ESTIMATE ANALYSIS OF A FULLY DISCRETE MIXED FINITE ELEMENT SCHEME FOR STOCHASTIC INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH MULTIPlicative NOISE

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Abstract. This paper is concerned with stochastic incompressible Navier-Stokes equations with multiplicative noise in two dimensions with respect to periodic boundary conditions. Based on the Helmholtz decomposition of the multiplicative noise, semi-discrete and fully discrete time-stepping algorithms are proposed. The convergence rates for mixed finite element methods based time-space approximation with respect to convergence in probability for the velocity and the pressure are obtained. By using the negative norm technique, the $H^1$ and $L^2$ convergence of this scheme for the velocity is derived.

Key words. Stochastic Navier-Stokes equations, multiplicative noise, Wiener process, Itô stochastic integral, mixed finite element methods, stability, error estimates

AMS subject classifications. 65N12, 65N15, 65N30,

1. Introduction. In this paper, we consider the following time-dependent stochastic incompressible Navier-Stokes equations:

\begin{align}
(1.1a) & \quad du = [\nu \Delta u - (u \nabla) u - \nabla p] dt + G(u) dW & \text{a.s. in } D_T := (0, T) \times D, \\
(1.1b) & \quad \text{div } u = 0 & \text{a.s. in } D_T, \\
(1.1c) & \quad u(0) = u_0 & \text{a.s. in } D,
\end{align}

where $T > 0$ denotes time, $\nu > 0$ is the viscosity of the fluid, $u$ and $p$ denote respectively the velocity and the pressure of the problem (1.1) which are spatially periodic with period $L > 0$, and $D = (0, L)^2 \subset \mathbb{R}^2$ is a period of the periodic domain with boundary $\partial D$ and $u_0$ denotes a given initial datum. Here we assume that $\{W(t); t \geq 0\}$ is an $[L^2(D)]^2$-valued $Q$-Wiener process.

The stochastic system (1.1) can take into account in a periodic domain, noise term in the sense of physical or numerical uncertainties and thermodynamical fluctuations. In [2], Bensoussan and Temam started to study the stochastic Navier-Stokes from mathematical investigation. The paper [16] by Flandoli and Gatarek developed a fully stochastic theory to prove the existence of a martingale solution. This paper [20] investigated the ergodic properties for the stochastic Navier-Stokes equations with degenerate noise. In the last twenty years, there is a large amount of literature about the analysis of problem (1.1). We refer to [1, 19, 3, 4, 10, 12, 14, 15] and the references therein for detailed discussions of the stochastic incompressible Navier-Stokes equations.

The paper [7] by Brzeźniak, et al. proposed two fully discrete finite element schemes for the stochastic Navier-Stokes equations with multiplicative noise, with using the compactness argument, the authors analyzed the convergence for the velocity field to weak martingale solutions in 3D and to strong solutions in 2D. In [10], Carelli and Prohl studied implicit and semi-implicit fully schemes for the stochastic Navier-Stokes problem. The result in [10] is convergence of rate (almost) $\frac{1}{4}$ in time and linear convergence in space for the velocity. However, the convergence of the pressure was not given. In work [3], the authors proposed an iterative splitting scheme for stochastic
Navier-Stokes equations and established a strong convergence in probability in the 2D case. In [4], the authors studied another time-stepping semi-discrete scheme and derived strong $L^2$ convergence for the velocity. In [17], Hausenblas and Randrianasolo proposed a time semi-discrete scheme of stochastic 2D Navier-Stokes equations with additive noise by using penalty-projection method. As noted in [17], the result is convergence of rate (almost) $\frac{1}{2}$ in time for the velocity and the pressure. In paper [14], Feng and Qiu developed a fully discrete mixed finite element scheme of the time-dependent stochastic Stokes equations with multiplicative noise and established strong convergence with rates not only for the velocity but also for the pressure. The paper [15], by Feng, et al. proposed a new fully discrete mixed finite element scheme of the time-dependent stochastic Stokes equations with multiplicative noise and obtained optimal strong convergence with rates for both the velocity and the pressure. In a very recent paper [3], Breit and Dodgson considered a fully discrete time-space finite element scheme and strong convergence with rates was proved for the velocity. The result in [5] is convergence of rate (almost) $\frac{1}{2}$ in time and linear convergence in space. The error estimate of the velocity field $u$ and its time-space numerical solution $u^h_k$ reads as: assume that $Lk \leq (-\epsilon \log h)^{-1}$ for some $L > 0$, then for any $\alpha < \frac{1}{2}$

\[
E\left[\Omega_{k,h}^2 \left( \max_{1 \leq n \leq N} \|u(t_n) - u^h_k\|^2_{L^2_x} + k \sum_{n=1}^N \|\nabla(u(t_n) - u^h_k)|^2_{L^2_x}\right) \right] \leq C(k^{2\alpha - \epsilon} + h^2),
\]

where $\Omega_{k,h}^{1} \subset \Omega$ with $P(\Omega \setminus \Omega_{k,h}^{1}) \to 0$ as $k, h \to 0$.

The primary goal of this paper is twofold. Our first goal is to develop an optimally convergent fully discrete finite element scheme with inf-sup stability. Our main idea, which is partly used from references [14] [15], is to apply the Helmholtz decomposition for the multiplicative noise at each step, and then solve velocity and pressure. We propose new semi-discrete and fully discrete time-stepping algorithms for problem [11] and prove the convergence of the velocity and the pressure for the fully discrete scheme for the stochastic Navier-Stokes equations. The second goal is to prove $L^2$ convergence of the scheme by using the negative norm technique. To the best of our knowledge, it is the first time that $L^2_x - H^1_x/L^2_x$ convergence of the discrete solution to a fully discrete system of the stochastic Navier-Stokes equations has been established. The highlight of this party (see section 4) is to derive the error estimates for the numerical solution $(u^h_k, p^h_k)$ as follows: for any $\alpha < \frac{1}{2}$

\[
E\left[1_{\Omega_{k,h}^{1}} \Omega_{k,h}^{2} \Omega_{k,h}^{3} \Omega_{k,h}^{4} \left( \|\nabla(u(t_n) - u^h_k)|^2_{L^2_x}\right) \right] \leq C(k^{2\alpha - \epsilon} + h^{2-2\epsilon}),
\]

\[
E\left[1_{\Omega_{k,h}^{1}} \Omega_{k,h}^{2} \left( \int_0^{t_m} p(s) \, ds - k \sum_{n=1}^m p^h_k \right)^2_{L^2_x}\right] \leq C(k^{2\alpha - \epsilon} + h^{2-2\epsilon}),
\]

\[
E\left[1_{\Omega_{k,h}^{1}} \Omega_{k,h}^{2} \Omega_{k,h}^{3} \Omega_{k,h}^{4} \left( \|u(t_n) - u^h_k|^2_{L^2_x}\right) \right] \leq C(\kappa_0, \kappa_1)(k^{2\alpha - 3\epsilon} + h^{4-7\epsilon}).
\]

The remainder of this paper is organized as follows. In Section 2 we introduce some function and space notation for problem [1.1] and obtain a few preliminary results. In Section 3 we establish the semi-discrete scheme for problem [1.1] and derive some optimal error estimates for both the velocity and pressure approximations. Section 4 proves some optimal error estimates for the fully discrete scheme for problem [1.1] by using the negative norm technique.
2. Preliminaries.

2.1. Notation and assumptions. Standard function and space notation will be adopted in this paper. Let $H^m(D)$ ($m \geq 0$) denote the standard Sobolev space, and $\| \cdot \|_{H^m}$ denotes its norm. Let $H^m_{per}(D)$ be the subspace of $H^m(D)$. Let $(\cdot, \cdot)_{D}$ denote the standard $L^2$-inner product. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a stochastic basis with a complete right continuous filtration. For a given random variable $v$ defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, let $\mathbb{E}[v]$ denote the expected value of $v$. Let $X$ denote a normed vector space $X$ with norm $\| \cdot \|_X$. Define the following Bochner space

$$L^p(\Omega, X) := \{ v : \Omega \to X; \mathbb{E}[\|v\|_X^p] < \infty \}$$

and the norm

$$\|v\|_{L^p(\Omega, X)} := \left( \mathbb{E}[\|v\|_X^p] \right)^{\frac{1}{p}}, \quad 1 < p < \infty.$$ 

We also define some special space notation as follows:

$${\mathcal{V}} := [H^1_{per}(D)]^2, \quad {\mathcal{W}} := \{ q \in L^2(D); (q, 1)_D = 0 \}, \quad \mathcal{V}_0 := \{ v \in \mathcal{V}; \text{div} v = 0 \text{ in } D \}.$$ 

Let $K := L_0(L^2_{per}(D); [L^2_{per}(D)]^2)$ denote the Banach space of linear operators from $L^2_{per}(D)$ to $[L^2_{per}(D)]^2$ with finite Hilbert-Schmidt norms denoted by $\| \cdot \|_K$. The stochastic integral $\int_0^t \varphi(s) \, dW(s)$ for $0 \leq t \leq T$ is defined as:

$$\left( \int_0^t \varphi(s) \, dW(s), \chi \right) = \lim_{j \to \infty} \sum_{j=1}^j \sqrt{\lambda_j} \int_0^t \langle \varphi(s) q_j, \chi \rangle \, d\beta_j(s) \quad \forall \chi \in [L^2_{per}(D)]^2, \text{ a.s.}$$

As it is noted [22] that the stochastic integral $\int_0^t \varphi(s) \, dW(s)$ is an $\{\mathcal{F}_t\}$-martingale and the following Itô’s isometry holds:

$$\mathbb{E}\left[ \left\| \int_0^T \varphi(s) \, dW(s) \right\|_{L^2_\mathbb{F}}^2 \right] = \mathbb{E}\left[ \int_0^T \|\varphi(s)\mathbb{P}^2 \, ds \right].$$

In this paper we assume that $G : [0, T] \times [L^2_{per}(D)]^2 \to L^2(\Omega, \mathcal{K})$ satisfies the following conditions:

(2.3a) $\|G(v) - G(w)\|_{L^2(\mathcal{K}, L^2_\mathbb{F})} \leq C\|v - w\|_{L^2_\mathbb{F}}, \quad \forall v, w \in [L^2_{per}(D)]^2,$

(2.3b) $\|G(v)\|_{L^2(\mathcal{K}, H^i_\mathbb{F})} \leq C(1 + \|v\|_{H^i_\mathbb{F}}), \quad \forall v \in [H^i_{per}(D)]^2, \quad i = 1, 2,$

(2.3c) $\|D^jG(v)\|_{L^2(\mathcal{K}, L^2([L^2(D)]^2); L^2(D))} \leq C, \quad \forall v \in [L^2_{per}(D)]^2, \quad j = 1, 2.$

We introduce the Helmholtz projection [18] $P_H : [L^2_{per}(D)]^2 \to \mathcal{V}$ defined by $P_H v = \eta$ for every $v \in [L^2_{per}(D)]^2$, where $(\eta, \xi) \in \mathcal{V} \times [H^1_{per}(D)]^2/\mathbb{R}$ is a unique decomposition such that

$$w = \eta + \nabla \xi,$$

and $\xi \in [H^1_{per}(D)]^2/\mathbb{R}$ satisfies the following problem:

(2.4) $$(\nabla \xi, \nabla q) = (v, \nabla q), \quad \forall q \in [H^1_{per}(D)]^2.$$
2.2. Definition of weak solutions. In this subsection we first recall the weak solution definition for problem (1.1), and refer to [9, 11, 14, 15]. We then give some regularity of pressure of various spatial norms.

**Definition 2.1.** Assume that \((Ω, \mathcal{F}, (\mathcal{F}_t)_{t≥0}, \mathbb{P})\) is a given stochastic basis and \(u_0\) is an \(\mathcal{F}_0\)-measurable random variable. Then \((u, p)\) is called a weak pathwise solution to problem (1.1) if

(i) the velocity and the pressure \((u, p)\) is \(\mathcal{F}_t\)-adapted and 
\[
\begin{align*}
    u &\in C([0, T]; [L^2_{per}(D)]^2) \cap L^2(0, T; \mathcal{V}), \quad \mathbb{P}\text{-a.s.,} \\
    p &\in W^{-1, ∞}(0, T; L^2_{per}(D)), \quad \mathbb{P}\text{-a.s.}
\end{align*}
\]

(ii) the problem (1.1) satisfies 
\[
\begin{align*}
    (u(t), v) + \int_0^t [a(u, v) + b(u, u, v)] \, ds - d\left( v, \int_0^t p(s) \, ds \right) \\
    = (u_0, v) + \left( \int_0^t G(u(s)) \, dW(s), v \right) \quad \forall v \in \mathcal{V},
\end{align*}
\]
holds \(\mathbb{P}\)-a.s. for all \(t \in (0, T]\). Where the bilinear forms \(a(\cdot, \cdot)\) and \(d(\cdot, \cdot)\) are defined 
\[
\begin{align*}
    a(v, w) &:= \nu(\nabla v, \nabla w), \quad \forall v, w \in \mathcal{V}, \\
    d(v, q) &:= (\nabla \cdot v, q), \quad \forall v \in \mathcal{V}, \ q \in \mathcal{W},
\end{align*}
\]
and the nonlinear form \(b(\cdot, \cdot, \cdot)\) is defined as follows: 
\[
b(w, u, v) := (w \cdot \nabla u, v) + \frac{1}{2}((\nabla \cdot w)u, v), \quad \forall w, u, v \in \mathcal{V}.
\]

Using the similar idea in [15], problem (2.5) can be considered as a mixed formulation for problem (1.1). Thus, we introduce a new pressure \(r := p - \xi(u(t))\) \(dW\), where we apply the Helmholtz decomposition \(G(u(t)) = \eta(u(t)) + \nabla \xi(u(t))\), where \(\xi(u(t)) \in \mathcal{V}, \ \mathbb{P}\text{-a.s.}\) such that 
\[
(\nabla \xi(u(t)), \nabla \phi) = (G(u(t)), \nabla \phi), \quad \forall \phi \in \mathcal{V}.
\]
By the elliptic regularity [18], we have
\[
\begin{align*}
    \|\nabla \xi(u(t))\|_{L^2} &\leq C\|G(u(t))\|_{L^2}, \\
    \|\nabla \xi(u(t))\|_{H^2/2} &\leq C\|\nabla \cdot G(u(t))\|_{L^2}.
\end{align*}
\]

**Definition 2.2.** Assume that \((Ω, \mathcal{F}, (\mathcal{F}_t)_{t≥0}, \mathbb{P})\) is a given stochastic basis and \(u_0\) is an \(\mathcal{F}_0\)-measurable random variable. Then \((u, r)\) is called a weak pathwise solution to problem (1.1) if

(i) the velocity and the pressure \((u, r)\) is \(\mathcal{F}_t\)-adapted and 
\[
\begin{align*}
    u &\in C([0, T]; [L^2_{per}(D)]^2) \cap L^2(0, T; \mathcal{V}), \quad \mathbb{P}\text{-a.s.,} \\
    r &\in W^{-1, ∞}(0, T; L^2_{per}(D)), \quad \mathbb{P}\text{-a.s.}
\end{align*}
\]

(ii) the problem (1.1) satisfies 
\[
\begin{align*}
    (u(t), v) + \int_0^t [a(u, v) + br(u, u, v)] \, ds - d\left( v, \int_0^t p(s) \, ds \right) \\
    = (u_0, v) + \left( \int_0^t G(u(s)) \, dW(s), v \right) \quad \forall v \in \mathcal{V},
\end{align*}
\]
holds \(\mathbb{P}\)-a.s. for all \(t \in (0, T]\). Where the bilinear forms \(a(\cdot, \cdot)\) and \(d(\cdot, \cdot)\) are defined 
\[
\begin{align*}
    a(v, w) &:= \nu(\nabla v, \nabla w), \quad \forall v, w \in \mathcal{V}, \\
    d(v, q) &:= (\nabla \cdot v, q), \quad \forall v \in \mathcal{V}, \ q \in \mathcal{W},
\end{align*}
\]
and the nonlinear form \(br(\cdot, \cdot, \cdot)\) is defined as follows: 
\[
br(w, u, v) := (w \cdot \nabla u, v) + \frac{1}{2}((\nabla \cdot w)u, v), \quad \forall w, u, v \in \mathcal{V}.
\]
(ii) the problem \[1.1\] satisfies

\[(2.8a)\]

\[
(\mathfrak{L}(t), v) + \int_0^t [a(u, v) + b(u, u, v)] \, ds - d\left( v, \int_0^t r(s) \, ds \right) = (u_0, v) + \left( \int_0^t \eta(u(t)) \, dW(s), v \right) \quad \forall v \in \mathcal{V},
\]

\[(2.8b)\]

\[d(u, q) = 0, \quad \forall q \in \mathcal{W},\]

holds \(\mathbb{P}\)-a.s. for all \(t \in (0, T]\), where \(\eta(u(t)) := G(u(s)) - \nabla \xi(u(t))\).

The next Lemma follows from \[3\].

Lemma 2.3. (i), Let \(u_0 \in L^1(\Omega, [L^2_0(D)]^2)\) for some \(l \geq 2\) and let \(G\) satisfy \[2.3\]. Then there exists a constant \(C > 0\), such that

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u\|_{L^2}^2 \, ds \right] \leq C.
\]

(ii), Let \(u_0 \in L^1(\Omega, [V])\) for some \(l \geq 2\) and let \(G\) satisfy \[2.3\]. Then there exists a constant \(C > 0\), such that

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \|\nabla^2 u\|_{L^2}^2 + \int_0^t \|\nabla^3 u\|_{L^2}^2 \, ds \right] \leq C.
\]

(iii), Let \(u_0 \in L^1(\Omega, [H^2(D)]^2 \cap \mathcal{V}) \cap L^2(\Omega, \mathcal{V})\) for some \(l \geq 2\) and let \(G\) satisfy \[2.3\]. Then there exists a constant \(C > 0\), such that

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \|\nabla^2 u\|_{L^2}^2 + \int_0^t \|\nabla^3 u\|_{L^2}^2 \, ds \right] \leq C.
\]

We next give some regularity of pressure of various spatial norms.

Lemma 2.4. (i), Let \(u_0 \in L^1(\Omega, L^2(D))\) for some \(l \geq 2\) and let \(G\) satisfy \[2.3\]. Then there exists a constant \(C > 0\), such that

\[(2.9)\]

\[\mathbb{E}\left[ \left\| \int_0^t p(s) \, ds \right\|_{L^2_2}^2 \right] \leq C,
\]

\[(2.10)\]

\[\mathbb{E}\left[ \left\| \int_0^t \nabla p(s) \, ds \right\|_{L^2_2}^2 \right] \leq C.
\]

where \((u, p)\) is the weak pathwise solution to \[1.1\], cf. Definition \[2.2\].

(ii), Let \(u_0 \in L^1(\Omega, L^2(D))\) for some \(l \geq 2\) and let \(G\) satisfy \[2.3\]. Then there exists a constant \(C > 0\), such that

\[(2.11)\]

\[\mathbb{E}\left[ \left\| \int_0^t r(s) \, ds \right\|_{L^2_2}^2 \right] \leq C,
\]

\[(2.12)\]

\[\mathbb{E}\left[ \left\| \int_0^t \nabla r(s) \, ds \right\|_{L^2_2}^2 \right] \leq C.
\]

where \((u, r)\) is the weak pathwise solution to \[1.1\], cf. Definition \[2.2\].

Proof. For \[2.9\], from \[2.8a\], we have

\[(2.13)\]

\[d\left( v, \int_0^t p(s) \, ds \right) = (u(t), v) - (u_0, v) + \int_0^t \left[ a(u(s), v) + \xi(u(s)) \right] \, ds - \left( \int_0^t G(u(s)) \, dW(s), v \right).
\]
Using the Young’s inequality, the Poincaré inequality and the Hölder inequality, one finds that
\[
d\left( v, \int_0^t p(s) \, ds \right) \leq C \left( \|u_0\|_{L^2} + \|u(t)\|_{L^2} \right) \|v\|_{L^2} + C \int_0^t \|\nabla u(s)\|_{L^2} \|\nabla v\|_{L^2} \, ds
+ C \int_0^t \|u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2 \, ds \|\nabla v\|_{L^2} \\
+ C \left\| \int_0^t G(u(s)) \, dW(s) \right\|_{L^2} \|v\|_{L^2}.
\]
By the well-known inf-sup condition [13], it follows that
\[
\beta \left\| \int_0^t p(s) \, ds \right\|_{L^2} \leq C \left( \|u_0\|_{L^2} + \|u(t)\|_{L^2} \right) + C \int_0^t \|\nabla u(s)\|_{L^2} \, ds
+ C \int_0^t \left( \|u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2 \right) \, ds \\
+ C \left\| \int_0^t G(u(s)) \, dW(s) \right\|_{L^2}.
\]
Taking the expectation, using Itô’s isometry, (2.3b) and Lemma 2.3, which lead to the desired result.

For (2.10), from (2.5a) and using the Hölder inequality, the Young’s inequality and the Poincaré inequality, we get
\[
\left( \int_0^t \nabla p(s) \, ds, v \right) = \left[ (u(t), v) - (u_0, v) + \int_0^t [a(u(s), v) + b(u(s), u(s), v)] \, ds - \int_0^t G(u(s)) \, dW(s), v \right]
\leq C \left( \|u_0\|_{L^2} + \|u(t)\|_{L^2} \right) \|v\|_{L^2} + C \int_0^t \|\nabla^2 u(s)\|_{L^2} \, ds \|v\|_{L^2}
+ C \int_0^t \left( \|u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2 + \|\nabla^2 u(s)\|_{L^2}^2 \right) \, ds \|v\|_{L^2}
+ C \left\| \int_0^t G(u(s)) \, dW(s) \right\|_{L^2} \|v\|_{L^2}.
\]
With the definition of $L^2$ norm, we obtain
\[
\left\| \int_0^t \nabla p(s) \, ds \right\|_{L^2} = \sup_{v \neq 0 \in L^2} \frac{\left( \int_0^t \nabla p(s) \, ds, v \right)}{\|v\|_{L^2}}
\leq C \left( \|u_0\|_{L^2} + \|u(t)\|_{L^2}^2 \right) + C \int_0^t \|\nabla^2 u(s)\|_{L^2} \, ds \\
+ C \int_0^t \left( \|u(s)\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2 + \|\nabla^2 u(s)\|_{L^2}^2 \right) \, ds \\
+ C \left\| \int_0^t G(u(s)) \, dW(s) \right\|_{L^2}.
\]
Taking the expectation, using Itô’s isometry and (2.3b) and Lemma 2.3, we get the desired result.
3. Semi-discrete time-stepping scheme. In this section we establish semi-
discrete time-stepping scheme for the mixed formulation (2.8). Then we analyze the
error estimate for the velocity and the pressure.

Let \(N\) be a positive integer and \(0 = t_0 < t_1 < \ldots < t_N = T\) be a uniform partition
of \([0,T]\), with \(k = t_{i+1} - t_i\) for \(i = 0,\ldots,N-1\), Set \(u^0 := u_0\). Our semi-discrete
time-stepping scheme for (1.1) is defined as follows:

Algorithm 1:
Step I: Find \(\xi(u^{n-1}) \in L^2(\Omega, \mathcal{V})\) by solving

\[
\begin{equation}
(\nabla \xi(u^{n-1}), \nabla \phi) = (G(u^{n-1}), \nabla \phi), \ \forall \phi \in \mathcal{V}, \ a.s.
\end{equation}
\]

Step II: Denote \(\eta(u^{n-1}) := G(u^{n-1}) - \nabla \xi(u^{n-1})\), and find \((u^n, r^n) \in L^2(\Omega, \mathcal{V}) \times
L^2(\Omega, \mathcal{W})\) by solving

\[
\begin{align}
(3.2a) & \quad (u^n, v) + k a(u^n, v) - k d(v, r^n) + k b(u^n, u^n, v) \\
(3.2b) & \quad d(u^n, q) = 0 \quad \forall q \in \mathcal{W}, \ a.s.,
\end{align}
\]

where \(\Delta W_n := W(t_n) - W(t_{n-1}) \sim \mathcal{N}(0, kQ)\).

Step III: Denote \(p^n := r^n + k^{-1} \xi(u^{n-1}) \Delta W_n\).

The following lemmas establish some stability results for the discrete processes
\([u^n, r^n]; 0 \leq n \leq N\).

Lemma 3.1. Let \(1 \leq p < \infty\) be a natural number. Assume \(u^0 \in L^{2p}(\Omega, \mathcal{V}_0)\) with
\(\|u^0\|_{L^2} \leq C\). Then there exists a sequence \([u^n, r^n]; 1 \leq n \leq N\), which for all \(\omega \in \Omega\),
solves Algorithm 1 and has the following properties:

\[
\begin{align}
(3.3) & \quad \mathbb{E} \left[ \max_{1 \leq n \leq N} \|u^n\|_{L^2}^{2p} + \nu k \sum_{n=1}^{N} \|u^n\|_{L^2}^{2p-1} \|\nabla u^n\|_{L_2}^2 \right] \leq C, \\
(3.4) & \quad \mathbb{E} \left[ \max_{1 \leq n \leq N} \|\nabla u^n\|_{L^2}^p + \nu k \sum_{n=1}^{N} \|\nabla u^n\|_{L^2}^{p-1} \|\nabla^2 u^n\|_{L_2}^2 \right] \leq C, \\
(3.5) & \quad \mathbb{E} \left[ \sum_{n=1}^{N} \|\nabla (u^n - u^{n-1})\|_{L_2}^2 \|\nabla u^n\|_{L_2}^2 \right] \leq C, \\
(3.6) & \quad \mathbb{E} \left[ \left( \sum_{n=1}^{N} \|\nabla (u^n - u^{n-1})\|_{L_2}^2 \right)^4 + \left( k \sum_{n=1}^{N} \|\nabla^2 u^n\|_{L_2}^2 \right)^4 \right] \leq C, \\
(3.7) & \quad \mathbb{E} \left[ k \sum_{n=1}^{N} \|r^n\|_{L_2}^2 \right] \leq C, \\
(3.8) & \quad \mathbb{E} \left[ k \sum_{n=1}^{N} \|\nabla r^n\|_{L_2}^2 \right] \leq C.
\]

Proof. Since the proofs of (3.3)–(3.6) were derived in [10]. Here we prove (3.7)–
By applying the summation operator $\sum_{n=1}^{N}$ to both sides of (3.2a), we obtain

\begin{equation}
3.9 \quad k \sum_{n=1}^{N} d(v, r^n) = (u^N, v) - (u^0, v) + k \sum_{n=1}^{N} \left[ a(u^n, v) + b(u^n, u^n, v) \right] - \sum_{n=1}^{N} (\eta(u^{n-1}) \Delta W_n, v).
\end{equation}

Using the Poincaré’s inequality and the Young’s inequality on the right hand side, one finds that

\begin{align*}
 k \sum_{n=1}^{N} d(v, r^n) & \leq C\left( \|u^0\|_{L^2_x} + \|u^N\|_{L^2_x} \right) \|\nabla v\|_{L^2_x} + Ck \sum_{n=1}^{N} \|\nabla u^n\|_{L^2_x} \|\nabla v\|_{L^2_x} \\
 & \quad + Ck \sum_{n=1}^{N} \left( \|u^n\|_{L^2_x}^{2} + \|\nabla u^n\|_{L^2_x}^{2} \right) \|\nabla v\|_{L^2_x} \\
 & \quad + C \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \eta(u^{n-1}) dW(s) \right| \|v\|_{L^2_x}.
\end{align*}

By the inf-sup condition, we get

\begin{align*}
 \beta k \sum_{n=1}^{N} \|r^n\|_{L^2_x} & \leq C\left( \|u^0\|_{L^2_x} + \|u^N\|_{L^2_x} \right) + Ck \sum_{n=1}^{N} \|\nabla u^n\|_{L^2_x} \\
 & \quad + Ck \sum_{n=1}^{N} \left( \|u^n\|_{L^2_x}^{2} + \|\nabla u^n\|_{L^2_x}^{2} \right) + C \left| \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \eta(u^{n-1}) dW(s) \right| \|v\|_{L^2_x}.
\end{align*}

With the definition of $\eta(u^{n-1})$ and (2.6), it follows that

\begin{align*}
 & \mathbb{E}[\|\eta(u^{n-1})\|_K] \leq \mathbb{E}\left[ \|G(u^{n-1})\|_{L^2_x} + \|\nabla \xi(u^{n-1})\|_{L^2_x} \right] \\
 & \quad \leq C \mathbb{E}\left[ \|G(u^{n-1})\|_{L^2_x} \right].
\end{align*}

Hence, taking the expectation and using Itô’s isometry and (3.4), which lead to the desired result.

For (3.8), setting $v = \nabla r^n$ in (3.2a), by using the Poincaré’s inequality and the Young’s inequality, one finds that

\begin{align*}
 k \sum_{n=1}^{N} (\nabla r^n, \nabla r^n) & = (u^N, \nabla r^n) - (u^0, \nabla r^n) + k \sum_{n=1}^{N} \left[ a(u^n, v) + b(u^n, u^n, \nabla r^n) \right] \\
 & \quad - \sum_{n=1}^{N} (\eta(u^{n-1}) \Delta W_n, \nabla r^n) \\
 & \leq C\left( \|u^0\|_{L^2_x} + \|u^N\|_{L^2_x} \right) \|\nabla r^n\|_{L^2_x} + Ck \sum_{n=1}^{N} \|\nabla u^n\|_{L^2_x} \|\nabla r^n\|_{L^2_x} \\
 & \quad + C \sum_{n=1}^{N} \left( k \|\nabla u^n\|_{L^2_x}^{2} + \|\nabla^2 u^n\|_{L^2_x} \right) \|\nabla r^n\|_{L^2_x}.
\end{align*}
With a standard calculation, we have

\[ k \sum_{n=1}^{N} \| \nabla r^n \|^2_{L^2} \leq C \left( \| u^0 \|^2_{L^2} + \| u^N \|^2_{L^2} \right) + Ck \sum_{n=1}^{N} \| \nabla^2 u^n \|^2_{L^2} + Ck \sum_{n=1}^{N} \left( \| \nabla u^n \|^2_{L^2} + \| \nabla^2 u^n \|^2_{L^2} \right). \]

Taking the expectation, using Itô’s isometry, (2.3b) and (3.4), we get the desired result. The proof is complete. \( \square \)

**Lemma 3.2.** Assume \( u^0 \in L^{2q}(\Omega, V_0) \) for some \( 1 \leq q < \infty \). Then there exists a sequence \( \{ u^n \}; 1 \leq n \leq N \), which for all \( \omega \in \Omega \), solves Algorithm 1 and satisfies the following properties:

\[
(3.11) \quad \mathbb{E} \left[ \max_{1 \leq n \leq N} \| Au^n \|^2_{L^2} + k \sum_{n=1}^{N} \| Au^n \|^{2p-2}_{L^2} \| A^\frac{q}{2} u^n \|^2_{L^2} \right] \leq C, \]

\[
(3.12) \quad \mathbb{E} \left[ \max_{1 \leq n \leq N} \| u^n \|^2_{L^2} + k \sum_{n=1}^{N} \| u^n \|^{2p-2}_{L^2} \| u^n \|^2_{L^2} \right] \leq C, \]

\[
(3.13) \quad \mathbb{E} \left[ \left( \sum_{n=1}^{N} \| A(u^n - u^{n-1}) \|^2_{L^2} \right)^4 + \left( k \sum_{n=1}^{N} \| A^\frac{q}{2} u^n \|^2_{L^2} \right)^4 \right] \leq C, \]

where \( A : V \cap [H^2(D)]^d \to V_0 \) denotes the Stokes operator (cf. 23).

**Proof.** For the first assertion, taking \( v = A^2 u^n \in V_0 \) and \( q_n = 0 \) in (3.2), we get

\[
(3.14) \quad \frac{1}{2} \left( \| Au^n \|^2_{L^2} - \| Au^{n-1} \|^2_{L^2} + \| A(u^n - u^{n-1}) \|^2_{L^2} \right) + k\nu \| A^\frac{q}{2} u^n \|^2_{L^2} = k\nu \| u^n \|_{L^2} + (A\eta(u^n - u^{n-1})) \\
+ (A\eta(u^{n-1}))W_n, A(u^n - u^{n-1})). \]

By Lemma 2.1.20 in 221 and the Young’s inequality, the first term on the right hand of (3.14) can be estimated as

\[
k\nu \| u^n \|_{L^2} \leq Ck\| \nabla^3 u^n \|_{L^2} \| \nabla u^n \|_{L^2} \| u^n \|_{L^2} \leq \frac{k\nu}{2} \| \nabla^3 u^n \|^2_{L^2} + Ck\| \nabla u^n \|_{L^2} \| u^n \|_{L^2} \]

\[
\leq \frac{k\nu}{2} \| \nabla^3 u^n \|^2_{L^2} + Ck(\| \nabla u^n \|^4_{L^2} + \| u^n \|^4_{L^2}). \]

The last term on the right hand of (3.14) vanishes when taking its expectation. Applying the Young’s inequality and the tower property for conditional expectations to
the second term on the right hand. Summing up then leads to

\[(3.15)\]
\[
\mathbb{E}\left[\|Au^n\|_{L^2}^2\right] + \sum_{n=1}^{N} \mathbb{E}\left[\|A(u^n - u^{n-1})\|_{L^2}^2\right] + \mathbb{E}\left[\nu \sum_{n=1}^{N} k\|A^{\frac{3}{2}}u^n\|_{L^2}^2\right] \\
\leq C \mathbb{E}\left[\sum_{n=1}^{N} k\left(\|\nabla u^n\|_{L^2}^2 + \|u^n\|_{L^2}^2\right)\right] + \mathbb{E}\left[\sum_{n=1}^{N} k\|Au^{n-1}\|_{L^2}^2\right].
\]

Using Lemma 3.1 and the discrete Gronwall's lemma leads to

\[(3.16)\]
\[
\max_{1 \leq n \leq N} \mathbb{E}\left[\|Au^n\|_{L^2}^2\right] + \mathbb{E}\left[\nu \sum_{n=1}^{N} k\|A^{\frac{3}{2}}u^n\|_{L^2}^2\right] \leq C.
\]

To derive the first inequality in (3.11), using the Young's inequality and Lemma 3.1, one finds that

\[(3.17)\]
\[
\mathbb{E}\left[\max_{1 \leq n \leq N} \|Au^n\|_{L^2}^2\right] \leq \mathbb{E}\left[\|Au^0\|_{L^2}^2\right] + C \mathbb{E}\left[\sum_{n=1}^{N} k\left(\|\nabla u^n\|_{L^2}^2 + \|u^n\|_{L^2}^2\right)\right] \\
+ C \mathbb{E}\left[\sum_{n=1}^{N} \|AP_{H\eta}(u^{n-1})\Delta W_n\|_{L^2}^2\right] \\
+ C \mathbb{E}\left[\max_{1 \leq n \leq N} \sum_{t=1}^{n} (AP_{H\eta}(u^{t-1})\Delta W_n, Au^{t-1})\right].
\]

The second term on the right hand side may be controlled by Lemma 3.1, the third term may be estimated by the tower property for conditional expectations, the fourth term is bounded with using the Burkholder-Davis-Gundy inequality. Thus, (3.11) holds for \(p = 2\). For \(p \geq 3\), by the similar line in [7], we may derive the desired result. Here we skip it.

For the second assertion, taking \(v = A^{-1}u^n \in V_0\) and \(q_h = 0\) in (3.2), one finds that

\[(3.18)\]
\[
\frac{1}{2} \left(\|u^n\|_{L^2}^2 - \|u^{n-1}\|_{L^2}^2 + \|u^n - u^{n-1}\|_{L^2}^2\right) + k\|u^n\|_{L^2}^2 \\
= kb(u^n, u^n, A^{-1}u^n) + (A^{-\frac{1}{2}}\eta(u^{n-1})W_n, A^{-\frac{1}{2}}(u^n - u^{n-1})) \\
+ (A^{-\frac{1}{2}}\eta(u^{n-1})W_n, A^{-\frac{1}{2}}(u^n - u^{n-1})).
\]

By the Young's inequality and the Hölter inequality, the first term on the right hand of (3.18) can be bounded

\[
k\|u^n, u^n, A^{-1}u^n) \leq C k\|u^n\|_{L^2} \|
abla u^n\|_{L^2} \|u^n - u^{n-1}\|_{-1} + C k\|u^n\|_{L^2} \|
abla u^n\|_{L^2} \|u^{n-1}\|_{-1} \\
\leq \frac{1}{4} \|u^n - u^{n-1}\|_{L^2}^2 + C k^2\|u^n\|_{L^2}^2 \|
abla u^n\|_{L^2}^2 + C k\|u^n\|_{L^2} \|
abla u^n\|_{L^2} \|u^{n-1}\|_{-1}.
\]
it follows that To obtain the first inequality in (3.12), using the Young’s inequality and Lemma 3.1, the second term on the right hand. Summing up then leads to

\[(3.19) \quad \mathbb{E}[\|u^n\|_{L^2}^2] + \mathbb{E}\left[\sum_{n=1}^{N} \|u^n - u^{n-1}\|_{L^2}^2\right] + \mathbb{E}\left[\nu \sum_{n=1}^{N} k\|u^n\|_{L^2}^2\right]
\leq C\mathbb{E}\left[\sum_{n=1}^{N} k^2\|u^n\|_{L^2}^2 \|\nabla u^n\|_{L^2}^2\right] + C\mathbb{E}\left[\sum_{n=1}^{N} k\|u^n\|_{L^2}^2 \|\nabla u^n\|_{L^2}^2 \|u^{n-1}\|_{\mathbb{H}^1}^2\right]
\leq C\left(\sum_{n=1}^{N} k\|u^n\|_{L^2}^4\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} k\|\nabla u^n\|_{L^2}^4\right)^{\frac{1}{2}}
\ + C\left(\max_{1 \leq n \leq N} \|u^n\|_{L^2}^4\right)^{\frac{1}{2}} \left(\sum_{1 \leq n \leq N} \|\nabla u^n\|_{L^2}^4\right)^{\frac{1}{2}} + \mathbb{E}\left[\sum_{n=1}^{N} k\|u^n\|_{L^2}^2\right].
\]

Using Lemma 3.1 and the discrete Gronwall’s lemma, one finds that

\[(3.20) \quad \max_{1 \leq n \leq N} \mathbb{E}[\|u^n\|_{L^2}^2] + \mathbb{E}\left[\nu \sum_{n=1}^{N} k\|u^n\|_{L^2}^2\right] \leq C.
\]

To obtain the first inequality in (3.12), using the Young’s inequality and Lemma 3.1 it follows that

\[(3.21) \quad \mathbb{E}\left[\max_{1 \leq n \leq N} \|u^n\|_{L^2}^2\right] \leq \mathbb{E}[\|u^0\|_{L^2}^2] + C\mathbb{E}\left[\sum_{n=1}^{N} k\|\nabla u^n\|_{L^2}^2\right]
\ + C\mathbb{E}\left[\sum_{n=1}^{N} \|A^{-\frac{1}{2}} P_H \eta(u^{n-1}) \Delta W_n\|_{L^2}^2\right]
\ + C\mathbb{E}\left[\max_{1 \leq n \leq N} \sum_{l=1}^{n} (A^{-\frac{1}{2}} P_H \eta(u^{l-1}) \Delta W_n, A^{-\frac{1}{2}} u^{l-1})\right].
\]

The second term and the third term on the right hand side may be controlled by Lemma 3.1, the fourth term may be estimated by the tower property for conditional expectations, the fifth term is bounded with using the Burkholder-Davis-Gundy inequality. Thus, (3.11) holds for $p = 2$. For $p \geq 3$, using the similar line in [7], we skip it.

For the third assertion, using the similar line in [7], it follows that

\[(3.22) \quad \left(\sum_{n=1}^{N} \|A(u^n - u^{n-1})\|_{L^2}^2\right)^{\frac{1}{4}} + \left(\sum_{n=1}^{N} k\nu\|A^{\frac{1}{2}} u^n\|_{L^2}^2\right)^{\frac{1}{4}}
\leq C\left(\sum_{n=1}^{N} k\|\nabla u^n\|_{L^2}^{10} + \|u^n\|_{L^2}^{10}\right)^{\frac{1}{4}} + C\left(\sum_{n=1}^{N} \|A P_H \eta(u^{n-1}) \Delta W_n\|_{L^2}^2\right)^{\frac{1}{4}}
\ + C\left(\sum_{n=1}^{N} (A P_H \eta(u^{n-1}) \Delta W_n, A u^{n-1})\right)^{\frac{1}{4}} + C\|A u^0\|_{L^2}^8.
\]
Taking the expectation, the second term and the third term on the right hand can be bounded as in [7], it follows that

\[\mathbb{E}\left(\sum_{n=1}^{N} \left|\sum_{n'=1}^{n} k\nu^{n'}\|A\|^{\frac{2}{3}}u^{n'}\|^2\right|^4\right) \leq C\mathbb{E}\left(\sum_{n=1}^{N} k\|\nabla u^n\|^4_{L^4} + \|u^n\|^4_{L^8}\right) + C\mathbb{E}\left(\sum_{n=1}^{N} k\|A u^n\|^8_{L^8}\right)\]

Thanks to Lemma 3.1, the desired result (3.13) holds. The proof is complete. □

Following [10], for \(\epsilon > 0\)

\[\Omega^n_\epsilon = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} \max_{t_{n-1} \leq s \leq t_n} \|A u^n\|^2_{L^2} \leq -\epsilon \log k \right\}

such that

\[\mathbb{P}(\Omega^n_\epsilon) \geq 1 - \frac{\mathbb{E}[\omega \in \Omega \mid \max_{1 \leq n \leq N} \max_{t_{n-1} \leq s \leq t_n} \|A u^n\|^2_{L^2}]}{-\epsilon \log k} \geq 1 + \frac{C}{\epsilon \log k} \]

and

\[\Omega^n_\epsilon = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} (\|A u^n\|^2_{L^2}, \max_{t_{n-1} \leq s \leq t_n} \|A u(s)\|^2_{H^2}) \leq -\epsilon \log k \right\}

such that

\[\mathbb{P}(\Omega^n_\epsilon) \geq 1 - \frac{\mathbb{E}[\omega \in \Omega \mid \max_{1 \leq n \leq N} (\|A u^n\|^2_{L^2}, \max_{t_{n-1} \leq s \leq t_n} \|A u(s)\|^2_{H^2})]}{-\epsilon \log k} \geq 1 + \frac{C}{\epsilon \log k} \]

By using the similar line in [10] [5], the following theorem states and derives the optimal order error estimate for \(\{u^n; 1 \leq n \leq N\}\).

**Theorem 3.3.** Assume that (2.3) holds and that \(u_0 \in L^8(\Omega, \mathcal{F}_0)\) is a measurable random variable. Let \((u, r)\) be the unique strong solution to (2.3) in the sense of Definition 2, Assume that

\[(3.28) \quad \mathbb{E}\left[\|u\|_{C^0(0,T; L^2)}\right] \leq C, \quad \mathbb{E}\left[\|u\|_{C^0(0,T; H^1)}\right] \leq C, \quad \mathbb{E}\left[\|u\|_{C^0(0,T; H^2)}\right] \leq C

for some \(\alpha \in (0, \frac{1}{2})\). Then, provided that 0 < k < k_0 with k_0 sufficiently small, the following error estimate holds:

\[(3.29) \quad \mathbb{E}\left[\left(\max_{1 \leq n \leq N} \|u(t^n) - u^n\|^2_{L^2} + \nu k \sum_{n=1}^{N} \|\nabla (u(t^n) - u^n)\|^2_{L^2}\right)\right] \leq C k^{2\alpha - \epsilon},

\[(3.30) \quad \mathbb{E}\left[\left(\max_{1 \leq n \leq N} \|\nabla (u(t^n) - u^n)\|^2_{L^2} + \nu k \sum_{n=1}^{N} \|A (u(t^n) - u^n)\|^2_{L^2}\right)\right] \leq C k^{2\alpha - \epsilon},

where C is a positive constant independent of k.
Proof. For every $n \geq 1$, denote $e^n_u := u(t_n) - u^n$, from (2.8a) and (3.2a) over $1 \leq n \leq m(\leq N)$, we get

\begin{equation}
\label{3.31}
(e^n_u - e^{n-1}_u, v) + \int_{t_{n-1}}^{t_n} a(u(s), v) - a(u^n, v) \, ds
= \int_{t_{n-1}}^{t_n} b(u^n, u^n, v) - b(u, u, v) \, ds
+ \int_{t_{n-1}}^{t_n} (\eta(u(s)) - \eta(u^{n-1})dW, v) \quad \forall v \in V_0.
\end{equation}

Setting $v = e^n_u$ in (3.31), using some standard calculations, it follows that

\begin{equation}
\label{3.32}
\frac{1}{2} \left( \|e^n_u\|_{L_x^2}^2 - \|e^{n-1}_u\|_{L_x^2}^2 + \|e^n_u - e^{n-1}_u\|_{L_x^2}^2 \right) + k\nu\|\nabla e^n_u\|_{L_x^2}^2
= \int_{t_{n-1}}^{t_n} a(u(s) - u(t^n), e^n_u) ds + \int_{t_{n-1}}^{t_n} b(u^n, u^n, e^n_u) - b(u(s), u(s), e^n_u) ds
+ \int_{t_{n-1}}^{t_n} (\eta(u(s)) - \eta(u^{n-1})dW, e^n_u)
= B_1 + B_2 + B_3.
\end{equation}

With the Poincaré inequality and the Young’s inequality, one finds that

\begin{equation}
\label{3.33}
B_1 = \int_{t_{n-1}}^{t_n} a(u(s) - u(t^n), e^n_u) ds
\leq C \int_{t_{n-1}}^{t_n} \|\nabla(u(s) - u(t^n))\|_{L_x^2}^2 ds + \frac{k\nu}{16} \|\nabla e^n_u\|_{L_x^2}^2
\end{equation}

For the nonlinear term $B_2$ can decomposed as follows:

\begin{equation}
\label{3.34}
B_2 = \int_{t_{n-1}}^{t_n} b(u^n, u^n, e^n_u) - b(u(s), u(s), e^n_u) ds
= \int_{t_{n-1}}^{t_n} \left( b(u^n - u(t_n), u^n, e^n_u) + b(u(t_n) - u(s), u^n, e^n_u)
+ b(u(s), u(t_n) - u(s), e^n_u) \right) ds
= B_{2,1} + B_{2,2} + B_{2,3}.
\end{equation}
With the Young’s inequality and the Sobolev inequality, we get

\[
B_{2,1} = \int_{t_{n-1}}^{t_n} b(u^n - u(t_n), u^n, e^n_u) \, ds
\]

\[
\leq C \left( \|e^n_u\|_{L^2} \|\nabla e^n_u\|_{L^2} \|\nabla u^n\|_{L^2} \right) L^2_x
\]

\[
\leq \frac{k\nu}{16} \|\nabla e^n_u\|_{L^2}^2 + Ck \|e^n_u\|_{L^2}^2 \|\nabla u^n\|_{L^2}^4,
\]

\[ (3.35) \]

\[
B_{2,2} = \int_{t_{n-1}}^{t_n} b(u(t_n), u^n, e^n_u) \, ds
\]

\[
\leq \frac{k\nu}{16} \|\nabla e^n_u\|_{L^2}^2 + C \int_{t_{n-1}}^{t_n} \|u(t_n) - u(s)\|_{L^2}^2 \|\nabla u^n\|_{L^2}^2 \, ds,
\]

\[ (3.36) \]

\[
B_{2,3} = \int_{t_{n-1}}^{t_n} b(u(s), u(t_n), e^n_u) \, ds
\]

\[
\leq \frac{k\nu}{16} \|\nabla e^n_u\|_{L^2}^2 + C \int_{t_{n-1}}^{t_n} \|\nabla u(t_n) - u(s)\|_{L^2}^2 \|\nabla u^n\|_{L^2}^2 \, ds.
\]

\[ (3.37) \]

Inserting estimates (3.33), (3.35)–(3.37), into (3.32), applying the summation operator \( \sum_{n=1}^{N} \) and taking the expectation, using (3.28) and Lemma 3.1, we arrive at

\[
E \left[ \Omega_k \left( \max_{1 \leq n \leq N} \|e^n_u\|_{L^2}^2 + \sum_{n=1}^{N} \|e^n_u - e^{n-1}_u\|_{L^2}^2 + \frac{1}{2} \sum_{n=1}^{N} kv_k \|\nabla e^n_u\|_{L^2}^2 \right) \right]
\]

\[
\leq Ck^{2\alpha} + CE \left[ \Omega_k \left( \sum_{n=1}^{N} k \|e^n_u\|_{L^2}^2 \|\nabla u^n\|_{L^2}^2 \right) \right]
\]

\[
+ E \left[ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\eta(u(s)) - \eta(u^{n-1}) dW_e^n) \right].
\]

\[ (3.38) \]

The last term use the similar arguments in [10] [15], it follows that

\[
E \left[ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\eta(u(s)) - \eta(u^{n-1}) dW_e^n) \right]
\]

\[
\leq Ck^{2\alpha} + \frac{1}{2} E \left[ \max_{1 \leq n \leq N} \|e^n_u\|_{L^2}^2 \right]
\]

\[
+ Ck \left( \sum_{n=1}^{N} k \|e^{n-1}_u\|_{L^2}^2 \right)
\]

\[
+ \frac{1}{2} E \left[ \sum_{n=1}^{N} \|e^n_u - e^{n-1}_u\|_{L^2}^2 \right].
\]

\[ (3.39) \]
Substituting (3.39) into (3.38), we have

\[
\begin{align*}
(3.40) \quad E \left[ 1_{\Omega_k} \left( \max_{1 \leq n \leq N} \| e_u^n \|_{L^2}^2 + \sum_{n=1}^N \| e_u^n - e_u^{n-1} \|_{L^2}^2 + \sum_{n=1}^N k \nu \| \nabla e_u^n \|_{L^2}^2 \right) \right] \\
\leq C k^{2\alpha} + C_1 \log k^{-\epsilon} \left[ \sum_{n=1}^N k \| e_u^n \|_{L^2}^2 \right] + C_2 E \left[ \sum_{n=1}^N k \| e_u^{n-1} \|_{L^2}^2 \right] \\
+ C k E \left[ \sum_{n=1}^N \| e^n \|_{L^2}^2 \| \nabla (u^n - u^{n-1}) \|_{L^2}^2 \right].
\end{align*}
\]

The term \( \Theta_0 \) may be controlled by Lemma 3.1. If \( 0 < k \leq k_0, k_* := \frac{1}{2C_1 \log k_0} < \frac{1}{C_1 \log k_0}, \) since \( 1 \leq \frac{1}{1 - C_1 \log (k - \epsilon) k} \leq 2, \) it follows that

\[
(3.41) \quad E \left[ 1_{\Omega_k} \left( \max_{1 \leq n \leq N} \| e_u^n \|_{L^2}^2 + \sum_{n=1}^N \| e_u^n - e_u^{n-1} \|_{L^2}^2 + \sum_{n=1}^N k \nu \| \nabla e_u^n \|_{L^2}^2 \right) \right] \\
\leq C k^{2\alpha} + \frac{C_1 \log k^{-\epsilon}}{1 - C_1 k \log k^{-\epsilon}} E \left[ \sum_{n=1}^N k \| e_u^{n-1} \|_{L^2}^2 \right] \\
+ \frac{C_2}{1 - C_1 k \log k^{-\epsilon}} E \left[ \sum_{n=1}^N k \| e_u^{n-1} \|_{L^2}^2 \right] \\
\leq C k^{2\alpha} + 2(C_1 \log k^{-\epsilon} + C_2) E \left[ \sum_{n=1}^N k \| e_u^{n-1} \|_{L^2}^2 \right].
\]

By using the discrete Gronwall inequality, the result (3.29) holds.

For (3.34) setting \( v = Ae_u^n \) in (3.31), one finds that

\[
(3.42) \quad \frac{1}{2} \left( \| \nabla e_u^n \|_{L^2}^2 - \| \nabla e_u^{n-1} \|_{L^2}^2 + \| \nabla e_u^n - e_u^{n-1} \|_{L^2}^2 \right) + k \nu \| Ae_u^n \|_{L^2}^2 \\
= \int_{t_{n-1}}^{t_n} a(u(s) - u(t^n), Ae_u^n) ds + \int_{t_{n-1}}^{t_n} b(u^n, u^n, e_u^n) - b(u(s), u(s), Ae_u^n) ds \\
+ \int_{t_{n-1}}^{t_n} (\nabla \eta(u(s)) - \eta(u^{n-1}) \| dW, \nabla e_u^n \) \\
= D_1 + D_2 + D_3.
\]

With the Poincaré inequality and the Young’s inequality, the term \( D_1 \) can be bounded

\[
(3.43) \quad D_1 = \int_{t_{n-1}}^{t_n} a(u(s) - u(t^n), Ae_u^n) ds \\
\leq C \int_{t_{n-1}}^{t_n} \| \nabla^2 (u(s) - u(t^n)) \|_{L^2}^2 ds + \frac{k \nu}{16} \| Ae_u^n \|_{L^2}^2
\]
For the nonlinear term $D_2$ can decomposed as follows:

\begin{equation}
D_2 = \int_{t_{n-1}}^{t_n} b(u^n, u^n, Ae_u^n) - b(u(s), u(s), Ae_u^n) ds
\end{equation}

\begin{align*}
&= \int_{t_{n-1}}^{t_n} \left( b(u^n - u(t_n), u^n, Ae_u^n) + b(u(t_n) - u(s), u^n, Ae_u^n) 
+ b(u(s), u(t_n) - u(s), Ae_u^n) + b(u(s), u^n - u(t_n), Ae_u^n) \right) ds \\
&= D_{2,1} + D_{2,2} + D_{2,3} + D_{2,4}.
\end{align*}

With the Young’s inequality and the Sobolev inequality, we get

\begin{align*}
D_{2,1} &= \int_{t_{n-1}}^{t_n} b(u^n - u(t_n), u^n, Ae_u^n) ds \\
&\leq \frac{k\nu}{16} \| Ae_u^n \|_{L^2}^2 + C k \| \nabla e^n \|_{L^2}^2 \| Au^n \|_{L^2}^2, \\
D_{2,2} &= \int_{t_{n-1}}^{t_n} b(u(t_n) - u(s), u^n, Ae_u^n) ds \\
&\leq \frac{k\nu}{16} \| Ae_u^n \|_{L^2}^2 + C \int_{t_{n-1}}^{t_n} \| \nabla (u(t_n) - u(s)) \|_{L^2}^2 \| Au^n \|_{L^2}^2 ds, \\
D_{2,3} &= \int_{t_{n-1}}^{t_n} b(u(s), u(t_n) - u(s), Ae_u^n) ds \\
&\leq \frac{k\nu}{16} \| Ae_u^n \|_{L^2}^2 + C \int_{t_{n-1}}^{t_n} \| \nabla u(t_n) - u(s) \|_{L^2}^2 \| u(s) \|_{H^2}^2 ds, \\
D_{2,4} &= \int_{t_{n-1}}^{t_n} b(u(s), u^n - u(t_n), Ae_u^n) ds \\
&\leq \frac{k\nu}{16} \| Ae_u^n \|_{L^2}^2 + C k \| \nabla e^n \|_{L^2}^2 \max_{t_{n-1} \leq s \leq t_n} \| u(s) \|_{H^2}^2.
\end{align*}

Inserting estimates (3.43), (3.45)–(3.48) into (3.42), applying the summation operator $\sum_{n=1}^{N}$ and taking the expectation, using (3.28) and Lemma 3.1 we arrive at

\begin{equation}
\mathbb{E} \left[ 1_{\Omega_T} \left( \max_{1 \leq n \leq N} \| \nabla e_u^n \|_{L^2}^2 + \sum_{n=1}^{N} \| \nabla (e_u^n - e_u^{n-1}) \|_{L^2}^2 + \frac{1}{2} \sum_{n=1}^{N} k\nu \| Ae_u^n \|_{L^2}^2 \right) \right]
\leq C k^{2\alpha} + C \mathbb{E} \left[ 1_{\Omega_T} \sum_{n=1}^{N} k \| \nabla e^n \|_{L^2}^2 \left( \| Au^n \|_{L^2}^2 + \max_{t_{n-1} \leq s \leq t_n} \| u(s) \|_{H^2}^2 \right) \right] \\
+ \mathbb{E} \left[ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\nabla \eta(u(s)) - \eta(u^{n-1}))[dW_t, \nabla e_u^n] \right] .
\end{equation}
Using (2.21), (2.24), (3.4) and (3.29), the last term \(D_3\) can be estimated

\[
E \left[ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left( \nabla [\eta(u(s)) - \eta(u^{n-1})] dW, \nabla e_u^n \right) \right]
\]

\[
\leq C k^{2\alpha} + \frac{1}{2} E \left[ \max_{1 \leq n \leq N} \| \nabla e_u^n \|_{L_2}^2 \right] + CE \left[ \sum_{n=1}^{N} k \| \nabla e_u^{n-1} \|_{L_2}^2 \right]
\]

\[
+ \frac{1}{2} E \left[ \sum_{n=1}^{N} \| \nabla (e_u^n - e_u^{n-1}) \|_{L_2}^2 \right].
\]

Substituting (3.50) into (3.49), we have

\[
E \left[ 1_{\Omega_k} \left( \max_{1 \leq n \leq N} \| \nabla e_u^n \|_{L_2}^2 + \sum_{n=1}^{N} \| \nabla (e_u^n - e_u^{n-1}) \|_{L_2}^2 + \sum_{n=1}^{N} k \nu \| A e_u^n \|_{L_2}^2 \right) \right]
\]

\[
\leq C k^{2\alpha} + \overline{C}_1 \log k^{-\epsilon} E \left[ \sum_{n=1}^{N} k \| \nabla e_u^n \|_{L_2}^2 \right] + \overline{C}_2 E \left[ \sum_{n=1}^{N} k \| \nabla e_u^{n-1} \|_{L_2}^2 \right]
\]

\[
+ C k E \left[ \sum_{n=1}^{N} \| \nabla e_u^n \|_{L_2}^2 \right] A(u^n - u^{n-1}) \|_{L_2}^2 \right] + C k E \left[ \sum_{n=1}^{N} \| \nabla e_u^n \|_{L_2}^2 \right] \max_{t_{n-1} \leq s \leq t_n} \| u(s) - u(s - k) \|_{L_2}^2 \right].
\]

The terms \(\Theta_1\) and \(\Theta_2\) may be controlled by Lemmas 3.1, 3.2. If \(0 < k \leq k_0, k_* := \frac{1}{2 \overline{C}_1 \log k_0} < \frac{1}{2 \overline{C}_1 \log k_0} \leq 2\), it follows that

\[
E \left[ 1_{\Omega_k} \left( \max_{1 \leq n \leq N} \| \nabla e_u^n \|_{L_2}^2 + \sum_{n=1}^{N} \| \nabla (e_u^n - e_u^{n-1}) \|_{L_2}^2 + \sum_{n=1}^{N} k \nu \| A e_u^n \|_{L_2}^2 \right) \right]
\]

\[
\leq C k^{2\alpha} + \frac{C_1 \log k^{-\epsilon}}{1 - \overline{C}_1 k \log k^{-\epsilon}} E \left[ \sum_{n=1}^{N} k \| \nabla e_u^{n-1} \|_{L_2}^2 \right]
\]

\[
+ \frac{C_2}{1 - \overline{C}_1 \log k^{-\epsilon}} E \left[ \sum_{n=1}^{N} k \| e_u^{n-1} \|_{L_2}^2 \right]
\]

\[
\leq C k^{2\alpha} + 2 \left( \overline{C}_1 \log k^{-\epsilon} + \overline{C}_2 \right) E \left[ \sum_{n=1}^{N} k \| \nabla e_u^{n-1} \|_{L_2}^2 \right].
\]

By using the discrete Gronwall inequality, the result (3.30) holds. The proof is complete. \(\square\)

The second result of this section is stated in the following theorems which give an optimal error estimate for the pressure \(\{r^n; 1 \leq n \leq N\}\) and \(\{p^n; 1 \leq n \leq N\}\).

**Theorem 3.4.** Let the assumptions of Theorem 3.3 be satisfied. Let \(\{r^n; 1 \leq n \leq N\}\) be the pressure approximation defined by Algorithm 1. Then the following error estimate holds for \(m = 1, 2, \ldots, N\)

\[
E \left[ 1_{\Omega_k} \left( \left\| \int_0^t r(s) ds - \sum_{n=1}^{m} r^n \|_{L_2}^2 \right) \right] \right] \leq C k^{2\alpha - \epsilon},
\]
finds that

\[
(u^m, v) + k \sum_{n=1}^{m} a(u^n, v) - k \sum_{n=1}^{m} d(v, r^n) + k \sum_{n=1}^{m} b(u^n, u^n, v) = (u^0, v) + \sum_{n=1}^{m} (\eta(u^{n-1}) \Delta W_n, v) \quad \forall v \in \mathcal{V}, \text{ a.s.}
\]

Subtracting (2.8a) (with \(t = t_n\)) from (3.54) and noting that \(u^0 = u(0)\), we obtain

\[
d(v, k \sum_{n=1}^{m} r^n - \int_0^t r(s) \, ds) = \nu \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} a(u(s) - u^n, v) \, ds \\
+ \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} b(u(s), u(s), v) - b(u^n, u^n, v) \, ds \\
+ \sum_{n=1}^{m} \left( \int_{t_{n-1}}^{t_n} (\eta(u^{n-1}) - \eta(u(s))) \, dW(s), v \right) + (u(t^m) - u^m, v) \\
= \nu \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} (\nabla u(s) - \nabla u(t^n) + \nabla u(t^n) - \nabla u^n, \nabla v) \, ds \\
+ \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} b(u(s), u(s), v) - b(u^n, u^n, v) \, ds \\
+ \sum_{n=1}^{m} \left( \int_{t_{n-1}}^{t_n} (\eta(u^{n-1}) - \eta(u(s))) \, dW(s), v \right) + (u(t^m) - u^m, v).
\]

By using the Poincaré inequality, the Hölder inequality and the inf-sup condition, one finds that

\[
\beta \left\| \sum_{n=1}^{m} r^n - \int_0^t r(s) \, ds \right\|_{L^p} \\
\leq C \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| \nabla u(s) - \nabla u(t^n) \|_{L^p} \, ds \\
+ \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} \| u - u(t^n) \|_{L^p} \| \nabla u \|_{L^p} \| \nabla (u - u(t^n)) \|_{L^p} \, ds \\
+ C \sum_{n=1}^{m} \left( \| \nabla u^n \|_{L^p} \max_{1 \leq t \leq m} \| \nabla u(t) \|_{L^p} + \max_{1 \leq t \leq m} \| \nabla u(t) \|_{L^p} \| \nabla e^n \|_{L^p} \right) \\
+ C \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} (\eta u^{n-1} - \eta u(s)) \, dW(s) \|_{L^p} + C \| u(t^m) - u^m \|_{L^p}.
\]
Taking the expectation, using (3.28) and Theorem 3.3, it follows that

\[
(3.57) \quad \mathbb{E} \left[ I_{\Omega_k} \left( \left\| k \sum_{n=1}^{m} r^n - \int_0^t r(s) \, ds \right\|_{L^2}^2 \right) \right] \leq C k^\alpha \\
+ C \left( \mathbb{E} \left[ I_{\Omega_k} \sum_{n=1}^{m} k \| \nabla e_u^n \|_{L^2}^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \max_{1 \leq l \leq m} \| \nabla u(t_l) \|_{L^2}^2 \right] \right)^{\frac{1}{2}} \\
+ C \left( \mathbb{E} \left[ \max_{1 \leq l \leq m} \| \nabla d^n \|_{L^2}^2 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ I_{\Omega_k} \sum_{n=1}^{m} k \| \nabla e_u^n \|_{L^2}^2 \right] \right)^{\frac{1}{2}} \\
+ \mathbb{E} \left[ \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} (\eta(u^n) - \eta(u(s))) \, dW(s) \right\|_{L^2} \right].
\]

By using (2.2), (2.3), (2.6) and (3.28), the last term can be bounded

\[
(3.58) \quad \mathbb{E} \left[ \left\| \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} (\eta(u^n) - \eta(u(s))) \, dW(s) \right\|_{L^2} \right] \leq C k^{\alpha - \frac{1}{2}}.
\]

Making use of the Lemmas 2.3, 3.1 and Theorem 3.3, the result (3.53) holds. The proof is complete. \( \square \)

**Theorem 3.5.** Let the assumptions of Theorem 3.3 be satisfied. Let \( \{p^n; 1 \leq n \leq N\} \) be the pressure approximation defined by Algorithm 1. Then the following error estimate holds for \( m = 1, 2, \ldots, N \)

\[
(3.59) \quad \mathbb{E} \left[ I_{\Omega_k} \left( \left\| \int_0^{t_m} p(s) \, ds - k \sum_{n=1}^{m} p^n \right\|_{L^2}^2 \right) \right] \leq C k^{2\alpha - \epsilon},
\]

where \( C \) is a positive constant independent of \( k \).

**4. Fully discrete mixed finite element scheme.** In this section we propose and analyze a fully discrete time-stepping scheme for the mixed formulation (2.8). The error estimates in strong norms for both the velocity and pressure approximations are obtained. Furthermore, we derive \( L^2 \) convergence of the scheme by using the negative norm technique.

Suppose that \( \mathcal{T}_h \) is a quasi-uniform family of triangulation of the periodic domain \( D \subset \mathbb{R}^2 \). We define three finite element spaces as follows:

\[
\mathcal{V}_h = \{ v_h \in H^1_{\text{per}}(D); v_h|_K \in P_2(K) \forall K \in \mathcal{T}_h \}, \\
\mathcal{W}_h = \{ q_h \in L^2_{\text{per}}(D); q_h|_K \in P_1(K) \forall K \in \mathcal{T}_h \}, \\
\mathcal{S}_h = \{ w_h \in H^1_{\text{per}}(D); w_h|_K \in P_1(K) \forall K \in \mathcal{T}_h \},
\]

where \( P_l(K) \) (\( l = 1, 2 \)) denotes the set of polynomials of degree less than or equal to \( l \) over the element \( K \in \mathcal{T}_h \).

In addition, we consider the weakly discrete divergence-free subspace \( \mathcal{V}_{0h} \subset \mathcal{V}_h \)

\[
(4.1) \quad \mathcal{V}_{0h} = \left\{ v_h \in \mathcal{V}_h; d(q_h, v_h) = 0, \forall q_h \in \mathcal{W}_h \right\}.
\]

As it is noted [3] that the finite element space pair \((\mathcal{V}_h, \mathcal{W}_h)\) is stable in the sense that the following discrete inf-sup condition holds, i.e., there exists an \( h \)-independent positive constant \( \gamma \) such that

\[
(4.2) \quad \sup_{v_h \in \mathcal{V}_h} \frac{d(v_h, q_h)}{\| \nabla v_h \|_{L^2}} \geq \gamma \| q_h \|_{L^2} \quad \forall q_h \in \mathcal{W}_h.
\]
We define the $L^2(D)$ projections $\rho_h : L^2_{\text{per}}(D) \to W_h$, $\Pi_h : L^2_{\text{per}}(D)^2 \to V_h$ and the $L^2$ Ritz-projection $\sigma_h : H^1_{\text{per}}(D)^2 \to S_h$ such that

\[
(\varphi - \rho_h \varphi, \psi_h) = 0, \quad \forall \psi_h \in W_h, \\
(v - \Pi_h v, w_h) = 0, \quad \forall w_h \in V_h, \\
(\nabla(\phi - \sigma_h \phi), \nabla \chi_h) = 0, \quad \forall \chi_h \in S_h.
\]

The following approximation properties are well-known \cite{18, 6, 11, 13}.

\[
\begin{align*}
(4.3) \quad & \| \varphi - \rho_h \varphi \|_{L^2_2} + h \| \nabla(\varphi - \rho_h \varphi) \|_{L^2_2} \leq C h^r \| \varphi \|_{H^r} \quad \forall \varphi \in H^r_{\text{per}}(D), \\
(4.4) \quad & \| v - \Pi_h v \|_{L^2_2} + h \| \nabla(v - \Pi_h v) \|_{L^2_2} \leq C h^s \| v \|_{H^s} \quad \forall v \in H^s_{\text{per}}(D)^2, \\
(4.5) \quad & \| \phi - \sigma_h \phi \|_{L^2_2} + h \| \nabla(\phi - \sigma_h \phi) \|_{L^2_2} \leq C h^s \| \phi \|_{H^s} \quad \forall \phi \in H^s_{\text{per}}(D)^2/\mathbb{R},
\end{align*}
\]

where $C$ is a positive constant independent of $h$.

Our fully discrete finite element algorithm for (2.8) is defined as follows.

**Algorithm 2:**

Set $u^n_0 \in L^2(\Omega, V_h)$, for $n = 1, \ldots, N$, we define the following steps:

**Step I:** Find $\xi(u_n^{n-1}) \in L^2(\Omega, S_h)$ by solving

\[
(\nabla \xi(u_n^{n-1}), \nabla \phi) = (G(u_n^{n-1}), \nabla \phi), \quad \forall \phi \in S_h \text{ a.s.}
\]

**Step II:** Denote $\eta(u_n^{n-1}) := G(u_n^{n-1}) - \nabla \xi(u_n^{n-1})$, and find $(u^n_n, r^n_n) \in L^2(\Omega, V_h) \times L^2(\Omega, W_h)$ by solving

\[
\begin{align*}
(4.7a) \quad & (u^n_n, v) + k a(u^n_n, v) - k d(v, r^n_n) + k b(u^n_n, u^n_n, v) \\
& = (u_n^{n-1}, v) + (\eta(u_n^{n-1}) \Delta W_n, v) \quad \forall v \in V_h, \text{ a.s.} \\
(4.7b) \quad & d(u^n_n, q) = 0 \quad \forall q \in W_h, \text{ a.s.}
\end{align*}
\]

**Step III:** Denote $p^n_n := r^n_n + k^{-1} \xi(u_n^{n-1}) \Delta_n W$.

We now give the following stabilities for $u^n_n$ and $r^n_n$, but omit their proofs because they are similar to semi-discrete scheme given in \cite{10} \cite{14}.

**Lemma 4.1.** Let $1 \leq p < \infty$ be a natural number. Assume $u^n_0 \in L^2(\Omega, \mathcal{V}(\Omega))$ with $\| u^n_0 \|_{L^2_2} \leq C$. Let $\{(u^n_n, r^n_n, p^n_n); 1 \leq n \leq N\}$ be a solution to Algorithm 2, then there hold

\[
\begin{align*}
(4.8a) \quad & \mathbb{E} \left[ \max_{1 \leq n \leq N} \| u^n_n \|_{L^2_2}^{2p} + \nu k \sum_{n=1}^N \| u^n_n \|_{L^2_2}^{2p-1} \| \nabla u^n_n \|_{L^2_2}^2 \right] \leq C, \\
(4.8b) \quad & \mathbb{E} \left[ k \sum_{n=1}^N \| r^n_n \|_{L^2_2}^2 \right] \leq C.
\end{align*}
\]

**Lemma 4.2.** Let $1 \leq p < \infty$ be a natural number. Assume $u^n_0 \in L^2(\Omega, \mathcal{V}(\Omega))$ with $\| u^n_0 \|_{L^2_2} \leq C$. Then there exists a sequence $\{u^n_n\}_{n \geq 1}$ of $\mathcal{V}$-valued random variables,
which for all $\omega \in \Omega$, solves Algorithm 2 and has the following stability estimates:

\begin{align}
(4.9a) \quad & \mathbb{E}\left[ \max_{1 \leq n \leq N} \|\nabla u^n_h\|_{L^2_x}^2 + \nu k \sum_{n=1}^N \|\nabla u^n_h\|_{L^2_x}^{2p-1} \|\nabla^2 u^n_h\|_{L^2_x} \right] \leq C, \\
(4.9b) \quad & \mathbb{E}\left[ \sum_{n=1}^N \|\nabla(u^n_h - u^{n-1}_h)\|_{L^2_x}^2 \|\nabla u^n_h\|_{L^2_x} \right] \leq C, \\
(4.9c) \quad & \mathbb{E}\left[ \left( \sum_{n=1}^N \|\nabla(u^n_h - u^{n-1}_h)\|_{L^2_x}^2 \right)^4 + \left( k \sum_{n=1}^N \|\nabla^2 u^n_h\|_{L^2_x} \right)^4 \right] \leq C.
\end{align}

We introduce for $\epsilon > 0$ the sample set

\begin{equation}
(4.10) \quad \Omega^\epsilon_h = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} \|\nabla u^n\|_{L^2_x} + \|u^n_h\|_{L^2_x} \leq -\epsilon \log(h^2 + k) \right\}
\end{equation}

such that

\begin{equation}
(4.11) \quad P(\Omega^\epsilon_h) \geq 1 - \frac{\mathbb{E}[\omega \in \Omega] \max_{1 \leq n \leq N} (\|\nabla u^n\|_{L^2_x} + \|u^n_h\|_{L^2_x})}{-\epsilon \log(h^2 + k)} \geq 1 + \frac{C}{\epsilon \log(h^2 + k)}.
\end{equation}

We are now in a position to state and prove the first main theorem of this section.

**Theorem 4.3.** Set $u^0 = u_0$ and let $\{u^n, 1 \leq n \leq N\}$ and $\{u^n_h, 1 \leq n \leq N\}$ be the solutions of Algorithm 1 and Algorithm 2, respectively. Then, provided that $0 < k < k_0$ and $0 < h < h_0$ with $k_0$ and $h_0$ sufficiently small, the following error estimate holds:

\begin{align}
(4.12) \quad & \mathbb{E}\left[ \Omega_h \left\{ \max_{1 \leq n \leq N} \|u^n - u^n_h\|_{L^2_x} + k \sum_{n=1}^N \|\nabla(u^n - u^n_h)\|_{L^2_x} \right\} \right] \leq C(h^{2-2\epsilon} + k^{1-\epsilon}).
\end{align}

**Proof.** For every $n \geq 1$, let $e^{n,h} := u^n - u^n_h$ and $e^{n,\Pi} := r^n - r^n_h$, it is easy to check that the following error equations hold:

\begin{align}
(4.13a) \quad & (e^{n,h}_u - e^{n-1,h}_u, v_h) + k a(e^{n,h}_u, v_h) + k d(v_h, e^{n,h}_r) \\
& \quad + k b(u^n, u^n, v_h) - k b(u^n_h, u^n_h, v_h) \\
& \quad = (\eta(u^{n-1}) - \eta(u^{n-1}_h))\Delta W_n, v_h), \\
(4.13b) \quad & d(e^{n,h}_u, q_h) = 0.
\end{align}

Setting $v_h = \Pi_h e^{n,h}_u$ and $q_h = \rho_h e^{n,h}_r$, we have

\begin{align}
(4.14) \quad & (e^{n,h}_u - e^{n-1,h}_u, \Pi_h e^{n,h}_u) + k a(e^{n,h}_u, \Pi_h e^{n,h}_u) - k d(\Pi_h e^{n,h}_u, e^{n,h}_r) \\
& \quad + k b(u^n, u^n, \Pi_h e^{n,h}_u) - k b(u^n_h, u^n_h, \Pi_h e^{n,h}_u) \\
& \quad = (\eta(u^{n-1}) - \eta(u^{n-1}_h))\Delta W_n, \Pi_h e^{n,h}_u).
\end{align}
By using the identity $a \cdot (a - b) = \frac{1}{2}(|a|^2 - |b|^2 + |a|^2 - |b|^2)$, we gain

\begin{equation}
\frac{1}{2} \left( \| \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} - \| \Pi v e_{u}^{n-1,h} \|_{L_{2}^{2}}^{2} + \| \Pi v e_{u}^{n-1,h} - \Pi v e_{u}^{n-1,h} \|_{L_{2}^{2}}^{2} \right) + k \nu \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} = k \alpha (u^{n} - v_{h}^{n}, \Pi v e_{u}^{n,h}) + k \beta \left( \Pi v e_{u}^{n,h}, e^{p,h} \right) + k b(u^{n-1}, u_{h}^{n}, \Pi v e_{u}^{n,h}) + k b(u^{n-1}, u_{h}^{n}, \Pi v e_{u}^{n,h}) + \left( |(u^{n-1}) - (u_{h}^{n})| \right) \Delta W_{n}, \Pi v e_{u}^{n,h} \\
= \sum_{i=1}^{4} I_{i}.
\end{equation}

For terms $I_{1}$ and $I_{2}$, thanks to the Young’s inequality, (13) and (14), we obtain

\begin{equation}
I_{1} \leq \frac{\nu k}{8} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} + \| \nabla (u^{n} - \Pi v u^{n}) \|_{L_{2}^{2}}^{2} \\
\leq \frac{\nu k}{8} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} + C kh^{2} \| \nabla v^{n} \|_{L_{2}^{2}}^{2},
\end{equation}

\begin{equation}
I_{2} \leq \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}} \| r^{n} - \rho_{h} r^{n} \|_{L_{2}^{2}} \\
\leq \frac{\nu k}{8} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} + C kh^{2} \| \nabla r^{n} \|_{L_{2}^{2}}^{2}.
\end{equation}

For nonlinear term $I_{3}$, we can decomposed as follows:

$$
I_{3} = -k b(u^{n} - \Pi v u^{n}, u^{n}, \Pi v e_{u}^{n,h}) - k b(\Pi v e_{u}^{n,h}, \Pi v e_{u}^{n,h}, u^{n}) - k b(\Pi v e_{u}^{n,h}, \Pi v e_{u}^{n,h}, u^{n} - \Pi v u^{n}) + k b(\Pi v u^{n}, \Pi v e_{u}^{n,h}, u^{n} - \Pi v u^{n})
$$

$$
= \sum_{i=1}^{4} I_{3,i}.
$$

Using the Poincaré inequality, the Young’s inequality, the embedding inequality and (14), one finds that

\begin{equation}
I_{3,1} \leq C k \| u^{n} - \Pi v u^{n} \|_{L_{2}^{2}}^{2} \| \nabla (u^{n} - \Pi v u^{n}) \|_{L_{2}^{2}}^{2} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \\
= \left( \frac{\nu k}{16} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} + C kh^{3} \| \nabla v^{n} \|_{L_{2}^{2}}^{2} \| \nabla u^{n} \|_{L_{2}^{2}}^{2},
\end{equation}

\begin{equation}
I_{3,2} \leq k \| \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \| \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \| u^{n} \|_{L_{2}^{2}}^{2} \\
\leq \left( \frac{\nu k}{16} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} + C k \| \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \| \nabla u^{n} \|_{L_{2}^{2}}^{2},
\end{equation}

\begin{equation}
I_{3,3} \leq k \| \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \| \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \| u^{n} - \Pi v u^{n} \|_{L_{2}^{2}}^{2} \\
\leq \left( \frac{\nu k}{16} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} + C k \| \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \| \nabla u^{n} \|_{L_{2}^{2}}^{2},
\end{equation}

\begin{equation}
I_{3,4} \leq k \| \nabla \Pi v u^{n} \|_{L_{2}^{2}} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}} \| u^{n} - \Pi v u^{n} \|_{L_{2}^{2}} \\
\leq \left( \frac{\nu k}{16} \| \nabla \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} + C k \| \Pi v e_{u}^{n,h} \|_{L_{2}^{2}}^{2} \| \nabla u^{n} \|_{L_{2}^{2}}^{2}.
\end{equation}
Inserting estimates (4.16)–(4.21) into (4.15), we arrive at

\begin{align}
&\frac{1}{2} \left( \|\Pi h e_u \|_{L^2}^2 - \|\Pi h e_u - \Pi h e_u^{n-1,h} \|_{L^2}^2 + \|\Pi h e_u^{n-1,h} - \Pi h e_u^{n,h} \|_{L^2}^2 \right) \\
&+ \frac{k\nu}{2} \|\nabla \Pi h e_u \|_{L^2}^2 \leq C kh^2 \|\nabla^2 u_n \|_{L^2}^2 + C kh^2 \|\nabla r_n \|_{L^2}^2 \\
&+ C kh^3 \|\nabla^2 u_n \|_{L^2}^2 \|\nabla u_n \|_{L^2}^2 + C k \|\Pi h e_u \|_{L^2}^2 \|\nabla u_n \|_{L^2}^2 \\
&+ C kh^2 \|e_u^{n,h} \|_{L^2}^2 \|\nabla u_n \|_{L^2}^2 + C kh^3 \|\nabla u_n \|_{L^2}^2 \|\nabla^2 u_n \|_{L^2}^2 \\
&+ \left( [\eta(u_n^{n-1}) - \eta(u_n^{n-1})] \Delta W_n, \Pi h e_u^{n,h} \right)
\end{align}

Taking the expectation and applying the summation operator \( \sum_{n=1}^N \), one finds that

\begin{align}
&\mathbb{E} \left[ \frac{1}{2} \left( \|\Pi h e_u \|_{L^2}^2 + \frac{1}{2} \sum_{n=1}^N \|\Pi h e_u^{n,h} - \Pi h e_u^{n-1,h} \|_{L^2}^2 + \sum_{n=1}^N \frac{k\nu}{2} \|\nabla \Pi h e_u \|_{L^2}^2 \right) \right] \\
\leq &\frac{1}{2} \mathbb{E} \left[ \|\Pi h e_u \|_{L^2}^2 + C kh^2 \mathbb{E} \left[ k \sum_{n=1}^N \|\nabla^2 u_n \|_{L^2}^2 \right] + C h^3 \mathbb{E} \left[ k \sum_{n=1}^N \|\nabla r_n \|_{L^2}^2 \right] \\
&+ C h^3 \mathbb{E} \left[ k \sum_{n=1}^N \|\nabla^2 u_n \|_{L^2}^2 \|\nabla u_n \|_{L^2}^2 \right] + C h^2 \mathbb{E} \left[ k \sum_{n=1}^N \|\nabla^2 u_n \|_{L^2}^2 \|\nabla^2 u_n \|_{L^2}^2 \right] \\
&+ C \log(h^2 + k)^{-\gamma} \mathbb{E} \left[ \sum_{n=1}^N \|\Pi h e_u \|_{L^2}^2 \right] + \mathbb{E} \left[ \sum_{n=1}^N \left( [\eta(u_n^{n-1}) - \eta(u_n^{n-1})] \Delta W_n, \Pi h e_u^{n,h} \right) \right].
\end{align}

Now we explain how to estimate in expectation for \( \Lambda_i \) (\( i = 1, \ldots, 4 \)). Making use of the Lemmas 3.1 and 4.1, the terms \( \Lambda_i \) (\( i = 1, 3 \)) are uniformly bounded

\begin{align}
\Lambda_1 \leq &\mathbb{E} \left[ \left( \max_{1 \leq m \leq N} \|\nabla u^m \|_{L^2}^2 \right) \left( k \sum_{n=1}^N \|\nabla^2 u_n \|_{L^2}^2 \right) \right] \\
\leq &\left( \mathbb{E} \max_{1 \leq m \leq N} \|\nabla u^m \|_{L^2}^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \sum_{n=1}^N k \|\nabla^2 u_n \|_{L^2}^4 \right)^{\frac{1}{4}}, \\
\Lambda_3 \leq &\mathbb{E} \left[ \left( \max_{1 \leq m \leq N} \|e_u^{m,h} \|_{L^2}^2 \right) \left( k \sum_{n=1}^N \|\nabla u_n \|_{L^2}^2 \right) \right] \\
\leq &\left( \mathbb{E} \max_{1 \leq m \leq N} \|u^m \|_{L^2}^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \sum_{n=1}^N k \|\nabla u_n \|_{L^2}^4 \right)^{\frac{1}{4}} \\
&+ \left( \mathbb{E} \max_{1 \leq m \leq N} \|\eta \|_{L^2}^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \sum_{n=1}^N k \|\nabla u_n \|_{L^2}^8 \right)^{\frac{1}{4}}.
\end{align}
About the term \( A_2 \), using the Lemmas 3.1 and 4.4, we have

\[
E \left[ \sum_{n=1}^{N} ||e_{n,h}^{n}||_{L_2^2}^2 \right] \leq E \left[ \sum_{n=1}^{N} (||u^n||_{L_2^2}^2 + ||u_{n,h}||_{L_2^2}^2) \right] \leq E \left[ \sum_{n=1}^{N} (||u^n||_{L_2^2}^2 + ||u_{n,h}||_{L_2^2}^2) \right]
\]

and the term \( A_4 \) is uniformly bounded as follows:

\[
A_4 \leq E \left[ \left( \max_{1 \leq m \leq N} ||\nabla u^m||_{L_2^2}^2 \right) \left( k \sum_{n=1}^{N} ||\nabla^2 u^n||_{L_2^2}^2 \right) \right] \leq \left( E \max_{1 \leq m \leq N} ||\nabla u^m||_{L_2^2}^4 \right)^{1/4} \left( E \sum_{n=1}^{N} k ||\nabla^2 u^n||_{L_2^2}^4 \right)^{1/4}.
\]

For term \( I_4 \), using Itô’s isometry and the Young’s inequality, we have

\[
(4.24) \quad E \left[ \sum_{n=1}^{N} \left( [\eta(u^{n-1}) - \eta(u_{h}^{n-1})] \Delta W_n, \Pi_h e_{n}^{n,h} \right) \right] = E \left[ \sum_{n=1}^{N} \left( [\eta(u^{n-1}) - \eta(u_{h}^{n-1})] \Delta W_n, \Pi_h e_{n}^{n,h} - \Pi_h e_{n}^{n-1,h} \right) \right] \leq E \left[ k \sum_{n=1}^{N} \left( [\eta(u^{n-1}) - \eta(u_{h}^{n-1})] \Delta W_n \| \Pi_h e_{n}^{n,h} - \Pi_h e_{n}^{n-1,h} \|_{L_2^2} \right) \right] \leq \frac{1}{4} E \left[ \| \Pi_h e_{n}^{n,h} - \Pi_h e_{n}^{n-1,h} \|_{L_2^2}^2 \right] + E \left[ k \sum_{n=1}^{N} \left( [\eta(u^{n-1}) - \eta(u_{h}^{n-1})] \|_{L_2^2}^2 \right) \right].
\]

With the definition of \( \eta \), and using (2.2), (2.3), (2.6), (3.28) and (4.4), one finds that

\[
(4.25) \quad \| \eta(u^{n-1}) - \eta(u_{h}^{n-1}) \|_{L_2^2}^2 \leq C ||e_{n}^{n-1}||_{L_2^2}^2 + C h^2 \| \eta(u^{n-1}) \|_{H_2^2}^2 \leq C h^2 \| \nabla \cdot G(u^{n-1}) \|_{L_2^2}^2 + C ||e_{n}^{n-1}||_{L_2^2}^2 \leq C h^2 \| \nabla u^{n-1} \|_{L_2^2}^2 + C h^4 \| \nabla^2 u^{n-1} \|_{L_2^2}^2 + C \| \Pi_h e_{n}^{n-1,h} \|_{L_2^2}^2.
\]
Combining (4.24)–(4.25) into (4.23), we get

\[
(4.26) \quad \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \frac{1}{2} \left\| \Pi_h e_u^n, h \right\|_{L^2}^2 + \frac{1}{4} \sum_{n=1}^N \left\| \Pi_h e_u^n, h - \Pi_h e_u^{n-1, h} \right\|_{L^2}^2 + \sum_{n=1}^N \frac{k \nu}{2} \left\| \nabla e_u^n, h \right\|_{L^2}^2 \right) \right] 
\leq C(h^2 + k) + C_3 \log(h^2 + k)^{-\epsilon} \mathbb{E} \left[ \sum_{n=1}^N k \left\| \Pi_h e_u^n, h \right\|_{L^2}^2 \right] 
+ C_4 \mathbb{E} \left[ \sum_{n=1}^N k \left\| \Pi_h e_u^{n-1, h} \right\|_{L^2}^2 \right].
\]

If \(0 < h < h_0\) and \(0 < k \leq k_0, k^* := \frac{1}{2C_3 \log(h_0^2 + k_0) - \epsilon} < \frac{1}{C_3 \log(h_0^2 + k_0) - \epsilon}\), since \(1 \leq \frac{1}{1 - C_3 k \log(h_2 + k)} \leq 2\), it follows that

\[
(4.27) \quad \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \frac{1}{2} \left\| \Pi_h e_u^n, h \right\|_{L^2}^2 + \frac{1}{4} \sum_{n=1}^N \left\| \Pi_h e_u^n, h - \Pi_h e_u^{n-1, h} \right\|_{L^2}^2 + \sum_{n=1}^N \frac{k \nu}{2} \left\| \nabla e_u^n, h \right\|_{L^2}^2 \right) \right] 
\leq C(h^2 + k) + \frac{C_3 \log(h^2 + k)^{-\epsilon}}{1 - C_3 k \log(h^2 + k)} \mathbb{E} \left[ \sum_{n=1}^N k \left\| \Pi_h e_u^n, h \right\|_{L^2}^2 \right]
+ \frac{C_4}{1 - C_3 k \log(h^2 + k)^{-\epsilon}} \mathbb{E} \left[ \sum_{n=1}^N k \left\| \Pi_h e_u^{n-1, h} \right\|_{L^2}^2 \right]
\leq C(h^2 + k) + 2C_3 \log(h^2 + k)^{-\epsilon} + C_4 \mathbb{E} \left[ \sum_{n=1}^N k \left\| \Pi_h e_u^{n-1, h} \right\|_{L^2}^2 \right].
\]

Then (4.12) follows from an application of the discrete Gronwall inequality and the triangle inequality. □

The second result of this section is the following error estimate for the pressure approximation \(\{r^n_1, 1 \leq n \leq N\}\) and \(\{p^n_1; 1 \leq n \leq N\}\).

**Theorem 4.4.** Let the assumptions of Theorem 3.3 be satisfied. Let \(\{r^n_1; 1 \leq n \leq N\}\) be the pressure approximation defined by Algorithm 2. Then the following error estimate holds for \(m = 1, 2, \cdots, N\)

\[
(4.28) \quad \mathbb{E} \left[ \mathbf{1}_{\Omega_h} \left( \left\| \sum_{n=1}^N (r^n - r^n_1) \right\|_{L^2}^2 \right) \right] \leq C(h^{2-2\epsilon} + k^{1-\epsilon}),
\]

where \(C\) is a positive constant independent of \(h\) and \(k\).

**Proof.** Summing (4.7a) over \(1 \leq n \leq m(\leq N)\) and subtracting the resulting equation from (3.54), we have

\[
(4.29) \quad (e_u^m, h) + k \sum_{n=1}^m a(e_u^n, h, v_h) - k \sum_{n=1}^m d(v_h, e_r^n)
+ k \sum_{n=1}^m [b(u^n, v_h, v_h) - b(u^n, u^n, v_h)] 
= (e_u^0, v_h) + \sum_{n=1}^m [(\eta(u^n) - \eta(u^{n-1})) \Delta W_n(v_h), \forall v_h \in \mathcal{V}_h, a.s.]
\]
Here $e_{u}^{m,h}$ and $e_{r}^{n,h}$ are the same as in the proof of the Theorem 4.3. Using the Poincaré inequality, the Hölder inequality and the embedding inequality, it follows that

\begin{equation}
\begin{aligned}
&d(v_h, k \sum_{n=1}^{m} e_{r}^{n,h}) = (e_{0}^{h} - e_{u}^{n,h}, v_h) - k \sum_{n=1}^{m} a(e_{u}^{n,h}, v_h) \\
&\quad + k \sum_{n=1}^{m} [b(u_{h}^{n}, u_{h}^{n}, v_h) - b(u^n, u^n, v_h)] \\
&\quad + \sum_{n=1}^{m} ([\eta(u^{n-1}) - \eta(u_{h}^{n-1})] \Delta W_{n}, v_h) \\
&\leq C \left( \|e_{0}^{0,h}\|_{L^2} + \|e_{u}^{n,h}\|_{L^2} + \sum_{n=1}^{m} k \|\nabla e_{u}^{n,h}\|_{L^2} \\
&\quad + \sum_{n=1}^{m} \|\nabla e_{u}^{n,h}\|_{L^2} \|\nabla u^n\|_{L^2} + \sum_{n=1}^{m} k \|\nabla u_{h}^{n}\|_{L^2} \|\nabla e_{u}^{n,h}\|_{L^2} \\
&\quad + \| \sum_{n=1}^{m} [\eta(u^{n-1}) - \eta(u_{h}^{n-1})] \Delta W_{n} \|_{L^2} \right) \|\nabla v\|_{L^2}.
\end{aligned}
\end{equation}

Applying the discrete inf-sup condition (4.2), we obtain

\begin{equation}
\begin{aligned}
\gamma \left\| k \sum_{n=1}^{m} e_{r}^{n,h} \right\|_{L^2} &\leq C \left( \|e_{0}^{0,h}\|_{L^2} + \|e_{u}^{n,h}\|_{L^2} + \sum_{n=1}^{m} k \|\nabla e_{u}^{n,h}\|_{L^2} \\
&\quad + \sum_{n=1}^{m} \|\nabla e_{u}^{n,h}\|_{L^2} \|\nabla u^n\|_{L^2} + \sum_{n=1}^{m} k \|\nabla u_{h}^{n}\|_{L^2} \|\nabla e_{u}^{n,h}\|_{L^2} \\
&\quad + \| \sum_{n=1}^{m} [\eta(u^{n-1}) - \eta(u_{h}^{n-1})] \Delta W_{n} \|_{L^2} \right).
\end{aligned}
\end{equation}

Taking the expectation, one finds that

\begin{equation}
\begin{aligned}
\mathbb{E} \left[ 1_{\Omega_{h}} \left( \left\| k \sum_{n=1}^{m} e_{r}^{n,h} \right\|_{L^2} \right) \right] &\leq C \mathbb{E} \left[ \|e_{0}^{0,h}\|_{L^2} \right] + C \mathbb{E} \left[ \|e_{u}^{n,h}\|_{L^2} \right] + C \mathbb{E} \left[ \sum_{n=1}^{m} k \|\nabla e_{u}^{n,h}\|_{L^2} \right] \\
&\quad + C \mathbb{E} \left[ 1_{\Omega_{h}} \left( \sum_{n=1}^{m} k \|\nabla e_{u}^{n,h}\|_{L^2} \right) \left( \max_{1 \leq n \leq m} \|\nabla u^n\|_{L^2} \right) \right] \\
&\quad + C \mathbb{E} \left[ 1_{\Omega_{h}} \left( \sum_{n=1}^{m} k \|\nabla e_{u}^{n,h}\|_{L^2} \right) \left( \max_{1 \leq n \leq m} \|\nabla u_{h}^{n}\|_{L^2} \right) \right] \\
&\quad + C \mathbb{E} \left[ \left\| \sum_{n=1}^{m} [\eta(u^{n-1}) - \eta(u_{h}^{n-1})] \Delta W_{n} \right\|_{L^2} \right].
\end{aligned}
\end{equation}
By a standard calculation, it follows that

\[ (4.33) \]
\[
E \left[ \mathbf{1}_{\Omega_h} \left( \sum_{n=1}^{m} e_n^{n,h} \right) \right] \leq C E \left[ \| u_0^{0,h} \|^2_{L_2^2} \right] + C E \left[ \| e_u^{n,h} \|^2_{L_2^2} \right] + \sum_{n=1}^{m} k \| \nabla e_u^{n,h} \|^2_{L_2^2} \\
+ C \left( E \left[ \mathbf{1}_{\Omega_h} \left( \sum_{n=1}^{m} k \| \nabla e_u^{n,h} \|^2_{L_2^2} \right) \right] \right)^{\frac{1}{2}} \left( E \left[ \max_{1 \leq n \leq m} \| \nabla u^n \|^2_{L_2^2} \right] \right)^{\frac{1}{2}} \\
+ C \left( E \left[ \mathbf{1}_{\Omega_h} \left( \sum_{n=1}^{m} k \| \nabla e_u^{n,h} \|^2_{L_2^2} \right) \right] \right)^{\frac{1}{2}} \left( E \left[ \max_{1 \leq n \leq m} \| \nabla u^n \|^2_{L_2^2} \right] \right)^{\frac{1}{2}} \\
+ C E \left[ \sum_{n=1}^{m} \left( \eta (u^{n-1}) - \eta (u_h^{n-1}) | \Delta W_n | \right) \right].
\]

With using Lemma 3.1 and (4.9a), the last term in (4.33) can be bounded as (4.24)-(4.28) which gives the desired result (4.28). The proof is complete.

**Theorem 4.5.** Let the assumptions of Theorem 4.3 be satisfied. Let \( \{ p^n_h ; 1 \leq n \leq N \} \) be the pressure approximation defined by Algorithm 2. Then the following error estimate holds for \( m = 1, 2, \cdots, N \)

\[ (4.34) \]
\[
E \left[ I_{\Omega_h} \left( \left( \sum_{n=1}^{N} (p^n - p_h^n) \right) \right) \right] \leq C (h^{2-2\epsilon} + k^{1-\epsilon}),
\]

where \( C \) is a positive constant independent of \( h \) and \( k \).

We introduce for \( \epsilon > 0 \) the sample set

\[ (4.35) \]
\[
\Omega_{h,\epsilon} = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} \left( \| \nabla^2 u^n \|^4_{L_2^2} + \| \nabla u_h^n \|^4_{L_2^2} \right) \leq (h^2 + k)^{-\epsilon} \right\}
\]

such that

\[ (4.36) \]
\[
P(\Omega_{h,\epsilon}) \geq 1 - \frac{E \left[ \omega \in \Omega \mid \max_{1 \leq n \leq N} \left( \| \nabla^2 u^n \|^4_{L_2^2} + \| \nabla u_h^n \|^4_{L_2^2} \right) \right]}{(h^2 + k)^{-\epsilon}} \geq 1 - \frac{C}{(h^2 + k)^{-\epsilon}}.
\]

The following theorems state and derive \( L^2 \) convergence of the scheme by using the negative norm technique.

**Theorem 4.6.** Set \( u^0 = u_0 \) and let \( \{ u^n ; 1 \leq n \leq N \} \) and \( \{ u_h^n ; 1 \leq n \leq N \} \) be the solutions of Algorithm 1 and Algorithm 2, respectively. Then, provided that \( 0 < k < k_0 \) and \( 0 < h < h_0 \) with \( k_0 \) and \( h_0 \) sufficiently small, the following error estimate holds:

\[ (4.37) \]
\[
E \left[ I_{\Omega_{h,\epsilon}}^n \left( \sum_{1 \leq n \leq N} \left( \| \nabla (u^n - u_h^n) \|^2_{L_2^2} + k \sum_{n=1}^{N} \| A(u^n - u_h^n) \|^2_{L_2^2} \right) \right) \right] \leq C (h^{2-3\epsilon} + k^{1-2\epsilon}),
\]

where \( C \) is a positive constant independent of \( h \) and \( k \).
Proof. Taking \( v_h = \Pi_h A_h e_u^{n,h} \in V_h \) and \( q_h = 0 \) in (4.13), we have

\[
(4.38) \quad \frac{1}{2} (\| \nabla \Pi_h e_u^{n,h} \|^2_{L_2} - \| \nabla \Pi_h e_u^{n-1,h} \|^2_{L_2} + \| \nabla \Pi_h e_u^{n-1,h} - \nabla \Pi_h e_u^{n-1,h} \|^2_{L_2}) + k \nu \| A_h \Pi_h e_u^{n,h} \|^2_{L_2} = k \nu (A_h(u^n - u_h^n), A_h \Pi_h e_u^{n,h}) + k b(u^n, u_h^n, A_h \Pi_h e_u^{n,h}) - k b(u^n, u^n, \Pi_h A_h e_u^{n,h}) + ([\eta(u^{n-1}) - \eta(u_h^{n-1})] \Delta W_n, \Pi_h A_h e_u^{n,h}) \\
= II_1 + II_2 + III.
\]

For term \( II_1 \), thanks to the Young’s inequality and (4.14), we obtain

\[
(4.39) \quad II_1 \leq \frac{\nu k}{8} \| A_h \Pi_h e_u^{n,h} \|^2_{L_2} + \| A_h[u^n - \Pi_h u^n] \|^2_{L_2} \leq \frac{\nu k}{8} \| A_h \Pi_h e_u^{n,h} \|^2_{L_2} + C k h^2 \| \nabla^3 u^n \|^2_{L_2}.
\]

For nonlinear term \( II_2 \), we can decomposed as follows:

\[
II_2 = -k b(u^n - u_h^n, u^n, A_h \Pi_h e_u^{n,h}) - k b(u_h^n, u^n - u_h^n, A_h \Pi_h e_u^{n,h}) \\
= II_{2,1} + II_{2,2}.
\]

Using the Poincaré inequality, the Young’s inequality, the embedding inequality and (4.3) , one finds that

\[
(4.40) \quad II_{2,1} \leq C k \| u^n - u_h^n \|^2_{L_2} + \| \nabla (u^n - u_h^n) \|^2_{L_2} \| A_h u^n \|_{L_2} \| A_h \Pi_h e_u^{n,h} \|_{L_2} \\
\leq \frac{k^2}{8} \| A_h \Pi_h e_u^{n,h} \|^2_{L_2} + C k \| u^n - u_h^n \|^2_{L_2} + C k \| \nabla (u^n - u_h^n) \|^2_{L_2} \| A_h u^n \|_{L_2}^2,
\]

\[
(4.41) \quad II_{2,2} \leq k \| \nabla u_h^n \|^2_{L_2} \| \nabla (u^n - u_h^n) \|^2_{L_2} \| A_h(u^n - u_h^n) \|^2_{L_2} \| A_h \Pi_h e_u^{n,h} \|^2_{L_2} \\
\leq \frac{k^2}{8} \| A_h \Pi_h e_u^{n,h} \|^2_{L_2} + C k \| \nabla (u^n - u_h^n) \|^2_{L_2} \| u_h^n \|_{L_2}^4 + C k h^2 \| \nabla^3 u^n \|^2_{L_2}.
\]

Inserting estimates (4.39)–(4.41) into (4.38), we have

\[
(4.42) \quad \frac{1}{2} (\| \nabla \Pi_h e_u^{n,h} \|^2_{L_2} - \| \nabla \Pi_h e_u^{n-1,h} \|^2_{L_2} + \| \nabla \Pi_h e_u^{n-1,h} - \nabla \Pi_h e_u^{n-1,h} \|^2_{L_2}) + k \nu \| A_h \Pi_h e_u^{n,h} \|^2_{L_2} \leq C k h^2 \| \nabla^3 u^n \|^2_{L_2} + C k \| u^n - u_h^n \|^2_{L_2} \\
+ C k \| \nabla (u^n - u_h^n) \|^2_{L_2} \| A_h u^n \|^2_{L_2} + C k \| \nabla (u^n - u_h^n) \|^2_{L_2} \| u_h^n \|^4_{L_2} + \left( \eta(u^{n-1}) - \eta(u_h^{n-1}) \right) \Delta W_n, \Pi_h A_h e_u^{n,h}).
\]
Taking the expectation and applying the summation operator \( \sum_{n=1}^{N} \), one finds that

\[
(4.43) \quad \mathbb{E}\left[ 1_{\Omega_{k,h}^{n,h}} \left( \frac{1}{2} \| \nabla \Pi_h e_u^{n,h} \|_{L_2}^2 + \frac{1}{2} \sum_{n=1}^{N} \| \nabla \Pi_h e_u^{n,h} - \Pi_h e_u^{n-1,h} \|_{L_2}^2 + \sum_{n=1}^{N} \frac{k\nu}{2} \| A_h \Pi_h e_u^{n,h} \|_{L_2}^2 \right) \right]
\]

\[
\leq C(h^{2-2\epsilon} + k^{1-\epsilon})(h^2 + k)^{-\epsilon} + \frac{1}{2} \mathbb{E}[\| \nabla \Pi_h e_u^{0,h} \|_{L_2}^2] + C h^2 \mathbb{E}\left[ k \sum_{n=1}^{N} \| \nabla \Lambda u^n \|_{L_2}^2 \right]
\]

\[
+ C k \mathbb{E}\left[ \frac{1}{2} \sum_{n=1}^{N} \| \nabla (u^n - u_h^n) \|_{L_2}^2 \| A_h (u^n - u^{n-1}) \|_{L_2}^4 \right]
\]

\[
+ C k \mathbb{E}\left[ \sum_{n=1}^{N} \| \nabla (u^n - u_h^n) \|_{L_2}^2 \| \nabla (u_h^n - u_h^{n-1}) \|_{L_2}^4 \right]
\]

\[
+ \mathbb{E}\left[ \sum_{n=1}^{N} \left( [\eta(u^{n-1}) - \eta(u_h^{n-1})] \Delta W_n, A_h \Pi_h e_u^{n,h} \right) \right].
\]

Now we explain how to estimate in expectation for \( \Lambda_5 \) and \( \Lambda_6 \). Using the Lemmas 3.1, 3.2 and Lemma 4.2, the terms \( \Lambda_5 \) and \( \Lambda_6 \) are uniformly bounded

\[
\Lambda_5 \leq \mathbb{E}\left[ \left( \max_{1 \leq m \leq N} \| \nabla (u^n - u_h^n) \|_{L_2}^2 \right) \left( \sum_{n=1}^{N} \| A_h (u^n - u^{n-1}) \|_{L_2}^4 \right) \right]
\]

\[
\leq \left( \mathbb{E} \max_{1 \leq m \leq N} \| \nabla u^n \|_{L_2}^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \sum_{n=1}^{N} \| A_h (u^n - u^{n-1}) \|_{L_2}^4 \right)^{\frac{1}{4}}
\]

\[
+ \left( \mathbb{E} \max_{1 \leq m \leq N} \| \nabla u_h^n \|_{L_2}^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \sum_{n=1}^{N} \| A_h (u^n - u^{n-1}) \|_{L_2}^8 \right)^{\frac{1}{4}}.
\]

and

\[
\Lambda_6 \leq \mathbb{E}\left[ \left( \max_{1 \leq m \leq N} \| \nabla (u^n - u_h^n) \|_{L_2}^4 \right) \left( \sum_{n=1}^{N} \| \nabla (u_h^n - u_h^{n-1}) \|_{L_2}^4 \right) \right]
\]

\[
\leq \left( \mathbb{E} \max_{1 \leq m \leq N} \| \nabla u^n \|_{L_2}^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \sum_{n=1}^{N} \| \nabla (u_h^n - u_h^{n-1}) \|_{L_2}^4 \right)^{\frac{1}{4}}
\]

\[
+ \left( \mathbb{E} \max_{1 \leq m \leq N} \| \nabla u_h^n \|_{L_2}^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \sum_{n=1}^{N} \| \nabla (u_h^n - u_h^{n-1}) \|_{L_2}^8 \right)^{\frac{1}{4}}.
\]
For term $II_3$, using the Itô’s isometry and the Young’s inequality, we have

\begin{align}
(4.44) & \quad \mathbb{E} \left[ \sum_{n=1}^{N} \left( \nabla [\eta(u^{n-1}) - \eta(u_n^{n-1})] \Delta W_n, \nabla \Pi_h e_n^{n,h} \right) \right] \\
& = \mathbb{E} \left[ \sum_{n=1}^{N} \left( \nabla [\eta(u^{n-1}) - \eta(u_n^{n-1})] \Delta W_n, \nabla \Pi_h e_n^{n,h} - \nabla \Pi_h e_n^{n-1,h} \right) \right] \\
& \leq \mathbb{E} \left[ k \sum_{n=1}^{N} \| \nabla [\eta(u^{n-1}) - \eta(u_n^{n-1})] \Delta W_n \|_{L^2} \| \nabla (\Pi_h e_n^{n,h} - \Pi_h e_n^{n-1,h}) \|_{L^2} \right] \\
& \leq \frac{1}{4} \mathbb{E} \left[ \| \nabla (\Pi_h e_n^{n,h} - \Pi_h e_n^{n-1,h}) \|_{L^2}^2 \right] + \mathbb{E} \left[ k \sum_{n=1}^{N} \| \nabla [\eta(u^{n-1}) - \eta(u_n^{n-1})] \|_{L^2} \right].
\end{align}

By the definition of $\eta$ and using (4.23), (2.23), (2.10), (5.25) and (4.4), it follows that

\begin{align}
(4.45) & \quad \| \nabla [\eta(u^{n-1}) - \eta(u_n^{n-1})] \|_{L^2}^2 \\
& \leq C \| \nabla e_n^{n-1} \|_{L^2}^2 + Ch^2 \| \eta(u^{n-1}) \|_{H^2}^2 \\
& \leq Ch^2 \| \nabla^2 G(u^{n-1}) \|_{L^2}^2 + C \| \nabla e_n^{n-1} \|_{L^2}^2 \\
& \leq Ch^2 \| \nabla^2 u^{n-1} \|_{L^2}^2 + Ch^2 \| \nabla^3 u^{n-1} \|_{L^2}^2 + C \| \nabla \Pi_h e_n^{n-1,h} \|_{L^2}^2.
\end{align}

Combining (4.44)–(4.45) into (4.43), we get

\begin{align}
(4.46) & \quad \mathbb{E} \left[ \Omega_{h}^{n} \cap \Omega_{\tilde{h}}^{n} \left( \frac{1}{2} \| \nabla \Pi_h e_n^{n,h} \|_{L^2}^2 \right) \\
& + \frac{1}{4} \sum_{n=1}^{N} \| \nabla (\Pi_h e_n^{n,h} - \Pi_h e_n^{n-1,h}) \|_{L^2}^2 \\
& + \sum_{n=1}^{N} \frac{k}{2} \| A_h e_n^{n,h} \|_{L^2}^2 \right) \leq C(h^{2-3\epsilon} + k^{1-2\epsilon}) + C \mathbb{E} \left[ \sum_{n=1}^{N} k \| \nabla \Pi_h e_n^{n-1,h} \|_{L^2}^2 \right].
\end{align}

Then the (4.37) follows from an application of the discrete Gronwall inequality and the triangle inequality. \qed

We introduce for $\epsilon > 0$ the sample set

\begin{align}
(4.47) & \quad \Omega_{\epsilon,h}^{n} = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} \left( \| \nabla^2 u^n \|_{L^2} + \| \nabla \nabla^3 u_n \|_{L^2} \right) \leq -\epsilon \log(h^4 + k) \right\}
\end{align}

such that

\begin{align}
(4.48) & \quad \mathbb{P}(\Omega_{\epsilon,h}^{n}) \geq 1 - \frac{\mathbb{E} \left[ \omega \in \Omega \mid \max_{1 \leq n \leq N} \left( \| \nabla^2 u^n \|_{L^2} + \| \nabla \nabla^3 u_n \|_{L^2} \right) \right]}{-\epsilon \log(h^4 + k)} \geq 1 + \frac{C}{\epsilon \log(h^4 + k)}.
\end{align}

For $\kappa_0 > 0$, the following sample set is defined as

\begin{align}
(4.49) & \quad \Omega_{\kappa_0}^{n} = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} \| u^n - u_n^{n,h} \|_{L^2}^2 \leq \kappa_0(h^{2-2\epsilon} + k^{1-2\epsilon}) \right\}.
\end{align}

**Theorem 4.7.** Set $u^0 = u_0$ and let \{u^n; 1 \leq n \leq N\} and \{u_n^{n,h}; 1 \leq n \leq N\} be the solutions of Algorithm 1 and Algorithm 2, respectively. Assume that
\[
\begin{align*}
k \log(h^4 + k)^{-\epsilon} &\leq C. \text{ Then, provided that } 0 < k < k_0 \text{ and } 0 < h < h_0 \text{ with } k_0 \text{ and } h_0 \\
\text{sufficiently small, the following error estimate holds:}
\end{align*}
\]
\[
(4.50)
\]
\[
\mathbb{E}[I_{\Omega_n}(e_h^n, u_h^n, A_h e_h^n) \leq C(k_0)(h^{4-\epsilon} + k^{1-3\epsilon}),
\]

where \( C(k_0) \) is a positive constant independent of \( h \) and \( k \).

Proof. Setting \( v_h = \Pi_h A_h^{-1} e_h^n \in \mathcal{V}_h \) and \( q_h = 0 \) in (4.13), we gain
\[
(4.51)
\]
\[
\frac{1}{2} \left( \| \Pi_h A_h^{-\frac{1}{2}} e_h^n \|_2^2 + \| \Pi_h A_h^{-\frac{1}{2}} e_h^n - \Pi_h A_h^{-\frac{1}{2}} e_h^{n-1} \|_2^2 + \| \Pi_h A_h^{-\frac{1}{2}} e_h^n - \Pi_h A_h^{-\frac{1}{2}} e_h^{n-1} \|_2 \right)
\]
\[
+ \frac{k}{8} \| \Pi_h e_h^n \|_2^2 + \| u^n - \Pi_h u^n \|_2^2
\]
\[
\leq \frac{k}{8} \| \Pi_h e_h^n \|_2^2 + Ck^4 \| \nabla u^n \|_2^2.
\]

For nonlinear term \( III_2 \), we can decomposed as follows:
\[
III_2 = -k b(u^n - u^n_h, u_h^n, \Pi_h A_h^{-1} e_h^n) - k b(u^n_h, u^n - u^n_h, \Pi_h A_h^{-1} e_h^n)
\]
\[
= III_{2,1} + III_{2,2}.
\]

Using the Poincaré inequality, the Young’s inequality and the embedding inequality, one finds that
\[
(4.53)
\]
\[
III_{2,1,1} \leq \frac{k}{4} \| \Pi_h e_h^{n-1, h} \|_2^2 + Ck \| \nabla u^n \|_2 \| u^n - \Pi_h u^n \|_2
\]
\[
+ Ck \| \nabla u^n \|_2 \| \Pi_h e_h^{n, h} \|_2^2 + Ck \| \nabla u^n \|_2 \| \Pi_h e_h^{n-1, h} \|_2^2
\]
\[
+ \frac{k}{8} \| \Pi_h e_h^n \|_2^2 + Ck \| \nabla u^n \|_2 \| (u^n - u_h^n) \|_2^2.
\]

Inserting estimates (4.52) into (4.51), we have
\[
(4.55)
\]
\[
\frac{1}{2} \left( \| \Pi_h e_h^n \|_2^2 - \| \Pi_h e_h^{n-1, h} \|_2^2 + \| \Pi_h e_h^n - \Pi_h e_h^{n-1, h} \|_2 \right)
\]
\[
+ \frac{k}{2} \| \Pi_h e_h^n \|_2^2 \leq Ck^4 \| \nabla u^n \|_2 + Ck \| \nabla u^n_h \|_2 \| \Pi_h e_h^n \|_2^2
\]
\[
+ Ck \| \Pi_h e_h^n \|_2 \| (u^n - u_h^n) \|_2 + Ck \| \nabla u^n_h \|_2 \| (u^n - u_h^n) \|_2^2
\]
\[
+ ((\eta(u^{n-1}) - \eta(u^{n-1})) \Delta W_n, \Pi_h A_h^{-1} e_h^n).
\]
Taking the expectation and applying the summation operator $\sum_{n=1}^{N}$, one finds that

\begin{equation}
\mathbb{E}
\left[
\sum_{n=1}^{N} \left(\frac{1}{2} \left| \Pi_k e_n^{u,h} \right|^2_{-1} + \frac{1}{2} \sum_{n=1}^{N} \left| \Pi_k e_n^{u,h} - \Pi_k e_{n-1}^{u,h} \right|^2_{-1} + \sum_{n=1}^{N} \frac{k\nu}{4} \left| \Pi_k e_n^{u,h} \right|^2_{L_2^2}\right)
\right]
\leq \frac{1}{2} \mathbb{E} \left[ \left| \Pi_k e_u^{0,h} \right|^2_{L_2^2} \right] + C h_4 \mathbb{E} \left[ k \sum_{n=1}^{N} \left| \nabla u_n \right|^2_{L_2^2} \right] + C \log(h^4 + k)^{-\epsilon} \mathbb{E} \left[ k \sum_{n=1}^{N} \left| \Pi_k e_n^{u,h} \right|^2_{L_2^2} \right]
\end{equation}

\begin{equation}
+ C(k)(h^{2-3\epsilon} + k^{1-2\epsilon}) + C k^2 \mathbb{E} \left[ \sum_{n=1}^{N} \left| \nabla(u_n - u_{n-1}) \right|^2_{L_2^2} \right] \left| \nabla(u_n - u_n) \right|^2_{L_2^2}
\end{equation}

\begin{equation}
+ \mathbb{E} \left[ \sum_{n=1}^{N} \left( A_h^{-\epsilon} [\eta(u_{n-1}) - \eta(u_{n-1})] \Delta W_n, A_h^{-\epsilon} \Pi_k e_n^{u,h} \right) \right].
\end{equation}

Now we explain how to estimate in expectation for $\Lambda_7$ and $\Lambda_8$. Making use of the Lemma 3.2 and Lemma 4.1, the terms $\Lambda_7$ and $\Lambda_8$ are uniformly bounded

\begin{align}
\Lambda_7 \leq & \mathbb{E} \left[ \left( \max_{1 \leq m \leq N} \left| u_m - u_h \right|^2 \left| \sum_{n=1}^{N} \left| \nabla^2(u_h - u_{h-1}) \left| L_2^2 \right|^2 \right) \right) \right] \\
\leq & \left( \mathbb{E} \max_{1 \leq m \leq N} \left| u_m \right|^4 \right)^{\frac{1}{2}} \left( \mathbb{E} \sum_{n=1}^{N} \left| \nabla^2(u_h - u_{h-1}) \left| L_2^4 \right|^4 \right) \right)^{\frac{1}{2}} \\
+ & \left( \mathbb{E} \max_{1 \leq m \leq N} \left| u_h \right|^4 \right)^{\frac{1}{2}} \left( \mathbb{E} \sum_{n=1}^{N} \left| \nabla^2(u_h - u_{h-1}) \left| L_2^4 \right| \right|^2 \right)^{\frac{1}{2}},
\end{align}

\begin{align}
\Lambda_8 \leq & \mathbb{E} \left[ \left( \max_{1 \leq m \leq N} \left| \nabla(u_m - u_h) \left| L_2^2 \right|^2 \right) \left| \sum_{n=1}^{N} \left| \nabla(u - u_{n-1}) \left| L_2^2 \right|^2 \right) \right) \right] \\
\leq & \left( \mathbb{E} \max_{1 \leq m \leq N} \left| \nabla u_m \right| L_2^4 \right)^{\frac{1}{2}} \left( \mathbb{E} \sum_{n=1}^{N} \left| \nabla(u - u_{n-1}) \left| L_2^4 \right|^4 \right) \right)^{\frac{1}{2}} \\
+ & \left( \mathbb{E} \max_{1 \leq m \leq N} \left| \nabla u_h \right| L_2^4 \right)^{\frac{1}{2}} \left( \mathbb{E} \sum_{n=1}^{N} \left| \nabla(u - u_{n-1}) \left| L_2^4 \right| \right|^4 \right)^{\frac{1}{2}}.
\end{align}
For term $III_3$, using the Itô’s isometry and the Young’s inequality, we have

\begin{equation}
\mathbb{E} \left[ \sum_{n=1}^{N} \left( A_h^{-1/2} [\eta(u^{n-1}) - \eta(u_h^{n-1})] \Delta W_n, \Pi_h A_h^{-1/2} e^{u,n,h}_u \right) \right] = \mathbb{E} \left[ \sum_{n=1}^{N} \left( A_h^{-1/2} [\eta(u^{n-1}) - \eta(u_h^{n-1})] \Delta W_n, \Pi_h A_h^{-1/2} e^{n,h}_u - \Pi_h A_h^{-1/2} e^{n-1,h}_u \right) \right] \leq \mathbb{E} \left[ \sum_{n=1}^{N} \| A_h^{-1/2} [\eta(u^{n-1}) - \eta(u_h^{n-1})] \Delta W_n \|_{L_2} \| \Pi_h A_h^{-1/2} e^{n,h}_u - \Pi_h A_h^{-1/2} e^{n-1,h}_u \|_{L_2} \right] \leq \frac{1}{4} \mathbb{E} \left[ \| \Pi_h e^{n,h}_u - \Pi_h e^{n-1,h}_u \|_{L_2}^2 \right] + \mathbb{E} \left[ \sum_{n=1}^{N} \| [\eta(u^{n-1}) - \eta(u_h^{n-1})] \|_{L_2}^2 \right].
\end{equation}

By the definition of $\eta$ and using (4.22), (2.20), (2.6), (3.28) and (4.4), one finds that

\begin{equation}
\| [\eta(u^{n-1}) - \eta(u_h^{n-1})] \|_{L_2}^2 \leq C \| e^{n-1}_u \|_{L_2}^2 + C h^2 \| [\eta(u^{n-1})] \|_{H_1^2} \leq C h^4 \| \nabla \cdot G(u^{n-1}) \|_{L_2}^2 + C \| e^{n-1}_u \|_{L_2}^2 \leq C h^4 \| \nabla u^{n-1} \|_{L_2}^2 + C h^4 \| \nabla^2 u^{n-1} \|_{L_2}^2 + C \| \Pi_h e^{n-1,h}_u \|_{L_2}^2.
\end{equation}

Combining (4.57)–(4.58) into (4.56), we get

\begin{equation}
\mathbb{E} \left[ \sum_{n=1}^{N} \left( A_h^{-1/2} [\eta(u^{n-1}) - \eta(u_h^{n-1})] \Delta W_n, \Pi_h A_h^{-1/2} e^{n,h}_u - \Pi_h A_h^{-1/2} e^{n-1,h}_u \right) \right] \leq C(k_0)(h^{4-6\epsilon} + k^{1-2\epsilon}) + C \log(h^4 + k) \mathbb{E} \left[ \sum_{n=1}^{N} \| \Pi_h e^{n,h}_u \|_{L_2}^2 \right].
\end{equation}

Using the similar line in the proof of Theorem 4.3, the (4.59) follows from an application of the discrete Gronwall inequality and the triangle inequality. \(\square\)

For $k > 0$, we introduce the following sample set

\begin{equation}
\Omega_k = \left\{ \omega \in \Omega \mid \max_{1 \leq n \leq N} \| \nabla (u^n - u_h^n) \|_{L_2}^2 \leq k(h^{2-2\epsilon} + k^{1-2\epsilon}) \right\}.
\end{equation}

**Theorem 4.8.** Set $u^0 = u_0$ and let $\{ u^n; 1 \leq n \leq N \}$ and $\{ u^n_h; 1 \leq n \leq N \}$ be the solutions of Algorithm 1 and Algorithm 2, respectively. Then, provided that $0 < k < k_0$ and $0 < h < h_0$ with $k_0$ and $h_0$ sufficiently small, the following error estimate holds:

\begin{equation}
\mathbb{E} \left[ \sum_{n=1}^{N} \left( A_h^{-1/2} [\eta(u^{n-1}) - \eta(u_h^{n-1})] \Delta W_n, \Pi_h A_h^{-1/2} e^{n,h}_u - \Pi_h A_h^{-1/2} e^{n-1,h}_u \right) \right] \leq C(k_0, \kappa)(h^{4-7\epsilon} + k^{1-3\epsilon}),
\end{equation}

where $C(k_0, \kappa)$ is a positive constant independent of $h$ and $k$.  

Proof. Taking $v_h = \Pi_h e_u^{n,h} \in V_{0h}$ and $q_h = 0$, we have

\begin{equation}
\frac{1}{2}\left(\|\Pi_h e_u^{n,h}\|_{L^2}^2 - \|\Pi_h e_u^{n-1,h}\|_{L^2}^2 + \|\Pi_h e_u^{n,h} - \Pi_h e_u^{n-1,h}\|_{L^2}^2 + k \nu \|\nabla \Pi_h e_u^{n,h}\|_{L^2}^2 = k a(u^n - u_h^n, \Pi_h e_u^{n,h}) + k b(u^n - u_h^n, \Pi_h e_u^{n,h}) + k \nu \|\nabla \Pi_h e_u^{n,h}\|_{L^2}^2 = IV_1 + IV_2 + IV_3.
\end{equation}

For term $IV_1$, thanks to the Young’s inequality, (4.3) and (4.4), we obtain

\begin{equation}
IV_1 \leq \frac{\nu k}{4} \|\nabla \Pi_h e_u^{n,h}\|_{L^2}^2 + \|\nabla [u^n - \Pi_h u^n]\|_{L^2}^2 \\
\leq \frac{\nu k}{4} \|\nabla \Pi_h e_u^{n,h}\|_{L^2}^2 + C k h^4 \|\nabla^3 u^n\|_{L^2}^2.
\end{equation}

For nonlinear term $IV_2$, using the Poincaré inequality, the Young’s inequality and the embedding inequality, one finds that

\begin{equation}
IV_2 = -k b(u^n - u_h^n, u^n, \Pi_h e_u^{n,h}) - k b(u^n - u_h^n, \Pi_h e_u^{n,h}) \\
\leq \frac{k \nu}{4} \|\nabla \Pi_h e_u^{n,h}\|_{L^2}^2 + C k \|\nabla^2 u^n\|_{L^2}^2 \|u^n - u_h^n\|_{L^2}^2 \\
+ C k \|\nabla(u^n - u_h^n)\|_{L^2}^2 + (\|u^n - u_h^n\|_{L^2}^2 + (\|u^n - u_h^n\|_{L^2}^2).
\end{equation}

Inserting estimates (4.63)–(4.64) into (4.62), we have

\begin{equation}
\frac{1}{2}\left(\|\Pi_h e_u^{n,h}\|_{L^2}^2 - \|\Pi_h e_u^{n-1,h}\|_{L^2}^2 + \|\Pi_h e_u^{n,h} - \Pi_h e_u^{n-1,h}\|_{L^2}^2 \right) \\
+ \frac{k \nu}{2} \|\nabla \Pi_h e_u^{n,h}\|_{L^2}^2 \leq C k h^4 \|\nabla^3 u^n\|_{L^2}^2 + C k \|\nabla^2 u^n\|_{L^2}^2 \|u^n - u_h^n\|_{L^2}^2 \\
+ C k \|\nabla(u^n - u_h^n)\|_{L^2}^2 + (\|u^n - u_h^n\|_{L^2}^2 + (\|u^n - u_h^n\|_{L^2}^2).
\end{equation}

Taking the expectation and applying the summation operator $\sum_{n=1}^N$, one finds that

\begin{equation}
E\left[\frac{1}{2} \|\Pi_h e_u^{n,h}\|_{L^2}^2 + \frac{1}{2} \sum_{n=1}^N \|\Pi_h e_u^{n,h} - \Pi_h e_u^{n-1,h}\|_{L^2}^2 \right] \\
\leq \frac{1}{2} E\left[\|\Pi_h e_u^{0,h}\|_{L^2}^2 + \sum_{n=1}^N \|\nabla^3 u^n\|_{L^2}^2 + \sum_{n=1}^N \|\nabla^2 u^n\|_{L^2}^2 \right] + C E\left[\sum_{n=1}^N \|\nabla(u^n - u_h^n)\|_{L^2}^2 \right] \\
+ C(k_0, \|u^n - u_h^n\|_{L^2}) + C(k_0, \|u^n - u_h^n\|_{L^2}) + C k \|\nabla(u^n - u_h^n)\|_{L^2}^2 \\
\leq \frac{1}{2} E\left[\|\Pi_h e_u^{0,h}\|_{L^2}^2 + \sum_{n=1}^N \|\nabla(u^n - u_h^n)\|_{L^2}^2 \right] + C(k_0, \|u^n - u_h^n\|_{L^2}) + C k \|\nabla(u^n - u_h^n)\|_{L^2}^2 \\
+ \sum_{n=1}^N \|\nabla(u^n - u_h^n)\|_{L^2}^2 \right].
Using of the Lemma 3.3 and Lemma 4.1, the term $\Lambda_3$ is uniformly bounded, we obtain (4.67)
\[
\mathbb{E}\left[1_{\Omega_{k,h}^v \cap \Omega_{h}^v \cap \Omega_{h}^\kappa \cap \Omega_{\tau_0}} \left( \frac{1}{2} \| \Pi_h e_{n,h} - e_{n-1,h} \|_{L^2}^2 + \frac{1}{4} \sum_{n=1}^{N} \| \Pi_h e_{n,h} - e_{n-1,h} \|_{L^2}^2 \right) \right] 
\leq C(k_0, \kappa)(h^{4-7\epsilon} + k^{1-3\epsilon}) + C \mathbb{E} \left[ \sum_{n=1}^{N} \| k \| \Pi_h e_{n-1,h} \|_{L^2}^2 \right].
\]

Then (4.61) follows from an application of the discrete Gronwall inequality and the triangle inequality.

Theorems 3.3, 3.4, 3.5, 4.4, 4.5, 4.6 and Theorem 4.8 and the triangle inequality infer the global error estimates, which are the main results of this paper.

**Theorem 4.9.** Under the assumptions of Theorems 3.3, 3.4, 3.5, 4.4, 4.5, 4.6 and Theorem 4.8, there hold the following error estimates:

\[
\mathbb{E}\left[1_{\Omega_{k,h}^v \cap \Omega_{h}^v \cap \Omega_{h}^\kappa \cap \Omega_{\tau_0}} \left( \| \nabla(u(t_n) - u_{h,n}) \|_{L^2}^2 \right) \right] \leq C(k^{2\alpha-2\epsilon} + h^{2-3\epsilon}),
\]

\[
\mathbb{E}\left[1_{\Omega_{k,h}^v \cap \Omega_{h}^v \cap \Omega_{h}^\kappa \cap \Omega_{\tau_0}} \left( \left\| \int_0^{t_m} r(s) ds - k \sum_{n=1}^{m} r_h^n \right\|_{L^2}^2 + \left\| \int_0^{t_m} p(s) ds - k \sum_{n=1}^{m} p_h^n \right\|_{L^2}^2 \right) \right] \leq C(k^{2\alpha-2\epsilon} + h^{2-2\epsilon}),
\]

\[
\mathbb{E}\left[1_{\Omega_{k,h}^v \cap \Omega_{h}^v \cap \Omega_{h}^\kappa \cap \Omega_{\tau_0}} \left( \| u(t_n) - u_{h,n} \|_{L^2}^2 \right) \right] \leq C(k_0, \kappa) \left( k^{2\alpha-3\epsilon} + h^{4-7\epsilon} \right),
\]

where $C(k_0, \kappa)$ are two positive constants independent of $h$ and $k$.

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