Analytical study of quality-biased competition dynamics for memes in social media

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The spreading of news, memes and other pieces of information occurring via online social platforms has a strong and growing impact on our modern societies, with enormous consequences, that may be beneficial but also catastrophic. In this work we consider a recently introduced model for information diffusion in social media taking explicitly into account the competition of a large number of items of diverse quality. We map the meme dynamics onto a one-dimensional diffusion process that we solve analytically, deriving the lifetime and popularity distributions of individual memes. We also present a mean-field type of approach that reproduces the average stationary properties of the dynamics. In this way we understand and control the role of the different ingredients of the model, opening the path for the inclusion of additional, more realistic, features.

I. INTRODUCTION

Understanding how information spreads in social media is a topic of uttermost interest, as it is fundamental for devising strategies aimed at fostering the diffusion of beneficial information or contrasting the dangerous spread of fake news \[1\]–[3]. Activity in this area has boomed in recent years \[4]–[12]. From the point of view of statistical physics, information spreading is a prominent example of a collective macroscopic phenomenon emerging in a self-organized manner from the spontaneous activity of a large number of individual elements \[13]–[14]. The investigation of information spreading is particularly challenging both from an empirical point of view and from a theoretical one. The existence of many different social media platforms, each characterized by different features often changing over time, provides a wealth of data but leaves the issues of universality and reproducibility wide open. From the modeling point of view, the identification of a limited number of relevant mechanisms and crucial observable quantities is highly nontrivial.

The topology of the interaction pattern among users in online social media, which is usually very heterogeneous, is one of the ingredients usually taken into account. Another fundamental factor affecting the way news, memes or rumors are diffused is information overload. When online, individuals are hit by a steady and overwhelming flow of messages; the finite attention and limited memory strongly influence what information is propagated further and how. This results in a competition among a large number of items diffusing simultaneously, which is a key ingredient of many models for information spreading \[15]–[18]. A third ingredient that plays a role in determining the fate of messages in online media is the variability of the “quality” of the item: some pieces of information may be intrinsically more appealing and thus more likely to be shared by online users. A very recent work by Qiu et al. \[19\] considered together these three elements to study the interplay of an heterogeneous quality distribution and information overload in online social media (with particular reference to Twitter), with the goal of investigating whether a good tradeoff between discriminative power and quality diversity is possible.

Although highly stilized, the model for meme dynamics introduced in Ref. \[19\] contains several relevant ingredients of the real phenomenon and in particular the original element that the competition among different memes favors those having a higher intrinsic quality. For this reason we call it the quality-biased competition (QBC) model. In this paper we study the QBC dynamics in detail, by considering some carefully devised simplifications, which make possible an analytical treatment providing explicit formulas for the behavior of the main observables. In this way we achieve a full understanding of the model phenomenology and of its dependence on the value of the different parameters.

II. THE QBC MODEL

We consider the model for meme spreading introduced in Ref. \[19\]. \(N_{\alpha}\) agents (or users), each of them equipped with a memory containing at most \(\alpha\) memes, are the nodes of a static network. Memories are ordered lists from \(\alpha\) to 1. At each time step an individual is selected uniformly at random and transmits a meme to all her neighbors. With probability \(1-\mu\), the transmitted meme is an existing one, taken from the agent memory; otherwise, with probability \(\mu\), a new meme is created. In both cases, the transmitted meme is put at the top (position \(\alpha\)) of the memory of the agents involved (both the transmitter and the receivers) shifting all other memes downward. Each meme is attributed randomly, upon its creation, a fitness value \(f_{i}\) between 0 and 1, a proxy of its quality. When a user selects an old meme for transmission, the probability to select meme \(i\) is proportional to \(f_{i}\). In this way high fitness increases the chance of the meme...
to be spread. Apart from this bias, the dynamics can be seen as the competition among many susceptible-infected spreading processes in a metapopulation framework [20].

From the initial configuration with all empty memories, memes are introduced and copied until some of the memories fill up. When all slots in a memory are occupied and a new meme must enter, the item in the last position is removed and forgotten by the agent. Memories thus work according to a "first-in first-out" rule, mimicking what happens on users feeds of some social networks, such as Twitter. After an initial transient, a steady state is reached where all \( N_u \alpha \) memory slots in the system are occupied. Memes are continuously created, diffuse over the network and get eventually extinct. Quantities characterizing the dynamics of a meme are its lifetime, i.e., the time passed between the creation of a meme and its extinction, and its popularity, defined as the total number of times the meme is transmitted, throughout its lifetime, from an agent to one of her neighbors.

### III. ROBUSTNESS WITH RESPECT TO THE TOPOLOGY

We first check how much the model phenomenology depends on details of the interaction pattern, by performing numerical simulations on several types of network (see Supplementary Material, SM). It turns out that the distributions of the main observables are qualitatively robust with respect to changes of the underlying network (see Fig. [1]). Both distributions have broad power-law tails, cutoff exponentially over a scale growing when \( \mu \), the rate of creation of new items, goes to zero. The lifetime distribution also exhibits a peak for \( l \) of the order of \( \alpha \), corresponding to the average time needed for a meme that is not shared to disappear from the memory of the agent that created it. The average values of the popularity and of the lifetime strongly grow with the fitness when \( \mu \) is small. The effect of the parameter \( \alpha \) is very weak (see SM). The overall picture remains the same even if the contact pattern is an annealed random regular graph where each node has a single connection. This suggests that a mean-field approach, which effectively considers a regular annealed network as contact pattern, may provide an accurate description of the model dynamics.

### IV. A MICROSCOPIC APPROACH

We focus now on the behavior of an individual meme of fitness \( f \). We define as \( 0 \leq N_i(t) \leq \alpha \) the position of meme \( i \) in the memory of agent \( j \) at time \( t \): \( N_{ij} = \alpha \) corresponds to the top position (a newly created or transmitted meme), while \( N_{ij} = 1 \) means that the meme is about to be forgotten. If meme \( i \) does not appear in the memory of agent \( j \), then \( N_{ij} = 0 \). We neglect the case in which an agent has more copies of the same meme in his feed. The quantity \( N_i(t) = \sum_j N_{ij}(t) \) cumulates the positions of the meme in all users’ feeds, thus providing information about its overall diffusion. For simplicity we assume that each user is in contact with a single randomly chosen other user and that, when with probability \( \mu \) a user produces a new meme, she simply puts it on top of her memory, without immediately sharing it. For the same reason we assume that, when an existing meme is selected for transmission, it is left in the original position in the transmitter feed, without putting it at the top of the memory. We checked that both these assumptions have negligible effects. The quantity \( N_i(t) \) performs over time a one dimensional random walk in the interval \([0, \alpha N_u]\). \( N_i = 0 \) is an absorbing boundary condition (after extinction a meme will never reappear) and \( N_i = \alpha N_u \) is a semi reflecting boundary (because of our approximation, if the meme is in the first position of all feeds, \( N_i \) cannot grow further). The initial condition is \( N_i = \alpha \). At each time step the elementary events are:

\[
N_i(t) = \begin{cases} 
    n \to n + \alpha & \text{with prob. } R_n \\
    n \to n - 1 & \text{with prob. } L_n \\
    n \to n & \text{with prob. } S_n = 1 - R_n - S_n.
\end{cases}
\]

(1)

Apart from different expressions close to the boundaries (see SM for details), the probabilities are:

\[
R_n = (1 - \mu) \frac{C_n}{N_u} \left(1 - \frac{C_n}{N_u}\right) \frac{f}{\alpha},
\]

(2)

and

\[
L_n = \frac{C_n}{N_u},
\]

(3)

where \( f \) is the fitness of the considered meme and \( C_n \) is the number of individuals possessing \( i \) in their memory.

Eq. (2) is derived based on the consideration that \( N_i \) is increased by \( \alpha \) if a transmission event takes place (it happens with probability \( 1 - \mu \)), if meme \( i \) is present in the feed of the transmitting user (probability \( C_n/N_u \)) and not present in the feed of the receiver (1 - \( C_n/N_u \)) and if meme \( i \) is selected for transmission among all memes in the feed. This last event occurs with probability \( f_i/ \sum_{j \in M_u} f_j \), which we approximate with \( f/\alpha \). \( C_n \) is the number of individuals possessing \( i \) in their memory, that we approximate as

\[
C_n = \left\lfloor \frac{n + \alpha - 1}{\alpha} \right\rfloor,
\]

(4)

where \( \lfloor x \rfloor \) represents the integer part (floor) of \( x \).

With regard to \( L_n \), the value of \( N_i(t) \) decreases because the insertion of a new meme in a user feed causes the downward shift of all other memes. The insertion occurs at each time step, irrespective of whether the inserted meme is new or transmitted. Hence \( L_n = \frac{C_n}{N_u} \), the likelihood that meme \( i \) is present in the involved memory. From the expressions of the probabilities it is immediately clear that nothing depends on \( f \) and \( \mu \) separately, but only through the combination \( \beta = (1 - \mu)f \).
We simulate numerically this random walk description of meme dynamics. In the SM we show that the popularity and lifetime distributions obtained match very closely those found for the original QBC model.

In order to make the analytical treatment easier, we further simplify the random-walk description. In particular, we remove the floor function from Eq. (4), we set equal to 1 the term $(1 - C_n/N_u)$ in Eq. (2) and we introduce a numerical constant $\gamma = (\alpha + 1)\frac{\alpha - 1}{\alpha^2 N_u}$ in the denominator of Eq. (4). See the SM for the justification of these modifications. Again we numerically check the distributions generated by this simplified random-walk description and find (see SM) that they are essentially equal to those of the original QBC dynamics.

At this point we can write down the master equation for the modified random walk, which reads

$$P_n(t + \Delta t) = S_n P_n(t) + L_{n+1} P_{n+1}(t) + R_{n-\alpha} P_{n-\alpha}(t)$$  \hspace{1cm} (5a)

$$P_{\alpha N_u}(t + \Delta t) = (1 - \mu)P_{\alpha N_u}(t) + \sum_{j=0}^{\alpha - 1} R_{\alpha N_u - j} P_{\alpha N_u - j}(t)$$  \hspace{1cm} (5b)

where Equation (5a) holds for $n = 0, 1, ... \alpha N_u - 1$ provided one considers $R_{n - \alpha} = 0$ for $n = 0, 1, ... \alpha$ and $\Delta t = N_u^{-1}$.

By setting $x_n = n/\gamma \alpha N_u$ with $x_n$ ranging between 0 and $1/\gamma$ and taking the thermodynamic limit $N_u \to \infty$, from the master equation we obtain (see SM) the Fokker-Planck (FP) equation for the probability $\rho(x,t)$ that the walker is in position $x$ at time $t$:

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{1 - \beta}{\gamma \alpha} \frac{\partial}{\partial x} x \rho(x,t) + \frac{1 + \beta \alpha}{2 \gamma^2 \alpha^2 N_u} \frac{\partial^2}{\partial x^2} x \rho(x,t).$$  \hspace{1cm} (6)

For large $N_u$ we have $\gamma = (\alpha + 1)/\alpha$. 

Figure 1. a) Average popularity as a function of fitness in the QBC model on an annealed random regular graph with degree distribution $P(k) = \delta_{k,1}$ for different values of $\mu$ with fixed $\alpha = 10$ and $N_u = 10^5$. Averages are performed over $10^5$ memes. b) Popularity probabilities for the same system. c) Average lifetime for the same system. d) Lifetime PDF for the same system.
A. Purely diffusive dynamics

In the limit $\beta \to 1$ the drift term in Eq. (6) vanishes. We are left with the FP equation of a purely diffusive stochastic process:

$$\frac{\partial}{\partial t} \rho(x, t) = D_0 \frac{\partial^2}{\partial x^2} x \rho(x, t),$$

(7)

where

$$D_0 = \frac{1 + \alpha}{2\gamma^2 \alpha^2 N_u} = \frac{1}{2(1 + \alpha)N_u},$$

(8)

which differs from standard diffusion because of the space-dependent diffusion coefficient. The limit $\mu \to 0$ changes also the boundary conditions: the boundary in $x = 1/\gamma$ is semireflecting because $N_i(t) = N_u \alpha$ can decrease with probability $\mu$ or remain unchanged, with probability $1 - \mu$. Thus in the case $\mu = 0$ both the boundary condition in $x = 0$ and in $x = 1/\gamma$ are absorbing: $\rho(0, t) = \rho(1/\gamma - 1, t) = 0$. The initial condition is $\rho(x, t) = 0$ with $x_\alpha = 1/(\gamma N_u)$.

It is possible to find the solution of this equation as an eigenfunction expansion of the operator $L_{FP} = D_0 \frac{\partial^2}{\partial x^2} x$ (see SM for details), obtaining:

$$\rho(x, t) = \frac{\pi^2 \gamma}{2} \sqrt{x} \sum_{n=1}^{\infty} nJ_1(\pi n \sqrt{x}) J_1(\pi n \sqrt{x}) e^{-t/\tau_n},$$

(9)

where $J_1(z)$ is a Bessel function of the first kind. The characteristic time scale of each eigenfunction is

$$\tau_n = \frac{8\gamma \alpha^2 N_u}{(1 + \alpha)j_{1,n}^2} \sim \frac{8\gamma \alpha N_u}{\pi^2 n^2}.$$

(10)

where the $j_{1,n}$, the zeros of $J_1(z)$, are approximated as $j_{1,n} \approx \pi n$. Using this expression, it is possible to compute (see SM for details) the survival probability in the limit $\mu \to 0$, which turns out to be

$$S(t) \approx \begin{cases} 1 & t \ll \alpha \\ \alpha t^{-1} e^{-t/\tau} & t \gg \alpha \end{cases}$$

(11)

where $\tau = 8\alpha N_u / \pi^2$ is $\tau_1$ after the approximation $j_{1,n} \to \pi n$ is made. Based on this result the lifetime distribution can be computed (see SM). In the limit of large $N_u$, i.e., diverging $\tau$, it reads

$$F(l) = -\frac{dS}{dl} \bigg|_{l=1} \approx \begin{cases} 0 & l \ll \alpha \\ \alpha l^{-2} & l \gg \alpha \end{cases}$$

(12)

This expression of $F(l)$ accounts for the most important feature observed in simulations: for $l \ll \tau$ (notice that $\tau$ diverges with $N_u$) the distribution decays as a power-law with exponent $\eta_l = 2$. Simulations of the QBC model with all memes having fitness $f = 1$ agree with this analytical prediction (see Fig. 2). By means of the standard argument connecting the exponents of power-law tails for scaling variables (see SM) it is possible to relate $\eta_l$ with the analogous exponent $\eta_p$ for the popularity distribution: $\eta_l = s(\eta_p - 1) + 1$, where $p \sim l^s$. Simulations yield a value close to $s = 2$, from which $\eta_p = 3/2$, in good agreement with simulations (see SM).

B. Pure drift

The opposite limit for the FP equation (6) is the pure drift case, which always holds in the large $N_u$ limit, as $D_0 \propto N_u^{-1}$, unless $\mu = 0$ and $f = 1$:

$$\frac{\partial}{\partial t} \rho(x, t) = \frac{1}{\tau_d} \frac{\partial}{\partial x} x \rho(x, t)$$

(13)

where

$$\frac{1}{\tau_d} = \frac{1 - \beta}{\gamma \alpha} = [1 - (1 - \mu) f](\alpha + 1).$$

(14)

This equation describes a deterministic motion

$$x(t) = x_\alpha e^{-t/\tau_d},$$

(15)

i.e., the meme position drifts exponentially toward $x = 0$; in other words the systematic drift attracts walkers toward the absorbing boundary. This introduces an additional exponential cutoff in the lifetime distribution, which can be globally written as

$$F(l) \propto \alpha l^{-2} e^{-l/\tau_d}$$

(16)

in agreement with simulations (see Fig. 2, inset).
C. Average over the fitness

In the original definition of the QBC model the fitness is a random variable uniformly distributed between 0 and 1. Using Eq. (10) it is possible to compute the lifetime distribution also in this case, by averaging over f (see SM) and obtaining, in the limit $\mu \to 0$:

$$F(f)(t) \approx \alpha(\alpha + 1)t^{-3} \left[1 - e^{-(\alpha+1)t}\right].$$

(17)

The exponent of the lifetime distribution is then $\eta_l = 3$, in reasonable agreement with Fig. [1]. A similar conclusion can be drawn for the popularity distribution, predicted to decay as $p^{-2}$.

In summary, by means of a mapping of QBC dynamics onto a random-walk description, we have derived expressions for for the lifetime and popularity distributions, which account for the phenomenology observed in numerical simulations.

V. A MACROSCOPIC APPROACH

The microscopic approach allows to determine the dependence of the average lifetime on the fitness and hence estimate the average number $N_f$ of memes with given $f$ in the steady state. However, the same quantities can be derived much more easily by a simple approach of mean-field type, focused directly on the temporal evolution of the $N_f$. For simplicity we assume that fitness values are discretized in $F$ classes and, again, that the degree of each agent is 1. We define $N_f(t)$ as the average number of memes with fitness $f$ present in the system at time $t$. This quantity changes over time because of two possible gain and two possible loss processes. The creation of a new meme, occurring at rate $\mu$, increases $N_f$ by 1 with a probability $1/F$ (if the created meme has exactly fitness $f$), but it may also reduce $N_f$ by 1 if the agent creating the new meme forgets a meme of fitness $f$. This last event occurs with probability $N_f/(N_a\alpha)$. The transmission of an existing meme, occurring at rate $1-\mu$, increases $N_f$ if the transmitting agent has a meme with fitness $f$ in her feed (probability proportional to $N_f$) and the meme is selected (probability proportional to $f$). Overall the normalized probability of the event is $fN_f/[\sum_{f'}f'N_{f'}]$. Finally also the transmission event may lead to an agent forgetting a meme with fitness $f$ with probability $N_f/(N_a\alpha)$. The temporal evolution of the $N_f$ is then given by the set of coupled equations

$$\dot{N}_f(t) = \mu \left[\frac{1}{F} - \frac{N_f(t)}{N}\right] + (1-\mu) \left[\frac{fN_f(t)}{\sum_{f'}fN_{f'}(t)} - \frac{N_f(t)}{N}\right].$$

(18)

which conserves the total number $\sum_f N_f$. Straightforward numerical integration of Eq. (15) allows to determine the stationary values of the $N_f$ and hence of the densities $n_f = N_f/(N_a\alpha/F)$, where $N_a\alpha/F$ is the average number of memes with fitness $f$ if all $F$ classes were populated uniformly. The comparison with the outcome of numerical simulations (see Fig. [1]) confirms a satisfactory agreement.

VI. CONCLUSIONS

In this paper we have studied the model for information diffusion recently introduced in Ref. [19]. We have been able to derive analytically the lifetime distribution and other properties for a simplified version of the dynamics, which reproduces the phenomenology of the original model.

Our treatment of the QBC model allows to understand how broad tails in the lifetime and popularity distributions, observed empirically, arise. A power-law distribution with an exponent $\eta_p \approx 2$ is in agreement with the observations of Ref. [19], where hashtags are used to identify Twitter memes. On the other hand, other studies using hashtags give quite different results from the QBC model predictions. In Ref. [16] a power-law decay for meme lifetime has been observed, with an exponent $\eta_l \approx 2.5$. This value is not far but distinct from the value $\eta_l$ predicted by the QBC model in the case of uniformly distributed fitness. Moreover, the strong correlation between meme lifetime and popularity (see SM) is not observed in Twitter data [8] [18], even if proxies different from hashtags are used to identify memes [8]. A stringent empirical validation of models of online information spreading is itself a difficult task because of the apparent lack of universality. Referring to Twitter data, the identification of memes as URLs leads to a lognormal distribution of popularity [3], the analysis of retweet cascades leads to a size distribution with exponent $\eta_s \approx 2.3$ [12] with possibly an exponential cutoff [3] and reply trees
give $\eta_s \approx 4^{[10]}$. Looking at other data sources, the landscape is even more varied: the popularity distribution, estimated from Facebook data, exhibits a power-law decay with exponent $\eta_s \approx 2.1^{[9]}$, while popularity-lifetime correlations are shown to be different between Digg and Youtube data $^{[21]}$. One could easily change, within the QBC model, the fitness distribution to achieve a better agreement with these observations. In any case it is clear that the QBC model is a gross oversimplification of the real meme diffusion process in online social media. To make the QBC dynamics less unrealistic several hypotheses underlying the present version of model could be lifted. Some of them, such as a nonuniform fitness distribution or a nonlinear dependence on $f$ of the probability of selecting a meme, can be easily treated within the present analytical approach. Other fundamental generalizations, such as agent-dependent values of $\alpha$ and $\mu$ or heterogeneous rates of individual activation, can be investigated by means of straightforward numerical simulations. One of the ingredients adding realism to the QBC dynamics is the consideration of agents that do not accept in their feeds (and thus do not spread further) memes they have already seen in the past. The effect of this long-term memory is briefly discussed in the SM, but the main result is the change of the popularity and lifetime distributions, that lose their power-law tail. At a more general level, one of the weak points of the QBC model is its insensitivity with respect to changes of the contact pattern topology. While this feature allows our mean-field approach to be successful, empirical data contradict this result: one of the main pieces of evidence is the existence of influential spreaders, i.e. users which, because of their position in the social network have a disproportionate effect on meme dynamics $^{[7,8,22]}$. The investigation of increasingly sophisticated models for information spreading and the comparison with the ever larger body of empirical data available remains a challenging avenue for future research.

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Supplementary Material for
Analytical study of quality-biased competition dynamics for memes in social media

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I. ROBUSTNESS OF THE QBC PHENOMENOLOGY

In order to understand the role played by the underlying network structure, we perform numerical simulations of the QBC dynamics on several types of network. A quenched scale-free network with degree distribution $P(k) \propto k^{-2.5}$ and a quenched regular random network with $P(k) = \delta_{k,20}$, both undirected and built according to the configuration model, are employed. The dynamics are investigated also on an annealed random regular network with $P(k) = \delta_{k,1}$. Figure 1 shows that the distributions are virtually independent from the topology of the substrate. The only small variation concerns width of the peak of the lifetime distribution.

In the QBC model, as defined in Ref. [1], at each time step the transmitted meme is put at the top of the memory of the agents involved (both the transmitter and the receivers). For simplicity, in the theoretical analysis we change this rule of the model, putting the meme at the top of the memory of the receiving agents only. It turns out that the distributions of the main observables are qualitatively robust with respect to this modification of meme dynamics.

The other parameter of the dynamics is the number $\alpha$ of memes that can be stored in the memory of each user. The effect of variations of $\alpha$ is very simple (Fig. 2): the popularity distribution has practically no dependence on $\alpha$; the peak of the lifetime distribution is instead shifted towards the right, the broad tail for long lifetimes remaining unchanged. These results are easily interpreted. The intrinsic temporal scale of the life of a meme is the time passing from creation until extinction of a meme that is never shared. This scale, clearly proportional to $\alpha$, determines the position of the peak and of the rest of the distribution. The popularity of a meme depends on how many times it is shared, hence it is not influenced by the size of user memories.

II. PROBABILITIES FOR THE RANDOM WALK ELEMENTARY EVENTS

We recall that $N_i(t) = \sum_j N_{ij}(t)$, where $N_{ij}(t)$ is the position of meme $i$ in the memory of agent $j$ at time $t$. Given this variable the elementary events of the random walk description of the dynamics are

$$N_i(t) = \begin{cases} n \to n + \alpha & \text{with prob. } R_n \\ n \to n - 1 & \text{with prob. } L_n \\ n \to n & \text{with prob. } S_n = 1 - R_n - L_n. \end{cases}$$

The probabilities for the event increasing $n$ are

$$R_n = \begin{cases} 0 & \text{if } n = \alpha N_u \\ 0 & \text{if } n = 0 \\ (1 - \mu) \frac{C_n}{N_u} (1 - \frac{C_n}{N_u}) \frac{L}{\alpha} & \text{if } 0 < n < \alpha N_u. \end{cases}$$

The justification of Eq. (2) for the case $0 < n < \alpha N_u$ is already presented in the main text. When $n = 0$, $R_n = 0$ because when a meme becomes extinct it cannot eventually reappear. When $n = \alpha N_u$ instead, $R_n = 0$ because we have assumed that no meme can appear twice in the feed of an agent.

$$L_n = \begin{cases} \mu & \text{if } n = \alpha N_u \\ 0 & \text{if } n = 0 \\ \frac{C_n}{N_u} & \text{if } 0 < n < \alpha N_u, \end{cases}$$

where

$$C_n = \left\lfloor \frac{n + \alpha - 1}{\alpha} \right\rfloor$$

is an approximation of the number of individuals possessing $i$ in their memory. Also the motivation for Eq. (3) for $0 < n < \alpha N_u$ is presented in the main text. The boundary condition in $n = 0$ is trivial. For $n = \alpha N_u$, $L_n = \mu$ because if a new meme is created it necessarily occupies the top position of the memory of a user, thus shifting downwards the previous meme in the same position.

III. COMPARISON BETWEEN THE QBC MODEL THE RANDOM WALK DESCRIPTION

In the context of the random walk description, the popularity of a meme is defined as the number of steps in the positive direction during its entire lifetime. In Figure 3(a-d) we show that the popularity and the lifetime distributions obtained within the random walk model match very closely those found for the QBC model when the latter is simulated with all memes having the same fitness $f = 1$. 
The first simplifying assumption we make in the random walk description is that at most one copy of a meme can be in the feed of a single agent. As a consequence, $N_{ij}$
FIG. 2: (a) Average popularity as a function of fitness in the QBC model on an annealed regular network with degree distribution $P(k) = \delta_{k,1}$ for different values of $\alpha$ with fixed $\mu = 0.1$ and $N_u = 10^3$. Averages are performed over $10^5$ memes. b) Popularity probabilities for the same system. c) Average lifetime for the same system. d) Lifetime PDF for the same system.

can be at most equal to $\alpha$ and thus the maximum value of $N_i = \sum_{ij} N_{ij}$ is $\alpha N_u$. Apart from its convenience for calculations, this choice is realistic for online social media, that do not present more than once the same meme in the feed of an individual.

An additional simplification to make the analytical treatment easier is the elimination of the nonlinearity introduced by the floor function in Eq. (4). Since already Eq. (4) is a rather crude approximation of the real value of $C_n$ and nevertheless it does not introduce major inaccuracies (see Fig. 3), we expect this further simplification not to affect significantly the results.

The other, more important, simplification is the removal of the factor $(1 - C_n/N_u)$ in Eq. (2) for $0 < n < \alpha N_u$. This is equivalent to assume that the number of individuals having a given meme in their memories is always much smaller than the total number of agents $N_u$. The reason for this choice is that it strongly simplifies the mathematical treatment. However, it can be a posteri-ori justified (see below). Unfortunately, this approximation has a drawback, as it may lead to negative values of $S_n = 1 - R_n - L_n$ for large $n$. For this reason a numerical constant $\gamma$ is introduced in the denominator of $C_n$. This factor ensures that $S_n \geq 0$ for any $n$, provided that $\gamma$ is equal to $(\alpha + 1) \frac{\alpha N_u + \alpha - 1}{\alpha N_u}$, which in the large $N_u$ limit becomes $\gamma = (\alpha + 1)/\alpha$.

The distributions generated by this simplified random-walk description, shown in Figure 3(e-f), are essentially those of the original random-walk model and of the QBC dynamics.
\[ P_{\alpha N_u}(t + \Delta t) = (1 - \mu) P_{\alpha N_u}(t) + \sum_{j=0}^{j=\alpha} R_{\alpha N_{u-j}} P_{\alpha N_{u-j}}(t). \]

V. THE FOKKER-PLANCK EQUATION

Given the elementary events, the master equation of the random-walk is

\[ P_n(t + \Delta t) = S_n P_n(t) + L_{n+1} P_{n+1}(t) + R_{n-\alpha} P_{n-\alpha}(t) \]

(5a)

and fitness \( f = 1 \). Averages over \( 10^5 \) memes. (b) Lifetime PDF for the same system. (c) Popularity probabilities in the random-walk model for different values of \( \mu \) with fixed \( \alpha = 10 \), \( N_u = 10^3 \) and \( f = 1 \). Averages over \( 5 \cdot 10^6 \) walkers. (d) Lifetime PDF for the same system. (e) Popularity probabilities in the simplified random-walk model for different values of \( \mu \) with fixed \( \alpha = 10 \), \( N_u = 10^3 \) and \( f = 1 \). Averages over \( 5 \cdot 10^6 \) walkers. (f) Lifetime PDF for the same system.
By noting that \( R_n = \beta L_n/\alpha \) and recalling that \( S_n = 1 - R_n - L_n \), Eq (5a) can be rewritten as
\[
P_n(t + \Delta t) - P_n(t) = L_{n+1}P_{n+1}(t) + \beta L_{n-\alpha}P_{n-\alpha}(t)/\alpha + \cdots - (1 + \beta/\alpha)L_nP_n(t).
\]
(6)
Defining \( x_n \equiv (n/\alpha N_u) \) and taking the thermodynamic limit \( N_u \to \infty \) we obtain \( L_n = x_n + \frac{\alpha - 1}{\gamma \alpha N_u} \to L(x) = x \).
Recalling that \( \Delta t = N_u^{-1} \), the left hand side of Eq. (6) is
\[
P_n(t + \Delta t) - P_n(t) \to \rho(x, t + \Delta t) - \rho(x, t) \simeq \Delta t \frac{\partial}{\partial t} \rho(x, t),
\]
while the right hand side can be rewritten as
\[
f \left( x + \frac{1}{\gamma \alpha N_u} \right) + \frac{\beta}{\alpha} f \left( x - \frac{1}{\gamma N_u} \right) - \left( 1 + \frac{\beta}{\alpha} \right) f(x)
\]
where \( f(x) = L(x)\rho(x, t) \). Expanding the function \( f(x) \) to second order with respect to \( x \) in the first and second term we obtain the Fokker-Plank (FP) equation for the probability \( \rho(x, t) \) that the walker is in position \( x \) at time \( t \):
\[
\frac{\partial}{\partial t} \rho(x, t) = \frac{1 - \beta}{\gamma} \frac{\partial}{\partial x} x \rho(x, t) + \frac{1 + \beta \alpha}{2 \gamma^2 \alpha^2 N_u} \frac{\partial^2}{\partial x^2} x \rho(x, t).
\]
(7)

VI. PURELY DIFFUSIVE DYNAMICS

In the limit \( \beta \to 1 \) Eq. 7 becomes
\[
\frac{\partial}{\partial t} \rho(x, t) = D_0 \frac{\partial^2}{\partial x^2} x \rho(x, t),
\]
(8)
where
\[
D_0 = \frac{1 + \alpha}{2 \gamma^2 \alpha^2 N_u} = \frac{1}{2(\alpha + 1) N_u}.
\]
(9)
Because of the expressions (2) and (3) for \( R_n \) and \( L_n \), the boundary in \( x = \gamma^{-1} \) becomes absorbing for \( \mu \to 0 \). Thus we search for a solution of Eq. (8) with boundary conditions \( \rho(x = 0, t) = \rho(x = \gamma^{-1}, t) = 0 \) \( \forall t \) and with initial condition \( \rho(x, t = 0) = \delta(x - x_\alpha) \) where \( x_\alpha = 1/\gamma N_u \).
We look for a solution in the form of an eigenfunction expansion [2]
\[
\rho(x, t) = \sum_{\lambda} A_\lambda \rho_\lambda(x) e^{-\lambda t},
\]
(10)
with
\[
A_\lambda = D_0 x_\alpha \rho_\lambda(x_\alpha).
\]
(11)
Inserting this expression into Eq. (8) we obtain the equation for the eigenfunctions \( \rho_\lambda(x, t) \)
\[
\frac{\partial^2}{\partial x^2} x \rho_\lambda(x) + \frac{\lambda}{D_0} \rho_\lambda(x) = 0.
\]
(12)
Redefining the eigenvalues as \( \lambda^2 \equiv \frac{\lambda}{\alpha} \) and defining \( Q_\lambda(x) = x \rho_\lambda(x) \) we obtain an equation for \( Q_\lambda \) whose general (real) solution is [3] (pag 362, 9.1.50)
\[
Q_\lambda(x) = b_1 \sqrt{x} J_1(\Lambda \sqrt{x}) + b_2 \sqrt{x} Y_1(\Lambda \sqrt{x}),
\]
(13)
where \( J_1(z) \) and \( Y_1(z) \) are Bessel functions of the first and second kind, respectively, and \( b_1 \) and \( b_2 \) real coefficients. The boundary conditions \( Q_\lambda(x = 0) = 0 \) and \( Q_\lambda(x = \gamma^{-1}) = 0 \) imply that \( b_2 = 0 \) and that the eigenvalues \( \Lambda \) are discretized \( \Lambda = \gamma^{1/2} j_{1,n} \) (where the \( j_{1,n} \) are the zeros of \( J_1(z) \)), so that
\[
Q_n(x) = x \rho_\lambda(x) = b_{1,n} \sqrt{x} J_1(j_{1,n} \sqrt{x}), \quad n = 1, 2, \ldots
\]
(14)
In this way we find
\[
\rho(x, t) = \sum_{n=1} A_n \frac{J_1(j_{1,n} \sqrt{x})}{\sqrt{x}} e^{-t/\tau_n},
\]
(15)
where the coefficients \( b_{1,n} \) for each of the \( \rho_n(x) \) are still to be determined by the initial condition and where we have defined
\[
\tau_n = \frac{1}{N_u} = \frac{4}{D_0 a^2 n} = \frac{8 \gamma a^2 N_u}{(1 + \alpha)^2 j_{1,n}^2}.
\]
(16)
Given the coefficients in Eq. (11), the solution for \( t = 0 \) has the form
\[
\rho(x, 0) = D_0 \sqrt{\frac{x_\alpha}{x}} \sum_{n=1} b_{1,n} J_1(j_{1,n} \sqrt{x_\alpha}) J_1(j_{1,n} \sqrt{x}) =
\]
\[
D_0 \sqrt{\frac{x_\alpha}{x}} \sum_{n=0} b_{1,n} J_1(j_{1,n} \sqrt{x_\alpha}) J_1(j_{1,n} \sqrt{x}) \to
\]
\[
D_0 \sqrt{\frac{x_\alpha}{x}} \int_0^\infty b_{1,n} J_1(j_{1,n} \sqrt{x_\alpha}) J_1(j_{1,n} \sqrt{x}) \, dn.
\]
(17)
We note that the zeros of \( J_1(z) \) are very well approximated by the expression \( j_{1,n} = \pi n \) (see Fig. 4), in particular for the large values of \( n \) we are interested in. Replacing the expression \( j_{1,n} = \pi n \) in Eq. (17) we obtain
\[
\rho(x, 0) = D_0 \sqrt{\frac{x_\alpha}{x}} \int_0^\infty b_{1,n} J_1(\pi n \sqrt{x_\alpha}) J_1(\pi n \sqrt{x}) \, dn.
\]
(18)
Exploiting the representation of the Dirac delta function in terms of Bessel functions
\[
\delta(a - b) = a \int_0^\infty t J_t(a t) J_t(b t) \, dt.
\]
(19)
it is not difficult to see that \( \rho(x, 0) = \delta(x - x_\alpha) \), provided \( b_{1,n}^2 = \frac{\pi^2}{2 D_0 \alpha} \).
In summary, we have found the solution of the FP Equation (8) satisfying the imposed boundary and initial conditions is
\[
\rho(x, t) = \frac{\pi^2 \gamma}{2} \sqrt{\frac{x_\alpha}{x}} \sum_{n=1} n J_1(\pi n \sqrt{x_\alpha}) J_1(\pi n \sqrt{x}) e^{-t/\tau_n},
\]
(20)
FIG. 4: First $10^4$ zeros $j_{1,n}$ of the Bessel function $J_1(x)$ as a function of $n$ (symbols). The line depicts the approximate expression $\pi n$.

FIG. 5: Plot of the density of walkers $\rho(x,t)$ [from Eq. (20), the sum is truncated after the first $10^3$ terms] for a time $t = 100$ such that the lifetime distribution is in the power-law regime. Other parameters are $\mu = 0$, $f = 1$, $N_u = 10^3$, $\alpha = 2$.

with the characteristic time scale of each eigenfunction

$$\tau_n = \frac{8\gamma\alpha^2N_u}{(1 + \alpha)\pi^2n^2}. \quad (21)$$

This result allows us to justify a posteriori the removal of the factor $(1 - C_n/N_n)$ from Eq. (2). Indeed, even in the case of pure drift ($\mu = 0, f = 1$), which is the most favorable for reaching high values of $x$, the density of walkers over temporal scales of interest is practically vanishing for values of $x$ well below the boundary $\gamma^{-1}$ (see Fig. 5).

FIG. 6: First $10^3$ values of $J_0(\pi n)$.

VII. THE SURVIVAL PROBABILITY AND THE LIFETIME DISTRIBUTION

Given the density [Eq. 20], the survival probability

$$S(t) = \int_0^{\gamma^{-1}} \rho(x,t) \, dx \quad (22)$$

has the form $\sum_{n=1}^\infty S_n e^{-t/\tau_n}$ with

$$S_n = \frac{\pi^2\gamma\sqrt{\alpha}}{2} n J_0(\pi n \sqrt{\gamma x\alpha}) \int_0^{1/\gamma} J_1(\pi n \sqrt{\gamma x}) \, dx. \quad (23)$$

The integral can be performed analytically to find

$$S_n = \pi \sqrt{\gamma x\alpha} J_0(\pi n \sqrt{\gamma x\alpha}) \left[ 1 - J_0(\pi n) \right]. \quad (24)$$

To simplify this expression we note that $x\alpha \propto N_u^{-1}$ so the factor $\sqrt{\gamma x\alpha} J_0(\pi n \sqrt{\gamma x\alpha})$ can be estimated as its limit for small argument, yielding

$$S_n = \frac{\pi^2\gamma\alpha n}{2} \left[ 1 - J_0(\pi n) \right]. \quad (25)$$

We note that $J_0(\pi n)$ is a succession of values that oscillate around zero with monotonically decreasing absolute value (Fig. 6). Given this shape we can neglect the contribution of the term $J_0(\pi n)$ compared to 1 in Eq. (25)

$$S_n = \frac{\pi^2\gamma\alpha}{2} n. \quad (26)$$

From Eq. (21), the ratio $t/\tau_n$ can be rewritten as

$$\frac{t}{\tau_n} = \frac{(1 + \alpha)\pi^2 n^2}{8\gamma\alpha^2N_u} \equiv \theta n^2. \quad (27)$$

The full expression of $S(t)$ is then

$$S(t) = \sum_{n=1}^\infty S_n e^{-t/\tau_n} \rightarrow \frac{\pi^2\gamma\alpha}{2} \int_1^\infty n e^{-\theta n^2} \, dn = \frac{\pi^2\gamma\alpha e^{-\theta}}{2\theta} = \alpha e^{-1} e^{-t/\tau}, \quad (28)$$
where \( \tau \equiv \tau_1 = 8\alpha^2Nu \left(\frac{1}{1 + \alpha}\right)\pi^2 = \frac{8\alpha Nu}{\pi^2} \), and the expression for \( \gamma \) in the limit of large \( Nu \) has been used.

We note that this expression diverges for \( t \to 0 \), while obviously \( S(t = 0) = 1 \). This is a consequence of the fact that we have assumed the initial condition \( x_\alpha \propto N_u^{-\frac{1}{2}} \) infinitely close to zero. The expression (28) of \( S(t) \) is actually valid only after a certain temporal scale \( t_m \), that is the time needed by a walker starting in \( x_\alpha \) to reach the boundary in zero: before that time, \( S(t) = 1 \). This time can be estimated by imposing, in Eq. (28), \( S(t_m) = 1 \), yielding \( t_m = \alpha \). The full expression of the survival probability is thus given by

\[
S(t) \simeq \begin{cases} 
1 & t \ll \alpha \\
\alpha l^{-1}e^{-t/\tau} & t \gg \alpha 
\end{cases}
\]  

(29)

Given the expression for the survival probability, computing the lifetime distribution is straightforward:

\[
F(l) = -\frac{dS}{dt}
\bigg|_{t=l} \simeq \begin{cases} 
0 & l \ll \alpha \\
\alpha l^{-2}[1 + \frac{l}{\tau}]e^{-l/\tau} & l \gg \alpha 
\end{cases}
\]  

(30)

VIII. THE POPULARITY DISTRIBUTION

The popularity of a meme and its lifetime are connected by a scaling relation (see Fig. 7): \( p \sim t^s \), with \( s \approx 2 \). Both quantities are power-law distributed with the popularity distribution decaying as \( p^{-\eta p} \) and the lifetime distribution decaying as \( l^{-\eta} \). Conservation of probability implies that the exponents are related by

\[
\eta = s(\eta_p - 1) + 1.
\]  

(31)

The value of the exponent \( \eta = 2 \) for the lifetime allows us to predict for the popularity distribution a decay with an exponent

\[
\eta_p = 3/2.
\]  

(32)

This value is in good agreement with the simulations of the model, shown in Figure 3(a).

IX. AVERAGE OVER THE FITNESS

From the most general expression for the lifetime distribution in the case the fitness is fixed

\[
F(l) = \alpha l^{-2}e^{-l/\tau_d},
\]

(33)

where \( 1/\tau_d = [(1 - (1 - \mu)f)/(\alpha + 1)] \), it is possible to calculate the distribution for the case of heterogeneously distributed fitnesses. In the original definition of the QBC the fitness is a random variable uniformly distributed between 0 and 1. Hence by averaging over \( f \), we get

\[
F(f)(l) = \int_0^1 F(l) df = \alpha l^{-2} \int_0^1 e^{l/\tau_d} df = \frac{\alpha(\alpha + 1)}{1 - \mu} l^{-3} \left[ e^{-l(\alpha + 1)} - e^{-l(\alpha + 1)} \right].
\]  

(34)

In the limit \( \mu \to 0 \),

\[
F(f)(l) = \alpha(\alpha + 1)l^{-3} \left[ 1 - e^{-l(\alpha + 1)} \right].
\]  

(35)

The exponent of the lifetime distribution is then \( \eta_l = 3 \), in reasonable agreement with simulations of the original QBC model, shown in Figure 1(f).

The scaling exponent \( s \) in the relation \( p \sim t^s \) is found to be \( s = 2 \) also in the QBC model with uniform random fitness: popularity as a function of lifetime results in a plot (not shown) which is very similar to Figure 7. Using the scaling relation Eq. (31) and the value \( \eta_l = 3 \), the popularity distribution is predicted to decay with an exponent \( \eta_p = 2 \).

X. LONG TERM MEMORY

One of the possible ingredients adding realism to the QBC dynamics is the consideration of agents that do not accept in their feeds (and thus do not spread further) memes they have already seen in the past. This modification of the dynamics considerably changes the phenomenology observed for small values of \( \mu \): the broad tails of the lifetime and popularity distribution do not appear any more. In particular the lifetime distribution shows a second peak and the popularity distribution is characterized by a plateau of constant value. Both the distributions decay exponentially after a certain value. These results are illustrated in Fig. 8. It is important to note that the cutoffs are not due to the finite size of the system. This different behaviour with respect to the original QBC model can be interpreted in term of the new role the fitness plays. Its role in the QBC model is well explained by our analytical approach, that shows how a fitness \( f < 1 \) is analogous to \( \mu \neq 0 \): memes with
a generic fitness $f$ can have any value of the popularity and of the lifetime until the cutoff introduced by the drift term in Eq. (7) is reached. This ceases to be true with the long-term memory addition: the new ingredient introduces a strong correlation between fitness and popularity, shown in Fig 9. In Ref. [1] a classification of real data in high and low quality doesn’t show this kind of correlation and, on the contrary, it shows that low and high quality memes have similar chances of going viral.

[1] Qiu, X., F.M. Oliveira D., Sahami Shirazi A., Flammini A. and Menczer F., Nature Human Behavior, 1, 0132 (2017).
[2] H. Risken, “The Fokker-Planck Equation. Methods of Solution and Applications” Springer series in synergetics, 18 (1989).
[3] M. Abramowitz I. Stegun, “Handbook of mathematical functions: with formulas, graphs, and mathematical tables”, Courier Corporation, 55, (1964).
[4] Digital Library of Mathematical Functions, Integral and Series Representations of the Dirac Delta http://dlmf.nist.gov/1.17
FIG. 8: (a) Popularity probabilities in the modified QBC model with long-term memory effect on a scale free network with degree distribution $P(k) = k^{-2.5}$ for different values of $\mu$ with fixed $\alpha = 10$ and $N_u = 10^4$. Averages over $2 \cdot 10^5$ memes. (b) Lifetime PDF for the same system.

FIG. 9: Popularity probabilities in the modified QBC model with long-term memory effect on a scale free network with degree distribution $P(k) = k^{-2.5}$ with fixed $\mu = 0.01$, $\alpha = 10$ and $N_u = 10^4$. Here data are categorized according to the fitness. Averages over $2 \cdot 10^5$ memes.