A NOTE ON THE VIABILITY OF GAUSS-BONNET COSMOLOGY

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In this paper, we analyze the viability of a vacuum Gauss-Bonnet cosmology by examining the dynamics of the homogeneous and anisotropic background in 4+1 dimensions. The trajectories of the system either originate from the standard singularity or from non-standard type, the later is characterized by the divergence of time derivative of the Hubble parameters for its finite value. At the onset, the system should relax to Einstein phase at late times as the effect of Gauss-Bonnet term becomes negligible in the low energy regime. However, we find that most of the trajectories emerging from the standard big-bang singularity lead to future re-collapse whereas the system beginning its evolution from the non-standard singularity enters the Kasner regime at late times. This leads to the conclusion that the measure of trajectories giving rise to a smooth evolution from a standard singularity to the Einstein phase is negligibly small for generic initial conditions.

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I. INTRODUCTION

Modified theories of gravity are under active consideration at present in cosmology. Efforts are being made to mimic late time acceleration from large scale modification of gravity without resorting to exotic forms of matter dubbed dark energy[1, 2]. The extra dimensional effects can give rise to modification of gravity; similar effects can be induced by adding a generic function of Ricci scalar to Einstein-Hilbert action giving rise to f(R) gravity (see Ref.[3] and references therein). The quantum effects can also lead to higher order curvature corrections to Einstein-Hilbert action. These corrections can be systematically computed in perturbative regime of string theory. Amongst all the higher derivative corrections which might arise quantum mechanically, the Gauss-Bonnet (GB) correction has distinguished features[4]. In this case, the equations of motion continue to be of second order thereby ensuring the uniqueness of their solutions. However, in 3+1 dimensions, the GB term is topological in nature; it acquires dynamics only in higher dimensions. Nevertheless, it can influence the 4 dimensional physics if it is coupled to a dynamically evolving scalar field(s). The pure GB term being in the higher dimensional bulk can also lead to modification of Einstein equations on the brane[24].  

Attempts have recently been made to derive current acceleration using the GB term coupled to a scalar field[5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17]. The model exhibits remarkable property that it does not disturb the scaling regime and can give rise to late time transition from matter regime to late time acceleration[7, 8]. This beautiful result comes with a cost: the coupling of GB curvature invariant to scalar field gets large at late times and can not be justified within the perturbative regime the curvature corrections are obtained; the model is also under pressure from nucleosynthesis constraint[2, 8]. On the theoretical ground, these models are faced with other serious problems related to stability against perturbations about FRW background[14]. Similar situation is expected to persist in the case of higher order Euler densities coupled to scalar (dilaton/modulus) fields. Of course, one can argue that these fields should be stabilized sufficiently early in order to respect the nucleosynthesis constraints. It is, nevertheless, important to examine the viability of Gauss-Bonnet cosmology in general.  

In this paper, we take a different route; we consider a vacuum 4+1 dimensional GB cosmology in a homogeneous and anisotropic background and study the structure of generic singularities in the model. Though 4+1 theories without compactification have no direct applications to our Universe, study of their properties is important for better understanding of gravity in four dimensions, showing its specific properties in comparison with other cases. It is known, for example, that in five dimensional Einstein gravity the uniqueness theorem for a stationary black hole configurations is no longer valid[18]. Another classical example is related to the disappearance of Mixmaster cosmological chaotic behavior in 10+1 dimensions[19]. These results have been formulated in the framework of Einstein gravity. The Gauss-Bonnet term can further modify traditional results known for 3+1 dimensional Einstein theory. The main goal of the present paper is to study the modifications of cosmological singularity due to Gauss-Bonnet term in
multidimensional cosmology. In the low energy regime one might expect the system to relax to 4+1 dimensional Kasner geometry. We shall examine the cosmological dynamics of the system under consideration and investigate the measure of trajectories which might connect to Einstein phase at late times.

II. EVOLUTION EQUATIONS

We consider a 4+1 dimensional theory with the action

\[ S = \int \sqrt{-g} \left( R + \alpha R_{GB}^2 \right) \, d^4x, \tag{1} \]

where \( R_{GB}^2 \) is the Gauss–Bonnet term

\[ R_{GB}^2 = R^{iklm} R_{iklm} - 4 R^{ik} R_{ik} + R^2. \]

In what follows we shall be interested in the dynamics of the system described by (1) in the homogeneous and anisotropic flat background with the metric

\[ g_{ik} = \text{diag}(−n^2(t), a^2(t), b^2(t), c^2(t), d^2(t)). \tag{2} \]

This metric provides us a simplest modification of the standard geometry allowing the realization of new dynamical regimes absent in both Einstein gravity and isotropic Gauss–Bonnet modified Einstein theory of gravity (for a complete survey of possible 5-dimensional cosmological backgrounds, see Ref. [20]).

It would be convenient to introduce Hubble parameters with respect to four spatial dimensions \( H_{a,b,c,d} = \frac{\dot{a}, \dot{b}, \dot{c}, \dot{d}}{a, b, c, d} \). In the background described by the metric (2), the action (1) is a functional of of the scale factors and the lapse function along with their time derivatives. Varying the action (1) with respect to the lapse function \( n = 1 \) thereafter we find the constraint equation

\[ 2\dot{H}_a H_b + 2\dot{H}_a H_c + 2\dot{H}_a H_d + 2\dot{H}_b H_c + 2\dot{H}_b H_d + 2\dot{H}_c H_d + 24\alpha \dot{H}_a H_b H_c H_d = 0, \tag{3} \]

which is the analogue of Friedmann equation in case of the geometry given by (2). Variation of (1) with respect to scale factors leads to the system of four dynamical equations,

\[ 2(\dot{H}_b + H_b^2) + 2(\dot{H}_c + H_c^2) + 2(\dot{H}_d + H_d^2) + 2\dot{H}_b H_c + 2\dot{H}_b H_d + 2\dot{H}_c H_d + \]

\[ + 8\alpha \left( (\dot{H}_b + H_b^2) H_c H_d + (\dot{H}_c + H_c^2) H_b H_d + (\dot{H}_d + H_d^2) H_b H_c \right) = 0, \tag{4} \]

\[ 2(\dot{H}_a + H_a^2) + 2(\dot{H}_c + H_c^2) + 2(\dot{H}_d + H_d^2) + 2\dot{H}_a H_c + 2\dot{H}_a H_d + 2\dot{H}_c H_d + \]

\[ + 8\alpha \left( (H_a + H_a^2) \dot{H}_c H_d + (H_c + H_c^2) \dot{H}_a H_d + (H_d + H_d^2) \dot{H}_a H_c \right) = 0, \tag{5} \]

\[ 2(\dot{H}_a + H_a^2) + 2(\dot{H}_b + H_b^2) + 2(\dot{H}_d + H_d^2) + 2\dot{H}_a H_b + 2\dot{H}_a H_d + 2\dot{H}_b H_d + \]

\[ + 8\alpha \left( (H_a + H_a^2) \dot{H}_b H_d + (H_b + H_b^2) \dot{H}_a H_d + (H_d + H_d^2) \dot{H}_a H_b \right) = 0, \tag{6} \]

\[ 2(\dot{H}_a + H_a^2) + 2(\dot{H}_b + H_b^2) + 2(\dot{H}_c + H_c^2) + 2\dot{H}_a H_b + 2\dot{H}_a H_c + 2\dot{H}_b H_c + \]

\[ + 8\alpha \left( (H_a + H_a^2) \dot{H}_b H_c + (H_b + H_b^2) \dot{H}_a H_c + (H_c + H_c^2) \dot{H}_a H_b \right) = 0. \tag{7} \]

The evolution equations, in general, look cumbersome for analytical investigations. In what follows we shall investigate the dynamical regimes of the model numerically.
The presence of Gauss-Bonnet (GB) term allows some specific dynamical regimes absent in pure Einstein gravity. First of all, the volume of a flat Universe can have local extrema in this background. The another new feature is associated with the possible existence of a nonstandard singularity, found in Ref.\[21\] (this type of singularity was also found previously in another context in Ref.\[22\], similar situation can also arise in 3+1-dimensional cosmology with GB-term in presence of a dynamical dilaton \[23, 24, 25\]). Interestingly, the GB brane worlds with the curvature term on the brane can also give rise to this type of singularity\[26\]. The non-standard singularity, under consideration, is characterized by $\dot{H}_i \to \infty$ ($H_i = H_{a,b,c,d}$), for finite values of Hubble parameters. It occurs when the major determinant of the system (4) – (7) vanishes.

We note that the generalized Kasner regime, being the solution of vacuum equation motion for Bianchi I Einstein Universe, remains intact in the low-energy regime, when Gauss-Bonnet contribution can be neglected. This solution has the form $ds^2 = -dt^2 + \sum t^{2p_i} dx_i^2$ with two known condition on the power indices

\[
p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1,
\]

\[
p_1 + p_2 + p_3 + p_4 = 1.
\]

In the high-energy regime, the Gauss-Bonnet term becomes important. However, in 4 + 1 Universe there are no pure Gauss-Bonnet nontrivial vacuum solutions, similar to found recently for the 5 + 1 dimensional case \[27\]. To illustrate this point, let us consider Eq. (3). We observe that there is only one term originating from the Gauss-Bonnet contribution (the last term on the LHS), so this term and the remaining Einstein contribution (first three terms of the LHS) are equal in absolute values and opposite in sign. This means that though a standard singularity (when all Hubble parameters tend to infinity) is still possible, the dynamics in its vicinity is more complicated than in 3 + 1 Einstein or 5 + 1 Gauss–Bonnet cases: the Einstein and Gauss-Bonnet term are equally important in the 4 + 1 case near a singularity. However, one particular regime can be studied easily. If three Hubble parameters are equal and large ($H_a = H_b = H_c = H$), the equation (3) tells us that $H_d$ should tend to zero near a standard singularity ($H_d = H/(1 + 4\alpha H^2)$), and we have

\[
\dot{H} = -\frac{6H^2 + 36\alpha H^4 + 48\alpha^2 H^6}{3 + 48\alpha^2 H^4 + 24\alpha H^2}, \quad \dot{H}_d = \frac{-3H^2 + 12\alpha H^4}{3 + 48\alpha^2 H^4 + 24\alpha H^2}.
\]

In this asymptotic regime the denominator is always positive, and we can not meet the nonstandard singularity. Numerical studies confirm that this singular regime can be smoothly matched with the low-energy Kasner asymptotic.
FIG. 2: Each point in the figure corresponds to a possible trajectory. The backward evolution leads to two possible outcomes, the standard singularity and the non-standard singularity. The figure shows that around 40 percent of the trajectories originate from the standard singularity and the rest have their origin in the non-standard singularity.

FIG. 3: Each point in the figure represents a trajectory originated from the standard singularity. In forward evolution, different trajectories evolve into low energy Kasner regime, non-standard singularity and re-collapse. Around 95 percent of the trajectories lead to re-collapse, about 1.5 percent encounter non-standard singularity, and the rest fall into low energy Kasner regime. Trajectories with three equal Hubble parameters go smoothly to Kasner regime.

Interestingly, our numerical integrations show that it is the only case when a Universe can evolve from a Big Bang singularity to low-energy regime when Gauss-Bonnet contribution is negligible. All other initial conditions lead either to trajectories originating in the nonstandard singularity (or meeting this singularity in their future evolution) or experience re-collapse back towards a singularity.

Let us now spell out the details of the numerical investigations of the system described by the system of equations (1)-(7) for a large set of initial conditions in the range $-1 < H_i < 1$ (we fixed three Hubble parameters and found the fourth from the constraint equation). As we consider only positive values of the coupling constant $\alpha$ in the present
paper, we set $\alpha = 1$ in our numerical work. Our simulations reveal that: (a) During forward evolution, around half of the trajectories evolve to low energy Kasner regime whereas in the case of the other half, we found that the system re-collapses back to singularity. A negligible number of trajectories lead to non-standard singularity in this case, see Fig.1. (b) In the backward evolution, we distinguish two cases representing two possible starting points of 4+1-dimensional universe, namely, a standard singularity and a nonstandard singularity. We have found that more than 60% initial conditions lead to non-standard singularity; in 40% cases, we find trajectories evolving to standard singularity, see Fig.2.

We further observe interesting features while connecting the history of universe with its future evolution:

- For a particular set of initial condition with three equal Hubble parameters, the system evolves to low curvature Einstein regime starting from a standard singularity. Except for the aforementioned particular case, we find that:
  - The trajectories originating from a standard singularity lead to re-collapse for most of the initial conditions in their future evolution (Fig.3 and Fig.4); the non-standard singularity is also possible in rare cases.
  - Trajectories reaching a low-curvature Einstein regime originate from a nonstandard singularity.

As the measure of trajectories with three equal Hubble parameters is zero, and our numerical results show that initial anisotropy of the order of $10^{-5}$ is enough to destroy the smooth evolution and that most of the trajectories reaching low-curvature regime must have their origin in nonstandard singularity. Due to instabilities of the numerical procedure near a singularity we can treat this number as only an upper limit of possible anisotropy allowing a smooth evolution. Similar result (no Big-Bang singularity) is obtained for the induced brane gravity with the Gauss-Bonnet contribution in 5-dimensional bulk [26].

IV. CONCLUSIONS

In the present paper we examined the dynamics of a flat anisotropic 4+1 dimensional universe in Gauss-Bonnet modified Einstein gravity. The GB gravity in 4+1 dimensions in the FRW background leaves standard big bang singularity unaltered. One could naively expect that the dynamical system under consideration would settle to Einstein phase in the low energy regime thereby leading to standard description in 4 dimensional space time in the Kaluza-Klein compactification scheme. It is really interesting that the introduction of small anisotropy is capable of destroying the smooth evolution and can lead to new dynamical regimes unknown to Einstein or GB modified Einstein gravity in the homogeneous and isotropic background. We have investigated the underlying dynamics of the
system numerically and observed several interesting features of the dynamics. In case, $H_a = H_b = H_c$, our numerical results show that trajectories starting from standard singularity can evolve to low energy Kesner regime, however, the measure of such trajectories is negligibly small. Excluding this particular case, we find that all the trajectories beginning from standard singularity end up in the non-standard singularity or encounter re-collapse in future. Within this class of initial conditions, the trajectories which reach the Einstein regime at late times are found to have their origin in the non-standard singularity.

We thus conclude that the nonstandard singularity discussed earlier in Ref. [21] is not a very particular case of the dynamics, it rather represents a typical feature of the cosmological dynamics which frequently occurs during the evolution. The measure of the system trajectories, which can smoothly connect the past standard singularity with the low energy Einstein regime, is very small and possibly can be zero. Further investigation are required to understand whether this result is connected with specific properties of 4+1 Gauss-Bonnet gravity or it is a more general feature of Lovelock theory.

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