Distances between quantum states in the
tomographic-probability representation

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Abstract
Distances between quantum states are reviewed within the framework of the
tomographic-probability representation. The tomographic approach is based on observed
probabilities and is straightforward for data processing. Different states are distinguished by
comparing corresponding probability-distribution functions. Fidelity as well as other distance
measures are expressed in terms of tomograms.

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1. Introduction
Comparing quantum states is of crucial importance for the
theory of quantum information processing. At any step of
computation one should be aware of a system state and
know how much it deviates from desired evolution. Moreover,
it is necessary to compare different states and quantify
the distance between them by virtue of experimental data.
Measurements enable us to inquire some information about
the system, but such information is presented in the form
of observed probabilities. For this reason, one needs a way
of comparing quantum states with the help of measured
probability distributions, and try not to employ the density
matrix formalism.

Probability-distribution functions enable one to recon-
struct quantum states [1–5]. On the other hand, quantum states
can be identified with observed probability distributions rather
than density operators or Wigner functions [6, 7]. According
to this representation, quantum states are associated with fair probability-distribution functions called tomograms. The
tomographic representation of quantum states with continuous
variables (position, momentum) is introduced in [1] and that
of states with discrete variables (spins) is introduced in [4, 5].
Such an approach describes not only states, but can also be
developed to build the whole tomographic picture of quantum
mechanics (see the reviews [8, 9]).

In this paper, we use peculiarities of the tomographic-
probability representation to express distances between spin
and light states in terms of quantum tomograms. The main
idea of this consideration is that two states are close to
each other if the corresponding probability distributions differ
slightly, and are far apart from each other if their tomograms
do not match significantly. These basic ideas were outlined
and successfully applied to Fock’s, coherent, squeezed and
Schrödinger cat states in [10, 11].

The paper is organized as follows. In section 2, we aim to
recall the tomographic representation of spin and light states
and observables. The star product of tomographic symbols
is also introduced in a concise manner. In section 3, we
consider conventional distance measures which are often used
in the quantum information theory. Tomographic analogues
of these distances are introduced for spins (qubits, qudits) in
section 3.1 and for photon states in section 3.2. In section 4,
conclusions and prospects are presented.

2. State tomograms
To begin with, whatever system is under investigation (spin,
qubit, qudit, light, particle) can be associated with the
tomographic symbol of the form
\[
u(x) = \text{Tr}(\hat{\rho} \hat{U}(x)),\]
where \(\hat{\rho}\) is the density operator and \(\hat{U}(x)\) is a dequantizer
operator depending on a particular set of parameters \(x\).

The inverse mapping of tomographic symbols onto
density operators is
\[
\hat{\rho} = \int dx \, w(x) \hat{D}(x),
\]
where \(\hat{D}(x)\) is a quantizer operator.
If operator $\hat{A}$ is given, it is also possible to construct the tomographic symbol of this operator. The only thing one should do is to replace density operator $\hat{\rho}$ by $\hat{A}$. If one knows the symbols $w_\alpha(x)$ and $w_\beta(x)$ of operators $\hat{A}$ and $\hat{B}$, respectively, then the symbol of operator $\hat{A}\hat{B}$ is equal to the star-product of separate tomographic symbols. Namely

$$w_{AB}(x) = \int w_A(x_1) w_B(x_2) K(x, x_1, x_2) \, dx,$$

where $K(x, x_1, x_2)$ is called tomographic star-product kernel. General and specific tomographic star-product schemes are discussed in [12–15]. The problem of spin tomographic kernels for spins was attacked from different perspectives, e.g. in [16–18].

The particular form of quantizer and dequantizer operators depends on the system in question. Further, we epitomize these operators for spin and light systems.

2.1. Spin tomography

By convention, basis states are eigenvectors $| jm \rangle$ of angular momentum operators $\hat{J}_z$ and $\hat{J}^2$. Let $u$ be an element of group $SU(2)$ or $SU(N)$ with $N = 2j + 1$. We use the following notation:

$$\mathbf{x} = (m, u),$$

$$\int dx = \sum_{m=\pm j} \frac{1}{8\pi^2} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\psi,$$

where the latter equation implies that $u \in SU(2)$ is parameterized by Euler angles $\alpha, \beta$ and $\gamma$. Then scanning and reconstruction procedures of spin tomography are given by dequantizer and quantizer operators of the form [17]

$$\hat{D}(\mathbf{x}) = u^\dagger | m \rangle \langle m | u,$$

$$\hat{U}(\mathbf{x}) = \frac{1}{2} \hat{R}_x(\alpha) \hat{R}_y(\beta) \hat{R}_z(\gamma) \hat{U}(\mathbf{x}) \hat{R}_z(\gamma) \hat{R}_y(\beta) \hat{R}_x(\alpha),$$

where

$$\hat{R}_x(\alpha) = \sum_{j=-1}^{j=1} u^\dagger j, m+1 \rangle \langle j, m | u,$$

$$\hat{R}_y(\beta) = \sum_{m=-j}^{m=j} u^\dagger j, m+1 \rangle \langle j, m | u.$$

2.2. Photon-number tomography

The quantum state of light is uniquely determined by the photon-number tomogram. In this case Fock states $| n \rangle$ form a basis, displacement operator $\hat{D}(\alpha) = \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}]$ plays the role of unitary matrix in the spin tomography. In other words, the notation is

$$\mathbf{x} = (n, \alpha), \quad \int dx = \sum_{n=0}^{\infty} \int d^3 \alpha / \pi,$$

and tomographic procedures are given by operators [19–22]

$$\hat{U}(\mathbf{x}) = \hat{D}(\alpha) | n \rangle \langle n | \hat{D}(\alpha)$$

$$\hat{D}(\mathbf{x}) = \frac{2}{1+s^2} \left( \frac{s-1}{s+1} \right)^{(\hat{a}^\dagger + \alpha^* \hat{a} - n)},$$

where $s$ is an arbitrary ordering parameter [23].

3. Distances between states in view of their tomograms

The extent to which quantum states are similar to each other is usually expressed in terms of their density operators $\hat{\rho}_1$ and $\hat{\rho}_2$. On the other hand, unlike the tomogram the density matrix is not observed directly in experiment. This fact makes it reasonable to express basic distance measures between states in terms of their tomograms. In this paper, we are going to explore the following standard quantities (see e.g. [24–28]):

1. Hilbert–Schmidt distance $\| \rho_1 - \rho_2 \|_{HS} \equiv \sqrt{\frac{1}{2} \text{Tr}(\rho_1 - \rho_2)^2};$
2. trace distance $\frac{1}{2} \text{Tr} | \rho_1 - \rho_2 |,$ where $| A \rangle \equiv \sqrt{A^\dagger A};$
3. fidelity $F(\rho_1, \rho_2) = \text{Tr} \left[ \sqrt{\rho_1 \rho_2} \sqrt{\rho_2 \rho_1} \right]^{1/2};$
4. operator norm $\| \rho_1 - \rho_2 \| = \sup_{\| \psi \| = 1} \| \rho_1 - \rho_2 \psi \|.$

If the reconstruction procedure is given, one is tempted to use quantizer operator $\hat{D}(\mathbf{x})$ instead of the density operator whenever it is possible. As far as distance measures are concerned, such a method does not give us any advantage as compared to density operators. Here we develop an alternative approach which is based on probability-distribution functions only, is straightforward for computation and sheds some light on the relationship between measures for classical and quantum information.

Both spin and photon-number tomograms can be considered as probability vectors, due to normalization conditions $\sum_{m=-j}^{m=j} w(m, u) = 1$ and $\sum_{n=0}^{\infty} w(n, \alpha) = 1$. Indeed, splitting different numbers $m$ and $n$, we obtain

$$\overline{w}(u) = (w(j, u)) \quad w(j = 1, u) \quad \ldots \quad w(j = n, u),$$

$$\overline{w}(\alpha) = (w(0, \alpha)) \quad w(1, \alpha) \quad w(2, \alpha) \quad \ldots \quad w(\alpha).$$

We will discuss spin systems (qubits, qudits) first and then will also pay attention to the photon-number tomogram.

3.1. Distances in terms of spin tomograms

In paper [29], it is shown that the Hilbert–Schmidt distance between qubit ($j = 1/2$) states is equal to the maximal possible Euclidean distance between the corresponding tomographic-probability vectors with respect to rotations in the Hilbert space. Here, we extend this claim to higher dimensions.
Proposition 3.1. For arbitrary qudit states $\rho_1$ and $\rho_2$, the following relation takes place $(N = 2j + 1)$:

$$\|\rho_1 - \rho_2\|_{HS} = \max_{u \in SU(N)} \left[ \frac{1}{2} \sum_{m=-j}^{j} \left( w_1(m, u) - w_2(m, u) \right)^2 \right]^{1/2}.$$  

Proof. The tomogram of any Hermitian operator $\hat{A}$ can be rewritten in terms of its eigenvalues $A_l$, $l = 1, \ldots, N$ and unitary matrix $u_A$ composed of its eigenvectors as follows [9]:

$$\bar{w}_A(u) = \begin{pmatrix} |(uu_A)_{11}|^2 \ldots |(uu_A)_{1N}|^2 \end{pmatrix} \begin{pmatrix} A_1 \ldots A_N \end{pmatrix}.$$  

(16)

By $M$ denote $N \times N$ matrix in (16). Note that $M$ is bistochastic, since all rows and all columns sum up to 1. It means that each component of vector $\bar{w}_A(u)$ is a convex sum of eigenvalues $A_k$. Moreover, the sum of all components of vector $\bar{w}_A(u)$ equals the sum of $A_k$. Due to these facts, the maximal value of quantity

$$\left( \bar{w}_A(u), \bar{w}_A(u) \right) = \sum_{k=1}^{N} |w_{A,k}(u)|^2 = \sum_{k=1}^{N} \left( \sum_{l=1}^{N} M_{kl} A_l \right)^2$$  

(17)

is achieved when $M$ is the identity matrix or, equivalently, $u = u_A$. In other words,

$$\max_{u \in SU(N)} \left( \bar{w}_A(u), \bar{w}_A(u) \right) = \left( \bar{w}_A(u_A), \bar{w}_A(u_A) \right) = \sum_{k=1}^{N} A_k^2 = \text{Tr} A^2.$$  

(18)

If we replace $A$ by $(\rho_1 - \rho_2)$ and recall $\|\rho_1 - \rho_2\|_{HS} = \left[ \frac{1}{2} \text{Tr}(\rho_1 - \rho_2)^2 \right]^{1/2}$, we obtain the statement of the proposition. \hfill $\Box$

In addition to this relation, it is worth emphasizing that rotations by elements $u \in SU(2)$ result in smaller distance

$$0 \leq \max_{u \in SU(2)} \left[ \frac{1}{2} \sum_{m=-j}^{j} \left( w_1(m, u) - w_2(m, u) \right)^2 \right]^{1/2} \leq \|\rho_1 - \rho_2\|_{HS};$$  

(19)

however, it can also serve as a measure of distance, because left inequality becomes equality iff $\rho_1 \equiv \rho_2$.

Proposition 3.2. Trace distance between spin states $\rho_1$ and $\rho_2$ is expressed in terms of tomograms as follows:

$$\frac{1}{2} \text{Tr} |\rho_1 - \rho_2| = \max_{u \in SU(N)} \left[ \frac{1}{2} \sum_{m=-j}^{j} |w_1(m, u) - w_2(m, u)| \right].$$  

(20)

Proof. Arguing as before, we obtain the similar equation for operator $A = \rho_1 - \rho_2$. Namely

$$\sum_{k=1}^{N} |w_{A,k}(u)| = \sum_{k=1}^{N} \sum_{l=1}^{N} M_{kl} A_l.$$  

(21)

This function of variables $M_{kl}$ has the same properties as function (17) and is harmonic. For these reasons it also achieves its maximal value at the boundary determined by bistochastic matrix $M$, with the maximum being equal to

$$\sum_{k=1}^{N} |A_k| = \text{Tr} |A|.$$  

(22)

In other words, trace distance is equal to the maximal possible Kolmogorov distance between tomographic-probability distributions. Note that the maximum is attained by the same element $u$ of SU(N) as in case of the Hilbert–Schmidt distance.

Let us now consider fidelity.

Proposition 3.3. Tomographic-probability version of fidelity reads

$$\text{Tr} \left[ \sqrt{\rho_1 \rho_2} \sqrt{\rho_1} \right]^{1/2} = \min_{u \in SU(N)} \left[ \sum_{m=-j}^{j} \sqrt{w_1(m, u) w_2(m, u)} \right].$$  

(23)

Proof. To prove this proposition, one can follow, step-by-step, the proof of the known formula (see e.g. [25])

$$F(\rho_1, \rho_2) = \min_{E_k} \sum_{k} \sqrt{\text{Tr}(\rho_1 E_k) \text{Tr}(\rho_2 E_k)},$$  

(24)

where the minimum is taken over all positive operator-valued measures $\{E_k\}$. To prove that the minimum is achieved the authors [25] use effects $E_k$ in the form of projectors $|\psi_k\rangle \langle \psi_k|$. In our case, we can employ $E_m = \sqrt{E_m} u|m \rangle \langle m |u^\dagger$ with matrix $u$ such that $|\psi_k\rangle = u|k\rangle$. \hfill $\Box$

It is worth noting that fidelity is the minimal value of the Bhattacharyya coefficient [30] (see also review [28]) of two tomographic-probability distributions.

Another way to consider fidelity is to use symmetric form $F = \text{Tr} \sqrt{\rho_1^2 \rho_2^2}$ and tomographic symbols of operators $\sqrt{\rho_1}$ and $\sqrt{\rho_2}$. Question arises as to how to express tomogram $w_\rho(m, u)$ of positive operator $\sqrt{\rho}$ if we know the tomogram of $w_\rho(m, u)$ of state $\rho$. Using spectral decomposition of density operator

$$\rho = \sum_{m=-j}^{j} \rho_m u_m |m\rangle \langle m| u_m^\dagger,$$  

(25)

it is easy to express tomogram

$$w_\rho(m, u) = \langle m| \rho u_m |m\rangle = \sum_{m=-j}^{j} \rho_m |\langle m| u_m |m\rangle|^2.$$  

(26)
From this it follows that \( \rho_{m'} = w_\rho(m', u_m) \). The spectral decomposition of operator \( \sqrt{\rho} \) is obtained from (25) by replacing \( \rho_{m'} \) by \( \sqrt{\rho_{m'}} \). Then we have

\[
w_{\sqrt{\rho}}(m, u) = \sum_{m'' = -j}^{j} \sqrt{w_{\rho}(m', u_m)} \left| \langle m|uu_{m'}|m' \rangle \right|^2.
\]

(27)

Tomographic symbol of operator \( \sqrt{\rho_1 \rho_2} \) is the star-product of corresponding symbols. Kernel of this star-product for qudits is calculated, e.g. in [16, 18]. Taking advantage of (20), we obtain

\[
\mathrm{Tr} |\sqrt{\rho_1 \rho_2}| = \max_{u \in \mathrm{SU}(N)} \left[ \sum_{m = -j}^{j} (w_{\sqrt{\rho_1}} \ast w_{\sqrt{\rho_2}})(m, u) \right].
\]

(28)

If we compare (24) and (28), we reveal a new property of spin tomograms. In fact,

\[
\min_{u \in \mathrm{SU}(N)} \left[ \sum_{m = -j}^{j} \sqrt{w_{\rho_1}(m, u) w_{\rho_2}(m, u)} \right] = \max_{u \in \mathrm{SU}(N)} \left[ \sum_{m = -j}^{j} (w_{\sqrt{\rho_1}} \ast w_{\sqrt{\rho_2}})(m, u) \right].
\]

(29)

As far as operator norm \( \|\rho_1 - \rho_2\| \) is concerned, it is equal to the maximal eigenvalue of operator \( |\rho_1 - \rho_2| \). Consequently, this norm is

\[
\|\rho_1 - \rho_2\| = \max_{m = -j, \ldots, j; \ u \in \mathrm{SU}(N)} |w_1(m, u) - w_2(m, u)|.
\]

(30)

### 3.2. Distances in terms of photon-number tomograms

Using the analogy of spin and photon-number tomograms, one can readily extend these results to the Hilbert–Schmidt distance between light states. For instance, Hilbert–Schmidt distance is

\[
\|\rho_1 - \rho_2\|_{\text{HS}} = \max_{D(\rho_1 - \rho_2) \in \mathbb{C}} \left[ \frac{1}{2} \sum_{n=0}^{\infty} |D(\rho_1 - \rho_2)D^\dagger n|^2 \right]^{1/2}.
\]

(31)

So as not to resort to group \( \mathrm{SU}(\infty) \), we will only formulate inequalities in terms of conventional photon-number tomograms:

\[
0 \leq \max_{u \in \mathbb{C}} \left[ \frac{1}{2} \sum_{n=0}^{\infty} \left( w_1(n, \alpha) - w_2(n, \alpha) \right) \right]^{1/2} \leq \|\rho_1 - \rho_2\|_{\text{HS}},
\]

(32)

\[
0 \leq \max_{u \in \mathbb{C}} \left[ \frac{1}{2} \sum_{n=0}^{\infty} |w_1(n, \alpha) - w_2(n, \alpha)| \right] \leq \frac{1}{2} \mathrm{Tr} |\rho_1 - \rho_2|,
\]

(33)

\[
F(\rho_1, \rho_2) \leq \min_{u \in \mathbb{C}} \left[ \sum_{n=0}^{\infty} \sqrt{w_1(n, \alpha) w_2(n, \alpha)} \right] \leq 1,
\]

(34)

It is worth mentioning that the left sides of inequalities (32), (33) and (35) turn into equalities if and only if two states coincide, because there is a one-to-one correspondence between photon-number tomograms \( w(n, \alpha) \) and density operators. As far as the right sides of relationships (32), (33) and (35) are concerned, they turn into equalities for states \( \rho_1 \) and \( \rho_2 \) such that their difference can be represented in the form \( \rho_1 - \rho_2 = \sum_{k=0}^{\infty} a_k D(\beta) |k\rangle \langle k| D^\dagger(\beta) \) with \( a_k \in \mathbb{R}, \beta \in \mathbb{C} \). As to fidelity, if the right side of (34) becomes a valid equality, then \( \rho_1 \equiv \rho_2 \). The left side of (34) turns into equality for states \( \rho_1 = \sum_{k=0}^{\infty} a_k D(\beta) |k\rangle \langle k| D^\dagger(\beta), i = 1, 2 \), e.g. for a coherent state \( |\psi_1\rangle = |\beta\rangle = D(\beta)|0\rangle \) and a displaced excited state \( |\psi_2\rangle = D(\beta)|1\rangle \). Moreover, the right sides of (32), (33) and (35) as well as the left side of (34) turn into equalities for arbitrary thermal equilibrium states.

### 4. Conclusions

To conclude, we summarize the main results of the paper.

Conventional distance measures between quantum states and fidelity, which are usually formulated for density matrices, are expressed in terms of quantum tomograms. It is demonstrated that the Hilbert–Schmidt distance is related to the maximal Euclidean distance of tomographic-probability vectors, trace distance is related to the maximal Kolmogorov distance of tomograms, fidelity is related to the minimal Bhattacharyya coefficient and operator norm is related to the maximum of the residual tomographic symbol. Analysing photon-number tomography, these results are also extended to the case of the infinite Hilbert space of Fock states and formulated in the form of inequalities. We believe the introduced quantities to be used as alternatives to distance measures based on density matrices. An interesting problem for further consideration is to develop an analogue approach for continuous variables quantum systems.

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