Pricing Exchange Rate Options and Quanto Caps in the Cross-Currency Random Field LIBOR Market Model

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Abstract

To hedge investments made in different economies, investors make use of cross-currency derivatives to manage risks associated with fluctuations of exchange rates. In this paper, we set forth the discussion of extending the random field LIBOR market model (RFLMM) to the cross-currency setting. We assume that forward LIBOR rates of the domestics and foreign economies are lognormal and derive an approximate closed-form pricing solution for a Quanto cap written on a foreign forward LIBOR rate. Further, we derive an exact pricing formula for an option written on the spot exchange rate between the domestic and a foreign economy.

1 Introduction

The London Inter-bank offered Rates (LIBOR) are reference interest rates at which banks lend loans to other banks in London [19]. Many authors have worked and developed closed-form derivative prices based on LIBOR rates by modeling uncertainties with a finite dimensional Brownian motion (for example see [5, 9, 10, 14]). Despite having many advantages, there are very few articles based on pricing interest-rate derivatives using random field interest models and the RFLMM. The main contribution of this paper is the construction of the cross-currency RFLMM. The cross-currency LIBOR market models were first developed by Mikkelsen [13] and Schlogl [18] while the RFLLM was developed by Wu and Xu [20]. The uncertainty in the model developed by Schlogl is driven by a finite dimensional Brownian motion. Scholg's model was extended to the multi-factor setting by Amin [1] and Brenner et al. [2]. Beveridge et al. [3] develop a discussion on displaced diffusion.
LIBOR market model in the cross-currency setting to price Bermudan options. The main result by Schögl [18] is that if all LIBOR rates in the domestic and foreign markets are assumed to be lognormal, the forward exchange rate is lognormal only for a single maturity time. We will show that this also holds true in the random field setting.

The structure of the article is as follows. In section 2, we briefly discuss the random field term structure model developed by Goldstein [7], and the random field LIBOR market model (RFLMM) developed by Wu and Xu [20]. Our main goal is to apply these two frameworks to extend the cross-currency LMM [18] to the random field setting. We construct the cross-currency RFLMM in section 3 and prove that even in the random field setting, it is not possible to assume that all the volatility terms for all maturities of the forward exchange rates are deterministic when all forward LIBOR rates are assumed to be lognormal. In section 4, we find an approximate valuation formula for the price of a Quanto cap in terms of the domestic currency even though the volatility terms of the forward exchange rates are stochastic. Finally in section 5, we construct an exact pricing formula for a call option written on the spot exchange rate. A similar formula in the finite dimensional Brownian motion case can be found in Musiela [15]. All closed-form derivative prices we state and prove in this paper has only so far been derived by other authors when the volatility terms are driven by a finite dimensional Brownian motion.

2 The Random Field Cross-Currency LIBOR Market Model

We begin this section by stating the dynamics of the domestic and the foreign instantaneous forward rates with respect to the the domestic and the foreign risk-neutral measures \( Q \) and \( Q_F \). Working in the Goldstein framework [7] we state the following:

\[
    df(t, T) = \mu^Q(t, T)dt + \sigma(t, T)dZ^Q(t, T),
\]

\[
    df_F(t, T) = \mu^{Q_F}(t, T)dt + \sigma_F(t, T)dZ^{Q_F}(t, T),
\]

where \( Z^Q \) and \( Z^{Q_F} \) are random fields satisfying the following conditions (see [17]):

1. \( Z^F(t, T) \) is continuous for all \( t, T \).
2. For any fixed $T$, the process $\{Z^P(t, T)\}_{t \leq T}$ is a $P$–martingale such that $E^P[dZ^P(t, T)] = 0$ and $\text{var}^P[dZ(t, T)] = dt$ for all $t \leq T$.

3. $dZ^P(t_1, T) dZ^P(t_2) = c(T_1, T_2) dt$ where $c$ is the correlation function of the random field which satisfies $C(T, T) = 1$.

We assume that the above assumptions are true for $P = Q, Q^F$ and that $c$ is a deterministic function. We also make the additional assumption that $dZ^Q(t_1) dZ^Q(t_2) = c(T_1, T_2) dt$. We also define a filtration generated by the random fields by $\mathcal{F}_t = \sigma\{Z(u, v) \mid u \leq t, u \leq v\}$.

For a fixed $T$, the process $Z^P(t, T)$ is a continuous martingale with quadratic variation $d < Z^P(t, T), Z^P(t, T) >= dt$. By Levy’s characterization theorem (see theorem 3.16 in [11]), $Z^P(t, T)$ is a Brownian motion (with respect to $P$) for all $T \geq 0$. Goldstein [4] proves that the no-arbitrage drift $\mu^P(t, T) = \sigma(t, T) \int_t^T \sigma(u, v) c(T, v) dv$. The above model is a generalization of the Heath-Jarrow-Morton (HJM) Model [8] to a random field. The HJM model was first constructed with the volatility term driven by a finite (but arbitrary) number of independent Brownian motions. This was first generalized to a Gaussian random field setting by Kennedy [12].

By $B(t, T)$ and $B_F(t, T)$ we denote the prices of the domestic and foreign zero-coupon bonds. These prices are calculated by

$$B(t, T) = \exp\left(-\int_t^T f(t, u) du\right),$$

$$B_F(t, T) = \exp\left(-\int_t^T f_F(t, u) du\right).$$

By a straightforward application of Ito’s lemma ([16] theorem 4.1.2) gives the $Q$–dynamics of the bond price as

$$\frac{dB(t, T)}{B(t, T)} = r(t) dt - \int_t^T \sigma(t, u) dZ^Q(t, u) du,$$

$$\frac{dB_F(t, T)}{B_F(t, T)} = r_F(t) dt - \int_t^T \sigma_F(t, u) dZ^Q_F(t, u) du$$

where $r(t) = f(t, t)$ and $r_F(t) = f_F(t, t)$. Goldstein [7] identifies the $T$–forward measure $Q_T$ (by taking $B(t, T)$ to the the numeraire asset) by the relationship
\[ dZ^Q_T(t, u) = dZ^Q(t, u) + \left( \int_t^T \sigma(t, v)c(u, v) \right) dt, \]  

\[ dZ^{Q_F}_T(t, u) = dZ^{Q_F}(t, u) + \left( \int_t^T \sigma_F(t, v)c(u, v) \right) dt \]  

where \( dZ^Q_T(t, T) \) and \( dZ^{Q_F}_T(t, T) \) are \( Q_T \) and \( Q^F_T \) random fields respectively. Therefore, the price of a derivative with expiration date \( T \) at time \( t \) in terms of \( Q_T \) is:

\[ V(t) = B(t, T)E^Q_T[V(T)]. \]

The Radon-Nikodym derivative of \( Q_T \) with respect to \( Q \) can be expressed as follows:

\[ \frac{dQ_T}{dQ} = \exp \left( - \int_0^T \int_t^T \sigma(s, u)dZ^Q(s, u)du - \frac{1}{2} \int_0^T \int_t^T \int_t^T \sigma(s, u)\sigma(s, v)c(u, v)dudvd \right). \]

**Theorem 2.1.** The dynamics of \( B(t, U)/B(t, T) \) with respect to \( Q_T \) is given by

\[ d \left( \frac{B(t, U)}{B(t, T)} \right) = \frac{B(t, U)}{B(t, T)} \left[ \int_U^T \sigma(t, u)dZ^Q_T(t, u)du \right]. \]

**Proof.** Observe that

\[ d \left( \frac{1}{B(t, T)} \right) = \frac{1}{B(t, T)} \left[ -r(t)dt + \int_t^T \sigma(t, u)dZ^Q(t, u)du + \int_t^T \int_t^T \sigma(t, u)\sigma(t, v)c(u, v)dudvdt \right]. \]

Applying Ito’s product rule to \( B(t, U)/B(t, T) \) yields the following:

\[
\begin{align*}
    d \left( \frac{B(t, U)}{B(t, T)} \right) &= B(t, U)d \left( \frac{1}{B(t, T)} \right) + \frac{1}{B(t, T)}dB(t, U) + d \left( \frac{1}{B(t, T)}, B(t, U) \right) \\
    &= \left( \frac{B(t, U)}{B(t, T)} \right) \left[ r(t)dt - \int_t^U \sigma(t, u)dZ^Q(t, u)du - r(t)dt + \int_t^T \sigma(t, u)dZ^Q(t, u)du \\
    &+ \int_t^T \int_t^T \sigma(t, u)\sigma(t, v)c(u, v)dudvdt - \int_t^T \int_t^U \sigma(t, u)\sigma(t, v)c(u, v)dudvdt \right].
\end{align*}
\]

Applying (3) to the above equation gives the desired result. \( \square \)

**3 The Cross-Currency RFLMM**

Fix a tenor \( 0 \leq T_1 < T_2 < ... < T_N \leq T \). Schlogl defines a process \( X(t) \) as the spot exchange rate at time \( t \) in terms of the domestic currency per one unit of foreign currency. \( X(t) \) is assumed
to be strictly positive martingale with respect to the $T-$ forward measure $Q_T$. The $T_i-$forward exchange rate is defined as

$$X(t, T_i) := \frac{B_F(t, T_i)X(t)}{B(t, T_i)} \tag{6}$$

where $B_F(t, T_i)$ here denotes the price of a bond quoted in some foreign currency. $X(t, T_i)$ can be interpreted as the spot exchange rate at a future date $T_i$ as seen from time $t$. For the purposes of this paper, we will not be directly specifying the dynamics of the spot exchange rate $X(t)$ and instead we will work with the dynamics of the forward exchange rate. The dynamics of $X(t)$ in the finite dimensional Brownian motion case can be found in [15], [4].

Without loss of generality, assume that the dynamics of $X(t, T_N)$ with respect to $Q_{T_N}$ is given by

$$dX(t, T_N) = X(t, T_N) \int_t^{T_N} \sigma_{X_N}(t, u)dZ_{Q_{T_N}}(t, u)du. \tag{7}$$

We also assume that $\sigma_{X_N}$ is a deterministic function. In the next section we will show that $\sigma_{X_i}$ for all $i = 1, 2, ..., N - 1$ is stochastic if lognormal dynamics is assumed for all LIBOR rates in all economies.

### 3.1 The dynamics of the foreign LIBOR rate with respect to the domestic forward measure.

Now we state the key results developed Wu and Xu [20] and extend it to the cross-currency format. Fix a discrete tenor $0 < T_0 < T_1 < ... < T_N \leq T$. The domestic forward LIBOR rate $L(t, T_i)$ is defined in terms of bond prices in the following way:

$$1 + \delta_{i+1}L(t, T_i) = \frac{B(t, T_i)}{B(t, T_{i+1})}$$

where $\delta_{i+1} = T_{i+1} - T_i$. Similarly, the foreign forward LIBOR rate $L_F(t, T_i)$ is defined by

$$1 + \delta_{i+1}L_F(t, T_i) = \frac{B_F(t, T_i)}{B_F(t, T_{i+1})}.$$ 

As a consequence of theorem (2.1), $L(t, T_i)$ and $L_F(t, T_i)$ are martingales with respect to $Q_{T_{i+1}}$ and $Q^F_{T_{i+1}}$. Using (3) and (1), we can see that $Q_{T_{i+1}}$, $Q_T$, and $Q^F_{T_{i+1}}$, $Q^F_{T_1}$ are related by

$$dZ_{Q_{T_{i+1}}}(t, u) = dZ_{Q_T}(t, u) + \left(\int_{T_i}^{T_{i+1}} \sigma(t, v)c(u, v)dv\right) dt \tag{8}$$
Applying Ito’s rule to the definitions of the domestic and foreign LIBOR rates together with the equations (8) and (9), it can be shown that

\[
dL(t, T_i) = \frac{B(t, T_i)}{\delta_{i+1} B(t, T_{i+1})} \int_{T_i}^{T_{i+1}} \sigma(t, u) dZ^{Q_{T_{i+1}}}(t, u) du \tag{10}
\]

and

\[
dL_F(t, T_i) = \frac{B_F(t, T_i)}{\delta_{i+1} B_F(t, T_{i+1})} \int_{T_i}^{T_{i+1}} \sigma_F(t, u) dZ^{Q_{T_{i+1}}^F}(t, u) du. \tag{11}
\]

Since \(L(t, T_i)\) and \(L_F(t, T_i)\) are \(Q_{T_{i+1}}\) and \(Q_{T_{i+1}}^F\) martingales respectively, by the martingale representation theorem [16], there exist \(\mathcal{F}_t\) adapted function \(\xi_i\) and \(\xi_i^F\) such that

\[
dL(t, T_i) = \int_{T_i}^{T_{i+1}} \xi_i(t, u) dZ^{Q_{T_{i+1}}}(t, u) du \tag{12}
\]

and

\[
dL_F(t, T_i) = \int_{T_i}^{T_{i+1}} \xi_i^F(t, u) dZ^{Q_{T_{i+1}}^F}(t, u) du. \tag{13}
\]

Observe that

\[
\xi_i(t, u) = \frac{B(t, T_i)}{\delta_{i+1} B(t, T_{i+1})} \sigma(t, u),
\]

\[
\xi_i^F(t, u) = \frac{B_F(t, T_i)}{\delta_{i+1} B_F(t, T_{i+1})} \sigma_F(t, u).
\]

Now define the functions \(\lambda_i, \lambda_i^F\) such that

\[
L(t, T_i) \lambda_i(t, u) = \xi_i(t, u),
\]

\[
L_F(t, T_i) \lambda_i^F(t, u) = \xi_i^F(t, u).
\]

Therefore,
\[ \lambda_i(t, u) = \frac{1 + \delta_{i+1}L(t, T_i)}{\delta_{i+1}L(t, T_i)} \sigma(t, u), \]  
\[ \lambda_i^F(t, u) = \frac{1 + \delta_{i+1}L_F(t, T_i)}{\delta_{i+1}L_F(t, T_i)} \sigma_F(t, u). \]  

It is clear from (14) and (15) that the functions \( \lambda_i \) and \( \lambda_i^F \) are stochastic in general. If we assume \( \lambda_i \) and \( \lambda_i^F \) are deterministic, this leads to the lognormal RFLMM. However, according to equations (14) and (15), it is clear that the volatility functions \( \sigma \) and \( \sigma_F \) are stochastic functions in the lognormal RFLMM. We can restate the dynamics of the lognormal LIBOR rates as follows:

\[ dL(t, T_i) = L(t, T_i) \int_{T_i}^{T_{i+1}} \lambda_i(t, u)dZ^{Q_{T_i+1}}(t, u)du \]  

and

\[ dL_F(t, T_i) = L_F(t, T_i) \int_{T_i}^{T_{i+1}} \lambda_i^F(t, u)dZ^{Q_{T_i+1}^F}(t, u)du. \]  

In order to express (13) in terms of \( Z^{Q_{T_i+1}} \), we need to first find a relationship between \( Z^{Q_{T_i+1}} \) and \( Z^{Q_{T_i+1}^F} \). Hence we state and prove the following theorem:

**Theorem 3.1.** The random fields under the measures \( Q_{T_i} \) and \( Q_{T_i}^F \) are related by

\[ dZ^{Q_{T_i}^F}(t, u) = dZ^{Q_{T_i}}(t, u) - \left( \int_t^{T_i} \sigma_X(t, v)c(u, v)dv \right) dt. \]  

**Proof.** Recall that \( X(t, T_i) \) is a martingale with respect to \( Q_{T_i} \). Therefore,

\[ \frac{1}{X(t, T_i)} = \frac{B(t, T_i)}{B_F(t, T_i)} \]

is a martingale with respect to the foreign \( T_i \)-forward measure \( Q_{T_i+1}^F \).

Applying Ito’s lemma to \( \frac{1}{X(t, T_i)} \) yields that

\[ d\left( \frac{1}{X(t, T_i)} \right) = -\frac{1}{X(t, T_i)^2}dX(t, T_i) + \frac{1}{X(t, T_i)}d(X(t, T_i), X(t, T_i)) \]

\[ = \frac{1}{X(t, T_i)} \left[ -\int_t^{T_i} \sigma_X(t, u)dZ^{Q_{T_i}}(t, u)du + \int_t^{T_i} \int_t^{T_i} \sigma_X(t, u)\sigma_X(t, v)c(u, v)dudvdt \right]. \]  

Observe that the volatility term of \( \frac{1}{X(t, T_i)} \), \( \sigma_{\frac{1}{X_i}}(t, u) = -\sigma_X(t, u) \). Since \( \frac{1}{X(t, T_i)} \) is a martingale under \( Q_{T_i}^F \),

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Comparing (19) with (20) proves the desired result.

Now that we have a relationship between the foreign and local random fields, we can state the dynamics of the foreign LIBOR rate $L_F(t, T_i)$ with respect to the domestic measure $Q_{T_{i+1}}$ as follows.

**Theorem 3.2.** Fix $i \in \{1, 2, ..., N - 1\}$. For every $t \in [0, T_i]$,

$$L_F(T_i, T_i) = L_F(t, T_i) \exp(-\alpha(t, T_i, T_{i+1})) \exp \left( \int_t^{T_i} \int_{T_i}^{T_{i+1}} \lambda^F_s(s, T_i) dZ^{Q_{T_{i+1}}}(s, u) du \right. \right.$$ 

$$- \frac{1}{2} \int_t^{T_i} \int_{T_i}^{T_{i+1}} \int_{T_i}^{T_{i+1}} \lambda^F_s(s, u) \lambda^F_s(s, v) c(u, v) dudvds \left. \right)$$

(21)

where

$$\alpha(t, T_i, T_{i+1}) := \int_t^{T_i} \int_{T_i}^{T_{i+1}} \int_s^{T_{i+1}} \lambda^F_s(s, u) \sigma_{X_{i+1}}(s, v) c(u, v) dvduds.$$ (22)

**Proof.** An application of Ito’s lemma to $\ln(L_F(t, T_i))$ together with (18) yields (21). \hfill \Box

Notice that $\alpha(t, T_i, T_{i+1})$ is a stochastic function when $i \neq N - 1$. We will show that we can only find an approximate formula for the price of a Quanto cap because the function $\alpha(t, T_i, T_{i+1})$ is not deterministic. To find a suitable approximation for $\alpha(t, T_i, T_{i+1})$, we recursively find a relationship between $\sigma_{X_i}$ and $\sigma_{X_N}$ for all $i = 0, 1, 2, ..., N - 1$.

Subtracting (8) and (9) and applying (18) to the resulting equation yields

$$\int_t^{T_i} \sigma_{X_i}(t, v) c(u, v) dv = \int_t^{T_{i+1}} (\sigma_F(t, v) - \sigma(t, v)) c(u, v) dv + \int_t^{T_{i+1}} \sigma_{X_{i+1}}(t, v) c(u, v) dv$$

(23)

for all $i = 0, 1, 2, ..., N - 1$. Hence, recursively we can show that

$$\int_t^{T_i} \sigma_{X_i}(t, v) c(u, v) dv = \sum_{j=1}^{N-1} \int_{T_j}^{T_{j+1}} (\sigma_F(t, v) - \sigma(t, v)) c(u, v) dv + \int_t^{T_N} \sigma_{X_N}(t, v) c(u, v) dv.$$ (24)
According to (24), it is clear that once $\sigma_N$ is chosen to be deterministic, it is no longer possible for $\sigma_X$ to be deterministic for $i \neq N$ because $\sigma$ and $\sigma_F$ are stochastic functions. This result in the Brownian motion setting was proved by Schlogl [18]. Therefore, (24) is the generalization of the main result in [18] to the random field setting.

Now we find an approximation $\tilde{\alpha}(t, T_i, T_{i+1})$ for $\alpha(t, T_i, T_{i+1})$ which will be used to approximate the valuation formula for a Quanto cap. To simplify notation, we define the following two functions:

$$A(t, T_j) = \frac{\delta_{j+1}L(t, T_j)}{1 + \delta_{j+1}L(t, T_j)},$$

$$A_F(t, T_j) = \frac{\delta_{j+1}L_F(t, T_j)}{1 + \delta_{j+1}L_F(t, T_j)}.$$

By substituting the above functions, (14), and (15) into (24) we get

$$\int_t^{T_i} \sigma_X(t, v)c(u, v)dv = \sum_{j=i+1}^{N-1} A_F(t, T_j) \int_{T_j}^{T_{j+1}} \lambda_j^F(t, u)c(u, v)dv$$

$$- \sum_{j=i+1}^{N-1} A(t, T_j) \int_{T_j}^{T_{j+1}} \lambda_j(t, u)c(u, v)dv + \int_t^{T_N} \sigma_X(t, v)c(u, v)dv.$$

By substituting (25) into (22) we get

$$\alpha(t, T_i, T_{i+1}) = \int_t^{T_i} \int_{T_i}^{T_{i+1}} \lambda_i^F(s, u) \sum_{j=i+1}^{N-1} A_F(s, T_j) \int_{T_j}^{T_{j+1}} \lambda_j^F(s, v)c(u, v)dvduds$$

$$- \int_t^{T_i} \int_{T_i}^{T_{i+1}} \lambda_i^F(s, u) \sum_{j=i+1}^{N-1} A(s, T_j) \int_{T_j}^{T_{j+1}} \lambda_j(s, v)c(u, v)dvduds$$

$$+ \int_t^{T_i} \int_{T_i}^{T_{i+1}} \int_{T_N}^{T_N} \lambda_i^F(s, u)\sigma_X(s, v)c(u, v)dvduds.$$

Now we find an approximation $\tilde{\alpha}(t, T_i, T_{i+1})$ to $\alpha(t, T_i, T_{i+1})$ by freezing the $L(s, T_i)$ and $L_F(s, T_i)$ at time $t$ where $s \in [t, T_i]$. 

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\[
\tilde{\alpha}(t, T_i, T_{i+1}) = \sum_{j=i+1}^{N-1} A_F(t, T_j) \int_t^{T_i} \int_{T_j}^{T_{j+1}} \lambda_F^F(s, u) \lambda_j^F(s, v) c(u, v) dv du ds
\]

\[
- \sum_{j=i+1}^{N-1} A(t, T_j) \int_t^{T_i} \int_{T_j}^{T_{j+1}} \lambda_j^F(s, u) \lambda_j^F(s, v) c(u, v) dv du ds
\]

\[
+ \int_t^{T_i} \int_{T_i}^{T_{i+1}} \int_s^{T_N} \lambda_j^F(s, u) \sigma_X N(s, v) c(u, v) dv du ds.
\]  

Freezing approximation methods have been used to approximate valuation formulas for swaps and swaptions in [6] and [20]. While \( \alpha(t, T_i, T_{i+1}) \) and \( \tilde{\alpha}(t, T_i, T_{i+1}) \) are both stochastic processes, the striking difference between the two processes is that \( \tilde{\alpha}(t, T_i, T_{i+1}) \) is \( F_t \) measurable and \( \alpha(t, T_i, T_{i+1}) \) is not.

4 Pricing Quanto Caps in the Cross-Currency RFLMM

A Quanto is a type of a derivative where the underlying asset is given in terms of one currency and the price of the derivative is quoted in terms of another currency. In this section, we find an approximate valuation formula for a cap written on a foreign LIBOR rate priced in terms of the domestic currency. We first state the price of a Quanto caplet (a single period Quanto cap).

**Theorem 4.1 (The Price of a Quanto Caplet).** Consider a caplet with a reset date \( T_i \), settling date \( T_{i+1} \), and a strike rate \( \kappa \) written on a foreign LIBOR rate \( L_F \) with a payoff \( \delta_{i+1} X (L_F(T_i, T) - \kappa)^+ \) at time \( T_{i+1} \) where \( X \) is a predetermined fixed exchange rate. The price of the caplet at time \( t \leq T_i \) is given by

\[
QCapl(t, T_i, T_{i+1}) \approx \delta_{i+1} X B(t, T_{i+1}) [L_F(t, T_i) N(\tilde{d}_1(t, T_i, T_{i+1}))/2 - N(\tilde{d}_2(t, T_i, T_{i+1}))/2]
\]

where

\[
\tilde{d}_1(t, T_i, T_{i+1}) = \frac{\ln \left( \frac{L_F(t, T_i)}{\kappa} \right) - \tilde{\alpha}(t, T_i, T_{i+1}) + \frac{1}{2} \tilde{\Omega}(t, T_i, T_{i+1})}{\sqrt{\tilde{\Omega}(t, T_i, T_{i+1})}}
\]

\[
\tilde{d}_2(t, T_i, T_{i+1}) = \frac{\ln \left( \frac{L_F(t, T_i)}{\kappa} \right) - \tilde{\alpha}(t, T_i, T_{i+1}) - \frac{1}{2} \tilde{\Omega}(t, T_i, T_{i+1})}{\sqrt{\tilde{\Omega}(t, T_i, T_{i+1})}}
\]

and
\[ \tilde{\Omega}(t, T_i, T_{i+1}) = \int_t^{T_i} \int_{T_i}^{T_{i+1}} \int_{T_i}^{T_{i+1}} \lambda_i^F(s, u) \lambda_i^F(s, v) c(u, v) du dv ds. \]

**Proof.** The price of this caplet at a time \( t \in [0, T_i] \) with respect to the \( QT_{i+1} \) forward measure (corollary 9.6.1) can be expressed as follows:

\[
Q_{\text{capl}}(t, T_i, T_{i+1}) = \delta_{i+1} \bar{X}B(t, T_{i+1}) E_t^{QT_{i+1}} [(L_F(T_i, T_i) - k)^+] = \delta_{i+1} \bar{X}B(t, T_{i+1})[I_1 - kI_2]
\]

where

\[
I_1 = E_t^{QT_{i+1}} [L_F(T_i, T_i)I_D],
\]

\[
I_2 = E_t^{QT_{i+1}} [I_D]
\]

and \( D = \{L_F(T_i, T_i) > k\} \) and \( E_t[\cdot] \) denotes the conditional expectation with respect to \( F_t \).

By taking the natural log of both sides of (21) and replacing \( \alpha(t, T_i, T_{i+1}) \) with \( \tilde{\alpha}(t, T_i, T_{i+1}) \) we get that

\[
\ln L_F(T_i, T_i) \approx \ln L_F(t, T_i) - \tilde{\alpha}(t, T_i, T_{i+1}) + \int_t^{T_i} \int_{T_i}^{T_{i+1}} \lambda_i^F(s, u) dZ^{QT_{i+1}}(s, u) du - \frac{1}{2} \int_t^{T_i} \int_{T_i}^{T_{i+1}} \lambda_i^F(s, u) \lambda_i^F(s, v) c(u, v) du dv ds.
\]

(28)

Therefore, the \( QT_{i+1} \) conditional variance and expectation of \( \ln L_F(T_i, T_i) \) are as follows:

\[
Var_t^{QT_{i+1}}(\ln L_F(T_i, T_i)) \approx \int_t^{T_i} \int_{T_i}^{T_{i+1}} \int_{T_i}^{T_{i+1}} \lambda_i^F(s, u) \lambda_i^F(s, v) c(u, v) dv ds := \tilde{\Omega}(t, T_i, T_{i+1}).
\]

(29)

\[
E_t^{QT_{i+1}}(\ln L_F(T_i, T_i)) \approx \ln L_F(t, T_i) - \tilde{\alpha}(t, T_i, T_{i+1}) - \frac{1}{2} \tilde{\Omega}(t, T_i, T_{i+1}).
\]

(30)

Since \( \ln L_F(T_i, T_i) \mid F_t \) is approximately normally distributed,

\[
I_2 \approx N \left( \frac{\ln \left( \frac{L_F(t, T_i)}{k} \right) - \tilde{\alpha}(t, T_i, T_{i+1}) - \frac{1}{2} \tilde{\Omega}(t, T_i, T_{i+1})}{\sqrt{\tilde{\Omega}(t, T_i, T_{i+1})}} \right)
\]

(31)

where \( N(\cdot) \) denotes the cumulative distribution function of a standard normal distribution. To evaluate \( I_1 \) we first define an equivalent measure \( \hat{Q}_{T_{i+1}} \sim Q_{T_{i+1}} \) by
\[
\frac{d\hat{Q}_{T+1}}{dQ_{T+1}} = \eta(T, T, T+1)
\]

where

\[
\eta(t, T, T+1) := \exp\left( \int_t \lambda^F(s, u) dZ^{Q_t} (s, u) du - \frac{1}{2} \int_t \int_t^{T+1} \lambda^F(s, u) \lambda^F(s, v) c(u, v) du dv \right).
\]

The process \(Z^{\hat{Q}_{T+1}}\) defined by

\[
dZ^{\hat{Q}_{T+1}}(s, u) = dZ^Q (s, u) - \left( \int_{T+1}^T \lambda^F(s, v) c(u, v) dv \right) ds
\]

is a \(\hat{Q}_{T+1}\) random field based on Girsanov theorem ([16] theorem 8.6.4). Under \(\hat{Q}_{T+1}\), (28) transforms to

\[
\ln L_F(T, T) \approx \ln L_F(t, T) - \tilde{\alpha}(t, T, T+1) + \frac{1}{2} \tilde{\Omega}(t, T, T+1).
\]

This implies that

\[
E_t^{\hat{Q}_{T+1}}[\ln L_F(T, T)] \approx \ln L_F(t, T) - \tilde{\alpha}(t, T, T+1) + \frac{1}{2} \tilde{\Omega}(t, T, T+1).
\]

Notice that the conditional variance remains unchanged under a change measure. Using Bayes' rule ([16] lemma 8.6.2) we evaluate \(I_1\) as follows:

\[
I_1 = E_t^{Q_{T+1}}[L(T, T) \mathbb{I}_D]
\]

\[
= E_t^{Q_{T+1}}[L(t, T) \eta(T, T, T+1) \eta(t, T, T+1)^{-1} \mathbb{I}_D]
\]

\[
= L(t, T) E_t^{Q_t}[\eta(T, T, T+1)^{-1} \mathbb{I}_D]
\]

\[
= L(t, T) E_t^{Q_t}[\mathbb{I}_D]
\]

\[
\approx L(t, T) N\left( \frac{\ln L_F(t, T) - \tilde{\alpha}(t, T, T+1) + \frac{1}{2} \tilde{\Omega}(t, T, T+1)}{\sqrt{\tilde{\Omega}(t, T, T+1)}} \right).
\]
The next result follows immediately from the preceding theorem.

**Corollary 4.1. (The price of a Quanto Cap)**

The price of a Quanto cap with reset dates $T_i$ for $i = 0, 1, 2, \ldots N - 1$ and settling dates $T_i$ for $i = 1, 2, \ldots N$ with a strike rate $\kappa$ at time $t \leq T_0$ equals

$$QCap(t) = \sum_{i=0}^{N-1} QCap(t, T_i, T_{i+1}).$$

## 5 Exchange Rate Options

In this section we focus on deriving a closed-form pricing formula for an option written on the spot exchange rate $X(t)$. Consider a call option written on the spot exchange rate $X$ with an expiration date $T = T_N$, strike rate $K$, and principal amount of 1 unit of domestic currency. The payoff at time $T$ can be written as

$$C_X^T = (X(T) - K)^+ = (X(T, T) - K)^+. $$

**Theorem 5.1.** Consider a call option written on the spot exchange rate $X$ which expires at time $T$ and strike rate $k$. Then the arbitrage price of this call option at time $t$ denoted by $C_t^X$ is given by

$$C_t^X = B(t, T)(X(t, T)N(d_1) - kN(d_2)) \quad (35)$$

where

$$d_1 := \frac{\ln \left( \frac{X(t, T)}{k} \right) + \frac{1}{2} \gamma(t, T)}{\sqrt{\gamma(t, T)}},$$

$$d_2 := \frac{\ln \left( \frac{X(t, T)}{k} \right) - \frac{1}{2} \gamma(t, T)}{\sqrt{\gamma(t, T)}},$$

and

$$\gamma(t, T) := \int_t^T \int_s^T \int_s^T \sigma_{XT}(s, u)\sigma_{XT}(s, v)c(u, v)du dv ds.$$
and the dynamics of \(X(t, T)\) is given by

\[
dX(t, T) = X(t, T) \int_t^T \sigma_{X_T}(t, u)dZ^{Q_T}(t, u)du.
\]

(36)

By Ito’s rule

\[
d\ln X(t, T) = \int_t^T \sigma_{X_T}(t, u)dZ^{Q_T}(t, u)du - \frac{1}{2} \int_t^T \int_t^T \sigma_{X_T}(t, u)\sigma_{X_T}(t, v)c(u, v)du dv dt.
\]

Integrating the above equation yields

\[
X(T, T) = X(t, T) \exp \left( \int_t^T \int_s^T \sigma_{X_T}(s, u)dZ^{Q_T}(s, u)du 
- \frac{1}{2} \int_t^T \int_s^T \int_s^T \sigma_{X_T}(s, u)\sigma_{X_T}(s, v)c(u, v)du dv ds \right)
\]

(37)

The price with respect to the forward measure \(Q_T\) is

\[
C_t^X = B(t, T) E^{Q_T}_{t} [(X(T, T) - k)^+]
= B(t, T)[I_1 - k I_2]
\]

where

\[
I_1 := E^{Q_T}_{t}[X(T, T) \mathbb{1}_D]
\]

\[
I_2 := E^{Q_T}_{t}[\mathbb{1}_D]
\]

and \(D = \{X(T, T) > k\}\). Let us first evaluate \(I_2\). For this, we have to find the mean and the variance of \(\ln X(T, T)\) with respect to \(Q_T\) conditioned on \(\mathcal{F}_t\). By (37),

\[
\ln X(T, T) = \ln X(t, T) + \int_t^T \int_s^T \sigma_{X_T}(s, u)dZ^{Q_T}(s, u)du 
- \frac{1}{2} \int_t^T \int_s^T \int_s^T \sigma_{X_T}(s, u)\sigma_{X_T}(s, v)c(u, v)du dv ds.
\]

(38)

Therefore,

\[
Var^{Q_T}_{t}[\ln X(T, T)] = \int_t^T \int_s^T \int_s^T \sigma_{X_T}(s, u)\sigma_{X_T}(s, v)c(u, v)du dv ds 
:= \gamma(t, T),
\]

(39)
and

\[ E_t^{QT} [\ln X(T, T)] = \ln X(t, T) - \frac{1}{2} \int_t^T \int_s^T \int_s^T \sigma_{X_T}(s, u) \sigma_{X_T}(s, v) c(u, v) dudvds \]

\[ = \ln X(t, T) - \frac{1}{2} \gamma(t, T). \]  \hspace{1cm} (40)

Since \( \ln X(T, T) | \mathcal{F}_t \) is normally distributed,

\[ I_2 = N \left( \frac{\ln \left( \frac{X(t, T)}{k} \right) - \frac{1}{2} \gamma(t, T)}{\sqrt{\gamma(t, T)}} \right). \]

To evaluate \( I_1 \), we define a new random field \( \hat{Z}^{QT} \) by

\[ d\hat{Z}^{QT}(s, u) = dZ^{QT}(s, u) - \left( \int_s^T \sigma_X(s, v)c(u, v) dv \right) ds. \]  \hspace{1cm} (41)

Then by Girsanov theorem, there exists a measure \( \hat{Q}_T \sim Q_T \) such that \( \{ Z(t, T) \}_t \) is a \( \hat{Q}_T \) Brownian motion. In particular, the Radon-Nikodym derivative of \( \hat{Q}_T \) with respect to \( Q_T \) is given by

\[ \frac{d\hat{Q}_T}{dQ_T} = \eta(T, T) \]

where

\[ \eta(t, T) = \exp \left( \int_0^t \int_s^T \sigma_{X_T}(s, u) dZ^T(s, u) du - \frac{1}{2} \int_0^t \int_s^T \int_s^T \sigma_{X_T}(s, u) \sigma_{X_T}(s, v) c(u, v) dudvds \right) \]

is a strictly positive martingale with respect to \( Q_T \). In particular, \( \eta(t, T) = E_t^{QT} [\eta(T, T)] \) for all \( t \leq T \). Applying (41) to (37) can express the dynamics of \( X(T, T) \) under the measure \( \hat{Q}_T \) as

\[ X(T, T) = X(t, T) \exp \left( \int_t^T \int_s^T \sigma_X(s, u) d\hat{Z}^{QT} \\
+ \frac{1}{2} \int_t^T \int_s^T \sigma_X(s, u) \sigma_X(s, v) c(u, v) dudvds \right). \]  \hspace{1cm} (42)

Therefore, using Bayes’ rule we show that

\[ I_1 = E_t^{QT} [X(T, T) \mathbb{I}_D] \]

\[ = E_t^{QT} [X(t, T) \eta(T, T) \eta(t, T)^{-1} \mathbb{I}_D] \]

\[ = X(t, T) E_t^{QT} \left[ \eta(T, T) \mathbb{I}_D \right] \]

\[ = X(t, T) E_t^{QT} \left[ \mathbb{I}_D \right]. \]
To evaluate $E_t^Q [1_D]$, observe that

$$
E_t^Q [\ln X(T, T) \mid \mathcal{F}_t] = \ln X(t, T) + \frac{1}{2} \int_t^T \int_s^T \sigma_X(s, u) \sigma_X(s, v) c(u, v) du dv ds
$$

\begin{equation}
= \ln X(t, T) + \frac{1}{2} \gamma(t, T).
\end{equation}

(43)

Therefore,

$$
I_1 = X(t, T) N \left( \frac{\ln \left( \frac{X(t, T)}{\kappa} \right) + \frac{1}{2} \gamma(t, T)}{\sqrt{\gamma(t, T)}} \right).
$$

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