Fermi coordinates and static observer in Schwarzschild spacetime.

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Abstract

In this paper we construct the Fermi coordinates along any arbitrary line in simple analytical way without use the orthogonal frames and their parallel transport. In this manner we extend the Eddington approach to the construction of the Fermi metric in terms of the Riemann tensor. In the second part of the present article we show how the proposed approach works practically by applying it for deriving the Fermi coordinates for the static observer in the Schwarzschild spacetime.
I. INTRODUCTION

It is known that for any metric and any line exists a set of Fermi coordinates \[1\] in which all Christoffel symbols are zero at points of this line (this is the definition of Fermi coordinates). However, the elimination of the Christoffel symbols on a line does not fix completely the corresponding coordinate transformations which means that there is an infinity of the Fermi coordinates associated to a given line. To make a concrete choice it is reasonable to search for some additional coordinate restrictions (not violating the vanishing of Christoffel symbols on line) appropriate from a physical point of view. The natural physical support have been proposed by Arthur Eddington \[2\] who also developed the way for the corresponding analytical calculations. Eddington did this for the case of the Riemann coordinates in the neighborhood of a point in 4-dimensional spacetime. His idea was to specify the coordinate transformations so as to represent the quadratic terms of the expansion of the metric near such point by the components of the Riemann tensor. It turn out that the generalization of Eddington approach to the case of Fermi coordinates in the neighborhood of an arbitrary line is straightforward. Such extension is the target of the first part of the present paper. It should be stressed that it is done for any original metric and any given curve, no matter what is its geometric character (geodesic or not, timelike, spacelike or null) and in pure analytical way without necessity to use orthogonal frames and their parallel (or Fermi-Walker) transport. Such simplified universal method has some value, because the majority of papers in the literature have been dedicated only to some specific type of the line and have been essentially based on the use of parallel transported frames (for example, \[3\] did this for geodesics and \[4\] for null curves).

In the second part of the present article we show the proposed approach in practical action by applying it for construction of the Fermi coordinates for the static observer in the Schwarzschild spacetime. This result is new since the known analogous constructions have been restricted to a quasi-Fermi system defined by Synge \[5\] when not all Christoffel symbols on the world line of interest disappear (for example, see \[6\] and references there in).
A. Construction of Fermi coordinates in general

It is known that for any metric \(g_{ik}(x)\) (by symbol \(x\) we denote the set of 4 coordinates \(x^0, x^1, x^2, x^3\) in 4-dimensional spacetime) and any line

\[ x^\alpha = f^\alpha(x^0) \]  

(1)

exists a set of Fermi coordinates \(\dot{x}\) (that is \(\dot{x}^0, \dot{x}^1, \dot{x}^2, \dot{x}^3\) in which all Christoffel \(\tilde{\Gamma}\)-symbols are zero at points of this line. For the corresponding coordinates transformation \(\dot{x}^i = \dot{x}^i(x^0, x^1, x^2, x^3)\) we denote the Jacobi matrix by \(A^i_k\):

\[ A^i_k(x) = \frac{\partial \dot{x}^i}{\partial x^k}. \]  

(2)

The transformation of \(\Gamma\)-symbols can be written as

\[ \Gamma^i_{kl} A^q_i = \tilde{\Gamma}^q_{nm} A^m_i A^n_k + A^q_{k,l}. \]  

(3)

From the last formula follows that \(\tilde{\Gamma}^q_{nm}\) in Fermi coordinates vanish on the line (1) if matrix \(A^i_k\) satisfy the differential equation:

\[ [A^i_{k,l}]_L = [\Gamma^i_{nm} A^m_i]_L, \]  

(4)

where \([F]_L\) means the value of any function \(F\) on the line (1), that is

\[ [F(x^0, x^1, x^2, x^3)]_L = F[x^0, f^1(x^0), f^2(x^0), f^3(x^0)]. \]  

(5)

It is easy to see that equation (4) represents the set of ordinary differential equations with respect to the variable \(x^0\). Indeed, in the vicinity of the line (1) the transformation between Fermi and original coordinates can be represented in form of an expansion with respect to the three small deviations \(x^\alpha - f^\alpha(x^0)\) from the line:

\[ \dot{x}^m = X^m(x^0) + Y^m_\alpha(x^0) [x^\alpha - f^\alpha(x^0)] \]

\[ + Z^m_{\alpha\beta}(x^0) [x^\alpha - f^\alpha(x^0)] [x^\beta - f^\beta(x^0)] + O(3), \]  

(6)

where \(O(n)\) means collection of terms of the order \(n\) and higher with respect to the small functional parameters \(x^\alpha - f^\alpha(x^0)\). From (3) and definition (2) follows expansion for the components of matrix \(A^m_k\):

\[ A^m_0 = \frac{dX^m}{dx^0} - Y^m_\alpha \frac{df^\alpha}{dx^0} + \left( \frac{dY^m_\beta}{dx^0} - 2Z^m_{\alpha\beta} \frac{df^\alpha}{dx^0} \right) (x^\beta - f^\beta) + O(2), \]  

(7)
\[ A_m^\alpha = Y_m^\alpha + 2 Z_m^{\alpha \beta} (x^\beta - f^\beta) + O(2). \]  

(8)

Consequently on the line the components \( A_m^k \) are:

\[ [A_m^0]_L = \frac{dX_m}{dx^0} - Y_m^\alpha \frac{df^\alpha}{dx^0}, \]

(9)

\[ [A_m^\beta]_L = Y_m^\alpha . \]

(10)

From (7) and (8) follows values of the partial derivatives \( A_m^{k,l} \) of matrix \( A_m^k \) on line:

\[ [A_m^{0,0}]_L = \frac{d}{dx^0} \left( \frac{dX_m}{dx^0} - Y_m^\alpha \frac{df^\alpha}{dx^0} \right) - \left( \frac{dY_m^\beta}{dx^0} - 2 Z_m^{\alpha \beta} \frac{df^\alpha}{dx^0} \right) \frac{df^\beta}{dx^0} , \]

(11)

\[ [A_m^{0,\beta}]_L = \frac{dY_m^\beta}{dx^0} - 2 Z_m^{\alpha \beta} \frac{df^\alpha}{dx^0} , \]

(12)

\[ [A_m^{\beta,0}]_L = \frac{dY_m^\beta}{dx^0} - 2 Z_m^{\alpha \beta} \frac{df^\alpha}{dx^0} , \]

(13)

\[ [A_m^{\alpha,\beta}]_L = 2 Z_m^{\alpha \beta} . \]

(14)

It is convenient to use for the quantity \([A_m^0]_L\) from (9) the special notation \( \Lambda^m \):

\[ \Lambda^m = \frac{dX_m}{dx^0} - Y_m^\alpha \frac{df^\alpha}{dx^0} . \]

(15)

After substitution expressions (9)-(14) into equation (4) we find that this equation is equivalent to the following system:

\[ \frac{d\Lambda^m}{dx^0} = \left\{ \left[ \Gamma^{0}_0 \right]_L \frac{df^\beta}{dx^0} + \left[ \Gamma^{0}_0 \right]_L \right\} \Lambda^m + \left\{ \left[ \Gamma^\alpha_0 \right]_L \frac{df^\beta}{dx^0} + \left[ \Gamma^\alpha_0 \right]_L \right\} Y_m^\alpha , \]

(16)

\[ \frac{dY_m^\beta}{dx^0} = \left\{ \left[ \Gamma^\alpha_0 \right]_L \frac{df^\alpha}{dx^0} + \left[ \Gamma^\alpha_0 \right]_L \right\} \Lambda^m + \left\{ \left[ \Gamma^\gamma_0 \right]_L \frac{df^\alpha}{dx^0} + \left[ \Gamma^\gamma_0 \right]_L \right\} Y_m^\gamma , \]

(17)

\[ Z_{\alpha \beta} = \frac{1}{2} \left[ \Gamma^0_{\alpha \beta} \right]_L \Lambda^m + \frac{1}{2} \left[ \Gamma^\gamma_{\alpha \beta} \right]_L Y_m^\gamma , \]

(18)

\[ \frac{dX_m}{dx^0} = \Lambda^m + Y_m^\alpha \frac{df^\alpha}{dx^0} . \]

(19)

Because all \( \Gamma \)-symbols of the original metric and functions \( f^\alpha \) are given equations (16) and (17) represent the closed linear system of the ordinary differential equations of first order with respect to the variable \( x^0 \) for coefficients \( \Lambda^m(x^0) \) and \( Y_m^\alpha(x^0) \) in expansion (3). These solutions should be substituted to the right hand side of the equation (18) which gives coefficients \( Z_{\alpha \beta}(x^0) \). After that we need to substitute \( \Lambda^m(x^0) \) and \( Y_m^\alpha(x^0) \) into the equation (19) where from we obtain the last coefficients \( X_m(x^0) \) by quadrature.
This is the general procedure how to construct the Fermi coordinates for the any metric in vicinity of any given curve. There is also a possibility to specialize the Fermi coordinates in such a way that the metric in these coordinates in the first two approximations will be Minkowskian:

$$\dot{g}_{ik}(\dot{x}) = \eta_{ik} + O(2) ,$$

where $\eta_{ik}$ is Minkowski metric tensor. This can be done by choosing in special way the arbitrary constants of integration which contain the general solution of the equations \((16)-(17)\) and \((19)\) (there are 20 such constants 10 of which should be fixed in order to obtain the form \((20)\) and another 10 will remain arbitrary reflecting the Poincarè symmetry of the Minkowskian spacetime).

II. METRIC IN FERMI COORDINATES

The same line \((1)\) in Fermi coordinates $\dot{x}$ has equation of the similar form:

$$\dot{x}^\alpha = F^\alpha (\dot{x}^0) .$$

The functions $F^\alpha (\dot{x}^0)$ follows from transformation \((6)\). This transformation tells that on the line $\dot{x}^0 = X^0(x^0)$ and $\dot{x}^\alpha = X^\alpha(x^0)$. Then

$$F^\alpha (\dot{x}^0) = [X^\alpha (\zeta)]_{\zeta=(arcX^0)(\dot{x}^0)} ,$$

where $arcX^0$ is function inverse to $X^0$.

Because in Fermi coordinates

$$[\dot{g}_{ik} (\dot{x})]_{\mathcal{L}} = c_{ik} , \quad \left[ \frac{\partial \dot{g}_{ik} (\dot{x})}{\partial \dot{x}^l} \right]_{\mathcal{L}} = 0 , \quad c_{ik} = \text{const}$$

the expansion for metric near the line has the form:

$$\dot{g}_{ik} (\dot{x}) = c_{ik} + \frac{1}{2} \left[ \frac{\partial^2 \dot{g}_{ik} (\dot{x})}{\partial \dot{x}^\alpha \partial \dot{x}^\beta} \right]_{\mathcal{L}} [\dot{x}^\alpha - F^\alpha (\dot{x}^0)] [\dot{x}^\beta - F^\beta (\dot{x}^0)] + O(3) .$$

Then to obtain this metric we need the second derivatives of the metric tensor with respect to the space coordinates $\dot{x}^\alpha$ on the line. However, these second derivatives depend on the cubic terms $O(3)$ in expansion \((6)\) and up to now remain completely arbitrary. To make a choice for this cubic addend it is necessary to accept some additional coordinate restrictions which will not violate conditions \((23)\). We already mentioned in Introduction that the natural
physical arguments for such a choice have been proposed by A. Eddington and here we will follow his proposal, that is we will specify the cubic addends in coordinates transformation to the Fermi coordinates so as to represent the second derivatives in metric \( \Gamma_i^{m} \) in terms of the Riemann tensor. Eddington showed that Riemann coordinates can be further specified in such a way that cyclic combination \( \Gamma_i^{m} \) of derivatives of \( \Gamma \)-symbols at point where \( \Gamma_i^{m} \) are zero also vanish. Under this condition it is simple matter to express second derivatives of the metric at this point in terms of the components of the Riemann tensor.

In case of Fermi coordinates we described in the preceding section the full 4-dimensional Eddington condition cannot be accepted because it contradicts to the equations \( (16)-(19) \). However, it is possible to restrict the choice of Fermi coordinates by the following reduced version of the same condition:

\[
\frac{\partial \Gamma_i^{m}}{\partial \dot{x}^l} \left( x \right) + \frac{\partial \Gamma_i^{m}}{\partial \dot{x}^k} \left( x \right) + \frac{\partial \Gamma_i^{m}}{\partial \dot{x}^l} \left( x \right) = 0 ,
\]

where the upper index remains 4-dimensional and all three lower indices are 3-dimensional. The proof of the possibility of this restriction we placed in Appendix B.

Under the restriction \( \Gamma_i^{kl} \left( x \right) = 0 \) from the general expression for the Riemann tensor we have:

\[
\left[ \hat{\Gamma}_i^{kl} \left( x \right) \right]_{\mathcal{L}} = \left[ \frac{\partial \Gamma_i^{kl}}{\partial \dot{x}^l} \left( x \right) - \frac{\partial \Gamma_i^{kl}}{\partial \dot{x}^m} \left( x \right) \right]_{\mathcal{L}} .
\]

Let’s apply this formula for the 3-dimensional indices \( (k, l, m) = (\nu, \lambda, \mu) \) that is:

\[
\left[ \hat{\Gamma}_i^{\nu\lambda\mu} \left( x \right) \right]_{\mathcal{L}} = \left[ \frac{\partial \Gamma_i^{\nu\lambda}}{\partial \dot{x}^l} \left( x \right) - \frac{\partial \Gamma_i^{\nu\lambda}}{\partial \dot{x}^\mu} \left( x \right) \right]_{\mathcal{L}} .
\]

By simple manipulation with indices it is easy to show that the last expression with the help of condition \( (25) \) can be inverted:

\[
\left[ \frac{\partial \Gamma_i^{\nu\lambda}}{\partial \dot{x}^\mu} \left( x \right) \right]_{\mathcal{L}} = -\frac{1}{3} \left[ \hat{\Gamma}_i^{\nu\lambda\mu} \left( x \right) + \hat{\Gamma}_i^{\nu\mu\lambda} \left( x \right) \right]_{\mathcal{L}} .
\]

Now from the identity \( [\dot{g}_{ik} \left( x \right)]_{;lm} = 0 \), taking into account the restriction \( \Gamma_i^{kl} \left( x \right) = 0 \), one can express the second derivatives of the metric tensor on the line \( \mathcal{L} \) in Fermi coordinates in the form:

\[
\left[ \frac{\partial \dot{g}_{ik} \left( x \right)}{\partial \dot{x}^\lambda \partial \dot{x}^\mu} \right]_{\mathcal{L}} = \left[ \frac{\partial \Gamma_i^{k\lambda}}{\partial \dot{x}^\mu} \left( x \right) \dot{g}_{ik} \left( x \right) + \frac{\partial \Gamma_i^{k\mu}}{\partial \dot{x}^\lambda} \dot{g}_{ki} \left( x \right) \right]_{\mathcal{L}} .
\]
From this formula we have:

$$\left[ \frac{\partial \hat{g}_{00} (\dot{x})}{\partial \dot{x}^\lambda \partial \dot{x}^\mu} \right]_L = 2 \left[ \frac{\partial \hat{\Gamma}_{\alpha \lambda}^i (\dot{x})}{\partial \dot{x}^\mu} \hat{g}_{0i} (\dot{x}) \right]_L,$$  \hspace{1cm} (30)

$$\left[ \frac{\partial \hat{g}_{0\alpha} (\dot{x})}{\partial \dot{x}^\lambda \partial \dot{x}^\mu} \right]_L = \left[ \frac{\partial \hat{\Gamma}_{0\lambda}^i (\dot{x})}{\partial \dot{x}^\mu} \hat{g}_{i\alpha} (\dot{x}) + \frac{\partial \hat{\Gamma}_{\alpha \lambda}^i (\dot{x})}{\partial \dot{x}^\mu} \hat{g}_{0i} (\dot{x}) \right]_L,$$ \hspace{1cm} (31)

$$\left[ \frac{\partial \hat{g}_{\alpha\beta} (\dot{x})}{\partial \dot{x}^\lambda \partial \dot{x}^\mu} \right]_L = \left[ \frac{\partial \hat{\Gamma}_{\alpha\lambda}^i (\dot{x})}{\partial \dot{x}^\mu} \hat{g}_{i\beta} (\dot{x}) + \frac{\partial \hat{\Gamma}_{\beta\lambda}^i (\dot{x})}{\partial \dot{x}^\mu} \hat{g}_{\alpha i} (\dot{x}) \right]_L.$$ \hspace{1cm} (32)

The first two of these formulas show that in order to express all second derivatives of the metric in terms of the Riemann tensor the relation (28) is not enough. It is necessary to find analogous expression also for the quantity $\partial \hat{r}_{0\alpha} (\dot{x}) / \partial \dot{x}^\mu$ on the line. To do this let’s take the general 4-dimensional relation (26) for indices $k = \nu, l = \lambda, m = 0$ and sum it with equation (27) being multiplied by the derivative $dF^\mu (\dot{x}^0) / d\dot{x}^0$. In the right hand side of this sum will appear the quantity

$$\left[ \frac{\partial \hat{\Gamma}_{\nu\lambda}^i (\dot{x})}{\partial \dot{x}^0} \right]_L + \left[ \frac{\partial \hat{\Gamma}_{\nu\mu}^i (\dot{x})}{\partial \dot{x}^\mu} \right]_L \frac{dF^\mu (\dot{x}^0)}{d\dot{x}^0},$$ \hspace{1cm} (33)

which is zero because for any function $\hat{\Psi} (\dot{x})$ which is zero along line $L$, that is which satisfy the restriction $\hat{\Psi} [\dot{x}^0, F^1 (\dot{x}^0), F^2 (\dot{x}^0), F^3 (\dot{x}^0)] = 0$, the ordinary derivative of its value on the line with respect to $\dot{x}^0$ is also zero and due this evident fact we deduce:

$$\frac{d}{d\dot{x}^0} \hat{\Psi} [\dot{x}^0, F^1 (\dot{x}^0), F^2 (\dot{x}^0), F^3 (\dot{x}^0)] = \left[ \frac{\partial \hat{\Psi} (\dot{x})}{\partial \dot{x}^0} \right]_L + \left[ \frac{\partial \hat{\Psi} (\dot{x})}{\partial \dot{x}^\mu} \right]_L \frac{dF^\mu (\dot{x}^0)}{d\dot{x}^0} = 0.$$

Then the resulting sum gives the following equation:

$$\left[ \hat{R}_{\nu\lambda 0} (\dot{x}) \right]_L + \left[ \hat{R}_{\nu\lambda\mu} (\dot{x}) \right]_L \frac{dF^\mu (\dot{x}^0)}{d\dot{x}^0} = \left[ \frac{\partial \hat{\Gamma}_{\nu\lambda 0}^i (\dot{x})}{\partial \dot{x}^\lambda} \right]_L + \left[ \frac{\partial \hat{\Gamma}_{\nu\mu}^i (\dot{x})}{\partial \dot{x}^\lambda} \right]_L \frac{dF^\mu (\dot{x}^0)}{d\dot{x}^0},$$ \hspace{1cm} (35)

where from the quantity $[\partial \hat{\Gamma}_{\nu\lambda 0}^i (\dot{x}) / \partial \dot{x}^\lambda]$ can be represented in terms of the Riemann tensor since for the derivatives $[\partial \hat{\Gamma}_{\nu\mu}^i (\dot{x}) / \partial \dot{x}^\lambda]$, we already have such representation, see formula (28). The result is:

$$\left[ \frac{\partial \hat{\Gamma}_{\nu\lambda 0}^i (\dot{x})}{\partial \dot{x}^\lambda} \right]_L = \left[ \hat{R}_{\nu\lambda 0} (\dot{x}) \right]_L + \frac{1}{3} \left[ \hat{R}_{\nu\mu\lambda}^i (\dot{x}) - 2 \hat{R}_{\nu\lambda\mu}^i \right]_L \frac{dF^\mu (\dot{x}^0)}{d\dot{x}^0}.$$

$$\left(36\right)$$
Now from (24) and (30)-(32) (using definition \( R_{iklm} = g_{in} R_{klm}^n \)) we obtain the final general result for the canonical (Eddington’s terminology) metric in Fermi coordinates:

\[
g_{00}(\dot{x}) = c_{00} + \left[ \dot{R}_{0\lambda\mu 0}(\dot{x}) - \frac{2}{3} \dot{R}_{0\lambda\nu \mu}(\dot{x}) \frac{dF^\nu(\dot{x}^0)}{dx^0} \right]_L \left[ \dot{x}^\lambda - F^\lambda(\dot{x}^0) \right] \left[ \dot{x}^\mu - F^\mu(\dot{x}^0) \right] + O(3) ,
\]

\[
g_{0\alpha}(\dot{x}) = c_{0\alpha} + \frac{2}{3} \dot{R}_{0\alpha\lambda\mu}(\dot{x}) \frac{dF^\nu(\dot{x}^0)}{dx^0} \left[ \dot{x}^\lambda - F^\lambda(\dot{x}^0) \right] \left[ \dot{x}^\mu - F^\mu(\dot{x}^0) \right] + O(3) ,
\]

\[
g_{\alpha\beta}(\dot{x}) = c_{\alpha\beta} + \frac{1}{3} \dot{R}_{\alpha\lambda\mu\beta}(\dot{x}) \left[ \dot{x}^\lambda - F^\lambda(\dot{x}^0) \right] \left[ \dot{x}^\mu - F^\mu(\dot{x}^0) \right] + O(3) .
\]

**III. FERMI COORDINATES FOR STATIC OBSERVER IN SCHWARZSCHILD SPACETIME**

Let’s take the Schwarzschild metric in its standard form:

\[
- ds^2 = - \left( 1 - \frac{2m}{r} \right) dt^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) ,
\]

with following designation for coordinates:

\[
t, r, \theta, \varphi = x^0, x^1, x^2, x^3.
\]

The world line of a static observer is:

\[
x^\alpha = x^\alpha_*
\]

where \( x^\alpha_* = (x^1_*, x^2_*, x^3_*) = (r_*, \theta_*, \varphi_*) \) are arbitrary constants. The transformation to Fermi coordinates \( \dot{x} \) along this line is given by the formula (1), that is:

\[
\dot{x}^m = X^m(t) + Y^m_\alpha(t)(x^\alpha - x^\alpha_*) + Z^m_\alpha\beta(t)(x^\alpha - x^\alpha_*)(x^\beta - x^\beta_*) + O(3) .
\]

In equations (16)-(17) and (19) all terms containing \( df^\alpha/dx^0 \) disappear and among those \( \Gamma \)–symbols which are present in these equations there are only two non-zero, namely

\[
[\Gamma^1_{00}]_L = \frac{m}{r^2_*} \left( 1 - \frac{2m}{r_*} \right) , \quad [\Gamma^0_{10}]_L = \frac{m}{r^2_*} \left( 1 - \frac{2m}{r_*} \right)^{-1} .
\]
Under these conditions equations (16)-(17) and (19) become very simple and can be integrated easily. The solution for the functions $\Lambda_m(t)$ is $\Lambda_m = C_1^m e^{\omega t} + C_2^m e^{-\omega t}$ and for coefficients $X^m(t)$ and $Y_\alpha^m(t)$ we have:

$$X^m = \omega^{-1} \left( C_1^m e^{\omega t} - C_2^m e^{-\omega t} \right) + C_3^m,$$

$$Y_1^m = \frac{\left( 1 - \frac{2m}{r_*} \right)^{-1}}{1} \left( C_1^m e^{\omega t} - C_2^m e^{-\omega t} \right),$$

$$Y_2^m = C_4^m, \quad Y_3^m = C_5^m,$$

where $C_1^m, ..., C_5^m$ are arbitrary constants of integration and

$$\omega = \frac{m}{r_*^2}. \tag{48}$$

Without loss of generality we can chose constants $C_1^m, C_2^m, C_4^m, C_5^m$ in the following way:

$$C_1^m = \left( C_0^1, C_1^1, C_2^1, C_3^1 \right) = (\lambda, \lambda, 0, 0), \tag{49}$$

$$C_2^m = \left( C_0^2, C_1^2, C_2^2, C_3^2 \right) = (\lambda, -\lambda, 0, 0), \tag{50}$$

$$C_4^m = \left( C_0^4, C_1^4, C_2^4, C_3^4 \right) = (0, 0, r_*, 0), \tag{51}$$

$$C_5^m = \left( C_0^5, C_1^5, C_2^5, C_3^5 \right) = (0, 0, 0, r_* \sin \theta_*), \tag{52}$$

where quantity $\lambda$ is defined by the relation

$$\lambda^2 = \frac{1}{4} \left( 1 - \frac{2m}{r_*} \right). \tag{53}$$

This choice for free parameters fixes the arbitrary constants $c_{ik}$ in the metric (37)-(39) as

$$c_{00} = -1, \quad c_{0\alpha} = 0, \quad c_{\alpha\beta} = \delta_{\alpha\beta}, \tag{54}$$

that is in the first two approximations the metric is Minkowskian in Fermi coordinates.

Now we substitute the constants (49)-(52) into expressions (45)-(47) to obtain the final form for coefficients $X^m(t), Y_\alpha^m(t)$ and after that insert them together with Schwarzschild $\Gamma$-symbols $[\Gamma^{\alpha}_{\alpha\beta}]_L$ and $[\Gamma^{\gamma}_{\alpha\beta}]_L$ into the right hand side of the equation (18). This gives the coefficients $Z_{\alpha\beta}(t)$ after which we can write the final form of transformation to Fermi coordinates along the world line of static Schwarzschild observer:

$$x^0 = C_3^0 + \frac{2\lambda}{\omega} \sinh \omega t + \frac{1}{2\lambda} (r_r - r_*) \sinh \omega t$$

$$- \left[ \frac{\omega}{16\lambda^3} (r_r - r_*)^2 + r_* \lambda (\theta - \theta_*)^2 + r_* \lambda \sin^2 \theta_* (\varphi - \varphi_*)^2 \right] \sinh \omega t + O(3). \tag{55}$$
\[ \dot{x}^1 = C_3^1 + \frac{2\lambda}{\omega} \cosh \omega t + \frac{1}{2\lambda} (r - r_*) \cosh \omega t \]
\[ - \left[ \frac{\omega}{16\lambda^3} (r - r_*)^2 + r_* \lambda (\theta - \theta_*)^2 + r_* \lambda \sin^2 \theta_* (\varphi - \varphi_*)^2 \right] \cosh \omega t + O(3) , \]
\[ \dot{x}^2 = C_3^2 + r_* (\theta - \theta_*) + (r - r_*) (\theta - \theta_*) - \frac{1}{2} r_* \sin \theta_* \cos \theta_* (\varphi - \varphi_*)^2 + O(3) , \]
\[ \dot{x}^3 = C_3^3 + r_* \sin \theta_* (\varphi - \varphi_*) + \sin \theta_* (r - r_*) (\varphi - \varphi_*) \]
\[ r_* \cos \theta_* (\theta - \theta_*) (\varphi - \varphi_*) + O(3) . \]

Metric for the static Schwarzschild observer in canonical Fermi coordinates follows from formulas (37)-(39). The arbitrary constants \( c_{ik} \) we already specified, see (54). Now we need to find the functions \( F^\alpha (\dot{x}^0) \) and components of the Riemann tensor \( \dot{R}_{iklm} (\dot{x}) \). The equation of the Schwarzschild static world line in the Fermi coordinates can be extracted from transformation (55)-(58). On the line we have
\[ \dot{x}^0 = C_3^0 + \frac{2\lambda}{\omega} \sinh \omega t , \quad \dot{x}^1 = C_3^1 + \frac{2\lambda}{\omega} \cosh \omega t , \quad \dot{x}^2 = C_3^2 , \quad \dot{x}^3 = C_3^3 . \]

Then functions \( F^\alpha (\dot{x}^0) \) are:
\[ F^1 (\dot{x}^0) = C_3^1 + \sqrt{a + (\dot{x}^0 - C_3^0)^2} , \quad F^2 = C_3^2 , \quad F^3 = C_3^3 , \]
where
\[ a = \frac{r_*^4}{m^2} \left( 1 - \frac{2m}{r_*} \right) . \]

The arbitrary constants \( C_3^i \) are not important, they can be eliminated by the shift of the origin of the Fermi coordinates.

The Riemann tensor \( \dot{R}_{iklm} (\dot{x}) \) can be found by transformation (55)-(58) from its known counterpart \( R_{iklm} (x) \) for the Schwarzschild metric (40) which has the following non-zero components:
\[ R_{0101} = R_{rrtr} = -\frac{2m}{r^3} , \]
\[ R_{0202} = R_{t\theta t\theta} = \frac{m}{r} \left( 1 - \frac{2m}{r} \right) , \]
\[ R_{0303} = R_{t\varphi t\varphi} = \frac{m}{r} \left( 1 - \frac{2m}{r} \right) \sin^2 \theta , \]
\[ R_{1212} = R_{r\theta r\theta} = -\frac{m}{r} \left( 1 - \frac{2m}{r} \right)^{-1} , \]
\[ R_{1313} = R_{r\varphi r\varphi} = -\frac{m}{r} \left( 1 - \frac{2m}{r} \right)^{-1} \sin^2 \theta, \quad (66) \]

\[ R_{2323} = R_{\theta\varphi\theta\varphi} = 2mr \sin^2 \theta. \quad (67) \]

We do not included in this list those non-zero components of \( R_{iklm}(x) \) which can be obtained from (62)-(67) by application of all symmetries of the Riemann tensor. These components transform to the components of \( \hat{R}_{iklm}(\hat{x}) \) by the usual tensor law and on the line this transformation take the form:

\[(\hat{R}_{psqn})_L = [R_{iklm}Q^i_pQ^k_sQ^l_qQ^m_n]_L , \quad (68)\]

where matrix \( Q^i_k \) is inverse to the Jacobian matrix \( A^k_i \) introduced in (2), see also (B3). For the transformation (55)-(58) these matrices calculated on the line \( L \) (the upper index numerates the matrix lines and lower index corresponds to the columns) are:

\[ [A^k_i]_L = \begin{pmatrix}
2\lambda \cosh \omega t & (2\lambda)^{-1} \sinh \omega t & 0 & 0 \\
2\lambda \sinh \omega t & (2\lambda)^{-1} \cosh \omega t & 0 & 0 \\
0 & 0 & r_* & 0 \\
0 & 0 & 0 & r_* \sin \theta_* \\
\end{pmatrix}, \quad (69)\]

\[ [Q^i_k]_L = \begin{pmatrix}
(2\lambda)^{-1} \cosh \omega t & - (2\lambda)^{-1} \sinh \omega t & 0 & 0 \\
-2\lambda \sinh \omega t & 2\lambda \cosh \omega t & 0 & 0 \\
0 & 0 & (r_*)^{-1} & 0 \\
0 & 0 & 0 & (r_* \sin \theta_*)^{-1} \\
\end{pmatrix}. \quad (70)\]

Calculations of \( [\hat{R}_{psqn}]_L \) from (68) using \( [Q^i_k]_L \) from (70) and \( [R_{iklm}]_L = R_{iklm}(r_*, \theta_*) \) from (62)-(67) gives:

\[ [\hat{R}_{0101}]_L = -\frac{2m}{r_*^3}, \quad [\hat{R}_{0202}]_L = \frac{m}{r_*^3}, \quad [\hat{R}_{0303}]_L = \frac{m}{r_*^3}, \quad (71)\]

\[ [\hat{R}_{1212}]_L = -\frac{m}{r_*^3}, \quad [\hat{R}_{1313}]_L = -\frac{m}{r_*^3}, \quad [\hat{R}_{2323}]_L = \frac{2m}{r_*^3}. \quad (72)\]

We see that on line \( L \) the Riemann tensor in the Fermi coordinates contains the same set of non-zero components as in Schwarzschild coordinates but their values are simpler. We again do not included in formulas (71)-(72) those non-zero components of \( [\hat{R}_{iklm}]_L \) which can be obtained by application of symmetries of the Riemann tensor.
To write down the final form of the metric it is convenient to introduce shifting Fermi coordinates $\tau, u, v, w$: 

$$
\tau = \dot{x}^0 - C^0_3, \quad u = \dot{x}^1 - C^1_3, \quad v = \dot{x}^2 - C^2_3, \quad w = \dot{x}^3 - C^3_3.
$$

(73)

Collecting all information on the constants $c_{ik}$ (54), functions $F^\alpha (\dot{x}^0)$ (60), and components of the Riemann tensor $[\dot{R}_{iklm}(\dot{x})]_L$ (71)-(72) we obtain from (37)-(39) the final form of the metric for the static Schwarzschild observer in Fermi coordinates $\tau, u, v, w$ (73):

$$
-ds^2 = \dot{g}_{ik}(\dot{x})\ddot{x}^i\ddot{x}^k = \dot{g}_{\tau\tau}d\tau^2 + 2\dot{g}_{\tau u}d\tau du + 2\dot{g}_{\tau v}d\tau dv + 2\dot{g}_{\tau w}d\tau dw
$$

$$
+\dot{g}_{uu}du^2 + \dot{g}_{vv}dv^2 + \dot{g}_{ww}dw^2 + 2\dot{g}_{uv}du dv + 2\dot{g}_{uw}du dw + 2\dot{g}_{vw}dv dw,
$$

(74)

where components of the metric tensor (up to the quadratic terms with respect to the three small deviations $u - \sqrt{\tau^2 + a}, v, w$ from the line) are:

$$
\dot{g}_{\tau\tau} = -1 + \frac{m}{r^3} \left[ 2 \left( u - \sqrt{\tau^2 + a} \right)^2 - v^2 - w^2 \right],
$$

(75)

$$
\dot{g}_{uu} = 1 + \frac{m}{3r^3} \left( v^2 + w^2 \right),
$$

(76)

$$
\dot{g}_{vv} = 1 + \frac{m}{3r^3} \left( u - \sqrt{\tau^2 + a} \right)^2 - 2w^2 \right],
$$

(77)

$$
\dot{g}_{ww} = 1 + \frac{m}{3r^3} \left( u - \sqrt{\tau^2 + a} \right)^2 - 2v^2 \right],
$$

(78)

$$
\dot{g}_{uv} = \frac{m\tau}{3r^3\sqrt{\tau^2 + a}} v, \quad \dot{g}_{uw} = \frac{m\tau}{3r^3\sqrt{\tau^2 + a}} w, \quad \dot{g}_{vw} = \frac{2m}{3r^3} vw.
$$

(79)

IV. ACKNOWLEDGMENT

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[1] E. Fermi "Sopra i fenomeni che avvengono in vicinanza di una linea oraria", Rendiconti dei Lincei, Serie 5, 31-1, 51-52 (1922). In: Enrico Fermi, Collected Papers (Note e Memorie), Accademia Nazionale dei Lincei - The University of Chicago Press (1962).
The Latin indices run through the 4 values 0,1,2,3 and Greek indices take 3 values 1,2,3. The simple partial derivatives we denote by comma and covariant derivatives by semicolon. We stress that in general $x^0$ has no obligation to be timelike. One can chose any distribution of signs in signature of the metric tensor.

Of course, the particular case for geodesic lines is also included. If the line (11) is geodesics then functions $f^\alpha(x^0)$ cannot be arbitrary but should follow from the geodesic equations in metric $g_{ik}(x)$. Without loss of generality we can consider the metric in Fermi coordinates (37)-(39) having the form $\hat{g}_{ik} = \eta_{ik} + O(2)$ where $\eta_{ik}$ is Minkowski tensor. Then near the line the spacetime is flat and geodesics will be just a stright line. We can use the Lorentz rotation and shift of the origin of coordinates (which group do not changes $\eta_{ik}$ and zero value of $\Gamma$-symbols on line) in order to make this geodesics coincides with coordinate line of the variable $\dot{x}^0$ starting from the origin of the Fermi coordinate system. This means that equations of line will be $\dot{x}^\alpha = 0$ [which means that transformation coefficients $X^\alpha(x^0)$ in (6) vanish] and all functions $F^\alpha(\dot{x}^0)$ in metric (37)-(39) disappear. In this case some part of the equations (16), (17) and (19) will be equivalent to the geodesic equations for the line $x^\alpha = f^\alpha(x^0)$ in metric $g_{ik}(x)$ and all other relations will define the rest of the transformation coefficients in (6). The results will be the same as in paper [3] (with the difference that the Riemann tensor in our paper has defined with opposite sign).
Appendix A: Standard formulas.

We use notations of the book [7]. In any spacetime with coordinates \( x^i \) and metric tensor \( g_{ik} \) the \( \Gamma \)-symbols and Riemann tensor are:

\[
\Gamma^i_{~kl} = \frac{1}{2} g^{im} (g_{mk,l} + g_{lm,k} - g_{kl,m}) ,
\]

(A1)

\[
R^i_{~klm} = \Gamma^i_{~km,l} - \Gamma^i_{~kl,m} + \Gamma^i_{~nl} \Gamma^n_{~km} - \Gamma^i_{~nm} \Gamma^n_{~kl} ,
\]

(A2)

\[
R_{iklm} = g_{in} R^n_{~klm} .
\]

(A3)

There are 4 symmetry identities for Riemann tensor:

\[
R^i_{~klm} = - R^i_{~kml} , \quad R^i_{~iklm} = - R^i_{~ikml} , \quad R^i_{~iklm} = R^i_{~lmik} ,
\]

(A4)

\[
R^i_{~klm} + R^i_{~imkl} + R^i_{~ilmk} = 0 .
\]

(A5)

From definitions (A1)-(A3) follows another representation for \( R_{iklm} \):

\[
R^i_{~klm} = \frac{1}{2} (g_{im,kl} + g_{kl,im} - g_{il,km} - g_{km,il}) + g_{np} (\Gamma^m_{~kl} \Gamma^p_{~im} - \Gamma^n_{~km} \Gamma^p_{~il}) .
\]

(A6)

Appendix B: On the reduced Eddington coordinates restriction.

The transformation (6) with cubic terms is:

\[
\dot{x}^m = X^m(x^0) + Y^m_{\alpha}(x^0) \left[ x^\alpha - f^\alpha(x^0) \right] \\
+ Z^m_{\alpha\beta}(x^0) \left[ x^\alpha - f^\alpha(x^0) \right] \left[ x^\beta - f^\beta(x^0) \right] \\
+ W^m_{\alpha\beta\gamma}(x^0) \left[ x^\alpha - f^\alpha(x^0) \right] \left[ x^\beta - f^\beta(x^0) \right] \left[ x^\gamma - f^\gamma(x^0) \right] + O(4) ,
\]

(B1)

where coefficients \( W^m_{\alpha\beta\gamma} \) are symmetric with respect to the transposition of any two of the lower indices. Then we have 40 (ten for each 4-dimensional index \( m \)) independent coefficients \( W^m_{\alpha\beta\gamma} \). Now we apply the 4-dimensional partial derivative \( \partial / \partial x^s \) to the general transformation of \( \Gamma \)-symbols (3) and restrict the result to the line \( \mathcal{L} \) (taking into account that all \( \dot{\Gamma}^q_{~nm} \) are zero on this line). This operation gives:

\[
\left[ \frac{\partial}{\partial x^s} (\Gamma^i_{~kl} A^q_i) = \left( \frac{\partial}{\partial x^s} \dot{\Gamma}^q_{~nm} \right) A^p_k A^m_l + \frac{\partial^3 \dot{x}^q}{\partial x^k \partial x^l \partial x^s} \right] \mathcal{L} .
\]

(B2)
Let's denote the 4-dimensional matrix inverse to $A^i_k$ by $Q^k_i$, that is:

$$Q^k_i A^i_k = \delta^k_i ,$$  \hspace{1cm} (B3)

and multiply relation (B2) by $(Q^s_{\alpha \beta} Q^q_l)\mathcal{L}$ with all three lower indices 3-dimensional. We obtain:

$$\left[ Q^s_{\alpha \beta} Q^k_l Q^l_q \frac{\partial}{\partial x^s} (\Gamma^{i}_{kl} A^i_q) = \frac{\partial}{\partial x^{\alpha}} \Gamma^q_{\beta \gamma} + \frac{\partial^3 \hat{x}^q}{\partial x^k \partial x^l \partial x^s} Q^s_{\alpha \beta} Q^k_l \Gamma^q_{\beta \gamma} \right] \mathcal{L} . \hspace{1cm} (B4)$$

Then we repeat this relation two times more with cyclic permutation of the 3-dimensional indices $\beta, \gamma, \alpha \rightarrow \alpha, \beta, \gamma \rightarrow \gamma, \alpha, \beta$ and sum all three expressions. In result we have:

$$\left[ \frac{\partial}{\partial x^{\alpha}} \Gamma^q_{\beta \gamma} + \frac{\partial}{\partial x^{\beta}} \Gamma^q_{\alpha \gamma} + \frac{\partial}{\partial x^{\gamma}} \Gamma^q_{\alpha \beta} \right] = - \left\{ 3 Q^s_{\alpha \beta} Q^k_l \frac{\partial^3 \hat{x}^q}{\partial x^k \partial x^l \partial x^s} \right\} \mathcal{L} \hspace{1cm} (B5)$$

$$+ \left\{ Q^s_{\alpha \beta} Q^k_l Q^l_q \left[ \frac{\partial}{\partial x^s} (\Gamma^{i}_{kl} A^i_q) + \frac{\partial}{\partial x^l} (\Gamma^{i}_{sk} A^i_q) + \frac{\partial}{\partial x^k} (\Gamma^{i}_{ls} A^i_q) \right] \right\} \mathcal{L} .$$

Consequently the 3-dimensional Eddington condition (25) will be satisfied if we chose the cubic addend in transformation (B3) to satisfy the requirement:

$$\left\{ Q^s_{\alpha \beta} Q^k_l Q^l_q \frac{\partial^3 \hat{x}^q}{\partial x^k \partial x^l \partial x^s} \right\} \mathcal{L} \hspace{1cm} (B6)$$

$$= \frac{1}{3} \left\{ Q^s_{\alpha \beta} Q^k_l Q^l_q \left[ \frac{\partial}{\partial x^s} (\Gamma^{i}_{kl} A^i_q) + \frac{\partial}{\partial x^l} (\Gamma^{i}_{sk} A^i_q) + \frac{\partial}{\partial x^k} (\Gamma^{i}_{ls} A^i_q) \right] \right\} \mathcal{L} .$$

The left and right sides in relation (B6) are symmetric with respect to the transposition of any two of the indices $\alpha, \beta, \gamma$, consequently this relation represents 40 independent equations for 40 unknown coefficients $W^m_{\alpha \beta \gamma}$ which enter the third derivatives of $\hat{x}^q$. No other quantity in (B6) contain these $W^m_{\alpha \beta \gamma}$. It is important that terms $(\partial^3 \hat{x}^q / \partial x^k \partial x^l \partial x^s)\mathcal{L}$ are linear with respect to $W^m_{\alpha \beta \gamma} (x^0)$ and do not contain $x^0$-derivatives of these functions. Then the system (B6) is the set of the linear algebraic equations with respect to the unknowns $W^m_{\alpha \beta \gamma}$. Indeed, only the last term in expansion (B1) for Fermi coordinates $\hat{x}^q$ contains quantities $W^m_{\alpha \beta \gamma}$ and it is easy to show that the left hand side of equation (B6) has the structure:

$$\left[ Q^s_{\alpha \beta} Q^k_l Q^l_q \frac{\partial^3 \hat{x}^q}{\partial x^k \partial x^l \partial x^s} \right] \mathcal{L} = 6 W^q_{\mu \lambda \nu} \left[ N^\mu_\alpha N^\lambda_\beta N^\nu_\gamma \right] \mathcal{L} + \ldots \hspace{1cm} (B7)$$

where dots mean all terms which do not contain coefficients $W^q_{\mu \lambda \nu}$ and $3 \times 3$ matrix $(N^\alpha_\beta)\mathcal{L}$ is:

$$\left[ N^\alpha_\beta \right] \mathcal{L} = \left[ Q^\alpha_\beta - Q^0_\beta \frac{df^\alpha_\beta}{dx^0} \right] \mathcal{L} . \hspace{1cm} (B8)$$

Then using matrix inverse to $(N^\alpha_\beta)\mathcal{L}$ the system (B6) can be uniquely resolved with respect to the unknown coefficients $W^q_{\mu \lambda \nu}$. This is the proof of the possibility to specialize the Fermi
coordinate in the way to achieve the 3-dimensional analogue of the Eddington coordinates condition \(^{(25)}\).