Markov’s Theorem for Weight Functions on the Unit Circle

Kenier Castillo\(^1\)

Received: 21 February 2020 / Revised: 14 November 2020 / Accepted: 19 January 2021 / Published online: 7 April 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract

The aim of this paper is to prove that Markov’s theorem on variation of zeros of orthogonal polynomials on the real line (Markoff in Math Ann 27:177–182, 1886) remains essentially valid in the case of paraorthogonal polynomials on the unit circle.

Keywords

Paraorthogonal polynomials on the unit circle · Weight functions on the unit circle · Markov’s theorem · Zeros

Mathematics Subject Classification

30C15 · 42C05

1 Introduction

In 1886, A. A. Markov proved a remarkable theorem concerning the dependence of the zeros of the elements of a sequence of orthogonal polynomials \((p_n)_{n=1}^{\infty}\) on a real parameter \(t\) which appears in the weight function \(\omega\) defined on the real interval \([a, b]\) (see [40, p. 178]). Szegő devotes two sections of his classical book to expose Markov’s work (see [51, Sections 6.12 and 6.21]) and, in a more recent monograph on the subject, Ismail refers to Markov’s theorem as “an extremely useful theorem” (see [31, p. 203]). The beauty and wide applicability of this result rest on its powerful simplicity:

\[
\text{Under suitable conditions, the zeros of } p_n(\cdot; t) \text{ are increasing functions of } t \text{ sprovided that}
\]

\[
\frac{1}{\omega(x; t)} \frac{\partial \omega}{\partial t}(x; t)
\]

is an increasing function of \(x\) on \((a, b)\).

Communicated by Sergey Denisov.

\(^1\) CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
As a direct consequence of his result, Markov himself showed that the zeros of
Jacobi polynomials, with weight function \( \omega(x; \alpha, \beta) = (1-x)^\alpha(1+x)^\beta \) on \([-1, 1] \)
for \( \alpha, \beta \in (-1, \infty) \), are decreasing functions of \( \alpha \) and increasing functions of \( \beta \). Indeed,

\[
\frac{1}{\omega(x; \alpha, \beta)} \frac{\partial \omega(x; \alpha, \beta)}{\partial \alpha} = \log(1-x),
\]
\[
\frac{1}{\omega(x; \alpha, \beta)} \frac{\partial \omega(x; \alpha, \beta)}{\partial \beta} = \log(1+x).
\]

Markov also attempts a general theorem to deal with the ultraspherical case \( \alpha = \beta \),
but his proof is incorrect. A proof of Markov’s theorem for even weight functions
on \([-1, 1]\)—easy once you realize that by mapping \((-1, 1)\) into \((0, 1)\) the problem
is reduced to the known case—can be found in [34, Corollary 2] in a more general
context.

Over the years, there were many extensions to the classical theory of orthogonal
polynomials on the real line (OPRL). After the influential works by Delsarte and Genin
[15–17] and Jones et al. [32] about the nowadays called paraorthogonal polynomials on
the unit circle (POPUC)—in many senses the appropriate complex analog of OPRL,—
this collection of polynomials and their zeros have received considerable attention
from two disparate audiences, namely researchers in orthogonal polynomials and
researchers in numerical linear algebra (see for instance [1,2,7–9,11–13,15–17,26–
29,33,41,43–49,52,53]). It must be said that rarely in the numerical linear algebra
context, the name POPUC is used; however, the reader has to proceed with caution in
the literature because many results on POPUC were first discovered in this framework.
As we will see below, POPUC are closely related with orthogonal polynomials on
the unit circle (OPUC) and, therefore, with weight functions on the unit circle. But
unfortunately Markov’s theorem can not deal with it. In Sect. 3, we discuss this question
and investigate the extent to which Markov’s theorem remains valid in the case of
weight functions on the unit circle. Unlike what happens in the case of OPRL (see
the proof of [51, Theorem 6.12.1.] and the hint of [22, Problem 15, Chapter III]),
we cannot use quadrature for our purpose because Szegő quadrature is much weaker
than Gaussian quadrature. In Sect. 4, we apply our results to some specific families of
polynomials, but first some preliminary definitions and basic results are needed (see
[46,49] for more details).

## 2 Preliminaries

Let \( d\mu(\theta) \) be a finite nonnegative measure with infinite support on the unit circle
parametrized by \( z = e^{i\theta} \) and

\[
c_j = \int e^{-ij\theta} d\mu(\theta) \quad (j = 0, 1, 2, \ldots)
\]
its moments. We will use $c_j(d\mu)$ if we want the $d\mu$ dependence to be explicit. Let $(Q_n)_{n=0}^\infty$ be the unique sequence of monic OPUC associated with $d\mu$, that is, polynomials $Q_n(z; d\mu) = Q_n(z) = z^n + \cdots$ which satisfy
\[
\int Q_n(e^{i\theta}) \overline{Q_m(e^{i\theta})} \, d\mu(\theta) = 0 \quad (n \neq m = 0, 1, 2, \ldots),
\]
\[
\int |Q_n(e^{i\theta})|^2 \, d\mu(\theta) \neq 0.
\]
Define $c_j$ for $j = -1, -2, -3, \ldots$ by $c_j = c_{-j}$. We mention the following explicit representation of $Q_n$ sometimes called Heine’s formula:
\[
Q_n(z) = D_{n-1}(d\mu)^{-1} \det \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_{-1} \\ 1 & z & \cdots & z^n \end{pmatrix} \quad (n = 1, 2, \ldots) \tag{1}
\]

where $D_{n-1}(d\mu) = \det(c_{k-j})_{j,k=0}^{n-1} > 0$ by the Carathéodory-Toeplitz theorem.
Define the normalized OPUC by $q_n(z) = \kappa_n z^n + \cdots$ where $\kappa_n = \|Q_n\|^{-1}$. The CD kernel is defined for $w, z \in \mathbb{C}$ by
\[
K_n(w, z; d\mu) = K_n(w, z) = \sum_{j=0}^{n} q_j(w)q_j(z).
\]

For any polynomial $f$ of degree at most $n$, we have
\[
\int f(e^{i\theta}) K_n(e^{i\theta}, w) \, d\mu(\theta) = f(w), \tag{2}
\]

often called the reproducing property.
Denote by $S^1(c)$ the boundary of the open disk $D_r(c)$ of radius $r > 0$ with center $c$. Since the unit disk with center at the origin plays a distinguished role in the theory of OPUC, we use the notation $D = D_1(0)$ and $S^1 = S^1_1(0)$. Fix $n \in \{1, 2, \ldots\}$ and $b \in S^1$. The monic POPUC of degree $n$ associated with $d\mu$ and $b$ is defined by (see [49, p. 115])
\[
P_n(z; b; d\mu) = P_n(z) = zQ_{n-1}(z) - \overline{b} Q^*_n(z), \tag{3}
\]

where $Q^*_n(z) = z^n \overline{Q(1/z)}$. The normalized POPUC is given by $p_n(z; b; d\mu) = p_n(z) = zq_{n-1}(z) - \overline{b} q^*_n(z)$. Another appropriate denomination for POPUC is quasi-orthogonal polynomials on the unit circle, in part because
\[
\int p_n(e^{i\theta}) g(e^{i\theta}) \, d\mu(\theta) = 0, \tag{4}
\]
for any polynomial $g$ of degree at most $n - 1$ vanishing at the origin, and in part because, as Geronimus pointed out (see [25, Footnote 10, p. 12]), *this property is analogous to a fundamental property of the so-called quasi-orthogonal polynomials of M. Riesz.* The “quasi-orthogonality” condition (4) gives rise to some interesting properties of POPUC. Suppose that $P_n(\zeta) = 0$ and let $h$ be a nonzero polynomial of degree at most $n - 1$. Since $h(z) - h(\zeta)$ has a zero of multiplicity at least one at $z = \zeta$, 

$$\frac{zh(z) - zh(\zeta)}{z - \zeta}$$

is a polynomial of degree $n - 1$ vanishing at the origin. From (4), we have

$$0 = -\frac{1}{\zeta} \int P_n(e^{i\theta}) \frac{h(e^{i\theta}) - h(\zeta)}{e^{i\theta}(e^{i\theta} - \zeta)} d\mu(\theta) = \int P_n(e^{i\theta}) \frac{h(e^{i\theta}) - h(\zeta)}{e^{i\theta} - \zeta} d\mu(\theta).$$

Hence,

$$\int \frac{P_n(e^{i\theta})}{e^{i\theta} - \zeta} h(e^{i\theta}) d\mu(\theta) = \frac{h(\zeta)}{\zeta} \int \frac{P_n(e^{i\theta})}{e^{i\theta} - \zeta} d\mu(\theta),$$

(5)

for any polynomial $h$ of degree at most $n - 1$. Moreover, since there exists $C \in \mathbb{C} \setminus \{0\}$ (cf. [53, p. 284]) such that

$$P_n(z) = C(z - \zeta)K_{n-1}(\zeta, z),$$

(2) shows that

$$C = \int \frac{P_n(e^{i\theta})}{e^{i\theta} - \zeta} d\mu(\theta) \neq 0.$$  

(6)

Denote by $a_j = -\overline{Q_{j+1}(0)}$ the Verblunsky coefficients. Set

$$\Theta_j = \Theta(a_j) = \begin{pmatrix} \overline{a}_j & r_j \\ r_j & -a_j \end{pmatrix},$$

where $r_j = (1 - |a_j|^2)^{1/2}$. Define $G_j = \text{diag}(I_j, \Theta_j, I_{n-j-2})$ and $G_{n-1} = \text{diag}(I_{n-1}, \overline{b})$. (Here, $I$ denotes the identity matrix, whose order is made explicit with a subindex.) It is well-known that $P_n$ is the characteristic polynomial of the GGT unitary matrix (see for instance [17, (4.19)])

$$G = G_0G_1 \cdots G_{n-1}.$$  

(7)

---

1 See also [24, Remark I].

2 As we will see later $\zeta \in S^1$, and so $\zeta \neq 0$.

3 The value of the constant $C$ can be written explicitly and appears in the proof of [38, Lemma 4.4].
In practical work, it is not always necessary to write this matrix explicitly, but it is important to know that G is a unitary upper Hessenberg matrix with positive subdiagonal elements. Therefore, the zeros of POPUC have two very attractive properties: (1) All the zeros of \( P_n \) lie on \( S^1 \); (2) The zeros of \( P_n \) are all simple (see a different proof in [25, Theorem 9.1.1]).

3 Main Results

Let us introduce the notation \( C_r(c) = \mathbb{D}_r(c) \cap S^1 \) and \( I_r(c) = \mathbb{D}_r(c) \cap \mathbb{R} \). In what follows, we shall use (explicitly or implicitly) the following result.

Proposition 3.1 Let \( d\mu(\theta; t) = \omega(\theta; t) d\mu(\theta) \) be a finite nonnegative measure with infinite support on the unit circle parametrized by \( z = e^{i\theta} (\theta \in [\theta_0, \theta_0 + 2\pi]) \) and depending on a parameter \( t \) varying in a real open interval containing \( t_0 \). Suppose that for almost all \( \theta \in [\theta_0, \theta_0 + 2\pi) \), \( \omega(\theta; t) \) is finite and admits partial derivative with respect to \( t \). Suppose furthermore that there exists a \( \mu \)-integrable function \( \alpha \) such that
\[
\left| \frac{\partial \omega}{\partial t}(\theta; t) \right| \leq \alpha(\theta),
\]
almost everywhere in \([\theta_0, \theta_0 + 2\pi)\). Let \( P(z; t) \) be a nonconstant monic POPUC associated with \( d\mu(\theta; t) \). Assume that \( P(\zeta_0; t_0) = 0 \). Then there exist \( \epsilon > 0 \) and \( \delta > 0 \) and a unique function, \( \zeta : I_\epsilon(t_0) \to C_\delta(\zeta_0) \), differentiable on \( I_\epsilon(t_0) \), with \( \zeta(t_0) = \zeta_0 \), and such that
\[
P(\zeta(t); t) = 0 \quad (8)
\]
for each \( t \in I_\epsilon(t_0) \).

Proof Assume that \( P \) has fixed positive degree \( n \). From (1), we see that the coefficients of \( P \) are rational functions of \( c_j(d\mu) (j = -n, \ldots, n - 2, n - 1) \), where the denominator is the determinant \( D_{n-1}(d\mu) \). Under our hypotheses, we can differentiate
\[
c_j(d\mu(\cdot; t)) = \int e^{-ij\theta} \omega(\theta; t) d\mu(\theta)
\]
under the integral sign (cf. [18, pp. 124-125]); we see immediately then that the coefficients of \( P(\cdot; t) \) are differentiable functions for each \( t \). Moreover, \( P(\xi_0; t_0) = 0 \); from this it follows that
\[
\left. \frac{\partial P}{\partial z}(z; t) \right|_{z=\xi_0,t=t_0} \neq 0,
\]
and the result is a direct consequence of the implicit function theorem (see [4, Theorem 2.4] and its remark).

We shall refer to Theorem 3.1 as circular Markov theorem with a fixed zero.
Theorem 3.1 Assume the hypotheses and notation of Proposition 3.1. Assume also that
\[ P(e^{i\theta}; t) = 0 \] for each \( t \in I_\epsilon(t_0) \) with \( \zeta_0 \neq e^{i\theta_0} \). Suppose that \( \omega(\theta; t) \) is positive and continuous for each \( \theta \in [\theta_0, \theta_0 + 2\pi) \) and \( t \in I_\epsilon(t_0) \). Suppose furthermore that the partial derivative of \( \omega(\theta; t) \) with respect to \( t \) is continuous for each \( \theta \in [\theta_0, \theta_0 + 2\pi) \) and \( t \in I_\epsilon(t_0) \). Then, \( \zeta(t) \) moves strictly clockwise along \( S^1 \) as \( t \) increases on \( I_\epsilon(t_0) \), provided that
\[
\frac{1}{\omega(\theta; t)} \frac{\partial \omega}{\partial t}(\theta; t)
\]
is a strictly increasing function of \( \theta \) on \( (\theta_0, \theta_0 + 2\pi) \).

Proof Assume that \( P \) has fixed degree \( n \geq 2 \) and write \( P_n \) instead of \( P \). By the implicit function theorem, we have
\[
\zeta'(t) = -\frac{\partial P_n}{\partial t}(\zeta(t); t) \frac{\partial P_n}{\partial z}(\zeta(t); t)
\]
for each \( t \in I_\epsilon(t_0) \). Since the leading coefficient of \( P_n(\cdot; t) \) does not depend on \( t \), (5) and (6) make it obvious that
\[
\frac{\partial P_n}{\partial t}(\zeta(t); t) = \int \frac{P_n(e^{i\theta}; t)}{e^{i\theta} - \zeta(t)} \frac{\partial P_n}{\partial \zeta}(\zeta(t); t) d\mu(\theta; t) \]
\[
\int \frac{P_n(e^{i\theta}; t)}{e^{i\theta} - \zeta(t)} d\mu(\theta; t) \]
Define the polynomial of degree \( n \) in \( z \),
\[
R(z; t) = P_n(z; t) - \frac{\partial P_n}{\partial z}(\zeta(t); t)(z - \zeta(t)).
\]
Since \( R(z; t) \) has a zero of multiplicity at least two at \( z = \zeta(t) \),
\[
\frac{zR(z; t)}{(z - \zeta(t))^2}
\]
is a nonzero polynomial of degree \( n - 1 \) in \( z \) vanishing at the origin. Therefore,
\[
0 = -\frac{1}{\zeta(t)} \int P_n(e^{i\theta}; t) \frac{R(e^{i\theta}; t)}{e^{i\theta}(e^{i\theta} - \zeta(t))} d\mu(\theta; t)
\]
\[
= \int \left[ \frac{P_n(e^{i\theta}; t)}{e^{i\theta} - \zeta(t)} \right]^2 d\mu(\theta; t) - \frac{\partial P_n}{\partial z}(\zeta(t); t) \int \frac{P_n(e^{i\theta}; t)}{e^{i\theta} - \zeta(t)} d\mu(\theta; t) \]
by (4). Combining (11) with (12) we can rewrite (10) as

\[
\zeta'(t) = -\frac{\int P_n(e^{i\theta}; t) \frac{\partial P_n}{\partial t}(e^{i\theta}; t) d\mu(\theta; t)}{\int \left| \frac{P_n(e^{i\theta}; t)}{e^{i\theta} - \zeta(t)} \right|^2 d\mu(\theta; t)}.
\]  

(13)

Write \( \xi = e^{i\theta_0} \). From (4), we also get

\[
0 = \int \frac{P_n(e^{i\theta}; t)}{e^{i\theta} - \xi} \frac{\partial P_n}{\partial t}(e^{i\theta}; t) d\mu(\theta; t).
\]  

(14)

Write \( \zeta(t) = e^{i\varphi(t)} \) (\( \varphi(t) \in [\theta_0, \theta_0 + 2\pi] \)) and let \( C(t) \) denote the denominator of the right hand side of (13). Note that

\[
\frac{i\xi}{e^{i\theta} - \xi} - \frac{i\xi(t)}{e^{i\theta} - \zeta(t)} = \frac{i(\zeta(t) - \xi(t)) e^{i\theta}}{(e^{i\theta} - \xi)(e^{i\theta} - \zeta(t))}.
\]

If (13) and (14) are multiplied by \(-i\zeta'(t)\) and \(-i\xi\), respectively, and the resulting equations are added, we have

\[
C(t)\varphi'(t) = \int \frac{i(\zeta(t) - \xi)e^{i\theta}}{(e^{i\theta} - \xi)(e^{i\theta} - \zeta(t))} P_n(e^{i\theta}; t) \frac{\partial P_n}{\partial t}(e^{i\theta}; t) d\mu(\theta; t).
\]  

(15)

Since

\[
\frac{z\cdot P_n(z, t)}{(z - \xi)(z - \zeta(t))}
\]

is a nonzero polynomial of degree \( n - 1 \) in \( z \) vanishing at the origin, (4) yields

\[
0 = \int \frac{e^{i\theta}}{(e^{i\theta} - \xi)(e^{i\theta} - \zeta(t))} |P_n(e^{i\theta}; t)|^2 d\mu(\theta; t).
\]  

(16)

Taking the partial derivative of (16) with respect to \( t \) and using (4) leads to

\[
\int \frac{e^{i\theta}}{(e^{i\theta} - \xi)(e^{i\theta} - \zeta(t))} P_n(e^{i\theta}; t) \frac{\partial P_n}{\partial t}(e^{i\theta}; t) d\mu(\theta; t)
\]

\[
= -\int \frac{e^{i\theta}}{(e^{i\theta} - \xi)(e^{i\theta} - \zeta(t))} |P_n(e^{i\theta}; t)|^2 \frac{\partial \omega}{\partial t}(\theta; t) d\mu(\theta).
\]  

(17)

Define the real-valued function

\[
\sigma(\theta; t) = \frac{1}{\omega(\theta; t)} \frac{\partial \omega}{\partial t}(\theta; t) - \frac{1}{\omega(\varphi(t); t)} \frac{\partial \omega}{\partial t}(\varphi(t); t).
\]
Combining (16) with (17) we deduce that

\[- \int \frac{e^{i\theta}}{(e^{i\theta} - \xi)(e^{i\theta} - \zeta(t))} P_n(e^{i\theta}; t) \frac{\partial P_n}{\partial t}(e^{i\theta}; t) d\mu(\theta; t)\]

\[= \int \frac{e^{i\theta}}{(e^{i\theta} - \xi)(e^{i\theta} - \zeta(t))} |P_n(e^{i\theta}; t)|^2 \sigma(\theta; t) d\mu(\theta; t). \tag{18}\]

Substituting (18) into (15), we can assert that

\[C(t)\varphi'(t) = \int \frac{i(\xi - \zeta(t)) e^{i\theta}}{(e^{i\theta} - \xi)(e^{i\theta} - \zeta(t))} |P_n(e^{i\theta}; t)|^2 \sigma(\theta; t) d\mu(\theta; t). \tag{19}\]

Observe that for each \(t \in I_\epsilon(t_0)\), the real-valued function

\[s(\theta; t; \theta_0) = \frac{i(\xi - \zeta(t)) e^{i\theta}}{(e^{i\theta} - \xi)(e^{i\theta} - \zeta(t))} = -\frac{1}{2} \sin \left(\frac{\varphi(t) - \theta}{2}\right) \sin \left(\frac{\theta_0 - \theta}{2}\right)\]

is negative for \(\theta \in (\theta_0, \varphi(t))\) and positive for \(\theta \in (\varphi(t), \theta_0 + 2\pi)\). Since, for each \(t \in I_\epsilon(t_0)\), \(\sigma(\theta; t)\) is positive for \(\theta \in (\theta_0, \varphi(t))\) and negative for \(\theta \in (\varphi(t), \theta_0 + 2\pi)\), \(\varphi'(t)\) is negative (see Fig. 1), and the theorem is proved. \(\square\)

**Remark 3.1** Theorem 3.1 specializes to [39, Theorem 3] if \(\theta_0 = 0\) (or what is the same, \(P(1; t) = 0\)) and \(d\mu(\theta) = d\theta\), the Lebesgue measure. It is important to highlight that unlike [39], where several previous results related to the particular case considered are needed, our arguments make use only of the condition (4). We also note that virtually

![Fig. 1 s and \(\sigma\) for the circular Markov theorem with a fixed zero](image-url)
[39, Theorem 1] and the main sentence of [39, Theorem 2] are already proved in [15, Section 5] \(^4\) and [10, Theorem B], \(^5\) respectively.

Even when the integrand of (19) change sign in the interval of integration, \( \varphi' \) may have a constant sign in \( I_\epsilon(t_0) \). We illustrate this possibility by proving the following result, which we will use later in Sect. 4.

**Corollary 3.1** Assume the hypotheses and notation of Theorem 3.1 and its proof, except that \( P(e^{i\theta}; t) = 0 \). Suppose that \( P(1; t) = 0 \). Set \( \theta_0 = -\pi \). Assume that \( d\mu(\theta) = d\mu(-\theta) \). Assume also that \( \omega(-\theta; t) \geq \omega(\theta; t) \) and

\[
\frac{1}{\omega(-\theta; t)} \frac{\partial \omega}{\partial t}(-\theta; t) = \frac{1}{\omega(\theta; t)} \frac{\partial \omega}{\partial t}(\theta; t)
\]

for almost all \( \theta \in (0, \varphi(t)) \) and \( t \in I_\epsilon(t_0) \). Suppose furthermore that (9) is a strictly decreasing function of \( \theta \) on \((-\pi, 0)\) and a strictly increasing function of \( \theta \) on \((0, \pi)\). Then, either \( \xi(t) \) moves strictly clockwise along \( S^1 \) as \( t \) increases on \( I_\epsilon(t_0) \) if \( \varphi(t) \in (-\pi, 0) \) or else \( \xi(t) \) moves strictly counterclockwise if \( \varphi(t) \in (0, \pi) \).

**Proof** Set \( W(\theta; t) = s(\theta; t; 0) \left| P_n(e^{i\theta}; t) \right|^2 \sigma(\theta; t) \omega(\theta; t) \). Suppose that \( \varphi(t) \in (0, \pi) \) for each \( t \in I_\epsilon(t_0) \). Observe that \( s(\theta; t; 0) \) is positive for each \( \theta \in (-\pi, 0) \cup (\varphi(t), \pi) \). Since \( \sigma(\theta; t) \) is positive for each \( \theta \in (-\pi, -\varphi(t)) \cup (\varphi(t), \pi) \), \( W(\theta; t) \) is positive for \( \theta \in (-\pi, -\varphi(t)) \cup (\varphi(t), \pi) \) (see Fig. 2). Moreover,

\[
s(-\theta; t; 0) = -\frac{\sin \left( \frac{\varphi(t) - \theta}{2} \right)}{\sin \left( \frac{\varphi(t) + \theta}{2} \right)} s(\theta; t; 0) < -s(\theta; t; 0)
\]

for each \( \theta \in (0, \varphi(t)) \). Hence

\[
C(t)\varphi'(t) > \int_{-\varphi(t)}^{\varphi(t)} W(\theta; t) d\mu(\theta)
\]

\[
= \int_{0}^{\varphi(t)} \left( W(-\theta; t) + W(\theta; t) \right) d\mu(\theta) > 0,
\]

and so \( \varphi' > 0 \).

The proof for \( \varphi(t) \in (-\pi, 0) \) is similar. \( \square \)

---

\(^4\) The reader must recall that the recurrence relation [39, (1.1)] can be transformed into the simplest form [15, (2.12)] by a normalization process (see [16, pp. 226-227]). In any case, [39, Theorem 1] is proved in a more general setting in [49, Corollary 2.14.5] (see in this regard Remark 4.1 below).

\(^5\) A refined version of [10, Theorem B] can be found in [11, Corollary 3.2], see also preprint available at arXiv:1706.05709 (2017).
Remark 3.2 We can go even further, however. Note that the result we want to prove is $S = I_1 + I_2 + I_3 + I_4 > 0$ (see Fig. 2), where

$$
I_1 = \int_{-\pi}^{-\varphi(t)} W(\theta; t) d\mu(\theta) > 0, \quad I_3 = \int_{0}^{\varphi(t)} W(\theta; t) d\mu(\theta) > 0,
$$

$$
I_2 = \int_{0}^{\varphi(t)} W(\theta; t) d\mu(\theta) < 0, \quad I_4 = \int_{\varphi(t)}^{\pi} W(\theta; t) d\mu(\theta) > 0,
$$

and, although under our hypothesis $I_2 + I_3 > 0$, there may be cases in which $I_2 + I_3 < 0$ and still $S > 0$.

Remark 3.3 POPUC with a fixed zero (see for instance [15, (2.11–2.13)]) are widely used in practice. This collection of polynomials is closely related with certain CD kernels. Indeed, given $\xi \in S^1$ and a measure $d\mu$ as defined in Sect. 2, the corresponding normalized POPUC of degree $n$ with parameter

$$
b(\xi) = \frac{q_{n-1}(\xi)}{q^{*}_{n-1}(\xi)} \tag{20}
$$

is given by (see [53, (3.7)–(3.8)] and [49, Theorem 2.14.3. (ii)])

$$
p_n(z; b(\xi); d\mu) = -\frac{b(\xi)}{q^{*}_{n-1}(\xi)} (1 - z\xi) K_{n-1}(\xi, z).
$$

Therefore, the zeros of $K_{n-1}(\xi, \cdot)$ are precisely the zeros of $p_n(\cdot; b(\xi); d\mu)$ other than $\xi$, and the zeros of $p_n(\cdot; b(\xi); d\mu)$ are $\xi$ plus the zeros of $K_{n-1}(\xi, \cdot)$.

With Theorem 3.1 under our belt, the following consequence essentially follows as for the case of OPRL (see [51, Theorem 6.12.2]).
Corollary 3.2 Let $d\mu_1(\theta) = \omega_1(\theta) \, d\mu(\theta)$ and $d\mu_2(\theta) = \omega_2(\theta) \, d\mu(\theta)$ be two non-negative measures with infinite support on the unit circle parametrized by $z = e^{i\theta}$ ($\theta \in [\theta_0, \theta_0 + 2\pi)$) and satisfying the hypotheses of Theorem 3.1. Suppose that $\omega_1(\theta)$ and $\omega_2(\theta)$ are finite, positive and continuous for almost all $\theta$. Let $\omega_2(\theta)/\omega_1(\theta)$ be a strictly increasing function on $[\theta_0, \theta_0 + 2\pi)$. Fix $n \geq 2$ and let $\theta_0 + 2\pi > \theta_{1,1} > \cdots > \theta_{n,1} \geq \theta_0$ and $\theta_0 + 2\pi > \theta_{1,2} > \cdots > \theta_{n,2} \geq \theta_0$ denote the arguments of the zeros of the POPUC of degree $n$ associated with $d\mu_1$ and $d\mu_2$, respectively. Then if $\theta_{k,1} = \theta_{k,2}$ for some $k \in \{1, \ldots, n\}$, we have

$$\theta_{j,1} < \theta_{j,2}$$

for each $j \neq k$.

Proof Define $d\sigma(\theta; t) = \omega(\theta; t) \, d\mu(\theta)$, where $\omega(\theta; t) = (1 - t)\omega_1(\theta) + t \, \omega_2(\theta)$ and $t \in [0, 1]$. Now one has that $\omega(\theta; t)$ is finite for almost all $\theta \in [\theta_0, \theta_0 + 2\pi)$ and admits partial derivative with respect to $t$ by construction; moreover, since

$$\left| \frac{\partial \omega}{\partial t} (\theta; t) \right| \leq \omega_1(\theta) + \omega_2(\theta),$$

$d\sigma$ satisfies the hypotheses of Proposition 3.1. By virtue of Remark 3.3, we can always construct a POPUC of degree $n$ associated with $d\sigma$ with a zero at $\theta_{k,1}$. We also see that

$$\frac{1}{\omega(\theta; t)} \frac{\partial \omega}{\partial t} (\theta; t) = \frac{1}{t} - \frac{1}{1 - t + \frac{\omega_2(\theta)}{\omega_1(\theta)} \, t}$$

is a strictly increasing function of $\theta$ on $[\theta_0, \theta_0 + 2\pi)$ for each $t \in (0, 1)$. Finally, since $\omega(\theta; 0) = \omega_1(\theta)$ and $\omega(\theta; 1) = \omega_2(\theta)$, the result is a consequence of Theorem 3.1. □

We will use the same arguments, as in the proof of Theorem 3.1, to prove the next theorem, which we will refer as the circular Markov theorem for complex conjugate zeros.

Theorem 3.2 Assume the hypotheses and notation of Theorem 3.1 and its proof, except that $P(e^{i\theta}; t) = 0$. Set $\theta_0 = -\pi$. Suppose that $P(\xi(t); t) = 0$ and $\phi(t) \in (0, \pi)$ (mod $[-\beta, \beta]$) for each $t \in I_e(t_0)$. Then, $\zeta(t)$ moves strictly counterclockwise along $S^1$ as $t$ increases on $I_e(t_0)$, provided that (9) is a strictly decreasing function of $\theta$ on $(-\pi, 0)$ and a strictly increasing function of $\theta$ on $(0, \pi)$.

Proof Replacing $\zeta(t)$ in (13) by $\overline{\zeta(t)}$, we get

$$\overline{\zeta'(t)} = - \int \frac{P_n(e^{i\theta}; t) \, \frac{\partial P_n}{\partial \overline{\xi(t)}} (e^{i\theta}; t) \, d\mu(\theta; t)}{|e^{i\theta} - \overline{\xi(t)}|^2} \, d\mu(\theta; t).$$

(21)
Now let $C(t)$ denotes the sum of the denominators of the right-hand sides of (13) and (21). Note that $-i\zeta(t)\zeta'(t) = \varphi'(t) = i\zeta(t)\zeta'(t)$ and

$$\frac{i\zeta(t)}{z - \zeta(t)} - \frac{i\zeta(t)}{z - \zeta(t)} = 2\text{Im}(\zeta(t))\frac{z}{(z - \zeta(t))(z - \zeta(t))}.$$  

If (13) and (21) are multiplied by $-i\zeta(t)$ and $i\zeta(t)$, respectively, and the resulting equations are added, we have

$$C(t)\varphi'(t) = -2\text{Im}(\zeta(t)) \int \frac{e^{i\theta}}{(e^{i\theta} - \zeta(t))(e^{i\theta} - \zeta(t))} P_n(e^{i\theta}; t) \frac{\partial P_n(e^{i\theta}; t)}{\partial t} d\mu(\theta; t).$$

(22)

Replacing $\xi$ in (17) by $\zeta(t)$, we get

$$\int \frac{e^{i\theta}}{(e^{i\theta} - \zeta(t))(e^{i\theta} - \zeta(t))} P_n(e^{i\theta}; t) \frac{\partial P_n(e^{i\theta}; t)}{\partial t} d\mu(\theta; t) = - \int \frac{e^{i\theta}}{(e^{i\theta} - \zeta(t))(e^{i\theta} - \zeta(t))} |P_n(e^{i\theta}; t)|^2 \frac{\partial \omega}{\partial t} (\theta; t) d\mu(\theta; t).$$

Replacing $\xi$ in (18) by $\zeta(t)$, we obtain

$$\int \frac{e^{i\theta}}{(e^{i\theta} - \zeta(t))(e^{i\theta} - \zeta(t))} P_n(e^{i\theta}; t) \frac{\partial P_n(e^{i\theta}; t)}{\partial t} d\mu(\theta; t) = - \int \frac{e^{i\theta}}{(e^{i\theta} - \zeta(t))(e^{i\theta} - \zeta(t))} |P_n(e^{i\theta}; t)|^2 \sigma(\theta; t) d\mu(\theta; t).$$

(23)

Substituting (23) into (22), we can assert that

$$\varphi'(t) = -2\frac{\text{Im}(\zeta(t))}{C(t)} \int \frac{e^{i\theta}}{(e^{i\theta} - \zeta(t))(e^{i\theta} - \zeta(t))} |P_n(e^{i\theta}; t)|^2 \sigma(\theta; t) d\mu(\theta; t).$$

Observe that, for each $t \in I_\epsilon(t_0)$, the real-valued function

$$\frac{e^{i\theta}}{(e^{i\theta} - \zeta(t))(e^{i\theta} - \zeta(t))} = \frac{1}{2} \frac{1}{\cos \theta - \cos \varphi(t)},$$

is negative for $\theta \in (-\pi, -\varphi(t)) \cup (\varphi(t), \pi)$ and positive for $\theta \in (-\varphi(t), \varphi(t))$. Since, for each $t \in I_\epsilon(t_0)$, $\sigma(\theta; t)$ is positive for $\theta \in (-\pi, -\varphi(t)) \cup (\varphi(t), \pi)$ and negative for $\theta \in (-\varphi(t), 0) \cup (0, \varphi(t)), \varphi'(t)$ is positive (see Fig. 3), which proves the theorem.

$\square$

**Remark 3.4** If $d\mu(\theta; t) = d\mu(-\theta; t)$ for each $\theta \in [-\pi, \pi]$ (symmetric measure) and $b = \pm 1$, then the nonreal zeros of the corresponding POPUC occur in complex
conjugate pairs. Under this more restrictive condition, Theorem 3.2 ‘coincides’ with Corollary 3.1 in case of \( \omega(-\theta; t) = \omega(\theta; t) \).

We can now rephrase Corollary 3.2 as follows.

**Corollary 3.3** Let \( d\mu_1(\theta) = \omega_1(\theta) d\mu(\theta) \) and \( d\mu_2(\theta) = \omega_2(\theta) d\mu(\theta) \) be two non-negative symmetric measures with infinite support on the unit circle parametrized by \( z = e^{i\theta} \) (\( \theta \in [-\pi, \pi) \)) and satisfying the hypotheses of Theorem 3.1. Suppose that \( \omega_1(\theta) \) and \( \omega_2(\theta) \) are finite, positive and continuous for almost all \( \theta \). Let \( \omega_2(\theta)/\omega_1(\theta) \) be a strictly decreasing function on \((-\pi, 0)\) and a strictly increasing function on \((0, \pi)\). Fix \( n \geq 2 \) and let \( \pi > \theta_{1,1} > \cdots > \theta_{n,1} \geq -\pi \) and \( \pi > \theta_{1,2} > \cdots > \theta_{n,2} \geq -\pi \) denote the arguments of the zeros of the POPUC of degree \( n \) associated with \( d\mu_1 \) and \( d\mu_2 \), respectively. Then,

\[ \theta_{j,1} < \theta_{j,2} \]

for each \( j \in \{1, \ldots, \lfloor n/2 \rfloor \} \).

**Proof** We proceed in the same manner as in the proof of Corollary 3.2, but now we construct a POPUC of degree \( n \) associated with \( d\sigma \) (defined in Corollary 3.2), \( P_n(\cdot; b; d\sigma) \), whose parameter \( b \) is equal to \( \pm 1 \). \( \square \)

The next proposition is nothing more than a direct consequence of a result by V. B. Lidskii [37] (see also [6, Section V.6.]).

**Proposition 3.2** Let \( d\mu(\theta) \) be a finite nonnegative measure with infinite support on the unit circle parametrized by \( z = e^{i\theta} \). Let \( b \) be a function of a real variable \( t \) defined on a real open interval containing \( t_0 \) with values in \( S^1 \). Assume the existence of the derivative of \( b(t) \) near \( t = t_0 \). Let \( P \) be a monic POPUC defined as in (3) for \( b = b(t) \). Suppose that \( P(\xi_0; t_0) = 0 \). Then, there exist \( \epsilon > 0 \) and \( \delta > 0 \) and a unique function, \( \xi : I_\epsilon(t_0) \to C_\delta(\xi_0) \), differentiable on \( I_\epsilon(t_0) \), with \( \xi(t_0) = \xi_0 \), and such that (8) holds.
for each \( t \in I_e(t_0) \). Furthermore, \( \zeta(t) \) moves strictly counterclockwise along \( S^1 \) as \( t \) increases on \( I_e(t_0) \), provided that

\[
i \frac{b(t)}{b'(t)}
\]

is strictly positive.

**Proof** Clearly, the first two statements of the theorem follow as in Theorem 3.1. Throughout the proof, the matrix-valued function \( G(t) \) denotes the matrix (7) for \( a_j = a_j(t) \) and \( b = b(t) \). In view of the implicit function theorem, we can choose a normalized eigenpair \( \{ \zeta(t), p(t) \} \) that depends differentiably on \( t \). Write \( \zeta(t) = e^{i\phi(t)} \).

Since \( G(t) \) is normal (in particular, unitary), \( \zeta(t) = (p(t), G(t)p(t)) \). Moreover, since \( \zeta(t) \) is a simple eigenvalue of \( G(t) \), \( \zeta'(t) = (p(t), G'(t)p(t)) \) (see [35, pp. 134–135]). Thus

\[
\phi'(t) = (p(t), -i G^*(t)G'(t)p(t)) = (p(t), -i G_n^*(t)G_n'(t)p(t))
\]

\[
= i \frac{b(t)}{b'(t)}|p_{n-1}(t)|^2,
\]

\( p_{n-1}(t) \) being the last component of \( p(t) \). Finally, the result follows because \( p_{n-1}(t) \) is nonzero. \( \Box \)

In this section, we have given readers a taste of the flexibility of our arguments, hoping that they can easily adapt it to a wide variety of situations not considered in this work.

### 4 Examples

In this section, we consider some applications of the results of Sect. 3 to specific weight functions on the unit circle. The reader should convince himself that the hypotheses of Proposition 3.1 are fulfilled.

#### 4.1 Degree One Bernstein–Szegö Polynomials

Let

\[
d\mu_\zeta(\theta) = \frac{1 - |\zeta|^2}{|1 - \zeta e^{i\theta}|^2} \frac{d\theta}{2\pi}
\]

for \( \zeta \in \mathbb{D} \) and \( \theta \in [0, 2\pi) \). If one set \( \zeta = re^{i\varphi} \) for \( \varphi \in [0, 2\pi) \) and defines the Poisson kernel by

\[
P_r(\theta, \varphi) = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \varphi)},
\]

\[\text{This is true for any unreduced Hessenberg matrix (cf. [36, Lemma 2.1]), although in this case all the components of p are nonzero (see [46, Chapter 4]).}\]
then we may write \( d\mu_\zeta(\theta) = P_r(\theta, -\varphi) \frac{d\theta}{2\pi} \). The OPUC for this measure are given by (cf. [46, Example 1.6.2])

\[
Q_n(z; d\mu_\zeta) = z^n - \bar{\zeta} z^{n-1} \quad (n = 1, 2, \ldots).
\]

An easy computation shows that

\[
A(\theta; r, \varphi) = \frac{1}{P_r(\theta, -\varphi)} \frac{\partial P_r}{\partial r}(\theta, -\varphi) = \frac{4r - 2(r^2 + 1) \cos(\theta + \varphi)}{(r^2 - 1)(r^2 - 2r \cos(\theta + \varphi) + 1)},
\]

\[
B(\theta; r, \varphi) = \frac{1}{P_r(\theta, -\varphi)} \frac{\partial P_r}{\partial \varphi}(\theta, -\varphi) = -\frac{2r \sin(\theta + \varphi)}{r^2 - 2r \cos(\theta + \varphi) + 1}.
\]

And once we have reached this point, the first thing we must do is to verify if the functions \( A \) and \( B \) increase (and decrease) at most once on \((0, 2\pi)\). (This is a necessary, although not sufficient, condition for a successful use of the results of the previous section.) For illustration, consider the case \( A(\theta; 0.1, 0) \) (solid line), \( A(\theta; 0.5, 0) \) (dash line), \( A(\theta; 0.5, \pi) \) (dotted line) and \( A(\theta; 0.1, 3\pi/2) \) (dash-dotted line), and \( B(\theta; 0.1, 0) \) (solid line), \( B(\theta; 0.5, 0) \) (dash line), \( B(\theta; 0.5, \pi) \) (dotted line) and \( B(\theta; 0.1, 3\pi/2) \) (dash-dotted line) displayed in Fig. 4. In all these cases, the function \( B \) does not fulfill the required conditions.

In what follows, for simplicity, we will specialize to the case \( r \in (0, 1) \) and \( \varphi = 0 \), that is, \( \zeta = r \). In this case,

\[
A(\theta; r; 0) = \frac{4r - 2(r^2 + 1) \cos \theta}{(r^2 - 1)(r^2 - 2r \cos \theta + 1)}
\]

is a strictly decreasing function of \( \theta \) on \((0, \pi)\) and a strictly increasing function of \( \theta \) on \((\pi, 2\pi)\). Fix \( \xi \in S^1 \). From Remark 3.3, it may be concluded that the zeros of

\[
P_{n+1}(z; \xi^n - 1, \frac{\xi - r}{\bar{\xi} - r}; d\mu_\zeta) = z^{n+1} - rz^n + r \left( \frac{\xi^n - 1}{\bar{\xi} - r} z - \frac{\xi^n - 1}{\xi - r} \right)
\]

are \( \xi \) plus the zeros of \( K_n(\xi, \cdot; d\mu_\zeta) \). Hence, the nonreal zeros of \( K_n(\pm 1, \cdot; d\mu_\zeta) \) occur in complex conjugate pairs. Thus, by the circular Markov theorem for complex conjugate zeros, these zeros move strictly clockwise on the upper semicircle as \( r \) increases.
Fig. 5 Zeros of $K_{14}(1, \cdot; d\mu_\xi)$ (left plot) and $K_{14}(i, \cdot; d\mu_\xi)$ (right plot) for certain values of $r$

on $(0, 1)$. We can evidently not expect to obtain information about the behaviour of the zeros of $K_n(\xi, \cdot; d\mu_\xi)$ for each $\xi \in \mathbb{S}^1$. Figure 5 shows the behaviour of the zeros of $K_{14}(1, \cdot; d\mu_\xi)$ and $K_{14}(i, \cdot; d\mu_\xi)$ for $r = 0.1$ (discs), $r = 0.5$ (squares) and $r = 0.9$ (diamonds). The zeros of $K_{14}(1, \cdot; d\mu_\xi)$ behave exactly as predicted, but on the other hand the zeros of $K_{14}(i, \cdot; d\mu_\xi)$ do not behave in the same way.

4.2 Single Nontrivial Moment

Let

$$d\mu_r(\theta) = (1 - r \cos \theta) \frac{d\theta}{2\pi}$$

for $r \in (0, 1)$ and $\theta \in [0, 2\pi)$. The OPUC for this measure are given by (see [46, Example 1.6.4])

$$Q_n(z; d\mu_r) = \frac{1}{d_n} \sum_{j=0}^n d_j z^j \quad (n = 0, 1, \ldots).$$

where

$$d_j = \frac{d_{j+1}^+ - d_{j+1}^-}{d_+ - d_-},$$

and $d_\pm$ are the roots of $rd^2 - 2d + r = 0$, that is,

$$d_\pm = \frac{1}{r} \pm \sqrt{\frac{1}{r^2} - 1}.$$
Now we proceed as in Sect. 4.1. Indeed, since the function

\[ A(\theta; r) = -\frac{\cos \theta}{1 - r \cos \theta} \]

is a strictly increasing function of \( \theta \) on \((0, \pi)\) and a strictly decreasing function of \( \theta \) on \((\pi, 2\pi)\), by the circular Markov theorem for complex conjugate zeros, we can conclude that the nonreal zeros of

\[ P_{n+1}(z; \pm 1; d\mu_r) = \frac{1}{d_n} \sum_{j=0}^{n+1} (d_{j-1} \mp d_{n-j}) z^j \]

move strictly counterclockwise on the upper semicircle as \( r \) increases on \((0, 1)\). By virtue of Corollary 3.3, we may also compare the zeros of \( P_{n+1}(\cdot; \pm 1; d\mu_r) \) with those of \( P_{n+1}(z; \pm 1; d\mu_\zeta) \) for \( \varphi = 0 \). Let \( 2\pi > \theta_1(d\mu_\zeta) > \cdots > \theta_{n-1}(d\mu_\zeta) \geq 0 \) and \( 2\pi > \theta_1(d\mu_r) > \cdots > \theta_n(d\mu_r) \geq 0 \) denote the arguments of the zeros of \( P_{n+1}(z; \pm 1; d\mu_\zeta) \) and \( P_{n+1}(z; \pm 1; d\mu_r) \), respectively. Define \( \omega(\theta; r) = (1 - r \cos \theta) \). An easy calculation reveals that the function

\[ \frac{P_r(\theta; 0)}{\omega(\theta; r)} = \frac{1 - r^2}{(1 + r^2 - 2r \cos \theta)(1 - r \cos \theta)} \]

is a strictly decreasing function of \( \theta \) on \((0, \pi)\) and a strictly increasing function of \( \theta \) on \((\pi, 2\pi)\). Thus, Corollary 3.2 implies that

\[ \theta_j(d\mu_r) < \theta_j(d\mu_\zeta) \quad (24) \]

when \( \theta_j(d\mu_r) \in (0, \pi) \). Figure 6 shows the behaviour of the zeros of \( K_{14}(1, \cdot; d\mu_\zeta) \) (discs) and \( K_{14}(1, \cdot; d\mu_r) \) (squares) for \( r = 0.8 \) and \( P_{15}(\cdot; i; d\mu_\zeta) \) (discs) and \( P_{15}(\cdot; i; d\mu_r) \) (squares) for \( r = 0.8 \). Although the zeros of \( K_{14}(1, \cdot; d\mu_\zeta) \) and \( K_{14}(1, \cdot; d\mu_r) \) behave exactly as predicted, the zeros of \( P_{15}(\cdot; i; d\mu_\zeta) \) and \( P_{15}(\cdot; i; d\mu_r) \), as expected, do not satisfy (24).

4.3 Jacobi–Szegő Polynomials

Let

\[ d\mu^{(r,s)}(\theta) = \frac{\Gamma(r + is + 1)^2}{\Gamma(2r + 1)} \frac{(2 - 2 \cos \theta)^r}{2\pi} \left( -e^{i\theta} \right)^s d\theta \]

for \( r \in (-1/2, \infty), s \in (-\infty, \infty), \) and \( \theta \in [-\pi, \pi) \). There are a variety of specific problems, particularly in statistical physics, which are closely related to this measure. Indeed, \( d\mu^{(r,s)} \) belongs to a class of measures introduced by Fisher and Hartwig in [21] which has been the subject of numerous investigations (see [14] and the references
given there). The following alternative expression for \( \theta \in [0, 2\pi) \) is also found in the literature (see [30]):

\[
d\mu^{(r,s)}(\theta) = \frac{|\Gamma(r + is + 1)|^2}{\Gamma(2r + 1)} 2^{2r} e^{(\pi - \theta) s} \left( \sin \frac{\theta}{2} \right)^{2r} \frac{d\theta}{2\pi}.
\]

The OPUC for this measure are given by (see [30, Sections 1.1 and 1.2], [5, Section 3] and [42, Theorem 5.1] for more details)

\[
Q_n(z; d\mu^{(r,s)}) = \binom{2r + 1}{r + is + 1} \binom{-n}{2r + 1} 2F1 \left( -n, r + is + 1; 1 - z \right) (n = 0, 1, \ldots).
\]

These polynomials can be expressed in terms of Heisenberg polynomials,\(^7\) which live on the Heisenberg group (see [30, (1.7)]), that is,

\[
Q_n(e^{i\theta}; d\mu^{(r,s)}) = \binom{n!}{r + is + 1} e^{in\theta/2} C_n^{(r-is, r+is+1)}(e^{i\theta/2}).
\]

Define \( \omega_1(\theta; r, s) = (2 - 2 \cos \theta)^r (-e^{i\theta})^s \) and \( \omega_2(\theta; r, s) = 2^r e^{(\pi - \theta) s} \left( \sin \frac{\theta}{2} \right)^{2r} \).

Hence

\[
A(\theta; r, s) = \frac{1}{\omega_1(\theta; r, s)} \frac{\partial \omega_1(\theta; r, s)}{\partial r} = \log(2 - 2 \cos \theta),
\]

\[
B(\theta; r, s) = \frac{1}{\omega_2(\theta; r, s)} \frac{\partial \omega_2(\theta; r, s)}{\partial s} = \pi - \theta.
\]

\(^7\) We use the notation, now standard, of [23,30].
We can therefore apply the results of Sect. 3 to study the variation of zeros of certain POPUC associated with $d\mu^{(r, s)}$.

Given any $\xi = e^{i\theta_{lr}(r, s)}$, we define

$$b^{(r, s)}(\xi) = \frac{(r + is + 1)_{n+1}}{(r - is + 1)_{n+1}} \frac{2F1 \left( \begin{array}{c} -n, r - is + 1 \\ 2r + 1 \end{array} ; 1 - \xi \right)}{2F1 \left( \begin{array}{c} -n, r - is \\ 2r + 1 \end{array} ; 1 - \xi \right)}.$$

By Remark 3.3, $P_{n+1}(z; b^{(r, s)}(\xi); d\mu^{(r, s)})$ has a zero at $z = \xi$. Assume $\xi = 1$ (or, what is the same, $\theta_0 = 0$). Since $B(\theta; r, s)$ is a strictly decreasing function of $\theta$ on $(0, 2\pi)$, by the circular Markov theorem with a fixed zero, the nonreal zeros of $P_{n+1}(z; b^{(r, s)}(1); d\mu^{(r, s)})$ move strictly clockwise along $\mathbb{S}^1$ as $s$ increases on $(-\infty, \infty)$. This is the main result of [19] (see Theorem 1.2 therein). Indeed, since

$$b^{(r, s)}(1) = \frac{(r + is + 1)_{n+1}}{(r - is + 1)_{n+1}},$$

we may conclude that

$$P_{n+1}(z; b^{(r, s)}(1); d\mu^{(r, s)}) = \frac{(2r + 2)_{n}}{(r + is + 1)_{n}} (z - 1) 2F1 \left( \begin{array}{c} -n, r + is + 1 \\ 2r + 2 \end{array} ; 1 - z \right).$$

Thus, for each $r \in (1/2, \infty)$, the zeros of the polynomial

$$f_n(z; r, s) = 2F1 \left( \begin{array}{c} -n, r + is \\ 2r \end{array} ; 1 - z \right)$$

move strictly clockwise along $\mathbb{S}^1$ as $s$ increases on $(-\infty, \infty)$. But this is also true whenever $r \in (0, 1/2)$ (see [11, Example 3.1]). Thus warned, the reader should recall that our conditions are only sufficient.

**Remark 4.1** Since

$$P_{n+1}(z; \frac{r + is}{r - is} b^{(r, s)}(1); d\mu^{(r, s)}) = \frac{(2r)_{n+1}}{(r + is)_{n+1}} f_{n+1}(z; r, s)$$

$$= \frac{(n + 1)!}{(r + is)_{n+1}} e^{i(\theta_{lr}(r, s)/2)} C_{n+1}^{(r-is, r+is)} (e^{i\theta/2}),$$

whenever $r \in (-1/2, \infty) \setminus \{0\}$ and $s \in (-\infty, \infty)$, it follows (for example by contradiction and using [3, (2.5.16)]) that $f_{n+1}(\cdot; r, s)$ and $f_{n+2}(\cdot; r, s)$ are “consecutive” coprime POPUC; whence [49, Corollary 2.14.5] shows that their zeros strictly interlace (in the sense explained in [13, Definition 1.2]) on $\mathbb{S}^1$. This specializes to the result
of [19, Theorem 1.1] if \( r \in (0, \infty) \). For \( r = 0 \), we have

\[
P_{n+1} \left( z; b^{(0,s)}(1); d\mu^{(0,s)} \right) = \frac{(n+1)!}{(is)_{n+1}} (z-1) g_n(z; s),
\]

where

\[
g_n(z; s) = \binom{-n, is + 1}{2} \binom{1-z}
\]

and so, by the argument above, it can be also shown that the zeros of \( g_{n+1}(\cdot; s) \) and \( g_{n+2}(\cdot; s) \) strictly interlace on \( \mathbb{S}^1 \).

As far as we know, the dependence of the zeros of \( f_n(\cdot; r, s) \) on \( r \) has been studied only when \( s = 0 \) (see [20, Theorem 2]). However, the case \( d\mu^{(r)} = d\mu^{(r,0)} \) (see [31, Example 8.2.5]) is especially simple because there is a direct connection with the ultrashperical polynomials.\(^8\) Indeed, by (27), we have

\[
f_n(e^{i\theta}; r, 0) = \frac{n!}{(2r)^n} e^{in\theta/2} C^{(r,r)}_n(e^{i\theta/2}) = \frac{n!}{(2r)^n} e^{in\theta/2} C^{(r)}_n \left( \cos \frac{\theta}{2} \right),
\]

where \( C^{(r)} \) denotes an ultrashperical polynomial (see [30, (1.9)]). In any case, since the nonreal zeros of

\[
P_n(\cdot; -1; d\mu^{(r)}) = \frac{(2r)_n}{(r)_n} f_n(\cdot; r, 0)
\]

occur in complex conjugate pairs, by the circular Markov theorem for complex conjugate zeros, we can conclude that the zeros of this polynomial move strictly counterclockwise on the upper semicircle and strictly clockwise on the lower semicircle as \( r \) increases on \((-1/2, \infty)\). We now turn to the general case \( s \in (-\infty, \infty) \).

\(^8\) This allows us to use an old result due to Stieltjes (see [50, p. 389]).
Since \( A(\theta; r, s) = A(-\theta; r, s) \) is a strictly decreasing function of \( \theta \) on \( (-\pi, 0) \) and a strictly increasing function of \( \theta \) on \( (0, \pi) \) and \( s \in [0, \infty) \), Corollary 3.1 implies that for each \( s \in [0, \infty) \) the zero of \( f_n(\cdot; r, s) \) move strictly counterclockwise on the upper semicircle as \( r \) increases on \( (1/2, \infty) \). In exactly the same way we may show that for each \( s \in (-\infty, 0] \) the zero of \( f_n(\cdot; r, s) \) move strictly clockwise on the lower semicircle as \( r \) increases on \( (1/2, \infty) \). Figure 7 shows the behaviour of the zeros of \( f_{10}(\cdot; r, 1) \) and \( f_{10}(\cdot; r, -2) \) for \( r = 0.1 \) (discs), \( r = 1 \) (squares) and \( r = 17 \) (diamonds). Note that the zeros of \( f_{10}(\cdot; r, 1) \) whose arguments lie between 0 and \( \pi \) and the zeros of \( f_{10}(\cdot; r, -2) \) whose arguments lie between \(-\pi\) and 0 behave exactly as predicted; however, the remaining zeros are not necessarily monotone functions of \( r \).

Acknowledgements

The author thanks the anonymous reviewers. Their pertinent questions and comments have improved the manuscript. This work is supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/ MCTES.

References

1. Ammar, G., Gragg, W., Reichel, L.: Constructing a unitary Hessenberg matrix from spectral data. In: Numerical Linear Algebra, Digital Signal Processing and Parallel Algorithms (Leuven, 1988), NATO Adv. Sci. Inst. Ser. F Compt. Systems Sci., vol. 70, pp. 385–395. Springer, Berlin (1991)
2. Ammar, G.S., He, C.: On an inverse eigenvalue problem for unitary Hessenberg matrices. Linear Algebra Appl. 15, 263–271 (1995)
3. Andrews, G.E., Askey, R., Roy, R.: Special Functions, Encyclopedia of Mathematics and its Applications, vol. 71. Cambridge University Press, Cambridge (1999)
4. Ashbaugh, M.S., II, E.M.H.: Perturbation theory for shape resonances and large barrier potentials. Comm. Math. Phys. 83, 151–170 (1982)
5. Askey, R.: Some open problems about special functions and computations. In: International Conference on Special Functions: Theory and Computation (Turin, 1984), vol. Special Volume, pp. 1–22 (1985)
6. Atkinson, F.V.: Discrete and continuous boundary problems, Mathematics in Science and Engineering, vol. 8. Academic Press, New York-London (1964)
7. Bohnhorst, B.: Beiträge zur numerischen Behandlung des unitären Eigenwertproblems. Ph.D. thesis, Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany (1993)
8. Breuer, J., Seelig, E.: On the spacing of zeros of paraorthogonal polynomials for singular measures. arXiv:1908.06737 (2019)
9. Bunse-Gerstner, A., He, C.: On a Sturm sequence of polynomials for unitary Hessenberg matrices. SIAM J. Matrix Anal. Appl. 16, 1043–1055 (1995)
10. Castillo, K.: Monotonicity of zeros for a class of polynomials including hypergeometric polynomials. Appl. Math. Comput. 266, 183–193 (2015)
11. Castillo, K.: On monotonicity of zeros of paraorthogonal polynomials on the unit circle. Linear Algebra Appl. 580, 475–490 (2019)
12. Castillo, K., Cruz-Barroso, R., Perdomo-Pío, F.: On a spectral theorem in para-orthogonality theory. Pac. J. Math. 208, 71–91 (2016)
13. Castillo, K., Petronilho, J.: Refined interlacing properties for zeros of paraorthogonal polynomials on the unit circle. Proc. Am. Math. Soc. 146, 3285–3294 (2018)
14. Deift, P., Its, A., Krousovsky, I.: Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher–Hartwig singularities. Ann. Math. 174, 1243–1299 (2011)
15. Delsarte, P., Genin, Y.: The tridiagonal approach to Szegő orthogonal polynomials, Toeplitz linear systems, and related interpolation problems. SIAM J. Math. Anal. 19(3), 718–735 (1988)
16. Delsarte, P., Genin, Y.: Tridiagonal approach to the algebraic environment of Toeplitz matrices. I. Basic results. SIAM J. Matrix Anal. Appl. 12(2), 220–238 (1991)
17. Delsarte, P., Genin, Y.: Tridiagonal approach to the algebraic environment of Toeplitz matrices. II. Zeros and eigenvalue problems. SIAM J. Matrix Anal. Appl. 12(3), 432–448 (1991)
18. Dieudonné, J.: Treatise on analysis. Vol. II. Translated from the French by I. G. Macdonald. Pure and Applied Mathematics, vol. 10-II. Academic Press, New York-London (1970)
19. Dimitrov, D.K., Ranga, A.S.: Zeros of a family of hypergeometric para-orthogonal polynomials on the unit circle. Math. Nachr. 65, 41–52 (2013)
20. Driver, K.: Zeros of the hypergeometric polynomials \( F(-n; b; 2b; z) \). Indag. Math. N.S. 11(1), 43–51 (2000)
21. Fisher, M.E., Hartwig, R.E.: Toeplitz determinants. Some applications, theorems and conjectures. Adv. Chem. Phys. 15, 333–353 (1968)
22. Freud, G.: Orthogonal Polynomials. Pergamon Press, Oxford (1971)
23. Gasper, G.: Orthogonality of certain functions with respect to complex valued weights. Can. J. Math. XXXIII 1261–1270 (1981)
24. Geronimus, Y.L.: On the trigonometric moment problem. Ann. Math. 47(2), 742–761 (1946)
25. Geronimus, Y.L.: Polynomials orthogonal on a circle and their applications. In: Series and Approximation, I, vol. 3, pp. 1–78. American Mathematical Society (1962)
26. Gragg, W.B.: Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and the Gauss quadrature on the unit circle (in Russian). Numerical Methods in Linear Algebra. Moskov. Gos. Univ., Moscow, pp. 16–32 (1982)
27. Gragg, W.B.: The QR algorithm for unitary Hessenberg matrices. J. Comput. Appl. Math. 16, 1–8 (1986)
28. Gragg, W.B.: Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and the Gauss quadrature on the unit circle. J. Comput. Appl. Math. 46, 183–198 (1993)
29. Gragg, W.B., Reichel, L.: A divide and conquer method for unitary and orthogonal eigenproblems. Numer. Math. 57, 695–718 (1990)
30. Greiner, P.C., Koornwinder, T.H.: Variations on the Heisenberg spherical harmonics. Report ZW 186/83, Mathematisch Centrum, Amsterdam (1983)
31. Ismail, M.E.H.: Classical and quantum orthogonal polynomials in one variable, Encyclopedia of Mathematics and Its Applications, vol. 98. Cambridge University Press, Cambridge (2005)
32. Jones, W.B., Njåstad, O., Thron, W.J.: Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle. Bull. Lond. Math. Soc. 21, 113–152 (1989)
33. Killip, R., Nenciu, I.: CMV: The unitary analogue of Jacobi matrices. Commun. Pure Appl. Math. LX 1148–1188 (2007)
34. Kroó, A., Peherstorfer, F.: On the zeros of polynomials of minimal \( L_p \) norm. Proc. Am. Math. Soc. 101, 652–656 (1987)
35. Lax, P.D.: Linear algebra and its applications, second edn. Pure and Applied Mathematics (Hoboken). Wiley-Interscience, John Wiley & Sons, Hoboken, NJ (2007)
36. Lehoucq, R.B.: Analysis and implementation of an implicitly restarted Arnoldi iteration. Ph.D. thesis, Rice University, Houston, Texas (1995)
37. Lidskii, V.B.: Oscillation theorems for a canonical system of differential equations (in Russian). Dokl. Akad. Nauk SSSR (N.S.) 102, 877–880 (1955)
38. Lubinsky, D.S.: Local asymptotics for orthogonal polynomials on the unit circle via universality. To appear in J. Anal. Math.
39. Lun, Y.C.: On zeros of paraorthogonal polynomials. Proc. Am. Math. Soc. 8, 3389–3399 (2019)
40. Markoff, A.: Sur les racines de certaines équations (second note). Math. Ann. 27, 177–182 (1886)
41. Martínez-Finkelshtein, A., Simanek, B., Simon, B.: Poncelet’s theorem, paraorthogonal polynomials and the numerical range of compressed multiplication operators. Adv. Math. 349, 992–1035 (2019)
42. Ranga, A.S.: Szegő polynomials from hypergeometric functions. Proc. Am. Math. Soc. 138, 4243–4247 (2010)
43. Simanek, B.: Zeros of non-Baxter paraorthogonal polynomials on the unit circle. Constr. Approx. 35, 107–121 (2012)
44. Simanek, B.: An electrostatic interpretation of the zeros of paraorthogonal polynomials on the unit circle. SIAM J. Math. Anal. 48, 2250–2268 (2016)
45. Simanek, B.: Zero spacings of paraorthogonal polynomials on the unit circle. arXiv:1907.01604 (2019)
46. Simon, B.: Orthogonal polynomials on the unit circle. Part I. Classical Theory. American Mathematical Society College Publications, vol. 54. American Mathematical Society, Providence, RI (2005)
47. Simon, B.: CMV matrices: five years after. J. Comput. Appl. Math. 208, 120–154 (2007)
48. Simon, B.: Rank one perturbations and the zeros of paraorthogonal polynomials on the unit circle. J. Math. Anal. Appl. 329, 376–382 (2007)
49. Simon, B.: Szegő’s theorem and its descendants: spectral theory for $L^2$ perturbations of orthogonal polynomials. M. B. Porter Lectures. Princeton University Press, Princeton (2011)
50. Stieltjes, T.J.: Sur les racines de l’equation $X_n = 0$. Acta Math. 9, 385–400 (1887)
51. Szegő, G.: Orthogonal polynomials, vol. 23, 4th edition, : edn, p. 1939. Math. Soc. Coll. Publ., Amer. Math. Soc., Providence, RI (1975)
52. Watkins, D.S.: Some perspectives on the eigenvalue problem. SIAM Rev. 35(3), 430–471 (1993)
53. Wong, M.L.: First and second kind paraorthogonal polynomials and their zeros. J. Approx. Theory 146, 282–293 (2007)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.