ON EQUICONTINUITY OF MAPPINGS
WITH A FINITE INTEGRAL OVER SPHERES BY PRIME ENDS

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Abstract

We study mappings that satisfy the inverse Poletsky-type inequality in a domain of Euclidean space. Under some conditions, it was proved that such mappings have a continuous extension to the boundary in terms of prime ends and form equicontinuous families of mappings on it. Here we consider the case of bad domains between which the mappings under study act.

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1 Introduction

In our joint publication [SSD], we have obtained some results on the local and boundary behavior of maps with inverse Poletsky inequality. In particular, we obtained theorems on a continuous extension to the boundary of these mappings and the equicontinuity of their families on the boundary. We emphasize that such results were related to the case when some majorant $Q$ participating in the defining inequality of the capacity-modulus type was integrable in some domain. In this article, we will show a little more, namely that these results are met not only for integrable $Q$, but also for those that have finite integrals on spheres centered at a fixed point on a set of radii some "not very small" measure. Let us point to examples of non-integrable functions that have these finite integrals by spheres and mappings that correspond to them (see, for example, [SevSkv3, Examples 1,2]).

Let us turn to the definitions. In what follows, $M_p(\Gamma)$ denotes the $p$-modulus of a family $\Gamma$ (see [Va, Section 6]). We write $M(\Gamma)$ instead $M_n(\Gamma)$. Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A = A(y_0, r_1, r_2) = \{y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2\}.$$  

(1.1)
Given $x_0 \in \mathbb{R}^n$, we put

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}, \quad \mathbb{B}^n = B(0, 1),$$

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}.$$  

Given sets $E, F \subset \mathbb{R}^n$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ a family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}}^n$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in [a, b]$. Given a mapping $f : D \to \mathbb{R}^n$, a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$, and $0 < r_1 < r_2 < r_0 = \sup_{y \in f(D)} |y - y_0|$, we denote by $\Gamma_f(y_0, r_1, r_2)$ a family of all paths $\gamma$ in $D$ such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function. We say that $f$ satisfies the inverse Poletsky inequality at a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^n(|y - y_0|) \, dm(y) \quad (1.2)$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \quad (1.3)$$

Using the inversion $\psi(y) = \frac{y}{|y|^2}$, we may also define the relation (1.2) at the point $y_0 = \infty$. A mapping $f : D \to \mathbb{R}^n$ is called discrete if the pre-image $\{f^{-1}(y)\}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and open if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. A mapping $f$ of $D$ onto $D'$ is called closed if $f(E)$ is closed in $D'$ for any closed set $E \subset D$ (see, e.g., [Vu, Chapter 3]). Let $h$ be a chordal metric in $\overline{\mathbb{R}}^n$,

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}},$$

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}} \quad x \neq \infty \neq y. \quad (1.4)$$

and let $h(E) := \sup_{x,y \in E} h(x, y)$ be a chordal diameter of a set $E \subset \overline{\mathbb{R}}^n$ (see, e.g., [Va, Definition 12.1]). Everywhere further the boundary $\partial A$ of the set $A$ and the closure $\overline{A}$ should be understood in the sense extended Euclidean space $\overline{\mathbb{R}}^n$. A continuous extension of the mapping $f : D \to \mathbb{R}^n$ also should be understood in terms of mapping with values in $\overline{\mathbb{R}}^n$ and relative to the metric $h$ in (1.4) (if a misunderstanding is impossible). Recall that a domain $D \subset \mathbb{R}^n$ is called locally connected at the point $x_0 \in \partial D$, if for any neighborhood $U$ of a point $x_0$ there is a neighborhood $V \subset U$ of $x_0$ such that $V \cap D$ is connected. A domain $D$ is locally connected at $\partial D$, if $D$ is locally connected at any point $x_0 \in \partial D$. The boundary of the domain $D$ is called weakly flat at the point $x_0 \in \partial D$, if for any $P > 0$ and for any neighborhood $U$ of a point $x_0$ there is a neighborhood $V \subset U$ of the same point such that
M(Γ(E, F, D)) > P for any continua E, F ⊂ D, which intersect ∂U and ∂V. The boundary of the domain D is called weakly flat if the corresponding property is fulfilled at any point of the boundary D.

Recall some definitions (see, for example, [KR1] and [KR2]). Let ω be an open set in \( \mathbb{R}^k \), \( k = 1, \ldots, n - 1 \). A continuous mapping \( σ: ω \rightarrow \mathbb{R}^n \) is called a \( k \)-dimensional surface in \( \mathbb{R}^n \). A surface is an arbitrary \((n - 1)\)-dimensional surface \( σ \) in \( \mathbb{R}^n \). A surface \( σ \) is called a Jordan surface, if \( σ(x) ≠ σ(y) \) for \( x ≠ y \). In the following, we will use \( σ \) instead of \( σ(ω) \) in \( \mathbb{R}^n \), \( \overline{σ(ω)} \) instead of \( \overline{σ(ω)} \) \( σ(ω) \). A Jordan surface \( σ: ω \rightarrow D \) is called a cut of \( D \), if \( σ \) separates \( D \), that is \( D \setminus σ \) has more than one component, \( ∂σ \cap ∂D = \emptyset \) and \( ∂σ \cap ∂D ≠ \emptyset \).

A sequence of cuts \( σ_1, σ_2, \ldots, σ_m, \ldots \) in \( D \) is called a chain, if:

(i) the set \( σ_{m+1} \) is contained in exactly one component \( d_m \) of the set \( D \setminus σ_m \), wherein \( σ_{m-1} \subset D \setminus (σ_m \cup d_m) \);

(ii) \( \bigcap_{m=1}^{∞} d_m = \emptyset \).

Two chains of cuts \( \{σ_m\} \) and \( \{σ_k\} \) are called equivalent, if for each \( m = 1, 2, \ldots \) the domain \( d_m \) contains all the domains \( d_k \), except for a finite number, and for each \( k = 1, 2, \ldots \) the domain \( d_k \) also contains all domains \( d_m \), except for a finite number.

The end of the domain \( D \) is the class of equivalent chains of cuts in \( D \). Let \( K \) be the end of \( D \) in \( \mathbb{R}^n \), then the set \( I(K) = \bigcap_{m=1}^{∞} \overline{d_m} \) is called the impression of the end \( K \). Throughout the paper, \( Γ(E, F, D) \) denotes the family of all paths \( γ: [a, b] \rightarrow \overline{\mathbb{R}^n} \) such that \( γ(a) \in E \), \( γ(b) \in F \) and \( γ(t) \in D \) for every \( t \in [a, b] \). In what follows, \( M \) denotes the modulus of a family of paths, and the element \( dm(x) \) corresponds to the Lebesgue measure in \( \mathbb{R}^n \), \( n ≥ 2 \), see [Ya]. Following [Na2], we say that the end \( K \) is a prime end, if \( K \) contains a chain of cuts \( \{σ_m\} \) such that \( \lim_{m \rightarrow ∞} M(Γ(C, σ_m, D)) = 0 \) for some continuum \( C \) in \( D \). In the following, the following notation is used: the set of prime ends corresponding to the domain \( D \), is denoted by \( E_D \), and the completion of the domain \( D \) by its prime ends is denoted \( \overline{D}_p \).

We say that the boundary of the domain \( D \) in \( \mathbb{R}^n \) is locally quasiconformal, if each point \( x_0 ∈ ∂D \) has a neighborhood \( U \) in \( \mathbb{R}^n \), which can be mapped by a quasiconformal mapping \( ϕ \) onto the unit ball \( \mathbb{B}^n \subset \mathbb{R}^n \) so that \( ϕ(∂D \cap U) \) is the intersection of \( \mathbb{B}^n \) with the coordinate hyperplane.

For a given set \( E \subset \mathbb{R}^n \), we set \( d(E) := \sup_{x, y ∈ E} |x − y| \). The sequence of cuts \( σ_m, m = 1, 2, \ldots \), is called regular, if \( \overline{σ_m} \cap \overline{σ_{m+1}} = \emptyset \) for \( m ∈ \mathbb{N} \) and, in addition, \( d(σ_m) \rightarrow 0 \) as \( m \rightarrow ∞ \). If the end \( K \) contains at least one regular chain, then \( K \) will be called regular. We say that a bounded domain \( D \) in \( \mathbb{R}^n \) is regular, if \( D \) can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in \( \mathbb{R}^n \), and, besides that, every prime end in \( D \) is regular. Note that space \( \overline{D}_p = D \cup E_D \) is metric, which can be demonstrated as follows. If \( g : D_0 \rightarrow D \) is a quasiconformal mapping of a domain \( D_0 \) with
a locally quasiconformal boundary onto some domain \( D \), then for \( x, y \in \overline{D}_P \) we put:

\[
\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|,
\]

where the element \( g^{-1}(x), x \in E_D \), is to be understood as some (single) boundary point of the domain \( D_0 \). The specified boundary point is unique and well-defined by \([S_2]\) Theorem 2.1, Remark 2.1], cf. \([Na_2]\) Theorem 4.1]. It is easy to verify that \( \rho \) in (1.5) is a metric on \( \overline{D}_P \), and that the topology on \( \overline{D}_P \), defined by such a method, does not depend on the choice of the map \( g \) with the indicated property.

We say that a sequence \( x_m \in D, m = 1, 2, \ldots \), converges to a prime end of \( P \in E_D \) as \( m \to \infty \), if for any \( k \in \mathbb{N} \) all elements \( x_m \) belong to \( d_k \) except for a finite number. Here \( d_k \) denotes a sequence of nested domains corresponding to the definition of the prime end \( P \).

In what follows, given sets \( A, B \subset \mathbb{R}^n \) we put \( h(A, B) = \inf_{x \in A, y \in B} h(x, y) \), where \( h \) denotes the chordal metric in \( \mathbb{R}^n \) defined in (1.3). Given a number \( \delta > 0 \), domains \( D, D' \subset \mathbb{R}^n \), \( n \geq 2 \), a continuum \( A \subset D' \) and a Lebesgue measurable function \( Q : D' \to [0, \infty] \) denote by \( \mathcal{G}_{\delta, A, Q}(D, D') \) a family of all open discrete and closed mappings \( f \) of \( D \) onto \( D' \) satisfying the relation (1.2) for any \( y_0 \in D' \) such that \( h(f^{-1}(A), \partial D) \geq \delta \). The following statement holds (see Theorem 2 in \([Sev]\) for the case of the corresponding integrable functions \( Q \)).

**Theorem 1.1.** Let \( D \) be a domain with a weakly flat boundary, and let \( D' \) be a regular domain. Suppose that, for each point \( y_0 \in \overline{D}' \) and for any \( 0 < r_1 < r_2 < r_0 = r_0(y_0) := \sup_{y \in D'} |y - y_0| \) there is a set \( E \subset [r_1, r_2] \) of a positive linear Lebesgue measure such that the function \( Q \) is integrable on \( S(y_0, r) \) for any \( r \in E \). Then any \( f \in \mathcal{G}_{\delta, A, Q}(D, D') \) has a continuous extension \( \overline{f} : \overline{D} \to \overline{D}_P \), while \( \overline{f}(\overline{D}) = \overline{D}_P \) and a family \( \mathcal{G}_{\delta, A, Q}(\overline{D}, \overline{D}') \) of all extended mappings \( \overline{f} : \overline{D} \to \overline{D}_P \) is equicontinuous in \( \overline{D} \).

## 2 Proof of the main result

Let \( D \subset \mathbb{R}^n, f : D \to \mathbb{R}^n \) be a discrete open mapping, \( \beta : [a, b] \to \mathbb{R}^n \) be a path, and \( x \in f^{-1}(\beta(a)) \). A path \( \alpha : [a, c] \to D \) is called a *maximal f-lifting* of \( \beta \) starting at \( x \), if

1. \( \alpha(a) = x \);
2. \( f \circ \alpha = \beta|_{[a, c]} \);
3. for \( c < c' \leq b \), there is no a path \( \alpha' : [a, c'] \to D \) such that \( \alpha = \alpha'|_{[a, c]} \) and \( f \circ \alpha' = \beta|_{[a, c']} \).

The following statement holds (see \([Ri]\) Corollary II.3.3]).

**Proposition 2.1.** Let \( f : D \to \mathbb{R}^n \) be a discrete open mapping, \( \beta : [a, b] \to \mathbb{R}^n \) be a path, and \( x \in f^{-1}(\beta(a)) \). Then \( \beta \) has a maximal f-lifting starting at \( x \). If \( \beta : (a, b) \to f(D) \) is a path, and \( x \in f^{-1}(\beta(b)) \), then \( \beta \) has a maximal f-lifting ending at \( x \).

A path \( \alpha : [a, b] \to D \) is called a *total f-lifting* of \( \beta \) starting at \( x \), if

1. \( \alpha(a) = x \);
2. \( (f \circ \alpha)(t) = \beta(t) \) for any \( t \in [a, b] \).

In the case when the mapping \( f \) is also closed, we have a strengthened version of Proposition 2.1 (see, for example, \([Vu]\) Lemma 3.7]).
Proposition 2.2. Let \( f : D \to \mathbb{R}^n \) be a discrete open and closed mapping, \( \beta : [a, b) \to f(D) \) be a path, and \( x \in f^{-1}(\beta(a)) \). Then \( \beta \) has a total \( f \)-lifting starting at \( x \).

The following statement holds.

Theorem 2.1. Let \( D \subset \mathbb{R}^n, n \geq 2 \), be a domain with a weakly flat boundary, and let \( D' \subset \mathbb{R}^n \) be a regular domain. Suppose that \( f \) is open discrete and closed mapping of \( D \) onto \( D' \) satisfying the relation (1.2) at any point \( y_0 \in \partial D' \). Suppose that, for each point \( y_0 \in \partial D' \), there is a set \( E \subset [r_1, r_2] \) of a positive linear Lebesgue measure such that for any \( 0 < r_1 < r_2 < r_* \), there exists \( E \subset [r_1, r_2] \) of a positive linear Lebesgue measure such that the function \( Q \) is integrable on \( S(y_0, r) \) for any \( r \in E \). Then \( f \) has a continuous extension \( \tilde{f} : \overline{D} \to \overline{D'}, \) while \( \tilde{f}(\overline{D}) = \overline{D'} \).

The following lemma was proved in [ISS] Lemma 2.1.

Lemma 2.1. Let \( D \subset \mathbb{R}^n, n \geq 2 \), be a regular domain, and let \( x_m \to P_1, y_m \to P_2 \) as \( m \to \infty \), \( P_1, P_2 \in \overline{D_2}, P_1 \neq P_2 \). Suppose that \( d_m, g_m, m = 1, 2, \ldots \), are sequences of descending domains, corresponding to \( P_1 \) and \( P_2 \), \( d_1 \cap g_1 = \emptyset \), and \( x_0, y_0 \in \overline{D \setminus (d_1 \cup g_1)} \). Then there are arbitrarily large \( k_0 \in \mathbb{N} \), \( M_0 = M_0(k_0) \in \mathbb{N} \) and \( 0 < t_1 = t_1(k_0), t_2 = t_2(k_0) < 1 \) for which the following condition is fulfilled: for each \( m \geq M_0 \) there are non-intersecting paths

\[
\gamma_{1,m}(t) = \begin{cases} \tilde{\alpha}(t), & t \in [0, t_1], \\ \tilde{\alpha}_m(t), & t \in [t_1, 1] \end{cases}, \quad \gamma_{2,m}(t) = \begin{cases} \tilde{\beta}(t), & t \in [0, t_2], \\ \tilde{\beta}_m(t), & t \in [t_2, 1] \end{cases},
\]

such that:

1) \( \gamma_{1,m}(0) = x_0, \gamma_{1,m}(1) = x_m, \gamma_{2,m}(0) = y_0 \) and \( \gamma_{2,m}(1) = y_m \);
2) \( |\gamma_{1,m}| \cap \overline{g_{k_0}} = \emptyset = |\gamma_{2,m}| \cap \overline{d_{k_0}} \);
3) \( \tilde{\alpha}_m(t) \in d_{k_0 + 1} \) for \( t \in [t_1, 1] \) and \( \tilde{\beta}_m(t) \in g_{k_0 + 1} \) for \( t \in [t_2, 1] \) (see Figure 1).

![Figure 1: To the statement of Lemma 2.1](image-url)

The following lemma was established in [Sev] Lemma 2] for the case of integrable functions \( Q \); see also the case of homeomorphisms established in [ISS] Lemma 2.2.

Lemma 2.2. Let \( D \) and \( D' \) be domains in \( \mathbb{R}^n, n \geq 2 \), let \( D' \) be a regular domain and let \( f \) be an open discrete and closed mapping of \( D \) onto \( D' \) which satisfies the condition (1.2) at
any point \( y_0 \in \overline{D'} \) and for some \( Q \). Let \( d_m \) be a sequence of decreasing domains corresponding to the chain of cuts \( \sigma_m, m = 1, 2, \ldots, \) lying on spheres \( S(x_0, r_m) \) and such that \( x_0 \in \partial D' \), while \( r_m \to 0 \) as \( m \to \infty \). Suppose that, for any point \( y_0 \in \overline{D'} \) there is \( 0 < r_\ast = r_\ast(y_0) < \sup_{y \in \overline{D'}} |y - y_0| \) such that, for any \( 0 < r_1 < r_2 < r_\ast \), there is a set \( E \subset [r_1, r_2] \) of a positive linear Lebesgue measure such that the function \( Q \) is integrable on \( S(y_0, r) \) for any \( r \in E \). Then, under conditions of Lemma 2.1, we may find a number \( |N| < \infty \), non depending on \( m \) and \( f \) such that

\[
M(\Gamma_m) \leq N, \quad m \geq M_0 = M_0(k_0),
\]

where \( \Gamma_m \) denotes a family of paths \( \gamma : [0, 1] \to D \) in \( D' \) such that \( f(\gamma) \in \Gamma(|\gamma_1, m|, |\gamma_2, m|, D') \).

**Proof.** We mostly follow the outline of the proof of Lemma 2 in [Sev]. Let \( k_0 \) be an arbitrary number for which the statement of Lemma 2.1 holds. By the definition of a path \( \gamma_1, m \) and a family \( \Gamma_m \) we may write

\[
\Gamma_m = \Gamma^1_m \cup \Gamma^2_m,
\]

(2.1)

where \( \Gamma^1_m \) is a family of paths \( \gamma \in \Gamma_m \) such that \( f(\gamma) \in \Gamma(|\alpha|, |\gamma_2, m|, D') \) and \( \Gamma^2_m \) is a family of paths \( \gamma \in \Gamma_m \) such that \( f(\gamma) \in \Gamma(|\alpha|, |\gamma_2, m|, D') \).

Taking into account the notions of Lemma 2.1, we put

\[
\varepsilon_0 := \min \{ \text{dist} \left( |\alpha|, \frac{y_0}{2} \right), \text{dist} \left( |\alpha|, |\beta| \right) \} > 0.
\]

Let us consider the covering of a set \( |\alpha| \) of the following type: \( \bigcup \limits_{x \in |\alpha|} B(x, \varepsilon_0/4) \). Since \( |\alpha| \) is a compactum in \( D' \), there are numbers \( i_1, \ldots, i_{N_0} \) such that \( |\alpha| \subset \bigcup \limits_{i=1}^{N_0} B(z_i, \varepsilon_0/4) \), where \( z_i \in |\alpha| \) for \( 1 \leq i \leq N_0 \). Due to [Ku, Theorem 1.1.5.46]

\[
\Gamma(|\alpha|, |\gamma_2, m|, D') > \bigcup \limits_{i=1}^{N_0} \Gamma(S(z_i, \varepsilon_0/4), S(z_i, \varepsilon_0/2), A(z_i, \varepsilon_0/4, \varepsilon_0/2)).
\]

(2.2)

Fix \( \gamma \in \Gamma^1_m, \gamma : [0, 1] \to D, \gamma(0) \in |\alpha|, \gamma(1) \in |\gamma_2, m| \). By the relation (2.2) it follows that \( f(\gamma) \) has a subpath \( f(\gamma)_1 := f(\gamma)|_{[p_1, p_2]} \) such that

\[
f(\gamma)_1 \in \Gamma(S(z_i, \varepsilon_0/4), S(z_i, \varepsilon_0/2), A(z_i, \varepsilon_0/4, \varepsilon_0/2))
\]

for some \( 1 \leq i \leq N_0 \). Then \( \gamma|_{[p_1, p_2]} \) is a subpath of \( \gamma \), and belongs to \( \Gamma_f(z_i, \varepsilon_0/4, \varepsilon_0/2) \), because

\[
f(\gamma)|_{[p_1, p_2]} = f(\gamma)|_{[p_1, p_2]} \in \Gamma(S(z_i, \varepsilon_0/4), S(z_i, \varepsilon_0/2), A(z_i, \varepsilon_0/4, \varepsilon_0/2)).
\]

Thus

\[
\Gamma^1_m > \bigcup \limits_{i=1}^{N_0} \Gamma_f(z_i, \varepsilon_0/4, \varepsilon_0/2).
\]

(2.3)
Set
\[ \eta_i(t) = \begin{cases} \frac{1}{t q_z(t)} & t \in [\varepsilon_0/4, \varepsilon_0/2], \\ 0 & t \notin [\varepsilon_0/4, \varepsilon_0/2] \end{cases}, \]
where
\[ I_i = \int_{\varepsilon_0/4}^{\varepsilon_0/2} \frac{dt}{t q_z(t)} \]
and \( q_z(r) \) is defined in
\[ q_z_0(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(z_0, r)} Q(y) dH^{n-1}(y), \quad (2.4) \]
and \( \omega_{n-1} \) denotes the area of the unit sphere \( \mathbb{S}^{n-1} \) in \( \mathbb{R}^n \). Observe that a function \( \eta \) satisfies the condition (1.3). Then, by the definition of the mapping \( f \) in (1.2) and by the relation (2.3), due to the subadditivity of the modulus of families of paths (see [Va, Theorem 6.2]) and by the Fubini theorem, we obtain that
\[ M(\Gamma^1_m) \leq \sum_{i=1}^{N_0} M(\Gamma_f(z_i, \varepsilon_0/4, \varepsilon_0/2)) \leq \sum_{i=1}^{N_0} \frac{\omega_{n-1}}{I_i} < \infty, \quad m \geq M_0, \quad (2.5) \]
Here we have used the fact that, \( q_{z_0}(t) \) is finite for all \( t \in E \) that follows due to the assumption of the lemma. Further, by [Ku, Theorem 1.I.5.46], \( \Gamma^2_m > \Gamma_f(\bar{x}_0, r_{k_0+1}, r_{k_0}) \). Arguing similarly to the above, we set
\[ \eta_0(t) = \begin{cases} \frac{1}{I_0 t q_{\bar{x}_0}(t)} & t \in [\varepsilon_0/4, \varepsilon_0/2], \\ 0 & t \notin [\varepsilon_0/4, \varepsilon_0/2] \end{cases}, \]
where
\[ I_0 = \int_{\varepsilon_0/4}^{\varepsilon_0/2} \frac{dt}{t q_{\bar{x}_0}(t)} \]
Then the latter relation implies that
\[ M(\Gamma^2_m) \leq \frac{\omega_{n-1}}{I_0} < \infty, \quad m \geq M_0. \quad (2.6) \]
Thus, by (2.1), (2.5) and (2.6), due to the subadditivity of the modulus of families of paths, we obtain that
\[ M(\Gamma_m) \leq \sum_{i=1}^{N_0} \frac{\omega_{n-1}}{I_i} + \frac{\omega_{n-1}}{I_0} < \infty, \quad m \geq M_0. \]
The right side of the latter inequality does not depend on \( m \), therefore we may put \( N := \sum_{i=1}^{N_0} \frac{\omega_{n-1}}{I_i} + \frac{\omega_{n-1}}{I_0} < \infty \). Lemma 2.2 is proved. 

Proof of Theorem 1.1. The proof of this theorem completely coincides with the proof of Theorem 2 in [Sev], however, for the sake of completeness, we present it in full in the
text. Note that, the possibility of a continuous extension of any \( f \in \mathfrak{G}_{\delta,A,Q}(D, D') \) to the boundary of \( D \) follows by Theorem 2.1. The equicontinuity of \( \mathfrak{G}_{\delta,A,Q}(D, D') \) at inner points of \( D \) is proved in [SevSkv3, Theorem 1.1].

Let us to prove the equicontinuity of the family \( \mathfrak{G}_{\delta,A,Q}(\overline{D}, \overline{D}) \) in \( \partial D \). Assume the contrary. Then there is a point \( z_0 \in \partial D \), a number \( \varepsilon_0 > 0 \), a sequence \( z_m \in \overline{D} \) and a mapping \( \overline{f}_m \in \mathfrak{G}_{\delta,A,Q}(\overline{D}, \overline{D'}) \) such that \( z_m \to z_0 \) as \( m \to \infty \), while

\[
\rho(\overline{f}_m(z_m), \overline{f}_m(z_0)) \geq \varepsilon_0, \quad m = 1, 2, \ldots, \tag{2.7}
\]

were \( \rho \) is one of the possible metrics in \( \overline{D} \), which is defined in (1.3). Since \( f_m = \overline{f}_m|_D \) has a continuous extension onto \( \overline{D} \), we may assume that \( z_m \in D \) and, in addition, there is another sequence \( z_m' \in D \), \( z_m' \to z_0 \) as \( m \to \infty \), such that \( \rho(f_m(z_m'), \overline{f}_m(z_0)) \to 0 \) as \( m \to \infty \). In this case, it follows from (2.7) that

\[
\rho(f_m(z_m), f_m(z_m')) \geq \varepsilon_0/2, \quad m \geq m_0. \tag{2.8}
\]

Since \( D' \) is regular, the metric space \( \overline{D} \) is compact. Thus, we may assume that the sequences \( f_m(z_m) \) and \( f_m(z_m') \) converge to some elements \( P_1, P_2 \in \overline{D} \), \( P_1 \neq P_2 \), as \( m \to \infty \). Let \( d_m \) and \( g_m \) be sequences of decreasing domains which correspond to prime ends \( P_1 \) and \( P_2 \), respectively. Due to [LS1, Lemma 3.1], cf. [KR2] Lemma 1], we may consider that a sequence of cuts \( \sigma_m \), which corresponds to domains \( d_m, m = 1, 2, \ldots \), lying on the spheres \( S(\overline{x}_0, r_m) \), where \( \overline{x}_0 \in \partial D' \) and \( r_m \to 0 \) as \( m \to \infty \). Put \( x_0, y_0 \in A \) such that \( x_0 \neq y_0 \) and \( x_0 \neq P_1 \neq y_0 \), where \( A \subset D' \) is a continuum from the conditions of Theorem 1.1. Without loss of generality, we may assume that \( d_1 \cap g_1 = \emptyset \) and \( x_0, y_0 \notin d_1 \cup g_1 \).

Be Lemmas 2.1 and 2.2 there are disjoint paths \( \gamma_{1,m} : [0, 1] \to D' \) and \( \gamma_{2,m} : [0, 1] \to D' \) and a number \( N > 0 \) such that \( \gamma_{1,m}(0) = x_0, \gamma_{1,m}(1) = f_m(z_m), \gamma_{2,m}(0) = y_0, \gamma_{2,m}(1) = f_m(z_m'), \) while

\[
M(\Gamma_m) \leq N, \quad m \geq M_0, \tag{2.9}
\]

where \( \Gamma_m \) consist from the paths \( \gamma \) in \( D \), for which \( f_m(\gamma) \in \Gamma(\gamma_{1,m}, \gamma_{2,m}, D') \) (see Figure 2). On the other hand, let \( \gamma_{1,m}^*, \gamma_{2,m}^* \) be total \( f_m \)-liftings of \( \gamma_{1,m} \) and \( \gamma_{2,m} \) starting a the points \( z_m \) and \( z_m' \), respectively (such liftings exist due to [Ym1, Lemma 3.7]). Then \( \gamma_{1,m}^*(1) \in f_m^{-1}(A) \) and \( \gamma_{2,m}^*(1) \in f_m^{-1}(A) \) and, since \( h(f_m^{-1}(A), \partial D) > \delta > 0, m = 1, 2, \ldots \), we may consider that

\[
\begin{align*}
\quad h(\gamma_{1,m}^*) &\geq h(z_m, \gamma_{1,m}^*(1)) \geq (1/2) \cdot h(f_m^{-1}(A), \partial D) > \delta/2, \\
\quad h(\gamma_{2,m}^*) &\geq h(z_m', \gamma_{2,m}^*(1)) \geq (1/2) \cdot h(f_m^{-1}(A), \partial D) > \delta/2
\end{align*} \tag{2.10}
\]

for sufficiently large \( m \in \mathbb{N} \). We chose a ball \( U := B_h(z_0, r_0) = \{ z \in \mathbb{R}^n : h(z, z_0) < r_0 \} \), where \( r_0 > 0 \) and \( r_0 < \delta/4 \). Note that, \( \gamma_{1,m}^* \cap U \neq \emptyset \neq \gamma_{1,m}^* \cap (D \setminus U) \) for sufficiently large \( m \in \mathbb{N} \), because \( h(f_m(\gamma_{1,m})), z_m \to z_0 \) as \( m \to \infty \). Arguing similarly,
we may make a conclusion that $|\gamma^*_2, m| \cap U \neq \emptyset \neq |\gamma^*_2, m| \cap (D \setminus U)$. Since $|\gamma^*_1, m|$ and $|\gamma^*_2, m|$ are continua, by [Ku, Theorem 1.1.5.46]

$$|\gamma^*_1, m| \cap \partial U \neq \emptyset, \quad |\gamma^*_2, m| \cap \partial U \neq \emptyset. \quad (2.11)$$

Put $P := N > 0$, where $N$ is a number from the relation $\text{(2.9)}$. Since the boundary of a domain $D$ is weakly flat, there is a neighborhood $V \subset U$ of a point $z_0$ such that the relation

$$M(\Gamma(E, F, D)) > N \quad \text{(2.12)}$$

holds for any continua $E, F \subset D$ with $E \cap \partial U \neq \emptyset \neq E \cap \partial V$ and $F \cap \partial U \neq \emptyset \neq F \cap \partial V$. Observe that

$$|\gamma^*_1, m| \cap \partial V \neq \emptyset, \quad |\gamma^*_2, m| \cap \partial V \neq \emptyset \quad \text{(2.13)}$$

for sufficiently large $m \in \mathbb{N}$. Indeed, $z_m \in |\gamma^*_1, m|$ and $z'_m \in |\gamma^*_2, m|$, where $z_m, z'_m \to z_0 \in V$ as $m \to \infty$. Thus, $|\gamma^*_1, m| \cap V \neq \emptyset \neq |\gamma^*_2, m| \cap V$ for sufficiently large $m \in \mathbb{N}$. Besides that, $h(V) \leq h(U) = 2r_0 < \delta/2$ and, by (2.10), $h(|\gamma^*_1, m|) > \delta/2$. Therefore, $|\gamma^*_1, m| \cap (D \setminus V) \neq \emptyset$. Then $|\gamma^*_1, m| \cap \partial V \neq \emptyset$ (see [Ku, Theorem 1.1.5.46]). Similarly, $h(V) \leq h(U) = 2r_0 < \delta/2$. Now, by (2.10) $h(|\gamma^*_2, m|) > \delta/2$. Thus, $|\gamma^*_2, m| \cap (D \setminus V) \neq \emptyset$. By [Ku, Theorem 1.1.5.46] we obtain that $|\gamma^*_1, m| \cap \partial V \neq \emptyset$. The relation (2.13) is proved. It follows from (2.12), (2.11) and (2.13) that

$$M(\Gamma(|\gamma^*_1, m|, |\gamma^*_2, m|, D)) > N. \quad (2.14)$$

The inequality (2.14) contradicts with (2.9) because $\Gamma(|\gamma^*_1, m|, |\gamma^*_2, m|, D) \subset \Gamma_m$ and, consequently,

$$M(\Gamma(|\gamma^*_1, m|, |\gamma^*_2, m|, D)) \leq M(\Gamma_m) \leq N.$$

The contradiction obtained above disproves the assumption in (2.7). Theorem is proved. □

3 Lemma on a continuum

One of the versions of the following statement is established in [SevSkv1] item v, Lemma 2] for homeomorphisms and "good" boundaries, see also [SevSkv2] Lemma 4.1]. Let us also point
out the case relating to mappings with branching and good boundaries, see [SSD, Lemma 6.1], as well as the case of bad boundaries and homeomorphisms, see [ISS, Lemma 2.13]. The statement below seems to refer to the most general situation when some function $Q$ participating in inequality (1.2), generally speaking, is not integrable; however, it has finite averages over spheres.

**Lemma 3.1.** Let $n \geq 2$, let $D$ and $D'$ be a domains in $\mathbb{R}^n$, while $D$ has a weakly flat domain none of the components of which degenerates into a point, and $D'$ is a regular domain. Let $A$ be some non-degenerate continuum in $D'$ and $\delta > 0$. Assume that, $f_m$ is a sequence of open discrete and closed mappings of $D$ onto $D'$ such that, for any $m = 1, 2, \ldots$ there is a continuum $A_m \subset D$, $m = 1, 2, \ldots$, such that $f_m(A_m) = A$ and $h(A_m) \geq \delta > 0$. Suppose also that, for any $m = 1, 2, \ldots$, a mapping $f_m$ satisfies the relation (1.2) at any point, and, in addition, for any $y_0 \in \overline{D'}$ and for any $0 < r_1 < r_2 < r_0 = r_0(y_0) := \sup_{y \in D'} |y - y_0|$ there is a set $E \subset [r_1, r_2]$ of a positive linear Lebesgue measure such that the function $Q$ is integrable on $S(y_0, r)$ for any $r \in E$. Then there is $\delta_1 > 0$ such that

$$h(A_m, \partial D) > \delta_1 > 0 \quad \forall m \in \mathbb{N}.$$ 

**Proof.** The proof of this assertion is very close to the proof of Lemma 1 in [SevIl]. Since the space $\mathbb{R}^n$ is compact, the boundary of $D$ is not empty, and is a compact set. Thus, $h(A_m, \partial D)$ is well-defined.

Let us do a proof by the contradiction. Assume that the statement of the lemma is not true. Then, for any $k \in \mathbb{N}$ there is a number $m = m_k$ such that $h(A_{m_k}, \partial D) < 1/k$. We may consider that a sequence $m_k$ is increasing by $k$. Since $A_{m_k}$ is a compact set, there are $x_k \in A_{m_k}$ and $y_k \in \partial D$ such that $h(A_{m_k}, \partial D) = h(x_k, y_k) < 1/k$ (see Figure 3). Since $\partial D$ is a compact set, we may assume that $y_k \to y_0 \in \partial D$ as $k \to \infty$. Then also $x_k \to y_0 \in \partial D$ as $k \to \infty$. Let $K_0$ be a component of $\partial D$ which contains $y_0$. Obviously, $K_0$ is a continuum in $\mathbb{R}^n$. Since $\partial D$ is weakly flat, by Theorem 2.1 $f_{m_k}$ has a continuous extension $\overline{f}_{m_k}: \overline{D} \to \overline{D'}$. Let $\rho$ be one of the metrics in (1.5), and let $g: D_0 \to D'$ be a quasiconformal mapping of some domain $D_0$ with a locally quasiconformal boundary onto $D'$, which corresponds to the metric $\rho$ in (1.3). Since $\overline{f}_{m_k}$ is a continuous on a compact set $\overline{D}$, a mapping $\overline{f}_{m_k}$ is uniformly continuous in $\overline{D}$ with a respect to the metric $\rho$ for any $k$. In other words, for any $\varepsilon > 0$ there is $\delta_k = \delta_k(\varepsilon) < 1/k$ such that

$$\rho(\overline{f}_{m_k}(x), \overline{f}_{m_k}(x_0)) < \varepsilon$$

$$\forall x, x_0 \in \overline{D}, \quad h(x, x_0) < \delta_k, \quad \delta_k < 1/k,$$

where, as usually, $h$ is a chordal metric in $\mathbb{R}^n$. Choose $\varepsilon > 0$ such that

$$\varepsilon < (1/2) \cdot \text{dist} (\partial D_0, g^{-1}(A)),$$
where \( \text{dist} \) denotes the chordal distance between \( A \) and \( B \) in \( \mathbb{R}^n \). Set \( B_h(x_0, r) = \{ x \in \mathbb{R}^n : h(x, x_0) < r \} \). Given \( k \in \mathbb{N} \), we set

\[
B_k := \bigcup_{x_0 \in K_0} B_h(x_0, \delta_k), \quad k \in \mathbb{N}.
\]

Since \( B_k \) is a neighborhood of \( K_0 \), by [HK, Lemma 2.2] there is neighborhood \( U_k \) of \( K_0 \) such that \( U_k \subset B_k \) and \( U_k \cap D \) is connected. We may assume that \( U_k \) is open, so that \( U_k \cap D \) is path connected (see [MRSY, Proposition 13.1]). Let \( h(K_0) = m_0 \). Then there are \( z_0, w_0 \in K_0 \) such that \( h(z_0, w_0) = m_0 \). Thus, there are sequences \( y_k \in U_k \cap D \), \( z_k \in U_k \cap D \) and \( w_k \in U_k \cap D \) such that \( z_k \to z_0 \), \( y_k \to y_0 \) and \( w_k \to w_0 \) as \( k \to \infty \). We may assume that

\[
h(z_k, w_k) > m_0/2 \quad \forall k \in \mathbb{N}.
\]

Since the set \( U_k \cap D \) is path connected, we may join the points \( z_k, y_k \) and \( w_k \) using by some path \( \gamma_k \) in \( D' \). If \( x \in |\gamma_k| \) then there is \( x_0 \in K_0 \) such that \( x \in B(x_0, \delta_k) \). Put \( \omega \in A \subset D \). Since \( x \in |\gamma_k| \) and, besides that, \( x \) is an inner point of \( D \), we may use the notation \( f_{m_k}(x) \) instead of \( f_{m_k}( \gamma_k ) \). Due to the relations (3.1) and (3.2), by the triangle inequality, we obtain that

\[
\rho(f_{m_k}(x), \omega) \geq \rho(\omega, f_{m_k}(x_0)) - \rho(f_{m_k}(x_0), f_{m_k}(x)) \geq
\]

\[
\geq \text{dist}(\partial D_0, g^{-1}(A)) - (1/2) \cdot \text{dist}(\partial D_0, g^{-1}(A)) =
\]

\[
= (1/2) \cdot \text{dist}(\partial D_0, g^{-1}(A)) > \varepsilon
\]

for sufficiently large \( k \in \mathbb{N} \), where

\[
\text{dist}(\partial D_0, g^{-1}(A)) := \inf_{x \in \partial D_0, y \in g^{-1}(A)} |x - y|.
\]
Passing to the inf in (3.4) over \( x \in |\gamma_k| \) and \( \omega \in A \), we obtain that

\[
\rho(f_{m_k}(|\gamma_k|), A) > \varepsilon, \quad k = 1, 2, \ldots.
\]  

(3.5)

Now let us to show that, there is \( \varepsilon_1 > 0 \) such that

\[
\text{dist} \ (f_{m_k}(|\gamma_k|), A) > \varepsilon_1, \quad \forall \ k = 1, 2, \ldots,
\]  

(3.6)

where \( \text{dist} \ (A, B) \), as usually, denotes the Euclidean distance between sets \( A, B \subseteq \mathbb{R}^n \). Indeed, let the relation (3.6) fails. Now, for a number \( \varepsilon_l = 1/l, \ l = 1, 2, \ldots \) there are \( \xi_l \in |\gamma_k| \) and \( \zeta_l \in A \) such that

\[
|f_{m_k}(\xi_l) - \zeta_l| < 1/l, \quad l = 1, 2, \ldots
\]  

(3.7)

We may assume that the sequence \( k_l, \ l = 1, 2, \ldots, \) is increasing. Since \( A \) is a compactum, we may also consider that the sequence \( \zeta_l \) converges to \( \zeta_0 \in A \) as \( l \to \infty \). By the triangle inequality and due to the relation (3.7) we obtain that

\[
|f_{m_k}(\xi_l) - \zeta_0| \to 0, \quad l \to \infty
\]  

(3.8)

On the other hand, let us recall that \( \rho(f_{m_k}(x), \omega) = |g^{-1}(f_{m_k}(x)) - g^{-1}(\omega)| \), where \( g : D_0 \to D' \) is some quasiconformal mapping of \( D_0 \) onto \( D' \), see (1.5). In particular, the mapping \( g^{-1} \) is continuous in \( D' \), thus, by the triangle inequality and by (3.8) we obtain that

\[
|g^{-1}(f_{m_k}(\xi_l)) - g^{-1}(\zeta_l)| \leq |g^{-1}(f_{m_k}(\xi_l)) - g^{-1}(\zeta_0)| + |g^{-1}(\zeta_0) - g^{-1}(\zeta_l)| \to 0, \quad l \to \infty.
\]  

(3.9)

However, by the definition of the metric \( \rho \) and by (3.9) we obtain that

\[
\rho(f_{m_k}(|\gamma_k|), A) \leq \rho(f_{m_k}(\xi_l), \zeta_l) = |g^{-1}(f_{m_k}(\xi_l)) - g^{-1}(\zeta_l)| \to 0, \quad l \to \infty,
\]

that contradicts with (3.5). The contradiction obtained above proves the relation (3.6).

We cover the set \( A \) with the balls \( B(x, \varepsilon_1/4), \ x \in A \). Since \( A \) is a compactum, we may assume that \( A \subseteq \bigcup_{i=1}^{M_0} B(x_i, \varepsilon_1/4), \ x_i \in A, \ i = 1, 2, \ldots, M_0, \ 1 \leq M_0 < \infty \). By the definition, \( M_0 \) depends only on \( A \), in particular, \( M_0 \) does not depend on \( k \). Put

\[
\Gamma_k := \Gamma(A_{m_k}, |\gamma_k|, D).
\]  

(3.10)

Let \( \Gamma_{ki} := \Gamma_{f_{m_k}(x_i, \varepsilon_1/4, \varepsilon_1/2)} \), in other words, \( \Gamma_{ki} \) consists from all paths \( \gamma : [0, 1] \to D \) such that \( f_{m_k}(\gamma(0)) \in S(x_i, \varepsilon_1/4), \ f_{m_k}(\gamma(1)) \in S(x_i, \varepsilon_1/2) \) and \( \gamma(t) \in A(x_i, \varepsilon_1/4, \varepsilon_1/2) \) for \( 0 < t < 1 \). Let us to show that

\[
\Gamma_k > \bigcup_{i=1}^{M_0} \Gamma_{ki}.
\]  

(3.11)
Indeed, let $\tilde{\gamma} \in \Gamma_k$, in other words, $\tilde{\gamma} : [0,1] \to D$, $\tilde{\gamma}(0) \in A_{mk}$, $\tilde{\gamma}(1) \in |\gamma_k|$ and $\tilde{\gamma}(t) \in D$ for $0 \leq t \leq 1$. Then $\gamma^* := f_{mk}(\tilde{\gamma}) \in \Gamma(A,f_{mk}(|\gamma_k|),D')$. Since the balls $B(x_i,\varepsilon_1/4)$, $1 \leq i \leq M_0$, form a covering of a compactum $A$, there is $i \in \mathbb{N}$ such that $\gamma^*(0) \in B(x_i,\varepsilon_1/4)$ and $\gamma^*(1) \in f_{mk}(|\gamma_k|)$. By (3.6), $|\gamma^*| \cap B(x_i,\varepsilon_1/4) \neq \emptyset \neq |\gamma^*| \cap (D' \setminus B(x_i,\varepsilon_1/4))$. Thus, by [Ku] Theorem 1.1.5.46 there is $0 < t_1 < 1$ such that $\gamma^*(t_1) \in S(x_i,\varepsilon_1/4)$. We may assume that $\gamma^*(t) \not\in B(x_i,\varepsilon_1/4)$ for $t > t_1$. Set $\gamma_1 := \gamma^*|_{[t_1,1]}$. It follows from (3.6) that $|\gamma_1| \cap B(x_i,\varepsilon_1/2) \neq \emptyset \neq |\gamma_1| \cap (D' \setminus B(x_i,\varepsilon_1/2))$. Thus, by [Ku] theorem 1.1.5.46 there is $t_1 < t_2 < 1$ such that $\gamma^*(t_2) \in S(x_i,\varepsilon_1/2)$. We may assume that $\gamma^*(t) \in B(x_i,\varepsilon_1/2)$ for any $t < t_2$. Setting $\gamma_2 := \gamma^*|_{[t_1,t_2]}$, we observe that a path $\gamma_2$ is a subpath of $\gamma^*$ which belongs to $\Gamma(S(x_i,\varepsilon_1/4),S(x_i,\varepsilon_1/2),A(x_i,\varepsilon_1/4,\varepsilon_1/2))$. Finally, $\tilde{\gamma}$ has a subpath $\tilde{\gamma}_2 := \tilde{\gamma}|_{[t_1,t_2]}$ such that $f_{mk} \circ \tilde{\gamma}_2 = \gamma_2$, while $\gamma_2 \in \Gamma(S(x_i,\varepsilon_1/4),S(x_i,\varepsilon_1/2),A(x_i,\varepsilon_1/4,\varepsilon_1/2))$. Thus, the relation (3.11) is proved. Put Set

$$\eta_i(t) = \begin{cases} \frac{1}{I(t,q^1_{n-1}(t))}, & t \in [\varepsilon_1/4,\varepsilon_1/2], \\ 0, & t \not\in [\varepsilon_1/4,\varepsilon_1/2], \end{cases}$$

where

$$I_i = \int_{\varepsilon_1/4}^{\varepsilon_1/2} \frac{dt}{t q^1_{n-1}(t)}$$

and $q_n(r)$ is defined in (2.4). Observe that a function $\eta$ satisfies the condition (1.3). Then, by the definition of the mapping $f$ in (1.2) and by the relation (2.3) and by the Fubini theorem, we obtain that

$$M(\Gamma_{f_{mk}}(x_i,\varepsilon_1/4,\varepsilon_1/2)) \leq \frac{\omega_{n-1}}{I_i} < \infty. \quad (3.12)$$

Due to the subadditivity of the modulus of families of paths, it follows from (3.11) and (3.12) that

$$M(\Gamma_k) \leq \sum_{i=1}^{M_0} \frac{\omega_{n-1}}{I_i} < \infty. \quad (3.13)$$

On the other hand, since $D$ has a weakly flat boundary, due to the condition (3.3), we obtain that $M(\Gamma_k) \to \infty$ as $k \to \infty$, that contradicts with (3.13). This contradiction proves the lemma. □

4 On mappings with a fixed point

Given domains $D,D' \subset \mathbb{R}^n$, $n \geq 2$, points $a \in D$, $b \in D'$ and a Lebesgue measurable function $Q : D' \to [0,\infty]$, denote by $\mathcal{S}_a,b,Q(D,D')$ a family of all open discrete and closed mappings $f$ of $D$ onto $D'$ satisfying the relation (1.2) for any $y_0 \in D'$ such that $f(a) = b$. The following result in a somewhat weaker version was proved in [SevII] Theorem 1].
Theorem 4.1. Assume that, \( D \) has a weakly flat boundary, none component of which degenerate into a point, and that \( D' \) is regular. Assume that, for any \( y_0 \in \overline{D'} \) and for any \( 0 < r_1 < r_2 < r_0 = r_0(y_0) := \sup_{y \in D'} |y - y_0| \) there is a set \( E \subset [r_1,r_2] \) of a positive linear Lebesgue measure such that the function \( Q \) is integrable on \( S(y_0,r) \) for any \( r \in E \). Then any \( f \in \mathcal{G}_{a,b,Q}(D,D') \) has a continuous extension \( \overline{f} : \overline{D} \to \overline{D'}_P \), while \( \overline{f}(D) = \overline{D'}_P \) and, in addition, a family \( \mathcal{G}_{a,b,Q}(\overline{D},\overline{D'}) \) of all extended mappings \( \overline{f} : \overline{D} \to \overline{D'}_P \) is equicontinuous in \( \overline{D} \).

Proof. The possibility of continuous extension \( \overline{f} : \overline{D} \to \overline{D'}_P \) follows by Theorem 2.1. It remains to establish the equicontinuity of the family of extended mappings \( \overline{f} : \overline{D} \to \overline{D'}_P \) in \( \overline{D} \). Let us to prove this by contradiction. Assume that \( f \mathcal{G}_{a,b,Q}(\overline{D},\overline{D'}) \) is not equicontinuous at some point \( x_0 \in \partial D \). Then there are points \( x_m \in D \) and mappings \( f_m \in \mathcal{G}_{a,b,Q}(\overline{D},\overline{D'}) \), \( m = 1,2,\ldots \), such that \( x_m \to x_0 \) as \( m \to \infty \), while, for some \( \varepsilon_0 > 0 \)

\[
h(f_m(x_m), f_m(x_0)) > \varepsilon_0, \quad m = 1,2,\ldots \tag{4.1}
\]

Put in arbitrary way a point \( y_0 \in D' \), \( y_0 \neq b \), and join it with a point \( b \) by some path in \( D' \), which is denoted by \( \alpha \). Set \( A := |\alpha| \). Let \( A_m \) be a whole \( f_m \)-lifting of \( \alpha \) starting at the point \( a \) (which exists by [Vil] Lemma 3.7). Observe that, \( h(A_m, \partial D) > 0 \) by the closeness of \( f_m \) (since, in particular, the pre-image of a compactum is a compact set under open discrete and closed mappings, see [Vil, Theorem 3.3(4)]). Now, the following two cases are possible: either \( h(A_m) \to 0 \) as \( m \to \infty \), or \( h(A_{m_k}) \geq \delta_0 > 0 \) as \( k \to \infty \) for some increasing sequence of numbers \( m_k \) and some \( \delta_0 > 0 \).

Obviously, in the first of these cases, \( h(A_m, \partial D) \geq \delta > 0 \) for some \( \delta > 0 \). Then the family \( \{f_m\}_{m=1}^\infty \) is equicontinuous at \( x_0 \) by Theorem 1.1 which contradicts with (4.1).

In the second case, if \( h(A_{m_k}) \geq \delta_0 > 0 \) as \( k \to \infty \), we also have that \( h(A_{m_k}, \partial D) \geq \delta_1 > 0 \) for some \( \delta_1 > 0 \) by Lemma 3.1. Again, by Theorem 1.1 the family \( \{f_{m_k}\}_{k=1}^\infty \) is equicontinuous at \( x_0 \), which contradicts with (4.1). This proves the theorem. \( \square \)

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