SEQUENTIALLY $S_r$ SIMPLICIAL COMPLEXES AND SEQUENTIALLY $S_2$ GRAPHS

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Abstract. We introduce sequentially $S_r$ modules over a commutative graded ring and sequentially $S_r$ simplicial complexes. This generalizes two properties for modules and simplicial complexes: being sequentially Cohen-Macaulay, and satisfying Serre’s condition $S_r$. In analogy with the sequentially Cohen-Macaulay property, we show that a simplicial complex is sequentially $S_r$ if and only if its pure $i$-skeleton is $S_r$ for all $i$. For $r = 2$, we provide a more relaxed characterization. As an algebraic criterion, we prove that a simplicial complex is sequentially $S_r$ if and only if the minimal free resolution of the ideal of its Alexander dual is componentwise linear in the first $r$ steps. We apply these results for a graph, i.e., for the simplicial complex of the independent sets of vertices of a graph. We characterize sequentially $S_r$ cycles showing that the only sequentially $S_2$ cycles are odd cycles and, for $r \geq 3$, no cycle is sequentially $S_r$ with the exception of cycles of length 3 and 5. We extend certain known results on sequentially Cohen-Macaulay graphs to the case of sequentially $S_r$ graphs. We prove that a bipartite graph is vertex decomposable if and only if it is sequentially $S_2$. We provide some more results on certain graphs which in particular implies that any graph with no chordless even cycle is sequentially $S_2$. Finally, we propose some questions.

1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over a field $k$. For finitely generated graded $R$-modules, Stanley has defined the sequentially Cohen-Macaulay property [16, Chapter III, Definition 2.9] and has studied the corresponding simplicial complexes. Here we consider sequentially $S_r$ graded modules, i.e., finitely generated graded $R$-modules which satisfy Serre’s $S_r$ condition sequentially. Then we study the corresponding simplicial complexes, sequentially $S_r$ simplicial complexes. Duval has shown that a simplicial complex is sequentially Cohen-Macaulay if and only if its pure $i$-skeleton is Cohen-Macaulay for all $i$ [5, Theorem 3.3]. We prove the analogue result for sequentially $S_r$ simplicial complexes (see Theorem 2.6). For $r = 2$, we show that a simplicial complex is sequentially $S_2$ if and only if its pure $i$-skeletons are connected for all $i \geq 1$ and the link of every singleton is sequentially $S_2$ (see Theorem 2.9). A major result of Eagon and Reiner states that a simplicial complex is Cohen-Macaulay if and only if the Stanley-Reisner ideal of its Alexander dual has a linear resolution [6, Theorem 3]. Later, Herzog and Hibi generalized this result by proving that a simplicial complex is sequentially Cohen-Macaulay if and only if the minimal free resolution of the Stanley-Reisner ideal of...
its Alexander dual is componentwise linear [10, Theorem 2.9]. The result of Eagon and Reiner has been generalized in another direction by Yanagawa (with N. Terai) showing that a simplicial complex is $S_r$ if and only if the minimal free resolution of its Alexander dual is linear in the first $r$ steps [22, Corollary 3.7]. We adopt the above two results to show that a simplicial complex is sequentially $S_r$ if and only if the minimal free resolution of the Stanley-Reisner ideal of its Alexander dual is componentwise linear in the first $r$ steps (see Corollary 3.3).

As the first application of our results, we characterize sequentially $S_r$ cycles. It is known that the only cycles which are sequentially Cohen-Macaulay are $C_3$ and $C_5$ [8, Proposition 4.1] and the only cycles which are $S_2$, are $C_3$, $C_5$ and $C_7$ [9, Proposition 1.6]. We extend these results by showing that $C_n$ is sequentially $S_2$ if and only if $n$ is odd and the only sequentially $S_3$ cycles are $C_3$ and $C_5$, i.e., the Cohen-Macaulay cycles (see Theorem 4.1 and Proposition 4.2).

Van Tuyl and Villarreal [20] have studied sequentially Cohen-Macaulay graphs. We extend some of their results and generalize a result of Francisco and Hà [7, Theorem 4.1] on graphs with whiskers (see Corollary 4.6).

Van Tuyl [19, Theorem 2.10] has recently proved that a bipartite graph is vertex decomposable if and only if it is sequentially Cohen-Macaulay. We prove that a bipartite graph is vertex decomposable if and only if it is sequentially $S_2$ (see Theorem 4.7 and Corollary 4.8). This result also generalizes the authors’ result which states that a bipartite graph is Cohen-Macaulay if and only if it is $S_2$ [9, Theorem 1.3].

Woodroofe [21, Theorem 1.1] has proved that a graph with no chordless cycles other that cycles of length 3 and 5 is sequentially Cohen-Macaulay. We extend this result for $S_2$ property (see Theorem 4.9). This in particular implies that any graph with no chordless even cycle is sequentially $S_2$ (see Corollary 4.10).

At the end of this paper we propose two questions on sequentially $S_r$ property of join of two simplicial complexes and topological invariance of $S_r$, respectively.

The motivation behind our work is the general philosophy that Serre’s $S_r$ condition plays an important role, not only in algebraic geometry and commutative algebra, but also in algebraic combinatorics (e.g. see [15], [22], [17]).

2. Criteria for sequentially $S_r$ simplicial complexes

In this section we give some basic definitions and criteria for sequentially $S_r$ property on simplicial complexes. We prove that a simplicial complex is sequentially $S_r$ if and only if its pure skeletons are all $S_r$, a generalization of Duval’s result on sequentially Cohen-Macaulay simplicial complexes [5, Theorem 3.3]. We show that a simplicial complex is sequentially $S_2$ if and only if its pure $i$-skeletons are connected for all $i \geq 1$ and the link of every singleton is sequentially $S_2$.

Recall that a finitely generated graded module $M$ over a Noetherian graded $k$-algebra $R$ is said to satisfy the Serre’s condition $S_r$ if

$$\text{depth } M_p \geq \min(r, \dim M_p),$$

for all $p \in \text{Spec } (R)$.

First we bring the definition of sequentially $S_r$ modules.
**Definition 2.1.** Let $M$ be a finitely-generated $\mathbb{Z}$-graded module over a standard graded $k$-algebra $R$ where $k$ is a field. For a positive integer $r$ we say that $M$ is sequentially $S_r$ if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$$

of $M$ by graded submodules $M_i$ satisfying the two conditions

(a) Each quotient $M_i/M_{i-1}$ satisfies the $S_r$ condition of Serre.

(b) $\dim(M_1/M_0) < \dim(M_2/M_1) < \ldots < \dim(M_t/M_{t-1})$.

We say that a simplicial complex $\Delta$ on $[n] = \{1, \ldots, n\}$ is sequentially $S_r$ (over a field $k$) if its face ring $k[\Delta] = k[x_1, \ldots, x_n]/I_{\Delta}$, as a module over $R = k[x_1, \ldots, x_n]$, is sequentially $S_r$.

This is a natural generalization of a $S_r$ simplicial complex, i.e., when $k[\Delta]$ satisfies the $S_r$ condition of Serre.

Since $k[\Delta]$ is a reduced ring, it always satisfies $S_1$ condition. Thus, throughout this paper we will always deal with $S_r$ for $r \geq 2$.

Using a result of Schenzel [15, Lemma 3.2.1] and Hochster’s formula on local cohomology modules, N. Terai has formulated the analogue of Reisner’s criterion for Cohen-Macaulayness of a relative simplicial complex in the case of $S_r$ simplicial complexes [17, page 4, following Theorem 1.7]. According to this formulation, a simplicial complex $\Delta$ of dimension $d - 1$ is $S_r$ if and only if for all $-1 \leq i \leq r - 2$ and all $F \in \Delta$ (including $F = \emptyset$) with $\#F \leq d - i - 2$ we have $H_i(lk_{\Delta}F; k) = 0$, where $lk_{\Delta}F = \{ G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset \}$. For $i = -1$ the vanishing condition is equivalent to purity of $\Delta$ and for $i = 0$ it is equivalent to the connectedness of $lk_{\Delta}F$ [17, page 4 and 5].

By this characterization of $S_r$ simplicial complexes it follows that the $S_r$ property carries over links.

**Lemma 2.2.** Let $\Delta$ be a $d - 1$ dimensional simplicial complex which satisfies the $S_r$ condition. Then for each $F \in \Delta$ the simplicial complex $lk_{\Delta}F$ also satisfies $S_r$.

**Proof.** Let $\#F = j$, then $\dim lk_{\Delta}F \leq d - j - 1$. By the above characterization of $S_r$ simplicial complexes, it suffices to show that for all $i \leq r - 2$ and every $G \in lk_{\Delta}F$, with $\#G \leq d - j - i - 2$, the reduced homology module $H_i(lk_{\Delta}F; G; k)$ is zero. This follows from the facts that $lk_{\Delta}F(G) = lk_{\Delta}(F \cup G)$, $\#(F \cup G) \leq d - i - 2$ and $\Delta$ is $S_r$.

Recall that a relative simplicial complex is a pair of simplicial complexes $(\Delta, \Gamma)$ where $\Gamma$ is a subcomplex of $\Delta$. For a relative simplicial complex $(\Delta, \Gamma)$ define $I_{\Delta, \Gamma}$ to be the ideal in $k[\Delta]$ generated by the monomials $x_{i_1}x_{i_2}\ldots x_{i_r}$ with $F = \{i_1, \ldots, i_r\} \in \Delta \setminus \Gamma$. A relative simplicial complex is said to be $S_r$ if $I_{\Delta, \Gamma}$ is $S_r$ as a module over $R = k[x_1, \ldots, x_n]$. Let $\Delta^*_i$ be the subcomplex of $\Delta$ generated by its $i$-dimensional facets. Following [3, Appendix II], it turns out that $\Delta$ is sequentially $S_r$ if and only if the relative simplicial complex $(\Delta^*_i, \Delta^*_i \cap (\Delta^*_{i+1} \cup \ldots \cup \Delta^*_{\dim(\Delta)})$ is $S_r$ for all $i$.

For a relative simplicial complex $(\Delta, \Gamma)$, let $\tilde{H}_i(\Delta, \Gamma; k)$ denote the $i$th reduced relative homology group of the pair $(\Delta, \Gamma)$ over $k$ (see [16] Chapter III, §7). Reisner’s criterion for Cohen-Macaulayness of a relative simplicial complex is similar to the one for a simplicial complex [16, Chapter III, Theorem 7.2]. Likewise, in an exact analogy, Terai’s formulation for a $S_r$ simplicial complex carries over for the relative case. In other words, a relative simplicial complex $(\Delta, \Gamma)$ is $S_r$ if and only
if $\bar{H}_i(\{k\Delta; F; k\}) = 0$ for all $−1 ≤ i ≤ r − 2$ and all $F ∈ ∆$ (including $F = \emptyset$) with $#F ≤ d − i − 2$, where $d − 1 = \dim(∆)$.

For a relative simplicial complex $(∆, ∆)$, as an $R$-module, $I_\Delta$ only depends on the difference $∆ \setminus ∆$ (see the remarks following Lemma 3.1). In particular, if $∆(i)$ is the $i$-skeleton of $∆$ and $∆(i)$ is the pure $i$-skeleton of $∆$, then

$$\Delta^i \setminus (\Delta^i \cap \Delta^i \cup \ldots \cup \Delta^i_{\dim(∆)}) = \Delta^{[i]} \setminus (\Delta^{[i+1]}(i)).$$

Duval makes the above observation and proves that the relative simplicial complex $\Delta^{[i]} \setminus (\Delta^{[i+1]}(i))$ for $∆$ is Cohen-Macaulay for all $i$ if and only if every pure $i$-skeleton $\Delta^{[i]}$ is Cohen-Macaulay (see the remarks following Lemma 3.1). We follow his proof step by step to show that the same result is true if we replace the Cohen-Macaulay property with $S_r$. To do this we need some preliminary results.

It is known that if $∆$ is a Cohen-Macaulay simplicial complex, then so is $∆(i)$, the $i$-skeleton of $∆$. We generalize this result for the property $S_r$.

**Proposition 2.3.** If $∆$ satisfies Serre’s condition $S_r$, then $∆(i)$ satisfies this condition $(2 ≤ r ≤ i + 1)$.

**Proof.** We check Terai’s criterion for $S_r$ simplicial complexes. To prove the assertion for $∆(i)$, we use induction on $r ≥ 2$. Assume that $∆$ satisfies Serre’s condition $S_2$. Then $∆$ is pure, hence $∆(i)$ is pure. Furthermore, for $F ∈ ∆$ with $#F ≤ d − 2$ $lk_{∆} F$ is connected. Let $F ∈ ∆(i)$ and $#F ≤ i − 1$. It is enough to show that $lk_{∆(i)} F$ is connected, or equivalently, path connected. Let $\{u\}, \{v\} ∈ lk_{∆(i)} F$. Then $\{u\}, \{v\} ∈ lk_{∆} F$ which is connected. Hence, there exists a sequence of vertices of $∆$, $u_0 = u, u_1, \ldots, u_t = v$, such that $\{u_i, u_{i+1}\} ∈ lk_{∆} F$, $j = 0, \ldots, t − 1$. Thus, $\{u_j, u_{j+1}\} \cap F = \emptyset$ and $\{u_j, u_{j+1}\} \cup F ∈ ∆$. Since $#(\{u_j, u_{j+1}\} \cup F) ≤ i + 1$, $\{u_j, u_{j+1}\} \cup F ∈ ∆(i)$ and hence $\{u_j, u_{j+1}\} ∈ lk_{∆(i)} F$.

Now assume that $∆$ satisfies Serre’s condition $S_r$ for $r > 2$. Then $∆$ satisfies Serre’s condition $S_j$ for $j = 1, \ldots, r$. Thus by induction hypothesis $∆(i)$ satisfies Serre’s condition $S_j$ for $j = 1, \ldots, r − 1$. Therefore, for $q ≤ r − 3$ and $F ∈ ∆(i)$ with $#F ≤ i − q − 1$, $H_q(lk_{∆(i)} F; k) = 0$. Thus it remains to show that for $#F ≤ i − r + 1$, $H_{r−2}(lk_{∆(i)} F; k) = 0$. To prove this, since $lk_{∆(i)} F ⊂ lk_{∆} F$, it is enough to show that for $q ≤ r − 1$, any $q$-dimensional face $H$ of $lk_{∆} F$ lies in $lk_{∆(i)} F$. But $#(H ∪ F) ≤ i + 1$, and hence, $H ∪ F ∈ ∆(i)$, and consequently, $H ∈ lk_{∆(i)} F$.

Now we adopt Duval’s results for the case of sequentially $S_r$ simplicial complexes.

**Lemma 2.4.** (see [5] Lemma 3.1). Let $F$ be a face of a $(d − 1)$-dimensional simplicial complex $∆$ and let $Γ$ be either the empty simplicial complex or a $S_r$ simplicial complex of the same dimension as $∆$. Then

$$\bar{H}_i(lk_{∆} F; k) = \bar{H}_i(lk_{∆} F; lk_{Γ} F; k)$$

for all $i ≤ r − 2$ and all $F ∈ ∆$ with $#F ≤ d − i − 2$.

**Proof.** The proof is the same as the proof of the similar lemma of Duval [5] Lemma 3.1. If $lk_{Γ} F$ is an empty set, then the equality is obvious. Otherwise one only needs to change the range of the index $i$ with the one given above, impose the condition on $#F$ and replace Cohen-Macaulay property with $S_r$. \qed
Corollary 2.5. (see [5 Corollary 3.2]). Let $\Delta$ be a simplicial complex, and let $\Gamma$ be either the empty simplicial complex or a $S_r$ simplicial complex of the same dimension as $\Delta$. Then $\Delta$ is $S_r$ if and only if $(\Delta, \Gamma)$ is relative $S_r$.

Proof. Similar to the corresponding corollary by Duval [5 Corollary 3.2], it follows from Lemma 2.4.

Theorem 2.6. (see [5 Theorem 3.3]). Let $\Delta$ be a $(d-1)$-dimensional simplicial complex. Then $\Delta$ is sequentially $S_r$ if and only if its pure $i$-skeleton $\Delta[i]$ is $S_r$ for all $-1 \leq i \leq d-1$.

Proof. The proof is the same as the one given by Duval [5 Theorem 3.3]. The only item needed is that each $i$-skeleton of a $S_r$ simplicial complex is again $S_r$ for all $i$. But this is proved in Proposition 2.3.

The following is an immediate bi-product of this theorem.

Corollary 2.7. A simplicial complex $\Delta$ is $S_r$ if and only if it is sequentially $S_r$ and pure.

Proof. Since $S_r$ condition implies purity, one implication is clear. Assume that $\Delta$ is pure and $\dim \Delta = d - 1$. Then $\Delta[d-1] = \Delta$ and the assertion follows by Theorem 2.6.

Remark 2.8. Some authors define a simplicial complex to be sequentially Cohen-Macaulay if its pure $i$-skeleton is Cohen-Macaulay for all $i$. Likewise, we might take a similar statement as the definition of sequentially $S_r$ simplicial complexes. But we preferred to begin with the algebraic definition given in this section and prove that both definitions are equivalent.

We end this section with the following characterization of sequentially $S_2$ simplicial complexes which will be used in the last section.

Theorem 2.9. Let $\Delta$ be a simplicial complex with vertex set $V$. Then $\Delta$ is sequentially $S_2$ if and only if the following conditions hold:

(i) $\Delta[i]$ is connected for all $i \geq 1$

(ii) $\text{lk}_\Delta(x)$ is sequentially $S_2$ for all $x \in V$

Proof. Let $\Delta$ be a sequentially $S_2$. Then $\Delta[i]$ is $S_2$ for all $-1 \leq i \leq d - 1$. Thus $\Delta[i]$ is connected for all $i \geq 1$. On the other hand $\text{lk}_{\Delta[i]}(x)$ is $S_2$ for all $-1 \leq i \leq d - 1$ and so $\text{lk}_{\Delta[i+1]}(x) = (\text{lk}_\Delta(x))^{[i]}$ is $S_2$ for all $-1 \leq i \leq d - 2$. Therefore $\text{lk}_\Delta(x)$ is sequentially $S_2$.

Now let $\Delta$ satisfies the conditions (i) and (ii). Since $\text{lk}_\Delta(x)$ is sequentially $S_2$ for all $x \in V$ we have that $(\text{lk}_\Delta(x))^{[i]} = \text{lk}_{\Delta[i+1]}(x)$ is $S_2$ for all $-1 \leq i \leq d - 2$ and so $\text{lk}_{\Delta[i]}(x)$ is $S_2$ for all $-1 \leq i \leq d - 1$. Now the connectedness of $\Delta[i]$ for $i \geq 1$ implies that $\Delta[i]$ is $S_2$ for all $-1 \leq i \leq d - 1$. Indeed, for $F \neq \emptyset$ and $x \in F$, $\text{lk}_{\Delta[i]}F = \text{lk}_{\text{lk}_{\Delta[i]}(x)}F$ for $G = F \setminus \{x\}$. Therefore $\Delta$ is sequentially $S_2$.

3. Alexander dual of sequentially $S_r$ simplicial complexes

In this section we show that a simplicial complex is sequentially $S_r$ if and only if the minimal free resolution of the Stanley-Reisner ideal of its Alexander dual is componentwise linear in the first $r$ steps. This result resembles a result of Herzog
and Hibi [10] Proposition 1.5] on sequentially Cohen-Macaulay simplicial complexes. And, our proof would be a modification of the sequentially Cohen-Macaulay case together with an application of Theorem 2.6

We first adopt the following definitions from [22] Definition 3.6] and [10] §1.

Consider \( R = k[x_1, \ldots, x_n] \) with \( \deg(x_i) = 1 \) for all \( i \). If \( I \) is a homogenous ideal of \( R \) and \( r \geq 1 \), then \( I \) is said to be linear in the first \( r \) steps, if for some integer \( d \), \( \beta_{i,r+t}(I) = 0 \) for all \( 0 \leq i < r \) and \( t \neq d \). We write \( I_{<j>} \) for the ideal generated by all homogenous polynomials of degree \( j \) belonging to \( I \). We say that a homogenous ideal \( I \subset R \) is componentwise linear if \( I_{<j>} \) has a linear resolution for all \( j \). The ideal \( I \) is said to be \textit{componentwise linear in the first \( r \) steps} if for all \( j \geq 0 \), \( I_{<j>} \) is linear in the first \( r \) steps. A simplicial complex \( \Delta \) on \( [n] \) is said to be linear in the first \( r \) steps, componentwise linear and componentwise linear in the first \( r \) steps, if \( I_{\Delta} \) satisfies either of these properties, respectively.

Now let \( I \subset R \) be an ideal generated by squarefree monomials. Then for each degree \( j \) we write \( I_{[j]} \) for the ideal generated by the squarefree monomials of degree \( j \) belonging to \( I \). We say that \( I \) is squarefree componentwise linear if \( I_{[j]} \) has linear resolution for all \( j \). The ideal \( I \) is said to be \textit{squarefree componentwise linear in the first \( r \) steps} if \( I_{[j]} \) has a resolution which is linear in the first \( r \) steps for all \( j \).

Below we adopt a result of Herzog and Hibi [10] Proposition 1.5] for the case of componentwise linearity in the first \( r \) steps.

**Proposition 3.1.** Let \( I \) be a squarefree monomial ideal in \( R \). Then \( I \) is componentwise linear in the first \( r \) steps if and only if \( I \) is squarefree componentwise linear in the first \( r \) steps.

**Proof.** The proof is the same as [10] Proposition 1.5] with just a restriction on the index \( i \) used in the proof of Herzog and Hibi. Here we need to assume that \( i < r \). Also one needs to observe that when \( I \) has a linear resolution in the first \( r \) steps, the ideal \( mI \) has a linear resolution in the first \( r \) steps too. Here \( m = (x_1, \ldots, x_n) \) is the irrelevant maximal ideal.

We may now generalize a result of Herzog and Hibi [10] Theorem 2.1]. As we already mentioned, Yanagawa and Terai proved that \( \Delta \) is \( S_r \) if and only if \( I_{\Delta} \) is linear in the first \( r \) steps.

**Theorem 3.2.** The Stanley-Reisner ideal of \( \Delta \) on \( [n] \) is componentwise linear in the first \( r \) steps if and only if \( \Delta^\vee \), the Alexander dual of \( \Delta \), is sequentially \( S_r \).

**Proof.** The proof is an adaptation of the proof of part (a) of [10] Theorem 2.1] with the following additional remarks: Let \( I = I_{\Delta} \). Then by Proposition 3.1, \( I \) is squarefree componentwise linear in the first \( r \) steps if and only if \( I \) is componentwise linear in the first \( r \) steps. By [22] Corollary 3.7] for every \( j \), \( I_{[j]} \) is linear in the first \( r \) steps if and only if \( (\Delta^\vee)^{[n-j-1]} \) is \( S_r \). Therefore, \( I \) is componentwise linear in the first \( r \) steps if and only if \( (\Delta^\vee)^{[q]} \) is \( S_r \) for every \( q \). But by Theorem 2.6, this is equivalent to the sequentially \( S_r \) property for \( \Delta^\vee \).

Van Tuyl and Villarreal [20] Theorem 3.8 (a]) state the dual version of [10] Theorem 2.1] for sequentially Cohen-Macaulay simplicial complexes. Dualizing the statement of the above theorem we get a similar generalization for sequentially \( S_r \) simplicial complexes.
Corollary 3.3. A simplicial complex $\Delta$ is sequentially $S_r$ if and only if the Stanley-Reisner ideal of the Alexander dual of $\Delta$ is componentwise linear in the first $r$ steps.

4. SOME CHARACTERIZATIONS OF SEQUENTIALLY $S_r$ CYCLES AND SEQUENTIALLY $S_2$ BIPARTITE GRAPHS

In this section, we provide some applications of the results of the previous sections. We first classify sequentially $S_r$ cycles and show that a cycle $C_n$ is sequentially $S_2$ if and only if $n$ is odd and no cycles are sequentially $S_3$ except those which are Cohen-Macaulay, i.e., $C_3$ and $C_5$. This generalizes a result of Francisco and Villarreal [8, Theorem 4.1]. Then we generalize some results of Van Tuyl and Villarreal [20] for sequentially $S_r$ graphs. We also extend a result of Francisco and Hà [7, Theorem 4.1] on graphs with whiskers. Then we generalize a result of Van Tuyl [19, Theorem 2.10] who proves that a bipartite graph is vertex decomposable if and only if it is sequentially Cohen-Macaulay. We prove that a bipartite graph is vertex decomposable if and only if it is sequentially $S_2$. This result also generalizes the authors’ result which states that a bipartite graph is Cohen-Macaulay if and only if it is $S_2$ [9, Theorem 1.3]. Woodroofe [21, Theorem 1] proved that a graph with no chordless cycles other that cycles of length 3 and 5 is sequentially Cohen-Macaulay. We provide some results which extend this statement for $S_2$ property. In particular they imply that any graph with no chordless even cycle is sequentially $S_2$.

At the end of this section we pose two questions on sequentially $S_r$ property of join of two simplicial complexes and topological invariance of $S_r$, respectively.

Recall that to a simple graph $G$ one associates a simplicial complex $\Delta_G$ on $V(G)$, the set of vertices of $G$, whose faces correspond to the independent sets of vertices of $G$. A graph $G$ is said to be $S_r$ if $\Delta_G$ is a $S_r$ simplicial complex. Likewise, $G$ is Cohen-Macaulay, sequentially Cohen-Macaulay and shellable if $\Delta_G$ satisfies either of these properties, respectively. We adopt the definition of shellability in the nonpure sense of Björner-Wachs [1].

We also recall the definition of a vertex decomposable simplicial complex. A simplicial complex $\Delta$ is recursively defined to be vertex decomposable if it is either a simplex or else has some vertex $v$ so that

1. Both $\Delta \setminus \{v\}$ and $\text{lk} \Delta(v)$ are vertex decomposable, and
2. No face of $\text{lk} \Delta(v)$ is a facet of $\Delta \setminus \{v\}$.

The notion of vertex decomposability was introduced in the pure case by Provan and Billera [13] and was extended to nonpure complexes by Björner and Wachs [1].

Sequentially Cohen-Macaulay cycles have been characterized by Francisco and Van Tuyl [8, Proposition 4.1]. They are just $C_3$ and $C_5$. Woodroofe has given a more geometric proof for this result [21, Theorem 3.1]. In [9, Proposition 1.6] it is shown that the only $S_2$ cycles are $C_3$, $C_5$ and $C_7$. We now generalize this result and prove that the odd cycles are sequentially $S_2$ and they are the only sequentially $S_2$ cycles.

Theorem 4.1. The cycle $C_n$ is sequentially $S_2$ if and only if $n$ is odd.

Proof. Let $n = 2k$. Then $\Delta = \Delta_{C_n}$ has only two facets of dimension $2k - 1 (= \dim \Delta)$, namely, $\{1, 3, \ldots, 2k - 1\}$ and $\{2, 4, \ldots, 2k\}$. Thus $\Delta^{[k-1]}$ is the union of
of simplicial complexes we have depth(k) in the first 3 steps. Hence if J is generated in degree 4. Thus the resolution of sequential S

Let n be an odd integer. To show that C

Thus by Corollary 3.3, it is enough to show that ∆[1] is connected. Therefore (∆[i])[1] = ∆[1]. On the other hand ∆[i] is connected if and only if (∆[i])[1] is connected. Therefore it is enough to show that ∆[1] is connected. Let x and y be two elements in V. If x is not adjacent to y, then \{x, y\} ∈ ∆. If x is adjacent to y, then there exists z ∈ V such that x is not adjacent to z and y is not adjacent to z. Therefore \{x, z\} and \{y, z\} belong to ∆ and hence ∆[1] is connected.

The following result completes the characterization of cycles with respect to the property S

Proposition 4.2. The cycle C

Proof. The first proof: Let ∆ = ∆C

For n = 7, again we follow the proof of [8] Theorem 4.1. In this case, the ideal J is generated in degree 4. Thus the resolution of J is the same as the resolution of J[4]. Moreover, the resolution of J is linear in the first 2 steps but not in step 3. Thus by Corollary 3.3 C7 is not sequentially S3.

The second proof: Let n = 2t + 1 ≥ 7. By [21] Lemma 3.2, ∆[k−1] is homotopic to the circle S1. Thus H1(∆[k−1]; k) = k. By Hochster’s formula on Betti numbers of simplicial complexes we have depth(k[∆[k−1]]) ≤ 2. Thus k[∆[k−1]] does not satisfy S3. Hence Cn is not sequentially S3.

We now outline the analogue statements of [20] Section 3 to conclude the sequentially S

Lemma 4.3. (see [20, Lemma 3.9]) Let $G$ be a bipartite graph with bipartition \{\(x_1, \ldots, x_m\)\} and \{\(y_1, \ldots, y_n\)\}. If $G$ is sequentially $S_r$, then there is a vertex \(v \in V(G)\) with $\deg(v) = 1$.

Proof. The only modification needed in the proof of [20, Lemma 3.9] is to justify that the kernel of the linear map $f$ used in that proof, is generated by linear syzygies. But under the hypothesis of the lemma, this is proved in Proposition 3.1. □

For a graph $G$ and a vertex $x \in V_G$, the set of neighbors of $x$ will be denoted by $N_G(x)$. For $F \in \Delta_G$ we set:

\[ N_G(F) = \bigcup_{x \in F} N_G(x) \]

We also use the following notation:

\[ N_G[x] = \{x\} \cup N_G(x) \]
\[ N_G[F] = F \cup N_G(F) \]

The following lemma gives a recursive procedure to check if a graph fails to be sequentially $S_r$. It is a sequentially $S_r$ version of [20, Theorem 3.3].

Lemma 4.4. Let $G$ be a graph and $x$ a vertex of $G$. Let $G' = G \setminus N_G[x]$. If $G$ is sequentially $S_r$, then $G'$ is also sequentially $S_r$.

Proof. The proof is identical with that of [20, Lemma 3.3]. We only need to use Theorem 2.6 and Lemma 2.2 instead of their sequentially Cohen-Macaulay and Cohen-Macaulay counterparts, respectively. □

Lemma 4.4 could be extended further.

Corollary 4.5. Let $G$ be a graph which is sequentially $S_r$. Let $F$ be an independent set in $G$. Then the graph $G' = G \setminus N_G[F]$ is sequentially $S_r$.

Proof. This follows by repeated applications of Lemma 4.4. □

The following generalizes a result of Francisco and Hà [7, Theorem 4.1] which is also proved by Van Tuyl and Villarreal by a different method (see [20, Corollary 3.5]).

Recall that for a subset $S = \{y_1, \ldots, y_m\}$ of a graph $G$, the graph $G \cup W(S)$ is obtained from $G$ by adding an edge (whisker) \(\{x_i, y_i\}\) to $G$ for all $i = 1, \ldots, m$, where $x_1, \ldots, x_m$ are new vertices.

Corollary 4.6. Let $S \subset V(G)$ and suppose that the graph $G \cup W(S)$ is sequentially $S_r$, then $G \setminus S$ is sequentially $S_r$.

Proof. This also follows by repeated applications of Lemma 4.4. □

Van Tuyl [19, Theorem 2.10] has proved that a bipartite graph is vertex decomposable if and only if it is sequentially Cohen-Macaulay. We now generalize this result. Observe that our result also generalizes the authors’ result which states that a bipartite graph is Cohen-Macaulay if and only if it is $S_2$ [9, Theorem 1.3].

First we need a more general result.

Theorem 4.7. Let $G = (V, E)$ be a graph. Suppose $H = G \setminus N_G[F]$ satisfies one of the conditions (i), (ii), and (iii) for any $F \in \Delta_G$ which is not a facet:

(i) $H$ has no chordless even cycle.
(ii) $H$ has a simplicial vertex, i.e., for some $z \in V(H)$, $N_H[z]$ is a complete graph.

(iii) For some $t \geq 2$, $H$ has a chordless $(2t + 1)$-cycle which has $t$ independent vertices of degree 2 in $H$.

Then $G$ is sequentially $S_2$.

Proof. We prove the theorem by induction on $n$, the number of vertices of $G$. The assertion holds for $n \leq 3$. Now we assume that $n \geq 4$. Set $\Delta = \Delta_G$ and let $x \in V(G)$. Observe that $G' = G \setminus N_G[x]$ satisfies the statement of the theorem. Thus it is sequentially $S_2$ by the induction hypothesis. Hence by [20, Lemma 2.5], $\text{lk}_G(x) = \Delta_G$ is sequentially $S_2$. Thus by Theorem [23] it is enough to show that $\Delta[i]$ is connected for $1 \leq i \leq \dim \Delta$. We show that for any $X, Y \in \Delta$ with $\dim X = \dim Y = i$, there is a chain $X = X_0, X_1, \ldots, X_s = Y$ of $i$-faces of $\Delta$ such that $X_{j-1} \cap X_j \neq \emptyset$ for $j = 1, 2, \ldots, s$. We may assume $X \cap Y = \emptyset$. For simplicity we set $X = \{x_1, x_2, \ldots, x_{i+1}\}$ and $Y = \{y_1, y_2, \ldots, y_{i+1}\}$.

We assume that the condition (i) is satisfied for $G$. Set $B = G_{X \cup Y}$, the restriction of $G$ to $X \cup Y$. Then $B$ is a bipartite graph on the partition $X \cup Y$. Since $B$ is bipartite, $B$ has no odd cycle. Since $B$ has no (chordless) even cycle by the condition (i), $B$ is a forest. Then there exists a vertex with degree 0 or 1 in $B$. We may assume that $x_1$ is such a vertex and that $x_1$ is adjacent at most to $y_1$. Set $X_1 = \{x_1, y_2, \ldots, y_{i+1}\}$. Then $X, X_1, Y$ is a desired chain.

We assume that the condition (ii) is satisfied for $G$. Then using the hypotheses for $F = \emptyset$, there is a simplicial vertex $z$ in $G$. Assume $z \not\in X \cup Y$. Since $z$ is simplicial, $z$ is adjacent to at most one vertex in $X$. We may assume that $z$ is not adjacent to $x_2, \ldots, x_{i+1}$. Similarly, we may assume that $z$ is not adjacent to $y_2, \ldots, y_{i+1}$. Set $X_1 = \{z, x_2, \ldots, x_{i+1}\}$ and $X_2 = \{z, y_2, \ldots, y_{i+1}\}$. Then $X, X_1, X_2, Y$ is a desired chain. Assume $z \in X \cup Y$. We may assume $z = y_1$. Then $X, X_1, Y$ is a desired chain.

Next we assume that the condition (iii) is satisfied for $G$. Then for some $t \geq 2$ there exists a chordless $(2t + 1)$-cycle $C$ which has $t$ vertices of degree 2 in $G$ which are independent in $G$. Let $\{z_1, z_2, \ldots, z_t\} \subset V(C)$ be an independent set of vertices of $G$ such that $\deg_G z_j = 2$ for $j = 1, 2, \ldots, t$.

Case I. $X \cup Y \subset V(C)$. As in the case that the condition (i) is satisfied, $B = G_{X \cup Y}$ is a bipartite graph on the partition $X \cup Y$. Since $C$ has no chord, $B$ is a disjoint union of paths. Then we can construct a desired chain as in the above case.

Case II. $X \subset V(C)$, and $Y \cap (V(G) \setminus V(C)) \neq \emptyset$. We may assume that $y_1 \in V(G) \setminus V(C)$. Note that $i + 1 \leq t$. Set $Y_1 = \{y_1, z_2, \ldots, z_{i+1}\}$ and $Z = \{z_1, z_2, \ldots, z_{i+1}\}$. Then $Y, Y_1, Z$ is a chain with $Y \cap Y_1 \neq \emptyset$, $Y_1 \cap Z \neq \emptyset$. Between $Z$ and $X$, we have a desired chain as in Case I. Hence we have a desired chain between $X$ and $Y$.

Case III. $X \cap (V(G) \setminus V(C)) \neq \emptyset$ and $Y \subset V(C)$. As in Case II.

Case IV. $X \cap (V(G) \setminus V(C)) \neq \emptyset$ and $Y \cap (V(G) \setminus V(C)) \neq \emptyset$. As in Case II, we can construct desired chains between $X$ and $Z$ and between $Y$ and $Z$. Thus we have a desired chain between $X$ and $Y$ via $Z$.

\[ \square \]

**Corollary 4.8.** Let $G$ be a bipartite graph. The following conditions are equivalent:

(i) $G$ is vertex decomposable.

(ii) $G$ is shellable.

(iii) $G$ is sequentially Cohen-Macaulay.
(iv) For any $F \in \Delta_G$ (including $\emptyset$) which is not a facet, there is a vertex $v \in G \setminus N_G[F]$ such that $\deg_{G \setminus N_G[F]}(v) \leq 1$.

(v) $G$ is sequentially $S_2$.

Proof. (i) $\Rightarrow$ (ii): Follows from [2] Theorem 11.3.

(ii) $\Rightarrow$ (iii): Follows from [10] Chap. III, §2.

(iii) $\Rightarrow$ (i): Follows from [19] Theorem 2.10.

(iv) $\Rightarrow$ (iv): For $F \in \Delta_G$ which is not a facet, set $H := G \setminus N_G[F]$. If $H$ has no edge, every vertex $v \in V(H)$ satisfies $\deg_H(v) = 0$. Therefore, we may assume that the bipartite graph $H$ has an edge. Applying [20] Theorem 3.3 repeatedly, we know $H$ is sequentially Cohen-Macaulay. Hence by [20] Lemma 3.9 there is a vertex $v \in V(H)$ such that $\deg_H(v) = 1$.

(iii) $\Rightarrow$ (v): This is trivial.

(iv) $\Rightarrow$ (ii): Suppose for any $F \in \Delta_G$ which is not a facet, there is a vertex $v \in H := G \setminus N_G[F]$ such that $\deg_H(v) \leq 1$. Hence $H[v]$ is the 1-complete graph or the 2-complete graph. This means $v$ is a simplicial vertex in $H$. By Theorem 4.7 (ii), $G$ is sequentially $S_2$.

(v) $\Rightarrow$ (ii): The proof is by induction on the number of vertices of $G$. Now let $G$ be a sequentially $S_2$ graph. By Lemma 1.3 there exists a degree one vertex $x_1 \in V(G)$. Assume that $N_G(x_1) = \{y_1\}$. Let $G_1 = G \setminus N_G[x_1]$ and $G_2 = G \setminus N_G[y_1]$. By Lemma 1.3 both of these graphs are sequentially $S_2$, hence by the induction hypothesis they are both shellable. Therefore, by [20] Theorem 2.9 $G$ is shellable.

In [21] Theorem 1.1 it is shown that a graph $G$ with no chordless cycles of length other than 3 or 5 is sequentially Cohen-Macaulay. In the following we extend this result on a larger class graphs for the sequentially $S_2$ property.

Theorem 4.9. Let $G$ be a graph. Suppose that a vertex in each chordless even cycle in $G$ has a whisker. Then $G$ is sequentially $S_2$.

Proof. We prove the theorem by induction on $n$, the number of vertices of $G$. The assertion holds for $n \leq 3$. Now we assume $n \geq 4$. Set $\Delta = \Delta_G$ and let $x \in V$. Since $G \setminus N_G[x]$ satisfies the condition of the theorem, it is sequentially $S_2$ by the induction hypothesis. Hence by Theorem 2.9 it is enough to show that $\Delta^i$ is connected for $1 \leq i \leq \dim \Delta$. We show that for any $X, Y \in \Delta$ with $\dim X = \dim Y = i$, there is a chain $X = X_0, X_1, \ldots, X_s = Y$ of $i$-faces of $\Delta$ such that $X_{j-1} \cap X_j \neq \emptyset$ for $j = 1, 2, \ldots, s$. We may assume $X \cap Y = \emptyset$. For simplicity we set $X = \{x_1, x_2, \ldots, x_{i+1}\}$ and $Y = \{y_1, y_2, \ldots, y_{i+1}\}$. Let $x_1$ have a whisker, that is, there exists $z \in V(G)$ such that $\deg z = 1$ and $\{x_1, z\} \in E$. Assume $z \notin Y$. Set $X_1 = \{z, x_2, \ldots, x_{i+1}\}$ and $X_2 = \{z, y_2, \ldots, y_{i+1}\}$. Then $X, X_1, X_2, Y$ is a desired chain. Assume $z = y_1 \in Y$. Then $X, X_1, Y$ is a desired chain.

Hence we may assume that no vertex in $X \cup Y$ has a whisker in $G$. Set $B = G_{X \cup Y}$, the restriction of $G$ to $X \cup Y$. Then $B$ is a bipartite graph on the partition $X \cup Y$. Since $B$ is bipartite, $B$ has no odd cycle. Since any vertices in $X \cup Y$ do not have a whisker in $G$, $B$ has no (chordless) even cycle. Hence $B$ is a forest. Then there exists a vertex with degree 0 or 1 in $B$. We may assume that $x_1$ is such a vertex and that $x_1$ is connected at most to $y_1$. Set $X_1 = \{y_1, x_2, \ldots, x_{i+1}\}$. Then $X, X_1, Y$ is a desired chain. □
Corollary 4.10. If $G$ is a graph with no chordless even cycle, then $G$ is sequentially $S_2$.

Remark 4.11. Corollary 4.10 gives an alternative proof for the fact that any odd cycle is sequentially $S_2$, the significant part of Theorem 4.7.

We end this section by proposing two questions.

Let $\Delta$ and $\Gamma$ be two simplicial complexes over disjoint vertex sets. In [14] it is shown that $\Delta \ast \Gamma$ is sequentially Cohen-Macaulay if and only if $\Delta$ and $\Gamma$ are both sequentially Cohen-Macaulay. By [18, Theorem 6], it follows that for $r \leq t$, if $\Delta$ is $S_r$ but not $S_{r+1}$ and $\Gamma$ is $S_t$ then $\Delta \ast \Gamma$ is $S_r$ but not $S_{r+1}$. One may study similar question for sequentially $S_2$ complexes.

Question 4.12. Let $\Delta$ and $\Gamma$ be two simplicial complexes. Is it true that $\Delta \ast \Gamma$ is sequentially $S_2$ if and only if $\Delta$ and $\Gamma$ are both sequentially $S_2$?

In particular, it is tempting to show that the join of the simplicial complexes of two disjoint odd cycles is $S_2$.

Munkres [12, Theorem 3.1] showed that Cohen-Macaulayness of a simplicial complex is a topological property. Stanley [16, Chap. III, Proposition 2.10] proved that sequentially Cohen-Macaulayness is also a topological property. Recently, Yanagawa [23, Theorem 4.5(d)] proved that Serre’s condition $S_r$ is a topological property as well. Therefore it is natural to pose the following question.

Question 4.13. Is sequentially $S_r$ a topological property on simplicial complexes?

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