Optical tomography of Fock state superpositions

S N Filippov and V I Man’ko

1 Moscow Institute of Physics and Technology, Moscow, Russia
2 P N Lebedev Physical Institute, Moscow, Russia

E-mail: sergey.filippov@phystech.edu and manko@sci.lebedev.ru

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Abstract

We consider optical tomography of photon Fock state superpositions (FSS) in connection with recent experimental achievements. The emphasis is put on the fact that it suffices to represent the measured tomogram as a main result of the experiment. We suggest a test for checking the correctness of experimental data. Explicit expressions for optical tomograms of FSS are given in terms of Hermite polynomials. Particular cases of vacuum and low-photon-number state superposition are considered and the influence of thermal noise on state purity is studied.

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((Some figures in this article are in colour only in the electronic version.)

Recent decades have seen a continuous growth of interest in quantum phenomena. This research has now become interdisciplinary and covers many fields ranging from quantum optics and particle physics to quantum information processing and foundations of quantum mechanics. The preparation, evolution and measurement of quantum states were focused on in numerous theoretical and experimental investigations. Also, a great deal of effort was made in characterizing quantum state properties such as nonlocality and entanglement. Both the explanation and prediction of experimental results resort to a mathematical description of quantum states. From a mathematical point of view, the optical tomogram is nothing else but the Radon transform of the Wigner function. The crucial point is that the Radon transform provides probability distributions (see e.g. [3]) and such a distribution turns out to be measurable directly via homodyne detection of photon states. The experimental output of such a detection is exactly the optical tomogram. Originally, it was used [4, 5] as a technical tool for reconstructing the Wigner function, which is still usually identified with the photon quantum state. But the optical tomogram contains complete information about all the properties of a quantum state. In view of this fact, it is pointed out that the accent in homodyne detection experiments has to be on the most accurate measurement of the optical tomogram. According to the relation ‘quantum state \( \iff \) measurable tomogram’, it would be enough to present, as the result of the experiment, a plot of the tomogram.

Once the optical tomogram is measured, there is a need to ascertain whether the output data are indeed a tomogram of the quantum state. In order to test the correctness of the measured tomogram, one can utilize peculiar properties of quantum states. For example, one can check that the experimental results do not contradict tomographic entropic inequalities [6], uncertainty relations [7] or purity constraints [8]. In other words, one must check that the tomogram fulfills specific requirements derived in [6–9].
On the other hand, the optical tomogram was shown to coincide with this propagator. The optical tomogram as a primary notion of quantum states makes it possible to compare output data with predicted ones, estimating the purity and thermal noise presented, and adjusting and calibrating the setup.

The optical tomogram of the state given by the density operator \( \hat{\rho} \) reads

\[
w(X, \theta) = \int G(X, \theta; z, 0) G^*(X, \theta; z', 0) \langle z | \hat{\rho} | z' \rangle \, dz \, dz',
\]

where \( G(q, t; q', 0) \) is a conventional Green’s function of the harmonic oscillator with the Hamiltonian \( \hat{H} = (\hat{p}^2 + \hat{q}^2)/2 = (\hat{a}^\dagger \hat{a} + 1)/2 \). Hereafter, we use dimensionless units, namely, the Planck constant \( \hbar = 1 \), the Boltzmann constant \( k_B = 1 \), etc. The relation of the optical tomogram to the propagator of the quantum oscillator was discussed in [18] (formulae (27), (34) and (35) therein), where the kernel of fractional Fourier transform was shown to coincide with this propagator. On the other hand, the optical tomogram was shown to be the modulus squared of the fractional Fourier transform of the wave function in [19]. Thus the optical tomogram can be expressed in terms of the oscillator propagator and the density operator as equation (1) shows.

Formula (1) can be treated as a reconstruction of the quantum state of the wavepacket from position probability distributions measured during the packet’s motion in the harmonic oscillator potential [20, 21]. In the case of optical tomography, a timelike evolution of the electromagnetic field is brought about by shifting the phase \( \theta \) of the local oscillator. The tomogram \( w(X, \theta) \) is the marginal distribution of the quadrature component \( X \) of the electric field strength, with \( X \) being rotated by angle \( \theta \) in the quadrature phase space.

Let us consider the photon state of the one-mode electromagnetic field \( |\psi\rangle = \sum_{n=0}^{N} c_n |n\rangle \), where \( |n\rangle \) is the photon Fock state, \( N < \infty \). The density operator of such a pure state is

\[
\hat{\rho} = |\psi\rangle \langle \psi| = \sum_{n=0}^{N} |c_n|^2 |n\rangle \langle n| + \sum_{n<k} (c_n^* c_k |k\rangle \langle n| + \text{c.c.}) \quad \text{(2)}
\]

The Green’s function \( G(q, t; q', 0) \) determines the evolution of Hamiltonian eigenstates \( |n\rangle \) as follows:

\[
\int G(q, t; q', 0) \langle q'|n\rangle \, dq' = (q|n)e^{-i\pi n + 1/2}t = \frac{1}{\sqrt{\sqrt{2\pi n!}}} H_n(q)e^{-q^2/2 - i\pi n + 1/2}t,
\]

where \( H_n(q) \) is the Hermite’s polynomial of degree \( n \). Substituting (2) and (3) into (1), we readily obtain the optical tomogram of Fock state superposition

\[
w_{\text{FSS}}(X, \theta) = \frac{e^{-X^2}}{\sqrt{\pi}} \left[ \sum_{n=0}^{N} |c_n|^2 H_n^2(X) \right. \\
\left. + \sum_{n<k} (c_n^* c_k e^{i(n-k)\theta} + \text{c.c.}) H_n(X) H_k(X) \right] \\
\times H_n(X) H_k(X) \]

where we have extracted phases from the coefficients \( c_n = |c_n|e^{i\psi_n} \) and \( c_k = |c_k|e^{i\psi_k} \).

Examples of FSS optical tomograms are illustrated in figure 1. It is readily observed that the tomograms satisfy the relation \( w(X, \theta) = w(-X, \theta + \pi) \), which also follows from analytical consideration. This simple constraint on experimental data allows checking the accuracy of the measured tomogram \( w_{\text{meas}}(X, \theta) \). Indeed, the difference \( |w_{\text{meas}}(X, \theta) - w_{\text{meas}}(-X, \theta + \pi)| \) must be zero for an exactly measured tomogram. The deviation of this quantity from the zero level for all the local oscillator phases, e.g. \( \sup_{X \in \mathbb{R}, \theta \in [0, 2\pi]} |w_{\text{meas}}(X, \theta) - w_{\text{meas}}(-X, \theta + \pi)| \), is suggested to be used as an indicator of homodyne detection precision.

It is worth mentioning that the Pauli problem can be constructively solved for FSS [22]. In other words, for FSS it is sufficient to measure \( w(X, \theta = 0) \) and \( w(X, \theta = \pi/2) \) to find the initial state, up to the well-known twofold ambiguity. This result is based on the fact that we know \textit{a priori} that the state of the system is pure and comprises a finite number of Fock states. However, in real experiments, perfectly pure states are hardly achievable. For instance, optical tomography of the microwave electromagnetic field exploits linear amplifiers that unavoidably introduce a noise. We will treat this noise as a thermal one with the effective temperature \( T \). We find it reasonable to consider the influence of such thermal noise on the purity of the desired FSS. Moreover, the purity can be directly calculated via the measured optical tomogram [8] avoiding the density matrix formalism. A comparison of the obtained and theoretically predicted values of purity can be used to test the correctness of data and to calibrate and adjust the apparatus.
For the sake of simplicity, we consider the following model. The density operator of thermal noise has the form
\[ \hat{\rho}_\text{th} = Z^{-1} e^{-\hat{H}/T}, \]
where \( Z = \text{Tr}[e^{-\hat{H}/T}] = \frac{1}{2} (\sinh \frac{1}{2T})^{-1} \) and \( T \) is the temperature. The associated optical tomogram reads
\[ w_T(X, \theta) = \left( \frac{2\pi\sigma^2}{\sqrt{\pi}} \right)^{-1} \exp\left[ -\frac{X^2}{2\sigma^2} \right], \]
where \( \sigma^2 = \frac{1}{2} \coth \frac{1}{2T} \). A mixture of the superposition of Fock states (2) and thermal noise is given by the density operator
\[ \hat{\rho}_\text{mix} = (1 - p) \hat{\rho} + p \hat{\rho}_\text{th}, \]
where the parameter \( 0 \leq p \leq 1 \) shows how much noise is added. The corresponding optical tomogram reads
\[ w_\text{mix}(X, \theta) = (1 - p) w_{\text{FSS}}(X, \theta) + pw_\text{th}(X, \theta). \]
The purity of this state is
\[ \mu_\text{mix} \equiv \text{Tr}[\hat{\rho}_\text{mix}^2] = (1 - p)^2 + 2(1 - p)p \langle \psi | \hat{\rho}_T | \psi \rangle + p^2 \text{Tr}[\hat{\rho}_T^2]. \]
\[ = (1 - p)^2 + 4(1 - p)p \sinh \left( \frac{1}{2T} \sum_{n=0}^{N} |c_n|^2 e^{-\pi(n/2)/T} \right) + p^2 \tanh \left( \frac{1}{2T} \right), \]
where we have taken into account the diagonal form of thermal noise \( \hat{\rho}_T \) in the basis of Fock states. The dependence
of the state purity on thermal noise is depicted for some examples in figure 2. If $T \rightarrow \infty$, then $\mu_{\text{mix}} \approx (1 - p)^2 + p^2/2T$. If $T \rightarrow 0$, then the thermal noise reduces to the vacuum mode and its effect on purity depends on two factors: the noise strength $p$ and whether the vacuum state was included in the initial FSS. If the vacuum state is absent from FSS, then $\mu_{\text{mix}}(T = 0)$ is the same for all such FSS. This means that the parameter $p$ can be evaluated by the one-photon state. It is worth mentioning that there is a nonmonotonic dependence of the one-photon state purity on $T$ for a fixed $p < \frac{1}{2}$, because $\mu_{\text{mix}} \approx (1 - p)^2 + p^2 + 2p(1 - 2p)e^{-1/T}$ if $T \ll 1$ (see figure 2(b)). In Militello’s paper [23], the effect of purity oscillations is discussed. In the present work, we note that the formula for purity in terms of optical tomograms can also be used to describe such phenomena.

To conclude, we summarize the main results of this paper. We studied the optical tomograms of FSS in view of several experiments [10–13] devoted to measuring the homodyne quadrature distributions in some nonclassical photon states [14]. We point out that the efforts of experiments should be focused on as precise a measurement of optical tomograms as possible. The optical tomogram provides complete information about a quantum state and the correctness of the measured tomogram can be checked by a series of constraints, one of which is introduced in this paper. Other tests on the precision of homodyne detecting photon states can also be used [6–9]. We emphasize that there is no need to convert the tomogram into the Wigner function or other quasi-probability functions. When applied to FSS, the optical tomogram is obtained in an explicit form (4). We have also analyzed the purity of a Fock state superposition mixed with a thermal noise unavoidably presented in experiments.

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