On the relationship between Uhlig extended and beta-Bartlett processes

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Abstract

Stochastic volatility processes are used in multivariate time-series analysis to track time-varying patterns in covariance structure. Uhlig extended and beta-Bartlett processes are especially useful for analyzing high-dimensional time-series because they are conjugate with Wishart likelihoods. In this article, we show that Uhlig extended and beta-Bartlett processes are closely related, but not equivalent: their hyperparameters can be matched so that they have the same forward-filtered posteriors and one-step ahead forecasts, but different joint (smoothed) posterior distributions. Under this circumstance, Bayes factors can’t discriminate the models and alternative approaches to model comparison are needed. We illustrate these issues in a retrospective analysis of volatilities of returns of foreign exchange rates. Additionally, we provide a backward sampling algorithm for the beta-Bartlett process, for which retrospective analysis had not been developed.

Keywords: Stochastic volatility, state-space models, Bayesian model comparison
1 Introduction

Time-series data with time-varying covariance structure arise in fields as diverse as finance, neuroimaging or online marketing. In such applications, stochastic volatility processes are necessary for successful forecasting and decision making. As illustrated in West (2020), models with conjugate sequential updates are particularly attractive for analyzing high-dimensional time-series, since implementing richly-parametrized models that require Markov chain Monte Carlo methods for posterior inference (e.g., Lopes et al. (2010), Nakajima and West (2012), Kastner et al. (2017), and Shirota et al. (2017)) may not be computationally feasible.

Two classes of stochastic volatility processes that are conjugate with Wishart likelihoods coexist in the literature: matrix-beta processes, which build upon Uhlig (1997), and beta-Bartlett processes, which were first used in Quintana et al. (2003). To this date, the most flexible matrix-beta process is the Uhlig extended process (Windle and Carvalho, 2014), which is the one we consider herein.

Our main contributions are (1) studying the relationship between Uhlig extended and beta-Bartlett processes (Section 3) and (2) providing the first backward sampler in the literature for beta-Bartlett processes (Section 4). The hyperparameters of the processes can be matched so that they yield the same marginal likelihoods, so Bayes factors can’t be used to distinguish them. In Section 5 we describe approaches to model comparison that don’t regard the processes as equivalent, which we then apply in a foreign exchange rates application in Section 6. We end the article with conclusions in Section 7.

2 Notation and Bartlett decomposition

We use the notation $1 : n$ to compactly denote the set $\{1, 2, \ldots, n\}$. Following Prado and West (2010), we use the notation $\mathcal{D}_t$ for our “information set” at time $t$. Before we observe any data, our prior knowledge is denoted $\mathcal{D}_0$. At time $t \in 1 : T$, the information set is $\mathcal{D}_t = \{\mathcal{D}_0, y_{1:t}\}$. We denote $q$-dimensional normal random variables with mean $\mu$ and covariance matrix $\Sigma$ as $N_q(\mu, \Sigma)$, chi-squared random variables with $k > 0$ degrees of free-
dom as $\chi^2_k$, and Beta random variables with shape $a > 0$ and scale $b > 0$ as $\text{Beta}(a,b)$.

The less common Wishart$_q(k,A)$ and MatrixBeta$_q(n/2,k/2)$ distributions are as defined in [Windle and Carvalho (2014)] and the supplementary material to this article. For extrema, we use the notation $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$. Finally, we use the notation $\text{uchol}(\cdot)$ for the function that returns the upper-triangular Cholesky factor of a symmetric positive-definite matrix.

The models we study rely heavily on the Bartlett decomposition of Wishart-distributed matrices, which we now review. Let $W \sim \text{Wishart}_q(k,A)$ be a $q \times q$ random matrix with $k > 0$ and symmetric positive-definite $A$. Its Bartlett decomposition is $W = (UP)'UP$, where $P = \text{uchol}(A)$ and upper-triangular $U = (u_{ij})_{i,j \in 1:q}$ with $u_{ij} \sim \text{N}_1(0,1)$ for $i < j$, which are independent of $u_{ii} \sim \chi^2_{k-i+1}$ for $i,j \in 1:q$.

### 3 Uhlig extended and beta-Bartlett processes

[Windle and Carvalho (2014)] extend a model that was originally proposed in Uhlig (1997). Given $(q \times q)$-dimensional symmetric positive-definite matrices $\{y_t\}_{t \in 1:T}$, their model can be written as

$$y_t \mid \Phi_t^U \sim \text{Wishart}_q(k,(k\Phi_t^U)^{-1}), \quad \Phi_t^U = (U_{t-1}^UP_{t-1})' \Psi_t U_{t-1}^UP_{t-1} / \lambda,$$

where $\Psi_t \sim \text{MatrixBeta}_q(n/2,k/2)$ and $U_{t-1}^U$ and $P_{t-1}^U$ come from the Bartlett decomposition $\Phi_t^{U_{t-1}} = (U_{t-1}^UP_{t-1})'U_{t-1}^UP_{t-1}$. The model is completed with the prior $\Phi_0^U \mid D_0 \sim \text{Wishart}_q(n+k,(kD_0^U)^{-1})$. The hyperparameters are $0 < \lambda < 1$, $n > q - 1$, and $k$, which is either a positive integer less than $q$ or a real number greater than $q - 1$. We refer to the process on $\{\Phi_t^U\}_{t \in 0:T}$ implied by the model above as to the Uhlig extended (UE) process.

The prior distributions and forward-filtered posteriors, as derived in [Windle and Carvalho (2014)], are given in Table 1.

In contrast, the beta-Bartlett (BB) stochastic volatility process [Quintana et al., 2003] can be written as

$$y_t \mid \Phi_t^B \sim \text{Wishart}_q(k,(k\Phi_t^B)^{-1}), \quad \Phi_t^B = (\tilde{U}_tP_{t-1}^B)\tilde{U}_tP_{t-1}^B / b,$$

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where $P_{t-1}^B$ is defined via the Bartlett decomposition $\Phi_{t-1}^B = (U_{t-1}^BP_{t-1}^B)/U_{t-1}^BP_{t-1}$ and $\tilde{U}_t = (\tilde{u}_{i,j,t})_{i,j \in 1:q}$ is constructed by modifying the diagonal elements of $U_{t-1}^B = (u_{i,j,t-1})_{i,j \in 1:q}$ as explained in Table 1. The hyperparameters of the model are $k > 0$, $0 < \beta < 1$, $0 < b < 1$, and $k_0 > 0$, which appears in the prior $\Phi_0 | \mathcal{D}_0 \sim \text{Wishart}(k_0, (kD_0^B)^{-1})$ for symmetric positive-definite $D_0^B$. We refer to the process defined on $\{\Phi^B_t\}_{t \in 0:T}$ as to the beta-Bartlett (BB) process. The prior distributions and forward-filtered posteriors with this model can be found in Table 1.

| Table 1: Comparison of Uhlig extended and beta-Bartlett models. |
|-----------------------------------------------|
| **Uhlig extended** | **beta-Bartlett** |
| Likelihood | $y_t | \Phi_t^U \sim \text{Wishart}_q(k, (k\Phi_t^U)^{-1})$ | $y_t | \Phi_t^B \sim \text{Wishart}_q(k, (k\Phi_t^B)^{-1})$ |
| State evol. | $\Phi_t^U = (U_{t-1}^U P_{t-1}^U)' \Psi_t U_{t-1}^U P_{t-1}^U/\lambda$ | $\Phi_t^B = (\tilde{U}_t P_{t-1}^B)/U_{t-1}^BP_{t-1}$ |
| Error | $\Psi_t \sim \text{MatrixBeta}(n/2, k/2)$ | $\tilde{u}_{i,j,t} = u_{i,j,t-1}$ ($i \neq j$) |
| Post. at $t-1$ | $\Phi_t^U | \mathcal{D}_{t-1} \sim \text{Wishart}_q(n + k, (kD_t^U)^{-1})$ | $\Phi_t^B | \mathcal{D}_{t-1} \sim \text{Wishart}_q(k_{t-1}, (kD_{t-1}^B)^{-1})$ |
| Prior at $t$ | $\Phi_t^U | \mathcal{D}_{t-1} \sim \text{Wishart}_q(n, (k\lambda D_t^U)^{-1})$ | $\Phi_t^B | \mathcal{D}_{t-1} \sim \text{Wishart}_q(\beta k_{t-1}, (\beta k_{t-1}D_{t-1}^B)^{-1})$ |
| Post. at $t$ | $\Phi_t^U | \mathcal{D}_t \sim \text{Wishart}_q(n + k, (kD_t^U)^{-1})$ | $\Phi_t^B | \mathcal{D}_t \sim \text{Wishart}_q(k_t, (kD_t^B)^{-1})$ |
| $D_t^U = \lambda D_t^U + y_t$ | $D_t^B = bD_{t-1}^B + y_t$ and $k_t = \beta k_{t-1} + k$ |

The priors, forward-filtered posteriors, and one-step ahead forecast distributions of the models defined in Equations (1) and (2) coincide under the condition

$$k_0 = n + k, \quad \beta = n/(n + k), \quad b = \lambda \quad \text{and} \quad D_0^B = D_0^U. \quad (3)$$

The change of variables is bijective, so if the hyperparameters are set by maximizing the marginal likelihoods of the models, the condition is satisfied.

However, the two processes are not equivalent under Equation (3) because the conditionals $\Phi_t^U | \mathcal{D}_{t-1}$ and $\Phi_t^B | \mathcal{D}_{t-1}$ differ. Assume Equation (3) holds, $\Phi_{t-1}^B = \Phi_{t-1}^U = (U_{t-1}P_{t-1})'U_{t-1}P_{t-1}$ and $\mathcal{D}_{t-1}$ are given, and $U_{t-1} = (u_{i,j,t-1})_{i,j \in 1:q}$. If the conditionals were equal in distribution, they would be equal in expectation. Using results
In general, according to Konno (1988),

$$E(\Phi^U_t - \Phi^B_t | \Phi_{t-1}, D_{t-1}) = \frac{1}{\lambda} P'_{t-1} \left[ \frac{n}{n+k} U_{t-1} U_{t-1}' - E(\tilde{U}_{t} \tilde{U}_t | \Phi_{t-1}, D_{t-1}) \right] P_{t-1}$$

and the conditionals

$$E[\tilde{U}_{t} \tilde{U}_t]_{ij} | \Phi_{t-1}, D_{t-1}] = \sum_{l=1}^{i\wedge j-1} u_{l;i,t-1} u_{l;j,t-1} + \delta_{ij} \left( \frac{(n-i+1)u_{ii,t-1}^2}{n-i+1+k} + (1-\delta_{ij})g(i,j) \right)$$

$$g(i,j) = \frac{\Gamma \left( \frac{n-i+j}{2} \right) \Gamma \left( \frac{n-i+j+k+1}{2} \right)}{\Gamma \left( \frac{n-i+j+1}{2} \right) \Gamma \left( \frac{n-i+j+k}{2} \right)} u_{i\wedge j,i\wedge j,t-1} u_{i\wedge j,i\wedge j,t-1},$$

where $(\tilde{U}_{t} \tilde{U}_t)_{ij}$ is the $(i,j)$th element of $\tilde{U}_{t} \tilde{U}_t$, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

In general, $E[(\tilde{U}_{t} \tilde{U}_t)]_{ij} | \Phi_{t-1}, D_{t-1}]$ and $nU_{t-1}U_{t-1}/k$ aren’t equal: if $U_{t-1}$ is diagonal, $E[(\tilde{U}_{t} \tilde{U}_t)]_{ii} | \Phi_{t-1}, D_{t-1}] = (n-i+1)u_{ii,t-1}^2/(n-i+1+k) \neq nu_{ii,t-1}^2/(n+k)$ for $i \in 2 : q$.

This distinction affects the (smoothed) posterior distributions $\Phi^U_{0:T} | D_T$ and $\Phi^B_{0:T} | D_T$. Dropping process superscripts, the posterior distribution can be factorized as

$$p(\Phi_{0:T} | D_T) = p(\Phi_T | D_T) \prod_{t=1}^{T-1} p(\Phi_t | \Phi_{t+1}, D_t),$$

and the conditionals $p(\Phi_t | \Phi_{t+1}, D_t)$ of UE and BB processes are different. For the UE model, we have $\Phi_t = \lambda \Phi_{t+1} + Z_t$, where $Z_t \sim \text{Wishart}_q(k,(kD^B_t)^{-1})$; for the BB model, see Section 4. Below, we compare the expectations and variances of the conditionals in a concrete example to prove our claim, as well as to gain some intuition.

**Example 1** Assume Equation (3) holds, let $k = 1$, $P_t = \text{uchol}((kD_t)^{-1})$ be the identity matrix, and $\Gamma = \text{uchol}(\Phi_{t+1}) = (v_{ij})_{i,j \in 1:q}$. Then,

$$E[(\Phi^U_t)]_{ij} | \Phi_{t+1}, D_t] = \lambda \sum_{l=1}^{i\wedge j} v_{il}v_{lj} + \delta_{ij}; \quad V[(\Phi^U_t)]_{ij} | \Phi_{t+1}, D_t] = 1 + \delta_{ij},$$

where $(\Phi^U_t)_{ij}$ is the $(i,j)$th element of $(\Phi^U_t)$. Similarly, for the BB process:

$$E[(\Phi^B_t)]_{ij} | \Phi_{t+1}, D_t] = \lambda \sum_{l=1}^{i\wedge j-1} v_{il}v_{lj} + \delta_{ij}(\lambda v_{ii}^2 + 1) + (1-\delta_{ij})h(i,j)$$

$$V[(\Phi^B_t)]_{ij} | \Phi_{t+1}, D_t] = 2\delta_{ij} + (1-\delta_{ij})[\lambda^2 v_{ii,j}^2 v_{jj,i}^2 + \lambda v_{i\wedge j,i\wedge j}^2 - h(i,j)^2],$$

with $h(i,j) = \sqrt{2\lambda v_{i\wedge j,i\wedge j}} U(-1/2, 0, 2 \lambda v_{i\wedge j,i\wedge j}^2)$, where $U(a, b, z)$ is Tricomi’s confluent hypergeometric function (see e.g. Abramowitz et al. (1988)). The expressions for the
diagonal elements coincide but that need not be the case for the off-diagonal elements: $\nu_{ij} > 0$ implies $E[(\Phi_t)_{ij}^U | \Phi_{t+1}, D_t] > E[(\Phi_t)_{ij}^U | \Phi_{t+1}, D_t]$ and $\lambda\nu_{ij}^2 < 1$ implies $V[(\Phi_t)_{ij}^U | \Phi_{t+1}, D_t] < V[(\Phi_t)_{ij}^U | \Phi_{t+1}, D_t]$.

4 Backward sampling for beta-Bartlett processes

Forward-filtered posteriors and forecast distributions for BB processes were derived in Quintana et al. (2010), but a backward sampler was not developed. Our sampler uses the factorization of $\Phi_0$: $\Phi_t | D_t$ in Section 3 and it consists in drawing $\Phi^*_t \sim \text{Wishart}_{q}(k_T, (kD^B_T)^{-1})$ and iteratively sampling $\Phi^*_t \sim \Phi_t | \Phi^*_{t+1}, D_t$.

Given $\Phi_{t+1}$ and $D_t$, consider the decomposition $\Phi_{t+1} = \tilde{U}^*_{t+1}P_t\tilde{U}^*_{t+1}P_t/b$. That is, $\tilde{U}^*_{t+1} = \text{uchol}(b(P_t^{-1})'\Phi_{t+1}P_t^{-1})$. Then, we can generate $U^*_t = (u^*_{i,j,t})_{i,j\in1:q}$ as follows. The off-diagonal elements are $u^*_{ij,t} = \tilde{u}^*_{ij,t+1}$ for $i < j$ and the diagonal elements are $(u^*_{ii,t})^2 = (\tilde{u}^*_{ii,t+1})^2 + \theta_{it}$, where $\theta_{it} \sim \chi^2_{(1-\beta)k_t}$. Finally, we can set $\Phi_t = (U^*_tP_t)'U^*_tP_t$.

The expression for the conditional of $(u^*_{ii,t})^2$ given $(\tilde{u}^*_{ii,t+1})^2$ can be justified using standard results for the univariate gamma-beta discount model (see e.g. Exercise 4 in Section 4.6 of Prado and West (2010)). To relate $U^*_t$ to $\tilde{U}^*_t$, observe that $\Phi_t = (U^*_tP_t)'U^*_tP_t = (\tilde{U}^*_tP_{t-1})'\tilde{U}^*_tP_{t-1}/b$. Therefore,

$$\tilde{U}^*_t = \text{uchol}(b(P_t'^{-1})\Phi_tP_t^{-1}) = \text{uchol}(b(U^*_tP_tP_{t-1}'U^*_tP_tP_{t-1}') = \sqrt{b} U^*_tP_tP_{t-1}'.

The matrix $P_{t-1}$ is upper-triangular, so it can be inverted at $O(n^2)$ computational cost using back-substitution. The backward sampler for the UE process requires simulating Wishart random matrices for all $t \in 0 : T$, whereas the BB process only requires sampling a Wishart random matrix for $t = T$. For $t \in 0 : (T - 1)$, the backward sampler of the BB process requires $q$ chi-squared random variates. Explicit pseudocode for the backward sampler can be found in Algorithm I. The sampler can also be used in multivariate dynamic linear models with BB stochastic volatilities (as in Section 10.4.8 in Prado and West (2010)).
Algorithm 1: Backward sampler for $\Phi_{0:T}^B | D_T$

Input: $b$, $\beta$, $k_t$, and $P_t = \text{uchol}((kD_t^B)^{-1})$ from $\Phi_t^B|D_t \sim \text{Wishart}(k_t, (kD_t^B)^{-1})$, $t \in 0:T$. 
Output: $\Phi_{0:T}^B \sim \Phi_{0:T}^B | D_T$. 

$U_T^* = (u_{T,i:j}^*), i:j \in 1:q$; $u_{T,i:i}^* \sim \chi^2_{k_{T-i+1}}$; $u_{T,i:j}^* \sim N_1(0, 1)$, $i \in 1:q$ and $i < j \leq q$; 
$\Phi_T^B = (U_T^*P_T)U_T^*P_T$; 
$\tilde{U}_T^* = \sqrt{b}U_T^*P_TP_T^{-1}$; 
for descending $t \in T : 1$ do 

$\theta(t-1),i \sim \chi^2_{1-\beta k_{t-1}}$ for $i \in 1:q$; 
$U_{t-1}^* = (u_{(t-1),i:j}^*), i:j \in 1:q$; 
$u_{(t-1),i:i}^* = \sqrt{(\tilde{u}_{t,i:i}^*)^2 + \theta(t-1),i}$; $u_{(t-1),i:j}^* = \tilde{u}_{t,i:j}^*$, $i \in 1:q$ and $i < j \leq q$; 
$\Phi_{t-1}^B = (U_{t-1}^*P_{t-1})U_{t-1}^*P_{t-1}^{-1}$; 
if $t \geq 2$ then 

$\tilde{U}_{t-1}^* = \sqrt{b}U_{t-1}^*P_{t-1}P_{t-1}^{-1}$; 
return $\Phi_{0:T}^B$.

5 Model comparison

If Equation (3) is satisfied, the marginal likelihoods of UE and BB models are equal and we can’t use Bayes factors (or posterior model probabilities) to compare them. However, the difference of $\Phi_{0:T}^U | D_T$ and $\Phi_{0:T}^B | D_T$ can be substantial in practice, as we see in Section 6.

Instead of Bayes factors, we can use posterior likelihood ratios (Aitkin, 1991) and posterior predictive checks (Gelman et al., 1996) to compare the models. Both of these approaches can be implemented given posterior draws, but they have been criticized for, among other reasons, using the data twice (Gelman et al., 2013). Alternatively, Kamary et al. (2014) propose comparing models via mixtures, which here amounts to fitting

$$y_t | \alpha, \Phi_t^U, \Phi_t^B \sim \alpha \text{Wishart}_q(k, (k\Phi_t^U)^{-1}) + (1 - \alpha) \text{Wishart}_q(k, (k\Phi_t^B)^{-1}),$$

where $\alpha \sim \text{Beta}(a_0, b_0)$, and studying the posterior distribution of the mixture weight $\alpha$.

6 Illustration: foreign exchange rates

We perform a retrospective analysis of volatilities of daily returns of exchange rates of three currencies measured in US dollars: euros (EUR), British pounds (GBP), and Canadian dollars (CAD), observed from January 2008 to October 2010 ($T = 739$). The vector
of returns $r_t$ can be turned into a rank-1 symmetric matrix by computing $y_t = r_tr_t'$. Our observational model is $y_t \mid \Phi_t \overset{\text{ind.}}{\sim} \text{Wishart}_q(1, \Phi^{-1}_t)$; that is, we set $k = 1$, which is equivalent to modeling $r_t \mid \Phi_t \overset{\text{ind.}}{\sim} \mathcal{N}_q(0_q, \Phi^{-1}_t)$.

The estimate of the volatility matrix at the starting point, $D_0$, is computed as the sample average of the data in 2007. The other hyperparameters are obtained by maximizing the marginal likelihood of the model and are $n = 5$ and $\lambda = 0.799$.

While UE and BB yield similar point estimates, the processes are markedly different in their uncertainty quantification. This difference is apparent in the posterior of correlations displayed in Figure 1.

Figure 1: Posterior medians and 95% credible intervals of correlations computed from sampled $(\Phi_t^U)^{-1}$ and $(\Phi_t^B)^{-1}$ for the UE (top row) and BB (bottom) models.

The logarithm of the posterior likelihood ratio of the UE model to the BB model is

$$\ell_{UB} = \log \left\{ \frac{E[L(\Phi_{0:T}^U)|D_T]}{E[L(\Phi_{0:T}^B)|D_T]} \right\}, \quad L(\Phi_{0:T}) = \prod_{t=1}^T \mathcal{N}_q(r_t \mid 0_q, \Phi_t^{-1}),$$

where the expectations are computed by posterior samples $\Phi_{0:T}^U \mid D_T$ and $\Phi_{0:T}^B \mid D_T$. In this application, $\ell_{UB} = -35.230$, which clearly favors the BB model.

We implement a mixture model to compare the models, as proposed in Kamary et al. (2014), by running a missing-data augmented Gibbs sampler (see supplementary material).
for $N = 10^4$ iterations. A mixture weight $\alpha$ close to 0 favors the BB model, whereas a mixture weight near 1 favors UE. Starting with $\alpha \sim \text{Beta}(1, 1)$, we estimate $E(\alpha \mid D_t) = 0.498$ with an estimated standard error of 0.00026 and $P(\alpha < 0.5 \mid D_t) = 0.533$ with an estimated standard error of 0.0055165 (we used batch means estimators for the standard errors; see e.g. Geyer (1992)). The simulation error is small enough to be confident that the mixture model prefers BB, but the evidence isn’t nearly as overwhelming as it is with the logarithm of the posterior likelihood ratio.

We also compare the models with posterior predictive checks. We found the length of 95% posterior predictive intervals at each time point, while monitoring their coverage. The BB model has smaller interval lengths at essentially no cost; the cumulative coverage rates of the intervals are higher than 95% and comparable to those of the UE model.

7 Conclusion

UE and BB processes can be parametrized so that they yield the same forecasts and marginal likelihoods. Therefore, practitioners who are only concerned with forecasting and Bayesians who compare models using Bayes factors can treat these processes as equivalent. On the other hand, Section 6 shows that the smoothed posteriors can be rather different, and that approaches to model comparison such as posterior likelihood ratios don’t see the models as equivalent. This example calls for further investigation to discover when this phenomenon occurs, and how to proceed when it does.

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Supplementary material

In this supplementary document, we set the notation for the distributions we use in the main text, give an additional example to compare the forward conditional distributions of Uhlig extended and beta-Bartlett processes, and provide technical details and extra results for the foreign exchange rates application.

Distributions

The definitions of Wishart and matrix Beta distributions can be found, for instance, in Windle and Carvalho (2014) and Prado and West (2010). We include them here for completeness.

**Wishart:** Let $A$ be a $q \times q$ symmetric positive definite matrix. Then, $A \sim \text{Wishart}_q(h, S)$ if its probability density function is

$$p(A) = 2^{-(hq)/2} |S|^{-h/2} \Gamma_q(h/2) |A|^{(h-q-1)/2} \exp \left\{ -\frac{1}{2} \text{tr}(S^{-1}A) \right\},$$

where $h > q - 1$ and $\Gamma_q(h/2)$ is the multivariate gamma function evaluated at $h/2$. The definition can be extended to $h \leq q - 1$, in which case $A$ is rank-deficient; see e.g. Windle and Carvalho (2014) for details.

**Matrix beta distribution:** Let $A_1 \sim \text{Wishart}_q(n_1, \Sigma^{-1})$ and $A_2 \sim \text{Wishart}_q(n_2, \Sigma^{-1})$ be independent where $\Sigma$ is symmetric positive-definite, $n_2 > q - 1$ and either $n_1 < q$ is an integer or $n_1 > q - 1$ is real-valued. Let $T = \text{uchol}(A_1 + A_2)$ and $B = (T^{-1})'A_1T^{-1}$. Then, $B \sim \text{MatrixBeta}_q(n_1/2, n_2/2)$.

Additional example: conditional distributions

Let $\Phi_{t+1} = \text{diag}(\phi_{1:q})$ and $D_t^{-1} = \text{diag}(d_{1:q})$. For simplicity, we let $k = 1$, although the same computations could be done for $k \neq 1$. For the UE process,

$$E(\Phi_t^U | \Phi_{t+1}, D_t) = \text{diag}(\lambda \phi_{1:q} + d_{1:q}); \quad V[(\Phi_t^U)_{ij} | \Phi_{t+1}, D_t] = \delta_{ij} d_i^2 + d_i d_j,$$

where $\delta_{ij}$ is Kronecker’s delta function. For the BB process, we have

$$E(\Phi_t^B | \Phi_{t+1}, D_t) = \text{diag}(\lambda \phi_{1:q} + d_{1:q}); \quad V[(\Phi_t^B)_{ij} | \Phi_{t+1}, D_t] = \delta_{ij} 2d_i^2.$$
The conditionals are equal in expectation, but the variance of the off-diagonal elements don’t coincide. With the BB process, the off-diagonal elements are 0 with probability 1, whereas with the UE process the off-diagonal elements aren’t identically equal to 0.

**Foreign exchange rates: technical details**

As we mentioned in the main text, we take \( k = 1 \), which implies that we can simply work with normal likelihoods for the returns. We use normal likelihoods in our implementation.

In Windle and Carvalho (2014), the discounting parameter \( \lambda \) is automatically chosen to satisfy

\[
\lambda - 1 = 1 + k/(n - q - 1).
\]

This constraint not only reduces the number of parameters to estimate, but also guarantees that \( E[\Phi_t^{-1}|D_t] = E[\Phi_{t+1}^{-1}|D_t] \), a property the authors deem desirable. In contrast, we directly assess the maximization of marginal likelihood in \((n, \lambda)\) under no constraint. The marginal likelihood is the product of one-step ahead forecast densities, each of which is the multivariate-\( t \) distribution defined by

\[
p(r_t|D_{t-1}) = \int N_q(r_t | 0, \Phi_t^{-1}) W_q(\Phi_t | n, (\lambda D_{t-1})^{-1}) (d\Phi_t)
\]

\[
= \frac{\Gamma(n/2)}{\Gamma((n + 1 - q)/2)} \left| \lambda D_{t-1} \right|^{-1/2} \pi^{q/2} (1 + r_t D_{t-1}^{-1} r_t / \lambda)^{-\frac{n+1}{2}}.
\]

In addition, the determinant of \( D_t \) is sequentially updated using the convenient relation

\[
\log |D_t| = \log(1 + r_t D_{t-1}^{-1} r_t / \lambda) + q \log(\lambda) + \log |D_{t-1}|,
\]

so the evaluation of marginal likelihood isn’t computationally demanding. For maximizing the marginal likelihood, we evaluate it at \( n \in \{3, 4, \ldots, 20\} \) and \( \lambda \in \{0.600, 0.601, \ldots, 0.990\} \).

The posterior likelihood ratio (Aitkin, 1991) can be hard to estimate numerically, but its logarithm is stable. To see this, recall that

\[
\tilde{\ell}_{UB} = \log \left\{ E[L(\Phi_{0:T}^U)|D_T]/E[L(\Phi_{0:T}^B)|D_T] \right\}, \quad L(\Phi_{0:T}) = \prod_{t=1}^{T} N_q(r_t | 0_q, \Phi_t^{-1}).
\]

Now, based on Monte Carlo samples \( \Phi_{1:N}^U \sim \Phi_{0:T}^U \mid D_T \) and \( \Phi_{1:N}^B \sim \Phi_{0:T}^B \mid D_T \),

\[
\tilde{\ell}_{UB} \approx \text{LSE}(\ell(\Phi_{1:N}^U)) - \text{LSE}(\ell(\Phi_{1:N}^B)),
\]

where \( \ell \) is log \( L(\Phi_{0:T}) \) and LSE is the log-sum-exp function, which can be implemented in a numerically stable way.
We implement the mixture model approach proposed in Kamary et al. (2014) through a missing-data augmented Gibbs sampler. The target model is defined by the mixture of likelihoods,

\[ r_t \mid \alpha, \Phi_t^U, \Phi_t^B \sim \alpha N_q(0, (\Phi_t^U)^{-1}) + (1 - \alpha)N_q(0, (\Phi_t^B)^{-1}). \]

We implement the following augmented model:

\[ r_t = z_t r_t^U + (1 - z_t) r_t^B \]

\[ r_t^M \mid \Phi_t^M \sim N_q(0, (\Phi_t^M)^{-1}), \quad M \in \{U, B\} \]

\[ z_t \text{iid} \sim \text{Bernoulli}(\alpha) \]

\[ \alpha \sim \text{Beta}(a_0, b_0) \]

The actual observed return, \( r_t \), is defined separately from the inputs of two models, \( r_t^U \) and \( r_t^B \). At each iteration of Gibbs sampler, conditional on \( z_t \), we decide which model is fed by \( r_t \), and which model is “missing” its observation. The notable advantage of this approach is that the missing observation, either \( r_t^U \) or \( r_t^B \), is a parameter, so it is sampled through the course of the Gibbs sampler. As a result, the sampling of \( \Phi_t^M \) is based on a full sequence of observations \( r_t^M \) and we can apply the forward filtering equations and backward sampler we described in the main text.

The Gibbs sampler consists in iteratively sampling from the following full-conditional distributions:

- Sample \( z_t \) from Bernoulli distribution with probability

\[
P[z_t = 1|\cdot] \propto \alpha N_q(r_t^U|0, (\Phi_t^U)^{-1})
\]

\[
P[z_t = 0|\cdot] \propto (1 - \alpha)N_q(r_t^B|0, (\Phi_t^B)^{-1})
\]

In computation, one can utilize the log-scale,

\[
\log P(z_t = 1|\cdot) = c + \log(\alpha) + \frac{1}{2} \log |\Phi_t^U| - \frac{1}{2}(r_t^U)'\Phi_t^U r_t^U
\]

\[
\log P(z_t = 0|\cdot) = c + \log(1 - \alpha) + \frac{1}{2} \log |\Phi_t^B| - \frac{1}{2}(r_t^B)'\Phi_t^B r_t^B
\]

where \( c \) is the common constant.

- Define \( r_t^U \) and \( r_t^B \) as follows:
- If $z_t = 1$, then set $r^U_t = r_t$ and generate $r^B_t \sim N_q(0, (\Phi^B_t)^{-1})$.
- If $z_t = 0$, then generate $r^U_t \sim N_q(0, (\Phi^U_t)^{-1})$ and set $r^U_t = r_t$.

- Sample $\alpha$ from $\text{Beta}(a_1, b_1)$,
  \[ a_1 = a_0 + \sum_{t=1}^{T} z_t, \quad b_1 = b_0 + \sum_{t=1}^{T} (1 - z_t) \]
- Sample $\{\Phi^U_{1:T}\}$ and $\{\Phi^B_{1:T}\}$ using the forward-filtering equations and the backward sampler described in the main text.

### Foreign exchange rates: additional results and figures

Figure 4 shows the original series of returns. Figure 5 shows the contours of the marginal likelihood, along with the maximizer $(n, \lambda) = (5, 0.799)$ indicated by the red circle. In this figure, we also show the maximizer under the constraint that was used in Windle and Carvalho (2014): $(n, \lambda) = (10, 0.857)$ indicated by the blue box. The posterior and predictive analysis in this study is based on the former choice.

Figure 2 visualizes the results with posterior-likelihood ratios and the mixture model approach of Kamary et al. (2014). The top plot shows that the posterior log-likelihood of the beta-Bartlett model is clearly higher than that of the Uhlig extended model. The bottom plot shows that the mixture approach does prefer the beta-Bartlett model as well, but to a much lesser extent.

Figure 3 shows lengths of 95% posterior predictive intervals and their cumulative empirical coverage over time. The beta-Bartlett process reports shorter intervals most of the time, and their empirical coverage is always above 95%.

In the main text, we presented our results with the mixture model approach in Kamary et al. (2014) with $\alpha \sim \text{Beta}(1, 1)$. We also tried other hyperparameters to test out the effect of the prior. For example, consider $\alpha \sim \text{Beta}(10, 1)$, so the UE model is strongly preferred a priori. The results are shown in Figure 6. The estimated posterior mean is 0.505123 (standard error: 0.000261316). The estimated posterior probability of having $\alpha < 0.5$ is 0.4124 (standard error: 0.00537318). This shows that the prior concentrates towards roughly 0.5 even if the starting point is far away from it.
Figure 2: Top: Posterior log-likelihoods with Uhlig extended (UE; solid red) and beta-Bartlett (BB; dashed blue). Bottom: Prior on mixture weight $\alpha$ (dashed blue) and posterior (red histogram). Vertical line at 0.5.

Figure 3: Length of 95% posterior predictive intervals and cumulative empirical coverage rates.
Figure 4: Time series of daily returns from EUR, GBP and CAD in US dollars.

Figure 5: Contour plots of marginal likelihoods as the functions of \((n, \lambda)\). The red circle indicates the maximizer \((5, 0.799)\). The blue box shows \((10, 0.857)\), the maximizer under the constraint.
Figure 6: Histogram and sample path of $\alpha$ for prior $\alpha \sim Be(10, 1)$