Research Article

Feedback linearization-based tracking control of a tilt-rotor with cat-trot gait plan

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Abstract
With the introduction of the laterally bounded forces, the tilt-rotor gains more flexibility in the controller design. Typical feedback linearization methods utilize all the inputs in controlling this vehicle; the magnitudes as well as the directions of the thrusts are maneuvered simultaneously based on a unified control rule. Although several promising results indicate that these controllers may track the desired complicated trajectories, the tilting angles are required to change relatively fast or in large scale during the flight, which turns to be a challenge in application. The recent gait plan for a tilt-rotor may solve this problem; the tilting angles are fixed or vary in a predetermined pattern without being maneuvered by the control algorithm. Carefully avoiding the singular decoupling matrix, several attitudes can be tracked without changing the tilting angles frequently. While the position was not directly regulated in that research, which left the position-tracking still an open question. In this research, we elucidate the coupling relationship between the position and the attitude. Based on this, we design the position-tracking controller, adopting feedback linearization. A cat-trot gait is further designed for a tilt-rotor to track the reference; three types of references are designed for our tracking experiments: set point, uniform rectilinear motion, and uniform circular motion. The significant improvement with less steady state error is witnessed after equipping with our modified attitude–position decoupler. It is also found that the frequency of the cat-trot gait highly influenced the steady state error.

Keywords
Quadcopter, tilt-rotor, feedback linearization, gait plan, tracking control, cat trot, stability

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Introduction
Comparing with the conventional quadrotor,1–4 the tilt-rotors5–9 provide the lateral forces, which are not applicable to the collinear/coplanar platforms (e.g. conventional quadrotors). The additional mechanical structures (usually tilting motors) mounted on the arms of the tilt-rotor provide the possibility of changing the direction of each thrust or “tilting.” As a consequence, the number of inputs increases to eight (four magnitudes of the thrust and four directions of the thrusts).

One of the systematic methods to solve tracking problem for a conventional quadrotor is feedback linearization (dynamic inversion), which transfers the nonlinear system to a linear one applicable to implement the linear controllers. Hereon, each output of interest is manipulated individually. Since the number of the inputs in conventional
quadrotors (four) is less than the number of degrees of freedom (six), a controller can independently stabilize four outputs at most. Typical choices of these four outputs can be attitude–altitude\(^{10-14}\) and position–yaw\(^{6,15-18}\).

On the contrary, since the number of inputs in Ryll’s tilt-rotors (eight) is larger than the number of degrees of freedom, the vehicle becomes an over-actuated system. This property intrigues the research on fully tracking the entire degrees of freedom; the tilt-rotor\(^7\) not only tracks the desired time-specified position but also the desired attitude relying on the eight inputs.\(^{19,20}\) Although the promising results yield acceptable tracking result, the tilting angles vary greatly or over-rapidly, which sharpens the feasibility of implementing the relevant controllers.

To solve this problem, our previous research\(^{21}\) plans the gait of the tilt-rotor before applying feedback linearization. The tilting angles are assigned beforehand, rather than manipulated by a unified control rule. Hereby, the magnitudes of the thrusts are the only inputs.

However, several challenges may hinder the application of this method. One of them is the singular decoupling matrix\(^{22}\) of feedback linearization. Considerable attention has been paid to this issue for the conventional quadrotor.\(^{10,23}\) It has been proved that the decoupling matrix is always invertible for a tilt-rotor with an over-actuated control scenario.\(^7\) However, this matrix may be singular if we take the magnitudes of the thrusts as the only inputs after the gait plan for a tilt-rotor. Further discussions on avoiding the singular decoupling matrix can refer to our previous work.\(^{21}\)

On the other hand, though the previous research\(^{21}\) witnessed promising result in tracking attitude and altitude, the position \((X, Y)\) is not successfully stabilized. One may notice that the relationship between the acceleration \((\ddot{X}, \ddot{Y})\) and the attitude has been elucidated by an attitude–position decoupler in a conventional quadrotor.\(^{12,14,20,24-26}\) While this decoupler does not hold for a tilt-rotor. A modified attitude–position decoupler applicable to a tilt-rotor is deduced in this article.

Similar to the gait plan problem in four-legged robots,\(^{27-29}\) we\(^{21}\) proposed several gaits for a tilt-rotor, averting the singular decoupling matrix. However, the gait in that research\(^{21}\) did not consider the gait patterns with varying tilting angles; all the gaits analyzed were the combinations of the fixed tilting angles. Another contribution of this research is to adopt the time-specified varying gait, which is inspired by cat trot.\(^{30-32}\)

Three types of references are designed to track for a tilt-rotor in simulation (Simulink). Notable improvements in tracking result are witnessed after the application with our modified attitude–position decoupler.

The rest of the article is structured as follows: The second section briefs the dynamics of the tilt-rotor. The third section explores the relationship coupled in attitude and position in a tilt-rotor before proposing the modified attitude–position decoupler, which is further used to design a controller. The fourth section introduces a gait for a tilt-rotor which is inspired by a cat-trot gait. The discussions upon the stability of the relevant controller are addressed in the fifth section. The sixth section introduces the settings of the simulation tests, the results of which are displayed in the seventh section. Finally, we make conclusions and discussions in the eighth section.

**Dynamics of the tilt-rotor**

Figure 1\(^{21}\) sketches Ryll’s tilt-rotor. This tilt-rotor\(^7,21\) can adjust the direction of each thrust during the flight; tilting each arm changes the direction of the thrust, which is...
restricted in the relevant yellow plane in Figure 1. \( \alpha_i \) (\( i = 1, 2, 3, 4 \)) represents the tilting angles.

The position \( P = [X \ Y \ Z]^T \) is ruled by

\[
\dot{P} = \begin{bmatrix}
0 \\
0 \\
-g
\end{bmatrix} + \frac{1}{m} W_R \cdot F(\alpha) \cdot \begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4 \\
w_5
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
-g
\end{bmatrix} + \frac{1}{m} W_R \cdot F(\alpha) \cdot w
\]

where \( m \) is the total mass, \( g \) is the gravitational acceleration, \( \omega_i \) (\( i = 1, 2, 3, 4 \)) is the angular velocity of the propeller: \( \omega_{13} < 0, \omega_{24} > 0 \) with respect to the propeller-fixed frame, \( W_R \) is the rotational matrix\(^4\) from the inertial frame, \( O_EX_EY_EZ_E \), to the body-fixed frame, \( O_BX_BY_BZ_B \)

\[
W_R = \begin{bmatrix}
c c \sin \phi \ c \sin \psi & s \phi \ c \psi - c \phi \ s \psi & c \phi \ c \psi + s \phi \ s \psi \\
c \cos \phi & c \phi \ c \psi + s \phi \ s \psi & s \phi \ c \psi - c \phi \ s \psi \\
-s \theta & s \phi & c \phi
\end{bmatrix}
\]

The relationship between the angular velocity of the body, \( \omega_B \), and the attitude rotation matrix (\( W_R \)) is given\(^3\) by

\[
W_R = \hat{W} R = \hat{\omega}_B R
\]

where "\( \hat{\cdot} \)" is the hat operation to produce the skew matrix.

We refer the readers to the literature\(^5\)\(^-\)\(^7\)\(^9\)\(^\)\(^10\)\(^\)\(^11\)\(^\)\(^12\)\(^\)\(^13\)\(^\)\(^14\)\(^\)\(^15\) for the detail in modeling the coefficients of the thrust, drag moment, and inertia moment, \( I_B \). They are assumed to be constant in this research.

The parameters of this tilt-rotor are specified as follows:

\( m = 0.429 \text{ kg}, \quad L = 0.1785 \text{ m}, \quad g = 9.8 \text{ N/kg}, \quad I_B = \text{diag}([2.24 \times 10^{-3}, 2.99 \times 10^{-3}, 4.80 \times 10^{-3}]) \text{ kg} \cdot \text{m}^2. \)

### Attitude–position decoupler and feedback linearization

The controller for the tilt-rotor comprises three parts: modified attitude–position decoupler, feedback linearization, and high-order PD controller.

#### Scenario of the controller

As mentioned, this controller can only control four degrees of freedom independently at most. Noticing that picking position and yaw may introduce the singular decoupling matrix,\(^10\)\(^25\) the independently controlled variables decided in this research are attitude and altitude.\(^11\)\(^12\)\(^20\)\(^25\)

The rest of the degrees of freedom (position \( X, Y \)) are tracked by adjusting the attitude, relying on the coupled relationship between the position and attitude ("Modified attitude–position decoupler" section). This nested structure can be found in Figure 2 (green part).

After dealing with the coupling relationship in position and attitude, an animal-inspired gait is designed for the tilt-rotor, which is detailed in the "Cat-trot-inspired gait" section.

Finally, feedback linearization ("Feedback linearization" section) is applied to accommodate a linear controller ("Third-order PD controller" section).

### Modified attitude–position decoupler

The relationship (conventional attitude–position decoupler) between attitude and position in a conventional quadrotor\(^10\)\(^\)\(^14\)\(^\)\(^17\)\(^\)\(^20\)\(^\)\(^25\)\(^\)\(^26\)\(^\)\(^38\)\(^\)\(^40\) is

\[
\phi = \frac{1}{g} \cdot (\dot{X} \cdot s\psi - \ddot{Y} \cdot c\psi)
\]
This relationship is deduced by linearizing the dynamics of a conventional quadrotor near hovering. However, this attitude–position decoupler does not hold for the tilt-rotor if the tilting angles are not all zero. Proposition 1 gives the modified attitude–position decoupler for a tilt-rotor.

Proposition 1 (Modified attitude–position decoupler). The attitude and the position of a tilt-rotor can be approximately decoupled as

\[ q = \frac{1}{g} \left( \vec{X} \cdot c\psi + \vec{Y} \cdot s\psi \right) \]  

(8)

\[ \phi = \frac{1}{g} \left( \vec{X} \cdot s\psi - \vec{Y} \cdot c\psi \right) + \frac{F_Y}{mg} \]  

(9)

\[ \theta = \frac{1}{g} \left( \vec{X} \cdot c\psi + \vec{Y} \cdot s\psi \right) - \frac{F_X}{mg} \]  

(10)

where \( F_X \) and \( F_Y \) are defined by

\[
\begin{bmatrix}
F_X \\
F_Y
\end{bmatrix} = K_f \cdot \begin{bmatrix}
0 & s2 & 0 & -s4 \\
-s1 & 0 & -s3 & 0
\end{bmatrix} 
\cdot \begin{bmatrix}
I_B^{-1} \cdot \tau(\alpha) \\
K_f \cdot \begin{bmatrix}
0 & 0 & 1 & |\alpha|_F(\alpha)
\end{bmatrix}^{-1} \cdot \begin{bmatrix}
0 \\
0 \\
g
\end{bmatrix}
\end{bmatrix} 
\]  

(11)

Proof. Linearize equation (1) at the equilibrium states

\[ \hat{\psi} = I_B^{-1} \cdot \tau(\alpha) \cdot \omega = 0 \]  

(12)

\[ \dot{Z} = 0 \]  

(13)

Ignoring the high-order infinitesimal terms (e.g. \( \phi \cdot \theta \), \( \phi^2 \), \( \theta^2 \)) yields equation (11).

Formulas (12) and (13) guarantee the angular accelerations and the vertical acceleration of the tilt-rotor are zero, given the assumption of the attitudes in formula (14). The components of the thrust along \( \vec{X} \) and \( \vec{Y} \), with respect to the body-fixed frame, are then calculated in formula (11) as \( F_X \) and \( F_Y \).

Remark 2. Given \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \), we receive \( F_X = F_Y = 0 \). Subsequently, equations (9) and (10) degrade to equations (7) and (8) in this special case. In other words, the attitude–position decoupler for the conventional quadrotor is a special case of the modified attitude–position decoupler for the tilt-rotor.

Feedback linearization

The degrees of freedom for the controller to track independently are attitude and altitude.

Define

\[ \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix} = \begin{bmatrix}
\phi \\
\theta \\
\psi \\
Z
\end{bmatrix} \]  

(15)

Since \( \omega_{1,3} < 0 \), \( \omega_{2,4} > 0 \), we have

\[ (\omega_{1,3} \cdot |\omega_i|)^2 = 2 \cdot \omega_i \cdot |\omega_i| \]  

(16)

Assuming

\[ \dot{\alpha}_i = 0, i = 1, 2, 3, 4 \]  

(17)

calculating the third derivative of equation (18) yields...
where $\Delta$ is called decoupling matrix.\textsuperscript{22} $[\phi_1 \phi_2 \phi_3 \phi_4]^T \triangleq U$ is the updated input vector.

Observing equation (18), one receives the decoupled relationship in equation (19), compatible with the controller design process, which will be detailed in the “Third-order PD controller” section.

$$
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3 \\
\dot{y}_4 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\Delta
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4 \\
\end{bmatrix} + Ma
$$

(19)

$$
4.000 \cdot c_1 \cdot c_2 \cdot c_3 \cdot c_4 + 5.592 \cdot (c_1 \cdot c_2 \cdot c_3 \cdot s_4 - c_1 \cdot c_2 \cdot s_3 \cdot c_4 + c_1 \cdot s_2 \cdot c_3 \cdot s_4 + c_1 \cdot s_2 \cdot c_3 \cdot c_4 - s_1 \cdot c_2 \cdot c_3 \cdot c_4) + 0.9716 \cdot \\
( + c_1 \cdot c_2 \cdot s_3 \cdot s_4 + c_1 \cdot s_2 \cdot s_3 \cdot c_4 + c_1 \cdot c_2 \cdot s_3 \cdot s_4 + c_1 \cdot s_2 \cdot c_3 \cdot c_4) + 2.000 \cdot ( - c_1 \cdot s_2 \cdot c_3 \cdot s_4 - s_1 \cdot c_2 \cdot s_3 \cdot c_4) + 0.1687 \cdot \\
( - c_1 \cdot s_2 \cdot s_3 \cdot s_4 + s_1 \cdot c_2 \cdot s_3 \cdot s_4 - s_1 \cdot s_2 \cdot c_3 \cdot s_4 + s_1 \cdot s_2 \cdot c_3 \cdot c_4) \neq 0
$$

(20)

where $s II = \sin\alpha II, c II = \cos\alpha II, (II = 1, 4), s A = \sin\Lambda, c A = \cos\Lambda, (\Lambda = \phi, \theta)$.

\textbf{Proof.} See our previous research.\textsuperscript{21} In this research, we adopt the cat-trot gait defined by

$$
\begin{align*}
\alpha_1 &= -\rho \\
\alpha_2 &= -\rho \\
\alpha_3 &= \rho \\
\alpha_4 &= \rho
\end{align*}
$$

(21) (22) (23) (24)

where $\rho$ is a time-specified gait, which will be specified in the “Cat-trot-inspired gait” section.

\textbf{Proposition 3.} When the roll angle and pitch angle of the tilt-rotor are close to zero, the cat-trot gait introduces no singular problem.
\[ \ddot{y}_{ar} = \ddot{y}_{ar} + K_{PZ_1} \cdot (\ddot{y}_{ar} - \ddot{y}_a) + K_{PZ_2} \cdot (\dot{y}_{ar} - \dot{y}_a) + K_{PZ_1} \cdot (\dot{y}_a - y_a) \]  

(27)

where \( K_{PZ_i} \) (i = 1, 2, 3) is the 3-by-3 diagonal control coefficient matrix, \( K_{PZ_i} \) (i = 1, 2, 3) is the control coefficient (scalar), \( y_j \) (j = 1, 2, 3, 4) is the state, \( y_{rj} \) (j = 1, 2, 3, 4) is the reference, \( y_{3r} \) represents the reference of the yaw angle, which is kept zero in this research; that is \( \ddot{y}_{3r} = y_{3r} = \dot{y}_{3r} = y_{3r} = 0 \). \( y_{ar} \) represents the reference of the altitude, which is also kept zero in this research; that is \( \ddot{y}_{ar} = \ddot{y}_{ar} = \dot{y}_{ar} = y_{ar} = 0 \). The derivatives of the references of the roll angle and pitch angle are set zero; that are \( \ddot{y}_{1r} = \ddot{y}_{1r} = y_{1r} = 0 \) and \( \ddot{y}_{2r} = \ddot{y}_{2r} = y_{2r} = 0 \). As for the reference of the roll angle and pitch angle, \( y_{1r} (\phi_r) \) and \( y_{2r} (\theta_r) \) are defined in formulas (28) to (30) to track the position \((X, Y)\).

The control parameters in the above controller are specified as: \( K_{PZ_1} = \text{diag}([1, 1, 1]) \), \( K_{PZ_2} = \text{diag}([10, 10, 1]) \), \( K_{PZ_3} = \text{diag}([50, 50, 1]) \), \( K_{PZ_2} = 10 \), \( K_{PZ_3} = 5 \), \( K_{PZ_3} = 10 \).

As for tracking the rest degrees of freedoms \((X, Y)\), firstly design the PD controller

\[
\begin{bmatrix}
\dot{X}_d \\
\dot{Y}_d
\end{bmatrix} = \begin{bmatrix}
\dot{X}_r \\
\dot{Y}_r
\end{bmatrix} + \begin{bmatrix}
K_{X_1} & 0 \\
0 & K_{Y_1}
\end{bmatrix} \cdot \begin{bmatrix}
\dot{X}_r - X_r \\
\dot{Y}_r - Y_r
\end{bmatrix} + \begin{bmatrix}
K_{X_2} & 0 \\
0 & K_{Y_2}
\end{bmatrix} \cdot \begin{bmatrix}
X_r - X \\
Y_r - Y
\end{bmatrix}
\]

(28)

where \( K_{X_1} = K_{Y_1} = K_{X_2} = K_{Y_2} = 1 \). \( X_r, \dot{X}_r, X_r, \dot{Y}_r, Y_r, \dot{Y}_r \) are the references of the position and their higher derivatives. They are different in each simulation depending on the type of the reference and will be detailed in the “Trajectory-tracking experiments” section.

The output (left side) of this PD controller is the desired position, which will be transferred to the desired attitude by modified attitude–position decoupler,

\[
\phi_r = \frac{1}{g} \cdot \left( \dot{X}_d \cdot s\psi_r - \dot{Y}_d \cdot c\psi_r \right) - \frac{F_y}{mg}
\]

(29)

\[
\theta_r = \frac{1}{g} \cdot \left( \dot{X}_d \cdot c\psi_r + \dot{Y}_d \cdot s\psi_r \right) - \frac{F_x}{mg}
\]

(30)

where \( \psi_r \) is the reference of the yaw angle \( (y_{3r}) \), which is kept zero in this research.

Notice that the attitude–altitude controller in (26) is designed much faster than the position-tracking controller in (28); the control coefficients, \( K \), are designed much larger in (26) than the ones in (28).

### Cat-trot-inspired gait

This research adopts three different gaits (fixed tilting angles, cat trot with instant switch, and cat trot with continuous switch).

#### Fixed tilting angles

The gait with the fixed tilting angles is defined by

\[
\alpha_1 = \alpha_2 = -\rho \quad (31)
\]

\[
\alpha_3 = \alpha_4 = \rho \quad (32)
\]

The tilting angles remains constant \((\rho \text{ remains constant})\) during the entire flight. In the simulations, we compare the results with different \( \rho \) \( (\rho = 0.65, \rho = 0.65/2, \rho = 0, \rho = -0.65/2, \rho = -0.65) \).

#### Cat trot with instant switch

Typical cat gaits can be categorized as walk gait,30 trot gait,31,32 and gallop gait.31,32

Parallel to the trot gait of a cat, we plan the cat-trot-inspired gait, which is specified as

\[
\begin{align*}
t - n \cdot T & \in \left[ 0, \frac{T}{2} \right] : \rho = 0.65 \\
t - n \cdot T & \in \left( \frac{T}{2}, T \right] : \rho = -0.65
\end{align*}
\]

(33)

where \( \rho \) determines the tilting angles in (21) to (24), \( T \) is the period of the gait, \( t \) represents the current time, \( n \) is the floor of \( t/T \). Note that the tilting angles change instantly in this gait plan. Thus, we call this cat-trot gait with instant switch.

We compare the result in the simulations with different periods, \( T \).

#### Cat trot with continuous switch

In the cat-trot gait with instant switch, the tilting angles change discretely. This section provides a continuous cat-trot gait.
where \( \rho \) determines the tilting angles in (21) to (24), \( T \) is the period of the gait, \( t \) represents the current time, \( n \) is the floor of \( t/T \). We call this cat-trot gait with continuous switch.

Also, we compare the result in the simulations with different periods, \( T \).

**Stability analysis**

In this section, we address the discussion on the stability of our controller. Firstly, we provide the stability proof to our controller. Secondly, we make comments on the potential state errors.

**Stability proof**

Noticing that the singular decoupling matrix is avoided (see Proposition 3), the original nonlinear system is stable if the linearized system is proved stable, given that the constraints (e.g. non-negative constraints of the thrusts, upper bound of the thrusts, etc.) are not activated; advanced stability analyses are demanded if the constraints are activated. \(^41\) In this research, these constraints are not activated, which simplifies our discussion on the stability.

**Proposition 4 (Exponential stability of attitude and altitude).** The attitude and altitude are exponentially stable if

\[
\begin{bmatrix}
K_{P1} \\
K_{PZ_1}
\end{bmatrix} > 0 \quad (35)
\]

\[
\begin{bmatrix}
K_{P2} \\
K_{PZ_1}
\end{bmatrix} > 0 \quad (36)
\]

\[
\begin{bmatrix}
K_{P3} \\
K_{PZ_1}
\end{bmatrix} > 0 \quad (37)
\]

\[
\begin{bmatrix}
K_{P1} \\
K_{PZ_1}
\end{bmatrix}, \begin{bmatrix}
K_{P2} \\
K_{PZ_2}
\end{bmatrix} - \begin{bmatrix}
K_{P3} \\
K_{PZ_3}
\end{bmatrix} > 0 \quad (38)
\]

**Proof.** Applying Hurwitz Criterion or Routh Criterion to (26) and (27) yields (35) to (38).

**Proposition 5 (Exponential stability of position \((X, Y)\)).** The position \((X, Y)\) is exponentially stable if

\[
\begin{bmatrix}
K_{X_1} & 0 \\
0 & K_{Y_1}
\end{bmatrix} > 0 \quad (39)
\]

\[
\begin{bmatrix}
K_{X_2} & 0 \\
0 & K_{Y_2}
\end{bmatrix} > 0 \quad (40)
\]

**Remarks to the stability proof**

In the “Stability proof” section, we proved that the attitude and altitude are exponentially stable applying this controller, which has been verified by simulations. \(^21\) However, the references in that study \(^21\) were attitude and altitude. The position \((X, Y)\) was not required to be stabilized.

Though the modified attitude–position decoupler in formulas (29) and (30) and the position-tracking controller in (28) may be used to track a position, proved in Proposition 4 and 5, there can be steady state error in position using this controller.

**Remark 6 (Steady state error in position \((X, Y)\) for tilt-rotors).** There can be steady state error with the application of the controllers in (26) to (30). This is because that the modified attitude–position decoupler was deduced by linearization at the state defined in (12) to (14). However, the states in (12) to (14) may not be satisfied simultaneously; generating zero angular accelerations and vertical acceleration in (12) and (13) may only be possible for non-zero roll angles and pitch angles, indicating that the conditions in (14) are violated.

As the consequence, the system may stabilized at a state near the state for linearization rather than on it. This unprecise relationship between the position and the attitude can introduce unexpected steady state error.

**Remark 7 (Steady state error in position \((X, Y)\) for conventional quadrotors).** There are no steady state errors in position \((X, Y)\) for a conventional quadrotor applying PD controllers in a position-tracking problem. \(^42, 43\) The equilibrium state for linearization to decouple the conventional quadrotor’s attitude and position is also at (12) to (14). While the conventional quadrotor stabilizes at this state, which is picked for linearization. Consequently, no bias is introduced in the linearization process.

Figure 3 also helps demonstrate the reason why there are no steady state error for a conventional quadrotor. We linearize (green dash line) the dynamics of a quadrotor at the equilibrium for a conventional quadrotor (red point), where both the roll angle and pitch angle are zero. On the other
hand, this state (red point) is exactly the quadrotor is supposed to be stabilized at.

**Trajectory-tracking experiments**

The tilt-rotor is expected to track three types of trajectories in our experiments. They are set point, uniform rectilinear motion, and uniform circular motion.

**Set point**

One may notice that our previous research\textsuperscript{21} tracks the set point using the similar controller. However, the set point set in that work\textsuperscript{21} is the attitude–altitude reference (e.g. \(\phi, \theta, \psi, Z = (0, 0, 0, 0)\)). The position \((X, Y)\) is not stabilized; stabilizing at \((\phi, \theta, \psi, Z) = (0, 0, 0, 0)\) produces accelerations along \(X\) and \(Y\) for the tilt-rotor.

In modified attitude–position decoupler (Proposition 1), we explicit the relationship between the position \((X, Y)\) and attitude, providing the possibility of position-tracking for the tilt-rotor.

The set point reference designed in this experiment is position–yaw pair

\[
(X, Y, Z, \psi) = (0, 0, 0, 0)
\]

(41)

Thus, the reference settings in formula (28) are \(X_r = \dot{X}_r = \dot{X}_r = 0, Y_r = \dot{Y}_r = \dot{Y}_r = 0\).

We track this set point adopting the gaits with the different fixed tilting angles \(\rho (\rho = 0.65, \rho = 0.65/2, \rho = 0, \rho = -0.65, \rho = -0.65/2)\) (“Fixed tilting angles” section).

As discussed in Remark 6, the steady state error is taken to evaluate the result of each experiment. We compare the resulting steady state errors between the controller adopting the conventional attitude–position decoupler in formulas (7) and (8) and the controller adopting the modified attitude–position decoupler in Proposition 1 for each gait.

**Uniform rectilinear motion**

The thorough introductions on the cat gaits can be referred to specialized studies on cats,\textsuperscript{30–32} where cat-walk gait is typically adopted at the moving speed of 0.54–0.74 m/s,\textsuperscript{30} cat-trot gait and cat-gallop gait are typically adopted at the moving speeds of 2.12 m/s and 3.5 m/s, respectively.\textsuperscript{31}

Referring to this, we pair our reference in (42) with the speed around 2.12 m/s.

\[
\begin{aligned}
X_r &= 1.5 \cdot t \\
Y_r &= 1.5 \cdot t
\end{aligned}
\]

(42)

Besides formula (42), further reference settings in formula (28) are \(X_r = \dot{X}_r = 0, Y_r = \dot{Y}_r = 1.5, \dot{Y}_r = 0\).

Three different kinds of the gaits are analyzed: the fixed tilting angle gaits \((\rho = 0.65)\) in the “Fixed tilting angles” section, the cat trot with instant switch with different periods in the “Cat trot with instant switch” section, the cat trot with continuous switch with different periods in the “Cat trot with continuous switch” section.

Also, the steady state error is taken to evaluate the result of each experiment. We compare the resulting steady state errors between the controller adopting the conventional attitude–position decoupler in formulas (7) and (8) and the controller adopting the modified attitude–position decoupler in Proposition 1 for each gait.

**Uniform circular motion**

A circular trajectory (uniform circular motion) is designed as

\[
\begin{aligned}
X_r &= 5 \cdot \cos(0.1 \cdot t) \\
Y_r &= 5 \cdot \sin(0.1 \cdot t)
\end{aligned}
\]

(43)

\[
\begin{aligned}
\dot{X}_r &= -0.1 \times 5 \cdot \sin(0.1 \cdot t) \\
\dot{Y}_r &= 0.1 \times 5 \cdot \cos(0.1 \cdot t)
\end{aligned}
\]

(44)
This indicates that the radius of the desired trajectory is 5 meters. The period is 20π seconds.

Specially, besides formulas (43) and (44), further reference settings in formula (28) are \( \dot{X}_r = 0 \) and \( \dot{Y}_r = 0 \). Although the derivatives of formula (44) are not zero, \( \dot{X}_r \) and \( \dot{Y}_r \) are set zero to avoid introducing the steady state error from high order.

Same as the previous reference, three different kinds of the gaits are analyzed in this reference: the fixed tilting angle gaits (ρ = 0.65) in the “Fixed tilting angles” section, the cat trot with instant switch with different periods in the “Cat trot with instant switch” section, the cat trot with continuous switch with different periods in the “Cat trot with continuous switch” section.

Similarly, the steady state error is taken to evaluate the result of each experiment. We compare the resulting steady state errors between the controller adopting the conventional attitude–position decoupler in formulas (7) and (8) and the controller adopting the modified attitude–position decoupler in Proposition 1 for each gait.

**Initial condition**

The absolute value of each initial angular velocity of the propellers is 300 rad/s in all the simulations. This angular velocity is not sufficient to compensate the effect of the gravity and will cause unstable in attitude if maintaining this speed. The tilt-rotor is expected to track the desired trajectory at this speed. The tilt-rotor degrades to the conventional quadrotor in this gait.

**Results**

This section displays the results of the tracking experiments with different references.

**Set point**

The set point are tracked by the gaits with the conventional attitude–position decoupler (marked as B in Figure 4) and with the modified attitude–position decoupler (marked as A in Figure 4).

The steady state errors along X and Y (\( e_x, e_y \)) are reported in Figure 4.

\[
A_1(0.65), A_2(\frac{0.65}{2}), A_3(0), A_4(-\frac{0.65}{2}), A_5(0.65) \]

in Figure 4 represent the resulting steady state errors adopting the relevant gaits equipped with the conventional attitude–position decoupler. Similarly, \( B_1(0.65), B_2(\frac{0.65}{2}), B_3(0), B_4(-\frac{0.65}{2}), B_5(0.65) \) in Figure 4 represent the resulting steady state errors adopting the relevant gaits equipped with the modified attitude–position decoupler.

Notice that the gait \( \rho = 0 \) receives zero dynamic state error for both cases, with the conventional attitude–position decoupler (marked as \( B_3 \)) and with the modified attitude–position decoupler (marked as \( A_3 \)). This is because that the tilt-rotor degrades to the conventional quadrotor in this gait. For the rest gaits, the modified attitude–position decoupler notably reduces the steady state error in the same gait.

**Uniform rectilinear motion**

In tracking a uniform rectilinear motional reference defined in the “Uniform rectilinear motion” section, Figure 5
displays the resulting dynamic state errors, $e_x = x_r - x$ and $e_y = y_r - y$, adopting the gait $\rho = 0.65$ (fixed tilting angles), equipped with the controller with the conventional attitude–position decoupler, denoted by $e_x$ (without) and $e_y$ (without), and with the controller with the modified attitude–position decoupler, denoted by $e_x$ (with) and $e_y$ (with).

It can be clearly seen that the yellow curve, $e_x$ (with), and the purple curve, $e_y$ (with), are closer to zero comparing with the blue curve, $e_x$ (without), and the red curve, $e_y$ (without), respectively, after sufficient time. This indicates that the steady state errors are reduced after equipping with the modified attitude–position decoupler.

The tracking result (dynamic state errors) for the uniform rectilinear motional reference adopting the cat-trot gait with instant switch ($T = 2s$), defined in the “Cat trot with instant switch” section, is plotted in Figure 6. The blue and red curves represent the dynamic state errors along $X$ and $Y$, respectively, equipped with the conventional attitude–position decoupler. The yellow and purple curves represent the dynamic state errors along $X$, respectively, equipped with the modified attitude–position decoupler.

It can be concluded that the controller equipped with the modified attitude–position decoupler receives smaller supreme of the dynamic state errors, calculating from sufficient time, for both controllers, with the conventional attitude–position decoupler and the modified attitude–position decoupler, with different periods $T$ ($T = 1$, $T = 2$, $T = 3$), we display the supreme of the relevant results in Figures 8 and 9, where Figure 8 shows the results adopting cat-trot gait with instant switch and Figure 9 shows the results adopting cat-trot gait with continuous switch.

The red curves in both figures represent the results of the supreme of the dynamic state error, $(e_x^2 + e_y^2)\frac{1}{2}$, calculating from sufficient time, equipped with the conventional attitude–position decoupler. While the blue curves in both figures represent the results of the supreme of the dynamic state error, $(e_x^2 + e_y^2)\frac{1}{2}$, calculating from sufficient time, equipped with the modified attitude–position decoupler.

Figure 7 demonstrates the dynamic state errors for the uniform rectilinear motional reference adopting the cat-trot gait with continuous switch ($T = 2s$), defined in the “Cat trot with continuous switch” section. Identical to the previous denotations, the blue and red curves represent the dynamic state errors along $X$ and $Y$, respectively, equipped with the conventional attitude–position decoupler. The yellow and purple curves represent the dynamic state errors along $X$ and $Y$, respectively, equipped with the modified attitude–position decoupler.

Clearly, with the increase of the period, the gait receives larger dynamic state error (supremum) for both cat-trot gaits with instant switch and with continuous switch.

Figure 6. The resulting dynamic state errors, adopting the cat-trot gait with instant switch ($T = 2s$), equipped with the conventional attitude–position decoupler (blue curve: the dynamic state error along $X$, red curve: the dynamic state error along $Y$) and with the modified attitude–position decoupler (yellow curve: the dynamic state error along $X$, purple curve: the dynamic state error along $Y$).

Figure 7. The resulting dynamic state errors, adopting the cat-trot gait with continuous switch ($T = 2s$), equipped with the conventional attitude–position decoupler (blue curve: the dynamic state error along $X$, red curve: the dynamic state error along $Y$) and with the modified attitude–position decoupler (yellow curve: the dynamic state error along $X$, purple curve: the dynamic state error along $Y$).
The modified attitude–position decoupler significantly reduces the dynamic state error (supremum), especially for the gait with long period (e.g. $T = 3$) for both cat-trot gaits with instant switch and with continuous switch. While no significant differences in the dynamic state error (supremum) are reported for the gaits whose period is short for both cat-trot gaits with instant switch and with continuous switch.

In addition, the dynamic state error (supremum) received by the cat-trot gait with continuous switch (Figure 9) is smaller than the corresponding dynamic state error (supremum) received by the cat-trot gait with instant switch.

The modified attitude–position decoupler significantly reduces the dynamic state error (supremum), especially for the gait with long period (e.g. $T = 3$) for both cat-trot gaits with instant switch and with continuous switch. While no significant differences in the dynamic state error (supremum) are reported for the gaits whose period is short for both cat-trot gaits with instant switch and with continuous switch.

In addition, the dynamic state error (supremum) received by the cat-trot gait with continuous switch (Figure 9) is smaller than the corresponding dynamic state error (supremum) received by the cat-trot gait with instant switch.

Uniform circular motion

In tracking a uniform circular motional reference defined in the “Uniform circular motion” section, Figure 10 displays the resulting dynamic state errors, $e_x = x_r - x$ and $e_y = y_r - y$, adopting the gait $\rho = 0.65$ (fixed tilting angles), equipped with the controller with the conventional attitude–position decoupler, denoted by $e_x$ (without) and $e_y$ (without), and with the controller with the modified attitude–position decoupler, denoted by $e_x$ (with) and $e_y$ (with).

Similar results can be found that the yellow curve, $e_x$ (with), and the purple curve, $e_y$ (with), are closer to zero comparing with the blue curve, $e_x$ (without), and the red curve, $e_y$ (without), respectively, after sufficient time. This indicates that the steady state errors are reduced after equipping with the modified attitude–position decoupler.

The tracking result (dynamic state errors) for the uniform circular motional reference adopting the cat-trot gait with instant switch ($T = 2\pi$), defined in the “Cat trot with instant switch” section, is plotted in Figure 11. The blue and red curves represent the dynamic state errors along $X$ and $Y$, respectively, equipped with the conventional attitude–position decoupler. The yellow and purple curves represent the dynamic state errors along $X$ and $Y$, respectively, equipped with the modified attitude–position decoupler.

Similarly, the results show that the controller equipped with the modified attitude–position decoupler receives smaller supreme of the dynamic state errors, calculating from sufficient time; the maximums of the yellow and purple curves, calculating from sufficient time, are smaller than the maximums of the blue and red curves, respectively.
Figure 12 demonstrates the dynamic state errors for the uniform rectilinear motional reference adopting the cat-trot gait with continuous switch ($T = 2s$), defined in the “Cat trot with continuous switch” section. Identical to the previous denotations, the blue and red curves represent the dynamic state error along $X$, red curve: the dynamic state error along $Y$) and with the modified attitude–position decoupler (yellow curve: the dynamic state error along $X$, purple curve: the dynamic state error along $Y$).

Figure 12. The resulting dynamic state errors, adopting the cat-trot gait with continuous switch ($T = 2s$), equipped with the conventional attitude–position decoupler (blue curve: the dynamic state error along $X$, red curve: the dynamic state error along $Y$) and with the modified attitude–position decoupler (yellow curve: the dynamic state error along $X$, purple curve: the dynamic state error along $Y$).

Figure 13. Supremum of the dynamic state error, $(e_x^2 + e_y^2)^{1/2}$, calculating from sufficient time, in cat-trot gait with instant switch with different periods ($T = 1$, $T = 2$, $T = 3$). The red curve represents the results equipped with the conventional attitude–position decoupler. The blue curve represents the results equipped with the modified attitude–position decoupler.

Figure 13. Supremum of the dynamic state error, $(e_x^2 + e_y^2)^{1/2}$, calculating from sufficient time, in cat-trot gait with instant switch with different periods ($T = 1$, $T = 2$, $T = 3$). The red curve represents the results equipped with the conventional attitude–position decoupler. The blue curve represents the results equipped with the modified attitude–position decoupler.

Figure 14. Supremum of the dynamic state error, $(e_x^2 + e_y^2)^{1/2}$, calculating from sufficient time, in cat-trot gait with continuous switch with different periods ($T = 1$, $T = 2$, $T = 3$). The red curve represents the results equipped with the conventional attitude–position decoupler. The blue curve represents the results equipped with the modified attitude–position decoupler.

Figure 14. Supremum of the dynamic state error, $(e_x^2 + e_y^2)^{1/2}$, calculating from sufficient time, in cat-trot gait with continuous switch with different periods ($T = 1$, $T = 2$, $T = 3$). The red curve represents the results equipped with the conventional attitude–position decoupler. The blue curve represents the results equipped with the modified attitude–position decoupler.

Figure 12 demonstrates the dynamic state errors for the uniform rectilinear motional reference adopting the cat-trot gait with continuous switch ($T = 2s$), defined in the “Cat trot with continuous switch” section. Identical to the previous denotations, the blue and red curves represent the dynamic state errors along $X$ and $Y$, respectively, equipped with the conventional attitude–position decoupler. The yellow and purple curves represent the dynamic state errors along $X$ and $Y$, respectively, equipped with the modified attitude–position decoupler.

Also, the similar results are concluded for the cat-trot gait with continuous switch ($T = 2s$) in this case that the controller equipped with the modified attitude–position decoupler receives smaller supreme of the dynamic state errors, calculating from sufficient time; the maximums of the yellow and purple curves, calculating from sufficient time, are smaller than the maximums of the blue and red curves, respectively.

Likewise, to compare the supremum of the dynamic state error, $(e_x^2 + e_y^2)^{1/2}$, calculating from sufficient time, for both controllers, with the conventional attitude–position decoupler and the modified attitude–position decoupler, with different periods $T$ ($T = 1$, $T = 2$, $T = 3$), we display the supremum of the relevant results in Figures 13 and 14, where Figure 13 shows the results adopting cat-trot gait with instant switch and Figure 14 shows the results adopting cat-trot gait with continuous switch.
The red curves in both figures represent the results of the supremum of the dynamic state error, \((e_x^2 + e_y^2)^{1/2}\), calculating from sufficient time, equipped with the conventional attitude–position decoupler. While the blue curves in both figures represent the results of the supremum of the dynamic state error, \((e_x^2 + e_y^2)^{1/2}\), calculating from sufficient time, equipped with the modified attitude–position decoupler.

The similar results here show that, with the increase of the period, the gait receives larger dynamic state error (supremum) for both cat-trot gaits with instant switch and with continuous switch.

Also, the modified attitude–position decoupler significantly reduces the dynamic state error (supremum), especially for the gait with long period (e.g. \(T = 3\)) for both cat-trot gaits with instant switch and with continuous switch. While no significant differences in the dynamic state error (supremum) are reported for the gaits whose period is short for both cat-trot gaits with instant switch and with continuous switch.

In addition, the dynamic state error (supremum) received by the cat-trot gait with continuous switch (Figure 14) is smaller than the corresponding dynamic state error (supremum) received by the cat-trot gait with instant switch (Figure 13), given the same period and type of the attitude–position decoupler.

Conclusions and discussions

The decoupling matrix is invertible in feedback linearization (attitude–altitude) for a tilt-rotor, adopting cat-trot gait. All the references considered in this research (set point, uniform rectilinear motion, uniform circular motion) are successfully tracked with acceptable steady state error by cat-trot gait.

The relationship between position and attitude of the tilt-rotor is elucidated. The modified attitude–position decoupler is invented for position-tracking problem for a tilt-rotor. It significantly reduces the dynamic state error (or steady state error for the point reference) comparing with the conventional attitude–position decoupler.

The length of the period of the cat-trot gait highly influences the dynamic state error in tracking a uniform rectilinear reference and a uniform circular reference. Specifically, a gait with a shorter period tends to receive a smaller supremum of the dynamic state error, calculating after sufficient time. On the other hand, the modified attitude–position decoupler notably reduces the dynamic state error for the cat-trot gait with a large period.

In general, the continuous cat-trot gait receives smaller supremum of the dynamic state error, calculating after sufficient time, than the discrete cat-trot gait, given same period and the type of the attitude–position decoupler.

There are several questions remaining to be answered further.

Firstly, equation (17) takes the tilting angles constant; they are assumed to be constant during the entire flight. While the switching process in the discrete gait and the entire period in the continuous gait violate this condition. Addressing the discussions on the underlying robustness is beyond the scope of this research, which can be a further step.

Secondly, though the steady state errors and the supremum of the dynamic state errors, calculating after sufficient time, are found reduced after applying the modified attitude–position decoupler, further reduction of the dynamic state error may be possible by not ignoring the high-order infinitesimal terms while deducing the modified attitude–position decoupler.

Also, analysis on other different cat gaits (e.g. gallop and walk) can be another further step for the gait plan for the tilt-rotor.

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