QUANTUM DOUBLE OF $U_q((\mathfrak{sl}_2)_{\leq 0})$

JUN HU AND YINHUO ZHANG

Abstract. Let $U_q(\mathfrak{sl}_2)$ be the quantized enveloping algebra associated to the simple Lie algebra $\mathfrak{sl}_2$. In this paper, we study the quantum double $D_q$ of the Borel subalgebra $U_q((\mathfrak{sl}_2)_{\leq 0})$ of $U_q(\mathfrak{sl}_2)$. We construct an analogue of Kostant–Lusztig $Z[v, v^{-1}]$-form for $D_q$ and show that it is a Hopf subalgebra. We prove that, over an algebraically closed field, every simple $D_q$-module is the pullback of a simple $U_q(\mathfrak{sl}_2)$-module through certain surjection from $D_q$ onto $U_q(\mathfrak{sl}_2)$, and the category of finite dimensional weight $D_q$-modules is equivalent to a direct sum of $|k^*|$ copies of the category of finite dimensional weight $U_q(\mathfrak{sl}_2)$-modules. As an application, we recover (in a conceptual way) Chen’s results [2] as well as Radford’s results [20] on the quantum double of Taft algebra. Our main results allow a direct generalization to the quantum double of the Borel subalgebra of the quantized enveloping algebra associated to arbitrary Cartan matrix.

1. Preliminaries

Let $k$ be a field. Let $q$ be an invertible element in $k$ satisfying $q^2 \neq 1$. The quantized enveloping algebra associated to the simple Lie algebra $\mathfrak{sl}_2$ is the associative $k$-algebra with the generators $E, F, K, K^{-1}$ and the relations (cf. [6] and [9], [10]):

\begin{align*}
KE &= q^2 KE, \quad KF = q^{-2} FK, \quad KK^{-1} = 1 = K^{-1}K, \\
EF - FE &= K^{-1} - K^{-1}
\end{align*}

We denote it by $U_q(\mathfrak{sl}_2)$ or just $U_q$ for simplicity. The algebra $U_q$ is a quantum analogue of the universal enveloping algebra $U(\mathfrak{sl}_2)$ associated to the simple Lie algebra $\mathfrak{sl}_2$. It is a Hopf algebra with comultiplication, counit and antipode given by:

\begin{align*}
\Delta(E) &= E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K) = K \otimes K, \\
\varepsilon(E) &= 0 = \varepsilon(F), \quad \varepsilon(K) = 1 = \varepsilon(K^{-1}), \\
S(E) &= -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}.
\end{align*}

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We are actually working with De Concini–Kac’s version of specialized quantum algebra, see [4].
Let $U_q^+$ (resp. $U_q^-$) be the $k$-subalgebra of $U_q$ generated by $E$ (resp. by $F$). Let $U_q^0$ be the $k$-subalgebra of $U_q$ generated by $K, K^{-1}$. Then the elements $\{E^a\}$ (resp. $\{F^b\}$), where $a, b \in \mathbb{N} \cup \{0\}$, form a $k$-basis of $U_q^+$ (resp. of $U_q^-$). The elements $\{K^c\}$, where $c \in \mathbb{Z}$, form a $k$-basis of $U_q^0$. Moreover, the natural $k$-linear map $U_q^+ \otimes U_q^0 \otimes U_q^- \to U_q$ given by multiplication is a $k$-linear isomorphism. The basis $\{E^a K^c F^b \mid a, b \in \mathbb{N} \cup \{0\}, c \in \mathbb{Z}\}$ is called the PBW basis of $U_q$. We define $U_{q}^{\geq 0} := U_q^+ U_q^0, U_{q}^{\leq 0} := U_q^- U_q^0$. Then both $U_{q}^{\geq 0}$ and $U_{q}^{\leq 0}$ are Hopf $k$-subalgebras of $U_q$. For any monomials $E^a K^b, F^c K^d$, we endow them the degree $a, c$ respectively.

Let $v$ be an indeterminate over $\mathbb{Z}$. We consider the quantized enveloping algebra $U_v = U_v(\mathfrak{sl}_2)$ with parameter $v$ and defined over $\mathbb{Q}(v)$. It is well-known (see [11, 17] and [25]) that there exists a unique pairing $\varphi : U_{v}^{\geq 0} \times U_{v}^{\leq 0} \to \mathbb{Q}(v)$ such that

1. $\varphi(1, 1) = 1$, $\varphi(1, K) = 1 = \varphi(K, 1)$
2. $\varphi(x, y) = 0$, if $x, y$ are homogeneous with different degree,
3. $\varphi(E, F) = \frac{1}{v^2 - 1}, \varphi(K, K) = v^2, \varphi(K, K^{-1}) = v^{-2},$
4. $\varphi(x, y' y'') = \varphi(\Delta^0(x), y' \otimes y'')$, for all $x \in U_{v}^{\geq 0}, y', y'' \in U_{v}^{\leq 0},$
5. $\varphi(x' x', y'') = \varphi(x \otimes x', \Delta(y''))$, for all $x, x' \in U_{v}^{\geq 0}, y'' \in U_{v}^{\leq 0},$
6. $\varphi(S(x), y) = \varphi(x, S^{-1} (y))$, for all $x \in U_{v}^{\geq 0}, y \in U_{v}^{\leq 0}.$

One usually call $(U_{v}^{\geq 0}, U_{v}^{\leq 0}, \varphi)$ a skew Hopf pairing (cf. [18]). Then, one can make $D(U_{v}^{\geq 0}, U_{v}^{\leq 0}) := U_{v}^{\geq 0} \otimes U_{v}^{\leq 0}$ into a Hopf $\mathbb{Q}(v)$-algebra, which is called the quantum double of $(U_{v}^{\geq 0}, U_{v}^{\leq 0}, \varphi)$. As a $\mathbb{Q}(v)$-coalgebra, $D(U_{v}^{\geq 0}, U_{v}^{\leq 0}) = U_{v}^{\geq 0} \otimes U_{v}^{\leq 0}$, the tensor product of two coalgebras. The algebra structure of $D(U_{v}^{\geq 0}, U_{v}^{\leq 0})$ is determined by

$$
(x \otimes y)(x' \otimes y') = \sum \varphi(x_{(1)} y_{(1)})(x x'_{(2)} \otimes y_{(2)} y') \varphi(x'_{(3)} S^{-1}(y_{(3)})),$$

for all $x, x' \in U_{v}^{\geq 0}, y, y' \in U_{v}^{\leq 0}$, where Sweedler’s sigma summation $\Delta^2(y) = \sum y_{(1)} \otimes y_{(2)} \otimes y_{(3)}$ is used. Note that the quantum double we described here actually arises from a 2-cocycle twist (see [3] for more detail). The representation results in the paper are related to the recent work of Radford and Schneider (see [21] and [22]). We thank the referee for pointing out these references. For simplicity, we shall write $D_v$ instead of $D(U_{v}^{\geq 0}, U_{v}^{\leq 0})$.

Let $A = \mathbb{Z}[v, v^{-1}, (v - v^{-1})^{-1}]$. Let $U_{A, v}^{\leq 0}$ be the associative $A$-algebra defined by the generators $F, K, K^{-1}$ and relations

$$KF = v^{-2} FK, \quad KK^{-1} = 1 = K^{-1} K.$$
Specializing \( v \) to \( q \), we make \( k \) into an \( A \)-algebra. Clearly \( U_{q}^{\leq 0} \cong k \otimes_{A} U_{A,v}^{< 0} \).

Similarly, we can define \( U_{A,v}^{> 0} \), and we have \( U_{q}^{\geq 0} \cong k \otimes_{A} U_{A,v}^{\geq 0} \).

The previous construction of skew Hopf pairing clearly gives rise to a pairing \( \varphi : U_{A,v}^{\geq 0} \times U_{A,v}^{\leq 0} \to A \), and hence gives rise to a pairing \( \varphi : U_{q}^{\geq 0} \times U_{q}^{\leq 0} \to k \).

**Lemma 1.1.** The pairing \( \varphi \) gives rise to a Hopf algebra map \( \theta \) from the Hopf algebra \( U_{q}^{\geq 0} \) to the Hopf algebra \( (U_{q}^{\leq 0})^{*} \) as well as a Hopf algebra map \( \theta' \) from the Hopf algebra \( U_{q}^{\geq 0} \) to the Hopf algebra \( (U_{q}^{\leq 0})^{*} \). Moreover, \( \theta, \theta' \) are injective if \( q \) is not a root of unity.

**Proof.** The maps \( \theta, \theta' \) are defined by

\[
\theta(y)(x) = \varphi(x, y),
\theta'(x)(y) = \varphi(x, y),
\]

for any \( x \in U_{q}^{\geq 0}, y \in U_{q}^{\leq 0} \).

Now the first statement follows directly from the definition of \( \varphi \). The second statement can be proved by using a similar argument in the proof of Lemma 4.1. \( \square \)

Now we can construct the quantum double \( D(U_{q}^{\geq 0}, U_{q}^{\leq 0}) := U_{q}^{\geq 0} \otimes U_{q}^{\leq 0} \) of \((U_{q}^{\geq 0}, U_{q}^{\leq 0}, \varphi)\) in a similar way, making it into a Hopf \( k \)-algebra. Henceforth, we shall write \( D_{q} \) instead of \( D(U_{q}^{\geq 0}, U_{q}^{\leq 0}) \). We have the following.

**Theorem 1.2.** As a \( k \)-algebra, \( D_{q} \) can be presented by the generators

\[
E, F, K, K^{-1}, \tilde{K}, \tilde{K}^{-1},
\]

and the following relations:

\[
KE = q^{2}EK, \quad KF = q^{-2}FK, \quad \tilde{K}E = q^{2}E\tilde{K}, \quad \tilde{K}F = q^{-2}F\tilde{K},
KK^{-1} = K^{-1}K = 1 = \tilde{K}\tilde{K}^{-1} = \tilde{K}^{-1}\tilde{K}, \quad K\tilde{K} = \tilde{K}K,
EF - FE = \frac{K - \tilde{K}^{-1}}{q - q^{-1}}.
\]

**Proof.** Let \( D_{q}' \) be an abstract \( k \)-algebra defined by the generators and relations as above. One checks directly that \( D_{q} \) is generated by the following elements

\[
E \otimes 1, \quad 1 \otimes qF, \quad K^{\pm 1} \otimes 1, \quad 1 \otimes K^{\pm 1},
\]

and these elements satisfy the above relations. In other words, there is a surjective \( k \)-algebra homomorphism \( \psi : D_{q}' \to D_{q} \) such that

\[
\psi(E) = E \otimes 1, \quad \psi(F) = 1 \otimes qF, \quad \psi(K^{\pm 1}) = K^{\pm 1} \otimes 1, \quad \psi(\tilde{K}^{\pm 1}) = 1 \otimes K^{\pm 1}.
\]

On the other hand, we claim that the monomials

\[
E^{a}K^{c}\tilde{K}^{d}F^{b}, \quad a, b, c, d \in \mathbb{Z}, \quad a, b \geq 0,
\]
form a basis of $D'_q$. We prove this by using a similar argument as in the proof of [8, Theorem 1.5]. We consider a polynomial ring $k[T_1, T_2, T_3, T_4]$ in four indeterminate $T_1, T_2, T_3, T_4$ and its localization $A' = k[T_1, T_2^{-1}, T_3, T_3^{-1}, T_4]$. Then all monomials $T_1^aT_2^cT_3^dT_4^b$ with $a, b, c, d \in \mathbb{Z}, a, b \geq 0$ are a basis of $A'$. We define linear endomorphisms $e, f, h, \tilde{h}$ of $A'$ by letting

$$e(T_1^aT_2^cT_3^dT_4^b) = T_1^{a+1}T_2^cT_3^dT_4^b,$$

$$f(T_1^aT_2^cT_3^dT_4^b) = \begin{cases} -q^{a-1}[a-1]_q T_1^aq^{-1}K - q^{1-a}K^{-1}T_2^cT_3^dT_4^b, & \text{if } a \geq 1; \\ q^{2c+2dT_1^aq^2T_2^cT_3^dT_4^b+1}, & \text{if } a = 0, \end{cases}$$

$$h(T_1^aT_2^cT_3^dT_4^b) = q^{2a}T_1^aT_2^cT_3^dT_4^b,$$

$$\tilde{h}(T_1^aT_2^cT_3^dT_4^b) = q^{2a}T_1^aT_2^cT_3^dT_4^b,$$

where

$$[a-1]_q := \frac{q^{a-1} - q^{1-a}}{q - q^{-1}}.$$

One can check that the above definition gives rise to a representation of $D'_q$ on $A'$ by taking $E$ to $e$, $F$ to $f$, $K, K^\pm1$ to $h, \tilde{h}$, and $K^{-1}$ to $\tilde{h}^{-1}$. So it takes a monomial $E^aK^c\bar{K}^dF^b$ to the monomial $e^ah^c\tilde{h}^df^b$. Note that

$$e^ah^c\tilde{h}^df^b(1) = T_1^aT_2^cT_3^dT_4^b,$$

which implies that the $e^ah^c\tilde{h}^df^b$ are linearly independent, hence the monomials $E^aK^c\bar{K}^dF^b$ must be linear independent as well. This proves our claim.

By definition of $D_q$, we know that the monomials

$$(E \otimes 1)^a(K \otimes 1)^c(1 \otimes \bar{K})^d(1 \otimes F)^b, \quad a, b, c, d \in \mathbb{Z}, a, b \geq 0,$$

are a basis of $D_q$. Therefore, $\psi$ maps a basis of $D'_q$ onto a basis of $D_q$. It follows that $\psi$ must be an isomorphism, as required.

**Lemma 1.3.** The map which sends $E$ to $E$, $F$ to $F$, $K^\pm1$ to $K^\pm1$ and $\bar{K}^\pm1$ to $\bar{K}^\pm1$ extends uniquely to a surjective Hopf algebra homomorphism $\pi : D_q \twoheadrightarrow U_q$.

**Proof.** This is obvious. Note that the kernel of $\pi$ is the two-sided ideal of $D_q$ generated by $K - \bar{K}$, which is in fact a Hopf ideal of $D_q$. \hfill \Box

The algebra $D_q$ will be the primary interest to us in this paper. It turns out that this algebra behaves quite similar to the quantized enveloping algebra $U_q$ in many ways. In the following sections we shall see that many constructions and equalities in the structure and representation theory of $U_q$ carry over to the algebra $D_q$. 


2. An analogue of Kostant–Lusztig $\mathbb{Z}[v, v^{-1}]$-form

The purpose of this section is to construct an analogue of Kostant–Lusztig $\mathbb{Z}[v, v^{-1}]$-form for the quantum double $D_q$.

Let $R$ be an integral domain. Let $v$ be an indeterminate over $R$. Let $\mathcal{A} = R[v, v^{-1}]$, the ring of Laurent $R$-polynomials in $v$. We consider the quantum double $D_v$ (with parameter $v$) defined over the quotient field of $\mathcal{A}$. We shall define an analogue of Kostant–Lusztig $\mathcal{A}$-form (see [13], [15], [16]) for the algebra $D_v$. For each positive integer $N$, we define

$$[N] := \frac{v^N - v^{-N}}{v - v^{-1}}, \quad [N]^! := [N][N - 1] \cdots [2][1].$$

For any integers $m, n$ with $n \geq 0$, we define

$$\binom{m}{n} = \frac{[m]^!}{[n]^![m - n]^!}.$$

Then it is well-known that $\binom{m}{n} \in \mathcal{A}$ (e.g., see [17, (1.3.1.d)]).

Let $D_{\mathcal{A}}$ be the $\mathcal{A}$-subalgebra of $D_v$ generated by (compare with [13, (3.1)])

$$E^{(N)} = \frac{E^N}{[N]^!}, \quad F^{(N)} = \frac{F^N}{[N]^!}, \quad K^\pm, \quad \tilde{K}^\pm,$$

$$\begin{bmatrix} K, \tilde{K} \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{Kv^{-s+1} - \tilde{K}^{-1}v^{s-1}}{v^s - v^{-s}},$$

where $N, t \in \mathbb{N} \cup \{0\}$.

For any integers $c, t$ with $t \geq 0$, we define the analogue of $\begin{bmatrix} K, c \\ t \end{bmatrix}$ (see [13, (4.1)])

$$\begin{bmatrix} K, \tilde{K}, c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{Kv^{c-s+1} - \tilde{K}^{-1}v^{c+s-1}}{v^s - v^{-s}}.$$

We have the following (compare it with [14, (4.3.1)]).

**Lemma 2.1.** For any non-negative integers $a, b$, we have that

$$E^{(a)}F^{(b)} = \sum_{0 \leq t \leq \min(a, b)} F^{(b-t)} \begin{bmatrix} K, \tilde{K}, 2t - a - b \\ t \end{bmatrix} E^{(a-t)}.$$

**Proof.** The equality is proved in a similar way to the proof of [14, (4.3.1)]. We show it by induction on $a$. For $a = 0$ (or $b = 0$), the claim is trivial. For $a = 1$ and $b > 0$, it also follows from a straightforward verification. Suppose
the formula holds for integers \( a, b \). Then we obtain

\[
E^{(a+1)} F^{(b)} = \frac{1}{a+1} E E^{(a)} F^{(b)}
\]

\[
= \frac{1}{a+1} \sum_{0 \leq t \leq \min(a,b)} E F^{(b-t)} \left[ K, \tilde{K}, 2t - a - b \right] E^{(a-t)}
\]

\[
= \frac{1}{a+1} \left\{ \sum_{0 \leq t \leq \min(a,b)} F^{(b-t)} E \left[ K, \tilde{K}, 2t - a - b \right] E^{(a-t)} + \sum_{0 \leq t \leq \min(a,b)} F^{(b-t)} E \left[ K, \tilde{K}, 2t - a - b \right] E^{(a-t)} \right\}
\]

\[
= \frac{1}{a+1} \left\{ \sum_{0 \leq t \leq \min(a,b)} F^{(b-t)} \left[ K, \tilde{K}, 2t - a - b \right] E^{(a-t)} + \sum_{0 \leq t \leq \min(a,b)} F^{(b-t)} \left[ K, \tilde{K}, 2t - a - b \right] E^{(a-t)} \right\}
\]

Thus we have to show that the equality

\[
\frac{1}{a+1} \left\{ \sum_{0 \leq t \leq \min(a,b)} F^{(b-t)} \left[ K, \tilde{K}, 2t - a - b \right] E^{(a-t)} + \sum_{0 \leq t \leq \min(a,b)} F^{(b-t)} \left[ K, \tilde{K}, 2t - a - b \right] E^{(a-t)} \right\}
\]

holds when \( 0 \leq t \leq \min(a,b) \), and that the equality

\[
\frac{1}{a+1} F^{(b-a-1)} E^{(a+1-t)} \left[ K, \tilde{K}, a - b \right] E^{(a+1-t)}
\]

holds when \( b \geq a + 1 \).

The verification of the second equality is straightforward while the first equality follows from the following equation which can be calculated easily:

\[
\left[ K, \tilde{K}, 2t - a - b \right] = \frac{1}{a+1} \left\{ \sum_{0 \leq t \leq \min(a,b)} F^{(b-t)} \left[ K, \tilde{K}, 2t - a - b \right] E^{(a-t)} + \sum_{0 \leq t \leq \min(a,b)} F^{(b-t)} \left[ K, \tilde{K}, 2t - a - b \right] E^{(a-t)} \right\}.
\]

\[\Box\]
Let $D^+_A$ (resp. $D^-_A$) be the $A$-subalgebra of $D_A$ generated by $E^{(a)}$ (resp. $F^{(b)}$), where $a, b \in \mathbb{N} \cup \{0\}$. Let $D^0_A$ be the $A$-subalgebra of $D_A$ generated by $K^{\pm 1}, \tilde{K}^{\pm 1}$, $\left[ K, \tilde{K} \right]_t$, $t = 0, 1, 2, \cdots$. We have the following analogues of \cite{13} (2.3),(g8),(g9),(g10)) and \cite{13} (4.1),(d)].

**Lemma 2.2.** 1) For any integers $c, t, p$ with $t \geq 0, 0 \leq p \leq t$, we have

$$v^{-pt} \left[ K, \tilde{K}, c \right]_t = \sum_{j=0}^{p} \left[ K, \tilde{K}, c - p \right]_t \left[ K - j \right]_t v^{-cj}.$$ 

In particular, for any integer $c, t$ with $0 \leq c \leq t$, we have

$$\left[ K, \tilde{K}, c \right]_t = \sum_{j=0}^{c} \left[ K, \tilde{K} \right]_t \left[ c \right]_{t-j} \tilde{K}^{-j} v^{c(t-j)}.$$ 

2) For any integers $c, t, p$ with $t \geq 0, p \geq 1$, we have

$$v^{-pt} \left[ K, \tilde{K}, -c \right]_t = \sum_{j=0}^{t} \left( -1 \right)^j \left[ K, \tilde{K}, p - c \right]_t \left[ p + j - 1 \right]_t \tilde{K}^j v^{-cj}.$$ 

In particular, for any integer $c, t$ with $c \geq 1, t \geq 0$, we have

$$\left[ K, \tilde{K}, -c \right]_t = \sum_{j=0}^{t} \left( -1 \right)^j \left[ K, \tilde{K} \right]_t \left[ c + j - 1 \right]_t \tilde{K}^j v^{c(t-j)}.$$ 

3) For any $c \in \mathbb{Z}, t \in \mathbb{N} \cup \{0\}$, $\left[ K, \tilde{K}, c \right]_t \in D^0_A$.

4) For any non-negative integers $t, t'$ with $t \geq 1$, we have that

$$\left[ t + t' \right]_t \left[ K, \tilde{K} \right]_t = \sum_{0 \leq j \leq t'} \left( -1 \right)^j v^{t'(t'-j)} \left[ t + j - 1 \right]_t \tilde{K}^j \left[ K, \tilde{K} \right]_t \left[ t', t' - j \right]_t.$$ 

**Proof.** 1) The second statement follows from induction on $t$. It suffices to prove the first statement. We show it by induction on $p$. The case where $p = 0$ is trivial. For $p = 1$, we have the following

$$\left[ K, \tilde{K}, c - 1 \right]_t = \left( \prod_{s=1}^{t-1} K v^{c-1-s+1} - \tilde{K}^{-1} v^{-c+1+s-1} \right) \left( K v^{c-t} - \tilde{K}^{-1} v^{-c+t} \right) v^t - v^{c-t} + \tilde{K}^{-1} v^{-c}$$

$$= v^{-t} \left[ K, \tilde{K}, c \right]_t,$$

Note that in \cite{13} (2.3),(g10)], the range of $j$ in the summation should be $0 \leq j \leq c$ instead of $0 \leq j \leq t$. 

$^3$
as required. Suppose now the equality holds for \( p = N \), we consider the case where \( p = N + 1 \). We get

\[
v^{-(N+1)t} \begin{bmatrix} K, \tilde{K}, c \end{bmatrix}_t = v^{-Nt} v^{-t} \begin{bmatrix} K, \tilde{K}, c \end{bmatrix}_t
\]

\[
= v^{-Nt} \left\{ \begin{bmatrix} K, \tilde{K}, c - 1 \end{bmatrix}_t + \begin{bmatrix} K, \tilde{K}, c - 1 \end{bmatrix}_{t-1} \tilde{K}^{-1} v^{-c} \right\}
\]

\[
= v^{-Nt} \begin{bmatrix} K, \tilde{K}, c - 1 \end{bmatrix}_t + v^{-N(t-1)} \begin{bmatrix} K, \tilde{K}, c - 1 \end{bmatrix}_{t-1} \tilde{K}^{-1} v^{-c-N}
\]

\[
= \sum_{j=0}^{N} \left\{ \begin{bmatrix} K, \tilde{K}, c - N - 1 \end{bmatrix}_{t-j} \left[ N \right]_j \tilde{K}^{-j} v^{-(c-1)j} + \begin{bmatrix} K, \tilde{K}, c - N - 1 \end{bmatrix}_{t-1-j} \left[ N \right]_j \tilde{K}^{-j-1} v^{-(c-1)j-c-N} \right\}
\]

\[
= \sum_{j=0}^{N+1} \begin{bmatrix} K, \tilde{K}, c - N - 1 \end{bmatrix}_{t-j} \left[ N+1 \right]_j \tilde{K}^{-j} v^{-cj} ,
\]

as desired.

2) Now we use induction on \( t \). The case where \( t = 0 \) is trivial. For \( t = 1 \), we obtain the equations

\[
\sum_{j=0}^{1} (-1)^j \begin{bmatrix} K, \tilde{K}, p - c \end{bmatrix}_{1-j} \left[ p + j - 1 \right]_j K^{j} v^{-cj}
\]

\[
= K v^{p-c} - \tilde{K}^{-1} v^{c-p} \frac{v^p - v^{-p}}{v - v^{-1}} \tilde{K}^{-c}
\]

\[
= v^{-p} K v^{c} - \tilde{K}^{-1} v^{c} \frac{v^p - v^{-p}}{v - v^{-1}} = v^{-p} \begin{bmatrix} K, \tilde{K}, -c \end{bmatrix}_1,
\]

as required.

Suppose the equality holds for \( t \), we now consider the equality for \( t + 1 \). We get then

\[
v^{-p(t+1)} \begin{bmatrix} K, \tilde{K}, -c \end{bmatrix}_{t+1} = v^{-pt} \begin{bmatrix} K, \tilde{K}, -c \end{bmatrix}_t v^{-p} K v^{-c-t} - \tilde{K}^{-1} v^{c+t} \frac{v^{t+1} - v^{-t-1}}{v^{t+1} - v^{-t-1}}
\]

\[
= \sum_{j=0}^{t} (-1)^j \begin{bmatrix} K, \tilde{K}, p - c \end{bmatrix}_{t-j} \left[ p + j - 1 \right]_j K^{j} v^{-cj-p} K v^{-c-t} - \tilde{K}^{-1} v^{c+t} \frac{v^{t+1} - v^{-t-1}}{v^{t+1} - v^{-t-1}}
\]

\[
= \begin{bmatrix} K, \tilde{K}, p - c \end{bmatrix}_{t+1} - \begin{bmatrix} K, \tilde{K}, p - c \end{bmatrix}_t K v^{-c-t} \frac{v^p - v^{-p}}{v^{t+1} - v^{-t-1}}
\]

\[
+ \sum_{j=1}^{t} (-1)^j \begin{bmatrix} K, \tilde{K}, p - c \end{bmatrix}_{t-j} \left[ p + j - 1 \right]_j K^{j} v^{-cj-p} K v^{-c-t} - \tilde{K}^{-1} v^{c+t} \frac{v^{t+1} - v^{-t-1}}{v^{t+1} - v^{-t-1}}.
\]
Note that

\[(-1)^{j} \left[ K, \tilde{K}, p - c \right]_{t - j} \left[ p + j - 1 \right]_{j} \frac{K^{j} v^{-c - p} \left( K_{t}^{v^{-c + t}} - \tilde{K}_{t}^{-1} v^{c + t} \right)}{v^{t+1} - v^{-t-1}}\]

\[= (-1)^{j} \left[ K, \tilde{K}, p - c \right]_{t-j} \left[ p + j - 1 \right]_{j} \frac{K^{j} v^{-c + p + t + j - \tilde{K}^{-1} v^{c - p + t - j}}}{v^{t+1} - v^{-t-1}}\]

\[\frac{v^{t+1} - v^{-t+2j-1}}{v^{t+1} - v^{-t-1}} - K_{t}^{v^{c - t} v^{p + 2j} - v^{-p}}\]

\[= (-1)^{j} \left[ K, \tilde{K}, p - c \right]_{t+1 - j} \left[ p + j - 1 \right]_{j} K^{j} v^{-c \left( j + 1 \right)} \frac{v^{t+1} - v^{-t+2j-1}}{v^{t+1} - v^{-t-1}} +\]

\[(-1)^{j+1} \left[ K, \tilde{K}, p - c \right]_{t - j} \left[ p + j - 1 \right]_{j} K^{j} v^{c - t} +\]

\[\frac{v^{p + 2j - t - 2} - v^{-p - t}}{v^{t+1} - v^{-t-1}}\]

\[= (-1)^{j} \left[ K, \tilde{K}, p - c \right]_{t+1 - j} \left[ p + j - 1 \right]_{j} K^{j} v^{-c j}.\]

3) follows from 1) and 2).

4) follows from 2) and the following equality:

\[\left[ K, \tilde{K} \right]_{t + t'} = \left[ K, \tilde{K} \right]_{t} \left[ K, \tilde{K}, -t \right]_{t'}.\]

Let \(\theta\) be the algebra automorphism of \(D_{v}\) which is defined on generators by

\[\theta(E) = E, \quad \theta(F) = F, \quad \theta(K^{\pm 1}) = \tilde{K}^{\pm 1}, \quad \theta(\tilde{K}^{\pm 1}) = K^{\pm 1}.\]

Since

\[\theta\left(\left[ K, \tilde{K} \right]_{t} \right) = \prod_{s=1}^{t} \frac{\tilde{K}^{v^{-s+1} - K^{-1} v^{s-1}}}{v^{s} - v^{-s}} = K^{-t} \tilde{K}^{t} \left[ K, \tilde{K} \right]_{t},\]
it follows that $\theta$ restricts to an $\mathcal{A}$-algebra automorphism of $\mathbb{D}_\mathcal{A}$. Henceforth, we write

$$\left[ \tilde{K}, \tilde{K} \right]_t := \theta \left( \left[ K, \tilde{K} \right]_t \right), \quad \left[ \tilde{K}, K, c \right]_t := \theta \left( \left[ K, \tilde{K}, c \right]_t \right).$$

Then one can get a second version of our previous two lemmas by applying the automorphism $\theta$.

**Lemma 2.3.** With the notations as above, we have that the $\mathcal{A}$-algebra $\mathbb{D}_\mathcal{A}^+$ (resp. $\mathbb{D}_\mathcal{A}^-$) is a free $\mathcal{A}$-module, and the set $\{ E^{(a)} \}$ (resp. the set $\{ F^{(b)} \}$), where $a, b \in \mathbb{N} \cup \{0\}$, form an $\mathcal{A}$-basis of $\mathbb{D}_\mathcal{A}^+$ (resp. of $\mathbb{D}_\mathcal{A}^-$).

**Proof.** This follows from the fact that the set $\{ E^{(a)} \}$ (resp. the set $\{ F^{(a)} \}$) is a basis of $\mathbb{D}_\mathcal{A}^+$ (resp. of $\mathbb{D}_\mathcal{A}^-$) and the following equalities:

$$E^{(a)} E^{(b)} = \begin{bmatrix} a + b \\ b \end{bmatrix} E^{(a+b)}, \quad F^{(a)} F^{(b)} = \begin{bmatrix} a + b \\ b \end{bmatrix} F^{(a+b)}.$$

We define $\mathbb{D}_{\mathcal{A}}^{\geq 0} := \mathbb{D}_{\mathcal{A}}^+ \mathbb{D}_{\mathcal{A}}^0$, $\mathbb{D}_{\mathcal{A}}^{\leq 0} := \mathbb{D}_{\mathcal{A}}^- \mathbb{D}_{\mathcal{A}}^0$. According to [8, Remark 3.1], we know that for each positive integer $N$,

$$\Delta(E^{(N)}) = \sum_{i=0}^{N} v^{i(N-i)} E^{(N-i)} K^i \otimes E^{(i)},$$

$$\Delta(F^{(N)}) = \sum_{i=0}^{N} v^{i(N-i)} F^{(i)} \otimes F^{(N-i)} \tilde{K}^{-i},$$

$$S(E^{(N)}) = (-1)^N v^{(1-N)N} K^{-N} E^{(N)},$$

$$S(F^{(N)}) = (-1)^N v^{(N-1)N} F^{(N)} \tilde{K}^N.$$ 

**Lemma 2.4.** For any positive integer $t$, we have

$$\Delta \left( \left[ K, \tilde{K} \right]_t \right) = \sum_{a=0}^{t} \left[ K, \tilde{K} \right]_{t-a} \tilde{K}^{-a} \otimes K^{t-a} \left[ K, \tilde{K} \right]_a.$$
Proof. We use induction on \( t \). If \( t = 1 \), we have that

\[
\Delta \left( \begin{bmatrix} K, \tilde{K} \\ 1 \end{bmatrix} \right) = \Delta \left( \frac{K - \tilde{K}^{-1}}{v - v^{-1}} \right) = \frac{K \otimes K - \tilde{K}^{-1} \otimes \tilde{K}^{-1}}{v - v^{-1}} = \frac{K - \tilde{K}^{-1}}{v - v^{-1}} \otimes K + \tilde{K}^{-1} \otimes \frac{K - \tilde{K}^{-1}}{v - v^{-1}} = \left[ K, \tilde{K} \right] \otimes K + \tilde{K}^{-1} \otimes \left[ K, \tilde{K} \right],
\]

as required.

Now assume that the equality holds for \( t = N \). We consider the case where \( t = N + 1 \). We have that

\[
\Delta \left( \begin{bmatrix} K, \tilde{K} \\ N + 1 \end{bmatrix} \right) = \Delta \left( \prod_{s=1}^{N+1} \frac{Kv^{-s+1} - \tilde{K}^{-1}v^{-s-1}}{v^s - v^{-s}} \right) = \Delta \left( \begin{bmatrix} K, \tilde{K} \\ N \end{bmatrix} \right) \Delta \left( \frac{Kv^{-N} - \tilde{K}^{-1}v^N}{v^{N+1} - v^{-N-1}} \right) = \sum_{a=0}^{N} \left( \begin{bmatrix} K, \tilde{K} \\ N - a \end{bmatrix} \tilde{K}^{-a} \otimes K^{N-a} \left[ K, \tilde{K} \right] \right) \left( \frac{Kv^{-N} \otimes K - \tilde{K}^{-1}v^N \otimes \tilde{K}^{-1}}{v^{N+1} - v^{-N-1}} \right)
\]

\[
= \sum_{a=0}^{N} \left( \begin{bmatrix} K, \tilde{K} \\ N - a \end{bmatrix} \tilde{K}^{-a} \otimes K^{N-a} \left[ K, \tilde{K} \right] \right) \left( \frac{Kv^{-N} - \tilde{K}^{-1}v^N}{v^{N+1} - v^{-N-1}} \otimes K + \frac{v^N \tilde{K}^{-1}}{[N + 1]} \otimes \left[ K, \tilde{K} \right] \right)
\]

\[
= \sum_{a=0}^{N} \left( \begin{bmatrix} K, \tilde{K} \\ N - a \end{bmatrix} \tilde{K}^{-a} \otimes K^{N-a} \left[ K, \tilde{K} \right] \right) \left( \frac{Kv^{-N} - \tilde{K}^{-1}v^N}{v^{N+1} - v^{-N-1}} \otimes K + \frac{v^N}{[N + 1]} \tilde{K}^{-a} \otimes K^{N-a} \left[ K, \tilde{K} \right] \left[ K, \tilde{K} \right] \right)
\]
This proves the lemma. \(\square\)

Note that a special case of the above result appeared in the proof of [1, Lemma 1.1(ii)]. However, we did not find any specific reference to the above calculations.

**Corollary 2.5.** With the notations as above, \(D_A\) is a Hopf \(A\)-subalgebra of \(D_v\), and both \(D_{\geq 0}^A\) and \(D_{\leq 0}^A\) are Hopf subalgebras of \(D_A\).

It is easy to see that \(D_{\geq 0}^A \cong D_{\geq 0}^A \otimes D_0^A\) and \(D_{\leq 0}^A \cong D_{\leq 0}^A \otimes D_0^A\). For any field \(k\) which is an \(A\)-algebra, we define \(D_k := k \otimes_A D_A\), \(D_{\geq 0}^k := k \otimes_A D_{\geq 0}^A\), and \(D_{\leq 0}^k := k \otimes_A D_{\leq 0}^A\).

**Remark 2.6.** It would be interesting to know if \(D_0^A\) is a free \(A\)-module and whether there is a triangular decomposition for the \(A\)-algebra \(D_A\).

**Corollary 2.7.** We consider \(Q(v)\) as an \(A\)-algebra in a natural way. Then the natural map \(Q(v) \otimes_A D_A \rightarrow D_v\), \(a \otimes x \mapsto ax\), \(\forall a \in Q(v), x \in D_A\), is a \(Q(v)\)-algebra isomorphism.
3. Representations of the algebra $D_q$

Throughout this section, we assume that $k$ is an algebraically closed field.

Let $M$ be a $D_q$-module such that $\text{End}_{D_q}(M) = k$. Note that the element $K\tilde{K}^{-1}$ is invertible and central in $D_q$. Therefore, there is an element $0 \neq z \in k$ such that $K\tilde{K}^{-1}$ acts as the scalar $z$ on $M$. For each $0 \neq z \in k$, we fix a square root $\sqrt{z}$ of $z$. Let $\pi_+^z$ be the $k$-algebra homomorphism $D_q \to U_q$ which is defined on generators by

$$
\pi_+^z(E) = \sqrt{z}E, \quad \pi_+^z(F) = F, \quad \pi_+^z(K) = \sqrt{z}K, \quad \pi_+^z(\tilde{K}) = \sqrt{z}^{-1}K.
$$

It is easy to check that $\pi_+^z$ is well-defined. Moreover, the kernel of $\pi_+^z$, which is the ideal generated by $K\tilde{K}^{-1} - z$, annihilates the module $M$. It follows that $M$ becomes a module over the algebra $U_q$ in a natural way. Similarly, we have a well-defined $k$-algebra homomorphism $\pi_-^z : D_q \to U_q$ which is defined on generators by

$$
\pi_-^z(E) = -\sqrt{z}E, \quad \pi_-^z(F) = F, \quad \pi_-^z(K) = -\sqrt{z}K, \quad \pi_-^z(\tilde{K}) = -\sqrt{z}^{-1}K.
$$

Note that $\pi_+^z$ is a Hopf algebra map, but in general, both $\pi_+^z$ and $\pi_-^z$ are not Hopf algebra maps.

We call a $D_q$-module $M$ a weight $D_q$-module if both $K$ and $\tilde{K}$ act semisimply on $M$. In that case, $K\tilde{K}^{-1}$ acts semisimply on $M$ as well. Similarly, we call a $U_q$-module $N$ a weight $U_q$-module if $K$ acts semisimply on $N$.

**Lemma 3.1.** Every finite dimensional simple (resp. indecomposable weight) $D_q$-module is the pull-back of a finite dimensional simple (resp. indecomposable weight) $U_q$-module through the algebra homomorphisms $\pi_+^z$ for some $0 \neq z \in k$.

**Proof.** For any finite dimensional $D_q$-module $M$, we consider the eigenspace of $K\tilde{K}^{-1}$ on $M$. Since $K\tilde{K}^{-1}$ is central in $D_q$, we deduce that each such eigenspace must be a $D_q$-submodule of $M$. Therefore, if $M$ is a simple $D_q$-module or an indecomposable weight $D_q$-module, then $K\tilde{K}^{-1}$ can have only one eigenvalue on $M$. This proves that $K\tilde{K}^{-1}$ acts as a scalar on $M$, hence the lemma follows immediately from the previous discussion. \qed

Let $M$ be a $U_q$-module. For any $0 \neq \lambda \in k$. We denote by $M_\lambda^+$ (resp. $M^-\lambda$) the pull-back of $M$ through the algebra homomorphism $\pi_+\lambda$ (resp. $\pi_-\lambda$).

**Theorem 3.2.** The category $\tilde{C}$ of finite dimensional weight $D_q$-modules is equivalent to a direct sum of $|k^\times|$ copies of the category $C$ of finite dimensional weight $U_q$-modules.

**Proof.** By definition, every object $M$ of $C$ is of the form $\oplus_{\lambda \in k^\times} M(\lambda)$, where for each $\lambda$, $M(\lambda)$ is a finite dimensional indecomposable weight $U_q$-module,
and $|\{\lambda \in k^\times \mid M(\lambda) \neq 0\}| < \infty$. We use $\theta^+$ to denote the functor from $\mathcal{C}$ to $\tilde{\mathcal{C}}$ such that 
\[
\theta^+ \left( \bigoplus_{\lambda \in k^\times} M(\lambda) \right) := \bigoplus_{\lambda \in k^\times} M(\lambda)_\lambda^+.
\]
The action of $\theta^+$ on the set of morphisms is defined in an obvious way. Then applying Lemma 3.1, we see that $\theta^+$ is an equivalence of categories. In a similar way, if we define $\theta^-$ to be the functor from $\mathcal{C}$ to $\tilde{\mathcal{C}}$ satisfying 
\[
\theta^- \left( \bigoplus_{\lambda \in k^\times} M(\lambda) \right) := \bigoplus_{\lambda \in k^\times} M(\lambda)_\lambda^-,
\]
and the action of $\theta^-$ on the set of morphisms is defined in an obvious way, then $\theta^-$ is also an equivalence of categories. \hfill \Box

Let $M$ be a $U_q$-module. Let $0 \neq z \in k$. Let $\varepsilon^+_z$ (resp. $\varepsilon^-_z$) be the one-dimensional representation of $D_q$ which is defined on generators by 
\[
\varepsilon^+_z(E) = 0 = \varepsilon^+_z(F), \quad \varepsilon^+_z(K) = \sqrt{z}, \quad \varepsilon^+_z(\bar{K}) = \sqrt{\bar{z}}^{-1}.
\]
(resp. $\varepsilon^-_z(E) = 0 = \varepsilon^-_z(F), \quad \varepsilon^-_z(K) = -\sqrt{z}, \quad \varepsilon^-_z(\bar{K}) = -\sqrt{\bar{z}}^{-1}$). It is easy to check that both $\varepsilon^+_z$ and $\varepsilon^-_z$ are well-defined. For any $z, z' \in k^\times$, we have that
\[
\varepsilon^+_z \otimes \varepsilon^+_z' \cong \varepsilon^+_z \otimes \varepsilon^+_z' \cong \begin{cases} 
\varepsilon^+_{zz'}, & \text{if } \sqrt{z} \sqrt{z'} = \sqrt{zz'}; \\
\varepsilon^-_{zz'}, & \text{if } \sqrt{z} \sqrt{z'} = -\sqrt{zz'};
\end{cases}
\]
and
\[
\varepsilon^+_z \otimes \varepsilon^-_z' \cong \varepsilon^-_z \otimes \varepsilon^+_z' \cong \begin{cases} 
\varepsilon^+_{zz'}, & \text{if } \sqrt{z} \sqrt{z'} = -\sqrt{zz'}; \\
\varepsilon^-_{zz'}, & \text{if } \sqrt{z} \sqrt{z'} = \sqrt{zz'}.
\end{cases}
\]

**Lemma 3.3.** Let $0 \neq z \in k$, let $M$ be a $U_q$-module. Then
1) $M^+_z \cong \varepsilon^+_z \otimes M_1$;
2) $\varepsilon^+_z \otimes M_1 \cong M_1 \otimes \varepsilon^+_z$ if and only if $\varepsilon^-_z \otimes M_1 \cong M_1 \otimes \varepsilon^-_z$, in that case, for any $0 \neq z' \in k$ and any $U_q$-module $N$, we have that
\[
M^+_z \otimes N^\pm_{z'} \cong \begin{cases} 
(M \otimes N)^+_\pm_{zz'}, & \text{if } \sqrt{z} \sqrt{z'} = \sqrt{zz'}; \\
(M \otimes N)_\pm^-_{zz'}, & \text{if } \sqrt{z} \sqrt{z'} = -\sqrt{zz'};
\end{cases}
\]
and
\[
M^+_z \otimes N^\pm_{z'} \cong \begin{cases} 
(M \otimes N)^+_\pm_{zz'}, & \text{if } \sqrt{z} \sqrt{z'} = -\sqrt{zz'}; \\
(M \otimes N)_\pm^-_{zz'}, & \text{if } \sqrt{z} \sqrt{z'} = \sqrt{zz'};
\end{cases}
\]
3) $\varepsilon^+_z \otimes M_1 \cong M_1 \otimes \varepsilon^+_z$ if and only if $\varepsilon^+_z \otimes \theta^+(M) \cong \theta^+(M) \otimes \varepsilon^+_z$. The same is true if we replace “$\theta^+$” by “$\theta^-$”.

**Proof.** The first statement follows from direct verification. The second and the third statements follow from the associativity of the tensor product and the previous discussion. \hfill \Box
Note that in general, the assumption $\varepsilon_+^z \otimes M_1 \cong M_1 \otimes \varepsilon_+^z$ in Lemma 3.3 (2) may not hold. However, it does hold in the following two cases:

Case 1. $q$ is not a root of unity, $M$ is an integrable weight module over $U_q$, i.e., both $E, F$ act locally nilpotently on $M$ and both $K, K^{-1}$ act semisimply on $M$. In this case, we claim that the $D_q$-modules $\varepsilon_+^z \otimes M_1$ and $M_1 \otimes \varepsilon_+^z$ are isomorphic to each other. In fact, since every integrable weight module over $U_q$ is completely reducible, we can reduce the proof to the case where $M$ is an irreducible highest weight module over $U_q$. Then $\varepsilon_+^z \otimes M_1 \cong M_2^+$ is an irreducible $D_q$-module. Note that the central element $KK^{-1}$ acts as the same scalar on both $\varepsilon_+^z \otimes M_1$ and $M_1 \otimes \varepsilon_+^z$. It follows that the $D_q$-action on both of these two modules can factored through the surjective homomorphism $\pi_{z'}$, for some $z' \in k^\times$. So these two modules can be naturally regarded as integrable weight modules over $U_q$. On the other hand, it is well-known that the isomorphism class of an integrable weight module over $U_q$ may not hold. However, it does hold in the following two cases: Case 2. $q$ is a primitive $d$th root of unity, $(\sqrt{z})^d = 1$, $M$ is a weight $D_q$-module such that all of the elements $E^d, F^d, K^d - 1$ act as 0 on $M$. In this case, both $\varepsilon_+^z$ and $M_1$ can be regarded as modules over the quotient algebra 

$$D_q/\langle E^d, F^d, K^d - 1, \bar{K}^d - 1 \rangle.$$  

Note that the algebra $D_q/\langle E^d, F^d, K^d - 1, \bar{K}^d - 1 \rangle$ is actually a Hopf algebra, and the natural homomorphism from $D_q$ onto $D_q/\langle E^d, F^d, K^d - 1, \bar{K}^d - 1 \rangle$ is indeed a Hopf algebra homomorphism, and $D_q/\langle E^d, F^d, K^d - 1, \bar{K}^d - 1 \rangle$ is indeed isomorphic to the Drinfel’d double of a Taft algebra. Hence it is a quasi-triangular Hopf algebra. As a consequence, both $M_1 \otimes \varepsilon_+^z$ and $\varepsilon_+^z \otimes M_1$ are isomorphic to each other as modules over $D_q/\langle E^d, F^d, K^d - 1, \bar{K}^d - 1 \rangle$, and hence are also isomorphic to each other as $D_q$-modules.

We use $\tilde{C}_0$ to denote the full subcategory of all the finite dimensional weight $D_q$-modules $\tilde{M}$ satisfying $\varepsilon_+^z \otimes \tilde{M} \cong \tilde{M} \otimes \varepsilon_+^z$ for any $z \in k^\times$, and we use $C_0$ to denote the full subcategory of all the finite dimensional weight $U_q$-modules $M$ satisfying $\varepsilon_+^z \otimes M_1 \cong M_1 \otimes \varepsilon_+^z$ for any $z \in k^\times$.

Lemma 3.3 (2) provides a very easy solution to the problem of decomposing the tensor product of certain $D_q$-modules, i.e., reducing them to the corresponding problem for $U_q$-modules, where it has been extensively studied and the results are well-known, see [19], [23] and [26]. Therefore, a large part of the representations (including all irreducible representations) of the quantum double $D_q$ can be realized as certain pullback from the representations of the quantized enveloping algebra $U_q$. Note that the representations of the
quantized enveloping algebra $U_q$ is well-understood (cf. [S]). In particular, the tensor product of finite dimensional simple $D_q$-modules is determined.

In the following, we shall summarize some results and corollaries for the algebra $D_q$. We mainly follow the formulation given in [S]. We fix a $0 \neq z \in k$. For each $0 \neq \lambda \in k$, let $M(\lambda)$ be the $U_q$-module defined in [S (2.4)]. By pulling back through $\pi_q^+$, we get a $D_q$-module $M_q^\pm(\lambda)$. We call it the Verma modules associated to $(z, \lambda)$. We have the following two results concerning Verma modules and simple modules over $D_q$ (compare with [S (2.4), (2.5)]).

**Corollary 3.4.** With the notations as above,

$$M_q^+(\lambda) \cong D_q/(D_qE + D_q(K - \sqrt{z}\lambda) + D_q(\bar{K} - \sqrt{z}^{-1}\lambda)),$$

and there is a $k$-basis $\{m_i\}_i \supseteq 0$ of $M_q(\lambda)$ such that for all $i$,

$$Km_i = \sqrt{z}\lambda q^{-2i}m_i,$$

$$Fm_i = m_{i+1},$$

$$Em_i = \begin{cases} 0, & \text{if } i = 0, \\ [i]_q \sqrt{z}\lambda q^{-1-i} - \lambda^{-1}q^{i-1} - q^{-1}m_{i-1}, & \text{otherwise}, \end{cases}$$

where

$$[i]_q := \frac{q^i - q^{-i}}{q - q^{-1}}.$$

The result for $M_q^-(\lambda)$ is similar.

**Corollary 3.5.** Suppose that $q$ is not a root of unity in $k$ and $0 \neq \lambda \in k$. If $\lambda \neq \pm q^n$ for all integers $n \geq 0$, then the $D_q$-module $M_q^\pm(\lambda)$ is simple. If $\lambda = \pm q^n$ for some integers $n \geq 0$, then $M_q^\pm(\lambda)$ has a unique maximal proper submodule which is spanned by all $m_i$ with $i \geq n + 1$ and is isomorphic to $M^\pm_q(q^{-2(n+1)}\lambda)$. In this case, the quotient of $M^\pm_q(\lambda)$ modulo the maximal proper submodule is an $(n + 1)$-dimensional simple $D_q$-module.

Suppose that $q$ is not a root of unity in $k$. By Corollary 3.5, we know that if $\lambda = q^n$ for some integers $n \geq 0$, we get two $(n + 1)$-dimensional simple $D_q$-modules, we denote it by $L^+_q(n, +), L^-_q(n, -)$; while if $\lambda = -q^n$ for some integers $n \geq 0$, we get two $(n + 1)$-dimensional simple $D_q$-modules, denoted by $L^+_q(n, -), L^-_q(n, +)$. Note that $L^+_q(n, -) \cong L^-_q(n, -), L^-_q(n, +) \cong L^+_q(n, -)$. Therefore, we define

$$L_q(n, +) := L^+_q(n, +), \quad L_q(n, -) := L^+_q(n, -).$$

In fact, the simple $D_q$-module $L_q(n, +)$ (resp. $L_q(n, -)$) is just the pullback of simple $U_q$-module $L(n, +)$ (resp. $L(n, -)$) through the $k$-algebra homomorphism $\pi^+_q$, see [S Theorem 2.6] for the definitions of $L(n, +)$ and $L(n, -)$. 


By construction, $L_z(n, +)$ has a basis $\{m_i\}_{i=0}^{n}$ such that

$$Km_i = z^{1/2}q^{-2i}m_i, \quad Km_i = z^{-1/2}q^{n-2i}m_i,$$

$$Fm_i = m_{i+1},$$

$$Em_i = \begin{cases} 0, & \text{if } i = 0, \\ z^{1/2}[i]q[n-1-i]m_{i-1}, & \text{otherwise}. \end{cases}$$

Similarly, $L_z(n, -)$ has a basis $\{m'_i\}_{i=0}^{n}$ such that

$$Km'_i = -z^{1/2}q^{-2i}m'_i, \quad Km'_i = -z^{-1/2}q^{n-2i}m'_i,$$

$$Fm'_i = m'_{i+1},$$

$$Em'_i = \begin{cases} 0, & \text{if } i = 0, \\ -z^{1/2}[i]q[n-1-i]m'_{i-1}, & \text{otherwise}. \end{cases}$$

Note that $L_z(n, +) \not\cong L_z(n, -)$. In fact, $L_z(n, +) \cong \varepsilon_1^{-1} \otimes L_z(n, -)$, where $\varepsilon_1^{-1}$ is the one-dimensional representation of $D_q$ which is defined on generators by $\varepsilon_1^{-1}(E) = 0 = \varepsilon_1^{-1}(F)$, $\varepsilon_1^{-1}(K) = -1 = \varepsilon_1^{-1}(\tilde{K})$.

**Corollary 3.6.** Suppose that $q$ is not a root of unity in $k$. If $ch k \neq 2$, then the set

$$\left\{ L_z(n, +), L_z(n, -) \mid 0 \neq z \in k, n \in N \cup \{0\} \right\},$$

is a complete set of pairwise inequivalent finite-dimensional simple weight $D_q$-modules; while if $ch k = 2$, then the set

$$\left\{ L_z(n, +) = L_z(n, -) \mid 0 \neq z \in k, n \in N \cup \{0\} \right\},$$

is a complete set of pairwise inequivalent finite-dimensional simple weight $D_q$-modules.

**Proof.** This follows from Lemma 3.1 and [8, Theorem 2.6].

Radford in [20] constructed a class of simple Yetter–Drinfel’d (shortly YD) modules for a graded Hopf algebra $H = \oplus_{n=0}^{\infty}H_n$ with $H_0$ both commutative and cocommutative. When $H$ is finitely graded over an algebraically closed field and $H_0$ is the group algebra of a finite abelian group, then all simple YD $H$-modules are in Radford’s class of simple YD modules. The Borel subalgebra $U_q^{\leq 0}$, denoted $H_\omega$ ($\omega = q^{-2}, a = K^{-1}$) in [20], is a simple pointed graded Hopf algebra, but not finitely graded. Thus we don’t know whether Radford’s class forms a proper subset of simple $D_q$-modules. Recall the Hopf algebra map $\theta'$ we introduced in Lemma 1.1. Using $\theta'$ and noting that the multiplication rule for our quantum double is compatible with the multiplication rule given in [12, Chapter IX, (4.3)] for the Drinfel’d quantum double of finite dimensional Hopf algebras, one sees easily that every
YD $U_q^{\leq 0}$-module naturally becomes a $D_q$-module. To keep in accordance with the notations used in [20] Proposition 4, we set $a = K^{-1}, x = F, g = a^l, \omega = q^{-2},$ and let $\beta : U_q^{\leq 0} \to K$ be an algebra homomorphism.

**Proposition 3.7.** With the notations as above and suppose that $q$ is not a root of unity in $k$, then

1) if $\beta(a) \neq \omega^{l+n}$ for any integer $n \geq 0$, then $\lambda^2 \neq q^{2n}$ for any integer $n \geq 0$, where $\lambda := \sqrt{\beta(a)}^{-1} q^{-l}, z := \beta(a)q^{-2l},$ in this case, the module $H_{\beta,kg}$ defined in [20] Corollary 1] is isomorphic (as $D_q$-module) to the infinite dimensional simple $D_q$-module $M_z^{+}(\lambda);$ 

2) if $\beta(a) = \omega^{l+n}$ for some integer $n \geq 0$, then we set $z = q^{-(n+2l)}$, then the module $H_{\beta,kg}$ defined in [20] Corollary 1] is isomorphic (as $D_q$-module) to the $(n+1)$-dimensional simple $D_q$-module $L_z(n, +)$ if $q^{-n-2l} = \sqrt{z}$; or to the $(n+1)$-dimensional simple $D_q$-module $L_z(n, -)$ if $q^{-n-2l} = -\sqrt{z}$.

*Proof.* 1) Let $\lambda := \sqrt{\beta(a)}^{-1} q^{-l}, z := \beta(a)q^{-2l}$. Then it is obvious that $\beta(a) \neq \omega^{l+n}$ for any integer $n \geq 0$ if and only if $\lambda^2 \neq q^{2n}$ for any integer $n \geq 0$. In this case, we know that (by Corollary 3.5) $M_z^{+}(\lambda)$ is a simple $D_q$-module. By [20] Proposition 4, (a) and the formula given in the paragraph below [20] Proposition 4], we have that

$$
\begin{cases}
\tilde{K}^{-1} \bullet g = a \bullet g = \beta(a)g, & K \bullet g = \varphi(K, K^{-1})g = q^{-2l}g,
\end{cases}
$$

On the other hand, by the formula given in the paragraph above Corollary 3.5, we have that

$$
\begin{cases}
\tilde{K}^{-1}m_0 = z\lambda^{-1}m_0 = \beta(a)m_0, & Km_0 = \lambda m_0 = q^{-2l}m_0,
\end{cases}
$$

By the universal property of the $D_q$-module $M_z^{+}(\lambda)$ (see Corollary 3.5), we deduce that the map which sends $m_0$ to $g$ can be uniquely extended to a homomorphism $\eta$ from $M_z^{+}(\lambda)$ to $H_{\beta,kg}$. Comparing the action of $F$ on the basis $\{x^i \bullet g\}_{i=0}^\infty$ given in the paragraph below [20] Proposition 4] and the action of $F$ on the basis $\{m_i\}_{i=0}^\infty$ given in the paragraph above Corollary 3.5, we know that $\eta(m_i) = x^i \bullet g$ for each $i \geq 0$, hence $\eta$ is an isomorphism, as required.

2) We consider only the case where $q^{-n-2l} = \sqrt{z}$, the other case is similar.

By the formula given in the paragraph below [20] Proposition 4], we have that

$$
\begin{cases}
\tilde{K}^{-1} \bullet g = a \bullet g = \omega^{l+n}g = q^{-2(l+n)}g, \\
K \bullet g = \varphi(K, K^{-1})g = q^{-2l}g, \\
E \bullet g = \varphi(E, K^{-1})g = 0.
\end{cases}
$$
On the other hand, by the formula given in the second paragraph below Corollary 3.5, we have that
\[
\tilde{K}^{-1}m_0 = q^{-2(l+n)}, \quad Km_0 = q^{-2l}m_0, \quad Em_0 = 0.
\]
By the universal property of the \(D_q\)-modules \(M^+_z(q^{-2l})\), \(L_z(n, +)\) (see Corollary 3.5), we deduce that the map which sends \(m_0\) to \(g\) can be uniquely extended to a homomorphism \(\eta'\) from \(M^+_z(q^{-2l})\) to \(H_{\beta, kg}\) and hence gives rises to a homomorphism \(\eta'\) from \(L_z(n, +)\) to \(H_{\beta, kg}\). Comparing the action of \(F\) on the basis \(\{x^i \cdot \beta g\}_{i=0}^{n}\) given in the paragraph below [20] Proposition 4 and the action of \(F\) on the basis \(\{m_i\}_{i=0}^{n}\) given in the second paragraph below Corollary 3.5, we know that \(\eta'(m_i) = x^i \cdot \beta g\) for each \(i \geq 0\), hence \(\eta'\) is an isomorphism, as required.

It would be interesting to know if every simple \(D_q\)-module comes from a simple YD \(U^{\otimes 0}_q\)-module when \(q\) is not a root of unity. If this is the case, we would know all simple YD \(U^{\otimes 0}_q\)-modules. Taking the advantage of the well established representation theory of the quantized enveloping algebra \(U_q(\mathfrak{sl}_2)\), we can easily obtain the decomposition of the tensor product of two finite dimensional \(D_q\)-modules while it might be difficult for the YD module setting of Radford in [20].

**Theorem 3.8.** Suppose that \(q\) is not a root of unity in \(k\). Let \(z, z' \in k^\times\).

Let \(m, n \in \mathbb{N} \cup \{0\}\). Then there is a decomposition of \(D_q\)-modules:

\[
L_z(m, \pm) \otimes L_{z'}(n, \pm) \cong \begin{cases} 
\bigoplus_{i=0}^{\min(m,n)} L_{zz'}(m + n - 2i, +) & \text{if } \sqrt{z} \sqrt{z'} = \sqrt{zz'} \\
\bigoplus_{i=0}^{\min(m,n)} L_{zz'}(m + n - 2i, -) & \text{if } \sqrt{z} \sqrt{z'} = -\sqrt{zz'}
\end{cases}
\]

\[
L_z(m, \pm) \otimes L_{z'}(n, \mp) \cong \begin{cases} 
\bigoplus_{i=0}^{\min(m,n)} L_{zz'}(m + n - 2i, -) & \text{if } \sqrt{z} \sqrt{z'} = \sqrt{zz'} \\
\bigoplus_{i=0}^{\min(m,n)} L_{zz'}(m + n - 2i, +) & \text{if } \sqrt{z} \sqrt{z'} = -\sqrt{zz'}
\end{cases}
\]

**Proof.** The theorem follows easily from Lemma 3.3 and the Clebsch–Gordan formula for \(L(n, \pm) \otimes L(m, \pm)\). \(\square\)

Many results for the quantized enveloping algebra \(U_q\) have their analogues for the algebra \(D_q\). For example, it is not hard to show that the element

\[
C = FE + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2}
\]

is equal to \(EF + \frac{Kq^{-1} + K^{-1}q}{(q - q^{-1})^2}\), and \(C\) is in the center of \(D_q\) (compare with [8 (2.7)]). In [8] §2.13, the classification of the finite dimensional simple \(U_q\)-modules are given when \(q\) is a primitive \(l\)-th root of unity with \(l\) odd (the even case is also similar). As a consequence, we have an analogous classification result for the algebra \(D_q\). For example, when \(q^2\) is a primitive \(d\)-th root of unity in \(k\) with \(d > 1\), we still have the well-defined simple \(D_q\)-modules \(L_z(n, \pm)\) for \(0 \neq z \in k\) and any integer \(n\) with \(0 \leq n < d\).
Lemma 3.9. Let $q^2$ be a primitive $d$-th root of unity in $k$ with $d > 1$. If $M$ is a finite dimensional simple weight $D_q$-module such that both $E^d$ and $F^d$ act as 0 on $M$, then $M$ is isomorphic to one of the following modules:

$$Z_{0,z}^\pm(\lambda), \; L_z(n, +), \; L_z(n, -), \; 0 \neq z \in k, \; 0 \leq n < d,$$

where $0 \neq z \in k, 0 \neq \lambda \in k$ with $\lambda^{2d} \neq 1$, and

$$Z_{0,z}^\pm(\lambda) := M_{z}^\pm(\lambda)/(D_q m_d).$$

4. Connections with the Drinfel’d double of the Taft algebra

Throughout this section, we assume that $k$ is an algebraically closed field, $1 < d \in \mathbb{N}$ and that $q^2 \in k$ is a primitive $d$-th root of unity.

We consider the quantized enveloping algebra $U_q$. It is well-known that the elements $E^d, F^d, K^d$ are central in the algebra $U_q$. Let $U_q^{\geq 0}$ (resp. $U_q^{\leq 0}$) be the quotient of the algebra $U_q^{\geq 0}$ (resp. $U_q^{\leq 0}$) modulo the ideal generated by $E^d, K^d - 1$ (resp. by $F^d, K^d - 1$). It is well-known that the ideal generated by $E^d, K^d - 1$ (resp. by $F^d, K^d - 1$) is a Hopf ideal. Hence the algebra $U_q^{\geq 0}$ (resp. the algebra $U_q^{\leq 0}$) is a quotient Hopf algebra of $U_q^{\geq 0}$ (resp. of $U_q^{\leq 0}$).

Recall the skew Hopf pairing between $U_q^{\geq 0}$ and $U_q^{\leq 0}$ defined in Section 1.

Lemma 4.1. The elements $E^d, K^d - 1 \in U_q^{\geq 0}, F^d, K^d - 1 \in U_q^{\leq 0}$ lie in the radical of the skew Hopf pairing. Moreover, the induced skew Hopf pairing between $U_q^{\geq 0}$ and $U_q^{\leq 0}$ is non-degenerate.

Proof. For convenience, we still denote by $E^a K^b$ the canonical image of $E^a K^b \in U_q^{\geq 0}$ in $U_q^{\geq 0}$, and do the same for the elements $F^a K^b \in U_q^{\leq 0}$. Note that the monomials $\{ E^a K^b \}_{0 \leq a, b < d}$ (resp. $\{ F^a K^b \}_{0 \leq a, b < d}$) form a $k$-basis of $U_q^{\geq 0}$ (resp. $U_q^{\leq 0}$). Recall that for the skew Hopf pairing $\varphi$ between $U_q^{\geq 0}$ and $U_q^{\leq 0}$,

$$\varphi(E^a K^b, F^{a'} K^{b'}) = 0$$

unless $a = a'$ (cf. [17, Proposition 1.2.3(d)])

With this in mind, the first statement of the lemma follows from a direct verification. It remains to show that the induced skew Hopf pairing is non-degenerate.

Suppose that $x := \sum_{0 \leq a, b < d} \lambda_{a,b} E^a K^b \in U_q^{\geq 0}$ (where $\lambda_{a,b} \in k$ for each $a, b$) lies in the radical of the induced skew Hopf pairing between $U_q^{\geq 0}$ and $U_q^{\leq 0}$. We want to show that $\lambda_{a,b} = 0$ for all $a, b$. 
Let $0 \leq a < d$ be a fixed integer. By assumption, we have that
\[ 0 = \varphi(x, F^a K^{b'}) = \sum_{0 \leq b < d} \lambda_{a,b} \varphi(E^a K^b, F^a K^{b'}) \text{ for } b' = 0, 1, 2, \ldots, d - 1. \]
It is not hard to calculate that
\[ \varphi(E^a K^b, F^a K^{b'}) = q^{-2bb'}[a]_q^1 \left( \frac{1}{1 - q^2} \right)^a. \]
Since $q^2$ is a primitive $d$th root of unity, it follows that $[a]_q^1 \neq 0$. Hence we get that
\[ \sum_{0 \leq b < d} \lambda_{a,b} q^{-2bb'} = 0, \text{ for } b' = 0, 1, 2, \ldots, d - 1. \]
Note that the coefficient matrix of the above system of linear equations is the Vandermonde matrix:
\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & q^{-2} & q^{-4} & \cdots & q^{-2(d-1)} \\
1 & (q^{-2})^2 & (q^{-4})^2 & \cdots & (q^{-2(d-1)})^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (q^{-2})^{d-1} & (q^{-4})^{d-1} & \cdots & (q^{-2(d-1)})^{d-1}
\end{pmatrix},
\]
which has the non-zero determinant. It follows that $\lambda_{a,b} = 0$ for all $0 \leq a, b < d$ and hence $x = 0$ as desired. In a similar way, one can prove that
\[ y := \sum_{0 \leq a, b < d} \lambda_{a,b} F^a K^b \in \mathcal{U}_q^{\leq 0} \] (where $\lambda_{a,b} \in k$ for each $a, b$) lies in the radical of the induced skew Hopf pairing between $\mathcal{U}_q^{\geq 0}$ and $\mathcal{U}_q^{\leq 0}$, then $y = 0$. This completes the proof of the lemma. \qed

Since $\mathcal{U}_q^{\leq 0}$ is of finite dimension, we have the following consequence of Lemma 4.1.

**Corollary 4.2.** With the above induced skew Hopf pairing, the associated quantum double of $\mathcal{U}_q^{\geq 0}$ and $\mathcal{U}_q^{\leq 0}$ is isomorphic to the usual Drinfel’d double (cf. [7], [12]) of $\mathcal{U}_q^{\leq 0}$ as a finite-dimensional $k$-Hopf algebra.

Denote by $\mathcal{D}_q$ the quantum double of $\mathcal{U}_q^{\geq 0}$ and $\mathcal{U}_q^{\leq 0}$ under the above skew Hopf pairing. Note that the ideal generated by $E^d$, $F^d$, $K^d - 1$, $\tilde{K}^d - 1$ is a Hopf ideal of $\mathcal{D}_q$.

**Theorem 4.3.** As a Hopf algebra, $\mathcal{D}_q$ is isomorphic to the quotient of $\mathcal{D}_q$ modulo the ideal generated by $E^d$, $F^d$, $K^d - 1$, $\tilde{K}^d - 1$.

Note that $\mathcal{U}_q^{\geq 0}$ and $\mathcal{U}_q^{\leq 0}$ are isomorphic as $k$-Hopf algebras. Thus $\mathcal{U}_q^{\leq 0}$ is a self-dual Hopf algebra. This Hopf algebra is usually called the Taft algebra, denoted by $T_d(q^{-2})$, as it was constructed in [24] as an interesting class of pointed Hopf algebras.
In [2], Chen classified the irreducible representations of the Drinfel’d double of \( T_d(q^{-2}) \) and studied their tensor products. We remark that most of the results obtained in [2] can be recovered easily from our Lemma 3.3 and the discussion below Lemma 3.3, and the corresponding known results for \( U_q(\mathfrak{sl}_2) \). For example, our Lemma 3.9 recovers the classification of simple \( D(T_d(q^{-2})) \)-modules obtained in [2, Proposition 2.4, Theorem 2.5]. The decomposition formula for the tensor product of two simple \( D(T_d(q^{-2})) \)-modules obtained in [2, Theorem 3.1] follows easily from our Lemma 3.3 and [23, Theorem 4.5] (see also [19]). Moreover, some results about finite dimensional indecomposable representations of \( D(T_d(q^{-2})) \) in [3] can be recovered from Theorem 4.3 and the results from [26].

5. Generalization to the case of arbitrary Cartan matrix

Our main results in Section 3 allow a direct generalization to the case of arbitrary Cartan matrix. To be precise, let \( A = (a_{i,j})_{1 \leq i,j \leq n} \) be a \( n \times n \) matrix with entries in \( \{-3, -2, -1, 0, 2\} \), \( a_{i,i} = 2 \) and \( a_{i,j} \leq 0 \) for \( i \neq j \). Suppose \( (d_1, \ldots, d_n) \) is a vector with entries \( d_i \in \{1, 2, 3\} \) such that the matrix \( (d_i a_{i,j}) \) is symmetric and positive definite. Then \( A \) is a Cartan matrix. Let \( \alpha_1, \ldots, \alpha_n \) be the set of simple roots in the corresponding root system.

Let \( k \) be a field. Let \( q \) be an invertible element in \( k \) satisfying \( q^{2d_i} \neq 1 \) for every \( 1 \leq i \leq n \). The quantized enveloping algebra \( U_q \) associated to the Cartan matrix \( A = (a_{i,j})_{1 \leq i,j \leq n} \) (cf. [6], [9] and [10]) is the associative \( k \)-algebra with generators \( E_i, F_i, K_i, K_i^{-1} (1 \leq i \leq n) \) and the relations:

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i, \\
K_i E_j = q^{d_i a_{i,j}} E_j K_i, \quad K_i F_j = q^{-d_i a_{i,j}} F_j K_i, \\
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_j} - q^{-d_i}}, \\
\sum_{r+s=1-a_{i,j}} (-1)^s \left[ \frac{1 - a_{i,j}}{s} \right] q^{d_i} E_i^r E_j E_i^s = 0, \quad \text{if } i \neq j, \\
\sum_{r+s=1-a_{i,j}} (-1)^s \left[ \frac{1 - a_{i,j}}{s} \right] q^{d_i} F_i^r F_j F_i^s = 0, \quad \text{if } i \neq j.
\]

\( U_q \) is a Hopf algebra with comultiplication, counit and antipode given by:

\[
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i, \\
\varepsilon(E_i) = 0 = \varepsilon(F_i), \quad \varepsilon(K_i) = 1 = \varepsilon(K_i^{-1}), \\
S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_i) = K_i^{-1}.
\]
Let $U_+^q$ (resp. $U_-^q$) be the $k$-subalgebra of $U_q$ generated by $E_i, 1 \leq i \leq n$ (resp. by $F_i, 1 \leq i \leq n$). Let $U_0^q$ be the $k$-subalgebra of $U_q$ generated by $K_i, K_i^{-1}, 1 \leq i \leq n$. Let $U_q^{\geq 0} := U_+^q U_0^q, U_q^{\leq 0} := U_q^{-1} U_q^0$. Then both $U_q^{\geq 0}$ and $U_q^{\leq 0}$ are Hopf $k$-subalgebras of $U_q$. For any monomials
\[
\left( \prod_{1 \leq i \leq n} F_i^{d_i} \right) \left( \prod_{1 \leq i \leq n} K_i^{b_i} \right) \in U_+^q, \quad \left( \prod_{1 \leq i \leq n} F_i^{a_i} \right) \left( \prod_{1 \leq i \leq n} K_i^{b_i} \right) \in U_-^q,
\]
we endow them the weights $\sum_{i=1}^n a_i \alpha_i, -\sum_{i=1}^n a_i \alpha_i$ respectively. Like the Hopf pair $(U_q(sl_2)^{\geq 0}, U_q(sl_2)^{\leq 0})$, there exists a unique pairing $\varphi : U_q^{\geq 0} \times U_q^{\leq 0} \rightarrow k$ (see [17], [11] and [25]) such that
\begin{enumerate}
\item $\varphi(1, 1) = 1, \varphi(1, K_i) = 1 = \varphi(K_i, 1)$, for $1 \leq i \leq n$,
\item $\varphi(x, y) = 0$, if $x, y$ are homogeneous with different weights,
\item $\varphi(E_i, F_j) = \delta_{ij} \frac{1}{q^{d_i} - 1}$, for $1 \leq i, j \leq n$,
\item $\varphi(K_i, K_j) = q^{d_i a_i, j} \varphi(K_i, K_j^{-1}) = q^{-d_i a_i, j}$, for $1 \leq i, j \leq n$,
\item $\varphi(x, y') = \varphi(\Delta^{op}(x), y' \otimes y'')$, for all $x \in U_q^{\geq 0}, y', y'' \in U_q^{\leq 0},$
\item $\varphi(x', y'') = \varphi(x \otimes x', \Delta(y''))$, for all $x, x' \in U_q^{\geq 0}, y'' \in U_q^{\leq 0},$
\item $\varphi(S(x), y) = \varphi(x, S^{-1}(y))$, for all $x \in U_q^{\geq 0}, y \in U_q^{\leq 0}$.
\end{enumerate}
In other words, $(U_q^{\geq 0}, U_q^{\leq 0}, \varphi)$ forms a skew Hopf pairing. Thus (as in Section 1) we can make $D(U_q^{\geq 0}, U_q^{\leq 0}) := U_q^{\geq 0} \otimes U_q^{\leq 0}$ into a Hopf $k$-algebra, called the quantum double of $(U_q^{\geq 0}, U_q^{\leq 0}, \varphi)$. For simplicity, we write $D_q$ instead of $D(U_q^{\geq 0}, U_q^{\leq 0})$.

**Theorem 5.1.** As a $k$-algebra, $D_q$ can be presented by the generators
\[
E_i, F_i, K_i, K_i^{-1}, \tilde{K}_i, \tilde{K}_i^{-1}, \quad (1 \leq i \leq n),
\]
and the following relations:
\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i,
\]
\[
\tilde{K}_i \tilde{K}_j = \tilde{K}_j \tilde{K}_i, \quad \tilde{K}_i \tilde{K}_i^{-1} = 1 = \tilde{K}_i^{-1} \tilde{K}_i, \quad K_i \tilde{K}_j = \tilde{K}_j K_i,
\]
\[
K_i E_j = q^{d_i a_i, j} E_j K_i, \quad K_i F_j = q^{-d_i a_i, j} F_j K_i,
\]
\[
\tilde{K}_i E_j = q^{d_i a_i, j} E_j \tilde{K}_i, \quad \tilde{K}_i F_j = q^{-d_i a_i, j} F_j \tilde{K}_i,
\]
\[
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - \tilde{K}_i^{-1}}{q^{d_i} - q^{-d_i}},
\]
\[
\sum_{r+s=1-a_i,j} (-1)^s \begin{bmatrix} 1 - a_i, j \\ s \end{bmatrix} q^{d_i} E_i^r E_j^s = 0, \quad \text{if } i \neq j,
\]
\[
\sum_{r+s=1-a_i,j} (-1)^s \begin{bmatrix} 1 - a_i, j \\ s \end{bmatrix} F_i^r F_j^s = 0, \quad \text{if } i \neq j.
\]
Let $M$ be a $D_q$-module such that $\text{End}_{D_q}(M) = k$. Note that the elements $K_i\tilde{K}_i^{-1}, i = 1, 2, \cdots, n$, are invertible central elements in $D_q$. Therefore, there is a vector $\vec{z} = (z_1, \cdots, z_n) \in (k^\times)^n$, such that for every $1 \leq i \leq n$, $K_i\tilde{K}_i^{-1}$ acts as the scalar $z_i$ on $M$. For each $0 \neq z \in k$, we fix a square root $z^{1/2}$ of $z$. Let $\pi^+_z$ be the $k$-algebra homomorphism $D_q \to U_q$ which is defined on generators by

$$\pi^+_z(E_i) = z_i^{1/2}E_i, \ \pi^+_z(F_i) = F_i, \ \pi^+_z(K_i) = z_i^{1/2}K_i, \ \pi^+_z(\tilde{K}_i) = z_i^{-1/2}K_i,$$

for every $1 \leq i \leq n$. It is easy to check that $\pi^+_z$ is well-defined. Moreover, the kernel of $\pi^+_z$, which is the ideal generated by $K_i\tilde{K}_i^{-1} - z_i, i = 1, 2, \cdots, n$, annihilates the module $M$. It follows that $M$ naturally becomes a module over the algebra $U_q$. Note that $\pi^+_z$ is in general not a Hopf algebra map unless $\vec{z} = (1, 1, \cdots, 1)$.

We call a $D_q$-module $M$ a weight $D_q$-module if $K_1, \cdots, K_n, \tilde{K}_1, \cdots, \tilde{K}_n$ all act semisimply on $M$. In that case, each $K_i\tilde{K}_i^{-1}$ acts semisimply on $M$ as well. Similarly, we call a $U_q$-module $N$ a weight $U_q$-module if $K_1, K_2, \cdots, K_n$ all act semisimply on $N$.

**Lemma 5.2.** Every finite dimensional simple (resp. indecomposable weight) $D_q$-module is the pull-back of a finite dimensional simple (resp. indecomposable weight) $U_q$-module through the algebra homomorphism $\pi^+_z$ for some $\vec{z} = (z_1, \cdots, z_n) \in (k^\times)^n$.

Let $M$ be a $U_q$-module. Let $\vec{z} = (z_1, \cdots, z_n) \in (k^\times)^n$. We use $M^+_{\vec{z}}$ to denote the pull-back of $M$ through the algebra homomorphism $\pi^+_z$. Let $\varepsilon^+_{\vec{z}}$ be the one-dimensional representation of $D_q$ which is defined on generators by

$$\varepsilon^+_{\vec{z}}(E_i) = 0 = \varepsilon^+_{\vec{z}}(F_i), \ \varepsilon^+_{\vec{z}}(K_i) = z_i^{1/2}, \ \varepsilon^+_{\vec{z}}(\tilde{K}_i) = z_i^{-1/2}, i = 1, 2, \cdots, n.$$

It is easy to check that $\varepsilon^+_{\vec{z}}$ is well-defined.

**Theorem 5.3.** The category $\tilde{C}$ of finite dimensional weight $D_q$-modules is equivalent to a direct sum of $|\{(k^\times)^n\}|$ copies of the category $C$ of finite dimensional weight $U_q$-modules.

With those one dimensional representations $\varepsilon^+_{\vec{z}}$ in mind, one can also formulate a version of Lemma $3.3$ in the context of arbitrary Cartan matrix. As before, this provides an easy solution to the problem of decomposing the tensor product of certain $D_q$-modules, i.e., reducing them to the corresponding problem for $U_q$-modules.

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Department of Applied Mathematics, Beijing Institute of Technology, Beijing, 100081, P.R. China

E-mail address: junhu303@yahoo.com.cn

School of Mathematics, Statistics and Computer Science, Victoria University of Wellington, PO Box 600, Wellington, New Zealand

E-mail address: yinhuo.zhang@vuw.ac.nz