ARITHMETIC MULTIVARIATE DESCARTES’ RULE

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Abstract. Let \( \mathcal{L} \) be any number field or \( p \)-adic field and consider \( F := (f_1, \ldots, f_k) \) where \( f_i \in \mathcal{L}[x_1^{\pm1}, \ldots, x_n^{\pm1}] \setminus \{0\} \) for all \( i \) and there are exactly \( m \) distinct exponent vectors appearing in \( f_1, \ldots, f_k \). We prove that \( F \) has no more than \( 1 + (\sigma mn(m - 1)^2 \log m)^n \) geometrically isolated roots in \( \mathcal{L}^n \), where \( \sigma \) is an explicit and effectively computable constant depending only on \( \mathcal{L} \). This gives a significantly sharper arithmetic analogue of Khovanski’s Theorem on Fewnomials and a higher-dimensional generalization of an earlier result of Hendrik W. Lenstra, Jr. for the case of a single univariate polynomial. We also present some further refinements of our new bounds and briefly discuss the complexity of finding isolated rational roots.

1. Introduction

A consequence of Descartes’ Rule (a classic result dating back to 1637) is that any real univariate polynomial with exactly \( m \geq 1 \) monomial terms has at most \( 2m - 1 \) real roots.

This has since been generalized by Askold G. Khovanski during 1979–1987 (see [Kho80] and

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to certain systems of multivariate sparse polynomials and even fewnomials\(^1\) Here we provide ultrametric and thereby arithmetic analogues for both results: we give explicit upper bounds, independent of the degrees of the underlying polynomials, for the number of isolated roots of sparse polynomial systems over any \(p\)-adic field and, as a consequence, over any number field. For convenience, let us henceforth respectively refer to these cases as the local case and the global case.

Suppose \(f_1, \ldots, f_k \in \mathcal{L}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \setminus \{0\}\) where \(\mathcal{L}\) is a field to be specified later, and \(m\) is the total number of distinct exponent vectors appearing in \(f_1, \ldots, f_k\) (assuming all polynomials are written as sums of monomials). We call \(F := (f_1, \ldots, f_k)\) an \(m\)-sparse \(k \times n\) polynomial system over \(\mathcal{L}\). Khovanski’s results take \(\mathcal{L} = \mathbb{R}\) and yield an explicit upper bound for the number of non-degenerate roots, in the non-negative orthant, of any \(m\)-sparse \(n \times n\) polynomial system [Kho80,Kho91]. With a little extra work (e.g., [Roj00a, cor. 3.2]) his results imply an upper bound of \(2^{O(n)} n^{O(m)} 2^{O(m^2)}\) on the number of isolated\(^2\) roots of \(F\) in \(\mathbb{R}^n\), and this is asymptotically the best general upper bound currently known.

In particular, since it is easy to show that the last bound can in fact be replaced by 1 when \(m \leq n\) (see, e.g., [LRW01, thm. 3, part (b)]), one should focus on better understanding the behavior of the maximum number of isolated real roots for \(n\) fixed and \(m \geq n + 1\). For example, is the dependence on \(m\) in fact polynomial for fixed \(n\)? This turns out to be an

\(^1\) Sparse polynomials are sometimes also known as lacunary polynomials and, over \(\mathbb{R}\), are a special case of fewnomials — a more general class of real analytic functions [Kho91].

\(^2\) We say a root of \(F\) is geometrically isolated iff it is a zero-dimensional component of the underlying scheme over the algebraic closure of \(\mathcal{L}\) defined by \(F\). For the case of \(\mathcal{L} = \mathbb{R}\) one can in fact use the slightly looser definition that a point is topologically isolated iff it is a connected component of the underlying real zero set. Unless otherwise mentioned, all our isolated roots will be geometrically isolated.
open question, but we can answer the arithmetic analogue (i.e., where \( L \) is any \( p \)-adic field or any number field) affirmatively and explicitly:

**Theorem 1** Let \( p \) be any (rational) prime and \( d, \delta \) positive integers. Suppose \( L \) is any degree \( d \) algebraic extension of \( \mathbb{Q}_p \) or \( \mathbb{Q} \), and let \( L^* := L \setminus \{0\} \). Also let \( F \) be any \( m \)-sparse \( k \times n \) polynomial system over \( L \) and define \( B(L,m,n) \) to be the maximum number of isolated roots in \((L^*)^n\) of such an \( F \) in the local case, counting multiplicities\(^3\) if (and only if) \( k = n \).

Then \( B(L,m,n) = 0 \) (if \( m \leq n \) or \( k < n \)) and

\[
B(L,m,n) \leq u(m,n) \left\{ c(m-1)n(p^d - 1) \left[ 1 + d \log_p \left( \frac{d(m-1)}{\log_p} \right) \right] \right\}^n \quad \text{if} \quad m \geq n + 1 \text{ and } k \geq n,
\]

where \( u(m,n) \) is \( m - 1, 4(m-1)^2, \text{ or } (m(m-1)/2)^n \) according as \( n = 1, n = 2, \text{ or } n \geq 3 \);

\( c := \frac{e}{e-1} \leq 1.582 \) and \( \log_p(\cdot) \) denotes the base \( p \) logarithm function. Furthermore, moving to the global case, let us say a root \( x \in \mathbb{C}^n \) of \( F \) is of degree \( \leq \delta \) over \( L \) iff every coordinate of \( x \) lies in an extension of degree \( \leq \delta \) of \( L \), and let us define \( A(L,\delta,m,n) \) to be the maximum number of isolated roots of \( F \) in \((\mathbb{C}^*)^n\) of degree \( \leq \delta \) over \( L \), counting multiplicities\(^3\) if (and only if) \( k = n \).

Then \( A(L,\delta,m,n) = 0 \) (if \( m \leq n \) or \( k < n \)) and

\[
A(L,\delta,m,n) \leq 2u(m,n) \left\{ c(m-1)2^{d\delta} \left[ 1 + 2d^2\delta^2 \log_2 \left( \frac{d^2\delta^2(m-1)}{\log 2} \right) \right] \right\}^n \quad \text{if} \quad m \geq n + 1 \text{ and } k \leq n.
\]

Our bounds can be sharpened even further, and this is detailed in corollaries \( \text{I} \) and \( \text{II} \) of sections \( \text{II} \) and \( \text{III} \) respectively.

**Remark 1** At the expense of underestimating\(^4\) some multiplicities, we can easily obtain a bound for the number of isolated roots of \( F \) in \( L^n \) (in the local case) or roots in \( \mathbb{C}^n \) of degree \( \leq \delta \) over \( L \) (in the global case): By simply setting all possible subsets of variables to

\(^3\) The multiplicity of any isolated root here, which we take in the sense of intersection theory for a scheme over the algebraic closure of \( L \) [Ful98], turns out to always be a positive integer (see, e.g., [Smi97, Roj99]).

\(^4\) e.g., roots on the coordinate hyperplanes may have multiplicities > 1 counted as 1 instead.
zero, we easily obtain respective bounds of $1 + \sum_{j=1}^{n} \binom{n}{j} B(L, m, j) \leq 1 + 2^n B(L, m, n)$ and $1 + \sum_{j=1}^{n} \binom{n}{j} A(L, \delta, m, j) \leq 1 + 2^n A(L, \delta, m, n)$. Of course, since many of the monomial terms of $F$ will vanish upon setting an $x_i$ to 0, these bounds will usually be larger than really necessary. ♦

**Example 1** Consider the following $2 \times 2$ system over $\mathbb{Q}_2$:

\[
\begin{align*}
f_1(x_1, x_2) &:= \alpha_1 + \alpha_2 x_1^{a_1} x_2^{a_2} + \alpha_3 x_1^{\nu_1} x_2^{\nu_2} \\ f_2(x_1, x_2) &:= \beta_1 + \beta_2 x_1^{a_{11}} x_2^{a_{22}} + \cdots + \beta_\mu x_1^{a_{1\mu}} x_2^{a_{2\mu}}
\end{align*}
\]

which is $m$-sparse for some $m \leq \mu + 2$. Theorem 1 and an elementary calculation then tell us that such an $F$ has no more than

\[
41(\mu + 1)^4 \left(1 + \log_2 \left(\frac{\mu + 1}{0.693}\right)\right)^2
\]

isolated roots, counting multiplicities, in $(\mathbb{Q}_2^\ast)^2$ (and $(\mathbb{Q}^\ast)^2$ as well, via the natural embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_2$). For instance, $\mu = 3 \implies F$ is at worst 5-sparse and has no more than 127645 roots in $(\mathbb{Q}_2^\ast)^2$. Explicit bounds independent of the total degrees of $f_1$ and $f_2$ appear to have been unknown before. However, if we replace $\mathbb{Q}_2$ by $\mathbb{R}$ throughout, then the best previous upper bounds were $4(2^\mu - 2)$ for all $\mu \geq 1$ and a bound of 20 in the special case $\mu = 3$. Interestingly, the latter bounds, which follow easily from [LRW01, thm. 1], in fact allow us to take real exponents and count topologically isolated roots, but without multiplicities.  

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\footnotesize{The numerical calculations throughout this paper were done with the assistance of Maple, and the code for these calculations is available from the author’s web-page.

\footnotesize{Khovanski’s Theorem on Fewnomials [Kho91, cor. 7, sec. 3.12], which only counts roots with non-singular Jacobian, implies an upper bound of 995328 for $\mu = 3$.}}
The real analytic upper bound exceeds our arithmetic bound for all \( \mu \geq 29 \), where both bounds begin to exceed 1.3 billion. A sharper bound, based on a refinement of theorem [?], appears in example [? of section [?.

Example 2 Another consequence of theorem [?] is that for fixed \( \mathcal{L} \), we now know \( \log B(\mathcal{L}, m, n) \) and \( \log A(\mathcal{L}, m, n) \) to within a constant factor: For \( m \geq 2 \) consider the \( m \)-sparse \( n \times n \) polynomial system \( F = (f_1, \ldots, f_n) \) where \( f_i = \prod_{j=1}^{m-1} (x_i - j) \) for all \( i \). Clearly then, this \( F \) has exactly \( (m-1)^n \) isolated roots in \( \mathbb{N}^n \). It is curious that the analogous growth-rate is unknown if \( \mathcal{L} \) is replaced by the usual Archimedean completion \( \mathbb{R} \) of \( \mathbb{Q} \).

A weaker version of theorem [?] with non-explicit bounds was derived earlier in [Roj01b]. In particular, explicit bounds were known previously only in the case \( k = n = 1 \) [Len99b, thm. 1 and thm. 2], and all our bounds (save the global case) match the bounds of [Len99b] in this special case. Philosophically, the approach of [Len99b] was more algebraic (low degree factors of polynomials) while our point of view here is more geometric (isolated rational points of low degree in a hypersurface intersection). The only other results known for \( k > 1 \) or \( n > 1 \) were derived via rigid analytic geometry and model theory, and in our notation yield a non-effective bound of \( B(\mathbb{Q}_p, m, n) < \infty \) (see the seminal works [DV88, Lip88]).

\[ \text{7 In order to streamline the proof of our number field generalization, we left our bound on } A(\mathcal{L}, \delta, m, n) \text{ in theorem [?] a bit loose: for } n = 1 \text{ our bound reduces to } \mathcal{O} \left( d^2 \delta^2 m^{2d^2} \log(d\delta m) \right), \text{ while the older univariate result yields } \mathcal{O} \left( d\delta m^{2d^2} \log(d\delta m) \right) \text{ in our notation. A sharper bound, agreeing with Lenstra’s univariate bound when } n = 1, \text{ appears in corollary [?] of section [?].} \]

\[ \text{8 Lenstra has also considered a higher-dimensional generalization but in a different direction: bounds for the number of rational hyperplanes in a hypersurface defined by a single } m \text{-sparse } n \text{-variate polynomial [Len99b, prop. 6.1].} \]
Our approach is simpler and is based on a higher-dimensional generalization (theorem 2 of the next section) of a result of Hendrik W. Lenstra, Jr. for univariate sparse polynomials over certain algebraically closed fields \[\text{Len99b, thm. 3}\]. Indeed, aside from the introduction of some higher-dimensional convex geometry, our proof of theorem 2 is structurally quite similar to Lenstra’s proof of the \(k = n = 1\) case in \[\text{Len99b}\]: reduce the global case to the local case, then reduce the local case to a refined result over the \(p\)-adic complex numbers.

We now describe two results used in our proofs which may be of broader interest. We also point out that connections between our results and complexity theory, including the question of whether we can find isolated rational roots in polynomial time, is described in section 3.

1.1. The Distribution of \(p\)-adic Complex Roots

For any (rational) prime \(p\), let \(\mathbb{C}_p\) denote the completion (with respect to the \(p\)-adic metric) of the algebraic closure of \(\mathbb{Q}_p\). Theorem 2 follows from a careful application of two results on the distribution of roots of \(F\) in \((\mathbb{C}_p^*)^n\). The first result strongly limits the number of roots that can be \(p\)-adically close to the point \((1, \ldots, 1)\). The second result strongly limits the number of distinct valuation vectors which can occur for the roots of \(F\).

**Theorem 2** Let \(F\) be any \(m\)-sparse \(k \times n\) polynomial system over \(\mathbb{C}_p\). Also let \(r_1, \ldots, r_n > 0, r := (r_1, \ldots, r_n),\) and let \(\text{ord}_p : \mathbb{C}_p \rightarrow \mathbb{Q} \cup \{+\infty\}\) denote the usual exponential valuation, normalized\(^9\) so that \(\text{ord}_p(p) = 1\). Finally, let \(C_p(m, n, r)\) denote the maximum number of isolated roots \((x_1, \ldots, x_n)\) of \(F\) in \(\mathbb{C}_p^n\) with \(\text{ord}_p(x_i - 1) \geq r_i\) for all \(i\), counting multiplicities

\(^9\) So, for example, \(\text{ord}_p(0) = +\infty\) and \(\text{ord}_p(p^k r) = k\) whenever \(r\) is a unit in \(\mathbb{Z}_p\) and \(k \in \mathbb{Q}\).
if (and only if) \( k = n \). Then \( C_p(m, n, r) = 0 \) (if \( m \leq n \) or \( k < n \)) and

\[
C_p(m, n, r) \leq \left\{ c(m - 1) \left[ r_1 + \cdots + r_n + \log_p \left( \frac{(m - 1)^n}{r_1 \cdots r_n \log^n p} \right) \right] \right\}^n \prod_{i=1}^{n} r_i
\]

(if \( m \geq n + 1 \) and \( k \geq n \)), where \( c := \frac{e}{e-1} \leq 1.582 \). Furthermore, if we restrict to those \( F \) where \( k = n \) and \( f_i \) has exactly \( m_i \) monomial terms for all \( i \), then we have the sharper bounds of \( C_p(m, n, r) = 0 \) (if \( m_i \leq 1 \) for some \( i \)) and

\[
C_p(m, n, r) \leq c^n \prod_{i=1}^{n} \left\{ (m_i - 1) \left[ r_1 + \cdots + r_n + \log_p \left( \frac{(m_i - 1)^n}{r_1 \cdots r_n \log^n p} \right) \right] \right\} / r_i
\]

(if \( m_1, \ldots, m_n \geq 2 \)).

These bounds appear to be new: the only previous results in this direction appear to have been Lenstra’s derivation of the special case \( n = 1 \) [Len99b, thm. 3] and an earlier observation of Leonard Lipshitz [Lip88, thm. 2] equivalent to the non-explicit bound \( C_p(m, n, (1, \ldots, 1)) < \infty \).

Our last bound over \( \mathbb{C}_p^n \) is based on a toric arithmetic-geometric result of Smirnov, stated below.

**Definition 1** For any \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \), let \( x^a := x_1^{a_1} \cdots x_n^{a_n} \). Writing any \( f \in \mathbb{L}[x_1, \ldots, x_n] \) as \( \sum_{a \in \mathbb{Z}^n} c_a x^a \), we call \( \text{Supp}(f) := \{ a \mid c_a \neq 0 \} \) the support of \( f \). Also, let \( \pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \) be the natural projection forgetting the \( x_{n+1} \) coordinate and, for any \( n \)-tuple of polytopes \( P = (P_1, \ldots, P_n) \), define \( \pi(P) := (\pi(P_1), \ldots, \pi(P_n)) \).

**Definition 2** For any \( k \times n \) polynomial system \( F \) over \( \mathbb{C}_p \), its \( k \)-tuple of \( p \)-adic Newton polytopes, \( \text{Newt}_p(F) = (\text{Newt}_p(f_1), \ldots, \text{Newt}_p(f_k)) \), is defined as follows: \( \text{Newt}_p(f_i) := \text{Conv}(\{ (a, \text{ord}_p c_a) \mid a \in \text{Supp}(f_i) \}) \subset \mathbb{R}^{n+1} \), where \( \text{Conv}(S) \) denotes the convex hull of \( S \).
set \( S \subseteq \mathbb{R}^{n+1} \). Also, for any \( w \in \mathbb{R}^n \) and any closed subset \( B \subset \mathbb{R}^n \), let the face of \( B \) with inner normal \( w \), \( B^w \), be the set of points \( x \in B \) which minimize the inner product \( w \cdot x \).

Finally, let \( \text{Newt}_p^w (F) := (\text{Newt}_p^w (f_1), \ldots, \text{Newt}_p^w (f_k)) \). \( \diamond \)

**Smirnov’s Theorem** [Smi97, thm. 3.4] For any \( n \times n \) polynomial system \( F \) over \( \mathbb{C}_p \), the number of isolated roots \((x_1, \ldots, x_n)\) of \( F \) in \((\mathbb{C}_p^*)^n\) satisfying \( \text{ord}_p x_i = r_i \) for all \( i \) (counting multiplicities) is no more than \( M(\pi(\text{Newt}^\hat{r}_p(F))) \), where \( \hat{r} := (r_1, \ldots, r_n, 1) \), \( M(\cdot) \) denotes mixed volume [BZ88] (normalized so that \( M(\text{Conv}(\{0, e_1, \ldots, e_n\}), \ldots, \text{Conv}(\{0, e_1, \ldots, e_n\}))=1 \)), and \( e_i \) is the \( i \)-th standard basis vector of \( \mathbb{R}^n \). \( \blacksquare \)

**Remarks 2**

0. Explicit examples of the preceding constructions are illustrated in [Roj01b].

1. The number of possible distinct valuation vectors for a root of an \( n \)-variate polynomial system \( F \) can thus be combinatorially bounded from above as a function depending solely on \( n \) and the number of monomial terms (cf. section \( \|$).

2. The number of roots of \( F \) in \((\mathbb{C}_p^*)^n\) with given valuation vector thus depends strongly on the individual exponents of \( F \) — not just on the number of monomial terms.

3. It is thus only the lower faces of the \( p \)-adic Newton polytopes that matter in counting roots or valuation vectors. \( \diamond \)

We prove theorem \( \|$ in section \( \|$. However, let us first show how theorem \( \|$ implies theorem \( \|$: We will begin by examining the local case in the next section, and then complete our proof by deriving the global case from the local case in section \( \|$.  

\( ^{11} \) Those with positive \( x_{n+1} \) coordinate for their inner normals...
2. The Local Case of Theorem [1]

Here we will assume that \( \mathcal{L} \) is any degree \( d \) algebraic extension of \( \mathbb{Q}_p \). The following lemma will help us reduce to the case \( k = n \).

**Lemma 1.** (See [Roj01b, lemma 1].) Following the notation of theorem 1, there is a matrix \([a_{ij}] \subset \mathbb{Z}^{n \times k}\) such that the zero set of \( G := (a_{11}f_1 + \cdots + a_{1k}f_k, \ldots, a_{n1}f_1 + \cdots + a_{nk}f_k) \) in \( \mathbb{C}_p^n \) is the union of the zero set of \( F \) in \( \mathbb{C}_p^n \) and a finite (possibly empty) set of points. ■

**Proof of the Local Case of Theorem [1]:** It is clear that there are no isolated roots whatsoever if \( k < n \), since the underlying algebraic set over \( \mathbb{C}_p^n \) is positive-dimensional. So we can assume \( k \geq n \). In the event that \( k > n \), lemma [1] then allows us to replace \( F \) by a new \( n \times n \) polynomial system (with no new exponent vectors) which has at least as many isolated roots as our original \( F \). So we can assume \( k = n \) and observe that root multiplicities are preserved if lemma [1] was not used (i.e., if we already had \( k = n \) in our input). Since an \( m \)-sparse \( n \times n \) polynomial system clearly has no geometrically isolated roots whatsoever when \( m \leq n \), we can clearly assume that \( m \geq n + 1 \). (Indeed, upon dividing each equation by a suitable monomial, \( m \leq n \) clearly implies that we can obtain \( n \) linear equations in \( \leq n - 1 \) non-constant monomial terms.)

The well-known classification of when mixed volumes vanish [BZ88] then yields that \( \mathcal{M}(\pi(\text{Newt}_p^{\hat{r}}(F))) > 0 \iff \) there are linearly independent vectors \( v_1, \ldots, v_n \), with \( v_i \) an edge of \( \text{Newt}_p^{\hat{r}}(f_i) \) for all \( i \). So let \( \varepsilon_i \) be the number of edges of \( \text{Newt}_p(f_i) \). If \( n = 1 \) then we clearly have \( \varepsilon_i \leq m - 1 \) for all \( i \), and this is a sharp bound for all \( m \). If \( n > 2 \) then we have the obvious bound of \( \varepsilon_i \leq m(m - 1)/2 \) for all \( i \), and it is not hard to generate examples showing that this bound is sharp for all \( m \) as well [Edde87, thm. 6.5, pg. 101]. If \( n = 2 \) then note that the number of edges of \( \text{Newt}_p(f_i) \) is clearly not decreased if we triangulate the boundary of
Newt$_p(f_i)$. Since each 2-face of the resulting complex is incident to exactly 2 edges, Euler’s relation [Edel87, thm. 6.8, pg. 103] then immediately implies that $\varepsilon_i \leq 2m - 2$ for all $i$. (This bound is easily seen to be sharp for all $m \geq 4$.)

We thus obtain that there are no more than $\varepsilon_1 \cdots \varepsilon_n \leq u(m, n)$ possible values for an $r \in \mathbb{R}^n$ with $\hat{r} = (r, 1)$ and $\mathcal{M}(\pi(\text{Newt}_p^\hat{r}(F))) > 0$. In particular, by Smirnov’s Theorem, this implies that the number of distinct values for the valuation vector $(\text{ord}_p x_1, \ldots, \text{ord}_p x_n)$, where $(x_1, \ldots, x_n) \in (\mathbb{C}_p^*)^n$ is a root of $F$, is no more than $u(m, n)$. So let us temporarily fix $(r_1, \ldots, r_n) := r$ and see how many roots of $F$ in $(\mathbb{L}^*)^n$ can have valuation vector $r$.

Following the notation of theorem 2, let $R_p := \{ x \in \mathbb{C}_p \mid |x|_p \leq 1 \}$ be the ring of algebraic integers of $\mathbb{C}_p$, let $M_p := \{ x \in \mathbb{C}_p \mid |x|_p < 1 \}$ be the unique maximal ideal of $R_p$, $\mathbb{F}_L := (R_p \cap \mathbb{L})/(M_p \cap \mathbb{L})$, and let $\rho$ be any generator of the principal ideal $M_p \cap \mathbb{L}$ of $R_p \cap \mathbb{L}$. Also let $e_L := \max_{y \in L} \{|\text{ord}_p y|^{-1}\}$ and $q_L := \# \mathbb{F}_L$. (The last two quantities are respectively known as the ramification degree and residue field cardinality of $\mathbb{L}$, and satisfy $e_L, \log_p q_L \in \mathbb{N}$ and $e_L \log_p q_L = d$ [Kob84, ch. III].) Since $\text{ord}_p \rho = 1/e_L$, it is clear that $r$ a valuation vector of a root of $F$ in $(\mathbb{L}^*)^n \Rightarrow r \in (\mathbb{Z}/e_L)^n$.

Fixing a set $A_L \subset R_p$ of representatives for $\mathbb{F}_L$ (i.e., a set of $q_L$ elements of $R_p \cap \mathbb{L}$, exactly one of which lies in $M_p$, whose image mod $M_p \cap \mathbb{L}$ is $\mathbb{F}_L$), we can then write any $x_i \in \mathbb{L}$ uniquely as $\sum_{j=\epsilon_L}^{+\infty} a_j^{(i)} \rho^j$ for some sequence of $a_j^{(i)} \in A_L$ [Kob84, corollary, pg. 68]. Note in particular that $\frac{x_i}{a_j^{(i)} \rho^j \epsilon_L^r}$ thus lies in $R_p \setminus M_p$ for any $a_j^{(i)} \in A_L \setminus M_p$.

Theorem 2 thus implies that the number of isolated roots $(x_1, \ldots, x_n)$ of $F$ in $(\mathbb{C}_p^*)^n$ satisfying $(\text{ord}_p x_1, \ldots, \text{ord}_p x_n) = r$ and $\frac{x_1}{a_1^{(i)} \rho_1^j \epsilon_L^r + r} \equiv \cdots \equiv \frac{x_n}{a_n^{(i)} \rho_n^j \epsilon_L^r + r} \equiv 1 \pmod{M_p}$ is no more than $C_p(m, n, (1/e_L, \ldots, 1/e_L))$. Furthermore, since $M_p \cap \mathbb{L} \subset M_p$, we obtain the same statement if we restrict to roots in $(\mathbb{L}^*)^n$ and use congruence $\pmod{M_p \cap \mathbb{L}}$ instead.
Since there are $q_L - 1$ possibilities for each $a^{(i)}_0$, our last observation tells us that the number of isolated roots $(x_1, \ldots, x_n)$ of $F$ in $(L^*)^n$ satisfying $(\text{ord}_p x_1, \ldots, \text{ord}_p x_n) = r$ is no more than $(q_L - 1)^n C_p(m, n, (1/e_L, \ldots, 1/e_L))$. So the total number of isolated roots of $F$ in $(L^*)^n$ is no more than $u(m, n)(q_L - 1)^n C_p(m, n, (1/e_L, \ldots, 1/e_L))$. Since $e_L \leq d$ and $q_L \leq p^d$, an elementary calculation yields our desired bound.

A simple consequence of our last proof is that there is a natural injection from the set of possible valuation vectors of an isolated root of $F$ to the set of lower facets of a particular polytope. In particular, we can define $\hat{\Sigma}_p(F)$ to be Newt$_p\left(\sum_{i=1}^k f_i\right)$ or the Minkowski sum $\sum_{i=1}^n \text{Newt}_p(f_i)$, according as $k > n$ or $k = n$, and immediately obtain the following corollary.

**Corollary 1** Following the notation above, we have

$$B(L, m, n) \leq F(F)(q_L - 1)^n C_p(m, n, (1/e_L, \ldots, 1/e_L)),$$

where $F(F)$ is the number of lower facets of $\hat{\Sigma}_p(F)$, and $q_L$ and $e_L$ are respectively the residue field cardinality and ramification index of $L$.

**Example 3** Returning to example 1, observe that $f_1$ has $\leq 3$ monomial terms (so Newt$_2(f_1)$ has $\leq 3$ edges) and $f_2$ has $\leq \mu$ monomial terms (so Newt$_2(f_2)$ has $\leq 2\mu$ edges (cf. our use of Euler’s formula in the proof of the local case of theorem 1)). So we in fact have $F(F) \leq 6\mu$ (for all $\mu \geq 4$) and $F(F) \leq 9$ (for $\mu = 3$). Corollary 1 then implies improved upper bounds of

$$304(\mu - 1)\mu \left(1 + \log_2 \left(\frac{\mu - 1}{0.693}\right)\right) \quad \text{for all } \mu \geq 4 \quad \text{and} \quad 2304 \quad \text{for } \mu = 3$$

\[\text{cf. part 3 of remark 2 of section 1.1. \quad Recall that a facet of a } d\text{-dimensional polytope is simply a face of dimension } d-1.\]
for the number of roots of $F$ in $(\mathbb{Q}_2^*)^2$. Also, our refined bound is smaller than the aforementioned real analytic bound (cf. example 4 of section 4) for all $m \geq 17$, where the two bounds begin to exceed 456800.

\textbf{Example 4} It is entirely possible that the maximum number of roots in $(\mathcal{L}_2)^n$ of an $m$-sparse $n \times n$ polynomial system over $\mathcal{L}$ is actually \textbf{larger} for $\mathcal{L} = \mathbb{Q}_2$ than for $\mathcal{L} = \mathbb{R}$, for small $m$ and $n$. In particular, a univariate trinomial over $\mathbb{R}$ clearly has at most 4 nonzero real roots. However, $3x_1^{10} + x_1^2 - 4$ has exactly 6 nonzero roots in $\mathbb{Q}_2$ and this is the maximum possible number of roots in $\mathbb{Q}_2^*$ for univariate trinomials over $\mathbb{Q}_2$ [Len99b, prop. 9.2].

\section{3. The Global Case of Theorem 1}

Let us start with a construction from [Len99b, sec. 8] for the univariate case: First, fix a group homomorphism $\mathbb{Q} \rightarrow \mathbb{C}_2^*$, written $r \mapsto 2^r$, with the property that $2^1 = 2$. To construct $2^r$ for an arbitrary rational $r$, choose $2^{1/n!}$ inductively to be an $n$th root of $2^{1/(n-1)!}$, and then define $2^{a/n!}$ to be the $a$th power of $2^{1/n!}$ for any $a \in \mathbb{Z}$. Clearly, ord$_2(2^r) = r$ for each $r \in \mathbb{Q}$.

For $j, e \in \mathbb{N}$ we then define the subgroups $U_e$ and $T_j$ of $\mathbb{C}_2^*$ by $U_e := \{x \mid \text{ord}_p(x - 1) \geq 1/e\}$ and $T_j := \{\zeta \mid \zeta^{2^j - 1} = 1\}$. Note that $U_e \subseteq U_e'$ if $e \leq e'$, and $T_j \subseteq T_{j'}$ if $j$ divides $j'$.

What we now show is that in addition to having few roots of bounded degree over $\mathbb{Q}_2$, $F$ has few roots in another suprisingly large piece of $(\mathbb{C}_2^*)^n$.

\textbf{Lemma 2.} Let $e, j, k \in \mathbb{N}$, and let $F$ be an $m$-sparse $n \times n$ polynomial system over $\mathbb{C}_2$. Then $F$ has at most $\mathcal{F}(F)(2^j - 1)^n C_2(m, n, (1/e, \ldots, 1/e))$ roots in the subgroup $(2^Q \cdot T_j \cdot U_e)^n$ of $(\mathbb{C}_2^*)^n$, where $\mathcal{F}(F)$ is as defined in corollary 4 of section 4.
Proof: First note that the case \( n = 1 \), in slightly different notation, is exactly lemma 8.2 of [Len99b]. The proof there generalizes quite easily to our higher-dimensional setting. Nevertheless, for the convenience of the reader, let us give a succinct but complete proof.

First note that by theorem 2, \( F \) has no more than \( C_2(m, n, (1/e, \ldots, 1/e)) \) roots in \( U_e^n \).

By the change of variables \((x_1, \ldots, x_n) \mapsto (\alpha_1 y_1, \ldots, \alpha_n y_n)\) we then easily obtain the same upper bound for the number of roots of \( F \) in any coset of \( U_e^n \).

Since \( T^n_j \) clearly has order \((2^j - 1)^n\), \( F \) thus has no more than \((2^j - 1)^nC_2(m, n, (1/e, \ldots, 1/e))\) roots in any coset \((2^{r_1}T_jU_e) \times \cdots \times (2^{r_n}T_jU_e)\). Since Smirnov’s Theorem implies, via our proof of the local case of theorem 1 (cf. section 2), that a root \( x \in \mathbb{C}^* \) of \( F \) can produce no more than \( F(F) \) possible distinct values for \((r_1, \ldots, r_n) = (\text{ord}_2 x_1, \ldots, \text{ord}_2 x_n)\), we are done.■

To at last prove the global case of theorem 1, let us quote another useful result of Hendrik W. Lenstra, Jr. Recall that \( \lceil x \rceil \) is the least integer greater than \( x \).

**Lemma 3.** [Len99b, lemma 8.3] Let \( n \in \mathbb{N} \) and let \( L \) be an extension of \( \mathbb{Q}_2 \) of degree \( \leq D \). Then there is a \( j \in \{1, \ldots, D\} \) such that \( L^* \subseteq 2^{\lceil D/j \rceil} \mathbb{Q}_2 T_j U_{\lceil D/j \rceil} \). ■

**Proof of the Number Field Case of Theorem 1:**

Since \( \mathbb{Q} \) naturally embeds in \( \mathbb{Q}_2 \), we can assume \( L \) is a subfield of \( \mathbb{C}_2 \) of finite degree over \( \mathbb{Q}_2 \). Then every root of \( F \) in \((\mathbb{C}_2^*)^n\) of degree \( \leq \delta \) over \( \mathbb{Q}_2 \) lies in \((L^*)^n\), where \( L' \) is an extension of \( \mathbb{Q}_2 \) of degree at most \( D := d\delta \). So by lemma 3, any such root of \( F \) also lies in \( \bigcup_{j=1}^D (2^\lceil D/j \rceil \mathbb{Q}_2 T_j U_{\lceil D/j \rceil}) \).

From lemma 3 it now follows that the number of roots of \( F \) of degree \( \leq \delta \) over \( \mathbb{Q}_2 \) is no more than \( \sum_{j=1}^D F(F)(2^j - 1)^nC_2(m, n, (\frac{1}{\lceil D/j \rceil}, \ldots, \frac{1}{\lceil D/j \rceil})) \). Since \( 2^j - 1 \leq 2^j \), \( F(F) \leq u(m, n) \) (cf. the proof of the local case of theorem 1 in section 2), and \( C_2(m, n, (r, \ldots, r)) \) is a decreasing function of \( r \), we thus obtain by geometric series that \( A(L, \delta, m, n) \leq \)


2^{nd\delta+1} u(m, n) C_2(m, n, (\frac{1}{\Delta \sigma}, \ldots, \frac{1}{\Delta \sigma}))$. So by theorem 3 and an elementary calculation we are done. ■

By leaving the last sum in our proof above unsimplified, we immediately obtain the following improvement of theorem 3.

**Corollary 2** We have

$$A(L, \delta, m, n) \leq F(\mathcal{F}(L, \delta, m, n)) \sum_{j=1}^{d\delta} (2^j - 1)^n C_2\left(m, n, \left(\frac{1}{|d\delta/j|d\delta}, \ldots, \frac{1}{|d\delta/j|d\delta}\right)\right),$$

where $\mathcal{F}(F)$ is as defined in corollary 4 of section 3. ■

### 4. Proving Theorem 2

We begin with a clever observation of Hendrik W. Lenstra, Jr. on binomial coefficients, factorials, and least common multiples. Recall that $a|b$ means that $a$ and $b$ are integers with $a$ dividing $b$, and that $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

**Definition 3** \cite[sec. 2]{Len99b} For any nonnegative integers $m$ and $t$ define $d_m(t)$ to be the least common multiple of all integers that can be written as the product of at most $m$ pairwise distinct positive integers that are at most $t$ (and set $d_m(t) := 1$ if $m = 0$ or $t = 0$).

Finally, for any $a \in \mathbb{Z}$, let us define $\binom{a}{t} := \prod_{i=0}^{t-1} \frac{a-i}{t-i}$ (and set $\binom{a}{0} := 1$). ◆

**Lemma 4.** \cite[sec. 2]{Len99b} Following the notation of definition 3, we have...

(a) $d_m(t) | n!$

(b) $m \geq t \implies d_m(t) = t!$

(c) $0 \leq i \leq m < t \implies i! | d_m(t)$

(d) $t \geq 1 \implies \text{ord}_p d_m(t) \leq m \lfloor \log_p t \rfloor$
Furthermore, if \( A \subseteq \mathbb{Z} \) is any set of cardinality \( m \), then there are rational numbers \( \gamma_0(A, t), \ldots, \gamma_{m-1}(A, t) \) such that:

1. the denominator of \( \gamma_j(A, t) \) divides \( d_{m-1}(t)/j! \) if \( t \geq m \) and \( \gamma_j(A, t) = \delta_{jt} \) otherwise.\(^\text{13}\)

\[
\binom{a}{t} = \sum_{j=0}^{m-1} \gamma_j(A, t) \binom{a}{j} \quad \text{for all } a \in A. \quad \blacksquare
\]

Our proof of theorem \( 2 \) will consist of a careful application of Smirnov’s Theorem to the “shifted” polynomial system \( G(x_1, \ldots, x_n) := F(1 + x_1, \ldots, 1 + x_n) \). (So roots of \( F \) close to \((1, \ldots, 1)\) are simply translations of roots of \( G \) close to \((0, \ldots, 0)\).) Since the \( g_i \) can be highly non-sparse, one might not expect Smirnov’s Theorem to give bounds independent of the degrees of the \( f_i \) on the number of roots of \( G \) close to \((0, \ldots, 0)\). However, lemma \( 4 \) and lemmata \( 5 \) and \( 6 \) below, save the day.

**Lemma 5.** Let \( \mathbf{c} := \frac{\mathbf{x}}{\mathbf{c}} \) (so \( c \leq 1.582 \)) and \( t_1, r_1, \ldots, t_n, r_n > 0 \). Then

\[
\sum_{i=1}^{n} (r_i t_i - (m-1) \log_p t_i) \leq (m-1) \sum_{i=1}^{n} r_i \implies \sum_{i=1}^{n} r_i t_i \leq c(m-1) \left[ \left( \sum_{i=1}^{n} r_i \right) + \log_p \left( \frac{(m-1)^n}{r_1 \cdots r_n \log^* p} \right) \right].
\]

**Proof:** Here we make multivariate extensions of some observations of Lenstra from [Len99b, prop. 7.1]: First note that it is easily shown via basic calculus that \( 1 - \frac{\log x}{x} \) assumes its minimum (over the positive reals), \( 1/c \), at \( x = e \). So for all \( x > 0 \) we have \( x \geq (\log x) + x/c \).

Letting \( t, r > 0 \), \( w := \frac{m-1}{r \log p} \) and \( x := t/w \), we then obtain

\[
rt \geq rwx \geq r w ((\log x) + x/c) = rw(\log w) - rt(\log w) + rt/c = (m-1)(\log_p t) - (m-1) \log_p \left( \frac{m-1}{r \log p} \right) + rt/c.
\]

Substituting \( r = r_i, t = t_i \), and summing over \( i \) then implies

\[
(*) \sum_{i=1}^{n} r_i t_i \geq (m-1) \left( \sum_{i=1}^{n} \log_p t_i \right) - (m-1) \log_p \left( \frac{(m-1)^n}{r_1 \cdots r_n \log^* p} \right) + \frac{1}{c} \sum_{i=1}^{n} r_i t_i.
\]

\(^{13}\) \( \delta_{ij} \) denoting the Kronecker delta, which is 0 when \( i \neq j \) and 1 when \( i = j \).
Now suppose that
\[(\star \star) \sum_{i=1}^{n} r_i t_i > c(m-1) \left[ \left( \sum_{i=1}^{n} r_i \right) + \log_p \left( \frac{(m-1)^n}{r_1 \cdots r_n \log^{n} p} \right) \right].\]

Substituting \((\star \star)\) into the last sum of the right hand side of our inequality \((\star)\) then tells us
\[\sum_{i=1}^{n} r_i t_i > (m-1) \left( \sum_{i=1}^{n} \log_p t_i \right) + (m-1) \left[ \sum_{i=1}^{n} r_i \right] + \log_p \left( \frac{(m-1)^n}{r_1 \cdots r_n \log^{n} p} \right).\]
So we obtain \[\sum_{i=1}^{n} r_i t_i > (m-1) \left( \sum_{i=1}^{n} \log_p t_i \right) + (m-1) \left( \sum_{i=1}^{n} r_i \right),\] which can be rearranged into
\[(\star \star \star) \sum_{i=1}^{n} (r_i t_i - (m-1) \log_p t_i) > (m-1) \sum_{i=1}^{n} r_i.\]
So \((\star) \implies (\star \star \star)\), and we conclude simply by taking the contrapositive. ■

The following lemma is a simple consequence of the basic properties of polytopes, their faces, and their mixed volumes \[\text{[BZ88]}.\]

**Lemma 6.** Following the notation of \[1.1\], let \(G := (g_1, \ldots, g_n)\) be any \(n \times n\) polynomial system and let \(r := (r_1, \ldots, r_n)\) be such that \(r_i > 0\) for all \(i\). Also let \(w(g_i, r) := \pi \left( \bigcup_{\hat{s} := (s_1, \ldots, s_n, 1) \atop s_i \geq r_i \text{ for all } i} \text{Newt}^\hat{s}_p(g_i) \right)\) for all \(i\). Then
\[\sum_{\hat{s} := (s_1, \ldots, s_n, 1) \atop s_i \geq r_i \text{ for all } i} \mathcal{M}(\pi(\text{Newt}^\hat{s}_p(G))) \leq \mathcal{M}(\text{Conv}(w(g_1, r), \ldots, \text{Conv}(w(g_n, r)))).\]
In particular, if \(Q = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid r_1 t_1 + \cdots r_n t_n \leq 1 \text{ and } t_j \geq 0 \text{ for all } j\}\), then
\[\mathcal{M}(Q, \ldots, Q) = 1 / \prod_{i=1}^{n} r_i.\]

**Proof of Theorem 2:**

First note that just as in the proof of the local case of theorem \[1\] (cf. section \[2\]), we have that \(k < n\) or \(m \leq n\) implies that there are no isolated roots whatsoever. So we can assume that \(m \geq n + 1\). Also, again like in the proof of the local case of theorem \[1\] we can safely assume
via lemma 4 that $k = n$ and observe that root multiplicities are preserved if we already had $k = n$ in our original input. Furthermore, if $m_i$ is the number of monomial terms occurring in $f_i$ for all $i$, then it is easily checked that $m_i \leq 1$ for any $i$ implies that there are no isolated roots at all. So we can also assume that $m_1, \ldots, m_n \geq 2$.

Let us now set $g_i(x_1, \ldots, x_n) := f_i(1 + x_1, \ldots, 1 + x_n)$ for all $i$ and $G := (g_1, \ldots, g_n)$. It is then clear that the number of isolated roots of $F$ with $\text{ord}_p(x_i - 1) \geq r_i$ for all $i$ is the same as the number of isolated roots of $G$ with $\text{ord}_p x_i \geq r_i$ for all $i$, and multiplicities are preserved by this change of variables. Smirnov’s Theorem tells us that the latter number (counting multiplicities) is exactly

$$\sum_{\hat{s} : (s_1, \ldots, s_n, 1) \in S} \mathcal{M}(\pi(\text{Newt}_{\hat{s}}^p(G))).$$

Now let us define the following scaled standard simplex:

$$S(m, n, r) := \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n \bigg| \sum_{j=1}^n r_j t_j \leq c(m - 1) \left[ \left( \sum_{j=1}^n r_j \right) + \log_p \left( \frac{(m-1)^n}{n \prod_{j=1}^n r_j \cdot \log p} \right) \right] \text{ and } t_j \geq 0 \text{ for all } j \right\}.$$ 

Note then that by lemma 6, $\mathcal{M}(S(m_1, n, r), \ldots, S(m_n, n, r))$ is exactly

$$c^n \prod_{i=1}^n \left\{ (m_i - 1) \left[ \left( \sum_{j=1}^n r_j \right) + \log_p \left( \frac{(m_i - 1)^n}{n \prod_{j=1}^n r_j \cdot \log^2 p} \right) \right] \right\},$$

since mixed volume is multihomogeneous with respect to scalings [BZ88]. Since $S(m, n, r)$ is clearly always convex, and since $w(g_i, r)$ is a union of convex hulls of subsets of $\text{Supp}(g_i)$, we also have that $w(g_i, r) \cap \text{Supp}(g_i) \subseteq S(m_i, n, r) \implies \text{Conv}(w(g_i, r)) \subseteq S(m_i, n, r)$ for all $i$.

Since mixed volume is monotonic with respect to containment [BZ88], lemma 6 then clearly implies that...

\footnote{Note that the sum over $s$ is actually infinite, but has only finitely many nonzero summands. This is because any polytope has only finitely many inner facet normals with last coordinate 1, and it is only these terms which can possibly be nonzero.}
To prove theorem 2, we need only show that \( w(g_i, r) \cap \text{Supp}(g_i) \subseteq S(m_i, n, r) \) for all \( i \).

To do this, we will first prove that the valuations of the coefficients of any \( g_i \) satisfy a "slow decay" condition, and then use convexity of the gently sloping lower faces of the \( p \)-adic Newton polytopes \( \text{Newt}_p(g_i) \) to prove that \( w(g_i, r) \cap \text{Supp}(g_i) \subseteq S(m_i, n, r) \) for all \( i \).

Let us temporarily abuse notation slightly to avoid a profusion of indices and respectively write \( f, g, \) and \( m \) in place of \( f_i, g_i, \) and \( m_i \) (for some arbitrary fixed \( i \)). Letting \( D_i := \deg_{x_i} f \), it is clear that we can write \( g(x) := \sum_{j \in \prod_{i=1}^n \{0, \ldots, D_i \}} b_j x^j \), where \( b_j := \sum_{a \in A} c_a \prod_{i=1}^n \left( \frac{a_i}{j_i} \right) \),

\[
f(x) = \sum_{a = (a_1, \ldots, a_n) \in A} c_a x^a \quad \text{(with every } c_a \text{ nonzero), } j = (j_1, \ldots, j_n), \text{ and } A := \text{Supp}(f).\]

Since \( f \neq 0 \) we have \( g \neq 0 \) and thus not all the \( b_j \) vanish. Note also that \( D_i = 0 \implies \text{Supp}(g) \subseteq \{ x \in \mathbb{R}^n \mid x_i = 0 \} \). Letting \( \pi_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1} \) denote the natural orthogonal projection forgetting the \( i \)-th coordinate, it is then clear that

\[
\pi_i(S(m, n, (r_1, \ldots, r_n)) \cap \{ x \in \mathbb{R}^n \mid x_i = 0 \}) \supseteq S(m, n - 1, (r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n)),
\]

and thus \( D_i = 0 \) implies that we can reduce to a case where \( n \) is smaller. So we can assume henceforth that \( D_1, \ldots, D_n > 0 \).

By lemma 2 there are rational numbers \( \{ \gamma_j(i)(t_i) \} \), with \( (i, j) \in \{1, \ldots, n\} \times \{0, \ldots, m - 1\} \), such that for all \( a = (a_1, \ldots, a_n) \in A \) we have

\[
\left( \begin{array}{c} a_i \\ t_i \end{array} \right) = \sum_{j=0}^{m-1} \gamma_j(i)(t_i) \left( \begin{array}{c} a_i \\ j \end{array} \right)
\]

and the denominators of the \( \{ \gamma_j(i)(t_i) \} \) are not too divisible by \( p \). (We will make the latter assertion precise in a moment.)

We thus obtain that for all \( t := (t_1, \ldots, t_n) \in \prod_{i=1}^n \{0, \ldots, D_i \} \),

\[
b_t = \sum_{a \in A} c_a \prod_{i=1}^n \left( \frac{a_i}{t_i} \right) = \sum_{a \in A} c_a \prod_{i=1}^n \sum_{j_i=0}^{m-1} \left( \gamma_j(i)(t_i) \left( \begin{array}{c} a_i \\ j_i \end{array} \right) \right) = \sum_{a \in A} c_a \sum_{j \in \{0, \ldots, m-1\}^n} \prod_{i=1}^n \left( \gamma_j(i)(t_i) \left( \begin{array}{c} a_i \\ j_i \end{array} \right) \right).
\]
\[
= \sum_{j \in \{0, \ldots, m-1\}} \left( \prod_{i=1}^{n} \gamma_{j_i}^{(i)}(t_i) \right) \sum_{a \in A} \prod_{i=1}^{n} a_{i_j} = \sum_{j \in \{0, \ldots, m-1\}} \left( \prod_{i=1}^{n} \gamma_{j_i}^{(i)}(t_i) \right) b_j.
\]

So the coefficients \( \{b_t\}_{t \in \prod_{i=1}^{n} \{0, \ldots, D_i\}} \) of \( g \) are completely determined by a smaller set of coefficients corresponding to the exponents of \( g \) lying in \( \{0, \ldots, m-1\}^n \). Even better, lemma \( \star \) tells us that \( t_i \leq m - 1 \implies \gamma_{j_i}^{(i)}(t_i) = 0 \) for all \( j_i \neq t_i \). So we in fact have that

\[(\bigtriangledown) \quad t_i \leq m - 1 \implies \text{the recursive sum for } b_t \text{ has no terms corresponding to any } j \text{ with } j_i \neq t_i.\]

Given this refined recursion for \( b_t \) we can then derive that the \( p \)-adic valuation of \( b_t \) decreases slowly and in a highly controlled manner: First note that our recursion, combined with \( (\bigtriangledown) \) and the ultrametric inequality, implies that

\[
(\star) \quad \ord_p b_t \geq \min_{j \in \text{Newt}_p^{(s, 1)}} \left\{ \ord_p (b_j) + \sum_{i=1}^{n} \ord_p \gamma_{j_i}^{(i)}(t_i) \right\} \quad \text{for all } t \in \prod_{i=1}^{n} \{0, \ldots, D_i\},
\]

where \( M_t \) is the subset of \( \{0, \ldots, m-1\}^n \) obtained by the intersection, over all \( i \) with \( t_i \leq m - 1 \), of the hyperplanes \( \{t \in \mathbb{R}^n \mid t_i = j_i\} \). Then, by the definition of a face with inner normal \( (s, 1) \), we have \( (t, b_t) \in \text{Newt}_p^{(s, 1)}(g) \implies (\sum_{i=1}^{n} s_i t_i) + \ord_p b_t \leq (\sum_{i=1}^{n} s_i j_i) + \ord_p b_j \) for all \( j \in \prod_{i=1}^{n} \{0, \ldots, D_i\} \). So for all such \( j \) we must have \( \ord_p b_j \geq \ord_p b_t + \sum_{i=1}^{n} s_i (t_i - j_i) \).

In particular, we obtain that

\[(\star\star) \quad [(t, b_t) \in \text{Newt}_p^{(s, 1)}(g) \text{ and } t_i \geq j_i \text{ and } s_i \geq r_i \text{ for all } i] \implies \ord_p b_j \geq \ord_p b_t + \sum_{i=1}^{n} r_i (t_i - j_i).\]

Since \( t \in \text{Supp}(g) \) and \( (t, \ord_p b_t) \in \text{Newt}_p^{(s, 1)}(g) \) implies that \( \ord_p b_t < \infty \), we can thus combine \( (\star) \) and \( (\star\star) \) to obtain that

\[
t \in w(g, r) \cap \text{Supp}(g) \implies \ord_p b_t \geq \min_{j \in \text{Newt}_p^{(s, 1)}} \left\{ \ord_p (b_j) + \sum_{i=1}^{n} (r_i (t_i - j_i) + \ord_p \gamma_{j_i}^{(i)}(t_i)) \right\}.
\]
Cancelling and rearranging terms, we thus obtain that $t \in w(g, r) \cap \text{Supp}(g) \implies$

\[
\sum_{i=1}^{n} r_i t_i \leq \max_{j \in M_t} \left\{ \sum_{i=1}^{n} \left( j_i r_i - \text{ord}_p(\gamma_{j_i}^{(i)}(t_i)) \right) \right\} \leq \max_{j \in \{0, \ldots, m-1\}^n} \left\{ \sum_{i=1}^{n} \left( j_i r_i - \text{ord}_p(\gamma_{j_i}^{(i)}(t_i)) \right) \right\}.
\]

Since lemma 4 tells us that $-\text{ord}_p(\gamma_{j_i}^{(i)}(t_i)) \leq (m - 1)(\log p t_i) - \text{ord}_p(j_i!)$ for all $i$, we then obtain

(♣) $\sum_{i=1}^{n} (r_i t_i - (m - 1) \log_p t_i) \leq \max_{j \in \{0, \ldots, m-1\}^n} \left\{ \sum_{i=1}^{n} (j_i r_i - \text{ord}_p(j_i!)) \right\} \leq (m - 1) \sum_{i=1}^{n} r_i.$

So by lemma 5 we obtain that $w(g, r) \cap \text{Supp}(g) \subseteq S(m, n, r)$, and thus $w(g_i, r) \cap \text{Supp}(g_i) \subseteq S(m_i, n, r)$ for all $i$. ■

5. Connections to Complexity Theory

Thanks to our results, we now know in particular that the maximum number of isolated rational roots of a $k \times n$ polynomial system over $\mathbb{Q}$ depends polynomially on the number of distinct exponent vectors, for fixed $n$. Here we note that it would be of considerable interest to know if this polynomiality persists relative to even more efficient encodings of polynomials.

In particular, instead of monomial expansions (a.k.a. the sparse encoding), consider the straight-line program (SLP) encoding for a univariate polynomial [BCSS98, sec. 7.1]: That is, suppose we have $p \in \mathbb{Z}[x_1]$ expressed as a sequence of the form $(1, x_1, q_2, \ldots, q_N)$, where $q_N = p$ and for all $i \geq 2$ we have that $q_i$ is a sum, difference, or product of some pair of elements $(q_j, q_k)$ with $j, k < i$. Let $\tau(p)$ denote the smallest possible value of $N - 1$, i.e., the smallest length, for such a computation of $p$. Clearly, $\tau(p)$ is no more than the number of monomial terms of $p$, and is often dramatically smaller.
Theorem 3 \([BCSS98, \text{ thm. 3, pg. 127}]\) Suppose there is an absolute constant \(\kappa\) such that for all nonzero \(p \in \mathbb{Z}[x_1]\), the number of distinct roots of \(p\) in \(\mathbb{Z}\) is no more than \((\tau(p) + 1)^{\kappa}\).

Then \(\mathbb{P}_C \neq \mathbb{NP}_C\). ■

In other words, an analogue (regarding complexity theory over \(\mathbb{C}\)) of the famous unsolved \(\mathbb{P} \overset{?}{=} \mathbb{NP}\) question from computer science (regarding complexity theory over the ring \(\mathbb{Z}/2\mathbb{Z}\)) would be settled. The question of whether \(\mathbb{P}_C \overset{?}{=} \mathbb{NP}_C\) remains open as well but it is known that \(\mathbb{P}_C = \mathbb{NP}_C \implies \mathbb{NP} \subseteq \mathbb{BPP}\). (This observation is due to Steve Smale and was first published in \([Shu93]\).) The complexity class \(\mathbb{BPP}\) is central in randomized complexity and the last inclusion (while widely disbelieved) is also an open question. The truth of the hypothesis of theorem 3, also known as the \(\tau\)-conjecture, is yet another open problem, even for \(\kappa = 1\).

One can reasonably suspect that a sufficiently good upper bound for the number of integral roots of an \(m\)-sparse \(k \times n\) polynomial system could be applied to settling the \(\tau\)-conjecture.\(^{15}\) Indeed, any computation of \(p\) of length \(\tau(p)\) can be specialized to obtain a computation of the same length for \(p(k)\) for any integer \(k\). Finding an integral root of \(p\) can then be reinterpreted as finding all possible values for the first coordinate of an integral root of the \(2\tau(p)\)-sparse \(\tau(p) \times (\tau(p) + 1)\) polynomial system defined by the corresponding computational sequence for \(p\). However, the number of variables grows linearly with \(\tau(p)\), so this route toward an application of theorem 3 would at best give us an upper bound exponential in \(\tau(p)\). For better or worse, we thus arrive at a Diophantine problem currently

\(^{15}\) i.e., Diophantine results for multivariate polynomial systems in the sparse encoding can be useful for Diophantine problems involving univariate polynomials in the SLP encoding.
out of our grasp: finding sharp bounds on the number of integral points on certain algebraic sets defined by quadratic binomials and linear trinomials.

A reasonable alternative approach would be to use the embedding of $\mathbb{Q}$ in another complete field — $\mathbb{R}$, in particular. Over $\mathbb{R}$ there are results for univariate polynomials involving an even sharper encoding: For any $p \in \mathbb{R}[x]$, let its **additive complexity**, $\sigma(p)$, be the minimal number of additions and subtractions necessary to express $p$ as an elementary algebraic expression with **constant** exponents. e.g., $p(x) = (1 - (x + 2)^{100})^{97} + 243(x - 7)^{999}$ has $\sigma(p) \leq 4$, and it is clear that $\tau(p) \geq 4$. More generally, it is easily checked that $\sigma(p) \leq \tau(p)$ for all $p \in \mathbb{Z}[x_1]$. Remarkably, one can bound the number of **real** roots of $p$ solely in terms of $\sigma(p)$. This was known since the work of Allan Borodin and Stephen A. Cook around 1974 [BC76], and the best current upper bound is Jean-Jacques Risler's $C^{\sigma(p)^2}$, for some absolute constant $C \in (1, 32)$ [Gri82, Ris85]. Unfortunately, there are examples of $p \in \mathbb{Z}[x_1]$ with $\sigma(p) = O(r)$ and at least $2^r$ real roots (all of which are irrational) [Roj00b, sec. 3, pg. 13]. So additive complexity is too efficient an encoding to be useful in settling the $\tau$-conjecture, at least over $\mathbb{R}$.

Whether analogous (hopefully polynomial) bounds in terms of a sharper encoding exist in our **arithmetic** setting is an open question, even for $n = 1$. In particular, it is interesting to note that the only obstructions to refining theorem 1 to a sharper encoding are (a) the strong dependence of the quantity $F(F)$, arising from our application of Smirnov’s Theorem, on the number of monomial terms, and (b) the existence of an analogue of theorem 2 for a sharper encoding.

As for actually **finding** all the isolated rational roots of $F$, there is both good news and bad news: The bad news is that one can **not** have a polynomial time algorithm (relative to
the sparse encoding) for \( n > 1 \). The good news is that there is a polynomial time algorithm (relative to the sparse encoding) for \( n = 1 \), and that the counter-examples for \( n > 1 \) are very simple.

In particular, if we take \( \mathcal{L} = \mathbb{Q} \) and measure the input size simply as the number of digits needed to write the coefficients and exponents of \( F \) in, say, binary; then it possible for an isolated rational root of \( F \) to have bit size\(^{16}\) exponential in the bit size of \( F \): Simply consider \( k = n = 2, \ m = 4, \) and \( F := (x_1 - x_2^2, x_2 - 2) \). This particular example clearly has bit size \( O(\log D) \) but its one rational root \( (2^D, 2) \) has a first coordinate of bit size \( D \) — exponential in the bit size of \( F \). Thus one can’t even write the output in polynomial time relative to the sparse encoding. Similar examples with bit size \( O(n \log D) \) and having a single rational root, but with root coordinates of bit size \( \Omega(D^n) \), are easy to construct for all \( n \geq 3 \) via the same recursive idea \([\text{Roj00b}, \text{pg. 16, complication Q2}]\). With a bit more work one can even show that such roots of “excessively large” bit size occur not only in a worst case sense but also in an average case sense.

On the other hand, it is a fortunate accident that the absolute logarithmic height of a complex root of \( F \) of degree \( \leq \delta \) over \( \mathcal{L} \) (and thus equivalently, the bit size of such a root) is polynomial in the bit size of \( F \) for \( n = 1 \) and \( \mathcal{L} \) a number field \([\text{Len99a}, \text{prop. 2.3}]\). This is what permits a clever polynomial time algorithm for solving \( F \) when \( n = 1 \) and \( \mathcal{L} \) and \( \delta \)

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\(^{16}\) The bit size of an integer is thus implicitly the number of digits in its binary expansion, and the bit size of a rational number can be taken as the maximum of the bit sizes of its numerator and denominator (written in lowest terms).
are fixed [Len99a, first theorem]. For $n > 1$ it thus appears that the only way to achieve a polynomial time algorithm would be to allow a more efficient encoding of the output than expanding into digits. In particular, it is an open question, even for $n = 2$, whether one can always find SLP’s, of length polynomial in the bit size of $F$, for the isolated rational roots of $F$.

Alternatively, one can simplify the question of solving and simply ask how many isolated rational roots $F$ has, or whether $F$ has any isolated rational roots at all. This is addressed in [Roj01a, thms. 1.3 and 1.4], where it is shown that the truth of the Generalized Riemann Hypothesis implies that detecting a strong form of non-solvability over the rationals (transitivity of the underlying Galois group) can be done within the complexity class $\mathbb{NP}^{\mathbb{NP}}$, provided the underlying complex zero set is finite. In the latter result, $n$ is allowed to be part of the input and can thus vary.

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