A Quantum Approach to Stock Price Fluctuations

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Abstract
A simple quantum model explains the Lévy-unstable distributions for individual stock returns observed by ref.[1]. The probability density function of the returns is written as the squared modulus of an amplitude. For short time intervals this amplitude is proportional to a Cauchy-distribution and satisfies the Schrödinger equation with a non-hermitian Hamiltonian. The observed power law tails of the return fluctuations imply that the "decay rate", $\gamma(q)$ asymptotically is proportional to $|q|$, for large $|q|$. The wave number $q$, the Fourier-conjugate variable to the return $x$, is interpreted as a quantitative measure of "market sentiment". On a time scale of less than a few weeks, the distribution of returns in this quantum model is shape stable and scales. The model quantitatively reproduces the observed cumulative distribution for the short-term normalized returns over 7 orders of magnitude without adjustable parameters. The return fluctuations over large time periods ultimately become Gaussian if $\gamma(q \sim 0) \propto q^2$. The ansatz $\gamma(q) = bT \sqrt{m^2 + q^2}$ is found to describe the positive part of the observed historic probability of normalized returns for time periods between $T = 5$ min and $T \sim 4$ years over more than 4 orders of magnitude in terms of one adjustable parameter $s_T = mb_T \propto T$. The Sharpe ratio of a stock in this model has a finite limit as the investment horizon $T \to 0$. Implications for short-term investments are discussed.

1 Introduction
The quest for a quantitative statistical description of stock prices began more than a century ago with Bachelier’s thesis in which he described price movements by a random walk. Although Bachelier’s random walk has since been modified, the effort to model stock price movements by a stochastic process continues. Computerized historic financial records have recently made a high-precision statistical analysis of short-term returns possible. For time scales that are less than a few weeks it is increasingly difficult to reconcile this wealth of new information with the conjecture that price fluctuations are described by a stochastic process with independent and identically distributed increments (iid).
To estimate the probability density function (pdf), $p_T^{(i)}(x_T^{(i)})$, for the return $x_T^{(i)}$ (the change in the logarithm of the price $S^{(i)}$ of the stock $(i)$ over a time period $T$),

$$x_T^{(i)} := \ln[S_{t+T}^{(i)}/S_t^{(i)}] = \ln[S_t^{(i)}] - \ln[S_t^{(i)}],$$  \hspace{1cm} (1)

the frequency of such changes was analyzed in ref.[1], using historical records that typically cover several years. Reducing the time horizon $T$ to minutes offered the advantage of suppressing the effects of a changing macro-economic environment while at the same time improving the statistics.

Fig. 1. Fits to the observed average cumulative distribution for short-term normalized returns. The historic probabilities for time periods of $T = 5$ minutes (dots) and $T = 1$ day $\sim 390$ minutes (squares) are from the analysis of ref.[1]. Only the positive tails of the slightly skew distributions are shown. The lines correspond to cumulative distributions of the form given in Eq. (13) for three different ratios, $r = (a/b) = 0, 1, 3$, of the parameters $a$ and $b$.

The analysis of ref.[1] revealed that the distribution of returns for different stocks and for different time intervals $T < 2$ weeks are all shape-similar and exhibit a somewhat unexpected power law fall off. More precisely: if $\mu_T^{(i)}$ denotes the mean and $v_T^{(i)}$ the standard deviation of the returns $x_T^{(i)}$ over a period $T$ for the stock $(i)$, the cumulative distribution of the normalized returns $g$,

$$N_T^{(i)}(g) = E[x_T^{(i)} - \mu_T^{(i)} > v_T^{(i)} g],$$  \hspace{1cm} (2)
was found to not depend on the stock \((i)\) nor on the time horizon \(T\) for 5 min \(< T < 2\) weeks. The mean, \(\mu_T^{(i)}\), and standard deviation, \(\nu_T^{(i)}\), thus were found to be the only statistically significant characteristics of the historical distribution of returns of any individual stock on time scales up to a few weeks. Scaling in the following denotes the fact that the distributions of the normalized returns are very similar. Although this would also be true for a random walk or any other Lévy stable process, it is somewhat unexpected that non-Gaussian return distributions with a finite variance are shape-similar over vastly different time horizons \(T\), for very different companies and in very different economic environments.

The empirically observed scaling of the distributions permits a significant improvement of the statistics by averaging over the distributions of the normalized returns of many individual stocks\(^1\),

\[
N_T(g) := \frac{1}{\# \text{ stocks}} \sum_{i=1}^{\# \text{ stocks}} N_T^{(i)}(g). \tag{3}
\]

One thus can reliably estimate the probability for return fluctuations that are about 100 times larger than average.

The shortest time horizon investigated by ref.[1] is \(T = 5\) min. The analysis in this case follows the stocks of the 1000 US companies with the largest market capitalization over a 2-year period from January 1994 to December 1995. The average cumulative probabilities for the normalized 5-minute returns found by ref.[1] is reproduced in the log-log plot of Fig. 1. Also shown is the average cumulative distribution of the normalized daily returns. The latter were extracted\(^1\) from the records of stock prices for about 16,000 individual companies (binned by market capitalization) over the entire 35-year interval 1962-96.

The interested reader is referred to the original analysis in ref.[1] for further details. But I would like to emphasize that the observed scaling of the distributions of individual companies (and of groups of companies with different market capitalization) is crucial to the interpretation of the analysis. The average normalized cumulative distribution of Eq. (3) otherwise would differ qualitatively from the normalized distribution of any individual stock. Without scaling, the error in the mean normalized distribution of Eq. (3) would be large and the data base of about \(4 \times 10^4\) events per company would not allow any conclusions about the frequency of events that occur only a few times out of \(10^7\).

Fig. 1 shows that the averaged distributions for different time intervals \(T < 1\) day also scale extremely well. It is remarkable that normalized data sets
for time intervals that differ by a factor of about $78 \sim (1 \text{ trading day})/5 \text{ min}$
and involve different companies in different economic periods should show no
statistically significant difference.

Apart from this shape-similarity over vastly different time scales and many
different companies, the most striking feature of the empirical probability distri-
bution is the pronounced power law over $\sim 5$ orders of magnitude,

$$N_T(g) \propto g^{-\alpha}, \text{ for } 2 < g < 100 \text{ and } T < 1 \text{ week}.$$ (4)

The exponent $\alpha$ empirically is close to 3. For $T = 5 \text{ min}$, the best estimate for
$\alpha$ is $3.10 \pm 0.03$ for the positive tail and $2.84 \pm 0.12$ for the negative tail of the
observed distribution. For short time intervals $T < 2 \text{ weeks}$ the distributions
are only slightly skewed, but this feature becomes more pronounced in the
data with increasing $T$. The observed asymmetry of the distributions could be
caused by a number of factors, such as the discreteness of prices or bankruptcy
regulations, that all tend to mainly distort its negative tail. We do not model
such effects and for the purpose of this investigation replace the empirical
distribution by a symmetric one with the same positive part.

The empirical estimates of the power law exponent are well outside the
region for Lévy-stable distributions, which would require that $0 < \alpha \leq 2$.

The stability of the power law tails with an exponent $\alpha \sim 3$ refute Bachelier’s
conjecture and its generalizations. The observed distributions cannot result
from an iid process. The pdf for $T = 390 \text{ min} \sim 1 \text{ day}$ otherwise would have
to differ considerably in shape from the pdf for $T = 5 \text{ min}$; the $390/5 = 78-$
fold convolution of the latter is rather close to a Gaussian pdf and not at all
shape-similar to itself.

The observed auto-correlation time is of the order of a few minutes only.
Memory effects that could explain the persistence of power law tails over days
therefore probably are of higher order. Although GARCH-processes in princi-
ple model such correlations, it is quite difficult to obtain a shape-stable tempo-
ral evolution with the observed power law in this manner. A GARCH-process
furthermore depends on parameters that determine the shape of the marginal
distribution. In order to reproduce the observed scaling, one would have to
contend that these shape degrees of freedom somehow are strongly correlated
and very similar for different companies, macro-economic conditions, etc.

Power law tails with exponents $\alpha \sim 3$ have also been observed for the cu-
mulative distributions of market indices and of commodity prices. Although
the existence of fat tails was recognized early on, an economic explanation of
the power law with an exponent $\alpha \sim 3$ has only recently been proposed. In
this micro-economic analysis, the asymptotic power law reflects the distribu-
tion and trading behavior of the largest investors in a stock, such as mutual-
and/or pension-funds. However, such an asymptotic analysis cannot, by itself, explain the actual magnitude of the tails nor the observed scaling of the distributions.

If one ignores the problem of temporal stability, the distribution of returns on a stock (including the power law tails) for any fixed (short) time interval \( T \) can be modelled by a subordinated stochastic process. Clark\(^9\) originally proposed a subordinated stochastic process in his attempt to explain daily returns on cotton futures. He contended that such a process would give a better description of cotton futures than Mandelbrot and Taylor’s suggestion\(^10\) of a Lévy stable process. Only a limited amount of data (\( \sim 2000 \) prices) on the daily closing of cotton futures was analyzed and the issue of shape-stability of the distribution of returns over longer (or shorter) time intervals than 1 day was not addressed. The daily returns on cotton futures indeed are better reproduced by a subordinated stochastic process than by a Lévy stable one, mainly because the latter can give a leptokurtic distribution with finite variance.

More recently, the Variance Gamma process\(^11\), a subordinated stochastic process with finite kurtosis, has been proposed to value options. A subordinated stochastic process with finite variance but infinite kurtosis that reproduces the observed distribution of returns on a stock for any fixed time interval \( T < 2 \) weeks is constructed in the appendix. However, this \( iid \) process with finite variance converges rapidly to its Gaussian fixed point and does not explain the shape-stability of the observed distribution on different time scales \( T \).

In short, the observed price fluctuations pose at least three theoretical challenges:

1. To explain the scaling of the distribution of short-term returns of many individual stocks over \( \sim 7 \) orders of magnitude.

2. To explain the power law behavior with \( \alpha \sim 3 \) of the tails of the cumulative distribution for the returns.

3. To explain the apparent temporal shape-stability of the distribution for the returns over time horizons from a few minutes to a few weeks and the slow convergence to a Gaussian distribution on much longer time scales.

The functional form of the observed normalized distribution of short-term returns is accurately reproduced by the ansatz of the next section. In section 3 the temporal evolution of the amplitude (the square root of the pdf) is found to be naturally shape-stable in a quantum model and the linear evolution operator (effective Hamiltonian) that reproduces the observed temporal evolution for
T < 1 day is obtained. Section 4 discusses and interprets a slightly modified effective Hamiltonian. It semi-quantitatively describes the temporal evolution of the distribution of returns over time scales up to four years. Section 5 explores some of the implications of this quantum mechanical description and the results are summarized in section 6.

2 Power Law Tails with $\alpha = 3$ in Quantum Finance

The recently proposed quantum description of financial markets\textsuperscript{12} offers a surprisingly simple and transparent explanation for the observed distributions. The exponent $\alpha \sim 3$ for the power law tails, in particular, is quite natural in this framework and is temporally stable.

$\alpha = 3$ in the cumulative distribution of Eq. (2) implies the asymptotic behavior,

$$p_T(x^2 \sim \infty) = -\frac{\partial}{\partial x} N_T(x/v) \bigg|_{x^2 \sim \infty} \propto x^{-4},$$

(5)

of the pdf of a representative stock’s return. The essence of a quantum description is that the pdf is interpreted as the squared magnitude of a (possibly complex) amplitude $\phi_T(x)$,

$$p_T(x) = |\phi_T(x)|^2.$$

(6)

If the temporal evolution (and thus the pdf) is invariant under the transformation# $x - \mu \rightarrow \mu - x$, the amplitude either is symmetric or antisymmetric in $x - \mu$. Since the probability of the mean return, $p_T(\mu)$, does not vanish, $\phi_T(x)$ in this case is symmetric and a function of $(x - \mu)^2$ only. The asymptotic behavior of the pdf in Eq. (5) implies

$$\phi_T(x^2 \sim \infty) \propto x^{-2},$$

(7)

for the asymptotic behavior of the amplitude. Since $\phi_T(x)$ falls off like a power law for large values of $x$ with an exponent that is (close to) a negative integer, we model the amplitude by a rational function, i.e. a function that is analytic in the whole complex plane apart from a finite number of poles. The position and strength of the poles in this case constitute the set of parameters that describe the pdf of an individual stock. The observed approximate scaling implies that the distribution of an individual stock’s returns to first approximation is specified by its variance and mean. Our ansatz therefore should not involve too many poles. Every additional parameter is related to additional\textsuperscript{*} statistical

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#This is an (approximate) “parity”-symmetry for reflection about the mean $\mu$.

*With more than two poles, the pdf can, for instance, be skewed.
properties that in general will break the scaling and distinguish between the 
shapes of the distribution of returns of individual stocks. An amplitude with 
a single pole corresponds to a pdf with infinite variance. The observed finite 
variance thus requires an amplitude with at least two poles and that the sum 
of residues must vanish. The amplitude then asymptotically falls off at least as 
fast as in Eq. (7). Note that only asymptotic power laws with an exponent \( \alpha \) 
that is an odd integer can be modelled in this fashion and that an odd exponent 
\( \alpha > 3 \) would require more than two poles \(^1\).

The simplest rational amplitude corresponding to a pdf of finite variance 
therefore has two poles of opposite strength. It automatically has the 
asymptotic behavior of Eq. (7). The requirement that the amplitude is a 
symmetric function of \( x - \mu \) further constrains the position of the two poles to 
\( z = \mu \pm (a + ib) \). One thus is led to consider the ansatz,

\[
\phi_T(x) = \frac{N}{(x - \mu)^2 - (a + ib)^2},
\]

for the amplitude. The real parameters \( \mu \), \( a \) and \( b > 0 \) generally will depend 
on the stock, economic era and the time interval \( T \). Normalizing the pdf 
determines the constant \( N \) up to an irrelevant phase,

\[
|N|^2 = (a^2 + b^2) \frac{2b}{\pi}.
\]

Note that the pdf is not normalizable at \( b = 0 \). The normalized pdf corre-
spending to the ansatz of Eq. (8) is,

\[
p_T(x) = \frac{2b(a^2 + b^2)}{\pi((x - \mu)^2 + b^2 - a^2)^2 + 4a^2b^2)}.
\]

The three real parameters, \( \mu \), \( a \) and \( b > 0 \) on which it depends are related to 
the mean, variance and the curvature at the mode of the density. The variance 
of \( p_T(x) \) is just the square of the distance of the complex poles from the mean,

\[
v^2 = |z - \mu|^2 = a^2 + b^2.
\]

The second (independent) parameter is the ratio \( r = a/b \), or, equivalently, the 
phase of \( z - \mu \). The curvature of the pdf at \( x = \mu \) is

\[
\rho = \frac{\partial^2}{\partial x^2} p_T(x) \bigg|_{x=\mu} = \frac{4b}{\pi v^4} \frac{a^2 - b^2}{a^2 + b^2}.
\]

\(^1\)For \( \alpha \)'s that are not odd integers, the amplitude would have branch cuts
\( \rho \) is negative for \( a^2 < b^2 \) and the pdf in this case peaks at \( x = \mu \). For \( a^2 > b^2 \) the pdf is double-humped with two distinct maxima at \( x_{\text{max}} = \mu \pm \sqrt{a^2 - b^2} \).

\( \rho v^3 \) is a function of the ratio \( r = a/b \) only.

The distribution for the normalized returns that corresponds to the pdf of Eq. (10) is,

\[
N_T(g) = \frac{1}{2\pi} \left[ \pi + \arctan \left( \frac{a - vg}{b} \right) - \arctan \left( \frac{a + vg}{b} \right) + \frac{b}{2a} \ln \left( \frac{1 + \left( \frac{a - vg}{b} \right)^2}{1 + \left( \frac{a + vg}{b} \right)^2} \right) \right].
\]

(13)

With Eq. (11), \( N_T(g) \) is seen to depend on the shape parameter \( r^2 = (a/b)^2 \) only. For large and small values of \( g \), \( N_T(g) \) has the expansions,

\[
N_T(g \sim \infty) \sim \frac{2b}{3\pi v} g^{-3} \left[ 1 + \frac{3\rho v^4}{10b} g^{-2} + O(g^{-4}) \right]
\]

\[
N_T(g \sim 0) \sim \frac{1}{2} - \frac{2b}{\pi v} g - \frac{\rho v^3}{3} g^3 + O(g^5).
\]

(14)

[The relative correction to an asymptotic power law \( \propto g^{-3} \) is never much more than \( 6/(5g^2) \) (or less than 1\% for \( g > 11 \)). The leading correction vanishes altogether for \( \rho = 0 \), or equivalently, \( r = 1 \).

Fig. 1 shows distributions \( N_T(g) \) of Eq. (13) for ratios \( r = 0, 1, 3 \), together with the empirical data. Apart from the overall change in the normalization of the tails, the parameter \( r \) qualitatively changes the shape of the distribution of normalized returns near \( g = 0 \). Note that the theoretical distributions, by construction, all have the same power law tails with an exponent \( \alpha = 3 \), but that the normalization of the tails of the distribution for \( r = 3 \) is off by a factor of \( \sim 20 \). Any asymptotic analysis that predicts only the exponent\(8 \), by itself, therefore cannot provide an explanation of the observed distribution of returns.

There is no convincing reason why parameters like \( r \) should be the same for different companies and/or economic eras or different time intervals \( T \). Large variations in \( r \) would break the observed shape similarity of the distributions. Within the accuracy of the analysis of ref.[1], the ratio \( r = a_T/b_T \) does not appear to depend strongly on \( T \) and should be similar for most stocks. The empirical data prefers small \( r < 1 \) and is consistent with \( r = 0 \), i.e. with an amplitude whose two poles have similar real parts. In this case the amplitude describing a mean return of \( \mu_T \) with standard deviation \( v_T = b_T \) has two complex conjugate poles of opposite strength located at \( z_T = \mu_T + iv_T \) and \( z_T^* = \mu_T - iv_T \). This is the minimal number of independent parameters the distribution of returns of an individual stock can depend upon. The normalized
probability distribution corresponding to this special case is the solid line in Fig 1, a student t-distribution with 3 degrees of freedom,

\[ p_T(x) = \frac{v_T}{2\pi} \left| \frac{1}{x - z_T} - \frac{1}{x - z_T^*} \right|^2 = \frac{2v_T^3}{\pi((x - \mu_T)^2 + v_T^2)^2}. \]  

It fits the data exceedingly well. For \( T < 2 \) weeks, the returns of a stock in this sense are statistically described by the path the pole \( z_T \) takes in the complex half-plane with \( \text{Im} z_T > 0 \). The path begins at the origin \( z_{T=0} = 0 \) and long term investments are encouraged if the standard deviation \( v_T \) is a concave function of the mean return \( \mu_T \).

3 Quantum Stability

The good description of the empirical data by an ansatz for the pdf that is the square of the modulus of a rational amplitude could be considered fortuitous and in itself does not require a quantum model. But a quantum model of the dynamics does explain the apparent stability of the shape of this distribution over vastly different time intervals without the need to explicitly model memory effects. It also casts some doubt on the notion that stock market fluctuations are predictable if they do not follow an iid process.

Quantum dynamics primarily describes the temporal evolution of the amplitude – the evolution of the corresponding pdf follows from the relation in Eq. (6) and is non-linear. Stochastic processes that give Lévy-unstable distributions of the type Eq. (10) (with the correct power law tails) over an extended period of time tend to be arbitrary in the sense that other distributions may almost equally well have been obtained. They either depend on quite a few carefully adjusted parameters or are directed by a process whose fractal dimension would have to be a time independent characteristic of financial markets. Unlike for physical phenomena, such "hidden" characteristics of financial markets are expected to slowly change over time and the distributions thus would depend on the economic era. The ansatz of Eq. (8) for the amplitude on the other hand turns out to be absolutely shape stable under a relatively simple and quite natural quantum dynamics.

To better see this, consider the Fourier-transform of the amplitude of Eq. (8) with the normalization Eq. (9). Contour integration about the simple poles of the ansatz at \( z = \mu_T \pm (a_T + ib_T) \) with \( b_T > 0 \) is elementary and gives the Fourier-transform \( \tilde{\phi}_T(q) \),

\[ \tilde{\phi}_T(q) := \int_{-\infty}^{\infty} \phi_T(x) e^{iqx} \, dx = \sqrt{2\pi b_T} e^{i(q\mu_T + |q|a_T)} e^{i|q|b_T}. \]
For very short times, when the mean and variance of the distribution are small, $\tilde{\phi}_T(q)$ essentially is constant over a wide range of wave numbers $q$. In this case, the corresponding amplitude Eq. (8) for the return is well localized about $x = \mu$, i.e. the expected log-price. The variance of the pdf grows with the time horizon if $a_T$ and/or $b_T$ grow in magnitude. As noted before, the cumulative distribution of the normalized returns remains strictly the same only if the parameters $a_T$ and $b_T$ grow proportionally and the ratio $r = (a_T/b_T)$ does not depend on the time interval $T$.

The temporal evolution of a state $|\varphi\rangle$ in the Hilbert space of a quantum model is generated by some effective Hamiltonian operator $\hat{H}_{\text{eff}}(t)$,

$$|\varphi\rangle_T = T e^{-\frac{i}{\hbar} \int_0^T dt \hat{H}_{\text{eff}}(t)} |\varphi\rangle_0$$

(17)

where the symbol $T$ denotes time ordering of the exponential factors (redundant if, as in the present case, the Hamiltonian operators for different times commute). To conserve probability in a complete Hilbert space, $\hat{H}_{\text{eff}}(t)$ necessarily would have to be hermitian and states would evolve by a unitary rotation. However, if $\hat{H}_{\text{eff}}(t)$ describes only the evolution in a subspace of the Hilbert space – such as that spanned by a single share – probability can be “lost” to the complementary part of the full Hilbert space. $\hat{H}_{\text{eff}}(t)$ in this case is not hermitian.

Comparing Eq. (17) with Eq. (16) suggests that the effective Hamiltonian for the time evolution of a share is diagonal in the Fourier-conjugate basis of $|q\rangle$-states, with matrix elements,

$$\langle q|\hat{H}_{\text{eff}}(t)|q'\rangle = 2\pi \delta(q-q')(q\mu_t - |q|\tilde{\alpha} - i|q|\tilde{\beta})$$

(18)

where the dot is shorthand for the derivative with respect to time, i.e. $\dot{\mu}_t = d\mu_t/dt$, etc.

In the language of particle physics, $\tilde{\phi}_T(q) \propto \langle q|\varphi\rangle_T$ is proportional to the wave-function of a (massless) particle whose frequency $\omega(q)$ and decay rate $\gamma(q)$ are both proportional to the magnitude of its wave-number $q$. If $r = a/b$ does not depend on time, one can interpret $b_T$ as proportional to the ”proper” time of a stock, i.e. as the monotonically increasing parameter that characterizes the evolution of the stock’s price distribution. $b_T$ need not be proportional to physical time or even to trading time. The proper time of a share could be proportional to some monotonically increasing quantity that is relevant to the actual trading dynamics of the stock, such as the overall number of trades or the number of traded shares. We will further examine this issue in section 5.

From the available data one in fact cannot conclude that the frequency $\omega(q)$ and decay rate $\gamma(q)$ of a share are proportional. As Fig. 1 shows, the
data implies only that the term proportional to $a_T$ is small compared to $b_T|q|$. Since $a_T$ and $b_T$ are the strengths of two commuting terms of the evolution operator, the contribution $\omega(q)a_T$ may simply be negligible. $a_T = 0$ is a trivial fixed point of the evolution that reproduces the empirical data rather well. [Note that this would not be the case if one could not separate the effects of the two parameters in $H_{\text{eff}}$.] Assuming $|a_T| \ll b_T$ in this sense is a "natural" approximation to the evolution that does not require any fine tuning.

The time dependent proportionality constant $\sqrt{2\pi b_T}$ in the amplitude $\tilde{\phi}_T(q)$ of Eq. (16) ensures that the pdf remains normalized at all times when $b_T > 0$. It does not appear in Eq. (17) and the norm of the state $|\phi_T\rangle$ decays with $T$ if $b_T$ increases with time. The time dependent normalization factor arises because the effective "one-particle" Hamiltonian $H_{\text{eff}}$ describes the temporal evolution of a particular share of a stock. We are therefore computing the probability of a return $x$, if a certain share traded at time $t$ is again traded at time $t + T > t$. The probability that any particular share is again traded (for any price) after a time interval $T$ is just the normalization $\langle \phi_T|\phi_T\rangle = (2\pi b_T)^{-1} \langle 0|\phi_T|0\rangle$ of the state of the state $|\phi_T\rangle$ in Eq. (17). The probability that a particular share is traded after a time interval $T$ thus decreases like $1/b_T$, but the probability for a particular share’s return under the condition that it is again traded after a time $T$ of course remains normalized for all $T$. [The calculation is analogous to that of the probability of a decaying particle’s change in position if it is observed after a time $T$.]

4 Ultimate Convergence to a Gaussian Fixed Point

The exponent $\alpha \sim 3$ and the apparent temporal shape stability of the distribution thus are linked by the fact that quantum theory primarily describes the temporal evolution of the amplitude, rather than of the transition probability. Although the evolution of an amplitude of Cauchy type such as the one in Eq. (8) is shape stable, there are other possible fixed points. Of some interest is the Gaussian one, because one does expect the pdf to eventually approach a Gaussian distribution. Note that the Fourier-transform of a Gaussian amplitude is a Gaussian and that a Gaussian amplitude implies a Gaussian pdf.

The pdf $p_T(x)$ of individual stocks empirically resembles a Gaussian distribution only after several years. The results of the analysis of ref.[1] for $T = 16$ days to $T = 1024$ days, or up to approximately 4 trading years, are shown in Fig. 2. The slow rate of convergence to a Gaussian fixed point sug-

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8We will see in section 5 that it is conceptually more appropriate to include a factor $2\pi b$ in the relation of Eq. (6) between the conditional probability density and the square of the amplitude, instead of normalizing the square of the amplitude.
gests that it may be due to macro-economic and other factors such as stock splits, buy-backs, etc., that are irrelevant for the short-term dynamics of a stock. Note that the positive and negative tails of the cumulative distribution do not approach the Gaussian fixed point at the same rate\(^1\). Although the approximation by a symmetric distribution becomes questionable on long time scales, it nevertheless should be possible to consistently describe the crossover from one (almost) fixed point to another within a single effective quantum model. For simplicity and because it is a fixed point and consistent with the short-term data, we consider only the case \(a_T = 0\), i.e. a stock whose evolution is completely dominated by its “decay rate” \(\gamma(q)\).

\[ T = 16 - 1024 \text{ days} \]

Fig. 2. The cumulative distribution for the normalized returns over time intervals from \(T = 16\) to \(T = 1024\) trading days. The empirical probabilities are from the analysis of ref.[1]. The full lines are distributions that correspond to the extended ansatz of Eq. (20) that best fit the data. They depend on the value of the parameter \(s = bm\) only.

For long time horizons, the variance of the pdf and thus of the amplitude are large. The Fourier-conjugate amplitude \(\tilde{\phi}_T(q)\) therefore is concentrated about \(q \sim 0\), and the approach to a Gaussian fixed point can be ensured if the function \(\gamma_0(q) = |q|\) in Eq. (16) is modified to one where the cusp at \(q = 0\) is replaced by a smooth quadratic dependence on \(q\). It was argued previously\(^1\), that the effective decay rate \(\gamma(q)\) should be a quadratic function of \(q\) in the long wavelength limit, \(q \sim 0\). The proposed modification for small values of \(q\)
will not affect the short-term power law behavior over a wide range of \( x \) if \( \gamma(q) \) for \( |q| \sim \infty \) approaches the function \( |q| \) sufficiently rapidly. Replacing \( |q| \) by the upper branch of a hyperbola with these asymptotes thus is a possibility,

\[
|q| \to \gamma_m(q) \propto \sqrt{q^2 + m^2 - m}.
\]

The parameter \( m \) controls the transition between the Gaussian and power law regimes. For \( m = 0 \) one recovers the previous case, whereas the density essentially is Gaussian for all but the shortest times when \( m \) is large. For \( m > 0 \) the ultimate fixed point thus is Gaussian, but the convergence to it is slow for small \( m \). The particular choice for the interpolating function in Eq. (19) is ad hoc, but a square-root dependence of the decay rate on the wave-number can be the result of diagonalizing a (anti-)hermitian \( 2 \times 2 \) matrix – such square-roots are quite generic for quantum systems near level crossings.

![Graph showing the conditional expectation of the number of trades](image)

**Fig. 3.** The conditional expectation \( \langle N \rangle_\Phi \) of the number of trades for a given number imbalance \( \Phi \). The data points are reproduced from Fig. 3a) of the analysis of ref.[15]. The solid line represents the function \( \sqrt{\Phi^2 + 0.49} - 0.7 \).

Empirical support for the particular form of \( \gamma_m(q) \) in Eq. (19) is provided by the analysis in ref.[15] of how the mean number of trades \( \langle N \rangle_{\Phi_T} \) in a time interval of \( T = 15 \) min depends on the imbalance between the buyer- and seller-initiated trades. Whether a trade is buyer- or seller-initiated was determined\(^{15}\) from the prevailing quote shortly before execution of the trade using the procedure of C. Lee and M. Ready\(^{16}\); the imbalance \( \Phi_T \) then is the
difference in the number of buyer- and seller-initiated trades in the time interval. The data points in Fig. 3 are the results of ref. [15] from the analysis of the trading records of the 116 most frequently traded US stocks during the 2-year period 1994-1995.

If one assumes that the average volume of shares per trade does not depend strongly on $\Phi_T$, the observations of ref. [15] can be interpreted and understood within the quantum model. Because the time interval of $T = 15$ min is relatively short, $\langle N \rangle_{\Phi_T}$ in this case is proportional to the "decay" rate $\gamma_m(q)$ of a share in state $|q\rangle$. The observations in Fig. 3 indeed appear to lie on a symmetric hyperbolic curve. Furthermore, there appears to be little trading activity at vanishing imbalance $\Phi_T \sim 0$. It therefore is tempting to assume that the wave number $q$ and the imbalance $\Phi_T$ are roughly proportional. This interpretation would associate the market dynamics captured by the imbalance $\Phi_T$ with the Fourier conjugate variable to the return $x_T$ (up to a proportionality constant). It has potentially interesting and observable consequences in a quantum model. The pdf of $\Phi_T$ in this case is related to the pdf of $x_T$: the corresponding amplitudes are Fourier conjugate to each other (up to a scale). The amplitude $\tilde{\phi}_T(q)$ of Eq. (16) implies that for time intervals $T < 2$ weeks the normalized distribution of $\Phi_T$ is $E[\Phi > 0 | \Phi_T > 0] \sim \frac{1}{\sqrt{2}} e^{-g/\sqrt{2}}$, where $v_{\Phi_T} \propto 1/b_T$ is the standard deviation of $\Phi_T$. The model thus predicts that the probability of a large imbalance in the trades falls off exponentially, rather than like a power law. [Note that $v_{\Phi_T} \propto 1/b_T$ is one version of Heisenberg’s uncertainty relation. It here implies that periods with large fluctuations in the imbalance should show relatively small fluctuations in the returns, and vice versa.]

Returning to the temporal evolution of the distribution of the returns, the modification of Eq. (19) leads to an amplitude of the form $^\kappa$,

$$\phi_T(x; m) = \frac{1}{\mathcal{N}} \int_0^\infty dq \frac{\cos(qx)}{\sqrt{\pi}} \exp[-b\sqrt{q^2 + m^2}] = \frac{mb}{\pi b_N} K_1(m\sqrt{x^2 + b^2}) K_1(0),$$

(20)

where $K_1(z)$ is the modified Bessel function of the third kind with the asymptotic behavior

$$K_1(z \sim 0) \sim 1/z, \quad K_1(z \sim \infty) \sim \frac{\sqrt{\pi}}{2z} e^{-z}.$$  

(21)

$^\kappa$We assume that $a_T \ll b_T$ and neglect any dependence on the parameter $a_T$. This approximation is consistent with the short-term results.
Normalization of the transition probability requires that,

\[ |\mathcal{N}|^2 = \int_0^\infty dq \frac{e^{-2b\sqrt{q^2+m^2}}}{\pi} = \frac{m}{\pi} K_1(2mb). \tag{22} \]

For a mean return \( \mu \), the pdf for the return on a stock in this model then becomes,

\[ p_T(x; \mu, b, m) = |\phi_T(x-\mu; m)|^2 = \frac{mb^2}{\pi K_1(2mb)} \frac{K_1^2(m\sqrt{(x-\mu)^2+b^2})}{(x-\mu)^2 + b^2}, \tag{23} \]

and depends on three, possibly \( T \) dependent, parameters \( \mu_T, b_T \) and \( m_T \). The variance of the returns is

\[ \nu_T^2 = \frac{b^2}{mK_1(2mb)} \int_0^\infty \frac{q^2 dq}{q^2 + m^2} e^{-2b\sqrt{q^2+m^2}}. \tag{24} \]

For long time horizons \( (mb \gg 1) \), the limiting density is Gaussian with a variance \( \nu_T^2(mb \gg 1) \sim b/(2m) \). For short time intervals \( (mb \ll 1) \) the variance to first approximation does not depend on \( m, \nu_T^2(mb \ll 1) = b^2(1 - \pi mb + \ldots) \).

Since the variance of normalized returns is unity by definition, the cumulative distributions for the normalized returns depends on the parameter combination \( s = mb \) only. As shown in Fig. 2, this extended ansatz qualitatively and to some extent even quantitatively reproduces the observed positive tails of the (average) cumulative distributions of the normalized returns. The distributions for time periods of \( T = 16, 64, 256 \) and \( T = 1024 \) trading days correspond to values of the parameter \( s = mb = 0.04, 0.16, 0.64 \) and \( s = 2.56 \). We obtain that

\[ s_T = m_T b_T \approx T/(400 \text{days}) \]. \tag{25} \]

This determination of the time scale unfortunately is not very accurate and depends among other things on our assumption that all stocks are characterized by the same \( m_T \) (see below). The approach to a Gaussian fixed point of the negative tails is even slower. The fact that we are approximating skewed distributions by symmetric ones may partly explain some of the discrepancies visible in Fig. 2.

Systematic deviations also arise if the individual stock distributions scale less than perfectly for large \( T \). If the parameter \( m_T \) depends on the stock, the scaling among companies is broken on large time scales and the ”lightest” stocks (the ones with the smallest \( m_T \)) in this case dominate the average of Eq. (3) at large normalized returns. The observed flattening out of the tails of the averaged cumulative distributions for \( T > 16 \) days in this case could be due to some ”lighter than average” stocks.
The overall quality of this one-parameter fit down to observed cumulative probabilities of $10^{-5}$ is encouraging. The fact that $s_T$ is approximately proportional to the trading time $T$ furthermore suggests that the effective Hamiltonian could be time independent. For sufficiently long time intervals one would hope this to be the case for any parameterization of the evolution – most monotonically increasing quantities that might be relevant for the evolution of a stock’s price eventually do become proportional to the physical- (and to the trading-) time.

5 Quantum Interference and Short Term Risk and Return

For long time horizons $T$ one expects that the mean return $\mu_T$ and the variance $v^2_T \sim b_T/(2m_T)$ of the returns are both proportional to $T$. We found that the time dependence of the distribution fits the empirical one if $s = m_T b_T$ is approximately proportional to $T$. Taken together, this would imply that on time scales of $T \sim \text{years}$,

$$\mu_T \propto b_T \propto T, \quad m_T = m \sim \text{const.} \quad (26)$$

If one assumes time homogeneity of the returns, the quantum model in fact completely specifies the $T$-dependence of the parameters $\mu_T$, $b_T$ and $m_T$ for all $T$. Assuming that the distribution of historic returns of a stock over a time interval $T$ does not depend on the initial time $t$ is not very reasonable for single companies, but perhaps is tenable for short time intervals $T$ once companies are grouped by their market capitalization – the procedure followed in ref.[1]. The conditional probability density $p_T(y|x)$ that the stock has a return $y - x$ in the time interval $T$ in this case is,

$$p_T(y|x) \propto |G(y, t + T; x, t)|^2, \quad (27)$$

where $G(y, t + T; x, t)$ is the transition amplitude that a particular share traded at time $t$ for a log-price $x$ is traded at time $t' = t + T > t$ for a log-price between $y$ and $y + dy$. [The proportionality constant in general depends on $T$ and $t$ and is obtained from $1 = \int p(y, t + T|x, t) dy$.]

The transition amplitude in the present model is\footnote{The states $|x\rangle$ and $|q\rangle$ are normalized eigenvectors of the hermitian operator $\hat{x}$ that corresponds to the log-price. The scalar product\footnote{with the previously introduced state $|q\rangle$ is $\langle q|x\rangle = e^{iqx}$.} with the previously introduced state $|q\rangle$ is $\langle q|x\rangle = e^{iqx}$.},

$$G(y, t + T; x, t) = \langle q|T e^{-i \int_t^{t+T} \hat{H}_{\text{eff}}(\xi) d\xi} |x\rangle$$

$$= \int_{-\infty}^{\infty} dq \frac{2\pi}{2\pi} \exp \left[ i(x - y)q + i\mu_T(t)q - b_T(t)\sqrt{q^2 + m_T^2(t)} \right]. \quad (28)$$
An effective Hamiltonian that is diagonal in the wave number implies that $G(y, t + T; x, t)$ and $p_T(y|x)$ depend on the return $y - x$ only. If the returns were homogeneous in time, $G(y, t + T; x, t)$ would not depend on the initial time $t$. This is an oversimplification of the dynamics and we will consider the possibility that the parameters in Eq. (28) vary slowly with the initial time, i.e. that the returns are homogeneous on time scales of the horizon $T$.

The definition of Eq. (28) and the completeness of the basis of $|y\rangle$-states gives Trotter’s formula for the transition amplitudes,

$$
G(z, T_1 + 2 T_2 + t; x, t) = \langle z | \mathcal{T} e^{-i \int_{t}^{t + T_1 + T_2} \hat{H}_{eff}(\xi) d\xi} e^{-i \int_{t}^{t + T_1} \hat{H}_{eff}(\xi) d\xi} | x \rangle
= \int_{-\infty}^{\infty} dy \langle z | \mathcal{T} e^{-i \int_{t}^{t + T_1 + T_2} \hat{H}_{eff}(\xi) d\xi} | y \rangle \langle y | \mathcal{T} e^{-i \int_{t}^{t + T_1} \hat{H}_{eff}(\xi) d\xi} | x \rangle
= \int_{-\infty}^{\infty} dy \ G(z, T_1 + 2 T_2 + t; y, T_1 + t) G(y, T_1 + t; x, t) .
$$

Eq. (29) is the mathematical expression of Huygens’ principle describing the propagation of waves. Using Eq. (28) and performing the integral over the intermediate log-price $y$, one finds that for Eq. (29) holds only if,

$$
\mu_{T_1 + T_2}(t) = \mu_{T_1}(t) + \mu_{T_2}(t + T_1)
\quad b_{T_1 + T_2}(t) = b_{T_2}(t) + b_{T_1}(t + T_1)
\quad m_{T_1 + T_2}(t) = m_{T_1}(t) = m_{T_2}(t + T_1) = m(t) .
$$

If the parameters depend little on the initial time (at least on time scales $T_1$ of interest), $\mu_{T_2}(t + T_1) \sim \mu_{T_2}(t)$ and $b_{T_1}(t + T_1) \sim b_{T_2}(t)$. The solution to Eq. (30) in this case is that $\mu_T(t)$ and $b_T(t)$ are both proportional to the time interval $T$,

$$
\mu_T(t) \sim T \dot{\mu}(t), \quad b_T(t) \sim T \dot{b}(t), \quad m_T(t) = m(t) .
$$

For notational clarity, the possible dependence of parameters on the initial time $t$ will again be suppressed in the following.

On a time scale of years, the $T$-dependence of the parameters is consistent with Eq. (26) and implies that both, the mean and the variance $\sigma^2_T \sim T \dot{b}/(2m)$ of long-term returns is proportional to $T$. On long time scales, Sharpe’s ratio ($\dot{r}$ is the risk-free return rate),

$$
\lambda_T = \frac{\mu_T - \dot{r} T}{\sigma_T} \propto T^{1/2} \text{ for } T m \dot{b} \gg 1
$$

is a monotonically increasing function of the investment horizon that is similar to that predicted by an $iid$ process. On short time scales, the Sharpe ratio
of the present model does not increase as fast as for an iid process. Since the standard deviation \( v_T \sim T b \) is itself proportional to \( T \), Sharpe’s ratio in particular does not vanish for \( T \to 0 \),

\[
\lambda_0 = \lim_{T \to 0} \frac{T(\mu - \hat{\mu})}{v_T} = \frac{\mu - \hat{\mu}}{b} > 0 .
\] (33)

If returns were described by an iid process, short-term trading strategies would be extremely risky and should not find rational investors. The persistence of day-trading and of ”excessive” trading by institutional investors\(^{18}\) suggests that short-term strategies may not be prohibitively risky. The prolonged existence of such phenomena is more plausible if the specific risk, \( 1/\lambda_T \), remains finite for \( T \to 0 \). Although the risk per unit of excess return does rise, short-term investments are not penalized excessively by the price dynamics of the quantum model. [Short term investments nevertheless can be very expensive due to transaction costs.]

**Sharpe’s Ratio**

![Sharpe's Ratio diagram](image)

Fig. 4: The time dependence of Sharpe’s ratio. The points are from ref.[19] and give the observed Sharpe ratio of the Dow-Jones Industrials index. The solid line is the Sharpe ratio the quantum model predicts for an individual stock. The time scale has been calibrated by Eq. (25) and corresponds to the one of Fig. 2. The overall normalization of the Sharpe ratio of the stock was adjusted to the same order of magnitude as the index. For comparison, the dashed line gives the time dependence \( \lambda(T) \propto \sqrt{T} \) of the Sharpe ratio that an iid process would predict.

The Sharpe ratio of the historic returns on the Dow-Jones Industrials index
(DJI) in fact is fairly constant for periods up to $T < 100$ days\textsuperscript{9} (and perhaps even declines somewhat). The results from Peters’ analysis of the DJI index are shown in Fig. 4 and compared with the time dependence of the Sharpe ratio of a single representative stock in the quantum model. Although Peters studied the Sharpe ratio of an index rather than of individual stocks, his findings do indicate that the specific risk does not become prohibitively large as the investment period is shortened.

To better see what a constant Sharpe ratio may mean to investors, compare the following two short-term investment strategies ($0 \leq \alpha \leq 1$):

S1 An amount $\alpha W$ is invested in asset $A$ and the remainder, $(1 - \alpha)W$, is invested in asset $B$ over a period $T$.

S2 The full amount $W$ is invested in asset $A$ for a time $\alpha T$ and for the remaining time, $(1 - \alpha)T$, the full amount is reinvested in asset $B$.

If the mean and the standard deviation of the returns on the individual assets $A$ and $B$ are uncorrelated and proportional to the investment period, both strategies carry the same short-term risk (as measured by the standard deviation of the overall returns) and have the same expected overall return. The equivalence of the two strategies does not depend on the ratio $\alpha$ nor on the standard deviations and mean returns of the individual assets\textsuperscript{**}. If the Sharpe ratio approaches a finite value for $T \to 0$, a short-term investor therefore could be (almost) indifferent to choosing asset diversification (strategy S1) or time diversification (strategy S2). Since time diversification offers the option of choosing the reinvestment time at a later point in time, it will often be preferred for short-term investments. [Note that the transaction costs incurred by both strategies could be similar and that time diversification may only be penalized by the fact that $\hat{\lambda}_0 > 0$.]

Less direct evidence for $b_T \propto T$ is provided by the probability density at zero return, $p_T(0) = p_T(x|x)$. Eq. (15) implies that for $T < 2$ weeks, $p_T(0) \sim \frac{2}{\pi v_T}$ in this model. With $v_T \sim b_T \propto T$, the probability density at zero return on short time scales therefore is inversely proportional to the period, $p_T(0) \propto T^{-1}$. $v_T \propto \sqrt{T}$ on the other hand would imply $p_T(0) \propto T^{-1/2}$. Neither of these power laws corresponds very well to $p_T^{S&P500}(0) \propto T^{-0.71 \pm 0.03}$ observed for the probability density at zero return of the S&P500 index\textsuperscript{20}. But the stocks in an index are not perfectly correlated. Diversification decreases the variance of the index relative to the variance of the single-name stocks. Since correlations tend to decrease over time, the probability density at zero

\textsuperscript{**}The overall return and risk of either strategy does of course depend on the returns on the individual assets and in general is optimal for a particular value of $\alpha$ only.
return on the index therefore increases relative to that of a single-name stock. A slower decay over time of the probability density at zero return for the index than for a single stock thus is more plausible than a faster one\footnote{The probability density at zero return decays equally fast ($\propto 1/\sqrt{T}$) for the stock and the index in the (uncorrelated) Gaussian case}. The prediction of the present model that $p_T(0) \propto T^{-1}$ for an individual stock thus may well be compatible with a probability density at zero return $p_{S&P500}^T(0) \propto T^{-0.71\pm0.03}$ observed for the S&P500.

One should emphasize that with the definition Eq. (27), Trotter’s relation Eq. (29) in general is not compatible with the relation for the transition probability densities of a Markov chain,

$$\rho_{T_1+T_2}(z|x) = \int \rho_{T_2}(z|y)\rho_{T_1}(y|x) \, dy ,$$

because the absolute square of a sum of terms in general is not the sum of absolute squares of the individual terms. The difference is known as quantum interference. A Gaussian transition amplitude (and consequently a Gaussian transition probability density) is an important exception where Eq. (29) and Eq. (34) both hold. The absence of interference effects characterizes the (incoherent) classical limit of a quantum system. Applying this criterion to the observed return distributions, the classical description of equity markets only becomes accurate on a timescale of years.

Note that Eq. (34) may be violated in quantum theory even if the (quantum) process is memoryless and the Markov property holds for the conditional probability densities, i.e. if $p(z,t''|y,t';x,t;\ldots) = p(z,t''|y,t')$ for all $t'' > t' > t$. This does not lead to any logical contradiction if price measurements at intermediate times have a non-negligible effect on the distribution. In a financial context this would mean that trading a share to determine its price at an intermediate time results in a different distribution for the final outcome than if it were not traded.

Although such back-coupling effects are well known and observable on financial markets, it is generally difficult to quantitatively include them in stochastic models. The quantum framework incorporates “measurement”-effects in a consistent and tractable (but perhaps rather narrow) fashion. There may be no need to model the impact from trading separately.

6 Summary

The observed cumulative distributions of the short-term returns on single-name stocks are accurately modelled by pdf’s of the form in Eq. (15). These
densities are the square of an amplitude that is an analytic function of the returns apart from two complex conjugate poles of opposite strength. The poles are located at \( z = \mu_T + iv_T \) and \( z^* = \mu_T - iv_T \), where \( \mu_T \) is the mean return and \( v_T \) is the standard deviation of the returns over the time period \( T < 2 \) weeks. This mathematically rather concise description trivially accounts for the observed power law tails (with an exponent \( \alpha = 3 \)) and also implies scaling between the distributions of different individual companies.

The fact that the distribution of short-term returns with finite variance approximately retains its shape over time scales \( T < 2 \) weeks rules out an iid process and is thus challenging to describe stochastically. The temporal evolution of the corresponding amplitude on the other hand was found to be given by the remarkably simple linear operator \( \hat{H}_{\text{eff}}(t) \) of Eq. (18) that is proportional to the wave-number \( q \). [The temporal evolution of the corresponding pdf (the absolute square of the amplitude) is not linear.] Since a dependence of the effective Hamiltonian on fractional powers of the wave number is unusual for quantum systems, \( \alpha = 3 \) in this sense is a "natural" exponent for distributions with stable power law tails.

The proposed quantum model gives a surprisingly quantitative and transparent explanation of the observed shape and temporal stability of the observed stock price fluctuations on short time scales. As Fig. 1 shows, the model quantitatively reproduces the average cumulative distribution observed for normalized 5 minute and daily returns over 7 orders of magnitude with no free parameters.

By regulating the cusp of \( |q| \) at \( q = 0 \) and thus modifying the non-hermitian part of the Hamiltonian to a quadratic function of \( q \) near \( q = 0 \), the shape similarity of the evolution is broken and eventual convergence to a Gaussian probability density over time horizons of several months to years is assured. With the modification of Eq. (19) the model qualitatively reproduces the observed distributions for the normalized returns down to cumulative probabilities of \( 10^{-4} \) for time periods up to 4 trading years. As discussed in section 4, the deterioration in the quality of the fit on long time scales may partly be due to the skewness of the observed distributions, which is more severe for longer time periods. It could also be due to some "lighter-than-average" stocks if the scaling of the individual stock distributions for \( T > 16 \) days is less than perfect. Such stocks would dominate the average of Eq. (3) at high normalized returns and explain the flattening out of the observed tails.

Comparing the prediction of this model for the "decay-", or trading-rate of a share with the observed dependence of the trading rate on the imbalance

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\[ \text{It is tempting to associate these singularities of the amplitude with long and short positions in the stock} \]
between buyer- and seller-initiated trades, suggest that the imbalance number of ref. [15] is proportional to the wave-number. The imbalance in this case is the Fourier conjugate variable to the return $x_T$ (up to an overall scale). The quantum model then predicts a definite form for the distribution of the imbalance number – it in particular should have exponential, rather than power law, tails.

By including sufficiently many "hidden variables", that is factors, stochastic models may be able to reproduce the historic observations with similar accuracy. Indeed, the (possible) existence of hidden variables was at the heart of the early critique of quantum theory. For physical quantum phenomena, this alternative explanation has only recently been rejected by experimental verification of Bell’s inequalities 21.

It is quite impossible to perform similar high-precision experiments on equities in order to reject the existence of "hidden variables". There in fact is no need to because it is almost self-evident that the return on a stock depends on many factors that have not been modelled. The question nevertheless is not just one of having a more efficient description of the dynamics. Unlike hidden variables describing physical phenomena, the factors that influence the dynamics of a stock are expected to change over time. It furthermore is not clear how economic factors like the gross national product influence the value of any given stock at any given point in time. The observed scaling of the return distributions for various stocks in different economic environments strongly suggests that all these "hidden" factors find their expression in the mean return and the variance of the returns. The dynamics that determines the shape of the return distributions on the other hand must be self-consistent and largely immune to the influence of "hidden" variables that are specific to a company and the economic and political climate.

The conceptual advantage of a quantum description is that this consistent framework describes the observed return fluctuations rather well without separately modelling all the possibly influential, time-dependent, and hard-to-measure factors. Contrary to a purely descriptive "hidden variable" model, the consistency of a quantum model leads to a number of predictions. Their verification or falsification will ultimately decide whether this is a useful approach.

Acknowledgement: I would like to thank several members of the applied mathematics department of the Courant Institute of New York University for their hospitality and for organizing the very informative and accessible seminar in mathematical finance where I became aware of the empirical analysis discussed here. I am very much indebted to Larry Spruch for his personal support.
A Subordinated Stochastic Process for the 5 minute Returns

In section 2 the \( T = 5 \) minute returns of a stock were found to be well described by the pdf of Eq. (15) with \( a = 0 \) and standard deviation \( b = v_T = 5\text{min} \) (without loss in generality, we consider only the case of vanishing mean return). To characterize the directing process of a subordinated stochastic process\(^2\) that would give this pdf, consider the following integral representation,

\[
\frac{2b^3}{\pi(x^2 + b^2)^2} = \frac{2b^3}{\pi} \int_0^\infty \lambda d\lambda e^{-\lambda(b^2 + x^2)}
\]

(35)

\[
= \int_0^\infty \frac{e^{1/(2t)}}{\sqrt{2\pi t^5}} \frac{\exp \left[-\frac{x^2}{2t}\right]}{\sqrt{2\pi b^2 t}} .
\]

(36)

One thus can interpret the pdf for the returns as the result of a random walk with variance \( t v_T^2 \), where \( t \) is itself a random variable drawn from the positive distribution

\[
q(t) = \frac{e^{1/(2t)}}{\sqrt{2\pi t^5}} .
\]

(37)

Clark interprets the stochastic process \( \tau(t) \) whose increments are drawn from the distribution \( q(t) \) as "operational" time\(^9\). Note that \( q(t) \) has unit mean and infinite variance and that the random variable \( 1/t \) is drawn from a \( \Gamma \) distribution. In fact, the probability for a large variance, respectively operational time interval, itself falls off as a power law with an exponent of \(-3/2\).

Although the subordinate stochastic process described here reproduces the observed distribution of returns for \( T = 5\text{min} \) by construction, it does not describe the observed time dependence of this distribution: the temporal evolution is not shape stable and a Gaussian fixed point is approached rather quickly.

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