Quantum Fluctuations of Effective Fields and the Correspondence Principle

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Abstract

The question of Bohr correspondence in quantum field theory is considered from a dynamical point of view. It is shown that the classical description of particle interactions is inapplicable even in the limit of large particles’ masses because of finite quantum fluctuations of the fields produced. In particular, it is found that the relative value of the root mean square fluctuation of the Coulomb and Newton potentials of a massive particle is equal to \(1/\sqrt{2}\). It is shown also that in the case of a macroscopic body, the quantum fluctuations are suppressed by a factor \(1/\sqrt{N}\), where \(N\) is the number of particles in the body. An adequate macroscopic interpretation of the correspondence principle is given.

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I. INTRODUCTION

The Bohr correspondence principle is one of the basic principles underlying quantum theory. An essential of this principle is the formulation of quasi-classical transition from the quantum theory to its classical original, \textit{i.e.}, indication of the conditions under which a given system can be considered classically.

Identification of the quasi-classical conditions is a twofold problem. Roughly speaking, it consists of the kinematical and dynamical parts. The former is concerned with the motion of a system, while the latter with its interactions. From the point of view of quantum kinematics, determination of the quasi-classical conditions is a quantum mechanical problem. The motion of a system can be considered classically if

\[ \frac{\bar{S}}{\hbar} \gg 1, \]  

where \( \bar{S} \) is the characteristic value of the system action. This is the case, for instance, for a sufficiently heavy particle. On the other hand, the dynamical part is essentially a quantum field problem. It requires examination of the fundamental interactions of the system.

At first sight, the above division of the whole problem into two parts is artificial. In quantum electrodynamics, for instance, the interaction of two charged particles takes the classical form of the Coulomb law if the particles’ masses tend to infinity, since all radiative corrections to the electromagnetic form factors disappear in this limit. This is not the case, however, in quantum gravity, where the radiative corrections do not disappear in the large mass limit, because the value of the particle mass determines the strength of its gravitational interactions. In fact, the relative value of the logarithmic contribution (which is of the order \( \hbar \)) to the gravitational form factors of the scalar particle is independent of the scalar particle mass \( [1] \). Moreover, there are contributions of the order \( \hbar^0 \) which are proportional to the particle mass and have the form of the ordinary post-Newtonian corrections \( [2] \).

The question of the correspondence between classical and quantum theories of gravity was considered in detail in Refs. \([3,4]\), where it was shown that the correct correspondence can only be established in the \textit{macroscopic} limit. Namely, the following formulation of the correspondence principle was suggested: the effective gravitational field produced by a macroscopic body of mass \( M \) consisting of \( N \) particles turns into the corresponding solution of the classical equations in the limit \( M \to \infty, N \to \infty \). This interpretation is underlined by an observation that the \( n \)-loop radiative contribution to the post-Newtonian correction of a given order to the gravitational field of a body with mass \( M \), consisting of \( N = M/m \) elementary particles with mass \( m \), contains an extra factor of \( 1/N^n \) in comparison with the corresponding tree contribution. As this fact is central in what follows, let us take as an example the first post-Newtonian approximation. In this case, there are two types of contributions to the gravitational field of the body. The first is the usual post-Newtonian correction predicted by general relativity, reproduced in quantum theory by the tree diagrams \textit{bilinear} in the energy-momentum tensor \( T^{\mu\nu} \) of the particles, see Fig. \( [4] \)(a). The second is the one-loop contribution shown in Fig. \( [4] \)(b). Since this diagram has two particle operators attached, it is only \textit{linear} in \( T^{\mu\nu} \). Therefore, when evaluated between \( N \)-particle states, the former is proportional to \( (m \cdot N) \cdot (m \cdot N) = M^2 \), while the latter, to \( m^2 \cdot N = M^2/N \).

Thus, in the case of gravity, classical consideration of both the kinematics and dynamics of a system is justified only for macroscopic systems. This property formally displays the
gravitational interaction as an exception among other fundamental interactions. The aim of this paper is to show that despite this seemingly natural conclusion, the above formulation of the correspondence principle must be extended to all interactions. For this purpose we will investigate quantum fluctuations of the effective electromagnetic and gravitational fields. Obviously, system interactions can be considered classically only in the case of vanishing fluctuation of the field produced by the system. As we will see, this requirement puts all fundamental interactions on an equal footing.

The paper is organized as follows. A general formula for the correlation function of arbitrary fields is written down in Sec. II, which is then used in the investigation of quantum fluctuations of electromagnetic and gravitational fields in Secs. III and IV, respectively. The results obtained are discussed in Sec. V. Some technical results used in the text are collected in two appendices.

Condensed notations of DeWitt are in force throughout this paper. Also, right and left derivatives with respect to the fields and the sources, respectively, are used. The dimensional regularization of all divergent quantities is assumed.

II. TREE CONTRIBUTIONS AND QUANTUM FLUCTUATIONS

Before we proceed to actual calculations, let us consider the question of quantum fluctuations of the effective fields in more detail.

Let a given system of interacting fields \( \varphi_a, a = 1, \ldots, A \), be described by the action \( S[\varphi] \). In the case when the system possesses gauge symmetries (which is of primary concern for us), in addition to the matter and gauge fields the set \{\( \varphi \)\} contains a number of auxiliary fields (the Faddeev-Popov ghosts, fields introducing the gauge etc.). Denoting by \( j^a \) the source for the field \( \varphi_a \), we write the generating functional of Green functions

\[
Z[j] = \int d\varphi \exp \left\{ \frac{i}{\hbar} (S[\varphi] + j^a \varphi_a) \right\},
\]

where \( d\varphi \) stands for the product of all \( d\varphi_a(x) \):

\[
d\varphi = \prod_{a=1}^A \prod_x d\varphi_a(x).
\]

As was mentioned in the Introduction, our aim is to investigate the correspondence between classical and quantum theories from the point of view of quantum fluctuations of the effective fields. Thus, we assume that the kinematics of our system is such that the inequality \( (2) \) takes place, so the leading contribution to the functional integral \( (2) \) is determined by the stationary “point” \( \varphi_a = \varphi_a^{(0)} \) satisfying the classical equations of motion

\[
\frac{\delta S}{\delta \varphi_a} = -j^a.
\]

Therefore, the generating functional of Green functions takes the following quasi-classical form

\[
Z^{(0)}[j] = \exp \left\{ \frac{i}{\hbar} \left( S[\varphi_a^{(0)}] + j^a \varphi_a^{(0)} \right) \right\}.
\]
With the help of this equation one finds the quasi-classical mean field

\[ \langle \varphi_a \rangle^{(0)} = -i\hbar \frac{\delta \ln Z^{(0)}[j]}{\delta j^a} = \frac{\delta \varphi_a^{(0)}}{\delta j^a} (-1)^{P_a} \left( \frac{\delta S[\varphi]}{\delta \varphi_b} + j^b \right)_{\varphi = \varphi^{(0)}} + \varphi_a^{(0)} = \varphi_a^{(0)}, \tag{5} \]

where \( P_a \) is the Grassmann parity of the field \( \varphi_a \). As expected, the mean field coincides at the tree level with the corresponding classical solution. Now, let us calculate the correlation function of two fields \( \varphi_a(x), \varphi_b(y) \). Using Eqs. (3), (5), one has

\[ \langle [\varphi_a(x) - \varphi_a^{(0)}(x)][\varphi_b(y) - \varphi_b^{(0)}(y)] \rangle^{(0)} = -\frac{\hbar^2}{Z^{(0)}[j]} \frac{\delta^2 Z^{(0)}[j]}{\delta j^a(x) \delta j^b(y)} - \varphi_a^{(0)}(x) \varphi_b^{(0)}(y) \]

\[ = -i\hbar \frac{\delta \varphi_b^{(0)}(y)}{\delta j^a(x)} = i\hbar (-1)^{P_a} \left[ \frac{\delta^2 S[\varphi]}{\delta \varphi_b(y) \delta \varphi_a(x)} \right]^{-1}_{\varphi = \varphi^{(0)}}. \tag{6} \]

Equation (6) represents zero order approximation of \( Z(j) \) in the formal expansion with respect to \( S/\hbar \). In this approximation, therefore, the right hand side of Eq. (6) is to be set zero, concluding that all field fluctuations disappear in the limit \( \hbar/S \to 0 \), as it should be in classical theory. Informally, this conclusion is only justified as long as \( \hbar S_{,ab}^{-1} \to 0 \) follows from \( \hbar/S \to 0 \). In quantum mechanics, this is indeed the case. Our purpose below will be to decide to what extent this is true in quantum field theory.

III. QUANTUM FLUCTUATIONS OF ELECTROMAGNETIC FIELD

Let us take the set \( \{ \varphi \} \) to consist of a single charged scalar field \( \phi \) and electromagnetic field \( A_\mu \). The action for this system

\[ S[\phi, A] = \int d^4x \left\{ (\partial_\mu \phi^* + ieA_\mu \phi^*) (\partial^\mu \phi - ieA^\mu \phi) - \left( \frac{mc}{\hbar} \right)^2 \phi^* \phi \right\} - \frac{1}{4} \int d^4xF_{\mu\nu}F^{\mu\nu}, \tag{7} \]

where \( e, m \) are the charge and mass of scalar field quanta, and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Indices are raised and lowered with the help of Minkowski metric \( \eta_{\mu\nu} = \text{diag}\{+1, -1, -1, -1\} \).

As was explained in the Introduction, our aim is to investigate the properties of particle interactions in the limiting case where their kinematics can be considered classically. Thus, we assume the scalar particles sufficiently heavy, so that to neglect uncertainty in their positions and velocities.

Consider a particle in the system, and let \( a(q) \) be its momentum probability distribution. We are interested in the fluctuations of electromagnetic field produced by this particle in a given spacetime point. According to Eq. (6), correlation of \( A_\mu, A_\nu \) at the same spacetime point \( x \) has the form

\[ B_{\mu\nu}(x) \equiv \langle [A_\mu(x) - A_\mu^{(0)}(x)][A_\nu(x) - A_\nu^{(0)}(x)] \rangle^{(0)} = i\hbar \left[ \frac{\delta^2 S[\phi, A]}{\delta A_\mu(x) \delta A_\nu(x)} \right]^{-1}_{\varphi = \varphi^{(0)}}. \tag{8} \]

To the leading order in the coupling constant \( e \), the right hand side of Eq. (8) is represented by the one-loop diagrams of Fig. 2. In the tree approximation, the loop occurs because the operators \( A_\mu \) and \( A_\nu \) are both taken in the same spacetime point.
Even without detailed calculation, it is clear that the diagram of Fig. 2(b) turns into zero in the limit $\hbar/\bar{S} \to 0$. Indeed, taking into account the structure of the $A^2\phi^2$ vertex [see Eq. (7)], one readily sees that this diagram $\sim \hbar e^2/m \to 0$, for $m \to \infty$.

Thus, it remains only to calculate the diagram of Fig. 2(a). Its analytical expression in the momentum space

$$I_{\mu\nu}^{(a)}(p) = -ie^2\mu' \int \frac{d^4-k}{(2\pi)^4} (2q_\alpha - p_\alpha + k_\alpha)G^\phi(q+k)(2q_\beta + k_\beta)G^{\alpha\mu}(p+k)G^{\beta\nu}(k), \quad (9)$$

where $G^\phi$ is the scalar particle propagator,

$$G^\phi(k) = \frac{1}{m^2 - k^2},$$

$G_{\mu\nu}$ the photon propagator,

$$G_{\mu\nu}(k) = \frac{\eta_{\mu\nu}}{k^2}, \quad (10)$$

$\epsilon_q = \sqrt{m^2 + q^2}$, $\mu$ arbitrary mass scale, and $\epsilon = 4 - d$, $d$ being the dimensionality of spacetime.

The photon propagator is taken in the Feynman gauge. However, it is shown in Appendix A that a change of the gauge conditions gives rise to terms $\sim 1/m$ which disappear in the large particle mass limit (classical particle kinematics).

Introducing the Schwinger parametrization

$$\frac{1}{k^2} = -\int_0^\infty dy \exp\{yk^2\}, \quad \frac{1}{(k+p)^2} = -\int_0^\infty dx \exp\{x(k+p)^2\},$$

$$\frac{1}{k^2 + 2(kq)} = -\int_0^\infty dz \exp\{z[k^2 + 2(kq)]\}, \quad (11)$$

the loop integrals are evaluated using

$$\int d^d k \exp\{k^2(x + y + z) + 2k^\mu(xp_\mu + zq_\mu) + p^2x\}$$

$$= i \left( \frac{\pi}{x + y + z} \right)^{d/2} \exp\left\{ \frac{p^2xy - m^2z^2}{x + y + z} \right\}. \quad (12)$$

Then, changing the integration variables $(x, y, z)$ to $(t, u, v)$ via

$$x = \frac{t(1 + t + u)v}{m^2(1 + atu)}, \quad y = \frac{u(1 + t + u)v}{m^2(1 + atu)}, \quad z = \frac{(1 + t + u)v}{m^2(1 + atu)}, \quad \alpha = -\frac{p^2}{m^2}, \quad (13)$$

integrating $v$ out, subtracting the ultraviolet divergence.

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1 To simplify calculations, in this section, we choose the units in which $\hbar = c = 1$.

2 A consistent treatment of divergences of this kind is the problem of the renormalization theory of composite operators. We are not going into details of the subtraction procedure, since the divergence $[7]$ (as well as divergence $[26]$ in the gravitational case below) does not interfere with the terms of the order $\hbar^0$ we are interested in.
\[ I^{(a)\text{div}}_{\mu\nu} = -\frac{e^2 \eta_{\mu\nu}}{128\pi^2 \sqrt{\varepsilon q^{-\mu} q^{-\rho}}} \frac{1}{\varepsilon} \left( \frac{\mu}{m} \right)^\epsilon, \] (14)

and setting \( \epsilon = 0 \), we obtain

\[
I_{\mu\nu} \equiv (I^{(a)}_{\mu\nu} - I^{(a)\text{div}}_{\mu\nu})_{\epsilon=0} = \frac{e^2}{32\pi^2 \sqrt{\varepsilon q^{-\mu} q^{-\rho}}} \int_0^{+\infty} \int_0^{+\infty} dudt \frac{1}{2m^2 DH^3} \\
\times \left\{ p_\mu p_\nu \left[ 2H^2 - 3H + \frac{2m^2}{p^2} (D - 1) + 1 \right] + (2q_\mu q_\nu - p_\nu q_\mu - p_\mu q_\nu)(2H - 1)^2 \right\},
\]

\[ H \equiv 1 + u + t, \quad D \equiv 1 + \alpha ut. \] (15)

The remaining \( u, t \)-integrals are evaluated in Appendix B. Using Eq. (B3), and retaining only terms finite in the limit \( m \to \infty \), we find

\[
I_{\mu\nu} = e^2 \frac{16}{16m \sqrt{\varepsilon q^{-\mu} q^{-\rho}}} \frac{q_\mu q_\nu}{\sqrt{-p^2}}.
\] (16)

Taking into account the wave packet spreading, and going back to the coordinate space we finally arrive at the following expression for the correlation function

\[
B_{\mu\nu}(x) = \frac{e^2}{16m} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} e^{-ipx} a^*(q)a(q-p)q_\mu q_\nu \frac{\delta_{0\mu}\delta_{0\nu}}{\sqrt{\varepsilon q^{-\mu} q^{-\rho}}},
\] (17)

where \( p_0 = \varepsilon_q - \varepsilon_{q-p} \).

Let us apply this result to the static case to find the fluctuations of the Coulomb potential. Thus, we take \( a(q) \) such that

\[
\int \frac{d^3q}{(2\pi)^3} a^*(q)a(q)q = 0.
\]

The probability distribution function \( a(q) \) is generally of the form

\[ a(q) = b(q)e^{iqr_0}, \]

where \( r_0 \) is the mean particle position, and \( b(q) \) (which can be taken real) describes the shape of the wave packet. Since the particle is assumed sufficiently heavy, one has up to terms \( \sim 1/m, \varepsilon_{q-p} \approx \varepsilon_q \approx m, \) and therefore, \( p_0 \approx 0 \). By the same reason, one can neglect the space components of the particle momentum in comparison with its time component. Furthermore, as long as we are concerned with fluctuations of the Coulomb potential, we can substitute \( b(q-p) \) by \( b(q) \) : this amounts to neglecting multipole moments of the charge distribution. Taking all this into account, we rewrite Eq. (17) as

\[
B_{\mu\nu}(x) = \frac{e^2}{16} \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip(x-r_0)}}{|p|} \int \frac{d^3q}{(2\pi)^3} b^2(q)\delta_{0\mu}\delta_{0\nu} = \delta_{0\mu}\delta_{0\nu} \frac{e^2}{32\pi^2 r^2},
\] (18)

where \( r = |x-r_0| \). Thus, the root mean square fluctuation of the static potential \( \Phi^e \equiv A_0 \) of a massive particle is

\[
\sqrt{\langle \Delta \Phi^e(r)^2 \rangle} = \frac{e}{\sqrt{32\pi} r}.
\] (19)

This is to be compared with the Coulomb potential

\[ \Phi^e(r) = \frac{e}{4\pi r}. \] (20)
IV. QUANTUM FLUCTUATIONS OF GRAVITATIONAL FIELD

In this section, we will investigate quantum fluctuations of the gravitational field. As in the preceding section, we consider a system of quantized scalar matter interacting with the quantized gravitational field. This time the scalar particles are assumed real. The action of this system

\[ S[\phi, h] = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ g^\mu\nu \partial_\mu \phi \partial_\nu \phi - \left( \frac{mc}{\hbar} \right)^2 \phi^2 \right\} - \frac{c^3}{k^2} \int d^4x \sqrt{-g} R , \]  

where \( k^2 = 16\pi G, G \) is the Newton gravitational constant, and \( h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \) are chosen as the dynamical variables.

According to Eq. (6), correlation function of the components of gravitational field taken at the same spacetime point \( x_B^{\mu\nu,\alpha\beta}(x) \equiv \langle [h_{\mu\nu}(x) - h_{\mu\nu}^{(0)}(x)] [h_{\alpha\beta}(x) - h_{\alpha\beta}^{(0)}(x)] \rangle^{(0)} \) is

\[ i\bar{h} \left[ \delta^2 S[\phi, h] \right]^{(0)}_{h=h^{(0)}} . \]

Calculation of this function follows the same steps as in the electromagnetic case, but is more tedious. The leading contribution in the large particle mass limit is a gain contained in the diagram of Fig. 2(a), where the indices attached to the point of observation \( x \) are now replaced by the pair of double indices \( \mu\nu, \alpha\beta \).

\[ I^{(a)}_{\mu\nu,\alpha\beta}(p) = \frac{-i\mu^\epsilon}{\sqrt{2\varepsilon q^2\varepsilon q-p}} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{1}{2} W^{\gamma\delta\rho\tau} (q_\rho - p_\rho) (k_\tau + q_\tau) - \frac{m^2}{2} \eta^{\gamma\delta} \right\} \]

\[ \times G^\phi(q+k) \left\{ \frac{1}{2} W^{\sigma\lambda\xi\zeta} q_\zeta(k_\xi + q_\xi) - \frac{m^2}{2} \eta^{\sigma\lambda} \right\} G_{\mu\nu\gamma\delta} (k+p) G_{\alpha\beta\sigma\lambda}(k) , \]

where

\[ G_{\mu\nu\sigma\lambda}(k) = \frac{W_{\mu\nu\sigma\lambda}}{k^2} \]

is the graviton propagator, and

\[ W^{\alpha\beta\gamma\delta} = \eta^{\alpha\beta}\eta^{\gamma\delta} - \eta^{\alpha\gamma}\eta^{\beta\delta} - \eta^{\alpha\delta}\eta^{\beta\gamma} . \]

The graviton propagator is taken in the most convenient DeWitt gauge. As in the case of electrodynamics, a change of the gauge conditions fixing general covariance gives rise to terms irrelevant in the classical limit.

The tensor multiplication in Eq. (23) is conveniently performed with the help of the tensor package \[3\] for the REDUCE system.

\[ ^3\text{Our notation is } R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu} = \partial_\nu \Gamma^\alpha_{\mu\nu} - \cdots, \quad R \equiv R_{\mu\nu} g^{\mu\nu}, \ g \equiv \det g_{\mu\nu}, \ g_{\mu\nu} = \text{sgn}(+,-,-,-). \]

\[ ^4\text{In this section, we choose the units in which } k = c = \hbar = 1. \]
Using Eq. (B3) of Appendix B, we find the leading at large particle mass limit, which is contained entirely in the diagram of Fig. 2(a), is symmetric with respect to this interchange. For convenience, the integrand in Eq. (27) is rewritten in an explicitly symmetric form. See also the footnote 2.

\[ I_{m,\alpha\beta}^{(a)} = \frac{-i\mu^\epsilon}{\sqrt{2\pi^2\varepsilon}} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{(k+p)^2} \frac{1}{m^2-(k+q)^2} \times \left\{ m^4\eta_{\alpha\beta} - 4m^2\eta_{\alpha\beta}q_{\alpha}q_{\beta} - 4m^2\eta_{\mu\nu}q_{(\alpha}k_{\beta)} \\
+ 2m^2\eta_{\alpha\beta}(p_{(\mu}q_{\nu)} + p_{(\nu}k_{\alpha)}) - 4p_{(\mu}q_{\nu)}q_{(\alpha}k_{\beta)} \\
- 4q_{\alpha}q_{\beta}(p_{(\mu}q_{\nu)} + p_{(\nu}k_{\alpha)}) - 4p_{(\mu}k_{\nu)}q_{(\alpha}k_{\beta)} \\
+ 8q_{(\mu}q_{\nu)}q_{\alpha}q_{\beta} + 4q_{(\mu}k_{\nu)}q_{(\alpha}q_{\beta)} + 4q_{\mu}q_{\nu}q_{\alpha}q_{\beta} \right\}, \tag{25} \]

where \((\mu_1\mu_2\cdots\mu_n)\) denotes symmetrization over indices enclosed in parentheses,

\[
(\mu_1\mu_2\cdots\mu_n) = \frac{1}{n!} \sum_{\{i_{12}\cdots in\}=\text{perm}(12\cdots n)} \mu_i \mu_i \cdots \mu_i.
\]

The loop integral in Eq. (25) is again ultraviolet divergent:

\[
I_{m,\alpha\beta}^{(a)\text{div}} = \frac{1}{256\pi^2 \sqrt{\varepsilon q^2}} \frac{1}{(m^2-DH^3)} \left( \frac{\mu}{m} \right)^\epsilon \left( \eta_{\mu\alpha}p_{\nu}q_{\beta} + \eta_{\nu\beta}p_{\alpha}q_{\mu} - \eta_{\alpha\beta}p_{\mu}q_{\nu} \right.
\]

\[
- \eta_{\alpha\mu}q_{\nu}q_{\beta} - \eta_{\alpha\nu}q_{\mu}q_{\beta} - \eta_{\nu\beta}p_{\mu}q_{\alpha}.
\tag{26} \]

Introducing the Schwinger parametrization Eq. (11), calculating the loop integrals with the help of Eq. (12), subtracting the ultraviolet divergence (26), and changing the integration variables according to Eq. (13), we obtain the following expression\(^5\) for \(I_{m,\alpha\beta}^{(a)} \equiv (I_{m,\alpha\beta}^{(a)} - I_{m,\alpha\beta}^{(a)\text{div}})_{\epsilon=0}\), after a trivial integration over \(v\):

\[
I_{m,\alpha\beta} = \frac{1}{32\pi^2 \sqrt{\varepsilon q^2}} \int_0^{+\infty} \int_0^{+\infty} \frac{dudt}{2m^2DH^3} \times \left\{ 2\eta_{\mu\nu}m^4H^2 + 4(\eta_{\mu\nu}p_{(\alpha}q_{\beta)} + \eta_{\alpha\beta}p_{(\mu}q_{\nu)})m^2H(H-1) \\
- 4(\eta_{\mu\nu}q_{(\alpha}q_{\beta)} \right\} (H-1)^2 + \frac{2m^2}{p^2}(D-1) \\
+ 2(p_{\mu}p_{\nu}p_{(\alpha}q_{\beta)} + p_{\alpha}p_{\beta}p_{(\mu}q_{\nu)}) (H-1)^2 + 8p_{(\nu}q_{\alpha}p_{(\alpha}q_{\beta)} [(H-1)^2 + \frac{m^2}{p^2}(D-1)]

\]

\[
- 8(p_{(\mu}q_{\nu)}q_{\alpha}q_{\beta} + p_{(\alpha}q_{\beta)}q_{\mu}q_{\nu})(H-1)^2 + 8q_{(\mu}q_{\nu)}q_{(\alpha}q_{\beta)} (H-1)^2 \right\}. \tag{27} \]

Using Eq. (B3) of Appendix B, we find the leading at \(m \to \infty\) contribution

\(^5\)By itself, the right hand side of Eq. (25) is not symmetric with respect to the interchange \((\mu\nu) \leftrightarrow (\alpha\beta)\), in particular, its divergent part (26) is not. However, the part remaining finite in the large particle mass limit, which is contained entirely in the diagram of Fig. 2(a), is symmetric with respect to this interchange. For convenience, the integrand in Eq. (27) is rewritten in an explicitly symmetric form. See also the footnote 2.
\[ I_{\mu\nu,\alpha\beta} = \frac{(2q_\mu q_\nu - m^2 \eta_{\mu\nu})(2q_\alpha q_\beta - m^2 \eta_{\alpha\beta})}{64m \sqrt{\varepsilon q^2 q - p^2}}. \]  

(28)

If \( a(q) \) is the momentum probability distribution for a given particle, then the correlation function of the fluctuating gravitational field produced by this particle takes the form, in the ordinary units,

\[ B_{\mu\nu,\alpha\beta}(x) = \frac{4G^2 \pi^2}{mc^2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{e^{-ipx}}{\sqrt{\varepsilon q^2 q - p^2}} \frac{a^*(q)a(q-p)}{\varepsilon q^2 q - p^2} \left( \frac{2q_\mu q_\nu}{c^2} - m^2 \eta_{\mu\nu} \right) \left( \frac{2q_\alpha q_\beta}{c^2} - m^2 \eta_{\alpha\beta} \right), \]  

(29)

where \( p_0 = \varepsilon_q - \varepsilon_{q-p} \).

Following discussion in the preceding section, we apply this result to the static case. Setting \( \varepsilon_{q-p} \approx \varepsilon_q \approx mc^2 \), \( p_0 \approx 0 \), we see that only the components with \( \mu = \nu, \alpha = \beta \) survive in the limit \( m \to \infty \), being all equal to each other:

\[ B_{\mu\nu,\alpha\beta}(x) = \frac{2G^2m^2}{r^2c^4} \delta_{\mu\nu} \delta_{\alpha\beta}, \]  

(30)

where \( r = |x - r_0| \) is the distance between the observation point \( x \) and the mean particle position \( r_0 \). In particular, the root mean square fluctuation of \( \mu\nu \)-component of the metric

\[ \sqrt{\langle \Delta h_{\mu\nu}(r)^2 \rangle} = \delta_{\mu\nu} \frac{\sqrt{2} Gm}{rc^2}. \]  

(31)

The nonrelativistic gravitational potential \( \Phi^g \) is related to the 00-component of metric as \( \Phi^g = h_{00}c^2/2 \), thus,

\[ \sqrt{\langle \Delta \Phi^g(r)^2 \rangle} = \frac{Gm}{\sqrt{2} r}. \]  

(32)

We see that in both electromagnetic and gravitational cases, the relative value of the root mean square fluctuation of the potential is equal to \( 1/\sqrt{2} \).

V. DISCUSSION AND CONCLUSIONS

Equations (19) and (32) describing quantum fluctuations of the electromagnetic and gravitational fields of a massive particle, respectively, lead us to an important conclusion that dynamics of an elementary system which kinematics can be considered classically are nevertheless essentially quantum. The relative value of the root mean square fluctuation of the particle potential turns out to be equal to \( 1/\sqrt{2} \) in both electromagnetic and gravitational cases. Despite the fact that the uncertainty in the position and velocity of a sufficiently massive particle can be completely neglected, its interactions remain essentially quantum.

Quantum character of particle interactions makes the classical consideration inadequate in the case of systems governed by the interaction of the constituent particles. It must be mentioned in this connection that there is a deep-rooted belief in the literature that the quantum field description of interacting remote systems, each of which consists of many
particles, is equivalent to that in which these systems are replaced by elementary particles with masses and charges equal to the total masses and charges of the systems. In other words, the familiar notion of a point particle is carried over from the classical mechanics to the quantum field theory. This point of view is adhered, for instance, in the classic paper by Iwasaki [2] where it is applied to the solar system to calculate the shift of the Mercury perihelion, considered as a “Lamb shift”.

The Sun and Mercury are regarded in Ref. [2] as scalar particles. As we saw in Sec. IV, the root mean square fluctuation of the gravitational potential of the Sun in this case would be 71% of its value. Fortunately, such fluctuations are not observed in reality. This is because the Sun is composed of a huge number of elementary particles each of which contributes to the total gravitational field. To find the resulting quantum fluctuation of the total field, we turn back to the arguments presented in the Introduction. As far as the $\phi$-lines are concerned, the diagram of Fig. 2(a) is of the same structure as that of Fig. 1(b). Therefore, in the case when the gravitational field is produced by a $N$-particle body, this diagram is proportional to $m^2N$, where $m$ is the mass of a constituent particle. Correspondingly, the root mean square fluctuation of the potential is proportional to $m\sqrt{N} = M/\sqrt{N}$ ($M$ is the total mass of the body), while its relative value, to $1/\sqrt{N}$. If the solar gravitational field is considered, the quantum fluctuation turns out to be suppressed by a factor of the order $\sqrt{m_{\text{proton}}/M_\odot} \approx 10^{-28}$.

Another example of attempts to recover the nonlinearity of a classical theory through the radiative corrections can be found in Ref. [7]. The authors of [7] claim that the electromagnetic corrections of the order $e^2$ to the classical Reissner-Nordström solution are reproduced by the diagram of Fig. 1(b) in which the internal wavy lines correspond to the virtual photons. However, as we have shown, it is meaningless to try to establish the correspondence between classical and quantum theories in terms of elementary particles, because the quantum fluctuations of the electromagnetic and gravitational fields produced by such particles are of the order of the fields themselves. On the other hand, dependence of the diagram 1(b) on the number of particles is inappropriate to reproduce the classical physics in the macroscopic limit. This can be shown using the same argument as in the case of purely gravitational interaction. Namely, given a body with the total electric charge $Q$, consisting of $N = Q/q$ particles with charge $q$, the contribution of the diagram 1(b) is proportional to $N \cdot q^2 = Q^2/N$ turning into zero in the macroscopic limit. The relevant contribution correctly reproducing the $e^2$-correction to the Reissner-Nordström solution is given by the tree diagrams of Fig. 1(a) in which internal wavy lines correspond to virtual photons.

Thus, we arrive at the conclusion that the requirement of vanishing of the field fluctuations in the classical limit forces us to extend the macroscopic formulation of the correspondence principle, suggested in Ref. [3] in the case of gravity, to all interactions.

Finally, in the light of the above discussion, a natural question arises whether sufficiently massive objects which can be considered as elementary particles actually exist in our Universe. An example of such objects is probably supplied by the black holes. As is well known, black holes of certain types do behave like normal elementary particles [8]. Further discussion of this and related issues can be found in Refs. [9,10].
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APPENDIX A: GAUGE INDEPENDENCE OF THE CORRELATION FUNCTION

Up to an additive constant, the quantities $\Phi^e(r)$, $\Phi^g(r)$ determine the potential energy of interacting particles. Their fluctuations are thus of direct physical significance, and therefore, expected to be independent of the gauge conditions chosen to fix the gauge invariance. Let us show that the correlation functions (8), (22) remain unchanged under variations of the gauge conditions indeed. We consider only variations which do not alter the potentials $\Phi^e(r)$, $\Phi^g(r)$ themselves, since the question of gauge independence of their fluctuations would be meaningless otherwise. This restriction, of course, is not a loss of generality. To allow for completely arbitrary variations of the gauge conditions, we should have been dealing with the fluctuations of gauge invariant quantities built from potentials, rather than the potentials themselves, from the very beginning.

Variations of the gauge conditions which leave the Coulomb potential unchanged are those satisfying

$$\delta G_{\mu\nu}(x) = \partial_\mu \partial_\nu d(x). \quad (A1)$$

Indeed, in this case only

$$\delta A_\mu(x) = \int d^4 y \, \delta G_{\mu\nu}(x - y) j^\nu(y) = \int d^4 y \, \partial_\mu^\nu d(x - y) \partial_\nu^\rho j^\rho(y) = 0,$$

in view of the current conservation. In the case of gravity, the corresponding variation of the propagator is more complicated:

$$\delta G_{\mu\nu\alpha\beta}(x) = \partial_\nu \partial_\beta d_{\mu\alpha}(x) + \partial_\mu \partial_\alpha d_{\nu\beta}(x) + \partial_\nu \partial_\alpha d_{\mu\beta}(x) + \partial_\mu \partial_\beta d_{\nu\alpha}(x), \quad d_{\mu\nu} = d_{\nu\mu}. \quad (A2)$$

Such a variation induces no variation of the metric (in particular, of the Newton potential) because of the energy-momentum conservation.

It is not difficult to verify that Eqs. (A1) and (A2) can be rewritten as

$$\delta G_{\mu\nu} = \partial_\mu \xi_\nu \equiv D^{(0)}_{\mu\nu} \xi_\nu, \quad \xi_\nu = \partial_\nu d, \quad (A3)$$

and

$$\delta G_{\mu\nu\alpha\beta} = \eta_{\nu\gamma} \partial_\alpha \xi^\gamma_{\beta\alpha} + \eta_{\nu\gamma} \partial_\mu \xi^\gamma_{\beta\alpha} \equiv D^{(0)}_{\mu\nu\gamma} \xi^\gamma_{\beta\alpha}, \quad \xi^\gamma_{\beta\alpha} = 2 \partial_{(\beta} d^\gamma_{\alpha)}, \quad (A4)$$

respectively. The operators $D^{(0)}$ are nothing but the generators of the gauge transformations of free electromagnetic and gravitational fields. To tackle both cases simultaneously, we denote the gauge field collectively by $Z_A$ with a single Latin capital index, and rewrite Eqs. (A3), (A4) uniquely as

$$\delta G_{AB} = D^{(0)}_{A|\gamma} \xi^\gamma_B = D^{(0)}_{B|\gamma} \xi^\gamma_A. \quad (A5)$$
(in the electromagnetic case, $\gamma$ takes only one value.) Using this representation, it is easy to show that the results (17), (29) are invariant with respect to the gauge transformations (A5). We will prove this fact even in a more general setting when the gauge fields are produced by an arbitrary species of particles, bosons or fermions, denoted collectively by $\phi_i$, $i = 1, \ldots, I$.

It follows from Eq. (A5) that the gauge dependent part of the gauge field propagators in Fig. 2(a) is attached to the matter line through the generator $D^{(0)}$. On the other hand, the action $S(\phi, Z)$ is gauge invariant

$$\delta S(\phi, Z) \over \delta \phi_i D_{i|\gamma} + \delta S(\phi, Z) D_{A|\gamma} = 0,$$

where $D_{A|\gamma} = D_{A|\gamma}(Z)$ and $D_{i|\gamma} = D_{i|\gamma}(\phi)$ are generators of the gauge transformations of the gauge and matter fields, respectively [$D_{A|\gamma}(0) \equiv D^{(0)}_{A|\gamma}$].

Differentiating this identity with respect to $\phi_k$, setting $Z_A = 0$, and taking into account that the external $\phi$-lines are on the mass shell

$$\delta S^{(2)}(\phi, 0) \over \delta \phi_i = 0,$$

where $S^{(2)}$ denotes the part of $S(\phi, Z)$ bilinear in $\phi$, the $\phi^2 Z$ vertex can be rewritten as

$$\delta^2 S^{(2)}(\phi, Z) \over \delta Z_A \delta \phi_k \bigg|_{Z=0} D^{(0)}_{A|\gamma} = - \delta^2 S^{(2)}(\phi, 0) \over \delta \phi_i \delta \phi_k D_{i|\gamma}.$$

Thus, under contraction with the vertex factor, the $\phi$-particle propagator, $G^{\phi}_{ik}$, satisfying

$$\delta^2 S^{(2)}(\phi, 0) \over \delta \phi_i \delta \phi_k G^{\phi}_{kl} = - \delta^i_l,$$

cancels out

$$G^{\phi}_{kl} \delta^2 S^{(2)}_{\phi} \over \delta Z_A \delta \phi_k \bigg|_{Z=0} D^{(0)}_{A|\gamma} = D_{i|\gamma}. (A8)$$

The root singularity responsible for the $r^{-2}$-behavior of the correlation function occurs because of the virtual $\phi$-particle propagation near its mass shell. Thus, the cancellation of the $\phi$-propagator in the gauge dependent part of the diagram 2(a) implies the gauge independence of the right hand sides of Eqs. (17), (29).

**APPENDIX B: ROOT SINGULARITIES OF FEYNMAN INTEGRALS**

The root singularity with respect to the momentum transfer, responsible for the $r^{-2}$ fall off of the correlation function, is contained in the integrals

$$J_{nm} \equiv \int_0^\infty \int_0^\infty \frac{dudt}{(1 + t + u)^n(1 + atu)^m},$$

where $n, m$ are positive integers.
encountered in Secs. III, IV. It can be extracted as follows. Consider the auxiliary quantity

\[ J(A, B) = \int_0^\infty \int_0^\infty \frac{dudt}{(A + t + u)(B + \alpha tu)}, \]

where \( A, B > 0 \) are parameters eventually set equal to 1. Performing an elementary integration over \( u \), we get

\[ J(A, B) = \int_0^\infty \frac{\ln B - \ln\{\alpha t(A + t)\}}{B - \alpha t(A + t)}. \]

Now consider the integral

\[ \tilde{J}(A, B) = \oint_C dz f(z, A, B), \quad f(z, A, B) = \frac{\ln B - \ln\{\alpha z(A + z)\}}{B - \alpha z(A + z)}, \quad (B1) \]

taken over the contour \( C \) shown in Fig. 3. \( \tilde{J}(A, B) \) is zero identically. On the other hand,

\[ \tilde{J}(A, B) = \int_{-A}^A dw \frac{\ln B - \ln\{\alpha w(A + w)\}}{B - \alpha w(A + w)} + \int_0^A dw \frac{\ln B - \ln\{-\alpha w(A + w)\}}{B - \alpha w(A + w)} + i\pi \]

\[ + pv \int_0^\infty dw \frac{\ln B - \ln\{\alpha w(A + w)\}}{B - \alpha w(A + w)} + 2i\pi \sum_{z_+,z_-} \text{Res} f(z, A, B), \]

“pv” denoting the principal value, and \( z_\pm \) the poles of the function \( f(z, A, B) \),

\[ z_\pm = -\frac{A}{2} \pm \sqrt{\frac{B}{\alpha} + \frac{A^2}{4}}. \]

Change \( w \to -A - w \) in the first integral. A simple calculation then gives

\[ J(A, B) = \frac{\pi^2}{2\sqrt{\alpha}} B^{-1/2} \left( 1 + \frac{\alpha A^2}{4B} \right)^{-1/2} - \frac{1}{2} \int_0^A dt \frac{\ln B - \ln\{\alpha t(A - t)\}}{B + \alpha t(A - t)}. \quad (B2) \]

The roots are contained entirely in the first term on the right of Eq. (B2). The integrals \( J_{nm} \) are found by repeated differentiation of Eq. (B2) with respect to \( A, B \). Expanding \( (1 + \alpha A^2/4B)^{-1/2} \) in powers of \( \alpha \), we find the leading terms needed in Eqs. (15), (27)

\[ J_{11}^{\text{root}} = \frac{\pi^2}{2\sqrt{\alpha}}, \quad J_{21}^{\text{root}} = \frac{\pi^2}{8\sqrt{\alpha}}, \quad J_{31}^{\text{root}} = -\frac{\pi^2}{16\sqrt{\alpha}}. \quad (B3) \]
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FIG. 1. Diagrams contributing to the first post-Newtonian correction. (a) The tree diagrams occurring because of the gravitational self-interaction, and gravitational particle interactions. (b) The one-loop radiative correction. Wavy lines represent gravitons, solid lines constituent particles.

FIG. 2. The leading ($\sim e^2$) contribution to the correlation function $B_{\mu\nu}(x)$.
FIG. 3. Contour of integration in Eq. (B1).