Asymptotic growth of trajectories of multifractional brownian motion, with statistical applications to drift parameter estimation

Marco Dozzi, Yuriy Kozachenko, Yuliya Mishura, Kostiantyn Ralchenko

To cite this version:
Marco Dozzi, Yuriy Kozachenko, Yuliya Mishura, Kostiantyn Ralchenko. Asymptotic growth of trajectories of multifractional brownian motion, with statistical applications to drift parameter estimation. Statistical Inference for Stochastic Processes, 2018, 21, pp.21-52. 10.1007/s11203-016-9147-z. hal-02937619

HAL Id: hal-02937619
https://hal.science/hal-02937619
Submitted on 14 Sep 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
ASYMPTOTIC GROWTH OF TRAJECTORIES OF MULTIFRACTIONAL BROWNIAN MOTION, WITH STATISTICAL APPLICATIONS TO DRIFT PARAMETER ESTIMATION

MARCO DOZZI, YURIY KOZACHENKO, YULIYA MISHURA, AND KOSTIANTYN RALCHENKO

ABSTRACT. We construct the least-square estimator for the unknown drift parameter in the multifractional Ornstein–Uhlenbeck model and establish its strong consistency in the non-ergodic case. The proofs are based on the asymptotic bounds with probability 1 for the rate of the growth of the trajectories of multifractional Brownian motion (mBm) and of some other functionals of mBm, including increments and fractional derivatives. As the auxiliary results having independent interest, we produce the asymptotic bounds with probability 1 for the rate of the growth of the trajectories of the general Gaussian process and some functionals of it, in terms of the covariance function of its increments.

1. Introduction

The goal of the present paper is twofold. First, we get the asymptotic bounds with probability 1 for the rate of the growth of the trajectories of multifractional Brownian motion (mBm) and of some other functionals of mBm, including increments and fractional derivatives. Second, we apply these bounds to construct consistent estimators of the unknown drift parameter in the linear and Ornstein–Uhlenbeck model involving mBm. As the auxiliary results having independent interest, we produce the asymptotic bounds with probability 1 for the rate of the growth of the trajectories of the general Gaussian process and some functionals of it, in terms of the covariance function of its increments. The results obtained generalize the respective results concerning asymptotic bounds with probability 1 for the rate of the growth of the trajectories of fractional Brownian motion (fBm) from [9] and numerous results concerning consistent estimators of the unknown drift parameter in the linear and Ornstein–Uhlenbeck model involving fBm. The extended survey of these results is contained, e.g., in the paper [10]. The methods of constructing the estimators and their properties in the fractional Brownian case essentially depend on factors such as the value of Hurst index \( H \), more precisely, cases \( H > 1/2 \) and \( H < 1/2 \) differ substantially; on the sign of unknown drift parameter \( \theta \) and also on whether the continuous and discrete observations. The MLE estimators of the unknown drift parameter for fractional Ornstein–Uhlenbeck process with \( H \geq 1/2 \) and any \( \theta \in \mathbb{R} \) were constructed in [8] with the help of so called Molchan fundamental martingale. The same estimator was studied in

2010 Mathematics Subject Classification. 60G15, 60G22, 62F10, 62F12.

Key words and phrases. Gaussian process, multifractional Brownian motion, parameter estimation, consistency, strong consistency, stochastic differential equation.
the paper [16] for $H < 1/2$. In the paper [7] the analog of the least-square estimator of the form \( \hat{\theta}_T = \frac{\int_0^T X_t^2 dt}{\int_0^T X_t dt} \) was constructed for $H \geq 1/2$ and $\theta < 0$ (the ergodic case) was studied in the supposition that the integral $\int_0^T X_t dX_t$ is the divergence-type one. As an alternative estimator, $\left( \frac{1}{T} \int_0^T X_t^2 dt \right)^{1/2}$ was proposed, and its properties are essentially based on the ergodic properties of the fractional Ornstein-Uhlenbeck process with negative drift. In the papers [10], [11] and [12] the discretized estimators were proposed. Another approach to discretization was studied in [2]. Note that the MLE is hardly discretized because of singular kernels and one should choose the nonstandard estimators for discretization, that was done in these papers. Mention also that the discretized estimator proposed in [10] for $H < 1/2$, in reality works properly only for $\theta \geq 0$ and apparently does not work in the ergodic case. In general, the problem of the discretization for $H < 1/2$ and $\theta < 0$ is open. Contrary to fractional case, multifractional Ornstein–Uhlenbeck model was not considered in its entirety, even though these models are gaining increasing popularity now. We can mention only the paper [6], where the least square estimator is studied for the non-ergodic Ornstein–Uhlenbeck process with some special Gaussian process, the case that includes not only fractional but, e.g., subfractional and bi-fractional Brownian motions. In the present paper we consider Ornstein–Uhlenbeck multifractional processes when the index $H_t$ of multifractionality is bounded from below by some constant exceeding $\frac{1}{2}$, and observations are continuous in time. We consider non-ergodic case, because the asymptotical bounds for the growth of Ornstein–Uhlenbeck multifractional process work properly in the non-ergodic case. The problem of the drift parameter estimation in the multifractional Ornstein–Uhlenbeck model is still open. Note that the linear model with unknown drift parameter is considered, and the properties of the estimator are based on the asymptotic growth of the trajectories of mBm. The paper is organized as follows. Section 2 contains auxiliary results for the asymptotic growth of Gaussian processes defined on arbitrary parameter set, on the half-axis, and in the strip on the plane. In Section 3 we establish the asymptotic growth with probability 1 of mBm and its increments. In Section 4 we investigate two statistical models with mBm: the linear model and the multifractional Ornstein–Uhlenbeck process. For these models we propose estimators for an unknown drift parameter and prove their strong consistency.

2. EXPONENTIAL MAXIMAL BOUNDS AND ASYMPOTOTIC GROWTH OF TRAJECTORIES OF GAUSSIAN PROCESSES

Let $T$ be a parameter set and $X = \{X(t), t \in T\}$ be a centered Gaussian process. Introduce the notation

$$\rho_X(t, s) = \left( \mathbb{E}(X(t) - X(s))^2 \right)^{1/2}, s, t \in T.$$ 

Evidently, $\rho_X$ is a pseudometric on $T$. Also, denote

$$m(T) = \sup_{t \in T} \left( \mathbb{E}|X(t)|^2 \right)^{1/2}.$$ 

Throughout the section we assume that the following conditions hold.

(A1) $m(T) < \infty$.

(A2) The space $(T, \rho_X)$ is separable and the process $X$ is separable on this space.
2.1. Exponential maximal upper bound for Gaussian process in terms of metric massiveness. In this subsection we present the general results concerning exponential maximal upper bound for Gaussian process defined on an arbitrary parameter set, in terms of metric massiveness. Let \( N(u), u > 0 \) be the metric massiveness of the space \((T, \rho_X)\), that is, \( N(u) \) is the number of open balls in the minimal \( u \)-covering of \((T, \rho_X)\). Consider the function \( r(x), x \geq 1 \) satisfying the following properties:

(i) \( r \) is non-negative and nondecreasing;

(ii) \( r(e^y), y \geq 0 \) is a convex function.

Introduce one more notation: let \( I_r(x) = \int_0^x r(N(v))dv, x > 0 \).

**Theorem 2.1.** Let \( I_r(m(T)) < \infty \). Then the following bounds hold:

(i) For any \( \theta \in (0, 1) \) and any \( \lambda > 0 \)

\[
\mathbb{E} \exp \left\{ \lambda \sup_{t \in T} |X(t)| \right\} \leq 2A_1(\lambda, \theta),
\]

where

\[
A_1(\lambda, \theta) = \exp \left\{ \frac{\lambda^2 m^2(T)}{2(1 - \theta)^2} \right\} r^{(-1)} \left( \frac{I_r(\theta m(T))}{\theta m(T)} \right),
\]

\( r^{(-1)}(t) \) is the generalized inverse function of \( r(t) \) that is

\[
r^{(-1)}(t) = \sup \{ u \geq 0 : r(u) \leq t \}.
\]

(ii) For any \( \theta \in (0, 1) \) and any \( \mu > 0 \)

\[
\mathbb{P} \left\{ \sup_{t \in T} |X(t)| \geq \mu \right\} \leq 2A_2(\mu, \theta),
\]

where

\[
A_2(\mu, \theta) = \exp \left\{ \frac{\mu^2 (1 - \theta)^2}{2m^2(T)} \right\} r^{(-1)} \left( \frac{I_r(\theta m(T))}{\theta m(T)} \right).
\]

**Proof.** (i) First, we simplify the notation: let \( m := m(T) \). Now our goal is to establish the following bound: for arbitrary \( \theta \in (0, 1) \) and any sequence \( \tau_n > 0 \) such that \( \sum_{n=1}^{\infty} \tau_n = 1 \),

\[
\mathbb{E} \exp \left\{ \lambda \sup_{t \in T} |X(t)| \right\} \leq \prod_{n=1}^{\infty} \left[ 2N(m \theta^n) \exp \left\{ \frac{\lambda^2 m^2 \theta^{2n-1} \tau_n}{2} \right\} \right]^{\frac{1}{\tau_n}}. \tag{3}
\]

Let \( \theta \in (0, 1) \) and put \( u_n = m \theta^n, n \geq 0 \). Denote by \( S_n \) a minimal \( u_n \)-net in the set \( T \) with respect to the pseudometric \( \rho_X \) and put \( S = \bigcup_{n=0}^{\infty} S_n \). According to condition (A2), the set \( S \) is countable and everywhere dense in \( T \) with respect to the pseudometric \( \rho_X \), and the process \( X \) is continuous in probability in \((T, \rho_X)\).

Therefore the set \( S \) is a \( \rho_X \)-separability set for the process \( X \) and moreover

\[
\sup_{t \in T} |X(t)| = \sup_{t \in S} |X(t)|
\]

with probability 1.

Suppose that \( t \in S \). Then there exists a number \( n(t) \) such that \( t \in S_{n(t)} \). Define a function \( \alpha_k : S \to S_k, k \geq 0 \) as \( \alpha_k(x) = x \) if \( x \in S_k \) and \( \alpha_k(x) \) is the point of \( S_k \) closest to \( x \) if \( x \notin S_k \). If there is more than one closest point then we may choose any of these points. The family of maps \( \{\alpha_k, k \geq 0\} \) is called the \( \alpha \)-procedure for
choosing points in \( S \). Using the \( \alpha \)-procedure we can choose a sequence of points 
\[ t_{n(t)} = t, \ t_{n(t)-1} = \alpha_{n(t)-1}(t_{n(t)}), \ldots, \ t_1 = \alpha_1(t_2) \text{ such that } t_k \in S_k, \ k = 1, \ldots, n(t) \text{ and } \rho_X(t_k, \alpha_{k-1}(t_k)) \leq u_{k-1}. \]
Evidently,
\[
X(t) = X(t_1) + \sum_{k=2}^{n(t)} (X(t_k) - X(t_{k-1})).
\]
Therefore we have an upper bound
\[
\sup_{t \in S} |X(t)| \leq \max_{s \in S_1} |X(s)| + \sum_{n=2}^{\infty} \max_{s \in S_n} |X(s) - X(\alpha_{n-1}(s))|.
\]
Take any sequence of numbers \( r_n > 0, \ n \geq 1 \) such that \( \sum_{n=1}^{\infty} r_n^{-1} = 1 \). It follows from the Hölder inequality that for any \( \lambda > 0 \)
\[
E \exp \left\{ \lambda \sup_{t \in S} |X(t)| \right\} \leq E \exp \left\{ \lambda \left( \max_{s \in S_1} |X(s)| + \sum_{n=2}^{\infty} \max_{s \in S_n} |X(s) - X(\alpha_{n-1}(s))| \right) \right\}
\]
\[
\leq \left[ E \exp \left\{ \lambda r_1 \max_{s \in S_1} |X(s)| \right\} \right] ^{\frac{1}{\lambda r_1}} \prod_{n=2}^{\infty} \left[ E \exp \left\{ \lambda r_n \max_{s \in S_n} |X(s) - X(\alpha_{n-1}(s))| \right\} \right] ^{\frac{1}{\lambda r_n}}.
\]
(4)

Furthermore, all the multipliers in the right-hand side of (4), except the 1st one, can be estimated as
\[
E \exp \left\{ \lambda r_n \max_{s \in S_n} |X(s) - X(\alpha_{n-1}(s))| \right\}
\]
\[
\leq N(u_n) \max_{s \in S_n} E \exp \left\{ \lambda r_n |X(s) - X(\alpha_{n-1}(s))| \right\},
\]
and, in addition,
\[
\left( E |X(s) - X(\alpha_{n-1}(s))|^2 \right) ^{\frac{1}{2}} = \rho_X(s, \alpha_{n-1}(s)) \leq u_{n-1} = \theta^{n-1} u_0
\]
for \( s \in S_n \). Therefore, for any \( n \geq 2 \)
\[
\max_{s \in S_n} E \exp \left\{ \lambda r_n |X(s) - X(\alpha_{n-1}(s))| \right\} \leq \max_{s \in S_n} \left( E \exp \left\{ \lambda r_n |X(s) - X(\alpha_{n-1}(s))| \right\} \right)
\]
\[
+ E \exp \left\{ - \lambda r_n |X(s) - X(\alpha_{n-1}(s))| \right\}
\]
\[
= \max_{s \in S_n} 2 \exp \left\{ \frac{1}{2} r_n^2 E |X(s) - X(\alpha_{n-1}(s))|^2 \right\} \leq 2 \exp \left\{ \frac{1}{2} r_n^2 (\theta^{n-1} u_0)^2 \right\}.
\]
(6)

Now we estimate the first multiplier in the right-hand side of (4):
\[
E \exp \left\{ \lambda r_1 \max_{s \in S_1} |X(s)| \right\} \leq N(u_1) \max_{s \in S_1} E \exp \left\{ \lambda r_1 |X(s)| \right\}
\]
\[
\leq 2N(u_1) \exp \left\{ \frac{\lambda r_1^2}{2} \max_{s \in S_1} |X(s)|^2 \right\} \leq 2N(u_1) \exp \left\{ \frac{\lambda r_1^2}{2} (\theta u_0)^2 \right\}.
\]
(7)

Taking into account that it follows from the separability of \( X \) that
\[
E \exp \left\{ \lambda \sup_{t \in T} |X(t)| \right\} = E \exp \left\{ \lambda \sup_{t \in S} |X(t)| \right\},
\]
we get inequality (3) from (4)–(7). It follows from (3) that
\[
E \exp \left\{ \lambda \sup_{t \in T} |X(t)| \right\} \leq 2 \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{r_n} H(m \theta^n) + \sum_{n=1}^{\infty} \frac{\lambda^2 m^2}{2} \theta^{2(n-1)} r_n \right\},
\]
(8)
where \( H(u) = \log N(u) \) is the metric entropy. Now, choose \( r_n = \frac{1}{\theta^{n-1}(1-\theta)} \). Then
\[
E \exp \left\{ \lambda \sup_{t \in T} |X(t)| \right\} \leq 2 \exp \left\{ (1 - \theta) \sum_{n=1}^{\infty} \theta^{n-1} H(m \theta^n) + \frac{\lambda^2 m^2}{2(1 - \theta)^2} \right\}.
\]
(9)
Since \( r(e^u) \) is a convex function, we have that
\[
r^{(-1)} \left( r \left( \exp \left\{ \sum_{n=1}^{\infty} (1 - \theta) \theta^{n-1} H(\theta^n m) \right\} \right) \right)
\]
\[
\leq r^{(-1)} \left( \sum_{n=1}^{\infty} (1 - \theta) \theta^{n-1} r \left( \exp \left\{ H(\theta^n m) \right\} \right) \right)
\]
\[
= r^{(-1)} \left( \sum_{n=1}^{\infty} (1 - \theta) \theta^{n-1} r \left( N(\theta^n m) \right) \right)
\]
\[
\leq r^{(-1)} \left( \sum_{n=1}^{\infty} (1 - \theta) \theta^{n-1} \frac{1}{m \theta^n (1 - \theta)} \int_{\theta^n m}^{m \theta^n} r(N(u)) du \right)
\]
\[
= r^{(-1)} \left( \frac{1}{m \theta} \int_{0}^{m \theta} r(N(u)) du \right).
\]
(10)
Now, inequality (1) follows from (9) and (10).
(ii) Now we are in position to establish inequality (2). Let \( u > 0 \), \( 0 < \theta < 1 \), \( \lambda > 0 \). Then Chebyshev’s inequality and (1) yield that
\[
P \left\{ \sup_{t \in T} |X(t)| \geq u \right\} \leq E \exp \left\{ \lambda \sup_{t \in T} |X(t)| \right\} \exp \left\{ -\lambda u \right\}
\]
\[
\leq 2 r^{(-1)} \left( \frac{L_r(\theta m)}{\theta m} \right) \exp \left\{ \frac{\lambda^2 m^2}{2(1 - \theta)^2} - \lambda u \right\}.
\]
Minimizing \( \frac{\lambda^2 m^2}{2(1 - \theta)^2} - \lambda u \) with respect to \( \lambda > 0 \), we note that minimum is achieved at the point \( \lambda = \frac{u(1-\theta)^2}{m^2} \), whence (2) immediately follows.

Applying this result to the parameter set \( T = [a, b] \), we get the following result.

**Corollary 2.2.** Let \( T = [a, b] \), \( X = \{X(t), t \in [a, b]\} \) be a centered separable Gaussian process and \( m := m([a, b]) = \sup_{t \in [a, b]} \left( E \|X(t)\|^2 \right)^{1/2} < \infty \). Assume that there exists a strictly increasing function \( \sigma = \{\sigma(h), h > 0\} \) such that \( \sigma(h) > 0, h > 0, \sigma(h) \downarrow 0 \) as \( h \downarrow 0 \), and
\[
\sup_{|t-s| < h} \left( E \|X(t) - X(s)\|^2 \right)^{1/2} \leq \sigma(h).
\]
Then for any \( \theta \in (0, 1) \) and any \( \lambda > 0 \)
\[
E \exp \left\{ \lambda \sup_{t \in [a, b]} |X(t)| \right\} \leq 2 A_4(\lambda, \theta),
\]
(11)
where
\[ A_3(\lambda, \theta) = \exp \left\{ \frac{\lambda^2 m^2}{2(1 - \theta)^2} \right\} r(-1) \left( \frac{\hat{I}_r(\theta m)}{\theta m} \right), \]
(12)
and
\[ \hat{I}_r(x) = \int_0^x r \left( \frac{b - a}{2\sigma^{(1)}(v)} + 1 \right) dv. \]
Indeed, in this case condition (A2) holds and \( N(v) \leq \frac{b - a}{2\sigma^{(1)}(v)} + 1 \), whence (11)-(12) immediately follow.

**Corollary 2.3.** Let we can put \( \sigma(h) = ch^\beta \) with \( c > 0 \), \( 0 < \beta \leq 1 \) in Corollary 2.2. Then for any \( \theta \in (0, 1) \) and any \( \lambda > 0 \)
\[ \mathbb{E} \exp \left\{ \lambda \sup_{t \in [a, b]} |X(t)| \right\} \leq 2^{\frac{\beta}{1 - \beta}} \exp \left\{ \frac{\lambda^2 m^2}{2(1 - \theta)^2} \left( \frac{2^{\beta - 1}(b - a)c^{1/\beta}}{(\theta m)^{1/\beta}} + 1 \right) \right\}. \]
(13)
Indeed, consider \( r(x) = x^\alpha - 1 \), \( x \geq 1 \), where \( 0 < \alpha < \beta \). Since \( \sigma^{(1)}(s) = (s^{1/\beta}) \), we have
\[ \hat{I}_r(\theta m) = \int_0^{\theta m} \left( \frac{b - a}{2\sigma^{(1)}(s)} + 1 \right)^\alpha ds \leq \int_0^{\theta m} \left( \frac{b - a}{2\sigma^{(1)}(s)} \right)^\alpha ds \]
\[ = \int_0^{\theta m} \left( \frac{(b - a)c^{1/\beta}}{2s^{1/\beta}} \right)^\alpha ds = \left( \frac{(b - a)c^{1/\beta}}{2} \right)^\alpha \left( \frac{1}{1 - \frac{\alpha}{\beta}} \right) \left( \theta m \right)^{1 - \alpha/\beta}. \]
Therefore in this case
\[ A_3(\lambda, \theta) \leq \exp \left\{ \frac{\lambda^2 m^2}{2(1 - \theta)^2} \left( \frac{(b - a)c^{1/\beta}}{2} \right)^\alpha \left( \frac{1}{1 - \frac{\alpha}{\beta}} \right) \left( \theta m \right)^{1 - \alpha/\beta} + 1 \right\}^{\frac{1}{\beta}}. \]
Applying the elementary inequality \( (a + b)^p \leq 2^{p-1}(a^p + b^p) \), \( p \geq 1 \), we get
\[ A_3(\lambda, \theta) \leq \exp \left\{ \frac{\lambda^2 m^2}{2(1 - \theta)^2} \right\} \cdot 2^{\frac{1}{p} - 1} \left( \frac{(b - a)c^{1/\beta}}{2 \left( 1 - \frac{\alpha}{\beta} \right)^{1/\beta} \left( \theta m \right)^{1/\beta}} + 1 \right). \]
(14)
Now (13) follows from (14) if we put \( p = \frac{\beta}{1 - \beta} \).

2.2. Exponential maximal upper bound for the weighted Gaussian process defined on the half-axis. Now, let \( X = \{X(t), t \geq 0\} \) be a centered Gaussian process and \( a(t) > 0 \) be a continuous strictly increasing function such that \( a(t) \to \infty \) as \( t \to \infty \). Introduce the sequence \( b_0 = 0, b_{k+1} > b_k, b_k \to \infty \) as \( k \to \infty \).

Denote \( a_k = a(b_k) \) and \( m_k = m([b_k, b_{k+1}]) = \sup_{t \in [b_k, b_{k+1}]} \left( \mathbb{E} |X(t)|^2 \right)^{1/2} \). Our goal is to get exponential maximal upper bound for the weighted Gaussian process \( X(t)/a(t) \), applying the above results, in particular, Corollary 2.3.

**Theorem 2.4.** Let the following conditions hold:
(i) There exist \( c_k > 0 \) and \( 0 < \beta < 1 \) such that
\[ \sup_{t \in [b_k, b_{k+1}]} \left( \mathbb{E} |X(t) - X(s)|^2 \right)^{1/2} \leq c_k h^\beta, \]
where
\[ A_3(\lambda, \theta) = \exp \left\{ \frac{\lambda^2 m^2}{2(1 - \theta)^2} \right\} r(-1) \left( \frac{\hat{I}_r(\theta m)}{\theta m} \right). \]
and
\[ \hat{I}_r(x) = \int_0^x r \left( \frac{b - a}{2\sigma^{(1)}(v)} + 1 \right) dv. \]
(ii) \[ 0 < m_k < \infty \text{ and } A = \sum_{k=0}^{\infty} \frac{m_k}{a_k} < \infty; \] (15)

(iii) There exists \( 0 < \gamma \leq 1 \) such that
\[ \sum_{k=0}^{\infty} \frac{m_k^{-\gamma/\beta} (b_{k+1} - b_k)^\gamma c_k^{1/\beta}}{a_k} < \infty. \] (16)

Then for any \( \theta \in (0, 1) \) and any \( \lambda > 0 \)
\[ I(\lambda) = \mathbb{E} \exp \left\{ \lambda \sup_{t > 0} \frac{|X(t)|}{a(t)} \right\} \leq 2^{\frac{1}{\gamma} - 1} \exp \left\{ \frac{\lambda^2 A^2}{2(1 - \theta)^2} \right\} A_4(\theta, \gamma), \]
where
\[ A_4(\theta, \gamma) = \exp \left\{ \frac{1}{\gamma A} \left( \sum_{k=0}^{\infty} \frac{m_k^{1-\gamma/\beta} (b_{k+1} - b_k)^\gamma c_k^{1/\beta}}{a_k} \right) \left( \frac{2^{2/\beta - 1}}{\theta^{1/\beta}} \right)^\gamma \right\}. \]

Proof. Let \( r_k > 0, k = 0, 1, 2 \ldots \) and \( \sum_{k=0}^{\infty} \frac{1}{r_k} = 1 \). Then for any \( \lambda > 0 \)
\[ I(\lambda) \leq \mathbb{E} \exp \left\{ \lambda \sum_{k=0}^{\infty} \sup_{t \in [b_k, b_{k+1}]} \frac{|X(t)|}{a(t)} \right\} \leq \prod_{k=0}^{\infty} \left( \mathbb{E} \exp \left\{ \lambda r_k \sup_{t \in [b_k, b_{k+1}]} \frac{|X(t)|}{a(t)} \right\} \right)^{r_k}. \]

It follows from Corollary 2.3 that for any \( \theta \in (0, 1) \)
\[ \mathbb{E} \exp \left\{ \lambda r_k \sup_{t \in [b_k, b_{k+1}]} \frac{|X(t)|}{a(t)} \right\} \leq \mathbb{E} \exp \left\{ \lambda r_k \sup_{t \in [b_k, b_{k+1}]} \frac{|X(t)|}{a(t)} \right\} \leq 2^{\frac{1}{\gamma} - 1} \exp \left\{ \frac{\lambda^2 r_k^2 m_k^2}{2(1 - \theta)^2 a_k^2} \left( 1 + \frac{b_{k+1} - b_k}{\theta^{1/\beta}} \right)^{2/\beta - 1} \left( \frac{c_k}{m_k} \right)^{1/\beta} \right\}. \]
Therefore,
\[ I(\lambda) \leq 2^{\frac{1}{\gamma} - 1} \exp \left\{ \frac{\lambda^2}{2(1 - \theta)^2} \sum_{k=0}^{\infty} \frac{r_k m_k^2}{a_k^2} \right\} \prod_{k=0}^{\infty} \left[ 1 + \frac{b_{k+1} - b_k}{\theta^{1/\beta}} \left( \frac{c_k}{m_k} \right)^{1/\beta} \right]^{r_k} = 2^{\frac{1}{\gamma} - 1} \exp \left\{ \frac{\lambda^2}{2(1 - \theta)^2} \sum_{k=0}^{\infty} \frac{r_k m_k^2}{a_k^2} \right\} \times \exp \left\{ \sum_{k=0}^{\infty} \frac{1}{r_k} \log \left[ 1 + \frac{b_{k+1} - b_k}{\theta^{1/\beta}} \left( \frac{c_k}{m_k} \right)^{1/\beta} \right] \right\}. \]

Recall the elementary inequality: for \( 0 < \gamma \leq 1 \) and \( x \geq 0 \),
\[ \log(1 + x) = \frac{1}{\gamma} \log(1 + x)^{\gamma} \leq \frac{\gamma}{2}. \] (17)

Taking this into account, we continue with the upper bound for \( I(\lambda) \):
\[ I(\lambda) \leq 2^{\frac{1}{\gamma} - 1} \exp \left\{ \frac{\lambda^2}{2(1 - \theta)^2} \sum_{k=0}^{\infty} \frac{r_k m_k^2}{a_k^2} \right\} \times \exp \left\{ \frac{1}{\gamma} \sum_{k=0}^{\infty} \frac{1}{r_k} \left( \frac{b_{k+1} - b_k}{\theta^{1/\beta}} \left( \frac{c_k}{m_k} \right)^{1/\beta} \right)^{\gamma} \right\}. \]
Let \( r_k = \frac{A \mu_k}{m_k} \). Then we get immediately the claimed upper bound:

\[
I(\lambda) \leq 2^{\frac{\lambda}{\beta}} \exp \left\{ \frac{\lambda^2 A^2}{2(1 - \theta)^2} \right\} \exp \left\{ \frac{1}{\gamma A} \sum_{k=0}^{\infty} m_k \frac{b_{k+1} - b_k}{\theta^{1+\beta}} \left( \frac{c_{k+1}}{c_k} \right)^{1/\beta} \right\}
\]

\[
= 2^{\frac{\lambda}{\beta}} \exp \left\{ \frac{\lambda^2 A^2}{2(1 - \theta)^2} \right\} \exp \left\{ \frac{1}{\gamma A} \left( \sum_{k=0}^{\infty} m_k \frac{1}{\theta^{1+\beta}} \left( c_k \right)^{1/\beta} \right)^{2/\beta-1} \right\}.
\]

\( \square \)

**Corollary 2.5.** Let the assumptions of Theorem 2.4 hold. Then for any \( \theta \in (0, 1) \) and any \( u > 0 \) the following inequality holds:

\[
P \left( \sup_{t>0} \frac{|X(t)|}{a(t)} > u \right) \leq 2^{\frac{\lambda}{\beta}} \exp \left\{ \frac{-u^2(1 - \theta)^2}{2A^2} \right\} A_4(\theta, \gamma).
\]

(18)

Indeed, from Chebyshev’s inequality we get that

\[
P \left( \sup_{t>0} \frac{|X(t)|}{a(t)} > u \right) \leq \frac{\mathbb{E} \exp \left\{ \frac{\lambda \sup_{t>0} |X(t)|}{a(t)} \right\}}{\exp \left\{ \lambda u \right\}} \leq 2^{\frac{\lambda}{\beta}} \exp \left\{ \frac{-\lambda^2 A^2}{2(1 - \theta)^2} - \lambda u \right\} A_4(\theta, \gamma).
\]

(19)

The inequality (18) follows from (19) if we put \( \lambda = \frac{u(1 - \theta)^2}{A^2} \).

**Corollary 2.6.** Let the assumptions of Theorem 2.4 hold. Then for any \( u > A \) we can get the following bound:

\[
P \left( \sup_{t>0} \frac{|X(t)|}{a(t)} > u \right) \leq 2^{\frac{\lambda}{\beta}} \sqrt{\epsilon} \exp \left\{ \frac{-u^2}{2A^2} \right\} A_4 \left( 1 - \sqrt{1 - \frac{A^2}{u^2}}, \gamma \right).
\]

(20)

Indeed, the inequality (20) follows from (18) if we put \( \theta = 1 - \sqrt{1 - \frac{A^2}{u^2}} \).

**Corollary 2.7.** Let the assumptions of Theorem 2.4 hold. Then for all \( t > 0 \) we have with probability 1 the following bound:

\[
|X(t)| \leq a(t) \xi,
\]

where \( \xi \) is a non-negative random variable whose distribution has the tail admitting the following upper bound: for any \( u > A \)

\[
P \{ \xi > u \} \leq 2^{\frac{\lambda}{\beta}} \sqrt{\epsilon} \exp \left\{ \frac{-u^2}{2A^2} \right\} A_4 \left( 1 - \sqrt{1 - \frac{A^2}{u^2}}, \gamma \right).
\]

2.3. **Exponential maximal upper bound for Gaussian process in the bounded strip on the plane.** Additionally, we need in exponential maximal upper bound for Gaussian process defined in the bounded strip on the plane. We get it, applying Theorem 2.1. So, let \( 0 \leq a < b < \infty, \Delta > 0, \)

\[
T_{a,b,\Delta} = \{ t = (t_1, t_2) \in \mathbb{R}_+^2 : a \leq t_1 \leq b, t_1 - \Delta \leq t_2 \leq t_1 \},
\]

\[
d(t, s) = \max \{|t_1 - s_1|, |t_2 - s_2|\} \text{ for } t, s \in T_{a,b,\Delta}.
\]

Let \( r(x), x \geq 1, \) be a non-negative nondecreasing function such that \( r(e^y), y \geq 0 \) is a convex function.

**Theorem 2.8.** Assume that \( X = \{X(t), t \in T_{a,b,\Delta}\} \) is a centered separable Gaussian process satisfying following conditions:


(iv) 
\[ m(T_{a,b,\Delta}) = \sup_{t \in T_{a,b,\Delta}} (E(X(t))^2)^{1/2} < \infty; \]

(v) 
\[ \sup_{d(t,s) \leq h \atop t,s \in T_{a,b,\Delta}} (E(X(t) - X(s))^2)^{1/2} \leq \sigma(h), \]
where \( \sigma = \{ \sigma(h), h > 0 \} \) is an increasing continuous function, \( \sigma(h) \geq 0, \sigma(0) = 0. \)

If 
\[ \tilde{I}_r(m(T_{a,b,\Delta})) = \int_0^{m(T_{a,b,\Delta})} r \left( \frac{(b - a)\Delta}{4} \left( \frac{1}{\sigma^{-1}}(v) + d \right)^2 \right) dv < \infty, \]
where \( d = \max \left\{ \frac{\Delta}{2}, \frac{\Delta}{2} \right\} \), then for any \( \theta \in (0,1) \) and any \( \lambda > 0 \)

\[ E \exp \left\{ \lambda \sup_{t \in T_{a,b,\Delta}} |X(t)| \right\} \leq 2A_\varepsilon(\lambda, \theta), \]
where
\[ A_\varepsilon(\lambda, \theta) = \exp \left\{ \frac{\lambda^2 m^2(T_{a,b,\Delta})}{2(1 - \theta)^2} \left( \tilde{I}_r(\partial m(T_{a,b,\Delta})) \right) \right\}. \]

Proof. The statement follows from Theorem 2.1, since in this case
\[ N(v) \leq \left( \frac{b - a}{2\sigma^{-1}(v)} + 1 \right) \left( \frac{\Delta + 2\sigma^{-1}(v)}{2\sigma^{-1}(v)} + 1 \right) \]
\[ = \left( \frac{b - a}{2\sigma^{-1}(v)} + 1 \right) \left( \frac{\Delta}{2\sigma^{-1}(v)} + 2 \right) \]
\[ = \frac{b - a}{2} \left( \frac{\Delta}{\sigma^{-1}(v)} + \frac{2}{b - a} \right) \left( \frac{1}{\sigma^{-1}(v)} + \frac{4}{\Delta} \right) \]
\[ \leq \frac{(b - a)\Delta}{4} \left( \frac{1}{\sigma^{-1}(v)} + d \right)^2. \]

Corollary 2.9. Let in Theorem 2.8 \( \sigma(h) = ch^\beta \) for some \( c > 0 \) and \( \beta \in (0,1] \), \( m = m(T_{a,b,\Delta}). \) Then for all \( \lambda > 0, 0 < \varepsilon < \beta, 0 < \theta < 1 \)

\[ E \exp \left\{ \lambda \sup_{t \in T_{a,b,\Delta}} |X(t)| \right\} \leq 2^{2-2}(b - a)\Delta \exp \left\{ \frac{\lambda^2 m^2}{2(1 - \theta)^2} \right\} \times \left( \frac{c^{2/\beta}}{\left( 1 - \frac{\varepsilon}{\beta} \right)} \left( \frac{1}{\theta m^{2/\beta}} + d^2 \right) \right), \]
where \( d = \max \left\{ \frac{d^2}{2}, \frac{\Delta}{2} \right\}. \) Indeed, we can choose
\[ r(s) = \begin{cases} 0, & 1 \leq s \leq \frac{d^2(b - a)\Delta}{4} \\ s^{\varepsilon / 2} - \left( \frac{d^2(b - a)\Delta}{4} \right)^{\varepsilon / 2}, & s > \frac{d^2(b - a)\Delta}{4}. \end{cases} \]
Then \( \nu^{-1}(t) = \left( t + \left( \frac{(d^2(b-a)\Delta)}{4} \right)^{1/2} \right)^2 \), and

\[
\tilde{L}_r(\theta m) = \int_0^{\theta m} r \left( \frac{(b-a)\Delta}{4} \left( \frac{c^{1/\beta}}{v^{1/\beta}} + d \right)^{1/2} \right) dv
\]

\[
= \int_0^{\theta m} \left( \frac{(b-a)\Delta}{4} \left( \frac{c^{1/\beta}}{v^{1/\beta}} + d \right)^{1/2} \right) - \left( \frac{(d^2(b-a)\Delta)}{4} \right)^{1/2} dv
\]

\[
= \left( \frac{(b-a)\Delta}{4} \right) \int_0^{\theta m} \left( \frac{c^{1/\beta}}{v^{1/\beta}} + d \right)^{1/2} dv
\]

\[
\leq \left( \frac{(b-a)\Delta}{4} \right) \int_0^{\theta m} \frac{c^{1/\beta}}{v^{1/\beta}} dv = \left( \frac{(b-a)\Delta}{4} \right) \left( \frac{c^{1/\beta}(\theta m)^{1-1/\beta}}{1-\frac{1}{\beta}} \right).
\]

Hence,

\[
A_{\epsilon}(\lambda, \theta) = \exp \left\{ \frac{\lambda^2 m^2}{2(1-\theta)^2} \left( \frac{\tilde{L}_r(\theta m)}{\theta m} + \frac{(d^2(b-a)\Delta)}{4} \right)^{1/2} \right\}
\]

\[
\leq \exp \left\{ \frac{\lambda^2 m^2}{2(1-\theta)^2} \frac{(b-a)\Delta}{4} \left( \frac{c^{1/\beta}}{1-\frac{1}{\beta}} (\theta m)^{1-1/\beta} + d^2 \right)^{1/2} \right\}
\]

\[
\leq \exp \left\{ \frac{\lambda^2 m^2}{2(1-\theta)^2} \frac{(b-a)\Delta}{4} 2^{\frac{1}{2}} - 1 \left( \frac{c^{1/\beta}}{(1-\frac{1}{\beta})^{1/2} (\theta m)^{1/\beta}} + d^2 \right) \right\}.
\]

Now we get the upper bound for the weighted Gaussian process defined on the bounded strip on the plane, similarly to getting Theorem 2.4 from Theorem 2.1.

Let \( T_\Delta = \{ t = (t_1, t_2) \in \mathbb{R}^2_+ : t_1 - \Delta \leq t_2 \leq t_1 \}, \Delta > 0, \)

\[
d(t, s) = \max \{ |t_1 - s_1|, |t_2 - s_2| \}, \quad (t, s) \in T_\Delta.
\]

Also, let \( b_l \) be an increasing sequence such that \( b_0 = 0, b_{l+1} - b_l \geq 1, b_1 \rightarrow \infty, l \rightarrow \infty, \) and \( a(t) > 0 \) is a continuous increasing function and denote \( a_l = a(b_l), \)

\( T_{b_l, b_{l+1}, \Delta} = \{ t = (t_1, t_2) \in \mathbb{R}^2_+ : b_l \leq t_1 \leq b_{l+1}, t_1 - \Delta \leq t_2 \leq t_1 \}. \)

**Theorem 2.10.** Let \( 0 < \Delta \leq 2(b_{l+1} - b_l), l \geq 0, X = \{ X(t) : t \in T_\Delta \} \) be a centered Gaussian process satisfying following conditions:

\( (vi) \ m_l = m(T_{b_l, b_{l+1}, \Delta}) = \sup_{t \in T_{b_l, b_{l+1}, \Delta}} \left( \mathbb{E}(X(t))^2 \right)^{1/2} < \infty; \)

\( (vii) \) There exist \( \beta \in (0, 1] \) and constants \( c_l > 0 \) such that

\[
\sup_{d(t,s) \leq h_{b_l, b_{l+1}, \Delta}} \left( \mathbb{E}(X(t) - X(s))^2 \right)^{1/2} \leq c_l h^\beta.
\]
(viii) \( A = \sum_{l=0}^{\infty} \frac{m_l}{a_l} < \infty, \sum_{l=0}^{\infty} \frac{m_l \log(b_{l+1} - b_l)}{a_l} < \infty, \) and for some \( \gamma \in (0, 1] \)
\[ \sum_{l=0}^{\infty} \frac{m_l^{1-2\gamma/\beta}}{a_l^{2\gamma/\beta}} < \infty. \]

Then for any \( \theta \in (0, 1), \varepsilon \in (0, \beta) \) and \( \lambda > 0 \)

\[
I(\lambda) = \mathbb{E} \exp \left\{ \lambda \sup_{t \in T_{\Delta}} \frac{|X(t)|}{a(t)} \right\} \leq \exp \left\{ \frac{\lambda^2 A^2}{2(1-\theta)^2} \right\} A_\varepsilon(\theta, \gamma, \varepsilon),
\]

where

\[
A_\varepsilon(\theta, \gamma, \varepsilon) = \frac{2^{\frac{\varepsilon+2}{\varepsilon}}}{{\Delta}} \exp \left\{ \lambda \sum_{l=0}^{\infty} \frac{m_l \log(b_{l+1} - b_l)}{a_l} \right\} \times \exp \left\{ \frac{\Delta^{2\gamma}}{\gamma A \left(1 - \frac{\varepsilon}{\beta} \right)^{2\gamma/\varepsilon}} \sum_{l=0}^{\infty} \frac{m_l^{2\gamma/\beta}}{a_l^{4\gamma/\beta}} \right\},
\]

Proof. The theorem follows from Corollary 2.9. Indeed, let \( r_l > 0, \sum_{l=0}^{\infty} \frac{1}{r_l} = 1 \). Then we easily get the following upper bounds

\[
I(\lambda) \leq \mathbb{E} \exp \left\{ \lambda \sum_{l=0}^{\infty} \frac{1}{a_l} \sup_{t \in T_{l+1}} |X(t)| \right\} \leq \prod_{l=0}^{\infty} \left( \mathbb{E} \exp \left\{ \lambda \frac{r_l}{a_l} \sup_{t \in T_{l+1}} |X(t)| \right\} \right)^{1/r_l}
\]

\[
\leq 2 \prod_{l=0}^{\infty} \exp \left\{ \frac{\lambda^2 m_l^2 r_l}{2a_l^2 (1-\theta)^2} \right\} (b_{l+1} - b_l) \Delta^{2\frac{\varepsilon}{\beta} - 3} \left( \frac{c_l^{2/\beta}}{\left(1 - \frac{\varepsilon}{\beta} \right)^{2/\varepsilon}} (\theta m_l)^{2/\beta} + \left( \frac{A}{4} \right)^2 \right)^{1/r_l}
\]

\[
= 2^{\frac{\varepsilon+2}{\varepsilon}} \prod_{l=0}^{\infty} \exp \left\{ \frac{\lambda^2 m_l^2 r_l}{2a_l^2 (1-\theta)^2} \right\} (b_{l+1} - b_l)^{1/r_l} \left( \frac{4}{A} \right)^{1/r_l}
\]

\[
\times \left( \frac{c_l^{2/\beta}}{\left(1 - \frac{\varepsilon}{\beta} \right)^{2/\varepsilon}} (\theta m_l)^{2/\beta} + 1 \right)^{1/r_l}
\]

\[
= \frac{2^{\frac{\varepsilon+2}{\varepsilon}}}{\Delta} \exp \left\{ \lambda^2 \sum_{l=0}^{\infty} \frac{r_l m_l^2}{a_l^2} \right\} \exp \left\{ \sum_{l=0}^{\infty} \frac{\log(b_{l+1} - b_l)}{r_l} \right\}
\]

\[
\times \left( \frac{c_l^{2/\beta}}{\left(1 - \frac{\varepsilon}{\beta} \right)^{2/\varepsilon}} (\theta m_l)^{2/\beta} + 1 \right) \right\},
\]

\[
= \left( \frac{C}{\Delta} \right)^{1/r_l} \exp \left\{ \sum_{l=0}^{\infty} \frac{\log(b_{l+1} - b_l)}{r_l} \right\}
\]

\[
\times \exp \left\{ \lambda^2 \sum_{l=0}^{\infty} \frac{r_l m_l^2}{a_l^2} \right\} \exp \left\{ \sum_{l=0}^{\infty} \frac{\log(b_{l+1} - b_l)}{r_l} \right\}
\]

\[
\leq \exp \left\{ \lambda^2 A^2 \right\} A_\varepsilon(\theta, \gamma, \varepsilon),
\]

\[
= \exp \left\{ \frac{\lambda^2 A^2}{2(1-\theta)^2} \right\} A_\varepsilon(\theta, \gamma, \varepsilon),
\]

\[
= \exp \left\{ \frac{\lambda^2 A^2}{2(1-\theta)^2} \right\} A_\varepsilon(\theta, \gamma, \varepsilon),
\]

\[
= \exp \left\{ \frac{\lambda^2 A^2}{2(1-\theta)^2} \right\} A_\varepsilon(\theta, \gamma, \varepsilon),
\]
Applying (17), we get
\[
I(\lambda) \leq \frac{2^{2+2^2}}{\Delta} \exp \left\{ \frac{\lambda^2}{2(1-\theta)^2} \sum_{t=0}^{\infty} \frac{r_t m_t^2}{a_t^2} \right\} \exp \left\{ \frac{\sum_{t=0}^{\infty} \log(a_{t+1} - b_t)}{r_t} \right\} \times \exp \left\{ \frac{c_t^{2\gamma/\beta} \Delta^{2\gamma}}{\gamma \lambda \left( 1 - \frac{\varepsilon}{\beta} \right)^{2\gamma/\varepsilon} (\theta m_t)^{2\gamma/\beta} \Delta^{2\gamma} a_t} \right\}.
\]

Now we choose \( r_t = \frac{A_i}{m_t} \). Then
\[
I(\lambda) \leq \frac{2^{2+2^2}}{\Delta} \exp \left\{ \frac{\lambda^2 A_t^2}{2(1-\theta)^2} \right\} \exp \left\{ \frac{1}{A_t} \sum_{t=0}^{\infty} \frac{m_t \log(b_{t+1} - b_t)}{a_t} \right\} \times \exp \left\{ \frac{\Delta^{2\gamma}}{\gamma A_t \left( 1 - \frac{\varepsilon}{\beta} \right)^{2\gamma/\varepsilon} \theta^{2\gamma/\beta} A_t^{2\gamma} a_t} \right\}.
\]

\( \square \)

**Corollary 2.11.** Let the assumptions of Theorem 2.10 hold. Then for all \( \theta \in (0,1) \), \( \varepsilon \in (0,\beta) \) and \( u > 0 \),
\[
P \left\{ \sup_{t \in T_\Delta} \frac{|X(t)|}{a(t)} > u \right\} \leq \exp \left\{ -\frac{u^2(1-\theta)^2}{2A^2} \right\} A_\varepsilon(\theta, \gamma, \varepsilon).
\]

**Corollary 2.12.** Let the assumptions of Theorem 2.10 hold. Then for any \( \varepsilon \in (0,\beta) \) and \( u > A \) we have that
\[
P \left\{ \sup_{t \in T_\Delta} \frac{|X(t)|}{a(t)} > u \right\} \leq \sqrt{\varepsilon} \exp \left\{ -\frac{u^2}{2A^2} \right\} A_\varepsilon \left( 1 - \sqrt{1 - \frac{A^2}{u^2}}, \gamma, \varepsilon \right).
\]

**Corollary 2.13.** Let the assumptions of Theorem 2.10 hold. Then for all \( t \in T_\Delta \)
\[
|X(t)| \leq a(t) \xi \quad a.s.,
\]
where \( \xi \) is such non-negative random variable that
\[
P \{ \xi > u \} \leq \exp \left\{ -\frac{u^2(1-\theta)^2}{2A^2} \right\} A_\varepsilon(\theta, \gamma, \varepsilon),
\]
and for \( u > a \)
\[
P \{ \xi > u \} \leq \sqrt{\varepsilon} \exp \left\{ -\frac{u^2}{2A^2} \right\} A_\varepsilon \left( 1 - \sqrt{1 - \frac{A^2}{u^2}}, \gamma, \varepsilon \right).
\]

3. **Asymptotic Growth with Probability 1 of Multifractional Brownian Motion**

In this section we apply the results of Section 2 to get the asymptotic growth of the trajectories of multifractional Brownian motion.
3.1. Definition and assumptions. Let $H: \mathbb{R}_+ \to (0, 1)$ be a continuous function. The (harmonizable) multifractional Brownian motion (mBm) with functional parameter $H$ was introduced in [4]. It is defined by

$$
Y(t) = \int_{\mathbb{R}} \frac{e^{itu} - 1}{|u|^H t^{H+1/2}} \tilde{W}(du), \quad t \geq 0,
$$

where $\tilde{W}(du)$ is the “Fourier transform” of the white noise $W(du)$, that is a unique complex-valued random measure such that for all $f \in L^2(\mathbb{R})$

$$
\int_{\mathbb{R}} f(u) W(du) = \int_{\mathbb{R}} \hat{f}(u) \tilde{W}(du) \quad a.s.,
$$

see [4, 15].

In what follows we assume that the function $H$ satisfies the following conditions:

(H1) There exist constants $0 < h_1 < h_2 < 1$ such that for any $t \geq 0$

$$
h_1 \leq H_t \leq h_2.
$$

(H2) There exist constants $D > 0$ and $\kappa \in (0, 1]$ such that for all $t \geq s > 0$

$$
|H_t - H_s| \leq D |t - s|^{\kappa}.
$$

It is known [1] that

$$
\left( \mathbb{E} |Y(t)|^2 \right)^{1/2} = C(H_t) t^{H_t},
$$

where $C(H) = \left( \frac{\kappa}{2\Gamma(2H) \sin(\pi H)} \right)^{1/2}$. Since the function $C(H)$ is bounded on $[h_1, h_2]$, we have under assumptions (H1)-(H2)

$$
\left( \mathbb{E} |Y(t)|^2 \right)^{1/2} \leq K_1 t^{h_2}, \quad t \geq 1,
$$

for some $K_1 > 0$.

3.2. Upper bounds for the incremental variances of mBm. The first result gives the upper bound for the distance variance function for multifractional Brownian motion.

Lemma 3.1. Under the assumption (H1), there exists a constant $K_2 > 0$ such that for all $t \geq s \geq 0$

$$
\mathbb{E} (Y(t) - Y(s))^2 \leq K_2 |t - s|^{2H_t} + K_2 (H_t - H_s)^2 z^2(s),
$$

where

$$
z(s) = \begin{cases} 
    s^{h_2} (\log^2 s + 1)^{1/2}, & s \geq 1, \\
    1, & 0 < s < 1;
\end{cases}
$$
Proof. By the isometry property,

\[
E(Y(t) - Y(s))^2 = E \left( \int_{\mathbb{R}} \left( \frac{e^{itu} - 1}{|u|H_t + 1/2} - \frac{e^{isu} - 1}{|u|H_s + 1/2} \right) \hat{W}(du) \right)^2
\]

= \int_{\mathbb{R}} \left| \frac{e^{itu} - 1}{|u|H_t + 1/2} - \frac{e^{isu} - 1}{|u|H_s + 1/2} \right|^2 du

= \int_{\mathbb{R}} \left| \frac{e^{itu} - 1}{|u|H_t + 1/2} - \frac{e^{isu} - 1}{|u|H_s + 1/2} + \frac{e^{isu} - 1}{|u|H_t + 1/2} - \frac{e^{isu} - 1}{|u|H_s + 1/2} \right|^2 du

\leq 2(I_1 + I_2),

where

\[
I_1 = \int_{\mathbb{R}} \left| \frac{e^{itu} - e^{isu}}{|u|^{2H_t+1}} \right|^2 du, \\
I_2 = \int_{\mathbb{R}} \left| e^{isu} - 1 \right|^2 \left( |u|^{-H_t-1/2} - |u|^{-H_s-1/2} \right)^2 du.
\]

Consider now \(I_1\) and \(I_2\) separately. For \(I_1\) we get the following bound

\[
I_1 = \int_{\mathbb{R}} \left| \frac{e^{itu} - 1}{|u|^{2H_t+1}} \right|^2 du = \int_{\mathbb{R}} \frac{4 \sin^2 \frac{u}{2} \nu^2}{|v|^{2H_t+1}} dv.
\]

Hence, \(I_1 \leq C_1 |t - s|^{2H_t}\), where

\[
C_1 = \int_{|v| < 1} \frac{4 \sin^2 \frac{\nu}{2} \nu^2}{|v|^{2H_t+1}} dv + \int_{|v| > 1} \frac{4 \sin^2 \frac{\nu}{2} \nu^2}{|v|^{2H_t+1}} dv
\]

\[
\leq \int_{|v| < 1} \frac{1}{|v|^{2H_t+1}} dv + \int_{|v| > 1} \frac{4}{|v|^{2H_t+1}} dv < \infty.
\]

Consider \(I_2\). By the mean value theorem,

\[
|u|^{-H_t-1/2} - |u|^{-H_s-1/2} = -|u|^{-h(u)-1/2} \log |u| (H_t - H_s),
\]

where \(h(u) \in [h_1, h_2]\). Therefore,

\[
I_2 = (H_t - H_s)^2 \int_{\mathbb{R}} \left| e^{isu} - 1 \right|^2 |u|^{-2h(u)-1} \log^2 |u| dv
\]

\[
= (H_t - H_s)^2 \int_{\mathbb{R}} \left| e^{iv} - 1 \right|^2 |v|^{-2h(v/s)+1} \log^2 |v| dv - \log s)^2 dv
\]

\[
\leq \begin{cases} 
\nu^{2h_1} (H_t - H_s)^2 I_3, & 0 \leq s < 1, \\
\nu^{2h_2} (H_t - H_s)^2 I_3, & s \geq 1,
\end{cases}
\]
where
\[
I_3 = \int_{\mathbb{R}} \frac{\left| e^{itv} - 1 \right|^2}{|v|^{2h(v/s)+1}} (\log |v| - \log s)^2 dv \\
\leq 2 \left( \int_{\mathbb{R}} \frac{4 \sin^2 \frac{v}{2}}{|v|^{2h(v/s)+1}} \log^2 |v| dv + \log^2 s \int_{\mathbb{R}} \frac{4 \sin^2 \frac{v}{2}}{|v|^{2h(v/s)+1}} dv \right) \\
\leq 2 (C_2 + C_1 \log^2 s),
\]
\[
C_2 = \int_{|v|<1} \frac{4 \sin^2 \frac{v}{2}}{|v|^{2h_2+1}} \log^2 |v| dv + \int_{|v|>1} \frac{4 \sin^2 \frac{v}{2}}{|v|^{2h_1+1}} \log^2 |v| dv < \infty.
\]

Note that \( f(s) = s^{2h_1} (C_2 + C_1 \log^2 s), s > 0, \) is a continuous function and \( f(s) \to 0 \) as \( s \downarrow 0. \) Therefore, \( f \) is bounded on \([0, 1], \) whence \( I_2 \leq C_3 (H_t - H_s)^2 z^2(s) \) for some \( C_3 > 0. \)

\[ \square \]

Remark 1. (a) Denote \( h_3 = \min \{h_1, \kappa \}. \) It immediately follows from (23),
from the fact that multifractional process is Gaussian, and from the Kolmogorov theorem that under conditions (H1) and (H2) process \( Y \) with probability 1 has Holder trajectories up to order \( h_3 \) on any finite interval.
(b) Bound (23) is inconvenient in the sense that it contains two different exponents of \( |t - s| \) and therefore one should every time relate corresponding terms depending upon the value of \( |t - s|. \) To avoid this technical difficulty, we establish the next result. Denote \( h_a = \max \{h_2, \kappa \}, \) \( h_5 = h_a - h_3. \)

Lemma 3.2. Assume that the Hurst function \( H \) satisfies the conditions (H1) and 
(H2). Let \( a, b \in \mathbb{R}, b - a \geq 1. \) Then
(a) for all \( t, s \in [a, b] \) such that \( |t - s| \leq 1, \)
\[
\left( \mathbb{E} (Y(t) - Y(s))^2 \right)^{1/2} \leq K_3 |t - s|^{h_3} z(b),
\]
where \( K_3 = K_2^{1/2} (1 + D^2)^{1/2}. \)
(b) for all \( t, s \in [a, b], \)
\[
\left( \mathbb{E} (Y(t) - Y(s))^2 \right)^{1/2} \leq K_3 |t - s|^{h_3} (b-a)^{h_5} z(b);
\]
(c) for all \( t_1, t_2, s_1, s_2 \in [a, b], \)
\[
\left( \mathbb{E} (Y(t_1) - Y(t_2) - Y(s_1) + Y(s_2))^2 \right)^{1/2} \leq 2K_3 \max \{|t_1 - s_1|, |t_2 - s_2|\}^{h_3} (b-a)^{h_5} z(b).
\]

Proof. It follows from the assumptions (H1), (H2) and Lemma 3.1 that
\[
\mathbb{E} (Y(t) - Y(s))^2 \leq K_2 (t - s)^{2H_t} + K_2 D^2 |t - s|^{2h_3} z^2(s).
\] (24)

(a) In the case \(|t - s| \leq 1, \) we have that
\[
\mathbb{E} (Y(t) - Y(s))^2 \leq K_2 |t - s|^{2h_3} + K_2 D^2 |t - s|^{2h_3} z^2(s) \\
= K_2 |t - s|^{2h_3} (1 + D^2 z^2(s)) \leq K_3 |t - s|^{2h_3} z^2(b).
\]
(b) For arbitrary values of arguments, inequality (24) implies that
\[
\mathbb{E}(Y(t) - Y(s))^2 \\
\leq K_2(b - a)^{2H_t} \left| \frac{t - s}{b - a} \right|^{2H_t} + K_2 D^2 (b - a)^{2\gamma} \left| \frac{t - s}{b - a} \right|^{2\gamma} z^2(s) \\
\leq K_2(b - a)^{2\gamma} \left| \frac{t - s}{b - a} \right|^{2\gamma} (1 + D^2 z^2(s)) \\
\leq K_3(b - a)^{2\gamma + 2\kappa} |t - s|^{2\gamma} z^2(s).
\]
(c) Proof follows immediately from the part (b), since by Minkowski's inequality,
\[
\left( \mathbb{E}(Y(t_1) - Y(t_2) - Y(s_1) + Y(s_2))^2 \right)^{1/2} \\
\leq (\mathbb{E}(Y(t_1) - Y(s_1))^2)^{1/2} + (\mathbb{E}(Y(t_2) - Y(s_2))^2)^{1/2}. \quad \square
\]

3.3. Asymptotic growth of the trajectories of mBM with probability 1

Now we apply the results of Section 2 to multifractional Brownian motion. The first result gives the maximal exponential bound for the weighted mBM. In order to get it, introduce the following notations. Let \( b_k, k \geq 0, \) be a sequence such that \( b_0 = 0, b_{k+1} - b_k \geq 1, \) and let \( a(t) > 0 \) be an increasing continuous function such that \( a(t) \to \infty \) as \( t \to \infty, \) \( a_k = a(b_k). \)

**Theorem 3.3.** Let the Hurst function \( H \) satisfy the conditions (H1) and (H2). Assume that there exists \( 0 < \gamma \leq 1 \) such that
\[
\sum_{k=0}^{\infty} \frac{b_{k+1}^{2\gamma} \left( \log^2 b_{k+1} + 1 \right)^{\frac{\gamma}{\kappa}}}{a_k} < \infty. \tag{25}
\]
Then for all \( 0 < \theta < 1, \) \( u > 0 \)
\[
P \left\{ \sup_{t > 0} \frac{|Y(t)|}{a(t)} > u \right\} \leq 2^\frac{\kappa}{\gamma} - 1 \exp \left\{ - \frac{u^2 (1 - \theta)^2}{2 A^2} \right\} A_7(\theta, \gamma), \tag{26}
\]
where \( A = K_1 \sum_{k=0}^{\infty} \frac{b_{k+1}^{2\gamma}}{a_k}, \)
\[
A_7(\theta, \gamma) = \exp \left\{ K_1^{1-\frac{\kappa}{\gamma}} K_3^{\frac{\kappa}{\gamma}} \sum_{k=0}^{\infty} \frac{b_{k+1}^{2\gamma} \left( \log^2 b_{k+1} + 1 \right)^{\frac{\kappa}{\gamma}}}{a_k} \left( \frac{2^\frac{\kappa}{\gamma} - 1}{\theta^\frac{\kappa}{\gamma}} \right)^\gamma \right\},
\]
or for all \( u > A \)
\[
P \left\{ \sup_{t > 0} \frac{|Y(t)|}{a(t)} > u \right\} \leq 2^\frac{\kappa}{\gamma} - 1 \sqrt{c} \exp \left\{ - \frac{u^2}{2 A^2} \right\} A_7 \left( 1 - \sqrt{1 - \frac{A^2}{u^2}}, \gamma \right). \tag{27}
\]

**Proof.** First, we check the conditions (i)–(iii) of Theorem 2.4. Lemma 3.2(b) implies that the assumption (i) holds with \( c_k = K_3(b_{k+1} - b_k)^{\beta} (b_{k+1}) \) and \( \beta = h_0. \)

According to (22), we can choose \( m_k = K_1 b_{k+1}^{\beta}. \) Then
\[
\sum_{k=0}^{\infty} \frac{m_k}{a_k} = K_1 \sum_{k=0}^{\infty} \frac{b_{k+1}^{2\gamma}}{a_k} < \infty,
\]
by (25), and the condition (ii) is satisfied. Finally, the condition (iii) follows from (25), because in our case
\[
m_k^{1 - \gamma / 2} (b_{k+1} - b_k)^{\gamma / 2} c_k^{1 / 2} = K_1^{1 - \frac{\gamma}{2}} K_{b_{k+1}}^{(b_{k+1} - b_k)^{\gamma / 2}} (\log^2 b_{k+1} + 1)^{\frac{\gamma}{2}}.
\]
\[
\leq K_1^{1 - \frac{\gamma}{2}} K_{b_{k+1}}^{h_{k+1} + \frac{\gamma}{2}} (\log^2 b_{k+1} + 1)^{\frac{\gamma}{2}}.
\]
where we used the equality \( h_5 = h_4 = h_3 \). Thus, the assumptions of Theorem 2.4 are satisfied. Now the statements (26) and (27) follow from Corollaries 2.5 and 2.6 respectively.

Now we present the first main result of this section, namely, the power upper bound for the asymptotic growth of the trajectories of mBm with probability 1.

**Theorem 3.4.** For any \( \delta > 0 \) there exists a nonnegative random variable \( \xi = \xi(\delta) \) such that for all \( t > 0 \)
\[
|Y(t)| \leq (t^{b_2 + \delta} \lor 1) \xi \quad \text{a.s.},
\]
and there exist positive constants \( C_1 = C_1(\delta) \) and \( C_2 = C_2(\delta) \) such that for all \( u > 0 \)
\[
P(\xi > u) \leq C_1 e^{-C_2 u^2}.
\]

**Proof.** Put in Theorem 3.3 \( a(t) = t^{b_2 + \delta} \lor 1, b_0 = 0, b_k = e^{b_k}, k \geq 1, \) and arbitrary \( \gamma \in \left(0, \frac{b_0}{b_2} \lor 1\right) \), \( \theta \in (0, 1) \). Then \( a_0 = 1, a_k = e^{b_k + \delta}, k \geq 1, \)
\[
\sum_{k=0}^{b_{k+1} + \frac{\gamma}{2}} (\log^2 b_{k+1} + 1)^{\frac{\gamma}{2}} = e^{b_2 + \frac{\gamma}{2}} \left(2^{\frac{\gamma}{2}} + \sum_{k=1}^{\infty} e^{k b_{k+1} + \frac{\gamma}{2}} (k+1)^{\frac{\gamma}{2}} \right) < \infty.
\]
Now the result follows from Theorem 3.3, if we additionally put
\[
\xi = \sup_{t > 0} \frac{|Y(t)|}{t^{b_2 + \delta} \lor 1}, \quad C_1 = 2^{\frac{\gamma}{2} - 1} A_7(\theta, \gamma), \quad C_2 = \frac{(1 - \theta)^2}{2 A^2}.
\]

### 3.4. Asymptotic growth with probability 1 of the increments of mBm.

Let \( \Delta \in (0, 1] \). Consider the increment of mBm \( Z(t) = Y(t_1) - Y(t_2), t \in T_\Delta \). Let \( b_k, k \geq 0, \) be a sequence such that \( b_0 = 0, b_{k+1} - b_k \geq 1, \) and let \( a(t) > 0 \) be an increasing continuous function such that \( a(t) \to \infty \) as \( t \to \infty, a_k = a(b_k). \)

**Theorem 3.5.** Let the Hurst function \( H \) satisfy the conditions (H1) and (H2). Assume that there exists \( 0 < \gamma \leq 1 \) such that
\[
\sum_{k=0}^{\infty} \frac{b_k^{\frac{\gamma}{2} \Delta}}{a_k} < \infty.
\]
Then for all \( \theta \in (0, 1), \varepsilon \in (0, \varepsilon_3) \) and \( \lambda > 0 \)
\[
\mathbb{E} \left\{ \lambda \sup_{t \in T_\Delta} \frac{|Z(t)|}{a(t_1)} \right\} \leq \frac{1}{\Delta} \exp \left\{ \frac{\lambda^2 A^2 \Delta^{2 b_0}}{2(1 - \theta)^2} \right\} A_8(\theta, \gamma, \varepsilon),
\]

\[\]
where \( A = K_3 \sum_{t=0}^{\infty} \frac{z(b_{t+1})}{a_t} \),

\[ A_8(\theta, \gamma, \epsilon) = 2^{\frac{3}{2}+2} \exp \left\{ \frac{K_3}{4} \sum_{t=0}^{\infty} \frac{z(b_{t+1}) \log(b_{t+1}-b_t)}{a_t} \right\} \times \exp \left\{ \frac{K_3}{\gamma A^2 \gamma \left(1 - \frac{\epsilon}{\gamma^2}\right)^{\frac{\gamma}{2} \epsilon}} \sum_{t=0}^{\infty} \frac{z(b_{t+1})(b_{t+1}-b_t)^{\frac{\gamma^3}{6}}}{a_t} \right\} . \]

**Proof.** We need to verify the assumptions of Theorem 2.10 for the process \( Z \). By Lemma 3.2(a), for all \( t \in T_{b_1, b_{t+1}} \),

\[ (E(Z(t))^2)^{\frac{1}{2}} = (E(Y(t_1) - Y(t_2))^2)^{\frac{1}{2}} \leq K_3 z(b_{t+1}) \Delta^{h_3}. \]

Hence, the condition (vi) is satisfied with \( m_1 = K_3 z(b_{t+1}) \Delta^{h_3} \). Further, Lemma 3.2(c) implies

\[ \sup_{t \in T_{b_1, b_{t+1}}, s \leq h_3} (E(Z(t) - Z(s))^2)^{\frac{1}{2}} \leq 2K_3 (b_{t+1} - b_t)^{h_3} z(b_{t+1}) \Delta^{h_3}. \]

Thus, the condition (vii) holds true with \( c_1 = 2K_3 (b_{t+1} - b_t)^{h_3} z(b_{t+1}), \beta = h_3 \). It is not hard to see that in this case the condition (viii) is equivalent to the condition

\[ \sum_{t=0}^{\infty} \frac{z(b_{t+1})}{a_t} < \infty, \quad \sum_{t=0}^{\infty} \frac{z(b_{t+1}) \log(b_{t+1}-b_t)}{a_t} < \infty, \]

\[ \sum_{t=0}^{\infty} \frac{z(b_{t+1})(b_{t+1}-b_t)^{\frac{\gamma^3}{6}}}{a_t} < \infty. \]

Obviously, these three series converge when (29) holds. Now the result follows from Theorem 2.10.

Let \( d_k, k \geq 0 \), be a strictly decreasing sequence such that \( d_0 = 1, d_k \downarrow 0 \) as \( k \to \infty \). Let \( g: (0, 1] \to (0, \infty) \) be a continuous function and \( g_k, k \geq 0 \), be such a sequence that \( 0 < g_k \leq \min_{d_{k+1} \leq \xi \leq d_k} g(\xi) \).

**Theorem 3.6.** Assume that the assumptions of Theorem 3.5 hold and

\[ \sum_{k=0}^{\infty} \frac{d_k^3 \log d_k}{g_k} < \infty. \]

Then for all \( \theta \in (0, 1), \epsilon \in (0, h_3) \) and \( \lambda > 0 \)

\[ I(\lambda) = E \exp \left\{ \lambda \sup_{0 \leq t_1 < t_2 \leq t_3} \frac{|Z(t)|}{a(t_1)g(t_1 - t_2)} \right\} \leq \exp \left\{ \frac{\lambda^2 A^4 B^2}{2(1 - \theta)^2} \right\} A_9(\theta, \gamma, \epsilon), \]

where

\[ B = \sum_{k=0}^{\infty} \frac{d_k^3}{g_k}, \quad A_9(\theta, \gamma, \epsilon) = \exp \left\{ \frac{1}{B} \sum_{k=0}^{\infty} \frac{d_k^3 \log d_k}{g_k} \right\} A_8(\theta, \gamma, \epsilon). \]
Proof. Denote $T^{(k)} = \{(t_1, t_2) \in \mathbb{R}^2 : d_{k+1} < t_1 - t_2 \leq d_k\}$. Then $T^{(k)} \subset T_{d_k}$, and $\bigcup_{k=0}^{\infty} T^{(k)} = T_{d_0}$. Therefore

$$\sup_{0 \leq t_2 < t_1 \leq t_2 + 1} \frac{|Z(t)|}{a(t_1)g(t_1 - t_2)} \leq \sum_{k=0}^{\infty} \sup_{t \in T^{(k)}} \frac{|Z(t)|}{a(t_1)g(t_1 - t_2)} \leq \sum_{k=0}^{\infty} \frac{1}{g_k} \sup_{t \in T_{d_k}} \frac{|Z(t)|}{a(t_1)} \leq \sum_{k=0}^{\infty} \frac{1}{g_k} \sup_{t \in T_{d_k}} \frac{|Z(t)|}{a(t_1)}.$$ 

Let $r_k > 0$, $\sum_{k=0}^{\infty} \frac{1}{r_k} = 1$. Then

$$I(\lambda) \leq E \exp \left( \sum_{k=0}^{\infty} \frac{\lambda}{g_k} \sup_{t \in T_{d_k}} \frac{|Z(t)|}{a(t_1)} \right) \leq \prod_{k=0}^{\infty} \left( E \exp \left( \frac{\lambda r_k}{g_k} \sup_{t \in T_{d_k}} \frac{|Z(t)|}{a(t_1)} \right) \right)^{\frac{1}{r_k}}.$$ 

By Theorem 3.5, we get

$$I(\lambda) \leq \prod_{k=0}^{\infty} \left( \frac{1}{d_k} \exp \left( \frac{\lambda^2 A^2 d_k^2}{2(1 - \theta)^2 g_k^2} \right) A_\delta(\theta, \gamma, \epsilon) \right)^{\frac{1}{r_k}} = A_\lambda(\theta, \gamma, \epsilon) \exp \left( \frac{\lambda^2 A^2 \sum_{k=0}^{\infty} \frac{r_k d_k^2 h_3}{g_k}}{2(1 - \theta)^2} \right) \exp \left( -\sum_{k=0}^{\infty} \frac{\log d_k}{r_k} \right).$$

Put $r_k = \frac{\mu d_k}{d_k^2}$. Then

$$I(\lambda) \leq A_\mu(\theta, \gamma, \epsilon) \exp \left( \frac{\lambda^2 A^2 B^2}{2(1 - \theta)^2} \right) \exp \left( -\frac{1}{B} \sum_{k=0}^{\infty} \frac{d_k h_3 \log d_k}{g_k} \right).$$

\[ \square \]

**Corollary 3.7.** Let the assumptions of Theorem 3.6 hold. Then for all $\theta \in (0, 1)$, $\epsilon \in (0, h_3)$ and $u > 0$,

$$P \left\{ \sup_{0 \leq t_2 < t_1 \leq t_2 + 1} \frac{|Z(t)|}{a(t_1)g(t_1 - t_2)} > u \right\} \leq \exp \left( -\frac{u^2(1 - \theta)^2}{2A^2 B^2} \right) A_3(\theta, \gamma, \epsilon). \quad (30)$$

Indeed, by Chebyshev’s inequality,

$$P \left\{ \sup_{0 \leq t_2 < t_1 \leq t_2 + 1} \frac{|Z(t)|}{a(t_1)g(t_1 - t_2)} > u \right\} \leq \exp \left( -\frac{u^2}{\lambda^2 A^2} \right) A_3(\theta, \gamma, \epsilon).$$

If we put $\lambda = \frac{u(1 - \theta)^2}{A^2 B^2}$, we get (30).

With the help of Corollary 3.7, we can now state the second main result of this section, which is the following upper bound for the asymptotic growth of the increments of mBm with probability 1.

**Theorem 3.8.** For any $\epsilon > 0$ and any $p > 2$ there exists a nonnegative random variable $\eta = \eta(\epsilon, p)$ such that for all $0 \leq t_2 < t_1 \leq t_2 + 1$

$$|Z(t)| \leq \left( t_1^{1+\epsilon} + 1 \right) \langle t_1 - t_2 \rangle^{h_3} \langle (\log(t_1 - t_2))^{p} + 1 \rangle \eta \quad a.s., \quad (31)$$
and there exist positive constants $C_1 = C_1(\varepsilon, p)$ and $C_2 = C_2(\varepsilon, p)$ such that for all $u > 0$

$$P(\eta > u) \leq C_1 e^{-C_2 u^2}.$$ 

Proof. Put in Theorem 3.5 $a(t) = t^{b_2+\varepsilon} \vee 1$, $b_0 = 0$, $b_l = \epsilon^l$, $l \geq 1$. Then $a_0 = 1$, $a_l = e^{t(b_2+\varepsilon)}$, $l \geq 1$, and

$$\sum_{l=0}^{\infty} \frac{b_{l+1}}{a_l} z(b_{l+1}) = e^{b_{l+1} \vee 1} \left( \sqrt{2} + \sum_{l=1}^{\infty} e^{l(\frac{b_{l+1}-\varepsilon}{\epsilon})} ((l+1)^2 + 1)^{1/2} \right).$$

Therefore, (29) holds, if we choose $\gamma = \left(0, \frac{b_2}{2b_5} \vee 1\right)$.

Further, put in Theorem 3.6 $g(t) = t^{l_3} \log t^{p}$, $d_k = e^{-k}$, $k \geq 0$, $g_0 = e^{-h_3}$, $g_k = d_{k+1} \log d_k = e^{-(k+1)h_3 k^p}$, $k \geq 1$. We have

$$\sum_{k=0}^{\infty} \frac{d_k}{g_k} = e^{h_3} \left( 1 + \sum_{k=1}^{\infty} \frac{1}{k^p} \right) < \infty,$n

$$\sum_{k=0}^{\infty} \frac{d_k}{g_k} \log d_k = e^{h_3} \sum_{k=1}^{\infty} \frac{1}{k^{p+1}} < \infty.$$

Thus, the conditions of Theorem 3.6 are satisfied. The result follows from Corollary 3.7.

3.5. Pathwise integration with respect to multifractional Brownian motion. To describe the statistical model, we need to introduce the pathwise integrals w.r.t. mBm. Consider two non-random functions $f$ and $g$ defined on some interval $[a, b] \subset \mathbb{R}^+$. Let $\alpha > 0$. Denote the Riemann–Liouville left- and right-sided fractional integrals on $(a, b)$ of order $\alpha$ by

$$(I^\alpha_a f)(t) := \frac{1}{\Gamma(\alpha)} \int_t^a f(s)(s-t)^{\alpha-1} ds,$$

and

$$(I^\alpha_b g)(s) := \frac{1}{\Gamma(\alpha)} \int_s^b g(t)(t-s)^{\alpha-1} dt,$$

respectively. Suppose also that the the following limits exist:

$$f(u+) := \lim_{\delta \downarrow 0} f(u + \delta) \quad \text{and} \quad g(u-) := \lim_{\delta \downarrow 0} g(u - \delta), \quad a \leq u \leq b.$$ 

Let $f_{a+}(s) := (f(s) - f(a+))1_{(a,b)}(s)$, $g_{a-}(s) := (g(b-) - g(s))1_{(a,b)}(s)$. Suppose that $f_{a+} \in I^\alpha_{a+}(L_p(a,b))$, $g_{a-} \in I^{-\alpha}_{a-}(L_p(a,b))$ for some $p > 1$, $q \geq 1$, $1/p + 1/q \leq 1$, $0 \leq \alpha \leq 1$. Introduce the fractional derivatives

$$(D^\alpha_{a+} f_{a+})(s) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f_{a+}(s)}{s-a} \right)^{1+\alpha} + \alpha \int_a^s \frac{f_{a+}(s) - f_{a+}(u)}{(s-u)^{1+\alpha}} du 1_{(a,b)}(s)$$

$$(D^\alpha_{b-} g_{a-})(s) = \frac{1}{\Gamma(\alpha)} \left( \frac{g_{a-}(s)}{b-s} \right)^{1-\alpha} + (1 - \alpha) \int_s^b \frac{g_{a-}(s) - g_{a-}(u)}{(s-u)^{2-\alpha}} du 1_{(a,b)}(s).$$
It is known that $\mathcal{D}^\alpha_a f_{a+} \in L_p[a,b]$, $\mathcal{D}^{1-\alpha}_b g_{b-} \in L_q[a,b]$. Under above assumptions, the generalized (fractional) Lebesgue-Stieltjes integral $\int_a^b f(x)dg(x)$ is defined as

$$\int_a^b f(x)dg(x) := e^{i\pi\alpha} \int_a^b (\mathcal{D}^\alpha_a f_{a+})(x)(\mathcal{D}^{1-\alpha}_b g_{b-})(x)dx + f(a+) (g(b-) - g(a+)),$$

and for $\alpha p < 1$ it can be simplified to

$$\int_a^b f(x)dg(x) := e^{i\pi\alpha} \int_a^b (\mathcal{D}^\alpha_a f)(x)(\mathcal{D}^{1-\alpha}_b g_{b-})(x)dx,$$

see [17, 18].

Assume that the Hurst function satisfies the conditions (H1)–(H2) and, additionally, $h_3 = \min \{h_1, \kappa\} > 1/2$. In this case, according to Remark 1, process $Y$ with probability 1 has Hölder trajectories up to order $h_3$ on any finite interval $[0,T]$. As follows from [14], for any $1 - h_3 < \alpha < 1$ there exists the fractional derivative $\mathcal{D}^{1-\alpha}_b Y_{b-} \in L_p[a,b]$ for any $0 \leq a < b \leq T$. Let us have another process, say $Z = \{Z_t, t \in [0,T]\}$, also having Hölder trajectories up to some order $h$ with $h + h_3 > 1$. In particular, it can be $h = h_3$. Then, according to [17], there exists an integral $\int_a^b Z \, dY$, which is the limit a.s. of the Riemann sums and has the standard properties (so called path-wise integral). This integral is defined as

$$\int_a^b Z \, dY := e^{i\pi\alpha} \int_a^b (\mathcal{D}^\alpha_a Z)(x)(\mathcal{D}^{1-\alpha}_b Y_{b-})(x)dx. \quad (32)$$

An evident estimate follows immediately from (32):

$$\left| \int_a^b Z \, dY \right| \leq \sup_{a \leq x \leq b} \left| (\mathcal{D}^{1-\alpha}_b Y_{b-})(x) \right| \int_a^b \left| (\mathcal{D}^\alpha_a Z)(x) \right| dx. \quad (33)$$

4. DRIFT PARAMETER ESTIMATION IN STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY MBM

4.1. Linear model. Consider the process

$$X_t = \theta t + Y_t, \quad t \geq 0, \quad (34)$$

where $\theta \in \mathbb{R}$ is an unknown parameter, $Y_t$ is an MBM with the Hurst function $H_t$ satisfying the conditions (H1)–(H2). Assume that our aim is to estimate the parameter $\theta$ by the observations of $X_t$. Let us introduce the estimator

$$\hat{\theta}_T = \frac{X_T}{T} = \theta + \frac{Y_T}{T}.$$  

**Theorem 4.1.** 1) The estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

2) For all $T > 0$,

$$\frac{T^{1-H_T}}{C(H_T)} \left( \hat{\theta}_T - \theta \right) \overset{\text{a.s.}}{\sim} N(0,1),$$

where $C(H) = \left( \frac{\pi}{4H(2H)^{\sin(\pi H)}} \right)^{1/2}$. Consequently, a confidence interval of level $1 - \alpha$ is given by

$$\hat{\theta}_T \pm \frac{C(H_T)}{T^{1-H_T}} z_{1-\alpha/2}$$

where $z_p$ denotes the p-quantile of the standard normal distribution.
Proof. 1) By Theorem 3.4, for all $T > 1$ and $\delta > 0$

$$|Y_T| \leq T^{-h_2+\delta} \xi \quad a.s.,$$

where $\xi = \xi(\delta)$ is some nonnegative random variable. Hence, if we choose $\delta < 1 - h_2$, then we get

$$\frac{|Y_T|}{T} \leq \frac{\xi}{T^{1-h_2-\delta}} \to 0, \quad a.s. \text{ as } T \to \infty.$$

2) Note that one-dimensional distributions of mBm $Y_t$ are centered Gaussian with standard deviation $C(H_\tau)\tau^{H_\tau}$, see (21). Therefore,

$$\frac{T^{1-H_\tau}}{C(H_\tau)} \left( \hat{\theta}_T - \tilde{\theta} \right) = \frac{Y_T}{C(H_\tau)T^{H_\tau}} \approx N(0,1).$$

\[\square\]

4.2. Multifractional Ornstein–Uhlenbeck process. Let, as in subsection 3.5, $h_3 > 1/2$. In this subsection we consider the estimation of the unknown parameter $\theta$ by observations of the process $X = \{X_t, t \geq 0\}$ that is a solution of the stochastic differential equation of Langevin type,

$$X_t = x_0 + \theta \int_0^t X_s \, ds + Y_t,$$  \hspace{1cm} (35)

where $x_0 \in \mathbb{R}$ is a known constant, $Y = \{Y_t, t \geq 0\}$ is an mBm. This solution exists and is unique, see [13, Th. 4.1].

Note that the trajectories of the processes $Y$ and consequently $X$ are a.s. Hölder continuous up to order $h_3$. Therefore, according to subsection 3.5, path-wise integrals $\int_0^T X_s \, dX_s$ and $\int_0^T X_s \, dY_s$ are well defined. One can verify that the solution of (35) can be represented in the following form

$$X_t = e^{\theta t} \left( x_0 + \int_0^t e^{-\theta s} \, dY_s \right).$$

Using the integration-by-parts, this process can be written as follows

$$X_t = x_0 e^{\theta t} + \theta e^{\theta t} \int_0^t e^{-\theta s} Y_s \, ds + Y_t.$$  \hspace{1cm} (36)

We call the process $X = \{X_t, t \geq 0\}$ multifractional Ornstein–Uhlenbeck process.

Let, more precisely, our goal be to estimate the unknown drift parameter $\theta \in \mathbb{R}$ by the continuous-time observations on the interval $[0,T]$. Consider the estimator

$$\tilde{\theta}_T = \frac{\int_0^T X_s \, dX_s}{\int_0^T X_s^2 \, ds}.$$  \hspace{1cm} (37)

Remark 2. In the case of the equation driven by ordinary fBm the estimator (37) was studied in [3, 7, 9]. Hu and Nualart [7] proved that in the ergodic case ($\theta < 0$) it is strongly consistent for all $H \geq \frac{1}{2}$ and asymptotically normal for $H \in [\frac{1}{2}, \frac{3}{4})$.

They considered $\int_0^T X_s \, dX_s$ in (37) as a divergence-type integral. In [3, 9] the corresponding non-ergodic case $\theta > 0$ was investigated and the strong consistency of the estimator (37) was proved for $H \geq \frac{1}{2}$. It was also obtained in [3] that $e^{\theta t} \left( \tilde{\theta}_T - \theta \right)$ converges in law to $2\theta C(1)$ as $t \to \infty$, where $C(1)$ is the standard Cauchy distribution. In [6] the more general situation was studied, namely the non-ergodic Ornstein-Uhlenbeck process driven by a Gaussian process.
Since by (35), \(dX_s = \theta X_s \, ds + dY_s\), we have that \(\hat{\theta}_T\) admits the following stochastic representation

\[
\hat{\theta}_T = \theta + \int_0^T X_s \, dY_s.
\]

Denote by \(\mathcal{F}\) a class of random variables \(\zeta \geq 0\) with the following property: there exist positive constants \(C_1\) and \(C_2\) not depending on \(T\) such that for all \(u > 0\)

\[
P(\zeta > u) \leq C_1 e^{-C_2 u^2}.
\]

**Lemma 4.2.** Let \(\epsilon > 0\), \(T > 1\), \(\theta > 0\). Then there exists such \(\zeta \in \mathcal{F}\) that

\[
\left| \int_0^T X_s \, dY_s \right| \leq \zeta^2 T^{\theta_3 + \epsilon + 1} e^{\theta T}.
\]  

(38)

**Proof.** By (36),

\[
\sup_{0 \leq s \leq t} |X_s| \leq |x_0| e^{\theta s} + \theta e^{\theta s} \int_0^s e^{-\theta u} \sup_{0 \leq u \leq s} |Y_u| \, ds + \sup_{0 \leq s \leq t} |Y_s|.
\]  

(39)

Then (35) implies that for \(t_1 > t_2 \geq 0\)

\[
|X_{t_1} - X_{t_2}| \leq \theta \int_{t_2}^{t_1} \left( |x_0| e^{\theta s} + \theta e^{\theta s} \int_0^s e^{-\theta u} \sup_{0 \leq u \leq v} |Y_u| \, dv + \sup_{0 \leq u \leq s} |Y_u| \right) \, ds
\]  

(40)

Furthermore, using Theorems 3.4 and 3.8 we get for \(t \geq 0\) and \(\delta > 0\)

\[
\sup_{0 \leq s \leq t} |Y_s| \leq (t^{\theta_3 + \epsilon} + 1) \zeta \quad \text{a.s.,}
\]  

(41)

and for \(0 \leq t_2 < t_1 \leq t_2 + 1\)

\[
|Y_{t_1} - Y_{t_2}| \leq \left( t^{\theta_3 + \epsilon} + 1 \right) \left( t_1 - t_2 \right)^{\theta_3} (|\log(t_1 - t_2)|^p + 1) \eta
\]

\[
= \left( t^{\theta_3 + \epsilon} + 1 \right) \left( (t_1 - t_2)^{\theta_3} |\log(t_1 - t_2)|^p + (t_1 - t_2)^{\theta_3} \right) \eta
\]  

(42)

where \(0 < \epsilon < \theta_3 - 1/2\). Then by (41),

\[
\int_0^t e^{-\theta s} \sup_{0 \leq u \leq s} |Y_u| \, ds \leq \xi \int_0^t e^{-\theta s} \left( s^{\theta_3 + \delta} + 1 \right) \, ds \leq C \xi
\]

Therefore, from (39) we obtain

\[
\sup_{0 \leq s \leq t} |X_s| \leq |x_0| e^{\theta s} + \theta e^{\theta s} C \xi + \left( t^{\theta_3 + \delta} + 1 \right) \xi.
\]

It follows from (40) and (42) that

\[
|X_{t_1} - X_{t_2}| \leq \theta \int_{t_2}^{t_1} \left( |x_0| e^{\theta s} + \theta e^{\theta s} C \xi + \left( s^{\theta_3 + \delta} + 1 \right) \xi \right) \, ds
\]

\[
+ C \left( t^{\theta_3 + \epsilon} + 1 \right) \left( t_1 - t_2 \right)^{\theta_3 - \eta} \eta.
\]
These formulas can be rewritten using simplified notation as follows:

\[
\sup_{0 \leq s \leq t} |X_s| \leq (e^{\theta t} + e^{h_3 + \delta}) \zeta, \quad (43)
\]

\[
|X_{t_1} - X_{t_2}| \leq \zeta (e^{\theta t_1} + e^{h_3 + \delta}) (t_1 - t_2) + \zeta \left( e^{h_2 + \varepsilon} + 1 \right) (t_1 - t_2)^{h_3 - r}. \quad (44)
\]

In order to estimate \( \int_0^T X_s \, dY_s \) we write

\[
\left| \int_0^T X_s \, dY_s \right| \leq \sum_{k=0}^{[T]+1} \left| \int_k^{k+1} X_s \, dY_s \right|
\]

\[
\leq \sum_{k=0}^{[T]+1} \sup_{k \leq s \leq k+1} \left( (D_{k+1}^{1-\alpha} - Y_{k+1_-} ) (s) \right) \int_k^{k+1} (D_{k+1}^{\alpha} X_{k+1}) (s) \, ds, \quad (45)
\]

where \( 1 - h_3 + r < \alpha < h_3 - r \), see (33). Now we need to estimate the fractional derivatives. By (42),

\[
\left| (D_{k+1}^{1-\alpha} - Y_{k+1_-} ) (s) \right| \leq \frac{1}{\Gamma(\alpha)} \left( \frac{|Y_{k+1} - Y_s|}{(k+1-s)^{1-\alpha}} + (1-\alpha) \int_s^{k+1} \frac{|Y_u - Y_s|}{(u-s)^{2-\alpha}} \, du \right)
\]

\[
\leq \zeta \left( ((k+1)^{h_2 + \varepsilon} + 1) (k+1-s)^{h_3 - r - 1 + \alpha}
\right.

\left. + \int_s^{k+1} (u^{h_2 + \varepsilon} + 1) (u-s)^{h_3 - r - 2 + \alpha} \, du \right)
\]

\[
\leq \zeta \left( ((k+1)^{h_2 + \varepsilon} + 1) (k+1-s)^{h_3 - r - 1 + \alpha} + \int_s^{k+1} (u-s)^{h_3 - r - 2 + \alpha} \, du \right)
\]

\[
\leq \zeta \left( ((k+1)^{h_2 + \varepsilon} + 1) (k+1-s)^{h_3 - r - 1 + \alpha} \right.
\]

\[
\leq \zeta \left( ((k+1)^{h_2 + \varepsilon} + 1) \right). \quad (46)
\]

Applying (43)–(44), we get

\[
\left| (D_{k+1}^\alpha X) (s) \right| \leq \frac{1}{\Gamma(1-\alpha)} \left( \frac{|X_s|}{(s-k)^{\alpha}} + \alpha \int_k^s \frac{|X_u - X_s|}{(s-u)^{\alpha+1}} \, du \right)
\]

\[
\leq \zeta \left( e^{\theta s} + s^{h_3 + \delta} \right) (s-k)^{-\alpha}
\]

\[
+ \int_k^s \left( e^{\theta u} + s^{h_3 + \delta} \right) (s-u)^{-\alpha} + (s^{h_2 + \varepsilon} + 1) (s-u)^{h_3 - r - 1 - \alpha} \, du \right)
\]

\[
\leq \zeta \left( e^{\theta (s-k)} + (s-k)^{1-\alpha} + \left( s^{h_2 + \varepsilon} + 1 \right) (s-k)^{h_3 - r - \alpha} \right).
\]

Then

\[
\int_k^{k+1} \left| (D_{k+1}^\alpha X) (s) \right| \, ds \leq \zeta \left( e^{\theta (k+1)} + (k+1)^{h_3 + \delta} + (k+1)^{h_2 + \varepsilon} + 1 \right). \quad (47)
\]
Combining (45)-(47), we get

\[ \left| \int_0^T X_s \, dY_s \right| \leq \zeta^2 \sum_{k=0}^{[T]+1} ((k + 1)^{h_2 + \varepsilon} + 1) \times \left( e^{\theta(k+1)} + (k + 1)^{h_2 + \delta} + (k + 1)^{h_2 + \varepsilon} + 1 \right). \]

Now, each summand in the right-hand side of the latter inequality can be bounded by \( CT^{h_2 + \varepsilon} e^{\theta T} \), whence (38) follows.

\[ \square \]

**Theorem 4.3.** Let \( \theta > 0 \). Then the estimator \( \hat{\theta}_T \) is strongly consistent as \( T \to \infty \).

**Proof.** By (38),

\[ |\hat{\theta}_T - \theta| = \left| \frac{\int_0^T X_s \, dY_s}{\int_0^T X_s^2 \, ds} \right| \leq \zeta^2 \frac{T^{h_2 + \varepsilon + 1} e^{\theta T}}{\int_0^T X_s^2 \, ds}. \]

Applying L'Hôpital's rule and (36), we get

\[ \lim_{T \to \infty} \frac{T^{h_2 + \varepsilon + 1} e^{\theta T}}{\int_0^T X_s^2 \, ds} = \lim_{T \to \infty} \frac{\left( (h_2 + \varepsilon + 1) T^{h_2 + \varepsilon} + T^{h_2 + \varepsilon + 1} \theta \right) e^{\theta T}}{X_T^2} \]

\[ = \lim_{T \to \infty} \frac{\left( (h_2 + \varepsilon + 1) T^{h_2 + \varepsilon} + T^{h_2 + \varepsilon + 1} \theta \right)}{e^{\theta T} \left( x_0 + \theta \int_0^T e^{-\theta s} Y_s \, ds + e^{-\theta T} Y_T \right)^2}. \]

Using the bound (28), we obtain that \( e^{-\theta T} Y_T \to 0 \) a.s. as \( T \to \infty \). Moreover, with probability 1 there exists the limit \( \lim_{T \to \infty} \int_0^T e^{-\theta s} Y_s \, ds = \int_0^\infty e^{-\theta s} Y_s \, ds \). Obviously, this limit is a Gaussian random variable. This implies that

\[ \lim_{T \to \infty} \left( x_0 + \theta \int_0^T e^{-\theta s} Y_s \, ds + e^{-\theta T} Y_T \right)^2 = \left( x_0 + \theta \int_0^\infty e^{-\theta s} Y_s \, ds \right)^2 > 0 \quad \text{a.s.} \]

Therefore, it follows from (48) that

\[ \lim_{T \to \infty} \frac{T^{h_2 + \varepsilon + 1} e^{\theta T}}{\int_0^T X_s^2 \, ds} = 0 \quad \text{a.s.} \]

This completes the proof.

\[ \square \]

**Theorem 4.4.** Let \( \theta = 0 \). Then the estimator \( \hat{\theta}_T \) is consistent as \( T \to \infty \).

**Proof.** In this case \( X_t = x_0 + Y_t \). Hence,

\[ \int_0^T X_s \, dY_s = x_0 Y_T + \int_0^T Y_s \, dY_s = x_0 Y_T + \frac{1}{2} Y_T^2, \]

see formula (32) in [17]. Then for \( T > 1 \),

\[ |\hat{\theta}_T - \theta| = \left| \frac{\int_0^T X_s \, dY_s}{\int_0^T X_s^2 \, ds} \right| \leq \frac{x_0 |Y_T| + \frac{1}{2} Y_T^2}{\int_0^T X_s^2 \, ds} \leq \zeta^2 \frac{T^{2h_2 + 2\delta}}{\int_0^T X_s^2 \, ds}. \]
by Theorem 3.4. It follows from the Cauchy–Schwarz inequality that

\[ \int_0^T X_s^2 \, ds \geq \frac{1}{T} \left( \int_0^T X_s \, ds \right)^2 = \frac{1}{T} \left( \int_0^T (x + Y_s) \, ds \right)^2 = T \left( x_0 + \frac{1}{T} \int_0^T Y_s \, ds \right)^2 \]

where \( \sigma^2_T \) denotes the variance of the centered Gaussian random variable \( \frac{1}{T} \int_0^T Y_s \, ds \), \( \mathcal{N}(0, 1) \) is the standard normal random variable. Therefore, it suffices to show that

\[ \frac{T^{2h_2 + 2\delta - 1}}{(x_0 + \sigma_T \mathcal{N}(0, 1))^2} \overset{p}{\to} 0 \quad \text{as} \quad T \to \infty. \]

In order to establish this convergence, we will bound \( \sigma^2_T \) from below. We have

\[ \sigma^2_T = \frac{1}{T^2} \mathbb{E} \left( \int_0^T Y_s \, ds \right)^2 = \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E} Y_s Y_u \, ds \, du \]

\[ = \frac{1}{T^2} \int_0^T \int_0^T D(H_s, H_u) \left( (s + H_s + u + H_u) - (s + u) \right) \, ds \, du, \]

where \( D(x, y) = \frac{\pi}{\Gamma(\delta)} \sum (x+y)^{\delta} \), see [1] or [15, p. 213]. Further,

\[ \sigma^2_T = \frac{1}{T^2} \int_0^T \int_0^T D(H_s, H_u) \left( (s + H_s + u + H_u) - (s - u) \right) \, ds \, du \]

\[ + \frac{1}{T^2} \int_0^T \int_0^T D(H_s, H_u) \left( (s + H_s + u + H_u) - (u - s) \right) \, ds \, du \]

\[ \geq \frac{1}{T^2} \left( \int_0^T \int_0^T D(H_s, H_u) u_{H_s + H_u} \, ds \, du + \int_0^T \int_0^T D(H_s, H_u) s_{H_s + H_u} \, ds \, du \right), \]

Since \( D(x, y) \) is positive and stays bounded away from 0 for \( x, y \in [h_1, h_2] \), we have

\[ \sigma^2_T \geq \frac{C}{T^2} \left( \int_0^T \int_0^T u_{H_s + H_u} \, ds \, du + \int_0^T \int_0^T s_{H_s + H_u} \, ds \, du \right) \]

\[ = \frac{2C}{T^2} \int_0^T \int_0^T u_{H_s + H_u} \, ds \, du + \frac{2C}{T^2} \int_0^T \int_0^T s_{H_s + H_u} \, ds \, du \]

\[ \geq \frac{2C}{T^2} \int_0^T \int_0^T T^{2h_2} \left( \frac{u}{T} \right)^{2h_2} \, ds \, du = \frac{2C T^{2h_2}}{(2h_2 + 1)(2h_2 + 2)} = C_1 T^{2h_1} \]

for \( T > 1 \).

Thus, for any \( \varepsilon > 0 \)

\[ P \left\{ \frac{T^{2h_2 + 2\delta - 1}}{(x_0 + \sigma_T \mathcal{N}(0, 1))^2} > \varepsilon^2 \right\} \leq P \left\{ \frac{x_0}{\sigma_T} + \mathcal{N}(0, 1) < \frac{T^{h_2 - h_1 + \delta - 1/2}}{\varepsilon C_1^{1/2}} \right\} \]

\[ \leq P \left\{ \frac{x_0}{\sigma_T} + \mathcal{N}(0, 1) < \frac{T^{h_2 - h_1 + \delta - 1/2}}{\varepsilon C_1^{1/2}} \right\} \to 0 \]

as \( T \to \infty \) for \( 0 < \delta < 1/2 - h_2 + h_1 \).
ASYMPTOTIC GROWTH OF TRAJECTORIES OF MBM

REFERENCES

[1] A. Aysche, S. Cohen, J. Lévy Véhel. The covariance structure of multifractional Brownian motion, with application to long range dependence. 2000 IEEE International Conference on Acoustics, Speech, and Signal Processing – ICASSP 2000, Istanbul, Turkey.

[2] Ahammad, E., Morlanes, I. (2013). Drift parameter estimation for fractional Ornstein–Uhlenbeck process of the Second Kind. Statistics: A Journal of Theoretical and Applied Statistics.

[3] R. Belfaïdi, K. Es-Sebaiy and Y. Ouknine. Parameter estimation for fractional Ornstein–Uhlenbeck processes: Non-ergodic case. Frontiers in Science and Engineering, 11:1-16, 2011.

[4] A. Benassi, S. Jaffard, and D. Roux. Gaussian processes and pseudodifferential elliptic operators. Revista Mathematica Iberoamericana, 13(1):19–89, 1997.

[5] V.V. Buldygin and Yu.V. Kozachenko. Metric characterization of random variables and random processes. Providence, RI: AMS, American Mathematical Society, 2000.

[6] M. El Machkouri, K. Es-Sebaiy, Y. Ouknine, Least squares estimator for non-ergodic Ornstein–Uhlenbeck processes driven by Gaussian processes. Journal of the Korean Statistical Society, 2015, DOI: 10.1016/j.jkss.2015.12.001 (In press).

[7] Y. Hu and D. Nualart. Parameter estimation for fractional Ornstein–Uhlenbeck processes. Stat. Probab. Lett. 80:1030–1038, 2010.

[8] Kleptsoy, M. L., Le Breton, A. (2002). Statistical analysis of the fractional Ornstein-Uhlenbeck type process. Statistical Inference for stochastic processes, 5(4), 229-248.

[9] Y. Kozachenko, A. Melnikov and Y. Mishura. On drift parameter estimation in models with fractional Brownian motion. Statistics, 49(1):35–62, 2015.

[10] K. Kubilius, Y. Mishura, K. Račkenko and O. Shevchenko Consistency of the drift parameter estimator for the discretized fractional Ornstein–Uhlenbeck process with Hurst index $H \in (0, 1/2)$. Electron. J. Statist., 9(2):1799–1825, 2015.

[11] Mishura, Y., Račkenko, K. (2014). On Drift Parameter Estimation in Models with Fractional Brownian Motion by Discrete Observations. Austrian Journal of Statistics, 43(3), 218-228.

[12] Mishura, Y., Račkenko, K., Shevchenko, G. (2014). Asymptotic properties of drift parameter estimator based on discrete observations of stochastic differential equation driven by fractional Brownian motion. In Modern Stochastics and Applications (pp. 303-318). Springer International Publishing.

[13] K.V. Račkenko. Approximation of multifractional Brownian motion by absolutely continuous processes. Theory Probab. Math. Stat., 82:115-127, 2011.

[14] Samko, S., Kilbas, A., Marichev, O.: Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach Science Publishers, New York, (1993).

[15] S.A. Stoer, M.S. Taequ. How rich is the class of multifractional Brownian motions? Stochastic Processes and their Applications, 116(2):200–221, 2006.

[16] Tudor, C. A., Viens, F. G. (2007). Statistical aspects of the fractional stochastic calculus. The Annals of Statistics, 1183-1212.

[17] M. Zähle. Integration with respect to fractal functions and stochastic calculus I. Probab. Theory Related Fields, 111(3):333–373, 1998.

[18] Zähle, M.: On the link between fractional and stochastic calculus. In: Stochastic dynamics, (Bremen, 1997), 305-325 (1999)
