Control design for discrete-time state-multiplicative noise stochastic systems

Dušan Krokavec and Anna Filasová
Department of Cybernetics and Artificial Intelligence, Faculty of Electrical Engineering and Informatics, Technical University of Košice, Letná 9, 042 00 Košice, Slovakia
E-mail: dusan.krokavec@tuke.sk; anna.filasova@tuke.sk

Abstract. Design conditions for existence of the $H_\infty$ linear state feedback control for discrete-time stochastic systems with state-multiplicative noise and polytopic uncertainties are presented in the paper. Using an enhanced form of the bounded real lemma for discrete-time stochastic systems with state-multiplicative noise, the LMI-based procedure is provided for computation of the gains of linear, as well as nonlinear, state control law. The approach is illustrated on an example demonstrating the validity of the proposed method.

1. Introduction
The control design for systems with stochastic uncertainties is an area of study for several decades (see [7] and the references therein) and so for this practically and theoretically attractive field were derived numerous solutions. Modeling some parameter uncertainties as white noise processes in a linear system, this approach was applied for discrete-time [10], as well as for continuous-time systems [3], [14], [17]. Regrettably, these ways of doing it on linear quadratic optimal control principle led mainly to nonstandard Riccati equations. Representing the system uncertainty by unknown disturbance signal, $H_\infty$ principle stayed the most prominent method of dealing with the corresponding control problem of disturbance relaxation [21]. Combining the both mentioned representations of uncertainty, adapting $H_\infty$ approach towards the disturbance attenuation for stochastic systems and including those that ensure a performance bound in the $H_\infty$ sense, these efforts have been contributed in design conditions. By that way, new accessions have been made in diverse practical problems for finite and infinite time horizon for continuous-time [4], [9], as well as for discrete-time stochastic systems [2], [5], denoting later such system as linear stochastic systems with multiplicative noise. It should be noted, the stochastic control for linear and nonlinear systems with multiplicative noises has been finding its applications in many fields in control theory, mainly in constrain control [11] and in gain scheduling when the scheduling parameters are corrupted with measurement noise [19].

Considering new results in control of systems with polytopic uncertainties [1], [8], [18], as well as in bounded real lemma forms for discrete-time stochastic systems [6], [12], [15], new design conditions for $H_\infty$ linear control for linear discrete-time stochastic systems with state-multiplicative noise and polytopic uncertainties are derived in the paper. To present this, the paper is divided in these sections. Following the introduction in Sec. 1, the control design task for discrete-time stochastic systems with state-multiplicative noise is presented in Sec. 2. The preliminary results focused on two bounded real lemma forms for such stochastic systems are...
presented in Sec. 3 and, subsequently, in Sec. 4 there are derived the design conditions for $H_\infty$ linear control of state-multiplicative noise stochastic systems with polytopic uncertainties. Sec. 5 illustrates the constrained control design task by a numerical solution and Sec. 6 draws some conclusions.

Throughout the paper, the notations are narrowly standard in such a way that $x^T$, $X^T$ denotes the transpose of the vector $x$ and matrix $X$, respectively, $X = X^T > 0$, $(\geq 0)$, means that $X$ is a symmetric positive definite (semi-definite) matrix, diag$[\cdot]$ designates a block diagonal matrix, the symbol $I_n$ indicates the $n$-th order identity matrix, $E_o[\cdot]$ remits expectation with respect to the variable $o$, $\mathbb{Z}_+$ is the set of all positive integers, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^{n\times r}$ refers to the set of all $n \times r$ real matrices and $L_2(0, +\infty)$ entails the space of square summable discrete vector random sequence over $(0, +\infty)$.

2. Problem formulation

Throughout the paper, the task is concerned with the feedback design to control the stochastic discrete-time linear dynamic system given by the set of equations

\[ q(i + 1) = (F + V o(i))q(i) + Gu(i) + Wd(i), \]
\[ y(i) = Cq(i), \]

where $q(i) \in \mathbb{R}^n$, $u(i) \in \mathbb{R}^r$, $y(i) \in \mathbb{R}^m$ is the state, input, and output vector, respectively, $d(i) \in \mathbb{R}^{r_w}$ is an exogenous disturbance vector, and $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{n\times r_w}$, $V \in \mathbb{R}^{n\times n}$ are real matrices.

It is assumed that the multiplicative noise $o(i)$, $0 \leq i \leq j$ satisfies the conditions

\[ E_o\{o(i)\} = 0, \quad E_o\{o(i) o(j)\} = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \]

where $E_o\{\cdot\}$ denotes expectation with respect to $o(i)$ and $\delta_{ij}$ is the Kronecker delta function. Disturbance is a non-anticipative process, where $\{d(i)\} \in L_2(0, \infty); \mathbb{R}^{r_w})$.

The problem of the interest is to design in the mean square sense stable the closed-loop system by the state feedback control.

3. Basic preliminaries

In order to create more convenient space for new design task conditions, two new modifications of the stability solution for stochastic systems (compare, e.g., [3], [9], [4], [5]), are presented first. Since these modifications constitute the first stage in the solution of the considered problem, to the best of authors’ belief, inclusion of the complete proofs was an easier way than to interpret in detail how to adapt the original bounded real lemma (BRL) formulation for the discrete-time stochastic systems with multiplicative noise.

Lemma 1 [12] If $X$ is a symmetric positive definite matrix, $L_i$ are matrices of appropriate dimension, $a_i \geq 0$ is a real scalar, $l = 1, 2, \ldots, s$ and $s$ is a positive integer, then

\[ \left( \sum_{h=1}^{s} a_h L_h^T \right) X \left( \sum_{k=1}^{s} a_k L_k \right) \leq \sum_{h=1}^{s} a_h L_h^T X L_h \left( \sum_{k=1}^{s} a_k \right) \]

and if $\sum_{k=1}^{s} a_k = 1$,

\[ \left( \sum_{h=1}^{s} a_h L_h^T \right) X \left( \sum_{k=1}^{s} a_k L_k \right) \leq \sum_{h=1}^{s} a_h L_h^T X L_h. \]
Solving for the deterministically unforced system, then input differential equation
\[ d = \ldots \] (compare [13]) Let the Lyapunov function candidate take the form
\[ \text{Proposition 1 (BRL for discrete-time MIMO state-multiplicative noise stochastic systems):} \]
The autonomous system (1), (2) is stable in the mean square sense and with the quadratic performance \( \gamma \) if there exists a positive definite matrix \( P \in \mathbb{R}^{n \times n} \) and a positive scalar \( \gamma \in \mathbb{R} \) such that
\[ P = P^T > 0, \quad \gamma > 0, \] (6)
\[ \begin{bmatrix}
-\gamma I_{r_w} & * & * & * \\
0 & -\gamma I_{r_w} & * & * \\
PF & PW & -P & * \\
PV & 0 & 0 & -P \\
C & 0 & 0 & 0 & -I_m \\
\end{bmatrix} < 0, \] (7)

Hereafter, \( * \) denotes the symmetric item in a symmetric matrix.

Proof: (compare [13]) Let the Lyapunov function candidate take the form
\[ v(q(i)) = q^T(i)Pq(i) + \sum_{j=0}^{i-1} (y^T(j)y(j) - \gamma d^T(j)d(j)), \] (8)
where \( \gamma \in \mathbb{R} \) is square of the \( H_{\infty} \) norm of the transfer function matrix defined for the disturbance input \( d \) and the system output \( y \). Taking expectation with respect to \( o(i) \) it yields
\[ E_o\{v(q(i+1))\} - v(q(i)) = E_o\{q^T(i+1)Pq(i+1)\} - q^T(i)Pq(i) \\
+ y^T(i)y(i) - \gamma d^T(i)d(i) < 0. \] (9)

Solving for the deterministically unforced system, then
\[ E_o\{q^T(i+1)Pq(i+1)\} = d^T(i)W^TPWd(i) \\
+ q^T(i)(F^TPF + E_o\{o^2(i)\}VTPV)q(i) \\
+ q^T(i)FTPWd(i) + d^T(i)W^TPFq(i) \] (10)
and, since \( E_o\{o^2(i)\} = 1 \), substituting (2) and (10) into (9), then also it yields
\[ E_o\{v(q(i+1))\} - v(q(i)) = q^T(i)(-P + F^TPF + V^TPV + C^TC)q(i) \\
+ q^T(i)FTPWd(i) + d^T(i)W^TPFq(i) \\
+ d^T(i)(W^TPW - \gamma I_{r_w})d(i) \] (11)

Defining the composed vector
\[ q^oT(i) = \begin{bmatrix} q^T(i) & d^T(i) \end{bmatrix}, \] (12)
the inequality
\[ E_o\{v(q(i+1))\} - v(q(i)) = q^oT(i)P^oq^o(i) < 0 \] (13)
is satisfied only if
\[ P^o = \begin{bmatrix}
-\gamma I_{r_w} + W^TPW & 0 \\
0 & 0 \\
\end{bmatrix} < 0. \] (14)

Writing (14) as follows
\[ \begin{bmatrix}
-\gamma I_{r_w} + W^TPW & 0 \\
0 & 0 \\
\end{bmatrix} < 0. \] (15)
Proof: (compare [13]) Since the system (1) in the autonomous mode implies

respectively. Adding (21) and its transposition to (9) gives

then, with an arbitrary symmetric positive definite matrix \( Q \)

Applying twice the Schur complement property, (16) implies (7). This concludes the proof. \( \Box \)

**Proposition 2 (enhanced BRL for discrete-time state-multiplicative noise stochastic systems)**

The autonomous system (1), (2) is stable in the mean square sense and with the quadratic performance \( \gamma \) if there exist positive definite matrices \( P, Q \in \mathbb{R}^{n \times n} \) and a positive scalar \( \gamma \in \mathbb{R} \) such that

\[
P = P^T > 0, \quad Q = Q^T > 0, \quad \gamma > 0,
\]

\[
\begin{bmatrix}
-P + V^T PV + C^T C & 0 & F^T P \\
0 & -\gamma I_n & W^T P \\
PF & PW & -P
\end{bmatrix} < 0. \tag{16}
\]

Proof: (compare [13]) Since the system (1) in the autonomous mode implies

\[
Fq(i) + Vo(i)q(i) + Wd(i) - q(i + 1) = 0,
\]

then, with an arbitrary symmetric positive definite matrix \( Q \in \mathbb{R}^{n \times n} \), it yields

\[
\left( q^T(i + 1) + 0.5q^T(i)V^T o(i) \right) Q \left( Fq(i) + Vo(i)q(i) + Wd(i) - q(i + 1) \right) = 0,
\]

\[
E_o\{q^T(i + 1)\}Q(Fq(i) + Wd(i) - E_o\{q(i + 1)\}) + 0.5E_o\{o^2(i)\}q^T(i)V^T QV q(i) = 0, \tag{21}
\]

respectively. Adding (21) and its transposition to (9) gives

\[
E_o\{v(q(i + 1))\} - v(q(i)) = E_o\{q^T(i + 1)\}P E_o\{q(i + 1)\} + q^T(i)(C^T C - P)q(i) + E_o\{q^T(i + 1)\}Q(Fq(i) + Wd(i) - E_o\{q(i + 1)\}) + (Fq(i) + Wd(i) - E_o\{q(i + 1)\})^T Q E_o\{q(i + 1)\} + q^T(i)V^T QV q(i) - \gamma d^T(i)d(i) < 0. \tag{22}
\]

Defining the composed vector

\[
q^{\star T}(i) = \begin{bmatrix} q^T(i) & d^T(i) & E_o\{q^T(i + 1)\} \end{bmatrix}, \tag{23}
\]

then (22) can be written as

\[
E_o\{v(q(i + 1))\} - v(q(i)) = q^{\star T}(i)P^{\star} q^{\star}(i) < 0, \tag{24}
\]

where

\[
P^{\star} = \begin{bmatrix}
-P + C^T C + V^T QV & 0 & F^T Q \\
0 & -\gamma I_n & W^T Q \\
QF & QW & P - 2Q
\end{bmatrix} < 0. \tag{25}
\]

To find the LMI form, (25) can be rewritten using Schur complement property as (18). This concludes the proof. \( \Box \)

The inequality (18) is an enhanced representation of BRL for the given class of stochastic systems. It is linear with respect to the system variables but does not involve any product of the Lyapunov matrix \( P \) and the system matrices \( F, V, W \). This offers mainly its possibility to be applied in a singular task analysis.
4. Uncertain state-multiplicative noise stochastic systems

The importance of Proposition 2 is that it separates \( P \) from \( F, V, W \) and, consequently, also from \( G \). This enables to derive design conditions for a system with polytopic uncertainties.

Assuming that the matrices \( F, G, V, W \) of (1) are not precisely known but belong to a polytopic uncertainty domain

\[
\mathcal{O} := \left\{ (F, G, V, W)(a) : (F, G, V, W)(a) = \sum_{l=1}^{s} a_l (F_l, G_l, V_l, W_l) \right\}, \quad (26)
\]

where \( \mathcal{Q} \) is the unit simplex, \( F_l, G_l, V_l \) and \( W_l \) are constant matrices with appropriate dimensions, and \( a_l, l = 1, 2, \ldots, s \) are time-invariant uncertainties.

Since \( a \) is constrained to the unit simplex as (27), the matrices \( (F, G, V, W)(a) \) are affine functions of the uncertain parameter vector \( a \in \mathbb{R}^n \) described by the convex combination of the vertex matrices \( (F_l, G_l, V_l, W_l), l = 1, 2, \ldots, s \).

**Theorem 1** The uncertain autonomous system (1), (2), (26) is stable in the mean square sense and with the quadratic performance \( \gamma \) if there exist symmetric positive definite matrices \( P, Q \in \mathbb{R}^{n \times n} \) and a positive scalar \( \gamma \in \mathbb{R} \) such that

\[
P = P^T > 0, \quad Q = Q^T > 0, \quad \gamma > 0, \quad (28)
\]

\[
\begin{bmatrix}
-P & * & * & * & *
-\gamma \mathbf{I}_r & * & * & *
-QF_l & QW_l & P - 2Q & * & *
-QV_l & 0 & 0 & -Q & *
C & 0 & 0 & 0 & -I_m
\end{bmatrix} < 0. \quad (29)
\]

for all \( l = 1, 2, \ldots, s \).

**Proof:** Now, since \( \sum_{l=1}^{s} a_l = 1 \), the first state difference equation of the uncertain system (1), (26) in the autonomous mode implies

\[
\sum_{l=1}^{s} a_l \left( F_l q(i) + V_l o(i) q(i) + W_l d(i) - q(i + 1) \right) = 0, \quad (30)
\]

Then, with an arbitrary symmetric positive definite matrix \( Q \in \mathbb{R}^{n \times n} \), it yields

\[
\sum_{k=1}^{s} a_k \left( q^T(i + 1) + 0.5 q^T(i) V^T_k o(i) \right) \sum_{l=1}^{s} a_l \left( F_l q(i) + V_l o(i) q(i) + W_l d(i) - q(i + 1) \right) = 0, \quad (31)
\]

which implies

\[
0 = E_o q^T(i + 1) Q \sum_{l=1}^{s} a_l \left( F_l q(i) + W_l d(i) - E_o q(i + 1) \right)
+ 0.5E_o \{o^2(i)\} \sum_{k=1}^{s} a_k a_l q^T(i) V^T_k QV_l q(i). \quad (32)
\]
Since the Lyapunov function candidate can take the form (8) then also yields (9). Thus, by adding (32) and its transposition to (9), it can obtain

$$
E_\alpha\{v(q(i + 1))\} - v(q(i)) = E_\alpha\{q^T(i + 1)\}P E_\alpha\{q(i + 1)\} + q^T(i)(C^T C - P)q(i) + \sum_{l=1}^{s} a_l E_\alpha\{q^T(i + 1)\}Q (F_l q(i) + W_l d(i) - E_\alpha\{q(i + 1)\}) + \sum_{l=1}^{s} a_l (F_l q(i) + W_l d(i) - E_\alpha\{q(i + 1)\})^T Q E_\alpha\{q(i + 1)\} + \sum_{k=1}^{s} \sum_{l=1}^{r} a_k a_l q^T(i)V_k^T Q V_k q(i) - \gamma d^T(i)d(i) < 0.
$$

Using the inequality (5) double summation elements can be eliminated from (33 and so

$$
E_\alpha\{v(q(i + 1))\} - v(q(i)) \leq E_\alpha\{q^T(i + 1)\}P E_\alpha\{q(i + 1)\} + q^T(i)(C^T C - P)q(i) + \sum_{l=1}^{s} a_l E_\alpha\{q^T(i + 1)\}Q (F_l q(i) + W_l d(i) - E_\alpha\{q(i + 1)\}) + \sum_{l=1}^{s} a_l (F_l q(i) + W_l d(i) - E_\alpha\{q(i + 1)\})^T Q E_\alpha\{q(i + 1)\} + \sum_{l=1}^{s} a_l q^T(i)V_k^T Q V_k q(i) - \gamma d^T(i)d(i) < 0.
$$

Exploiting the composed vector (23) then (34) can be written as

$$
E_\alpha\{v(q(i + 1))\} - v(q(i)) \leq \sum_{l=1}^{s} a_l q^{*T}(i)P_l^* q^*(i) < 0,
$$

where

$$
P_l^* = \begin{bmatrix}
-P + C^T C + V_l^T Q V_l & 0 & F_l^T Q \\
0 & -\gamma I_{r_w} & W_l^T Q \\
Q F_l & Q W_l & P - 2Q
\end{bmatrix} < 0.
$$

Thus, using Schur complement property, the inequality (36) can be rewritten as (29). This concludes the proof.

The following parts address the problem of finding the state-feedback control law that stabilize the system (1), (2), (26) and achieve the optimal level of unknown input disturbance attenuation.

Considering the linear state-feedback control law

$$
u(i) = -K q(i)
$$

where $K \in \mathbb{R}^{r \times n}$ is the feedback controller gain matrix, the design conditions are given by the following theorem.

**Theorem 2** The control law (37) to the system (1), (2), (26) exists if there exist symmetric positive definite matrices $X, Z \in \mathbb{R}^{n \times n}$, a matrix $Y \in \mathbb{R}^{r \times n}$ and a positive scalar $\gamma \in \mathbb{R}$ such that for all $l = 1, 2, \ldots, s$

$$
X = X^T > 0, \quad Z = Z^T > 0, \quad \gamma > 0,
$$

$$
\begin{bmatrix}
-Z & * & * & * \\
0 & -\gamma I_{r_w} & * & * \\
F_l X - G_l Y & W_l & Z - 2X & * \\
V_l X & 0 & 0 & -X \\
CX & 0 & 0 & 0 -I_{m}
\end{bmatrix} < 0.
$$

When the above conditions hold,

$$
K = Y X^{-1}.
$$
Proof: Since it was supposed that $Q$ is a positive definite matrix, defining the transform matrix

$$\mathbf{T} = \text{diag} \left[ \mathbf{X} \quad \mathbf{I}_{r_v} \quad \mathbf{X} \quad \mathbf{I}_m \right], \quad \mathbf{X} = Q^{-1},$$  \hfill (41)

then, pre-multiplying the left-hand side and post-multiplying the right-hand side of (29) by (41), it yields

$$\begin{bmatrix}
-X \mathbf{P} \mathbf{X} & * & * & * & * \\
0 & -\gamma \mathbf{I}_{r_v} & * & * & * \\
\mathbf{F}_l \mathbf{X} & \mathbf{W}_l & X \mathbf{P} \mathbf{X} - 2 \mathbf{X} & * & * \\
\mathbf{V}_l \mathbf{X} & 0 & 0 & -\mathbf{X} & * \\
\mathbf{C} \mathbf{X} & 0 & 0 & 0 & -\mathbf{I}_m
\end{bmatrix} < 0. \hfill (42)
$$

Replacing $\mathbf{F}_l$ in (29) by the closed-loop system matrix

$$\mathbf{F}_{cl} = \mathbf{F}_l - \mathbf{G}_l \mathbf{K},$$  \hfill (43)

then (42) takes the form

$$\begin{bmatrix}
-X \mathbf{P} \mathbf{X} & * & * & * & * \\
0 & -\gamma \mathbf{I}_{r_v} & * & * & * \\
\mathbf{F}_l \mathbf{X} - \mathbf{G}_l \mathbf{K} \mathbf{X} & \mathbf{W}_l & X \mathbf{P} \mathbf{X} - 2 \mathbf{X} & * & * \\
\mathbf{V}_l \mathbf{X} & 0 & 0 & -\mathbf{X} & * \\
\mathbf{C} \mathbf{X} & 0 & 0 & 0 & -\mathbf{I}_m
\end{bmatrix} < 0 \hfill (44)
$$

and with the notations

$$\mathbf{Z} = X \mathbf{P} \mathbf{X}, \quad \mathbf{Y} = \mathbf{K} \mathbf{X},$$  \hfill (45)

(44) implies (39). This concludes the proof. \hfill \blacksquare

If the uncertain stochastic state-multiplicative systems with polytopic uncertainties (1), (2), (26) under control (37) is not stable for given set of $\alpha_i$, the weighted sum of linear state-feedback control laws (the parallel distributed compensation) of the form

$$\mathbf{u}(i) = -\sum_{h=1}^{s} \alpha_h \mathbf{K}_h \mathbf{q}(i)$$  \hfill (46)

is proposed, where $\mathbf{K}_h \in \mathbb{R}^{r \times n}$, $h = 1, 2, \ldots, s$ is the set of feedback controller gain matrices to be designed.

**Theorem 3** The control law (46) to the system (1), (2), (26) exists if there exist symmetric positive definite matrices $\mathbf{X}_h, \mathbf{Z}_h \in \mathbb{R}^{n \times n}$, matrices $\mathbf{Y}_h \in \mathbb{R}^{r \times n}$ and a positive scalar $\gamma \in \mathbb{R}$ such that for all $l, h = 1, 2, \ldots, s$

$$\begin{bmatrix}
-\mathbf{Z}_h & * & * & * & * \\
0 & -\gamma \mathbf{I}_{r_v} & * & * & * \\
\mathbf{F}_l \mathbf{X}_h - \mathbf{G}_l \mathbf{Y}_h & \mathbf{W}_l & \mathbf{Z}_h - 2 \mathbf{X}_h & * & * \\
\mathbf{V}_l \mathbf{X}_h & 0 & 0 & -\mathbf{X}_h & * \\
\mathbf{C} \mathbf{X}_h & 0 & 0 & 0 & -\mathbf{I}_m
\end{bmatrix} < 0. \hfill (48)
$$

When the above conditions hold,

$$\mathbf{K}_h = \mathbf{Y}_h \mathbf{X}_h^{-1}.$$  \hfill (49)
Proof: With respect to (46), the closed-loop system matrix $F_l$ is considered in the form

$$F_{clh} = F_l - G_lK_h,$$  \hfill (50)

and so (34) can be modified to describe the closed loop stability as

\[
E_o\{v(q(i+1)) - v(q(i))\} \leq E_o\{q^T(i+1)\}PE_o\{q(i+1)\} + q^T(i)(C^TC - P)q(i) + \sum_{l=1}^{s} \sum_{h=1}^{s} a_{lh}E_o\{q^T(i+1)\}Q_h(F_{clh}q(i) + W_l d(i) - E_o\{q(i+1)\}) + \sum_{l=1}^{s} \sum_{h=1}^{s} a_{lh}q^T(i) V_l^T Q_h V_l q(i) - \gamma d^T(i)d(i) < 0.
\]  \hfill (51)

Exploiting the composed vector (23) then the condition (51) can be reformulated as

\[
E_o\{v(q(i+1)) - v(q(i))\} \leq \sum_{l=1}^{s} \sum_{h=1}^{s} a_{lh}q^T(i)P_{clh}^*q(i) < 0,
\]  \hfill (52)

where

$$P_{clh}^* = \begin{bmatrix} -P + C^TC + V_l^T Q_h V_l & 0 & F_{clh}^T Q_h \\ 0 & -\gamma I_{r_v} & W_l^T Q_h \\ Q_h F_{clh} & W_l Q_h & P - 2Q_h \end{bmatrix} < 0$$  \hfill (53)

and, using the Schur complement property, (53) can be rewritten as

\[
\begin{bmatrix} -P & 0 & F_{clh}^T Q_h & V_l^T Q_h & C^T \\ 0 & -\gamma I_{r_v} & W_l^T Q_h & 0 & 0 \\ Q_h F_{clh} & W_l Q_h & P - 2Q_h & 0 & 0 \\ C & 0 & 0 & 0 & -I_m \end{bmatrix} < 0.
\]  \hfill (54)

Since it was supposed that $Q_h$ is a positive definite matrix, defining the transform matrix

$$T_h = \text{diag} \left[ X_h \quad I_{r_v} \quad X_h \quad X_h \quad I_m \right], \quad X_h = Q_h^{-1},$$  \hfill (55)

then, pre-multiplying the left-hand side and post-multiplying the right-hand side of (54) by (55), and substituting (50) it yields

\[
\begin{bmatrix} -X_hPX_h & 0 & X_h F_l^T - X_h K_h^T G_l^T & X_h V_l^T & X_h C^T \\ 0 & -\gamma I_{r_v} & W_l^T & 0 & 0 \\ F_l X_h - G_l K_h X_h & W_l & X_h PX_h - 2X_h & 0 & 0 \\ V_l X_h & 0 & 0 & -X_h & 0 \\ C X_h & 0 & 0 & 0 & -I_m \end{bmatrix} < 0.
\]  \hfill (56)

Thus, with the notations

$$Z_h = X_h PX_h, \quad Y_h = K_h X_h,$$  \hfill (57)

then (56) implies (48). This concludes the proof.

Generally speaking, there is no method which enables a Lyapunov function to be chosen from all possible candidate functions. Thus, Lyapunovs theory leads to sufficient stability conditions, the conservatism of which is dependent on the particular form $v_p(q(i)) = q^T(i)Pq(i)$ and on the
structure of the system. The disadvantage of application of this type of particular form function for multiple models is that very conservative stability conditions are obtained.

Note, the above systems belong to a class of systems called bilinear stochastic systems, since the system is linear in the states and in the stochastic multiplicative terms. Such systems, analyzed in the sense of negative definiteness of the difference \( E_q \{ v(q(i + 1)) - v(q(i)) \} \) can be only asymptotically mean square stable, and so only a solution of the stationary BRL can be achieved. It is evident that while the particular form of Lyapunov function accounts mean square properties, the second part of Lyapunov function reflects the worst case of the disturbance effect.

Consider the case \( r = m \) (square plants), where with each output signal is associated a reference signal. Such regime is called the forced regime and is defined as follows:

**Definition 1** The forced regime for (1), (2) with the controller (46) is given by the control policy

\[
u(i) = -\sum_{h=1}^{s} a_h K_h q(i) + \sum_{l=1}^{s} \sum_{h=1}^{s} a_l a_h G_{lh} w(i),
\]

where \( w(i) \in \mathbb{R}^m \) is desired output signal vector, and \( G_{lh} \in \mathbb{R}^{m \times m} \) is the set of signal gain matrices.

**Theorem 4** If a square system (1), (2) is stabilizable by the control policy (58) and for all \( l \) are satisfied the conditions [20]

\[
\text{rank} \left[ \begin{array}{cc} A_l & B_l \\ C & 0 \end{array} \right] = n + m,
\]

then the matrices \( G_{lh} \) take the forms

\[
G_{hl} = \left( C(I_n - (A_l - B_l K_h))^{-1} B_l \right)^{-1}.
\]

**Proof:** In a steady-state, the disturbance-free equations (1), (2) and the control law (58) imply

\[
E_q \{ v(q(i + 1)) \} = q_s = \sum_{l=1}^{s} \sum_{h=1}^{s} a_l a_h (A_l - B_l K_h) q_s + \sum_{l=1}^{s} \sum_{h=1}^{s} a_l a_h B_l G_{lh} w_s,
\]

\[
y_s = C q_s,
\]

where \( q_s, y_s, w_s \) are steady-state values of the vectors \( q(t), y(t), w(t) \), respectively.

Since in a steady-state (61) implies

\[
0 = -\sum_{l=1}^{s} \sum_{h=1}^{s} a_l a_h (I_n - (A_l - B_l K_h)) q_s + \sum_{l=1}^{s} \sum_{h=1}^{s} a_l a_h B_l G_{lh} w_s,
\]

then (63) gives the equation

\[
q_s = \sum_{l=1}^{s} \sum_{h=1}^{s} a_l a_h (I_n - (A_l - B_l K_h))^{-1} B_l G_{lh} w_s
\]

and, according to (62),

\[
y_s = \sum_{l=1}^{s} \sum_{h=1}^{s} a_l a_h C (I_n - (A_l - B_l K_h))^{-1} B_l G_{lh} w_s.
\]

Thus, considering \( y_s = w_s \), then (65) implies (60). This concludes the proof. 

Note, the static gains realized by the \( G_{lh} \) matrices are ideal in control if the plant parameters, on which the values of \( W_{lh} \) depend, are known and do not vary with time. The forced regime is basically designed for constant references and is very closely related to shift of origin.
5. Illustrative Example

The considered system is represented by the model (1), (2), (26) with the system model parameters and the sampling period $T_s$.

\[
F_a = \begin{bmatrix} 1.0393 & 0.0073 & 0.0106 \\ 0.0224 & 1.0402 & 0.0364 \\ 0.0365 & 0.0114 & 1.0303 \end{bmatrix}, \quad F_b = \begin{bmatrix} 1.0457 & 0.0160 & 0.0227 \\ 0.0244 & 1.0537 & 0.0564 \\ 0.0742 & 0.0395 & 1.0345 \end{bmatrix},
\]

\[
G_a = \begin{bmatrix} 0.0403 \\ 0.0245 \\ 0.0180 \end{bmatrix}, \quad G_b = \begin{bmatrix} 0.0622 \\ 0.0393 \\ 0.0377 \end{bmatrix}, \quad W_a = \begin{bmatrix} 0.0095 \\ 0.0243 \\ 0.0312 \end{bmatrix}, \quad W_b = \begin{bmatrix} 0.0004 \\ 0.0010 \\ 0.0013 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 0.020 \\ 0 & 0 & 0 \\ 0.015 & 0 & 0 \end{bmatrix}, \quad T_s = 0.05 \, s,
\]

so that considering for $s = 4$, the parameter matrices were included in the form

\[
(F_1, G_1, W_1) = (F_a, G_a, W_a), \quad (F_2, G_2, W_2) = (F_b, G_a, W_a),
\]

\[(F_3, G_3, W_3) = (F_a, G_b, W_b), \quad (F_4, G_4, W_4) = (F_b, G_b, W_b).\]

where

\[a_1 = 0.1, \quad a_2 = 0.4, \quad a_3 = 0.2, \quad a_4 = 0.3.\]

Note, such limitation was chosen to define less than $N = 25$ inequalities with respect to (47), (48) (really, $N = 4 + 4 + 4^2 = 24$).

Solving the variables $\gamma, X_i, Z_i, Y_i, i = 1, 2, 3, 4$, satisfying (47), (48) via the LMI technique using toolbox SeDuMi [16], the following results were obtained for $\gamma = 6.8564$:

\[
K_1 = K_2 = K_3 = K_4 = K = \begin{bmatrix} -29.9607 & 80.8889 & 0.6409 \end{bmatrix},
\]

guaranteeing the stable eigenvalues spectra of all local system matrices in such a way that

\[
\rho(A_{41}) = \rho(A_1 - G_1 K) = \{0.4363, \ 0.9096, \ 0.9779\}, \quad \rho(A_{42}) = \rho(A_2 - G_2 K) = \{0.4899, \ 0.9291 \pm 0.1039 i\},
\]

\[
\rho(A_{43}) = \rho(A_3 - G_3 K) = \{-0.1275, \ 0.9068, \ 0.9910\}, \quad \rho(A_{44}) = \rho(A_4 - G_4 K) = \{-0.0620, \ 0.9282 \pm 0.0715 i\}.
\]

It is evident that the uncertain state-multiplicative noise stochastic systems with the above given matrix parameters can be stabilized by the linear state controller.

Applying the designed results to the system model with the initial conditions

\[
q^T(0) = [0.1 \ -0.1 \ 0.0], \quad \sigma_u^2 = 0.05, \quad \sigma_o^2 = 1,
\]

the simulation result of the system at the second vertex of vertices is stated in Fig. 1.

To illustrate the system behavior as the control parameters were computed using Theorem 3, the system state response of the closed-loop system in the equilibrium state control with impact of the disturbance and multiplicative noise is shown. Since analogous results were obtained for all system vertices structure, as well as for system with prescribed $a$, in view of this, it can be concluded that, besides the rise-time, generally all the design specifications were reasonably well satisfied. It is simply verifiable that adequate results can be obtained for the forced mode.
6. Concluding remarks

The paper presents new control design principle for discrete-time stochastic multi-variable dynamic systems with state-multiplicative noise. The stability of the control scheme in the mean square sense and with the quadratic performance is established using an enhanced representation of bounded real lemma, where, in the resulting LMIs, are decoupled potential pairs of the Lyapunov matrix and the system parameter matrices. This provides a suitable way for determination of linear state controllers by solving this naturally affine LMI tasks. Since the required state feedback gains has been obtained by solving an LMI feasibility problem, comparing with the previous results, the number of assumptions is reduced since the solutions obtained according to the conditions of both presented theorems tends to receive a linear state controller.

Presented applications can be considered as a task concerned the class of $H_{\infty}$ stabilization control problems where the design conditions were newly formulated. This formulation poses the problem as a stabilization problem whose control law gain matrices take no special structures and allows to find a solution (if exists) to the control law without restrictive assumptions and additional specifications on the design parameters. The procedures used here for discrete time stochastic systems with state-multiplicative noise could be similarly extended for the continuous-time case. Moreover, a reasonable tracking performance could now be achieved.

Acknowledgments

The work presented in this paper was supported by VEGA, the Grant Agency of the Ministry of Education and the Academy of Science of Slovak Republic under Grant No. 1/0348/14. This support is very gratefully acknowledged.

References

[1] Cai G Hu C Yin B He H and Han X 2014 Gain-scheduled $H_2$ controller synthesis for continuous-time polytopic LPV systems Mathematical Problems in Engineering 2014 Article ID 972624 14p
[2] El Bouhtouri A Hinrichsen D and Pritchard A J 1999 $H_{\infty}$-type control for discrete-time stochastic systems Int. J. Robust and Nonlinear Control 9(13) 923-48
[3] El Ghaoui L 1995 State-feedback control of systems with multiplicative noise via linear matrix inequalities Systems & Control Letters 24(3) 223–8
[4] Gao H and Wang C 2004 Improved bounded real lemma for continuous-time stochastic systems with polytopic uncertainties Proc. 2004 American Control Conference Boston MA USA 2654-8
[5] Gershon E and Shaked U 2008 $H_\infty$ output-feedback control of discrete-time systems with state-multiplicative noise *Automatica* **44**(2) 574-9

[6] Gershon E and Shaked U 2013 *Advanced Topics in Control and Estimation of State-Multiplicative Noisy Systems* (London:Springer)

[7] Gershon E Shaked U and Yaesh I 2005 $H_\infty$ control and Estimation of State-multiplicative Linear Systems (London:Springer)

[8] He Y Wu M and She J H 2005 Improved bounded-real-lemma representation and $H_\infty$ control of systems with polytopic uncertainties *IEEE Tran. Circuits and Systems* **52**(7) 380-3

[9] Hinrichsen D and Pritchard A J 1998 Stochastic $H_\infty$ *SIAM J. Control and Optimization* **36**(5) 1504-38

[10] Joshi S M 1976 On optimal control of linear systems in the presence of multiplicative noise *IEEE Tran. Aerospace and Electronic Systems* **12**(1) 80-5

[11] Krokavec D and Filasová A 2008 Constrained control of discrete-time stochastic systems *Proc. 17th World Congress IFAC* Seoul Korea 15315-20

[12] Krokavec D and Filasová A 2014 $H_\infty$ enhanced control design of discrete-time Takagi-Sugeno state-multiplicative noisy systems *Mathematical Problems in Engineering* **2014** Article ID 151095 12p

[13] Krokavec D and Filasová A 2014 $H_\infty$ control of discrete-time stochastic state-multiplicative systems constrained in state by equality constraints *Proc. 19th World Congress IFAC* Cape Town South Africa 8699-704

[14] McLane P 1971 Optimal stochastic control of linear systems with state- and control-dependent disturbances *IEEE Tran. Automatic Control* **16**(6) 793-8

[15] Niamsup P and Rajchakit G 2013 New results on robust stability and stabilization of linear discrete-time stochastic systems with convex polytopic uncertainties *J. Applied Mathematics* **2013** Article ID 368259 10p

[16] Peaucelle D Henrion D Labit Y and Taitz K 2002 *User’s Guide for SeDuMi Interface 1.04* (Toulouse:LAAS-CNRS)

[17] Phillis Y A 1985 Controller design of systems with multiplicative noise *IEEE Tran. Automatic Control* **30**(10) 1017-9

[18] de Souza W A Teixeira M C M Santim M P A Cardim R and Assuncao E 2013 On switched control design of linear time-invariant systems with polytopic uncertainties *Mathematical Problems in Engineering* **2013** Article ID 595029 10p

[19] Todorov E 2005 Stochastic optimal control and estimation methods adapted to the noise characteristics of the sensorimotor system *Neural Computation* **17**(5) 1084–1108

[20] Wang Q G 2003 *Decoupling Control* (Berlin:Springer)

[21] Zhou K Doyle J C and Glover K 1996 *Robust and Optimal Control* (Englewood Cliffs:Prentice Hall)