Gamma-Limit of a Model for the Elastic Energy of an Inextensible Ribbon

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Abstract A $\Gamma$-convergence result involving the elastic bending energy of a narrow inextensible ribbon is established. As a consequence of the result, the energy is reduced to a one-dimensional integral, over the centerline of the ribbon, in which the aspect ratio of the ribbon appears as a small parameter. That integral is observed to increase monotonically with the aspect ratio. The $\Gamma$-limit of the family of energies is taken in a Sobolev space of centerlines with nonvanishing curvature. In that space, it is shown that the $\Gamma$-limit is a functional first proposed by Sadowsky in the context of narrow ribbons that form Möbius bands. The results obtained here do not apply to such ribbons, since the centerline of a Möbius band must have at least one inflection point. As a first step toward dealing with such inflection points, a result concerning the lower semicontinuity of the Sadowsky functional with inflection points comprising a set of measure zero within the domain of an arclength parameterization is presented.

Keywords Low-dimensional media · Dimensional reduction · Curvature elasticity · Sadowsky functional · Torsion · Sequential lower semicontinuity · Weak convergence

Mathematics Subject Classification 49Q10 · 49S05 · 82B21

1 Introduction

An inextensible ribbon is modeled as a two-dimensional surface that is geometrically constrained to be isometric to a rectangle of given length $\ell$ and width $2w$. The dimensionless...
parameter $\varepsilon = 2w/\ell$ is referred to as the aspect ratio of the ribbon. As Giomi and Mahadevan [10] note, certain biopolymers, such as DNA, and graphene and silicene nanoribbons are examples of ribbons with very small aspect ratios. Granted that the curvature $\tilde{\kappa}$ of the centerline $C$ of the ribbon is nonvanishing, the geometric constraint yields a parametrization of the ribbon in terms of $C$.

To determine the equilibrium shape of an elastic, inextensible ribbon subject to imposed end conditions, it suffices to minimize its net potential energy. Here, it is assumed that the elastic energy density $\phi$ of the ribbon is an isotropic, quadratic function of the Weingarten map, and thereby a symmetric, quadratic function of the principle curvatures of the ribbon. Upon completing the square, $\phi$ admits a representation in terms of the mean and Gaussian curvatures $H$ and $K$ of the ribbon of the form

$$\phi = \frac{D}{2} (H - H_0)^2 + C K,$$

(1)

where $D$ and $C$ are constant moduli and $H_0$ is the spontaneous mean curvature. The expression (1) was proposed by Germain [9]. The particular version of (1) considered here, in which $H_0$ is taken to be zero, was considered by Poisson [12].

In the limit $\varepsilon \to 0$ of vanishing aspect ratio, Sadowsky [15] argued that the energy of a ribbon forming a Möbius band should be proportional to

$$\mathcal{F} = \int_C \tilde{\kappa}^2 (1 + \eta^2)^2 \, d\xi,$$

(2)

where $\eta$ is the ratio $\tilde{\tau}/\tilde{\kappa}$, with $\tilde{\tau}$ being the torsion of the centerline $C$, and where $\xi$ denotes arclength along $C$. The properties of the functional (2) were studied in some detail by Wunderlich [19]. Recently, Starostin and van der Heijden [16] used the variational bi-complex formalism to investigate the equilibrium equations for the problem associated with minimizing the functional

$$\mathcal{F}_{\varepsilon} = \int_C \tilde{\kappa}^2 (1 + \eta^2)^2 \frac{1}{\varepsilon \ell \dot{\eta}} \ln \left( \frac{2 + \varepsilon \ell \dot{\eta}}{2 - \varepsilon \ell \dot{\eta}} \right) \, d\xi,$$

(3)

for $\varepsilon > 0$, where a superposed dot indicates differentiation with respect to the arclength parameter $\xi$. Upon inspection, it is evident that the Sadowsky functional $\mathcal{F}$ is the pointwise limit of the elastic energy $\mathcal{F}_{\varepsilon}$ as $\varepsilon \to 0$. However, the question of whether the Sadowsky functional (2) is the proper variational limit (that is, the $\Gamma$-limit) of the elastic energy (3) of a ribbon with a given centerline remains unanswered. This question is settled herein for curves with nonvanishing curvature that are parametrized by arclength and are elements of certain Sobolev spaces.

As Randrup and Røgen [13] remark, the centerline of a nonorientable developable, like a developable Möbius band, must have at least one point at which the curvature vanishes. The problem of establishing the Sadowsky functional as the $\Gamma$-limit for a space of centerlines containing those corresponding to a Möbius band is left for future work. However, a result in this direction is provided.

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The first steps in the analysis are identical to those appearing in the papers of Wunderlich [19] and Starostin and van der Heijden [16] and also in the thesis of Yong [20]. These steps deliver an expression for the elastic energy of the ribbon in terms of the shape of its centerline and depending parametrically on the aspect ratio $\varepsilon$ of the ribbon. In particular, the energy is given by

$$E = \frac{\varepsilon \ell D}{2} \int_0^\ell \tilde{\kappa}^2(\xi)(1 + \eta^2(\xi))^2 g(\varepsilon \ell \dot{\eta}(\xi)) \, d\xi,$$

(4)