A DETAILED RECALCULATION OF THE SPECTRUM OF THE RADIATION EMITTED DURING GRAVITATIONAL COLLAPSE OF A SPHERICALLY SYMMETRIC STAR

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Abstract

We address the question of radiation emission from a collapsing star. We consider the simple model of a spherical star consisting of pressure-free dust and we derive the emission spectrum via a systematic asymptotic expansion of the complete Bogolubov amplitude. Inconsistencies in derivations of the black body spectrum are pointed out.

I. INTRODUCTION

More than a quarter of a century has passed since Hawking’s remarkable suggestion that a star collapsing to a black hole gives rise to radiation emission at a steady rate characterized by the black body spectrum (Hawking 1975). That particle creation takes place is not in itself surprising. Let us take the case of a spherical star with radius initially larger than its Schwarzschild radius that eventually collapses contracting to a point (according to classical gravitation). If we consider a quantized scalar photon field the quantum spaces of the \( \text{in} \) states (before the initiation of collapse) and of the \( \text{out} \) states (after collapse is completed) are certainly different. Hence particle creation clearly takes place and is determined by the Bogolubov \( \alpha(\omega, \omega') \) and \( \beta(\omega, \omega') \) amplitudes. The fact that the spectrum is that of a black body is indeed noticeable, and ties up with the somewhat earlier results on black hole thermodynamics.

From the mathematical point of view the black body result is due to the special behaviour of the photon modes near the stellar surface just before the horizon is formed. Hawking in his derivation makes heavy use of the asymptotic (i.e. near the horizon) form of the modes. The singular behaviour of the modes in this regime has in turn given rise to various statements in the literature that are not strictly correct. For example there have been references to “parts of \( \alpha(\omega, \omega') \) and \( \beta(\omega, \omega') \) that relate to the steady-state regime at late times” ((DeWitt 1975), p. 327). However the Bogolubov amplitudes are global constructs and the distinction between early and late times does not make strict sense. Hawking too talks about particle production that depends on the details of the collapse and that such particles “will disperse”, thus leaving only the thermal part at late times (Hawking (1975), p. 207). The point of view we are adopting in this paper is the one dictated by quantum-mechanical orthodoxy, namely that particle production is a global process ((Hawking 1975) p. 216) and that one should start with the full standard expression for \( \beta(\omega, \omega') \). Of course at some point in the mathematical handling of the amplitude \( \beta(\omega, \omega') \) the special role of the horizon will show up. The above remarks are certainly not meant to imply that one cannot calculate local quantities like \( \langle T_{\mu\nu} \rangle \); such quantities certainly behave very differently during the various phases of the collapse.

Shortly after Hawking’s work on black hole evaporation two classic papers (Fulling and Davies 1976) and (Davies and Fulling 1977) were published (the latter shall be quoted as DF in what follows). The authors demonstrated an illuminating analogy, physical as well as mathematical, between gravitational collapse and the seemingly rather different problem of a perfect mirror starting from rest and accelerating for an infinite time. The renormalized matrix element of the \( T_{uu} \) component of the energy momentum tensor, defined as

\[
T_{uu} = (\partial_u \phi)^2
\]

is calculated in DF. The result is that \textit{asymptotically for} \( t \to \infty \)

\[
\langle T_{uu} \rangle \to \kappa^2 \frac{1}{48\pi}
\]
astounding intuition shown by the early workers on the subject. Our findings, summarized in the concluding section, in no way do they diminish the profound problem of loss of information during collapse (see e.g. Preskill 1992, Helfer 2003). There are steps in the derivation of the black body spectrum that both obscure the mathematics and create confusion regarding quantum gravitational effects have to come into play. Such matters are not discussed here. Nor do we touch upon mirrors and black holes is presented; $\kappa$ is a constant characterizing the trajectory. Equation (1) shows that there is a constant energy flux at late times, analogous to the thermal energy flux found in (Hawking 1975). Davies and Fulling (1977) also calculated the Bogolubov amplitude following Hawking’s lines and arrived at the black body spectrum relevant to an accelerating mirror

$$|\beta(\omega,\omega')|^2 = \frac{1}{2\pi \omega' \kappa} \frac{1}{e^{2\pi \omega/\kappa} - 1}$$

Certain technicalities in the latter calculation have been elucidated in a previous paper by the author (Calogeracos 2002, hereafter referred to as I). In the present paper we intend to show that similar points may be raised in connection to the standard black hole literature (see (Birrell and Davies 1982), chapter 8 and references therein). There are steps in the derivation of the black body spectrum that both obscure the mathematics and create confusion regarding the physics of the problem. Our findings, summarized in the concluding section, in no way do they diminish the astounding intuition shown by the early workers on the subject.

In section 2 we examine the simple model of a collapsing spherical star consisting of pressure-free dust and we review the fundamental features of the collapse. The problem is widely treated in the literature and some of the results are included so that the paper be self-contained. In section 3 we write down the photon modes and the full expression for the Bogolubov amplitude $\beta(\omega,\omega')$. In section 4 we show that the asymptotic behaviour of the latter for large $\omega'$ is

$$\beta(\omega,\omega') \approx (\omega')^{-\frac{1}{2}} + O\left((\omega')^{-N}\right) (N > 1)$$

The $1/\omega'$ in (2) leads to a logarithmic divergence in (3) and this signals the production of particles at a finite rate for an infinite time. One often refers to the ultraviolet divergence by saying that large $\omega'$ frequencies dominate. Our analysis emphasizes two points, stressed also in I: (a) the thermal result depends crucially on the behaviour of the photon modes near the horizon, (b) for a consistent derivation one must consider the whole collapse and not just the late phase. The truth of statement (a) is usually taken as common knowledge. However the importance of statement (b) is often not appreciated. Comparison of our approach with the somewhat more conventional one is presented in Appendix A.

The ultraviolet divergence previously mentioned is widely taken to signify that one cannot fully resolve the problem within the context of classical gravitation. It is well known that the constants we have at our disposal are $c = 3 \times 10^{10}$ cm/s$^{-1}$, $\hbar = 1.05 \times 10^{-27}$ g·cm$^2$·s$^{-1}$, $G = 6.67 \times 10^{-8}$ cm$^3$·s$^{-2}$·g$^{-1}$ and that we can form three quantities with dimensions of mass, length, and time respectively: $M = (\hbar c/G)^{1/2} = 2 \times 10^{-5}$ g, $L = (\hbar G/c^3)^{1/2} = 1.6 \times 10^{-33}$ cm, $T = (\hbar G/c^5)^{1/2} = 5 \times 10^{-44}$ s. It is clear that when the radius of the contracting star becomes of order $L$ then quantum gravitational effects have to come into play. Such matters are not discussed here. Nor do we touch upon the profound problem of loss of information during collapse (see e.g. Preskill 1992, Helfer 2003).

Note: in what follows $\hbar = G = c = 1$.

**II. A COLLAPSING SPHERE OF DUST**

We consider a collapsing spherical star consisting of pressure-free dust and follow the approach of (Weinberg 1972), chapter 11, sections 8 and 9. Since each dust particle falls freely the spacetime geometry inside the star is most appropriately described in a comoving frame. Let $M$ be the mass of the star. The metric reads

$$ds^2 = dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right]$$

In the comoving frame each dust particle is labelled by a unique set of $(r, \theta, \varphi)$ and the time $t$ stands for the proper time registered by the particle in question. The radius of the star is specified in the comoving frame by $r = a$ where $a$ is by definition constant. The quantities $k$ and $R(t)$ refer to details of the model and their meaning shall be explained shortly. The collapse is initiated at time $t = 0$, and we normalize $R(t)$ by requiring
Outside the star spacetime is described by the Schwarzschild metric

\[
\text{ds}^2 = \left(1 - \frac{2M}{\bar{r}}\right)dt^2 - \left(1 - \frac{2M}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2 d\theta^2 - \bar{r}^2 \sin^2 \theta d\phi^2
\]

The match of the two metrics at the surface implies (Weinberg 1972)

\[
\bar{r} = R(t), \theta = \bar{\theta}, \phi = \bar{\phi}
\]

Relation (6) together with the first of (8) imply that the quantity \(a\) stands for the radius of the star in Schwarzschild spacetime at the moment when collapse starts. Thus the radius of the star in Schwarzschild coordinates is given by

\[
\bar{R} = a R
\]

The constant \(k\) is proportional to the star’s density and is related to the mass via

\[
2M = ka^3
\]

At times we shall consider the case of an initially very large star \(a >> M\), which implies via (9)

\[
ka^2 << 1
\]

The relation between \(\bar{t}\) and \(t\) is a rather more intricate one and is treated in detail by Weinberg op. cit. (see the remark after (14) below). It has already been mentioned that since the pressure inside the star vanishes the motion of the surface corresponds to that of a free falling particle in the gravitational field of a body of mass \(M\). The proper time \(t\) registered by an observer located on the surface of the collapsing star and the quantity \(\bar{R}\) are best given parametrically in terms of the cycloidal variable \(\eta\) (Weinberg 1972):

\[
\bar{R}(\eta) = a R(\eta), R(\eta) = \frac{1}{2} (1 + \cos \eta)
\]

\[
t(\eta) = \frac{1}{2\sqrt{k}} (\eta + \sin \eta)
\]

where \(0 \leq \eta \leq \pi\). Collapse starts at \(\eta = 0\) and is completed at \(\eta = \pi\). Thus according to (12) the total time of the collapse as measured by an observer comoving with the star’s surface is given by \(\pi / (2\sqrt{k})\). The black hole is formed when \(\bar{R} = 2M\). According to (11) this happens at a value \(\eta_0\) of the parameter given by

\[
\frac{\eta_0}{2} = \cos^{-1} a \sqrt{k}
\]

The Schwarzschild time \(\bar{t}\) for a particle on the stellar surface is given in terms of \(\eta\) by the expression (see (Chandrasekhar 1983) chapter 3, section 19) :

\[
\bar{t} = 2a \sqrt{1 - ka^2} \left[\frac{1}{2} (\eta + \sin \eta) + ka^2 \eta\right] + 2M \ln \left[\frac{\tan \frac{\eta_0}{2} + \tan \frac{\eta}{2}}{\tan \frac{\eta_0}{2} - \tan \frac{\eta}{2}}\right]
\]

Radial null geodesics in Schwarzschild spacetime satisfy (see (7))

\[
d\bar{t}^2 = \frac{1}{\left(1 - \frac{2M}{\bar{r}}\right)^2} d\bar{r}^2
\]

This leads to the definition of the Regge-Wheeler radial coordinate \(r^*\)

\[
r^* = \bar{r} + 2M \ln \left(\frac{\bar{r}}{2M} - 1\right)
\]

so that (15) may be written as \(d\bar{t}^2 - dr^{*2} = 0\), and to the introduction of the incoming and outgoing Eddington-Finkelstein coordinates

3
\[ v = \tilde{t} + r^*, \quad u = \tilde{t} - r^* \]  \hspace{1cm} (17)

The quantities \( v \) and \( u \) label photon trajectories in Schwarzschild spacetime.

In order not to confuse the reader with the proliferation of radial variables let us clarify: \( \bar{R} \) stands for the stellar radius in Schwarzschild coordinates, \( R^* \) is the corresponding Regge-Wheeler coordinate, \( \bar{R} \) is the auxiliary function given by the second of (11) and appearing in (5), \( \bar{r} \) is the radial Schwarzschild coordinate of an arbitrary point outside the star, \( r^* \) the corresponding Regge-Wheeler coordinate, and \( r \) the radial coordinate inside the star in the comoving frame.

A property that one must establish is that as the star approaches collapse

\[ \frac{\bar{R}}{2M} - 1 \simeq A e^{-\tilde{t}/2M} \]  \hspace{1cm} (18)

where \( A \) is a calculable constant. This expression is of course widely known. It agrees with the remark following (14) that as far as the Schwarzschild observer is concerned the collapse takes an infinitely long time. A similar exponential appears in the calculation of the red shift as observed by the Schwarzschild observer; see (Weinberg 1972), p. 348 (the asymptotic form of the mirror trajectory given by DF, equation (2.3), has a form identical to that of (18)). To establish the validity of (18) and also calculate \( A \) one takes logarithms of both sides and uses (11), (14). The logarithmic divergences cancel and \( A \) is expressed in terms of \( \eta_0 \) (see Appendix A). The quantity \( A \) depends on the parameters \( a \) and \( M \) of the collapse and does not turn up in the expression for the spectrum. Note that relations (17) define parametrically (via \( \eta \)) a function

\[ u = f_{st}(v) \]  \hspace{1cm} (19)

and its inverse

\[ v = p_{st}(u) \]  \hspace{1cm} (20)

which give the trajectory of the stellar surface in terms of \( u, v \) coordinates.

In the limit \( \eta \to \eta_0 \) both \( R^* \) and \( \bar{R} \) are singular (as is obvious from (14) and (16)), however their combination in the Eddington-Finkelstein coordinate \( \bar{v} \) is analytic. The cancellation of the logarithmic singularities is hardly surprising since \( \nu_0 \) must have a well-defined value labelling a null line; for details see Appendix A. For given values of \( M \) and \( a \) the value of \( \eta_0 \) is readily calculable via (13) and then so is \( \nu_0 \) after some algebra (setting \( \eta = \eta_0 \) in (81)). Thus \( \nu_0 - \nu \) admits a Taylor expansion near \( \eta_0 \) and of course so does \( 2M - \bar{R} \) which is analytic for all \( 0 < \eta < \pi \). Writing

\[ 2M - \bar{R} \simeq C_1 (\eta_0 - \eta), \quad \nu_0 - \nu \simeq C_2 (\eta_0 - \eta) \]  \hspace{1cm} (21)

we can read the positive constants \( C_1, C_2 \) off (83), (82). Then

\[ \bar{R} - 2M \simeq C (\nu_0 - \nu), \quad C = C_1/C_2 \]  \hspace{1cm} (22)

Combining the above with (18) we get

\[ \nu \simeq \nu_0 - \frac{2MA}{C} e^{-\tilde{t}/2M} \]  \hspace{1cm} (23)

For \( \eta \to \eta_0 \) we can combine the two relations (17) and write

\[ \tilde{t} \simeq \frac{u + \nu_0}{2} \]  \hspace{1cm} (24)

Then (23) reads

\[ \nu \simeq \nu_0 - Be^{-u/4M}, \quad B \equiv \frac{2MA}{C} \]  \hspace{1cm} (25)

This defines \( p_{st}(u) \) in (20).

In what follows we shall need the metric (7) expressed in terms of Kruskal coordinates; see e.g. (Townsend 1997), (Misner, Thorne, Wheeler 1973). We introduce

\[ U = -E e^{-u/4M}, \quad V = \frac{e^{u/4M}}{E} \]  \hspace{1cm} (26)
and we transform the metric to
\[ ds^2 = \frac{32M^3}{\bar{r}} e^{-r/2M} dU dV - \bar{r}^2 d\Omega \]  \hspace{1cm} (27)
where \( r \) is given in terms of \( U, V \) implicitly via
\[ UV = \frac{\bar{r} - 2M}{2M} e^{\bar{r}/2M} \]  \hspace{1cm} (28)

At early times spacetime is taken to be essentially flat (cf remark preceding (35)) and the metric is simply expressed in terms of advanced and retarded Eddington-Finkelstein coordinates
\[ ds^2 = du dv - \bar{r}^2 d\Omega \]  \hspace{1cm} (29)

We now turn to ray-tracing and consider light rays obeying the condition that they be reflected at the centre of the star \( r = 0 \). The rationale for the boundary condition at \( r = 0 \) is reviewed in the next section. An incident ray corresponding to a certain \( v \) at early times upon reflection becomes an outgoing ray corresponding to a certain \( u \) at late times, thus defining a function \( u = f(v) \). The inverse function is given by \( v = p(u) \). Note that the function \( u = f(v) \) is by construction quite distinct from the function \( u = f_{st}(v) \) (19) (the former involves reflection of the ray at the centre of the star whereas the latter at the surface). The importance of ray tracing lies in the fact that once we determine the function \( f(v) \) (or its inverse) we can immediately construct the photon modes via (45), (46).

Note also that these expressions involve the Eddington-Finkelstein coordinates rather than the Kruskal ones. It is convenient to exhibit collapse via a Kruskal diagram, where radial null lines are at 45° on the plane of the paper. The line \( l(1) \) represents a ray that crosses the star’s surface at point B, is reflected at the centre of the star at spacetime point C, and after reflection crosses the star’s surface at A (reflected ray labelled by \( l'(1) \)). By definition ray \( l(1) \) is the last incoming ray that manages to escape before the black hole is formed, and corresponds to an incoming Eddington-Finkelstein coordinate \( v_H \). At A the star’s surface crosses the Schwarzschild radius, and the coordinates of A correspond to the value \( \eta_0 \) of the cycloidal parameter \( \eta \) in (11), (12). The extension of the line CA is the future horizon and corresponds to the value \( U = 0 \) of the Kruskal coordinate. Let \( l'(i) \) be an outgoing ray slightly preceding \( l(1) \), corresponding to an incoming ray \( l(i) \). We also trace the incoming light ray \( m(1) \) which becomes \( l(1) \) upon reflection on the star’s surface at A without entering the star (this is an auxiliary ray and the reflection mechanism is an entirely hypothetical one, not connected to partial reflection that may take place on the star’s surface). Similarly let \( m(i) \) be an incoming ray slightly preceding \( m(1) \). The problem is to relate the \( u \) of the outgoing \( l'(i) \) to the \( v \) of the incoming \( l(i) \), thus determining the function \( u = f(v) \). We review the argument in (Hawking 1975); see also (Townsend 1997), p. 125. Recall ((Townsend 1997), p. 29 for proof) that \( U \) is an affine parameter for the null geodesic \( U = \text{constant} \). (This means that \( L \cdot DL^\mu = 0 \) rather than \( L \cdot DL^\mu \propto L^\mu \).) Instead of \( U = e^{-u/4M} \) one may also use any parameter \( \lambda = -E e^{-u/4M} \), \( E \) being some constant; we exploit this freedom and will comment on the value of \( E \) later on. Let \( \mathbf{T} \) be a null vector on \( U = 0 \) parallel to the horizon and pointing along the radial spatial direction, and let \( \mathbf{N} \) be a null vector pointing to the future and along the radial spatial direction and satisfying \( \mathbf{N} \cdot \mathbf{T} = -1 \). The above requirements are consistent (see (Townsend 1997), p. 109) if \( \mathbf{N} \) is parallely transported along the geodesic defined by \( \mathbf{T} \). Consider a neighbouring null geodesic labelled by \( U = -\varepsilon \) where \( \varepsilon \) is small. This corresponds to constant \( u \) according to (26):
\[ \varepsilon = E e^{-u/4M} \Rightarrow \ln \varepsilon = \ln E - \frac{\varepsilon}{4M} \]  \hspace{1cm} (30)

The two geodesics are thus connected by the displacement vector \( -\varepsilon \mathbf{N} \). We parallely transport the pair \( \mathbf{N}, \mathbf{T} \) to the point where the \( U = 0 \) geodesic cuts the trajectory of centre of the star. We then reflect \( \mathbf{N} \) and \( \mathbf{T} \) as in figure 2 and parallely transport the new pair to past infinity where the stellar radius satisfies (35). Then spacetime becomes essentially flat, the metric is given by (29) and the affine parameter is given by \( v \) for a general geodesic \( l(i) \) (figures 1, 2) and by \( v_H \) for the special geodesic \( l(1) \). We thus have
\[ \varepsilon = v_H - v \]  \hspace{1cm} (31)

Comparing (30) and (31) we obtain
\[ v_H - v = E e^{-u/4M} \] or
\[ v \simeq v_H - E e^{-u/4M} \]  

(32)

(the \( \simeq \) symbol above refers to the fact that this approximation is valid near the horizon). Note that the preceding argument involving \( N, T \) can be equally well applied to the incoming rays \( m(i), m(1) \) which upon reflection on the surface of the star coincide with \( l'(1), l'(i) \). In particular the affine distance between \( l(1) \) and \( l(i) \) equals the distance between \( m(1) \) and \( m(i) \). This implies that the quantity \( E \) appearing in (32) is equal to the \( B \) appearing in (25). (The identification of \( E \) with \( B \) does not affect the derivation of the black body spectrum and we shall keep \( E \) in notation.)

Note also that the quantity

\[ d \equiv v_0 - v_H \]  

(33)

is calculable; see appendix A, the remark following (89). Inverting (32) we get

\[ u = f(v) \simeq -\frac{1}{4M} \ln \left( \frac{v_H - v}{E} \right) \]  

(34)

We now turn to the early stage of the collapse (figure 3). We assume that the collapse is initiated at \( \bar{t} = 0 \) (and that for \( \bar{t} < 0 \) the star is held stable by some external means). Let us assume large initial stellar radius

\[ a \gg M \]  

(35)

Expanding (11), (12) for early times, i.e. small \( \eta \), we obtain

\[ \bar{t} \simeq \frac{\eta}{\sqrt{k}}, R \simeq 1 - \frac{\eta^2}{4} \]

Hence the radius as a function of time at the early stage is given by

\[ \bar{R} = a \left( 1 - \frac{k \eta^2}{4} \right) \]  

(36)

Recalling the definition (9) we observe that the star’s surface initially contracts according to non-relativistic kinematics with acceleration \( M/a^2 \) (in accordance with Newton’s law of gravity). Given the assumption (35) the initial gravitational field is sufficiently weak so the light rays in the Schwarzschild geometry are straight lines. Let us consider a \( \bar{t} - \bar{R} \) diagram ( \( \bar{t}, \bar{R} \) being Schwarzschild coordinates; at early times when (10) is satisfied there is little difference between radial and Regge-Wheeler coordinates) and take an incoming light ray \( q \) which has \( v = 0 \). The reflected ray \( q' \) has \( u = 0 \) and takes time \( a \) to travel from the centre of the star to the surface. During this time the gravitational field is still weak. This may be seen by setting \( t = a \) in (36); then (10) implies that by the time the ray emerges from the star’s surface \( \bar{R} \) still is almost equal to \( a \). To summarize the function \( u = f(v) \) has the properties

\[ f(v) = v = 0, \text{ initially} \]

(37)

\[ f(v) \simeq -4M \ln \left( \frac{v_H - v}{E} \right), \quad v \rightarrow v_H \]

(38)

III. CONSTRUCTION OF THE IN AND OUT STATES

We follow (Birrell and Davies 1982), (Brout et al. 1995). A scalar mode corresponding to angular momentum \( l \) and satisfying the Klein-Gordon equation is written in the form

\[ \psi_l = \left( \sqrt{4\pi r} \right)^{-1} \phi_l(r) Y_l^m(\theta, \varphi) \]  

(39)

where

\[ \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - V_l(r) \right) \phi_l = 0 \]  

(40)
with

\[ V_l(r) = \left(1 - \frac{2M}{r}\right) \left(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right) \]  

(41)

As far as conceptual purposes and technical details are concerned it suffices to restrict ourselves to \( s \) waves (following the references in the beginning of this section). Note that according to (41) there is a centrifugal barrier even for \( s \) waves which does not affect either the high frequency modes or the modes that are present at the early stages of the collapse (when the gravitational field is weak). Again following the standard treatments of the problem we are going to neglect the barrier. For a discussion of its effect on the modes see (DeWitt 1975), section 5.2. Thus (40) becomes

\[ \left(\frac{\partial^2}{\partial \tilde{t}^2} - \frac{\partial^2}{\partial r^*^2}\right) \phi = 0 \]  

(42)

(in what follows the index 0 in \( \phi \) is suppressed). Since \( \psi_0(r = 0) \) must be finite it follows from (39) that

\[ \phi(r = 0) = 0 \]  

(43)

Relations (42) and (43) define the problem.

Relation (42) can be written in the form

\[ \frac{\partial^2 \phi}{\partial u \partial v} = 0 \]  

(44)

Hence any function that depends only on \( u \) or \( v \) (or the sum of two such functions) is a solution of (44). We can now make contact with the accelerating mirror problem (DF,I). One set of modes satisfying (44) and the boundary condition (43) is given by

\[ \varphi_\omega(u, v) = \frac{i}{2\sqrt{\pi} \omega} (\exp(-i\omega v) - \exp(-i\omega p(u))) \]  

(45)

Another set of modes satisfying the boundary condition is immediately obtained from (45)

\[ \bar{\varphi}_\omega(u, v) = \frac{i}{2\sqrt{\pi} \omega} (\exp(-i\omega f(v)) - \exp(-i\omega u)) \]  

(46)

The modes \( \varphi_\omega(u, v) \) of (45) describe sinusoidal waves incident from \( r^* = \infty \) as it is clear from the sign of the exponential in the first term; the second term represents the outgoing part which has a rather complicated behaviour depending on the motion of the mirror. These modes constitute the \( in \) space and should obviously be absent before collapse,

\[ a(\omega') |0in\rangle = 0 \]  

(47)

Similarly the modes \( \bar{\varphi}_\omega(u, v) \) describe sinusoidal outgoing waves travelling towards \( r^* = \infty \) (emitted by the star) as can be seen from the exponential of the second term. Correspondingly the first term is complicated.

Recall that apart from the modes \( \varphi_\omega(u, v) \) we should include modes \( \bar{q}_\omega(u, v) \) that contain no outgoing component; instead they are confined inside the black hole (the letter \( q \) complies with (Hawking 1975, 1976)). The modes \( \bar{q}_\omega(u, v) \) are undetectable by an outside observer, but they are needed to make the set \( \varphi_\omega, \bar{q}_\omega \) complete. Correlations between the two sets is part of the information problem alluded to in the Introduction.

The \( \varphi_\omega(u, v) \) modes should be absent from the \( out \) vacuum

\[ \bar{a}(\omega) |0out\rangle = 0 \]  

(48)

The state \( |0out\rangle \) corresponds to the state where there are no outgoing particles detectable by an outside observer. The two representations are connected by the Bogolubov transformation

\[ \bar{a}(\omega) = \int_0^\infty d\omega' (\alpha(\omega, \omega') a(\omega') + \beta^*(\omega, \omega') a^\dagger(\omega')) \]  

(49)

Using (49) and its hermitean conjugate we may immediately verify that the expectation value of the number of excitations of the mode \( \varphi_\omega \) in the \( |0in\rangle \) vacuum.
\[
(0in \mid N(\omega) \mid 0in) = \int_0^\infty d\omega' |\beta(\omega, \omega')|^2
\]  

(50)

The matrix element \(\beta(\omega, \omega')\) is given by (21) of I (we use \(r\) or \(r^*\) as the case may be to ease up on notation and make contact with I)

\[
\beta(\omega, \omega') = -i \int dz \varphi(\omega, z, 0) \frac{\partial}{\partial \omega} \bar{\varphi}(\omega, z, 0) + i \int dz \left( \frac{\partial}{\partial \omega} \varphi(\omega', z, 0) \right) \bar{\varphi}(\omega, z, 0)
\]  

(51)

The integration in (51) can be over any spacelike hypersurface. Since collapse starts at \(t = 0\) the choice \(t = 0\) for the hypersurface is convenient. The \(in\) modes evaluated at \(t = 0\) are given by the simple expression (45) (i.e. \(p(u) = u\))

\[
\varphi(\omega, u, v) = \frac{i}{2\sqrt{\pi \omega}} \left( \exp(-i\omega z) - \exp(i\omega z) \right) \theta(z)
\]  

(52)

where the presence of \(\theta(z)\) emphasizes the fact that \(z\) is a radial coordinate. The \(\bar{\varphi}\) modes are given by (46) with \(f\) depending on the history of the collapse. Relation (51) is rewritten in the form (the endpoints of integration shall be stated presently)

\[
\beta(\omega, \omega') = -i \int dz \frac{i}{2\sqrt{\pi \omega}} \left\{ e^{-i\omega' z} - e^{i\omega' z} \right\} \theta(z) \frac{\omega}{2\sqrt{\pi \omega}} \left\{ f e^{-i\omega' f} - e^{i\omega' f} \right\} + i \int dz \frac{\omega'}{2\sqrt{\pi \omega}} \left\{ e^{-i\omega' z} - e^{i\omega' z} \right\} \theta(z) \frac{i}{2\sqrt{\pi \omega}} \left\{ e^{-i\omega f} - e^{i\omega f} \right\}
\]  

(53)

The above expression may be rearranged in the form

\[
\beta(\omega, \omega') = \frac{1}{4\pi \sqrt{\omega \omega'}} \int_0^{\nu_H} dz \left\{ e^{i\omega' z} - e^{-i\omega' z} \right\} \theta(z) \left\{ \omega e^{-i\omega f} - \omega f e^{-i\omega f} \right\} + \frac{(\omega - \omega')}{4\pi \sqrt{\omega \omega'}} \int_0^\infty dz \left\{ e^{i\omega' z} - e^{-i\omega' z} \right\} \theta(z) e^{i\omega z}
\]  

(54)

where the limits of integration are displayed (we took into account the fact that the argument of \(f\) runs up to \(\nu_H\)). The rays with \(v < 0\) do not affect the amplitude due to the presence of the \(\theta\) function. The first and second integral in the above relation will be denoted by \(\beta_1(\omega, \omega')\) and \(\beta_2(\omega, \omega')\) respectively. Note that the second integral is of kinematic origin and totally independent of the collapse. We also quote the expression for the \(\alpha(\omega, \omega')\) amplitude

\[
\alpha(\omega, \omega') = -i \int_0^\infty dz \varphi(\omega, z, 0) \frac{\partial}{\partial \omega} \bar{\varphi}(\omega, z, 0) + i \int_0^\infty dz \left( \frac{\partial}{\partial \omega} \varphi(\omega', z, 0) \right) \bar{\varphi}(\omega, z, 0)
\]  

(55)

Observe the unitarity condition

\[
\int_0^\infty d\omega \left( \alpha(\omega_1, \omega) \alpha^* (\omega_2, \omega) - \beta(\omega_1, \omega) \beta^* (\omega_2, \omega) \right) = \delta(\omega_1 - \omega_2)
\]  

(56)

which is a consequence of the fact that the set of \(in\) states is complete.

Recall from I that we can introduce quantities \(A(\omega, \omega')\), \(B(\omega, \omega')\) that are analytic functions of the frequencies via

\[
\alpha(\omega, \omega') = \frac{A(\omega, \omega')}{\sqrt{\omega \omega'}} \beta(\omega, \omega') = \frac{B(\omega, \omega')}{\sqrt{\omega \omega'}}
\]  

(57)

The quantity \(B(\omega, \omega')\) is read off (53) (and \(A(\omega, \omega')\) from the corresponding expression for \(\alpha(\omega, \omega')\)). From the definitions of the Bogolubov coefficients, the explicit form (53) of the overlap integral and expressions (45) and (46) for the field modes one can deduce that

\[
B^*(\omega, \omega') = A(-\omega, \omega'), A^*(\omega, \omega') = B(-\omega, \omega')
\]  

(58)

The above relations allow the calculation of \(\alpha(\omega, \omega')\) once \(\beta(\omega, \omega')\) is determined.
IV. CALCULATION OF THE BOGOLUBOV AMPLITUDES

The strategy we adopt in handling (54) is as follows. The first integral will be evaluated via an asymptotic expansion in negative powers of \( \omega' \), which will in fact show that the \( \omega' \) integration in (3) is logarithmically divergent. We start with the second integral in (54):

\[
\beta_2(\omega, \omega') = \frac{(\omega - \omega')}{4\pi \sqrt{\omega' \omega}} \left\{ \int_0^\infty dze^{i(\omega + \omega')z} - \int_0^\infty dze^{i(\omega - \omega')z} \right\}
\]

(59)

The integrals in (59) are readily evaluated in terms of the function \( \zeta \) and its complex conjugate \( \zeta^* \) (see e.g. Heitler (1954), pages 66-71):

\[
\zeta(x) \equiv -i \int_0^\infty e^{ix\kappa} d\kappa = \frac{1}{x} - i\pi \delta(x)
\]

(60)

Since we are interested in the asymptotic limit \( \omega' \to \infty \) the argument of the \( \delta \) function in (60) never vanishes so it is only the first term in (60) that is operative as far as the calculation of the \( \beta_2(\omega, \omega') \) goes. (The \( \delta \) proportional term is relevant in the calculation of the \( \alpha(\omega, \omega') \) amplitude via relations (57), (58).) Thus asymptotically in the said limit

\[
\beta_2(\omega, \omega') \approx \frac{1}{2\pi i \sqrt{\omega'}}
\]

(61)

We turn to the first integral \( \beta_1(\omega, \omega') \) in (54). Rather than splitting it to four integrals we perform an integration by parts to get (54) in the form

\[
\beta_1(\omega, \omega') = -\frac{1}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_0^{v_H} dze^{-i\omega f(z)-i\omega' z} + \frac{1}{2\pi \sqrt{\omega' \omega}} \sin(\omega' v_H)e^{-i\omega f(v_H)}
\]

(62)

This is the same integration by parts that was used to obtain (35) of I and also the one that is used in DF to go from their (2.10a) to (2.10b). The lower limit contribution of the integration by parts vanishes (being proportional to \( \sin \omega' z \)), quite irrespectively of the value of \( f(0) \). The second term in (62) (originating from the upper limit) oscillates rapidly since the exponent tends to infinity. Thus the term tends distributionally to zero it may be neglected as in DF (also it is one power of \( \omega' \) down compared to the first term). So asymptotically in \( \omega' \) we are entitled to write

\[
\beta_1(\omega, \omega') \approx \frac{-1}{2\pi i \sqrt{\omega'}}
\]

(63)

To bring the singularity in the integral to zero we make the change of variable

\[
z = v_H - x
\]

(64)

and rewrite \( \beta_1(\omega, \omega') \) in the form

\[
\beta_1(\omega, \omega') = -\frac{e^{-i\omega H}}{2\pi} \sqrt{\frac{\omega'}{\omega}} \int_0^{v_H} dxe^{-i\omega g(x)}e^{i\omega' x}
\]

(65)

where the function \( g(x) \equiv f(v_H - z) \) is defined in the range \( 0 < x < v_H \) and has the properties that follow from (37), (38):

\[
g(x) \approx -4M \ln \left( \frac{z}{E} \right), x \to 0
\]

(66)

\[
g(v_H) = f(0)
\]

(67)

We isolate the integral

\[
I \equiv \int_0^{v_H} dxe^{-i\omega g(x)}e^{i\omega' x}
\]

(68)
To obtain the asymptotic behaviour of (68) for \( \omega' \) large we adopt the standard technique of deforming the integration path to a contour in the complex plane; see (Bender and Orszag 1978), chapter 6; (Ablowitz and Fokas 1997), chapter 6; (Morse and Feshbach 1953), p. 610 where a very similar contour is used in the study of the asymptotic expansion of the confluent hypergeometric. The deformed contour runs from 0 up the imaginary axis till \( iT \) (we eventually take \( T \to \infty \)), then parallel to the real axis from \( iT \) to \( iT + v_H \), and then down again parallel to the imaginary axis from \( iT + v_H \) to \( v_H \). The contribution of the segment parallel to the real axis vanishes exponentially in the limit \( T \to \infty \). We thus get

\[
I = i \int_{0}^{\infty} ds e^{-s} e^{-i\omega'(is)} - i \int_{0}^{\infty} ds e^{i\omega'(v_H + is)} e^{-i\omega(v_H + is)}
\]  

(69)

In both integrations the dominant contribution comes from the region where \( s \simeq 0 \). The second integral (including the minus sign in front) takes the form using (67)

\[
-i e^{i\omega'v_H} e^{-i\omega f(0)} \int_{0}^{\infty} ds e^{-s} = -i \frac{e^{i\omega'v_H} e^{-i\omega f(0)}}{\omega'}
\]  

(70)

In the first integral we use the asymptotic form (66) valid for small \( x \) and write

\[
\exp (-i\omega g(is)) = \exp \left( i4M\omega \ln \left( \frac{i8}{E} \right) \right) = \left( \frac{i8}{E} \right)^{4M\omega} = \exp \left( -\frac{\pi}{2} 4M\omega \right) \left( \frac{s}{E} \right)^{4M\omega}
\]

where we took the branch cut of the function \( x^i \omega \) to run from zero along the negative \( x \) axis, wrote \( x^i \omega = \exp (i\omega \ln (x + i2N\pi)) \) and chose the branch \( N = 0 \). Thus

\[
i \int_{0}^{\infty} ds e^{-s} e^{-i\omega g(is)} = i \exp \left( -\frac{\pi}{2} 4M\omega \right) E^{-i4M\omega} \int_{0}^{\infty} ds e^{-s} (s)^{4M\omega} = i \exp \left( -\frac{\pi}{2} 4M\omega \right) E^{-i4M\omega} \frac{\Gamma (1 + i4M\omega)}{(\omega')^{i4M\omega}}
\]  

(71)

We substitute (70) and (71) in (69) and then in (65) to get

\[
\beta_1(\omega, \omega') = -\frac{e^{-i\omega v_H}}{2\pi \sqrt{\omega \omega'}} \exp \left( -\frac{\pi}{2} 4M\omega \right) E^{-i4M\omega} \frac{\Gamma (1 + i4M\omega)}{(\omega')^{i4M\omega}} - \frac{e^{-i\omega f(0)}}{i2\pi \sqrt{\omega \omega'}} + \frac{1}{2\pi \omega'}
\]  

(72)

Collecting (61) and (72) we get

\[
\beta(\omega, \omega') = -\frac{e^{-i\omega v_H}}{2\pi \sqrt{\omega \omega'}} \exp \left( -\frac{\pi}{2} 4M\omega \right) E^{-i4M\omega} \frac{\Gamma (1 + i4M\omega)}{(\omega')^{i4M\omega}} - \frac{e^{-i\omega f(0)}}{i2\pi \sqrt{\omega \omega'}} + \frac{1}{2\pi \omega'}
\]  

(73)

The day is saved by (37) which causes an exact cancellation of the last two terms. The first term on its own immediately leads to the black body spectrum. Taking its modulus, squaring, and using the property

\[
|\Gamma (1 + iy)|^2 = \pi y / \sinh (\pi y)
\]

we get

\[
|\beta(\omega, \omega')|^2 = \frac{4M}{2\pi \omega'} e^{8\pi \omega M} - 1
\]

V. CONCLUSION

The objective of the paper was to prove that the Bogolubov amplitude \( \beta(\omega, \omega') \) has the asymptotic form (4) and that the radiation emitted has the spectrum of a black body. Standard quantum mechanics dictate that we should specify the initial and final states before calculating the transition amplitude, and to this end we considered the photon in states before the collapse and the out states after collapse has taken place (the final state does not have a \( \tilde{q}_\omega(u, v) \) component; cf the remark following (47)). We consider the case where the gravitational field is initially weak so that we may use the simple modes (52) as in modes; this is standard practice and leads to expression (65)
for the amplitude. Unfortunately the term (59) and the consequent (61) are missing from the standard treatments. Term (59) does not depend on the kinematics of the collapse and is certainly there in order to give a diagonal $S$ matrix in the trivial case where collapse is never initiated and nothing is produced. A correct treatment of the large $\omega'$ asymptotics of the amplitude (65) yields one contribution that leads to the black body spectrum and a second contribution (which again is missed in the standard treatments; see Appendix A) that precisely cancels (61). However this second contribution does depend on the collapse and the exact cancellation takes place only in the case of an initially weak gravitational field. In the light of the above remarks the black body result is indeed independent of the details of the collapse (as often asserted) but does depend on the assumption of an initially weak gravitational field. The question as to what would happen in the case of an initial photon state in a gravitational background corresponding to an advanced state of collapse is not answered either here or in the standard treatments of Hawking radiation. The question may be of academic interest in the context of gravity, but it may be relevant in other cases where the analog of Hawking radiation is expected to occur.

The derivation of the black body spectrum presented in this note is based on the calculation of the Bogolubov $\beta(\omega, \omega')$ amplitude. As emphasized in the Introduction this quantity is by definition time-independent, and thus the question as to where and when the photons are produced simply does not arise. It is certainly true that were it not for the singularity on the horizon the thermal spectrum would not arise. There are of course arguments based on classical arguments related to the red shift during collapse ([Weinberg 1972], p. 347). However such statements may be misleading in connection to the quantum mechanical calculation of global ("time independent") quantities. Similarly attempts to distinguish between "transient" and "steady state" radiation at the level of the $\alpha$ and $\beta$ amplitudes are bound to fail; the emphasis in the literature on the behaviour of the amplitude near the horizon has unfortunately led to such statements. One of the main conclusions of this note is that the correct derivation of the thermal result requires the consideration of the function $f(v)$ throughout its range and not just of its asymptotic part.

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Appendix A: Comparison with previous derivations

The black body result is often obtained via a sequence of somewhat peculiar mathematical steps. One often starts (see e.g. (Birrell and Davies 1982) p. 108 and also references cited therein) with expression (65) thus wrongly rewriting (75) to get (76) in the form

$$\beta_1(\omega, \omega') \approx \int_0^{\rho_H} dx \left( \frac{x}{E} \right)^{i\omega M} e^{i\omega' x} \quad (74)$$

That (74) is a wrong approximation to the original integral may easily be seen by the fact that the use of (66) has changed the behaviour of the integrand at $x = \rho_H$ (the non-singular end) and an asymptotic estimate similar to that given in section 4 would not lead to the cancellation that took place between the last two terms in (73). One then proceeds to rewrite (74) by rescaling $\omega' x \rightarrow x$

$$\beta_1(\omega, \omega') \approx E^{-i\omega M} (\omega')^{-i\omega M-1} \int_0^{\omega' \rho_H} dx x^{i\omega M} e^{i\omega x} \quad (75)$$

Since one is chasing the ultraviolet divergence one simply sets $\omega' \rho_H = \infty$, changes variable $\rho = i\sigma$ and rotates in the complex plane to get (75) in the form

$$\beta_1(\omega, \omega') \approx E^{-i\omega M} (\omega')^{-i\omega M-1} e^{-\frac{\omega}{2} \omega M} \int_0^{\infty} d\sigma e^{-\sigma \omega} \quad (76)$$

Note that setting $\omega' \rho_H = \infty$ certainly does not amount to a systematic expansion in $(\omega')^{-1}$. The $\sigma$ integration yields $\Gamma(1 + i\omega)$ and one thus obtains the form for the $\beta_1$ amplitude leading to the black body spectrum. On the other hand the step from (75) to (76) is again incorrect. Integral (74) can be performed exactly in terms of the confluent hypergeometric function and the asymptotic estimate for large $\omega'$ may be examined afterwards. Indeed let us rescale the variable in (74) $x \rightarrow x/\rho_H$ and rewrite

$$\beta_1(\omega, \omega') \approx \rho_H^{i\omega M+1} E^{-i\omega M} \int_0^1 dx e^{i\omega' \rho_H x} x^{i\omega M} = \quad (77)$$

$$= \rho_H^{i\omega M+1} E^{-i\omega M} \frac{1}{i\omega + 1} M (1 + i\omega M, 2 + i\omega M, i\omega \rho_H)$$
where $M$ is the confluent hypergeometric. We can now examine the asymptotic limit of (77) for large $\omega'$. The asymptotic limit of the confluent $M(a, b, i\,z)$ for large values of $|z|$ is given by item 13.5.1 of (Abramowitz and Stegun 1972) ($z \equiv \omega'u_M$). In the case $b = a + 1$ some simplifications occur and we get

$$M (1 + i\omega, 2 + i\omega, i\,z) \simeq (1 + i\omega) e^{|i\omega|} \frac{i}{|z|} + i\Gamma (2 + i\omega) \frac{e^{-\frac{|z|}{|1 + i\omega|}}}{|z|^{|1 + i\omega|}}$$ (78)

(other terms are down by higher powers of $1/|z|$). The second term of the above relation combined with the prefactors in (77) does feature the $\Gamma (1 + i\omega) e^{-\frac{|z|}{|1 + i\omega|}}$ factor characteristic of the black body spectrum. The reason for the discrepancy between (76) and (78) lies in the fact that one should first evaluate the integral in terms of the confluent and then take the $\omega' \to \infty$ limit rather than take the limit first. The rotation in the complex plane stumbles upon the Stokes phenomenon for the confluent (different limits for $|z| \to \infty$ depending on $\arg z$). In short the black body spectrum (76) is obtained by (a) incorrectly dropping (61), (b) incorrectly approximating (65) by (74), (c) wrongly estimating the $\omega'$ asymptotics of the latter.

**Appendix B: Technical remarks on the collapse of a sphere of dust**

We write down the expression for the Regge-Wheeler coordinate of the stellar radius as given by (11), (16):

$$R^* (\eta) = \frac{a}{2} (1 + \cos \eta) + 2M \ln \left[ \frac{a}{4M} (1 + \cos \eta) - 1 \right]$$

It takes a sequence of trivial trigonometric transformations to bring this to the form

$$R^* (\eta) = \frac{a}{2} (1 + \cos \eta) + 2M \ln \left[ \frac{\sin \frac{\eta + \gamma}{2} \sin \frac{\eta - \gamma}{2}}{\cos^2 \frac{\eta}{2}} \right]$$ (79)

We similarly transform the argument of the logarithm in (14):

$$\bar{t} (\eta) = 2a\sqrt{1 - ka^2} \left[ \frac{1}{2} (\eta + \sin \eta) + ka^2 \eta \right] + 2M \ln \left[ \frac{\sin \frac{\eta + \gamma}{2}}{\sin \frac{\eta - \gamma}{2}} \right]$$ (80)

Both $R^* (\eta)$ and $\bar{t} (\eta)$ diverge as $\eta \to \eta_0$ (i.e. as the stellar radius approaches the Schwarzschild radius), but the combination $v = \bar{t} + R^*$ does not:

$$v = 2a\sqrt{1 - ka^2} \left[ \frac{1}{2} (\eta + \sin \eta) + ka^2 \eta \right] + a \cos^2 \frac{\eta}{2} + 2M \ln \left[ \frac{\sin^2 \frac{\eta + \gamma}{2}}{\cos^2 \frac{\eta}{2}} \right]$$ (81)

The quantity $C_2$ in (21) is the derivative of the above evaluated at $\eta_0$:

$$C_2 = 2a\sqrt{1 - ka^2} \left( \frac{1}{2} + \frac{1}{2} \cos \eta_0 \right) - \frac{a}{2} \sin \eta_0 + 2M \cot \eta_0$$ (82)

The quantity $C_1$ in (21) is trivially obtained by differentiating (12):

$$C_1 = \frac{a}{2} \sin \eta_0$$ (83)

To calculate $A$ in (18) we rewrite the latter in the form

$$\frac{R - 2M}{2M} \simeq Ae^{-\frac{i}{2M}}$$ (84)

We take logarithms of both sides of (84) to rewrite it in the form

$$\ln \left( \frac{R - 2M}{2M} \right) \simeq \ln A - \frac{i}{2M}$$ (85)

We now use (9), (13), (85) and a sequence of trigonometric identities to obtain

$$\ln \left[ \frac{2}{ka^2} \sin \eta_0 \right] = \ln A - \frac{a}{M} \sqrt{1 - ka^2} \left[ \frac{1}{2} (\eta_0 + \sin \eta_0) + ka^2 \eta_0 \right]$$ (86)
thus determining $A$. The crucial step in the calculation of $A$ lies in the fact that although both sides of (85) diverge in the limit $\eta \to \eta_0$, there is precise cancellation of a term

$$\ln \left( \sin \frac{\eta_0 - \eta}{2} \right)$$

on each side thus leading to the finite result (86).

For a given value of $\eta_0$ the corresponding value $t(\eta_0)$ given by (12). We take advantage of the fact that in the comoving frame the endpoints B and A of the ray BCA lie at $r = a$. Thus the unknown is the time $t_H$ in the comoving frame where ray $l_1$ hits the star’s surface. The latter may be determined via the equation of a null geodesic inside the star obtained through (5). The path of a light ray BCA propagating inside the star is given by

$$\int_{t_H}^{t_0} dt \frac{R(t)}{R(t)} = 2 \int_0^a \frac{dr}{\sqrt{1 - kr^2}}$$

(87)

To evaluate the left hand side of (87) we change variable according to (12)

$$\frac{dr}{d\eta} = \frac{1}{2\sqrt{k}} (1 + \cos \eta)$$

Using (11) for $R(t)$ the left hand side of (87) immediately yields

$$\frac{1}{4\sqrt{k}} (\eta_0 - \eta_H)$$

(88)

The right hand side of (87) yields

$$\frac{2}{\sqrt{k}} \arcsin \left( \sqrt{ka} \right)$$

which in the limit (10) reduces to $2a$. Combining with (88) we get

$$\eta_0 - \eta_H = 8\sqrt{ka}$$

(89)

In the limit (10) we get that $v_H$ lies very close to $v_0$.

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