Research Article
Closed-Form Equations for Contact Force and Moment in Elastic Contact of Rough Surfaces

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It is reasonable to expect that, when two nominally flat rough surfaces are brought into contact by an applied resultant force, they must support, in addition to the compressive load, an induced moment. The existence of a net applied moment would imply noneven distribution of contact force so that there are more asperities in contact over one region of the nominal area. In this paper, we consider the contact between two rectangular rough surfaces that provide normal and tangential contact force as well as contact moment to counteract the net moment imposed by the applied forces. The surfaces are permitted to develop slight angular misalignment, and thereby contact moment is derived. Through this scheme, it is possible to also define elastic contribution to friction since the half-plane tangential contact force on one side of an asperity is no longer balanced by the half-plane tangential force component on the opposite side. The elastic friction force, however, is shown to be of a much smaller order than the contact normal force. Approximate closed-form equations are found for contact force and moment for the contact of rough surfaces.

1. Introduction

The seminal work of Greenwood and Tripp [1] and Greenwood and Williamson [2] established the early models for elastic contact of nominally flat rough surfaces. These models have been used extensively in the realm of tribology and contact mechanics of rough surfaces. Many published works have appeared [3–35] since the publication of GW and GT models that, in one way or another, are inspired by [1, 2]. Almost an entire body of work [3–26, 28–35] ignores altogether the effect of contact moment. Hess and Soom [27] are perhaps the first and only ones to address contact moment. But their treatment of contact moment was limited to the application of the GW model in which only normal contact force is considered. Sepehri and Farhang [28] developed a three-dimensional model of elastic interaction between two rough surfaces. In this model asperity shoulder-shoulder contact is considered leading to the formulation of both half-plane tangential and normal components of contact force.

This paper develops the governing equations for the elastic contact of two nominally flat rough surfaces subject simultaneously to applied force and moment. To treat contact moment, the surfaces are allowed to sustain a slight angular misalignment so as to counteract the effect of applied moment. It is shown that the existence of a net moment induces a net tangential force as the half-plane tangential forces imposed on the surfaces no longer cancel. The net tangential force due to elastic interaction is found to be several orders of magnitude smaller than the normal contact force. Approximate equations are found, using a multistep optimization procedure that provides accuracy within seven percent.

2. Elastic Contact

When two rough surfaces are brought into contact by an applied resultant force, the surfaces must support, in addition to the compressive load, an induced moment. The existence of a net applied moment would imply nonuniform distribution of contact force so that there are more asperities in contact over one region of the nominal area. It is assumed that the moment would result in a slight angular misalignment of the mean planes of the two surfaces.
We consider the contact between two rectangular \((a \times b)\) rough surfaces that provide normal and tangential contact force as well as contact moment to counteract the net moment imposed. Thus, the mean plane separation is no longer constant and is described below using a linear function of the following form:

\[
d = d_0 - m(x - x_0),
\]

where \(d_0\) denotes the separation when the normal force is applied symmetrically so that no net moment is supported by the surfaces. When a net moment is additionally imposed by either the application of an applied tangential force or asymmetric application of the normal force or both, the surfaces would experience a slight relative rotation so that the normal pressure redistributes to accommodate the net applied moment. Let \(x_0\), measured from the center of the plate nominal area of contact along \(x\), correspond to the location along the mean plane, at which the mean plane separation is \(d_0\) in the relative tilted configuration (Figure 1). Let \(m\) be the mean plane slope of surface \(S_1\) with respect to \(S_2\) after applying the external moment.

We begin by examining the contact between two asperities as shown in Figure 2. An asperity on surface \(S_1\) interacts with an asperity on surface \(S_2\). Such an interaction occurs, in general, obliquely as shown in the figure so that the Hertz contact force along the common normal to the two asperities is not orthogonal to the mean planes of the two surfaces. Therefore, the Hertz contact force for interaction of two asperities can be expressed as

\[
f_u = \frac{4}{3}E'\beta(r)^{1/2}w^{3/2},
\]

where \(E'\) is the combined Young’s modulus of elasticity for the two surfaces

\[
\frac{1}{E'} = \frac{1}{E_1} + \frac{1}{E_2},
\]

using a linear approach along the normal to the mean planes. For \(r = 0\), asperities interfere along an oblique line, as shown in an enlarged view of the interference in Figure 3.

For the asperity interaction shown in Figures 2 and 3, the interference as well as equivalent radius of curvature must be found corresponding to the mid-point of contact, located approximately midway between the intersections of undeformed asperities. Denoting by \(r_1\) and \(r_2\), respectively, the tangential asperity summits, whereas the second represents

\[
\beta(r) = \frac{1}{\beta_1} + \frac{1}{\beta_2}.
\]

Alternatively, the offset may be written as follows:

\[
r_1 = \frac{\beta_1}{\beta}, \quad r_2 = \frac{\beta_2}{\beta}, \quad \beta = \beta_1 + \beta_2,
\]
the equivalent radius of curvature at a contact of the two asperities. Using a quadratic approximation for asperity shape near its summit, we find

\[ \beta(r) = \beta\left(1 + \frac{r^2}{\beta^2}\right)^{3/2}. \quad (7) \]

The approach, \( w \), is (Figures 2 and 3)

\[ w = w_1 \cos \alpha, \quad (8) \]

where \( w_1 \) is the interference defined by Greenwood and Williamson [2]. It is found in terms of the local mean plane separation, \( d' \), and asperity heights and offset as follows:

\[ w_1 = z_1 + z_2 - d' - f_1(r_1) - f_2(r_2), \quad (9) \]

\[ \cos \alpha = \left(1 + \frac{r^2}{\beta^2}\right)^{-1/2}, \quad \sin \alpha = \left(1 + \frac{r^2}{\beta^2}\right)^{-1/2} \frac{r}{\beta}. \quad (10) \]

\( f_1 \) and \( f_2 \) in (9) are the asperity shape functions of surfaces \( S_1 \) and \( S_2 \), respectively, and \( d' \) the local mean plane separation:

\[ d' = d - mr \cos \theta, \quad (11) \]

where \( \theta \) is the local orientation of an asperity on surface \( S_2 \) relative to the asperity on surface \( S_1 \), as shown in Figure 4. Note that in (9) the change of \( z_1 \) due to mean plane tilt (order of \( 10^{-6} \)) is negligible compared to that of the local mean plane separation (order of \( 10^{-3} \)). That is, the change in separation due to relative angular rotation has a most profound effect through its influence on changing the mean plane separation. Its effect through relative rotation of two asperities is negligible. Using a quadratic approximation near the summit of each asperity, (9) reduces to

\[ w_1 = z_1 + z_2 - d' - \frac{r^2}{2\beta^2}. \quad (12) \]

Therefore, the asperity interference can be found by combining (8), (10), (11), and (12) that yield

\[ w = \left(z_1 + z_2 - d + mr \cos \theta - \frac{r^2}{2\beta^2}\right) \left(1 + \frac{r^2}{\beta^2}\right)^{-1/2}. \quad (13) \]

Finally, substitution from (7) and (13) in (2) yields

\[ f_n = 4E'\beta^{1/2}\left(z_1 + z_2 - d + mr \cos \theta - \frac{r^2}{2\beta^2}\right)^{3/2}. \quad (14) \]

The asperity contact force in (14) is directed along the normal to the contact patch. It yields two components as shown in Figure 3. Denoting by \( f_N \) and \( f_i \) the components of the asperity contact force along the normal and tangential directions, respectively, we find, with the help of (10) and (14),

\[ f_N = 4E'\beta^{1/2}\left(z - d + mr \cos \theta - \frac{r^2}{2\beta^2}\right)^{3/2}\left(1 + \frac{r^2}{\beta^2}\right)^{-1/2}, \]

\[ f_i = 4E'\beta^{1/2}\left(z - d + mr \cos \theta - \frac{r^2}{2\beta^2}\right)^{3/2}\left(1 + \frac{r^2}{\beta^2}\right)^{-1/2} \frac{r}{\beta}, \quad (15) \]

where \( z = z_1 + z_2 \) the height sum of two asperities.

3. Normal Force

All the normal components of various contact forces \( (dF'z_1) \) are parallel (Figure 4) and can be algebraically summed by statistical means to obtain the total normal force of one surface on another.

Let \( dA \) (Figure 4), be an infinitesimal area on surface \( S_2 \) located at \( r \) from an asperity on an infinitesimal area \( dA' \) Figure 5 on surface \( S_1 \) located at \( x \) from the middle of the two rectangular surfaces:

\[ dA = r dr d\theta, \quad (16) \]

\[ dA' = b dx. \]

The total normal force exerted on an asperity of the differential area \( dA' \) at height \( z_1 \) by all asperities on surface \( S_2 \) is obtained by the following sum:

\[ (F_n)_{z_1} = 4E' \eta_2 \beta^{1/2} \int_{z_1} \int_\pi \int_\pi \left(z - d + mr \cos \theta - \frac{r^2}{2\beta^2}\right)^{3/2} \left(1 + \frac{r^2}{\beta^2}\right)^{-1/2} \phi_2(z_1) r dr d\theta dz_2, \quad (17) \]

where \( \phi_2(z_1) \) is the density function associated with the asperity height distribution on surface \( S_2 \). The above triple
integral could be simplified if an averaging method is used for the integration over the angular coordinate \( \theta \) in the range of \(-\pi/2\) to \(\pi/2\). We present the average representation of the integrand as follows:

\[
\int_{-\pi/2}^{\pi/2} \left( z - d + mr \cos \theta - \frac{r^2}{2\beta_x} \right)^{3/2} d\theta
\approx \left( z - d + \frac{2}{3} mr - \frac{r^2}{2\beta_x} \right)^{3/2} \pi.
\] (18)

The approximation provides relative error within 1.16 percent. So (17) yields

\[
(F_N)_{z_1} = \frac{4}{3} E' \eta_1 \beta^{1/2} \int_{z_2} \int_{z_1} \left( z - d + \frac{2}{3} mr - \frac{r^2}{2\beta_s} \right)^{3/2} \left( 1 + \frac{r^2}{\beta_s^2} \right)^{-1/2} \phi_2(z_2) r dr dz_2.
\] (19)

d\(F_N\), the normal force on all asperities of the differential area \(dA\) at height \(z_1\) by all asperities on surface \(S_1\), can be found by

\[
dF_N = \frac{4}{3} E' \eta_1 \beta^{1/2} \int_{z_1} \int_{z_2} \left( z - d + \frac{2}{3} mr - \frac{r^2}{2\beta_s} \right)^{3/2} \left( 1 + \frac{r^2}{\beta_s^2} \right)^{-1/2} \phi_1(z_1) \phi_2(z_2) r dr dz_1 dz_2 dx.
\] (20)

The total normal force on all asperities of the differential area \(dA\) due to all asperities on surface \(S_2\) is obtained as follows

\[
(F_N)_{dA} = \frac{4}{3} E' \eta_1 \beta^{1/2} \int_{x} \int_{r} \left( z - d + \frac{2}{3} mr - \frac{r^2}{2\beta_s} \right)^{3/2} \left( 1 + \frac{r^2}{\beta_s^2} \right)^{-1/2} \phi(z) r dr dz dx,
\] (21)

where \(\phi(z)\) is the density function for summit height sum distribution of asperity summits on \(S_1\) and \(S_2\). The total normal force on all asperities of surface \(S_1\) due to all asperities on surface \(S_2\) can be found by integration along \(x\)

\[
F_N = \frac{4}{3} E' \eta_1 \beta^{1/2} \int_{x} \int_{r} \left( z - d + \frac{2}{3} mr - \frac{r^2}{2\beta_s} \right)^{3/2} \left( 1 + \frac{r^2}{\beta_s^2} \right)^{-1/2} \phi(z) r dr dz dx.
\] (22)

Every length parameter in (22) is normalized with respect to the standard deviation of asperity height sum. Define

\[
s = z/\sigma, h = d/\sigma \text{ and consider hereafter } r, \beta, \text{ and } \beta_s \text{ as normalized values using } \sigma \text{ as the normalization parameter.}
\]

The limits of \(r\) are determined by the roots of the following equation (setting interference, \(w\), equal to zero)

\[
s - h + \frac{2}{3} mr - \frac{r^2}{2\beta_s} = 0.
\] (23)

We find

\[
r_1 = \frac{2}{3} \beta_s m + \sqrt{\left( \frac{2}{3} \beta_s m \right)^2 + 2\beta_s(s - h)},
\] (24)

and

\[
r_2 = \frac{2}{3} \beta_s m - \sqrt{\left( \frac{2}{3} \beta_s m \right)^2 + 2\beta_s(s - h)},
\] (25)

where

\[
h = h_0 - m(x - x_0).
\]

\(h_0\) is the normalized value of \(d_0\) and \(x\) and \(x_0\) are now used to denote normalized values using \(\sigma\) as the normalization parameter. For a Gaussian distribution of the asperity height sum we find the following normalized form for the equation describing total normal contact force between surfaces \(S_1\) and \(S_2\)

\[
F_N(h_0, \beta_s, m, a) = \frac{C \pi}{2} I_N(h_0, \beta_s, m, a),
\] (26)

where

\[
C = \frac{8}{3 \sqrt{2\pi}} E' \eta_1 \beta^{1/2} \sigma^3,
\]

\[
I_N(h_0, \beta_s, m, a) = \int_{\beta_s}^{\infty} \left( \int_{h_0}^{\beta} \left( s - h + \frac{2}{3} mr - \frac{r^2}{2\beta_s} \right)^{3/2} \right.
\]
\[
\times \left( 1 + \frac{r^2}{\beta_s^2} \right)^{-1/2} e^{-r^2/2} dr dr ds
\]
\[
+ \int_{h_0}^{\infty} \left( s - h + \frac{2}{3} mr - \frac{r^2}{2\beta_s} \right)^{3/2}
\]
\[
\times \left( 1 + \frac{r^2}{\beta_s^2} \right)^{-1/2} e^{-r^2/2} dr dr ds
\] (27)

In the absence of a moment, the total normal force is [28]

\[
F_N(h_0, \beta_s) = C \pi \alpha I_N(h_0, \beta_s),
\] (28)

where

\[
I_N(h_0, \beta_s) = \int_{h_0}^{\infty} \sqrt{\frac{2\beta}{(s - h)}} \left( s - h - \frac{r^2}{2\beta_s} \right)^{3/2}
\]
\[
\times \left( 1 + \frac{r^2}{\beta_s^2} \right)^{-1/2} e^{-r^2/2} dr dr ds.
\] (29)
So we have

\[ I_N(h_0, \beta_s, m, a) = 2aI_N(h_0, \beta_s). \]  

(30)

The results of numerical solution confirm the above equation. For a plate of known size \( a \times b \) and prescribed normal force and moment, one can solve the above equation along with the moment equation to find \( x_0 \) and \( m \). \( x_0 \) is the location at which the mean plane separation equals that with no applied moment. We refer to this as the position of initial separation. It is essential that the relation between \( x_0 \) and other parameters is established since it greatly facilitates the solution of problems involving contact moment. \( x_0 \) only depends on \( h_0, m, a \), and it is not a function of \( \beta_s \). We find, using a multistep optimization procedure, the following approximate equation for \( x_0 \) over \( h_0 = 1 \) to \( 4 \), \( m = 3 \times 10^{-5} \) to \( 3 \times 10^{-5} \), and \( a = 50000 \) to \( 250000 \)

\[ x_0(h_0, m, a) = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{5} c_{i,j,k} (m \times 10^5)^{i-1} (h_0)^{j-1} (a \times 10^{-4})^{k-1}. \]  

(31)

Recall that the large values assigned to \( a \) are due to their normalization with respect to the standard deviation of asperity height sum, \( \sigma \). \( c_{i,j,k} \) values are given in the Table 1.

Figures 6, 7, and 8, respectively, illustrate \( x_0(h_0, m) \) for \( a = 150000 \), \( x_0(h_0, a) \) for \( m = 2 \times 10^{-5} \), and \( x_0(m, a) \) for \( h_0 = 1 \). The approximate function in (31) provides estimates of \( x_0 \) to within 0.6 percent accuracy over the entire domain of \( h_0, m, \) and \( a \).

4. Tangential Force

The tangential components due to various interactions cannot be algebraically added as they are projections of contact force onto the mean plane and depend on circumferential position of asperities on surface \( S_2 \) (Figures 9 and 10). We are interested in formulating the cumulative effect of \( x \)-component of tangential force after applying the moment. Hereafter as we generate result for the \( x \)-component of the tangential force, we will refer to this as the “tangential force” and denote the force component \( F_x \). The goal here is to account for the tangential force that an asperity would experience on each side. Accumulation or summation of such forces would establish the total tangential load on an asperity from both sides, that is, due to all contacts at positive slopes and negative slopes.

It can be shown that the total tangential force (along \( x \)) on an asperity of the differential area \( dA' \) at height \( z_1 \) by all
asperities on surface \(S_2\) is found by

\[
(F_x)_{z_1} = \frac{4}{3} \frac{\eta_2}{\beta_s} \beta_s^{1/2} \int_{z_1} \int_{\theta} \left( z - d + mr \cos \theta - \frac{r^2}{2\beta_s} \right)^{3/2} \left( 1 + \frac{r^2}{\beta_s^2} \right)^{-1/2} \cos \theta \phi_z(z_1) r^2 \, d\theta \, dz_2.
\]

(32)

The integral (32) can be reduced through the following approximation of the \(\theta\)-integration over the range of \( -\pi/2 \) to \( \pi/2 \):

\[
\int_{-\pi/2}^{\pi/2} \left( z - d + mr \cos \theta - \frac{r^2}{2\beta_s} \right)^{3/2} \cos \theta \, d\theta \\
\approx 2 \left( z - d + \frac{4}{5} mr - \frac{r^2}{2\beta_s} \right)^{3/2}.
\]

(33)
The approximation provides a relative error less than 0.83 percent. We find

\[
(F_x)_{z_1} = \frac{8}{3} E' \eta_1 \beta^{1/2} \int_{z_1} \left( z - d + 4.5m \rho - \frac{r^2}{2\beta} \right) 3/2 \times \left( 1 + \frac{r^2}{\beta} \right)^{-1/2} \phi(z) r^2 dr dz.
\]

where

\[
I_x(h_0, \beta, m, a) = \frac{C}{\beta_s} I_x(h_0, \beta_s, m, a),
\]

and C and r₁, r₂ are defined in (27) and (24), respectively.

5. Contact Moment

Contact moment about the y-axis consists of two terms: (1) sum of moments about each asperity (M₁) and (2) moment of normal force about the y-axis (M₂) due to movement of the point of exertion of the resultant normal force (x_N), as shown in Figure 1. First we consider M₁ to find total moment about all asperities. Moment about the y-axis on an asperity in the differential area dA’ at height z₁ due to all asperities at height z₂ confined in (rdr, dθ) and located at r is

\[
(M_1)_{z_1} = \frac{4}{3} E' \eta_1 \beta^{1/2} \int_{z_1} \left( z - d + 4.5m \rho - \frac{r^2}{2\beta} \right) 3/2 \times \left( 1 + \frac{r^2}{\beta} \right)^{-1/2} \cos \phi(z) r^2 dr d\theta.
\]

The total moment carefully found for all asperities on surface S_2 is given by

\[
(M_2)_{z_1} = \frac{8}{3} E' \eta_1 \beta^{1/2} \int_{z_1} \left( z - d + 4.5m \rho - \frac{r^2}{2\beta} \right) 3/2 \times \left( 1 + \frac{r^2}{\beta} \right)^{-1/2} \phi(z) r^2 dr dz.
\]

For a Gaussian distribution of asperity height sum the total tangential force between surfaces S₁ and S₂ in normalized form is

\[
F_x(h_0, \beta, m, a) = \frac{C}{\beta_s} F_x(h_0, \beta_s, m, a).
\]
The total moment on all asperities in the differential area \( dA \) due to all asperities on surface \( S_2 \) is obtained as follows:

\[
(M_1)_{dA} = \frac{8}{3} E' \eta_1 \eta_2 b \beta^{1/2} \
\times \left( z - d + \frac{4}{5} mr - \frac{r^2}{2 \beta} \right)^{3/2} \
\times \left( 1 + \frac{r^2}{\beta^2} \right)^{-1/2} \phi(z) r^2 dr dz dx.
\]  

(43)

The total moment on all asperities of surface \( S_1 \) due to all asperities on surface \( S_2 \) can be found by integration along \( x \)

\[
M_1 = \frac{8}{3} E' \eta_1 \eta_2 b \beta^{1/2} \
\times \int_x \int_z \int_r \left( z - d + \frac{4}{5} mr - \frac{r^2}{2 \beta} \right)^{3/2} \
\times \left( 1 + \frac{r^2}{\beta^2} \right)^{-1/2} \phi(z) r^2 dr dz dx.
\]

(44)

For a Gaussian distribution of asperity height sum the total moment between surfaces \( S_1 \) and \( S_2 \) is

\[
M_1(h_0, \beta_s, m, a) = C \sigma I_{M_1}(h_0, \beta_s, m, a),
\]

(45)

where \( I_{M_1}(h_0, \beta_s, m, a) \) is given by (39) and \( C \) and \( r_1, r_2 \) are defined in (27) and (24), respectively. The second term of moment \( (M_2) \) due to shifting of the location of the resultant normal load is

\[
M_2(h_0, \beta_s, m, a) = \frac{C \pi a}{2} I_{M_2}(h_0, \beta_s, m, a),
\]

(46)

where

\[
I_{M_2}(h_0, \beta_s, m, a) = \int_{-a/2}^{a/2} \left( \int_{h_0}^{\infty} \int_{0}^{r_1} \left( s - h + \frac{2}{3} mr - \frac{r^2}{2 \beta} \right)^{3/2} \
\times \left( 1 + \frac{r^2}{\beta^2} \right)^{-1/2} e^{-r'^2/2} r dr ds 
\right. \\
+ \int_{h_0}^{\infty} \int_{0}^{r_2} \left( s - h + \frac{2}{3} mr - \frac{r^2}{2 \beta} \right)^{3/2} \
\times \left( 1 + \frac{r^2}{\beta^2} \right)^{-1/2} e^{-r'^2/2} r dr ds \right),
\]

(47)

Therefore, the total moment between surfaces \( S_1 \) and \( S_2 \) can be written as

\[
M(h_0, \beta_s, m, a) = C \sigma I_M(h_0, \beta_s, m, a),
\]

(48)

where

\[
I_M(h_0, \beta_s, m, a) = I_{M_1}(h_0, \beta_s, m, a) + \frac{\pi}{2} I_{M_2}(h_0, \beta_s, m, a).
\]

(49)

Numerical results show that the first moment term \( I_{M_1} \) (order of \( 10^5 \)) is negligible in comparison to the second term \( I_{M_2} \) (order of \( 10^{12} \)). The force center, \( x_N \), is obtained by

\[
x_N(h_0, \beta_s, m, a) = \frac{I_{M_1}(h_0, \beta_s, m, a)}{2 \sigma I_N(h_0, \beta_s)}. \hspace{1cm} \text{(50)}
\]

Friction coefficient due to elastic deformation is equal to

\[
\mu(h_0, \beta_s, m, a) = \frac{I_N(h_0, \beta_s, m, a)}{\pi a \beta_s I_N(h_0, \beta_s)}. \hspace{1cm} \text{(51)}
\]

\( I_N(h_0, \beta_s), I_s(h_0, \beta_s, m, a), \) and \( I_M(h_0, \beta_s, m, a) \) in the above two equations are given by (29), (39), and (49), respectively.

### 6. Results

Figures 11 and 12, respectively, depict the dimensionless normal and tangential contact force components, \( I_N \) and \( I_s \), versus dimension length, \( x \), for \( h_0 = 1 \), \( \beta_s = 500 \), and plate dimensionless length \( a = 250000 \). Since each slope corresponds to an applied moment, the figures show the pressure and shear distribution for various applied moments. Figures 13, 14, and 15 illustrate the dependence of \( I_{M_1}(h_0, \beta_s, m, a) \), in each case holding two of the parameters fixed. Figure 13 shows \( I_{M_1}(h_0, \beta_s) \) for \( m = 2 \times 10^{-7} \) and \( a = 150000 \), Figure 14 depicts \( I_{M_1}(h_0, m) \) for \( \beta_s = 500 \) and \( a = 150000 \), and Figure 15 represents \( I_{M_1}(h_0, a) \) for \( \beta_s = 500 \) and \( m = 2 \times 10^{-7} \). In these figures \( h_0 \) corresponds to an applied normal force such that a small value of \( h_0 \) represents a large normal force and large \( h_0 \) corresponds to a small applied normal force. Therefore, for large applied normal force it is shown in Figures 13, 14, and 15 that the sensitivity of dimensionless contact moment increases with asperity radius of curvature sum \( \beta_s \), slope \( m \), and nominal dimension \( a \). Similar observation can be made about the dimensionless
contact force $I_x$. Figures 16, 17 and 18, respectively, show $I_x(h_0, \beta_s), I_y(h_0, m)$ and $I_z(h_0, a)$ for the same cases.

Figures 19, 20, and 21, respectively, demonstrate the relation between the location of the resultant normal contact force, $x_N(h_0, \beta_s, m, a)$, while, as before, holding two of the parameters constant. These are shown as $x_N(h_0, \beta_s), x_N(h_0, m)$, and $x_N(h_0, a)$. Figures 22, 23, and 24, respectively, represent $\mu(h_0, \beta_s), \mu(h_0, m)$, and $\mu(h_0, a)$. Figures 19 and 22 show that the force center and elastic friction coefficient are independent of $\beta_s$.

7. Approximate Equations

In this section we introduce approximate equations for the integral functions of contact moment and tangential force. We find, using a multistep optimization procedure, the following general approximate equations for $I_M$ and $I_x$:

$$I_M(h_0, \beta_s, m, a) = \alpha_1(\beta_s, m, a) e^{\alpha_2(m,a) h_0^{\alpha_3(m,a)}},$$

$$\alpha_1(\beta_s, m, a) = f_1(m)(a \times 10^{-4})^f(m) \beta_s$$
$$- f_3(m)(a \times 10^{-4}) f_5(m),$$

$$\alpha_2(m, a) = - (f_5(m)(a \times 10^{-4}) + f_6(m)),$$

$$\alpha_3(m, a) = f_7(m)(a \times 10^{-4}) + f_8(m),$$

$$\alpha_5(m, a) = f_9(m)(a \times 10^{-4}) f_{10}(m).$$

FIGURE 12: Dimensionless tangential force, $I_x$, versus $x$ for $h_0 = 1$, $\beta_s = 500$, and $a = 250000$.

FIGURE 14: $I_M(h_0, m)$ for $\beta_s = 500$ and $a = 150000$.

FIGURE 15: $I_M(h_0, a)$ for $\beta_s = 500$ and $m = 2 \times 10^{-5}$. 

FIGURE 13: $I_M(h_0, \beta_s)$ for $m = 2 \times 10^{-5}$ and $a = 150000$. 

FIGURE 16: The dependence of $I_M$ on $h_0$. 

FIGURE 17: The dependence of $I_M$ on $\beta_s$. 

FIGURE 18: The dependence of $I_M$ on $a$. 

FIGURE 19: The dependence of $x_N$ on $h_0$. 

FIGURE 20: The dependence of $x_N$ on $\beta_s$. 

FIGURE 21: The dependence of $x_N$ on $a$. 

FIGURE 22: The dependence of $\mu$ on $h_0$. 

FIGURE 23: The dependence of $\mu$ on $\beta_s$. 

FIGURE 24: The dependence of $\mu$ on $a$. 

FIGURE 25: The dependence of $f_1$ on $m$. 

FIGURE 26: The dependence of $f_2$ on $m$. 

FIGURE 27: The dependence of $f_3$ on $m$. 

FIGURE 28: The dependence of $f_4$ on $m$.
where

\[ f_1(m) = 6.30326 \times 10^{12} m^{1.33929}, \]
\[ f_2(m) = -2.03970 \times 10^4 m + 3.05009, \]
\[ f_3(m) = 5.96853 \times 10^{11} m^{1.20289}, \]
\[ f_4(m) = -2.94562 \times 10^4 m + 3.22846, \]
\[ f_5(m) = 6.52400 \times 10^2 m - 1.31000 \times 10^{-3}, \]
\[ f_6(m) = 9.49280 \times 10^{-1} m^{-1.38200 \times 10^{-2}}, \]
\[ f_7(m) = -6.08955 \times 10^6 m^{1.93643}, \]
\[ f_8(m) = 1.65800 \times 10^5 m + 1.61790, \]

as well as

\[ I_{xa}(h_0, \beta_s, m, a) = \alpha_1(\beta_s, m, a) e^{\alpha_2(m,a) h_0^\alpha_3(m,a)}, \]
\[ \alpha_1(\beta_s, m, a) = g_1(m)(a \times 10^{-4})^{g_2(m)}(g_3(m)^{g_4(m)}, \]
\[ \alpha_2(m, a) = - \left( g_4(m)(a \times 10^{-4} - g_5(m))^2 + g_6(m) \right), \]
\[ \alpha_3(m, a) = g_7(m)(a \times 10^{-4} - g_8(m))^2 + g_9(m), \]

where

\[ g_1(m) = 9.39400 \times 10^{-2}(m \times 10^5)^{8.73060 \times 10^{-1}}, \]
\[ g_2(m) = 9.92810 \times 10^{-1}(m \times 10^5)^{7.14000 \times 10^{-2}}, \]
\[ g_3(m) = -3.88000 \times 10^{-4}(m \times 10^5) + 2.00278, \]
To assess the accuracy of the approximation in (52), we define the following error between the dimensionless critical contact moment and its approximation in percent error form:

\[ EM(h_0, \beta_s, m, a) = \frac{I_M(h_0, \beta_s, m, a) - I_{Ma}(h_0, \beta_s, m, a)}{I_M(h_0, \beta_s, m, a)} \times 100. \]  

The error is plotted over the ranges \( h_0 = 1 \) to 4, and \( \beta_s = 100 \) to 2000 for \( m = 2 \times 10^{-5} \) and \( a = 150000 \) in Figure 25, demonstrating that the accuracy of approximation is within seven percent (7%). Indeed the approximate function in (52) yields accuracy to within 7 percent over the entire domain of \( h_0, \beta_s, m, \) and \( a \).

A similar study for error associated with the approximation of \( E_z(h_0, \beta_s, m, a) \) yields accuracy within 7 percent over the entire range of \( h_0, \beta_s, m, \) and \( a \). Figure 26 depicts
The approximate functions in (57) provide estimates of $x_N$ and $\mu$ to within 5%.

### 8. Concluding Remarks

This paper has addressed a methodology for treating contact moment in the interaction of two nominally flat rough surfaces. The method is based on an extension of GT model [28] in which the asperity elastic shoulder-shoulder contact is considered, yielding resultant asperity contact force in a slanted orientation with respect to the mean planes of the surfaces.

In consideration of contact moment, the method allows a slight relative angular rotation (tilting) of the mean planes. Hence the applied moment is balanced by a nonuniform distribution of normal contact pressure. In addition to the nonuniform distribution of normal force, relative tilting of the mean planes is shown to result in imbalance of the $+x$ and $-x$ half-plane tangential forces, leading to a definition of elastic friction force. The results have shown that friction force due to elastic contact is very small so that the coefficient of friction is found to be less than $3 \times 10^{-5}$.

Approximate equations were forwarded for the integral functions of contact moment and tangential force. These equations were shown to provide accuracy within seven percent over ranges of mean plane separation, asperity summit radius of curvature sum, and slope and plate dimension. The approximate equations greatly simplify solution of problems involving elastic contact of rough surfaces.

### Nomenclature

- $f_a$: Hertz contact force
- $E'$: Combined Young’s modulus
- $\beta_{1,2}$: Dimensionless average summit radius of asperities on the surfaces 1 and 2
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