INTRODUCTION

Climate change has serious impacts on countries around the world mainly because of the emissions of polluted airs, which severely affect and degrade people's life quality. In developing countries such as Asia, in order to realize their rapid economic development, more than half emissions of global greenhouse gases are generated due to the inadequate combustion of fossil fuels.\(^1,\!^2\) Meanwhile, fossil fuel combustion not only causes global warming, but also leads to energy shortages because fossil fuels are limited. In order to solve above problems perfectly, it is imperative to develop green and renewable energy. Solar energy and wind energy are two kinds of clean energy, which have been utilized in many countries. However, these energies have inevitable problems, for example, solar energy is inefficient in rainy days, and wind energy is invalid in windless days. Wave energy is an idea of promising energy for its higher energy density and ceaseless undulation. In addition, it is effective at any time and undeveloped commercially up to now.\(^3\) It is reported that wave energy has a net power between 40 and 70 kW/m, which is 4-30 times more powerful than that of wind energy.\(^4\) It is also pointed out that the development of wave energy generation will be able to effectively alleviate the power crisis and promote the sustainable development of human beings, and has important military and economic value.\(^5\) Therefore, it is significant and urgent to explore the efficient technologies of wave energy generation.

A study on a near-shore cantilevered sea wave energy harvester with a variable cross section

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Abstract
A cantilevered piezoelectric coupled energy harvester with variable cross section is proposed to obtain the electric power from the longitudinal motion of the near-shore sea wave. In order to describe the practical process of the sea wave energy harvesting, a mathematical model of the harvester is established based on Airy linear wave theory and Bessel equations to calculate the output charge, voltage, current, and power. Compared to traditional cantilever harvesters with uniform cross section, this harvesting device has a larger energy harvesting efficiency for its more uniform and bigger surface strain in the action of the sea wave. The computation results show a root mean square (RMS) of electric power of 132.6 W can be reached up with the cantilever height, the wave height, and the wavelength of 2.5, 2, and 12.5 m, respectively, which is 2.41 times of those of the traditional cantilever harvester with uniform cross section. In addition, the output powers of the harvester in tiding and in array are also discussed. This research provided a new practical and efficient method of harvesting near-shore sea wave energy to supply the electric power demand of the households near seaside.

KEYWORDS
cantilevered piezoelectric coupled harvester, longitudinal sea wave, root mean square (RMS), variable cross section

1 | INTRODUCTION

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In recent years, with the development of piezoelectric materials, piezoelectric technologies have been used widely in Microelectromechanical Systems (MEMS) field,\textsuperscript{5–9} sensing,\textsuperscript{10–12} vibration control,\textsuperscript{13,14} damage detection,\textsuperscript{15–18} and structural health monitoring.\textsuperscript{19–21} However, only a few reports can be seen from the research papers on the piezoelectric energy harvesting in large-scale vibration and large power generation.\textsuperscript{22–25} Actually, in the three mains of energy conversion, the piezoelectric conversion is an optimal choice to be used for energy harvesting since it has a higher energy conversion density than that of electrostatic conversion and electromagnetic conversion.\textsuperscript{26} Therefore, some scholars have explored the application of piezoelectric materials in energy harvesting including large power generation.\textsuperscript{27–29} Due to simple structure and cushy integration, piezoelectric coupled cantilever structures (unimorph and bimorph), as one of the most popular energy harvesters, are studied by many scholars and applied in a wide range of fields.\textsuperscript{30,31} The following literatures focus on the research of cantilever-style energy harvesters for the same kind of study object in this research. Hwang et al\textsuperscript{32} designed a piezoelectric coupled cantilevered harvesting device with a magnet proof mass, which is excited by a moving ball in the cylinder fixed on the cantilever tip. The research findings indicate that this design can effectively harvest ocean wave with low frequency. Xie et al\textsuperscript{33} proposed a wave collecting device made of two piezoelectric coupled cantilever board with dimensions of 1.2 m × 1.0 m × 0.007 m to harvest the sea wave energy in transversal direction, which can generate 30 W electric power in the wave condition near seashore. In the sea wave motion, when the wavelength is more than two times of the sea depth, the wave energy in the longitudinal direction is more than that in the transversal direction, in view of which Xie et al\textsuperscript{34} also developed a piezoelectric coupled cantilevered harvester atop fixed a proof mass and its computation model for the output of electrical parameters from the longitudinal sea wave energy. The computation research shows that this harvester can generate a power up to 55 W for an actual sea condition with a sea depth of 3 m, a wave height of 2 m, and a wavelength of 15 m. Erturk et al\textsuperscript{35} proposed a bimorph cantilever piezoelectric energy harvester with a tip mass and validated its effectiveness of output electric power by experimental method. Na et al\textsuperscript{36} presented a bimorph cantilever harvester working in a steady and turbulent sea current. A multiphysical analysis was carried out to estimate the output performance of the harvester, which was verified by experiments.

Although the aforementioned harvesters have a certain of energy harvesting effect, the piezoelectric cantilever structures just can harvest vibration energy efficiently at a certain frequency, and the effective bandwidth of harvesting energy is narrow. Therefore, it is difficult to maintain a high efficiency in variable and low-frequency environment vibrations. In view of these drawbacks, a series of multistable cantilevered harvesters were proposed to increase the energy harvesting bandwidth and convert up the excitation frequency.

Zhou et al\textsuperscript{37–39} theoretically studied a series of asymmetric tristable energy harvesters and gave their nonlinear dynamic mechanism of how to improve their energy harvesting efficiency under different excitations. Later on, Huang et al\textsuperscript{40} further researched the multistable harvesting devices by a mathematical model with high-order stiffness terms to display their working mechanism and increase their harvesting efficiency. Wang et al\textsuperscript{41} developed an impact-driven quintuple tristable energy harvester including one central cantilever with magnet proof mass and four piezoelectric coupled cantilevers with angles. This harvester has a notable energy harvesting effectiveness in a low and big bandwidth of excitation frequency for its quintuple-well potential. Shah et al\textsuperscript{42} proposed a bistable nonlinear piezoelectric energy harvester excited by magnetic repulsion between a proof mass magnet atop a cantilever and an external magnet. Its energy harvesting property has been researched by numeral and experimental methods. Zhou et al\textsuperscript{43} developed a bistable energy harvester made of a piezoelectric coupled cantilever with a proof magnet, two curved wings, and two fixed magnets, which can efficiently harvest wind energy in a wide range of wind speeds. Fu and Yeatman\textsuperscript{44} designed a piezoelectric harvester excited by the magnetic repulsion between the magnet atop the cantilever and the magnet fixed on a rotating magnet on a revolving circular plate, which is effective for low-speed rotational energy harvesting by using its frequency upconversion design. Li et al\textsuperscript{45} proposed a generalized multimode harvester of a piezoelectric coupled cantilever attached with many small cantilevers with tip masses. This harvester is efficient for low-amplitude broadband and low-frequency ambient excitations. Nabavi et al\textsuperscript{46} fixed two piezoelectric coupled cantilevers sharing one proof mass on a movable base and then installed them in the offshore buoys. This harvester is more efficient in the action of raging wave with low frequency. Usharani et al\textsuperscript{47} designed an energy harvester made of a multistage cantilever with distributional rectangular holes, which can generate electric power efficiently in the range of its inclusive 1st and 2nd modes.

For piezoelectric harvesters above, the energy harvesting efficiency is improved notably since they can effectively work in a larger range and higher excitation frequency. However, there is still a drawback that restricted the efficiency of the piezoelectric materials; namely, traditional cantilevered harvesters with a uniform cross section would just have a notable strain near its fixed end in the vibration process, which leads to an inadequate application of the positive effect of the piezoelectric materials mounted on the cantilever surfaces.

In order to solve above problem improving the energy harvesting efficiency, piezoelectric harvesters with variable cross section were proposed. Thein and Liu\textsuperscript{48} demonstrated a novel method for optimizing the bimorph cantilever harvester
to make the output power density maximum and the structural volume minimum simultaneously by a shape and then an opening hole optimization. Xie et al.\(^4\) developed a tapered shape piezoelectric energy harvesting device and the corresponding differential solution of its mathematical model, which can achieve an output power at tens of times higher than the cantilever harvesters with a constant cross section. Based on the tapered shape mathematical model, Xie et al.\(^5\) designed two composite energy harvesters made of a series of piezoelectric coupled beam with variable cross section, which was proven to have a bigger energy harvesting efficiency from ocean wave. Ramalingam et al.\(^5\) introduced an energy harvester with tapered cavity using piezoelectric mechanism and validated the output power from the analytical model through comparison with the measured results. Keshmiri et al.\(^5\) presented an analytical model for piezoelectric energy harvesting device with nonlinearly tapered FGM (functionally graded material). The research results show that it can generate an output power up to 19.76 times higher than that of the traditional uniform design. However, these studies about the tapered harvesters are based on test and/or numerical methods, which restrict the research range and the solution precision of the harvester models. In view of this, it is necessary to present an accurate mathematical model with an analytical solution for variable cross-section cantilevered energy harvester and conduct a compressive research.

Therefore, a cantilevered harvester with variable thicknesses subjected to longitudinal wave pressure is proposed and its analytical solution is developed based on Airy linear wave theory and Bessel equations. In this research, the influences of some design factors, such as wave height, wave periods, ratio of wavelength to height of the harvester, thickness ratio of fixed end to free end of the harvester, and width and height of the harvester, on the output power are discussed. In addition, the output powers of the harvester in tiding and in array are also discussed. The results show that the device has a great potential to provide the power supply for coastal residents.

2 | INTRODUCTION OF THE MATHEMATICAL MODEL

The harvester is composed of a tapered cantilever host beam and some piezoelectric patches with same surface dimensions, where the thicknesses of the fixed end and the free end and the height of the cantilever beam are \(h_0\), \(h_1\), and \(L_b\), respectively (shown in Figure 1). In order to make the best of the action of the longitudinal wave motion, \(f_H\), the highest point of the cantilever beam is set at the trough of the wave. Piezoelectric patches made of same materials have same dimensions including length of \(a\), width of \(b\), and thickness of \(h_p\). These patches are mounted one by one on both sides of the beam, which has the width of \(b\) and the thickness of \(h\) varying linearly along its height. In addition, the waves in rising tide and ebbing tide are also shown in Figure 1 with red dotted line and green dotted line, respectively.

3 | DEVELOPMENT OF THE MATHEMATICAL MODEL

3.1 | Longitudinal wave pressure exerting on the harvester

A mathematical model of the cantilevered piezoelectric energy harvester with a variable cross section is developed to study the effect of the longitudinal sea wave movement on the output power of the harvester. Based on Airy linear wave theory, which gives a linearized description of the propagation of gravity waves on the surface of a homogeneous fluid layer and assumes the fluid layer has a uniform mean depth, and the fluid is inviscid, incompressible, and irrotational, the longitudinal wave force on the tapered cantilever at the height of \(z\) can be expressed as (see Figure 1)\(^5\):

\[
f_H = \frac{1}{2} C_D \rho b \left( u_x - \frac{\partial w(z,t)}{\partial t} \right) \left| u_x - \frac{\partial w(z,t)}{\partial t} \right| + C_M \rho b a_x - \frac{b h}{c_m} \frac{\partial^2 w(z,t)}{\partial t^2}.
\]
where \( C_D, C_M, \) and \( C_m \) are respectively the resistance coefficient, the inertia force coefficient, and the additional mass coefficient of the piezoelectric coupled cantilever; \( \rho \) is the density of the sea water; \( x \) is the horizontal position of the water particle; \( w(z, t) \) is the horizontal displacement of the cantilever beam relative to its equilibrium position, in which \( z (-d \leq z \leq -H/2) \) is the vertical position of the cantilever beam and \( t \) is the time, where \( H \) and \( d = L_o + H/2 \) are the wave heights and sea depth, respectively; \( u_i = (\alpha H/T) (\cosh k(z + d)/\sinh kd) \cos(kx - \omega t) \) and \( a_i = (2\pi^2\hbar^2/H^2) (\cosh k(z + d)/\sinh kd) \sin(kx - \omega t) \) are the longitudinal velocity and acceleration of the seawater particles, in which \( T, k, \) and \( \omega \) are periods, wavenumber, and angular frequency of the sea wave, respectively. In practical working condition, \( u_i \gg \partial w(z,t)/\partial t; \) therefore, the Equation 1 can be simplified as:

\[
f_H = \frac{1}{2} C_D \rho b u_x \left| u_i \right| + C_M \rho bh a_x - C_m \rho bh \frac{\partial^2 w(z,t)}{\partial t^2}, \tag{2}
\]

Substituting the expressions of \( u_i \) and \( a_i \) into Equation 2 with re-arrangement gives,

\[
f_H = K_1(z) \cos \omega t \left| \cos \omega t \right| - K_2(z) \sin \omega t - K_3 \frac{\partial^2 w(z,t)}{\partial t^2}, \tag{3}
\]

where the coefficients are

\[
K_1(z) = \frac{1}{2} C_D \rho b (\pi^2 \hbar^2 / T^2) (\cosh k(z + d)/\sinh kd),
\]

\[
K_2(z) = C_M \rho bh (2\pi^2 \hbar^2 / T^2) (\cosh k(z + d)/\sinh kd), \text{ and}
\]

\[
K_3 = C_m \rho bh.
\]

### 3.2 Mathematical model of the harvester

Based on the Euler-Bernoulli beam theory, the governing equation of the harvester in the action of the longitudinal vibration as below:

\[
[K_3 + m(z)] \frac{\partial^2 w(z,t)}{\partial t^2} + \frac{\partial^2 w(z,t)}{\partial z^2} \left[ E I(z) \frac{\partial^2 w(z,t)}{\partial z^2} \right] = 0.
\]

In order to solve the displacement function in Equation 4, we should firstly obtain the natural frequencies and corresponding modes by solving the governing equation of its free vibration as below:

\[
[K_3 + m(z)] \frac{\partial^2 w(z,t)}{\partial t^2} + \frac{\partial^2 w(z,t)}{\partial z^2} \left[ E I(z) \frac{\partial^2 w(z,t)}{\partial z^2} \right] = 0. \tag{5}
\]

It is noted that the flexural rigidity of the piezoelectric patches is ignored in the computation because its rigidity and thickness are very small compared to those of the host beam made of stainless steel. In the design, we suppose the thickness of the host beam is \( h(z) = rz + a_1, \) where \( r \) and \( a_1 \) are a scaling factor and a constant, respectively. For the convenience of formula derivation and computation, let \( z = x_1 a_1/r x_1 \) denote a variable and changes in the range between \( a_1/r - d \) and \( a_1/r - H/2. \) Therefore, the thickness, mass, and moment of inertia of the host beam can be expressed as \( h(z) = rx_1, \ m(z) = \rho a_1 h x_1, \) and \( I(z) = \pi^3 h x_1^3 / 12, \) respectively, where \( \rho \) is the density of the host beam. Let \( w(z, t) = W(x_1, t) \) and substitute it in Equation 5, then the following equation can be obtained:

\[
x_1^2 \frac{d^4 W(x_1)}{dx_1^4} + 6x_1 \frac{d^3 W(x_1)}{dx_1^3} + 6 \frac{d^2 W(x_1)}{dx_1^2} - \lambda^2 W(x_1) = 0, \tag{6}
\]

where \( \lambda^2 = c \omega^2 \) and \( c = 12(\rho' + C_m \rho)/E^2_r, \) \( \omega \) is the natural frequency of the harvester in the unit of rad/s, and \( W(x_1) \) is the modal shape of the harvester. The solution of Equation 6 can be expressed as \( 55:
\]

\[
W(x_1) = \frac{1}{\sqrt{x_1}} [C_1J_1(2\sqrt{\lambda x_1}) + C_2Y_1(2\sqrt{\lambda x_1}) + C_3I_1(2\sqrt{\lambda x_1}) + C_4K_1(2\sqrt{\lambda x_1})], \tag{7}
\]

where \( J_1 \) is the first kind of the 1st-order Bessel function; \( Y_1 \) is the second kind of the 1st-order Bessel function; \( I_1 \) is the first kind of the 1st-order modified Bessel function; \( K_1 \) is the second kind of the 1st-order modified Bessel function; and \( C_1, C_2, C_3, \) and \( C_4 \) are the corresponding unknown coefficients.

For the fixed end of the harvester, the value of the normal mode and its first-order differential should be equal to zero, namely.

\[
\begin{cases}
W(x_1) = 0, \\
W'(x_1) = 0, \\
x_1 = a_1/r - d.
\end{cases} \tag{8}
\]

For the free end of the harvester, its 2nd- and 3rd-order differential should be zero, namely

\[
\begin{cases}
W''(x_1) = 0, \\
W'''(x_1) = 0, \\
x_1 = a_1/r - H/2.
\end{cases} \tag{9}
\]

Substituting Equation 8 and Equation 9 into Equation 7, we can have four equations of boundary conditions:

\[
\begin{align*}
C_1J_1(2\sqrt{\lambda(a_1/r - d)}) + C_2Y_1(2\sqrt{\lambda(a_1/r - d)}) \\
+ C_3I_1(2\sqrt{\lambda(a_1/r - d)}) + C_4K_1(2\sqrt{\lambda(a_1/r - d)}) &= 0.
\end{align*}\tag{10}
\]
\[ C_1 J_2(2\sqrt{\lambda(a_1/r-d)}) + C_2 Y_2(2\sqrt{\lambda(a_1/r-d)}) - C_3 J_2(2\sqrt{\lambda(a_1/r-d)}) + C_4 K_2(2\sqrt{\lambda(a_1/r-d)}) = 0, \quad (11) \]

\[ C_1 J_3(2\sqrt{\lambda(a_1/r-H/2)}) + C_2 Y_3(2\sqrt{\lambda(a_1/r-H/2)}) + C_3 J_3(2\sqrt{\lambda(a_1/r-H/2)}) + C_4 K_3(2\sqrt{\lambda(a_1/r-H/2)}) = 0, \quad (12) \]

\[ C_1 J_4(2\sqrt{\lambda(a_1/r-H/2)}) + C_2 Y_4(2\sqrt{\lambda(a_1/r-H/2)}) - C_3 J_4(2\sqrt{\lambda(a_1/r-H/2)}) + C_4 K_4(2\sqrt{\lambda(a_1/r-H/2)}) = 0, \quad (13) \]

where \( J_{2,4} \) is the first kind of the 2nd- to 4th-order Bessel function, respectively; \( Y_{2,4} \) is the second kind of the 2nd- to 4th-order Bessel function, respectively; \( I_{2,4} \) is the first kind of the 2nd- to 4th-order modified Bessel function, respectively; and \( K_{2,4} \) is the second kind of the 2nd- to 4th-order modified Bessel function, respectively. The expressions of the Bessel functions above can be seen from Appendix.

From equations above, the nth normal mode of the harvester, \( W_n \), corresponding to the nth natural angular frequency can be easily obtained.

Let's assume the displacement function of the forced vibration of the harvester as follows:

\[ w(z,t) = \sum_{n=1}^{\infty} W_n(x_1)q_n(t), \quad (14) \]

By substituting Equation (14) into Equation (4), the generalized coordinate of \( q_n \) can be obtained as:

\[ q_n(t) = \left\{ \begin{array}{ll}
\frac{-1}{\rho} \left[ J_0(\omega_1 a_1) Q_1(x_1) - J_1(\omega_1 a_1) Q_2(x_1) \right] & \text{if } \omega_1 > 0 \\
\frac{-1}{\rho} \left[ J_0(\omega_1 a_1) Q_1(x_1) - J_1(\omega_1 a_1) Q_2(x_1) \right] & \text{if } \omega_1 < 0
\end{array} \right. \quad (15) \]

where \( B = \frac{1}{\rho} \int_{x_1}^{x_1} W_n(x_1) dx_1 \),
\[ Q_{n1}(x_1) = \int_{x_1}^{x_1} \frac{1}{h\rho} \left[ K_1(x_1) W_n(x_1) dx_1 \right] \]
\[ Q_{n2}(x_1) = \int_{x_1}^{x_1} \frac{1}{h\rho} \left[ K_1(x_1) W_n(x_1) dx_1 \right] \]

### 3.3 Verification of the mathematical model

The key of the verification of the mathematical model of the harvester is the natural frequencies, which decide the validity and precision of the vibration response of the harvester, so we use the results from Rayleigh-Ritz method with 5 terms of polymerization to verify the precision of the natural frequencies calculated by the mathematical model. The results of the 1st natural frequencies of the harvesters computed by the Rayleigh-Ritz method and the mathematical model above are displayed in Table 1. From Table 1, we can see the error of the 1st natural frequency of the tapered cantilever with a length of 3, 4, and 5 m computed by this model is 1.8e-6, 1.6e-6, and 0.6e-6, respectively, compared to that computed by Rayleigh-Ritz method. The results show that this model has an idea accuracy and the accuracy will be higher with the increase in the tapered cantilever height.

#### 3.4 Computation of the output electricity of the harvester

In the vibration process of the harvester excited by longitudinal wave, the charge \( Q_g \), the voltage \( V_g \), and the current \( I_g \) generated by one of the piezoelectric patches are derived as below:

\[ Q_g = -\frac{e_{31} b}{2} \int_z^{z+a} \frac{d^2 w(z,t)}{dx^2} (h+h_p) dx, \quad (16) \]

\[ V_g = \frac{Q}{C_V} = -\frac{e_{31}}{2C_V} \int_z^{z+a} \frac{d^2 w(z,t)}{dx^2} (h+h_p) dx, \quad (17) \]

\[ I_g = \frac{dQ}{dt} = -\frac{e_{31} b q'(t)}{2} \int_z^{z+a} \frac{d^2 W(z)}{dx^2} (h+h_p) dx, \quad (18) \]

where \( e_{31} \) is the piezoelectric constant; \( C_V \) is the capacitance of the piezoelectric patches; \( C_i \) is the capacitance per unit width of piezoelectric patches; and \( q'(t) \) is the 1st-order differential of the generalized coordinate of \( q(t) \).

Now, by substituting Equation 14 into Equations 16-18 and replacing the variable of \( z \) with \( x_1 \), the charge \( Q_g \), the voltage \( V_g \), and the current \( I_g \) generated by the jth piezoelectric patch mounted on the beam surface at time \( t \) in the action of longitudinal wave can be obtained as:

\[ Q_g(t) = -\frac{e_{31} b}{2} \sum_{n=1}^{\infty} q_n(t) \int_{x_1}^{x_1} \frac{d^2 W_n(x_1)}{dx^2} (r_1+h_p) dx_1, \quad (19) \]

| \textbf{The parameters} | \textbf{L}_a = 3 \text{ m} | \textbf{L}_a = 4 \text{ m} | \textbf{L}_a = 5 \text{ m} |
|-------------------------|------------------|------------------|------------------|
| Rayleigh-Ritz solutions | 44.2293          | 24.8789          | 15.9222          |
| Present solutions      | 44.2285          | 24.8785          | 15.9223          |
| Tolerance              | 1.8e-5           | 1.6e-5           | 0.6e-5           |

Note: \( E = 2.1e11, b = 1 \text{ m}, h_0 = 0.05 \text{ m}, h_1 = 0 \text{ m}, \rho = 7800 \text{ kg/m}^3 \).
\[ V_g(t) = -\frac{e_{31}}{2C_v} \sum_{n=1}^{\infty} q_n(t) \int_{-d+a_i/r+(j-1)a}^{-d+a_i/r+j} (rx_1 + h_p) \frac{d^2W_n(x_1)}{dx_i^2} dx, \quad (20) \]

\[ I_g(t) = -\frac{e_{31} b}{2} \sum_{n=1}^{\infty} q_n(t) \int_{-d+a_i/r+(j-1)a}^{-d+a_i/r+j} (rx_1 + h_p) \frac{d^2W_n(x_1)}{dx_i^2} dx, \quad (21) \]

where \( 1 \leq j \leq m_1 \), and \( m_1 = [L_s/a] \) is the number of piezoelectric patches mounted on one surface of host beam, \( q_n(t) \) is the 1st-order derivative of the generalized coordinate of \( q_n(t) \).

Finally, the RMS of the output electric power in a wave period of \( T_i \) is given as:

\[ P_{em}^{rms} = \sqrt{\frac{1}{T_i} \int_0^{T_i} [P_e(t)]^2 dt}, \quad (22) \]

in which \( P_e(t) \) is the total power generated by the harvester at time \( t \) \( (0 \leq t \leq T_i) \), \( P_e(t) = 2 \sum_{n=1}^{m_1} I_g(t)V_g(t) \).

By using the MATLAB software, the dynamic response and the output electric energy of the harvester with variable thickness can be computed through the programs based on the mathematical model above, and the computation flowchart is shown in Figure 2.

**FIGURE 2** Dynamic response and output performance flowchart of the harvester

**TABLE 2** Sizes and material properties of the tapered piezoelectric energy harvester

| Parameters          | Host beam (steel) | Piezoelectric patches (PZT4) |
|---------------------|-------------------|-----------------------------|
| \( L_w \) (m)       | 1 ~ 2.5           |                             |
| \( h_0 \) (m)       | 0.006 ~ 0.04      |                             |
| \( h_p \) (m)       | 0.001             |                             |
| \( b \) (m)         | 0.1 ~ 1           | 0.1 ~ 1                     |
| \( a \) (m)         | 0.1               |                             |
| Young’s modulus (N/m²) | 2.1e11           | 7.8e10                      |
| Mass density (kg/m³) | 7800             | 7500                        |
| \( e_{31} \) (C/m²) |                   | −2.8                        |
| \( C_v \) (nF)      | 0.75 for the piezoelectric patch with the size of 0.01 × 0.06 × 0.0003 m³ |

**RESULTS AND DISCUSSION**

Through the established programs, the charge, voltage, and the RMS of output power generated by the harvester can be obtained. Based on the computation results, the performance of energy harvesting of the harvester was deeply analyzed through its output power in different conditions. In particular, we study the effects of the width \( (b) \), the thickness ratio \( (h_i/h_0) \) of free end to fixed end, the fixed end thickness \( (h_0) \) of the harvester, the ratio of the wavelength to the height of the harvester \( (L_w/L_b) \), the wave height \( (H) \), and the wave period \( (T) \) on the RMS of the output power. The sizes and material properties of the harvester in the computation are shown in Table 2. The harvester includes the stainless steel tapered cantilever and PZT4 (lead zirconate titanate) piezoelectric patches mounted on its surfaces. The sizes and material properties of the tapered cantilever and the PZT4 include the length, the width, the thickness, Young’s modulus, and the mass density. In addition, the piezoelectric constant \( (e_{31}) \) and the capacitance \( (C_v) \) of the piezoelectric patches are also shown in Table 2.

The 1st natural frequency of the harvester is a critical factor affecting the absorbing efficiency of sea wave energy. That is to say, if the 1st natural frequency is near the excitation frequency of the sea wave, the energy harvesting efficiency will be very high; otherwise, it is very small. For the convenience of optimization of the design parameters of the harvester for a more efficient harvesting, we first explore the influences of the changes in the thickness ratio \( (h_i/h_0) \) of free end to fixed end and the thickness \( (h_0) \) of the fixed end of the harvester on the 1st natural frequency of the harvester as shown in Figure 3. The parameters in the computation are taken as: \( L_w = 2.5 \) m, \( b = 1 \) m, \( h_p = 0.001 \) m, and \( a = 0.1 \) m. It is shown that the 1st natural frequency decreases from 6.296 rad/s to 4.885 rad/s
and 41.79 rad/s to 32.57 rad/s with the change of \( h_1/h_0 \) from 0.1 to 0.8 at \( h_0 \) of 0.006 mm and 0.04 mm, respectively. It can be observed that the change of \( h_0 \) has a more obvious effect on the 1st natural frequency than that of the change of \( h_1/h_0 \). In addition, with the increasing of \( h_0 \), the influence of \( h_1/h_0 \) on the 1st frequency increases obviously.

Figure 4 demonstrates the influence of the thickness ratio \((h_1/h_0)\) of free end to fixed end of the harvester on the RMS of output power with the dimensions as below: \( h_0 = 0.006m, h_p = 0.001m, b = 1m, L_b = 2.5m, d = 3.5m, L_w = 5L_b, H = 2m, \) and \( a = 0.1m \). From the figure, we can find the RMS of the output power of the harvester decreases first and then increases. For the harvester, the factors affecting its output power are mainly its 1st natural frequency and its surface strain. With the increase in the thickness ratio of \( h_1/h_0 \), the 1st natural frequency will decrease and approach to the sea wave frequency (shown in Figure 3), which leads to the increase in the output power (indicated in Equations 15 and 20-22); however, the surface strain of the harvester will decrease, which causes the decrease in the output power. In the case of Figure 4, when the ratio increases from 0.1 to 0.25, the RMS decreases from 104.6 W to 83.36 W because the influence of the surface strain of the harvester on the RMS is more notable than that of the ratio of \( h_1/h_0 \) in this range; when the ratio increases from 0.25 to 0.8, the RMS increases from 83.4 W to 132.6 W because in this range, the situation is opposite. This figure indicates that a very small ratio or a very big ratio is a good choice for a high efficient output power of the harvester from sea waves.

In order to further study the comprehensive effects of the thickness ratio of \( h_1/h_0 \) and the wave period on the RMS, the corresponding nephogram of the RMS of output power of the harvester is displayed in Figure 5. The parameters in the computation are taken as: \( L_b = 2.5m, h_0 = 0.006m, b = 1m, h_p = 0.001m, H = 2m, d = 3.5m, \) and \( a = 0.1m \). The computation results show that the RMS of the output power changes nonlinearly with the changes in the thickness ratio and the wave period. For instance, the RMS changes with the increase in the thickness ratio of \( h_1/h_0 \) from 0.1 to 0.8 at the period of 3 s (shown in Figure 4); however, with the increase in the wave period, the tendency of first decreasing and then increasing of the RMS along the increase of the ratio fades away, and the RMS thoroughly decreases from 5.38 W to 1.677 W with the increase in the thickness ratio of \( h_1/h_0 \) from 0.1 to 0.8 at the period of 9 s. This phenomenon can be explained as: with the increase in the wave period, the influence of the 1st natural frequency on the RMS gradually becomes weak, which leads to the decreases in the generalized coordinate and the output power (seen from Equations 15 and 20-22), and the influence of the surface strain of the harvester on the RMS becomes relatively stronger than that of the thickness ratio.

Figure 6 shows the change in the RMS of the output power of the harvester alongside the changes in the harvester height of \( L_w \) and the ratio of the wavelength to the harvester height of \( L_w/L_p \). The parameters in this simulation are taken as: \( h_0 = 0.006m, h_1/h_0 = 0.8, b = 1m, h_p = 0.001m, d = L_b+H/2, H = 2m, \) and \( a = 0.1m \). Obviously, the RMS of the obtained power increases nonlinearly with the decrease in \( L_w/L_p \) and the increase in \( L_w \). For example, when \( L_w/L_p \) changes from 10 to 5, the RMS of the output power increases from 0.20 W to 0.56 W, and 5.11 W to 132.6 W at \( h_0 \) of 1 m and 2.5 m, respectively. This is because when the height \( (L_w) \) of the harvester is set, the decrease in the ratio of \( L_w/L_p \) means the decrease in the wavelength, which would cause the increase in
The excitation frequency and further lead to the increase in
the output power. This reason can be seen from Equations
15 and 20–22. In addition, with the increase in the harvester
height, its 1st natural frequency would decrease to the ex-
citation frequency, and its vibration amplitude and surface
strain would increase, undoubtedly causing the increase in
the output power, which can be hinted from the Equations
15 and 20–22. From Figure 6, we can also observe that when
$Lw/Lb = 5$ and $Lb = 2.5 \text{ m}$, the RMS of obtained power can
reach up to 132.6 W, which can meet the demand of many
household appliances.

Figure 7 graphically displays the relationship between the
fixed end thickness ($h_0$) of the harvester and the RMS of the
output power. The parameters used in the program are set to
be: $h_p = 0.001 \text{ m}$, $b = 1 \text{ m}$, $Lb = 2.5 \text{ m}$, $d = 3.5 \text{ m}$,
$Lw = 5Lb$, $h_1/h_0 = 0.8$, and $a = 0.1 \text{ m}$. As can be seen from
the figure, the RMS of the obtained power sharply decreases
from 132.6 W to 5.391 W when the fixed end thickness in-
creases from 0.006 m to 0.008 m. This is mainly because the
harvester with the thickness in this range has a 1st natural
frequency near the wave excitation, which leads to the sen-
sitivity of the thickness change to the RMS. And the RMS
has no obvious decrease from 5.391 W to 0.043 W with the
increase in the fixed end thickness from 0.008 m to 0.02 m.
This is because along the increase in the fixed end thickness,
the flexural rigidity of the harvester will increase, which
causes its 1st natural frequency far away from the wave exci-
tation frequency and accordingly causes the small output of

Figure 8 illustrates the variation of the RMS of the output
power versus the width ($b$) of the harvester with the parame-
ter conditions below: $h_0 = 0.006 \text{ m}$, $h_p = 0.001 \text{ m}$, $Lb = 2.5 \text{ m}$,
$H = 2 \text{ m}$, $d = 3.5 \text{ m}$, $Lw = 5Lb$, $h_1/h_0 = 0.8$, and $a = 0.1 \text{ m}$. The sim-
ulation displays that the RMS of the output power raises
nonlinearly with the raise of the wave height of $H$. This phe-
nomenon indicates that the output power of the harvester
is very sensitive to the change in the wave height, which is
caused by a massive change in the wave energy. Specifically,
the RMS of the output power of the harvester increases from

Figure 9 depicts the relationship between the wave height
$H$ and the RMS of the output power with the following pa-
rameters: $h_0 = 0.006 \text{ m}$, $h_p = 0.001 \text{ m}$, $Lb = 2.5 \text{ m}$, $b = 1 \text{ m}$,
d = 3.5 m, $Lw = 5Lb$, $h_1/h_0 = 0.8$, and $a = 0.1 \text{ m}$. The sim-
ulation displays that the RMS of the output power raises
nonlinearly with the raise of the wave height of $H$. This phe-
nomenon indicates that the output power of the harvester
is very sensitive to the change in the wave height, which is
caused by a massive change in the wave energy. Specifically,
7.03 W to 132.6 W when the wave height increases from 1 m to 2 m, which means the notable enlargement of the longitudinal wave pressure $f_{H}$ (shown as Equation 3). It is worth noting that in the same sea condition, the maximum RMS of output power (132.6 W) for a tapered cantilever harvester with a length of 2.5 m and a width of 1 m is 2.41 times of that (55 W) for a uniform cantilever harvester with a length of 3 m and a width of 1 m. This is obvious that the tapered cantilever harvester is much better than the uniform cantilever harvester in sea wave energy harvesting.

Tide is a periodic movement with tremendous kinetic energy on the sea/ocean, in order to study the influence of the tide on the output power of the cantilevered harvester, some cases on ebbing tide and rising tide are considered for prediction of the output power, and the RMSs of generated power are listed in Table 3. From Table 3, we can find the RMS of the output power of the harvester decreases from 132.6 W to 93.1 W and increases from 132.6 W to 163.8 W with the tidal range from 0 m to 1 m in the process of ebbing tide and rising tide, respectively. This is because the velocity and acceleration of the tide wave increase with the increase in the tidal range of the rising tide and opposite for the ebbing tide, which can be indicated by Equation 1. These computation results also show that the cantilevered harvester is effective to harvest the near-shore sea wave energy in rising tide and ebbing tide.

From the study above, we can see that the output power from one cantilevered harvester is too small to meet the power demand of the near-shore residents. In order to harvest more electric energy, we further studied the output power of the cantilevered harvester in array (shown in Figure 10). The interval space of the harvester in array and the ratios of the RMS of the output power from each harvester in array to that from individual harvester are listed in Table 4. From Table 4, we can see that the ratio of the RMS decreases from 1.5 to 1.1 with the increase in the ratio of $l/b$ from 2 to 4 in the direction of perpendicular to the wave. However, the change in the ratio of $l/b$ from 2 to 4 has no influence on the ratio of the RMS in the direction of parallel to the wave. This tendency indicates that dense array of the harvesters in the direction of perpendicular to the wave is benefit to harvesting more wave energy because the interference effect of the wave and the arrangement of the harvesters in the direction of parallel to the wave basically have no influence on the output power of the harvester.

### Table 3: The harvested electric power at rising tide and ebbing tide

| Tidal range(m) | 0    | 0.2  | 0.4  | 0.6  | 0.8  | 1    |
|---------------|------|------|------|------|------|------|
| Ebbing tide   | 132.6| 124.7| 116.5| 108.4| 100.5| 93.1 |
| Rising tide   | 132.6| 140.1| 147.0| 153.3| 158.9| 163.8|

Note: $h_0 = 0.006$ m, $h_p = 0.001$ m, $b = 1$ m, $L_b = 2.5$ m, $H = 2$ m, $d = 3.5$ m, $L_w = 5L_b$, $h_1/h_0 = 0.8$, $a = 0.1$ m.

### Table 4: The ratio of RMS of output power from harvester in array to that from individual harvester

| R | (1,1) | (1,2) | (2,1) | (2,2) |
|---|-------|-------|-------|-------|
| l/b |      |       |       |       |
| 2  | 1.5   | 1.5   | 1.0   | 1.0   |
| 3  | 1.2   | 1.2   | 1.0   | 1.0   |
| 4  | 1.1   | 1.1   | 1.0   | 1.0   |

Note: $C$ denotes the coordinate of the cantilevered harvester; $R$ denotes the ratio of the RMS from one array harvester to that from individual harvester; $l$ denotes the distance between two interfacing harvesters; $b$ denotes the width of the harvester.

### Figure 10: The array of harvesters
energy harvester has a great future in sea wave energy harvesting near shore.

CONFLICTS OF INTEREST
The authors declare no conflicts of interest.

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APPENDIX

\[ J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(1+k)} \left( \frac{x}{2} \right)^{1+2k} ; ~ J_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2+k)} \left( \frac{x}{2} \right)^{2+2k} ; \]

\[ J_3(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(3+k)} \left( \frac{x}{2} \right)^{3+2k} ; ~ J_4(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(4+k)} \left( \frac{x}{2} \right)^{4+2k} ; \]

\[ Y_1(x) = \lim_{a \to 1} \frac{J_a(x) \cos(\alpha \pi) - J_{-a}(x)}{\sin(\alpha \pi)} ; ~ Y_2(x) = \lim_{a \to 2} \frac{J_a(x) \cos(\alpha \pi) - J_{-a}(x)}{\sin(\alpha \pi)} ; \]

\[ Y_3(x) = \lim_{a \to 3} \frac{J_a(x) \cos(\alpha \pi) - J_{-a}(x)}{\sin(\alpha \pi)} ; ~ Y_4(x) = \lim_{a \to 4} \frac{J_a(x) \cos(\alpha \pi) - J_{-a}(x)}{\sin(\alpha \pi)} ; \]

\[ I_1(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{1+2k}}{\Gamma(k+1)\Gamma(k+2)} ; ~ I_2(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2+2k}}{\Gamma(k+1)\Gamma(k+3)} ; \]

\[ I_3(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{3+2k}}{\Gamma(k+1)\Gamma(k+4)} ; ~ I_4(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{4+2k}}{\Gamma(k+1)\Gamma(k+5)} ; \]

\[ K_1(x) = \frac{1}{x} + \sum_{r=0}^{\infty} \frac{(x/2)^{1+2r}}{\Gamma(r+1)\Gamma(r+2)} \left[ \ln \frac{x}{2} + 0.5772 - \frac{1}{2} \left( \sum_{m=1}^{1+r} \frac{1}{m} + \sum_{m=1}^{r} \frac{1}{m} \right) \right] ; \]

\[ K_2(x) = -\frac{2}{x^2} - \sum_{r=0}^{\infty} \frac{(x/2)^{2+2r}}{\Gamma(r+1)\Gamma(r+3)} \left[ \ln \frac{x}{2} + 0.5772 - \frac{1}{2} \left( \sum_{m=1}^{2+r} \frac{1}{m} + \sum_{m=1}^{r} \frac{1}{m} \right) \right] ; \]

\[ K_3(x) = \frac{8}{x^2} - \frac{1}{x} + \frac{8}{x^3} + \sum_{r=0}^{\infty} \frac{(x/2)^{3+2r}}{\Gamma(r+1)\Gamma(r+4)} \left[ \ln \frac{x}{2} + 0.5772 - \frac{1}{2} \left( \sum_{m=1}^{3+r} \frac{1}{m} + \sum_{m=1}^{r} \frac{1}{m} \right) \right] ; \]

\[ K_4(x) = \frac{48}{x^4} - \frac{4}{x^3} - \frac{x^2}{48} - \sum_{r=0}^{\infty} \frac{(x/2)^{4+2r}}{\Gamma(r+1)\Gamma(r+5)} \left[ \ln \frac{x}{2} + 0.5772 - \frac{1}{2} \left( \sum_{m=1}^{4+r} \frac{1}{m} + \sum_{m=1}^{r} \frac{1}{m} \right) \right] . \]