Measurements and kernels for source-structure inversions in noise tomography

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\textbf{SUMMARY}

Seismic noise cross-correlations are used to image crustal structure and heterogeneity. Typically, seismic networks are anisotropically illuminated by seismic noise, a consequence of the non-uniform distribution of sources. Here, we study the sensitivity of such a seismic network to structural heterogeneity in a 2-D setting. We compute finite-frequency cross-correlation sensitivity kernels for traveltime, waveform-energy and waveform-difference measurements. In line with expectation, wave speed anomalies are best imaged using traveltimes and the source distribution using cross-correlation energies. Perturbations in attenuation and impedance are very difficult to image and reliable inferences require a high degree of certainty in the knowledge of the source distribution and wave speed model (at least in the case of transmission tomography studied here). We perform single-step Gauss-Newton inversions for the source distribution and the wave speed, in that order, and quantify the associated Cramér-Rao lower bound. The inversion and uncertainty estimate are robust to errors in the source model but are sensitive to the theory used to interpret measurements. We find that when classical source–receiver kernels are used instead of cross-correlation kernels, errors appear in both the inversion and uncertainty estimate, systematically biasing the results. We outline a computationally tractable algorithm to account for distant sources when performing inversions.

\textbf{Key words:} Theoretical seismology; Wave scattering and diffraction; Wave propagation.

1 INTRODUCTION

Terrestrial seismic noise fluctuations are observed over a broad range of temporal frequencies, excited by anthropogenic activity, storms and oceanic microseisms among other mechanisms (e.g. Longuet-Higgins 1950; Nawa et al. 1998; Rhie & Romanowicz 2004; Kedar & Webb 2005; Stehly et al. 2006). Seismic noise has been used as a successful alternative to earthquake tomography to image the crust. It has shown great promise as a means of generating new crustal constraints and characterizing its temporal variations (e.g. Brenguier et al. 2007; Wegler & Sens-Schonfelder 2007; Brenguier et al. 2008; Rivet et al. 2011; Zaccarelli et al. 2011).

The dominant measurement in seismic noise tomography is the ensemble-averaged cross-correlation. The cross-correlation measurement, by virtue of being a product of wavefields, possesses different physical attributes when compared with its classical tomographic analogue, that is wavefield displacement. Under certain ideal circumstances, such as when the source distribution is uniform, or when wave attenuation and source generation are collocated, it is possible to derive representation theorems (e.g. Fleury et al. 2010) which state that the cross-correlation is effectively Green’s function between the stations. Based on this reasoning, practitioners in this field view cross-correlation measurements through the lens of classical tomography. Earth noise sources are, however, typically highly anisotropically distributed, as a consequence of which these theorems are broken and a more rigorous interpretation becomes necessary.

The theoretical treatment of terrestrial seismic noise has connections with the wavefields of stars, specifically, the Sun. The use of cross-correlations of the wavefield of the Sun to probe the interior 3-D structure of the Sun was pioneered by Duvall et al. (1993). Gizon & Birch (2002), based on the finite-frequency theory of Marquering et al. (1999), derived a recipe to compute kernels for cross-correlations of helioseismic noise arising from a distribution of sources. Building on this work, Tromp et al. (2010) derived an adjoint theory for cross-correlations for the terrestrial case. The theory allows for treating distributions of sources as opposed to a discrete number of them (e.g. Larose et al. 2006; Tsai 2009). The results of Tromp et al. (2010) enable the prediction of cross-correlations on a rigorous basis, for a given earth model and source distribution (this problem has been studied extensively by, e.g. Chevrot
et al. 2007; Pedersen et al. 2007; Yang & Ritzwoller 2008; Weaver et al. 2009; Cupillard & Capdeville 2010; Froment et al. 2010; Tsai 2010).

However, complex random heterogeneities in Earth’s crust induce multiple scattering (e.g. Wegler & Sens-Schonfelder 2007) and introduce theoretical complications in the forward calculation. In a scenario where the randomness in the medium is ergodic, that is sampling some finite area of the medium is sufficient to extract its full properties, and when the probability density function describing the heterogeneities is spatially translationally invariant, a variety of methods can be brought to bear on the forward problem, including periodic homogenization (e.g. Papanicolaou & Varadhan 1982) and radiative transfer (e.g. Sato et al. 2012). Modelling the heterogeneous Earth, whose probability density function is likely to vary from one point to another, is considerably more difficult (e.g. Guillot et al. 2010; Sato et al. 2012).

The first Born approximation treats a scatterer as a source of a new wave (where the type and amplitude of the new source, i.e. monopolar, dipolar, etc. depends on the type of scatterer) and therefore, roughly speaking, medium randomness tends to ‘isotropize’ the directionality of the sources. Consequently, the impact of structural heterogeneities on the source distribution can certainly be modelled when inverting for the azimuthal distribution of source amplitudes. Structural heterogeneities outside of the aperture of the network are unlikely to affect the local measurement process and may, for all practical purposes, be lumped into an effective source term.

The purpose of this paper is to investigate the robustness of noisetomographic inferences to incomplete theoretical models (such as treating noise as if it were a classical tomographic measurement) and ignoring the anisotropy of illumination. We do so using the framework of a 2-D wave propagation problem with heterogeneities such as wave speed and attenuation illuminated by anisotropic sources. The governing equation is outlined in Section 2, followed by a discussion of the cross-correlation (Section 2.1), choice of measurements (Section 2.2) and a means to compute sensitivity kernels for the parameters of density, attenuation, wave speed and source distribution for a given measurement. We calculate the response of a network of stations to anomalies in wave speed and attenuation for uniform and non-uniform source distributions and demonstrate that the ability to image attenuation and density is sensitively dependent on the knowledge of the sources. Using cross-correlation energy and travel times as measurements, we pose inverse problems for the source distribution and wave speed, solving them in that order. We invert for these parameters using a single-step Gauss-Newton method. The appendices outline the procedure applied to compute kernels. Synthetics around the heterogeneous model show that the misfit falls by a factor of more than 4. Finally, we quantify the uncertainty of the inversion by computing the diagonal of Hessian matrix, to which we have full access.

## 2 THE 2-D FORWARD PROBLEM

We consider wave propagation in a 2-D heterogeneous medium described by

\[
\rho \frac{\partial^2 \phi}{\partial t^2} + \rho \Gamma \frac{\partial \phi}{\partial t} = -\nabla \cdot (\rho c^2 \nabla \phi) = f(x, t),
\]

where \(\phi(x, t)\) is the wavefield, \(\rho(x)\) is the density of the medium, \(c(x)\) is the wave speed, \(\Gamma(x)\) is the attenuation (measured in units of inverse time), \(\nabla\) is the covariant spatial derivative, \(t\) is time, \(\partial / \partial t\) is the derivative with respect to time, \(f\) is the source and \(x\) is the spatial coordinate. The form of attenuation in eq. (1) is simplistic, does not fully capture the mechanisms that govern wave damping in Earth, and is only adopted for the ease it affords in studying the sensitivity of noise to attenuation. See, for example Zhu et al. (2013), for details on a more rigorous treatment of wave attenuation. Eq. (1) is solved using a pseudo-spectral scheme. Fast Fourier transforms are applied to compute spatial derivatives and temporal evolution of the system is achieved using an optimized second-order Runge-Kutta scheme (Hu et al. 1996). Green’s functions are computed around each station and cross-correlations are estimated according to the method described by, for example Tromp et al. (2010), Hanasoge (2013). We discuss the method here for the sake of completeness in Section 2.1. See eq. (2) for the general relationship between Green’s functions and the cross-correlation.

### 2.1 Computing noise cross-correlations

The first step in the inverse problem for noise is to have a means of predicting cross-correlations. The formal interpretation of noise cross-correlations in terms of Green’s functions of the medium for an arbitrary distribution of sources has been described by, for example Woodard (1997), Gizon & Birch (2002), Hanasoge et al. (2011) for helioseismology and by, for example Tromp et al. (2010), Hanasoge (2013) for noise tomography. We summarize the theoretical results here since they will be used frequently, referring the interested reader to these references for more detailed expositions. For a temporally stochastic and spatially stationary and uncorrelated distribution of sources, a situation relevant in some cases to Earth ambient noise, it may be shown that the expected cross-correlation is connected to Green’s functions of the medium thus (invoking identity [A4])

\[
C(x_i, x_j, \omega) = \int dx' \ G(x_i, x'; \omega) \ G(x_j, x'; \omega) S(x') \mathcal{P}(\omega),
\]

where \(\mathcal{P}(\omega)\) is the temporal power spectral distribution of the sources, \(S(x')\) is the spatial distribution of sources, \(G(x, x'; \omega)\) is Green’s function measured at point \(x\) due to a source at \(x'\). The quantity \(C\) is the cross-correlation of signals measured at receivers \(x_i, x_j\). Note that \(C\) is the expected value of the cross-correlation, and that assumptions of stationarity and ergodicity have been invoked in justifying its existence (e.g. Hanasoge 2013). Eq. (2) is written in frequency domain for the sake of simplicity. The corresponding form in time domain is more tedious, and we will avoid it in this paper.

Source–receiver seismic reciprocity gives us,

\[
G(x, x'; \omega) = G(x', x; \omega),
\]

since we have an unbounded medium. This result allows us to rewrite eq. (2), thus

\[
C(x_i, x_j, \omega) = \int dx' \ G(x', x_i; \omega) \ G(x, x_j; \omega) S(x') \mathcal{P}(\omega),
\]

an important manipulation that greatly simplifies the computation since all we need to estimate the predicted cross-correlation is to calculate Green’s functions due to sources placed at the receivers and integrate the product weighted by the source distribution.

For a uniform background medium, which we use as the starting model, the following operative relation holds:

\[
G(x, x'; \omega) = G(|x - x'|; \omega).
\]

This relationship makes it computationally feasible to compute spatial convolutions of Green’s functions by transforming to the
Measurements and kernels for noise tomography

Figure 1. Example cross-correlation and the reference measured at a pair of stations separated by a distance of 120 km, where the wave speed of the medium is 3 km s$^{-1}$. The stations are placed symmetrically across a disc-shaped wave speed anomaly of $\pm 10$ per cent, with no density or attenuation contrasts. The horizontal axis is time lag measured in seconds. The positive and negative branches correspond to positive and negative time lags, respectively. Traveltimes measured in each branch are denoted by $\tau^+$ and $\tau^-$. The average and difference of these two traveltimes are termed mean and difference times, respectively (eq. 7).

Figure 2. Example wave speed kernels for energy, waveform difference, mean and difference traveltime measurements for a station pair located 120 km apart. The units of the traveltime kernels are s km$^{-2}$ and the energy kernels are km$^{-2}$ (since we are dealing with a 2-D problem). The integral of the difference traveltime kernel is zero and is not sensitive to wave speed anomalies. The energy kernel is interesting in that it does not have a ‘doughnut hole’ in the centre, which the traveltime kernels evidently possess (as Dahlen & Baig 2002; Nolet et al. 2005, have noted). Waveform differences are also sensitive to the underlying anomaly, also appearing to not possess a ‘hole’. The source distribution is uniform in this case, resulting in strong symmetries about the bisector between and line joining the two stations.
Figure 3. Example impedance kernels for energy, waveform difference, mean and difference traveltime measurements for a station pair located 120 km apart. The units of the traveltime kernels are s km$^{-2}$ and the energy kernels are km$^{-2}$ (since we are dealing with a 2-D problem). Not surprisingly, it is seen from the colour bars that the impedance kernels do not contribute as significantly to the misfit as the wave speed kernels. The source distribution is uniform in this case, resulting in strong symmetries about the bisector between and line joining the two stations.

Figure 4. Source-time function and power spectrum for all the kernels shown and the inversions. In this problem, we focus only on this specific frequency band.

Fourier domain. Note that eq. (5) does not hold for heterogeneous models.

2.2 Measurements

The dominantly used measurement in noise tomography is the ensemble-averaged cross-correlation, defined by

$$C_{ij}(t) = \frac{1}{T} \int_0^T dt' \phi(x_i, t + t') \phi(x_j, t').$$

where $C_{ij}(t)$ is the cross-correlation at time lag $t$ between signals recorded at stations located at $x_i$ and $x_j$ using a recording length $T$. To improve the signal-to-noise ratio of the measurement, an ensemble average over several temporal epochs of length $T$ is taken. Processing methods such as pre-whitening (e.g. Seats et al. 2012) and one-bit filtering (e.g. Aki 1957) are typically used to make sensible measurements. From these measurements, wave traveltimes, amplitudes and cross-correlation energies may be extracted. Here, we study the impact of this standard suite of measurements, that is traveltimes, cross-correlation energies and waveforms, on the
Measurements and kernels for noise tomography

Figure 5. The starting model consists of a ring-like uniform distribution sources (light blue ring) surrounding the network. The ray path coverage of the network is excellent within the central wave speed anomaly, which consists of positive and negative perturbations to the tune of 20 per cent, suggesting that the eventual inversion will likely be good. Stations, of which there are 24, are marked by diamonds. Note that despite the uniform illumination, the ring of sources is too close to the network for representation theorems to accurately apply (e.g. Fleury et al. 2010).

Inversions. The traveltime shift $\tau$, formally defined in Appendix B, has been studied by a number of authors (e.g. Luo & Schuster 1991; Woodard 1997; Marquering et al. 1999; Gizon & Birch 2002; Tromp et al. 2005). The cross-correlation function has two branches, the positive-lag branch ($t > 0$) and the negative-lag branch ($t < 0$). Thus, two traveltimes may be extracted from each cross-correlation, and we denote $\tau^+$ as the traveltime obtained from the positive branch and $\tau^-$ as the traveltime associated with the negative branch (see illustration in Fig. 1). In the inversion, we will use mean and difference traveltimes, which are defined as (e.g. Gizon & Birch 2002; Tromp et al. 2010):

$$\tau^w = \frac{\tau^+ + \tau^-}{2}, \quad \tau^d = \frac{\tau^+ - \tau^-}{2}. \quad (7)$$

The energy of the cross-correlation is given by (e.g. Dahlen & Baig 2002; Hanasoge 2013):

$$s = \sqrt{\frac{1}{T} \int dt \, w(t) C^2}, \quad (8)$$

and the waveform difference is $w(t)(C_{ref} - C)$. Thus, we have four measurements from a cross-correlation: mean traveltime, difference traveltime, energy and waveform difference. In order, we define misfit functionals for mean traveltime shifts, difference traveltime shifts, energy and waveform:

$$\chi^w = \frac{1}{2N} \sum_{ij} (\tau^w_{ij})^2, \quad (9)$$

$$\chi^d = \frac{1}{2N} \sum_{ij} (\tau^d_{ij})^2, \quad (10)$$

$$\chi^e = \frac{1}{2N} \sum_{ij} \left[ \ln \left( \frac{\varepsilon_{ref}}{\varepsilon_{ij}} \right) \right]^2, \quad (11)$$

$$\chi^w = \frac{1}{2N} \sum_{ij} \int dt \, w(t)(C_{ref} - C)^2, \quad (12)$$

where $\varepsilon_{ref}$ is the energy of the reference cross-correlation and $N$ is the number of measurements. Studying variations of these misfit
Figure 6. Total gradients of misfit functions comprising different measurements with respect to model parameters. The true anomaly is in wave speed. This is a best-case scenario of sorts because we assume that the true source distribution is the same as in Fig. 5. The sum of all event kernels, in some ways, is the inversion, since it is the gradient of the entire misfit functional. A 3-point Gaussian smoothing filter was applied to the kernels. There are four parameters that we have imperfect knowledge of and that we wish to invert for: wave speed, impedance, attenuation and source distribution (rows, top to bottom). We have four different measurements, mean and difference traveltimes, energies and waveform misfit (columns, left to right). The true model contains positive and negative wave speed perturbations to the tune of ±20 per cent within the central region. It is seen that the mean traveltime is able to infer the wave speed. That said, waveform differences also indicate attenuation and impedance anomalies, both of which are non-existent. The different magnitudes of the total gradient generally indicate the relative significance of each kernel (with the exception of waveform difference which has not been normalized).

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We analyse the variation of the positive traveltime shift keeping in mind that a similar procedure is applied to the negative time shift. Substituting the formal definition of the traveltime from eq. (B1), dropping the ‘+’ superscript for convenience and invoking identity (A5),

\[ \delta \tau = \frac{\int dt \, w(t) \mathcal{C}_0 \delta \mathcal{C}(t)}{\int dt \, w(t) \mathcal{C}_0^2} = \int d\omega \, \mathcal{W}(\omega) \delta \mathcal{C}(\omega), \]

(14)

where the windowing function, reference cross-correlation and the denominator have been subsumed into a weight function, whose temporal Fourier transform is written as \( \mathcal{W}(\omega) \). The transformation to Fourier domain was accomplished according to the Fourier convention described in Appendix A. From this point, we only broadly sketch the details of how to compute the kernel. Details of the derivation are found in, for example Hanasoge (2013) and in Appendices B and C. The variation of the cross-correlation function is further expressed in terms of variations of Green’s functions, that is
Measurements and kernels for noise tomography

Figure 7. Total gradients of misfit functions comprising different measurements with respect to model parameters. The true anomaly is in attenuation. This is a best-case scenario of sorts because the true source distribution is as in Fig. 5. There are four parameters that we have imperfect knowledge of and that we wish to invert for: wave speed, impedance, attenuation and source distribution (rows, top to bottom). We have four different measurements, mean and difference traveltimes, energies and waveform misfit (columns, left to right). The true anomaly is a 20 per cent increase in attenuation in the central region of Fig. 5, with excellent ray coverage. And yet, looking at the total gradients suggests the inversion will not meet with the success evidenced in Fig. 6. The mean traveltimes indicate a wave speed anomaly rather than an attenuation anomaly although there appears to be a trade-off with density. Attenuation and density, as is well known, are indeed difficult parameters to infer because for propagating waves, wave speed is overwhelmingly influential. A 3-point Gaussian smoothing filter was applied to the kernels.

\[ \delta C = \delta C_{\text{structure}} + \delta C_{\text{source}}, \]  
\[ \delta C_{\text{structure}} = \int dx G'(x, x', \omega) \, G(x, x'; \omega) + G(x, x'; \omega) \, \delta G(x, x'; \omega)] \, S(x') \, \mathcal{P}, \]  
\[ \delta C_{\text{source}} = \int dx' G'(x, x', \omega) \, G(x, x'; \omega) \, \delta S(x') \, \mathcal{P}. \]

The variation of Green’s function as described by the single-scattering first-Born approximation is

\[ \delta G(x, x'; \omega) = - \int dx'' G(x, x'') (\delta \mathcal{L} G)(x'', x'), \]

where \( \delta \mathcal{L} \) is the variation of the wave operator. Some algebraic manipulation is required to derive expressions for the kernels; see Appendix C for a more detailed discussion on this. The kernels are designed to address non-dimensional quantities and are connected to the variation of the misfit (for a given measurement) thus

\[ \delta \chi = \int dx \, K_c(x) \, \delta \ln c + K_\rho(x) \, \delta \ln \rho + K_\Gamma(x) \, \delta \ln \Gamma + K_S(x) \, \delta S, \]

where \( K_c, K_\rho, K_\Gamma, K_S \) are the wave speed, impedance, attenuation and source kernels, respectively (see Appendix C for full expressions). In Fig. 2, we show examples of wave speed kernels for all the measurements discussion in Section 2.2. Similarly in Fig. 3 we show impedance kernels for these measurements.

4 THE TOTAL MISFIT GRADIENT

Kernels, measurements and ‘data’ (numerically computed around the heterogeneous model) in hand, we are ready to study the inverse
problem. For all the problems, the starting structure model is a homogeneous uniform medium and the source model consists of a uniform ring. In the first pass at an inverse problem, we compute the total misfit gradient, which is an indicator of how well the inversion is likely to work. We explore three variants here:

(i) Only wave speed perturbation, true source distribution is uniform, starting source distribution model = true source distribution,
(ii) Only attenuation perturbation (+20 per cent) within the central region, true source distribution is uniform, starting source distribution model = true source distribution.
(iii) Only wave speed perturbation, true source distribution is non-uniform, starting source distribution model = uniform.

The specific frequency band of 6-12 s is considered here. The source-time function and the power spectrum are shown in Fig. 4. Note that, when considering dispersive surface waves on Earth, appropriate filters may be applied to narrow the frequency range, thereby allowing for defining the measurements with greater clarity. The theory discussed here directly applies to frequency-filtered measurements, only requiring the filter to be incorporated in the definition of the cross-correlation (see, e.g. Tromp et al. 2010, for details).

4.1 Uniform distribution of sources

The first case we consider is one in which there is a ±20 per cent wave speed perturbation within the central region while the density is uniform and the attenuation level is fixed at a quality factor of $Q = 100$. Fig. 5 shows a uniform distribution of sources surrounding a network of 24 stations which in turn surround the anomaly. This scenario is ideal because of the illumination and coverage. We emphasize that despite the uniform illumination, the ring of sources is too close to the network for representation theorems, which allow for the cross-correlation to be written as a linear combination of Green’s functions, to accurately apply (e.g. Fleury et al. 2010).

In Fig. 6, we show the misfit gradients of the four measurements (mean and difference traveltimes, amplitudes and waveform difference) with respect to four model parameters (wave speed, impedance, attenuation and source distribution) for case (i), illustrated in Fig. 5. Mean traveltimes successfully point to the anomaly and waveform differences are problematic, pointing misleadingly
Measurements and kernels for noise tomography

Figure 9. Total gradients of misfit functions comprising different measurements with respect to model parameters. The true anomaly is in wave speed. The true source distribution is as shown in Fig. 8 whereas the starting source model is as in Fig. 5. The sum of all event kernels, in some ways, is the inversion, since it is the gradient of the entire misfit functional. There are four parameters that we have imperfect knowledge of and that we wish to invert for: wave speed, impedance, attenuation and source distribution (rows, top to bottom). We have four different measurements, mean and difference traveltimes, energies and waveform misfit (columns, left to right). The true model contains positive and negative wave speed perturbations to the tune of ±20 per cent within the central region. Mean traveltimes are able to recover the wave speed structure while the amplitude wave speed kernels fail, owing to the fact that the starting source model isotropic whereas the true source model is anisotropic. The difference traveltime gradient is aligned along north-west—south-east direction, indeed suggesting that anisotropic source distributions can create differences in traveltimes between the positive and negative branches (e.g. Hanaoge 2013). The source-amplitude kernels suggest an increase in source magnitude in the south-east direction, which is what we would expect. Waveform differences, as before, are not to be trusted. A 3-point Gaussian smoothing filter was applied to the kernels.

to an attenuation anomaly. A 3-point Gaussian smoothing filter was applied to the kernels.

The next case addresses the imaging of a +20 per cent attenuation perturbation within the central region of Fig. 5, that is case (ii). The assumed and true source distributions are identical, as shown in Fig. 5. Even in this ideal scenario, Fig. 7 highlights a mixed outcome, with traveltimes measurements indicating an increase in wave speed whereas amplitude measurements appear to implicate both density and attenuation anomalies, suggesting a trade-off between the two. Attenuation and density are indeed difficult to convincingly image, even with perfect source and wave speed models (see also, e.g. Tsai 2009).

4.2 Non-uniform distribution

Fig. 8 shows a non-uniform configuration of sources illuminating a network of 24 stations which in turn surround the wave speed anomaly.

The set of total gradients for case (iii) are shown in Fig. 9. Once again, mean traveltimes are successful at imaging the wave speed anomaly, although not to the same degree as when the illumination was uniform, that is in Fig. 6. The cross-correlation amplitude measurements suggest a deficit in source magnitude in all directions except the south-east where it points to an increment.

Based on these simple tests, we form the following strategy:

(i) Use cross-correlation energies to infer the azimuthal dependence of the source distribution (if not imaging its exact location),
(ii) This will serve as the starting model for the source distribution,
(iii) With this source-distribution, recompute kernels,
(iv) With these kernels and mean traveltimes, invert for the wave speed,
(v) Possibly, after both of these steps, invert for attenuation/density.
WAVE SPEED INVERSIONS

For all the problems, the starting structure model is a homogeneous uniform medium. We use a Gauss–Newton method to precondition the total gradient. A one-step inversion is performed; see Appendix D for details on the procedure. The model variance is computed using the Hessian matrix, which we have full access to.

The experiments of Section 4 emphasize the well known, namely that density and attenuation are indeed very difficult to invert for when the exact source model is poorly known. In general, however, the sensitivity to attenuation is relatively weak, making it a difficult quantity to measure. In this section, we consider inverting only for source distribution followed by wave speed and show that the correct theory can substantially improve the recovery of anomalies. We study two more cases here:

(iv) Only wave speed perturbation, true source distribution is non-uniform (as in Fig. 8), comparison between source-wave speed and only wave speed inversions,

(v) Only wave speed perturbation, true source distribution is non-uniform (as in Fig. 8), comparison between cross-correlation and classical inversions.

In case (iv), we study the impact of the source distribution on the reducing the misfits. It is well known that a uniform ring of sources placed far away from a network mimics a general uniform source distribution (e.g. Larose et al. 2004). Based on this argument, our strategy for inverting for the source distribution is the following:

(i) For most networks, the noise sources are too far away to image accurately (e.g. Hanasoge 2013),

(ii) We, therefore, seek to reduce the dimensionality of the problem, parametrizing the sources as a ring with some azimuthal modulation in magnitude,

(iii) Construct a uniform ring of as large a radius as feasible surrounding the network,

(iv) Estimate the cross-correlation energy as a function of azimuth, interpolate, smooth and multiply the uniform ring by this function (e.g. Stehly et al. 2006),

(v) Recompute the synthetics, perform a source inversion and retain only the azimuthal information, multiplying the uniform ring by this new azimuthal dependence.

Fig. 10 compares two cases with and without source inversions prior to the wave speed inversion. The difference, in this particular configuration, appears to not be significant, although the total misfit of
Measurements and kernels for noise tomography

Figure 11. Case (v): a one-step Gauss–Newton inversion for the wave speed using 141 measurements of mean traveltimes, comparison between correlation and classical approaches. The network, wave speed anomaly and non-uniform source illumination are as in Fig. 8 whereas the starting assumed source model is uniform, that is as in Fig. 5. The true model contains positive and negative wave speed perturbations to the tune of ±20 per cent within the central region. We invert only for wave speed using the method described in Appendix D using correlation and classical approaches. For the correlation approach, we compute cross-correlation kernels and for the classical approach, we treat the measurement as if it were a traveltime derived from raw wavefield measurements. The error in the theoretical model in the classical inversion propagates into the inversion and the misfit is larger. Applying the correct theoretical description is therefore quite important. The Cramér-Rao lower bound (Appendix D) is shown in the lower most panels for both inversions. The uncertainty in inferring \( \delta c / c \) is underestimated when using classical theory to interpret cross-correlation measurements.

Finally, in Fig. 11, we compare the classical and correlation interpretations and find the classical approach to underperform the correlation approach significantly. Indeed, the correct theoretical description outweighs other aspects such as inverting for the source distribution first.

6 CONCLUSIONS

Through the construction of a 2-D inverse problem, we have attempted to characterize the sensitivity of cross-correlation measurements to parameters such as wave speed, density and attenuation. Using a Gauss–Newton method, we perform inversions for source and wave speed and demonstrate the utility of using cross-correlation theory as opposed to applying classical interpretations to the measurements. We summarize our conclusions here

(i) Density and attenuation are very difficult to measure, being sensitive to the source and structure model of choice.

(ii) The apertures of most small networks are insufficient to image sources that are typically very far away. We describe a parametrization and method to address this issue while still accounting for anisotropic illumination.

(iii) We recommend inverting for sources using cross-correlation energies followed by using mean traveltimes of waves to invert for the wave speed. Performed in this order, we saw a factor of 2 reduction in amplitude misfit and a factor of 4 reduction in traveltime misfit,
(iv) Most of all, we found that a significant source of bias comes from not interpreting measurements correctly, that is treating cross-correlations as if they were classical measurements.
(v) A dominant source of uncertainty in the inversion is the sparsity of the station network, especially when using well-defined arrivals for which we have a good starting model. We compute the uncertainty of the inversion using the diagonal of the approximate Hessian (see Section D). This estimate of the uncertainty may be considered a lower bound to the full uncertainty.

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APPENDIX A: FOURIER CONVENTION

The following temporal Fourier transform convention is utilized:

\[
\int_{-\infty}^{\infty} \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \! \!
where $\delta(x - x')$ is the delta function centred around $x = x'$. The variation of Green’s function as described in the single-scattering regime of the first Born approximation is given by

$$\delta\mathcal{L}G + \mathcal{L}\delta G = 0,$$

and we have therefore

$$\mathcal{L}\delta G = -\delta\mathcal{L}G,$$

and upon using Green’s theorem,

$$\delta G(x, x') = -\int dx'' G(x, x'') \delta\mathcal{L}G(x'', x').$$

Let us first consider perturbations to the bulk modulus $\rho c^2$, that is $\delta\mathcal{L} = -\nabla \cdot (\delta(\rho c^2) \nabla)$, where the spatial gradients and the bulk modulus are functions $x''$. Recalling eqs (14) and (16) and invoking identity (A5),

$$\delta\tau_{ij} = \int dw \mathcal{W}_{ij} \int dx' [G^+(x, x') G(x, x')] S(x') P.$$

For simplicity, we consider one of the terms in the variation, with the analysis described here equally applicable to the other,

$$\int dw \mathcal{W}_{ij} \int dx' G^+(x, x') \delta G(x, x') S(x') P$$

$$\times \int dx G(x, x) \nabla \cdot [\delta(\rho c^2) \nabla G(x, x')].$$

The inner integral over $x$ may be integrated by parts and using the fact that the medium is unbounded, boundary terms are dropped. This allows us to free $\delta(\rho c^2)G(x')$ from within the derivative. The result of this operation provides

$$\int dw \mathcal{W}_{ij} \int dx \delta(\rho c^2)$$

$$\times \int dx' G^+(x, x') S(x') \mathcal{P} \nabla G(x, x) \times \mathcal{P} \nabla G(x, x'),$$

which is rewritten as

$$\int dx \delta(\rho c^2) \left\{ \int dw \mathcal{W}_{ij} \mathcal{P} \rho c^2 \nabla G(x, x)$$

$$\times \int dx' G^+(x, x') \nabla G(x, x') S(x') \mathcal{P} \right\}.$$

Invoking seismic reciprocity and the assumption of a uniform starting background model, we recast eq. (C9)

$$\int dx \delta(\rho c^2) \left\{ \int dw \mathcal{W}_{ij} \mathcal{P} \rho c^2 \nabla G(x, x)$$

$$\times \int dx' G^+(x, x') \nabla G(x, x') S(x') \mathcal{P} \right\}.$$

where the term within the parentheses contributes to the bulk modulus kernel. The inner integral over $x'$ is the crux of the scattering problem, and represents the primary challenge in computing these kernels. Because we have assumed translation invariance, we may rewrite this in spatial Fourier domain, converting the convolution to a Fourier product (see Appendix A, specifically relation [A9]) and transforming back to the spatial domain. We apply similar analyses to the density and attenuation terms. The expressions for the three kernels are as follow:

$$K_{\rho c^2}(x) = \rho c^2 \int dw \mathcal{W}_{ij} \mathcal{P} \left[ \nabla G(x, x) \times \int dx' S(x') G^+(x', x)$$

$$\times \nabla G((|x - x'|) + \nabla G^+(x, x) \times \int dx' S(x') G(x', x)$$

$$\times \nabla G^+((|x - x'|)) \right].$$
\[
K_{\rho\tau}(x) = -\rho \Gamma \int \text{d}x' i \omega \mathcal{W}_{ij}' \mathcal{P} \left[ \mathcal{G}(x, x') \int \text{d}x'' \mathcal{G}''(x'', x) \times \mathcal{G}''(x - x'') \right] \times \mathcal{G}(x - x) + \mathcal{G}''(x', x) \int \text{d}x' \mathcal{S}(x') \mathcal{G}(x', x) \times \mathcal{G}''(x - x') \right].
\]

\[
K_\gamma(x) = -\rho \int \text{d}x'' \omega \mathcal{W}_{ij}' \mathcal{P} \left[ \mathcal{G}(x, x') \int \text{d}x'' \mathcal{S}(x') \mathcal{G}(x', x) \times \mathcal{G}''(x - x') \right].
\]

All put together, we recover the following relation between travel-time shifts and the kernels \( \delta \tau \)

\[
\delta \tau = \int \text{d}x \delta \ln c + K_\rho \delta \ln \Gamma + (K_\rho + K_\gamma) \delta \ln \rho + K_\gamma \delta S.
\]

and define the following as the wave speed, attenuation and impedance kernels, respectively:

\[
K_c = 2K_\rho c, \quad K_\gamma = K_\rho \Gamma, \quad K_\rho' = K_\rho + K_\gamma + K_\gamma.
\]

The simplest kernel of all is the source kernel (see, e.g. Hanasoge 2013), which has no scattering terms

\[
K_S(x) = \int \text{d}x' \mathcal{W}_{ij}' \mathcal{P} \mathcal{G}''(x, x') \mathcal{G}(x, x).
\]

All of these kernels put together, the variation of the traveltime reduces to

\[
\delta \tau = \int \text{d}x K_\gamma \delta \ln c + K_\gamma \delta \ln \Gamma + K_\gamma' \delta \ln \rho + K_\gamma \delta S.
\]

In order to validate traveltime kernels, we consider a fractional uniform change to the wave speed, for example \( \delta \ln c = 0.001 \). The measured change in the traveltime must then be equal to the integral of the wave speed kernel, that is

\[
\delta \tau = 0.001 \int \text{d}x K_\gamma.
\]

Similarly, for the impedance kernel, a fractional uniform change in density can lead only to a correspondingly small change in the amplitude of the cross-correlation, with there being virtually no change in the traveltime as defined in eq. (B1). Consequently, the integral of the impedance kernel is virtually zero. We use both of these tests to verify the kernel computation procedure. A test for the source kernels computed for the energy measurement was described in Hanasoge (2013), requiring

\[
\int \text{d}x K_S S = 1,
\]

where \( S \) is the source distribution.

### Appendix D: Gaussian-Newton Method and Hessian-Derived Uncertainties

For simplicity’s sake, let us rewrite the misfit \( \chi \) as a sum of \( M \) individual measurement misfits \( \mu_k \), that is \( \chi = \sum_k \lambda_k \mu_k/2 = \mu^T A \mu/2 \), where \( \mu \) is an \( M \times 1 \) vector and \( A \) is an \( M \times M \) sized inverse of the data-noise covariance matrix. Expanding the misfit functional, which is dependent on the properties of the medium \( m(x) \), around small variations \( \delta m(x) \),

\[
\chi(m + \delta m) = \chi(m) + \frac{\partial \chi}{\partial m} \delta m + \frac{1}{2} \frac{\partial^2 \chi}{\partial m^2} \delta m \delta m + O(||f||^3),
\]

where \( m_i = m(x_i) \) in which we have \( N \) points on the grid \( x \) and we invoke Einstein’s convention of summing over repeated indices. We denote the Jacobian \( \frac{\partial \chi}{\partial m_k} = -K(x_i) \), where the right side is the kernel for measurement \( f \) evaluated at location \( j \). The gradient of the misfit function with respect to model parameters is

\[
\frac{\partial \mu}{\partial m} = K,
\]

which is an \( M \times N \) matrix and \( f \) is an \( N \times 1 \) vector. The first term of the Taylor expansion when written in matrix notation is

\[
\frac{\partial \chi}{\partial m} \delta m = \delta m^T A \mu,
\]

is a \( 1 \times N \) vector. The second-order term is

\[
\frac{\partial^2 \chi}{\partial m \partial \delta m} \approx \delta m^T K^T A K \delta m.
\]

Thus, the Gaussian–Newton update (upon seeking the stationary point of \( \delta \chi \)) becomes

\[
K^T A K \delta m = -K^T \sqrt{\Lambda} \mu.
\]

The covariance matrix is positive definite and so it has a real (symmetric) square root. Defining \( \hat{K} = \sqrt{\Lambda} K \), we obtain

\[
\hat{K}^T \hat{K} \delta m = -\hat{K}^T \sqrt{\Lambda} \mu.
\]

The right side is the negative of the total gradient, or sum of all kernels. The left side may be solved using a sparse matrix inversion method. Consider \( M \) measurements \( \leq N \) spatial points. So the full kernel matrix is \( M \times N \). Performing a singular value decomposition \( [K] = U \Sigma V^T \) and we have \( \hat{K}^T \hat{K} = V \Sigma^2 V^T = \sum_k \sigma_k^2 v_k v_k^T \). We note that \( v_k^T v_j = \delta_{kj} \). Adding damping \( d \) to the singular values we obtain the update

\[
\delta \mu = -\sum_k (\sigma_k + d)^{-2} v_k v_k^T K^T \sqrt{\Lambda} \mu,
\]

The diagonal of the inverse of the Hessian \( \text{diag}([K^T A K]^{-1}) \) gives us the total model variance \( \sigma_m^2 \) as a function of space (assuming multivariate Gaussian statistics, e.g. Tarantola 1987),

\[
\sigma_m^2 = \text{diag} \left[ \sum_k (\sigma_k + d)^{-2} v_k v_k^T \right].
\]

The model variance provided by eq. (D8) is the Cramér-Rao bound where the Fisher information matrix is the Hessian. Note that the full model covariance is obtained by evaluating the inverse of the Hessian.