Abstract

We present here an application of the standard Langevin dynamics to the problem of perturbative expansions on the Lattice QCD. This method can be applied in the computation of the most general observables. In this work we will concentrate in particular on the computation of the perturbative terms of the $1 \times 1$ Wilson loop, up to fourth order. It is shown that a stochastic gauge fixing is a possible solution to the problem of divergent fluctuations which affect higher order coefficients.

Since its introduction in 1981 by Parisi and Wu\cite{1}, Langevin dynamics has been extensively used for Monte Carlo simulations.

Basically it consists in a stochastic dynamical system on the field configuration space dictated by the general equation

$$\frac{\partial \phi(x,t)}{\partial t} = - \frac{\partial S[\phi]}{\partial \phi(x,t)} + \eta(x,t), \tag{1}$$

where $\phi$ is the field, $S[\phi]$ the action and $\eta$ a Gaussian random noise satisfying to the normalization

$$< \eta(x,t)\eta(x',t') > = 2\delta(x'-x)\delta(t'-t). \tag{2}$$

* We warmly thank P.Rossi, G.Parisi, G.C.Rossi, N.Cabibbo and C.Destri for interesting discussions and precious suggestions.
As a matter of fact, stochastic dynamics is devised in such a way that time averages on the noise converge to averages on the Gibbs measure

$$< \frac{1}{T} \int_0^T dt O[\phi(t)] >_\eta \rightarrow \frac{1}{Z} \int D\phi O[\phi] e^{-S[\phi]}.$$  

(3)

To obtain from equation (1) a useful expression for computer simulations, one can take $t$ discrete with time step $dt = \epsilon$:

$$\phi(x, t_{n+1}) = \phi(x, t_n) - f_x[\phi, \eta],$$  

(4)

where

$$f_x = \epsilon \frac{\partial S}{\partial \phi(x, t_n)} + \sqrt{\epsilon} \eta(x, t_n)$$  

(5)

and now $\eta$ is normalized by:

$$< \eta(x_i, t_i) \eta(x_j, t_j) > = 2 \delta_{x_i x_j} \delta_{t_i t_j}.$$  

(6)

In this discrete form, Langevin equation has to be regarded solely as an approximation of equation (1), valid only for $\epsilon \rightarrow 0$.

The method has been widely adopted as an alternative to Metropolis, Heat Bath and other Monte Carlo algorithms for scalar fields and gauge theories. In 1985 Batrouni et al. [2] presented a new analysis for lattice fields theories. Given the standard Wilson action

$$S = -\frac{\beta}{2n} \sum_P Tr(U_P + U_P^\dagger),$$  

(7)

where the sum is over the plaquettes $P$, in a four dimensional periodic lattice. $U_P$ are ordered products of the link gauge variables $U_\mu(x)$, which are $SU(3)$ matrices. Here $\mu = 1, \ldots, 4$ and $x$ is a point of the lattice. Each configuration is then described by $4 \times \text{volume} \ 3 \times 3 \ \text{complex matrices}$.

For each link $U = U_\mu(x)$, one adopts the evolution

$$U(t_{n+1}) = e^{-F(t_n)} U(t_n),$$  

(8)

where

$$F(t_n) = \frac{\epsilon \beta}{4n} \left[ \sum_{U_P \supset U_\mu} (U_P - U_P^\dagger) - \frac{1}{n} \sum_{U_P \supset U_\mu} Tr(U_P - U_P^\dagger) \right] + \sqrt{\epsilon} H(t_n),$$  

(9)

and $H$ is a traceless antihermitian noise matrix with normalization given by

$$< H_{ik}(x, t) \overline{H}_{jm}(x', t') >_H = [\delta_{il} \delta_{km} - \frac{1}{n} \delta_{ik} \delta_{lm}] \delta_{x,x'} \delta_{t,t'}.$$  

(10)

Langevin approach was originally formulated for perturbation theory also on the continuum. What we present here is the application of this idea to compute the weak coupling expansion directly in the lattice. The problem is well known and has been considered by
diagrammatic technique (see [4]), which allows the calculation of the expansion coefficients up to $g^4$ (and in some cases $g^6$). Since gauge fields are written as

$$ U_\mu = e^{gA_\mu}, \quad A^{\dagger}_\mu = -A_\mu, \quad Tr A_\mu = 0 $$

where $g$ is the coupling constant, the Langevin equation takes the following form:

$$ e^{gA'_\mu} = e^{-F} e^{gA_\mu}. $$

(12)

The fields $A_\mu$ can be expanded in series of $g$

$$ A_\mu = \sum_k g^k A^{(k)}_\mu. $$

(13)

In the same manner, recalling that $\beta = 1/g^2$ and imposing $\epsilon = g^2 \tau$, the drift (9) becomes;

$$ F_\mu = \frac{\tau}{12} \left[ \sum_{U_P > U_\mu} (U_P - U_P^\dagger) - \frac{1}{3} \sum_{U_P > U_\mu} Tr(U_P - U_P^\dagger) \right] + g\sqrt{\tau} H(t_n) = \sum_k g^k F^{(k)}, $$

(14)

The main point is to apply to equation (12) the Baker - Campbell - Hausdorff formula and to extract the contributions order by order in $g$.

At present, we have implemented the simulation computing the evolutions for the gauge fields $A_\mu$ up to fourth order in $g$. At this order, perturbative coefficients of many observables have been computed analytically. Thus, in order to check our lattice formulation of the above Langevin dynamics, the terms of the standard $1 \times 1$ plaquette have been measured (always to the order $g^4$), confirming the analytical results.

While the original motivation of the Langevin approach was to make it possible to calculate in perturbation theory without fixing a gauge, it is known that some divergent fluctuations (averaging to zero) may plague high order terms. We confirm this phenomenon in the case of the plaquette expansion. We observe indeed that, from the third order in $g$, the errors associated to our observables grow in time, even if the mean value remains always stable around its known value. In higher orders, this spurious fluctuation may completely hide the signal. As a way out we have applied a technique which goes back to Zwanziger and has been more recently implemented on the lattice [5] in the form of “stochastic gauge fixing”. We may think to a new source in the Langevin equation, responsible of a stochastic gauge fixing, so that the actual algorithm implemented is (for the gauge links)

$$ U^{N'}_{\mu}(x) = e^{F[U^{N'}_{\mu},x,\mu]} U^{N}_{\mu}(x) $$

$$ U^{N+1}_{\mu}(x) = e^{w[U^{N'}_{\mu},x]} U^{N'}_{\mu}(x) e^{-w[U^{N'}_{\mu},x-\mu]}.$$

with

$$ w[U, x] = \alpha \sum_\mu \left( \Delta_{-\mu} \left[ U_\mu(x) - U_\mu^\dagger(x - \mu) \right] \right)_{\text{traceless}} $$

$$ \Delta_{-\mu} U_\nu(x) \equiv U_\nu(x) - U_\nu(x - \mu) $$

(15)

(16)
α being a free parameter.
As a matter of fact, the result is very impressive. We report, for example, the term in $g^4$ of
the plaquette, measured (in figure 1) without gauge fixing, and (in figure 2) with the above
gauge fixing. By means of this essential reduction of the noise, the result we obtained for
the reported term, with $\tau$ and $\alpha$ set to 0.02, is $c_4 = 1.211 \pm 0.003$, through an average over
2048 iterations.

All the numerical experiments have been done on Connection Machine CM2, using CM
Fortran, with a great contribution of the CMIS assembler. This kind of computation is
indeed very expensive: 2.455 Gflops are needed to complete a single Langevin iteration on
a lattice size of $8^4$, at the fourth order, with the measure of the corresponding coefficient of
the plaquette. For the same lattice size, the program uses about 10 MBytes of memory.

The work is in progress to go to higher orders and test a wide class of observables of
interest for Lattice gauge Theories.
References

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Figure 1: Term in $g^4$ of the plaquette, measured without gauge fixing
Figure 2: Term in $g^4$ of the plaquette, measured with gauge fixing