On the Differentials of the Spectral Sequence of a Fibre Bundle

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The theorem of W. Shi, [1], which in its turn generalizes the theorem of Faddell-Hurewicz [2], states the following: if the structure group $G$ of a fibre bundle $F \to E \to B$ is $(n-1)$-connected, then in the spectral sequence of this bundle the differentials $d^r = 0$ for $r < n$ and for $n \leq r \leq 2n - 2$ are determined by certain characteristic classes of the associated principal bundle. The above mentioned Faddell-Hurewicz theorem calculates only the first nontrivial differential $d^n$.

For simplicity, to avoid difficulties with signs, all homologies below we consider over $\mathbb{Z}_2$ although it is enough to assume them free.

In this paper we generalize the Shis result for higher dimensions: we construct some cochains from $C^*(B, H_*(G))$ and certain polylinear operations which define all differentials $d^r$. In dimensions $n \leq r \leq 2n - 2$ these cochains are cocycles form the classes from Shis theorem and the above mentioned polylinear operations is just $\sim$ product.

In papers [3], [4] N.A. Berikashvili has constructed cochain $h = h^2 + h^3 + \ldots$, $h^r \in C^r(B, Hom(H_*(F), H_*(F)))$ (representative of the predifferential of the fibration) which determines all differentials of the spectral sequence. From this point of view the Shis theorem states that for $(n - 1)$-connected structure group one has $h^r = 0$ for $r < n$ and for $n \leq r \leq 2n - 2$ the components $h^r$ can be expressed in terms of some cocycles from $C^r(B, H_{r-1}(G))$.

Let $\xi = (X, p, B, G)$ be a principal $G$-bundle, $F$ be a $G$ space and $\eta = (E, p, B, F)$ be the associated bundle with the fiber $F$.  

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Let $C^* = C^*(H_*(G), H_*(G))$ be the Hochschild cochain complex of the ring $H_*(G)$ with coefficients in itself: cochains in $C^i$ are homomorphisms $f^i : H_*(G) \otimes ... (i\text{-times}) \otimes H_*(G) \to H_*(G)$ and the coboundary operator is given by $\delta f^i(a_1, ..., a_{i+1}) = a_1 \cdot f^i(a_2, ..., a_{i+1}) + \sum_k f^i(a_1, ..., a_k \cdot a_{k+1}, ..., a_{i+1}) + f^i(a_1, ..., a_i) \cdot a_{i+1}$.

There are $\sim$ and $\sim_1$ products in the Hochschild complex $C^*$ defined as follows: for $f \in C^i$ and $g \in C^j$ let

$$f \sim g(a_1, ..., a_{i+j}) = f(a_1, ..., a_i) \cdot g(a_{i+1}, ..., a_{i+j});$$

$$f \sim_1 g(a_1, ..., a_{i+j-1}) = \sum_k f(a_1, ..., a_k, g(a_{k+1}, ..., a_{k+j}), a_{k+j+1}, ..., a_{i+j-1}).$$

The standard formulas $\delta(f \sim g) = \delta f \sim g + f \sim \delta g$, $\delta(f \sim_1 g) = \delta f \sim_1 g + f \sim_1 \delta g + g \sim_1 f - f \sim_1 g$ are valid for so defined $\sim$ and $\sim_1$.

Now let $C^* = C^*(C^*(B, H_*(G)), C^*(B, H_*(G)))$ be the Hochschild complex of the ring $C^*(B, H_*(G))$. There exists a map $\mu : C^* \to C^*$ which assigns to $f^i \in C^i$ the following homomorphism: for $b^k \in C^*(B, H_*(G))$ let $(\mu f^i)(b_1, ..., b_i)$ be the $i$ fold $\sim$ product of elements $b_1, ..., b_i$ when the coefficients are multiplied by $f^i$.

Let us define a **Hochschild twisting cochain** as an element $f = f^3 + f^4 + ...$, $f^k \in C^k$ satisfying the condition $\delta f = f \sim_1 f$. The set of all such twisting cochains we denote by $N$.

Let us define the subset $L \subset N \times C^*(B, H_*(G))$ as $L = \{(f, h_0), f \in N, h_0 \in C^*(B, H_*(G)), \text{s.t. } h_0 = h_0 \cdot h_0 + \sum_k (\mu f^k)(h_0, ..., h_0)\}$.

**Definition of the map** $\alpha : L \to D^*(B, H_*(G))$. First recall [3], [4] that $D^*(B, H_*(G))$ is defined as the factorset of the set of twisting cochains $M(B, H_*(G)) = \{h = h^2 + h^3 + ... , h^k \in C^k(B, \text{Hom}(H_*(G), H_*(G))), \delta h = h \sim h\}$. Note that since the modules $H_i(G)$ are assumed free, an element $h$ is determined by a collection $\{h(a) \in C^*(B, H_*(G)), a \in I\}$ where $I$ is the set of free generators of $H_*(G)$. Now for an element $(f, h_0) \in L$ we define $h \in M(B, H_*(G))$ as $h = \{h(a), a \in I, h(a) = h_0 \cdot a + \sum_i (\mu f^i)(h_0, ..., h_0, a)\}$. Inspection shows that $\delta h = h \sim h$. Finally we define $\alpha(f, h_0)$ as the class of $h$ in $D(B, H_*(G))$.

Now let us consider the set of triples $\bar{L} = \{(f, h_0, \bar{f})\}$, where $(f, h_0) \in L$ and $\bar{f} = \bar{f}^3 + \bar{f}^4 + ...$, $\bar{f}^i : H_*(G) \otimes ... ((i-1)\text{-times}) \otimes H_*(G) \otimes H_*(F) \to$
As above each polynomial map $\bar{f}^i$ induces the map

$$(\bar{\mu}, \bar{f}^i) : C^*(B, H_*(G)) \otimes \ldots \otimes C^*(B, H_*(G)) \otimes C^*(B, H_*(F)) \rightarrow C^*(B, H_*(F)).$$

**Definition of the map $\beta : \bar{L} \to D^*(B, H_*(F))$.** We define $\beta(f, f_0, \bar{f})$ as the class of the twisting cochain $h \in M(B, H_*(F)) = \{h = h^2 + h^3 + \ldots, h^k \in C^k(B, \text{Hom}(H_*(F), H_*(F)), \delta h = h \circ h\}$, defined as follows: $\bar{h} = \{\bar{h}(x); x \in J\}$ (here $J$ is the set of free generators of $H_*(F)$), where $\bar{h}(x) = h_0 \cdot x + \sum_i(\bar{\mu} \bar{f}^i)(h_0, \ldots, h_0, x)$. The condition (1) allows to check that $\delta \bar{h} = \bar{h} \circ \bar{h}$.

The group $G$ and, respectively, the action $G \times F \to F$, define (non uniquely) certain Hochschild twisting cochain $f$, and respectively the element $\bar{f}$, for which the condition (1) is satisfied. This can be done as follows.

Since the groups $H_*(G)$ are assumed free, it is possible to fix a cycle choosing homomorphism $g : H_*(G) \to Z_*(G)$. Besides let us fix also a homomorphism $\delta^{-1} : B_*(G) \to C_*(G)$ which satisfies $\delta \delta^{-1} = \text{id}$. Let us define also a homomorphism $\phi : Z_*(G) \to C_*(G)$ by $\phi(z) = \delta^{-1}(z - g(cl(z)))$, where $cl(z)$ is the homology class of $z$.

We construct by induction a sequences of multioperations $f^i = f^3 + f^4 + \ldots$ and homomorphisms $A_i : H_*(G) \otimes \ldots \otimes H_*(G) \to C_*(G)$ using the following conditions:

1) $A_2(a_1, a_2) = g(a_1) \cdot g(a_2)$;
2) $A_i(a_1, \ldots, a_i) \in Z_*(G)$;
3) $f^i(a_1, \ldots, a_i) = cl(A_i(a_1, \ldots, a_i)) \in H_*(G)$;
4) $A_{i+1}(a_1, \ldots, a_{i+1}) = g(a_1) \cdot \phi A_i(a_2, \ldots, a_{i+1}) + \phi A_i(a_1, \ldots, a_i) \cdot g(a_{i+1}) + \sum_{s,t} [A_s(a_1, \ldots, a_s, f^i(a_{k+1}, \ldots, a_{k+t}), \ldots, a_{i+1}) + \sum_{s,t} [A_s(a_1, \ldots, a_s) \cdot \phi A_t(a_{s+1}, \ldots, a_{i+1})].$

Now we construct $\bar{f}^i$. Let $\bar{g} : H_*(F) \to Z_*(F)$ be a cycle choosing homomorphism; $\delta^{-1} : B_*(F) \to C_*(F)$, $\delta \delta^{-1} = \text{id}$; and $\psi : Z_*(F) \to C_*(F)$.

$$H(F)$$ such that the following condition is satisfied

$$\sum_{s+t=i+1} f^s(a_1, \ldots, a_k, f^t(a_{k+1}, \ldots, a_{k+t}), \ldots, a_{i-1}, x) + \sum_{s+t=i+1} f^s(a_1, \ldots, a_{i-t}, f^t(a_{i-t+1}, \ldots, a_{i-1}, x)) = \delta \bar{f}^i(a_1, \ldots, a_{i-1}, x).$$ (1)
is given by $\psi(z) = \delta^{-1}(z - \bar{g}(\text{cl}(z)))$. As above we construct by induction a sequence of multioperations $\bar{f} = \bar{f}^3 + \bar{f}^4 + \ldots$ and homomorphisms $\bar{A}_i : H_*(G) \otimes \ldots \otimes H_*(G) \otimes H_*(F) \to C_*(F)$ using the following conditions:

1) $\bar{A}_2(a, x) = g(a) \cdot \bar{g}(x)$;
2) $\bar{A}_i(a_1, \ldots, a_{i-1}, x) \in Z_i(F)$;
3) $\bar{f}^i(a_1, \ldots, a_{i-1}, x) = \text{class}(\bar{A}^i(a_1, \ldots, a_{i-1}, x)) \in H_*(F)$;
4) $\bar{A}_{i+1}(a_1, \ldots, a_i, x) = g(a_1) \cdot \psi \bar{A}^i(a_2, \ldots, a_{i-1}, x) + \phi A_i(a_1, \ldots, a_i) \cdot \bar{g}(x) + \sum_{s,t,k} \psi \bar{A}_s(a_1, \ldots, a_k, \bar{f}^i(a_{k+1}, \ldots, a_{i+t}), \ldots, a_i, x) + \sum_{s,t} \psi \bar{A}_s(a_1, \ldots, a_{i-t}, \bar{f}^i(a_{i-t+1}, \ldots, a_i, x)) + \sum_{s,t} \phi A_s(a_1, \ldots, a_s) \cdot \psi \bar{A}_t(a_{s+1}, \ldots, a_i, x)$.

**Theorem.** For the above constructed $f$ and $\bar{f}$ there exists a cochain $h_0 = h_0^3 + h_0^4 + \ldots$, $h_0^r \in C^r(B, H_{r-1}(G))$ such that $(f, h_0, \bar{f}) \in L$, $\alpha(f, h_0)$ is the predifferential of the principal bundle $\xi$, and $\beta(f, h_0, \bar{f})$ is the predifferential of the associated bundle $\eta$.

In particular if the group $G$ is $(n - 1)$-connected, we obtain the result of Shih: $h_0^r = 0$ for $r < n$, and consequently $h_0^r = 0$; as for $n \leq r \leq 2n - 2$ we have $h_0^r(a) = h_0^r \cdot a$, and $h_0^r$ are cocycles.

**References**

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