Quantum state stability against decoherence

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We study the stability of the coherence of a state of a quantum system under the effect of an interaction with another quantum system at short time. We find an expression for evaluating the order of magnitude of the time scale for the onset of instability as a function of the initial state of both involved systems and of the sort of interaction between them. As an application we study the spin-boson interaction in the dispersive interaction regime, driven by a classical field. We find, for this model, that the behavior of the time scale for the onset of instability, with respect to the boson bath temperature, changes depending on the intensity of the classical field.

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I. INTRODUCTION

The question as to whether or not a pure quantum state can persist in the macroscopic world has been considered since Schrödinger introduced his gedanken experiment known as Schrödinger’s cat [1] and it was in this way that the entanglement concept was introduced into the quantum world. Decoherence is a word to indicate that the state of a quantum system is not pure. In general, a hamiltonian interaction which generates entanglement between two quantum systems produces reversible decoherence in each of the involved systems. However, when a system interacts with an infinite number of systems or a single system interacting with an infinite number of degrees of freedom, the concept of irreversible decoherence arises, which is usually called decoherence only. Thus, decoherence has become the terminology for the irreversible evolution of a quantum state due to its interaction with an environment [2]. In this context numerous works allow us to gain intuition about the dynamical behavior of open quantum systems. For instance, spontaneous emission arises from the coupling of a system to a noise vacuum environment [3]. Dalvit et al. [4] studied various measures of classicality of the pointer states of open quantum systems subjected to decoherence. Also J. I. Cirac et al. [5] found a dark state of a single two-level ion trapped in a harmonic potential by controlling its motion degree of freedom. This is achieved by generating a squeezed motion state. T. A. Costi and R. H. McKenzie [6] gave a quantitative description of the entanglement between a two-level system and an environment for an ohmic coupling. D. DiVincenzo and D. Loss [7] provided an exact analysis of the weak coupling limit of the spin-boson model for an ohmic heat bath in the low temperature limit, using non-Markovian and Born approximations. S. Bose et al. [8] studied the enforcer of entanglement between a two-level system and a quantized mode in a thermal state. Decoherence-free subspaces in cavity QED have been found [9]. Also a method has been developed which achieves the slowing down of decoherence and relaxation by fast frequency modulation of the system-heat-bath coupling [10]. In addition A. G. Kofman and G. Kurizki [11] developed a unified theory of dynamically suppressed decay and decoherence by an external field in qubits to arbitrary thermal bath and dephasing sources. The problem of stability of a quantum state under a class of Lindblad dissipative dynamics has been also studied [12].

In this article we study, in a simple form, the problem of coherence stability, at short time, of a quantum state without Markov nor Born approximations. We have found an expression which allows us to estimate the time scale order of onsetting the decoherence for a quantum system when it interacts only with another single quantum system. The expression can also be applied when the second interacting system is in a thermal equilibrium or is in an adiabatic dynamics regimen with respect to the studied one. We apply our result to the following cases: (i) the reversible pure-dephasing interaction with a numerical simulation, (ii) a cavity mode driven by a thermal light, and (iii) the spin-boson dispersive interaction driven by a resonant classical field.

II. DECOHERENCE RATE

The stability against decoherence is understood to be the process where quantum coherence is preserved along evolution [12]. So, we can say that an initial pure state $\ket{\psi}$ is stable against decoherence during the time $t_d$ if $tr\rho^2(t) \approx 1$ for all $t \leq t_d$. The decoherence of a state, represented by $\rho$, is measured by means of the first-order entropy: $s(t) = 1 - tr\rho^2(t)$. In order to consider the stability of the coherence, we assume that $s(t)$ is an analytic function of $t$ [12]. To find out the $t_d$ stability time scale order for which $s(t)$ remains being approximately $s(0) = 0$, it can be expanded as a Taylor series, in such a way that the time scale order will be given by $t_d = 1/\sqrt{s_n(0)}$, where $s_n(0)$ denotes the $n$th derivative of $s(t)$ evaluated at $t = 0$, and $s_n(0)$ is the lowest order derivative differ-
ent from zero. The first two derivatives of the first-order entropy are: \( s_1 = -2tr(\rho \dot{\rho}) \) and \( s_2 = -2tr(\dot{\rho}^2 + \rho \ddot{\rho}) \), with \( \dot{\rho} \) and \( \ddot{\rho} \) denoting, respectively, the first and the second derivatives of \( \rho \) with respect to \( t \) at time \( t \).

Now let us suppose that a system under study, labeled by \( a \), is interacting with the \( R \) system through the \( V(t) \) Hamiltonian. We consider the whole \( a-R \) system to be isolated or: \( R \) being in a thermal equilibrium or in an adiabatic dynamics regime. Initially the \( a \) system is in a pure state \( |\psi \rangle \) and the \( R \) system is in the state \( \rho_R \). In the whole tensorial product Hilbert space, \( \mathcal{H}_a \otimes \mathcal{H}_R \), the dynamics of the composite system state \( \rho(t) \) is driven by the Master equation \( \dot{\rho}(t) = -i[V(t), \rho(t)] \), whereas the dynamics of the partial density operator of the \( a \) system, \( \rho_a(t) = tr_R(\rho(t)) \), is governed by \( \dot{\rho}_a(t) = -itr_R[V(t), \rho(t)] \), where \( tr_R \) denotes the tracing up over the \( R \) system.

To find out if the initial pure state \( |\psi \rangle \) is stable under the dynamics described by the 1 and 2 Eqs., and besides, to obtain the expression for the \( t_d \), we consider the change in \( s(t) \) by calculating its derivatives at \( t = 0 \). Since we suppose initially the state of the system \( a \) to be pure, then \( s(0) = 0 \) and \( s_1(0) = 0 \); hence the second derivative of the first-order entropy, which could be different from zero, is given by

\[
s_2(0) = 2t \langle [V, |\psi \rangle \langle \psi |] \rangle_R - \langle [V, \langle \psi | \psi \rangle] \rangle_R \alpha, \tag{3}
\]

where we have denoted the average on the system \( a \) (\( R \)) by the subindex \( a \) (\( R \)), and they are taken at \( t = 0 \), and \( V = V(0) \). Thus, when the \( |\psi \rangle \) pure state does not commute with \( V(0) \), i.e. \( [V, |\psi \rangle \langle \psi |] \neq 0 \), the value of \( t_d = 1/\sqrt{s_2(0)} \) gives a time scale order for the onset of decoherence. We can also see from Eq. 3 that, in principle, \( t_d \) will be a functional of both the initial pure state of \( a \) as the initial state of the \( R \) system, and is proportional to \( 1/g \). It is important to point out that \( s_2(0) \), Eq. 3, is the positive defined operator-correlation in the \( R \) system between the \( V \) and \([V, |\psi \rangle \langle \psi |]\) operators, averaged on \( |\psi \rangle \). When the \( |\psi \rangle \) state commutes with \( V(0) \), one must calculate the third derivative of the first-order entropy at \( t = 0 \) in order to have the time scale order.

Here it is worth emphasizing that \( t_d \) was found using neither Markovian nor Born approximations. It is well to recall that the Markov approximation is a coarse grained dynamics description, in the sense that the time scale in which the system is observed is much longer that the characteristic correlation time of the reservoir. Since, in that case, fine temporal structure can not be seen, \( s_1(0) \neq 0 \) in general. Our description of the dynamics of the system corresponds to a time scale smaller than the correlation time of the reservoir.

By way of examples first let us consider familiar interaction models which generate reversible decoherence, that is, a far from the resonance interaction which is known as simplest pure-dephasing mechanisms \([14, 15, 16]\) described by the \( V = g(b + b^\dagger)\sigma_z \) Hamiltonians \([15]\), where \( \sigma_z \) is the \( z \)-component of the \( \sigma \) spin-1/2 operator with eigenstates \( |0 \rangle \) and \( |1 \rangle \). \( b \) and \( b^\dagger \) are the boson annihilation and creation operators respectively, and \( g \) gives account of the effective coupling strength. For this case the decoherence time scale order is given by

\[
t_d = \frac{1}{2g\sqrt{\langle (\Delta(b + b^\dagger))^4 \rangle}_R \sqrt{1 - \langle |\sigma_z |^2 \rangle^2}}, \tag{4}
\]

where \( \sqrt{\langle (\Delta(b + b^\dagger))^4 \rangle}_R \) is the root-mean-square deviation of the \( b + b^\dagger \) boson quadrature at \( t = 0 \).

We can see that the initial coherence of the boson state plays an important role, i.e., for a fixed \( |\psi \rangle \) state, an initial boson squeezed state \([13, 14]\) causes a decoherence time scale smaller than one caused by an initial Fock state or by a thermal state with equal average boson number. In this effective model, \( |0 \rangle \) and \( |1 \rangle \) states are affected only by a phase and each one is stable under this pure-dephasing mechanism. So, from 11 we can see that, for a fixed boson state, the states on the equator of the Bloch sphere have a smaller decoherence time scale than the one for states being near to the poles. It is worth noting that, for this pure-dephasing mechanism, any two states, \( |\psi \rangle \) and its orthogonal \( |\psi \rangle \), have the same \( t_d \) \([20]\). For the Jaynes-Cummings resonance \([17]\) interaction model, one can show that two states, \( |\psi \rangle \) and its orthogonal \( |\psi \rangle \), have different \( t_d \) \([20]\).

Fig. 11 shows the exact evolution of the \( s(t) \) first-order entropy of the two-level system under an effective pure-dephasing interaction with a boson field mode. The two-level system is initially in the \( |+ \rangle \) eigenstate of \( \sigma_x \) and the boson mode is in: a Fock state (solid), thermal state (dash), and vacuum squeezed state (dot), each one with the same average boson number \( \langle n \rangle = 3 \). The \( r \) squeeze parameter is such that \( \sqrt{\langle (\Delta(b + b^\dagger))^4 \rangle}_R = \sqrt{7 - 4\sqrt{3}} \approx 0.26795 \) and for thermal and Fock state \( \sqrt{\langle (\Delta(b + b^\dagger))^4 \rangle}_R = \sqrt{7} \approx 2.64575 \). Thus, in this particular case the \( t_d \) time scale orders differ in one order of magnitude as can be seen in Fig. 11.

As a second example, let us consider the explicit model which consists of a cavity mode driven by a thermal light \([12, 13]\). In the Markov limit and Born approximation this model accounts for cavity losses. The Hamiltonian in the interaction picture of this physical model is given by

\[
V(t) = \sum_j g_j (a^\dagger_j e^{i(\omega_j - \omega)t} + a_j e^{-i(\omega_j - \omega)t}), \tag{5}
\]

where \( \omega \) is the mode frequency and \( \omega_j \) is the frequency of the \( j \)th mode of the thermal light. \( a \) and \( a^\dagger \) are the annihilation and creation operators respectively, and \( r_j \) and \( r_j^\dagger \) are the annihilation and creation operators respectively of the \( j \)th mode of the thermal light. \( g_j \) account for the effective coupling strength between the main mode.
and the \( j \)th mode of the thermal light. For this model the decoherence time scale is

\[
t_d = \frac{1}{2\sqrt{\gamma + 2\gamma_T}}(\langle a^\dagger a \rangle - \langle a^\dagger \rangle\langle a \rangle) + \gamma_T,
\]

being \( \gamma = \sum_j |g_j|^2 \) which gives account of the whole magnitude of effective couplings strength, \( \gamma_T = \sum_j |g_j|^2 \langle n_j \rangle \), and they are the reservoir correlation functions \[13\]. From Eq. (6) we can see that, at zero temperature \( \langle n_j \rangle = 0 \) \( \forall j \), a coherent state is stable at this short time scale. Retamal and Zagury \[12\] show that this remains valid under the Markovian and Born approximations (Lindblad). At finite temperature a coherent state has a decoherence rate time scale given by \( 1/\gamma_T \). We also see that, for this irreversible dissipation mechanism only the initial field coherence, \( \langle a^\dagger \rangle\langle a \rangle \), allows to increase the decoherence time scale.

III. SPIN-BOSON MODEL

Now let us consider a two-level system interacting with an external laser mode of frequency \( \omega_f \) and with a bosonic bath modeled by an infinite collection of quantized harmonic oscillators having frequencies \( \omega_k \). The Hamiltonian which drives the unitary dynamics of the whole system, in the \( H_R = \omega_f (\sigma_z + \sum_k b_k^\dagger b_k) \) rotating wave frame has the form \( (\hbar = 1) \):

\[
H = \Delta \sigma_z + \sum_k \Delta_k b_k^\dagger b_k + \Omega (\sigma_+ + \sigma_-) + \sum_k g_k (b_k \sigma_+ + b_k^\dagger \sigma_-),
\]

where \( \Delta = \Delta_G - \omega_f \) and \( \Delta_G = \omega_1 - \omega_0 \) represent the energy difference between the upper state \( |1 \rangle \) and the lower state \( |0 \rangle \) of the two-level system. Operators \( \sigma_\pm \), obey the standard SU(2) commutation relations. \( \Delta_k = \omega_k - \omega_f \), and \( g_k \)’s are the respective coupling constants for the dipolar interaction between the \( k \)th mode and the two-level system. \( \Omega \) stands for the Rabi frequency which determines the coupling to the external classical field. \( b_k^\dagger \) and \( b_k \) are the creation and the annihilation bosonic operators of the \( k \)th mode, respectively. We assume the external field to be near resonance with the transition of the two-level system whereas the boson modes are considered to be far off resonance in such a way that \( \Delta_k \gg \omega_k \).

Therefore, boson mediated transitions, described by the fourth term on the right hand side of (7), can be strongly suppressed. Thus, the effective Hamiltonian approximately describing the interaction process can be obtained from the \( \hat{H} \) Hamiltonian by using the method of Lie rotations \[21,22\], namely applying to the \( \hat{H} \) Hamiltonian the unitary transformation: \( U = \exp[\sum_k \epsilon_k (b_k \sigma_+ + b_k^\dagger \sigma_-)] \), with \( \epsilon_k = g_k/\Delta_k \ll 1 \). Neglecting terms of order higher than \( \epsilon_k \) we obtain the following \( H_{eff} \) effective Hamiltonian:

\[
H_{eff} = \Delta \sigma_z + \sum_k \Delta_k b_k^\dagger b_k + \Omega (\sigma_+ + \sigma_-) + \sum_k \frac{g_k^2}{\Delta_G} \sigma_+ \sigma_- + \sum_k \frac{g_k^2}{\Delta_G} \sum_{k',k''} \epsilon_k \epsilon_{k'} b_k^\dagger b_k' + \Omega \sum_k \frac{g_k}{\Delta_G} \left( b_k + b_k^\dagger \right) \sigma_z,
\]

where we have considered that \( \Delta_G - \omega_k \approx \Delta_G \). Thus, off-resonant transition terms have been eliminated and in their stead appear three diagonal terms in the \( \sigma_z \) representation: a constant shift only for the upper level (a classical Stark shift); a random phase shift term similar to that which is described in Ref. \[13\]; and a pure-

FIG. 1: Evolution of \( s(t) \) for the two-level system under an effective pure-dephasing interaction with a boson field mode. The initial state of the boson mode is: a Fock state (solid), a thermal state (dash), and a vacuum squeezed state with quantum noise reduced in the \( b + b^\dagger \) quadrature.
dephasing term (that depends on the intensity of the bath modes) which gives rise to a randomized phase between the two levels, but what is striking about it is that here the coupling constant includes the Rabi frequency of the field whereas, in the standard spin-boson modes hamiltonian, an external field is not coupled at all. Thus, from first principles, an effective Hamiltonian for describing the dispersive interaction of a two-level system with a bosonic reservoir is obtained.

The transformation of $H_{eff}$ into the interaction picture separates the motion generated by

$$H_0 = \Delta \sigma_z + \sum_k \Delta_k b_k^{\dagger} b_k + \sum_k \frac{g_k^2}{\Delta G} \sigma_+ \sigma_-$$  \hspace{1cm} (9)

from the motion generated by the interaction term $\hat{V} = H_{eff} - H_0$, so that

$$V = e^{iH_0t} \hat{V} e^{-iH_0t} = \sigma(t) + \hat{B}(t) \sigma_z,$$  \hspace{1cm} (10)

where $\sigma(t) = \Omega(\sigma_+ e^{i\Delta t} + \sigma_- e^{-i\Delta t})$, and

$$\hat{B}(t) = 2\Omega \sum_k \frac{g_k}{\Delta G} \left( b_k^\dagger e^{i\omega_k t} + b_k e^{-i\omega_k t} \right)$$

$$+ \sum_{k,k'} \frac{g_k g_{k'}}{\Delta G} \left( b_k^\dagger b_{k'} e^{i(\omega_k - \omega_{k'})t} - \bar{n}_k \delta_{k,k'} \right)$$  \hspace{1cm} (11)

with $\bar{n}_k$ being the average boson number of the kth mode. Here we emphasize that the effective interaction hamiltonian $\hat{V}$ has the same form of the ohmic spin-boson model.

Considering Eqs. (8) and (11) we obtain the following $t_d$ time scale order for this model:

$$t_d = \frac{1}{2\sqrt{\langle \hat{B}^2 \rangle_R} \left( 1 - \langle \psi | \sigma_z | \psi \rangle^2 \right)},$$  \hspace{1cm} (12)

where we have supposed that each boson mode is initially in a thermal state at absolute temperature $T$. Under that condition the operator $\hat{B}$ satisfies $\langle \hat{B} \rangle_R = 0$, and

$$\langle \hat{B}^2 \rangle_R = \frac{4\Omega^2}{\Delta G} \sum_k g_k^2 (2\bar{n}_k + 1) + \sum_{k,k'} \frac{g_k^2 g_{k'}^2}{\Delta G^2} (\bar{n}_{k'} + 1) \bar{n}_k.$$

For a fixed bath state the eigenstates of $\sigma_z$, that is, the poles of the Bloch sphere, are stable under decoherence effects. Meanwhile, the states on the equator of the Bloch sphere are the less stable. If there are no spin-boson interactions, that is $g_k = 0 \forall k$, all the $|\psi\rangle$ states are stable since the classical field drives a unitary evolution. The classical field affects decreasing the $t_d$ time scale order through an effective dispersive interaction similar to the pure-dephasing mechanism. We can distinguish two regimes: the first one appears at weak field regime, this is, $\Omega \ll g_k$; the second one turns up at strong field limit, $\Omega \gg g_k$. At the high $T$ temperature limit: the strong field regime has associated a $t_d$ given by:

$$t_d,\Omega \gg g_k = \frac{\Delta G}{4\Omega \sqrt{2kT} \gamma (1 - \langle \psi | \sigma_z | \psi \rangle^2)}$$

whereas in the weak classical field limit the $t_d$ becomes

$$t_d,\Omega \ll g_k = \frac{\Delta G}{2kT} \gamma \sqrt{1 - \langle \psi | \sigma_z | \psi \rangle^2}$$

where $\gamma = \sum_k g_k^2 / \omega_k$. Here we obtain an important difference for the $t_d$ behavior in the two regimes: in the strong field limit $t_d$ is proportional to $1/\sqrt{T}$; meanwhile, in the weak field regime $t_d$ is proportional to $1/T$, both at high temperature limit.

IV. CONCLUSIONS

In summary we have found a general expression which allows us to estimate the decoherence time scale order for which the instability of a quantum state is onset. This decoherence time scale is a functional of the initial states of both interacting subsystems, and of the kind of interaction between them. It is worth pointing out that the found decoherence time scale is not symmetric with respect to the involved subsystems.

Specifically, for a spin-boson weak and dispersive interaction driven by a classical field we have found diverse behaviors of $t_d$ with respect to the absolute temperature, depending on the intensity of the external classical field.

One physical system which for instance fits these conditions would be a semiconductor quantum dot, where the uppermost valence band and the lowest conduction band can be represented by the ground and the excited eigenstates of $\sigma_z$. It is well known that the energy of a phonon is smaller than the transition energy of a two-level quantum dot system. Thus, the far from resonance constraint that we have imposed on the spin-boson model would be well satisfied in a semiconductor quantum dot.

Further studies could involve other natural interactions and other atomic configurations as well as a resonant quantum driven field mode. Besides, one could study a more general model for the decoherence mechanism, which would include both dephasing and dissipation, that is, considering two infinite sets of modes, one set around the resonance and the other one far from the resonance.

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