Polytopes and Knots

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Abstract

A construction of polytopes is given based on integers. These geometries are constructed through a mapping to pure numbers and have multiple applications, including statistical mechanics and computer science. The number form is useful in topology and has a mapping to one-dimensional knot contours.
1 Introduction

Geometric discrete surfaces are commonplace in physics and mathematics. Their use in path integrals on discrete spaces, e.g. in statistical mechanics or Regge calculus, is well known. In mathematics discrete surfaces are sometimes used to characterize topologies, and they have many applied uses.

The construction of multi-dimensional surfaces in terms of simplicial complexes is standard practice in labeling surfaces [1]. Simplicial complexes are not usually written in a convenient form for practical computations. A definition of a simplicial complex, or polytope, that is in one to one correspondence with integers is provided in this work. The numbering of the surfaces is useful for calculations in mathematics and has applications in applied physics including statistical physics.

A primary example of the use of the polytopic definition presented here is many body discrete systems. Statistical mechanical models require the summation of surfaces, i.e. polytopes, weighted in a manner with the coupling constants. The number theoretic definition of the polytopes reduces the many body problem of summing the individual lattice sites to one variable. Then, the counting of the zeros (e.g. [3]) of an associated polynomial generate the solution of statistical mechanics models in various dimensions at low temperature, in a well-defined expansion.

Another important use of the number definition of the polytopes is in the construction of alternative computing languages based on geometric surfaces. The gluing and assembly of solids, in a real object oriented sense, is relevant to handling of data, but in a number represented form. The encoding of data in a geometrically high dimensional sense is also useful for information theory and cryptography, and for transcendental calculations in mathematics.

2 Polytope construction: Spatial

The polytopes considered are labeled by taking a lattice and inserting 0s and 1s in all of its points. The 1s then label a surface. Further colorings, e.g. a fiber on the tangent space, on the surface are obtained by expanding the base 2 to base $M$. This surface is illustrated in figure 1.

The polytopes (simplicial complexes) considered are constructed via a set of integers that label the points and faces parameterizing the surface. The integers may be given a matrix representation that permits a polynomial interpretation, and hence maps to knot(s) invariant(s).
Figure 1: An illustration of a polytope (e.g. simplicial complex) in two dimensions.
The polytopes considered are rectangular in the lattice. That is the solids have edges at right angles in all dimensions. A straightforward generalization alleviates this condition to permit non right angles, either by a rotation or a different definition of the polytopic surface (and volume). Also, ‘boundary’ surfaces, i.e. polytopes constructed with only 2-dimensional surfaces can be defined, generated with a different algorithm than used for the space filling surfaces.

Take a series of numbers \( a_1a_2 \ldots a_n \) corresponding to the digits of an integer \( p \), with the base of the individual number being \( 2^n \); this number \( a_j \) could be written in base 10 by the usual digits. In this way, upon reduction to base 2 the digits of the base reduced number spans a square with \( n + 1 \) entries. Each number \( a_j \) parameterizes a column with ones and zeros in it. The lift of the numbers could be taken to base 10 with minor modifications, by converting the base of \( p \) to 10 (with possible remainder issues if the number does not 'fit' well).

The individual numbers \( a_i \) decompose as \( \sum a_i^m 2^m \) with the components \( a_i^m \) being 0 or 1. Then map the individual number to a point on the plane,

\[
\vec{r}_i^m = a_i^m \times m\hat{e}_1 + a_i^m \times i\hat{e}_2 ,
\]

with the original number mapping to a set of points on the plane via all of the entries in \( a_1a_2 \ldots a_m \). In doing this, a collection of points on the plane is spanned by the original number \( p \), which could be a base 10 number. The breakdown of the number to a set of points in the plane is represented in figure 1.

In the case of a rectangular region spanned by the vectors \( \vec{r}_i^m \) no additional vector is required to delimit the region, as opposed to the general case with non-orthogonal sides only bounding the region. The vectors \( \vec{r}_i^m \) label points on the plane, and between any two points which are adjacent in the \( \vec{e}_1 \) direction, a line is drawn between them (adjacent means on the same \( x_1 \)-axis). Similarly, between any two points adjacent in the \( \vec{e}_2 \) direction a line is drawn. This integer \( p \) then defines a bounded region in the plane, with general disconnected components. An alternative would be to fill in the entire rectangular region with points, and have the number parameterize all of the points. The two representations are equivalent, but generate different numbers \( p \).

A set of further integers \( p_j = a_1^{(j)}a_2^{(j)} \ldots a_m^{(j)} \) are used to label a stack of coplanar lattices with the same procedure to fill in the third dimension. The spacial filling of the disconnected polhedron is assembled through the stacking of the base reduced integers.
Colored polytopes are introduced by base reducing the integers \( p_j \) into the based reduced \( a_j^{(k,m)} \) into base \( N \). The individual entries in the lattice spanned by,

\[
\vec{r} = r_i^m = a_i^{m,k} \times m\hat{e}_1 + a_i^{m,k} \times i\hat{e}_2 + a_i^{m,k} \times k\hat{e}_3 .
\]

(2.2)

The based reduced entries may be attributed into 'colors' or group theory indices labeling a representation.

Next the volume \( V \) and the \( \partial V \) surface area of the polytope region is deduced from the entries \( a_i^{m,k} \). The volume is the sum of the individual entries \( a_i^{m,k} \) over the entire lattice,

\[
p_j = a_i^{k,2^i} \quad V_s = \sum_{i,k,m} a_i^{k,m} .
\]

(2.3)

The surface area of the polytope is a region bounded by the entries of the entries \( a_i^{m,k} \). The bounded region is found via the differences of the entries \( a_i \); in two dimensions,

\[
V_{sf} = \sum_{ij} |a_i^j - a_i^{j-1}| - \sum_{ij} |a_i^j - a_i^{j-1}| .
\]

(2.4)

The region bounding the polytope is deduced from the differences in the integers.

The terms in both series, \( V_s \) and \( V_{sf} \), are defined or computed via the expansions,

\[
P_1^i = \sum M_{(1)}^{ij} p^j \quad P_2^i = M_{(2)}^{ij} p^j = \sum |a_i - a_{i-1}|_{\text{pint}} ,
\]

(2.5)

\[
P_1^i = \sum a_i |_{\text{pint}} ,
\]

(2.6)

defined for the integer \( p \) configuration. Even though the the individual terms \( |a_i - a_i| \) in the summations involved the expansion are absolute value, the entire sum is found via a summation over the individual numbers \( p \) parameterizing the lattice and its configuration. (A computation of Ising model partition function in one dimension allows the matrices \( M_1 \) and \( M_2 \) to be computed indirectly for particular lattices).

The 'colored' boundary is given a boundary via the same formalism, but with a generalized difference \( |a_i - a_{i-1}| \); group theory or 'color' differences found with a
different inner product are possible. The summations for these numbers may also be inverted to obtain the values $a_i$ in terms of $p^j$ and an associated matrix.

An example list of this variables is given in the following table,

\[
\begin{pmatrix}
p & a_i & p_1 & p_2 \\
1 & 1 & 1 & 1 \\
2 & 01 & 1 & 2 \\
3 & 11 & 2 & 0 \\
4 & 001 & 1 & 2 \\
5 & 101 & 2 & 2 \\
6 & 011 & 2 & 2 \\
7 & 111 & 3 & 0 \\
\end{pmatrix}
\]

(2.7)

The number $p$ is listed, followed by the binary format; the integers $p_1$ and $p_2$ are the sums $\sum a_i$ and $\sum |a_i - a_{i+1}|$, in a cyclic fashion around the numbers $p$.

The polyhedron is constructed by the single numbers spanning the multiple layers in 3-d, or by one number with the former grouped as $p_1 p_2 \ldots p_n$. The generalization to multiple dimensions is straightforward.

The gluing of the polyhedra is clear. For example, the numbers $p_1$ and $p_2$ that label two polytopes in two dimensions may be joined by adding their base two reduced forms. The vectors $\vec{r}_1$ and $\vec{r}_2$ are added together to find $\vec{r}_3$; then $\vec{r}_3$ is modified to another number $p_3$. If the overlap of the two initial vectors results in a 2 in the base two form, then there is intersection; there should be an arithmetic operation on the two integers $p_1$ and $p_2$ to find this answer. For example, if $a_j^{(1)} + a_j^{(2)}$ results in a number greater than $2^n$ then there is overlap; this is for a base $2^n$ number parameterizing a column of a square of dimension $(n+1) \times (n+1)$. The individual numbers $a_j$ in $p_1$ and $p_2$ add without overlap into the number $p$.

The rotations and translations of the individual polytopes may also be formulated presumably as a functions operation on the number $p$. These operations have a direct application on the base 2 form, by treating the solids as a collection of vector points $\vec{r}$ and taking the usual actions. Changing the colors, when colored, is another operation.

These polytopic operations have many applications when the individual numbers $p$ (i.e. the geometry) take on a dynamic setting, for example in computing and cryptography, or when these numbers represent simplicial complexes in a more physical application.
3 Statistical Mechanics

The summation over surfaces is required in statistical mechanics in order to compute the free energy and correlations. This usually involves the summation of variables at large numbers of lattice sites. The dimensionality of the lattice, and the couplings in these dimensions of the lattice points, complicates the solutions of these models. The summation of variables at the lattice sites can be converted into a summation of random surfaces; the latter is made simpler by the polytopic definition and the conversion of the many body lattice integral into a discrete sum of integers that label all of the surfaces. This is demonstrated in the following.

The high temperature limit of the models is changed into a low temperature limit, in this formalism, via the solution of the counting problem of the zero set to the level polynomials \( P(z) = q \). This is described in this section. The mathematics is partially addressed in [3].

Consider the \( Z_N \) models defined by the Hamiltonian,

\[
H_0 = \sum \sigma_i \gamma_i^+ \tag{3.1}
\]

\[
H_1 = \sum \sigma_i \gamma_i^+ + \sum \mu \sigma_i \tag{3.2}
\]

which in polytope language is,

\[
H = \sum_\Delta e^{-V_s \gamma_i^+ + (V - V_s) \gamma_i^- - 2V_s^4 \gamma_i^+ + 2V_s^4 \gamma_i^-} \tag{3.3}
\]

The solids and surfaces count the + and − configurations, in which the islands of +s and −s are polyhedra. In the 2-d case, these models are typically solved, in a restricted coupling sense and for nearest neighbors, through transfer matrix methods. An alternative solution is via resolvants of the lattice configurations into the algebraic forms, i.e. integers, labeling them.

The partition function derived from these lattice configurations is,

\[
Z[\gamma_i, \mu] = \sum_{\sigma_i} e^{-\beta H} \tag{3.4}
\]

and via the partition sum, the free energy is,
\[ Z[\gamma_i, \mu] = \sum_p e^{-\beta H(p)} . \]  

(3.5)

The pieces of the Hamiltonian are generated through the forms,

\[ H_0(p) = -(\gamma_0^+ - \gamma_0^-) \sum M_{(1)}^{ij} p^i + (N^d \gamma_1^- - 4\gamma_1^+) \sum M_{(1)}^{ij} p^j \]  

(3.6)

\[ H_1(p) = \gamma_1^- \sum M_{(2)}^{ij} p^j . \]  

(3.7)

and

\[ H_2(p) = -2\mu \sum M_{(1)}^{ij} p^j + N^d \mu \]  

(3.8)

The polynomials in \( p \) label the volumes and surface areas of the polytopes involved in the sum. In this approach the infinite number of variables is reduced to the single summation indexed by the polytope configuration variable \( p \). The polynomials \( M_{ij}^{p^j} \) are found in the previous section; the coefficients of the polynomials dictate the geometry of the lattice and also the form and number of interactions such as nearest neighbor and non-nearest neighbor. A modification of the polynomials can incorporate all of these various interactions.

The summations are expandable into,

\[ F(T; \gamma_i, \mu) = \sum_i \Delta(i) e^{-\gamma(i)} \]  

(3.9)

a partition into quasi-modular forms. The function \( \Delta(i) \) counts the repetitions of the \( \gamma(i) \) in the exponential expansion of the partition function \( 3.5 \). Basically,

\[ \sum_p \prod_{i=1}^M e^{b_ip^i} = \sum_p N_p e^{-\gamma(p)} , \]  

(3.10)

and the counting \( N_p \) is found via solving for the zero set to the polynomials in the exponent of \( 3.5 \). These zeros are found from the solutions to

\[ \sum b_i p^i = \gamma(p) , \]  

(3.11)
with \( p \) an integer. The countings of \( p \) requires solving for zeros of polynomial equations in one variable, with the degree of the polynomial set by the number of lattice sites. The form is derived from \( M(i) \), describing the interactions.

Consider the scenario of \( \gamma_0^+ = -\gamma_0^- = \gamma_1 = \gamma \), and no magnetic field. Scaling the coupling constant out of the partitions would generate an expansion in terms of \( e^{m\beta\gamma} \). The function \( \Delta(i) \) is not coupling dependent, and the function \( \gamma(i) \) is \( \beta\gamma m \). The explicit coupling dependence is

\[
\sum_p \prod_{i=1}^{M} e^{-\gamma\beta\gamma p^i} = \sum N_n e^{-\beta\gamma(n)},
\]

(3.12)

with \( \gamma(p) = \gamma p \). The polynomial solutions to \( \beta_i p^i = n \) generate the high-temperature solution. The polynomial nature also makes the automorphicity somewhat apparent, because counting the solutions have to be done.

The expansion generalizes to further interactions with,

\[
F(T; \gamma, i) = \sum_i \Delta_\gamma(i) \Delta_\rho(i) e^{-\gamma(i) - \rho(i)}.
\]

(3.13)

This occurs in the case of multiple interactions in the Hamiltonian, for example, when the magnetic field is turned on.

Before closing this section, a few comments are made. Most importantly, the multiple summations on the spin variables \( \sigma_i \), which is large (near infinite), have been traded in for one variable, an integer. This seems to be quite a simplification; however, the zeros of a polynomial equation have to be performed (some progress along the lines in [3] is required in order to make this explicit). The reduction of the system to one variable is quite important in the solution to these models in this approach.

Conversely, a solution to these models allows one to find zeros to polynomials in special cases, e.g. model dependent. The known solutions for the models may be used, such as the Ising model in two dimensions. However, a more general statistical mechanical model is required to find more general zero level sets of polynomials.

Multiple interactions may be included in these models. For example, the non-nearest neighbor interaction

\[
H_{n.n.} = \sum_i \gamma_s \sigma_i \sigma_i \pm s
\]

(3.14)
may be incorporated into the model, with the $i \pm s$ meaning that there are interactions spaced a distance $s$ apart in specified directions. The complication in including these interactions is in changing the polynomial equation $b_ip_i$. The interactions $|a_i - a_{i+s}|$ require new matrices $M_{(i)}$ which change the $b_i$ matrices. The matrix equations $p^{(i)}_k = M^{(i)}_{(k)}p^j$ require the $M_{(k)}$ to be computed. Of course, the end result for the partition function with the additional interactions requires only one sum to be computed.

The matrices $M_{(i)}$ are not included here for these interactions; however, they may be found for various lattices. The solution of two-dimensional models is useful in the derivation for more general models. Potentially all of the non-nearest neighbor interactions may be included, when the $M$ matrices are found that represent these couplings.

Models with non-integer spin degrees of freedom may be examined, such as rational $p/q$ ones. Also, perturbations with defect singularities, such as required with the Hubbard, can be placed in the models by changing the matrices $M_{(i)}$ appropriate to the defect couplings.

The upshot of the analysis in the solution to the free energy of these models is that the two mathematical steps are required to be completed:

1. derivation of the appropriate matrices $M_{(i)}$
2. derivation of the count to level sets of polynomials $P(z) = q$

These two steps are mathematically well-posed. Their solution is important to solve most statistical mechanical models, and to uncover the structure beneath them.

4 Numbers and the Polytopes to the Knots

First a brief review of the definition and construction of a polynomial invariant that uniquely characterizes the topology of a contour in three dimensions is given (i.e. a knot invariant).

The contour of the knot is labeled by an oriented line that self-intersects (over and under) at a number of points. The knot configuration is made mathematically precise by labeling all of the oriented self-intersections. These four types of oriented intersections are labeled by two by two matrices; the collection of the matrices is put into a polynomial form by collecting the information in a systematic fashion. These matrices are,
\[ M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]  

\[ M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]  

(4.1)

There are a total of \( n \) intersections in the knot configuration, which through a single closed contour are passed through twice each in traversing the loop. These matrices are assembled into a \( 2n \) by \( 2n \) matrix \( M \) via block form by inserting at position \((i,j)\) the two by two matrix associated with the \( (i,j) \) node along the contour; this fills up all but the diagonal elements. The diagonal entries along \((i,i)\) are given an empty two by two matrix. Following the arrows along the contour, the lower triangular two by two matrices are the transpose of the upper triangular ones and the matrix satisfies \( M = M^T \). (\( M_1 \) is the geometric transpose of \( M_4 \)).

This matrix is a member of \( \text{Sp}(2n) \) and gives a projection onto the adjoint representation, \( M = \sum_i a_i T^i \). One could put minus signs in the upper triangular portion so that the final matrix satisfies \( M = -M^T \) to make it belong to \( \text{SO}(2n) \). The \( \text{Sp}(2n) \) (or \( \text{SO}(2n) \)) generators could be given the standard form,

\[(M_{ab})^{ij} = \delta^i_a \delta^j_b \pm \delta^i_b \delta^j_a. \]  

(4.3)

Via the projection \( M = \sum a_i T^i \) a polynomial is made that labels the knot. The coefficients \( a_i \) are assembled into the form \( P(z) \),

\[ P(z) = \sum_{i=1}^{2n} a_i z^i, \]  

(4.4)
a polynomial in the parameter \( z \).

As an example, the trefoil knot’s polynomial in figure 1 is given. This configuration has three intersections and \( M_t \) is dimension twelve. The matrix \( M_t \) in block form with the \( M_j \) matrices is,

\[ M_t = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \end{pmatrix}. \]  

(4.5)
Figure 2: (1) The four types of intersections. (2) A sample trefoil knot.
The decomposition of this trefoil’s $M_t$ is $a_{8,1} = 1$, $a_{9,4} = 1$, and $a_{12,5} = 1$ and is symmetrized. The polynomial $P_t(z)$ is, via the decomposition of the generators through $z^{(j-1)\times 2n+i}$,

$$P_t(z) = z^8 + z^{40} + z^{60}.$$  \hspace{1cm} (4.6)

This example describes the procedure for finding $M$ and $P(z)$.

The knot polynomials described in [2] may be further reduced and put into the form of invariants, including the Reidemeister moves. The polynomials have both number theory and group theory properties.

Equivalent knots under the Reidemester moves are grouped into polynomials of infinite degree,

$$Q(z) = \sum b_i z^i,$$  \hspace{1cm} (4.7)

with the $b_i$ numbers representing the individual knot topologies. Each $Q(z)$ represents an equivalence class of the topologies via the three Reidemeister moves, as discussed in [2]. By definition, all of the $b_i$ are distinct numbers in not just the individual equivalence classes, but in all of the $Q(z)$ classes. The number form of the knot invariant, as presented in [2], is required; other knot invariants may also be placed in a number form.

A use of the number theoretic knot form is that the geometries labeled by the polytopes may be mapped into the one-dimensional knot configurations embedded in three dimensions. The use of such a mapping, basically from all topologies in more than one dimension to one-dimensional knot topologies is not entirely clear analytically, but can be used to classify topologies in higher dimensional geometries. The reduction of the $d$-dimensions to $d = 1$ might emphasizes the importance of knot mathematics and physics.

The reduction to knot topologies of the higher dimensional topologies emphasizes further characterizations of the latter in terms of, for example, Reidemeister moves. There could be unrecognized symmetries in the specifications of dimensionally varying topologies using the knot groups and knot characterizations.

Also, developments in transcendental computing based on number theoretic forms of higher dimensional geometries can be further classified and and possibly reduced in terms of these line elements and the symmetries inherit in them. (Three-dimensional gauge physics and their correlations could be of use in this regard.)
5 Concluding remarks

Polytopes, i.e. polyhedra, within a volume $V$ are mapped to integer numbers. The algebraic nature of the mapping of the topologies to the integers is relevant to descriptions of the topologies and their properties. As an example, a function map of the cohomologies and homotopy based on the integers could be possible.

The characterization of the multi-dimensional topologies can be mapped to one-dimensional knot configurations. This property could lead to the manifestation of hidden symmetries in the topology, with relations to the former knot descriptions. It seems possible that without the knot configurations a transcendental description of topology can be based in number theory and algebra also.

The application to physics is clear especially in the field of statistical mechanics. These models require means to sum over many variables at the the individual lattice sites and the polytopic definition can reduce this sum to a one variable summation. Generalizations from nearest neighbor interactions to non-nearest neighbor interactions is clear from an algebraic standpoint.

Required in the solution to the statistical mechanical models is that the two mathematical steps are required to be completed:

(1) derivation of the appropriate matrices $M(i)$

(2) derivation of the count to level sets of polynomials $P(z) = q$

These two steps are mathematically well-posed. Their solution is important to solve most statistical mechanical models, and to uncover the structure beneath them.

The description of the polytopes in terms of integers has applications to math and physics. There could be alternative descriptions of the surfaces that would lead to a simpler derivation of their uses; simpler means less computational transcendentally.

The use in computational programming allows for an inherent parallel processing not described in this text based on geometric routing of data. The integer description might be useful in this regard both in the assimilation and building of data streams. Generalized RSA keys are also simple to construct based on fitting and placing of interlocking geometric solids.
References

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