A self-dual boundary phase transition of the 2d $\mathbb{Z}_N$ topological order

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Abstract

In this work, we construct an enriched fusion category to describe a critical point of a 1d self-dual boundary phase transition between two gapped boundaries of the $\mathbb{Z}_N$ topological order. To verify that the enriched fusion category actually describes this critical point, we also construct a lattice model to recover the mathematical data of this enriched fusion category. The construction further shows that the symmetry of the boundary is determined by the topological defect in the bulk. This work as a concrete example shows that the mathematical theory of the gapless edges of 2d topological orders developed in arXiv:1905.04924 and arXiv:1912.01760 by Kong and Zheng is a powerful tool to study general phase transitions.

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1 Introduction

The study of topological orders (TOs) has attracted the attention in condensed matter physics and mathematical physics, see review [Wen17, Wen19]. Even more interesting is that these exotic topological phases have recently been generalized to the 1+1D CFT-type gapless case [GV12, RA17, SPV17, VJP18, JLSL18, PSV18, KZ18, IV19, JSW20, Ver20, KZ20b, KZ21, CJK+20, JW20b, JW20a, VTJP21, WJX21, CW22, MMT22]. If a gapless phase is not stable against certain symmetry-preserving perturbations, meaning that it is gappable, then it can be viewed as a (multi) critical point that describes continuous phase transitions between 1d gapped quantum phases. In this paper, we focus on a 1d gappable non-chiral CFT-type gapless phase (or a phase transition described by a non-chiral CFT) with the non-invertible gravitational anomaly $Z_1(\text{Rep}(Z_N))$. The anomaly $Z_1(\text{Rep}(Z_N))$ means that the phase transition actually occurs on the boundary of the 2d $Z_N$ topological order (TO) [Wen13, JW21]. Due to the anomaly being fixed, the phase transition occurs purely at the boundary without changing the bulk. So, a 1d pure boundary phase transition with gravitational anomaly $Z_1(\text{Rep}(Z_N))$ should be nothing but a gappable non-chiral gapless boundary of the 2d anomaly-free topological order $Z_1(\text{Rep}(Z_N))$.

Here we would like to interpret our motivation for studying boundary phase transitions. On the one hand, a common CFT-type gapless state will decompose into some decoupled gapless sectors restricted to the symmetric Hilbert subspaces in the low-energy limit. Due to each sector only corresponding to the part of Hilbert space, each sector is often anomalous. These (emergent) anomalies can be described by non-invertible gravitational anomalies (or topological orders in one higher dimension). Therefore, these gapless sectors are the gapless boundary states of the corresponding 2d TO [JW20b]. The boundary phase transitions provide concrete examples for the study of anomalous CFTs. On the other hand, a phase transition of a 1d quantum chain can actually be viewed as a special case of the boundary phase transitions in the sense that the 1d

\*we use $nd$ to represent $n$ spatial dimension and $(n + 1)D$ to represent $n + 1$ spacetime dimension.
quantum chain has a trivial 2d bulk (or gravitational anomaly). Therefore, it is natural to study the boundary phase transitions of 2d topological orders.

In a concrete lattice model, the authors of [CJK20] demonstrate that a critical point of a boundary phase transition of the $\mathbb{Z}_2$ TO (toric code) can be mathematically described by a so-called enriched fusion category. The mathematical theory of a gapless boundary of a 2d anomaly-free TO used in their paper was developed in [KZ18, KZ20b, KZ21]. This theory states that all long-wave-length-limit physical observables on a gapped/gapless boundary form an enriched fusion category. Therefore, the mathematical structure of a critical point of a boundary topological phase transition is an enriched fusion category.

Remark 1.1. To be more precise, the mathematical structure on a 1d gapped/gapless boundary can be split into two parts: a local quantum symmetry $U$ which encodes the information of local observables; and a $\text{Mod}_U$-enriched fusion category $\text{Mod}_U M$ which encodes all the topological defects. The enriched fusion category $\text{Mod}_U M$ is called the topological skeleton [KZ20a].

In this work, we demonstrate this statement explicitly in a more general case: A self-dual boundary phase transition between two gapped boundaries $\text{Rep}(\mathbb{Z}_N)$ ($m$-condensed) and $\text{Vec}_{\mathbb{Z}_N}$ ($e$-condensed) of the $\mathbb{Z}_N$ topological order can be mathematically described by an enriched fusion category. Here, the word “self-dual” means that the critical point has an emergent $\mathbb{Z}_2$ symmetry from the $e$-$m$ duality. More explicitly, we construct an enriched fusion category to mathematically describe a gappable non-chiral gapless boundary which corresponds to a critical point of this phase transition. And we construct a lattice model to realize this gappable non-chiral gapless boundary, including all data from the corresponding enriched fusion category. The process of the construction magically shows that the local operators of the boundary are determined by the string operators of the bulk. It is natural because the anyon traveling to the boundary can cause the fluctuation of the boundary [CJK20, JW20a, JW20b, CW22].

In Section 2 we briefly review some mathematical basics of a gapped 2d anomaly-free TO, including the categorical description of a 2d anomaly-free TO by a unitary modular tensor category (UMTC), and the anyon condensation theory of 2d anomaly-free TO [Kon14]. We also give two examples of unitary modular tensor categories: the $\mathbb{Z}_N$ parafermion UMTC $\text{PF}_N$ and the $\mathbb{Z}_N$ quantum double UMTC $\mathcal{Z}_1(\text{Rep}(\mathbb{Z}_N))$, both of which are related to this work. In Section 3 we first review the mathematical description of gapless boundaries introduced in [KZ20b, KZ21], including a refresher of the rational conformal field theory (RCFT) and the ingredients of an enriched fusion category. Then, we find a gapped domain wall between the double $\mathbb{Z}_N$ parafermion TO and the $\mathbb{Z}_N$ quantum double TO by anyon condensation theory. Next, using a holographic duality called the topological Wick rotation [KZ20b], we construct a gappable non-chiral gapless boundary:

\[
(V_{\text{PF}_N} \otimes_{C} V_{\text{PF}_N}) \mathcal{Z}_1(\text{PF}_N)R
\]  

(1.1)
of the $\mathbb{Z}_N$ quantum double TO $\mathcal{Z}_1(\text{Rep}(\mathbb{Z}_N))$, where

- $V_{\text{PF}_N}$ is the $\mathbb{Z}_N$ parafermion vertex operator algebra (VOA) whose module category is $\text{PF}_N$.
- $V_{\text{PF}_N} \otimes_{C} V_{\text{PF}_N}$ is the full field algebra (FFA) that describes the local quantum symmetry on the 1+1D world sheet of this gapless boundary.
- $B = \bigoplus_{u=0}^{n-1} \mathbb{N}_{2u,0} \boxtimes \mathbb{N}_{2u,0}$ is a condensable algebra such that the condensed phase of the double $\mathbb{Z}_N$ parafermion TO via $B$ is the $\mathbb{Z}_N$ quantum double TO.
- $\mathcal{Z}_1(\text{PF}_N)R$ is the category of right $B$-modules in $\mathcal{Z}_1(\text{PF}_N)$ that mathematically describe the particle-like excitation on the 1d gapped domain wall between the double $\mathbb{Z}_N$ parafermion TO and the $\mathbb{Z}_N$ quantum double TO.
- $\mathcal{Z}_1(\text{PF}_N)$ is the enriched fusion category that describes the topological skeleton of the gappable non-chiral gapless boundary, where
- $Z_1(\text{PF}_N)_B$ consists of the boundary topological excitations (or boundary conditions).
- $Z_1(\text{PF}_N)$ demonstrates the internal homs $M_{x,y}$ between boundary topological excitations, or equivalently, the domain walls between two boundary CFT’s with boundary conditions $x$ and $y$.

We also compute the partition functions of $M_{x,y}$ exactly. It is expected that the enriched fusion category $(V_{\text{PF}_N}, \otimes, \text{Vec}_{Z_N}, Z_1(\text{PF}_N)_B)$ indeed describes a critical point of a self-dual boundary phase transition of $Z_N$ TO. Therefore, in Section 4, we construct a lattice model to realize a critical point of a boundary phase transition between two gapped boundaries, $\text{Rep}(Z_N)$ ($m$-condensed) and $\text{Vec}_{Z_N}$ ($e$-condensed) of the $Z_N$ topological order. And we further calculate the low-energy effective theory at the critical point. This phase transition is described by a set of modular covariant partition functions that recover the partition functions of internal homs $M_{x,y}$ between local $B$-modules. Finally, some examples for $Z_2$, $Z_3$, and $Z_4$ cases are listed. The results in [CJK'20] can be regarded as a special case $Z_2$ of our work.

This work shows that the mathematical theory developed in [KZ'18, KZ'20b, KZ'21] is powerful, and provides a potential tool to study more complicated topological phase transitions.

## 2 Categorical Preliminaries

Our goal in this paper is to find a gapless boundary that can describe a critical point of a self-dual topological phase transition on the boundary of the 2d $Z_N$ topological order. Before we discuss an anomalous 1d gapless phase, it is necessary to review some mathematical basics of 2d gapped TOs, including a categorical description of a 2d anomaly-free TO and the anyon condensation theory. In particular, the anyon condensation theory, which provides a description of the domain walls between two TOs, plays an important role in the construction of the gapless boundary we need.

In this section, we briefly review the notion of a unitary modular tensor category, which describes the particle-like excitations of a 2d anomaly-free TO and the anyon condensation theory.

### 2.1 Unitary modular tensor categories

A gapped quantum liquid without symmetry is called a topological order (TO) [Wen90, ZW15, SM16]. It is known that a 2d anomaly-free TO can be described by a unitary modular tensor category (UMTC) $\mathcal{C}$ with a central charge $c$ [Kit06, KZ22], hence we will denote a TO by a pair $(\mathcal{C}, c)$. We won’t give an explicitly mathematical definition of a UMTC here. Instead, we’ll introduce the ingredients and some properties of a UMTC $\mathcal{C}$.

- $\mathcal{C}$ has finitely many objects/anyons $x, y, z, \ldots$. We denote $\text{Irr}(\mathcal{C})$ the set of isomorphic classes of simple objects. Each object can be written as a direct sum of simple objects.
- For two anyons $x, y$, the hom space $\text{hom}_{\mathcal{C}}(x, y)$ is a finite-dimensional Hilbert space. Elements in a hom space are called instantons. There is a map

  $$\circ : \text{hom}_{\mathcal{C}}(y, z) \otimes \text{hom}_{\mathcal{C}}(x, y) \to \text{hom}_{\mathcal{C}}(x, z),$$

  for each triple $x, y$ and $z$ called the composition map, which describes the fusion of instantons along the time axis. The composition map shall be associative.
- There is a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, that describes the fusion of two anyons $x, y$ into the anyon $x \otimes y$. This fusion process should be associative, so that for all $x, y, z \in \mathcal{C}$ there exists an isomorphism $\alpha_{x,y,z} : x \otimes (y \otimes z) \to (x \otimes y) \otimes z$ satisfying the necessary coherence conditions.
• There is a distinguished object $1$ called the tensor unit (or vacuum), which is simple, together with unit isomorphisms $l_x : 1 \otimes x \to x$ and $r_x : x \otimes 1 \to x$ for all $x$ satisfying necessary coherence conditions.

• Each object $x \in \mathcal{C}$ has a dual $x^*$, together with the creation and annihilation morphisms $v_x : x^* \otimes x \to 1$ and $u_x : 1 \to x \otimes x^*$ satisfying some coherence conditions.

• It has a braiding structure, i.e. for all $x, y \in \mathcal{C}$ there is an isomorphism $c_{x,y} : x \otimes y \to y \otimes x$ satisfying Yang-Baxter equations. The braiding is non-degenerate; that is, the S-matrix, which is the trace of double braiding of simple objects, is non-degenerate.

• Each object $x$ has a twist (topological spin), which is an isomorphism $\theta_x : x \to x$ representing the self statistics of simple objects.

• It has a unitary structure: for each morphism $f : x \to y$, there is an adjoint morphism $f^* : y \to x$ such that $(g \otimes h)^* = g^* \otimes h^*$ for any $g : v \to w$. And the coherence data $a, l, r, c$ are both unitary.

The simplest example of a UMTC is the category of finite-dimensional Hilbert space denoted by $\mathcal{H}$. It has a unique simple object given by the one-dimensional Hilbert space $\mathbb{C}$. In particular, the pair $(\mathcal{H}, 0)$ describes the trivial 2d TO.

**Remark 2.1.** For readers who are interested in the bootstrap analysis that a 2d TO can be described by a UMTC, we recommend [KZ22]. For readers who are interested in the rigorous mathematical definition of a UMTC, we recommend [EGNO16, Mug00].

2.1 UMTC $3_1(\text{Rep}(\mathbb{Z}_N))$

Consider the category $\text{Rep}(\mathbb{Z}_N)$ of finite-dimensional representations of $\mathbb{Z}_N$, where $\mathbb{Z}_N$ is the cyclic group of order $N$. The Drinfeld center $3_1(\text{Rep}(\mathbb{Z}_N))$ of the $\text{Rep}(\mathbb{Z}_N)$ is a UMTC [BK01], which describes the particle-like topological excitations of the $\mathbb{Z}_N$ quantum double model [Kit03]. We list some of its ingredients below,

• The simple objects are $O_{\alpha, \beta}$ labeled by $\alpha, \beta$ with $0 \leq \alpha, \beta < N$.

• The fusion rule of two simple objects $O_{\alpha_1, \beta_1}$ and $O_{\alpha_2, \beta_2}$ is

$$O_{\alpha_1, \beta_1} \otimes O_{\alpha_2, \beta_2} \simeq O_{\alpha_1 + \alpha_2, \beta_1 + \beta_2}, \quad (2.1)$$

The tensor unit is $1 := O_{0,0}$.

• The braiding is given by:

$$c_{\alpha_1, \beta_1; \alpha_2, \beta_2} : O_{\alpha_1, \beta_1} \otimes O_{\alpha_2, \beta_2} \to O_{\alpha_2, \beta_2} \otimes O_{\alpha_1, \beta_1},$$

for $O_{\alpha_1, \beta_1}$ and $O_{\alpha_2, \beta_2}$.

• The twist is given by: $\theta_{\alpha, \beta} = e^{-2\pi i \alpha \beta}$ for simple object $O_{\alpha, \beta}$.

In physics, $e := O_{1,0}$ denotes the elementary charge and $m := O_{0,1}$ denotes the elementary flux. Hence, we will use $e^\alpha m^\beta$ to denote the simple object $O_{\alpha, \beta}$. The unitary fusion category (UFC) $\text{Rep}(\mathbb{Z}_N)$ describes the 1d $m$-condensed boundary of the 2d $\mathbb{Z}_N$ TO $(3_1(\text{Rep}(\mathbb{Z}_N)), 0)$. And the 1d $e$-condensed boundary is $\text{Vec}_{\mathbb{Z}_N}$, the category of finite-dimensional $\mathbb{Z}_N$-graded vector spaces. In general, the gapped boundaries of $(3_1(\text{Rep}(\mathbb{Z}_N)), 0)$ are classified by the subgroups of $\mathbb{Z}_N$, in which $\text{Rep}(\mathbb{Z}_N)$ corresponds to $\mathbb{Z}_N$ itself, and $\text{Vec}_{\mathbb{Z}_N}$ corresponds to the trivial group $|e|$. 

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Example 2.2. When $N = 2$, the UMTC $Z_1(\text{Rep}(\mathbb{Z}_2))$ describes the toric code model [Kit03]. It has 4 simple objects denoted by $\mathbb{1}, e, m$ and $f$ with the fusion rules:

$$e \otimes e \simeq \mathbb{1} \simeq m \otimes m, \quad e \otimes m \simeq f.$$  \hfill (2.2)

It has two gapped boundaries, $m$-condensed boundary $\text{Rep}(\mathbb{Z}_2)$ and $e$-condensed boundary $\text{Vec}_{\mathbb{Z}_2}$.

In this paper, we will construct a gapless boundary that describes a critical point of a topological phase transition between $\text{Rep}(\mathbb{Z}_N)$ and $\text{Vec}_{\mathbb{Z}_N}$.

Remark 2.3. As fusion categories, $\text{Rep}(\mathbb{Z}_N)$ and $\text{Vec}_{\mathbb{Z}_N}$ are equivalent. But in general, when $G$ is a finite group, $\text{Rep}(G)$ is not equivalent to $\text{Vec}_G$. For example, consider the permutation group $S_3$, $\text{Rep}(S_3)$ has 3 simple objects, but $\text{Vec}_{S_3}$ has 6 simple objects. The equivalence holds when $G$ is abelian.

Remark 2.4. In Landau’s paradigm, the topological phase transition between two specific 1d gapped boundaries $\text{Rep}(\mathbb{Z}_N)$ and $\text{Vec}_{\mathbb{Z}_N}$ in this work can be viewed as a completely spontaneous symmetry breaking process.

2.1.2 UMTC $\text{PF}_N$ and $Z_1(\text{PF}_N)$

In this subsection, we first introduce the parafermion UMTC $\text{PF}_N$. It is the category of modules over a unitary rational vertex operator algebra $V_{\text{PF}_N}$ called the parafermion VOA [DW11, DL12, DLWY10, ADJR18]. For the details of the parafermion VOA and the mathematical details of this UMTC, one can refer to Appendix A. Here we list some ingredients and properties of this UMTC.

- Simple objects are denoted by $M_{\ell,m}$, with $0 \leq \ell \leq N$, $0 \leq m < 2N$ and $\ell + m \equiv 0 \pmod{2}$. Notice that there are $N(N+1)/2$ inequivalent simple objects in $\text{PF}_N$. The equivalence relation between simple objects is given by

$$M_{\ell,m} \sim M_{\ell',m'}, \quad \text{if} \quad \ell = N - \ell', m = N + m'.$$

- The fusion rule is given by:

$$M_{\ell,m} \otimes M_{\ell',m'} = \bigoplus_{r=\max(\ell + \ell' - N,0)} \min(\ell,\ell') \quad M_{\ell + \ell' - 2r,m + m'}.$$  \hfill (2.3)

- The double braiding of two simple objects $M_{\ell,m}$ and $M_{\ell',m'}$ is given by:

$$c_{\ell,m',\ell' m} \otimes c_{\ell m,\ell',m'} = \bigoplus_{s=\max(\ell + \ell' - N,0)} e^{2\pi i (h_{\ell + \ell' - 2s} - h_{\ell} + h_{\ell'})} e^{\frac{\pi}{N} m m'} id_{\ell + \ell' - 2s,m + m'},$$  \hfill (2.4)

where $h_t := \frac{t(t+1)}{2(N+1)}$.

- The twist of the simple object $M_{\ell,m}$ is:

$$\theta_{\ell,m} = e^{2\pi i \frac{h_{\ell} - h_{\ell}}{N}}.$$  

The quantum dimension of $\text{PF}_N$ is $\frac{N(N+2)}{4 \sin^2 \left(\frac{\pi}{2N}\right)}$. 

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Example 2.5. When $N = 2$, we have $\mathcal{Is} = \mathbb{F}_2$, where $\mathcal{Is}$ is the module category over the Ising VOA consisting of the simple objects $[\mathbb{1}, \sigma, \psi]$ with the fusion rules:

$$\psi \otimes \psi = \mathbb{1}, \quad \sigma \otimes \sigma = 1 \otimes \psi.$$ 

We have the following correspondence of the simple objects between these two categories:

$$1 = \mathcal{M}_{0,0}, \quad \sigma = \mathcal{M}_{1,1}, \quad \psi = \mathcal{M}_{2,0}.$$ 

It is not hard to check by the above formulas that the fusion rules, braiding, and the twists of simple objects of $\mathbb{F}_2$ coincide with those of $\mathcal{Is}$.

Now we consider the Drinfeld center $\mathcal{Z}(\mathbb{F}_N)$ of the parafermion UMTC which describes the double $Z_N$ parafermion TO. Since $\mathbb{F}_N$ is non-degenerate, we have $\mathcal{Z}(\mathbb{F}_N) = \mathbb{F}_N \boxtimes \overline{\mathbb{F}_N}$ [Mug03]. Here, $\mathbb{F}_N$ has the same objects and fusion rules with $\mathbb{F}_N$ but has a different braiding defined by $\beta_{x,y} := c_{-1}^{\frac{1}{N}}$ for $x, y \in \mathcal{Z}(\mathbb{F}_N)$. And $\boxtimes$ is the Deligne tensor product of $\mathbb{C}$-linear categories. Simple objects in $\mathbb{F}_N \boxtimes \overline{\mathbb{F}_N}$ are $\mathcal{M}_{\ell,m} \boxtimes \overline{\mathcal{M}_{\ell,m'}}$ where $\mathcal{M}_{\ell,m}$ is a simple object in $\mathbb{F}_N$ and $\overline{\mathcal{M}_{\ell,m'}}$ is a simple object in $\overline{\mathbb{F}_N}$. Recall that $\mathbb{F}_N$ has $N(N+1)/2$ inequivalent simple objects, hence there are $N^2(N+1)^2/4$ inequivalent simple objects in $\mathcal{Z}(\mathbb{F}_N)$. For two simple objects $\mathcal{M}_{\ell_1,m_1} \boxtimes \overline{\mathcal{M}_{\ell_1,m_1'}}$ and $\mathcal{M}_{\ell_2,m_2} \boxtimes \overline{\mathcal{M}_{\ell_2,m_2'}}$, the fusion rule is given by

$$\left(\mathcal{M}_{\ell_1,m_1} \boxtimes \overline{\mathcal{M}_{\ell_1,m_1'}} \right) \otimes \left(\mathcal{M}_{\ell_2,m_2} \boxtimes \overline{\mathcal{M}_{\ell_2,m_2'}} \right) = \left(\mathcal{M}_{\ell_1,m_1} \otimes \mathcal{M}_{\ell_2,m_2} \right) \boxtimes \left(\overline{\mathcal{M}_{\ell_1,m_1'}} \otimes \overline{\mathcal{M}_{\ell_2,m_2'}} \right).$$

Example 2.6. When $N = 2$, $\mathcal{Z}(\mathbb{F}_2) = \mathcal{Z}(\mathcal{Is}) \boxtimes \overline{\mathcal{Is}}$, where $\mathcal{Z}(\mathcal{Is})$ is the double Ising category that describes the double Ising TO. $\mathcal{Z}(\mathcal{Is})$ has 9 simple objects and consists of $\mathbb{1} \boxtimes \sigma$ for all $\mathbb{1} \in \mathcal{Is}$ and $\sigma \in \mathcal{Is}$. For instance, $\mathbb{1} \boxtimes 1$ and $\psi \boxtimes \psi$ are both simple objects of $\mathcal{Z}(\mathcal{Is})$.

### 2.2 Anyon condensation theory

For topological order, a phase transition phenomenon called anyon condensation may occur [BSS02, BSS03, BSS09, BW10]. After anyon condensation, the Hilbert space of the condense phase are the energy-favorable subspace of the original Hilbert space. Hence, there are fewer topological defects and the system is still gapped. Also, there should be a domain wall between the original phase and the condensed phase. Some defects can move across the domain wall and others will be confined to it. The anyon condensation has a systematical mathematic description developed in [Kon14]. This theory tells us how to condense bosonic anyons in a known TO and obtain a new TO. In addition, we have an explicit description of the domain wall between these two TOs. Domain walls between TOs play an important role in the description of gapless boundaries, which we will see in the following section.

As shown in Fig. 1, given a 2d anomaly-free TO described by a UMTC $\mathcal{C}$, we need to find a condensable algebra $A$ in $\mathcal{C}$ that controls which anyons in $\mathcal{C}$ can move across the domain wall freely and which anyons should be confined to the domain wall. Anyons that can move freely across the wall correspond to the so-called local $A$-modules in $\mathcal{C}$. They are the topological defects in the condensed phase. We denote the category of local right $A$-modules in $\mathcal{C}$ as $\mathcal{C}_A^{loc}$. Mathematically, it has been proved that $\mathcal{C}_A^{loc}$ is still a UMTC [BEK00, KO02]. Hence $\mathcal{C}_A^{loc}$ describes a new 2d anomaly-free TO. The 1d gapped domain wall between $\mathcal{C}$ and $\mathcal{C}_A$ can be described by a UFC $\mathcal{C}_A$, which is the category of right $A$-modules. For mathematical details of the anyon condensation theory, please refer to Appendix B.

Example 2.7. We can find a condensable algebra $B = (\mathbb{1} \boxtimes 1) \oplus (\psi \boxtimes \psi) = (\mathcal{M}_{0,0} \boxtimes \mathcal{M}_{0,0}) \oplus (\mathcal{M}_{2,0} \boxtimes \mathcal{M}_{2,0})$ in the double Ising $\mathcal{Z}(\mathcal{Is})$ such that $\mathcal{Z}(\mathcal{Is})_B^{loc} \cong \mathcal{Z}(\text{Rep}(\mathbb{Z}_2))$ [CJK+20].
Figure 1: An illustration of anyon condensation process. $\mathcal{C}$ is the original phase, and $\mathcal{C}^{\text{loc}}_A$ is the condensed phase. The domain wall between these two phases is $\mathcal{C}_A$. The anyon $a$ in $\mathcal{C}$ can freely cross the domain wall; it becomes the anyon $x$ in $\mathcal{C}^{\text{loc}}_A$ after condensation. The anyon $b$ in $\mathcal{C}$ cannot move across the domain wall and is confined to the domain wall; it becomes the anyon $y$ in $\mathcal{C}_A$.

3 A Gappable Gapless Boundary of 2d $\mathbb{Z}_N$ Topological Order

In this section, we will construct a gappable gapless boundary of the $\mathbb{Z}_N$ topological order step by step: In Section 3.1, we begin with a review of a mathematical description of gapless boundaries [KZ18, KZ20b, KZ21]; in Section 3.2, we construct a gapped domain wall between the double parafermion topological order and the $\mathbb{Z}_N$ TO; in Section 3.3, we obtain a gappable gapless boundary of the $\mathbb{Z}_2$ topological order by “dual” the domain wall [KZ21]. Finally, to verify that this gappable non-chiral gapless boundary actually describes the critical point of the boundary phase transition, we compute the physical observables in Section 3.4 and list some examples.

3.1 A mathematical theory of gapless boundaries

Imagine a 2d TO living on a disk propagating vertically along the temporal dimension. This 2d TO has a 1d phase living on the boundary of the disk whose trajectory in spacetime is a $1+1$D worldsheet. This boundary can be either gapped or gapless. The gapped boundaries of 2d TO has been systematically studied [BK98, BSW11, KK12, HWW17, HLP+18]. More interestingly, a mathematical theory of $1+1$D gapless boundaries has been developed recently [KZ20b, KZ21].

At the IR fixed point of the renormalization group flow, the gapless boundary modes on the $1+1$D worldsheet should be described by a $1+1$D conformal field theory (CFT) [BPZ84, MS89]. For a chiral $1+1$D CFT, it can be described by a vertex operator algebra (VOA) $V$ [Hua05, Sch03, FRS02, FFRS07]; for a non-chiral $1+1$D CFT, it can be described by a so-called full field algebra (FFA) $W$ [HK07, Kon07]. Besides these observables on the worldsheet, there should be topological excitations on the $1+0$D boundary, and they must be compatible with the observables on the $1+1$D worldsheet. As a result, a $1+1$D gapless boundary of a $2+1$D gapped TO can be described by a pair $(U, \text{Mod}_U \mathcal{M})$ [KZ20b], where

- $U$ is the local quantum symmetry living on the $1+1$D worldsheet,
- $\text{Mod}_U \mathcal{M}$ is a (unitary) $\text{Mod}_U$-enriched fusion category $\mathcal{M}$ where $\text{Mod}_U$ is the background category that captures the gapless boundaries modes and $\mathcal{M}$ is the underlying category formed by the topological excitations on the 1d boundary.

Here the local quantum symmetry $U$ is assumed to be a unitary rational VOA $V$ [DL14], which corresponds to a chiral gapless boundary, or a unitary rational FFA $W$, which corresponds to a non-chiral gapless boundary [KZ21].

Remark 3.1. When the VOAs or rational FFAs are rational, it has been proved that the corresponding module categories are modular tensor categories [Hua08b, Hua08a].
Remark 3.2. In general, for a VOA $V$, it is not clear if $\text{Mod}_V$ is a UMTC when $V$ is unitary. See [Gui19b, Gui19c, Gui19a] for a discussion of the relation between the unitarity of a VOA $V$ and that of $\text{Mod}_V$.

Remark 3.3. If we regard $\mathbb{C}$ as a trivial VOA, then we can write a VOA $V$ as a FFA $W := V \otimes \mathbb{C}$, hence each chiral gapless boundary can be regarded as a special case of non-chiral gapless boundaries.

Here we wouldn’t give a detailed mathematical definition of a (unitary) enriched fusion category. But we would introduce a geometrical intuition of enriched fusion categories instead. As shown in Fig. 2 there are some topological defect lines on the 1+1D worldsheet.

- The bullets on the 1+0D boundary represents the boundary topological excitations (or boundary conditions) such as $x$ and $x'$.
- The segments of these topological defect lines are 1D topological defects on the worldsheet and are labeled by $M_{x,x'}$.
- The little white squares on these topological defects lines are 0D topological defects on the worldsheet. They are 0D domain walls between two 1D defects, so they are labeled by the label of two adjacent segments. For instance, the 0D domain wall between $M_{x,x}$ and $M_{y,y}$ is labeled by $M_{x,y}$. Physically, $M_{x,y}$ consists of boundary condition changing operators.
- The local quantum symmetry $U$ living on the worldsheet can transparently move across topological defect lines except for those 0D defects. This induces an $U$-action on each 1D defect, i.e. $\iota_U : U \hookrightarrow M_{x,x}$. Thus each defect $M_{x,y}$ is an $U$-module and hence is an object in $\text{Mod}_U$.
- As shown in the vertical process in Fig. 2 0D defects can fuse vertically, which corresponds to the composition of instantons. For example, consider two 0D defects, $M_{x,y}$ and $M_{y,z}$.

![Figure 2: Topological Defect Lines on 1+1D Worldsheet](image)
Their composition is given by the following data:

\[ \circ : M_{y,z} \otimes U M_{x,y} \to M_{x,z} \]

where \( M_{y,z} \otimes U M_{x,y} \) is the tensor product (monoidal structure) in UMTC Mod\(_U\).

- Also, as shown in the horizontal process in Fig. 2, the topological defect line can fuse horizontally, and the 0D defects on the line also fuse at the same time. This corresponds to the monoidal structure of enriched fusion categories. For instance, the horizontal fusion of \( M_{x,y} \) and \( M_{x',y'} \) is given by

\[ M_{x,y} \otimes U M_{x',y'} \to M_{x \otimes x', y \otimes y'} \]

Therefore, we can conclude that an enriched fusion category consists of the topological defect lines with their fusions. For more mathematical details of enriched fusion categories, please see [MP19, KZ20b]. For readers who are interested in the bootstrap analysis that an 1d gapless boundary can be mathematically described by an enriched fusion category with a local quantum symmetry, we recommend [KZ20b].

**Remark 3.4.** Actually, we can regard \( M_{x,x} \) as a 0D domain wall between \( M_{y,x} \) and \( M_{z,x} \) itself. Hence, the 1D defect \( M_{x,x} \) can be regarded as a combination of infinitely many 0D defects \( M_{x,x} \). Based on this view, on a 1D defect \( M_{x,x} \), the composition \( \circ : M_{x,x} \otimes U M_{x,x} \to M_{x,x} \) induces an algebra structure on \( M_{x,x} \). This algebra structure is highly non-trivial, indeed, it is a so-called open string vertex operator algebra that describes an open (boundary) CFT [HK04].

**Remark 3.5.** A gapped domain wall \( \mathcal{M} \) can also be described by a pair \((\mathcal{C}, \mathcal{H}_M)\) where \( \mathcal{C} \) can be regarded as a trivial VOA whose module category is \( \mathcal{H} \). Therefore, gapped and gapless boundaries are unified in the language of enriched categories. This finishes the unified story of enriched category language on 1d quantum phases (without symmetry).

![Diagram](image)

Recall that our goal is to find a gapless boundary to describe a critical point of a self-dual topological phase transition on the 1d boundary of \( Z_N \) TO. Note that the \( Z_N \) quantum double TO is non-chiral, which does not admit a chiral gapless boundary. Hence, we need to consider a non-chiral gapless boundary

\[ (W, Mod_W M) \]

where \( W \) is a FFA and \( M \) is an Mod\(_W\)-enriched fusion category. Moreover, the non-chiral gapless boundary describing the critical point should be gappable because this gapless boundary will become one of the gapped boundaries under a small perturbation. During this process, the bulk is unchanged, so this gapless boundary and the two gapped boundaries share the same bulk. Mathematically, it is equivalent to say that the center of this gapless boundary and the center of the two gapped boundaries are the same. As shown in Picture 3.1 using \( N \) denote one of the gapped boundaries, we have \( Z_1(\text{Mod}_W M) = Z_1(\text{Rep}(Z_N)) = Z_1(N) \) by the boundary-bulk relations [KWZ17, KYZZ21].
3.2 A gapped wall between $Z_N$ quantum double and $Z_N$ double parafermion

In Section 2.2 and Appendix B we review the anyon condensation theory and its mathematical description. To find a gapped domain wall between the double $Z_N$ parafermion TO ($\mathcal{A}(\text{Rep}(Z_N))$, 0) and the $Z_N$ TO ($\mathcal{A}(\text{Rep}(Z_N))$, 0), we need to find a condensable algebra $B$ in $\mathcal{A}(\text{Rep}(Z_N))$ such that the category $\mathcal{A}(\text{Rep}(Z_N))$ of local $B$-modules is equivalent to $\mathcal{A}(\text{Rep}(Z_N))$ as UMTC. Then the category $\mathcal{A}(\text{Rep}(Z_N))$ of right $B$-modules in $\mathcal{A}(\text{Rep}(Z_N))$ describes the gapped domain wall we need.

![Diagram of categories]

We claim that

$$B := \bigoplus_{u=0}^{[\pi/2]} \mathcal{M}_{2u,0} \otimes \mathcal{M}_{2u,0}$$

(3.3)

is a condensable algebra in $\mathcal{A}(\text{Rep}(Z_N))$. To see this, consider the fusion subcategory $\mathcal{P} \subset \text{Rep}(Z_N)$ generated by simple objects $\mathcal{M}_{2u,0}$, $u = 0, \ldots, [N/2]$. Let $\otimes : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}$ be the tensor functor, then there is a lagrangian algebra in $\mathcal{P} \otimes \mathcal{P}$ given by $\otimes^R(\mathcal{M}_{0,0}) = \bigoplus_{u=0}^{[N/2]} \mathcal{M}_{2u,0} \otimes \mathcal{M}_{2u,0}$ [KR09], where $\otimes^R$ is the right adjoint of the tensor functor. Since $\mathcal{P} \otimes \mathcal{P}$ is a fusion subcategory of $\mathcal{A}(\text{Rep}(Z_N))$, thus a condensable algebra of $\mathcal{P} \otimes \mathcal{P}$ also is a condensable algebra of $\mathcal{A}(\text{Rep}(Z_N))$, therefore $B$ is a condensable algebra in $\mathcal{A}(\text{Rep}(Z_N))$.

It is clear that the dimension of $B$ is $\frac{N(N^2+2)}{4\sin^2\frac{\pi}{N+1}}$. Recall that the dimension of $\mathcal{A}(\text{Rep}(Z_N))$ is $\frac{N(N^2+2)}{4\sin^2\frac{\pi}{N+1}}$. Thus the dimension of $\mathcal{A}(\text{Rep}(Z_N))$ is $N^2$ which is equivalent to the dimension of $\mathcal{A}(\text{Rep}(Z_N))$. It suggests that these two categories $\mathcal{A}(\text{Rep}(Z_N))$ and $\mathcal{A}(\text{Rep}(Z_N))$ might be equivalent. What remains is to prove that the equivalence holds. However, we have to discuss it separately because the properties of $\mathcal{A}(\text{Rep}(Z_N))$ are very different for $N$ as odd and even cases.

3.2.1 Odd case

To determine the category $\mathcal{A}(\text{Rep}(Z_N))$, we need to determine all the simple objects and the coherence data of these simple objects. Recall that the objects of $\mathcal{A}(\text{Rep}(Z_N))$ are local $B$-modules in $\mathcal{A}(\text{Rep}(Z_N))$, thus we begin with a discussion of $B$-modules.

A right $B$-module is a pair $(x, \rho_x)$ where $x$ is an object in $\mathcal{A}(\text{Rep}(Z_N))$ and $\rho_x : x \otimes B \to x$ is the $B$-action on $x$ satisfying some coherence conditions. Hence, we need not only to determine an object $x$ in $\mathcal{A}(\text{Rep}(Z_N))$ but also to find a $B$-action on $x$. In other words, there might be multiple inequivalent $B$-actions such as $\rho_x^1 : x \otimes B \to x$ and $\rho_x^2 : x \otimes B \to x$, resulting in $(x, \rho_x^1)$ and $(x, \rho_x^2)$ being two inequivalent $B$-modules. Therefore, it is better to consider some specific $B$-modules called the free $B$-modules first.

By a right free $B$-module, we mean a right $B$-module like $x \otimes B$ for some $x \in \mathcal{A}(\text{Rep}(Z_N))$, where the condensable algebra $B$ acts on $x \otimes B$ from right, and the $B$-module action is induced by the multiplication $m_B : B \otimes B \to B$ of $B$, that is, $\rho_{x \otimes B} : x \otimes B \otimes B \xrightarrow{id \otimes m_B} x \otimes B$. In other words, the $B$-action on a free $B$ module $x \otimes B$ has no relation with $x$. Also, we can prove that each

\[ \text{(3.3)} \]
simple $B$-modules are direct summand of some free $B$-modules [DMNO13]. This is even more motivating to consider the free $B$-modules.

Fortunately, when $N$ is odd, that is, $N = 2k + 1$ for some $k \in \mathbb{N}$, all simple local $B$-modules in $\mathcal{Z}_1(\text{PF}_{2k+1})$ are free $B$-modules. We claim that

$$X_{a,b} := (\mathcal{M}_{0,2a} \boxtimes \mathcal{M}_{0,2b}) \otimes B, a, b = 0, \ldots, 2k,$$ \hspace{1cm} (3.4)

exhaust all simple local $B$-modules in $\mathcal{Z}_1(\text{PF}_{2k+1})$. To see this, we need the following adjunction between the tensor functor $- \otimes B : \mathcal{Z}_1(\text{PF}_N) \rightarrow \mathcal{Z}_1(\text{PF}_N)_B$ which sends each object $x$ in $\mathcal{Z}_1(\text{PF}_N)$ to the free module $x \otimes B$, and the forgetful functor $U : \mathcal{Z}_1(\text{PF}_N)_B \rightarrow \mathcal{Z}_1(\text{PF}_N)$ which forgets the $B$-action $\rho_x$ for each $B$-module $(y, \rho_y)$.

**Proposition 3.6.** For any $x, y$ in $\text{PF}_N$, there is a (natural) isomorphism:

$$\text{hom}_{\mathcal{Z}_1(\text{PF}_N)_B}(x \otimes B, y \otimes B) \cong \text{hom}_{\mathcal{Z}_1(\text{PF}_N)}(x, y \otimes B).$$ \hspace{1cm} (3.5)

We use this adjunction to check if a free $B$-module is simple and local, and if two simple free $B$-modules are equivalent to each other.

1. For a free $B$-module $x \otimes B$, we can check whether it is a simple $B$-module by the above adjunction. By Schur’s lemma, $x \otimes B$ is simple if and only if $\dim \text{hom}_{\mathcal{Z}_1(\text{PF}_N)_B}(x \otimes B, x \otimes B) = 1$ [EGNO16]. However, it is hard to calculate the $B$-module homomorphism directly, so we would rather calculate the dimension of $\text{hom}_{\mathcal{Z}_1(\text{PF}_N)}(x, x \otimes B)$ so that we only need to count the number of $x$ in $x \otimes B$.

2. Moreover, for two simple free $B$-modules $x \otimes B$ and $y \otimes B$, we can also use this adjunction to determine if they are equivalent to each other. If $\dim \text{hom}_{\mathcal{Z}_1(\text{PF}_N)_B}(x \otimes B, y \otimes B) = 0$, then due to Schur’s lemma [EGNO16], $x \otimes B$ and $y \otimes B$ must be inequivalent. As above, $\dim \text{hom}_{\mathcal{Z}_1(\text{PF}_N)}(x, y \otimes B)$ is equivalent to the number of $x$ in $y \otimes B$, hence we only need to calculate $y \otimes B$ and count $x$.

3. For locality, recall that a $B$-module $(x, \rho_x)$ is local if $\rho_x \circ c_{B,x} \circ c_{x,B} = \rho_x$ holds. When we only consider the free $B$-modules, we can just check that if the double braiding $c_{B,x} \circ c_{x,B}$ is trivial. If so, then $x \otimes B$ is a local $B$-module.

Indeed, by procedure 1, it is easy to see that each $(\mathcal{M}_{0,2a} \boxtimes \mathcal{M}_{0,2b}) \otimes B = \bigoplus_{i=0}^{k} \mathcal{M}_{2i,2a} \boxtimes \mathcal{M}_{2i,2b}$ has only one $\mathcal{M}_{0,2a} \boxtimes \mathcal{M}_{0,2b}$ berm, hence it must be simple. By procedure 2, two free $B$-modules $(\mathcal{M}_{0,2a} \boxtimes \mathcal{M}_{0,2b}) \otimes B$ and $(\mathcal{M}_{0,2c} \boxtimes \mathcal{M}_{0,2d}) \otimes B$ with $a \neq c, b \neq d$ are not equivalent to each other because we cannot find $\mathcal{M}_{0,2a} \boxtimes \mathcal{M}_{0,2b}$ in $(\mathcal{M}_{0,2c} \boxtimes \mathcal{M}_{0,2d}) \otimes B$ and vice versa. By procedure 3, and Eq. (2.4) of the double braiding of $\text{PF}_N$, it is clear that the double braiding of $\mathcal{M}_{0,2a} \boxtimes \mathcal{M}_{0,2b}$ with the condensable algebra $B$ is 1, hence each free $B$-module $(\mathcal{M}_{0,2a} \boxtimes \mathcal{M}_{0,2b}) \otimes B$ is local.

In summary, we have found $(2k + 1)^2$ inequivalent simple local $B$-modules, recall that there are $(2k + 1)^2$ inequivalent simple objects in $\mathcal{Z}_1(\text{Rep}(\mathcal{Z}_{2k+1}))$, it suggests us that we have exhausted all the simple local $B$-modules in $\mathcal{Z}_1(\text{PF}_N)$. Indeed, we have the following theorem:

**Theorem 3.7.** The category $\mathcal{Z}_1(\text{PF}_{2k+1})^\text{loc}$ is equivalent to the category $\mathcal{Z}_1(\text{Rep}(\mathcal{Z}_{2k+1}))$ as modular tensor categories. The correspondence of simple objects of two categories is given by

$$O_{a+b,a-b} \sim X_{a,b} = \bigoplus_{i=0}^{k} \mathcal{M}_{2i,2a} \boxtimes \mathcal{M}_{2i,2b}.$$ \hspace{1cm} (3.6)

The mathematical details of the proof of the equivalence are shown in Appendix C.
Recall the physical notation $e^a m^\beta$ of simple objects $O_{a,b}$. Let $a = a + b \mod (2k + 1)$ and $\beta = a - b \mod (2k+1)$, we can write the correspondence of simple objects between $\mathcal{Z}_{2k+1}$ and $\mathbb{Z}_N$ as follows,

$$
e^a m^\beta = \begin{cases} 
\mathcal{Z}_{2k+1} \times \mathbb{Z}_N, & \text{if } a + \beta \equiv 0 \mod 2, \\
\mathbb{Z}_{2k+1-(a+\beta)} \times \mathbb{Z}_{2k+1-(a-\beta)}, & \text{if } a + \beta \equiv 1 \mod 2.
\end{cases} \tag{3.7}
$$

The above formula is a little complicated, but we can rewrite it in a compact form. First, we consider the case with only charges $e^a$. We can rewrite the expansion of $\mathcal{Z}_{a,b}^\alpha$ such that the second index of each term is $\alpha$:

$$e^a := \bigoplus_{\ell + \alpha \equiv 0 \mod 2} M_{\ell,\alpha} \otimes \overline{M}_{\ell,\alpha}. \tag{3.8}$$

Similarly, for the case with only flux $m^\beta$, we can rewrite the expansion of $\mathcal{Z}_{a,b}^\alpha$ as follows:

$$m^\beta = \bigoplus_{\ell + \beta \equiv 0 \mod 2} M_{\ell,\beta} \otimes \overline{M}_{\ell,\beta \mod (2k+1)-2\ell}. \tag{3.9}$$

Recall that $e^a m^\beta = e^a \otimes m^\beta$, therefore by Eq. (3.8) and Eq. (3.9) we have a general formula:

$$e^a m^\beta = \bigoplus_{m = a + \beta \mod (2k+1)} M_{\ell,m} \otimes \overline{M}_{\ell,m-2\beta}. \tag{3.10}$$

Here we choose $m$ to be the remainder of $a + \beta$ modulo $(2k+1)$ to avoid double counting.

In summary, we prove that the condensed phase $\mathcal{Z}_{2k+1}$ is the $\mathbb{Z}_N$ quantum double TO $\mathcal{Z}_N(\mathbb{Z}_N)$. According to the anyon condensation theory, the gapped domain wall between double $\mathbb{Z}_N$ parafermion TO and $\mathbb{Z}_N$ quantum double TO is $\mathcal{Z}_N(\mathbb{Z}_N)$.

### 3.2.2 Even case

When $N = 2k$ for some $k \in \mathbb{N}$, the task of determining all simple local $B$-modules is quite challenging. The argument based on Eq. (3.3) only works for free $B$-modules. With the notation in the above subsection, $\{X_{a,b} \mid a, b = 0, \ldots, 2k - 1\}$ is still a set of simple local $B$-modules. However, the elements in this set satisfy an equivalence relation $X_{a,b} \equiv X_{a+b,k+b} (\mod 2k+1)$, hence at least half of the simple local $B$-modules are free $B$-modules as listed in $\{X_{a,b} \mid a = 0, \ldots, 2k - 1, b = 0, \ldots, k - 1\}$. For the rest of the simple local $B$-modules, it is very possible that they are not free $B$-modules.

In fact, when we consider $N = 2$, there are 4 inequivalent simple local $B$-modules:

$$1 := (M_{0,0} \otimes M_{0,0}) \otimes B, \quad f := (M_{2,0} \otimes M_{0,0}) \otimes B, \quad e := M_{1,1} \otimes M_{1,1}, \quad m := (M_{1,1} \otimes M_{1,1})^{\text{tw}}.$$

which correspond to 4 simple objects in $\mathcal{Z}_1(\mathbb{Z}_2)$ respectively. Notice that $M_{1,1} \otimes M_{1,1}$ and $(M_{1,1} \otimes M_{1,1})^{\text{tw}}$ are not free $B$-modules. Here $(M_{1,1} \otimes M_{1,1})^{\text{tw}}$ is a $B$-module whose object is $M_{1,1} \otimes M_{1,1}$ with a twisted $B$-action. For a general $N = 2k > 2$, it is difficult to determine the simple local $B$-modules that are not free because we need to consider the $B$-actions on modules now. One possible way is to guess a $B$-module and its $B$-action, check if it is a local $B$-module, and then check if it is simple. However, for two $B$-modules, they may be isomorphic to each other, which raises the difficulty of finding simple local $B$-modules. We do not give all the simple local $B$-modules in the even cases, and also no proof for the equivalence between UMTCs $\mathcal{Z}_1(\mathbb{Z}_2)$ and $\mathcal{Z}_1(\mathbb{Z}_N)$. But we believe that the equivalence: $\mathcal{Z}_1(\mathbb{Z}_N)^{\text{loc}} = \mathcal{Z}_1(\mathbb{Z}_N)$ still holds for $N = 2k$, since we confirm that half of the simple objects $\{e^a m^\beta \mid a + \beta \equiv 0 \mod 2\}$ in $\mathcal{Z}_1(\mathbb{Z}_N)$ indeed correspond to the simple objects in $\mathcal{Z}_1(\mathbb{Z}_N)$.
3.3 A gappable non-chiral gapless boundary of $Z_N$ topological order

In the previous subsection, we have found a gapped domain wall between $\mathcal{Z}_1(\text{Rep}(Z_N))$ and $\mathcal{Z}_1(\text{Rep}(Z_N))$. By a holographic duality called the topological Wick rotation [KZ20b], we can “dual” this gapped domain wall to a gappable non-chiral gapless boundary of $Z_N$ TO.

One can imagine that flipping $\mathcal{Z}_1(\text{PF}_N)$ from the spatial dimension to the temporal dimension. The vertical part becomes a background of the $Z_N$ quantum double TO. By the classification theorem of non-chiral gapless edge [KZ21], we obtain an enriched fusion category $\mathcal{Z}_1(\text{PF}_N)\mathcal{Z}_1(\text{PF}_N)$.

Recall that $\text{PF}_N = \text{Mod}_{V_{\text{PF}_N}}$, thus we have $\mathcal{Z}_1(\text{PF}_N) = \mathcal{PF}_N \boxtimes \mathcal{PF}_N = \text{Mod}_{V_{\text{PF}_N} \otimes V_{\text{PF}_N}}$ which is the module category over the FFA $V_{\text{PF}_N} \otimes_{\mathbb{C}} V_{\text{PF}_N}$. The enriched fusion category $\mathcal{Z}_1(\text{PF}_N)\mathcal{Z}_1(\text{PF}_N)$ together with the FFA $W := V_{\text{PF}_N} \otimes_{\mathbb{C}} V_{\text{PF}_N}$ form a non-chiral gapless boundary of 2+1D $Z_N$ quantum double TO ($\mathcal{Z}_1(\text{Rep}(Z_N))$, $\mathcal{Z}_1(\text{PF}_N)$).

\begin{equation}
(V_{\text{PF}_N} \otimes_{\mathbb{C}} V_{\text{PF}_N}, \mathcal{Z}_1(\text{PF}_N)\mathcal{Z}_1(\text{PF}_N)).
\end{equation}

Remark 3.8. Such a folding process is physically impossible. Imagine that we cut out the left half of a 2d topological order by brutal force. It takes the system in the neighborhood of the boundary away from a RG fixed point. As time goes by, the boundary will undergo a self-healing process by flowing to a new RG fixed point. One of the possible RG fixed points is the canonical chiral gapless boundary ($V^B_{\mathbb{C}}$). We believe that this folding process is not just an ad hoc bookkeeping trick. It should have some deep physical meanings which need to be further studied.

Remark 3.9. When we regard the gapped domain wall $\mathcal{Z}_1(\text{PF}_N)_B$ as an enriched fusion category $^{H}_\mathcal{Z}_1(\text{PF}_N)_B$, there is another interpretation of the gapless boundary ($V_{\text{PF}_N} \otimes_{\mathbb{C}} V_{\text{PF}_N}, \mathcal{Z}_1(\text{PF}_N)_B$).

It can be regarded as a fusion of the canonical gapless boundary ($W, \mathcal{Z}_1(\text{PF}_N)_B$) with the gapped domain wall ($C, \mathcal{Z}_1(\text{PF}_N)_B$), as shown in the following picture.

\begin{equation}
(\mathcal{Z}_1(\text{PF}_N), \mathcal{Z}_1(\text{PF}_N)_B) \otimes_{\mathbb{C}} \mathcal{Z}_1(\text{PF}_N)_B.
\end{equation}

The enriched fusion category $^{H}_\mathcal{Z}_1(\text{PF}_N)_B$ can be described in more detail when $N = 2k+1$.

- Objects (topological boundary excitations) are right $B$-modules in $\mathcal{Z}_1(\text{PF}_N)_B$, including the simple local $B$-modules $\mathcal{X}_{a,b}$. They can fuse horizontally by the fusion rule of $\mathcal{Z}_1(\text{PF}_N)_B$.

- For two objects $x$ and $y$ in $\mathcal{Z}_1(\text{PF}_N)_B$, the 0D defects $M_{x,y}$ between them are given by $(x \otimes y)^*$ [Os03], which should be viewed as an object in $\mathcal{Z}_1(\text{PF}_N)_B$. Here $y^*$ is the dual of $y$, since $\mathcal{Z}_1(\text{PF}_N)_B$ is a pointed fusion category, the dual of $\mathcal{X}_{a,b}$ is its inverse $\mathcal{X}_{2k+1-a,2k+1-b}$. In more detail, we can calculate the 0D defects for any simple local $B$-modules, and therefore any local $B$-modules. For two local $B$-modules $\mathcal{X}_{a,b}$ and $\mathcal{X}_{c,d}$, we have

$$
(\mathcal{X}_{a,b} \otimes_B \mathcal{X}_{c,d})^* = (\mathcal{X}_{a,b} \otimes_B \mathcal{X}_{2k+1-a,2k+1-d})^* = (\mathcal{X}_{2k+1-(c-a),2k+1-(d-b)})^* = \mathcal{X}_{c-a,d-b}.
$$
This non-chiral gapless boundary is gappable. To see this, we need to prove that the center of $3\otimes N$ is $3\otimes \text{Rep}(Z_N)$. By the definition in [KYZZ21], it is given by the centralizer of $3\otimes N$ in $3\otimes N$, i.e.

$$Z_2(3\otimes N, 3\otimes N) = (3\otimes N, 3\otimes N)\otimes (3\otimes N, 3\otimes N).$$

Recall by anyon condensation theory [KO02, Kon14], there are equivalences between UMTCs,

$$3\otimes N \otimes 3\otimes N \otimes N \otimes N = 3\otimes \text{Rep}(Z_N).$$

Finally, by the centralizer Theorem [Müg03] proved in [Müg03], we can prove that

$$3\otimes N = 3\otimes N \otimes N \otimes N \otimes N,$$

which ensures that the non-chiral gapless boundary ($W$) has the same bulk as the gapped boundary ($C$, $H\otimes \text{Rep}(Z_N)$).

### 3.4 The partition functions of $M_{x,y}$

Now we have constructed a gappable non-chiral gapless boundary ($V_{PF_N}$) $3\otimes N$ of $Z_N$ quantum double TO. It is natural to ask if it describes the critical point of the topological phase transition on the 1d gapped boundary of $Z_N$ TO. To verify this, it is enough to check the partition functions of $M_{x,y}$. In physics, the partition function of $M_{x,y}$ can be computed by the path integral over an annulus obtained by compactifying the time axis, and fixing the two boundary conditions to be $x$ and $y$. By the fact that $M_{x,y} = M_{x,y} \otimes \text{Rep}(Z_N)$ as objects in $3\otimes N$, it is easy to see that the partition function of $M_{x,y}$ is the same as that of $M_{x,y} \otimes \text{Rep}(Z_N)$. Thus, calculating partition functions of $M_{x,y}$ is enough for us to get all partition functions of $M_{x,y}$. We denote the partition function of $M_{x,y}$ by $Z(x)$ in the following.

Note that $x = M_{x,y}$ can be viewed as the simple objects in $3\otimes N$. Therefore, the partition function $Z(x)$ can be expressed in terms of the characters $\chi_{\ell,m}$ of simple objects $\mathcal{Q}_{\ell,m}$ in the parafermion CFT. Recall the expansion of simple local $B$-modules Eq. (3.10), we have

$$Z(e^m \bar{B}) = \sum_{m=\pm 1, \ell \equiv -1 \mod 2} \chi_{\ell,m} \bar{\chi}_{\ell,m, \bar{B}}.$$  

However, when $N = 2k$, since we only figure out half of the simple objects in $3\otimes N$, the other half of the partition functions given by Eq. (3.17) cannot be verified as correct. Fortunately, all partition functions can be recovered from the lattice model construction in Section 4.

### 3.5 Some examples

In this subsection, we list some examples of $Z_N$. Recall that the condensable algebra in $3\otimes N$ is $B = \bigoplus_{\ell=0}^{[N/2]} \mathcal{Q}_{\ell,0} \otimes \mathcal{Q}_{\ell,0}$. 

**Notation 3.10.** For the convention of the following discussion, we use $[\ell; m]$ denote the simple objects $\mathcal{Q}_{\ell,m}$ in $PF_N$. And use $[\ell; m | \ell'; m']$ denote the simple objects $\mathcal{Q}_{\ell,m} \otimes \mathcal{Q}_{\ell',m'}$ in $3\otimes N$.

#### 3.5.1 Example $Z_2$

Let $N = 2$, as Example 2.6 shows, there are 9 simple objects in double Ising TO. The condensable algebra is $B = \{0; 0 | 0; 0\} \oplus \{2; 0 | 2; 0\}$. There are four local $B$-modules:

$$1 := (\mathcal{Q}_{0,0} \otimes \mathcal{Q}_{0,0}) \otimes B, \ e := \mathcal{Q}_{1,1} \otimes \mathcal{Q}_{1,1}, \ m := (\mathcal{Q}_{1,1} \otimes \mathcal{Q}_{1,1})^{\text{ev}}, \ f := (\mathcal{Q}_{2,0} \otimes \mathcal{Q}_{2,0}) \otimes B.$$
The corresponding partition function is given by
\[ Z(1) = X_{(0),0} \overline{X}_{(0),0} + X_{(2),0} \overline{X}_{(2),0}, \]
\[ Z(e) = X_{(1),1} \overline{X}_{(1),1}, \]
\[ Z(\mathfrak{m}) = X_{(1),1} \overline{X}_{(1),1}, \]
\[ Z(f) = X_{(2),0} \overline{X}_{(0),0} + X_{(0),0} \overline{X}_{(2),0}. \]

This recovers the result in \([\mathcal{C}][\mathbb{K}^*20]\).

### 3.5.2 Example \(Z_3\)

Let \(N = 3\), for simple objects \(\{\ell; m \mid \ell'; m'\}\), the index \(\ell, \ell'\) has the range \(\{0, 1, 2, 3\}\), \(m, m'\) has the range \(\{0, 1, 2, 3, 4, 5\}\). There are 6 inequivalent simple objects \(\text{Irr}(\text{PF}_3) = \{(0; 0), (0; 2), (0; 4), (2; 0), (2; 2), (2; 4)\}\) in \(\text{PF}_3\). Thus there are 36 inequivalent simple objects in double parafermion \(\text{Z}_3(\text{PF}_3)\) given by \(\text{Irr}(\text{PF}_3) \times \text{Irr}(\text{PF}_3)\).

The index \(t\) of direct summand of the condensable algebra \(B\) has range \(\{0, 1\}\), then the condensable algebra in \(\text{Z}_3(\text{PF}_3)\) is \(B = \{0; 0 \mid 0; 0, 0; 0 \oplus 0; 0, 2; 0\}\). Recall that in odd case, the set of free \(B\)-modules \(\{X_{a,b} \mid 0 \leq a, b \leq 2\}\) gives all simple local \(B\)-modules. Under the equivalence (see Appendix[C]) between two modular tensor categories \(\text{Z}_3(\text{PF}_{2k+1})\) and \(\text{Z}_3(\text{Rep}(\text{Z}_{2k+1}))\), we have \(a = a + b \mod 3\) and \(\beta = a - b \mod 3\). Then we can write all the simple local \(B\)-modules as follows,

\[
\begin{align*}
1 & := \{0; 0 \mid 0; 0\} \otimes B = \{0; 0 \mid 0; 0\} + \{2; 0 \mid 2; 0\}, \\
\text{em}^2 & := \{0; 0 \mid 0; 2\} \otimes B = \{0; 0 \mid 0; 2\} + \{2; 0 \mid 2; 2\}, \\
\text{em} & := \{0; 2 \mid 0; 0\} \otimes B = \{0; 2 \mid 0; 0\} + \{2; 2 \mid 2; 0\}, \\
\text{em}^2 & := \{0; 2 \mid 0; 2\} \otimes B = \{0; 2 \mid 0; 2\} + \{2; 2 \mid 2; 2\}, \\
\text{m}^2 & := \{0; 4 \mid 0; 4\} \otimes B = \{0; 4 \mid 0; 4\} + \{2; 4 \mid 2; 4\}, \\
\text{em} & := \{0; 4 \mid 0; 2\} \otimes B = \{0; 4 \mid 0; 2\} + \{2; 4 \mid 2; 2\}, \\
\text{e} & := \{0; 4 \mid 0; 4\} \otimes B = \{0; 4 \mid 0; 4\} + \{2; 4 \mid 2; 4\}.
\end{align*}
\]

This gives the partition function of simple objects in \(\text{Z}_3(\text{Rep}(Z_3))\).

\[
\begin{align*}
Z(1) & = X_{(0),0} \overline{X}_{(0),0} + X_{(2),0} \overline{X}_{(2),0}, \\
Z(e) & = X_{(1),1} \overline{X}_{(1),1}, \\
Z(\text{em}) & = X_{(1),1} \overline{X}_{(1),1}, \\
Z(e^2) & = X_{(0),0} \overline{X}_{(0),0} + X_{(2),0} \overline{X}_{(2),0}, \\
Z(\text{em}^2) & = X_{(0),0} \overline{X}_{(0),0} + X_{(2),0} \overline{X}_{(2),0}, \\
Z(e^2\text{m}) & = X_{(0),0} \overline{X}_{(0),0} + X_{(2),0} \overline{X}_{(2),0}, \\
Z(\text{em}^2\text{m}) & = X_{(0),0} \overline{X}_{(0),0} + X_{(2),0} \overline{X}_{(2),0}. \\
\end{align*}
\]

### 3.5.3 Example \(Z_4\)

When \(N = 4\), \(Z_4\) is the second primary even case. In this case the index \(\ell, \ell'\) has the range \(\{0, 1, 2, 3, 4\}\), \(m, m'\) has the range \(\{0, 1, 2, 3, 4, 5, 6, 7\}\). There are 10 inequivalent simple objects.
Irr(PF_4) := \{0; 0\}, \{2; 0\}, \{4; 0\}, \{0; 2\}, \{2; 2\}, \{4; 2\}, \{1; 1\}, \{3; 1\}, \{1; 3\}, \{3; 3\}\} in \text{PF}_4. \text{ Thus there are 100 inequivalent simple objects in double parafermion } \mathcal{Z}_4(\text{PF}_4) \text{ given by } \text{Irr}(\text{PF}_4) \times \text{Irr}(\text{PF}_4).

The index \( t \) of direct summand of the condensable algebra \( B \) has range \{0, 1, 2\}, then the condensable algebra in \( \mathcal{Z}_4(\text{PF}_4) \) is \( B = \{0; 0 | 0; 0\} \oplus \{2; 0 | 2; 0\} \oplus \{4; 0 | 4; 0\}. \text{ In principle, there should be 16 simple objects, or simple local } B\text{-modules. The set of free } B\text{-modules } \{\chi_{a,b}\} \text{ gives 8 inequivalent simple local } B\text{-modules. We also have the relation } \alpha = a + b \mod 4 \text{ and } \beta = a - b \mod 4.

\[
\begin{align*}
\mathbb{1} & := \{0; 0 | 0; 0\} \oplus \{2; 0 | 0; 0\} \oplus \{2; 0 | 0; 0\} \oplus \{4; 0 | 4; 0\}, \\
\mathbf{em}^3 & := \{0; 2 | 0; 2\} \oplus \{0; 0 | 2; 0\} \oplus \{2; 2 | 2; 0\} \oplus \{4; 2 | 4; 2\}, \\
\mathbf{e}^2 \mathbf{m} & := \{4; 0 | 0; 2\} \oplus \{0; 0 | 4; 2\} \oplus \{2; 0 | 2; 0\} \oplus \{4; 0 | 0; 2\}, \\
\mathbf{e}^3 \mathbf{m}^3 & := \{0; 2 | 4; 0\} \oplus \{2; 2 | 2; 0\} \oplus \{4; 2 | 0; 0\}, \\
\mathbf{m}^2 & := \{0; 2 | 4; 2\} \oplus \{0; 2 | 4; 2\} \oplus \{2; 2 | 2; 2\} \oplus \{4; 2 | 0; 2\}, \\
\mathbf{e}^2 \mathbf{m}^2 & := \{0; 0 | 4; 0\} \oplus \{0; 0 | 4; 0\} \oplus \{2; 0 | 2; 0\} \oplus \{4; 0 | 0; 0\}.
\end{align*}
\]

The remaining 8 simple local modules cannot be determined by the above algorithm. In fact, they are not free \( B\)-modules but should be the direct summand of some free \( B\)-modules. We can now write the partition functions as we did in the \( \mathbb{Z}_3 \) case, which we omit here.

\section{A Lattice Model Realization}

In this section, we give a lattice model realization of the self-dual critical point of the pure topological order, and recover all the ingredients of the gappable non-chiral gapless boundary constructed in Section \ref{sec:topological_order}. We choose the string-net model to be the lattice realization of the 2d \( \mathbb{Z}_N \) topological order, because we hope that the construction in this paper can be further extended to a more general case. For the convenience of readers in the mathematical background, we review the string-net model in detail.

\subsection{\( \mathbb{Z}_N \) Toric Code Model}

It is well known that an arbitrary 2d non-chiral topological order can be realized by Levin-Wen model \cite{LW05}, which is a machine that imports a fusion category \( \mathcal{C} \) and outputs the corresponding Drinfeld center \( \mathcal{Z}_1(\mathcal{C}) \). When the input category \( \mathcal{C} = \text{Rep}(\mathbb{Z}_N) \), we can obtain the \( \mathbb{Z}_N \) TO as the \( \mathbb{Z}_N \) generalized version of the toric code \cite{HWW12, SDO12}.

The Levin-Wen model is usually defined on a 2d honeycomb lattice. The edges of the lattice are labeled by the oriented simple-type strings, which are the simple objects of the input fusion category \( \mathcal{C} \). And a vertex \( v \) connecting three edges labelled by simple objects \( i, j, k \) is associated to the hom space \( V_{ij}^k = \text{Hom}_{\mathcal{C}}(i \otimes j, k) \) of \( \mathcal{C} \). Therefore, the Hilbert space of Levin-Wen model can be organized as \( \mathcal{H} = \bigotimes_v \mathcal{H}_v \), where \( \mathcal{H}_v = \bigoplus_{i,j,k} V_{ij}^k \). The Hamiltonian is supposed to be exactly solvable and is written as,

\[
H_{\text{Levin-Wen}} = -\sum_v A_v - \sum_p B_p
\]

where \( A_v \mid \gamma_i^\ell \rangle = \delta_{i,k} \mid \gamma_i^\ell \rangle \) is the vertex operator for imposing the particular string-net branching rule, such as the \( \mathbb{Z}_N \) branching rules: \( i + j + k = 0 \mod N \). And \( B_p = \sum_{s=0}^N a_s B_p^s \) is the
“magnetic–flux” operator defined for each hexagonal plaquette of the string–net lattice, with \( a_s = d_s/D \) and \( D = \sum_{i=0}^{N} d_i^2 \). In general, a plaquette operator is constituted by a product of 6\( j \) symbols \( F \),

\[
B_p^s \left| \begin{array}{cccccc}
\ell_1 & \ell_2 & \ell_3 & \ell_4 & \ell_5 & \ell_6 \\
\alpha & \alpha & \alpha & \alpha & \alpha & \alpha \\
\end{array} \right| = \sum_e \prod_{e'\in e^+} F_{s \ell_{e'}\ell_{e'}} \prod_{e''\in e^-} F_{s' \ell_{e''\ell_{e''}}} ,
\]

which is obviously very complicated.

However, for the \( \mathbb{Z}_N \) topological order, there are \( N \) different string types, labeled by 0,...,\( N-1 \). In addition, the string is oriented. A natural way to fix the orientation is to use the sub-lattice of the honeycomb, \( L_B \) and \( L_W \), whose sites are colored black and white respectively as shown in Fig. 3. We require that the arrow always points from a black vertex toward a white vertex to uniquely fix the orientation of each edge. After fully fixing the direction of each edge, the degrees of freedom of the original string can be viewed as the \( \mathbb{Z}_N \) spin degrees of freedom lying at the edge [HW12, SDO12]. In the following, we use the \( N \) orthonormal basis \( |n\rangle \), to label the associated spin degrees of freedom on the edge \( i \), where \( n \in \mathbb{Z}_N \). Obviously, there is \( a_s = 1/N \) since the quantum dimension \( d_i \) of the abelian topological orders is 1. And \( F \) symbols satisfying the \( \mathbb{Z}_N \) branching rule: \( s + e_i = e_i' \mod N \) are 1, the others are 0. Thus the action of \( B_p^s \) on the plaquette is simply shifting the string/spin type \( e_i \) of every edge of the plaquette by \( s \mod N \). This suggests that \( B_p^s \) can be more naturally expressed as a product of spin shifting operators acting on the \( \mathbb{Z}_N \) spin of plaquette’s edges

\[
B_p^s = \prod_{i\in p^+} (X_i^s) \prod_{j\in p^-} (X_j^s) .
\]

Here we introduce the spin shifting operators \( X_i \), which generalize the Pauli operator \( \sigma^x \) to \( N \)-dimensional space. For the edges in opposite orientation, the shifting actions need to be
conjugated. For convenience, we also introduce operators \( Z_i \) to measure the value of spin at the edge \( i \), which generalize the Pauli operator \( \sigma^z \) to \( N \)-dimensional space. They are defined by,

\[
Z_i |n\rangle_i = \omega^n |n\rangle_i \quad \text{and} \quad X_i |n\rangle_i = |n-1\rangle_i, \quad \text{for } \omega = e^{2\pi i/N}.
\]

Instead of anti-commutation relations of Pauli operators, the operators \( X_i \) and \( Z_i \) on the same site satisfy the Weyl algebra’s relations \([Wey50]\),

\[
Z_iX_i = \omega X_i Z_i, \quad Z_i^N = 1, \quad Z_i^N = Z_i^{N-1}, \quad X_i = X_i^{N-1}.
\]

while operators at different sites commute. Also, the operator \( A_v \) can be expressed as a product of \( Z_i \): \( A_v = \frac{1}{N} \sum_{i} Z_i \prod_{\ell \in v} Z_{\ell} \) to make the groundstates satisfy branching rules. Finally, we obtain the \( Z_N \) Levin-Wen model written by spin operators,

\[
H_{\text{bulk}}^{Z_N} = \frac{1}{N} \sum_{v} \sum_{s=0}^{N-1} \prod_{i \in v} Z_i^s - \frac{1}{N} \sum_{p} \sum_{s=0}^{N-1} \prod_{j \in p} (X_j)^s \prod_{j \in p} (X_j)^s
\]

\[
= -\frac{1}{N} \sum_{v} \sum_{s=0}^{N-1} A_v^s - \frac{1}{N} \sum_{p} \sum_{s=0}^{N-1} B_p^s.
\]

It is easy to check that this model is indeed exactly solvable by using the relations between \( Z_i \) and \( X_i \). Since all the terms in the Hamiltonian \( H_{\text{bulk}}^{Z_N} \) mutually commute, the groundstate subspace can be stabilized by simultaneously satisfying: \( A_v^s |\text{GS}\rangle = 1 \) \& \( B_p^s |\text{GS}\rangle = 1 \), \( \forall v, p \) and \( s \). When some \( A_v^s \) or \( B_p^s \)’s eigenvalues violate the above condition, these eigenstates with higher energy at local vertices or plaquettes can be regarded as some particle-like topological excitations. According to the different eigenvalues \( \{e^{\pm \pi s/N}, e^{\pm \pi t/N}\} \) of \( A_v^s \) and \( B_p^s \) carried by these particles, they are usually known as \( Z_N \) charge \( e^\alpha \) and \( Z_N \) flux \( m^\beta \), where \( \alpha \) and \( \beta \) can be chosen from 0, \ldots, \((N - 1)\)’. Obviously, these charges and fluxes naturally correspond to the objects \( O_{\alpha \beta} \) and \( C_{0\beta} \) in UMTC \( 3_{1}(\text{Rep}(Z_N)) \). They are not created by local perturbations but can be generated in pairs by string operators. String operators \( S^a \) and \( S^a \) can be defined by collecting the corresponding spin operators along the string, but note that the positive and negative orientation of the string \( S^a \) correspond to \( X^+ \) and \( X \), respectively.

### 4.2 Boundary Lattice Model

In this subsection, we consider the case that the model is equipped with a boundary. It is well known that the \( Z_N \) topological order can admit many gapped boundaries. The lattice realizations of many different kinds of gapped boundaries have been given in [BSW11][CCW17]. Here, we will try to construct a general boundary Hamiltonian to recover many of the gapped boundaries of \( H_{\text{bulk}}^{Z_N} \) by tuning the parameters. In addition, it is predictable that the phase transitions between these different gapped boundaries naturally correspond to the gappable gapless boundaries. That’s our final goal.

Now let’s begin with considering the model with a boundary \( H_{\text{bulk}}^{Z_N} + H_{\text{bdy}} \) on a cylinder, as shown in Fig. 4. We only deal with the boundary at the bottom; the upper part is the bulk of infinite length. Moreover, the spins of edges in black are artificially and rigidly defined as boundary degrees of freedom. Hence, our boundary Hamiltonian \( H_{\text{bdy}} \) is assumed to only contain the local operators acting on these black edges. Finally, these operators must commute with \( A_v^s \) and \( B_p^s \) in order to preserve the exact solvability of \( H_{\text{bulk}}^{Z_N} \). If the boundary Hamiltonian \( H_{\text{bdy}} = 0 \), the whole system will have a very large degeneracy that depends on the number \( n \) of edges at the boundary. In principle, we want to remove most of the degeneracy from the
boundary by including the fewest local operators into $H_{\text{bdy}}$. The remaining difficulty is how to find and represent all of the local operators that meet the aforementioned criteria.

Usually, any operator that commutes with $\{A_s^v, B_s^p\}$ consists of closed-string operators in the bulk. But in the case with a boundary, open strings $S(a, b)$ with two endpoints $a, b$ outside the bulk are also available. Therefore, all we need to do is push all allowed strings onto the boundary so that we can get all local operators on the boundary that commute with $\{A_s^v, B_s^p\}$, as shown in Fig. 5. In the current model, only pushing open strings $S(a, b)$ can yield non-trivial $S_{\text{bdy}}(a, b)$. As the boundary is merely 1d, closed-string operators on the boundary must be identity, except for two non-contractible loop operators $L^e$ and $L^m$, as shown in Fig. 8. However, the boundary operators generated by $L^e$ and $L^m$ are not local. There are two classes of string operators $S^e$ and $S^m$ in the $Z_N$ TO. When we push them onto the boundary, there are only two types of allowed configurations to ensure that the endpoints are outside the bulk and only act on the boundary spin, as shown in Fig. 6. All valid configurations can be achieved by the product of a number of the two shortest boundary strings $S^e_{\text{bdy}}(\quad)$ and $S^m_{\text{bdy}}(\quad)$. So far, we have obtained all the local operators on the boundary that satisfy the criteria. And they can be represented by $S^e_{\text{bdy}}(\quad)$ and $S^m_{\text{bdy}}(\quad)$. In other words, any legal $H_{\text{bdy}}$ can be viewed as a combination and product of $S^e_{\text{bdy}}(\quad)$ and $S^m_{\text{bdy}}(\quad)$. A typical example is $H^m_{\text{bdy}} = -\sum_{j \text{odd link}} (Z_j Z_{j+1} + Z_j^1 Z_j^1)$ that realizes the $m$-condensed boundary [BSW11, KK12], as shown in Fig. 7. $H^m_{\text{bdy}}$ can be viewed as the sum of $S^m_{\text{bdy}}(\quad)$ that describes the fluctuation of $m$ along the boundary. Indeed, these terms give rise to the $m$ condensation. As we expect, $N^m$-fold degeneracy from the boundary is lifted by local operators $S^m_{\text{bdy}}(\quad)$ so that only the topological $N$-fold degeneracy from the semi-infinite open string $S^m_{\text{inf}}$ remains. We will discuss this later.

It is more interesting for us to consider the model only with nearest neighbor interactions. Therefore, the general boundary Hamiltonian $H_{\text{bdy}}^{Z_N}$ with periodic conditions along the circle can be written down,

$$H_{\text{bdy}}^{Z_N} = -\sum_{i=1}^{N-1} a_i (S^e_{\text{bdy}}(\quad))^i - \sum_{i=1}^{N-1} b_i (S^m_{\text{bdy}}(\quad))^i$$

$$= -\sum_{i \text{even link}} a_i (X_i^1 X_{i+1})^i - \sum_{j \text{odd link}} b_j (Z_j Z_{j+1})^j, \quad \text{with } X_{2n+1} = X_1,$$

(4.6)
Figure 5: Local boundary operators can be represented by the string operators $S_{\text{bdy}}(a, b)$ labeled by blue string, where $S_{\text{bdy}}(a, b)$ can be obtained by pushing $S(a, b)$ labeled by dashed blue string to the boundary.

Figure 6: Configurations of boundary strings $S^e_{\text{bdy}}$ and $S^m_{\text{bdy}}$ that are in purple and cyan, respectively. Upper part: two types of allowed configurations, marked with green check mark. Bottom part: invalid configurations are marked by red cross, because the end points are inside the bulk or they have action on the internal degrees of freedom.
where the lattice translation symmetry has been applied. Since $a_i$ and $b_i$ are complex number, so the conditions on the couplings $a_i = a_{n-i}$ and $b_i = b^*_{n-i}$ are necessary to make the Hamiltonian hermitian. It is clear that the model captures many of the boundaries of the $Z_1 (\text{Rep}(Z_N))$ constructed before. In addition to the $m$-condensed boundary ($b_1 = b_{n-1} = 1$, others = 0) and the $e$-condensed boundary ($a_1 = a_{n-1} = 1$, others = 0), the gapped boundaries induced by the subgroups of $Z_N$ are also included [BSW11]. In the Landau’s paradigm, they are seen as partially symmetry broken phases. Thus, it can be expected that the model (4.6) will have a rich phase diagram.

Before proceeding further, we make an important observation about the boundary model. Obviously, the model has two different $Z_N$ symmetries generated by $S_{\text{bdy}}^e$ and $S_{\text{bdy}}^m$ operators respectively. The corresponding symmetry operators are written as $\prod_{j=1}^n X^{1}_{2j} X^{2j+1}_{2i}$ and $\prod_{j=1}^n Z^i_{2j} Z^{2j+1}_i$. We call them $Z_N$ and $Z_N^\nu$. Indeed, they are the loop operators $L^e_{\text{bdy}}$ and $L^m_{\text{bdy}}$ (as shown in Fig. 8) respectively, from the viewpoint of bulk. It means that the two $Z_N$ symmetries of the 1+1D boundary theory come from the large gauge transformations of the $Z_N$ gauge theory in the 2+1D bulk. The symmetry sectors of the boundary Hilbert space $\mathcal{H}_{\text{bdy}}$ should match up with the topological sectors (anyons) of the bulk Hilbert space $\mathcal{H}_{\text{bulk}}$. Therefore, the Hilbert space $\mathcal{H}_{\text{bdy}}$ has a decomposition as $\mathcal{H}_{\text{bdy}} = \bigoplus_{\alpha,\beta} \mathcal{H}_{\text{bdy}}^{\alpha,\beta}$, where $\alpha$ and $\beta$ are the charge and flux number. Let’s recall the example $m$-condensed boundary $H^m_{\text{bdy}} = - \sum_{\text{o/odd link}} (Z_j Z_{j+1}^\dagger + Z_j^\dagger Z_{j+1})$ with $Z_N$ symmetry; another $\nu$ symmetry is identity for the periodic case. For this example, there are $N$-fold degeneracy states $|\Psi_\alpha\rangle$ labeled by the eigenvalue $\alpha$ of the $Z_N$ symmetry operator $L^e$. These different symmetry sector $\alpha$ of $Z_N$ can be constructed by inserting a semi-infinite open strings $S_{\text{Inf}}^m$, as shown in Fig. 8. Because $S_{\text{Inf}}^m$ commutes with $H_{\text{bulk}}$ and $H^m_{\text{bdy}}$, it won’t change the energy of the system. Viewed from the boundary, the $N$-fold degeneracy can be kept until a phase transition occurs as long as the $Z_N$ is not broken. In Landau’s paradigm, the degeneracy is attributed to the spontaneous symmetry breaking. On the other hand, the viewpoint from bulk provides a topological interpretation. Since $S_{\text{Inf}}^m$ only can be detected by large loop $L^e$ rather than any local operators in the bulk, the $N$-fold degeneracy is topological robust against any local perturbation in bulk. The two statements can be related when we push $L^e$ onto the boundary. $L^e$ will become the symmetry constraint on the local perturbations of the boundary. That shows a holographic point of view: symmetry originates from the topology in one higher dimension [CJL20, JW20a, JW20b, KKLW20, LTL21, CW22]. If we insert another semi-infinite open string $(S_{\text{Inf}}^e)^\nu$ into the system instead of $S_{\text{Inf}}^m$, the energy of the system will increase because $S_{\text{bdy}} (S^e)^\nu |\Psi_{\text{bdy}}\rangle = \alpha^\nu S^e |\Psi_{\text{bdy}}\rangle$ at the cross point. But if we also replace the $S^m_{\text{bdy}}$ of $H^m_{\text{bdy}}$ at the
Figure 8: Left part: two kinds of non-contractible loop operators $L^e$ and $L^m$ wrapped around the cylinder. Right part: the semi-infinite open string $S_{\text{INF}}^e$ can be detected by loop $L^e$.

cross point with $\omega - \beta S_{\text{INF}}^e \langle \Psi_{\text{bdy}} \rangle$ will be the one with lower energy. It actually assumes a twisted boundary condition at the cross point. We'll clarify this later.

Let’s return to the general case. In the $H_{\alpha,1}^{\text{bdy}}$, the model Eq. (4.6) can be exactly mapped to the $Z_N$ chiral clock model through the Kramers-Wannier transformation:

$$X_i^\dagger X_{i+1} \rightarrow \tilde{X}_{i+1/2}, \quad Z_j Z_{j+1} \rightarrow \tilde{Z}_{j+1/2} \tilde{Z}_{j+1/2},$$

$$Z_N \rightarrow \prod \tilde{X}_{i+1/2}, \quad Z_N^\vee \rightarrow \text{identity},$$

$$\tilde{H}_{\text{bdy}}^Z = -\sum_{i \in \text{even link}} a_s (\tilde{X}_{i+1/2})^s - \sum_{j \in \text{odd link}} b_s (\tilde{Z}_{j+1/2} \tilde{Z}_{j+1/2})^s,$$

where $Z_N^\vee = L_m^{\text{bdy}} = \mathbb{1}$ after transformation. That’s why we have to work inside the $H_{\alpha,1}^{\text{bdy}}$. But if we first remove the identification of $\frac{1}{2}$-site and $(2n + \frac{1}{2})$-site, an extra degree of freedom can help us realize the mapping in every $H_{\alpha,\beta}^{\text{bdy}}$. At this point, we have

$$Z_N^\vee \rightarrow \tilde{Z}_{2n+1/2} \tilde{Z}_{2n+1/2}.$$

In addition to the condition of the symmetry sector: $Z_N |\Psi_{\text{bdy}}\rangle = \prod \tilde{X}_{i+1/2} |\Psi_{\text{bdy}}\rangle = \alpha |\Psi_{\text{bdy}}\rangle$, we also have a twisted boundary condition: $Z_N^\vee |\Psi_{\text{bdy}}\rangle = \tilde{Z}_{2n+1/2} \tilde{Z}_{2n+1/2} |\Psi_{\text{bdy}}\rangle = \beta |\Psi_{\text{bdy}}\rangle$, for the subspace $H_{\alpha,\beta}^{\text{bdy}}$. Therefore, the mapping is exact only in every subspace $H_{\alpha,\beta}^{\text{bdy}}$. The $(\alpha, \beta)$ of $H_{\alpha,\beta}^{\text{bdy}}$ corresponds to the symmetry sector $\alpha$ and twisted boundary condition $\beta$ of the mapped model $\tilde{H}_{\text{bdy}}^Z$. Working in the $H_{\alpha,\beta}^{\text{bdy}}$ is more convenient. In addition to that, it also has real physical significance. Assuming that the cylinder has a finite length, we imagine a process in which a pair of $e$ are created in the bulk and one of them is moved to the boundary. The boundary state must be in the corresponding sector $H_{\alpha,\beta}^{\text{bdy}}$. We can also directly change the sign of $A_s^v$ and $B_s^p$ in the $H_{\text{bulk}}^Z$ to make the system favor any topological sector $(\alpha, \beta)$ that we want. Therefore, the symmetry of the boundary is not independent but is determined by the topological defect in the bulk [CJK20, LTL21]. The model is essentially a $(\alpha, \beta)$-constrained $Z_N$ chiral clock model as depicted in Fig. 9 which is
distinct from the pure 1+1D $\mathbb{Z}_N$ chiral clock model. It is a sign that such a theory describing the boundary of 2+1D $\mathbb{Z}_N$ topological order has a non-invertible gravitational anomaly [JW20a].

By tuning the parameters $a_s$ and $b_s$, the boundary model may undergo a phase transition described by non-chiral conformal field theory. However, $H_{\text{bdy}}$ actually has many different critical points for large $N$, which have many different additional symmetries. We need to further specify the system’s FFA $W$, i.e., local quantum symmetry. It is very difficult to find all of them. In this paper, we consider a special case, a $\mathbb{Z}_N$ symmetric integrable parameter point: $a_s = b_s = \frac{1}{\sin \frac{\pi}{N}}$, which is the self-dual Fateev-Zamolodchikov spin chain described by $\mathbb{Z}_N$ parafermion CFT [FZ82, ZF85, JMO86]. For any $N$, it always can be a critical point between $\mathbb{Z}_N$ spontaneous symmetry breaking phase (e-condensed) and $\mathbb{Z}_N$ symmetric phase (m-condensed) of the boundary. In order to handle this model analytically, we will only work with the Fateev-Zamolodchikov critical point in the following. Such a known result helps us determine the local quantum symmetry of the boundary as $(V_{\text{PF}}, \otimes_c V_{\text{PF}})$. We expect that the observables of this critical point can be captured by the gapless boundary theory of TO.

4.3 Low-Energy Effective Field Theory and Partition Functions

In this section, we’ll try to obtain the low-energy effective field theory of the self-dual Fateev-Zamolodchikov critical point at the boundary. As mentioned before, the Fateev-Zamolodchikov spin chain is an integrable model which can be extended to the $\mathbb{Z}_N$ parafermion CFT by taking the thermodynamic limit [FZ82, ZF85, JMO86]. It will be more convenient to calculate the partition function within the framework of CFT. And the parafermion CFT can be constructed through the $\hat{\text{SU}}(2)_N/\hat{\text{U}}(1)_{2N}$ coset procedure [GKO86, FMS12], hence it is also known as the $\hat{\text{SU}}(2)_N/\hat{\text{U}}(1)_{2N}$ coset CFT with central charge,

$$c(\hat{\text{SU}}(2)_N/\hat{\text{U}}(1)_{2N}) = c(\hat{\text{SU}}(2)_N) - c(\hat{\text{U}}(1)) = \frac{3N}{N+2} - 1. \quad (4.10)$$

The coset formalism will be useful for us to construct the partition functions on the subspace $\mathcal{Z}^{a,b}_{\text{bdy}}$. For a simple introduction to the coset construction, please see the Appendix. Here
we directly list the results of the coset construction. The representation \( \ell \) of \( \tilde{\text{SU}}(2)_N \) can be decomposed into a direct sum of representations \( m \) of \( \text{U}(1)_{N} \). In physical, the \( \text{SU}(2)_N \) WZW model can be viewed as composed of two building blocks: a \( \tilde{\text{SU}}(2)_N/\text{U}(1) \) piece associated with the parafermions and a \( \text{U}(1) \) factor associated with a free boson.

\[
(\ell) \mapsto \bigoplus_{m=-\ell}^{\ell} (m)_\ell \quad \text{(branching condition: } \ell - m = 0 \text{ mod } 2),
\]

\[
\mathcal{H}_{\text{SU}(2)_N\text{wzw}}^{\ell} = \mathcal{H}_{\text{para}}^{\ell,m} \times \mathcal{H}_{\text{boson}}^{m},
\]

\[0 \leq \ell \leq N, \quad -N + 1 \leq m \leq N.
\]

where \( \mathcal{H}_{\text{SU}(2)_N\text{wzw}}^{\ell} \) denotes the subspace of the Hilbert space of the \( \tilde{\text{SU}}(2)_N \) current algebra which contains the states in the highest weight representation with isospin \( \ell \). Also, \( \mathcal{H}_{\text{para}}^{\ell,m} \) denotes the parafermionic states obtained from the highest weight parafermionic state which have \( Z_N \) charge \( m \) mod \( N \). \( \mathcal{H}_{\text{boson}}^{m} \) is a subspace of the bosonic theory containing the states of momenta \( m \).

In the following, we use \( \ell, m \) and \( \{\ell; m\} \) to label the \( \tilde{\text{SU}}(2)_N \) representations, the \( \tilde{\text{U}}(1)_{2N} \) representations and the coset field (parafermion primary field), respectively. To avoid confusion with the \( \tilde{\text{SU}}(2)_N \) characters \( \chi_\ell^{(N)}(q) \), the \( \tilde{\text{U}}(1)_{2N} \) characters will be indicated by \( K_{\ell}^{(N)}(q) \), where \( q = e^{2\pi i} \) and \( \tau \) is modular parameter.

Following the procedure of coset construction \( \check{g}_k/\check{p}_{\pm,k} \) in Appendix.D, the character decomposition is

\[
\chi_\ell^{(N)}(q) = \sum_{m=-N+1}^{N} \chi_{\ell; m}(q) K_m^{(N)}(q),
\]

with character identity,

\[
\chi_{\ell; m}(q) = \chi_{|N-\ell; m+N|}(q) = \chi_{|N-\ell; m-N|}(q).
\]

The fractional part of conformal weight of \( \{\ell; m\} \) is,

\[
h_\ell \text{ mod } 1 = h_\ell - h_m = \frac{\ell(\ell + 2)}{4(N + 2)} - \frac{m^2}{4N}.
\]

Modular transformation matrices of coset characters \( \chi_{\ell; m} \),

\[
\begin{align*}
\mathcal{S}_{\text{SU}(2)_N/\tilde{\text{U}}(1)_{2N}}^{\ell} & = \mathcal{S}_{\ell; \ell'}^{\ell} \mathcal{S}_{m,m'}^{\ell}, \\
\mathcal{T}_{\text{SU}(2)_N/\tilde{\text{U}}(1)_{2N}}^{\ell} & = \mathcal{T}_{\ell; \ell'}^{\ell} \mathcal{T}_{m,m'}^{\ell}, \\
\mathcal{S}_{\ell; \ell'}^{\ell} & = \sqrt{\frac{2}{2N + 1}} \sin \frac{\pi(\ell(\ell + 1)(\ell' + 1))}{N + 2}, \\
\mathcal{S}_{m,m'}^{\ell} & = \frac{1}{\sqrt{2N}} e^{\frac{\pi i m^2}{2N}}, \\
\mathcal{T}_{\ell; \ell'}^{\ell} & = e^{\frac{\pi i m^2}{2N}} \delta_{\ell,\ell'}, \\
\mathcal{T}_{m,m'}^{\ell} & = e^{\frac{\pi i m^2}{2N}} \delta_{m,m'}.
\end{align*}
\]

To obtain the partition functions on the subspace \( \mathcal{H}_{\text{wzw}}^{\ell,b}, \) we need to consider the so-called twisted partition function \( Z_{\ell,b} \) (\( \ell \) and \( t \) are defined modulo \( N \)), which describes the CFT on the torus with \( r \)-twisted spatial boundary condition and \( t \)-twisted temporal boundary condition. Let’s...
begin with a modular invariant untwisted coset partition function. For simplicity, we take the
fully diagonal modular invariant solution as an example,
\[
Z_{0,0} = \frac{1}{2} \sum_{\{\ell, m\}} \chi(\ell; m) \chi(\pi, m) = \frac{1}{2} \sum_{\{\ell, m\}} \chi(\ell; m) \overline{\chi(\ell; m)}. \tag{4.16}
\]

The twisted partition functions \(Z_{0,t}\) is easier to calculate than the general case. The \(t\) twist in
temporal direction can be interpreted as a \(Z_N\) symmetry line defect wrapped around the spatial
circle of the torus. Therefore, we just need to insert the \(Z_N\) symmetry operator \(Q^t\) into the trace
over the Hilbert space \(\mathcal{H}_{\text{para}}^{t,m}\) to implement this symmetry defect line,
\[
Q^t \quad Z_{0,t} = \text{Tr}_{\mathcal{H}_{\text{para}}} Q^t q_{J_0 - \pi}^t \mathcal{H}_{\text{para}}^t = \frac{1}{2} \sum_{\{\ell, m\}} e^{2\pi i t m} \chi(\ell; m) \overline{\chi(\ell; m)}. \tag{4.17}
\]

As the \(Z_N\) symmetry operator, \(Q^t\) measures the \(Z_N\) charge of the field \(\{\ell, m\}\) with \(e^{2\pi i t m}\) as the result. Since the \(S\) transformation can exchange the temporal and spatial directions of the torus,
the twisted partition functions \(Z_{0,t}\) will transform to \(Z_{-1,0}\),
\[
Z_{0,t} \quad \xrightarrow{\text{S}} \quad Z_{-1,0}.
\]

By the modular transformation, we can get all the twisted partition functions \(Z_{r,t}\) \([\text{GQ87}]\).
\[
Z_{r,t} = \frac{1}{2} e^{-2\pi i t N} \sum_{\{\ell, m\}} e^{2\pi i t m} \chi(\ell; m) \overline{\chi(\ell; m-2t)} \tag{4.18}
\]

The gapless boundary of \(Z_N\) topological order is described by a set of \(Z_N\) parafermion
partition functions on the \(H_{\text{bdy}}^{m,\beta}\). In physical, it comes from the fact that the boundary partition
function will respond to the different topological defects in the bulk. For example, pushing the \(e^a\)
particle of \(Z_N\) topological order to the boundary can change the \(Z_N\) symmetric boundary
Hilbert space from \(0\) symmetry sector to \(a\) sector (here, symmetry sector \(a\) corresponds to
the charge \(m\) of \(\tilde{U}(1)_{2N}\) CFT in the \(\{\ell, m\}\) notation). This is reflected in the change of
the partition functions of the boundary theory. Therefore in contrast to the pure 1+1D parafermion
CFT with trivial bulk that is described by only one modular invariant partition function, the
boundary theory with a topological order as bulk requires a set of partition functions. Obviously,
because \(\mathcal{H}_{\text{bdy}} = \bigoplus_{a,\beta} \mathcal{H}_{\text{bdy}}^{a,\beta}\), these boundary partition functions can be labeled by \(Z_N\) topological
excitations: \(Z(e^a m^\beta)\).

We know that \(m^\beta\) determines the spatial direction with the \(\beta\) twisted condition and \(e^a\) determines
the symmetry sector (\(Z_N\) charge) \(a\) mod \(N\). Thus, the partition function with topological
defect \(e^a\) simply needs to collect all characters \(\chi(\ell; m)\) with the charge \(a\),
\[
Z(e^a) = \sum_{\{\ell, m\}} \chi(\ell; m) \overline{\chi(\ell; m)}. \tag{4.19}
\]

For the spatial \(\beta\) twisted case, the \(Z_N\) symmetry acts on the coset field \(\{\ell, m, \ell, m - 2\beta\}\) will
get the charge \((m - \beta)\). Thus,
\[
Z(e^a m^\beta) = \sum_{\{\ell, m\}} \chi(\ell; m) \overline{\chi(\ell; m-2\beta)}. \tag{4.20}
\]
Obviously, the partition functions in the anyon basis can also be obtained through a basis transformation from the twisted partition functions $Z^\text{para}_{r,t}$. To make this clearer, we reorganized the partition functions based on the $\bar{Z}_N$ charge basis $\alpha, \beta$,

$$Z_{0,t} = \sum_{\alpha \in \bar{Z}_N} e^{2\pi i\alpha/N} Z(e^\alpha),$$

$$Z_{r,t} = \sum_{\alpha, \beta \in \bar{Z}_N} e^{2\pi i\alpha/N} Z(e^{\alpha} m^{\beta}) \delta_{\beta, r}.$$

Since Eq. (4.21) above is the discrete Fourier series expansion, the $r, t$ twisted basis and the anyon $e^\alpha, m^\beta$ basis are related by a discrete Fourier transformation,

$$Z(e^\alpha) = 1/N \sum_{s \in \bar{Z}_N} e^{-2\pi i s/N} Z_{0,s},$$

$$Z(e^{\alpha} m^{\beta}) = 1/N \sum_{r, s \in \bar{Z}_N} e^{-2\pi i s/N} Z_{r,s} \delta_{\beta, r}.$$

So far, we have recovered the partition functions $Z(e^{\alpha} m^{\beta})$ directly from the lattice model of critical point. These partition functions Eq. (4.20) all agree with the result Eq. (3.17) from the mathematical theory, as expected. Therefore, the boundary model Eq. (4.6) with Fateev-Zamolodchikov parameters indeed can be described by a gappable gapless boundary of $\bar{Z}_N$ TO.

In the following, we discuss the behavior of these partition functions $Z(e^{\alpha} m^{\beta})$ under modular transformations. By

$$\chi(t; m) = \mathcal{S}^{\mathbb{U}(2)/\mathbb{U}(1)}_{(m)} \chi(t; m')$$

$$\chi(t; m) = \mathcal{T}^{\mathbb{U}(2)/\mathbb{U}(1)}_{(m)} \chi(t; m'),$$

it is not difficult to verify that $Z(e^{\alpha} m^{\beta})$ satisfy the following equations under the modular transformation,

$$Z(e^{\alpha} m^{\beta}) = S^{Z_N}_{[\alpha; \beta]} Z(e^{\alpha'} m^{\beta'})$$

$$Z(e^{\alpha} m^{\beta}) = T^{Z_N}_{[\alpha; \beta]} Z(e^{\alpha'} m^{\beta'})$$

$$S^{Z_N}_{[\alpha; \beta]} = \frac{1}{N} e^{\pi i (\alpha' + \beta)}$$

$$T^{Z_N}_{[\alpha; \beta]} = e^{-\pi i (\beta)} \delta_{\alpha, \alpha'} \delta_{\beta, \beta'}$$

where $S^{Z_N}$ and $T^{Z_N}$ are the modular $S$ and $T$ matrices of the $\bar{Z}_N$ topological order. In contrast to the modular invariant partition function of the pure 1+1D parafermion CFT, the partition functions $Z(e^{\alpha} m^{\beta})$ is modular covariant.

To confirm the validity of our result, here we give a few examples of specific $N$.

### 4.3.1 $N = 2$, Ising model

For $N = 2$ case, it is the well-known Ising model. Thus $0 \leq \ell \leq 2$, $-1 \leq m \leq 2$ and $0 \leq \alpha, \beta \leq 1$. There are six distinct parafermion coset fields,

$$\{0; 0\} = \{2; 2\} \quad h = 0 \mod 1$$

$$\{0; 2\} = \{2; 0\} \quad h = \frac{1}{2} \mod 1$$

$$\{1; -1\} = \{1; -1\} \quad h = \frac{1}{16} \mod 1.$$
We identify the $Z_2$ parafermion coset characters $\chi_{[\ell; m]}$ with Virasoro characters of Ising CFT: $\{\chi_{[\ell; m]}^{1s}, \chi_{[\ell; m]}^{1s}, \chi_{[\ell; m]}^{2s}\}$,

$$\begin{align*}
\chi_{[0; 0]} &= \chi_{[1]}^{1s} \\
\chi_{[0; 2]} &= \chi_{[1]}^{1s} \\
\chi_{[1; -1]} &= \chi_{[0]}^{1s}.
\end{align*}$$

(4.26)

And using Eq. (4.20), we can obtain the partition functions in the anyon basis $(e^m m^\beta)$,

\[
\begin{align*}
Z(\emptyset) &= \chi_{[0; 0]} \overline{\chi}_{[0; 0]} + \chi_{[0; 2]} \overline{\chi}_{[0; 2]} = |\chi_{[1]}^{1s}|^2 + |\chi_{[1]}^{1s}|^2 \\
Z(e) &= \chi_{[1; -1]} \overline{\chi}_{[1; -1]} = |\chi_{[0]}^{1s}|^2 \\
Z(m) &= \chi_{[1; -1]} \overline{\chi}_{[0; 2]} = |\chi_{[0]}^{1s}|^2 \\
Z(em) &= \chi_{[0; 0]} \overline{\chi}_{[2; 0]} + \chi_{[0; 2]} \overline{\chi}_{[0; 0]} = \chi_{[1]}^{1s} \chi_{[1]}^{1s} + \chi_{[0]}^{1s} \chi_{[1]}^{1s}. 
\end{align*}
\]

(4.27)

This recovers the result in [CJ]K+20).

4.3.2 $N = 3$, Three potts model

For $N = 3$ case, $Z_3$ parafermion CFT can be realized by the so-called three Potts model. Thus $0 \leq \ell \leq 3, -2 \leq m \leq 3$ and $0 \leq \alpha, \beta \leq 1$. There are six distinct parafermion coset fields,

\[
\begin{align*}
|0; 0| &= \{3; 3\} \quad h = 0 \mod 1 \\
|0; -2| &= \{3; 1\} \quad h = \frac{2}{3} \mod 1 \\
|0; 2| &= \{3; -1\} \quad h = \frac{1}{3} \mod 1 \\
|1; 3| &= \{2; 0\} \quad h = \frac{1}{3} \mod 1 \\
|1; 1| &= \{2; -2\} \quad h = \frac{2}{3} \mod 1 \\
|1; -1| &= \{2; 2\} \quad h = \frac{1}{3} \mod 1.
\end{align*}
\]

(4.28)

We identify the $Z_3$ parafermion coset characters $\chi_{[\ell; m]}$ with Virasoro characters of the minimal model: $M(6, 5) \{\chi_{[1]}^{(6,5)}, \chi_{[\ell]}^{(6,5)}, \chi_{[\ell]}^{(6,5)}, \chi_{X}^{(6,5)}, \chi_{Y}^{(6,5)}, \chi_{Z}^{(6,5)}\}$,

\[
\begin{align*}
\chi_{[0; 0]} &= \chi_{[1]}^{(6,5)} + \chi_{Y}^{(6,5)} \\
\chi_{[1; 3]} &= \chi_{Z}^{(6,5)} + \chi_{X}^{(6,5)} \\
\chi_{[0; -2]} &= \chi_{[0; 2]} = \chi_{Z}^{(6,5)} \\
\chi_{[1; -1]} &= \chi_{[1; 1]} = \chi_{Y}^{(6,5)}.
\end{align*}
\]

(4.29)

And using Eq. (4.20), we can obtain the partition functions in the anyon basis $(e^m m^\beta)$,

\[
\begin{align*}
Z(\emptyset) &= \chi_{[0; 0]} \overline{\chi}_{[0; 0]} + \chi_{[0; 3]} \overline{\chi}_{[1; 3]} = |\chi_{[1]}^{(6,5)}|^2 + |\chi_{Y}^{(6,5)}|^2 + |\chi_{Z}^{(6,5)}|^2 \\
Z(e) &= \chi_{[0; -2]} \overline{\chi}_{[0; 2]} + \chi_{[1; 1]} \overline{\chi}_{[1; 1]} = |\chi_{Z}^{(6,5)}|^2 + |\chi_{X}^{(6,5)}|^2 \\
Z(m) &= \chi_{[0; -2]} \overline{\chi}_{[0; 2]} + \chi_{[1; 1]} \overline{\chi}_{[0; 2]} = |\chi_{Z}^{(6,5)}|^2 + |\chi_{Y}^{(6,5)}|^2 \\
Z(em) &= \chi_{[0; 0]} \overline{\chi}_{[2; 0]} + \chi_{[0; 3]} \overline{\chi}_{[3; 0]} = \chi_{Z}^{(6,5)} \overline{\chi}_{Z}^{(6,5)} + \chi_{Y}^{(6,5)} \overline{\chi}_{Y}^{(6,5)} + \chi_{X}^{(6,5)} \overline{\chi}_{X}^{(6,5)} + \chi_{Z}^{(6,5)} \overline{\chi}_{X}^{(6,5)} + \chi_{Y}^{(6,5)} \overline{\chi}_{Y}^{(6,5)} + \chi_{X}^{(6,5)} \overline{\chi}_{X}^{(6,5)} \\
Z(e^2 m) &= \chi_{[0; 0]} \overline{\chi}_{[0; 0]} + \chi_{[0; 3]} \overline{\chi}_{[1; 3]} = \chi_{Z}^{(6,5)} \overline{\chi}_{Z}^{(6,5)} + \chi_{Y}^{(6,5)} \overline{\chi}_{Y}^{(6,5)} + \chi_{X}^{(6,5)} \overline{\chi}_{X}^{(6,5)} + \chi_{Y}^{(6,5)} \overline{\chi}_{Z}^{(6,5)} + \chi_{X}^{(6,5)} \overline{\chi}_{Y}^{(6,5)} + \chi_{Z}^{(6,5)} \overline{\chi}_{X}^{(6,5)} \\
Z(e^2 m^2) &= \chi_{[0; 2]} \overline{\chi}_{[0; 2]} + \chi_{[1; 1]} \overline{\chi}_{[1; 1]} = |\chi_{Z}^{(6,5)}|^2 + |\chi_{Y}^{(6,5)}|^2 + |\chi_{X}^{(6,5)}|^2. 
\end{align*}
\]

(4.30)
4.3.3 $N = 4$, $c = 1$ $Z_2$ orbifold CFT with radius $R_{\text{orbifold}} = \sqrt{3}/2$

$Z_4$ parafermion CFT is the $Z_2$ orbifold CFT with radius $R_{\text{orbifold}} = \sqrt{3}/2$. For $N = 4$ case, $0 \leq \ell \leq 4$, $-3 \leq m \leq 4$ and $0 \leq \alpha, \beta \leq 3$. There are ten distinct parafermion coset fields,

\[
\begin{align*}
[0; 0] &= \{4; 4\} \quad h = 0 \mod 1 \\
[0; -2] &= \{4; 2\} \quad h = \frac{3}{4} \mod 1 \\
[0; 2] &= \{4; -2\} \quad h = \frac{1}{4} \mod 1 \\
[0; 4] &= \{4; 0\} \quad h = 1 \mod 1 \\
[1; -3] &= \{3; 1\} \quad h = \frac{3}{4} \mod 1 \\
[1; -1] &= \{3; 3\} \quad h = \frac{1}{2} \mod 1 \\
[1; 1] &= \{3; -3\} \quad h = \frac{3}{4} \mod 1 \\
[1; 3] &= \{3; -1\} \quad h = \frac{1}{2} \mod 1 \\
[2; -2] &= \{2; 2\} \quad h = \frac{3}{4} \mod 1 \\
[2; 0] &= \{2; 4\} \quad h = \frac{1}{2} \mod 1.
\end{align*}
\] (4.31)

And using Eq. (4.20), we can obtain the partition functions in the anyon basis $(e^a m^b)$,

\[
\begin{align*}
Z(1) &= \chi(0; 0) \overline{\chi}(0; 0) + \chi(0; 4) \overline{\chi}(0; 4) + \chi(2; 0) \overline{\chi}(2; 0) \\
Z(\text{e}) &= \chi(1; -3) \overline{\chi}(1; -3) + \chi(1; 1) \overline{\chi}(1; 1) \\
Z(\text{e}^2) &= \chi(0; -2) \overline{\chi}(0; -2) + \chi(0; 2) \overline{\chi}(0; 2) + \chi(2; -2) \overline{\chi}(2; -2) \\
Z(\text{e}^3) &= \chi(1; -1) \overline{\chi}(1; -1) + \chi(1; 3) \overline{\chi}(1; 3) \\
Z(\text{m}) &= \chi(1; -3) \overline{\chi}(1; 3) + \chi(1; 1) \overline{\chi}(1; 1) \\
Z(\text{m}^2) &= \chi(0; 2) \overline{\chi}(0; 2) + \chi(0; 0) \overline{\chi}(0; 0) + \chi(2; 2) \overline{\chi}(2; 2) \\
Z(\text{m}^3) &= \chi(1; -1) \overline{\chi}(1; 1) + \chi(1; 3) \overline{\chi}(1; 3) \\
Z(\text{em}) &= \chi(1; -3) \overline{\chi}(1; 3) + \chi(1; 1) \overline{\chi}(1; 1) \\
Z(\text{em}^2) &= \chi(0; 2) \overline{\chi}(0; 2) + \chi(0; 0) \overline{\chi}(0; 0) + \chi(2; 0) \overline{\chi}(2; 0) \\
Z(\text{em}^3) &= \chi(1; -1) \overline{\chi}(1; 1) + \chi(1; 3) \overline{\chi}(1; 3) \\
Z(\text{em}^3) &= \chi(0; -2) \overline{\chi}(0; 0) + \chi(0; 2) \overline{\chi}(0; 4) + \chi(2; -2) \overline{\chi}(2; 0).
\end{align*}
\] (4.32)

5 Conclusion and Discussions

In summary, we study a pure boundary phase transition of two specific gapped boundaries, $\text{Rep}(Z_N)$ and $\text{Vec}_{Z_N}$ of the $Z_N$ TO in this paper. The critical point of this boundary phase transition corresponds to a gappable non-chiral gapless boundary, which is mathematically described by the pair

\[
(V_{\text{PF}_N} \otimes_C \overline{V_{\text{PF}_N}}; 3_1(0\text{PF}_N)3_1(\text{PF}_N))\beta.
\] (5.1)
The $Z_N$ parafermion FFA $V_{PF_N} \otimes \overline{V_{PF_N}}$ encodes the local quantum symmetry and the enriched fusion category $\mathcal{Z}_1(\mathcal{P}F_N)_{B}$ encodes the information of topological defects. All the ingredients of the enriched fusion category $\mathcal{Z}_1(\mathcal{P}F_N)_{B}$, i.e., $M_{x,y}$ Eq. (3.10) match up with the low-energy effective theory Eq. (4.20) based on the lattice model construction Eq. (4.6). As an example, this work shows that the mathematical theory \[KZ20b, KZ21\] of gapless edges of 2d topological orders is effective in studying general phase transitions. However, for a general $N$, there are more than two kinds of gapped boundaries of $Z_N$ topological order, unlike the $Z_2$ case \[BK98, KK12, BSW11\].

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**A Derivation of category $PF_N$**

By introducing the parafermion VOA, we give a detailed description of $PF_N$, the category of modules over the parafermion VOA.

**A.1 Parafermion vertex operator algebras**

A rational CFT can be described by a rational vertex operator algebra (VOA) and its modules \[MS89\]. Let $V$ be a rational VOA, the category of $V$-modules is a MTC \[Hua08a\]. The parafermion CFT \[ZF85\] is described by the so called parafermion VOA \[DL12, DW11\].

The parafermion CFT can be given by the following coset construction \[GKO86, FMS12\]

$$\frac{\hat{SU}(2)_N}{U(1)_{2N}},$$

where $\hat{SU}(2)_N$ is the affine Lie algebra $\hat{sl}(2)$ at level $N$. For readers in physical background, please refer to Appendix \[D\].

It is known that for $\hat{sl}(2)$, its Weyl module $V_{\hat{sl}(2)_N}$ has a VOA structure with central charge $c_{\hat{sl}(2)_N} = \frac{3N}{N+2}$ \[LL04\]. It is worth mentioning that $V_{\hat{sl}(2)_N}$ has a sub-VOA $V_H$ with a different central charge, called the Heisenberg VOA. The central charge of $V_H$ is 1. There is a theorem \[FZ92\] tells us that the commutant $C_{V_{\hat{sl}(2)_N}}(V_H)$ also has a VOA structure with the central charge

$$c_{PF} = \frac{3N}{N+2} - 1 = \frac{2(N-1)}{N+2}.$$

We are close enough to get the parafermion VOA. Technically, after quotient the commutant $C_{V_{\hat{sl}(2)_N}}(V_H)$ by the maximal ideal, we will obtain a new VOA $V_{PF_N}$, called the parafermion VOA. Unfortunately, the category of modules of the Heisenberg VOA is not finite \[LL04\]. However,
there is another lattice VOA \( V_L \) whose module category is finite and \( C_{V_{\frac{\text{ad}}{(2)}}} (V_L) = V_{\text{PF}_N} \) \([\text{ALY19}]\). Therefore, we can also obtain the parafermion VOA by replacing \( V_L \) with \( V_H \).

### A.2 UMTC PF\(_N\)

It was proved that \( V_{\text{PF}_N} \) is a rational VOA \([\text{DR17}]\), thus the category \( \text{Mod}_{V_{\text{PF}_N}} \) is a MTC. Since the affine VOA with integral level and Heisenberg VOA is unitary, and the centralizer of a unitary sub VOA in a unitary VOA is still unitary \([\text{DL14}]\), the parafermion VOA is unitary.

We use \( \text{Rep}_N(\hat{\mathfrak{sl}}(2)) \) to denote the representation category of the affine Lie algebra \( \hat{\mathfrak{sl}}(2) \). The coset construction can be given by the category of local right \( A \)-modules \( \text{Rep}_N(\hat{\mathfrak{sl}}(2)) \bigotimes \text{Mod}_{V_L}^{\text{loc}} \) of the category \( \text{Rep}_N(\hat{\mathfrak{sl}}(2)) \bigotimes \text{Mod}_{V_L} \) for some condensable algebra \( A \) \([\text{FFRS04}]\). Here \( \text{Mod}_{V_L} \) is the category of modules over the lattice VOA \( V_L \), \( \bigotimes \) is the Deligne tensor product whose physical meaning is stacking two topological orders together. Now we want to write the category PF\(_N\) more explicitly. We begin with the category \( \text{Rep}_N(\hat{\mathfrak{sl}}(2)) \). From \([\text{DNO13}]\), the category \( \text{Rep}_N(\hat{\mathfrak{sl}}(2)) \) consists of

- simple objects \( U_{\ell}, 0 \leq \ell \leq N \),
- the fusion rule:
  \[
  U_{\ell} \otimes U_{\ell'} = \bigoplus_{r = \max(\ell + \ell' - N, 0)} U_{\ell + \ell' - 2r},
  \]
- the double braiding of \( U_{\ell} \) and \( U_{\ell'} \) is:
  \[
  c_{\ell, \ell'} \circ c_{\ell', \ell} = \bigoplus_{s = \max(\ell + \ell' - N, 0)} e^{2\pi i (h_{\ell, \ell' - 2s - h_{\ell, \ell'}})},
  \]
  where \( h_{\ell} = \frac{(\ell + 2)}{4(N + 2)} \).
- the twist:
  \[
  \theta_{\ell} = e^{2\pi i \frac{\ell(\ell + 2)}{4(N + 2)}},
  \]

The FP dimensions of simple objects are

\[
\text{FPdim}(U_{\ell}) = \frac{q^{\ell + 1} - q^{-\ell - 1}}{q - q^{-1}},
\]

where \( q = e^{\pi i N} \).

The category \( \text{Mod}_{V_L} \) consists of

- simple objects \( W_m, 0 \leq i < 2N \);
- the fusion rule is given by:
  \[
  W_m \otimes W_{m'} \simeq W_{m + m'};
  \]
- the double braiding of simple objects \( W_m \) and \( W_{m'} \) is:
  \[
  c_{m', m} \circ c_{m, m'} = e^{2\pi i m m'};
  \]

\[31\]
• the twist is:
\[ \theta_m = e^{2\pi i (-m^2/2N)}. \]

The condensable algebra \( A \) in \( \text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1} \) is classified by the quintuple \((A_1, A_2, C_1, D_1, \phi)\) by [DNO13], where \( A_1 \) is a condensable algebra in \( \text{Rep}_N(\hat{sl}(2)) \), \( A_2 \) is a condensable algebra in \( \text{Mod}_{V_1} \), \( C_1 \) is the fusion subcategory of \( \text{Rep}_N(\hat{sl}(2)) \), \( D_1 \) is the fusion subcategory of \( \text{Mod}_{V_1} \) and \( \phi \) is a braided equivalence between \( D_1 \) and \( C_1 \). If we find such a quintuple, we can write the condensable algebra explicitly:

\[ A = \otimes^R(1) = \bigoplus_{x \in \text{Irr}(C_1)} x^* \boxtimes x, \]

where \( \otimes^R \) is the right adjoint of the tensor functor \( \otimes : C_1 \boxtimes \overline{C_1} \to C_1 \).

Let \( C_1 \) be the fusion subcategory generated by the simple objects \( U_0 \) and \( U_N \) and let \( D_1 \) be the fusion subcategory generated by the simple objects \( W_0 \) and \( W_N \). Notice that the braiding of \( W_N \) is \( e^{2\pi i} \) and the braiding of \( U_N \) is \( e^{-2\pi i} \), then there is a natural braided equivalence \( \phi : C_1 \to D_1 \) which gives a condensable algebra

\[ A = (U_0 \boxtimes W_0) \oplus (U_N \boxtimes W_N). \]

Now we need to find all local right \( A \)-modules of \( \text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1} \). First we will examine all so called free \( A \)-modules. Consider the adjunction

\[ \text{hom}_{(\text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1})^A} (F(x), y) \simeq \text{hom}_{\text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1}} (x, G(y)), \]

where \( F(x) = x \boxtimes A \) and \( G \) is the forgetful functor. We have

\[ \text{hom}_{(\text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1})^A} ((U_\ell \boxtimes W_m) \boxtimes A, (U_{\ell'} \boxtimes W_{m'}) \boxtimes A) = \text{hom}_{\text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1}} (U_\ell \boxtimes W_m, (U_{\ell'} \boxtimes W_{m'}) \boxtimes A). \]

It is clear that all free modules like \( x \boxtimes A \) are simple, and two free modules \((U_\ell \boxtimes W_m) \boxtimes A \) and \((U_{\ell'} \boxtimes W_{m'}) \boxtimes A \) are equivalent iff \( \ell = \ell' \), \( m = m' \) or \( \ell = N - \ell' \), \( m = N + m' \). So there are \( N(N + 1) \) inequivalent simple objects in \( \text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1} \).

Let \( \mathcal{M}_{\ell,m} = (U_\ell \boxtimes W_m) \boxtimes A \) denote the simple objects in \( \text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1} \). We know that for two free modules \( x \boxtimes y \boxtimes A \) and \( x \boxtimes y \boxtimes A \), their relative tensor product \( (x \boxtimes A) \boxtimes (y \boxtimes A) \) is isomorphic to \( (x \boxtimes y) \boxtimes A \). Then the fusion rule of \( \text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1} \) is actually the fusion rule of \( \text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1} \).

\[ \mathcal{M}_{\ell,m} \boxtimes \mathcal{M}_{\ell',m'} = \bigoplus_{r = \max((\ell' - N,0))}^{\min(\ell',\ell')} \mathcal{M}_{\ell + \ell' - 2r, m + m'}. \]

We can say we have known the fusion rule of \( PF_N \), we just need to figure out all simple objects of \( PF_N \).

The double braiding of \( \text{Rep}_N(\hat{sl}(2)) \boxtimes \text{Mod}_{V_1} \) is given by [DNO13].

\[ c_{\ell,m,\ell',m'} = \bigoplus_{s = \max(\ell + \ell' - N,0)}^{\min(\ell',\ell')} e^{2\pi i(h_{\ell + \ell' - 2s} - h_{\ell} - h_{\ell'})} e^{2\pi i m' s} \text{id}_{\ell + \ell' - 2s, m + m'}, \]

where \( h_1 : = \frac{h_{\ell + \ell' - 2s}}{2N}. \)

We know that for free module \( x \boxtimes A \), it is local iff the double braiding \( c_{A,x} \boxtimes c_{x,A} \) is trivial. Notice that the double braiding of \( U_\ell \boxtimes W_m \) with \( A \) is trivial if \( \ell + m \) is even. Then we find all simple objects \( \mathcal{M}_{\ell,m} \) of \( PF_N \) labeled by \( 0 \leq \ell \leq N, 0 \leq m \leq 2N - 1 \) and \( \ell + m \mod 2 = 0 \). There are \( N(N + 1)/2 \) inequivalent simple objects in \( PF_N \).
B Anyon condensation theory

We give a mathematical description of anyon condensation theory [Kon14, KO02].

Definition B.1. Let $\mathcal{C}$ be a UMTC, an algebra $A$ in $\mathcal{C}$ is an object equipped with two morphisms $m : A \otimes A \to A$ and $i : 1_\mathcal{C} \to A$ satisfying
\[
m \circ (m \otimes \text{id}_A) = m \circ (\text{id}_A \otimes m),
\]
\[
m \circ (i \otimes \text{id}_A) = \text{id}_A = m \circ (\text{id}_A \otimes i).
\]
The algebra is called commutative if $m = m \circ c_{A,A}$.

Definition B.2. An algebra $A$ is called

- separable if $m : A \otimes A \to A$ splits as a $A$-$A$-bimodule homomorphism.
- connected if $\dim \text{hom}_\mathcal{C}(1_\mathcal{C}, A) = 1$;
- condensable if $A$ is commutative connected separable;
- lagrangian if $A$ is condensable and $\dim(A)^2 = \dim(\mathcal{C})$.

Remark B.3. The connectedness means the vacuum is unique.

Definition B.4. Let $A$ be an algebra in $\mathcal{C}$. A right $A$-module in $\mathcal{C}$ is a pair $(M, \mu_M)$, where $M$ is an object in $\mathcal{C}$ and $\mu_M : M \otimes A \to M$ is a morphism in $\mathcal{C}$ such that
\[
\mu_M \circ (\mu_M \otimes \text{id}_A) = \mu_M \circ (\text{id}_M \otimes m),
\]
\[
\text{id}_A = \mu_M \circ (\text{id}_M \otimes i).
\]

Definition B.5. A right $A$-module $M$ in $\mathcal{C}$ is called a local $A$-module if $\mu_M \circ c_{A,M} \circ c_{M,A} = \mu_M$.

Let $A$ be a condensable algebra in a UMTC $\mathcal{C}$, we denote the category of right $A$-modules in $\mathcal{C}$ by $\mathcal{C}_A$ and denote the category of local right $A$-modules in $\mathcal{C}$ by $\mathcal{C}^\text{loc}_A$. Then we have the following theorem [KO02][BEK00].

Theorem B.6. Let $\mathcal{C}$ be a UMTC, $A$ be a condensable algebra in $\mathcal{C}$. Then $\mathcal{C}_A$ is a UFC and $\mathcal{C}^\text{loc}_A$ is a UMTC.

Theorem B.7. Let $\mathcal{C}$ be a UMTC, $A$ be a condensable algebra in $\mathcal{C}$.
\[
\dim(\mathcal{C}_A) = \frac{\dim(\mathcal{C})}{\dim(A)},
\]
\[
\dim(\mathcal{C}^\text{loc}_A) = \frac{\dim(\mathcal{C})}{\dim(A)^2}.
\]

Corollary B.8. Let $\mathcal{C}$ be a UMTC, $A$ be a lagrangian algebra in $\mathcal{C}$. Then $\mathcal{C}^\text{loc}_A \simeq \text{Hilb}$ where $\dim(\text{Hilb}) = 1$.

C Proof of equivalence for odd case

C.1 Pre-metric group and pointed braided fusion categories

We first recall the notion of a pre-metric group [DGNO10].

Definition C.1. A pre-metric group is a pair $(G, q)$ where $G$ is a finite abelian group and $q : G \to k^*$ is a quadratic form. A metric group is a pre-metric group such that $q$ is non-degenerate.
Definition C.2. An orthogonal homomorphism between pre-metric groups \((G, q)\) and \((G', q')\) is a group homomorphism \(f : G \to G'\) such that \(q' \circ f = q\).

Proposition C.3. Pre-metric groups and orthogonal homomorphisms form a category.

Pre-metric groups can be used to describe pointed braided fusion categories.

Definition C.4. A tensor category \(C\) is pointed if every simple object of \(C\) is invertible.

Let \(C\) be a pointed braided fusion category. The isomorphism class of simple objects of \(C\) forms a finite abelian group \(G\). For \(x \in G\), we define \(q_C(x) = c_{x,x}\) where \(c_{x,x}\) is the braiding of \(x\) with itself.

Lemma C.5. The function \(q_C : G \to \mathbb{K}^\times\) is a quadratic form.

Proof. See [EGNO16]. □

This lemma shows that each pointed braided fusion category gives a pre-metric group. Conversely, we have

Lemma C.6. For any pre-metric group \((G, q)\) there exists a unique up to a braided equivalence pointed braided fusion category \(\mathcal{C}(G, q)\) such that the group of isomorphism classes of simple objects is \(G\) and the associated quadratic form is \(q\).

And we can eventually prove that

Theorem C.7 ([JS93]). The category of pointed braided monoidal categories is equivalent to the category of pre-metric groups.

Moreover if the pointed braided fusion category \(\mathcal{C}\) has a twist such that it becomes a pre-modular category, the corresponding pre-metric group \((G, q)\) should also be equipped with a character \(\chi : G \to \mathbb{K}^\times\) to capture the information of twist [DGNO10]. The twist \(\theta\) and character is related by the following formula

\[
\theta(x) = q(x)\chi(x). \tag{C.1}
\]

We will call the triple \((G, q, \chi)\) the twisted pre-metric group and denote the corresponding pointed pre-modular category as \(\mathcal{C}(G, q, \chi)\). There is also a bijection between isomorphism classes of twisted pre-metric group and pointed pre-modular categories up to pre-modular equivalence.

When the pointed pre-modular category is non-degenerate, that is, a pointed modular tensor category, the corresponding twisted pre-metric group is a twisted metric group. We can prove that if there is an isomorphism between two twisted metric groups, then the corresponding pointed modular tensor categories are equivalent.

Example C.8. It is clear \(\mathcal{Z}_1(\text{Rep}(\mathbb{Z}_N))\) is pointed, i.e. \(O_{\alpha,\beta}\) has inverse \(O_{N-\alpha,N-\beta}\), and all the simple objects and fusion rules of \(\mathcal{Z}_1(\text{Rep}(\mathbb{Z}_N))\) is described by the abelian group \(\mathbb{Z}_N \times \mathbb{Z}_N\). For a pointed fusion category with more structure added, we can also add corresponding structures to the group. As the consequence, the pointed modular tensor category \(\mathcal{Z}_1(\text{Rep}(\mathbb{Z}_N))\) is described by a twisted metric group \((\mathbb{Z}_N \times \mathbb{Z}_N, q, \tau)\) where \(q(\alpha, \beta) = e^{2\pi i \frac{\alpha \beta}{N}}\) is a non-degenerate quadratic form on \(\mathbb{Z}_N \times \mathbb{Z}_N\) and \(\tau(\alpha, \beta) = e^{-\frac{4\pi i}{N}\alpha \beta}\) is a character. The correspondence is listed as follows:

- \(O_{\alpha,\beta} \sim (\alpha, \beta) \in \mathbb{Z}_N \times \mathbb{Z}_N\);
- fusion rule of \(\mathcal{Z}_1(\text{Rep}(\mathbb{Z}_N))\) is the group multiplication of \(\mathbb{Z}_N \times \mathbb{Z}_N\);
- the braiding \(c\) is corresponding to the quadratic form \(q\);
- the twist \(\theta\) is corresponding to the character \(\tau\).
C.2 The main theorem

In this section, we prove that there is an equivalence of UMTCs, \( Z_1(\text{PF}_N)^\text{loc} \cong Z_1(\text{Rep}(\hat{\text{sl}}(2))) \) for \( N = 2k + 1 \).

**Definition C.9.** Let \( \mathcal{C} \) be a braided monoidal category, \( \mathcal{D} \) be its subcategory, the *centralizer* \( Z_2(\mathcal{D}, \mathcal{C}) \) of \( \mathcal{D} \) in \( \mathcal{C} \) consists of all objects of \( \mathcal{C} \) that commutes with each object in \( \mathcal{D} \).

**Definition C.10.** Let \( \mathcal{C} \) be a braided monoidal category, then the Müger center \( Z_2(\mathcal{C}) \) of \( \mathcal{C} \) is the centralizer \( Z_2(\mathcal{C}, \mathcal{C}) \).

**Definition C.11.** A braided fusion category \( \mathcal{C} \) is called *non-degenerate* if the Müger center is trivial, i.e. \( Z_2(\mathcal{C}) = \text{Vec} \).

Let \( \mathcal{X} \) denote the braided fusion category generated by \( U_0, U_N \in \text{Rep}_N(\hat{\text{sl}}(2)) \).

**Proposition C.12.** For \( N = 2k + 1 \), \( \mathcal{X} \) is non-degenerate.

*Proof.* The only possibly non-trivial double braiding in \( \mathcal{X} \) is \( c_{N, N} \circ c_{N, N} = e^{2\pi i N} \text{id}_0 \). For \( N = 2k + 1 \), this double braiding equals to \(-1\). Then the Müger center of \( \mathcal{X} \) is isomorphic to \( \text{Vec} \). \( \square \)

**Proposition C.13.** The centralizer of \( \mathcal{X} \) in \( \text{Rep}_N(\hat{\text{sl}}(2)) \) is the full subcategory denoted by \( \mathcal{R}_{\text{even}} \) with the simple objects \( U_{2s}, s = 0, \ldots, k \).

*Proof.* The double braiding of \( U_\ell \) and \( U_N \) in \( \text{Rep}_N(\hat{\text{sl}}(2)) \) is
\[
c_{\ell, N} \circ c_{N, \ell} = e^{2\pi i (h_\ell - h_N - h_0)} \text{id}_{N-\ell} = (-1)^\ell \text{id}_{N-\ell}.
\]
When \( \ell = 2s \), the double braiding is trivial. \( \square \)

**Theorem C.14** ([Müg03b]). Let \( \mathcal{C} \) be a modular category and \( \mathcal{D} \) be a modular subcategory of \( \mathcal{C} \). Then there is an equivalence of ribbon categories:

\[
\mathcal{C} \cong \mathcal{D} \boxtimes Z_2(\mathcal{D}, \mathcal{C}).
\]

**Corollary C.15.** There is an equivalence of ribbon categories:

\[
\text{Rep}_{2k+1}(\hat{\text{sl}}(2)) \cong \mathcal{X} \boxtimes \mathcal{R}_{\text{even}}.
\]

Now we have \( \text{PF}_{2k+1} \cong (\mathcal{R}_{\text{even}} \boxtimes \mathcal{X} \boxtimes \text{Mod}_{\text{loc}})_A \). Under the equivalence \( \text{Rep}_{2k+1}(\hat{\text{sl}}(2)) \cong \mathcal{R}_{\text{even}} \boxtimes \mathcal{X} \), we have

\[
U_0 \mapsto U_0 \boxtimes U_0,
\]
\[
U_{2k+1} \mapsto U_0 \boxtimes U_{2k+1}.
\]

Respectively, the condensable algebra \( A = U_0 \boxtimes U_0 \oplus U_{2k+1} \boxtimes U_{2k+1} \) in \( \text{PF}_{2k+1} \) becomes

\[
A' = (U_0 \boxtimes U_0 \boxtimes W_0) \oplus (U_0 \boxtimes U_{2k+1} \boxtimes W_{2k+1})
\]
\[
= U_0 \boxtimes ((U_0 \boxtimes W_0) \oplus (U_{2k+1} \boxtimes W_{2k+1}))
\]
\[
= U_0 \boxtimes A_0,
\]

\[35\]
where \( A_0 := (\mathcal{U}_0 \boxtimes \mathcal{U}_0) \boxtimes (\mathcal{U}_{2k+1} \boxtimes \mathcal{U}_{2k+1}) \) is a condensable algebra in \( \mathcal{X} \boxtimes \text{Mod}_V \). So we can rewrite \( \text{PF}_{2k+1} \) as

\[
\text{PF}_{2k+1} \cong (\mathcal{R}^{\text{even}} \boxtimes (\mathcal{X} \boxtimes \text{Mod}_V))^{\text{loc}}_{A_0} \cong (\mathcal{R}^{\text{even}})^{\text{loc}}_{A_0} \boxtimes (\mathcal{X} \boxtimes \text{Mod}_V)^{\text{loc}}_{A_0}.
\]

It is clear that \( (\mathcal{R}^{\text{even}})^{\text{loc}}_{A_0} \cong \mathcal{R}^{\text{even}} \) because \( \mathcal{U}_0 \) is the trivial condensable algebra (tensor unit) in \( \mathcal{R}^{\text{even}} \).

And the modular tensor category \( (\mathcal{X} \boxtimes \text{Mod}_V)^{\text{loc}}_{A_0} \) is actually a pointed modular tensor category and thus can be describe by a twisted metric group.

**Proposition C.16.** There is an equivalence of modular tensor categories

\[
(\mathcal{X} \boxtimes \text{Mod}_V)^{\text{loc}}_{A_0} \cong C(Z_{2k+1}, p, \xi).
\]

**Proof.** The simple objects in \( \mathcal{X} \boxtimes \text{Mod}_V \) are \( \mathcal{U}_a \boxtimes W_m \) or \( \mathcal{U}_{2k+1} \boxtimes W_m' \). After a short calculation we can check \( (\mathcal{U}_0 \boxtimes W_{2m}) \boxtimes A | a = 0, \ldots, 2k + 1 \) exhausts all simple objects of \( (\mathcal{X} \boxtimes \text{Mod}_V)^{\text{loc}}_{A_0} \). Then the simple objects in \( \text{PF}_{2k+1} \) can also be labeled by \( \mathcal{W}_{\ell, m} \) where \( m = 0, \ldots, 2k + 1 \) and \( \ell = 0, \ldots, k \). Also notice that the fusion rule of \( (\mathcal{X} \boxtimes \text{Mod}_V)^{\text{loc}}_{A_0} \) is actually the fusion rule of \( W_{2a} \in \text{Mod}_V \). Then there is an equivalence

\[
(\mathcal{X} \boxtimes \text{Mod}_V)^{\text{loc}}_{A_0} \cong C(Z_{2k+1}, p, \xi),
\]

where \( p(a) = \exp(\frac{2\pi i}{2k+1}a^2) \) and \( \xi(a) = \exp(-\frac{4\pi i}{2k+1}a^2) \). \( \square \)

**Corollary C.17.** There is an equivalence of modular tensor categories

\[
\text{PF}_{2k+1} \cong \mathcal{R}^{\text{even}} \boxtimes C(Z_{2k+1}, p, \xi).
\]

Denote \( \mathcal{X}_{a, b} := (\mathcal{W}_{0, 2a} \boxtimes \mathcal{W}_{0, 2b}) \boxtimes B, a, b = 0, \ldots, N - 1 \). We can check that each \( \mathcal{X}_{a, b} \) is a simple object with quantum dimension 1 in \( \mathcal{Z}_1(\text{PF}_n)^{\text{loc}}_B \). For \( N = 2k + 1 \), it is clear that \( \mathcal{X}_{a, b} \) is inequivalent to each other. The quantum dimension of \( \mathcal{Z}_1(\text{PF}_{2k+1})^{\text{loc}}_B \) is \( (2k + 1)^2 \), then we have found all simple objects of \( \mathcal{Z}_1(\text{PF}_{2k+1})^{\text{loc}}_B \).

Let \( \mathcal{P} \) denote the subcategory of \( \text{PF}_{2k+1} \) generated by \( \mathcal{W}_{2a} \). It is clear that \( \mathcal{P} \) is the image of the following embedding

\[
\mathcal{R}^{\text{even}} \boxtimes \text{Vec} \rightarrow \mathcal{R}^{\text{even}} \boxtimes C(Z_{2k+1}, p, \xi).
\]

**Proposition C.18.** Then we have

\[
(\text{PF}_{2k+1} \boxtimes \text{PF}_{2k+1})^{\text{loc}}_B \cong C(Z_{2k+1}, p, \xi) \boxtimes C(Z_{2k+1}, p, \xi) \cong C(Z_{2k+1} \times Z_{2k+1}, p \times p, \xi \times \xi).
\]

Recall that in Appendix [C.1], we have shown that \( \mathcal{Z}_1(\text{Rep}(\mathbb{Z}_N)) \) is a pointed modular tensor category \( C(\mathbb{Z}_N \times \mathbb{Z}_N, q, \tau) \). Thus if there exists an isomorphism between two twisted metric groups, the corresponding pointed modular tensor categories would be equivalent to each other.

**Theorem C.19.** There is an equivalence of modular tensor categories

\[
\mathcal{Z}_1(\text{PF}_{2k+1})^{\text{loc}}_B \cong \mathcal{Z}_1(\text{PF}_{2k+1}^{(1)}).
\]

**Proof.** For \( (a, b) \in Z_{2k+1} \times Z_{2k+1} \), we have \( (p \times p)(a, b) = \exp(\frac{2\pi i}{2k+1}(a^2 - b^2)) \). We know that \( \mathcal{Z}_1(\text{PF}_{2k+1}^{(1)}) \cong C(Z_{2k+1} \times Z_{2k+1}, q, \tau) \), where \( q(a, b) = \exp(\frac{2\pi i}{2k+1}ab) \).

It is not hard to find there is a group automorphism \( f : (a, b) \mapsto (a + b, a - b) \) on \( Z_{2k+1} \times Z_{2k+1} \) such that \( f \) is an isomorphism of pre-metric groups \( (Z_{2k+1} \times Z_{2k+1}, p \times p) \) and \( (Z_{2k+1} \times Z_{2k+1}, q) \). It is also clear that \( \xi \times \xi(a, b) = \tau(a + b, a - b) = \tau \circ f(a, b) \).

Then by the equivalence theorem of category of pointed modular tensor categories and category of twisted metric groups, this isomorphism induce an equivalence between \( \mathcal{Z}_1(\text{PF}_{2k+1})^{\text{loc}}_B \) with \( \mathcal{Z}_1(\text{PF}_{2k+1}^{(1)}) \). \( \square \)
D Preliminaries of $\hat{g}_k / \hat{p}_{x,k}$ coset construction

Coset construction of Goddard–Kent–Olive [GKO86] is a well-known method to construct rational conformal field theories with known chiral data, such as conformal weights, fusion rules, braiding and fusing matrices. Given an affine Lie algebra $\hat{g}$ together with a choice $k$ of levels, i.e. a positive integer for each simple ideal of $\hat{g}$, and subalgebra $\hat{p}$ of $\hat{g}$ with level $x_k$, that determined by the embedding index $x_k$, we can show that $L^g_{m} = L^g_{m} - L^p_{m}$ also satisfies the Virasoro algebra and further construct the other chiral data, where $L^g_{m}$ and $L^p_{m}$ are the Virasoro modes of $\hat{g}$ and $\hat{p}$, respectively. In the following, we directly list the basic ingredients of $\hat{g}_k / \hat{p}_{x,k}$ coset construction. For more details, please refer to Coset Virasoro algebra:

\[
\begin{align*}
L^g_{m} &= L^g_{m} - L^p_{m} \\
\left[ L^g_{m}, L^g_{n} \right] &= \left[ L^g_{m}, L^p_{n} \right] \\
&= (m - n)L^g_{m+n} + (c(\hat{g}_k) - c(\hat{p}_{x,k})) \left( \frac{m^3 - m}{12} \right) \delta_{m+n,0}
\end{align*}
\]

coset central charge:
\[
c(\hat{g}_k / \hat{p}_{x,k}) = c(\hat{g}_k) - c(\hat{p}_{x,k}) = \frac{k \dim g}{k + g} \frac{x_k k \dim p}{x_k k + p}
\]

where $g / p$ is the dual Coxeter number of group $g / p$.

Because the various representation $\hat{\lambda}$ of $\hat{g}$ can decompose into a direct sum of representations $\hat{\mu}$ of $\hat{p}$
\[
\hat{\lambda} \mapsto \bigoplus_{\hat{\mu}} b_{\hat{\lambda}, \hat{\mu}},
\]

the corresponding coset characters can be obtained from this decomposition,
\[
\chi_{\hat{p}, \hat{\lambda}} = \sum_{\hat{\mu}} \chi_{\hat{\lambda}, \hat{\mu}} \chi_{\hat{\mu}}.
\]

Conformal weight of coset primary field,
\[
h_{\hat{\lambda}, \hat{\mu}} = h_{\hat{\lambda}} - h_{\hat{\mu}} + n,
\]

where $n$ is the grade in the representation of $g$ which the tip of $\hat{\mu}$ representation lies at.

Modular transformation matrices of coset characters $\chi_{\hat{\lambda}, \hat{\mu}}$,
\[
\begin{align*}
S_{\hat{\lambda}, \hat{\mu}} &= S_{\hat{\lambda}, \hat{\mu}}^{(k \chi_{\hat{\lambda}})} \\
T_{\hat{\lambda}, \hat{\mu}} &= T_{\hat{\lambda}, \hat{\mu}}^{(k \chi_{\hat{\lambda}})}
\end{align*}
\]

Modular invariant partition function in the coset theory is simply to take, for the coset mass matrix $M$, the product
\[
\mathcal{M} = M^{(k \chi_{\hat{\lambda}})} M^{(k \chi_{\hat{\mu}})}
\]

However, the branching conditions will impose constraints on the summations of partition function. Furthermore, field identifications must be considered to divide the partition function by $N$ (If all orbits of field identifications have length $N$). The partition function is then
\[
Z = \frac{1}{N} \sum_{\hat{\lambda}, \hat{\mu}, \hat{\lambda'}, \hat{\mu'}} \chi_{\hat{\lambda}, \hat{\mu}}(\tau) M^{(k \chi_{\hat{\lambda}})} M^{(k \chi_{\hat{\mu}})} \chi_{\hat{\mu'}, \hat{\mu}}(\bar{\tau})
\]

where $P \lambda - \mu = P \lambda' - \mu' = 0 \mod Q$ is the branching condition, $Q$ is the root lattice of $g$. 37
References

[ADJR18] Chunrui Ai, Chongying Dong, Xiangyu Jiao, and Li Ren. The irreducible modules and fusion rules for the parafermion vertex operator algebras. Transactions of the American Mathematical Society, 370(8):5963–5981, 2018.

[ALY19] Tomoyuki Arakawa, Ching Hung Lam, and Hiromichi Yamada. Parafermion vertex operator algebras and w-algebras. Transactions of the American Mathematical Society, 371(6):4277–4301, 2019.

[BEK00] Jens Böckenhauer, David E Evans, and Yasuyuki Kawahigashi. Chiral structure of modular invariants for subfactors. Communications in Mathematical Physics, 210(3):733–784, 2000.

[BK98] Sergey B Bravyi and A Yu Kitaev. Quantum codes on a lattice with boundary. arXiv preprint quant-ph/9811052, 1998.

[BK01] Bojko Bakalov and Alexander A Kirillov. Lectures on tensor categories and modular functors, volume 21. American Mathematical Soc., 2001.

[BPZ84] Alexander A Belavin, Alexander M Polyakov, and Alexander B Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Physics B, 241(2):333–380, 1984.

[BS09] FA Bais and JK Slingerland. Condensate-induced transitions between topologically ordered phases. Physical Review B, 79(4):045316, 2009.

[BSS02] F Alexander Bais, Bernd J Schroers, and Joost K Slingerland. Broken quantum symmetry and confinement phases in planar physics. Physical review letters, 89(18):181601, 2002.

[BSS03] Alexander F Bais, Bernd J Schroers, and Joost K Slingerland. Hopf symmetry breaking and confinement in (2+ 1)-dimensional gauge theory. Journal of High Energy Physics, 2003(05):068, 2003.

[BSW11] Salman Beigi, Peter W Shor, and Daniel Whalen. The quantum double model with boundary: condensations and symmetries. Communications in mathematical physics, 306(3):663–694, 2011.

[BW10] Maissam Barkeshli and Xiao-Gang Wen. Anyon condensation and continuous topological phase transitions in non-abelian fractional quantum hall states. Physical review letters, 105(21):216804, 2010.

[CCW17] Iris Cong, Meng Cheng, and Zhenghan Wang. Hamiltonian and algebraic theories of gapped boundaries in topological phases of matter. Communications in Mathematical Physics, 355(2):645–689, 2017.

[CJK*20] Wei-Qiang Chen, Chao-Ming Jian, Liang Kong, Yi-Zhuang You, and Hao Zheng. Topological phase transition on the edge of two-dimensional z 2 topological order. Physical Review B, 102(4):045139, 2020.

[CW22] Arkya Chatterjee and Xiao-Gang Wen. Holographic theory for the emergence and the symmetry protection of gaplessness and for continuous phase transitions. arXiv preprint arXiv:2205.06244, 2022.

[DGNO10] Vladimir Drinfeld, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. On braided fusion categories i. Selecta Mathematica, 16(1):1–119, 2010.
[DL12] Chongying Dong and James Lepowsky. *Generalized vertex algebras and relative vertex operators*, volume 112. Springer Science & Business Media, 2012.

[DL14] Chongying Dong and Xingjun Lin. Unitary vertex operator algebras. *Journal of algebra*, 397:252–277, 2014.

[DLWY10] Chongying Dong, Ching Hung Lam, Qing Wang, and Hiromichi Yamada. The structure of parafermion vertex operator algebras. *Journal of Algebra*, 323(2):371–381, 2010.

[DMNO13] Alexei Davydov, Michael Müger, Dmitri Nikshych, and Victor Ostrik. The Witt group of non-degenerate braided fusion categories. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2013(677):135–177, 2013.

[DNO13] Alexei Davydov, Dmitri Nikshych, and Victor Ostrik. On the structure of the Witt group of braided fusion categories. *Selecta Mathematica*, 19(1):237–269, 2013.

[DR17] Chongying Dong and Li Ren. Representations of the parafermion vertex operator algebras. *Advances in Mathematics*, 315:88–101, 2017.

[DW11] Chongying Dong and Qing Wang. Parafermion vertex operator algebras. *Frontiers of Mathematics in China*, 6(4):567–579, 2011.

[EGNO16] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205. American Mathematical Soc., 2016.

[Fen12] Paul Fendley. Parafermionic edge zero modes in zn-invariant spin chains. *Journal of Statistical Mechanics: Theory and Experiment*, 2012(11):P11020, 2012.

[FFRS04] Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. Algebras in tensor categories and coset conformal field theories. *Fortschritte der Physik: Progress of Physics*, 52(6-7):672–677, 2004.

[FFRS07] Jürg Fröhlich, Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. Duality and defects in rational conformal field theory. *Nuclear Physics B*, 763(3):354–430, 2007.

[FMS12] Philippe Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Springer Science & Business Media, 2012.

[FRS02] Jürgen Fuchs, Ingo Runkel, and Christoph Schweigert. Tft construction of rcft correlators i: Partition functions. *Nuclear Physics B*, 646(3):353–497, 2002.

[FZ82] VA Fateev and AB Zamolodchikov. Self-dual solutions of the star-triangle relations in zn-models. *Physics Letters A*, 92(1):37–39, 1982.

[FZ92] Igor B Frenkel and Yongchang Zhu. Vertex operator algebras associated to representations of affine and virasoro algebras. *Duke Mathematical Journal*, 66(1):123–168, 1992.

[GKO86] Peter Goddard, Adrian Kent, and David Olive. Unitary representations of the virasoro and super-virasoro algebras. *Communications in Mathematical Physics*, 103(1):105–119, 1986.

[GQ87] Doron Gepner and Zongan Qiu. Modular invariant partition functions for parafermionic field theories. *Nuclear Physics B*, 285:423–453, 1987.

[Gui19a] Bin Gui. Energy bounds condition for intertwining operators of types b, c, and g2 unitary affine vertex operator algebras. *Transactions of the American Mathematical Society*, 372(10):7371–7424, 2019.
[JW20b] Wenjie Ji and Xiao-Gang Wen. Metallic states beyond the tomonaga-luttinger liquid in one dimension. *Physical Review B*, 102(19):195107, 2020.

[JW21] Wenjie Ji and Xiao-Gang Wen. A unified view on symmetry, anomalous symmetry and non-invertible gravitational anomaly. *arXiv preprint arXiv:2106.02069*, 2021.

[Kit03] A Yu Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2–30, 2003.

[Kit06] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2–111, 2006.

[KK12] Alexei Kitaev and Liang Kong. Models for gapped boundaries and domain walls. *Communications in Mathematical Physics*, 313(2):351–373, 2012.

[KLW+20] Liang Kong, Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, and Hao Zheng. Algebraic higher symmetry and categorical symmetry: A holographic and entanglement view of symmetry. *Physical Review Research*, 2(4):043086, 2020.

[KO02] Alexander Kirillov and Viktor Ostrik. On a q-analogue of the mckay correspondence and the ade classification of sl2 conformal field theories. *Advances in Mathematics*, 171(2):183–227, 2002.

[Kog79] John B Kogut. An introduction to lattice gauge theory and spin systems. *Reviews of Modern Physics*, 51(4):659, 1979.

[Kon07] Liang Kong. Full field algebras, operads and tensor categories. *Advances in Mathematics*, 213(1):271–340, 2007.

[Kon14] Liang Kong. Anyon condensation and tensor categories. *Nuclear Physics B*, 886:436–482, 2014.

[KR09] Liang Kong and Ingo Runkel. Cardy algebras and sewing constraints, i. *Communications in Mathematical Physics*, 292(3):871–912, 2009.

[KWZ17] Liang Kong, Xiao-Gang Wen, and Hao Zheng. Boundary-bulk relation in topological orders. *Nuclear Physics B*, 922:62–76, 2017.

[KYZZ21] Liang Kong, Wei Yuan, Zhi-Hao Zhang, and Hao Zheng. Enriched monoidal categories i: centers. *arXiv preprint arXiv:2104.03121*, 2021.

[KZ18] Liang Kong and Hao Zheng. Gapless edges of 2d topological orders and enriched monoidal categories. *Nuclear Physics B*, 927:140–165, 2018.

[KZ20a] Liang Kong and Hao Zheng. Categories of quantum liquids i. *arXiv e-prints*, pages arXiv–2011, 2020.

[KZ20b] Liang Kong and Hao Zheng. A mathematical theory of gapless edges of 2d topological orders. part i. *Journal of High Energy Physics*, 2020(2):1–62, 2020.

[KZ21] Liang Kong and Hao Zheng. A mathematical theory of gapless edges of 2d topological orders. part ii. *Nuclear Physics B*, 966:115384, 2021.

[KZ22] Liang Kong and Zhi-hao Zhang. An invitation to topological orders and category theory. *arXiv preprint arXiv:2205.05565*, 2022.

[LL04] James Lepowsky and Haisheng Li. *Introduction to vertex operator algebras and their representations*, volume 227. Springer Science & Business Media, 2004.
[LTL*21] Tsuf Lichtman, Ryan Thorngren, Netanel H Lindner, Ady Stern, and Erez Berg. Bulk anyons as edge symmetries: Boundary phase diagrams of topologically ordered states. *Physical Review B*, 104(7):075141, 2021.

[LW05] Michael A Levin and Xiao-Gang Wen. String-net condensation: A physical mechanism for topological phases. *Physical Review B*, 71(4):045110, 2005.

[MMT22] Heidar Moradi, Seyed Faroogh Moosavian, and Apoorv Tiwari. Topological holography: Towards a unification of landau and beyond-landau physics. *arXiv preprint arXiv:2207.10712*, 2022.

[MP19] Scott Morrison and David Penneys. Monoidal categories enriched in braided monoidal categories. *International Mathematics Research Notices*, 2019(11):3527–3579, 2019.

[MS89] Gregory Moore and Nathan Seiberg. Classical and quantum conformal field theory. *Communications in Mathematical Physics*, 123(2):177–254, 1989.

[Müg00] Michael Müger. Galois theory for braided tensor categories and the modular closure. *Advances in Mathematics*, 150(2):151–201, 2000.

[Müg03a] Michael Müger. From subfactors to categories and topology ii: The quantum double of tensor categories and subfactors. *Journal of Pure and Applied Algebra*, 180(1-2):159–219, 2003.

[Müg03b] Michael Müger. On the structure of modular categories. *Proceedings of the London Mathematical Society*, 87(2):291–308, 2003.

[Ost03] Victor Ostrik. Module categories, weak hopf algebras and modular invariants. *Transformation groups*, 8(2):177–206, 2003.

[PSV18] Daniel E Parker, Thomas Scaffi di, and Romain Vasseur. Topological luttinger liquids from decorated domain walls. *Physical Review B*, 97(16):165114, 2018.

[RA17] Jonathan Ruhman and Ehud Altman. Topological degeneracy and pairing in a one-dimensional gas of spinless fermions. *Physical Review B*, 96(8):085133, 2017.

[Sch03] Christoph Schweigert. Category theory for conformal boundary conditions. *Vertex operator algebras in mathematics and physics*, 39:25, 2003.

[SDO+12] Marc Daniel Schulz, Sébastien Dusuel, Roman Orus, Julien Vidal, and Kai Phillip Schmidt. Breakdown of a perturbed topological phase. *New Journal of Physics*, 14(2):025005, 2012.

[SM16] Brian Swingle and John McGreevy. Renormalization group constructions of topological quantum liquids and beyond. *Physical Review B*, 93(4):045127, 2016.

[SPV17] Thomas Scaffi di, Daniel E Parker, and Romain Vasseur. Gapless symmetry-protected topological order. *Physical Review X*, 7(4):041048, 2017.

[Ver20] Ruben Verresen. Topology and edge states survive quantum criticality between topological insulators. *arXiv preprint arXiv:2003.05453*, 2020.

[VJP18] Ruben Verresen, Nick G Jones, and Frank Pollmann. Topology and edge modes in quantum critical chains. *Physical review letters*, 120(5):057001, 2018.

[VTJP21] Ruben Verresen, Ryan Thorngren, Nick G Jones, and Frank Pollmann. Gapless topological phases and symmetry-enriched quantum criticality. *Physical Review X*, 11(4):041059, 2021.
[Wen90] Xiao-Gang Wen. Topological orders in rigid states. *International Journal of Modern Physics B*, 4(02):239–271, 1990.

[Wen13] Xiao-Gang Wen. Classifying gauge anomalies through symmetry-protected trivial orders and classifying gravitational anomalies through topological orders. *Physical Review D*, 88(4):045013, 2013.

[Wen17] Xiao-Gang Wen. Colloquium: Zoo of quantum-topological phases of matter. *Reviews of Modern Physics*, 89(4):041004, 2017.

[Wen19] Xiao-Gang Wen. Choreographed entanglement dances: Topological states of quantum matter. *Science*, 363(6429):eaal3099, 2019.

[Wey50] Hermann Weyl. *The theory of groups and quantum mechanics*. Courier Corporation, 1950.

[WJX21] Xiao-Chuan Wu, Wenjie Ji, and Cenke Xu. Categorical symmetries at criticality. *Journal of Statistical Mechanics: Theory and Experiment*, 2021(7):073101, 2021.

[ZF85] AB Zamolodchikov and VA Fateev. Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in zn-symmetric statistical systems. *Sov. Phys. JETP*, 62(2):215–225, 1985.

[ZW15] Bei Zeng and Xiao-Gang Wen. Gapped quantum liquids and topological order, stochastic local transformations and emergence of unitarity. *Physical Review B*, 91(12):125121, 2015.