A new variants of quasi-newton equation based on the quadratic function for unconstrained optimization

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ABSTRACT
The focus for quasi-Newton methods is the quasi-Newton equation. A new quasi-Newton equation is derived for quadratic function. Then, based on this new quasi-Newton equation, a new quasi-Newton updating formulas are presented. Under appropriate conditions, it is shown that the proposed method is globally convergent. Finally, some numerical experiments are reported which verifies the effectiveness of the new method.

Keywords: Global convergence
New quasi-Newton equation
Quasi-Newton methods

1. INTRODUCTION
The quasi-Newton (QN) method is a willing device to find the minimum value of problem. Let us think over problem for the following form:

\[ \text{Min } f(x), \ x \in \mathbb{R}^n \]  

where \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is continuously differentiable. For more details can be found in [1].

The basic form of a quasi-Newton method for solving (1). and it is written as:

\[ x_{k+1} = x_k + \alpha_k d_k \]  

where \( \alpha_k \) is the step length which satisfies certain Wolfe conditions:

\[ f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k \]  

\[ d_k^T g(x_k + \alpha_k d_k) \geq \sigma d_k^T g_k \]

for some parameters \( 0 < \delta < \sigma < 1 \). For details see [2].
The search direction of the quasi-Newton method is defined by:

\[ \alpha_k d_k + g_k = 0 \]

(5)

where \( B_k \) is an approximation matrix of the Hessian, and \( g_k \) denotes \( g(x_k) \). The matrix \( \{B_k\} \) are positive definite and they satisfy the quasi-Newton equation:

\[ \alpha_k s_k = y_k \]

(6)

where \( s_k = x_{k+1} - x_k \) and \( y_k = g_{k+1} - g_k \), is satisfied. For details see [3].

One possibility to obtain \( B_{k+1} \) from \( B_k \) by an update formula is to have a BFGS update:

\[ B_{k+1}^{BFGS} = B_k - \frac{B_k s_k s_k^T B_k^T}{s_k^T B_k s_k} + \frac{y_k y_k^T}{s_k^T y_k} \]

(7)

Precisely, if \( H_k^{-1} = B_k \), \( B_{k+1} = U(B_k, y_k, s_k) \) and \( H_{k+1} = U(H_k, s_k, y_k) \) then \( H_{k+1}^{-1} = B_{k+1} \), this property called duality transformation. Applying this relation to the BFGS method, we will get the dual of the BFGS formula:

\[ H_{k+1}^{BFGS} = H_k - \frac{H_k y_k s_k^T + s_k y_k^T H_k}{s_k^T y_k} + \left[ 1 + \frac{y_k^T H_k y_k}{s_k^T y_k} \right] \frac{s_k y_k^T}{s_k^T y_k} \]

(8)

For details see [4].

Different quasi-Newton methods correspond to different ways of updating the matrix \( H_{k+1} \) to include the new curvature information obtained during the \( k^{th} \) iteration. Different Improving of these methods are made, in the aim to develop them in [5-8]. We are going to present a new quasi-Newton equation based on quadratic function and show that these methods are the best for solving unconstrained.

2. DERIVING QUASI-NEWTON EQUATION BASED ON THE QUADRATIC FUNCTION:

Before presenting new quasi-Newton equation, we shall derive an estimate of a step size. Assume that the function is defined a quadratic function of the form:

\[ f_{k+1} = f_k + \alpha_k g_k^T d_k + \frac{1}{2} \alpha_k^2 d_k^T Q d_k \]

(9)

Then the minimum point \( \alpha_k \) of above function is given by:

\[ \alpha_k = -\frac{g_k^T d_k}{d_k^T Q d_k} \]

(10)

Now, we shall derive alternate quasi-Newton equation. Therefore, from above equation we get:

\[ \alpha_k d_k^T Q d_k = -g_k^T d_k \]

(11)

The basic idea is to approximate either \( Q \) by another matrix \( B_{k+1} \) to obtain a higher accuracy, can be expressed as:
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\[ s_k^T B_{k+1} s_k = - s_k^T g_k \]  
(12)

By the following definition of the different gradient \( y_k \) we get:

\[ y_k^T s_k = g_k^T s_k - g_k^T s_k \]  
(13)

By putting (13) in (12) we get:

\[ s_k^T B_{k+1} s_k = y_k^T s_k - g_k^T s_k \]  
(14)

One of the possible choices in approximation of \( B_{k+1} s_k \) can be given by:

\[ B_{k+1} s_k = y_k, \quad y_k = y_k - \frac{g_k^T s_k}{s_k^T z_k} z_k \]  
(15)

where \( s_k^T z_k \neq 0 \) and \( z_k \) is any vector.

The value of \( z_k \) are not unwavering in a unique technique, but the suitable choice are \( z_k = y_k \) and \( z_k = g_k \).

If \( f \) is non-convex may mislay the positive definiteness. We need for some extra assumptions on the update.

In order to prove the following theorems, we define the index set \( K \) as:

\[ K = \left\{ k \ : \ s_k^T y_k \geq \beta \| g_k \|^2 \right\} \]  
(16)

where \( \beta > 0 \) is constant and \( \delta > 0 \) is bounded.

Now, we give the algorithm of the new method.

**Assumptions:** \( x_0 \in \mathbb{R}^n \). Let \( k = 0 \).

Stage 1: if \( \| g_k \| = 0 \), then stop.

Stage 2: Solve \( H_k^{-1} d_k = - g_k \) for \( d_k \).

Stage 3: Determine the step size \( \alpha_k \) such that Wolfe line search rules hold.

Stage 4: Set \( x_{k+1} = x_k + \alpha_k d_k \). If \( s_k^T y_k > 0 \), update \( H_{k+1} \) by the formula (8) and (15), otherwise let \( H_{k+1} = H_k \).

Stage 5: \( k = k + 1 \), return to stage 1.

The next theorem is very important. We prove the condition \( s_k^T y_k > 0 \) which is known as the curvature condition

**Theorem 1.**

The new update (8) and (15) retains positive definiteness if and only if \( s_k^T y_k > 0 \).

**Proof:**

Having in view that the definition of the different gradient by:

\[ \bar{y}_k = y_k - \frac{g_k^T s_k}{s_k^T z_k} z_k \]  
(17)
Multiplying above equation by $s_k^T$, we have:

$$s_k^T y_k = s_k^T y_k - s_k^T s_k = -s_k^T g_k$$

From (2) we get $s_k = \alpha_k d_k$. In fact, the search direction of a QN is descent, we noting that $s_k^T g_k = \alpha_k d_k^T g_k < 0$, such that:

$$s_k^T y_k = -\alpha_k d_k^T g_k > 0$$

The proof is complete.

3. CONVERGENCE ANALYSIS

In order to prove the convergence, we consider the following assumptions: The level set $D = \{x \mid f(x) \leq f(x_0)\}$, with $x_0$ is an initial point of iterative method is restricted.

Assumption A. Using Lipschitz continuous; that is exist constants $L$ and $\gamma$, such that:

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\|, \quad \forall \ x, \ y \in D$$

And

$$\|\nabla f(x)\| \leq \gamma, \quad \forall \ x \in D$$

Since $\{f(x_k)\}$ is a decreasing, which ensures $\{x_k\}$ is contained in $D$ and the existence of $x^*$ we get:

$$\lim_{k \to \infty} f(x_k) = f(x^*)$$

In reality, that sequence $x_k$ is restricted, there exist some positive constant $\mu$ such that for all $k$:

$$\|s_k\| = \|x - x_k\| \leq \|x\| + \|x_k\| \leq \mu$$

For more details see [9,10].

Existing the handy theorem to explain that our method is globally convergent.

Theorem 2.

If $\|\nabla f(x)\| \leq \gamma$ is not holds for all $k$. Let $\{x_k\}$ be generated by new methods, and the following inequality holds:

$$\|B_k s_k\| \leq a_1 \|s_k\| \quad \text{and} \quad s_k^T B_k s_k \geq a_2 \|s_k\|^2,$$

where $a_1 > 0$ and $a_2 > 0$ are constants. For infinitely $k$, then we have:

$$\lim_{k \to \infty} \inf |g_k| = 0.$$
**Proof:**

By (4) of the Wolfe conditions we obtain:

\[(g_{k+1} - g_k)^T d_k \geq -(1-\sigma) g_k^T d_k \quad (26)\]

Moreover, from Lipschitz condition we obtain:

\[(g_{k+1} - g_k)^T d_k \leq L \alpha_k \|d_k\|^2 \quad (27)\]

Combining above two equation we get:

\[\alpha_k \geq \frac{- (1-\sigma) g_k^T d_k}{L \|d_k\|^2} = \frac{(1-\sigma) d_k^T B_k d_k}{L \|d_k\|^2} \geq \frac{(1-\sigma) a_k}{L} \quad (28)\]

using (22), we obtain:

\[\sum_{k=1}^{\infty} (f_k - f_{k+1}) = \lim_{N \to \infty} \sum_{k=1}^{N} (f_k - f_{k+1}) = \lim_{N \to \infty} (f_1 - f_{N+1}) = f_1 - f^* \quad (29)\]

Therefore,

\[\sum_{k=1}^{\infty} (f_k - f_{k+1}) \leq + \infty, \quad (30)\]

which combining with Wolfe condition (3) yields:

\[\sum_{k=1}^{\infty} \alpha_k g_k^T d_k \leq + \infty \quad (31)\]

Then:

\[\lim_{k \to \infty} \alpha_k g_k^T d_k = 0 \quad (32)\]

together with (31) provide that:

\[\lim_{k \to \infty} d_k^T B_k d_k = \lim_{k \to \infty} - g_k^T d_k = 0 \quad (33)\]

Combining (33) with (24) we obtain the conclusion (25). The proof is finished.

For proof a global convergence for non-convex problems, we state a lemma due to Powell [11].

**Lemma 1.**

“If BFGS method with Wolfe condition is applied to a continuously differentiable function \( f \) that is bounded below, and if there exists a constant \( M \) such that the inequality holds”:

\[\left\|y_k\right\| \leq M \quad (34)\]

then:

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\[
\lim \inf_{k \to \infty} \|g_k\| = 0
\]  
(35)

For more details can be found in [12].

**Theorem 3.**

Let \( \{x_k\} \) be generated by the new method. Then we have:

\[
\lim \inf_{k \to \infty} \|g_k\| = 0
\]  
(36)

**Proof:**

If \( K \) is a finite set, then \( B_k \) is a constant-matrix, obviously, (24) satisfies. Now, if \( K \) is a infinite we will deduce a contradiction with there exists \( \varepsilon > 0 \) such that:

\[
\|g_k\| > \varepsilon
\]  
(37)

It follows from (28) and (16) that:

\[
s_k^T y_k \geq \beta \varepsilon \|s_k\|
\]  
(38)

By the definition of \( y_k \), we have:

\[
\|y_k\| = \left\| y_k - \frac{g_k^T s_k}{s_k^T z_k} z_k \right\|
\leq \|y_k\| + \left\| \frac{g_k}{s_k} \|s_k\| \right\|
\]  
(39)

It follows from (20), (23), (28) and (39) that:

\[
\|y_k\| \leq L\|s_k\| + \|g_{k+1}\| \leq L\mu + \gamma
\]  
(40)

This, together with (38), lead to:

\[
\frac{s_k^T y_k}{\|s_k\|} \leq M
\]  
(41)

Using lemma 1, to the sub \( \{B_k\}_{k \in K} \), obviously, existence \( a_1 \) and \( a_2 \) we get (24) for infinitely many \( k \). By using theorem 2 the proof is Finished.

4. **Tests Numerically for Methods and Discussions**

In this section, some numerical tests are performed in order to illustrate the implementation and efficiency of the proposed method. In our tests, we consider the unconstrained optimization problems from the set provided by [13]. Other test functions have been used in various research such as [14-24]. In solved the problems required the number of iterations (NI) and the number of function evaluations (NF), respectively, which contain in tables. All runs reported in this paper terminate when:
“If \( |f(x_k)| > 10^{-5} \), let \( \text{stop } 1 = \left| f(x_k) - f(x_{k+1}) \right| / |f(x_k)|; \) Otherwise, let \( \text{stop } 1 = \left| f(x_k) - f(x_{k+1}) \right| . \)

For every problem, if \( \|g_k\| < \varepsilon \) or \( \text{stop } 1 < 10^{-5} \) is satisfied, the program will be stopped”. For more details can be found in [25]. Comparison the new methods with the standard BFGS method as shown in Table 1.

### Table 1. Comparison the new methods with the standard BFGS method

| Problems | BFGS algorithm | BFGS with \( z_k = y_k \) | BFGS with \( z_k = g_k \) |
|----------|----------------|--------------------------|--------------------------|
|          | N | NI | NF | N | NI | NF | N | NI | NF |
| Rose     | 2 | 35 | 140 | 33 | 127 | 24 | 82 |
| Froth    | 2 | 9 | 26 | 8 | 23 | 8 | 23 |
| Badscp   | 2 | 43 | 166 | 34 | 133 | 36 | 129 |
| Badscp   | 2 | 3 | 30 | 3 | 30 | 3 | 30 |
| Beale    | 2 | 15 | 50 | 14 | 47 | 12 | 37 |
| Jensam   | 2 | 2 | 27 | 2 | 27 | 2 | 27 |
| Helix    | 3 | 34 | 113 | 33 | 105 | 29 | 88 |
| Bard     | 3 | 16 | 54 | 18 | 58 | 16 | 50 |
| Gauss    | 3 | 2 | 4 | 2 | 4 | 2 | 4 |
| Gulf     | 3 | 2 | 27 | 2 | 27 | 2 | 27 |
| Box      | 3 | 2 | 27 | 2 | 27 | 2 | 27 |
| Sing     | 4 | 20 | 60 | 19 | 58 | 18 | 53 |
| Wood     | 4 | 19 | 61 | 20 | 62 | 13 | 41 |
| Kowosb   | 4 | 21 | 65 | 22 | 98 | 21 | 69 |
| Bd       | 4 | 17 | 54 | 16 | 49 | 15 | 48 |
| Osb1     | 5 | 2 | 27 | 2 | 27 | 2 | 27 |
| Biggs    | 6 | 25 | 72 | 25 | 77 | 7 | 43 |
| Osb2     | 11 | 3 | 31 | 3 | 31 | 3 | 31 |
| Watson   | 20 | 31 | 102 | 32 | 101 | 26 | 78 |
| Singx    | 400 | 64 | 209 | 51 | 170 | 16 | 49 |
| Pen1     | 400 | 2 | 27 | 2 | 27 | 53 | 164 |
| Pen2     | 200 | 2 | 5 | 2 | 5 | 2 | 5 |
| Vardm    | 100 | 2 | 27 | 2 | 27 | 2 | 27 |
| Tng      | 500 | 9 | 33 | 9 | 32 | 8 | 24 |
| Bv       | 500 | 2 | 4 | 2 | 4 | 2 | 4 |
| Ic       | 500 | 6 | 16 | 7 | 19 | 7 | 19 |
| Band     | 500 | 57 | 281 | 36 | 180 | 11 | 65 |
| Lin      | 500 | 2 | 4 | 2 | 4 | 2 | 4 |
| Lin1     | 500 | 3 | 7 | 3 | 7 | 3 | 7 |
| Lino     | 500 | 3 | 7 | 3 | 7 | 3 | 7 |

| Total    | 453 | 1756 | 409 | 1593 | 350 | 1289 |

Based on the above comparisons, it indicates our method has improved rate (9-26) % in total number of the iterations and (9-27) % in total number of the function evaluation by compare with the BFGS method. Relative efficiency of the between algorithms as shown in Table 2.

### Table 2. Relative efficiency of the between algorithms

|          | BFGS algorithm | BFGS with \( z_k = y_k \) | BFGS with \( z_k = g_k \) |
|----------|----------------|--------------------------|--------------------------|
| N1       | 100%           | 90.28%                   | 77.26%                   |
| NF       | 100%           | 90.71%                   | 73.40%                   |

5. CONCLUSIONS

Based on a quadratic function we deriving new quasi-Newton equation. The effectiveness of the proposed methods have been shown by some numerical examples. Exciting the remarkable performance of method with choice are \( z_k = y_k \) and \( z_k = g_k \) on these test problems, it is okay to guess on \( z_k = g_{k+1} \).

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_A new variants of quasi-Newton equation based on the quadratic function for... (Basim A. Hassan)_
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