Role of dissipation in the stability of a parametrically driven quantum harmonic oscillator

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Abstract
We study the dissipative dynamics of a single quantum harmonic oscillator subjected to parametric driving within an effective Hamiltonian approach. Using the Liouville–von Neumann approach, we show that the time evolution of a parametrically driven dissipative quantum oscillator has a strong connection with the classical, damped Mathieu equation. Based on the numerical analysis of the Monodromy matrix, we demonstrate that the dynamical instability generated by the parametric driving is reduced by the effect of dissipation. Furthermore, we obtain a closed relationship between the localization of the Wigner function and the stability of the damped Mathieu equation.

Keywords Quantum harmonic oscillator · Liouville–von Neumann approach · Parametrically driven · Damped Mathieu equation · Wigner function

1 Introduction
The quantum harmonic oscillator is one of the simplest, but important, models of microscopic and macroscopic systems in many branches of physics, such as quantum optics [1], ion-traps [2], quantum dots [3], quantum information [4], and superconductivity [5]. In real physical situations, damping is a key issue which makes analysis very difficult. However, two main approaches can be used to understand dissipation in quantum systems at a fundamental level. The first one is to put a system in a bath of many harmonic oscillators, where the bath oscillators interact with the system through certain coupling [6]. The system and the bath oscillators together make a conservative system, and the rule of quantization is applied to that conservative system. Then, the dissipation in the system is obtained by eliminating the bath’s degrees of freedom. Thus, the quantum mechanical fluctuation–dissipation relation is established due to the coupling, where the bath oscillators exert a force on the system as a fluctuation, and the system gives back its stored energy to the bath oscillators through coupling as dissipation. The second approach is based on the formulation of an effective Hamiltonian, which basically represents the classical model of a quantum damped harmonic oscillator (QDHO). In the 1940s, Caldirola [7] and Kanai [8] independently introduced the QDHO model Hamiltonian known as the CK Hamiltonian, which is a one-dimensional system with an exponentially increasing mass. The advantage of the CK Hamiltonian is that it is an open system whose parameters, such as mass and frequency are time-dependent. One can easily derive the equation of motion of a harmonic oscillator subjected to a frictional force from the CK Hamiltonian. However, of note is that no external force is involved in the system.

At long time, the quantization of the CK Hamiltonian gives rise of a violation of uncertainty principle, because the system is losing energy exponentially with time [9]. To overcome this unphysical violation of uncertainty principle, one can add an external force in a certain way to the CK Hamiltonian [10]. Thus, the CK Hamiltonian, which only takes into account the damping, is not sufficient to describe a quantum dissipative system. Related to the CK model, a detailed theory of Brownian motion of a quantum oscillator was developed in 1971 [11]. The author had employed the phase-space distribution functions obtained from the density operator via certain rules of mapping. The resulting equation for the reduced phase-space
distribution function is found to be of the Fokker–Planck type. The system was also to reach equilibrium as $t \to \infty$. More recently, the dissipative dynamics of a harmonic oscillator was studied using a nonperturbative approach [12]. In the limit of vanishing oscillator frequency of the system, the authors recover the results of for a free Brownian particle.

An immediate interesting study would be to understand the role of dissipation for the CK Hamiltonian in the presence of parametric driving. In general, a system is said to be parametrically driven if one, or more of its parameters are varied periodically in time; the phenomenon of parametric resonance can also occur at multiple frequencies. Under parametric resonance of a harmonic oscillator, stable points of the undriven system become unstable and vice versa for specific values of the period of the parameter variation. Parametric resonance occurs in a wide variety of systems, such as classical oscillators, nonlinear optics, systems governed by non-linear Schrödinger equation, and Hamiltonian chaotic systems. A cigar-shaped Bose–Einstein condensate (BEC) in a harmonic trap exhibits parametric excitations in the form of Faraday waves when the frequency of the trap is periodically modulated in time [13, 14]. These parametric excitations create instability in the system which tries to delocalize the BEC. In another context, the quantum phase transitions from a normal to a superradiant phase have been shown to become prolific in a periodically driven Dicke model. However, the stability of those phases are sensitive to the parameters of the time-periodic coupling [15]. Usually, for parametrically driven systems, the dynamics are governed by a Mathieu equation, and the usual technique for dealing with such systems is the Floquet formalism [16]. However, the role of dissipation in the context of parametrically driven quantum systems has still not been explored.

In this paper, we have studied the dynamics of a dissipative quantum harmonic oscillator for a CK Hamiltonian whose parameters are periodically driven with time. The annihilation and the creation operators associated with this dissipative parametrically driven quantum oscillator are time-dependent, but satisfy the Liouville–von Neumann quantum canonical equation. We have shown that in the quantum domain, the dynamical behavior of the system is fully determined by the solution of the corresponding classical damped Mathieu equation. Our main aim is to investigate the effect of dissipation on the stability of the system which is very essential in the context of parametrically driven systems. For that purpose, we performed the stability analysis by constructing the Monodromy matrix to check whether or not damping increases the stability for parametrically driven quantum systems. Furthermore, we studied the dynamical behavior of the Wigner functions for the same model and found that the behaviors of Wigner functions are quite consistent with the stability profile of the system.

2 Parametrically driven CK Hamiltonian

We consider a CK Hamiltonian of a single quantum harmonic oscillator with time-dependent parameters:

$$\hat{H}_D(t) = \frac{1}{2} [e^{-\gamma t} \hat{P}^2 + e^{\gamma t} \omega^2(t) \hat{Q}^2],$$

where $\hat{P}$ and $\hat{Q}$ are dynamical variables and $\gamma$ is the damping constant. The condition $\gamma > 0$ ensures the depletion of energy from the system. For driving the system periodically, we assume $\omega^2(t) = g - h \cos(\phi t)$, where $g$ is a static contribution. Here, $h$ and $\phi$ are the amplitude and the frequency of time-modulated parts, respectively. From Eq. (1), one can easily derive the equation of motion describing a parametrically driven damped classical harmonic oscillator:

$$\ddot{Q} + \gamma \dot{Q} + \omega^2(t) Q = 0.$$

The system can be driven out of equilibrium by a change of the time-dependent parameters of the model. To deal with such systems, we will follow the Liouville–von Neumann (LvN) quantum canonical formalism [17]. We assume that the non-equilibrium dynamics of the system is still governed by standard the Schrödinger equation [18]:

$$i \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H}_D(t) |\psi(t)\rangle.$$

In the Lewis–Riesenfeld canonical method, the invariant operators ($\hat{O}$) should satisfy the time-dependent LvN equation:

$$\frac{d\hat{O}}{dt} = \frac{\partial \hat{O}}{\partial t} + i[\hat{H}_D(t), \hat{O}] = 0.$$

The operator $\hat{O}$ satisfying Eq. (4) can be used to construct the exact quantum states of the Schrödinger equation, Eq. (3), which is given by

$$|\psi(t)\rangle = \sum_n c_n e^{i\theta_n} |\lambda_n(t)\rangle,$$

where

$$\hat{O} |\lambda_n(t)\rangle = \lambda_n |\lambda_n(t)\rangle$$

and

$$\theta_n(t) = \int_0^t dt' \langle \lambda_n(t') | i \frac{\partial}{\partial t} - \hat{H}_D(t) | \lambda_n(t)\rangle.$$

Next, we introduce time dependent annihilation and creation operators, $\hat{a}_D(t)$ and $\hat{a}_D^\dagger(t)$, given by

$$\hat{a}_D(t) = A(t) \hat{Q} + B(t) \hat{P},$$

and

$$\hat{a}_D^\dagger(t) = A^*(t) \hat{Q} + B^*(t) \hat{P}.$$

The time dependence in $\hat{a}_D(t)$ and $\hat{a}_D^\dagger(t)$ comes from the complex coefficients $A(t)$ and $B(t)$. For all times, the commutation relation between $\hat{a}_D(t)$ and $\hat{a}_D^\dagger(t)$ is
\[ \left[ \hat{a}_p(t), \hat{a}_p^\dagger(t) \right] = 1. \]  
(8)

From Eq. (8), we find that
\[ A(t)B^*(t) - B(t)A^*(t) = -i. \]  
(9)

Using Eq. (6) in Eq. (4), we obtain
\[ \dot{A}(t)\dot{Q} + B(t)\dot{P} = -A(t)e^{-\tau t}\dot{P} + \alpha^2(t)e^{\tau t}B(t)\dot{Q}. \]  
(10)

Thus, we get
\[ \ddot{B}(t) = -A(t)e^{-\tau t}. \]  
(11)

and
\[ \dot{A}(t) = B(t)e^{\tau t}\alpha^2(t). \]  
(12)

By combining Eqs. (11) and (12), we can write
\[ \ddot{B}(t) + \gamma \dot{B}(t) + \alpha^2(t)B(t) = 0. \]  
(13)

Hence, from Eq. (9), the Wronskian condition will be
\[ [B^*(t)B(t) - B^*(t)\dot{B}(t)] = i e^{-\tau t}. \]  
(14)

Equation (13) is called the damped Mathieu equation; hence, it is not an area preserving ordinary differential equations (ODE). It should be noted that Eq. (13) above is structurally the same as Eq. (2), the only difference being that it is for the complex coefficient, \( B(t) \). However, the Wronskian condition (Eq. 14) survives due to by the time-dependent nature of \( B(t) \) from the \( e^{\tau t} \) term in Eq. (14). This is similar to the observation that the positivity condition has to be satisfied in order to maintain the uncertainty relation in the problem of quantum Brownian motion [12]. The positivity condition is only satisfied above a certain breakdown temperature that depends directly on the oscillator’s damping and inversely on its frequency [12].

The ground state of the time-dependent oscillator (\( |0, t\rangle \)) must satisfy the condition:
\[ \hat{a}_p|0, t\rangle = 0. \]  
(15)

In the coordinate representation, the ground state can be written as \( \psi_0(x, t) = \langle x|0, t\rangle \), and Eq. (15) becomes
\[ \left( \ddot{B}(t)e^{\tau t}x + iB(t)\frac{\partial}{\partial x} \right)\psi_0(x, t) = 0. \]  
(16)

The solution of Eq. (16) is given by
\[ \psi_0(x, t) = C_1 e^{\int \frac{iB(t)e^{\tau t}}{2B(t)} x^2}. \]  
(17)

The reader should note that \( B(t) \) is a complex quantity. With Eq. (14), the normalized solution of Eq. (16) will be
\[ \psi_0(x, t) = \left( \frac{1}{2\pi|B(t)|^2} \right)^{1/2} e^{iB(t)e^{\tau t}x^2}. \]  
(18)

### 3 Stability of the damped Mathieu equation

To explore the stability of the solutions of Eq. (13), we apply the Floquet theorem to the damped Mathieu equation [16]. The two-dimensional first-order differential equation associated with Eq. (13) can be presented in matrix form as
\[ x' = K(t)x, \]  
(19)

where \( x = [B(t), B(t)]^\dagger \) and \( ' \dagger ' \) denotes first order differentiation, and \( K(t) \) is a periodic function of time \( t \) with periodicity \( \tau = \frac{2\pi}{\phi} \). Using the Floquet theorem, one can write the general solution of Eq. (19) as
\[ x(t) = e^{\phi t}P(t), \]  
(20)

where \( P(t + \tau) = P(t) \). Let \( \{ \lambda_i \} \) be the eigenvalues of \( R \). Then the solution of Eq. (19) will be stable if \( \text{Re}(\lambda_i) < 0 \). Next, we will construct the Monodromy matrix \( (M) \) to calculate \( \{ \lambda_i \} \), because \( \{ \lambda_i \} \) are related to the eigenvalues of \( M, \{ \mu_i \} \), as \( \mu_i = e^{\phi \tau} \) [19]. We will construct \( M \) below.

We can formally express \( B(t) \) as
\[ B(t) = B^{(1)}(t) + B^{(2)}(t), \]  
(21)

and
\[ \ddot{B}(t) = \dot{B}^{(1)}(t) + \dot{B}^{(2)}(t). \]  
(22)

Using Eq. (21) and (22), we construct the Monodromy matrix \( (M) \) for Eq. (19):
\[ M = F^{-1}(0)F(\tau), \]  
(23)

where \( F(\tau) = \begin{pmatrix} B^{(1)}(\tau) & B^{(2)}(\tau) \\ B^{(1)}(\tau) & B^{(2)}(\tau) \end{pmatrix} \) We have numerically constructed the matrix \( M \) for the initial conditions \( B(0) = 1 \) and \( \dot{B}(0) = 0 \) and plotted the real part of \( \{ \lambda_i \} \) in Fig. 1 for \( \phi = 2 \) (or \( \tau = \pi \)). The shaded band-like regions in the \( g - h \) plane of Fig. 1 correspond to the stable zones, whereas the white regions correspond to the unstable zones. When the damping coefficient is increased, the band-like regions spreads more in the \( g - h \) plane of Fig. 1b in comparison with Fig. 1a. This implies, damping increases the stability of the system [20]. A similar result was also found in the work of Agarwal [11], where the damped harmonic oscillator system was shown to reach equilibrium for large time. For further investigation, we have chosen two different points from Fig. 1b: one is from the stable zone for which \( g = 4.0, h = 2.0 \) (blue diamond) and the other is from the unstable zone for which \( g = 4.0, h = 2.5 \) (red circle). In
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4 Wigner function analysis

To get a deeper understanding of the physics underlying this dissipative system, we consider now the problem from another point of view: the Wigner function representation which is defined by the position representation of the density matrix [21] as

\[ W(q, p; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq e^{ipq} \left< q - \frac{s}{2} | \hat{\rho} | q + \frac{s}{2} \right> . \]  

The density matrix is defined as \( \hat{\rho} = |0, t\rangle \langle 0, t| \), and in the coordinate representation, it will be

\[ \hat{\rho}(x, x_1, t) = \psi_0(x, t) \psi_0^\dagger(x_1, t) \]

\[ = \left( \frac{1}{2\pi |B(t)|^2} \right) \frac{1}{2} \exp \left( \frac{iB(t)e^{\gamma t}}{2B(t)} \right) \exp \left( \frac{-iB^\dagger(t)e^{\gamma t}}{2B^\dagger(t)} x_1^2 \right) . \]  

(25)

The Wigner function can take on negative, as well as positive, values. However, for the pure state, the Wigner function is Gaussian and, hence, positive definite. Because of the positivity of the Wigner function, it can be interpreted as a “phase space distribution”, similar to a probability distribution of classical particles. Using Eq. (25), we get the following result:

\[ W(q, p; t) = \frac{1}{\pi} \exp \left[ -\frac{q^2}{2|B(t)|^2} - \frac{p^2}{2} \left( \frac{\dot{B}(t)}{B(t)} + \frac{\dot{B}^\dagger(t)}{B^\dagger(t)} \right) \right] \]  

(26)

However, the term \( \left( \frac{\dot{B}(t)}{B(t)} + \frac{\dot{B}^\dagger(t)}{B^\dagger(t)} \right) \) is a pure real number. In Figs. 3 and 4, we plot the evolution of \( W(q, p, t) \) at three different time intervals for the stable (blue diamond) and the
unstable (red circle) points, respectively. In both the stable and the unstable zones, $W(q, p, t)$ evolves in such a way that it is stretched along a dynamical axis with a consequent contraction along its perpendicular axis in phase space. This happens because of the time-dependent nature of uncertainty principle. In comparison, the behaviour of $W(q, p, t)$ for the stable zone is different from that for the unstable zone. For the unstable zone, the stretching of the $W(q, p, t)$ is observed for all time (Fig. 4a–c). However, for the stable zone, $W(q, p, t)$ stretches along the diagonal axis at intermediate time (Fig. 3b) and regains its shape at the final time (Fig. 3c).
Hence, the overall squeezing of $W(q, p, t)$ in the stable zone is less than that in the unstable zone.

5 Conclusion

In this work, we have studied the effective Hamiltonian dynamics of a single quantum harmonic oscillator in the presence of dissipation and parametric driving. The time-dependent parametric frequency drives the system perpetually out of equilibrium, and the stable points of the undriven system become unstable, or an unstable point may become a stable one. However, the role of dissipation is always unidirectional, which means it always pushes the system from an unstable zone to a stable one. By tuning the parameters of the system, one can have control over the competition between dissipation and parametric driving. In addition, we have studied the relationship between dissipation and the localization of the system’s wave function in the Wigner function representation. Dissipation modifies the functional form of the Wigner function. In phase space, the delocalization of the Wigner function decreases with increasing dissipation, because dissipation stabilizes the system. Hence, a direct correspondence between the dynamical stability of the time-dependent parameters and the temporal behaviors of the Wigner function is observed in an isolated quantum damped harmonic oscillator system with parametric driving.

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