GENERALIZED NIKOLSKII’S PROPERTY AND ASYMPTOTIC EXPONENT IN MARKOV’S INEQUALITY

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Abstract. We introduce an asymptotic Markov’s exponent and show that it is equal to Markov’s exponent for a wide class of norms. However it is not true for all norms in the space of polynomials, as it will be presented in few examples. We shall prove an important inequality $m(q) \geq m(E)$, where $q$ is a norm in $P(C^N)$ with Nikolski’s property related to $E$. As a consequence we obtain a lower bound for the optimal exponent in Markov’s inequality considered with the $L^p$ norms and other norms possessing Nikolski type property.

Keywords Nikolski property, Markov properties, Markov exponent, polynomial inequalities.

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1. Introduction

By $P(K^N)$ ($P_d(K^N)$), respectively we shall denote the vector space of all polynomials of $N$ variables with coefficients in the field $K$ (with total degree $\leq d$). Let us recall the multivariate Markov’s inequality

\begin{equation}
\left\| \frac{\partial}{\partial x_j} P \right\|_E \leq M (\deg P)^m \|p\|_E
\end{equation}

where $\| \cdot \|_E$ is the supremum norm on $E$.

A compact set $E$ with the above property is called the Markov’s set. It is a generalization of the classical inequality proven by A. A. Markov in 1889, which gives such estimate on $[-1,1]$. The development of the theory of generalizations of this inequality is still continuing. More information about the various generalizations of Markov’s inequality can be found in [39], [33], [40], [28], [42], [41]. It is important to know more about the best exponent in this inequality for a given set $E$. The notion
\(m(E)\) called Markov’s exponent was defined in [10]. For a Markov set \(E\) it is \(m(E) := \inf \{ s > 0 : E \text{ is Markov’s set with exponent } s \}\). If \(E\) is not Markov’s set, we put \(m(E) := \infty\). It is known that \(m(E) \geq 2\) in the real case and \(m(E) \geq 1\) in the complex one. The surprising fact, proved in [7], is that Markov’s inequality does not have to fulfilled with Markov’s exponent (see also [27]).

Similarly we define Markov’s exponent with respect to other norms, if \(q\) is a norm on \(\mathbb{P}(\mathbb{K}^N)\) we can define Markov’s exponent for the norm \(q\) as

\[
m(q) = \inf \{ s > 0 : \exists C > 0 \forall P \in \mathbb{P}(\mathbb{K}^N) \forall 1 \leq j \leq N \| \frac{\partial}{\partial x_j} P \| \leq C (\deg P)^s q(P) \}.
\]

The Markov type inequalities were also considered in \(L^p\) norms (cf. [29],[6],[12],[16],[21],[24],[25],[26],[38]). In this case a progression in research seems to be slower except \(L^2\) norms are considered (cf. e.g. [13],[14],[15],[19],[20],[2],[1]). In particular, an example of a compact set in \(\mathbb{R}^N\) with cusps for which Markov’s exponent (with respect to the Lebesgue measure) is calculated, is still out of reach.

We can consider Markov’s inequality for any other norm. Then Markov’s exponent can even be equal to 0.

**Example 2.** For the norm \(\| P \| = \sum_{k=0}^{\infty} |P^{(k)}(0)|\) we have

\[
\| P' \| = \sum_{k=1}^{\infty} |P^{(k)}(0)| \leq \| P \|.
\]

However, if we have a spectral norm \(q\) (it means for every polynomial \(P \in \mathbb{P}(\mathbb{C}^N)\), \(q(P^n) = (q(P))^n\)) and Markov’s inequality holds for this norm, then the exponent \(m(q)\) has to be not less than 1. Indeed, let us consider polynomials \(P_j(x) = x^n_j\), for \(j \in \{1,2,\ldots,N\}\). Then

\[
\| x_j \|^n = \| P_j \| \geq \frac{1}{M^n(n!)^m} \| \frac{\partial^n P_j}{\partial x_j^n} \| = \frac{n!}{M^n(n!)^m} \| 1 \| = \frac{1}{M^n(n!)^{1-m}}.
\]

Hence

\[
\| x_j \| \geq \frac{1}{M} (n!)^{(1-m)^{\frac{1}{n}}}.
\]

This inequality is possible only for \(m \geq 1\).

**Remark 3.** It was proved in [8], the Markov type condition

\[
\left\| \frac{\partial}{\partial z_j} P \right\|_E \leq M(\deg P)^m \| P \|_E, \ j = 1,\ldots,N, P \in \mathbb{P}(\mathbb{C}^N)
\]
with positive constants $M$ and $m$ is equivalent to the inequality
\[
\left\| \sum_{j=1}^{N} \frac{\partial^{2l} P}{\partial z_j^{2l}} \right\|_E \leq M'(\deg P)^{2m} \|P\|_E, \quad P \in \mathbb{P}(\mathbb{C}^N)
\]
with some positive constant $M'$. Here $E \subset \mathbb{R}^N$ and $l \in \mathbb{Z}_+$ is fixed. In particular, Markov’s property with exponent $m$ is equivalent to the bound $\|\Delta P\|_E \leq M'(\deg P)^{2m} \|P\|_E$.

2. NIKLSSII’S PROPERTY

**Definition 4.** Let $E$ be a compact subset of $\mathbb{C}^N$. A norm $q = \| \cdot \|$ on $\mathbb{P}(\mathbb{C}^N)$ is $E$-admissible or has Nikolskii’s property if there exist constants: positive $A, B$ and nonnegative $a, b$ such that for every $P \in \mathbb{P}(\mathbb{C}^N)$ with $\deg P \geq 1$ we have
\[
\|P\|_E \leq A(\deg P)^a \|P\| \quad \text{and} \quad \|P\| \leq B(\deg P)^b \|P\|_E.
\]
If $q = \| \cdot \|$ is $E$-admissible then
\[
\|P\|_E = \lim_{s \to \infty} \|P^s\|^{1/s}.
\]
Since the supremum norm is the main example of spectral norm (see [50]) we can generalize the above definition.

**Definition 5.** A norm $q = \| \cdot \|$ on $\mathbb{P}(\mathbb{C}^N)$ is spectral admissible or has the generalized Nikolskii’s property if there exist a spectral norm $\| \cdot \|_\sigma$ and constants: positive $A, B$ and nonnegative $a, b$ such that for every $P \in \mathbb{P}(\mathbb{C}^N)$ with $\deg P \geq 1$ we have
\[
\|P\|_\sigma \leq A(\deg P)^a \|P\| \quad \text{and} \quad \|P\| \leq B(\deg P)^b \|P\|_\sigma.
\]
The spectral norm is given by the formula
\[
q_\sigma(P) = \|P\|_\sigma = \lim_{s \to \infty} \|P^s\|^{1/s}.
\]
By way of illustration, here are examples of such norms.

**Example 6.** Let $E$ be a compact subset of $\mathbb{C}$ and $r > 0$ be fixed. Put (cf. [3])
\[
\|P\| = \sum_{k=0}^{\infty} \frac{1}{k!} \|P^{(k)}\|_E r^k.
\]
Then
\[
\lim_{n \to \infty} \|P^n\|^{1/n} = \max_{|\zeta| \leq r} \|P(x + \zeta)\|_E.
\]
Moreover for $\|P\|_\sigma := \max_{|\zeta| \leq r} \|P(x + \zeta)\|_E$ we have
\[
\|P\|_\sigma \leq \|P\| \leq (\deg P + 1) \|P\|_\sigma.
\]
Example 7. If $\mu$ is a probabilistic measure on $E$, then for $1 \leq s < \infty$ the norm

$$||P|| = ||P||_E + \left( \int_E |P(z)|^s d\mu(z) \right)^{1/s}$$

is $E$-admissible on $\mathbb{P}(\mathbb{C}^N)$ with

$$\lim_{n \to \infty} ||P^n||^{1/n} = \max(||P||_E, \text{ess sup} |P|) = ||P||_E.$$

Example 8. In the classical case of the interval $[-1, 1]$ we have S.M. Nikolskii’s inequalities (cf. [35], [44], [33], [43])

$$\left( \frac{1}{2} \int_{-1}^1 |P(x)|^p dx \right)^{1/p} \leq ||P||_{[-1,1]} \leq (2(p + 1)n^2)^{1/p} \left( \frac{1}{2} \int_{-1}^1 |P(x)|^p dx \right)^{1/p}.$$

Example 9. (A generalization of Nikolskii’s inequality) Let $\mu$ be a probabilistic measure on $E$ such that for a system of orthonormal polynomials we have the inequality $||P||_E \leq B(\deg P)^\beta$ with some positive $\beta$, which is equivalent to the fact that for each polynomial $P$, $\deg P \geq 1$,

$$(2) \quad ||P||_E \leq B_1(\deg P)^\beta_1||P||_2$$

with some positive constants $B_1, \beta_1$. Indeed, if $(P_\alpha)_{\alpha \in \mathbb{N}^N}$ is an orthonormal system such that $\deg P_\alpha = |\alpha|$ then for each polynomial $P$ with $\deg P \geq 1$, $||P||_E \leq \left( \frac{(n+N)}{n} \right) \max_{|\alpha| \leq n} |c_\alpha|B|\alpha|^\beta$, where $c_\alpha = \int_E P(z)\overline{P_\alpha(z)}d\mu(z)$, so we can take $B_1 = B_2^{2N}/N$, $\beta_1 = \beta + N$.

Let us also note that the condition $||P||_E \leq B_1(\deg P)^\beta_1||P||_2$ implies the inequality $||P||_E \leq B_3^{2/s}(\lceil s \rceil)^2\beta_1^{2/s}(\deg P)^\beta/s||P||_s$, $s \geq 1$. In particular, $||P||_E = ||P||_\infty = \text{ess sup} |P|$.

Then for all $p \geq 1$ each norm $||P||_p = \left( \int_E |P(z)|^p d\mu(z) \right)^{1/p}$ is an $E$-admissible norm.
Remark 10. If $\mu$ is the normalized Lebesgue measure on a fat compact set $E \subset \mathbb{R}^N$ then Nikolskii's inequality implies Markov's property of $E$. It is a consequence of main results of [3], [5] and [45] (cf. [46, 47]) in one dimensional case. Hence, if we want to show that a given compact subset of $\mathbb{R}^N$ possesses Markov's property, it suffices to show Nikolskii's inequality as in the example above. Generally, it is a very difficult task to check Markov's property. Recently, a nontrivial result in this topic has been obtained by R. Pierzchala [38]. His remarkable result relates to a class of sets with a special parametric property introduced by himself. This property implies Nikolskii's inequality and thus Markov's property, as it was noticed above (but it was not considered in [38]).

Example 11. Let $E \subset \mathbb{R}^N$ and $\mu$ be a probabilistic measure on $E$ with the following density condition:

$$\exists G, \gamma > 0 \ \forall x \in E, r > 0 \quad \mu(\{E \cap \overline{B}(x, r)\}) \geq Gr^\gamma.$$ 

Assuming $E$ has Markov’s property, one can prove (2) for $E$. This method was used in the proof of Nikolskii’s inequality in the classical case (cf. [35, 44]) as well as in more general situations investigated by A. Zeriahi [49], P. Goetgheluck [23] and A. Jonsson [30] (cf. also [31]). Goetgheluck in [23] proved that each UPC set in $\mathbb{R}^N$ (this wide family of sets was introduced by W. Pawłuck and W. Pleśniak in [36]) satisfies the density condition and also by [36] has Markov’s property. Therefore each UPC set (in particular each compact fat subanalytic subset of $\mathbb{R}^N$, cf. [36, 37] for this deep result) satisfies the generalized Nikolskii’s inequality with respect to the normalized Lebesgue measure $\mu$ and Markov’s inequality in $L^p(\mu)$. However, no example is known of a set with cusp for which Markov’s exponent (in $L^p(\mu)$, $1 \leq p < \infty$) is calculated.

Example 12. Let $E \subset \mathbb{R}^N$. Put

$$||P|| = ||P||_E + \int_{\text{int}(E)} |D_j P(x)| dx. \tag{3}$$

Since

$$\int_{\text{int}(E)} |D_j P(x)| dx \leq \sqrt{N} \pi^N (\text{diam}(E))^{N-1} (\deg P) ||P||_E,$$

the norm $\|\cdot\|$ defined by (3) is $E$-admissible. The last inequality follows from [3], [5] and [45].
Example 13. Let $\|P\| = \sup_{x \in [-1, 1]} |P(x)|/\sqrt{1 - x^2}$ be Schur’s norm. Since $\|P\| \leq \|P\|_{[-1,1]} \leq (\deg P + 1)\|P\|$, Schur’s norm is $[-1, 1]$-admissible. Similarly, if we put $\|P\|_{\alpha} = \sup_{x \in [-1, 1]} |P(x)|((1 - x^2)^\alpha$, $\alpha > 0$, then (cf. [6] for $\alpha \geq 1/2$) the norm $\|\cdot\|_{\alpha}$ is $[-1, 1]$-admissible. Moreover, if we replace the interval $[-1, 1]$ by the unit closed ball $B := \{x \in \mathbb{R}^N : \|x\|_* \leq 1\}$ with respect to a fixed norm $\|\cdot\|_*$ in $\mathbb{R}^N$ then the norm defined by $\|P\|_{\alpha} = \sup_{x \in B} |P(x)|(1 - \|x\|^2)^\alpha$ is $B$-admissible. A more general situation is contained in the following way (cf. [4]). Let $\Omega$ be a bounded, star-shaped (with respect to the origin) and symmetric domain in $\mathbb{R}^N$ and let $E = \overline{\Omega}$. Let $v \in S^{N-1}$ be a fixed direction, we assume that $\rho_v(tx) \geq M(1 - |t|)^m$, $t \in [-1, 1], x \in \partial E$. Then for any $\alpha > 0$, the norm $\|P\|_{\alpha} = \sup\{|P(tx)|(1 - |t|)^\alpha : x \in \partial E, t \in [-1, 1]\}$ is $E$-admissible. Note that in this case $m(E, v) \leq 2m$ (cf. the first definition in the next section with $D = v_1 D_1 + \cdots + v_N D_N$).

Remark 14. The Schur inequality in Example 13 is a special case of the division type inequality, which is often called the Schur type inequality. It was proved in [11] that on the complex plane properties related to Markov’s and Schur’s inequalities are equivalent.

Remark 15. If we have some norms with the generalized Nikolskii’s property (GNP), we can easily construct many other norms with this property. For example, if $q_1, q_2$ have GNP (with spectral norms $q_{1,\sigma}, q_{2,\sigma}$) then $q(P) = (q_1(P)^p + q_2(P)^p)^{1/p}$, $1 \leq p \leq \infty$ has GNP with $q_{\sigma} = \max(q_{1,\sigma}, q_{2,\sigma})$.

Remark 16. Let $\|\cdot\|_0$ be a spectral norm in $\mathbb{P}(\mathbb{C}^N)$ and let $\|\cdot\|_1$ be a GNP norm with respect to $\|\cdot\|_0$. If $\alpha_j \in \mathbb{Z}_+, j = 1, \ldots, l$ are fixed then we can consider $\|P\| = \|P\|_0 + \max_{1 \leq j \leq l} \|\alpha_j D \|_1$.

We have $\lim_{n \to \infty} \|P^s\|_{1/s} = \|P\|_0$ but GNP will be satisfied if and only if we have a Markov-Nikolskii type bound $\max_{1 \leq j \leq l} \|\alpha_j D \|_1 \leq C(\deg P)^\gamma \|P\|_0$.

Let us give two examples.
If $||P|| = ||P||_{[-1,1]\cup(2)} + ||P||_{[-1,1]\cup(2)}$ then this norm does not satisfy GNP.

Now we define $||P|| = ||P||_E + ||\frac{\partial P}{\partial x}||_E$, where $E = \{(x,y) \in \mathbb{R}^2 : |x| < 1, \ |y| \leq \exp(-1/(1 - |x|))\} \cup \{(-1,0),(1,0)\}$. Since (cf. [1])

$$||\frac{\partial P}{\partial x}||_E \leq 2(\deg P)^2 ||P||_E,$$

the norm $|| \cdot ||$ possesses GNP.

3. ASYMPTOTIC EXPONENT IN MARKOV’S INEQUALITY

Let $\varphi = (\varphi_1, \ldots, \varphi_N) \in C^\infty(\mathbb{R}^N)^N$ (if $\mathbb{K} = \mathbb{C}$ we understand that $\varphi_j \in C^\infty(\mathbb{R}^{2N})$). We assume that $\varphi_j$ can take complex values. In particular, we can consider $\varphi_j \equiv v_j \in \mathbb{C}$ for $j \in \{1, \ldots, N\}$ and then $\varphi = v \in \mathbb{C}^N$. Define

$$D = D_\varphi = \varphi_1 D_1 + \cdots + \varphi_N D_N : C^\infty(\mathbb{R}^N) \longrightarrow C^\infty(\mathbb{R}^N)$$

and put $D^{(k)} = D \circ \cdots \circ D$ $k$-times.

Let us recall a deep identity (cf. [34], [9], [8])

$$(4) \quad (D(f))^k = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} f^j D^{(k)}(f^{k-j}).$$

**Definition 17.** Let $q = ||\cdot||$ be a norm in $\mathbb{P}(\mathbb{K}^N)$. If $H$ is a homogenous polynomial of $N$ variables of degree $k \geq 1$ then we consider a differential operator $D = H(D_1, \ldots, D_N)$ and define

$$m(H, q) = \inf \{s > 0 : \exists M > 0 \forall P \in \mathbb{P}(\mathbb{K}^N) \ | |DP|| \leq M(\deg P)^s ||P|| \}.$$ 

For $\alpha \in \mathbb{N}^N$ and $H_\alpha(x) = x^\alpha$, $x \in \mathbb{K}^N$, we put $m(\alpha, q) = M(H_\alpha, q)$ and $m(q) = \max_{1 \leq j \leq N} m(e_j, q)$. For $k \geq 1$ we put $m_k(q) = \max \{m(\alpha, q) : |\alpha| = k\}$. In particular, $m_1(q) = m(q)$ is Markov’s exponent for a norm $q$.

**Remark 18.** In a special case, if $q(P) = ||P||_E$, where $E$ is a compact subset of $\mathbb{K}^N$, then we define $m(H, E) = m(H, q)$, $m(\alpha, E) = m(\alpha, q)$, $m_k(E) = m_k(q)$, $m(E) = m(q)$. Moreover the last one is Markov’s exponent of $E$ which was recalled in the first section and if $m(E) < \infty$ we say that $E$ has Markov’s property. Let us note the equality (for subsets of $\mathbb{R}^N$, cf. [8])

$$m(H_k, E) = km(E),$$

where $H_k(x_1, \ldots, x_N) = x_1^k + \cdots + x_N^k$ ($k$ is a fixed positive even integer).
Since
\[ m(\alpha, q) \leq m(e_1, q)\alpha_1 + \cdots + m(e_N, q)\alpha_N \leq \max_{1 \leq j \leq N} m(e_j, q)|\alpha| = m(q)|\alpha|, \]
we get the inequality
\[ m_k(q) \leq km(q) \Rightarrow \frac{1}{k}m_k(q) \leq m(q). \]

Remark 19. From [34] we have \( \frac{1}{k}m_k(E) = m(E) \). Therefore
\[ \lim_{k \to \infty} \frac{1}{k}m_k(E) = m(E). \]

Definition 20. Let \( q \) be a norm in \( \mathbb{P}(\mathbb{K}^N) \). We define the asymptotic exponent for \( q \),
\[ m^*(q) := \limsup_{k \to \infty} \frac{1}{k}m_k(q). \]

Remark 21. Let us note a few basic properties of the above notion.

a) If \( q_1 \) and \( q_2 \) are two norms on \( \mathbb{P}(\mathbb{K}^N) \) such that
\[ q_1(P) \leq A(\deg P)^a q_2(P), \quad q_2(P) \leq B(\deg P)^b q_1(P), \quad \deg P \geq 1, \]
then \( m^*(q_1) = m^*(q_2) \).

b) If \( q_1(P) = \text{essup}_m|P| \) then \( m^*(q_2) = m(q_1) \).

c) We have \( m^*(q) \leq m(q) \). In general, these exponents do not need to be equal.

Now, we give an example of the norms for which \( m^*(q) < m(q) \). First, we need the following

Proposition 22. For \( || \cdot ||_0 \) a seminorm on \( \mathbb{P}(\mathbb{C}) \), \( m > 0 \) and \( s \in \mathbb{N}_1 \) we define the norm
\[ q_{m,s}(P) = ||P||_{m,s} = \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m}||P^{(rs)}||_0. \]

If for every \( s \geq 2 \) there exist positive constants \( A, B \) such that for every \( j \in \{1, \ldots, s\} \) and \( P \in \mathbb{P}(\mathbb{C}) \), \( ||P^{(j)}||_0 \leq A||P||_0 + B||P^{(s)}||_0 \), then \( m_k(q_{m,s}) \leq sm\left[\frac{k}{s}\right] \) for every \( k \in \mathbb{N}_1 \).
Proof. For every $m > 0$, $t, s \in \mathbb{N}_1$, $j \in \{1, \ldots, s\}$ and $P \in \mathbb{P}(\mathbb{C})$ we obtain

$$
\|P^{(st+j)}\|_{m,s} = \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m} \|P^{(rs+st+j)}\|_0
$$

$$
\leq A \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m} \|P^{(rs+st)}\|_0
$$

$$
+ \max\{B, 1\} \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m} \|P^{(rs+st+s)}\|_0
$$

$$
\leq A \sum_{r=0}^{[\deg P]_s} \frac{1}{((rs)!)^m} \|P^{(rs+st)}\|_0
$$

$$
+ \max\{B, 1\} \sum_{r=0}^{[\deg P]_s} \frac{1}{((rs)!)^m} \|P^{(rs+st+s)}\|_0
$$

$$
= A \sum_{r=t}^{[\deg P]_s+t} \frac{1}{(((r-t)s)!)^m} \|P^{(rs)}\|_0
$$

$$
+ \max\{B, 1\} \sum_{r=t+1}^{[\deg P]_s+t+1} \frac{1}{(((r-t-1)s)!)^m} \|P^{(rs)}\|_0
$$

$$
\leq (A + \max\{B, 1\})(\deg P)^{s(t+1)m} \sum_{r=0}^{\infty} \frac{1}{((rs)!)^m} \|P^{(rs)}\|_0
$$

$$
= (A + \max\{B, 1\})(\deg P)^{s(t+1)m} \|P\|_{m,s}.
$$

$\Box$

Example 23. Let us consider the norms $q_{m,s}$ defined like in Proposition 22 with seminorm $\|P\|_0 = \sum_{l=0}^{s-1} \frac{1}{l!} |P^{(l)}(0)|$, $s \in \mathbb{N}_1$. Then for every
\( P \in \mathbb{P}(\mathbb{C}) \) and \( j \in \{1, \ldots, s - 1\} \) we have

\[
\|P^{(j)}\|_0 = \sum_{l=0}^{s-1} \frac{1}{l!} |P^{(j+l)}(0)| = \sum_{l=j}^{s-1} \frac{l!}{l!(l-j)!} |P^{(l)}(0)| \\
+ \sum_{l=0}^{j-1} \frac{l!}{l!(s+l-j)!} |P^{(s+l)}(0)|
\]

\[
\leq (s-1)^{s-1} \sum_{l=0}^{s-1} \frac{1}{l!} |P^{(l)}(0)| + \sum_{l=0}^{s-1} \frac{1}{l!} |P^{(s+l)}(0)| \\
\leq (s-1)^{s-1} \|P\|_0 + \|P^{(s)}\|_0
\]

From Proposition 22 for \( m > 0 \) and \( s \in \mathbb{N}_1 \) we obtain \( m_k(q_{m,s}) \leq sm\left[ \frac{k}{s} \right] \).

On the other hand for every \( m > 0 \) and \( s, n \in \mathbb{N}_1 \) we have

\[
\|x^{sn}\|_{m,s} = \sum_{r=0}^{\infty} \left( \frac{1}{(rs)!} \right) \sum_{l=0}^{m} \frac{1}{l!} |(x^{sn})^{(rs+l)}(0)| = \frac{1}{(sn)!^{m-1}}.
\]

and

\[
\| (x^{sn})^{(st+j)} \|_{m,s} = \sum_{r=0}^{\infty} \left( \frac{1}{(rs)!} \right) \sum_{l=0}^{m} \frac{1}{l!} |(x^{sn})^{(rs+st+j+l)}(0)| \\
= \frac{(sn)!}{(s-j)!(sn-st-s)!^m}.
\]

Hence for every \( k \in \mathbb{N}_1 \) we have \( m_k(q_{m,s}) = sm\left[ \frac{k}{s} \right] \), where for \( x \in \mathbb{R} \), \([x]\) is the smallest integer greater than or equal to \( x \). From this it follows that \( m^*(q_{m,s}) = m \) and \( m(q_{m,s}) = sm \).

Now we formulate main results of this paper.

**Theorem 24.** Let \( q \) be a spectral admissible norm for some spectral norm \( q_\sigma \). Then

\[
m^*(q) = \lim_{k \to \infty} \frac{1}{k} m_k(q) = m(q_\sigma).
\]

In particular, \( m(q_\sigma) \leq m(q) \).

**Corollary 25.** Let \( q \) be an \( E \)-admissible norm. Then

\[
m^*(q) = \lim_{k \to \infty} \frac{1}{k} m_k(q) = m(E).
\]

In particular, \( m(E) \leq m(q) \).
Proof. Firstly, we prove that \( m_k(q_\sigma) = km(q_\sigma), \ k \geq 1. \) If for every \( j \in \{1, \ldots, N\} \) there exist positive constants \( M_j, m_j \) such that for every polynomial \( P \in \mathbb{P}^n_\sigma(K^N), \)

\[
\|D_j P\| \leq M_j n^{m_j} \| P \|
\]

then for \( \alpha \in \mathbb{N}_0^N \) such that \( |\alpha| = k \) we have

\[
\|D^{(\alpha_1, \ldots, \alpha_N)} P\| \leq M_1^{\alpha_1} \cdots M_N^{\alpha_N} n^{\alpha_1 m_1 + \cdots + \alpha_N m_N} \| P \|
\]

\[
\leq \left( \max_{j \in \{1, \ldots, N\}} M_j \right)^k n^{k m} \| P \|
\]

where \( m = \max_{j \in \{1, \ldots, N\}} m_j. \) Hence \( m_k(q) \leq km(q) \) for every norm \( q. \)

On the other hand

\[
(D_j P)^k = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} P^j \frac{\partial^k}{\partial x_j^k} P^{k-j}
\]

The norm \( q_\sigma \) is spectral and by the Theorem in [18] it is submultiplicative. Hence, if an \( \varepsilon > 0 \) is fixed,

\[
\|(D_j P)^k\|_\sigma \leq \text{const.}(\varepsilon) \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \| P \|_\sigma^j (n(k-j))^{m_k(q_\sigma)+\varepsilon} \| P \|_\sigma^{k-j}
\]

\[
\leq \text{const.}(\varepsilon) \frac{2^k}{k!} (nk)^{m_k(q_\sigma)+\varepsilon} \| P \|_\sigma^k,
\]

which shows that \( m(q_\sigma) \leq m_k(q_\sigma)/k + \varepsilon/k. \) Letting \( \varepsilon \to 0^+ \) we get the inequality \( m(q_\sigma) \leq m_k(q_\sigma)/k \) and finally \( m_k(q_\sigma) = km(q_\sigma), \ k \geq 1. \)

Now, let \( s > m(\alpha, q_\sigma) \). Then

\[
\|D^\alpha P\| \leq B(\deg P)^b \|D^\alpha P\|_\sigma \leq BM_s(\deg P)^{b+s} \| P \|_\sigma
\]

\[
\leq BM_s A(\deg P)^{b+a+s} \| P \|
\]

which gives

\[
m(\alpha, q) \leq b + a + s \Rightarrow m(\alpha, q) \leq b + a + m(\alpha, q_\sigma)
\]

and therefore \( m_k(q) \leq b + a + m_k(q_\sigma) = b + a + km(q_\sigma). \) Hence

\[
m^*(q) = \limsup_{k \to \infty} \frac{1}{k} m_k(q) \leq m(q_\sigma).
\]

Analogously, let \( s > m(\alpha, q). \) Then

\[
\|D^\alpha P\|_\sigma \leq A(\deg P)^a \|D^\alpha P\| \leq AM'_s(\deg P)^{a+s} \| P \|
\]

\[
\leq ABM'_s(\deg P)^{a+b+s} \| P \|_\sigma,
\]
which implies $m(\alpha, q_{\sigma}) \leq a + b + s$ and $m(\alpha, q_{\sigma}) \leq a + b + m(\alpha, q)$, whence $km(q_{\sigma}) = m_k(q_{\sigma}) \leq a + b + m_k(q)$. Hence

$$m(q_{\sigma}) \leq \lim \inf_{k \to \infty} \frac{1}{k} m_k(q) \leq \lim \sup_{k \to \infty} \frac{1}{k} m_k(q) \leq m(q_{\sigma}).$$

□

The second statement in the following important corollary is very useful. Also the third statement gives new result.

**Corollary 26.**

a) If a norm $q$ has GNP with the spectral norm $q_{\sigma}$ then

$$m^*(q) = m(q) \iff m(q) = m(q_{\sigma}).$$

b) If for a norm $q = || \cdot ||$ we have Markov’s inequality

$$||D_j P|| \leq M(deg P)^{m(q_{\sigma})} ||P||, \ j = 1, \ldots, N$$

then the exponent $m(q_{\sigma})$ is the best possible. In particular, $m(q) = m(q_{\sigma})$.

c) If $E$ is an UPC subset of $\mathbb{R}^N$, then $m_p(E) \geq m(E)$, where $m_p(E)$ is Markov’s exponent with respect to the Lebesgue measure.

**Remark 27.** In papers where Markov’s inequality in $L^p$ norms was proved with the best possible exponent, usually it was difficult and time-consuming to prove the optimality of the exponent, which is Markov’s exponent for such kind of norms (cf. [29], [22], [17], [32]). By applying the above corollary it is done automatically.

Let us consider another (simple) example. By Bernstein’s inequality

$$||\sqrt{1 - x^2} P'(x)||_{[-1,1]} \leq (\deg P) ||P||_{[-1,1]}$$

and by Schur’s inequality

$$||P||_{[-1,1]} \leq (\deg P + 1)||\sqrt{1 - x^2} P(x)||_{[-1,1]}$$

we get Markov’s inequality with respect to Schur’s norm

$$||\sqrt{1 - x^2} P'(x)||_{[-1,1]} \leq \deg P(\deg P + 1)||\sqrt{1 - x^2} P(x)||_{[-1,1]},$$

with exponent 2, which is, by the corollary above, the best possible.
4. Markov’s exponent for a sequence of polynomials in \( P(\mathbb{C}) \).

**Definition 28.** Fix a compact set \( E \subset \mathbb{C} \) and a sequence of polynomials \( \hat{P} = (\hat{P}_n)_{n \geq 0} \subset P(\mathbb{C}) \), \( \deg \hat{P}_n = n \). Put, for \( k \geq 1 \),

\[
m_k(\hat{P}) := \limsup_{n \to \infty} \frac{\log(||\hat{P}_n||_E/||\hat{P}_n||_E)}{\log n}
\]

and \( m^*(\hat{P}) := \limsup_{k \to \infty} \frac{1}{k} m_k(\hat{P}) \).

**Theorem 29.** Let \( \hat{P} = (\hat{P}_n)_{n \geq 0} \) be an orthonormal system (with respect to a probabilistic measure \( \mu \) supported on \( E \)) such that

\[
\limsup_{n \to \infty} \frac{\log ||\hat{P}_n||_E}{\log n} = \alpha < \infty.
\]

Then \( m(E) = m^*(\hat{P}) \).

**Proof.** It is clear that \( m^*(\hat{P}) \leq m(E) \). Assume that \( m^*(\hat{P}) =: \gamma < \infty \). Fix an \( \varepsilon > 0 \). There exists \( k_0 \) such that for all \( k \geq k_0 \), \( m_k(\hat{P}) \leq k(\gamma + \varepsilon) \).

If \( k \geq k_0 \) is fixed then for \( n \geq n_0 \) we have an estimation

\[
||\hat{P}_n^{(k)}||_E \leq n^{m_k(\hat{P})+\varepsilon} ||\hat{P}_n||_E \leq An^{k(\gamma+\varepsilon)+\alpha+2\varepsilon},
\]

where \( A \) is a positive constant. Let \( \langle , \rangle \) be a scalar product in \( L^2(\mu) \). It is well known that for \( P \in P_n(\mathbb{C}) \) we have

\[
P = \sum_{j=0}^{n} \langle P, \hat{P}_j \rangle \hat{P}_j.
\]

Hence

\[
||P^{(k)}||_E \leq \sum_{j=0}^{n} ||\langle P, \hat{P}_j \rangle|| ||\hat{P}_j^{(k)}||_E \leq ||P||_E \sum_{j=0}^{n} ||\hat{P}_j^{(k)}||_E
\]

\[
\leq ABk^{k(\gamma+\varepsilon)+\alpha+1+2\varepsilon} ||P||_E.
\]

Here \( B_k \) is a positive constant. Now we can write

\[
m_k(E)/k \leq (\gamma + \varepsilon) + (\alpha + 1 + 2\varepsilon)/k,
\]

which gives

\[
m(E) = m^*(E) = \lim_{k \to \infty} m_k(E)/k \leq \gamma + \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, we get \( m(E) \leq \gamma \) which finishes the proof. \( \square \)
Example 30. \( m([-1,1]) = m_1(\hat{P}_{\alpha,\beta}) \), where \( \hat{P}_{\alpha,\beta} \) is the family of normalized Jacobi polynomials.

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