Social Security Benefit Valuation, Risk, and Optimal Retirement

Yassmin Ali¹, Pablo A. A. Sota¹, Ming Taylor¹, Stephen Taylor *¹,², and Xun Wang¹

¹New Jersey Institute of Technology, Martin Tuchman School of Management, 3000 Central Avenue Building (CAB), Newark, New Jersey 07102, USA
²Visiting Asst. Prof. Department of Probability and Mathematical Statistics, Faculty of Mathematics and Physics, Charles University, Sokolovska 83, 186 75 Prague, Czech Republic.

August 30, 2019

Abstract

We develop techniques to estimate the present day value of the future social security benefits of a retiree based upon their chosen date of retirement, the term structure of interest rates, and life expectancy forecasts. These valuation methods are then used to determine the optimal retirement time of a beneficiary given a specific wage history and health profile in the sense of maximizing the present day value of future cashflows. We then examine how a number of risk factors including interest rates, disease diagnosis, and population life table risks impact the current value of future payments. Specifically, we utilize principal component analysis in order to assess interest rate and population life expectancy variation risks. We then examine how such risks range over distinct income and demographic groups and finally summarize future research directions.

Keywords: social security, principal component analysis, pension risk.

1 Introduction and Overview

Labor force participants typically rely upon accumulated wealth during their working years to finance retirement. Regular deductions are taken from American employee wages to fund a variety of retirement oriented investment vehicles including mandatory programs such as Social Security and state administered 401(A) plans. In addition, employees may also supplement retirement savings through voluntarily contribution programs, the most prominent being employer sponsored 401(k) plans. Finally, excess savings tend to be invested in an individual’s discretion between investment retirement, personal brokerage, or retail banking accounts. Given all these options, social security retirement benefits are the most important form of savings for future retirement expenses for the majority of Americans [10].

While planning for retirement and allocating capital between different investment options, it is crucial to estimate the present day value of each investment under consideration. For non-tradeable fixed income securities such as defined benefit plans, this may be achieved by discounting future cashflows one expects to receive. Pension benefits often comprise a substantial portion of the fixed income component of an

*Corresponding author. smt-at-njit-dot-edu

Electronic copy available at: https://ssrn.com/abstract=3438080
individual’s retirement portfolio, which should be considered in tandem with securities held say in a brokerage account when designing a retirement portfolio from a holistic perspective. However, valuing pension benefits involves several additional complications over traditional fixed income securities. The most notable feature of pension valuation is that the date of death of the beneficiary is unknown and it must be carefully modeled as under or over estimation will significantly influence the number of future pension related cashflows received and as a result the estimated present day value of all such cashflows.

We focus on the valuation of the future benefits from a pensioner’s perspective below and do not consider broader program solvency issues. A number of approaches have been considered for pension valuation. In [28], the authors develop stochastic process based techniques to model the relationship between a worker’s wage history, the risk free rate, and pension value. Public pension future liabilities for distinct age groups are considered in [20] under different accrual methods which are aggregated to determine program level liabilities. Program cost differences between pensioners initiating benefits at early and late ages are considered in [19]. In [27], the authors examine pension valuation in the setting of studying the opportunity cost of continued work versus when to retire and receive payouts from a single firm’s pension plan. Longevity risk is considered in [11], where the authors use annuity valuation techniques to argue that state level mandates to annuitize retirements assets can be beneficial for both adverse asset selection and longevity risk of an individual. In [6], similar annuity valuation questions are considered in the context of longevity insurance varying over different cohorts with distinct mortality rates.

We proceed in a similar manner to a number of these works by studying defined benefit plan value from the point of view of the beneficiary receiving a future stream of cashflows from retirement until death. In particular, the motivating application we consider will be to determine the optimal retirement date and pension value for a Social Security beneficiary based on their wage history and life expectancy in addition to external factors such as the term structure of interest rates. We note however that the valuation and risk techniques developed below are not limited only to defined benefit plans and have applicability to a wide range of fixed income investments. In addition, we will incorporate granular lifetable information into the forecasting model developed in [18] in order to project future life expectancy of the beneficiary on an annual basis, which substantively impacts pension present day value. Finally, we develop a PCA based model to understand the impact of interest rate risk on pension value and also develop a health risk modeling framework to enable a pensioner to gauge the change in present day value of future pension cashflows after contracting a disease.

Life expectancy forecasting is a widely studied topic, c.f. [4] for a survey. The Lee-Carter model [15] became a widely utilized technique for lifetable forecasting especially after the Social Security Administration’s adoption, this model for their internal program solvency studies as well as its utilization by the Census Bureau for population forecasting [14]. Since the publication of this seminal work, a number of wide ranging extensions have been developed which for example involve restricting to specific cohorts as in [22], or extending to more complex predictive models with the aim of achieving stronger predictive performance [23]. We will consider one such minimal extension developed by [18] where the authors remain within the Lee-Carter framework; however, demonstrate that if one estimates the model using mortality rate changes as opposed to absolute rates as originally specified, significant performance improvements may be realized while retaining the complexity of the original model. We apply this refined model to yearly lifetable data below to multiple cohorts in below applications for the purpose of forecasting pensioner life expectancy.

In addition, we consider the two major risk factors associated with pension value which include interest rate and health risk. Principal component analysis (PCA) is a widely utilized technique to decompose the time variation of interest rate and commodity curves [8, 16] into their main directions of variation. We first perform a PCA on the discount curve used to present day value future pension cashflows in order to understand the direction of maximal interest rate risk. Then we consider scenario analyses by perturbing
the current discount curve in this direction in proportion to the factors’ observed standard deviation and estimate associated changes in pension value for several risk scenarios. Secondly, we examine health related risks including the event that a beneficiary contracts a mortal disease. In particular, given a survival curve conditional on the pensioner’s age and severity of diagnosis, we estimate an associated updated life expectancy distribution and compare the new pension value to the prior value. We consider three disease examples including, hearth failure, pancreatic cancer, and Alzheimer’s disease, but note that these techniques extend to any survival curve. We do not consider program solvency risks in the form of benefit reductions, increased retirement age, or other plan parameter changes and leave these issues for subsequent studies.

Finally, we carry out a number of numerical studies to demonstrate these valuation and risk techniques. In particular, we first demonstrate the differences in future social security payment present day value as a function of the date of death of the beneficiary. This in turn is used with life expectancy forecasts data from the entire American population to determine the present day value for each possible retirement date of the beneficiary which allows one to determine the optimal retirement date. Next, we compare differences in pension value depending on when the beneficiary makes the best and worst possible retirement date decisions as a function of income. We next examine how such optimal retirement decisions vary for beneficiaries who have different health profiles. Then we consider similar studies for distinct demographic cohorts in order to understand which groups are most impacted by sub-optimal decision making. In addition, we develop a number of risk scenarios based on PCA techniques for both yield curves and life tables.

This article contains a number of novel contributions which include, to the best of our knowledge, the first integration of life table forecasting techniques for purposes of studying optimal social security retirement. In addition, we develop the first application of PCA based methods for risk related scenario generation to defined benefit plans. Also, the new health and demographic studies considered below serve as precursors to broader program solvency and reform studies. Finally, we contribute new PCA based population lifetable risk metrics which may prove to be a useful tool for further actuarial applications.

This paper is organized as follows: In Section 2, we fix notation and describe how to value future social security benefit cashflows. In Section 3, we review the construction, estimation, and forecasting of life expectancies and their incorporation into pension valuation. Then in Section 4, we summarize the social security valuation algorithm and associated methodology that will be utilized in the numerical studies below. Next in Section 5, we develop techniques for quantifying interest rate and life expectancy risks associated with receiving these Social Security benefits. Then in Section 6, we provide a description of a number of data sources that were required to carry out several valuation and risk numerical studies considered throughout this section. Finally, in Section 7, we summarize our main findings as well as discussing a number of futures research directions.

2 Social Security Benefit Valuation

Defined benefit plans are funded during the course of an employee’s career and pay regular benefits upon retirement. They are typically received either as a lump sum, or as an annuity in the form of a monthly benefit which terminates upon the death of the beneficiary. The monthly benefit is determined in large part by the employee’s salary and length of work history.

In the case of Social Security, the beneficiary is able to select the date when they receive their first payment which we refer to as the retirement date. There are constraints on the retirement date, specifically one may only start receiving pension payments after reaching age 62 and much receive such payments after reaching age 70. If the beneficiary chooses to delay receipt of their initial payment, then the amount received for their monthly benefit will increase; however, they will receive fewer payments over their lifetime. We
will investigate the tradeoff between these two factors while also taking into account the health profile of the beneficiary when considering optimal retirement date studies below.

To fix notation, we let \( t_v = 0 \) denote the pension valuation date and let \( t_r \geq t_v \) represent the time until the pensioner’s retirement date, on which the first pension payment is received. We let \( t_m \) and \( t_M \) denote the minimum and maximum times that the beneficiary may initiate payments, which in the case of social security corresponds to the time when the beneficiary reaches age 62 and age 70 respectively.

After retirement benefit initiation, the retiree will receive monthly cashflows at times \( t_0 = t_r, t_1, \ldots, t_n = t_d \) until the month prior to \( t_d \) which we call the death date or date of death for simplicity. We denote the cashflows associated with \( t_i \) by \( c_i \) for \( i = 1, \ldots, n \). All times may be summarized graphically on the timeline of Figure 1. Note that the entirety of the pension payments may be thought of as a coupon-only bearing bond with variable maturity date \( t_d \). In addition, \( t_d \) typically ranges between twenty and thirty years. As a result, long term interest rates changes have a significant effect on the pension value as is the case with zero-coupon or long maturity bonds.

Given a fixed date \( t_d \), one can compute the present day value of all future pension payments assuming continuous compounding in the usual fashion. In particular, let \( r(t) \) denote the spot discount curve on the valuation date \( t_v \). Then, the present day value of the future cashflows is given by

\[
P(t_v) = \sum_{i=1}^{n} e^{-r(t_i) t_i} c_i,
\]

where we note that \( t_i \) typically occur at monthly intervals.

Note that equation (1) contains all the essential risk factors associated with pension plans that will be further examined below. First, the date of death \( t_d \) impacts the total number of payments received corresponding to the number of terms in the sum. Second, increases in the discount curve \( r(t) \) will reduce the value of future cash flows. Finally, the size of the cash flows \( c_i \) depends on the retirement date and the constraints \( t_r \in [t_m, t_M] \) may be altered by the pension administrator in the case of program solvency issues. Our aim will be to investigate how these variables influence the price and risk of the present day value of future pension payments.

3 Life Expectancy Forecasting

The life expectancy of the beneficiary is the predominant factor associated with the present day value of future pension payments. For example, all other variables being comparable, the present day value of the benefits of a beneficiary with a long life expectancy is typically several times greater than that of an individual with a short life expectancy. Life expectancy data is typically presented in the form of lifetables.
which contain probability of survival over a given time window (usually 1 or 5 years), conditional on survival to the first age in this interval.

Specifically, let \( p_i \) denote the probability of death prior to age \( i + 1 \) conditional on living to age \( i \); a lifetable contains a collection of these \( p_i \) values for a specific demographic group. Given that these occur at yearly frequencies, one may determine the distribution of the year of death of the beneficiary from the \( p_i \) values. In particular, conditioned on living to age \( i \), there is a \( 1 - p_i \) chance of surviving between ages \( i \) and \( i + 1 \). Similarly, the probability of death between ages \( i \) and \( i + 2 \) is then \( (1 - p_i)p_{i+1} \) where the first factor accounts for the necessary survival from ages \( i \) to \( i + 1 \). Continuing, the probability of death between years \( j \) and \( j + 1 \) is \( w_j = p_j \prod_{i=1}^{j-1}(1 - p_i) \) which vanishes every year after the first year that \( p_i = 1 \). Given a lifetable for an individual, we may construct a distribution for their death date given their current age from this process. We plot several example date of death distributions in Figure 2 conditioned upon surviving to various ages. Next, we note that for a given cohort, the \( p_i \) vary over time and, in particular,

![Figure 2: Probability of death for the average American conditional on living to age 0, 30, 50, 65, 75, and 85 given data in Table 1. Life table for total population: United States, 2015 in [1], the Center for Disease Control’s 2015 annual study.](https://ssrn.com/abstract=3438080)

...generally decline over long timeframes. We denote time varying lifetable probabilities by \( p^t_i \) which represents the probability in year \( t \) of living from age \( i \) to \( i + 1 \). Then as before, the probability of living to age \( j \) is \( w_j = p^t_j \prod_{i=1}^{j-1}(1 - p^t_i) \). The process of estimating future \( p^t_j \) values is referred to as lifetable forecasting; we review a few techniques designed to handle this task below.
3.1 Life Table Forecasting

Life expectancies evolve over time and estimates for their rate of change for different cohorts and age groups needs to be incorporated into a pension valuation model. In particular, death rates generally decrease given that life expectancy tends to increase; however, there are situations where death rates for certain age groups increase as we demonstrate in a numerical study below, typically over a few years time. Since pension payments tend to occur over several decades, incorporating forecasts of future life expectancy is essential to accurately estimating the future value of benefits one will potentially receive in the distant future.

It is not uncommon for pension payments to persist to the beneficiary for several decades beyond initiation during which time life expectancies may significantly increase. Accounting for this increased longevity is important when estimating the date of death distribution of the beneficiary. There is an extensive literature on life table forecasting, c.f. [21] for a survey. The seminal Lee-Carter forecasting model [13, 15] has gained widespread adoption for both its relative simplicity and strong forecasting accuracy. A number of extensions have been developed to further improve the predictive performance of the Lee-Carter model which vary in complexity from simple single factor cohort additions [22] to neural network based generalizations of the model [23]. We consider one such extension which focuses on model simplicity while simultaneously gaining significant performance improvements initially developed in [18] which we now summarize.

We first define the central mortality rate \( m(x,t) \) at age \( x \) and time \( t \), by considering the number of people who after reaching age \( x \) died during the subsequent twelve month period. Then \( m(x,t) \) is this number of deaths divided by the total sample size at the beginning of this time interval. The variable \( t \) represents the year for which this value was estimated. For example, mortality rates in the 50-51 age group differ significantly if estimated in \( t = 1950 \) as opposed to \( t = 2015 \). Life table forecasting models estimate how the \( m(x,t) \) evolve based upon historical realized values for these mortality rates.

The Lee-Carter model achieves this by fitting historical mortality data to an exponential model of the following form

\[
m(x,t) = \exp\left(a(x) + b(x)\kappa(t) + \epsilon(x,t)\right),
\]

where here \( a(x) \) captures the general shape of the mortality curve over time, \( \kappa(t) \) is a temporal mortality index which represents the evolution of death rates and \( b(x) \) describes the response of each age group to changes in mortality rates. In particular, it is assumed that the relationship splits into two factors contained in the second term of the exponential which separates age group and time dependence. Note that if there are \( M \) total age groups, and \( N \) years considered in \( m(x,t) \), then this may be viewed as a \( 2M + N \) parameter model.

One main limitation of the Lee-Carter model is that mortality rates are used directly in its estimation procedure. Instead, in [18], the authors take the approach of modeling changes in mortality rates, by altering the Lee-Carter model to

\[
m(x,t + 1) = m(x,t) \exp(\alpha(x) + \beta(x)k(t) + \epsilon(x,t)),
\]

which may be re-expressed as a linear model of the log mortality differences

\[
\ln m(x,t + 1) - \ln m(x,t) = \alpha(x) + \beta(x)k(t) + \epsilon(x,t).
\]

This model is not identifiable, for example one can inversely scale \( \beta(x) \) and \( k(t) \). In addition there are additional symmetries that go beyond rescaling, c.f. [13], for a more detailed description of these symmetries as well as their impact on estimation.
In order to fit this model to mortality data, first, define a matrix of log mortality differences
\begin{equation}
M(x, t) = \ln(m(x, t + 1)) - \ln(m(x, t)).
\end{equation}
The \(\alpha\) parameter is defined to be the time average of mortality rates for each age group which may be estimated through
\begin{equation}
\hat{\alpha}(x) = \frac{1}{n} \sum_{i=1}^{n} M(x, t_i), \quad \hat{M}(x, t) = M(x, t) - \hat{\alpha}(x),
\end{equation}
where here we have defined the demeaned mortality rates \(\hat{M}(x, t)\). Now, to estimate model parameters, we perform a singular value decomposition on \(\hat{M}\) to find unitary matrices \(U, V\), and a non-negative definite diagonal matrix \(S\). Here \(\hat{k}(t)\) is a scalar multiple times the first column of \(U\) times the largest singular value, and \(\hat{\beta}(x)\) is the first column of \(V\) divided by the same scalar.

Now that the model parameters have been estimated for a specific cohort, the model may be used to forecast future mortality rates. This consists of specifying a distribution for the time varying factor \(k(t)\) as well as the residuals \(\epsilon(x, t)\). Following Section 4 of [18], we fit a normal-inverse Gaussian distribution to \(\hat{k}(t)\), using maximum likelihood estimation. Specifically, this distribution is given by
\begin{equation}
\phi(x; \delta, \theta, \lambda, \mu) = \exp \left( \frac{\lambda}{\theta} + \mu \sqrt{\hat{x}^2 - \lambda} \right) \frac{C}{\hat{x}} K_1 \left( \frac{\sqrt{\gamma}}{\hat{x}} \right),
\end{equation}
where here we define \(\hat{x}^2 = \lambda + (x - \delta)^2\) and \(\gamma = \lambda + \mu^2 \theta^2\).

This distribution is fit to the \(\hat{k}(t)\) time series by maximizing the log-likelihood function
\begin{equation}
\ln \phi(x_i; \Theta) = \frac{\lambda}{\theta} + \mu \sqrt{\hat{x}_i^2 - \lambda} + \ln \frac{C}{\hat{x}_i} + \ln K_1 \left( \frac{\sqrt{\gamma}}{\hat{x}_i} \right),
\end{equation}
where here we let \(\Theta = (\delta, \theta, \lambda, \mu)\), \(\mu, \delta \in \mathbb{R}\) and \(\theta, \lambda > 0\). In addition, \(K_\nu\) is a modified Bessel function with \(\nu = 1\). The maximum likelihood parameters associated with this model are then defined by log-likelihood maximization:
\begin{equation}
\hat{\Theta} = \max_{\Theta} \sum_{i=1}^{n} \ln \phi(x_i; \Theta) \equiv \max_{\Theta} l(\Theta).
\end{equation}

The coupling between the model parameters results in a system of non-linear maximum likelihood equations if one attempts to directly identify the optimal parameters by setting \(\partial l(\Theta)/\partial \Theta = 0\), i.e. these equations are not directly invertible. In order to determine their values, we thus consider direct optimization of the log-likelihood function. We found a variant of the Nelder-Mead method provides a robust means of determining the value of the maximum likelihood parameters for this model over a wide variety of mortality datasets.

We then forecast future lifetable rates by recursively sampling from equation (4) treating both \(k(t)\) and \(\epsilon(x, t)\) as random variables to generate future life expectancy forecasts. This simulation is repeated ten thousand times for each age group and the mean life expectancy per year is used in the below numerical experiments as a forecast for future death rates. For example, in Figure 3, in the left subplot, we display a histogram of the fitted \(k(t)\) values with two overlaid model distributions fit using maximum likelihood estimation. In black, we plot the best fit normal distribution, and in red the best fit NIG distribution. In the left subplot, we display actual historical death rates for the 55-56 age group over the entire United States.
population from 1997 to 2015. The model in equation (4) was fit to this data and the above Monte Carlo simulation was used to forecast death rates until 2065. We note that the NIG distribution fits the estimated $\hat{k}(t)$ data more strongly than the normal distribution in this particular example as well as in dozens of additional examples that we considered. This confirms a claim made in [18] that the NIG distribution offers an improvement over the normal distribution as a model for $k(t)$. In addition, the right subplot demonstrates how forecasting error increases over time. In particular, the one standard deviation forecasting error in 2020 is approximately 2.5% of the expected value whereas it is 6.5% of the mean value in 2060. This plot also illustrates the importance of incorporating life expectancy forecasts in pension valuation calculations as the approximate 20% difference between the 2015 and forecasted 2060 death rates is significant and a major factor in the determination of the number of future monthly payments the beneficiary will received.

4 Social Security Valuation Methodology

We now turn to combining the topics discussed in the prior sections to develop a methodology to value future social security payments. First, we summarize the process set by the Social Security Administration (SSA) to determine the monthly benefit payment for a beneficiary. We then forecast the date of death distribution as previously described, and determine the present date value of future cashflows for each date of death of the beneficiary. Finally, we take an expectation over these values with respect to the date of death distribution of the beneficiary. Each step of this procedure is now described in further detail.

The procedure utilized by the Social Security Administration to determine the monthly benefit amount one will receive largely depends upon career earnings and one’s age upon benefit initiation. More specifically, the main variables that influence the monthly benefit payment amount are the salaries during the top thirty-five working years of the retiree, and the year and month of life which when one retires.

We follow the benefits calculation form specified in Appendix D of SSA’s Annual Statistical Supplement.
to estimate the monthly benefit payment for a retiree. There are subtle differences between the actual social security benefit calculation and the output of this form; however, for practical purposes and in particular the applications considered below, such differences are marginal. We now provide a brief description of the content of this procedure.

The initial step involves the specification of the number of computation years for which the beneficiaries payroll taxes will be considered in the benefit calculation. Then in the subsequent steps two and three, the top thirty-five years of employment of the retiree are obtained and one takes the minimum between these values and the maximum taxable amount of wages per year for which social security was collected. Next, we compute the average indexed monthly earnings (AIME) of the beneficiary by dividing the sum of these contributions by 420 and rounding down to the nearest dollar.

Next in the fourth step, the first and second bend points are defined; they are determined by the year that the retiree reaches age sixty-two. Using this information, the primary insurance amount (PIA) is calculated based upon the values of these tie points and the process outlined in step four. Finally, in step five, the monthly benefit is determined by adjusting the PIA based upon early or delayed retirement of the beneficiary.

This resulting estimated monthly benefit is one of the central inputs into the subsequent numerical studies below. It constitutes a baseline monthly payment upon retirement that the beneficiary will receive until their month of death. A premature or prolonged date of death will significantly influence the value of these payments.

4.1 Valuation Algorithm Outline

We now describe how the preceding ideas of pension valuation, life table forecasting, and monthly benefit calculation may be combined to develop a procedure to estimate the present day value of all future social security payments. The sequential descriptions below provide an outline for such a process, variations of which will be utilized in the below numerical studies. Specifically, given the inputs previously described, the present day value of future social security payments to a beneficiary is determined according to this procedure.

1. First, we assume that the retiree has reached the age of age 62 at time $t_v$ and would like to determine which month $t_r$ to start receiving social security benefits over the subsequent eight year until reaching age 70, i.e. $t_r > t_v$, with $t_r - t_v < 8$. We note that when restricting to this age range, that monthly benefit calculations are completely specified in Appendix D of [26] whereas one would need to include additional assumptions in order to consider $t_v < 62$ for retirement planning purposes. Since the minimum retirement age $t_m = 62$, this condition is not exceedingly restrictive.

2. We next consider lifetable data reported by the Center for Disease Control (CDC) for distinct cohorts [1]; specifically, all combinations of gender and four races: African American, Asian, Caucasian, and Hispanic. We note that the CDC lifetables do not extend beyond age 100; this is accounted for in CDC lifetable data by assigning a probability of 1 for death over the subsequent year of life to any person that survives to age 100. For example, in 2015, according to the CDC, if the average American lives to age 98, the probability of death prior to reaching age 99 is 31.1%; however, the probability of living to age 101 conditional on living to age 100 is 1. Although effects on present day value estimated associated with assumption are minor during years prior to 2015, when one forecasts out life expectancies to later years, e.g. 2065, then the probability of living to 100 is no longer negligible. In addition, the jump in death probability from approximately 30% to 1, biases dependent present day value calculations. With this in mind, we linearly extrapolate death rates based on the 98-99 and 99-100 rates with a maximum value of 1 to compensate for this issue.
3. Now, the lifetable forecasting model in equation (4) is used to estimate death rate evolution for the subsequent fifty years (until 2065) for the cohort under consideration using the method described in Section 3.1; this results in a death rate forecast for the survival probability of each future one year interval of life for the beneficiary. We finally construct a two-dimensional linear interpolation of these forecasted death rates to ensure we can sample them on a monthly frequency.

4. Now we denote forecasted lifetable values by \( p_{jt} \) which represent the probability at time \( t \) conditioned on living to age \( a_j \) of living one year longer to age \( a_{j+1} \). From this, we construct weights

\[
(10) \quad w_j = p_{jt+j} \prod_{i=1}^{j-1} (1 - p_{it+i}),
\]

which we note may be done recursively through

\[
(11) \quad \frac{w_{j+1}}{w_j} = \frac{p_{jt+j+1}}{p_{jt+j}} (1 - p_{jt+j}).
\]

We assume that \( p_{jt} = 1 \) beyond a sufficiently large \( j \) for all \( t \) so that there is an \( m \) where \( w_j = 0 \) for all \( j \geq m \). From these weights, we are able to construct the date of death distribution \( \phi(x; a_t) \) of the beneficiary conditional on living to age \( a_t \).

5. Let \( f(t, T, S) \) denote the forward rate curve at time \( t \) which represents the rate of return of a risk-free loan from time \( T \) to time \( S > T \). Let \( r(s) = f(t, t, s) \) denote the associated spot rate curve. We will use the forward rate curve \( r(t, t_r, s) \) to discount future cashflows back to the retirement date \( t_r \) to determine the future value on the retirement date of social security payments. Then we present day value these cashflows back to the valuation date \( t_v \). Specifically, we estimate the future cashflows that the beneficiary will receive based upon their date of death distribution \( \phi(x; a_t) \) of the beneficiary conditional on living to age \( a_t \).

We estimate this factor by averaging the social security cost of living increases over the twenty years prior to 2018 to be \( \gamma = 0.002195 \). Thus we recursively define monthly cashflows by \( c_{t+12} = (1 + \gamma)c_t \) taking \( c_0 \) to be the monthly benefit estimate as determined by the process described above. We then compute the value of these cashflows at time \( t_v \) for one month prior to the death date, denoted \( t_k \), by

\[
(12) \quad P_k(t_v) = e^{-r(\tau)} \sum_{j} c(t_j) \exp(-f(t, t_a, t_j)(t_j - t_r)),
\]

where here \( \tau = t_r - t_v \) and the forward curve is constructed from the current discount curve \( r(t) \) through

\[
(13) \quad f(t, T, S) = \ln q(t, T) - \ln q(t, S), \quad q(t, T) = e^{-r(T)t}.
\]

for times \( S > T \geq t_v \) where we assume continuous compounding in all the above.

6. We finally estimate the present day value of future social security payments through linearly interpolating the date of death distribution, denoted \( \phi(t) \), and present day value functions in order to align
disparate frequencies between these two datasets and estimate the present day value of future cashflows at the valuation date

\[ PV(t_v) = \mathbb{E}(P(t)) = \int P_k(T)\phi(T)dT, \]

through an adaptive numerical quadrature method.

This process will form the basis for valuing social security benefits in different applications below. For example, when estimating the optimal age to initiate social security benefits, we will replicate this computation for each retirement month possible for the beneficiary between ages 62 and 70 to determine which date results in the greatest present day value. In another study, we will incorporate lifetable forecasts for different cohorts and repeat the above. In summary, the preceding social security valuation technique integrates the SSA’s monthly benefit calculation and lifetable forecasting methods to estimate the expected value of future social security benefits.

5 Interest Rate and Life Expectancy Risk

Next, we consider the two most prominent risk factors which influence the present day value of social security benefits. First, we examine the impact of changes in the discount curve to the value of future benefit payments. Given that the collection of social security payments may be viewed as a long dated coupon only bearing bond with variable maturity, it exhibits similar interest rate risk to long duration fixed income securities. In particular, the value is inversely related to interest rates. We quantify this relationship through a principal component analysis (PCA) study of the historical movements of a discount curve composed of swap rates. After identifying the directions in which the rate curve vary most significantly, we design related yield curve scenario analyses from the principal components to examine the impact of such changes on social security benefit values.

Next, we consider the question of how disease diagnosis impacts the present day value of future social security payments. In particular, when life expectancy considerably shortens due to the contraction of a disease, we address how one can incorporate the survival curve associated with the disease into a modified present day value calculation. The numerical examples below provide a further demonstration of how such considerations affect the optimal date to initiate social security payments.

5.1 Yield Curve PCA Risk

Interest rate risk is a primary concern in fixed income portfolio management. When interest rates rise, the value of future cashflows declines and vice versa. The degree of change depends upon the reference time for the rate, for example, three month, two year, and thirty year rates will have distinct percentage changes over any given day; although, as we will see below, they will be highly correlated. PCA provides a means of determining how the interest rate curve varies as a whole based upon how it has historically changed. This information may be used to determine the associated price risk of future social security benefit cashflows. We first review the method of principal component analysis in a general setting and then discuss its application to yield curve risk.

Let \( X \) denote an \( n \times m \) data matrix which for application purposes we will take to be a collection of \( m \) interest rate time series with \( n \) data points each. The singular value decomposition of \( X \) is given by

\[ X = USV^*, \quad UU^* = U^*U = I_m, \quad VV^* = V^*V = I_n, \]

Electronic copy available at: https://ssrn.com/abstract=3438080
where here $U, V$ are $n \times n$ and $m \times m$ unitary matrices respectively, $S$ is an $n \times m$ matrix whose diagonal consists of the singular values of $X$, $I_n$ denotes the identity matrix, and $U^*$ denotes the Hermitian conjugate of $U$.

The principal components of $X$ are defined in terms of this decomposition by

$$\text{(16)} \quad XV = USV^*V = US.$$ 

We may represent $X$ as

$$\text{(17)} \quad X = \sum_{i=1}^{r} \sigma_i u_i v_i^*,$$

where here $u_i, v_i$ denote the columns of $U$ and $V$, respectively and $\sigma_i$ denote the singular values which are elements of $S$.

However, in the case of computing the principal components of the yield curve, which is our main interest, the spectral decomposition theorem simplifies this expression. In particular, if $\tilde{X}$ is a matrix of historical rate curves with components $X_{ij}^t$, let $X_i^j = \tilde{X}_{ij}^t - \tilde{X}_{ij}^{t-1}$ denote the absolute differences between rates. These are invariants for fixed income markets [17] and are more stable than directly modeling interest rates themselves. The empirical covariance matrix $\Sigma_X$ for these rate differences is given by

$$\text{(18)} \quad \Sigma_X = \frac{1}{n-1} XX^T,$$

and it is the object from which we will compute the principal components. Next, the eigen-decomposition of $X$ may be expressed as $\Sigma = EAE^{-1} = E\Lambda E^T$, where here $\Lambda$ denotes the diagonal matrix of eigenvalues of $\Sigma$ and $E$ is the associated eigenvector matrix whose columns $e_i$ correspond to eigenvalues $\lambda_i$. We note that since $\Sigma$ is a positive definite symmetric matrix, the eigenvalues have an ordering $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

By analogy to the singular value decomposition, here $E$ corresponds to the unitary matrices $U, V$ and $\Lambda$ is a diagonal matrix of eigenvalues equivalent to the corresponding singular value matrix. In addition, the principal components $p_i$ of $X$ are the eigenvectors of $XX^T$ and the variance along the $p_i$-th direction is the normalized eigenvalue $\lambda_i/(n-1)$, c.f. [24] for a more detailed review.

In order to incorporate the yield curve principal components into an interest rate risk framework, we recall that the first principal component can be interpreted as the direction of maximal variation of data being considered. The second principal component depicts the direction of maximal variation on the $n-1$ dimensional subspace which is the orthogonal complement of the first. In addition all principal components are independent in this covariance matrix setting. With this interpretation in mind, we define a perturbed yield curve,

$$\text{(19)} \quad r_c(t; c, m) = r(t) + c \sum_{i=1}^{m} \sqrt{\lambda_i} e_i,$$

in the direction of the first $m$ principal components with magnitude given by the square root of the eigenvalues that are scaled by a factor of $c > 0$. In subsequent interest rate risk applications, we will revalue future benefit payments using the perturbed yield curve and examine the impact on the total value of all such payments when compared with the current market yield curve.
5.2 Life Expectancy Risk

The most significant variable impacting the present day value of future social security payments is the life expectancy of the beneficiary. Typically, a pensioner expects to receive monthly benefit payments for two to three decades after initiating retirement. However, if the beneficiary contracts a disease that significantly shortens their life expectancy one may only received a fraction of the expected payments.

If a beneficiary who was otherwise healthy is diagnosed with a disease, we account for this in the preceding benefit valuation model by updating their date of death distribution to the survival function of the illness which is largely a function of the age of the beneficiary at the time of diagnosis and the severity of diagnosis. In the numerical studies below, we will examine to what extent social security valuation is impacted by contracting three common mortal diseases.

Secondarily, a pensioner is subject to longevity risks related to the entire population. In a similar manner to the yield curve risk model we described above, we consider a PCA based life table risk measure. In particular, as noted in [2, 3, 13], the Lee-Carter model can be viewed as a single factor principal component model. We wish to examine higher order risks associated to lifetable changes from a principal component study.

More specifically, denote a lifetable function \( l(t, x) \), where here \( x \) represents the age group variable, which can be viewed as an interpolated version of lifetable data previously considered \((t_i, p_i)\) where here \( p_i \) denote the conditional probabilities of death over the subsequent period for \( i = 1, \ldots, n \). We are interested in understanding the relative change of social security payment values due to a change in the life curve. Let \((\lambda_i, e_i)\) denote the eigenvalues and associated eigenvectors of a time series of historical life tables. Given a base expected lifetable \( l(t) \), we can then perturb in the direction of the principal components through

\[
l_c(t; c, m) = l(t) + \epsilon \sum_i \sqrt{\lambda_i} P_i,
\]

where here we denote the scale constant by \( \epsilon \) and eigenvectors by \( P_i \) in order to prevent confusion with the interest rate example. Then, we compute the social security benefit present day value change through

\[
\Delta_l PV = \frac{PV(l_c) - PV(l)}{PV(l)}.
\]

These relative changes in value are calculated in a number of numerical studies below. Finally, we note that one may perform this lifetable PCA on different time frequencies, i.e. instead of just one year intervals, one may consider five or ten year intervals to ascertain long term trends in lifetable changes.

6 Numerical Studies

We now turn to several numerical studies to provide demonstrations of the above social security valuation and risk techniques. The studies rely upon data gathered from a variety of sources which we summarize in the first subsection below. Next, we identify how one may determine the optimal year and month to initiate social security benefits based upon the wage history and health profile of the beneficiary. Building upon this study, we then investigate the difference in present day value of future social security payments between the worst and best times to initiate benefits. This is considered for a range of health and income profiles. Next, we examine how these minimum and maximum present day value bounds vary over different cohorts including Caucasian, African and Hispanic males and females as well as provide explanations for the qualitative differences between these values.
Then we develop a number of risk scenarios based upon the PCA techniques described above. First, we develop interest rate swap curve simulations by first computing and depicting the principal components of a discount curve composed of swap rates on daily, monthly, quarterly, and semi-annually frequencies. We then revalue future social security cashflows based upon perturbations of the discount curve in the direction of the principal components and determine the percentage change in present day value of these future payments. We finally turn to health risk considerations by first incorporating the survival curves of pancreatic cancer, heart failure, and Alzheimer’s disease into the valuation procedure. We demonstrate a dramatic decrease in value if one contracts any of these diseases upon retirement time. Finally, we examine population level lifetable risk by computing the principal components of mortality rates for the full American population and developing associated risk scenarios as in the case of the swap rate example.

6.1 Data and Implementation Description

We first summarize the content, aggregation, and cleaning process for all datasets required in the subsequent numerical studies. In particular, we utilize five main data sources including information related to the specification of the SSA’s monthly benefit calculation, historical mean American annualized wages, lifetable data for multiple age group frequencies for the cohorts under consideration, swap rate time series for discounting and rate risk studies, and finally survival curve information related to the diseases under consideration for health risk studies.

The calculation of the amount of one’s social security monthly benefit as specified according to the form in Appendix D of [26], relies upon multiple tables provided in this reference. This includes two worksheets to compute the indexed earnings and primary insurance amount of the beneficiary. In addition, it involves the use of tables that provide indexed earning factors, a type of inflation adjustment, cumulative cost of living adjustments, and maximum reduction limits given that the beneficiary takes early retirement.

Next, in order to carry out income related studies, we gather data representing the average American’s annual wages that are subject to federal income and social security taxes. This information is contained in [26] as well and updated annually by the SSA. We note that it differs from mean income data gathered by other governmental organizations, e.g. the Census Bureau or Department of Labor in that it only represents the portion of income that is subject to payroll taxes.

The most involved data gathering task consists of obtaining historical life table data for varying cohorts. This information is available in the annually compiled United States life tables which are part of the National Vital Statistics Reports Series published by the Center Disease Control, c.f. [1]. Specifically, these lifetables contain the probability of survival over the following year for an individual who has lived to a fixed age; this information is available for various age, gender, and race as well as geographically distinct groups. The availability of data depends on the group being considered as well as the desired age group frequency. For example, we are only able to obtain life expectancy data for yearly age groups for the Hispanic population from 2007 to 2015 whereas the same data is available from 1996 to 2015 for the Caucasian and African American male and female cohorts.

In addition, we gather life table data for the entire population collected during the decennial census surveys from 1900 to 2010. The associated survivor and log death rate curves are displayed in Figure 4. Note here that younger age group, in particular infants, have seen dramatic increases whereas such gains are less pronounced for older age groups. The varying changes per age group displayed in these graphs is precisely what the life table forecasting models previously discussed try to capture.

Next, we construct a dataset of interest rate time series for the principal component and interest rate risk studies considered below. Specifically, since one expects to receive social security payments over multi-decade time frames, we download United States swap rate data with 1 through 10, 12, 15, 20, 25, 30, 40, and 50 year

Electronic copy available at: https://ssrn.com/abstract=3438080
maturities from 2005 through the end of 2015. We download these time series using Bloomberg’s Python API in addition along with the Python packages tia and pandas. We then restrict to dates for which rate data is available for all rates under consideration.

Lastly, for the health risk studies discussed below, we source survival curve data for the three diseases under consideration from the medical literature. Such information ranges from only a few datapoints to involved hazard rate model function fit to actual survival data. We consider pancreatic cancer survival data from Kaplan-Meier survival curves fit in [30], one, five, and ten year survival rates for patients encountering heart failure in [29], and piecewise constant survival curves for Alzheimer’s diagnosis from [5].

Finally, we note that all numerical studies are implemented in the Python programming language. Data is first read, formatted, cleaned, and merged with the pandas package. Then we utilize numpy and scipy’s optimization model for model fitting tasks. Finally, graphical results are displayed using the matplotlib plotting package.

6.2 Optimal Benefit Initiation Time

We first consider an experiment designed to determine the optimal retirement age of an American with an average health profile and historical annual wage in the sense of maximizing the present day value of their future social security benefits. Specifically, we assume that the beneficiary has the average American population lifetable as of 2015 given in Table 1 of [1]. In addition, we assume that they have earned the average American wage over the past thirty-five years using wage data from the national average wage index.

Moreover, we assume that the beneficiary is presently age 62 and is seeking to understand which month within the next eight years to initiate social security benefits. First, in Figure 5, in the left subplot we display the present day value of future social security cashflows given that the beneficiary has a deterministic death date indicated by values on the $x$-axis, assuming social security payments are initiated at age 62.
example, if the beneficiary dies just after reaching age 62, then they will receive a $0 benefit. However, if the beneficiary lives to age 80, the benefits are worth approximately $200,000, whereas at age 100, they are worth approximately $400,000. The right subplot of Figure 5 is the corresponding date of death distribution of the beneficiary conditioned on living to age 62; assuming death rates according to the average American lifetable. In order to determine the expected present day value of the beneficiary, we integrate the product of these two functions. The associated value is approximately $293,000 which is the present day value of future social security payments given that the beneficiary chooses to retire at the age of 62; this is displayed in the left subplot of Figure 6. In other words, the values in the left subplot of Figure 6 can be thought of as the market value of all future social security payments given a date of retirement between age 62 and 70. The main reason for the large increase in value is due to the monthly benefit increase for delayed retirement depicted in the left subplot of Figure 6. If a beneficiary waits until reaching age 70, the present day value of future social security payments will be approximately $357,000, e.g. a $64,000 difference compared with initiating retirement at the earliest date possible. Notice that one can see the influence of the two tie points in the social security monthly benefit calculator in this graph between retirement ages 63 and 64 as well as near age 66. After both, the rate of increase of present day value of future benefits increases more rapidly than before. In addition, in the monthly benefits subplot to the right, we display the values of the original monthly benefit dependent upon which month the beneficiary decided to initiate retirement. Note that one can see both tie points there as well, and the rate of increase of the value of the monthly benefit increases after each tie point.

6.3 Optimal vs Worst Retirement Decision

One of the most striking conclusions from the prior study is that there is approximately a 23% difference between making the best and worst retirement date initiation decisions. Our next aim is to explore how such percentages vary as a function of the historical income and health profile of the beneficiary.

Figure 5: The left subplot contains the present day value of future social security cashflows given a deterministic death date. The right plot is the average american death distribution conditioned on living to age 62.
We first parametrize beneficiary income and health levels. We consider the national average wage index as a baseline income measure. This index is intended to represent the average annual wages per worker that are subject to federal payroll taxes and is also used as an inflation index when the SSA considers cost of living adjustments on a yearly basis [25]. In this study, we consider seven possible historical income scenarios for a beneficiary who is presently aged 62 who first started working in 1970 and earned 25%, 50%, 75%, 1, 1.5, 2, and 3 times the national wage index. Time series of the non-inflation adjusted wages for each case are displayed in the right subplot of Figure 7.

Next, we construct eight possible health scenarios for the beneficiary based upon the forecasted average American date of death distribution conditioned upon living to age 62 constructed in Section 6.2. Specifically, we first estimate the deciles of this distribution and secondly rescale its domain and the corresponding functional form for the distribution so that the resulting median value matches the associated deciles. The result is a collection of nine distributions whose median values are the quantiles of the average American date of death distribution. In addition, the general shape of the average distribution is retained through the rescaling procedure and we plot the probability density functions of each such distribution in the left subplot of Figure 7. Here the blue distribution represents the lowest life expectancy situation with the dark yellow representing the longest living scenario.

The distinct combinations of historical wage an future death distribution scenarios will comprise the main data content of this study. We first compute the expected present day value of future payments assuming the average American forecasted lifetable of future social security and associated monthly benefits based upon the selected age of retirement and display the results in Tables 1 and 2 respectively.

Note that the relative difference between the greatest and least present day value corresponding to initiating benefits age 70 rather than age 62 is nearly constant across varying incomes. In particular, there is a 18.4% difference in value between retiring at the age which maximized present day value vs that which minimizes this amount.

We display corresponding monthly benefit amounts for each income and retirement age combination in
Table 2, which also have a constant difference across income level between age 62 and 70 given by 43.2%. Stated another way, the penalty one pays for taking early retirement is equivalent on a relative basis across income although it increases from the perspective of an absolute dollar amount.

Now, we examine the joint effects of health and income on present day value of future social security cashflows. We wish to understand how the percentage difference between the best and worst retirement times from a present day value point of view varies as a function of these two variables. We first note that it does not depend upon income level just as we saw in the specific case above. With this in mind, in Table 3, we display the best an worst retirement ages for each health level as well as the associated relative difference in social security present day value. Note that in the case of very poor 10%-ile health, that it is optimal to take social security as quickly as possible at age 62 due to short life expectancy expectations. Moreover the difference between the worst and best months to initiate benefits is quite substantial being nearly a factor of seven. This is due to the fact that one is unlikely to live to age seventy and if one delays benefit initiation to this date, they are likely to have zero value. Next, for near median health levels, note that the percentages are much lower which may provide justification for a beneficiary to initiate benefits earlier at a smaller penalty. For healthy individuals with longer life expectancies, it is in their best interest to prolong retirement until the maximum age in order to realize the increased monthly benefits over several decades to some.

Next, we graphically depict the absolute dollar difference between the best and worst decisions as a function of health and income levels in Figure 8. Here in the left subplot, the domain represents the seven income scenarios considered in increasing order and the line plots depict the absolute dollar difference between making the best and worst retirement date decisions for varying health profiles. The left subplot contains the same information, but now varies over nine health scenarios and displays one line plot per income level. Note that as a function of income, level losses rise with increased slope. In addition, one can see the influence of the two tie points in the social security benefits estimators in each such plot. When viewed as a function of health level, the plots show that for very low or high income, there is significant variance between losses between the best and worst decisions. However, for near median health levels, in particular between the
Table 1: Present day value of future social security benefits based upon age of retirement and income level as a multiple of the national average wage index.

| Age/Income | 25%  | 50%  | 75%  | 1x   | 1.5x | 2x   | 3x   |
|------------|------|------|------|------|------|------|------|
| 62         | 136.2| 188.6| 241.1| 293.5| 379.3| 428.5| 472.7|
| 63         | 139.6| 193.4| 247.2| 300.9| 389.4| 439.4| 484.7|
| 64         | 144.7| 200.5| 256.2| 311.9| 403.2| 455.4| 502.3|
| 65         | 149.5| 207.0| 264.6| 322.1| 416.3| 470.2| 518.7|
| 66         | 153.3| 212.3| 271.4| 330.3| 427.0| 482.3| 532.0|
| 67         | 157.9| 218.7| 279.4| 340.2| 439.8| 496.7| 547.9|
| 68         | 161.8| 224.1| 286.4| 348.6| 450.6| 509.0| 561.5|
| 69         | 164.7| 228.2| 291.6| 355.0| 458.9| 518.3| 571.7|
| 70         | 166.8| 231.1| 295.3| 359.5| 464.6| 524.8| 578.9|

Table 2: Monthly benefit amounts according to SSA’s benefit estimator for varying age of retirement and historical income scenarios. The column headers indicate the multiple of the national average wage index considered in each scenario.

| Age/Income | 25%  | 50%  | 75%  | 1x   | 1.5x | 2x   | 3x   |
|------------|------|------|------|------|------|------|------|
| 62         | 619  | 857  | 1095 | 1333 | 1723 | 1947 | 2147 |
| 63         | 660  | 915  | 1169 | 1423 | 1840 | 2078 | 2292 |
| 64         | 714  | 989  | 1263 | 1538 | 1988 | 2246 | 2477 |
| 65         | 769  | 1066 | 1362 | 1657 | 2143 | 2420 | 2670 |
| 66         | 825  | 1143 | 1460 | 1778 | 2298 | 2596 | 2863 |
| 67         | 890  | 1233 | 1575 | 1918 | 2479 | 2800 | 3088 |
| 68         | 957  | 1325 | 1693 | 2061 | 2665 | 3010 | 3319 |
| 69         | 1023 | 1417 | 1812 | 2205 | 2850 | 3220 | 3551 |
| 70         | 1090 | 1510 | 1930 | 2349 | 3036 | 3429 | 3782 |

30%-ile and 50%-ile, the differences are minor.

6.4 Cohort Studies

We now turn to considering how social security present day value varies over different demographic cohorts. Specifically, we consider African American, Caucasian, and Hispanic male and female groups. We omit Asian American’s as we were unable to obtain sufficient data in order to estimate the life table forecasting model accurately. Data for the Hispanic population was available going back to 2007, and to 2016 for African Americans and Caucasians.

Although life expectancies generally increase for the majority of age groups for each cohort over this time period, we found a few interesting exceptions. For example, in Figure 9 below, we display actual and forecasted life table rates for the 15-16, 35-36, 55-56, and 65-66 age groups for Hispanic males. Prior to 2015, we plot actual death probabilities and after this date, we plot forecasted probabilities along with associated standard deviation bounds. Three of the age groups exhibit typical forecasting behavior; namely a steady decline of death probabilities over time. Note that in these three example, the standard deviation of the forecast is largely determined by the variance of the actual lifetable data. Now, the 35-36 age group demonstrates non-standard behavior. In particular, the death rate for this age group starting at
Table 3: Best and worst age to initiate social security benefits and associated percentage difference in the present day value of future social security payments based upon these days.

| Health %-ile | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |
|-------------|----|----|----|----|----|----|----|----|----|
| **Best Age** | 62 | 62 | 62 | 65.2 | 68.2 | 69.3 | 70 | 70 | 70 |
| **Worst Age** | 70 | 70 | 70 | 70 | 62 | 62 | 62 | 62 | 62 |
| **Rel. Diff** | 692% | 71.3% | 20.7% | 5.4% | 7.7% | 13.4% | 19.0% | 23.5% | 27.4% |

Figure 8: The left subplot contains the absolute dollar difference between the present day value of future social security benefits as a function of income level for varying health profiles. The right subplot contains the same difference for varying health levels for each income scenario considered.

2007 decreases until 2012, and the increases beyond the 2007 level in 2015. As a result, the associated forecasting curves slightly increase with large variation. We found similar effects for older ages in our sample, specifically, mid-nineties age groups. However, overall they have marginal contributions to the present day value calculations as the probability of living to such ages is quite low.

Next, we examine how the minimum and maximum possible present day value of future social security payments varies for the different demographic groups we are considering. In particular, for each of the three races we compute the present day value for the female and male subpopulations as well as the entire population over all possible retirement dates. We then report the minimum and maximum present day value along with the associated absolute and relative differences in Table 4. Here we assume that all groups earn the national average wage index over the past 35 years so that only health effects are under consideration.

We first note that there is rather considerable variation between values between the different groups. Female present day values exceed those of males for each group due to longer expected female life expectancies. Interestingly, the Hispanic population has a considerably longer life expectancy over the timeframe we are considering than either the Caucasian or African American groups. There are a number of studies that examine this issue; for example, in [9], the authors partially attribute this increased longevity to the fact that a large portion of the United States Hispanic population is migratory and that it is more likely those in good health would ensure the stresses of migration than those who are in poor health. We finally note that the percentage differences between making the best and worst retirement decisions are roughly uniform across the different groups with Caucasians having the least loss at 18.1% and the Hispanic population having the
Figure 9: Hispanic male actual and expected forecasted lifetables for the 15-16, 35-36, 55-56, and 65-66 age groups.

largest losses at 20.2%.

6.5 Interest Rate Risk

Next, we examine how risk associated with yield curve changes impacts the value of future pension payments. Since these payments often are received over several decades, it is important to understand how the full term structure of rates may evolve. Our aim is to develop a scenario analysis based upon plausible future changes of the yield curve. In particular, we pursue a PCA based framework for identifying the directions in which the yield curve moves that correspond to the most severe rate risks.

First, we study how interest rate risk impacts the present day value of future social security payments. Benefit payments may be viewed as a coupon only bearing bond, the interest rate risk profile exhibits similarities to traditional bonds. Specifically, as interest rates rise, the present day value of future cashflows decreases and vice-versa.

We seek to design realistic risk scenarios by understanding the historical evolution of the discount curve used to present day value future cashflows. We consider a subset of rates from Bloomberg’s US23 swap curve for this example. Specifically, we consider the swap rates at annual intervals from year 1 to 10 as well as the 12, 15, 20, 25, 30, 40, and 50 year swap rates on a daily basis from the beginning of 2005 to the end of 2015. In addition, we will consider rate curve changes on different frequencies; daily, monthly, quarterly, and annually. As described in Section 5, denote spot rates by $r_{ij}$ where here $j = 1, \ldots, m$ ranges over tenors and $i = 1, \ldots, n$ ranges over the frequency rates being considered, e.g. daily, monthly, etc. The covariance matrix and principal components are constructed for each rate dataset which will be used to define the perturbed
Table 4: Display of the present day value (in units of $1,000) of future social security payments for nine cohorts determined by gender and race. We assume that each group earned the national average wage index so that the below study examines average health differences between cohorts. In all cases, the minimum value was at 62 and maximum at age 70.

| Model          | Min PV | Max PV | Abs. Diff | Pct. Diff. |
|----------------|--------|--------|-----------|------------|
| Caucasian Male | 267.3  | 320.6  | 53.3      | 16.6%      |
| Female         | 319.0  | 391.5  | 72.5      | 18.5%      |
| All            | 295.4  | 360.8  | 65.4      | 18.1%      |
| African Male   | 234.4  | 287.6  | 53.2      | 18.5%      |
| Female         | 289.2  | 358.4  | 69.2      | 19.3%      |
| All            | 265.3  | 329.3  | 64.1      | 19.4%      |
| Hispanic Male  | 316.9  | 393.9  | 77.1      | 19.6%      |
| Female         | 359.5  | 450.1  | 90.6      | 20.1%      |
| All            | 341.4  | 427.9  | 86.5      | 20.2%      |

yield curve $l_1(t; \epsilon, m)$ in equation (20). Here, we will take $m = 3$ as this accounts for nearly all the variability of the swap curve we are considering.

We consider changes in the yield curve on four different frequencies. First, we compute absolute rate changes based on end of day, month, quarter, and semi-annual intervals. In each case, this is done on a rolling daily basis from 2005 to 2015. The principal components are then calculated from the covariance matrix of absolute rate changes and its corresponding eigenvalues and eigenvectors. We plot these principal components in the case of daily changes in Figure 10 along side an associated percentage variance explained graph. In addition, we consider how the first principal component changes over time in order to qualitatively assess its stability. Note that the greatest three principal components estimated over the full ten year dataset have associated interpretations. In particular, the first principal component is nearly a constant function with the exception of slight variations for shorter term rates The second principal component is a decreasing line (similarly it may be represented as an increasing line since the principal components are only defined up to a sign), and lastly the third principal component represents a bowing effect with high positive weight on short and long term rates with opposite negative weight on mid-maturity rates. Next, in the middle subplot we display the cumulative percentage variance explained by the first five principal components. Note that the bulk of the variation of the yield curve occurs in the direction of the first principal component which explains approximately 90% of the variance. The first two principal components explain 97% and the first three combined explain approximately 98% with marginal subsequent increases. This indicates that although many rates go into the construction of the yield curve term structure, the majority of its variation occurs in two directions. In the third subplot, we change the estimation procedure to only consider one year of prior rate data when computing the covariance matrix. We then store the first principal component and move forward one year and repeat this process. Note that these curves exhibit reasonable stability over time. However, there is a significant decrease in the values of the first principal component for short rates starting in 2010. This corresponds to the time when interest rates were lowered to near zero values after the 2008 financial crisis. Since rates were held at these values for several subsequent years, there was very little variation that occurred in short term rates as is reflected in this plot. However, overall, these curves are reasonably stable over time which provides motivation for their utilization in pension value interest rate scenario analyses.

We now consider the change in present day values for a social security beneficiary with the average
population health profile who earned the national average wage index for the past 35 years in the below yield curve risk scenarios. We consider two types of scenarios based upon the first and greatest three principal components. Specifically, we shift the yield curve in the direction of the first principal component with magnitude given by the square root of the associated eigenvalue in the former case and shift by the sum of the top three principal components in the latter. We shift the yield curve down for each scenario and scale the shifts by factors associated with which range from minor shift with a 0.25 to a major 3 factor.

In Table 5, we display the downward shift in basis points of the yield curve in each of the eight scenarios considered for the 5, 10, 20, 30 and 50 year rates. We perform the PCA and revaluation considering different frequencies of yield curve data. In particular, we examine daily, monthly, quarterly, and semi-annual interest rate changes. In the PC1 scenarios, we only adjust the current swap curve by shifting by the first principal component of the absolute rate changes in the downward direction. In the Full PCA scenarios considered in the bottom portion of Table 5, we perturb in the direction of a sum of the first three principal components. Since these components are only defined up to a sign, we consider both positive and negative shifts of the second principal component which can be identified by the values of the rates described in the scenarios below. Specifically, note the full PCA daily and monthly scenarios have decreasing rates as maturity increases whereas the quarterly and semi-annual rates increase. This is reflective of the fact that we perturbed by a decreasing second principal component in the first two cases and an increasing one in the later two.

Note that scenarios that only consider the first principal component are nearly uniform across the different rates considered whereas the full three principal component scenarios exhibit more variation due to the influence of the second principal component. We display results of relative changes to the present day value of future social security payments in Table 7 for each scenario. Here we scale the yield curve changes by a factor $\epsilon$ described in the column header of the table to consider yield curve movements of increasing severity. For example, if $\epsilon = 2$, then the thirty year rate would be shifted by 11.9 basis points in the Daily PC1 example.
Table 5: Basis point decreases in swap curve risk scenarios for one and three principal components cases for the 5, 10, 20, 30, and 50 year swap rates. Here PC1 scenarios only utilize the first principal component whereas the Full scenarios are constructed from sums of the first three principal components. Here principal components are estimated from daily, monthly, quarterly, and semi-annual changes in interest rates.

| Scen./BP Shift | 5Y  | 10Y | 20Y | 30Y | 50Y |
|----------------|-----|-----|-----|-----|-----|
| Daily PC1      | 5.92| 6.29| 6.01| 5.95| 5.75|
| Monthly PC1    | 27.8| 28.6| 25.6| 24.6| 25.5|
| Quarterly PC1  | 39.1| 40.4| 38.7| 38.8| 39.3|
| Semi Ann. PC1  | 62.8| 64.0| 59.3| 57.8| 56.9|
| Full Daily     | 7.87| 6.48| 4.48| 3.64| 2.80|
| Full Monthly   | 31.5| 25.3| 19.9| 18.9| 19.6|
| Full Quarterly | 23.0| 41.6| 53.4| 56.8| 58.0|
| Full Semi Ann. | 41.2| 65.2| 81.3| 84.8| 83.5|

Table 6: Percentage change in the value of future social security payments for each scenario where the perturbation by the principal components is scaled by the value in the column header.

| Scen./ε | 25% | 50% | 1x  | 2x  | 3x  |
|---------|-----|-----|-----|-----|-----|
| PC1 Daily | 0.26| 0.53| 1.06| 2.13| 3.21|
| PC1 Monthly | 1.16| 2.33| 4.73| 9.70| 14.94|
| PC1 Quarterly | 1.69| 3.42| 6.97| 14.5| 22.62|
| PC1 Semi Ann. | 2.65| 5.38| 11.09| 23.58| 37.64|
| Full Daily | 0.23| 0.45| 0.91| 1.83| 2.75|
| Full Monthly | 0.92| 1.86| 3.75| 7.65| 11.71|
| Full Quarterly | 2.12| 4.30| 8.83| 18.67| 29.64|
| Full Semi Ann. | 3.27| 6.68| 13.94| 30.37| 49.84|

Note that since social security payments are expected to be long dated, even relatively short rate changes can have significant impact of the present day value of future social security payments. Relatively small single digit basis point changes in the yield curve result in approximately a one percent change in future payment present day value. Only in the more extreme scenarios over quarterly or semiannual time periods do we find large changes in pension value.

6.6 Life Expectancy Risk

We finally turn to the examination of the sensitivity of social security payment value to changes in the life expectancy of the beneficiary. There are two distinct varieties of life table related risk that we examine. First, we wish to understand the impact of a major change in the health status of the beneficiary on the optimal retirement date as well as value of future expected payments. We incorporate this change in health into our framework through the survival curve of the beneficiary that results from a contracted disease. In particular, we consider three specific examples for diseases with high mortality rates, including pancreatic cancer, heart failure, and Alzheimer’s disease.

Next, we study how changes in broader population life table death rates impact the present day value of future payments of the beneficiary. Recall that in [2, 3, 13], the authors mention that the Lee-Carter method
can be viewed as a single factor principal component analysis. With this in mind, we compute higher order principal components and design an associated scenario analysis as in the case of the yield curve.

6.6.1 Individual Health Risk

When a beneficiary is diagnosed with a disease with a high mortality rate, the present day value of their future social security benefits is significantly reduced. The amount of decline depends largely upon the type and severity of the disease, which is quantified through a survival curve, in addition to the age of the beneficiary. We will consider three examples of diseases with available survival curve data including pancreatic cancer, heart failure, and Alzheimer’s disease.

In order to understand the impact of being diagnosed with one of these diseases on social security value, we will replace the life distribution with the associated survival curve of the disease and then revalue future payments. Our central aim is to understand the expected decline in pension value due to such a change.

We source survival curve data for the three diseases under consideration from the medical literature. This information ranges from only a few data points per curve to involved hazard rate model function fits to actual survival data. We consider pancreatic cancer survival data from Kaplan-Meier survival curves fit in [30], one, five, and ten year survival rates for patients encountering heart failure in [29], and piecewise constant survival curves for Alzheimer’s diagnosis from [5]. We display the survival curves considered in Figure 11, where here we use piecewise linear interpolation to define rates between data points. We finally extrapolate future survival probabilities in each case using linear extrapolation based upon the final two survival data points of each disease. These three diseases were considered to examine the impact of differences between a high mortality disease like pancreatic cancer with a moderate disease such as Alzheimer’s and a relatively high survival probability example like heart failure. However, losses are quite significant in each case where we estimate a loss of 87.7% in the case of Alzheimer’s, 83.5% for heart failure, and 96.6% in the case of pancreatic cancer. Note that the median survival times for each of these diseases are approximately five years, six years, and four months respectively. By contrast, a healthy individual has a median life expectancy of
approximately 27 years conditioned on living to age sixty-two. This large decline in life expectancy is the main contributing factor for the associated very large losses in social security present day value.

6.6.2 Population Life Table Risk

Lastly, we examine how broader population lifetable changes may influence social security present value by applying PCA techniques to historical life tables for the entire American population. We note that similar methods have been applied to life table modeling, c.f. [31]. For this study, we compute relative changes for each age group from the all population lifetables from 1997 to 2015. We then compute the principal components of this dataset and plot them alongside the associated percentage variance explained in Figure 12. First note that the cumulative percentage variance explained right subplot qualitatively differs from that of the prior swap curve example that we considered. In particular, it increases more gradually and there is not a clear distinction between signal and noise components. However, approximately 80% of the variance is explained by the first three principal components which will be utilized for life table risk scenario design below. The values of the principal components are displayed in the left subplot.

The most notable features of these plots are the spikes on the second and third principal components near the ten year age as well as the large increase in variance for older ages. For social security present day value estimation purposes, we are assuming that the beneficiary has lived to age 62, so the artifacts for younger ages will not impact associated risk scenarios. Beyond this age, the second and third principal components are relatively stagnant whereas the first has a sizeable slope.

We consider two types of risk scenarios associated with these principal components in Table 7 below. Here we add on the first principal component scaled by a factor represented in the associated column header onto each life table in order simulate an increase or decline of the average population life expectancies. Here we use the six positive and negative combinations of one, two and three as multiples to design the scenarios. After the lifetables have been shifted, we revalue future pension cashflows in order to estimate the percentage change in the pension value associated with the optimal retirement decision when compared
with those estimated in Table 1. We note that these changes are relatively minor when compared with the
disease risk scenarios considered above. In addition, note that this table is not symmetric as, for example,
one realizes a large 6.9% increase for the three full principal component multiple in the event of increased
life expectancies vs a 5.2% decrease in present day value when we decrease life expectancies by three times
the full principal component sum.

7 Conclusions and Future Extensions

We have presented a collection of techniques for the valuation and risk assessment of the present day value
of future social security that incorporate the retirement age and health profile of the beneficiary, life table
forecasting models, and the SSA’s monthly benefit specifications. The associated risk metrics are based
upon the principal components of both the discount curve and historical life tables for the cohort under
consideration.

In addition, we considered a number of numerical studies which included the determining of the optimal
retirement age of a beneficiary based upon their current health profile and prior wage history. For further
work, we would like to develop a public calculator and dashboard visualization to make these techniques
accessible to wealth managers and those who need to estimate the value of their future social security benefits
for asset allocation purposes. We also seek to incorporate survival curves for major diseases with associated
high mortality rates in order to better advise individuals on when to start receiving social security benefits
as well as estimating expected losses associated with sub-optimal initiation decisions.

Also, we seek to refine the manner in which disease survival curves are incorporated into the social security
present day valuation framework. Specifically, if the beneficiary had a greater than average life expectancy
prior to contracting a treatable disease, then it is likely that they would also have an above average life
expectancy post disease. We seek to apply Bayesian techniques to incorporate the current health profile of
the beneficiary as prior information into the estimation of the survival curve of the diseases specific to the
beneficiary. This in turn would be incorporated into the present day value framework.

Lastly, we seek extend the methods developed in this article to investigate solvency issues related to
the Social Security program. In particular, we seek to forecast the future American workforce demographic
distribution and utilize the present day valuation methods developed in this article to determine upper and
lower bounds on future program liabilities based upon the best and worst times that beneficiaries from each
cohort can retire from a present day value perspective. This will be compared alongside future revenue
expectations to estimate program solvency duration.

Acknowledgements The authors would like to thank Runhuan Feng for comments related to improving
the present day value calculation methodology considered in this article.
References

[1] (2015) Arias, E., and Xu J. (2018). United States Life Tables, 2015 National Vital Statistics Reports 67 (7).

[2] Bozik, J.E. and Bell, W.R. (1987). Forecasting Age Specific Fertility Using Principal Components. Proceedings of the American Statistical Association, Social Statistics Section. pp. 396401.

[3] Bell, W.R. and Monsell B.C. (1991). Using Principal Components in time Series modeling and Forecasting of Age-Specific Mortality Rates. Proceedings of the American Statistical Association, Social Statistics Section pp. 154159.

[4] Booth, H. and Tickle, L. (2008). Mortality modelling and forecasting: A review of methods. Annals of Actuarial Science 3(1-2) p.3-43.

[5] Brookmeyer R., Corranada MM, Curriero, FC. Kawas C (2002). Survival following a diagnosis of Alzheimer disease. Arch Neurol. 59 (11). p. 1764-1767.

[6] Brown, J. (2003). Redistribution and Insurance: Mandatory Annuitzation With Mortality Heterogeneity. Journal of Risk and Insurance, 70 p.17-41.

[7] Centers for Disease Control Provisional monthly and 12-month ending number of live births, deaths, and infant deaths and rates: United States, Jan. 2009 - Jan. 2014.

[8] Cortazar, G., Schwartz, E., (1994). The valuation of commodity contingent claims. Journal of Derivatives 1, p. 2739.

[9] Diaz, C. J., Koning, S. M., and Martinez, A.P.D. (2016). Moving Beyond Salmon Bias: Mexican Return Migration and Health Selection. Demography 53 (6), p. 2005-2030.

[10] Feldstein, M. (1974). Social Security, Induced Retirement, and Aggregate Capital Accumulation. The Journal of Political Economy 82(5), p. 905-926.

[11] Fong, J., Mitchell, O., and Koh, B. Longevity Risk Management in Singapore’s National Pension System. Journal of Risk and Insurance 78 (4), p.961-982.

[12] U.S. Census Bureau, Real Median Household Income in the United States [MEHOINUSA672N], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/MEHOINUSA672N.

[13] Girosi, Federico and King, Gary. Understanding the Lee-Carter Mortality Forecasting Method.

[14] Hollmann, F.W., T.J. Mulder and J.E. Kallan. 2000. Methodology and Assumptions for the Population Projections of the United States: 1999 to 2100. Working Paper 38, Population Division, U.S. Bureau of Census.

[15] Lee, Ronald D. and Carter, Lawrence R. (1992), Modeling and Forecasting U.S. Mortality. Journal of the American Statistical Association. 87 (419), p. 659-671.

[16] Litterman, R., Scheinkman, J., 1991. Common factors affecting bond returns. Journal of Fixed Income 1, 5461.
[17] Meucci, A (2005). Risk and Asset Allocation. Springer Finance.

[18] Mitchell, D., Brockett, P., Mendoza-Arriaga, R., Muthuraman, K. Forecasting mortality rates. Insurance: Mathematica and Economics. 52 (2013) p. 275-285.

[19] Nalebuff, B., and Zeckhauser, R. (1984). Pensions and the Retirement Decision. NBER Working Paper Series. Working Paper No. 1285.

[20] Novy-Marx, R., and Rauh, J. (2011) Pension Promises: How Big Are They and What are They Worth? Journal of Finance LXVI (4). p.1211-1249.

[21] Ramirez, M. V. (2015). Mortality: Modelling, Socio-Economic Differences and Basis Risk. Doctoral Thesis: City University London.

[22] Renshaw, A.E. and Haberman S. (2006). A cohort-based extension to the LeeCarter model for mortality reduction factors. Insurance: Mathematics and Economics 38 (3) p. 556-570.

[23] Richman, R. and Wüthrich, M. (2018). A Neural Network Extension of the Lee-Carter Model to Multiple Populations. Working Paper. SSRN: abstract 3270877.

[24] Shlens, J. (2005), A Tutorial on Principal Component Analysis.

[25] Clingman, M. and Kunkel, J. (1992). Average Wages for 1985-90 for Indexing under the Social Security Act. Social Security Administration Notes 133.

[26] Annual Statistical Supplement to the Social Security Bulletin, 2017 (2018). SSA Publication No. 13-11700.

[27] Stock, J. and Wise, D. (1990), Pensions, the Option Value of Work, and Retirement. Econometrica 58 (5). p. 1151-1180.

[28] Sundaresan, Suresh and Zapatero, Fernando. Valuation, Optimal Asset Allocation and Retirement Incentives of Pension Plans (1997). The Review of Financial Studies. 10 (3).

[29] Taylor, C. J., et. al (2017). Survival following a diagnosis of heart failure in primary care. Family Practice 34 (2) p. 161-168.

[30] Wahutu. M. et. al. (2016). Pancreatic Cancer: A Survival Analysis Study in Oklahoma. J Okla State Med Assoc. 109 7-8, p. 391-398.

[31] Yang, S. S., Yue, J. C., Huang, H. C. (2010) Modeling longevity risks using a principal component approach: A comparison with existing stochastic mortality models. Insurance: Mathematics and Economics. 46 (1), p. 254-270.