Hamiltonian cycles in 4-connected planar and projective planar triangulations with few 4-separators

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Abstract

Whitney proved in 1931 that every 4-connected planar triangulation is hamiltonian. Later in 1979, Hakimi, Schmeichel and Thomassen conjectured that every such triangulation on \( n \) vertices has at least \( 2(n - 2)(n - 4) \) hamiltonian cycles. Along this direction, Brinkmann, Souffriau and Van Cleemput established a linear lower bound on the number of hamiltonian cycles in 4-connected planar triangulations. In stark contrast, Alahmadi, Aldred and Thomassen showed that every 5-connected triangulation of the plane or the projective plane has exponentially many hamiltonian cycles. This gives the motivation to study the number of hamiltonian cycles of 4-connected triangulations with few 4-separators. Recently, Liu and Yu showed that every 4-connected planar triangulation with \( O(n/\log n) \) 4-separators has a quadratic number of hamiltonian cycles. By adapting the framework of Alahmadi et al. we strengthen the last two aforementioned results. We prove that every 4-connected planar or projective planar triangulation with \( O(n) \) 4-separators has exponentially many hamiltonian cycles.

1 Introduction

A classical theorem of Whitney in 1931 proved that every 4-connected planar triangulation has a hamiltonian cycle [19]. In 1956, this result was extended by Tutte [18] to 4-connected planar graphs. One may subsequently ask how many hamiltonian cycles a 4-connected planar triangulation or planar graph may have. In 1979, Hakimi, Schmeichel and Thomassen [9] proposed the following conjecture:

Conjecture 1 ([9]). Every 4-connected planar triangulation \( G \) on \( n \) vertices has at least \( 2(n - 2)(n - 4) \) hamiltonian cycles, with equality if and only if \( G \) is the double-wheel graph on \( n \) vertices, that is, the join of a cycle of length \( n - 2 \) and an empty graph on two vertices.

In the same paper, they also proved that every 4-connected planar triangulation on \( n \) vertices has at least \( n/\log_2 n \) hamiltonian cycles. This lower bound was recently improved by Brinkmann, Souffriau and Van Cleemput [3] to a linear bound of \( 12(n - 2)/5 \), which was then refined to \( 161(n - 2)/60 \) for \( n \geq 7 \) by Cuvelier [6]. For 4-connected planar graphs, Sander [16] showed that there exists a hamiltonian cycle containing any two prescribed edges in any 4-connected planar graph. This implies that every 4-connected planar graph \( G \) has at least \( (\Delta + 1)^2 \) hamiltonian cycles, where \( \Delta \) denotes the maximum degree of \( G \). However, it assures only a constant lower bound as there are infinitely many 4-connected planar graphs of maximum degree upper bounded by 4.

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Generalizing the method used in [3], Brinkmann and Van Cleemput [4] gave a linear lower bound on the number of hamiltonian cycles in 4-connected planar graphs.

We will focus on the number of hamiltonian cycles of 4-connected triangulations of the plane or the projective plane with a bounded number of 4-separators. Interestingly, if Conjecture 1 held, we would have that a 4-connected planar triangulation has a minimum number of hamiltonian cycles if and only if it has a maximum number of 4-separators, as the double-wheel graphs are the 4-connected planar triangulations that maximize the number of 4-separators [8]. Trivially, 5-connected planar triangulations have a minimum number of 4-separators among 4-connected planar triangulations as they have no 4-separators. Indeed, Alahmadi, Aldred and Thomassen [1] proved that every 5-connected triangulation embedded on the plane or on the projective plane has $2^{\Omega(n)}$ hamiltonian cycles. Following the approach of Alahmadi et al., it was shown in [12] that every 4-connected planar triangulation has at least $\Omega((n/\log n)^2)$ hamiltonian cycles if it has only $O(\log n)$ 4-separators. Recently, Liu and Yu [11] improved this result by showing that every 4-connected planar triangulation with $O(n/\log n)$ 4-separators has $\Omega(n^2)$ hamiltonian cycles. These results, in a sense, give evidence supporting that triangulations of the plane or the projective plane with fewer 4-separators may have more hamiltonian cycles. We will prove the following result, indicating that a 4-connected planar or projective planar triangulation may have exponentially many hamiltonian cycles as long as it has at most a linear number of 4-separators, thereby extending the results mentioned above.

**Theorem 2.** Let $G$ be a 4-connected planar or projective planar triangulation on $n$ vertices and let $c$ be an arbitrary constant less than $1/324$. If $G$ has at most $cn$ 4-separators, then it has $2^{\Omega(n)}$ hamiltonian cycles.

2 Results

In this section we first prepare some lemmas which will be used to construct a vertex subset with several properties (see also Lemma 8). The proof of Theorem 2 will be given at the end of this section.

For notation and terminology not explicitly defined in this paper, we refer the reader to [2, 13]. A vertex subset or a subgraph of a connected graph is *separating* if its removal disconnects the graph. We call a separating vertex set on $k$ vertices a *$k$-separator*. A *$k$-cycle* is a cycle of length $k$. Let $S$ be an independent set. We say $S$ *saturates* a 4- or 5-cycle $C$ if $S$ contains two vertices of $C$. A *diamond-6-cycle* is the graph depicted in Figure 1, where the white vertices are called *crucial*. We say $S$ *saturates* a diamond-6-cycle $D$ if $S$ contains three crucial vertices of $D$. Recall that the *Euler genus* $eg(\Sigma)$ of a surface $\Sigma$ is defined to be $2 - \chi(\Sigma)$, where $\chi(\Sigma)$ denotes the Euler characteristic of $\Sigma$. A graph $H$ is *$d$-degenerate* if every induced subgraph of $H$ has a vertex of degree at most $d$. It is well known that every $d$-degenerate graph $H$ is $(d+1)$-colorable and hence has an independent set of at least $|V(H)|/(d+1)$ vertices.

![Figure 1: A diamond-6-cycle with six crucial vertices (white).](image_url)
The following tool is due to Alahmadi et al. [1], which helps finding homotopic curves from a sufficiently large family of curves.

**Lemma 3** ([1, Corollary 2]). Let $Σ$ be a surface of Euler genus $σ$ and let $C$ be a family of simple closed curves on $Σ$ with the property that every $C ∈ C$ can have at most one point that is contained in other curves in $C$. Let $r$ be a positive integer. If $|C| ≥ 5(r − 1)σ + 1$, then there are $r$ homotopic curves in $C$.

The number of the vertices adjacent to at least three vertices of a 4-cycle on a surface can be bounded as follows.

**Lemma 4.** Let $G$ be a triangulation of a surface of Euler genus $σ$, and $C$ be a 4-cycle in $G$. Denote $V_C := \{ v ∈ V(G) : |N_G(v) ∩ V(C)| ≥ 3 \}$. Then $|V_C| ≤ 8(σ + 1)$.

**Proof.** Suppose to the contrary that $|V_C| > 8(σ + 1)$. We have that $G$ contains $K_{3,q}$ as a subgraph, where $q = 2(σ + 1) + 1$. This is however impossible since, by a theorem of Ringel [14, 15], the Euler genus of $K_{3,q}$ equals $eg(K_{3,q}) = \lceil \frac{q^2 - 2}{4} \rceil = σ + 1 > σ$.

The next three lemmas aim at finding a vertex set that saturates no 4-cycle, or 5-cycle, or diamond-6-cycle. The main idea of the proofs comes from [1].

**Lemma 5.** Let $G$ be a triangulation of a surface of Euler genus $σ$ and let $S ⊆ V(G)$ be an independent set of vertices of degree at most 6. If $S$ saturates no separating 4-cycle in $G$, then $S$ has a subset of size at least $|S|/c$ that saturates no 4-cycle, where $c := (\frac{3}{2})(10σ + 1) + 1$.

**Proof.** Let $H$ be the graph on the vertex set $S$ such that two vertices are adjacent if they saturate some 4-cycle in $G$. It suffices to show that $H$ has maximum degree at most $d := c - 1 = (\frac{3}{2})(10σ + 1)$ and hence chromatic number at most $c$. Suppose, to the contrary, there exists $v ∈ S = V(H)$ with $d_H(v) > d$. As $d_G(v) ≤ 6$, there are $u, w ∈ N_G(v)$ such that the path $uvw$ are contained in at least $\lceil (d + 1)/(\frac{3}{2}) \rceil = (10σ + 1) + 1$ 4-cycles in $G$. We may contract the path $uvw$ and apply Lemma 3 to show that there are at least three homotopic 4-cycles containing the path $uvw$ and saturated by $S$. This yields a separating 4-cycle saturated by $S$ and hence contradicts our assumption. Thus the lemma follows.

**Lemma 6.** Let $G$ be a triangulation of a surface of Euler genus $σ$ and let $S ⊆ V(G)$ be an independent set of vertices of degree at most 6. If $S$ saturates no 4-cycle in $G$, then $S$ has a subset of size at least $|S|/c$ that saturates no 5-cycle, where $c := 2(\frac{6}{2})(40σ + 1) + 1$.

**Proof.** Let $H$ be the graph on the vertex set $S$ in which two vertices are adjacent if they saturate some 5-cycle in $G$. Let $d := c - 1 = 2(\frac{6}{2})(40σ + 1)$. It suffices to show that $H$ is $d$-degenerate. Let $K$ be any induced subgraph of $H$. We will show that $K$ has a vertex of degree at most $d$.

We first consider the following. Let $v$ be a vertex in $K$ with $d_K(v) ≥ d$. Since $d_G(v) ≤ 6$, there exist $u, w ∈ N_G(v)$, distinct vertices $x_1, ..., x_{2t} ∈ N_K(v)$ and $y_1, ..., y_{2t} ∈ V(G) \setminus S$, where $t := 40σ + 1$, such that for every $1 ≤ i ≤ 2t$ either $wwx_iy_ju$ or $wwy_iy_jx_iu$ is a 5-cycle saturated by $V(K)$. Denote by $C_i$ the 5-cycle saturated by $v$ and $x_i$, and denote by $P_i$ the path obtained from $C_i$ by deleting $v$ ($1 ≤ i ≤ 2t$). We claim that at least $t$ vertices of $y_1, ..., y_{2t}$ are distinct. Otherwise there are $1 ≤ i < j < k ≤ 2t$ such that $y_i = y_j = y_k$. Then $u$ or $w$ is adjacent to two of $x_i, x_j, x_k$, say $x_i, x_j ∈ N_G(w)$. However, this implies that $wx_iy_jx_kw$ is a 4-cycle saturated by $V(K)$, which contradicts our assumption. Therefore, we may assume that the paths $P_1, ..., P_t$ are pairwise internally disjoint. By contracting the path $uvw$ and applying Lemma 3, we may obtain nine homotopic curves from $P_1, ..., P_t$, say $P_1, ..., P_9$. Relabelling if necessary, we may
further assume that the closed disc $D_v$ bounded by $P_1 \cup P_9$ contains $P_1, \ldots, P_9$, and the closed disc bounded by $P_1 \cup P_5$ contains $P_1, \ldots, P_5$ but not $P_6, \ldots, P_9$.

We now show that $K$ has minimum degree at most $d$. Suppose to the contrary that $d_K(v) > d$ for every $v \in V(K)$. We choose a vertex $v \in V(K)$ and the associated nine homotopic curves such that the number of vertices of $G$ contained in $D_v$ is minimum.

Let $C$ be a 5-cycle in $G$ containing $x_5$ and another vertex $v' \in V(K)$ outside of $D_v$. We claim that $v' = v$. Let $P$ be the minimal path in $C$ containing $x_5$ with end-vertices in $P_1 \cup P_9$. Notice that $C$ has length five and $x_5$ lies in the interior but not the boundary of the disc $D_v$. So by the arrangement of $P_1, \ldots, P_9$, the end-vertices of $P$ must be $u$ and $w$. Since $V(K)$ is an independent set of $G$ and $v, x_5 \in V(K)$, we have $v' \notin N_G(x_5) \cup \{u, w\}$. Therefore $P$ has length less than four. If $P$ has length two, then $uvwx_5u$ would be a 4-cycle saturated by $V(K) \subseteq S$, which contradicts our assumption. If $P$ has length three, then we have $v' = v$ (otherwise $uvw'u$ would be a 4-cycle saturated by $V(K)$). This thus establishes our claim.

Our previous claim implies that all 5-cycles in $G$ containing $x_5$ and another vertex of $V(K)$ other than $v$ must lie in $D_v$. As $d_{K-v}(x_5) = d_K(x_5) - 1 \geq d$, we may apply our discussion above to $K - v$ and $x_5$ (instead of $K$ and $v$) to obtain nine homotopic curves associated with $x_5$ and a closed disc containing them. Then we may have $D_{x_5} \subseteq D_v$. Moreover, it is not hard to see that $D_{x_5}$ does not contain $x_1$. Therefore the number of vertices in $D_{x_5}$ is strictly less than that of $D_v$. This contradicts our choice of $v$ and $D_v$ and hence the result follows.

The proof of following lemma is omitted as it can be readily deduced from the proof of [1, Lemma 10].

**Lemma 7** ([1, Lemma 10]). Let $G$ be a triangulation of a surface of Euler genus $\sigma$ and let $S \subseteq V(G)$ be an independent set of vertices of degree at most 6. If $S$ saturates no 4-cycle in $G$, then $S$ has a subset of size at least $|S|/c$ that saturates no diamond-6-cycle, where $c$ is a positive constant depending only on $\sigma$.

One of the key ingredients of the proof of exponential lower bound on the number of hamiltonian cycles in 5-connected triangulations given by Alahmadi et al. [1] is to find many edge sets $F \subseteq E(G)$ so that $G - F$ is 4-connected. Their approach was refined by [12, 11] for 4-connected planar triangulations. In order to prove our result for triangulations of the projective plane, we need to generalize a lemma given by Liu and Yu [11] to surfaces of higher genus.

Let $G$ be a triangulation of any surface and $A \subseteq V(G)$ be a 3-separator of $G$. It is shown in the proof of [1, Lemma 1] and its subsequent discussion that $G[A]$ is a surface separating 3-cycle. Using this fact and a theorem of Thomas and Yu [17] that every 4-connected projective planar graph is hamiltonian, the proof of [11, Lemma 2.1] can be easily modified to show the following result. We omit the proof.

**Lemma 8** ([11, Lemma 2.1]). Let $G$ be a 4-connected triangulation of a surface $\Sigma$ of Euler genus $\sigma$. Let $S \subseteq V(G)$ be a vertex subset satisfying the following conditions:

(i) $d_G(v) \leq 6$ for any $v \in S$;

(ii) $S$ is an independent vertex set;

(iii) no vertex in $S$ is contained in any separating 4-cycle in $G$;

(iv) no vertex in $S$ is adjacent to three vertices of any separating 4-cycle in $G$; and

(v) $S$ saturates no 4-, 5- or diamond-6-cycle.
Let $F \subseteq E(G)$ be any edge subset such that $|F| = |S|$ and for any $v \in S$ there is precisely one edge in $F$ incident with $v$. Then $G - F$ is 4-connected. Moreover, if $\Sigma$ is the plane or the projective plane, then $G$ has $2^{\Omega(|S|)}$ hamiltonian cycles.

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** To prove the theorem, it suffices to construct a vertex set $S \subseteq V(G)$ of size $\Omega(n)$ satisfying conditions (i) to (v) in Lemma 8.

Since $G$ has average degree less than 6 and minimum degree at least 4, the set $S_1$ of vertices of degree at most 6 has size at least $n/3$. We may subsequently obtain a vertex set $S_2 \subseteq S_1$ of size at least $|S_1|/6 \geq n/18$ satisfying (i) and (ii) of Lemma 8, as planar and projective planar graphs are 6-colorable.

Let $C$ be any separating 4-cycle in $G$. Since $S_2$ is an independent set, it follows from Lemma 4 that $S_2$ has at most $16 + 2 = 18$ vertices that are contained in $C$ or adjacent to three vertices of $C$. Deleting these vertices from $S_2$ for every separating 4-cycle, we obtain a vertex set $S_3 \subseteq S_2$ satisfying (i) to (iv) of Lemma 8 and $|S_3| \geq n/18 - 18cn = (1/18 - 324c/18)n$. Recall that $c$ is a constant less than $1/324$. This means that $1/18 - 324c/18$ is a positive constant and hence $|S_3| = \Omega(n)$.

As no vertex in $S_3$ is contained in any separating 4-cycle, no 4-cycle saturated by $S_3$ is separating. Successively applying Lemmas 5, 6 and 7, we obtain a vertex set $S \subseteq S_3$ of size $\Omega(n)$ satisfying (i) to (v) of Lemma 8, implying that $G$ has $2^{\Omega(n)}$ hamiltonian cycles. This completes our proof.

We remark that Grünbaum [7] and Nash-Williams [5] independently conjectured that every 4-connected toroidal graph is hamiltonian. The truth of this conjecture (respectively, an analogue for the Klein bottle) would extend Theorem 2 to triangulations of the torus (respectively, the Klein bottle). Note that there are non-hamiltonian 4-connected graphs that are embedded in the double torus or in the surface obtained from the sphere by attaching three crosscaps (see [10]).

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