Nonlinear Schrödinger-Poisson definition
of fine-structure-constant’s value $\sim 1/137$

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Abstract. By numerically investigating the nonlinear Schrödinger-Poisson eigenstates of a condensed Bose gas of charged particles that is confined in a two-dimensional axisymmetric parabolic potential $\frac{1}{2}m_\text{e}\omega^2r^2$ (e.g. quantum-dot helium), it is shown that the probability amplitude between two nonlinear—and hence non-orthogonal—eigenstates displays an interference pattern scaled (within 0.03 %) by $\sqrt{\pi\alpha} = 0.15141 \ldots$ Since $\alpha = e^2/\hbar c \sim 1/137$ is the fine structure constant, this stunning result—indeed velocity of light $c$ does not enter the present non-relativistic model—is tentatively explained by the existence of a “nonlinear” bound state of the trapped particle-particle interaction Coulomb field whose energy $E = \frac{1}{2}\pi\hbar\omega$ defines the induced emission or absorption equilibrium processes between two appropriate chemical potentials. Besides, a non-decoherence quantum-classical transition with increasing nonlinearity is pointed out. As a possible experimental test for the present theory, the $0s^2 \rightarrow 1s^2$ nonlinear transition in an $\hbar\omega = 1.66$ meV GaAs quantum-dot helium is emphasized.

1. Introduction
The numerical value $\sim 1/137$ of the fine-structure-constant $\alpha = e^2/\hbar c$ has been a mystery ever since it was discovered [1]. “It’s one of the greatest damn mysteries of physics: a magic number that comes to us with no understanding by man. We know what kind of a dance to do experimentally to measure this number very accurately but we don’t know what kind of a dance to do on a computer to make this number come out—without putting it in secretly!” [1], adds R. P. Feynman in his unique kidding style. The present paper suggests an original—and, remarkably enough: non-relativistic—scenario for such an “computer dance”. It makes use of a Schrödinger-Poisson (SP) differential model for single-particle nonlinear eigenstates in a particular two-electron physical system, namely quantum-dot helium (QDH). However, we insist that the present SP description is also quite appropriate for any harmonically confined and condensed Bose gas of charged particles, the present QDH system being only a very simple special case with experimental interest.

Quantum-dots can be viewed as artificially structured, disk-like shaped atoms in heterojunctions or metal-oxide-semiconductor devices where few electrons are confined to a length comparable to the mesoscopic effective Bohr radius $a_\text{B}$ ($a_\text{B} \sim 10^{-2}$ μm in the case of GaAs) [2–4]. Since electron numbers $N$ as low as one or two per dot have already been realized, QDH consisting of two electrons trapped in the two-dimensional (2D) axisymmetric harmonic potential $V(r) = \frac{1}{2}M\omega^2r^2$ (where $r^2 = x^2 + y^2$ and $M$ is the effective electron mass) is actually considered as the simplest realistic model for an interacting quantum system [5–6]. As itself or amongst
other such few-electron systems, it has been extensively studied in the relationship with the development of nanotechnologies [7–10]. Both its exact 2D-3D analytical [9] or 2D numerical [6] solutions are known.

Nonlinear eigenstates are as a rule not orthogonal [11]. The probability amplitude \(< \Psi_{NL}^i | \Psi_{NL}^j > \neq 0\) of a couple of stationary (time-independent) nonlinear (hence the subscripts) normalized SP eigenstates \(\Psi_{NL}^{i,j}\) yields the occupation number \(w_{NL}^{i,j} = \frac{|< \Psi_{NL}^i | \Psi_{NL}^j >|^2}{\Psi_{NL}^i | \Psi_{NL}^j >}\) of eigenstate \(\Psi_{NL}^i\) when the system is assumed to be in \(\Psi_{NL}^i\), or reverse. Therefore the SP quantum system is in a mixed state: its appropriate description in terms of the density matrix \(\rho\) is

\[
\rho = \frac{1}{1 + w_{NL}^{i,j}} |\Psi_{NL}^i > < \Psi_{NL}^i| + \frac{w_{NL}^{i,j}}{1 + w_{NL}^{i,j}} |\Psi_{NL}^j > < \Psi_{NL}^j|.
\]

or reverse. The paradoxical property, compared with standard linear theory, is that although the system is in the mixed state described by Eq. (1) and not in a superposition state (indeed this later is not a solution of the SP differential system, due to nonlinearity), it nevertheless allows interference between the two eigenstates \(|\Psi_{NL}^i >\) to occur, according to the following theorem:

\[
< \Psi_{NL}^i | \Psi_{NL}^j > = \frac{\Phi_{j,i}^{i,j}}{\mu_i - \mu_j} + \frac{\Phi_{i,j}^{i,j}}{\mu_j - \mu_i}.
\]  

The subscripts in \(\Phi_{j,i}^{i,j}\) and \(\Phi_{i,j}^{i,j}\) define the matrix elements—calculated by use of the SP nonlinear eigenstates \(\Psi_{NL}^{i,j}\)—of the corresponding particle-particle interaction potential \(\Phi^{i,j}\) while \(\mu_{i,j}\) are those nonlinear eigenvalues (or chemical potentials) which respectively define \(\Psi_{NL}^{i,j}\). Eq. (2) is a direct consequence of the Hermiticity of the Laplacian operator in the Schrödinger equation of the parabolically-confined SP differential system [12]. The term on the rhs of Eq. (2) defines the probability amplitude for the system being in the nonlinear eigenstate \(|\Psi_{NL}^i >\) to be also in the nonlinear eigenstate \(|\Psi_{NL}^j >\), due to the interaction potential \(\Phi\) defined by the probability density \(|\Psi_{NL}^i|^2\) through the Poisson equation, while the second term of Eq. (2) defines the reverse process, namely, the probability amplitude for the system being in \(|\Psi_{NL}^j >\) to be also in \(|\Psi_{NL}^i >\) as a consequence of the interaction potential \(\Phi^{i,j}\) defined by \(|\Psi_{NL}^i|^2\). Eq. (2) is exact. Therefore no perturbative-like ordering in \(\Phi^{i,j}\) is needed although, of course, a straightforward time-independent perturbation scheme that considers \(\Phi^{i,j}\) as perturbative potential in the respective \((i,j)\) Schrödinger equation of the SP system (for instance when nonlinearity \(N\) is weak) allows one to recover it.

The two amplitudes in Eq. (2) interfere in the build-up of the probability amplitude \(< \Psi_{NL}^i | \Psi_{NL}^j >\) because the two \(i \leftrightarrow j\) processes are indistinguishable from the point of view of the (numerical) value of \(< \Psi_{NL}^i | \Psi_{NL}^j >\). Therefore this later defines the nonlinear quantum coherence in our two-level system. The resulting interference pattern is achieved by the \(\omega \to 0\) progressive “flattening” of the parabolic trap or, equivalently, by a progressive increase of the nonlinearity in the system as numerically shown by figure 3. The quantum coherence allows energy to be exchanged between the two energy levels or, more accurately, between the two nonlinear-eigenvalue chemical potentials. The present paper proposes that these energy exchanges be quantized in terms of the “nonlinear” (or “renormalized”) non-integer bunch \(E = \frac{1}{2}\pi\hbar\omega\) of harmonic quanta \(\hbar\omega\). It will be shown that the coupling probability per particle \(\sigma\) to the surrounding harmonic quanta—assumed to be in thermodynamical equilibrium between the emission and absorption processes—equals \(1/137.17\) in the present state of the numerical art. This result should be compared with the fine-structure-constant’s value \(1/137.04\).

The explanation for such an amazing coincidence—recall that the whole SP model is non-relativistic—might be due to the fundamental equivalence between i) the classical non-relativistic description à-la-Rutherford of an elastic electron Coulomb scattering; and ii) its quantum electrodynamics (QED) analogue defined (in the lowest-order in \(\alpha\)) by the probability amplitude

\[
\frac{1}{1 + w_{NL}^{i,j}} |\Psi_{NL}^i > < \Psi_{NL}^i| + \frac{w_{NL}^{i,j}}{1 + w_{NL}^{i,j}} |\Psi_{NL}^j > < \Psi_{NL}^j|.
\]
related to the single-vertex electron-photon coupling process [13]. In the former classical case that corresponds to the Poisson definition of the Coulomb potential, there is no photon at all (and hence obviously no velocity of light c) while in the later QED case, the photon (and hence c) crucially mediates the interaction through the coupling constant α. Although c disappears at the end of the QED calculation in the non-relativistic limit, the present work shows that the nonlinear probability amplitude $<\Psi_{\text{NL}}|\Psi_{\text{NL}}>$ seems to keep a memory of this α (and hence c) dependence even in the non-relativistic limit that defines the SP differential system.

2. Nonlinear quantum-classical Schrödinger-Poisson transition

We start from the stationary Schrödinger equation with the classical potential $W(x,y)$ that defines the 2D orbital eigenstate wave function $\Psi(x,y)$ per particle, together with its corresponding energy eigenvalue $\mu$:

$$-\frac{\hbar^2}{2M} \nabla^2 \Psi(x,y) + W(x,y)\Psi(x,y) = \mu \Psi(x,y).$$

(3)

Then we assume that $W(x,y) \equiv V(x,y) + \Phi(x,y)$ is but the mere addition of both classical potentials per particle, namely the external parabolic confining potential $V(x,y) = V(r) = \frac{1}{2} M \omega^2 r^2$ and the mutual electrostatic repulsive interaction potential $\Phi(x,y)$ between the two particles which is defined by Poisson equation:

$$\nabla^2 \Phi(x,y) = -4\pi \varepsilon |\Psi(x,y)|^2.$$

(4)

Since $|\Psi|^2 \propto [\text{length}]^{-2}$ and the two above potentials $V$ and $\Phi$ are actually potential energies, we must introduce, for dimensionless reasons related to Eq. (4), a characteristic energy $\varepsilon$ which is a typical measure of the SP nonlinearity and which will be specified as a functional of $\Psi$, $\Phi$ and $\mu$ later on. Note that the differential system of Eqs. (3–4) is only relevant for particles in the same orbital state $\Psi$: e.g. a Bose-Einstein condensate of charged particles. It also yields a non-relativistic Schrödinger description of a Cooper-like pair of opposite-spin electrons trapped in the same orbital bound state [11,14] whose physical meaning is transparent. Indeed, one electron, say electron a with orbital wave function $\Psi_a$, “feels” the repulsive electrostatic potential $\Phi_a$ that is being created by its fellow electron b with orbital wave function $\Psi_b$. This potential $\Phi_a$ is classically defined by the Poisson equation $\nabla^2 \Phi_a \propto -|\Psi_a|^2$ while $\Psi_a$ is solution of the one-particle Schrödinger equation including both classical potentials, namely the Coulomb interaction potential $\Phi_0$ and the external confining potential $V(r)$. The system is closed by the “orbital bosonic assumption” $\Psi_a \equiv \Psi_b$, which obviously agrees with the Pauli exclusion principle for two opposite-spin fermion electrons.

Due to axisymmetry, we have $\Psi(x,y) = \Psi(r, \phi) = \psi(r) e^{im\phi}$. The one-particle wavefunction $\Psi$ is thus eigenstate of the angular-momentum operator $-i\hbar \partial/\partial \phi$ related to its eigenvalue $m\hbar$. Its radial part $\psi(r)$, which describes the 2D confinement of the Bose gas, is defined by the following dimensionless ordinary-differential system:

$$\ddot{u} + \frac{1}{X} \dot{u} + \left[ C - \frac{X^2}{4} \right] u = 0;$$

$$\ddot{C} + \frac{1}{X} \dot{C} + \frac{4m^2}{X^4} = u^2,$$

(5a, b)

if $l_0 = \sqrt{\hbar/M\omega}$ is the characteristic harmonic length and if

$$X = \sqrt{2} \frac{r}{l_0}; \quad u = \frac{1}{\omega} \sqrt{\frac{2\pi \varepsilon}{M}} \psi; \quad C = \frac{\mu}{X^2} - \Phi.$$ 

(6a, b, c)

The dot stands for derivation with respect to the (dimensionless) radius $X$ and the tilde superscript labels energy in units of $\hbar \omega$. The one-particle probability of presence $|\Psi|^2$ must be
normalized to unity. Hence the corresponding norm $\mathcal{N}$ of the dimensionless solution $u$, which provides a first definition of $\mathcal{E}$, is given by

$$\int |\Psi|^2 d^2x = 1 \rightarrow \int_0^\infty u^2 X dX = \mathcal{N} = 2\tilde{\mathcal{E}}.$$  \hspace{1cm} (7)

Eqs. (5) yield $u(r)$ and $C(r)$ as functionals of $\mathcal{N}$, once given the (actually spatial) “initial conditions” $u_0 = u(0)$, $\dot{u}_0 = \dot{u}(0)$, $C_0 = C(0)$ and $\dot{C}_0 = \dot{C}(0)$. Amongst them, $u_0$ and $C_0$ are left free and will be numerically chosen by dichotomy in order to yield regular bound-state eigenstates defined by $u(X) \rightarrow 0$ for $X \rightarrow \infty$. We consider in the present work and for the sake of simplicity the two first $m = 0$ zero-angular-momentum nonlinear eigenmodes per particle, namely $u_1(X)$ and $u_3(X)$, for which $\dot{C}_0 = \dot{u}_0 = 0$ (the nonlinear SP differential system works equally well with $m \neq 0$ eigenstates, as shown by figure 2). The subscripts (1) and (3) refer to the $u_0 \sim \sqrt{\mathcal{N}} \rightarrow 0$ linear limit of their energy eigenvalues [6]:

$$E_{n,m}^{\text{lin}} = (2n + |m| + 1)\hbar\omega.$$ \hspace{1cm} (8)

Figure 1 displays the $C_0$ versus $u_0$ spiraling trajectories for $u_1$ (circles) and $u_3$ (stars). These trajectories are parametrized with respect to increasing values of the nonlinear parameter $\mathcal{N}$ defined by Eq. (7), i.e. with decreasing values of the trap harmonicity $\omega$.

![Figure 1](image_url)

**Figure 1.** The convergence of the discrete SP nonlinear system towards the quasi-classical continuum Thomas-Fermi regime defined by the fixed point $u \equiv 1$ and $C_0 = \frac{1}{4} [X^2]_{X=0} = 0$ when $\mathcal{N}$ increases from $\sim 10^{-2}$ to $\sim 10^2$ in the $C_0$ vs $u_0$ boundary-condition phase space for the two first $m = 0$ nonlinear eigenmodes. This modes are defined by their corresponding linear quantum numbers, namely (cf. Eq. (8)) $n = 0$, $m = 0$ ($u_1$: circles) and $n = 1$, $m = 0$ ($u_3$: stars).

Indeed electron-electron Coulomb interaction becomes (relatively to quantum kinetic and external potential energy) more and more important when the two electrons are less and less confined (i.e. when the trap is “flattened”). Actually $\mathcal{N}$ varies in figure 1 from $10^{-2}$ ($u_0 \sim 0.1$) to $10^2$
(\(u_0 \sim 1\)) where one then reaches the quasi-classical asymptotic Thomas-Fermi regime. This regime is defined by neglecting the quantum kinetic Laplacian derivative terms in Eq. (5a), thus yielding \(C(X) \sim X^2/4\) and hence \(C_0 = C(0) = 0\), while \(u(X) \equiv 1\) through Eq. (5b) with \(m = 0\). Therefore the “initial conditions” for the two discrete modes \(u_1\) and \(u_3\) converge for \(N \to \infty\) towards the same Thomas-Fermi fixed point \(\{u_0 = 1; C_0 = 0\}\) as evidenced by figure 1. Physically, this means that there is a continuous transition, through the increase of nonlinearity in the system (i.e. through \(\omega \to 0\) flattening of the trap), from the \(N \leq \pi\) “pure” quantum regime towards the \(N \gg 1\) classical one where the dimensionless Schrödinger equation (5a) reduces to its last-bracket classical-energy term. As a consequence, the \(N \to \infty\) highly nonlinear case leads to the progressive merging of the two discrete energy levels corresponding to \(u_1\) and \(u_3\) into the single asymptotic one defined by the fixed point in figure 1. Quantum energy discreteness disappears, which is the hallmark of the classical regime.

3. Nonlinear eigenvalue: chemical potential and corresponding energy

The SP virial energy \(E\) per particle corresponding to the nonlinear eigenstate \(u(X)\) is twice the quantum expectation value \(\langle \frac{1}{2} M \omega^2 r^2 \rangle\) of the external parabolic potential energy \([15]\). On the other hand, the energy eigenvalue \(\mu\) defined by Eq. (3) is that energy which is needed in order to add the second electron to the single-electron quantum-dot (Koopman’s theorem \([16]\)). Therefore it is indeed the chemical potential and can truly be regarded as the nonlinear eigenvalue of the SP differential system related to the corresponding nonlinear eigenstate \(\Psi\) (or \(u\) in the reduced units Eqs. (6)). Therefore we have:

\[
\tilde{E} = \frac{1}{2N} \int_0^\infty u^2 X^3 dX = \frac{1}{2} \left( \tilde{E}_{\text{lin}} + \tilde{\mu} \right),
\]

where \(\tilde{E}_{\text{lin}}\) is defined by Eq. (8). The nonlinear integrodifferential system Eqs. (5–9) is closed by the use of Eq. (6c) at \(X = 0\),

\[
\tilde{\mu} = C(0) + \tilde{\Phi}(0) = C_0 + \frac{e^2}{\hbar \omega} \int \frac{|\Psi|^2}{r^2} d^3 r = C_0 + \frac{\sqrt{2}}{N} \int_0^\infty u^2 dX,
\]

where we recall that \(C_0\) is that “initial condition” which defines, in addition to \(u_0\), the nonlinear eigenstate \(u(X)\). The parameter \(k = l_0/\alpha^*\) is the usual dimensionless dot size that consists in measuring the characteristic harmonic length \(l_0 = \sqrt{\hbar/M\omega}\) in units of the effective Bohr radius \(\alpha^* = \hbar^2/Mc^2\) (ranging from \(\alpha^* = 67\) nm for InSb to \(\alpha^* = 9.8\) nm for GaAs).

Eq. (5–10) self-consistently define, for any given value of the trap harmonic frequency \(\omega\) (or, equivalently, its reduced size \(k\)), the solution \(u \equiv u_\omega(X)\), its norm \(\mathcal{N} = \mathcal{N}(\omega)\) as well as its corresponding energy \(\tilde{E} \equiv \tilde{E}(\omega)\) per particle, together with the chemical potential \(\tilde{\mu} \equiv \tilde{\mu}(\omega)\). In \([6]\), for instance, where \(\hbar \omega = 3.37\) meV for a GaAs parabolic quantum dot (\(M = 0.067\) electron mass while the charge is \(1/2.74\) electron charge), we have \(\alpha^* = 9.79\) nm and \(l_0 = 18.5\) nm. Hence \(k = 1.89\). Then Eqs. (9–10) respectively yield \(\mathcal{N} = 2.53\) and \(\tilde{E} = \tilde{E}_{\text{per particle}} = 1.83\). Therefore \(E_{\text{QDH}} = 2(1.83) \times 12.33\) meV, which must be compared with the authors’ value 12.28 meV. Similarly, in \([10]\), \(\hbar \omega = 2\) meV yields \(k = 2.43\). Hence \(\mathcal{N} = 2.97\) and \(\tilde{E} = 2.00\), which yields \(\tilde{\mu} = 2\tilde{E} - 1 = 3.00\) in accordance with Eq. (9), and therefore \(\mu = 3.00 \hbar \omega = 6.00\) meV which is in complete agreement with the Coulomb-interaction case \((d_1 = d_2 = \infty)\) of that reference.

Therefore we recover an important property that has already been emphasized in the \(N = 2\) Coulomb-trap case, both for highly-compressed giant-planet atomic helium \([14]\) and for astrophysical \(H^-\) hydrogen ion \([11]\), namely, that the SP nonlinear differential description yields surprisingly accurate values for the ground state energy when compared to the corresponding mean-field Hartree-Fock ones. Note that the use of nonlinear eigenstates and their related eigenvalues, together with Eqs. (7–9), yields in the quite opposite physical situation of a rotating
condensed Bose gas of $10^5$ neutral Rb atoms with hard-sphere particle-particle interactions the correct vortex thresholds, as compared with the corresponding experimental results [17].

A remarkable property of the SP nonlinear eigensolutions is their “universal” limit behavior for small $\mathcal{N}$, whatever the actual state’s quantum numbers $n$ and $m$ in Eq. (8) are. Indeed we have

$$\tilde{E} \sim \tilde{E}^{\text{lin}} + \frac{1}{4} \mathcal{N}$$

for $\mathcal{N} < 1 \iff k < \frac{1}{2}$, (11)

(see figure 2). Therefore the characteristic energy $\tilde{E} = \frac{1}{2} \mathcal{N} \hbar \omega$ (cf. Eq. (7)), which was introduced for dimensional reasons into the Poisson equation (4), is simply the smallest additional particle-particle interaction energy with respect to the linear regime that is due to nonlinearity. Indeed Eq. (11) yields for QDH’s energy whose $\mathcal{N} \to 0$ non-interaction limit is $2E_{\text{lin}}$

$$\lim_{\mathcal{N} \to 0} 2E = 2E_{\text{lin}} + \mathcal{E}. \quad (12)$$

Figure 2. Illustration of Eq. (11) (dotted lines) for the $1 \leq \tilde{E}^{\text{lin}} \leq 5$ nonlinear eigenmodes (as defined by their linear $\mathcal{N} \to 0$ quantum numbers $n$ and $m$: see Eq. (8)). From bottom to top ($a$ and $b$ superscripts label linear degenerated states whose degeneracy is lifted by nonlinearity): $\{n = 0, m = 0\}$, $\{n = 0, m = 1\}$, $\{n = 1, m = 0\}^a$, $\{n = 0, m = 2\}^a$, $\{n = 1, m = 1\}$, $\{n = 2, m = 0\}^b$, $\{n = 1, m = 2\}^b$. The stars display the corresponding values of the first amplitude maxima reached at $X = 0$ by the nonlinear modes as $\mathcal{N}$ increases from its linear limit $\mathcal{N} \sim 0$ (see figure 1).
4. Nonlinear probability amplitude and renormalized state

Let us define the normalized probability amplitude

\[ \langle u_1 | u_3 \rangle = \frac{1}{\sqrt{N_1 N_3}} \int_0^\infty u_1 u_3 X dX \]  

(13)

of the two zero-angular-momentum eigenstates \( u_1(X) \) and \( u_3(X) \), together with their “matching condition”

\[ k(\omega) = \frac{N_1 (\bar{\mu}_1 - C_0^{(1)})}{\sqrt{2} \int_0^\infty u_1^2 dX} = \frac{N_3 (\bar{\mu}_3 - C_0^{(3)})}{\sqrt{2} \int_0^\infty u_3^2 dX} \]  

(14)

Eq. (14) is the trivial condition that the trap profile defined by \( \omega \)—or equivalently its dimensionless length \( k = \sqrt{\hbar/M\omega/(\hbar^2/Mc^2)} \)—be identical for the two modes \( u_{1,3} \) that enter the calculation of the Hilbertian scalar product (13): see Eq. (10). Let \( P_{1,3} = \langle u_1 | u_3 \rangle^2 \) (the eigenstates \( u_{1,3} \) being real-valued, taking the modulus like for \( w_{NL} \) in Eq. (1) is unnecessary). According to Eq. (1), we define the following respective population numbers of levels 1,3:

\[ n_1 = \frac{1}{1 + P_{1,3}} ; \quad n_3 = \frac{P_{1,3}}{1 + P_{1,3}} = n_1 P_{1,3}. \]  

(15a, b)

Eq. (15b) can be loosely regarded as stating that the probability for the one-particle system to be in the nonlinear excited state \( |u_3> \) is but its probability to be in the ground state \( |u_1> \) multiplied by the mere nonlinear transition probability \( P_{1,3} \) from \( |u_1> \) to \( |u_3> \). However we should keep in mind that the very concept of transition probability in a time-independent description of a stationary quantum system is questionable. Hence the concept of a mixed state like in Eq. (1) for the description of our nonlinear quantum system is much more appropriate.

Black-body thermodynamical equilibrium between the two levels is achieved if

\[ n_1 n_1^{eq} = n_3 (n_3^{eq} + 1), \]  

(16)

where \( n_1^{eq} \) is that number of parabolic \( \hbar \omega \) quanta which allows equal rates of stimulated emission and absorption processes between the two levels. Therefore Eqs. (15) yield

\[ n_1^{eq} = \frac{P_{1,3}}{1 - P_{1,3}}. \]  

(17)

Defining

\[ N_1^{\text{eff}} = N_1 + 2 n_1^{eq} \]  

(18)

as the effective ground-state nonlinear parameter \( N_1 \)—or “renormalized nonlinear parameter” (renormalized actually by the equilibrium harmonic quanta which play here the role of QED vacuum photon fluctuations)—and scaling the probability amplitude (13–14) by use of \( \gamma = \pi\alpha = \pi/137.036 = 2.2925 \ldots \times 10^{-2} \), we obtain the remarkable interference pattern

\[ \langle u_1 | u_3 \rangle = -\sqrt{\gamma} \sin \left( \frac{N_1^{\text{eff}}}{2} \right) + o(\gamma^{3/2}), \]  

(19)

(see figure 3). The factor 2 in Eq. (18) originates from the corresponding renormalized nonlinear quantum defined by Eqs. (4) and (7),

\[ \mathcal{E}^{\text{eff}} = \frac{1}{2} N_1^{\text{eff}} \hbar \omega = \mathcal{E} + n_1^{eq} \hbar \omega, \]  

(20)

which now appears as “dressed” by the equilibrium harmonic quanta in much the same way as fermion lines are “dressed” by emission-reabsorption processes of one virtual photon in the QED calculation of second-order radiative corrections to Coulomb scattering [18].
Figure 3 also displays the following chemical-potential resonance at the maximum of $\mathcal{P}_{1,3}$
that occurs at $N_1^{\text{eff}} \sim \pi$ (hence the subscripts),

$$[\tilde{\mu}_3 - \tilde{\mu}_1]_\pi \sim \tilde{E}_\pi^{\text{eff}}, \quad (21)$$

where $\tilde{E}_\pi^{\text{eff}} = \frac{1}{2} N_1^{\text{eff}} \sim \frac{\pi}{2}$ in accordance with Eqs. (19–20).

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**Figure 3.** Eq. (19)'s square scalar product $\langle u_1 | u_3 \rangle^2 / \pi \alpha$, as compared with its $\sin^2$ approximation (dotted line), together with the nonlinear resonance condition defined by Eq. (21) (in “tilde” units of $\hbar \omega$: intersection of the two upper plots; for some reasons, $\mathcal{E}$ is labelled $\Delta$ in the plot). Note the intersection of the dashed-dotted lower plot with the zero-energy value that yields $\tilde{E}_1 \sim \tilde{E}_3$, i.e. $N_1 \sim N_3$ at the resonance $\omega = \omega_\pi$ where $N_1^{\text{eff}} \sim \pi$.

This remarkable nonlinear resonance physically means that the quantum coherence maximum,
reached at that particular value $\omega_\pi$—or $N_1^{\text{eff}} = \pi$—of the trap parabolicity $\omega$, is achieved when
the renormalized quantum (20) equals the chemical-potential difference. It is thus tempting to
extrapolate to the present case the standard picture of a discrete quantum jump, the energy
quantum exchanged between the two corresponding “levels” (actually the two chemical potential
nonlinear eigenvalues) being $\tilde{E}_\pi^{\text{eff}}$. From Eqs. (9–10), we numerically obtain $\hbar \omega_\pi \sim 0.14 \epsilon$ where
$\epsilon = M e^4 / \hbar^2$ is the effective quantum-dot’s atomic energy unit: $\epsilon = 11.86 \text{ meV}$ for GaAs, thus
yielding $\hbar \omega_\pi \sim 1.66 \text{ meV}$ and $E_\pi^{\text{eff}} \sim 2.61 \text{ meV}$ while $\epsilon = 27.21 \text{ eV}$ if the dielectric constant of the
bulk material equals unity, then yielding $\hbar \omega_\pi \sim 3.80 \text{ eV}$ and hence $E_\pi^{\text{eff}} \sim 5.97 \text{ eV}$. Therefore Eq.
(21) provides parameters related to a possible experimental verification of the present SP theory.
Note that $N_1 \sim N_3 \sim \pi$ at $\omega = \omega_\pi$ (dashed-dotted lower plot in figure 3). Therefore $E_\pi^{\text{eff}}$ is
the common nonlinear quantum of the two eigenstates $u_{1,3}$ at their quantum-coherence resonance.

According to Eq. (7), Eq. (21) leads to the corresponding quantization rule for the total
(two-particle) quantum-dot energy at $\omega \sim \omega_\pi$:

$$2(E_3 - E_1) \sim 2 \hbar \omega_\pi + E_\pi^{\text{eff}} \sim \left(2 + \frac{\pi}{2}\right) \hbar \omega_\pi. \quad (22)$$
Eqs. (11), (21) and (22) show that the characteristic energy $\mathcal{E}$ which actually scales the electrostatic particle-particle interaction through the nonlinear differential Poisson equation (2) is in fact a true “nonlinear quantum”. Indeed, on the one hand, it is the smallest particle-particle interaction energy that is present in the system at vanishing nonlinearity $\mathcal{N} \to 0$. On the other hand, the maximum of $\mathcal{P}_{1,3}$ “transition probability” between the two nonlinear eigenstates is achieved by the $e_{\text{eff}}\pi$ resonant quantum jump: i) either between the chemical-potential nonlinear eigenvalues $\mu_1$ and $\mu_3$; ii) or between both corresponding values $2E_{1,3}$ of the QDH energy, then with the addition of two standard “linear” radial quanta $h\omega_\pi$.

![Figure 4](image_url)

Figure 4. Numerical illustration of Eq. (23) which yields the fine-structure-constant’s value within 0.1% (the experimental value 1/137.036 would correspond in this figure to the small circle whose respective abcissa and ordinate are $\pi$ and 1). The numerical calculations are done by use of standard tools [20] and consist in increasing nonlinearity through a three-loop iterative scheme from the $\mathcal{N} \ll 1$ linear regime. The two first loops define each eigenstate $u_{1,3}$ with a $10^{-8}$ accuracy at $X \sim 8$ (see figure 5) while the third one evaluates the matching condition Eq. (14) within $10^{-6}$ and then calculates the scalar product given by Eq. (13).

![Figure 5](image_url)

Figure 5. The nonlinear eigenstate profiles $u_1(X)$ and $u_3(X)$ of respective maximum amplitudes $\sim 1$ and $\sim 1.5$ corresponding to figure 4.

The nonlinear “quantum-transition” probability $\mathcal{P}_{1,3}$ takes into account both possible mechanisms related to Eq. (21) that interfere in accordance with Eq. (2), namely, $u_1 \to u_3$ (“absorption”: ↑) or $u_3 \to u_1$ (“emission”: ↓). Therefore either of these two processes has the probability...
\( \mathcal{P}_{11} = \frac{1}{2} \mathcal{P}_{13} \). At the nonlinear resonance defined by Eq. (21), the corresponding effective number of resonant harmonic quanta that enters the “transition” is \( \Delta_{\text{eff}}^\pi = \frac{1}{2} N_{\text{max}}^\pi \) by use of Eq. (20), where \( N_{\text{max}}^\pi = [N_{\text{eff}}^\pi]_{\omega^\pi} \sim \pi \) is the abcissa of \( \mathcal{P}_{13} = [\mathcal{P}_{13}]_{\text{max}} = 1.0005 \pi \alpha \) (see figure 4). Figure 5 displays the two corresponding nonlinear eigenstates.

Therefore the elementary probability \( \sigma \) per particle and per harmonic quantum for either “induced transition” process (“emission” or “absorption”: like in QED, it does not make any difference which) is numerically given by

\[
\sigma = \frac{[\mathcal{P}_{11}]_{\text{max}}}{\Delta_{\text{eff}}^\pi} = \frac{1}{2}[\mathcal{P}_{13}]_{\text{max}} = 0.3180 \gamma = \frac{1}{137.17} .
\]

(23)

This value of \( \sigma \) approaches the numerical value 1/137.036 of Sommerfeld’s fine structure constant \( \alpha \) within 0.1 %.

5. Conclusion

The present work emphasizes the physical properties of—and specifically the interference between—the SP nonlinear quantum eigenstates as displayed by figure 5. It is useful to bear in mind the hierarchy of physical effects present in the system. Besides each particle’s quantum kinetic effect defined by the Laplacian term in the Schrödinger equation (3) and its external parabolic confinement, the next significant effect is the particle-particle Coulomb interaction described by Poisson equation (4). Consequently, one may wish to define new quantum tools, namely, stationary eigenstates that take all these three dominant effects into account as already suggested by Cohen-Tannoudji [19]. The price to pay is high: abandoning the linearity of the quantum description. But it may be fruitful. New interference effects are indeed predicted that seem to be related to an “\( \alpha \)-memory” in resonant energy-exchange nonlinear processes. If these later can be experimentally tested (we provide corresponding relevant parameters in the case of GaAs QDH), the crucial question raised by the present work—namely, which physical effect, if any, is described by the probability amplitude \( \langle u_1 | u_3 \rangle \)?—would be positively answered: \( \langle u_1 | u_3 \rangle \) would then provide a nonlinear and, remarkably enough, non-relativistic definition of the numerical value of the QED coupling constant \( \sqrt{\alpha} \).

At the present stage of the numerical investigations, we find \( \sigma = 1/137.17 \), see Eq. (23). It seems hard to ascribe all nonlinear quantum properties that yield this result to mere numerical coincidences. On the other hand, there is definitely no velocity of light \( c \) in the original differential system Eqs. (3–4) and its subsequent solution. Therefore, if numerical coincidences are discarded, \( \alpha \) would then be a universal number, say of the Feigenbaum type in period-doubling transition to deterministic chaos, that would be related to some fundamental nonlinear quantum properties which the present particular model—quite fortunately actually—happened to hit. Then the numerical values of the elementary charge \( e \), Planck’s constant \( h \) and \( c \) would be related by this number.

This discovery would echo Feynman’s concern for both further research concerning \( \alpha \)’s numerical value recalled in section 1 and fears about QED’s mathematical self-consistency: “What is certain is that we do not have a good mathematical way to describe the theory of quantum electrodynamics…” [1]. By pointing out the specific role of \( \textit{ab-initio} \) nonlinear quantum effects [11,14,17] as well as by using them in the present differential description of a 2D harmonically-trapped condensed Bose gas of charged particles, it is hoped that such a particular nonlinear quantum system might help progressing in the research trend advocated by Feynman [1], namely, “all good theoretical physicists should put this number \( [\alpha] \) up on their wall and worry about it”.

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References
[1] Feynman R P 1985 *QED: the strange theory of light and matter* Chap. 4 (Princeton: University Press)
[2] Sikorski C and Merkt U 1989 *Phys. Rev. Lett.* 62 2164; Sikorski C and Merkt U 1990 *Phys. Rev. Lett.* 64 3100
[3] Maksym P A and Chakraborty T 1990 *Phys. Rev. Lett.* 65 108
[4] Ashoori R C, Stormer H L, Weiner J S, Pfeiffer L N, Baldwin K W and West K W 1993 *Phys. Rev. Lett.* 71 613
[5] Merkt U, Huser J and Wagner M 1991 *Phys. Rev. B* 43 7320
[6] Pfannkuche D, Gudmundsson V and Maksym P A 1993 *Phys. Rev. B* 47 2244
[7] Maksym P A 1996 *Phys. Rev. B* 53 10871
[8] Wagner M, Merkt U and Chaplik A V 1992 *Phys. Rev. B* 45 1951
[9] Dineykhan M and Nazmitdinov R G 1997 *Phys. Rev. B* 55 13707
[10] Hallam L D, Weis J and Maksym P A 1996 *Phys. Rev. B* 53 1452
[11] Reinisch G 2004 *Phys. Rev. A* 70 033613
[12] Bec J 2010 private communication
[13] see for instance Landau L and Lifchitz E 1989 *Électrodynamique quantique* 2nd Ed. (Moscow: MIR)
[14] Reinisch G, de Freitas Pacheco J and Valiron P 2001 *Phys. Rev. A* 63 042505
[15] Park D 1990 *Classical Dynamics and its Quantum Analogues* 2nd Ed., Sect. 4.5 (Berlin: Springer)
[16] Szabo A and Ostlund N S 1996 *Modern Quantum Chemistry: Introduction to Advanced Electronic Structure Theory* (New York: Dover)
[17] Reinisch G 2007 *Phys. Rev. Lett.* 99 120402
[18] Mandl F 1959 *Introduction to Quantum Field Theory* Chap. 17 (London: Interscience)
[19] Cohen-Tannoudji C, Dupont-Roc Jand Gryenberg G 1988 *Processus d’intéraction entre photons et atomes* (Paris: InterEditions/Editions du CNRS)
[20] MathWorks 2004 MATLAB Version 7.0.1.24704 (R14), options = odeset(’RelTol’,1e-10) (MathWorks)