On the Gauss map of finite geometric type surfaces.

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Abstract

Surfaces of finite geometric type are complete, immersed into the tree-dimensional Euclidean space with finite total curvature and Gauss map extending to an oriented compact surface as a smooth branched covering map over the unit sphere of the Euclidean three dimensional space. In a recent preprint J. Jorge and F. Mercuri gave a geometric proof that the Gauss map can not omit three or more points if the immersion is minimal and no flat. Here we give a topological proof of this result in the class of no flat finite geometric type surfaces and also give a topological classification when the Gauss map is a regular covering map. This facts are easy applications of our main result, a generalization of the little Picard theorem for the class of branched covering of a finite geometric type surface into the unit sphere of the tree dimensional Euclidean space. A finite geometric type surface given by a compact surface minus a finite set of points has the following property: any branched covering from the surface to the unit Euclidean sphere having a $C^0$ extension to the compact surfaces can miss at most 2 points. This is a generalization of the little Picard theorem to the class of finite geometric type surfaces.

1 Introduction

Finite geometric type surfaces was introduced in [1] as those immersions $\varphi: M \to \mathbb{R}^3$ of a surface $M$ such that $M$ is complete in the induced metric and

1. $M$ is diffeomorphic to a compact oriented surface $\overline{M}$ minus a finite set of points, $E_m = \{w_1, \ldots, w_m\}$,

2. the Gaussian curvature vanishes only at a finite number of points,

3. the Gauss map $G$ extends to a smooth branched covering, denoted by the same symbol, $G: \overline{M} \to S^2$.

The points $w_i$, or sometimes punctured neighborhoods of these points are called the ends on $\varphi$. The authors in [1] proved that the cardinality of $S^2 \setminus G(M)$ is at most 3 like the minimal case proved by Osserman [8]. In Jorge and Mercuri preprint [4] the authors shows that in the subclass of minimal immersions into $\mathbb{R}^3$ of finite geometric type the number of missing points of the Gauss map
is at most 2 and this number is sharp. In the paper of L. Rodríguez [9] the author consider the classification of minimal immersions of finite geometric type that are embedded. He answer affirmatively a particular case of the following questions:

(Q1) Is the catenoid the only embedded one?
(Q2) If the Gauss map of such surface omits two or more points, then must it be a covering of the catenoid?
(Q3) If the Gaussian curvature is always strictly negative then must the surface be catenoid?

He proves the following result.

**Theorem 1.1 ([9]).** If the surface is minimal, embedded with Gaussian curvature strictly negative and of finite geometric type then it is the catenoid.

We will consider in this notes finite geometric type surfaces whose Gauss map omit two or more points and is a covering map, that is, the Gaussian curvature has no zeros. Before we go further let’s introduce the examples of [7] and [2]. In [7] the authors present examples of minimal immersions of \( S^2 \setminus E \) whose image of Gauss map is exactly \( S^2 / \{a, b\} \), \( a, b \in S^2 \) and of tori \( T^2 \setminus E_3 \) where the Gauss map image is \( S^2 \setminus \{a\} \), \( a \in S^2 \). In [2] there are examples of families \( M \setminus E \) into \( \mathbb{R}^3 \) with the Gauss map omitting a par of antipodal points of \( S^2 \) for any genus. By other side we can bend a minimal immersion of finite geometric type preserving the cardinality of the set of omitted points by the Gauss map. If \( M \) is a finite geometric type surface with non empty set \( Y = S^2 \setminus G(M) \), we define

\[
Y_\varepsilon = \bigcup_{y \in Y} B^{S^2}_\varepsilon(y), \quad 0 < \varepsilon < \min \{\text{dist}^{S^2}(y_1, y_2) \mid y_1, y_2 \in Y, y_1 \neq y_2\}.
\] (1)

If \( \sharp Y = 1 \) take \( Y_\varepsilon \) as one small ball inside the image of neighborhood of ends that are graphic over an unbounded annulus. In section (2) we prove that it is possible to bend \( M \) to get new immersion \( \tilde{M} \) satisfying the following.

**Theorem 1.** Let \( M \) be a finite geometric type surface with set \( Y = \{y_j \mid 1 \leq j \leq 3\} \) of omitted points by the Gauss map \( G \) of \( M \). Then there is \( \varepsilon_0 > 0 \) such that for each \( 0 < \varepsilon \leq \varepsilon_0 \), then \( \tilde{Y} = \{\tilde{y}_j\} \) with \( \tilde{y}_j \) in the connected component of \( Y_\varepsilon \), exist one finite geometric type surface \( \tilde{M} \) with Gauss map \( \tilde{G} \) omitting exactly \( \tilde{Y} \). The surface \( \tilde{M} \) is one bent of \( M \).

Therefore it is possible to give only topological classification of finite geometric type surfaces. For example, assuming that the Gauss map of a finite geometric type surface \( M \) is a covering map, or equivalently, the Gaussian curvature has no zero on \( M \) our conclusion is the following.

**Theorem 2.** Let \( M \) be an non flat finite geometric type surface with the Gauss map \( G \) a covering map. Then

1. There is no example if \( S^2 \setminus G(M) \) has tree points,
2. If \( S^2 \setminus G(M) \) has two points then \( M \) is a covering map of \( S^2 \setminus \{a, b\} \), \( a \neq b \).
3. If \( S^2 \setminus G(M) \) has one point then \( M \) is diffeomorphic to \( \mathbb{C} \).
The items 2 and 3 are manipulations of Riemann-Hurwitz (RH) and total curvature (TC) formulas. The proof of item 1 is included in the next result. In fact we give a topological prove of the same result of [4] but for the class of all no flat finite geometric type immersions. The idea follows like this. First we prove that the cardinality \( \sharp(S^2 \setminus F(M)) \leq 3 \) for any branched covering \( F: M \to S^2 \) having a \( C^0 \) extension to a branched map \( F: \overline{M} \to S^2 \) (see theorem [24]). Given a set \( Y \subset S^2 \), \( \sharp Y = 3 \), it is possible to find a branched covering map \( f: S^2 \to S^2 \) of degree 4 such that \( f^{-1}(Y) = X \), \( \sharp X = 6 \), and for each \( y \in Y \) the set \( f^{-1}(y) \) has two points, one with order of branching 0 and the other with order 2 and the induced map \( f_* \) between the homotopy groups \( \pi_1(S^2 \setminus X) \) to \( \pi_1(S^2 \setminus Y) \) is over (see lemma [22]). Further, the map \( f: S^2 \setminus X \to S^2 \setminus Y \) is a regular covering map. Then if \( M \) is a finite geometric type surface and \( F: M \to S^2 \) is a branched covering with \( Y = S^2 \setminus F(M) \) and \( \sharp Y = 3 \) we can do a lifting of \( F \) by \( f \) to a new map \( F: M \to S^2 \setminus X \) having continuous extension to \( \overline{M} \) where \( X = S^2 \setminus F(M) \) and \( \sharp X = 6 \), getting a contradiction with theorem [24]. It is important to light that the cardinality \( \sharp \) is true only for \( Y \) with \( \sharp Y \geq 3 \).

In fact this generalize the little Picard theorem in the case the entire function is rational for a new class of surfaces and for branched covering (see §3 for details). We show that any branched covering map \( F: M \to S^2 \) with continuous extension to \( \overline{M} \) can not miss more then 2 points unless it is constant, without use of the conformal structure (\( F \) do not need to be holomorphic nor \( M \) to be Riemann surfaces). A particular case is that the Gauss map of one finite geometric type surfaces can not miss 3 or more points unless it is constant, generalizing the theorem of [3]. We get

**Theorem 3 (Generalization of little Picard theorem).** Let \( M = \overline{M} \setminus E_m \) be a finite geometric type surface and \( F: M \to S^2 \) be a non constant branched covering with finite fiber that has a \( C^0 \) extension to a branched covering \( F: \overline{M} \to S^2 \). Then \( \sharp(S^2 \setminus F(M)) \leq 2 \). In particular if \( G \) is the Gauss map of \( M \) and \( M \) not flat then \( \sharp(S^2 \setminus G(M)) \leq 2 \).

In [7] the authors ask who are the embedded minimal surfaces of finite geometric type whose Gauss map misses two points (question Q1). The same question appear in Conjecture 17.0.33, item 3 of [6]. In [6] we present examples of Gauss maps that have not continuous extensions. Those examples should have at least one end \( w \in E \) with \( G(w) \) a missed point of \( G|M \) and order \( \beta(w) \) of branch such that \( 1 + \beta(w) \) is not divided by 3 (see §6).

### 2 Finite Geometric Type.

We will recall some facts about the behavior of an immersion of finite geometric type near the ends. Since the Gauss map is defined at a point \( w \in E \), we have a tangent plane at \( w \), namely \( G(w)^\perp \). It follows from the above properties that the image of the immersion of a small punctured neighborhood of \( w \) projects (orthogonally) onto the complement of a disk in \( G(w)^\perp \) as a finite covering map of order \( I(w) \). The number \( I(w) \) is called the geometric index of \( w \), see [3].

Since the branching points of \( G \), i.e. the points of zero Gaussian curvature and, possibly, the ends, are isolated, a punctured neighborhood of such a point \( v \) is mapped onto its image, as a covering map of order \( \nu(v) \). The number
\( \beta(v) = \nu(v) - 1 \) is called the \textit{branching number} at \( v \). Observe that if \( v \) is not a branching point then \( \beta(v) = 0 \). We have the following topological relations:

**RH (Riemann-Hurwitz):** \( 2 \deg(G) = \chi(M) + \sum_{w \in M} \beta(w) \),

**TC (Total curvature):** \( 2 \deg(G) = -\chi(M) + \sharp E + \sum_{i=1}^{m} I(w_i) \),

where \( \chi(M) \) is the Euler characteristic of \( M \), \( I(w_i) \) is the geometric index of \( w_i \in E \) and \( \deg(G) \) is the degree of the Gauss map \( G \) (the cardinality of \( G^{-1}(y) \) for almost all \( y \) in the image).

The first relation is a well known fact in covering space theory. The second one was obtained, as an inequality, for the case of minimal surfaces, by Osserman [8] and in the above form, by Jorge and Meeks [3] (see also [1]).

The two relations RH and TC seems not to be enough to prove the sharp bound \( \sharp Y \leq 2 \), but are enough to assert theorem 3 in the class of finite geometric type surfaces. We will use the following notations for one immersion \( f \) of finite geometric type:

\[
E^\infty = G^{-1}(Y), \quad E_0 = E \setminus E^\infty, \quad \ell_0 = \sharp G(E) \setminus Y, \ell = \sharp Y, \\
\ell n = -\deg(G), \quad \beta(W) = \sum W \beta(w), \quad I(W) = \sum W I(w).
\]

Since \( G: M \rightarrow \mathbb{S}^2 \) is a branch covering we have

\[
\ell n = \sharp E^\infty + \beta(E^\infty)
\]

Combining RH and TC we obtain the following theorem (see also [1]):

**Theorem 2.1.** If \( f: M \rightarrow \mathbb{R}^3 \) is a surface of finite geometric type, then

\[
0 \leq \ell \leq 3
\]

If \( \ell = 3 \) then \( \chi(M) \leq 0 \). In addition if \( \ell = 3 \) and \( \chi(M) = 0 \) then \( E = E^\infty \), \( \beta(M) = 0 \), \( \beta(E) = 2n \), \( n = \sharp E = I(E) \) and all ends are embedded.

**Proof.** Adding (RH) and (TC) we get

\[
4n = \ell n + \sharp E_0 + \beta(E_0 \cup M) + I(E)
\]

Substituting equation (2) we get

\[
4n = \ell n + \sharp E_0 + \beta(E_0 \cup M) + I(E)
\]

Hence

\[
(4 - \ell)n = \sharp E_0 + \beta(M \cup E_0) + I(E) > 0.
\]

proving that \( 0 \leq \ell \leq 3 \). Now assume \( \ell = 3 \). The (RH) and \( \beta(E^\infty) + \sharp E^\infty = 3n \) gives \( \sharp E^\infty = \chi(M) + n + \beta(E_0 \cup M) \). Using the (TC) we get

\[
2n = -\chi(M) + \sharp E + I(E) = n + \sharp E_0 + \beta(E_0 \cup M) + I(E) = 2n + \chi(M) + 2\sharp E_0 + 2\beta(E_0 \cup M) + I_0
\]
where
\[ I_0 = \sum_I (I(w) - 1) \geq 0 \]

Then
\[ \chi(M) + 2\varepsilon E_0 + 2\beta(E_0 \cup M) + I_0 = 0 \]  
(8)
giving \( \chi(M) \leq 0 \) and if \( \chi(M) = 0 \) then \( E_0 = \emptyset \) and \( \beta(M) = 0 \) proving the result.

\textbf{Lemma 2.2.} Given three points \( a_1 = \infty, a_2 = 0, x_1, x_1 \notin \{a_1, a_2\} \) of \( S^2 = \mathbb{C} \cup \{\infty\} \), the points \( x_2 = -3x_1, y_1 = -x_1, y_2 = 3x_1, w = 16x_1^3 \) and the rational function
\[ f(z) = \frac{(z-x_1)^3(z-x_2)}{z}, \quad z \in S^2 \]
then
\[ \deg f = 4, \quad \beta_f(a_j) = \beta_f(x_j) = \beta_f(y_j) = \begin{cases} 2, & j = 1, \\ 0, & j = 2 \end{cases} \]  
(10)
In particular if \( X = \{0, \infty, x_1, x_2, y_1, y_2\} \) then
\[ f: S^2 \setminus X \to S^2 \setminus \{0, w, \infty\} \]  
(11)
is a regular covering map of degree 4 and \( \sharp X = 6 \). Fixed appropriated basics points we have that the map \( f_* \) induced by \( f \) between the fundamental groups \( \pi_1(S^2 \setminus X) \) and \( \pi_1(S^2 \setminus Y) \) is over.

Reciprocally, given a branched covering \( h: S^2 \to S^2 \) and subsets \( Y, X = h^{-1}(Y) \), such that
\begin{enumerate}
\item The set \( B_h \) of branching points of \( h \) is a subset of \( X \),
\item \( h_*(\pi_1(S^2 \setminus X)) = \pi_1(S^2 \setminus Y) \)
\end{enumerate}
then \( \sharp Y = 3 + m, m \geq 0 \) an integer, \( \sharp X = 6 + 4m, \beta_h(x) = 2 \) for all \( x \in B_h \), \( \deg h = 4 \), and \( \sharp B_h = 3 \). We have splits \( X = X_0 \cup X_1, \quad Y = Y_0 \cup Y_1 \) with \( \sharp X_0 = 6, \sharp Y_0 = 3, \sharp X_1 = \sharp Y_1 = m, \) and \( B_h \subset X_0 \).

Further, if \( \psi \) is the Möbius transform such that \( \psi(Y_0) = \{0, w, \infty\} \), and \( X_f = f^{-1}(\psi(Y)) \), where \( f \) is defined by the choose \( x_1 = (w/16)^{1/3} \), then there is a diffeomorphism preserving fiber \( \phi: S^2 \setminus X_f \to S^2 \setminus X \) such that \( \psi \circ h \circ \phi = f \).

\textbf{Proof.} The function \( f(z) = (z-x_1)^2(z-x_2)/z, \quad z \in S^2, \) satisfy the conditions for \( z \in \{a_j, x_j\} \) \( j = 1, 2 \), \( \deg f = 4 \) and \( f(y_1) = f(y_2) = w \). Since
\[ f'(z) = 3(\frac{(z-x_1)^2(z-y_1)^2}{z^2}, \]
it follows that \( f \) fulfill all conditions of the lemma. Since \( \sum_{z \in S^2} \beta_f(z) = 6 \) there is no more branch for \( f \) unless \( \infty, x_1, y_1 \) proving that \( f: S^2 \setminus X \to S^2 \setminus \{0, w, \infty\} \) is a regular covering. If \( \gamma \) is a small circle around \( x \in X \) then \( f(\gamma) \) is a small closed curve around \( y = f(x) \in \{0, w, \infty\} \) giving \( 1 + \beta_f(x) \) loops up to orientation. Hence the induced map \( f_* \) between the first fundamental groups \( \pi_1(S^2 \setminus X) \) and \( \pi_1(S^2 \setminus \{0, w, \infty\}) \) satisfies
\[ f_*([\gamma_x]) = [\delta_y]^{1+\beta_f(x)}, \quad x \in X, \quad y \in \{0, w, \infty\}, \]
where $\gamma_x, \delta_y$ are generators. Since each point $y \in \{0, w, \infty\}$ is image of a point $x \in X$ with $\beta_f(x) = 0$ we get that all generators of $\pi_1(S^2 \setminus \{0, w, \infty\})$ belongs to $f_* (\pi_1(S^2 \setminus X))$, that is,

$$\pi_1(S^2 \setminus \{0, w, \infty\}) \subset f_* (\pi_1(S^2 \setminus X)),$$

completing one way of proof.

Assume now the existence of a branched covering $h : S^2 \to S^2$ and sets $Y, X = h^{-1}(Y')$ with the branchings of $h$ into the set $X$ such that $h_* (\pi_1(S^2 \setminus X)) = \pi_1(S^2 \setminus Y)$. Set $Y_0 = h(B_h), X_0 = h^{-1}(Y_0)$, and the splits $Y = Y_0 \cup Y_1$, $X = X_0 \cup X_1$. Hence $h_*$ send generator into generator implying that $X_0$ has a subset $X'_0$ with $\sharp X'_0 = \sharp Y_0$ having zero branching points. Then $X_0 = X'_0 \cup B_h$ and $\sharp X_0 = \sharp X'_0 + \sharp B_h$. We have

$$\sharp Y_0 \deg(h) = \sharp X_0 + \beta(B_h) = \sharp Y_0 + \sharp B_h + \beta(B_h).$$

Subtracting the $(R - H)$ we get

$$\left(\sharp Y_0 - 2\right) \left(\deg(h) - 1\right) = \sharp B_h + \beta(B_h) > 0. \quad (12)$$

implying that $\sharp Y_0 \geq 3$ and $\deg(h) > 1$. Choose a Möbius transform $\psi_0$ such that $\psi_0(Y_0) = \{0, w_0, \infty\}$ and $W = \psi_0(Y) = \{0, w_0, \infty\} \cup Y'$. Let $f : S^2 \to S^2$ be the map of the first part of the lemma defined by $x_1$, $x_2 = w_0/16$. If $X_f = f^{-1}(W)$ the map $f : S^2 \setminus X_f \to S^2 \setminus W$ is a regular covering and $f_*$ is over.

Then we have two lifting $\tilde{h} : S^2 \setminus X \to S^2 \setminus X_f$ of $\psi_0 \circ h$ by $f$ and $\tilde{f} : S^2 \setminus X_f \to S^2 \setminus X$ of $f$ by $\psi_0 \circ h$. Since $h = f \circ \tilde{h}$ and $f = h \circ \tilde{f}$ we get $\deg \tilde{h} \cdot \deg \tilde{f} = 1$ implying that both are diffeomorphism and $\deg h = \deg f = 4$, $\sharp X = \sharp X_f$. Then

$$\sharp f^{-1}(\{0, w_0, \infty\}) = 3 \deg f - \beta_f(f^{-1}(\{0, w_0, \infty\})) = 3 \times 4 - 6 = 6.$$

and

$$\sharp X_f = \sharp f^{-1}(\{0, w_0, \infty\} \cup Y') = f^{-1}(Y') + \sharp f^{-1}(\{0, w_0, \infty\}), \quad (13)$$

implying that

$$6 + 4 \sharp Y' = \sharp X_f = \sharp X. \quad (14)$$

Let $D_1, D_2, D_3$ be 3 disks without the center and $f : \overline{D_1} \to \overline{D_2}$ and $F : \overline{D_3} \to \overline{D_2}$ branched covering maps with one branching at the center of each disk of order $\beta_f$ and $\beta_F$. We consider $f$ and $F$ of class $C^1$, $l \geq 2$, inside $D_j$ and continuous in $\overline{D_j}$. The first homotopy group $\pi_1(D_j)$ is an infinite cyclic group with generators $\gamma_j$. The existence of a continuous lift $\tilde{F} : D_3 \to D_1$ of $F$ by $f$ is equivalent to

$$F_* (\pi_1(D_3)) \subset f_* (\pi_1(D_1)),$$

where the subindex means the induced group homomorphism between the fundamental groups. Since $f_* [\gamma_1] = [\gamma_2]^{1+\beta_f}$ and $F_* [\gamma_3] = [\gamma_2]^{1+\beta_F}$ the existence of $\tilde{F}$ is equivalent to have

$$f_* [\gamma_1]^k = F_* [\gamma_3], \quad k \in \mathbb{Z}, \quad k \geq 1,$$

or yet $1 + \beta_F = k(1 + \beta_f), \quad k \geq 1$. 

\[\square\]
Lemma 2.3. Let $D_1$, $D_2$, and $D_3$ be 3 disks without the center and $f: \overline{D}_1 \to \overline{D}_2$ and $F: \overline{D}_3 \to \overline{D}_2$ branched covering maps with one branching at the center of each disk of order $\beta_f$ and $\beta_F$. We consider $f$ and $F$ of class $C^1$, $l \geq 2$, inside $D_j$ and continuous in $\overline{D}_j$. Then there exist a continuous lifting $\tilde{F}: \overline{D}_3 \to \overline{D}_1$ iff

$$\frac{1 + \beta_F}{1 + \beta_f} = k = 1 + \beta_F, \quad k \in \mathbb{Z}, \quad k \geq 1.$$  \hspace{1cm} (15)

In that case there exist a $k$-root $\zeta_j: D_3 \to \tilde{D}_1$ of $\tilde{F}$ such that $\zeta_j^k = \tilde{F}$ and $\tilde{F}$ has the same differentiability of $f$ and $F$.

Proof. We can assume that  

$$F(z) = z^{1+\beta_F}, \quad z \in D_3,$$

and $f(w) = w^{1+\beta_f}Q(w)$, where $Q(w) \neq 0$, $w \in D_1$. The existence of $\tilde{F}$ is equivalent to $1 + \beta_F = k(1 + \beta_f)$. Then the degree of $\tilde{F}$ is $k$ implying on the existence of a $k$-root $\zeta_j$. This shows that $\tilde{F}$ is smooth once $\zeta_j^{1+\beta_F} = (zQ_1(z))^{1+\beta_F}$ and $\zeta_j \circ \tau = zQ_1(z)$ where $\tau$ is one deck transform. Then $\tilde{F}$ has the same differentiability of $f$.

The next result is similar to theorem 2.1 but is a weak version of the little Picard’s theorem in the case of rational functions. It means that we can change the Gauss map with any other branched covering $F: M \to S^2$ and the same conclusion about the number of missing points holds. It is consequence that (RH) and (TC) formulas are true for $F$.

Definition 2.1. Let $F: M \to S^2$ be a $C^2$ branched covering that has $C^0$ extension to $F: \overline{M} \to S^2$. Let $Z \subset \overline{M}$ be the subset where this extension is made. If $Z \neq \emptyset$ we define the order of the branching $\beta_F(z)$, $z \in Z$, by

$$\beta_F(z) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\partial D_\epsilon} k_g - 1$$

where $D_\epsilon \subset \overline{M}$ are small neighborhoods of the end $z \in Z \subset E_m$ endowed with the metric of $S^2$ by $F$ and the geodesic curvature with respect to this metric.

Theorem 2.4. Let $M = \overline{M} \setminus E_m$ be a finite geometric type surface. Let $F: M \to S^2$ be an at least $C^2$ branched covering map having $C^0$ extension to a branched covering map denoted by $F: \overline{M} \to S^2$. Then the (T-C) and (R-H) formulas holds

$$2\deg F = -\chi(M) + \sharp E_m + I(E_m), \hspace{1cm} (16)$$

$$2\deg F = \chi(M) + \beta_F(M). \hspace{1cm} (17)$$

In particular

$$\sharp Y \leq 3$$

for $Y = S^2 \setminus F(M)$. 

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Proof. Set $Z = F(E_m)$, where $E_m$ is the set of ends of $M$. Consider $W = F^{-1}(Z)$. Over $\overline{M} \setminus W$ take one appropriated $\epsilon > 0$ such that:

1. All the balls $B_{\epsilon}^2(z)$, $z \in Z$, are disjoints
2. The collection $U_\epsilon = F^{-1}(B_{\epsilon}^2(z) \setminus \{z\})$, $F(w) = z \in Z$, $w \in W$, are disjoints topological annulus isolating the points of $W$ and $\text{diam}(U_w) \to 0$ if $\epsilon \to 0$ where $U_w$ are the connected component of $U_\epsilon$.

Set $V_\epsilon = \{B_{\epsilon}^2(z) \mid z \in Z\}$. Observe that $\overline{M} \setminus W \subset M$ and $F|_{M \setminus W}$ is a regular covering map. If we endowed $S^2 \setminus Z$ with the metric of $M$ we get

$$2 \deg(F) - \text{area}(V_\epsilon) = -\frac{1}{2\pi} \int_{M \setminus U_\epsilon} K dM$$

$$= -\left(\chi(\overline{M}) - \sharp W\right) + \frac{1}{2\pi} \int_{\partial(M \setminus U_\epsilon)} k_g,$$

$$= -\chi(\overline{M}) + \sharp W - \frac{1}{2\pi} \int_{\partial U_\epsilon} k_g,$$

(18)

(19)

(20)

where $K$ is the Gaussian curvature of $M$ and $k_g$ is the geodesic curvature of $\partial U_\epsilon$. Hence

$$\frac{1}{2\pi} \int_{\partial U_\epsilon} k_g = \sum_{w \in W} \frac{1}{\pi} \int_{\partial U_w} k_g.$$  

(21)

If in some disk $D_\epsilon = B_{\epsilon}^2(0)$ we have a metric $\lambda|dz| = |z|^s \nu(z)|dz|$, where $\nu(z)$ has no zeros, a straightforward calculation gives

$$\lim_{\epsilon \to 0} \int_{\partial D_\epsilon} k_g = 2\pi(1 + s).$$  

(22)

For isothermal parameter $\psi: D_\epsilon \to M \subset \overline{M}$ with $\psi(0) = w \in W$ and $w \in E_m$ we get $s = -1 - I(w)$ where $I(w)$ is the geometric index of the end. Hence

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} k_g = -2\pi I(w),$$  

(23)

If $w \in W \cap M$ then the metric has $s = 0$ and

$$\lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\partial U_w} k_g = 1.$$  

(24)

Making $\epsilon \to 0$ we see that the TC formula holds for $F$, or

$$2 \deg F = -\chi(\overline{M}) + \sharp W - \sharp(M \cap W) + I(E_m),$$  

(25)

$$= -\chi(\overline{M}) + \sharp E_m + I(E_m).$$  

(26)

As we did before, if $Y = S^2 \setminus F(M)$ then $F^{-1}(Y) \subset E_m$ once that $F$ extends to the branching and ends. Then

$$\sharp Y \deg(F) = \sharp E_m^\infty + \beta_F(E_m^\infty), \quad E_m^\infty = E_m \cap F^{-1}(Y).$$  

(27)
In similar way the map \( F: M \setminus W' \to \mathbb{S}^2 \setminus F(E_m \cup B) \), \( B \) the set of branching, is an immersion and we can do the pull back of the metric of \( \mathbb{S}^2 \) by \( F \). Following the proof of (16) we get

\[
2 \deg(F) - \text{area}(V_e) = \frac{1}{2\pi} \int_{M \setminus U_e} KdM \tag{28}
\]

\[
= \chi(M) - 2W - \frac{1}{2\pi} \int_{\partial(M \setminus U_e)} k_g, \tag{29}
\]

\[
= \chi(M) - 2W + \frac{1}{2\pi} \int_{\partial U_e} k_g, \tag{30}
\]

where \( K \equiv 1 \) is the Gaussian curvature and \( k_g \) is the geodesic curvature in the metric \( \lambda(F)[dF][dz] \) and \( \lambda(w)[dw] \) is the metric of the sphere. The equations (21) and (22) holds and

\[
s = \beta_F(x), \quad x \in B. \tag{31}
\]

Hence

\[
2 \deg F = \chi(M) + \beta_F(M). \tag{32}
\]

The equation \( 2W \deg F = 2F^{-1}(W) + \beta_F(V) \) is true if \( W \cap F(E_m) = 0 \). Since this number is one integer and constant over \( \mathbb{S}^2 \setminus F(Z) \) it is the same over all \( W \). Adding the TC and RH formulas for \( F \) we get

\[
(4 - 2Y) \deg(F) = 2(E_m \setminus E^\infty_m) + \beta_F(M \setminus E^\infty_m) + I(E_m) > 0, \tag{33}
\]

implying that \( 2Y \leq 3 \) as we wish.

**Remark 2.1.** Let \( M = \overline{M} \setminus E_m \) a finite geometric type surface and \( G: M \to \mathbb{S}^2 \) a branched covering map with \( C^0 \) extension to \( \overline{M} \to \mathbb{S}^2 \) and not empty set of missing points \( \mathbb{S}^2 \setminus G(M) \). Assume there are a regular covering map \( h : \mathbb{S}^2 \setminus X \to \mathbb{S}^2 \setminus Y \) such that

\[
h_\ast(\pi_1(\mathbb{S}^2 \setminus X)) = \pi_1(\mathbb{S}^2 \setminus Y), \quad \mathbb{S}^2 \setminus G(M) \subset Y. \tag{34}
\]

Let \( F : M \to \mathbb{S}^2 \setminus X \) the lifting of \( G \) by \( h \).

**Question 1.** When the map \( F \) can be extended continuously to \( \overline{M} \)?

For all point \( x \in h^{-1}(y) \) where \( \beta_h(x) = 0 \) the map \( F \) can be extended to one point in the fiber \( G^{-1}(y) \) smoothly. For \( y \in Y \) and \( x \in h^{-1}(y) \) with \( \beta_h(x) > 0 \) we need the condition given by lemma (2.3). If it do not happen there is no \( C^0 \) extension.

### 2.1 Proof of theorem 3.

Let \( M = \overline{M} \setminus E_m \) be a finite geometric type surface and \( G: M \to \mathbb{S}^2 \) be a non constant branched covering with finite fiber that has a \( C^0 \) extension to a branched covering \( G: \overline{M} \to \mathbb{S}^2 \). Set \( Y = (\mathbb{S}^2 \setminus G(M)) \) and suppose that \( 2Y = 3 \). Hence it is possible to find \( h: \mathbb{S}^2 \setminus X \to \mathbb{S}^2 \setminus Y \) satisfying (33) and proving the existence of the lifting \( F: M \to \mathbb{S}^2 \setminus X \) of \( G \) by \( h \) and \( F \) has continuous extension to \( \overline{M} \). Since \( X = \mathbb{S}^2 \setminus F(M) \) and \( 2X = 6 \) we get a contradiction with theorem (2.4) proving the theorem 3.
3 The proof of theorem 1.

The space finite geometric type surfaces is closed for an operation of bending the ends. Take $f : M \to \mathbb{R}^3$ in FGT class with Gauss map $G$ and set of ends $E_k = \{w_1, \ldots, w_k\}$. By [3] we can choose $R_0 > 0$ such that $f$ satisfies

(i) $f^{-1}(\mathbb{R}^3 \setminus (B_R^2(0) \times \mathbb{R})) = \cup_j V_{jR}$, where $V_{jR}$ is a neighborhood of each end $w_j \in E$, $G(w_j) = y_j \in Y$, in $\overline{M}$, for all $R \geq R_0$ and all $i$,

(ii) $F_i = P \circ f : (V_{jR} \setminus \{w_j\}) \to \mathbb{R}^2 \setminus B_R^2(0)$ is a covering with fibre’s cardinality the geometric index $I(w_j)$, for all $R \geq R_0$ and for all $w_j \in E$ and all $i$.

Choose $\delta_0 > 0$ such that

$$B_{\delta_0}^2(y_j) \subset G(V_{jR0}), \quad y_j = G(w_j), \quad w_j \in E,$$

and all balls disjoint. There is $R > R_0$ such that

$$G(V_{jR}) \subset B_{\delta_0}^2(y_j), \quad w_j \in E. \quad (35)$$

Fix some $y_j \in G(E)$ and take $Z = \cup V_{jR}'$ where $\{w_j'\} = V_{jR}' \cap E$ satisfies $G(w_j') = y_j$. Now we are in position to bend the set $Z$ to get a new $C^3$ immersion $\tilde{f}$ that just move $y_j$ to a point $y$ close to $y_j$. Choose some $\delta_1 > 0$ such that

$$\sin(\delta_1) < \text{dist}_{C^2}(y_j, f(\partial Z))/4$$

and $y \in B_{\delta_1}^2(y_j)$. Take a $C^\infty$ function $\psi : \mathbb{R}^3 \to \mathbb{R}$, $0 \leq \psi \leq 1$, with $\psi \equiv 0$ for $|x| \leq 2R$ and $\psi \equiv 1$ for $|x| \geq 3R$. Define $h : \mathbb{R}^3 \to \mathbb{R}^3$ by $h(x) = (1 - \psi(x))x + \psi(x)A_4x$ where $\delta$ is the angle between $y_j$ and $y$ and $A_4$ is the rotation of angle $\delta > 0$ moving $y_j$ to $y$. Let $\tilde{f} : M \to \mathbb{R}^3$ be defined by $\tilde{f}(x) = f(x)$ if $x \in M \setminus Z$ and $\tilde{f}(x) = h(f(x))$ for $x \in Z$. Since $f$ is proper the set $K = f^{-1}(B_R^2(0))$ is compact in $M$. If we take $\delta_1$ small enough with no points with zero Gaussian curvature in the gluing area. Hence we have the following result.

**Proposition 3.1.** Let $f$ a finite geometric type surface with projection $F = P \circ f$ and let $\tilde{f}$ be the bending constructed above. Let $G$ and $\tilde{G}$ be the respective Gauss maps. Then

1. $G(x) = \tilde{G}(x)$ for all $x \notin Z$ and $\tilde{G}(Z) \subset B_{\delta_1}^2(y_j)$,
2. $\tilde{G}(w_j') = y$ for all $w_j' \in G^{-1}(y_j)$ for any chose of $y \in B_{\delta_1}^2(y_j)$,
3. if $G$ misses $y_j$ then $\tilde{G}$ misses $y$,
4. $G(w) = \tilde{G}(w)$ for all $w \in E \setminus G^{-1}(y_j)$,
5. the number of missing points of $G$ and $\tilde{G}$ are the same,
6. the geometric index of $f$ and $\tilde{f}$ are the same at each end,
7. the order of branching points of $G$ and $\tilde{G}$ are the same.

The above results proves Theorem 1.
4 Gauss map not branched: proof of theorem 2, items 2 and 3.

Let $M = \overline{M} \setminus E_m$ a finite geometric type surface with $\overline{M}$ of genus $\mu$. We can represent $\overline{M}$ as one regular rectangle of $R_\mu$ of $4\mu$ sides with boundary ordered by the relation $a_i b_i a_i^{-1} b_i^{-1}$, $1 \leq i \leq \mu$. The vertexes are identified and the sides $a_i$ and $b_i$ defining curves $\alpha_i$ and $\gamma_i$, witch are generators of the fundamental group $\pi_1(\overline{M})$. When we take $M = \overline{M} \setminus E_m$ we add more $m - 1$ generators to the first fundamental group. From now on we assume that the Gauss map

$$G: M \to S^2 \setminus Y, \quad Y = S^2 \setminus G(M)$$

is a regular covering map, that is, there are no branching. In particular we have $E^\infty_m = E_m$, $E^0_m = B_0 = \emptyset$, and $B^\infty \subset E_m$. The R-H formula gives

$$n(2 - 2Y) = \chi(M) = \chi(\overline{M}) - 2E_m, \quad (36)$$

or

$$\chi(\overline{M}) = 2E_m + n(2 - 2Y). \quad (37)$$

If $2Y = 1$ then $2E_m + n = 2$ and $M = S^2 \setminus \{\infty\}$ proving item 3 of theorem 2. If $2Y = 2$ we get $2E_m = 2$ proving item 2 of theorem 2.

5 New proof of a Lopes-Ros theorem.

In [5] the authors proved the following result.

**Theorem 5.1 (Lopes-Ros, [5]).** Let $M$ be a non flat complete minimal surface with finite total curvature and embedded. If $M = S^2 \setminus E_m$ then $M$ is a catenoid.

The original proof of this theorem use deformation of small pieces of $M$ under some special flow to get the conclusion. Theorem 3 implies this result easily. In fact, consider $M$ in some position that $\infty$ is the image of some end. Then we can consider $M = C \setminus E$ and $2E < \infty$. If $x: M \to R^3$ is such that minimal embedded the map then $f = x^{-1}: M \to C$ is well defined and is a branched covering. By theorem 3 we get $2C \setminus f(M) \leq 1$, or $f$ is constant. Hence $2E_m = 2$ and classical results grantee that $M$ is a catenoid.

6 Remarks on Gauss map missing two points.

Let $M = \overline{M} \setminus E_m$ be a finite geometric type minimal surface whose Gauss map $G$ misses two points $S^2 \setminus G(M) = \{a_1, a_2\}$. The existence of those examples are guarantee by [7] for genus 1 and [2] for arbitrary genus but all are not embedded. Since $2\deg(G) = 2E^\infty_m + \beta(E^\infty_m)$ the R-H formula gives

$$\chi(\overline{M}) = 2E^\infty_m - \beta(M \cup E^0_m), \quad (38)$$

where $E^\infty_m = G^{-1}(\{a_1, a_2\})$ and $E^0_m = E_m \setminus E^\infty_m$. Take $w \in S^2 \setminus \{a_1, a_2\}$, $Y = \{a_1, a_2, w\}$ and $f: S^2 \setminus X = S^2 \setminus Y$ the map of lemma [2.2] with $X = f^{-1}(Y)$. Then $f|S^2 \setminus X$ is a regular covering and there is the lifting $F: M \setminus G^{-1}(Y) \to S^2 \setminus X$. By Remark [2.4] we have $C^0$ extension of $f$ to $\overline{M}$ if and only if 3 divides all $1 + \beta_G(z) \geq 2$, $z \in E^\infty_m \cup G^{-1}(w)$. But $2S^2 \setminus F(M) = 4$ what is impossible by theorem [2.3]. Then for those immersions we can not have continuous extensions $F: \overline{M} \to S^2$ for any $Y \supset \{a_1, a_2\}$, $2Y \geq 3$, no matter $G$ has.
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