Singularity dominated strong fluctuations for some random matrix averages

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The circular and Jacobi ensembles of random matrices have their eigenvalue support on the unit circle of the complex plane and the interval (0, 1) of the real line respectively. The averaged value of the modulus of the corresponding characteristic polynomial raised to the power $2\mu$ diverges, for $2\mu \leq -1$, at points approaching the eigenvalue support. Using the theory of generalized hypergeometric functions based on Jack polynomials, the functional form of the leading asymptotic behaviour is established rigorously. In the circular ensemble case this confirms a conjecture of Berry and Keating.

1 Introduction

Random matrices from the classical groups — $N \times N$ unitary matrices $U(N)$, $N \times N$ real orthogonal matrices $O(N)$, and $N \times N$ unitary matrices with real quaternion elements embedded as $2N \times 2N$ complex unitary matrices $Sp(N)$ — play a special role in the application of random matrix theory to number theory (see e.g. the recent review [21]). Of particular interest in such applications are the random matrix averages of the modulus of the characteristic polynomial raised to some power $2\mu$ say. Thus in [20] it was shown how knowledge of

$$\langle |\det(zI - U)|^{2\mu} \rangle_{U \in U(N)}$$

for $|z| = 1$ (in this case (1.1) is in fact independent of $z$) allows the mean value of the $2\mu$-th power of the modulus of the Riemann zeta function on the critical line to be predicted. Knowledge of the analogue of (1.1) for the classical groups $O(N)$ and $Sp(N)$ allows for similar predictions in the case of families of $L$-functions [5, 19, 6].

Our interest is in the asymptotic behaviour of (1.1) and its analogues as $|z|$ approaches unity. Consider in particular the generalization of (1.1)

$$\langle \prod_{l=1}^{N} |z - e^{i\theta_l}|^{2\mu} \rangle_{\text{C}^\beta\text{E}_N},$$

where $C^\beta\text{E}_N$ (circular $\beta$-ensemble) refers to the eigenvalue probability density function proportional to

$$\prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta, \quad -\pi \leq \theta_1 < \pi.$$  

The cases $\beta = 1$ and $\beta = 4$ are well known in random matrix theory as the COE (circular orthogonal ensemble) and CSE (circular symplectic ensemble) respectively, while the case $\beta = 2$ is just (1.1). For $2\mu \leq -1$ it has been argued by Berry and Keating [3] that (1.2) diverges as $|1 - |z||^{-\delta}$ where, provided $N \gg |\mu|$, $\delta$ as a function of $\mu$ is the non-smooth function

$$\delta = \text{int}\left[\frac{k-1}{\beta}\right] + 1\left(k - 1 + \frac{\beta}{2} - \frac{\beta}{2}\text{int}\left[\frac{k-1}{\beta}\right] + 1\right), \quad k = 2|\mu|,$$

with $\text{int}[\cdot]$ denoting the integer part. It was also suggested that for $k$ values such that $(k-1)/\beta$ is an integer, there is a logarithmic correction to the behaviour of (1.2) which then diverges as $|1 - |z||^{-\delta}\log |1 - |z||$. 

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The intricate behaviour exhibited by (1.4) is due to a phenomenon termed singularity dominated strong fluctuations. Thus for $2\mu \leq -1$ and $|z|$ close to unity, degeneracies of the spectrum of varying order are being probed. A similar mechanism has been identified in the analysis of twinkling starlight, van Hove-type singularities, and the influence of classical periodic orbit bifurcations on quantum energy level and wavefunction statistics.

The arguments presented in [3] leading to (1.4) were heuristic. Subsequently Fyodorov and Keating [11] gave a rigorous derivation for the exponent (1.4) in the case $\beta = 1, 2\mu \in \mathbb{Z}_{<0}$, for the Gaussian orthogonal ensemble analogue of (1.2),

$$\langle |\det(xI - X)|^{2\mu} \rangle_{X \in \text{GOE}_N}. \quad (1.5)$$

Thus they proved that for $N \gg \mu$ and $k = 2|\mu| \in \mathbb{Z}_{>0}$

$$\langle |\det(xI - X)|^{2\mu} \rangle_{X \in \text{GOE}_N} \sim \epsilon^{-k(k-1)/2} \log 1/\epsilon, \quad (1.6)$$

(the explicit evaluation of $C_{N,k}$ was also given). In this paper we will give a rigorous derivation of the exponent (1.4), together with the logarithmic corrections when present, for the average (1.2) in the case of general $\beta > 0$ and general $2\mu \leq -1$. Our analysis relies on identifying (1.2) as a special generalized hypergeometric function based on Jack polynomials [25, 17]. We are then able to use known asymptotic properties of the latter to deduce the sought asymptotic behaviour of (1.2).

Also studied will be the $x \to 1^+$ asymptotic behaviour of

$$\langle \prod_{l=1}^{N} |x - x_l|^{2\mu} \rangle_{J^\beta E_N} \quad (1.7)$$

where $J^\beta E_N$ (Jacobi $\beta$-ensemble) refers to the eigenvalue probability density function proportional to

$$\prod_{l=1}^{N} x_l^a (1 - x_l)^b \prod_{1 \leq j < k \leq N} |x_k - x_j|^{\beta}, \quad 0 < x_l < 1. \quad (1.8)$$

In the special case $\beta = 2$, (1.7) includes the analogue of (1.1) in the cases of the classical groups $O(2N+1)$, $O(2N)$ and $Sp(N)$. Again our analysis proceeds via identifying (1.2) as a special generalized hypergeometric function based on Jack polynomials.

As a final issue, it is pointed out in [12] that for fixed $|z| \neq 1$ and $N \to \infty$, Szegö’s theorem [24] on the asymptotic behaviour of Toeplitz determinants with smooth generating functions implies (1.1) is a simple Gaussian in $\mu$,

$$\lim_{N \to \infty} \langle |\det(zI - U)|^{2\mu} \rangle_{U \in U(N)} = e^{-\mu^2 \log |1 - |z|^2|}, \quad |z| < 1. \quad (1.9)$$

This gives for the exponent characterizing the divergence as $|z| \to 1^-$, $\delta = \mu^2$ and so does not reproduce (1.4). Therefore the limit $N \to \infty$ with $|z| \neq 1$ fixed has the effect of smoothing out the singularity dominated strong fluctuations. Note that in this limit the point $z$ is a macroscopic distance away from the eigenvalue support, as measured in units of the inter-eigenvalue spacing. A theorem of Johansson [13, 14] will be used to establish the analogue of (1.9) for the average (1.2).
2 Generalized hypergeometric functions based on Jack polynomials

2.1 Definitions

The Gauss hypergeometric function $2F_1$ is, for $|x| < 1$, defined by the series

$$2F_1(a, b; c; x) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k. \quad (2.1)$$

A multivariable generalization of $(2.1)$ can be defined in which the sum over the non-negative integers $k$ is replaced by a sum over partitions $\kappa := (\kappa_1, \ldots, \kappa_N)$, $\kappa_i \geq \kappa_j$ ($i < j$) and $\kappa_i \in \mathbb{Z}_{\geq 0}$; the function

$$(u)_k := u(u+1) \cdots (u+k-1) = \frac{\Gamma(u+k)}{\Gamma(u)}$$

is replaced by

$$[u]^{(\alpha)}_\kappa := \prod_{j=1}^{N} \frac{\Gamma(u-(j-1)/\alpha + \kappa_j)}{\Gamma(u-(j-1)/\alpha)};$$

$k!$ is replaced by $\kappa! := (\sum_{j=1}^{N} \kappa_j)!$; and $x^k$ is replaced by the homogeneous symmetric polynomial $C_\kappa^{(\alpha)}(x_1, \ldots, x_N)$ which in turn is proportional to the Jack polynomial $P_\kappa^{(\alpha)}(x_1, \ldots, x_N)$. Thus for $|x_1| < 1, \ldots, |x_N| < 1$ one defines $[25, 17]$

$$2F_1^{(\alpha)}(a, b; c; x_1, \ldots, x_N) := \sum_{\kappa} \frac{[a]^{(\alpha)}_\kappa [b]^{(\alpha)}_\kappa}{[c]^{(\alpha)}_\kappa \kappa!} C_\kappa^{(\alpha)}(x_1, \ldots, x_N). \quad (2.2)$$

Let us say a little on the definition of the Jack polynomials $P_\kappa^{(\alpha)}(x_1, \ldots, x_N) := P_\kappa^{(\alpha)}(x)$ and their renormalized version $C_\kappa^{(\alpha)}(x_1, \ldots, x_N) := C_\kappa^{(\alpha)}(x)$. The former are the unique homogeneous polynomials of degree $|\kappa|$ with the structure

$$P_\kappa^{(\alpha)}(z) = m_\kappa + \sum_{\mu < \kappa} a_{\kappa\mu} m_\mu \quad (2.3)$$

(the $a_{\kappa\mu}$ are some coefficients in $\mathbb{Q}(\alpha)$) and which satisfy the orthogonality

$$\langle P_\kappa^{(\alpha)}, P_\rho^{(\alpha)} \rangle^{(\alpha)} \propto \delta_{\kappa,\rho}$$

where

$$\langle f, g \rangle^{(\alpha)} := \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_N \int_{-1/2}^{1/2} d\theta_{1N} f(z_1, \ldots, z_N) g(z_1, \ldots, z_N) \prod_{1 \leq j < k \leq N} |z_k - z_j|^{2/\alpha}, \quad z_j := e^{2\pi i \theta_j}. $$

In $(2.3)$ $m_\kappa$ denotes the monomial symmetric function (polynomial) in $x_1, \ldots, x_N$ corresponding to $\kappa$, and $\mu < \kappa$ means $\sum_{j=1}^{p} \kappa_j \geq \sum_{j=1}^{p} \mu_j$ for each $p = 1, \ldots, N$. Regarding the definition of $C_\kappa^{(\alpha)}(x_1, \ldots, x_N)$, let

$$d'_\kappa = \prod_{(i,j) \in \kappa} \left( a(i,j) + 1 + l(i,j) \right), \quad (2.4)$$

where the notation $(i,j) \in \kappa$ refers to the diagram of $\kappa$, in which each part $\kappa_i$ becomes the nodes $(i,j)$, $1 \leq j \leq \kappa_i$ on a square lattice labelled as is conventional for a matrix. The quantity $a(i,j)$ is the so called arm length (the number of nodes in row $i$ to the right of column $j$), while $l(i,j)$ is the leg length (number of nodes in column $j$ below row $i$). In terms of $d'_\kappa$ we have

$$C_\kappa^{(\alpha)}(z) := \frac{\alpha^{|\kappa|}|\kappa|!}{d'_\kappa} P_\kappa^{(\alpha)}(z). \quad (2.5)$$
2.2 Integration formulas

Introduce the Selberg integral

\[ S_N(\lambda_1, \lambda_2, \lambda) := \int_0^1 dt_1 \cdots \int_0^1 dt_N \prod_{l=1}^N t_1^{\lambda_1} (1 - t_l)^{\lambda_2} \prod_{1 \leq j < k \leq N} |t_k - t_j|^{2\lambda} \]

and the Morris integral

\[ M_N(a, b, \lambda) := \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_N \prod_{l=1}^N e^{\pi i \theta_l(a-b)} |1 + e^{2\pi i \theta_l} a + b| \prod_{1 \leq j < k \leq N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^{2\lambda} \]

(both these integrals have gamma function evaluations — see e.g. [7] — although these will not be needed here). Fundamental to our ability to express (2.6) and (2.7) in terms of the generalized hypergeometric function \( \text{2F1} \) are the generalized Selberg integral evaluation [22] [16] [17]

\[
\frac{1}{S_N(\lambda_1, \lambda_2, 1/\alpha)} \int_0^1 dt_1 \cdots \int_0^1 dt_N \prod_{l=1}^N t_1^{\lambda_1} (1 - t_l)^{\lambda_2} P^{(\alpha)}_\kappa(t_1, \ldots, t_N) \prod_{1 \leq j < k \leq N} |t_k - t_j|^{2/\alpha}
= P^{(\alpha)}_\kappa(1^N) \frac{[\lambda_1 + (N-1)/\alpha + 1]_\kappa}{[\lambda_1 + \lambda_2 + 2(N-1)/\alpha + 2]_\kappa} \tag{2.6}
\]

and the generalized Morris integral evaluation [9]

\[
\frac{1}{M_N(a, b, 1/\alpha)} \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_N \prod_{l=1}^N e^{\pi i \theta_l(a-b)} |1 + e^{2\pi i \theta_l} a + b| P^{(\alpha)}_\kappa(-e^{2\pi i \theta_1}, \ldots, -e^{2\pi i \theta_N}) \times \prod_{1 \leq j < k \leq N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^{2/\alpha} = P^{(\alpha)}_\kappa(1^N) \frac{[-b]_\kappa(1^N)}{[1 + a + (N-1)/\alpha]_\kappa} \tag{2.7}
\]

(2.6 and 2.7 are in fact equivalent [10]), where

\[
P^{(\alpha)}_\kappa(1^N) := P^{(\alpha)}_\kappa(x_1, \ldots, x_N) \bigg|_{x_1 = \ldots = x_N = 1}.
\]

Equally important is the generalized binomial summation formula [25] [28]

\[
\text{1F0}^{(\alpha)}(a; x_1, \ldots, x_N) := \sum_{\kappa} \frac{[a]_\kappa^{(\alpha)}}{[\kappa]!} C^{(\alpha)}_\kappa(x_1, \ldots, x_N) = \prod_{j=1}^N (1 - x_j)^{-a} \tag{2.8}
\]

Thus we see by multiplying both sides of (2.6) and (2.7) by

\[
\left( x^\alpha \right)^{[\kappa]!} \frac{1}{d^{(\alpha)}_\kappa}
\]

and summing over \( \kappa \), using (2.5) on the left hand sides and using the definition (2.2) on the right hand sides together with the fact that \( C^{(\alpha)}_\kappa \) is homogeneous of degree \( |\kappa| \), that [5]

\[
\frac{1}{S_N(\lambda_1, \lambda_2, 1/\alpha)} \int_0^1 dt_1 \cdots \int_0^1 dt_N \prod_{l=1}^N t_1^{\lambda_1} (1 - t_l)^{\lambda_2} (1 - xt_l)^{-r} \prod_{1 \leq j < k \leq N} |t_k - t_j|^{2/\alpha}
= 2F_1^{(\alpha)}(r, 1/\alpha; (N-1) + \lambda_1 + 1; \frac{2}{\alpha}(N-1) + \lambda_1 + \lambda_2 + 2; t_1, \ldots, t_N) \bigg|_{t_1 = \ldots = t_N = x} \tag{2.9}
\]

and

\[
\frac{1}{M_N(a, b, 1/\alpha)} \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_N \prod_{l=1}^N e^{\pi i \theta_l(a-b)} |1 + e^{2\pi i \theta_l} a + b| (1 + xe^{2\pi i \theta_l})^{-r} \times \prod_{1 \leq j < k \leq N} |e^{2\pi i \theta_k} - e^{2\pi i \theta_j}|^{2/\alpha} = 2F_1^{(\alpha)}(r, -b/\alpha; (N-1) + a + 1; \frac{2}{\alpha}(N-1) + a + 1; t_1, \ldots, t_N) \bigg|_{t_1 = \ldots = t_N = x} \tag{2.10}
\]
3 Random matrix averages as generalized hypergeometric functions

Here we will express (1.2) and (1.7) in terms of \( \mathfrak{F}_1^{(n)} \). Consider first (1.2). By definition and simple manipulation

\[
\left\langle \prod_{l=1}^{N} |z - e^{i\theta_l}|^{2\mu} \right\rangle_{C\beta E_N} = \frac{1}{M_N(0, 0, \beta/2)} \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_N \prod_{l=1}^{N} |1 + ze^{2\pi i\theta_l}|^{2\mu} \prod_{1 \leq j < k \leq N} |e^{2\pi i\theta_k} - e^{2\pi i\theta_j}|^{\beta}.
\]

Now

\[|1 + ze^{2\pi i\theta_l}|^{2\mu} = (1 + ze^{2\pi i\theta_l})^\mu (1 + \bar{z}e^{-2\pi i\theta_l})^\mu.\]

Regarding the integrals over \( \theta_l \in [-1/2, 1/2] \) as parametrizing the unit circles \( w_l = e^{2\pi i\theta_l} \) in the complex plane, then deforming each \( w_l \) by \( w_l \to \bar{z}e^{2\pi i\theta_l} \) (this is immediate for \( 2\mu, \beta/2 \in \mathbb{Z}_{\geq 0} \) by Cauchy’s theorem; it remains valid for general complex \( \mu \) and \( \beta \) by Carlson’s theorem) we see that we can write

\[
\left\langle \prod_{l=1}^{N} |z - e^{i\theta_l}|^{2\mu} \right\rangle_{C\beta E_N} = \frac{1}{M_N(0, 0, \beta/2)} \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_N \prod_{l=1}^{N} |1 + |z|^2 e^{2\pi i\theta_l}|^{\mu} (1 + e^{-2\pi i\theta_l})^\mu
\times \prod_{1 \leq j < k \leq N} |e^{2\pi i\theta_k} - e^{2\pi i\theta_j}|^{\beta}
\]

\[
= \frac{1}{M_N(0, 0, \beta/2)} \int_{-1/2}^{1/2} d\theta_1 \cdots \int_{-1/2}^{1/2} d\theta_N \prod_{l=1}^{N} |1 + |z|^2 e^{2\pi i\theta_l}|^{\mu} e^{-\pi i\theta_l} |1 + e^{-2\pi i\theta_l}|^\mu
\times \prod_{1 \leq j < k \leq N} |e^{2\pi i\theta_k} - e^{2\pi i\theta_j}|^{\beta}
\]

\[= 2F_1^{(2/\beta)}(-\mu, -\mu; \frac{\beta}{2} (N - 1) + 1; t_1, \ldots, t_N) \bigg|_{t_1 = \cdots = t_N = |z|^2} \tag{3.1}\]

where the third equality follows from (2.10), and is valid for \( |z|^2 \leq 1 \). For \( |z|^2 \geq 1 \) one notes the simple symmetry

\[
\left\langle \prod_{l=1}^{N} |z - e^{i\theta_l}|^{2\mu} \right\rangle_{C\beta E_N} = |z|^{2\mu N} \left\langle \prod_{l=1}^{N} \frac{1}{z} - e^{i\theta_l}|^{2\mu} \right\rangle_{C\beta E_N},
\]

thus relating this case back to the case \( |z|^2 \leq 1 \). We remark that in the case \( \beta = 2 \) the same generalized hypergeometric function evaluation (3.1) has recently been given in [10].

Consider next (1.7). By definition and simple manipulation, for \( x \) real and greater than or equal to unity

\[
\left\langle \prod_{l=1}^{N} |x - x_l|^{2\mu} \right\rangle_{J\beta E_N} = \frac{x^{2\mu N}}{S_N(a, b, \beta/2)} \int_0^1 dt_1 \cdots \int_0^1 dt_N \prod_{l=1}^{N} t_l^a (1 - t_l)^b (1 - t_l/x)^{2\mu} \prod_{1 \leq j < k \leq N} |t_k - t_j|^{\beta}
\]

\[= x^{2\mu N} 2F_1^{(2/\beta)}(-2\mu, \frac{\beta}{2} (N - 1) + a; \beta(N - 1) + a + b + 2; t_1, \ldots, t_N) \bigg|_{t_1 = \cdots = t_N = 1/x} \tag{3.2}\]
where the second equality follows from (2.9). The task now is to analyze the generalized hypergeometric functions in (3.1) and (3.2) as $|z| \to 1$ and $x \to 1^+$ respectively. Fortunately, as will be presented in the next section, the required asymptotic formulas are available in the literature.

4 Asymptotic forms

Using the series definition (2.2), Yan [25] has analyzed the behavior of the function $\binom{\alpha}{\nu}$ as $x_1, \ldots, x_N \to 1^-$. Thus according to Proposition 4.4 of [25] the following result holds, where the notation $A(x) \approx B(x)$ means that there exists two positive numbers $C_1$ and $C_2$ independent of $x$ such that

$$C_1 \leq \frac{A(x)}{B(x)} \leq C_2.$$  

**Proposition 1.** Let

$$\gamma := a + b - c,$$

and suppose for all $k$

$$\frac{|a|^{(k)}|b|^{(k)}}{|c|^{(k)}} > 0.$$  

We have for $-1 < x_i < 1$ ($i = 1, \ldots, N$)

(i) If $\gamma > (N - 1)\beta/2$, then

$$2F_1(\alpha; a, b; c; x_1, \ldots, x_N) \approx \prod_{i=1}^{N}(1 - x_i)^{-\gamma}. \quad (4.3)$$

(ii) If $\gamma < -(N - 1)\beta/2$, then there exists a constant $C$ such that

$$2F_1(\alpha; a, b; c; x_1, \ldots, x_N) \leq C. \quad (4.4)$$

(iii) If $\gamma = \beta(-\frac{N-1}{2} + j - 1)$, $j = 1, \ldots, N$, then for $x_1 = \cdots = x_N = t$

$$2F_1(\alpha; a, b; c; x_1, \ldots, x_N) \Bigg|_{x_1=\cdots=x_N=t} \approx (1 - t)^{-j(\beta/2)} \log \frac{1}{1-t}. \quad (4.5)$$

(iv) If $\beta(-\frac{N-1}{2} + j - 1) < \gamma < \beta(-\frac{N-1}{2} + j)$, $j = 1, \ldots, N - 1$, then for $x_1 = \cdots = x_N = t$

$$2F_1(\alpha; a, b; c; x_1, \ldots, x_N) \Bigg|_{x_1=\cdots=x_N=t} \approx (1 - t)^{-j(\gamma + (N-j)\beta/2)}. \quad (4.6)$$

Application of Proposition 1 to the $2F_1(\alpha)$ evaluations (3.1) and (3.2) gives the desired functional form of the singular behavior of (1.2) and (1.7) in the case of a negative exponent $2\mu$.

**Corollary 1.** Consider the circular ensemble average (1.2). Suppose $\mu < 0$ and $|z| < 1$. For

$$2|\mu| = \beta(j - 1) + 1 \quad (j = 1, \ldots, N)$$

we have

$$\left\langle \prod_{i=1}^{N}|z - e^{i\theta_i}|^{2\mu} \right\rangle_{C/\beta E_N} \approx (1 - |z|)^{-(j-1)\beta/2} \log \frac{1}{1 - |z|} \quad (4.7)$$

while for

$$\beta(j - 1) + 1 < 2|\mu| < \beta j + 1 \quad (j = 1, \ldots, N)$$




and thus
\[ j = \text{int}[(2|\mu| - 1)/\beta + 1], \quad (2|\mu| - 1)/\beta + 1 \notin \mathbb{Z} \]
we have
\[ \langle \prod_{l=1}^{N} [z - e^{i\theta_l}]^{2\mu} \rangle_{C_{\beta E_N}} \approx (1 - |z|)^{-j(2|\mu| - 1 - (j - 1)\beta/2)}. \tag{4.8} \]

Consider the Jacobi ensemble average \( (1.7) \). Suppose \( \mu < 0 \) and \( x > 1 \). For \( 2|\mu| = \beta(j - 1) + 2 + b \)
we have
\[ x^{-2\mu N} \langle \prod_{l=1}^{N} [x - x_l]^{2\mu} \rangle_{J_{\beta E_N}} \approx (1 - 1/x)^{-(j-1)\beta/2} \log 1 - \frac{1}{x} \tag{4.9} \]
while for
\[ \beta(j - 1) + b + 2 < 2|\mu| < \beta j + b + 2 \quad (j = 1, \ldots, N) \]
and thus
\[ j = \text{int}[(2|\mu| - b - 2)/\beta + 1], \quad (2|\mu| - b - 2)/\beta + 1 \notin \mathbb{Z} \]
we have
\[ x^{-2\mu N} \langle \prod_{l=1}^{N} |x - x_l|^{2\mu} \rangle_{J_{\beta E_N}} \approx (1 - 1/x)^{-j(2|\mu| - b - 2 - (j - 1)\beta/2)}. \tag{4.10} \]

The asymptotic behaviour \( (4.8) \) indeed exhibits the exponent \( (1.4) \) predicted by Berry and Keating \( [3] \), and the existence of a logarithmic correction in the cases that \( (2|\mu| - 1)/\beta \) is an integer is confirmed.

Finally, let us apply the results \( (4.9) \) and \( (4.10) \) to deduce the asymptotic behaviours of
\[ \langle |\det(zI - U)|^{2\mu} \rangle_{U \in G}, \quad \mu < 0 \]
for \( G \) the classical groups \( Sp(N), O^+ (2N) \) and \( O^- (2N) \) (we leave the cases of \( O^+ (2N-1) \) and \( O^- (2N-1) \) as an exercise for the interested reader). Consider first \( Sp(N) \). The eigenvalues come in complex conjugate pairs \( e^{\pm i\theta_j}, (j = 1, \ldots, N) \) so we have
\[ \det(zI - U) = (2z)^N \prod_{j=1}^{N} \left( x^2 + \frac{1}{2z} - \cos \theta_j \right). \tag{4.11} \]
Furthermore, we know (see e.g. \( [7] \) ) that in the variable \( \cos \theta_j := x_j \) the eigenvalue probability density function is of the form \( (1.8) \) with \( a = b = 1/2, \beta = 2 \). Thus setting \( z = 1 + \epsilon, 0 < \epsilon \ll 1 \) and noting \( (z^2 + 1)/2z \sim 1 + \epsilon^2/2 \) it follows from \( (4.9) \) that for
\[ 2|\mu| = 2j + 2 \quad (j = 1, \ldots, N) \]
we have
\[ \langle |\det((1 + \epsilon)I - U)|^{2\mu} \rangle_{U \in Sp(N)} \approx \epsilon^{-2(j-1)\beta/2} \log 1 - \frac{1}{\epsilon} \tag{4.12} \]
while for
\[ 2j + 1/2 < 2|\mu| < 2(j + 1) + 1/2 \]
we have
\[ \langle |\det((1 + \epsilon)I - U)|^{2\mu} \rangle_{U \in Sp(N)} \approx \epsilon^{-2j(2|\mu| - 3/2 - j)}. \tag{4.13} \]
For the classical group $O^+(2N)$ the eigenvalues again come in complex conjugate pairs $e^{\pm i\theta_j}$ ($j = 1, \ldots, N$), so (4.11) remains valid. In the variable $\cos \theta_j =: x_j$ the eigenvalue probability density function is proportional to $|x_j^1 - \beta = 2$ (see e.g. [7]). Thus setting $z = 1 + \epsilon$, $0 < \epsilon \ll 1$ if follows from (4.13) that for
\begin{equation}
2|\mu| = 2j - 1/2 \quad (j = 1, \ldots, N)
\end{equation}
we have
\begin{equation}
\left\langle \left| \det((1 + \epsilon)I - U)\right|^{2\mu} \right\rangle_{U \in O^+(2N)} \approx \epsilon^{-2(j - 1)j} \log \frac{1}{\epsilon}
\end{equation}
while for
\begin{equation}
2j - 1/2 < 2|\mu| < 2(j + 1) - 1/2
\end{equation}
we have
\begin{equation}
\left\langle \left| \det((1 + \epsilon)I - U)\right|^{2\mu} \right\rangle_{U \in O^+(2N)} \approx \epsilon^{-2j(2|\mu| - 1/2 - j)}.
\end{equation}

In the case of the classical group $O^-(2N)$, there is a pair of fixed eigenvalues at $\pm 1$, with the remaining eigenvalues coming in complex conjugate pairs $e^{\pm i\theta_j}$ ($j = 1, \ldots, N - 1$). Hence in this case
\begin{equation}
\det(zI - U) = (z^2 - 1)(2z)^{N-1} \prod_{j=1}^{N-1} \left( \frac{z^2 + 1}{2z} - \cos \theta_j \right).
\end{equation}
In the variable $\cos \theta_j =: x_j$ the eigenvalue probability density function is proportional to $|x_j|^{2\mu}$ with $N \rightarrow N - 1$ and $a = 1/2$, $b = -1/2$, $\beta = 2$. As only the value of $b$ enters in (4.13) and (4.14), the sole modification of (4.14) and (4.15) is multiplication by $\epsilon$ to account for the factor of $z^2 - 1$ in (4.16). Thus for
\begin{equation}
2|\mu| = 2j - 1/2 \quad (j = 1, \ldots, N - 1)
\end{equation}
we have
\begin{equation}
\left\langle \left| \det((1 + \epsilon)I - U)\right|^{2\mu} \right\rangle_{U \in O^-(2N)} \approx \epsilon^{-2j(j - 1) + 1} \log \frac{1}{\epsilon}
\end{equation}
while for
\begin{equation}
2j - 1/2 < 2|\mu| < 2(j + 1) - 1/2 \quad (j = 1, \ldots, N - 1)
\end{equation}
we have
\begin{equation}
\left\langle \left| \det((1 + \epsilon)I - U)\right|^{2\mu} \right\rangle_{U \in O^-(2N)} \approx \epsilon^{-2j(2|\mu| - 1/2 - j) + 1}.
\end{equation}

5 The macroscopic limit

Consider the general Toeplitz determinant
\begin{equation}
D_N[e^{a(\theta)}] := \det \left[ \frac{1}{2\pi} \int_0^{2\pi} e^{a(\theta)} e^{i(j-k)\theta} d\theta \right]_{j,k=1,\ldots,N}.
\end{equation}
Such determinants are related to averages over $U(N)$ by the simple to establish formula
\begin{equation}
D_N[e^{a(\theta)}] = \left\langle \prod_{l=1}^{N} e^{a(\theta_l)} \right\rangle_{U(N)}.
\end{equation}
With $a(\theta) = \sum_{p=-\infty}^{\infty} a_p e^{ip\theta}$ and the Fourier coefficients falling off fast enough that
\begin{equation}
\sum_{p=-\infty}^{\infty} |p||a_p|^2 < \infty,
\end{equation}
an asymptotic formula of Szegö [24] gives

\[ D_N[e^{a(\theta)}] \sim_{N \to \infty} \exp \left( N a_0 + \sum_{p=1}^{\infty} p a_p a_{-p} + o(1) \right). \]  

(5.4)

Now, in the case

\[ a(\theta) = 2\mu \log |z - e^{i\theta}|, \quad |z| < 1, \]

(5.5)

the Fourier coefficients have the explicit form

\[ a_p = -\frac{\mu z^p}{p} \quad (p > 0), \quad a_p = \frac{\mu z^{-p}}{p} \quad (p < 0), \quad a_0 = 0. \]  

(5.6)

Since we are assuming \(|z| < 1\) these coefficients satisfy (5.3). Substituting (5.6) in (5.4) gives (1.9).

In this section, as our final issue, we will make use of a generalization of (5.4) due to Johansson to establish the generalization of (1.9) when the average over \(U(N)\) is replaced by an average over \(C\beta E_N\). But before doing so, following [12], we make note of the interpretation of (1.9) as specifying the distribution of the linear statistic

\[ A(z) = N \sum_{j=1}^{N} \log |z - e^{i\theta_j}|^2, \quad |z| < 1. \]

(5.7)

Thus let \(P_z(t)\) denote the probability density that \(A(z)\) takes on the value \(t\) after averaging over the eigenvalue distribution of \(U(N)\). Then we have

\[ P_z(t) = \left\langle \delta(t - A(z)) \right\rangle_{U(N)} \]

(5.8)

and consequently

\[ \tilde{P}_z(k) := \int_{-\infty}^{\infty} P_z(t)e^{itk} dt = \left\langle \prod_{l=1}^{N} e^{2ik\log|z-e^{i\theta_l}|} \right\rangle_{U(N)}. \]  

(5.9)

Thus \(\tilde{P}_z(k)\) is precisely \(5.2\) with the substitution \(5.4\), and \(\mu\) therein set equal to \(ik\). Consequently

\[ \lim_{N \to \infty} P_z(t) = e^{-k^2\sigma^2/2}, \quad \sigma^2 = -\log |1 - |z|^2|^2, \]  

(5.10)

telling us that the limiting distribution of (5.7) is a Gaussian with an \(O(1)\) variance taking the explicit value \(-\log |1 - |z|^2|^2\).

To study \(5.8\) with the average over \(U(N)\) replaced by the probability density function \(C\beta E_N\), we make note of the following generalization of the Szegö theorem \(5.4\) due to Johansson [13, 15].

**Proposition 2.** Suppose \(a(\theta) = \sum_{p=-\infty}^{\infty} a_p e^{ip\theta}\) and the Fourier coefficients fall off fast enough that the inequality \(5.3\) holds. We have

\[ \left\langle \prod_{l=1}^{N} e^{a(\theta_l)} \right\rangle_{C\beta E_N} \sim_{N \to \infty} \exp \left( N a_0 + \frac{2}{\beta} \sum_{p=1}^{\infty} p a_p a_{-p} + o(1) \right). \]

(5.11)

As an immediate corollary, by substituting \(5.6\) for the Fourier coefficients the sought generalization of \(1.9\) can be deduced.

**Corollary 2.** For \(|z| < 1\) we have

\[ \lim_{N \to \infty} \left\langle \prod_{l=1}^{N} |z - e^{i\theta_l}|^{2\mu} \right\rangle_{C\beta E_N} = e^{-(2\mu^2/\beta) \log |1 - |z|^2|}. \]  

(5.12)
Of course the result \((5.12)\) affords an interpretation as a fluctuation formula as in \((5.10)\), the only difference being that \(\sigma^2 = -(2/\beta) \log |1 - |z||^2\).

An obvious question is to seek the analogue of \((5.12)\) for the average \((1.7)\) with \(x \notin [0, 1]\). Johansson \[14\] has derived the analogue of \((5.11)\), but only in the case \(\beta = 2\) and further with the restriction that \(a(\theta)\) is a polynomial in \(\theta\), which is not the case for \((5.5)\).

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