CONFINEMENT AND THE TRANSVERSE LATTICE*

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The status of the transverse lattice formulation of light-front QCD is reviewed. It is explained how confinement arises in this formulation for large lattice spacing. The nonperturbative renormalization procedure is outlined in general and a more detailed discussion is given for the case of $QCD_{2+1}(N_C \to \infty)$.

1 Introduction

Deep inelastic lepton-hadron scattering has played a fundamental role in the investigation of hadron structure. For example, the discovery of Bjorken scaling confirmed the existence of point-like charged objects inside the nucleon (quarks). Besides such fundamental discoveries, deep inelastic scattering (DIS) revealed surprising and interesting details about the structure of nucleons and nuclei, such as

- the nuclear EMC effect
- the spin crisis in polarized DIS experiments (EMC, NMC, SMC, SLAC, E665)
- the isospin asymmetry of the nucleon’s Dirac sea (NMC)
  $\leftrightarrow$ failure of the Gottfried sum-rule

More interesting data is expected in the near future: For example, the experiments by the HERMES collaboration at HERA will provide measurements of polarized structure functions (leading and higher twist) with unprecedented precision. Chirally odd parton distributions will be accessible both at HERMES and in proton-proton collisions at the RHIC facility. Future experiments at CEBAF-II and the planned European high energy accelerator ELFE will provide additional information about semi-inclusive DIS and higher-twist effects.

Perturbative QCD evolution has been successfully applied to correlate large amounts of experimental data. However, progress in understanding nonperturbative features of the parton distributions in these experiments has been slow. In fact, the theoretical understanding of the surprising results listed above is mostly limited to ad hoc models with little connection to the underlying quark and gluon degrees of freedom.

Part of the difficulty in describing parton distributions nonperturbatively derives from the fact that parton distributions measured in DIS are dominated by correlations along the light-cone ($x^2 = 0$). For example, this makes calculations of parton distributions on a Euclidean lattice, where all distances are space-like, exceedingly difficult. Furthermore, in an equal time quantization scheme, deep inelastic structure functions are described by real time response functions which are not only very difficult to interpret but also to calculate.

Light-Front (LF) quantization seems is a promising tool to describe the immense wealth of experimental information about structure functions for a variety of reasons:

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• correlations along the light-cone become “static” observables in this approach [i.e. equal 
  \( x^+ \equiv (x^0 + x^3)/\sqrt{2} \) observables]

• structure functions are easy to evaluate from the LF wavefunctions

• structure functions are easily interpreted as LF momentum densities

Further advantages of the LF formalism derive from the simplified vacuum structure (nontrivial 
vacuum effects can only appear in zero-mode degrees of freedom) which provides a physical 
basis for the description of hadrons that stays close to intuition.

2 The Transverse Lattice

Before one can apply the LF formalism to QCD one has to remove the divergences first (i.e. 
regularize and renormalize). Then one has to formulate bound state problems into a form that 
can be solved numerically with a reasonable effort. The basic idea of the transverse lattice — an idea originally suggested by Bardeen, Pearson and Rabinovici [1] — is very simple: one 
keeps two directions (the time and the z-direction) continuous but discretizes the transverse 
space coordinates (Fig. 1). The metric is Minkowskian. Two immediate advantages of this 
construction are

• manifest boost and translational invariance in the longitudinal direction — thus keeping 
  parton distributions easily accessible

• a gauge invariant cutoff for divergences associated with large transverse momenta

Several steps are required before one can use this construction to actually calculate parton 
distribution functions:

First, one has to write down a discretized approximation to the QCD action on a \( \perp \) lattice. 
This has been done in the original work [1]. Since space is kept continuous in the \( \perp \) direction, 
the longitudinal component of the gauge field is non-compact. However, in order to maintain 
gauge invariance in the \( \perp \) direction, one introduces compact \( SU(N_C) \) fields to represent the
component of the gluon field. One can thus think of the $\perp$ lattice as a large number of gauged, nonlinear $\sigma$-model fields in 1+1 dimensions coupled together. This picture will prove useful when one performs nonperturbative numerical calculations. Naive fermions are straightforward to implement: the only difference between the continuum action and the $\perp$ lattice action is the replacement of $\perp$ derivatives by finite differences with appropriate gauge field links to maintain gauge invariance.

Note that, since species doubling occurs only for the two latticized transverse directions, species doubling is a lesser problem than on a Euclidean lattice and only 4 species of fermions arise if one employs a naive fermion action. This can be easily dealt with, using staggered fermions. An alternative to staggering the fermion degrees of freedom is introducing additional terms in the action that lift the degeneracy between the naive species (Wilson-formulation). There are some subtle differences between chiral symmetry on the LF and what one usually considers chiral symmetry (the difference arises from the fact that on the LF only half the degrees of freedom are treated as dynamical degrees of freedom). It turns out that Wilson fermions on the LF do not break the LF-chiral symmetry, which makes them very appealing for practical calculations. However, I should emphasize that this does not solve the general problem of formulating chiral theories on a lattice since “chiral” transformations in the two component LF-formulation are slightly different from usual chiral transformations.

In the next step, one applies LF quantization to the transverse lattice action. Once one has thus derived a LF Hamiltonian, one can proceed to the final step and apply powerful numerical techniques to solve for the low lying eigenstates (= low lying hadrons) of the LF-Hamiltonian. From the ground state (LF-) wavefunctions one can then easily extract parton distribution functions and other light-like correlation functions (both leading and higher twist!).

It is interesting to see how confinement emerges on the transverse lattice in the limit of large lattice spacing: In this limit, the coupling between the sheets is weak and the energy scale associated with link field excitations is high. Thus, when one separates two test charges in the longitudinal direction, the transverse lattice behaves similar to QCD$_{1+1}$ and linear confinement results trivially. For transverse separations between the charges, a different mechanism is at work. Gauge invariance demands that the two charges are connected by a string of link fields. For large spacing the link fields fluctuate only little and the energy of such a configuration can be estimated by counting the number of link fields needed to connect the charges, which again yields linear confinement. Of course, as the spacing gets smaller, confinement becomes a dynamical question. However, it is a tremendous practical advantage if there is a limit which is technically simple but still very close to the expected physics.

### 3 Monte Carlo Techniques

In a recent paper, I have demonstrated that transverse lattice Hamiltonians can be solved using powerful Monte-Carlo techniques. The basic idea is very simple: since the LF Hamiltonian is local in the $\perp$ direction (only nearest neighbor interactions) one can set up the problem of finding ground states in such a way that one first solves the Hamiltonian numerically exactly on one link or one plaquette (this is relatively easy, since it implies only solving a 1+1 dimensional field theory). Then one includes the coupling in the transverse direction by means of a random walk in Fock space governed by ensemble projector Monte Carlo techniques. Such algorithms eliminate the need for Tamm-Dancoff approximations and thus allow

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1 The number is equal to the number of transverse links.

2 Several methods to solve 1+1 dimensional field theories in LF quantization have been described in the literature. In the context of the $\perp$ lattice, DLCQ seems to be quite useful.
to solve reasonably large $\perp$ lattices on realistic computers. For example, in Ref.\cite{7}, I solved the $\perp$ lattice LF Hamiltonian for self-interacting scalars on 64 $\perp$ sites on a workstation. Even if one accounts for the increased complexity of the Hamiltonian in a gauge theory, it thus seems feasible to perform QCD calculations on a similar size $\perp$ lattice on available supercomputers.

4 Renormalization

Due to the lack of full covariance (specifically rotational invariance), renormalization on a $\perp$ lattice is anything but trivial. Since the cutoffs violate rotational invariance one must allow for counter terms which (by themselves) violate rotational invariance in order to achieve rotational invariance in physical observables\cite{3}. The richer counter term structure is compensated by an increase in renormalization conditions. The infinite parts of various counter terms are often related when a symmetry is hidden. This phenomenon has been called coupling constant coherence and has been used in the context of light-front Tamm Dancoff\cite{3}. It should also be of use in the context of the transverse lattice. The finite part of the symmetry breaking (or restoring!) counter terms can be determined by making use of the increased number of renormalization conditions. At least in principle, one can demand full covariance for each physical observable. This provides a very large number (in principle infinite) of renormalization conditions which one can use to determine the (finite parts of) non-covariant counter terms. An observable which is particularly useful in this context is the potential. Currently, I am performing numerical test calculations on a 2+1 dimensional lattice for $QCD(N_C \to \infty)$ using a Lanczos algorithm. In these calculations I am fine-tuning the finite part of the counter terms such that one obtains a rotationally invariant $\bar{Q}Q$ potential.

Vacuum effects, i.e. effects that are associated with zero-modes in the LF formulation, were largely ignored in Ref.\cite{1}. There has been substantial progress in understanding how zero modes affect LF Hamiltonians in the continuum limit (see e.g. Refs.\cite{1, 2, 3, 4} and references therein). The general result of these studies is that the LF vacuum in the continuum limit is trivial but that one has to pay for this advantage with a more complicated Hamiltonian. However, on the $\perp$ lattice only a few additional terms are necessary.

In addition to these renormalization issues that are typical for any LF field theory, one must also address issues that are specific for the transverse lattice formulation, such as scale setting. This will be illustrated for the example of $QCD_{2+1}(N_C \to \infty)$\cite{5} where the LF Hamiltonian reads\cite{5}

$$ P^- = \sum_n \left[ -c_3 \int dx^- dy^- J^{ij}_{n,n+1}(x^-)|x^- - y^-|J^{ji}_{n,n+1}(y^-) \right. \right.$$

$$ + c_4 \int dx^- tr \left( U_n(x^-)U^\dagger_n(x^-)U_n(x^-)U^\dagger_n(x^-) \right) + c_2 \int dx^- tr \left( U_n(x^-)U^\dagger_n(x^-) \right) \left. \right] ,$$

where $U_n(x^-) \equiv U^{ij}_n(x^-)$ is the link field, represented by a complex $N_C \times N_C$ matrix,

$$ J^{ij}_{n,n+1}(x^-) = U^\dagger_n(x^-) \partial_+ U_n(x^-) - U_{n+1}(x^-) \partial_+ U^\dagger_{n+1}(x^-) $$

is the current that couples to the longitudinal gauge field at each site and the terms multiplied by the constants $c_4$ and $c_2$ are introduced to enforce the $SU(N_C)$ constraint on the link fieldsquare brackets

\footnote{Glueball spectra for $QCD_{2+1}(N_C \to \infty)$ in the Euclidean formulation are available by taking numerical results from $N_C = 2, 3, 4$\cite{5} and extrapolating them to $N_C \to \infty$}
$U_{ij}$. $c_g$ is the gauge coupling within the 1+1 dimensional sheets. In the naive continuum limit, it is related to the 2+1 dimensional gauge coupling via $c_g = g^2/a$, i.e., it carries dimension $L^{-2}$.

The coupling constants are chosen as follows: First of all, $c_4$ and $c_2$ are supposed to enforce the SU($N_C$) constraint, which implies in the classical action $c_2 = -2c_4$ and $c_4 \rightarrow \infty$. Due to renormalization effects (tadpoles etc.) $c_2$ picks up additional pieces so that this procedure has to be modified a little: one still takes (gradually) $c_4 \rightarrow \infty$. For every value of $c_4$, $c_2$ is chosen near the critical point, where the transverse string tension vanishes in lattice units: For large values of $c_2$, the energy of a transversely separated static quark anti-quark pair grows roughly linearly with the number of sites in between the $QQ$ pair. As one decreases $c_2$, the growth with the number of sites becomes slower and slower, which means that the lattice spacing, in units of the physical string tension, decreases. The continuum limit, i.e., vanishing lattice spacing $a$, has been reached when the energy of the $QQ$ pair no longer increases when their separation increases in lattice units, which is identical with the critical point beyond which the LF spectrum becomes tachionic. In practice, this means that if one is not interested in the actual numerical value of the lattice spacing in physical units, then one does not need to evaluate the transverse string tension! One only has to make sure that one is close to the critical point. Of course, the transverse string tension is nevertheless interesting because one would like to understand how confinement arises on the transverse lattice in the continuum limit.

So far, we have not discussed how one fixes the longitudinal gauge coupling $c_g$. It plays a dual role: first, at least if one measures the other couplings in units of $c_g$, it merely fixes the overall mass scale, and its numerical value is irrelevant as long as one always forms dimensionless combinations of physical observables, such as ratios of masses and/or the string tension. The second role played by $c_g$ is that it largely determines the longitudinal string tension. For a $QQ$ pair separated on the same site in the longitudinal direction no gluons are required to maintain residual gauge invariance. Since the valence approximation is usually rather good in LF calculations this implies that the longitudinal string tension roughly equals $c_g$. In summary, this implies the following nonperturbative renormalization procedure:

1. Pick an arbitrary value for $c_g$
2. Pick an arbitrary value for $c_4$
3. Solve the spectrum as a function of $c_2$ and tune $c_2$ such that $c_2$ stays slightly above the critical point.
4. Increase $c_4$ and continue with step 3 until one is close enough to the continuum limit.

There are at least two ways to verify whether or not one is close to the continuum. One is to actually calculate the string tension in lattice units. The other is to study numerical convergence of dimensionless quantities, such as mass ratios.

In practice, a different approach is also conceivable: instead of trying to make the lattice spacing as small as possibly (i.e., working very close to the critical line), it may be more efficient to look for the perfect LF-Hamiltonian for finite spacing (i.e., coupling constants away from the critical line). This implies that one has to add more terms to the LF Hamiltonian and the nonperturbative renormalization procedure becomes more complex, which is certainly a disadvantage. However, the advantage of such a procedure would be that one stays away from the critical regime. Since the Fock space content of hadrons is known to become infinitely

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4See [10] for an algorithm to calculate longitudinal and transverse string tensions from a LF Hamiltonian.
complicated very close to the critical regime, this implies that a perfect LF-Hamiltonian can be more easily solved numerically and in the end, be more economical.

5 Summary

The transverse lattice is a hybrid technique (a shotgun marriage between lattice field theory and light-front field theory) to solve relativistic field theories. Its main advantages over conventional lattice gauge theories is that parton distributions are much more easily calculable and interpretable. The disadvantage compared to conventional lattice gauge theories is that one inherits some (not all) problems of light-front field theories. The advantages over conventional light-front field theory is manifest invariance under a class of residual gauge transformation, less infrared problems and simple ways to implement Monte Carlo algorithms. Furthermore, confinement is “built in”, which means that the zeroth order approximation is already close to the expected physics in QCD. The disadvantages compared to other approaches to light-front field theory (say LF Tamm Dancoff) is an increase in the complexity of the wave functions. Current results on the transverse lattice are still limited to simple model studies in 2 + 1 dimensions, but the reason is more lack of manpower than any physics reason. Considering the substantial experimental effort to explore the parton structure of hadrons, it is definitely worthwhile to investigate further, to what extent one can describe deep inelastic structure functions using the transverse lattice formulation of QCD.

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