A UNIVERSAL ACCELERATED PRIMAL-DUAL METHOD FOR CONVEX OPTIMIZATION PROBLEMS

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Abstract. This work presents a universal accelerated first-order primal-dual method for affinely constrained convex optimization problems. It can handle both Lipschitz and Hölder gradients but does not need to know the smoothness level of the objective function. In line search part, it uses dynamically decreasing parameters and produces approximate Lipschitz constant with moderate magnitude. In addition, based on a suitable discrete Lyapunov function and tight decay estimates of some differential/difference inequalities, a universal optimal mixed-type convergence rate is established. Some numerical tests are provided to confirm the efficiency of the proposed method.

Key words. Convex optimization, primal-dual method, mixed-type estimate, optimal complexity, Bregman divergence, Lyapunov function

AMS subject classifications. 65B99, 68Q25, 90C25

1. Introduction. Consider the minimization problem

\[ \min_{x \in Q} \{ f(x) := h(x) + g(x) : Ax = b \}, \]

where \((A, b) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m, Q \subset \mathbb{R}^n\) is a simple closed convex subset, and \(f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\}\) is properly closed and convex, with smooth part \(h\) and nonsmooth part \(g\). The model problem (1.1) arises from many practical applications, such as compressed sensing [5], image processing [7] and decentralized distributed optimization [3].

In the literature, existing algorithms mainly include Bregman iteration [4, 28, 74], quadratic penalty method [34, 35], augmented Lagrangian method (ALM) [25, 26, 27, 32, 41, 58, 62, 63], and alternating direction method of multipliers [20, 21, 30, 36, 40, 51, 54, 59, 60, 66, 67, 72]. Generally speaking, these methods have sublinear rate \(O(1/k)\) for convex problems and can be further accelerated to \(O(1/k^2)\) for (partially) strongly convex objectives. We also note that primal-dual methods [6, 18, 24, 29, 61, 64, 65, 68] and operator splitting algorithms [12, 16, 17, 43] can be applied to (1.1) with two-block structure.

However, among these works, it is rare to see the optimal mixed-type convergence rate, i.e., the lower complexity bound [52]

\[ \min \left\{ \frac{\|A\|}{\epsilon}, \frac{\|A\|}{\sqrt{\mu\epsilon}} \right\} + \min \left\{ \sqrt{L/\epsilon}, \sqrt{L/\mu} \cdot |\ln \epsilon| \right\}, \]

where \(\mu \geq 0\) is the convexity parameter of \(f\) and \(L\) is the Lipschitz constant of \(\nabla h\). Both Nesterov’s smoothing technique [47] and the accelerated primal-dual method in [11] achieve the lower bound for convex case \(\mu = 0\). The inexact ALM framework in [73] possesses the optimal complexity (1.2) but involves a subroutine for inexactely solving the subproblem.

We mention that the second part of (1.2) corresponds to the objective \(f\) and agrees with the well-known lower complexity bound of first-order methods for solving unconstrained convex problems with Lipschitz gradients. The intermediate non-Lipschitz case is also of interest to be considered [44, 46]. Particularly, when \(\nabla f\) is Hölder continuous (cf.(2.4)) with exponent \(\nu \in [0, 1)\), Nesterov [49] presented a universal fast gradient method (FGM) that did not require à priori knowledge of the smoothness parameter \(\nu\) and the Hölderian constant

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Let \( M_\nu(f) \). A key ingredient of FGM is that Hölderian gradients can be recast into the standard Lipschitz case but with inexact computations \([13, 55, 56, 57]\), and it achieves the optimal complexity \([45]\)

\[
\left( \frac{M_\nu(f)}{\epsilon} \right)^\frac{1}{\nu}
\]

(1.3)

More extensions of FGM can be found in \([22, 23, 31]\).

The dual problem of (1.1) reads equivalently as

\[
\min_{\lambda \in \mathbb{R}^m} \left\{ \varphi(\lambda) := \langle b, \lambda \rangle + \max_{x \in Q} \{ \langle -A^T \lambda, x \rangle - f(x) \} \right\}
\]

(1.4)

If \( f \) is uniformly convex of degree \( p \geq 2 \) (see \([49\), Definition 1]), then \( \nabla \varphi \) is Hölder continuous with exponent \( \nu = 1/(p - 1) \) (cf. \([49\), Lemma 1]). The methods in \([14, 37]\) work for strongly convex problems, i.e., the Lipschitzian case \( \nu = 1 \). Yurtsever et al. \([75]\) proposed an accelerated universal primal-dual gradient method (AccUniPDGrad) for general Hölderian case \( \nu < 1 \) and established the complexity bound (1.3) for objective residual and feasibility violation, with \( M_\nu(f) \) being replaced with \( M_\nu(\varphi) \). Similarly with the spirit of FGM, the proposed method utilizes the “inexactness” property of \( \nabla \varphi \) and applies FISTA \([2]\) to (1.4) with a backtracking line search procedure.

In this work, we propose a universal accelerated primal-dual method (see Algorithm 3.1) for solving (1.1). Compared with existing works, the main contributions are highlighted as follows:

- It is first-order black-box type for both Lipschitz and Hölder cases but does not need to know the smoothness level priorly.
- It is equipped with the Bregman divergence and can handle the non-Euclidean setting.
- In line search part, it adopts dynamically decreasing tolerance while FGM \([49]\) and AccUniPDGrad \([75]\) use the desired fixed accuracy.
- By using the tool of Lyapunov function and tight decay estimates of some differential/difference inequalities, we prove the universal mixed-type estimate that achieves the optimal complexity (including (1.2) as a special case).

We also provide some numerical tests to validate the practical performance. It is confirmed that: (i) the proper choice of Bregman distance is crucial indeed; (ii) our method outperforms FGM and AccUniPDGrad especially for non-Lipschitz problems and smooth problems with large Lipschitz constants, as the automatically decreasing tolerance leads to approximate Lipschitz constants with moderate magnitude.

Our method here is motivated from an implicit-explicit time discretization of a novel accelerated Bregman primal-dual dynamics (see (3.10)), which is an extension of the previous accelerated primal-dual flow \([39]\) to the non-Euclidean case. For unconstrained problems, there are some existing continuous dynamics \([33, 69, 70]\) with Bregman divergence. For linearly constrained case, we see an accelerated primal-dual mirror model \([76]\), which is inspired by the accelerated mirror descent \([33]\) and primal-dual dynamical approach \([19]\) but without numerical discretizations.

The rest of the paper is organized as follows. In section 2 we provide some preliminaries including Bregman divergence and Hölder continuity. Then the main algorithm together with its universal mixed-type estimate is presented in section 3, and rigorous proofs of two technical lemmas are summarized in sections 4 and 5, respectively. Finally, some numerical results are reported in section 6.

2. Preliminary.
2.1. Notations. Let $\langle \cdot , \cdot \rangle$ be the usual inner product of vectors and $\| \cdot \|$ be the standard Euclidean norm (of vectors and matrices). Given a proper function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the subdifferential of $g$ at any $x \in \mathbb{R}^n$ is the set of all subgradients:

$$\partial g(x) := \{ \xi \in \mathbb{R}^n : g(y) \geq g(x) + \langle \xi, y - x \rangle \quad \forall y \in \mathbb{R}^n \}.$$ 

Recall that $Q \subset \mathbb{R}^n$ is a nonempty closed convex subset. We denote by $\iota_Q(\cdot)$ the indicator function of $Q$ and let $N_Q(\cdot) := \partial \iota_Q(\cdot)$ be its normal cone.

Introduce the Lagrangian for the model problem \eqref{eq:3.2}:

$$\mathcal{L}(x, \lambda) := f(x) + \iota_Q(x) + \langle \lambda, Ax - b \rangle \quad \forall (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m.$$ 

We say $(x^*, \lambda^*) \in Q \times \mathbb{R}^m$ is a saddle point of $\mathcal{L}$ if

$$\mathcal{L}(x^*, \lambda) \leq \mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x, \lambda^*) \quad \forall (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m,$$

which also implies the optimality condition:

$$Ax^* - b = 0, \quad \partial f(x^*) + N_Q(x^*) + A^\top \lambda^* \succeq 0.$$ 

2.2. Bregman divergence. Let $\phi : Q \to \mathbb{R}$ be a smooth prox-function and consider the corresponding Bregman divergence

$$D_\phi(x, y) := \phi(x) - \phi(y) - \langle \nabla \phi(y), x - y \rangle \quad \forall x, y \in Q.$$ 

Suppose $\phi$ is $1$-strongly convex, which means

$$D_\phi(x, y) \geq \frac{1}{2} \|x - y\|^2 \quad \forall x, y \in Q. \tag{2.1}$$

Particularly, $\phi(x) = 1/2 \|x\|^2$ leads to $D_\phi(x, y) = D_\phi(y, x) = 1/2 \|x - y\|^2$, which boils down to the standard Euclidean setting. In addition, we have the following three-term identity; see [8, Lemma 3.2] or [15, Lemma 3.3].

**Lemma 2.1** ([8, 15]). For any $x, y, z \in Q$, it holds that

$$\langle \nabla \phi(x) - \nabla \phi(y), y - z \rangle = D_\phi(z, x) - D_\phi(z, y) - D_\phi(y, x). \tag{2.2}$$

If $\phi(x) = 1/2 \|x\|^2$, then

$$\langle \nabla \phi(x) - \nabla \phi(y), y - z \rangle = 2 \langle x - y, y - z \rangle = \|x-z\|^2 - \|y-z\|^2 - \|x-y\|^2. \tag{2.3}$$

2.3. Hölder continuity. Let $h$ be any differentiable function on $Q$. For $0 \leq \nu \leq 1$, define

$$M_\nu(h) := \sup_{x \neq y} \frac{\|\nabla h(x) - \nabla h(y)\|}{\|x - y\|^\nu}.$$ 

If $M_\nu(h) < \infty$, then $\nabla h$ is Hölder continuous with exponent $\nu$:

$$\|\nabla h(x) - \nabla h(y)\| \leq M_\nu(h) \|x - y\|^\nu \quad \forall x, y \in Q, \tag{2.4}$$

and this also implies

$$h(x) \leq h(y) + \langle \nabla h(y), x - y \rangle + \frac{M_\nu(h)}{1 + \nu} \|x - y\|^{1+\nu} \quad \forall x, y \in Q. \tag{2.5}$$
For $\nu = 1$, $M_1(h)$ corresponds to the Lipschitz constant of $\nabla h$, and we also use the conventional notation $L_h = M_1(h)$.

According to [49, Lemma 2], the estimate (2.5) can be transferred into the usual gradient descent inequality, with “inexact computations”. Based on this, (accelerated) gradient methods can be used to minimize functions with Hölder continuous gradients [13, 55, 56, 57].

**Proposition 2.2 ([49]).** Assume $M_\nu(h) < \infty$ and define

$$M(\nu, \delta) := \delta^{\frac{\nu}{\nu+1}} [M_\nu(h)]^{\frac{1}{\nu+1}} \quad \forall \delta > 0.$$  

Then for any $M \geq M(\nu, \delta)$, we have

$$h(x) \leq h(y) + \langle \nabla h(y), x - y \rangle + \frac{M}{2} \| x - y \|^2 + \frac{\delta}{2} \quad \forall x, y \in Q.$$

3. **Main Algorithm.** Throughout, we make the following assumption on $f = h + g$:

Assumption 3.1. The nonsmooth part $g$ is properly closed and convex on $Q$. The smooth part $h$ satisfies $\inf_{0 \leq \nu \leq 1} M_\nu(h) < \infty$ and is $\mu$-convex on $Q$ with $\mu \geq 0$, i.e.,

$$h(x) \geq h(y) + \langle \nabla h(y), x - y \rangle + \mu D_\phi(x, y) \quad \forall x, y \in Q.$$

**Algorithm 3.1** Universal Accelerated Primal-Dual (UAPD) Method

**Input:** $\beta_0 = 1$, $\gamma_0$, $M_0 > 0$, $\mu \geq 0$ and $\| A \|$.  
1. Initialization: $x_0, v_0 \in Q$ and $\lambda_0 \in \mathbb{R}^m$.  
2. for $k = 0, 1, \ldots$ do  
3. Set $i_k = 0$, $M_{k,0} = M_k$ and $S_k = \{x_k, v_k, \lambda_k, \beta_k, \gamma_k\}$.  
4. $(y_{k,i_k}, x_{k,i_k}, v_{k,i_k}, \alpha_{k,i_k}, \delta_{k,i_k}, \Delta_{k,i_k}) = \text{sub-UAPD}(k, S_k, M_{k,i_k})$.  
5. while $h(x_{k,i_k}) - \Delta_{k,i_k} > \delta_{k,i_k}/2$ do {Line search}  
6. Set $i_k = i_k + 1$ and $M_{k,i_k} = 2^i M_{k,0}$.  
7. $(y_{k,i_k}, x_{k,i_k}, v_{k,i_k}, \alpha_{k,i_k}, \delta_{k,i_k}, \Delta_{k,i_k}) = \text{sub-UAPD}(k, S_k, M_{k,i_k})$.  
8. end while  
9. Set $\alpha_k = \alpha_{k,i_k}$, $M_{k+1} = M_{k,i_k}$ and $\delta_{k+1} = \delta_{k,i_k}$.  
10. Update $\gamma_{k+1} = (\gamma_k + \mu \alpha_k)/(1 + \alpha_k)$ and $\beta_{k+1} = \beta_k/(1 + \alpha_k)$.  
11. Update $x_{k+1} = x_{k,i_k} + \gamma_k A v_{k,i_k}$ and $\lambda_{k+1} = \lambda_k + \alpha_k/\beta_k (A v_{k,i_k} - b)$.  
12. end for

**Algorithm 3.2** $(\bar{y}_k, \bar{x}_k, \bar{v}_k, \bar{\alpha}_k, \bar{\delta}_k, \bar{\Delta}_k) = \text{sub-UAPD}(k, S_k, \bar{M}_k)$

**Input:** $k \in \mathbb{N}$, $M_k > 0$ and $S_k = \{x_k, v_k, \lambda_k, \beta_k, \gamma_k\}$.  
1. Choose the step size $\tilde{\alpha}_k = \sqrt{\beta_k \gamma_k}/\sqrt{\beta_k M_k + \| A \| ^2}$.  
2. Set $\tilde{\beta}_k = \beta_k/(1 + \tilde{\alpha}_k)$ and $\tilde{\delta}_k = \tilde{\beta}_k/(k + 1)$.  
3. Set $\tilde{y}_k = (x_k + \tilde{\alpha}_k v_k)/(1 + \tilde{\alpha}_k)$ and $\tilde{\lambda}_k = \lambda_k + \tilde{\alpha}_k/\beta_k (A v_k - b)$.  
4. Update $\tilde{x}_k = (x_k + \tilde{\alpha}_k \tilde{v}_k)/(1 + \tilde{\alpha}_k)$ with

$$\tilde{v}_k = \underset{v \in Q}{\arg \min} \left\{ g(v) + \langle \nabla \tilde{h}(\tilde{y}_k), A^T \tilde{\lambda}_k, v \rangle + \mu D_\phi(v, \tilde{y}_k) + \frac{\gamma_k}{\tilde{\alpha}_k} D_\phi(v, v_k) \right\}.$$  
5. Compute $\bar{\Delta}_k = h(\bar{y}_k) + \langle \nabla h(\bar{y}_k), \bar{x}_k - \bar{y}_k \rangle + \frac{\bar{M}_k}{2} \| \bar{x}_k - \bar{y}_k \|^2$.  

Our main algorithm, called Universal Accelerated Primal-Dual (UAPD) method, is summarized in Algorithm 3.1, where the subpart sub-UAPD in lines 4 and 7 has been given by Algorithm 3.2. Note that we do not require priorly the smoothness constant $M(h)$ but perform a line search procedure.

3.1. Line search. From line 5 of Algorithm 3.1, we find that $i_k$ is the smallest integer such that
\[
h(x_{k,i_k}) \leq h(y_{k,i_k}) + \langle \nabla h(y_{k,i_k}), x_{k,i_k} - y_{k,i_k} \rangle + \frac{M_{k,i_k}}{2} \|x_{k,i_k} - y_{k,i_k}\|^2 + \frac{\delta_{k,i_k}}{2}.
\]

We claim that $i_k$ is finite for each $k \in \mathbb{N}$. Indeed, $M_{k,i_k} = 2^{i_k} M_{k,0}$ increases as $i_k$ does, and the step size
\[
\alpha_{k,i_k} = \frac{\sqrt{\beta_k \gamma_k}}{\sqrt{\beta_k M_{k,i_k} + \|A\|^2}}
\]
has to be decreasing. Thus the tolerance
\[
\delta_{k,i_k} = \frac{1}{k+1} \cdot \frac{\beta_k}{1 + \alpha_{k,i_k}}
\]
is increasing and by (2.6), $M(\nu, \delta_{k,i_k})$ is decreasing. This together with Proposition 2.2 and Assumption 3.1 concludes that either $i_k = 0$ or $1 \leq i_k \leq s^* + 1$ where $s^* \geq 0$ solves $M_{k,s^*} = M(\nu, \delta_{k,s^*})$; see Figure 1. Moreover, we notice that
\[
M_{k,s^*} = 2^{s^*} M_{k,0} \leq M(\nu, \delta_{k,0}) \implies s^* \leq \log_2 \frac{M(\nu, \delta_{k,0})}{M_{k,0}} < \infty.
\]

**Remark 3.2.** In the line search part, Algorithm 3.1 adopts dynamically decreasing tolerance (3.2), i.e., $\delta_k = \beta_k / k$. However, the methods in [49] and [75, Algorithm 2] chose $\delta_k = \epsilon / k$, where $\epsilon$ is the desired accuracy. Hence, by Proposition 2.2, the approximate smoothness constant $M_k$ of our algorithm is smaller than these two methods, especially for Hölderian case. This will be verified by numerical experiments.
Below, we give an upper bound of $M_k$ and the total number of line search steps. By Theorem 3.5, $\beta_k$ corresponds to the convergence rate of Algorithm 3.1 and admits explicit decay estimate with respect to $k$ (see Lemma 3.7). If the desired accuracy $\beta_{k+1} = O(\epsilon)$ is given, then the term $|log_2 \beta_{k+1}|$ in (3.5) can also be replaced by $|log_2 \epsilon|$.

**Lemma 3.3.** For any $k \in \mathbb{N}$, we have

$$M_{k+1} \leq \max \left\{ 2\sqrt{2} M(\nu, \delta_{k+1}), M_0 \right\},$$

and consequently, it holds that

$$\sum_{j=0}^{k} i_j \leq k + 1 + \max \left\{ 1, \log_2 \frac{M(\nu, \delta_{k+1})}{M_0/2\sqrt{2}} \right\},$$

where

$$\log_2 \frac{M(\nu, \delta_{k+1})}{M_0/2\sqrt{2}} = \log_2 \left[ \frac{M(\nu, h)}{M_0/2\sqrt{2}} \right] + \frac{1 - \nu}{1 + \nu} \left[ \log_2 (k + 1) + |log_2 \beta_{k+1}| \right].$$

**Proof.** See Appendix A.

### 3.2. Time discretization interpretation.

Below, we provide a time discretization interpretation of Algorithm 3.1. Given the $k$-th iterations $(x_k, v_k, \lambda_k)$ and the parameters $(\gamma_k, \beta_k, M_k)$, the line search procedure produces $(y_k, x_{k+1}, v_{k+1})$ that satisfy

\begin{align}
\gamma_k \frac{\nabla \phi(v_{k+1}) - \nabla \phi(v_k)}{\alpha_k} &\in \mu \left[ \nabla \phi(y_k) - \nabla \phi(v_{k+1}) \right] - G(y_k, v_{k+1}, \lambda_k), \\
\alpha_k &\frac{x_{k+1} - x_k}{\alpha_k} = v_{k+1} - x_{k+1}, \\
\beta_k &\frac{\lambda_{k+1} - \lambda_k}{\alpha_k} = Av_{k+1} - b,
\end{align}

where $G(y_k, v_{k+1}, \lambda_k) := \nabla h(y_k) + \partial g(v_{k+1}) + N_Q(v_{k+1}) + A^\top \hat{\lambda}_k$ with $\hat{\lambda}_k = \lambda_k + \alpha_k/\beta_k (Av_k - b)$, and the step size $\alpha_k$ solves (cf. (3.1))

$$\alpha_k^2 (\beta_k M_{k+1} + \|A\|^2) = \gamma_k \beta_k.$$

Besides, $y_k$ and $x_{k+1}$ satisfy

$$h(x_{k+1}) \leq h(y_k) + \left( \nabla h(y_k), x_{k+1} - y_k \right) + \frac{M_{k+1}}{2} \|x_{k+1} - y_k\|^2 + \frac{\delta_{k+1}}{2},$$

and the parameters $(\gamma_{k+1}, \beta_{k+1})$ are governed by

$$\gamma_{k+1} - \gamma_k \frac{\alpha_k}{\alpha_k} = \mu - \gamma_{k+1}, \quad \beta_{k+1} - \beta_k \frac{\alpha_k}{\alpha_k} = -\beta_{k+1},$$

with $\beta_0 = 1$ and $\gamma_0 > 0$.

As one can see, $y_k$ in (3.6a) is an intermediate which provides a “prediction”, and then the “correction” step (3.6c) is used to update $x_{k+1}$. From (3.6a), (3.6b), and (3.6c), it is not hard to find that $y_k, v_{k+1}, x_{k+1} \in Q$, as long as $x_k, v_k \in Q$. Therefore, with $x_0, v_0 \in Q$, it holds that $\{x_k, y_k, v_k\}_{k \in \mathbb{N}} \subset Q$. 
Furthermore, we mention that the reformulation (3.6) admits an implicit-explicit time discretization for the following primal-dual dynamics:

\[
\begin{aligned}
&x' = v - x, \\
&\gamma \frac{d}{dt} \nabla \phi(v) \in \mu (\nabla \phi(x) - \nabla \phi(v)) - (\partial f(x) + N_Q(x) + A^T \lambda), \\
&\beta \lambda' = Av - b,
\end{aligned}
\]

where \(\gamma\) and \(\beta\) are governed by continuous analogues to (3.9):

\[
\gamma' = \mu - \gamma, \quad \beta' = -\beta.
\]

We call (3.10) the Accelerated Bregman Primal-Dual (ABPD) flow. In the standard Euclidean setting \(\phi(x) = 1/2 \|x\|^2\), it amounts to the accelerated primal-dual flow proposed in [39]. For well-posedness and exponential decay estimate of (3.10) with smooth objective \(f\) and general prox-function \(\phi\), we refer to Appendix B.

3.3. A universal estimate. Let \(\{(x_k, v_k, \lambda_k, \gamma_k, \beta_k)\}_{k \in \mathbb{N}}\) be the sequence generated from Algorithm 3.1. We introduce the discrete Lyapunov function

\[
E_k := \mathcal{L}(x_k, \lambda^*) - \mathcal{L}(x^*, \lambda_k) + \gamma_k D_\phi(x^*, v_k) + \frac{\beta_k}{2} \|\lambda_k - \lambda^*\|^2.
\]

A one-step estimate is presented below.

**Lemma 3.4.** Under Assumption 3.1, we have

\[
E_{k+1} - E_k \leq -\alpha_k E_{k+1} + \frac{\delta_{k+1}}{2}(1 + \alpha_k) \quad \forall k \in \mathbb{N}.
\]

**Proof.** See section 4. \(\square\)

Using this lemma, we obtain the following theorem, which says the final convergence rate is given by the sharp decay estimate of the sequence \(\{\beta_k\}_{k \in \mathbb{N}}\); see Lemma 3.7.

**Theorem 3.5.** Under Assumption 3.1, we have \(\{x_k, v_k\}_{k \in \mathbb{N}} \subset Q\) and

\[
\begin{align*}
&\|Ax_k - b\| \leq \beta_k T_0,k, \\
&|f(x_k) - f(x^*)| \leq \beta_k W_{0,k}, \\
&\mathcal{L}(x_k, \lambda^*) - \mathcal{L}(x^*, \lambda_k) \leq \beta_k R_{0,k},
\end{align*}
\]

for all \(k \in \mathbb{N}\), where \(R_{0,k} := E_0 + \ln(k + 1)\), \(T_0,k := \|Ax_0 - b\| + 2 \sqrt{2R_{0,k}}\) and \(W_{0,k} := R_{0,k} + \|\lambda^*\| T_0,k\). Moreover, if \(\mu > 0\), then

\[
\gamma_{\min} \|v_k - x^*\|^2 + \mu \|x_k - x^*\|^2 \leq 2\beta_k R_{0,k},
\]

where \(\gamma_{\min} := \min\{\gamma_0, \mu\}\).

**Proof.** From (3.9) and the contraction estimate (3.13) follows immediately that

\[
E_{k+1} \leq \frac{1}{1 + \alpha_k} E_k + \frac{\delta_{k+1}}{2} \quad \Rightarrow \quad E_k \leq \beta_k E_0 + \frac{\beta_k}{2} \sum_{i=0}^{k-1} \delta_{i+1}.
\]

By (3.2), we have \(\delta_{k+1} = \beta_{k+1}/(k + 1)\), which further implies

\[
E_k \leq \beta_k E_0 + \frac{\beta_k}{2} \sum_{i=0}^{k-1} \frac{1}{i + 1} \leq \beta_k [E_0 + \ln(k + 1)].
\]

This proves (3.16) and (3.17). Following the proof of [39, Theorem 3.1], it is not hard to establish (3.14) and (3.15). Hence, we conclude the proof of this theorem. \(\square\)
REMARK 3.6. Note that the choice (3.2) can be replaced with
\[
\delta_{k,i_k} = \frac{\delta}{k+1} \cdot \frac{\beta_k}{1 + \alpha_{k,i_k}}, \quad \delta > 0.
\]
Then the item \( \mathcal{R}_{\alpha,k} \) in Theorem 3.5 becomes \( \mathcal{R}_{\alpha,k} = \mathcal{E}_0 + \delta \ln(k+1) \) and \( \delta = 1/\ln(K+1) \) cancels the logarithm factor, where \( K \in \mathbb{N} \) is the number of iterations chosen in advance.

It remains to establish the decay estimate of \( \{\beta_k\}_{k \in \mathbb{N}} \). From (3.7) and (3.9), we obtain
\[
(3.19) \quad \beta_{k+1} - \beta_k = -\sqrt{\gamma_k \beta_k \beta_{k+1}} / \sqrt{\beta_k M_{k+1} + \|A\|^2}.
\]
A careful investigation into this difference equation gives the desired result.

**LEMMA 3.7.** Assume that \( M_0 \leq [M_\nu(h)]^{\frac{1}{\nu+\gamma}} \) and \( \max\{\gamma_0, \mu\} \leq \|A\|^2 \). If \( \mu = 0 \), then
\[
(3.20) \quad \beta_k \leq C_\nu \left( \frac{\|A\|}{\gamma_0^k} + \frac{M_\nu(h)}{\gamma_0^k k^{\frac{1}{\nu+\gamma}}} \right) \quad \forall \ k \geq 1,
\]
and if \( \mu > 0 \), then for all \( k \geq 1 \), we have
\[
(3.21) \quad \beta_k \leq C_\nu \begin{cases}
\frac{\|A\|^2}{\gamma_{\min} k^2} + \frac{[M_\nu(h)]^{\frac{1}{\nu+\gamma}}}{\gamma_{\min}^{\frac{1}{\nu+\gamma}}} & \text{if } \nu < 1, \\
\frac{\|A\|^2}{\gamma_{\min} k^2} + \exp \left( -\frac{k}{8\sqrt{3}} \sqrt{\frac{\gamma_{\min}}{L_h}} \right) & \text{if } \nu = 1,
\end{cases}
\]
where \( \gamma_{\min} = \min\{\gamma_0, \mu\} \) and \( C_\nu > 0 \) depends only on \( \nu \).

**Proof.** By Lemma 3.3, we have
\[
M_k \leq \max \left\{ 2\sqrt{2} M(\nu, \delta_k), M_0 \right\} \quad \forall \ k \geq 1.
\]
In view of \( \delta_1 = \beta_1 = 1/(1 + \alpha_0) \leq 1 \), it follows immediately that
\[
M(\nu, \delta_1) \geq M(\nu, 1) = [M_\nu(h)]^{\frac{1}{\nu+\gamma}} \geq M_0.
\]
Since \( \delta_k = \beta_k/k \) and \( \beta_k \) is decreasing, it holds that \( M(\nu, \delta_1) \leq M(\nu, \delta_k) \) and \( M_k \leq 2\sqrt{2} M(\nu, \delta_k) \). Plugging this into (3.19) gives
\[
(3.22) \quad \beta_{k+1} - \beta_k \leq -\frac{\sqrt{\gamma_k \beta_k \beta_{k+1}}}{\sqrt{2\sqrt{2} \beta_k M(\nu, \delta_{k+1}) + \|A\|^2}}.
\]
Based on this difference inequality, we obtain (3.20) and (3.21). Missing proofs are provided in section 5.

According to the universal mixed-type estimate established in Lemma 3.7, our Algorithm 3.1 achieves the optimal complexity bound for both the unconstrained case \( A = O \) and affinely constrained case \( A \neq O \), with Hölderian smoothness exponent \( \nu \in [0, 1] \). Detailed comparisons with existing results are summarized in order.

**REMARK 3.8.** Consider first the unconstrained case: \( A = O \).
\begin{itemize}
  \item **The Lipschitzian case** $\nu = 1$:
    \[
    \min \left\{ \sqrt{\frac{L}{\mu}} \cdot |\ln \epsilon|, \sqrt{\frac{L}{\mu}} \cdot |\ln \epsilon| \right\}.
    \]
    This is the well-known optimal complexity bound (cf.\cite{46, 50}) of first-order methods for smooth convex functions with Lipschitz continuous gradients; see \cite{9, 10, 38, 42, 48}.
  \item **The Hölderian case** $0 \leq \nu < 1$:
    \[
    \min \left\{ \left( \frac{M_\nu(h)}{\epsilon} \right)^{\frac{2}{1+3\nu}}, \left( \frac{M_\nu(h)}{\mu} \right)^{\frac{2}{1+3\nu}} \cdot \left( \frac{\mu}{\epsilon} \right)^{\frac{1}{1+3\nu}} \right\}.
    \]
    This matches the lower bound in \cite{44, 45}. The convex case $\mu = 0$ has been obtained by the methods in \cite{31, 44, 49}, and the restarted schemes in \cite{31, 53} attained the complexity bound for $\mu > 0$. Besides, Guminov et al. \cite{23} obtained (3.23) for nonconvex problems, with an additional 1D line search.
  \end{itemize}

**Remark 3.9.** Then let us focus on the affine constraint case: $A \neq O$.

\begin{itemize}
  \item **The Lipschitzian case** $\nu = 1$:
    \[
    \min \left\{ \frac{\|A\|}{\epsilon} + \sqrt{\frac{L}{\mu}} \cdot |\ln \epsilon|, \frac{\|A\|}{\sqrt{\mu \epsilon}} + \sqrt{\frac{L}{\mu}} \cdot |\ln \epsilon| \right\}.
    \]
    This coincides with the lower complexity bound in \cite{52}. The methods in \cite{11, 47, 73} achieved the bound for convex case $\mu = 0$, and the strongly convex case $\mu > 0$ can be found in \cite{73}.
  \item **The Hölderian case** $0 \leq \nu < 1$:
    \[
    \min \left\{ \frac{\|A\|}{\epsilon} + \left( \frac{M_\nu(h)}{\epsilon} \right)^{\frac{2}{1+3\nu}}, \frac{\|A\|}{\sqrt{\mu \epsilon}} + \left( \frac{M_\nu(h)}{\mu} \right)^{\frac{2}{1+3\nu}} \cdot \left( \frac{\mu}{\epsilon} \right)^{\frac{1}{1+3\nu}} \right\}.
    \]
    Similarly with (3.24), this universal mixed-type estimate has optimal dependence on $\|A\|$ (corresponding to the affine constraint), and the remainder agrees with (3.23), which is optimal with respect to $\mu$ and $M_\nu(h)$ (related to the objective $f$).
  \end{itemize}

4. Proof of Lemma 3.4. Let us start from the difference $\mathcal{E}_{k+1} - \mathcal{E}_k = I_1 + I_2 + I_3$, where

\[
\begin{aligned}
I_1 &:= \mathcal{L}(x_{k+1}, \lambda^*) - \mathcal{L}(x_k, \lambda^*), \\
I_2 &:= \gamma_{k+1} D_\phi(x^*, v_{k+1}) - \gamma_k D_\phi(x^*, v_k), \\
I_3 &:= \frac{\beta_{k+1}}{2} \|\lambda_{k+1} - \lambda^*\|^2 - \frac{\beta_k}{2} \|\lambda_k - \lambda^*\|^2.
\end{aligned}
\]

Notice that $x_k, x_{k+1} \in Q$ and the first term is easy to handle:

\[
I_1 = f(x_{k+1}) - f(x_k) + \langle \lambda^*, A(x_{k+1} - x_k) \rangle.
\]

We derive the estimate of $I_2$ in subsection 4.1 and finish the proof of (3.13) in subsection 4.2.
4.1. Estimate of $I_2$. Invoking the three-term identity (2.2) and the difference equation of $\{\gamma_k\}_{k \in \mathbb{N}}$ in (3.9), we split the second term $I_2$ as follows:

$$I_2 = (\gamma_{k+1} - \gamma_k)D_\phi(x^*, v_{k+1}) + \gamma_k [D_\phi(x^*, v_{k+1}) - D_\phi(x^*, v_k)]$$

$$= \alpha_k (\mu - \gamma_{k+1})D_\phi(x^*, v_{k+1}) - \gamma_k D_\phi(v_{k+1}, v_k) + \gamma_k (\nabla_\phi(v_{k+1}) - \nabla_\phi(v_k), v_{k+1} - x^*).$$

Let us prove

$$\mu \alpha_k D_\phi(x^*, v_{k+1}) + \gamma_k (\nabla_\phi(v_{k+1}) - \nabla_\phi(v_k), v_{k+1} - x^*)$$

$$\leq h(x_k) - h(y_k) - \alpha_k [h(y_k) - h(x^*) + \langle \lambda_k, A v_{k+1} - b \rangle]$$

$$- \alpha_k [g(v_{k+1}) - g(x^*) + \langle \nabla h(y_k), v_{k+1} - v_k \rangle],$$

which leads to the desired estimate of $I_2$:

$$I_2 \leq - \alpha_k \gamma_{k+1} D_\phi(x^*, v_{k+1}) - \gamma_k D_\phi(v_{k+1}, v_k) - \alpha_k \langle \lambda_k, A v_{k+1} - b \rangle$$

$$- \alpha_k [g(v_{k+1}) - g(x^*) + h(y_k) - h(x^*)]$$

$$+ h(x_k) - h(y_k) - \alpha_k \langle \nabla h(y_k), v_{k+1} - v_k \rangle.$$

To do this, define $\zeta_{k+1}$ by that

$$\gamma_k [\nabla_\phi(v_{k+1}) - \nabla_\phi(v_k)]$$

$$= \mu \alpha_k [\nabla_\phi(y_k) - \nabla_\phi(v_{k+1})] - \alpha_k [\nabla h(y_k) + \zeta_{k+1} + A^T \lambda_k].$$

Observing (3.6b), it follows that $\zeta_{k+1} \in \partial g(v_{k+1}) + N_Q(v_{k+1})$ and

$$- \alpha_k \langle \zeta_{k+1}, v_{k+1} - x^* \rangle \leq - \alpha_k [g(v_{k+1}) - g(x^*)].$$

Thanks to (2.2), we have the decomposition

$$\mu \alpha_k (\nabla_\phi(y_k) - \nabla_\phi(v_{k+1}), v_{k+1} - x^*)$$

$$= \mu \alpha_k [D_\phi(x^*, y_k) - D_\phi(x^*, v_{k+1}) - D_\phi(v_{k+1}, y_k)],$$

and invoking (3.6a) leads to

$$- \alpha_k \langle \nabla h(y_k), v_{k+1} - x^* \rangle$$

$$= - \alpha_k \langle \nabla h(y_k), v_{k+1} - v_k \rangle - \langle \nabla h(y_k), y_k - x_k \rangle - \alpha_k \langle \nabla h(y_k), y_k - x^* \rangle.$$

Since $x_k, y_k \in Q$, by Assumption 3.1 we obtain

$$- \langle \nabla h(y_k), y_k - x_k \rangle - \alpha_k \langle \nabla h(y_k), y_k - x^* \rangle$$

$$\leq h(x_k) - h(y_k) - \alpha_k [h(y_k) - h(x^*) + \mu D_\phi(x^*, y_k)].$$

Hence, combining the above estimates with (4.4) proves (4.2).

4.2. Proof of (3.13). Similarly as before, by (2.2), (3.6d) and (3.9), the third term $I_3$ is rearranged by that

$$I_3 = - \frac{\alpha_k \beta_{k+1}}{2} \|\lambda_{k+1} - \lambda^*\|^2 - \frac{\beta_k}{2} \|\lambda_{k+1} - \lambda_k\|^2$$

$$+ \alpha_k \langle Av_{k+1} - b, \lambda_{k+1} - \lambda^* \rangle.$$
To match the cross term \(-\alpha_k \langle \hat{\lambda}_k, Av_{k+1} - b \rangle\) in the estimate of \(I_2\) (cf.\( (4.3)\)), we rewrite the last term as follows
\[
\alpha_k \langle Av_{k+1} - b, \lambda_{k+1} - \lambda^* \rangle = \alpha_k \langle Av_{k+1} - b, \lambda_{k+1} - \hat{\lambda}_k \rangle + \alpha_k \langle Av_{k+1} - b, \hat{\lambda}_k - \lambda^* \rangle.
\]
In view of \((2.3)\) and \((3.6d)\), we get
\[
\alpha_k \langle Av_{k+1} - b, \lambda_{k+1} - \hat{\lambda}_k \rangle = \beta_k \left( \lambda_{k+1} - \lambda_k, \lambda_{k+1} - \hat{\lambda}_k \right)
= \frac{\beta_k}{2} \|\lambda_{k+1} - \lambda_k\|^2 + \frac{\beta_k}{2} \|\lambda_{k+1} - \hat{\lambda}_k\|^2 - \frac{\beta_k}{2} \|\lambda_k - \hat{\lambda}_k\|^2,
\]
which gives
\[
I_3 \leq -\frac{\alpha_k \beta_{k+1}}{2} \|\lambda_{k+1} - \lambda^*\|^2 + \frac{\beta_k}{2} \|\lambda_{k+1} - \hat{\lambda}_k\|^2 + \alpha_k \langle Av_{k+1} - b, \hat{\lambda}_k - \lambda^* \rangle.
\]
Therefore, collecting this with \((4.1)\) and \((4.3)\) yields
\[
E_{k+1} - E_k \leq -\alpha_k E_{k+1} + \frac{\beta_k}{2} \|\lambda_{k+1} - \hat{\lambda}_k\|^2 - \gamma_k D_{\phi}(v_{k+1}, v_k)
+ (1 + \alpha_k) \left[ h(x_{k+1}) - h(y_k) \right] - \alpha_k \langle \nabla h(y_k), v_{k+1} - v_k \rangle
+ (1 + \alpha_k) g(x_{k+1}) - g(x_k) - \alpha_k g(v_{k+1}).
\]
From \((3.6c)\), we see that \(x_{k+1}\) is a convex combination of \(x_k\) and \(v_{k+1}\), which implies
\[
(1 + \alpha_k) g(x_{k+1}) \leq g(x_k) + \alpha_k g(v_{k+1}).
\]
Thanks to \((3.8)\) and the relation \(\alpha_k (v_{k+1} - v_k) = (1 + \alpha_k) (x_{k+1} - y_k)\) (cf.\( (3.6a)\) and \((3.6c)\)), we obtain
\[
(1 + \alpha_k) \left[ h(x_{k+1}) - h(y_k) \right] - \alpha_k \langle \nabla h(y_k), v_{k+1} - v_k \rangle
\leq \frac{\alpha_k^2 M_{k+1}}{2 + 2\alpha_k} \|v_{k+1} - v_k\|^2 + \frac{\delta_{k+1}}{2} (1 + \alpha_k).
\]
Consequently, applying \((2.1)\) leads to
\[
E_{k+1} - E_k \leq -\alpha_k E_{k+1} + \frac{\delta_{k+1}}{2} (1 + \alpha_k) + \frac{\beta_k}{2} \|\lambda_{k+1} - \hat{\lambda}_k\|^2 + \frac{\alpha_k^2 M_{k+1} - \gamma_k (1 + \alpha_k)}{1 + \alpha_k} D_{\phi}(v_{k+1}, v_k).
\]
Recall that \(\hat{\lambda}_k = \lambda_k + \alpha_k/\beta_k (Av_k - b)\), which together with \((3.6d)\) gives \(\lambda_{k+1} - \hat{\lambda}_k = \alpha_k/\beta_k A(v_{k+1} - v_k)\) and
\[
E_{k+1} - E_k \leq -\alpha_k E_{k+1} + \frac{\delta_{k+1}}{2} (1 + \alpha_k) + \frac{1}{\beta_k} \left[ \alpha_k^2 (\beta_{k+1} M_{k+1} + \|A\|^2) - \gamma_k \beta_k \right] D_{\phi}(v_{k+1}, v_k).
\]
Since \(\beta_{k+1} \leq \beta_k\) (cf.\( (3.9)\)), the desired estimate \((3.13)\) follows immediately from \((3.7)\). This finishes the proof of Lemma 3.4.
5. Proof of Lemma 3.7. The key to complete the proof of Lemma 3.7 is the difference inequality (3.22). In subsection 5.1, we shall introduce an auxiliary differential inequality (cf.(5.1)) that can be viewed as a continuous analogue to (3.22). Later in subsections 5.2 and 5.3, we finish the proofs of (3.20) and (3.21) by using the asymptotic estimate of (5.1).

5.1. A differential inequality. Let $$\eta, R \geq 0$$ and $$\theta > 1$$ be real constants such that $$\eta \leq \theta - 1$$. Assume $$y \in W^{1,\infty}(0, \infty)$$ is positive and satisfies the differential inequality

$$y'(t) \leq -\frac{\sigma(t)y^\theta(t)}{\sqrt{\varphi(t)}y^{2\eta}(t) + R^2} \quad y(0) = 1,$$

where $$\sigma \in L^1(0, \infty)$$ is nonnegative, and $$\varphi \in C^1[0, \infty)$$ is positive and nondecreasing. Plugging the trivial estimate $$\sqrt{\varphi(t)}y^{2\eta}(t) + R^2 \leq \sqrt{\varphi(t)}y^{\eta}(t) + R$$ into (5.1) gives

$$\left(\frac{\sqrt{\varphi(t)}}{y^{\eta - \eta}(t)} + \frac{R}{y^\theta(t)}\right)y'(t) \leq -\sigma(t).$$

The decay estimate of $$y(t)$$ is given below. Detailed proof can be found in Appendix C.

**Lemma 5.1.** Assume $$y \in W^{1,\infty}(0, \infty)$$ is positive and satisfies (5.1). Then for all $$t > 0$$, we have

$$y(t) \leq C_{\theta,\eta} \begin{cases} \left(\frac{\sqrt{\varphi(t)}}{\Sigma(t)}\right)^{\eta - \frac{1}{\theta - 1}} + \left(\frac{R}{\Sigma(t)}\right)^{\frac{1}{\theta - 1}} & \text{if } \eta < \theta - 1, \\ \exp\left(-\frac{\Sigma(t)}{2\sqrt{\varphi(t)}}\right) + \left(\frac{R}{\Sigma(t)}\right)^{\frac{1}{\theta - 1}} & \text{if } \eta = \theta - 1, \end{cases}$$

where $$\Sigma(t) := \int_0^t \sigma(s)\,ds$$ and $$C_{\theta,\eta} > 0$$ depends only on $$\theta$$ and $$\eta$$.

5.2. Proof of (3.20). In this case, by (3.9), we have $$\gamma_k = \gamma_0\beta_k$$ and (3.22) becomes

$$\beta_{k+1} - \beta_k \leq -\frac{\sqrt{\gamma_0\delta_k\beta_k}}{\sqrt{2\sqrt{2}\beta_kM(\nu, \delta_{k+1}) + \|A\|^2}},$$

where $$M(\nu, \delta_{k+1}) = \delta_{k+1}^{\nu-1} [M_\nu(h)]^{\nu-1}$$ with $$\delta_{k+1} = \beta_{k+1}/(k + 1)$$.

Define a piecewise continuous linear interpolation

$$y(t) := \beta_k(k + 1 - t) + \beta_{k+1}(t - k) \quad \forall t \in [k, k + 1), k \in \mathbb{N}.$$

Clearly, $$y \in W^{1,\infty}(0, \infty)$$ is positive and $$0 < y(t) \leq y(0) = 1$$. In particular, we have $$\beta_k = y(k)$$ for all $$k \in \mathbb{N}$$, and the decay estimate of $$\beta_k$$ is transferred into the asymptotic behavior of $$y(t)$$, which satisfies

$$y'(t) \leq -\frac{\sqrt{\gamma_0/2}y^2(t)}{\sqrt{8\sqrt{2}\varphi(t)|y(t)|^{\frac{1}{\nu - 1}} + \|A\|^2}},$$

where $$\varphi(t) := (t + 1)^{\frac{\nu - 1}{\nu}} [M_\nu(h)]^{\nu-1}$$. Thus, utilizing Lemma 5.1 gives

$$\beta_k = y(k) \leq C_{\nu} \left(\frac{\|A\|}{\sqrt{\gamma_0k}} + \frac{M_\nu(h)}{\gamma_0^{1+\frac{1}{\nu-1}}k^{1+\frac{3}{\nu}}}\right) \quad \forall k \geq 1,$$
where $C_{\nu} > 0$ depends only on $\nu$. This establishes (3.20).

Below, let us verify (5.5). Since $\gamma_k \leq \max\{\gamma_0, \mu\} \leq \|A\|^2$, from (3.7) we find that

$$\alpha_k \leq \sqrt{\gamma_k \beta_k / \|A\|} \leq 1 \quad \forall k \in \mathbb{N}.$$ 

For any $t \in (k, k + 1)$, it is clear that

$$1 \geq \frac{\beta_{k+1}}{y(t)} \geq \frac{\beta_{k+1}}{\beta_k} = \frac{1}{1 + \alpha_k} \geq \frac{1}{2},$$

and $1 \leq \frac{\beta_k}{\beta_{k+1}} \leq 2$,

which implies

$$\beta_k M(\nu, \delta_{k+1}) = \varphi(k)\beta_k\beta_{k+1}^{1/2} \leq 2\sqrt{\nu} \varphi(t)[y(t)]^{1/2}.\$$

Since $y'(t) = \beta_{k+1} - \beta_k$, plugging the above estimate into (5.3) proves (5.5).

5.3. Proof of (3.21). By (3.9), we have $\gamma_k \geq \gamma_{\min} = \min\{\gamma_0, \mu\}$, and (3.22) becomes

$$\beta_{k+1} - \beta_k \leq -\sqrt{\gamma_{\min} \beta_{k+1}}/ \sqrt{2\sqrt{2} \beta_k M(\nu, \delta_{k+1}) + \|A\|^2}.$$

Recall the piecewise interpolation $y(t)$ defined by (5.4). Similarly with (5.5), we claim that

$$y'(t) \leq -\sqrt{\gamma_{\min}/ 2} y^{3/2}(t) / \sqrt{\sqrt{8} \varphi(t)[y(t)]^{3/2} + \|A\|^2},$$

and invoking Lemma 5.1 again gives

$$\beta_k \leq C_{\nu} \left\{ \begin{array}{ll}
\|A\|^2 / \gamma_{\min} k^2 + \left[M_{\nu}(h)\right]^{1/2} / \gamma_{\min} k^{1/2} & \text{if } \nu < 1, \\
\|A\|^2 / \gamma_{\min} k^2 + \exp\left(-k / 8\sqrt{3} \sqrt{\gamma_{\min}/ L_h}\right) & \text{if } \nu = 1,
\end{array} \right.$$

which proves (3.21) and completes the proof of Lemma 3.7.

6. Numerical Examples. In this part, we provide several numerical tests to validate the performance of our Algorithm 3.1 (denoted shortly by UAPD). It is compared with Nesterov’s FGM [49] and the AccUniPDGrad method [75], respectively for unconstrained and affinely constrained problems.

Both UAPD and FGM involve the proximal mapping of the nonsmooth part $g$ under Bregman distance. However, FGM performs one more proximal calculation for updating $v_k$, and in line search part, FGM and AccUniPDGrad use the tolerance $\delta_k = \epsilon \tau_k$ with $\tau_k = O(1/k)$, which is smaller than ours $\delta_k = \beta_k/k$. As discussed previously in Remark 3.2, this will lead to over-estimate issue, especially for Hölderian case (cf. subsection 6.1) and smooth problems with large Lipschitz constants (cf. subsection 6.2).

6.1. Matrix game. The problem reads as

$$\min_{x \in \Delta_n} \max_{y \in \Delta_m} \langle x, Py \rangle = \min_{x \in \Delta_n} \left\{ h(x) := \max_{1 \leq j \leq m} \langle p_j, x \rangle \right\},$$

where
where $P = (p_1, p_2, \cdots, p_m) \in \mathbb{R}^{n \times m}$ is the given payoff matrix and $\Delta_x$ denotes the standard simplex with $x = m$ or $n$. According to von Neumann’s minimax theorem [1, Corollary 15.30], it is also equivalent to

$$
\max_{y \in \Delta_m} \left\{ \min_{1 \leq i \leq n} \langle e_i, Py \rangle \right\} = -\min_{y \in \Delta_m} \left\{ \min_{1 \leq i \leq n} \langle e_i, P y \rangle \right\} = -\min_{y \in \Delta_m} \left\{ g(y) := \max_{1 \leq i \leq n} \langle q_i, y \rangle \right\},
$$

where $P^T = -(q_1, q_2, \cdots, q_n) \in \mathbb{R}^{m \times n}$. As we do not the know the optimal value of $h$ and $g$, it is more convenient to consider

$$
(6.2) \quad \min_{x \in \Delta_n, y \in \Delta_m} \{ f(x, y) := h(x) + g(y) \}.
$$

Clearly, this problem is nonsmooth ($f$ is only Lipschitz continuous) and the minimal value is zero. A natural prox-function for this problem is the entropy $\phi(x) = \langle x, \ln x \rangle$.

---

**Fig. 2. Numerical performances of FGM and UAPD on the matrix game problem with $m = 100$, $n = 400$.**

We record (i) the decay behavior of the objective residual $|f_k| = |f(x_k)|$ (with respect to iteration number $k$ and running time $t$ in seconds), (ii) the total number $\#i_k$ of the line search step $i_k$, and (iii) the approximate Lipschitz constant $M_k$. The pay off matrix $P$ is generated from normal distribution and for FGM, we set the accuracy parameter $\epsilon = 1e-5$.

Numerical results are displayed in Figure 2, from which we see that our UAPD outperforms FGM, with faster convergence and smaller Lipschitz constants. The total number $\#i_k$ is close to each other. But, FGM produces over-estimated Lipschitz parameters with dramatically growth behavior since it adopts smaller tolerance $\epsilon/k$ for line search procedure.
Besides, we investigate the difference between Euclidean distance $\phi(x) = 1/2 \|x\|^2$ and entropy function $\phi(x) = \langle x, \ln x \rangle$. It is observed that these two cases are very similar in line search procedure but entropy function leads to better convergence rate.

6.2. Regularized matrix game problem. The problem (6.1) admits an approximation

$$f_\sigma(x) := \sigma \ln \left( \sum_{j=1}^{m} e^{(p_j, x)/\sigma} \right),$$

where $\sigma > 0$ denotes the smoothing parameter. This regularized objective is smoother than the original one. According to [47, Eq.(4.8)], we choose $\sigma = \epsilon/(2 \ln m)$, and the Lipschitz constant of $\nabla f_\sigma$ is $L_\sigma = \max_{i,j} |P_{i,j}|^2/(4\sigma)$. 

**Fig. 3.** Numerical results of UAPD on the matrix game problem with different prox-functions.

**Fig. 4.** Numerical performances of FGM and UAPD on regularized matrix game with $m = 100$, $n = 400$. 
We then apply UAPD and FGM (with $\epsilon = 1e-5$) to the smooth problem (6.3) and report the numerical outputs in Figure 4. The optimal value $f^*$ is obtained by running UAPD with enough iterations. Similarly as before, our UAPD is superior to FGM in convergence and approximate Lipschitz constant. Also, we plot the objective residuals of the original matrix game and find that with smoothing technique both two methods perform better than before.

6.3. Continuous Steiner problem. Let us consider one more unconstrained problem

\[ \min_{x \in \mathbb{R}^n} f(x) = \sum_{j=1}^{m} \|x - a_j\|, \]

where $a_j \in \mathbb{R}^n$ denotes a given location. Note that the objective is actually quite smooth far away from each location $a_j$. We generate $a_j$ from normal distribution and run UAPD with enough iterations to obtain an approximated optimal value $f^*$. Numerical results in Figure 5 show that both FGM (with $\epsilon = 1e-8$) and UAPD work well and possess similar convergence behaviors. Moreover, as $\nabla f$ is almost Lipschitz continuous and the magnitude of the Lipschitz constant $L$ is not so large, the over-estimated issue of FGM is negligible, and the approximated constant $M_k$ is the same as that of UAPD.

![Graphs showing performance comparison between FGM and UAPD](image)

**Fig. 5.** Numerical performances of FGM and UAPD on the continuous Steiner problem with $m = 800$, $n = 400$.

6.4. Basis pursuit problem. In the last example, we move to the basis pursuit problem

\[ \min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \ Ax = b, \]

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. To be compatible with the problem setting of AccUmiPDGrad, we consider an equivalent formulation

\[ \min_{x \in \mathbb{R}^n} \frac{1}{2} \|x\|_2^2 \quad \text{s.t.} \ Ax = b. \]
The dual problem reads as
\[
\min_{\lambda \in \mathbb{R}^m} \left\{ \varphi(\lambda) := \langle b, \lambda \rangle + \frac{1}{2} \| A^T \lambda \|_2 \right\}.
\]

Note that existing accelerated Bregman method \cite{28} and accelerated ALM \cite{71} can be applied to this problem with theoretical rate \(O(1/k)\). But we only focus on the comparison between UAPD and AccUniPDGrad \cite{75}, as black-box type methods with line search procedure. We mention that the AccUniPDGrad method also uses smaller tolerance \(\epsilon/k\) as that in FGM. Numerical results are showed in Figure 6, which indicate that (i) our UAPD has smaller objective residual and feasibility violation, and (ii) the line search procedure is more efficient with smaller total number \(#i_k\) and Lipschitz constant \(M_k\).

**Figure 6.** Numerical performances of AccUniPDGrad and UAPD on the basis pursuit problem with \(m = 100, n = 500\). The desired accuracy for AccUniPDGrad is \(\epsilon = 1e-3\).

**Appendix A. Proof of Lemma 3.3.** Let us first prove (3.3). Recall that \(i_k\) is the smallest integer such that
\[
h(x_{k,i_k}) - \Delta_{k,i_k} \leq \frac{\delta_{k,i_k}}{2}.
\]
If \(i_k = 0\), then \(M_{k+1} = M_k\). If \(i_k \geq 1\), then we claim that
\[
M_{k,i_k} \leq 2M(\nu, \delta_{k,i_k-1}). \quad \text{(A.1)}
\]
Otherwise, we have \(M_{k,i_k-1} = M_{k,i_k}/2 > M(\nu, \delta_{k,i_k-1})\). According to Proposition 2.2, this implies immediately that
\[
h(x_{k,i_k-1}) - \Delta_{k,i_k-1} \leq \frac{\delta_{k,i_k-1}}{2},
\]
which yields a contradiction and thus verifies the estimate (A.1). Additionally, by (3.1), we have \(\alpha_{k,i_k} \leq \alpha_{k,i_k-1} \leq \sqrt{2} \alpha_{k,i_k}\). Thus, using (2.6), (3.2), and (A.1) leads to
\[
M_{k+1} = M_{k,i_k} \leq 2\sqrt{2}M(\nu, \delta_{k,i_k}) = 2\sqrt{2}M(\nu, \delta_{k+1}). \quad \text{(A.2)}
\]
This implies that for all \( k \geq 0 \), we have

\[(A.3) \quad M_{k+1} \leq \max \{2\sqrt{2}M(\nu, \delta_{k+1}), M_k\}.\]

Note that \( \delta_k = \beta_k/k \) and \( \beta_k \) is decreasing. Thus \( \delta_i \geq \delta_k \) and \( M(\nu, \delta_i) \leq M(\nu, \delta_{k+1}) \) for all \( 1 \leq i \leq k \), this indicates that

\[M_{k+1} \leq \max \{2\sqrt{2}M(\nu, \delta_{k+1}), M_i\} \quad \forall 1 \leq i \leq k.\]

Taking \( i = 1 \) and using (A.3) with \( k = 0 \), we obtain

\[M_{k+1} \leq \max \{2\sqrt{2}M(\nu, \delta_{k+1}), M_0\},\]

which proves (3.3).

Then, let us verify (3.4). Observing that

\[M_{k+1} = M_{k,i_k} = 2^{i_k-1}M_k \quad \Rightarrow \quad i_k = 1 + \log_2 \frac{M_{k+1}}{M_k},\]

we get

\[\sum_{j=0}^{k} i_j = k + 1 + \log_2 \frac{M_{k+1}}{M_0} \leq k + 1 + \max \left\{1, \log_2 \frac{M(\nu, \delta_{k+1})}{M_0/2\sqrt{2}}\right\}.\]

Since \( M(\nu, \delta_{k+1}) = \delta_{k+1} \overline{M}_\nu(h) \), we complete the proof of Lemma 3.3.

Appendix B. Accelerated Bregman Primal-Dual Flow. Recall the conjugate function

\[\phi^*(\xi) := \sup_{x \in Q} \{\langle \xi, x \rangle - \phi(x)\} \quad \forall \xi \in \mathbb{R}^n.\]

We have the relation: \( \xi = \nabla \phi(x) \iff x = \nabla \phi^*(\xi) \); see [1, Theorem 16.23]. Therefore, by introducing \( w = \nabla \phi(v) \), we obtain an alternative first-order formulation of (3.10):

\[(B.1) \quad \begin{cases} x' = \nabla \phi^*(w) - x, \\ \gamma w' \in \mu(\nabla \phi(x) - w) - (\partial f(x) + N_Q(x) + A^\top \lambda), \\ \beta \lambda' = A\nabla \phi^*(w) - b. \end{cases}\]

**B.1. Well-posedness and exponential decay.** Let us focus on the smooth setting: \( Q = \mathbb{R}^n \) and \( f \in C^1 \) satisfies

\[f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \mu D_\phi(x, y) \quad \forall x, y \in \mathbb{R}^n,\]

with \( \mu \geq 0 \). Then our ABPD flow dynamics (3.10) becomes

\[(B.2a) \quad x' = v - x, \]

\[(B.2b) \quad \gamma \frac{d}{dt} \nabla \phi(v) = \mu(\nabla \phi(x) - \nabla \phi(v)) - \nabla f(x) - A^\top \lambda, \]

\[(B.2c) \quad \beta \lambda' = A\nabla \phi^*(w) - b. \]

By (B.1), this is also equivalent to

\[(B.3) \quad \begin{cases} x' = \nabla \phi^*(w) - x, \\ \gamma w' = \mu(\nabla \phi(x) - w) - \nabla f(x) - A^\top \lambda, \\ \beta \lambda' = A\nabla \phi^*(w) - b. \end{cases}\]
Recall that $\gamma$ and $\beta$ are governed by (3.11), which actually admits explicit solutions
\[
\beta(t) = \beta_0 e^{-t}, \quad \gamma(t) = \mu + (\gamma_0 - \mu) e^{-t}.
\]
Since $\phi$ is 1-strongly convex (cf. (2.1)), $\nabla \phi^*$ is 1-Lipschitz continuous. Consequently, if both $\nabla f$ and $\nabla \phi$ are Lipschitz continuous, then by standard theory of ordinary differential equations, we conclude that the dynamical system (B.3) admits a unique classical $C^1$ solution $(x, w, \lambda)$. This also promises that our ABPD flow (B.2) exists a unique solution $(x, v, \lambda)$ with $v = \nabla \phi^*(w)$ being continuous.

We then introduce a Lyapunov function
\[
E(x, v, \lambda) := L(x, x^*) - L(x^*, \lambda) + \gamma D_\phi(x^*, v) + \frac{\beta}{2} \| \lambda - \lambda^* \|^2,
\]
which is a continuous analogue to the discrete one (3.12).

**Theorem B.1.** Let $(x, v, \lambda) \in C^1(\mathbb{R}_+; \mathbb{R}^n) \times C^0(\mathbb{R}_+; \mathbb{R}^n) \times C^1(\mathbb{R}_+; \mathbb{R}^m)$ be the unique solution to the ABPD flow (B.2). Then we have
\[
\frac{d}{dt} E(x, v, \lambda) \leq -E(x, v, \lambda) - \mu D_\phi(x, v),
\]
which implies the exponential decay rate
\[
e^t E(x(t), v(t), \lambda(t)) + \mu \int_0^t e^s D_\phi(v(t), x(t)) \, ds \leq E(x_0, v_0, \lambda_0),
\]
for all $t \geq 0$.

**Proof.** Taking the derivative with respect to the time variable gives
\[
\frac{d}{dt} E(x, v, \lambda) = \langle \nabla_x L(x, \lambda^*), x' \rangle + \gamma' D_\phi(x^*, v) + \gamma \frac{d}{dt} D_\phi(x^*, v)
+ \frac{\beta'}{2} \| \lambda - \lambda^* \|^2 + \beta \langle \lambda - \lambda^*, \lambda' \rangle.
\]
Since $w = \nabla \phi(v) \in C^1(\mathbb{R}_+; \mathbb{R}^n)$, we see that $D_\phi(x^*, v)$ is continuous differentiable in terms of $t$ and by (B.2b), we have
\[
\gamma \frac{d}{dt} D_\phi(x^*, v) = \langle \gamma \frac{d}{dt} \nabla \phi(v), v - x^* \rangle
= \mu \langle \nabla \phi(x) - \nabla \phi(v), v - x^* \rangle - \langle \nabla f(x) + A^T \lambda, v - x^* \rangle.
\]
Then using the three-term identity (2.2) and following the proof of [39, Lemma 2.1], we can verify (B.5) and complete the proof.

**Appendix C. Proof of Lemma 5.1.**

**C.1.** The case $\eta = \theta - 1$. The estimate (5.2) becomes
\[
\sqrt{\phi(t)} y'(t) \frac{y'(t)}{y(t)} + R \frac{y'(t)}{y^2(t)} \leq -\sigma(t).
\]
Since $y(0) = 1$ and $y'(t) \leq 0$, it holds that $0 < y(t) \leq 1$ for all $t \geq 0$. As $\phi(t)$ is positive and nondecreasing, we obtain
\[
(\sqrt{\phi} \ln y)' = \frac{\phi'}{2\sqrt{\phi}} \ln y + \sqrt{\phi} \frac{y'}{y} \leq \sqrt{\phi} \frac{y'}{y}.
\]
Combining this with (C.1) gives
\[
\left( \sqrt{\varphi(t)} \ln y(t) + \frac{R}{1-\theta} y^{1-\theta}(t) \right)' \leq -\sigma(t),
\]
and integrating over \((0, t)\) leads to
\[
(C.2) \quad \sqrt{\varphi(t)} \ln \frac{1}{y(t)} + \frac{R}{\theta - 1} \left( y^{1-\theta}(t) - 1 \right) \geq \int_0^t \sigma(s) \, ds = \Sigma(t).
\]
Define
\[
(C.3) \quad Y_1(t) := \exp \left( -\frac{\Sigma(t)}{2\sqrt{\varphi(t)}} \right) \quad \text{and} \quad Y_2(t) := \left( 1 + \frac{\theta - 1}{2R} \Sigma(t) \right)^{\frac{1}{\theta-\eta}}.
\]
Then one finds that
\[
\begin{cases}
\sqrt{\varphi(t)} \ln \frac{1}{Y_1(t)} = \frac{1}{2} \Sigma(t), & Y_1(0) = 1, \\
\frac{R}{\theta - 1} \left( Y_2^{1-\theta}(t) - 1 \right) = \frac{1}{2} \Sigma(t), & Y_2(0) = 1.
\end{cases}
\]
This also implies
\[
(C.4) \quad \sqrt{\varphi(t)} \ln \frac{1}{Y(t)} + \frac{R}{\theta - 1} \left( Y^{1-\theta}(t) - 1 \right) \leq \Sigma(t),
\]
where \(Y(t) := Y_1(t) + Y_2(t)\). For fixed \(t > 0\), the function
\[v \rightarrow \sqrt{\varphi(t)} \ln \frac{1}{v} + \frac{R}{\theta - 1} \left( v^{1-\theta} - 1 \right)\]
is monotonously decreasing in terms of \(v \in (0, \infty)\). Collecting (C.2) and (C.4) yields that
\[
y(t) \leq Y(t) = \exp \left( -\frac{\Sigma(t)}{2\sqrt{\varphi(t)}} \right) + \left( 1 + \frac{\theta - 1}{2R} \Sigma(t) \right)^{\frac{1}{\theta-\eta}}.
\]
This completes the proof of Lemma 5.1 with \(\eta = \theta - 1\).

**C.2. The case \(\eta < \theta - 1\).** The proof is in line with the previous case. We have
\[
\left( \sqrt{\varphi(t)} y_{\eta+1-\theta}^\nu \right)' = \sqrt{\varphi(t)} y_{\eta+1-\theta}^\nu \left( \frac{\nu}{\eta+1-\theta} \right)' + \frac{y_{\eta+1-\theta}^\nu}{\eta+1-\theta} \cdot \frac{\varphi(t) y_\nu'}{2\sqrt{\varphi(t)}} \leq \frac{\sqrt{\varphi(t)} y_\nu'}{y_{\eta+1-\theta}},
\]
which together with (5.2) gives
\[
\left( \sqrt{\varphi(t)} y_{\eta+1-\theta}^\nu(t) + \frac{R y_{1-\theta}^\nu(t)}{1-\theta} \right)' \leq -\sigma(t) \quad \Rightarrow \quad G(\varphi(t), y(t)) \geq \Sigma(t),
\]
where \(G : [0, \infty) \times (0, \infty) \rightarrow \mathbb{R}\) is define by
\[
G(w, v) := \frac{\sqrt{w}}{\theta - \eta - 1} (v_{\eta+1-\theta}^\nu - 1) + \frac{R}{\theta - 1} (v_{1-\theta}^\nu - 1),
\]
for all \( w \geq 0 \) and \( v > 0 \). In addition to \( Y_2(t) \) defined in (C.3), we introduce

\[
Y_3(t) := \left( 1 + \frac{\theta - \eta - 1}{2\sqrt{\varphi(t)}} \right)^{\frac{1}{\theta - \eta - 1}}.
\]

Since \( G(w, \cdot) \) is monotonously decreasing and

\[
G(\varphi(t), Y_2(t) + Y_3(t)) \leq \Sigma(t) \leq G(\varphi(t), y(t)),
\]

we obtain

\[
y(t) \leq \left( 1 + \frac{\theta - 1}{2R} \Sigma(t) \right)^{\frac{1}{\theta - 1}} + \left( 1 + \frac{\theta - \eta - 1}{2\sqrt{\varphi(t)}} \Sigma(t) \right)^{\frac{1}{\theta - \eta - 1}}.
\]

This concludes the proof of Lemma 5.1 with \( \eta < \theta - 1 \).

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