Periods of Mixed Tate Motives over Real Quadratic Number Rings

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Abstract

Recently, the author defined multiple Dedekind zeta values [5] associated to a number field and a cone $C$. In this paper we construct explicitly non-trivial examples of mixed Tate motives over the ring of integers in $K$, for a real quadratic number field $K$ and a particular cone $C$. The period of such a motive is a multiple Dedekind zeta values at $(s_1, s_2) = (1, 2)$, associated to the pair $(K; C)$, times a nonzero element of $K$.

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1 Introduction

The Riemann zeta function

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s}$$

is widely used in number theory, algebraic geometry and quantum field theory. Euler’s multiple zeta values

$$\zeta(s_1, \ldots, s_m) = \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{s_1} \cdots n_m^{s_m}},$$

where $s_1, \ldots, s_m$ are positive integers and $s_m \geq 2$, appear as values of some Feynman amplitudes, and in algebraic geometry, as periods of mixed Tate motives over $\text{Spec}(\mathbb{Z})$ (see [4], [3], [1], [7]).

Dedekind zeta values

$$\zeta_K(s) = \sum_{a \neq (0)} \frac{1}{N(a)^s},$$

are a generalization of the Riemann zeta function to a number field $K$. In some Feynman amplitudes one of the summands is $\log(1 + \sqrt{2})$ or $\log \left( \frac{1 + \sqrt{5}}{2} \right)$. These values are essentially the residues at $s = 1$ of Dedekind zeta functions over $\mathbb{Q}(\sqrt{2})$ and over $\mathbb{Q}(\sqrt{5})$, respectively. For $s = 2, 3, 4, \ldots$ the values $\zeta_K(s)$ are periods of mixed Tate motives over the ring of algebraic integers in $K$ with ramification only at the discriminant of $K$ (see [2]).
In [5], the author has constructed multiple Dedekind zeta values, which are a generalization of Euler’s multiple zeta values to number fields in the same way as Dedekind zeta values generalizes Riemann zeta values. For a quadratic number field $K$, the key examples of multiple Dedekind zeta values are

$$
\zeta_{K;C}(s_1, \ldots, s_1; \ldots; s_m, \ldots, s_m) = \sum_{\alpha_1, \ldots, \alpha_m \in C} \frac{1}{N(\alpha_1)^{s_1}N(\alpha_1 + \alpha_2)^{s_2} \cdots N(\alpha_1 + \cdots + \alpha_m)^{s_m}}, \quad (1)
$$

where $s_1, \ldots, s_m$ are positive integers and $s_m \geq 2$ and $C$ is a cone generated by a totally positive unit $\beta$ in $K$ and 1, defined by

$$
C = \mathbb{N}\{1, \beta\} = \{\gamma \in K \mid \gamma = a + b\beta, \text{ for positive integers } a \text{ and } b\}.
$$

Similar types of cones were considered by Zagier in [8] and [9].

In [5], the author has proven that multiple Dedekind zeta values can be interpolated to multiple Dedekind zeta functions, which have meromorphic continuation to all complex values of the variables $s_1, \ldots, s_m$.

In this paper we prove the following theorem.

**Theorem 1** Let $K$ be a real quadratic field, and let $C$ be a cone generated by a totally positive unit $\beta$ in $K$ and 1. Then the multiple Dedekind zeta values

$$
(\beta_2 - \beta_1)^3 \zeta_{K;C}(1, 2)
$$

is a period of a mixed Tate motive over the ring of integers in $K$. In particular, it is unramified over the primes dividing the discriminant $\sqrt{D}$.

**Remark:** The proof of the Theorem can easily be generalized to all

$$
(\beta_2 - \beta_1)^{s_1 + \cdots + s_m} \zeta_{K;C}(s_1, \ldots, s_m)
$$

for the same cone $C$. The details for the general case will be completed in a sequel to this paper. The choice of considering $\zeta_{K;C}(1, 2)$ in this paper is two-fold. First, this is among the simplest non-trivial example of a multiple Dedekind zeta value. Second, for any other (multiple) Dedekind zeta value, the proof of the corresponding statement is essentially the same.

## 2 Background

### 2.1 Multiple zeta values

The Riemann zeta function at the value $s = 2$ can be expressed in term of an iterated integral in the following way
\[
\int_0^1 \left( \int_0^y \frac{dx}{1-x} \right) \frac{dy}{y} = \int_0^1 \left( \int_0^y \left( 1 + x^2 + x^3 \ldots \right) dx \right) \frac{dy}{y} = \int_0^1 \left( y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} \ldots \right) \frac{dy}{y} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots = \zeta(2).
\]

Let us examine the domain of integration of the iterated integral. Note that \(0 < x < y\) and \(0 < y < 1\). We can put both inequalities together. Then we obtain the domain \(0 < x < y < 1\), which is a simplex. Thus, we can express the iterated integral as

\[
\zeta(2) = \int_0^1 \left( \int_0^y \frac{dx}{1-x} \right) \frac{dy}{y} = \int_{0<x<y<1} \frac{dx}{1-x} \wedge \frac{dy}{y}.
\]

Moreover, Goncharov and Manin [4] have expressed all multiple zeta values as periods of motives related to the moduli space of curves of genus zero with \(n + 3\) marked points, \(\mathcal{M}_{0,n+3}\). In particular, \(\zeta(2)\) can be expressed as a period of the motive \(H^2(\overline{\mathcal{M}}_{0,5} - A, B - A \cap B)\) by pairing of \([\Omega_A] \in Gr_W^1 H^2(\overline{\mathcal{M}}_{0,5} - A)\) for \(\Omega_A = \frac{dz}{1-x} \wedge \frac{dy}{y}\), with \([\Delta_B] \in (Gr_0^W H^2(\overline{\mathcal{M}}_{0,5} - B))^\vee\). The Deligne-Mumford compactification \(\overline{\mathcal{M}}_{0,5}\) of the moduli space \(\mathcal{M}_{0,5}\) can be obtained by three blow-ups of \(\mathbb{P}^1 \times \mathbb{P}^1\) at the points \((0, 0)\), \((1, 1)\) and \((\infty, \infty)\). Let us name the exceptional divisors at the three points by \(E_0\), \(E_1\) and \(E_\infty\), respectively. Then \(A = (x = 1) \cup (y = 0) \cup (x = \infty) \cup (y = \infty) \cup E_\infty\) and \(B = (x = 0) \cup (x = y) \cup (y = 1) \cup E_0 \cup E_1\).

Similarly, one can express \(\zeta(3)\) and \(\zeta(1, 2)\) as iterated integrals

\[
\zeta(3) = \int_0^1 \left( \int_0^z \left( \int_0^y \frac{dx}{1-x} \right) \frac{dy}{y} \right) \frac{dz}{z} = \int_{0<x<y<z<1} \frac{dx}{1-x} \wedge \frac{dy}{y} \wedge \frac{dz}{z},
\]

\[
\zeta(1, 2) = \int_0^1 \left( \int_0^z \left( \int_0^y \frac{dx}{1-x} \right) \frac{dy}{1-y} \right) \frac{dz}{z} = \int_{0<x<y<z<1} \frac{dx}{1-x} \wedge \frac{dy}{1-y} \wedge \frac{dz}{z}.
\]

Again, \(\zeta(3)\) and \(\zeta(1, 2)\) can be expressed as periods of motives related to \(\mathcal{M}_{0,6}\). In the same paper, Goncharov and Manin prove that the motives associated to multiple zeta values (MZVs) are mixed Tate motives unramified over \(Spec(\mathbb{Z})\).

A few years later, Francis Brown [1] proved that periods of mixed Tate motives unramified over \(Spec(\mathbb{Z})\) can be expressed as a \(\mathbb{Q}\)-linear combination of MZVs times an integer power of \(2\pi i\).

### 2.2 Multiple Dedekind zeta values (MDZVs)

We recall the construction of MDZVs over a real quadratic field \(K\). (See [3] for definition of MDZVs over any number field.) Let \(\mathcal{O}_K\) be the ring of integers in \(K\).

And let \(\beta\) be a totally positive unit in \(\mathcal{O}_K\). Let \(C\) be the cone defined as \(\mathbb{N}\)-linear combination of 1 and \(\beta\), that is,

\[
C = \{\gamma \in \mathcal{O}_K \mid \gamma = a + b\beta, \text{ for } a, b \in \mathbb{N}\}.
\]
Let \( f_0(C; t_1, t_2) = \sum_{\gamma \in C} \exp(-t_1 \gamma_1 - t_2 \gamma_2) \), where \( \gamma_1 \) and \( \gamma_2 \) are two real embeddings of \( \gamma \). We express \( \zeta_{K;C}(2) \), \( \zeta_{K;C}(3) \) and \( \zeta_{K;C}(1, 2) \) as iterated integrals on a membrane. See [5] and [6], for more examples and properties of iterated integrals on membranes.

\[
\int_0^\infty \int_0^\infty \left( \int_0^\infty \int_0^\infty f_0(C; t_1, t_2) dt_1 \wedge dt_2 \right) du_1 \wedge du_2.
\]

\[
= \int_0^\infty \int_0^\infty \left( \int_0^\infty \int_0^\infty \left( \sum_{\gamma \in C} \exp(-t_1 \gamma_1 - t_2 \gamma_2) \right) dt_1 \wedge dt_2 \right) du_1 \wedge du_2.
\]

\[
= \int_0^\infty \int_0^\infty \left( \sum_{\gamma \in C} \frac{\exp(-u_1 \gamma_1 - u_2 \gamma_2)}{\gamma_1 \gamma_2} \right) du_1 \wedge du_2
\]

\[
= \sum_{\gamma \in C} \frac{1}{(\gamma_1 \gamma_2)^2} = \sum_{\gamma \in C} \frac{1}{N(\gamma)^2} = \zeta_{K;C}(2).
\]

Similarly,

\[
\zeta_{K;C}(3) = \sum_{\gamma \in C} \frac{1}{N(\gamma)^3}
\]

\[
= \int_0^\infty \int_0^\infty \left( \int_0^\infty \int_0^\infty \left( \int_0^\infty \int_0^\infty f_0(C; t_1, t_2) dt_1 \wedge dt_2 \right) du_1 \wedge du_2 \right) dv_1 \wedge dv_2,
\]

and

\[
\zeta_{K;C}(1, 2) = \sum_{\gamma \in C} \frac{1}{N(\gamma)^3 N(\gamma + \delta)^2} = \int_0^\infty \int_0^\infty \left( \int_0^\infty \int_0^\infty \left( \int_0^\infty \int_0^\infty f_0(C; t_1, t_2) dt_1 \wedge dt_2 \right) \right.
\]

\[
\times f_0(C; u_1, u_2) du_1 \wedge du_2 \Big) dv_1 \wedge dv_2.
\]

3 Transition to Algebraic Geometry

We can write the infinite sum in the definition of \( f_0 \) as a product of two geometric series

\[
f_0(C; t_1, t_2) = \sum_{\gamma \in C} \exp(-\gamma_1 t_1 - \gamma_2 t_2)
\]

\[
= \sum_{a=1}^\infty \sum_{b=1}^\infty \exp[-(a \alpha_1 + b \beta_1) t_1 - (a \alpha_2 + b \beta_2) t_2]
\]

\[
= \sum_{a=1}^\infty \sum_{b=1}^\infty \exp[-a(\alpha_1 t_1 + \alpha_2 t_2)] \exp[-b(\beta_1 t_1 + \beta_2 t_2)]
\]

\[
= \frac{\exp[-(\alpha_1 t_1 + \alpha_2 t_2)]}{1 - \exp[-(\alpha_1 t_1 + \alpha_2 t_2)]} \times \frac{\exp[-(\beta_1 t_1 + \beta_2 t_2)]}{1 - \exp[-(\beta_1 t_1 + \beta_2 t_2)]}
\]
Let \( x_1 = e^{-t_1} \) and \( x_2 = e^{-t_2} \). Then

\[
f_0(C; t_1, t_2) = \frac{x_1 x_2}{1 - x_1 x_2} \cdot \frac{x_1^{\beta_1} x_2^{\beta_2}}{1 - x_1^{\beta_1} x_2^{\beta_2}}.
\] (3)

Now we are going to express \( f_0 \) algebraically. At this point there is a problem of raising the variable \( x \) to an integer algebraic power. Note that \( \beta_1 \) and \( \beta_2 \) are algebraic integers (in fact totally positive units), which are not rational integers.

How do we raise \( x \) to power \( \beta_1 \) and to \( \beta_2 \)? We introduce new variables

\[
y_1 = x_1^{\beta_1} \quad \text{and} \quad y_2 = x_2^{\beta_2}.
\]

Then \( x_1^{a+b\beta_1} = x_1^a y_1^b \), where \( a \) and \( b \) are integers.

We are going to use the variables \( x_1, x_2 \). For each of them we introduce \( y_1, y_2 \), so that we write \( y_1 \) instead of \( x_1^{\beta_1} \) and \( y_2 \) instead of \( x_2^{\beta_2} \). In terms of \( x_1, x_2, y_1 \) and \( y_2 \), we can express \( f_0 \) as

\[
f_0(C; t_1, t_2) = \frac{x_1 x_2}{1 - x_1 x_2} \cdot \frac{x_1^{\beta_1} x_2^{\beta_2}}{1 - x_1^{\beta_1} x_2^{\beta_2}} = \frac{x_1 x_2}{1 - x_1 x_2} \cdot \frac{y_1 y_2}{1 - y_1 y_2}.
\]

Let us also define \( \omega_1 = \frac{d(x_1 x_2)}{y_1 y_2} \wedge \frac{d(y_1 y_2)}{x_1 x_2} \) and let \( \omega_0 = \frac{d(x_1 x_2)}{y_1 y_2} \wedge \frac{d(y_1 y_2)}{x_1 x_2} \).

Key Remark: The differential forms \( \omega_0 \) and \( \omega_1 \) will be used for both algebraic geometry on moduli spaces and for defining multiple Dedekind zeta values.

Lemma 2 If we substitute \( x_1 = e^{-t_1} \), \( x_2 = e^{-t_2} \), \( y_1 = e^{-\beta_1 t_1} \) and \( y_2 = e^{-\beta_2 t_2} \), then

\[
\omega_0 = (\beta_2 - \beta_1) dt_1 \wedge dt_2.
\]

Proof: Consider \( x_1, x_2, y_1 \) and \( y_2 \) as functions of \( t_1 \) and \( t_2 \). Then

\[
y_1 y_2 = x_1^{\beta_1} x_2^{\beta_2}
\]

and

\[
d(y_1 y_2) = d(x_1^{\beta_1} x_2^{\beta_2}) = \beta_1 \frac{dx_1}{x_1} + \beta_2 \frac{dx_2}{x_2} = -\beta_1 dt_1 - \beta_2 dt_2
\]

Similarly,

\[
d(x_1 x_2) = -dt_1 - dt_2.
\]

Again, as functions of \( t_1 \) and \( t_2 \), we have

\[
\omega_0 = \frac{d(x_1 x_2)}{x_1 x_2} \wedge \frac{d(y_1 y_2)}{y_1 y_2} = (dt_1 + dt_2) \wedge (\beta_1 dt_1 + \beta_2 dt_2) = (\beta_2 - \beta_1) dt_1 \wedge dt_2.
\]

Now let us write \( \omega_0(x_1, x_2) \) and \( \omega_1(x_1, x_2) \), when we want to specify the dependence on the variables. In fact, both forms depend also on \( y_1 \) and \( y_2 \); however, we will take care of that by choosing a region of integration together with tangential base points.
4 Tangential base points

Let \( x_1 = e^{-t_1} \) and let \( y_1 = e^{-\beta_1 t_1} \). We would like to find an algebraic relation among the variables \( x_1 \) and \( y_1 \) when they approach \((0, 0)\) or when they approach \((1, 1)\). That occurs when \( t_1 \) approaches \( \infty \) or when \( t_1 \) approaches 0, respectively. If \( \beta_1 > 1 \) then

\[
\lim_{t_1 \to \infty} \frac{dy_1}{dx_1} = \lim_{t_1 \to \infty} \frac{de^{-\beta_1 t_1}}{de^{-t_1}} = \lim_{t_1 \to \infty} \frac{\beta_1 e^{t_1}}{(e^{t_1})^{\beta_1}} = 0.
\]

Also

\[
\lim_{t_1 \to 0} \frac{dy_1}{dx_1} = \lim_{t_1 \to 0} \beta_1 \frac{e^{-\beta_1 t_1}}{e^{-t_1}} = \beta_1.
\]

Let

\[
\gamma_1 : (0, \infty) \to \mathcal{M}_{0, 5},
\]

\[
\gamma_1(t_1) = (e^{-t_1}, e^{-\beta_1 t_1}) = (x_1, y_1).
\]

For a vector \( v = (a, b) \), consider \([v] = [a : b]\) as an element of \( \mathbb{P}^1 \).

We have proven the following lemma.

**Lemma 3**

(a) \( \lim_{t_1 \to \infty} \left[ \frac{d\gamma_1}{dt_1} \right] = [1 : 0] \),

(b) \( \lim_{t_1 \to 0} \left[ \frac{d\gamma_1}{dt_1} \right] = [1 : \beta_1] \).

Similarly, we have \( x_2 = e^{-t_2} \) and \( y_2 = e^{-\beta_2 t_2} \) with \( 0 < \beta_2 < 1 \). Let

\[
\gamma_2 : (0, \infty) \to \mathcal{M}_{0, 5},
\]

\[
\gamma_2(t_2) = (e^{-t_2}, e^{-\beta_2 t_2}) = (x_2, y_2).
\]

The following Lemma could be proven in the same way.

**Lemma 4**

(a) \( \lim_{t_2 \to \infty} \left[ \frac{d\gamma_2}{dt_2} \right] = [0 : 1] \),

(b) \( \lim_{t_2 \to 0} \left[ \frac{d\gamma_2}{dt_2} \right] = [1 : \beta_2] \).

**Remark:** The paths \( \gamma_1 \) and \( \gamma_2 \) can be used to define a membrane \( m = \gamma_1 \times \gamma_2 \) by taking a Cartesian products of both the domains and the targets

\[
m = \gamma_1 \times \gamma_2 : (0, 1)^2 \to (\mathcal{M}_{0, 5})^2.
\]

The definition of multiple Dedekind zeta values via iterated integrals on a membrane use exactly the membrane \( m \) in the case of quadratic fields (see \[3\]).

**Proposition 5**

With the above choice of tangential base points, we have

\[
\int_{0 < x_1 < x_3 < 1; 0 < x_2 < x_4 < 1} \omega_1(x_1, x_2) \wedge \omega_0(x_3, x_4) = (\beta_2 - \beta_1)^2 \zeta_{K, C}(2).
\]
Proof: The differential forms $\omega_0$ and $\omega_1$ are closed. Thus we can vary the paths $\gamma_1$ and $\gamma_2$ without changing the value of the integral as long as the tangential base points remain the same. Thus, we can choose the parametrization $x_i = e^{-t_i}$ and $y_i = e^{-\beta_it_i}$, keeping the tangential points fixed. Using Formulas (2) and (3), we obtain

$$\frac{d(x_3x_4)}{x_3x_4} \wedge \frac{d(y_3y_4)}{y_3y_4} = (\beta_2 - \beta_1)dt_3 \wedge dt_4$$

Similarly, we have that

$$\frac{x_1x_2}{1 - x_1x_2} \cdot \frac{y_1y_2}{1 - y_1y_2} \cdot \left( \frac{d(x_3x_4)}{x_3x_4} \wedge \frac{d(y_3y_4)}{y_3y_4} \right) = f_0(C; t_1, t_2)(\beta_2 - \beta_1)dt_1 \wedge dt_2.$$ 

Thus, with the above choice of tangential base points, we have

$$\int_{0<x_1<x_3<1; 0<x_2<x_4<1} \omega_1(x_1, x_2) \wedge \omega_0(x_3, x_4)$$

$$= (\beta_2 - \beta_1)^2 \int_{t_1>t_4>0; t_2>t_4>0} f_0(C; t_1, t_2)dt_1 \wedge dt_2 \wedge dt_3 \wedge dt_4$$

$$= (\beta_2 - \beta_1)^2 \zeta_{K; C}(2).$$

Corollary 6  With the above choice of tangential base points, we have

$$(\beta_2 - \beta_1)^3 \zeta_{K; C}(1, 2)$$

$$= \int_{0<x_1<x_3<x_5<1; 0<x_2<x_4<x_6<1} \omega_1(x_1, x_2) \wedge \omega_1(x_3, x_4) \wedge \omega_0(x_5, x_6).$$

Theorem 7  In Corollary 6, the integral on the right hand side is a period of a mixed Tate motive unramified over a real quadratic number ring.

Proof: In this proof we are going to follow closely the paper by Goncharov and Manin [1]. The period will be a pairing between $[\Omega_A] \in Gr^W H^6(\overline{M}_{0,15} - A)$ and $[\Delta_B] \in (Gr^W H^6(\overline{M}_{0,15} - B))^\vee$ associated to a mixed Tate motive $H^b(\overline{M}_{0,15} - A; B - A \cap B)$.

Let the $(4n)$-coordinates $x_{2i-1}, y_{2i-1}, z_{2i-1}, w_{2i-1}$ for indices $i = 1, 2, \ldots, n$, be a coordinate of a point on $M_{0,4n+3}$. One can think of $M_{0,4n+3}$ as $(\mathbb{P}^1)^{4n} - D$ where the divisor $D$ is obtained by setting any of the coordinates to be $0, 1, \infty$ or setting any two of the coordinates to be equal. Let us define

$$x_{2i} = \frac{1}{z_{2i-1}} \text{ and } y_{2i} = \frac{1}{w_{2i-1}}.$$ 

Now the coordinates of any point on $M_{0,4n+3}$ can be written as $(x_1, y_1, x_2, y_2, \ldots, x_{2n}, y_{2n})$. In terms of the new coordinates, we have the following components of $D$: 

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\[
x_i = 0, \quad x_i = 1, \quad x_i = \infty,
\]
\[
y_i = 0, \quad y_i = 1, \quad y_i = \infty,
\]
\[
x_1 = x_3, \quad x_3 = x_5, \quad y_1 = y_3, \quad y_3 = y_5, \quad x_1 x_2 = 1, \quad x_3 x_4 = 1, \quad y_1 y_2 = 1, \quad y_3 y_4 = 1.
\]

The last four components can be realized in terms of the previous coordinates as:
\[
x_1 = z_1, \quad x_3 = z_3, \quad y_1 = w_1, \quad y_3 = w_3.
\]

Let \( n = 3 \). Let \( \overline{M}_{0, 4n+3} = \overline{M}_{0, 15} \) be the Deligne-Mumford compactification of the moduli space of curves of genus 0 with 15 marked points. The ambient space will be \( M_{0, 15} \). From it we will remove a divisor \( A \) whose components occur as poles of the differential forms under the integral. Explicitly, the differential forms are
\[
\omega_1(x_1, x_2) = \frac{d(x_1 x_2)}{1 - x_1 x_2} \wedge \frac{d(y_1 y_2)}{1 - y_1 y_2},
\]
\[
\omega_1(x_3, x_4) = \frac{d(x_3 x_4)}{1 - x_3 x_4} \wedge \frac{d(y_3 y_4)}{1 - y_3 y_4},
\]
\[
\omega_0(x_5, x_6) = \frac{d(x_5 x_6)}{x_5 x_6} \wedge \frac{d(y_5 y_6)}{y_5 y_6}.
\]

The components of the divisor \( A \) consists of the union of:
\[
(x_1 x_2 = 1), \quad (y_1 y_2 = 1), \quad (x_3 x_4 = 1), \quad (y_3 y_4 = 1),
\]
\[
(x_5 = 0), \quad (x_6 = 0), \quad (y_5 = 0), \quad (y_6 = 0),
\]
\[
(x_i = \infty), \quad (y_i = \infty), \quad \text{for } i = 1, 2, \ldots, 6,
\]

together with the exceptional divisors obtained via blow-up at the intersections of two components that both contain the same variable or the same constant 0, 1 or \( \infty \) on the right hand side of the equalities.

Thus, the differential form
\[
\Omega_A = \omega_1(x_1, x_2) \wedge \omega_1(x_3, x_4) \wedge \omega_0(x_5, x_6)
\]
is well-defined on \( \overline{M}_{0, 15} - A \).

Now we proceed to defining \( B \). The key part will be to include the tangential base points in the definition of \( B \).

The components of \( B \) consist of a union of codimension 1 subvarieties and codimension 2 subvarieties. The latter ones correspond to the tangential base points.

The codimension 1 components are the following:
\[
(x_1 = 0), \quad (x_1 = x_3), \quad (x_3 = x_5), \quad (x_5 = 1),
\]
\[
(x_2 = 0), \quad (x_2 = x_4), \quad (x_4 = x_6), \quad (x_6 = 1),
\]
\[
(y_1 = 0), \quad (y_1 = y_3), \quad (y_3 = y_5), \quad (y_5 = 1),
\]
\[
(y_2 = 0), \quad (y_2 = y_4), \quad (y_4 = y_6), \quad (y_6 = 1),
\]

together with the exceptional divisors of the blow-up at an intersection of two subvarieties such that the two polynomials contain the same variable or the same constant
0 or 1 on the right hand side of the equalities, except the following 4 double intersections of components

\((x_1 = 0)\) and \((y_1 = 0)\),
\((x_2 = 0)\) and \((y_2 = 0)\),
\((x_5 = 1)\) and \((y_5 = 1)\),
\((x_6 = 1)\) and \((y_6 = 1)\),

to which we associate a codimension 2 subvarieties of \(\overline{M}_{0,15}\), using the tangential base points.

For the blow-up at the intersection \((x_1 = 0)\) and \((y_1 = 0)\) we choose a divisor \(B_1\) on the exceptional divisor defined by \([x_1 : y_1] = [1 : 0]\). Note that \(B_1\) is of codimension 2 in \(\overline{M}_{0,15}\).

For the blow-up at the intersection \((x_2 = 0)\) and \((y_2 = 0)\) we choose a divisor \(B_2\) on the exceptional divisor defined by \([x_2 : y_2] = [0 : 1]\).

For the blow-up at the intersection \((x_5 = 1)\) and \((y_5 = 1)\) we choose a divisor \(B_5\) on the exceptional divisor defined by \([x_5 : y_5] = [1 : \beta_1]\).

For the blow-up at the intersection \((x_6 = 1)\) and \((y_6 = 1)\) we choose a divisor \(B_6\) on the exceptional divisor defined by \([x_6 : y_6] = [1 : \beta_2]\).

The tangential base points define the components \(B_1, B_2, B_5, B_6\). Thus, \((\beta_2 - \beta_1)^3 \zeta_{K,C}(1,2)\) occurs as a period of \(H^6(\overline{M}_{0,15} - A; B - A \cap B)\) when \([\Omega_A] \in Gr^W_{12} H^6(\overline{M}_{0,15} - A)\) is paired with \([\Delta_B] \in (Gr^W_0 H^6(\overline{M}_{0,15} - B))^\vee\).

Note that \(B_1\) and \(B_2\) are defined over \(\mathbb{Z}\), and \(B_5\) and \(B_6\) are defined over the ring of integers \(\mathcal{O}_K\) of the field \(K\). Each of them is naturally isomorphic to \(\overline{M}_{0,13}\) as a variety over \(\mathcal{O}_K\). Similarly, any intersection of the components of \(B\) is isomorphic over \(\mathcal{O}_K\) to \(\overline{M}_{0,n}\) for some integer \(n\). Using that \(H^i(\overline{M}_{0,n})\) is a mixed Tate motive over \(Spec(\mathcal{O}_K)\), we obtain that the motivic cohomology of the components of \(B\) are mixed Tate motives. Using Proposition 1.7 from Deligne and Goncharov, \([3]\), we conclude that for \(l \neq \text{char}(\nu)\) the \(l\)-adic cohomology of the reduction of \(B_j\) modulo \(\nu\) of the motive \(H^i(B_j)\) is unramified for any component \(B_j\) of \(B\), since \(B_j\) is isomorphic to \(\overline{M}_{0,n}\) over \(Spec(\mathcal{O}_K)\) for some \(n\). We conclude that for \(l \neq \text{char}(\nu)\) the \(l\)-adic cohomology of the reduction modulo any \(\nu \in Spec(\mathcal{O}_K)\) of the motive \(H^6(\overline{M}_{0,15} - A; B - A \cap B)\) is unramified. Thus, \(H^6(\overline{M}_{0,15} - A; B - A \cap B)\) is a mixed Tate motive unramified over \(Spec(\mathcal{O}_K)\).

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