One-loop corrections to the string tension of the vortex in the Abelian Higgs model.

Jürgen Baacke
Fachbereich Physik, Technische Universität Dortmund
D - 44221 Dortmund, Germany

and

Nina Kevlishvili
Dipartimento di Fisica, Universita degli studi di Ferrara
I-44100 Ferrara, Italy
INFN, Sezione di Ferrara, I-44100 Ferrara, Italy
Andronikashvili Institute of Physics, GAS, 0177 Tbilisi, Georgia

Abstract

We present an exact numerical computation of the one-loop correction of the string tension for the Nielsen-Olesen vortex in the Abelian Higgs model. The computations proceed via the computation of the Euclidean Green's function for the gauge, Higgs and Faddeev-Popov fields using mode functions, and taking the appropriate trace. Renormalization is an essential part of this computation, it is done by removing leading order contributions from the numerical results so as to make these finite, and to add the divergent parts back, after suitable regularization and renormalization. We encounter and solve some problems which are specific to gauge theories and topological solutions. The corrections to the energy are found to be sizeable, but still smaller than the classical energy as long as $g^2$ is smaller than unity.

1e-mail: juergen.baacke@tu-dortmund.de
2e-mail: nkevli@fe.infn.it
1 Introduction

Extended classical solutions that can be interpreted as strings exist in various realistic and semi-realistic models of particle physics \[1, 2\]. Their possible rôle in cosmology has been evocated long ago \[3\], see \[4, 5\] for reviews. In confrontation with the recent wealth of cosmological observations they present a very interesting and active, but still controversial field of research, see e.g. \[6, 7\].

In the present investigation we consider the string that is made up by the vortex solution of the Abelian Higgs model in 3 + 1 dimensions, well-known from superconductivity \[8\], and commonly denoted in particle physics as the Nielsen-Olesen vortex \[9\]. The 1 + 1 dimensional version of this solution represents an instanton solution, that has been widely considered in the context of baryon number violation.

There have been various investigations of the one-loop correction to the string tension. The fermionic corrections to the energy of the Nielsen-Olesen vortex have been computed exactly in Ref. \[10\]. Fermionic corrections to strings in 2 + 1 and 3 + 1 dimensions have also been considered by various other authors \[11, 12, 13, 14\]. Such calculations may be important in the context of the instability of the electroweak string \[15\].

Here we consider the energy corrections arising from the gauge-Higgs and Faddeev-Popov sector. There are some previous investigations, using heat kernel techniques \[16, 17\]. Here we attempt an exact computation, using techniques that have been developed previously in Refs. \[18, 19, 20\] and applied in various semiclassical computations, of one-loop energy corrections and one-loop prefactors to transition rates. The computations have three essential ingredients: the use of mode functions (“Jost functions”, see e.g., \[21\]) in order to compute exact results, the use of perturbative subtractions, so as to make these results finite, and the computation of the subtracted parts using a covariant regularization and renormalization scheme. Similar approaches have been used recently by other authors, see e.g. \[22, 23, 24, 25\]. In a gauge theory as considered here one finds some complications due to the fact that there are cancellations of divergences between graphs of a different number of vertices. This will discussed in detail in the context of renormalization.

Whether the energy corrections are small or big depends, in the present case, of the gauge coupling. The classical string tension is proportional to \(\nu^2 = m_W^2/g^2\), where \(g\) is the gauge coupling, while the corrections are pro-
portional to $m_W^2$ multiplied by a function of $\xi = m_H/m_W$. So the corrections are necessarily small relative to the classical string tension if $g$ is sufficiently small. If ever the corrections are big then this signals the breakdown of the semiclassical method. In intermediate situations one may have recourse to a Hartree type approximation by including the back-reaction of the quantum fluctuations to the classical solution. The methods we use here are suitable for such investigations [26, 27], or even for self-consistent calculations without classical solution [28].

The text is organized as follows: In section 2 we present the model, the classical vortex solution and the classical string tension. In section 3 we relate in general the fluctuation operator to the one-loop correction to the string tension, and we explicitly derive the fluctuation operator. Its partial wave reduction is presented in section 4. This is the basis for the actual numerical computation of the one-loop string tension, which is described in section 5. Renormalization is discussed in some detail in section 6. In section 7 we give some details of the numerical implementation and present the results. We conclude with a summary in section 8. Some technical details are discussed in Appendices A-G.

2 Basic relations

The Abelian Higgs model in (3+1) dimensions is defined by the Lagrange density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu \phi)^* D^\mu \phi - \frac{\lambda}{4} (|\phi|^2 - v^2)^2 .$$

(2.1)

Here $\phi$ is a complex scalar field and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,$$

(2.2)

$$D_\mu = \partial_\mu - igA_\mu .$$

(2.3)

The particle spectrum consists of Higgs bosons of mass $m_H^2 = 2\lambda v^2$ and vector bosons of mass $m_W^2 = g^2 v^2$. The model allows for vortex type solutions, representing strings with a magnetic flux, the Nielsen-Olesen vortices [8, 9].
The cylindrically symmetric ansatz for this solution is given by

\[ A_i^{cl\perp}(x, y, z) = \frac{\epsilon_{ij} x_j}{gr^2} A(r) \quad i = 1, 2, \quad (2.4) \]

\[ \phi^{cl}(x, y, z) = v f(r) e^{i \varphi(x)} , \quad (2.5) \]

where \( r = \sqrt{x^2 + y^2} \) and \( \varphi \) is the polar angle. Furthermore \( A_3^{cl} = A_0^{cl} = 0 \). In order to have a purely real Higgs field one performs a gauge transformation

\[ \phi \rightarrow e^{-i \varphi} \phi, \quad (2.6) \]

\[ A_i^{\perp} \rightarrow A_i^{\perp} - \nabla_i^{\perp} \varphi / g \quad (2.7) \]

to obtain the instanton fields in the singular gauge

\[ A_i^{cl\perp}(x, y, z) = \frac{\epsilon_{ij} x_j}{gr^2} [A(r) + 1] \quad i = 1, 2 \quad (2.8) \]

\[ \phi^{cl}(x, y, z) = v f(r) . \quad (2.9) \]

With this ansatz the energy per unit length, or string tension \( \sigma \) takes the form

\[
\sigma_{cl} = \pi v^2 \int_0^\infty dr \left\{ \frac{1}{r m_W^2} \left[ \frac{dA(r)}{dr} \right]^2 + r \left[ \frac{df(r)}{dr} \right]^2 + \frac{f^2(r)}{r} [A(r) + 1]^2 \right. \\
+ \left. \frac{r m_W^2}{4} [f^2(r) - 1]^2 \right\} .
\]

(2.10)

The magnetic flux is given by

\[ \Phi_M = \int d^2 x B_3 = - \int dx \ dy F_{12} . \quad (2.11) \]

Explicitly we find

\[
\Phi_M = \int d^2 x (\nabla_1^{\perp} A_2^{cl\perp} - \nabla_2^{\perp} A_1^{cl\perp}) = \int d\varphi dr \left[ -\frac{1}{gr} A'(r) \right] = \frac{2\pi}{g} [A(0) - A(\infty)] .
\]

(2.12)

For the case \( m_H = m_W \) an exact solution to the variational equation is known \[29\], for which the classical string tension takes the value \( \sigma_{cl} = \pi v^2 \). We here

\[ \text{We use Euclidean notation for the transverse components, so } A_i^+ \equiv A_i = -A_i \text{ etc.} \]
will consider the general case $m_W \neq m_H$, for which the classical equations of motion

\[
\begin{align*}
\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{[A(r) + 1]^2}{r^2} - \frac{m_H^2}{2} [f^2(r) - 1] \right\} f(r) &= 0, \quad (2.13) \\
\left\{ \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{m_W^2}{r^2} f^2(r) \right\} [A(r) + 1] &= 0 \quad (2.14)
\end{align*}
\]

have to be solved numerically.

Imposing the boundary conditions on the profile functions

\[
\begin{align*}
A(r) &\xrightarrow{r \to 0} \text{const} \cdot r^2, \quad A(r) &\xrightarrow{r \to \infty} -1, \\
f(r) &\xrightarrow{r \to 0} \text{const} \cdot r, \quad f(r) &\xrightarrow{r \to \infty} 1, \quad (2.15)
\end{align*}
\]

the magnetic flux is $\Phi_M = 2\pi/g$, the Dirac magnetic flux quantum, and the action is finite.

Since we have to consider fluctuations around these solutions a good numerical accuracy for the profile functions $f(r)$ and $A(r)$ is required. As in previous publications [31, 32] we have the method of Bais and Primack [33].

The values for the classical string tension are given in Table 1.

### 3 Fluctuation operator and one-loop string tension

The fluctuation operator is defined in general form as

\[
\mathcal{M} = \left. \frac{\delta^2 S}{\delta \psi^*_i(x) \delta \psi_j(x')} \right|_{\psi_k = \psi_k^{cl}}, \quad (3.1)
\]

where $\psi_i$ denotes the fluctuating fields and $\psi_i^{cl}$ the “classical” background field configuration; here these will be the vortex and the vacuum configurations. If the fields are expanded around the background configuration as $\psi_i = \psi_i^{cl} + \phi_i$ and if the Lagrange density is expanded accordingly, then the fluctuation operator is related to the second order Lagrange density via

\[
\mathcal{L}^{II} = \frac{1}{2} \phi^*_i \mathcal{M}_{ij} \phi_j, \quad (3.2)
\]
In terms of the fluctuation operators $\mathcal{M}$ on the vortex and $\mathcal{M}^0$ on the vacuum backgrounds, the effective action is defined as

$$S_{\text{eff}} = \frac{i}{2} \ln \left\{ \frac{\det \mathcal{M} + i\epsilon}{\det \mathcal{M}^0 + i\epsilon} \right\}.$$  \hspace{1cm} (3.3)

As the background field is time-independent and also independent of $z$ the fluctuation operators take the form

$$\mathcal{M} = -\partial_0^2 + \partial_3^2 - \mathcal{M}_\perp,$$  \hspace{1cm} (3.4)

where $\mathcal{M}_\perp$ is a positive-definite operator describing the transversal fluctuations. As is well known the logarithm of the determinant can be written as the trace of the logarithm. One can do the trace over $p_0$, the momentum associated with the time variable, by integrating over $T \int dp_0/2\pi$, where $T$ is the lapse of time. One then obtains

$$S_{\text{eff}} = -iT \sum_\alpha \left[ E_\alpha - E_\alpha^{(0)} \right],$$  \hspace{1cm} (3.5)

where $E_\alpha$ are square roots of the eigenvalues of the positive definite operator

$$-\partial_3^2 + \mathcal{M}_\perp,$$  \hspace{1cm} (3.6)

and likewise $E_\alpha^{(0)}$ are those of the analogous operator in the vacuum

$$-\partial_3^2 + \mathcal{M}^0 = -\partial_3^2 - \vec{\nabla}_\perp^2 + m^2.$$  \hspace{1cm} (3.7)

Here $m^2 = \text{diag}(m_1^2, \ldots, m_n^2)$ is the diagonal mass squared operator for the various fluctuations.

So the effective action becomes equal to the difference between the zero point energies of the fluctuations around the vortex and in the vacuum, multiplied by $-T$. We can also do the trace over the variable $p_3$ by integration over $L \int dp_3/2\pi$. We then obtain

$$S_{\text{eff}} = -iT L \sum_\alpha \int \frac{dk_3}{2\pi} \frac{1}{2} \left[ \sqrt{k_3^2 + \mu_\alpha^2} - \sqrt{k_3^2 + \mu_\alpha^{(0)}^2} \right],$$  \hspace{1cm} (3.8)

where $\mu_\alpha^2$ are the eigenvalues of the operator $\mathcal{M}_\perp$ and $\mu_\alpha^{(0)}^2$ those of $-\vec{\nabla}_\perp^2 + m^2$. In the same way the classical action becomes

$$S_{\text{cl}} = -TL\sigma_{\text{cl}}$$  \hspace{1cm} (3.9)
where $\sigma_{cl}$ is the classical string tension. So our goal reduces to computing the one-loop approximation to the string tension given by

$$
\sigma_{1\text{-loop}} = \sigma_{cl} + \sigma_{fl},
$$

where the fluctuation part of the string tension is given by

$$
\sigma_{fl} = \sum_\alpha \int \frac{dk_3}{2\pi} \frac{1}{2} \left[ \sqrt{k_3^2 + \mu_0^2} - \sqrt{k_3^2 + \mu_0^{(0)}^2} \right].
$$

Of course all expressions are formal, the integrals do not exist before a suitable regularization. Anyway we do not plan to compute any eigenvalues of the fluctuation operators but will reduce these expressions to traces over Euclidean Green’s functions, where renormalization will be done properly. However the formal identities will allow us to trace the way in which the counter terms in the original Lagrangean enter the final expressions that are going to be computed numerically.

The fluctuation operator has been derived previously [31], in the context of quantum corrections to the Abelian instanton; we here recall this derivation. The gauge and Higgs fields are expanded as

$$
A^\mu = A^\mu_{cl} + a^\mu, \quad \phi = \phi_{cl} + \varphi.
$$

In the following we will drop the superscript $\text{cl}$, so the letters $A^\mu$ and $\phi$ will denote the background field, and $a^\mu$ and $\varphi$ the quantum fluctuations.

In order to eliminate the gauge degrees of freedom we introduce, as in Ref. [31], the background gauge function

$$
\mathcal{F}(a) = \partial_\mu a^\mu - \frac{ig}{2} (\phi^* \varphi - \phi \varphi^*).
$$

We note that for the background field $\partial^\mu A_\mu = 0$. In the Feynman background gauge we get the gauge-fixing Lagrange density

$$
\mathcal{L}_{GF}^H = - \left( \frac{1}{2} \mathcal{F}^2(a) \right)^H
$$

$$
= - \frac{1}{2} (\partial_\mu a^\mu)^2 - \frac{ig}{2} a^\mu (\varphi \partial_\mu \phi + \bar{\phi} \partial_\mu \varphi - \varphi^* \partial_\mu \bar{\phi} - \bar{\varphi} \partial_\mu \phi^*)
$$

$$
+ \frac{g^2}{8} \phi^2 (\varphi - \varphi^*)^2.
$$
The associated Faddeev-Popov Lagrangean becomes

\[ L_{FP} = \frac{1}{2} \eta^* (-\partial^2 - g^2 \phi^2) \eta. \] (3.16)

In terms of the real components \( \varphi = \varphi_1 + i \varphi_2 \) and \( \eta = (\eta_1 + i \eta_2) \) the second order Lagrange density now becomes

\[
(L + L_{GF} + L_{FP})^{II} = -a_\mu \frac{1}{2} (-\Box + g^2 \phi^2) a^\mu \\
+ \varphi_1 \frac{1}{2} \left[ -\Box + g^2 A_\mu A^\mu - \lambda (3 \phi^2 - v^2) \right] \varphi_1 \\
+ \varphi_2 \frac{1}{2} \left[ -\Box + g^2 A_\mu A^\mu - g^2 \phi^2 - \lambda (\phi^2 - v^2) \right] \varphi_2 \\
+ \varphi_2 (g A_\mu \partial_\mu) \varphi_1 + \varphi_1 (-g A_\mu \partial_\mu) \varphi_2 \\
+ a^\mu (2g^2 A_\mu \phi) \varphi_1 + a^\mu (2g \partial_\mu \phi) \varphi_2 \\
+ \eta_1 \frac{1}{2} (-\Box - g^2 \phi^2) \eta_1 + \eta_2 \frac{1}{2} (-\Box - g^2 \phi^2) \eta_2,
\] (3.17)

where we have omitted the superscript from \( \phi^d \) and \( A^d_\mu \). We now specify the fluctuating fields \( (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \) as \( (a_1^\perp, a_2^\perp, \varphi_1, \varphi_2, \eta_{12}) \),

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 \\
\psi_5
\end{pmatrix} =
\begin{pmatrix}
a_1^\perp \\
a_2^\perp \\
\varphi_1 \\
\varphi_2 \\
\eta_{12}
\end{pmatrix},
\] (3.18)

where we have used Euclidean notation for the transverse gauge field components. We furthermore write (see also above)

\[ M = -\partial_0^2 + \partial_3^2 - M^\perp, \] (3.19)

separating the trivial part from the one that is modified by the background field. With these preliminaries we obtain the following nonvanishing components of the overall fluctuation operator \( M_{ij}^\perp \):
\[
\begin{align*}
\mathcal{M}^{\perp}_{11} &= -\Delta^{\perp} + g^2 \phi^2 \\
\mathcal{M}^{\perp}_{13} &= 2g^2 A^{\perp}_1 \phi \\
\mathcal{M}^{\perp}_{23} &= 2g^2 A^{\perp}_2 \phi \\
\mathcal{M}^{\perp}_{33} &= -\Delta^{\perp} + g^2 A^{\perp}_2 + g^2 \phi^2 + \lambda (\phi^2 - v^2) \\
\mathcal{M}^{\perp}_{44} &= -\Delta^{\perp} + g^2 A^{\perp}_2 + \lambda (3 \phi^2 - v^2) \\
\mathcal{M}^{\perp}_{55} &= -\Delta^{\perp} + g^2 \phi^2 .
\end{align*}
\]

\[\mathcal{M} = \mathcal{M}^0 + \mathcal{V}. \quad (3.21)\]

This potential will be specified below after partial wave decomposition.

## 4 Partial wave decomposition

The fluctuation operator \(\mathcal{M}^{\perp}\) can be decomposed into partial waves with respect to the polar angle \(\varphi\), and the string tension decomposes accordingly. We introduce the following partial wave decomposition for fields

\[
\begin{align*}
\vec{a} &= \sum_{n=-\infty}^{+\infty} b_n(r) \left( \frac{\cos \varphi}{\sin \varphi} \right) \frac{e^{in\varphi}}{\sqrt{2\pi}} + i c_n(r) \left( \frac{-\sin \varphi}{\cos \varphi} \right) \frac{e^{in\varphi}}{\sqrt{2\pi}}, \\
\varphi_1 &= \sum_{n=-\infty}^{+\infty} h_n(r) \frac{e^{in\varphi}}{\sqrt{2\pi}}, \\
\varphi_2 &= \sum_{n=-\infty}^{+\infty} \tilde{h}_n(r) \frac{e^{in\varphi}}{\sqrt{2\pi}}, \\
\eta_{12} &= \sum_{n=-\infty}^{+\infty} g_n(r) \frac{e^{in\varphi}}{\sqrt{2\pi}}.
\end{align*}
\]
After inserting these expressions into the Lagrange density and using the reality conditions for the fields one finds that the following combinations are real relative to each other and make the fluctuation operators symmetric:

\[ F_n^1(r) = \frac{1}{2}(b_n(r) + c_n(r)), \]
\[ F_n^2(r) = \frac{1}{2}(b_n(r) - c_n(r)), \]
\[ F_n^3(r) = \tilde{h}_n(r), \]
\[ F_n^4(r) = i\tilde{h}_n(r), \]
\[ F_n^5(r) = g_n(r). \] (4.2)

Writing the partial fluctuation operators as

\[ M^\perp = M^\perp_0 + V, \] (4.3)
the free operators \( M^\perp_0 \) become diagonal matrices with elements

\[ M^\perp_{0,ii} = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{n_i^2}{r^2} + m_i^2, \] (4.4)

where \((n_i) = (n-1, n+1, n, n, n)\) and \((m_i) = (m_W, m_W, m_W, m_H, m_W)\). The potential \( V \) takes the elements

\[ V^n_{11} = m_W^2 (f^2 - 1) \quad V^n_{12} = 0 \]
\[ V^n_{13} = \sqrt{2}m_W f' \quad V^n_{14} = \sqrt{2}m_W f \frac{A + 1}{r} \]
\[ V^n_{22} = V^n_{11} \quad V^n_{23} = V^n_{13} \]
\[ V^n_{24} = -V^n_{14} \quad V^n_{33} = \frac{(A + 1)^2}{r^2} + \left( \frac{m_H^2}{2} + m_W^2 \right) (f^2 - 1) \]
\[ V^n_{34} = -2A + 1 \quad V^n_{44} = \frac{(A + 1)^2}{r^2} + \frac{3}{2} m_H^2 (f^2 - 1) \]
\[ V^n_{55} = m_W^2 (f^2 - 1) \quad V^n_{55} = 0. \] (4.5)

Choosing the dimensionless variable \( m_W r \) one realizes that the fluctuation operator only depends on the ratio \( m_H/m_W \) up to an overall factor \( m_W^2 \) which cancels in the ratio with the free operator.
5 Computation of the effective string tension

The method for computing the effective string tension used here is based on the Euclidean Green’s function of the fluctuation operator. This Green’s function is defined by

\[
\nu^2 + k_3^2 + \mathcal{M}^\perp \right) \mathcal{G}(\vec{x}_\perp, \vec{x'}_\perp, k_3) = \delta(\vec{x}_\perp - \vec{x'}_\perp).
\]

(5.1)

and similarly for the operator \( \mathcal{M}^0 \). It contains the information on the eigenvalues \( \lambda_\alpha^2 \) of the fluctuation operator \( \mathcal{M}^\perp \) via

\[
\int d^2x \right) \text{Tr} \left( \mathcal{G}(\vec{x}_\perp, \vec{x'}_\perp, k_3) - \mathcal{G}^0(\vec{x}_\perp, \vec{x'}_\perp) \right) = \sum_\alpha \frac{1}{\lambda_\alpha^2 + k_3^2 + \nu^2}.
\]

(5.2)

We define a function \( F(k_3, \nu) \) as

\[
F(k_3, \nu) = \int d^2x \right) \text{Tr} \left( \mathcal{G}(\vec{x}_\perp, \vec{x'}_\perp, k_3, \nu) - \mathcal{G}^0(\vec{x}_\perp, \vec{x'}_\perp, \nu) \right).
\]

(5.3)

We then find

\[
-\int_0^{\infty} \frac{d\nu \nu^2}{2\pi} F(k_3, \nu) = \sum_\alpha \frac{1}{2} \left[ \sqrt{k_3^2 + \lambda_\alpha^2} - \sqrt{k_3^2 + (\lambda_\alpha^0)^2} \right].
\]

(5.4)

This expression is still to be integrated over \( k_3 \) and is by itself already linearly divergent. The sum over eigenvalues becomes an integral \( \int d^2k \perp \) and the difference of the energies behaves asymptotically as \( 1/|k_3| \). So regularization is required. This will be discussed in the next section. Assuming that it has been achieved, we can do the \( \nu \) and \( k_3 \) integrations at once, using the fact that \( F(k_3, \nu) \) only depends on \( p = \sqrt{k_3^2 + \nu^2} \), i.e., \( F(k_3, \nu) = F(0, p) \). We then obtain for the one-loop string tension

\[
\sigma_{\text{fl}} = -\int_0^{\infty} \frac{dpp^3}{4\pi} F(0, p).
\]

(5.5)

After these more formal considerations we will present the way in which we actually compute \( F(k_3, \nu) \). We first use the partial wave decomposition to write

\[
F(k_3, \nu) \equiv F(p) = \sum_{n=-\infty}^{+\infty} F_n(p),
\]

(5.6)
where

\[ F_n(p) = \int drr \text{Tr}(G_n(r, r, p) - G_n^0(r, r, p)), \quad (5.7) \]

and where the partial wave Green functions are defined by

\[ (M_n + p^2)G_n(r, r', p) = \frac{1}{r} \delta(r - r') . \quad (5.8) \]

For \( M_n^0 \) the Green function is simply a diagonal matrix with elements

\[ G_n^{0 \, ii}(r, r', p) = I_{ni}(\kappa_ir)K_{ni}(\kappa_i r) , \quad (5.9) \]

where \( \kappa_i = \sqrt{m_i^2 + p^2} \). For the Green function of the operator \( M_n \) the matrix elements similarly become

\[ G_n^{\alpha j}(r, r', p) = f_{\alpha - i}(r) f_{\alpha + j}(r') \quad (5.10) \]

where the functions \( f_{\alpha \pm \, ni} \) form a fundamental system of linearly independent solutions of (5.8), regular as \( r \to 0 \) for the minus sign and as \( r \to \infty \) for the plus sign. The correct normalization is obtained by imposing the boundary conditions

\[ f_{\alpha - \, ni}(r) \approx \delta_{\alpha i} I_{ni}(\kappa_ir) , \]
\[ f_{\alpha + \, ni}(r) \approx \delta_{\alpha i} K_{ni}(\kappa_ir) , \quad (5.11) \]

as \( r \to \infty \). Actually we have solved numerically the differential equations for the functions \( h_{\alpha \pm \, ni} \) defined by

\[ f_{\alpha \pm \, ni} = B_{\alpha \pm \, ni}(\kappa_ir)(\delta_{\alpha \pm \, i} + h_{\alpha \pm \, ni}(r)) \quad (5.12) \]

where \( B_{\alpha + \, ni} = K_{ni} \) and \( B_{\alpha - \, ni} = I_{ni} \) are the appropriate Bessel functions, and with the boundary conditions \( h_{\alpha \pm \, i} \to 0 \) as \( r \to \infty \). In this way one keeps track of the free contribution \( \propto \delta_{\alpha i} \) and

\[ \text{Tr} \left[ G_n(r, r, \nu) - G_n^0(r, r, \nu) \right] = \left[ h_{\alpha - i}(r) + h_{\alpha + i}(r) + h_{\alpha - i}(r)h_{\alpha + i}(r) \right] I_{ni}(\kappa ir)K_{ni}(\kappa ir) , \quad (5.13) \]

to be inserted into (5.6).
6 Renormalization

Our numerical results are computed in such a way that we first compute the contributions of the subsystems for fixed $n$, then sum over $n$ and finally integrate over $p$. The subtractions necessary to make the summation over $n$ and the integration over $p$ convergent are done in the partial waves. There are two ways of doing this: either one plainly subtracts in the partial waves the first order contributions $\text{Tr} G_n^{(1)} = -\text{Tr} G_0^n V_{ii} G_0^n$ and the second order contributions $\text{Tr} G_n^{(2)} = \text{Tr} G_0^n V_{ij} G_0^n V_{ji} G_0^n$, or one does the subtractions already in the functions $h_{n,i}^{a\pm}$ as described in detail in Refs. [20]. For the Faddeev-Popov sector we have used both methods and the results agree very well. In the gauge-Higgs sector the singularity of the external gauge field at $r = 0$ leads to specific difficulties, to be discussed below. These are easier to handle with the first method, and so we decided to work with the plain subtractions throughout.

The essential problems occurring here are twofold: (i) the external gauge field is not square integrable, due to its singularity at $r = 0$ and (ii) there are cancellations between graphs of different order in the external vertices, the well-known cancellation of the quadratic divergence in the vacuum polarization involves seagull terms with one vertex and graphs with two vertices. Such cancellations occur in higher orders as well, so one has to be careful, some contributions with two external vertices do not have to be subtracted, because their divergences are cancelled in higher order.

In the following we will treat the various contributions of first and second order term by term, comparing the contributions which are subtracted with the corresponding Feynman graphs. The subtracted contributions are then added back in covariant form. In this way we preserve covariance, which would be violated if we would introduce noncovariant cutoffs (e.g. in $p$).

The Feynman rules are formulated in the vacuum sector, with a gauge fixing analogous to the one for the vortex sector: we define the background gauge function

$$\mathcal{F}_v(A) = \partial_\mu a^\mu - \frac{ig}{2} [(v + \varphi_1)^* \varphi - (v + \varphi_1) \varphi^*]$$ (6.1)
and the gauge fixing Lagrangean

\[
L_{\text{GF}}^\mu = -\left(\frac{1}{2}F^2_v(A)\right)^\mu \\
= -\frac{1}{2} (\partial_\mu a^\mu)^2 + ga^\mu (\varphi_2 \partial_\mu \varphi_1 + (v + \varphi_1) \partial_\mu \varphi_2) \\
- \left(\frac{g^2}{2} v^2 + g^2 v \varphi_1 + \frac{g^2}{2} \varphi_1^2 \right) \varphi_2^2.
\] (6.2)

Then the Lagrangean, including the gauge fixing and Faddeev-Popov terms, takes the form

\[
L = -\frac{1}{2} \partial_\mu a_\nu \partial^\mu a^\nu + \frac{1}{2} m_W^2 a_\mu a^\mu \\
+ \frac{1}{2} \partial_\mu \varphi_1 \partial^\mu \varphi_1 - \frac{1}{2} m_H^2 \varphi_1^2 \\
+ \frac{1}{2} \partial_\mu \varphi_2 \partial^\mu \varphi_2 - \frac{1}{2} m_W^2 \varphi_2^2 \\
+ 2 g a^\mu \varphi_2 \partial_\mu \varphi_1 \\
+ g^2 a_\mu a^\mu v \varphi_1 + \frac{1}{2} g^2 a_\mu a^\mu \left(\varphi_1^2 + \varphi_2^2\right) \\
- \frac{\lambda}{4} \left(\varphi_1^4 + \varphi_2^4 + 2 \varphi_1^2 \varphi_2^2\right) - \lambda v \varphi_1^3 - \lambda v \varphi_1 \varphi_2^2 \\
- g^2 v \varphi_1 \varphi_2 - \frac{g^2}{2} \varphi_1^2 \varphi_2^2 \\
+ \sum_{i=1}^2 \eta_i \frac{1}{2} \left(-\Box - g^2 (v + \varphi_1)^2\right) \eta_i.
\] (6.3)

The vertices for the Feynman rules are presented in Figs. 1 and 2.

We denote as Feynman graphs the vacuum graphs with the external sources which are provided by the classical Higgs and gauge fields: \(\varphi_1(x) = v[f(r) - 1]\), \(\varphi_2 = 0\), \(a_\mu = 0\) for \(\mu = 0, 1\) and \(a_i = \epsilon_{ik} x_k (A(r) + 1)/gr^2\) for \(i = 3, 4\). Vacuum graphs with external fields \(\varphi_2\) will not be displayed as they do not contribute.

The vertex graphs originate from the expansion of the Green’s function. Starting with

\[
(-\Delta_2 + M_0) G(x, x') = -V(x) G(x, x') \quad \text{,(6.4)}
\]

\[
(-\Delta_2 + M_0) G^0(x, x') = \delta^2(x - x') \quad \text{,(6.5)}
\]
Figure 1: Vertices for Feynman graphs derived from the Lagrangean \((6.3)\): gauge-Higgs sector. We have not included combinatorial factors for permutations of the external lines.
Figure 2: Vertices for Feynman graphs derived from the Lagrangean (6.3): Faddeev-Popov sector, without combinatorial factors for permutations of external lines.

\[ ig^2 v \]

\[ -\frac{i}{2} g^2 \]

Figure 3: The one-vertex graph. The cross symbolizes the point where the first order Green's function is contracted.

we get the expansion

\[
G(x, x') = G^0(x, x') - \int d^2 x'' G^0(x, x'') V(x'') G^0(x'', x') + ... \quad (6.6)
\]

which we recall mainly in order to keep track of the signs. The vertex graphs are obtained by contracting the Green's function according to Eq. (5.7) in the partial waves, or, after summation, according to Eq. (5.3). The first order and second order graphs are depicted in Figs. 3 and 4, respectively.

6.1 Graphs with one vertex

The graphs with one vertex all come with in a combination analogous to the one displayed in Fig. 5 except for the graphs with the seagull vertex. The external lines \( v \phi_1 \) and \( \phi_1^2 \) combine as \( v^2(f(r) - 1) + v^2(f - 1)^2/2 = v^2(f^2 - 1)/2 \). They then are in correspondence with one of the vertex graphs.
with the external potential $V_{ii}(r)$. In order to illustrate the relation between the Feynman and vertex graphs we consider the graphs with a Higgs field loop. The corresponding vertex graph is the one with the vertex $V_{44}$. We use the subscript 44 for identifying this contribution.

The Feynman rules yield the contribution

$$3i\lambda v^2 \frac{m_H^2}{32\pi^2} (L_\epsilon + 1 - \ln \frac{m_H^2}{\mu^2}) LT \int d^2x \left[ (f(r) - 1) + \frac{1}{2}(f(r) - 1)^2 \right]$$

$$= iLT \frac{3m^2_H}{64\pi^2} (L_\epsilon + 1 - \ln \frac{m_H^2}{\mu^2}) \int d^2x \left[ f^2(r) - 1 \right] \equiv -iLT \Delta \sigma_{fl,44}$$ (6.7)

where $L_\epsilon = 2/\epsilon - \gamma + \ln 4\pi$ and where we have used $\lambda v^2 = m^2_H/2$. The factor $LT$ comes from the trivial integrations over time and along the string axis. The Feynman graph constitutes a correction to $iS$, where $S$ is the action, so as an energy correction it receives a minus sign.

The contribution we subtract is given, in the partial waves, by

$$F_{44}^n = - \int_0^\infty r \, dr \int_0^\infty r' \, dr' V_{44}^H(r) G_{n,0}(r, r', \kappa_H) G_{n,0}(r', r, \kappa_H) .$$ (6.8)
The corresponding vertex graph is of the type shown in Fig. 3. As announced above, we do not include the \((A(r) + 1)^2/r^2\) term and denote the restricted potential by \(V_{44}^H\). We also leave out the index \(n\) as this potential does not depend on it. The \(r'\) integration can be carried out and one finds

\[
F_{n,44} = \left[ \frac{d}{dm_H^2} \right] \int_0^\infty r \, dr V_{44}(r) G_n^0(r, r, \kappa_H) .
\]

This can be summed up to yield

\[
F_{44}(p) = \left[ \frac{d}{dm_H^2} \right] \int_0^\infty r \, dr V_{44}^H(r) G^0(x, x, \kappa_H)
= \frac{1}{2p \, dp} \left[ \frac{d^2 k}{(2\pi)^2} \right] \left[ \frac{k^2}{k^2 + p^2 + m_H^2} \right] \int d^2 x V_{44}^H(r)
\]

(6.9)

This is discussed in some more detail in Appendix B. Integration over \(dpp^3/4\pi\) leads, after integration by parts, to

\[
\Delta \sigma_{fi,44} = - \int_0^\infty \frac{p^3 \, dp}{4\pi} F_{44}(p) = \int_0^\infty \frac{p \, dp \, d^2 k}{4\pi} \left[ \frac{1}{(2\pi)^2} \right] \left[ \frac{1}{k^2 + p^2 + m_H^2} \right] \int d^2 x V_{44}^H(r)
= \frac{1}{2} \int \frac{d\nu dk_3 d^2 k}{(2\pi)^4} \left[ \frac{1}{k^2 + k_3^2 + \nu^2 + m_H^2} \right] \int d^2 x V_{44}^H(r)
\]

(6.10)

If we use dimensional regularization applied to the Euclidean four momentum integration \(d\nu dk_3 d^2 k = d^4 k\) we obtain

\[
\Delta \sigma_{fi,44} = - \frac{m_H^2}{32\pi^2} \left[ L_\epsilon + 1 - \ln \frac{m_H^2}{\mu^2} \right] \int d^2 x V_{44}^H(r)
= -3 \frac{m_H^4}{64\pi^2} \left[ L_\epsilon + 1 - \ln \frac{m_H^2}{\mu^2} \right] \int d^2 x (f^2(r) - 1)
\]

(6.11)

in agreement with the result from the Feynman graph. In this case it was possible to check in detail the relation between the subtracted part, the vertex graph and the Feynman graphs.

Obviously the procedure is analogous for the other graphs with one vertex. We obtain, both by using the Feynman graphs of Fig. 3 as also the vertex graph:

\[
\Delta \sigma_{fi,33} = - \frac{m_H^2}{32\pi^2} \left[ L_\epsilon + 1 - \ln \frac{m_H^2}{\mu^2} \right] \int d^2 x V_{33}^H(r)
= -(m_H^2 + 2m_W^2) \frac{m_H^2}{64\pi^2} \left[ L_\epsilon + 1 - \ln \frac{m_W^2}{\mu^2} \right] \int d^2 x (f^2(r) - 1)
\]

(6.12)
For the contribution of $V_{11}$ and $V_{12}$ in the sector of angular momentum $n$ one notes that they involve the Green's functions $G_{n+1}^0$. In the sum the shifts compensate each other and the result is, both from the Feynman graphs in Fig. 7 as from the vertex graphs,

$$\Delta \sigma_{fl,11+22} = -\frac{m_W^2}{32\pi^2} \left[ L_\epsilon + 1 - \ln \frac{M_W^2}{\mu^2} \right] \int d^2 x \left[ V_{11}(r) + V_{22}(r) \right]$$

$$= -\frac{m_W^4}{16\pi^2} \left[ L_\epsilon + 1 - \ln \frac{m_W^2}{\mu^2} \right] \int d^2 x \left( f^2(r) - 1 \right). \quad (6.14)$$

These are cancelled exactly by the Faddeev-Popov graphs of Fig. 8.
6.2 Graphs with two vertices: $V_{ii} \times V_{ij}$

We have subtracted in the partial waves the second order contributions to $F_n^{(2)}(p)$ of the type

$$F_{n,ij,ji}(p) = \int dr' r' \int dr r \int dr'' r'' V_{ij}(r) G^0_n(r, r', \kappa_j) V_{ji}(r') G^0_n(r'', r, \kappa_i),$$

(6.15)

as represented graphically in Fig. 4. The $r''$ integration can be done, see Appendix C, with the result

$$F_{n,ij,ji}(p) = -\frac{d}{dm_i^2} \int dr' r' \int dr r V^0_n(r) G^0_n(r, r', \kappa_j) V^0_n(r') G^0_n(r', r, \kappa_i).$$

(6.16)

We recall that $\kappa^2_j = p^2 + m_j^2$. The one-to-one relation with Feynman graphs

Figure 9: Feynman graphs corresponding to the vertex graph with $V_{33}H \times V_{44}H$ is most obvious for the diagonal parts where $i = j$, and we will treat these first. For $V_{33}$ and $V_{44}$ we again discard the gauge field parts $V_{33}^g = V_{44}^g = (A(r) + 1)/r^2$, restricting them to the Higgs field contributions $V_{33}^H$ and $V_{44}^H$. This will be justified later. As now $\kappa_i = \kappa_j$ we can apply the derivative with respect to $m_i^2$ to both Green’s functions and compensate this double action by a factor 1/2. Furthermore, we can replace this derivative by the derivative with respect to $dp^2 = 2p dp$, applied to the whole graph. Then

Figure 10: Feynman graphs corresponding to the vertex graph with $V_{33}^H \times V_{33}^H$
\[ -\int \frac{dp}{4\pi^3} V_{n,ii,ii}(p) = \frac{1}{2} \int \frac{p^3 dp}{4\pi} \frac{1}{2p dp} \int dr' r' \int dr \ r V_{ii}^n(r) G_n^0(r, r', \kappa_i) V_{ii}^n(r') G_n^0(r', r, \kappa_i) \]
\[ = -\int \frac{dp}{8\pi} \int dr' r' \int dr \ r V_{ii}^n(r) G_n^0(r, r', \kappa_i) V_{ii}^n(r') G_n^0(r', r, \kappa_i). \tag{6.17} \]

As the potentials under consideration do not depend on \( n \) we may sum over \( n \) to obtain
\[ -\int \frac{dp}{8\pi} \int d^2 x \int d^2 x' V_{ii}(r) G^0 (x, x', \kappa_i) V_{ii}(r') G^0 (x', x, \kappa_i), \tag{6.18} \]
and this is just the conventional Feynman graph with external sources. As in the case with one vertex the integral over \( pdp/2\pi \) can be rewritten as an integral over \( dk_3 d\nu/(2\pi)^2 \), and the Green’s function \( G^0 \) in two dimensions involves the integral over \( dk^2_\perp \), so altogether we have an integration over \( d^4k/(2\pi)^4 \), times a factor 1/4. In momentum space we obtain
\[ \frac{1}{4} \int \frac{d^2q}{(2\pi)^2} \left| \tilde{V}_{ii}(q) \right|^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 + m^2_i][(k + q)^2 + m^2_j]}, \tag{6.19} \]
where \( q = (0, 0, q) \) has only the transversal components.

Figure 11: Feynman graphs corresponding to the vertex graphs with \( V_{11} \times V_{11} \) and \( V_{22} \times V_{22} \)

The Euclidean Feynman integral is logarithmically divergent. In dimensional regularization it is given by
\[ \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{1}{[k^2 + m^2_i][(k + q)^2 + m^2_j]} \]
\[ = \frac{1}{16\pi^2} \left\{ L_\epsilon - \int_0^1 d\omega \ln \frac{\omega(1-\omega)q^2 + \omega m^2_i + (1-\omega)m^2_j}{\mu^2} \right\}. \tag{6.20} \]
This is to be folded with the Fourier transform of \( f^2 - 1 \):

\[
[f^2 - 1]^{-1}(q) = 2\pi \int_0^\infty dr r(f^2(r) - 1)J_0(qr) \equiv \chi_{f^2-1}(q) ,
\]

 Altogether we find

\[
\Delta \sigma_{fl,44,44} = - \frac{9m_H^4}{256\pi^2} \int \frac{d^2q}{(2\pi)^2} \left\{ L_\epsilon - \int_0^1 d\omega \ln \frac{\omega(1-\omega)q^2 + m_H^2}{\mu^2} \right\} |\chi_{f^2-1}(q)|^2 .
\]

The integral over the logarithm is discussed in Appendix C. The Fourier transform of the potential, which actually is a Fourier-Bessel transform, is discussed in Appendix F. The analogous diagonal contributions, displayed in Figs. 10, 11 and 12 are given by

\[
\Delta \sigma_{fl,33,33} = \frac{(m_W^2 + m_{H/2}^2)^2}{64\pi^2} \int \frac{d^2q}{(2\pi)^2} \int_0^1 d\omega \ln \frac{\omega(1-\omega)q^2 + m_W^2}{\mu^2} |\chi_{f^2-1}(q)|^2 ,
\]
\[\Delta \sigma_{11,11} + \Delta \sigma_{22,22} = \frac{m_W^4}{32\pi^2} \int \frac{d^2q}{(2\pi)^2} \int_0^1 d\omega \ln \frac{\omega(1-\omega)q^2 + m_H^2}{\mu^2} |\chi_{f^2-1}(q)|^2, \quad (6.26)\]

and

\[\Delta \sigma_{55,55} = -\frac{m_W^4}{32\pi^2} \int \frac{d^2q}{(2\pi)^2} \int_0^1 d\omega \ln \frac{\omega(1-\omega)q^2 + m_H^2}{\mu^2} |\chi_{f^2-1}(q)|^2, \quad (6.27)\]

respectively. The latter two contributions, those of the transverse gauge field and the Faddeev-Popov fluctuations, cancel each other, as for the first order contributions.

### 6.3 Graphs with two vertices: \(V_{13} \times V_{31}\) and \(V_{23} \times V_{32}\)

The vertex graphs with \(V_{13} \times V_{31}\) and \(V_{23} \times V_{32}\) correspond to the Feynman graph of Fig. 13. The Feynman integral is given by

\[-\frac{ig^2}{8\pi^2}q^2 \left\{ L_\epsilon - \int_0^1 d\omega \ln \frac{-\omega(1-\omega)q^2 + m_W^2}{\mu^2} \right\}. \quad (6.28)\]

![Figure 13: Feynman graph corresponding to the vertex graphs with \(V_{13} \times V_{31}\) and \(V_{23} \times V_{32}\).](image)

The external legs are given by \(v(f - 1)\). Denoting the Fourier transform by

\[[v(f - 1)]^\sim(q) \equiv v\chi_{f-1}(q) = 2\pi v \int_0^\infty dr r(f(r) - 1)J_0(qr) \quad (6.29)\]

we obtain for the Feynman graph including the external legs

\[iLT\frac{m_W^2}{8\pi^2} \int \frac{d^2q}{(2\pi)^2} q^2 |\chi_{f-1}(q)|^2 \left\{ L_\epsilon - \int d\omega \ln \frac{\omega(1-\omega)q^2 + m_W^2}{\mu^2} \right\}. \quad (6.30)\]
The infinite part corresponds to the wave function renormalization of the Higgs field. In the $\overline{MS}$ scheme the finite correction to the string tension is given by

$$\Delta \sigma_{fl,(12)3(12)3} = \frac{m_W^2}{8\pi^2} \int \frac{d^2 q}{(2\pi)^2} q^2 |\chi_{f-1}(q)|^2 \int d\omega \ln \frac{\omega(1-\omega)q^2 + m_W^2}{\mu^2}.$$  

(6.31)

### 6.4 Graphs with two vertices: $V_{14} \times V_{41}$ and $V_{24} \times V_{42}$

These nondiagonal terms correspond to the Feynman graphs of Fig. 14. The external Higgs field legs combine as

![Feynman Graphs](image)

Figure 14: The Feynman graphs corresponding to the vertex graphs with $V_{14} \times V_{41}$ and $V_{24} \times V_{42}$.

$$1 + (f(r) - 1) + (f(r') - 1) + (f(r) - 1)(f(r') - 1) = f(r)f(r').$$  

(6.32)

The Feynman integral is given by

$$-ig_{\mu
u} g^4 4\pi \left\{ L_\epsilon - \int_0^1 d\omega \ln \frac{\omega(1-\omega)q^2 + \omega m_W^2 + (1-\omega)m_H^2}{\mu^2} \right\}. \quad (6.33)$$

The external legs combine to $A_\mu \phi$, which restricts to a two-dimensional vector $A_i^+ v f(r)$. We write the Fourier transform as

$$[A_i^+ v f]_i^* = 2\pi \frac{v}{g} \epsilon_{ij} \hat{q}_j \int_0^\infty dr (A(r) + 1)f(r) J_1(qr) = \frac{v}{g} \epsilon_{ij} \hat{q}_j \chi_{Af}(q).$$  

(6.34)

The Feynman graph with external legs then is given by

$$iLT \frac{m_W^2}{8\pi^2} \int \frac{d^2 q}{(2\pi)^2} |\chi_{Af}(q)|^2 \left\{ L_\epsilon - \int_0^1 d\omega \ln \frac{\omega(1-\omega)q^2 + \omega m_W^2 + (1-\omega)m_H^2}{\mu^2} \right\}$$.  

(6.35)
The infinite term corresponds to the coupling constant renormalization in the term $g^2 |\phi|^2 A_\mu A_\mu / 2$. For the finite correction to the string tension we obtain

$$
\Delta \sigma_{\text{fl}}(12) = \int d^2 q (2\pi)^2 |\chi_{Af}(q)|^2 \int_0^1 d\omega \ln \frac{\omega(1-\omega)q^2}{\mu^2} + \omega m^2_W + (1-\omega)m^2_H.
$$

6.5 Graphs with two vertices: $V_{34} \times V_{43}$

The graphs containing $V_{34} \times V_{43}$ were the hardest stumbling block for this computation. As is well known, the quadratic divergence of the second order Feynman graph of Fig. 15 cancels with the one of the first order seagull diagrams displayed in the same figure. The problem we have here is the fact that the quadratic divergence is proportional to $A_\mu A_\mu$ and this has a nonintegrable singularity at $r = 0$. While the ultraviolet divergence appears only after summation over $n$ and integration over $p$, the radial integrations already appear in the partial waves. On the other hand, all we have to ensure is the fact that the final ultraviolet divergence is a wave function renormalization and does not contain terms proportional to $A_\mu A_\mu$. Furthermore our subtraction must render the total result finite. The solution of this problem was found essentially by trial and error and amounts to subtracting in the partial waves a second order term

$$
\int r dr \int r' dr' V_{34}(r) G_n^0(r, r', \kappa_W) \left[ V_{43}(r') - \frac{r^2}{r'^2} V_{43}(r) \right] \left[ G_n^0(r', r, \kappa_H) \right]^2
$$

$$
= \int r dr \int r' dr' 2n \frac{(A(r) + 1)}{r^2} G_n^0(r, r', \kappa_W)
\times 2n \frac{(A(r') - A(r))}{r'^2} \left[ G_n^0(r', r, \kappa_H) \right]^2.
$$

Figure 15: Vacuum polarization Feynman graphs. They are equivalent to the vertex graphs with $V_{34} \times V_{43}$ and $V_{43} \times V_{34}$, and to the one-vertex graphs with $V_{33}^g$ and $V_{44}^g$, respectively.
and analogously for $V_{43} \times V_{34}$ or $\kappa_W \leftrightarrow \kappa_H$. This contains already the subtraction of first order divergences that would arise from the gauge field parts of $V_{33}$ and $V_{44}$. So the numerical results get finite without subtraction of those first order terms. However, this subtraction does not correspond exactly to the subtraction of the Feynman graphs of Fig. 15; it only takes care in the correct way of the divergent terms. The finite terms will be different. So when we add back the subtracted term after covariant regularization, we have to compute it in the same way as we did the subtraction, not by evaluating the Feynman graphs with the external gauge field legs. This we will describe now.

The Feynman integral corresponding to the second order diagram is given, in our gauge, by

$$g^2 \int \frac{d^4 p}{(2\pi)^4} \frac{4p_\mu p_\nu}{(p^2 - m_H^2 + i\epsilon)((p - q)^2 - m_W^2 + i\epsilon)},$$

where $q_\mu$ is the external momentum. The graph evaluates to

$$i \frac{g^2}{4\pi^2} \int_0^1 d\omega \left[ \frac{2Q^2}{4 - \epsilon} \left\{ L_\epsilon + \frac{1}{2} - \ln \frac{Q^2}{\mu^2} \right\} + q_\mu q_\nu \omega^2 \left\{ L_\epsilon - \ln \frac{Q^2}{\mu^2} \right\} \right],$$

with $Q^2 = \omega m_W^2 + (1 - \omega) m_H^2 - \omega(1 - \omega) q^2$. Our gauge potential is transverse, so in the following we do not have to keep track of the $q_\mu q_\nu$ terms. The external fields as we have used them in the subtraction can be written as

$$\epsilon_{ik}\hat{x}_k\epsilon_{jl}\hat{x}_l \frac{A(r) + 1}{gr} \frac{A(r') - A(r)}{gr'},$$

$$= \epsilon_{ik}\hat{x}_k\epsilon_{jl}\hat{x}_l \left[ \frac{A(r) + 1}{gr} \frac{A(r') - 1}{gr'} - \frac{(A(r) + 1)^2}{gr} \frac{1}{gr'} \right].$$

Taking the Fourier transform and setting $q = q'$ because of momentum conservation we have

$$\frac{1}{g^2} \epsilon_{ik}\hat{q}_k\epsilon_{jl}\hat{q}_l (2\pi)^2 \left[ \int dr J_1(qr)(A(r) + 1) \int dr' J_1(qr')(A(r') + 1) - \int dr J_1(qr)(A(r) + 1)^2 \int dr' J_1(qr') \right].$$

$$- \int dr J_1(qr)(A(r) + 1)^2 \int dr' J_1(qr').$$
Contracting with $g_{\mu\nu}$ we have

$$\frac{-1}{g^2(2\pi)^2} \left[ \int dr J_1(qr)(A(r) + 1) \int dr' J_1(qr')(A(r') + 1) ight. \\ \left. - \int dr J_1(qr)(A(r) + 1)^2 \int dr' J_1(qr') \right] .$$  \hspace{1cm} (6.42)

At first we consider the divergent part of the Feynman graph. It is given by

$$i \frac{g^2}{16\pi^2} g_{\mu\nu} L_\epsilon \left[ \frac{1}{3} q^2 - (m_H^2 + m_W^2) \right] .$$  \hspace{1cm} (6.43)

This is to be multiplied with the external leg factor given above and to be integrated with $d^2q/(2\pi)^2$. The part independent of $q^2$ vanishes upon this integration:

$$\int q dq \left[ \int dr J_1(qr)(A(r) + 1) \int dr' J_1(qr')(A(r') + 1) \\ - \int dr J_1(qr)(A(r) + 1)^2 \int dr' J_1(qr') \right] = \int dr J_1(qr)(A(r) + 1) \int dr' J_1(qr') \frac{1}{r} \delta(r - r') = 0 ,$$  \hspace{1cm} (6.44)

where we have used

$$\int q dq J_n(qr) J_n(qr') = \frac{1}{r} \delta(r - r') .$$  \hspace{1cm} (6.45)

This cancellation implies, as it should, the absence of a divergent term proportional to $A_\mu A^\mu$ in our subtraction. The remaining divergent term is proportional to $q^2$. We note that now $q^2 = -q^2$. Furthermore, $\int dr J_1(qr) = 1/q$, $\int dq^2 J_1(qr) = 0$ and $qJ_1(qr) = -dJ_0(qr)/dr$. Including all factors we find

$$i \frac{L_\epsilon}{16\pi^2} \frac{1}{3} 2\pi^2 \int q^3 dq \left[ \int dr J_1(qr)(A(r) + 1) \int dr' J_1(qr')(A(r') + 1) \\ - \frac{1}{q} \int dr J_1(qr)(A(r) + 1)^2 \right]$$

$$= i \frac{L_\epsilon}{16\pi^2} \frac{1}{3} 2\pi^2 \int q dq \int dr \frac{dJ_0(qr)}{dr} (A(r) + 1) \int dr' \frac{dJ_0(qr')}{dr'} (A(r') + 1)$$

$$= i \frac{L_\epsilon}{16\pi^2} \frac{1}{3} 2\pi^2 \int \frac{dr}{r} [A'(r)]^2 .$$  \hspace{1cm} (6.46)
This is just the kinetic term of the gauge field, multiplied with the wave function renormalization factor \((g^2/3)L_\varepsilon/8\pi^2\). In the \(\overline{MS}\) scheme this term is dropped. We are left with the finite part. Denoting
\[
2\pi \int dr J_1(qr)(A(r) + 1) = \chi_A(q), \quad (6.47)
\]
\[
2\pi \int dr J_1(r)(A(r) + 1)^2 = \chi_{A^2}(q), \quad (6.48)
\]
the finite part is given by
\[
\Delta\sigma_{h,3434} = \frac{1}{8\pi^2} \int \frac{d^2q}{(2\pi)^2} \int_0^1 d\omega Q^2 \left[ \ln \frac{Q^2}{\mu^2} - 1 \right] \left[ \chi_A^2(q) - \frac{2\pi}{q} \chi_{A^2}(q) \right],
\]
with \(Q^2 = \omega m_H^2 + (1 - \omega)m_W^2 + \omega(1 - \omega)q^2\). The integration over \(d\omega\) can be done analytically, we do not display the somewhat lengthy result.

6.6 Graphs with two vertices: gauge field contributions in \(V_{33}\) and \(V_{44}\).

There are some contributions with two vertices which we have left out. Denoting the gauge part of \(V_{33}\) with \(V_{33}^g\) with \(V_{33}^g(r) = (A(r) + 1)^2/r^2\) and analogously for \(V_{44}\) we have left out second order the diagonal parts \(V_{33}^g \times V_{33}^H\) and \(V_{33}^g \times V_{33}^g\), and likewise for the \(44\) (Higgs) channel. These diagrams with seagull type vertices combine with higher order graphs to give finite results. So they do not have to be subtracted. The corresponding Feynman graphs are displayed in Fig. [16] and [17]. If one looks at the Feynman graphs there are graphs with more than three vertices that are superficially divergent. In fact these combine in such a way that their divergences are cancelled. So with the subtractions described in the previous subsections we have done the necessary steps towards computing the finite result. Indeed with these subtractions the numerical results get finite, and we have already presented the finite expressions by which these graphs are to be replaced.

7 Numerical results

We have carried out the numerical programme as described in the previous sections, for values of \(\xi = m_H/m_W\) between 0.5 and 2.
Figure 16: Feynman graphs corresponding to the vertex graph with $V^g_{33} \times V^H_{33}$ and the three-vertex graphs that cancel their logarithmic divergences. The cancellation for the graphs related to $V^g_{44} \times V^H_{44}$ is analogous, with the replacement $\phi_1 \rightarrow \varphi_2$ in the internal lines.

The programme starts with computing a fundamental system of the mode functions $h^{\alpha \pm}_{n,i}(r)$ for the $4 \times 4$ gauge-Higgs and the Faddeev-Popov sector. We have used 2000 grid points in $r$, up to $r_{\text{max}} = 30$, as we did already for the classical profiles $f(r)$ and $A(r)$. These are the basis for the partial wave Green’s functions. This computation is identical to the one performed already in Refs. [31, 32].

We have computed the partial wave Green’s function and the related integrals up to $n = \bar{n} = 35$. Contributions of higher $n$ were included by fitting

Figure 17: Feynman graphs corresponding to the vertex graphs with $V^g_{33} \times V^g_{33}$ and $V^g_{44} \times V^g_{44}$ and the higher order graphs that cancel their logarithmic divergences.
the data between \( \bar{n} - 5 \) and \( \bar{n} \) using power fits \( An^{-3} + Bn^{-4} + Cn^{-5} \), and by appending the sum for \( \bar{n} < n < \infty \) on the basis of these fits. The perturbative subtractions were done in the partial waves, the sum over the unsubtracted and the subtracted partial wave contributions constitute the unsubtracted and subtracted functions \( F(p) \). These are displayed, for \( \xi = 1 \) in Figs. 18 and 19 for the gauge-Higgs and the Faddeev-Popov sector, respectively. All the general features of these figures are similar for other values of \( \xi \). In these figures we also present the perturbative contributions of first and second order, summed separately. These can and have been used for cross-checks against semi-analytical results:

The numerical sum over the first-order subtractions can be checked against the result obtained by summing the partial wave contributions analytically. One finds

\[
\sum_{i=1,4} F_{ii}(p) = - \left[ \frac{3m_W^2 + m_H^2/2}{2\kappa_W^2} + \frac{3m_H^2/2}{2\kappa_H^2} \right] \int r \, dr(f^2(r) - 1) . \tag{7.1}
\]

The Faddeev-Popov contribution, which was computed separately, behaves (including the factor -2) as

\[
F_{55}(p) = \frac{m_W^2}{\kappa_W^2} \int r \, dr(f^2(r) - 1) . \tag{7.2}
\]

The integrals over \( r \), here and below, are computed numerically using the classical profiles.

For the second-order contributions we cannot do the sum over partial waves analytically, this involves the numerical Fourier transforms of various functions of the classical profiles; but we can easily check the asymptotic behaviour. It is given by

\[
\sum_{ij} F_{ijij}(p) = \left[ (m_W^4 + (m_W^2 + m_H^2/2)^2 + 9m_H^4/4) \int r \, dr(f^2(r) - 1)^2 \right. \\
+ 8m_W^2 \int r \, dr f'(r)^2 + 8m_W^2 \int r \, dr f^2(r) \frac{(A(r) + 1)^2}{r^2} \\
+ 4 \int r \, dr \frac{[A'(r)]^2}{r^2} \left. \right] \frac{1}{4p^4} . \tag{7.3}
\]

The first term collects the diagonal contributions that can be derived analytically by summing up the partial waves. The three other contributions are
retrieved from the Feynman graphs by doing the transverse loop momentum integrations only. They collect the 1313 and 2323 contributions, the 1414 and 2424 contributions, and the 3434 contribution, respectively. Of course these can also be computed and verified separately. The asymptotic behaviour of the Faddeev-Popov contribution is given by

\[
F_{5555}(p) = 2 \frac{m_W^4}{4p^4} \int r \, dr \left( f^2(r) - 1 \right)^2.
\]  

(7.4)

These semianalytic results can be and have been checked against the numerical subtractions, thus verifying prefactors and signs.

---

Figure 18: The integrand function \( F(p) \) defined in Eq. (5.6), for the gauge-Higgs sector: circles: the unsubtracted function; dashed line: one vertex contribution; dash-dotted line: two-vertex contribution; squares: subtracted function; straight line at small \( p \): translation mode pole \( 2/p^2 \); solid line at large \( p \): asymptotic fit.
After the subtractions the function $F_{\text{sub}}(p)$ behaves as $p^{-6}$ and the integral over $p^4dp$ can be done in order to obtain the subtracted part of the string tension. Finally we have to add back the subtracted terms in a covariantly regularized and renormalized form given in the previous section.

As we have already mentioned, the integrands for the $p$ integration are displayed in Figs. 18 and 19 for the gauge-Higgs and the Faddeev-Popov sector, respectively. The figures show the unsubtracted functions, the first order and second order contributions and the subtracted functions. Note that the functions differ, at large $p$, by several orders of magnitude, so these subtractions are quite delicate.

The gauge-Higgs sector displays a $2/p^2$ behaviour for small $p$, which is of course related to the 2 translation modes of the two-dimensional solution (the “instanton”). Here it merges into a continuum of transverse string oscillations. The small-$p$ pole remains of course after subtractions and causes
the gauge Higgs sector to be much more important than the Faddeev-Popov sector. Indeed, the finite parts obtained after integration with \(p^3 dp/4\pi\) are much larger for the gauge-Higgs than for the Faddeev-Popov sector. Obviously this important contribution of the translation mode is related to the transversal quantum oscillation of the string.

At large \(p\) the subtracted integrand should behave as \(p^{-6}\). For the Faddeev-Popov sector this is realized in ideal form. For the gauge-Higgs sector there are deviations for \(p > 6\), which remain even if higher partial waves are included. So they seem to be caused by some very small numerical deficiencies in the low partial waves. Their origin is difficult to localize. As these contributions only appear above \(p \simeq 6\) we have done least-square fits of the form \(A/p^6 + B/p^8\) based on the data points between \(p = 2\) and \(p = 6\). These fits then were used in order to append the integrals from \(p = 6\) to \(\infty\). The fits are displayed in Figs. 18 and 19 for the gauge-Higgs and Faddeev-Popov sectors, respectively.

The integrals over the subtracted part of the function \(F(p)\), including the asymptotic tail based on the fits are given in Table 1, they are denoted as \(\Delta\sigma_{\text{sub}}\).

Now that we have computed the subtracted integrals we have to add back the regularized and renormalized divergent contributions. Those with one vertex are given, in unrenormalized form, by

\[
\Delta\sigma^{(1)} = -\frac{m_W^2}{32\pi^2} \left\{ (L_\epsilon + 1 - \ln \frac{m_W^2}{\mu^2})(1 + 2\xi^2) - \frac{3}{2}\xi^2 \ln \xi^2 \right\} \int d^2 x (f^2(r) - 1) .
\]  

(7.5)

One may choose the renormalization such as to omit these tadpole contributions entirely, as one would do in the 1 + 1 dimensional theory, using normal ordering the field operators. Here we use the strict \(\overline{MS}\) scheme and choose the renormalization scale as \(\mu^2 = m_W^2\). Therefore we have to add back

\[
\Delta\sigma^{(1)}_{\text{fin}} = -\frac{m_W^2}{32\pi^2} \left\{ (1 - \ln \frac{m_W^2}{\mu^2})(1 + 2\xi^2) - \frac{3}{2}\xi^2 \ln \xi^2 \right\} \int d^2 x (f^2(r) - 1).
\]  

(7.6)

As we separately present this contribution in Table 1 the reader may easily change this contribution according to her/his own preferences.

The separate results for the graphs with one vertex, \(\Delta\sigma^{(1)}_{\text{fin}}\), for the graphs with two vertices \(\Delta\sigma^{(2)}_{\text{fin}}\), the subtracted contribution \(\Delta\sigma_{\text{sub}}\) and the total one-loop contribution \(\Delta\sigma_{\text{tot}}\) are listed in Table 1 and displayed in Fig. 20.

We should like to recall that the subtracted contribution still includes some
gauge field graphs with two vertices, which by themselves are divergent, but whose divergences are cancelled by higher order vertex graphs, as discussed in subsection 6.6.

Figure 20: The correction to the string tension as a function of $\xi = m_H/m_W$. Squares: contributions with one vertex, $\Delta \sigma^{(1)}$; triangles: contributions with two vertices $\Delta \sigma^{(2)}$; diamonds: subtracted contribution $\Delta \sigma_{\text{sub}}$; circles: total one-loop correction $\Delta \sigma_{\text{tot}}$, asterisks: the classical string tension multiplied by $g^2/\pi$. All string tensions are in units of $m_W^2$.

8 Summary

We have presented here an exact, albeit numerical, computation of the one loop corrections to the string tension of the the Nielsen-Olesen vortex in
the $3 + 1$ dimensional Abelian Higgs model, taking into account the fluctuations of gauge, the Higgs and the Faddeev-Popov fields. One of the main complications arise from the fact, that the divergences of graphs with external gauge field loops require cancellations of Feynman graphs with different numbers of external vertices. This has required an extensive discussion of vertex graphs and vacuum Feynman graphs. On the numerical side, we had to adapt a previously developed computation scheme to this new situation. We had to find and have found a way of implementing the necessary cancellations, which are relatively straightforward when done analytically, into the numerical procedure.

The size of the corrections of course depends on the renormalization scheme and renormalization conditions. We here have adapted the $\overline{MS}$ scheme with $m^2_W$ as the renormalization scale. The corrections are sizeable, but small with respect to the classical string tension as long as the gauge coupling $g$ is smaller than unity, which may be considered as a reasonable assumption. Unlike in the case of the fermionic corrections we do not have at our disposal an extra parameter, the Yukawa coupling, that could render the corrections important for heavy fermions. The parameters are fixed already at the classical level.

Of course, in the present situation of cosmic string phenomenology a precise information on the string tensions and corrections to it cannot be considered as very important. Indeed we consider as our main result that we presented a method for computing such corrections in a gauge theory with all its technical complications. The situation may be different if these corrections are computed at finite temperature, as it may be realistic in primordial cosmology. Such computations, using similar techniques, have been done for bubble nucleation in Ref. [35]. In the high-temperature approximation to the electroweak theory the corrections to the transition rate were found to be huge [36, 37].

A further application of the methods presented here may be the investigation of the rôle of quantum fluctuations for the electroweak string, in particular in the context of its stabilization [38].
Appendix

A Partial wave decomposition of the free Green’s function

We shortly recall the partial wave decomposition of the free Green’s function, a decomposition that is used repeatedly in the subtraction procedure. We here consider the Green’s function in two dimensions, with an effective mass 

\[ \kappa^2 = m^2 + p^2 \]

the one in four dimensions is obtained by replacing \( p^2 = \nu^2 + k_3^2 \) and by further integrations over \( \nu \) and \( k_3 \). We further omit the subscript \( \perp \), replacing \( x_{\perp} \rightarrow x \) and \( k_{\perp} \rightarrow k \). The free Green’s function \( G_0(x, x', \kappa) \) is defined as

\[
G_0(x, x', \kappa) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik \cdot (x - x')}}{k^2 + \kappa^2}.
\]  

(A.1)

It can be computed readily, using formulas (9.1.18) and (11.4.44) of Ref. [39]:

\[
G_0(x, x', \nu) = \frac{1}{4\pi^2} \int_0^\infty dk \int_0^\pi d\phi \frac{2 \cos(kR \cos \phi)}{k^2 + \kappa^2} = \frac{1}{2\pi} J_0(kR)
\]

(A.2)

with \( R = |x - x'| \). Furthermore this may be expanded, using Eq. (4) in section 7.6.1 of Ref. [40] as

\[
G_0(x, x', \nu) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} K_n(\kappa R_>) I_n(\kappa R_<) e^{i\nu \phi}.
\]  

(A.3)
\section*{B \ First order contributions to the partial waves}

Using the formula for the free Green’s function derived in Appendix \ref{app:free_green} the first order contribution to the Green’s function in two dimensions is given by

\begin{align}
G^{(1)}(\mathbf{x}, \mathbf{x}', p) &= -\int d^2x'' G_0(\mathbf{x}, \mathbf{x}'', \kappa) V(r'')G_0(\mathbf{x}'', \mathbf{x}', \kappa) \\
&= -\frac{1}{4\pi^2} \int dr''r''d\phi'' \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} K_n(\kappa r_>) I_n(\kappa r_<) e^{i n(\phi-\phi'')} \\
&\quad \times V(r'') K_{n'}(\kappa r_>') I_{n'}(\kappa r_<') e^{i n'(\phi''-\phi')} \quad (B.1)
\end{align}

where \( r_> = \max\{r, r''\} \), \( r_>' = \max\{r', r''\} \) and similarly for \( r_< \) and \( r_<' \). The “potential” \( V(r) \) subsumes all vertex contributions appearing in first order.

We define

\[ G^{(1)}(\mathbf{x}, \mathbf{x}', p) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{i n(\phi-\phi')} G^{(1)}_n(r, r', p) \quad (B.2) \]

and therefore get

\[ G^{(1)}_n(r, r, p) = -\int dr''r'' K_n^2(\kappa r_>) I_n^2(\kappa r_<) V(r'') . \quad (B.3) \]

Using the integrals of Appendix \ref{app:integrals} this can be integrated over \( r \) \( dr \) with the result

\[
\int_0^\infty dr \ r \ G^{(1)}_n(r, r, p) = -\int_0^\infty dr' r' V(r') \\
\times \frac{z}{2\kappa^2} \left\{-2n \left[I_{n+1}(z)I_n(z)K_n^2(z) + K_{n+1}(z)K_n(z)I_{n}^2(z)\right] \right. \\
\left. + z \left[K_{n+1}^2(z)I_n^2(z) - I_{n+1}^2(z)K_n^2(z)\right]\right\} , \quad (B.4)
\]

with \( z = \kappa r' \). Using the Wronskian relation

\[ I_n(z)K_{n+1}(z) + I_{n+1}(z)K_n(z) = \frac{1}{z} \quad (B.5) \]
this can be rewritten as
\[
\int_0^\infty dr \ r \ G_n^{(1)}(r,r,p) = - \int_0^\infty dr' V(r') \frac{1}{2\kappa^2} \{ -2nI_n(z)K_n(z) \\
+ z [K_{n+1}(z)I_n(z) - I_{n+1}(z)K_n(z)] \}
\tag{B.6}
\]
\[
= - \int_0^\infty dr' r' V(r') \frac{1}{2\kappa} \frac{d}{d\kappa} I_n(z)K_n(z) .
\]

This is the contribution of one partial wave to the sum over partial waves, which is then to be integrated with respect to \(\nu\) and \(k_3\) to yield the string tension in first order. However, the sum over partial waves and the integrals are divergent; the perturbative contribution has to be subtracted from the full, nonperturbative contribution of each partial wave in order to obtain convergent summations and integrations. We may check that formally the first order perturbative contributions can be summed up and integrated so as to reobtain the Feynman integral. The sum over partial waves of this expression yields

\[
- \sum_{n=-\infty}^{\infty} \frac{1}{p} \frac{d}{dp} \int_0^\infty dr' r' V(r') I_n(z)K_n(z)
\]
\[
= -2\pi \frac{1}{2p} \frac{d}{dp} \int_0^\infty dr' r' V(r')G_0(x,\kappa)
\]
\[
= -\frac{1}{2p} \frac{d}{dp} \int d^2xV(|x|) \int \frac{d^2k}{(2\pi)^2} \frac{1}{k_3^2 + m^2 + p^2} . \tag{B.7}
\]

Going back to section 5 we see that we have integrate this with \(- \int p^3 dp/4\pi\). We obtain

\[
\sigma^{(1)} = -\frac{1}{2} \int d^2xV(|x|) \int \frac{p^3 d^2k}{(2\pi)^3} \frac{1}{2p d\kappa k_3^2} \frac{1}{k_3^2 + p^2 + m^2}
\]
\[
= -\frac{1}{2} \int d^2xV(|x|) \int \frac{p d^2k}{(2\pi)^3} \frac{1}{k_3^2 + p^2 + m^2}
\]
\[
= -\frac{1}{2} \int d^2xV(|x|) \int \frac{d\nu d^3k}{(2\pi)^4} \frac{1}{k^2 + \nu^2 + m^2} , \tag{B.8}
\]

where in the last step we have substituted \(p^2 = k_3^2 + \nu^2\). The result is indeed the first order vacuum graph.
C Second order contribution to the partial waves

Using similar steps as in Appendix B and using again the integrals in Appendix D and Wronskian relations, we find that the second order contribution to the partial waves is given by

\[ \int dr G^{(2)}_n(r, r, p) = \sum_{ij} \int dr' r' V_{ij}^n(r') \int dr'' r'' V_{ji}^n(r'') I_{ni}(\kappa_i r_r(<)) K_{ni}(\kappa_i r_r(>)) \times \left( -\frac{1}{2\kappa_j} \frac{d}{d\kappa_j} \right) I_{nj}(\kappa_j r_r(<)) K_{nj}(\kappa_j r_r(>)) , \]  

where \( r_r(=) \min(r', r'') \) and \( r_r(>) = \max(r', r'') \). The computation is quite lengthy, as we start with an expression where the values \( r, r' \) and \( r'' \) appear in six different orderings. Here we have written all relevant indices, as the two propagators have, in general, different masses and angular momenta. We note that the differentiation with respect to \( \kappa_j = \sqrt{p^2 + m_j^2} \) can be written as a differentiation with respect to \( p \) is the combinations \( V_{ij} V_{ji} \) and \( V_{ji} V_{ij} \) are combined. If applied to the sum over \( i \) and \( j \), this results in a double counting that has to be compensated by a factor \( 1/2 \). So

\[ \int dr r G^{(2)}_n(r, r, p) = \frac{1}{2} \left( -\frac{1}{2p} \frac{d}{dp} \right) \sum_{ij} \int dr' r' V_{ij}^n(r') \int dr'' r'' V_{ji}^n(r'') \times I_{ni}(\kappa_i r_r(<)) K_{ni}(\kappa_i r_r(>) I_{nj}(\kappa_j r_r(<)) K_{nj}(\kappa_j r_r(>)) . \]  

D Some integrals

In our calculations we repeatedly need some integrals over Bessel functions which we write down here, without derivation. They can be obtained from Eq. (11.3.29) of Ref. [39] by taking the limit \( k \to l \) (see there). The formulas
\[
\int_0^r dr r I_n^2(\kappa r) = \frac{z}{2\kappa^2} \left\{ -2n I_{n+1}(z) I_n(z) + z \left[ I_n^2(z) - I_{n+1}^2(z) \right] \right\} , \quad (D.1)
\]
\[
\int_r^\infty dr r K_n^2(\kappa r) = \frac{z}{2\kappa^2} \left\{ -2n K_{n+1}(z) K_n(z) - z \left[ K_n^2(z) - K_{n+1}^2(z) \right] \right\} , (D.2)
\]
\[
\int r dr I_n(\kappa r) K_n(\kappa r) = \frac{z}{2\kappa^2} \left\{ n \left[ K_{n+1}(z) I_n(z) - K_n(z) I_{n+1}(z) \right] + z \left[ I_n(z) K_n(z) + I_{n+1}(z) K_{n+1}(z) \right] \right\} , \quad (D.3)
\]
with \( z = \kappa r \). Note the limits of integration, the third integral is an indefinite one.

### E Some useful sums

We define the two-dimensional Green’s function \( G_2^{(0)}(x, x', \nu^2) \) as the solution of the equation
\[
[-\Delta^2 + m^2 + \nu^2] G_2^{(0)}(0) = \delta^2(x - x') . \quad (E.1)
\]
We readily obtain
\[
G_2^{(0)}(x, x', \nu^2) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{ik(x-x')}}{k^2 + m^2 + \nu^2} . \quad (E.2)
\]
The integral may be done explicitly with the result
\[
G_2^{(0)}(x, x', \nu^2) = \frac{1}{2\pi} K_0(\kappa R) , \quad (E.3)
\]
with
\[
R = |x - x'| = \sqrt{r^2 + r'^2 - 2rr' \cos(\varphi - \varphi')} . \quad (E.4)
\]
Furthermore the Gegenbauer expansion of the modified Bessel function yields
\[
G_2^{(0)}(x, x', \nu^2) = \frac{1}{2\pi} \sum_{n=-\infty}^\infty e^{in(\varphi - \varphi')} I_n(\kappa r_<) K_n(\kappa r_>) . \quad (E.5)
\]
The limit \( x \to x' \) of the Green’s function does not exist. However, what we need is
\[
- \frac{1}{2\kappa} \frac{d}{dk} \sum_{n=-\infty}^\infty I_n(\kappa r) K_n(\kappa r) . \quad (E.6)
\]
We note that
\[- \frac{1}{2\kappa} \frac{d}{d\kappa} K_0(\kappa R) = \frac{R}{2\kappa} K_1(\kappa R). \tag{E.7}\]
The limit $R \to 0$ exists and one obtains
\[- \frac{1}{2\kappa} \frac{d}{d\kappa} \sum_{n=-\infty}^{\infty} I_n(\kappa r) K_n(\kappa r) = \frac{1}{2\kappa^2}. \tag{E.8}\]

\section*{F Fourier-Bessel transforms}

For the evaluation of the perturbative contributions we need the Fourier-transform of the external sources. Let us consider at first a scalar field $\Phi(r_\perp)$, independent of Euclidean time $\tau$ and $z$. In our application $\phi$ will be $f(r_\perp) - 1, f^2(r_\perp) - 1$ etc.. We have

\[
\tilde{\Phi}(q_0, q_z, q_\perp) = \int d^4 x e^{-ip \cdot x} \Phi(r_\perp)
\]

\[
= (2\pi)^2 \delta(p_0) \delta(p_z) \int_0^\infty dr \int_{-\pi}^{\pi} d\varphi \Phi(r) e^{-iq_\perp r \cos \varphi} \tag{F.1}
\]

\[
= (2\pi)^3 \delta(p_0) \delta(p_z) \int_0^\infty rdr \Phi(r) J_0(q_\perp r). \tag{F.2}
\]

In the following we will omit the trivial factor $(2\pi)^2 \delta(p_0) \delta(p_z)$ originating from the $\tau$ and $z$ integrations and write

\[
\tilde{\Phi}(q) = \chi_{\Phi}(q) = 2\pi \int_0^\infty rdr \Phi(r) J_0(qr). \tag{F.3}
\]

implying $q \sim q_\perp$. The relation between the Fourier-Bessel transformation and the inverse transformation implies the relation

\[
\int qdq J_l(qr) J_l(qr') = \frac{1}{r} \delta(r - r') \tag{F.4}
\]

for arbitrary $l$. We deduce the Parseval equation

\[
\int \frac{d^2q}{(2\pi)^2} |\chi_{\Phi}(q)|^2 = \int \frac{dq}{2\pi} |\chi_{\Phi}(q)|^2 = 2\pi \int_0^\infty rdr |\Phi(r)|^2 = \int d^2x |\Phi(r)|^2. \tag{F.5}
\]
which can be used as a numerical cross check. If one disregards $V_{34} = V_{13}$ and the gauge field parts of $V_{33}$ and $V_{44}$ these Fourier-Bessel transforms are unproblematic. We shortly discuss the various components.

In the diagonal we have the Higgs field parts which are proportional to $f^2(r) - 1$. This function goes to zero exponentially as $r \to \infty$, so the Fourier transform exists. It is found to decrease exponentially at large $q$.

Considering the contribution of $V_{13} = V_{23} = \sqrt{2}m_W f'$ the function $f'(r)$ decays exponentially as $r \to \infty$ and goes to a constant as $r \to 0$. So $\tilde{V}_{13}(q)$ exists. It is found to decrease as $1/q^2$.

The function $V_{14}(r) = \sqrt{2}m_W f(A + 1)/r$ goes to a constant as $r \to 0$ and decreases exponentially as $r \to \infty$. So again the Fourier transform exists, and again it decreases as $1/q^2$.

All the Fourier transforms discussed up to now have to be folded with the Feynman integral kernels (see Appendix G) which behave as $\ln q^2$ at large $q$. One easily convinces oneself, that the integrals over the kernels times the squared Fourier transforms are well convergent.

The Fourier transform of the gauge field

$$A_i^\perp = \epsilon_{ij} \hat{x}_j \frac{A(r) + 1}{r}$$

is problematic. The function is not square integrable, due to the singularity at $r = 0$ its norm diverges logarithmically. So the Fourier transform is not square integrable either. Nevertheless it can be computed. The Fourier transform must be of the form

$$\tilde{A}_i^\perp(q) = \epsilon_{ij} \hat{q}_j \chi_A(q).$$

We have

$$\chi_A(q) = \epsilon_{ij} \hat{q}_j \tilde{A}_i^\perp(q)$$

$$= \epsilon_{ij} \epsilon_{ik} \int d^2 x \frac{A(r) + 1}{r} e^{-i\vec{q} \cdot \vec{x}}$$

$$= \int_0^\infty r \, dr \frac{A(r) + 1}{r} e^{-iqr} \int_{-\pi}^{\pi} e^{-iqr \cos \varphi} \cos \varphi$$

$$= 2\pi \int_0^\infty r \, dr \frac{A(r) + 1}{r} J_1(qr).$$

The function $\chi_A(q)$ behaves as $\beta q$ as $q \to 0$ with

$$\beta = \pi \int_0^\infty r \, dr (A(r) + 1),$$

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and like \(2\pi/q\) as \(r \to \infty\). This asymptotic behaviour makes the square norm of the Fourier transform logarithmically divergent, as a reflection of the logarithmic divergence of the norm of the field \(A_1^+(r)\). The handling of the graphs with two external gauge field legs is discussed in some detail in subsection 6.5.

**G The second order kernels**

The finite parts of the second order graphs all contain an integral

\[
I(m_a^2, m_b^2, q^2) = \int_0^1 d\omega \log[\omega(1 - \omega)q^2 + \omega m_a^2 + (1 - \omega)m_b^2]. \tag{G.1}
\]

We generally assume \(m_a^2 > 0, m_b^2 > 0\). If furthermore \(q^2 > 0\) then we define

\[
\omega_\pm = \frac{1 + m_a^2 - m_b^2}{2q^2} \pm \sqrt{\frac{(1 + m_a^2 - m_b^2)^2 + 4m_b^2q^2}{4q^4}}. \tag{G.2}
\]

One easily realizes that \(\omega_+ > 1\) and \(\omega_- < 0\). With this definition \(I\) is given by

\[
I(m_a^2, m_b^2, q^2) = \omega_+ \ln \omega_+ + \omega_- \ln(-\omega_-) + (1 - \omega_+) \ln(\omega_+ - 1) + (1 - \omega_-) \ln(1 - \omega_-) + \ln q^2 - 2. \tag{G.3}
\]

This expression is symmetric in \(m_a^2\) and \(m_b^2\).

If \(q^2 = 0\) the integral is given, for \(m_a^2 \neq m_b^2\) by

\[
I(m_a^2, m_b^2, 0) = \frac{m_a^2 \ln m_a^2 - m_b^2 \ln m_b^2}{m_a^2 - m_b^2} - 1. \tag{G.4}
\]

Finally, if \(m_a^2 = m_b^2\) we have

\[
I(m_a^2, m_a^2, 0) = \ln m_a^2. \tag{G.5}
\]

For small \(q^2\) the integral behaves as \(c_1 + c_2q^2\) where the constant \(c_1\) is of course identical to \(I^2(m_a^2, m_b^2, 0)\). For large \(q^2\) the integral behaves as \(\ln q^2 + \text{const.} + O(1/q^2)\).

The correction to the gauge boson propagator, the term \(\Delta\sigma_{\ell,3434}\) contains an integral

\[
\mathcal{J} = \int_0^1 d\omega Q^2 \left[ \ln \frac{Q^2}{\mu^2} - 1 \right], \tag{G.6}
\]
with $Q^2 = \omega m_H^2 + (1 - \omega) m_W^2 + \omega(1 - \omega) q^2$. Using the same conventions for $\omega_-$ and $\omega_+$ with $m_a = m_w, m_b = m_H$. It is given by

$$J = q^2 \left[ (9\omega_- + 9\omega_+ - 18\omega_-\omega_+ - 6) \ln(-\omega_-\omega_+ + \omega_+ + \omega_- - 1) \right. \right.$$ \right.
$$+ (9\omega_- + 9\omega_+ - 18\omega_-\omega_+ - 6) \ln q^2$$
$$+ 3\omega_-^2 (3\omega_+ - \omega_-) \ln \frac{1 - \omega_-}{-\omega_-} +$$
$$+ 3\omega_+^2 (3\omega_- - \omega_+) \ln \frac{\omega_+ - 1}{\omega_+}$$
$$- 3\omega_-^2 + 18\omega_-\omega_+ - 6\omega_- - 3\omega_+^2 - 6\omega_+ + 4 \right] / 18.$$
Table 1: The correction to the string tension as a function of $\xi = m_H/m_W$. We present the finite part of the contributions with one vertex, $\Delta\sigma_{\text{fin}}^{(1)}$, the finite contributions from graphs with two vertices $\Delta\sigma_{\text{fin}}^{(2)}$, the sum of higher order terms $\Delta\sigma_{\text{sub}}$, and the total one-loop correction $\Delta\sigma_{\text{tot}}$. We also include the classical string tensions. All entries are in units of $m_W^2$.

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