ON THE PARAMETERIZATION OF PRIMITIVE IDEALS IN AFFINE PI ALGEBRAS

EDWARD S. LETZTER

ABSTRACT. We consider the following question, concerning associative algebras $R$ over an algebraically closed field $k$: When can the space of (equivalence classes of) finite dimensional irreducible representations of $R$ be topologically embedded into a classical affine space? We provide an affirmative answer for algebraic quantum groups at roots of unity. More generally, we give an affirmative answer for $k$-affine maximal orders satisfying a polynomial identity, when $k$ has characteristic zero. Our approach closely follows the foundational studies by Artin and Procesi on finite dimensional representations. Our results also depend on Procesi’s later study of Cayley-Hamilton identities.

1. Introduction

1.1. Let $k$ be an algebraically closed field, and let $R$ be an associative $k$-algebra with generators $X_1, \ldots, X_s$. In the foundational studies of Artin [1], in 1969, and Procesi [9], in 1974, it was shown that the semisimple $n$-dimensional representations of $R$ (over $k$) were parametrized up to equivalence by a closed subset of $\text{Max}(n, s)$, where $\mathbb{T}(n, s)$ is the affine (i.e., finitely generated) commutative $k$-algebra generated by the coefficients of the characteristic polynomials of $s$-many generic $n \times n$ matrices. It was further shown by Artin and Procesi in [1; 9] that $\text{Prim}_n R$, the set of kernels of $n$-dimensional irreducible representations of $R$, is homeomorphic to a locally closed subset of $\text{Max}(n, s)$. (Here and throughout, the Jacobson/Zariski topology is employed.) In particular, when the irreducible representations of $R$ all have dimension $n$ (e.g., when $R$ is an Azumaya algebra of rank $n$, by what is now known as the Artin-Procesi theorem [1; 9]), the space $\text{Prim} R$ of kernels of irreducible representations of $R$ is homeomorphic to a locally closed subset of affine space. In this note we examine generalizations of this embedding for more general classes of $k$-affine PI (i.e., polynomial identity) algebras. Our analysis closely follows the above cited work of Artin and Procesi, and also depends on the later study by Procesi of Cayley-Hamilton identities [7].

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1.2. Our main result, proved in (5.4):

**Theorem.** Let $A$ be a prime affine PI algebra over an algebraically closed field $k$ of characteristic zero, and suppose that $A$ is a maximal right (or left) order in a simple artinian ring $Q$. Then $\text{Prim } A$ is homeomorphic to a constructible subset of the affine space $k^N$, for a suitable choice of positive integer $N$.

Examples to which the theorem applies include algebraic quantum groups at roots of unity. Recent studies of quantum groups from this general point of view include [4].

1.3. For an arbitrary $k$-affine PI algebra $R$, in arbitrary characteristic, we are able to construct a closed bijection from $\text{Prim } R$ onto a constructible subset of $k^N$, again for a suitable choice of $N$. We therefore ask whether the conclusion of the preceding theorem holds for all $k$-affine PI algebras.

1.4. We assume that the reader is familiar with the basic theory of PI algebras; general references include [6, Chapter 13], [8], and [11].

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2. Constructing the injection $\Psi$

Our goal in this section is to construct an injection, specified in (2.11), from $\text{Prim } R$ into the maximal spectrum of a suitable “trace ring.” The approach is directly adapted from [9], with some added bookkeeping.

2.1 $(R, d, N)$ First Notation and Conventions. The following will remain in effect throughout this paper.

(i) Set

$$R = k \left\langle \hat{X}_1, \hat{X}_2, \ldots, \hat{X}_s \right\rangle / \left\langle \hat{f}_1, \hat{f}_2, \ldots \right\rangle,$$

the factor of the free associative $k$-algebra in the generators $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_s$ modulo the (not necessarily finitely many) relations $\hat{f}_1, \hat{f}_2, \ldots$. Let $X_\ell$ denote the canonical image of $\hat{X}_\ell$ in $R$, for each $\ell$. Assume that $R$ satisfies a (monic) polynomial identity.

(ii) All $k$-algebra homomorphisms mentioned will be assumed to be unital. A representation is a $k$-algebra homomorphism into the algebra of linear operators on a $k$-vector space. If $\Gamma$ is a $k$-algebra we will assume that the sets $\text{Prim } \Gamma$ of (left) primitive ideals and $\text{Max } \Gamma$ of maximal ideals are equipped with the Jacobson/Zariski topology: The closed sets have the form $V_\Gamma(I) = \{ P : P \supseteq I \}$ for ideals $I$ of $\Gamma$.

(iii) Recall from Kaplansky’s Theorem and standard PI theory that there exists a positive integer $d$ such that every irreducible representation of $R$ has $(k)$-dimension no greater than $d$. Let $N$ be a common multiple of $1, 2, \ldots, d$. (Note that our choices of $d$ and $N$ remain valid when $R$ is replaced with a homomorphic image.)
(iv) Repeatedly-used non-standard notation will be listed (within parentheses) at the beginning of the subsection in which it is introduced.

2.2 \((\hat{C}_n, \hat{x}_{ij}^{(\ell,n)}, \mathbb{G}(n,s), \mathbb{T}(n,s), \hat{M}_n)\). Let \(n\) be a positive integer. Set

\[
\hat{C}_n = k \left[ x_{ij}^{(\ell,n)} : 1 \leq i, j \leq n, \ell = 1, 2, \ldots \right],
\]

the commutative polynomial \(k\)-algebra in the variables \(\hat{x}_{ij}^{(\ell,n)}\). Also, set \(\hat{M}_n = M_n(\hat{C}_n)\), the \(k\)-algebra of \(n\times n\) matrices with entries in \(\hat{C}_n\). Identify \(\hat{C}_n\) with the \(\hat{C}_n\)-scalar matrices in \(\hat{M}_n\); in other words, identify \(\hat{C}_n\) with the center \(Z(\hat{M}_n)\) of \(\hat{M}_n\).

Let \(\mathbb{G}(n,s)\) denote the \(k\)-subalgebra of \(\hat{M}_n\) generated by the generic matrices

\[
\left( \hat{x}_{ij}^{(1,n)} \right)_{n \times n}, \ldots, \left( \hat{x}_{ij}^{(s,n)} \right)_{n \times n}.
\]

The \(k\)-subalgebra of \(\hat{C}_n\) generated by the coefficients of the characteristic polynomials of the elements of \(\mathbb{G}(n,s)\) will be denoted \(\mathbb{T}(n,s)\). It is a well known consequence of Shirshov’s Theorem that \(\mathbb{T}(n,s)\) is \(k\)-affine [9, 3.1]. It is also well known that \(\mathbb{T}(n,s)\) is generated, in characteristic zero, by the traces of the elements of \(\mathbb{G}(n,s)\).

2.3 \((\text{Rel}(\hat{C}_n), \text{Rel}(\hat{M}_n))\). Now consider the \(k\)-algebra homomorphism

\[
k \{ \hat{X}_1, \ldots, \hat{X}_s \} \xrightarrow{\hat{\pi}_n} \hat{M}_n,
\]

mapping

\[
\hat{X}_\ell \mapsto \left( \hat{x}_{ij}^{(\ell,n)} \right)_{n \times n},
\]

for each \(\ell\). Let \(\text{Rel}(\hat{C}_n)\) be the ideal of \(\hat{C}_n\) generated by the entries of

\[
\hat{\pi}_n \left( \hat{f}_1 \right), \hat{\pi}_n \left( \hat{f}_2 \right), \ldots,
\]

and let \(\text{Rel}(\hat{M}_n)\) be the ideal of \(\hat{M}_n\) generated by \(\text{Rel}(\hat{C}_n)\). Then

\[
\text{Rel} \left( \hat{M}_n \right) = M_n \left( \text{Rel} \left( \hat{C}_n \right) \right), \quad \text{and} \quad \text{Rel} \left( \hat{C}_n \right) = \text{Rel} \left( \hat{M}_n \right) \cap \hat{C}_n.
\]

2.4 \((C_n, M_n, x_{ij}^{(\ell,n)}, \pi_n, T_n)\). Set

\[
C_n = \hat{C}_n / \text{Rel} \left( \hat{C}_n \right), \quad \text{and} \quad M_n = M_n( C_n ) \cong \hat{M}_n / \text{Rel} \left( \hat{M}_n \right).
\]

Denote the natural image of each \(\hat{x}_{ij}^{(\ell,n)}\) in \(C_n\) by \(x_{ij}^{(\ell,n)}\). We obtain a \(k\)-algebra homomorphism

\[
\pi_n : R \xrightarrow{X_\ell} \left( x_{ij}^{(\ell,n)} \right)_{n \times n} \xrightarrow{M_n}.
\]

Note that \(\pi_n(R)\) is a natural image of \(\mathbb{G}(n,s)\).

Identify \(C_n\) with \(Z(M_n)\), and let \(T_n = T_n(R)\) denote the \(k\)-subalgebra of \(C_n\) generated by the coefficients of the characteristic polynomials of the elements of \(\pi_n(R)\). Observe that \(T_n\) is a natural image of \(\mathbb{T}(n,s)\).
2.5. Say that a \( k \)-algebra homomorphism \( h: M_n \to M_n(k) \) is matrix unital if \( h \) restricts to the identity map on \( M_n(k) \subseteq M_n \). Letting \( e_{ij} \) denote the \( ij \)th matrix unit of \( M_n(k) \), we see that \( h \) is matrix unital if and only if \( h(e_{ij}) = e_{ij} \) for all \( i \) and \( j \).

2.6 (\( \tilde{\rho} \)). Now let \( \rho: R \to M_n(k) \) be a representation. Observe that there is a unique matrix unital \( k \)-algebra homomorphism \( \tilde{\rho}: M_n \to M_n(k) \) such that the following diagram commutes:

\[
\begin{array}{c}
R \xrightarrow{\pi_n} M_n & \xleftarrow{\text{inclusion}} & C_n & \xleftarrow{\text{inclusion}} & T_n \\
\| & \downarrow{\tilde{\rho}} & |\tilde{\rho}|_{C_n} & \downarrow{\tilde{\rho}|_{T_n}} \\
R \xrightarrow{\rho} M_n(k) & \xleftarrow{\text{inclusion}} & k & \xrightarrow{=} & k
\end{array}
\]

Of course, every \( k \)-algebra homomorphism \( C_n \to k \) produces a representation \( R \to M_n(k) \) in an obvious way.

2.7 (\( \Theta \)). Let \( \text{Rep}_n R \) denote the set of \( n \)-dimensional representations of \( R \) (without identifying equivalence classes), and let \( \text{Alg}(T_n, k) \) denote the set of \( k \)-algebra homomorphisms from \( T_n \) onto \( k \). We have a function

\[
\Theta_n: \text{Rep}_n(R, k) \xrightarrow{\rho \mapsto \tilde{\rho}|_{T_n}} \text{Alg}(T_n, k) \cong \text{Max} T_n.
\]

For a given representation \( \rho: R \to M_n(k) \), let \( \text{semisimple}(\rho) \) denote the unique equivalence class of semisimple \( n \)-dimensional representations corresponding to \( \rho \) (i.e., the semisimple representations obtained from the direct sum of the Jordan-Hölder factors of the \( R \)-module associated to \( \rho \)). We now recall:

**Theorem.** (Artin [1, §12]; Procesi [9, §4]) (a) \( \Theta_n \) is surjective. (b) \( \Theta_n(\rho) = \Theta_n(\rho') \) if and only if \( \text{semisimple}(\rho) = \text{semisimple}(\rho') \).

2.8 (\( \gamma_P, \Phi_m \)). (i) Let \( \text{Prim}_m R \) denote the set of (left) primitive ideals of rank \( m \) (i.e., the set of kernels of \( m \)-dimensional irreducible representations of \( R \)). Note that \( 1 \leq m \leq d \).

Equip \( \text{Prim}_m R \) with the relative topology, viewing it as a subspace of \( \text{Prim} R \). As noted in [1, §12] and [9, §5], \( \text{Prim}_m R \) is a locally closed subset of \( \text{Prim} R \).

(ii) Choose \( P \in \text{Prim}_m R \). Then \( P \) uniquely determines an equivalence class of irreducible \( m \)-dimensional representations; choose \( \rho: R \to M_n(k) \) in this equivalence class. Let \( \gamma_P \) denote the \( k \)-algebra homomorphism \( \tilde{\rho}|_{T_m}: T_m \to k \). By (2.7), \( \gamma_P \) depends only on \( P \), and we obtain an injection

\[
\Phi_m: \text{Prim}_m R \xrightarrow{P \mapsto \ker \gamma_P} \text{Max} T_m.
\]

(iii) It follows from [1, §12] and [9, §5] that the image of \( \Phi_m \) is an open subset of \( \text{Max} T_m \) and that \( \Phi_m \) is homeomorphic onto its image.
2.9 ($\rho_N$). Now choose a positive integer $m$ no greater than $d$, and let $\rho: R \to M_m(k)$ be a representation. We will use $\rho_N: R \to M_N(k)$ to denote the associated $N$-dimensional diagonal representation, mapping

$$r \mapsto \begin{bmatrix} \rho(r) \\ \rho(r) \\ \vdots \\ \rho(r) \end{bmatrix},$$

for $r \in R$.

2.10 ($C, \pi, M, T, x_{ij}^{(\ell)}$). In the remainder of this note we mostly will be concerned with the case when $n = N$, and so we will set $C = C_N$, $\pi = \pi_N$, $M = M_N$, $T = T_N = T_N(R) = T(R)$, and

$$(x_{ij}^{(\ell)}) = (x_{ij}^{(\ell, N)})_{n \times n}.$$

2.11 ($\gamma_{N,P}, \Psi$) The injection. Now let $P$ be a primitive ideal of $R$. Proceeding as before, $P$ uniquely determines an equivalence class of irreducible $m$-dimensional representations for some $1 \leq m \leq d$; choose $\rho: R \to M_m(k)$ in this equivalence class. Combining (2.6) and (2.9), let $\gamma_{N,P}$ denote the $k$-algebra homomorphism $(\overline{\rho}_N)|_T: T \to k$. We can now define an injection:

$$\Psi: \text{Prim } R \xrightarrow{P \mapsto \ker \gamma_{N,P}} \text{Max } T$$

In §3 the image of $\Psi$ will be described. In §4 it will be proved that $\Psi$ is open (and closed) onto its image. In §5 it will be seen, in certain special cases, that $\Psi$ is homeomorphic onto its image.

Note now, however, that implicit in the preceding is a natural (and obvious) homeomorphism between Prim $R$ and Prim $\pi(R)$.

2.12. Choose $P$, $m$, and $\rho$ as in (2.11). Up to equivalence, there is exactly one $N$-dimensional representation of $R$ with kernel $P$, namely, the representation corresponding to the unique (up to isomorphism) semisimple $R/P$-module of length $N/m$. Therefore, by (2.7), $\gamma_{N,P}$ depends only on $P$ and not our specific choice $\rho_N$ of $N$-dimensional representation.

3. The image of $\Psi$

Retain the notation of the preceding section. Throughout this section, $m$ will denote a positive integer no greater than $d$. The main result of this section, (3.7), explicitly determines the image of $\Psi$; in particular, the image is a constructible subset.

3.1. Given an $N \times N$ matrix, the $(N/m)$-many $m \times m$ blocks running consecutively down the main diagonal will form the $m$-block diagonal. An $N \times N$ matrix with only zero entries off the $m$-block diagonal will be referred to as an $m$-block diagonal matrix.
3.2. Consider the $k$-algebra homomorphism $\hat{C} \to \hat{C}_m$ mapping the $ij$th entry of $\left(\hat{x}_{ij}^{(\ell)}\right) \in \hat{M}$ to the $ij$th entry of the $m$-block diagonal matrix

$$\begin{bmatrix}
\left(\hat{x}_{ij}^{(\ell,m)}\right)_{m \times m} \\
\vdots \\
\left(\hat{x}_{ij}^{(\ell,m)}\right)_{m \times m}
\end{bmatrix} \in M_N(\hat{C}_m).$$

We obtain a commutative diagram of $k$-algebra homomorphisms:

$$\begin{array}{cccc}
\hat{C} & \longrightarrow & \hat{C}_m \\
\text{projection} & \downarrow & \text{projection} \\
C & \longrightarrow & C_m \\
\text{inclusion} & \uparrow & \text{inclusion} \\
T & \longrightarrow & T_m
\end{array}$$

We will refer to the horizontal maps as specializations.

3.3. Note that the preceding maps $\hat{C} \to \hat{C}_m$ and $C \to C_m$ are surjective. In characteristic zero, $T$ and $T_m$ are generated, respectively, by the traces of the matrices contained in $\pi(R)$ and $\pi_m(R)$, and it follows in this situation that the specialization $T \to T_m$ is surjective. In arbitrary characteristic, it is not hard to see that the specialization $T \to T_m$ is an integral ring homomorphism.

3.4 ($H_m$, $I_m$, $J_m$). Let $H_m$ denote the kernel of the specialization $C \to C_m$. In other words, $H_m$ is the ideal of $C$ generated by the sets

$$\left\{ x_{ij}^{(\ell)} \mid x_{ij}^{(\ell)} \text{ is not within the } m\text{-block diagonal of } \left(x_{ij}^{(\ell)}\right); \quad 1 \leq i, j \leq N; \ell = 1, 2,\ldots \right\}$$

and

$$\left\{ x_{ij}^{(\ell)} - x_{i'j'}^{(\ell)} \mid x_{ij}^{(\ell)} \text{ and } x_{i'j'}^{(\ell)} \text{ are within the } m\text{-block diagonal of } \left(x_{ij}^{(\ell)}\right); \quad i = i' \text{ (mod } m) \text{ and } j = j' \text{ (mod } m); 1 \leq i, j \leq N; \ell = 1, 2,\ldots \right\}.$$ 

Set $I_m = MH_m = M_N(H_m)$. Let $J_m = H_m \cap T = I_m \cap T$ denote the kernel of the specialization $T \to T_m$. 
3.5. Now suppose that $P$ is the kernel of an irreducible representation $\rho: R \to M_m(k)$. Recalling the notation of (2.6) and (2.9), we see that the kernel of $\tilde{\rho}_N: M \to M_N(k)$ contains $I_m$, and so the kernel of $\tilde{\rho}|_T$ contains $J_m$. We conclude that $\Psi$ maps $\text{Prim}_m R$ into the set $V_T(J_m)$ of maximal ideals of $T$ containing $J_m$.

3.6 ($E_m$). We now proceed in a fashion similar to [9, §5], employing the central polynomials of Formanek [5] or Razmyslov [10]. To start (see, e.g., [6, §13.5] or [11, §1.4] for details), we can construct a polynomial $p_m$ in noncommuting indeterminates with the following two properties, holding for all commutative rings $\Lambda$ with identity: First, $p_m(M_m(\Lambda)) \subseteq Z(M_m(\Lambda))$, and second, $p_m(M_m(\Lambda))$ generates $M_m(\Lambda)$ as an additive group. (Here, $p_m(M_m(\Lambda))$ refers to all evaluations of $p_m$ where the indeterminates have been substituted with elements of $M_m(\Lambda)$.) Hence, $\pi(p_m(R)) = p_m(\pi(R))$ is contained, modulo $I_m$, within the center of $M$. Moreover, a representation $R \to M_m(k)$ is irreducible if and only if $p_m(R)$ is not contained in the kernel, if and only if the image of $p_m(R)$ generates as an additive group the full set of scalar matrices in $M_m(k)$.

Next, modulo $I_m$, the characteristic polynomial of $c \in \pi(p_m(R))$ is $(\lambda - c)^N$, and so $c^N \in T + I_m$. We can therefore choose a set $E_m \subseteq T$ of transversals in $M$ for

$$\{ c^N + I_m : c \in \pi(p_m(R)) \},$$

with respect to $I_m$.

Now let $\varphi: R \to M_m(k)$ be a representation. As in (3.5), $I_m \subseteq \ker \varphi_N$ and $J_m \subseteq \ker \varphi|_T$. Observe that

$$\varphi \text{ is irreducible} \iff \varphi(p_m(R)) \neq 0 \iff \tilde{\varphi}_N(\pi(p_m(R))) \neq 0$$

$$\iff \tilde{\varphi}_N(E_m) \neq 0 \iff \tilde{\varphi}_N|_T(E_m) \neq 0.$$

3.7. Let $K_m$ be the ideal of $T$ generated by $E_m$ and $J_m$.

Theorem. (i) $\Psi$ maps $\text{Prim}_m R$ onto $V_T(J_m) \setminus V_T(K_m)$.

(ii) The image of $\Psi$ is

$$\bigcup_{m=1}^d V_T(J_m) \setminus V_T(K_m).$$

In particular, the image of $\Psi$ is a constructible subset of $\text{Max} T$.

Proof. Immediate from (3.5) and (3.6). □

3.8. We ask: Can the image of $\Psi$ be described in a simpler fashion? Is there a simple way to specify how the locally closed subsets in (3.7) fit together?
4. **Ψ is open and closed onto its image**

Retain the notation of §2 and §3. We now begin to consider the topological properties of Ψ. In (4.2) it is shown that Ψ is open and closed onto its image.

4.1. Let $R \to R'$ be a $k$-algebra homomorphism. As described in [9, pp. 177–178], the construction in (2.11) is functorial in the following sense.

(i) To start, we have a commutative diagram:

\[
\begin{array}{ccc}
R & \xrightarrow{\pi} & M & \xleftarrow{} & T(R) \\
\downarrow & & \downarrow & & \downarrow \\
R' & \xrightarrow{\pi'} & M' & \xleftarrow{} & T(R')
\end{array}
\]

Moreover, if $R \to R'$ is surjective then so too is $T(R) \to T(R')$.

(ii) Next, assuming that $I$ is an ideal of $R$, that $R' = R/I$, and that $R \to R'$ is the natural projection, we obtain a commutative diagram:

\[
\begin{array}{ccc}
\text{Prim } R & \xrightarrow{\Psi} & \text{Max } T \\
P/I \hookrightarrow P & & (M\pi(I)M) \cap T \xrightarrow{m} m \\
\text{Prim } R' & \xrightarrow{\Psi'} & \text{Max } T'
\end{array}
\]

Each arrow represents an injection, and each vertical arrow represents a topological embedding onto a closed subset.

The kernel of the homomorphism $T \to T'$ is $(M\pi(I)M) \cap T$.

4.2. Now let $I$ be an arbitrary ideal of $R$, with corresponding closed subset $V_R(I)$ of Prim $R$. Set

\[J = (M\pi(I)M) \cap T.\]

**Proposition.** Ψ is open and closed onto its image. In particular,

\[\Psi(V_R(I)) = \text{Image } \Psi \cap (V_T(J)), \quad \text{and } \Psi(W_R(I)) = \text{Image } \Psi \cap (W_T(J)),\]

where $W_R(I)$ denotes the complement of $V_R(I)$.

**Proof.** This follows from (4.1ii) and the injectivity of Ψ. □
4.3. Combining (2.8) and (3.2), we have a commutative diagram:

\[
\begin{array}{c}
\text{Prim} R & \xrightarrow{\Psi} & \text{Max} T \\
\text{Prim}_m R & \xrightarrow{\Phi_m} & \text{Max}_{T_m}
\end{array}
\]

We conclude from (2.8iii) that the restriction of \( \Psi \) to \( \text{Prim}_m R \) is continuous. Recalling from (2.8iii) that \( \text{Prim}_m R \) is a locally closed subset of \( \text{Prim} R \), we may view this last conclusion as an assertion that \( \Psi \) is “piecewise continuous.” Also, we can conclude that the preimage under \( \Psi \) of a constructible subset of \( \text{Max} T \) is constructible.

4.4. We ask: Is \( \Psi \) necessarily continuous? More generally, is \( \text{Prim} R \) homeomorphic to a subspace of affine \( n \)-space, for sufficiently large \( n \)? A partial answer is given in §5.

5. Applications to Algebras Satisfying Cayley-Hamilton Identities

We retain the notation of the previous sections, but assume in this section that \( k \) has characteristic zero. Our approach closely follows [7].

5.1 [7, §2] Formal Traces and Cayley-Hamilton Identities. Let \( \Gamma \) be a \( k \)-algebra.

(i) [7, 2.3] Say that \( \Gamma \) is equipped with a (formal) trace (over \( k \)) provided there exists a \( k \)-linear function \( \text{tr}: \Gamma \to \Gamma \) such that for all \( a, b \in \Gamma \),

\[
\text{tr}(a)b = b \text{tr}(a), \quad \text{tr}(ab) = \text{tr}(ba), \quad \text{and} \quad \text{tr(tr}(a)b) = \text{tr}(a) \text{tr}(b).
\]

(ii) [7, 2.4] Suppose that \( \Gamma \) is equipped with a trace \( \text{tr} \). For each \( r \in \Gamma \), set

\[
\chi^{(n)}_r(t) = \prod_{i=1}^n (t - t_{r,i}),
\]

where the \( t_{r,i} \) are “formal eigenvectors” for \( r \) satisfying

\[
\sum_{i=1}^n t^j_{r,i} = \text{tr}(r^j),
\]

for all non-negative integers \( j \). Say that \( \Gamma \) satisfies the \( n \)-th Cayley-Hamilton identity if \( \chi^{(n)}_r(r) = 0 \) for all \( r \in \Gamma \).

(iii) Suppose that \( \Gamma \) is equipped with a trace \( \text{tr} \). By [7, Theorem], there exists a commutative \( k \)-algebra \( \Lambda \), and a trace compatible \( k \)-algebra embedding of \( \Gamma \) into \( M_n(\Lambda) \), if and only if \( \Gamma \) satisfies the \( n \)-th Cayley-Hamilton identity.

(iv) (Cf. [4, 3.10].) Let \( p \) be a positive integer, let \( \Lambda \) be a commutative \( k \)-algebra, and suppose that there is a trace compatible \( k \)-algebra embedding \( \Gamma \to M_n(\Lambda) \). The block diagonal embedding of \( M_n(\Lambda) \) into \( M_{pn}(\Lambda) \) then provides a trace compatible embedding of \( \Gamma \) into \( M_{pn}(\Lambda) \).

(v) Suppose that \( \Gamma \) is equipped with a trace. We conclude from (iii) and (iv) that if \( \Gamma \) satisfies the \( n \)-th Cayley-Hamilton identity then \( \Gamma \) satisfies the Cayley-Hamilton identity for all positive multiples of \( n \).
5.2. Returning to the setting of the previous sections (but with $k$ now having characteristic zero), suppose that $R$, as in (2.1), satisfies the $N$-th Cayley-Hamilton identity. It then follows directly from [7, 2.6] (the main theorem in [7]) that $T$ is contained in $\pi(R)$, the image of $\pi: R \to M$. Since $T$ must be central in $\pi(R)$, and since every irreducible representation of $R$ is finite dimensional over $k$, it follows from well known arguments that the function

$$\text{Prim } \pi(R) \xrightarrow{P \mapsto P \cap T} \text{Max } T$$

is continuous. Furthermore, as noted in (2.11), $\pi$ produces a natural homeomorphism between $\text{Prim } R$ and $\text{Prim } \pi(R)$. However, the composition

$$\text{Prim } R \xrightarrow{\text{natural homeomorphism}} \text{Prim } \pi(R) \xrightarrow{P \mapsto P \cap T} \text{Max } T$$

is precisely the function $\Psi$ of (2.11), which must therefore be continuous.

We obtain:

5.3 Proposition. (Recall that $k$ has characteristic zero.) Suppose that $R$ satisfies the $N$-th Cayley-Hamilton identity. Then $\Psi: \text{Prim } R \to \text{Max } T$ is homeomorphic onto its image.

Proof. That $\Psi$ is closed onto its image follows from (4.2). That $\Psi$ is continuous onto its image follows from (5.2). □

Combining (5.3) with our previous analysis produces our main result:

5.4 Theorem. Let $A$ be a prime affine PI algebra over an algebraically closed field $k$ of characteristic zero, and suppose that $A$ is a maximal right (or left) order in a simple artinian ring $Q$. Further suppose that $Q$ has rank $d$, that $A$ is a maximal order in $Q$, and that $N$ is a common multiple of $1, 2, \ldots, d$. Then $\text{Prim } A$ is homeomorphic to a constructible subset of the affine space $k^N$.

Proof. To start, the irreducible representations of $A$ all have dimension no greater than $d$. Next, $A$ is equipped with both a trace and a trace compatible embedding into $d \times d$ matrices over a commutative ring, since $A$ is a maximal order in $Q$; see (e.g.) [6, §13.9]. Hence, by (5.1.iii), $A$ satisfies the $d$-th Cayley-Hamilton identity, and so, by (5.1.iv), $A$ satisfies the $N$-th Cayley-Hamilton identity. The theorem now follows from (3.7) and (5.3). □

5.5 Quantum groups. For suitable complex roots of unity $\epsilon$, the quantum enveloping algebras $U_\epsilon$ and quantum function algebras $F_\epsilon$ are prime affine PI $\mathbb{C}$-algebras and are maximal orders; see [2] and (e.g.) [3]. In particular, (5.4) applies to these algebras.
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Department of Mathematics, Temple University, Philadelphia, PA 19122
E-mail address: letzter@math.temple.edu