RADIAL FRACTIONAL LAPLACE OPERATORS
AND HESSIAN INEQUALITIES

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Abstract. In this paper we deduce a formula for the fractional Laplace operator \((-\Delta)^s\) on radially symmetric functions useful for some applications. We give a criterion of subharmonicity associated with \((-\Delta)^s\), and apply it to a problem related to the Hessian inequality of Sobolev type:

\[
\int_{\mathbb{R}^n} \left| (-\Delta)^{k+1} u \right| dx \leq C \int_{\mathbb{R}^n} -u F_k[u] dx,
\]

where \(F_k\) is the \(k\)-Hessian operator on \(\mathbb{R}^n\), \(1 \leq k < \frac{n}{2}\), under some restrictions on a \(k\)-convex function \(u\). In particular, we show that the class of \(u\) for which the above inequality was established in [FFV] contains the extremal functions for the Hessian Sobolev inequality of X.-J. Wang [W1]. This is proved using logarithmic convexity of the Gaussian ratio of hypergeometric functions which might be of independent interest.

1. Introduction

Let \(n \geq 2\). For every \(s \in (0,1)\) and a locally integrable function \(u : \mathbb{R}^n \to \mathbb{R}\) such that

\[
\int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|^2)^{\frac{n+2s}{2}}} dx < +\infty,
\]

we define the \(s\)-fractional Laplace operator as follows:

\[
(-\Delta)^s u(x) = c_{s,n} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,
\]

where \(c_{s,n}\) is a positive normalization constant. The integral is convergent (in the principal value sense if \(\frac{1}{2} \leq s < 1\)) if \(u\) is, for instance, a bounded \(C^2\) function. For any \(u\) satisfying (1.1), \((-\Delta)^s u\) is defined in the sense of distributions:

\[
\langle (-\Delta)^s u, h \rangle = \int_{\mathbb{R}^n} u (-\Delta)^s h dx,
\]

for all test functions \(h \in C_0^\infty(\mathbb{R}^n)\). This is a linear nonlocal operator, that is, roughly speaking, if \(u\) satisfies \((-\Delta)^s u(x) = 0\) in a domain \(\Omega \subseteq \mathbb{R}^n\), then the value of \(u\) at any point of \(\Omega\) depends not only on the neighborhood of the point itself but also on the behavior of the function in the entire space. We refer to [Lan] and [CaS] for further details.

Some new relationships with a class of local nonlinear operators, the \(k\)-th Hessian operators, have been pointed out in [FFV]. We recall that the \(k\)-th Hessian operator \(F_k\) can be defined in several equivalent ways (see e.g. [W2]). For a \(C^2\) function \(u\) in a domain \(\Omega \subseteq \mathbb{R}^n\), we define \(F_k[u], k \in \mathbb{N}, 1 \leq k \leq n\), as the \(k\)-th symmetric elementary function of the eigenvalues of \(D^2u(x)\), the Hessian matrix of \(u\). Just to give an idea of this family of operators we note that \(F_1[u] = \Delta u\) when \(k = 1\), and \(F_n[u] = \det(D^2u)\) when \(k = n\). A function \(u \in C^2(\Omega)\) is called \(k\)-convex if \(F_j[u] \geq 0\) for all \(j = 1,2,\ldots,k\). This definition was extended to general upper semicontinuous
functions by Trudinger and Wang [TW1], [TW2]. In particular, \( u \) is 1-convex if and only if \( u \) is subharmonic, while \( u \) is \( n \)-convex if it is convex in the usual sense.

Clearly, \( F_k \) are local operators, since it is possible to calculate \( F_k[u] \) pointwise by using second order partial derivatives of \( u \). Moreover, \( k \)-th Hessian operators are fully nonlinear while \( s \)-fractional Laplace operators are linear. Nevertheless, these operators are closely related. It was shown in [FFV] (Theorem 2.1) that, under certain assumptions on \( u \) discussed below, the following inequality holds (see Proposition 5.8 in this paper for the detailed statement):

\[
\int_{\mathbb{R}^n} |(-\Delta)^s u|^{k+1} \, dx \leq C_{k,n} \int_{\mathbb{R}^n} |u|^{k+1} \, dx,
\]

where \( s = \frac{k}{k+1} \), \( 1 \leq k < \frac{n}{2} \), and \( C_{k,n} \) is positive constant depending only on \( k \) and \( n \). The converse inequality holds as well with a different constant, for all \( k \)-convex \( C^2 \) functions \( u \) vanishing at infinity (FFV, Theorem 3.1).

In this paper we want to improve our knowledge of the \( s \)-fractional Laplace operator by studying \( s \)-subharmonic functions that are analogous to subharmonic functions for the Laplace operator, and in particular the class of \( k \)-convex functions introduced in [FFV] for which \( \text{(1.4)} \) holds. In fact we will prove below that this class contains extremal functions of the Hessian Sobolev inequality due to X.-J. Wang [W1], [W2]:

\[
\left( \int_{\mathbb{R}^n} |u|^{q} \, dx \right)^{\frac{1}{q}} \leq C'_{k,n} \left( \int_{\mathbb{R}^n} |u|^{k+1} \, dx \right)^{\frac{1}{k+1}},
\]

where \( 1 \leq k < \frac{n}{2} \), \( q = \frac{n(k+1)}{n-2k} \), and \( u \) is a \( k \)-convex \( C^2 \) function on \( \mathbb{R}^n \) vanishing at \( \infty \). We remark that \text{(1.4)} implies \text{(1.5)} by the classical Sobolev embedding theorem.

Our approach is to study the \( s \)-fractional Laplace operator and the corresponding notion of \( s \)-subharmonicity for radially symmetric functions \( u(x) = u(r) \), \( r = |x| \). We will prove the following result.

**Theorem 1.1.** Let \( s \in (0,1) \). For every radial \( C^2 \) function \( u \) such that

\[
\int_0^{+\infty} \frac{|u(r)|}{(1+r)^{n+2s}} \, r^{n-1} \, dr < +\infty,
\]

the following formula holds:

\[
(-\Delta)^s u(r) = c_{s,n} r^{-2s} \int_1^{+\infty} \left( u(r) - u(r\tau) + (u(r) - u(r)) \tau^{-n+2s} \right) \tau^{\frac{n}{2}} \, d\tau,
\]

where \( r = |x| > 0 \), \( x \in \mathbb{R}^n \), and

\[
H(\tau) = 2\pi \alpha_n \int_0^{\tau} \sin^{n-2} \theta \left( \frac{\sqrt{\tau^2 - \sin^2 \theta} + \cos \theta} {\sqrt{\tau^2 - \sin^2 \theta}} \right)^{1+2s} \, d\theta, \quad \tau \geq 1, \quad \alpha_n = \frac{n \pi}{\Gamma(\frac{n-1}{2})}.
\]

Clearly, \( H(\tau) \) is a positive continuous function on \([1, +\infty)\), with \( H(\tau) \simeq \tau^{2s} \) as \( \tau \to +\infty \). One can express \( H(\tau) \) in terms of the Gaussian hypergeometric function:

\[
\tau(\tau^2 - 1)^{-1-2s} \, H(\tau) = \frac{2\pi \frac{n}{2}} {\Gamma(\frac{n}{2})} \tau^{-1-2s} \, 2F_1(a, b, c, \tau^{-2}) \simeq (\tau - 1)^{-1-2s}, \quad \tau \geq 1,
\]

where

\[
a = \frac{n + 2s}{2}, \quad b = 1 + s, \quad c = \frac{n}{2}.
\]

Note that \( 2F_1(a, b, c, \tau^{-2}) \) has a singularity of order \( (\tau - 1)^{-1-2s} \) at \( \tau = 1 \), while \( H(\tau) \) is continuous at \( \tau = 1 \). The integral in (1.7) is convergent for any radial \( C^2 \) function \( u \) satisfying (1.6).
For every $s \in (0,1]$ and $u$ satisfying (1.1), we say that $u$ is $s$-subharmonic in $\Omega$ if $u : \Omega \to [-\infty, +\infty]$ is upper semicontinuous in $\Omega$, and
\[-(-\Delta)^s u(x) \geq 0 \quad \text{in } D'(\Omega),\]
in other words, $-(-\Delta)^s u$ is a positive Borel measure in $\Omega$.

**Theorem 1.2.** Let $s \in (0,1]$. Let $u(r)$ be a radial upper semicontinuous function in $\mathbb{R}_+$ such that (1.6) holds. If, for all $r > 0$ and $\tau \geq 1$,
\[(1.8) \quad u(r) - u(r\tau) + (u(r) - u(\frac{r}{\tau}))\tau^{-n+2s} \leq 0,
\]
then $u$ is $s$-subharmonic in $\mathbb{R}^n \setminus \{0\}$.

If in Theorem 1.2 one assumes additionally that $u$ is bounded above in a neighborhood of $0$, or more generally $\limsup_{r \to 0} u(r) r^n - 2s \leq 0$, then $u$ has a removable singularity at $0$ (see [BH], p. 379, Corollary 10.2), i.e., $u$ is $s$-subharmonic in the entire space $\mathbb{R}^n$ (with $u(0) = \limsup_{r \to 0} u(r)$).

We observe that condition (1.8) is of interest for any real $s$. The following theorem provides a more convenient pointwise characterization of (1.8). Combined with Theorem 1.2, it gives a useful sufficient condition for a radially symmetric function to be $s$-subharmonic when $0 < s < 1$.

**Theorem 1.3.** Let $s \in \mathbb{R}$. Let $u \in C^2(\mathbb{R}_+)\,$ be a radial function. Then (1.8) holds if and only if, for all $r > 0$,
\[(1.9) \quad u''(r) + (n - 2s + 1)\frac{u'(r)}{r} \geq 0.
\]

Condition (1.9) is not necessary for a radially symmetric function $u$ to be $s$-subharmonic when $0 < s < 1$ (see examples below). However, the class of functions obeying (1.9) is quite rich, and contains many interesting $s$-subharmonic functions.

In the special case $s = \frac{n(k-1)}{2k} + 1$, condition (1.9) coincides with the following characterization of $k$-convex radially symmetric $C^2$ functions (see [W1], [W2]):
\[(1.10) \quad u''(r) + \frac{n-k}{k} \frac{u'(r)}{r} \geq 0,
\]
where $k = 1,2,\ldots,n$.

For $\beta > 0$, let us consider
\[f_\beta(x) = -(1+|x|^2)^{-\frac{\beta}{2}}, \quad x \in \mathbb{R}^n.
\]
It is easy to see (cf. Corollary 1.4 below) that for any $s \in \mathbb{R}$, $f_\beta$ satisfies (1.8), and consequently (1.9), if and only if $\beta \leq n - 2s$. Moreover, if $0 < s \leq 1$ and $0 < \beta < n$, then $f_\beta$ is $s$-subharmonic in $\mathbb{R}^n$ if and only if $\beta \leq n - 2s$.

Now let $\beta = \frac{n}{k} - 2 > 0$, where $k = 1,2,\ldots,\frac{n}{2}$. Then by (1.10), $f_\beta$ is $k$-convex, and is known to be an extremal function for the Hessian Sobolev inequality (1.5); see [W1], [W2], (A new proof of this inequality, but without the sharp constant, is available; see [V]). Our main goal is to show that $u = f_\beta$ satisfies the following condition introduced in [FFV]:
\[(1.11) \quad (-\Delta)^s[-(-\Delta)^s u]^k \geq 0 \quad \text{in } \mathbb{R}^n,
\]
where $s = \frac{k}{k+1}$, $1 \leq k < \frac{n}{2}$. Under this condition the enhanced Hessian Sobolev inequality (1.11) was proved in [FFV] (Theorem 2.1). The following theorem demonstrates that, in particular, this class of functions is nontrivial.

**Theorem 1.4.** Let $s = \frac{k}{k+1}$ and $\beta = \frac{n}{k} - 2$, where $1 \leq k < \frac{n}{2}$. Then $f_\beta$ satisfies condition (1.11).
The proof of Theorem 1.4 for \( k \geq 2 \) relies on Theorems 1.2 and 1.3, along with a new logarithmic convexity property of the Gaussian quotient of hypergeometric functions \( \frac{2F_1(a,b,c,x)}{2F_1(a+1,b,c+1,x)} \).

It is worth observing that when \( k = 1 \), the function \(-u = (-\Delta)^{1/2} f_\beta \) with \( \beta = n - 2 \) \((n \geq 3)\), fails to satisfy condition (1.9) with \( s = 1/2 \); however, \(-u\) is still \( 1/2 \)-subharmonic. The latter is easily checked using the superposition property of fractional Laplacians:

\[
(-\Delta)^{1/2}(-\Delta)^{1/2} f_\beta = \Delta f_\beta \geq 0.
\]

The paper is organized as follows. The proofs of Theorems 1.1 and 1.2 are given in Section 2. In Section 3 we prove Theorem 1.3 and discuss the \( s \)-subharmonicity property for radially symmetric functions. Some examples are given in Section 4. In Section 5 we discuss some further applications of our results. In particular, we prove a Liouville theorem, the maximum principle for radial \( s \)-subharmonic functions, and a derivative formula involving the fractional Laplacian. In Section 6 we prove Theorem 1.4 using a convexity property of a certain ratio of hypergeometric functions. Finally, in Section 7 we prove logarithmic convexity of \( \frac{2F_1(a,b,c,x)}{2F_1(a+1,b+1,c+1,x)} \) and convexity of \( \frac{2F_1(a,b,c,x)}{2F_1(a+1,b+1,c+1,x)} \) in \((-\infty,1)\) (Theorem 7.1 and Corollary 7.4) under certain restrictions on the parameters \( a, b, c \) using a method developed recently in [KSI1] (see also [KSI2]). The proofs generalize to ratios of generalized hypergeometric functions \( q+1F_q \) for \( q \geq 2 \).

2. Proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** For the sake of convenience, we will drop the normalization constant \( c_{s,n} \) in \((1.2)\) when it does not lead to a confusion. Let \( u \) be a radial \( C^2 \) function in \( \mathbb{R}^n \) satisfying \((1.6)\). If \( u(x) = u(|x|), \ x = r'x', \ y = \rho y', \) and \( |x'| = |y'| = 1, \) then

\[
(-\Delta)^s u(x) = \int_0^{+\infty} \left( \int_{|y'|=1} \frac{u(r) - u(\rho)}{r^x' - \rho y'} \rho^{n-1} dH^{n-1}(y') \right) d\rho - \int_0^{+\infty} \left( \int_{|y'|=1} \frac{1}{r^x' - \rho y'} \rho^{n-1} dH^{n-1}(y') \right) d\rho
\]

\[
= \int_0^{+\infty} \frac{u(r) - u(\rho)}{r^{n+2s}} \rho^{n-1} \left( \int_{|y'|=1} \frac{1}{x' - \tau y'} |y'|^{n+2s} dH^{n-1}(y') \right) d\tau
\]

\[
= r^{1-2s} \int_0^{+\infty} \frac{(u(r) - u(\tau r)) \tau^{n-1}}{r^{2s}} \left( \int_{|y'|=1} \frac{1}{x' - \tau y'} |y'|^{n+2s} dH^{n-1}(y') \right) d\tau
\]

Notice that

\[
\int_{|y'|=1} \frac{1}{x' - \tau y'} |y'|^{n+2s} dH^{n-1}(y')
\]

is independent of \( x' \in \{ |y'| = 1 \} \). Indeed, suppose that \( z' \in \{ |y'| = 1 \} \). Then there exists a unitary matrix \( Q \) such that \( z' = Qx' \). Thus, performing a change of variables such that \( y' = Qw' \), we get:

\[
\int_{|y'|=1} \frac{1}{z' - \tau y'} |y'|^{n+2s} dH^{n-1}(y') = \int_{|w'|=1} \frac{1}{Qx' - \tau Qw'} |w'|^{n+2s} dH^{n-1}(w')
\]

\[
= \int_{|w'|=1} \frac{1}{x' - \tau w'} |w'|^{n+2s} dH^{n-1}(w'),
\]

because \( |\text{det}Q| = 1 \) and \( |Qv_1 - Qv_2| = |v_1 - v_2| \) for every \( v_1, v_2 \in \mathbb{R}^n \).
Moreover,
\[\int_{|y'|=1} \frac{1}{|z' - \tau y'|^{\frac{n+2s}{2}}} dH^{n-1}(y') = \int_{|y'|=1} \frac{1}{(1 - 2\tau(y', z') + \tau^2)^{\frac{n+2s}{2}}} dH^{n-1}(y') = 2\pi \alpha_n \int_0^\pi \frac{\sin^{n-2}\theta}{(1 - 2\tau \cos \theta + \tau^2)^{\frac{n+2s}{2}}} d\theta,\]
where
\[\alpha_n = \prod_{k=1}^{n-3} \int_0^\pi \sin^k \theta d\theta = \frac{\pi^{\frac{n-3}{2}}}{\Gamma\left(\frac{n-3}{2}\right)}.
\]
Notice also that
\[1 - 2\tau \cos \theta + \tau^2 \geq 1 - 2\tau + \tau^2 = (1 - \tau)^2.
\]
Thus, for every \(\tau \neq 1\),
\[\int_0^\pi \frac{\sin^{n-2}\theta}{(1 - 2\tau \cos \theta + \tau^2)^{\frac{n+2s}{2}}} d\theta
\]
is bounded, while for \(\tau = 1\),
\[\sin^n \theta \sim \theta^{n-2} \theta^{n-2s} \sim \theta^{2-2s}, \quad \text{as } \theta \to 0.
\]
We denote
\[(2.1) \quad K(\tau) = 2\pi \alpha_n \int_0^\pi \frac{\sin^{n-2}\theta}{(1 - 2\tau \cos \theta + \tau^2)^{\frac{n+2s}{2}}} d\theta,
\]
and study the behavior of the integral with respect to \(\tau\).

Let \(\tau \geq 1\). We perform a change of variable as follows:
\[\frac{\sin \theta}{\sqrt{\tau^2 - 2\tau \cos \theta + 1}} = \frac{\sin \psi}{\tau}.
\]
Consequently,
\[\cos \theta = \frac{\sin^2 \psi \pm \cos \psi \sqrt{\tau^2 - \sin^2 \psi}}{\tau}.
\]
Hence
\[\tau \sin \theta \frac{d\theta}{d\psi} = \tau^{-1} \sin \psi \left(2 \cos \psi \mp \sqrt{\tau^2 - \sin^2 \psi} \mp \frac{\cos^2 \psi}{\sqrt{\tau^2 - \sin^2 \psi}}\right)
\]
(2.2)
\[= \frac{\sin \psi}{\tau \sqrt{\tau^2 - \sin^2 \psi}} \left(\cos \psi \mp \sqrt{\tau^2 - \sin^2 \psi}\right)^2.
\]
Moreover
\[\tau \sin \theta = \sin \psi \sqrt{\tau^2 - 2\tau \cos \theta + 1}
\]
(2.3)
\[= \sin \psi \sqrt{\tau^2 - 2(\sin^2 \psi \pm \cos \psi \sqrt{\tau^2 - \sin^2 \psi}) + 1}
\]
\[= \sin \psi \sqrt{\tau^2 - \sin^2 \psi \mp 2 \cos \psi \sqrt{\tau^2 - \sin^2 \psi} + 1 - \sin^2 \psi}
\]
\[= \sin \psi \sqrt{\tau^2 - \sin^2 \psi \mp 2 \cos \psi \sqrt{\tau^2 - \sin^2 \psi} + \cos^2 \psi}
\]
\[= \sin \psi \sqrt{\sqrt{\tau^2 - \sin^2 \psi \mp \cos \psi}^2} = \sin \psi \sqrt{\tau^2 - \sin^2 \psi \mp \cos \psi}.
\]
We obtain, by plugging the relation (2.3) into (2.2):
$$\frac{d\theta}{d\psi} = -\frac{1}{|\sqrt{\tau^2 - \sin^2 \psi \mp \cos \psi}| \sqrt{\tau^2 - \sin^2 \psi}} \left(\cos \psi \mp \sqrt{\tau^2 - \sin^2 \psi} \right)^2$$

(2.4)

Hence,

$$\int_0^\pi \frac{\sin^{n-2} \theta}{(1 - 2\tau \cos \theta + \tau^2)^{n/2}} d\theta = \int_0^\pi \left(\frac{\sin \psi}{\tau}\right)^{n+2s} \sin^{2s} \theta \left(1 \mp \frac{\cos \psi}{\sqrt{\tau^2 - \sin^2 \psi}}\right) d\psi$$

$$= \tau^{2s} \int_0^\pi \left(\frac{\sin \psi}{\tau}\right)^{n+2s} \sin^{2s} \theta \left(\sqrt{\tau^2 - \sin^2 \psi \mp \cos \psi}\right)^{2-2s} \left(1 \mp \frac{\cos \psi}{\sqrt{\tau^2 - \sin^2 \psi}}\right) d\psi$$

(2.5)

$$= \tau^{2-n} \int_0^\pi \sin^{n-2} \psi \left(\sqrt{\tau^2 - \sin^2 \psi \mp \cos \psi}\right)^{1+2s} \frac{1}{\tau^2 - \sin^2 \psi} d\psi$$

$$= \tau^{2-n} (\tau^2 - 1)^{-2s} \int_0^\pi \frac{\sin^{n-2} \psi \sqrt{\tau^2 - \sin^2 \psi \mp \cos \psi}}{\sqrt{\tau^2 - \sin^2 \psi}} d\psi. $$

Let us denote

$$H(\tau) = 2\pi a \int_0^\pi \frac{\sin^{n-2} \psi \sqrt{\tau^2 - \sin^2 \psi \mp \cos \psi}}{\sqrt{\tau^2 - \sin^2 \psi}} d\psi. $$

Then $H(\tau) = \tau^{n-2}(\tau^2 - 1)^{1+2s} K(\tau)$. Clearly, $H(\tau)$ is a positive continuous function on $[1, +\infty)$ such that $H(\tau) \approx \tau^{2s}$ as $\tau \to +\infty$. Moreover, it follows from (2.1) that, as was mentioned in the Introduction, $H(\tau)$ can be expressed in terms of the hypergeometric function $\frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu}{2})} \frac{2\pi^\nu}{\Gamma(\frac{\nu}{2})} 2F_1(a, b, c, \tau^{-2})$, $\tau \geq 1$,

where

$$a = \frac{n + 2s}{2}, \quad b = 1 + s, \quad c = \frac{n}{2}. $$

On the other hand

$$(-\Delta)^s u(r) = \int_0^{+\infty} \frac{u(r) - u(r\tau)}{r^{2s}} r^{n-1} K(\tau) d\tau$$

(2.7)

$= \int_0^1 \frac{u(r) - u(r\tau)}{r^{2s}} r^{n-1} K(\tau) d\tau + \int_1^{+\infty} \frac{u(r) - u(r\tau)}{r^{2s}} r^{(\tau^2 - 1)^{1-2s}} H(\tau) d\tau.$

Let us consider the following integral

$$\int_0^1 \frac{u(r) - u(r\tau)}{r^{2s}} r^{n-1} K(\tau) d\tau,$$
and perform the following change of variable: \( \xi = \frac{1}{r} \). Then

\[
\int_0^1 \frac{u(r) - u(r\tau)}{r^{2s}} r^{n-1} K(\tau) d\tau = \int_1^{+\infty} \frac{u(r) - u\left(\frac{r}{\xi}\right)}{r^{2s}} \xi^{-n+1} \xi^{-2K\left(\frac{1}{\xi}\right)} d\xi
\]

(2.8)

\[
= \int_1^{+\infty} \frac{u(r) - u\left(\frac{r}{\xi}\right)}{r^{2s}} \xi^{-n-1} K\left(\frac{1}{\xi}\right) d\xi.
\]

Notice that

\[
K\left(\frac{1}{\xi}\right) = 2\pi \alpha_n \xi^{n+2s} \int_0^\pi \frac{\sin^{-2} \theta}{(\xi^2 - 2\xi \cos \theta + 1)^{n+2s}} d\theta.
\]

Hence,

(2.9)

\[
\int_0^1 \frac{u(r) - u(r\tau)}{r^{2s}} r^{n-1} K(\tau) d\tau = \int_1^{+\infty} \frac{u(r) - u\left(\frac{r}{\xi}\right)}{r^{2s}} \xi^{-n+1} \xi^{-2K\left(\frac{1}{\xi}\right)} d\xi
\]

\[
= \int_1^{+\infty} \frac{u(r) - u\left(\frac{r}{\xi}\right)}{r^{2s}} \xi^{-n+2s} K(\xi) d\xi = \int_1^{+\infty} \frac{u(r) - u\left(\frac{r}{\xi}\right)}{r^{2s}} \xi^{-1+2s} \xi^{2-n(\xi^2 - 1)^{-1-2s}} H(\xi) d\xi
\]

\[
= \int_1^{+\infty} \frac{u(r) - u\left(\frac{r}{\xi}\right)}{r^{2s}} \xi^{1-n+2s}(\xi^2 - 1)^{-1-2s} H(\xi) d\xi.
\]

Thus

(2.10) \((-\Delta)^s u(r) = r^{-2s} \int_1^{+\infty} \left(u(r) - u(r\xi) + (u(r) - u\left(\frac{r}{\xi}\right))\xi^{-n+2s}\right) \xi^{(\xi^2 - 1)^{-1-2s}} H(\xi) d\xi.

The convergence of the integral in (2.10) is discussed below in Sec. 3.

□

We now deduce Theorem 1.2 from Theorem 1.1 using mollification defined by means of Mellin’s convolution which preserves condition (1.8).

**Proof of Theorem 1.2.** Suppose \( u \) is a radially symmetric upper semicontinuous function satisfying the conditions

(2.11)

\[
\int_0^{+\infty} \frac{|u(r)|}{(1 + r)^{n+2s}} r^{n-1} dr < +\infty,
\]

and

(2.12)

\[
u(r) - u(r\xi) + (u(r) - u\left(\frac{r}{\xi}\right))\xi^{-n+2s} \leq 0, \text{ for all } r > 0, \xi \geq 1.
\]

Let us show that \( u \) is \( s \)-subharmonic, i.e., \(-(-\Delta)^s u \geq 0\) in the sense of distributions. Let \( \phi \in C_0^\infty(\mathbb{R}) \) so that \( \phi \geq 0, \phi \) is supported in the interval \(|r| \leq r_0\), where \( 0 < r_0 < +\infty \), and \( \int_{\mathbb{R}} \phi(y) dy = 1 \). For \( \epsilon > 0 \), define the approximate identity on \( \mathbb{R}_+ \) by:

(2.13)

\[
\phi_\epsilon(r) = \frac{1}{\epsilon} \phi \left( \frac{\log r}{\epsilon} \right), \quad r > 0.
\]

Then clearly, for every \( \epsilon > 0 \),

(2.14)

\[
\int_0^{+\infty} \phi_\epsilon(r) \frac{dr}{r} = 1.
\]

We observe that (2.12) is invariant under the Mellin convolution:

(2.15)

\[
u_\epsilon(\tau) = \int_0^{+\infty} \phi_\epsilon \left( \frac{\tau}{t} \right) u(t) \frac{dt}{t}, \quad \tau > 0.
\]
Indeed, integrating both sides of (2.13) against \( \phi_{c}(\vec{r}) \frac{dr}{r} \), we obtain:

(2.16) \( u_{c}(\tau) - u_{e}(\tau \xi) + (u_{c}(\tau) - u_{e}(\tau \xi)) \xi^{-n+2s} \leq 0, \) for all \( \tau > 0, \; \xi \geq 1 \).

Moreover, obviously \( u_{c} \in C^{\infty}(\mathbb{R}^{+}) \), and

(2.17) \[
\int_{0}^{+\infty} \frac{|u_{c}(r)| \cdot r^{n-1}}{(1 + r)^{n+2s}} dr \leq C \int_{0}^{+\infty} \frac{|u(r)| \cdot r^{n-1}}{(1 + r)^{n+2s}} dr < +\infty,
\]

where \( C = C(\epsilon, r_{0}, s, n) \) is a positive constant. Indeed, by Fubini’s theorem,

\[
\int_{0}^{+\infty} \frac{|u_{c}(r)| \cdot r^{n-1}}{(1 + r)^{n+2s}} dr \leq \int_{0}^{+\infty} |u(t)| \int_{0}^{+\infty} \frac{r^{n}}{(1 + r)^{n+2s}} \phi_{c}(\frac{r}{t}) \frac{dr}{r} dt,
\]

where the last estimate follows since \( \phi_{c}(\lambda) \) vanishes outside the interval \((e^{-\epsilon r_{0}}, e^{\epsilon r_{0}})\), and (2.14) holds.

Let \( h \in C_{0}^{\infty}(\mathbb{R}^{n} \setminus \{0\}) \) be a nonnegative test function supported in \( 0 < |x| \leq R < +\infty \). We observe that, since \( h \) is compactly supported, it follows from (1.2) that \((-\Delta)^{s} h \in C^{\infty}(\mathbb{R}^{n})\), and

(2.18) \[
|(-\Delta)^{s} h(x)| \leq \frac{C}{(1 + |x|)^{n+2s}}, \quad x \in \mathbb{R}^{n}.
\]

In particular,

\[
\langle (-\Delta)^{s} u, h \rangle = \int_{\mathbb{R}^{n}} u (-\Delta)^{s} h dx
\]

is well-defined in terms of distributions (see [Lan], Sec. 1.6). By Theorem 1.1 and (2.16), \((-\Delta)^{s} u_{c} \geq 0\), and hence, using (2.11) and recalling that \( u_{c} \in C^{\infty}(\mathbb{R}^{n} \setminus \{0\}) \), we obtain:

(2.19) \[
\langle (-\Delta)^{s} u_{c}, h \rangle = \langle u_{c}, (-\Delta)^{s} h \rangle = \int_{\mathbb{R}^{n}} u_{c} (-\Delta)^{s} h dx \geq 0,
\]

for every \( h \in C_{0}^{\infty}(\mathbb{R}^{n} \setminus \{0\}), \; h \geq 0 \).

It remains to prove the approximation property:

(2.20) \[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^{n}} u_{c} (-\Delta)^{s} h dx = \int_{\mathbb{R}^{n}} u (-\Delta)^{s} h dx.
\]

Denote by \( \psi(t) \) the spherical mean of \((-\Delta)^{s} h\):

\[
\psi(t) = \int_{|x'| = 1} (-\Delta)^{s} h(tx') dH^{n-1}(x'), \quad t > 0.
\]

Then

\[
\int_{\mathbb{R}^{n}} [u(x) - u_{c}(x)] (-\Delta)^{s} h dx = \int_{0}^{+\infty} [u(t) - u_{c}(t)] \psi(t) t^{n} \frac{dt}{t}.
\]

We estimate:

\[
\left| \int_{0}^{+\infty} [u(t) - u_{c}(t)] \psi(t) t^{n} \frac{dt}{t} \right| = \left| \int_{0}^{+\infty} u(t) \left[ \psi(t) t^{n} - \int_{0}^{+\infty} \phi_{c}(\frac{r}{t}) \psi(r) r^{n} \frac{dr}{r} \right] \frac{dt}{t} \right| 
\]

\[
\leq \int_{0}^{+\infty} |u(t)| \int_{0}^{+\infty} \left| t^{n} \psi(t) - r^{n} \psi(r) \right| \phi_{c}(\frac{r}{t}) \frac{dr}{r} \frac{dt}{t}.
\]

We recall that, since \( \phi(y) \) is supported in the interval \(|y| \leq r_{0}\), it follows that \( \phi_{c}(\frac{r}{t}) \) is supported in the interval where \(|\log \frac{t}{r}| \leq \epsilon_{0}\). Hence, in the above estimates we can assume:

\[
e^{-\epsilon_{0}} - 1 \leq \frac{r}{t} - 1 \leq e^{\epsilon_{0}} - 1.
\]
From this we deduce:

\[(2.21) \quad |r - t| \leq \delta t,\]

where \( \delta = e^{r_{\tau_0}} - 1 \to 0 \) as \( \epsilon \to 0. \)

We notice that, since \((-\Delta)^s h \in C^\infty(\mathbb{R}^n)\), it follows that its spherical mean \( \psi \) is infinitely differentiable on \([0, +\infty)\) (see, e.g., [Lan], Sec. I.6). We will need the following estimates:

\[(2.22) \quad |\psi(t)| \leq \frac{C}{(1 + t)^{n+2s}}, \quad |\psi'(t)| \leq \frac{C}{(1 + t)^{n+2s+1}}, \quad t \geq 0,
\]

where \( C \) depends on \( h \). Indeed, \( h(x) = 0 \) for \( |x| > R \), and consequently \( \psi \) yields, for \( t > R \):

\[
\psi(t) = -c_{s,n} \int_{|y| \leq R} h(y) \int_{|x'| = 1} \frac{1}{|y - tx'|^{n+2s}} dH^{n-1}(x') dy,
\]

\[
\psi'(t) = c_{s,n}(n+2s) \int_{|y| \leq R} h(y) \int_{|x'| = 1} \frac{t - x' \cdot y}{|y - tx'|^{n+2s+2}} dH^{n-1}(x') dy.
\]

Here \( |y - tx'| \geq t - R > \frac{t}{2} \) for \( t > 2R \), which yields \( 2.22 \) for \( t > 2R \). Since \( \psi \) is infinitely differentiable on \([0, +\infty)\), it follows that \( 2.22 \) holds for all \( t \geq 0. \)

Let \( \psi_1(t) = t^n \psi(t) \). Clearly, estimates \( 2.22 \) yield:

\[(2.23) \quad |\psi_1(t)| \leq \frac{C t^n}{(1 + t)^{n+2s}}, \quad |\psi_1'(t)| \leq \frac{C t^{n-1}}{(1 + t)^{n+2s}}, \quad t \geq 0.
\]

Invoking the mean value inequality we estimate:

\[|t^n \psi(t) - r^n \psi(r)| = |\psi_1(t) - \psi_1(r)| \leq |r - t| |\psi_1'(\lambda)| \leq C |r - t| \frac{C t^{-1}}{(1 + \lambda)^{n+2s}},\]

for some \( \lambda \) between \( t \) and \( r \). Assuming by \( 2.21 \) that \( |r - t| < \delta t \), we see that \( t(1 - \delta) \leq \lambda \leq t(1 + \delta) \). Combining the preceding estimates, we obtain:

\[|t^n \psi(t) - r^n \psi(r)| \leq C \delta (1 + \delta)^{-1} \frac{t^n}{(1 - \delta)^{n+2s}} \frac{t^n}{(1 + t)^{n+2s}}.
\]

Using this together with \( 2.11 \), we conclude:

\[
\int_0^\infty |u(t)| \int_0^{\infty} |t^n \psi(t) - r^n \psi(r)| \frac{dr}{r} dt \leq C \delta (1 + \delta)^{-1} \frac{t^n}{(1 - \delta)^{n+2s}} \frac{t^n}{(1 + t)^{n+2s}} \int_0^\infty |u(t)| \frac{t^{n-1}}{(1 + t)^{n+2s}} dt,
\]

where the right-hand side is finite by \( 2.11 \). Letting \( \epsilon \to 0 \), and hence \( \delta \to 0 \), we conclude the proof of the approximation property \( 2.20 \). \( \square \)

3. Fractional subharmonicity for radially symmetric functions

In this section we study further the condition

\[(3.1) \quad u(r) - u(r \xi) + (u(r) - u(\xi)^n + 2s \leq 0, \quad \text{for all } r > 0, \quad \xi \geq 1,\]

which ensures that a radially symmetric \( C^2 \) function \( u \) is \( s \)-subharmonic, i.e., \(-(-\Delta)^s u \geq 0, \) for \( 0 < s \leq 1 \). However, this condition makes sense for any real \( s \). We will show that for \( C^2 \) functions \( u \) it is equivalent to:

\[u''(r) + (n + 1 - 2s) \frac{u'(r)}{r} \geq 0, \quad \text{for all } r > 0,\]

which is stated as Theorem \( 1.3 \) in the Introduction.
Suppose \( u \) is a radial \( C^2 \) function. It is worth noting that in a neighborhood of 1, whenever \( \xi \to 1, \xi \geq 1 \),

\[
u(r) - u(r\xi) + (u(r) - u(r))\xi^{-n+2s} = -u'(r) - (\xi - 1)^2(\xi_1 - 1) - r^2 u''(r) + o(\xi - 1)^2(1 + \frac{1}{\xi^{n-2s}}) \\
+ \xi^{n+2s} u((\xi - 1)^2) = \frac{1}{2}r^2(\xi - 1)^2(\xi_1 - 1) + (u''(r) + \frac{2\xi(\xi_1 - 1)u'(r)}{r(1 + \xi^{n+2s})}) + o(\xi - 1)^2 \\
= \frac{1}{2}r^2(\xi - 1)^2(\xi_1 - 1) + (u'' + (n + 1 - 2s) u' + o(r, (\xi - 1)^2)) \\
+ o(r, (\xi - 1)^2) = u'' + (n + 1 - 2s)(1 + h(\xi - 1)) u',
\]

where \( \lim_{\xi \to 1^+} h(\xi - 1) = 1. \)

In particular, the integral in (1.11) is convergent in a neighborhood of 0, since for any fixed \( r > 0 \) it behaves as \( (1 - \xi)^{-2s} \) that converges whenever \( 0 < s < 1 \). Moreover, by (2.11) it converges when \( \xi \to +\infty \). Hence the integral in (1.11) is convergent for every \( C^2 \) function \( u \) satisfying (2.11).

The above calculation demonstrates that condition (3.1) is closely related to condition (3.2) for any \( s \in \mathbb{R} \), which is proved below.

Proof of Theorem 7.3 Let \( s \in \mathbb{R} \). Suppose \( r > 0 \) and \( \xi \geq 1 \). Dividing both sides of (3.1) by \( (\xi - 1)^2 \) and passing to the limit as \( \xi \to 1^+ \), we deduce (3.2).

Conversely, suppose that (3.2) holds, and \( u'(r) \geq 0 \). Let \( \gamma = n - 2s \). Making a substitution \( r = e^t, \xi = e^x \), where \( t \in \mathbb{R} \), and \( x \geq 0 \), we rewrite (3.1) in the equivalent form:

\[
v(t + x) - v(t) = e^{\gamma x} v(t - x) - v(t) \geq 0, \quad \text{for all} \ t \in \mathbb{R}, \ x \geq 0,
\]

where \( v(t) = u(e^t) \), while (3.2) is equivalent to

\[
v'' + \gamma v' \geq 0, \quad \text{for all} \ t \in \mathbb{R}.
\]

We next fix \( t \in \mathbb{R} \), and let \( \phi(x) = v(x + t) - v(t) \), where \( x \in \mathbb{R} \). Then clearly \( \phi(0) = 0, \phi'(0) = u'(t) \), and (3.4) is equivalent to

\[
\phi''(x) + \gamma \phi'(x) \geq 0, \quad \text{for all} \ x \in \mathbb{R}.
\]

We need to prove:

\[
\phi(x)e^{\gamma x} + \phi(-x) \geq 0, \quad x \geq 0,
\]

which is equivalent to (3.3). It is not difficult to deduce (3.6) from (3.5) using Gronwall’s inequality (see [Har], Sec. 3.1, Theorem 1.1) in the case \( \gamma > 0 \).

Let us prove this directly for all \( \gamma \in \mathbb{R} \). By (3.5), it follows that \( \phi'(x) + \gamma \phi(x) \) is non-decreasing on \( \mathbb{R} \). Since \( \phi(0) = 0 \), we obtain:

\[
\phi'(x) + \gamma \phi(x) \geq \phi'(0), \quad \text{for all} \ x \geq 0.
\]
Let
\[ \psi(x) = \phi(-x) + \phi(x)e^{\gamma x}, \quad x \in \mathbb{R}. \]
We next show that \(\psi'(x) \geq 0\) for all \(x \geq 0\). By (3.6),
\[ \psi'(x) = -\phi'(-x) + (\phi'(x) + \gamma \phi(x)) e^{\gamma x} \geq -\phi'(-x) + \phi'(0)e^{\gamma x}, \quad \text{for all } x \geq 0. \]
It remains to prove the inequality
(3.8) \[ \phi'(-x) \leq \phi'(0) e^{\gamma x}, \quad \text{for all } x \geq 0. \]
Let \(g(x) = e^{-\gamma x} \phi'(-x)\). Then
\[ g'(x) = -\gamma e^{-\gamma x} \phi'(-x) - e^{-\gamma x} \phi''(-x) = -e^{-\gamma x}(\phi''(-x) + \gamma \phi'(-x)) \leq 0, \]
by (3.5). Hence \(g\) is nonincreasing, and consequently \(g(x) \leq g(0)\) for all \(x \geq 0\). This proves (3.8). Thus \(\psi'(x) \geq 0\) for all \(x \geq 0\), and since \(\psi(0) = 0\), we deduce \(\psi(x) \geq 0\) for all \(x \geq 0\). This proves (3.6), which in its turn yields (3.1).

\[ \blacksquare \]

4. Examples of \(s\)-subharmonic functions and further remarks

We consider the fundamental solution of the \(k\)-Hessian operator \(F_k\),
\[ u(x) = -|x|^{-(\frac{\beta}{2})}, \quad x \in \mathbb{R}^n, \]
for \(1 \leq k \leq \frac{n}{2}\), which satisfies the equation
\[ F_k[u] = \delta_0, \]
in the viscosity sense (see [1], [W2], [W2]).

It is not difficult to verify directly that \(u\) is a radial function satisfying the condition
(4.1) \[ u(r) - u(r\xi) + (u(r) - u(\xi))|\xi|^{-n+2s} \leq 0, \quad \text{for all } r > 0, \xi \geq 1, \]
if \(s = \frac{n(k-1)}{2k} + 1\), or equivalently \(n-2s = \frac{n}{k} - 2\).

More generally, for \(\beta > 0\), let us consider the function
\[ u(x) = -|x|^{-\beta}, \quad x \in \mathbb{R}^n. \]
If \(\beta = n-2s\), we have:
\[ \frac{u(r) - u(r\xi)}{u(\xi) - u(r)} = \frac{-r^{-(n-2s)} + r^{-(n-2s)}\xi^{-(n-2s)}}{-r^{-(n-2s)}\xi^{n-2s} + r^{-(n-2s)}} = -1 + \frac{\xi^{-(n-2s)}}{-\xi^{n-2s} + 1} = \xi^{-(n-2s)}. \]
It follows that, for \(\beta = n-2s\) and \(s < \frac{n}{k}\), condition (4.1) turns into an equality for all \(r > 0, \xi \geq 1\). Unfortunately the \(k\)-energy of \(u(x) = -|x|^{-(\frac{n}{k}-2)}\) is unbounded, and so such functions cannot be used in Hessian inequalities of the type (1.4) or (1.5) discussed above.

This suggests considering the function
\[ f_\beta(x) = -(1 + |x|^{\frac{2}{k}})^{-\frac{\beta}{2}}, \quad x \in \mathbb{R}^n, \]
for \(\beta > 0\). When \(\beta = \frac{n}{k} - 2\), \(f\) is a \(k\)-convex function of finite \(k\)-energy. Moreover, \(f_\beta\) is known to be an extremal function for the Hessian Sobolev inequality (1.5) (see [W1], [W2]).

**Corollary 4.1.** Let \(s \in \mathbb{R}\) and \(\beta > 0\). Then \(f_\beta\) satisfies (1.8), and consequently (1.9), if and only if \(\beta \leq n-2s\). Moreover, if \(0 < s \leq 1\) and \(0 < \beta < n\), then \(f_\beta\) is \(s\)-subharmonic in \(\mathbb{R}^n\) if and only if \(\beta \leq n-2s\).
Proof. Let $r = |x|$. It is easy to see that
\[ f'_\beta(r) = -\beta r (r^2 + 1)^{-\frac{\beta}{2} - 1}, \quad f''_\beta(r) = \beta (r^2 + 1)^{-\frac{\beta}{2} - 2} [(\beta + 1)r^2 - 1]. \]
Hence,
\[ f''_\beta(r) + \frac{n - 2s + 1}{r} f'_\beta(r) = \beta (r^2 + 1)^{-\frac{\beta}{2} - 2} [(\beta - n + 2s)r^2 - (n - 2s + 2)]. \]
Suppose $0 < \beta < n - 2s$. Then the right-hand side of the preceding inequality is negative, and \((1.9)\) holds. If $0 < s < 1$, this implies that $f_\beta$ is $s$-subharmonic.

If $\beta > n - 2s$, then $f''_\beta(r) + \frac{n - 2s + 1}{r} f'_\beta(r)$ is either positive or changes sign. Hence, \((1.9)\) fails in this case.

Moreover, by formula \((6.5)\) discussed below,
\[ \phi(r) = -(-\Delta)^s f_\beta(r) = C(\alpha, \beta, n) F(a, b, c, -r^2), \]
where $C(\alpha, \beta, n)$ is a positive constant, and
\[ a = \frac{n + 2s}{2}, \quad b = \frac{2s + \beta}{2}, \quad c = \frac{n}{2}. \]
where $F(x) = \text{F}_{21}(a, b, c, x)$ is the hypergeometric function. Notice that $F(0) = 1$ and by a formula due to Gauss (see [AAR], Theorem 2.2.2, p. 66):
\[ F(1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \text{if } c > a + b. \]

By Pfaff’s transformation,
\[ F(a, b, c, -r^2) = (1 + r^2)^{-b} F(c - a, b, c, \frac{r^2}{r^2 + 1}). \]
It follows:
\[ F(-r^2) \sim (1 + r^2)^{-b} \frac{\Gamma(n)^2\Gamma\left(\frac{n - \beta}{2}\right)}{\Gamma(n + \frac{\beta}{2})\Gamma\left(\frac{n - 2s}{2} - s\right)} \quad \text{as } r \to +\infty. \]
Hence, when $n - 2s < \beta < n$, $0 < s < 1$, $\phi(r) < 0$ for $r$ large, and so changes sign. In other words, $f_\beta$ fails to be $s$-subharmonic in this case.

5. Applications

We first recall the following definitions. Let $s \in (0, 1]$. Let $u$ be any continuous (or more generally upper semicontinuous) function $u : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ such that
\[ \int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|^2)^{\frac{n + 2s}{2}}} dx < +\infty. \]
We say that $u$ is $s$-subharmonic in $\mathbb{R}^n$ if
\[ -(-\Delta)^s u \geq 0 \quad \text{in } D'(\mathbb{R}^n). \]
Analogously we shall say that $u$ is $s$-superharmonic in $\mathbb{R}^n$
\[ -(-\Delta)^s u \leq 0 \quad \text{in } D'(\mathbb{R}^n). \]
Whenever $u$ is both $s$-subharmonic and $s$-superharmonic in $\mathbb{R}^n$ we shall say that $u$ is $s$-harmonic in $\mathbb{R}^n$.

We remark that from the representation given in Theorem \((1.1)\) we can deduce a Liouville theorem for radial $s$-subharmonic functions.
Corollary 5.1. Assume that $u$ is a positive continuous function, radially decreasing and $s \in (0, 1]$. Suppose that
\[
\int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|^2)^{n+2s}} dx < +\infty,
\]
and $-(-\Delta)^s u \geq 0$. If $\lim_{r \to +\infty} u(r) = 0$ then $u = 0$.

Proof. We remark that if $u$ is radially decreasing then $u(r) \geq u(r\xi)$ and $u(r) \leq u(\xi)$ for every $\xi \geq 1$. Hence, recalling Theorem 1.2 inequality 1.3, for every $\xi \geq 1$ and for every positive $r$ it results
\[
\frac{u(r) - u(\xi)}{u(r\xi) - u(r)} \geq \xi^{n-2s}.
\]
On the other hand, for every fixed $r$, if $\xi \to +\infty$, then
\[
\lim_{\xi \to +\infty} \frac{u(r) - u(\xi)}{u(r\xi) - u(r)} = +\infty.
\]
We recall that $\lim_{r \to +\infty} u(r) = 0$, so we get
\[
\lim_{\xi \to +\infty} \frac{u(r) - u(0)}{-u(r)} = +\infty,
\]
which implies a contradiction whenever $u$ is bounded and not identically zero. \hfill $\square$

The following result gives a derivative rule for fractional Laplacians.

Theorem 5.2. Let $s \in [0, 1]$. For every differentiable radial function $u$ such that
\[
\int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|^2)^{n+2s}} dx < +\infty,
\]
the following formula holds:
\[
(5.1) \quad \frac{d(-\Delta)^s u(r)}{dr} = \frac{1}{r} (-\Delta)^s (-2su + ru').
\]
Proof. Differentiating (1.7), we obtain:
\[
\frac{d(-\Delta)^s u(r)}{dr} = -\frac{2s}{r} (-\Delta)^s u(r)
\]
\[
+ r^{-1-2s} \int_1^{+\infty} \left( (u'(r) - u'(r\xi))\xi + (u'(r) - \frac{u'(\xi)}{\xi})\xi^{-n+2s} \right) \xi(\xi^2 - 1)^{-1-2s} H(\xi) d\xi
\]
\[
= -\frac{2s}{r} (-\Delta)^s u(r)
\]
\[
+ r^{-2s} \int_1^{+\infty} \left( r^{-1} \left( ru'(r) - u'(r\xi)\xi + (ru'(r) - \frac{u'(\xi)}{\xi})\xi^{-n+2s} \right) \xi(\xi^2 - 1)^{-1-2s} H(\xi) d\xi
\]
\[
= -\frac{2s}{r} (-\Delta)^s u(r) + \frac{1}{r} (-\Delta)^s (ru'(r)) = \frac{1}{r} (-\Delta)^s (-2su + ru'),
\]
\hfill $\square$

If
\[
f_k(x) = -(1 + r^2)^{-\frac{2s}{2k} - 1},
\]
then
\[
f'_k(r) = 2\left(\frac{n}{2k} - 1\right)(1 + r^2)^{-\frac{2s}{2k} - 1} - \frac{2s}{2k} r.
\]
and recalling formula \((5.1)\) we get
\[
-2sf_k(r) + rf'_k(r) = 2s(1 + r^2)^{-\frac{n}{2k}} + 2\left(\frac{n}{2k} - 1\right) + (1 + r^2)^{-\frac{n}{2k}} r^2
\]
\[
= 2(1 + r^2)^{-\frac{n}{2k}} \left(s(1 + r^2) + \left(\frac{n}{2k} - 1\right)r^2\right) = 2(1 + r^2)^{-\frac{n}{2k}} \left(s + \left(\frac{n}{2k} - 1\right)r^2\right).
\]
Thus
\[
d\left(-\Delta\right)^s f_k(r) = \frac{1}{r} \left(-\Delta\right)^s (-2sf_k + rf'_k) = \frac{2}{r} \left(-\Delta\right)^s \left((1 + r^2)^{-\frac{n}{2k}} \left(s + \left(\frac{n}{2k} - 1\right)r^2\right)\right)
\]
\[
= \frac{2}{r} \left(-\Delta\right)^s \left((1 + r^2)^{-\frac{n}{2k}} \left(s + \left(\frac{n}{2k} - 1\right)r^2\right)\right)
\]
\[
+ \frac{2}{r} \left(-\Delta\right)^s \left((1 + r^2)^{-\frac{n}{2k}} \left(s - \left(\frac{n}{2k} - 1\right)r^2\right)\right) = \frac{2}{r} \left((1 + r^2)^{-\frac{n}{2k}} - k\right) \left(1 + r^2)^{-\frac{n}{2k}} - k\right).
\]

In the next theorem we deduce the maximum principle for radial \(s\)-subharmonic functions.

**Theorem 5.3.** Suppose that \(u\) is a radial \(s\)-subharmonic function \(s \in (0,1]\). Then either \(u\) is constant or \(u\) can not realize a maximum in \(\mathbb{R}^n\).

**Proof.** Indeed, by contradiction, suppose that \(u\) realizes an absolute maximum in \(\bar{r} \in \mathbb{R}^n\). Hence for every \(\xi \geq 1\), \(u(\bar{r}) - u(\bar{r} \xi) \geq 0\), \(u(\bar{r}) - u(\frac{\bar{r}}{\xi}) \geq 0\) and as a consequence
\[
u(\bar{r}) - u(\bar{r} \xi) + \left(u(\bar{r}) - u(\frac{\bar{r}}{\xi})\right) \xi^{2s-n} \geq 0.
\]
On the other hand
\[-(-\Delta)^s(\bar{r}) \geq 0,
\]
and in particular this implies \(u \equiv u(\bar{r})\) because \(u(\bar{r}) - u(\bar{r} \xi) \geq 0\) and \((u(\bar{r}) - u(\frac{\bar{r}}{\xi})) \geq 0\).

**Corollary 5.4.** If \(u\) is a radial \(s\)-harmonic function, then either \(u\) is constant or \(u\) does not realize either an absolute maximum or an absolute minimum.

**Corollary 5.5.** Suppose \(u_1\) and \(u_2\) are two radial \(s\)-harmonic functions. If \(u_1 \geq u_2\), then \(u_1 > u_2\), or \(u_1 \equiv u_2\), i.e., \(u_1\) cannot intersect with \(u_2\); otherwise \(u_1\) coincides with \(u_2\).

**Proof.** Indeed, if there exists \(\bar{r}\) such that \(u_1(\bar{r}) = u_2(\bar{r})\), then \(u_1 - u_2\) has a minimum in \(\bar{r}\) and \(u_1 - u_2\) is \(s\)-harmonic. Then \(u_1 \equiv u_2\) by the previous result.

**Corollary 5.6.** Suppose that \(u\) is a radial continuous \(s\)-harmonic function vanishing at infinity. Then \(u \equiv 0\).

**Proof.** If \(\sup_{\mathbb{R}^n} u = 0\) then \(u\) realizes its minimum in \(\mathbb{R}^n\). Then we can apply Corollary 5.4 obtaining that \(u\) is constant and necessarily \(u \equiv 0\). Analogously if \(\inf_{\mathbb{R}^n} u = 0\) then \(u\) has to attain its maximum in \(\mathbb{R}^n\) otherwise \(u \equiv 0\). Thus by Corollary 5.4 we conclude that \(u \equiv 0\).

**Corollary 5.7.** Let \(u_1, u_2\) be continuous radially symmetric functions such that
\[-(-\Delta)^s u_1 = (-\Delta)^s u_2.
\]
If \(\lim_{r \to +\infty} (u_1 - u_2) = 0\) then \(u_1 \equiv u_2\).
Proof. Recalling the linearity of the fractional Laplace operator we get:

\[ (-\Delta)^s(u_1 - u_2) = 0. \]

Moreover \( \lim_{r \to +\infty} (u_1 - u_2) = 0 \). It follows by Corollary 5.7 that \( u_1 \equiv u_2 \). \qed

Our results make it possible to provide an answer to a question which was left open in [FFV]. More precisely, in [FFV], Theorem 2.1, an inequality involving \( k \)-convex functions and the fractional Laplace operator was proved. For reader’s convenience we state below the result in [FFV].

**Proposition 5.8** (Ferrari-Franchi-Verbitsky). 1 \( \leq k < \frac{\alpha}{2} \), and let \( \alpha = \frac{2k}{k+1} \). Suppose \( u \in C^2(\mathbb{R}^n) \) is a \( k \)-convex function on \( \mathbb{R}^n \) vanishing at \( \infty \). If

(i) \(-(-\Delta)^{\alpha/2}u \geq 0 \),

(ii) \(-\Delta)^{\alpha/2}[-(-\Delta)^{\alpha/2}u]^k \geq 0 \),

then there exists a positive constant \( C_{k,n} \) such that

\[ \int_{\mathbb{R}^n} \left(-(-\Delta)^{\alpha/2}u\right)^{k+1} dx \leq C_{k,n} \int_{\mathbb{R}^n} -u F_k[u] dx. \] (5.2)

It was unclear if, for \( 1 \leq k < \frac{\alpha}{2} \), the set

\[ \mathcal{F}C_k = \{ u \in C^2(\mathbb{R}^n) : u, \ k - \text{convex, vanishing at } \infty, \ (i) \text{ and } (ii) \text{ hold} \}, \]

was not trivial. Obviously is not empty because for any \( k, \ 1 \leq k < \frac{\alpha}{2}, \ 0 \in \mathcal{F}C_k \).

Theorems 1.2 and 1.3 in this paper will be employed to give a positive answer to this question. Indeed,

\[ f_k(x) = -(1 + |x|^2)^{-(\frac{\alpha}{2} - 1)} \]

is \( k \)-convex, vanishing at \( \infty \), and \(-\Delta)^{\alpha/2}f_k \geq 0 \), for every \( k \in \mathbb{N} \) such that \( \frac{n}{2} \geq k \), i.e. \( f_k \)

satisfies (i). Moreover if \( k = 1 \), then \( \alpha = 1 \), and condition (ii) becomes

\[ (-\Delta)^{1/2}[-(-\Delta)^{1/2}f_1] = \Delta f_1 \geq 0. \]

The case \( k \geq 2 \), which is much harder, is considered in the next two sections.

6. The iterated fractional Laplacian condition

In this section we verify the iterated fractional Laplacian condition of Proposition 5.8

\[ (-\Delta)^{\alpha/2}[-(-\Delta)^{\alpha/2}u]^k \geq 0, \] (6.1)

where \( \alpha = \frac{2k}{k+1} \), \( 1 \leq k < \frac{\alpha}{2} \), for the \( k \)-convex function \( u(x) = -(1 + |x|^2)^{-\beta/2} \) on \( \mathbb{R}^n \) with \( \beta = \frac{n}{2} - 2 > 0 \). As explained above, the case \( k = 1 \) is trivial, and from now on we will assume \( k \geq 2 \). We observe that \( u \) is an extremal function in the important Hessian Sobolev inequality of X.-J. Wang [W1], [W2]. The Fourier transform of \( u \) is given by the radially symmetric function

\[ \hat{u}(\xi) = -C(\beta, n)\frac{K_{\frac{n-\beta}{2}}(|\xi|)}{|\xi|^{n-\beta}}, \ \xi \in \mathbb{R}^n, \] (6.2)

where \( C(\beta, n) \) is a positive constant, and \( K_{\gamma} \) is the modified Bessel function of order \( \gamma \).

It is possible to express \(-\Delta)^{\alpha/2}u \) in terms of the Gaussian hypergeometric function

\[ F(a, b, c, x) = _2F_1(a, b, c, x) \]

discussed in the next section. For radially symmetric functions \( f(r) \), where \( r = |x| \), the Fourier transform formula can be stated in the form:

\[ \hat{f}(s) = (2\pi)^{n/2} \int_0^{+\infty} J_{\frac{n-\alpha}{2}}(sr) \frac{f(r)}{r^{n-1}} dr, \] (6.3)
where $s = |\xi|$. Hence, from the preceding formulas we deduce:

\[
(6.4) \quad -(-\Delta)^{\alpha/2}u(s) = C(\alpha, \beta, n)\int_0^{+\infty} \phi(r) J_{\alpha/2} (sr) r^{\beta+\alpha} dr,
\]

where $C(\alpha, \beta, n)$ is a positive constant.

Using the well-known integral involving the product of Bessel functions $J_\alpha$ and modified Bessel functions $K_\gamma$, we obtain the explicit formula for $(-\Delta)^{\alpha/2}u$ (see e.g., [MOS], Sec. 3.8, p. 100):

\[
(6.5) \quad -(-\Delta)^{\alpha/2}u(x) = C(\alpha, \beta, n) F(a, b, c, -|x|^2), \quad x \in \mathbb{R}^n,
\]

where $C(\alpha, \beta, n)$ is a positive constant. The parameters $a, b$, and $c$ are given by:

\[
(6.6) \quad a = \frac{n + \alpha}{2}, \quad b = \frac{\alpha + \beta}{2}, \quad c = \frac{n}{2},
\]

where

\[
(6.7) \quad \alpha = \frac{2k}{k+1}, \quad \beta = \frac{n}{k} - 2, \quad n \geq 2k, \quad k \geq 2.
\]

Notice that $a > c > b > 0$ and $a > b + 1$ ($a = b + 1$ when $k = 1$) in our case. To verify (6.1), we need to demonstrate:

\[
(6.8) \quad (-\Delta)^{\alpha/2} \left[ F(a, b, c, -|x|^2) \right]^k \geq 0.
\]

Let

\[
(6.9) \quad \phi(x) = F(a, b, c, -|x|^2), \quad x \in \mathbb{R}^n.
\]

By (6.4), $\phi$ clearly satisfies condition (1.6). Invoking Theorems 1.3 and 1.2, we see that (6.8) follows from condition (1.9):

\[
((\phi(r))^k)'' + \frac{n - \alpha + 1}{r} ([\phi(r)]^k)' \leq 0, \quad r \geq 0.
\]

By direct differentiation,

\[
((\phi(r))^k)'' + \frac{n - \alpha + 1}{r} ([\phi(r)]^k)''' = k[\phi(r)]^{k-1} \left[ \phi''(r) + (k-1) \frac{[\phi'(r)]^2}{\phi(r)} + \frac{n - \alpha + 1}{r} \phi'(r) \right].
\]

Hence it suffices to verify the inequality

\[
(6.10) \quad \phi''(r) + (k-1) \frac{[\phi'(r)]^2}{\phi(r)} + \frac{n - \alpha + 1}{r} \phi'(r) \leq 0, \quad r > 0.
\]

Since $\phi(r) = F(-r^2)$, where $F(x) = F(a, b, c, x)$, we have $\phi'(r) = -2rF'(-r^2)$ and $\phi''(r) = 4r^2F''(-r^2) - 2F'(-r^2)$. It follows that the preceding inequality is equivalent to:

\[
(6.11) \quad F''(-r^2) + (k-1) \frac{[F'(-r^2)]^2}{F(-r^2)} - \frac{n - \alpha + 2}{2r^2} F'(-r^2) \leq 0, \quad r > 0.
\]

Letting $x = r^2$, $x > 0$, and noticing that $\frac{n - \alpha + 2}{2} = 2c - a + 1$ by (6.6), we rewrite the preceding inequality as:

\[
(6.12) \quad F''(-x) + (k-1) \frac{[F'(-x)]^2}{F(-x)} - \frac{2c - a + 1}{x} F'(-x) \leq 0, \quad x > 0.
\]

Using the hypergeometric equation ([AAR], p. 75):

\[
(6.13) \quad x(1 + x)F''(-x) - (c + (a + b + 1)x) F'(-x) + abF(-x) = 0,
\]

we eliminate the second derivative in (6.12):

\[
(k-1)[F'(-x)]^2 + \left[ \frac{c + (a + b + 1)x}{x(1 + x)} - \frac{2c - a + 1}{x} \right] F'(-x) F(-x) - \frac{ab}{x(1 + x)} [F(-x)]^2 \leq 0.
\]
Letting \( \lambda = \frac{F'(x)}{F(x)} \), where \( x \geq 0 \), we reduce the preceding estimate to the quadratic inequality

\[
(6.14) \quad x(1 + x)(k - 1)\lambda^2 + [(2a + b - 2c)x + a - c - 1] \lambda - ab \leq 0,
\]

for \( x > 0 \). From the differentiation formula ([AAR], p. 94),

\[
(6.15) \quad F'(a, b, c, x) = \frac{ab}{c} F(a + 1, b + 1, c + 1, x),
\]

it follows:

\[
(6.16) \quad \lambda = \frac{ab F(a + 1, b + 1, c + 1, -x)}{F(a, b, c, x)} > 0, \quad x > 0.
\]

Solving inequality (6.14) for \( \lambda > 0 \), we obtain:

\[
(6.17) \quad 0 < \lambda \leq \frac{-(d_1 x - d_2) + \sqrt{(d_1 x - d_2)^2 + d_3 x(1 + x)}}{2(k - 1)x(1 + x)}, \quad x \geq 0.
\]

where

\[
(6.18) \quad d_1 = 2a + b - 2c, \quad d_2 = c - a + 1, \quad d_3 = 4ab(k - 1) = 4a(2c - a - b)
\]

by (6.7). Equivalently,

\[
(6.19) \quad 0 < \lambda \leq \frac{2ab}{d_1 x - d_2 + \sqrt{(d_1 x - d_2)^2 + d_3 x(1 + x)}}, \quad x > 0,
\]

Using (6.16), we see that (6.14) is equivalent to:

\[
(6.20) \quad \frac{F(a, b, c, -x)}{F(a + 1, b + 1, c + 1, -x)} \geq \frac{d_1 x - d_2 + \sqrt{(d_1 x - d_2)^2 + d_3 x(1 + x)}}{2c}, \quad x > 0.
\]

We next use Pfaff’s transformation ([AAR], Theorem 2.2.5, p. 68):

\[
(6.21) \quad F(a, b, c, y) = (1 - y)^{-b} F(c - a, b, c, \frac{y}{y - 1}), \quad \text{for } y < 1.
\]

Letting \( y = -x \), \( x \geq 0 \), and applying (6.21) to both \( F(a, b, c, -x) \) and \( F(a + 1, b + 1, c + 1, -x) \), we get

\[
(6.22) \quad \frac{F(a, b, c, -x)}{F(a + 1, b + 1, c + 1, -x)} = (1 + x) \frac{F(c - a, b, c, \frac{x}{x+1})}{F(c - a, b + 1, c + 1, \frac{x}{x+1})}, \quad x > 0.
\]

Letting \( t = \frac{x}{x+1} \), we rewrite (6.20) in the equivalent form:

\[
(6.23) \quad \frac{F(c - a, b, c, t)}{F(c - a, b + 1, c + 1, t)} \geq \frac{d_1 t - d_2 (1 - t) + \sqrt{(d_1 t - d_2 (1 - t))^2 + d_3 t}}{2c}, \quad 0 < t \leq 1,
\]

where \( d_1, d_2, d_3 \) are given by (6.18). We note that since \( (c-a) + b < c \) and \( (c-a) + (b+1) < c+1 \), it follows that the both hypergeometric functions in the ratio on the left-hand side are finite at \( t = 1 \). By (6.3),

\[
(6.24) \quad F(c - a, b, c, 1) = \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)}, \quad F(c - a, b + 1, c + 1, 1) = \frac{\Gamma(c + 1)\Gamma(a - b)}{\Gamma(a + 1)\Gamma(c - b)}.
\]

Consequently,

\[
(6.25) \quad \frac{F(c - a, b, c, 1)}{F(c - a, b + 1, c + 1, 1)} = \frac{a}{c}.
\]

In fact, since \( a > b + 1 \) for \( k \geq 2 \), both hypergeometric functions in the ratio are twice continuously differentiable for \( t \leq 1 \), including the endpoint \( t = 1 \) when (see [AAR], Theorem 2.3.2, p. 78).
To prove (6.23), we express it in the form:

\[(6.26) \quad f(t) \geq g(t), \quad 0 \leq t \leq 1,\]

where \(f\) is the Gaussian ratio:

\[(6.27) \quad f(t) = \frac{F(c - a, b, c, t)}{F(c - a, b + 1, c + 1, t)}, \quad -\infty < t \leq 1,\]

and

\[(6.28) \quad g(t) = \frac{d_1 t - d_2 (1 - t) + \sqrt{(d_1 t - d_2 (1 - t))^2 + d_3 t}}{2c}, \quad t \geq 0.\]

The proof of (6.26) relies on the important observation that \(f\) is convex (in fact, even logarithmically convex, as is proved in the next section) while \(g\) is concave:

**Lemma 6.1.** The function \(g\) defined by (6.28) is concave on \([0, +\infty)\).

**Proof.** Let \(h(t) = \sqrt{At^2 + Bt + C}\), then

\[h'(t) = \frac{2At + B}{2\sqrt{At^2 + Bt + C}}, \quad \text{and} \quad h''(t) = \frac{4AC - B^2}{4(At^2 + Bt + C)^{3/2}}.\]

Hence, \(h\) is concave if \(B^2 - 4AC \geq 0\). Let \(At^2 + Bt + C = (d_1 t - d_2 (1 - t))^2 + d_3 t\), where \(d_1, d_2, d_3\) are defined by (6.18). Then clearly, \(A = (d_1 - d_2)^2, B = d_3 - 2d_2 (d_1 + d_2), C = d_2^2,\) and \(At^2 + Bt + C \geq 0\) for \(t \geq 0\) since \(d_3 \geq 0\). Clearly,

\[B^2 - 4AC = [d_3 - 2d_2 (d_1 + d_2)]^2 - 4(d_1 + d_2)^2 d_2^2 = d_3[d_3 - 4d_2 (d_1 + d_2)],\]

where

\[d_3 - 4d_2 (d_1 + d_2) = a(2c - a - b) - (c - a + 1)(a + b - c + 1) = (c + 1)(c - b - 1) \geq 0.\]

Notice that \(c - b - 1 \geq 0\). Indeed, from (6.7) it follows that \(n(k - 1) \geq 2k\) since \(n \geq 2k\) and \(k \geq 2\). Hence,

\[(6.29) \quad c - b - 1 = \frac{n(k - 1) - 2k + \alpha}{2k} > \frac{n(k - 1) - 2k}{2k} \geq 0.\]

Thus, \(B^2 - 4AC > 0\), and consequently \(h\) is concave. Since \(g'' = \frac{1}{2c} h'' < 0\), it follows that \(g\) is concave as well. \(\Box\)

Since \(f\) is convex by Theorem 7.1 below, and \(g\) is concave by Lemma 6.1, it follows that the graph of \(f\) lies above the tangent line at \(t = 1\), while the graph of \(g\) lies below the tangent line at \(t = 1\). In other words, \(f(t) \geq f(1) + (t - 1) f'(1)\), while \(g(t) \leq g(1) + (t - 1) g'(1)\) for \(0 \leq t \leq 1\). Recall that by (6.25), \(f(1) = \frac{a}{c}\). Notice that \(g(1) = \frac{a}{c}\) as well. Indeed,

\[g(1) = \frac{d_1 + \sqrt{d_1^2 + d_3}}{2c},\]

where

\[d_1^2 + d_3 = a^2 - 2a(2c - a - b) + (2c - a - b)^2 + 4a(2c - a - b)\]
\[= a^2 + 2a(2c - a - b) + (2c - a - b)^2 = (2c - b)^2.\]

Since \(2c > b\) by (6.7), we obtain:

\[(6.30) \quad \sqrt{d_1^2 + d_3} = 2c - b,\]

and consequently, by (6.18),

\[g(1) = \frac{d_1 + 2c - b}{2c} = \frac{2a + b - 2c + 2c - b}{2c} = \frac{a}{c}.\]
Hence,
Simplifying further the right-hand side, we rewrite it as follows:

\begin{equation}
 f'(1) = \frac{a(a - c)}{c(a - b - 1)}.
\end{equation}

Invoking the differentiation formula (6.15), we obtain:

\begin{equation}
 f'(t) = \frac{(c - a) b}{c} \frac{F(c - a + 1, b + 1, c + 1, t)}{F(c - a, b + 1, c + 1, t)} - \frac{(c - a)(b + 1)}{c + 1} \frac{F(c - a + 1, b + 2, c + 2, t)}{F(c - a, b + 1, c + 1, t)}.
\end{equation}

Using (6.30), we deduce from the preceding formula:

\begin{equation}
 f'(t) = \left(\frac{(c - a) b}{c} \frac{F(c - a + 1, b + 1, c + 1, 1)}{F(c - a, b + 1, c + 1, 1)} - \frac{(c - a)(b + 1)}{c + 1} \frac{F(c - a + 1, b + 2, c + 2, 1)}{F(c - a, b + 1, c + 1, 1)}\right).
\end{equation}

Applying Gauss’s formula (6.24) for \( F(a, b, c, 1) \) \((\text{AAR}, \text{Theorem 2.2.2, p. 66})\), we obtain

\begin{equation}
 f'(1) = \frac{(c - a) b}{c} \frac{\Gamma(c + 1) \Gamma(a - b - 1) \Gamma(a + 1) \Gamma(c - b)}{\Gamma(a) \Gamma(c - b) \Gamma(c + 1) \Gamma(a - b)} - \frac{a(c - a)(b + 1)}{c + 1} \frac{\Gamma(c + 2) \Gamma(a - b - 1) \Gamma(a + 1) \Gamma(c - b)}{\Gamma(a + 1) \Gamma(c - b) \Gamma(c + 1) \Gamma(a - b)} - \frac{(c - a) ab}{c(a - b - 1)} - \frac{a(c - a)(b + 1)}{c(a - b - 1)}.
\end{equation}

This proves (6.31).

On the other hand, by direct differentiation,

\begin{equation}
 g'(t) = \frac{1}{2c} \left( d_1 + d_2 + \frac{2(d_1 + d_2)(d_1 t - d_2(1 - t)) + d_3}{2\sqrt{(d_1 t - d_2(1 - t))^2 + d_3}} \right).
\end{equation}

Hence,

\begin{equation}
 g'(1) = \frac{1}{2c} \left( d_1 + d_2 + \frac{2(d_1 + d_2)(d_1 + d_3)}{2\sqrt{d_1^2 + d_3}} \right).
\end{equation}

We next show that, after simplification, we get:

\begin{equation}
 g'(1) = \frac{a(c + 1)}{c(2c - b)}.
\end{equation}

To prove this, recall that \( \sqrt{d_1^2 + d_3} = 2c - b \) by (6.30), and \( d_1 = 2a + b - 2c = a - (2c - a - b) \), \( d_1 + d_2 = a + b - c + 1 \), and \( d_3 = 4a(2c - a - b) \) by (6.18). It follows:

\begin{equation}
 g'(1) = \frac{1}{2c} \left( a + b - c + 1 + \frac{(2a + b - 2c)(a + b - c + 1) + 2a(2c - a - b)}{2c - b} \right).
\end{equation}

Simplifying further the right-hand side, we rewrite it as follows:

\begin{equation}
 g'(1) = \frac{(2a + b - 2c)(a + b - c + 1) + 2a(2c - a - b)}{2c(2c - b)}.
\end{equation}

This proves (6.34).

Now it is not difficult to see that \( g'(1) \geq f'(1) \), i.e.,

\begin{equation}
 \frac{a(a - c)}{c(a - b - 1)} \leq \frac{a(c + 1)}{c(2c - b)}.
\end{equation}
Indeed, by \((6.7)\), \(2c > b > 0 \) and \(a - b - 1 = \frac{n(k-1)}{2k} > 0\) for \(k \geq 2\). Hence \((6.35)\) is equivalent to:

\[
(a - c)(2c - b) - (c + 1)(a - b - 1) \leq 0.
\]

By factoring, we rewrite this as:

\[
(a - c)(2c - b) - (c + 1)(a - b - 1) = ac - 2c^2 - ab + 2bc + c - a + b + 1
\]

\[
= c(a - 2c) - b(a - 2c) - (c - b - 1) = (a - 2c - 1)(c - b - 1) \leq 0.
\]

By \((6.29)\), \(c - b - 1 \geq 0\). On the other hand, since \(1 < a < 2\), we have:

\[
a - 2c - 1 = -\frac{n + 2 - \alpha}{2} < 0,
\]

This proves \((6.35)\), and consequently \(g'(1) \geq f'(1)\), which yields \((6.26)\).

These computations have been verified using \textit{Mathematica}, which was also used to check that inequality \((6.23)\) holds for many concrete values of \(k\) and \(n\). In the next section, we will complete the proof of this estimate by showing that \(f(t)\) is convex if \(t \leq 1\).

\textbf{Remark 6.2.} When \(k = 1\) and \(\alpha = 1\), the function \(u(x) = -(\Delta)^{\frac{1}{2}}(1 + |x|^2)^{2-n}, \ n \geq 3,\) is \(\frac{1}{2}\)-subharmonic, i.e. \((6.1)\) holds, but inequality \((1.9)\) with \(s = \frac{1}{2}\) fails for \(r\) large. In other words, \((1.9)\) is sufficient but not necessary for a \(C^2\) radial function to be \(s\)-subharmonic.

Indeed, if \(k = 1\), then \(a = b + 1\), and consequently by \((6.24)\) and \((6.31)\), it follows that \(f'(1) = +\infty\), where

\[
f(t) = \frac{F(a, b, c, t)}{F(a, b + 1, c + 1, t)}, \quad -\infty < t \leq 1,
\]

Note that inequality \((6.12)\), and hence \((6.14)\), is equivalent to \((6.23)\), i.e., \(f(t) \geq g(t)\) for \(t \leq 1\), where \(g(t)\) is defined by \((6.28)\) with \(d_3 = 0\). Since \(f(1) = g(1) = \frac{a}{c}\), and \(g'(1) < +\infty\) by \((6.31)\), this contradicts \(f'(1) = +\infty\).

\section{Logarithmic convexity of the ratio of hypergeometric functions}

Let \(F(a, b, c, x) = {}_2F_1(a, b, c, x)\) denote the hypergeometric function. We refer to \textit{AAR} and \textit{MOS} for the relevant theory. For \(c > b > 0\), it can be defined by \((\text{AAR}, \ p, \ 65)\):

\[
(7.1) \quad F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 s^{b-1}(1-s)^{c-b-1}\frac{(1-sz)^a}{(1-z)^n}ds,
\]

in the complex plane with a cut from 1 to \(\infty\) along the positive real axis. We will show in this section that the Gaussian ratio \(\varphi(x) = \frac{F(a, b, c, x)}{F(a, b + 1, c + 1, x)}\) is logarithmically convex, and hence convex, for \(x \in (-\infty, 1)\) if \(c > b > 0\) and \(-1 < a < 0\). Since \(a + b < c\), it follows by \((6.24)\) that \(\varphi\) is continuous at \(x = 1\) as well, and hence \(\varphi\) is convex in \((-\infty, 1]\), including the end-point. Applying this to \(f(x) = \frac{F(c-a, b, c, x)}{F(c-a, b + 1, c + 1, x)}\) where \(a, b, c\) are given by \((6.6)\) so that \(c > b > 0\), and \(c - a = -\frac{2}{c} \in (-1, 0)\), we will conclude the proof of Theorem \((1.4)\).

The proof of the following theorem employs a method developed recently in \textit{KS1} (see also \textit{KS2}) to prove monotonicity and log-concavity in parameters \(a, b, c\) of generalized hypergeometric functions and their ratios.

\textbf{Theorem 7.1.} Let \(-1 < a < 0\) and \(c > b > 0\). Then the function \(\varphi(x) = \frac{F(a, b, c, x)}{F(a, b + 1, c + 1, x)}\) is logarithmically convex in \((-\infty, 1)\).

\textbf{Proof.} Let \(x \in (-\infty, 1)\). Let \(f_1(x) = F(a, b, c, x)\) and \(f_2(x) = F(a, b + 1, c + 1, x)\). Then \(\log \varphi'(x) = \frac{f_1'(x)}{f_1(x)} - \frac{f_2'(x)}{f_2(x)},\) and

\[
(\log \varphi)''(x) = \frac{f_1''(x)}{f_1(x)} - \frac{f_2''(x)}{f_2(x)} - \left(\frac{f_1'(x)}{f_1(x)}\right)^2 - \left(\frac{f_2'(x)}{f_2(x)}\right)^2
\]
Clearly, \( f_1(x) > 0 \) and \( f_2(x) > 0 \) by (6.15). Since \( a < 0 \) and \( b > 0, c > 0 \), it follows by the differentiation formula (6.15):

\[
f_1'(x) = \frac{ab}{c} F(a + 1, b + 1, c + 1, x) < 0, \quad f_2'(x) = \frac{ab}{c} F(a + 1, b + 2, c, x) < 0,
\]

and consequently,

\[
f_1'(x) f_2(x) + f_2'(x) f_1(x) < 0, \quad -\infty < x < 1.
\]

Hence to show that \((\log \varphi)''(x) > 0\), it suffices to prove the following inequalities:

\[
(7.2) \quad f_1(x) f_2(x) - f_2'(x) f_1(x) > 0, \quad f_1''(x) f_2(x) - f_2''(x) f_1(x) > 0, \quad -\infty < x < 1.
\]

It will be more convenient to make a substitution \( t = -x \), and work with \( g_1(t) = f_1(-t), g_2(t) = f_2(-t) \) for \( t \in (-1, +\infty) \). Then the preceding inequalities are equivalent to:

\[
(7.3) \quad g_1'(t) g_2(t) - g_2'(t) g_1(t) < 0, \quad g_1''(t) g_2(t) - g_2''(t) g_1(t) > 0, \quad -1 < t < +\infty.
\]

We next employ the integral representation (7.1) to express both \( g_1 \) and \( g_2 \) in the form of Stieltjes-type transformations:

\[
g_1(t) = \int_0^1 s^{b-1} (1 + st)^{-a} w(s) ds, \quad g_2(t) = \frac{c}{b} \int_0^1 s^{b} (1 + st)^{-a} w(s) ds, \quad -1 < t < +\infty,
\]

where

\[
w(s) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} (1 - s)^{c-b-1}, \quad 0 < s < 1,
\]

is a positive weight function. Then we have:

\[
g_1'(t) = -a \int_0^1 s^{b} (1 + st)^{-a-1} w(s) ds, \quad g_2'(t) = -a \frac{c}{b} \int_0^1 s^{b+1} (1 + st)^{-a-1} w(s) ds,
\]

\[
g_1''(t) = a(a + 1) \int_0^1 s^{b+1} (1 + st)^{-a-2} w(s) ds, \quad g_2''(t) = a(a + 1) \frac{c}{b} \int_0^1 s^{b+2} (1 + st)^{-a-2} w(s) ds,
\]

where \(-1 < t < +\infty\).

To prove the first inequality in (7.3), i.e., \( g_1'(t) g_2(t) < g_2'(t) g_1(t) \), we rewrite it in the equivalent form:

\[
-a \frac{c}{b} \left( \int_0^1 s^{b} (1 + st)^{-a-1} w(s) ds \right) \left( \int_0^1 s^{b} (1 + st)^{-a} w(s) ds \right)
\]

\[
< -a \frac{c}{b} \left( \int_0^1 s^{b+1} (1 + st)^{-a-1} w(s) ds \right) \left( \int_0^1 s^{b-1} (1 + st)^{-a} w(s) ds \right).
\]

Similarly, the second inequality in (7.3), i.e., \( g_1''(t) g_2(t) > g_2''(t) g_1(t) \), is equivalent to:

\[
a(a + 1) \frac{c}{b} \left( \int_0^1 s^{b+1} (1 + st)^{-a-2} w(s) ds \right) \left( \int_0^1 s^{b} (1 + st)^{-a} w(s) ds \right)
\]

\[
> a(a + 1) \frac{c}{b} \left( \int_0^1 s^{b+2} (1 + st)^{-a-2} w(s) ds \right) \left( \int_0^1 s^{b-1} (1 + st)^{-a} w(s) ds \right).
\]

Fix \( t \in (-1, +\infty) \). Since \( c > b > 0 \), and \(-1 < a < 0\), the first inequality above can be expressed in the form (see [KSI], p. 341):

\[
(7.4) \quad \left( \int_0^1 h(s) p(s) ds \right) \left( \int_0^1 q(s) p(s) ds \right) < \left( \int_0^1 q(s) h(s) p(s) ds \right) \left( \int_0^1 p(s) ds \right),
\]

where \( h(s) = \frac{s}{1 + st}, p(s) = s^{b-1}(1 + st)^{-a}w(s), q(s) = s \) and \( w(s) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} (1 - s)^{c-b-1} \).
Analogously, the second inequality is equivalent to:

\begin{equation}
\left( \int_0^1 h_1(s) p(s) \, ds \right) \left( \int_0^1 q(s) p(s) \, ds \right) < \left( \int_0^1 q(s) h_1(s) p(s) \, ds \right) \left( \int_0^1 p(s) \, ds \right),
\end{equation}

where \( h_1(s) = \frac{s^2}{(1+st)^2} \), \( p(s) \), \( q(s) \) and \( w(s) \) are as above. Since \( p(s) > 0 \), and the functions \( q(s) \), \( h(s) \), and \( h_1(s) \) are increasing on \((0,1)\) for any fixed \( t > -1 \), both \((7.4)\) and \((7.5)\) follow from Chebyshev’s inequality for monotone functions (see [HLP]).

**Remark 7.2.** The condition \( c > b > 0 \) in Theorem 7.1 can be extended to \( c > \min(a,b) > 0 \) using the symmetry of \( F(a,b,c,x) \) in \( a \) and \( b \).

**Remark 7.3.** Theorem 7.1 holds for the ratio of hypergeometric functions \( q+1F_q((a,b),c,x) \) in place of \( 2F_1(a,b,c,x) \), where \(-1 < a < 0 \) and \( c_k > b_k > 0 \), \( k = 1,2,\ldots,q \), \( q \geq 2 \).

The proof of the logarithmic convexity of the ratio of \( q+1F_q \) is similar to the proof of Theorem 7.1 using the corresponding Stieltjes type integral representation ([KSI], Lemma 1).

**Corollary 7.4.** Let \( 0 < a - c < 1 \), and \( c > b > 0 \). Then the ratio \( \frac{F(a,b,c,x)}{F(a+1,b+1,c+1,x)} \) is convex in \((-\infty,1)\).

**Proof.** Let \( \varphi_1(-x) = \frac{F(a,b,c,x)}{F(a+1,b+1,c+1,x)} \), \( x < 1 \). By \((6.22)\),

\[ \varphi_1(x) = (1 + x) \varphi_0\left( \frac{x}{x+1} \right), \quad x > -1, \]

where

\[ \varphi_0(y) = \frac{F(c-a,b,c,y)}{F(c-a,b+1,c+1,y)}, \quad y < 1. \]

We deduce:

\[ \varphi_1'(x) = \varphi_0\left( \frac{x}{x+1} \right) + \varphi_0'\left( \frac{x}{x+1} \right) \frac{1}{1 + x}, \quad \varphi_1''(x) = \varphi_0''\left( \frac{x}{x+1} \right) \frac{1}{(x+1)^3} \geq 0, \]

where \( \varphi_0 \) is logarithmically convex by Theorem 7.1 and consequently convex, so that \( \varphi_0''(y) \geq 0 \), \( y < 1 \). Then, obviously, \( \varphi_1(-x) \) is convex as well. \(\Box\)

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