Trudinger-Moser inequalities on a closed Riemannian surface with the action of an finite isometric group

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Abstract

Let \((\Sigma, g)\) be a closed Riemannian surface, \(W^{1,2}(\Sigma, g)\) be the usual Sobolev space, \(G\) be a finite isometric group acting on \((\Sigma, g)\), and \(\mathcal{H}_G\) be a function space including all functions \(u \in W^{1,2}(\Sigma, g)\) with \(\int_\Sigma udv_g = 0\) and \(u(\sigma(x)) = u(x)\) for all \(\sigma \in G\) and all \(x \in \Sigma\). Denote the number of distinct points of the set \(\{\sigma(x) : \sigma \in G\}\) by \(I(x)\) and \(\ell = \inf_{x \in \Sigma} I(x)\). Let \(A^G_1\) be the first eigenvalue of the Laplace-Beltrami operator on the space \(\mathcal{H}_G\). Using blow-up analysis, we prove that if \(\alpha < A^G_1\) and \(\beta \leq 4\pi\ell\), then there holds

\[
\sup_{u \in \mathcal{H}_G} \left( \int_\Sigma |\nabla u|^2 dv_g - \alpha \int_\Sigma u^2 dv_g \right) \leq 1 \int_\Sigma e^{\beta u^2} dv_g < \infty;
\]

if \(\alpha < A^G_1\) and \(\beta > 4\pi\ell\), or \(\alpha \geq A^G_1\) and \(\beta > 0\), then the above supremum is infinity; if \(\alpha < A^G_1\) and \(\beta \leq 4\pi\ell\), then the above supremum can be attained. Moreover, similar inequalities involving higher order eigenvalues are obtained. Our results partially improve original inequalities of J. Moser [17], L. Fontana [9] and W. Chen [4].

Key words: Trudinger-Moser inequality, isometric group action
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1. Introduction

Let \(\Omega \subset \mathbb{R}^n\) be a smooth bounded domain, \(W^{1,n}_{0}(\Omega)\) be the usual Sobolev space, and \(\omega_{n-1}\) be the area of the unit sphere in \(\mathbb{R}^n\). It was proved by Moser [17] that for any \(\alpha \leq \alpha_n = \frac{\omega_{n-1}}{\omega_n^{1/(n-1)}}\), there holds

\[
\sup_{u \in W^{1,n}_{0}(\Omega), \int_\Omega |\nabla u|^n dx \leq 1} \int_\Omega e^{\alpha u^n}dx < \infty.
\]

Moreover, \(\alpha_n\) is the best constant in the sense that if \(\alpha > \alpha_n\), the integrals in the above are still finite, but the supremum is infinity. Such kind of inequalities are known as the Trudinger-Moser inequalities in literature. Earlier contributions are due to Yudovich [34], Pohozaev [21], Peetre [20] and Trudinger [24]. Let \(A_1(\Omega)\) be the first eigenvalue of the Laplace operator with respect...
to the Dirichlet boundary condition. Adimurthi-Druet [1] proved that for any \( \alpha < \lambda_1(\Omega) \), there holds

\[
\sup_{u \in W^{1,2}_0(\Omega), \|\nabla u\|_2^2 \leq 1} \int_\Omega e^{4\alpha x} |(1 + |u|)^2| dx < \infty;
\]

moreover, if \( \alpha \geq \lambda_1(\Omega) \), then the above supremum is infinity, where \( \|u\|_2^2 = \int_\Omega u^2 dx \). The inequality (2) improves (1) and was extended by the second named author [26] to higher dimensional case. Later, Tintarev [23] proved among other results that for any \( \alpha < \lambda_1(\mathbb{B}_R(0)) \), there holds

\[
\sup_{u \in W^{1,2}_0(\Omega), \|\nabla u\|_2^2 \rightarrow 0} \int_\Omega e^{4\alpha x} dx < \infty,
\]

where \( \mathbb{B}_R(0) \) denotes the ball centered at 0 with radius \( R \) and its measure is equal to that of \( \Omega \). As one expected, \( \lambda_1(\mathbb{B}_R(0)) \) can be replaced by \( \lambda_1(\Omega) \), which is a consequence of [28, Theorem 1].

One can ask whether the supremum in (1) can be attained or not. Existence of extremal functions was proved first by Carleson-Chang [3] in the case that \( \Omega \) is the unit ball, then by Struwe [22] in the case that \( \Omega \) is close to a ball in the sense of measure, later by Flucher [8] when \( \Omega \) is a planar domain, and finally by Lin [13] when \( \Omega \) is a domain in \( \mathbb{R}^n \). In [25], the second named author claimed that the supremum in (2) can be attained for all \( 0 \leq \alpha < \lambda_1(\Omega) \). We remark that there is a mistake during that test function computation ([25], page 338, line 8). In fact, in two dimensions, extremal function for (2) exists only for sufficiently small \( \alpha \), see for example [27]. Concerning extremal functions for inequalities of the type (2), we refer the reader to [14, 15, 6, 29, 30, 10, 35, 32, 33, 19]. While in [28], it was proved that the supremum in (3) can be attained for all \( \alpha < \lambda_1(\Omega) \). It is remarkable that (4) is stronger than (2), however, there is no relation on existence of extremal functions between (2) and (3).

Let \((S^2, g_0)\) be the 2-dimensional sphere \( x_1^2 + x_2^2 + x_3^2 = 1 \) with the metric \( g_0 = dx_1^2 + dx_2^2 + dx_3^2 \) and the corresponding volume element \( dv_{g_0} \). According to Moser [17], one can find a constant \( C \) such that for all functions \( u \) with \( \int_{S^2} |\nabla_{g_0} u|^2 dv_{g_0} \leq 1 \) and \( \int_{S^2} udv_{g_0} = 0 \),

\[
\int_{S^2} e^{4\alpha u} dv_{g_0} \leq C.
\]

Concerning all even functions \( u \), it was indicated by Moser [18] that the best constant \( \alpha_2 = 4\pi \) would double. Namely, there exists a constant \( C \) such that for all functions \( u \) satisfying \( u(-x) = u(x) \), \( \forall x \in S^2 \), \( \int_{S^2} |\nabla_{g_0} u|^2 dv_{g_0} \leq 1 \), and \( \int_{S^2} udv_{g_0} = 0 \), there holds

\[
\int_{S^2} e^{8\pi u} dv_{g_0} \leq C.
\]

Later, by using an isoperimetric inequality on closed Riemannian surfaces with conical singularities, Chen [4] proved a Trudinger-Moser inequality for a class of “symmetric” functions, which particularly generalized (4) and (5).

Let \((M, g)\) be a closed \( n \)-dimensional Riemannian manifold. Among other results, it was proved by Fontana [9] that there exists a constant \( C \), depending only on \((M, g)\), such that if \( u \in W^{1,n}(M, g) \) satisfies \( \int_M |\nabla_{g} u|^n dv_g \leq 1 \) and \( \int_M udv_g = 0 \), then

\[
\int_M e^{\alpha \|u\|^{n/(n-1)}} dv_g \leq C.
\]
The existence of extremal functions for (6) was obtained by Li [11, 12]. Precisely, there exists some \( u_0 \in W^{1,2}(M) \cap C^1(M) \) with \( \int_M |\nabla_x u_0|^2 dv_g = 1 \) and \( \int_M u_0dv_g = 0 \) such that
\[
\int_M e^{\alpha|u_0|_{W^{1,2}}^2} dv_g = \sup_{u \in W^{1,2}(M), \int_M |\nabla_x u|^2 dv_g \leq 1, \int_M udv_g = 0} \int_M e^{\alpha|u|_{L^2}^2} dv_g. \tag{7}
\]
Obviously (7) implies (6). In [27], the inequality (2) was generalized to a closed Riemannian surface version, namely for any \( \alpha \) with \( 0 \leq \alpha < \lambda_1(\Sigma) = \inf \|\mu\|_1 = \inf \int_\Sigma |\nabla u|^2 dv_g \),
\[
\sup_{u \in W^{1,2}(\Sigma, g), \int_\Sigma |\nabla u|^2 dv_g \leq 1, \int_\Sigma udv_g = 0} \int_\Sigma e^{n(1+|\mu|^2)\alpha} dv_g < \infty; \tag{8}
\]
moreover, the supremum in (8) can be attained for sufficiently small \( \alpha \). However, in a recent work [28], an analog of (3) was also established on a closed Riemannian surface, say for any \( \alpha < \lambda_1(\Sigma) \),
\[
\sup_{u \in W^{1,2}(\Sigma, g), \int_\Sigma |\nabla u|^2 dv_g \leq 1, \int_\Sigma udv_g = 0} \int_\Sigma e^{n\alpha u^2} dv_g < \infty. \tag{9}
\]
Moreover, the above supremum can be attained for any \( \alpha < \lambda_1(\Sigma) \). Further, this kind of inequalities involving higher order eigenvalues of the Laplace-Beltrami operator has been studied.

In this paper, our aim is to establish Trudinger-Moser inequalities for “symmetric” functions and prove the existence of their extremal functions on a closed Riemannian surface with the action of a finite isometric group. They can be viewed as a “combination” of (5) and (9). We believe that such inequalities would play an important role in the study of prescribing Gaussian curvature problem and mean field equations. Before ending this introduction, we mention Mancini-Martiniazzi [14], who studied the classical Trudinger-Moser inequality by estimating the energy of extremals for subcritical functionals.

2. Notations and main results

Let \((\Sigma, g)\) be a closed Riemannian surface and \(G = \{\sigma_1, \cdots, \sigma_N\}\) be an isometric group acting on it, where \(N\) is some positive integer. By definition, \(G\) is a group and each \(\sigma_i : \Sigma \to \Sigma\) is an isometric map, particularly \(\sigma_i^*g = g_{\sigma_i(x)}\) for all \(x \in \Sigma\). Let \(u : \Sigma \to \mathbb{R}\) be a measurable function, we say that \(u \in \mathcal{H}_G\) if \(u\) is \(G\)-invariant, namely \(u(\sigma_i(x)) = u(x)\) for any \(1 \leq i \leq N\) and almost every \(x \in \Sigma\). We denote \(W^{1,2}(\Sigma, g)\) the closure of \(C^{\infty}(\Sigma)\) under the norm
\[
\|u\|_{W^{1,2}(\Sigma, g)} = \left( \int_\Sigma (|\nabla_g u|^2 + u^2) dv_g \right)^{1/2},
\]
where \(\nabla_g\) and \(dv_g\) stand for the gradient operator and the Riemannian volume element respectively. Define a Hilbert space
\[
\mathcal{H}_G = \left\{ u \in W^{1,2}(\Sigma, g) \cap \mathcal{H}_G : \int_\Sigma udv_g = 0 \right\} \tag{10}
\]
with an inner product
\[
\langle u, v \rangle_{\mathcal{H}_G} = \int_\Sigma \langle \nabla_g u, \nabla_g v \rangle dv_g,
\]
where $\langle \nabla_g u, \nabla_g v \rangle$ stands for the Riemannian inner product of $\nabla_g u$ and $\nabla_g v$. Let $\Delta_g = -\text{div}_g \nabla_g$ be the Laplace-Beltrami operator, and

$$\lambda^G_1 = \inf_{u \in \mathcal{H}_G, u \neq 0} \frac{\int_{\Sigma} |\nabla_g u|^2 \, dv_g}{\int_{\Sigma} u^2 \, dv_g} \quad (11)$$

be the first eigenvalue of $\Delta_g$ on the space $\mathcal{H}_G$. For any $x \in \Sigma$, we set $I(x) = \sharp G(x)$, where $\sharp A$ stands for the number of all distinct points in the set $A$, and $G(x) = \{\sigma_1(x), \ldots, \sigma_N(x)\}$. Let

$$\ell = \inf_{x \in \Sigma} I(x). \quad (12)$$

Clearly we have $1 \leq \ell \leq N$ since $1 \leq I(x) \leq N$ for all $x \in \Sigma$. As one will see, the best constant in the Trudinger-Moser inequality for “symmetric” functions would be $4\pi \ell$. Precisely we state the following theorem.

**Theorem 1.** Let $(\Sigma, g)$ be a closed Riemannian surface and $G = \{\sigma_1, \ldots, \sigma_N\}$ be an isometric group acting on it. Assume $\mathcal{H}_G$, $\lambda^G_1$ and $\ell$ are defined by (10), (11) and (12) respectively. Then we have the following assertions:

(i) For any $\alpha < \lambda^G_1$ and $\beta \leq 4\pi \ell$, there holds

$$\sup_{u \in \mathcal{H}_G} \int_{\Sigma} e^{\beta u^2} \, dv_g \leq \int_{\Sigma} e^{\alpha u^2} \, dv_g < \infty; \quad (13)$$

(ii) If $\alpha < \lambda^G_1$ and $\beta > 4\pi \ell$, or $\alpha \geq \lambda^G_1$ and $\beta > 0$, then the supremum in (13) is infinity;

(iii) If $\alpha < \lambda^G_1$ and $\beta \leq 4\pi \ell$, then the supremum in (13) can be attained by some function $u_0 \in \mathcal{H}_G \cap C^1(\Sigma)$ with $\int_{\Sigma} |\nabla_g u_0|^2 \, dv_g - \alpha \int_{\Sigma} u_0^2 \, dv_g = 1$.

As in [23], we may consider the effect of higher order eigenvalues on the Trudinger-Moser inequality. For this purpose, we define the first eigenfunction space with respect to $\lambda^G_1$ by

$$E^G_{\lambda^G_1} = \left\{ u \in \mathcal{H}_G : \Delta_g u = \lambda^G_1 u \right\}.$$ 

By an induction, the $j$-th ($j \geq 2$) eigenvalue and eigenfunction space will be defined as

$$\lambda^G_j = \inf_{u \in \mathcal{H}_G, u \neq 0} \frac{\int_{\Sigma} |\nabla_g u|^2 \, dv_g}{\int_{\Sigma} u^2 \, dv_g} \quad (14)$$

and

$$E^G_{\lambda^G_j} = \left\{ u \in E^G_{\lambda^G_{j-1}} : \Delta_g u = \lambda^G_j u \right\}$$

respectively, where $E_{\lambda^G_1} = E^G_{\lambda^G_1} \oplus \cdots \oplus E^G_{\lambda^G_1}$ and

$$E^G_{\lambda^G_j-1} = \left\{ u \in \mathcal{H}_G : \int_{\Sigma} u v \, dv_g = 0, \forall v \in E_{\lambda^G_{j-1}} \right\}. \quad (15)$$

Then higher order eigenvalues of $\Delta_g$ affect the Trudinger-Moser inequality in the following way.
Theorem 2. Let $(\Sigma, g)$ be a closed Riemannian surface and $G = \{\sigma_1, \ldots, \sigma_n\}$ be an isometric group acting on it. Assume $\mathcal{H}_G$, $\ell$, $\lambda^G$ and $E^G_{j}$ are defined by (11), (12), (14) and (15) respectively, $j \geq 2$.

(i) For any $\alpha < \lambda^G$ and $\beta \leq 4\pi\ell$, there holds

$$\sup_{u \in E^G_{j-1}, \int |\nabla u|^2 dv_G \geq \alpha \int u^2 dv_G \leq 1} \int_{\Sigma} e^{\beta u^2} dv_G < \infty; \quad (16)$$

(ii) If $\alpha < \lambda^G$ and $\beta > 4\pi\ell$, or $\alpha \geq \lambda^G$ and $\beta > 0$, then the supremum in (16) is infinity.

(iii) For any $\alpha < \lambda^G$ and $\beta \leq 4\pi\ell$, the supremum in (16) can be attained by some function $u_0 \in E^G_{j-1} \cap C^1(\Sigma, g)$ with $\int |\nabla u_0|^2 dv_G \leq \alpha \int u_0^2 dv_G = 1$.

Let us give several examples for the finite isometric group $G$ acting on a closed Riemannian surface $(\Sigma, g)$. (a) If $G = \{Id\}$, where $Id$ denotes the identity map, then $G$ is a trivial isometric group action, and Theorems 1 and 2 are reduced to (28), Theorems 3 and 4. (b) Let $(S^2, g_0)$ be the standard 2-sphere given as in the introduction, and $G = \{Id, \sigma_0\}$, where $\sigma_0(x) = -x$ for any $x \in S^2$. Then we have $\#G(x) = \#(x, -x) = 2$ for any $x \in S^2$, and thus $\ell = 2$. Hence Moser’s inequality (5) for even functions is a special case of our theorems. (c) If $G$ has a fixed point, namely there exists some point $p \in \Sigma$ such that $\sigma(p) = p$ for all $\sigma \in G$, then we have $\ell = \#G(p) = 1$, and whence both of the best constants in (13) and (16) are $4\pi$.

From now on, to simplify notations, we write

$$\|u\|_{1, \alpha} = \left( \int_{\Sigma} |\nabla u|^2 dv_G - \alpha \int u^2 dv_G \right)^{1/2}, \quad (17)$$

provided that the right hand side of the above equality makes sense, say, if $\alpha < \lambda^G$ and $u \in \mathcal{H}_G$, then $\|u\|_{1, \alpha}$ is well defined. For the proof of Theorems 1 and 2, we follow the lines of (28) and thereby follow closely (11). Pioneer works are due to Carleson-Chang (3), Ding-Jost-Li-Wang (7), and Adimurthi-Struwe (2). Since both of them are similar, we only give the outline of the proof of Theorem 1. Firstly, we prove that the best constant in (13) is $4\pi\ell$, which is based on Moser’s original inequality and test function computations; Secondly, a direct method of variation shows that every subcritical Trudinger-Moser functional has a maximizer, namely for any $\epsilon > 0$, there exists some $u_{\epsilon} \in \mathcal{H}_G$ with $\|u_{\epsilon}\|_{1, \alpha} = 1$ satisfying

$$\int_{\Sigma} e^{(4\pi\ell - \epsilon)u_{\epsilon}^2} dv_G = \sup_{u \in \mathcal{H}_G, \|u\|_{1, \alpha} \leq 1} \int_{\Sigma} e^{(4\pi\ell - \epsilon)u^2} dv_G,$$

where $\alpha < \lambda^G$ and $\|u\|_{1, \alpha}$ is defined as in (17). Thirdly, we use blow-up analysis to show that if $\sup_{u \in \Sigma} |u| \to \infty$ as $\epsilon \to 0$, then

$$\sup_{u \in \mathcal{H}_G, \|u\|_{1, \alpha} \leq 1} \int_{\Sigma} e^{4\pi\ell u^2} dv_G \leq \text{Vol}_G(\Sigma) + \pi \epsilon e^{1+4\pi\ell A_\infty},$$

where $A_\infty$ is a constant related to certain Green function (see (61) below); Finally, we construct a sequence of functions $\phi_\epsilon \in \mathcal{H}_G$ with $\|\phi_\epsilon\|_{1, \alpha} \leq 1$ such that

$$\int_{\Sigma} e^{4\pi\ell \phi_\epsilon^2} dv_G > \text{Vol}_G(\Sigma) + \pi \epsilon e^{1+4\pi\ell A_\infty},$$

and therefore, we have a contradiction.
provided that $\epsilon > 0$ is chosen sufficiently small. Combining the above two estimates, we get a contradiction, which implies that $u_\epsilon$ must be uniformly bounded. Then applying elliptic estimates to the equation of $u_\epsilon$, we get a desired extremal function.

In the remaining part of this paper, we will prove Theorems 1 and 2. Throughout this paper, we do not distinguish sequence and subsequence. Moreover we often denote various constants by the same $C$, but the dependence of $C$ will only be given if necessary. Also we use symbols $|O(Re)| \leq CR\epsilon$, $o_\delta(1) \to 0$ as $\epsilon \to 0$, $o_\delta(1) \to 0$ as $\delta \to 0$, and so on.

3. Proof of Theorem 1

In this section, we prove Theorem 1. In the first subsection, we show that the best constant in (12) is equal to $4\pi\ell$. The essential tools we use are subcritical Trudinger-Moser inequality and Moser’s sequence of functions. Also we prove (ii) of Theorem 1. In the second subsection, we consider the existence of maximizers for subcritical Trudinger-Moser functionals and study their energy concentration phenomenon. In the third subsection, assuming blow-up occurs, we derive an upper bound of the supremum in (12), which obviously leads to (i) of Theorem 1. In the final subsection, we construct a sequence of test functions to show that the upper bound we obtained in the third subsection is not really an upper bound. Therefore blow-up cannot occur and elliptic estimates lead to existence of extremal function. This concludes (iii) of Theorem 1.

3.1. The best constant

In view of (11), one can see that $\lambda_1^{G^\alpha} > 0$ by using a direct method of variation. For any fixed $\alpha < \lambda_1^{G^\alpha}$, if $u \in \mathcal{G}_G$ satisfies $||u||_{1,\alpha} \leq 1$, then $||\nabla_G u||_{G}^2 \leq \lambda_1^{G^\alpha}/(\lambda_1^{G^\alpha} - \alpha)$. By Fontana’s inequality, there exists a positive constant $\beta_0$ depending only on $\lambda_1^{G^\alpha}$ and $\alpha$ such that

$$\sup_{u \in \mathcal{G}_G, ||u||_{1,\alpha} \leq 1} \int_{\Sigma} e^{\beta u^2} d\nu_g < \infty.$$ 

Now we define

$$\beta^* = \sup \left\{ \beta : \sup_{u \in \mathcal{G}_G, ||u||_{1,\alpha} \leq 1} \int_{\Sigma} e^{\beta u^2} d\nu_g < \infty \right\}.$$ 

Lemma 3. Let $\ell$ and $\beta^*$ be defined as in (12) and (18) respectively. Then $\beta^* = 4\pi\ell$.

Proof. We divide the proof into two steps.

Step 1. There holds $\beta^* \leq 4\pi\ell$.

In view of (12), there exists some point $x_0 \in \Sigma$ satisfying $\ell = [G(x_0) = \{ \sigma_1(x_0), \ldots, \sigma_N(x_0) \}$. Without loss of generality, we assume that $\sigma_1 = Id$ is the identity map, and that $G(x_0) = \{ \sigma_i(x_0) \}_{i=1}^\ell$. Take

$$r_0 = \frac{1}{4} \min_{1 \leq i < j \leq \ell} d_g(\sigma_i(x_0), \sigma_j(x_0)),$$

where $d_g(\sigma_i(x_0), \sigma_j(x_0))$ denotes the Riemannian distance between $\sigma_i(x_0)$ and $\sigma_j(x_0)$. Since every $\sigma_i : \Sigma \to \Sigma$ is an isometric map, we can see that for all $0 < r \leq r_0$,

$$B_i(\sigma_i(x_0)) = \sigma_i(B_r(x_0)), \quad 1 \leq i \leq \ell,$$

where $B_r(x)$ stands for the geodesic ball centered at $x \in \Sigma$ with radius $r$. 


Fixing $p \in \Sigma$, $k \in \mathbb{N}$ and $0 < r \leq r_0$, we take a sequence of Moser functions by

$$M_{p,k} = M_{p,k}(x, r) = \begin{cases} 
\log k & \text{when } \rho \leq rk^{-1/4} \\
4 \log \frac{2}{\rho} & \text{when } rk^{-1/4} < \rho \leq r \\
0 & \text{when } \rho > r,
\end{cases} \quad (20)$$

where $\rho$ denotes the Riemannian distance between $x$ and $p$. Define

$$\tilde{M}_k = \tilde{M}_k(x, r) = \begin{cases} 
M_{\tau((k),j)}(x, r), & x \in B_{\tilde{r}_k}(\sigma_j(x_0)), \ 1 \leq i \leq \ell \\
0, & x \in \Sigma \setminus \bigcup_{i=1}^{\ell} B_{\tilde{r}_k}(\sigma_j(x_0))
\end{cases} \quad (21)$$

If $x \in B_{\tilde{r}_k}(\sigma_j(x_0))$ for some $i$, then it follows from (19) that for any $j = 1, \cdots, N$, $\sigma_j(x) \in B_{\tilde{r}_k}(\sigma_j(x_0))$ and $d_\Sigma(\sigma_j(x), \sigma_j(x_0)) = d_\Sigma(x, \sigma_j(x_0))$. In view of (20) and (21), one can easily check that

$$\tilde{M}_k(\sigma_j(x), r) = \tilde{M}_k(x, r), \ \forall x \in B_{\tilde{r}_k}(\sigma_j(x_0)), \ 1 \leq i \leq \ell, \ 1 \leq j \leq N. \quad (22)$$

A straightforward calculation shows

$$\int_\Sigma |\nabla_g \tilde{M}_k|^2 dv_g = (1 + O(r))8\pi \ell \log k, \quad (24)$$

$$\int_\Sigma \tilde{M}_k^m dv_g = O(1), \ m = 1, 2. \quad (25)$$

Denote $\overline{M}_k = \frac{1}{\text{Vol}(\Sigma)} \int_\Sigma \tilde{M}_k dv_g$ and define

$$M'_k = M'_k(x, r) = \frac{\tilde{M}_k(x, r) - \overline{M}_k}{\|\tilde{M}_k - \overline{M}_k\|_{1,\sigma}} \quad (26)$$

In view of (23), we have $M'_k \in \mathcal{H}_\alpha$. Note that $\|M'_k\|_{1,\alpha} = 1$. If $\beta > 4\pi \ell$, then there are two constants $\beta_1 > 4\pi \ell$ and $C > 0$ such that

$$\int_\Sigma e^{\beta M'_k^2} dv_g \leq C$$

for all $k \in \mathbb{N}$. By (24) and (25),

$$\|\tilde{M}_k - \overline{M}_k\|_{1,\sigma} = (1 + O(r))8\pi \ell \log k + O(1).$$

Hence

$$\int_{B_{\tilde{r}_k}(\sigma_0)} e^{\beta M'_k} dv_g \geq \int_{B_{\tilde{r}_k-1/4}(\sigma_0)} e^{\beta \frac{1}{\sqrt{\text{Vol}(\Sigma)}} \frac{\|\tilde{M}_k - \overline{M}_k\|}{\|\tilde{M}_k - \overline{M}_k\|_{1,\sigma}} dv_g}
= e^{\frac{\beta (1+O(r))8\pi \ell \log k}{7}} \frac{\pi r^2 k^{-1/2}}{}(1 + o_k(1)).$$
Choosing \( r > 0 \) sufficiently small and then passing to the limit \( k \to \infty \) in the above estimate, we conclude

\[
\int_{B_o(x_0)} e^{\beta |x|^2} |u^k|^2 \, dv_g \to \infty \quad \text{as} \quad k \to \infty,
\]

which contradicts (26). Therefore \( \beta^* \leq 4 \pi \ell \).

**Step 2. There holds** \( \beta^* \geq 4 \pi \ell \).

Suppose \( \beta^* < 4 \pi \ell \). Then for any \( k \in \mathbb{N} \), there is a \( u_k \in \mathcal{H}_G \) with \( \|u_k\|_{1,\ell} \leq 1 \) such that

\[
\int_{\Sigma} e^{(\beta^*+k^{-1})|x|^2} |u^k|^2 \, dv_g \to \infty \quad \text{as} \quad k \to \infty.
\]  

(27)

Since \( \beta < 3 \ell \), we can see that \( u_k \) is bounded in \( W^{1,2}(\Sigma, g) \). Up to a subsequence, we can assume that \( u_k \) converges to some function \( u_0 \) weakly in \( W^{1,2}(\Sigma, g) \), strongly in \( L^q(\Sigma, g) \), \( \forall q > 1 \), and for almost every \( x \in \Sigma \). Clearly \( u_0 \in \mathcal{H}_G \) and \( \|u_0\|_{1,\ell} \leq 1 \). We now claim that \( u_0 \equiv 0 \). For otherwise, we have

\[
\|u_k - u_0\|_{1,\ell}^2 \leq 1 - \|u_0\|_{1,\ell}^2 + o_k(1) \leq 1 - \frac{1}{2} \|u_0\|_{1,\ell}^2 < 1
\]  

(28)

for sufficiently large \( k \). Given any \( \varepsilon > 0 \). We calculate

\[
\int_{\Sigma} e^{(\beta^*+k^{-1})|x|^2} |u^k|^2 \, dv_g \leq \int_{\Sigma} e^{(\beta^*+k^{-1})(1+\varepsilon)|u_k|^2 + C_{\ell}^2} \, dv_g
\]

\[
\leq C \left( \int_{\Sigma} e^{(\beta^*+k^{-1})(1+2\varepsilon)|u_k|^2} \, dv_g \right)^{\frac{1}{1+2\varepsilon}},
\]

(29)

where \( C \) is a constant depending only on \( u_0, \beta^* \) and \( \varepsilon \). In view of (28), one can find a small \( \varepsilon > 0 \) and a large integer \( k_0 \) such that when \( k \geq k_0 \), there holds

\[
(\beta^* + k^{-1})(1 + 2\varepsilon)\|u_k - u_0\|_{1,\ell}^2 \leq \beta^*(1 - 8^{-1}\|u_0\|_{1,\ell}^2).
\]

This together with (29) leads to

\[
\int_{\Sigma} e^{(\beta^*+k^{-1})|x|^2} |u^k|^2 \, dv_g \leq C,
\]

contradicting (27). This confirms our claim \( u_0 \equiv 0 \).

For any fixed \( x \in \Sigma \), we let \( I = I(x) = \mathcal{B}_r(x) \). Without loss of generality, we assume that \( \sigma_1 = I_d \) and that \( G(x) = \{\sigma_1(x), \cdots, \sigma_I(x)\} \). There exists sufficiently small \( r_1 > 0 \) such that \( \bigcap_{i=1}^I B_{r_1}(\sigma_i(x)) = \emptyset \). Since \( \sigma_i \)'s are all isometric maps, if \( 0 < r \leq r_1 \), then we have

\[
\int_{B_{i}(\sigma_i(x))} |\nabla_g u_k|^2 \, dv_g = \int_{B_{i}(x)} |\nabla_g u|^2 \, dv_g, \quad \forall 1 \leq i \leq I.
\]

Noting that \( I \geq \ell, \|u_k\|_{1,\ell} \leq 1 \) and \( u_0 \equiv 0 \), we have for \( 0 < r \leq r_1 \),

\[
\int_{B_{i}(x)} |\nabla_g u|^2 \, dv_g \leq \frac{1}{\ell} + o_k(1).
\]

(30)

Let \( \zeta \in C^1_0(B_r(x)), 0 \leq \zeta \leq 1, \zeta \equiv 1 \) on \( B_{r/2}(x) \) and \( |\nabla_g \zeta| \leq \frac{2}{r} \). This together with (30) and \( u_0 \equiv 0 \) implies that \( \zeta u_k \in W^{1,2}_0(B_r(x)) \) and

\[
\int_{B_{i}(x)} |\nabla_g (\zeta u_k)|^2 \, dv_g \leq \frac{1}{\ell} + o_k(1).
\]
Choosing sufficiently large \( K > 0 \) and sufficiently small \( r > 0 \), we have by using Moser’s inequality that
\[
\int_{B_r(x)} e^{(\beta^* + k-1)u^2} dv_g \leq C
\]
for some constant \( C \) and all \( k \geq K \). Since \( (\Sigma, g) \) is compact, there exists some constant \( C \) such that for all \( k \),
\[
\int_{\Sigma} e^{(\beta^* + k-1)u^2} dv_g \leq C.
\]
This contradicts (27) again. Hence \( \beta^* \geq 4\pi\ell \).

We finish the proof of the lemma by combining Steps 1 and 2. □

We now clarify the proof of \((ii)\) of Theorem 1, which is partially implied by Lemma 3.

\textbf{Proof of \((ii)\) of Theorem 1.} If \( \alpha < \lambda_{G1} \) and \( \beta > 4\pi\ell \), then Step 1 of the proof of Lemma 3 gives the desired result. In the following, we assume \( \alpha \geq \lambda_{G1} \) and \( \beta > 0 \). By a direct method of variation, one can find a function \( u_0_nequivalence 0 \) satisfying
\[
\int_{\Sigma} |\nabla g u_0|^2 dv_g = \lambda_{G1} \int_{\Sigma} u_0^2 dv_g.
\]
For any \( t \in \mathbb{R} \), we have \( tu_0 \in \mathcal{H}_G \) and
\[
\int_{\Sigma} |\nabla g (tu_0)|^2 dv_g - \alpha \int_{\Sigma} (tu_0)^2 dv_g \leq 0.
\]
Moreover, there holds
\[
\int_{\Sigma} e^{\beta(tu_0)} dv_g \to \infty \quad \text{as} \quad t \to \infty.
\]
Again this gives the desired result. □

3.2. Maximizers for subcritical functionals

Let \( \alpha \prec \lambda^G_{1} \). As in (28), page 3183, by Lemma 3 and a direct method of variation, we can prove that for any \( 0 < \epsilon < 4\pi\ell \), there exists some \( u_\epsilon \in \mathcal{H}_G \) with \( ||u_\epsilon||_{1,\alpha} = 1 \) such that
\[
\int_{\Sigma} e^{(4\pi\ell - \epsilon)u_\epsilon^2} dv_g = \sup_{u \in \mathcal{H}_G, ||u||_{1,\alpha} \leq 1} \int_{\Sigma} e^{(4\pi\ell - \epsilon)u^2} dv_g.
\]
The Euler-Lagrange equation for the maximizer \( u_\epsilon \) reads
\[
\begin{cases}
\Delta_g u_\epsilon - \alpha u_\epsilon = \frac{1}{\lambda_{G1}} u_\epsilon e^{(4\pi\ell - \epsilon)u_\epsilon^2} - \frac{\mu_\epsilon}{\lambda_{G1}} \\
u_\epsilon \in \mathcal{H}_G, \ ||u_\epsilon||_{1,\alpha} = 1 \\
\lambda_\epsilon = \frac{1}{\lambda_{G1}} \int_{\Sigma} u_\epsilon^2 e^{(4\pi\ell - \epsilon)u_\epsilon^2} dv_g \\
\mu_\epsilon = \frac{1}{\lambda_{G1}} \int_{\Sigma} u_\epsilon e^{(4\pi\ell - \epsilon)u_\epsilon^2} dv_g.
\end{cases}
\]
Regularity theory implies that \( u_\epsilon \in C^1(\Sigma, g) \). Using an argument of (29), page 3184, one has
\[
\liminf_{\epsilon \to 0} \lambda_\epsilon > 0, \quad |\mu_\epsilon|/\lambda_\epsilon \leq C.
\]
By (31), one can easily see that

$$\lim_{\epsilon \to 0} \int_{\Sigma} e^{i|\epsilon(x)-\epsilon|} d\nu_{\bar{\epsilon}} = \sup_{u \in \mathcal{H}_G, \|u\|_{L^1} \leq 1} \int_{\Sigma} e^{i\epsilon(x)} d\nu_{\bar{\epsilon}}. \quad (34)$$

Note that we do not assume the supremum on the right hand side of (34) is finite. If \(|u_\epsilon| \leq C\), in view of (33), applying elliptic estimate to (32), we obtain \(u_\epsilon \to u^*\) in \(C^{1}(\Sigma, g)\), which implies that \(u^* \in \mathcal{H}_G\) and \(\|u^*\|_{L^1} = 1\). In view of (34), we know that \(u^*\) is a desired extremal function.

From now on, we assume \(c_\epsilon = \max_{\Sigma} |u_\epsilon| \to +\infty\) as \(\epsilon \to 0\). Noting that \(-u_\epsilon\) also satisfies (31) and (32), we may assume with no loss of generality that

$$c_\epsilon = \max_{\Sigma} |u_\epsilon| = \max_{\Sigma} u_\epsilon = u_\epsilon(x_\epsilon) \to +\infty \quad (35)$$

and that

$$x_\epsilon \to x_0 \in \Sigma \quad \text{as} \quad \epsilon \to 0. \quad (36)$$

To proceed, we need the following energy concentration phenomenon of \(u_\epsilon\).

**Lemma 4.** Under the assumptions (35) and (36), we have

(i) \(u_\epsilon\) converges to 0 weakly in \(W^{1,2}(\Sigma, g)\), strongly in \(L^2(\Sigma, g)\), and almost everywhere in \(\Sigma\);

(ii) \(I(x_0) = \#G(x_0) = \ell\);

(iii) \(\liminf_{\epsilon \to 0} \int_{B_{\epsilon}(x_0)} |\nabla u_\epsilon|^2 d\nu_{\bar{\epsilon}} = 1/\ell\).

**Proof.** (i) Since \(\alpha < \lambda_1^G\) and \(\|u_\epsilon\|_{L^1} = 1\), \(u_\epsilon\) is bounded in \(W^{1,2}(\Sigma, g)\). Hence we may assume \(u_\epsilon\) converges to \(u_0\) weakly in \(W^{1,2}(\Sigma, g)\), strongly in \(L^2(\Sigma, g)\), and almost everywhere in \(\Sigma\). If \(u_0 \neq 0\), then

$$\|u_\epsilon - u_0\|^2_{L^1} = 1 - \|u_0\|^2_{L^1} + o_\epsilon(1) \leq 1 - \frac{1}{2} \|u_0\|^2_{L^1},$$

provided that \(\epsilon\) is sufficiently small. It follows from Lemma 3 that \(e^{i|\epsilon(x)-\epsilon|} u_\epsilon^2\) is bounded in \(L^2(\Sigma, g)\) for some \(g > 1\). Then applying elliptic estimate to (32), we have that \(\|u_\epsilon\|_{L^1(\Sigma)} \leq C\), which contradicts (35). Therefore \(u_0 \equiv 0\).

(ii) Since \(\ell = \inf_{x \in \Sigma} I(x)\), we have \(I(x_0) \geq \ell\). Suppose \(I = I(x_0) > \ell\). Without loss of generality, we may assume \(\sigma_1 = \text{Id}\) and \(\sigma_1, \sigma_2, \ldots, \sigma_I(x_0)\) are distinct. Take

$$n_0 = \frac{1}{4} \min_{1 \leq i \leq I} d_{\bar{G}}(\sigma_i(x_0), \sigma_j(x_0)). \quad (37)$$

Then \(\cap_{i=1}^I B_{n_0}(\sigma_i(x_0)) = \emptyset\). Moreover, we have \(B_{n_0}(\sigma_i(x_0)) = \sigma_i(B_{n_0}(x_0)), 1 \leq i \leq I\). Since \(u_\epsilon \in \mathcal{H}_G\), there holds

$$\int_{B_{n_0}(\sigma_i(x_0))} |\nabla_g u_\epsilon|^2 d\nu_{\bar{\epsilon}} = \int_{B_{n_0}(x_0)} |\nabla_g u_\epsilon|^2 d\nu_{\bar{\epsilon}}, \quad 1 \leq i \leq I. \quad (38)$$

By (i), we have

$$\int_{\Sigma} |\nabla_g u_\epsilon|^2 d\nu_{\bar{\epsilon}} = \int_{\Sigma} |\nabla_g u_\epsilon|^2 d\nu_{\bar{\epsilon}} + \alpha \int_{\Sigma} u_\epsilon^2 d\nu_{\bar{\epsilon}} = 1 + o_\epsilon(1). \quad (39)$$

Combining (38) and (39), we obtain

$$\int_{B_{n_0}(x_0)} |\nabla_g u_\epsilon|^2 d\nu_{\bar{\epsilon}} \leq \frac{1}{\ell} + o_\epsilon(1). \quad (40)$$
For any \(0 < r \leq r_0\), take \(\zeta \in C^1_0(B_r(x_0))\) such that \(0 \leq \zeta \leq 1\), \(\zeta \equiv 1\) on \(B_{r/2}(x_0)\), and \(|\nabla_x \zeta| \leq 2/r\). It follows from (i) and (31) that \(\zeta u \in W^{1,2}_0(B_r(x_0))\) and
\[
\int_{B_r(x_0)} |\nabla_x (\zeta u)|^2 dv_g \leq \frac{1}{I} + o_\epsilon(1).
\]
Hence
\[
(4\pi \ell - \epsilon) \|\nabla_x (\zeta u)\|_{L^2(B_r(x_0))}^2 \leq 4\pi \ell + o_\epsilon(1) \leq 4\pi \ell + \frac{I}{2I}
\]
for sufficiently small \(\epsilon > 0\). By the classical Trudinger-Moser inequality (1), \(e^{4\pi \ell - \epsilon} \) is bounded in \(L^q(B_{r/2}(x_0))\) for some \(1 < q < \frac{2\ell}{\epsilon} 2\), provided that \(r\) is sufficiently small. Applying elliptic estimate to (32), we have that \(u_\epsilon\) is uniformly bounded in \(B_{r/4}(x_0)\). This contradicts (35). Therefore \(I(x_0) = \ell\).

(iii) By (ii), there exists some \(r_0 > 0\) such that \(\|\nabla_x u_\epsilon\|_{L^q(B_{r_0}(x_0))} \leq \frac{1}{1} + o_\epsilon(1)\). It follows that
\[
\lim_{\epsilon \to 0} \lim_{r \to 0} \int_{B_r(x_0)} \|\nabla_x u_\epsilon\|^2 dv_g \leq \frac{1}{\ell}.
\]
We claim that the equality of (41) holds. For otherwise, there exist two positive constants \(\nu\) and \(r_1\) with \(0 < r_1 < r_0\) such that
\[
\int_{B_{r_1}(x_0)} \|\nabla_x u_\epsilon\|^2 dv_g \leq \frac{1}{\ell} - \nu.
\]
Similarly as we did in the proof of (ii), we have that \(e^{4\pi \ell - \epsilon} \) is bounded in \(L^q(B_{r_1/2}(x_0))\) for some \(q > 1\). Then applying elliptic estimate to (32), we obtain \(u_\epsilon\) is uniformly bounded in \(B_{r_1/4}(x_0)\), which contradicts (35). This concludes our claim and (iii) holds.

3.3. Blow-up analysis

Set
\[
\ell_\epsilon = \frac{\sqrt{\ell}}{c_\ell} e^{-2\pi \ell - \epsilon/2} r_\epsilon^2.
\]
For any \(0 < a < 4\pi \ell\), by Lemma (ii) and the H"older inequality and (i) of Lemma (iii) one has
\[
\lambda_\epsilon = \int_\Sigma u_\epsilon^a e^{4\pi \ell - \epsilon} dv_g = \epsilon^{-2} \int_\Sigma u_\epsilon^a e^{4\pi \ell - \epsilon} dv_g = \epsilon^{-2} \lambda_1(1).
\]
It then follows that
\[
r_\epsilon^2 \ell_\epsilon^2 e^{4\pi \ell - \epsilon} dv_g = o_\epsilon(1).
\]
In particular, \(r_\epsilon \to 0\) as \(\epsilon \to 0\). Let \(0 < \delta < \frac{1}{4} i_\ell(\Sigma)\) be fixed, where \(i_\ell(\Sigma)\) is the injectivity radius of \((\Sigma, g)\). For \(y \in B_{\delta r_\epsilon}(0) \subset \mathbb{R}^2\), we define \(\psi_\epsilon(y) = c_\ell^{-1} u_\epsilon(\exp_{x_\epsilon}(r_\epsilon y))\), \(\varphi_\epsilon(y) = c_\ell u_\epsilon(\exp_{x_\epsilon}(r_\epsilon y)) - c_\epsilon\) and \(g_\epsilon(y) = (\exp_{x_\epsilon} g)(r_\epsilon y)\), where \(B_{\delta r_\epsilon}(0)\) is the Euclidean ball of radius \(\delta r_\epsilon^{-1}\) centered at 0 and \(\exp_{x_\epsilon}\) is the exponential map at \(x_\epsilon\). Note that \(g_\epsilon\) converges to \(g_0\) in \(C^2_{\text{loc}}(\mathbb{R}^2)\) as \(\epsilon \to 0\), where \(g_0\) denotes the standard Euclidean metric. By (32), we have on \(B_{\delta r_\epsilon}(0)\),
\[
\Delta_{g_\epsilon} \psi_\epsilon(y) = ar_\epsilon^2 \psi_\epsilon(y) + c_\epsilon^{-2} \psi_\epsilon(y) e^{4\pi \ell - \epsilon} (\exp_{x_\epsilon}(r_\epsilon y) - c_\epsilon) - r_\epsilon^{-1} \frac{\Delta_{g_\epsilon} u_\epsilon}{\lambda_\epsilon}
\]
\[
\Delta_{g_\epsilon} \varphi_\epsilon(y) = ar_\epsilon^2 c_\epsilon^2 \psi_\epsilon(y) + \varphi_\epsilon(y) e^{4\pi \ell - \epsilon} (\exp_{x_\epsilon}(r_\epsilon y) - c_\epsilon) - r_\epsilon^{-2} \frac{\Delta_{g_\epsilon} u_\epsilon}{\lambda_\epsilon}.
\]
In view of (43), applying elliptic estimate to (44) respectively, we have
\[ \psi_\epsilon \to 1 \text{ in } C^{1}_{\text{loc}}(\mathbb{R}^2), \] (46) and
\[ \phi_\epsilon \to \phi \text{ in } C^{1}_{\text{loc}}(\mathbb{R}^2), \] (47)
where \( \phi \) satisfies
\[
\begin{cases}
-\Delta \phi = e^{8\pi \ell \phi} \text{ in } \mathbb{R}^2 \\
\phi(0) = 0 = \sup_{\mathbb{R}^2} \phi \\
\int_{\mathbb{R}^2} e^{8\pi \ell \phi(y)} \, dy < \infty.
\end{cases}
\]
By a result of Chen-Li [5], we have
\[ \phi(y) = -\frac{1}{4\pi \ell} \log(1 + \pi \ell |y|^2), \]
which leads to
\[ \int_{\mathbb{R}^2} e^{8\pi \ell \phi(y)} \, dy = \frac{1}{\ell}. \] (48)
By (42), (46) and (47), there holds for any \( R > 0, \)
\[ \int_{B_R(0)} e^{4\pi \ell \phi(y)} \, dy = \lim_{\epsilon \to 0} \int_{B_R(0)} e^{4\pi \ell \epsilon (u_\epsilon + \epsilon \sigma_i(x_\epsilon)) - \epsilon^2} \, dy \\
= \lim_{\epsilon \to 0} \frac{c_i^2}{\lambda_\epsilon} \int_{B_{R_\epsilon}(x_\epsilon)} e^{4\pi \ell \epsilon \sigma_i(x_\epsilon)} \, dv_g \\
= \lim_{\epsilon \to 0} \frac{1}{\lambda_\epsilon} \int_{B_{R_\epsilon}(x_\epsilon)} u_\epsilon^2 e^{4\pi \ell \epsilon \sigma_i(x_\epsilon)} \, dv_g. \]
This together with (48) gives
\[ \lim_{R \to \infty} \lim_{\epsilon \to 0} \frac{1}{\lambda_\epsilon} \int_{B_{R_\epsilon}(x_\epsilon)} u_\epsilon^2 e^{4\pi \ell \epsilon \sigma_i(x_\epsilon)} \, dv_g = \frac{1}{\ell}. \] (49)
By (ii) of Lemma 4 and (36), one can choose \( \epsilon \) sufficiently small such that
\[ \cap_{i=1}^\ell B_{R_\epsilon}(\sigma_i(x_\epsilon)) = \emptyset. \] (50)
Noting that \( u_\epsilon \in \mathcal{H}_G, \) we have
\[ \int_{B_{R_\epsilon}(\sigma_i(x_\epsilon))} u_\epsilon^2 e^{4\pi \ell \epsilon \sigma_i(x_\epsilon)} \, dv_g = \int_{B_{R_\epsilon}(x_\epsilon)} u_\epsilon^2 e^{4\pi \ell \epsilon \sigma_i(x_\epsilon)} \, dv_g, \quad 1 \leq i \leq \ell. \]
This together with (49) and (50) leads to
\[ \lim_{R \to \infty} \lim_{\epsilon \to 0} \frac{1}{\lambda_\epsilon} \int_{B_{R_\epsilon}(\sigma_i(x_\epsilon))} u_\epsilon^2 e^{4\pi \ell \epsilon \sigma_i(x_\epsilon)} \, dv_g = \frac{1}{\ell}, \quad 1 \leq i \leq \ell. \] (51)
By definition of \( \lambda_\epsilon \) in (32), we conclude from (51) that
\[ \lim_{R \to \infty} \lim_{\epsilon \to 0} \frac{1}{\lambda_\epsilon} \int_{\bigcup_{i=1}^\ell B_{R_\epsilon}(\sigma_i(x_\epsilon))} u_\epsilon^2 e^{4\pi \ell \epsilon \sigma_i(x_\epsilon)} \, dv_g = 0. \] (52)
Similar to [11, 1], \( 0 < \beta < 1, \) we let \( u_{\epsilon, \beta} = \min\{u_\epsilon, \beta c_\epsilon\}. \)
Lemma 5. \( \forall 0 < \beta < 1, \) there holds
\[
\lim_{e \to 0} \int_{\Sigma} |\nabla u_{e, \beta}|^2 dv_g = \beta.
\]
Proof. Multiplying (52) by \( u_{e, \beta}, \) we have
\[
\int_{\Sigma} |\nabla u_{e, \beta}|^2 dv_g = \int_{\Sigma} \nabla u_{e, \beta} \nabla u_{e} dv_g = \frac{1}{\lambda_e} \int_{\Sigma} u_{e, \beta} u_{e} e^{(4\pi t - e) \mu_e} dv_g + \alpha \int_{\Sigma} u_{e, \beta} u_{e} dv_g - \frac{\mu_e}{\lambda_e} \int_{\Sigma} u_{e, \beta} dv_g
\]
\[
= \frac{1}{\lambda_e} \sum_{\ell=1}^{\ell} \int_{B_{R_e}(\sigma_{\ell}(x_i))} u_{e, \beta} u_{e} e^{(4\pi t - e) \mu_e} dv_g + \frac{1}{\lambda_e} \int_{\Sigma \setminus \bigcup_{\ell=1}^{\ell} B_{R_e}(\sigma_{\ell}(x_i))} u_{e, \beta} u_{e} e^{(4\pi t - e) \mu_e} dv_g + o_1(1).
\]
Note that \( 0 \leq u_{e, \beta} u_{e} \leq u_{e}^2 \) on \( \Sigma, \) and \( u_{e, \beta} = \beta(1 + o_e(1)) u_{e} \) on \( B_{R_e}(\sigma_{\ell}(x_i)) \) for \( 1 \leq \ell \leq \ell. \) In view of (55), (52), and (53), letting \( e \to 0 \) first and then \( R \to \infty, \) we conclude the lemma.

Lemma 6. There holds \( \lim \inf_{e \to 0} \lambda_e/c_e^2 > 0. \)
Proof. Let \( 0 < \beta < 1. \) In view of Lemma 5, we have by using the Hölder inequality
\[
\int_{\Sigma \cap B_{R_e}} u_{e}^2 e^{(4\pi t - e) \mu_e} dv_g \leq \int_{\Sigma} u_{e}^2 e^{(4\pi t - e) \mu_e} dv_g = o_1(1).
\]
Similarly
\[
\frac{\lambda_e}{c_e^2} \geq \beta^2 \int_{u_e > c_e} e^{(4\pi t - e) \mu_e} dv_g + o_1(1)
\]
\[
\geq \beta^2 \left( \int_{\Sigma} e^{(4\pi t - e) \mu_e} dv_g - \int_{\Sigma} e^{(4\pi t - e) \mu_e} dv_g \right) + o_1(1)
\]
\[
= \beta^2 \int_{\Sigma} e^{(4\pi t - e) \mu_e} - 1) dv_g + o_1(1).
\]
This together with (54) ends the proof of the lemma.

Lemma 7. For any \( 1 < q < 2, \) we have \( c_e u_e \) converges to \( G \) weakly in \( W^{1,q}(\Sigma, g), \) strongly in \( L^{2q/(2-q)}(\Sigma), \) and almost everywhere in \( \Sigma, \) where \( G \) is a Green function satisfying
\[
\begin{cases}
\Delta_g G - \alpha G = \frac{1}{2} \sum_{i=1}^{\ell} \delta_{\sigma_i(x_i)} - \frac{1}{\vol(\Sigma)} \\
\int_{\Sigma} G dv_g = 0 \\
G(\sigma_i(x)) = G(x), \ x \in \Sigma \setminus \{\sigma_j(x_0)\}_{j=1}^{\ell}, 1 \leq i \leq \ell.
\end{cases}
\]
Proof. By (52),
\[
\Delta_g(c_e u_e) - \alpha(c_e u_e) = h_e = \frac{1}{13} c_e u_e e^{(4\pi t - e) \mu_e} - \frac{c_e \mu_e}{\lambda_e}.
\]
It follows from Lemmas 5 and 6 that for any $0 < \beta < 1$,
\[
\int_{\Sigma} \frac{c}{\lambda} |u_{e}| e^{(4\pi^2 - \epsilon)u_e^2} dv_g = \frac{c}{\lambda} \int_{B_{1/2}(e)} |u_{e}| e^{(4\pi^2 - \epsilon)u_e^2} dv_g + \frac{c}{\lambda} \int_{B_{1/2}(e)} u_{e} e^{(4\pi^2 - \epsilon)u_e^2} dv_g \\
\leq \frac{c}{\lambda} \int_{\Sigma} |u_{e}| e^{(4\pi^2 - \epsilon)u_e^2} dv_g + \frac{1}{\beta} \\
\leq \frac{1}{\beta} + o(1),
\]
and that
\[
\frac{c_{\epsilon}|u_{e}|}{\lambda_{e}} \leq \frac{1}{\text{Vol}(\Sigma)} \frac{c_{\epsilon}}{\lambda} \int_{B_{1/2}(e)} |u_{e}| e^{(4\pi^2 - \epsilon)u_e^2} dv_g + \frac{1}{\text{Vol}(\Sigma)} \frac{c_{\epsilon}}{\lambda} \int_{B_{1/2}(e)} u_{e} e^{(4\pi^2 - \epsilon)u_e^2} dv_g \\
\leq \frac{1}{\text{Vol}(\Sigma)} \frac{1}{\beta} + o(1).
\]
Hence $d_{e}$ is bounded in $L^1(\Sigma, g)$. Then by (31), Lemma 2.11, we have $c_{\epsilon}u_{e}$ is bounded in $W^{1, q}(\Sigma, g)$ for any $1 < q < 2$. Up to a subsequence, for any $1 < q < 2$ and $1 < s < 2q/(2 - q)$, $c_{\epsilon}u_{e}$ converges to $G$ weakly in $W^{1, q}(\Sigma)$, strongly in $L^q(\Sigma, g)$, and almost everywhere in $\Sigma$.

We calculate
\[
\int_{B_{1/2}(e)} c_{\epsilon} u_{e} e^{(4\pi^2 - \epsilon)u_e^2} dv_g = \frac{1}{\lambda} \frac{c_{\epsilon}}{\lambda} \int_{B_{1/2}(e)} u_{e} e^{(4\pi^2 - \epsilon)u_e^2} dv_g = \frac{1}{\beta} + o(1),
\]
where $o(1) \to 0$ as $\epsilon \to 0$ first and then $R \to \infty$. Integrating the equation \((56)\), we have by combining \((57)\) - \((59)\),
\[
\frac{c_{\epsilon} u_{e}}{\lambda_{e}} \text{Vol}(\Sigma) = \int_{\Sigma} \frac{c_{\epsilon}}{\lambda} u_{e} e^{(4\pi^2 - \epsilon)u_e^2} dv_g = 1 + o(1).
\]
In view of \((57)\) - \((59)\), again testing the equation \((56)\) by $\phi \in C^2(\Sigma)$ and passing to the limit $\epsilon \to 0$, we have
\[
\int_{\Sigma} G\Delta_{g}\phi dv_g - \alpha \int_{\Sigma} G\phi dv_g = \frac{1}{\ell} \sum_{i=1}^{\ell} \phi(\sigma_{i}(x_0)) - \frac{1}{\text{Vol}(\Sigma)} \int_{\Sigma} \phi dv_g.
\]
Since $c_{\epsilon}u_{e} \in \mathcal{H}_{\Phi}$, we have $\int_{\Sigma} Gdv_g = 0$ and $G(\sigma_{i}(x)) = G(x)$ for all $x \in \Sigma \setminus [\sigma_{1}(x_0), \cdots, \sigma_{\ell}(x_0)]$ and all $1 \leq i \leq \ell$.

Let
\[
\psi(x) = G(x) + \frac{1}{2\pi \ell} \sum_{i=1}^{\ell} \log d_{e}(\sigma_{i}(x_0), x).
\]
It follows from \((55)\) that the distributional Laplacian of $\psi$ belongs to $L^s(\Sigma, g)$ for some $s > 2$. Then we have by elliptic estimates that $\psi \in C^1(\Sigma, g)$. Let $n_{0}$ be defined as in \((57)\), where $I = \ell$.

For $x \in B_{n_{0}}(x_0)$, the Green function $G$ can be decomposed as
\[
G(x) = -\frac{1}{2\pi \ell} \log d_{e}(x, x_0) + A_{x_0} + \tilde{\psi}(x),
\]
where $\bar{\phi} \in C^1(B_0(x_0))$, $\bar{\phi}(x_0) = 0$ and

$$A_{x_0} = \lim_{x \to x_0} \left( G(x) + \frac{1}{2\pi \ell} \log d_g(x, x_0) \right) = \lim_{x \to x_0} \left( \psi(x) - \frac{1}{2\pi \ell} \sum_{j=2}^{\ell} \log d_g(\sigma_j(x_0), x) \right)$$  \hspace{1cm} (61)

By (58), we have

$$\int_{\Sigma(\cup_{j=1}^{\ell} B_0(\sigma_j(x_0))))} |\nabla_s G|^2 dv_g = \alpha \int_{\Sigma(\cup_{j=1}^{\ell} B_0(\sigma_j(x_0))))} G^2dv_g - \int_{\Sigma(\cup_{j=1}^{\ell} B_0(\sigma_j(x_0))))} G \frac{\partial G}{\partial v} dv_g$$

$$- \frac{1}{\text{Vol}_g(\Sigma)} \int_{\Sigma(\cup_{j=1}^{\ell} B_0(\sigma_j(x_0))))} G dv_g$$

$$= - \frac{1}{2\pi \ell} \log \delta + A_{x_0} + \alpha \int_{\Sigma} G^2 dv_g + o_{\delta}(1).$$

Hence

$$\int_{\Sigma(\cup_{j=1}^{\ell} B_0(\sigma_j(x_0))))} |\nabla_s u_{x_0}^2| dv_g = \frac{1}{c_\ell} \left( \frac{1}{2\pi \ell} \log \delta + A_{x_0} + \alpha \int_{\Sigma} G^2 dv_g + o_{\delta}(1) + o_{\epsilon}(1) \right).$$

It follows that

$$\int_{\cup_{j=1}^{\ell} B_0(\sigma_j(x_0)))} |\nabla_s u_{x_0}^2| dv_g = 1 + \alpha \int_{\Sigma} u_{x_0}^2 dv_g - \int_{\Sigma(\cup_{j=1}^{\ell} B_0(\sigma_j(x_0))))} |\nabla_s u_{x_0}^2| dv_g$$

$$= 1 - \frac{1}{c_\ell} \left( \frac{1}{2\pi \ell} \log \delta + A_{x_0} + o_{\delta}(1) + o_{\epsilon}(1) \right).$$

Let $s_\epsilon = \sup_{B_0(x_0)} u_\epsilon$ and $u_\epsilon = (u_{x_0} - s_\epsilon)^+$. Then $u_\epsilon \in W^{1,2}_0(B_0(x_0))$, and satisfies

$$\int_{\cup_{j=1}^{\ell} B_0(\sigma_j(x_0)))} |\nabla_s \tilde{u}_{x_0}^2| dv_g \leq \tau_\epsilon = 1 - \frac{1}{c_\ell} \left( \frac{1}{2\pi \ell} \log \delta + A_{x_0} + o_{\delta}(1) + o_{\epsilon}(1) \right).$$

Now we choose an isothermal coordinate system $(U, \phi; \{x^1, x^2\})$ near $x_0$ such that $B_0(x_0) \subset U$, $\phi(x_0) = 0$, and the metric $g = \phi^4(dx^1)^2 + dx^2^2$ for some function $h \in C^1(\phi(U))$ with $h(0) = 0$. Clearly, for any $\delta > 0$, there exists some $c(\delta) > 0$ with $c(\delta) \to 0$ as $\delta \to 0$ such that $\sqrt{\tau} \leq 1 + c(\delta)$ and $\phi(B_\delta(p)) \subset B_{0(1+c(\delta))}(0) \subset \mathbb{R}^2$. Noting that $u_\epsilon = 0$ outside $B_\delta(p)$ for sufficiently small $\delta$, we have

$$\int_{\mathbb{R}^2} |\nabla_{x_0} (\tilde{u}_\epsilon \circ \phi^{-1})|^2 dx = \int_{\mathbb{R}^2} \phi^{-1} (\tilde{u}_\epsilon \circ \phi^{-1})(0) |\nabla_{x_0} \tilde{u}_\epsilon|^2 dv_g = \int_{B_{\delta(x_0)}} |\nabla_{x_0} \tilde{u}_\epsilon| dv_g \leq \frac{\tau_\epsilon}{\ell}.$$

This together with a result of Carleson-Chang [3] leads to

$$\limsup_{\epsilon \to 0} \int_{B_\delta(p)} (e^{4\pi \ell \tau_\epsilon/\ell} - 1) dv_g \leq \limsup_{\epsilon \to 0} \left( 1 + c(\delta) \right) \int_{\mathbb{R}^2} (e^{4\pi \ell \tau_\epsilon/\ell} - 1) dx$$

$$\leq \pi \delta^2 (1 + c(\delta))^3 \epsilon.$$  \hspace{1cm} (62)
Therefore
\[
\int_{B_{Rr}(x)} e^{(4\pi\ell - \epsilon)u^2} \, dv_g \leq \delta^{-2} e^{4\pi\ell A_0 + o(1)} \int_{B_{Rr}(x)} e^{4\pi\ell A_0/\tau} \, dv_g \\
= \delta^{-2} e^{4\pi\ell A_0 + o(1)} \int_{B_{Rr}(x)} (e^{4\pi\ell A_0/\tau} - 1) \, dv_g + o(1)
\]
and
\[
(4\pi\ell - \epsilon)u^2 \leq 4\pi\ell (u_x + s_x)^2 \\
\leq 4\pi\ell u_x^2 + 8\pi\ell s_x u_x + o_x(1) \\
\leq 4\pi\ell u_x^2 - 4\log \delta + 8\pi\ell A_0 + o(1)
\]
Therefore
\[
\limsup_{\epsilon \to 0} \int_{B_{Rr}(x)} e^{(4\pi\ell - \epsilon)u^2} \, dv_g \leq \pi e^{1 + 4\pi\ell A_0}.
\]
Therefore
\[
\limsup_{\epsilon \to 0} \int_{\cup_{i=1}^{j} B_{Rr}(x_{i})} e^{(4\pi\ell - \epsilon)u^2} \, dv_g \leq \pi e^{1 + 4\pi\ell A_0}.
\]
Proposition 8. Under the assumptions (35) and (56), there holds
\[
\sup_{u \in \mathcal{H}_0, \|u\| \leq 1} \int_{\Sigma} e^{4\pi\ell u^2} \, dv_g = \lim_{\epsilon \to 0} \int_{\Sigma} e^{(4\pi\ell - \epsilon)u^2} \, dv_g \leq \text{Vol}_g(\Sigma) + \pi e^{1 + 4\pi\ell A_0}.
\]
Proof. We calculate
\[
\int_{B_{Rr}(x)} e^{(4\pi\ell - \epsilon)u^2} \, dv_g = (1 + o_x(1)) \int_{\mathbb{S}^2} e^{(4\pi\ell - \epsilon)w^2} (\exp_{x}(r_i))^2 \, dy \\
= (1 + o_x(1)) \frac{\lambda_i}{C_i} \left( \int_{\mathbb{S}^2} e^{4\pi \ell w^2} \, dy + o_x(1) \right).
\]
In view of (43) and (64),
\[
\lim_{R \to \infty} \lim_{\epsilon \to 0} \int_{B_{Rr}(x)} e^{(4\pi\ell - \epsilon)u^2} \, dv_g = \frac{1}{\ell} \lim_{\epsilon \to 0} \frac{\lambda_i}{c_i^2}.
\]
Hence
\[
\lim_{R \to \infty} \lim_{\epsilon \to 0} \int_{\cup_{i=1}^{j} B_{Rr}(x_{i})} e^{(4\pi\ell - \epsilon)u^2} \, dv_g = \lim_{\epsilon \to 0} \frac{\lambda_i}{c_i^2}.
\]
By (54), we have
\[
\lim_{\epsilon \to 0} \int_{\Sigma} (e^{(4\pi\ell - \epsilon)u^2} - 1) \, dv_g \leq \frac{1}{16} \lim_{\epsilon \to 0} \frac{\lambda_i}{c_i^2}, \quad \forall \beta < 1.
\]
Letting $\beta \to 1$, we obtain
\[
\lim_{\epsilon \to 0} \int_{\Sigma} (e^{4\epsilon \phi} - 1) d\nu_\Sigma \leq \lim_{\epsilon \to 0} \frac{A_\epsilon^\beta}{\epsilon^\alpha}.
\]
This together with (64) and (65) gives the desired result. \(\square\)

3.4. Test function computation

We shall construct a function sequence $\phi_\epsilon$ satisfying $\phi_\epsilon \in \mathcal{H}_G$,
\[
\int_{\Sigma} |\nabla \phi_\epsilon|^2 d\nu_\Sigma = \alpha \int_{\Sigma} \phi_\epsilon^2 d\nu_\Sigma = 1 \quad (66)
\]
and
\[
\int_{\Sigma} e^{4\epsilon \phi} d\nu_\Sigma > \text{vol}_g(\Sigma) + \pi \epsilon e^{1+14\pi A_0} \quad (67)
\]
for sufficiently small $\epsilon > 0$, where $x_0$ and $A_{x_0}$ are defined as in (46) and (60) respectively. If there exists such a sequence $\phi_\epsilon$, then we have by Proposition 8 that $c_\epsilon$ must be bounded. Applying elliptic estimates to (32), we conclude the existence of the desired extremal function.

To do this, we define a sequence of functions by
\[
b_\epsilon(x) = \begin{cases} 
c + \frac{\log(1 + \pi \epsilon \psi_\epsilon + B)}{\epsilon}, & x \in B_{R_\epsilon}(x_0) \\
G - \frac{c}{\epsilon}, & x \in B_{2R_\epsilon}(x_0) \setminus B_{R_\epsilon}(x_0),
\end{cases} \quad (68)
\]
where $\psi$ is defined as in (60), $\zeta \in C_0^\infty(B_{2R_\epsilon}(x_0))$ satisfies that $\zeta \equiv 1$ on $B_{R_\epsilon}(x_0)$ and $\|\nabla \zeta\|_{L^\infty} = O(1/(R \epsilon))$, $r = r(x) = \text{dist}_g(x, x_0)$, $R = -\log \epsilon$, $B$ and $c$ are constants depending only on $\epsilon$ to be determined later. Define another sequence of functions
\[
\eta_\epsilon(x) = \begin{cases} 
b_\epsilon(x), & x \in B_{2R_\epsilon}(x_0) \\
b_\epsilon(\sigma_i^{-1}(x)), & x \in B_{2R_\epsilon}(\sigma_i(x_0)), \, 2 \leq i \leq \ell \\
G - \frac{c}{\epsilon}, & x \in \Sigma \setminus \bigcup_{i=1}^\ell B_{2R_\epsilon}(\sigma_i(x_0)).
\end{cases} \quad (69)
\]
Noting that $G(\sigma_i(x)) = G(x)$ for all $x \in \Sigma \setminus [\sigma_1(x_0), \cdots, \sigma_\ell(x_0)]$, one can easily check that
\[
\eta_\epsilon(\sigma_i(x)) = \eta_\epsilon(x), \quad \forall x \in \Sigma, \, \forall 1 \leq i \leq \ell. \quad (70)
\]
In view of (68) and (69), in order to ensure that $\eta_\epsilon \in W^{1,2}(\Sigma, g)$, we set
\[
c + \frac{1}{\pi \ell} \left( \frac{1}{4\pi \ell} \log(1 + \pi \ell R^2) + B \right) = \frac{1}{c} \left( \frac{1}{2\pi \ell} \log(\pi \ell) + A_{x_0} \right),
\]
which gives
\[
2\pi \ell c^2 = -\log \epsilon - 2\pi \ell B + 2\pi \ell A_{x_0} + \frac{1}{2} \log(\pi \ell) + O\left(\frac{1}{R^2}\right). \quad (71)
\]
Noting that \( \int G dv_g = 0 \), we have
\[
\int_{\Sigma \cup \mathcal{B}_0(\sigma, (x_0))} |\nabla_g G|^2 dv_g = \int_{\Sigma \cup \mathcal{B}_0(\sigma, (x_0))} G \Delta_g G dv_g - \int_{\Sigma \cup \mathcal{B}_0(\sigma, (x_0))} \frac{\partial G}{\partial v} dv_g - \frac{1}{4 \pi} \int_{\Sigma \cup \mathcal{B}_0(\sigma, (x_0))} G dv_g \]
\[
= \alpha \int_{\Sigma \cup \mathcal{B}_0(\sigma, (x_0))} G^2 dv_g - \frac{1}{4 \pi} \log R - \frac{1}{4 \pi} + O(\frac{1}{R^2}).
\]

Since \( \tilde{\psi} \in C^1(\Sigma, g) \) and \( \tilde{\psi}(x_0) = 0 \), we have
\[
\int_{\mathcal{B}_2(x_0) \cap \mathcal{B}_3(x_0)} |\nabla_g \tilde{\psi}|^2 dv_g = O(R^2),
\]
\[
\int_{\mathcal{B}_2(x_0) \cap \mathcal{B}_3(x_0)} \nabla_g G \nabla_g \tilde{\psi} dv_g = O(\alpha),
\]
\[
\int_{\mathcal{B}_2(x_0) \cap \mathcal{B}_3(x_0)} |\nabla_g \eta| dv_g = \ell \int_{\mathcal{B}_2(x_0)} |\nabla_g \eta| dv_g.
\]

Combining (72) and noting that
\[
\int_{\mathcal{B}_2(x_0) \cap \mathcal{B}_3(x_0)} |\nabla_g \eta| dv_g = \ell \int_{\mathcal{B}_2(x_0)} |\nabla_g \eta| dv_g,
\]
we obtain
\[
\int_{\Sigma} |\nabla_g \eta| dv_g = \frac{1}{4 \pi \ell^2} \left( \frac{2}{\pi} \log \frac{1}{\ell} + \log(\pi \ell) - 1 + 4 \pi \ell A_{x_0} + 4 \pi \ell \alpha \int_{\Sigma} G^2 dv_g + \frac{1}{4 \pi} \log R + O(\frac{1}{R^2}) \right). 
\]

Observe
\[
\int_{\Sigma} \eta dv_g = \frac{1}{c} \left( \int_{\Sigma \cup \mathcal{B}_0(\sigma, (x_0))} G dv_g + O(\alpha \log(\alpha)) \right)
\]
\[
= \frac{1}{c} \left( \int_{\Sigma \cup \mathcal{B}_0(\sigma, (x_0))} G dv_g + O(\alpha \log(\alpha)) \right)
\]
\[
= \frac{1}{c} \left( \int_{\Sigma \cup \mathcal{B}_0(\sigma, (x_0))} G dv_g + O(\alpha \log(\alpha)) \right)
\]
\[
= \frac{1}{c} O(\alpha \log(\alpha)),
\]
we have
\[
\bar{\eta} = \frac{1}{\text{Vol}_g(\Sigma)} \int_{\Sigma} \eta dv_g = \frac{1}{c} O(\alpha \log(\alpha)).
\]

Hence
\[
\int_{\Sigma} (\eta - \bar{\eta})^2 dv_g = \int_{\Sigma} \eta^2 dv_g - 2 \bar{\eta} \int_{\Sigma} \eta dv_g + \bar{\eta}^2 \text{Vol}_g(\Sigma)
\]
\[
= \frac{1}{c^2} \left( \int_{\Sigma} G^2 dv_g + O(\alpha \log(\alpha)) \right).
\]
This together with (76) yields
\[ \| \eta_e - \eta_\ell \|_{1,\alpha} = \int_\Sigma |\nabla_\ell \eta_\ell|^2 v_\ell = \alpha \int_\Sigma (\eta_e - \eta_\ell)^2 v_\ell \]
\[ = \frac{1}{4\pi \ell c^2} \left( 2 \log \frac{1}{\epsilon} + \log(\pi \ell) - 1 + 4\pi \ell A_\ell + O\left(\frac{1}{R^2}\right) + O(Re \log(Re)) \right). \] (77)

Now we choose \( B \) in (71) such that
\[ \| \eta_e - \eta_\ell \|_{1,\alpha} = 1. \] (78)

Combining (77) and (78), we have
\[ c^2 = -\frac{\log \epsilon}{2\pi \ell} + \frac{\log(\pi \ell)}{4\pi \ell} + A_\ell + O\left(\frac{1}{R^2}\right) + O(Re \log(Re)). \] (79)

It then follows from (71) and (79) that
\[ B = \frac{1}{4\pi \ell} + O\left(\frac{1}{R^2}\right) + O(Re \log(Re)). \] (80)

Let
\[ \phi_\ell = \eta_e - \eta_\ell. \] (81)

In view of (70), (81) and the fact that \( \eta_e \in W^{1,2}(\Sigma, g) \), we have \( \phi_\ell \in \mathcal{H}_\ell \). Moreover, the equality (78) is exactly \( \| \phi_\ell \|_{1,\alpha} = 1 \), and thus (66). A straightforward calculation shows on \( B_{Re}(x_0) \),
\[ 4\pi \ell \phi_\ell^2 \geq 4\pi \ell c^2 - 2 \log(1 + \pi \ell e^2) + 8\pi B + O(Re \log(Re)). \]

This together with (79) and (80) yields
\[ \int_{B_{2\ell}(x_0)} e^{4\pi \ell \phi_\ell^2} d\ell v_\ell \geq \pi e^{1 + 4\pi \ell A_\ell} + O\left(\frac{1}{(\log \epsilon)^2}\right), \]
which immediately leads to
\[ \int_{\cup_{i=1}^\ell B_{2\ell}(\sigma_i(x_0))} e^{4\pi \ell \phi_\ell^2} d\ell v_\ell \geq \pi e^{1 + 4\pi \ell A_\ell} + O\left(\frac{1}{(\log \epsilon)^2}\right). \] (82)

On the other hand,
\[ \int_{\Sigma \setminus \cup_{i=1}^\ell B_{2\ell}(\sigma_i(x_0))} e^{4\pi \ell \phi_\ell^2} d\ell v_\ell \geq \int_{\Sigma \setminus \cup_{i=1}^\ell B_{2\ell}(\sigma_i(x_0))} \left(1 + 4\pi \ell \phi_\ell^2\right) d\ell v_\ell \]
\[ \geq \text{vol}_\ell(\Sigma) + 4\pi \ell \frac{\|G\|^2}{c^2} + o\left(\frac{1}{c^2}\right). \] (83)

Recalling (79) and combining (82) and (83), we conclude (67) for sufficiently small \( \epsilon > 0 \). This completes the proof of Theorem 1. □
4. Proof of Theorem 2

Since the proof of Theorem 2 is analogous to that of Theorem 1, we only give its outline.

Let $j \geq 2$, $\lambda_j^G$ and $E_{j-1}^\perp$ be defined as in (14) and (15) respectively. For $\alpha < \lambda_j^G$, we define

$$
\beta_j^* = \sup \left\{ \beta : \sup_{u \in E_{j-1}^\perp, \|u\| \leq 1} \int_{\Sigma} e^{\beta u^2} dv_g < \infty \right\}.
$$

Comparing (18) with (84), similar to Lemma 3, we have $\beta_j^* = 4\pi \ell$, where $\ell$ is defined as in (12).

We now prove (ii) of Theorem 2. If $\alpha \geq \lambda_j^G$ and $\beta > 0$, we take $u_j \in H \cap C^1(\Sigma, g)$ satisfies $\Delta_j u_j = \lambda_j^G u_j$, and $u_j \neq 0$. It follows that

$$
\int_{\Sigma} |\nabla_j (tu_j)|^2 dv_g - \alpha \int_{\Sigma} (tu_j)^2 dv_g \leq 0, \quad \forall t \in \mathbb{R}
$$

and that

$$
\int_{\Sigma} e^{\beta (tu_j)^2} dv_g \to \infty \quad \text{as} \quad t \to \infty.
$$

Then (85) and (86) implies that the supremum in (16) is infinity.

If $\alpha < \lambda_j^G$ and $\beta > 4\pi \ell$, then we will prove the supremum in (16) is infinity. To do this, we let \( \{e_i\}_{i=1}^{m_{j-1}} \subset H \cap C^1(\Sigma, g) \) be an orthonormal basis of $E_{j-1} = E_{\lambda_j^G} \oplus \cdots \oplus E_{\lambda_j^G}$, with respect to the inner product in $L^2(\Sigma, g)$, namely, $E_{j-1} = \text{span}\{e_1, \cdots, e_{m_{j-1}}\}$ and

$$
(e_i, e_k) = \int_{\Sigma} e_i e_k dv_g = \delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}
$$

for all $i, k = 1, \cdots, m_{j-1}$. Let $\tilde{M}_k$ be defined as in (21). Set

$$
Q_k = \tilde{M}_k - \frac{1}{\text{Vol}_g(\Sigma)} \int_{\Sigma} \tilde{M}_k dv_g = \sum_{i=1}^{m_{j-1}} (\tilde{M}_k, e_i) e_i.
$$

Then $Q_k \in E_{j-1}^\perp$. By a straightforward calculation, $\|Q_k\|^2_{1,0} = (1 + O(r))8\pi \ell \log k + O(1)$. Thus there holds for any fixed $\beta > 4\pi \ell$

$$
\int_{\Sigma} e^{\beta Q_k^2} dv_g \geq \int_{B_{4\pi \ell / \log k}(Q_k)} e^{\beta \lambda_j^G \|Q_k\|^2_{1,0}} dv_g
$$

$$
= e^{\beta \lambda_j^G (1/\log k)} \pi \ell^2 k^{-1/2} (1 + o_k(1)).
$$

Choosing $r > 0$ sufficiently small and then passing to the limit $k \to \infty$ in the above estimate, we conclude

$$
\int_{\Sigma} e^{\beta Q_k^2} dv_g \to \infty \quad \text{as} \quad k \to \infty.
$$

Hence the supremum in (16) is infinity.
In the following, we sketch the proof of (i) and (iii) of Theorem 2.

Let \( \alpha < 4\pi \ell \). By a variational direct method, one can see that for any \( 0 < \epsilon < 4\pi \ell \), there exists some \( u_\epsilon \in E_{j-1}^1 \) with \( \|u_\epsilon\|_{1,\alpha} = 1 \) such that

\[
\int_\Sigma e^{4\pi\ell(-\epsilon)u^2} \, dv_\epsilon = \sup_{u \in E_{j-1}^1, \|u\|_{1,\alpha} \leq 1} \int_\Sigma e^{4\pi\ell(-\epsilon)u^2} \, dv.
\]

Clearly, \( u_\epsilon \) satisfies the Euler-Lagrange equation

\[
\begin{align*}
\Delta u_\epsilon - a u_\epsilon = & \frac{1}{\epsilon} u_\epsilon e^{(4\pi\ell(-\epsilon)u^2)} - \frac{\epsilon}{4\pi\ell} \sum_{k=1}^{m_{j-1}} \gamma_k e_k \\
u_\epsilon & \in E_{j-1}^1, \|u_\epsilon\|_{1,\alpha} = 1 \\
\lambda_\epsilon & = \int_\Sigma u_\epsilon^2 e^{(4\pi\ell(-\epsilon)u^2)} \, dv \\
\gamma_\epsilon & = \int_\Sigma e u_\epsilon e^{(4\pi\ell(-\epsilon)u^2)} \, dv.
\end{align*}
\]  

(87)

Without loss of generality, we assume \( c_\epsilon = u_\epsilon(x_\epsilon) = \sup_{x \in \Sigma} |u_\epsilon| \to +\infty \) and \( x_\epsilon \to x_0 \) as \( \epsilon \to 0 \). Let \( r_\epsilon \) be the blow-up scale defined as in (42) and \( \varphi_\epsilon(y) = c_\epsilon(u_\epsilon(x_\epsilon) - c_\epsilon) \) for \( y \in B_{r_\epsilon}(0) \), where \( 0 < \delta < \frac{1}{2} \delta\epsilon_\epsilon(\Sigma) \). As before, we can derive

\[
\varphi_\epsilon(y) \to \varphi(y) = -\frac{1}{4\pi\ell} \log(1 + \pi\ell|y|^2) \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^2).
\]

Moreover, we can prove that \( \forall 1 < q < 2 \), \( c_\epsilon u_\epsilon \) converges to a Green function \( G \) weakly in \( W^{1,q}(\Sigma, g) \), strongly in \( L^\infty(\Sigma, g) \), and almost everywhere in \( \Sigma \). In this case, \( G \) satisfies

\[
\begin{align*}
\Delta G - a G &= \frac{1}{\epsilon} \sum_{j=1}^{m_{j-1}} \delta_{A(x_0)} - \frac{1}{(\omega_{j-1})^{\frac{1}{2}}} \sum_{j=1}^{m_{j-1}} \epsilon_{j}(x_0) e_k \\
\int_\Sigma G \psi dv_\epsilon &= 0, \quad \forall \psi \in E_{j-1} \\
\int_\Sigma G(\sigma(x)) &= G(x), \quad \forall x \in \Sigma \setminus G(x_0), \forall \sigma \in G.
\end{align*}
\]  

(88)

Similarly, \( G \) has a decomposition (60) near \( x_0 \) and \( A(x_0) \) is defined as in (61). Analogous to Proposition 8, we arrive at

\[
\sup_{u \in E_{j-1}^1, \|u\|_{1,\alpha} \leq 1} \int_\Sigma e^{4\pi\ell(-\epsilon)u^2} \, dv_\epsilon \leq \text{Vol}_g(\Sigma) + \pi\ell e^{1+4\pi\ell A(x_0)}.
\]  

(89)

This particularly leads to (i) of Theorem 2.

Finally, we construct a sequence of functions to show that the estimate (89) is not true. This implies that blow-up can not occur and elliptic estimates on (87) give the desired extremal function. To do this, we let \( \eta_\epsilon, \phi_\epsilon \) be defined respectively as in (62) and (61) satisfying \( \eta_\epsilon \in W^{1,2}(\Sigma, g) \) and \( \|\phi_\epsilon\|_{1,\alpha} = 1 \). Note that the constants \( c \) and \( B \) in definitions of \( \eta_\epsilon \) and \( \phi_\epsilon \) are given by (72) and (80) respectively. It then follows that

\[
\int_\Sigma e^{4\pi\ell(-\epsilon)u^2} \, dv_\epsilon \geq \text{Vol}_g(\Sigma) + \pi\ell e^{1 + 4\pi\ell A(x_0)} + \frac{4\pi\ell|\text{Vol}_g(\Sigma)|}{\pi\ell \log \epsilon} + o\left(\frac{1}{\log \epsilon}\right).
\]  

(90)
Let
\[ \tilde{\phi}_e = \phi_e - \sum_{i=1}^{m_j-1} (\phi_e, e_i) e_i. \]  
(91)

Obviously \( \tilde{\phi}_e \in E_{j-1}^\perp \). Since \( G \) satisfies (88) and
\[ \int_{\Sigma} G e_i dv_g = \lim_{\epsilon \to 0} \int_{\Sigma} c_i u_i e_i dv_g = 0, \quad \forall 1 \leq i \leq m_j - 1, \]
we calculate
\[ (\phi_e, e_i) = \int_{\cup \ell_i B_{2R}(\tau_i (x_0))} (\eta_i - \eta_i) e_i dv_g + \int_{\Sigma \backslash \cup \ell_i B_{2R}(\tau_i (x_0))} \left( \frac{G}{c} - \eta_i \right) e_i dv_g = o \left( \frac{1}{\log^2 \epsilon} \right). \]

This together with (91) leads to
\[ \tilde{\phi}_e = \phi_e + o \left( \frac{1}{\log^2 \epsilon} \right), \quad \| \tilde{\phi}_e \|^2_{1,\alpha} = 1 + o \left( \frac{1}{\log^2 \epsilon} \right). \]
(92)

It follows from (90) and (92) that
\[ \int_{\Sigma} e^{4\pi \frac{\partial}{\partial t} \eta_i} dv_g = \int_{\Sigma} e^{4\pi \frac{\partial}{\partial t} \eta_i} dv_g \]
\[ \geq \left( 1 + o \left( \frac{1}{\log \epsilon} \right) \right) \left( \text{Vol}_g(\Sigma) + \pi \ell e^{1+4\pi |A_{\Sigma}|} + \frac{4\pi \ell |G|^2_{L^2(\Sigma,e)}}{c^2} \right) \]
\[ \geq \text{Vol}_g(\Sigma) + \pi \ell e^{1+4\pi |A_{\Sigma}|} + \frac{4\pi \ell |G|^2_{L^2(\Sigma,e)}}{- \log \epsilon} + o \left( \frac{1}{\log \epsilon} \right), \]
which implies that (89) does not hold. This completes the proof of (iii) of Theorem 2.

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