Fully Off-shell Effective Action and its Supersymmetry in Matrix Theory

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Abstract

As a step toward clarification of the power of supersymmetry (SUSY) in Matrix theory, a complete calculation, including all the spin effects, is performed of the effective action of a probe D-particle, moving along an arbitrary trajectory in interaction with a large number of coincident source D-particles, at one loop at order 4 in the derivative expansion. Furthermore, exploiting the SUSY Ward identity developed previously, the quantum-corrected effective supersymmetry transformation laws are obtained explicitly to the relevant order and are used to verify the SUSY-invariance of the effective action. Assuming that the agreement with 11-dimensional supergravity persists, our result can be regarded as a prediction for supergravity calculation, which, yet unavailable, is known to be highly non-trivial.

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1 Introduction

One of the main themes of string theory in the past few years is the correspondence between supergravity/superstring theories in the bulk and certain types of supersymmetric theories defined on the worldvolume of various brane configurations or “on the boundary” of the space-time. The proposal of Maldacena\cite{1}, termed AdS/CFT correspondence, has by now been extended to much wider class of systems than was originally envisaged and has spurred wide variety of new developments\cite{2}.

Although conjectured prior to this proposal, the Matrix theory for M-theory, put forward by Banks, Fischler, Shenker and Susskind \cite{3}, can be thought of as a prime example of this correspondence. Re-interpreted in the framework of discrete light-cone quantization \cite{4}, it has enjoyed numerous successes. Just to mention only the direct comparison with eleven dimensional supergravity, complete agreement for the multi-graviton scattering (including the recoil effects) at 2-loop\cite{5, 6} and that for the two-body potential between arbitrary fermionic as well as bosonic objects at 1-loop \cite{7} should be regarded as highly non-trivial and remarkable.

Just as the mechanism of the general bulk-boundary correspondence has not been clearly identified, the origin of these successes in the Matrix theory is yet to be fully understood. Now in string theory, it is often the case that symmetry principles play decisive roles, much more so than in local field theories, in determining the dynamics of the system, and the situation should be the same in Matrix theory, which is an explicit representation of M-theory that unifies all the string theories.

Indeed there have been a number of studies\cite{8, 9, 10, 11, 12, 13} which point to the assertion that, in particular, the high degree of supersymmetry, namely with the maximally allowed 16 supercharges, is powerful enough to determine the effective action of the D-particles, which can be directly compared with the corresponding supergravity calculation. If it is indeed the case, it is rather surprising since usually such a global symmetry can only give certain relations among the correlation functions and not more, and consequently the dynamical significance of Matrix theory would have to be reconsidered. However, upon close examinations, even restricted to the simple so-called “source-probe” situation, one can argue that the existing analysis is not quite complete. This is largely due to the fact that for a system with maximal supersymmetry unconstrained superfield formulation does not exist and therefore a clear-cut off-shell analysis is not possible: One is forced to deal with the component formalism, where the supersymmetry algebra gets intertwined with gauge symmetry and does not close without the aid of equations of motion.
In a previous work [14], we gave a rather extensive discussion on the existing literature and emphasized the importance of the consistency of the approximation scheme in studying the symmetry of the effective action, which is an off-shell entity. We argued that the only consistent procedure is to deal with the trajectory, including the spin degrees of freedom, with arbitrary time dependence and adopt the derivative expansion according to the concept of order, defined as the number of derivatives plus half the number of fermions. Only this scheme is free of total derivative ambiguities inevitably present in the effective action. For more discussions, see the Section 3 of [14].

Following this philosophy, we derived in [14] a completely off-shell SUSY Ward identity in the background gauge (which is naturally intertwined with the BRST symmetry) and applied it to the effective action at order 2 to study the power of supersymmetry. Our conclusion was that at this order the effective action is indeed determined by the Ward identity alone. As we already remarked there, however, such a result was to be expected since at order 2 the higher derivatives, such as the acceleration etc., can be eliminated from the effective action by integration by parts and our analysis was essentially the same as in the existing literature. The full significance of the completely off-shell analysis becomes apparent starting from the next order, i.e. from order 4, where complete elimination of higher derivatives is no longer possible.

Unlike the case of order 2, at which the 1-loop effective action was already available, the only fully off-shell results so far known at order 4 are the famous bosonic part [15, 16, 17] given (in Euclidean formulation) by $-N \int d\tau (15/16)v^4/r^7$, with $N$ the number of source D-particles, and a part of the fermionic contributions containing 8 powers of the fermion field $\theta$ [18]. The latter is the simplest among the fermionic contributions since it does not involve any derivatives. A calculation of $O(\theta^2)$ contribution, which is already formidable, was attempted in [19] but was not fully completed. As for $O(\theta^4)$ and $O(\theta^6)$ contributions, no attempt has been made to date.

In this work, as a necessary step toward clarification of the power of supersymmetry in Matrix theory, we perform a complete calculation of the effective action for a probe D-particle, including all the spin effects, at one-loop at order 4, with the aid of the algebraic manipulation program Mathematica. Furthermore, exploiting the SUSY Ward identity developed previously, the quantum-corrected effective supersymmetry transformation laws are obtained explicitly to the relevant order and are used to verify the SUSY-invariance of the effective action so obtained. Since the corresponding supergravity calculation is available only up to $O(\theta^2)$ [20], we cannot at present time test if our result agrees with supergravity. Rather, provided that the agreement with supergravity persists, our result should be regarded as a prediction until such a calculation will have been made.
Organization of the rest of this article is as follows: We start in Section 2 with a brief summary of the Matrix theory and its symmetries, mainly to set the notations. Section 3 is devoted to the calculation of the effective action at one-loop. After explaining the derivative expansion scheme used and reviewing the existing results for the effective action in Section 3.1, we describe the calculational procedures in Section 3.2. The result of the calculation, together with how it was simplified using $SO(9)$ Fierz identities, is presented in Section 3.3. Since the actual calculation, performed by Mathematica, involved an enormous number of steps, it is desirable to have an independent check. For this purpose, as well as for its own interest, we compute in Section 4 the quantum-corrected SUSY transformation laws and check the invariance of the effective action under these transformations: Following a brief review of the SUSY Ward identity in Section 4.1, we sketch the calculational procedure in Section 4.2. Then in Section 4.3 we describe how the invariance under SUSY was verified. Finally, we give a summary and discussions in Section 5. Three appendices, A~C are provided to supply some details of the calculations. In appendix A we describe a new efficient algorithm for generating $SO(9)$ Fierz identities. Non-trivial two-point functions needed for the calculation of the SUSY transformation laws are collected in Appendix B and the quantum-corrected SUSY transformation laws are displayed in Appendix C.

2 Preliminaries

We begin with a very brief summary of the Matrix theory and its symmetries, mainly to set our notations.

The classical action for the $U(N + 1)$ Matrix theory in the Euclidean formulation is given by

$$\tilde{S}_0 = \text{Tr} \int d\tau \left\{ \frac{1}{2} [D_\tau, X_m]^2 - \frac{g^2}{4} [X_m, X_n]^2 + \frac{1}{2} \Theta^T [D_\tau, \Theta] - \frac{1}{2} g \Theta^T \gamma^m [X_m, \Theta] \right\}. \quad (2.1)$$

In this expression, $X_{ij}^m(\tau), \tilde{A}_{ij}(\tau)$ and $\Theta_{\alpha,ij}(\tau)$ are the $(N + 1) \times (N + 1)$ hermitian matrix fields, representing the bosonic part of the D-particles, the gauge fields, and the fermionic part of the D-particles, respectively. $D_\tau = \partial_\tau - ig\tilde{A}$ is the covariant derivative, $\gamma^m$ are the real symmetric $16 \times 16 SO(9)$ $\gamma$-matrices, and the vector index $m$ runs from 1 to 9. We put a tilde on some relevant symbols to remind us of Euclidean formulation.

This action is known to possess a number of important symmetries. The first is the
obvious global Spin(9) invariance, which nevertheless is quite non-trivial in the fermionic sector being responsible for various Fierz identities to be used extensively later. The second is the CPT symmetry inherited from the 10-dimensional super Yang-Mills theory from which the above action can be obtained by dimensional reduction. The third is the invariance under the $U(N+1)$ gauge transformations given, with the gauge parameter matrix $\Lambda$, by

$$\delta_\Lambda \tilde{A} = [D_\tau, \Lambda], \quad \delta_\Lambda X_m = ig [\Lambda, X_m], \quad \delta_\Lambda \Theta = ig [\Lambda, \Theta].$$

(2.2)

The fourth, and the main focus of this article, is the supersymmetry with 16 spinorial parameters $\epsilon_\alpha$. The transformation laws are

$$\delta_\epsilon \tilde{A} = \epsilon^T \Theta, \quad \delta_\epsilon X^m = -i \epsilon^T \gamma^m \Theta, \quad \delta_\epsilon \Theta = i \left( [D_\tau, X_m] \gamma^m + \frac{g}{2} [X_m, X_n] \gamma^{mn} \right) \epsilon.$$  

(2.3)

(2.4)

Although the algebra closes only on-shell up to field-dependent gauge transformations, $\tilde{S}_0$ itself is invariant without the use of equations of motion, i.e. off-shell.

In addition to these well-known symmetries, there is a so-called generalized conformal symmetry\[21, 22, 23\], which may be used to restrict the form of the effective action. Finally, the agreement with the 11-dimensional supergravity calculation for the multi-body processes\[5\] strongly suggests that the 11-dimensional Lorentz symmetry is hidden in $\tilde{S}_0$, awaiting to be disclosed.

In this article, we shall concentrate on the so-called source-probe situation, namely the configuration of a probe D-particle interacting with a large number, $N$, of the source D-particles all sitting at the origin. This is expressed by the splitting

$$X_m(\tau) = \frac{1}{g} B_m(\tau) + Y_m(\tau), \quad \Theta_\alpha(\tau) = \frac{1}{g} \theta_\alpha(\tau) + \Psi_\alpha(\tau),$$

(2.5)

$$B_m(\tau) = \text{diag} \left( r_m(\tau), 0, 0, \ldots, 0 \right), \quad \theta_\alpha(\tau) = \text{diag} \left( \theta_\alpha(\tau), 0, 0, \ldots, 0 \right),$$

(2.6)

where $B_m(\tau)$ and $\theta_\alpha(\tau)$ are the bosonic and the fermionic backgrounds expressing the positions and the spin degrees of freedom of the D-particles respectively and $Y_m(\tau)$ and $\Psi_\alpha(\tau)$ denote the quantum fluctuations around them. We will be interested in the general case where $r_m(\tau)$ and $\theta_\alpha(\tau)$ are arbitrary functions of $\tau$ not satisfying equations of motion.

In order to quantize the system and perform the calculation of the effective action of the probe D-particle, we must fix the gauge. Practically the only tractable choice, and
indeed the one used for all the calculations in the past, is the background gauge specified
by the gauge-fixing function $\tilde{G}$ given by

$$
\tilde{G} = -\partial_\tau \tilde{A} + i [B^m, X_m].
$$

(2.7)

The associated BRST transformations for the quantum part of the fields are given by

$$
\begin{align*}
\delta_B \tilde{A} &= [D_\tau, C], \\
\delta_B Y_m &= -ig [X_m, C], \\
\delta_B \Psi &= ig \{ C, \Theta \}, \\
\delta_B C &= ig C^2, \\
\delta_B \bar{C} &= ib, \\
\end{align*}
$$

(2.8)

where $\bar{C}$ and $b$ are, respectively, the ghost, the anti-ghost and the Nakanishi-Lautrup auxiliary field. As usual, the gauge-fixing and the ghost part of the action are given altogether by the BRST total variation

$$
\tilde{S}_{gg} = \delta_B \text{Tr} \int d\tau \left[ \frac{1}{i} \bar{C} \left( \tilde{G} - \frac{1}{2} b \right) \right].
$$

(2.9)

3 Calculation of the Off-shell Effective Action

We are now ready to start the computation of the off-shell effective action for the probe D-particle.

3.1 Derivative expansion and the existing results

As was already emphasized in the introduction and further elaborated in [14], it is important that the approximation scheme for computing the effective action must be consistent with the freedom of adding total derivatives. The only such scheme is the derivative expansion according to the concept of order defined as [24]

$$
\text{order} \equiv \text{number of } \tau\text{-derivatives} + \frac{1}{2} \text{number of fermions}.
$$

(3.10)

In other words, we assign order($r$) = 0, order($\partial$) = 1, and order($\theta$) = 1/2, where $\partial$ denotes the derivative with respect to $\tau$.

This concept can be applied loop-wise. For instance, the tree level action for $r_m$ and $\theta_\alpha$ is of the form (we use $v_m$ to denote $\dot{r}_m$, the superscript in parenthesis is the loop number and the subscript signifies the order)

$$
\Gamma^{(0)}_2 = \int d\tau \left( \frac{v^2}{2g^2} + \frac{\theta^T \dot{\theta}}{2g^2} \right),
$$

(3.11)
and is entirely of order 2. The order 2 part at 1 loop was first computed\footnote{Here and hereafter, all the results are in the background gauge.} in \cite{7} and is given by

\[ \tilde{\Gamma}^{(1)}_2 = N \int d\tau \frac{\theta^T \dot{\theta}}{r^3}. \] (3.12)

Consider now the order 4 part at 1-loop, which consists of the bosonic and the fermionic parts. The former was computed long ago in eikonal approximation in \cite{15} and more recently for fully off-shell case in \cite{16, 17} and has the well-known form

\[ \tilde{\Gamma}^{(4),b}_1 = -N \int d\tau \frac{15 v^4}{16 r^7}. \] (3.13)

As we shall see shortly, calculation of the fermionic part\footnote{Within the eikonal-type approximation, spin effects have been discussed in \cite{24, 25, 26, 27, 28, 29, 30}.} which can be further classified by the (even) number of $\theta$'s up to 8, is exceedingly more difficult. The $\mathcal{O}(\theta^8)$ part is relatively easy to compute \cite{18} since it cannot contain any derivatives. The calculation of the $\mathcal{O}(\theta^2)$ part (with three $\partial$'s) was attempted in \cite{19} but was not fully completed. As for the $\mathcal{O}(\theta^4)$ and $\mathcal{O}(\theta^6)$ contributions, even an attempt has not been made.

In the following, we perform the complete calculation of all the fermionic order 4 terms at 1 loop. This is made possible with the extensive use of Mathematica.

### 3.2 Calculational procedure

As we shall perform the calculation at 1 loop, we only need the part of the gauge-fixed action quadratic in the quantum fluctuations. To simplify the presentation as well as the subsequent calculations, define the $(1 + 9)$-component fermionic and bosonic vectors in the following way:

\[ \Xi_\alpha \equiv \left( \begin{array}{c} i\theta_\alpha \\ \theta_\beta \gamma^m_{\beta\alpha} \end{array} \right), \quad \Phi \equiv \left( \begin{array}{c} \tilde{A} \\ Y^m \end{array} \right), \] (3.14)

where the matrix indices are suppressed. Also define the kinetic operators $D_B$ and $D_F$ for bosonic and fermionic fields respectively as

\[ D_B \equiv \begin{pmatrix} \Delta^{-1} & -2i\nu^m \\ 2i\nu^m & \Delta^{-1}\delta_{nm} \end{pmatrix}, \quad D_F \equiv \partial + \gamma. \] (3.15)

Here, $\Delta^{-1} \equiv -\partial^2 + r^2(\tau)$ is the basic kinetic operator, and its inverse $\Delta$, the basic propagator, will appear frequently in the actual calculations. Note that the “mass” $r(\tau)$ is an arbitrary function of $\tau$ and this will make the computation non-trivial.
With these notations, the quadratic part of the action can be written as

\[ \tilde{S}^{(2)} = \int d\tau \sum_{i,j} \left[ \frac{1}{2} \Phi_{ij} D_B \Phi_{ji} + i \bar{C}_{ij} \Delta^{-1} C_{ji} \ight. \\
\left. + \frac{1}{2} \Psi_{ij} D_F \Psi_{ji} + \frac{1}{2} \Phi^T_{ji} \Xi \Phi_{ij} \right]. \]  

(3.16)

Performing the Gaussian integration, we obtain the formal expression for the 1-loop effective action \( \tilde{\Gamma}^{(1)} \) consisting of the bosonic and the fermionic parts:

\[ \tilde{\Gamma}^{(1)} = \tilde{\Gamma}^{(1)}_B + \tilde{\Gamma}^{(1)}_F, \]  

(3.17)

\[ \tilde{\Gamma}^{(1)}_B = \text{tr} \ln(1 - 4v^n \Delta v^n \Delta) - \frac{1}{2} \text{Tr} \ln(1 + \phi \Delta), \]

\[ \tilde{\Gamma}^{(1)}_F = -\text{Tr} \ln \left( 1 - D^{-1}_F \Xi^T D^{-1}_B \Xi \right) = \sum_{n=1}^{\infty} \frac{\text{Tr} \left( D^{-1}_F \Xi^T D^{-1}_B \Xi \right)^n}{n}. \]  

(3.18)

In the above, “tr” is the trace over the matrix indices and over the function space, while “Tr” further includes the trace over the spinor indices as well. At order 4, the bosonic part \( \tilde{\Gamma}^{(1)}_B \) reproduces the known result (3.13). Our interest is in the fermionic part \( \tilde{\Gamma}^{(1)}_F \) and we need to compute up to \( n = 4 \).

Now to proceed, we must expand the propagators \( D^{-1}_B \) and \( D^{-1}_F \) in powers of the derivatives. The explicit form of \( D^{-1}_B \) is given by

\[ D^{-1}_B = \left( \begin{array}{cc} \Delta(1 - 4v^\ell \Delta v^\ell \Delta)^{-1} & 2i \Delta(1 - 4v^\ell \Delta v^\ell \Delta)^{-1} v^m \Delta \\ -2i v^n \Delta(1 - 4v^\ell \Delta v^\ell \Delta)^{-1} & \Delta(\delta_{nm} - 4v^n \Delta v^m \Delta)^{-1} \end{array} \right), \]  

(3.19)

where the expansion of the following type should be substituted:

\[ (\delta_{nm} - 4v^n \Delta v^m \Delta)^{-1} \equiv \delta_{nm} + 4v^n \Delta v^m \Delta + 16v^2 \Delta v^\ell \Delta v^m \Delta + \cdots. \]  

(3.20)

The corresponding expansion for \( D_F \) is given by

\[ D^{-1}_F = (\partial + \phi)^{-1} = -(\partial - \phi)(1 + \Delta \phi)^{-1} \Delta \]

\[ = -(\partial - \phi)\Delta + (\partial - \phi)\Delta \phi \Delta + \cdots. \]  

(3.21)

When these expansions are implemented, evidently the expressions for \( \text{Tr} \left( D^{-1}_F \Xi^T D^{-1}_B \Xi \right)^n \) get rather involved. Below we only display the intermediate result (which still has to be
fully expanded) for the simplest case of $n = 1$ as an example:

$$
\text{Tr} D^{-1}_F \Xi^T D^{-1}_B \Xi = \text{Tr} \left[-(\partial - \hat{\phi})(1 - \Delta \phi + \Delta \phi \Delta \phi - \Delta \phi \Delta \phi)_{\alpha \epsilon} \Delta \\
\times \left(-\theta \epsilon \Delta \theta_{\alpha} - 4\theta \epsilon \Delta v^\xi \Delta v^\xi \Delta \theta_{\alpha} - 2\Delta \theta \epsilon \hat{\phi}_{\beta \alpha} \Delta \theta_{\delta} - 8\Delta \theta \epsilon \Delta v^\xi \Delta v^\xi \Delta \hat{\phi}_{\beta \alpha} \Delta \theta_{\delta} \\
+ 2\theta \gamma \Delta \hat{\phi}_{\gamma \epsilon} \Delta \theta_{\alpha} + 8\theta \gamma \Delta \hat{\phi}_{\gamma \epsilon} \Delta v^\xi \Delta v^\xi \Delta \theta_{\alpha} + \theta \gamma \gamma^n \Delta \theta_{\delta} \delta^n_{\beta \alpha} \\
+ 4\theta \gamma \Delta \hat{\phi}_{\gamma \epsilon} \Delta \hat{\phi}_{\beta \alpha} \Delta \theta_{\delta}\right) + O(\partial^4) .
\right]
$$

(3.22)

The number of terms are much larger for $n \geq 2$ and altogether they add up to about 1000.

The next step is the actual evaluation of the trace “Tr”, which here means tracing over the function space and over the spinor indices. While the latter, which is nothing but the $\gamma$-matrix algebra, is conceptually simple, the former step requires some explanation. To perform it efficiently, we employ the so-called “normal-ordering method”, an algorithm invented by Okawa in [17]. The idea is to bring the expressions involving $f(\tau)$ (some function of $\tau$), $\partial$ and $\Delta$ in various orders into a “normal-ordered” form $f(\tau) \partial^m \Delta^n$ by recursively using the formulas

$$
\partial f = f \partial + \hat{f} , \\
\Delta f = f \Delta + \Delta(\hat{f} + 2\hat{f} \partial) \Delta , \\
\Delta \partial = \partial \Delta + 2\Delta(\hat{r} \cdot r) \Delta .
$$

(3.23)

Further, by using the trivial relation $\partial^2 \Delta = r^2 \Delta - 1$, which follows from the definition of $\Delta$, one can reduce the number of $\partial$’s in the normal-ordered form down to either zero or one. Once each term is brought to this standard form, the remaining task is to compute the matrix elements $\langle \tau | \Delta^n | \tau' \rangle$ and $\langle \tau | \partial \Delta^n | \tau' \rangle$, where the latter is actually expressed in terms of the former as $\langle \tau | \partial \Delta^n | \tau' \rangle = \frac{1}{2} \partial \langle \tau | \Delta^n | \tau' \rangle$. Now employing an integral representation, one can write

$$
\langle \tau | \Delta^n | \tau' \rangle = \frac{1}{(-\partial^2 + r(\tau))^n} \delta(\tau - \tau') \\
= \frac{1}{(n - 1)!} \int_0^\infty d\sigma \sigma^{n-1} e^{-\sigma(-\partial^2 + r(\tau))^2} \delta(\tau - \tau') .
$$

(3.24)

Since the exponent of the exponential in the integrand contains the operator $\partial^2$ and the function $r^2(\tau)$ which are non-commuting, one needs to make use of the Baker-Campbell-Hausdorff formula up to the appropriate order in the derivative expansion, so that $e^{\sigma \partial^2}$ acts directly on $\delta(\tau - \tau')$. For more details, see [17, 31]. An example of the result of such
a calculation is
\[
\langle \tau | \Delta^4 | \tau \rangle = \frac{5}{32 r^7} + \frac{1155 (r \cdot \dot{r})^2}{256 r^{13}} - \frac{105 (r \cdot \ddot{r})}{128 r^{11}} - \frac{105 \dot{r}^2}{128 r^{11}} + \mathcal{O}(\partial^4).
\] (3.25)

3.3 Simplification procedure and the result

As already mentioned, the procedures described above were executed by developing an elaborate codes for Mathematica. The number of terms at the input stage is about 1000. As one performs the normal-ordering and the \(\gamma\)-matrix algebra, this number remains to be roughly of the same order. We then tried to simplify the results by bringing them to appropriate standard forms via integration by parts and identification of the same structures with different repeated indices. This manipulation brought the number down to about 70.

The final step is to simplify them as much as possible via the use of \(SO(9)\) Fierz identities. According to the terminology of [7], a Fierz identity in which \(n\) of the indices of the \(\gamma\)-matrices involved are not contracted among themselves is called an “\(n\)-free-index” identity. It turns out that, including the situations that occur in the next section where we examine the SUSY invariance of the effective action, we need up to 5-free-index identities in this parlance. Although some class of Fierz identities are known in the literature and a general procedure to generate all possible identities was described in [7], this was not enough for our purposes. The reason is two-fold: First, since we deal with \(\theta(\tau)\) with arbitrary \(\tau\) dependence, \(\theta, \dot{\theta}, \ddot{\theta}\), etc. must be treated as different spinors and hence we need general forms of the identities. The ones available in the literature do not cover such cases in sufficient generality. Second, the algorithm of [7] turned out to be prohibitively time-consuming when the number of free indices exceeds 3. To overcome this difficulty, we developed a new more efficient algorithm, which is described in the Appendix A. None the less, one must cope with the fact that the number of independent Fierz identities grows like \(\sim 5 \times 2^n\) and besides the length of each identity increases rapidly with \(n\) as well. Consequently, it was a difficult task to find the right identities to be applied for simplification.

However, when the right identities were applied, the result became remarkably simple. The following, we believe, are the simplest forms for the desired fermionic part of the 1-loop effective action at order 4:
under these transformations. To demonstrate that the effective action shown above is invariant under such transformations, we shall perform such a test by computing the effective SUSY transformation laws for the effective action and demonstrate that indeed the effective action is invariant under these transformations. It is certainly desirable to have an independent check. We shall perform such a test by computing the effective SUSY transformation laws for the effective action and demonstrate that indeed the effective action is invariant under these transformations. It is certainly desirable to have an independent check. We shall perform such a test by computing the effective SUSY transformation laws for the effective action and demonstrate that indeed the effective action is invariant under these transformations. It is certainly desirable to have an independent check. We shall perform such a test by computing the effective SUSY transformation laws for the effective action and demonstrate that indeed the effective action is invariant under these transformations. It is certainly desirable to have an independent check. We shall perform such a test by computing the effective SUSY transformation laws for the effective action and demonstrate that indeed the effective action is invariant under these transformations.

Here and for the rest of the article, we omit the overall common factor of $N$ for simplicity. This constitutes one of the main results of this work. We remark that \( \tilde{\Gamma}_{\theta^2}^{(1)} \) is slightly different from the one quoted in a previous attempt [19], while \( \tilde{\Gamma}_{\theta^4}^{(1)} \), which does not contain any derivatives, agrees completely with the calculation of [18]. Furthermore, if we drop the terms containing the derivatives of \( \theta \) and the acceleration \( a \), our result coincides (up to an overall constant) with the previous calculations [26, 29, 11] for this very special configuration. As for comparison with 11-dimensional supergravity, we shall make a remark in the final section.

In view of the fact that the calculation consisted of enormous number of steps and that we deal with fully off-shell configurations, the agreement cited above for special cases should only be regarded as a strong but not decisive evidence for the correctness of our calculation: It is certainly desirable to have an independent check. In the next section, we shall perform such a test by computing the effective SUSY transformation laws for the effective action and demonstrate that indeed the effective action shown above is invariant under these transformations.
4 Effective SUSY Transformation Laws and Invariance of the Effective Action

4.1 A brief review of the SUSY Ward identity

Since the calculation of the effective SUSY transformation laws will be based on the
supersymmetric Ward identity in the background gauge derived in a previous publication
[14], let us briefly review its salient features.

The derivation of the SUSY Ward identity in the path-integral formalism is more or
less a textbook matter, except for two non-trivialities. One stems from the intertwining of
the SUSY and BRST symmetries. In order to bring the SUSY Ward identity in a useful
form, one must make judicious uses of the BRST Ward identities. The other complication
is due to the two different origins of the $B_m$ dependence, one from the separation of the
background and the quantum fluctuation, namely $X_m = (1/g)B_m + Y_m$, and the other
from the gauge-fixing function $\tilde{G} = -\partial_r \tilde{A} + i [B^m, X_m]$. In order to obtain the correct
Ward identity, one must carefully distinguish these two origins. For more details, see
[14].

Through the procedure outlined above, one obtains the SUSY Ward identity in the
form where the effective SUSY transformation laws can be read off in closed forms.
Adapted to the source-probe situation under consideration, it takes the form

$$0 = \int d\tau \left( \Delta_\epsilon r_m(\tau) \frac{\delta \tilde{\Gamma}}{\delta r_m(\tau)} + \Delta_\epsilon \theta_\alpha(\tau) \frac{\delta \tilde{\Gamma}}{\delta \theta_\alpha(\tau)} \right),$$

(4.30)

where the effective SUSY transformation laws are given by

$$\Delta_\epsilon r_m(\tau) = \int d\tau' T^{-1}_{m,n}(\tau', \tau) \left( \langle \delta_\epsilon \tilde{y}_n(\tau') \rangle + \langle \delta_\beta \tilde{y}_n(\tau') \rangle O_\epsilon \right),$$

(4.31)

$$\Delta_\epsilon \theta_\alpha(\tau) = \langle \delta_\epsilon \tilde{\psi}_\alpha(\tau) \rangle - \langle \delta_\beta \tilde{\psi}_\alpha(\tau) \rangle O_\epsilon$$

$$- \int d\tau' d\tau'' T^{-1}_{m,n}(\tau'', \tau') \langle \delta_\beta \tilde{\psi}_\alpha(\tau) \rangle O_{\epsilon}(\tau') \times \langle \delta_\epsilon \tilde{y}_m(\tau'') + \delta_\beta \tilde{y}_m(\tau'') \rangle O_\epsilon \right).$$

(4.32)

In the expressions above, $\delta_\epsilon$ is the SUSY variation (2.3), $\delta_\beta$ is the BRST variation (2.8),
$\tilde{y}_m$ and $\tilde{\psi}_\alpha$ respectively denote $gY_{11}$ and $g\Psi_{\alpha,11}$ (i.e. the diagonal fluctuation of $r_m$ and
$\theta_\alpha$ respectively), $\langle \quad \rangle$ expresses the the expectation value, and the operators $O_\epsilon$, $O_m$ and
the kernel $T_{n,m}(\tau, \tau')$ are given as shown below:

$$O_{\epsilon} = -i \text{Tr} \int d\tau \, C\delta_{\epsilon}(\tau) A + i [B^m, X_m] , \quad (4.33)$$

$$O_m = \sum_{I=2}^{N+1} (C_{1I} Y_{m,1I} - \tilde{C}_{1I} Y_{m,1I}) , \quad (4.34)$$

$$T_{n,m}(\tau, \tau') \equiv \delta_{mn} \delta(\tau - \tau') - \langle \delta_B \tilde{y}_n(\tau') O_m(\tau) \rangle . \quad (4.35)$$

Note that these quantum-corrected transformation laws are much more involved than those at the tree level. Even the linear law $\delta_{\epsilon} X_m = -i \epsilon^T \gamma_m \Theta$ for the bosonic field gets modified non-trivially contrary to naive expectation. It is not simply the expectation value of the linear law itself because what is relevant is the effective law that acts on $\tilde{\Gamma}$.

### 4.2 Calculation of the effective SUSY transformation laws

Actual evaluation of $\Delta r_m$ and $\Delta \theta_\alpha$ at 1-loop to the relevant order is essentially straightforward but extremely cumbersome. Since similar calculations performed at order 2 were fully displayed in the previous work [14], we shall only give a sketch of the general procedure for the present case.

When the variations $\delta_{\epsilon}$ and $\delta_B$ are performed and the definitions of the composite operators $O_{\epsilon}$ and $O_m$ are substituted into (4.31) and (4.32), it is not difficult to see that, at 1-loop, terms contributing to $\Delta r_m$ and $\Delta \theta_\alpha$ can be expressed as Feynman diagrams composed of products of tree-level 2-point functions for various fields. These are of course computed from the quadratic part of the gauge-fixed action already displayed in (3.16). The only complication is that we must disentangle the mixings among fields, which, due to the presence of the fermionic background, includes those between bosonic and fermionic fields. The results for the non-trivial 2-point functions are given in the Appendix B.

Although most of these diagrams produce local expressions as desired, but there exist some which give non-local contributions. For example, a term contributing to $\Delta r_m$ is of the form

$$g^2 \int d\tau' d\tau'' \langle C(\tau) C^*(\tau') \rangle \langle C_{11}(\tau'') \tilde{C}_{11}(\tau') \rangle \langle -\tilde{A} + ir_n Y_n \rangle \langle \tilde{Y}_m^*(\tau) \rangle \epsilon_\beta \theta_\beta(\tau') , \quad (4.36)$$

where the second factor $\langle C_{11}(\tau'') \tilde{C}_{11}(\tau') \rangle$ going like $1/\partial$ produces unwanted non-locality. Fortunately, when one isolates all the terms of this sort, one notices that they precisely
cancel in the SUSY Ward identity due to the following BRST Ward identity:

\[ \langle \delta_B Q(\tau'') \rangle = -\int d\tau \left( \frac{\delta \tilde{\Gamma}}{\delta r_m(\tau)} \langle \delta B \tilde{y}_m(\tau) Q(\tau'') \rangle + \frac{\delta \tilde{\Gamma}}{\delta \theta_\alpha(\tau)} \langle \delta B \tilde{\psi}_\alpha(\tau) Q(\tau'') \rangle \right), \quad (4.37) \]

\[ Q(\tau'') \equiv -i\tilde{C}(\tau'') \left( -\tilde{A}^* - ir_n Y_n^* \right)(\tau'') + i\tilde{C}^*(\tau'') \left( -\tilde{A} + ir_n Y_n \right)(\tau''). \quad (4.38) \]

We omit the details of the demonstration.

Just as for the effective action, the calculation of the relevant diagrams for the SUSY transformation laws was performed with the aid of Mathematica. Even after various simplification procedures the results are quite complicated, with the longest expression, \( \Delta \theta_\alpha \) with 4 \( \theta \)'s, consisting of 56 terms. So we relegate them to the Appendix C in order not to interrupt the flow of the main text. A glance at this result would convince one that, when we prove the invariance of effective action under these transformations in the sequel, calculations of both the effective action and the SUSY transformations must be correct.

### 4.3 SUSY invariance of the effective action

Now we come to the last stage of this work, the check of the invariance of the effective action under the SUSY transformation laws just computed. Logically what we wish to demonstrate is quite simple: The tree level SUSY variation of the 1-loop effective action and the 1-loop level SUSY variation of the tree action must cancel. The difficulty is again that above \( O(\theta^4) \) we need judicious applications of various Fierz identities to effect such cancellations. The required Fierz identities are more involved than those used in the process of simplifying the effective action itself, since we have one more independent spinor, namely the SUSY transformation parameter \( \epsilon_\alpha \). Below we give a sketch of how such cancellations take place, choosing the case involving \( \tilde{\Gamma}^{(1)}_{\theta^3} \) as an example.

For the structure with one \( \epsilon \) and 3 \( \theta \)'s, what we wish to prove is

\[ (\delta \tilde{\Gamma})_{\epsilon,\theta^3} = \delta_{r}^{(0)} \tilde{\Gamma}_{\theta^3}^{(1)} + \delta_{Y}^{(0)} \tilde{\Gamma}_{\theta^3}^{(1)} + \delta_{m}^{(0)} \tilde{\Gamma}_{\theta^3}^{(1)} + \delta_{\theta}^{(1)} \tilde{\Gamma}_{\theta^3}^{(0)} = 0. \quad (4.39) \]

Substituting the SUSY transformation laws and bring the result into certain standard forms by integration by parts, \( \delta \tilde{\Gamma} \) turned out to contain 40 terms. They are classified into two groups, one with 3 free indices and the rest with 1 free index. We then apply three independent 3-free-index Fierz identities to reduce the terms in the first group down to 1-free-index type. Since the identities used are somewhat complicated, let us display only
one of them as an example:

\[
(e_{\gamma^{a_1k}} \dot{\theta}) (\theta \gamma^{a_1ij} \theta) = - (e_{\gamma^{a_1jk}} \dot{\theta}) (\theta \gamma^{a_1i} \theta) + (e_{\gamma^{a_1ik}} \dot{\theta}) (\theta \gamma^{a_1j} \theta) + (e_{\gamma^{a_1i}} \theta) (\theta \gamma^{a_1j} \theta) - (e_{\gamma^{a_1i}} \theta) (\theta \gamma^{a_1j} \theta) + 4(e_{\gamma^{ij}} \dot{\theta} \dot{\gamma}^{ik} \theta) + 2(e_{\gamma^{ik}} \dot{\theta} \dot{\gamma}^{ij} \theta) + 2(e_{\gamma^{ik}} \dot{\theta})(\theta \gamma^{i j} \theta) - 2(e_{\gamma^{ij}} \theta)(\dot{\theta} \dot{\gamma}^{ik} \theta) - 2(\dot{\theta}) (e_{\gamma^{ij}} \theta) \delta_{ik} + (e_{\gamma^{ai}} \theta) (\theta \gamma^{a_1j} \theta) \delta_{ik} + 2(\dot{\theta}) (e_{\gamma^{ij}} \theta) \delta_{jk} - (e_{\gamma^{ai}} \theta)(\theta \gamma^{a_1j} \theta) \delta_{jk},
\]

(4.40)

After this reduction, \(\delta \tilde{\Gamma}\) consists of 29 terms, all with one free index only. To this we apply the following relatively simple 1-free-index Fierz identities:

(i) \((e_{\gamma^{ai}} \dot{\theta}) (\theta \gamma^{a_1i} \theta) = 2(\dot{\theta}) (e_{\gamma^{ai}} \theta) - (e_{\gamma^{a_1i}} \theta) (\theta \gamma^{ai} \theta) - 2(e_{\gamma^{ai}} \theta) (e_{\gamma^{a_1i}} \theta),\)

(ii) \((e_{\gamma^{ai}} \dot{\theta} \dot{\gamma}^{a_1i} \theta) = 2(\dot{\theta}) (e_{\gamma^{a_1i}} \theta) - (e_{\gamma^{a_1i}} \theta) (\theta \gamma^{ai} \theta) - 2(\dot{\theta}) (e_{\gamma^{a_1i}} \theta),\)

(iii) \((e_{\gamma^{a_1i}} \theta) (\dot{\theta} \gamma^{a_1i} \dot{\theta}) = -2(\dot{\theta}) (e_{\gamma^{a_1i}} \theta) + (e_{\gamma^{a_1i}} \theta) (\dot{\theta} \gamma^{a_1i} \theta) + 2(\dot{\theta}) (e_{\gamma^{a_1i}} \theta),\)

(iv) \((e_{\gamma^{a_1i}} \theta) (\dot{\theta} \gamma^{a_1i} \dot{\theta}) = -2(\dot{\theta}) (e_{\gamma^{a_1i}} \theta) - 2(e_{\gamma^{a_1i}} \theta) (\dot{\theta} \gamma^{a_1i} \theta) - 6(e_{\gamma^{a_1i}} \theta) (\dot{\theta} \gamma^{a_1i} \theta),\)

(v) \((e_{\gamma^{a_1i}} \theta) (\dot{\theta} \gamma^{a_1i} \dot{\theta}) = -2(e_{\gamma^{a_1i}} \theta) (\dot{\theta} \gamma^{a_1i} \dot{\theta}).\)

Then \(\delta \tilde{\Gamma})_{e, \theta^a}\) can be brought to the following form, in which the 4 fermions involved are all different:

\[
(\delta \tilde{\Gamma})_{e, \theta^a} = \int d\tau \left( \frac{15 i r_{i_1} (\dot{\theta} \gamma^{a_1i} \theta)}{16 r^7} + \frac{15 i r_{i_1} (\dot{\theta} \gamma^{a_1i} \theta)}{16 r^7} - \frac{5 i r_{i_1} (e_{\gamma^{a_1i}} \theta) (\dot{\theta} \gamma^{a_1i} \theta)}{16 r^7} \right.
\]

\[
+ \frac{5 i r_{i_1} (e_{\gamma^{a_1i}} \theta)}{32 r^7} + \frac{5 i r_{i_1} (e_{\gamma^{a_1i}} \theta)}{16 r^7} + \frac{5 i r_{i_1} (e_{\gamma^{a_1i}} \theta)}{16 r^7} + \frac{5 i r_{i_1} (e_{\gamma^{a_1i}} \theta) (\dot{\theta} \gamma^{a_1i} \dot{\theta})}{16 r^7} \right).
\]

Finally, by a Fierz identity of the type described in the Appendix A, this vanishes identically.
The vanishing of \((\delta \tilde{\Gamma})_{\kappa,\theta}\) can be proved similarly with much less effort. On the other hand, for \((\delta \tilde{\Gamma})_{\kappa,\theta^5}\) and \((\delta \tilde{\Gamma})_{\kappa,\theta^7}\) we encounter a considerable difficulty: We needed up to 5-free-index identities which are in general quite complicated\(^3\) and moreover for expressions with more than six spinors there are many possibilities to choose the four among them to which to apply the Fierz. If the choice of which identity to use or which spinors to act on is not appropriate, the result immediately becomes more complicated, by dozens of terms, than the one prior to the application. For these reasons, even after considerable amount of trial and error efforts we could not fully identify all the Fierz identities responsible for the vanishing of the variation. To overcome this difficulty, we took the following strategy: First simplify the expression as much as possible by the judicious application of various Fierz identities. Then, when it gets reduced to a sufficiently simple form, check if it vanishes “numerically” by the use of explicit representations of the \(SO(9)\) \(\gamma\)-matrices. In this way, we succeeded in checking the full SUSY invariance of our effective action.

5 Summary and Discussions

In this paper, as a necessary step toward clarification of the power of supersymmetry in Matrix theory, we have performed a complete calculation of the 1-loop effective action for arbitrary off-shell trajectory and spin degrees of freedom for a probe D-particle at order 4 in the derivative expansion. Although the calculation was quite involved in the intermediate stages, the result presented in (3.26) \((3.29)\) turned out to be remarkably simple. Further, for an independent check of this result as well as for its own interest, we have computed the quantum-corrected SUSY transformation laws to the appropriate order based on the SUSY Ward identity developed previously[14]. This computation was again extremely cumbersome and we obtained rather complicated results shown in Appendix C. To test the invariance of the effective action under these transformations, it was crucial to apply a series of judicious Fierz identities, many of which had not been known. We developed a new efficient algorithm and generated the necessary identities. With the aid of these identities and some numerical computations, we succeeded in checking the desired invariance.

An obvious important question about our result for the effective action is whether it agrees with the supergravity calculation. Unfortunately, at present time this question cannot be answered for the following reason. A relevant supergravity calculation was performed in [20] up to \(O(\theta^2)\), adapting a technique developed for the case of 11-dimensional supermembrane in [32] to the same order. Subsequently it was argued in [19] that by

\(^3\)The longest 5-free-index identity we used consists of 109 terms.
appropriate re-definitions of the fields $r_m(\tau)$ and $\theta_\alpha(\tau)$ such an effective action can be made to agree with the Matrix theory calculation to that order\textsuperscript{[4]}. In order to check our complete result including all the spin effects at order 4, one needs a calculation on the supergravity side up to $O(\theta^8)$. Judging from the existing calculation at $O(\theta^2)$ level\textsuperscript{[32]}, which contains some ambiguities left unresolved, this appears to be quite a difficult task. Thus, assuming that the agreement with the supergravity calculation would persist, our result stands as a prediction (up to field re-definitions) until such a challenge will have been met.

We conclude by recalling the prime motivation that prompted this work, namely the study of the power of supersymmetry in Matrix theory. What is most curious is to see how much of the highly non-trivial yet remarkably simple structure of the completely off-shell order 4 effective action we computed is determined by supersymmetry alone. This requires enumerating the most general form of the effective action at this order and study how the coefficients of independent structures are restricted by the requirement of SUSY invariance. Such a work is now underway\textsuperscript{[33]} and we hope to communicate the result elsewhere.

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\textsuperscript{4} These re-definitions were not unique since any difference at $O(\theta^4)$ was neglected.
Appendix A: A new efficient algorithm for generating $SO(9)$ Fierz identities

In this appendix, we describe a new efficient algorithm for generating the $SO(9)$ Fierz identities and make a remark on the existence of a class of identities which are often overlooked.

Let us define two types of four fermion structures $f^n_m$ and $g^n_m$, with $n$ indices contracted and $m$ indices free, as follows:

\[ f^n_m = (\lambda_1^T \gamma^{a_1 \ldots a_n i_1 \ldots i_k} \lambda_2) (\lambda_3^T \gamma^{a_n \ldots a_1 i_{k+1} \ldots i_m} \lambda_4), \tag{A.1} \]
\[ g^n_m = (\lambda_2^T \gamma^{a_1 \ldots a_n i_1 \ldots i_k} \lambda_3) (\lambda_4^T \gamma^{a_n \ldots a_1 i_{k+1} \ldots i_m} \lambda_1). \tag{A.2} \]

The generic Fierz identities relate these two types of fermion bilinears. (In general, some of the free indices may be carried by factors of Kronecker $\delta$'s such as $\delta^{i_1 i_2}$, etc.) In [7] a systematic procedure to generate such identities was described. A part of this algorithm requires generation of tensor product identities for $\gamma$-matrices and this turned out to be progressively time-consuming as the number of free indices increases. Besides, the set of Fierz identities so obtained are highly redundant, including the repetition of relations already obtained for lower number of free indices.

The basic idea of our new algorithm is to make full use of the Fierz identities for $m$-free-index in the calculation of the $(m+1)$-free-index case, i.e. it is an inductive algorithm. Suppose we already have all the identities with up to $m$ free indices. Concentrating on the $\gamma$-matrix structure, we have $m+1$ $f$-type structures of the form

\[ \gamma^{a_1 \ldots a_n i_1 \ldots i_k} \gamma^{a_n \ldots a_1 i_{k+1} \ldots i_m} \]  

expressed in terms of the $g$-type structures. Now we try to add another free index $j$ to this relation. Clearly there are 4 places to insert $\gamma^j$. So we generate $4(m+1)$ relations. This is considerably smaller than $4^{m+1}$ relations generated by the method of [7].

For example, adding $\gamma^j$ to the left-most position, we get, in an obvious tensor product notation,

\[
\begin{align*}
&\gamma^j \gamma^{a_1 \ldots a_n i_1 \ldots i_k} \otimes \gamma^{a_n \ldots a_1 i_{k+1} \ldots i_m} \\
&= (-1)^n \gamma^{a_1 \ldots a_n j i_1 \ldots i_k} \otimes \gamma^{a_n \ldots a_1 i_{k+1} \ldots i_m} \\
&\quad + n \gamma^{a_1 \ldots a_n-1 i_1 \ldots i_k} \otimes \gamma^{a_{n-1} \ldots a_1 j i_{k+1} \ldots i_m} \\
&\quad + \sum_l (-1)^{n+l-1} \delta^{ji_l} \gamma^{a_1 \ldots a_n i_l \ldots i_k} \otimes \gamma^{a_n \ldots a_1 i_{k+1} \ldots i_m},
\end{align*}
\]

\(^5\text{Labels which distinguish the various ways the free indices are distributed are suppressed.}\)
where \( \hat{i}_l \) means that it is deleted. The first term contains \( m + 1 \) free indices, while the second sum contains structures with \( m - 1 \) indices, which can be reduced into \( g \)-type structures by the relations already computed.

Now on the right hand side of the original \( m \)-free-index Fierz relations, we have \( g \)-type structures of the form (** stands for a set of indices)

\[
\sum C_{\gamma bc}^{**} \gamma_{da}^{**}.
\]  

(A.5)

When \( \gamma^j \) is added in the manner above, this becomes

\[
\sum C_{\gamma bc}^{**} (\gamma^{**} \gamma^j)_{da},
\]  

(A.6)

which can easily be computed. This produces \( g \)-type structures with one more or one less free indices on the \( \gamma \)'s. Equating the left and the right hand sides, we produce a \( (m + 1) \)-free-index identity.

Now we make a cautionary remark in enumerating all possible Fierz identities when all the spinors involved are distinct. The Fierz identities discussed so far (and in [7]) are the ones where the order of the spinors are cyclically rotated. Schematically, the relation is of the type \( (\lambda_1 \gamma \lambda_2) (\lambda_3 \gamma \lambda_4) \rightarrow (\lambda_2 \gamma \lambda_3) (\lambda_4 \gamma \lambda_1) \). Further cyclic rotation of course does not produce new identities. However, there is an additional structure \( (\lambda_1 \gamma \lambda_3) (\lambda_2 \gamma \lambda_4) \), which cannot be reached by cyclic rotations from the original. Therefore, one must also add the transformation of the type \( (\lambda_1 \gamma \lambda_2) (\lambda_3 \gamma \lambda_4) \rightarrow (\lambda_1 \gamma \lambda_3) (\lambda_2 \gamma \lambda_4) \). One should further be aware that in general these two classes of identities may contain redundancy. So the correct procedure is to generate all the identities of both classes and then re-solve them as coupled equations to find the truly independent and complete set of identities. For example, the identity which states the vanishing of (4.41) was obtained only through this procedure.

In this way, starting from the 0-free-index identities given in [7] we have generated all the independent Fierz identities up to and including 4 free indices and a part of 5-free-index ones. Unfortunately, the result is too space-filling to be displayed in this article.

**Appendix B: Two-point functions needed for the calculation of SUSY transformation laws**

We list the non-trivial two-point functions which serve as the basic elements in the calculation of SUSY transformation laws. In the expressions below, \( \Xi_{\alpha M} \) and \( \Phi_M \) are the \((1 + 9)\)-component fermionic and bosonic vectors introduced in (3.14), and \( D_{BMN}^{-1} \) and \( D_{Fab}^{-1} \) are the bosonic and fermionic propagator matrices described in (3.19) and (3.21).
respectively, with the component indices $M, N$ etc. (running from 0 to 9) displayed explicitly for clarity. Further, $D_{B}^{\dagger-1}$ and $D_{F}^{\dagger-1}$, which were not explicitly given in the main text, are defined as

$$D_{BNM}^{\dagger-1} = \begin{pmatrix} \Delta^{-1} & 2iv^m \\ -2iv^n & \Delta^{-1}\delta_{nm} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \Delta(1 - 4v^\ell \Delta v^\delta) & -2i\Delta(1 - 4v^\ell \Delta v^\delta)^{-1}v^m \Delta \\ 2i\Delta v^n \Delta(1 - 4v^\ell \Delta v^\delta) & \Delta(\delta_{nm} - 4v^n \Delta v^m \Delta)^{-1} \end{pmatrix}, \quad (B.1)$$

$$D_{F}^{\dagger-1} = (\partial - \partial_{ij})^{-1} = -(\partial + \partial_{ij})(1 - \Delta \delta_{ij})^{-1} \Delta. \quad (B.2)$$

With these notations, the basic two point functions are given by

$$\langle \Phi_{M}^{\ast}(\tau)\Phi_{N}(\tau') \rangle = \langle \tau | (\delta_{ML} - D_{B}^{\dagger-1} \Xi_{K}D_{F}^{\dagger-1} \Xi_{L})^{-1}D_{BN}^{\dagger-1} | \tau' \rangle, \quad (B.3)$$

$$\langle \Psi_{\alpha}^{\ast}(\tau)\Phi_{M}(\tau') \rangle = \langle \tau | (\delta_{ML} - D_{B}^{\dagger-1} \Xi_{K}D_{F}^{\dagger-1} \Xi_{L})^{-1}D_{BL}^{\dagger-1} \Xi_{P}D_{F}^{\dagger-1} | \tau' \rangle, \quad (B.4)$$

$$\langle \Psi_{\alpha}^{\ast}(\tau)\Psi_{\beta}(\tau') \rangle = \langle \tau | (\delta_{\alpha \lambda} - D_{F}^{\dagger-1} \Xi_{\rho}D_{BK}^{\dagger-1} \Xi_{L})^{-1}D_{F}^{\dagger-1} | \tau' \rangle, \quad (B.5)$$

$$\langle \Phi_{N}^{\ast}(\tau)\Psi_{\beta}(\tau') \rangle = \langle \tau | (\delta_{\alpha \lambda} - D_{F}^{\dagger-1} \Xi_{\rho}D_{BK}^{\dagger-1} \Xi_{L})^{-1}D_{F}^{\dagger-1} \Xi_{P} | \tau' \rangle, \quad (B.6)$$

while their "conjugates" take the form

$$\langle \Phi_{M}(\tau)\Phi_{N}^{\ast}(\tau') \rangle = \langle \tau | (\delta_{ML} - D_{B}^{\dagger-1} \Xi_{K}D_{F}^{\dagger-1} \Xi_{L})^{-1}D_{BL}^{\dagger-1} | \tau' \rangle, \quad (B.7)$$

$$\langle \Psi_{\alpha}(\tau)\Phi_{M}^{\ast}(\tau') \rangle = \langle \tau | (\delta_{ML} - D_{B}^{\dagger-1} \Xi_{K}D_{F}^{\dagger-1} \Xi_{L})^{-1}D_{BL}^{\dagger-1} \Xi_{P}D_{F}^{\dagger-1} | \tau' \rangle, \quad (B.8)$$

$$\langle \Psi_{\alpha}(\tau)\Psi_{\beta}^{\ast}(\tau') \rangle = \langle \tau | (\delta_{\alpha \lambda} - D_{F}^{\dagger-1} \Xi_{\rho}D_{BK}^{\dagger-1} \Xi_{L})^{-1}D_{F}^{\dagger-1} | \tau' \rangle, \quad (B.9)$$

$$\langle \Phi_{N}(\tau)\Psi_{\beta}^{\ast}(\tau') \rangle = \langle \tau | (\delta_{\alpha \lambda} - D_{F}^{\dagger-1} \Xi_{\rho}D_{BK}^{\dagger-1} \Xi_{L})^{-1}D_{F}^{\dagger-1} \Xi_{P} | \tau' \rangle. \quad (B.10)$$

Of course in the actual calculation, we must expand them to the appropriate order in the derivative expansion.
Appendix C: Quantum-corrected SUSY transformation laws

In this appendix we display the 1-loop corrections to the supersymmetry transformation laws needed for checking the invariance of the order 4 effective action. The results are classified according to the number of $\theta$’s in excess of the tree-level laws.

**Corrections at $O(\theta^0)$:**

\[
\Delta r^m = - \frac{5 i g^2 v_i v_m (\epsilon \gamma^i \theta)}{2 r^7} + \frac{5 i g^2 v^2 (\epsilon \gamma^m \theta)}{4 r^7} - \frac{35 i g^2 (r \cdot v)^2 (\epsilon \gamma^m \theta)}{8 r^9} + \frac{5 i g^2 (r \cdot a) (\epsilon \gamma^m \theta)}{8 r^9} + \frac{5 i g^2 (r \cdot v) (\epsilon \gamma^m \theta)}{8 r^9} - \frac{i g^2 (\epsilon \gamma^m \theta)}{4 r^5}. \tag{C.1}
\]

\[
\Delta \theta^i = - \frac{5 i g^2 v^2 v_i (\epsilon \gamma^i \theta)}{4 r^7} - \frac{i g^2 \dot{a}^i (\epsilon \gamma^i \theta)}{4 r^5} + \frac{5 i g^2 a_i (r \cdot v) (\epsilon \gamma^i \theta)}{4 r^7} - \frac{35 i g^2 v_i (r \cdot v)^2 (\epsilon \gamma^i \theta)}{8 r^9} + \frac{5 i g^2 v_i (r \cdot a) (\epsilon \gamma^i \theta)}{4 r^7}. \tag{C.2}
\]

**Corrections at $O(\theta^2)$:**

\[
\Delta r^m = - \frac{15 i g^2 (\dot{\theta} \theta) (\epsilon \gamma^m \theta)}{32 r^7} + \frac{35 i g^2 r_m v_i (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta)}{64 r^9} - \frac{105 i g^2 r_i v_m (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta)}{64 r^9} + \frac{35 i g^2 r_i v_j (\epsilon \gamma^m \theta) (\theta \gamma^{ij} \theta)}{64 r^9} + \frac{35 i g^2 (r \cdot v) (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta)}{64 r^9} + \frac{5 i g^2 (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta)}{32 r^7} - \frac{5 i g^2 (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta)}{32 r^7} - \frac{105 i g^2 r_i v_j (\epsilon \theta) (\theta \gamma^{ij} \theta)}{64 r^9} - \frac{35 i g^2 r_i v_j (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta)}{64 r^9} + \frac{105 i g^2 r_i v_j (\epsilon \theta) (\theta \gamma^{ij} \theta)}{64 r^9} + \frac{105 i g^2 r_i v_j (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta)}{64 r^9} - \frac{15 i g^2 (\epsilon \theta) (\dot{\theta} \gamma^m \theta)}{32 r^7} - \frac{5 i g^2 (\epsilon \gamma^j \theta) (\dot{\theta} \gamma^{ij} \theta)}{32 r^7}. \tag{C.3}
\]
\[ \Delta \theta_\alpha = - \frac{175 \, i \, g^2 \, r_i \, v_j \, v_k \, (\theta \gamma^{jk} \theta) \, (\gamma^i_\alpha)}{64 \, r^9} - \frac{5 \, i \, g^2 \, v_i \, (\hat{\theta} \gamma^{ij} \theta) \, (\gamma^i_\alpha)}{8 \, r^7} + \frac{35 \, i \, g^2 \, r_i \, (r \cdot v) \, (\hat{\theta} \gamma^{ij} \theta) \, (\gamma^i_\alpha)}{16 \, r^9} \\
+ \frac{5 \, i \, g^2 \, r_i \, (\hat{\theta} \gamma^{ij} \theta) \, (\gamma^i_\alpha)}{32 \, r^7} - \frac{5 \, i \, g^2 \, r_i \, (\hat{\theta} \gamma^{ij} \theta) \, (\gamma^i_\alpha)}{16 \, r^7} - \frac{15 \, i \, g^2 \, v_i \, (\hat{\theta} \gamma^{ij} \theta) \, (\gamma^i_\alpha)}{32 \, r^7} \\
+ \frac{15 \, i \, g^2 \, v_i \, (\hat{\theta} \gamma^{ij} \theta) \, (\gamma^i_\alpha)}{64 \, r^7} + \frac{35 \, i \, g^2 \, r_i \, (r \cdot v) \, (\hat{\theta} \gamma^{ij} \theta) \, (\gamma^i_\alpha)}{32 \, r^9} + \frac{5 \, i \, g^2 \, r_i \, (\hat{\theta} \gamma^{ij} \theta) \, (\gamma^i_\alpha)}{64 \, r^7} \\
+ \frac{5 \, i \, g^2 \, r_i \, (\hat{\theta} \gamma^{ij} \theta) \, (\gamma^i_\alpha)}{32 \, r^7} + \frac{105 \, i \, g^2 \, r_i \, v_j \, v_k \, (\theta \gamma^{ikl} \theta) \, (\gamma^j_\alpha)}{64 \, r^9} + \frac{105 \, i \, g^2 \, v^2 \, r_i \, (\epsilon \gamma^j \theta) \, \theta_\alpha}{32 \, r^9} \\
- \frac{105 \, i \, g^2 \, a_i \, (\epsilon \gamma^i \theta) \, \theta_\alpha}{32 \, r^7} + \frac{245 \, i \, g^2 \, v_i \, (r \cdot v) \, (\epsilon \gamma^i \theta) \, \theta_\alpha}{16 \, r^9} - \frac{315 \, i \, g^2 \, r_i \, (r \cdot v)^2 \, (\epsilon \gamma^i \theta) \, \theta_\alpha}{8 \, r^{11}} \\
+ \frac{35 \, i \, g^2 \, r_i \, (r \cdot a) \, (\epsilon \gamma^i \theta) \, \theta_\alpha}{4 \, r^9} - \frac{15 \, i \, g^2 \, v_i \, (\epsilon \gamma^i \theta) \, \theta_\alpha}{32 \, r^7} + \frac{35 \, i \, g^2 \, r_i \, (r \cdot v) \, (\epsilon \gamma^i \theta) \, \theta_\alpha}{16 \, r^9} \\
- \frac{5 \, i \, g^2 \, r_i \, (\epsilon \gamma^i \theta) \, \theta_\alpha}{16 \, r^7} + \frac{105 \, i \, g^2 \, v^2 \, r_i \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{64 \, r^9} + \frac{25 \, i \, g^2 \, a_i \, (\epsilon \gamma^i \theta) \, (\gamma^j_\alpha)}{32 \, r^7} \\
- \frac{175 \, i \, g^2 \, v_i \, (r \cdot v) \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{32 \, r^9} + \frac{15 \, i \, g^2 \, v_i \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{32 \, r^7} - \frac{35 \, i \, g^2 \, r_i \, (r \cdot v) \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{16 \, r^9} \\
+ \frac{5 \, i \, g^2 \, r_i \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{16 \, r^7} - \frac{105 \, i \, g^2 \, v^2 \, r_i \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{64 \, r^9} + \frac{5 \, i \, g^2 \, a_i \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^9} \\
- \frac{35 \, i \, g^2 \, v_i \, (r \cdot v) \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^9} + \frac{5 \, i \, g^2 \, v_i \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^9} + \frac{35 \, i \, g^2 \, r_i \, (r \cdot v) \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^9} \\
- \frac{5 \, i \, g^2 \, r_i \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^9} - \frac{105 \, i \, g^2 \, r_j \, v_k \, (\epsilon \gamma^j \theta) \, (\gamma^{ik} \alpha)}{32 \, r^9} - \frac{5 \, i \, g^2 \, v_j \, (\epsilon \gamma^j \theta) \, (\gamma^{ik} \alpha)}{4 \, r^7} \\
+ \frac{35 \, i \, g^2 \, r_i \, (r \cdot v) \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{16 \, r^9} - \frac{5 \, i \, g^2 \, r_i \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{16 \, r^7} + \frac{5 \, i \, g^2 \, v_j \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{4 \, r^7} \\
- \frac{35 \, i \, g^2 \, r_i \, (r \cdot v) \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{16 \, r^9} + \frac{5 \, i \, g^2 \, r_i \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{16 \, r^7} - \frac{5 \, i \, g^2 \, v_j \, (\epsilon \gamma^i \theta) \, (\gamma^i_\alpha)}{16 \, r^7} \\
+ \frac{35 \, i \, g^2 \, r_i \, (r \cdot v) \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^9} - \frac{5 \, i \, g^2 \, r_i \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^7} - \frac{5 \, i \, g^2 \, r_j \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^7} \\
+ \frac{5 \, i \, g^2 \, r_i \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^7} - \frac{5 \, i \, g^2 \, r_i \, (\epsilon \gamma^j \theta) \, (\gamma^j_\alpha)}{16 \, r^7} \cdot \text{(C.4)} \]
Corrections at $\mathcal{O}(\theta^4)$:

$$
\Delta r^m = - \frac{63 i g^2 r_i r_j (\epsilon \gamma^k \theta) (\theta \gamma^{im} \theta) (\theta \gamma^{jk} \theta)}{128 r^{11}} + \frac{7 i g^2 (\epsilon \gamma^i \theta) (\theta \gamma^{ij} \theta) (\theta \gamma^{jm} \theta)}{128 r^9} - \frac{63 i g^2 r_i r_j (\epsilon \gamma^k \theta) (\theta \gamma^{im} \theta) (\theta \gamma^{jk} \theta)}{128 r^{11}} - \frac{63 i g^2 r_i r_j (\epsilon \theta) (\theta \gamma^{ik} \theta) (\theta \gamma^{jm} \theta)}{128 r^{11}}.
$$

$$
\Delta \theta^\alpha = - \frac{63 i g^2 r_i r_j v_k (\theta \gamma^{il} \theta) (\theta \gamma^{jk} \theta) (\gamma^i \epsilon)_\alpha}{256 r^{11}} - \frac{7 i g^2 v_i (\theta \gamma^{ik} \theta) (\theta \gamma^{jk} \theta) (\gamma^j \epsilon)_\alpha}{256 r^9} + \frac{63 i g^2 r_i (r \cdot v) (\theta \gamma^{ik} \theta) (\theta \gamma^{jk} \theta) (\gamma^j \epsilon)_\alpha}{128 r^{11}} - \frac{63 i g^2 r_i (r \cdot v) (\theta \gamma^{km} \theta) (\theta \gamma^{km} \theta) (\gamma^k \epsilon)_\alpha}{256 r^{11}} + \frac{7 i g^2 r_i (\theta \gamma^{ik} \theta) (\theta \gamma^{jk} \theta) (\gamma^j \epsilon)_\alpha}{32 r^9} + \frac{21 i g^2 r_i (\theta \gamma^{jk} \theta) (\hat{\theta} \gamma^{ik} \theta) (\gamma^j \epsilon)_\alpha}{256 r^9} - \frac{7 i g^2 r_i (\theta \gamma^{jk} \theta) (\hat{\theta} \gamma^{ik} \theta) (\gamma^j \epsilon)_\alpha}{32 r^9} - \frac{189 i g^2 r_i r_j v_k (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{256 r^{11}} - \frac{189 i g^2 r_i r_j v_k (\theta \gamma^{im} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{256 r^{11}} - \frac{7 i g^2 r_i (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{64 r^{11}} + \frac{7 i g^2 r_i (\theta \gamma^{il} \theta) (\hat{\theta} \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{64 r^{11}} - \frac{63 i g^2 r_i r_j v_k (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{256 r^{11}} + \frac{63 i g^2 r_i r_j v_k (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{256 r^{11}} + \frac{35 i g^2 r_i (\theta \gamma^{ij} \theta) (\hat{\theta} \gamma^{ik} \theta) (\gamma^j \epsilon)_\alpha}{128 r^9} + \frac{7 i g^2 r_i (\theta \gamma^{il} \theta) (\hat{\theta} \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{64 r^9} - \frac{21 i g^2 r_i (\theta \gamma^{il} \theta) (\hat{\theta} \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{128 r^9} + \frac{189 i g^2 r_i r_j v_k (\theta \gamma^{im} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{256 r^{11}} - \frac{189 i g^2 r_i r_j v_k (\theta \gamma^{im} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{256 r^{11}} - \frac{21 i g^2 r_i (\theta \gamma^{il} \theta) (\hat{\theta} \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{128 r^9} + \frac{189 i g^2 r_i r_j v_k (\theta \gamma^{im} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{256 r^{11}} - \frac{63 i g^2 r_i r_j v_k (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{256 r^{11}} + \frac{63 i g^2 r_i r_j v_k (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\gamma^j \epsilon)_\alpha}{256 r^{11}} + \frac{35 i g^2 r_i (\hat{\theta} \gamma^{il} \theta) (\gamma^j \epsilon)_\alpha}{64 r^{11}} - \frac{7 i g^2 v_i (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta) \theta_\alpha}{128 r^9} - \frac{35 i g^2 r_i (\hat{\theta} \gamma^{il} \theta) (\gamma^j \epsilon)_\alpha}{64 r^{11}} - \frac{63 i g^2 r_i (r \cdot v) (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta) \theta_\alpha}{128 r^{11}} - \frac{35 i g^2 r_i (\hat{\theta} \gamma^{il} \theta) (\gamma^j \epsilon)_\alpha}{64 r^{11}} - \frac{63 i g^2 r_i (r \cdot v) (\epsilon \gamma^j \theta) (\theta \gamma^{ij} \theta) \theta_\alpha}{128 r^{11}}$$. 

23
\[
\begin{align*}
&+ \frac{35 i g^2 r_i (e^\gamma \theta) (\dot{\theta}^\gamma \theta)}{64 r^9} + \frac{35 i g^2 r_i (e \theta) (\dot{\theta} \theta) (\gamma^i \theta)}{64 r^9} \\
&- \frac{189 i g^2 r_i r_j v_k (e \theta) (\theta \gamma^j k \theta) (\gamma^i \theta)}{64 r^{11}} + \frac{63 i g^2 r_i r_j v_k (e \gamma^j \theta) (\theta \gamma^j k \theta) (\gamma^i \theta)}{32 r^{11}} \\
&- \frac{35 i g^2 r_i (e^\gamma \theta) (\dot{\theta}^\gamma \theta) (\gamma^i \theta)}{64 r^9} - \frac{21 i g^2 v_i (e \theta) (\theta \gamma^i \theta) (\gamma^j \theta)}{128 r^9} \\
&- \frac{63 i g^2 r_i (r \cdot v) (e \theta) (\theta \gamma^i \theta) (\gamma^j \theta)}{32 r^{11}} - \frac{7 i g^2 r_i (e \theta) (\theta \gamma^i \theta) (\gamma^j \theta)}{64 r^9} \\
&+ \frac{7 i g^2 r_i (e^\gamma \theta) (\dot{\theta}^\gamma \theta) (\gamma^j \theta)}{64 r^9} - \frac{7 i g^2 r_i (e^\gamma \theta) (\dot{\theta} \theta) (\gamma^j \theta)}{64 r^9} - \frac{35 i g^2 r_i (e \theta) (\dot{\theta} \gamma^j \theta) (\gamma^i \theta)}{64 r^9} \quad \text{(C.6)} \\
&- \frac{35 i g^2 r_i (e^\gamma \theta) (\dot{\theta} \gamma^j \theta) (\gamma^i \theta)}{64 r^9} - \frac{7 i g^2 r_i (e \gamma^j \theta) (\theta \gamma^j k \theta) (\gamma^k \theta)}{64 r^9} \\
&- \frac{7 i g^2 r_i (e \theta) (\dot{\theta} \gamma^j \theta) (\gamma^i \theta)}{16 r^9} - \frac{7 i g^2 v_i (e \gamma^j \theta) (\theta \gamma^j k \theta) (\gamma^k \theta)}{128 r^9} \\
&- \frac{7 i g^2 r_i (e \gamma^j \theta) (\theta \gamma^j k \theta) (\gamma^k \theta)}{128 r^9} + \frac{63 i g^2 r_i r_j v_k (e \gamma^j \theta) (\theta \gamma^j l \theta) (\gamma^i \theta)}{32 r^{11}} \\
&- \frac{63 i g^2 r_i r_j v_k (e \gamma^j \theta) (\theta \gamma^j k \theta) (\gamma^i \theta)}{128 r^{11}} + \frac{189 i g^2 r_i r_j v_k (e \gamma^j \theta) (\theta \gamma^j l \theta) (\gamma^i \theta)}{128 r^{11}} \\
&- \frac{189 i g^2 r_i r_j v_k (e \gamma^j \theta) (\theta \gamma^j k \theta) (\gamma^i \theta)}{128 r^{11}} - \frac{63 i g^2 r_i r_j v_k (e \gamma^j \theta) (\theta \gamma^j k \theta) (\gamma^i \theta)}{128 r^{11}} \\
&- \frac{63 i g^2 r_i r_j v_k (e \gamma^j \theta) (\theta \gamma^j k \theta) (\gamma^i \theta)}{128 r^{11}} \quad \text{(C.6)} \\
&+ \frac{35 i g^2 r_i (e^\gamma \theta) (\theta \gamma^i j \theta) \dot{\theta}}{64 r^9} - \frac{7 i g^2 r_i (e \theta) (\theta \gamma^i \theta) (\gamma^j \theta)}{16 r^9} \\
&- \frac{7 i g^2 r_i (e^\gamma \theta) (\theta \gamma^i j \theta) (\gamma^k \theta)}{16 r^9} - \frac{7 i g^2 r_i (e \gamma^j \theta) (\theta \gamma^j k \theta) (\gamma^i \theta)}{64 r^9}.
\end{align*}
\]
Corrections at $\mathcal{O}(\theta^6)$:

$$
\Delta_i \theta_\alpha = \\
+ \frac{21 i g^2 r_i (\theta \gamma^{il} \theta) (\theta \gamma^{jk} \theta) (\theta \gamma^{kl} \theta) (\gamma^j \epsilon)_\alpha}{1024 r^{11}} - \frac{231 i g^2 r_i r_j r_k (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{km} \theta) (\gamma^l \epsilon)_\alpha}{1024 r^{13}} \\
+ \frac{231 i g^2 r_i r_j r_k (\theta \gamma^{im} \theta) (\theta \gamma^{jmn} \theta) (\theta \gamma^{klm} \theta) (\gamma^j \epsilon)_\alpha}{1024 r^{13}} + \frac{21 i g^2 r_i (\theta \gamma^{il} \theta) (\theta \gamma^{lm} \theta) (\theta \gamma^{ikm} \theta) (\gamma^j \epsilon)_\alpha}{1024 r^{11}} \\
+ \frac{21 i g^2 r_i (\theta \gamma^{jl} \theta) (\theta \gamma^{km} \theta) (\theta \gamma^{jln} \theta) (\gamma^j \epsilon)_\alpha}{2048 r^{11}} - \frac{231 i g^2 r_i r_j r_k (\theta \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{kmn} \theta) (\gamma^l \epsilon)_\alpha}{1024 r^{13}} \\
- \frac{231 i g^2 r_i r_j r_k (\theta \gamma^{iln} \theta) (\theta \gamma^{jmn} \theta) (\theta \gamma^{kmn} \theta) (\gamma^l \epsilon)_\alpha}{2048 r^{13}} + \frac{21 i g^2 r_i (\theta \gamma^{jl} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{ikm} \theta) (\gamma^j \epsilon)_\alpha}{512 r^{11}} \\
+ \frac{21 i g^2 r_i (\epsilon \gamma^{jl} \theta) (\theta \gamma^{kl} \theta) (\gamma^j \epsilon)_\alpha}{512 r^{13}} + \frac{231 i g^2 r_i r_j r_k (\epsilon \gamma^{il} \theta) (\theta \gamma^{im} \theta) (\theta \gamma^{jln} \theta) (\gamma^k \epsilon)_\alpha}{512 r^{11}} \\
+ \frac{21 i g^2 r_i (\epsilon \gamma^{jl} \theta) (\theta \gamma^{kln} \theta) (\gamma^j \epsilon)_\alpha}{512 r^{11}} + \frac{231 i g^2 r_i r_j r_k (\epsilon \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{jln} \theta) (\gamma^k \epsilon)_\alpha}{512 r^{13}} \\
- \frac{231 i g^2 r_i r_j r_k (\epsilon \gamma^{il} \theta) (\theta \gamma^{jm} \theta) (\theta \gamma^{km} \theta) (\gamma^l \epsilon)_\alpha}{512 r^{13}} + \frac{231 i g^2 r_i r_j r_k (\epsilon \gamma^{il} \theta) (\theta \gamma^{jim} \theta) (\theta \gamma^{kmn} \theta) (\gamma^l \epsilon)_\alpha}{512 r^{13}}.
$$

(C.7)
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