Fast Erasure-and-Error Decoding and Systematic Encoding of a Class of Affine Variety Codes*

Hajime Matsui
Toyota Technological Institute
Hisakata, Tenpaku, Nagoya 468–8511, Japan
Email: matsui@toyota-ti.ac.jp

Abstract—In this paper, a lemma in algebraic coding theory is established, which is frequently appeared in the encoding and decoding for algebraic codes such as Reed–Solomon codes and algebraic geometry codes. This lemma states that two vector spaces, one corresponds to information symbols and the other is indexed by the support of Gröbner basis, are canonically isomorphic, and moreover, the isomorphism is given by the extension through linear feedback shift registers from Gröbner basis and discrete Fourier transforms. Next, the lemma is applied to fast unified system of encoding and decoding erasures and errors in a certain class of affine variety codes.

Keywords: Berlekamp–Massey–Sakata algorithm, Gröbner basis, discrete Fourier transforms, order domain codes, evaluation codes.

I. INTRODUCTION

Despite many researches have been done for both of encoding and erasure-and-error decoding, any relation between them have never been found so far except maximum distance separable (MDS) codes, which satisfy \(d - 1 = n - k\), where \(n, k, d\) are the code length, dimension, and the minimum distance of the code, respectively. Since the correctable numbers of erasures and \(t\) of errors satisfy \(2t + u \leq d - 1\), it is obvious for MDS codes that, if \(t = 0\), then erasure-only decoding can determine \(n - k\) redundant symbols, that is, systematic encoding is done by erasure-only decoding. In general, algebraic geometry codes are not MDS codes; there has never been known no such case where erasure decoding can undertake systematic encoding.

In this paper, we first establish a lemma that is essential in algebraic coding theory. We observe that almost all manipulations on algebraic Goppa codes such as encoding and decoding are described in terms of the code, which provides an isomorphism between a certain pair of vector spaces over a finite field. The isomorphism of the lemma is written by the combined map of the extension by the linear recurrence relation from a Gröbner basis and \(N\)-dimensional inverse discrete Fourier transform (IDFT). Next, the lemma is applied to a class of affine variety codes [3, 13], which are essentially same as order domain codes or evaluation codes [11, 13, 4], and enables us to decode efficiently erasures and errors. Finally, we notice that, in a class of affine variety codes, systematic encoding can be viewed as a certain type of erasure-only decoding.

The rest of this paper is organized as follows. In Section II we prepare notations. In Section III we state the lemma. In Subsection III-A we generalize discrete Fourier transforms (DFTs) from on \((\mathbb{F}_q)^N\) into on \(\mathbb{F}_q^N\). In Subsection III-B two vector spaces are defined via Gröbner basis. In Subsection III-C we give an isomorphism between the vector spaces. In Section IV we apply the lemma. In Subsection IV-A we construct affine variety codes in terms of the lemma. In Subsection IV-B an erasure-and-error decoding algorithm and its relation with systematic encoding is described. In Section V the number of finite-field operations in our algorithm is estimated. Section VI concludes the paper.

II. NOTATIONS

Throughout this paper, \(\mathbb{N}_0\) is the set of non-negative integers and \(\alpha\) is a fixed primitive element of finite field \(\mathbb{F}_q\), where \(q\) is a prime power. For \(a, b \in \mathbb{N}_0\) with \(a \leq b\), let \([a, b] := \{a, a+1, \ldots, b\}\). For two sets \(A\) and \(B\), \(A \setminus B\) is defined as \(\{u \in A \mid u \notin B\}\). For arbitrary finite set \(S\), let \(V_S := \{(v_s)_{s \in S} \mid v_s \in \mathbb{F}_q\}\) denote a vector space over \(\mathbb{F}_q\) whose components are indexed by \(S\). For any arbitrary subset \(R \subset S\), the vector space \(V_R\) is considered as a subspace of \(V_S\) by \(V_R = \{(v_s)_{s \in S} \mid v_s = 0\} \text{ for all } s \in S \setminus R\).

III. MAIN LEMMA

A. Fourier-type transforms on \(\mathbb{F}_q^N\)

Let \(N\) be a positive integer and let

\[
A := [0, q-1]^N, \\
\Omega := \mathbb{F}_q^N = \{(\omega) \mid \omega = (\omega_1, \ldots, \omega_N) \mid \omega_1, \ldots, \omega_N \in \mathbb{F}_q\}. 
\]

In this subsection, Fourier-type transforms are defined as maps between two vector spaces, which are isomorphic to \(\mathbb{F}_q^N\),

\[
V_A := \{(h_\omega)_A \mid \omega \in \Omega, h_\omega \in \mathbb{F}_q\}, \\
V_\Omega := \{(c_\omega)_\Omega \mid \omega \in \Omega, c_\omega \in \mathbb{F}_q\}. 
\]
For \((c_\omega^a) \in V_\Omega\), discrete Fourier transform (DFT) \((\mathcal{F}_N c_\omega^a)_A \in V_A\) is defined as:

\[
\mathcal{F}_N c_\omega^a := \sum_{\omega \in \Omega} c_\omega^a \hat{\omega}_{\omega}^a \in V_A,
\]

where \(\hat{\omega}_{\omega}^a\) is defined by \(\omega^1 \cdots \omega^N\), and \(\omega^a\) is considered as the substituted value \(\omega^a := x^a\). Thus, \(\mathcal{F}_N c_\omega^a\) can be written as:

\[
\mathcal{F}_N c_\omega^a = \begin{cases} 
\sum_{\omega \in \Omega} c_\omega^a \omega^a & a \neq 0 \\
0 & a = 0.
\end{cases}
\]

Assume the next simplest case \(N = 2\). Then, for each \((a,b) = (a,b) \in A\), \(\mathcal{F}_2 c_{ab}\) can be directly written as:

\[
\mathcal{F}_2 c_{ab} = \begin{cases} 
\sum_{(\phi,\psi,\omega) \in \Omega} c_{\phi \psi \omega}^a \psi^b & ab \neq 0 \\
0 & a = 0
\end{cases}
\]

Assume \(N = 3\). Then, for each \((a,b,c) = (a,b,c) \in A\), \(\mathcal{F}_3 c_{abc}\) can be directly written as:

\[
\mathcal{F}_3 c_{abc} = \begin{cases} 
\sum_{(\phi,\psi,\omega,\theta) \in \Omega} c_{\phi \psi \omega \theta}^a \psi^b \theta^c & abc \neq 0 \\
0 & a = 0
\end{cases}
\]

In general, to write \(\mathcal{F}_N c_\omega^a\) directly, \(2^N\) equalities are required.

\[\square\]

On the other hand, for \((h_\omega^a) \in V_A\), inverse discrete Fourier transform (IDFT) \((\mathcal{F}_N^{-1} h_\omega^a)_\Omega \in V_\Omega\) is defined as follows:

\[
\mathcal{F}_N^{-1} h_\omega^a := \begin{cases} 
\sum_{l_i, \cdots, l_m = 1} q^{-1} h_{l_i} \omega^{l_i} \cdots \omega^{l_m} & \text{for } i \in [1,N] \setminus I \\
0 & \text{otherwise}
\end{cases}
\]

where \(\tilde{I}(I,J) := (b_1, \cdots, b_N) \in A\) and, for \(1 \leq i \leq N\),

\[
b_i := \begin{cases} 
l_i & i \in I \\
q - 1 & i \in J \\
0 & i \in [1,N] \setminus (I \cup J)
\end{cases}
\]

For example, if \(\omega_1, \cdots, \omega_N \neq 0\) for \(\omega = (\omega_1, \cdots, \omega_N) \in \Omega\), then \(I\) is equal to \([1, N]\), there is only one choice of \(J = 0\), and in this case the definition implies

\[
\mathcal{F}_N^{-1} h_\omega^a = (-1)^N \sum_{l_i, \cdots, l_m = 1} h(l_i, \cdots, l_m) \omega^{l_i} \cdots \omega^{l_m},
\]

in other words, \(\mathcal{F}_N^{-1}\) agrees with \(N\)-dimensional inverse discrete Fourier transform if \(\Omega\) is restricted to \((\mathbb{F}_q^N)\). In general, for each \(\omega \in \Omega\), \(\mathcal{F}_N^{-1} h_\omega^a\) is equal to a linear combination of inverse discrete Fourier transforms whose dimensions do not exceed \(N\).

Example 2: Assume \(N = 1\). If \(\omega \neq 0 \in \Omega\), then \(I = \{1\} \subseteq [1, 1]\); \(J = 0 \subseteq [1, 1]\); \(I = \emptyset\), and \(\mathcal{F}_N^{-1} h_\omega^a = (i)\); if \(\omega = 0\), then \(I = \{1\} \subseteq [1, 1]\); \(J = \emptyset\), \(I = \{1\} \subseteq [1, 1]\); \(I = \{2\}\), and \(\mathcal{F}_N^{-1} h_\omega^a = (i,0)\); \(i \neq 0\) \(i, q - 1\), respectively. Thus \(\mathcal{F}_N^{-1} h_\omega^a\) can be directly written as:

\[
\mathcal{F}_N^{-1} h_\omega^a = \begin{cases} 
\sum_{i=1}^{q-1} h_{i} \omega^{-i} & \omega \neq 0 \\
0 & \omega = 0
\end{cases}
\]

Assume \(N = 2\). For \((\omega_1, \omega_2) = (\psi, \phi) \in \Omega\), for example, \(\mathcal{F}_N^{-1} h_{\omega_1}^{\omega_2} = (i, j); \)\(i, j) = \emptyset\); if \(\psi = 0\), then \(I = \{1\} \subseteq [1, 1]\); \(J = \emptyset\), \(\tilde{I}(i,0)\); \(i \neq 0\); \(i, q - 1\), respectively. Thus \(\mathcal{F}_N^{-1} h_{\omega_1}^{\omega_2}\) can be directly written as:

\[
\mathcal{F}_N^{-1} h_{\omega_1}^{\omega_2} = \begin{cases} 
\sum_{i=1}^{q-1} h_{i} \omega^{-i} & \omega \neq 0 \\
0 & \omega = 0
\end{cases}
\]

In general, the summand in each condition of \(\omega\) consists of \(2^{N-m}\) terms, where \(m\) is the number of nonzero components in \(\omega\).

\[\square\]

**Proposition 1:** The two linear maps

\[
\begin{align*}
[ V_A \ni (h_\omega^a)_A & \mapsto (\mathcal{F}_N^{-1} h_\omega^a)_\Omega \in V_\Omega ] \\
[ V_\Omega \ni (c_\omega^a)_\Omega & \mapsto (\mathcal{F}_N c_\omega^a)_A \in V_A ]
\end{align*}
\]

are inverse each other, that is, \(\mathcal{F}_N \mathcal{F}_N^{-1} h_\omega^a = h_\omega^a\) and \(\mathcal{F}_N^{-1} \mathcal{F}_N c_\omega^a = c_\omega^a\).

\[\square\]

**B. Two vector spaces \(V_S\) and \(V_Q\)**

Let \(\Psi \subseteq \Omega\) and \(n := |\Psi|\). One of the two vector spaces in the lemma is given by

\[
V_\Psi := \left\{ (c_\omega^a) : \psi \in \Psi, c_\omega^a \in \mathbb{F}_q \right\},
\]

\[3\]For any finite set \(S\), the number of elements in \(S\) is represented by \(|S|\).
\[ F_2^{-1}h_{\psi\omega} := \begin{cases} \sum_{i,j=1}^{q-1} h_{ij} \psi^{-i} \omega^{-j} & \psi \omega \neq 0 \\ -\sum_{i=1}^{q-1} (h_{i,0} - h_{i,q-1}) \psi^{-i} & \psi \neq 0, \omega = 0 \\ -\sum_{j=1}^{q-1} (h_{0,j} - h_{q-1,j}) \omega^{-j} & \psi = 0, \omega \neq 0 \\ h_{0,0} - h_{0,q-1} - h_{q-1,0} + h_{q-1,q-1} & \psi = \omega = 0. \end{cases} \]

\[ F_3^{-1}h_{\phi\psi\omega} := \begin{cases} -\sum_{i,j,l=1}^{q-1} h_{ij} \phi^{-i} \psi^{-j} \omega^{-l} & \phi \psi \omega \neq 0 \\ \sum_{i,j=1}^{q-1} (h_{ij,0} - h_{ij,q-1}) \phi^{-i} \psi^{-j} & \phi \neq 0, \omega = 0 \\ \sum_{i,j=1}^{q-1} (h_{0,i,j} - h_{q-1,i,j}) \phi^{-i} \psi^{-j} & \psi \neq 0, \phi = 0 \\ \sum_{j=1}^{q-1} (h_{0,0,j} - h_{q-1,0,j} - h_{0,j,q-1} + h_{0,j,q-1}) \phi^{-i} \psi^{-j} & \psi = 0, \phi = 0 \\ -\sum_{i,j,l=1}^{q-1} (h_{0,0,i,j} - h_{0,0,j,i} - h_{0,j,0} + h_{0,j,0}) \phi^{-i} \psi^{-j} \omega^{-l} & \omega \neq 0, \phi = 0 \\ h_{0,0,0} - h_{0,0,q-1} - h_{0,q-1,0} - h_{q-1,0,0} + h_{0,q-1,1} + h_{q-1,0,1} + h_{q-1,q-1,1} & \phi = \psi = \omega = 0. \end{cases} \]

\[ V_\Psi := \{ f(\underline{a}) \in \mathbb{F}_q[\underline{x}] | f(\psi) = 0 \text{ for all } \psi \in \Psi \}. \]

We fix a monomial order \( \preceq \) of \( \{ \underline{a}^2 | \underline{a} \in N_0^N \} \). We denote, for \( f \neq 0 \in \mathbb{F}_q[\underline{x}] \),

\[
\text{LM}(f) := \max \{ \underline{a}^2 | \underline{a} \in N_0^N, f_{\underline{a}} \neq 0 \}.
\]

where \( \underline{a} := x_1^{s_1} \cdots x_N^{s_N} \) for \( \underline{s} = (s_1, \cdots, s_N) \in N_0^N \), and \( \text{LM}(f) \) is called the leading monomial of \( f(\underline{a}) \in \mathbb{F}_q[\underline{x}] \). Then the support \( S_\Psi = S \subseteq N_0^N \) of \( \Psi \) is defined by

\[
S_\Psi = S := N_0^N \setminus \{ \text{LM}(f) | f(\underline{a}) \in V_\Psi \},
\]

where \( \text{LM}(\underline{a}) := \underline{a} \in N_0^N \). Fortunately, \( S_\Psi \) has an intuitive description if a Gröbner basis \( \mathcal{G}_\Psi \) of \( Z_\Psi \) is obtained; it corresponds to the area surrounded by \( \text{LM}(\underline{a}) \). The support \( S_\Psi = S \subseteq N_0^N \) of \( Z_\Psi \) for \( \Psi \) is equivalently defined by

\[
\{ \underline{a}^2 | \underline{a} \in S_\Psi \} = \{ \underline{a}^2 | \underline{a} \in N_0^N \} \setminus \{ \text{LM}(f) \mid f(\underline{a}) \in V_\Psi \}.
\]

Then the other of the two vector spaces is given by

\[
V_S = V_{S_\Psi} := \{ (h_{\underline{a}})_{\Psi} = (h_{\underline{a}})_{S_\Psi} \mid \underline{a} \in S_\Psi, h_{\underline{a}} \in \mathbb{F}_q \},
\]

namely, \( V_S \) is the vector space over \( \mathbb{F}_q \) indexed by the elements of \( S_\Psi \). Since \( \{ \underline{a}^2 | \underline{a} \in S_\Psi \} \) is a basis of \( \mathbb{F}_q[\underline{x}]/Z_\Psi \) that is the quotient ring viewed as a vector space over \( \mathbb{F}_q \), \( V_S \) is isomorphic to \( \mathbb{F}_q[\underline{x}]/Z_\Psi \). It is known [5], [6] that the evaluation map

\[
\mathbb{F}_q[\underline{x}]/Z_\Psi \ni f(\underline{x}) \mapsto (f(\psi))_{\Psi} \in V_\Psi
\]

is an isomorphism between two vector spaces. Thus the map (7) is also written as

\[
V_S \ni (h_{\underline{a}})_S \mapsto \sum_{\underline{a} \in S} h_{\underline{a}} \psi(x) \in V_\Psi,
\]

which is denoted as \( \text{ev} : V_S \to V_\Psi \). In particular, it follows from the isomorphism (7) or (8) that \( |S_\Psi| = |\Psi| \) and \( \text{dimg}_s V_S = n \).

Since \( V_S \) and \( V_\Psi \) have the same dimension \( n \), it is trivial that \( V_S \) is isomorphic to \( V_\Psi \) as a vector space over \( \mathbb{F}_q \). However, this type of isomorphic maps depends on the choices of the bases of vector spaces; in addition, the normal orthogonal basis is not always convenient for encoding and decoding. Our lemma asserts that there is a canonical isomorphism that does not depend on the bases. As explained in Introduction, the isomorphic map \( V_S \to V_\Psi \) of the lemma is the composition map of the extension and IDFT, which are defined accurately in the next subsection C. On the other hand, the inverse map \( V_\Psi \to V_S \) can be written concisely; that is “DFT”

\[
V_\Psi \ni (c_{\underline{a}})_\Psi \mapsto \sum_{\underline{a} \in \Psi} c_{\underline{a}} \psi^{\underline{a}} \in V_S,
\]

which is actually the compound of DFTs in various dimensions. It is shown from the definitions that the matrices that represent two maps (8) and (9) are transposed each other if the bases of vector spaces are fixed.

C. Isomorphic map \( V_S \ni c_{\underline{a}} \mapsto v_{\underline{a}} \)

Let \( \mathcal{G}_\Psi \) be a Gröbner basis for the ideal \( Z_\Psi \) with respect to \( \preceq \). We assume that \( \mathcal{G}_\Psi \) consists of \( d+1 \) elements \( \{ g^{(n)} \}_{0 \leq n \leq d} \).
where
\[
g^{(u)} = g^{(u)}(x) = x^a + \sum_{u \in S_\Psi} g^{(u)}(x) \in \mathbb{F}_q[x] \quad \text{with} \quad a_n \in \mathbb{N}_0 \setminus S_\Psi.
\] (10)

**Definition 1:** We define that \((h_a)_A \in V_A\) satisfies the linear recurrence relation from \(G_\Psi\) if and only if there exists \((h_a)_S \in V_S\) such that, for all \(a \in A\) and all \(0 \leq u \leq d\),
\[
h_a + \sum_{u \in S_\Psi} g^{(u)}(a) h_{a+u} = 0,
\] (13)
where the indices \(i = (i_1, \ldots, i_N)\) of \(h_i\) are viewed within \(1 \leq (i_l \mod (q - 1)) < q \) if \(i_l \neq 0 \) for \(1 \leq l \leq N\). Then we denote that \((h_a)_A \rightarrow (h_a)_S\).

Namely, each \(h_a\) satisfies \(d + 1\) equations. Then we also say that \((h_a)_A\) is the extension of \((h_a)_S\). In fact, there is a one-to-one correspondence between arbitrary vectors \((h_a)_S\) and all vectors \((h_a)_A\) that satisfy the linear recurrence relation from \(G_\Psi\); from a given \((h_a)_S\), generate \((h_a)_A\) inductively by
\[
h_a := - \sum_{u \in S_\Psi} g^{(u)}(a) h_{a+u} \quad \text{for} \quad a \in A \setminus S_\Psi.
\] (14)

Then we obtain \((h_a)_A\) that satisfies \((13)\); the resulting values do not depend on the order of the generation because of the minimal property of Gröbner bases. Conversely, from a given \((h_a)_A\) that satisfies \((13)\), we obtain a vector \((h_a)_S\) by restricting \(A\) to \(S\). Thus all \((h_a)_A\) that satisfy the linear recurrence relation from \(G_\Psi\) are the extension of \((h_a)_S\) by \((14)\). Denote \(E : V_S \rightarrow V_A\) as the extension map \([V_S \ni (h_a)_S \mapsto (h_a)_A \in V_A]\), and moreover, denote \(R : V_O \rightarrow V_\Psi\) as the restriction map \([V_O \ni (c_a)_O \mapsto (c_a)_\Psi \in V_\Psi]\). The following lemma is frequently used in this paper.

**Main Lemma:** If \(\omega \in \Omega\), \(\Psi\), it holds, for \((h_a)_A \in E(V_S)\), that \(\mathcal{F}_N^{-1} h_\omega = 0\). Moreover, the composition map \(V_S \rightarrow V_\Psi\) in the following commutative diagram
\[
\begin{array}{c}
V_A \\
\xrightarrow{\mathcal{F}_N^{-1}} \\
\xrightarrow{E} \\
V_O \\
\xrightarrow{R} \\
\xrightarrow{C} \\
V_\Psi \\
\end{array}
\]
gives an isomorphism between \(V_S\) and \(V_\Psi\). The composition map is written as \(C : V_S \rightarrow V_\Psi\).

Note that, if we admit that \(V_S\) is isomorphic to \(V_O\) by \(\mathcal{F}_N^{-1}\), then the first assertion of the lemma “\(\mathcal{F}_N^{-1} h_\omega = 0 \quad \text{for} \quad \omega \notin \Psi\)” deduces the isomorphism \(V_S\) and \(V_\Psi\) since the image of \(E(V_S)\) by \(\mathcal{F}_N^{-1}\) agrees with \(V_\Psi\). We apply this lemma by putting \(\Psi\) as the set of rational points, the set of erasure-and-error locations, and the set of redundant locations of codewords.

### IV. Applications of main lemma

#### A. Affine variety codes

Let \(\Psi \subseteq \Omega\) and \(R \subseteq S_\Psi\). Consider two types \((11), (12)\) of affine variety codes \([5]\) with code length \(n := |\Psi|\), where \(\psi^\omega := \psi_1^\omega \cdots \psi_N^\omega\) is defined namely as in \((1)\). It follows from the isomorphic map \(ev : V_S \rightarrow V_\Psi\) of \((8)\) that
\[
C(R, \Psi) = ev(V_R),
\] (15)
and that \(\{\psi^\omega \mid \omega \in R\}\) is a linearly independent basis of \(C(R, \Psi)\). Since \(\sum_{\omega \in \Psi} c_\omega^\omega \psi^\omega\) in \((12)\) is the value of the inner product for \((c_\omega^\omega)\Psi\) in \(V_\Psi\), the dual code of \(C(R, \Psi)\) is equal to \(C^\perp(R, \Psi)\). Thus the dimension or the number of information symbols \(k\) of \(C^\perp(R, \Psi)\) is equal to \(n - |R|\), in other words, \(n - k = |R|\).

Consider a subspace \(V_{S \setminus R}\) of \(V_S\) with \(S := S_\Psi\) that has dimension \(n - |R|\). It follows from the isomorphic map \(C : V_S \rightarrow V_\Psi\) of the lemma that
\[
C^\perp(R, \Psi) = C(V_{S \setminus R}),
\] (16)
which is similar to \((15)\). While the definition \((12)\) of \(C^\perp(R, \Psi)\) is indirect and not constructive, the equality \((16)\) provides a direct construction and it corresponds to a non-systematic encoding of \(C^\perp(R, \Psi)\). Moreover, it is shown in the next subsection that the lemma also gives the systematic encoding for a class of such codes.

It is shown \([5]\) that \(C(R, \Psi)\) and \(C^\perp(R, \Psi)\) represent all linear codes over \(\mathbb{F}_q\) respectively. Furthermore, the decoding algorithm \([5]\) using Gröbner basis up to half the minimum distance \(|(d_{min} - 1)/2|\) is shown for \(C(R, \Psi)\). However, since this type of decoding belongs to the class of NP-complete problems \([2]\), it is strongly suggested that the algorithm in \([5]\) does not run in polynomial time.

There is another algorithm \([15]\) that decode all linear codes by \(t\)-error locating pair and solving system of linear equations. The algorithm \([15]\) can correct at least up to half the Feng–Rao minimum distance bound \(|(d_{FR} - 1)/2|\) and its computational complexity equals \(O(n^3)\), where \(n\) is the code length of the linear code and \(f(n) = O(g(n))\) means that \(|f(n)| \leq c|g(n)|\) for all \(n\) and some constant \(c > 0\).

It is also shown \([14]\) that all linear codes over \(\mathbb{F}_q\) are represented as algebraic geometric (AG) codes from algebraic curves. As for fast decoding, Sakata et al. \([16]\) showed fast algorithm for decoding up to \(|d_{FR} - 1)/2|\) applicable to AG codes of one-point type, a well-studied subclass of AG codes. Sakata et al. \([17]\) also showed that the similar algorithm to that in \([16]\) can decode erasures and errors up to \(|(d_{FR} - 1)/2|\) for one-point AG codes. O’Sullivan \([13, 14]\) generalized BMS algorithm for finding the Gröbner basis of error locator ideal of affine variety codes. However, fast decoding of affine variety codes including finding error values has been an open problem so far.
\[ C(R, \Psi) := \left\{ (c_\varphi)_\Psi \in V_\Psi \mid c_\varphi = \sum_{r \in R} h_r \varphi^r \text{ for some } (h_r)_R \in V_R \right\} \quad (11) \]

\[ C^\perp(R, \Psi) := \left\{ (c_\varphi)_\Psi \in V_\Psi \mid \sum_{\varphi \in \Psi} c_\varphi \varphi^s = 0 \text{ for all } s \in R \right\} \quad (12) \]

**B. DFT erasure-and-error decoding and systematic encoding**

Consider the encoding problem for \( C^\perp(R, \Psi) \). From the lemma, non-systematic encoding is obtained as follows. For \((h_\varphi)_S \in V_R\), let \((h_\varphi)_A \in E(V_R)\) be its extended vector. Then \((c_\varphi) := F_N^{-1}h_\varphi)_\Psi \in C^\perp(R, \Psi)\) holds since \(F_Nc_\varphi = F_NF_N^{-1}h_\varphi = h_\varphi = 0\) for all \(s \in R\). However, since error-correcting codes are usually encoded systematically, it is natural to consider the systematic encoding for \( C^\perp(R, \Psi) \), which is a certain type of erasure-only decoding as we observe in this subsection.

Let \( \Phi \subseteq \Psi \subseteq \Omega \) so that \( \Phi \) corresponds to the set of redundant positions and \( \Psi \setminus \Phi \) corresponds to the set of information positions. Then \( S_\Phi \subseteq S_\Psi \) holds since \( Z_\Phi \supseteq Z_\Psi \) and the definition \( \Phi \). From now on, consider the linear codes \( C := C^\perp(S_\Phi, \Psi) \), i.e.,

\[ C_c := \left\{ (c_\varphi)_\Psi \in V_\Psi \mid \sum_{\varphi \in \Psi} c_\varphi \varphi^s = 0 \text{ for all } s \in S_\Phi \right\}. \quad (17) \]

We choose \( \Phi \subseteq \Psi \) such that \( \Phi \neq \Psi \). Then \( k := \dim \Psi_n C = n - |\Phi| > 0 \) holds.

Suppose that erasure-and-error \((c_\varphi)_\Psi\) has occurred in a received word \((r_\varphi)_\Psi = (c_\varphi)_\Psi + (e_\varphi)_\Psi\) from the channel. Let \( \Phi_1 \subseteq \Psi \) be the set of erasure locations and let \( \Phi_2 \subseteq \Psi \) be the set of error locations; we suppose that \( \Phi_1 \) is known but \( \Phi_2 \) and \( (e_\varphi)_\Psi \) are unknown, that \( e_\varphi \neq 0 \Rightarrow \varphi \in \Phi_1 \cup \Phi_2 \), and that \( \varphi \in \Phi_2 \Rightarrow e_\varphi \neq 0 \). If \( u + 2t < d_{\text{FR}} \) with \( u := |\Phi_1| \) and \( t := |\Phi_2| \) holds, where \( d_{\text{FR}} \) is the Feng–Rao minimum distance bound \([1, 3]\), then it is known that the erasure-and-error version \([8, 17]\) of Berlekamp–Massey–Sakata (BMS) algorithm \([3, 4]\) Chapter 10) or multidimensional Berlekamp–Massey (BM) algorithm calculates the Gröbner basis \( G_{\Phi_1 \cup \Phi_2} \). Since \( \Phi_1 \) is known, \( G_{\Phi_1} \) can be calculated by the ordinary error-only version in advance and then \( G_{\Phi_1 \cup \Phi_2} \) can be calculated by the erasure-error version from the syndrome and the initial value \( G_{\Phi_1} \). By using the recurrence from \( G_{\Phi_1 \cup \Phi_2} \) and the lemma, the erasure-and-error decoding algorithm is realized as follows.

**Algorithm 1: DFT erasure-and-error decoding @**

**Input:** \((r_\varphi)_\Psi\) and \( \Phi_1 \)

**Output:** \((c_\varphi)_\Psi \in C \)

1. \((h_\varphi)_A := \left( \sum_{\varphi \in \Phi_1} v_\varphi^u \right)_A \)
2. Calculate \( G_{\Phi_1} \) from syndrome \((h_\varphi)_A \)
3. \((\tilde{r}_\varphi)_S_\phi := \left( \sum_{\varphi \in \Psi \setminus \Phi_1} v_\varphi^u \right)_S_\phi \)
4. Calculate \( G_{\Phi_1 \cup \Phi_2} \) from \((\tilde{r}_\varphi)_S_\phi \) and \( G_{\Phi_1} \)
5. \((\tilde{r}_\varphi)_A \leftarrow (\tilde{r}_\varphi)_S_\phi \) by \( G_{\Phi_1 \cup \Phi_2} \)
6. \((c_\varphi)_\Psi := \left( F_N^{-1}\tilde{r}_\varphi \right)_\Psi \)
7. \((c_\varphi)_\Psi := (r_\varphi)_\Psi - (c_\varphi)_\Psi \)

Next, we comment on the systematic encoding as erasure-only decoding. Systematic means that, for a given information \((h_\varphi)_\Psi, \varphi \in \Phi \), one finds \((c_\varphi)_\Psi \in C \) with \( c_\varphi = h_\varphi \) for all \( \varphi \in \Phi \setminus \Psi \). Since \( \Phi \) is known, systematic encoding can be viewed as an erasure-only decoding for \((c_\varphi)_\Psi := (\tilde{c}_\varphi)_\Psi \). However, since \( \varphi \neq 0 \) \( u + 2t < d_{\text{FR}} \) does not hold in general.

Nevertheless we can show that systematic encoding works as an erasure-only decoding. We calculate the Gröbner basis \( G_{\Phi} \) in advance, which has the role of generator polynomials in the case of RS codes.

**Algorithm 2: DFT systematic encoding @**

**Input:** \((h_\varphi)_\Psi \setminus \Phi \) and \( \Phi \)

**Output:** \((c_\varphi)_\Psi \in C \) with \((c_\varphi)_\Psi \setminus \Phi = (h_\varphi)_\Psi \setminus \Phi \)

1. \((\tilde{r}_\varphi)_S_\phi := \left( \sum_{\varphi \in \Psi \setminus \Phi} v_\varphi^u h_\varphi^\varphi \right)_S_\phi \)
2. \((\tilde{r}_\varphi)_A \leftarrow (\tilde{r}_\varphi)_S_\phi \) by \( G_{\Phi} \)
3. \((c_\varphi)_\Psi := \left( -F_N^{-1}\tilde{r}_\varphi \right)_\Psi \)

Thus the systematic encoding can be viewed as a special case of Algorithm 1 for \((r_\varphi)_\Psi = (h_\varphi)_\Psi \) with \( h_\varphi := 0 \) for \( \varphi \in \Phi \). Moreover, it can be obviously seen that any erasure-only \((c_\varphi)_\Psi \setminus \Phi \) can be decoded by Algorithm 1 if \( S_{\Phi_1} \subseteq S_\Phi \).

It is expected that not only erasure-only but also erasure-and-error can be often decoded beyond the erasure-and-error correcting bound \( u + 2t < d_{\text{FR}} \); in \([12]\), the improvement and the necessary and sufficient condition that succeeds in erasure-and-error decoding are obtained for Hermitian codes.

**V. ESTIMATION OF COMPLEXITY**

We estimate the number of finite-field operations, i.e., additions, subtractions, multiplications, and divisions, in Algorithm 1 for codes \([17]\) since systematic encoding algorithm can
be derived from Algorithm [1] Although there are various methods, e.g., fast Fourier transform (FFT), to reduce the manipulations in the algorithm, we mainly consider direct counting in their definitions because of conciseness. Moreover, we enumerate not the strict numbers but approximate bounds.

Summarizing the results, we evaluate the algorithm as follows, where \( n \) is code length, \( N \) is dimension of \( \Omega \), \( q \) is finite-field size, \( d \) is the number of elements in Gröbner bases.

| Algorithm | order of bound |
|-----------|----------------|
| Step 1    | \( nNq^N \)    |
| Step 2    | \( dn^2 \)     |
| Step 3    | \( n^2N \)     |
| Step 4    | \( dn^2 \)     |
| Step 5    | \( nq^N \)     |
| Step 6    | \( nNq^N \)    |
| Step 7    | \( n \)        |

Since \( N \leq d \), the total number of operations in Algorithm [1] has the order \( dn^2 + nNq^N \). In the proof [5] of \{linear codes\} = \{affine variety codes\}, \( q^N \) is chosen as \( q^N \geq n > q^{N−1} \), which leads \( qn > q^N \) and \( N \geq \log_q n > N−1 \). Thus \( dn^2 + nNq^N \) has the order of \( n^2(d + q \log n) \). Thus Algorithm [1] has the computational complexity of order \( n^{2+\varepsilon} \), where \( 0 \leq \varepsilon < 1 \) and the implied constant depends on finite-field size \( q \), which improves the order \( n^3 \) of that of Gaussian elimination for system of linear equations.

VI. CONCLUSION

In this paper, DFT and its inverse have been generalized to \( \mathbb{F}_q^N \) and their Fourier inversion formula has been shown. Moreover, a lemma for algebraic coding theory has been obtained. As applications of our lemma, the construction of affine variety codes has been described, and fast erasure-and-error decoding and systematic encoding of a class of affine variety codes have been proposed.

ACKNOWLEDGMENT

This work was supported in part by KAKENHI, Grant-in-Aid for Scientific Research (C) (23560478), and was supported in part by a research grant from Storage Research Consortium (SRC).

REFERENCES

[1] H. E. Andersen, O. Geil, “Evaluation codes from order domain theory,” Finite Fields and Their Applications, vol.14, Issue 1, pp.92–123, Jan. 2008.
[2] E. Berlekamp, R. McEliece, H. van Tilborg, “On the inherent intractability of certain coding problems,” IEEE Int. Symp. Inf. Theory, vol.24, Issue 3, pp.384–386, May 1978.
[3] M. Bras-Amorós, M. E. O’Sullivan, “The correction capability of the Berlekamp–Massey–Sakata algorithm with majority voting,” Applicable Algebra in Eng., Commun. Comput., vol.17, no.5, pp.315–335, Oct. 2006.
[4] D. A. Cox, J. Little, D. O’Shea, Using Algebraic Geometry, 2nd ed., Springer, 2005.
[5] J. Fitzgerald, R. F. Lax, “Decoding affine variety codes using Gröbner bases,” Designs, Codes and Cryptography, vol.13, no.2, pp.147–158, Feb. 1998.
[6] T. Hoholdt, “On (or in) the Blahut footprint,” Codes, Curves and Signals: Common Threads in Communications, pp.3–7, A. Vardy, Ed., Springer, 1998.
[7] H. E. Jensen, R. R. Nielsen, T. Hoholdt, “Performance analysis of a decoding algorithm for algebraic-geometry codes,” IEEE Trans. Inf. Theory, vol.45, no.5, pp.1712–1717, Jul. 1999.
[8] R. Kötter, “A fast parallel implementation of a Berlekamp–Massey algorithm for algebraic-geometric codes,” IEEE Trans. Inf. Theory, vol.44, no.4, pp.1353–1368, Jul. 1998.
[9] H. Matsui, S. Mita, “Footprint of polynomial ideal and its application to decoder for algebraic-geometric codes,” Int. Symp. on Inf. Theory and its Applications, pp.1473–1478, Parma, Italy, Oct. 10–13, 2004.
[10] H. Matsui, S. Mita, “Encoding via Gröbner bases and discrete Fourier transforms for several types of algebraic codes,” IEEE Int. Symp. Inf. Theory, pp.2656–2660, Nice, France, Jun. 24–29, 2007.
[11] H. Matsui, S. Mita, “A new encoding and decoding system of Reed–Solomon codes for HDD,” IEEE Trans. on Magnetics, vol.45, no.10, pp.3757–3760, Oct. 2009.
[12] H. Matsui, “Unified system of encoding and decoding erasures and errors for algebraic geometry codes,” Int. Symp. on Inf. Theory and its Applications, Taichung, Taiwan, pp.1001–1006, Oct. 17–20, 2010.
[13] S. Miura, “Linear codes on affine algebraic varieties,” (in Japanese) IEICE Trans. Fundamentals, vol.J81–A, no.10, pp.1386–1397, Oct. 1998.
[14] R. Pellikaan, B.-Z. Shen, G. J. M. van Wee, “Which linear codes are algebraic-geometric?” IEEE Trans. Inf. Theory, vol.37, no.3, pp.585–602, May 1991.
[15] R. Pellikaan, “On decoding by error location and dependent sets of error positions;” (A collection of contributions in honor of Jack van Lint,) Discrete Math., vol.106/107, pp.369–381, Sep. 1992.
[16] S. Sakata, H. E. Jensen, T. Høholdt, “Generalized Berlekamp–Massey decoding of algebraic geometric code up to half the Feng–Rao bound,” IEEE Trans. Inf. Theory, vol.41, no.6, Part I, pp.1762–1768, Nov. 1995.
[17] S. Sakata, D. A. Leonard, H. E. Jensen, T. Hoholdt, “Fast erasure-and-error decoding of algebraic geometry codes up to the Feng–Rao bound,” IEEE Trans. Inf. Theory, vol.44, no.4, pp.1558–1564, Jul. 1998.