APPROSSIMAZIONE DI \( m \)-SUBHARMONICHE FUNZIONI SULLE DOMINI LIMITATI IN \( \mathbb{C}^n \)

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Abstract. Let \( D \) be a bounded domain in \( \mathbb{C}^n \). We study approximation of (not necessarily bounded from above) \( m \)-subharmonic functions in \( D \) by continuous \( m \)-subharmonic ones defined on neighborhoods of \( \overline{D} \). We also consider the existence of a \( m \)-subharmonic function on \( D \) whose boundary values coincide with a given real valued continuous function on \( \partial D \) except for a sufficiently small subset of \( \partial D \).

1. Introduction

Subharmonic functions and plurisubharmonic functions are fundamental notions in potential theory and pluripotential theory respectively. The theory of \( m \)-subharmonic (\( m \)-sh., for short) functions was introduced and investigated thoroughly since the seminal work [Bł2]. This new class of functions encompasses the subharmonic and plurisubharmonic ones naturally. The definition of \( m \)-sh function is however a bit technical. Let \( D \) be an open subset of \( \mathbb{C}^n \), and \( u \) be a subharmonic function defined on \( D \), \( u \not\equiv -\infty \). We say that \( u \) is \( m \)-subharmonic (\( m \)-sh. for short) if the \((1,1)\) current \( dd^c u \) is \( m \)-positive in the weak sense, i.e., for \( \eta_1, \cdots, \eta_{m-1} \in \hat{\mathcal{C}}_m \) we have

\[
\forall \eta \in \mathcal{C}_{1,1} : \eta \wedge \omega^{n-1} \geq 0, \cdots, \eta^m \wedge \omega^{n-m} \geq 0.
\]

Here we define

\[
\hat{\mathcal{C}}_m := \{ \eta \in \mathcal{C}_{1,1} : \eta \wedge \omega^{n-1} \geq 0, \cdots, \eta^m \wedge \omega^{n-m} \geq 0 \},
\]

where \( \omega = dd^c |z|^2 \) is the standard Kähler form and \( \mathcal{C}_{1,1} \) denotes the space of \((1,1)\)-forms with constant coefficients. Moreover, \( u \) is said to be strictly \( m \)-sh. on \( D \) if for each relatively compact subdomain \( D' \) in \( D \) there exists a constant \( M > 0 \) such that \( u - M|z|^2 \) is \( m \)-sh. on \( D' \).

Thus, in the case \( m = 1 \) or \( m = n \) we recover the classes of subharmonic and plurisubharmonic functions respectively. We write \( \text{SH}_m(D) \) for the set of \( m \)-sh functions on \( D \). In this paper, we address the question of approximating an element \( u \in \text{SH}_m(D) \) by a sequence \( \{u_j\} \) of continuous \( m \)-sh functions on \( \text{neighborhoods of } \overline{D} \). If we ask for continuity and \( m \)-subharmonicity of \( u_j \) only on \( D \) then our problem has a satisfactory answer if \( D \) enjoys certain convexity condition. Namely we have the following result which reduces to a classical approximation theorem of Fornaess and Narasimhan in the case where \( m = n \).

Theorem 1.1. Assume that \( D \) has a \( m \)-sh. exhaustion function \( \varphi \), i.e., \( \{ \varphi < c \} \) is relatively compact in \( D \) for all \( c \in \mathbb{R} \). Then for every \( u \in \text{SH}_m(D) \), there exists a sequence \( \{u_j\} \) of smooth strictly \( m \)-sh functions on \( D \) such that \( u_j \downarrow u \) on \( D \).

On the the other hand, under the additional condition that \( u \) is bounded from above on \( D \) then the above problem may be approached by the use of Jensen measures associated to certain convex...
cones in $SH_m(D)$. The first results in this direction, in the special case $m = n$, are due to Wikström in [Wik] and then in [DW] and [Di]. In this work, we will exploit further the techniques developed in [Wik], [DW] and [Di] to work with the case where $u$ is not necessarily bounded from above on $D$. For the reader convenience, we will first sketch the general approach in the case $u \in SH_m(D)$ with $\sup_D u < \infty$. Then, after subtracting a large constant we may assume $u < 0$ on $D$. Now let $\mathcal{M}_m(D)$ be the convex cone of non-positive upper semicontinuous functions on $\overline{D}$ which are $m$–sh on $D$ and $\mathcal{M}_m^+(D)$ be the sub-cone of $\mathcal{M}_m^+(D)$ that consists of restrictions on $\overline{D}$ of continuous $m$–sh functions on neighborhoods of $\overline{D}$. Then, by a general duality theorem of Edwards we may express upper envelopes of plurisubharmonic functions taken in $\mathcal{M}_m^+(D)$, $\mathcal{M}_m^+(D)$ in terms of Jensen measures with respect to these cones. The general principle is that the approximation of elements in $\mathcal{M}_m^+(D)$ by elements in the smaller cone $\mathcal{M}_m^+(D)$ is possible when we have equality of the sets of Jensen measures with respect to these cones. In order to formulate our results properly, it is convenient to introduce the following notions pertaining to our work.

**Definition 1.2.** For a point $z \in \overline{D}$, we define below two classes of Jensen measures.

$$J_{m,z} := \{ \mu \in \mathcal{B}(\overline{D}) : u(z) \leq \int \overline{D} u d\mu, \forall u \in \mathcal{M}_m(D) \} ;$$

$$J_{m,z}^+ := \{ \mu \in \mathcal{B}(\overline{D}) : u(z) \leq \int \overline{D} u d\mu, \forall u \in \mathcal{M}_m^+(D) \};$$

where $\mathcal{B}(\overline{D})$ denotes the class of positive, regular Borel measures on $\overline{D}$.

**Remarks.** 1. If $\xi \in \partial D$ then $J_{m,z} = \{ \delta_\xi \}$. This is seen by applying Jensen’s inequality to the element $u \in \mathcal{M}_m(D)$ defined by $u(\xi) = 0$ whereas $u = -1$ on $\overline{D} \setminus \{ \xi \}$.

2. For $z \in D$, let $L$ be an affine complex subspace of dimension $n - m + 1$ passing through $z$, $\mathbb{B} \subset L$ be an open ball centered at $z$ and relatively compact in $L \cap D$. Then the normalized Lebesgue measure on $\partial \mathbb{B}$ belongs to $J_{m,z}$. This follows directly from Lemma 2.7 in the next section and the mean value inequality for subharmonic functions.

3. It is obvious that $J_{m,z} \subseteq J_{m,z}^+$. If $D$ is homogeneous, i.e., for $p, q \in D$ there exists an automorphism $\varphi : D \to D$ sending $p$ to $q$ and extends to a homeomorphism from $\overline{D}$ onto $\overline{D}$, then the set $\{ z \in D : J_{m,z} = J_{m,z}^+ \}$ equals either $D$ or the empty set.

The connection between Jensen measures and approximation of $m$–sh functions stems from the following fact which is a simple consequence of Fatou’s lemma.

**Proposition 1.3.** Let $D \subset \mathbb{C}^n$ be a bounded domain and $E$ be a subset of $D$. Assume that for every $u \in \mathcal{M}_m(D)$, there exists $\{ u_j \}_{j \geq 1} \subset \mathcal{M}_m(D)$ having the following properties:

(i) $u_j \to u$ pointwise on $E$.

(ii) $\overline{\lim}_{j \to \infty} u_j \leq u$ on $\overline{D}$.

Then $J_{m,z} = J_{m,z}^+$ for every $z \in E$.

In the opposite direction, the next result gives a sufficient condition so that point-wise approximation of functions in $\mathcal{M}_m(D)$ by elements in $\mathcal{M}_m^+(D)$ is possible. We need the following standard notation: If $u : \overline{D} \to [-\infty, \infty]$ then the upper regularization $u^*$ of $u$ is defined as $u^*(z) := \overline{\lim}_{\xi \to z, \xi \in \overline{D}} u(\xi)$, $\forall z \in \overline{D}$. If $u \in \mathcal{M}_m(D)$ then obviously $u \leq u^*$ on $\overline{D}$ while $u = u^*$ on $D$. 

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**Theorem 1.4.** Let $D \subset \mathbb{C}^n$ be a bounded domain. Assume that there exists a subset $X$ of $D$ with $\lambda_{2n}(X) = 0$ such that $J_{m,z} = J_{m,z}^c$ for all $z \in D \setminus X$. Let $E$ be any compact subset of $\partial D$ such that $J_{m,\xi} = \{\delta_{\xi}\}$, $\forall \xi \in E$. Then there exists a $m$–polar subset $Y$ of $D$ having the following property: For every $u \in \text{SH}_m(D)$, there exists a sequence $\{u_j\} \subset \text{SH}_m^*(D)$ having the following properties:

(i) $u_j$ converges pointwise to $u^*$ on $E \cup (D \setminus Y)$;

(ii) $\lim_{j \to \infty} u_j \leq u$ on $\overline{D}$.

Here by a $m$–polar set we mean singular the locus of a $m$–sh. function. We postpone to the next section a brief discussion of $m$–polar sets. In the case where $m = m$ and $X = \emptyset$, we cover Theorem 3.1 in [DW].

The next theorem, which is our main result, deals with approximation of $m$–sh. functions which are only assumed to be bounded from above on a portion (possibly empty) of $\partial D$. We will see, at the same time, that the exceptional set $Y$ mentioned in Theorem 1.4 might occur. For this purpose, the following piece of notation is useful.

**Definition 1.5.** Let $a \in D$. Then by $\partial D(a)$ we mean the set of limits points of sequences in $\overline{D} \cap h_t(\partial D)$ as $t \uparrow 1$, where $h_t(z) := t(z - a) + a$.

By an abuse of notation, we will sometimes write $h(t, z)$ instead of $h_t(z)$. Notice that $\partial D(a) \subset \partial D$ and $\partial D(a) = \emptyset$ if and only if $\overline{D} \cap h_t(\partial D) = \emptyset$ for $t$ close enough to 1.

**Theorem 1.6.** Let $D \subset \mathbb{C}^n$ be a bounded domain and $a \in D$. Suppose that there exist an open neighborhood $U$ in $\mathbb{C}^n$ of $\partial D(a)$ and a $m$–polar subset $E$ of $U \cap \partial D$ satisfying the following conditions:

(a) $J_{m,\xi} = \{\delta_{\xi}\}$ for every $\xi \in (U \cap \partial D) \setminus E$.

(b) $J_{1,\xi}^c = \{\delta_{\xi}\}$ for every $\xi \in U \cap \partial D$.

Then there exists a $m$–polar subset $E'$ of $D$ such that for every $u \in \text{SH}_m(D)$ satisfying $\sup_{U \cap \partial D} u^* < \infty$, there exists a sequence $\{u_j\} \subset \text{SH}_m^*(D)$ satisfying the following properties:

(i) $u_j \to u$ pointwise on $D \setminus E'$;

(ii) $\lim_{j \to \infty} u_j \leq u$ on $\overline{D}$;

(iii) $u_j(x) \to u^*(x)$ for every $x \in \partial D$ such that $u$ is continuous at $x$, i.e.,

$$\lim_{z \to x, z \in D} u(z) = u^*(x) \in [-\infty, \infty].$$

In particular, if $\partial D(a) = \emptyset$ then $J_{m,z} = J_{m,z}^c$ for all $z \in D$.

The proof of the above theorem is inspired by Theorem 3.2 in [DW]. Nevertheless, as we will see, besides the (possible) unboundedness from above of $u$, there is an additional technical difficulty coming from the fact that 1 may not be a thin point of the segment $t \mapsto tz(0 \leq t \leq 1)$ for $m$–sh. functions.

The structure of Jensen measures is particularly simple at boundary points which admits a sort of peak $m$–sh. function. We isolate them in the following definition.

**Definition 1.7.** Let $\xi \in \partial D$. Then we say that $\xi$ admits a local $m$–sh. barrier if there exist a small neighborhood $U$ of $\xi$ and $u \in \text{SH}_m(U)$ such that $u(\xi) = 0$ whereas $u < 0$ on $U \cap (\overline{D} \setminus \{\xi\})$. 

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Remarks. 1. The condition $J_{m,\xi}^c = \{\delta_\xi\}$ is fulfilled at $\xi \in \partial D$ if there is a local $m$--sh. barrier at $\xi$. Indeed, by shrinking $U$ we may achieve that $\sup_{\partial^cD \setminus U_\delta} u < 0$. So we may find an open neighborhood $V$ of $\overline{D} \setminus U$ such that $\delta := - \sup_{V \setminus U} u > 0$. Thus by the gluing lemma (Lemma 2.1(g)) the function $\tilde{u} := \max\{u, -\delta\}$ on $U$ and $\tilde{u} := -\delta$ on $V \setminus U$ is $m$--sh. on $U \cup V$, an open neighborhood of $\overline{D}$. Now let $\mu \in J_{m,\xi}^c$. By convolving $u$ with smoothing kernels we obtain a sequence $u_j$ of $C^\infty$--smooth $m$--sh. functions defined on neighborhoods of $\overline{D}$ that decrease to $u$ on $\overline{D}$. It follows that

$$u_j(\xi) \leq \int_{\overline{D}} u_j(z) d\mu.$$

By letting $j \to \infty$ and using Fatou's lemma we conclude that $\mu = \{\delta_\xi\}$. This reasoning is essentially contained in Proposition 1.4 in [Si].

2. Suppose that there exists an open set $U \subset \mathbb{C}^n$ and a continuous strictly $m$--sh. function $\varphi$ on $U$ such that $U \cap D = \{z \in U : \varphi(z) < 0\}$. Then for every $\xi \in U \cap \partial D$ we have $J_{m,\xi}^c = \{\delta_\xi\}$. Indeed, for $M > 0$ we set

$$u_M(z) := M \varphi(z) - |z - \xi|^2, \quad \forall z \in U.$$

Then for $M > 0$ large enough, $u_M$ is a local continuous $m$--sh. barrier at $\xi$. By the above remark we have $J_{m,\xi}^c = \{\delta_\xi\}$.

3. Our proof shows that $E'$ is included in the $m$--polar hull of $E$, i.e., intersection of all $m$--polar sets that contain $E$. In particular, if $E = \emptyset$ then $E' = \emptyset$.

In analogy with the concept of $B$--regular domains that was introduced and investigated thoroughly in [Si] (see also [Bl1]), we have the following definition.

Definition 1.8. A bounded domain $D$ in $\mathbb{C}^n$ is called $B_m$--regular if for every continuous function on $f$ on $\partial D$ we can find a $u \in SH_m(D) \cap \mathcal{C}(\overline{D})$ such that $u|_{\partial D} = f$.

If $D$ is $B_m$--regular then obviously for every boundary point $\xi \in \partial D$ there exists $u_\xi \in SH_m(D) \cap \mathcal{C}(\overline{D})$ such that $u_\xi(\xi) = \sup_{\partial D} u = E$ and $u_\xi(z) < 0$ elsewhere. We will provide a sort of converse to this statement in Theorem 1.10.

Using Theorem 1.4 and Theorem 1.6 we may derive the following consequences.

Corollary 1.9. Let $D$ be an intersection of a finite number of bounded $B_m$--regular domains with $C^1$--smooth boundary. Then for every $u \in SH_m^-(D)$, there exists a sequence $\{u_j\} \subset SH_m^-(D)$ such that $u_j \downarrow u$ on $\overline{D}$.

Remark. In the case where $m = n$ and $\partial D$ is $C^1$--smooth, the above result can be deduced by combining results in [Wik] and [FW]. Indeed, according to Theorem 4.1 in [Wik] $u^*$ may be approximated from above on $\overline{D}$ by a decreasing sequence $\{v_j\} \subset SH_n(D) \cap \mathcal{C}(\overline{D})$. It is now sufficient to use Theorem 1 in [FW] to approximate each $v_j$ uniformly on $\overline{D}$ by elements in $SH_n^*(D)$.

The result below illustrates a class of domains in $\mathbb{C}^n$ to which Theorem 1.6 is applicable.

Corollary 1.10. Let $\Omega$ be a bounded $B_m$--regular domain in $\mathbb{C}^n$ and $f$ be a $C^1$--smooth function defined on $\Omega$. Let

$$D := \{z \in \Omega : f(z) < 0\}.$$
For $a \in D$, we set
\[
K_a := \left\{ z \in \Omega \cap \partial D : \Re \left( z_1 - a_1 \right) \frac{\partial f}{\partial z_1}(z) + \cdots + \left( z_n - a_n \right) \frac{\partial f}{\partial z_n}(z) \geq 0 \right\}.
\]

Assume that there exists $a \in D$ such that the following conditions hold true:

(i) $f$ is strictly $m-$sh. on an open neighborhood $U$ of $K_a$.

(ii) There exists a $m-$polar subset $E$ of $U \cap \partial D$ such that for each point $\xi \in (U \cap \partial D) \setminus E$ we have $\xi = l_\xi \cap \partial D \cap \Omega$, where $l_\xi := \{ t \xi : t \in \mathbb{R} \}$.

Then $D$ satisfies the condition given in Theorem 7.6.

Remark. For a concrete application of the above corollary, consider the case where $n = 3, m = 2, \Omega = \mathbb{B}_3$ is the unit ball in $\mathbb{C}^3$ and
\[
f(z_1, z_2, z_3) = |z_1|^2 + |z_2|^2 + \varphi(|z_3|^2),
\]
where $\varphi$ is a $C^2$ smooth function on $\mathbb{R}$ satisfying the following conditions:

(a) $\varphi(0) < 0, \varphi(1) > \varphi'(1)$;

(b) $x\varphi''(x) + \varphi'(x) > -\delta$ for all $x \in [0, 1]$ such that $x\varphi'(x) \geq \varphi(x)$, where $\delta \in (0, 1/2)$ is a constant;

(c) $\varphi$ is real analytic on $(0, 1)$;

(d) For every $x \in (0, 1)$ there exists $t \in (0, 1)$ such that $\varphi(tx) \neq t\varphi(x)$.

Then the function $f$ satisfies the conditions (i) and (ii) of Corollary 1.10. To see this, we first compute
\[
\frac{dd^c f}{dz_1} = |z_1|^2 + |z_2|^2 + \varphi(|z_3|^2),
\]
where $\varphi$ is a $C^2$ smooth function on $\mathbb{R}$ satisfying the following conditions:

(a) $\varphi(0) < 0, \varphi(1) > \varphi'(1)$;

(b) $x\varphi''(x) + \varphi'(x) > -\delta$ for all $x \in [0, 1]$ such that $x\varphi'(x) \geq \varphi(x)$, where $\delta \in (0, 1/2)$ is a constant;

(c) $\varphi$ is real analytic on $(0, 1)$;

(d) For every $x \in (0, 1)$ there exists $t \in (0, 1)$ such that $\varphi(tx) \neq t\varphi(x)$.

Then the function $f$ satisfies the conditions (i) and (ii) of Corollary 1.10. To see this, we first compute
\[
\frac{dd^c f}{dz_1} = |z_1|^2 + |z_2|^2 + \varphi(|z_3|^2),
\]
In view of (a) we have $a = 0 \in D$. So an easy computation yields that
\[
K_a = \left\{ (z_1, z_2, z_3) \in \mathbb{B}_3 : |z_1|^2 + |z_2|^2 = -\varphi(|z_3|^2), |z_3|^2 \varphi''(|z_3|^2) \geq \varphi'(|z_3|^2) \right\}.
\]
By (b) and (1.1) we see that $f$ is strictly $2-$sh. on a small neighborhood $U$ of $K_a$. In view of the second condition in (a), we may obtain that $|z_3| < 1$ on $U$. Finally, given $\xi = (\xi_1, \xi_2, \xi_3) \in U \cap \partial D$, we claim that $\xi$ is an isolated point of $l_\xi \cap \partial D$. If this is false, then there exists a sequence $t_j \to 1$ such that $f(t_j \xi) = 0$. Using the assumption (c) on real analyticity of $\varphi$ on $(0, 1)$ we conclude that $f(t \xi) = 0$ for all $t \in (0, 1)$. This means that
\[
t^2(\xi_1^2 + \xi_2^2) + \varphi(t^2 \xi_3^2) = 0, \forall t \in (0, 1).
\]
Plugging $\xi_1^2 + \xi_2^2 = -\varphi(|\xi_3|^2)$ into the above equation we arrive at a contradiction to (d). The claim follows.

For an example of $\varphi$ having the above properties we take $\varphi(x) = \alpha(c - x)^3$, where $\alpha, c$ are real constants such that $\alpha > 0, -2 < c < 0, \alpha c^2 < \frac{\lambda}{4\lambda}$. We end up with the problem of finding a bounded continuous maximal $m-$sh function $u$ on $D$ such that the boundary values of $u$ coincides with a given continuous function defined on part of the boundary $\partial D$. Recall that $u \in SH_m(D)$ is said to be maximal if for every relatively compact open subset $U$ of $D$ and every $v \in SH_m(D)$ such that $v \leq u$ on $D \setminus U$ we have $v \leq u$ on $D$. This definition is analogous to the classical one, i.e., $m = n$ given by Sadullaev (see Proposition 3.1.1 in [KI]).
Theorem 1.11. Suppose that there is \( v \in SH_m^-(D), v > -\infty \) on \( D \) and a compact \( K \subset \partial D \) satisfying the following properties:

(i) \( \lim_{z \to \xi} v(z) = -\infty, \forall \xi \in K; \)

(ii) Every \( \xi \in (\partial D) \setminus K \) admits a local \( m \)-sh barrier.

Then for every \( \varphi \in \mathcal{E}(\partial D) \), there exists uniquely a bounded function \( u \in SH_m(D) \cap \mathcal{E}(D) \) having the following properties:

(a) \( \lim_{z \to \xi, \xi \in D} u(z) = \varphi(\xi), \forall \xi \in (\partial D) \setminus K; \)

(b) \( u \) is maximal on \( D \);

(c) \( u \) can be approximated uniformly on compact sets of \( \overline{D} \setminus K \) by elements in \( SH_m^*(D) \).

The above theorem appears to be new even in the case where \( m = n \) because we allow the existence of points on \( \partial D \) which may not admit continuous plurisubharmonic barriers. Theorem 1.11 also differs somewhat from Theorem 2.1 in [Si] and Theorem 1.7 in [Bł1] even in the case \( K = \emptyset \) and \( m = n \), since the solution \( u \) may be approximated uniformly on \( \overline{D} \) by continuous ones defined on neighborhoods of \( \overline{D} \). In the recent preprint [ACH], the authors use Jensen measures to develop some extension and approximation results for \( m \)-subharmonic functions. They, in particular, generalize several results in [FW] and [Wik] to the context of \( m \)-subharmonic functions. There is apparently, no overlap between the current paper and their work.

2. Preliminaries

Throughout this paper, unless otherwise specified, by \( D \) we always mean a bounded domain in \( \mathbb{C}^n \). We also fix an approximate of identity \( \{\rho_\delta\} \) in \( \mathbb{C}^n \), i.e., \( \rho_\delta(x) := \frac{1}{\delta^n} \rho(x/\delta) \), where \( \rho \) is a smooth radial function with compact support in the unit ball of \( \mathbb{C}^n \) and satisfies \( \int_{\mathbb{C}^n} \rho d\lambda_{2n} = 1 \) with \( \lambda_{2n} \) is the Lebesgue measure of \( \mathbb{C}^n \).

Our first lemma contains elementary facts about \( m \)-sh functions. The proof of these statements follows from either standard arguments in pluripotential theory (see [Kl]) or from direct computation (see [Bli]). The details are therefore omitted.

Lemma 2.1. (a) If \( u \in SH_m(D) \), then the standard regularization \( u_\delta := u * \rho_\delta \) is also \( m \)-sh in \( D_\delta := \{ z \in D : d(z, \partial D) > \delta \} \). Moreover, \( u_\delta \downarrow u \) as \( \delta \to 0; \)

(b) If \( u, v \in SH_m(D) \) then \( au + bv \in SH_m(D) \) for any \( a, b \geq 0 \), i.e. the class \( SH_m(D) \) represents a convex cone;

(c) \( PSH(D) = SH_0(D) \supset \cdots \supset SH_1(D) = SH(D); \)

(d) If \( \chi \) is a convex increasing function on \( \mathbb{R} \) and \( u \in SH_m(D) \), then \( \chi \circ u \in SH_m(D); \)

(e) The limit of a uniformly converging or decreasing sequence of \( m \)-sh functions is \( m \)-sh;

(f) The maximum of a finite number of \( m \)-sh functions is \( m \)-sh. More generally, for an arbitrary locally uniformly bounded from above family \( \{u_\alpha\}_{\alpha \in I} \subset SH_m(D) \) we have \( (\sup u_\alpha)^* \in SH_m(D). \)

(g) (gluing lemma) Let \( U \) be an open subset of \( D \) such that \( \partial U \cap D \) is relatively compact in \( D \). If \( u \in SH_m(D), v \in SH_m(U) \) and \( \lim_{x \to y} v(x) \leq u(y) \) for each \( y \in \partial U \cap D \), then the function \( w \) defined
by
\[ w := \begin{cases} u & \text{on } D \\ \max\{u, v\} & \text{on } D \setminus U. \end{cases} \]
is $m$-sh in $D$.

(h) If $u \in SH_m(D)$ then the restriction of $u$ on each $n - m + 1$ affine complex subspace is subharmonic.

(i) If $u \in SH_m(D)$ and $g(t) = at + b, t \in \mathbb{C}^n$ is an affine map then $u \circ g \in SH_m(g^{-1}(D))$. In other words, $m$–subharmonicity is invariant under translations and dilations.

(j) For $1 \leq m \leq n$ the function $H_m(z) = -|z|^{2 - \frac{2m}{n}}$ belongs to $SH_m(\mathbb{C}^n)$.

Notice, however, that $m$–subharmonicity is not invariant under composition of holomorphic mappings. We now have the useful notion of $m$–polar sets.

**Definition 2.2.** A subset $E$ of $\mathbb{C}^n$ is said to be $m$–polar if for every $z_0 \in E$ we may find a neighborhood $U$ of $z_0$ and $u \in SH_m(U)$ such that $u = -\infty$ on $E \cap U$.

The most basic properties of $m$–polar sets are collected below.

**Proposition 2.3.**

(a) For every $m$–polar subset $E$, there exists $u \in SH_m(\mathbb{C}^n)$ such that $u \equiv -\infty$ on $E$.

(b) Let $\{u_\alpha\}_{\alpha \in I}$ be a set of $m$–sh functions defined on $D$ which are locally uniformly bounded from above. Set $u := \sup_{\alpha \in I} u_\alpha$. Then the set $\{u < u^*\}$ is $m$–polar.

(c) Let $\{X_j\}_{j \geq 1}$ be a sequence of $m$–polar sets. Then $\bigcup_{j \geq 1} X_j$ is also $m$–polar.

In the special case where $m = n$, the above results are proved by Bedford and Taylor using the key notion of relative capacity. The general case can be attacked by the same method where the above notion of capacity is replaced by that of $m$–capacity (see [Lu] or [SA]). We should say that it is not so easy to construct $m$–polar sets which are not pluripolar ($1$–polar). The following result (Theorem 2.26 in [Lu]) enables us to construct a substantial class of such sets (see Example 2.27 in [Lu]).

**Proposition 2.4.** Let $H(r) := r^{2n - 2m}(1 \leq m < n)$. Then every subset $E \subset \mathbb{C}^n$ that satisfies
\[ \infty > \Lambda_H(E) := \lim_{\delta \to 0} \left( \inf \sum_k H(r_k) \right), \]
where the infimum is taken over all coverings of $E$ by balls $B_k$ of radii $r_k \leq \delta$, is $m$–polar.

Using the same arguments, we can even show that the class of $m$–polar sets is properly included in the set of $(m - 1)$–polar ones. The proof of Proposition 2.4 uses among other things, a formula for $m$–relative extremal functions between concentric balls, which requires Lemma 2.1(j).

A major technical tool that will be used throughout our work is a version of Edwards’ duality theorem which relates upper envelopes of upper semicontinuous functions defined on a compact metric space with lower envelopes of integrals with respect to certain classes of measures. To begin with, let us fix the notation. Let $X$ be a compact metric space, by $C(X)$ we denote the set of real-valued continuous functions on $X$. We also write $\mathcal{B}(X)$ for the class of positive, regular Borel measures on $X$. Let $\mathcal{F}$ be a convex cone of upper semicontinuous functions on $X$. \[ \text{(1.5)} \]
containing all the constants. If \( g : X \to [-\infty, \infty) \) is a Borel measurable function on \( X \) and \( z \in X \) then we define

\[
Sg(z) := \sup\{u(z) : u \in \mathcal{F}, u \leq g\}, \\
Ig(z) := \inf\{\int_X gd\mu : \mu \in J^F_z\}.
\]

Here \( J^F_z := \{\mu \in \mathcal{B}(X) : u(z) \leq \int_X ud\mu, \forall u \in \mathcal{F}\} \). It is easy to see that \( J^F_z \) is a convex subset of \( \mathcal{B}(X) \). Moreover, \( \mu(X) = 1 \) for every \( \mu \in J^F_z \) since \( \mathcal{F} \) contains the constants. In view of Banach-Alaooglu’s theorem and the fact that every upper semicontinuous function on \( X \) is the limit of a decreasing sequence of continuous function on \( X \), we can also check that \( J^F_z \) is weak-* compact in \( \mathcal{B}(X) \). Now we have the following basic duality theorem of Edwards (see [Ed], [Wik]).

**Theorem 2.5.** Let \( X, \mathcal{F} \) be as above. If \( g : X \to (-\infty, \infty] \) is lower semicontinuous, then \( Sg = Ig \).

Apparently the first use of Edwards’ duality theorem in pluripotential theory has been made in the seminal work [Si] where we can find a systematic study of domains in \( \mathbb{C}^n \) on which the Dirichlet problem for plurisubharmonic functions is solvable.

In our context, by applying the above theorem to the convex cones \( \text{SH}^*_m(D) \) and \( \text{SH}^*_m(D) \) we obtain the following result which will be referred to as Edwards’ duality theorem.

**Theorem 2.6.** (Edwards’ duality theorem) Let \( \varphi : \overline{D} \to (-\infty, +\infty] \) be a lower semicontinuous function. Then we have

\[
\inf_\mathcal{F}\left\{ \int \varphi d\mu, \mu \in J^F_{m,z}\right\} = \sup\{u(z) : u \in \text{SH}^*_m(D), u \leq \varphi \text{ on } \overline{D}\}, \forall z \in D \\
\inf_\mathcal{D}\left\{ \int \varphi d\mu, \mu \in J^F_{m,z}\right\} = \sup\{u(z) : u \in \text{SH}^*_m(D), u \leq \varphi \text{ on } \overline{D}\}, \forall z \in \overline{D}.
\]

Our next ingredients consists of a few standard facts about upper semicontinuous and lower semicontinuous functions on compact sets of \( \mathbb{C}^n \). First, we have an elementary yet useful result of Choquet (see Lemma 2.3.4 in [Kl]).

**Lemma 2.7.** Let \( \{u_\alpha\}_{\alpha \in \mathcal{A}} \) be a family of upper semicontinuous functions defined on a closed subset \( X \subset \mathbb{C}^n \), which is locally bounded from above. Then there exists a countable subfamily \( \mathcal{B} \) of \( \mathcal{A} \) such that

\[
(\sup\{u_\alpha : \alpha \in \mathcal{B}\})^* = (\sup\{u_\alpha : \alpha \in \mathcal{A}\})^*.
\]

If \( u_\alpha \) are lower semicontinuous then \( \mathcal{B} \) can be chosen so that

\[
\sup\{u_\alpha : \alpha \in \mathcal{B}\} = \sup\{u_\alpha : \alpha \in \mathcal{A}\}.
\]

The next simple lemma deal with monotone sequences of lower semicontinuous on subsets of \( \mathbb{C}^n \). The easy proof is left to the interested reader.

**Lemma 2.8.** Let \( X \) be a subset of \( \mathbb{C}^n \) and \( \{\varphi_j\}_{j \geq 1} \) be a sequence of lower semicontinuous functions on \( X \) that increases to a lower semicontinuous function \( \varphi \) on \( X \). Then for every sequence \( \{a_j\}_{j \geq 1} \subset X \) with \( a_j \to a \in X \) we have

\[
\varphi(a) \leq \lim_{j \to \infty} \varphi_j(a_j).
\]

We end up this preparatory section by presenting a useful result which permits approximation of continuous strictly \( m-\text{sh} \) functions by smooth ones. This lemma will be used only in the proof of Theorem 1.4.
Lemma 2.9. Let \( D \) be a domain in \( \mathbb{C}^n \). Assume that \( u \) is a continuous strictly \( m \)-sh function on \( D \). Then for every continuous positive function \( h \) on \( D \) we can find a smooth strictly \( m \)-sh function \( v \) on \( D \) such that \( u < v < u + h \) on \( D \).

Proof. In view of Lemma 2.1 we may modify easily the original proof of Richberg’s theorem in the case of \( m = n \) (see Theorem 1.3 in [Bł1]). The details are left to the interested reader. \( \square \)

3. PROOFS OF THE MAIN RESULTS

Proof. (of Theorem 1.1) We first show that \( D \) admits a smooth strictly \( m \)-sh exhaustion function which is larger than \( \varphi \). This will be done by an adaptation of the proof of Theorem 2.6.11 in [Hö]. For the reader convenience, we indicate some details. For each \( j \geq 1 \) we let

\[
\varphi_j(z) := (\varphi * \rho_{\delta_j})(z) + \delta_j|z|^2.
\]

Here \( \delta_j > 0 \) is chosen so small that \( \varphi_j \) is smooth and strictly \( m \)-sh on \( D_{j+1} \). Moreover, we may arrange so that \( \varphi_j > \varphi \) there. All this is possible in view of Lemma 2.1. Take a smooth convex increasing function \( \chi \) on \( \mathbb{R} \) such that \( \chi(t) = 0 \) for \( t < 0 \) and \( \chi'(t) > 0 \) when \( t > 0 \). Then the function \( \chi(\varphi_j - (j-1)) \) is positive, smooth and strictly \( m \)-sh. on the open set \( D_{j+1} \setminus \overline{D}_j \). Therefore we may choose inductively positive numbers \( \{a_j\} \) such that the function

\[
\psi_j := \sum_{l=1}^j a_l \chi(\varphi_l + 1 - l)
\]

is strictly \( m \)-sh and \( > \varphi \) on \( D_j \). By the choice of \( \chi \) we also have \( \psi_{j'} = \psi_j^{j'} \) on \( D_j \) if \( j < j' < j'' \). It follows that \( \psi := \lim_{j \to \infty} \psi_j \) is a smooth, strictly \( m \)-sh function on \( D \). Moreover, since \( \psi > \varphi \), we conclude that \( \psi \) exhausts \( D \). Next, in view of Lemma 2.1 and Richberg’s approximation lemma (cf. Lemma 2.9), we may repeat the proof of Theorem 5.5 in [FN] to produce the desired approximating sequence for \( u \). The details are omitted. \( \square \)

Proof. (of Proposition 1.3) Obviously \( J_{m,z} \subset J_{m,z}^c \), \( \forall z \in D \). Conversely, fix \( z \in E \) and \( \mu \in J_{m,z}^c \). For every \( u \in SH_m^w(D) \) we choose a sequence \( \{u_j\}_{j \geq 1} \subset SH_m^w(D) \cap C(D) \) that satisfy the conditions (i) and (ii). Then we have

\[
u_j(z) \leq \int u_j d\mu, \forall j \geq 1.
\]

By letting \( j \to \infty \) and making use of Fatou’s lemma we get

\[
u(z) \leq \int u d\mu.
\]

Thus \( \mu \in J_{m,z} \) as desired. \( \square \)

For the ease of exposition, we introduce the following notation: For each bounded function \( f \) on \( D \), we set

\[
Smf(z) := \sup\{u(z) : u \in SH_m^-(D), u^* \leq f \text{ on } \overline{D}\}, z \in D,
\]

\[
S'_mf(z) := \sup\{u(z) : u \in SH_m^+(D), u \leq f \text{ on } \overline{D}\}, z \in D.
\]

Proof. (of Theorem 1.4) We split the proof into two steps.
Step 1. We will show that there exists a $m-$polar subset $Y$ of $D$ such that for every $f \in \mathcal{C}(\overline{D})$ we have

$$S_m f = S_m^c f \text{ on } D \setminus Y.$$ Choose a countable dense subset $\{f_j\}$ of $\mathcal{C}(\overline{D})$. Fix $j \geq 1$. Then from Proposition 2.3, Edwards’ duality theorem and the fact that $J_{m,z} = J_{m,z}^*$, for $z \in D \setminus X$, we obtain

$$S_m f_j = S_m^c f_j \forall z \in D \setminus X.$$ Since $f_j$ is continuous on $\overline{D}$, we have $(S_m f_j)^* \leq f_j$ on $\overline{D}$. Hence $S_m f_j = (S_m f_j)^* \in \mathcal{H}_m(D)$. Since $S_m^c f_j = (S_m^c f_j)^*$ a.e. on $D$, we infer that $S_m f_j = (S_m^c f_j)^*$ a.e. on $D$. Since $S_m f_j$ and $(S_m^c f_j)^*$ are subharmonic on $D$ we deduce that

$$S_m f_j = (S_m^c f_j)^* \text{ on } D.$$ Notice that, by Proposition 2.3, the set $X := \{z : (S_m^c f_j)^*(z) > S_m^c f_j(z)\}$ is $m-$polar. Set $Y := \bigcup Y_j$. Then $Y$ is $m-$polar and

$$S_m^c f_j = (S_m^c f_j)^* \text{ on } D \setminus Y.$$ Now we choose a subsequence $\{f_{k_j}\}$ that converges uniformly to $f$ on $\overline{D}$. Since $S_m f_{k_j}$ (resp. $S_m f_j$) converges uniformly on $\overline{D}$ to $S_m f$ (resp. $S_m^c f_j$), we infer that $S_m f = S_m^c f$ on $D \setminus Y$.

Step 2. We will prove that $Y$ has the properties indicated in the theorem. To this end, we may assume that $Y$ is $G_\delta$. Fix $u \in \mathcal{H}_m^-(D)$. We now follow closely the arguments in Theorem 3.1 of [DW]. Choose a sequence of real valued continuous functions $\varphi_j$ on $\overline{D}$ such that $\varphi_j \downarrow u^*$ on $\overline{D}$. Then by Edwards’ duality theorem and the fact that $J_{m,z} = J_{m,z}^*$, for every $z \in D \setminus Y$ we infer

$$S_m^c \varphi_j = S_m \varphi_j \text{ on } D \setminus Y.$$ Since $\varphi_j$ is continuous on $\overline{D}$, we have $(S_m \varphi_j)^* \leq \varphi_j$. Therefore

$$S_m \varphi_j = (S_m \varphi_j)^* \in \mathcal{H}_m^-(D) \forall j \geq 1.$$ On the other hand, since $S_m \varphi_j$ is lower semicontinuous on $D$ we deduce that $(S_m \varphi_j)^*$ is continuous at every point in $D \setminus Y$.

It follows that the restriction of $S_m \varphi_j$ on $D \setminus Y$ is continuous. Observe also that $u \leq S_m \varphi_j \leq \varphi_j$ on $D$ for every $j$, so we get $S_m \varphi_j \downarrow u$ on $D$. Hence $S_m^c \varphi_j \downarrow u$ on $D' := D \setminus Y$. Since $D'$ is a $F_\sigma$ set, there exists an exhaustion of $D'$ by compact subsets $\{K_j\}_{j \geq 1}$. By Edwards’ duality theorem and the assumption that $J_{m,\xi} = \{\delta_\xi\}$ for each $\xi \in E$, we infer that $S_m \varphi_j = \varphi_j$ on $E$. In particular $S_m^c \varphi_j$ is continuous on $K_j := E \cup K_j$. For every $j \geq 1$, by Choquet’s lemma, we can find a sequence $\{v_{l(j),j}\}_{l \geq 1} \subset \mathcal{H}_m^-(D)$ that increases to $S_m^c \varphi_j$ on $\overline{D}$. By Dini’s theorem and continuity of $S_m \varphi_j$ on $K_j$, the convergence is uniform on $K_j$ as $l \rightarrow \infty$. Thus we can choose $v_{l(j),j} \in \mathcal{H}_m^-(D)$ such that

$$\|S_m \varphi_j - v_{l(j),j}\|_{K_j'} \leq \frac{1}{j}, v_{l(j),j} \leq \varphi_j \text{ on } \partial D.$$ It is then easy to check that $u_j := v_{l(j),j}$ converges pointwise to $u$ on $E \cup D'$ and

$$\overline{\lim}_{j \rightarrow \infty} u_j \leq \lim_{j \rightarrow \infty} \varphi_j = u^* \text{ on } \overline{D}.$$ The proof is thereby completed.

Remark. By the same proof as in the one given in Step 2, we can show that for $z \in D$, the equality $J_{m,z} = J_{m,z}^*$ implies that for each $u \in \mathcal{H}_m^-(D)$, there exists $\{u_j\}_{j \geq 1} \subset \mathcal{H}_m^-(D)$ such that $u_j(z) \rightarrow u(z)$ and $\overline{\lim}_{j \rightarrow \infty} u_j \leq u$ on $\overline{D}$.
Proof. (of Theorem 3.6) After subtracting a large constant and shrinking $U$ we may assume $\sup u^* < 0$. For $\delta > 0$ we set $D_\delta := \{z \in D : \text{dist}(z, \partial D) > \delta\}$. Fix an exhaustion sequence $\{K_j\}$ of $D$ by compact sets. We claim that for each $j \geq 1$, there exists $\delta_j \in (0, 1/j)$ such that

$$(\partial D) \setminus U \subset h_t(D_{\delta_j}), \quad \forall t \in (1, 1 + \delta_j].$$

Indeed, if the claim is false, then there exists a sequence $\{x_m\} \subset (\partial D) \setminus U$ but $x_m \notin h_{\alpha_m}(D_{1/m})$, where $\alpha_m := 1 + 1/m$. Hence

$$x_m = h_{\alpha_m}(y_m), y_m \notin D_{1/m}.$$After switching to a subsequence we may assume that $\{x_m\} \to x^* \in (\partial D) \setminus U$. It follows that $\{y_m\} \to x^*$. Now we take a sequence $b_m \uparrow 1$ such that for $m$ large enough we have

$$|h_{\alpha_m}(y_m) - h_{b_m}(y^*)| < \frac{1}{m}.$$It follows that $h_{b_m}(y^*) \to x^*$. Hence $x^* \in \partial D(a)$, which is a contradiction. The claim follows. Set

$$u_j(z) := (u \ast \rho_{\delta_j})(z), z \in D_{\delta_j}.$$We may also choose $\delta_j$ such that $\delta_j > \delta_{j+1}$ and that $K_j \subset h_t(D_{\delta_j})$ for all $t \in (1, 1 + \delta_j]$. Then for $z \in K_j \subset D \cap \psi_t(D_{\delta_j})$ and $t \in (1, 1 + \delta_j]$ we have

$$|u_j \circ h_t^{-1}(z) - u_j(z)| \leq \int |u_j(x)||\rho_{\delta_j}(h_t^{-1}(z) - x) - \rho_{\delta_j}(z - x)| d\lambda_2(x) \leq M_j ||u||_{L^1(K_j)} |t - 1|.$$Here $M_j > 0$ is a constant independent of $t$. Thus, we can choose $t_j \in (1, 1 + \delta_j]$ such that

$$||u_j \circ h_t^{-1}(z) - u_j(z)||_{K_j} < \frac{1}{j}. \quad \text{(3.2)}$$Let $\{\varphi_j\}$ be a sequence of negative continuous functions on $\partial D$ such that $\varphi_j \downarrow u^*$ on $\overline{U} \cap \partial D$. Let $K$ be a closed ball contained in $D$. Consider the envelopes

$$V(z) := \sup\{v(z) : v \in SH_m^*(D), v \leq -\chi_K\}, z \in \overline{D},$$

$$\Phi_j(z) := \sup\{v(z) : v \in SH_m^*(D) : v|_{\partial D} \leq \varphi_j\}, z \in \overline{D}.$$Using Edwards’ duality theorem and the assumptions (a) and (b) we obtain

$$\Phi_j = \varphi_j \text{ and } V = 0 \text{ on } \partial D \cap (\overline{U} \setminus E). \quad \text{(3.3)}$$

$$\Phi_j \leq S^*_t \varphi_j = \varphi_j \text{ on } \partial D \cap \overline{U}. \quad \text{(3.4)}$$Since $V^* \in SH_m^*(D)$ and since $V^* = -1$ on the interior of $K$, by the maximum principle we have $V \leq V^* < 0$ on $D$. Now using Choquet’s lemma, we may choose sequences $\{v_k\} \subset SH_m^*(D)$ and $\varphi_{k,j} \in SH_m^*(D)$ with $\varphi_{k,j} \uparrow \Phi_j$ and $v_k \uparrow V$ on $\overline{D}$. Moreover, by the assumption, there exists $\psi \in SH_m^*(D')$ with $\psi|_E \equiv -\infty$, where $D'$ is some open neighborhood of $\overline{D}$. Then for fixed $j \geq 1$ we claim that there exist $l_j \geq 1, \delta_j' \in (0, \delta_j)$ such that for $t \in (1, 1 + \delta_j']$ we have

$$v_{l_j}(\xi) + \frac{1}{j} \varphi_{l_j}(\xi) \geq \lim_{z \to \xi} \left(\frac{1}{j} u_j \circ h_t^{-1}(z) + \frac{1}{j^2} (\psi \ast \rho_{\delta_j'}) (\xi)\right) - \frac{1}{j^2} \forall \xi \in \overline{D} \cap \varphi_t(\partial D_{\delta_j'}),$$

where $u_j(z) := (u \ast \rho_{\delta_j'})(z), z \in D_{\delta_j'}$. If this is false then we can find sequences $k_l \uparrow \infty, \delta'_l \downarrow 0, t_l \downarrow 1$ and points $\{\xi_l\}$ such that $\xi_l \in \overline{D} \cap h_t(\partial D_{\delta_l'})$ and

$$v_{k_l}(\xi_l) + \frac{1}{j} \varphi_{k_l}(\xi_l) < \frac{1}{j} (u_j \circ h_t^{-1})(\xi_l) + \frac{1}{j^2} (\psi \ast \rho_{\delta_l'}) (\xi_l) - \frac{1}{j^2},$$
After passing to a subsequence, we may assume that \( \{ \xi_l \} \) converges to \( \xi^* \in \partial D(a) \subset U \). On one hand, by Lemma 2.8 we have
\[
\lim_{l \to \infty} \Phi_{k_l,j}(\xi_l) \geq \Phi_j(\xi^*) \quad \text{and} \quad \lim_{l \to \infty} v_{k_l}(\xi_l) \geq V(\xi^*).
\]
On the other hand,
\[
\frac{1}{j} \lim_{l \to \infty} (u_{j_l} \circ h_{\delta_j}^{-1})(\xi_l) + \frac{1}{j} \lim_{l \to \infty} (\psi * \rho_{\delta_j})(\xi_l) \leq \frac{1}{j} u^*(\xi^*) + \frac{1}{j} \psi(\xi^*)< \frac{1}{j} \Phi_j(\xi^*) + \frac{1}{j} \psi(\xi^*).
\]
Putting all this together, using (3.3) and the fact that \( \psi|_E = -\infty \), we obtain a contradiction. The claim is proved. Furthermore, using Dini’s theorem and (3.3) again we may choose \( l_j \) so large such that
\[
v_{l_j} \geq \frac{1}{j} \quad \text{on the compact set } (\overline{U} \cap \partial D) \setminus \{ \psi < -j \}.
\]
This implies that the function \( \tilde{u}_j \) defined by
\[
\tilde{u}_j := \begin{cases} 
\max\{u_{j_l} \circ h_{\delta_j}^{-1} + \frac{1}{j} \psi * \rho_{\delta_j}, j v_{l_j} + \Phi_{j_l,j}\} & \text{on } \overline{D} \cap h_{\delta_j}'(D_{\delta_j}) \\
 j v_{l_j} + \Phi_{j_l,j} & \text{on } \overline{D} \setminus h_{\delta_j}'(D_{\delta_j})
\end{cases}
\]
belongs to \( SH_m^*(D) \). Now we set
\[
L_j := \{ z \in K_j : u(z) \geq -j, \psi(z) \geq -j \}.
\]
Then \( L_j \) is a compact subset of \( K_j \). Set \( E' := \{ z \in D' : \psi(z) = -\infty \} \). Now we claim that \( \tilde{u}_j \to u \) on \( D \setminus E' \). Indeed, given \( z_0 \in D \) with \( \psi(z_0) > -\infty \)Consider first the case where \( u(z_0) > -\infty \). Then \( z_0 \in L_j \) for \( j \) large enough. Since
\[
j v_{l_j}(z_0) \leq j V(z_0) \to -\infty \quad \text{as } j \to \infty,
\]
from (3.2) we infer that for \( j \) sufficiently large
\[
\tilde{u}_j(z_0) = (u_{j_l} \circ h_{\delta_j}^{-1})(z_0) + \frac{1}{j} (\psi * \rho_{\delta_j})(z_0) \to -\infty.
\]
Therefore, using again (3.2) and the fact that \( u_{j_l}(z_0) \downarrow u(z_0) \) we see that \( \tilde{u}_j(z_0) \to u(z_0) \) as \( j \to \infty \). On the other hand, if \( u(z_0) = -\infty \) then by the same reasoning we have \( (u_{j_l} \circ h_{\delta_j}^{-1})(z_0) \to -\infty \).
Hence \( \lim_{j \to \infty} \tilde{u}_j(z_0) = -\infty = u(z_0) \). This proves (i). For (ii), we first note that if \( z \in \overline{D} \) with \( V(z) < 0 \) then
\[
\lim_{j \to \infty} (j v_{l_j}(z) + \Phi_{j_l,j}(z)) \leq \lim_{j \to \infty} j V(z) = -\infty.
\]
Thus, the preceding proof yields that \( \lim_{j \to \infty} \tilde{u}_j(z) \leq u(z) \). For \( z \in \overline{D} \) with \( V(z) = 0 \) we have \( z \in (\partial D) \cap U \). It follows, using (3.4) that
\[
\lim_{j \to \infty} (j v_{l_j}(z) + \Phi_{j_l,j}(z)) \leq \lim_{j \to \infty} \Phi_{j_l,j}(z) \leq \lim_{j \to \infty} \Phi_j(z) \leq \lim_{j \to \infty} \Phi_j(z) \leq u^*(z).
\]
This proves (ii). Next, we fix \( x \in (\partial D) \setminus E' \) such that \( u \) is continuous at \( x \). If \( x \notin U \), then \( V(x) < 0 \), so by the same reasoning as above we get
\[
\tilde{u}_j(x) = (u_{j_l} \circ h_{\delta_j}^{-1})(x) + \frac{1}{j} (\psi * \rho_{\delta_j})(x) \to u(x) \quad \text{as } j \to \infty.
\]
For the case $x \in U$ we observe that
\[
\lim_{j \to \infty} (jv_{1j}(x) + \varphi_{l_j,1j}(x)) = \lim_{j \to \infty} \varphi_{l_j,1j}(x) \geq \lim_{k \to \infty} \lim_{j \to \infty} \varphi_{k,j}(x) = u(x).
\]
Hence we get (iii). Finally, we note that if $u \in SH_m^c(D)$ then so is $\tilde{u}_j$. Thus, if $\partial D(a) = \emptyset$ we may choose $U = \emptyset$. It follows that $\lim_{j \to \infty} u_j \leq u$ on $\overline{D}$. Hence, by applying Proposition 1.3 we get that $J_{m,\xi}(D) = J_{m,\xi}^c$ for all $\xi \in D$.

For the proof of Corollary 1.9 we need the following lemma

**Lemma 3.1.** Let $D$ be as in Corollary 1.9 and $\xi \in \partial D$. Then $J_{m,\xi}^c(D) = \{\delta_\xi\}$.

**Proof.** We split the proof into two steps.

**Step 1.** We first show that $J_{m,\xi}^c(D \cap U) = \{\delta_\xi\}$ for some small neighborhood $U$ of $\xi$. To see this, we write $D = D_1 \cap \cdots \cap D_k$, where $D_1, \ldots, D_k$ are $B_m$-regular domains with $C^1$-smooth boundaries. Then $\xi \in \partial D_j$ for some $1 \leq j \leq k$. Since $D_j$ is $B_m$-regular, we can find $u \in SH_m^c(D_j) \cap C^1(\overline{D}_j)$ such that $u(\xi) = -1$ and $u < -1$ on $\overline{D}_j \setminus \{\xi\}$. Since $\partial D_j$ is $C^1$-smooth, we may find $\varepsilon > 0$ and open balls $B_1, B_2 \subset \mathbb{B}_2$ such that
\[
\overline{B}_1 \cap D_j \subset (B_1 \cap D) + \varepsilon n, \quad \forall \varepsilon \in (0, \varepsilon_0),
\]
where $n$ is the unit outward normal to $\partial D$ at $\xi$. We claim that $J_{m,\xi}^c(D') = \{\delta_\xi\}$, where $D' := D_j \cap \mathbb{B}_1$. For this, we set $u_\varepsilon(z) := u(z - \varepsilon n)$. Then $u_\varepsilon$ is $m$-sh on a neighborhood of $\partial D'$ for each $\varepsilon \in (0, \varepsilon_0)$. Fix $\mu \in J_{m,\xi}^c(D')$. Then for $\delta > 0$ small enough we have
\[
(u_\varepsilon \ast \rho_\delta)(\xi) \leq \int_{\partial D'} (u_\varepsilon + \rho_\delta) \mu.
\]
By letting $\delta \downarrow 0$ and $\varepsilon \downarrow 0$ and using Fatou’s lemma we obtain $-1 \leq \int_{\partial D'} u \mu$. This forces $\mu = \delta_\xi$ as claimed.

**Step 2.** Fix $v \in J_{m,\xi}^c(D)$ we will show that $v = \{\delta_\xi\}$. Consider the upper envelope
\[
\varphi(z) := \sup\{u(z) : u \in SH_m^c(D') : u \leq h \text{ on } \partial D'\},
\]
where $h(z) := -|z - \xi|$. Then by Edwards’ duality theorem and the result obtained in the previous step we obtain $\varphi(\xi) = 0$ whereas $\varphi < 0$ on $\partial D' \setminus \{\xi\}$. Thus, by Choquet’s lemma, we can find a sequence $\{u_k\} \subset SH_m^c(D')$ such that $u_k \uparrow \varphi$ as $k \to \infty$. In particular $u_k \leq h$ on $\partial D'$. Hence, we can find $\delta > 1$ such that $u_k < -\delta$ on $\partial D' \cap D$. Define for each $k$ the function
\[
\tilde{u}_k := \begin{cases} -\delta & \text{on } \partial D \setminus D' \\ \max\{u_k, -\delta\} & \text{on } D'.
\end{cases}
\]
By Lemma 2.4 we see that $\tilde{u}_k \in SH_m^c(D)$. Thus we have
\[
\tilde{u}_k(\xi) \leq \int_{\partial D} u_k d\nu.
\]
By letting $k \to \infty$ and using Lebesgue monotone convergence theorem we infer that $v = \{\delta_\xi\}$. This completes the proof of the lemma.

**Proof.** (of Corollary 1.9) In view of the above lemma, we may apply Theorem 1.6 to $U = \mathbb{C}^n, E = \emptyset$ and $a$ is an arbitrary point of $D$ to get the desired conclusions.
Proof. (of Corollary 1.10) First we show that \( \partial D(a) \) is included in \( K_a \). Indeed, fix \( \alpha \in \partial D(a) \). Then we can find a sequence \( t_j \uparrow 1 \) and a sequence \( p_j \in \partial D \) such that

\[
\overline{D} \ni \alpha_j := h(t_j, p_j) \to a.
\]

Recall that \( h(t, z) = h(z) = t(z - a) + a \). Set \( t'_j := 1/t_j > 1 \). It follows that \( g(h(t'_j, \alpha_j)) = 0 \) whereas \( g(h(1, \alpha_j)) \leq 0 \). Hence, there exists \( t''_j \in (1, t'_j) \) such that

\[
0 \leq \frac{\partial}{\partial t} f(h(t, \alpha_j)) \bigg|_{t = t''_j} = 2\Re\left( (\alpha_1 - a) \frac{\partial}{\partial z_1} h(t'_j, \alpha_1) + \cdots + (\alpha_n - a) \frac{\partial}{\partial z_n} h(t'_j, \alpha_n) \right).
\]

By letting \( j \to \infty \) we see that \( \alpha \in K_a \). Next, fix \( \bar{\xi} = (\xi_1, \ldots, \xi_n) \in (U \cap \partial D) \setminus E \), we will show that \( J_{m,c}^* (\xi) = \{ \delta_{\bar{\xi}} \} \). To see this, it suffices to construct a local \( m-\)sh barrier of \( D \) at \( \bar{\xi} \). Since \( \Omega \) is \( B_m \)-regular and \( \partial \Omega \) is \( C^1 \)-smooth, by Lemma 3.1 we may assume that \( \bar{\xi} \in \Omega \). Since \( f \) is strictly \( m-\)sh. on \( U \), after shrinking \( U \) and multiplying \( f \) with a constant we may assume

\[
f(z) = |z|^2 + g(z),
\]

where \( g \in SH_m(U) \cap C^1(U) \). Notice that \( g(\bar{\xi}) = -|\bar{\xi}|^2 < 0 \). For \( z \in U \) we define

\[
u_{\bar{\xi}}(z) = \Re(z_{1}z_{2}^{2} + \cdots + z_{n}^{2}z_{n}^{2}) - g(z)g(\bar{\xi}).
\]

By the hypothesis, we see that \( \nu_{\bar{\xi}} \) is continuous \( m-\)sh on \( U \) and \( \nu_{\bar{\xi}}(\bar{\xi}) = 0 \). Moreover, for \( \bar{\xi} \in \overline{D} \setminus \{ \bar{\xi} \} \), by Cauchy-Schwarz’s inequality we obtain

\[
\nu_{\bar{\xi}}(\bar{\xi}) \leq |\bar{\xi}| |z|^2 - g(z)g(\bar{\xi}) \leq 0.
\]

Here the equality occurs only if \( \bar{\xi} \in \partial D \) and \( \bar{\xi} = t\bar{\xi} \) for some constant \( t \geq 0, t \neq 1 \), i.e., \( \bar{\xi} \in I_{\bar{\xi}} \). This is impossible in view of (ii). Hence, \( \nu_{\bar{\xi}} \) is indeed a local \( m-\)sh barrier at \( \bar{\xi} \). The proof is thereby completed.

The proof of Theorem 1.11 requires the following fact.

Lemma 3.2. Let \( \bar{\xi} \in \partial D \) be a boundary point that admits a local \( m-\)sh. barrier. Let \( \varphi < 0 \) be a continuous function on \( \partial D \). Then for every sequence \( \{ \xi_j \} \subset D \) with \( \xi_j \to \bar{\xi} \) we have

\[
\lim_{j \to \infty} S_{m} \varphi(\xi_j) \leq \varphi(\bar{\xi}),
\]

where \( \varphi \) is extended to a lower semicontinuous function on \( \overline{D} \) by setting \( \varphi := +\infty \) on \( D \).

Proof. Let \( u \) be a local \( m-\)sh. barrier at \( \bar{\xi} \). By the argument given in the remark following Theorem 1.6 we may extend \( u \) to a \( m-\)sh. function \( \tilde{u} \) on a neighborhood of \( \overline{D} \) such that \( \tilde{u}(\bar{\xi}) = 0 \) while \( \tilde{u} < 0 \) on \( \overline{D} \setminus \{ \xi \} \). Let \( \{ \mu_j \}_{j \geq 1}, \mu_j \in J_m \) be a sequence of Jensen measures with compact support in \( \partial D \). We claim that \( \mu_j \) converges to \( \delta_{\bar{\xi}} \) in the weak \( *- \) topology. Let \( \mu^* \) be a cluster point of \( \{ \mu_j \} \). Then for \( \delta > 0 \) small enough and \( j \geq 1 \) we have

\[
(\tilde{u} * \rho_\delta)(z_j) \leq \int_{\partial D} (\tilde{u} * \rho_\delta)d\mu_j.
\]

By letting \( \delta \downarrow 0 \) and then \( j \to \infty \) we infer that \( \mu^* = \delta_{\bar{\xi}} \). This proves the claim. It follows that

\[
\lim_{j \to \infty} S_{m} \varphi(z_j) \leq \lim_{j \to \infty} \int_{\partial D} \varphi d\mu_j \leq \varphi(\bar{\xi}).
\]

This is the desired conclusion.
Proof. (of Theorem 1.11) We split the proof in two two parts.

Existence. After subtracting a large constant we may assume \( \varphi < 0 \) on \( \partial D \). Define

\[
S_m \varphi(z) := \sup \{ u(z) : u \in SH_m^-(D), u \leq \varphi \text{ on } \partial D \}, \quad z \in D;
\]

\[
S_m^c \varphi(z) := \sup \{ u(z) : u \in SH_m^+(D), u \leq \varphi \text{ on } \partial D \}, \quad z \in \overline{D}.
\]

Then by Lemma 2.1 (f), \( u := (S_m^c \varphi)^* \in SH_m^-(D) \). In view of the assumption (a) and the remark following Theorem 1.6 we also have \( J_{m, \xi}^c = \{ \delta_\xi \} \) for every \( \xi \in (\partial D) \setminus K \). So using Edwards’ duality theorem (with \( \varphi := +\infty \) on \( D \)) we obtain

\[
S_m^c \varphi = \varphi \text{ on } (\partial D) \setminus K.
\] (3.6)

Furthermore, by Theorem 1.6 we get \( J_{m, z} = J_{m, \xi}^c \) for every \( z \in D \). So applying again Edwards’ duality theorem we get

\[
S_m \varphi = S_m^c \varphi \text{ on } D.
\] (3.7)

On the other hand, by the assumption (ii) and Lemma 3.2 we get

\[
\lim_{z \to \xi, z \in D} u(z) \leq \varphi(\xi), \quad \forall \xi \in (\partial D) \setminus K.
\] (3.8)

Fix \( \varepsilon > 0 \) and set \( u_\varepsilon := u + \varepsilon v \). Then we infer from the last inequality and the assumption that \( (u_\varepsilon)^* \leq \varphi \) on \( \partial D \). This implies that \( u_\varepsilon \leq S_m \varphi \) on \( D \). By letting \( \varepsilon \downarrow 0 \) and noting that \( v > -\infty \) on \( D \) we get

\[
S_m^c \varphi \leq u \leq S_m \varphi = S_m^c \varphi \text{ on } D.
\]

Hence \( u = S_m \varphi \) on \( D \). In particular \( u \) is lower semicontinuous on \( D \). Therefore \( u \in SH_m(D) \cap \mathcal{C}(\overline{D}) \) and \( \|u\|_D \leq \|\varphi\|_{\partial D} \). Next we claim that \( u \) has the right boundary values off \( K \). Indeed, fix \( \xi \in (\partial D) \setminus K \) and a sequence \( \xi_j \to \xi, \xi_j \in D \). By the lower semicontinuity on \( \overline{D} \) of \( S_m^c \varphi \) and (3.6), (3.8) we have

\[
\varphi(\xi) = S_m^c \varphi(\xi) \leq \lim_{j \to \infty} u(\xi_j) \leq \lim_{j \to \infty} u(\xi_j) \leq \varphi(\xi).
\]

It follows that \( \lim_{z \to \xi} u(z) = \varphi(\xi) \). This proves our claim. By the same argument as the one given at the end of the proof of Theorem 1.4 (applying Choquet’s lemma and then Dini’s theorem on an exhaustion sequence of compact subsets of \( \overline{D} \setminus K \)) we conclude that \( u \) may be approximated uniformly on compact sets of \( \overline{D} \setminus K \) by elements in \( SH_m(D) \). Finally, for maximality of \( u \), let \( w \in SH_m(D) \) with \( w \leq u \) on \( D \setminus U \) for some open set \( U \) relatively compact in \( D \). Then the function

\[
\tilde{u}(z) := \begin{cases} 
\max \{ u(z), w(z) \}, & z \in U \\
u(z) & z \in D \setminus U 
\end{cases}
\]

belongs to \( SH_m(D) \). Moreover, \( \tilde{u} \) is a member in the defining family for \( S_m \varphi \). Therefore \( \tilde{u} \leq S_m \varphi = u \) on \( D \). In particular, \( w \leq u \) on \( U \). This proves the existence of the desired solution.

Uniqueness. Assume that there exist bounded continuous maximal functions \( u_1, u_2 \in SH_m(D) \) on \( D \) such that

\[
\lim_{z \to \xi, z \in D} u_1(z) = \lim_{z \to \xi, z \in D} u_2(z) = \varphi(\xi), \quad \forall \xi \in (\partial D) \setminus K.
\]

Let \( \{D_j\}_{j \geq 1} \) be a sequence of relative compact open subset of \( D \) with \( D_j \uparrow D \). Fix \( \varepsilon > 0 \), since \( u_2 \) is bounded from below on \( D \) we can find \( j_0 \geq 1 \) so large such that

\[
u_1 + \varepsilon v - \varepsilon \leq u_2 \text{ on } D \setminus D_{j_0}.
\]

It follows from maximality of \( u_2 \) that

\[
u_1 + \varepsilon v - \varepsilon \leq u_2 \text{ on } D.
\]
By letting $\varepsilon \downarrow 0$, we infer that $u_1 \leq u_2$ on $D$. Similarly we also get $u_2 \leq u_1$ on $D$. Therefore $u_1 = u_2$ on $D$.

The theorem is proved. \hfill \Box

**Remark.** If we do not assume that $v > -\infty$ on $D$ then a slight modification of the above proof (similar to the proof of Theorem 1.3) gives a maximal $m$--sh function $u$ on $D$ with boundary values $\varphi$ (on $(\partial D) \setminus K$) which is only continuous at every point $z \in D$ with $v(z) > -\infty$.

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