EXISTENCE OF AXIALLY SYMMETRIC WEAK SOLUTIONS TO STEADY MHD WITH NONHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. We establish the existence of axially symmetric weak solutions to steady incompressible magnetohydrodynamics with nonhomogeneous boundary conditions. The key issue is the Bernoulli’s law for the total head pressure \( \Phi = \frac{1}{2}(|u|^2 + |h|^2) + p \) to a special class of solutions to the inviscid, non-resistive MHD system, where the magnetic field only contains the swirl component.

Keywords. Existence, MHD equations, axially symmetric, Bernoulli’s law.

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1. Introduction and main results

Let \( \Omega \subset \mathbb{R}^3 \) be an axially symmetric domain with \( C^2 \)-smooth boundary \( \partial \Omega = \bigcup_{j=0}^{N} \Gamma_j \) consisting of \( N + 1 \) disjoint components \( \Gamma_j \), i.e.,

\[
\Omega = \Omega_0 \setminus \left( \bigcup_{j=1}^{N} \overline{\Omega_j} \right), \quad \overline{\Omega_j} \subset \Omega_0, j = 1, \ldots, N,
\]

where \( \Gamma_j = \partial \Omega_j \). Consider the steady magnetohydrodynamics (MHD) equations in \( \Omega \):

\[
\begin{aligned}
(u \cdot \nabla)u + \nabla p &= (h \cdot \nabla)h + \Delta u + \nabla \times f, \quad \forall x \in \Omega, \\
(u \cdot \nabla)h - (h \cdot \nabla)u &= \Delta h + \nabla \times g, \quad \forall x \in \Omega, \\
\text{div } u &= \text{div } h = 0, \quad \forall x \in \Omega, \\
u &= a, \quad h = b \quad \text{on } \partial \Omega.
\end{aligned}
\]

For the existence of weak solutions to the system (1.2), the following compatibility conditions are necessary:

\[
\begin{aligned}
\sum_{j=0}^{N} F_j := \sum_{j=0}^{N} \int_{\Gamma_j} a \cdot n ds &= 0, \\
\sum_{j=0}^{N} G_j := \sum_{j=0}^{N} \int_{\Gamma_j} b \cdot n ds &= 0,
\end{aligned}
\]

where \( n \) is the outward unit vector to the boundary \( \partial \Omega \).

If the magnetic field \( h \) is absent, then the system (1.2) is reduced to the famous steady Navier–Stokes equations

\[
\begin{aligned}
(u \cdot \nabla)u + \nabla p &= \Delta u + \nabla \times f, \quad \forall x \in \Omega, \\
\text{div } u &= 0, \\
u &= a \quad \text{on } \partial \Omega.
\end{aligned}
\]

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Leray [21] made fundamental contributions to the existence theory and showed the existence of a weak solution $u \in W^{1,2}(\Omega)$ to the system (1.5) under the stronger assumptions

$$\mathcal{F}_j = \int_{\Gamma_j} \mathbf{a} \cdot n dS = 0, \quad j = 0, 1, \ldots, N.$$  \hspace{1cm} (1.6)

Leray provided two different methods for the existence results in [21]. The first one reduced the nonhomogeneous case to homogeneous case by using the solenoidal extension of boundary value $\mathbf{a}$ into $\Omega$, which was successively completed and clarified in [6, 11, 20]). The second one is based on a clever contradiction argument, which was used in [1, 2, 12, 25]. However, the problem of whether the systems (1.5) and (1.3) admit a solution or not has been open for long time and is usually referred to as Leray’s problem in the literature. For sufficiently small fluxes $\mathcal{F}_j$, one can also obtain the existence of weak solutions [2, 6, 7, 9, 10, 12, 18, 25]. The existence was also known with certain symmetric restrictions on the domain and the boundary data and the forcing term (see [1, 8, 14, 22–24]). Recently, Korobkov, Pileckas, and Russo have made an important breakthrough in a series of papers [13, 15–17] on the existence theory without any restrictions on the fluxes. First, in [13], they obtained the existence for a plane domain $\Omega$ with two connected components of the boundary assuming only the inflow condition on the external component. The new ingredients of analysis in [13] are the weak one-sided maximum principle for the total head pressure $\Phi = \frac{1}{2} |\mathbf{u}|^2 + p$ obtained by the Bernoulli’s law for weak solutions to the Euler equations and a divergence form representation of $\Phi$. The Bernoulli’s law is based on the Morse–Sard theorem developed in [3]. The spatial axially symmetric case was investigated in [15], where the existence was established without any restrictions on the fluxes, if all components $\Gamma_j$ of $\partial \Omega$ intersect the axis of symmetry.

In [16], Korobkov, Pileckas, and Russo finally established the existence of weak solutions $u \in H^1(\Omega)$ to the steady Navier–Stokes with boundary values $\mathbf{a} \in W^{3/2,2}(\partial \Omega)$ and the force $\nabla \times \mathbf{f} \in H^1(\Omega)$ in 2-D bounded domain or 3-D axially symmetric domain with $C^2$-smooth boundary, assuming only the total fluxes are zero. By the Morse–Sard theorem proved in [3], almost all level sets of the stream function $\psi$ are finite unions of $C^1$ curves. Based on the clear understanding of the level sets of $\psi$ and $\Phi$, they can construct appropriate integration domains (bounded by smooth level lines) and estimate the upper bound of the $L^2$ of $\nabla \Phi$. On the other hand, the length of each of these level lines is bounded from below, and the coarea formula implies a lower bound for the $L^1$ norm of $\nabla \Phi$, from which they can derive a contradiction. In the proof given in [16], the Bernoulli’s law for the Euler equations plays an essential role.

In this paper, we adapt their idea in [16] to the steady MHD equations. More precisely, we will establish the existence of axially symmetric weak solutions $u(x) = u_r(r,z)e_r + u_\theta(r,z)e_\theta + u_z(r,z)e_z$ and $h(x) = h_\theta(r,z)e_\theta$ to the system (1.2) with nonhomogeneous boundary values in axially symmetric domains with $C^2$ smooth boundary. We introduce some notations. Let $O_{x_1}, O_{x_2}, O_{x_3}$ be coordinate axes in $\mathbb{R}^3$ and $\theta = \arctan(x_2/x_1), r = (x_1^2 + x_2^2)^{1/2}, z = x_3$ be cylindrical coordinates. Denote by $u_\theta, v_r, v_z$ the projections of the vector $\mathbf{v}$ on the axes $\theta, r, z$. A function $f$ is said to be axially symmetric if it does not depend on $\theta$. A vector-valued function $h = (h_r, h_\theta, h_z)$ is called axially symmetric if $h_r$, $h_\theta$, and $h_z$ do not depend on $\theta$. A vector-valued function $h' = (h_r, h_\theta, h_z)$ is called axially symmetric with no swirl if $h_\theta = 0$ while $h_r$ and $h_z$ do not depend on $\theta$.

We need to use the following symmetry assumptions:
Theorem 1.1. Assume that
Remark 1.1. In the case that
We will use standard notation for Sobolev spaces: $W^{k,q}(\Omega)$, $W_0^{k,q}(\Omega)$, $W^{\alpha,q}(\partial \Omega)$, where $\alpha \in (0,1)$, $k \in \mathbb{N}_0$, $q \in [1,\infty]$. Denote by $H(\Omega)$ the subspace of all solenoidal vector fields from $W_0^{1,2}(\Omega)$ equipped with the norm $\|u\|_{H(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$. Denote by $L^q_{AS}(\Omega)$ ($L^q_{AS,SO}(\Omega)$) the space of all axially symmetric vector-valued functions (without rotation) in $L^q(\Omega)$. Similarly, define the spaces $L^q_{AS,SO}(\Omega)$, $H_{AS}(\Omega)$, $H_{ASwR}(\Omega)$, $H^q_{AS,SO}(\Omega)$, $W^{1,2}_{AS}(\Omega)$, $W^{1,2}_{ASwR}(\Omega)$, $W^{3/2,2}_{AS}(\partial \Omega)$, $W^{3/2,2}_{ASwR}(\partial \Omega)$, $W^{3/2,2}_{ASwR}(\Omega)$, etc. We denote by $H^1$ the one-dimensional Hausdorff measure, i.e., $H^1(F) = \lim_{t \to 0+} t \cdot H^1_t(F)$, where

$$H^1_t(F) = \inf \left\{ \sum_{i=1}^{\infty} \text{diam } F_i : \text{diam } F_i \leq t, F \subset \bigcup_{i=1}^{\infty} F_i \right\}.$$ 

The main result of this paper is stated as follows.

**Theorem 1.1.** Assume that $\Omega \subset \mathbb{R}^3$ is a bounded axially symmetric domain of type (1.1) with $C^2$-smooth boundary $\partial \Omega$. If $(\nabla \times f, \nabla \times g) \in H_{AS}(\Omega) \times H_{ASoS}(\Omega)$, $(a, b) \in W^{3/2,2}(\partial \Omega) \times W^{3/2,2}_{ASoS}(\partial \Omega)$ and $a$ satisfy the compatibility condition (1.3). Then the system (1.2) admits at least one weak axially symmetric solution $(u, h) \in H_{AS}(\Omega) \times H_{ASoS}(\Omega)$. Moreover, if $\nabla \times f \in H_{ASwR}(\Omega)$ and $a \in W^{3/2,2}_{ASwR}(\partial \Omega)$ are axially symmetric with no swirl, then the system (1.2) admits at least one weak axially symmetric solution with $(u, h) \in H_{ASwR}(\Omega) \times H_{ASoS}(\Omega)$.

**Remark 1.1.** In the case that $b = b_0(r, z)e_\theta$, Equation (1.4) holds automatically since $e_\theta \cdot n \equiv 0$ on $\partial \Omega$.

For the stationary MHD equations (1.2), we can define the total head pressure $\Phi = \frac{1}{2}(|u|^2 + |h|^2) + p$. Suppose $(u, h, p)$ are a smooth solution to the inviscid, non-resistive MHD system. Then we only have

$$(u \cdot \nabla)\Phi = (h \cdot \nabla)(u \cdot h). \quad (1.7)$$

So even in the two-dimensional case, the right side is not zero in general. In particular, the level sets of the stream function $\psi$ and $\Phi$ do not coincide with each other; the Bernoullis’ law is lost. However, if we further restrict ourselves to the axially symmetric MHD case with the special solution form $u(x) = u_r(r, z)e_r + u_\theta(r, z)e_\theta + u_z(r, z)e_z$ and $h(x) = h_\theta(r, z)e_\theta$, then $(h \cdot \nabla)(u \cdot h) = \frac{h_\theta}{r} \partial_\theta (u \cdot h) \equiv 0$ and the Bernoulli’s law holds

$$(u \cdot \nabla)\Phi = 0. \quad (1.8)$$

This has been observed in our previous paper [4], where we have used this to prove some Liouville-type theorems for the steady MHD equations. Here, we will adapt the methods developed in [16] to establish the existence of axially weak solutions to the system (1.2).
This paper is organized as follows. We first prepare some preliminaries to reduce the existence problem to some uniform estimates needed in Lemma 2.5 and Lemma 2.6. Then, in Section 3.1, we first run the Leray's \textit{reductio ad absurdum} argument for the steady MHD equations. The Bernoulli’s law for the inviscid, nonresistive MHD equations is obtained in Section 3.2. Finally, we adapt the methods developed in [16] to the steady MHD equation to obtain a contradiction.

2. Preliminaries

The following lemmas concern the existence of solenoidal extensions of boundary values.

**Lemma 2.1.**

(i) If \( \mathbf{a} \in W^{3/2,2}_{AS}(\partial \Omega) \) and Equation (1.3) holds, then there exists an axially symmetric solenoidal extension \( \mathbf{A} \in W^{2,2}(\Omega) \) of \( \mathbf{a} \) with the estimate

\[
\| \mathbf{A} \|_{W^{2,2}(\Omega)} \leq c \| \mathbf{a} \|_{W^{3/2,2}_{AS}(\partial \Omega)}. \tag{2.1}
\]

Moreover, if conditions \((\text{ASwR})\) are prescribed, then \( \mathbf{A} \) can be chosen to have zero swirl component.

(ii) If \( \mathbf{b} \in W^{3/2,2}_{AoS}(\partial \Omega) \), then there exists a unique vector field \( \mathbf{H} \in W^{2,2}_{AoS}(\Omega) \) such that

\[
\Delta \mathbf{H} = 0 \quad \text{in} \quad \Omega, \quad \text{div} \, \mathbf{H} = 0 \quad \text{in} \quad \Omega, \quad \mathbf{H} = \mathbf{b} \quad \text{on} \quad \partial \Omega. \tag{2.2}
\]

We also have the estimate

\[
\| \mathbf{H} \|_{W^{2,2}_{AoS}(\Omega)} \leq c \| \mathbf{b} \|_{W^{3/2,2}_{AoS}(\partial \Omega)}. \tag{2.3}
\]

**Proof.** The conclusion (i) has been proved in [15]. (ii) Let \( \mathbf{b} \in W^{3/2,2}_{AoS}(\partial \Omega) \). Then there exists a unique vector field \( \mathbf{F} \in W^{2,2}_{AoS}(\partial \Omega) \) to the Laplace equation

\[
\Delta \mathbf{F} = 0 \quad \text{in} \quad \Omega, \quad \mathbf{F} = \mathbf{b} \quad \text{on} \quad \partial \Omega. \tag{2.4}
\]

By similar arguments as in Lemma 2.2 in [15], we can choose \( \mathbf{F} \) to be axially symmetric. By the standard formulas for \( \Delta \) in cylindrical coordinate system, one has for \( \mathbf{F} = (F_r, F_\theta, F_z) \)

\[
\Delta \mathbf{F} = (\Delta_2 - \frac{1}{r^2})F_r \mathbf{e}_r + (\Delta_2 - \frac{1}{r^2})F_\theta \mathbf{e}_\theta + (\Delta_2 F_z) \mathbf{e}_z = 0, \tag{2.5}
\]

where \( \Delta_2 = (\partial_r^2 + \frac{1}{r^2} \partial_\theta + \partial_z^2) \). Take \( \mathbf{H} = F_\theta \mathbf{e}_\theta \). Then \( \mathbf{H} \in W^{2,2}_{AoS}(\Omega) \), and it follows easily from Equation (2.5) that

\[
\Delta \mathbf{H} = 0.
\]

Since \( \mathbf{b} \in W^{3/2,2}_{AoS}(\partial \Omega) \), we still have \( \mathbf{H} = \mathbf{b} \) on \( \partial \Omega \); therefore, \( \mathbf{H} = \mathbf{F} \) by uniqueness. That is, \( F_r = F_z \equiv 0 \), which implies that

\[
\text{div} \, \mathbf{H} = \text{div} \, \mathbf{F} = \partial_r F_r + \frac{1}{r} F_r + \partial_z F_z = 0.
\]

**Remark 2.1.** The statement and proof of (ii) were suggested by one of the referees. The author would like to thank them for the important improvement.
Given a function \( F \in L^q(\Omega) \) with \( q \geq 6/5 \), consider the continuous linear functional
\( H(\Omega) \ni \eta \mapsto \int_{\Omega} F \cdot \eta \, dx \). By the Riesz representation theorem, there exists a unique function \( G \in H(\Omega) \) with
\[
\int_{\Omega} F \cdot \eta \, dx = \int_{\Omega} \nabla \eta \cdot \nabla G \, dx = \langle G, \eta \rangle_{H(\Omega)} \quad \forall \eta \in H(\Omega).
\]

Put \( G = T_0 F \). Evidently, \( T_0 \) is a continuous linear operator from \( L^q(\Omega) \) to \( H(\Omega) \). The following lemmas are easily verified.

**Lemma 2.2.** The operator \( T_0 : L^{3/2}(\Omega) \to H(\Omega) \) has the following symmetry properties:
\[
\forall F \in L^{3/2}_{AS}(\Omega) \quad T_0 F \in H_{AS}(\Omega),
\]
\[
\forall F \in L^{3/2}_{ASwR}(\Omega) \quad T_0 F \in H_{ASwR}(\Omega), \tag{2.6}
\]
\[
\forall F \in L^{3/2}_{ASoS}(\Omega) \quad T_0 F \in H_{ASoS}(\Omega).
\]

**Lemma 2.3.** The following inclusions are valid:
\[
\forall u, v \in H_{AS}(\Omega) \quad (u \cdot \nabla) v \in L^{3/2}_{AS}(\Omega),
\]
\[
\forall u, v \in H_{ASwR}(\Omega) \quad (u \cdot \nabla) v \in L^{3/2}_{ASwR}(\Omega), \tag{2.7}
\]
\[
\forall u \in H_{AS}(\Omega), v \in H_{ASoS}(\Omega) \quad (u \cdot \nabla) v - (v \cdot \nabla) u \in L^{3/2}_{ASoS}(\Omega),
\]
\[
\forall u, v \in H_{ASoS}(\Omega) \quad (u \cdot \nabla) v \in L^{3/2}_{ASwR}(\Omega).
\]

Suppose \( a \in W^{3/2,2}(\partial \Omega) \) and also the conditions (1.3) and (AS) (or (ASwR)) are fulfilled. Then we can find a weak axially symmetric solution \( U \in W^{2,2}(\Omega) \) to the Stokes problem in the sense that \( U - A \in H(\Omega) \cap W^{2,2}(\Omega) \), and the following formula is satisfied by \( U \):
\[
\int_{\Omega} \nabla U \cdot \nabla \eta \, dx = \int_{\Omega} (\nabla \times f) \cdot \eta \, dx, \quad \forall \eta \in H(\Omega).
\]
Moreover,
\[
\|U\|_{W^{2,2}(\Omega)} \leq c(\|a\|_{W^{3/2,2}(\partial \Omega)} + \|\nabla \times f\|_{L^2(\Omega)}).
\]
Put \( w = u - U \) and \( k = h - H \). Then the problem (1.2) is equivalent to
\[
\begin{align*}
-\Delta w + (U \cdot \nabla) w + (w \cdot \nabla) U &= -\nabla p - (U \cdot \nabla) U, \\
+ (H \cdot \nabla) k + (k \cdot \nabla) h + (H \cdot \nabla) H, & \text{in } \Omega, \\
-\Delta k + (U \cdot \nabla) k + (w \cdot \nabla) H - (k \cdot \nabla) U - (k \cdot \nabla) w - (H \cdot \nabla) w &= 0, \\
- (U \cdot \nabla) H + (H \cdot \nabla) U + \nabla \times g, & \text{in } \Omega, \\
\text{div } w &= \text{div } k = 0, \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

By a weak solution to the problem (1.2), we understand functions \((u, h)\) such that \( w = u - U \in H(\Omega), k = h - H \in H(\Omega) \) and for any \( \eta \in H(\Omega), \zeta \in W^{1,2}_0(\Omega) \)
\[
\langle w, \eta \rangle_{H(\Omega)} = -\int_{\Omega} (U \cdot \nabla) U \cdot \eta \, dx - \int_{\Omega} (U \cdot \nabla) w \cdot \eta \, dx - \int_{\Omega} (w \cdot \nabla) w \cdot \eta \, dx
\]
The operator \( \mathbb{T} \) has the following symmetry properties:

\[
\langle k, \zeta \rangle_{H(\Omega)} = -\int_{\Omega} (U \cdot \nabla) H \cdot \zeta dx - \int_{\Omega} (U \cdot \nabla) k \cdot \zeta dx - \int_{\Omega} (w \cdot \nabla) k \cdot \zeta dx
\]

Equation (2.9) follows from Lemma 2.2 and Lemma 2.3.

By the Riesz representation theorem, for any \( \begin{pmatrix} w \\ k \end{pmatrix} \in H(\Omega) \times H(\Omega) \) there exists a unique element \( T \begin{pmatrix} w \\ k \end{pmatrix} = \begin{pmatrix} T_1 \begin{pmatrix} w \\ k \end{pmatrix} \\ T_2 \begin{pmatrix} w \\ k \end{pmatrix} \end{pmatrix} \in H(\Omega) \times H(\Omega) \) such that the right-hand sides of Equation (2.9) are equivalent to \( \langle T_1 \begin{pmatrix} w \\ k \end{pmatrix}, \eta \rangle_{H(\Omega)} \) and \( \langle T_2 \begin{pmatrix} w \\ k \end{pmatrix}, \zeta \rangle_{H(\Omega)} \) for all \( \eta \in H(\Omega), \zeta \in W_{0}^{1,2}(\Omega) \), respectively. Obviously, \( T \) is a nonlinear operator from \( H(\Omega) \times H(\Omega) \) to \( H(\Omega) \times H(\Omega) \).

**Lemma 2.4.** The operator \( \mathbb{T} : H(\Omega) \times H(\Omega) \to H(\Omega) \times H(\Omega) \) is compact. Moreover, \( \mathbb{T} \) has the following symmetry properties:

\[
\forall \begin{pmatrix} w \\ k \end{pmatrix} \in H_{AS}(\Omega) \times H_{AS_{0}}(\Omega), \quad T_1 \begin{pmatrix} w \\ k \end{pmatrix} \in H_{AS}(\Omega),
\]

\[
\forall \begin{pmatrix} w \\ k \end{pmatrix} \in H_{AS_{wR}}(\Omega) \times H_{AS_{0}}(\Omega), \quad T_1 \begin{pmatrix} w \\ k \end{pmatrix} \in H_{AS_{wR}}(\Omega),
\]

\[
\forall \begin{pmatrix} w \\ k \end{pmatrix} \in H_{AS}(\Omega) \times H_{AS_{0}}(\Omega), \quad T_2 \begin{pmatrix} w \\ k \end{pmatrix} \in H_{AS_{0}}(\Omega).
\]

**Proof.** The compactness can be proved in a standard way as shown in [20], and Equation (2.10) follows from Lemma 2.2 and Lemma 2.3. \( \square \)

Hence, Equation (2.9) is equivalent to the operator equation

\[
\begin{pmatrix} w \\ k \end{pmatrix} = \mathbb{T} \begin{pmatrix} w \\ k \end{pmatrix}
\]

in the space \( H(\Omega) \times H(\Omega) \). Thus, we can apply the Leray–Schauder fixed point theorem to the compact operators \( \mathbb{T}|_{H_{AS}(\Omega) \times H_{AS_{0}}(\Omega)} \) and \( \mathbb{T}|_{H_{AS_{wR}}(\Omega) \times H_{AS_{0}}(\Omega)} \). Then the following statements hold.

**Lemma 2.5.** Let conditions \( \text{(AS)} \) and \( \text{(AS}_{0}\text{S}) \) and Equations (1.3)–(1.4) be fulfilled. Suppose all possible solutions \( \begin{pmatrix} w \\ k \end{pmatrix} \) to the equation \( \begin{pmatrix} w \\ k \end{pmatrix} = \lambda \mathbb{T} \begin{pmatrix} w \\ k \end{pmatrix} \) with \( \lambda \in [0,1] \) are uniformly bounded in \( H(\Omega) \times H(\Omega) \). Then problem (1.2) admits at least one weak axially symmetric solution \( (u,h) \in H_{AS}(\Omega) \times H_{AS_{0}}(\Omega) \).

**Lemma 2.6.** Let conditions \( \text{(AS}_{w\text{R}}\text{)} \) and \( \text{(AS}_{0}\text{S}) \) and Equations (1.3)–(1.4) be fulfilled. Suppose all possible solutions \( \begin{pmatrix} w \\ k \end{pmatrix} \) to the equation \( \begin{pmatrix} w \\ k \end{pmatrix} = \lambda \mathbb{T} \begin{pmatrix} w \\ k \end{pmatrix} \) with
$\lambda \in [0,1]$ are uniformly bounded in $H(\Omega) \times H(\Omega)$. Then problem (1.2) admits at least one weak axially symmetric solution $(u,h) \in H_{AWR}(\Omega) \times H_{ASoS}(\Omega)$.

3. Proof of Theorem 1.1

3.1. The reductio ad absurdum argument by Leray. We apply the reductio ad absurdum argument of Leray [21] to the stationary MHD equations. To prove the existence of a weak solution to the MHD system (1.2), by Lemma 2.5, and Lemma 2.6 it is sufficient to show that the weak solutions $(w,k)$ satisfying for any $(\eta,\zeta) \in H(\Omega) \times W^{1,2}_0(\Omega)$

$$
\langle w, \eta \rangle_{H(\Omega)} = -\lambda \int_\Omega (U \cdot \nabla) U \cdot \eta dx - \lambda \int_\Omega (U \cdot \nabla) w \cdot \eta dx
$$

$$
\langle k, \zeta \rangle_{H(\Omega)} = -\lambda \int_\Omega (U \cdot \nabla) H \cdot \zeta dx - \lambda \int_\Omega (U \cdot \nabla) k \cdot \zeta dx
$$

are uniformly bounded in $H(\Omega) \times H(\Omega)$ with respect to $\lambda \in [0,1]$. Assume that this is false. Then there exist sequences $\{\lambda_n\}_{n \in \mathbb{N}} \subset [0,1]$ and $\{\tilde{w}_n, \tilde{k}_n\}_{n \in \mathbb{N}} \in H(\Omega) \times H(\Omega)$ such that, for any $(\eta, \zeta) \in H(\Omega) \times W^{1,2}_0(\Omega)$,

$$
\int_\Omega \nabla \tilde{w}_n \cdot \nabla \eta dx - \lambda_n \int_\Omega ((\tilde{w}_n + U) \cdot \nabla) \eta \cdot \tilde{w}_n dx - \lambda_n \int_\Omega (\tilde{w}_n \cdot \nabla) \eta \cdot U dx
$$

$$
+ \lambda_n \int_\Omega ((\tilde{k}_n + H) \cdot \nabla) \eta \cdot \tilde{k}_n dx + \lambda_n \int_\Omega (\tilde{k}_n \cdot \nabla) \eta \cdot H dx
$$

$$
= \lambda_n \int_\Omega (U \cdot \nabla) \eta \cdot U dx - \lambda_n \int_\Omega (H \cdot \nabla) \eta \cdot H dx,
$$

$$
\int_\Omega \nabla \tilde{k}_n \cdot \nabla \zeta dx - \lambda_n \int_\Omega ((\tilde{w}_n + U) \cdot \nabla) \zeta \cdot \tilde{k}_n dx - \lambda_n \int_\Omega (\tilde{w}_n \cdot \nabla) \zeta \cdot H dx
$$

$$
+ \lambda_n \int_\Omega ((\tilde{k}_n + H) \cdot \nabla) \zeta \cdot \tilde{w}_n dx + \lambda_n \int_\Omega (\tilde{k}_n \cdot \nabla) \zeta \cdot U dx
$$

$$
= \lambda_n \int_\Omega ((U \cdot \nabla) \zeta) \cdot H dx - \lambda_n \int_\Omega (H \cdot \nabla) \zeta \cdot U dx - \lambda_n \int_\Omega \nabla H \cdot \nabla \zeta dx + \lambda_n \int_\Omega (\nabla \times g) \cdot \zeta dx
$$

(3.3)

and

$$
\lim_{n \to \infty} \lambda_n = \lambda_0 \in [0,1], \quad \lim_{n \to \infty} J_n^2 = \lim_{n \to \infty} (\|\tilde{w}_n\|_{H(\Omega)}^2 + \|\tilde{k}_n\|_{H(\Omega)}^2) = \infty.
$$

(3.4)

Denote $w_n = J^{-1}_n \tilde{w}_n, k_n = J^{-1}_n \tilde{k}_n$. Since $\|w_n\|_{H(\Omega)}^2 + \|k_n\|_{H(\Omega)}^2 = 1$, there exists a subsequence $\{w_{n_k}, k_{n_k}\}$ converging weakly in $H(\Omega)$ to vector fields $w, k \in H(\Omega)$. Because
of the compact embedding
\[ H(\Omega) \hookrightarrow L^r(\Omega) \quad \forall r \in [1,6), \]
the subsequence \( \{w_n,k_n\} \) converges strongly in \( L^r(\Omega) \). Replacing \( \zeta \) in Equation (3.3) by \( J^{-1}_n\zeta \) and letting \( n \to \infty \), we obtain
\[ \lambda_0 \int_{\Omega} [(w \cdot \nabla)k - (k \cdot \nabla)w] \cdot \zeta dx = 0. \quad (3.5) \]

Taking \( \eta = J^{-1}_n\tilde{w}_n, \zeta = J^{-1}_n\tilde{k}_n \) in Equations (3.2)–(3.3) and adding the above two identities, we get
\[
\int_{\Omega} |\nabla w_n|^2 + |\nabla k_n|^2 dx \\
= \lambda_n \int_{\Omega} [(w_n \cdot \nabla)w_n - (k_n \cdot \nabla)k_n] \cdot U dx - \lambda_n \int_{\Omega} [(w_n \cdot \nabla)k_n - (k_n \cdot \nabla)w_n] \cdot H dx \\
+ J^{-1}_n \lambda_n \int_{\Omega} [(U \cdot \nabla)w_n - (H \cdot \nabla)k_n] \cdot U dx + (U \cdot \nabla)k_n \cdot H - (H \cdot \nabla)k_n \cdot U] dx \\
- J^{-1}_n \lambda_n \int_{\Omega} [(\nabla \times g) \cdot k_n + \nabla H \cdot \nabla k_n] dx. \quad (3.6)
\]

Therefore, passing to a limit as \( n_l \to \infty \) in Equation (3.6) and using Equation (3.5), we obtain
\[ 1 = \lambda_0 \int_{\Omega} [(w \cdot \nabla)w - (k \cdot \nabla)k] \cdot U dx. \quad (3.7) \]

This implies \( \lambda_0 \in (0,1] \). Let us return to the integral identity (3.2). Consider the functional
\[
R_n(\eta) = \int_{\Omega} \nabla \tilde{w}_n \cdot \nabla \eta dx - \lambda_n \int_{\Omega} (\tilde{w}_n + U) \cdot \nabla \eta \cdot \tilde{w}_n dx - \lambda_n \int_{\Omega} (\tilde{w}_n \cdot \nabla)\eta \cdot U dx \\
+ \lambda_n \int_{\Omega} (\tilde{k}_n + H) \cdot \nabla \eta \cdot \tilde{k}_n dx + \lambda_n \int_{\Omega} (\tilde{k}_n \cdot \nabla)\eta \cdot H dx - \lambda_n \int_{\Omega} (U \cdot \nabla)\eta \cdot U dx \\
+ \lambda_n \int_{\Omega} (H \cdot \nabla)\eta \cdot H dx \quad \forall \eta \in W^{1,2}_0(\Omega).
\]

Obviously, \( R_k(\eta) \) is a linear functional and
\[
|R_n(\eta)| \leq c(\|\tilde{w}_n, \tilde{k}_n\|_{H(\Omega)} + \|\tilde{w}_n, \tilde{k}_n\|^2_{H(\Omega)} + \|a, b\|^2_{W^{3/2,2}(\partial \Omega)} + \|f\|^2_{W^{1,2}_0(\Omega)}) \|\eta\|_{H(\Omega)}
\]
with constant \( c \) independent of \( n \). It follows from Equation (3.2) that
\[ R_n(\eta) = 0 \quad \forall \eta \in H(\Omega). \]

Therefore, there exists an axially symmetric function \( \tilde{p}_n \in L^2(\Omega) = \{q \in L^2(\Omega): \int_{\Omega} q(x) dx = 0\} \) such that
\[ R_n(\eta) = \int_{\Omega} \tilde{p}_n \text{div} \eta dx \quad \forall \eta \in W^{1,2}_0(\Omega) \]
\[
\parallel \hat{\rho}_n \parallel_{L^2(\Omega)} \leq c(\parallel (\hat{\omega}_n, \hat{k}_n) \parallel_{H(\Omega)} + \parallel (\hat{\omega}_n, \hat{k}_n) \parallel^2_{H(\Omega)} + \parallel (a, b) \parallel^2_{W^{3/2,2}(\partial \Omega)} + \parallel f \parallel^2_{W^{1,2}_0(\Omega)}).
\]

(3.8)

The pair \((\hat{\omega}_n, \hat{k}_n, \hat{\rho}_n)\) satisfies the integral identity

\[
\int_{\Omega} \nabla \hat{\omega}_n \cdot \nabla \eta dx - \lambda_n \int_{\Omega} ((\hat{\omega}_n + U) \cdot \nabla) \eta \cdot \hat{\omega}_n dx - \lambda_n \int_{\Omega} (\hat{\omega}_n \cdot \nabla) \eta \cdot U dx \\
+ \lambda_n \int_{\Omega} (\hat{\omega}_n \cdot H) \nabla \eta \cdot \hat{k}_n dx + \lambda_n \int_{\Omega} (\hat{k}_n \cdot \nabla) \eta \cdot H dx + \lambda_n \int_{\Omega} (\hat{\omega}_n \cdot \nabla) \eta \cdot H dx \\
- \lambda_n \int_{\Omega} (U \cdot \nabla) \eta \cdot U dx = \int_{\Omega} \hat{\rho}_n \text{div} \ \eta dx, \ \forall \eta \in W^{1,2}_0(\Omega).
\]

(3.9)

Let \(\hat{u}_n = \hat{\omega}_n + U, \hat{h}_n = \hat{k}_n + H\). Then identity (3.9) reduces to

\[
\int_{\Omega} \nabla \hat{u}_n \cdot \nabla \eta dx - \int_{\Omega} \hat{\rho}_n \text{div} \ \eta dx = -\lambda_n \int_{\Omega} (\hat{u}_n \cdot \nabla) \hat{u}_n \cdot \eta dx \\
+ \lambda_n \int_{\Omega} (\hat{h}_n \cdot \nabla) \hat{h}_n \cdot \eta dx + \lambda_n \int_{\Omega} (\nabla \times f) \cdot \eta dx, \ \forall \eta \in W^{1,2}_0(\Omega).
\]

Thus \((\hat{u}_n, \hat{h}_n, \hat{\rho}_n)\) might be considered as a weak solution to the Stokes problem

\[
\begin{aligned}
-\Delta \hat{\omega}_n + \nabla \hat{\rho}_n &= -\lambda_n (\hat{u}_n \cdot \nabla) \hat{u}_n + \lambda_n (\hat{h}_n \cdot \nabla) \hat{h}_n + \lambda_n \nabla \times f := F_n \quad \text{in} \ \Omega, \\
-\Delta \hat{h}_n &= -\lambda_n (\hat{u}_n \cdot \nabla) \hat{h}_n + \lambda_n (\hat{h}_n \cdot \nabla) \hat{u}_n + \nabla \times g := H_n \quad \text{in} \ \Omega, \\
\text{div} \ \hat{u}_n &= \text{div} \ \hat{h}_n = 0 \quad \text{in} \ \Omega, \\
\hat{u}_n &= a, \hat{h}_n = b \quad \text{on} \ \partial \Omega.
\end{aligned}
\]

Obviously, \(F_n, H_n \in L^{3/2}(\Omega)\) and

\[
\parallel F_n \parallel_{L^{3/2}(\Omega)} \leq c(\parallel (\hat{u}_n, \nabla) \hat{u}_n \parallel_{L^{3/2}(\Omega)} + \parallel (\hat{h}_n, \nabla) \hat{h}_n \parallel_{L^{3/2}(\Omega)} + \parallel \nabla \times f \parallel_{L^{3/2}(\Omega)}
\]

\[
\leq c(\parallel \hat{u}_n \parallel_{L^6(\Omega)} \parallel \nabla \hat{u}_n \parallel_{L^2(\Omega)} + \parallel \hat{h}_n \parallel_{L^6(\Omega)} \parallel \nabla \hat{h}_n \parallel_{L^2(\Omega)} + \parallel f \parallel_{W^{1/2,2}_0(\Omega)})
\]

\[
\leq c(\parallel \hat{u}_n \parallel_{H(\Omega)}^2 + \parallel \hat{h}_n \parallel_{H(\Omega)}^2 + \parallel a \parallel_{W^{1/2,2}(\partial \Omega)}^2 + \parallel b \parallel_{W^{1/2,2}(\partial \Omega)}^2 + \parallel f \parallel_{W^{1/2,2}(\Omega)}^2),
\]

\[
\parallel H_n \parallel_{L^{3/2}(\Omega)} \leq c(\parallel (\hat{u}_n, \nabla) \hat{h}_n \parallel_{L^{3/2}(\Omega)} + \parallel (\hat{h}_n, \nabla) \hat{u}_n \parallel_{L^{3/2}(\Omega)} + \parallel \nabla \times g \parallel_{L^{3/2}(\Omega)}
\]

\[
\leq c(\parallel \hat{u}_n \parallel_{H(\Omega)}^2 + \parallel \hat{h}_n \parallel_{H(\Omega)}^2 + \parallel a \parallel_{W^{1/2,2}(\partial \Omega)}^2 + \parallel b \parallel_{W^{1/2,2}(\partial \Omega)}^2 + \parallel g \parallel_{W^{1/2,2}_0(\Omega)}),
\]

where \(c\) is independent of \(n\). By the well-known regularity results for the Stokes system (see [10, Theorem IV.6.1]), we have \(\hat{u}_n, \hat{h}_n \in W^{2,3/2}(\Omega), \hat{\rho}_n \in W^{1,3/2}(\Omega)\), and also the estimate

\[
\parallel \hat{u}_n \parallel_{W^{2,3/2}(\Omega)} + \parallel \hat{\rho}_n \parallel_{W^{1,3/2}(\Omega)} \leq c(\parallel F_n \parallel_{L^{3/2}(\Omega)} + \parallel a \parallel_{W^{3/2,2}(\partial \Omega)})
\]

\[
\leq c(\parallel \hat{u}_n \parallel_{H(\Omega)}^2 + \parallel \hat{h}_n \parallel_{H(\Omega)}^2 + \parallel a, b \parallel_{W^{3/2,2}(\partial \Omega)}^2 + \parallel f \parallel_{W^{1/2,2}(\Omega)}),
\]

(3.10)

\[
\parallel \hat{h}_n \parallel_{W^{2,3/2}(\Omega)} \leq c(\parallel H_n \parallel_{L^{3/2}(\Omega)} + \parallel b \parallel_{W^{3/2,2}(\partial \Omega)})
\]

\[
\leq c(\parallel \hat{u}_n \parallel_{H(\Omega)}^2 + \parallel \hat{h}_n \parallel_{H(\Omega)}^2 + \parallel (a, b) \parallel_{W^{3/2,2}(\partial \Omega)}^2 + \parallel (a, b) \parallel_{W^{3/2,2}(\Omega)} + \parallel g \parallel_{W^{1,2}_0(\Omega)}),
\]

(3.11)
Denote $u_n = J^{-1}_n \tilde{u}_n$, $h_n = J^{-1}_n \tilde{h}_n$ and $p_n = \lambda^{-1}_n J^{-2}_n \tilde{p}_n$. Then

$$
\begin{align*}
-\nu_n \Delta u_n + (u_n \cdot \nabla) u_n + \nabla p_n &= (h_n \cdot \nabla) h_n + \nabla \times f_n, & \text{in } \Omega, \\
-\nu_n \Delta h_n + (u_n \cdot \nabla) h_n - (h_n \cdot \nabla) u_n &= \nabla \times g_n, & \text{in } \Omega, \\
\text{div } u_n = \text{div } h_n &= 0, & \text{in } \Omega, \\
u_n = \lambda^{-1}_n J^{-1}_n, f_n = J^{-2}_n f, g_n = J^{-2}_n g \text{ and } a_n = J^{-1}_n a, b_n = J^{-1}_b.
\end{align*}
$$

(3.12)

where $\nu_n = \lambda^{-1}_n J^{-1}_n, f_n = J^{-2}_n f, g_n = J^{-2}_n g$ and $a_n = J^{-1}_n a, b_n = J^{-1}_b$.

It follows from Equation (3.10) that

$$
\|p_n\|_{W^{1,3/2}(\Omega)} \leq \text{const.}
$$

Hence, from the sequence $\{p_n\}$, we can extract a subsequence, still denoted by $\{p_n\}$, which converges weakly in $W^{1,3/2}(\Omega)$ to some function $p \in W^{1,3/2}(\Omega)$. Let $\phi \in C_0^\infty(\Omega)$.

Taking $\eta = J^{-2}_n \phi$ in Equation (3.9) and letting $n \to \infty$, we get

$$
-\lambda_0 \int_\Omega (w \cdot \nabla) \phi \cdot w dx + \lambda_0 \int_\Omega (k \cdot \nabla) \phi \cdot k dx = \lambda_0 \int_\Omega p \text{div } \phi dx \quad \forall \phi \in C_0^\infty(\Omega).
$$

Integrating by parts in the last equality, we derive

$$
\lambda_0 \int_\Omega [(w \cdot \nabla) w - (k \cdot \nabla) k] \cdot \phi dx = -\lambda_0 \int_\Omega \nabla p \cdot \phi dx \quad \forall \phi \in C_0^\infty(\Omega). \quad (3.13)
$$

Hence, the pair $(w, k, p)$ satisfies, for almost all $x \in \Omega$, the inviscid, nonresistive MHD equations

$$
\begin{align*}
(w \cdot \nabla) w + \nabla p &= (k \cdot \nabla) k, & \text{in } \Omega, \\
(w \cdot \nabla) k - (k \cdot \nabla) w &= 0, & \text{in } \Omega, \\
\text{div } w &= \text{div } k = 0, & \text{in } \Omega, \\
w &= k = 0, & \text{on } \partial \Omega.
\end{align*}
$$

(3.14)

We summarize the above results as follows.

**Lemma 3.1.** Assume that $\Omega \subset \mathbb{R}^3$ is a bounded axially symmetric domain of type (1.1) with $C^2$-smooth boundary $\partial \Omega$, $(\nabla \times f, \nabla \times g) \in W^{1,2}_{AS}(\Omega) \times W^{1,2}_{ASoS}(\Omega)$, $(a, b) \in W^{3/2,2}_{AS} \times W^{3/2,2}_{ASoS}(\partial \Omega)$ are axially symmetric, and $a$ and $b$ satisfy conditions (1.3)–(1.4). If the assertion of Theorem 1.1 is false, then there exist $w, k, p$ with the following properties:

**IMHD-AX** The axially symmetric functions $(w, k) \in H_{AS}(\Omega) \times H_{ASoS}(\Omega)$, $p \in W^{1,3/2}_{AS}(\Omega)$ satisfy the inviscid, nonresistive MHD system (3.14) and Equation (3.7).

**MHD-AX** There exist a sequence of axially symmetric functions $u_n \in W^{1,2}_{AS}(\Omega), h_n \in W^{1,2}_{ASoS}(\Omega), p_n \in W^{1,3/2}_{AS}(\Omega)$ and numbers $\nu_n \to 0+, \lambda_n \to \lambda_0 \in (0, 1]$ such that the norms $\|u_n\|_{W^{1,2}(\Omega)} + \|h_n\|_{W^{1,2}(\Omega)}$ and $\|p_n\|_{W^{1,3/2}(\Omega)}$ are uniformly bounded, the pair $(u_n, h_n, p_n)$ satisfies Equation (3.12), and

$$
\begin{align*}
\|\nabla u_n\|_{L^2(\Omega)} + \|\nabla h_n\|_{L^2(\Omega)} &\to 1, \\
u_n \to w, h_n \to k &\text{ in } W^{1,2}(\Omega), \quad p_n \to p \quad \text{in } W^{1,3/2}(\Omega).
\end{align*}
$$

Moreover, $(u_n, h_n) \in W^{3,2}_{loc}(\Omega)$ and $p_n \in W^{2,2}_{loc}(\Omega)$.
Assume that

\[ \Gamma_j \cap O_{x_3} \neq \emptyset, \quad j = 0, \ldots, M', \]

\[ \Gamma_j \cap O_{x_3} = \emptyset, \quad j = M' + 1, \ldots, N. \]

Let \( P_+ = \{(0, x_2, x_3) : x_2 > 0, x_3 \in \mathbb{R}\}, \; D = \Omega \cap P_+ \). Obviously, on \( P_+ \), the coordinates \( x_2, x_3 \) coincide with the coordinates \( r, z \). For a set \( A \subseteq \mathbb{R}^3 \), put \( \bar{A} := A \cap P_+ \), and for \( B \subseteq P_+ \) denote by \( \bar{B} \) the set in \( \mathbb{R}^3 \) obtained by rotation of \( B \) around the \( O_z \)-axis. Then \( (S_1) \; D \) is a bounded plane domain with Lipschitz boundary. Moreover, \( \bar{\Gamma}_j \) is a connected set for every \( j = 0, \ldots, N \). In other words, \( \{\bar{\Gamma}_j : j = 0, \ldots, N\} \) coincides with the family of all connected components of the set \( P_+ \cap \partial D \).

Hence \( w, k, \) and \( p \) satisfy the following system in the plan domain \( D \):

\[
\begin{cases}
  w_r \partial_r w_r + w_z \partial_z w_r - \frac{w_0^2}{r} + \partial_r p = -\frac{k_0^2}{r}, \\
  w_r \partial_r w_\theta + w_z \partial_z w_\theta + \frac{w_0 w_\theta}{r} = 0, \\
  w_r \partial_r w_z + w_z \partial_z w_z + \partial_z p = 0, \\
  w_r \partial_r k_\theta + w_z \partial_z k_\theta - \frac{w_0 k_\theta}{r} = 0, \\
  \partial_r (rw_r) + \partial_z (rw_z) = 0.
\end{cases}
\] (3.16)

These equations are satisfied for almost all \( x \in D \) and

\[ w(x) = k(x) = 0 \quad \text{for } \mathcal{H}^1 \text{-almost all } x \in P_+ \cap \partial D. \] (3.17)

We have the following integral estimates: \( w, k \in W^{1,2}_{\text{loc}}(D) \),

\[ \int_D (|\nabla w(r,z)|^2 + |\nabla k(r,z)|^2) r \, dr \, dz < \infty, \] (3.18)

and, by the Sobolev embedding theorem for three-dimensional domains, \( w, k \in L^6(\Omega) \), i.e.,

\[ \int_D (|w(r,z)|^6 + |k(r,z)|^6) r \, dr \, dz < \infty. \] (3.19)

Also, the condition \( \nabla p \in L^{3/2}(\Omega) \) can be written as

\[ \int_D |\nabla p(r,z)|^{3/2} r \, dr \, dz < \infty. \] (3.20)

Denote by \( \Phi = p + \frac{|w|^2}{2} + \frac{|k|^2}{2} \) the total head pressure corresponding to the solution \( (w, k, p) \). Obviously,

\[ \int_D r |\nabla \Phi(r,z)|^{3/2} dr dz < \infty. \] (3.21)

Hence,

\[ \Phi \in W^{1,3/2}(D) \quad \forall \epsilon > 0. \] (3.22)

We also have the important **Bernoulli’s law**: for almost all \( x \in D \),

\[ (w_r \partial_r + w_z \partial_z) \Phi = 0. \] (3.23)
3.2. Some results on inviscid MHD equations. Since \( w \) and \( k \) satisfy Equation (3.14), \( w = k \equiv 0 \) on \( \partial \Omega \), and \( \nabla p \in L^{3/2}(\Omega) \): then one can follow [1] and [12] to prove the following statement.

**Lemma 3.2.** If \((\text{IMHD-AX})\) are satisfied, then

\[
\forall j \in \{0,1,\ldots,N\} \; \exists p_j \in \mathbb{R} : \; p(x) \equiv p_j \quad \text{for } \mathcal{H}^2\text{-almost all } x \in \Gamma_j. \tag{3.24}
\]

In particular, by axial symmetry,

\[
p(x) \equiv p_j \quad \text{for } \mathcal{H}^1\text{-almost all } x \in \tilde{\Gamma}_j. \tag{3.25}
\]

We need a weak version of Bernoulli’s law for a Sobolev solution \((w,k,p)\) to the inviscid MHD Equations (3.16).

From the last equality in Equations (3.16) and from Equation (3.18), it follows that there exists a stream function \( \psi \in W^{2,2}_{\text{loc}}(D) \) such that

\[
\frac{\partial \psi}{\partial r} = -rw_z, \quad \frac{\partial \psi}{\partial z} = rw_r. \tag{3.26}
\]

Fix a point \( x_* \in D \). For \( \epsilon > 0 \), denote by \( D_\epsilon \) the connected component of \( D \cap \{(r,z): r > \epsilon\} \) containing \( x_* \). Since \( \psi \in W^{2,2}_{\text{loc}}(D_\epsilon) \) for \( \epsilon > 0 \), (3.27) by the Sobolev embedding theorem, \( \psi \in C(D_\epsilon) \). Hence \( \psi \) is continuous at points of \( D \setminus O_z = D \setminus \{(0,z): z \in \mathbb{R}\} \). By the definition of \( \psi \) and since \( w = k \equiv 0 \) on \( \partial \Omega \), we see that all the boundary components are level sets of \( \psi \).

**Lemma 3.3.** If \((\text{IMHD-AX})\) are satisfied, then there exist constants \( \xi_0,\ldots,\xi_N \in \mathbb{R} \) such that

\[
\psi(x) \equiv \xi_j \text{ on each curve } \tilde{\Gamma}_j, \; j = 0,\ldots,N.
\]

**Proof.** By virtue of Equations (3.17) and (3.26), we have \( \nabla \psi(x) = 0 \) for \( \mathcal{H}^1\)-almost all \( x \in \partial D \setminus O_z \). Then the Morse–Sard property (see [3]) implies that

\[
\text{for any connected set } C \subset \partial D \setminus O_z, \; \exists \alpha = \alpha(C) \in \mathbb{R} : \psi(x) \equiv \alpha \quad \forall x \in C.
\]

Hence since \( \tilde{\Gamma}_j \) are connected, the lemma follows. \( \square \)

By the properties of Sobolev functions \( w,k,\psi,\Phi \) (see [5]), we get the following.

**Lemma 3.4.** If conditions \((\text{IMHD-AX})\) hold, then there exists a set \( A_w \subset D \) such that

(i) \( \mathcal{H}^1(A_w) = 0 \);

(ii) for all \( x = (r,z) \in D \setminus A_w \),

\[
\lim_{\rho \to 0} \int_{B_{\rho}(x)} |w(y) - w(x)|^2 dy = \lim_{\rho \to 0} \int_{B_{\rho}(x)} |k(y) - k(x)|^2 dy = \lim_{\rho \to 0} \int_{B_{\rho}(x)} |\Phi(y) - \Phi(x)|^2 dy = 0;
\]

moreover, the function \( \psi \) is differentiable at \( x \), and

\[
\nabla \psi(x) = (-rw_z(x),rw_r(x));\text{ and}
\]

(iii) for every \( \epsilon > 0 \), there exists a set \( U \subset \mathbb{R}^2 \) with \( \mathcal{H}^1_\infty(U) < \epsilon, A_w \subset U \) and such that the functions \( w,k,\Phi \) are continuous on \( D \setminus (U \cup O_z) \).
Then one can mimic the proof in [15] to establish the following weak version of Bernoulli’s law.

**Lemma 3.5 (Bernoulli’s Law).** Let conditions (IMHD-AX) be valid, and let $A_w$ be a set from Lemma 3.4. For any compact connected set $K \subset \overline{D} \setminus O_z$, the following property holds: if

$$\psi|_{K} = \text{const},$$

then

$$\Phi(x_1) = \Phi(x_2) \quad \text{for all } x_1, x_2 \in K \setminus A_w.$$ (3.29)

In particular, we can denote by $\Phi(K)$ the uniform constant $c \in \mathbb{R}$ such that $\Phi(x) = c$ for all $x \in K \setminus A_w$ for any compact set $K \subset \overline{D} \setminus O_z$ with $\psi|_{K} = \text{const}$. Moreover, $\Phi$ has some continuity properties when $K$ approaches the singularity axis $O_z$.

**Lemma 3.6.** Assume that conditions (IMHD-AX) are satisfied. Let $K_i$ be a sequence of compact sets with the following properties: $K_i \subset \overline{D} \setminus O_z, \psi|_{K_i} = \text{const}$, and

$$\lim_{i \to \infty} \inf_{(r,z) \in K_i} r = 0, \liminf_{i \to \infty} \sup_{(r,z) \in K_i} r > 0.$$ Then $\Phi(K_i) \to p_0$ as $i \to \infty$.

**Lemma 3.7.** If conditions (IMHD-AX) are satisfied, then $p_0 = \cdots = p_{M'}$, where $p_j$ are the constants from Lemma 3.2.

Heuristically, one can imagine that the axis $Oz$ is an “almost” streamline. By Lemma 3.5, all the boundary components that intersects with the symmetry axis should share the same total head pressure $\Phi$, which immediately implies Lemma 3.7. Since the proof of lemmas 3.2–3.7 are quite similar to the proofs in [15], we omit the details.

### 3.3. Obtaining a contradiction

We consider three possible cases.

(a) The maximum of $\Phi$ is attained on the boundary component intersecting the symmetry axis:

$$p_0 = \max_{j=0, \ldots, N} p_j = \sup_{x \in \Omega} \Phi(x).$$ (3.30)

(b) The maximum of $\Phi$ is attained on a boundary component that does not intersect the symmetry axis:

$$p_0 < p_N = \max_{j=0, \ldots, N} p_j = \sup_{x \in \Omega} \Phi(x).$$ (3.31)

(c) The maximum of $\Phi$ is not attained on $\partial \Omega$:

$$\max_{j=0, \ldots, N} p_j < \sup_{x \in \Omega} \Phi(x).$$ (3.32)

#### 3.3.1. The case $\sup_{x \in \Omega} \Phi(x) = p_0$. Adding a constant to the pressure $p$, we can assume that

$$p_0 = \sup_{x \in \Omega} \Phi(x) = 0.$$ (3.33)

Since the identity $p_0 = p_1 = \cdots = p_N$ is impossible, we have that $p_j < 0$ for some $j \in \{M'+1, N\}$. Recall that, by Lemma 3.7, $p_0 = p_1 = \cdots = p_{M'} = 0$. From Equation (3.14), we obtain

$$0 = x \cdot \nabla p(x) + x \cdot (w(x) \cdot \nabla)w(x) - x \cdot (k(x) \cdot \nabla)k(x)$$
= \text{div} [xp(x) + (w(x) \cdot x)w(x) - (k(x) \cdot x)k(x)] - p(x)\text{div} x - |w(x)|^2 + |k(x)|^2

= \text{div} [xp(x) + (w(x) \cdot x)w(x) - (k(x) \cdot x)k(x)] - 3\Phi(x) + \frac{1}{2}|w(x)|^2 + \frac{5}{2}|k(x)|^2. \quad (3.34)

Integrating it over \partial\Omega and using Equation (3.33), we derive a contradiction as follows:

\begin{align*}
0 \geq \int_{\Omega} [3\Phi(x) - \frac{1}{2}|w(x)|^2 - \frac{5}{2}|k(x)|^2]dx &= \int_{\partial\Omega} p(x)(x \cdot n)ds = \sum_{j=0}^{N} p_j \int_{\Gamma_j} (x \cdot n)ds \\
&= \sum_{j=1}^{N} p_j \int_{\Omega_j} \text{div} x dx = -3 \sum_{j=1}^{N} p_j |\Omega_j| > 0.
\end{align*}

Hence, we exclude the first case.

**3.3.2. The case:** \( p_0 < p_N = \sup_{x \in \Omega} \Phi(x) \). We may assume that the maximum value is zero:

\( p_0 < p_N = \max_{j=0,\ldots,N} p_j = \sup_{x \in \Omega} \Phi(x) = 0. \quad (3.35) \)

Then \( p_0 = \cdots = p_{M'} = 0 \).

Change (if necessary) the numbering of the boundary components \( \Gamma_{M'+1}, \ldots, \Gamma_{N-1} \) so that

\begin{align*}
p_j &< 0, j = 0, \ldots, M, M \geq M', \quad (3.36) \\
p_{M'+1} = \cdots = p_N = 0. \quad (3.37)
\end{align*}

To remove a neighborhood of the singularity line \( O_z \) from our consideration, we take \( r_0 > 0 \) such that the open set \( D_\epsilon = \{(r, z) \in D : r > \epsilon\} \) is connected for every \( \epsilon \leq r_0 \) (i.e., \( D_\epsilon \) is a domain), and

\begin{align*}
\hat{\Gamma}_j \subset \overline{D_\epsilon} \quad \text{and} \quad \inf_{(r, z) \in \hat{\Gamma}_j} r \geq 2r_0, \quad j = M' + 1, \ldots, N, \\
\hat{\Gamma}_j \cap \overline{D_\epsilon} \quad \text{is a connected set and} \quad \sup_{(r, z) \in \hat{\Gamma}_j \cap \overline{D_\epsilon}} r \geq 2r_0, j = 0, \ldots, M', \epsilon \in (0, r_0]. \quad (3.38)
\end{align*}

Let a set \( C \subset \overline{D_\epsilon} \) separate \( \hat{\Gamma}_i \) and \( \hat{\Gamma}_j \) in \( D_\epsilon \) for some different indexes \( i, j \in \{0, \ldots, N\} \); i.e., \( \hat{\Gamma}_i \cap \overline{D_\epsilon} \) and \( \hat{\Gamma}_j \cap \overline{D_\epsilon} \) lie in different connected components of \( \partial D_\epsilon \setminus C \). Obviously, for \( \epsilon \in (0, r_0] \), there exists a constant \( \delta(\epsilon) > 0 \) (not depending on \( i, j, C \)) such that the uniform estimate \( \sup_{(r, z) \in C} r \geq \delta(\epsilon) \) holds. Moreover, the function \( \delta(\epsilon) \) is nondecreasing. In particular,

\[ \delta(\epsilon) \geq \delta(r_0), \quad \epsilon \in (0, r_0]. \quad (3.39) \]

In the following, we will construct an appropriate integration domain by using the level sets of \( \Phi \) and \( \Phi_n \). We need some information concerning the behavior of the limit total head pressure \( \Phi \) on stream lines. Following [16] and [19], we introduce some facts of topology. By continuum we mean a compact connected set. We understand connectedness in the sense of general topology. A subset of a topological space is called an arc if it is homeomorphic to the unit interval \([0, 1]\). Let \( Q = [0, 1] \times [0, 1] \) be a square in \( \mathbb{R}^2 \), and let \( f \) be a continuous function on \( Q \). Denote by \( E_t \) a level set of the function \( f \), i.e., \( E_t = \{ x \in Q : f(x) = t \} \). A connected component \( K \) of the level set \( E_t \) containing
a point \( x_0 \) is a maximal connected subset of \( E_t \) containing \( x_0 \). By \( T_f \) denote a family of all connected components of level sets of \( f \).

We apply Kronrod’s results to the stream function \( \psi|_{\bar{D}_{x_0}} \), \( \epsilon \in (0, r_0) \). Accordingly, \( T_{\psi, \epsilon} \) means the corresponding Kronrod tree for the restriction \( \psi|_{\bar{D}_{x_0}} \). Define the total head pressure on the Kronrod tree \( T_{\psi, \epsilon} \) as follows. Let \( K \in T_{\psi, \epsilon} \) with \( \text{diam } K > 0 \). Take any \( x \in K \setminus A_w \) and put \( \Phi(K) = \Phi(x) \). By the Bernoulli’s Law in Lemma 3.5, the value \( \Phi(K) \) is independent of the choice \( x \in K \setminus A_w \). Then \( \Phi \) has the following continuity properties on stream lines.

**Lemma 3.8** (See [16, Lemma 3.5]). Let \( A, B \in T_{\psi, \epsilon} \), where \( \epsilon \in (0, r_0) \), \( \text{diam } A > 0 \), and \( \text{diam } B > 0 \). Consider the corresponding arc \( [A, B] \subset T_{\psi, \epsilon} \) joining \( A \) to \( B \). Then the restriction \( \Phi|_{[A, B]} \) is a continuous function.

Denote by \( B'_0, \ldots, B'_N \) the elements of \( T_{\psi, \epsilon} \) such that \( B'_j \supset \tilde{\Gamma}_j \cap \partial \mathcal{D}_r, j = 0, \ldots, M' \), and \( B'_j \supset \tilde{\Gamma}_j, j = M' + 1, \ldots, N \). By construction, \( \Phi(B'_j) < 0 \) for \( j = 0, \ldots, M \), and \( \Phi(B'_j) = 0 \) for \( j = M + 1, \ldots, N \). For \( r > 0 \), let \( L_r \) be the horizontal straight line \( L_r = \{(r, z) : z \in \mathbb{R} \} \). Then, similar to [16, Lemma 4.6], we can find \( r_* \in (0, r_0) \) and \( C_j \in [B'_j, B'_N], j = 0, \ldots, M \), such that \( \Phi(C_j) < 0 \) and \( C_j \cap L_r = \emptyset \) for all \( C \in [C_j, B'_N] \).

We restrict our argument on the domain \( \mathcal{D}_{r_*} \) and put \( T_{\psi} = T_{\psi, r_*} \) and \( B_j = B'_j \). Since \( \partial \mathcal{D}_{r_*} \subset B_0 \cup \cdots \cup B_N \cup L_{r_*} \) and the set \( B_0, \ldots, B_N \subset T_{\psi} \) is finite, we can change \( C_j \) (if necessary) such that

\[
\forall j = 0, \ldots, M, \quad C_j \in [B_j, B_N], \quad \Phi(C_j) < 0, \quad (3.40)
\]

Denote \( C \cap \partial \mathcal{D}_{r_*} = \emptyset \) \( \forall C \in [C_j, B_N] \) (3.41).

Observe that \( \Gamma_j \cap L_{r_*} \neq \emptyset \) for \( j = 0, \ldots, M' \). Therefore, if a cycle \( C \in \mathcal{D}_{r_*} \) separates \( \Gamma_N \) from \( \Gamma_0 \) and \( C \cap \partial \mathcal{D}_{r_*} = \emptyset \), then \( C \) separates \( \Gamma_N \) from \( \Gamma_j \) for all \( j = 1, \ldots, M' \). So we can take \( C_0 = \cdots = C_M \) and consider only the Kronrod arcs \([C_M', B_N], \ldots, [C_M, B_N]\). Recall that a set \( Z \subset T_{\psi} \) has \( \text{T-measure zero} \) if \( \mathcal{H}^1(\{z \in Z \} : C \subset Z) = 0 \).

**Lemma 3.9.** For every \( j = M', \ldots, M \), \( T \)-almost all \( C \in [C_j, B_N] \) are \( C^1 \)-curves homeomorphic to the circle. Moreover, there exists a subsequence \( \Phi_n \) such that the sequence \( \Phi_n|_C \) converges to \( \Phi|_C \) uniformly \( \Phi_n|_C \Rightarrow \Phi|_C \) on \( T \)-almost all cycles \( C \in [C_j, B_N] \).

Without loss of generality, we assume that the subsequence \( \Phi_n \) coincides with \( \Phi \). Besides, cycles satisfying the assertion of Lemma 3.9 will be called *regular cycles*. From Lemma 3.9 and [16, Lemma 3.6], we can conclude that

\[
\mathcal{H}^1(\{z \in [C_j, B_N] : \Phi(C) \text{ is not a regular cycle} \}) = 0, j = M', \ldots, M. \quad (3.42)
\]

Setting \( \alpha = \max_{j = M', \ldots, M} \min_{C \in [C_j, B_N]} \Phi(C) \), by (3.40), \( \alpha < 0 \). By Equation (3.42), we can find a sequence of positive values \( t_i \in (0, -\alpha), i \in \mathbb{N} \) with \( t_{i+1} = \frac{1}{2} t_i \) such that the implication

\[
\Phi(C) = -t_i \Rightarrow C \text{ is a regular cycle}
\]

holds for every \( j = M', \ldots, M \) and for all \( C \in [C_j, B_N] \). Consider the natural order on the arc \([C_j, B_N]\), namely, \( C' < C'' \) if \( C' \) is closer to \( B_N \) than \( C'' \). For \( j = M', \ldots, M \) and \( i \in \mathbb{N} \), put

\[
A^i_j = \max\{C \in [C_j, B_N] : \Phi(C) = -t_i \}
\]

Then each \( A^i_j \) is a regular cycle and \( A^i_j \subset \mathcal{D}_{r_*} \). In particular, for each \( i \in \mathbb{N} \), the compact set \( \cup_{j = M'}^M A^i_j \) is separated from \( \partial \mathcal{D}_{r_*} \) and \( \text{dist}(\cup_{j = M'}^M A^i_j, \partial \mathcal{D}_{r_*}) > 0 \). Then for each \( i \)}
and for sufficiently small $h > 0$, we have the inclusion $\{x \in D_{r_{\ast}} : \text{dist}(x, \hat{\Gamma}_{N}) < h\} \subset D_{r_{\ast}} \setminus (\cup_{j=M}^{N} A_{j}^{M})$. Denote by $V_{i}$ the connected component of the open set $D_{r_{\ast}} \setminus (\cup_{j=M}^{M} A_{j}^{M})$ which encloses the set $\{x \in D_{r_{\ast}} : \text{dist}(x, \hat{\Gamma}_{N}) < h\}$. Then we have

$$\{x \in D_{r_{\ast}} : \text{dist}(x, \hat{\Gamma}_{N}) < h\} \cap \partial V_{i} = A_{i}^{M'} \cup \cdots \cup A_{i}^{M}.$$ 

By the construction, the sequence of domains $V_{i}$ is decreasing, i.e., $V_{i} \supset V_{i+1}$. Hence, the sequence of sets $(\partial D_{r_{\ast}}) \cap (\partial V_{i})$ is nonincreasing. Every set $(\partial D_{r_{\ast}}) \cap (\partial V_{i})$ consists of several components $\Gamma_{i}$ with $l > M$. Since there are only finitely many components $\Gamma_{i}$, we can conclude that, for sufficiently large $i$, the set $(\partial D_{r_{\ast}}) \cap (\partial V_{i})$ is independent of $i$. So we can assume that $(\partial D_{r_{\ast}}) \cap (\partial V_{i}) = \hat{\Gamma}_{K} \cup \cdots \cup \hat{\Gamma}_{N}$, where $K \in \{M+1, \ldots, N\}$. Hence,

$$\partial V_{i} = A_{i}^{M'} \cup \cdots \cup A_{i}^{M} \cup \hat{\Gamma}_{K} \cup \cdots \cup \hat{\Gamma}_{N}. \quad (3.43)$$

By Lemma 3.9, we have the uniform convergence $\Phi_{n} |_{A_{i}^{j}} \Rightarrow \Phi (A_{i}^{j})$ as $n \to \infty$. Then for each $i \in \mathbb{N}$ there exists $n_{i}$ such that for all $n \geq n_{i}$

$$\Phi_{n} |_{A_{i}^{j}} < -\frac{7}{8} t_{i}, \quad \Phi_{n} |_{A_{i+1}^{j}} > -\frac{5}{8} t_{i} \quad \forall j = M', \ldots, M.$$ 

Then

$$\forall t \in \left[\frac{5}{8} t_{i}, \frac{7}{8} t_{i}\right] \quad \forall n \geq n_{i} \quad \Phi_{n} |_{A_{i}^{j}} < -t, \quad \Phi_{n} |_{A_{i+1}^{j}} > -t \quad \forall j = M', \ldots, M.$$ 

Accordingly, for $n \geq n_{i}$ and $t \in \left[\frac{5}{8} t_{i}, \frac{7}{8} t_{i}\right]$, we can define $W_{in}^{j}(t)$ as the connected component of the open set $\{x \in V_{i} \setminus V_{i+1} : \Phi_{n}(x) > -t\}$ with $\partial W_{in}^{j}(t) \supset A_{i+1}^{j}$ and put

$$W_{in}(t) = \bigcup_{j=M'}^{M} W_{in}^{j}(t), \quad S_{in}(t) = (\partial W_{in}(t)) \cap (V_{i} \setminus V_{i+1}).$$

By construction, $\Phi_{n} \equiv -t$ on $S_{in}(t)$ and

$$\partial W_{in}(t) = S_{in}(t) \cup A_{i+1}^{M'} \cup \cdots \cup A_{i+1}^{M},$$

and the set $S_{in}(t)$ separates $A_{i}^{M'} \cup \cdots \cup A_{i}^{M}$ from $A_{i+1}^{M'} \cup \cdots \cup A_{i+1}^{M}$. Since $\Phi_{n} \in W_{loc}^{2,2}(\Omega)$, by the Morse–Sard theorem, for almost all $t \in \left[\frac{5}{8} t_{i}, \frac{7}{8} t_{i}\right]$, the level set $S_{in}(t)$ consists of finitely many $C^{1}$-cycles, and $\Phi_{n}$ is differentiable in the classical sense at every point $x \in S_{in}(t)$ with $\nabla \Phi_{n}(x) \neq 0$. We will say the values $t \in \left[\frac{5}{8} t_{i}, \frac{7}{8} t_{i}\right]$ having the above property are $(n, i)$-regular. Therefore, $S_{in}(t)$ is a finite union of smooth surfaces (tori), and by construction

$$\int_{S_{in}(t)} \nabla \Phi_{n} \cdot n dS = -\int_{S_{in}(t)} |\nabla \Phi_{n}| dS < 0, \quad (3.44)$$

where $n$ is the unit outward normal vector to $\partial W_{in}(t)$.

For $h > 0$, denote $\Gamma_{h} = \{x \in \Omega : \text{dist}(x, \hat{\Gamma}_{K} \cup \cdots \cup \hat{\Gamma}_{N}) = h\}$, $\Omega_{h} = \{x \in \Omega : \text{dist}(x, \hat{\Gamma}_{K} \cup \cdots \cup \hat{\Gamma}_{N}) < h\}$. Since the distance function $\text{dist}(x, \partial \Omega)$ is $C^{1}$-regular and the norm of its gradient is equal to one in the neighborhood of $\partial \Omega$, there is a constant $\delta_{0} > 0$ such that,
for every $h \leq \delta_0$, the set $\Gamma_h$ is a union of $N - K + 1$ $C^1$-smooth surfaces homeomorphic to the torus, and

$$
\mathcal{H}^2(\Gamma_h) \leq c_0 \quad \forall h \in (0, \delta_0],
$$

(3.45)

where the constant $c_0 = 3\mathcal{H}^2(\Gamma_K \cup \cdots \cup \Gamma_N)$ is independent of $h$.

**Lemma 3.10.** For any $i \in \mathbb{N}$, there exists $n(i) \in \mathbb{N}$ such that, for every $n \geq n(i)$ and for almost all $t \in [\frac{5}{8} t_i, \frac{7}{8} t_i]$, the inequality

$$
\int_{\mathcal{S}_{n}(t)} |\nabla \Phi_n|dS \leq \mathcal{F}t
$$

(3.46)

holds with the constant $\mathcal{F}$ independent of $t$, $n$, and $i$.

**Proof.** By a direct calculation, Equation (3.16) implies

$$
\nabla \Phi = \nabla \frac{1}{2} |w|^2 - (w \cdot \nabla)w + \nabla \frac{1}{2} |k|^2 + (k \cdot \nabla)k = [\nabla w - (\nabla w)^T] \cdot w + [\nabla k + (\nabla k)^T] \cdot k.
$$

(3.47)

Since $\Phi \neq \text{const}$ on $\tilde{V}_i$, (3.47) implies $\int_{\tilde{V}_i} |\nabla w - (\nabla w)^T|^2 + |\nabla k + (\nabla k)^T|^2 dx > 0$ for every $i$. Hence, from the weak convergence $\nabla u_n \rightharpoonup \nabla w$ and $\nabla h_n \rightharpoonup \nabla k$ in $L^2(\Omega)$ it follows that for any $i \in \mathbb{N}$, there exist constants $\epsilon_i > 0, \delta_i \in (0, \delta_0)$ and $k_i' \in \mathbb{N}$ such that

$$
\Omega_{\delta_i} \cap A_i^j = \Omega_{\delta_i} \cap A_{i+1}^j = \emptyset, \quad j = M', \ldots, M,
$$

and for all $n \geq n_i'$

$$
\int_{\tilde{V}_{i+1} \setminus \Omega_{\delta_i}} (|\nabla u_n - (\nabla u_n)^T|^2 + |\nabla h_n + (\nabla h_n)^T|^2) dx > \epsilon_i.
$$

(3.48)

Fix $i \in \mathbb{N}$. We assume that $n \geq n_i$. Since we have removed a neighborhood of the singularity line $O_z$, we can use the Sobolev embedding theorem in the plane domain $D_{r_a}$. The uniformly boundedness of $\|\Phi_n\|_{W^{1,3/2}(D_{r_a})}$ imply that the norm $\|\Phi_n\|_{L^6(D_{r_a})}$ and then $\|\Phi_n \nabla \Phi_n\|_{L^{6/5}(D_{r_a})}$ are also uniformly bounded. Finally, we have

$$
\|\Phi_n \nabla \Phi_n\|_{L^{6/5}(D_{r_a})} \leq \text{const}.
$$

(3.49)

Fix a sufficiently small $\sigma > 0$ (the exact value of $\sigma$ will be specified below), and take the parameter $\delta_0 \in (0, \delta_i]$ small enough to satisfy the following conditions:

$$
\Omega_{\delta_a} \cap A_i^j = \Omega_{\delta_a} \cap A_{i+1}^j = \emptyset, \quad j = M', \ldots, M,
$$

(3.50)

$$
\int_{\Gamma_h} \Phi_n^2 dS < \sigma^2 \quad \forall h \in (0, \delta_i] \quad \forall n \geq n'.
$$

(3.51)

The last estimate follows from the identity $\Phi|_{\Gamma_K \cup \cdots \cup \Gamma_N} = 0$, the weak convergence $\Phi_n \rightharpoonup \Phi$ in the space $W^{1,3/2}(\Omega)$, and (3.49).

By a direct calculation, (3.12) implies

$$
\nabla \Phi_n = -\nu_n \text{curl curl } u_n + [\nabla u_n - (\nabla u_n)^T] \cdot u_n \\
+ [\nabla h_n + (\nabla h_n)^T] \cdot h_n + \nabla \times f_n.
$$
Then, using Stokes’ theorem, we obtain
\[
\int_S \nabla \Phi_n \cdot \mathbf{n} \, dS = \int_S (|\nabla u_n - (\nabla u_n)^T| \cdot u_n) \cdot \mathbf{n} \, dS + \int_S \left( |\nabla h_n + (\nabla h_n)^T| \cdot h_n \right) \cdot \mathbf{n} \, dS.
\]

Now, fix a sufficiently small \( \epsilon > 0 \). The exact value of \( \epsilon \) will be specified below. For a given sufficiently large \( n \geq n' \), we follow the proof of [16, Lemma 3.8] to find a number \( \overline{h}_n \in (0, \delta_\sigma) \) such that the estimates
\[
\left| \int_{\Gamma_{\overline{h}_n}} \nabla \Phi_n \cdot \mathbf{n} \, dS \right| \leq 2 \int_{\Gamma_{\overline{h}_n}} \left( |u_n| \cdot |\nabla u_n| + |h_n| \cdot |\nabla h_n| \right) \, dS < \epsilon, \tag{3.52}
\]
\[
\int_{\Gamma_{\overline{h}_n}} (|u_n|^2 + |h_n|^2) \, dS \leq C_\epsilon \nu_n^2 \tag{3.53}
\]
hold, where \( C_\epsilon \) is independent of \( n \) and \( \sigma \).

Now, for \((n,i)\)-regular value \( t \in [\frac{2}{3} t_i, \frac{5}{8} t_i] \), consider the domain
\[
\Omega_{\overline{h}_n}(t) = \overline{W_{in}(t)} \cup (\overline{V_{i+1}} \setminus \Omega_{\overline{h}_n}).
\]

By construction, \( \partial \Omega_{\overline{h}_n}(t) = \Gamma_{\overline{h}_n} \cup \overline{S_{in}(t)} \). Also using Equation (3.12), we know
\[
\Delta \Phi_n = \Delta p_n + |\nabla u_n|^2 + |\nabla h_n|^2 + u_n \cdot \Delta u + h_n \cdot \Delta h_n \\
= -\text{div}(\nabla u_n \cdot \mathbf{n}) + \text{div}(\nabla h_n \cdot \mathbf{n}) + |\nabla u_n|^2 + |\nabla h_n|^2 \\
- \frac{1}{\nu_n} \left( (\nabla \times f_n) \cdot u_n + (\nabla \times g_n) \cdot h_n \right) \\
+ \frac{1}{\nu_n} \left( |u_n|^2 |h_n|^2 \right) - \nu_n \cdot \nu_n \cdot \left( (\nabla \cdot \nabla) u_n \right)
\]
\[
= -3 \sum_{i,j=1} \partial_i u_{nj} \partial_j u_{ni} + |\nabla u_n|^2 + |\nabla h_n|^2 + \sum_{i,j=1} \partial_i h_n \partial_j h_n + \frac{1}{\nu_n} (u_n \cdot \nabla \Phi_n) \\
- \frac{1}{\nu_n} (h_n \cdot \nabla) (u_n \cdot h_n) - \frac{1}{\nu_n} \left( (\nabla \times f_n) \cdot u_n + (\nabla \times g_n) \cdot h_n \right) \\
= \frac{1}{\nu_n} \text{div} (\Phi_n u_n) + \frac{1}{2} |\nabla u_n - (\nabla u_n)^T|^2 + \frac{1}{2} |\nabla h_n + (\nabla h_n)^T|^2 \\
- \frac{1}{\nu_n} \left( (\nabla \times f_n) \cdot u_n + (\nabla \times g_n) \cdot h_n \right), \tag{3.54}
\]
where we have used the special structure of \( u_n \) and \( h_n \), so that \( (h_n \cdot \nabla)(u_n \cdot h_n) \equiv 0 \).

Integrating Equation (3.54) over the domain \( \Omega_{\overline{h}_n}(t) \), we obtain
\[
\int_{\overline{S_{in}}} \nabla \Phi_n \cdot \mathbf{n} \, dS + \int_{\Gamma_{\overline{h}_n}} \nabla \Phi_n \cdot \mathbf{n} \, dS \\
= \int_{\Omega_{\overline{h}_n}(t)} \frac{1}{2} |\nabla u_n - (\nabla u_n)^T|^2 + \frac{1}{2} |\nabla h_n + (\nabla h_n)^T|^2 \, dx \\
- \frac{1}{\nu_n} \int_{\Omega_{\overline{h}_n}(t)} ((\nabla \times f_n) \cdot u_n + (\nabla \times g_n) \cdot h_n) \, dx + \frac{1}{\nu_n} \int_{\overline{S_{in}}} \Phi_n u_n \cdot \mathbf{n} \, dS + \frac{1}{\nu_n} \int_{\Gamma_{\overline{h}_n}} \Phi_n u_n \cdot \mathbf{n} \, dS
\]
Assume that Equation (3.35) lead to a contradiction. 3.9 of [16], i.e., replacing Hausdorff measure and \( \Phi(\sigma) < 0 \) with \( \epsilon < \), we have excluded the second case.

Therefore, we can derive a contradiction by using the Co-area formula. The proof of Lemma 3.11 can be obtained by slightly modifying the proof of Lemma 3.3.3. The case: Assume that Equation (3.32) is satisfied, and set \( \sigma = \max_{j=0, \ldots, N} p_j \). Then we can find a compact connected set \( F \subset D \setminus A \) such that \( \text{diam}(F) > 0 \), \( \psi|_F = \text{const} \), and \( \Phi(F) > \sigma \). We may assume that \( \sigma < 0 \) and \( \Phi(F) = 0 \). We still need to separate \( F \) from \( \partial D \) by regular cycles and take a number \( r_0 > 0 \) such that \( F \subset D_{r_0} \), the open set \( D_\varepsilon = \{ (r,z) \in D : r > \varepsilon \} \) is connected for every \( \varepsilon \leq r_0 \) and conditions (3.38) are satisfied. Then, for \( \varepsilon \in (0, r_0] \), we can consider the
behavior of \( \Phi \) on the Kronrod trees \( T_{\psi,r} \) corresponding to the restrictions \( \psi|_{\overline{B_j}} \). Denote by \( F^* \) the element of \( T_{\psi,r} \) containing \( F \). Using the same procedure as before, we can find \( r_* \in (0,r_0) \) and \( C_j \in [B_j^*, F^*] \), \( j = 0, \ldots, N \), such that \( \Phi(C_j) < 0 \) and \( C \cap L_{r_*} = \emptyset \) for all \( C \in [C_j, F^*] \).

Set \( T_b = T_{\psi, r_*}, F^* = F_{r_*} \), and \( B_j = B_{r_*}^j \), i.e., \( B_j \in T_{\psi} \) and \( B_j \supseteq \overline{\Gamma}_j \cap \overline{D_{r_*}} \). As above, we can change \( C_j \) so that

\[
\forall j = 0, \ldots, N \quad C_j \in [B_j, F^*], \quad \Phi(C_j) < 0, \\
C \cap \partial D_{r_*} = \emptyset \quad \forall C \in [C_j, F^*], \quad \text{and} \quad C_0 = \ldots = C_{M^*}.
\]

Similarly, we should construct an appropriate integration domain by using the level sets of \( \Phi \) and \( \Phi_n \). Take positive numbers \( t_i = 2^{-i} t_0 \), regular cycles \( A_i \in [C_j, F^*] \) with \( \Phi(A_i) = -t_i \), and the set \( S_{in}(t) \) with \( \Phi_n|_{S_{in}(t)} \equiv -t \) separating \( A_{i+1}^M \cup \ldots \cup A_i^N \) from \( A_{i+1}^M \cup \ldots \cup A_i^N \), etc. Argued in Lemma 3.10 and Lemma 3.11, we can derive a similar contradiction as before. Therefore, we have finished the proof of Theorem 1.1.

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