Lorentz invariant supersymmetric mechanism for non(anti)commutative deformations of space-time geometry

A.A. Zheltukhin$^{a,b}$

$^{a}$ Kharkov Institute of Physics and Technology, 61108 Kharkov, Ukraine

$^{b}$ Institute of Theoretical Physics, University of Stockholm
SE-10691, AlbaNova, Stockholm, Sweden

Abstract

A supersymmetric Lorentz invariant mechanism for superspace deformations is proposed. It is based on an extension of superspace by one $\lambda_a$ or several Majorana spinors associated with the Penrose twistor picture. Some examples of Lorentz invariant supersymmetric Poisson and Moyal brackets are constructed and the correspondence: $\theta_{mn} \leftrightarrow i\psi_m \psi_n$, $C_{ab} \leftrightarrow \lambda_a \lambda_b$, $\Psi^a_m \leftrightarrow \psi_m \lambda^a$ mapping the brackets depending on the constant background into the Lorentz covariant supersymmetric brackets is established. The correspondence reveals the role of the composite anticommuting vector $\psi_m = -\frac{i}{2}(\theta_m \lambda)$ as a covariant measure of space-time coordinate noncommutativity.

1 Introduction

The unification of physics and mathematics in the development of noncommutative quantum geometry \cite{1,2,3,4,5,6,7} and field theories \cite{3,9,10,11,12,13} resulted in new ideas and approaches (see reviews \cite{14,15} and additional references therein). One of them has come from string theory, where the noncommutativity of the bosonic string coordinates $x_m$ in the presence of the constant antisymmetric field $B_{mn}$ was observed \cite{10}. More recently, the noncommutativity between the components of the odd spinor coordinate $\theta_a$ in the presence of a constant graviphoton field $C_{ab}$ was considered in \cite{16}. The constant gravitino background $\Psi^a_m$ resulted in the noncommutativity between the $x_m$ and $\theta_a$ coordinates \cite{17,18}. These results have focused attention on the role of constant background fields in superspace deformations. Studying field/string theories and supersymmetry preservation in the superspaces deformed by the graviphoton background \cite{17,18} was further advanced in \cite{19,20}. A general approach to the construction of superspace deformations in a constant background based on the Moyal-Weyl quantization of the Poisson brackets was developed in \cite{11,17,21}. The presence of constant background fields in the much discussed deformed (anti)commutation relations for the (super)coordinate operators leads to the well-recognized problem of Lorentz symmetry breaking. The idea to overcome this problem by using twisted Hopf algebra was recently proposed in \cite{22} and its supersymmetric generalization was realized in \cite{23} and further developed in \cite{24,25}. Another way was observed in \cite{26}, where the Hamiltonian structure of free twistor-like model \cite{27} of super p-brane in $N = 1$ superspace extended by
tensor central charge coordinates was studied and the Dirac bracket-non(anti)commutativity of the brane (super)coordinates was established. The r.h.s. of these D.B’s. have been constructed from the components of auxiliary twistor-like dynamical variables which are Lorentz covariant and supersymmetric. It gives a hint that a hidden spinor structure, associated with the Penrose twistor picture \[28\, 29\, 30\, 31\, 32\] might be an alternative source for the non(anti)commutativity of the quantum space-time (super)coordinates. Accepting such a possibility we start here from the above mentioned spinor extension of the \[\theta\] by \[\bar{\theta}\] supersymmetric brackets, where nonanticommutativity occurs on the components of spin structure, non(anti)commutativity and supergravity. We find also Lorentz invariant and the more sophisticated cases considered below and points to a deep correlation between the B-(modulo the change from the two Majorana spinors \[\lambda, \theta\] with opposite chirality. The generalizations to the higher dimensions \(D = 4\) supersymmetry algebra in the presence of a spinor \(\theta\) composed from the two Majorana spinors \(\lambda, \theta\) and encoding primordial degrees of freedom presented by \(\theta\). In the simplest case there is a correspondence between the Lorentz invariant brackets in question and the known brackets including the constant background fields. That correspondence may be schematically illustrated as the map transforming the field dependent brackets into the new brackets and vice versa:

\[B_{mn} \leftrightarrow i\psi_m \psi_n, \quad C_{ab} \leftrightarrow \lambda_a \lambda_b, \quad \Psi^a_m \leftrightarrow \psi_m \lambda^a.\]
(modulo the change \(B_{mn} \leftrightarrow \theta_{mn} \equiv B_{mn}^{-1}\) etc). The schematical correspondence is preserved in the more sophisticated cases considered below and points to a deep correlation between the spin structure, non(anti)commutativity and supergravity. We find also Lorentz invariant and supersymmetric brackets, where nonanticommutativity occurs only for the components of \(\theta^a\) with opposite chirality. The generalizations to the higher dimensions \(D = 2, 3, 4(mod 8), N > 1\) and several additional spinors are discussed.

2 Supersymmetry algebra in the presence of a spinor coordinate

Using the agreements of \[20\] we accept here the \(D = 4\) \(N = 1\) supersymmetry transformation law in the presence of the twistor-like Majorana spinor coordinates \((\nu_a, \bar{\nu}_{\dot{a}})\) in the form

\[\delta \theta_a = \varepsilon_a, \quad \delta \bar{x}_{a\dot{a}} = 2i(\varepsilon_a \bar{\theta}_{\dot{a}} - \theta_a \bar{\varepsilon}_{\dot{a}}), \quad \delta \nu_a = 0,\]

(1)

The supercharges \(Q_\alpha\) and \(\bar{Q}_{\dot{\alpha}}\) of the superalgebra \[11\] are given by the differential operators

\[Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + 2i\bar{\theta}_a \partial^{a\dot{\alpha}}, \quad \bar{Q}_{\dot{\alpha}} = - (Q^\alpha)^\ast = \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + 2i\theta_a \partial^{a\dot{\alpha}}, \quad [Q_\alpha, \bar{Q}_{\dot{\alpha}}]_+ = 4i\partial^{a\dot{\alpha}},\]

(2)

where \(\partial^{a\dot{\alpha}} \equiv \frac{\partial}{\partial x_{a\dot{a}}}\) and the correspondent supersymmetric covariant derivatives \(D, \bar{D}\) are

\[D_\alpha = \frac{\partial}{\partial \theta^\alpha} - 2i\bar{\theta}_a \partial^{a\dot{\alpha}}, \quad \bar{D}_{\dot{\alpha}} = -(D^\alpha)^\ast = \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - 2i\theta_a \partial^{a\dot{\alpha}}, \quad [D_\alpha, \bar{D}_{\dot{\alpha}}] = -4i\partial^{a\dot{\alpha}},\]

(3)

\([Q_\alpha, \bar{D}_{\dot{\alpha}}]_+ = [Q^\alpha, \bar{D}^\dot{\alpha}]_+ = [\bar{Q}^\dot{\alpha}, D_\alpha]_+ = [\bar{Q}^\dot{\alpha}, \bar{D}^\alpha]_+ = 0.\]

The spinor coordinates \((\nu_a, \bar{\nu}_{\dot{a}})\) and the light-like real vector \(\varphi_{a\dot{a}}\) composed from them

\[\varphi_{a\dot{a}} \equiv \nu_a \bar{\nu}_{\dot{a}}, \quad \varphi_{a\dot{a}} \nu^\alpha = \varphi_{a\dot{a}} \bar{\nu}^\dot{\alpha} = 0, \quad \delta \varphi_{a\dot{\beta}} = 0\]

(4)
may be used to construct the Lorentz invariant differential operators $D, \bar{D}, \partial$

$$D = \nu_\alpha D^\alpha, \quad \bar{D} = \bar{\nu}_\dot{\alpha} \bar{D}^{\dot{\alpha}}, \quad \partial = \varphi_{\alpha\dot{\alpha}} \partial^{\alpha\dot{\alpha}}$$

which form a supersymmetric subalgebra of the algebra of derivatives

$$[D, \bar{D}]_+ = -4i\partial, \quad [D, D]_+ = [\bar{D}, \bar{D}]_+ = 0, \quad [D, \partial] = [\bar{D}, \partial] = [\partial, \partial] = 0. \quad (6)$$

Introducing $D_\pm$ combinations of the invariant derivatives $D, \bar{D}$

$$D_\pm \equiv D \pm \bar{D} \quad (7)$$

one can split the Lorentz invariant complex subalgebra (6) into two invariant and (anti)commuting subalgebras formed by the generators $(D_-, \partial)$ and $(D_+, \partial)$

$$[D_\pm, D_\pm]_+ = \mp 8i\partial, \quad [D_+ , D_-]_+ = 0, \quad [D_\pm, \partial] = [\partial, \partial] = 0. \quad (8)$$

A twistor-like character of the Majorana spinor $(\nu_\alpha, \bar{\nu}_{\dot{\alpha}})$ means that its dilatations, generated by the differential operator $\Delta$

$$\Delta = \nu_\alpha \frac{\partial}{\partial \nu_\alpha} + \bar{\nu}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\nu}_{\dot{\alpha}}},$$

have to be an additional symmetry of physical theories of massless fields. Taking into account this dilaton symmetry assumes extension of the superalgebra (6) by the dilaton generator $\Delta$

$$[D, \bar{D}]_+ = -4i\partial, \quad [D, D]_+ = [\bar{D}, \bar{D}]_+ = 0, \quad [\Delta, D] = D, \quad [\Delta, \bar{D}] = \bar{D}, \quad [\Delta, \partial] = 2\partial, \quad (10)$$

$$[\partial, D] = [\partial, \bar{D}] = [\partial, \partial] = [\Delta, \Delta] = 0,$$

which has two real anticommutative subalgebras formed by the generators $(D_{\pm}, \Delta)$

$$[D_{\pm}, D_{\pm}]_+ = \mp 8i\partial, \quad [\Delta, D_{\pm}] = D_{\pm}, \quad [\Delta, \partial] = 2\partial, \quad (11)$$

$$[D_+, D_-]_+ = [D_{\pm}, \partial] = [\partial, \partial] = [\Delta, \Delta] = 0.$$

The Lorentz invariant supersymmetric differential operators forming the superalgebras (10), (11) may be used as building blocks for the construction of Lorentz invariant and supersymmetric non(anti)commutative relations among quantum operators of (super)coordinates.

### 3 Lorentz invariant supersymmetric Poisson brackets: non(anti)commutativity of space-time coordinates

To clarify the role of $(\nu_\alpha, \bar{\nu}_{\dot{\alpha}})$ in the formation of Lorentz invariant non(anti)commutative relations among $x_{\alpha\dot{\alpha}}, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ we consider the Poisson bracket constructed from the three differential operators $(D_-, \partial, \Delta)$ forming the simplest superalgebra (11)

$$\{F,G\} = F \left[ -\frac{i}{4} \rightarrow \partial \rightarrow \partial \rightarrow D_- \rightarrow \Delta \rightarrow \Delta \right] G, \quad (12)$$

where $\{,\}_{P.B.} \equiv \{,\}$ and $F(x, \theta, \bar{\theta}, \nu, \bar{\nu}), G(x, \theta, \bar{\theta}, \nu, \bar{\nu})$ are generalized superfields depending on both the superspace coordinates $(x, \theta, \bar{\theta})$ and on the spinor coordinates $(\nu, \bar{\nu})$. The Lorentz
invariant and supersymmetric differential operators $\tilde{D}_-, \tilde{\partial}, \tilde{\Delta}$ define derivatives acting from
the right hand side. Conversely, the differential operators (11) act from the l.h.s and coincide
with the left derivatives $\tilde{D}_-, \tilde{\partial}, \tilde{\Delta}$

$$\tilde{D}_- G \equiv D_- G, \quad \tilde{\partial} G \equiv \partial G, \quad \tilde{\Delta} G \equiv \Delta G$$

The left and right invariant derivatives in (12) are connected by the relations

$$F \tilde{D} = (-1)^f \tilde{D} F, \quad F \tilde{\partial} = (-1)^f \tilde{\partial} F, \quad F \tilde{\Delta} = \Delta F,$$

where $f = 0, 1$ is the Grassmannian grading of the superfield $F$.

The action of $D, \tilde{D}, D_{\pm}$ on the composite coordinates $\psi$ and $\varphi$ is given by the relations

$$\tilde{D} \psi_{a\dot{a}} = -i\varphi_{a\dot{a}}, \quad \tilde{D} \psi_{a\dot{a}} = i\varphi_{a\dot{a}}, \quad \psi_{a\dot{a}} \equiv i(\nu_{a}\theta_{\dot{a}} - \theta_{a}\bar{\nu}_{\dot{a}}),$$

$$\tilde{D}_- \psi_{a\dot{a}} = -2i\varphi_{a\dot{a}}, \quad \tilde{D}_+ \psi_{a\dot{a}} = \tilde{D}_{\pm} \varphi_{a\dot{a}} = 0.$$  

After the substitutions of the (super)coordinates under discussion in the P.B. (12) we find
non(anti)commutative P.B.’s. for them.

The twistor-like coordinates have zero P.B.’s. among themselves

$$\{\nu_{a}, \nu_{\dot{a}}\} = \{\nu_{a}, \bar{\nu}_{\dot{a}}\} = \{\bar{\nu}_{a}, \bar{\nu}_{\dot{a}}\} = 0$$

and with the Grassmannian spinors $(\theta_{a}, \bar{\theta}_{\dot{a}})$

$$\{\nu_{a}, \theta_{\dot{a}}\} = \{\nu_{a}, \bar{\theta}_{\dot{a}}\} = \{\bar{\nu}_{a}, \theta_{\dot{a}}\} = \{\bar{\nu}_{a}, \bar{\theta}_{\dot{a}}\} = 0.$$  

However, they have non zero P.B.’s. with the space-time coordinates $x_{a\dot{a}}$

$$\{x_{a\dot{a}}, \nu_{\dot{a}}\} = \varphi_{a\dot{a}}\nu_{\dot{a}}, \quad \{x_{a\dot{a}}, \bar{\nu}_{\dot{a}}\} = \varphi_{a\dot{a}}\bar{\nu}_{\dot{a}}.$$  

The remaining non zero P.B.’s. define the P.B.’s. among the space-time coordinates $x_{a\dot{a}}$ and spinors $(\theta_{a}, \bar{\theta}_{\dot{a}})$

$$\{x_{a\dot{a}}, x_{\dot{b}\dot{b}}\} = -i\psi_{a\dot{a}}\varphi_{\dot{b}\dot{b}},$$

$$\{x_{a\dot{a}}, \theta_{\dot{b}}\} = \frac{i}{2}\psi_{a\dot{a}}\varphi_{\dot{b}}, \quad \{x_{a\dot{a}}, \bar{\theta}_{\dot{b}}\} = -\frac{i}{2}\psi_{a\dot{a}}\bar{\varphi}_{\dot{b}},$$

$$\{\theta_{a}, \theta_{\dot{b}}\} = \frac{i}{4}\varphi_{a\dot{b}}, \quad \{\theta_{a}, \bar{\theta}_{\dot{b}}\} = -\frac{i}{4}\varphi_{a\dot{b}}, \quad \{\bar{\theta}_{a}, \bar{\theta}_{\dot{b}}\} = \frac{i}{4}\bar{\varphi}_{a\dot{b}},$$

where $\varphi_{a\dot{b}}, \varphi_{a\dot{b}}, \bar{\varphi}_{a\dot{b}}$ are composite symmetric spin-tensors

$$\varphi_{a\dot{b}} \equiv \nu_{a}\nu_{\dot{b}}, \quad \varphi_{a\dot{b}} \equiv \bar{\nu}_{a}\bar{\nu}_{\dot{b}}, \quad \delta\varphi_{a\dot{b}} = \delta\varphi_{a\dot{b}} = 0$$

orthogonal to the vector $\varphi_{a\dot{a}}$ and to the composite Grassmannian vector $\psi_{a\dot{a}}$

$$\psi_{a\dot{a}} \equiv i(\nu_{a}\bar{\theta}_{\dot{a}} - \theta_{a}\bar{\nu}_{\dot{a}}), \quad \varphi_{a\dot{a}}\psi_{a\dot{a}} = \varphi_{a\dot{a}}\psi_{a\dot{a}} = \varphi_{a\dot{a}}\psi_{a\dot{a}} = 0, \quad \delta\psi_{a\dot{a}} = -i(\varepsilon_{a}\bar{\nu}_{\dot{a}} - \bar{\varepsilon}_{a}\nu_{a})$$

The appearance of the odd vector $\psi_{a\dot{a}}$ associated with the description of the spin
degrees of freedom of fermions in the r.h.s. of P.B.’s. (13) hints on a spin structure of super-spaces in back of the coordinate’s non(anti)commutativity. The Lorentz covariance of the Poisson brackets (16)-(19) is provided by the spinor, vector and spin-tensor representations of the Lorentz group involved in the r.h.s. of the Poisson brackets. These P.B.’s. are also supersymmetric by the construction.

In the next Section we prove the Jacobi identities for the P.B.’s. (16)-(19).
4 Proof of the Jacobi identities

The graded Jacobi identities for the considered P.B. algebra have the standard form

$$\{\{A, B\}, C\} + (-1)^{(b+c)a}\{\{B, C\}, A\} + (-1)^{(a+b)c}\{\{C, A\}, B\} = 0,$$

(22)

where $a = 0, 1$ is the Grassmannian grading of $A$. To prove these identities for the P.B.'s. (16)-(19) one needs to study the Poisson brackets of the composite vectors $\varphi_{a\dot{b}}$ (14), $\psi_{a\dot{a}}$ (21) and spin-tensors $\varphi_{a\dot{b}}, \bar{\varphi}_{\dot{a}b}$ (20) between themselves and with $x_{a\dot{a}}, \theta_{\alpha}, \bar{\theta}_{\dot{a}}$. The P.B.'s. (16), (17) together with the definitions (20), (21) show the P.B.-commutativity of $\varphi_{a\dot{b}}, \varphi_{a\dot{b}}, \bar{\varphi}_{\dot{a}b}$ between themselves and with $(\nu_{\alpha}, \nu_{\dot{a}})$, $(\theta_{\alpha}, \bar{\theta}_{\dot{a}})$ and $\psi_{a\dot{a}}$

$$\{\varphi_{*}, \nu_{\alpha}\} = \{\varphi_{**}, \nu_{\dot{a}}\} = \{\psi_{a\dot{a}}, \nu_{\beta}\} = \{\psi_{a\dot{a}}, \bar{\nu}_{\dot{b}}\} = 0,$$

$$\{\varphi_{*}, \varphi_{**}\} = \{\varphi_{**}, \theta_{\alpha}\} = \{\varphi_{**}, \bar{\theta}_{\dot{a}}\} = \{\varphi_{**}, \psi_{\gamma\bar{\gamma}}\} = 0,$$

(23)

where $\varphi_{*} \equiv (\varphi_{a\dot{b}}, \bar{\varphi}_{b\dot{a}}, \bar{\varphi}_{\dot{a}b})$ is a condensed symbol for the composite coordinates (14) and (20). However, the P.B. of the spin-tensors $\varphi_{a\dot{b}}, \bar{\varphi}_{b\dot{a}}, \bar{\varphi}_{\dot{a}b}$ with $x_{\gamma\bar{\gamma}}$ are different from zero

$$\{x_{a\dot{a}}, \varphi_{\beta\gamma}\} = 2\varphi_{a\dot{a}}\varphi_{\beta\gamma}, \quad \{x_{a\dot{a}}, \bar{\varphi}_{b\dot{a}}\} = 2\varphi_{a\dot{a}}\bar{\varphi}_{b\dot{a}}, \quad \{x_{a\dot{a}}, \bar{\varphi}_{\dot{a}b}\} = 2\varphi_{a\dot{a}}\bar{\varphi}_{\dot{a}b},$$

(24)

as well as, the P.B. between $x_{a\dot{a}}, \psi_{b\dot{b}}$ (and $(\theta_{\beta}, \bar{\theta}_{\dot{b}})$

$$\{x_{\dot{a}a}, \psi_{b\dot{b}}\} = \varphi_{a\dot{a}}\psi_{b\dot{b}} + \varphi_{b\dot{b}}\psi_{a\dot{a}},$$

$$\{\psi_{a\dot{a}}, \psi_{b\dot{b}}\} = -i\varphi_{a\dot{a}}\varphi_{b\dot{b}},$$

$$\{\psi_{a\dot{a}}, \theta_{\beta}\} = \frac{1}{2}\varphi_{a\dot{a}}\psi_{b\dot{b}}, \quad \{\psi_{a\dot{a}}, \bar{\theta}_{\dot{b}}\} = -\frac{1}{2}\varphi_{a\dot{a}}\bar{\psi}_{\dot{b}}.$$

(25)

A combination of the P.B. relations (23) together with ones (26) results in the relation

$$\{\{\psi_{a\dot{a}}, \psi_{b\dot{b}}\}, \psi_{\gamma\bar{\gamma}}\} = 0$$

(26)

which proves the graded Jacobi identity (12) for the case $A = B = C = \psi$

$$Cycle\{\psi_{a\dot{a}}, \psi_{b\dot{b}}\}, \psi_{\gamma\bar{\gamma}}\} = 0$$

(27)

The same result occurs for the Jacoby cycles cubic in $(\theta_{\alpha}, \bar{\theta}_{\dot{a}})$

$$Cycle\{\{\theta_{\alpha}, \theta_{\beta}\}, \theta_{\gamma}\} = ... = Cycle\{\{\bar{\theta}_{\dot{a}}, \bar{\theta}_{\dot{b}}\}, \bar{\theta}_{\bar{\gamma}}\} = 0,$$

(28)

as well as, for the cycles quadratic in $\theta_{\alpha}$ or $\psi_{a\dot{a}}$ and linear in $(\nu_{\gamma}, \bar{\nu}_{\bar{\gamma}})$

$$Cycle\{\{\theta_{\alpha}, \theta_{\beta}\}, \nu_{\gamma}\} = ... = Cycle\{\{\bar{\theta}_{\dot{a}}, \bar{\theta}_{\dot{b}}\}, \bar{\nu}_{\bar{\gamma}}\} = 0,$$

$$Cycle\{\{\psi_{a\dot{a}}, \psi_{b\dot{b}}\}, \nu_{\gamma}\} = Cycle\{\{\psi_{a\dot{a}}, \psi_{b\dot{b}}\}, \bar{\nu}_{\bar{\gamma}}\} = 0$$

(29)

and for other trivial Jacobi cycles cubic or quadratic in $(\nu_{\gamma}, \bar{\nu}_{\bar{\gamma}})$ and linear in $(\theta_{\alpha}, \bar{\theta}_{\dot{a}})$ or $\psi_{a\dot{a}}$. To calculate the Jacobi cycle cubic in $x_{a\dot{a}}$ we use the relation

$$\{\{x_{a\dot{a}}, x_{b\dot{b}}\}, x_{\gamma\bar{\gamma}}\} = 2i(\varphi_{a\dot{a}}\varphi_{b\dot{b}})\varphi_{\gamma\bar{\gamma}} + i(\varphi_{a\dot{a}}\varphi_{b\dot{b}} - \varphi_{b\dot{b}}\varphi_{a\dot{a}})\varphi_{\gamma\bar{\gamma}}$$

(30)

arisen from the P.B.’s. (19) and (25) and resulting in zero Jacobi cycle

$$Cycle\{\{x_{a\dot{a}}, x_{b\dot{b}}\}, x_{\gamma\bar{\gamma}}\} = 0.$$

(31)
It follows from the mutual cancellation between the contributions of first and last summands in the r.h.s. of the cyclic sum generated by Eq. (30). Next one can see that the Jacobi cycles quadratic in $x_{\alpha \bar{\alpha}}$ and linear in $\psi_{\alpha \bar{\alpha}}$ or $(\theta_{\alpha}, \bar{\theta}_{\alpha})$ are equal to zero

$$
Cycle\{\{x_{\alpha \bar{\alpha}}, x_{\beta \bar{\beta}}\}, \psi_{\gamma \bar{\gamma}}\} = 0,
$$

$$
Cycle\{\{x_{\alpha \bar{\alpha}}, x_{\beta \bar{\beta}}\}, \theta_{\gamma}\} = Cycle\{\{x_{\alpha \bar{\alpha}}, x_{\beta \bar{\beta}}\}, \bar{\theta}_{\gamma}\} = 0,
$$

because of the relations

$$
\{\{x_{\alpha \bar{\alpha}}, x_{\beta \bar{\beta}}\}, \psi_{\gamma \bar{\gamma}}\} = -(\psi_{\alpha \bar{\alpha}}\varphi_{\beta \bar{\beta}} - \varphi_{\alpha \bar{\alpha}}\psi_{\beta \bar{\beta}})\varphi_{\gamma \bar{\gamma}},
$$

$$
\{\{x_{\beta \bar{\beta}}, \psi_{\gamma \bar{\gamma}}\}, x_{\alpha \bar{\alpha}}\} - (\alpha \leftrightarrow \beta) = (\psi_{\alpha \bar{\alpha}}\varphi_{\beta \bar{\beta}} - \varphi_{\alpha \bar{\alpha}}\psi_{\beta \bar{\beta}})\varphi_{\gamma \bar{\gamma}},
$$

$$
\{\{x_{\alpha \bar{\alpha}}, x_{\beta \bar{\beta}}\}, \theta_{\gamma}\} = -\frac{i}{2}(\psi_{\alpha \bar{\alpha}}\varphi_{\beta \bar{\beta}} - \varphi_{\alpha \bar{\alpha}}\psi_{\beta \bar{\beta}})\nu_{\gamma},
$$

$$
\{\{x_{\beta \bar{\beta}}, \theta_{\gamma}\}, x_{\alpha \bar{\alpha}}\} - (\alpha \leftrightarrow \beta) = \frac{i}{2}(\psi_{\alpha \bar{\alpha}}\varphi_{\beta \bar{\beta}} - \varphi_{\alpha \bar{\alpha}}\psi_{\beta \bar{\beta}})\nu_{\gamma}
$$

and their complex conjugate following from the P.B.’s, (18), (19) and (25). A similar cancellation takes place in the Jacobi cycles quadratic in $\psi_{\alpha \bar{\alpha}}$ and linear in $x_{\alpha \bar{\alpha}}$ or $\theta_{\alpha}, \bar{\theta}_{\alpha}$

$$
Cycle\{\{\psi_{\alpha \bar{\alpha}}, \psi_{\beta \bar{\beta}}\}, x_{\gamma \bar{\gamma}}\} = 0.
$$

$$
Cycle\{\{\psi_{\alpha \bar{\alpha}}, \psi_{\beta \bar{\beta}}\}, \theta_{\gamma}\} = Cycle\{\{\psi_{\alpha \bar{\alpha}}, \psi_{\beta \bar{\beta}}\}, \bar{\theta}_{\gamma}\} = 0,
$$

as it follows from the P.B. relations

$$
\{\{\psi_{\alpha \bar{\alpha}}, \psi_{\beta \bar{\beta}}\}, x_{\gamma \bar{\gamma}}\} = 4i\varphi_{\alpha \bar{\alpha}}\varphi_{\beta \bar{\beta}}\varphi_{\gamma \bar{\gamma}},
$$

$$
\{\{x_{\gamma \bar{\gamma}}, \psi_{\beta \bar{\beta}}\}, \psi_{\alpha \bar{\alpha}}\} + (\alpha \leftrightarrow \beta) = 4i\varphi_{\alpha \bar{\alpha}}\varphi_{\beta \bar{\beta}}\varphi_{\gamma \bar{\gamma}},
$$

$$
\{\{\psi_{\alpha \bar{\alpha}}, \psi_{\beta \bar{\beta}}\}, \theta_{\gamma}\} = \{\{\psi_{\alpha \bar{\alpha}}, \theta_{\gamma}\}, \psi_{\beta \bar{\beta}}\} = 0.
$$

Next we prove the Jacobi identities for cycles quadratic in $\theta_{\alpha}, \bar{\theta}_{\alpha}$ and linear in $x_{\alpha \bar{\alpha}}$ or $\psi_{\alpha \bar{\alpha}}$

$$
Cycle\{\{\theta_{\alpha}, \theta_{\beta}\}, x_{\gamma \bar{\gamma}}\} = \{\{\theta_{\alpha}, \theta_{\beta}\}, \psi_{\gamma \bar{\gamma}}\} = 0,
$$

$$
Cycle\{\{\theta_{\alpha}, \bar{\theta}_{\beta}\}, x_{\gamma \bar{\gamma}}\} = \{\{\theta_{\alpha}, \bar{\theta}_{\beta}\}, \psi_{\gamma \bar{\gamma}}\} = 0,
$$

and for their complex conjugate using the relations

$$
\{\{\theta_{\alpha}, \theta_{\beta}\}, x_{\gamma \bar{\gamma}}\} = -\frac{i}{2}\varphi_{\alpha \beta}\varphi_{\gamma \bar{\gamma}},
$$

$$
\{\{\theta_{\alpha}, \bar{\theta}_{\beta}\}, x_{\gamma \bar{\gamma}}\} = \frac{i}{2}\varphi_{\alpha \beta}\varphi_{\gamma \bar{\gamma}},
$$

$$
\{\{x_{\gamma \bar{\gamma}}, \theta_{\beta}\}, \theta_{\alpha}\} + (\alpha \leftrightarrow \beta) = \frac{i}{2}\varphi_{\alpha \beta}\varphi_{\gamma \bar{\gamma}},
$$

$$
\{\{x_{\gamma \bar{\gamma}}, \bar{\theta}_{\beta}\}, \theta_{\alpha}\} + \{\{x_{\gamma \bar{\gamma}}, \bar{\theta}_{\beta}\}, \bar{\theta}_{\alpha}\} = \frac{i}{2}\varphi_{\alpha \beta}\varphi_{\gamma \bar{\gamma}}.
$$

together with the relations

$$
\{\{\theta_{\alpha}, \theta_{\beta}\}, \psi_{\gamma \bar{\gamma}}\} = \{\{\theta_{\alpha}, \psi_{\gamma \bar{\gamma}}\}, \theta_{\beta}\} = 0,
$$

$$
\{\{\theta_{\alpha}, \bar{\theta}_{\beta}\}, \psi_{\gamma \bar{\gamma}}\} = \{\{\theta_{\alpha}, \psi_{\gamma \bar{\gamma}}\}, \bar{\theta}_{\beta}\} = 0
$$

and their complex conjugate. The remaining nontrivial and also vanishing Jacobi cycles are formed by any three coordinates from the set $[x_{\alpha \bar{\alpha}}, \psi_{\alpha \bar{\alpha}}, (\theta_{\alpha}, \bar{\theta}_{\alpha}), (\nu_{\alpha}, \nu_{\bar{\alpha}})]$

$$
Cycle\{\{x_{\alpha \bar{\alpha}}, \psi_{\beta \bar{\beta}}\}, \theta_{\gamma}\} = Cycle\{\{x_{\alpha \bar{\alpha}}, \psi_{\beta \bar{\beta}}\}, \nu_{\gamma}\} = 0,
$$

$$
Cycle\{\{x_{\alpha \bar{\alpha}}, \theta_{\beta}\}, \nu_{\gamma}\} = Cycle\{\{\psi_{\alpha \bar{\alpha}}, \theta_{\beta}\}, \nu_{\gamma}\} = 0.
$$
Their complex conjugate cycles equal zero too. The proof of first and second Jacobi identities in (39) is based on the P.B. relations
\[
\{\{x_{\alpha\dot{\alpha}}, \psi_{\beta\dot{\beta}}\}, \theta_\gamma\} = -\frac{2}{3}\{\{\psi_{\beta\dot{\beta}}, \theta_\gamma\}, x_{\alpha\dot{\alpha}}\} = 2\{\{x_{\alpha\dot{\alpha}}, \theta_\gamma\}, \psi_{\beta\dot{\beta}}\} = \varphi_{\alpha\dot{\alpha}}\varphi_{\beta\dot{\beta}}\nu_\gamma, \\
\{\{x_{\alpha\dot{\alpha}}, \psi_{\beta\dot{\beta}}\}, \nu_\gamma\} = \{\{\psi_{\beta\dot{\beta}}, \nu_\gamma\}, x_{\alpha\dot{\alpha}}\} = \{\{\nu_\gamma, x_{\alpha\dot{\alpha}}\}, \psi_{\beta\dot{\beta}}\} = 0. 
\]
(40)

The proof of third and fourth Jacobi identitities in (39) uses the P.B. relations
\[
\{\{x_{\alpha\dot{\alpha}}, \theta_\beta\}, \nu_\gamma\} = \{\{\theta_\beta, \nu_\gamma\}, x_{\alpha\dot{\alpha}}\} = \{\{\nu_\gamma, x_{\alpha\dot{\alpha}}\}, \theta_\beta\} = 0, \\
\{\{\psi_{\alpha\dot{\alpha}}, \theta_\beta\}, \nu_\gamma\} = \{\{\theta_\beta, \nu_\gamma\}, \psi_{\alpha\dot{\alpha}}\} = \{\{\nu_\gamma, \psi_{\alpha\dot{\alpha}}\}, \theta_\beta\} = 0 
\]
(41)

which follow from the P.B. relations (17), (19), (23), (25). It completes the proof of the Jacobi identities for the above introduced Lorentz invariant Poisson brackets. The next step is to use them for the construction of the Moyal brackets.

5 Lorentz invariant and supersymmetric star product

A transition to the quantum picture based on the P.B. (12) may be done by using the well known Weyl-Moyal correspondence which establishes one to one correspondence between quantum field operators and their symbols acting on commutative space-time. Then the quantum information is encoded in the change of the usual product by the Moyal \( \star \)-product of their Weyl symbols. To realise this prescription here we note that the P.B. (12) may be presented as
\[
\{F, G\} = F \, \tilde{D}_\Lambda \, C^{\Lambda\Sigma} \, \tilde{D}_\Sigma \, G = \\
\bar{F}(\tilde{\partial}, \tilde{\Delta}, \tilde{D}_-) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} \begin{pmatrix} \tilde{\partial} \\ \tilde{\Delta} \\ \tilde{D}_- \end{pmatrix} G, 
\]
(42)

where the condensed notation \( \tilde{D}_\Lambda = (D_-, \partial, \Delta) \) was used for the invariant derivatives of the \((-\cdot)\)-superalgebra (11) numerated by the index \( \Lambda \) running over the even and odd variables. As a result, the superalgebra (11) is presented in a condensed form
\[
[D_\Lambda, D_\Sigma] = C^{\Lambda\Sigma\Xi} \, D_\Xi, 
\]
(43)

where \( C^{\Lambda\Sigma\Xi} \) are the structural constants defined by the explicit (anti)commutation relations (11) and \( C^{\Lambda\Sigma} \) is represented by the 3 \( \times \) 3 matrice
\[
C^{\Lambda\Sigma} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{i}{4} \end{pmatrix}. 
\]
(44)

The representation (12) defines the Moyal \( \star \)-product of the superfields \( F \) and \( G \)
\[
F \star G = Fe^{\tilde{D}_\Lambda C^{\Lambda\Sigma} \tilde{D}_\Sigma} G, 
\]
(45)
where the Planck constant and the velocity of light are chosen to be equal to unit. The definition (45) together with (12) yield the Moyal products of the (super)coordinates

\[ x_{\alpha\dot{\alpha}} \star x_{\beta\dot{\beta}} = x_{\alpha\dot{\alpha}} x_{\beta\dot{\beta}} - \frac{i}{2} \bar{\psi}_{\alpha\dot{\alpha}} \psi_{\beta\dot{\beta}}, \]

\[ x_{\alpha\dot{\alpha}} \star \theta_\beta = x_{\alpha\dot{\alpha}} \theta_\beta + \frac{i}{2} \bar{\psi}_{\alpha\dot{\alpha}} \nu_\beta, \quad x_{\alpha\dot{\alpha}} \star \bar{\theta}_{\dot{\beta}} = x_{\alpha\dot{\alpha}} \bar{\theta}_{\dot{\beta}} - \frac{i}{2} \bar{\psi}_{\alpha\dot{\alpha}} \bar{\nu}_{\dot{\beta}}, \]

\[ \theta_\alpha \star \theta_\beta = \theta_\alpha \theta_\beta + \frac{i}{8} \varphi_{\alpha\beta}, \quad \theta_\alpha \star \bar{\theta}_{\dot{\beta}} = \theta_\alpha \bar{\theta}_{\dot{\beta}} - \frac{i}{8} \bar{\varphi}_{\alpha\dot{\beta}}, \]

\[ \bar{\theta}_{\dot{\alpha}} \star \bar{\theta}_{\dot{\beta}} = \bar{\theta}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} + \frac{i}{8} \bar{\varphi}_{\dot{\alpha}\dot{\beta}}. \] (46)

Consequently, the (anti)commutators of the coordinate operators are replaced by the following Lorentz invariant and supersymmetric Moyal brackets

\[ [x_{\alpha\dot{\alpha}}, x_{\beta\dot{\beta}}]_* = x_{\alpha\dot{\alpha}} \star x_{\beta\dot{\beta}} - x_{\beta\dot{\beta}} \star x_{\alpha\dot{\alpha}} = -i \psi_{\alpha\dot{\alpha}} \psi_{\beta\dot{\beta}}, \]

\[ [x_{\alpha\dot{\alpha}}, \theta_\beta]_* = \frac{i}{2} \psi_{\alpha\dot{\alpha}} \nu_\beta, \quad [x_{\alpha\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}]_* = -\frac{i}{2} \bar{\psi}_{\alpha\dot{\alpha}} \bar{\nu}_{\dot{\beta}}, \] (47)

\[ [\theta_\alpha, \theta_\beta]_* = -\frac{i}{4} \varphi_{\alpha\beta}, \quad [\theta_\alpha, \bar{\theta}_{\dot{\beta}}]_* = -\frac{i}{4} \bar{\varphi}_{\alpha\dot{\beta}}, \quad [\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}]_* = \frac{i}{4} \bar{\varphi}_{\dot{\alpha}\dot{\beta}}. \]

which in turn are directly restored from the invariant P.B.'s. (46)-(49). The change \( \{,\} \rightarrow [\,,\,]_* \) restores the remaining Moyal brackets originated from the above considered P.B.'s that together with the brackets (47) may be used for the studying Lorentz invariant and supersymmetric quantum field models in non(anti)commutative superspace.

### 6 Noncommutativity of the twistor components

The twistor associated with \( \nu_\alpha \) and \( x_{\alpha\dot{\alpha}} \) is formed by the pair \( Z^A = (\omega^\alpha, \bar{\nu}_{\dot{\alpha}}) \), where the first twistor element \( \omega^\alpha \) is composed from \( \nu_\alpha \) and \( x_{\alpha\dot{\alpha}} \)

\[ \omega_\alpha = i x_{\alpha\dot{\alpha}} \bar{\nu}^{\dot{\alpha}}. \] (48)

The considered Poisson and Moyal brackets result in the commutativity between the twistor components \( \omega_\alpha, \nu_\beta \) and their complex conjugate

\[ \{\omega_\alpha, \nu_\beta\} = \{\omega_\alpha, \bar{\nu}_{\dot{\beta}}\} = \{\bar{\omega}_{\dot{\alpha}}, \nu_\beta\} = \{\bar{\omega}_{\dot{\alpha}}, \bar{\nu}_{\dot{\beta}}\} = 0, \] (49)

because of the P.B.'s. (16), (18) and the orthogonality relations

\[ \varphi_{\alpha\dot{\alpha}} \nu^\alpha = \varphi_{\alpha\dot{\alpha}} \bar{\nu}^{\dot{\alpha}} = 0. \] (50)

However, \( \omega_\alpha \) and \( \bar{\omega}_{\dot{\alpha}} \) have non zero brackets with \( x_{\alpha\dot{\alpha}} \)

\[ \{x_{\alpha\dot{\alpha}}, \omega_\beta\} = \varphi_{\alpha\dot{\alpha}} \omega_\beta - i \bar{\eta} \psi_{\alpha\dot{\alpha}} \nu_\beta, \quad \{x_{\alpha\dot{\alpha}}, \bar{\omega}_{\dot{\beta}}\} = \varphi_{\alpha\dot{\alpha}} \bar{\omega}_{\dot{\beta}} - i \eta \bar{\psi}_{\alpha\dot{\alpha}} \bar{\nu}_{\dot{\beta}}, \] (51)

as well as, with \( \theta_\alpha \) and \( \bar{\theta}_{\dot{\alpha}} \)

\[ \{\omega_\alpha, \theta_\beta\} = -\frac{i}{2} \bar{\eta} \varphi_{\alpha\beta}, \quad \{\omega_\alpha, \bar{\theta}_{\dot{\beta}}\} = \frac{i}{2} \bar{\eta} \bar{\varphi}_{\alpha\dot{\beta}}, \quad \eta \equiv \theta_\alpha \nu^\alpha, \] (52)

because of the P.B.'s. (17), (19), (49). The Grassmannian scalar \( \eta \) has zero P.B.'s. with \( \nu, \omega, \theta \)

\[ \{\eta, \nu_\alpha\} = \{\eta, \omega_\alpha\} = \{\eta, \theta_\alpha\} = 0 \] (53)
and their complex conjugate. The multiplication of the relations (51) by \((i\bar{\nu}\dot{\nu})\) together with using (49) and (50) yield zero brackets between the components \(\omega_\alpha\) and \(\omega_\beta\) of the same chirality
\[
\{\omega_\alpha, \omega_\beta\} = i\bar{\eta}\eta\varphi_{\alpha\beta} \equiv 0,
\]
but yields zero brackets with \(\bar{\omega}_{\dot{\beta}}\)
\[
\{\omega_\alpha, \bar{\omega}_{\dot{\beta}}\} = 0.
\]
Comparing the latter bracket with the bracket (19) for \(\theta_\alpha\) and \(\bar{\theta}_{\dot{\beta}}\) we get the relation
\[
\{\omega_\alpha, \bar{\omega}_{\dot{\beta}}\} = -4\eta\bar{\eta}\{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\}
\]
pointing out a connection of the \((\omega, \bar{\omega})\) – noncommutativity with the \((\theta, \bar{\theta})\) – nonanti-commutativity. On the other side, it shows the connection of the twistor complex structure with supersymmetry. Therefore, the choice of \((\theta, \bar{\theta})\) – nonanticommutative bracket induces the \((\omega, \bar{\omega})\) – noncommutative bracket. Such a correlation of the spin complex structure with supersymmetry and non(anti)commutativity deserves more careful studying.

7 Lorentz invariant brackets in higher dimensions

The brackets (17)-(19) get more compact form in the Majorana representation
\[
\nu_a = \left(\nu_\alpha \right)_{\bar{\nu}^\alpha}, \quad \theta_a = \left(\theta_\alpha \right)_{\bar{\theta}^\alpha}, \quad C^{ab} = \left(\begin{array}{cc} \bar{\varepsilon}^{\alpha\beta} & 0 \\ 0 & \bar{\varepsilon}_{\dot{\alpha}\dot{\beta}} \end{array} \right), \quad \chi^a = C^{ab}\chi_b
\]
for the considered Weyl spinors \(\nu_\alpha, \theta_\alpha\), their c.c. and the charge conjugation matrix \(C^{ab}\). Then the P.B’s. (16)-(18) are presented in a form suitable for generalizations. The P.B’s. (16)-(18) take the form
\[
\{\nu_a, \nu_b\} = 0, \quad \{\theta_a, \nu_b\} = 0, \quad \{x_m, \nu_a\} = \varphi_m\nu_a,
\]
where the real vectors \(x_m\) and \(\varphi_m\) are defined (37) by the relations
\[
\begin{align*}
\varphi_m &= -\frac{1}{2}(\bar{\sigma}_m)^{\dot{\alpha}\alpha}\varphi_{\dot{\alpha}} \equiv -\frac{1}{4}(\bar{\nu}\gamma_m\nu) \\
\varphi_m &= \frac{1}{2}(\bar{\sigma}_m)^{\dot{\alpha}\alpha} \varphi_{\dot{\alpha}} \equiv \frac{1}{4}(\bar{\nu}\gamma_m\nu)
\end{align*}
\]
and \(\gamma_m\) are the Dirac matrices in the Majorana representation.

To rewrite the rest of the P.B’s. in the Majorana representation it is convenient to change the Majorana spinor \(\nu_a\) by other Majorana spinor \(\lambda_a\)
\[
\lambda_a = \left(\begin{array}{c} \lambda_\alpha \\bar{\lambda}^\alpha \end{array} \right) \equiv (\gamma_5\nu)_a, \quad (\gamma_5)_a = \left(\begin{array}{cc} -i\bar{\delta}_\alpha^\beta & 0 \\ 0 & i\bar{\delta}^\beta_\alpha \end{array} \right)
\]
with the form of the P.B’s. (58). In terms of the real Majorana spinor \(\lambda_a\) and the composed vectors \(\varphi_m\) and \(\psi_m\)
\[
\varphi_m = \frac{1}{4}(\bar{\lambda}\gamma_m\lambda), \quad \psi_m = -\frac{1}{2}(\bar{\sigma}_m)^{\dot{\alpha}\alpha}\psi_{\dot{\alpha}} \equiv -\frac{1}{2}(\bar{\theta}\gamma_m\lambda)
\]
the P.B.’s. \((16)-(19)\) of the primordial coordinates \(x, \theta, \lambda\) are presented as follow
\[
\begin{align*}
\{\lambda_a, \lambda_b\} &= 0, \quad \{\theta_a, \lambda_b\} = 0, \quad \{x_m, \lambda_a\} = \varphi_m \lambda_a, \\
\{x_m, x_n\} &= -i \psi_n \psi_m, \quad \{x_m, \theta_a\} = -\frac{i}{2} \psi_m \lambda_a, \quad \{\theta_a, \theta_b\} = -\frac{i}{4} \lambda_a \lambda_b.
\end{align*}
\]

(62)

The P.B.’s. of the secondary composite vectors \(\psi_m\) and \(\varphi_m\) \((61)\) between themselves and with the primordial coordinates are presented in the form
\[
\begin{align*}
\{x_m, \psi_n\} &= \varphi_m \psi_n + \varphi_n \psi_m, \quad \{\psi_m, \theta_b\} = \frac{i}{2} \varphi_m \lambda_b, \quad \{\psi_m, \lambda_a\} = 0, \\
\{\psi_m, \psi_n\} &= -i \varphi_m \varphi_n, \quad \{\psi_m, \varphi_n\} = 0
\end{align*}
\]
and respectively
\[
\begin{align*}
\{x_m, \varphi_n\} &= 2 \varphi_m \varphi_n, \quad \{\theta_a, \varphi_m\} = \{\lambda_a, \varphi_m\} = \{\varphi_m, \varphi_n\} = 0.
\end{align*}
\]
(64)

The Poisson brackets \((62)-(64)\) originally derived for \(D = 4\) remain to be valid in \(D\)–dimensional spaces with \(D = 2, 3, 4(m0d8)\), where the Majorana spinors exist. Using the arguments given in the Section 5 one can restore the Moyal brackets originated from the P.B.’s. \((62)-(64)\) in the higher dimensions by the simple change \(\{, \} \to [ , ]_{\ast \ast}\).

8 Other Lorentz invariant brackets with one spinor

Using the Majorana spinor \(\nu_a\) one can constuct other simple supersymmetric and Lorentz invariant brackets. One of the possible invariant Poisson bracket might be
\[
\{F, G\} = F \left[ -\frac{i}{4} \left( \overleftrightarrow{D} \overleftrightarrow{D} + \overleftrightarrow{D} \overleftrightarrow{D} \right) + c(\partial \Delta - \Delta \partial) \right] G
\]
(65)

which changes the brackets \((19)\) to the brackets
\[
\begin{align*}
\{x_{\alpha a}, x_{\beta b}\} &= i(\varphi_{\alpha a} \bar{\theta}_a \theta_b + \varphi_{\beta b} \bar{\theta}_b \theta_a), \\
\{x_{\alpha a}, \theta_b\} &= -\frac{i}{2} \varphi_{\alpha a} \bar{\theta}_b, \quad \{x_{\alpha a}, \bar{\theta}_b\} = -\frac{i}{2} \varphi_{\alpha a} \theta_b, \\
\{\theta_a, \theta_b\} &= \frac{i}{4} \varphi_{\alpha a} \bar{\theta}_b, \quad \{\theta_a, \bar{\theta}_b\} = 0, \quad \{\bar{\theta}_a, \theta_b\} = \frac{i}{4} \varphi_{\alpha a} \bar{\theta}_b.
\end{align*}
\]
(66)

We see that the bracket of the spinor components \(\theta_a, \bar{\theta}_a\) having opposite chiralities is not deformed and remains equal zero. Moreover, the brackets \(x_{\alpha a}\) with \(\theta_a\) and \(\bar{\theta}_a\) don’t preserve the chiralities of \(\theta\) and \(\bar{\theta}\) spinors. As a result, one can find breaking of the Jacoby identity for the \((x, \theta, \bar{\theta})\)-cycle, because of the relation
\[
Cy cle\{\{x_{\alpha a}, \theta_b, \bar{\theta}_c\}\} = \{\{x_{\alpha a}, \theta_b, \bar{\theta}_c\} + \{x_{\alpha a}, \bar{\theta}_c, \theta_b\} = -\frac{i}{4} \varphi_{\alpha a} \varphi_{\beta b} \varphi_{\gamma c}.
\]
(67)

So, we conclude that the Lorentz invariant P.B. \((65)\) has to be excluded. But, the next supersymmetric and Lorentz invariant Poisson bracket
\[
\{F, G\} = F \left[ \frac{i}{4} \left( \overleftrightarrow{D} \overleftrightarrow{D} + \overleftrightarrow{D} \overleftrightarrow{D} \right) + \frac{1}{2} (\partial \Delta - \Delta \partial) \right] G
\]
(68)
is proved to be selfconsistent and yields the following invariant Poisson brackets
\[
\begin{align*}
\{x_{\alpha a}, x_{\beta b}\} &= -i(\varphi_{\alpha a} \bar{\theta}_a \theta_b - \varphi_{\beta b} \bar{\theta}_b \theta_a), \\
\{x_{\alpha a}, \theta_b\} &= \frac{i}{2} \varphi_{\beta b} \bar{\theta}_a, \quad \{x_{\alpha a}, \bar{\theta}_b\} = \frac{i}{2} \varphi_{\alpha a} \theta_b, \\
\{\theta_a, \theta_b\} &= \{\theta_a, \bar{\theta}_b\} = 0, \quad \{\theta_a, \bar{\theta}_b\} = -\frac{i}{4} \varphi_{\alpha a} \bar{\theta}_b.
\end{align*}
\]
(69)
added by the brackets

\[ \{\nu_\alpha, \nu_\beta\} = \{
\nu_\alpha, \bar{\nu}_\beta\} = \{
\nu_\alpha, \bar{\nu}_\beta\} = 0, \]
\[ \{\nu_\alpha, \theta_\beta\} = \{\nu_\alpha, \bar{\theta}_\beta\} = \{\bar{\nu}_\alpha, \theta_\beta\} = \{\bar{\nu}_\alpha, \bar{\theta}_\beta\} = 0, \]
\[ \{x_{a\alpha}, \nu_\beta\} = \frac{1}{2} \varphi_{a\alpha} \nu_\beta, \quad \{x_{a\alpha}, \bar{\nu}_\beta\} = \frac{1}{2} \varphi_{a\alpha} \bar{\nu}_\beta. \]

(70)

One can see that in contrast to the deformation (65) the new deformation (68) generates zero P.B.’s. for the \( \theta_a \) components with the same chiralities. The P.B.’s. (69), (70) satisfy the Jacobi identities and deserve to be studied in physical applications. The proof of the Jacobi identities for the P.B.’s. (69) and (70) is analogous to the proof presented in the Section 4.

The P.B.’s. (69) may be presented in the vector form as follows

\[ \{x_m, x_n\} = -\frac{i}{4} (x_m \bar{x}_n - x_n \bar{x}_m), \]
\[ \{x_m, \bar{x}_n\} = -\frac{1}{4} \bar{x}_m \nu_\beta, \quad \{x_m, \bar{\theta}_n\} = -\frac{1}{4} \bar{x}_m \bar{\nu}_\beta, \]
\[ \{\theta_a, \nu_\beta\} = -\frac{i}{8} (\nu_a^{(+) \nu}_b^{(-)} + \nu_b^{(+) \nu}_a^{(-)}), \]

(71)

where we introduced the complex Grassmannian vector \( \chi_m \) with the real and imaginary parts presented by \( \psi_{1m}, \psi_{2m} \) and the chiral components \( \theta^{(\pm)} \) and \( \nu^{(\pm)} \)

\[ \chi_m \equiv (\nu \sigma_m \bar{\theta}) \equiv -\bar{\nu} \gamma_m \frac{1+i\gamma_5}{2} \theta \equiv \psi_{1m} + i \psi_{2m}, \]
\[ \bar{\chi}_m \equiv (\chi_m)^* = -\bar{\nu} \gamma_m \frac{1-i\gamma_5}{2} \theta, \quad \psi_{1m} = -\frac{1}{2} (\bar{\theta} \gamma_m \nu), \quad \psi_{2m} = -\frac{1}{2} (\bar{\theta} \gamma_m \gamma_5 \nu), \]
\[ \theta^{(\pm)} = \frac{1}{2} (1 \pm i\gamma_5) \theta, \quad \nu^{(\pm)} = \frac{1}{2} (1 \pm i\gamma_5) \nu. \]

Then the P.B.’s. (71) are presented in the form directly generalizing the P.B.’s. (62)

\[ \{x_m, x_n\} = -\frac{i}{2} (\psi_{1m} \psi_{1n} + \psi_{2m} \psi_{2n}), \]
\[ \{x_m, \theta_a\} = -\frac{1}{4} (\psi_{1m} \nu_a + \psi_{2m} \bar{\nu}_a), \]
\[ \{\theta_a, \nu_\alpha\} = -\frac{i}{8} (\nu_a \nu_\alpha + \lambda_a \bar{\nu}_\alpha), \]

(73)

where \( \lambda_a \equiv (\gamma_5 \nu)_a \) (60). Comparing (73) with (62) we observe that the change of the P.B. (62) by (68) is equivalent to the complexification of the real Grassmannian vector \( \psi_m \) (61) accompanied by the appearance of the spinors \( \nu_a \) and \( (\gamma_5 \nu)_a \) in the r.h.s. of (73).

The P.B.’s. (68) may be generalized to the case of extended supersymmetries with \( N > 1 \)

\[ \{F, G\} = F \left[ \frac{1}{4} (\bar{D}_i D^i + \bar{D}^i D_i) + \frac{1}{2} (\bar{\Delta} \Delta - \bar{\Delta} \Delta) \right] G, \]

(74)

where \( D_i = \nu_a D^a_i \) and \( \bar{D}^i = \bar{\nu}_a D^{a\bar{a}}_i \), \( (i = 1, 2, ..., N) \). The P.B.’s. (74) generate the following brackets for the primordial space-time (super)coordinates

\[ \{x_{a\alpha}, x_{\beta\beta}\} = -i(\varphi_{a\alpha} \bar{\theta}_{\alpha\beta} \theta_{\beta\beta} - \varphi_{a\beta} \bar{\theta}_{\alpha\beta} \theta_{\alpha\alpha}) \]
\[ \{x_{a\alpha}, \bar{\theta}_\beta\} = \frac{1}{2} \varphi_{a\alpha} \theta_{\beta}, \quad \{x_{a\alpha}, \bar{\theta}_{\beta}\} = \frac{1}{2} \varphi_{a\beta} \bar{\theta}_{\alpha\alpha}, \]
\[ \{\theta_{a\alpha}, \theta_{\beta}\} = \{\bar{\theta}_{a\alpha}, \bar{\theta}_{\beta}\} = 0, \quad \{\theta_{a\alpha}, \bar{\theta}_{\beta}\} = i \frac{1}{4} \varphi_{a\beta} \delta^k_{\alpha}, \]
\[ \{\theta_{a\alpha}, \bar{\theta}_{\beta}\} = \{\bar{\theta}_{a\alpha}, \bar{\theta}_{\beta}\} = 0, \quad \{\theta_{a\alpha}, \bar{\theta}_{\beta}\} = i \frac{1}{4} \varphi_{a\beta} \delta^k_{\alpha}. \]

(75)

The rest of the P.B.’s for \( x_{a\alpha}, \nu_a, \theta_{a\alpha} \) coincides with the P.B.’s. (70).
9 More spinors - more Lorentz invariant brackets

The above consideration to construct the Lorentz covariant Poisson and Moyal brackets was restricted by the simplest case of one additional spinor coordinate which resulted in the appearance of the supersymmetric derivatives $D^\alpha, \bar{D}^{\dot{\alpha}}$ and $\partial^\alpha$ in the considered P.B.’s. only in the form of the scalars $[3]$. Using only these scalars for the construction of the invariant P.B. \[12\] restricts the class of admissible motions in superspace. To extend this class still preserving the Lorentz invariance and supersymmetry one can introduce additional independent spinor coordinates. In the case $D = 4$ it is enough to add only one new spinor coordinate $\mu_a$, because $\mu_a$ and $\nu_\alpha$ form the complete spinorial basis and may be identified with the Newman-Penrose dyad $[28]$

$$\mu^\alpha \nu_\alpha \equiv \mu^\alpha \varepsilon_{\alpha\beta} \nu^\beta = 1, \quad \mu_\alpha \nu_\beta - \mu_\beta \nu_\alpha = \varepsilon_{\alpha\beta}. \quad (76)$$

Then one can form four independent Lorentz invariant supersymmetric differential operators

$$D^{(\nu)} = \nu_\alpha D^\alpha, \quad \bar{D}^{(\dot{\nu})} = \bar{\nu}_\dot{\alpha} \bar{D}^{\dot{\alpha}}, \quad D^{(\mu)} = \nu_\alpha D^\alpha, \quad \bar{D}^{(\dot{\mu})} = \bar{\mu}_\dot{\alpha} \bar{D}^{\dot{\alpha}}, \quad (77)$$

two of which $D^{(\nu)}, \bar{D}^{(\dot{\nu})}$ coincide with the operators $D, \bar{D}$ \[5\]. Their linear combinations

$$D^{(\nu)}_{\pm} \equiv D^{(\nu)} \pm \bar{D}^{(\dot{\nu})}, \quad D^{(\mu)}_{\pm} \equiv D^{(\mu)} \pm \bar{D}^{(\dot{\mu})}, \quad (78)$$

form four Lorentz invariant and supersymmetric supersubalgebras

$$[D^{(\nu)}_{\pm}, D^{(\mu)}_{\pm}] = \mp 8i \partial^{(\nu)}, \quad [D^{(\nu)}_{\pm}, \partial^{(\mu)}] = [\partial^{(\nu)}, \partial^{(\mu)}] = 0, \quad \partial^{(\nu)} \equiv (\nu_\alpha \bar{\nu}_\dot{\alpha} \partial^{\alpha\dot{\alpha}}), \quad (79)$$

which are connected by the P.B. relations

$$[D^{(\nu)}_{\pm}, D^{(\mu)}_{\pm}] = \mp 4i \partial^{(\nu)}, \quad \partial^{(\nu)} \equiv (\nu_\alpha \bar{\mu}_\dot{\alpha} + \mu_\alpha \bar{\nu}_\dot{\alpha}) \partial^{\alpha\dot{\alpha}}. \quad (80)$$

It is easy to see that the Lorentz invariant and supersymmetric differential operators $D^{(\nu)}, D^{(\mu)}, \partial^{(\nu)}, \partial^{(\mu)}$ describe whole class of admissible motions in the superspace and together with the extended dilatation operator $\Delta' \equiv (\nu_\alpha \bar{\nu}^\alpha + \nu_\alpha \bar{\nu}_\dot{\alpha} \partial^{\alpha\dot{\alpha}})$

$$\Delta' = (\nu_\alpha \frac{\partial}{\partial \nu_\alpha} + \bar{\nu}_\dot{\alpha} \frac{\partial}{\partial \bar{\nu}_\dot{\alpha}}) - (\mu_\alpha \frac{\partial}{\partial \mu_\alpha} + \bar{\mu}_\dot{\alpha} \frac{\partial}{\partial \bar{\mu}_\dot{\alpha}}) \quad (81)$$

preserving the condition \[12\] may be used as invariant building blocks for the construction of more general Lorentz invariant supersymmetric Poisson and Moyal brackets.

Then the Lorentz invariant and supersymmetric Poisson bracket

$$\{F, G\} = F \left[ -\frac{i}{4} \left( D^{(\nu)} \rightarrow^{(\nu)} + \bar{D}^{(\dot{\nu})} \rightarrow^{(\dot{\nu})} \right) + c(\partial^{(\nu)} + \partial^{(\dot{\mu})}) \right] G. \quad (82)$$

might be considered as a candidate for the generalizations of \[12\]. The P.B. \[82\] yields the following coordinate P.B.’s.

$$\{x_m, x_n\} = -i \left( \psi^{(\nu)}_n \psi^{(\nu)}_m + \psi^{(\nu)}_m \psi^{(\nu)}_n \right), \quad (83)$$

$$\{x_m, \theta_a\} = -\frac{1}{2} \left( \psi^{(\nu)}_m \lambda^{(\nu)}_a + \psi^{(\nu)}_a \lambda^{(\nu)}_m \right),$$

$$\{\theta_a, \theta_b\} = -\frac{i}{4} \left( \lambda^{(\nu)}_a \lambda^{(\nu)}_b + \lambda^{(\nu)}_b \lambda^{(\nu)}_a \right).$$
for the primordial coordinates $x_m$ and $\theta_a$, where the additional Majorana spinor $\lambda_a^{(\mu)}$ and the Grassmannian vector $\psi_n^{(\mu)}$ are defined by the relations

$$
\psi_n^{(\nu)} \equiv \psi_n, \quad \lambda_a^{(\nu)} \equiv \lambda_a, \quad \psi_n^{(\mu)} \equiv \frac{1}{2}(\bar{\theta}_\gamma n \lambda^{(\mu)}), \quad \lambda_a^{(\mu)} \equiv (\gamma_5 \mu)_a
$$

The primordial Majorana spinors $\lambda_a^{(\nu)}$ and $\lambda_a^{(\mu)}$ have zero P.B.'s. between themselves and with $\theta_a, \psi_m^{(\nu)}, \psi_n^{(\mu)}$, but non zero P.B.'s. with $x_m$

$$
\{x_m, \lambda_a^{(\nu)}\} = c(\varphi_m^{(\nu)} + \varphi_m^{(\mu)}) \lambda_a^{(\nu)}, \quad \{x_m, \lambda_a^{(\mu)}\} = -c(\varphi_m^{(\nu)} + \varphi_m^{(\mu)}) \lambda_a^{(\mu)}
$$

The real constant $c$ has to be defined from the solution of the Jacobi identities. We intend to give back to the studying this P.B. and other possible generalizations of the P.B.’s in other place.

## 10 Conclusion

It was shown here that the extension of the $N = 1$ superspace ($x_m, \theta_a$) by commuting Majorana spinors may be used for the construction of supersymmetric and Lorentz invariant Poisson and Moyal brackets generating deformed non(anti)commutative relations for space-time (super)coordinates. To make clear the proposal we elaborate the case of one additional spinor $\lambda_a$ extending the standard $N = 1$ superspace to the non(anti)commutative superspaces free of background fields. The corresponding Lorentz invariant and supersymmetric coordinate brackets were presented. It was established that noncommutativity of $x_m$ with $x_n$ and $\theta_a$ is measured by the real or complex Grassmannian vectors $\psi_m$ composed from $\theta_a$ and $\lambda_a$, which are known as dynamical variables describing the spin degrees of freedom of spinning string or particle. At the same time, the nonanticommutativity of the $\theta_a$ components between themselves is measured by only additional spinor or its chiral components. These results hint on a hidden spinorial structure of space-time encoded in the Penrose twistor picture and its supersymmetric extensions as an alternative source of (super)coordinate non(anti)commutativity. In the simplest case corresponding to the P.B.'s a correlation between the spinorial structure and supergravity fields may be schematically illustrated by the correspondence:

$$
\theta_{mn} \leftrightarrow i\psi_m \psi_n, \quad C_{ab} \leftrightarrow \lambda_a \lambda_b, \quad \Psi^a \leftrightarrow \psi_m \lambda^a
$$

where $B_{mn} = \theta_{mn}^{-1}$, $C_{ab}$ and $\Psi^a_m$ are constant antisymmetric field, the graviphoton and the gravitino respectively. The map transforms the well-known field dependent bracket relations into the brackets and vice versa. On the other hand, such a correspondence hints on a connection with the known Feynman-Wheeler picture and its supersymmetric generalization, where the Maxwell supermultiplet fields arise as secondary objects constructed from the superspace coordinates. We outlined also a way to construction of more general supersymmetric Lorentz invariant brackets for the cases of $N$ extended supersymmetries and additional spinor coordinates using Lorentz invariant supersymmetric derivatives generalizing. Studying that generalizations and the corresponding deformations of quantum field models are under consideration.
11 Acknowledgements

The author thanks Fysikum at the Stockholm University for kind hospitality and I. Bengtsson
and B. Sundborg for useful discussions. The work was partially supported by the grant of the
Royal Swedish Academy of Sciences and by the SFFR of Ukraine under Project 02.07/276.

References

[1] H.S. Snyder, *Quantized Space-Time*, Phys. Rev. 71 (1947) 38.

[2] R. Casalbuoni, *The classical mechanics of Bose-Fermi systems*, Nuovo. Cim. A 33
(1976) 389.

[3] L. Brink and J.H. Schwarz, *Quantum superspace*, Phys. Lett. B 100 (1981) 310.

[4] J.H. Schwarz and P. van Nieuwenhuizen, *Speculations concerning a fermionic substructure of space-time* Lett. Nuovo. Cim. 34 (1982) 21;

[5] A. Connes, *Noncommutative Geometry*, Academic Press, London, 1990.

[6] Yu.I. Manin, *Quantum Groups and Non-Commutative Geometry*, Publications CRM,
Mantreal, 1989; Comm. Math. Phys. B 123 (1989) 163.

[7] M. Kontsevich, *Deformation quantization of Poisson manifolds.1*, arXiv:q-alg/9709040

[8] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a
conjecture*, Phys. Rev. D 55 (1997) 5112; arXiv:hep-th/9610043.

[9] A. Connes, M. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory:
compactification on tori*, JHEP 02 (1998) 003; arXiv:hep-th/9711162.

[10] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP 09 (1998)
032; arXiv:hep-th/9908142.

[11] S. Ferrara and M.A. Lledo, *Some aspects of deformations of supersymmetric field theo-
ries* JHEP 05 (2000) 008; arXiv: hep-th/0002084.

[12] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*,
Eur. Phys. J. C 16 (2000); arXiv: hep-th/0001203.

[13] P. Kosinski, J. Lukierski and P. Maslanka *Quantum deformations of space-time SUSY
and noncommutative superfield theory*; arXiv: hep-th/0012053.

[14] M. Douglas and N. Nekrasov, *Noncommutative field theories*, Rev. Mod. Phys. 73 (2001)
977; arXiv:hep-th/0106048.

[15] R.J. Szabo, *Quantum field theory on noncommutative spaces*; arXiv:hep-th/0109162.

[16] H. Ooguri and C. Vafa, *The c-deformation of gluino and non-planar diagrams* Adv.
Theor. Math. Phys. 7 (2003) 53; arXiv: hep-th/0104190. *Gravity induced c-deformation*
arXiv:hep-th/0303063.
[17] D. Klemm, S. Penati and L. Tamassia, *Non(anti)commutative superspace*, Class.Quant.Grav. **20** (2003) 2905; arXiv: [hep-th/0104190](https://arxiv.org/abs/hep-th/0104190).

[18] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, Phys. Lett. **B574** (2003) 98; arXiv: [hep-th/0302078](https://arxiv.org/abs/hep-th/0302078).

[19] N. Seiberg, *Noncommutative superspace, \(N = \frac{1}{2}\) supersymmetry, Field theory and string theory*, an, JHEP **06** (2003) 010; arXiv:hep-th/0305248.

[20] N. Berkovits and N. Seiberg, *Superstring in graviphoton background and \(N = \frac{1}{2} + \frac{3}{2}\)*, JHEP **07** (2003) 010; arXiv: [hep-th/0306226](https://arxiv.org/abs/hep-th/0306226).

[21] S. Ferrara, M.A. Lledo and O. Macia, *Supersymmetry in noncommutative spaces*, JHEP **09** (2003) 068; arXiv: [hep-th/0307039](https://arxiv.org/abs/hep-th/0307039).

[22] M. Chaichian, P.P. Kulish, K. Nishijima and A. Tureanu, *On a Lorentz-invariant interpretation of noncommutative space-time and its implication on noncommutative QFT*, Phys. Lett. **B 604** (2004) 98; arXiv:hep-th/0408069; M. Chaichian, P. Presnajder and A. Tureanu, *New conception of relativistic invariance in NC space-time: twisted Poincare symmetry and its implications*, Phys. Rev. Lett. **94** (2005) 151602; arXiv:hep-th/0408069.

[23] Y. Kobayashi and S. Sasaki, *Lorentz-invariant and supersymmetric interpretation of noncommutative quantum field theory*, arXiv: [hep-th/0410164](https://arxiv.org/abs/hep-th/0410164).

[24] B. M. Zupnik, *Twist-deformed supersymmetries in noncommutative superspaces*, arXiv: [hep-th/0506043](https://arxiv.org/abs/hep-th/0506043).

[25] M. Ihl and C. Sämann, *Drinfeld-Twisted Supersymmetry and Non-Anticommutative Superspace*; arXiv: [hep-th/0506057](https://arxiv.org/abs/hep-th/0506057).

[26] D.V. Uvarov and A.A. Zheltukhin, *Hamiltonian structure and noncommutativity in p-brane models with exotic supersymmetry*, JHEP **03** (2004) 063; arXiv: [hep-th/0310284](https://arxiv.org/abs/hep-th/0310284).

[27] A.A. Zheltukhin and D.V. Uvarov, *Exactly solvable p-brane model with extra supersymmetry*, Phys. Lett. **B 545** (2002) 183;

[28] R. Penrose and M.A.H. McCallum, *Twistor theory: an approach to the quantization of fields and space-time*, Phys. Rept. **6** (1972) 241.

[29] A. Ferber, *Supertwistors and conformal supersymmetry*, Nucl. Phys. **B 132** (1978) 55.

[30] E. Witten, *An interpretation of classical Yang-Mills theory*, Phys. Lett. **B77** (1978) 215; *Perturbative gauge theory as a string theory in twistor space*, Commun. Math. Phys. **252** (2004) 189; arXiv: [hep-th/0310284](https://arxiv.org/abs/hep-th/0310284).

[31] T. Shirafuji, *Lagrangian mechanics of massless particles with spin*, Prog. Theor. Phys. **70** (1983) 18.

[32] I. Bengtsson and M. Cederwal, *Particles, twistors and division algebras*, Nucl. Phys. **B 302** (1988) 81.
[33] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory* Cambridge University Press, Cambridge 1987.

[34] D.V. Volkov and A.A. Zheltukhin, *Extension of the Penrose representation and its use to describe supersymmetric models*, JETP Lett. 48 (1988) 63; *On the equivalence of the Lagrangians of massless Dirac and supersymmetrical particles*, Lett. Math. Phys. 17 (1989) 141; *Lagrangians for massless particles and strings with local and global supersymmetry*, Nucl. Phys. B 335 (1990) 723.

[35] L. Brink, Di Vechia, P. Howe and B. Zumino,, Phys. Lett. B 64 (1976) 435;

[36] A.A. Zheltukhin, *The Kaluza-Klein mechanism and a superfield description of spinning particle interactions*, Phys. Lett. B 168 (1986) 43;

[37] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton, USA, Univ. Press (1992).

[38] V.V. Tugai and A.A. Zheltukhin *A superfield generalization of the classical action-at-a-distance theory*, Phys. Rev. D 51 (1995) 4160; *Supersymmetry and minimality principle for electromagnetic interactions*, Phys. of Atomic Nucl. 61 (1998) 274.