Stationary vs. singular points in an accelerating FRW cosmology derived from six-dimensional Einstein-Gauss-Bonnet gravity

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Abstract

Six-dimensional Einstein-Gauss-Bonnet gravity (with a linear Gauss-Bonnet term) is investigated. This theory is inspired by basic features of results coming from string and M-theory. Dynamical compactification is carried out and it is seen that a four-dimensional accelerating FRW universe is recovered, when the two-dimensional internal space radius shrinks. A non-perturbative structure of the corresponding theory is identified which has either three or one stable fixed points, depending on the Gauss-Bonnet coupling being positive or negative. A much richer structure than in the case of the perturbative regime of the dynamical compactification recently studied by Andrew, Bolen, and Middleton is exhibited.

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1. Introduction. To realize that the expansion of the universe is accelerating was one of the most important scientific discoveries of the last century. There are several alternative explanations of this remarkable fact (what already means that it is not well understood yet). A quite appealing possibility for the gravitational origin of the dark energy responsible for this accelerated expansion is the modification of General Relativity (GR) or the corresponding Einsteinian gravity (see [1] for a review of these approaches). Such a well-established and successful theory cannot be modified without a very good reason, and much less in an arbitrary way. But observe that there is no compelling reason why standard GR should be trusted at cosmological scales. For a rather minimal modification of the same, one assumes that the gravitational action might contain some additional terms which would start to grow slowly with decreasing curvature. A particularly interesting formulation is obtained by using well-grounded geometrical arguments, specifically when the modified gravity action is endowed with a function of the Gauss-Bonnet (GB) topological invariant, $G$, as it was suggested in [2].

It must be noted that different types of dark energy may actually show up in different ways, at large distances. Cold dark matter is known to be localized near galaxy clusters but, quite on the contrary, dark energy distributes uniformly in the universe. The reason for that behavior could be explained by a difference in the equation of state parameter $w = p/\rho$. Moreover, the effect of gravity on the cosmological fluid turns out to depend on $w$ and so happens that, even when $-1 < w < 0$, gravity can act sometimes as a repulsive force. The effect of gravity on matter with $-1 < w < 0$ can be shown to be opposite to that on usual matter, which becomes dense near a star, while matter with $-1 < w < 0$ becomes less dense when approaching a star [2]. Dark energy contributes uniformly throughout the universe, which would be indeed consistent, since the equation of state parameter of dark energy is almost $-1$. If dark energy is of phantom nature ($w < -1$), its density becomes large near the cluster but if dark energy is of quintessence type ($-1 < w < -1/3$), then its density becomes smaller.

Another very important argument in favor of the GB modified theory is the fact that it can be seen as being inspired by string and/or M-theory [3]. In fact, specific models for string-inspired scalar GB gravity, considered as possible forms of dark energy, have been discussed in [4] and [5]. It was subsequently shown in [6], that scalar GB gravity can actually be represented as a modified GB gravity without scalars, and specifically that it can be equivalent to an ideal fluid with an homogeneous equation of state [7].

In the present paper we will study the case of 6-dimensional Einstein GB gravity, with a linear GB term. As already advanced, this situation is quite interesting, since the theory that we will thereby obtain can be shown to be inspired by what is derived from string/M-theory, after some specific compactification is carried out, when the scale factor of the 2-dimensional internal space goes to zero. In the course of our study we will discover a non-perturbative structure of the corresponding theory with either three (for positive GB coupling, $\epsilon$) or one (for negative GB coupling) stable fixed points. This will exhibit a much richer structure than the one that follows from the case of the perturbative regime of the dynamical compactification recently studied by Andrew et al. [8]

2. Dynamical compactification of 6-dimensional Einstein-Gauss-Bonnet gravity. We shall start from the following, string-inspired action in six dimensions

$$S = \int d^6x \sqrt{-g}(R + \epsilon L_{GB}),$$

where $\epsilon$ is a constant, and the metric is the product of the usual metric corresponding to the 4-dimensional FRW universe and a 2-dimensional surface, namely

$$ds^2 = -dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + b^2(t)[(dx^4)^2 + (dx^5)^2].$$
the scalar curvature is
\[ R = \frac{6\dot{a}^2}{a^2} + \frac{12\dot{a}\dot{b}}{ab} + \frac{2\dot{b}^2}{b^2} + \frac{6\ddot{a}}{a} + \frac{4\dddot{b}}{b}, \]  
while the four-dimensional and topologically invariant Gauss-Bonnet Lagrangian, \( L_{GB} \), has the form
\[ L_{GB} = \frac{48\dot{a}^3\dot{b}}{a^3b} + \frac{72\dot{a}\dot{b}^2}{a^2b^2} + \frac{24\dot{a}^2\ddot{b}}{a^3} + \frac{96\dot{a}\dddot{b}}{a^2b} + \frac{24\ddot{a}\dot{b}^2}{ab^2} + \frac{48\dot{a}^2\dddot{b}}{a^2b} + \frac{48\dddot{a}\dddot{b}}{ab^2}, \tag{4} \]
or, equivalently,
\[ L_{GB} = \frac{24}{a^3b^2}(2\dot{a}^3\dot{b} + 3a\dot{a}^2\dot{b}^2 + a^2\dddot{b}^2 + 4a\dddot{a}\dddot{b} + a\dddot{a}\dddot{b} + 2a^2\dddot{b} + 2a^2\dddot{b}). \tag{5} \]
Note that, unlike the six-dimensional compatification case of Elizalde et al. [9], the theory under consideration is not multiplicatively renormalizable. The corresponding equations of motion are obtained by variation of the action with respect to \( a \) and \( b \), what yields
\[ \dot{a}^2\dot{b}^2 + 4a\dddot{b} + a^2\dddot{b}^2 + 2a\dddot{a}\dddot{b} + 2a^2\dddot{b} + 12\dot{a}^2\dot{b} + 16\dot{a}\dddot{b} + 8\dot{a}\dddot{b} \\
+ 8\dot{a}\dddot{b} + 16\dot{a}\dddot{b} = 0, \]
\[ 3a^2\dot{b} + 3a^2\dot{a} + a^3\dot{b} + a\dot{a}^3 + 12\dot{a}^2\dot{b} + 24\dot{a}\dddot{a} + 12\dot{a}^2\dddot{b} = 0. \tag{6} \]
These equations can be easily rewritten in terms of the Hubble rates \( H = a'/a \) and \( h = b'/b \), namely
\[ 3h^2 + 4hH + 3H^2 + 2H + 16h^3 + 12\epsilon h^2H + 28\epsilon h^2H^2 + 16\epsilon h^3H^3 \\
+ 16\epsilon h^4H + 8\epsilon h^2H^2 + 8\epsilon h^2H + 16\epsilon hH = 0, \]
\[ h^2 + 3hH + 6H^2 + h + 3H + 12\epsilon^2 H^2 + 36\epsilon hH^2 + 12\epsilon H^4 \\
+ 12\epsilon h^3H + 24\epsilon hHH + 12\epsilon H^2H = 0. \tag{7} \]
In addition, variation over the metric in the above expressions gives the constraint equation
\[ h^2 + 6hH + 3H^2 + 36\epsilon h^2H^2 + 24\epsilon hH^3 = 0 \tag{8} \]
This equation helps to exclude \( h \) and \( h' \) from Eqs. (7). As a result, one gets an equation for \( H \) only:
\[ H' = 3H^2 \times \]
\[ \frac{\sqrt{6} + 4G + \epsilon(-22\sqrt{6} + 64G)H^2 - 24\epsilon^2(9\sqrt{6} - 52G)H^4 + 96\epsilon^3(17\sqrt{6} + 12G)H^6 - 8064\sqrt{6}\epsilon^4 H^8}{\sqrt{6} - 12\epsilon(\sqrt{6} + 16G)H^2 + 72\epsilon^2(3\sqrt{6} - 32G)H^4 - 2880\sqrt{6}\epsilon^3 H^6 + 31104\sqrt{6}\epsilon^4 H^8}, \tag{9} \]
where
\[ G = \sqrt{1 - 6\epsilon H^2 + 24\epsilon^2 H^4.} \tag{10} \]
One can check that this last equation obeys the fundamental relation (for \( \epsilon > 0 \)):
\[ H' = \frac{H^2(H^2 - p^2)}{(H^2 - q^2)(H^2 - r^2)} f(H), \tag{11} \]
where \( p, q \) and \( r \) are constants, and the function \( f(H) < 0 \). We start now with its numerical analysis.

3. **First case**: \( \epsilon > 0 \). It leads to the following values of the constants, corresponding to constant curvatures in the 4-dimensional and 2-dimensional spaces:

\[
p_0 \approx 0.7501/\sqrt{\epsilon}, \quad q_0 \approx 0.4842/\sqrt{\epsilon}, \quad r_0 \approx 0.1023/\sqrt{\epsilon}.
\] (12)

Note that \( p_0 \) is a stable fixed point while \( q_0 \) and \( r_0 \) are singular points. It turns out then, that the initial values of \( H(t) \) can be classified as belonging to four different regions, which are delimited by these values of \( p_0, q_0 \), and \( r_0 \). The behavior of \( H(t) \) in each of these regions can differ considerably, from one to the other. We will now consider the different possibilities in detail. The four different cases corresponding to \( \epsilon > 0 \) are as follows.

1. Case \( 0 < H(0) < r_0 \), then \( H' > 0 \), when \( H \to q_0 \), and in the limit it turns out that \( H' \to +\infty \). Thus, at \( H = q_0 \) a singularity develops, Figs. 1a, 1b (for a classification of the different types of future singularities, see e.g. [10]):

![Fig. 1a](image1a.png)

![Fig. 1b](image1b.png)

Using Eq. (8), we obtain the form of \( h(t) \) and \( h'(h) \), Figs. 1c, 1d (remember that \( h = \dot{b}/b \)):

![Fig. 1c](image1c.png)

![Fig. 1d](image1d.png)

We see that in this case the resulting FRW universe expands with acceleration while the compactification radius of the extra-dimensional space decreases with time. This indicates that the higher-dimensional GB term can indeed play the role of dark energy in this universe. As is the cases in the majority of the dark energy models currently available, a future singularity occurs.

2. Case \( r_0 < H(0) < q_0 \), then \( H' < 0 \), when \( H \to r_0 \), and in the limit one has that \( H' \to -\infty \). Again, \( H = r_0 \) corresponds to a singularity, and the curves are, respectively (Figs. 2a, 2b):

![Fig. 2a](image2a.png)

![Fig. 2b](image2b.png)
As in the case before, using Eq. (8) the behavior of $h(t)$ and $h'(h)$ can be obtained (Figs. 2c, 2d).

We see that in this case the size of the FRW universe decreases, while the 2-dimensional internal space scale factor may increase. This case cannot thus naturally describe the dark energy universe.

3. Case $q_0 < H(0) < p_0$, then $H' > 0$, when $H \to p_0$, and then $H$ tends to a constant, after an oscillatory regime (Figs. 3a, 3b).

Furthermore, the scale factor of the internal space can actually decrease, in fact (Figs. 3c, 3d)
Hence, the exact solution obtained at $H = p_0$ turns out to be a stationary stable point. Precise analysis corresponding to specific values of the parameters can be further carried out in a rather simple way.

4. Case $p_0 < H(0)$, then $H' < 0$ as $H \to p_0$, and the solution in this case is oscillatory (Figs. 4a, 4b).

As in the previous cases, using (8) the behaviors of $h(t)$ and $h'(h)$ can be obtained (Figs. 4c, 4d).

In this situation the FRW scale factor tends to a stationary point with decreasing $H$. The dark energy universe can correspond, in this case, to one of the branches of the oscillatory universe: expansion is turned into contraction, and vice-versa, with the repeated oscillations. Such universe would explicitly correspond to the general case described in [11]. The analytic solution for this case can be obtained.

Thus, it turns that in the FRW universe the scale factor behavior is much less complicated...
for these models than in the case of the absence of the GB term. This is certainly rewarding. In particular, for instance, in order to keep close to the Einstein regime, the condition of proportionality (perturbative regime) of $a(t)$ and $b(t)$ was used for the study of dynamical compactification in the Einstein-GB theory in Ref. [8]. For comparison, one can easily check that the situation there gets quite complicated and that different regimes for the Hubble rates appear, everything being much more simple and natural in our model above.

4. **Second case:** $\epsilon < 0$. In this case there is one stationary stable point $H = h = \pm 1/\sqrt{6|\epsilon|}$.

Notice here that solutions with close initial data tend to the stationary point in the oscillatory regime. However, both scale factors have the same sign and no dynamical compactification occurs! The universe, as a whole 6-dimensional object, either exponentially expands or shrinks. This indicates already that problems should be expected from a negative–$\epsilon$ model (indeed, in the higher-dimensional black hole case this may lead to negative entropy [14]).

5. **Third case:** $\epsilon = 0$. In this case one does recover (as it should be) an explicit solution. In fact, the equations for $a, b$ are

$$b'^2 a'^2 + 4ab'a'b' + a^2 b'^2 + 2a^2 b'' + 2a^2 bb'' = 0,$$

$$3aba'^2 + 3a^2 a'b' + 3a^2 ba'' + a^3 b'' = 0,$$

and, from here,

$$3h^2 + 4hH + 3H^2 + 2h' + 2H' = 0,$$

$$h'^2 + 3hH + 6H^2 + h' + 3H' = 0.$$  \hspace{1cm} (13)

From where one gets that

$$H' = (3 \pm 2\sqrt{6})H^2$$  \hspace{1cm} (15)
and the solution is given by

\[ H = -\frac{1}{\alpha t + C_1}, \]  

being

\[ \alpha = 3 \pm 2\sqrt{6}. \]  

Moreover, in terms of the scale factors:

\[ a = C_2[\pm(\alpha t + C_1)]^{-1/\alpha}, \]
\[ b = C_3[\pm(\alpha t + C_1)]^{3/\alpha}, \]  

where

\[ \beta = 3 \pm \sqrt{6}. \]  

6. Discussion. Summing up, we have investigated in this paper explicit non-perturbative dynamical compactification to a 4-dimensional FRW universe starting from a model of 6-dimensional Einstein-GB gravity. The number of stationary points obtained depends on the sign of the GB parameter, \( \epsilon \). They correspond in fact to exact solutions where the curvatures of the 2-dimensional and 4-dimensional spaces are constant. We have found a regime where the FRW universe does expand with acceleration, at the same time that the scale factor of the internal space goes to zero. But the most interesting regime discovered here is the oscillatory case, where a unification of an inflationary epoch with a late-time acceleration regime is indeed possible. This was suggested in [12] and has been explicitly realized here with concrete solutions and numbers, by combining numerical with analytical methods. For lack of space, in this letter we have just provided a brief presentation of the results. A more detailed study of the different accelerating universes that are derived from the model will be given elsewhere, in particular, details on the sequence of the matter dominated phase, the transition to acceleration, and the proper accelerating phases, as was done for other dark energy models of modified gravity in [13]. It will be also necessary to include in our considerations the effects of scalars (dilaton, moduli) in the low-energy sector of the string effective action.

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