TWO-LOOP AMPLITUDES FOR $e^+e^- \rightarrow q\bar{q}g$: 
THE $n_f$-CONTRIBUTION

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We discuss the calculation of the $n_f$-contributions to the two-loop amplitude for $e^+e^- \rightarrow q\bar{q}g$. The calculation uses an efficient method based on nested sums. The result is presented in terms of multiple polylogarithms with simple arguments, which allow for analytic continuation in a straightforward manner.

1. Introduction

Searches for new physics in particle physics rely to a large extend on our ability to constrain the parameters of the standard model. For instance, the strong coupling constant $\alpha_s$ can be measured by using the data for $e^+e^- \rightarrow 3$-jets. At present, the error on the extraction of $\alpha_s$ from this measurement is dominated by theoretical uncertainties [1], most prominently, by the truncation of the perturbative expansion at a fixed order.

The perturbative QCD calculation of $e^+e^- \rightarrow 3$-jets at next-to-next-to-leading order (NNLO) requires the tree-level amplitudes for $e^+e^- \rightarrow 5$ partons [2], the one-loop amplitudes for $e^+e^- \rightarrow 4$ partons [3, 4] as well as the two-loop amplitude for $e^+e^- \rightarrow q\bar{q}g$ together with the one-loop amplitude $e^+e^- \rightarrow q\bar{q}g$ to order $\epsilon^2$ in the parameter of dimensional regularization.

The helicity averaged squared matrix elements at the two-loop level for $e^+e^- \rightarrow q\bar{q}g$ have recently been given [5]. In contrast, having the two-loop amplitude available, one keeps the full correlation between the incoming $e^+e^-$ and the outgoing parton’s spins and momenta. Thus, one can study oriented event-shape observables. In addition, one has the option to investigate event-shape observables in polarized $e^+e^-$-annihilation at a future linear $e^+e^-$-collider TESLA.
2. Calculation

We are interested in the following reaction

\[ e^+ + e^- \rightarrow q + g + \bar{q}, \quad (1) \]

which we consider in the form, \( 0 \rightarrow q(p_1) + g(p_2) + \bar{q}(p_3) + e^-(p_4) + e^+(p_5), \)

with all particles in the final state, to be consistent with earlier work [3].

The kinematical invariants for this reaction are denoted by

\[ s_{ij} = (p_i + p_j)^2, \quad s_{ijk} = (p_i + p_j + p_k)^2, \quad s = s_{123}, \quad (2) \]

and it is convenient to introduce the dimensionless quantities

\[ x_1 = \frac{s_{12}}{s_{123}}, \quad x_2 = \frac{s_{23}}{s_{123}}. \quad (3) \]

Working in a helicity basis, it suffices to consider the pure photon exchange amplitude \( A_\gamma \) as it allows the reconstruction of the full amplitude with Z-boson exchange by adjusting the couplings. Furthermore, the complete information about \( A_\gamma \) is given by just one independent helicity amplitude, which we take to be \( A_\gamma(1^+, 2^+, 3^-, 4^+, 5^-) \). All other helicity configurations can be obtained from parity and charge conjugation.

We can write \( A_\gamma(1^+, 2^+, 3^-, 4^+, 5^-) \) in terms of coefficients \( c_2, c_4, c_6 \) and \( c_{12} \) for the various independent spinor structure as

\[ A_\gamma(1^+, 2^+, 3^-, 4^+, 5^-) = \frac{i \sqrt{2}}{s^3} \left[ \begin{array}{c} \langle 3 | \langle 5 | \langle 42 \left( (1 - x_1) \left( c_2 + \frac{2}{x_2} c_6 - c_{12} \right) + (1 - x_2) \left( c_4 - c_{12} \right) + 2 c_{12} \right) \\
- \langle 31 | \langle 43 | \langle 35 \left( c_2 + \frac{2}{x_2} c_6 - c_{12} \right) + [41] \langle 15 \left( c_4 - c_{12} \right) \right] \right), \quad (4) \right. \]

where we have introduced the short-hand notation for spinors of definite helicity, \( |i \pm \rangle = |p_i \pm \rangle = u_\pm(p_i), \quad \langle i \pm | = \langle p_i \pm | = \bar{u}_\pm(p_i) = \bar{v}_\pm(p_i), \)

and for the spinor products \( \langle pq \rangle = \langle p - |q \rangle \) and \( [pq] = \langle p + |q \rangle \).

The coefficients \( c_i \) depend on the \( x_1 \) and \( x_2 \) of eq.(3) and can be calculated in conventional dimensional regularization. To that end, we proceed as follows [6]-[8]. In a first step, with the help of Schwinger parameters [9], we map the tensor integrals to combinations of scalar integrals in various dimensions and with various powers \( \nu_i \) of the propagators. For every basic topology, these scalar integrals can be written as nested sums involving \( \Gamma \)-functions. The evaluation of the nested sums proceeds systematically with
the help of the algorithms of [6], which rely on the algebraic properties of the so called \(Z\)-sums,

\[
Z(n; m_1, \ldots, m_k; x_1, \ldots, x_k) = \sum_{n \geq i_1 > i_2 > \ldots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \ldots \frac{x_k^{i_k}}{i_k^{m_k}}.
\]

(5)

By means of recursion the algorithms allow to solve the nested sums in terms of a given basis in \(Z\)-sums to any order in \(\varepsilon\). \(Z\)-sums can be viewed as generalizations of harmonic sums [10] and an important subset of \(Z\)-sums are multiple polylogarithms [11],

\[
\text{Li}_{m_k, \ldots, m_1}(x_k, \ldots, x_1) = Z(\infty; m_1, \ldots, m_k; x_1, \ldots, x_k).
\]

(6)

All algorithms for this procedure have been implemented in FORM [12] and in the GiNaC framework [13, 14]. In this way, we could calculate all loop integrals contributing to the one- and two-loop virtual amplitudes very efficiently in terms of multiple polylogarithms.

The perturbative expansion in \(\alpha_s\) of the functions \(c_i\) is defined through

\[
c_i = \sqrt{4\pi\alpha_s} \left( c_i^{(0)} + \frac{\alpha_s}{2\pi} c_i^{(1)} + \left( \frac{\alpha_s}{2\pi} \right)^2 c_i^{(2)} + O(\alpha_s^3) \right).
\]

(7)

Then, after ultraviolet renormalization, the infrared pole structure of the renormalized coefficients \(c_i^{\text{ren}}\) agrees with the prediction made by Catani [15] using an infrared factorization formula. We use this formula to organize the finite part into terms arising from the expansion of the pole coefficients and a finite remainder,

\[
c_i^{(2),\text{fin}} = c_i^{(1),\text{ren}} - \left( I^{(1)}(\varepsilon)c_i^{(1),\text{ren}} - I^{(2)}(\varepsilon)c_i^{(0)} \right),
\]

for \(i = \{2, 4, 6, 12\}\), and with the one- and two-loop insertion operators \(I^{(1)}(\varepsilon)\) and \(I^{(2)}(\varepsilon)\) given in [15].

As an example, we present our result for \(n_fN\)-contribution to the finite part \(c_{12}^{(2),\text{fin}}\) at two loops,

\[
c_{12}^{(2),\text{fin}}(x_1, x_2) = n_f N \left( 3 \ln(x_1) \left( \frac{\ln(x_1)}{(x_1 + x_2)^2} + \frac{1}{4} \frac{\ln(x_2)^2 - 2 \text{Li}_2(1-x_2)}{x_1(1-x_2)} \right) + \frac{\zeta(2)}{12(1-x_2)x_1} - \frac{1}{18} \frac{13x_1^2 + 36x_1 - 10x_1x_2 - 18x_2 + 31x_2^2}{(x_1 + x_2)^2x_1(1-x_2)} \ln(x_2) + \frac{x_1^2 - x_2^2 - 2x_1 + 4x_2}{(x_1 + x_2)^4} \text{R}_1(x_1, x_2) - \frac{1}{12} \frac{\text{R}(x_1, x_2)}{x_1(x_1 + x_2)^2} \left[ 5x_2 + 42x_1 + 5 \right] \right)
\]

(9)
\[-\frac{(1+x_1)^2}{1-x_2} - 4 \frac{1-3x_1+3x_1^2}{1-x_1-x_2} - 72 \frac{x_1^2}{x_1+x_2} \right] + \left[ \frac{1}{12} \frac{x_1}{1-x_2} + \frac{6}{(x_1+x_2)^3} \right.
\left. - \frac{1+2x_1}{x_1(x_1+x_2)^2} \right] (\text{Li}_2(1-x_2) - \text{Li}_2(1-x_1)) - \frac{1}{(x_1+x_2)x_1} \]
\[-\frac{1}{2} \pi n_f N \frac{\ln(x_2)}{x_1(1-x_2)}.
\]

We have introduced the function \( R(x_1, x_2) \), which is well known from [16],

\[
R(x_1, x_2) = \left( \frac{1}{2} \ln(x_1) \ln(x_2) - \ln(x_1) \ln(1-x_1) + \frac{1}{2} \zeta(2) - \text{Li}_2(x_1) \right) + (x_1 \leftrightarrow x_2).
\]

In addition, it is convenient, to define the symmetric function \( R_1(x_1, x_2) \), which contains a particular combination of multiple polylogarithms [11],

\[
R_1(x_1, x_2) = \left( \ln(x_1) \text{Li}_{1,1} \left( \frac{x_1}{x_1+x_2}, x_1+x_2 \right) - \frac{1}{2} \zeta(2) \ln(1-x_1-x_2) \right) \]
\[
+ \text{Li}_3(x_1+x_2) - \ln(x_1) \text{Li}_2(x_1+x_2) - \frac{1}{2} \ln(x_1) \ln(x_2) \ln(1-x_1-x_2) \]
\[- \text{Li}_{1,2} \left( \frac{x_1}{x_1+x_2}, x_1+x_2 \right) - \text{Li}_{2,1} \left( \frac{x_1}{x_1+x_2}, x_1+x_2 \right) \right) + (x_1 \leftrightarrow x_2).
\]

We have made the following checks on our result. As remarked, the infrared poles agree with the structure predicted by Catani [15]. This provides a strong check of the complete pole structure of our result. In addition, we have tested various relations between the \( c_i \). For instance, the combination \( x_1 c_6 \) is symmetric under exchange of \( x_1 \) with \( x_2 \). Finally, we could compare with the result for the squared matrix elements, i.e. the interference of the two-loop amplitude with the Born amplitude, and the interference of the one-loop amplitude with itself. The results of [5] are given in terms of one- and two-dimensional harmonic polylogarithms, which form a subset of the multiple polylogarithms [11]. Thus, we have performed the comparison analytically and we agree with the results of [5].

3. Conclusions

Our result represents one contribution to the full next-to-next-to-leading order calculation of \( e^+ e^- \to 3\)-jets. It has been obtained by means of an efficient method based on nested sums and is expressed in terms of multiple polylogarithms with simple arguments. As a consequence, our result can be continued analytically and applies also to \((2+1)\)-jet production in deep-inelastic scattering or to the production of a massive vector boson in
hadron-hadron collisions. At the same time, it provides an important cross check on the results for the squared matrix elements [5] with a completely independent method.

After the results of section 2 had been presented at this conference, Garland et al. published results for the complete two-loop amplitude for $e^+e^- \to q\bar{q}g$. Our results are in agreement with ref. [17].

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