A note on semi-infinite program bounding methods

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Abstract

Semi-infinite programs are a class of mathematical optimization problems with a finite number of decision variables and infinite constraints. As shown by Blankenship and Falk [2], a sequence of lower bounds which converges to the optimal objective value may be obtained with specially constructed finite approximations of the constraint set. In [4], it is claimed that a modification of this lower bounding method involving approximate solution of the lower-level program yields convergent lower bounds. We show with a counterexample that this claim is false, and discuss what kind of approximate solution of the lower-level program is sufficient for correct behavior.

1 Introduction

This note discusses methods for the global solution of semi-infinite programs (SIP). Specifically, the method from [4] is considered, and it is shown with a counterexample that the lower bounds do not always converge. Throughout we use notation as close as possible to that used in [4], embellishing it only as necessary with, for instance, iteration counters.

Consider a SIP in the general form

\[ f^* = \inf_x f(x) \]
\( \text{s.t. } x \in X, \]
\( g(x, y) \leq 0, \forall y \in Y, \)

for subsets \( X, Y \) of finite dimensional real vector spaces and \( f : X \to \mathbb{R}, g : X \times Y \to \mathbb{R} \). We may view \( Y \) as an index set, with potentially uncountably infinite cardinality. Important to validating the feasibility of a point \( x \) is the lower-level program:

\[ \sup_y \{ g(x, y) : y \in Y \}. \]

Global solution of (SIP) often involves the construction of convergent upper and lower bounds. The approach in [4] to obtain a lower bound is a modification of the constraint-generation/discretization method of [2]. The claim is that the lower-level program may be solved approximately; the exact nature of the approximation is important to the convergence of the lower bounds and this is the subject of the present note.
2 Sketch of the lower bounding procedure and claim

The setting of the method is the following. The method is iterative and at iteration $k$, for a given finite subset $Y^{LBD,k} \subset Y$, a lower bound of $f^*$ is obtained from the finite program

$$f^{LBD,k} = \inf_x f(x)$$

\[ \text{s.t. } x \in X, \quad g(x, y) \leq 0, \forall y \in Y^{LBD,k}. \]

This is indeed a lower bound since fewer constraints are enforced, and thus (1) is a relaxation of (SIP). Assume that the lower bounding problem (1) is feasible (otherwise we can conclude that (SIP) is infeasible). Let $\bar{x}_k$ be a (global) minimizer of the lower bounding problem (1). In [4], Lemma 2.2 states that we either verify $\sup_y \{g(\bar{x}_k, y) : y \in Y\} \leq 0$, or else find $\bar{y}_k \in Y$ such that $g(\bar{x}_k, \bar{y}_k) > 0$. If $\sup_y \{g(\bar{x}_k, y) : y \in Y\} \leq 0$, then $\bar{x}_k$ is feasible in (SIP) and thus optimal (since it also solves a relaxation). Otherwise, set $Y^{LBD,k+1} = Y^{LBD,k} \cup \{\bar{y}_k\}$ and we iterate.

The precise statement of the claim is repeated here (again, with only minor embellishments to the notation to help keep track of iterations).

**Lemma 1** (Lemma 2.2 in [4]). Take any $Y^{LBD,0} \subset Y$. Assume that $X$ and $Y$ are compact and that $g$ is continuous on $X \times Y$. Suppose that at each iteration of the lower bounding procedure the lower-level program is solved approximately for the solution of the lower bounding problem $\bar{x}_k$ either establishing $\max_{y \in Y} g(\bar{x}_k, y) \leq 0$, or furnishing a point $\bar{y}_k$ such that $g(\bar{x}_k, \bar{y}_k) > 0$. Then, the lower bounding procedure converges to the optimal objective value, i.e. $f^{LBD,k} \to f^*$.

3 Correction

3.1 Counterexample

We now present a counterexample to the claim in Lemma [4]. Consider

$$\inf_x -x \quad \text{(CEx)}$$

\[ \text{s.t. } x \in [-1, 1], \quad 2x - y \leq 0, \forall y \in [-1, 1], \]

thus we define $X = Y = [-1, 1]$, $f : x \mapsto -x$, $g : (x, y) \mapsto 2x - y$. The behavior to note is this: We are trying to maximize $x$; The feasible set is

$$\{x \in [-1, 1] : x \leq (1/2)y, \forall y \in [-1, 1]\} = [-1, -1/2];$$

The infimum, consequently, is $1/2$. See Figure [1]

Beginning with $Y^{LBD,1} = \emptyset$, the minimizer of the lower bounding problem is $\bar{x}_1 = 1$. Now, assume that solving the resulting (LLP) approximately, we get $\bar{y}_1 = 1$ which we note satisfies

$$2\bar{x}_1 - \bar{y}_1 = 1 > 0$$

as required by Lemma [4]

The next iteration, with $Y^{LBD,2} = \{1\}$, adds the constraint $2x - 1 \leq 0$ to the lower bounding problem; the feasible set is $[-1, 1/2]$ so the minimizer is $\bar{x}_2 = 1/2$. Again, assume that solving the lower-level program approximately yields $\bar{y}_2 = 1/2$; again we get

$$2\bar{x}_2 - \bar{y}_2 = 1/2 > 0$$

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as required by Lemma 1.

The third iteration, with $Y^L_{BD,3} = \{1, 1/2\}$, adds the constraint $2x - 1/2 \leq 0$ to the lower bounding problem; the feasible set is $[-1, 1/4]$ so the minimizer is $\bar{x}^3 = 1/4$. Again, assume that solving the lower-level program approximately yields $\bar{y}^3 = 1/4$; again we get

$$2\bar{x}^3 - \bar{y}^3 = 1/4 > 0$$

as required by Lemma 1.

Proceeding in this way, we construct $\bar{x}^k$ and $\bar{y}^k$ so that $g(\bar{x}^k, \bar{y}^k) > 0$ and the lower bounds satisfy $f^{LBD,k} = -\bar{x}^k = -\frac{1}{2^k - 1}$, for all $k$. Consequently, they converge to 0, which we note is strictly less than the infimum of $1/2$.

### 3.2 Modified claim

We now present a modification of the claim in order to demonstrate what kind of approximate solution of the lower-level program suffices to establish convergence of the lower bounds. To state the result, let the optimal objective value of (LLP) as a function of $x$ be

$$g^*(x) = \sup_{y} \{g(x,y) : y \in Y\}.$$  

The proof of the following result has a similar structure to the original proof of [4, Lemma 2.2].

**Lemma 2.** Choose any finite $Y^{LBD,0} \subset Y$, and $\alpha \in (0,1)$. Assume that $X$ and $Y$ are compact and that $f$ and $g$ are continuous. Suppose that at each iteration $k$ of the lower bounding procedure (LLP) is solved approximately for the solution $\bar{x}^k$ of the lower bounding problem (I), either establishing that $g^*(\bar{x}^k) \leq 0$ or furnishing a point $\bar{y}^k$ such that

$$g(\bar{x}^k, \bar{y}^k) \geq \alpha g^*(\bar{x}^k) > 0.$$

Then, the lower bounding procedure converges to the optimal objective value, i.e. $f^{LBD,k} \to f^*.$

**Proof.** First, if the lower bounding problem (I) is ever infeasible for some iteration $k$, then (SIP) is infeasible and we can set $f^{LBD,k} = +\infty = f^*$. Otherwise, since $X$ is compact, $Y^{LBD,k}$ is finite, and $f$ and $g$ are continuous, for every iteration the lower bounding problem has a solution by
Weierstrass' (extreme value) theorem. If at some iteration \( k \) the lower bounding problem furnishes a point \( \bar{x}^k \) for which \( g^*(\bar{x}^k) \leq 0 \), then \( \bar{x}^k \) is feasible for (SIP), and thus optimal. The corresponding lower bound \( f^{LBD,k} \), and all subsequent lower bounds, equal \( f^* \).

Otherwise, we have an infinite sequence of solutions to the lower bounding problems. Since \( X \) is compact we can move to a subsequence \( (\bar{x}^k)_{k \in \mathbb{N}} \subset X \) which converges to \( x^* \in X \). By construction of the lower bounding problem we have

\[
g(\bar{x}^\ell, \bar{y}^k) \leq 0, \quad \forall \ell, k : \ell > k.
\]

By continuity and compactness of \( X \times Y \) we have uniform continuity of \( g \), and so for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
g(x, \bar{y}^k) < \epsilon, \quad \forall x : \|x - \bar{x}^\ell\| < \delta, \quad \forall \ell, k : \ell > k. \tag{2}
\]

Since the (sub)sequence \( (\bar{x}^k)_{k \in \mathbb{N}} \) converges, there is an index \( K \) sufficiently large that

\[
\|\bar{x}^\ell - \bar{x}^k\| < \delta, \quad \forall \ell, k : \ell > k \geq K. \tag{3}
\]

Using (3), we can substitute \( x = \bar{x}^k \) in (2) to get that for any \( \epsilon > 0 \), there exists \( K \) such that

\[
g(\bar{x}^k, \bar{y}^k) < \epsilon, \quad \forall k \geq K.
\]

By assumption \( g(\bar{x}^k, \bar{y}^k) > 0 \) for all \( k \), and so combined with the above we have that \( g(\bar{x}^k, \bar{y}^k) \to 0 \).

Combining \( g(\bar{x}^k, \bar{y}^k) \to 0 \) with \( g(\bar{x}^k, \bar{y}^k) \geq \alpha g^*(\bar{x}^k) > 0, \) for all \( k \), we see \( g^*(\bar{x}^k) \to 0 \). Meanwhile \( g^* : X \to \mathbb{R} \) is a continuous function, by classic parametric optimization results like [1, Theorem 1.4.16] (using continuity of \( g \) and compactness of \( Y \)). Thus

\[
g^*(x^*) = \lim_{k \to \infty} g^*(\bar{x}^k) = 0.
\]

Thus \( x^* \) is feasible in (SIP) and so \( f^* \leq f(x^*) \). But since the lower bounding problem is a relaxation, \( f^{LBD,k} = f(\bar{x}^k) \leq f^* \) for all \( k \), and so by continuity of \( f \), \( f(x^*) \leq f^* \). Combining these inequalities we see \( f^{LBD,k} \to f(x^*) = f^* \). Since the entire sequence of lower bounds is an increasing sequence, we see that the entire sequence converges to \( f^* \) (without moving to a subsequence).

\[\square\]

4 Remarks

The main contribution of [4] is a novel upper bounding procedure, which still stands, and combined with the modified lower bounding procedure from Lemma 2 or the original procedure from [2], the overall global solution method for (SIP) is still effective.

The counterexample that has been presented may seem contrived. However, as the lower bounding method for SIP from [4] is adapted to give a lower bounding method for generalized semi-infinite programs (GSIP) in [5], a modification of the counterexample reveals that similar behavior may occur (and in a more natural way) when constructing the lower bounds for a GSIP. Consequently, the lower bounds fail to converge to the infimum. See [3].

References

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