A complement on representations of Hom-Lie algebras

Xiong Zhen
Department of Mathematics and Computer, Yichun University, Jiangxi 336000, China
email:205137@jxycu.edu.cn

Abstract
In this paper, we give a new series of coboundary operators of Hom-Lie algebras. And prove that cohomology groups with respect to coboundary operators are isomorphic. Then, we revisit representations of Hom-Lie algebras, and prove that there is an one-to-one correspondence between Hom-Lie algebraic structure on vector space $g$ and these coboundary operators on $\Lambda^g \otimes V$.

1 Introduction
The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [1] as part of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. Some $q$-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra [1, 2]. Because of close relation to discrete and deformed vector fields and differential calculus [1, 3, 4], more people pay special attention to this algebraic structure. In particular, Hom-Lie algebras on semisimple Lie algebras are studied in [5]; Its geometric generalization is given in [6, 7]; Quadratic Hom-Lie algebras are studied in [8]; Representation theory, cohomology and homology theory are systematically studied in [9, 10, 11, 12, 13]; Bialgebra theory and Hom-(Classical) Yang-Baxter Equation are studied in [14, 15, 16, 17]; The notion of a Hom-Lie 2-algebra, which is a categorification of a Hom-Lie algebra, is introduced in [18], in which the relation with Hom-left-symmetric algebras [19] and symplectic Hom-Lie algebras are studied.

Let $(g, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra, $V$ be a vector space, $\rho : g \rightarrow \mathfrak{gl}(V)$ be a representation of $(g, [\cdot, \cdot], \alpha)$ on the vector space $V$ with respect to $\beta \in \mathfrak{gl}(V)$. The set of $k$-cochains on $g$ with values in $V$, which we denote by $C^k(g; V)$, is the set of skewsymmetric $k$-linear maps from $g \times \cdots \times g(k$-times) to $V$:

$$C^k(g; V) := \{ \eta : \wedge^k g \rightarrow V \text{ is a linear map} \}.$$ 

In [20], the authors define Hom-$k$-cochains: $C^k_{\alpha, \beta}(g; V) = \{ \eta \in C^k(g; V) | \eta \circ \alpha = \beta \circ \eta \}$, which is a subset of $C^k(g; V)$. There are a series of coboundary operators $\hat{d}^* \text{ define on } C^k_{\alpha, \beta}(g; V)$; in [21], the authors give a special coboundary operator of regular Hom-Lie algebras. For regular Hom-Lie
algebras, there are many works are done by the special coboundary operator [6, 21]. In this article, we give a new series coboundary operators on $k$-cochains $C^k(g; V)$, prove that cohomology groups with respect to these coboundary operators, are isomorphic. Then, we revisit representations of Hom-Lie algebras, and generalize the result 'If $\mathfrak{t}$ is a Lie algebra, $\rho : \mathfrak{t} \rightarrow \mathfrak{gl}(V)$ is a representation if and only if there is a degree-1 operator $D$ on $\Lambda^* \otimes V$ satisfying $D^2 = 0$, and

$$D(\xi \wedge \eta \otimes u) = d_k \xi \wedge \eta \otimes u + (-1)^k \xi \wedge D(\eta \otimes u), \quad \forall \xi \in \Lambda^k \ast, \ \eta \in \Lambda^l \ast, \ u \in V,$$

where $d_k : \Lambda^k \ast \rightarrow \Lambda^{k+1} \ast$ is the coboundary operator associated to the trivial representation.'

The paper is organized as follows. In Section 2, we first recall some necessary background knowledge: Hom-Lie algebras and their representations. Then, we show that $d^s$ is coboundary operators of Hom-Lie algebras and prove that cohomology groups with respect to these coboundary operators, are isomorphic(Theorem 2.6). In Section 3, we give some properties of $d^s$ and revisit representations of Hom-Lie algebras, and have Theorem 3.4.

## 2 Cohomology operators of Hom-Lie algebras

The notion of a Hom-Lie algebra was introduced in [1], see also [8, 19] for more information.

**Definition 2.1.** (1) A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ consisting of a vector space $\mathfrak{g}$, a skewsymmetric bilinear map (bracket) $[\cdot, \cdot] : \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$ and a linear transformation $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying $\alpha(x, y) = [\alpha(x), \alpha(y)]$, and the following Hom-Jacobi identity:

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g}. \quad (1)$$

A Hom-Lie algebra is called a regular Hom-Lie algebra if $\alpha$ is a linear automorphism.

(2) A subspace $\mathfrak{h} \subset \mathfrak{g}$ is a Hom-Lie sub-algebra of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ if $\alpha(\mathfrak{h}) \subset \mathfrak{h}$ and $\mathfrak{h}$ is closed under the bracket operation $[\cdot, \cdot]$, i.e. for all $x, y \in \mathfrak{h}$, $[x, y] \in \mathfrak{h}$.

(3) A morphism from the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ to the hom-Lie algebra $(\mathfrak{h}, [\cdot, \cdot], \gamma)$ is a linear map $\psi : \mathfrak{g} \rightarrow \mathfrak{h}$ such that $\psi([x, y]) = [\psi(x), \psi(y)]$ and $\psi \circ \alpha = \gamma \circ \psi$.

Representation and cohomology theories of Hom-Lie algebras are systematically introduced in [13, 10]. See [14] for homology theories of Hom-Lie algebras.

**Definition 2.2.** A representation of the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ on a vector space $V$ with respect to $\beta \in \mathfrak{gl}(V)$ is a linear map $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, such that for all $x, y \in \mathfrak{g}$, the following equalities are satisfied:

$$\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x); \quad (2)$$
$$\rho([x, y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x). \quad (3)$$

Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra, $V$ be a vector space, $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ on the vector space $V$ with respect to $\beta \in GL(V)$, where $\beta$ is invertible. In this paper, we just consider $\beta \in GL(V)$.

The set of $k$-cochains on $\mathfrak{g}$ with values in $V$, which we denote by $C^k(\mathfrak{g}; V)$, is the set of skewsymmetric $k$-linear maps from $\mathfrak{g} \times \cdots \times \mathfrak{g}(k\text{-times})$ to $V$:

$$C^k(\mathfrak{g}; V) := \{ \eta : \Lambda^k \mathfrak{g} \rightarrow V \text{ is a linear map} \}.$$
For $s = 0, 1, 2, \ldots$, define $d^s : C^k(g; V) \rightarrow C^{k+1}(g; V)$ by

$$d^s(\alpha_1, \ldots, \alpha_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \beta^{k+1+s} \rho(x_i) \beta^{-k-2-s} \eta(\alpha(x_i), \ldots, \alpha(x_{k+1}))$$

where $\beta^{-1}$ is the inverse of $\beta$, $\eta \in C^k(g; V)$.

**Proposition 2.3.** With the above notations, the map $d^s$ is a coboundary operator, i.e. $d^s \circ d^s = 0$.

**Proof.** For any $\eta \in C^k(g; V)$, by straightforward computations, we have

$$d^s \circ d^s(\alpha_1, \ldots, \alpha_{k+2}) = \sum_{i=1}^{k+2} (-1)^{i+1} \beta^{k+2+s} \rho(x_i) \beta^{-k-3-s} d^s \eta(\alpha(x_i), \ldots, \alpha(x_{k+2}))$$

$$+ \sum_{i<j} (-1)^{i+j} d^s \eta([x_i, x_j], \alpha(x_1), \ldots, \alpha(x_{k+2})).$$

And

$$d^s \eta(\alpha(1), \ldots, \alpha_{k+2})$$

$$= \sum_{i<j} (-1)^{j+1} \beta^{k+1+s} \rho(\alpha(x_j)) \beta^{-k-2-s} \eta(\alpha^2(x_1), \ldots, \alpha^2(x_{k+2}))$$

$$+ \sum_{i>j} (-1)^{j} \beta^{k+s} \rho(\alpha(x_i)) \beta^{-k-2-s} \eta(\alpha^2(x_1), \ldots, \alpha^2(x_{k+2}))$$

$$+ \sum_{m<n<i} (-1)^{m+n} \eta(\alpha([x_i, x_j]), \alpha^2(x_1), \ldots, \alpha^2(x_{k+2}))$$

$$+ \sum_{m<n<i} (-1)^{m+n-1} \eta(\alpha([x_i, x_j]), \alpha^2(x_1), \ldots, \alpha^2(x_{k+2}))$$

$$+ \sum_{i<m<n<i} (-1)^{m+n} \eta(\alpha([x_i, x_j]), \alpha^2(x_1), \ldots, \alpha^2(x_{k+2})).$$

At the same time, we have
By Hom-Jacobi identity:

\[ d^s \eta([x_i, x_j], \alpha(x_i), \cdots, \alpha(x_{k+2})) = \beta^{k+1+s} \rho([x_i, x_j]) \beta^{-k-2-s} \eta(\alpha^2(x_i), \cdots, \alpha^2(x_{k+2})) \]

\[ + \sum_{p<q} \rho(\alpha(x_p)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_i), \cdots, x_{p,i,j}) \]

\[ + \sum_{i<j} \rho(\alpha(x_p)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_i), \cdots, x_{i,j,p,j}) \]

\[ + \sum_{i<j} \rho(\alpha(x_p)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_i), \cdots, x_{i,j,p,j}) \]

\[ + \sum_{q<i<j} \rho(\alpha(x_q)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_i), \cdots, x_{q,i,j}) \]

\[ + \sum_{i<j} \rho(\alpha(x_q)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_i), \cdots, x_{i,q,j}) \]

\[ + \cdots \]

By Hom-Jacobi identity:

\[ 0 = (1) + \cdots. \]

and we have: \( 0 = \sum_{1}^{\infty} \).

By \( \rho(\alpha(x)) = \beta \rho(x) \) and \( \rho(\alpha(x)) = \beta \rho(x) \beta^{-1} \), we have

\[ d^s \circ d^s \eta(x_i, \cdots, x_{k+2}) = \sum_{i<j} \sum_{p<q} \rho(\alpha(x_p)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_i), \cdots, x_{p,i,j}) \]

\[ + \sum_{i<j} \sum_{p<q} \rho(\alpha(x_q)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_i), \cdots, x_{i,q,j}) \]

\[ + \cdots \]
By \( \rho([x, y])\beta = \rho(\alpha(x))\rho(y) - \rho(\alpha(y))\rho(x) \), we have
\[
\mathbf{S} + \mathbf{U} + \mathbf{U} = 0.
\]

About above equations, sum of the rest six equations is zero. So, we proof that \( d^s \circ d^s = 0 \).

**Remark 2.4.** In [20], the authors give a series coboundary operators \( \hat{d}^s : C_{\alpha,\beta}^k (g; V) \rightarrow C_{\alpha,\beta}^{k+1} (g; V) \), on the set of Hom-\( k \)-cochains \( C_{\alpha,\beta}^k (g; V) = \{ \eta \in C^k (g; V) | \eta \circ \alpha = \beta \circ \eta \} \). The coboundary operators \( d^s \) we define are not the same as those in [21], [21] just give coboundary operator for regular Hom-Lie algebras.

From \( d^s \), we know that the coboundary operator associated to the trivial representation is \( d : \wedge^k g^* \rightarrow \wedge^{k+1} g^* \),
\[
d \bar{\xi}(x_1, \cdots, x_{k+1}) = \sum_{i<j} (-1)^{i+j} \bar{\xi}([x_i, x_j], \alpha(x_1), \cdots, \widehat{x_{i,j}}, \cdots, \alpha(x_{k+1})).
\]

For Hom-Lie algebra \( (g, [\cdot, \cdot], \alpha) \) and representation \( \rho \) with respect with \( \beta, \alpha \) induces a map \( \tilde{\alpha} : C^l (g; V) \rightarrow C^l (g; V) \) via
\[
\tilde{\alpha}(\eta)(x_1, \cdots, x_l) = \eta(\alpha(x_1), \cdots, \alpha(x_l)).
\]
And \( \beta \) induces a map \( \tilde{\beta} : C^l (g; V) \rightarrow C^l (g; V) \) via
\[
\tilde{\beta}(\eta)(x_1, \cdots, x_l) = \beta \circ \eta(x_1, \cdots, x_l).
\]

\( C^* (g; V) = \bigoplus C^l (g; V) \) is a \( \wedge^* \oplus \Lambda^k g^* \)-module, where the action \( \circ : \wedge^k g^* \times C^l (g; V) \rightarrow C^{k+l}(g; V) \) is given by
\[
\xi \circ \eta(x_1, \cdots, x_{k+l}) = \sum_\kappa \text{sgn}(\kappa) \eta(x_{\kappa(1)}, \cdots, x_{\kappa(k)}) \eta(x_{\kappa(k+1)}, \cdots, x_{\kappa(k+l)}),
\]
where \( \xi \in \Lambda^k g^*, \eta \in C^l (g; V) \), and the summation is taken over \((k,l)\)-unshuffles.

Obviously, for \( \xi, \xi_1, \xi_2 \in \wedge^*, \eta \in C^* (g; V) \), we have
\[
\tilde{\alpha}(\xi_1 \wedge \xi_2) = \tilde{\alpha}(\xi_1) \wedge \tilde{\alpha}(\xi_2); \quad \tilde{\alpha}(\xi \circ \eta) = \tilde{\alpha}(\xi) \circ \tilde{\alpha}(\eta); \quad \tilde{\beta}(\xi \circ \eta) = \xi \circ \tilde{\beta}(\eta).
\]

Associated to the representation \( \rho \), we obtain the complex \( (C^k (g; V), d^s) \). Denote the set of closed \( k \)-cochains by \( Z^k (d^s) \) and the set of exact \( k \)-cochains by \( B^k (d^s) \). Denote the corresponding cohomology by
\[
H^k (d^s) = Z^k (d^s) / B^k (d^s).
\]
Now, we study the relation between \( H^k (d^s) \) and \( H^k (d^{s+1}) \).

**Proposition 2.5.** With the above notations, we have
\[
\tilde{\beta} \circ d^s = d^{s+1} \circ \tilde{\beta}.
\]
Proof. For \( \eta \in C^l(\mathfrak{g}; V) \), we have

\[
\tilde{\beta} \circ d^s \eta(x_1, \cdots, x_{l+1}) = \sum_{i=1}^{l+1} (-1)^{i+1} \beta^{l+2+s} \rho(x_i) \beta^{-1-2-s} \eta(\alpha(x_1), \cdots, \hat{x}_i, \cdots, \alpha(x_{l+1})) \\
+ \sum_{i<j} (-1)^{i+j} \beta \circ \eta(x_i, x_j, \alpha(x_1), \cdots, \hat{x}_{i,j}, \cdots, \alpha(x_{l+1})) \\
= \sum_{i=1}^{l+1} (-1)^{i+1} \beta^{l+2+s} \rho(x_i) \beta^{-1-3-s} \tilde{\beta}(\eta)(\alpha(x_1), \cdots, \hat{x}_i, \cdots, \alpha(x_{l+1})) \\
+ \sum_{i<j} (-1)^{i+j} \tilde{\beta}(\eta)(x_i, x_j, \alpha(x_1), \cdots, \hat{x}_{i,j}, \cdots, \alpha(x_{l+1}))
\]

which implies that \( \tilde{\beta} \circ d^s = d^{s+1} \circ \tilde{\beta} \).

Theorem 2.6. For \( s = 0, 1, 2, \ldots \), we have: \( H^k(d^s) \cong H^k(d^{s+1}) \).

Proof. By \( d^{s+1} \circ \tilde{\beta} = \tilde{\beta} \circ d^s \), for \( \eta \in Z^k(d^s) \), we have \( \tilde{\beta}(\eta) \in Z^k(d^{s+1}) \). On the other hand, for \( \eta_1 \in B^k(d^s) \), there is \( \omega \in C^{k-1}(\mathfrak{g}; V) \), such that: \( \eta_1 = d^s \omega \). So,

\[ \tilde{\beta}(\eta_1) = \tilde{\beta} \circ d^s \omega = d^{s+1} \circ \tilde{\beta}(\omega). \]

Obviously, \( \tilde{\beta}(\omega) \in C^{k-1}(\mathfrak{g}; V) \), then, \( \tilde{\beta}(\eta_1) \in B^k(d^{s+1}) \). Actually, we have proof:

\[ \tilde{\beta}(Z^k(d^s)) \subset Z^k(d^{s+1}), \quad \tilde{\beta}(B^k(d^s)) \subset B^k(d^{s+1}). \]

Next, for \( \beta^{-1} \), we define map \( \beta^{-1} : C^k(\mathfrak{g}; V) \to C^k(\mathfrak{g}; V) \) by

\[ \beta^{-1}(\eta)(x_1, \cdots, x_k) = \beta^{-1} \circ \eta(x_1, \cdots, x_k), \quad \forall \eta \in C^k(\mathfrak{g}; V). \]

For \( \eta \in Z^k(d^{s+1}) \), we have \( d^{s+1} \eta = 0 \). By

\[ \tilde{\beta} \circ d^s \circ \beta^{-1}(\eta) = d^{s+1} \circ \tilde{\beta} \circ \beta^{-1}(\eta) = d^{s+1} \eta = 0. \]

We have:

\[ d^s \circ \beta^{-1}(\eta) = 0, \]

then, we have:

\[ \beta^{-1}(\eta) \in Z^k(d^s). \]

On the other hand, for \( \eta_1 \in B^k(d^{s+1}) \), there is \( \omega \in C^{k-1}(\mathfrak{g}; V) \), such that \( \eta_1 = d^{s+1} \omega \). Then, we have:

\[ \tilde{\beta} \circ d^s \circ \beta^{-1}(\omega) = d^{s+1} \circ \tilde{\beta} \circ \beta^{-1}(\omega) = d^{s+1} \omega = \eta_1. \]

So,

\[ d^s \circ \beta^{-1}(\omega) = \beta^{-1} \circ \eta_1, \]

then

\[ \beta^{-1}(\eta_1) \in B^k(d^s). \]

Actually, we have proof:

\[ \beta^{-1}(Z^k(d^{s+1})) \subset Z^k(d^s); \quad \beta^{-1}(B^k(d^{s+1})) \subset B^k(d^s). \]

Now, we complete the proof. \( \blacksquare \)
3 Representations of Hom-Lie algebras-revisited

We first consider the coboundary operator $d$, which is associated to the trivial representation. The following is right.

**Proposition 3.1.** For $\xi_1 \in \wedge^k V^*$, $\xi_2 \in \wedge^l V^*$, we have

$$d(\xi_1 \wedge \xi_2) = d\xi_1 \wedge \bar{\alpha}(\xi_2) + (-1)^k \bar{\alpha}(\xi_1) \wedge d\xi_2.$$ 

**Proof.** This proof is similar to Proposition 3.2 in [20].

**Proposition 3.2.** For $\xi \in \wedge^k V^*, \eta \in C^l(V)$, we have

$$d^*(\xi \circ \eta) = d\xi \circ \bar{\alpha}(\eta) + (-1)^k \bar{\alpha}(\eta) \circ d^{*+k}\eta.$$ 

**Proof.** First let $k = 1$, then $\xi \circ \eta \in C^{l+1}(V)$. We have

$$d^n(\xi \circ \eta)(x_1, \cdots, x_{l+2}) = \sum_{i=1}^{k+2} (-1)^{i+1} \beta^{i+2+s} \rho(x_i) \beta^{-l-3-s} \xi \circ \eta(x_1, \cdots, \hat{x}_i, \cdots, x_{l+2})$$

$$+ \sum_{i<j} (-1)^{i+j} \xi \circ \eta([x_i, x_j], \alpha(x_1), \cdots, \hat{x}_{i,j}, \cdots, \alpha(x_{l+2}))$$

$$= \sum_{i<j} (-1)^{i+j} \xi([x_i, x_j])\eta(\alpha(x_1), \cdots, \hat{x}_{i,j}, \cdots, \alpha(x_{l+2}))$$

$$+ \sum_{i<j} (-1)^{i+j} \xi([x_i, x_j])\eta(\alpha(x_1), \cdots, \hat{x}_{i,j}, \cdots, \alpha(x_{l+2}))$$

$$+ \sum_{i<j} (-1)^{i+j} \xi([x_i, x_j])\eta(\alpha(x_1), \cdots, \hat{x}_{i,j}, \cdots, \alpha(x_{l+2}))$$

$$+ \sum_{i<j} (-1)^{i+j} \xi([x_i, x_j])\eta(\alpha(x_1), \cdots, \hat{x}_{i,j}, \cdots, \alpha(x_{l+2}))$$

$$+ \sum_{i<j} (-1)^{i+j} \xi([x_i, x_j])\eta(\alpha(x_1), \cdots, \hat{x}_{i,j}, \cdots, \alpha(x_{l+2}))$$

$$= d\xi \circ \bar{\alpha}(\eta)(x_1, \cdots, x_{l+2}) + (-1)^1 \bar{\alpha}(\xi) \circ d^{*+1}\varphi(x_1, \cdots, x_{l+2}).$$

Thus, when $k = 1$, we have

$$d^n(\xi \circ \eta) = d\xi \circ \bar{\alpha}(\eta) + (-1)^1 \bar{\alpha}(\xi) \circ d^{*+1}\eta.$$ 

By induction on $k$, assume that when $k = n$, we have

$$d^n(\xi \circ \eta) = d\xi \circ \bar{\alpha}(\eta) + (-1)^n \bar{\alpha}(\xi) \circ d^{*+n}\eta.$$ 


For $\omega \in g^*$, $\xi \wedge \omega \in \wedge^{n+1}g^*$, we have

$$d^s((\xi \wedge \omega) \circ \eta) = d^s(\xi \circ (\omega \circ \eta)) = d\xi \circ \tilde{\alpha}(\omega \circ \eta) + (-1)^n\tilde{\alpha}(\xi) \circ d^{s+n}(\omega \circ \eta) = (d\xi \wedge \omega) \circ \tilde{\alpha}(\eta) + (-1)^n\tilde{\alpha}(\xi) \circ (d\omega \circ \tilde{\alpha}(\eta)) + (-1)^n\tilde{\alpha}(\omega) \circ d^{s+n+1}\eta = (d\xi \wedge \tilde{\alpha}(\omega)) + (-1)^n\tilde{\alpha}(\xi) \wedge d\omega \circ \tilde{\alpha}(\eta) + (-1)^n\tilde{\alpha}(\omega) \circ d^{s+n+1}\eta = d(\eta \wedge \omega) \circ \tilde{\alpha}(\eta) + (-1)^n\tilde{\alpha}(\eta \wedge \omega) \circ d^{s+n+1}\eta.$$  

The proof is completed. ■

**Proposition 3.3.** With the above notations, we have

$$\tilde{\alpha} \circ d^s = d^{s+1} \circ \tilde{\alpha}.$$  

**Proof.** For any $\eta \in C^l(g; V)$, by $\rho(\alpha(x_i)) = \beta \circ \rho(x_i) \circ \beta$, we have

$$\tilde{\alpha} \circ d^s\eta(x_1, \cdots, x_{l+1}) = d^s\eta(\alpha(x_1), \cdots, \alpha(x_{l+1})) = \sum_{i=1}^{l+1}(-1)^{i+1}\beta^{l+1+s}(\alpha(x_i))\beta^{-l}(-s-\eta(\alpha^2(x_i), \cdots, \tilde{x}_i, \cdots, \alpha^2(x_{l+1}))) + \sum_{i<j}(-1)^{i+j}\eta(\alpha(x_i), \alpha(x_j)), \alpha^2(x_i), \cdots, \tilde{x}_i, \cdots, \alpha^2(x_{l+1})) = \sum_{i=1}^{l+1}(-1)^{i+1}\beta^{l+2+s}(\alpha(x_i))\beta^{-l-3-s}\tilde{\alpha}(\eta)(\alpha(x_1), \cdots, \tilde{x}_i, \cdots, \alpha(x_{l+1})) + \sum_{i<j}(-1)^{i+j}\tilde{\alpha}(\eta)(\alpha(x_1), \cdots, \tilde{x}_i, \cdots, \alpha(x_{l+1})) = d^{s+1}(\tilde{\alpha}(\eta))(x_1, \cdots, x_{l+1}),$$

which implies that $\tilde{\alpha} \circ d^s = d^{s+1} \circ \tilde{\alpha}$. ■

The converse of the above conclusions are also true. Thus, we have the following theorem, which generalize the result * If $t$ is a Lie algebra, $\rho: t \rightarrow gl(V)$ is a representation if and only if there is a degree-1 operator $D$ on $\Lambda^* \otimes V$ satisfying $D^2 = 0$, and

$$D(\xi \wedge \eta \otimes u) = d\xi \wedge \eta \otimes u + (-1)^k\xi \wedge D(\eta \otimes u), \quad \forall \xi \in \Lambda^k\ast, \eta \in \Lambda^l\ast, \ u \in V,$$

where $d_k: \Lambda^k\ast \rightarrow \Lambda^{k+1}\ast$ is the coboundary operator associated to the trivial representation.*

**Theorem 3.4.** Let $V$ be a vector space, $\beta \in GL(V)$. Then $(g, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra, and $\rho: g \rightarrow gl(V)$ is a representation of $(g, [\cdot, \cdot], \alpha)$ on the vector space $V$ with respect to $\beta$ if and only if there exists: $d^s: C^l(g; V) \rightarrow C^{l+1}(g; V)$, $s = 0, 1, 2, \ldots$ and such that:

i) $d^s \circ d^s = 0$;

ii) for any $\xi \in \Lambda^k g^*, \eta \in C^l(g; V)$, we have

$$d^s(\xi \circ \eta) = d\xi \circ \tilde{\alpha}(\eta) + (-1)^k\tilde{\alpha}(\xi) \circ d^{s+k}\eta;$$

where $d: \Lambda^k g^* \rightarrow \Lambda^{k+1} g^*$ is the coboundary operator associated to the trivial representation.
When \( \xi \) according to (13), (12) and iii), we have
\[
\forall v \in V, x \in g.
\] (11)

By straightforward computations, we have
\[
(\tilde{\alpha} \circ d^v(x)) = \beta^{1+s} \rho(\alpha(x)) \beta^{-2-s} v,
\]
\[
d^{s+1} \circ \tilde{\alpha}(v)(x) = \beta^{1+s} \rho(x) \beta^{-3-s} v.
\]

According to iii), we have:
\[
\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x).
\] (12)

When \( \xi \in g^* \), according to (13), we have
\[
\langle \xi, [x, y] \rangle = -d\xi(x, y).
\] (14)

According to (13), (12) and iii), we have
\[
\langle [\eta, [x, y]] \rangle = \beta^{2+s} \rho(\alpha(x)) \beta^{-3-s} \eta(\alpha(y)) - \beta^{2+s} \rho(\alpha(y)) \beta^{-3-s} \eta(\alpha(x)) - d^v(\eta(x, y)).
\]

So, we have
\[
\alpha([x, y]) = [\alpha(x), \alpha(y)].
\] (15)

When \( \eta \in C^0(g; V) = V \), by i), (13), (11) and (12), we have
\[
0 = d\xi \circ d^v(x, y)
\]
\[
= \beta^{2+s} \rho(x) \beta^{-3-s} \eta(\alpha(y)) - \beta^{2+s} \rho(\alpha(y)) \beta^{-3-s} \eta(\alpha(x)) - d^v([x, y])
\]
\[
= \beta^{1+s} \rho(x) \beta^{-3-s} v - 1 + \beta^{1+s} \rho(\alpha(y)) \rho(x) \beta^{-3-s} v - \beta^{1+s} \rho([x, y]) \beta^{-3-s} v.
\]

We get
\[
\rho(\alpha(x)) \rho(y) - \rho(\alpha(y)) \rho(x) = \rho([x, y]) \beta.
\] (16)

When \( \xi \in g^* \), \( \eta \in C^0(g; V) \), according to ii), (13) and (13), we have
\[
d^v(\xi \circ \eta)(x, y, z)
\]
\[
= d\xi \circ d^{s+1} \eta(x, y, z) + \xi \circ d^{s+1} \eta(x, y, z)
\]
\[
= \beta^{3+s} \rho(\alpha(y)) \beta^{-4-s} \xi \circ \eta(\alpha(x), \alpha(z)) - \beta^{3+s} \rho(\alpha(x)) \beta^{-4-s} \xi \circ \eta(\alpha(y), \alpha(z)) + \beta^{3+s} \rho(\alpha(z)) \beta^{-4-s} \xi \circ \eta(\alpha(y), \alpha(z))
\]
\[
+ \beta^{3+s} \rho(\alpha(z)) \beta^{-4-s} \xi \circ \eta(\alpha(x), \alpha(z)) - \xi \circ \eta([x, y], \alpha(z))
\]
\[
+ \xi \circ \eta([y, z], \alpha(y)) - \xi \circ \eta([y, z], \alpha(x)).
\]
So, for any $\omega \in C^2(g; V)$, we have

$$d^\sigma \omega(x, y, z) = \beta^3 + \rho(x) \beta^{-4} \omega(\alpha(y), \alpha(z)) - \beta^3 + \rho(y) \beta^{-4} \omega(\alpha(x), \alpha(z))$$

$$+ \beta^3 + \rho(z) \beta^{-4} \omega(\alpha(x), \alpha(y)) - \omega([x, y], \alpha(z))$$

$$+ \omega([x, z], \alpha(y)) - \omega([y, z], \alpha(x)).$$

For any $\eta \in C^1(g; V)$, according to i), we have

$$0 = d^\sigma \circ d^\tau \eta(x, y, z)$$

$$= \eta([[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)]).$$

Then, we have

$$[[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)] = 0.$$

(17)

So, according to (15) and (17), we have: $(g, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra;

according to (12) and (16), we have: $\rho$ is a representation of $(g, [\cdot, \cdot], \alpha)$ on the vector space $V$ with respect to $\beta$. ■

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