THE NUMBER OF VARIETIES IN A FAMILY WHICH CONTAIN A RATIONAL POINT

DANIEL LOUGHRAN

Abstract. We consider the problem of counting the number of varieties in a family over a number field which contain a rational point. A special focus is placed on products of Brauer-Severi varieties and closely related counting functions associated to Brauer group elements. Using harmonic analysis on toric varieties, we provide a positive answer to some special cases of a question posed by Serre on such counting functions. We also speculate about possible generalisations of Serre’s problem.

Contents

1. Introduction 1
2. Brauer groups and Brauer-Severi schemes 9
3. Virtual L-functions and Delange’s Tauberian theorem 21
4. Algebraic tori, toric varieties and their Brauer groups 25
5. Counting functions associated to Brauer group elements 37
6. Concluding remarks 48
References 51

1. Introduction

Given a variety over a number field $F$, a fundamental problem in number theory is to determine whether or not it contains a rational point. More generally one may consider a family of varieties over $F$, given as the fibres of some morphism $\pi : Y \to X$, for which we have the following fundamental questions.

(1) Is there a member of this family which contains a rational point?
(2) Is the collection of fibres which contain a rational point infinite, or even Zariski dense in $Y$?
(3) If there are infinitely many fibres which contain a rational point, can one give more precise quantitative estimates for the distribution of these fibres?

The main focus of this paper will be on families of Brauer-Severi varieties, i.e. varieties which become isomorphic to projective space over an algebraic closure $\overline{F}$ of $F$. The simplest example of a Brauer-Severi variety is a non-singular conic, and already here the problem is non-trivial. Indeed, it is an open problem whether or not an arbitrary conic bundle over $\mathbb{P}_F^1$ with at least one rational point necessarily has a Zariski dense set of rational points (see BMS13 for some recent work on this problem). To make Question (3) more precise, we shall use height functions. To any embedding $X \subset \mathbb{P}^n$ of $X$ over $F$ and any

2010 Mathematics Subject Classification. 14G05 (primary), 11D45, 14F22, 14M25 (secondary).
Here Serre [Ser90] has shown using the large sieve that for the family of hypersurfaces in a homogeneous space under coflasque tori [BB13b]), where it was again shown that a positive proportion of all such hypersurfaces contain a rational point. Other families have also recently been considered (e.g. Châtelet surfaces [BB13a] and certain principal homogeneous spaces under collasque tori [BB13a]), where it was again shown that a positive proportion of the varieties under consideration contain a rational point. 

For such hypersurfaces, Poonen and Voloch [PV04] have shown that\footnote{The height functions associated to such hypersurfaces are\footnote{Note that a point $x$ in $X(F)$ is counted by this function if and only if the fibre over $x$ contains a rational point.}\text{1.2}$. Here it is conjectured that if $X$ is Fano with $X(F) \neq \emptyset$ and $H$ is an anticanonical height function (i.e. a height function associated to the anticanonical bundle), then there exists an open subset $U \subset X$ and a constant $c_{U,H} > 0$ such that \footnote{One of the aims of this paper is to create a testing ground for a possible conjectural framework for the counting functions (1.2). Note that in the study of such counting functions it is also natural to restrict to sufficiently small open subsets, to avoid certain collections of “bad” fibres (e.g. the singular fibres). Counting functions of the form (1.2) have been considered before. For example let $Y_{d,n}$ denote the total space of the family of all non-singular hypersurfaces over $\mathbb{Q}$ of degree $d$ in $\mathbb{P}^n$ with $n, d \geq 2$. There is a natural projection $\pi_{d,n} : Y_{d,n} \to \mathbb{P}^N_{\mathbb{Q}}$ given by the coefficients of each hypersurface, where $N = \binom{n+d}{d}$. Then under the assumptions that $d < n$, that $(d, n) \neq (2, 2)$ and that the Brauer-Manin obstruction is the only one to the Hasse principle for such hypersurfaces, Poonen and Voloch [PV01] have shown that \footnote{Note that this result implies that a positive proportion of all such hypersurfaces contain a rational point. Other families have also recently been considered (e.g. Châtelet surfaces [BB13a] and certain principal homogeneous spaces under collasque tori [BB13a]), where it was again shown that a positive proportion of the varieties under consideration contain a rational point. Results of this type do not hold in the case where $(d, n) = (2, 2)$, i.e the case of conics. Here Serre [Ser90] has shown using the large sieve that for the family $\pi_{2,2}$ of all non-singular\text{1.1})} \begin{equation}
N(U, H, B) \sim c_{U,H} B (\log B)^{\rho(X)-1}, \quad \text{as } B \to \infty, \end{equation} where $\rho(X) = \text{rank} \text{Pic}.X$. Note that one needs to look at sufficiently small open subsets to avoid so called “accumulating subvarieties” whose contribution may dominate the counting problem (e.g. lines on cubic surfaces). This conjecture has been proven for various Fano varieties (e.g. Flag varieties [FMT89], certain del Pezzo surfaces [FMT89] and certain complete intersections via the circle method [Bir62]) and has also been shown to hold for other rationally connected varieties with sufficiently positive anticanonical bundle (e.g. toric varieties [BT98]). Nevertheless this conjecture is false as stated and there are counter-examples over any number field (see [BT96] and [Lou13]). The advantage of such a theory is that it is intrinsic, i.e. it does not depend on a choice of embedding. We define the counting function associated to a height function $H$ and $\pi$ to be \footnote{More generally, there is a theory of height functions associated to adelically metrised line bundles (see Section 4.10). The advantage of such a theory is that it is intrinsic, i.e. it does not depend on a choice of embedding. We define the counting function associated to a height function $H$ and $\pi$ to be}\text{1.1})$ \begin{equation}
N(X, H, \pi, B) = \# \{ x \in X(F) : H(x) \leq B, x \in \pi(Y(F)) \}. \end{equation} Note that a point $x \in X(F)$ is counted by this function if and only if the fibre over $x$ contains a rational point.

\text{1.1}) \begin{equation}
H(x) = \prod_{v \in \text{Val}(F)} \max\{|x_0|_v, \ldots, |x_n|_v\}. \end{equation} More generally, there is a theory of height functions associated to adelically metrised line bundles (see Section 4.10). The advantage of such a theory is that it is intrinsic, i.e. it does not depend on a choice of embedding. We define the counting function associated to a height function $H$ and $\pi$ to be

\begin{equation}
N(X, H, \pi, B) = \# \{ x \in X(F) : H(x) \leq B, x \in \pi(Y(F)) \}. \end{equation}

Note that a point $x \in X(F)$ is counted by this function if and only if the fibre over $x$ contains a rational point.

If every variety in the family contains a rational point, then $N(X, H, \pi, B) = N(X, H, B)$ is independent of $\pi$ and simply counts the number of rational points of bounded height on $X$, which can be quite a difficult problem in its own right. Manin and others (see [FMT89] and [BM90]) have formulated conjectures on the asymptotic behaviour of such counting functions in special cases (e.g. Fano varieties). Here it is conjectured that if $X$ is Fano with $X(F) \neq \emptyset$ and $H$ is an anticanonical height function (i.e. a height function associated to the anticanonical bundle), then there exists an open subset $U \subset X$ and a constant $c_{U,H} > 0$ such that

\begin{equation}
N(U, H, B) \sim c_{U,H} B (\log B)^{\rho(X)-1}, \quad \text{as } B \to \infty, \end{equation}

where $\rho(X) = \text{rank} \text{Pic}.X$. Note that one needs to look at sufficiently small open subsets to avoid so called “accumulating subvarieties” whose contribution may dominate the counting problem (e.g. lines on cubic surfaces). This conjecture has been proven for various Fano varieties (e.g. Flag varieties [FMT89], certain del Pezzo surfaces [FMT89] and certain complete intersections via the circle method [Bir62]) and has also been shown to hold for other rationally connected varieties with sufficiently positive anticanonical bundle (e.g. toric varieties [BT98]). Nevertheless this conjecture is false as stated and there are counter-examples over any number field (see [BT96] and [Lou13]). One of the aims of this paper is to create a testing ground for a possible conjectural framework for the counting functions (1.2). Note that in the study of such counting functions it is also natural to restrict to sufficiently small open subsets, to avoid certain collections of “bad” fibres (e.g. the singular fibres). Counting functions of the form (1.2) have been considered before. For example let $Y_{d,n}$ denote the total space of the family of all non-singular hypersurfaces over $\mathbb{Q}$ of degree $d$ in $\mathbb{P}^n$ with $n, d \geq 2$. There is a natural projection $\pi_{d,n} : Y_{d,n} \to \mathbb{P}^N_{\mathbb{Q}}$ given by the coefficients of each hypersurface, where $N = \binom{n+d}{d}$. Then under the assumptions that $d < n$, that $(d, n) \neq (2, 2)$ and that the Brauer-Manin obstruction is the only one to the Hasse principle for such hypersurfaces, Poonen and Voloch [PV01] have shown that

\begin{equation}
N(P^N_{\mathbb{Q}}, H, \pi_{d,n}, B) \sim c_{d,n} B, \quad \text{as } B \to \infty, \end{equation}

for some constant $c_{d,n} > 0$. Here $H$ denotes the anticanonical height function on $\mathbb{P}^N_{\mathbb{Q}}$ given by the $(N + 1)$-th power of the height function (1.1). Note that this result implies that a positive proportion of all such hypersurfaces contain a rational point. Other families have also recently been considered (e.g. Châtelet surfaces [BB13a] and certain principal homogeneous spaces under collasque tori [BB13a]), where it was again shown that a positive proportion of the varieties under consideration contain a rational point.

Results of this type do not hold in the case where $(d, n) = (2, 2)$, i.e the case of conics. Here Serre [Ser90] has shown using the large sieve that for the family $\pi_{2,2}$ of all non-singular
plane conics we have
\[ N(\mathbb{P}^5_Q, H, \pi_{2,2}, B) \ll \frac{B}{(\log B)^{1/2}}. \]
In particular this implies that “almost all” plane conics do not contain a rational point. Serre in fact proved a more general result about certain counting functions associated to Brauer group elements, which we now introduce. Let \( X \) be a non-singular variety over \( F \) equipped with a choice of height function \( H \) and let \( \mathcal{B} \subset \text{Br} X \) be a finite subset. Let
\[ X(F)_\mathcal{B} = \{ x \in X(F) : b(x) = 0 \in \text{Br} F \text{ for all } b \in \mathcal{B} \}, \]
denote the “zero-locus” of \( \mathcal{B} \). Note that without loss of generality one may assume that \( \mathcal{B} \) is a finite subgroup since we have the obvious equality \( X(F)_\mathcal{B} = X(F)_{\langle \mathcal{B} \rangle} \), where \( \langle \mathcal{B} \rangle \) denotes the subgroup generated by \( \mathcal{B} \). We define the associated counting function to be
\[ N(X, H, \mathcal{B}, B) = \# \{ x \in X(F)_\mathcal{B} : H(x) \leq B \}. \] (1.4)
Whilst at first glance it might not be clear how such counting functions are related to counting functions of the type \((1.2)\), this relationship becomes immediately clear once one is acquainted with the dictionary that relates Brauer group elements to families of Brauer-Severi varieties. For example, to any conic bundle \( \pi : Y \to X \) one may associate a quaternion algebra over \( F(X) \), which gives rise to an element \( Q_\pi \in \text{Br} U \). Here \( U \subset X \) is the open subset given by removing those points \( x \in X \) whose fibre \( \pi^{-1}(x) \) is singular. For example, if the generic fibre of \( \pi \) has the shape
\[ ax^2 + by^2 = z^2 \subset \mathbb{P}^2_{F(X)}, \]
with \( a, b \in F(X)^* \), then the associated quaternion algebra is simply \((a, b)\). Moreover given \( x \in U(F) \), we have \( Q_\pi(x) = 0 \in \text{Br} F \) if and only if the fibre over \( x \) contains a rational point.

In \([Ser90]\), Serre only considers the case where \( X = \mathbb{P}^n, F = \mathbb{Q} \) and \( \mathcal{B} = \{ b \} \) consists of a single element of order two. In this setting, he shows that for any open subset \( U \subset \mathbb{P}^n \) where \( b \) is defined we have
\[ N(U, H, b, B) \ll \frac{B}{(\log B)^{\Delta_{\mathbb{P}^n}(b)}} \text{ as } B \to \infty, \] (1.5)
where \( H \) is an anticanonical height function on \( \mathbb{P}^n \) and
\[ \Delta_{\mathbb{P}^n}(b) = \sum_{D \in (\mathbb{P}^n)^{(1)}} \left( 1 - \frac{1}{|\partial_D(b)|} \right). \]
Here, for any variety \( X \) we denote by \( X^{(1)} \) the set of codimension one points of \( X \), which is naturally in bijection with the set of irreducible divisors of \( X \). Also \( \partial_D(b) \) denotes the residue of \( b \) at \( D \) (for \( b \) of order two we have \( |\partial_D(b)| \in \{1, 2\} \)). The residue maps are explained in Section \([2]\) intuitively they measure the extent to which \( b \) fails to extend to a larger open subset than \( U \). In \([Ser90]\), Serre asked whether or not the bounds given in \((1.3)\) were sharp. To the author’s knowledge, the corresponding lower bounds have only been shown in two cases: For the family of all plane conics over \( \mathbb{Q} \) \([Hoo07]\) and the family of all plane diagonal conics over \( \mathbb{Q} \) \([Hoo93], [Guo95]\). Moreover no asymptotic formulae have been achieved for the problem as stated here (there are however results for related problems on integral points, for example \([Guo95]\) and classical work of Landau \([Lan08]\) on the number of integers which may be written as a sum of two squares).
1.1. Statement of results. The main theorem of this paper concerns counting functions of the type \( \langle \rangle \) for toric varieties. Recall that an algebraic torus \( T \) over a perfect field \( k \) is an algebraic group over \( k \) which becomes isomorphic to \( \mathbb{G}_m^n \) over an algebraic closure of \( k \), for some \( n \in \mathbb{N} \). We say that such a torus is anisotropic if it has trivial character group over \( k \), i.e. \( \text{Hom}(T, \mathbb{G}_m) = 0 \). A toric variety for \( T \) is a non-singular projective variety with an action of \( T \) which has an open dense orbit which contains a rational point. Rational points of bounded height on toric varieties have been considered by Batyrev and Tschinkel [BT95], [BT98]. Batyrev and Tschinkel constructed a special anticanonical height function for toric varieties which is particularly well-behaved (see Section 4.6). In the case of \( \mathbb{G}_m^n \subset \mathbb{P}^n \), this height function is simply the \((n+1)\)-th power of usual height function \((\langle \rangle)\).

Theorem 1.1. Let \( F \) be a number field and let \( T \) be an anisotropic torus over \( F \). Let \( X \) be a toric variety over \( F \) with respect to \( T \) and let \( U \subset X \) denote the open dense orbit. Let \( \mathcal{B} \subset \text{Br}_1 U \) be a finite subgroup and suppose that the zero-locus \( U(F)_{\mathcal{B}} \) of \( \mathcal{B} \) is non-empty. If \( H \) denotes the Batyrev-Tschinkel anticanonical height function on \( X \), then there exists a constant \( c_{X,\mathcal{B},H} > 0 \) such that

\[
N(U,H,\mathcal{B}) \sim c_{X,\mathcal{B},H} B \frac{(\log B)^{\rho(X)-1}}{(\log B)^{\Delta_X(\mathcal{B})}}, \quad \text{as } B \to \infty,
\]

where

\[
\Delta_X(\mathcal{B}) = \sum_{D \in X(1)} \left(1 - \frac{1}{|\partial_D(\mathcal{B})|}\right),
\]

and \( \rho(X) = \text{rank Pic} X \).

Here \( \partial_D \) denotes the residue map associated to \( D \) and \( \text{Br}_1 U = \ker(\text{Br} U \to \text{Br} U_T) \) denotes the algebraic Brauer group of \( U \). Note that the theorem implies the non-obvious fact that if \( U(F)_{\mathcal{B}} \neq \emptyset \) then \( U(F)_{\mathcal{B}} \) is infinite, and moreover we shall even show that \( U(F)_{\mathcal{B}} \) is Zariski dense in \( U \) (see the end of the introduction). The first step in the proof of Theorem 1.1 is to choose an embedding \( T \subset X \) so that we may identify \( T \cong U \) in such a way that \( \mathcal{B} \subset \text{Br}_1 T \) and moreover \( b(1) = 0 \) for each \( b \in \mathcal{B} \). A result of Sansuc [San81, Lem. 6.9] states that as each \( b \in \mathcal{B} \) is algebraic, the associated evaluation map \( T(F) \to \text{Br} F \) is a group homomorphism. In particular the zero-locus \( T(F)_{\mathcal{B}} \) of \( \mathcal{B} \) is a subgroup of \( T(F) \), and hence has a rich structure. This is one of the main reasons why we focus on algebraic Brauer group elements, as the analogue of Sansuc’s result fails for transcendental Brauer group elements and other methods will be required to handle these. We then proceed by introducing a height zeta function

\[
Z(s) = \sum_{t \in T(F)_{\mathcal{B}}} \frac{1}{H(t)^s},
\]

in a complex variable \( s \). The analytic properties of \( Z(s) \) can be related to the original counting problem via so-called Tauberian theorems. We study the analytic properties of this height zeta function using harmonic analysis (in particular Possion summation) and obtain a continuation of \( Z(s) \) to the line \( \text{Re } s = 1 \), away from \( s = 1 \). The fact that \( T \) is anisotropic greatly simplifies the Poisson summation formula as the quotient \( T(A_F)/T(F) \) is compact in this case.

This approach of using harmonic analysis and height zeta functions has already been put to good effect in the study of rational and integral points of bounded height on toric varieties by Batyrev and Tschinkel [BT95], [BT98] and Chambert-Loir and Tschinkel [CLT13].
detailed treatment of this approach can be found in the memoir [Bou11], which also simultaneously handles the function field case. In particular on taking $\mathcal{B} = 0$ in Theorem 1.1 we recover the main results of [BT95]. The height zeta functions which we encounter are not as well behaved as in the case $\mathcal{B} = 0$ (which admitted a meromorphic continuation past the line $\Re s = 1$), as at $s = 1$ our height zeta functions in general have branch point singularities of the shape $(s-1)^{-m}$ for some rational number $m$. Moreover, we are unable to say anything (unconditionally) about $Z(s)$ in the half plane $\Re s < 1$. Thankfully Delange [Del54] has developed a very general Tauberian theorem which can handle singularities of this type, and we may apply this result to deduce the required asymptotic formula in Theorem 1.1.

We plan to investigate Serre’s problem for non-anisotropic tori and transcendental Brauer group elements in another paper.

Specialising Theorem 1.1 to the case where $X = \mathbb{P}^n$, we provide a positive answer to Serre’s question on the sharpness of (1.5) in some special cases (note that $\text{Pic} \mathbb{P}^n = \mathbb{Z}$). Moreover not only is it the first result for which an asymptotic formula is achieved, rather than simply a lower bound, it is also the first such result which applies to varieties other than projective space and to number fields other than $\mathbb{Q}$. As already remarked, Theorem 1.1 leads to results for families of products of Brauer-Severi varieties. One may ask if in this case there is an interpretation of the factor $\Delta_X(\mathcal{B})$ appearing in Theorem 1.1 in terms of the geometry of the family. It turns out that the answer is yes. For technical reasons we work with special types of morphisms which we call almost smooth (see Definition 2.3). Any morphism between non-singular varieties is automatically almost smooth, so this condition is weaker than being non-singular.

**Theorem 1.2.** Let $F$ be a number field and let $T$ be an anisotropic torus over $F$. Let $X$ be a toric variety over $F$ with respect to $T$ and let $U \subset X$ denote the open dense orbit. Let $Y$ be a variety over $F$ equipped with a proper surjective almost smooth morphism $\pi : Y \to X$ such that $Y$ contains a rational point whose image lies in $U$. Suppose that

- $\pi$ admits a rational section over $F$.
- The restriction of $\pi$ to $U$ is isomorphic to a product of Brauer-Severi schemes over $U$.
- The fibre over each point of codimension one contains an irreducible component of multiplicity one.

For each $D \in X^{(1)}$, choose an irreducible component $D'$ of $\pi^{-1}(D)$ of multiplicity one such that $[F(D)_{D'} : F(D)]$ is minimal amongst all irreducible components of $\pi^{-1}(D)$ of multiplicity one, where $F(D)_{D'}$ denotes the algebraic closure of $F(D)$ inside $F(D')$. If $H$ denotes the Batyrev-Tschinkel anticanonical height function on $X$, then there exists a constant $c_{X,\pi,H} > 0$ such that

$$N(U, H, \pi, B) \sim c_{X,\pi,H} B^{\rho(X)-1} \left( \log B \right)^{\Delta_X(\pi)}, \quad \text{as } B \to \infty,$$

where

$$\Delta_X(\pi) = \sum_{D \in X^{(1)}} \left( 1 - \frac{1}{[F(D)_{D'} : F(D)]} \right),$$

and $\rho(X) = \text{rank Pic } X$.

Note that Theorem 1.2 naturally gives an answer to Question 3 which was posed at the beginning of the introduction. Moreover, we even see that as soon as there exists some $D'$ as in the theorem with $[F(D)_{D'} : F(D)] > 1$, then “almost all” of the elements of the family
do not contain a rational point. As an application, if \( \pi : Y \to X \) is a conic bundle which satisfies the conditions of Theorem 1.2, then one can easily show that

\[
\Delta_X(\pi) = \frac{1}{2} \cdot \# \{ D \in X(1) : \pi^{-1}(D) \text{ is non-split} \}.
\]

Here we say that a reduced conic over a perfect field is non-split if it is isomorphic to two intersecting lines which are conjugate over a quadratic field extension of that field (the assumptions of the theorem imply that non-reduced conics do not occur as the fibres over points of codimension one). In general, it is only the “non-split” fibres which contribute to \( \Delta_X(\pi) \) (see Lemma 2.2). We also give a description of the leading constant \( c_{X, B, H} \) appearing in Lemma 2.2, which formally resembles the leading constant \( c_{H, \text{Peyre}} \) conjectured to appear by Peyre \[\text{Pey95}\] in the context of Manin’s conjecture. We use this description to give examples of \( B \) where \( \Delta_X(\mathcal{B}) = 0 \) and \( c_{X, B, H} < c_{H, \text{Peyre}} \) (examples of these types cannot occur in the case of projective space considered by Serre in \[\text{Ser90}\]). To assist with the calculation of \( c_{X, B, H} \) we prove the following result, which is of independent interest.

**Theorem 1.3.** Let \( U \) be a principal homogeneous space under an algebraic torus over a number field \( F \) and let \( V \to U \) be a product of Brauer-Severi schemes over \( U \) which obtains a rational section over \( F \). Then the Brauer-Manin obstruction is the only obstruction to the Hasse principle and weak approximation for any non-singular proper model of \( V \).

We shall moreover show that the Brauer groups of the non-singular proper models appearing in Theorem 1.3 are finite (modulo constants). Theorem 1.3 therefore gives an effective method to determine the existence of a rational point on such varieties, hence allows one to answer Question 1 for such families. Moreover as soon as there is a rational point on \( V \), we see that \( V(F) \) is Zariski dense in \( V \), which answers Question 2 (this also shows that \( U(F) \) is Zariski dense in \( U \), in the notation of Theorem 1.1).

As a special case of Theorem 1.3, we obtain a new proof of the following fact: Let \( U \subset \mathbb{P}^1 \) be the complement of two rational points or a closed point of degree two. Then the Brauer-Manin obstruction is the only one to the Hasse principle and weak approximation for any non-singular proper model \( X \) of a product of Brauer-Severi schemes over \( U \) (see e.g. \[\text{Sko96}\] Thm. 0.4], \[\text{CTSSD98}\] Thm. 2.2.2] or \[\text{CTS00}\] Ex. 1].) Indeed, here it is well-known that \( U \) may be given the structure of an algebraic torus and moreover Tsen’s theorem implies that \( X \) obtains a rational section over \( F \). Our proof of this fact is genuinely different from previous proofs, as we do not need to construct explicit models of \( X \) over \( \mathbb{P}^1 \), nor do we use descent, the fibration method or variants of Dirichlet’s theorem on primes in arithmetic progressions. We also obtain similar results for products of Brauer-Severi schemes defined over certain open subsets of higher dimensional projective spaces (e.g. complements of \( n + 1 \) general Galois conjugate hyperplanes in \( \mathbb{P}^n \)). Note that here however, the existence of a rational section over \( F \) is not automatically guaranteed. To the author’s knowledge, all previous results of this latter type have been conditional on the assumption of Schinzel’s hypothesis (see e.g. \[\text{Wit07}\] Cor. 3.6]). We prove Theorem 1.3 by first showing that \( V \) itself is stably birational to a principal homogeneous space under an algebraic torus. This reduces the problem to the work of Sansuc \[\text{San81}\], who has already shown that the Brauer-Manin obstruction is the only one to the Hasse principle and weak approximation in this case.

To also aid with the calculation of the leading constant appearing in Theorem 1.1, we prove another result of independent interest (Theorem 2.15 ] which enables one to calculate the unramified Brauer groups of products of Brauer-Severi schemes, generalising a result of Colliot-Thélène and Swinnerton-Dyer \[\text{CTSD91}\] Thm. 2.2.1] (who handled the case where the base is an open subset of \( \mathbb{P}^1 \)). To do this we introduce “subordinate” Brauer group
elements, which were also considered by Serre in the case of $\mathbb{P}^1$ in the appendix of [Ser97, Ch. III]. In Section 2 we perform a detailed study of the theory of subordinate Brauer group elements, as the author expects that they shall feature heavily in the analysis of counting functions of the type (1.3).

1.2. An application. We now show that Theorem 1.1 is non-empty by giving some explicit examples of families of varieties over $\mathbb{P}^n$ which we can handle. Let $n \in \mathbb{N}$ and let $F \subset E$ be a field extension of degree $n + 1$. Then on choosing a basis for $F \subset E$, we obtain a norm form $N_{E/F} : \mathbb{A}^{n+1}_F \to \mathbb{A}^1_F$. The Weil restriction $R_{E/F} \mathbb{G}_m$ of $\mathbb{G}_m$ is an algebraic torus over $F$ which may be identified with the subset $N_{E/F}(t) \neq 0$ in $\mathbb{A}^{n+1}_F$ (see [BLR90, Ch. 7.6]). In particular, we may realise $\mathbb{P}^n$ as a toric variety with respect to the anisotropic torus $T_{E/F} = R_{E/F} \mathbb{G}_m/\mathbb{G}_m$, whose complement is the irreducible divisor $D_{E/F} = \{N_{E/F}(t) = 0\}$. Let now $r \in \mathbb{N}$ and let $F \subset E_i$ be cyclic field extensions of degree $n_i$ for each $i = 1, \ldots, r$, with associated norm forms $N_{E_i/F}$. For each $i = 1, \ldots, r$ let

$$Z_i : N_{E_i/F}(x_1, \ldots, x_{n_i}) = N_{E/F}(1, t_1, \ldots, t_n) \subset \mathbb{A}^{n_i} \times \mathbb{A}^n,$$

(1.6)

with associated projections $\pi_i : Z_i \to \mathbb{A}^n$. The fibres of $\pi_i$ over rational points are principal homogeneous space under the norm one torus

$$R_{E_i/F} \mathbb{G}_m : N_{E_i/F}(x) = 1.$$

Let $Z = Z_1 \times \mathbb{A}^{n_1} \cdots \times \mathbb{A}^{n_r} Z_r$ and let $\pi : Z \to \mathbb{A}^n$ denote the associated map. We now introduce the Brauer group elements which control the arithmetic of $\pi$. Let $\chi_i$ be a non-trivial character of $\text{Gal}(E_i/F)$ and let

$$b_i = (\chi_i, N_{E/F}(1, t_1, \ldots, t_n)),$$

be the associated cyclic algebra over $F(t_1, \ldots, t_n)$. If $n_i \mid (n + 1)$, then $b_i$ is unramified at $t_0 = 0$ and so we have $b_i \in \text{Br} T_{E/F}$ (one may use (2.3) to perform this simple residue calculation). Given a point $t \in T_{E/F}(F)$, we have $b_i(t) = 0$ if and only if $N_{E/F}(1, t_1, \ldots, t_n)$ is a norm of some element in $E_i$, i.e. if and only if the fibre $\pi_i^{-1}(t)$ contains a rational point. In particular we obtain an equality

$$N(T_{E/F}, H, \pi, B) = N(T_{E/F}, H, \{b_1, \ldots, b_r\}, B),$$

of counting functions for each height function $H$ on $\mathbb{P}^n$. The residue of $b_i$ at $D_{E/F}$ is exactly the element of $\text{Hom}(\text{Gal}(\overline{F}/E), \mathbb{Q}/\mathbb{Z})$ induced by $\chi_i$ via the maps $\text{Gal}(\overline{F}/E) \subset \text{Gal}(\overline{F}/F) \to \text{Gal}(E_i/F)$ (see (2.3)). Finally as $b_i(1) = 0$ and $b_i \otimes_F E_i = 0$ for each $i = 1, \ldots, r$, Theorem 1.1 applies and we deduce the following.

**Corollary 1.4.** Let $E_i, E, F$ and $\pi$ be as above with $n_i \mid (n + 1)$ for each $i = 1, \ldots, r$. Assume that $E, E_1, \ldots, E_r$ are linearly disjoint over $F$. If $H$ denotes the Batyrev-Tschinkel anticanonical height function for $T_{E/F} \subset \mathbb{P}^n$, then there exists a constant $c_{T_{E/F}, \pi, H} > 0$ such that

$$N(T_{E/F}, H, \pi, B) \sim c_{T_{E/F}, \pi, H} \frac{B}{(\log B)^{\Delta_{\mathbb{P}^n}(\pi)}},$$

as $B \to \infty$,

where

$$\Delta_{\mathbb{P}^n}(\pi) = 1 - \frac{1}{n_1 \cdots n_r}.$$

Other examples to which Theorem 1.1 applies may be obtained by replacing the right hand side of (1.6) with a *product* of norm forms. Note that in the special case where $r = 1$ and $F \subset E_1$ is quadratic, the equation (1.6) becomes

$$x_1^2 - ax_2^2 = N_{E/F}(1, t_1, \ldots, t_n),$$
for some \( a \in \mathbb{F}^* \) which is not a square in \( E \). This may visibly be extended to a conic bundle over \( \mathbb{P}^n \) and in particular Corollary 1.3 provides a positive answer to Serre’s question in the case where \( 2 \mid n + 1 \).

**Outline of the paper:** In Section 2, we begin by recalling various facts on Brauer groups, Brauer-Severi varieties and the Brauer-Manin obstruction. We then prove Theorem 1.2 (assuming Theorem 1.1) followed by Theorem 1.3. Next we introduce the notion of subordinate Brauer group elements and study some of their basic properties. As mentioned above, the author believes that Brauer group elements of this type will feature heavily in the study of the counting functions (1.4). Our main theorem on subordinate Brauer group elements is of independent interest, and concerns their use in calculating unramified Brauer groups of products of Brauer-Severi schemes, which generalises a result of Colliot-Thélène and Swinnerton-Dyer [CTSD94, Thm 2.2.1]. We also prove an analogue for subordinate Brauer group elements of a result of Harari [Har94, Thm. 2.1.1] on the specialisations maps of Brauer group elements over number fields, which makes clear the relevance of subordinate Brauer group elements in the study of zero-loci of Brauer group elements.

In Section 3, we gather various analytic results we will need on virtual L-functions and certain partial Euler products which shall naturally appear in the proof of Theorem 1.1. We also state here the version of Delange’s Tauberian which we shall use. This section may be read independently of Section 2.

In Section 4, we begin by gathering numerous facts on the adelic spaces of algebraic tori over number fields. We then proceed by giving a description of the algebraic Brauer groups of algebraic tori over number fields and their completions. The author could not find these latter results in the literature, but they are probably well-known to the experts and follow from a standard application of Tate-Nakayama duality. We then study toric varieties and their Brauer groups over number fields, in particular we derive an analogue for subordinate Brauer group elements of an exact sequence of Voskresenskiĭ [Vos69] on the unramified Brauer groups of algebraic tori.

In Section 5, we bring together all the work in the previous sections to prove Theorem 1.1. At the end of Section 5, we calculate the leading constant in Theorem 1.1 and provide an interpretation of this in terms of a certain Tamagawa measure associated to a virtual Artin representation built from the Picard group of \( X \) and \( \mathcal{B} \).

We finish the paper in Section 6, with a conjecture and some suggestions on possible further research directions.

**Acknowledgments:** Part of this work was completed whilst the author was attending the program “Arithmetic and Geometry” at the Hausdorff Research Institute for Mathematics in Bonn. The author would like to thank David Bourqui, Martin Bright, Tim Browning, Ariyan Javanpeykar, Arne Smeets and Yuri Tschinkel for useful comments. Special thanks go to Jean-Louis Colliot-Thélène, for many useful conversations and for his help with the proofs of Theorem 1.3 and Theorem 2.15. Thanks also go to Alexei Skorobogatov for his help with the proof of Lemma 2.5.

**Notation.**

*Algebra.* Given a ring \( R \), we denote by \( R^* \) the group of units of \( R \). For a subset \( A \subset G \) of a group \( G \) we denote by \( (A) \) the subgroup generated by \( A \). For any element \( g \in G \) of finite order, we denote by \( |g| \) its order. Given a topological group \( G \), we denote by \( G^\sim = \text{Hom}(G, S^1) \) the group of continuous characters of \( G \) and \( G^\sim = \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \) the group of continuous characters of finite order. We fix embeddings \( \mathbb{Q}/\mathbb{Z} \subset S^1 \subset \mathbb{C}^* \) so that we
have an embedding $G^\sim \subset G^A$. Throughout this paper we make frequent use of the following version of character orthogonality: Let $G$ be a compact Hausdorff topological group with a Haar measure $d g$. Then for any character $\chi$ of $G$ we have
\[
\int_G \chi(g) d g = \begin{cases} \text{vol}(G), & \chi = 1, \\ 0, & \text{otherwise.} \end{cases}
\]

**Number theory.** We shall often denote number fields with the letter $F$. We denote by $\mathcal{O}_F$ the ring of integers of $F$ and by $\text{Val}(F)$ the set of valuations of $F$. For any $v \in \text{Val}(F)$, we denote by $F_v$ (resp. $\mathcal{O}_v$) the completion of $F$ (resp. $\mathcal{O}_F$) with respect to $v$. Given a non-archimedean place $v$ of $F$, we denote by $q_v$ the size of the residue field of $F_v$. We choose absolute values on each $F_v$ such that $|x|_v = |N_{F_v/\mathbb{Q}_p}(x)|_p$, where $v|p \in \text{Val}(\mathbb{Q})$ and $| \cdot |_p$ is the usual absolute value on $\mathbb{Q}_p$. With these choices, we have the following product formula
\[
\prod_{v \in \text{Val}(F)} |x|_v = 1, \quad \text{for all } x \in F^\times. \quad (1.7)
\]

We choose Haar measures $d x_v$ on each $F_v$ à la Tate [CF10, Ch. XV]. The only facts we shall use about these choices is that $\int_{\mathcal{O}_v} d x_v = 1$, for almost all non-archimedean $v$ and that $\text{vol}(\mathbb{A}_F/F) = 1$ with respect to the associated Haar measure on the adele group $\mathbb{A}_F$ of $F$.

**Geometry.** For a field $k$, we denote by $\mathbb{P}^n_k$ and $\mathbb{A}^n_k$ projective $n$-space and affine $n$-space over $k$ respectively. We sometimes omit the subscript $k$ if the field is clear. A variety over $k$ is a separated geometrically integral scheme of finite type over $k$. For every perfect field $k$, we fix a choice of algebraic closure $\overline{k}$ and we denote by $G_k$ the absolute Galois group of $k$ with respect to $\overline{k}$. Given a variety $X$ over $k$, we shall denote by $\overline{X} = X \times_k \overline{k}$ and $X_v = X \times_k k_v$ if moreover $k$ is a number field and $v$ is a place of $k$. All cohomology will be taken with respect to the étale topology.

## 2. Brauer groups and Brauer-Severi schemes

In this section we begin by collecting various facts that we shall need on Brauer groups of algebraic varieties and the Brauer-Manin obstruction, before proving Theorem 1.2 and Theorem 1.3. We then introduce the notion of subordinate Brauer group elements in order to obtain a description of the unramified Brauer groups of families of Brauer-Severi varieties.

### 2.1. Brauer groups of varieties

We now recall various standard facts on Brauer groups which can be found in [Gro68], [CTSD94, Sec. 1] and [GS06]. Let $X$ be a non-singular variety over a field $k$ of characteristic zero. We define the (cohomological) Brauer group of $X$ to be $\text{Br} X = H^2(X, \mathbb{G}_m)$. This naturally embeds into $\text{Br} k(X)$, in particular $\text{Br} X$ is torsion group. We denote by $\text{Br}_0 X$ the image of the homomorphism $\text{Br} k \to \text{Br} X$ induced by the structure morphism $X \to \text{Spec} k$. We let $\text{Br}_1 X = \ker(\text{Br} X \to \text{Br} \overline{X})$ denote the algebraic Brauer group of $X$ and $\text{Br}_a X = \text{Br}_1 X/\text{Br}_0 X$. Elements of $\text{Br} X$ which are not contained in $\text{Br}_1 X$ are called transcendental.
2.1.1. The Brauer-pairing. Fundamental to this paper will be the pairing
\[ \text{Br} X \times X(k) \to \text{Br} k, \quad (b, x) \mapsto b(x). \] (2.1)

Here \( b(x) \) denotes the evaluation (or specialisation) of \( b \) at \( x \), namely the pull-back of \( b \) via the morphism \( \text{Spec} k \to X \) associated to the rational point \( x \). This pairing is additive on the left and functorial, namely if \( f : Y \to X \) is a morphism and \( b \in \text{Br} X \), then \( (f^*b)(y) = b(f(y)) \) for all \( y \in Y(k) \). Given a subset \( \mathcal{B} \subset \text{Br} X \), we shall denote by
\[ X(k)_{\mathcal{B}} = \{ x \in X(k) : b(x) = 0 \text{ for all } b \in \mathcal{B} \}, \] (2.2)
the “zero-locus” of \( \mathcal{B} \). We shall also denote by \( \mathcal{B} \) the indicator function of the set \( X(k)_{\mathcal{B}} \) (or simply \( \mathcal{B} \) if \( \mathcal{B} \) is clear).

2.1.2. Residues. For any discrete valuation \( v \) on the function field \( k(X) \) of \( X \) there is a residue map
\[ \partial_v : \text{Br} k(X) \to H^1(k(v), \mathbb{Q}/\mathbb{Z}), \]
where \( k(v) \) denotes the residue field of \( v \). If the discrete valuation is associated to some discrete valuation ring \( R \), we shall also denote the residue map by \( \partial_R \). Such discrete valuation rings arise naturally as the local rings of elements of \( X^{(1)} \), where \( X^{(1)} \) denotes the set of codimension one points of \( X \). The elements of \( X^{(1)} \) correspond bijectively to irreducible divisors on \( X \) and we shall also denote by \( \partial_D \) the residue map associated to such an irreducible divisor \( D \). There is an isomorphism \( H^1(k(v), \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(G_{k(v)}, \mathbb{Q}/\mathbb{Z}) \), in particular the residue of an element of \( \text{Br} k(X) \) may be thought of as a cyclic field extension of \( k(v) \) together with a choice of generator of the Galois group of that field extension.

In simple cases the residues are easy to calculate. For example, for \( \chi \in H^1(k, \mathbb{Q}/\mathbb{Z}) \) and \( a \in k(X)^* \) let \( (\chi, a) \), denote the associated cyclic algebra (see [GS06, Sec. 2.5] or [CTSD94, Sec. 1]). Then for a discrete valuation ring \( k \subset R \) with field of fractions \( k(X) \) and valuation \( v \), by [CTSD94, Prop. 1.1.3] we have
\[ \partial_v((\chi, a)) = v(a) \cdot \chi_v \in H^1(k(v), \mathbb{Q}/\mathbb{Z}), \] (2.3)
where \( \chi_v \) denotes the image of \( \chi \) under the map \( H^1(k, \mathbb{Q}/\mathbb{Z}) \to H^1(k(v), \mathbb{Q}/\mathbb{Z}) \). This formula can be used to verify the residue calculation in Section 1.2.

2.1.3. Purity and the unramified Brauer group. Grothendieck’s purity theorem implies that the residue maps give rise to an exact sequence
\[ 0 \to \text{Br} X \to \text{Br} k(X) \to \bigoplus_{D \in X^{(1)}} H^1(k(D), \mathbb{Q}/\mathbb{Z}). \]
By considering those elements of the above sequence which become trivial over \( \overline{k} \) (see e.g. [CTS77, Lem. 14]), we also obtain the exact sequence
\[ 0 \to \text{Br}_1 X \to H^2(k, \overline{k}(X)) \to \bigoplus_{D \in X^{(1)}} H^1(k(D), \mathbb{Q}/\mathbb{Z}), \]
where \( k_D = \overline{k} \cap k(D) \subset \overline{k}(D) \). Note that \( D \) is geometrically irreducible if and only if \( k_D = k \) (see Lemma 2.2). If \( U \subset X \) is the complement of a finite collection of divisors of \( X \), we also have
\[ 0 \to \text{Br} X \to \text{Br} U \to \bigoplus_{D \in (X \setminus U)^{(1)}} H^1(k(D), \mathbb{Q}/\mathbb{Z}), \] (2.4)
and
\[ 0 \to \text{Br}_1 X \to \text{Br}_1 U \to \bigoplus_{D \in (X \setminus U)^{(1)}} H^1(k(D), \mathbb{Q}/\mathbb{Z}). \] (2.5)
We define the unramified Brauer group $\text{Br}_{\text{nr}}(k(X)/k)$ of $k(X)/k$ to be the collection of those $b \in \text{Br} k(X)$ such that $\partial_v(b) = 0$ for all discrete valuations $v$ of $k(X)$ which are trivial on $k$. We shall write this group as $\text{Br}_{\text{nr}} X$ if $k$ is clear. Again by Grothendieck’s purity theorem, this is isomorphic to the Brauer group of any non-singular proper model of $X$ (see [CTSD94 Thm. 1.3.2]).

2.2. Brauer groups over number fields. If $F$ is a number field, local class field theory implies that for each place $v$ of $F$ there are canonical injective maps $\text{inv}_v : \text{Br} F_v \to \mathbb{Q}/\mathbb{Z}$ which are isomorphisms for non-archimedean $v$. We shall often identify $\text{Br} F_v$ with its image under the invariant map. Global class field theory implies that there is a short exact sequence

$$0 \to \text{Br} F \to \bigoplus_{v \in \text{Val}(F)} \text{Br} F_v \to \mathbb{Q}/\mathbb{Z} \to 0,$$

(2.6)

where the final map is the sum of the local invariants. If $X$ is a non-singular variety defined over $F$, we let

$$\mathfrak{B}(X) = \ker \left( \text{Br}_1 X \to \bigoplus_{v \in \text{Val}(F)} \text{Br}_1 X_v \right).$$

(2.7)

Lemma 2.1. Let $X$ be a non-singular variety over a number field $F$ and let $\mathcal{X}$ be a model of $X$ over $\mathcal{O}_F$. Then the local pairings

$$\text{Br} X_v \times X(F_v) \to \text{Br} F_v \subset \mathbb{Q}/\mathbb{Z},$$

are locally constant on the right for all $v \in \text{Val}(F)$. Any $b \in \text{Br} X$ pairs trivially with $\mathcal{X}(\mathcal{O}_v)$ for almost all places $v \in \text{Val}(F)$, in particular on taking the sum of local invariants we obtain a global pairing

$$\text{Br} X \times X(A_F) \to \mathbb{Q}/\mathbb{Z},$$

which is also locally constant on the right and trivial on $X(F)$. Any $b \in \text{Br}_{\text{nr}} X$ pairs trivially with $X(F_v)$ for almost all places $v \in \text{Val}(F)$, so again on taking the sum of local invariants we obtain a pairing

$$\text{Br}_{\text{nr}} X \times \prod_{v \in \text{Val}(F)} X(F_v) \to \mathbb{Q}/\mathbb{Z},$$

which is locally constant on the right and trivial on $X(F)$.

Proof. This result is very well-known, see [San81 Lem. 6.2] and [Sko01 Sec. 5.2] (note that the fact that the global pairings are trivial on $X(F)$ follows from the exact sequence (2.6)).

It follows from (2.6) that for any $x \in X(F)$ and any $b \in \text{Br} X$, we have $b(x) = 0$ if and only if $b_v(x) = 0$ for every place $v \in \text{Val}(F)$, where $b_v = b \otimes_F F_v \in \text{Br} X_v$. In particular, if for any subset $\mathcal{B} \subset \text{Br} X$ we let

$$X(A_F)_{\mathcal{B}} = \{(x_v) \in X(A_F) : b(x_v) = 0 \text{ for all } v \in \text{Val}(F) \text{ and all } b \in \mathcal{B}\},$$

denote the “adelic zero-locus of $\mathcal{B}$”, then we have a natural inclusion $X(F)_{\mathcal{B}} \subset X(A_F)_{\mathcal{B}}$ (see (2.2)) and the map

$$X(A_F) \to \{0,1\}, \quad (x_v) \mapsto \prod_{v \in \text{Val}(F)} \hat{b}_{x_v}(x_v),$$

is a continuous extension of $\hat{b}_{\mathcal{B}}$ to $X(A_F)$, which by abuse of notation we shall also denote by $\hat{b}_{\mathcal{B}}$. For any subset $Z \subset X(A_F)$, we shall write $Z_{\mathcal{B}} = Z \cap X(A_F)_{\mathcal{B}}$. 
2.2.1. The Hasse principle, weak approximation and the Brauer-Manin obstruction. We now recall some facts about the Brauer-Manin obstruction which can be found in [Sko01, Sec. 5] (this makes use of the pairings given in Lemma 2.1). We say that a non-singular variety $X$ over $F$ satisfies the Hasse principle if $X(A_F) \neq \emptyset$ implies that $X(F) \neq \emptyset$. We say that $X$ satisfies weak approximation if $X(F)$ is dense in $\prod_{v \in \Val(F)} X(F_v)$ with respect to the product topology. Most varieties which we shall consider in this paper will satisfy so-called weak weak approximation as soon as $X(F) \neq \emptyset$, namely there is a finite set of places $S$ such that $X(F)$ is dense in $\prod_{v \in S} X(F_v)$. In particular in the study of weak approximation, we shall instead often consider the closure $\overline{X(F)^w}$ of $X(F)$ in $X(A_F)$ with respect to the product topology. Most varieties which we shall consider in this paper will satisfy so-called weak weak approximation. Namely given a subset $B \subset \Br X$ we shall denote by $X(A_F)^B = \{(x_v) \in X(A_F) : \sum_{v \in \Val(F)} \inv_v b(x_v) = 0 \text{ for all } b \in B \}$. By Lemma 2.1 this is a closed subset of $X(A_F)$ with respect to the adelic topology (but not closed with respect to the product topology in general). We also let $X(A_F)^{Br} = X(A_F)^{Br X}$. If $B \subset Br_{nr} X$, then we also denote by $\left(\prod_{v \in \Val(F)} X(F_v)\right)^B = \{(x_v) \in \prod_{v \in \Val(F)} X(F_v) : \sum_{v \in \Val(F)} \inv_v b(x_v) = 0 \text{ for all } b \in B \}$. Again by Lemma 2.1 this is a closed subset of $\prod_{v \in \Val(F)} X(F_v)$. We shall say that the Brauer-Manin obstruction is the only one to Hasse principle and weak approximation if $X(F)$ is dense in $\prod_{v \in \Val(F)} X(F_v)^{Br_{nr} X}$. For most varieties under consideration in this paper $Br_{nr} X/Br_0 X$ will be finite, in which case the Brauer-Manin obstruction is the only one to Hasse principle and weak approximation if and only if $\overline{X(F)^w} = X(A_F)^{Br_{nr} X}$. Note that in this case if $X(F) \neq \emptyset$, then weak weak approximation holds and $X(F)$ is Zariski dense in $X$. We shall combine the above notations with those for zero-loci of Brauer group elements, namely for any $B_1, B_2 \subset Br X$ and any subset $Z \subset X(A_F)$ the notation $Z^{B_2}_{B_1}$ means $Z^{B_2}_{B_1} = \{(x_v) \in Z : b_1(x_v) = 0 \text{ for all } b_1 \in B_1 \text{ and all } v \in \Val(F), \sum_{v \in \Val(F)} \inv_v b_2(x_v) = 0 \text{ for all } b_2 \in B_2 \}$.  

2.3. Brauer-Severi schemes and their products. Let $k$ be a field of characteristic zero. Recall that a Brauer-Severi variety of dimension $n$ over $k$ is a variety over $k$ which becomes isomorphic to $\mathbb{P}^n$ over $\overline{k}$. For a variety $X$ over $k$, a Brauer-Severi scheme $Y$ of relative dimension $n$ over $X$ is a scheme over $X$ which is étale locally on $X$ isomorphic to $\mathbb{P}^n_X$. In particular $Y$ is smooth over $X$. A nice way to visualise $Y$ is as a family of Brauer-Severi varieties parametrised by the points of $X$. There is an important relationship between Brauer-Severi schemes and Brauer group elements which we shall frequently exploit in this paper (see [Gro68]). Let $Y$ be a Brauer-Severi scheme of relative dimension $n$ over a non-singular variety $X$. Then $Y$ corresponds
 uniquely, up to isomorphism, with an element of $H^1(X, \text{PGL}_{n+1})$. The exact sequence
\[ 1 \rightarrow \mathbb{G}_m \rightarrow \text{GL}_{n+1} \rightarrow \text{PGL}_{n+1} \rightarrow 1, \]
of group schemes gives rise to the exact sequence
\[ H^1(X, \text{GL}_{n+1}) \rightarrow H^1(X, \text{PGL}_{n+1}) \rightarrow \text{Br} X, \tag{2.8} \]
of pointed sets. We denote by $[Y]$ the corresponding element of $\text{Br} X$. A theorem of Gabber states that as $X$ is non-singular every element of $\text{Br} X$ is of the form $[Y]$ for some Brauer-Severi scheme $Y$ over $X$ (equivalently, every element of the Brauer group is represented by some Azumaya algebra).

Note that (2.8) implies that $[Y] = 0$ if and only if $Y$ is Zariski locally on $X$ isomorphic to $\mathbb{P}^F_k$. In particular, a Brauer-Severi variety over $k$ has trivial class in $\text{Br} k$ if and only if it is isomorphic to $\mathbb{P}^F_k$, which by a classical theorem Châtelet (see \cite[Thm. 5.1.3]{GS06}) occurs if and only if it contains a rational point. The inclusion $\text{Br} X \subset \text{Br} k(X)$ therefore implies that $[Y] = 0$ if and only if the morphism $Y \rightarrow X$ admits a rational section. Similar remarks apply to products of Brauer-Severi schemes. For example if $Y_1, \ldots, Y_r$ are Brauer-Severi schemes over $X$ and $Y = Y_1 \times_X \cdots \times_X Y_r$, then $Y \rightarrow X$ admits a rational section if and only if each $Y_i \rightarrow X$ admits a rational section, which occurs if and only if $[Y_i] = 0$ for each $i = 1, \ldots, r$.

2.4. Proof of Theorem 1.2. We now begin the proof of Theorem 1.2 (assuming Theorem 1.1). Note that the remarks in the previous section immediately imply that we have an equality
\[ N(U, H, \mathcal{B}, B) = N(U, H, \pi, B), \]
of counting functions, in the notation of Theorem 1.1 and Theorem 1.2. Therefore to prove the result, it suffices to show that $\Delta_X(\mathcal{B}) = \Delta_X(\pi)$. We begin by showing that it is only the “non-split” fibres of $\pi$ that contribute to $\Delta_X(\pi)$, and moreover that the field extensions $F(D)_D'$ appearing in Theorem 1.2 are indeed finite. Here, following Skorobogatov \cite[Def. 0.1]{Sko96}, we say that a scheme over a perfect field $k$ is split if it contains a geometrically integral open subscheme. As an example, if $C$ is a conic over $k$ then $C$ is split if and only if it is either non-singular or isomorphic to two intersecting lines which are both defined over $k$.

**Lemma 2.2.** Let $X$ be an integral scheme of finite type over a perfect field $k$ and denote by $k_X$ the algebraic closure of $k$ inside the function field of $X$. Then $k \subset k_X$ is a finite field extension. Moreover $k = k_X$ if and only if $X$ is geometrically integral, i.e. split.

**Proof.** We begin by showing that $k \subset k_X$ is finite. As the problem is local, we may assume that $X = \text{Spec} A$ where $A$ is a finitely generated $k$-algebra which is also an integral domain. Moreover as the integral closure of $A$ inside $k(X)$ is finite as an $A$-module, it suffices to prove the result when $A$ is integrally closed. In which case we have $k_X \subset A$. On choosing a maximal ideal $\mathfrak{m}$ of $A$, we obtain a $k$-algebra homomorphism $A \rightarrow K = A/\mathfrak{m}$. Since clearly $k_X \cap \mathfrak{m} = 0$, we deduce the result on noting that $k \subset K$ is a finite field extension by Zariski’s lemma \cite[Cor. 5.24]{AM69}.

We now prove the second part on the lemma. Note that $X$ is geometrically integral if and only if $k(X) \otimes_k K$ is an integral domain for all finite field extensions $k \subset K$. In particular if $F \neq k_X$, then $k(X) \otimes_k k_X$ is not an integral domain (it contains $k_X \otimes_k k_X$) so $X$ is not geometrically integral. Conversely, if on the other hand $k(X) \otimes_k k_X$ is an integral domain then we must have $k = k_X$ (this again follows by considering $k_X \otimes_k k_X$). This completes the proof. \hfill $\Box$
Our next aim is to show that $\Delta_X(\mathscr{B}) = \Delta_X(\pi)$. To do this we may work locally around each divisor, so we continue with some results on schemes over discrete valuation rings. Throughout this section we let $R$ be a discrete valuation ring with field of fractions $K$ and perfect residue field $k$. We shall work with the following types of schemes.

**Definition 2.3.** Let $X$ be a scheme of finite type over $R$. We say that $X$ is *almost smooth* if its generic fibre is smooth and for any étale $R$-algebra $R'$, each $R'$-point of $X$ lies in the smooth locus of $X$. We say that a scheme $X$ over a normal scheme $Y$ is *almost smooth* if it is almost smooth over the local ring of every point of codimension one.

In the language of [BLR90, Def. 3.1.1], we are asking that the identity map on $X$ be a smoothing. Examples of almost smooth schemes include smooth schemes (trivial) and regular schemes (see [BLR90, p. 61]). Note that it follows from [BLR90, Lem. 3.6.5] that if $X$ is almost smooth over $R$, then every $R'$-point of $X$ lies in the smooth locus of $X$ for every flat extension of discrete valuation rings $R \subset R'$ of ramification index one. One may obtain non-regular almost smooth schemes by taking fibre products of regular schemes.

**Lemma 2.4.** If $X_1$ and $X_2$ are almost smooth schemes over $R$, then $X_1 \times_R X_2$ is almost smooth over $R$.

*Proof.* Let $R'$ be an étale algebra over $R$. Any $R'$-point of $X_1 \times_R X_2$ is induced by $R'$-points of $X_1$ and $X_2$. As $X_1$ and $X_2$ are almost smooth, these points lie in the smooth locus. Since the fibre product of the smooth loci of $X_1$ and $X_2$ lies inside the smooth locus of $X_1 \times_R X_2$, this proves the result. \(\square\)

The proof of the following result is based on the proof of [Wit07, Lem. 3.8] (see also [SK96, Lem. 1.1]).

**Lemma 2.5.** Let $\pi_i : X_i \to \text{Spec } R$ be flat proper almost smooth integral schemes over $R$, for $i = 1, 2$. Assume that the generic fibres of $\pi_1$ and $\pi_2$ are isomorphic. Then for each irreducible component $D_1$ of multiplicity one of the special fibre of $\pi_1$ there exists an irreducible component $D_2$ of multiplicity one of the special fibre of $\pi_2$ such that

$$k_{D_2} \subset k_{D_1}.$$  

In particular, choose some such component $D_1$ (if it exists) for which $[k:k_{D_1}]$ is minimal amongst the irreducible components of multiplicity one of the special fibre of $\pi_1$. Then there exists an irreducible component $D_2$ of multiplicity one of the special fibre of $\pi_2$ such that

$$k_{D_1} \cong k_{D_2},$$

as $k$-algebras.

*Proof.* Note that as $\pi_1$ is flat, the special fibre of $\pi_1$ is indeed a divisor. Let $d_1$ denote the generic point of $D_1$. Let $R_1$ be the discrete valuation ring given by the local ring at $d_1$ and let $K_1$ be the field of fractions of $R_1$. Note that $K_1$ is the function field of $X_1$, in particular the inclusion of the generic point $\text{Spec } K_1 \to X_1$ yields a $K_1$-point of $X_1$. As $\pi_1$ and $\pi_2$ have isomorphic generic fibres, by the valuative criterion for properness [Har77, Thm. II.4.7] there exists a unique morphism $f_1 : \text{Spec } R_1 \to X_2$ extending the inclusion of the generic point of $X_2$. Let $c_2 = f_1(d_1)$. Then the residue field $k(c_2)$ of $c_2$ embeds $k$-linearly in the residue field $k(d_1)$ of $d_1$. Moreover since $R \subset R_1$ has ramification index one and $\pi_1$ is almost smooth, we see that $c_2$ lies in the smooth locus of $\pi_2$. Hence $c_2$ lies in a unique irreducible component $D_2$ of multiplicity one of the special fibre of $\pi_2$. Let $d_2$ denote the generic point of $D_2$. Note that the local rings of $d_2$ and $c_2$ have the same field of fractions and moreover the local ring at $c_2$ is integrally closed as it is a regular local ring. It follows that $k_{D_2}$ embeds
k-linearly into the local ring of $c_2$ and we obtain a $k$-linear embedding $k_{D_2} \hookrightarrow k(c_2)$. We have shown a sequence of inclusions

$$k \subset k_{D_2} \subset k(c_2) \subset k(d_1).$$

The required inclusion follows from the fact that $k \subset k_{D_2}$ is a finite field extension by Lemma 2.2. To prove the second part of the lemma, choose some $D_2$ such that $k_{D_2} \subset k_{D_1}$. Choose also an irreducible component $D'_1$ of multiplicity one of the special fibre of $\pi_1$ such that $k_{D'_1} \subset k_{D_2}$. As $[k_{D'_1} : k] \geq [k_{D_1} : k]$ by assumption, the result therefore again follows from the fact that $k_{D_1}$ and $k_{D_2}$ are finite dimensional as $k$-vector spaces.

This last result tells us that the data of interest in the definition of $\Delta_X(\pi)$ in Theorem 1.2 is independent of the choice of almost smooth model. The next result, which is a simple application of the work of Artin [Art82] and Frossard [Fro97], shows that there are almost smooth models for products of Brauer-Severi schemes which are particularly well-behaved.

**Lemma 2.6.** Let $V = V_1 \times_K \cdots \times_K V_r$ be a product of Brauer-Severi varieties over $K$. Then there exists a flat proper almost smooth integral scheme $V \to \text{Spec} R$ whose generic fibre is isomorphic to $V$ and whose special fibre is reduced. Moreover, the algebraic closure of $k$ in the function field of each irreducible component of the special fibre is the compositum of the cyclic field extensions determined by the residues $\partial_R([V_1]), \ldots, \partial_R([V_r])$.

**Proof.** Artin [Art82] Thm. 1.4] has constructed regular flat proper integral schemes $V_i \to \text{Spec} R$ whose generic fibres are isomorphic to $V_i$ and whose special fibres are integral, for each $i = 1, \ldots, r$. Frossard [Fro97] Prop. 2.3] has shown that the algebraic closure of $k$ inside the function field of the special fibre of $V_i$ is exactly the cyclic field extension of $k$ determined by the residue $\partial_R([V_i])$ (note that Frossard’s result is stated for complete discrete valuation rings, this does not cause problems for us as the residue field remains unchanged on passing to a completion). We now take $V = V_1 \times_R \cdots \times_R V_r$. This is obviously flat and proper over $R$ and integral. The special fibre is reduced as $k$ is perfect. Also by Lemma 2.4 it is almost smooth over $R$. The result therefore follows on noting that the tensor product of a collection of Galois field extensions is isomorphic to a direct power of the compositum of those field extensions.

Note that one cannot in general use the fact that the $V_i$ are regular to deduce that $V$ itself is regular (regularity can fail to hold even for the fibre product of a conic bundle with itself). It is for this reason that we have introduced the notion of almost smooth, as it allows us to avoid the need to construct an explicit desingularisation of $V$. Combining these results, we obtain the following.

**Lemma 2.7.** Let $X$ be a non-singular variety over a field $k$ of characteristic zero. Let $Y$ be an integral scheme together with a proper surjective almost smooth morphism $\pi : Y \to X$. Suppose that

- The generic fibre of $\pi$ is isomorphic to a product of Brauer-Severi varieties $V_1, \ldots, V_r$ over $k(X)$.
- The fibre over each point of codimension one contains an irreducible component of multiplicity one.

For each $D \in X^{(1)}$, choose an irreducible component $D'$ of $\pi^{-1}(D)$ of multiplicity one such that $[k(D)_{D'} : k(D)]$ is minimal amongst all irreducible components of $\pi^{-1}(D)$ of multiplicity one. Then for any codimension one point $D \in X^{(1)}$ we have

$$[k(D)_{D'} : k(D)] = |\langle \partial_D(\mathcal{R}) \rangle|,$$
where \( \mathcal{B} = \{[V_1], \ldots, [V_r]\} \subset \text{Br} k(X) \). In particular, if we let
\[
\Delta_X(\mathcal{B}) = \sum_{D \in X^{(i)}} \left( 1 - \frac{1}{\#(\partial D(\mathcal{B}))} \right), \quad \Delta_X(\pi) = \sum_{D \in X^{(i)}} \left( 1 - \frac{1}{[k(D)_{D'} : k(D)]} \right),
\]
then
\[
\Delta_X(\pi) = \Delta_X(\mathcal{B}).
\]

**Proof.** Note that as the generic fibre is smooth, the sum appearing in the definition of \( \Delta_X(\pi) \) is indeed finite. By Lemma 2.5, the integer \([k(D)_{D'} : k(D)]\) is independent of the choice of almost smooth model for \( Y \) locally near \( D \). The result therefore follows from Lemma 2.6. \( \square \)

This completes the proof of Theorem 1.2

2.5. **Proof of Theorem 1.3.** The aim of this section is to prove Theorem 1.3. We begin with some results on Brauer-Severi schemes which correspond to algebraic Brauer group elements.

Let \( U \) be a non-singular variety over a field \( k \) of characteristic zero and let \( V_i \to U \) for \( i = 1, \ldots, r \) be Brauer-Severi schemes over \( U \). Let \([V_i] \in \text{Br} U\) denote the corresponding Brauer group classes and assume that \([V_i] \in \text{Br}_k U\) for each \( i = 1, \ldots, r \). We fix a finite field extension \( K \) of \( k \) such that \([V_i] \otimes_k K = 0 \in \text{Br}_k\). We define \( S \) to be the algebraic torus over \( k \) given by the exact sequence
\[
1 \to \mathbb{G}_m^r \to (\text{R}_k/\mathbb{G}_m)^r \to S \to 1.
\]

Note that \( S \) is rational since it may be embedded as an open subset of a product of projective spaces. We also let \( V = V_1 \times_U \cdots \times_U V_r \). Applying étale cohomology to the above sequence we obtain an exact sequence
\[
(Pic U_K)^r \to H^1(U, S) \xrightarrow{\delta} (\text{Br} U)^r \to (\text{Br} U_K)^r. \tag{2.9}
\]

Here we have used the isomorphisms \( H^1(U, \text{R}_k/\mathbb{G}_m) \cong H^1(U_K, \mathbb{G}_m) \) for \( i = 1, 2 \), which follow from Shapiro’s lemma (see e.g. [KMR T98, Lem. 29.6]). As by assumption \(([V_i],[V_i])\) lies in the kernel of the last map in the sequence (2.9), there exists a (non-unique) \( U \)-torsor \( W \) under \( S \) with class \([W] \in H^1(U, S)\) such that \( \delta([W]) = ([V_1], \ldots, [V_r]) \). The specialisation maps for the \( V_i \) and \( W \) are closely related.

**Lemma 2.8.** Let \( k \subset L \) be any field extension and let \( u \in U(L) \). Then
\[
[W](u) = 0 \in H^1(L, S_L) \text{ if and only if } ([V_i](u), \ldots, [V_r](u)) = 0 \in (\text{Br} L)^r.
\]

**Proof.** By the functoriality of the exact sequence (2.9), the morphism \( U_L \to \text{Spec} L \) yields a commutative diagram
\[
\xymatrix{
(Pic(K \otimes_k L))^r \ar[r] & H^1(L, S_L) \ar[r] & (\text{Br} L)^r \\
H^1(U_L, S_L) \ar[r] & (\text{Br} U_L)^r
}
\]
with exact top row. It immediately follows that if \([W](u) = 0\), then certainly \([V_i](u) = 0\) for each \( i = 1, \ldots, r \). To prove the converse, it suffices show that \( Pic(K \otimes_k L) = 0 \). As \( k \subset L \) is separable, the algebra \( K \otimes_k L \) is étale over \( L \) hence is a product of finite field extensions of \( L \). The required vanishing therefore follows from Hilbert’s theorem 90. \( \square \)
We may use the previous lemma to show that \( W \) is stably birational to \( V \). Recall that we say that a variety \( V_1 \) is stably birational to a variety \( V_2 \) if there exists \( n_1 \) and \( n_2 \) such that \( V_1 \times \mathbb{P}^{n_1} \) is birational to \( V_2 \times \mathbb{P}^{n_2} \).

**Lemma 2.9.** \( V \) is stably birational to \( W \).

**Proof.** The style of argument which we shall use is well-known, cf. [GS06, Rem. 5.4.3]. To prove the result, it suffices to show that \( V_\eta \) is stably birational to \( W_\eta \), where \( \eta \) denotes the generic point of \( U \). Note that \( V_\eta \) has a \( k(V_\eta) \)-point, corresponding to the inclusion of the generic point of \( V_\eta \). In particular, it follows from Lemma 2.3 that \( W_\eta \) has a \( k(V_\eta) \)-point. However as \( W_\eta \) is a \( k(U) \)-torsor under \( S_k(U) \) and since \( S_k(U) \) is a rational variety, we see that \( W_\eta \times_{k(U)} k(V_\eta) \) is birational to \( \mathbb{P}^w \times_{k(U)} k(V_\eta) \) for some \( w \in \mathbb{N} \). This implies that \( W_\eta \times_{k(U)} V_\eta \) is birational to \( \mathbb{P}^w \times_{k(U)} V_\eta \). As \( V_\eta \) is a Brauer-Severi variety over \( k(U) \), we may apply the exact same argument again to deduce that \( W_\eta \times_{k(U)} V_\eta \) is also birational to \( W_\eta \times_{k(U)} \mathbb{P}^v \) for some \( v \in \mathbb{N} \). This completes the proof. \( \square \)

We now specialise to the case where \( U \) is a principal homogeneous space under some algebraic torus over \( k \) (keeping all previous notations).

**Lemma 2.10.** \( W \) is a principal homogeneous space under some algebraic torus.

**Proof.** As \( W \) is a \( U \)-torsor under \( S \) and \( \overline{S} \) and \( \overline{S} \) are algebraic tori, it follows from [CT08, Thm. 5.6] that \( \overline{W} \) admits the structure of an algebraic group together with a short exact sequence

\[
1 \rightarrow \overline{S} \rightarrow \overline{W} \rightarrow \overline{U} \rightarrow 1. \quad (2.10)
\]

I claim that \( \overline{W} \) is also an algebraic torus. Indeed by (2.10), we see that \( \overline{W} \) is solvable and moreover contains no non-trivial unipotent elements. Therefore by the structure theorem for solvable algebraic groups [Bor91, Thm. III.10.6], we see that \( \overline{W} \) is equal to its own maximal torus. The fact that this implies that \( W \) is itself a principal homogeneous space under some algebraic torus is well-known, see e.g. [Sko01, Lem. 2.4.4]. \( \square \)

It therefore follows from the main theorem of [San81] that if \( k \) is a number field, then the Brauer-Manin obstruction is the only obstruction to the Hasse principle and weak approximation for any non-singular proper model of \( W \). In light of Lemma 2.9 to deduce that the same holds for \( V \) we shall need the following lemma. This result is well-known so we only sketch a proof (a complete proof appears in [CTPS13]).

**Lemma 2.11.** Let \( X_1 \) and \( X_2 \) be non-singular proper varieties over a field \( k \) of characteristic zero, which are stably birational over \( k \). Then \( \text{Br} X_1 \cong \text{Br} X_2 \). If moreover \( k \) is a number field and \( \text{Br} X_1 / \text{Br} X \) is finite, then

\[
\overline{X_1(k)} = X_1(A_k)^\text{Br} \text{ if and only if } \overline{X_2(k)} = X_2(A_k)^\text{Br}. \]

**Proof.** We break the proof up into two parts. First let \( X \) be a non-singular proper variety over \( k \), let \( n \in \mathbb{N} \) and let \( Y = X \times \mathbb{P}^n \). Then the projection \( Y \rightarrow X \) induces an isomorphism \( \text{Br} Y \cong \text{Br} X \) (this may be proved using Theorem 2.15 below). Next suppose that \( k \) is a number field and that \( \overline{X}(k) = X(A_k)^\text{Br} \). We shall show that \( \overline{Y}(k) = Y(A_k)^\text{Br} \). If \( Y(A_k)^\text{Br} = \emptyset \) then certainly \( \overline{Y}(k) = Y(A_k)^\text{Br} \), so we may assume that \( Y(A_k)^\text{Br} \neq \emptyset \). Choose an adelic point \( (x_v) \times (z_v) \in Y(A_k)^\text{Br} = X(A_k)^\text{Br} \times \mathbb{P}^n(A_k) \). By assumption, there exists a rational point \( x \in X(k) \) arbitrarily close to \((x_v)\). Also, by the classical weak approximation theorem (see e.g. [Sko01, Thm. 5.1.2]), we know that there exists \( z \in \mathbb{P}^n(k) \) which is arbitrarily close to \((z_v)\). It follows that \( x \times z \) is arbitrarily close to \((x_v) \times (z_v)\), in particular \( \overline{Y}(k) = Y(A_k)^\text{Br} \). If on the other hand \( \overline{Y}(k) = Y(A_k)^\text{Br} \), then a similar argument
also shows that $X(k) = X(A_k)^{Br}$. Therefore to prove the lemma, without loss of generality we may assume that $X_1$ and $X_2$ are birationally equivalent. The result in this case is proven in [CTPS13] (it is here where the finiteness of $Br X_1/Br_0 X$ is used).

To apply this lemma to complete the proof of Theorem 1.3, it suffices to show that $Br_{nr}(k(V)/k)/Br_0 V$ is finite. By the previous lemma, this follows from the fact that $Br_{nr}(k(W)/k)/Br_0 W$ is finite as $W$ is geometrically rational [CTSD94, Prop. 1.3.1] (this finiteness also follows from Lemma 2.18 below). This completes the proof of Theorem 1.3.

### 2.6. Subordinate Brauer group elements

In the appendix of [Ser07] Ch. III, Serre defined the notion of subordinate Brauer group elements for $\mathbb{P}^1$. In this section, we shall consider generalisations of this to other varieties. Brauer group elements of this type will naturally arise in the calculation of the leading constant in Theorem 1.1 and they also greatly assist in the calculation of the unramified Brauer group of products of Brauer-Severi schemes (see Theorem 2.15).

**Definition 2.12.** Let $X$ be a non-singular proper variety over a field $k$ of characteristic zero and let $\mathcal{B} \subseteq Br k(X)$ be a finite subset. Then we say that $b \in Br k(X)$ is subordinate to $\mathcal{B}$ with respect to $X$ if $\partial_D(b)$ lies in the subgroup generated by the $\partial_D(b')$ for all $D \in X^{(1)}$ and all $b' \in \mathcal{B}$. We let

$$Sub(X, \mathcal{B}) = \{ b \in Br k(X) : \partial_D(b) \in \langle \partial_D(\mathcal{B}) \rangle \text{ for all } D \in X^{(1)} \},$$

denote the collection of all such elements.

By (2.4) this is a subgroup of $Br U$ for any open subset $U \subseteq X$ such that $\mathcal{B} \subseteq Br U$. We obviously have $Sub(X, \mathcal{B}) = Sub(X, \{ \mathcal{B} \})$ and also $Br X = Sub(X, 0) \subseteq Sub(X, \mathcal{B})$ is a subgroup of finite index, as $\mathcal{B}$ is finite. It is important to note that $Sub(X, \mathcal{B})$ depends on the choice of model for $k(X)$ in general, as the next example shows.

**Example 2.13.** Consider $\mathbb{A}^2 \subseteq \mathbb{P}^2$ and $\mathbb{A}^2 \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ embedded as an affine patch with coordinate functions $x$ and $y$. Suppose that $k$ contains a non-square element $a \in k^*$. Let $b \in Br k(x,y)$ be the class of the quaternion algebra $(xy, a)$. Then using the residue formula (2.3) one can check that $(x,a)$ is subordinate to $b$ with respect to $\mathbb{P}^1 \times \mathbb{P}^1$, but not with respect to $\mathbb{P}^2$. In particular $Sub(\mathbb{P}^2, b) \neq Sub(\mathbb{P}^1 \times \mathbb{P}^1, b)$. In fact one can show that

$$Sub(\mathbb{P}^2, b) = \{ Br k, (x, y, a) \}, \quad Sub(\mathbb{P}^1 \times \mathbb{P}^1, b) = \{ Br k, (x, a), (x, y, a) \},$$

in this case.

We shall therefore introduce another definition which will not depend on a choice of model.

**Definition 2.14.** Let $U$ be a non-singular variety over $k$ and let $\mathcal{B} \subseteq Br k(U)$ be a finite subset. We say that an element $b \in Br k(U)$ is subordinate to $\mathcal{B}$ with respect to $k(U)/k$ if $\partial_v(b)$ lies in the subgroup generated by the $\partial_v(b')$ for all $b' \in \mathcal{B}$ and all discrete valuations $v$ of $k(U)$ which are trivial on $k$. We let $Sub(k(U)/k, \mathcal{B})$ denote the collection of all such elements.

The group $Sub(k(U)/k, \mathcal{B})$ is clearly independent of the choice of non-singular model for $k(U)$. We shall almost always denote this group by $Sub(k(U), \mathcal{B})$; we only use the notation $Sub(k(U)/k, \mathcal{B})$ for emphasis if $k$ is not clear. Note that we also have $Sub(k(U), \mathcal{B}) = Sub(k(U), \{ \mathcal{B} \})$. If $U = X$ is proper, then $Sub(k(X), \mathcal{B}) \subseteq Sub(X, \mathcal{B})$ and

$$Sub(k(X), 0) = Sub(X, 0) = Br X,$$
but however \( \text{Sub}(k(X), \mathcal{B}) \neq \text{Sub}(X, \mathcal{B}) \) in general, as Example 2.13 shows.

In the case of \( \mathbb{P}^1 \) considered by Serre in the appendix of [Ser97 Ch. II], for all finite subsets \( \mathcal{B} \subset \text{Br} k(t) \) we have \( \text{Sub}(k(t), \mathcal{B}) = \text{Sub}(\mathbb{P}^1, \mathcal{B}) \), as \( \mathbb{P}^1 \) is the unique non-singular projective variety with function field \( k(t) \). Colliot-Thélène and Swinnerton-Dyer [CTSD94 Thm 2.2.1] also considered groups of this type in the case of \( \mathbb{P}^1 \). If \( V_1, \ldots, V_r \) are Brauer-Severi varieties over \( k(t) \), they showed that \( \text{Sub}(k(t), \mathcal{B}) = \text{Sub}(\mathbb{P}^1, \mathcal{B}) \) is isomorphic to the Brauer group of any non-singular proper model of \( V \) over \( k \). The following theorem is a generalisation of this result.

**Theorem 2.15.** Let \( U \) be a non-singular variety over a field \( k \) of characteristic 0 and let \( V_1, \ldots, V_r \) be a Brauer-Severi varieties over \( k(U) \). Then there is a short exact sequence

\[
0 \to \langle \mathcal{B} \rangle \to \text{Sub}(k(U), \mathcal{B}) \to \text{Br}_{nr}(k(V)/k) \to 0,
\]

where \( \mathcal{B} = \{[V_1], \ldots, [V_r]\} \subset \text{Br} k(t) \) and \( V = V_1 \times_{k(t)} \cdots \times_{k(t)} V_r \). Note that \( \text{Br}_{nr}(k(V)/k) \) is isomorphic to the Brauer group of any non-singular proper model of \( V \) over \( k \). The following theorem is a generalisation of this result.

**Proof.** The map in (2.11) is the map induced by the composition

\[
\text{Sub}(k(U), \mathcal{B}) \subset \text{Br}(k(U)) \xrightarrow{\gamma} \text{Br} V \subset \text{Br} k(U)(V) = \text{Br} k(V),
\]

where \( \gamma \) denotes the natural restriction map. The claim of the theorem is that the image of \( \text{Sub}(k(U), \mathcal{B}) \) in \( \text{Br} k(V) \) consists of exactly those elements of \( \text{Br}_{nr}(k(V)/k) \) and that the kernel is the subgroup generated by \( \mathcal{B} \). Our proof is based on the proof of [CTSD94 Thm 2.2.1].

For simplicity of notation, we let \( b_i = [V_i] \). A classical theorem of Amitsur (see e.g. [GS06 Thm. 5.4.1]) states that there is a short exact sequence

\[
0 \to \langle b_i \rangle \to \text{Br} k(U) \to \text{Br} V \to 0,
\]

for each \( i = 1, \ldots, r \). The same proof yields a short exact sequence

\[
0 \to \langle \mathcal{B} \rangle \to \text{Br} k(U) \xrightarrow{\gamma} \text{Br} V \to 0,
\]

which shows that the sequence (2.11) is exact on the left and in the middle. Fix some \( b \in \text{Sub}(k(U), \mathcal{B}) \). We shall now show that \( \gamma(b) \in \text{Br}_{nr}(k(V)/k) \). For all discrete valuation rings \( k \subset R \) with field of fractions \( k(U) \), there exists \( m_{i,R} \in \mathbb{Z} \) for each \( i = 1, \ldots, r \) such that

\[
\partial_R(b) = \sum_{i=1}^{r} m_{i,R} \partial_R(b_i).
\]

Let \( k \subset R' \) be a discrete valuation ring with field of fractions \( k(V) \). If \( k(U) \subset R' \), then we clearly we have \( \partial_R(\gamma(b)) = 0 \). Otherwise \( R' = k(U) \cap R' \) is a discrete valuation ring containing \( k \) with field of fractions \( k(U) \). Let \( e(R'/R) \) denote the ramification degree of \( R' \) over \( R \). Then as \( \gamma(b_i) = 0 \) for each \( i = 1, \ldots, r \) by (2.12), we have

\[
\partial_{R'}(\gamma(b)) = \partial_{R'}(\gamma(b)) - \sum_{i=1}^{r} m_{i,R} \partial_{R'}(\gamma(b_i)) = e(R'/R) \partial_R \left( b - \sum_{i=1}^{r} m_{i,R} b_i \right) = 0,
\]

on using [CTSD94 Prop. 1.1.1] followed by (2.13). Hence \( \gamma(b) \in \text{Br}_{nr}(k(V)/k) \) as required.

To finish the proof, it suffices to show that the map \( \text{Sub}(k(U), \mathcal{B}) \to \text{Br}_{nr}(k(V)/k) \) is surjective. Note that (2.12) shows that every element of \( \text{Br}_{nr}(k(V)/k) \) is of the form \( \gamma(b) \) for some \( b \in \text{Br} k(U) \). So let \( b \in \text{Br} k(U) \) suppose that \( b \notin \text{Sub}(k(U), \mathcal{B}) \), i.e. there exists a
discrete valuation ring $k \subset R$ with field of fractions $k(U)$ such that $\partial_R(b)$ does not lie in the subgroup generated by $\partial_R(\mathcal{B})$. In order to show that $\gamma(b)$ does not lie in $\text{Br}_v(k(V)/k)$, it suffices to construct a discrete valuation ring $k \subset R'$ with field of fractions $k(V)$ such that $\partial_R(\gamma(b)) \neq 0$.

To do this, we shall need to use the explicit model $V \to \text{Spec } R$ for $V$ given in Lemma 2.16. Let $D$ denote one of the irreducible components of the special fibre of $V$ and denote by $R'$ the corresponding discrete valuation ring. Let $v$ (resp. $v'$) denote the discrete valuation on $R$ (resp. $R'$). Applying the residue maps to the exact sequence (2.12) we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \langle \mathcal{B} \rangle & \longrightarrow & \text{Br } k(U) & \longrightarrow & \text{Br } V & \longrightarrow & 0 \\
& & \downarrow{\partial_R} & & \downarrow{\partial_R} & & \downarrow{\partial_{R'}} & & \\
0 & \longrightarrow & \langle \partial_R(\mathcal{B}) \rangle & \longrightarrow & H^1(k(v), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^1(k(v'), \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0
\end{array}
$$

with exact rows. Here the map $H^1(k(v), \mathbb{Q}/\mathbb{Z}) \to H^1(k(v'), \mathbb{Q}/\mathbb{Z})$ is the natural restriction map as $R \subset R'$ has ramification index one, since $D$ is integral (this again follows from [CTSD94 Prop. 1.1.1]). The exactness of the bottom row follows from a simple application of the inflation-restriction exact sequence (see e.g. [NSW00 Prop. 1.6.6]), on noting that by Lemma 2.6 the algebraic closure of $k(v)$ in $k(v')$ is the compositum of the field extensions given by the $\partial_R(b_i)$. By assumption $\partial_R(b)$ does not lie in $\langle \partial_R(\mathcal{B}) \rangle$, hence we see that $\partial_R(\gamma(b)) \neq 0$. This completes the proof. \hfill \Box

Using this theorem we shall be able to obtain a characterisation of subordinate Brauer group elements for varieties over number fields in terms of the pairing (2.1). First recall the following theorem of Harari [Har94 Thm. 2.1.1].

**Theorem 2.16** (Harari). Let $U$ be non-singular variety over a number field $F$ and let $b \in \text{Br } U$. Then $b \in \text{Br}_{nr} U$ if and only if the evaluation map $U(F_v) \to \text{Br } F_v$ induced by $b$ is trivial for all but finitely many places $v$ of $F$.

Note that the first implication follows easily on applying Lemma 2.1 to any non-singular proper model of $U$, while the reverse implication is a deep result. The following may be viewed as the analogue of Harari’s theorem for subordinate Brauer group elements, and makes clear their importance in the study of zero-loci of Brauer group elements.

**Theorem 2.17.** Let $U$ be non-singular variety over a number field $F$, let $\mathcal{B} \subset \text{Br } U$ be a finite subset and let $b \in \text{Br } U$. Then $b \in \text{Sub}(F(U), \mathcal{B})$ if and only if the evaluation map $U(F_v)_{\mathcal{B}} \to \text{Br } F_v$ induced by $b$ is trivial for all but finitely many places $v$ of $F$.

**Proof.** Let $\pi : V \to U$ be the product of Brauer-Severi schemes corresponding to the elements of $\mathcal{B}$. By the functoriality of the Brauer pairing, for any place $v$ of $F$ we have a commutative diagram

$$
\begin{array}{ccc}
V(F_v) & \longrightarrow & \text{Br } F_v \\
\downarrow{\pi^* b} & & \\
U(F_v)_{\mathcal{B}} & \longrightarrow & 
\end{array}
$$

given by pairing with $\pi^* b$ and $b$, respectively. Note that by definition the map $V(F_v) \to U(F_v)_{\mathcal{B}}$ is surjective for any place $v$ of $F$. By Theorem 2.16 we have $\pi^* b \in \text{Br}_{nr}(F(V)/F)$ if and only if the evaluation map $V(F_v) \to \text{Br } F_v$ induced by $\pi^* b$ is trivial for all but finitely
many places $v$ of $F$. By Theorem 2.15 we also know that $\pi^*b \in \Br_{nr}(F(V)/F)$ if and only if $b \in \Sub(F(U), R)$. This proves the required equivalence. □

We finish by giving some necessary conditions for the finiteness of $\Sub(X, R)/\Br_0 X$.

**Lemma 2.18.** Let $X$ be a non-singular proper variety over a field $k$ of characteristic zero and assume that $X$ is geometrically rational. Let $R \subset \Br_k(X)$ be a finite subset. Then $\Sub(X, R)/\Br_0 X$ is finite.

**Proof.** As $\Br X \subset \Sub(X, R)$ is a subgroup of finite index, it suffices to show that the group $\Br X/\Br_0 X$ is finite. This follows from the assumption that $X$ is geometrically rational (see e.g. [CTSD94] Prop. 1.3.1]). □

Note that if $U$ and $V$ are as in Theorem 2.15 and $U$ is geometrically rational, then Lemma 2.18 implies that $\Br_{nr}(k(V)/k)/\Br_0 V$ is finite.

3. Virtual L-functions and Delange’s Tauberian Theorem

In this section we gather various analytic results on virtual L-functions and certain partial Euler products which shall arise in the proof of Theorem 1.1.

3.1. Hecke L-functions. Let $F$ be a number field. Recall (see e.g. [CF10] or [Wei74]) that a Hecke character for $F$ is a character $A_F^* \to S^1 \subset \mathbb{C}^*$ which is trivial on $F^\times \subset A_F^*$, with respect to the diagonal embedding. Each Hecke character $\chi$ may be decomposed as a product of local characters $\chi_v : F_v^* \to S^1$ for each place $v$ of $F$. We say that $\chi$ is unramified at $v$ if the character $\chi_v$ is trivial on $\mathcal{O}_v^*$. Each Hecke character has a conductor $q(\chi) \in \mathbb{N}$, which measures the amount of ramification of $\chi$ at the finite places. Associated to $\chi$ is an L-function $L(\chi, s)$ in a complex variable $s$, which is given as a Euler product of local factors $L_v(\chi_v, s)$ for each non-archimedean place $v$ of $F$ for $\Re s > 1$. For our purposes it suffices to know that if $\chi$ is unramified at $v$, then the local factor takes the form

$$L_v(\chi_v, s) = \left(1 - \frac{\chi_v(\pi_v)}{q_v}ight)^{-1},$$

where $\pi_v$ is a uniformiser at $v$ and $q_v$ denotes the order of the residue field of $v$. The fact that $\chi$ is unramified at $v$ implies that this representation is independent of the choice of uniformising element. We denote by $\zeta_F(s) = L(1, s)$ the Dedekind zeta function of $F$.

A well-known theorem of Hecke gives a functional equation and meromorphic continuation of these L-functions (see e.g. [Wei74] Thm. 7.7.5]). Following Weil, we shall say that a Hecke character is principal if its restriction to the collection $A_F^{\ast,1}$ of norm one ideles is trivial, and non-principal otherwise. It is easy to see that the principal characters are exactly those of the form $|| \cdot ||^{i\theta}$ for some $\theta \in \mathbb{R}$, where

$$|| \cdot || : A_F^* \to S^1, \quad (t_v) \mapsto \prod_{v \in \Val(F)} |t_v|_v,$$

is the adelic norm map.

**Theorem 3.1.** Let $\chi$ be a non-principal Hecke character for $F$. Then $L(\chi, s)$ admits a holomorphic continuation to $\mathbb{C}$ with no zeros in the region $\Re s > 1$.

The Dedekind zeta function $\zeta_F(s)$ admits a meromorphic continuation to $\mathbb{C}$, with a pole of order one at $s = 1$ and no other zeros or poles in the region $\Re s \geq 1$.

**Proof.** A proof of these analytic properties can be found in [Wei74] Thm. 7.7.5. The fact that these L-functions do not vanish in the given regions is well-known, see e.g. [Col90] or [IK04] Thm. 5.10 for the known zero-free regions. □
Note that the condition that \( \chi \) be non-principal (rather than simply non-trivial) is certainly necessary in the statement of the theorem; for example the Hecke L-function \( L(|| \cdot ||^\varepsilon, s) = \zeta_F(s - i\theta) \) has a pole at \( s = 1 + i\theta \). L-functions of this type were overlooked in the statement of \([\text{BT}95]\) Thm. 3.1.2. We now state bounds for the growth rate of Hecke L-functions. Let \( \chi \) be a character of \( \prod_v F_v^* \) (e.g. \( \chi \) could come from a Hecke character).

Let \( v \) be an archimedean place of \( F \). Restricting \( \chi \) to the obvious subgroup \( \mathbb{R}_{>0} \subset F_v^* \) (as \( F_v^* \cong \mathbb{R}^* \) or \( \mathbb{C}^* \)), we obtain a continuous homomorphism \( \mathbb{R}_{>0} \to S^1 \). Such a homomorphism must be of the form \( x \mapsto |x|^{\kappa_v} \) for some \( \kappa_v \in \mathbb{R} \). In which case we define

\[
|| \chi || = \max_{v|\infty} |\kappa_v|.
\]

**Lemma 3.2.** Let \( \varepsilon > 0 \) and let \( \chi \) be a Hecke character of \( F \). If \( \chi \) is non-principal then

\[
L(\chi, s) \ll_{\varepsilon} q(\chi)^\varepsilon (1 + |\text{Im}(s)| + ||\chi||)^\varepsilon,
\]

uniformly for \( \text{Re} \, s \geq 1 \). If \( \chi = || \cdot ||^\theta \) is principal then

\[
\left( \frac{s - i\theta - 1}{s - i\theta + 1} \right) L(|| \cdot ||^\theta, s) \ll_{\varepsilon} (1 + |\text{Im}(s)| + |\theta|)^\varepsilon,
\]

uniformly for \( \text{Re} \, s \geq 1 \). In particular, given any compact subset \( C \) of the half plane \( \text{Re} \, s \geq 1 \) which avoids the pole of \( L(\chi, s) \) (if it exists), one has

\[
L(\chi, s) \ll_{\varepsilon, C} q(\chi)^\varepsilon (1 + ||\chi||)^\varepsilon,
\]

uniformly for \( s \in C \).

**Proof.** These bounds follow from \([\text{IK}04]\) (5.20) p. 100. \( \square \)

The non-trivial principal Hecke characters were overlooked in the statement of \([\text{BT}95]\) Thm. 3.2.3.

3.1.1. **Complex powers of Hecke L-functions.** During the course of the proof of Theorem 3.1 certain rational powers of Hecke L-functions shall occur. We record their analytic properties here, looking at the more general case of complex powers of Hecke L-functions for completeness. Note that complex powers of the Riemann zeta function commonly appear in applications of the Selberg-Delange method, the unfamiliar reader is advised to consult \([\text{Ten}95]\) Ch. II.5.

Given \( z \in \mathbb{C}^* \) and a Hecke character \( \chi \), the Euler product formula of \( L(\chi, s) \) for \( \text{Re} \, s > 1 \) shows that \( L(\chi, s)^z \) is a holomorphic function on the half-plane \( \text{Re} \, s > 1 \) (on taking a principal branch for the complex logarithm). We shall require the following result.

**Theorem 3.3.** Let \( \chi \) be a non-principal Hecke character for \( F \) and let \( z \in \mathbb{C}^* \). Then \( L(\chi, s)^z \) admits a holomorphic continuation with no zeros to the region \( \text{Re} \, s \geq 1 \).

The complex power of the Dedekind zeta function \( \zeta_F(s)^z \) admits a holomorphic continuation with no zeros to the region \( \text{Re} \, s \geq 1 \), apart from at \( s = 1 \). Here we have

\[
\zeta_F(s)^z = \frac{c_F}{(s - 1)^z} + O\left( \frac{1}{(s - 1)^{z-1}} \right),
\]

as \( s \to 1 \), where \( c_F \neq 0 \).

**Proof.** Let \( s \in \mathbb{C} \) be such that \( \text{Re} \, s = 1 \). By Theorem 3.1 we know that \( L(\chi, s) \) is non-zero at \( s \). Moreover as it is also holomorphic at \( s \), there exists an open neighbourhood of \( s \) where \( L(\chi, s) \) is non-zero. It follows that \( L(\chi, s)^z \) is well-defined and holomorphic in this neighbourhood, which proves the first part of the theorem. The second part follows in a similar manner on again using Theorem 3.1. \( \square \)
Note that using known zero-free regions for Hecke L-functions (see e.g. [Col90] or [IK04, Thm. 5.10]) one may obtain a continuation of $L(\chi, s)^z$ to an explicit region which contains the line $\Re s = 1$. We shall not have need of such a result.

3.2. Virtual Artin L-functions. For a number field $F$, a virtual Artin representation is a formal finite sum

$$V = \sum_{i=1}^{n} z_i V_i,$$

where $z_i \in \mathbb{C}$ and the $V_i$ are Artin representations, i.e. finite dimensional complex vector spaces which admit a continuous action of the absolute Galois group $G_F$ of $F$. We define $\text{rank} V = \sum_{i=1}^{n} z_i \text{rank} V_i$ and let $V^{G_F} = \sum_{i=1}^{n} z_i V_i^{G_F}$. We define the L-function of $V$ to be

$$L(V, s) = \prod_{i=1}^{n} L(V_i, s)^{z_i},$$

where $L(V_i, s)$ is the usual Artin L-function associated to $V_i$ (see [IK04, Ch. 5.13]). Note that $L(V, s)$ is not an L-functions in the traditional sense of the phrase (as in [IK04, Sec. 5]), as it does not in general admit a meromorphic continuation to $\mathbb{C}$. We shall content ourselves with the following theorem.

**Theorem 3.4.** Let $V$ be a virtual Artin representation over $F$. Then $L(V, s)$ admits a holomorphic continuation with no zeros to the region $\Re s \geq 1$ apart from possibly at $s = 1$.

Here we have

$$L(V, s) = \frac{c_V}{(s-1)^r} + O\left(\frac{1}{(s-1)^{r-1}}\right),$$

as $s \to 1$, where $r = \text{rank} V^{G_F}$ and $c_V \neq 0$.

**Proof.** Let

$$V = \sum_{i=1}^{n} z_i V_i,$$

be as in (3.2). By a well-known argument using the Brauer induction Theorem (see e.g. [CF10, Thm. VII.3.7]), we may write

$$L(V_i, s) = \zeta_F(s)^{\text{rank} V_i^{G_F}} \prod_j L(\chi_{ij}, s)^{n_{ij}},$$

where $n_{ij} \in \mathbb{Z}$ and the $\chi_{ij}$ are finite order non-trivial Hecke characters for some possibly larger field extensions. The result therefore follows from Theorem 3.3. □

In the notation of Theorem 3.4 we shall write

$$L^*(V, 1) = c_V.$$

3.3. Analytic properties of certain partial Euler products. Let $F$ be a number field and fix a finite group $\mathcal{R}$ of Hecke characters for $F$. For each place $v$ of $F$, let also

$$\hat{\rho}_v : F^*_v \to \{0, 1\}, \quad \hat{\rho}_v : t_v \mapsto \begin{cases} 1, & \text{if } \rho_v(t_v) = 0 \text{ for all } \rho \in \mathcal{R}, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $\hat{\rho}_v$ is simply the indicator function for $\bigcap_{\rho \in \mathcal{R}} \ker \rho_v$, in particular it follows from character orthogonality that

$$\hat{\rho}_v = \frac{1}{|\mathcal{R}|} \sum_{\rho \in \mathcal{R}} \rho_v.$$  \hspace{1cm} (3.3)
Let $\chi$ be a Hecke character of $F$. The partial Euler product of interest to us is
\[
L_{\mathfrak{A}}(\chi, s) = \prod_v \left( 1 - \frac{\rho_v(\pi_v)\chi_v(\pi_v)}{q_v^s} \right)^{-1}.
\] (3.4)

Here the product is only over those non-archimedean places $v \in \text{Val}(F)$ for which $\rho_v$ and $\chi_v$ are unramified for all $\rho \in \mathfrak{A}$. Also $q_v$ denotes the residue degree of $F_v$ and $\pi_v$ denotes a uniformiser at $v$. Note that if $\mathfrak{A} = 1$, then (3.4) is equal to the usual Hecke L-function $L(\chi, s)$ associated to the character $\chi$, up to finitely many Euler factors. In general, the Euler product (3.4) is over a certain collection of places of density $1/|\mathfrak{A}|$. In particular it is clear that the product in (3.4) is absolutely convergent for $\text{Re} s > 1/2$ and defines a holomorphic function without zeros on this domain. The analytic properties of functions of this type have been studied by numerous authors (see e.g. [Kur87] and [Has10]). For example it was shown in [Has10] that in general such functions admit a continuation (with various singularities) to the region $\text{Re} s > 0$, with a natural boundary at the line $\text{Re} s = 0$.

We shall content ourselves with the following elementary lemma.

**Lemma 3.5.** Let $\varepsilon > 0$ and let $\chi$ be a Hecke character for $F$. Then there exists a function $G(\mathfrak{A}, \chi, s)$ which is holomorphic, uniformly bounded with respect to $\chi$ and non-zero on the domain $\text{Re} s > 1/2 + \varepsilon$ such that
\[
L_{\mathfrak{A}}(\chi, s) |_{\mathfrak{A}} = G(\mathfrak{A}, \chi, s) \prod_{\rho \in \mathfrak{A}} L(\rho \chi, s),
\]
for $\text{Re} s > 1$.

If $\chi$ is not of the form $|| \cdot ||_\theta \rho$ for some $\theta \in \mathbb{R}$ and some $\rho \in \mathfrak{A}$, then $L_{\mathfrak{A}}(\chi, s)$ admits a holomorphic continuation to the line $\text{Re} s = 1$. If $\chi \in \mathfrak{A}$, then $L_{\mathfrak{A}}(\chi, s)$ admits a holomorphic continuation to the line $\text{Re} s = 1$, away from $s = 1$. Here we have
\[
L_{\mathfrak{A}}(\chi, s) = \frac{c_{\mathfrak{A}, \chi}}{(s - 1)^{1/|\mathfrak{A}|}} + O \left( \frac{1}{(s - 1)^{1/|\mathfrak{A}| - 1}} \right),
\]
as $s \to 1$, where $c_{\mathfrak{A}, \chi} \neq 0$.

**Proof.** To prove the expression, it suffices to compare the terms in the Euler products. For $\text{Re} s > 1/2 + \varepsilon$ and for almost all $v \in \text{Val}(F)$, the Euler factor at $v$ in the product of L-functions on the right hand side is
\[
\prod_{\rho \in \mathfrak{A}} \left( 1 - \frac{\rho_v(\pi_v)\chi_v(\pi_v)}{q_v^s} \right)^{-1} = 1 + \frac{\chi_v(\pi_v)}{q_v^s} \sum_{\rho \in \mathfrak{A}} \rho_v(\pi_v) + O(q_v^{-1-\varepsilon}).
\]
The first part therefore follows by (3.3). The stated analytic properties follow from Theorem 3.3. \qed

3.4. Delange’s Tauberian theorem. We now state the version of Delange’s Tauberian theorem which we shall need in the proof of Theorem 1.1. Delange’s result belongs to the realm of complex analysis, so we begin with a general statement in terms of Laplace transforms.

**Theorem 3.6** (Delange). Let $\alpha(x)$ be a positive non-decreasing real-valued function on $[0, \infty)$ such that the Laplace transform
\[
f(s) = \int_0^\infty e^{-sx} \alpha(x) \, dx,
\]
converges for $\Re s > 1$. Suppose that there exists some real number $\omega > 0$ and some $0 \leq \delta < 1$ such that the function $g(s) = f(s)(s - 1)^{-\omega}$ admits an extension to an infinitely differentiable function on the line $\Re s = 1$ with $g(1) \neq 0$ and that
\[
f(s) = \frac{g(1)}{(s - 1)^{-\omega}} + O\left(\frac{1}{(s - 1)^{-\omega - 1 + \delta}}\right),
\]
as $s \to 1$. Then
\[
\alpha(x) \sim \frac{g(1)}{\Gamma(\omega)} x^{\omega - 1}, \quad x \to \infty.
\]
\textbf{Proof.} This is a standard application of Delange’s Tauberian theorem [Del54, Thm. I]. The proof follows with minor modifications from the proof of [Del54, Thm. III] and the details are left the reader (Delange takes $\Psi(u) = 1$ in \textit{ibid.} and the exact same proof works in our case on taking $\Psi(u) = u^{-\delta}$.) \hfill $\square$

From this we obtain the following Tauberian theorem for Dirichlet series.

\textbf{Theorem 3.7.} Let $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ be a Dirichlet series with real non-negative coefficients which converges for $\Re s > 1$. Suppose that there exists some real number $\omega > 0$ and some $0 \leq \delta < 1$ such that the function $g(s) = f(s)(s - 1)^{-\omega}$ admits an extension to an infinitely differentiable function on the line $\Re s = 1$ with $g(1) \neq 0$ and that
\[
f(s) = \frac{g(1)}{(s - 1)^{-\omega}} + O\left(\frac{1}{(s - 1)^{-\omega - 1 + \delta}}\right),
\]
as $s \to 1$. Then
\[
\sum_{n \leq x} a_n \sim \frac{g(1)}{\Gamma(\omega)} x(\log x)^{\omega - 1}, \quad x \to \infty.
\]
\textbf{Proof.} This follows from Theorem 3.6 on taking $\alpha(x) = \sum_{n \leq x} a_n$. \hfill $\square$

4. Algebraic tori, toric varieties and their Brauer groups

In this section we gather various facts about algebraic tori and toric varieties over number fields. The main results of this section are a description of algebraic Brauer groups of algebraic tori over number fields and an analogue for subordinate Brauer group elements of a theorem of Voskresenskii [Vos69]. We finish by studying heights on toric varieties.

4.1. Algebraic tori over perfect fields. Let $k$ be a perfect field with fixed algebraic closure $\overline{k}$. Recall that an algebraic torus over $k$ is an algebraic group $T$ over $k$ such that $\overline{T} = T \times_k \overline{k}$ is isomorphic to $G^n$, for some $n \in \mathbb{N}$. We denote by $1 \in T(k)$ the identity element of $T$.

The category of algebraic tori is dual to the category of free $\mathbb{Z}$-modules with continuous $G_k$-action. This correspondence is given by associating to an algebraic torus $T$ its character group $X^*(T) = \text{Hom}(\overline{T}, G_m)$. Note that $X^*(T) = X^*(\overline{T})^{G_k}$ is the collection of characters of $\overline{T}$ which are defined over $k$. We denote by $X_*(T) = \text{Hom}(X^*(T), \mathbb{Z})$ the collection of cocharacters of $T$ and also let $X^*(T)_{\mathbb{R}} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X_*(T)_{\mathbb{R}} = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The splitting field of $T$ is defined to be the fixed field of the kernel of the representation $G_k \to GL(X^*(\overline{T}))$; it is the smallest Galois field extension of $k$ over which $T$ becomes isomorphic to $G^n_m$.

4.2. Algebraic tori over number fields. The standard references for this section are the papers [Ono61] and [Ono63]. Many of the facts presented here are the obvious generalisations of the case of $G_m$ studied in Tate’s thesis [CF10, Ch. XV].
4.2.1. The local points. Let $T$ be an algebraic torus of a number field $F$. For any place $v$ of $F$ we shall denote by $T(O_v)$ the maximal compact subgroup of $T(F_v)$. For non-archimedean $v$, we have a bilinear pairing

$$T(F_v) \times X^*(T_v) \rightarrow \mathbb{Z}, \quad (t, m) \mapsto \log |m(t)|_v \log q_v.$$ 

This pairing induces an exact sequence

$$0 \rightarrow T(O_v) \rightarrow T(F_v) \rightarrow X^*(T_v) \rightarrow 0. \quad (4.1)$$

The image of the map $T(F_v) \rightarrow X^*(T_v)$ has finite index, and moreover this map is surjective if $v$ is unramified in the splitting field of $T$. For archimedean $v$ we have a similar pairing

$$T(F_v) \times X^*(T_v) \rightarrow \mathbb{R}, \quad (t, m) \mapsto \log |m(t)|_v,$$

which induces a short exact sequence

$$0 \rightarrow T(O_v) \rightarrow T(F_v) \rightarrow X^*(T_v) \rightarrow 0. \quad (4.2)$$

The maps $T(F_v) \rightarrow X^*(T_v)$ and $T(F_v) \rightarrow X^*(T)$ admit canonical sections. The section to the first map is constructed in [Bou11, Lem. 2.18]. For the second map, it suffices to construct a canonical section of the map $X^*(T_v) \rightarrow X^*(T)$. This may be given by the dual of the map

$$X^*(T_v) \rightarrow X^*(T), \quad m \mapsto \frac{1}{|H|} \sum_{h \in H} m^{-1},$$

where $H$ is Galois group of the splitting field for $T$. We shall also denote by $T_\infty = \prod_{v \mid \infty} T_v$.

4.2.2. The adelic space. The local pairings give rise to an adelic pairing

$$T(A_F) \times X^*(T)_\mathbb{R} \rightarrow \mathbb{R}, \quad ((t_v), m) \mapsto \sum_{v \in \text{Val}(F)} \frac{\log |m(t)|_v}{\log q_v},$$

where we take $\log q_v = 1$ if $v$ is archimedean. If we denote by $T(A_F)^1$ the left kernel of this pairing, we have a short exact sequence

$$0 \rightarrow T(A_F)^1 \rightarrow T(A_F) \rightarrow X^*(T)_\mathbb{R} \rightarrow 0. \quad (4.3)$$

Under the diagonal embedding $T(F) \subset T(A_F)^1$ is a discrete and cocompact subgroup. The sequence (4.3) admits a splitting which is natural in the sense of category theory, though a choice still needs to be made. As explained in the previous section, for $v$ archimedean we may canonically identify $X^*(T)_\mathbb{R}$ with a subgroup of $T(F_v)$. We therefore choose the section given by

$$X^*(T)_\mathbb{R} \rightarrow T(A_F), \quad n \mapsto (n/r)_{v \mid \infty} \times (1)_{v \mid \infty},$$

where $r = \#\{v \in \text{Val}(F) : v \mid \infty\}$. It is clear that this choice gives a functorial isomorphism

$$T(A_F) \cong T(A_F)^1 \times X^*(T)_\mathbb{R}. \quad (4.4)$$
4.2.3. The Hasse principle and weak approximation. Let 

\[ \Omega(T) = \ker \left( H^1(F, T) \to \prod_{v \in \text{Val}(F)} H^1(F_v, T) \right). \]

Non-zero elements of \( \Omega(T) \) correspond to principal homogeneous spaces under \( T \) which fail the Hasse principle. This is finite and Sansuc \cite{San81} Prop. 8.3] constructed a canonical isomorphism

\[ \Omega(T) \cong B(T)^\sim, \tag{4.5} \]

where \( B(T) \) is given by \((2.7)\). Next let

\[ A(T) = T(A_F)/T(F)^w, \tag{4.6} \]

where \( T(F)^w \) denotes the closure of \( T(F) \) in \( T(A_F) \) with respect to the product topology. This group measures the failure of weak approximation for \( T \). This is finite and by a theorem of Voskresenski\i \cite{Vos69} (see also \cite{San81} Thm. 9.2) there is a short exact sequence

\[ 0 \to B(T) \to Br_{in} T/Br F \to A(T)^\sim \to 0. \tag{4.7} \]

Here we have used the isomorphisms \( Br_{in} T/Br F \cong H^1(F, \text{Pic} X) \cong Br_a X \), which hold for any non-singular proper model \( X \) of \( T \) (see e.g. \cite{San81} Lem. 6.3(iii)).

4.2.4. Characters. Given a place \( v \) of \( F \) and a character \( \chi_v \) of \( T(F_v) \), we shall say that \( \chi_v \) is unramified if it is trivial on \( T(\mathcal{O}_v) \). Any character \( \chi \) of \( T(A_F) \) may be decomposed as a product of local characters \( \chi_v \), almost all of which are unramified. We say that \( \chi \) is automorphic if it is trivial on \( T(F) \). Note that the automorphic characters of \( G_{in}(A_F) = A_F^* \) are exactly the Hecke characters of \( F \).

The inclusion \( X_*(T_\infty)_R \to T(A_F) \), determined by the canonical sections of the exact sequences \([4.2] \), yields a “type at infinity” map

\[ T(A_F)^w \to X^*(T_\infty)_R. \tag{4.8} \]

If we let \( K_T = \prod_{v \in \text{Val}(F)} T(\mathcal{O}_v) \subset T(A_F)^1 \), then the splitting \((4.4)\) induces a map

\[ (T(A_F)^1/T(F)K_T)^\sim \to X^*(T_\infty)_R. \]

This has finite kernel and moreover its image is a lattice whose codimension is rank \( X^*(T) \) \cite{Bon11} Lem. 4.52] (this may be viewed as a generalisation of Dirichlet’s unit theorem and the finiteness of the class number).

4.2.5. The Haar measure and the Tamagawa number. Let \( \omega \) be an invariant differential form on \( T \). By a classical construction (see e.g. \cite{CLT10} Sec. 2.1.7]), for each place \( v \) of \( F \) we obtain a measure (denoted \( |\omega|_v \)) on \( T(F_v) \), which is a Haar measure on \( T(F_v) \). If \( T = G_{in} \), then this measure is simply some multiple of the usual \( dt/|t|_v \). The product of these measures does not in general converge to give a measure on \( T(A_F) \), so we instead consider the measures

\[ \mu_v = c_v^{-1}|\omega|_v, \quad c_v = \begin{cases} L_v(X^*(T),1)^{-1}, & v \text{ non-archimedean}, \\ 1, & v \text{ archimedean}. \end{cases} \]

For almost all places \( v \) of \( F \) we have (see \cite{Ono61} Sec. 3.3])

\[ L_v(X^*(T),1) = \left( \int_{T(\mathcal{O}_v)} |\omega|_v \right)^{-1}. \]
hence $\mu_v(T(O_v)) = 1$ for almost all $v$. Therefore the product of the $\mu_v$ converges and we obtain a Haar measure $\mu$ on $T(A_F)$. This measure is independent of the choice of $\omega$ by the product formula (1.7). By the splitting (1.4) we obtain a Haar measure $\mu^1$ on $T(A_F)^1$, on equipping $X_s(T)_R$ with the unique Haar measure such that $X_s(T) \subset X_s(T)_R$ has covolume one. Ono’s works (Ono61 and Ono63) on Tamagawa numbers of algebraic tori imply that with respect to this measure we have

$$\text{vol}(T(A_F)^1/T(F)) = L^*(X^*(\overline{T}), 1) \cdot \frac{\text{Pic} T}{\text{III}(T)},$$

(4.9)

where $L^*(X^*(\overline{T}), 1)$ is as in Theorem 3.4.

4.3. Algebraic Brauer groups of algebraic tori.

4.3.1. Brauer groups over perfect fields. Let $T$ be an algebraic torus over a perfect field $k$. In this paper, we shall be particularly interested in the group

$$\text{Br}_e T = \{ b \in \text{Br}_1 T : b(1) = 0 \}.$$  

(4.10)

Note that we have a canonical isomorphism $\text{Br}_1 T \cong \text{Br}_0 T \bigoplus \text{Br}_e T$ and the homomorphism

$$\text{Br}_e T \to \text{Br}_a T, \quad b \mapsto b + \text{Br}_0 T$$

is an isomorphism. The following result is well-known.

**Lemma 4.1.** There are natural isomorphisms

$$\text{Pic} T \cong H^1(k, X^*(\overline{T})), \quad \text{Br}_e T \cong H^2(k, X^*(\overline{T})).$$

In particular Pic $T$ is finite.

**Proof.** This is essentially [San81, Lem. 6.9(ii)] (see also [San81, Lem. 6.3(ii)]). Sansuc uses the Hochshild-Serre spectral sequence to show the first isomorphism and also that $\text{Br}_a T \cong H^2(k, X^*(\overline{T}))$. This second isomorphism is given by choosing a rational point; taking the rational point to be the identity yields the result. □

Applying this lemma to $\mathbb{G}_m$ and using the isomorphisms $H^2(k, \mathbb{Z}) \cong H^1(k, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$ we obtain

$$\text{Br}_e \mathbb{G}_m \cong \text{Hom}(G_k, \mathbb{Q}/\mathbb{Z}).$$

(4.11)

This isomorphism can be made very explicit. Namely, elements of $\text{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$ correspond to cyclic extensions $K$ of $k$ together with a choice of generator of $\text{Gal}(K/k)$. Given an element $\chi \in \text{Hom}(G_k, \mathbb{Q}/\mathbb{Z})$, the corresponding Brauer group element may be represented by the cyclic algebra $(\chi, x)$, where $x$ is the usual coordinate function on $\mathbb{G}_m$. In particular $\text{Br}_e \mathbb{G}_m$ will be very large in general. Our interest in the group $\text{Br}_e T$ stems from the following result of Sansuc, which is one of the key reasons why we focus on algebraic Brauer group elements only in Theorem 1.1.

**Lemma 4.2.** The pairing

$$\text{Br}_e T \times T(k) \to \text{Br} k, \quad (b, t) \mapsto b(t),$$

is bilinear. In particular given $b \in \text{Br}_e X$, the map

$$T(k) \to \text{Br} k, \quad t \mapsto b(t),$$

is a homomorphism.

**Proof.** This is proven in the more general case of connected linear algebraic groups in [San81], Lem. 6.9. □
Note that applying Lemma 4.2 to \( G_m \) and using (4.11), we recover the familiar bilinearity property \([\chi, t_1 t_2] = [\chi, t_1][\chi, t_2]\) of cyclic algebras, where \( t_1, t_2 \in k^* \) and \( \chi \in \text{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \). Sansuc proved the result in Lemma 4.2 by showing that the functor \( \text{Br} \) is additive on the category of connected linear algebraic groups. For tori, it is possible to give another proof. Namely, as noted by Sansuc at the top of page 65 of [San81], we have a commutative diagram

\[
\begin{array}{ccc}
\text{Br}_e T & \times & T(k) \\
\downarrow & & \downarrow \\
H^2(k, X^*(\overline{T})) & \times & H^0(k, T(\overline{k})) \rightarrow & H^2(k, k^*)
\end{array}
\]  

(4.12) where here \( \rightarrow \) denotes the cup product and the map \( \text{Br}_e T \rightarrow H^2(k, X^*(\overline{T})) \) is the natural isomorphism given by Lemma 4.1. Since the cup product is bilinear, this implies the bilinearity of the pairing \( \text{Br}_e T \times T(k) \rightarrow \text{Br}_k \).

### 4.3.2. Brauer groups over number fields.

We now give a description of the algebraic Brauer groups of tori over number fields and their completions, via the Brauer pairing (2.1). Throughout this section \( T \) is an algebraic torus over a number field \( F \).

**Theorem 4.3.** For any place \( v \) of \( F \), the bilinear pairing

\[ \text{Br}_e T_v \times T(F_v) \rightarrow \text{Br} F_v \subset \mathbb{Q}/\mathbb{Z}, \]

is perfect, i.e. the induced map

\[ \text{Br}_e T_v \rightarrow \text{Hom}(T(F_v), \mathbb{Q}/\mathbb{Z}), \]

is an isomorphism of abelian groups.

**Proof.** As we may identify the above pairing with a cup product (4.12), this follows from local Tate duality (see [NSW00, Thm. 7.2.9] or [Mil06, Cor. 2.4] for the non-archimedean case and [Mil06, Thm. 2.13] for the archimedean case). \( \square \)

This theorem may be viewed as a generalisation of the main theorem of local class field theory; indeed taking \( T = G_m \) and using the isomorphism (4.11), we obtain an isomorphism between \( \text{Hom}(F_v^*, \mathbb{Q}/\mathbb{Z}) \) and \( \text{Hom}(G_{F_v}, \mathbb{Q}/\mathbb{Z}) \). The following theorem is the global analogue of Theorem 4.3.

**Theorem 4.4.** The pairing

\[ \text{Br}_e T \times T(\mathbb{A}_F) \rightarrow \mathbb{Q}/\mathbb{Z}, \]

is bilinear and induces a short exact sequence

\[ 0 \rightarrow \text{B}(T) \rightarrow \text{Br}_e T \rightarrow \text{Hom}(T(\mathbb{A}_F)/T(F), \mathbb{Q}/\mathbb{Z}) \rightarrow 0, \]

where \( \text{B}(T) \) is given by (2.7).

**Proof.** Bilinearity follows from Lemma 4.2 applied to the local pairings. To derive the exact sequence, we shall use Nakayama duality (see [NSW00, Thm. 8.4.1] or [Mil06, Cor. 4.7]). Let

\[ C_\mathbb{P} = \lim_{\substack{\text{F \subset E} \\
\rightarrow}} A^*_E/E^*, \quad T(C_\mathbb{P}) = \text{Hom}(X^*(\overline{T}), C_\mathbb{P}). \]

Then Nakayama duality implies that the pairing

\[ H^2(F, X^*(\overline{T})) \times H^0(F, T(C_\mathbb{P})) \rightarrow \mathbb{Q}/\mathbb{Z}, \]
given by composing the cup product with the natural surjection $H^2(F, C_T) \to \mathbb{Q}/\mathbb{Z}$, is perfect. Now, as the natural inclusion $A^*_F/F^* \subset C_T$ has Galois invariant image, we deduce from (4.12) a commutative diagram

$$
\begin{array}{c}
\Br_e T \times T(A_F)/T(F) \to \mathbb{Q}/\mathbb{Z} \\
H^2(F, X^*(T)) \times H^0(F, T(C_T)) \to \mathbb{Q}/\mathbb{Z}
\end{array}
$$

where $\Br_e T \to H^2(F, X^*(T))$ is the natural isomorphism given by Lemma 4.1. In particular, this implies that we have a commutative diagram

$$
\begin{array}{c}
\alpha \\
\Br_e T \\
\gamma
\end{array}
\begin{array}{c}
(T(A_F)/T(F))' \\
\beta \\
H^0(F, T(C_T))'
\end{array}
\quad (4.13)
$$

Since $\mathbb{Q}/\mathbb{Z}$ is divisible, we deduce from [Bou11, Sec. 2.3] the following short exact sequence

$$
0 \to \Sha(T) \to H^0(F, T(C_T))' \to \beta \to (T(A_F)/T(F))' \to 0. \quad (4.14)
$$

By Nakayama duality $\gamma$ is an isomorphism. Since $\beta$ is surjective by (4.14), we see from (4.13) that $\alpha$ is surjective. As $\gamma$ is an isomorphism, we deduce from (4.13) that $\ker \alpha$ is finite and $|\ker \alpha| = |\Sha(T)|$. However as clearly $\Br(T) \subset \ker \alpha$ and $|\Sha(T)| = |\Br(T)|$ by (4.5), we deduce the result. \hfill $\square$

For rational tori one can say more.

**Corollary 4.5.** Let $T$ be a rational torus over a number field $F$. Then $\Br(T) = 0$. In particular the pairing

$$
\Br_e T \times (T(A_F)/T(F)) \to \mathbb{Q}/\mathbb{Z},
$$

is perfect.

**Proof.** It follows from [San81, Lem. 6.1] that for any non-singular variety $U$ which is birational to $T$ over $F$, we have $\Br(T) \cong \Br(U)$. Since $T$ is rational, it suffices to show that $\Br(P^n) = 0$ for any $n \in \mathbb{N}$. This follows from the fact that $\Br(P^n) \cong \Br F$ and that $\Br(F) = 0$ by (2.6). The second part follows immediately from Theorem 4.4. \hfill $\square$

Note that one may similarly recover the main theorem of global class field theory, on applying Corollary 4.5 to the torus $\mathbb{G}_m$ and using the isomorphism (4.11).

### 4.4. Toric varieties over number fields.

Given an algebraic torus $T$ over a perfect field $k$, a toric variety for $T$ is a non-singular projective variety $X$ with an action of $T$ which has an open dense orbit which contains a rational point. In particular all our toric varieties will have a Zariski dense set of rational points. Note that our terminology differs from that of other authors, who sometimes allow toric varieties to be non-projective and singular (see e.g. the book [CLS11]). In this paper we avoid the use of fans and instead use the approach to toric varieties adopted in the paper [CLT13]. The complement of the open dense orbit is a divisor, whose irreducible components we call the boundary components of $X$. Much of the
geometry and the arithmetic of toric varieties are controlled by the boundary components, as we now explain.

4.4.1. The boundary components. We now fix a toric variety $X$ for an algebraic torus $T$ over a number field $F$. We let $\mathcal{A}$ denote the set of boundary components of $X$ and similarly define $\mathcal{B}$ (resp. $\mathcal{C}$) to be the set of boundary components of $X$ (resp. $X_v$ for a place $v$ of $F$). Given $\alpha \in \mathcal{A}$, we let $D_\alpha$ denote the corresponding irreducible divisor in $X$, we define $F_\alpha = \overline{F} \cap F(D_\alpha) \subset \overline{F}(D_\alpha)$ and let $f_\alpha = [F_\alpha : F]$. Note that by Lemma 2.2, we have $f_\alpha = 1$ if and only if $D_\alpha$ is geometrically irreducible. For $\overline{\alpha} \in \mathcal{B}$ and $\alpha_v \in \mathcal{C}_v$, we define $D_{\overline{\alpha}}$, $D_{\alpha_v}$, $F_{\alpha_v}$ and $f_{\alpha_v}$ similarly. Given a place $v$ of $F$, we say that an element $\alpha_v \in \mathcal{C}_v$ divides an element $\alpha \in \mathcal{A}$ (written $\alpha_v | \alpha$) if $D_{\alpha_v} \subset D_{\alpha}$. Note that for $v$ non-archimedean and $\alpha \in \mathcal{A}$, there is a bijective correspondence between those $\alpha_v \in \mathcal{C}_v$ such that $\alpha_v | \alpha$ and those places $w$ of $F_\alpha$ such that $w | v$. By [CLS11] Thm. 8.2.3], there is an isomorphism
\[
\omega_X \cong \bigotimes_{\alpha \in \mathcal{A}} \mathcal{O}_X(-D_\alpha),
\] (4.15)
where $\omega_X$ denotes the canonical bundle of $X$.

4.4.2. The Picard group and the Brauer group. Associated to each toric variety $X$, we have the following fundamental short exact sequence of Galois modules (see e.g. [CLS11] Thm. 4.2.1]),
\[
0 \to X^*(\overline{T}) \to \mathbb{Z}^{\mathcal{A}} \to \text{Pic } X \to 0,
\] (4.16)
where here $\mathbb{Z}^{\mathcal{A}}$ denotes the free abelian group generated by the elements of $\mathcal{A}$. The first map associates to a character of $\overline{T}$ its divisor (viewing the character as a rational function on $X$), and the second map is simply given by sending a divisor to its divisor class. The significance of this sequence is that it allows us to relate the Galois modules $X^*(\overline{T})$ and $\text{Pic } X$, which could potentially be quite complicated, to the Galois module $\mathbb{Z}^{\mathcal{A}}$, which is simply a permutation module. By duality of tori, we deduce from (4.16) the following short exact sequence
\[
0 \to T_{NS} \to \prod_{\alpha \in \mathcal{A}} T_{\alpha} \to T \to 0.
\] (4.17)
Here $T_{NS}$ denotes the Néron-Severi torus of $X$ and $T_\alpha = R_{F_\alpha/F} \mathbb{G}_m$ is the Weil restriction of $\mathbb{G}_m$ with respect to $F \subset F_\alpha$. Given an element $b \in \text{Br } T$, we may pull it back to obtain an element $b_\alpha \in \text{Br } T_\alpha$. Also, for a character $\chi \in T(A_F)^{\wedge}$ we denote by $\chi_\alpha$ its image in $T_\alpha(A_F)^{\wedge}$. If $\chi$ is automorphic, then we will often identify $\chi_\alpha$ with a Hecke character of $F_\alpha$ via the canonical isomorphism $T_\alpha(A_F) = A_{F_\alpha}$, given by the definition of the Weil restriction. Applying Galois cohomology to (4.17) we obtain a long exact sequence, the first part of which reads
\[
0 \to X^*(T) \to \mathbb{Z}^{\mathcal{A}} \to \text{Pic } X \to \text{Pic } T \to 0.
\] (4.18)
Here we have used the isomorphisms
\[
\text{Pic } X \cong (\text{Pic } X)^{G_F}, \quad H^1(F, X^*(\overline{T})) \cong \text{Pic } T, \quad H^1(F, \mathbb{Z}^{\mathcal{A}}) = 0.
\]
This first isomorphism follows from the fact that $X(F) \neq \emptyset$ (see e.g. [CN98] Cor. 1.3]), whilst the second isomorphism is proved in Lemma 4.1 and the third isomorphism follows on combing Shapiro’s lemma (see e.g. [KMRT98] Lem. 29.6]) with $H^1(F_\alpha, \mathbb{Z}) = 0$. Next fix an equivariant embedding $T \subset X$ and define
\[
\text{Br}_e X = \{ b \in \text{Br}_1 X : b(1) = 0 \}.
\]
Then the long exact sequence associated to (4.16) continues as
\[ 0 \rightarrow \text{Br}_e X \rightarrow \text{Br}_e T \rightarrow \bigoplus_{\alpha \in \mathcal{A}} \text{Br}_e T_\alpha, \quad (4.19) \]
where here we have used the isomorphism \( \text{Br}_e T \cong H^2(F, X^*(T)) \) of Lemma 4.1 and the isomorphism \( \text{Br}_e X \cong \text{Br}_a X \cong H^1(F, \text{Pic } X) \) of [San81, Lem. 6.3(iii)].

4.4.3. Weak approximation. A theorem of Voskresenski˘ı [Vos70] (see also [Bou11, Prop. 2.34]) implies that the sequence (4.17) gives rise to an exact sequence
\[ \prod_{\alpha \in \mathcal{A}} T_\alpha(A_F)/T_\alpha(F) \rightarrow T(A_F)/T(F) \rightarrow A(T) \rightarrow 0, \quad (4.20) \]
where \( A(T) \) is given by (4.6). From this we obtain the following exact sequence
\[ 0 \rightarrow A(T) \rightarrow (T(A_F)/T(F))^\wedge \rightarrow \prod_{\alpha \in \mathcal{A}} (T_\alpha(A_F)/T_\alpha(F))^\wedge. \quad (4.21) \]

4.4.4. Purity. The Grothendieck purity sequence (2.5) for the algebraic Brauer group of \( T \) reads
\[ 0 \rightarrow \text{Br}_1 X \rightarrow \text{Br}_1 T \rightarrow \bigoplus_{\alpha \in \mathcal{A}} H^1(F_\alpha, \mathbb{Q}/\mathbb{Z}). \quad (4.22) \]

Note that the sequences (4.19) and (4.21) formally resemble (4.22). The following lemma shows that these sequences are indeed compatible.

**Lemma 4.6.** The Brauer pairing yields a commutative diagram
\[ \begin{array}{cccc}
0 & \rightarrow & \mathcal{B}(T) & \rightarrow & \text{Br}_e T & \rightarrow & (T(A_F)/T(F))^\wedge & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \bigoplus_{\alpha \in \mathcal{A}} \text{Br}_e T_\alpha & \rightarrow & \prod_{\alpha \in \mathcal{A}} (T_\alpha(A_F)/T_\alpha(F))^\wedge & \rightarrow & 0
\end{array} \]
with exact rows. Moreover, the diagram
\[ \begin{array}{ccc}
\text{Br}_e T & \rightarrow & \bigoplus_{\alpha \in \mathcal{A}} \text{Br}_e T_\alpha \\
\downarrow & & \downarrow \\
\text{Br}_1 T & \rightarrow & \bigoplus_{\alpha \in \mathcal{A}} H^1(F_\alpha, \mathbb{Q}/\mathbb{Z})
\end{array} \]
commutes up to sign (the maps are explained in the proof).

**Proof.** The first commutative diagram follows from the functoriality of the Brauer pairing and Theorem 4.1 (note that \( \mathcal{B}(T_\alpha) = 0 \) by Corollary 4.5). In the second diagram, the top and bottom rows come from the exact sequences (4.19) and (4.22) respectively. The maps between them are the obvious inclusion combined with the isomorphisms \( \text{Br}_e T_\alpha \cong H^2(F, X^*(T_\alpha)) \) of Lemma 4.1 and \( H^2(F, X^*(T_\alpha)) \cong H^1(F_\alpha, \mathbb{Q}/\mathbb{Z}) \) given by Shapiro’s lemma and the definition of \( T_\alpha \). The fact that this diagram commutes up to sign is proven in [San81, Lem. 9.1]. \( \square \)
4.5. **Subordinate Brauer group elements for algebraic tori.** Let $T$ be an algebraic torus over a number field $F$ and let $X$ be a toric variety for $T$ together with a choice of equivariant embedding $T \subset X$. Then the exact sequence (4.7) beautifully encapsulates the fact that the Brauer-Manin obstruction is the only obstruction to weak approximation for $T$. Indeed we have $\text{Br}_T T / \text{Br} F \cong \text{Br}_e X$, and so we obtain
\[
\overline{T(F)}^{\mu} = \bigcap_{\chi \in \text{Br}_e X / B(T)} \ker \chi,
\]
where we have identified the elements of $\text{Br}_e X / B(T)$ with characters of $T(A_F)/T(F)$ via Theorem [1.4]. In this section we obtain an analogue of this result for subordinate Brauer group elements (see Section 2.6 for definitions). First we show that under suitable conditions, Brauer group elements which are subordinate to algebraic Brauer group elements are themselves algebraic.

**Lemma 4.7.** Let $V$ be a non-singular proper variety over a field $k$ of characteristic zero and let $U \subset V$ be an open subset. Let $\mathcal{B} \subset \text{Br}_1 U$ be a finite subset and suppose that $\text{Br} V = 0$. Then $\text{Sub}(V, \mathcal{B}) \subset \text{Br}_1 U$.

**Proof.** Let $b \in \text{Sub}(V, \mathcal{B})$. As $\mathcal{B} \subset \text{Br}_1 U$, the residues of each element of $\mathcal{B}$ become trivial over $k$, so the same holds for $\overline{b} = b \otimes_k k$. Hence by purity $\overline{b} \in \text{Br} V = 0$ and so $b \in \text{Br}_1 U$, as required. \hfill \Box

Fix a finite subset $\mathcal{B} \subset \text{Br}_e T$. By Lemma [4.7], we know that $\text{Sub}(X, \mathcal{B}) \subset \text{Br}_1 T$. We therefore let $\text{Sub}_e(X, \mathcal{B}) = \text{Br}_e T \cap \text{Sub}(X, \mathcal{B})$ and define $\text{Sub}_e(F(T), \mathcal{B})$ similarly.

**Theorem 4.8.** Suppose that $T(F)_{\mathcal{B}} \neq \emptyset$ and let
\[
A(T, \mathcal{B}) = T(A_F)_{\mathcal{B}} / \overline{T(F)}^{\mu}_{\mathcal{B}},
\]
where $\overline{T(F)}^{\mu}_{\mathcal{B}}$ denotes the closure of $T(F)_{\mathcal{B}}$ in $T(A_F)_{\mathcal{B}}$ with respect to the product topology. Then $A(T, \mathcal{B})$ is finite and there is a short exact sequence
\[
0 \to \{B(T), \mathcal{B}\} \to \text{Sub}_e(F(T), \mathcal{B}) \to A(T, \mathcal{B}) \wedge \to 0,
\]
where $B(T)$ is given by (2.7).

**Proof.** This theorem is proved using the tools developed during the proof of Theorem 1.3. Recall that in Section 2.5 we constructed a rational torus $S$ and a $T$-torsor $\pi : W \to T$ under $S$ such that
\[
\pi(W(F)) = T(F)_{\mathcal{B}} \quad \text{and} \quad \pi(W(A_F)) = T(A_F)_{\mathcal{B}},
\]
(4.23)
(these equalities follow from Lemma [2.5]). As $T(F)_{\mathcal{B}} \neq \emptyset$, on translating if necessary we may assume that $1 \in T(F)_{\mathcal{B}}$. Therefore as in the proof of Lemma [2.10] it follows from [CT08] Thm. 5.6 that we may equip $W$ with the structure of an algebraic torus such that $\pi$ induces a short exact sequence of algebraic tori
\[
0 \to S \to W \to T \to 0.
\]
(4.24)
First we show that $A(T, \mathcal{B})$ is finite. The finiteness of $A(W)$ implies that $\overline{W(F)}^{\mu}_{\mathcal{B}} \subset W(A_F)$ is a closed subgroup of finite index. It follows from (4.23) that $\pi(\overline{W(F)}^{\mu}_{\mathcal{B}}) \subset T(A_F)_{\mathcal{B}}$ is also a closed subgroup of finite index and hence $\pi(\overline{W(F)}^{\mu}_{\mathcal{B}}) = T(F)_{\mathcal{B}}^{\mu}$. The sequence (4.24)
therefore gives the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & S(F)^w \\
\downarrow & & \downarrow \\
0 & \rightarrow & W(F)^w \\
\downarrow & & \downarrow \\
0 & \rightarrow & T(F)_{\mathcal{B}}^w \\
\downarrow & & \downarrow \\
0 & \rightarrow & S(A_F) \\
\downarrow & & \downarrow \\
W(A_F) & \rightarrow & T(A_F)_{\mathcal{B}} \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

with exact rows and columns. As \(S\) is rational, the classical weak approximation theorem (see e.g. \cite[Thm. 5.1.2]{Sk01}) implies that \(S(F)^w = S(A_F)\) and therefore the snake lemma gives an isomorphism \(A(W) \cong A(T, \mathcal{B})\). This shows that \(A(T, \mathcal{B})\) is finite. Next we show that the map \(B(T) \rightarrow B(W)\) is surjective. The exact sequence (4.24) yields an exact sequence of finite abelian groups

\[\Pi(S) \rightarrow \Pi(W) \rightarrow \Pi(T),\]

which by the duality (4.5) yields an exact sequence

\[B(T) \rightarrow B(W) \rightarrow B(S).\]

The required surjection follows on noting that since \(S\) is rational, we have \(B(S) = 0\) by Corollary 4.6. Next by Lemma 2.9 we know that \(W\) is stably birational to the product of Brauer-Severi schemes associated to \(\mathcal{B}\), and hence by Lemma 2.11 and Theorem 2.15 we know how to calculate \(Br_{nr} W\) using subordinate Brauer group elements. Putting all these facts together and applying (4.7) to \(W\), we obtain the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & B(W) \\
\downarrow & & \downarrow \\
0 & \rightarrow & Br_{nr} W/ Br F \\
\downarrow & & \downarrow \\
B(T) & \rightarrow & Sub_{c}(F(T), \mathcal{B}) \\
\downarrow & & \downarrow \\
(\mathcal{B}) & \rightarrow & A(T, \mathcal{B})^\vee \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

with exact rows and column. Note that \(B(T) \subset Sub_{c}(F(T), \mathcal{B})\) by Theorem 2.17. A simple chase through diagram (4.20) shows that \(ker(Sub_{c}(F(T), \mathcal{B}) \rightarrow A(T, \mathcal{B})^\vee) = (B(T), \mathcal{B})\), which completes the proof of the theorem.

\[\square\]

Note that we recover (4.7) from Theorem 4.8 on taking \(\mathcal{B} = 0\). The next result gives a concrete description of \(T(F)_{\mathcal{B}}\).

**Corollary 4.9.** We have

\[T(A_F)_{Sub_{c}(F(T), \mathcal{B})} = T(F)_{\mathcal{B}}^w.\] (4.26)
Moreover, there exists a finite set of places $S$ of $F$ such that
\[ \overline{T(F)}_{\mathcal{B}} = \left( \overline{T(F)}_{\mathcal{B}} \cap \prod_{v \in S} T(F_v)_{\mathcal{B}} \right) \times T(A_F)^S_{\mathcal{B}}, \]

where we write $T(A_F)^S_{\mathcal{B}} = T(A_F)_{\mathcal{B}} \cap \prod_{v \in S} T(F_v)_{\mathcal{B}}$.

\begin{proof}
First we prove (4.26). It follows from Theorem 2.17 and the fact that $\text{Sub}_e(F(T), \mathcal{B})$ is finite (see Lemma 2.18) that $T(A_F)^{\text{Sub}_e(F(T), \mathcal{B})}_{\mathcal{B}} \subset T(A_F)_{\mathcal{B}}$ is closed in the product topology, in particular $\overline{T(F)}_{\mathcal{B}} \subset T(A_F)^{\text{Sub}_e(F(T), \mathcal{B})}_{\mathcal{B}}$. If $T(A_F)^{\text{Sub}_e(F(T), \mathcal{B})}_{\mathcal{B}} = \emptyset$ then (4.26) clearly holds, so assume that $T(A_F)^{\text{Sub}_e(F(T), \mathcal{B})}_{\mathcal{B}} \neq \emptyset$. We shall now show that $T(F)_{\mathcal{B}} \neq \emptyset$. Let $\pi : V \to T$ be the product of Brauer-Severi schemes over $T$ corresponding to the elements of $\mathcal{B}$. By Theorem 4.13 we know that
\[ V(F)_{\mathcal{B}} = V(A_F)^{\text{Br}}_{\mathcal{B}}, \]

Moreover, by Theorem 2.15 we also have
\[ \pi(V(A_F)^{\text{Br}}_{\mathcal{B}}) = T(A_F)^{\text{Sub}_e(F(T), \mathcal{B})}_{\mathcal{B}}. \]

It follows that $V(A_F)^{\text{Br}}_{\mathcal{B}} \neq \emptyset$, hence $V(F) \neq \emptyset$ and so $T(F)_{\mathcal{B}} \neq \emptyset$ as required. We may now apply Theorem 4.13 to deduce (4.26). The second part of the lemma follows from (4.26) and Theorem 2.17.
\end{proof}

Note that in the classical case where $\mathcal{B} = 0$, the group $A(T)$ also plays another rôle in the theory (4.21), namely its dual may be written as
\[ A(T)^{\vee} = \{ \chi \in (T(A_F)/T(F))^{\vee} : \chi_{\alpha} = 0 \text{ for all } \alpha \in \mathcal{A} \}, \]

(here $\mathcal{A}$ denotes the set of boundary components of $X$, see Section 4.4). This description was crucial in [BT95] when calculating the leading constant in the asymptotic formula (see [BT95] Thm. 3.4.6]). For subordinate Brauer group elements, a different group from $A(T, \mathcal{B})$ plays this rôle. To state the result, let $\mathcal{R} \subset (T(A_F)/T(F))^{\vee}$ denote the collection of characters obtained from $\mathcal{B}$ via Theorem 4.13. Let $\mathcal{R}_\alpha \subset \text{Br}_e T_\alpha$ and $\mathcal{R}_\alpha \subset (T_\alpha(A_F)/T_\alpha(F))^{\vee}$ for $\alpha \in \mathcal{A}$ denote the corresponding elements obtained via pull-back in the sequence (4.17).

**Theorem 4.10.** Let
\[ C(T, \mathcal{R}) = \{ \chi \in (T(A_F)/T(F))^{\vee} : \chi_{\alpha} \in \mathcal{R}_\alpha \text{ for all } \alpha \in \mathcal{A} \}. \]

Then $C(T, \mathcal{R})$ is finite and there is a short exact sequence
\[ 0 \to \mathcal{B}(T) \to \text{Sub}_e(X, \mathcal{B}) \to C(T, \mathcal{R}) \to 0, \]

where $\mathcal{B}(T)$ is given by (2.7).

\begin{proof}
First we show that every element of $C(T, \mathcal{R})$ has finite order. Let
\[ \mathcal{T} = \ker \left( \prod_{\alpha \in \mathcal{A}} T_\alpha(A_F)/T_\alpha(F) \to T(A_F)/T(F) \right), \]

where $\mathcal{B}(T)$ is defined by (2.7).
\end{proof}
where here the map is the one induced by (4.17). We obtain a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
(T(A_F)/T(F))^\sim & \longrightarrow & \bigoplus_{\alpha \in \mathcal{A}} (T_A(A_F)/T_A(F))^\sim \\
\downarrow & & \downarrow \\
(T(A_F)/T(F))^\wedge & \longrightarrow & \bigoplus_{\alpha \in \mathcal{A}} (T_A(A_F)/T_A(F))^\wedge \\
\downarrow & & \downarrow \\
\overline{T} & \longrightarrow & \overline{T}
\end{array}
\]

(4.27)

with exact rows and columns. Let \( \chi \in C(T, \mathcal{R}) \). Then by definition \((\chi_\alpha)_{\alpha \in \mathcal{A}}\) has finite order. The exactness and the commutativity of the diagram (4.27) implies that \((\chi_\alpha)_{\alpha \in \mathcal{A}}\) has trivial image in \( \overline{T} \) and hence comes from an element of \((T(A_F)/T(F))^\sim\), i.e. \( \chi \) has finite order.

Next, by Theorem 4.4 we know that \( C(T, \mathcal{R}) \) lies in the image of the map \( \text{Br}_e T \rightarrow (T(A_F)/T(F))^\sim \). I claim that this map restricts to a surjection \( \text{Sub}_e(X, \mathcal{R}) \rightarrow C(T, \mathcal{R}) \). Let \( b \in \text{Br}_e T \) with corresponding character \( \chi \in (T(A_F)/T(F))^\sim \). Then Lemma 4.6 implies that

\[
b \in \text{Sub}_e(X, \mathcal{R}) \iff \partial D_\alpha(b) \in \langle \partial D_\alpha(\mathcal{R}) \rangle \text{ for all } \alpha \in \mathcal{A},
\]

\[
\iff b_\alpha \in \mathcal{R}_\alpha \text{ for all } \alpha \in \mathcal{A},
\]

\[
\iff \chi_\alpha \in \mathcal{R}_\alpha \text{ for all } \alpha \in \mathcal{A},
\]

\[
\iff \chi \in C(T, \mathcal{R}),
\]

thus proving the claim. The finiteness of \( C(T, \mathcal{R}) \) therefore follows from the finiteness of \( \text{Sub}_e(X, \mathcal{R}) \) (see Lemma 2.18). The proof is complete on noting that Theorem 4.4 gives the equality \( \text{ker}(\text{Sub}_e(X, \mathcal{R}) \rightarrow C(T, \mathcal{R})) = B(T) \). \( \square \)

Note that again on taking \( \mathcal{R} = 0 \) in Theorem 4.10 and using (4.21), we recover (4.7).

4.6. Heights. Let \( X \) be a projective variety over a number field \( F \) and let \( L \) a line bundle on \( X \). For a place \( v \in \text{Val}(F) \), a \( v \)-adic metric \( || \cdot ||_v \) on \( L \) is a continuously varying family of \( v \)-adic norms on the fibres of \( L \). An adelic metric on \( L \) is a collection \( || \cdot || = (|| \cdot ||_v) \) of \( v \)-adic metrics on \( L \) for each place \( v \in \text{Val}(F) \), such that almost all of these \( v \)-adic metrics are defined by a fixed model of \( X \) over \( \mathcal{O}_F \) (see e.g. [CLT10]). We call the data \( \mathcal{L} = (L, || \cdot ||) \) an adelically metrised line bundle. Given a rational point \( x \in X(F) \), we define the height of \( x \) with respect to \( \mathcal{L} \) to be

\[
H_{\mathcal{L}}(x) = \prod_{v \in \text{Val}(F)} ||s(x)||_v^{-1},
\]

where \( s \) is any local section of \( L \) such that \( s(x) \neq 0 \). The product formula (4.7) implies that this definition is independent of the choice of \( s \).

4.6.1. Heights on toric varieties. Now suppose that \( X \) is a toric variety with respect to an algebraic torus \( T \) with a fixed equivariant embedding \( T \subset X \). We shall extend the previous construction in several ways; namely we want to be able to define the “height” of adelic point and we also want to define a system of “complex” height functions on every line bundle of \( T \) in a compatible manner. To do this we choose adelic metrics \( || \cdot ||_\alpha \) on the line bundles \( \mathcal{O}_X(D_\alpha) \) for each \( \alpha \in \mathcal{A} \) (see Section 4.4). For each \( \alpha \in \mathcal{A} \), let \( d_\alpha \) denote the global section
of $\mathcal{O}_X(D_\alpha)$ corresponding to the divisor $D_\alpha$. We then define the following local height pairing

$$H_v : T(F_v) \times \mathbb{C}^\alpha \to \mathbb{C}^*$$

$$H_v : (t_v; s) \mapsto \prod_{\alpha \in \mathcal{A}} ||d_\alpha(t_v)||^{-s_\alpha}_{v, \alpha},$$

and also the following global height pairing

$$H : T(\mathbb{A}_F) \times \mathbb{C}^\mathcal{A} \to \mathbb{C}^*$$

$$H : ((t_v); s) \mapsto \prod_{v \in \text{Val}(F)} H_v((t_v); s).$$

Here we are writing $s = (s_\alpha)_{\alpha \in \mathcal{A}}$. As an example, note that it follows from (4.15) that the adelic metrics on $\mathcal{O}_X(D_\alpha)$ induce an adelic metric on $\omega_X^{-1}$, and on taking $s_\alpha = s$ for each $\alpha \in \mathcal{A}$ we have

$$H(t; (s)_{\alpha \in \mathcal{A}}) = H_{\omega_X^{-1}}(t)^s,$$

for each $t \in T(F)$. In their original paper, Batyrev and Tschinkel [BT95, Def. 2.1.5] constructed "canonical" adelic metrics on the line bundles $\mathcal{O}_X(D_\alpha)$. In this paper we shall focus on these adelic metrics, as it greatly simplifies the harmonic analysis in the proof of Theorem 1.1. The key property of these metrics which we shall use is that the corresponding local height functions are $T(\mathcal{O}_v)$-invariant and trivial on $T(\mathcal{O}_v)$ for any place $v$ of $F$ [BT95, Thm. 2.16].

5. Counting functions associated to Brauer group elements

5.1. The set-up. We now begin the proof of Theorem 1.1. Throughout this section $X$ is a toric variety over a number field $F$ with respect to an algebraic torus $T$, with set of boundary components $\mathcal{A}$ (see Section 4.4 for our use of notation for toric varieties). We only specialise to the case where $T$ is anisotropic just before we apply the Poisson summation formula in Section 5.4.1. We fix a finite subgroup $\mathcal{B} \subset \text{Br}_1 U$ of the open dense orbit $U \subset X$ such that $U(F)_\mathcal{B} \neq \emptyset$. We fix a choice of equivariant embedding $T \subset X$ such that $1 \in U(F)_\mathcal{B}$. In particular we may identify $\mathcal{B}$ with a finite subgroup of $\text{Br}_x T$ (see (4.10)). We also equip the line bundles $\mathcal{O}_X(D_\alpha)$ with the canonical Batyrev-Tschinkel adelic metrics (see Section 4.6).

Let $\rho : T(F) \to \{0, 1\}$ denote the indicator function for the zero-locus $T(F)_\mathcal{B}$ of $\mathcal{B}$. By Lemma 2.4, this extends to a locally constant function (also denoted $\rho$) on $T(\mathbb{A}_F)$, which is given as a product of local indicator functions $\rho_v$. We denote by $\mathcal{B}$ the finite group of automorphic characters of $T(\mathbb{A}_F)$ associated to $\mathcal{B}$ via Theorem 4.3. Note that $\rho_v$ is the indicator function of $\bigcap_{\rho \in \mathcal{B}} \ker \rho$ for each place $v \in \text{Val}(F)$, in particular by character orthogonality we may write

$$\hat{\rho}_v = \frac{1}{|\mathcal{B}|} \sum_{\rho \in \mathcal{B}} \rho_v.$$  (5.1)

However $\rho$ is not the indicator function of $\bigcap_{\rho \in \mathcal{B}} \ker \rho \subset T(\mathbb{A}_F)$ in general; indeed as each character of $\mathcal{B}$ is automorphic we have $T(F) \subset \bigcap_{\rho \in \mathcal{B}} \ker \rho$.

Using the map (1.17), we may pull-back $\mathcal{B}$ to obtain a collection of Brauer group elements $\beta_\alpha \in \text{Br}_x T_\alpha$ for each $\alpha \in \mathcal{A}$. We let $\hat{\rho}_\alpha : T_\alpha(\mathbb{A}_F) \to \{0, 1\}$ denote the indicator function of $T_\alpha(\mathbb{A}_F)_{\beta_\alpha}$. We denote the residue map associated to the divisor $D_\alpha$ by $\partial_\alpha$. We may pull-back characters $\chi$ of $T(\mathbb{A}_F)$ to obtain characters $\chi_\alpha$ of $T_\alpha(\mathbb{A}_F)$. For automorphic $\chi$, we will often identify these with Hecke characters of $F_\alpha$ using the identification $T_\alpha(\mathbb{A}_F) = \mathbb{A}_{F_\alpha}^\ast$. 

THE NUMBER OF VARIETIES CONTAINING A RATIONAL POINT 37
(since $T_\alpha = \mathrm{R}_{F_\alpha/F} \mathbb{G}_m$). We let $\mathcal{R}_\alpha$ denote the collection of characters on $T_\alpha(\mathbb{A}_F)$ induced by the $\mathcal{R}_\alpha$ via Theorem 14.3. These coincide with the pull-back of $\mathcal{R}$ to $T_\alpha(\mathbb{A}_F)$, by the functoriality of the Brauer pairing. We identify those places of $F_\alpha$ which lie above a fixed non-archimedean place $v$ of $F$, with those elements $\alpha_v \in \mathcal{R}_F$ such that $\alpha_v | \alpha$. With respect to the identification $T_\alpha(\mathbb{A}_F) = \mathbb{A}_F^\times$, we may therefore write $\rho_\alpha$ and each $\chi_\alpha$ as a product of local factors $\rho_{\alpha_v}$ and $\chi_{\alpha_v}$ for each place $v$ of $F$ and each $\alpha_v | \alpha$.

We fix a finite set $S$ of places of $F$ that contains all archimedean places, all places which are ramified in the splitting field of $T$, all places $v$ for which $\rho_v$ is ramified for some $\rho \in \mathcal{R}$ and all places $v$ for which $\mathrm{vol}(\mathcal{O}_v) \neq 1$ with respect to our choice of Haar measure on $F_v$.

5.2. The height zeta function and the Poisson summation formula. Throughout $s = (s_\alpha) \in \mathbb{C}^S$ denotes a complex variable. Given some $c \in \mathbb{R}$, we shall say that $s$ satisfies $\text{Re } s > c$ if it lies in the complex tube domain $\{ s \in \mathbb{C}^S : \text{Re } s_\alpha > c \text{ for all } \alpha \in S \}$. The generating series for the rational points of interest is the following height zeta function

$$Z(s) = \sum_{t \in T(F)} \rho(t) H(t; -s).$$

As $\rho$ is the indicator function of $T(F)_{\neq S}$ this may also be written as

$$Z(s) = \sum_{t \in T(F) \setminus S} H(t; -s).$$

In what follows, we shall often be able to reduce the study of this zeta function to the works of [BT95] and [BT98]. For example since $|\rho(t)| \leq 1$ for each $t \in T(F)$, we deduce immediately from [BT98 Thm. 4.2] that the sum in (5.3) converges absolutely on the tube domain $\text{Re } s > 1$, and hence defines a holomorphic function on this domain. The key tool in the study of $Z(s)$ is the Poisson summation formula, which we now explain.

Let $f : T(\mathbb{A}_F) \to \mathbb{C}$ be a continuous function which is given as a product of local factors $f_v$, which for almost all places $v$ of $F$ take only the value 1 on $T(\mathcal{O}_v)$. In practice we will take $f = \rho$ or $f = 1$ (that $\rho$ satisfies the required properties will be shown in Lemma 5.3). We define the Fourier transform of a character $\chi \in T(\mathbb{A}_F)^\wedge$ with respect to $f$ to be

$$\hat{H}(f, \chi; -s) = \int_{T(\mathbb{A}_F)} f(t) \chi(t) H(t; -s) d\mu,$$

for those $s \in \mathbb{C}^S$ for which the integral exists. Our assumptions on $f$ imply that the Fourier transform decomposes as a product of local Fourier transforms

$$\hat{H}(f, \chi; -s) = \prod_{v \in \text{Val}(F)} \hat{H}_v(f_v, \chi_v; -s),$$

where

$$\hat{H}_v(f_v, \chi_v; -s) = \int_{T(F_v)} f_v(t_v) \chi_v(t_v) H_v(t_v; -s) d\mu_v.$$

Applying the Poisson summation formula to the height zeta function (see [BT98 Thm 4.4]) we obtain

$$Z(s) = \frac{1}{(2\pi)^{\text{rank } \chi(\mathbb{T})} \text{vol}(T(\mathbb{A}_F)^1/T(F))} \int_{\chi \in (T(\mathbb{A}_F)/T(F))^\wedge} \hat{H}(\rho, \chi; -s) d\chi,$$

provided that the integral in (5.4) exists. Note that in the case where $T$ is anisotropic, we have $T(\mathbb{A}_F) = T(\mathbb{A}_F)^1$ by [4.3] and hence $T(\mathbb{A}_F)/T(F)$ is compact. In particular $(T(\mathbb{A}_F)/T(F))^\wedge$ is discrete and therefore the above integral is really a sum over characters.
5.3. The local Fourier transforms. We begin by obtaining meromorphic continuations of the local Fourier transforms. To do this, note that it follows from (5.1) that we have
\[\hat{H}_v(p_v, \chi_v; -s) = \frac{1}{|\mathcal{R}|} \sum_{\rho \in \mathcal{R}} \hat{H}_v(1, \rho_v \chi_v; -s),\]
for any place \(v\) of \(F\). This expression will allow us to deduce the analytic properties of the local Fourier transform from the work of Batyrev and Tschinkel in [BT95] and [BT98], which corresponds to the case \(\mathcal{R} = 1\).

**Lemma 5.1.** Let \(v \in \text{Val}(F)\) and let \(\chi_v\) be a character of \(T(F_v)\). For any \(\varepsilon > 0\), the local Fourier transform \(\hat{H}_v(p_v, \chi_v; -s)\) converges absolutely and is uniformly bounded (in terms of \(\varepsilon\) and \(v\)) on the tube domain \(\text{Re}\, s > \varepsilon\). Moreover, \(\hat{H}_v(p_v, \chi_v; -s)\) admits a meromorphic continuation to \(\mathbb{C}^\mathbb{C}\).

**Proof.** The first part follows from the results of [BT95], since \(|p_v \chi_v| \leq 1\). The meromorphic continuation again follows from (5.5).

We also show that the local Fourier transform of the trivial character does not vanish.

**Lemma 5.2.** Let \(v \in \text{Val}(F)\). Then \(\hat{H}_v(p_v, 1; -s)\) is non-zero for any \(s \in \mathbb{R}^\mathbb{C}_{>0}\).

**Proof.** As \(p_v\) is locally constant and \(T(F_v)\) is locally compact, there exists some compact neighbourhood \(C_v\) of 1 such that \(p_v(C_v) = 1\) (since \(p_v(1) = 1\) by assumption). As \(s \in \mathbb{R}^\mathbb{C}_{>0}\), it follows that
\[\hat{H}_v(p_v, 1; -s) \geq \int_{C_v} H_v(t_v; -s) d\mu_v,\]
which exists and is non-zero since \(d\mu_v\) is a Haar measure.

5.3.1. The non-archimedean places. We now focus on the non-archimedean places. We first show that our height functions and \(p\) are well-behaved from a harmonic analysis perspective.

**Lemma 5.3.** For every non-archimedean place \(v \in \text{Val}(F)\), there exists a compact open subgroup \(K_v \subseteq T(\mathcal{O}_v)\) of finite index such that \(H_v(\cdot; -s)\) and \(p_v\) are \(K_v\)-invariant and trivial on \(K_v\). Moreover, one may take \(K_v = T(\mathcal{O}_v)\) for any place \(v \notin S\).

**Proof.** The result for \(p_v\) follows from (5.1) together with the corresponding result for \(\rho_v\) for each \(\rho \in \mathcal{R}\). The result for \(H_v(\cdot; -s)\) follows from the fact that we are using the Batyrev-Tschinkel height (see Section 4.3). Using this, we obtain the following.

**Lemma 5.4.** For any non-archimedean place \(v \in \text{Val}(F)\) and any character \(\chi_v\) of \(T(F_v)\) which is non-trivial on \(K_v\), we have
\[\hat{H}_v(p_v, \chi_v; -s) = 0.\]

**Proof.** By Lemma 5.3 we have
\[\hat{H}_v(p_v, \chi_v; -s) = \int_{T(F_v)} p_v(t_v)\chi_v(t_v)H_v(t_v; -s) d\mu_v\]
\[= \sum_{n_v \in T(F_v)/K_v} p_v(n_v)\chi_v(n_v)H_v(n_v; -s) \int_{K_v} \chi_v(t_v) d\mu_v\]
\[= 0,\]
on applying character orthogonality.
We emphasise now that the previous lemma is crucial to our analysis. It implies that in the Possion formula (5.4), only those automorphic characters \( \rho \) of \( T \) which are trivial on \( K_v \) for all \( v \) occur. This fact will allow us to deduce the convergence of the Possion integral. This is an important part of the argument where we use the fact that the Brauer group elements under consideration are algebraic, as the conclusion of Lemma \( \ref{lemma:alg} \) does not hold for indicator functions of transcendental Brauer group elements in general. One can see this for example by considering the quaternion algebra \( (1 + \mathbb{Z}_p) \mathbb{Q} \) and the quaternion algebra \( \mathbb{H} \) which is a maximal imaginary division algebra over \( \mathbb{R} \). The quaternion algebra \( \mathbb{H} \) is not trivial on the Brauer group elements \( \rho \) with \( \rho(1 + \mathbb{Z}_p) \mathbb{Q} = 0 \).

**Lemma 5.5.** Let \( \varepsilon > 0 \) be a place of \( F \) and let \( \chi_v \) be an unramified character of \( T(F_v) \). Then on the tube domain \( \text{Re } s > 1/2 + \varepsilon \) we have

\[
\hat{H}_v(\rho_v, \chi_v; -s) = \prod_{\alpha_v \in \mathcal{A}_v} \left( 1 - \frac{\rho_v(\pi_{\alpha_v}) \chi_v(\pi_{\alpha_v})}{\bar{q}_v^\epsilon} \right)^{-1} \left( 1 + O_{\varepsilon} \left( \frac{1}{\bar{q}_v^{1+\varepsilon}} \right) \right),
\]

where \( q_v \) denotes the size of the residue field of \( F_v \) and \( \pi_{\alpha_v} \) denotes a uniformiser of \( F_{\alpha_v} \).

**Proof.** The proof of [BT95] Thm. 3.1.3 shows that

\[
\hat{H}_v(1, \chi_v; -s) = \prod_{\alpha_v \in \mathcal{A}_v} \left( 1 - \frac{\chi_v(\pi_{\alpha_v})}{\bar{q}_v^\epsilon} \right)^{-1} \left( 1 + O_{\varepsilon} \left( \frac{1}{\bar{q}_v^{1+\varepsilon}} \right) \right),
\]

Combining this with (5.5) we obtain

\[
\hat{H}_v(\rho_v, \chi_v; -s) = 1 + \frac{1}{|\mathcal{A}|} \sum_{\rho \in \mathcal{R}} \sum_{\alpha_v \in \mathcal{A}_v} \rho_v(\pi_{\alpha_v}) \chi_v(\pi_{\alpha_v}) + O_{\varepsilon} \left( \frac{1}{\bar{q}_v^{1+\varepsilon}} \right)
\]

\[
= 1 + \sum_{\alpha_v \in \mathcal{A}_v} \frac{\chi_v(\pi_{\alpha_v})}{\bar{q}_v^\epsilon} \sum_{\rho \in \mathcal{R}} \rho_v(\pi_{\alpha_v}) + O_{\varepsilon} \left( \frac{1}{\bar{q}_v^{1+\varepsilon}} \right),
\]

on the above tube domain. By character orthogonality we deduce that

\[
\sum_{\rho \in \mathcal{R}} \rho_v(\pi_{\alpha_v}) \left| \mathcal{A}_v \right| = \hat{h}_{\alpha_v}(\pi_v),
\]

and the result follows. \( \Box \)

5.3.2. The archimedean places. If \( v \) is an archimedean place of \( F \) then \( \rho_v \) is very easy to describe. Namely if \( v \) is complex then \( \hat{h}_v = 1 \) as \( \text{Br } \mathbb{C} = 0 \). If \( v \) is real, then it is well-known that we have an isomorphism

\[
T(F_v) \cong (\mathbb{R}^*)^{r_1} \times (S^1)^{r_2},
\]

of topological groups, where \( r_1 = \text{rank } X^*(T_v) \) and \( r_1 + r_2 = \dim T \). On noting that \( h_v \) is locally constant and using (5.1), we see that \( h_v \) is simply the indicator function of an open and closed subgroup of \( T(F_v) \) whose index divides \( 2^{r_1} \). These remarks easily allow us to prove the archimedean analogues of Lemma \( \ref{lemma:alg} \) and Lemma \( \ref{lemma:alg} \).
Lemma 5.6. For every archimedean place \( v \in \text{Val}(F) \), there exists a compact subgroup \( K_v \subset T(O_v) \) of finite index such that \( H_v(\; ; -s) \) and \( p_v \) are \( K_v \)-invariant and trivial on \( K_v \).

Proof. The required property for \( H_v(\; ; -s) \) follows as we are using the Batyrev-Tschinkel height (see Section 4.6). The statement for \( p_v \) follows from the above remarks. \( \square \)

Lemma 5.7. Let \( v \) be an archimedean place of \( F \). Then for any character \( \chi_v \) of \( T(F_v) \) which is non-trivial on \( K_v \) we have

\[
\hat{H}_v(p_v, \chi_v; -s) = 0,
\]

Proof. Equip \( K_v \) with a Haar measure \( \kappa_v \) and denote by \( \mathfrak{m}_v \) the induced quotient measure on \( T(F_v)/K_v \). Then we have

\[
\begin{align*}
\hat{H}_v(p_v, \chi_v; -s) &= \int_{T(F_v)} h_v(t_v)\chi_v(t_v)H_v(t_v; -s)\,d\mu_v \\
&= \int_{n_v \in T(F_v)/K_v} h_v(n_v)\chi_v(n_v)H_v(n_v; -s)\,d\mathfrak{m}_v \int_{K_v} \chi_v(t_v)\,d\kappa_v \\
&= 0,
\end{align*}
\]

by character orthogonality. \( \square \)

In [BT95, Sec. 2.3] a detailed analysis of the archimedean Fourier transforms is performed in the case where \( S = 1 \). One may perform a very similar analysis in our case, as \( p_v \) is so simple for archimedean \( v \). For example it follows from [5.5] and [BT95, Prop. 2.3.1] that the local Fourier transform is a rational function in \( s \). We shall content ourselves with the following result, which shall be used to handle the sum appearing in the Poisson formula (5.4).

Lemma 5.8. Let \( L \subset X^*(T_\infty)_R \) be a lattice and choose a \( \mathbb{R} \)-vector space norm \( \| \cdot \| \) on \( X^*(T_\infty)_R \). Let \( C \) be a compact subset inside the tube domain \( \text{Re } s \geq 1 \) and let \( g : X^*(T_\infty)_R \times C \to \mathbb{C} \) be a function. Suppose that there exists some \( 0 < \delta < 1/\dim X \) such that

\[
|g(\psi, s)| \ll (1 + \|\psi\|)^\delta,
\]

for all \( \psi \in X^*(T_\infty)_R \) and all \( s \in C \). Then the sum

\[
\sum_{\psi \in L} \prod_{v|\infty} \hat{H}_v(p_v, \psi_v; -s)g(\psi, s),
\]

is absolutely and uniformly convergent on \( C \), where we write \( \psi = (\psi_v)_{v|\infty} \).

Proof. By [5.5] we obtain similar bounds for \( \hat{H}_v(p_v, \psi_v; -s) \) to those given in [BT95, Prop. 2.3.2] and hence the result follows from [BT95, Cor. 2.3.4]. \( \square \)

5.4. The global Fourier transform. Now that we have good control over the local Fourier transforms, the next step is to consider the global Fourier transform. We shall relate these to the partial Euler products considered in Section 3.3.

Lemma 5.9. For each \( \varepsilon > 0 \) and each automorphic character \( \chi \) of \( T(A_F) \), there exists a function \( \varphi(\chi; s) \) which is holomorphic and uniformly bounded with respect to \( \chi \) on \( \text{Re } s > 1/2 + \varepsilon \), such that

\[
\hat{H}(p, \chi; -s) = \prod_{v|\infty} \hat{H}_v(p_v, \chi_v; -s) \prod_{\alpha \in \mathfrak{a}} L_{\mathfrak{a}_\alpha}(\chi_\alpha, s_\alpha)\varphi(\chi; s),
\]

for \( \text{Re } s > 1 \).
Proof. Recalling the definition of the partial Euler product \((3.4)\), this follows from Lemma 5.1, Lemma 5.9 and the fact that for non-archimedean places \(v\), we have identified those elements \(\alpha_v \mid \alpha\) with those places \(w\) of \(F_\alpha\) such that \(w \mid v\). \(\square\)

5.4.1. The anisotropic case. So far, all our arguments have applied to arbitrary tori.

From now on we assume that \(T\) is anisotropic.

We also restrict to the complex line \(s_\alpha = s\) for all \(\alpha \in \mathcal{A}\) in the space \(\mathbb{C}^\mathcal{A}\). As explained in Section 4.6, the resulting height function \(H(\cdot; -s)\) is a complex power of the anticanonical height function. In this case we obtain the following.

Lemma 5.10. Let \(\chi\) be an automorphic character of \(T(\mathbf{A}_F)\). Then \(\hat{H}(p, \chi; -s)\) admits a holomorphic continuation to the line \(\Re s = 1\), apart from possibly at \(s = 1\). Here we have

\[
\hat{H}(p, \chi; -s) = c_{\mathcal{A}, \chi} \prod_{\alpha \in \mathcal{A}} \left( \frac{1}{(s - 1)^{1/|\mathcal{A}_\alpha|}} + O\left( \frac{1}{(s - 1)^{1/|\mathcal{A}_\alpha| - 1}} \right) \right),
\]

as \(s \to 1\), for some constant \(c_{\mathcal{A}, \chi}\) which is non-zero if \(\chi = 1\).

Proof. By Lemma 5.1 and Lemma 5.9 it suffices to study the analytic properties of the partial Euler products \(L_{\mathcal{A}_\alpha}(\chi_\alpha, s)\). By Lemma 5.3 we know that \(L_{\mathcal{A}_\alpha}(\chi_\alpha, s)\) has a singularity on the line \(\Re s = 1\) if and only if \(\chi_\alpha = m_\alpha \rho_\alpha\) for some \(\rho_\alpha \in \mathcal{A}_\alpha\) and some \(m_\alpha \in X^*(T_\alpha)\). As \(T\) is anisotropic however we have \(X^*(T) = 0\). It follows from the functorial isomorphism \((4.4)\) that for any such character we have \(m_\alpha = 0\), in particular \(\chi_\alpha \in \mathcal{A}_\alpha\). Hence Lemma 3.3 implies that the singularity of \(L_{\mathcal{A}_\alpha}(\chi_\alpha, s)\) occurs at \(s = 1\) and also proves the first part of the lemma.

The non-vanishing of \(c_{\mathcal{A}, 1}\) follows from the non-vanishing of the local Fourier transforms at the trivial character proved in Lemma 5.2 and Lemma 3.5 which shows that

\[
\lim_{s \to 1} (s - 1)^{\sum_{\alpha \in \mathcal{A}} 1/|\mathcal{A}_\alpha|} \prod_{\alpha \in \mathcal{A}} L_{\mathcal{A}_\alpha}(1, s) \neq 0.
\]

\(\square\)

5.5. The asymptotic formula. As \(T\) is anisotropic, it follows from \((4.3)\) that \(T(\mathbf{A}_F) = T(\mathbf{A}_F)^1\). Hence \(T(\mathbf{A}_F)/T(F)\) is compact and \((T(\mathbf{A}_F)/T(F))^\wedge\) is discrete. The Poisson summation formula \((5.4)\) therefore reads

\[
Z(s) = \frac{1}{\text{vol}(T(\mathbf{A}_F)/T(F))} \sum_{\chi \in (T(\mathbf{A}_F)/T(F))^\wedge} \hat{H}(p, \chi; -s).
\]

(5.6)

We may use this expression to deduce the following.

Theorem 5.11. Let

\[
\Omega(s) = Z(s)(s - 1)^{\sum_{\alpha \in \mathcal{A}} 1/|\mathcal{A}_\alpha|}.
\]

Then \(\Omega(s)\) admits an extension to an infinitely differentiable function on \(\Re s \geq 1\). Moreover we have

\[
Z(s) = \frac{\Omega(1)}{(s - 1)^{\sum_{\alpha \in \mathcal{A}} 1/|\mathcal{A}_\alpha| - 1}} + O\left( \frac{1}{(s - 1)^{\sum_{\alpha \in \mathcal{A}} 1/|\mathcal{A}_\alpha| - 1}}\right) + \sum_{\alpha_0 \in \mathcal{A}} \frac{1}{(s - 1)^{\sum_{\alpha \in \mathcal{A} \setminus \alpha_0} 1/|\mathcal{A}_\alpha|}},
\]

as \(s \to 1\).
Proof. Given Lemma 5.10 it suffices to show that the sum in (5.6) is absolutely and uniformly convergent on any compact subset $C$ of the half-plane $\Re s \geq 1$ such that $1 \notin C$. Let $K = \prod_{v \in \Val(F)} K_v$ and denote by $U$ the group of automorphic characters of $T$ which are trivial on $K$. Note that by Lemma 5.4 and Lemma 5.7 the sum in (5.6) may be taken only over those characters which lie in $U$. As $K \subset K_T$ is a subgroup of finite index, it follows that the type at infinity map (1.8) yields a homomorphism

$$U \to X^*(T_{\infty})_R, \quad \chi \mapsto \chi_{\infty},$$

which has finite kernel $\mathcal{K}$ and whose image $\mathcal{L}$ is a lattice of full rank. We obtain

$$\sum_{\chi \in U} \hat{H}(\chi, -s) = \sum_{\psi \in \mathcal{L}} \sum_{\chi \in U \chi_{\infty} = \psi} \hat{H}(\chi, -s) = \sum_{\psi \in \mathcal{L}} \prod_{v | \infty} \hat{H}_v(\psi_v, \psi_v; -s) \sum_{\chi \in U \chi_{\infty} = \psi} \prod_{v | \infty} \hat{H}_v(\chi_v, \psi_v; -s).$$

Therefore by Lemma 5.9 we have

$$\sum_{\chi \in U} |\hat{H}(\chi, -s)| \leq \sum_{\psi \in \mathcal{L}} \prod_{v | \infty} |\hat{H}_v(\psi_v, \psi_v; -s)| \sum_{\chi \in U \chi_{\infty} = \psi} \prod_{\alpha \in \mathcal{A}} |L_{\mathcal{A}_\alpha}(\chi, s)|.$$

Since $K \subset K_T$ has finite index, there exists some $Q > 0$ such that $q(\chi_\alpha) < Q$ for all $\chi \in U$ and all $\alpha \in \mathcal{A}$. Therefore Lemma 3.2 and Lemma 3.3 imply that for any $\varepsilon > 0$ we have

$$\sum_{\chi \in U} \prod_{\alpha \in \mathcal{A}} |L_{\mathcal{A}_\alpha}(\chi_\alpha, s)| \ll \varepsilon, C, \mathcal{A}, \mathcal{F} |K| \cdot Q^\varepsilon \cdot (1 + \max_{\alpha \in \mathcal{A}} ||\psi_\alpha||)^\varepsilon,$$

for each $\psi \in \mathcal{L}$. Here the $\psi_\alpha$ are the image of $\psi$ under the map $X^*(T_{\infty}) \to \prod_{\alpha \in \mathcal{A}} X^*(T_{\alpha, \infty})$ and $||\psi_\alpha||$ is defined as in (3.1). The result therefore follows from Lemma 5.8. \hfill \Box

One may wonder whether $Z(s)$ admits a holomorphic extension to the line $\Re s = 1$, away from $s = 1$ (i.e. differentiable in a neighbourhood of each such point). To obtain such a result, one would require uniform holomorphic continuations of the partial Euler products appearing in Lemma 3.5 which in turn would require uniform zero-free regions for Hecke $L$-functions over a fixed number field. For example, assuming the generalised Riemann hypothesis one can show that $Z(s)$ admits a holomorphic continuation to some half-plane past the line $\Re s = 1$, away from the branch cut at $s = 1$. Unfortunately, the current known unconditional zero-free regions for Hecke $L$-functions (see e.g. [Co90] or [IK04, Thm. 5.10]) approach the line $\Re s = 1$ as one varies the infinity type of the character, and hence it does not seem possible with current technology to obtain any kind of continuation of $Z(s)$ in the half-plane $\Re s < 1$. Thankfully these subtle complex analytic issues do not concern us, since Delange’s Tauberian theorem is sufficiently powerful that one only needs to know the behaviour of $Z(s)$ on the line $\Re s = 1$. In order to successfully apply Theorem 3.4 to deduce Theorem 1.1 we need to know that

$$\lim_{s \to 1} Z(s)(s - 1)^{\sum_{\alpha \in \mathcal{A}} 1 / |\mathcal{A}_\alpha|} \neq 0.$$  \hspace{1cm} (5.7)

It should be emphasised that (5.7) does not follow from what we have shown so far; since more than one character may give rise to the singularity of highest order, it is theoretically possible that cancellation may occur. We postpone the proof of (5.7) until the next section.
Since $T$ is anisotropic, by (4.18) we know that $\#\mathcal{A} = \rho(X)$. It also follows from Lemma 4.6 that

$$|\mathcal{R}_\alpha| = |\partial_{\alpha}(B)|, \quad \text{for all } \alpha \in \mathcal{A}.$$ 

Hence we deduce that

$$\sum_{\alpha \in \mathcal{A}} \frac{1}{|\mathcal{R}_\alpha|} = \rho(X) - \Delta_X(B),$$

(5.8)

where $\Delta_X(B)$ is as in Theorem 1.1. Therefore assuming (5.7), we may apply Theorem 3.7 and use Theorem 5.11 to deduce that

$$N(T, H, B) \sim \Omega(1) \Gamma(\rho(X) - \Delta_X(B)) B (\log B)^{\rho(X) - \Delta_X(B) - 1},$$

(5.9)

as $B \to \infty$.

which gives the required asymptotic formula for Theorem 1.1.

5.6. **Non-vanishing of the leading constant.** In this section we shall verify (5.7), showing that the leading constant appearing in the asymptotic formula (5.9) is indeed non-zero. It is at this point where subordinate Brauer group elements enter the picture, and hopefully it should soon become clear to the reader that the detailed study which we performed earlier in Section 2.6 and Section 4.5 was well worth the effort.

By Lemma 5.10, we know that the characters which gives rise to the singularity of $Z(s)$ of the highest order are the exactly the finite collection of characters $C(T, R)$, as defined in Theorem 4.10. Given (5.6), we therefore need to consider the sum

$$\sum_{\chi \in \text{Sub}_e(X, B)/B(T)} \hat{H}(\rho, \chi; -s),$$

where we have identified $C(T, R) \cong \text{Sub}_e(X, B)/B(T)$, on using Theorem 4.10. Note that it follows from character orthogonality that

$$\sum_{\chi \in \text{Sub}_e(X, B)/B(T)} \hat{H}(\rho, \chi; -s) = \frac{|\text{Sub}_e(X, B)|}{|B(T)|} \int_{T(A_F)_{\mathcal{B}(X, B)}} H(t; -s) d\mu.$$ 

Therefore in order to show (5.7), by (5.8) it suffices to prove that

$$\lim_{s \to 1} (s - 1)^{\rho(X) - \Delta_X(B)} \int_{T(A_F)_{\mathcal{B}(X, B)}} H(t; -s) d\mu \neq 0.$$ 

(5.10)

If $\text{Sub}(X, B) \neq \text{Sub}(F(X), B)$, then Theorem 2.17 implies that the set $T(A_F)_{\mathcal{B}(X, B)}$ can be quite complicated. Therefore rather than dealing with $T(A_F)_{\mathcal{B}(X, B)}$ directly, we shall show that the analogue of (5.10) holds for a certain subspace.

**Lemma 5.12.** The limit

$$\lim_{s \to 1} (s - 1)^{\rho(X) - \Delta_X(B)} \int_{T(A_F)_{\mathcal{B}(X, B)}} H(t; -s) d\mu,$$

exists and is non-zero.

**Proof.** First note that $T(A_F)_{\mathcal{B}(X, B)} \neq \emptyset$; indeed $1 \in T(A_F)_{\mathcal{B}(X, B)}$ as $b(1) = 0$ for all $b \in \text{Sub}_e(X, B)$, by definition. Moreover Sub$_e(X, B)$ is finite by Lemma 2.18. The integral
in the lemma is simply the Fourier transform \( \hat{\mu}_{\text{Sub}_b(X, \mathscr{B})} \), so applying Lemma 5.10 and (5.5) to \( \text{Sub}_b(X, \mathscr{B}) \) we deduce that
\[
\lim_{s \to 1} (s - 1)^\rho(X) - \Delta_X(\text{Sub}_b(X, \mathscr{B})) \int \sigma(T(A_F)_{\text{Sub}_b(X, \mathscr{B})}) H(t; s) d\mu \neq 0.
\]
The lemma is proved on noting that \( \Delta_X(\text{Sub}_b(X, \mathscr{B})) = \Delta_X(\mathscr{B}) \), since by definition \( \text{Sub}_b(X, \mathscr{B}) \) generates the same group of residues as \( \mathscr{B} \). \hfill \Box

As \( \mathscr{B} \subset \text{Sub}_b(X, \mathscr{B}) \), we obviously have \( T(A_F)_{\text{Sub}_b(X, \mathscr{B})} \subset T(A_F)_{\mathscr{B}} \). Hence for \( \sigma \in \mathbb{R}_{>1} \) we obtain
\[
\lim_{\sigma \to 1^+} (\sigma - 1)^\rho(X) - \Delta_X(\mathscr{B}) \int T(A_F)_{\mathscr{B}} H(t; -\sigma) d\mu \geq \lim_{\sigma \to 1^+} (\sigma - 1)^\rho(X) - \Delta_X(\mathscr{B}) \int T(A_F)_{\text{Sub}_b(X, \mathscr{B})} H(t; -\sigma) d\mu,
\]
and this latter limit is non-zero by Lemma 5.10. This shows (5.10) and hence (5.7), which completes the proof of Theorem 1.1.

5.7. Calculation of the leading constant. We now turn our attention towards calculating the leading constant which appears in Theorem 1.1. Peyre [Pey95] has formulated a general conjectural expression for the leading constant in the classical case of Manin’s conjecture where \( \mathscr{B} = 0 \). This expression was confirmed by Batyrev and Tschinkel [BT95 Cor. 3.4.7] for anisotropic tori and takes the shape
\[
\frac{\alpha(X) \beta(X) \tau(X)}{(\rho(X) - 1)!}.
\]
(5.11)

Here \( \alpha(X) \) is a certain rational number defined in terms of the cone of effective divisors of \( X \). For our purposes it suffices to know that for anisotropic tori we simply have \( \alpha(X) = 1/|\text{Pic} T| \) by [BT95 Ex. 2.4.9]. Also \( \beta(X) = |\text{Br} X/\text{Br} F| \) and \( \tau(X) \) is the Tamagawa number of \( X \), defined as the volume of \( X(F) \) inside \( X(A_F) \) with respect to a certain Tamagawa measure. In what follows the reader should keep the expression (5.11) in mind, as the expression which we will derive shall bear a striking resemblance to it.

5.7.1. A Tamagawa measure. We now define a certain Tamagawa measure which may be viewed as a generalisation of Peyre’s Tamagawa measure [Pey95] to the current setting. This measure is closely related to the measure constructed in Section 4.2.5, though here we shall take a different choice of convergence factors and also take into account of our choice of adelic metric on the canonical bundle \( \omega_X \).

Let \( \omega \) be an invariant differential form on \( T \) and for any place \( v \) of \( F \) let \( |\omega|_v \) denote the associated Haar measure on \( T(F_v) \). We define the local Tamagawa measure to be
\[
\tau_v = \frac{|\omega|_v}{||\omega||_v}.
\]

This definition is independent of the choice of \( \omega \), though depends on the choice of adelic metric on \( \omega_X \). Recalling the construction of \( \mu \) given in Section 4.2.5 we see that
\[
\tau_v = \frac{c_v \cdot \mu_v}{H_v(t_v)}. \quad (5.12)
\]

In particular we have
\[
\tau_v(T(F_v)) = c_v \cdot \hat{H}_v(1, 1; -1), \quad \tau_v(T(F_v)_{\mathscr{B}}) = c_v \cdot \hat{H}_v(b_v, 1; -1). \quad (5.13)
\]
In general the product of these measures does not converge to give an adelic measure so we introduce convergence factors. To do this, consider the following virtual Artin representation (see Section 3.2)

\[ \text{Pic}_\mathcal{B}(\mathcal{X})_\mathbb{C} = \text{Pic}(\mathcal{X})_\mathbb{C} - \sum_{\alpha \in \mathcal{A}} \left( 1 - \frac{1}{\partial_\alpha(\mathcal{B})} \right) \text{Ind}_{F_\alpha}^F \mathbb{C}. \] (5.14)

Here \( \text{Ind}_{F_\alpha}^F \) denotes the representation induced by the inclusion \( G_{F_\alpha} \subset G_F \) and \( \text{Pic}(\mathcal{X})_\mathbb{C} = \text{Pic}(\mathcal{X}) \otimes \mathbb{C} \). We also let \( \text{Pic}_\mathcal{B}(\mathcal{X})_\mathbb{C} = \text{Pic}_\mathcal{B}(\mathcal{X})^{G_F} \). Note that in the case where \( \mathcal{B} = 0 \), then (5.14) is simply \( \text{Pic}(\mathcal{X})_\mathbb{C} \). By the exact sequence (4.16), we may write the corresponding virtual Artin L-function as

\[ L(\text{Pic}_\mathcal{B}(\mathcal{X})_\mathbb{C}, s) = \frac{L(\text{Pic}(\mathcal{X})_\mathbb{C}, s)}{\prod_{\alpha \in \mathcal{A}} \zeta_{F_\alpha}(s)(1-1/\partial_\alpha(\mathcal{B}))} = \frac{\prod_{\alpha \in \mathcal{A}} \zeta_{F_\alpha}(s)^{1/\partial_\alpha(\mathcal{B})}}{L(X^*(\mathcal{T})_\mathbb{C}, s)}. \]

For each place \( v \in \text{Val}(F) \) we define

\[ \lambda_v = \begin{cases} L_v(\text{Pic}_\mathcal{B}(\mathcal{X})_\mathbb{C}, 1), & v \text{ non-archimedean,} \\ 1, & v \text{ archimedean.} \end{cases} \]

In light of (5.13), the calculation of the Fourier transforms given in Lemma 5.9 and the analytic properties of the partial Euler products proven in Lemma 3.5 together imply that these are a “family of convergence factors”, i.e. the measure

\[ \prod_{v \in \text{Val}(F)} \lambda_v^{-1} \tau_v, \]

converges to a measure on \( T(A_F) \) (note that this converges for arbitrary tori, not just for anisotropic tori). We define the Tamagawa measure on \( T(A_F) \) associated to \( \mathcal{B} \) to be

\[ \tau_\mathcal{B} = L^*(\text{Pic}_\mathcal{B}(\mathcal{X})_\mathbb{C}, 1) \prod_{v \in \text{Val}(F)} \lambda_v^{-1} \tau_v. \] (5.15)

Note that this depends on the choice of adelic metric on \( \omega_X \). Here \( L^*(\text{Pic}_\mathcal{B}(\mathcal{X})_\mathbb{C}, 1) \neq 0 \) is given by Theorem 5.3. We have not included a discriminant factor as in Peyre [Pey95, def. 2.1], since we have normalised our Haar measure on \( A_F \) so that \( \text{vol}(A_F/F) = 1 \).

5.7.2. The leading constant. We now show a more precise version of Theorem 1.1

**Theorem 5.13.** Under the same assumptions of Theorem 1.1 we have

\[ N(U, H, \mathcal{B}, B) \sim c_{X, \mathcal{B}, H} B (\log B)^{\rho(\mathcal{B})^2 - (X) - 1}, \quad \text{as } B \to \infty, \]

where \( \rho(\mathcal{B})(X) = \text{rank} \text{Pic}_\mathcal{B}(\mathcal{X})_\mathbb{C} \) and

\[ c_{X, \mathcal{B}, H} = \frac{\alpha(X) \cdot |\text{Sub}(X, \mathcal{B})/\text{Br} F| \cdot \tau_\mathcal{B} \left( T(A_F)_{\text{Sub}(X, \mathcal{B})} \right)}{\Gamma(\rho(\mathcal{B})(X))}. \]

**Proof.** The asymptotic formula shown follows from inserting the definition (5.14) of \( \text{Pic}_\mathcal{B}(\mathcal{X})_\mathbb{C} \) into Theorem 1.1. Note that as \( T \) is anisotropic, it follows from Theorem 3.4 that \( L(X^*(\mathcal{T}), s) \) is holomorphic on \( \mathbb{C} \) and non-zero in the half-plane \( \text{Re } s \geq 1 \). By (5.6), (5.9) and the work in Section 5.6 we know that the leading constant \( c_{X, \mathcal{B}, H} \) takes the form

\[ \frac{|\text{Sub}_\mathcal{B}(X, \mathcal{B})|}{\Gamma(\rho(\mathcal{B})(X)) \text{vol}(T(A_F)/T(F))] B(T)|} \lim_{s \to 1} (s - 1)^{\rho(\mathcal{B})^2 - (X)} \int_{T(A_F)_{\text{Sub}(X, \mathcal{B})}} H(t; -s) d\mu. \]
By (5.10) the limit
\[
\lim_{s \to 1} \frac{1}{\prod_{\alpha \in \mathcal{A}} \zeta_{F_\alpha}(s)^{1/\deg(\alpha)}} \int_{T(\mathbf{A}_F)_{\text{Sub}(X, \mathcal{B})}^{(s)}} H(t; -s) \, d\mu,
\]
exists and is non-zero. Therefore, on using (5.12) we see that
\[
\lim_{s \to 1} (s - 1)^{\rho(X) - \Delta_X(\mathcal{B})} \int_{T(\mathbf{A}_F)_{\text{Sub}(X, \mathcal{B})}^{(s)}} H(t; -s) \, d\mu
\]
\[
= \lim_{s \to 1} (s - 1)^{\rho(X)} \frac{L(\text{Pic}_\mathcal{B}(X), s)}{L(\text{Pic}_\mathcal{B}(X), s)} \int_{T(\mathbf{A}_F)_{\text{Sub}(X, \mathcal{B})}^{(s)}} \frac{d\mu}{H(t)^s}
\]
\[
= L(X^*(T), 1) \lim_{s \to 1} (s - 1)^{\rho(X)} \frac{L(\text{Pic}_\mathcal{B}(X), s)}{\prod_{\alpha \in \mathcal{A}} \zeta_{F_\alpha}(s)^{1/\deg(\alpha)}} \int_{T(\mathbf{A}_F)_{\text{Sub}(X, \mathcal{B})}^{(s)}} \prod_v \frac{L_v(X^*(T), 1) \, d\tau_v}{H_v(t_v)^{s-1}}
\]
\[
= L(X^*(T), 1) \tau(\mathbf{A}_F, \text{Sub}(X, \mathcal{B}))
\]
The result then follows from Ono’s formula (4.19) together with the fact that $|\mathcal{W}(T)| = |B(T)|$ by (4.20) and the fact that $\alpha(X) = 1/|\text{Pic}T|$, since $T$ is anisotropic (see [BT95, Ex. 2.4.9]).

5.7.3. The case where $\text{Sub}(X, \mathcal{B}) = \text{Sub}(F(X), \mathcal{B})$. If $\text{Sub}(X, \mathcal{B}) \neq \text{Sub}(F(X), \mathcal{B})$, then Theorem 2.17 implies that $T(\mathbf{A}_F)_{\text{Sub}(X, \mathcal{B})}$ can be quite complicated. In particular the volume $\tau(\mathbf{A}_F, \text{Sub}(X, \mathcal{B}))$ appearing in Theorem 5.13 can be difficult to calculate in general. So assume now that $\text{Sub}(X, \mathcal{B}) = \text{Sub}(F(X), \mathcal{B})$ and let $V = V_1 \times_T \cdot \times_T V_r$ be any product of Brauer-Severi schemes over $T$ such that $\langle [V_1], \ldots, [V_r] \rangle = \mathcal{B}$. Then
\[
c_{X, \mathcal{B}, H} = \frac{\alpha(X) \cdot |\mathcal{B}| \cdot |\text{Br}_{\mathcal{B}}(F(V)/F) | \cdot \text{Br}_F | \cdot \tau(\mathbf{A}_F, \mathcal{B})}{\Gamma(\rho(X))},
\]
where $\overline{T(F)_{\mathcal{B}}}$ denotes the closure of $T(F)_{\mathcal{B}}$ inside $T(\mathbf{A}_F, \mathcal{B})$ with respect to the product topology. Indeed the equality
\[
|\text{Sub}(F(X), \mathcal{B})/\text{Br} F| = |\mathcal{B}| \cdot |\text{Br}_{\mathcal{B}}(F(V)/F) | / \text{Br} F,
\]
follows from Theorem 2.13 and Corollary 4.9 implies that $T(\mathbf{A}_F)_{\text{Sub}(F(X), \mathcal{B})} = \overline{T(F)_{\mathcal{B}}}$. Corollary 4.9 also implies that there exists a finite subset $S \subset \text{Val}(F)$ such that
\[
\tau(\mathbf{A}_F, \mathcal{B}) = L(\text{Pic}_\mathcal{B}(X), 1) \prod_{v \in S} \tau_v T(F_v, \mathcal{B}) \prod_{v \in S} \tau_v \left( \overline{T(F_v)_{\mathcal{B}}} \right).
\]
In particular $\tau(\mathbf{A}_F, \mathcal{B})$, in a similar fashion to the classical case, is given as a product of “local densities” over almost all places, with a factor for the remaining places which measures the failure of weak approximation for $V$ (note that in the classical case the corresponding factor measures the failure of weak approximation for $T$ itself).

5.7.4. The case where $\mathcal{B} \subset \text{Br} X$. We finish our treatment of the leading constant by looking at the special case where $\mathcal{B} \subset \text{Br} X$ (keeping the notation of Theorem 1.1). Here we have $\Delta_X(\mathcal{B}) = 0$, in particular $\text{Sub}(X, \mathcal{B}) = \text{Sub}(F(X), \mathcal{B}) = \text{Br} X$ and the measure $\tau(\mathcal{B})$ is the Tamagawa measure $\tau$ defined by Peyre [Pey95]. The asymptotic formula obtained in Theorem 1.1 has the same order of magnitude as in the case of Manin’s conjecture, so one may wonder how the constant $c_{X, \mathcal{B}, H}$ obtained compares with the constant $c_{H, \text{Peyre}}$. We
give two types of phenomenon which occur for anisotropic tori and which cannot occur in the case of projective space considered by Serre [Ser90].

First suppose that \( B(T) \neq 0 \) and that \( \mathcal{B} \subset B(T) \) is non-zero, where \( B(T) \) is given by (2.7). Note that by the duality (4.5), this first condition is equivalent to \( W(T) \neq 0 \). This occurs for example for the biquadratic field extension \( \mathbb{Q}(\sqrt{13}, \sqrt{17})/\mathbb{Q} \) (see [CTS77] p224 – note that this torus is indeed anisotropic). As \( b \otimes_F F_v = 0 \) for all \( v \in \text{Val}(F) \) and all \( b \in \mathcal{B} \), the exact sequence (2.6) implies that \( b(x) = 0 \) for all \( x \in X(F) \), hence \( X(F)_{\mathcal{B}} = X(F) \). Thus we see that \( c_{X,\mathcal{B},H} = c_{H,\text{Peyre}} \), yet \( \mathcal{B} \neq 0 \). Geometrically, we obtain a product of Brauer-Severi schemes over \( X \) which is not Zariski locally trivial over \( F \), yet is Zariski locally trivial over every completion of \( F \).

Next suppose that \( \mathcal{B} \subset \text{Br}X \) is such that \( X(F)_{\mathcal{B}} \neq \emptyset \) but that \( X(F)_{\mathcal{B}} \neq X(F) \). For example, such a \( \mathcal{B} \) exists for the norm one torus given by the biquadratic field extension \( \mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q} \), as in this case there exist rational points which are not Brauer equivalent (again see [CTS77] p224). Note that such an \( X \) must necessarily fail weak approximation. By Lemma 2.1 there exists a finite set of places \( S \) such that

\[
\prod_{v \in S} X(F_v)_{\mathcal{B}X} \neq \prod_{v \in S} X(F_v)_{\text{Br}X},
\]

and such that \( X(F_v)_{\mathcal{B}X} = X(F_v) \) for all \( v \notin S \). The fact that the Brauer pairing is locally constant implies that the complement of \( \prod_{v \in S} X(F_v)_{\mathcal{B}X} \) in \( \prod_{v \in S} X(F_v)_{\text{Br}X} \) has positive measure with respect to \( \prod_{v \in S} \tau_v \) (open subsets have positive measure). In this case we therefore obtain an example where \( 0 < c_{X,\mathcal{B},H} < c_{H,\text{Peyre}} \).

6. Concluding remarks

We now speculate about which classes of varieties one might expect an analogue of Theorem 1.4 to hold. In what follows we only consider varieties for which a suitably uniform version of Manin’s conjecture holds, namely we assume the following.

Assumption. Assume that \( X \) is a non-singular projective variety over a number field \( F \) such that

1. The anticanonical bundle of \( X \) is big, \( \text{Pic} X \) is torsion free and

\[ H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0. \]

2. Manin’s conjecture holds for \( X \) with a uniform asymptotic constant. i.e. for any anticanonical height function \( H \) on \( X \), there exists an open subset \( U \subset X \) such that for all non-empty open subsets \( V \subset U \) we have

\[
N(V, H, B) \sim c_{U,H} B(\log B)^{\rho(X)-1},
\]

for some constant \( c_{U,H} > 0 \). Here \( \rho(X) = \text{rank Pic} X \).

Examples of varieties which satisfy condition (1) include all Fano varieties and all equivariant compactifications of connected linear algebraic groups and their homogeneous spaces [FZ13] Thm. 1.2. Such varieties are sometimes called “almost Fano”. Note that condition (2) implies that the rational points on \( X \) are Zariski dense. We begin with the following conjecture.

Conjecture 6.1. Let \( X \) be a variety over \( F \) which satisfies the above assumptions and let \( H \) be an anticanonical height function on \( X \). Let \( \mathcal{B} \subset \text{Br}F(X) \) be a finite subgroup and suppose that there exists some \( x \in X(F) \) such that each \( b \in \mathcal{B} \) is defined at \( x \) and that \( b(x) = 0 \) for all \( b \in \mathcal{B} \).
Then there exists a Zariski open subset $U \subset X$ with $\mathcal{B} \subset \text{Br} U$ and a constant $c_{U, \mathcal{B}, H} > 0$ such that

$$N(U, H, \mathcal{B}, B) \sim c_{U, \mathcal{B}, H} B \frac{(\log B)^{\rho(X)-1}}{(\log B)^{\Delta X(\mathcal{B})}}, \quad \text{as } B \to \infty,$$

where

$$\Delta X(\mathcal{B}) = \sum_{D \in X(1)} \left(1 - \frac{1}{|\partial_D(\mathcal{B})|}\right).$$

Note that in the case where $X = \mathbb{P}^n$, we conjecture a positive answer to the question posed by Serre in [Ser90]. The language of virtual Artin representations (see Section 3.2) provides a useful formalism to describe the factors appearing in this conjecture. Namely in the setting of Conjecture 6.1, the virtual Artin representation of interest is

$$\text{Pic}_{\mathcal{B}}(X)_C = \text{Pic}(X)_C - \sum_{D \in X(1)} \left(1 - \frac{1}{|\partial_D(\mathcal{B})|}\right) \text{Ind}^F_{F_D} \mathbb{C},$$

where $F_D$ denotes the algebraic closure of $F$ inside $F(D)$ (note that $\text{rank} \text{Pic}_{\mathcal{B}}(X)_C^{GF} = \rho(X) - \Delta X(\mathcal{B})$). We are also in a position to speculate about what form the leading constant in Conjecture 6.1 should take. As in the results obtained in Section 5.7, this should involve a factor given by the size of a group of subordinate Brauer group elements, which in special cases can be related to the size of the unramified Brauer group of the associated product of Brauer-Severi varieties. In our setting we also obtained a Tamagawa type number, whose convergence factors were provided by the local factors of the virtual Artin L-function $L(\text{Pic}_{\mathcal{B}}(X)_C, s)$. Our proof that these were indeed a family of convergence factors hinged on our calculation of the local Fourier transforms of the height functions. It would be interesting to see if one could prove that these local factors provide a family of convergence factors in the general case.

We now give some simple examples for which Conjecture 6.1 is not known, in terms of the counting functions associated to the corresponding Brauer-Severi schemes. Let $F$ be a number field and let $H$ be an anticanonical height function on $\mathbb{P}^1$. Let $f_0, f_1, f_2 \in F[t]$ have degrees $d_0, d_1, d_2$ and let $f = f_0 f_1 f_2$. Consider the variety

$$Y : f_0(t)x_0^2 + f_1(t)x_1^2 + f_2(t)x_2^2 = 0 \subset \mathbb{A}^1 \times \mathbb{P}^2.$$

This corresponds to the class $b$ of the quaternion algebra $(-f_1(t)/f_0(t), -f_2(t)/f_0(t))$ in $\text{Br} F(t)$. The natural projection $\mathbb{A}^1 \times \mathbb{P}^2 \to \mathbb{A}^1$ restricts to give a map $\pi : Y \to \mathbb{A}^1$, which realises $Y$ as a conic bundle over $\mathbb{A}^1$. Note that we may modify any finite collection of fibres of $\pi$ without changing the asymptotic formula obtained, in particular we may assume that $f$ is separable. Also by contracting Galois orbits of disjoint $(-1)$-curves, we may assume that $-f_j(t_i)/f_k(t_i)$ is not a square in the field $F(t_i)$ for any root $t_i$ of $f_j$ over $\overline{F}$ and any $\{i, j, k\} = \{0, 1, 2\}$. For simplicity, we also assume that $d_0 = d_1 = d_2 \mod 2$ (geometrically, this corresponds to asking that the fibre at infinity be smooth). In which case Conjecture 6.1 predicts the asymptotic formula

$$N(\mathbb{A}^1, H, \pi, B) \sim \frac{c_{\mathbb{A}^1, \pi, H} B}{(\log B)^{m/2}}, \quad \text{as } B \to \infty,$$

where $m$ is the number of irreducible factors of $f$ over $F$ and $c_{\mathbb{A}^1, \pi, H} > 0$ if $Y(F) \neq \emptyset$. Note that one can easily show that $\Delta(b) = m/2$ in this case by using Lemma 2.7 (there are $m$ non-split fibres). If $(d_0, d_1, d_2) = (2, 0, 0)$ then $Y$ admits a compactification which is a del Pezzo surface of degree six. If $f$ is irreducible, then (6.1) follows from Corollary 4.4. The case where $f$ is reducible is currently open, but it seems possible that the harmonic analysis
approach could be made to apply (here one would need to work with the torus $\mathbb{G}_m$). The next simplest cases where $(d_0, d_1, d_2) = (1, 1, 1), (0, 2, 2)$ or $(4, 0, 0)$ are wide open (in this last case one obtains a Châtelet surface).

One may also speculate about possible generalisations of the results obtained in Theorem 1.2 to the case where the generic fibre is no longer a product of Brauer-Severi varieties. Suppose for example that $X$ satisfies the above assumptions and that $Y$ is a variety with a Zariski dense set of rational points together with a proper surjective almost smooth morphism $\pi : Y \to X$ with geometrically integral generic fibre (see Definition 2.3 for the definition of almost smooth). Suppose that outside of some Zariski closed subset of $X$, all the fibres of $\pi$ over rational points satisfy the Hasse principle. Suppose also that given any codimension one point $D \in X^{(1)}$, the fibre $\pi^{-1}(D)$ over $D$ is reduced and the algebraic closure of the function field of $D$ inside the function field of each irreducible component of $\pi^{-1}(D)$ is Galois and independent of the choice of irreducible component (e.g. this happens if $\pi^{-1}(D)$ is itself irreducible). Denote this field by $F(D)_{\pi}$. Then does there exist a Zariski open subset $U \subset X$ and a constant $c_{U, \pi, H} > 0$ such that

$$N(U, H, \pi, B) \sim c_{U, \pi, H} B^{(\log B)^{\rho(X)-1} / (\log B)^{\Delta_X(\pi)}}, \quad \text{as } B \to \infty,$$

where

$$\Delta_X(\pi) = \sum_{D \in X^{(1)}} \left(1 - \frac{1}{[F(D)_{\pi} : F(D)]}\right) ?$$

Note that by Lemma 2.5, the number $\Delta_X(\pi)$ only depends on $X$ and the generic fibre of $\pi$. This is consistent with the fact that one would expect the asymptotic behaviour of the counting function $N(U, H, \pi, B)$ to depend only on $X$ and the generic fibre of $\pi$, for all sufficiently small open subsets $U \subset X$. The above question (essentially) applies to the case where the generic fibre is a product of Brauer-Severi varieties, as they have models of the above type locally around each divisor (see Lemma 2.0). Note that if the fibre over each point of codimension one is geometrically integral, then for this question to have a positive answer a positive proportion of all the varieties in the family need to contain a rational point. This is indeed the case for the families of hypersurfaces considered by Poonen and Voloch [PV04] (see Section 1), where it is simple to check that this geometric condition holds. Other examples of varieties to which this question applies include those of the form

$$N_{E/F}(x) = f(t) \subset \mathbb{A}^n \times \mathbb{A}^1,$$

where $F \subset E$ is a finite Galois field extension of degree $n$ and $f(t) \in F[t]$. Note that if this extension is cyclic, then here the problem is covered by Conjecture 1.1 (one introduces an appropriate cyclic algebra – see the construction used in the proof of Corollary 1.4). In general the Hasse principle does not hold for the fibres (Hasse’s norm theorem can fail for arbitrary extensions), however it is known by work of Sansuc [San81] that the Brauer-Manin obstruction is the only one to the Hasse principle and weak approximation for such principal homogeneous spaces under norm one tori. It seems reasonable to ask the same question in such cases.

There are many examples of families of varieties of interest which fail the above geometric condition, namely there may be some codimension one point $D \in X^{(1)}$ for which the algebraic closure of $F(D)$ inside the function field of the irreducible components of $\pi^{-1}(D)$ depends on the choice of irreducible component (e.g. quadric bundles of relative dimension two [Sko90] and torsors under multinorm tori). In such cases it would be interesting to see to
what extent the non-split fibres over points of codimension one determine the quantitative arithmetic of the family.

REFERENCES

[Art82] M. Artin, *Left ideals in maximal orders*, in Brauer groups in ring theory and algebraic geometry. Lecture Notes in Math., *917* Springer, Berlin-New York, 182–193, 1982.

[AM69] M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[BM90] V. V. Batyrev and Y. I. Manin, *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques*. Math. Ann., *286* (1990), 27–43.

[BT95] V. V. Batyrev and Y. Tschinkel, *Rational points of bounded height on compactifications of anisotropic tori*. Intern. Math. Research Notices, *12* (1995), 591–635.

[BT96] V. V. Batyrev and Y. Tschinkel, *Rational points on some Fano cubic bundles*. C. R. Acad. Sci., *323*, Ser. I, 41–46 (1996).

[BT98] V. V. Batyrev and Y. Tschinkel, *Manin’s conjecture for toric varieties*. J. Algebraic Geom., *7* (1) (1998), 15–53.

[Bir62] B. J. Birch, *Forms in many variables*, Proc. Roy. Soc. London Ser. A, *265* (1962), 245–263.

[Bor91] A. Borel, *Linear algebraic groups*. Second ed., Springer-Verlag, 1991.

[BLR90] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*. Springer, 1990.

[Bou11] D. Bourqui, *Fonctions zêta des hauteurs des variétés toriques non déployées*. Memoirs of the AMS, *211*, 2011.

[BB13a] R. de la Bretèche and T. Browning, *Density of Châtelet surfaces failing the Hasse principle*. Proc. London Math. Soc., to appear.

[BB13b] R. de la Bretèche and T. Browning, *Contre-exemples au principe de Hasse pour certains tores coflasques*. J. Théor. Nombres Bordeaux, to appear.

[Bro07] T. D. Browning, *An overview of Manin’s conjecture for del Pezzo surfaces*. Analytic number theory - A tribute to Gauss and Dirichlet (Goettingen, 20th June - 24th June, 2005), Clay Mathematics Proceedings *7* (2007), 39–56.

[BMS13] T. D. Browning, L. Matthiesen and A. Skorobogatov, *Rational points on pencils of conics and quadrics with many degenerate fibres*. arXiv:1209.0207.

[CF10] J. Cassels and A. Fröhlich, *Algebraic number theory*. London Mathematical Society, second ed. 2010.
[Del54] H. Delange, Généralisation du Théorème de Ikehara. Ann. Sc. E.N.S. (3) 71 (1954), fasc. 3, 213–242.

[FMT98] J. Franke, Y. I. Manin and Y. Tschinkel, Rational Points of Bounded Height on Fano Varieties. Invent. Math. 95 (1989), 421–435.

[Fro97] E. Frossard, Fibrés dégénérés des schémas de Severi-Brauer d’ordres. J. Algebra 198 (1997), no. 2, 362–387.

[FZ13] B. Fu and D.-Q. Zhang, A characterization of compact complex tori via automorphism groups. Math. Ann., to appear.

[GS06] P. Gille and T. Szamuely, Central simple algebras and galois cohomology. Cambridge Studies in Advanced Mathematics 101, 2006.

[Gro68] A. Grothendieck, Le groupe de Brauer. Dix exposés sur la cohomologie des schémas, North-Holland (1968), 46–188.

[Guo95] C. R. Guo, On solvability of ternary quadratic forms. Proc. London Math. Soc. (3) 70 (1995), no. 2, 241–263.

[Har94] D. Harari, Méthode des fibrations et obstructions de Manin. Duke Math. J. 75 (1994), 221–260.

[Har77] R. Hartshorne, Algebraic Geometry. Springer-Verlag, 1977.

[Has10] Y. Hashimoto, Partial zeta functions. Arch. Math. (Basel) 95 (2010), no. 4, 363–372.

[Hoo93] C. Hooley, On ternary quadratic forms that represent zero. Glasgow Math. J. 35 (1993), no. 1, 13–23.

[Hoo07] C. Hooley, On ternary quadratic forms that represent zero. II. J. Reine Angew. Math. 602 (2007), 179–225.

[IK04] H. Iwaniec and E. Kowalski, Analytic number theory. Amer. Math. Soc., 2004.

[KMRT98] M.-A. Knus, A. Merkurjev, M. Rost and J.-P. Tignol, The book of involutions. Amer. Math. Soc., 1998.

[Kur87] N. Kurokawa, On certain Euler products. Acta Arith. 48 (1987), no. 1, 49–52.

[Lan08] E. Landau, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate. Arch. der Math. und Physik (3), 13 (1908), p. 305–312 (= Collected Works, IV, p. 59–66).

[Lou13] D. Loughran, Rational points of bounded height and the Weil restriction. arXiv:math.NT/1210.1792 (2012).

[Mi06] J. S. Milne, Arithmetic duality theorems, second ed. BookSurge, LLC, Charleston, SC, 2006.

[NSW00] J. Neukirch, A. Schmidt and K. Wingberg, Cohomology of Number Fields, first ed. Springer-Verlag, 2000.

[Ono61] T. Ono, Arithmetic of algebraic tori, Ann. of Math. (2) 74 (1961), 101–139.

[Ono63] T. Ono, On the Tamagawa number of algebraic tori, Ann. of Math. (2) 78 (1963), 47–73.

[Pey95] E. Peyre, Hauteurs et mesures de Tamagawa sur les variétés de Fano. Duke Math. J., 79(1), 101–218 (1995).

[PV04] B. Poonen and P. Vojloch, Random Diophantine equations. Progr. Math., 226, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 175–184, Birkhäuser Boston, Boston, MA, 2004.

[San81] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, J. Reine Angew. Math. 327 (1981), 12–80.

[Ser90] J.-P. Serre, Spécialisation des éléments de Br2(Q(T1, . . . , Tn)), C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), no. 7, 397–402.

[Ser97] J.-P. Serre, Galois cohomology, corrected ed. Springer, 1997.

[Sko90] A. Skorobogatov, Arithmetic on certain quadric bundles of relative dimension 2, J. Reine Angew. Math. 407 (1990), 57–74.

[Sko96] A. Skorobogatov, Descent on fibrations over the projective line, Amer. J. Math. 118 (1996), 905–923.

[Sko01] A. Skorobogatov, Torsors and rational points. Cambridge University press, 2001.

[Ten95] G. Tenenbaum, Introduction to analytic and probabilistic number theory. Cambridge University press, 1995.

[Vos69] V. E. Voskresenski, The birational equivalence of linear algebraic groups, Dokl. Akad. Nauk SSSR 188 (1969), 978–981.

[Vos70] V. E. Voskresenski, Birational properties of linear algebraic groups, Izv. Akad. Nauk SSSR Ser. Mat. 34 (1970), 3–19.

[Wei74] A. Weil, Basic number theory. Springer, Third Ed. 1974.
[Wh07] O. Wittenberg, *Intersections de deux quadriques et pincesaux de courbes de genre 1*. Lecture Notes in Mathematics, 1901. Springer, Berlin, 2007.

DANIEL LOUGHRAN, DEPARTMENT OF MATHEMATICS, UNIVERSITY WALK, BRISTOL, UK, BS8 1TW. 
E-mail address: Daniel.Loughran@bristol.ac.uk