Hessian of the Ricci Calabi functional

SATOSHI NAKAMURA

Abstract

The Ricci Calabi functional is a functional on the space of Kähler metrics of Fano manifolds. Its critical points are called generalized Kähler Einstein metrics. In this article, we show that the Hessian of the Ricci Calabi functional is non-negative at generalized Kähler Einstein metrics. As its application, we give another proof of a Matsushima’s type decomposition theorem for holomorphic vector fields, which was originally proved by Mabuchi. We also discuss a relation to the inverse Monge-Ampère flow developed recently by Collins-Hisamoto-Takahashi.

1 Introduction

In his paper [11], Mabuchi extended the notion of Kähler Einstein metrics for Fano manifolds with non-vanishing Futaki invariant. In this paper, we call them generalized Kähler Einstein metrics. Let $X$ be an $n$-dimensional Fano manifold and $\omega \in 2\pi c_1(X)$ be a reference Kähler metric. We denote its volume $\int_X \omega^n$ by $V$. Let

$$\mathcal{M}(\omega) := \{ \phi \in C^\infty(X) \mid \omega_\phi := \omega + \sqrt{-1} \partial \bar{\partial} \phi > 0 \}$$

be the space of Kähler metrics in $[\omega] = 2\pi c_1(X)$. We usually identify the Kähler metric $\omega_\phi$ with its potential $\phi \in \mathcal{M}(\omega)$. Note that the tangent space $T_\phi \mathcal{M}(\omega)$ is nothing but

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We denote the Ricci form for $\phi$ by $\text{Ric}(\omega_\phi) = -\sqrt{-1} \partial \bar{\partial} \log \det \omega_\phi^n$. The Ricci potential $f_\phi$ for $\phi$ is a function satisfying
\begin{equation}
\text{Ric}(\omega_\phi) - \omega_\phi = \sqrt{-1} \partial \bar{\partial} f_\phi \quad \text{and} \quad \int_X e^{f_\phi} \omega_\phi^n = V.
\end{equation}
Then $\omega_\phi = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is called \textit{generalized Kähler Einstein} if the complex gradient vector field of $1 - e^{f_\phi}$ is holomorphic, that is,
\[\bar{\partial} \left( g^{i\bar{j}} \frac{\partial (1 - e^{f_\phi})}{\partial \bar{z}^j} \frac{\partial}{\partial z^i} \right) = 0.\]
If $X$ has no nontrivial holomorphic vector field, generalized Kähler Einstein metrics are nothing but Kähler Einstein metrics. In general, if the Futaki invariant of $X$ vanishes, these are Kähler Einstein metrics.

Yao \cite{Yao} gave a characterization of generalized Kähler Einstein metrics in terms of the \textit{Ricci Calabi functional} (Yao calls it the Ding energy). The Ricci Calabi functional $\mathcal{E}_{RC}$ is a functional on $\mathcal{M}(\omega)$ defined by
\[\mathcal{E}_{RC}(\phi) = \int_X (1 - e^{f_\phi})^2 \omega_\phi^n.\]
Yao observed that $\phi \in \mathcal{M}(\omega)$ is a critical point of $\mathcal{E}_{RC}$ if and only if it is a generalized Kähler Einstein metric (Yao calls it the Mabuchi metric). See section 2. Following Donaldson’s new GIT (Geometric Invariant Theory) picture \cite{Donaldson} for Fano manifolds, this functional $\mathcal{E}_{RC}$ can be seen as the norm squared of a moment map on the corresponding moduli space. This shows that generalized Kähler Einstein metrics can be seen as one of the canonical Kähler metrics for Fano manifolds. Note that they are not in general neither extremal Kähler metrics nor Kähler Ricci solitons.

The main result of this paper is a development of the observation by Yao \cite{Yao}. In fact, we show that generalized Kähler Einstein metrics are local minima of the Ricci Calabi functional. To state results more precisely, we fix more notations. We denote the (negative) Laplacian of $\omega_\phi \in \mathcal{M}(\omega)$ by $\Delta_\phi$. Let us define
\[Lu = \left( -\Delta_\phi u - \langle \partial u, \partial f_\phi \rangle - u + \frac{1}{V} \int_X u e^{f_\phi} \omega_\phi^n \right) e^{f_\phi}\]
as an operator on $C^\infty(X)$. We define its complex conjugate operator by $\overline{Lu} := \overline{Tu}$. More precisely we have
\[\overline{Lu} = \left( -\Delta_\phi u - \langle \partial u, \partial f_\phi \rangle - u + \frac{1}{V} \int_X u e^{f_\phi} \omega_\phi^n \right) e^{f_\phi}.\]
We define a natural inner product on $C^\infty(X)$ by
\[
\langle \langle u, v \rangle \rangle = \int_X u \overline{v} \omega^n.
\]

Note that operators $L$ and $\overline{L}$ and the inner product $\langle \cdot, \cdot \rangle$ depend on a Kähler metric $\phi \in \mathcal{M}(\omega)$.

The followings are main results of this paper.

**Theorem 1.1.** The Hessian of the Ricci Calabi functional $E_{RC}$ at every generalized Kähler Einstein metric $\phi$ along directions $\delta \phi_1, \delta \phi_2 \in T_\phi \mathcal{M}(\omega)$ is given by
\[
\text{Hess}(E_{RC})(\delta \phi_1, \delta \phi_2) = 2\langle \langle L\overline{L}\delta \phi_1, \delta \phi_2 \rangle \rangle = 2\langle \langle L\overline{L}\delta \phi_1, \delta \phi_2 \rangle \rangle.
\]

As a corollary, we can see that the Hessian $\text{Hess}(E_{RC})$ is non-negative at every generalized Kähler Einstein metric.

**Corollary 1.2.** At every generalized Kähler Einstein metric, operators $L$ and $\overline{L}$ are commutative. As the result, their composition $L\overline{L}$ is a self-adjoint non-negative operator on $T_\phi \mathcal{M}(\omega)$ with respect to the inner product $\langle \cdot, \cdot \rangle$.

Similar results for the Hessian formula and the non-negativity of the Hessian were observed for various functionals and its critical Kähler metrics. See for instance, [1] [6] [9] (for the Calabi functional and the extremal Kähler metric), [6] (for a Calabi’s type functional and the perturbed extremal Kähler metric), [4] (for the He functional and the Kähler Ricci soliton) and [8] (for a Calabi’s type functional and the $f$-extremal Kähler metric).

Let $\mathfrak{h}(X)$ be the the Lie algebra of holomorphic vector fields on $X$. For any $u \in C^\infty(X)$, we define the gradient vector field $\text{grad}_\phi u$ for a Kähler metric $\phi \in \mathcal{M}(\omega)$ by
\[
i_{\text{grad}_\phi u} \omega_\phi = \sqrt{-1} \partial u.
\]

As an application of Corollary 1.2, we give the following Matsushima’s type decomposition theorem for $\mathfrak{h}(X)$.

**Theorem 1.3.** Let $X$ be a Fano manifolds admitting a generalized Kähler Einstein metric $\phi \in \mathcal{M}(\omega)$. Then the Lie algebra $\mathfrak{h}(X)$ is, as a vector space, the direct sum
\[
\mathfrak{h}(X) = \sum_{\lambda \geq 0} \mathfrak{h}_\lambda(X),
\]
where $\mathfrak{h}_\lambda(X)$ is the $\lambda$-eigenspace of the adjoint action of $-\text{grad}_\phi e^{f_\phi}$. Furthermore, $\mathfrak{h}_0(X)$ is the complexification of the Lie algebra of Killing vector fields on $(X, \omega_\phi)$. In particular, $\mathfrak{h}_0(X)$ is reductive.

This theorem was originally proved by Mabuchi in [11, Theorem 4.1]. His proof heavily depends on Futaki-Mabuchi’s theory [7] of the extremal Kähler vector field. Our proof is done by a simple linear algebraic argument based only on the commutativity of $L$ and $\overline{\partial}$.

Since Kähler Einstein metrics are trivial generalized Kähler Einstein metrics, Theorem 1.3 includes the following classical Matsushima’s result [12] called the Matsushima’s obstruction.

**Corollary 1.4.** Let $X$ be a Fano manifolds admitting a Kähler Einstein metric. Then the holomorphic automorphism group of $X$ is reductive.

The work of Yao [17] brings many interests to study about related topics of generalized Kähler Einstein metrics. See [13, 14] (for the modified Ding functional of toric Fano manifolds), [10] (for the modified Ding functional of general Fano manifolds), [15] (for relative GIT stabilities). In particular, Collins-Hisamoto-Takahashi [2] developed a geometric flow, called the inverse Monge-Ampère flow, whose self similar solutions are generalized Kähler Einstein metrics. In section 4, we discuss a relation between Corollary 1.2 and this flow.

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## 2 Variations of the Ricci Calabi functional

The Ricci Calabi functional can be written in terms of $\phi \in \mathcal{M}(\omega)$ as

$$\mathcal{E}_{RC}(\phi) = \int_X (1 - e^{f_\phi})^2 \omega^n_\phi.$$

First we compute the variation of the Ricci potential to obtain the first variation of the Ricci Calabi functional.

**Lemma 2.1.** For any $\phi \in \mathcal{M}(\omega)$ and any direction $\delta \phi \in T_\phi \mathcal{M}(\omega)$, we have

$$\delta f_\phi = -\Delta_\phi \delta \phi - \delta \phi + \frac{1}{V} \int_X \delta \phi e^{f_\phi} \omega^n_\phi.$$
Proof. By taking the variation of the first equation in (1.1), we have
\[ \delta f + \Delta \phi = -\Delta \phi - \delta \phi + C \]
for some constant \( C \). The constant \( C \) is equal to \( \frac{1}{V} \int_X \bar{\phi} e^{f \phi} \omega^n_\phi \) by the variation
\[ \int_X (\delta f + \Delta \phi) e^{f \phi} \omega^n_\phi = 0 \]
of the second equation in (1.1). \( \square \)

Lemma 2.2. (c.f. [17, Proof of Theorem 1]) For any \( \phi \in M(\omega) \) and any direction \( \delta \phi \in T_\phi M(\omega) \), we have
\[ \delta \mathcal{E}_{RC}(\delta \phi) = 2 \langle L(e^{f \phi}), \delta \phi \rangle = 2 \langle \overline{L}(e^{f \phi}), \delta \phi \rangle. \]

Proof. By integrations by parts (see also (2.1) and (2.2) in Lemma 2.3), note that
\[
\int_X e^{2f \phi} \Delta \phi \omega^n_\phi = \int_X 2(\Delta \phi e^{f \phi} + \bar{\phi} e^{f \phi} \omega^n_\phi) \delta \phi e^{f \phi} \omega^n_\phi
\]
\[
= \int_X 2(\Delta \phi e^{f \phi} + \bar{\phi} e^{f \phi} \omega^n_\phi) \delta \phi e^{f \phi} \omega^n_\phi.
\]
We then have
\[ \delta \mathcal{E}_{RC}(\delta \phi) = \int_X e^{2f \phi} 2\Delta \phi \delta \phi \omega^n_\phi + e^{2f \phi} \Delta \phi \delta \phi \omega^n_\phi \]
\[ = \int_X 2e^{2f \phi} \left(-\Delta \phi \delta \phi - \delta \phi + \frac{1}{V} \int_X \delta \phi e^{f \phi} \omega^n_\phi\right) \omega^n_\phi + e^{2f \phi} \Delta \phi \delta \phi \omega^n_\phi \]
\[ = \int_X 2 \left(-\Delta \phi e^{f \phi} + \bar{\phi} e^{f \phi} \omega^n_\phi - e^{f \phi} + \frac{1}{V} \int_X e^{f \phi} \bar{\phi} \omega^n_\phi\right) \delta \phi e^{f \phi} \omega^n_\phi \]
\[ = 2 \langle L(e^{f \phi}), \delta \phi \rangle. \]
Similarly we have \( \delta \mathcal{E}_{RC}(\delta \phi) = 2 \langle \overline{L}(e^{f \phi}), \delta \phi \rangle. \) \( \square \)

The followings are fundamental properties for operators \( L \) and \( \overline{L} \).

Lemma 2.3. \( L \) and \( \overline{L} \) are self-adjoint non-negative operators on \( C_c^\infty(X) \) with respect to the inner product \( \langle \cdot, \cdot \rangle \).

Proof. The operator \( u \mapsto \Delta \phi u + \langle \partial u, \overline{\partial f \phi} \rangle \) can be seen as a (negative) Laplacian with respect to a weighted inner product \( \langle u, v \rangle_f := \int_X u \overline{v} e^{f \phi} \omega_\phi \) on \( C_c^\infty(X) \), and the operator \( u \mapsto \Delta \phi u + \langle \partial u, \partial f \phi \rangle \) is its complex conjugate. Indeed we have
\[
(2.1) \quad \langle \Delta \phi u + \overline{\partial u}, \partial f \phi \rangle_f = \int_X (-\overline{\langle \partial u, \overline{\partial (e^{f \phi})} \rangle} + \overline{\langle \partial u, \bar{\phi} (e^{f \phi}) \rangle}) \omega^n_\phi
\]
\[ = \int_X -\overline{\langle \partial u, \bar{\phi} v \rangle} e^{f \phi} \omega^n_\phi, \]
and

\[ \langle u, \Delta \phi \rangle = \int_X (-\langle \overline{\partial} (ue^{f \phi}), \overline{\partial} v \rangle + \langle \overline{\partial} e^{f \phi}, \overline{\partial} v \rangle) \omega^n. \]

It follows that \( L \) and \( \overline{L} \) are self-adjoint operators with respect to \( \langle \cdot, \cdot \rangle \). Furthermore the following Bochner type formula [5, Proof of Theorem 2.4.3] (see also [17, Proposition 2]) holds:

\[ \int_X |\Delta \phi u + \langle \overline{\partial} u, \overline{\partial} f \phi \rangle|^2 e^{f \phi} \omega^n = \int_X |\nabla_i \nabla_j \overline{\partial} u|^2 e^{f \phi} \omega^n + \int_X |\overline{\partial} u|^2 e^{f \phi} \omega^n. \]

Thus the first eigenvalue of the operator \( u \mapsto -\Delta \phi u - \langle \overline{\partial} u, \overline{\partial} f \phi \rangle \) is greater than or equal to 1. It follows that \( L \) and \( \overline{L} \) are non-negative operators with respect to \( \langle \cdot, \cdot \rangle \). \( \square \)

The following lemma is crucial for Theorem 1.1.

**Lemma 2.4.** There is an vector space isomorphism:

\[ \{ u \in C_\infty^\infty(X) \mid Lu = 0 \} / \mathbb{C} \cong \mathfrak{h}(X). \]

In particular, the isomorphism is given by taking the gradient \( u \mapsto \text{grad}_\phi u \).

**Proof.** By definition of \( L \), the space \( \{ u \in C_\infty^\infty(X) \mid Lu = 0 \} / \mathbb{C} \) can be identified with

\[ \{ u \in C_\infty^\infty(X) \mid \Delta \phi u + \langle \overline{\partial} u, \overline{\partial} f \phi \rangle = u \}
\]

through \( u \mapsto u - \frac{1}{V} \int_X u e^{f \phi} \omega^n \). By [5, Theorem 2.4.3] (see also [17, Corollary 4]), this can be identified with the space of holomorphic vector fields on \( X \) through \( u \mapsto \text{grad}_\phi u \). \( \square \)

**Remark 2.5.** In views of the above lemmas for the operator \( L \), it is natural to consider the operator \( u \mapsto -\Delta \phi u - \langle \overline{\partial} u, \overline{\partial} f \phi \rangle - u + \frac{1}{V} \int_X u e^{f \phi} \omega^n \) on the vector space with the weighted inner product \( (C_\infty^\infty(X), \langle \cdot, \cdot \rangle_{f}) \) instead of the operator \( L \) on \( (C_\infty^\infty(X), \langle \cdot, \cdot \rangle) \). However, it is technically essential to consider \( L \) on \( (C_\infty^\infty(X), \langle \cdot, \cdot \rangle) \) in the proofs of Theorem 1.1 and Theorem 1.3.

The Lemma 2.2 and Lemma 2.4 show that every critical point of the Ricci Calabi functional defines a generalized Kähler Einstein metric and vice versa (c.f. [17, Theorem 1]). Indeed, every critical point \( \phi \in \mathcal{M}(\omega) \) of \( \mathcal{E}_{RC} \) satisfies \( L(e^{f \phi}) = 0 \), and \( e^{f \phi} \) defines a holomorphic vector field on \( X \).
We now prove Theorem 1.1.

**Proof of Theorem 1.1.** We compute the variation \((\delta L)(e^f \phi)\) at a generalized Kähler Einstein metric \(\phi \in M(\omega)\) to obtain the Hessian

\[
\text{Hess}(\mathcal{E}_{RC})(\delta \phi_1, \delta \phi_2) = 2\langle \delta_1(L(e^f \phi)), \delta \phi_2 \rangle = 2\langle (\delta_1 L)(e^f \phi) + L(\delta_1 e^f \phi), \delta \phi_2 \rangle,
\]

where \(\delta_1\) means the variation along \(\delta \phi_1 \in T_\phi M(\omega)\). Since \(\omega_\phi = \omega + \sqrt{-1}\partial \bar{\partial} \phi\) is generalized Kähler Einstein, \(Z := \text{grad}_\phi e^f \phi\) is holomorphic. By Lemma 2.4, \(e^f \phi\) is in the Kernel of \(L\) for the Kähler metric \(\omega_\phi\). Note that \(Z\) can be also written by \(\text{grad}_\phi e^f \phi + t\delta \phi (e^f \phi + tZ(\delta \phi))\) for any small \(t \in (-\varepsilon, \varepsilon)\), since it is easy to see that

\[
i_Z(\omega_\phi + t\sqrt{-1}\partial \bar{\partial} \delta \phi) = \sqrt{-1}\partial \bar{\partial}(e^f \phi + tZ(\delta \phi)).
\]

Thus, by Lemma 2.4 again, a perturbation \(e^f \phi + tZ(\delta \phi)\) is in the Kernel of \(L\) for the Kähler metric \(\omega_\phi + t\sqrt{-1}\partial \bar{\partial} \delta \phi\). When we denote the operator \(L\) for the Kähler metric \(\omega_\phi + t\sqrt{-1}\partial \bar{\partial} \delta \phi\) by \(L_t\), we thus have

\[
L_t(e^f \phi + tZ(\delta \phi)) = 0.
\]

Taking derivative at \(t = 0\), we have

\[
(\delta L)(e^f \phi) = -L(Z(\delta \phi)) = -L(\partial \delta \phi, \partial (e^f \phi)).
\]

Therefore we obtain

\[
(\delta L)(e^f \phi) + L(\delta e^f \phi) = L(-\langle \partial \delta \phi, \partial f \phi \rangle e^f \phi + (-\Delta_\phi \delta \phi - \delta \phi + \frac{1}{V} \int_X \delta \phi e^f \phi \omega^n_\phi) e^f \phi) = L L(\delta \phi).
\]

Similarly we have \(\delta(\mathcal{L}(e^f \phi)) = \mathcal{L} L(\delta \phi)\). This completes the proof. \(\square\)

**Proof of Corollary 1.2** This is an immediate consequence of Theorem 1.1 and Lemma 2.3.

\(\square\)

### 3 Matsushima’s type decomposition theorem

As an application of Corollary 1.2, we give another proof of Theorem 1.3 which was originally proved by Mabuchi [11, Theorem 4.1]. In Mabuchi’s proof, it was essential to
apply the strict periodicity [7, Theorem F] for the real part of the extremal Kähler vector field \( \text{grad}_\phi \text{pr}(S(\omega_\phi) - \overline{S}) \), where we let

\[
\text{pr} : L^2(X, \omega_\phi) \to \left\{ f \in C^\infty(X) \mid \overline{\partial} \text{grad}_\phi f = 0 \text{ and } \int_X f \omega^n_\phi = 0 \right\}
\]

be the projection, and we let \( S(\omega_\phi) \) be the scalar curvature of \( \omega_\phi \) and \( \overline{S} \) its average. The vector field \(-\text{grad}_\phi e^{f_\phi}\) in Theorem [1.3] is nothing but the extremal Kähler vector field.

Indeed, we have \( 1 - e^{f_\phi} = \text{pr}(1 - e^{f_\phi}) \) by definition of the generalized Kähler Einstein metric, and we have \( \text{pr}(1 - e^{f_\phi}) = \text{pr}(S(\omega_\phi) - \overline{S}) \) by [11, Theorem 2.1].

We now prove Theorem 1.3 by only applying the commutativity of \( \mathcal{L} \) and \( \mathcal{T} \).

**Proof of Theorem 1.3.** Since operators \( \mathcal{L} \) and \( \mathcal{T} \) are commutative by Corollary 1.2, we see that \( \mathcal{T} \in \text{End}(\text{Ker} \mathcal{L}) \). Let \( E_\lambda \) be the \( \lambda \)-eigenspace of \( \mathcal{T}|_{\text{Ker} \mathcal{L}} \). Note that \( \lambda \geq 0 \), since the operator \( \mathcal{T} \) is non-negative. Let us take any \( u \in E_\lambda \). Since \( E_\lambda \subset \text{Ker} \mathcal{L} \), we have \( \text{grad}_\phi u \in \mathfrak{h}(X) \). We also have

\[
\lambda u = \mathcal{T}u = (\mathcal{T} - \mathcal{L})u = -\langle \partial u, \partial e^{f_\phi} \rangle + \langle \overline{\partial} u, \overline{\partial} e^{f_\phi} \rangle = \{ e^{f_\phi}, u \},
\]

where \( \{ \cdot, \cdot \} \) is the Poisson bracket defined by \( \{ u, v \} = (\text{grad}_\phi v)u - (\text{grad}_\phi u)v \). It is well-known that the map \( u \mapsto -\text{grad}_\phi u \) is a complex Lie algebra homomorphism from \( (C^\infty, \{ \cdot, \cdot \}) \) to \( (\Gamma(TX), [\cdot, \cdot]) \), where \( [\cdot, \cdot] \) is the Lie bracket defined by \( [Z, W] = ZW - WZ \). Thus it follows that

\[
\lambda \text{grad}_\phi u = [-\text{grad}_\phi e^{f_\phi}, \text{grad}_\phi u].
\]

Let \( \mathfrak{h}_\lambda(X) := \text{grad}_\phi(E_\lambda) \). Since by Lemma 2.4, every holomorphic vector field on \( X \) can be written as \( \text{grad}_\phi u \) for a function \( u \in \text{Ker} \mathcal{L} \) unique up to additive constant, we then have a decomposition

\[
\mathfrak{h}(X) = \sum_{\lambda \geq 0} \mathfrak{h}_\lambda(X).
\]

By the above computation, we see that each \( \mathfrak{h}_\lambda(X) \) is the \( \lambda \)-eigenspace of the action of \(-\text{ad}(\text{grad}_\phi e^{f_\phi})\).

Since \( E_0 = \text{Ker} \mathcal{L} \cap \text{Ker} \mathcal{T} \), if \( u \in E_0 \), then both the real part and the imaginary part of \( u \) are in \( E_0 \). It follows that

\[
E_0 = \{ u \in \sqrt{-1} C^\infty(X) \mid \text{grad}_\phi u \in \mathfrak{h}(X) \} \otimes \mathbb{C}.
\]
Thus
\[ h_0(X) = \{ \text{grad}_\phi u + \text{grad}_\phi u \mid u \in \sqrt{-1}C^\infty_\mathbb{R}(X) \text{ and grad}_\phi u \in h(X) \} \otimes \mathbb{C}. \]

By [5, Lemma 2.3.8], \( \{ \text{grad}_\phi u + \text{grad}_\phi u \mid u \in \sqrt{-1}C^\infty_\mathbb{R}(X) \text{ and grad}_\phi u \in h(X) \} \) is equal to the space of Killing vector fields on \( X \). Since the Lie algebra of Killing vector fields on \( X \) corresponds to the Isometry group of \( X \) which is a compact group, we see that \( h_0(X) \) is reductive. This completes the proof.

**Proof of Corollary 1.4.** For any Kähler Einstein metric \( \phi \in \mathcal{M}(\omega) \), the holomorphic vector field \( \text{grad}_\phi e^{f_\phi} \) vanishes. By Theorem 1.3 we thus have \( h(X) = h_0(X) \).

\[ \blacksquare \]

## 4 The inverse Monge-Ampère flow and generalized Kähler Einstein metrics

Very recently, Collins-Hisamoto-Takahashi [2] developed a geometric flow \( \{ \phi_t \}_{t \in [0, \infty)} \subset \mathcal{M}(\omega) \), called the inverse Monge-Ampère flow (written as the MA\(^{-1}\) flow for simplicity), defined by
\[ \frac{d\phi_t}{dt} = 1 - e^{f_{\phi_t}}. \]

They proved the existence of a long time solution for any initial Kähler metric. It is easy to see that self-similar solutions of the MA\(^{-1}\) flow are generalized Kähler Einstein metrics. Furthermore the Ricci Calabi functional \( \mathcal{E}_{RC} \) is monotonically decreasing along the MA\(^{-1}\) flow. Indeed, by Lemma 2.2 and Lemma 2.3 we have
\[ \frac{d}{dt} \mathcal{E}_{RC}(\phi_t) = -2\langle L(e^{f_{\phi_t}}), e^{f_{\phi_t}} \rangle \leq 0, \]
along the flow \( \phi_t \). Therefore, Corollary 1.2 suggests that for any Fano manifold admitting a generalized Kähler Einstein metric, the MA\(^{-1}\) flow with any initial Kähler metric converges to a generalized Kähler Einstein metric in some sense.
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Mathematical Institute
Tohoku University
Sendai 980-8578
Japan

E-mail: satoshi.nakamura.r8@dc.tohoku.ac.jp