limit $m \in \mathcal{N}_X(\hat{E})$. For each $j$, let $T_j \in \mathcal{L}^k_{\lambda, \mathcal{U}}(X, \hat{E})$ be such that $\nu_{X, \hat{E}}(T_j) = m_j$, and let $T \in \mathcal{L}^k_{\lambda, \mathcal{U}}(X, \hat{E})$ be such that $\nu_{X, \hat{E}}(T) = m$.

**Claim 3.11.1.** $(\{T_j\})_j$ is a Cauchy sequence.

Notice that it follows from this, and the fact that the quotient metric $\hat{d}_{X, \hat{E}}$ is complete (here we use that $X$ and $\hat{E}$ are Banach spaces), that $(\{T_j\})_j$ converges to some $[U]$; by the continuity of $\nu_{X, \hat{E}}$ we have that $\nu_{X, \hat{E}}(U) = m = \nu_{X, \hat{E}}(T)$, so $(\{T_j\})_j$ converges to $[U] = [T]$. Also, given a bounded subset $A$ of $\mathcal{N}_X(\hat{E})$, its closure $\text{cl}(A)$ is compact, so $\nu_{X, \hat{E}}$ and $\omega$ are uniformly equivalent on $\text{cl}(A)$, thus also on $A$. 

\[ \]
Let us prove the previous claim: Set $Y := T(X)$, normed as subspace of $\hat{E}$, and let $\theta : Y \to X$ be the inverse isometry of $T : X \to Y$, and fix $\varepsilon > 0$; since $\hat{E}$ is weak-Fraïssé, there is some $\delta > 0$ such that the canonical action $\text{Iso}(\hat{E}) \cap \text{Emb}_{\delta}(Y, \hat{E})$ is $\varepsilon$-transitive; let $j_0$ be such that $T_j \circ \theta \in \text{Emb}_{\delta}(Y, \hat{E})$ for every $j \geq j_0$; this means that if $j_1, j_2 \geq j_0$, then there is $\alpha \in \text{Iso}(\hat{E})$ such that $\|T_{j_1} - \alpha \circ T_{j_2}\| = \|T_{j_1} \circ \theta - \alpha \circ T_{j_2} \circ \theta\| \leq \varepsilon$, hence $\hat{d}_{X,E}([T_{j_1}], [T_{j_2}]) \leq \varepsilon$.

Let us prove now that $\hat{d}_{X,E}(m, n) = \hat{d}_{X,E}(m, n)$: Since $E$ is isometrically embedded into $\hat{E}$, we have that $\hat{d}_{X,E}(m, n) \leq \hat{d}_{X,E}(m, n)$. Now suppose that $\nu_{X,E}(T) = m$, $\nu_{X,E}(U) = n$, and $\varepsilon > 0$. To simplify the notation, let $X_n := (X, m)$ and $X_n := (X, n)$. We use the fact that $\text{Age}(E)$ is an amalgamation class to find $\delta > 0$ such that for every $Y, Z, V$ that can be isometrically embedded into $E$, with $\dim Y = k$, and every $\gamma \in \text{Emb}_\delta(Y, Z)$, there is $\hat{y} \in \text{Emb}_\delta(Y, V)$ such that $\|\hat{y} \circ \gamma - \hat{y}\| \leq \varepsilon$. Since $\text{Age}(E)_{\hat{E}}$ is $\Lambda_{\hat{E}}$-dense in $\text{Age}(\hat{E})$, we can find $Z \in \text{Age}(E)_{\hat{E}}$ such that there is $\theta \in \text{Emb}_\delta(Y, Z)$ such that $\|\theta - \hat{y}_{X,E}\| \leq \varepsilon$, where $Y := \text{Im} T + \text{Im} U$. Then $T_0 := \theta \circ T \in \text{Emb}_\delta(X_n, Z)$, $U_0 := \theta \circ U \in \text{Emb}(X_n, Z)$ and setting $K_n := \|\text{Id}_X\|_{X, X_n}$, $K_n := \|\text{Id}_X\|_{X, X_n}$, $\|T_0 - U_0\|_{X, Z} \leq \|T_0 - U_0\|_{X_n, K_n} + \|T - U\|_{X, \hat{E}} + \|U - U_0\|_{X_n, K_n} \leq \varepsilon(K_n + K_n) + \|T - U\|_{X, \hat{E}}$.

We use now that $\text{Age}(E)$ has the amalgamation property to find $V \in \text{Age}(E)_{\hat{E}}$ and $I \in \text{Emb}(Z, V)$, $T_1 \in \text{Emb}((X, m), V)$ and $T_1 \in \text{Emb}((X, n), V)$ such that $\|T_1 - I \circ T_0\|_{X_n, V}, \|U_1 - I \circ U_0\|_{X_n, V} \leq \varepsilon$. Thus,

$$
\|T_1 - U_1\|_{X, V} = \|T_1 - I \circ T_0\|_{X_n, E} K_n + \|I \circ T_0 - I \circ T_1\|_{X, \hat{E}} + \|U_1 - I \circ U_0\|_{X_n, E} K_n \leq (2 + K_n + K_n) + \|T - U\|_{X, \hat{E}}.
$$

Since $V \in \text{Age}(E)_{\hat{E}}$, and $\varepsilon > 0$ is arbitrary, we obtain that $\hat{d}_{X,E}(m, n) \leq \|T - U\|_{X, \hat{E}}$. Finally, because $\text{Age}(E)_{\hat{E}}$ is $\Lambda_{\hat{E}}$-dense in $\text{Age}(\hat{E})$, it follows that $\bigcup_{Y \in \text{Age}(E)_{\hat{E}}} L^k(X, Y)$ is dense in $\mathcal{L}^k(X, Y)$, so, $N_{X}(E)$ is dense in $N_{X}(\hat{E})$.

b): $\tau_{k, \hat{E}}$ is continuous: suppose that $V_n \to V$ for $n \to \infty$ in $\text{Gr}(k, E)$ in the opening metric $\Lambda_{\hat{E}}$. Let $T \in \mathcal{L}^k(F, E)$ be such that $\text{Im } T = V$, and for each $i < k$ and $n$ choose $x^i_n \in B_{V_n, |x^i|}$ such that $\|x^i_n - T(u_i)\| \to 0$ for $n \to \infty$. It is clear that, for $n$ large enough, $(T(x^i_n))_{i<k}$ are linearly independent, so the mapping $T_n : F^k \to E$, $T_n(u_i) = x^i_n$, belongs to $\mathcal{L}^k(F^k, E)$ and satisfies that $d_{X,E}(T_n - T) \to 0$ for $n \to \infty$, where $X = (F^k, \nu_{\mathcal{P}_k, E}(T))$. It follows from the continuity of $\nu_{X,E}$ that $\nu_{X,E}(T_n) \to \nu_{X,E}(T)$, so $\tau_{k, \hat{E}}(V_n) \to \tau_{k, \hat{E}}(V)$ for $n \to \infty$.

Suppose that $\tau_{k, E}(V) = \tau_{k, E}(W)$. By the approximately ultrahomogeneity of $E$, for a given $\varepsilon > 0$ we can find an isometry $g \in \text{Aut}(E)$ such that $\Lambda_{E}(V, g \cdot W) < \varepsilon$, and hence $V \in [W]$. The fact that $\hat{T}_{k, \hat{E}}$ is a homeomorphism follows from a).

c): We start with the continuity of $\nu_{\mathcal{P}_k, E}^2$. Suppose that $T_n \to T$ in norm. Then, $\text{Im}(T_n) \to \text{Im}(T)$ in the opening distance $\Lambda_{E}$. Now fix a basis $(e_j)_{j<k}$ of $\text{Im } T$, and let $(x_j)_{j<k}$ be a linearly independent sequence in $E$ such that $T = T_0 \circ T_1$, where $T_0 : F^k \to E$ is defined by $T_0(u_j) = e_j$ and $T_1 : (F^k)^* \to E$ by $T_1(u^*_j) = x_j$. For large enough $n$ choose a basis $(e^*_j)_{j<k}$ of $\text{Im } T_n$ such that $e^*_j \to e_j$ for every $j < k$. Similarly, we define $T_0^* : F^k \to (F^k)^*$, $T_0^*(u_j) := e^*_j$, and let $T_1^* : (F^k)^* \to E$ be such that $T_n = T_0^* \circ (T_1^*)^*$. Then $T_0^* \to T_0$, so by continuity of $\nu_{\mathcal{P}_k, E}$, it follows that $\nu_{\mathcal{P}_k, E}(T_0^*) \to \nu_{\mathcal{P}_k, E}(T_0)$. On the other hand, $T_0^*, T_0^*$ are $1$-1, so

$$
\|(T_0^*)^* - (T_1^*)^*\| \leq \|T_0^* - T_1^*\| \leq \|T_0^* - T_0\| + \|T_0^* - T_0\| \leq \|T_0\| - \|T_0^* - T_0\| + \|T_0\| - \|T_0\| = \|T_0\| - \|T_0^* - T_0\| = \|T_0\| - \|T_0\| = \|T_0\| - \|T_0\|.
$$

This implies that $T_0^* \to T_1^*$, and $\nu_{\mathcal{P}_k, E}^2(T_0^*) \to \nu_{\mathcal{P}_k, E}^2(T_1^*)$.

Suppose that $\nu_{\mathcal{P}_k, E}^2(T) = \nu_{\mathcal{P}_k, E}^2(U)$. Decompose $T = T_0 \circ T_1^*$ and $U = U_0 \circ U_1^*$ in a way that $\nu_{\mathcal{P}_k, E}(T_0) = \nu_{\mathcal{P}_k, E}(U_0)$ and $\nu_{\mathcal{P}_k, E}(T_1) = \nu_{\mathcal{P}_k, E}(U_1)$. As in the proof of 1), we can find
$g, h \in \text{Iso}(E)$ such that $\|g \circ T_0 - U_0\| \leq \varepsilon/(2\|T_1\|)$ and $\|h \circ T_1 - U_1\| \leq \varepsilon/(2\|U_0\|)$. Hence,
\[
\|g \circ T \circ h^* - U\| \leq \|g \circ T_0 - U_0\| \cdot \|T_1\| + \|h \circ T_1 - U_1\| \cdot \|U_0\| \leq \varepsilon.
\]
Since $\varepsilon > 0$ is arbitrary, $[U] = [T]$. We see now that $\tilde{\varphi}_{k,E}^2([T])$ is a homeomorphism. Suppose that $\tilde{\varphi}_{k,E}^2([T]) \to [\tilde{\varphi}_{k,E}^2([T])] \in D_k(E)$. Our goal is to find a subsequence of $(\{[T_n]\})_n$ that converges to $[T]$: We first decompose $T = T_0 \circ T_1$, with $T_0 \in \mathcal{L}^k(\mathbb{F}^k, E)$ and $T_1 \in \mathcal{L}^k((\mathbb{F}^k)^*, E)$, a subsequence $(T_{mn})_n$ and decompositions $T_{mn} = T_{0mn} \circ T_{1mn}$ in a way that both $\omega(\nu_{E_k^*, E}(T_0^m), \nu_{E_k^*, E}(T_0^m)) < m^{-1}$ and $\omega(\nu_{\ell^\infty_k, E}(T_1^m), \nu_{\ell^\infty_k, E}(T_1^m)) < m^{-1}$ for every $m \in \mathbb{N}$. It follows from a) that $[T_0^m] \to [T_0]$ and $[T_1^m] \to [T_1]$. This easily implies that $[T_{mn}] \to [T]$. The fact that $\varphi_{k,E}$ is uniformly equivalent to $\tilde{\omega}_2$ on $\tilde{\omega}_2$-bounded sets follows from the Heine-Borel property of $(D_k, \tilde{\omega}_2)$.

We finish with the following fact on bounded sets considered before.

**Lemma 3.12.** Suppose that $X = (X, \mathbb{m})$ is a normed space with dim $X = k$, $E$ is a Banach space and $\lambda \geq 1$. We have that $\nu_{E_k, E}^2(\mathcal{L}^{k, w^*}_{\lambda, k}(E^*, E)) = D_k(E; \lambda)$ and $\nu_{E_k, E}^2(\mathcal{L}^{k, w^*}_{\lambda, k}(E^*, E)) = D_k(E; <\lambda)$.

**Proof.** We will use the following simple fact.

**Claim 3.12.1.** If $V$ is a vector space with dim $V = k$, then $(\nu_{E_k, E}(T))^* (f) = \min \{\|g\|_{E^*} : T^*(g) = f\}$ for every $T \in \mathcal{L}(k, E)$ and $f \in X^*$.

**Proof of Claim:** We know that $T : Y := (V, \mathbb{n}) \to E$ is an isometry for $n := \nu_{E_k, E}(T)$. Fix $f \in X^*$, set $Z := (T(Y), \|\cdot\|_{E})$, and $U : Z \to Y$ be the inverse of $T$. Let $g_0 \in Z^*$ be such that $U^*(g_0) = f$, and let $g \in E^*$ be such that $\|g\| = \|g_0\|$. It is easily seen that $T^*(g) = f$, and since $\|T^*\|_{E^*, Y^*} = \|T\|_{Y, E} = 1$, we obtain the desired equality.

Fix $([m_0, m_1]) \in D_k(E; \lambda)$, and choose $T_0 \in \mathcal{L}^k(\mathbb{F}^k, E)$ and $T_1 \in \mathcal{L}^k((\mathbb{F}^k)^*, E)$ such that $\nu_{E_k^*, E}(T_0) = m_0$ and $\nu_{\ell^\infty_k, E}(T_1) = m_1$. We claim that $T_0 \circ T_1 \in \mathcal{L}^{k, w^*}_{\lambda, k}(E^*, E)$. Given $\|g\|_{E^*} = 1$,
\[
\|T_0(T_1^*(g))\|_E = \|m_0(T_1^*(g))\|_E \leq \lambda m_1^*(T_1^*(g)) \leq \lambda \|g\|_{E^*},
\]
where the last inequality holds by Claim 3.12.1. Now suppose that $\|T_0(T_1^*(g))\|_E \leq \lambda^{-1}$. It follows that $m_0(T_1^*(g)) \leq \lambda^{-1}$, so $m_1^*(T_1^*(g)) \leq 1$. Hence, by Claim 3.12.1, there is $h \in E^*$ such that $T_1^*(h) = T_1^*(g)$ and $\|h\|_{E^*} \leq 1$. This implies that Ball(Im$(T_0 \circ T_1^*) \leq \lambda \cdot (T_0 \circ T_1^*)(\text{Ball}(E^*))$. Similarly one shows that $\nu_{E_k^*, E}^2(\mathcal{L}^{k, w^*}_{\lambda, k}(E^*, E)) = D_k(E; <\lambda)$.

**Appendix A. Extrinsic metrics for $p = \infty$.**

The case $p = \infty$ is special because the Fraïssé limit that corresponds to $\ell^\infty_k$ is a universal space, the Gurarij space $G$. We are going to see that the $G$-extrinsic metrics are Lipschitz equivalent to the intrinsic ones on bounded sets. We start by analyzing $\partial_X G$. Given a finite dimensional normed space $X = (X, \|\cdot\|_X)$, another compatible, more geometrical, metric on $\mathcal{N}_X$ is the next. Having in mind that a norm is completely determined by its dual unit ball, let
\[
\alpha_X(m, n) := d_{H_1, \|\cdot\|_X^*}(\text{Ball}(X, m^*), \text{Ball}(X, n^*)),
\]
where $d_{H_1, \|\cdot\|_X^*}$ is the Hausdorff distance with respect to the norm distance induced by $\|\cdot\|_X^*$. In other words, $\alpha_X(m, n)$ measures the $d_{\|\cdot\|_X^*}$-distance between the unit balls of $(X^*, m^*)$ and of $(X^*, n^*)$. In the next we write $\text{Sph}(X) = \{x \in X : \|x\|_X = 1\}$ to denote the unit sphere of $X$.

**Proposition A.1.** Let $X = (X, \|\cdot\|_X)$ be a finite dimensional normed space and let $m, n \in N_X$.

a) If $m, n \in N_X(\ell^\infty_k)$, then $\partial_X \ell^\infty_k(m, n) = \alpha_X(m, n)$. Consequently, in general, $\partial_X G(m, n) = \alpha_X(m, n)$.

b) If $m, n \in B_\omega(\|\cdot\|_X; \lambda)$, then $\lambda^{-1} \cdot \omega(m, n) \leq \alpha_X(m, n) \leq \lambda \cdot \omega(m, n)$.
A.2.1 Proof. We show that $E^\ast$ is Lipschitz equivalent to the Banach-Mazur metric on $B_k$. Let $x, y \in E^\ast$ such that $\|x - y\|_{E^\ast} \leq \lambda \cdot \|x, y\|_{E^\ast}$. We assume that $d(\|x, y\|_{E^\ast}) > 0$. Let us show the first inequality. Without modification of $\gamma_G$, we may assume that $x \in \text{Sph}(E)$ such that $0 < d(\|x, y\|_{E^\ast}) = d_X(x, y) < 1$. Then $n(x) > 1$ and $d_H(\|x, y\|_{E^\ast}) \leq \|x - y\|_X \leq \lambda \cdot \|x - y\|_X$. Then follows that $n(x) \leq n(y) + n(x - y) \leq 1 + \lambda \cdot \|x - y\|_X$, and similarly, $n(y) \leq n(x) + n(x - y) \leq 1 + \lambda \cdot \|x - y\|_X$. Consequently, $\|x, y\|_{E^\ast} \geq \lambda \cdot \|x, y\|_{E^\ast}$. We see now that $\gamma_G$ is Lipschitz equivalent to the Banach-Mazur metric on $B_k$.

Corollary A.2. $d_{BM}$ and $\gamma_{k,G}$ are Lipschitz equivalent on $B_k$. In fact, for $m, n \in \mathbb{N}_k$, $\frac{1}{4k \log k} d_{BM}(\|m, n\|_{BM}) \leq \gamma(\|m, n\|_{BM}) \leq (\log k) d_{BM}(\|m, n\|_{BM})$.

Proof. We start with the following.

Claim A.2.1. $d_{BM}(\|m, n\|_{BM}) \leq 4k \log k \gamma_{k,B}(\|m, n\|_{BM})$ for every Banach space $E$ and $m, n \in \mathbb{N}_k$.

Proof of Claim: Suppose that $\gamma_{k,B}(\|m, n\|_{BM}) < 1/(3k)$. Let $V, W \in \text{Gr}(k, E)$ be such that $r_{k,B}(V) = [m, n]$, $r_{k,B}(W) = [m, n]$, and $\gamma_E([m, n]) = \Lambda_{E}(V, W)$. Let $(x_j)_{j<k}$ be an Auerbach basis of $(V, \|\cdot\|_E)$. For each $j < k$, let $y_j \in \text{Ball}(W, \|\cdot\|_E)$ be such that $\|x_j - y_j\|_E \leq \Lambda_E(V, W)$. Since

$$\left\| \sum_{j<k} \lambda_j y_j \right\|_E \geq \left\| \sum_{j<k} \lambda_j x_j \right\|_E - \left\| \sum_{j<k} \lambda_j (x_j - y_j) \right\|_E \geq \left\| \sum_{j<k} \lambda_j x_j \right\|_E - k \Lambda_E(V, W) \max_{j<k} |\lambda_j| \geq (1 - k \Lambda_E(V, W)) \left\| \sum_{j<k} \lambda_j x_j \right\|_E > 0.$$

we obtain that $(y_j)_{j<k}$ is a basis of $W$ and $T : V \to W$, $T(x_j) := y_j$, $j < k$ is invertible. In addition, from (4) we have that $\left\| T^{-1}(V, \|\cdot\|_{ET}, W, \|\cdot\|_{E}) \right\| \leq (1 - k \Lambda_E(V, W))^{-1}$, and similarly, $\left\| T^{-1}(V, \|\cdot\|_{ET}, W, \|\cdot\|_{E}) \right\| \leq 1 + k \Lambda_E(V, W)$. We use that $(1 + x)/(1 - x) \leq \exp(9x/4)$ for every $0 \leq x \leq 1/3$, to conclude that $d_{BM}(r_{k,B}(V), r_{k,B}(W)) \leq (9/4) k \Lambda_E(V, W) \leq 4k \log k \gamma_{k,B}(\|m, n\|_{BM})$.

Suppose that $\Lambda_E(V, W) \geq (3k)^{-1}$. Since the diameter of $B_k$ is at most $\log(k)$, we obtain that $d_{BM}(r_{k,B}(V), r_{k,B}(W)) \leq \log(k) \leq 3k \log(k) \Lambda_E(V, W)$. In any case, $d_{BM}(r_{k,B}(V), r_{k,B}(W)) \leq 4k \log(k) \Lambda_E(V, W)$.

Fix two norms $m, n \in \mathbb{N}_k$, set $X := (\mathbb{F}^k, m)$ and $Y := (\mathbb{F}^k, n)$. The following result is a slight modification of [15, Proposition 6.2].

Claim A.2.2. Suppose that $E$ and $F$ are two finite-dimensional normed spaces, and $T : F \to G$ is an $1$-$1$ linear operator. There is a normed space $H$ and $I \in \text{Emb}(F, H)$ and $J \in \text{Emb}(G, H)$ such that:

i) If $1 \leq \|T\|, \|T^{-1}\|$, then $\|I - J \circ T\| \leq \|T\| \cdot \|T^{-1}\| - 1$. 

ii) If \( \dim F = \dim G \) and \( \|T\| = 1 \), then \( \Lambda_H(\Im I, \Im J) \leq \|T^{-1}\| - 1 \).

**Proof of Claim:** Fix a 1-1 linear operator \( T : F \to G \). On the direct sum \( F \oplus G \) we define the seminorm
\[
m(x, y) := \max \left\{ \frac{T x}{\|T\|} + y \|G, \max_{g \in D} \frac{|g(y)| + (T^* g)(x)}{\|T^* g\|_{F^*}} \right\},
\]
where \( D \) is chosen so that \( (T^*)^{-1}(\|T^{-1}\| - 1) \cdot \text{Ext}(\text{Ball}(F^*))) \subseteq D \subseteq \text{Ball}(F) \), and where for a compact convex set \( K \), \( \text{Ext}(K) \) is the set of extreme points of \( K \). Let \( H \) by the quotient of \( F \oplus G \) by the kernel of \( m \), and let \( I : F \to H, J : G \to H \) be the two canonical injections \( I(x) := [(x, 0)], J(y) := [(0, y)] \). It is routine to check that \( I, J \) and \( H \) have the desired properties. \( \square \)

Let \( T : X \to Y \) be such that \( \|T\| \cdot \|T^{-1}\| = \exp(d_{BM}([m], [n])) \), and without loss of generality, we assume that \( \|T\| = 1 \). We apply Claim A.2.2 to \( T \), and we obtain a normed space \( Z \) and isometric embeddings \( I : X \to Z \) and \( J : Y \to Z \) such that (b) holds, that is, \( \Lambda_Z(\Im I, \Im J) \leq \exp(d_{BM}([m], [n])) - 1 \). Since \( d_{BM}([m], [n]) \leq \log k \), it follows that \( \exp(d_{BM}([m], [n])) - 1 \leq \log k \cdot d_{BM}([m], [n]) \). Thus \( \gamma([m], [n]) \leq \Lambda_Z(\Im X, \Im Y) \leq \log k \cdot d_{BM}([m], [n]) \). \( \square \)

We conclude by proving that \( \delta_{k,G} \) is Lipschitz equivalent to the following intrinsically defined metric on \( D_k(\lambda) \). Recall that \( \omega_2 \) is the compatible metric \( \omega_2((m_0, m_1), (n_0, n_1)) := \omega(m_0, n_0) + \omega(m_1, n_1) \), and that \( \bar{\omega}_2 \) is the corresponding quotient metric on \( D_k \).

**Proposition A.3.** \( \delta_{k,\ell^2}^\infty \) and \( \bar{\omega}_2 \) are Lipschitz equivalent on \( D_k(\ell^2_k; \lambda) \). In fact,
\[
\frac{1}{2 \log(\lambda k)} \bar{\omega}_2([m], [n]) \leq \delta_{k,\ell^2}^\infty([m], [n]) \leq k^2 \lambda^3 \bar{\omega}_2([m], [n]).
\]

**Proof.** We first estimate the \( \bar{\omega}_2 \)-diameter of \( D_k(\ell^2_k; \lambda) \).

**Claim A.3.1.** For every \( ([m_0, m_1], [n_0, n_1]) \in D_k(\ell^2_k; \lambda) \) one has that
\[
\bar{\omega}_2(([m_0, m_1], [n_0, n_1])) \leq 2(\log \lambda + \min\{d_{BM}([m_0], [n_0]), d_{BM}([m_1], [n_1])\}).
\]
Consequently, \( \text{diam}(D_k(\ell^2_k; \lambda)) \leq 2 \log(\lambda k) \).

**Proof of Claim:** Given \( ([m_0, m_1], [n_0, n_1]) \in D_k(\ell^2_k; \lambda) \), \( \omega(m_0, m_1) + \omega(n_0, n_1) \leq \log(\lambda) \). Choose \( \Delta \in \text{GL}(\mathbb{R}^k) \) with \( d_{BM}([m_0], [n_0]) = \omega(m_0, \Delta \cdot n_0) \). Then, \( \omega(m_1, \Delta \cdot n_1) \leq \omega(m_1, m_0) + \omega(m_0, \Delta \cdot n_0) + \omega(\Delta \cdot n_0, \Delta \cdot n_1) \leq 2 \log \lambda + d_{BM}(m_0, n_0) \). From here we get easily the inequality in (5). \( \square \)

**Claim A.3.2.** Let \( E \) be a Banach space. If \( T, U \in L^\infty_k(E^*, E) \) are such that \( \|T - U\| < 1/(\lambda \sqrt{k}) \), then \( \bar{\omega}_2(\nu_{k,E}(T), \nu_{k,E}(U)) \leq \lambda \sqrt{k} \|T - U\| \).

**Proof of Claim:** Let \( T_0, U_0 \in L^\infty_k(\mathbb{R}^k; E) \) and \( T_1, U_1 \in L^\infty_k((\mathbb{R}^k)^*, E) \) be such that \( T = T_0 \circ T_1^* \) and \( U = U_0 \circ U_1^* \). Define \( m_0 := \nu_{k,E}(T_0), m_1 := \nu_{(k^*)^*, E}(T_1) \), and \( n_0 := \nu_{k,E}(U_0), n_1 := \nu_{(k^*)^*, E}(U_1) \). We use the Kadets-Snobar Theorem—see for example [1, Theorem 12.1.6]—to fix a projection \( P : E \to \Im(T_1) \) of norm at most \( \sqrt{k} \). Then, \( T_1^* P = T_1^* \circ P^* \circ r, \) where \( r : E^* \to (\Im(T_1))^* \) is the restriction map \( r(g) := g \upharpoonright \Im(T_1) \), hence, the rank of \( T_1^* \mid \Im P^* \) is \( k \) and since the dimension of \( \Im P^* \) is also \( k \), it follows that \( \theta := T_1^* \mid \Im P^* \to (\mathbb{R}^k, m_0) \) is an isomorphism. We estimate some norms.

- \( |\theta| \leq \lambda \) and \( |\theta^{-1}| \leq \lambda \sqrt{k} : |\theta| \leq |T_1^*|_{E^*, (\mathbb{R}^k, m_0)} = |T|_{E^*, E} \leq \lambda \). Now fix \( g \in \Im P^* \), and suppose that \( 1 = m_0(T_1^*(g)) = |T(g)|_E \). Find \( h \in E^* \) with \( \|h\|_{E^*} \leq \lambda \) such that \( T_1^*(h) = T_1^*(g) \); then \( \theta(g) = T_1^*(g) = T_1^*(h_0) = \theta(h_0) \), where \( h_0 = P^*(h \upharpoonright \Im T_1) \). Since \( \theta \) is a bijection, \( g = h_0 \) and \( \|g\|_{E^*} \leq \|P^*\| \|h\|_{E^*} \leq \lambda \sqrt{k} \).
- \( U_1^* : \Im P^* \to (\mathbb{R}^k, n_0) \) is also isomorphism: First, \( \|U_1^*|_{E^*, (\mathbb{R}^k, m_0)} = \|U\|_{E^*, E} \leq \lambda \), and if \( g \in Sph(\Im P^*) \), then by (a), \( m_0(T_1^*(g)) \geq 1/(\lambda \sqrt{k}) \), and \( |m_0(U_1^*(g)) - m_0(T_1^*(g))| = \|T g\|_E - |U(g)|_E \leq \|T - U\| < 1/(\lambda \sqrt{k}) \). So, \( m_0(U_1^*(g)) > 0, \) hence \( U_1^*(g) \neq 0 \), and since \( \dim \Im P^* = k \), it follows that \( U_1^* \) is an isomorphism. Set \( \Delta := U_1^* \circ (T_1^*)^{-1} \in \text{GL}(\mathbb{R}^k) \).
• $\Delta \cdot m_1 = n_1$: If $f \in (F^k)^*$, then $(\Delta \cdot m_1)(f) = m_1(\Delta^*(f)) = \|T_1(T_1^{-1}(U_1(f)))\|_E = \|U_1(f)\|_E = n_1(f)$.

• $\omega(\Delta \cdot m_0, n_0) \leq \lambda \sqrt{k}\|T - U\|$: Fix $x \in F^k$ such that $m_0(x) = 1$, and set $g := (U_1^*)^{-1}(x)$. Notice that $\|g\|_{F^k} \leq \lambda \sqrt{k}$. Then $m_0(\Delta^{-1}(x)) - n_0(x) = \|T_0(T_0^*(g))\|_E - \|U_0(U_0^*(g))\|_E \leq \|[T - U]\|_{F^k} \|g\|_{F^k} \leq \lambda \cdot \sqrt{k} \cdot \|T - U\|_{F^k} E$. Similarly, $n_0(x) \leq (\lambda \cdot \sqrt{k} \cdot \|T - U\| + 1)\Delta \cdot m_0(x)$. It follows that 

$$\omega_2([(m_0, m_1)], [(n_0, n_1)]) \leq \omega_2(\Delta \cdot (m_0, m_1), (n_0, n_1)) = \omega(\Delta \cdot m_0, n_0) \leq \log(\lambda \cdot \sqrt{k} \cdot \|T - U\| + 1) \leq \lambda \cdot \sqrt{k} \cdot \|T - U\|.$$ 

From the previous two claims we have that $\omega_2([m], [n]) \leq 2\log(\lambda k)\lambda \sqrt{k} \delta_k, E([m], [n])$ for $[m], [n] \in D_k(E; \lambda)$.

**Claim A.3.3.** For every $[m], [n] \in D_k(E; \lambda)$ one has that $\delta_{k, E}([m], [n]) \leq k^2 \lambda^3 \omega_2([m], [n])$.

**Proof of Claim:** Let $\Delta \in GL(F^k)$ be such that $\omega_2([(m_0, m_1)], [(n_0, n_1)]) = \omega(m_0, \Delta \cdot n_0) + \omega(m_1, \Delta \cdot n_1)$. We use Proposition A.1 to find $T_0, U_0 \in L(F^k, E), T_1, U_1 \in L((F^k)^*, E_\lambda^\ast)$ such that:

- $m_0 = \nu_{F^k, E}^\ast(T_0)$,
- $\Delta \cdot n_0 = \nu_{F^k, E}^\ast(U_0)$,
- $m_1 = \nu_{(F^k)^*}, E_\lambda^\ast(T_1)$ and $\Delta \cdot m_1 = \nu_{(F^k)^*}, E_\lambda^\ast(U_1)$;

- $\alpha_{((F^k)^*, m_1)}(m_0, \Delta \cdot n_0) = \|T_0 - U_0\|_{F^k, E} 1_{m_1}^\ast, E_\lambda^\ast$ and $\alpha_{((F^k)^*, m_1)}(m_1, \Delta \cdot n_1) = \|T_1 - U_1\|_{((F^k)^*), m_1} E_\lambda^\ast$.

Let $T := T_0 \circ T_1^*$, $U := U_0 \circ U_1^*$, and $\alpha_{k, E}([m], [n]) \leq \|T - U\|_{F^k, E}$. Now let $g \in Sph(E_\lambda^\ast)$. Then 

$$\|T - U\|_{E} \leq 2 \|T_1 - T_1^*\|_{E} + \|T_0 - U_0\|_{((F^k)^*), m_0} \|U_1^\ast\|_{E_\lambda^\ast} + \|T_0 - U_0\|_{((F^k)^*), m_0} \|U_1^\ast\|_{E_\lambda^\ast} + \alpha_{((F^k)^*, m_1)}(m_0, \Delta \cdot n_0)$$

(6)

On the other hand, by Claim A.3.1, 

$$\max\{\|\Id\|_{\Delta^*, m_1} \cdot \|\Id\|_{\Delta^*, m_0} \} \leq \lambda \omega_2([m], [n]) \leq k^2 \lambda^3 \omega_2([m], [n]).$$

It follows from the inequality in (6) and Proposition A.1 b) that 

$$\delta_{k, E}([m], [n]) \leq k^2 \lambda^3 (\omega(m_1, \Delta \cdot n_1) + \omega(m_0, \Delta \cdot n_0)) \leq k^2 \lambda^3 \omega_2([m], [n]).$$

We do not know a similar explicit description of the extrinsic metrics for $p$’s other than $\infty$.

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