On lepton mixing in gauge models

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Abstract

We reexamine lepton mixing in gauge models by considering two theories within the type I seesaw mechanism, the Extended Standard Model, i.e. $SU(2)_L \times U(1)_Y$ with singlet right-handed heavy neutrinos, and the Left-Right Model $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. The former is often used as a simple heuristic approach to masses and mixing of light neutrinos and to leptogenesis, while we consider the latter as an introduction to other left-right symmetric gauge theories like $SO(10)$. We compare lepton mixing in both theories for general parameter space and discuss also some particular cases. In the electroweak broken phase, we study in parallel both models in the “current basis” (diagonal gauge interactions), and in the “mass basis” (diagonal mass matrices and mixing in the interaction), and perform the counting of $CP$ conserving and $CP$ violating parameters in both bases. We extend the analysis to the Pati-Salam model $SU(4)_C \times SU(2)_L \times SU(2)_R$ and to $SO(10)$. Although specifying the Higgs sector increases the predictive power, in the most general case one has the same parameter structure in the lepton sector for all the left-right symmetric gauge models. We make explicit the differences between the Extended Standard Model and the left-right models, in particular $CP$ violating and lepton-number violating new terms involving the $W_R$ gauge bosons. As expected, at low energy, the differences in the light neutrino spectrum and mixing appear only beyond leading order in the ratio of Dirac mass to right-handed Majorana mass.

LPT-Orsay-14-41

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1 Introduction

In the last years, an impressive experimental progress has been achieved on the neutrino spectrum and mixing. Using this information on the light neutrinos mass matrix $m_L$, one is tempted to use the inverse of the seesaw formula $M_R = -m_D^t m_D^{-1} m_D$, where $m_D$ is the Dirac neutrino mass matrix, as a window on high energy neutrino physics, i.e. on the heavy right-handed neutrino mass matrix $M_R$ [1, 2, 3, 4, 5].

To use the inverse seesaw formula one needs information on the crucial Dirac mass matrix $m_D$. It has been often suggested that theoretical information on this matrix can be guessed within the $SO(10)$ Grand Unification gauge theory [6]. In order to study the whole structure of $SO(10)$ as far as lepton mixing is concerned, we have realized that it is convenient to begin by considering simpler theories that also exhibit left-right (LR) symmetry (for a review, see ref. [7]).

The simplest gauge theory that has been builded to study lepton mixing is the one that we call Extended Standard Model (ESM), i.e. the Standard Model (SM) $SU(3) \times SU(2)_L \times U(1)_Y$ plus right-handed neutrinos $N_R$, one per generation, singlet under the SM gauge group. Although this scheme allows to introduce heavy right-handed neutrinos, it does not exhibit LR symmetry like $SO(10)$.

One main aim of the present paper is to compare lepton mixing in the ESM, on the one hand, with lepton mixing in left-right models like $SO(10)$. Lepton mixing in the ESM has been thoroughly studied in the literature [8, 9, 10, 11], specially in ref. [10] on which the present paper heavily relies, together with the comprehensive review paper [12].

To compare the ESM with left-right gauge theories we have found convenient to consider next the Left-Right Model (LRM) $SU(2)_L \times SU(2)_R \times U_{B-L}(1)$ [13, 14], that exhibits a number of interesting new features concerning lepton mixing [15, 16].

This gauge group has already an appreciable complexity that will be useful as an introduction for the study of larger LR gauge groups, like the Pati-Salam model $SU(4)_C \times SU(2)_L \times SU(2)_R$ [17], and the grand unified $SO(10)$ gauge group [6].

We will first consider completely general Dirac or Majorana mass matrices consistent with Lorentz invariance, that coincide with mass matrices arising from the most general Higgs structure. We then look for the parameters that can be rotated away, although in a different way in the ESM and the LRM. We will consider the
current basis, in which the interaction Lagrangian $L_w$ is diagonal, and the mass basis, in which the mass Lagrangian $L_m$ is diagonal, and we check that, for a given model, the final number of independent parameters, angles and phases, is the same in both bases.

Some main results exposed below are already known. The purpose of this paper is in part didactic, and in part the understanding a number of particular points. We think it is worth to explain in detail the differences between the Extended Standard model and the left-right gauge models as far as lepton mixing is concerned, specially the comparison of the interaction Lagrangians of both schemes in the mass basis.

Here below we expose briefly the fermion and gauge boson content of the ESM and LRM. In Sections 2 and 3 we perform the counting of the lepton sector parameters of the ESM and LRM in the current and in the mass bases. For the mass basis, special care is given to the approximation $m_D << M_R$, as compared with exact results, and in Section 4 we recall two different representations proposed in the literature for the Dirac mass matrix $m_D$. In Section 5 we briefly examine leptogenesis in the ESM and in the LRM. In Section 6 we summarize the differences between both models for lepton mixing. Section 7 is devoted to the extension of our results to other left-right theories, Pati-Salam and $SO(10)$, and in Section 8 we conclude. In the Appendix we present some details of the calculations.

1.1 Gauge boson and fermion content of the gauge models

We now expose the fermion and gauge boson content of the two gauge theories that we consider in detail, the Extended Standard model and the Left-Right model $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$.

1.1.1 Extended Standard Model

The Extended Standard Model (ESM) is just the Standard Model (SM) $SU(3) \times SU(2)_L \times U(1)_Y$ with the addition of one Majorana fermion $N_R$ per generation, singlet under the gauge group.
The fermion content of the model is for quarks

\[ \begin{pmatrix} u_L \\ d_L \end{pmatrix} \sim \begin{pmatrix} 3, 2, \frac{1}{3} \end{pmatrix}, \quad u_R \sim \begin{pmatrix} 3, 1, \frac{4}{3} \end{pmatrix}, \quad d_R \sim \begin{pmatrix} 3, 1, -\frac{2}{3} \end{pmatrix} \]  

and for leptons

\[ \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \sim \begin{pmatrix} 1, 2, -1 \end{pmatrix}, \quad e_R \sim \begin{pmatrix} 1, 1, -2 \end{pmatrix}, \quad N_R \sim \begin{pmatrix} 1, 1, 0 \end{pmatrix} \]

with

\[ Q = T_{3L} + \frac{Y}{2} \]  

The gauge bosons are the gluons \((8, 1, 0)\), the \(W_L\) bosons \((1, 3, 0)\) and the \(B\) boson \((1, 1, 0)\).

The Higgs sector needed to achieve the Spontaneous Symmetry Breaking (SSB) and give masses to the fermions is the usual doublet \(\phi \sim (1, 2, -1)\). The novelty in the ESM with respect to the SM is just the presence of the Majorana \(N_R\) singlet. The right-handed fermion \(N_R\) can have a large mass, of a different scale than the SM, that can be originated from a Higgs boson, singlet relatively to the Standard Model \(\Phi \sim (1, 1, 0)\), or simply be a bare mass term

\[ (1, 1, 0) f \times (1, 1, 0) f = (1, 1, 0) \]  

that, together with the Dirac mass terms

\[ (1, 2, -1) f \times (1, 2, 1) T \times (1, 2, -1) H = (1, 1, 0) + \ldots \]

gives the general neutrino mass matrix

\[ \mathcal{M} = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix} \]

where \(m_D\) and \(M_R\) are respectively general complex and complex symmetric matrices.
1.1.2 Left-Right Model

In the LRM model $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, the classification of $L$ and $R$ fermions is for quarks

$$
\begin{pmatrix}
u_L \\
e_L
\end{pmatrix} \sim (1, 2, 1, -1),
\begin{pmatrix} N_R \\
e_R
\end{pmatrix} \sim (1, 1, 2, -1)
$$

and for leptons

$$
\begin{pmatrix} u_L \\
d_L
\end{pmatrix} \sim (3, 2, 1, \frac{1}{3}),
\begin{pmatrix} u_R \\
d_R
\end{pmatrix} \sim (3, 1, 2, \frac{1}{3})
$$

with

$$Q = T_{3L} + T_{3R} + \frac{B - L}{2}$$

The gauge bosons are the gluons $(8, 1, 1, 0)$, the $W_L$ bosons $(1, 3, 1, 0)$, the $W_R$ bosons $(1, 1, 3, 0)$ and the $B - L$ singlet $(1, 1, 1, 0)$.

The Higgs fields needed to achieve SSB and the seesaw mechanism are the bidoublet $\phi \sim (1, 2, 2, 0)$ and the triplet $\Delta_R \sim (1, 1, 3, 2)$.

The bidoublet, written as

$$
\phi = \begin{pmatrix} \phi^0_1 & \phi^+_1 \\ \phi^-_2 & \phi^0_2 \end{pmatrix}
$$

breaks the SM group and gives masses to quarks and leptons through the Yukawa terms

$$
\begin{pmatrix} 3, 2, 1, \frac{1}{3} \end{pmatrix}_f \times \begin{pmatrix} 3, 1, 2, -\frac{1}{3} \end{pmatrix}_T \times (1, 2, 2, 0)_{H, \overline{H}} = (1, 1, 1, 0) + ...
$$

$$
(1, 2, 1, -1)_f \times (1, 1, 2, 1)_{\overline{H}} \times (1, 2, 2, 0)_{H, \overline{H}} = (1, 1, 1, 0) + ...
$$

with $H = \phi$ and $\overline{H} = \sigma_2 H^* \sigma_2$.

From the vacuum expectation values

$$< \phi^0_1 > = k_1, \quad < \phi^0_2 > = k_2$$

that can be complex, the Yukawa couplings give the Dirac masses, as in the SM, but with a different pattern. Quark mass matrices $m_u$, $m_d$ and the Dirac neutrino mass matrix $m_D$ read

$$m_u = pk_1 + qk^*_2, \quad m_d = pk_2 + qk^*_1$$

with

$$k_1 = \frac{B - L}{2}$$

and

$$k_2 = \frac{B - L}{2}$$

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$$

breaks the SM group and gives masses to quarks and leptons through the Yukawa terms

$$
\begin{pmatrix} 3, 2, 1, \frac{1}{3} \end{pmatrix}_f \times \begin{pmatrix} 3, 1, 2, -\frac{1}{3} \end{pmatrix}_T \times (1, 2, 2, 0)_{H, \overline{H}} = (1, 1, 1, 0) + ...
$$

$$
(1, 2, 1, -1)_f \times (1, 1, 2, 1)_{\overline{H}} \times (1, 2, 2, 0)_{H, \overline{H}} = (1, 1, 1, 0) + ...
$$

with $H = \phi$ and $\overline{H} = \sigma_2 H^* \sigma_2$.

From the vacuum expectation values

$$< \phi^0_1 > = k_1, \quad < \phi^0_2 > = k_2$$

that can be complex, the Yukawa couplings give the Dirac masses, as in the SM, but with a different pattern. Quark mass matrices $m_u$, $m_d$ and the Dirac neutrino mass matrix $m_D$ read

$$m_u = pk_1 + qk^*_2, \quad m_d = pk_2 + qk^*_1$$

with

$$k_1 = \frac{B - L}{2}$$

and

$$k_2 = \frac{B - L}{2}$$
\[ m_D = r k_1 + s k_2^*, \quad m_e = r k_1 + s k_2^* \]  \hfill (13)

where \( p, q, r \) and \( s \) are complex Yukawa coupling matrices.

The triplet \( H = \Delta_R \) breaks the LR model to the SM and, at the same time, gives a Majorana mass to the right-handed neutrino \( N_R \) through the Yukawa term

\[
(1, 1, 2, -1)_f \times (1, 1, 2, -1)_f \times (1, 1, 3, 2)_H = (1, 1, 1, 0) + \ldots \tag{14}
\]

\[
< \Delta_R^0 > = v_R, \quad M_R = t v_R, \quad M^t_R = M_R \tag{15}
\]

where \( t \) is a complex symmetric Yukawa coupling matrix.

The full neutrino mass matrix has the form

\[
\mathcal{M} = \begin{pmatrix}
0 & r k_1 + s k_2^* \\
* & t v_R
\end{pmatrix} \tag{16}
\]

i.e. it has the general form (6).

We consider this minimal Higgs content that is necessary in the LRM, and we do not introduce a possible left-handed triplet \( \Delta_L = (1, 3, 1, 2)_H \) that could in principle contribute to the light neutrino masses.

## 2 Current basis

In what follows, we consider the gauge models in the *electroweak broken phase*. We only make explicit the charged current terms in the interaction Lagrangians of both gauge models.

### 2.1 Extended Standard Model

The mass and interaction Lagrangians write, in an obvious compact notation

\[
\mathcal{L}_m = \bar{\nu}_L m_D N_R + \frac{1}{2} (N_R)^c M_R N_R + \bar{e}_L m_e e_R + h.c. \\
\mathcal{L}_w = \bar{\nu}_L \gamma_\mu e_L W^\mu_L + h.c. \tag{17}
\]

The matrices \( m_D \) and \( m_e \) are general complex, each has 9 complex parameters, while \( M_R \) is general complex symmetric with 6 complex parameters.
The lepton number assignment \( L(N_R) = -L((N_R)^c) = 1 \) implies that the Majorana mass term is \( |\Delta L| = 2 \) while, like for the other fermions, the Dirac mass term is \( |\Delta L| = 0 \).

From now on we adopt the following simplifying notation for the real parameters of an arbitrary square complex matrix \( M \), that has \( n(m) \) parameters, where \( n \) is the total number of real parameters, among which there are \( m \) (\( m < n \)) are phases:

\[
M \text{ has } n(m) \text{ real parameters } \leftrightarrow n \text{ real parameters, } m < n \text{ phases} \tag{18}
\]

In this example, \( m_D \) and \( m_e \) have 18(9) real parameters and \( M_R \) has 12(6) real parameters. Therefore, \textit{a priori} one has in this model 30(15) real parameters.

Let us see now that we can reduce the number of independent parameters without modifying the interaction Lagrangian \( \mathcal{L}_w \). Diagonalizing \( m_e \) and \( M_R \) by

\[
m_e = V_{eL}^{\dagger} m^\text{diag}_e V_{eR}, \quad M_R = U_R^{\dagger} M^\text{diag}_R U_R \tag{19}
\]

and redefining the fields

\[
U_R N_R \to N_R, \quad V_{eR} e_R \to e_R, \quad \begin{pmatrix} V_{eL} \nu_L \\ V_{eL} e_L \end{pmatrix} \to \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \tag{20}
\]

one gets

\[
\mathcal{L}_m = \overline{\nu}_L V_{eL} m_D U_R^{\dagger} N_R + \frac{1}{2} (N_R)^c M^\text{diag}_R N_R + \overline{\nu}_L m^\text{diag}_e e_R + h.c.
\]

\[
\mathcal{L}_w = \overline{\nu}_L \gamma_\mu e_L W^\mu_L + h.c. \tag{21}
\]

The simultaneous transformation of \( \nu_L \) and \( e_L \) in (20, 21) ensures the invariance of \( \mathcal{L}_w \), but then \( V_{eL} \) appears in the Dirac mass term. Since \( m_D \) is a general complex symmetric matrix, so is \( V_{eL} m_D U_R^{\dagger} \). Changing the notation

\[
V_{eL} m_D U_R^{\dagger} \to m_D \tag{22}
\]

one obtains

\[
\mathcal{L}_m = \overline{\nu}_L m_D N_R + \frac{1}{2} (N_R)^c M^\text{diag}_R N_R + \overline{\nu}_L m^\text{diag}_e e_R + h.c.
\]

\[
\mathcal{L}_w = \overline{\nu}_L \gamma_\mu e_L W^\mu_L + h.c. \tag{23}
\]
We can redefine the doublet $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ and the singlet $e_R$ by the same diagonal phase matrix $P_e$:

$$
\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \rightarrow \begin{pmatrix} P_e \nu_L \\ P_e e_L \end{pmatrix}, \quad e_R \rightarrow P_e e_R
$$

and one gets

$$
\mathcal{L}_m = \overline{\nu}_L P^* e_L m_D N_R + \frac{1}{2} (N_R)^c M^{diag}_R N_R + \overline{\nu}_L m^{diag}_e e_R + h.c.
$$

$$
\mathcal{L}_w = \overline{\nu}_L \gamma^\mu e_L W^\mu_L + h.c.
$$

Finally we can choose the phase matrix $P_e$ to cancel three phases of $m_D$ in $P^*_e m_D$:

$$
\mathcal{L}_m = \overline{\nu}_L m_D N_R + \frac{1}{2} (N_R)^c M^{diag}_R N_R + \overline{\nu}_L m^{diag}_e e_R + h.c.
$$

$$
\mathcal{L}_w = \overline{\nu}_L \gamma^\mu e_L W^\mu_L + h.c.
$$

where now the Dirac mass matrix $m_D$ is not a general complex matrix, but has 9 real parameters + 6 phases, i.e. 15(6) real parameters.

To summarize parameter counting, one is left in the current basis with 15(6) (from $m_D$) + 3(0) (from $m^{diag}_e$) + 3(0) (from $M^{diag}_R$) = 21(6) real parameters, i.e. among them 6 phases. This counting agrees with the one performed in ref. [18].

### 2.2 Left-Right Model

In the LRM, the Lagrangian in the lepton sector writes

$$
\mathcal{L}_m = \overline{\nu}_L m_D N_R + \frac{1}{2} (N_R)^c M^{diag}_R N_R + \overline{\nu}_L m^{diag}_e e_R + h.c.
$$

$$
\mathcal{L}_w = \overline{\nu}_L \gamma^\mu e_L W^\mu_L + \overline{N}_R \gamma^\mu e_R W^\mu_R + h.c.
$$

Notice that, to simplify the notation, possible $W_L - W_R$ mixing is for the moment neglected in the interaction term, that will be considered later. The matrices $m_D$ and $m_e$ are a priori general complex with 18(9) parameters each, and $M_R$ is a general complex symmetric matrix with 12(6) parameters.

An important remark is in order here. Parameter counting of the Left-Right Model in the "Current basis" means that we are assuming the whole interaction
Lagrangian $\mathcal{L}_w$ in (27) to be diagonal, both in the left and the right sectors. For low energy neutrino physics, it can seem academic to assume that the right-handed piece $\overline{N}_R \gamma_\mu e_R W^\mu_R + h.c.$ is kept diagonal, because it is an interaction term involving high energy degrees of freedom. However, this natural assumption in any LR gauge theory is not only a formal point since, to keep this piece diagonal amounts to assume that one assigns a lepton number to the $N_R$ neutrinos, just in the same way as it is done for the $\nu_L$ neutrinos in (27), and in consistency with the assignment $L(N_R) = -L((N_R)^c) = 1$ in the ESM. As we will see below, the diagonalization of the light neutrino mass matrix and of the right neutrino mass matrix will result in mixing matrices of the PMNS type for both the light and the heavy neutrinos.

Diagonalizing $m_e$ by (19) and redefining the fields

$$
\begin{pmatrix}
V_{eL} \nu_L \\
V_{eL} e_L
\end{pmatrix} \rightarrow \left( \begin{array}{c}
\nu_L \\
e_L
\end{array} \right),
\begin{pmatrix}
V_{eR} N_R \\
V_{eR} e_R
\end{pmatrix} \rightarrow \left( \begin{array}{c}
N_R \\
e_R
\end{array} \right)
$$

one gets

$$
\mathcal{L}_m = \bar{\nu}_L V_{eL} m_D V_{eR}^\dagger N_R + \frac{1}{2} (N_R)^c V_{eR}^* M_R V_{eR}^\dagger N_R + \bar{\nu}_L m_{e \text{diag}} e_R + h.c.
$$

$$
\mathcal{L}_w = \bar{\nu}_L \gamma_\mu e_L W^\mu_L + \overline{N}_R \gamma_\mu e_R W^\mu_R + h.c.
$$

Since $m_D$ is general complex, so is $V_{eL} m_D V_{eR}^\dagger$, and $M_R$ being complex symmetric, so is $V_{eR}^* M_R V_{eR}^\dagger$.

Changing the notation

$$
V_{eL} m_D V_{eR}^\dagger \rightarrow m_D, \quad V_{eR}^* M_R V_{eR}^\dagger \rightarrow M_R
$$

one obtains

$$
\mathcal{L}_m = \bar{\nu}_L m_D N_R + \frac{1}{2} (N_R)^c M_R N_R + \bar{\nu}_L m_{e \text{diag}} e_R + h.c.
$$

$$
\mathcal{L}_w = \bar{\nu}_L \gamma_\mu e_L W^\mu_L + \overline{N}_R \gamma_\mu e_R W^\mu_R + h.c.
$$

We can redefine the doublets by the same diagonal phase matrix $P_e$ :

$$
\begin{pmatrix}
\nu_L \\
e_L
\end{pmatrix} \rightarrow \left( \begin{array}{c}
P_e \nu_L \\
P_e e_L
\end{array} \right),
\begin{pmatrix}
N_R \\
e_R
\end{pmatrix} \rightarrow \left( \begin{array}{c}
P_e N_R \\
P_e e_R
\end{array} \right)
$$
and one gets
\[ \mathcal{L}_m = \overline{\nu}_L P^e \nu_D P e N_R + \frac{1}{2} (N_R)^c P e M_R P e N_R + \overline{\nu}_L m^{\text{diag}} e_R + h.c. \]
\[ \mathcal{L}_w = \overline{\nu}_L \gamma_\mu e_W^\mu L + \overline{N}_R \gamma_\mu e_R W^\mu R + h.c. \]  
(33)

We can chose the phase matrix \( P e \) to cancel three phases of \( m_D \) or three phases of \( M_R \), but not both at the same time. We choose to absorb 3 phases in \( M_R \). Changing the notation \( P^e m_D P e \rightarrow m_D \), one gets finally
\[ \mathcal{L}_m = \overline{\nu}_L m_D N_R + \frac{1}{2} (N_R)^c M_R N_R + \overline{\nu}_L m^{\text{diag}} e_R + h.c. \]
\[ \mathcal{L}_w = \overline{\nu}_L \gamma_\mu e_W^\mu L + \overline{N}_R \gamma_\mu e_R W^\mu R + h.c. \]  
(34)

where \( m_D \) is an arbitrary complex matrix with 18(9) parameters and \( M_R \) is complex symmetric with 9(3) parameters.

To summarize, one gets finally in the LRM: 18(9) parameters from \( m_D \) + 9(3) parameters from \( M_R \) + 3 eigenvalues in \( m^{\text{diag}}_e = 30(12) \) parameters.

Much more constrained models have been considered in the literature. For example, the Minimal LRM within supersymmetry with a Higgs content that implies \( m_e = m_D, m_u = m_d \) (up-down unification) \([19]\), that has a reduced number of parameters.

3 Mass basis

3.1 Extended Standard Model

For the diagonalization of the whole \( 6 \times 6 \) neutrino mass matrix, we proceed step by step, and we begin with (26), where \( m^{\text{diag}}_e \) and \( M^{\text{diag}}_R \) are diagonal and the Dirac mass matrix \( m_D \) has 15(6) parameters. So we can rewrite
\[ \mathcal{L}_m = \frac{1}{2} \left( \overline{\nu}_L, (N_R)^c \right) \mathcal{M} \begin{pmatrix} \nu^c_L \\ N_R \end{pmatrix} + \overline{\nu}_L m^{\text{diag}} e_R + h.c. \]
\[ \mathcal{L}_w = \overline{\nu}_L \gamma_\mu e_W^\mu L + h.c. \]  
(35)

where \( \mathcal{M} \) has the form
\[ \mathcal{M} = \begin{pmatrix} 0 & m_D \\ m^t_D & M^{\text{diag}}_R \end{pmatrix} \]  
(36)
This matrix has 18(6) parameters: 15(6) from $m_D$ and 3(0) from $M_{R}^{\text{diag}}$.

Let us now diagonalize $\mathcal{M}$ with the unitary matrix $V$ \cite{9,10,11}

$$\mathcal{M} = V \mathcal{M}^{\text{diag}} V^t$$

(37)

where

$$\mathcal{M}^{\text{diag}} = \begin{pmatrix} m_{L}^{\text{diag}} & 0 \\ 0 & M_{R}^{\text{diag}} \end{pmatrix}$$

(38)

$$V = \begin{pmatrix} K & R \\ S & T \end{pmatrix}$$

(39)

Notice that since $\mathcal{M}^{\text{diag}}$ has 6 eigenvalues, and $\mathcal{M}$ has 18(6) parameters, the $6 \times 6$ unitary matrix $V$ will have $18(6) - 6(0) = 12(6)$ parameters. Rewriting (35) under the form

$$L_{m} = \frac{1}{2} \left( \bar{\nu}_L, (N_R)^c \right) V \mathcal{M}^{\text{diag}} V^t \begin{pmatrix} \nu_L^c \\ N_R \end{pmatrix} + \bar{e}_L m_e^{\text{diag}} e_R + h.c.$$  

$$L_{w} = \left( \bar{\nu}_L, (N_R)^c \right) \gamma_{\mu} V^t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} e_L \\ e_L \end{pmatrix} W_{\mu}^L + h.c.$$  

(40)

and redefining

$$V^t \begin{pmatrix} \nu_L^c \\ N_R \end{pmatrix} \rightarrow \begin{pmatrix} \nu_L^c \\ N_R \end{pmatrix}, \quad \left( \bar{\nu}, (N_R)^c \right) V \rightarrow \left( \bar{\nu}, (N_R)^c \right)$$

(41)

one gets

$$L_{m} = \frac{1}{2} \left( \bar{\nu}_L, (N_R)^c \right) \mathcal{M}^{\text{diag}} \begin{pmatrix} \nu_L^c \\ N_R \end{pmatrix} + \bar{e}_L m_e^{\text{diag}} e_R + h.c.$$  

$$L_{w} = \left( \bar{\nu}_L, (N_R)^c \right) \gamma_{\mu} V^t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} e_L \\ e_L \end{pmatrix} W_{\mu}^L + h.c.$$  

(42)

or

$$L_{m} = \frac{1}{2} \nu_L m_{L}^{\text{diag}} (\nu_L)^c + \frac{1}{2} (N_R)^c M_{R}^{\text{diag}} N_R + \bar{e}_L m_e^{\text{diag}} e_R + h.c.$$  

$$L_{w} = \left( \bar{\nu}_L K^t e_L + (N_R)^c R^t e_L \right) \gamma_{\mu} W_{\mu}^L + h.c.$$  

(43)

The first term in $L_w$ describes the Standard Model $\Delta L = 0$ decay

$$W_L \rightarrow e_L \bar{\nu}_L$$  

(44)
while the second term corresponds to the well-known $\Delta L = 2$ process

$$(N_R)^c \rightarrow e_L W_L$$

$(L(N_R) = -L((N_R)^c) = L(e_L))$. The notation $(N_R)^c$ for the heavy neutrino makes explicit also the chirality conservation of the $V - A$ interaction.

Notice that only the $3 \times 3$ complex matrices $K$ and $R$ from the $6 \times 6$ unitary matrix $(39)$ are involved in the formula (43). Let us now count the parameters of these matrices. From the zero in the matrix $M$ (36) and the definitions (37-39) one finds eqn. (44)

$$K m^\text{diag}_L K^t + R M^\text{diag}_R R^t = 0$$

Using the unitarity of the matrix $V$ (39) one has

$$KK^\dagger + RR^\dagger = 1$$

Eqns. (46) and (47) are identities between $3 \times 3$ matrices involving only the mixing matrices $K$ and $R$ and not the whole matrix (39). Due to these relations, the matrices $K$ and $R$ are correlated.

The conditions (46) and (47) reduce the number of independent parameters. Equation (46) is self-transposed, and gives 12(6) constraints, while (47) is hermitian, giving 9(3) constraints. This reduces the number of parameters of the two complex matrices $K$ and $R$ from 36(18) down to 15(9).

Finally, redefining the charged lepton fields by a diagonal $3 \times 3$ phase matrix $Q_e$

$$e_L \rightarrow Q^\dagger_e e_L, \quad e_R \rightarrow Q^\dagger_e e_R$$

one gets, from (43),

$$\mathcal{L}_m = \frac{1}{2} \nu_L m^\text{diag}_L (\nu_L)^c + \frac{1}{2} (N_R)^c M^\text{diag}_R N_R + \bar{e}_L m^\text{diag}_e e_R + \text{h.c.}$$

$$\mathcal{L}_w = (\bar{\nu}_L (Q_e K)^\dagger + (N_R)^c (Q_e R)^\dagger) \gamma_\mu e_L W^\mu_L + \text{h.c.}$$

On the other hand, multiplying (46) on the left by $Q_e$ and on the right by $Q^\dagger_e$, and (47) on the left by $Q_e$ and on the right by $Q^\dagger_e$, these equations become

$$(Q_e K) m^\text{diag}_L (Q_e K)^\dagger + (Q_e R) M^\text{diag}_R (Q_e R)^\dagger = 0$$

$$(Q_e K)(Q_e K)^\dagger + (Q_e R)(Q_e R)^\dagger = 1$$
and we can absorb 3 phases of one of the matrices $K$ or $R$, but not of both matrices at the same time.

In summary, the matrices $K$ and $R$ have together 12(6) parameters, and adding the 9(0) parameters from $m_e^{\text{diag}}$, $m_L^{\text{diag}}$ and $M_R^{\text{diag}}$ one obtains a total of 21(6) parameters, the same number as in the current basis. In the ESM the matrices $K$ and $R$ are decoupled from $S$ and $T$ of \eqref{39}, and obey relations \eqref{46,47}.

We can now go somewhat further by considering first the whole matrix \eqref{39}, and assuming $m_D << M_R$.

### 3.1.1 The matrices $K$, $R$, $S$, $T$ in the Extended Standard Model

Starting from the Lagrangian in the current basis \eqref{26}, $m_D$ has now 15(6) parameters. Particularizing formulas \eqref{144,146} of the Appendix to the present case, we have:

\begin{align*}
Km_L^{\text{diag}}K^t + RM_R^{\text{diag}}R^t &= 0 \quad (52) \\
Sm_L^{\text{diag}}S^t + TM_R^{\text{diag}}T^t &= M_R^{\text{diag}} \quad (53) \\
Km_L^{\text{diag}}S^t + RM_R^{\text{diag}}T^t &= m_D \quad (54)
\end{align*}

Considering for the moment the unitarity of the matrix \eqref{39}, the number of independent parameters in the l.h.s. will be 36(21) from $(K, R, S, T) + 3(0)$ from $m_L^{\text{diag}} + 3(0)$ from $M_R^{\text{diag}} = 42(21)$.

The complex symmetric matrix equation (52) gives 12(6) constraints. On the other hand, $M_R^{\text{diag}}$ appears already in the r.h.s. of (53), and this equation implies 12(6) - 3(0) = 9(6) constraints. Since $m_D$ has now 15(6) free parameters, eq. (54) gives 3(3) constraints, giving a total of 12(6) + 9(6) + 3(3) = 24(15) constraints. Therefore the number of independent parameters is 42(21) - 24(15) = 18(6) parameters. Adding the 3(0) eigenvalues of $m_e^{\text{diag}}$ one gets 18(6) + 3(0) = 21(6) parameters, the same result as in the current basis.

Moreover, substracting from this total of 21(6) parameters the 9(0) mass eigenvalues $m_e^{\text{diag}}$, $m_L^{\text{diag}}$ and $M_R^{\text{diag}}$, the set of matrices $(K, R, S, T)$ has 12(6) parameters, the same number that we have found for $K$ and $R$, so that $S$ and $T$ are not independent.
Exact relations between the matrices $K$, $R$, $S$, $T$

On the other hand, from (36-39) one has

$$
\begin{pmatrix}
0 & m_D \\
m_D^t & M_R^{\text{diag}}
\end{pmatrix}
\begin{pmatrix}
K^* & R^* \\
S^* & T^*
\end{pmatrix}
= 
\begin{pmatrix}
K & R \\
S & T
\end{pmatrix}
\begin{pmatrix}
m_L^{\text{diag}} & 0 \\
0 & M_R^{\text{diag}}
\end{pmatrix}
$$

(55)

hence

$$
\begin{pmatrix}
m_D S^* \\
m_D^t K^* + M_R^{\text{diag}} S^* \\
m_D^t R^* + M_R^{\text{diag}} T^*
\end{pmatrix}
= 
\begin{pmatrix}
K m_L^{\text{diag}} & R M_R^{\text{diag}} \\
S m_L^{\text{diag}} & T M_R^{\text{diag}}
\end{pmatrix}
$$

(56)

and therefore one obtains the following exact expressions of the matrices $R$, $S$ in terms of $K$, $T$, $m_D$ and the mass eigenvalues:

$$
R = m_D T^* (M_R^{\text{diag}})^{-1}
$$

(57)

$$
S = (m_D^*)^{-1} K^* m_L^{\text{diag}}
$$

(58)

From inspection of the precedent equations, one sees that (57,58) are relations between the mass basis quantities ($K$, $R$, $S$, $T$, $m_L^{\text{diag}}$, $M_R^{\text{diag}}$) and the current basis matrices $m_D$, $M_R^{\text{diag}}$, since $M_R$ is diagonalized and appears in both bases. Eliminating $m_D$, one finds an exact relation between quantities in the mass basis:

$$
M_R^{\text{diag}} T^{-1} S = (R^*)^{-1} K^* m_L^{\text{diag}}
$$

(59)

The matrices $(K, R, S, T)$ for $m_D << M_R$

If $m_D << M_R$, one has the order of magnitude

$$
R \sim S \sim O \left( \frac{m_D}{M_R} \right)
$$

(60)

Neglecting in equations (138-143) of the Appendix the terms of $O \left( \frac{m_D^2}{M_R^2} \right)$ one gets the approximate unitarity conditions

$$
K K^\dagger \simeq K^\dagger K \simeq 1
$$

(61)

$$
T T^\dagger \simeq T^\dagger T \simeq 1
$$

(62)

Moreover, from (61,62), both equations (140) and (143) imply the same approximate relation between $R$ and $S$

$$
R \simeq -K S^\dagger T
$$

(63)
In conclusion, in the present approximation one gets two unitary matrices \(K\) and \(T\) and the matrix \(R\) given in terms of \((K, T, S)\) by (63).

On the other hand, neglecting terms of \(O\left(\frac{m_D^2}{M_R}\right)\) in (52,54), one gets

\[
K m_L^{\text{diag}} K^t + R M_R^{\text{diag}} R^t = 0 \tag{64}
\]

\[
T M_R^{\text{diag}} T^t \simeq M_R^{\text{diag}} \tag{65}
\]

\[
R M_R^{\text{diag}} T^t \simeq m_D \tag{66}
\]

Eqn. (65) implies

\[
T \simeq 1 \tag{67}
\]

Notice that (66) is identical to the relation (57) obtained above. On the other hand, combining (63) with the exact relation (59) one consistently obtains (64).

One can see that (63) gives just the seesaw formula. From (57,58,67), eqn. (63) implies, after some algebra

\[
K m_L^{\text{diag}} K^t \simeq -m_D (M_R^{\text{diag}})^{-1} m_D^t \tag{68}
\]

and from the general complex symmetric matrix \(m_L\),

\[
m_L = K m_L^{\text{diag}} K^t \tag{69}
\]

one gets the seesaw formula in the ESM:

\[
m_L \simeq -m_D (M_R^{\text{diag}})^{-1} m_D^t \tag{70}
\]

We see that \(K\) is the mixing matrix for light neutrinos, that appears in (43) in the basis in which \(m_e\) is diagonal.

On the other hand, relation (57) or (66), together with (67), implies

\[
R \simeq m_D (M_R^{\text{diag}})^{-1} \tag{71}
\]

and using the seesaw formula (70), relation (58) becomes

\[
S = -(M_R^{\text{diag}})^{-1} m_D^t K \tag{72}
\]

in consistency with (63).
The whole set $K$, $R$, $S$, $T$ has 12(6) parameters, implying from (67) that $K$, $R$ and $S$ have 12(6) independent parameters. Since according to (72) the matrix $S$ is not independent, the matrices $K$, $R$ that appear in the interaction Lagrangian (43), have together 12(6) parameters. From (71) and the 15(6) number of parameters of $m_D$, we see that $R$ will have 12(6) parameters. Since $K$ is unitary in the present approximation, we can choose 6(3) independent parameters within $R$ to provide the unitary matrix $K$ with 6(3) parameters, the physically relevant PMNS structure. Then $R$ will have other extra 6(3) parameters. However, other solutions are allowed, since $K$ is unitary, not necessarily of the PMNS type.

3.1.2 Summary of the parameter counting in the mass basis

In the mass basis, parameter counting in the physically relevant case is: 12(6) parameters from both the complex matrices $K$, $R$ (among these, 6(3) parameters from the PMNS-like matrix $K$) + 3(0) parameters from $M^\text{diag}_R$ + 3(0) parameters from $m^\text{diag}_{L}$ + 3(0) parameters from $m^\text{diag}_e = 21(6)$, the same counting as in the current basis.

The more constrained condition $m_D \ll M_R$ provides a particular case: $R$ has 12(6) parameters, among which one has to choose the 6(3) parameters of the PMNS matrix $K$.

3.2 Left-Right Model

Let us start from the Lagrangian (34) of the LRM. At this stage $M_R$ is complex symmetric with 9(3) parameters. We rewrite (34) under the form

$$\mathcal{L}_m = \frac{1}{2} \overline{\nu_L} (\nu_L)^c \mathcal{M} \begin{pmatrix} (\nu_L)^c \\ N_R \end{pmatrix} + \overline{e}_L m^\text{diag}_e e_R + \text{h.c.}$$

$$\mathcal{L}_w = \overline{\nu_L} (\nu_L)^c \gamma_\mu \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_L \\ e_L \end{pmatrix} W^\mu_L$$

$$+ \overline{N_R} (\nu_L)^c \gamma_\mu \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e_R \\ e_R \end{pmatrix} W^\mu_R + \text{h.c.} \quad (73)$$
where

\[ M = \begin{pmatrix} 0 & m_D \\ m_D^t & M_R \end{pmatrix} \] (74)

Unlike the case of the ESM, the complex symmetric block \( M_R \) is not diagonalized, it has 9(3) parameters since three phases have been rotated away.

Using the the unitary matrix \( V \) (36-39),

\[
\begin{pmatrix} \nu_L, (N_R)^c \end{pmatrix} \rightarrow \begin{pmatrix} \nu_L, (N_R)^c \end{pmatrix} V^t
\]

we obtain the following Lagrangian in the mass basis

\[ L_m = \frac{1}{2} \overline{\nu_L} m_{\nu_L} \text{diag} (\nu_L) + \frac{1}{2} (\overline{N_R})^c M_{\nu} \text{diag} N_R + e_{\nu_L} m_{\nu_R} \text{diag} e_R + h.c. \]

\[ L_w = (\overline{\nu_L} K^t + (\overline{N_R})^c R^t) \gamma_\mu e_L W_{\mu L}^\mu + (\overline{N_R} T^t + (\overline{\nu_L})^c S^t) \gamma_\mu e_R W_{\mu R}^\mu + h.c. \] (76)

The 3 × 3 matrices \( K \) and \( R \) enter in the left sector, while \( T \) and \( S \) enter in the right sector, in a symmetric way. Formula (76) has been already obtained, using a different notation, in [16].

It is important to point out that the terms dependent on \( K \) and \( T \) are lepton number conserving \( \Delta L = 0 \), while those that depend on \( R \) and \( S \) are lepton number violating \( \Delta L = 2 \).

The gauge bosons \( W_L \) and \( W_R \) are mixed in the Left-Right Model:

\[ W_L = \cos \zeta W_1 - \sin \zeta W_2, \quad W_R = e^{i\omega} (\sin \zeta W_1 + \cos \zeta W_2) \] (77)

where \( W_1 \) and \( W_2 \) are mass eigenstates, and the mixing angle \( \zeta \), in terms of the vacuum expectation values [12,15], is of the order [15]

\[ \zeta \simeq \pm g_L \frac{2 | k_1 k_2 |}{g_R | v_R|^2} \] (78)

From (76), it is interesting to write down the lightest mass vector boson \( W_1 \) couplings to leptons

\[ L_{w_1}^{W_1} = \left[ \cos \zeta \left( \overline{\nu_L} K^t + (\overline{N_R})^c R^t \right) \gamma_\mu e_L + e^{i\omega} \sin \zeta \left( \overline{N_R} T^t + (\overline{\nu_L})^c S^t \right) \gamma_\mu e_R \right] W_{1 \mu} + h.c. \] (79)
Besides the $\sim \cos \zeta$ term that describes the processes $\Delta L = 0$ and $\Delta L = 2$, as in the ESM case, the subleading term $\sim \sin \zeta$ describes the $\Delta L = 0$ process

$$N_R \to e_RW_1$$

and another term describing the lepton-number violating decay $\Delta L = 2$ of the gauge boson

$$W_1 \to \overline{\nu}_R(\nu_L)^c$$

$(L(\nu_R) = L((\nu_L)^c) = -L(e_R) = -L(\nu_L))$. However, the amplitude for this latter decay is very small, as we will see below.

On the other hand, the heavier vector boson $W_2$ couplings to leptons read:

$$L_{w2}^W = \left[-\sin \zeta \left(\overline{\nu}_L R^\dagger + (N_R)^c R^\dagger\right) \gamma_\mu e_L + e^{\omega} \cos \zeta \left(\overline{N_R} T^\dagger + (\nu_L)^c S^\dagger\right) \gamma_\mu e_R\right] W_2^\mu + h.c.$$  \hfill (82)

Here, the subleading $\sim \sin \zeta$ term describes the $\Delta L = 0$ process

$$W_2 \to \overline{\nu}_L \nu_L$$

and the $\Delta L = 2$ transition, assuming the mass of $W_2$ heavier that the one of $N_R$:

$$W_2 \to \overline{\nu}_L (N_R)^c$$

(84)

On the other hand, the leading $\sim \cos \zeta$ term describes the process $\Delta L = 0$

$$W_2 \to \overline{\nu}_R N_R$$

and the $\Delta L = 2$ involving light leptons:

$$W_2 \to \overline{\nu}_R (\nu_L)^c$$

(86)

Of course, the phenomenological relevance of the $\Delta L = 2$ decay involving the $W_R$ gauge boson depends on its mass scale. If one assumes that the mass scale of the LRM is low, it makes sense to look at the LHC for lepton-number violation processes through the search of $pp \to \ell\ell jj$ topologies, where the two leptons are of the same charge (see for example the recent refs. \cite{20, 21, 22}).

Indeed, using (82) there is the possibility of the $\Delta L = 2$ process

$$W_2^+ \sim W_R^+ \to e_R^+ N_R \to e_R^+ e_L^+ W_L^- \to e_R^+ e_L^+ jj$$

(87)
where $W_L^{-}$ decays into two hadronic jets, the subscripts in $e_R$ and $e_L$ mean the couplings to $W_R$ and $W_L$, and we use the notation $(e_R)^c = e_R^+$, $(e_L)^c = e_L^+$. The decay chain (87) is the very interesting Keung-Senjanović process proposed long time ago [23] that tests, at the same time, the decay of the gauge boson $W_R$ and the Majorana character of the right handed neutrino $N_R$.

3.2.1 The matrices $K, R, S, T$ in the Left-Right Model

Particularizing (144-146) to the case (74), we have:

$$Km_L^{\text{diag}}K^t + RM_R^{\text{diag}}R^t = 0$$  \hfill (88)

$$Sm_L^{\text{diag}}S^t + TM_R^{\text{diag}}T^t = M_R$$  \hfill (89)

$$Km_L^{\text{diag}}S^t + RM_R^{\text{diag}}T^t = m_D$$  \hfill (90)

Considering for the moment only the unitarity of the full matrix $V$ (39), that has 36(21) parameters, the number of independent parameters in the l.h.s. of the precedent equations will be 36(21) from $(K, R, S, T) + 3(0) + 3(0)$ from $M_R^{\text{diag}} = 42(21)$ parameters.

The complex symmetric matrix equation (88) gives 12(6) constraints. On the other hand, $M_R$ in the r.h.s. of (89) has 9(3) free parameters, and this equation implies 12(6) - 9(3) = 3(3) constraints. Finally, since $m_D$ is a general complex matrix, with 18(9) free parameters, eq. (90) does not give any constraint. This gives a total of 12(6) + 3(3) = 15(9) constraints. Therefore one has 42(21) - 15(9) = 27(12) independent parameters. Adding the 3(0) eigenvalues of $m_e^{\text{diag}}$, not counted up to now, one gets 27(12) + 3(0) = 30(12) parameters, the same result as in the current basis.

Moreover, substracting from this total number of 30(12) parameters the 9(0) mass eigenvalues $m_e^{\text{diag}}, m_L^{\text{diag}}$ and $M_R^{\text{diag}}$, we see that the set of matrices $(K, R, S, T)$, that appear in the interaction term (76), have a total of 21(12) parameters.

In the $SU(2)_L \times SU(2)_R \times U(1)$ Model one obtains also the exact relations between the matrices $K, R, S, T$ given above by eqns. (55-59).

The matrices $(K, R, S, T)$ for $m_D << M_R$
The relations given above within the approximation $m_D \ll M_R$ (61, 63) for the ESM also hold in the LR model.

Let us rewrite eqns. (88-90) neglecting terms of $O\left(\frac{m_D^2}{M_R^2}\right)$:

$$K m_L^{\text{diag}} K^t + R M_R^{\text{diag}} R^t = 0 \quad (91)$$

$$T M_R^{\text{diag}} T^t \simeq M_R \quad (92)$$

$$R M_R^{\text{diag}} T^t \simeq m_D \quad (93)$$

Eqn. (93) is the above obtained exact relation (57) if one neglects in the latter higher order terms. This means that in (90), the first term of the l.h.s., that is of $O(m_D^3/M_R^2)$, is compensated by higher order terms in the second term $R M_R^{\text{diag}} T^t$. On the other hand, combining (63) with the exact relation (59), one consistently obtains the exact relation (91).

According to (62) and (92), the matrix $T$ is the unitary mixing matrix of right-handed neutrinos, for which we can take 6(3) parameters, i.e. a matrix of the PMNS type. Eqn. (61) holds also in the LRM, and $K$ is the unitary mixing matrix of light left-handed neutrinos.

Since the whole set $K$, $R$, $S$ and $T$ has 21(12) parameters and the matrices $K$, $T$ have 6(3) parameters each, this implies that $R$ and $S$ can have together 9(6) extra independent parameters.

In the LR model, from relations (63) and (92) one obtains

$$K m_L^{\text{diag}} K^t \simeq -m_D T^*(M_R^{\text{diag}})^{-1} T^i m_D^i \quad (94)$$

i.e. the seesaw formula

$$m_L \simeq -m_D M_R^{-1} m_D^t \quad (95)$$

where $M_R$ is not diagonalized, to be compared with the seesaw formula (70) in the case of the ESM.

Notice the important point that in Section 1 we have disregarded the possibility in the LRM of a Higgs triplet $\Delta_L$ that in principle could also contribute to the mass of the light neutrinos (see for example [3, 25]), so that formula (95) is only correct in the LRM if one neglects this type II seesaw contribution.

Equation (93) implies, using the approximate unitarity of $T$:

$$R \simeq m_D T^*(M_R^{\text{diag}})^{-1} \quad (96)$$
to be distinguished from (71), that holds in the ESM case. We see that in the LR case the PMNS matrix \( T \) of the heavy neutrinos \( T \) enters in the matrix \( R \) and, on the other hand, the matrix \( S \) satisfies relation (72) that we found in the ESM.

### 3.2.2 Summary of the parameter counting in the mass basis

We have seen that the set of matrices \( K \), \( R \), \( S \) and \( T \) have together 21(12) parameters. Unlike the case of the ESM, in the LR model we have enough parameter space to accomodate two different PMNS matrices for \( K \) and \( T \), with 6(3) parameters each. Then, \( R \) and \( S \) can have together extra 9(6) parameters. However, this situation is not compulsory: there can be overlap between the parameters of all the four matrices \( K \), \( R \), \( S \) and \( T \).

In conclusion, the parameter counting in the physically interesting solution is as follows: 6(3) parameters from the PMNS-like unitary matrix \( K \) + 6(3) parameters from the PMNS-like matrix \( T \) + 9(6) extra parameters from the complex matrices \( R \), \( S \) + 3(0) from \( M_R^{\text{diag}} \) + 3(0) from \( m_L^{\text{diag}} \) + 3(0) from \( m_e^{\text{diag}} = 30(12) \) parameters, the same number as in the current basis.

### 4 Representations of the Dirac mass matrix

The Dirac mass matrix \( m_D \) is a crucial input in neutrino physics, making the link between the high and the low energy. We review now some useful representations of \( m_D \).

#### 4.1 Triangular parametrization

An interesting representation of the Dirac mass matrix \( m_D \) has been proposed by Branco et al. [10]:

\[
m_D = U m_\Delta
\]

(97)

where \( U \) is a unitary matrix with 6(3) parameters of the PMNS form, although not identical to it, and \( m_\Delta \) is a triangular matrix, with 3 off-diagonal vanishing elements, 3 real diagonal elements and 3 complex off-diagonal elements.
The factorization formula (97) is usually called in Mathematics QR Decomposition of a complex square matrix $M$. In Mathematica notation \[24\] QRDecomposition[M] gives the decomposition of a numerical complex matrix $M$ in terms of a unitary matrix $U$ and an upper triangular matrix $m_\Delta$, while \[9\] \[10\] refers to a lower triangular matrix, although this is not an essential point. This decomposition can be numerically very useful for texture models of the matrix $m_D$, since it isolates $m_\Delta$, and hence the parameters that are relevant for leptogenesis.

The counting of parameters for $m_D$ holds in (97): 15(6) parameters of $m_D = 6(3)$ parameters of $U + 9(3)$ parameters from the triangular matrix $m_\Delta$. Relation (97) also holds if $m_D$ is general complex and $U$ a general unitary matrix: 18(9) parameters of $m_D = 9(6)$ parameters of $U + 9(3)$ parameters from the triangular matrix $m_\Delta$. In the same way that 3 phases of $m_D$ can be rotated away by the transformation (24-26), one can consistently rotate away 3 phases of the general unitary matrix $U$ \[10\].

Relation (97) is non-trivial. Indeed, because of the unitarity of $U$ we see that $m_D^\dagger m_D$ is given by

$$m_D^\dagger m_D = m_\Delta^\dagger m_\Delta$$

and therefore the three CP phases of $m_\Delta$ control the amount of leptogenesis at high energies in the one-flavor approximation.

### 4.1.1 Extended Standard Model

With (97), equation (71) obtained within the seesaw, writes

$$R \simeq U m_\Delta (M_R^{\text{diag}})^{-1}$$

We have seen above that if we decide that $K$ is of the PMNS type with 6(3) parameters, then the parameters of $K$ have to be chosen among the ones of $R$. A solution satisfying this criterium is a Dirac mass matrix given by \[9\]

$$m_D = K m_\Delta, \quad R \simeq K m_\Delta (M_R^{\text{diag}})^{-1}$$

Besides its historical interest, this solution has the very nice feature of factorization of the Dirac mass matrix into two pieces, a low energy PMNS mixing matrix $K$ with
6(3) parameters, and a high energy mass matrix $m_\Delta$, that has 9(3) parameters and controls leptogenesis.

Another extreme case would be to assume that $U = 1$ [26, 27] that implies

\[ m_D = m_\Delta, \quad R \simeq m_\Delta (M_R^{\text{diag}})^{-1} \]  

(101)

This ansatz relates directly the CP-violating phase in leptogenesis and CP-violation at low energy in neutrino oscillations.

However, there are many other solutions, since in all generality one can choose the parameters of $K$ among the ones of the product $m_D = U m_\Delta$.

4.1.2 Left-Right Model

Equation (96), writes

\[ R = U m_\Delta T^* (M_R^{\text{diag}})^{-1} \]  

(102)

where we see that the matrix $T$, unlike the case of the ESM [99], enters in the definition of the matrix $R$, that controls leptogenesis.

4.2 The orthogonal parametrization

Another useful parametrization of $m_D$ has been proposed by Casas and Ibarra [28].

4.2.1 Extended Standard Model

Starting from the seesaw formula (70) and diagonalizing $m_L$ by the PMNS matrix $K$ (69),

\[ m_L^{\text{diag}} = -K^\dagger m_D (M_R^{\text{diag}})^{-1/2} m_D^t K^* \]  

(103)

As pointed out in [28], this relation implies,

\[ - (m_L^{\text{diag}})^{-1/2} K^\dagger m_D (M_R^{\text{diag}})^{-1/2} (M_R^{\text{diag}})^{-1/2} m_D^t K^* (m_L^{\text{diag}})^{-1/2} = 1 \]  

(104)

and therefore the matrix $i (m_L^{\text{diag}})^{-1/2} K^\dagger m_D (M_R^{\text{diag}})^{-1/2}$ is an orthogonal complex matrix $O$

\[ O = i (m_L^{\text{diag}})^{-1/2} K^\dagger m_D (M_R^{\text{diag}})^{-1/2} \]  

(105)

i.e. $OO^t = 1$. One finds the general expression for $m_D$ in terms of the matrix $O$

\[ m_D = -i K (m_L^{\text{diag}})^{1/2} O (M_R^{\text{diag}})^{1/2} \]  

(106)
One can check from this expression that $m_D = K m_{\Delta}$ is not the most general form for $m_D$ because $O$, being a general complex orthogonal matrix, the combination $-i(m_L^{\text{diag}})^{1/2}O(M_R^{\text{diag}})^{1/2}$ is not triangular in general.

The parametrization is very useful to analyze leptogenesis $CP$ asymmetries when taking flavor into account.

4.2.2 Left-Right Model

From eq. (94) one gets, instead of (104)

$$- (m_L^{\text{diag}})^{-1/2}K^t m_D T^* (M_R^{\text{diag}})^{-1/2} (M_R^{\text{diag}})^{-1/2} T^t m_D^t K^* (m_L^{\text{diag}})^{-1/2} = 1$$

(107)

that defines the orthogonal matrix

$$O' = i (m_L^{\text{diag}})^{-1/2}K^t m_D T^* (M_R^{\text{diag}})^{-1/2}$$

(108)

and $m_D$ is now in the LRM

$$m_D = -i K (m_L^{\text{diag}})^{1/2} O'(M_R^{\text{diag}})^{1/2} T^t$$

(109)

that includes the PMNS mixing matrix $T$ of right-handed neutrinos.

4.3 Relation between the triangular and orthogonal forms

The orthogonal parametrization of the Dirac mass matrix $m_D$ appears to be powerful because it includes low energy quantities, the light neutrino eigenvalues $m_L^{\text{diag}}$ and the PMNS mixing matrix $K$ and, on the other hand, high energy quantities, the heavy right-handed neutrino eigenvalues $M_R^{\text{diag}}$ and an unknown orthogonal complex matrix $O$. One can write down the relation between both representations.

In the ESM, from relation (106) one can write the QR decomposition of the matrix

$$- i (m_L^{\text{diag}})^{1/2} O (M_R^{\text{diag}})^{1/2} = V m_{\Delta}$$

(110)

where $V$ is another unitary matrix, and $m_{\Delta}$ a triangular matrix. We see therefore that the matrix $m_D$ has the form of the triangular parametrization (97) $m_D = U m_{\Delta}$, with the PMNS matrix $K$ being a factorizable part of the unitary matrix $U$, namely $U = KV$. Therefore, although one can set $U = 1$, i.e. $V = K^{-1}$, and then the low
energy phases are part of $m_\Delta$ and hence of leptogenesis, the natural solution seems to be that the PMNS matrix $K$ is a unitary factor of the matrix $U$, i.e. $U = KV$, $V$ being a unitary matrix.

5 Leptogenesis

The gauge models that we consider conserve $B - L$. As nicely pointed out by Strumia [29], the mere existence of sphalerons, that violate $B + L$ in the Standard Model at high temperature, suggests that baryogenesis can proceed via leptogenesis [30, 31]. From (43) or (76), we see that lepton number is naturally violated by the decays of heavy right-handed neutrinos, giving rise to a lepton asymmetry that is partially converted into a baryon asymmetry by the sphalerons. The out-of-equilibrium CP violating decays of heavy Majorana neutrinos, supplemented by sphaleron interactions, satisfy the three Sakharov criteria [32] to obtain baryogenesis.

In this section we consider leptogenesis in the electroweak broken phase, coming from the $CP$ violating $\Delta L = 2$ decay $(N_R)^c \rightarrow e_L W_L$ in the Lagrangians (43) of the ESM and (76) of the LRM.

The actual leptogenesis occurs at very high temperature, in the electroweak unbroken phase. The connection between cosmological $CP$ violation in the unbroken phase [38] with a single massless Higgs doublet, and in the broken phase has been underlined by Branco et al. [10]. In the case of the Left-Right model, this connection is not clear a priori because the massless Higgs fields in the unbroken case belong to the bidoublet [10]. As we emphasize below, this relation is worth to be investigated. For the moment, we are interested here in the possible differences between the ESM and the LRM in the broken phase, where the interaction Lagrangians (43) and (76) apply.

5.1 One-flavor approximation

5.1.1 Extended Standard Model

In this part on the ESM we reproduce the results of ref. [10], with the aim of comparing below with the LRM. The lepton number asymmetry from the decay of
the 1st, lightest heavy Majorana neutrino, in the broken electroweak phase and in
the one-flavor approximation is given by:

$$\epsilon_1 = \frac{g^2}{M_W^2} \frac{1}{16\pi} \frac{1}{(R^t R)_{11}} \sum_{k \neq 1} F(x_k) M_k^2 Im[(R^t R)_{1k}]^2$$

(111)

where

$$F(x_k) = \sqrt{x_k} \left[ 1 + (1 + x_k) \ln \left( \frac{x_k}{1 + x_k} \right) + \frac{1}{1 - x_k} \right] \left( x_k = \frac{M_k^2}{M_1^2} \right)$$

(112)

since, from (13), the matrix $R$ is responsible for the transition $(N_R)^c \rightarrow e_L W_L$, or
equivalently the decay $(N_R)^c \rightarrow e_L H$ above the phase transition.

As pointed out in ref. [10], from (71) $R \approx m_D (M_R^{diag})^{-1}$, that holds in the ESM
for $m_D << M_R$, one gets the lepton number asymmetry in terms of the Dirac mass
or, equivalently, in terms of the Yukawa couplings $m_D \sqrt{v}$ in the unbroken phase [38] :

$$\epsilon_1 = \frac{g^2}{M_W^2} \frac{1}{16\pi} \frac{1}{(m_D^t m_D)_{11}} \sum_{k \neq 1} F(x_k) Im[(m_D^t m_D)_{1k}]^2$$

(113)

While the expression of the lepton number asymmetry (111) depends only on quantities of the mass basis, namely on the matrices $R$, $M_R^{diag}$, expression (113) depends
only on quantities of the current basis, since the matrix $M_R$ is diagonalized from
the beginning in both bases.

Notice that, as exposed in [10], expression (113) has a well-defined limit for the
SM vacuum expectation value limit $v \rightarrow 0$, given in terms of Yukawa couplings
Corresponding to the decay in the unbroken electroweak phase $(N_R)^c \rightarrow e_L H$ [38].

In terms of the matrix $m_\Delta$ one gets

$$\epsilon_1 = \frac{g^2}{M_W^2} \frac{1}{16\pi} \frac{1}{(m_\Delta^t m_\Delta)_{11}} \sum_{k \neq 1} F(x_k) Im[(m_\Delta^t m_\Delta)_{1k}]^2$$

(114)

that depends only on the three phases of $m_\Delta$.

On the other hand, in terms of the orthogonal matrix $O$ defined in (105) the CP
asymmetry is given by

$$\epsilon_1 = \frac{g^2}{M_W^2} \frac{1}{16\pi} \frac{1}{M_1 \sum_i |O_{i1}|^2} \sum_{k \neq 1} F(x_k) M_1 M_k \text{ Im} \left[ \sum_j (m_j O_{j1})^2 \right]$$

(115)
5.1.2 Left-Right Model

In the LR model one has in principle two types of contributions to the light neutrino masses, through type I seesaw and type II seesaw, the latter arising from triplet Higgs exchange (see for example refs. 31 [25]). In this paper we consider only the contribution of type I seesaw.

In the LR case we have seen that the matrix responsible for the transitions \((N_R)^c \rightarrow e_L W_L\) is the matrix called also \(R\) in the mass basis Lagrangian (76). Then, the lepton number asymmetry from the decay of the 1st heavy Majorana neutrino, in the single flavor approximation, is given by the same formulas (111,112).

In the LR model we have now (96) \(R \approx m_D T^* (M^\text{diag}_R)^{-1}\), that yields the lepton number asymmetry in terms of the Dirac mass and the mixing matrix \(T\) of the heavy neutrinos:

\[
\epsilon_1 = \frac{g^2}{M_W^2} \frac{1}{16\pi} \left( T^* \bar{m}_D m_D T^* \right)_{11} \sum_{k \neq 1} F(x_k) \Im \left[ (T^* \bar{m}_D m_D T^*)_{1k} \right]^2
\]

Expression (116) for the CP asymmetry in the electroweak broken phase follows from the \(R\)-term in the interaction Lagrangian (76), responsible for the decay \((N_R)^c \rightarrow e_L W_L\). This is the expression that has been used precisely to compute the leptogenesis CP asymmetry within LRM (see for example refs. 25, 39).

In other terms, the matrix \(m_D T^* = m_D^\prime\) is the Dirac mass matrix in the basis in which \(M_R = t v_R\) (15).

In the unbroken electroweak phase, the Higgs bidoublet (10) would be massless, and one should consider both contributions \(N_1 \rightarrow e_\varphi_{1,2}\) to the leptogenesis asym-
metry, with both Higgses $\varphi_{1,2}$ contributing to the loops needed to interfere with the tree diagram to obtain $CP$ violation. This situation reminds the one of the Standard Model with several Higgs doublets [40]. The relation between the $CP$ asymmetries in the broken and unbroken phases of the LRM deserves a specific further investigation.

Since the matrix $m_D$ is general complex, so is $m_D^\ast$ and we can write a decomposition in terms of another general unitary matrix $U'$ and another triangular matrix $m'_{\Delta}$:

$$m'_{\Delta} = m_D T^* = U' m'_{\Delta} \quad (117)$$

The lepton asymmetry writes

$$\epsilon_1 = \frac{g^2}{M_W^2} \frac{1}{16\pi} \frac{1}{(m_{\Delta}^* m'_{\Delta})_{11}} \sum_{k \neq 1} F(x_k) \text{Im}[ (m_{\Delta}^* m'_{\Delta})_{1k} ]^2 \quad (118)$$

that now depends on the three CP phases of $m'_{\Delta}$.

On the other hand, notice that the interaction Lagrangian (76) contains also the $\Delta L = 2$ term $(\nu_L)^c S^T e_R W_R$ that could give a contribution to the lepton asymmetry through the decay

$$W_R \rightarrow \tau_R (\nu_L)^c \quad (119)$$

The masses $M_{W_R}$ and $M_i$ are both generated by the same Higgs triplet, and since one usually assumes that the Yukawa coupling of the heaviest neutrino $N_3$ is of $O(1)$, and then $M_{W_R} \gg M_i$ assuming a hierarchical spectrum for the heavy neutrinos. Hence, the lepton asymmetry generated by the decay of $W_R$ will be washed out and only the one due to the $N_1$ decays will survive. However, one should keep in mind in model building the possibility of leptogenesis through the decay (119).

### 5.2 Flavored leptogenesis

#### 5.2.1 Extended Standard Model

A crucial progress in leptogenesis has been achieved by taking into account flavor [33, 34, 35, 36]. At very high temperatures $T \geq 10^{12}$ GeV, all three $\tau, \mu$ and $e$ are out of equilibrium because their Yukawa couplings are weak relatively to the temperature. In this regime, the one-flavor approximation can be applied since the different lepton flavors are undistinguishable.
However, for "realistic" temperatures $T \simeq M_1$ such that $10^9 \leq T \leq 10^{12}$ GeV, the τ lepton doublet Yukawa coupling is large enough to be in thermal equilibrium, while the $\mu$ and $e$ doublets are out of equilibrium. The net result is that the leptogenesis CP violation splits into two pieces, $\epsilon_\tau$ and $\epsilon_2 = \epsilon_\mu + \epsilon_e$, since the flavors $\mu$ and $e$ remain undistinguishable. Then, in the range $10^9 \leq T \leq 10^{12}$ GeV, the final baryon asymmetry $Y_B$ is the sum of two contributions, given by the lepton CP asymmetries $\epsilon_\tau$ and $\epsilon_2$ affected by different wash-out factors $\eta_\tau$ and $\eta_2 : Y_B \propto \epsilon_\tau \eta_\tau + \epsilon_2 \eta_2$. A recent updated flavor covariant description of flavor effects in leptogenesis can be found in ref. [37].

The CP violating asymmetry for each flavour is given by the expression (see for example [5]):

$$
\epsilon_{1\ell} = \frac{g^2}{M_W^2} \frac{1}{16\pi} \frac{1}{(m_D^\dagger m_D)_{11}} \sum_{k \neq 1} F(x_k) \text{Im}[(m_D^\dagger)_{1\ell}(m_D)_{\ell k}(m_D^\dagger m_D)_{1k}]^2 \\
+ \frac{g^2}{M_W^2} \frac{1}{16\pi} \frac{1}{(m_D^\dagger m_D)_{11}} \sum_{k \neq 1} G(x_k) \text{Im}[(m_D^\dagger)_{1\ell}(m_D)_{\ell k}(m_D^\dagger m_D)_{k1}]^2
$$

(120)

where the second term corresponds to the lepton flavor violating but lepton number conserving self-energy diagram [35]. The function $F(x_k)$ is given by (112), and

$$
G(x_k) = \frac{1}{1 - x_k}
$$

(121)

The second term in (120) vanishes when summing over $\ell$, while the first term gives the one-flavor approximation expression (113), because $\sum_\ell \epsilon_{1\ell} = \epsilon_1$. On the other hand, the second term in (120) is subleading if one assumes $M_1 << M_2, M_3$.

The flavored wash-out factors read [36]

$$
\eta_\ell = \eta \frac{(m_D^\dagger)_{1\ell} m_D^{\dagger 1\ell}}{(m_D^\dagger m_D)_{11}}
$$

(122)

where $\eta$ is the wash-out factor in the single flavor approximation.

Concerning the link between low energy CP violation in the PMNS mixing matrix and leptogenesis CP violation, the situation is quite different if flavor is taken into account [36]. As an illustration, let us write the CP asymmetry $\epsilon_{1\ell}$, where the subindex 1 means decay of the lightest heavy Majorana neutrino $N_1$, by using the orthogonal parametrization (106). The flavor CP asymmetries $\epsilon_{1\ell}$ depend then on the low energy parameters, i.e. the light neutrino masses and the PMNS mixing.
matrix $K$. Assuming $M_1 << M_2 < M_3$, one finds from (106) and (120) the leptonic CP violation parameter $\epsilon_{1\ell}$ \[^{36}\] :

$$
\epsilon_{1\ell} \simeq -\frac{3g^2}{32\pi M_W^2} \frac{\text{Im} \left( \sum_{k,j} m_j m_k^{3/2} K_{\ell j}^{*} K_{\ell k}^{*} O_{j1}^{*} O_{k1}^{*} \right)}{\sum_{i} m_i \left| O_{i1} \right|^2}
$$

(123)

### 5.2.2 Left-Right Model

As we have seen in the LRM in the one-flavor approximation (formula (116)), $m_D$ is replaced by $m_D T^*$, and the formula for the lepton asymmetry in this approximation is the same as in the Extended Standard model with the replacement $m_D \to m'_D = m_D T^*$ where $m'_D$ is the Dirac mass matrix in the basis in which the mass matrix $M_R$ is diagonalized.

Because of (108), formulas for the CP asymmetry (120) and the wash-out factor (122) remain correct for the Left-Right model, with the replacement $m_D \to m'_D = m_D T^*$, where $m_D$ is given by (109), that has a complete left-right symmetry in the dependence on the mass eigenvalues $m^{diag}_L, M^{diag}_R$ as well as on the mixing matrices $K, T$. Then, the flavor asymmetry has the same form (123), with the replacement $O \to O'$.

### 6 Comparison between the Extended Standard Model and the Left-Right Model

We now summarize the comparison between the ESM and the LRM, as far as lepton mixing is concerned.

(a) In the current basis both models differ in the following way.

In the ESM the Dirac matrix $m_D$ has 15(6) parameters because one can rotated away 3 phases and one can diagonalize the right-handed mass matrix $M_R$. One has finally a total of 21(6) parameters.

In the LRM one cannot diagonalize $M_R$ without changing the interaction Lagrangian. On the other hand, one cannot rotate away phases in both $m_D$ and in $M_R$, but only three phases in one of these matrices, that we have chosen to be $M_R$. Then, one is left with a general complex $m_D$ with 18(9) parameters and a complex
symmetric $M_R$ with 9(3) parameters. With the $m_e$ mass eigenvalues, this gives a total of 30(12) parameters.

However, if in the LRM one diagonalizes $M_R$ from the start, the left-handed interaction term $\bar{\nu}_L \gamma_\mu e_L W_\mu^L$ remains diagonal, while the right-handed term $\bar{N}_R \gamma_\mu e_R W_\mu^R$ is modified. Also $m_D$ is modified to another Dirac mass term, that would eventually control leptogenesis. Therefore, as far as one considers the mass terms and the $W_L$ interaction, one has the same number of parameters as in the ESM. For physics at low energy and also for leptogenesis, if the latter is attributed to the decays of the lightest right-handed heavy neutrino $N_1$, one can disregard the $W_R$ interaction term, that involves heavier degrees of freedom.

(b) In the mass basis in the ESM without approximations one has two mixing matrices $K$ and $R$ in the left sector, that have together 12(6) parameters. For $m_D << M_R$ one has a priori 12(6) parameters for the set of matrices $K, R$ (mixing in the left sector), and $S, T$ (mixing in the right sector). The mixing matrix of the left-handed neutrinos is approximately unitary and can be chosen to be of the PMNS type, with 6(3) parameters. The model constrains the mixing matrix of the right-handed neutrinos to be $T \simeq 1$, the matrix $R$ (71) has a total of to 12(6) parameters and $S$ is not independent because of relation (72). The parameters of the PMNS mixing matrix for light neutrinos $K$ have to chosen among the ones of $R$. Adding the mass eigenvalues $m_\text{diag}_L, M_\text{diag}_R, m_\text{diag}_e$ one has a total of 21(6) parameters.

In the LRM in the mass basis one has more symmetry: two mixing matrices $K, R$ in the left sector and two $S, T$ in the right sector. These four matrices have together 21(12) parameters, that added to the mass eigenvalues $m_\text{diag}_L, M_\text{diag}_R, m_\text{diag}_e$ gives again a total of 30(12) parameters. In the approximation $m_D << M_R$ the mixing matrices $K$ (left sector) and $T$ (right sector) are unitary, and both can be chosen to be of the PMNS type, with 6(3) parameters each. This is different from the ESM for the right sector, where $T$ is trivial. This feature of the ESM seems unnatural, since physically one should expect a full PMNS matrix for the heavy right-handed neutrinos as well.

(c) Adopting the decomposition $m_D = U\Delta$ ($U$ unitary and $\Delta$ triangular complex), in the ESM the matrix $U$ has 6(3) parameters and $\Delta$ 9(3) parameters, corresponding to the 15(6) parameters of $m_D$. The natural solution is that the PMNS
matrix $K$ is a unitary factor of the matrix $U$, namely $U = KV$, $V$ being also unitary. In the LRM the situation is somewhat different: $m_D$ is a general complex matrix with $18(9)$ parameters, $U$ is a general unitary matrix with $9(6)$ parameters and $m_\Delta$ has also $9(3)$ parameters. The Dirac mass matrix in the basis in which $M_R$ is diagonal \((117)\) $m'_D = m_DT^*$ can be decomposed in the same way: $m'_D = U'm'_\Delta$.

(d) Concerning the lepton asymmetry relevant for leptogenesis, we find the following situation in both models.

In the ESM, in the one-flavor approximation, the asymmetry is dependent on matrix elements of the matrices $R'^\dagger R$ or $m_D^\dagger m_D$ or $m_\Delta^\dagger m_\Delta$, i.e. dependent on the $3$ CP phases of $m_\Delta$. In the flavored case, the asymmetry \((120)\) depends on the PMNS matrix $K$ and the three high energy phases of the orthogonal matrix $O$ \((105)\).

In the LRM, in the one-flavor approximation, the lepton asymmetry is dependent on $R'^\dagger R$ or $T'^\dagger m_D^\dagger m_D T^*$. Writing the product $m_D T^*$ as in \((117)\), the asymmetry depends on the three CP phases of the triangular matrix $m'_\Delta$ through $m'^\dagger_\Delta m'_\Delta$. In the flavored case, the asymmetry depends on the PMNS mixing matrix $K$ and on the three phases of $O'$ \((108)\).

As far as model building is concerned, the situation is different in both schemes. As an example, imagine that one has a model for the Yukawas with some ansatz for $m_D$ and $M_R$. In the ESM, $M_R$ is diagonalized and $m_D$ is enough to compute the lepton asymmetry. In the LRM one needs to compute the matrix $T$ that diagonalizes $M_R$, in order to get $m'_D$.

(e) A possible identification between low energy phases and leptogenesis phases is not possible in general. In the ESM one could imagine models in which the three CP phases of the light neutrinos mixing matrix $K$ are the same as the three phases of the triangular matrix $m_\Delta$, since one has to choose the parameters of $K$ among the ones of the matrix $R$ in the lepton asymmetry formula \((111)\). In the LRM one could choose the three phases of $K$ to be the same as the ones of $m'_\Delta$ \((117)\).

As to whether in general the leptogenesis CP asymmetry could depend on the low energy phases, in the flavored regime the usual argument that $\epsilon_{1\ell}$ in the ESM depend on the PMNS matrix $K$ and on the matrix $O$ \((105)\) extends to the LRM with another orthogonal matrix $O'$ \((108)\).
Relatively to the ESM, we have found that the LRM has some interesting new features:

- The non-trivial PMNS mixing matrix $T$ of the heavy neutrinos enters in the quantitative estimation of decay branching ratios of heavy neutrinos $N_R$ to various final states.

- On the other hand, in the calculation of the leptogenesis $CP$ asymmetries, the matrix $T$ is unobservable because the Dirac matrix that plays a role is now $m_D' = m_D T^*$, the Dirac matrix in the basis in which $M_R$ is diagonal.

- The term $(\nu_L)^c S^t e_R W_R$ in (76) could give a contribution to the cosmological lepton asymmetry through the $\Delta L = 2$ lepton number violating decay to light leptons $W_R \rightarrow \bar{e}_R (\nu_L)^c$. As we have indicated above, this latter possibility seems unlikely in reasonable left-right models because $W_R$ is heavier than the lightest neutrino $N_1$. However, one should keep in mind this possibility in model building.

- Considering the $W_1, W_2$ basis, i.e. without neglecting $W_L - W_R$ mixing, we have seen in Section 3.2 that there is a term involving the lighter $W_1$ boson $\sim \sin \zeta (\nu_L)^c S^t \gamma_\mu e_R W_1^\mu$, that allows for the subleading $\Delta L = 2$ lepton-number violating decay to light leptons $W_1 \rightarrow \bar{e}_R (\nu_L)^c$.

7 Extension to Pati-Salam and $SO(10)$

One can extend the precedent considerations to other left-right gauge models like the Pati-Salam gauge theory $SU(4)_C \times SU(2)_L \times SU(2)_R$ [17] or $SO(10)$ [6].

We can consider first each of these models in the current basis, with general mass terms determined only by the Dirac or Majorana character of the fermions, and perform the counting of the $CP$ conserving and $CP$ violating free parameters. In a second step, one can diagonalize the mass matrices and obtain mixing in the interaction terms and, in a third step, switch on the Higgs sector of each theory and see how, according to the different hypothesis on this sector, the predictive power of each scheme is improved. Of course, with the most general Higgs structure for each model, one populates the general parameter space of the mass terms obtained by imposing only Lorentz invariance.

Moreover, since in these theories leptons are related to quarks, lepton mixing
will be related to quark mixing, at least for some Higgs structures. This feature is interesting in view of increasing the predictive power of $SO(10)$ for leptogenesis, and has been used more or less quantitatively in the literature.

Let us give some details for the Pati-Salam model and for $SO(10)$. Consider first the general mass Lagrangian consistent with Lorentz invariance of Dirac and Majorana mass terms

$$L_m = \overline{\nu}_L m_D N_R + \frac{1}{2} (N_R)^c M_R N_R + \overline{\nu}_L m_e e_R + \overline{u}_L m_u u_R + \overline{d}_L m_d d_R + h.c. \tag{124}$$

For the moment the matrices $m_D, m_e, m_u$ and $m_d$ are general complex with 18(9) parameters each and $M_R$ is a general complex symmetric matrix with 12(6) parameters. This gives a priori a total of 84(42) parameters, while in the lepton sector one has 18(9) (from $m_D$) + 18(9) (from $m_e$) + 12(6) (from $M_R$) = 48(24) parameters.

In the Pati-Salam model and in $SO(10)$, the interaction Lagrangian has the general form

$$L_{int} = L_w + L_x \tag{125}$$

where one has in both models, keeping only the interesting flavor-changing terms:

$$L_w = \overline{\nu}_L \gamma_\mu \nu_L W_\mu^L + \overline{\nu}_R \gamma_\mu N_R W_\mu^R + \overline{d}_L \gamma_\mu u_L W_\mu^L + \overline{d}_R \gamma_\mu u_R W_\mu^R + h.c. \tag{126}$$

and the extra interaction term writes,

$$L_{PS} = \overline{\nu}_L \gamma_\mu d_L X_\mu^L + \overline{\nu}_R \gamma_\mu d_R X_\mu^R + \overline{\nu}_L \gamma_\mu u_L X_\mu^L + \overline{\nu}_R \gamma_\mu u_R X_\mu^R + h.c. \tag{127}$$

The colored gauge bosons have charges $|Q(X_L)| = |Q(X_R)| = \frac{2}{3}$, and in $SO(10)$ one has $[41, 42]$:

$$L_x^{SO(10)} = \left[ e^{ijk} (u_R^i)^c \gamma_\mu u_L^j + \overline{d}_L \gamma_\mu (d_R^k)^c \right] X^{k\mu}$$

$$+ \left[ e^{ijk} (u_R^i)^c \gamma_\mu u_L^j + \overline{d}_L \gamma_\mu (d_R^k)^c - u_L^i \gamma_\mu (e_R^j)^c \right] Y^{k\mu}$$

$$+ \left[ e^{ijk} (d_R^i)^c \gamma_\mu d_L^j + \overline{\nu}_L \gamma_\mu (u_R^k)^c - d_L^i \gamma_\mu (N_R^j)^c \right] Y^{ik\mu}$$

$$+ \left[ e^{ijk} (d_R^i)^c \gamma_\mu d_L^j + \overline{\nu}_L \gamma_\mu (u_R^k)^c - u_L^i \gamma_\mu (N_R^j)^c \right] X^{k\mu}$$

$$+ \left[ \overline{\nu}_L \gamma_\mu u_L^k + \overline{\nu}_L \gamma_\mu d_R^k - (d_R^k)^c \gamma_\mu (e_R^j)^c - (u_R^k)^c \gamma_\mu (N_R^j)^c \right] S^{k\mu} + h.c. \tag{128}$$
where \(i, j, k\) are color indices and the colored gauge bosons \(X, Y, Y', X_D, S\) have the charges: 

\[
|Q(X)| = \frac{4}{3},
|Q(Y)| = |Q(Y')| = \frac{1}{3},
|Q(X_D)| = |Q(S)| = \frac{2}{3}.
\]

Let us see how many parameters can be rotated away in both models. Analogously to the LRM, one can diagonalize \(m_\nu\) and absorb 3 phases in \(M_R\) in (124) while keeping \(L_w\) (126) invariant. However, as it is obvious from (127,128), \(L_x\) is changed under these transformations. In the pure lepton sector, leaving aside the quark-lepton terms in \(L_x\), the starting point for the diagonalization of the mass terms is the same as in the LRM (34), with 30(12) parameters in \(m_D, M_R\) and \(m_\nu^{diag}\). Diagonalizing (124) one gets the flavor-changing mixing in the interaction Lagrangian \(L_w + L_x\).

In the pure lepton sector our conclusions are the following. The diagonalization has the same form for \(SU(2)_L \times SU(2)_R \times U(1)\), Pati-Salam and \(SO(10)\) models. Separately, the \(3 \times 3\) matrices \(K\) and \(R\) enter in the left sector, while the \(3 \times 3\) matrices \(T\) and \(S\) enter in the right sector, like in the LRM, eqn. (76). In \(SU(2)_L \times SU(2)_R \times U(1)\), Pati-Salam and \(SO(10)\) models we have in the lepton sector the same counting of free parameters, i.e. 30 real parameters, among them 12 CP-violating phases.

Let us now make some remarks on masses and mixing in some particular cases in the interesting \(SO(10)\) case. Let us look at the product

\[
16 \times 16 = 10_S + \overline{126}_S + 120_A
\]

where \(10 + \overline{126}\) is the symmetric part and \(120\) the antisymmetric part. The representations \(10\) and \(120\) are real, \(126\) is complex, and the Yukawa terms that can give mass to the fermions are

\[
16_f \times 16_f \times 10_H = 1 + ...
\]

\[
16_f \times 16_f \times \overline{126}_H = 1 + ...
\]

\[
16_f \times 16_f \times 120_H = 1 + ...
\]

The Yukawa part of the Lagrangian reads

\[
L_Y = 16_f \left( Y_{10} 10_H + Y_{126} \overline{126}_H + Y_{120} 120_H \right) 16_f
\]
where a possible sum over Higgs representations and Yukawa coupling matrices in family space is implicit. After spontaneous symmetry breaking one gets the mass Lagrangian (see for example [43])

$$m_d = v_{10} d_{10} Y_{10} + v_{126} d_{126} Y_{126} + v_{120} d_{120} Y_{120}$$

$$m_u = v_{10} u_{10} Y_{10} + v_{126} u_{126} Y_{126} + v_{120} u_{120} Y_{120}$$

$$m_e = v_{10} e_{10} Y_{10} - 3 v_{126} e_{126} Y_{126} + v_{120} e_{120} Y_{120}$$

$$m_D = v_{10} D_{10} Y_{10} - 3 v_{126} D_{126} Y_{126} + v_{120} D_{120} Y_{120}$$

$$M_R = v_{120} R_{126}$$

(134)

where the Yukawa matrices $Y_{10}$ and $Y_{126}$ are complex symmetric, $Y_{120}$ is complex antisymmetric, and the $v$'s are Higgs vacuum expectation values. From the term (130) alone we obtain the well-known relations $m_e = m_d$ and $m_D = m_u$, while the term (131) alone would give the relations $m_e = -3m_d$ and $m_D = -3m_u$, and no relation from the term (132).

The vev's in (134) are in all generality complex numbers if we assume that $CP$ can be spontaneously broken (soft $CP$ violation). If $CP$ is not spontaneously broken the vevs are real and all $CP$ violation comes from the Yukawa couplings (hard $CP$ violation).

One could wonder how within SO(10) one can get the most general counting of parameters done above, i.e. 84(42) parameters for the whole mass sector (124), with 48(24) parameters in the lepton sector. As said above, this is simply achieved if all the representations $10_H, 126_H, 120_H$ in (133) are present and are different for each mass matrix, that becomes then completely general.

An interesting particular case is to consider only the $10$ and $126$ representations in (134), with $120$ absent:

$$m_d = m^d_{10} + m^d_{126}$$

$$m_u = m^u_{10} + m^u_{126}$$

$$m_e = m^d_{10} - 3m^d_{126}$$

$$m_D = m^u_{10} - 3m^u_{126}$$

$$M_R = m^R_{126}$$

(135)
In this situation, all mass matrices \( m_u, m_d, m_D, m_e \) and \( M_R \) are complex symmetric.

Let us count again the number of parameters under this hypothesis. The complex symmetric matrices \( m_{10}^d, m_{126}^d, m_{10}^u, m_{126}^u, m_{126}^R \), have 12(6) parameters each, that gives a total number of 60(30) parameters, a reduction relatively to the 84(42) total number of parameters of the general case. One can diagonalize the complex symmetric matrices \( m_d, ... M_R \) with unitary matrices \( V_d, ... V_R \). Because of relations \((135)\), the unitary matrices \( V_e, V_D, V_R \) are in principle given in terms of \( V_u \) and \( V_d \) and mass eigenvalues. Notice that, as discussed in the mass basis for the pure lepton sector, we can adopt without loss of generality the basis in which \( m_e = m_e^{diag} \). However, these relations give complicated equations between the elements of mixing matrices. Within this case of considering both \( 10 \) and \( 126 \), it seems hard to find relations between the mixing matrices in the quark and the lepton sector, at least in a model-independent way.

Let us consider two limiting cases: while the \( 126 \) contributes to \( M_R \), only the \( 10 \) or only the \( 126 \) contribute to \( m_d, m_u, m_e \) and \( m_D \).

From \((135)\) we see that in both cases one has quark-lepton symmetry in the mixing matrices, i.e. a relation between the left-handed neutrino Dirac mixing matrix \( V_L \), where \( m_D = V_L^\dagger m_D^{diag} V_R \), and the CKM quark matrix

\[
V_L = V_u V_d^\dagger = V_{CKM} \tag{136}
\]

This relation has been often used in a number of phenomenological schemes \([2, 4, 5]\). However, as it is well known, one needs both representations \( 10 \) and \( 126 \) to describe fermion masses in \( SO(10) \) \([44, 45]\), and therefore we must conclude that there is a clash between a good description of fermion masses and the one of obtaining quark-lepton symmetry in mixing.

Although the point of view of obtaining useful theoretical hints from \( SO(10) \) on the eigenvalues and mixing of the Dirac neutrino mass matrix has been advanced in a number of works \([1, 3, 2, 4, 5]\), it is worth to point out that there could be an alternative philosophy concerning the Dirac mass matrix. Within the Left-Right Model, if the \( W_R \) gauge boson and the lightest heavy neutrino \( N_R \) are light enough, there is the interesting possibility of a complete determination of the Dirac mass matrix from the experimental study of \( W_R \) and \( N_R \) decays \([46]\).
8 Conclusions

We have examined the parameter counting and structure of CP conserving and CP violating lepton mixing in two gauge models in the electroweak broken phase, the Extended Standard Model - i.e. the Standard Model plus one right-handed heavy neutrino per generation -, and the Left-Right Model $SU(2)_L \times SU(2)_R \times U_{B-L}(1)$. We have used both the "current basis", in which the gauge interactions are diagonal, and the "mass basis", where the mass matrices are diagonal and mixing appears in the charged current gauge-fermion part of the Lagrangian. On the other hand, we have distinguished between results that are exact and results that hold within the approximation of Dirac masses that are small relatively to right-handed neutrino masses, $m_D << M_R$.

We think that it is worth to compare these two models. One reason is that, for simplicity, in the literature people usually discuss lepton mixing within the simple ESM, while actually have in mind left-right Grand Unified Theories like $SO(10)$, that naturally include heavy right-handed neutrinos. The simplest LR model that we study in this paper is a kind of prototype for these more involved LR theories.

Although the outline of the parameter counting and structure of lepton mixing is rather close in both schemes, there are differences between the two models. In particular, the Extended Standard Model can accommodate a PMNS mixing matrix $K$ for light neutrinos, but there is no room in parameter space for a mixing matrix $T$ for the heavy neutrinos, the mixing matrix being close to the identity. On the other hand, as one could naturally expect, the Left-Right Model is consistent with PMNS mixing matrices for both light and heavy neutrinos. The lepton asymmetry relevant for leptogenesis depends, not only on the Dirac mass $m_D$, but also on the matrix $T$, that is non-trivial. But the lepton asymmetry is given in terms of the Dirac mass in the basis in which the right-handed heavy neutrino mass matrix is diagonal, while the interaction term in the right-handed sector is not diagonal anymore.

The connection between the lepton $CP$ asymmetry in the electroweak broken phase, coming from the decay $(N_R)^c \rightarrow W_L e_L$ and its $CP$ conjugate, and the one in the unbroken phase coming from the decay above the phase transition $N_R \rightarrow e \varphi$, where $\varphi$ is the Higgs bidoublet, is an open problem worth to be investigated.

As we have seen, this implies the existence of new terms that involve $\Delta L = 2$
CP violating interactions involving the $W_R$ gauge bosons. Considering the $W_L - W_R$ mixing, there are interesting new possible $\Delta L = 2$ processes with light leptons in the final state: the subleading decay $W_1 \rightarrow \nu_R(\nu_L)^c$ and the leading one $W_2 \rightarrow \nu_R(\nu_L)^c$. As emphasized above, it is worth to keep in mind, in model building, the possibility of the latter as a contribution to leptogenesis.

We have extended these results to other LR theories, namely the Pati-Salam model $SU(4)_C \times SU(2)_L \times SU(2)_R$ and the grand unified model $SO(10)$, for which we find that the structure of mixing in the lepton sector is, in the most general case, the same as in the Left-Right Model $SU(2)_L \times SU(2)_R \times U_{B-L}(1)$. The specification of the Higgs sector provides schemes that have more predictive power.

If one assumes both symmetric 10 and 126 Higgs representations, necessary to describe the quark mass spectrum, we emphasize that there is a clash between the description of this spectrum and the assumption that the left-handed Dirac mixing matrix is approximately given by the quark CKM matrix, as sometimes it is assumed in phenomenological models arguing naive quark-lepton symmetry.

Phenomenological analyses are done usually within these theoretical gauge models supplemented by simplifying hypotheses that give tractable schemes. But one should keep in mind that the general parameter space can yield other possibilities concerning the description of the interesting observables, i.e. neutrino spectrum, neutrino mixing and cosmological baryon asymmetry via leptogenesis. Concerning these main observables, there are no differences between the Extended Standard Model and Left-Right models at leading order in the $m_D/M_R$ order.

Appendix

A general digression on the matrices $K, R, S, T$

To count the number of independent parameters in each scheme, it is useful to consider the general case of diagonalization of a $6 \times 6$ complex symmetric matrix,

$$
\mathcal{M} = \begin{pmatrix}
m_L & m_D \\
m_D^t & M_R
\end{pmatrix}
$$  \hspace{1cm} (137)

where $m_L$ and $M_R$ are $3 \times 3$ complex symmetric. In general, a $6 \times 6$ complex
symmetric matrix has 42(21) real parameters.

Let us now diagonalize $\mathcal{M}$ with the unitary matrix $V$ (37)(39). The unitarity condition $VV^\dagger = 1$ is an hermitian relation that implies 36(15) constraints. A general complex $6 \times 6$ matrix has 72(36) parameters. Therefore, because of these constraints, $V$ must have $72(36) - 36(15) = 36(21)$ parameters, consistent with the number of $\frac{n(n-1)}{2}$ angles and $\frac{n(n+1)}{2}$ phases of a $n \times n$ unitary matrix. Since $\mathcal{M}^{\text{diag}}$ has 6(0) parameters, the r.h.s. of (37) has 36(21) (from $V$) + 6(0) = 42(21), in consistency with the counting of parameters of the matrix $\mathcal{M}$ (137).

The unitarity of the matrix $V$ (39) implies [9, 10]

\begin{align*}
KK^\dagger + RR^\dagger &= 1 \quad (138) \\
SS^\dagger + TT^\dagger &= 1 \quad (139) \\
KS^\dagger + RT^\dagger &= 0 \quad (140) \\
K^\dagger K + S^\dagger S &= 1 \quad (141) \\
R^\dagger R + T^\dagger T &= 1 \quad (142) \\
K^\dagger R + S^\dagger T &= 0 \quad (143)
\end{align*}

Let us do the exercise of counting again the number of parameters of the matrices $(K, R, S, T)$. If each of them were general complex, we would have for each 18(9) parameters, that gives for $(K, R, S, T)$ a total of 72(36) parameters. Relations (138) and (139) are hermitian, giving each 9(3) constraints, while (140) is general complex, giving 18(9) constraints. In total, we have again 9(3) + 9(3) + 18(9) = 36(15) constraints, and therefore, the set $(K, R, S, T)$ has 72(36) - 36(15) = 36(21) independent parameters, in agreement with the counting of independent parameters of the unitary matrix $V$.

On the other hand, the diagonalization of (137) reads

\begin{align*}
Km_L^{\text{diag}} K^t + RM_R^{\text{diag}} R^t &= mL \quad (144) \\
Sm_L^{\text{diag}} S^t + TM_R^{\text{diag}} T^t &= MR \quad (145) \\
Km_L^{\text{diag}} S^t + RM_R^{\text{diag}} T^t &= m_D \quad (146)
\end{align*}

Verifying again the counting of parameters, we have for the r.h.s. of (144), 12(6) + 12(6) + 18(9) parameters from respectively $m_L$, $M_R$ and $m_D$. This gives a
total of 42(21) independent parameters for the r.h.s., that is equal to the number of parameters of the l.h.s., 36(21) + 3(0) + 3(0) from, respectively ($K, R, S, T$), $m^\text{diag}_L$ and $M^\text{diag}_R$.

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Acknowledgements

We are grateful to Dr. V. Tello for reminding us the Keung-Senjanović effect [23] and for pointing out a theoretical study about the interesting possibility of
measuring the neutrino Dirac mass matrix within the Left-Right model [46]. We are also indebted to Dr. P. Bhupal Dev for calling our attention to a recent updated formulation of flavor effects in leptogenesis [37].