QUANTIZATION OF COMPACT RIEemannian SYMMETRIC SPACES

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ABSTRACT. The phase space of a compact, irreducible, simply connected, Riemannian symmetric space admits a natural family of Kähler polarizations parametrized by the upper half plane $S$. Using this family, geometric quantization, including the half-form correction, produces the field $H^{corr} \to S$ of quantum Hilbert spaces. We show that projective flatness of $H^{corr}$ implies that the symmetric space must be isometric to a compact Lie group equipped with a biinvariant metric. In the latter case the flatness of $H^{corr}$ was previously established.

0. Introduction. Suppose the configuration space of a classical mechanical system is an $m$–dimensional compact Riemannian manifold $M$ and the metric corresponds to twice the kinetic energy. The aim of geometric quantization is to construct a Hilbert space (the quantum Hilbert space) associated to this system, in a natural way.

According to the prescriptions of Kostant and Souriau [Ko1,So,Wo], the first step in this process, is to pass to phase space $(N,\omega)$, which for the moment we take $TM \approx T^*M$, a symplectic manifold with an exact symplectic form, and then to choose a Hermitian line bundle with connection $E \to N$, the so called prequantum line bundle, whose curvature is $-i\omega$. This bundle is unique when $M$ is simply connected.

The next step is a choice of a Kähler structure on $N$ with Kähler form $\omega$. This induces on $E$ the structure of a holomorphic line bundle and gives rise to the quantum Hilbert space $H$, consisting of holomorphic sections of $E$ that are $L^2$ with respect to the volume form $\omega^m/m!$. Often one includes in this construction the so called half-form correction. Suppose $\kappa$ is a square root of the canonical bundle $K_N$. Then the corrected quantum Hilbert space $H^{corr}$ consists of the $L^2$ holomorphic sections of $E \otimes \kappa$.

The quantum Hilbert space obtained this way depends on the choices made in this process and the question arises whether there is a canonical way to identify the quantum Hilbert spaces corresponding to the different choices. This question is a fundamental issue in geometric quantization, that has been studied from different perspectives in several papers, see e.g. [ADW, Bl1-2, Ch, F, FMN1-2, FU, Hal1-2, Hi, Ko2, KW, OW, R, S, Vi].

When $M$ is a real-analytic Riemannian manifold, there is a natural Kähler polarization defined at least in some neighborhood $X \subset TM$ of the zero section of $N$ ([GS, HK, Sz1]), the so called adapted complex structure, in which $\omega$ becomes a

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Kähler form. (See Sect. 1.1 for more details on adapted complex structures.) In good cases \(X = N\). One gets examples of this sort when \(M\) is a compact Riemannian symmetric space or more generally a compact normal Riemannian homogeneous space \([Sz1, Sz3]\), but there are nonhomogeneous examples as well, see [A].

In fact the adapted complex structure is just one member in a natural family of Kähler structures on \(N\) \([LSz2]\), that is parametrized by the upper half plane \(S\). This is the family of Kähler structures that respects the symmetries of \(N\). Here \(\text{Im} s, s \in S\) plays the role of Planck’s constant (cf. Sect. 1.1).

Suppose for the compact Riemannian manifold \(M\), the adapted complex structures \(J(s), s \in S\) exist on \(N\). Geometric quantization then produces a family of quantum Hilbert spaces. Our main concern is how (and when) one can define a natural (projective) isomorphism among these Hilbert spaces.

Whenever one has a family of Kähler structures on the phase space \(N\) parametrized by some smooth manifold \(S\), [ADW] and [Hi] suggest to view the corresponding collection \(\{H_s : s \in S\}\) (or \(H^{\text{corr}}\)) of Hilbert spaces as the fibers of a Hilbert bundle \(H \to S\) (resp. \(H^{\text{corr}} \to S\)) endowed with some Hermitian connection, the quantum connection. If this were true, parallel transport along a curve in \(S\) would yield a unitary map between different fibers. If the quantum connection were projectively flat, and \(S\) simply connected, parallel translation would even yield a path-independent canonical projective unitary map between \(H_{s_1}\) and \(H_{s_2}\) for arbitrary parameters \(s_1, s_2 \in S\). When \(S\) is a complex manifold and the bundle \(H \to S\) is holomorphic, there is a canonical choice of a Hermitian connection, the Chern connection.

Now in the uncorrected version one may try to implement the idea above as follows. For each \(s \in S\), \(H_s\) is a closed Hilbert subspace of \(K\), the \(L^2\) sections of the smooth prequantum line bundle \(E \to N\). Thus we could view the set \(H\), the set theoretical disjoint union of all the \(H_s\), as a subset of the trivial Hilbert bundle \(pr : K = S \times K \to S\), such that \(H_s\) is a closed Hilbert subspace of the fiber \(pr^{-1}(s)\) for each \(s \in S\). This way \(H\) certainly inherits a topology from \(K\). It is much less clear whether \(H\) inherits (or under what conditions) a complex (or even a smooth) manifold structure. It is even less clear why (and when) \(H\) is a subbundle (of some kind) of \(K\).

The corrected version is more complicated. Now in addition to the problems we faced in the uncorrected case, an extra complication arises. The corrected quantum Hilbert spaces \(H^{\text{corr}}\) are no longer subspaces of a fixed Hilbert space, but rather of a varying family \(K_s\) of Hilbert spaces, where \(K_s\) denotes the \(L^2\) sections of the corrected prequantum line bundle \(E \otimes \kappa_s\).

It is not clear if the \(K_s\) family itself forms a holomorphic (smooth) Hilbert bundle \(K \to S\) or not (whether \(H^{\text{corr}} \to S\) is a smooth (holomorphic) subbundle of \(K\)).

In fact it turns out that there are at least two equally natural but inequivalent ways to make \(K_s\) a smooth Hilbert bundle. That means on this family there are two natural and different smooth Hilbert bundle structures (see [Sz4]).

To avoid these difficulties it is better not to put any smooth structure on an object like \(H^{\text{corr}} \to S\) initially, rather try to understand its structure through its sections. This was the motivation for us to introduce in [LSz3] the notion of a smooth or analytic field of Hilbert spaces, generalizing Hilbert bundles that [ADW] worked with.

A field of Hilbert spaces is simply a map \(p : H \to S\) of sets with each fiber \(H_s = p^{-1}(s)\) endowed with the structure of a Hilbert space. Since we do not put
any topology on $H$, a section of $p : H \to S$ simply means any map $\varphi : S \to H$ with $\varphi(s) \in H_s$.

When $S$ is a manifold, one says that a smooth (resp. analytic) structure on $H$ is specified (with which $H$ becomes a smooth [analytic] field of Hilbert spaces) if a vector space $\Gamma^\infty$ (resp. $\Gamma^\omega$) of sections of $H$ is chosen, together with a connection like operation on it in such a way that they satisfy a natural set of axioms imitating as if $\Gamma^\infty$ (resp. $\Gamma^\omega$) was the vector space of smooth (real-analytic) sections of a smooth (real-analytic) Hilbert bundle equipped with a Hermitian connection (see [LSz3] and Sect. 1.2 for the precise definition and more details on fields of Hilbert spaces).

Although these objects are quite a bit more general than ordinary Hilbert bundles, still with the help of the connection-like operation built into their definition, it still makes sense to talk about its curvature. Similarly to the classical situation if the Hilbert field is analytic and the curvature turns out to be 0, or at least “central” (=projectively flat), then path independent parallel transport allows for canonical identification of the (projectivized) quantum Hilbert spaces [LSz3, Theorem 2.3.2, Theorem 2.4.2].

When $M$ is a compact, normal Riemannian homogeneous space, one can naturally endow the field $H^{corr} \to S$ with an analytic structure ([L-Sz3, Theorem 11.1.1]). Our main result is:

**Theorem 0.1.** Let $(M, g)$ be a compact, irreducible, simply connected, Riemannian symmetric space. Assume the corrected field of quantum Hilbert spaces $H^{corr} \to S$ is projectively flat. Then $M$ is isometric to a group manifold (i.e. a compact, connected, simple, simply connected Lie group equipped with a biinvariant metric).

We prove Theorem 0.1 in Sect. 6. Group manifolds were treated in [Lsz3, Theorem 11.3.1], where it was shown that whenever $M$ is isometric to a compact, simply connected Lie group with a biinvariant metric, the field $H^{corr} \to S$ is flat. Together with Theorem 0.1 we get Corollary 0.2.

**Corollary 0.2.** Let $(M, g)$ be a compact, irreducible, simply connected, Riemannian symmetric space. Then the corrected field of quantum Hilbert spaces $H^{corr} \to S$ is projectively flat if and only if $M$ is isometric to a group manifold. In the latter case the field $H^{corr} \to S$ is flat.

Corollary 0.2 shows that quantization is unique for group manifolds and for other symmetric spaces quantization does depend on the choice of the Kähler polarization.

Flatness implies that $H^{corr} \to S$ is a genuine Hilbert bundle (trivial in this case). It is not known whether this is also true when the Hilbert field is not projectively flat.

Theorem 0.1 generalizes [LSz3, Theorem 12.1.1], that dealt with spheres, and [Lsz4, Theorem 1.1] that treated rank $-1$ symmetric spaces.

The main scheme of the proof of the theorem is based on the rank $-1$ case [L-Sz4], but the situation here is much more complicated.

Writing $M$ in the usual $M = U/K$ form (see Sect. 1.3 for the notation), each irreducible $K$-spherical representation of $U$ gives rise to a certain integral on the positive Weyl chamber (2.2.6). The integrand involves the corresponding $K$-spherical function and it also depends on a real parameter $\tau$, that takes arbitrary positive values. Projective flatness of the Hilbert field is expressed as a simple relation among
these integrals (Theorem 2.1.1 (b)). Since the explicit value of these integrals is not known, one needs other ways to test projective flatness.

The idea in the rank−1 case ([LSz4]) was to tend with the parameter to zero resp. to infinity, calculate the asymptotic behavior of our integrals and compare the information obtained this way with the relation that holds among the integrals corresponding to different spherical representations. In the rank−1 situation $K$−spherical functions are quite explicit, they reduce to hypergeometric polynomials, greatly simplifying the situation.

In the higher rank case, the basic idea is the same, but the situation is more involved. We still want to calculate the asymptotic behavior of those (now multivariable) integrals as the parameter tends to zero, resp. to infinity. Spherical functions now correspond to multivariable Jacobi polynomials associated to the restricted root system ([H], [HO1-2], [HS]) and they are much more complicated functions to calculate with.

The key observation here is that despite this, their main contribution to the asymptotic behavior of our integrals (when $\tau \to \infty$) is simple. The Jacobi polynomials are actually exponential polynomials, where each term corresponds to a weight of the given $K$−spherical representation. The main contribution comes only from one term, that corresponds to the highest weight. We even know the coefficient of this term, it is Harish-Chandra’s $c$−function. This is the content of Proposition 3.1 and Theorem 5.2. As a consequence, projective flatness implies that a certain numerical quantity $Q(\delta)$ (see (5.7)) associated to every irreducible $K$−spherical representation $\delta$, that involves only the usual $\Gamma$ function, the restricted root system, the multiplicities and the highest weight of $\delta$, in fact is independent of the representation (Theorem 5.5).

Finally the question, for which spaces will this be true, can be translated to a problem about abstract root systems with multiplicities. This problem is treated in Theorem 6.2.2, after which the proof of Theorem 0.1 easily follows.

The organization of our paper is the following. After an introductory section, where we shortly summarize the necessary background, in Sect. 2 we discuss the curvature of the field $H^{corr} \to S$ when the manifold $M$ is a compact, irreducible, simply connected Riemannian symmetric space. In Sect. 3 we discuss spherical functions briefly. The asymptotics of our integrals as $\tau \to 0$ (resp. $\tau \to \infty$) is calculated in Sect. 4 (resp. Sect. 5). Sect. 6 is devoted to root systems with multiplicities and at the end we prove our main theorem.

1. Basic notations

1.1. Adapted complex structures. Here we review some important facts on adapted complex structures mainly from [LSz2]. Following Souriau’s philosophy ([So]), we define the phase space $N$ of a compact Riemannian manifold not as $TM \approx T^*M$, but rather as the manifold of parametrized geodesics $x : \mathbb{R} \to M$. Any $t_0 \in \mathbb{R}$ induces a diffeomorphism $N \ni x \mapsto \dot{x}(t_0) \in TM$, and the pull back of the canonical symplectic form of $TM \approx T^*M$ is independent of $t_0$; we denote it by $\omega$. We identify $M$ with the submanifold of zero speed geodesics in $N$. Affine reparametrizations $t \mapsto a + bt, a, b \in \mathbb{R}$, act on $N$ and define a right action of the Lie semigroup $\mathcal{A}$ of affine reparametrizations.

Given a complex manifold structure on $\mathcal{A}$, a complex structure on $N$ is called adapted if for every $x \in N$ the orbit map $\mathcal{A} \ni \sigma \mapsto x\sigma \in N$ is holomorphic. An
adapted complex structure on $N$ can exist only if the initial complex structure on $A$ is left invariant. The left invariant complex structures on $A$ are parametrized by the points of $\mathbb{C} \setminus \mathbb{R}$. (The points of $\mathbb{R} \subset \mathbb{C}$ correspond to left invariant real polarizations on $A$.) For each $s \in \mathbb{C} \setminus \mathbb{R}$ and corresponding left invariant complex structure $I(s)$ on $A$, if an $I(s)$ adapted complex structure $J(s)$ exists on $N$, then this structure is unique and if $J(i)$ exists, then $J(s)$ also exists for all $s \in \mathbb{C} \setminus \mathbb{R}$. This is the case for example when $(M,g)$ is a compact symmetric space or more generally, a compact, normal Riemannian homogeneous space ($[Sz1, Sz3]$). The original definition of adapted complex structures in [L-Sz1, Sz1] corresponds to the parameter $s = i$.

In fact the Kähler manifolds $(N, J(s))$ are all biholomorphic to a fixed one, $(N, J(i))$, but the biholomorphism maps the Kähler form $\omega$ to $\omega/\text{Im } s$; this is the content of [LSz3, (10.3.4)]. Thus $\text{Im } s$ plays the role of Planck’s constant.

Suppose we are in the situation when $J(s)$ exists. Denote by $\partial_s, \bar{\partial}_s$ the complex exterior derivations for this structure, and $L(x)$ the square of the speed of the geodesic $x \in N$. Then $i\omega = (\text{Im } s)\bar{\partial}_s \partial_s L$. In particular $L$ is a potential of a positive (resp. negative) Kähler structure with Kähler form $\omega$, when $s$ is in the upper (resp. lower) half plane. When $s$ is a point in the lower half plane, the only holomorphic $L^2$ section of the quantum line bundle $E \to (N, J(s))$ is the identically zero section. For this reason we are only interested in the $J(s)$ structures when $s$ is an element of the upper half plane, denoted by $S$.

It is important that the family of adapted complex structures $J(s), s \in S$ on $N$ can all be put together to form a “twistor space” like holomorphic fibration $\pi : Y \to S$, where the fibers $Y_s = \pi^{-1}s$ are biholomorphic to $(N, J(s))$. As a differentiable manifold, $Y = S \times N$, and the projection $pr : Y \to N$ realizes the biholomorphisms $Y_s \to (N, J(s))$ ([L-Sz2, Theorem 5]). The pull back $\tilde{\omega}$ of $\omega$ by $pr$ satisfies

\begin{equation}
(1.1.1) \quad i\tilde{\omega} := \bar{\partial}\partial (\text{Im } s) \quad \text{on } Y.
\end{equation}

Assuming now that $M$ is simply connected, with the help of the fibration $\pi : Y \to S$ one can perform geometric quantization simultaneously ([LSz3]). Because of (1.1.1), $\tilde{\omega}$ is an exact $(1,1)$ form. Therefore the unique Hermitian line bundle with connection $(E, h^E) \to Y$ whose curvature is $-i\tilde{\omega}$ becomes a holomorphic line bundle. The restriction of $E$ to $Y_s$ yields the prequantum line bundle corresponding to $(N, J(s), \omega)$. The restriction of the form $\nu = \tilde{\omega}^m/m!$ to a fiber $Y_s$ is a volume form. The spaces of holomorphic $L^2$-sections of $E|Y_s$ form the Hilbert field $H \to S$.

Since $M$ is simply connected, there is a unique Hermitian holomorphic line bundle $\kappa$ on $Y$ [LSz3, Sect. 10.4], so that $\kappa \otimes \kappa \approx K_\pi$ (the relative canonical bundle of $Y$ with $K_\pi|Y_s$ being the canonical bundle of $Y_s$). Let $(E^{corr}, h^{E^{corr}}) = (E \otimes \kappa, h^E \otimes h^\kappa)$. The spaces of holomorphic $L^2$-sections of $E^{corr}|Y_s$ form the corrected Hilbert field $H^{corr} \to S$, which at the moment is just a map of sets where all the fibers have a Hilbert space structure. The complex structure of $Y$ and the holomorphic fibration $\pi : Y \to S$ plays a crucial role in the construction of the extra structures (smooth and analytic) we need on the field $H^{corr} \to S$. This is discussed in the next section.

1.2. Fields of Hilbert spaces. Here we review some notions and results from [LSz3] concerning fields of Hilbert spaces. A field of Hilbert spaces is simply a map $p : H \to S$ of sets with each fiber $H_s = p^{-1}(s)$ endowed with the structure of a
Hilbert space. The inner products on the fibers, taken together, define a function

$$h: H \oplus H \to \mathbb{C}, \quad \text{where} \quad H \oplus H = \bigsqcup_{s \in S} H_s \oplus H_s.$$  

If \( v \in H \), we also write \( h(v) \) for \( h(v, v) \). Hilbert fields naturally arise as direct images of holomorphic vector bundles. Let \( \pi: Y \to S \) be a surjective holomorphic submersion of finite dimensional complex manifolds, where we do not assume that \( \pi \) is proper. Let \( \nu \) be a smooth form on \( Y \) that restricts to a volume form on each fiber \( Y_s = \pi^{-1} s \) and let \( (E, h^H) \to Y \) be a Hermitian holomorphic vector bundle of finite rank. Denote by \( H_s \) the Hilbert space of \( L^2 \) holomorphic sections of \( E|Y_s \). Then \( H_s \) forms a Hilbert field \( H \to S \). An example of this sort is \( H^\text{corr} \to S \) from Sect. 1.1.

Direct images tend to have looser structures than bundles. As it was hinted at in the introduction, instead of specifying some smooth structure on the total space \( H \), we try to understand their structure through their sections.

**Definition 1.2.1.** Let \( S \) be a smooth manifold. A smooth structure on a field \( H \to S \) of Hilbert spaces is given by specifying a set \( \Gamma^\infty \) of sections of \( H \), closed under addition and under multiplication by elements of \( C^\infty(S) \), together with linear operators \( \nabla_\xi: \Gamma^\infty \to \Gamma^\infty \) for each vector field \( \xi \) on \( S \), such that for \( f \in C^\infty(S), \varphi, \psi \in \Gamma^\infty \) and vector fields \( \xi, \eta \)

\[
\nabla_{\xi+\eta} = \nabla_\xi + \nabla_\eta, \quad \nabla_{f\xi} = f \nabla_\xi, \quad \nabla_\xi(f\varphi) = (\xi f)\varphi + f\nabla_\xi\varphi; \\
h(\varphi, \psi) \in C^\infty(S) \quad \text{and} \quad \xi h(\varphi, \psi) = h(\nabla_\xi\varphi, \psi) + h(\varphi, \nabla_\xi\psi); \\
\{\varphi(s): \varphi \in \Gamma^\infty\} \subset H_s \quad \text{is dense, for all } s \in S.
\]

The collection \( \nabla \) of the operators \( \nabla_\xi \) is called a connection on \( H \). A field of Hilbert spaces together with a smooth structure is called a smooth Hilbert field. The curvature of \( H \to S \) is defined by

$$R(\xi, \eta) = \nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi, \eta]} : \Gamma^\infty \to \Gamma^\infty.$$  

The field \( H \) is called flat if \( R = 0 \) and projectively flat if \( R(\xi, \eta) \) acts by multiplication by a function \( r(\xi, \eta): S \to \mathbb{C} \). In the latter case, similarly to vector bundles, \( r \) is in fact a smooth closed 2-form on \( S \), and a simple twisting will reduce projectively flat smooth Hilbert fields to flat ones.

As [LSz3, Example 2.3.4] shows, flatness of a smooth Hilbert field does not guarantee its local triviality. Here a Hilbert field with a smooth structure \( (H, \Gamma^\infty, \nabla) \) is called trivial if there exist a fixed Hilbert space \( V \), a fiber preserving and fiberwise unitary bijection \( T: H \to S \times V \), such that for any \( \varphi \in \Gamma^\infty \) and vector field \( \xi \) along \( S \), \( T\varphi \) will be a \( C^\infty \) section of \( S \times V \to S \) and \( T(\nabla_\xi\varphi) = \xi T\varphi \).

**Definition 1.2.2.** Let \( H \to S \) be a smooth Hilbert field over a real-analytic manifold \( S \).

(i) A section \( \varphi \in \Gamma^\infty \) is analytic if for any compact \( C \subset S \) and any finite set \( \Xi \) of vector fields, analytic in a neighborhood of \( C \), there is an \( \varepsilon > 0 \) such that

$$\sup \frac{\varepsilon^n}{n!} h(\nabla_{\xi_n} \cdots \nabla_{\xi_1}\varphi)(s)^{1/2} < \infty,$$
where the sup is taken over \( n = 0, 1, \ldots, \xi_j \in \Xi, \) and \( s \in C. \) The set of analytic sections is denoted by \( \Gamma^\omega \subset \Gamma^\infty. \)

(ii) \( H \to S \) is an analytic Hilbert field if \( \{ \varphi(s) : \varphi \in \Gamma^\omega \} \subset H_s \) is dense for all \( s \in S. \)

An analytic and flat Hilbert field \( H \to S \) will be locally trivial ([LSz3, Theorem 2.3.2]) and so parallel transport can be introduced that identifies the fibers locally in a canonical way.

For a surjective holomorphic submersion \( \pi : Y \to S \) and a Hermitian holomorphic vector bundle \( (E, h^H) \to Y, \) under appropriate conditions on \( Y \) and \( E, \) the direct image Hilbert field \( H \to S \) comes naturally endowed with a smooth structure ([LSz3, Sect. 6-7]). In the problem of geometric quantization by adapted complex structures, these conditions are known to be satisfied when \( M \) is a compact, simply connected, normal Riemannian homogeneous space. In this case \( H^{corr} \to S \) turns out to be analytic ([LSz3, Theorem 11.1.1]).

In the rest of this section we sketch the basic idea of the construction of the (quantum) connection on \( H^{corr} \to S. \) Recall from Sect. 1.1 the holomorphic submersion \( \pi : Y = S \times N \to S \) and Hermitian holomorphic vector bundle \( (E^{corr}, h^{E^{corr}}) \to Y. \) Sections \( \varphi \) of \( H^{corr} \to S \) are in one to one correspondence with sections \( \Phi \) of the bundle \( E^{corr} \) that are holomorphic and \( L^2 \) on each \( Y_s, \) the correspondence is \( \Phi(y) = \varphi(\pi y)(y), \) for \( y \in Y. \) Write \( \Phi = \hat{\varphi} \) and \( \varphi = \tilde{\Phi} \) to indicate this correspondence.

A lift of a smooth, vector field \( \xi \) defined on \( S, \) is a vector field \( \hat{\xi} \) on \( Y \) such that \( \pi_* \hat{\xi}(y) = \xi(\pi(y)), \) for \( y \in Y. \) Lifts are not unique, but we can at least require that if \( \xi \) is of type \( (1, 0) \) or \( (0, 1), \) the lift should be of the same type. If \( \hat{\xi}^1 \) and \( \hat{\xi}^2 \) are two lifts of the same vector field, then \( \hat{\xi}^1 - \hat{\xi}^2 \) will be vertical, i.e. tangential to the fibers \( Y_s. \)

Denote by \( \nabla \) the Chern connection of \( (E^{corr}, h^{E^{corr}}). \) This implies in particular that whenever \( Z \) is a smooth, vertical vector field on \( Y \) of type \( (0, 1) \) and \( \Phi \) a smooth section of \( E^{corr}, \) whose restriction to each \( Y_s \) is holomorphic, we have

\[
(1.2.3) \quad \nabla_Z \Phi = 0.
\]

If \( \hat{\xi} \) is a lift of a smooth vector field \( \xi \) on \( S \) of type \( (0, 1) \) and \( Z \) as before, we have

\[
(1.2.4) \quad \nabla_Z (\nabla_{\hat{\xi}} \Phi) = \nabla_{\hat{\xi}} (\nabla_Z \Phi) + \nabla_{[Z, \hat{\xi}]} \Phi - i \tilde{\omega}(Z, \hat{\xi}) \Phi = 0.
\]

This holds because each term on the right hand side of (1.2.4) is zero: the first because of (1.2.3), the second because (as one easily computes) \([Z, \hat{\xi}]\) will be also vertical and of type \((0, 1)\) and again we can use (1.2.3) with \( Z \) replaced by \([Z, \hat{\xi}]\), finally the last term vanishes because the form \( \tilde{\omega} \) is of type \((1, 1)\) and \( Z \) and \( \hat{\xi} \) are both of type \((0, 1).\)

Now (1.2.3) and (1.2.4) together imply the following. Let \( \xi \) be a smooth vector field of type \((0, 1)\) on \( S, \) \( \hat{\xi} \) an arbitrary lift (also of type \((0, 1)) \) to \( Y \) and \( \Phi \) a smooth section of \( E^{corr} \) whose restriction to each \( Y_s \) is holomorphic. Then \( \nabla_{\hat{\xi}} \Phi \) is a well defined (i.e. does not depend on how we chose the lift \( \hat{\xi} \)), smooth section of \( E^{corr} \to Y \) whose restriction to each fiber \( Y_s \) is holomorphic. Now if \( \nabla_{\hat{\xi}} \Phi \) happens to be \( L^2 \) along each fiber \( Y_s, \) then \( (\nabla_{\hat{\xi}} \Phi) \) yields a section of \( H^{corr} \to S. \) This
gives the idea how to try to define the quantum connection. The elements \( \varphi \in \Gamma^\infty \) should have the properties: \( \varphi \) is a smooth section of \( E^{corr} \), holomorphic and \( L^2 \) along each \( Y_s \) and for any smooth vector field \( \xi \) of type \( (0,1) \) on \( S \), \( (\tilde{\nabla}_\xi \varphi) \) should have the same properties. Then

\[
(1.2.5) \quad \nabla_\xi \varphi := (\tilde{\nabla}_\xi \varphi).
\]

would define the quantum connection for \((0,1)\) type vector fields.

The quantum connection is supposed to be a Hermitian connection, i.e. for a \((1,0)\) vector field \( \xi \) and (appropriate) section \( \varphi \) of \( H^{corr} \to S \), \( \nabla_\xi \varphi \) should be that section \( \psi \) of \( H^{corr} \to S \), which corresponds to the pointwise continuous linear functional \( \theta \mapsto \xi h(\varphi, \theta) - h(\varphi, \nabla_\xi \theta), \theta \in \Gamma^\infty \).

Finally if \( \xi \) is any smooth vector field on \( Y \) and \( \varphi \in \Gamma^\infty \),

\[
\nabla_\xi \varphi := \nabla_{\xi^{0.1}} \varphi + \nabla_{\xi^{1.0}} \varphi.
\]

For more details and precise statements see [LSz3, Sect. 6-9].

1.3. Symmetric spaces. Let \((M^m, g)\) be an \( m \)-dimensional, compact, irreducible, simply connected, Riemannian symmetric space. Then \( M \) is isometric to \( U/K \), where \( U \) is a compact, connected, simply connected, semisimple Lie group and \( K \) is the fixed point set (automatically connected) of a nontrivial involution \( \theta: U \to U \).

The metric on \( U/K \) is induced from a biinvariant metric on \( U \) ([He1]). Furthermore either \( U \) is simple or has the form \( U = G \times G \) where \( G \) is simple, \( \theta(g_1, g_2) = (g_2, g_1) \) and \( K \) is the diagonal in \( G \times G \). In the latter case \( M \) is isometric to \( G \) equipped with a biinvariant metric.

Let \( u \) be the Lie algebra of \( U \), \( u_\mathbb{C} \) its complexification and \( U_\mathbb{C} \) the simply connected complex group with Lie algebra \( u_\mathbb{C} \). Since \( U \) is compact, the canonical Lie algebra embedding \( \iota_*: u \to u_\mathbb{C} \) yields an embedding \( \iota: U \to U_\mathbb{C} \).

As a smooth manifold \( U_\mathbb{C} \) naturally identifies with the tangent bundle \( TU \). The complex structure on \( TU \) obtained using this diffeomorphism will be the adapted complex structure of a biinvariant metric on \( U \) (see [Sz2, Prop.3.5]). This is the complex structure that corresponds to the parameter \( i \) from Sect. 1.1.

\( \theta \) induces a Lie algebra involution \( \theta_*: u \to u \). Then \( u = \mathfrak{k} + \mathfrak{p}_* \), where \( \mathfrak{k} = \{ X \in u : \theta_*(X) = X \} \) and \( \mathfrak{p}_* = \{ X \in u : \theta_*(X) = -X \} \). Here \( \mathfrak{k} \) is the Lie algebra of \( K \) and \( \mathfrak{p}_* \) can be identified with \( T_{[K]} M \).

Let \( \mathfrak{p}_0 = i\mathfrak{p}_* \), \( \mathfrak{g}_0 = \mathfrak{k} + \mathfrak{p}_0 \) and denote by \( G_0 \) the analytic subgroup of \( U_\mathbb{C} \) with Lie algebra \( \mathfrak{g}_0 \). Then \( G_0 \) is closed in \( U_\mathbb{C} \) and \( K \subset G_0 \). Let \( \theta_\mathbb{C} \) be the holomorphic extension of \( \theta \) to \( U_\mathbb{C} \). Then \( \theta_\mathbb{C}|_{G_0} \) is a Cartan involution on \( G_0 \) with fixed point set \( K \). The corresponding symmetric space \( X = G_0/K \) is the noncompact dual of \( U/K \).

Let \( a_* \subset \mathfrak{p}_* \) be a maximal Abelian subspace. Its dimension \( r := \dim a_* \) is the rank of \( M \). Let \( a_0 := ia_* \) and \( \mathfrak{h}_0 \subset \mathfrak{g}_0 \) be a maximal Abelian subalgebra containing \( a_0 \). The complexification of \( \mathfrak{h}_0 \) (resp. of \( a_0 \)) is \( \mathfrak{h} \) (resp. \( a \)). Let \( \Delta \) be the set of nonzero roots corresponding to \((u_\mathbb{C}, \mathfrak{h})\) and \( \Sigma \) the set of restricted roots.

Let \( \mathfrak{h}_\mathbb{R} = \mathfrak{h}_0 \cap \mathfrak{k} \) and \( \mathfrak{h}_\mathbb{R} = a_0 + i\mathfrak{h}_\mathbb{R} \). The roots are real valued on \( \mathfrak{h}_\mathbb{R} \). Choose a compatible ordering in the dual spaces of \( a_0 \) and \( \mathfrak{h}_\mathbb{R} \). This yields an ordering of \( \Delta \) and \( \Sigma \). Let \( \rho_\Delta \) be half the sum of the positive roots and \( \rho \) its restriction to \( a \),
i.e. \( \rho = (1/2) \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \), where \( m_\alpha \) is the multiplicity of \( \alpha \). \( a_+ \subset a_0 \) denotes the positive Weyl chamber

\[
a_+ := \{ H \in a_0 : \alpha(H) > 0, \forall \alpha \in \Sigma^+ \}.
\]

The classification of compact, irreducible Riemannian symmetric spaces shows that the restricted root system together with the multiplicity function determines the symmetric space uniquely (see [He1]). In particular we have

**Proposition 1.3.1.** A compact, simply connected Riemannian symmetric space \( M \) is isometric to a compact, simply connected Lie group equipped with a biinvariant metric if and only if \( \Sigma \) is a reduced root system and each \( m_\alpha \) is equal to 2.

(See [L, Theorem 4.4, p.82]). We intend to use this characterization of Lie groups to prove Theorem 0.1.

2. Curvature calculations

2.1. Flatness and projective flatness.

Consider a compact, simply connected, irreducible Riemannian symmetric space \((M^m, g)\), given in the form \( M = U/K \) as in Sect. 1.3 and \( H^{corr} \to S \) the corresponding field of quantum Hilbert spaces (\( S \) being the complex upper half plane).

\( U \) acts on \((N, J(i))\) by biholomorphisms and this action induces a representation \( \hat{\pi} \) on \( \mathcal{O}(N, J(i)) \), by the formula \( av = (a^{-1})^* v \) (pull back by \( a^{-1} \)), where \( a \in U \), \( v \in \mathcal{O}(N, J(i)) \). The same formula defines a unitary representation \( \pi \) on \( L^2(M) \).

The restrictions \( V_\chi|_M \) of the isotypical subspaces of \( \hat{\pi} \) are precisely the isotypical subspaces of \( \pi \) and the latter are well known to be finite dimensional. Since \( M \) is a maximal dimensional, totally real submanifold in \( N \), we get that \( V_\chi \) are also finite dimensional.

The restrictions of \( \hat{\pi} \) to the isotypical subspaces \( V_\chi \) (or equivalently the restrictions of \( \pi \) to \( V_\chi|_M \)) are irreducible, they are precisely the irreducible \( K \)-spherical representations of \( U \) ([He2, Chap. V, Theorem 4.3]). Therefore from now on we use the spherical representations \( \delta \) themselves instead of their character \( \chi \), to label the objects (unlike in [L-Sz3]), for example \( V_\delta \) will replace \( V_\chi \).

Flatness of the field \( H^{corr} \to S \) can be understood in terms of certain Toeplitz operators \( P_\delta(s) \) on \( V_\delta \). They are \( U \)-equivariant, whence according to Schur’s lemma, have the form \( P_\delta(s) = p_\delta(s) Id_{V_\delta} \) with an appropriate function \( p_\delta \). \( H^{corr} \to S \) is flat (resp. projectively flat) if and only if \( \bar{\partial}\partial \log p_\delta(s) = 0 \) for all \( \delta \) (resp. \( \bar{\partial}\partial \log p_\delta(s) \) is independent of \( \delta \)), see [L-Sz3, Theorem 9.2.1].

In our situation according to [L-Sz3, Lemma 11.2.1] \( p_\delta(s) \) depends only on \( \tau = \text{Im} s \) and has the specific form

\begin{equation}
(2.1.1) \quad p_\delta(s) = C_\delta \tau^{-m/2} q_\delta(\tau),
\end{equation}

where \( m \) is the dimension of the space \( M \), \( C \) is some constant, \( c_\delta \) a constant for each representation \( \delta \) and \( q_\delta \) an appropriate function (see (2.2.6) for the precise form). As one easily sees, a factor like \( C \tau^{-m/2} \) that depends only on \( \tau = \text{Im} s \) but not on \( \delta \) does not affect the condition for projective flatness. In our case, in light of (2.1.1), the above mentioned characterization of (projective) flatness takes the form.
Theorem 2.1.1.
(a) \( H^{corr} \to S \) is flat iff for each \( \delta \), \( \log(p_\delta(s)) \) is harmonic.
(b) \( H^{corr} \to S \) is projectively flat iff for each \( \delta \) there exist constants \( A_\delta > 0 \), \( B_\delta \) with \( q_\delta(\tau) = A_\delta e^{B_\delta \tau} q_{\delta_0} \), where \( \delta_0 \) denotes the trivial representation.

Since we cannot compute \( q_\delta \) explicitly, we cannot check directly whether condition (b) in Theorem 2.1.1 holds or not. Therefore we shall apply the following strategy to prove Theorem 0.1. We shall investigate the asymptotic behavior of \( q_\delta(\tau) \) as \( \tau \) tends to 0 and to infinity. From the \( \tau \to 0 \) asymptotics we shall determine the values of \( A_\delta \) and \( B_\delta \) dictated by condition (b) in Theorem 2.1.1. Then do the same as \( \tau \to \infty \) and obtain possible different values for \( A_\delta \) and \( B_\delta \). If the values for \( A_\delta \) or \( B_\delta \) do not match as \( \tau \to 0 \) and as \( \tau \to \infty \), we can conclude that the Hilbert field is not projectively flat.

It turns out, that \( B_\delta \) does not help in determining the projective flatness of \( H^{corr} \to S \), for all rank-1 symmetric spaces the two asymptotics give the same value for \( B_\delta \) (see Remark 1, after Theorem 5.4). Theorem 0.1 is proved by showing that the \( \tau \to 0 \) asymptotics yields \( A_\delta = 1 \) for all \( \delta \) (see Theorem 4.2.2), on the other hand the \( \tau \to \infty \) asymptotic shows that if the coefficient \( A_\delta \) is independent of \( \delta \), the restricted root system of \( M \) must be reduced and all multiplicities of the roots are equal to two (see Sect. 5 and 6). But these properties characterize compact Lie groups among compact Riemannian symmetric spaces (see [L]) and Theorem 0.1 will follow.

2.2. The function \( q_\delta(\tau) \).

Now to implement the plan in Sect. 2.1, we need to recall first of all the precise form of \( p_\delta(s) \) (see [L-Sz3, Sect. 12.1], \( \tau = \text{Im} s \)).

\[
(2.2.1) \quad p_\delta(s) = \frac{c_\delta}{\tau^{m/2}} \int_{p_*} \int_{K} e^{-\frac{|\zeta|^2}{\tau}} \chi_\delta(k \exp(-2i\zeta)) \sqrt{\eta(\zeta)} dk d\zeta,
\]

where \( c_\delta \) is independent of \( s \), \( dk \) is normalized Haar measure on \( K \), \( d\zeta \) is the Lebesgue measure on \( p_* \) induced by the metric, \( \chi_\delta \) the character of \( \delta \) and

\[
(2.2.2) \quad \eta(\zeta) = \det \left( \frac{\sin 2ad\zeta}{ad\zeta} \right)_{\mathbb{C} \otimes p_*}.
\]

The function \( f_\delta(g) = \int_K \chi_\delta(kg^{-1}) dk \), \( g \in U \) is known as the \( K \)-spherical function ([Ha1,Ha2]), corresponding to the representation \( \delta \), see [He2, Theorem 4.2, p.417]. We denote by the same letter the holomorphic extension of \( f_\delta \) to the complexified group \( U_\mathbb{C} \). Thus we can rewrite (2.2.1) as an integral over \( p_0 \) and we get

\[
(2.2.3) \quad p_\delta(s) = \frac{c_\delta}{\tau^{m/2}} \int_{p_0} e^{-\frac{|H|^2}{\tau}} f_\delta(\exp(2H)) \sqrt{\eta(-iH)} dH.
\]

Every restricted root \( \alpha \in \Sigma \) is real valued on \( a_0 \). Furthermore the operator \( ad^2_H \), \( H \in p_0 \), is symmetric, has zero eigenvalue with multiplicity \( r = \dim a_0 \) and \( \alpha(H)^2 \) with multiplicity \( m_\alpha \). Thus from (2.2.2) and the identity \( \sin i2z/iz = \text{sh}2z/z \) we get

\[
(2.2.4) \quad \eta(-iH) = 2^r \prod_{\alpha \in \Sigma^+} \left( \frac{\text{sh}(2\alpha(H))}{\alpha(H)} \right)^{m_\alpha}.
\]
Let $C(a_0) := \{k \in K : Ad(k) \zeta = \zeta, \forall \zeta \in a_0\}$ be the centralizer of $a_0$ in $K$. Recall the following integral formula for the generalized polar coordinate map

$$\Phi : (K/C(a_0)) \times a_0 \to p_0, \quad \Phi(kC(a_0), H) := Ad(k)H,$$

\textbf{Theorem 2.2.1.} Let $f \in L^1(p_0)$ be an $Ad(K)$ invariant function. Then

$$\int_{p_0} f(H)dH = c \int_{a_+} f(H) \prod_{\alpha \in \Sigma^+} \alpha(H)^{m_\alpha}dH,$$

where $c$ is some constant, independent of $f$ and $a_+ \subset a_0$ denotes the positive Weyl chamber.

(See [He2, Theorem 5.17, p.195].)

From [L-Sz4, Prop. 2.1 and Prop. 2.2] we know that $f_\delta \circ \exp$ and $\eta$ are $Ad_K$ invariant on $p_0$. Thus Theorem 2.2.1, (2.2.3) and (2.2.4) yields the following formula.

$$p_\delta(s) = 2^r c_\delta c r^{-m/2} \int_{a_+} e^{-\frac{|H|^2}{r}} f_\delta(\exp(2H)) \prod_{\alpha \in \Sigma^+} (\alpha(H)\text{sh}(2\alpha(H)))^{\frac{m_\alpha}{2}} dH.$$

In the special case when $M$ is isometric to a compact Lie group $G$, let $U = G \times G$ and $K$ the diagonal in $U$. Then the $K$–spherical functions will have the form $f_\delta = \chi_\delta/d(\delta)$, where $\delta$ is an irreducible representation of $G$, $\chi_\delta$ its character and $d(\delta)$ denotes its dimension ([He2, p.407]). Thus $f_\delta$ is given by the Weyl character formula and since all $m_\alpha = 2$ the terms $\text{sh}(2\alpha(H))$ cancel out the Weyl denominator and we end up essentially integrating the product of a Gaussian and a harmonic polynomial. This yields that $\log(p_\delta(s)) = c_1 + c_2 \text{Im}s$, a harmonic function (see [LSz3, Theorem 11.3.1]). Thus in light of Theorem 2.1.1(a), the field $H^{\text{corr}} \to S$ will be flat.

To treat the other symmetric spaces, we introduce the essential part of $p_\delta$ as a function of $\tau > 0$:

$$q_\delta(\tau) := \int_{a_+} e^{-\frac{|H|^2}{r}} f_\delta(\exp(2H)) \prod_{\alpha \in \Sigma^+} (\alpha(H)\text{sh}(2\alpha(H)))^{\frac{m_\alpha}{2}} dH.$$

3. Spherical functions

In order to be able to handle the integral in (2.2.6), we shall need another description of spherical functions. Let $\delta : U \to GL(V)$ be an irreducible $K$–spherical representation. We can endow $V$ with a scalar product $\langle ., . \rangle$ that makes $\delta$ unitary. Let $v_K \in V$ be a $K$–fixed vector with unit length. Then the spherical function $f_\delta$ corresponding to $\delta$ is ([He2, Theorem 3.7, p.414])

$$f_\delta(g) := \langle \delta(g)v_K, v_K \rangle, \quad g \in U.$$

Since $\delta$ extends holomorphically to the complexified group $U_C$, the same formula yields the holomorphic extension of $f_\delta$ to $U_C$.

We would like to obtain some formula for the function $f_\delta \circ \exp$, occurring in (2.2.6), when we restrict it to the Cartan subalgebra $\mathfrak{h}$ in $u_C$. 
Let \( \Lambda(\delta) \) be the set of weights of \( \delta \) and for a weight \( \mu \), \( W_\mu \) the corresponding weight space.

The weight spaces give an orthogonal direct decomposition of \( V \), thus

\[
v_K = \sum_{\mu \in \Lambda(\delta)} w_\mu, \quad w_\mu \in W_\mu,
\]

where \( \|v_K\| = 1 \) implies \( \sum \|w_\mu\|^2 = 1. \)

Let \( H \in \mathfrak{h} \). Then (cf. [V])

\[
(3.1) \quad f_\delta(\exp 2H) = \langle \exp(\delta \ast 2H)v_K, v_K \rangle = \sum_{\mu \in \Lambda(\delta)} e^{2\mu(H)} \langle w_\mu, w_\mu \rangle.
\]

Later on we shall need to figure out which term in (3.1) has the dominating contribution when (3.1) is plugged into the formula (2.2.6) of \( q_\delta \). It is no surprise that the term corresponding to the highest weight will play this role. Theorem 5.2 will give the precise answer. That theorem will be based on Theorem 5.1, a general result on asymptotics of integrals of the form (2.2.6), where the function \( f_\delta \) is replaced by an exponential of a linear function, like the terms in (3.1). The result of Theorem 5.1 shall explain why we need Proposition 3.1.

Let \( \lambda \) be the highest weight of \( \delta \). Then \( \dim W_\lambda = 1 \). Let \( v_\lambda \in W_\lambda \) with \( \|v_\lambda\| = 1. \) Thus \( w_\lambda = a_\lambda v_\lambda \) with \( a_\lambda = \langle v_K, v_\lambda \rangle \). From the first formula in [He2, p.538] we know that \( a_\lambda \neq 0 \) and

\[
(3.2) \quad \langle w_\lambda, w_\lambda \rangle = |a_\lambda|^2 = c(-i\lambda - i\rho),
\]

where \( \rho = \rho \Delta |_{a_0} \) is half the sum of the positive restricted roots with multiplicity, \( X = G/K \) the noncompact dual symmetric space and \( c \) is the corresponding Harish-Chandra’s \( c \)-function of \( G \) ([Ha1, Ha2], [He2, (8), p.538]).

**Proposition 3.1.** Let \( \mu \in \Lambda(\delta), \mu \neq \lambda \). Then

\[
\| (\mu + \rho \Delta) |_{a_0} \| < \| (\lambda + \rho \Delta) |_{a_0} \|.
\]

**Proof.** We follow the steps of the proof of [He2, Theorem. 1.3, p.498], that is the same statement without taking restrictions to \( a_0 \). First we show that

\[
(3.3) \quad (\lambda - \mu) |_{a_0} \neq 0.
\]

Since \( \lambda \) is the highest weight of a \( K \)-spherical representation, the Cartan-Helgason theorem ([He2, Theorem 4.1 (1), p.535]) implies

\[
\lambda |_{ih^{\Gamma_0}} \equiv 0.
\]

Thus if (3.3) does not hold, we would get \( \langle \lambda - \mu, \lambda \rangle = 0 \) and then

\[
(3.4) \quad \langle \mu, \mu \rangle = \langle \mu - \lambda, \mu - \lambda \rangle + \langle \lambda, \lambda \rangle > \langle \lambda, \lambda \rangle,
\]

since \( \mu \neq \lambda \). But (3.4) contradicts the fact that for all weights \( \mu, \|\mu\| \leq \|\lambda\| \) (see [He2, Theorem 1.3 (7), p.498]) and so (3.3) is proved.
We need to show
\[ C := \|\lambda + \rho\|^2 - \|\mu\|_{a_0} + \rho \|^2 > 0. \]

But
\begin{equation}
(3.5) \quad C = \|\lambda\|^2 - \|\mu\|_{a_0}^2 + 2\langle \lambda - \mu\|_{a_0}, \rho \rangle \geq \|\lambda\|^2 - \|\mu\|^2 + 2\langle \lambda - \mu\|_{a_0}, \rho \rangle.
\end{equation}

And since \( \|\lambda\| \geq \|\mu\| \), it suffices to show that the last term in (3.5) is positive.

Let \( \alpha_1, \ldots, \alpha_l \) be a basis of the roots, compatible with \( \Sigma \), i.e. for \( 1 \leq j \leq r \), \( \alpha_j|_{a_0} \in \Sigma^+ \) forming a basis of \( \Sigma \). Since \( \mu \) is a weight, \( \exists \eta_j \in \mathbb{Z}_+^\Sigma \) with \( \mu = \lambda - \sum \eta_j \alpha_j \).

Now (3.3) implies that \( \exists \eta_j \) with \( 1 \leq j \leq r \) and \( \eta_j > 0 \). Proposition 3.2 shows that \( \langle \alpha_j|_{a_0}, \rho \rangle > 0 \) for \( 1 \leq j \leq r \). Hence
\[ \langle \lambda - \mu\|_{a_0}, \rho \rangle = \sum_{j=1}^{r} \eta_j \langle \alpha_j|_{a_0}, \rho \rangle > 0, \]
thus indeed \( C > 0 \). \( \square \)

**Proposition 3.2.** Let \( \alpha_1, \ldots, \alpha_r \in \Sigma^+ \) be a basis of the restricted roots \( \Sigma \) with multiplicities \( m_{\alpha_j} \). Then
\begin{equation}
(3.6) \quad \langle \rho, \alpha_j \rangle = (m_{\alpha_j}/2 + m_{2\alpha_j}) \langle \alpha_j, \alpha_j \rangle, \quad j = 1 \ldots, r
\end{equation}

where \( m_{2\alpha_j} \) is meant to be zero if \( 2\alpha_j \) is not a root.

**Proof.** Let \( \Sigma_j^+ = \Sigma^+ \setminus \{\alpha_j, 2\alpha_j\}, \rho_j = \frac{1}{2} \sum_{\alpha \in \Sigma_j^+} m_{\alpha} \alpha \) and \( S_{\alpha_j} \) the reflection on \( a_0 \), corresponding to \( \alpha_j \). As is well known ([He1, ChVII, Sect. 3, Lemma 2.21]) \( S_{\alpha_j} \) permutes the elements of \( \Sigma_j^+ \), hence \( S_{\alpha_j} \rho_j = \rho_j \). From their definitions we get
\[ \rho = \rho_j + \frac{m_{\alpha_j} \alpha_j + m_{2\alpha_j} 2 \alpha_j}{2}. \]

Thus
\[ S_{\alpha_j} \rho = \rho - m_{\alpha_j} \alpha_j - m_{2\alpha_j} 2 \alpha_j. \]

Since \( S_{\alpha_j} \) is an orthogonal transformation, we obtain
\[ \langle \rho, \alpha_j \rangle = \langle S_{\alpha_j} \rho, S_{\alpha_j} \alpha_j \rangle = \langle \rho - m_{\alpha_j} \alpha_j - m_{2\alpha_j} 2 \alpha_j, -\alpha_j \rangle \]

and (3.6) follows. \( \square \)

4. \( \tau \to 0 \) asymptotics

**4.1. A multivariable Watson lemma.**

**Proposition 4.1.1.** Let \( 0 < \tau, 0 < h, D \subset \mathbb{R}^n \) be a domain that is a homogeneous cone \( (\xi \in D, 0 < r \text{ implies } r\xi \in D) \), \( G := D \cap S^{n-1} \) (where \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \) ) and \( Q \) an \( h \)-homogeneous (for all \( \xi \in D, 0 < r, Q(r\xi) = r^h Q(\xi) \) ) continuous function defined on \( \overline{D} \). Then
\[ \int_D e^{-\frac{H^2}{r}} Q(H) dH = \frac{\Gamma(\frac{n+h}{2})}{2} \left( \int_G Q(\xi) d\xi \right) \tau^{\frac{n+h}{2}}, \]
where $\Gamma$ denotes the usual gamma function.

**Proof.** Using polar coordinates and the homogeneity of $Q$ we get

$$
\int_D e^{-\|H\|^2/\tau} Q(H) dH = \int_0^\infty \int_G e^{-r^2} r^{n+h-1} Q(\xi) d\xi dr.
$$

Substituting $r = \sqrt{\tau} t$ yields the formula. $\square$

**Proposition 4.1.2.** Let $\delta, \tau_0 > 0$, $D \subset \mathbb{R}^n$ be a domain, $D_\delta := D \cap \{\|H\| \geq \delta\}$ and $g$ a Lebesgue measurable function on $D$ with

$$
C := \int_D e^{-\|H\|^2/\tau_0} |g(H)| dH < \infty.
$$

Then for every $0 < \tau < \tau_0$

$$
\int_{D_\delta} e^{-\|H\|^2/\tau} |g(H)| dH \leq C e^{\delta^2(\frac{1}{\tau_0} - \frac{1}{\tau})}.
$$

**Proof.** Let $\delta \leq \|H\|$. Then

$$
e^{\|H\|^2(\frac{1}{\tau_0} - \frac{1}{\tau})} \leq e^{\delta^2(\frac{1}{\tau_0} - \frac{1}{\tau})}.
$$

Thus

$$
\int_{D_\delta} e^{-\|H\|^2/\tau} |g(H)| dH = \int_{D_\delta} e^{-\|H\|^2/\tau_0} |g(H)| e^{\|H\|^2(\frac{1}{\tau_0} - \frac{1}{\tau})} dH \leq \int_{D_\delta} e^{-\|H\|^2/\tau_0} |g(H)| e^{\delta^2(\frac{1}{\tau_0} - \frac{1}{\tau})} dH. \quad \square
$$

**Theorem 4.1.3.** Let $0 < a \leq \infty$, $G$ be a domain in $S^{n-1}$ (unit sphere), $0 < d$,

$$
G_a := \{r\xi : 0 < r < a, \xi \in G\}
$$

and $Q$ a $d$–homogeneous continuous function defined on $\overline{G_a}$. Suppose $f \in C(G_a)$ that is $C^\infty$ in a neighborhood of the origin. Assume that for some $0 < \tau_0$ the function $e^{-\|H\|^2/\tau_0} Q(H) f(H)$ is in $L^1(G_a)$. For $0 < \tau < \tau_0$ let $\Phi(\tau)$ be defined by

$$
\Phi(\tau) = \int_{G_a} e^{-\|H\|^2/\tau} Q(H) f(H) dH.
$$

Then $\Phi$ admits an asymptotic series expansion around 0:

$$
\Phi(\tau) \sim \sum_{j=0}^{\infty} \frac{\Gamma(n+d+1)}{2} \int_G QP_j d\xi \tau^{\frac{n+d+1}{2}}, \quad \tau \to 0,
$$
where $P_j$ is the $j$-th homogeneous polynomial term of the Taylor series of $f$ around the origin.

**Proof.** We follow the scheme of the proof of Watson’s lemma in one variable (cf. [M]). Let $0 < \delta \leq a$ be so small that $f$ is $C^\infty$ in a neighborhood of the ball $\mathbb{B}_\delta(0)$. Then $G_a \cap \mathbb{B}_\delta(0) = G_\delta$ and with $h(\tau, H) = e^{-\|H\|^2/\tau}Q(H)f(H)$

$$\Phi(\tau) = \int_{G_a \cap \{\|H\| \geq \delta\}} h(\tau, H)dH + \int_{G_\delta} h(\tau, H)dH =: \Phi_1(\tau) + \Phi_2(\tau).$$

With $g(H) = Q(H)f(H)$ and $C = \int_{G_a} e^{-\|H\|^2/\omega_0} |Q(H)f(H)|dH$, Proposition 4.1.2 implies

$$|\Phi_1(\tau)| \leq Ce^{\delta^2/\omega_0}e^{-\delta^2/\tau} = o(\tau^n), \quad \tau \to 0,$$

for all $n \in \mathbb{N}$. The Taylor formula with remainder term yields

$$(4.1.1) \quad f(H) = \sum_{j=0}^{N} P_j(H) + f_N(H), \quad \|H\| \leq \delta, \quad |f_N(H)| \leq C_N\|H\|^{N+1},$$

where $P_j$ is a $j$-homogeneous polynomial and $C_N$ an appropriate constant. Thus

$$\Phi_2(\tau) = \sum_{j=0}^{N} \int_{G_\delta} e^{-\|H\|^2/\tau}Q(H)P_j(H)dH + \int_{G_\delta} e^{-\|H\|^2/\tau}Q(H)f_N(H)dH$$

and

$$\int_{G_\delta} e^{-\|H\|^2/\tau}Q(H)P_j(H)dH = \int_{G_\infty} e^{-\|H\|^2/\tau}Q(H)P_j(H)dH - \int_{G_\infty \cap \{\|H\| \geq \delta\}} e^{-\|H\|^2/\tau}Q(H)P_j(H)dH.$$

In light of Proposition 4.1.1 the first integral on the right hand side is equal to

$$\frac{\Gamma(n+d+j)}{2} \left(\int_G QP_j d\xi\right) \tau^{n+d+j/2},$$

and Proposition 4.1.2 yields with $g = QP_j$, that the second integral is $o(\tau^n)$ for all $n \in \mathbb{N}$. Homogeneity of $Q$ implies $|Q(H)| \leq K\|H\|^d$ with some $K > 0$. Then from (4.1.1) and Proposition 4.1.1 we get

$$\left|\int_{G_\delta} e^{-\|H\|^2/\tau}Q(H)f_N(H)dH\right| \leq C_NK \int_{G_\infty} e^{-\|H\|^2/\tau} \|H\|^{N+d+1}dH =$$

$$= Vol(G)C_NK \frac{\Gamma(n+d+N+1)}{2} \tau^{n+d+N+1},$$

finishing the proof of the theorem. □
4.2. Determining $A_\delta$ and $B_\delta$ from $\tau \to 0$.

Let us get back to the symmetric space situation. Suppose $(M^m = U/K, g)$ is a compact, simply connected, irreducible, Riemannian symmetric space as in Sect. 1. As before, $m$ is the dimension of $M$. Let $\delta$ be an irreducible $K$–spherical representation and $f_\delta$ the corresponding spherical function. Then

\begin{equation}
(4.2.1) \quad f_\delta(\exp(2H)) = 1 + R_1(H) + R_2(H) + \ldots, \quad H \in \mathfrak{a}_0,
\end{equation}

where $R_j$ is the $j$–th homogeneous polynomial term of the Taylor series. Since $f_\delta \circ \exp$ is $\text{Ad}_K$ invariant on $\mathfrak{p}_0$ (see [L-Sz4, Proposition 2.1]), it is Weyl group invariant on $\mathfrak{a}_0$. Therefore every $R_j$ is Weyl group invariant as well. Since $M$ is irreducible, the Weyl group acts irreducible on $\mathfrak{a}_0$, thus $R_1 \equiv 0$ and $R_2$ must be of the form

\begin{equation}
(4.2.2) \quad R_2(H) = b_\delta \|H\|^2,
\end{equation}

with some $b_\delta \in \mathbb{R}$. (4.2.2) is true because up to a constant scalar, $\|H\|^2$ is the only Weyl group invariant quadratic polynomial on $\mathfrak{a}_0$. One can see this either as a corollary of Schur’s lemma, or as a corollary of Chevalley’s theorem (see [Hu, Sect. 3.5, 3.7]). For the trivial representation $\delta_0$, $f_{\delta_0} \equiv 1$ and $b_{\delta_0} = 0$.

**Proposition 4.2.1.** Assume that the rank of $M$ is 1 and $\lambda$ is the highest weight of $\delta$. Then

\[ b_\delta = \frac{2(\|\lambda + \rho\|^2 - \|\rho\|^2)}{m}. \]

**Proof.** If $\Sigma$ is nonreduced, $\Sigma^+ = \{\beta, \beta/2\}$ and $\Sigma^+ = \{\beta\}$ in the reduced case. The corresponding multiplicities are $m_\beta$ and $m_{\beta/2}$, where our convention is that the latter is zero when $\Sigma$ is reduced. Let $H_0 \in \mathfrak{a}_+$ with $\|H_0\| = 1$. Then $\beta(H_0) = \|\beta\|$. Recall that Gauss’ hypergeometric functions are given by

\[ F(a, b, c, z) := 1 + \frac{ab}{c} z + \ldots + \frac{a(a+1) \ldots (a+k-1)b(b+1) \ldots (b+k-1)}{k!c(c+1) \ldots (c+k-1)} z^k + \ldots \]

where $a, b, c \in \mathbb{C}$, $c \not\in \mathbb{Z} = \{0, -1, -2, \ldots\}$. The series converges at least in the unit disk. If $n \in \mathbb{Z}_+$, $b = -n$, $a \in \mathbb{C} \setminus \mathbb{Z}_+$, and $a = A + n$, then $F$ is a polynomial (in $z$) of degree $n$.

According to [He2, Theorem 4.1(ii), p. 535 and Sect. 3, p. 542] the highest weight of $\delta$ has the form $\lambda = n_\delta \beta$, where $n_\delta \in \mathbb{Z}_+$. Let

\[ a_\delta := \frac{1}{2} m_{\beta/2} + m_\beta + n_\delta, \quad c_\delta := \frac{m_{\beta/2} + m_\beta + 1}{2} = \frac{m}{2}. \]

Denote by $F_\delta$ the hypergeometric function (polynomial in this case), corresponding to these parameters

\[ F_\delta(x) = F(a_\delta, -n_\delta, c_\delta, x). \]

According to [He2, formula (25), p.543], the spherical function $f_\delta$ can be expressed as

\[ f_\delta(\exp(2H)) = F_\delta(-\text{sh}^2(\beta(H))), \quad H \in \mathfrak{a}_0. \]
Hence
\[ f_\delta(\exp(2H)) = 1 + \frac{a_\delta n_\delta}{c_\delta} \|\beta\|^2 \|H\|^2 + o(\|H\|^2). \]
Thus \( b_\delta = \frac{a_\delta m_\alpha}{c_\alpha} \|\beta\|^2. \) Now \( \rho = \frac{1}{2}(m_\beta/2\beta + m_\beta \beta), \) hence
\[ a_\delta n_\delta \|\beta\|^2 = 2\langle \rho, \lambda \rangle + \|\lambda\|^2, \]
and our statement follows. \( \Box \)

The \( \tau \to 0 \) asymptotics yields the following values for \( A_\delta, B_\delta \) in Theorem 2.1.1 (b).

**Theorem 4.2.2.** Suppose the corrected field of quantum Hilbert spaces \( H^{corr} \to S \) is projectively flat. Then for every irreducible \( K- \)spherical representation \( \delta \),
\[ A_\delta = 1, \quad B_\delta = \frac{m_\alpha}{2} b_\delta. \]

**Proof.** Easy calculation shows that
\[ (4.2.3) \quad F(H) := \prod_{\alpha \in \Sigma^+} \left( \frac{\text{sh}(2\alpha(H))}{\alpha(H)} \right)^{\frac{m_\alpha}{2}} = 1 + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha}{3} \alpha^2(H) + \ldots \]
From (4.2.2) and (4.2.3) we obtain that in the homogeneous polynomial series expansion of
\[ f_\delta(\exp(2H))F(H) = 1 + P_2^\delta(H) + P_3^\delta(H) + \ldots, \]
the quadratic term is
\[ (4.2.4) \quad P_2^\delta(H) = b_\delta \|H\|^2 + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha}{3} \alpha^2(H) = b_\delta \|H\|^2 + P_2^\delta_0(H). \]
Now \( Q(H) := \prod_{\alpha \in \Sigma^+} \alpha(H)^{m_\alpha} \) is a homogeneous polynomial of degree
\[ d = \sum_{\alpha \in \Sigma^+} m_\alpha = m - r, \]
where \( r = \dim a_0 \) is the rank of \( M \). Applying Theorem 4.1.3 with \( f, Q, a = \infty, G = a_+ \cap S^{r-1} \) we obtain
\[ (4.2.5) \quad q_\delta(\tau) = \frac{\Gamma(m/2)}{2} \int_G Q(\xi)d\xi \tau^\frac{m}{2} + \frac{\Gamma(m+2)/2}{2} \int_G Q(\xi)P_2^\delta(\xi)d\xi \tau^\frac{m+2}{2} + o(\tau^\frac{m+2}{2}). \]
Since the restricted roots are positive on the Weyl chamber \( a_+ \), we get \( \int_G Q(\xi)d\xi > 0. \)
Now writing out (4.2.5) for both \( \delta \) and the trivial representation \( \delta_0 \), comparing the coefficients of the \( \tau^\frac{m}{2} \) term in the asymptotic series and using Theorem 2.1.1 (b) we obtain \( A_\delta = 1. \) Then comparing the coefficients of the \( \tau^\frac{m+2}{2} \) as well, we obtain
\[ (4.2.6) \quad B_\delta \frac{\Gamma(m/2)}{2} \int_G Q(\xi)d\xi = \frac{\Gamma(m+2)/2}{2} \int_G Q(\xi)(P_2^\delta(\xi) - P_2^\delta_0(\xi))d\xi. \]
From (4.2.4) we get \( P_2^\delta(\xi) - P_2^\delta_0(\xi) = b_\delta \|\xi\|^2 = b_\delta, \) since \( G \) is part of the unit sphere. Thus (4.2.6) yields \( B_\delta = \frac{m_\alpha}{2} b_\delta. \) \( \Box \)
5. Asymptotics at infinity

The following setting is motivated by the system of restricted roots of a compact Riemannian symmetric space.

Let \((Z, \langle , \rangle)\) be a Euclidean space of dimension \(r\) and \(\Sigma^+ \subset Z^*\) a finite set so that

\[
Z_+ := \{ H \in Z \mid \alpha(H) > 0, \forall \alpha \in \Sigma^+ \}
\]

is nonempty. For each \(\alpha \in \Sigma^+\), let \(m_\alpha > 0\) be given and define

\[
m := r + \sum_{\alpha \in \Sigma^+} m_\alpha, \quad \rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha.
\]

For a linear functional \(l : Z \to \mathbb{R}\), define \(A_l \in Z\) by

\[
l(H) = \langle A_l, H \rangle, \quad H \in Z.
\]

Then \(\langle l, L \rangle := \langle A_l, A_L \rangle, l, L \in Z^*\), defines an inner product on \(Z^*\). Let \(f : Z_+ \to \mathbb{R}\) be any measurable function. Assuming the integral below is finite, introduce the following function, defined for \(\tau > 0\).

\[
q(\tau, f) = \int_{Z_+} e^{-\frac{\|H\|^2}{\tau}} f(H) \prod_{\alpha \in \Sigma^+} (\alpha(H) \text{sh}(2\alpha(H)))^{m_\alpha} dH.
\]

With \(\mu \in Z^*\), let \(I_\mu(\tau) := q(\tau, e^{2\mu})\). Even though it is impossible to calculate precisely this integral (except in some special cases), it is possible to determine the order of its magnitude as \(\tau \to \infty\), and that suffices for our purposes.

**Theorem 5.1.** For any \(\mu \in Z^*\)

\[
I_\mu(\tau) = \begin{cases} 
2^{r-m} \pi^{\frac{r}{2}} \prod_{\alpha \in \Sigma^+} \langle \mu + \rho, \alpha \rangle^{m_\alpha} \tau^{\frac{m}{2}} e^\tau \|\mu + \rho\|^2 (1 + o(1)), & A_{\mu + \rho} \in Z_+ \\
\tau^{m} e^\tau \|\mu + \rho\|^2 o(1), & A_{\mu + \rho} \in Z \setminus Z_+ 
\end{cases}
\]

as \(\tau \to \infty\).

**Proof.** Factoring out \(e^{m_\alpha \alpha(H)}\) from the product for each \(\alpha \in \Sigma^+\), we get

\[
I_\mu(\tau) = 2^{r-m} \int_{Z_+} e^{-\frac{\|H\|^2}{\tau} + 2\mu(H) + 2\rho(H)} \prod_{\alpha \in \Sigma^+} \alpha(H)^{m_\alpha} (1 - e^{-4\alpha(H)})^{m_\alpha} dH
\]

Now

\[-\|H\|^2/\tau + 2\mu(H) + 2\rho(H) = -\|H/\sqrt{\tau} - \sqrt{\tau} A_{\mu + \rho}\|^2 + \tau \|\mu + \rho\|^2.\]

Thus

\[
I_\mu(\tau) = \frac{e^{\|\mu + \rho\|^2}}{2m - r} \int_{Z_+} e^{-\|H/\sqrt{\tau} - \sqrt{\tau} A_{\mu + \rho}\|^2} \prod_{\alpha \in \Sigma^+} \alpha(H)^{m_\alpha} (1 - e^{-4\alpha(H)})^{m_\alpha} dH
\]

Let \(\Phi_\tau(Y)\) be the affine linear change of coordinates in \(Z\) defined by

\[
\Phi_\tau(Y) := \sqrt{\tau} Y + \tau A_{\mu + \rho}.
\]
Then \( \det \Phi' = \tau^2 \) and with \( H = \Phi(Y) \),

\[
\alpha(H) = \alpha(\sqrt{\tau}Y + \tau A_{\mu+\rho}) = \tau \alpha(Y/\sqrt{\tau} + A_{\mu+\rho}).
\]

Using the coordinate change \( \Phi \) the integral \( I_\mu \) is transformed to

\[
I_\mu(\tau) = \frac{\tau^m}{2^{m-r}} e^{\tau \|\mu+\rho\|^2} \int_{\Phi^{-1}(Z_+)} e^{-\|Y\|^2} \prod_{\alpha \in \Sigma^+} \alpha(\Phi(Y)/\tau)^{\frac{m_\alpha}{\tau}} \left( 1 - e^{-4\alpha(\Phi(Y))} \right)^{\frac{m_\alpha}{\tau}} dY.
\]

Let \( \chi(Y) \) be the characteristic function of the set \( \Phi^{-1}(Z_+) \) and let

\[
g(Y) := \chi(Y) \prod_{\alpha \in \Sigma^+} \alpha(Y/\sqrt{\tau} + A_{\mu+\rho}) \frac{m_\alpha}{\tau} \left( 1 - e^{-4\alpha(\Phi(Y))} \right)^{\frac{m_\alpha}{\tau}}
\]

that is now defined on the entire space \( Z \) and

\[
I_\mu(\tau) = \frac{\tau^m}{2^{m-r}} e^{\tau \|\mu+\rho\|^2} \int_Z e^{-\|Y\|^2} g(Y) dY.
\]

We want to show that the integral here has a limit as \( \tau \to \infty \). First we prove this for the function \( g(Y) \).

**Claim.** For all \( Y \in Z \)

\[
\lim_{\tau \to \infty} g(Y) = \begin{cases} 
\prod_{\alpha \in \Sigma^+} \langle \mu + \rho, \alpha \rangle^{\frac{m_\alpha}{\tau}}, & A_{\mu+\rho} \in Z_+ \\
0, & A_{\mu+\rho} \in Z \setminus Z_+
\end{cases}
\]

**Proof of the Claim.** First let \( A_{\mu+\rho} \in Z_+ \). Then \( \alpha(A_{\mu+\rho}) > 0 \), for all \( \Sigma^+ \). Let \( Y \in Z \) be arbitrary. Then with an appropriate \( \tau_0 \), \( \alpha(\sqrt{\tau}Y + \tau A_{\mu+\rho}) > 0 \) holds for every \( \tau \geq \tau_0 \). Thus \( Y \in \Phi^{-1}(Z_+) \) and so \( \chi(Y) = 1 \) for \( \tau \geq \tau_0 \). Also

\[
\lim_{\tau \to \infty} \alpha(Y/\sqrt{\tau} + A_{\mu+\rho}) = \alpha(A_{\mu+\rho}) = \langle \alpha, \mu + \rho \rangle > 0
\]

and hence \( \lim_{\tau \to \infty} \alpha(\Phi(\tau)) = \infty \). All these together prove our claim in this case.

Now let \( A_{\mu+\rho} \in Z \setminus Z_+ \). Suppose there is an \( \alpha \in \Sigma^+ \) with \( \alpha(A_{\mu+\rho}) < 0 \). Then for all \( Y \in Z \) there exists some \( \tau_0 > 0 \) so that for every \( \tau \geq \tau_0 \), \( \alpha(\sqrt{\tau}Y + \tau A_{\mu+\rho}) < 0 \) and consequently \( Y \notin \Phi^{-1}(Z_+) \) implying \( \chi(Y) = 0 = g(Y) \).

Now assume there is at least one \( \alpha \in \Sigma^+ \) with \( \alpha(A_{\mu+\rho}) = 0 \) and \( \beta(A_{\mu+\rho}) \geq 0 \) for all \( \beta \in \Sigma^+ \). Denote by \( \Sigma_{+0} \) those \( \beta \in \Sigma^+ \), for which \( \beta(A_{\mu+\rho}) = 0 \).

Let \( Y \in Z \). If there exists a \( \beta \in \Sigma_{+0} \) with \( \beta(Y) \leq 0 \), then \( \beta(\sqrt{\tau}Y + \tau A_{\mu+\rho}) \leq 0 \) and so \( \chi(Y) = 0 = g(Y) \) for all \( \tau > 0 \).

Suppose that for all \( \beta \in \Sigma_{+0}, \beta(Y) > 0 \). Then for all \( \tau > 0 \) and \( \beta \in \Sigma_{+0}, \beta(\sqrt{\tau}Y) = \beta(\Phi(\tau)) > 0 \) and so \( 0 < 1 - e^{-4\beta(\Phi(Y))} < 1 \). Also just as before: with an appropriate \( \tau_0 \), \( \beta(\sqrt{\tau}Y + \tau A_{\mu+\rho}) > 0 \) holds for every \( \tau \geq \tau_0 \) and \( \beta \in \Sigma^+ \setminus \Sigma_{+0} \). Thus for all \( \tau \geq \tau_0, \Phi(Y) \in Z_+ \) hence

\[
\chi(Y) = 1 \quad \text{and} \quad 0 < \prod_{\alpha \in \Sigma^+} (1 - e^{-4\alpha(\Phi(Y))})^{\frac{m_\alpha}{\tau}} < 1.
\]
But
\[
\lim_{\tau \to \infty} \prod_{\alpha \in \Sigma^+} (\alpha(Y/\sqrt{\tau} + A_{\mu + \rho}))^{\frac{m_\alpha}{2}} = \prod_{\alpha \in \Sigma^+} (\alpha(A_{\mu + \rho}))^{\frac{m_\alpha}{2}} = 0,
\]
proving that \( \lim_{\tau \to \infty} g_\tau(Y) = 0. \)

Now to finish the proof of the theorem we estimate \( g_\tau(Y) \). By its definition \( g_\tau(Y) \) vanishes outside of the set \( \Phi^{-1}(Z_+) \).

Hence the trivial estimate yields
\[
|g_\tau(Y)| \leq \prod_{\alpha \in \Sigma^+} \|\alpha\|^{\frac{m_\alpha}{2}} (\|Y\| + \|A_{\mu + \rho}\|)^{\frac{m_\alpha}{2}} =: C.
\]
Valid for all \( Y \in Z \) and \( \tau \geq 1 \). Thus \( Ce^{-\|Y\|^2/2} \) is an integrable majorant of \( g_\tau(Y) \) for all \( \tau \geq 1 \). Using Lebesgue’s dominated convergence theorem together with our claim and the fact that \( \int_Z e^{-\|Y\|^2/2} dY = \pi^{\frac{r}{2}} \) finishes the proof of the theorem.

Back to symmetric spaces again, let \( (M^m = U/K, g) \) be a compact, irreducible, simply connected, Riemannian symmetric space, \( \delta \) an irreducible unitary \( K \)-spherical representation of \( U \) with highest weight \( \lambda \). \( c \) denotes Harish-Chandra’s \( c \)-function associated to the dual symmetric space \( X = G/K \) and \( q_\delta \) is from (2.2.6).

**Theorem 5.2.**

\[
q_\delta(\tau) = 2^{r-m} \pi^{\frac{r}{2}} c(-i\lambda - i\rho) \left( \prod_{\alpha \in \Sigma^+} \langle \lambda + \rho, \alpha \rangle^{\frac{m_\alpha}{2}} \right) \tau^{\frac{r}{2}} e^{\tau \|\lambda + \rho\|^2} (1 + o(1)),
\]

as \( \tau \to \infty \).

**Proof.** It follows from the Cartan-Helgason theorem ([He2, Theorem 4.1, p.535]), that \( A_\lambda \in \mathfrak{a}_+ \). But then Proposition 3.2 implies with \( l = \lambda + \rho \), that \( A_l \in \mathfrak{a}_+ \). Thus if \( \mu \) is a weight of \( \delta \), different from \( \lambda \), Proposition 3.1 and Theorem 5.1 (with \( Z = a_0 \) and \( \Sigma^+ \) the set of positive restricted roots) yields \( I_\mu(\tau) = I_\lambda(\tau) o(1) \), as \( \tau \to \infty \). Now using (3.1) for the spherical function corresponding to \( \delta \) we get

\[
q_\delta(\tau) = \sum_{\mu \in \Lambda(\delta)} \langle w_\mu, w_\mu \rangle I_\mu(\tau)
\]

The discussion above implies, that \( I_\lambda(\tau) \) dominates all the other terms in (5.2). Therefore (3.2) and Theorem 5.1 finish the proof.

Since \( c(-i\rho) = 1 \), Theorem 2.1.1 and Theorem 5.2 yield Theorem 5.3.

**Theorem 5.3.** If the corrected field of quantum Hilbert spaces \( H^\text{corr} \to S \) is projectively flat, then for every irreducible \( K \)-spherical representation \( \delta \) with highest weight \( \lambda \),

\[
A_\delta = \frac{c(-i\lambda - i\rho) \prod_{\alpha \in \Sigma^+} \langle \lambda + \rho, \alpha \rangle^{\frac{m_\alpha}{2}}}{\prod_{\alpha \in \Sigma^+} \langle \rho, \alpha \rangle^{\frac{m_\alpha}{2}}}.
\]

(5.3)
and

\[ (5.4) \quad B_\delta = \|\lambda + \rho\|^2 - \|\rho\|^2. \]

Denote by \( \Sigma_0 \) the set of indivisible restricted roots, i.e. those \( \alpha \in \Sigma \), for which \( c_\alpha \in \Sigma \) implies \( c = \pm 1, \pm 2 \). Let \( \Sigma_0^+ = \Sigma_0 \cap \Sigma^+ \). As before, for an \( \alpha \in \Sigma \) we take \( m_{2\alpha} = 0 \) if \( 2\alpha \not\in \Sigma \) and \( \alpha_0 = \alpha / \langle \alpha, \alpha \rangle \). Now combining Theorem 4.2.1 with Theorem 5.3 we get.

**Theorem 5.4.** Assume the corrected field of quantum Hilbert spaces \( H^{\text{corr}} \to S \) is projectively flat. Let \( \delta \) be an irreducible \( K \)-spherical representation with highest weight \( \lambda \). Then \( A_\delta \) must be equal to \( 1 \), hence the quantity

\[ (5.5) \quad c(-i\lambda - i\rho) \prod_{\alpha \in \Sigma_0^+} \langle \lambda + \rho, \alpha_0 \rangle^{m_{\alpha} + m_{2\alpha}} \]

is independent of \( \delta \) and

\[ (5.6) \quad \|\lambda + \rho\|^2 - \|\rho\|^2 = \frac{m}{2} b_\delta, \]

where \( b_\delta \) is from (4.2.2).

**Remarks.** 1) Proposition 4.2.1 shows that when \( M = U/K \) is any compact, irreducible, simply connected Riemannian symmetric space of rank-1, (5-6) holds for every irreducible \( K \)-spherical representation of \( U \). Thus the constants \( B_\delta \) from Theorem 2.1.1 (b) do not help in deciding whether the field \( H^{\text{corr}} \to S \) is projectively flat or not. It is not clear whether (5-6) should always hold for the higher rank symmetric spaces as well, regardless of projective flatness.

2) If \( M \) is isometric to a compact Lie group \( U \) equipped with a biinvariant metric, we know from [L-Sz3, Theorem 11.3.1] that \( H^{\text{corr}} \to S \) is flat. Also it is well known in this case, that for all \( \alpha \in \Sigma \), \( m_\alpha = 2 \) and \( m_{2\alpha} = 0 \) (i.e. \( \Sigma \) is reduced). Now with

\[ \pi(\nu) := \prod_{\alpha \in \Sigma^+} \langle \nu, \alpha \rangle, \quad \nu \in a_0^*, \]

we have

\[ c(\nu) = \frac{\pi(\rho)}{\pi(i\nu)} \]

(see [He2, p. 447.]) and the quantity in (5.5) is equal to \( \pi(\rho) \), indeed independent of \( \delta \).

Next we express condition (5.5) purely in terms of the root system \( \Sigma \) and its multiplicities.

**Theorem 5.5.** Let \( \delta \) be an irreducible \( K \)-spherical representation with highest weight \( \lambda \). Suppose the corrected field of quantum Hilbert spaces \( H^{\text{corr}} \to S \) is projectively flat. Then the quantity

\[ (5.7) \quad Q(\delta) := \prod_{\alpha \in \Sigma_0^+} \frac{\Gamma(\frac{1}{4} m_\alpha + \frac{1}{2} \langle \lambda + \rho, \alpha_0 \rangle) \Gamma(\langle \lambda + \rho, \alpha_0 \rangle)}{\Gamma(\frac{1}{2} m_\alpha + \langle \lambda + \rho, \alpha_0 \rangle) \Gamma(\frac{1}{4} m_\alpha + \frac{1}{2} m_{2\alpha} + \frac{1}{2} \langle \lambda + \rho, \alpha_0 \rangle)} \]

is independent of \( \delta \).
is independent of $\delta$.

If $m_{2\alpha} = 0$ and $m_\alpha = 2$ for all $\alpha \in \Sigma_0^+$, then it is obvious that $Q(\delta)$ is in fact independent of $\delta$. This is the group manifold case.

**Proof.** The Gindikin-Karpelevič formula expresses Harish-Chandra’s $c-$function as a meromorphic function on $a_+^\ast$ (see [He2, p.447]),

\begin{equation}
(5.8) \quad c(\nu) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{(-i\nu, \alpha_0)} \Gamma(\langle i\nu, \alpha_0 \rangle)}{\Gamma\left(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\nu, \alpha_0 \rangle)\right) \Gamma\left(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\nu, \alpha_0 \rangle)\right)}.
\end{equation}

Here the constant $c_0$ is determined by $c(-i\rho) = 1$. Using the duplication formula

\begin{equation}
\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right),
\end{equation}

from (5.8) we get

\begin{equation}
(5.9) \quad c(-i\lambda - i\rho) = c_1 \prod_{\alpha \in \Sigma_0^+} \frac{\Gamma\left(\frac{1}{2}(\frac{1}{2}m_\alpha + \langle \lambda + \rho, \alpha_0 \rangle)\right) \Gamma(\langle \lambda + \rho, \alpha_0 \rangle)}{\Gamma\left(\frac{1}{2}m_\alpha + \langle \lambda + \rho, \alpha_0 \rangle\right) \Gamma\left(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle \lambda + \rho, \alpha_0 \rangle)\right)},
\end{equation}

where

\[ c_1 = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{m_\alpha/2}}{2\sqrt{\pi}}. \]

From (5.5) and (5.9) we see (since $(2\alpha_0) = \alpha_0/2$), that the quantity in (5.5) does not depend on $\delta$ iff $Q(\delta)$ is independent of $\delta$. $\Box$

6. Root systems and the proof of Theorem 0.1

6.1. $\Gamma$-related functions.

Here we take a closer look at the functions appearing in (5.7) to find out which compact symmetric spaces have the property that $Q(\delta)$ (from (5.7)) is independent of $\delta$.

Let $0 < a_j, 0 \leq b_j, c_j, d_j$, $j = 1, \ldots, N$ and $G(z) := \prod_{j=1}^N F(z, a_j, b_j, c_j, d_j)$.
Proposition 6.1.2. Assume that for some $s$, $\frac{a_s}{c_s} < \frac{a_j}{c_j}$, for all $j \neq s$ and there exists a constant $D \neq 0$ with $G(n) = D$ for all $n \in \mathbb{Z}_+$. Then $2b_s + d_s = 1$.

Proof. After renumbering we can assume that $s = 1$. From Proposition 6.1.1 we know that $G$ is a bounded holomorphic function in a neighborhood of $\overline{D}$. In light of Carlson’s theorem ([T., p.186]), our assumptions imply that $G \equiv D$ and so

$$ (2c_1z + 2a_1)^{2b_1 + d_1} \prod_{j=2}^{N} (2c_jz + 2a_j)^{2b_j + d_j} $$

$$ \equiv D \prod_{j=1}^{N} \frac{\Gamma(2c_jz + 2a_j + 2b_j)\Gamma(c_jz + a_j + b_j + d_j)}{\Gamma(c_jz + a_j + b_j)\Gamma(2c_jz + 2a_j)}. $$

Since $a_1/c_1 < a_j/c_j$, $1 < j$ and because $\Gamma$ is zero free and holomorphic in $\mathbb{C} \setminus \{0,-1,-2,\ldots\}$ and has first order poles in the nonpositive integers, the right hand side is holomorphic in a neighborhood $U$ of $\{\text{Re}z \geq -\frac{a_1}{c_1}\}$ and has a simple zero at $-\frac{a_1}{c_1}$. Furthermore $\prod_{j=2}^{N} (2c_jz + 2a_j)^{2b_j + d_j}$ is holomorphic and zero free in $U$. Hence $(2c_1z + 2a_1)^{2b_1 + d_1}$ should extend holomorphically to a neighborhood of $z_0 := -\frac{a_1}{c_1}$, with a first order zero at $z_0$. But this happens iff $2b_1 + d_1 = 1$. \square

6.2 Root systems.

Let $(Z,\langle,\rangle)$ be an $r$–dimensional Euclidean space. For $0 \neq \alpha \in Z$ let $\alpha_0 = \alpha/\langle \alpha,\alpha \rangle$.

Let $R \subset Z$ be a (possible nonreduced) root system. Choose a basis $\alpha_1, \ldots, \alpha_r$ of $R$ and let $R^+$ be the set of positive roots, $\mathbb{Z}_+: = \{0,1,2,\ldots\}$.

$$(6.2.1) \quad P_+ := \{\gamma \in Z : \langle \gamma,\alpha_0 \rangle \in \mathbb{Z}_+, \forall \alpha \in R^+\}.$$  

According to the Cartan-Helgason theorem ([He2, Theorem 4.1, p.535, Corollary 4.2, p.538]), when $Z = a_0^*$ and $R = \Sigma$ the set of restricted roots of a compact, simply connected Riemannian symmetric space $M = U/K$, the highest weights of the irreducible $K$-spherical representations of $U$ are precisely the elements of $P_+$.

A multiplicity function on $R$ is a map $m : R \to \mathbb{R}$, denoted by $\alpha \mapsto m_\alpha$ such that $m_\omega = m_\alpha$ for every Weyl group element $\omega$. Let $\rho := \frac{1}{2} \sum_{\alpha \in R^+_+} m_\alpha \alpha$. Denote by $R_0$ the set of indivisible roots and $R_0^+ = R^+ \cap R_0$. Inspired by the formula (5.7) for $Q(\delta)$, we define the analogous function for $\mu \in P_+$ as follows.

$$(6.2.2) \quad Q(\mu) := \prod_{\alpha \in R_0^+} \frac{\Gamma(\frac{1}{2}m_\alpha + \frac{1}{2}(\mu + \rho,\alpha_0))\Gamma(\langle \mu + \rho,\alpha_0 \rangle)\langle \mu + \rho,\alpha_0 \rangle \overline{m_\alpha + m_\alpha}}{\Gamma(\frac{1}{2}m_\alpha + \langle \mu + \rho,\alpha_0 \rangle)\Gamma(\frac{1}{2}m_\alpha + \frac{1}{2}m_\alpha + \frac{1}{2}(\mu + \rho,\alpha_0))}.$$  

(6.2.5) shows that this is a well defined quantity when all multiplicities are positive. Denote by $R_*$ the set of unmultipliable roots in $R$. A basis $\beta_1, \ldots, \beta_r$ of $R_*$ can be obtained by taking $\beta_j = \alpha_j$ if $2\alpha_j \not\in R$ and $\beta_j = 2\alpha_j$ if $2\alpha_j \in R$. Define $\mu_j \in Z$, $j = 1,\ldots,r$ by

$$(6.2.3) \quad \langle \mu_j,\beta_k,0 \rangle = \delta_{jk}, \quad j,k = 1,\ldots,r.$$  

Then

$$(6.2.4) \quad \mu \in P_+ \text{ if and only if } \mu = \sum_{j=1}^{r} n_j \mu_j \text{ with } n_j \in \mathbb{Z}_+.$$  

([He3, Proposition 4.23, p.150]).
Proposition 6.2.1. Suppose that $0 < m_\alpha$ for all $\alpha \in R$. Then

\begin{equation}
0 < \langle \rho, \alpha \rangle \quad \text{and} \quad 0 \leq \langle \mu, \alpha \rangle \quad \forall \alpha \in R^+, \forall \mu \in P_+.
\end{equation}

For a fixed $1 \leq j \leq r$, let $R_j^+ := \{ \alpha \in R_0^+ : 0 < \langle \mu_j, \alpha_0 \rangle \}$. Then

\begin{equation}
\frac{\langle \rho, \alpha_j, 0 \rangle}{\langle \mu_j, \alpha_j, 0 \rangle} < \frac{\langle \rho, \alpha_0 \rangle}{\langle \mu_j, \alpha_0 \rangle}, \quad \forall \alpha \in R_j^+, \quad \alpha \neq \alpha_j.
\end{equation}

Proof. The proof of Proposition 3.2 also works here, showing the first part of (6.2.5). The second part follows from (6.2.3) and (6.2.4). If $\alpha_j \neq \alpha \in R_j^+$, then $\alpha = \sum_1^r n_s \alpha_s$ with $n_s \in \mathbb{Z}_+$. From (6.2.3) we have $0 < \langle \mu_j, \alpha_j \rangle$ and

\begin{equation}
0 < \langle \mu_j, \alpha \rangle = n_j \langle \mu_j, \alpha_j \rangle.
\end{equation}

Hence $0 < n_j$. Since $\alpha$ is indivisible and is different from $\alpha_j$, there must be at least one more $s$ with $0 < n_s$. (6.2.5) then implies

\begin{equation}
\langle \rho, n_j \alpha_j \rangle < \langle \rho, \alpha \rangle.
\end{equation}

Now in light of (6.2.7), if we divide (6.2.8) by $n_j \langle \mu_j, \alpha_j \rangle$ we get (6.2.6). \Box

We call a multiplicity function $m : R \rightarrow \mathbb{R}$ geometric if it takes only positive integer values and satisfies the following property: if $\alpha \in R$ and $m_\alpha$ is odd, then $2\alpha \notin R$. For $\alpha \in R$ we use the convention as before: $m_\alpha = 0$ if $2\alpha$ is not a root. If $R = \Sigma$, a restricted root system of a compact, Riemannian symmetric space, its multiplicity function is geometric in this sense, see [Ar, Proposition 2.3] or [He1, p.530, 4F].

Theorem 6.2.2. Let $R$ be an irreducible root system with a geometric multiplicity function $m$. Suppose $Q(\mu)$, $\mu \in P_+$ is independent of $\mu$ ($Q(\mu)$ is from (6.2.2)). Then $R$ is reduced and for all $\alpha \in R$, $m_\alpha = 2$.

Proof. Let $\beta_j \in R$, $\mu_j \in Z$ as in (6.2.3) and fix a $j$ with $1 \leq j \leq r$. From (6.2.3) we have $n_j \mu_j \in P_+$ for all $n \in \mathbb{Z}_+$. Now let

$$H_j(z) := \prod_{\alpha \in R_0^+} F\left( z, \frac{\langle \rho, \alpha_0 \rangle}{2}, \frac{m_\alpha}{4}, \frac{\langle \mu_j, \alpha_0 \rangle}{2}, \frac{m_2\alpha}{2} \right),$$

where $F$ is from (6.1.1). Then from (6.2.2) we get

$$Q(n \mu_j) = H_j(n), \quad \forall n \in \mathbb{Z}_+.$$  

By our assumption on $Q$, $H_j(n)$ will be independent of $n$. For any values of the parameters $a, b, d$, the function $F(z, a, b, 0, d)$ from (6.1.1) is always a nonzero constant. Thus if we leave out from the definition of $H_j$ all those terms that correspond to a root $\alpha \in R_0^+$ with $\langle \mu_j, \alpha_0 \rangle = 0$, the result is still a function that is a nonzero
constant on the nonnegative integers. Let \( R_j^+ := \{ \alpha \in R_0^+ : \langle \mu_j, \alpha_0 \rangle > 0 \} \) be as in Proposition 6.2.1 and

\[
G_j(z) := \prod_{\alpha \in R_j^+} F\left(z, \frac{\langle \rho, \alpha_0 \rangle}{2}, \frac{m_\alpha}{4}, \frac{\langle \mu_j, \alpha_0 \rangle}{2}, \frac{m_{2\alpha}}{2}\right).
\]

Then we still have that \( G_j(n) \) is a nonzero constant when \( n \in \mathbb{Z}_+ \). This together with (6.2.6) and Proposition 6.1.2 implies

\[
(6.2.9) \quad m_{\alpha_j} + m_{2\alpha_j} = 2.
\]

Since \( m \) is a geometric multiplicity function, (6.2.9) yields \( m_{\alpha_j} = 2 \) and \( m_{2\alpha_j} = 0 \). Thus \( 2\alpha_j \) is not a root. Since \( R_0, R \) and \( m \) are Weyl group invariant, this yields that \( R \) is reduced and \( m \equiv 2 \). □

**Proof of Theorem 0.1.** If \((M, g)\) is an irreducible, simply connected, compact, Riemannian symmetric space, the set of restricted roots \( \Sigma \) in \( \mathfrak{a}_0^\ast \) forms an irreducible root system with a geometric multiplicity function. In light of Theorem 5.5 and Theorem 6.2.2, projective flatness of \( H_{corr} \to S \) implies \( \Sigma \) is reduced and all the multiplicities are equal to 2. But these conditions characterize compact Lie groups among compact Riemannian symmetric spaces ([L, Theorem 4.4, p.82]). □

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