\( \mathcal{N} = 2 \) GAUGE THEORIES ON SYSTEMS OF FRACTIONAL D3/D7 BRANES

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Abstract

We study a bound state of fractional D3/D7-branes in the ten-dimensional space \( \mathbb{R}^{1,5} \times \mathbb{R}^4/\mathbb{Z}_2 \) using the boundary state formalism. We construct the boundary actions for this system and show that higher order terms in the twisted fields are needed in order to satisfy the zero-force condition. We then find the classical background associated to the bound state and show that the gauge theory living on a probe fractional D3-brane correctly reproduces the perturbative behavior of a four-dimensional \( \mathcal{N} = 2 \) supersymmetric gauge theory with fundamental matter.

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1 Introduction

During the last few years, it has become more and more evident that the low-energy properties of D-branes can be studied in two complementary ways: one based on the fact that a D-brane is a classical solution of the string effective theory (supergravity) charged under a RR potential; the other based on the fact that a D-brane supports a gauge field theory on its world-volume. The realization of this two-fold interpretation, which in the literature goes under the name of gauge/gravity correspondence, has been one of the most significant results of the recent research in string theory. In fact, in view of this correspondence, one can exploit the classical geometrical properties of D-branes to study gauge theories or, vice-versa, use the quantum properties of the gauge theory defined on a D-brane to study the dynamics of non-perturbative extended objects.

In the case of the D3-branes of the type IIB string theory in flat space, it was possible to carry this correspondence much further by taking the so-called near-horizon limit, and observing that in this limit the gravity degrees of freedom (closed strings) propagating in the entire ten-dimensional space decouple from the gauge degrees of freedom (open strings) living on the four-dimensional world-volume of the D3-brane. This decoupling led Maldacena [1] to conjecture an exact duality between the $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, which is the conformal field theory living on the D3-brane world-volume, and type IIB string theory on $AdS_5 \times S_5$, which is the space-time geometry in the near horizon limit. This remarkable conjecture, which has been confirmed by all subsequent studies, has opened the way to the use of brane dynamics in the analysis of the strong coupling regime of four-dimensional gauge theories.

More recently, a lot of efforts have been made to find possible extensions of the Maldacena duality to less supersymmetric and non-conformal gauge theories with more realistic properties such as asymptotic freedom and a running coupling constant. The simplest theories with these features are those with $\mathcal{N} = 2$ su-
persymmetry that can be obtained, for instance, by studying fractional branes on orbifolds \cite{2, 3, 4, 5}. In particular, the world-volume theory defined on a stack of fractional D3-branes in $\mathbb{R}^{1,5} \times \mathbb{R}^3/\mathbb{Z}_2$ is a pure $\mathcal{N} = 2$ super Yang-Mills theory in four dimensions \cite{6}, and thus this is a very natural system to consider in order to study possible non-conformal extensions of the Maldacena duality. However, as shown in Ref.s \cite{7, 8} (and for more general orbifolds in Ref. \cite{9}) the supergravity solutions corresponding to these fractional D3-branes possess naked singularities of repulson type \cite{10}. The appearance of naked singularities is a quite general feature of the supergravity solutions that are dual to non-conformal gauge theories, but in the orbifold case, it seems that there exists a general mechanism to resolve them. Indeed, by analyzing the action of a probe D3-brane in the singular background, one can see that the probe becomes tensionless before reaching the repulson singularity on a hypersurface called enhançon \cite{11} (similar results hold also in the case of fractional branes on compact orbifolds, see Ref. \cite{12}). This fact suggests that at the enhançon the classical solution cannot be trusted anymore because additional light degrees of freedom come into play and the supergravity approximation is no longer correct. Therefore, the presence of the enhançon allows to consistently excise the singularity region and obtain a well-behaved solution \cite{13}, but at the same time it does not allow to easily take the decoupling limit anymore (for a discussion on the physics of the enhançon for these and more complicated systems see Ref.s \cite{14, 15}).

Despite these problems, the classical solution describing fractional D3-branes on orbifolds has been successfully used to study the perturbative dynamics of $\mathcal{N} = 2$ supersymmetric gauge theories and obtain their correct perturbative moduli space \cite{7, 8, 9}. Among other things, this analysis has also shown that the enhançon corresponds, in the gauge theory, to the scale where the gauge coupling constant diverges (the analogue of $\Lambda_{\text{QCD}}$ in QCD). These results, that seem to be in contrast with a duality interpretation à la Maldacena where the supergravity solution gives a good description of the gauge theory for large ’t Hooft coupling, can instead be easily understood if we regard the classical supergravity solution as an effective way of summing over all open string loops, as explained for example in Ref. \cite{16}. From this point of view, in fact, one does not take the near-horizon limit (i.e. $r \rightarrow 0$, where $r$ is the distance from the source branes), but rather expands the classical solution around $r \rightarrow \infty$ where the metric is almost flat and the supergravity approximation is valid. This expansion corresponds to summing closed string diagrams at tree level, but, because of the open/closed string duality, it is also equivalent to summing over open string loops. Therefore, expanding the supergravity solution around $r \rightarrow \infty$ is equivalent to perform an expansion for small ’t Hooft coupling. In view of these considerations, it is then not surprising that the supergravity solution of Ref.s \cite{7, 8, 9} encodes the perturbative properties of the $\mathcal{N} = 2$ gauge theory living on the world-volume of a fractional D3-brane, but at the same time it is also natural to expect that this approach does not include the non-perturbative instanton corrections to the moduli space. In conclusion we can say that the previous results are not a consequence of a Maldacena-like duality, but rather they follow di-
rectly from the gauge/gravity correspondence or, equivalently, from the open/closed string duality. On the other hand, the presence of the enhançon, which excises the region of space-time corresponding to scales where non-perturbative effects become relevant in the gauge theory and which is reminiscent of the curve separating strong from weak coupling in $\mathcal{N} = 2$ super Yang-Mills theory [20], is consistent with the above picture. To incorporate in this scenario also the non-perturbative instanton effects, presumably one must include D-instanton corrections already at the string level [21, 22], as done in Ref. [23] for the $\mathcal{N} = 4$ super Yang-Mills theory, or start directly from M-theory [24]. M-theory is also the starting point of a very interesting alternative approach pursued recently in Ref.s [25, 26] where the non-perturbative properties of $\mathcal{N} = 2$ supersymmetric gauge theories have been obtained by taking the near-horizon limit. More recently, the instanton corrections for systems of fractional D-branes have been discussed in Ref. [27], while alternative approaches to investigate supersymmetric $\mathcal{N} = 2$ gauge theories, at the perturbative level, have been carried out for instance in Ref.s [28, 29, 30, 31].

In a recent paper [32], the approach of Ref.s [7, 8] has been extended to the case of a system of fractional D3/D7-branes on the orbifold $\mathbb{R}^{1,5} \times \mathbb{R}^{4}/\mathbb{Z}_2$ and the perturbative properties of the $\mathcal{N} = 2$ supersymmetric gauge theory living on the D3-brane world-volume have again been recovered from supergravity. In this case the gauge theory includes also hypermultiplets in the fundamental representation, associated to the open strings stretched between the D3 and the D7-branes.

In this paper, motivated by the considerations discussed above and using essentially the information provided by the boundary state formalism, we discuss further properties of the fractional D7-branes and of the bound states of fractional D3/D7-branes in the orbifold background. In particular we construct the boundary action for a fractional D7-brane and find that in order to satisfy the no-force condition required for a supersymmetric system, terms of higher order in the twisted fields, which are not accounted by the boundary state, must be included. We then solve explicitly the supergravity field equations for the D3/D7 system and study the properties of the dual four-dimensional gauge theory, finding agreement with the analysis of Ref. [32]. An interesting feature of this system is that, unlike the case discussed in Ref.s [7, 8, 9], the twisted fields receive contribution from diagrams with an arbitrary number of open string loops or, equivalently, from closed string tree diagrams containing an arbitrary number of boundaries. However, as expected from the $\mathcal{N} = 2$ non-renormalization theorems, we find that in the gauge theory the twisted fields appear always in special combinations in which only the one-loop perturbative contribution is non-trivial.

The paper is organized as follows. In Sect. 2 we write the field equations of type IIB supergravity on the orbifold coupled to fractional branes. In Sect. 3 we use the boundary state to study the properties of the fractional D7-branes determining their couplings with the bulk fields and the first order approximation of their classical

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[1] Recent papers stressing the importance of the open/closed string duality in our context are in Ref.s [17, 18, 19].
solution. In Sect. 4 we extend the previous analysis to a bound state of fractional D3/D7-branes finding its complete classical solution. Using the no-force argument, we are also able to fix the complete form of the D7 brane boundary action. Finally in Sect. 5, by means of the probe analysis, we derive the perturbative behavior of the corresponding $\mathcal{N} = 2$ gauge theory and discuss its properties. A few technical results on the boundary state construction are reviewed in Appendix A.

2 Field equations for fractional D-branes

We consider type IIB supergravity in ten dimensions on the orbifold

$$\mathbb{R}^{1,5} \times \mathbb{R}^4 / \mathbb{Z}_2$$

(2.1)

where $\mathbb{Z}_2$ is the reflection parity along $x^6$, $x^7$, $x^8$ and $x^9$. Its action (in the Einstein frame) can be written as

$$S_{IIB} = \frac{1}{2 \kappa^2_{\text{orb}}} \left\{ \int d^{10} x \sqrt{-\text{det} G} \ R - \frac{1}{2} \int \left[ d\phi \wedge * d\phi + e^{-\phi} H_3 \wedge * H_3 + e^{2\phi} F_1 \wedge * F_1 \right. \\
+ e^{\phi} \bar{F}_3 \wedge * \bar{F}_3 + \frac{1}{2} \bar{F}_5 \wedge * \bar{F}_5 - C_4 \wedge H_3 \wedge F_3 \right\}$$

(2.2)

where

$$H_3 = dB_2 \ , \ F_1 = dC_0 \ , \ F_3 = dC_2 \ , \ F_5 = dC_4$$

(2.3)

are, respectively, the field strengths of the NS-NS 2-form and the 0-, 2- and 4-form potentials of the R-R sector, and

$$\bar{F}_3 = F_3 + C_0 \wedge H_3 \ , \ \bar{F}_5 = F_5 + C_2 \wedge H_3 \ .$$

(2.4)

Moreover, $\kappa^2_{\text{orb}} \equiv (2\pi)^7 g_s^2 \alpha'^4 = 2\kappa^2$ where $g_s$ is the string coupling constant, and the self-duality constraint $* \bar{F}_3 = \bar{F}_5$ has to be implemented on shell.

We are interested in obtaining classical solutions of the field equations descending from the action (2.2) that describe fractional D branes characterized by the presence of “twisted” scalar fields $b$ and $c$ defined as

$$C_2 = c \ \omega_2 \ , \ B_2 = b \ \omega_2$$

(2.5)

where $\omega_2$ is the anti-self dual 2-form associated to the vanishing 2-cycle of the orbifold ALE space. In our normalizations\footnote{Our conventions for curved indices and forms are the following: $\epsilon^{0...9} = +1$; signature $(-, + , +, +, +, +, +, +, +, +)$; $\mu, \nu = 0, \ldots, 9$; $\alpha, \beta = 0, \ldots, 3$; $i, j = 4, 5$; $\ell, m = 6, \ldots, 9$; $\omega_{(n)} = \frac{1}{n!} \omega_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}$, and $* \omega_{(n)} = \frac{1}{n! (10-n)!} \epsilon_{\mu_1 \ldots \mu_{10-n} \ell_1 \ldots \ell_n} \omega_{\mu_1 \ldots \mu_n} dx^{\ell_1} \wedge \ldots \wedge dx^{\ell_n}$.}, this form satisfies

$$\int_{\text{ALE}} \omega_2 \wedge \omega_2 = -1 \ .$$

(2.6)

\footnote{Notice that we use definitions where $B_2 \rightarrow -B_2$ with respect to Ref. \cite{ref}.}
Inserting eq. (2.3) in the action (2.2), we easily get
\[ S'_{\text{IIB}} = \frac{1}{2\kappa_{\text{orb}}^2} \left\{ \int d^{10} x \sqrt{-G} R - \frac{1}{2} \int \left[ d\phi \wedge^* d\phi + e^{2\phi} dC_0 \wedge^* dC_0 + \frac{1}{2} \tilde{F}_5 \wedge^* \tilde{F}_5 \right] \right. \\
- \frac{1}{2} \int \left[ e^{-\phi} db \wedge^* db + e^\phi (dc + C_0 db) \wedge^* (dc + C_0 db) + C_4 \wedge db \wedge dc \right] \right\} \] (2.7)
where the index 6 refers to the six-dimensional space orthogonal to the orbifold directions.

The field equations for a fractional D-brane are obtained by varying the total action
\[ S'_{\text{IIB}} + S_b \] (2.8)
where \( S_b \) describes the coupling of the bulk supergravity fields with the brane; we recall however, that only the linear part of the boundary action \( S_b \) is relevant to yield the source terms for the inhomogeneous field equations [16]. For the moment we do not specify the form of \( S_b \) which instead will be discussed in the following sections for the specific cases of the fractional D3 and D7 branes. Defining
\[ \Omega_4 \equiv \delta(x^6) \cdots \delta(x^9) \ dx^6 \wedge \cdots \wedge dx^9 \] (2.9)
the field equation for the dilaton is
\[ d^* d\phi - e^{2\phi} dC_0 \wedge^* dC_0 + \frac{1}{2} \left[ e^{-\phi} db \wedge^* db \\
- e^\phi (dc + C_0 db) \wedge^* (dc + C_0 db) \right] \wedge \Omega_4 + 2\kappa_{\text{orb}}^2 \frac{\delta S_b}{\delta \phi} = 0 \] (2.10)
and the one for the axion is
\[ d (e^{2\phi} * dC_0) - e^{\phi} db \wedge^* (dc + C_0 db) \wedge \Omega_4 + 2\kappa_{\text{orb}}^2 \frac{\delta S_b}{\delta C_0} = 0 \] (2.11)
The field equations for the two twisted fields \( b \) and \( c \) are respectively
\[ d \left[ e^{-\phi} * db + C_0 e^{\phi} * (dc + C_0 db) \right] + \tilde{F}_5 \wedge dc + 2\kappa_{\text{orb}}^2 \frac{\delta S_b}{\delta b} = 0 \] (2.12)
and
\[ d \left[ e^\phi * (dc + C_0 db) \right] - \tilde{F}_5 \wedge db + 2\kappa_{\text{orb}}^2 \frac{\delta S_b}{\delta c} = 0 \] (2.13)
Finally, the field equation for the untwisted 4-form \( C_4 \) is
\[ d^* \tilde{F}_5 - db \wedge dc \wedge \Omega_4 + 2\kappa_{\text{orb}}^2 \frac{\delta S_b}{\delta C_4} = 0 \] (2.14)
and the one for the metric tensor is
\[ R_{\mu
u} = \frac{1}{4} \cdot 4! (\tilde{F}_5)_{\mu\rho\sigma\delta} (\tilde{F}_5)_{\nu}^{\rho\sigma\delta} + 2\kappa_{\text{orb}}^2 \frac{\delta S_b}{\delta G_{\mu\nu}} = \frac{1}{2} \left[ \partial_\mu \phi \partial_\nu \phi + e^{2\phi} \partial_\mu C_0 \partial_\nu C_0 \right] + T^{(b,c)}_{\mu\nu} \] (2.15)
where $T_{\mu\nu}^{(b,c)}$ is the stress energy tensor of the scalars $b$ and $c$ whose explicit expression is not really needed in the following.

To analyze these equations it is convenient to introduce the following complex quantities

$$
\tau = C_0 + i e^{-\phi} \quad ; \quad \gamma = c + \tau b \quad ; \quad G_1 = dc + \tau db \ .
$$

In fact, with simple manipulations the four equations (2.10)–(2.13) can be combined into two complex differential equations for $\tau$ and $\gamma$, which are

$$
d^*d\tau + i e^\phi d\tau \wedge *d\tau + \frac{i}{2} G_1 \wedge G_1 \wedge \Omega_4 - 2 i \kappa_{\text{orb}}^2 \left[ e^{-\phi} \frac{\delta S_b}{\delta \phi} + i e^{-2\phi} \frac{\delta S_b}{\delta C_0} \right] = 0 \ , \quad (2.17)
$$

and

$$
d^*d\gamma + i e^\phi d\tau \wedge *G_1 - b d^*d\tau + i \tilde{F}_5 \wedge G_1 + 2 i \kappa_{\text{orb}}^2 \left[ \frac{\delta S_b}{\delta b} - \tau \frac{\delta S_b}{\delta c} \right] = 0 \ . \quad (2.18)
$$

In the following we are going to solve these equations for bound states made of fractional D3 and D7 branes. In particular we will consider configurations in which the D7 branes extend in the directions $x^0, \ldots, x^3, x^6, \ldots x^9$ (i.e. entirely along the orbifold) and the D3 branes in the directions $x^0, \ldots, x^3$ (i.e. transversely to the orbifold). With this arrangement, the twisted fields $b$ and $c$, which are stuck at the orbifold fixed point, are functions only of the transverse coordinates $x^4$ and $x^5$. Moreover, since the D3 branes emit neither the dilaton $\phi$ nor the axion $C_0$, these two fields are produced only by the D7 branes and thus they too are functions only of the transverse coordinates $x^4$ and $x^5$. For the remaining fields, the metric $G_{\mu\nu}$ and the self-dual field strength $\tilde{F}_5$, we take the standard Ansatz for a D7/D3 system

$$
ds^2 = H^{-1/2} \eta_{\alpha\beta} dx^\alpha dx^\beta + H^{1/2} \left( \delta_{\ell m} dx^\ell dx^m + e^{-\phi} \delta_{ij} dx^i dx^j \right) \ , \quad (2.19)
$$

$$
\tilde{F}_5 = d \left( H^{-1} dx^0 \wedge \ldots \wedge dx^3 \right) + d \left( H^{-1} dx^0 \wedge \ldots \wedge dx^3 \right) \ , \quad (2.20)
$$

where the warp factor $H$ is a function of all coordinates that are transverse to the D3 brane ($x^4, \ldots, x^9$).

A drastic simplification occurs by analyzing the supersymmetry transformation rules of the gravitinos and dilatinos and asking for the solution to be supersymmetric. Plugging the Ansatz (2.19) and (2.20) in the variations of the gravitinos and dilatinos, one can show [32] that the existence of a Killing spinor implies that both $G_1$ and $\tau$ are holomorphic functions of $z \equiv x^4 + i x^5$, i.e.

$$
\partial_z G_1 = \partial_{\bar{z}} \tau = 0 \ . \quad (2.21)
$$

4The relation between $G_1$ and the usual complex 3-form of type IIB supergravity is $G_3 \equiv F_3 + \tau H_3 = G_1 \wedge \omega_2$.

5The case of constant $\tau$ was discussed in Ref.s [34, 35] for the case of singular spaces and in Ref. [36] for the blown-up case, while that for vanishing G-flux was discussed in Ref. [37]. This kind of type IIB supersymmetric solutions are dual to those discussed in Ref. [38].
The analyticity of $G_1$ and $\tau$ in turn implies that $\gamma$ is analytic too (see eq.(2.16)); thus the equations for $\tau$ and $\gamma$ drastically simplify and reduce to

$$d^*d\tau - 2i\kappa^2_{\text{orb}} \left[ e^{-\phi} \frac{\delta S_b}{\delta \phi} + ie^{-2\phi} \frac{\delta S_b}{\delta C_0} \right] = 0 \quad ,$$

(2.22)

and

$$d^*d\gamma + iF_b \wedge G_1 - b d^*d\tau + 2i\kappa^2_{\text{orb}} \left[ \frac{\delta S_b}{\delta b} - \tau \frac{\delta S_b}{\delta c} \right] = 0 \quad .$$

(2.23)

Finally, we can see that the field equations for the metric and $C_4$ are satisfied provided that $H$ be a solution of the following equation

$$\left( \delta^{ij} \partial_i \partial_j H + e^{-\phi} \delta^{lm} \partial_l \partial_m H \right) V_6 - db \wedge dc \wedge \Omega_4 + 2\kappa^2_{\text{orb}} \frac{\delta S_b}{\delta C_4} = 0 \quad (2.24)$$

where $V_6$ denotes the six-dimensional volume form $dx^4 \wedge \cdots \wedge dx^9$.

As anticipated, we want to find the solution of these field equations for bound states of fractional D7/D3 branes. However, before doing this, we present a discussion of the pure D7 branes on orbifold from a string theory point of view, using the boundary state formalism.

### 3 The fractional D7-branes

In this section we will analyze in some detail the D7 branes of type IIB in the orbifold background (2.1). In order to realize a BPS brane of type IIB, the number $s$ of orbifold directions along its world-volume must be even [39]. Thus, for a 7 brane we have only two possibilities: $s = 2$ and $s = 4$. Here we discuss only the case $s = 4$, i.e. when the brane extends throughout the entire orbifold, since this is the relevant case to yield four-dimensional $\mathcal{N} = 2$ gauge theories in the presence of fractional D3 branes.

The fact that D7 branes with $s = 4$ extend entirely along the orbifold has some peculiar consequences that we would like to emphasize. From a string theory point of view, these branes are sources not only of those fields that are typical of a D7 brane (i.e. the metric $G_{\mu\nu}$, the dilaton $\phi$ and the 8-form R-R potential $C_8$), but also of twisted fields, which, specifically, comprise a scalar $\tilde{b}$ from the twisted NS-NS sector and a 4-form potential $A_4$ from the twisted R-R sector. It is interesting to observe that these twisted fields are the same as those emitted by the fractional D3 branes studied for example in Refs [7, 8]. Moreover, the charge of these D7 branes under $C_8$ is a half of that carried by the D7 branes in flat space. Thus, it is natural to regard these configurations as fractional branes, despite the fact that, contrarily to what happens for the fractional branes of lower dimension, they cannot be interpreted as wrapped branes. Finally, the charge of these D7 branes under the twisted 4-form potential is a quarter of that carried by the fractional D3 branes.
All these features can be clearly seen by computing the vacuum energy $Z$ of the open strings stretched between two such D7 branes that is given by

$$Z = \int_0^\infty \frac{ds}{s} \text{Tr}_{\text{NS-R}} \left[ P_{\text{GSO}} \left( \frac{1 + g}{2} \right) e^{-2\pi s(L_0 - a)} \right]$$

where $P_{\text{GSO}}$ is the GSO projection, $g$ is the orbifold $\mathbb{Z}_2$ parity, and $a = 1/2$ in the NS sector and $a = 0$ in the R sector. When one takes the $1$ inside the bracket, one gets half of the contribution of the open strings stretched between two D7 branes in flat space, whereas when one takes the $g$ inside the bracket one obtains $1/16$ times the contribution of the twisted sectors of the fractional D3 branes of Refs. [7, 8] (see Appendix A for more details). After performing the modular transformation $s \to 1/s$ and factorizing the resulting expression in the closed string channel, one can derive the boundary state $|D7\rangle$ associated to the D7 brane along the orbifold (for a review of the boundary state formalism and its applications see, for example, Ref. [11]; for an analysis of the boundary state in orbifold theories see, for example, Refs. [11, 39, 42]). The explicit expression of $|D7\rangle$ and a discussion of its properties can be found in appendix A. Here we just mention that this boundary state contains both an untwisted and a twisted part:

$$|D7\rangle = |D7\rangle^U + |D7\rangle^T$$

The untwisted part $|D7\rangle^U$ is the same as that of the D7 branes in flat space but with a normalization differing by a factor of $1/\sqrt{2}$; the twisted part $|D7\rangle^T$ is, instead, similar to that of the fractional D3 branes but with a normalization differing by a factor of $1/4$. By saturating the boundary state $|D7\rangle$ with the massless closed string states of the various sectors, one can determine which are the fields that couple to the fractional D7 brane. In particular, following the procedure found in Ref. [43] and reviewed in Ref. [11], one can find that in the untwisted sectors the D7 brane emits the graviton $h_{\mu\nu}$, the dilaton $\phi$ and the 8-form potential $C_8$. The couplings of these fields with the boundary state are explicitly given by [12]

$$\langle D7|h_\mu \rangle = -\frac{T_7}{\sqrt{2}} h^a_\mu V_8^a,$$
$$\langle D7|\phi \rangle = -\frac{T_7}{\sqrt{2}\kappa_{\text{orb}}} \phi V_8,$$
$$\langle D7|C_8 \rangle = \frac{T_7}{\sqrt{2}\kappa_{\text{orb}}} C_{01...67} V_8$$

where $T_p = \sqrt{\pi} \left( 2\pi \sqrt{\alpha'} \right)^{(3-p)}$, appearing in the normalization of the boundary state, is related to the brane tension in units of the gravitational coupling constant [43, 44]. $V_8$ is the (infinite) world-volume of the D7 brane, and the index $a$ labels its eight longitudinal directions. Notice that there is no coupling of the boundary state $|D7\rangle$.

\footnote{We recall that the graviton field and the metric are related by $G_{\mu\nu} = \eta_{\mu\nu} + 2\kappa_{\text{orb}} h_{\mu\nu}$.}
with the untwisted 4-form $C_4$, in agreement with the observation \cite{32} that the D7 branes do not carry charge under $C_4$.

By doing this same analysis in the twisted sectors, we find that, as advertised before, the boundary state $|\text{D7}\rangle$ emits a massless scalar $\tilde{b}$ from the NS-NS sector and a 4-form potential $A_4$ from the R-R sector. These fields exist only at the orbifold fixed hyperplane $x^6 = x^7 = x^8 = x^9 = 0$, and their couplings with the boundary state turn out to be given by \cite{12}

$$
\langle \text{D7}|\tilde{b}\rangle = \frac{T_3}{4\sqrt{2}\kappa_{\text{orb}}} \frac{1}{2\pi^2\alpha'} \tilde{b} V_4 ,
$$

$$
\langle \text{D7}|A_4\rangle = -\frac{T_3}{4\sqrt{2}\kappa_{\text{orb}}} \frac{1}{2\pi^2\alpha'} A_{0123} V_4
$$

(3.4)

where $V_4$ is the (infinite) world-volume of the 7 brane that lies outside the orbifold.

From the explicit couplings (3.3) and (3.4), it is possible to infer the form of the world-volume action of a fractional D7 brane. Of course, the boundary state approach allows to obtain only the terms of the world-volume action that are linear in the bulk fields. However, terms of higher order can be determined with other methods. For example, from our previous considerations, it is natural to write for the untwisted fields the same world-volume action of the D7 branes in flat space but with an extra overall factor of $1/\sqrt{2}$ as dictated by the boundary state. Therefore, we write (in the Einstein frame)

$$
S^{b}_{\text{D7}}|_{U} = -\tau_7 \int d^8x\, e^\phi \sqrt{-\det g_{ab}} \, \tilde{b} + \tau_7 \int C_8
$$

(3.5)

where $\tau_p \equiv T_p/(\sqrt{2}\kappa_{\text{orb}})$ and $g_{ab}$ is the induced metric. It is easy to check that this action correctly accounts for the couplings (3.3). Furthermore, recalling that $\kappa_{\text{orb}} = \sqrt{2}\kappa$, we can see that the $C_8$ charge of our D7 brane is a half of that carried by the D7 branes in flat space.

For the twisted fields, instead, things are slightly more complicated. Using the couplings (3.4) and defining $\tau_3 \equiv T_3/(2\sqrt{2}\pi^2\alpha'\kappa_{\text{orb}})$, one can write

$$
S^{b}_{\text{D7}}|_{T} \simeq \frac{\tau_3}{4} \int d^4x \sqrt{-\det g_{\alpha\beta}} \, \tilde{b} - \frac{\tau_3}{4} \int A_4 + ...
$$

(3.6)

where in the first term the four-dimensional induced metric has been inserted to enforce reparametrization invariance on the world-volume, and the ellipses stand for terms of higher order which are not accounted by the boundary state approach but which, in principle, can be present. In the following section we will show that such higher order terms are indeed present in the complete world-volume action of the fractional D7 branes.

As explained in Ref. \cite{43}, the boundary state formalism allows also to compute the asymptotic behavior of the various fields in the classical brane solution. Applying this technique to the case of a stack of N coincident fractional D7 branes, we
find that, to leading order in $N g_s$, the metric is

$$ds^2 \simeq \eta_{ab} dx^a dx^b + \left( 1 - \frac{N g_s}{2\pi} \log \frac{\rho}{\epsilon} \right) \delta_{ij} dx^i dx^j + \ldots$$  \hspace{1cm} (3.7)

where $\rho = \sqrt{(x^4)^2 + (x^5)^2}$ and $\epsilon$ is a regulator, while the dilaton is

$$\phi \simeq \frac{N g_s}{2\pi} \log \frac{\rho}{\epsilon} + \ldots ,$$  \hspace{1cm} (3.8)

the 8-form R-R potential is

$$C_8 \simeq \frac{N g_s}{2\pi} \log \frac{\rho}{\epsilon} dx^0 \wedge \cdots \wedge dx^7 + \ldots .$$  \hspace{1cm} (3.9)

and the 4-form potential $C_4$ vanishes at linear order. The asymptotic behavior of the twisted fields is instead given by

$$\tilde{b} \simeq - N g_s \pi \alpha' \log \frac{\rho}{\epsilon} + \ldots ,$$  \hspace{1cm} (3.10)

$$A_4 \simeq - N g_s \pi \alpha' \log \frac{\rho}{\epsilon} dx^0 \wedge \cdots \wedge dx^3 + \ldots .$$  \hspace{1cm} (3.11)

If we insert these expressions into the world-volume action

$$S_{b}^{D7} = S_{b}^{D7} \big|_U + S_{b}^{D7} \big|_T$$  \hspace{1cm} (3.12)

and consistently retain only terms of first order in $N g_s$, we obtain a constant result, thus verifying the no-force condition at first order.

To extend this analysis to all orders, one needs to solve the complete field equations which we have derived in the previous section. However, to do this it is first necessary to establish the correct relation between the fields that the string description suggests and those appearing in the supergravity field equations. In particular we have to find how $\tilde{b}, A_4$ and $C_8$ are related to $b, c$ and $C_0$. It turns out that $\tilde{b}$ represents the fluctuation of the scalar $b$ of eq.(2.5) around the background value of the $Z_2$ orbifold [47], i.e.

$$b = \frac{1}{2} \left( 2\pi \sqrt{\alpha'} \right)^2 + \tilde{b} .$$  \hspace{1cm} (3.13)

The 4-form potential $A_4$ is instead the Hodge dual (in the six dimensional sense) to the twisted scalar $c$, while the 8-form potential $C_8$ is the dual (in the ten dimensional sense) to the axion $C_0$. These duality relations, which can be obtained from eq.s (2.11) and (2.13) remembering the analyticity of $\gamma$ and $\tau$ and the absence of source terms, are

$$dC_8 = - e^{2\phi} * dC_0 ,$$  \hspace{1cm} (3.14)

$$dA_4 = e^{\phi} * \epsilon^6 (dc + C_0 db) - C_4 \wedge db .$$  \hspace{1cm} (3.15)
The absence of source terms is due to the fact that the fields \( c \) and \( C_0 \) are not coupled to the boundary state of a D7-brane (see eqs. (3.3) and (3.4)).

Using the asymptotic expressions for the various fields in these relations, one can easily find that

\[
C_0 \simeq \frac{Ng_s}{2\pi} \theta + \ldots , \quad (3.16)
\]

\[
c \simeq Ng_s\pi \alpha' \theta + \ldots \quad (3.17)
\]

where \( \theta = \tan^{-1}(x^5/x^4) \). In the next section we are going to determine the higher order terms and find the complete classical solution with the asymptotic behavior described above. In particular we will discover that the untwisted 4-form \( C_4 \) and the metric along the world-volume directions of the fractional D7 brane will develop a non trivial profile at higher order.

4 The fractional D7/D3 bound state

Since the world volume of a 7-brane is eight-dimensional, a stack of \( N \) fractional D7 branes is not immediately useful to yield information on gauge theories in four dimensions. To obtain a classical solution capable of describing a four-dimensional field theory we must include also some D3-branes, and hence it is natural to study a bound state made of \( N \) fractional D7-branes and \( M \) fractional D3-branes. As shown in Ref. [32], this is a BPS configuration which preserves eight of the sixteen supersymmetries of the type IIB theory on the orbifold (2.1). Our task is then to solve the field equations derived in Section 2 and specify the appropriate boundary action \( S_b \) for this configuration. For the D3 brane components, we can simply take \( M \) times the action introduced in Ref. [7], and thus we can write

\[
S_{D3}^b = -M\tau_3 \int d^4x \sqrt{-\det g_{\alpha\beta}} \left( 1 + \frac{1}{2\pi^2\alpha'} \tilde{b} \right) + M\tau_3 \int C_4 \left( 1 + \frac{1}{2\pi^2\alpha'} \tilde{b} \right) + M\tilde{\tau}_3 \int A_4
\]

where \( \tau_3 \) and \( \tilde{\tau}_3 \) are defined after eq.(3.5) and before eq.(3.6). For the D7 brane components, instead, we can take as boundary action \( N \) times the sum of (3.5) and (3.6). As mentioned in the previous section, the twisted part (3.6) may be non-complete, but it is certainly correct at linear order and thus yields the right source terms in the various field equations. Using these ingredients and the Ansatz (2.19) and (2.20), one can show that eqs. (2.22) and (2.23) become

\[
\delta^{ij} \partial_i \partial_j \tau + i 2\kappa_{\text{orb}}^2 \tau_7 N \delta(x^4) \delta(x^5) = 0 , \quad (4.2)
\]

\[
\delta^{ij} \partial_i \partial_j \gamma - i \kappa_{\text{orb}}^2 \tilde{\tau}_3 (2M - N) \delta(x^4) \delta(x^5) = 0 . \quad (4.3)
\]
The holomorphic solutions to these equations can be immediately found and are (see also Ref. [32])

\[ \tau = i \left( 1 - \frac{N g_s}{2\pi} \log \frac{z}{\epsilon} \right), \quad (4.4) \]

and

\[ \gamma = i 2\pi \alpha' g_s \left[ \frac{\pi}{g_s} + (2M - N) \log \frac{z}{\epsilon} \right], \quad (4.5) \]

where we have chosen the integration constants to enforce the appropriate background values. Written in terms of the real supergravity fields, eqs. (4.4) and (4.5) become

\[ e^{\phi} = \frac{1}{1 - \frac{N g_s}{2\pi} \log \frac{2}{\epsilon}}, \quad (4.6) \]

\[ C_0 = \frac{N g_s}{2\pi} \theta, \quad (4.7) \]

\[ b = \frac{(2\pi \sqrt{\alpha'})^2}{2} \left[ 1 + \frac{(2M-N)g_s}{\pi} \log \frac{2}{\epsilon} \right] \left[ 1 - \frac{N g_s}{2\pi} \log \frac{2}{\epsilon} \right], \quad (4.8) \]

\[ c = -(2\pi \alpha') \theta g_s \left( 2M - N \right) \left[ 1 - \frac{N g_s}{2\pi} \log \frac{2}{\epsilon} \right]. \quad (4.9) \]

Notice that for \( N = 0 \) this solution reduces to the one of pure fractional D3-branes discussed in Refs. [7, 8]. It is interesting to observe, on the other hand, that putting \( M = 0 \) and expanding in powers of \( \frac{N g_s}{2\pi} \), we recover at first order the solution (3.8), (3.16) and (3.17) obtained from the boundary state approach. In this respect, we observe that the axion \( C_0 \) does not receive corrections to higher orders while the twisted fields \( b \) and \( c \) acquire an infinite tail of logarithmic terms. This is to be contrasted with the solution of the pure fractional D3 branes [7, 8] where the twisted scalars had, instead, only terms at first order. Thus, if one wants to determine the classical profile of the twisted scalars using the boundary state formalism in the presence of fractional D7 branes, it is not sufficient to consider contributions with just one boundary, but it is necessary to sum over all contributions with an arbitrary number of boundaries as explained in Ref. [16], which, due to the open/closed string duality, is equivalent to sum over an arbitrary number of open-string loops.

Finally, if we insert the above expressions for \( b \) and \( c \) into eq. (2.24), we obtain the following equation for the warp factor \( H \):

\[ \left( \delta^{ij} \partial_i \partial_j + e^{-\phi} \delta^{\ell m} \partial_{\ell} \partial_{m} \right) H + 2\kappa_{\text{orb}}^2 \tau_3 \delta(x^4) \cdots \delta(x^9) \]

\[ + \left( 2\pi \alpha' g_s \right)^2 \left[ \frac{(2M-N)^2}{\rho^2} \left( \frac{N g_s}{2\pi} \right)^3 \delta(x^6) \cdots \delta(x^9) \right] = 0. \quad (4.10) \]

In general, it is not possible to find an explicit solution of this equation in terms of elementary functions. For \( N = 0 \) this equation was solved exactly in Ref. [1],
whereas for \( N = 4M \), i.e. when the last term vanishes, this equation becomes of the same form that was considered in Ref. 33. It is also interesting to observe that eq. (4.10) remains non-trivial even for \( M = 0 \). This fact means that for a system made of only D7 branes on orbifold both the longitudinal metric and the 4-form \( C_4 \) are not trivial, contrarily to what happens for D7 branes in flat space. However, it should be realized that these fields start developing only at the second order in the string coupling constant, as is clear from the structure of eq. (4.10) for \( M = 0 \).

Using the full solution (1.6)-(1.9) in the duality relations (3.14) and (3.15), and recalling, apart irrelevant additional terms, that

\[
C_4 = \left( H^{-1} - 1 \right) dx^0 \wedge \cdots \wedge dx^3,
\]

we can easily obtain the complete expressions for the 8-form \( C_8 \) and for the twisted 4-form \( A_4 \) which are more natural from a stringy perspective. After some algebra, we find

\[
C_8 = \frac{Ng_s}{2\pi} \log \frac{2}{1 - \frac{Ng_s}{2\pi} \log \frac{2}{\epsilon}} \ dx^0 \wedge \cdots \wedge dx^7 = \left( e^\phi - 1 \right) dx^0 \wedge \cdots \wedge dx^7,
\]

and

\[
A_4 = \left( 2\pi \sqrt{\alpha'} \right)^2 \frac{(4M-N)Ng_s}{2\pi} \log \frac{2}{1 - \frac{Ng_s}{2\pi} \log \frac{2}{\epsilon}} \ dx^0 \wedge \cdots \wedge dx^3 = \tilde{b} \ dx^0 \wedge \cdots \wedge dx^3.
\]

Having the complete solution, we can verify the no-force condition and check the structure of the world-volume action of the bound state. If we substitute our solution into the D3-brane component (1.1) of the boundary action, we find, as expected, that all terms depending on the transverse coordinates cancel, leaving a constant result. Doing the same thing for the D7-brane components (3.5) and (3.6), we see that in the twisted part not all terms cancel, thus indicating the presence of a non-zero force. However, this result is unacceptable in view of the BPS properties of our solution. This problem is easily overcome if we recall that the boundary action (3.6) is actually justified only at the linear level, and thus may be non-complete. To construct the full boundary action we can start from the standard expansion of the WZ part of the action for a D7 brane, namely

\[
S_{WZ} = \tau_7 \int \left( \hat{C}_8 + \hat{C}_6 \wedge B_2 + \frac{1}{2} \hat{C}_4 \wedge B_2 \wedge B_2 \right) + \ldots
\]

where the ellipses stand for curvature terms. We now decompose the forms \( \hat{C}_8, \hat{C}_6 \) and \( \hat{C}_4 \) into untwisted components (denoted by \( C \)) and into twisted components along the 2-form \( \omega_2 \) (denoted by \( A \)). For the case under consideration, the relevant expressions are

\[
\hat{C}_8 = C_8 + x \left( 2\pi \sqrt{\alpha'} \right)^2 A_4 \wedge \omega_2 \wedge \omega_2,
\]

\[
\hat{C}_6 = y A_4 \wedge \omega_2,
\]

\[
\hat{C}_4 = C_4.
\]
where $x$ and $y$ are numerical coefficients which will be determined later. If we substitute eq. (4.15) into the action (4.14) and recall that $B_2 = b \omega_2$ with $b$ given by eq. (3.13), after some simple manipulations we get

$$S_{WZ} = \tau_7 \int C_8 - \frac{\tau_3}{4} \int A_4 \left[ (2x + y) + \frac{y}{2\pi^2 \alpha'} \tilde{b} \right] - \frac{\tau_3}{4} \int C_4 \tilde{b} \left( 1 + \frac{\tilde{b}}{4\pi^2 \alpha'} \right).$$  

(4.16)

Notice that in writing the last expression we have used the fact that the (understood) curvature contribution exactly cancels the term linear in $C_4$. This fact, shown in Ref. [32], is consistent with the boundary state of a fractional D7 brane which indeed does not couple to $C_4$. Instead, it couples to the twisted 4-form $A_4$, and matching the corresponding charge with the boundary state result (see eq. (3.6)) fixes

$$2x + y = 1.$$  

(4.17)

If we substitute the classical solution (4.12)-(4.13) in eq. (4.16) and require no force, we can see that to cancel the contribution of $C_8$ we must add to the boundary action the expected DBI term

$$- \tau_7 \int d^8 x \ e^\phi \sqrt{-\det g_{ab}},$$  

(4.18)

while to cancel the contribution of $C_4$ we must add a term like

$$\frac{\tau_3}{4} \int d^4 x \ \sqrt{-\det g_{\alpha\beta}} \ \tilde{b} \left( 1 + \frac{\tilde{b}}{4\pi^2 \alpha'} \right)$$  

(4.19)

and fix $y = 1/2$. In this way the no-force condition is fully satisfied, as it should be. We thus conclude that the world-volume action of a fractional D7 brane consists of an untwisted part given by eq. (3.5) and a twisted part given by

$$S^{D7}_b |_T = \frac{\tau_3}{4} \int d^4 x \ \sqrt{-\det g_{\alpha\beta}} \ \tilde{b} \left( 1 + \frac{\tilde{b}}{4\pi^2 \alpha'} \right) - \frac{\tau_3}{4} \int A_4 \left( 1 + \frac{\tilde{b}}{4\pi^2 \alpha'} \right) - \frac{\tau_3}{4} \int C_4 \tilde{b} \left( 1 + \frac{\tilde{b}}{4\pi^2 \alpha'} \right) \ .$$  

(4.20)

It would be interesting to confirm the structure of this boundary action with geometrical considerations and also with explicit calculations of closed string scattering amplitudes on a disk with boundary conditions appropriate for the fractional D7 brane, similarly to what has been done in Ref. [15] for the fractional D3 branes.
5 The probe action and the $\mathcal{N} = 2$ gauge theory

The supergravity solution found in the previous section can provide non-trivial information on its dual four-dimensional gauge theory. To see this we use the probe technique (for a review see Ref. [46]) and consider a probe fractional D3-brane carrying a gauge field $F_{\alpha\beta}$ and slowly moving in the supergravity background produced by $M$ D3 and $N$ D7 fractional branes. We then fix the static gauge and study the world-volume action of the probe, regarding the transverse coordinates as Higgs fields $\Phi^i = (2\pi\alpha')^{-1/2}x^i$, and expanding up to quadratic terms in derivatives. From a gauge theory point of view, the resulting action describes the $SU(M) \times U(1)$ Coulomb phase of a $SU(M + 1)$ gauge theory in which the symmetry breaking corresponds to taking one of the D3 branes (the probe) away from the others at a distance $\rho = |z|$ related to the energy scale where the theory is defined.

Applying this technique to our case, we find that the action of a probe fractional D3-brane can be written as

$$S = S_0 + S_{\text{gauge}}$$

(5.1)

where $S_0$ is given by eq. (4.1) with $M = 1$ and

$$S_{\text{gauge}} = -\frac{1}{8\pi g_s} \int d^4x \sqrt{-\det G_{\alpha\beta}} \left\{ \frac{1}{4} e^{-\phi} G^{\alpha\gamma} G^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \right\}$$

$$+ \frac{1}{2} G_{ij} G^{\alpha\beta} \partial_\alpha \Phi^i \partial_\beta \Phi^j \left( 1 + \frac{\tilde{b}}{2\pi^2 \alpha'} \right) + \frac{1}{8\pi g_s} \int d^4x \frac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \left( c + C_0 b \right)$$

(5.2)

where $\tilde{F}^{\alpha\beta} = (1/2) \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$. Inserting in $S_0$ the solution for the closed string fields obtained in the previous section (i.e. eqs (4.4)-(4.5) and the Ansatz (2.19)-(2.20)), we see that $S_0$ becomes independent of the distance between the probe and the source branes that yield the classical solution. This is in agreement with the fact that there is no interaction between a fractional D3-brane and a system of fractional D3/D7-branes.

Considering now eq. (5.2), we see that the dependence on the function $H$ drops out in this case too, while the kinetic terms for the gauge field strength $F_{\alpha\beta}$ and the scalar fields $\Phi^i$ have the same coefficient, in agreement with the fact that the gauge theory living on the brane has $\mathcal{N} = 2$ supersymmetry. Indeed one gets

$$S_{\text{gauge}} = -\frac{1}{g_{YM}^2(\mu)} \int d^4x \left\{ \frac{1}{2} \partial_\alpha \Phi^i \partial^\alpha \Phi^i + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right\} + \frac{\theta_{YM}}{32\pi^2} \int d^4x F_{\alpha\beta} \tilde{F}^{\alpha\beta}$$

(5.3)

where

$$\frac{1}{g_{YM}^2(\mu)} = \frac{1}{g_{YM}^2} + \frac{2M - N}{8\pi^2} \log \mu \ ; \ g_{YM}^2 = 8\pi g_s$$

(5.4)

$$\theta_{YM} = (2M - N) \theta$$

(5.5)
are the effective Yang-Mills gauge coupling and $\theta$-angle, respectively. The renormalization group scale is defined by $\mu \equiv |z|/\epsilon$, while $g_{YM}^2$ is the bare coupling, i.e. the value of the gauge coupling at the ultraviolet cutoff $\mu = 1$.

Eq. (5.4) clearly shows that $g_{YM}(\mu)$ is the running coupling constant of an $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $SU(M)$ and $N$ hypermultiplets in the fundamental representation. This is precisely the field theory living on the system of $M$ D3-branes and $N$ D7-branes, where the gauge vector multiplet corresponds to open strings stretched between two fractional D3-branes, while the hypermultiplets correspond to strings stretched between the D3 and the D7-branes. The reason why the hypermultiplet kinetic term is absent in $S_{\text{gauge}}$ is just because the probe is a D3-brane only, and therefore there are no 3-7 strings that can give rise to massless fields on the probe world-volume. Of course, this theory is ultraviolet free only for $N \leq 2M$.

From eq. (5.4) one sees that on the geometric locus defined by

$$
\rho < |z_e| = \epsilon e^{-\pi/(2M-N)g_s},
$$

the D3-brane probe becomes tensionless, thus indicating the presence of an enhançon. At distances smaller than $|z_e|$ the probe has negative tension, while at the enhançon extra light degrees of freedom come into play [8]. This means that the supergravity approximation leading to the solution described in section 4 is not valid anymore, and that the region of space-time $\rho < |z_e|$ is excised. Notice that the vanishing of the tension of the probe at the enhançon is consistent with the fact that fractional branes are tension-full because of the presence of a non-vanishing $B_{(2)}$ flux, $b$ [17]. Indeed, by using eq. (5.4), one can write $\gamma$ as follows

$$
\gamma = 2\pi i \alpha' g_s (2M - N) \log z / |z_e|,
$$

and see that the quantity $\text{Im} \gamma \equiv e^{-\theta} b$, which is proportional to the probe tension, vanishes at the enhançon, since there the fluctuation of the $b$ field cancels precisely its background value.

From eq. (5.4) one can immediately recognize what is the meaning of the enhançon from the gauge theory point of view. This is the scale where the gauge coupling diverges (which in QCD is called $\Lambda_{\text{QCD}}$) and where non-perturbative corrections become relevant. This means that, as discussed in the Introduction, the supergravity solution is only able to reproduce the perturbative moduli space of the gauge theory, while the appearance of the enhançon prevents from using the classical solution to analyze the strong-coupling properties of the gauge theory. The translational dictionary between supergravity and gauge quantities can then be summarized as follows

$$
\frac{4\pi}{g_{YM}^2(\mu)} = \frac{1}{(2\pi \sqrt{\alpha'})^2} e^{-\phi} b, \quad \frac{\theta_{YM}}{2\pi} = -\frac{1}{(2\pi \sqrt{\alpha'})^2} \frac{1}{g_s} (c + C_0 b),
$$

$$
\Lambda_{\text{UV}} = (2\pi \alpha')^{-1} \epsilon, \quad \Lambda_{\text{QCD}} = (2\pi \alpha')^{-1} |z_e|,
$$
where in eq. (5.8) the dilaton includes also its background value. It is interesting to observe that the presence of D7-branes lowers the enhançon radius. This can be seen explicitly from eq. (5.6). In particular, when $N = 2M$ the enhançon vanishes, the gauge coupling constant does not run anymore and the gauge theory becomes conformal, as expected for a supersymmetric $\mathcal{N} = 2$ gauge theory with gauge group $SU(M)$ and with $2M$ hypermultiplets transforming according to the fundamental representation of $SU(M)$. Notice that also in the conformal case the contribution of the twisted fields in eq. (4.10) for $H$ does not vanish making the solution of eq. (4.10) quite not trivial. The twisted field contribution vanishes, however, for $N = 4M$ and in this case the field equation for $H$ reduces to the one discussed in Ref. [33]. The vanishing of the twisted contribution is a consequence of the fact that the coupling of a fractional D7-brane to the twisted fields is a factor $1/4$ smaller than that of a fractional D3-brane. In this case, however, the theory is not ultraviolet free.

A distinctive feature of the D3/D7 system with respect to that of pure fractional D3-branes of Ref. [7] is that the scalar fields given in eqs (4.6)-(4.9) are expressed as an infinite series in the open string coupling. However, the scalar field combinations which have a meaning at the gauge theory level, namely those appearing in eq. (5.8), are exact at one-loop, as expected for a $\mathcal{N} = 2$ super Yang-Mills theory. This non-trivial cancellation is a (higher loop) check of the validity of the gauge/gravity correspondence.

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A The boundary state description of the Dp brane

The boundary state for a Dp brane with $r$ directions of its world-volume outside and $s = p - r$ directions along the orbifold $\mathbb{R}^4/\mathbb{Z}_2$, can be derived by factorizing the one-loop vacuum amplitude of the open strings stretching between two such branes. This amplitude is given by

$$ Z = Z_1 + Z_g $$

where

$$ Z_1 = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr}_{\text{NS}} \left[ P_{\text{GSO}} e^{-2\pi s (L_0 - a)} \right] $$

$$ = \frac{1}{2} \frac{V_{p+1}}{(8\pi^2 \alpha')^{(p+1)/2}} \int_0^\infty \frac{ds}{s^{(p+3)/2}} \frac{1}{2} \left[ \frac{f_3^\delta(q) - f_5^\delta(q) - f_2^\delta(q)}{f_1^\delta(q)} \right], \quad (A.2) $$
\[ Z_g = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr}_{\text{NS-R}} \left[ g P_{\text{GSO}} e^{-2\pi s (L_0 - a)} \right] \]

\[ = \frac{V_{r+1}}{2^s (8\pi^2 \alpha')^{(r+1)/2}} \int_0^\infty \frac{ds}{s^{(r+3)/2}} \left[ \frac{f_3^4(q) f_4^4(q)}{f_1(q) f_2^4(q)} - \frac{f_3^4(q) f_4^4(q)}{f_1(q) f_2^4(q)} \right] \]  

\text{(A.3)}

where \( P_{\text{GSO}} \) is the GSO projection, \( g \) is the orbifold parity, \( q = e^{-\pi s} \), the \( f \)'s are the standard one-loop modular functions and the intercept \( a \) is 1/2 \([0]\) in the NS \([R]\) sector. Notice the appearance of the important factor \( 2^{-s} \) in eq.\( (A.3) \) that is due to the integration over the bosonic zero modes along the orbifolded directions \([42]\).

After performing the modular transformation \( s \to t = 1/s \), \( Z_1 \) and \( Z_g \) can be interpreted as tree level closed string amplitudes between two untwisted and two twisted boundary states respectively, that is

\[ Z_1 = \frac{\alpha' \pi}{2} \int_0^\infty dt \ U\langle Dp | e^{-\pi t (L_0 + \tilde{L}_0 - 2a)} | Dp \rangle^U \ , \]  

\[ Z_g = \frac{\alpha' \pi}{2} \int_0^\infty dt \ T\langle Dp | e^{-\pi t (L_0 + \tilde{L}_0)} | Dp \rangle^T \ . \]  

\text{(A.4) (A.5)}

From eq.\( (A.2) \) it is immediate to realize that \( Z_1 \) is one half of the amplitude for \( Dp \)-branes in flat space, and therefore the untwisted part of the boundary state is simply

\[ |Dp\rangle^U = \frac{T_p}{2\sqrt{2}} \left( |Dp\rangle_{\text{NS}}^U + |Dp\rangle_{\text{R}}^U \right) \]  

\text{(A.6)}

where \( |Dp\rangle_{\text{NS}}^U \) and \( |Dp\rangle_{\text{R}}^U \) are the usual boundary states for a bulk \( Dp \)-brane given in Refs. \([43, 44]\).

From eq.\( (A.3) \) we can see that the twisted amplitude for a fractional \( Dp \)-brane with \( s \) directions along the orbifold is the same as the one for a fractional \( Dr \)-brane entirely outside the orbifold, apart from a factor \( 2^{-s} \). Therefore, using eq.\( (A.3) \), we can deduce that the boundary state \( |Dp\rangle^T \) is similar to the boundary state for a fractional \( Dr \)-brane transverse to the orbifold, but with an extra factor of \( 2^{-s/2} \) in its normalization. In conclusion, we get

\[ |Dp\rangle^T = -\frac{1}{2^{s/2} 2\sqrt{2} \pi^2 \alpha'} \left( |Dp\rangle_{\text{NS}}^T + |Dp\rangle_{\text{R}}^T \right) \]  

\text{(A.7)}

where

\[ |Dp\rangle_{\text{NS,R}}^T = \frac{1}{2} \left( |Dp, +\rangle_{\text{NS,R}}^T + |Dp, -\rangle_{\text{NS,R}}^T \right) \ , \]  

\text{(A.8)}

and the Ishibashi states \( |Dp, \eta\rangle_{\text{NS,R}}^T \) are

\[ |Dp, \eta\rangle_{\text{NS}}^T = |Dp_{\psi}^X\rangle^T |Dp_{\psi}, \eta\rangle_{\text{NS}}^T \]  

\text{(A.9)}

in the NS-NS twisted sector, and

\[ |Dp, \eta\rangle_{\text{R}}^T = |Dp_{\psi}^X\rangle^T |Dp_{\psi}, \eta\rangle_{\text{R}}^T \]  

\text{(A.10)}
in the R-R twisted sector \[1\] with

\[
|Dp_X|^T = \delta^{(g-r)} (q^i - y^i) \prod_{n=1}^{\infty} e^{- \frac{i}{8} \alpha_{-n} \bar{\alpha}_n} \prod_{n=1}^{\infty} e^{\frac{i}{8} \alpha_{-n} \bar{\alpha}_n} \prod_{r=\frac{1}{2}}^{\infty} e^{- \frac{i}{8} \alpha_r \bar{\alpha}_r} |p_\beta = 0 \rangle \prod_i |p_i = 0 \rangle,
\]

\[
|Dp_\psi, \eta\rangle^{(0)}_{NS} = \prod_{r=\frac{1}{2}}^{\infty} e^{i \eta \psi^\alpha_{-r} \bar{\psi}^\beta_{-r}} \prod_{r=\frac{1}{2}}^{\infty} e^{-i \eta \psi^\alpha_{-r} \bar{\psi}^\beta_{-r}} \prod_{n=1}^{\infty} e^{i \eta \psi^\alpha_{-n} \bar{\psi}^\beta_{-n}} |Dp_\psi, \eta\rangle^{(0)}_{NS}|T\rangle,
\]

\[
|Dp_\psi, \eta\rangle^{(0)}_R = \prod_{n=1}^{\infty} e^{i \eta \psi^\alpha_{-n} \bar{\psi}^\beta_{-n}} \prod_{n=1}^{\infty} e^{-i \eta \psi^\alpha_{-n} \bar{\psi}^\beta_{-n}} \prod_{r=\frac{1}{2}}^{\infty} e^{i \eta \psi^\alpha_{-r} \bar{\psi}^\beta_{-r}} |Dp_\psi, \eta\rangle^{(0)}_R|T\rangle.
\]

In these expressions the longitudinal indices \(\alpha, \beta\) take values \(0, 1, \ldots, r\), the transverse index \(i\) takes values \(r+1, \ldots, 5\), while the index \(\ell\) labels the orbifold directions (to avoid further clutter, in the above formulas we have explicitly considered only the case in which all these orbifold directions are longitudinal). The zero-mode part of the boundary state has a non trivial structure in both sectors; in the NS-NS sector it is given by \[4\]

\[
|Dp_\psi, \eta\rangle^{(0)}_{NS} = \left( \hat{\gamma}^6 \cdots \hat{\gamma}^{5+s} \frac{1+i\eta \hat{\gamma}^s}{1+i\eta} \right)_{LM} |L\rangle |\hat{M}\rangle
\]

(A.11)

where \(\hat{\gamma}^\ell\) are the gamma matrices and \(\hat{C}\) the charge conjugation matrix of \(SO(4)_\text{NS}\), \(\hat{\gamma} = \hat{\gamma}^6 \cdots \hat{\gamma}^9\), and, finally, \(|L\rangle\) and \(|\hat{M}\rangle\) are spinors of \(SO(4)_\text{NS}\). The matrices of \(SO(4)_\text{NS}\) satisfy the following relations under transposition

\[
\hat{C}^t = \hat{C} , \quad \hat{\gamma}^\ell t = \hat{C} \hat{\gamma}^\ell \hat{C}^{-1} .
\]

(A.12)

In the R-R sector, instead, we have

\[
|Dp_\psi, \eta\rangle^{(0)}_R = \left( \hat{\gamma}^0 \cdots \hat{\gamma}^{r} \frac{1+i\eta \hat{\gamma}^r}{1+i\eta} \right)_{AB} |A\rangle |\hat{B}\rangle
\]

(A.13)

where \(\hat{\gamma}^\alpha\) are the gamma matrices and \(\hat{C}\) the charge conjugation matrix of \(SO(1,5)_\text{R}\), \(\hat{\gamma} = \hat{\gamma}^0 \cdots \hat{\gamma}^{5}\), and, finally, \(|A\rangle\) and \(|\hat{B}\rangle\) are spinors of \(SO(1,5)_\text{R}\). The matrices of \(SO(1,5)_\text{R}\) satisfy the following relations under transposition

\[
C^t = -C , \quad \hat{\gamma}^\alpha t = -C \hat{\gamma}^\alpha C^{-1} .
\]

(A.14)

In order to compute the fermionic zero-mode contribution to \(Z_g\) in eq.(A.3) it is convenient to write explicitly the conjugate vacuum states, which are given by \[14\]

\[
\langle Dp_\psi, \eta|^{(0)T}_{NS} \langle \hat{M} | \langle L| \left( \hat{\gamma}^6 \cdots \hat{\gamma}^{5+s} \frac{1-i\eta \hat{\gamma}^s}{1-i\eta} \right)_{LM}
\]

(A.15)
for the twisted NS-NS sector, and

$$\langle (0)^T_R (Dp\psi, \eta_1)|Dp\psi, \eta_2\rangle_{R} = -4\delta_{\eta_1, \eta_2}$$  \hspace{1cm} (A.18)

for the R-R sector. Using the previous expressions and performing some straightforward algebra, it is possible to show that

$$\langle (0)^T_{NS} (Dp\psi, \eta_1)|Dp\psi, \eta_2\rangle_{NS} = 4\delta_{\eta_1, \eta_2}$$  \hspace{1cm} (A.17)

for the NS-NS sector, and

$$\langle (0)^T_{NS} (Dp\psi, \eta_1)|Dp\psi, \eta_2\rangle_{NS} = 4\delta_{\eta_1, \eta_2}$$  \hspace{1cm} (A.17)

for the R-R sector.

Finally, by saturating the boundary states described above with the untwisted closed string states explicitly given in Refs. [1, 4, 5, 12], one obtains eq.s (3.3), while by saturating the twisted components with the twisted states given in Ref. [12] one gets eq.s (3.4).

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