WEAK PSEUDOCONCAVITY AND THE MAXIMUM MODULUS PRINCIPLE

C.Denson Hill and Mauro Nacinovich

In this paper we focus on the maximum modulus principle and weak unique continuation for CR functions on an abstract almost CR manifold $M$. It is known that some assumption must be made on $M$ in order to have either of these: it suffices to consider the standard CR structure on the sphere $S^3$ in $\mathbb{C}^2$ to see that the maximum modulus principle is not valid in the presence of strict pseudoconvexity. For weak unique continuation, Rosay [R] has shown by an example that there is a strictly pseudoconvex CR structure on $\mathbb{R}^3$, which is a perturbation of the aforementioned standard CR structure on $S^3$, such that there exists a smooth CR function $u$, $u \neq 0$, with $u \equiv 0$ on a nonempty open set. However positive results were obtained in [DCN] under the assumption of pseudoconcavity and in [HN] under the assumption of essential pseudoconcavity (and also finite kind for the maximum modulus principle).

Here we investigate these matters under the assumption of weak pseudoconcavity on $M$, which is a more general notion than that of essential pseudoconcavity, insofar as it drops the minimality (and the finite kind) hypothesis on $M$. We obtain sharp results involving propagation along Sussmann leaves. The core of our argument is that on a weakly pseudoconcave $M$ the square of the modulus of a CR function $u$ is subharmonic with respect to a degenerate-elliptic operator $P$ on $M$. We employ a maximum principle for real valued functions which is in the spirit of [Hf], [Ni], [B], [H].

In order to understand our motivation in considering the weak pseudoconcavity condition on $M$, the reader is referred to the examples in [HN].

§1 Weak pseudoconcavity of almost CR manifolds

An abstract smooth almost CR manifold of type $(n, k)$ consists of: a connected smooth paracompact manifold $M$ of dimension $2n + k$, a smooth subbundle $HM$ of $TM$ of rank $2n$, and a smooth complex structure $J$ on the fibers of $HM$.

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Let \( T^{0,1}M \) be the complex subbundle of the complexification \( \mathbb{C}HM \) of \( HM \), which corresponds to the \(-\sqrt{-1}\) eigenspace of \( J \):

\[
T^{0,1}M = \{ X + \sqrt{-1} JX \mid X \in HM \}.
\]

We say that \( M \) is a CR manifold if, moreover, the formal integrability condition

\[
[C^\infty(M, T^{0,1}M), C^\infty(M, T^{0,1}M)] \subset C^\infty(M, T^{0,1}M)
\]

holds.

Next we define \( T^{*,1,0}M \) as the annihilator of \( T^{0,1}M \) in the complexified cotangent bundle \( \mathbb{C}T^*M \). We denote by \( Q^{0,1}M \) the quotient bundle \( \mathbb{C}T^*M/T^{*,1,0}M \), with projection \( \pi_Q \). It is a rank \( n \) complex vector bundle on \( M \), dual to \( T^{0,1}M \). The \( \bar{\partial}_M \)-operator acting on smooth functions is defined by \( \bar{\partial}_M = \pi_Q \circ d \). A local trivialization of the bundle \( Q^{0,1}M \) on an open set \( U \) in \( M \) defines \( n \) smooth sections \( \bar{L}_1, \bar{L}_2, \ldots, \bar{L}_n \) of \( T^{0,1}M \) in \( U \); hence

\[
\bar{\partial}_M u = (\bar{L}_1 u, \bar{L}_2 u, \ldots, \bar{L}_n u),
\]

where \( u \) is a function in \( U \). Solutions \( u \) of \( \bar{\partial}_M u = 0 \) are called CR functions.

The characteristic bundle \( H^0M \) is defined to be the annihilator of \( HM \) in \( T^*M \).

Its purpose it to parametrize the Levi form: recall that the Levi form of \( M \) at \( x \) is defined for \( \xi \in H^0_xM \) and \( X \in H_xM \) by

\[
\mathcal{L}(\xi; X) = d\tilde{\xi}(X, JX) = \langle \xi, [J\tilde{X}, \tilde{X}] \rangle,
\]

where \( \tilde{\xi} \in C^\infty(M, H^0M) \) and \( \tilde{X} \in C^\infty(M, HM) \) are smooth extensions of \( \xi \) and \( X \). For each fixed \( \xi \) it is a Hermitian quadratic form for the complex structure \( J_x \) on \( H_xM \).

Denote by \( H^{1,1}M \) the smooth subbundle of the tensor bundle \( HM \otimes_M HM \) whose fiber \( H^{1,1}_xM \) at \( x \in M \) is the real vector subspace of \( H_xM \otimes H_xM \) generated by the tensors of the form \( v \otimes v + (Jv) \otimes (Jv) \) for \( v \in H_xM \). \( H^{1,1}M \) is the bundle of Hermitian symmetric tensors in \( HM \otimes_M HM \). For each \( x \in M \) and \( \xi \in H^0M \) the Levi form \( \mathcal{L}(\xi, \cdot) \) defines a linear form \( \mathcal{L}_\xi : H^{1,1}M \to \mathbb{R} \) such that

\[
\mathcal{L}_\xi (v \otimes v + (Jv) \otimes (Jv)) = \mathcal{L}(\xi, v) \quad \forall v \in H_xM.
\]

For \( x \in M \) let us denote by \( \Gamma H^{1,1}_xM \) the convex hull of \( \{ v \otimes v + (Jv) \otimes (Jv) \mid v \in H_xM \} \) and by \( \Gamma H^{1,1}M \) its interior (in \( H^{1,1}_xM \simeq \mathbb{R}^{2n^2} \)). They are the closed cone of nonnegative Hermitian symmetric tensors and the open cone of positive Hermitian symmetric tensors of \( H_xM \otimes H_xM \), respectively. The disjoint union \( \Gamma H^{1,1}M = \bigcup_{x \in M} \Gamma H^{1,1}_xM \) is an open subset of \( H^{1,1}M \) and the restriction of the projection onto the base:

\[
\pi : \Gamma H^{1,1}M \to M
\]
is a smooth fiber bundle, whose fibers are open convex cones in $\mathbb{R}^{n^2}$. Note that the choice of a smooth Hermitian metric $h$ on the fibers of $HM$ defines an exponential map

$$\exp_h : H^{1,1}M \to \Gamma H^{1,1}M,$$

giving a smooth bundle isomorphism between $\Gamma H^{1,1}M$ and $H^{1,1}M$.

**Definition** We say that an abstract almost CR manifold $M$ is *weakly pseudoconcave* iff for every $x \in M$ there is an open neighborhood $U$ of $x$ in $M$ and a smooth section $\Omega \in C^\infty(U, \Gamma H^{1,1}M)$ such that

$$\mathcal{L}_\xi(\Omega) = 0 \quad \forall x \in U, \xi \in H_x^0M. \quad (1.8)$$

**Remark** Every abstract almost CR manifold, whose Levi form vanishes identically, is trivially weakly pseudoconcave. However, when $k > 0$, such a manifold is not necessarily *essentially pseudoconcave* in the sense of Definition A of [HN].

An abstract almost CR manifold of type $(n,0)$ is the same thing as an *almost complex manifold*; such manifold can be regarded as being essentially pseudoconcave, and hence weakly pseudoconcave. In this case the CR functions will be called *almost holomorphic functions*.

We shall need the following results from [HN]:

**Proposition 1.1** Let $M$ be an abstract almost CR manifold of type $(n,k)$. Then $M$ is weakly pseudoconcave if and only if there exists a smooth Hermitian metric $h$ on the fibers of $HM$ such that

$$\text{trace}_h (\mathcal{L}(\xi, \cdot)) = 0, \quad \forall \xi \in H^0M. \quad (1.9)$$

**Proposition 1.2** Let $M$ be an abstract almost CR manifold of type $(n,k)$. If $M$ is weakly pseudoconcave then

$$\begin{aligned}
&\text{For each } \xi \in H^0M \text{ the Levi form } \mathcal{L}(\xi, \cdot) \text{ is either 0} \\
&\text{or has at least one positive and one negative eigenvalue.}
\end{aligned} \quad (1.10)$$

If $D := C^\infty(M, HM) + [C^\infty(M, HM), C^\infty(M, HM)]$ is a distribution of constant rank, then (1.10) is also sufficient for $M$ to be weakly pseudoconcave.
Proposition 1.3 Under the assumptions of Proposition 1.1, let $U$ be an open subset of $M$ on which $X_1, \ldots, X_n \in C^\infty(U, HM)$ give at each point $y \in U$ an h-orthonormal basis of the complex Hermitian vector space $H_y M$. Set $\bar{L}_j = X_j + iJX_j$ and $L_j = X_j - iJX_j$, for $j = 1, \ldots, n$. Then there are smooth complex valued functions $\beta^r$ ($1 \leq r \leq n$) on $U$ such that

\[(1.11) \quad i \sum_{j=1}^n [L_j, \bar{L}_j] = \sum_{r=1}^n (\beta^r L_r + \bar{\beta}^r \bar{L}_r) \quad \text{in} \quad U.\]

Let $L = X - iJX$ be one of the $L_j$’s from Proposition 1.3. We have

\[(1.12) \quad \Re L\bar{L} = X^2 + (JX)^2 \quad \Im L\bar{L} = [X, JX].\]

Let $u$ be a CR function in $U$, and consider $|u|^2 = u\bar{u}$. Since

\[(1.13) \quad \bar{L} |u|^2 = (Lu) \bar{u} + u \bar{L} \bar{u},\]

and $\bar{L} u = 0$, we obtain

\[(1.14) \quad L\bar{L} |u|^2 = |Lu|^2 + u [L, \bar{L}] \bar{u}.\]

It follows that

\[(1.15) \quad \left( \sum_{j=1}^n L_j L_j \right) |u|^2 = \sum_{j=1}^n |L_j u|^2 + u \left( \sum_{j=1}^n [L_j, \bar{L}_j] \right) \bar{u} = \sum_{j=1}^n |L_j u|^2 + \bar{u} \left( \sum_{j=1}^n \beta^r \bar{L}_r \right) \bar{u} = \sum_{j=1}^n |L_j u|^2 + \frac{1}{i} \left( \sum_{r=1}^n \beta^r \bar{L}_r \right) |u|^2,\]

because of (1.11). Hence

\[(1.16) \quad \left\{ \Re \left( \sum_{j=1}^n L_j \bar{L}_j \right) + \Im \left( \sum_{j=1}^n \beta^j L_j \right) \right\} |u|^2 = \sum_{j=1}^n |L_j u|^2 \geq 0.\]

A similar calculation shows that

\[(1.17) \quad \left\{ \Re \left( \sum_{j=1}^n L_j \bar{L}_j \right) + \Im \left( \sum_{j=1}^n \beta^j L_j \right) \right\} \Re u = 0.\]
Let $P_U$ denote the real operator inside the curly brackets. It has the form

\begin{equation}
\sum_{j=1}^{n} \left( X_j^2 + (JX_j)^2 \right) + X_0,
\end{equation}

where the $X_1, \ldots, X_n, JX_1, \ldots, JX_n$ provide a basis for $HM$ at each point of $U$, and $X_0 \in C^\infty(U, HM)$.

**Proposition 1.4** Let $M$ be a weakly pseudoconcave almost CR manifold of type $(n, k)$. Then one can construct a smooth real linear second order partial differential operator $P$ on $M$ such that:

(i) each $x_0 \in m$ has a neighborhood $U$ in which $P$ can be written in the form (1.18);

(ii) if $u$ is a $C^2$ CR function on $M$, then $Pu = 0$ and $P|u|^2 \geq 0$ on $M$.

**Proof** It suffices to take

\begin{equation}
P = \sum_U \psi_U P_U,
\end{equation}

where $\{\psi_U\}$ is a nonnegative partition of unity subordinate to a covering $\{U\}$ of $M$ by open sets $U$, as in Proposition 1.3. Indeed (ii) is then obvious, while (i) follows because $P_U$ and $P_V$ have the same principal symbol on $U \cap V$.

§2 **Sussmann leaves**

In this section we collect the results which we shall need concerning the Sussmann leaves of an arbitrary set $D$ of smooth real vector fields on a smooth paracompact manifold $M$ of real dimension $N$. In our final application, $M$ will be an abstract almost CR manifold, and $D = C^\infty(M, HM)$. However, in our discussion of the maximum principle for real valued functions, in the next section, we shall be in this more general situation.

Let $x_0 \in M$ and $\Omega$ be an open subset of $M$ containing $x_0$. The Sussmann leaf $\mathcal{F}(x_0, \Omega)$ of $D$ in $\Omega$ through $x_0$ is defined to be the set of points $x \in \Omega$ for which there exist finitely many smooth curves $s_j : [0, 1] \to \Omega$, for $j = 1, \ldots, \ell$, such that:

\begin{equation}
\begin{cases}
\dot{s}_j(t) \in D_{s_j(t)} & \text{for } 0 \leq r \leq 1 \text{ and } j = 1, 2, \ldots, \ell; \\
s_j(0) = x_0, \quad s_j(0) = s_{j-1}(1) & \text{for } j = 2, \ldots, \ell \text{ and } s_\ell(1) = x.
\end{cases}
\end{equation}

Note that $\mathcal{F}(x, \Omega) = \mathcal{F}(x_0, \Omega)$ for all $x \in \mathcal{F}(x_0, \Omega)$. Sussmann proved in [S] that $\mathcal{F}(x_0, \Omega)$ is always a smooth immersed (but not necessarily embedded) submanifold
of $\Omega$. Note also that $T_x \mathcal{F}(x_0, \Omega) \supset D_x$ for all $x \in \mathcal{F}(x_0, \Omega)$. We say that $M$ is minimal at $x_0$ in $M$ iff for every open neighborhood $U$ of $x_0$ in $M$, the Sussmann leaf $\mathcal{F}(x_0, U)$ contains an open neighborhood of $x_0$ in $M$. The manifold $M$ is said to be minimal if it is minimal at each point. This condition is equivalent to the nonexistence of a lower dimensional smooth submanifold $S$ of $M$ with $x_0 \in S$ and $T_x S \supset D_x$ for every $x \in S$.

Next we recall the definition of the set $N_e F$ of exterior conormals to a closed subset $F$ of $M$: it is the subset of $T^* M$ consisting of all the nonzero $\xi_0 \in T^*_{x_0} M$, with $x_0 \in F$, for which there exists a smooth real valued function $f$ on $M$ with $df(x_0) = \xi_0$ and $f(x) \leq f(x_0)$ for all $x \in F$.

In what follows we shall use the well known trapping lemma (see for instance [Ho I, Theorem 8.5.11, p.304]):

**Proposition 2.1** Let $F$ be a closed subset of $M$. If

\[ \xi(X) = 0 \quad \text{for all} \quad \xi \in N_e F \quad \text{and all} \quad X \in D, \]

then $\mathcal{F}(x, M) \subset F$ for every $x \in F$.

### §3 A maximum principle for real valued functions

Let $M$ and $D$ be as in section 2. We shall consider a smooth real second order linear partial differential operator $P$ on $M$ with the following property: Given $x_0 \in M$, there is an open neighborhood $U$ of $x_0$ in $M$, and $Y_0, Y_1, \ldots, Y_\ell \in D$ such that

\[ \begin{cases} 
Y_1, \ldots, Y_\ell & \text{generate } D \text{ in } U, \\
Y_0 &= \sum_{j=1}^{\ell} Y_j^2 + Y_0 \quad \text{in } U.
\end{cases} \]

**Theorem 3.1** Let $\Omega$ be an open subset of $M$, $x_0 \in \Omega$, $u \in C^2(\Omega, \mathbb{R})$ and $Pu \geq 0$ along $\mathcal{F}(x_0, \Omega)$. If $u(x) \leq u(x_0)$ for all $x \in \mathcal{F}(x_0, \Omega)$, then $u$ is constant along $\overline{\mathcal{F}(x_0, \Omega)} \cap \Omega$.

**Proof** For the proof we can, without loss of generality, assume that $\Omega = M = \mathcal{F}(x_0, \Omega)$ and $Pu \geq 0$ on $M$.

Let $F$ denote the closed subset $\{x \in M \mid u(x) = u(x_0)\}$. We want to show that $F = M$. Assume by contradiction that $F \neq M$; i.e., that $F$ does not contain $\mathcal{F}(x_0, M)$. By Proposition 2.1 there exist $x_1 \in \partial F$, $\xi \in T^*_{x_1} M$ with $\xi \in N_e F$ and
$Y \in \mathcal{D}$ such that $\xi(Y) \neq 0$. This implies the following: there is a coordinate patch $U \simeq \{y \in \mathbb{R}^N \mid |y| < R\}$ containing $x_1$, with $0 < |y(x_1)| = r < R$, such that

\[(i) \quad P = \sum_{j=1}^{\ell} Y_j^2 + Y_0 \text{ in } U \text{ with } Y_0, Y_1, \ldots, Y_\ell \in \mathcal{D};\]

\[(ii) \quad Y_{j_0}(|y|^2) \neq 0 \text{ at } x_1 \text{ for some } j_0 \text{ with } 1 \leq j_0 \leq \ell;\]

\[(iii) \quad u(x) < u(x_0) = u(x_1) \text{ if } x \in U \text{ and } |y(x)| \leq r, \ x \neq x_1.\]

Let $\gamma > 0$. Then

\[
P \left( e^{-\gamma |y|^2} \right) = e^{-\gamma |y|^2} \left\{ \gamma^2 \sum_{j=1}^{\ell} |Y_j (|y|^2)|^2 + O(\gamma) \right\}
\]

is positive on a neighborhood of $x_1$ for $\gamma > 0$ sufficiently large. Fix $\gamma > 0$ and $\epsilon > 0$ in such a way that $0 < \epsilon < R - r$ and $P(\exp(-\gamma |y|^2)) > 0$ when $x \in U$ and $|y(x) - y(x_1)| \leq \epsilon$. For $\delta > 0$ set $v_\delta = u + \delta \left( e^{-\gamma |y|^2} - e^{-\gamma r^2} \right)$. Then $Pv_\delta > 0$ for $|y(x) - y(x_1)| \leq \epsilon$. Note that $v_\delta(x) < u(x)$ when $|y(x)| > r$. On the other hand, if $|y(x)| \leq r$ and $|y(x) - y(x_1)| = \epsilon$. Thus for $\delta > 0$ sufficiently small, we obtain that $v_\delta(x) < u(x_0) = u(x_1)$ on the boundary of $\omega = \{x \in U \mid |y(x) - y(x_1)| < \epsilon\}$. Since $v_\delta(x_1) = u(x_1) = u(x_0)$, the restriction of $v_\delta$ to $\overline{\omega}$ has a maximum at some point $x_2 \in \omega$. But at $x_2$ we would then have that $Pv_\delta(x_2) \leq 0$, which contradicts the inequality $Pv_\delta > 0$ we have established in $\omega$. Thus $F = M$ and the theorem is proved, after using continuity of $u$ to pass to the closure of the Sussmann leaf.

§4 Weak unique continuation

In this section we return to a smooth manifold $M$ which is an abstract almost CR manifold of type $(n, k)$, and $\mathcal{D}$ will be $\mathcal{C}^\infty(M, HM)$. In this situation, for any open $\Omega \subset M$ and $x_0 \in M$, the Sussmann leaf $\mathcal{F}(x_0, \Omega)$ is itself a smooth abstract almost CR manifold of type $(n, h)$ for some $h \leq k$.

The next theorem is an improvement of the weak unique continuation result of [DCN, Theorem 4.1], [HN, Theorem 5.1].

**Theorem 4.1** Assume that $M$ is weakly pseudoconcave. Let $u \in L^2_{\text{loc}}(M)$ satisfy the following:

\[
\text{for every } \tilde{L} \in \mathcal{C}^\infty(M, T^{0,1}M), \tilde{L}u \in L^2_{\text{loc}}(M) \quad \text{and there exists } \kappa_\tilde{L} \in L^\infty_{\text{loc}}(M) \text{ such that } \quad |\tilde{L}u(x)| \leq \kappa_\tilde{L}(x)|u(x)| \quad \text{a.e. in } M.
\]

Then $\mathcal{F}(x, M) \subset \text{supp } u$ for every $x \in \text{supp } u$. 
We use again Proposition 2.1. Indeed under the contrary assumption, there exists a \( \xi \in N_e(supp \, u) \) such that \( \xi(X) \neq 0 \) for some \( X \in H M \). We obtain a contradiction by using the Carleman type estimate given by the following theorem.

**Theorem 4.2** Let \( M \) be a weakly pseudoconcave abstract almost CR manifold of type \((n, k)\). Let \( \phi \) be a real valued smooth function on \( M \) and \( x_0 \in M \) a point where \( \phi(x_0) = 0 \) and \( d\phi(x_0) \notin H^0 M \). Then we can find \( A > 0 \), \( C > 0 \), \( \tau_0 > 0 \) and an open neighborhood \( U \) of \( x_0 \) in \( M \) such that:

\[
\sqrt{\tau} \cdot \|f \cdot \exp(\tau(\phi + A\phi^2))\|_0 \leq c\|\bar{\partial} \, f \cdot \exp(\tau(\phi + A\phi^2))\|_0 \\
\forall f \in C_0^\infty(U), \quad \forall \tau \geq \tau_0.
\]

Here the \( L^2 \)-norms \( \| \cdot \|_0 \) are computed using any smooth Riemannian metric on \( M \) and any smooth Hermitian metric on the fibers of \( Q^{0,1} M \).

Theorem 4.2 is just Theorem 5.2 of [HN], with “weakly pseudoconcave” replacing “essentially pseudoconcave” in the hypothesis. In fact the proof of Theorem 5.2 in [HN] does not use the minimality assumption on \( M \), which is part of the definition of essential pseudoconcavity, but only uses the weak pseudoconcavity.

**Corollary 4.3** Assume that \( M \) is weakly pseudoconcave. Let \( u \) be a continuous CR function on \( M \), and \( x_0 \in M \). Let \( \omega \) be an open neighborhood of \( x_0 \) in \( M \). If \( u \equiv 0 \) on \( F(x_0, M) \cap \omega \), then \( u \equiv 0 \) along \( F(x_0, M) \).

**Proof** We obtain the Corollary from Theorem 4.2, after replacing \( M \) by \( F(x_0, M) \).

**Corollary 4.4** Let \( M \) be a weakly pseudoconcave smooth abstract CR manifold of type \((n, k)\). Let \( \mathcal{L} \xrightarrow{L^2} M \) be a smooth complex CR line bundle over \( M \), and \( u \) be a continuous CR section of \( \mathcal{L} \) over \( M \). If \( x_0 \in M \) and \( \omega \) is an open neighborhood of \( x_0 \) such that \( u \equiv 0 \) on \( F(x_0, M) \cap \omega \), then \( u \equiv 0 \) along \( F(x_0, M) \).

**Proof** For the notion of a complex CR line bundle we refer to section 7 of [HN]. The corollary follows from Theorem 4.2 because, according to formula (7.4) in [HN], the representative of the section \( u \), in any smooth (not necessarily CR) local trivialization of \( \mathcal{L} \), satisfies (4.1).

§ 5 The maximum modulus principle

In this section we have: \( M \) is a smooth abstract almost CR manifold of type \((n, k)\), \( \Omega \) is an open subset of \( M \), and \( \mathcal{D} = C^\infty(M, H M) \). Fix a point \( x_0 \in \Omega \) and set \( \mathcal{F} = F(x_0, \Omega) \).
**Lemma 5.1** Let \( u \in C^1(\Omega) \) be a CR function in \( \Omega \). Assume that \( u \big|_\mathcal{F} \) has values which lie along a piecewise \( C^1 \)-regular curve in \( \mathbb{C} \). Then \( u(x) = u(x_0) \) for every \( x \in \mathcal{F} \).

**Proof** It suffices to show that \( u \) is locally constant along \( \mathcal{F} \), and we can also assume that the values of \( u \) lie on a \( C^1 \)-regular curve in \( \mathbb{C} \). Let \( \gamma \) be the \( C^1 \)-regular curve in \( \mathbb{C} = \mathbb{R}^x \times \mathbb{R}^y \). Let \( p_0 \in \mathcal{F} \) and \( \omega \) be a connected open neighborhood of \( u(p_0) \) in \( \gamma \). If we take \( \omega \) sufficiently small, then there is an open neighborhood \( \Omega \) of \( u(p_0) \) in \( \mathbb{C} \), and a real valued \( C^1 \) function \( F(x,y) \) in \( \Omega \) such that

\[
\omega = \{ x + iy \in \Omega \mid F(x,y) = 0 \}, \quad dF \neq 0 \quad \text{in} \quad \Omega.
\]

Choose a connected open neighborhood \( V \) of \( p_0 \) in \( \mathcal{F} \) such that \( u(V) \subset \omega \). Then \( F(\Re u, \Im u) = 0 \) on \( V \), so

\[
0 = \overline{\partial_F} F = F_u \overline{\partial_F} u + F_\bar{u} \overline{\partial_F} \bar{u} = F_u \overline{\partial_F} \bar{u} \quad \text{and} \quad F_u \neq 0;
\]

hence \( Xu = 0 \) in \( V \) for every \( X \in \mathcal{D} \). This in turn implies that \( u \) is constant along \( \mathcal{F} \) in \( V \), and hence along \( \mathcal{F} \).

**Remark** The lemma remains valid if we assume \( u \in C^1(\mathcal{F}) \) and \( u \) is CR on the almost CR manifold \( \mathcal{F} \).

**Theorem 5.2** Let \( M \) be a smooth abstract weakly pseudoconcave almost CR manifold of type \((n,k)\). Consider an open subset \( \Omega \) of \( M \) and a point \( x_0 \in \Omega \). Let \( u \in C^2(\mathcal{F}(x_0,\Omega)) \) be a CR function on the almost CR manifold \( \mathcal{F}(x_0,\Omega) \). Assume that

\[
|u(x_0)| = \sup_{\mathcal{F}(x_0,\Omega)} |u|.
\]

Then \( u \) is constant along \( \overline{\mathcal{F}(x_0,\Omega)} \cap \Omega \).

**Proof** We observe that \( \mathcal{F}(x_0,\Omega) \) is a smooth abstract almost CR manifold of type \((n,k)\) for some \( h \leq k \). By Proposition 1.4 there is a smooth real linear second order operator \( P \) on \( \mathcal{F}(x_0,\Omega) \) of the form \((3.1)\) such that \( P|u|^2 \geq 0 \). By Theorem 3.1 the real valued function \( |u|^2 \) is constant along \( \mathcal{F}(x_0,\Omega) \). According to Lemma 5.1, \( u \) is constant along \( \mathcal{F}(x_0,\Omega) \).

**Theorem 5.3** Let \( M \) be a smooth abstract weakly pseudoconcave almost CR manifold of type \((n,k)\). Consider a nonempty open subset \( \Omega \) of \( M \) and a point \( x_0 \in \Omega \). Let \( u \in C^2(\mathcal{F}(x_0,\Omega)) \) be a CR function on the almost CR manifold
Theorem 5.2. Assume that $M$ is minimal at $x_0$ and that $|u|$ has a local weak maximum at $x_0$. Then $u$ is constant along $\mathcal{F}(x_0, \Omega) \cap \Omega$.

Proof By our assumption $\mathcal{F}(x_0, \Omega)$ is an open neighborhood of $x_0$ in $\Omega$. Hence there is an open subset $\omega$ of $\Omega$, containing $x_0$, such that

$$|u(x_0)| = \sup_{\omega}|u|.$$  

By Theorem 5.2 it follows that $u$ is constant along $\mathcal{F}(x_0, \omega)$, which is a neighborhood of $x_0$ in $\Omega$. Corollary 4.3 then implies that the function $u - u(x_0)$ is identically zero along $\mathcal{F}(x_0, \Omega)$.

Recall that the notion of essential pseudoconcavity in [HN] is weak pseudoconcavity plus minimality. Thus we obtain the following improvement of Theorem 6.4 in [HM]:

**Corollary 5.4** Assume that $M$ is a smooth connected essentially pseudoconcave abstract almost CR manifold of type $(n, k)$. Let $u \in C^2(M)$ be a CR function on $M$. If $|u|$ has a weak local maximum at some point $x_0$ of $M$, then $u$ is constant on $M$.

**Remark 1** In the statement of Theorem 5.2, Theorem 5.3, and Corollary 5.4 one can substitute $\Re u$ in place of $|u|$, because of (1.17). In particular if $M$ is as in Corollary 5.4, a $C^2$ CR function on $M$, which is real valued on a neighborhood of a point of $M$, is constant on $M$.

**Remark 2** Suppose $M$ is an almost complex manifold. Then, according to Corollaries 4.3, 4.4, 5.4, the almost holomorphic functions on $M$ obey weak unique continuation, and enjoy the usual form of the maximum modulus principle. However in this situation the almost holomorphic functions obey strong unique continuation, because of (1.17), according to Theorem 17.2.6 in [Ho III].

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WEAK PSEUDOCONCAVITY ...

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C.Denson Hill - Department of Mathematics, SUNY at Stony Brook, Stony Brook NY 11794, USA
E-mail address: dhill@math.sunysb.edu

Mauro Nacinovich - Dipartimento di Matematica ”L. Tonelli” - via F.Buonarroti, 2 - 56127 PISA, Italy
E-mail address: nacinovi@dm.unipi.it