$K^0 - \bar{K}^0$ Mixing in the $1/N_c$ Expansion

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Abstract

We present the result for the invariant $\hat{B}_K$ factor of $K^0 - \bar{K}^0$ mixing in the chiral limit and to next-to-leading order in the $1/N_c$ expansion. We explicitly demonstrate the cancellation of the renormalization scale and scheme dependences between short- and long-distance contributions in the final expression. Numerical estimates are then given, by taking into account increasingly refined short- and long-distance constraints of the underlying QCD Green’s function which governs the $\hat{B}_K$ factor.

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1 Introduction

The Standard Model predicts strangeness changing transitions with $\Delta S = 2$ via two virtual $W$-exchanges between quark lines, the so-called box diagrams. The low-energy physics of these transitions is governed by an effective Hamiltonian which is proportional to the local four-quark operator, (summation over colour indices within brackets is understood and $q_L = \frac{1 - \gamma_5}{2} q_i$)

$$Q_{\Delta S = 2}(x) \equiv (\bar{s}_L(x)\gamma^\mu d_L(x)) (\bar{s}_L(x)\gamma_\mu d_L(x)),$$

modulated by a quadratic form of the flavour mixing matrix elements $\lambda_q = V_{qd} V_{qs}$, $q = u, c, t$, with coefficient functions $F_{1,2,3}$ of the heavy masses of the fields $t$, $Z^0$, $W^\pm$, $b$, and $c$ which have been integrated out:\n
$$\mathcal{H}_{\text{eff}}^{\Delta S = 2} = \frac{G_F^2 M_W^2}{4\pi^2} \left[ \lambda^2 F_1 + \lambda^2 F_2 + 2\lambda_c \lambda_t F_3 \right] C_{\Delta S = 2}(\mu) Q_{\Delta S = 2}(x).$$

The operator $Q_{\Delta S = 2}$ is multiplicatively renormalizable and has an anomalous dimension $\gamma(\alpha_s)$ which in perturbative QCD (pQCD) is defined by the equation

$$\frac{\mu^2}{d\mu} < Q_{\Delta S = 2} > = -\frac{1}{2} \gamma(\alpha_s) < Q_{\Delta S = 2} >,$$

$$\gamma(\alpha_s) = \frac{\alpha_s}{\pi} \gamma_1 + \left( \frac{\alpha_s}{\pi} \right)^2 \gamma_2 + \cdots.$$  (1.3)

with $(\beta_1 = \frac{1}{4}(11N_c + 2n_f))$

$$\gamma_1 = \frac{3}{2} \left( 1 - \frac{1}{N_c} \right)$$

and

$$\gamma_2 = \frac{1}{32} \left( 1 - \frac{1}{N_c} \right) \left[ -\frac{19}{3} \frac{N_c}{4} + \frac{4}{3} n_f - 21 + 5\frac{N_c}{6} \kappa \right],$$

(1.4)

where $\kappa = 0$ in the naive dimensional renormalization scheme (NDR) and $\kappa = -4$ in the 't Hooft–Veltman renormalization scheme (HV). The renormalization $\mu$-scale dependence of the Wilson coefficient $C_{\Delta S = 2}(\mu)$ in Eq. (1.2) is then, $(\beta_2 = -\frac{17}{12} N_c^2 + \frac{1}{4} N_c n_f + \frac{5}{12} N_c n_f)$,

$$\begin{aligned}
C_{\Delta S = 2}(\mu) &= \left[ 1 + \frac{\alpha_s(\mu)}{\pi} \frac{1}{\beta_1} \left( \gamma_2 + \frac{\gamma_1}{\beta_1} \beta_2 \right) \left( \frac{1}{\alpha_s(\mu)} \right)^{\frac{\gamma_1}{\beta_1}} \right] \\
&\quad \rightarrow \left[ 1 + \frac{\alpha_s(\mu)}{\pi} \left( \frac{1433}{1936} + \frac{1}{8} \kappa \right) \left( \frac{1}{\alpha_s(\mu)} \right)^{\frac{\gamma_1}{\beta_1}} \right],
\end{aligned}$$

(1.5)

where the second line gives the result to next-to-leading order in the $1/N_c$ expansion, which is the approximation at which we shall be working here. The matrix element

$$\langle K^0 | Q_{\Delta S = 2}(0) | K^0 \rangle \equiv \frac{4}{3} f_K^2 M_K^2 B_K(\mu),$$

(1.7)

defines the so-called $B_K$-parameter of $K^0 - \bar{K}^0$ mixing at short-distances, which is one of the crucial parameters in the phenomenological studies of CP-violation in the Standard Model. In the large-$N_c$ limit of QCD, the four quark operator $Q_{\Delta S = 2}$ factorizes into a product of two current operators. Each of these currents, to lowest order in chiral perturbation theory, has a simple bosonic realization:

$$\langle \bar{s}_L \gamma^\mu d_L \rangle = \frac{\delta \mathcal{L}_{\text{QCD}}(x)}{\partial \mu(x)} \Rightarrow \frac{i}{2} F_0^2 \text{tr} \left[ \lambda_{32} (D^\mu U^\dagger) U \right] = \frac{1}{\sqrt{2}} F_0 \partial^\mu K^0 + \cdots,$$

(1.8)

where $F_0$ denotes the coupling constant of the pion in the chiral limit and $U$ is the $3 \times 3$ unitary matrix in flavour space, $U = 1 + \cdots$, which collects the Goldstone fields and which under chiral rotations transforms like $U \rightarrow V_R U_L^\dagger$. The large-$N_c$ approximation, with inclusion of the chiral corrections in the factorized contribution, leads to the result:

$$B_K|_{N_c \to \infty} = \frac{3}{4}$$

and

$$C_{\Delta S = 2}(\mu)|_{N_c \to \infty} = 1.$$  (1.9)

\footnote{For a detailed discussion, see e.g. refs. [4, 5] and references therein.}
In full generality, the bosonization of the four–quark operator $Q_{\Delta S=2}(x)$ to lowest order in the chiral expansion is described by an effective operator which is of $O(p^2)$:

$$Q_{\Delta S=2}(x) \Rightarrow -\frac{F_0^4}{4} g_{\Delta S=2}(\mu) \text{ tr} \left[ \lambda_{32}(D^\mu U^\dagger)U \lambda_{32}(D_\mu U^\dagger)U \right],$$  \hspace{1cm} (1.10)

where $\lambda_{32}$ denotes the matrix $\lambda_{32} = \delta_{35} \delta_{2j}$ in flavour space and $g_{\Delta S=2}(\mu)$ is a dimensionless coupling constant which depends on the underlying dynamics of spontaneous chiral symmetry breaking (S\_SB) in QCD. The relation between the coupling $g_{\Delta S=2}(\mu)$ and the phenomenological $B_K$ factor defined in Eq. (1.7) is simply

$$B_K(\mu, m_{u,d,s} \to 0) = \frac{3}{4} g_{\Delta S=2}(\mu).$$  \hspace{1cm} (1.11)

Notice that the coupling $g_{\Delta S=2}(\mu)$, and hence $B_K$, is perfectly well defined in the chiral limit, while the matrix element in Eq. (1.7) vanishes in the chiral limit as a chiral power. To lowest order in the chiral expansion, the relation to the so called invariant $B_K$–factor is as follows:

$$\hat{B}_K = \frac{3}{4} C_{\Delta S=2}(\mu) \times g_{\Delta S=2}(\mu),$$  \hspace{1cm} (1.12)

which means that the coupling $g_{\Delta S=2}(\mu)$ must have a $\mu$ dependence (and a scheme dependence) which must cancel with the $\mu$ dependence (and scheme dependence) of the Wilson coefficient $C_{\Delta S=2}(\mu)$. The purpose of this note is to show how the mechanism of cancellations works in practice within the framework of the $1/N_c$ expansion. This will allow us to give a numerical result for $\hat{B}_K$ which is valid to lowest order in the chiral expansion and to next–to–leading order in the $1/N_c$ expansion. We wish to emphasize that the novel feature of this work is that, to our knowledge, it is the first calculation of $\hat{B}_K$ which explicitly shows the cancellations of scale and scheme dependences.

2 Bosonization of Four–Quark Operators

The QCD Lagrangian in the presence of external chiral sources $l_\mu$, $r_\mu$ of left– and right– currents, but with neglect of scalar and pseudoscalar sources which is justified in the chiral limit approximation we shall be working here, has the form

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^{(a)}_{\mu \nu} G^{(a)\mu \nu} + i\bar{q} D_{\chi} q,$$  \hspace{1cm} (2.1)

with $D_{\chi}$ the Dirac operator

$$D_{\chi} = \gamma^\mu \left( \partial_\mu + i G_\mu \right) - i \gamma^\mu \left[ l^\mu \frac{1 - \gamma_5}{2} + r^\mu \frac{1 + \gamma_5}{2} \right].$$  \hspace{1cm} (2.2)

The bosonization of the four–quark operator $Q_{\Delta S=2}(x)$ is formally defined by the functional integral $^3$

$$\langle Q_{\Delta S=2}(x) \rangle = \text{ Tr } D_{\chi}^{-1} i \frac{\delta D_{\chi}}{\delta l_\mu(x)} \text{ Tr } D_{\chi}^{-1} i \frac{\delta D_{\chi}}{\delta r_\mu(x)}$$

$$- \int d^4 y \frac{d^4 q}{(2\pi)^4} e^{-i q \cdot (x-y)} \text{ Tr } \left( D_{\chi}^{-1} i \frac{\delta D_{\chi}}{\delta l_\mu(y)} D_{\chi}^{-1} i \frac{\delta D_{\chi}}{\delta l_\mu(x)} \right),$$  \hspace{1cm} (2.3)

where the trace Tr here also includes the functional integration over gluons in large $N_c$. The first term corresponds to the factorized pattern and gives the contributions of $O(N_c^2)$; the second term corresponds to the unfactorized pattern and it involves an integral, (which is regularization dependent,) over all virtual momenta $q$. This is the term which gives the next–to–leading $O(N_c)$ contribution we are interested in.

$^3$See e.g. refs. [1, 3]
To proceed further, it is convenient to use the Schwinger’s operator formalism. With $\mathbf{P}$ the conjugate momentum operator, in the absence of external chiral fields, the full quark propagator is

$$\langle x | \frac{1}{\mathcal{D}_x} | y \rangle = \langle x | \frac{i}{\mathbf{P} + \gamma^\alpha [l_\alpha \frac{1}{2} + r_\alpha \frac{1}{2}]} | y \rangle \quad \text{with} \quad \langle x | \mathbf{P} | y \rangle = \gamma^\mu \left[ i \partial_\mu - G_\mu \right] \delta(x - y). \quad (2.4)$$

The chiral expansion is then defined as an expansion in the $l_\alpha$ and $r_\alpha$ external sources. The precise relation between the formal bosonization in Eqs. (2.3) and the explicit chiral realization in Eq. (1.1) can best be seen from the fact that: $D_\mu U^\dagger U = \partial_\mu U^\dagger U + iU^\dagger \tau \mu U - i\tau \mu$. This shows that there are a priori six different ways to compute the constant $g_{\Delta S=2}(\mu)$, although they are all related by chiral gauge invariance. One possible choice is the term $\mathcal{L}^\Delta_{\Delta=2}(x) = -\frac{F_0^4}{4} g_{\Delta S=2}(\mu) \{ \cdots - \text{tr} |\lambda_2 U^\dagger r \mu U \lambda_2 U^\dagger r \mu U | + \cdots \}$.

$$\mathcal{L}^\Delta_{\Delta=2}(x) = -\frac{F_0^4}{4} g_{\Delta S=2}(\mu) \{ \cdots - \text{tr} |\lambda_2 U^\dagger r \mu U \lambda_2 U^\dagger r \mu U | + \cdots \}. \quad (2.5)$$

This is a convenient choice because the underlying QCD Green’s function is the four–point function, $[ L_{sd}^\mu(x) = \sum_a s^a(x) \gamma^\mu \frac{1}{2} d^a(x) \text{ and } R_{ds}^\mu(x) = \sum_a d^a(x) \gamma^\mu \frac{1}{2} s^a(x)]$

$$\mathcal{W}^{\mu\nu\beta}_{LRLR}(q,l) = \lim_{i \rightarrow 0} i^3 \int d^4 x d^4 y d^4 z e^{i q \cdot x} e^{i l \cdot y} e^{-i l \cdot z} \langle 0 | T \{ L_{sd}^\mu(x) R_{ds}^\alpha(y) L_{sd}^\nu(z) | R_{ds}^\beta(0) \} | 0 \rangle, \quad (2.6)$$

In fact, what we need, as seen in Eq. (2.3), is the integral of the unfactorized four–point function $\mathcal{W}^{\mu\nu\beta}_{LRLR}(q,l)$ over the four–vector $q$ with the Lorentz indices of the two left–currents contracted. This is a quantity which is a good order parameter of $\SigmaSB$. The integral over the solid angle has the form, $(Q^2 \equiv -q^2)$.

$$\int d\Omega_q g_{\mu\nu} \mathcal{W}^{\mu\nu\beta}_{LRLR}(q,l) | \text{unfactorized} = \left( \frac{\rho^\beta}{t^2} - g^{\alpha\beta} \right) \mathcal{W}^{(1)}_{LRLR}(Q^2), \quad (2.7)$$

where the transversality in the four–vector $l$ follows from current algebra Ward identities. We are still left with an integral of the invariant function $\mathcal{W}^{(1)}_{LRLR}(Q^2)$ over the full euclidean range: $0 \leq Q^2 \leq \infty$ which has to be done in the same renormalization scheme as the calculation of the short–distance Wilson coefficient $C_{\Delta S=2}(x)$ in Eq. (1.2) has been done, i.e. in the $\overline{MS}$–scheme. The coupling constant $g_{\Delta S=2}(\mu)$ defined with dimensional regularization in $d = 4 - \epsilon$ dimensions is then given by the following integral,

$$g_{\Delta S=2}(\mu, \epsilon) = 1 - \frac{\mu^2_{\text{had.}}}{32\pi^2 F_0^2} \frac{(4 \pi \mu^2 / \mu^2_{\text{had.}})^{\epsilon/2}}{\Gamma(2 - \epsilon/2)} \int_0^\infty dz \ z^{\epsilon/2} \left( W[z] \equiv \frac{\rho^2_{\text{had.}}}{F_0^2} \mathcal{W}^{(1)}_{LRLR}(z \mu^2_{\text{had.}}) \right), \quad (2.8)$$

where, for convenience, we have normalized $Q^2$ to a characteristic hadronic scale $\mu^2_{\text{had.}}, (z \equiv Q^2 / \mu^2_{\text{had.}})$ which in practice we shall choose within the range: $1 \text{ GeV} \leq \mu_{\text{had.}} \leq 2 \text{ GeV}.$

3 $\hat{B}_K$ to Next–to–Leading Order in the $1/N_c$ Expansion

In full generality, Green’s functions in QCD at large $N_c$ are given by sums over an infinite number of hadronic poles. Since the function $\mathcal{W}^{(1)}_{LRLR}(Q^2)$ is an order parameter of spontaneous chiral symmetry breaking it receives no contribution from the perturbative continuum and satisfies an unsubtracted dispersion relation. After repeated use of partial fractions decomposition, one can see that for the particular case of the function $W[z]$, [see Eqs. (2.8), (2.7) and (2.6)] the most general ansatz in large–$N_c$ QCD is an infinite sum of poles, double poles and triple poles of the form

$$W[z] = 6 - \sum_{i=1}^{\infty} \frac{\alpha_i}{\rho_i} - \sum_{i=1}^{\infty} \frac{\beta_i}{\rho_i^2} - \sum_{i=1}^{\infty} \frac{\gamma_i}{(z + \rho_i)} + \sum_{i=1}^{\infty} \left\{ \frac{\alpha_i}{(z + \rho_i) + \gamma_i} \left( \frac{1}{z + \rho_i} \right)^2 + \frac{\beta_i}{(z + \rho_i)^2} + \frac{\gamma_i}{(z + \rho_i)^3} \right\}, \quad (3.1)$$

4Other choices are indeed possible but, in general, the underlying QCD Green’s functions have pieces which are not order parameters and spurious contributions which depend on the regularization. We have shown the equivalence among the various possible choices, but we postpone the detailed discussion which is rather technical to a longer publication.

5We thank J. Bijnen and J. Prades for pointing out to us the existence of triple poles, which were ignored in the previous version of this manuscript.
where \( \rho_i = M_i^2/\mu^2_\text{had.} \) and \( M_i \) is the mass of the \( i \)-th narrow hadronic state. Triple poles result from the combination of two odd–parity couplings involving vector mesons and Goldstone particles. The first term on the r.h.s. is the contribution from the Goldstone poles to the \( W[z] \) function. This constant term is known from previous calculations, (see refs. [7, 8, 9]) with which we agree. Constant terms, however, do not contribute to the integral in Eq. (2.8) defined in dimensional regularization. In the absence of a solution of large–\( N_c \) QCD, the actual values of the mass ratios \( \rho_i \), and residues \( \alpha_i \), \( \beta_i \) and \( \gamma_i \) remain unknown. As discussed below, there are, however, short–distance and long–distance constraints that the function \( W[z] \) has to obey. With \( W[z] \) given by Eq. (3.1), the integral in Eq. (2.8) can be done analytically, with the result

\[
g_\Delta S=2(\mu, \epsilon) = 1 - \frac{\mu^2_\text{had}}{32\pi^2 F_0} \left\{ \left( \frac{2}{\epsilon} + \log 4\pi - \gamma_E + \log \mu^2 / \mu^2_\text{had} \right) + 1 \right\} \sum_{i=1} \alpha_i \\
- \sum_{i=1} \alpha_i \log \rho_i + \sum_{i=1} \frac{\beta_i}{\rho_i} + \frac{1}{2} \sum_{i=1} \frac{\gamma_i}{\rho_i^2} \right\} . \tag{3.2}
\]

We shall next explore the short–distance properties and long–distance properties which are presently known about the function \( W[z] \), and hence about the parameters \( \alpha_i, \beta_i, \gamma_i \) and \( \rho_i \).

- The large–\( Q^2 \) behaviour of the function \( W^{(1)}_{LRLR}(Q^2) \) is governed by the OPE. We find the result

\[
\lim_{Q^2 \to \infty} W^{(1)}_{LRLR}(Q^2) = 24\pi^2 \frac{\alpha_s}{\pi} \left[ 1 + \frac{\epsilon}{2} (5 + \kappa) + O\left( \frac{\alpha_s^2}{\pi^2} \right) \right] \frac{F_0}{F^2} + \ldots , \tag{3.3}
\]

with \( \kappa = 0 \) in the NDR scheme and \( \kappa = -4 \) in the HV scheme. The fact that the residue of the \( \frac{1}{Q^2} \) power term in the OPE is known, entails the constraint

\[
\sum_{i=1} \alpha_i = R + \frac{\epsilon}{2} S , \tag{3.4}
\]

with

\[
R = \left[ 24\pi^2 \frac{\alpha_s}{\pi} + O\left( \frac{N_c \alpha_s^2}{\pi^2} \right) \right] \frac{F^2}{\mu^2_\text{had.}} \quad \text{and} \quad S = \left[ 4\pi^2 (5 + \kappa) \frac{\alpha_s}{\pi} + O\left( \frac{N_c \alpha_s^2}{\pi^2} \right) \right] \frac{F^2}{\mu^2_\text{had.}} . \tag{3.5}
\]

This allows us to define the renormalized coupling constant \( g^{(r)}_\Delta S=2(\mu) \) in the \( \overline{MS} \) scheme:

\[
g^{(r)}_\Delta S=2(\mu) = 1 - \frac{\mu^2_\text{had}}{32\pi^2 F_0} \left\{ R \log \mu^2 / \mu^2_\text{had.} + R + S + \sum_{i=1} \left[ -\alpha_i \log \rho_i + \frac{\beta_i}{\rho_i} + \frac{1}{2} \frac{\gamma_i}{\rho_i^2} \right] \right\} \tag{3.6}
\]

\[
\Rightarrow \left( \frac{\alpha_s(\mu)}{\alpha_s(\mu_\text{had.})} \right) \frac{1}{\log \frac{\mu}{\mu_\text{had.}}} \left\{ 1 - \frac{\mu^2_\text{had}}{2 \cdot 16\pi^2 F_0} \left( R + S + \sum_{i=1} \left[ -\alpha_i \log \rho_i + \frac{\beta_i}{\rho_i} + \frac{1}{2} \frac{\gamma_i}{\rho_i^2} \right] \right) \right\} . \tag{3.7}
\]

where in the second line we have written the renormalization group improved result. We insist on the fact that this result, contrary to the various large–\( N_c \) inspired calculations which have been published so far [10, 11, 12, 13, 14, 15] is the full result to next–to–leading order in the \( 1/N_c \) expansion. The renormalization \( \mu \) scale dependence as well as the scheme dependence in the factor \( S \) cancel exactly, at the next–to–leading log approximation, with the short distance \( \mu \) and scheme dependences in the Wilson coefficient \( C_\Delta S=2(\mu) \) in Eq. (1.1). Therefore, the full expression of the invariant \( B_K \) factor to lowest order in the chiral expansion and to next–to–leading order in the \( 1/N_c \) expansion which takes into account the hadronic contribution from light quarks below a mass scale \( \mu_\text{had.} \) is then

\[
\hat{B}_K = \left( \frac{1}{\alpha_s(\mu_\text{had.})} \right)^{\frac{3}{4}} \left\{ 1 - \frac{\alpha_s(\mu_\text{had.})}{\pi} \frac{1229}{1936} + O\left( \frac{N_c \alpha_s^2(\mu_\text{had.})}{\pi^2} \right) \right\} \\
- \frac{\mu^2_\text{had}}{32\pi^2 F_0} \sum_{i=1} \left[ -\alpha_i \log \rho_i + \frac{\beta_i}{\rho_i} + \frac{1}{2} \frac{\gamma_i}{\rho_i^2} \right] \right\} . \tag{3.8}
\]
The numerical choice of the hadronic scale $\mu_{\text{had.}}$ is, a priori, arbitrary. In practice, however, $\mu_{\text{had.}}$ has to be sufficiently large so as to make meaningful the truncated pQCD series in the first line. Our final error in the numerical evaluation of $\hat{B}_K$ will include the small effect of fixing $\mu_{\text{had.}}$ within the range $1 \text{ GeV} \leq \mu_{\text{had.}} \leq 3 \text{ GeV}$.

- The fact that there is no $1/Q^2$ term in the OPE expansion of the function $W_{\text{OPE}}^{(1)}(Q^2)$ implies the sum rule
  \[
  \sum_{i=1}^{\infty} \frac{\alpha_i}{\rho_i} + \sum_{i=1}^{\infty} \frac{\beta_i}{\rho_i^2} + \sum_{i=1}^{\infty} \frac{\gamma_i}{\rho_i^4} = 6. \tag{3.9}
  \]
  This sum rule plays an equivalent rôle to the 1st Weinberg sum rule in the case of the $\Pi_{LR}(Q^2)$ two–point function. Like in this case, it guarantees the cancellation of quadratic divergences in the integral in Eq. (2.8) and it relates the contribution from Goldstone particles to a specific sum of resonance state contributions. What this means in practice is that, although the Goldstone pole does not contribute to the integral in Eq. (2.8) defined in dimensional regularization, the normalization of the function $W[z]$ at $z = 0$ is, precisely, fixed by the residue of the Goldstone pole contribution; i.e., $W[0] = 6$.

- The slope at the origin of the function $W[z]$ is fixed by a linear combination of coupling constants of the $O(p^4)$ Gasser–Leutwyler Lagrangian [14], with the result
  \[
  \sum_{i=1}^{\infty} \frac{\alpha_i}{\rho_i} + 2 \sum_{i=1}^{\infty} \frac{\beta_i}{\rho_i^2} + 3 \sum_{i=1}^{\infty} \frac{\gamma_i}{\rho_i^4} = 24\frac{\mu_{\text{had.}}^2}{F^2_0} [2L_1 + 5L_2 + L_3 + L_9]. \tag{3.10}
  \]

The numerical evaluation of $\hat{B}_K$ in Eq. (3.8) requires the input of the mass ratios $\rho_i$ and couplings $\alpha_i$, $\beta_i$ and $\gamma_i$ of the narrow states which fill the details of the hadronic spectrum of light flavours. Our goal is to find the minimal hadronic ansatz of pole terms of the form shown in Eq. (3.1) which satisfies the short–distance and long–distance constraints discussed above, and use then this minimal hadronic ansatz to evaluate the integral in Eq. (2.8). The fact that the function $W[z]$ is an order parameter of $S_{\chi SB}$ ensures that it has a smooth behaviour in the ultraviolet and it is then reasonable to approximate it by a few states. (There is no need here for an infinite number of narrow states to reproduce the asymptotic behaviour of pQCD, as it would be the case with a Green’s function which is not an order parameter of $S_{\chi SB}$.) This procedure has been successfully tested in other examples, like the calculation of the electroweak $\pi^+ - \pi^0$ mass difference [13] and the decay of pseudoscalars into lepton pairs [15], and it has been applied [16] to the evaluation of matrix elements of electroweak penguin operators as well [17].

4 Numerical Evaluation and Conclusions

We now proceed to the numerical evaluation of the analytic result for $\hat{B}_K$ in Eq. (2.8) by successive improvements in the hadronic input.

The crudest approximation is the one where the only terms of the next–to–leading order in the $1/N_c$ expansion which are retained are those from the first line in Eq. (3.8). This implicitly assumes that all the way down to the hadronic $\mu_{\text{had.}}$ scale there is only pQCD running and that the hadronic contribution below the $\mu_{\text{had.}}$ scale due to light quarks is negligible. One can see that from the integral in Eq. (2.8), if we write it in an equivalent cut–off form:

\[
\int_0^1 dz W[z] + \int_1^{\mu_{\text{had.}}^2} dz W_{\text{OPE}}[z],
\tag{4.1}
\]

\[\text{See e.g. ref. [13]}\]

\[\text{The r.h.s. in Eq. (3.10) agrees with the expression reported in ref. [1].}\]

\[\text{See also refs. [17, 18, 19] and refs. [20, 21, 22] for other recent evaluations of electroweak penguin operators.}\]
with \( \Lambda^2 = \mu^2 \exp \left(1 + \frac{z}{3} \right) \), and where in the second integral we use the asymptotic OPE expression in Eq. (3.3). The approximation we are discussing is the one where the first low–energy integral is simply neglected. The result of this next–to–leading log (nll) approximation is to bring up the large–\( N_c \) prediction in Eq. (4.1) to a value which, including an estimate of higher order corrections, (we use \( \Lambda^{(3)}_{\overline{MS}} = (372 \pm 40) \text{ MeV} \)) but ignoring the systematic errors involved in the neglect of the hadronic contribution, would be

\[
\hat{B}_K |_{\text{nll}} \sim 0.96 \pm 0.04 \quad \text{for} \quad \mu_{\text{had.}} = 1.4 \text{ GeV}.
\] (4.2)

The problem, however, with this simple estimate is that the underlying assumption of neglecting entirely the low–energy hadronic integral does not satisfy any of the long–distance matching constraints discussed in the previous section.

One can considerably improve on the previous estimate by taking into account the contribution from the lowest hadronic state to the low–energy hadronic integral, the \( \rho \) vector meson. It turns out that, in the presence of possible triple poles in Eq. (3.1), this is the minimal hadronic ansatz approximation at which one can fix the large–\( N_c \) hadronic expression in Eq. (3.1) to satisfy the two matching constraints in Eqs. (3.9) and (3.10) as well as the first non–trivial constraint from the OPE in Eq. (3.3). In fact, it has been shown that the phenomenological values of the \( L_i \) constants which appear on the r.h.s. of Eq. (3.10) agree well with those obtained from the integration of the lowest vector state \([23, 24, 27]\) alone. In this approximation \([28]\), the sum rule in Eq. (3.10) becomes simply

\[
\alpha V \rho^2 + 2 \beta V \rho \nu + 3 \gamma V = 21 \rho^2 \, \chi, \quad \text{with} \quad \rho V = M^2_V / \mu^2_{\text{had.}}.
\] (4.3)

Using this constraint, together with Eqs. (3.4) and (3.9) restricted to the lowest hadronic state, we can then solve for \( \alpha V \), \( \beta V \) and \( \gamma V \) in terms of \( \rho V \) with the result:

\[
\alpha V = 24 \pi \alpha_s \rho V F^2_0 / M^2_V, \quad \beta V = - \rho V (3 \rho V + 2 \alpha V), \quad \text{and} \quad \gamma V = \rho^2 V (9 \rho V + \alpha V).
\] (4.4)

The resulting \( W[z] \) function, normalized to its value at the origin \( W[0] = 6 \), is plotted in Fig. 1 below as a function of \( z = Q^2 / \mu^2_{\text{had.}} \). Also shown in the same plot are the chiral perturbation behaviour of the \( W[z] \) function from the knowledge of its value and the slope at the origin (the green line) and the corresponding behaviour from the OPE expression in Eq. (3.3) (the blue line). We can now make an improved estimate of \( \hat{B}_K \) by calculating the integral

\[
\int_0^{\hat{z}} dz \, W[z] + \int_{\hat{z}}^{\Lambda^2 / \mu^2_{\text{had.}}} dz \, W_{\text{OPE}}[z],
\] (4.5)

with \( W[z] \) in the first integral approximated by the minimal hadronic ansatz just discussed. The choice of \( \hat{z} \) which minimizes the dependence of \( \hat{B}_K \) on \( \hat{z} \) is the one at which the two curves \( W[z] \) and \( W_{\text{OPE}}[z] \) intersect. In terms of this \( \hat{z} \), which separates the long–distance part of the integral estimated with the minimal hadronic ansatz and the short–distance part of the integral estimated with the leading OPE behaviour, the resulting expression for \( \hat{B}_K \) is

\[
\hat{B}_K = \left( \frac{1}{\alpha_s(\mu_{\text{had.}})} \right)^2 \frac{3}{4} \left\{ 1 - \frac{\alpha_s(\mu_{\text{had.}})}{\pi} \left[ \frac{1229}{1936} - \frac{3}{4} \log \hat{z} \right] + \mathcal{O} \left( \frac{N_c \alpha_s^2(\mu_{\text{had.}})}{\pi^2} \right) \right\},
\] (4.6)

As seen in Fig. 1, which corresponds to \( \mu_{\text{had.}} \approx 1.4 \text{ GeV}, F_0 = 85.3 \text{ MeV} \) and \( M_V = 770 \text{ MeV} \), the intersection of the hadronic curve and the OPE curve happens at a value \( \hat{z} \approx 0.39 \) (i.e. \( Q \approx 0.88 \text{ GeV} \)), at which value Eq. (4.6) gives

\[
\hat{B}_K \approx 0.38.
\] (4.7)

The stability of \( \hat{B}_K \) versus \( \hat{z} \) is rather good, and it is shown in Fig. 2 for the same input values as in Fig. 1.

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\textsuperscript{9}The constant \( L_3 \) gets also an extra contribution from the lowest scalar state, but it is rather small.
Figure 1: Plot of the hadronic function $W[z]$ in Eq. (3.1) versus $z = Q^2/\mu^2_{\text{had.}}$ for $\mu_{\text{had.}} = 1.4$ GeV. The red curve is the normalized function $W[z]$ corresponding to the minimal hadronic ansatz discussed in the text, with a vector meson of mass $M_V = 770$ MeV. The green line represents the low-energy chiral behaviour and the blue curve the prediction from the OPE.

Notice that, if we had had a perfect matching between the minimal hadronic ansatz and the OPE, we could have taken the limit $\hat{z} \to \infty$ in Eq. (4.6) and find again the same result as in Eq. (3.8) with the sum over hadronic states restricted to the lowest vector state. The fact that, as seen in Fig. 1, the matching is not perfect is not surprising; it is due to the restriction of the infinite sum in Eq. (3.1) to just the couplings of the lowest vector state. It is quite remarkable that, already at this approximation, the quality of the matching is so good. The advocated choice of a $\hat{z}$ which minimizes the value of $B_K$ at the separation between a low $Q^2$ region and a high $Q^2$ region implicitly assumes that the leading term of the OPE controls reasonably well the behaviour of the underlying Green’s function $W[z]$ down to $Q$ values in the region $0.8 \text{ GeV} \lesssim Q \lesssim 1 \text{ GeV}$.

In order to quantify the errors in Eq. (4.7) we proceed as follows: for every choice of $\mu_{\text{had.}}$ one finds the corresponding value of $\hat{z}$ for which the hadronic ansatz and the OPE intersect. This value of $\hat{z}$ is then used in Eq. (4.4) to obtain $\hat{B}_K$. The error is estimated by allowing for a reasonable variation of the input parameters: $M_V = 770 \pm 30$ MeV and $1 \text{ GeV} \leq \mu_{\text{had.}} \leq 3 \text{ GeV}$. The corresponding values for $\hat{B}_K$ are given in Table 1. They correspond to the spread of values:

$$0.33 \leq \hat{B}_K \leq 0.44$$

One way to quantify the systematic error of the minimal hadronic ansatz approximation of this calculation would be to include in the analysis e.g. higher order terms in Eq. (3.3). Until we do that we suggest as a cautious rule-of-thumb estimate of this uncertainty to round off the spread in Eq. (4.8) to an overall 30% of the central value, with the result

$$\hat{B}_K = 0.38 \pm 0.11.$$
Figure 2: Plot of $\hat{B}_K$ in Eq. (4.4) versus $\hat{z}$, for $\mu_{\text{had.}} \simeq 1.4 \, \text{GeV}$, $F_0 = 85.3 \, \text{MeV}$ and $M_V = 770 \, \text{MeV}$. Notice the vertical scale in the figure.

Table 1: Summary of $\hat{B}_K$ results for different input values of $\mu_{\text{had.}}$ and $M_V$

| $M_V$   | $\mu_{\text{had.}}^2$ (1 GeV$^2$) | $\mu_{\text{had.}}^2$ (2 GeV$^2$) | $\mu_{\text{had.}}^2$ (3 GeV$^2$) | $\mu_{\text{had.}}^2$ (4 GeV$^2$) | $\mu_{\text{had.}}^2$ (5 GeV$^2$) | $\mu_{\text{had.}}^2$ (6 GeV$^2$) | $\mu_{\text{had.}}^2$ (7 GeV$^2$) | $\mu_{\text{had.}}^2$ (8 GeV$^2$) | $\mu_{\text{had.}}^2$ (9 GeV$^2$) |
|---------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| 740 MeV | 0.387                            | 0.413                            | 0.421                            | 0.426                            | 0.429                            | 0.431                            | 0.433                            | 0.434                            | 0.435                            |
| 770 MeV | 0.388                            | 0.413                            | 0.422                            | 0.426                            | 0.429                            | 0.432                            | 0.433                            | 0.435                            | 0.436                            |
| 800 MeV | 0.329                            | 0.342                            | 0.346                            | 0.347                            | 0.348                            | 0.349                            | 0.350                            | 0.350                            | 0.350                            |

- Our result in Eq. (4.9) does not include the error due to next-to-leading terms in the $1/N_c$ expansion nor the error due to chiral corrections in the unfactorized contribution, but we consider it a rather robust prediction of $\hat{B}_K$ in the chiral limit and at the next-to-leading order in the $1/N_c$ expansion.

- Our calculation shows a crucial qualitative issue, which is the fact that the low-energy hadronic contribution below a mass scale $1 \, \text{GeV} \lesssim \mu_{\text{had.}} \lesssim 3 \, \text{GeV}$ brings down, rather dramatically, the value of $\hat{B}_K|_{\text{full}}$ in Eq. (1.2) evaluated at that scale. We can already conclude that any calculation which "ignores" the details of the low-energy hadronic contribution will give an overestimate of $\hat{B}_K$.

- The result in Eq. (4.9) is compatible with the current algebra prediction [26] which, to lowest order in chiral perturbation theory, relates the $B_K$ factor to the $K^+ \rightarrow \pi^+\pi^0$ decay rate. In fact, our calculation of the $B_K$ factor can be viewed as a successful prediction of the $K^+ \rightarrow \pi^+\pi^0$ decay rate!

- As discussed in ref. [27] the bosonization of the four-quark operator $Q_{\Delta S=2}$ and the bosonization of the operator $Q_2 - Q_1$ which generates $\Delta I = 1/2$ transitions are related to each other in the combined chiral limit and next-to-leading order $1/N_c$ expansion. Lowering the value of $B_K$ from the large-$N_c$ prediction in Eq. (1.9) to the result in Eq. (4.9) is correlated with an increase of the coupling constant in the lowest order effective chiral Lagrangian which generates $\Delta I = 1/2$ transitions, and provides a first step towards a quantitative understanding of the dynamical origin of the $\Delta I = 1/2$ rule.

- Finally, we wish to point out that the techniques applied here can be extended as well to include...
higher order terms in the chiral expansion. It remains to be seen if chiral corrections to the
unfactorized term in Eq. (2.3) could be so large, (of order 100%?) as to increase our result
in Eq. (4.9) to the values favoured by the lattice QCD determinations \(^{11}\) as well as by recent
phenomenological arguments \(^{29}\).

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\(^{11}\)See e.g. the latest review in ref. \(^{29}\) and references therein.
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