Sutured Khovanov homology distinguishes braids from other tangles

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Abstract
We show that the sutured Khovanov homology of a balanced tangle in the product sutured manifold $D \times I$ has rank 1 if and only if the tangle is isotopic to a braid.

1 Introduction

In [11], Khovanov constructed a categorification of the Jones polynomial that assigns a bigraded abelian group to each link in $S^3$. Sutured Khovanov homology is a variant of Khovanov’s construction that assigns

- to each link $L$ in the product sutured manifold $A \times I$ (see Section 2.1) a triply-graded vector space $SKh(L)$ over $F := \mathbb{Z}/2\mathbb{Z}$ [11][19], and

- to each balanced, admissible tangle $T$ in the product sutured manifold $D \times I$ (see Section 2.2) a bigraded vector space $SKh(T)$ over $F$ [12][5].

Khovanov homology detects the unknot [14] and unlinks [8][3], and the sutured annular Khovanov homology of braid closures detects the trivial braid [2]. In this note, we prove that the sutured Khovanov homology of balanced tangles distinguishes braids from other tangles.

Theorem 1.1. Let $T \subset D \times I$ be a balanced, admissible tangle. Then $SKh(T)$ has rank 1 if and only if $T$ is isotopic to a braid in $D \times I$.

Theorem 1.1 is one of many results about the connection between Floer homology and Khovanov homology, starting with the work of Ozsváth and
This theorem is an analogue of the fact that sutured Floer homology detects product sutured manifolds \cite{7,10}, which is also an ingredient in our proof. Other ingredients include a spectral sequence relating sutured Khovanov homology and sutured Floer homology \cite{5}, Meeks–Scott’s theorem on finite group actions on product manifolds \cite{15}, and Kronheimer–Mrowka’s theorem that Khovanov homology is an unknot detector \cite{14}.

Given a link \( L \subset A \times I \), the \textit{wrapping number} of \( L \) is the minimal geometric intersection number of all links isotopic to \( L \) with the meridional disk of \( A \times I \).

Theorem 1.1 combined with the observations in \cite{6} imply:

**Corollary 1.2.** Let \( L \subset A \times I \) be a link with wrapping number \( \omega \), then the group

\[
SKh(L; \omega) = \bigoplus_{i,j} SKh^i(L; j, \omega)
\]

has rank 1 if and only if \( L \) is isotopic to a closed braid in \( A \times I \).

This corollary is an analogue of the fact that knot Floer homology detects fibered knots.

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## 2 Preliminaries

In this section, we will review the basics about sutured manifolds \cite{4} and sutured Khovanov homology \cite{1,19,5,6}.

**Definition 2.1.** A \textit{sutured manifold} \((M, \gamma)\) is a compact oriented 3–manifold \(M\) together with a set \( \gamma \subset \partial M \) of pairwise disjoint annuli \( A(\gamma) \) and tori \( T(\gamma) \).

Every component of \( R(\gamma) = \partial M - \text{int}(\gamma) \) is oriented. Define \( R^+(\gamma) \) (or \( R^-(\gamma) \)) to be the union of those components of \( R(\gamma) \) whose normal vectors point out of (or into) \( M \).

As an example, let \( S \) be a compact oriented surface, \( M = S \times I, \gamma = (\partial S) \times I, R^-(\gamma) = S \times 0, R^+(\gamma) = S \times 1 \), then \((M, \gamma)\) is a sutured manifold. In this case we say that \((M, \gamma)\) is a \textit{product sutured manifold}.

**Definition 2.2.** \cite[Definition 2.2]{9} A \textit{balanced sutured manifold} is a sutured manifold \((M, \gamma)\) satisfying

1. \( M \) has no closed components.
2. Every component of \( \partial M \) intersects \( \gamma \) nontrivially, \( T(\gamma) = \emptyset \).
3. \( \chi(R^+(\gamma)) = \chi(R^-(\gamma)) \).

If \((M, \gamma)\) is a balanced, sutured manifold, then \( SFH(M, \gamma) \) will denote its \textit{sutured Floer homology}, as defined by Juhász in \cite{9}. Whenever \( \gamma \) is implicit (e.g., when \( M \) is a product), we shall omit it from the notation.
We will be interested in Khovanov-type invariants for certain links and tangles in product sutured manifolds.

2.1 Sutured Khovanov homology of links in $A \times I$

Sutured annular Khovanov homology, originally defined in [1], [19] (see also [6]) associates to an oriented link $L$ in the product sutured manifold $A \times I$ a triply-graded vector space

$$\text{SKh}(L) = \bigoplus_{i,j,k} \text{SKh}^i(L; j, k),$$

which is an invariant of the oriented isotopy class of $L \subset A \times I$.

To define it, one chooses a diagram $D_L$ of $L$ on $A \times \{1/2\}$. By filling in the inner (resp., outer) boundary component of $A \times \{1/2\}$ with a disk marked with a basepoint $X$ (resp., $O$) at its center, one obtains a diagram on $S^2 - \{X, O\}$. Ignoring the $X$ basepoint yields a diagram on $\mathbb{R}^2 = S^2 - \{O\}$ from which the ordinary bigraded Khovanov chain complex

$$\text{CKh}(D_L) := \text{CKh}^i(D_L; j)$$

can be constructed from a cube of resolutions. Here, $i$ and $j$ are the homological and quantum gradings, respectively. The basepoint $X$ gives rise to a filtration on $\text{CKh}(D_L)$, and $\text{SKh}(L)$ is the homology of the associated graded object.

To define this filtration, choose an oriented arc from $X$ to $O$ missing all crossings of $D_L$. As described in [7, Sec. 4.2], the generators of $\text{CKh}(D_L)$ are in one-to-one correspondence with oriented resolutions. The "$k$" grading of an oriented resolution is defined to be the algebraic intersection number of this resolution with our oriented arc. Roberts proves ([19, Lem. 1]) that the Khovanov differential does not increase this extra grading.

One therefore obtains a bounded filtration,

$$0 \subseteq \ldots \subseteq F_{n-1}(D_L) \subseteq F_n(D_L) \subseteq F_{n+1}(D_L) \subseteq \ldots \subseteq \text{CKh}(D_L),$$

where $F_n(D_L)$ is the subcomplex of $\text{CKh}(D_L)$ generated by oriented resolutions with $k$ grading at most $n$. Let

$$F_n(D_L; j) = F_n(D_L) \cap \bigoplus_i \text{CKh}^i(D_L; j).$$

The sutured annular Khovanov homology groups of $L$ are defined to be

$$\text{SKh}^i(L; j, k) := H^i \left( \frac{F_k(D_L; j)}{F_{k-1}(D_L; j)} \right).$$

It is an immediate consequence of the definitions that if $L$ has wrapping number $\omega$, then $\text{SKh}^i(L; j, k) \cong 0$ for $k \not\in \{-\omega, -2, \ldots, -\omega - 2, \omega\}$.

We shall denote by $\Sigma(A \times I, L)$ the sutured manifold obtained as the double cover of $A \times I$ branched along $L$ (cf. [6] Rmk. 2.6).
2.2 Sutured Khovanov homology of balanced tangles in \(D \times I\)

A tangle \(T\) in the product sutured manifold \((D \times I, \gamma)\) is said to be \textit{admissible} if \(\partial T \cap \gamma = \emptyset\) and \textit{balanced} if \(|T \cap R_+| = |T \cap R_-|\). In order to make sense of tangle composition (stacking), we will fix an identification of \(D\) with the standard unit disk in \(\mathbb{C}\) and assume that \(\partial T\) intersects both \(D_+\) and \(D_-\) along the real axis.

The sutured Khovanov homology of an admissible, balanced tangle in \(D \times I\) was defined by Khovanov \([12]\) in the course of constructing a categorification of the reduced \(n\)-colored Jones polynomial. Khovanov’s construction was later related to sutured Floer homology in \([5]\). The theory associates to an oriented, admissible, balanced tangle in \(D \times I\) a bigraded vector space

\[
\text{SKh}(T) = \bigoplus_{i,j} \text{SKh}^i(T; j)
\]

that is invariant under oriented isotopy rel boundary of \(T \subset D \times I\).

To define it, one chooses a diagram \(D_T\) of \(T\) on \([-1, 1] \times I\) and constructs a cube of resolutions and a Khovanov quotient complex

\[
\text{CKh}(D_T) := \text{CKh}^i(D_T; j)
\]

as follows.

First, consider the complex whose underlying vector space is freely generated by all \textit{oriented} resolutions of \(T\). The boundary map on this complex is defined in the usual way, with upward (resp., downward) orientations on non-closed components treated like CCW (resp., CW) orientations in the merge/split maps defining the differential. Now take the quotient by the subcomplex generated by those oriented resolutions containing at least one downward-oriented non-closed component.

Note that the underlying vector space of the resulting quotient complex has a basis given by oriented resolutions whose non-closed components all point \textit{upward} at the boundary. Informally, the non-closed components are treated as base-pointed strands in Khovanov’s \textit{reduced} theory. It is evident that if a resolution \textit{backtracks}, i.e., if one of its non-closed components has both boundary points on \(R_+\), then the vector space assigned to the corresponding vertex in the cube of resolutions is 0.

\textbf{Remark 2.3.} If \(T\) is an admissible \(n\)-balanced tangle in \(D \times I\), then we can alternatively associate to \(T\) a left \(H^n\)–module, \(\mathcal{F}(T)\), as in \([13]\), by viewing \(T\) as a tangle with \(2n\) left endpoints. The chain complex computing \(\text{SKh}(T)\) may then be identified with \(\bar{\nu}_- \otimes_{H^n} \mathcal{F}(T)\), where \(\bar{\nu}_-\) is the right \(H^n\) module constructed as follows. Let \(b\) denote the fully-nested crossingless match on \(2n\) points; then \(\bar{\nu}_-\) is the two-sided ideal of the \(H^n\) module \(\mathcal{F}(W(b)b)\) corresponding to the generator whose strands are all oriented clockwise (i.e., labeled with a \(v_-\)).

Comparing the above description with the description of the sutured annular Khovanov invariant in the previous section, we have:
Proposition 2.4. [6, Thm. 3.1] If $L \subset A \times I$ is an oriented annular link with wrapping number $\omega$, and $T_\theta$ is the oriented, admissible balanced tangle obtained by decomposing $A \times I$ along a meridional disk $D_\theta$ for which $|L \cap D_\theta| = \omega$, then

$$SKh_i(L; j, \omega) \cong SKh_i(T_\theta; j).$$

3 Proof of the main theorem

Definition 3.1. A tangle $T \subset D \times I$ is a string link if it consists of proper arcs, each of which has one end on $D \times \{0\}$ and the other end on $D \times \{1\}$. In other words, $T$ contains no closed components, and $T$ does not backtrack.

Lemma 3.2. Let $T \subset D \times I$ be a balanced tangle, then rank $SKh(T)$ is odd if and only if $T$ is a string link.

Proof. We observe that if two tangles $T_+, T_-$ differ by a crossing change, then the corresponding chain complexes $CKh(T_+)$ and $CKh(T_-)$ have the same set of generators, thus the parities of the ranks of their homology are the same.

If a tangle $T$ has closed components, after crossing changes we can transform $T$ to a tangle $T'$ with a diagram $D'$ containing a trivial loop. This loop persists in any complete resolution of $D'$, so it follows from the construction that the rank of $CKh(D')$ is even, hence rank $SKh(T)$ is even.

If $T$ backtracks, after crossing changes we can transform $T$ to a tangle $T'$ with an arc which can be isotoped rel boundary into $D \times \{0\}$ or $D \times \{1\}$ without crossing other components. We can find a diagram $D'$ of $T'$ such that any complete resolution of $D'$ backtracks. So $CKh(D') = 0$, and rank $SKh(T)$ is even.

If $T$ is a string link, after crossing changes we can transform $T$ to a braid $B$. Since $SKh(B) \cong \mathbb{F}$, rank $SKh(T)$ is odd.

Definition 3.3. A tangle $T \subset D \times I$ is split, if there exists a 3–ball $B \subset D \times I$, such that $L_2 = T \cap B$ is a link and $L_2 \neq T$. In this case, let $T_1 = T - L_2$, then we write $T = T_1 \sqcup L_2$. We say $T$ is nonsplit if it is not split.

A tangle $T \subset D \times I$ is nonprime, if there exists a 3–ball $B \subset D \times I$, such that $T_2 = T \cap B$ is a $(1, 1)$–tangle in $B$, and $T_2$ does not cobound a disk with any arc in $\partial B$. In this case, Let $T_1 \subset D \times I$ be the tangle obtained by replacing $T_2$ with a trivial arc in $B$, and let $L_2$ be the link obtained from $T_2$ by connecting the two ends of $T_2$ by an arc in $\partial B$. We denote $T = T_1 \# L_2$. We say $T$ is prime if there does not exist such a $B$.

Lemma 3.4. Let $(M, \gamma)$ be the sutured manifold which is the double branched cover of $T$. Then $M$ is irreducible if and only if $T$ is nonsplit and prime.

Proof. The conclusion follows from the Equivariant Sphere Theorem [16] by the same argument as in [8, Proposition 5.1].

Lemma 3.5. If $T = T_1 \# L_2$ is a nonprime string link, then

$$SKh(T) \cong SKh(T_1) \otimes Kh_r(L_2).$$
In the above, $Kh_r(L_2)$ denotes the reduced Khovanov homology of $L_2$.

Proof. We appeal to Proposition 2.4 to choose a diagram of $T$ realized as the composition of $T_1$ and $L_2^{*,n}$, where $L_2^{*,n}$ is an $(n,n)$ tangle obtained from $L_2$ by removing a neighborhood of a point near the connected sum region and adjoining $n-1$ trivial strands as pictured in Figure 3.

We then have a chain isomorphism

$$\tilde{v}^- \otimes_{H_n} F(T) \cong (\tilde{v}^- \otimes_{H_n} F(T_1)) \otimes (\tilde{v}^- \otimes_{H_n} F(L_2^{*,n})).$$

Moreover, $\tilde{v}^- \otimes_{H_n} F(L_2^{*,n})$ is canonically chain isomorphic to $v^- \otimes_{H_1} F(L_2^{*,1})$, and the homology of the latter complex is the reduced Khovanov homology of $L_2$ with $F$ coefficients. The desired conclusion then follows from the K"unneth formula.

**Proposition 3.6.** Suppose that $T \subset D^2 \times I$ is a balanced tangle. If the double branched cover of $T$ is a product sutured manifold, then $T$ is isotopic to a braid.

**Proof.** Let $\pi: F \times I \rightarrow D^2 \times I$ be the double branched cover of $T$, then the nontrivial deck transformation $\rho$ is an involution on $F \times I$ that preserves $F \times \partial I$ setwise. By Meeks–Scott [15, Theorem 8.1], $\rho$ is conjugate to a map preserving the product structure. In particular, $\pi^{-1}(T)$, being the set of fixed points of $\rho$, is homeomorphic to $P \times I \subset F \times I$ for some finite set $P \subset F$, via a homeomorphism of $F \times I$ which preserves $F \times \partial I$. It follows that $T$ is isotopic to a braid.

**Proof of Theorem 1.1.** By Lemma 3.2, if $SKh(T)$ has rank 1, then $T$ is a string link. In particular, $T$ must be nonsplit.

If $T = T_1 \# K_2$ is nonprime, where $K_2$ is a nontrivial knot, it follows from Lemma 3.5 that $Kh_r(K_2)$ has rank 1. Using a theorem of Kronheimer and Mrowka [14], we conclude that $K_2$ is the unknot, a contradiction.
From now on we assume $T$ is nonsplit and prime, then Lemma 3.4 implies that $\Sigma(D \times I, T)$ is irreducible. Suppose that $SKh(T)$ has rank 1. By [5, Proposition 5.20], the sutured Floer homology group $SFH(\Sigma(D \times I, T))$ also has rank 1. By [17, 10], $\Sigma(D \times I, T)$ is a product sutured manifold. Proposition 3.6 then implies that $T$ is isotopic to a braid.

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