METRIC CURVATURES AND THEIR APPLICATIONS 2: 
METRIC RICCI CURVATURE AND FLOW

EMIL SAUCAN

ABSTRACT. In this second part of our overview of the different metric curvatures and their various applications, we concentrate on the Ricci curvature and flow for polyhedral surfaces and higher dimensional manifolds, and we largely review our previous studies on the subject, based upon Wald’s curvature. In addition to our previous metric approaches to the discretization of Ricci curvature, we consider yet another one, based on the Haantjes curvature, interpreted as a geodesic curvature. We also try to understand the mathematical reasons behind the recent proliferation of discretizations of Ricci curvature. Furthermore, we propose another approach to the metrization of Ricci curvature, based on Forman’s discretization, and in particular we propose on that uses our graph version of Forman’s Ricci curvature.

CONTENTS

1. Introduction: Why Ricci Curvature? 2
2. Metric Ricci Curvature and Flow for PL Manifolds 4
  2.1. Metric Ricci Flow for Surfaces 5
  2.2. Metric Ricci Curvature for PL manifolds 13
  2.3. A Different Approach: From Geodesic Curvature to Ricci Curvature 23
3. An Alternative Approach: Forman’s Ricci Curvature 26
  Acknowledgement 34
  References 34

Date: February 12, 2019.
1. Introduction: Why Ricci Curvature?

The natural question that would probably arise in the reader’s mind is why, after surveying in the first part of this paper [76] a variety of essential notions of classical (smooth) curvature, we dedicate a separate paper solely to the metric discretizations of Ricci curvature? - After all, till relatively recently, this notion was relegated to the inner pages of Riemannian curvature textbooks and to the mathematical study of General Relativity [12]. True, a tremendous amount of interest has been raised by Hamilton’s [42], [43] and Perelman’s [65], [66] work on the Ricci flow, and its application to the settling Thurston’s Geometrization Conjecture (hence to the Poincaré conjecture). In particular, Chow and Luo’s approach to Discrete Ricci curvature [28], based on circle packings, has proved to be not only theoretically interesting and important, but also extremely useful and influential in various applications – see e.g. [40], [34], [101]. Therefore, a drive to obtain a purely metric (in the sense of the first paper in this series [76]) Ricci flow is only natural. Still, being driven only by the quest for a metric analogue of the Combinatorial Ricci flow and its possible applications would have been, from the Metric Geometry viewpoint, somewhat mercenary.

Indeed, another motivation for the geometer in understanding and generalizing Ricci curvature to discrete settings stems from a different direction, namely from the observation that Ricci curvature controls the volume growth of a manifold, a fact whose wide range of implications has been pointed out first by Gromov [35] (see also the deep results of Cheeger and Colding [24], as well as Fukaya [33]). This insight conducted researchers towards a generalized notion of Ricci curvature bounds for Riemannian manifolds with density and, more generally, for metric measure spaces, namely the so called \textit{curvature-dimension condition} \(\text{CD}(K,N)\), that was obtained by Lott-Villani [54] and Sturm [91]. (We should note that these authors obtained a number of very strong results that confirm the correctness of these definition as a synthetic version of the notion of Ricci curvature bounded from above, such as locality, generalized versions of Meyer’s and Bishop-Gromov’s Theorems, generalized Poincaré inequalities, and stability under Gromov-Hausdorff limits, to number just a few; see [96] for further details.) Naturally, the quest for a suitable version of this notion for a typical discrete case followed shortly [15]. A second strategy towards defining a Ricci
curvature for weighted manifolds and graphs has its roots in the work of Bakry, Emery and Ledoux [4], [5]. It is derived from the so called Bochner-Weitzenböck formula (see for instance [46]), that relates curvature to the classical (Riemannian) Laplace operator. (We shall see that a quite different framework for the discretization of Ricci curvature is also derived from the Bochner-Weitzenböck formula [32].) As with the Lott-Villani and Sturm approach, within this framework one is able to formulate a curvature dimension condition. Furthermore, in contrast with the Lott-Villani and Sturm definition, the one of Bakry and Emery easily lends itself to adaptation to graphs, as demonstrated by the works of Yau, Jost and their collaborators [50], [6], [47]. (see also [59]). Finally, a third approach to Ricci curvature for weighted manifolds, due to Morgan and his students [58], [30] originates – yet again – from another paper of Gromov, namely [36] (the ideas of Bakry and Emery also playing a significant role in its development). Of the approaches to Ricci curvature of weighted structures it is the most direct and, perhaps, with the most evident geometric vein. Although powerful, this method employs simple formulae that represent straightforward generalizations of classical ones. Even so, only one attempt has been made so far (at least to best of or knowledge) to employ this approach in an applicative context – see [51].

However, yet another, more geometric, view of Ricci curvature, proved itself to be most useful and intuitive for the geometrization of such discrete spaces as graphs. This approach, due to Ollivier [63], [64], captures, in the discrete setting, the quantization of the growth of infinitesimal balls that represents one of the fundamental properties of Ricci curvature. Also, akin to the method of Lott-Villani and Sturm, it is based upon the notion of optimal mass transportation. It confirmed to be a powerful, yet intuitive, tool for the understanding of geometric (as well as analytic) properties of graphs and related structures (see, for instance, [7], [47], [53]). In consequence, Ollivier’s Ricci curvature represents an excellent apparatus for the intelligence of complex networks, in their various aspects, be they communication [62], [97], biological [92], economical [93] or transportation [19] ones.

Moreover, besides these somewhat interrelated generalizations and discretizations of Ricci curvature, two approaches, both quite different from the ones above, as well as distinct from each other need to be noted, namely the ones due, to Stone [88], [89], and Forman [32]. Since these works are
strongly related to our own research that we shall detail below, we shall discuss each of them in more detail in the sequel.

It would seem, however, that we only managed to replace the original question (e.g. "Why should we devote a separate paper solely to discretizations of Ricci curvature?") with an equally frustrating one, namely whether the multiplication of the various discretizations and generalizations of the classical notions is due, as one might perhaps suspect, to the excessive zeal of researchers who might have wished to contribute their own ideas on an attractive and "hot" topic, or is there an intrinsic reason why a "definitive" notion of discrete Ricci curvature proves to be so evasive?

It turns out that, there exists, at least in part, quite a deep scientific justification for this variety of discrete Ricci curvatures and that, in some sense, there is no possible "ultimate" discretization. More precisely, one is confronted with the following (unfortunately not known or acknowledged enough) result, due to Bernig [11], Proposition 8.1: For the large class of so called singular spaces (that includes the polyhedral manifolds that constitute the object of our interest in this paper) – see e.g. [13] for the technical definition and further details, the exists no continuous, intrinsic tensor-valued distribution (measure) generalizing the Ricci tensor. In fact, no such generalization of the Riemann tensor is possible, either. It follows, therefore, that the quest for “perfect” discretization of these tensorial classical curvatures to the context of polyhedral meshes, so dear to Graphics and Imaging community, is a forlorn one, and to such ends, one has to do only with as good as possible approximations at most. More importantly, from a mathematical point of view, the result above proves, once more, that when passing from the classical (smooth) case to a discrete one, there is no choice but trying to capture one of the essential properties of the classical notion and use it as the basis definition in the discrete setting. Let us note here (with no little chagrin) that, while for geometers this is a clear and accepted paradigm, the Applied Mathematics (Imaging and related fields included) community is still reluctant to accept it.

2. **Metric Ricci Curvature and Flow for PL Manifolds**

In this section we present some novel developments regarding Ricci curvature and flow for polyhedral manifolds, intrinsically based upon the Wald metric curvature discussed in detail in the first part of this study of metric
curvatures in modern, computer related applications [76]. We begin by introducing a metric Ricci flow for surfaces (stemming from our paper [75]) and some of its applications and continue by proposing (based on [38]) a metric Ricci curvature for PL (polyhedral) manifolds, and test it, from the viewpoint of Synthetic Differential Geometry by proving a fitting variant (or rather variants — see below) of the Bonnet-Myers Theorem.

2.1. **Metric Ricci Flow for Surfaces.** We begin with what represents not only the simplest case, but also the most important one, in view of the aforementioned works of Chow-Luo and Gu, namely that of the Ricci flow for surfaces.

2.1.1. **Smoothings and Metric Curvatures.** We begin by noticing that the Gromov-Hausdorff convergence results mentioned in [76] may appear to many a bit too general, at least when considering their applications to Imaging and related fields. One would like – and not only in the applied context above – wish to obtain similar results where “second order” curvature is also preserved (i.e. well approximated), not just “first order geometry”, that is distances and where the approximating spaces are themselves smooth. This is, indeed, a highly natural question, to which one would like to give better, easier to use answers than the one provided by the observations brought (with anticipation) in Remark 2.16 of [76]. One such answer is given by a results of Brehm and Kühnel [17] as well as [75], where a sketch of proof and its relation to Proposition 2.1 below.

While less elementary than the one adopted by Brehm and Kühnel, our approach is, perhaps, more natural for geometers and topologists, as well as for Imaging applications, since it uses only standard analytic tools. In addition, it applies to a larger class of surfaces. Moreover, it captures better the meaning of the notion of Hausdorff convergence and its interplay with curvature: Instead of building, as in the Brehm-Kühnel approach, the smooth surfaces from a set of “standard elements” (cylinders, etc.), we consider instead smoothings $S^2_m$ (see, e.g. [61]).

Since, by [61], Theorem 4.8, such smoothings are $δ$-approximations, that is metric approximations, and therefore, by [61], Lemma 8.7, they also are, for $δ$ small enough, $α$-approximations, i.e. angle approximations of the given piecewise-linear surface $S^2_{Pol}$, they approximate arbitrarily well both distances and angles on $S^2_{Pol}$. Therefore angles, hence defects, are arbitrarily
Figure 1. A smooth analytic surface (left) as the smoothing of one of its \( PL \) approximations (right).

also well approximated. (For the precise technical definitions and proofs, see [61].)

The observations above amount, in fact, to a positive answer to the question – not posed so far, to the best of our knowledge – whether the metric curvature version of Brehm and Hühnels’s basic result also holds, namely we have proved:

**Proposition 2.1** ([75]). Let \( S^2_{Pol} \) be a compact polyhedral surface without boundary. Then there exists a sequence \( \{S^2_m\}_{m \in \mathbb{N}} \) of smooth surfaces, (homeomorphic to \( S^2_{Pol} \)), such that

1. \( S^2_m = S^2_{Pol} \) outside the \( \frac{1}{m} \)-neighbourhood of the 1-skeleton of \( S^2_{Pol} \);
2. The sequence \( \{S^2_m\}_{m \in \mathbb{N}} \) converges to \( S^2_{Pol} \) in the Hausdorff metric;
3. \( K(S^2_{Pol}) \to K_W(S^2_{Pol}) \), where the convergence is in the weak sense.

The result above shows that, in fact, the metric approach to curvature is essentially equivalent to the combinatorial (angle-based) one, as far as polyhedral surfaces (in \( \mathbb{R}^3 \)) are concerned.\(^1\) In particular, as far as approximations of smooth surfaces in \( \mathbb{R}^3 \) are concerned, both approaches render, in the limit, the classical Gauss curvature.) We should also stress that, in fact, the metric approach is more general, since it can be applied to a very large class of metric spaces (see discussion at the end of this section).

\(^1\)This holds, of course, up to the specific type of convergence for the metric and combinatorial curvature, namely pointwise and in measure, respectively.
Remark 2.2. The converse implication – namely that Gaussian curvature $K(\Sigma)$ of a smooth surface $\Sigma$ may be approximated arbitrarily well by the Wald curvatures $K_W(\Sigma_{Pol,m})$ of a sequence of approximating polyhedral surfaces $\Sigma_{Pol,m}$ – is, as we have already mentioned above, quite classical. (For other approaches to curvatures convergence, see, amongst the extensive literature dedicated to the subject, [25] and [16], [27], for the theoretical and applicative viewpoints, respectively.)

Remark 2.3. The metric approach adopted here renders, in fact, a somewhat stronger result than that given in [17], since no embedding in $\mathbb{R}^3$ is apriorily assumed, just in some $\mathbb{R}^N$ (even though as we have already noted above, this represents only a slight improvement). Even more importantly, no change in the geometry of the 1-skeleton is made, not even in the neighbourhoods of the vertices.

2.1.2. A Metric Ricci Flow. From the metric and curvature approximations results above, it follows that one can study the properties of the metric Ricci flow via those of its smooth counterpart, by passing to a smoothing of the polyhedral surface. The heavier machinery of metric curvature considered above pay off, in the sense that, by using it, the flow is purely metric and, moreover, the curvature at each stage (that is, for every “$t$”) is given – as in the classical context – in an intrinsic manner, i.e. solely in terms of the metric.

The Flow. In direct analogy with the classical flow

\[
\frac{dg_{ij}(t)}{dt} = -2K(t)g_{ij}(t),
\]

we define the metric Ricci flow as

\[
\frac{dl_{ij}}{dt} = -2K_i l_{ij},
\]

where $l_{ij} = l_{ij}(t)$ denote the edges (1-simplices) of the triangulation ($PL$ or piecewise flat surface) incident to the vertex $v_i = v_i(t)$, and $K_i = K_i(t)$ denotes the curvature at the same vertex. We shall discuss below in detail what proper notion of curvature should be chosen to render this type of metric flow meaningful.

We also consider the normalized flow

\[
\frac{dg_{ij}(t)}{dt} = (K - K(t))g_{ij}(t),
\]
and its metric counterpart

\[ \frac{dl_{ij}}{dt} = (\bar{K} - K_i)l_{ij}, \]

where \( K, \bar{K} \) denote the average classical, respectively metric, sectional (Gauss) curvature of the initial surface: \( S_0, \bar{K} = \int_{S_0} K(t) dA/\int_{S_0} dA, \) and \( \bar{K} = \frac{1}{|V|} \sum_{i=1}^{|V|} K_i, \) respectively. (Here \(|V|\) denotes, as usually, the number of the vertex set of \( S_{Pol}. \))

Our approach to this type of metric flow should be evident in light of the previous discussion: Using smoothings, we approximate the metric Ricci curvature by its classical counterpart, thus replacing the metric flow with its smooth model.

Remark 2.4. Since for PL surfaces (hence for their smoothings) \( K_W(p) = 0 \) for all points apart from vertices, the ensuing Ricci flow is stationary, except at vertices where the change rate is quite drastic. In this aspect, the metric Ricci flow introduced here resembles the combinatorial, rather than the smooth (classical) one. This should not be too surprising, given that, as already stated, the initial motivation of considering the metric flow (and, a fortiori, smoothings) was to gain a better understanding of some of the properties of the combinatorial flow.

An advantage of the approach introduced here is that it automatically solves, by passing to infinitesimal distances, the asymmetry in equation 2.2, that is caused by the fact that the curvature on two different vertices acts, so to say, on the same edge. We shall address this issue again shortly.

The main theoretical disadvantage of the approach embraced here is that it is not easy to extend it to higher dimensional manifolds. Indeed, it is not clear how to correctly define a flow for general high-dimensional manifolds (both for theoretical reasons and because of their applications, such as Medical Imaging, Video, etc.), not least because 3-dimensional analogues of all the relevant results on the Ricci flow for smooth surfaces have yet to be obtained. Moreover, even defining a metric Ricci curvature in dimension 3 and higher is a non-trivial endeavor. See, however, the discussion and results in the following sub-section.

Therefore, developing a purely metric Ricci flow – that is one that does not make appeal to smoothings – is highly desirable. One basic observation that has to be made is that one hast to take care of the lack of symmetry...
that we mentioned when we first introduced the metric flow. From symmetry reasons, a natural way of defining the flow while addressing this issue is (using the same notation as before):

\[
\frac{dl_{ij}}{dt} = -\frac{K_i + K_j}{2}l_{ij},
\]

(2.5)

where in this case, \(K_i, K_j\) denote, of course, the Wald curvature at the vertices \(v_i\) and \(v_j\), respectively. (Note that, in fact, this expression appears also in the practical method of computing the combinatorial curvature, where it is derived via the use of a conformal factor \([40]\).)

An important benefit of such a purely metric flow would be that, as in the case of combinatorial Ricci flow of Chow and Luo \([28]\), equation (2.1) becomes – due to the fact that \(K_i\) depends only the lengths of the edges \(l_{ij}\), and not on their derivatives – an ODE, instead of a PDE, thence they are easier to study and enjoy better properties. In particular (and of importance in applications) the metric flow will have the backward existence property.

For further observations regarding the metric flow, as well as for some other possible improvements, see \([75]\).

**Existence and Uniqueness.** For the classical Ricci flow \(dg_{ij}(t)/dt = 2K(t)g_{ij}(t)\) the (local) existence and uniqueness hold, on some maximal time interval \([0, T); 0 < T \leq \infty\) (see, e.g. \([29]\), as well as \([41]\), for the original, different proof\(^2\)). Moreover, the backward uniqueness of a solution (if existing) has been proven by Kotschwar \([48]\) (see also \([94]\) for a sketch of the proof). Beyond the theoretical importance, the existence and uniqueness of the backward flow would allow us to find surfaces in the conformal class of a given circle packing (Euclidean or Hyperbolic).

More importantly, the use of purely metric approach, based on the Wald curvature (or any of other equivalent metric curvatures, for that matter), rather than the combinatorial (and metric) approach of \([28]\), allows us to give a first, tentative, purely theoretical at this point, answer to Question 2, p. 123, of \([28]\), namely whether there exists a Ricci flow defined on the space of all piecewise constant curvature metrics (obtained via the assignment of lengths to a given triangulation of 2-manifold). Since, by Hamilton’s results \([41]\) (and those of Chow \([23]\), for the case of the sphere), the Ricci flow

\(^2\)See also \([94]\), Theorems 5.2.1 and 5.2.2 and the discussion following them for short exposition of the main steps of the proof.
exists for all compact surfaces, it follows from the arguments above that
the fitting metric flow exits for surfaces of piecewise constant curvature. In
consequence, given a surface of piecewise constant curvature (usually a mesh
with edge lengths satisfying the triangle inequality for each triangle), one can
evolve it by the Ricci flow, either forward, as in the papers mentioned above,
to obtain, after the suitable area normalization, the polyhedral surface of
constant curvature conformally equivalent to it; or backwards – if possible – to
find the “primitive” family of surfaces, including the “original” surface
conformally equivalent to the given one, (where, by “original”, we mean the
surface obtained via the backwards Ricci flow, at time $T$).

Remark 2.5. Note that is not necessarily true that all the surfaces obtained
via the backwards flow are embedded (or, indeed, embeddable) in $\mathbb{R}^3$ – a see
the discussion below.

Proposition 2.6. Let $(S^2_{Pol}, g_{Pol})$ be a compact polyhedral 2-manifold
without boundary, having bounded metric curvature. Then there exists $T > 0$
and a smooth family of polyhedral metrics $g(t), t \in [0, T]$, such that

\[
\begin{align*}
\frac{\partial g}{\partial t} &= -2K(t)g(t) \quad t \in [0, T] ; \\
g(0) &= g_{Pol}.
\end{align*}
\]

(Here $K(t)$ denotes the Wald curvature induced by the metric $g(t)$.)

Moreover, both the forwards and the backwards (when existing) Ricci flows
have the uniqueness of solutions property, that is, if $g_1(t), g_2(t)$ are two Ricci
flows on $(S^2_{Pol})$, such that there exists $t_0 \in [0, T]$ such that $g_1(t_0) = g_2(t_0)$,
then $g_1(t) = g_2(t)$, for all $t_0 \in [0, T]$.

In fact, by combining our method with a result of Shi \[84\], we can extend
the proposition above to complete polyhedral surfaces, as follows:

Proposition 2.7. Let $(S^2_{Pol}, g_{Pol})$ be a complete polyhedral surface, such
that $0 < K_W \leq K_0$. Then there exists a (small) $T$ as above, such that there
exists a unique solution of (2.1) for any $t \in [0, T]$.

\[\text{see [44], [75]}\]
Convergence Rate. A further type of result, highly important both from the theoretical viewpoint and for computer-driven applications, is that of the convergence rate. For the Ricci flow mostly applied in such settings, namely for the combinatorial Ricci flow, it was proven in [28] that, in the case of background Euclidean (Theorem 1.1) or Hyperbolic (Theorem 1.2) metric, the solution – if it exists – converges, without singularities, exponentially fast to a metric of constant curvature. Using the classical results of [41] and [23], since we already know that the solution exists and it is unique (see the subsection below for the nonformation of singularities), we are able to control the convergence rate of the curvature:

Theorem 2.8. Let \((S^2_{Pol}, g_{Pol})\) be a compact polyhedral 2-manifold without boundary. Then the normalized metric Ricci flow converges to a surface of constant metric curvature. Moreover, the convergence rate is

- (1) exponential, if \(K < 0; \chi(S^2_{Pol}) < 0\);
- (2) uniform; if \(K = 0\);
- (3) exponential, if \(K > 0\).

Recall that convergence rate of solutions is defined as follows:

**Definition 2.9.** A solution of \((2.4)\) is said to be convergent iff

1. \(\lim_{t \to \infty} K_i(t) = K_i(\infty), \) for all \(1 \leq u \leq |V|, \) where \(K_i(\infty) \in (0, 2\pi)\);
2. \(\lim_{t \to \infty} l_{ij}(t) = l_{ij}(\infty), \) \(l_{ij}(\infty) > 0.\)

A convergent solution is said to converge exponentially fast iff there exists constants \(c_1, c_2,\) such that, fora any \(t \geq 0,\) the following inequalities hold:

1. \(|K_i(t) - K_i| \leq c_1 e^{-c_2 t};\)
2. \(|l_{ij}(t) - l_{ij}| \leq c_1 e^{-c_2 t} .\)

(The fitting definition for the flow \((2.2)\) is immediate.)

**Remark 2.10.** A more realistic model for (gray-scale as well as color) images should be based on surfaces with boundary. Similar results can be obtained for this type of surfaces (see [18], [19]), however we defer for further study the detailed analysis, in this model, of the metric Ricci flow of images.

**Singularities Formation.** In the classical (smooth) case, by [41], Theorems 1.1 and 5.1, the Ricci flow evolves without singularities formation, even for surfaces of low genus. Also, by [28], Theorem 5.1, the combinatorial Ricci flow evolves without singularities on compact surfaces of genus \(\geq 2.\)
However, on surfaces of low genus, that are extremely important in Graphics and Imaging, singularities do form [37].

Combining our smoothing technique with Hamilton’s classical results mentioned above, as well was with a more recent result of Topping [95], we obtain the following results:

**Proposition 2.11.** Let \((S^2_{Pol}, g_{Pol})\) be a complete polyhedral 2-manifold, with at most a finite number of hyperbolic cusps (punctures), having bounded metric curvature and satisfying the noncollapsing condition below.

There exists \(r_0 > 0\), such that, for all \(x \in M\) the following holds:

\[
\text{Vol}_g (B_g(x, r_0)) \geq \varepsilon > 0.
\]

(Here, as usual, \(B_g(x, r_0)\) denotes the open ball, in the metric \(g\), of center \(x\) and radius \(r_0\).)

Then there exists a unique Ricci flow that contracts the cusps. Furthermore, the curvature remains bounded at all times during the flow.

**Embeddability in \(\mathbb{R}^3\).** In Graphics, where one of the main problems solved via the combinatorial Ricci flow is that of registration, by producing, via the flow, a conformal mapping from the given surface to one of the model surfaces (see, e.g. [40], [45]), the embeddability – not necessarily isometric – is both trivial and not of real interests. However, this aspect is highly significant in Image Processing (see [3]), and, in fact, the results below were motivated precisely by these applicative aspects of the Ricci flow.

Here we mainly consider a problem regarding smooth surfaces, since, by now, the connection with the version for polyhedral surfaces is, we hope, quite clear. We should note that, by [61], Theorem 8.8, any \(\delta\)-approximation of an embedding is also an embedding, for small enough \(\delta\). Since, as we have already mentioned, smoothing represent \(\delta\)-approximations, the possibility of using results regarding smooth surfaces to deduce facts regarding polyhedral embeddings is proven. (The reverse implication – namely from smooth to PL and polyhedral manifolds – follows from the fact that the secant approximation is a \(\delta\)-approximation if the simplices of the PL approximation satisfy a certain nondegeneracy condition – see [61], Lemma 9.3.)

In the following \(S^2_0\) denotes a smooth surface of positive Gauss curvature, and let \(S^2_t\) denote the surface obtained at time \(t\) from \(S^2_0\) via the Ricci flow.
Proposition 2.12. Let $S^2_0$ be the unit sphere $\mathbb{S}^2$, equipped with a smooth metric $g$, such that $K(g) > 0$. Then the surfaces $S^2_t$ are (uniquely, up to a congruence) isometrically embeddable in $\mathbb{R}^3$, for any $t \geq 0$.

In fact, the result above can be slightly strengthened as Corollary 2.13. Let $S^2_0$ be a compact smooth surface. If $\chi(S^2_0) > 0$, then there exists some $t_0 \geq 0$, such that the surfaces $S^2_t$ are isometrically embeddable in $\mathbb{R}^3$, for any $t \geq t_0$.

In contrast, for (complete) surfaces uniformized by the hyperbolic plane we have only a negative result:

Proposition 2.14. Let $(S^2_0, g_0)$ be a complete smooth surface, and consider the normalized Ricci flow on it. If $\chi(S^2) < 0$, then there exists some $t_0 \geq 0$, such that the surfaces $S^2_t$ are not isometrically embeddable in $\mathbb{R}^3$, for any $t \geq t_0$.

For further related facts and comments, as well as some experimental confirmation of these results, see [75].

2.2. Metric Ricci Curvature for PL manifolds. Our approach here stems from Formula (3.53) of [75] and it involves a combination of Wald’s curvature and the earlier work of Stone [88], [89]. The main (in fact essentially the only) part of Stone’s work on which we rely upon is in his method of determining the relevant two sections and, of course, to decide what a direction at a vertex of a PL manifold is. We begin our discussion of PL Ricci curvature with this basic fact, as well as with related preliminaries. Next we explore various generalizations of the Bonnet-Myers Theorem, and we conclude by introducing a fitting version of scalar curvature.

2.2.1. Definition and Basic Results. In Stone’s work, combinatorial Ricci curvature is defined both for the given simplicial complex $\mathcal{T}$, and also for its dual complex $\mathcal{T}^*$. In the later case, cells – here playing the role of the planes in the classical setting of which sectional curvatures are to be averaged – are considered. Unfortunately, his approach for the given complex, where one computes the Ricci curvature $\text{Ric}(\sigma, \tau_1 - \tau_2)$ of an $n$-simplex $\sigma$ in the direction of two adjacent $(n-1)$-faces, $\tau_1, \tau_2$, is less natural in a geometric context (even if useful in his purely combinatorial one), except for the 2-dimensional case, where it coincides with the notion of Ricci curvature in a
direction (i.e., in this case, an edge). On the other hand, passing to the dual complex will not confine us, since \((T^*)^* = T\) and, moreover – and more importantly – considering thick triangulations enables us to compute the more natural metric curvature for the dual complex and use the fact that the dual of a thick triangulation is thick, as we shall detail below. Moreover, working only with thick triangulations does not restrict us, however, at least in dimension \(\leq 4\), since any triangulation admits a “thickening” – see [70].

To define and compute the Ricci curvature of \(T\) and \(T^*\) and the connection between them, we have to essentially employ the thickness of the given complex. Before proceeding further, it is imperative that we emphasize again the fundamental role of thickness in the sequel: Thickness ensures, by its definition, the fact that no degeneracy of the simplices occurs, hence no collapse and degeneracy of the metric can take place. Moreover, in its absence no uniform estimates for the edge lengths can be made, hence convergence of (dual) meshes and, as we shall see shortly, of their metric Ricci curvatures, can not be guaranteed. Keeping in mind Munkres’ definition of thickness [73] cf. [61], we begin by noting that, since the length of the edge \(l^*_{ij}\), dual to the edge \(l_{ij}\) common to the faces \(f_i, f_j\) equals \(r_i + r_j\), the first barycentric subdivision of a thick triangulation is thick. (For planar triangulations, and also for higher dimensional complexes embedded in some \(\mathbb{R}^N\), one can realize the dual complex (also in \(\mathbb{R}^N\)) by constructing the dual edges \(l^*_{ij}\) orthogonal to the middle of the respective \(l_{ij}\)-s. To prove the thickness of the dual simplices, one has also to make appeal to the characterization of thickness in terms of dihedral angles (4.2) in [75]. To be sure, the notion of thickness also makes sense for for general cells:

**Definition 2.15.** Let \(c = c^k\) be a \(k\)-dimensional cell. The thickness (or fatness) of \(c\) is defined as:

\[
\varphi(c) = \min_b \frac{\text{Vol}(b)}{\text{diam}^l(b)},
\]

This holds, as already mentioned, for any \(PL\) manifold of dimension \(\leq 4\), and in all dimensions for smoothable \(PL\) manifolds, as well for any manifold of class \(\geq C^1\). Since the proof of one of our main results, regarding manifolds of dimension higher than 3, holds only for manifolds admitting smoothings, restricting ourselves only to such manifolds does not represent any additional obstruction.

\(^5\)needed in the construction of the dual complex – see e.g. [44]
where the minimum is taken over all the \( l \)-dimensional faces of \( c \), \( 0 \leq k \). (If \( \dim b = 0 \), then \( \text{Vol}(b) = 1 \), by convention.)

The following facts now follow immediately:

**Lemma 2.16.** Let \( T \) be a thick (simplicial) complex, and let \( T^* \) denote its dual. Then the following hold:

1. \( T^* \) is thick.
2. Let \( \delta(T), \delta(T^*) \) denote the mesh of \( T, T^* \). Then

\[
\lim_{\delta(T) \to 0} (T) = \lim_{\delta(T^*) \to 0} (T^*),
\]

where the convergence is in the Gromov-Hausdorff metric.

We can now return to the definition of Ricci curvature for simplicial complexes: Given a vertex \( v_0 \), in the dual of a \( n \)-dimensional simplicial complex, a direction at \( v_0 \) is just an oriented edge \( e_1 = v_0v_1 \). Since, there exist precisely \( n \) 2-cells, \( c_1, \ldots, c_n \), having \( e_1 \) as an edge and, moreover, these cells form part of \( n \) relevant variational (Jacobi) fields (see [88]), the Ricci curvature at the vertex \( v \), in the direction \( e_1 \) is simply

\[
\text{(2.10) } \text{Ric}(v) = \sum_{i=1}^{n} K(c_i).
\]

Note that the index “\( i \)” in the definition \( \text{(2.10)} \) above runs from 1, and not from 2, as expected judging from the classical (smooth) setting. This is due to the fact that we defined Ricci curvature by passing to the dual complex, with its simple but demanding (so to say) combinatorics. This fact has further implications – see Theorem \( \text{2.34} \) and Remark \( \text{2.35} \) below.

To determine – using solely metric considerations – the sectional curvatures \( K(c_i) \) of the cells \( c_i \), we shall employ the (modified) Wald curvature \( K_W \). Let us first note the role of the abstract open sets \( U \) in the definition of Wald curvature (Definition 3.26 of [76]) is naturally played by the cells \( c_i \). We can now formulate the following definition:

**Definition 2.17.** Let \( c \) be a cell with vertex set \( V_c = \{v_1, \ldots, v_p\} \). The (metric) curvature \( K(c) \) of \( c \) is defined as:

\[
\text{(2.11) } K(c) = \min_{\{i,j,k,l\} \subseteq \{1, \ldots, p\}} \kappa(v_i, v_j, v_k, v_l).
\]
Remark 2.18. In the definition above we presume that cells in the dual complex have at least 4 vertices. However, except for some totally degenerate (planar) cases, this condition always holds. However, even in this case Ricci curvature can be computed using a slightly different approach – see the following remark.

Remark 2.19. Note that by choosing to work with the dual complex we have restricted ourselves largely to considering solely submanifolds of $\mathbb{R}^N$, for some $N$ sufficiently large. However, in the case of 2-dimensional $PL$ manifolds this does not represent restriction, since, by a result of Burago and Zalgaller [22] (see also [74] and the references therein) such manifolds admit isometric embeddings in $\mathbb{R}^3$, embeddings that, furthermore, are unique (up to isometries of the ambient space, of course).

Remark 2.20. It should be emphasized that we have followed [88] only in determining the variational fields, but not in his definition of Ricci curvature. However, it is still possible (by dualization) to compute Ricci curvature according, more-or-less, to Stone’s ideas, at least for the 2-dimensional case. (For more details see [38].)

2.2.2. A Bonnet-Myers Theorem. Having introduced a metric Ricci curvature for $PL$ manifolds, one naturally wishes to verify that this represents, indeed, a proper notion of Ricci curvature, and not just an approximation of the classical notion. According to the synthetic approach to Differential Geometry (see, e.g. [35], [96]), a proper notion of Ricci curvature should satisfy adapted versions of the main, essential theorems that hold for the classical notions. Amongst such theorems the first and foremost is Myers’ Theorem (see, e.g., [8]). And, indeed, fitting versions for combinatorial cell complexes and weighted cell complexes were proven, respectively, by Stone [88], [89], and Forman [32]. Moreover, the Bonnet part of the Bonnet-Myers theorem, that is the one appertaining to the sectional curvature, was also proven for $PL$ manifolds, again by Stone – see [90], [87].

The 2-dimensional case. In the degenerate – but of main importance in applications (see [23], [39], [88]) – case of 2-dimensional manifolds, such a result is easy to formulate and prove, due to the fact that Ricci and sectional curvature essentially coincide:
Theorem 2.21 (Bonnet-Myers for PL 2-manifolds – Metric). Let $M^2_{PL}$ be a complete, connected 2-dimensional PL manifold such that

(i’) There exists $d_0 > 0$, such that $\text{mesh}(M^2_{PL}) \leq d_0$, (where $\text{mesh}(M^2_{PL})$ denotes the mesh of the 1-skeleton of $M^2_{PL}$, i.e. the supremum of the edge lengths).

(ii’) $K_W(M^2_{PL}) \geq K_0 > 0$.

Then $M^2_{PL}$ is compact and, moreover

\begin{equation}
\text{diam}(M^2_{PL}) \leq \frac{\pi}{\sqrt{K_0}}.
\end{equation}

Remark 2.22. Condition (i), that ensures that the set of vertices of the PL manifold is “fairly dense\(^6\) is nothing but the necessary and common density condition for good approximation of both distances and of curvature measures, as we have already expounded in [75], Section 4. The mere existence of such a $d_0$ is evident for a compact manifold, however it can’t be apriori be supposed for a general manifold, hence has do be postulated. In addition, to ensure a good approximation of curvature in secant approximation, this density factor has to be properly chosen (see, e.g. [80]), therefore tighter estimates for the mesh of the triangulation can be obtained using better curvature approximation. In this context we should also note that, apparently, the bound for diameter given by the proof above, is tighter than the one obtained by Stone in [90], Theorem 3. Nevertheless, we should keep in mind that, in practice, one is more likely to encounter PL surfaces as approximations of smooth ones\(^7\) However, the larger the mesh of the approximating surface (i.e. the “rounder” the approximation), the larger the deviation of the approximating triangles from the tangent planes (at the vertices), hence the more likely is to obtain large combinatorial curvature. Hence, there is a correlation between size of the simplices and curvature, even though not a straightforward one. No less importantly, an adequate choice of the vertices of the triangulation, also ensures, via the thickness property, the non-degeneracy of the manifold (and of its curvature measures), as we have detailed in Section 4 of [75].

Remark 2.23. Using the same approach a similar result for the combinatorial (defect) Gauss curvature of $M^2_{PL}$, $K_{Comb}(v_i) = 2\pi - \sum_{p=1}^{n_i} \alpha_p(v_i)$ – see [38],

\(^6\)in Stone’s formulation ([87], p. 1062).

\(^7\)and, obviously, PL surfaces are PL approximations of their own smoothings
Remark 2.24. The result above extends easily to polyhedral manifolds, since Wald curvature does not take into account the number of sides of the faces incident to a vertex, but only their lengths.

The $n$-dimensional case, $n \geq 3$. The proof of above does not extend immediately to higher dimensions, since, in general, no smoothing of a $PL$ manifold exists in dimension higher than $n > 4$ and, even if it exists, it is not necessarily unique, for $n \geq 4$ – see [60]. However, if such a smoothing exists, the proof of does extend to any dimension, and we obtain the following $PL$ (metric) versions of the classical results:

**Theorem 2.25 (PL Bonnet-Myers – metric).** Let $M^n_{PL}$ be a complete, $n$-dimensional $PL$, smoothable manifold without boundary, such that

(i") There exists $d_0 > 0$, such that $\text{mesh}(M^n_{PL}) \leq d_0$;

(ii") $K_W(M^n_{PL}) \geq K_0 > 0$,

where $K_W(M^n_{PL})$ denotes the sectional curvature of the “combinatorial sections”, i.e. the cells $c_i$.

Then $M^n_{PL}$ is compact and, moreover

\[ \text{diam}(M^n_{PL}) \leq \frac{\pi}{\sqrt{K_0}}. \]

**Remark 2.26.** An approach similar to the one used in the proof above was also employed by Cheeger [23] in a rather similar context. (It should be emphasized here that, as a byproduct of the results in [38] (and in this section), we also address – using our own methods – a problem posed by Cheeger in [23], Remark 3.5.)

Before passing further on, we should underline the fact that, if we adopt the viewpoint of $PL$ (secant) approximations of smooth manifolds, then a number of problems arise. Indeed, as we have already emphasized in Section 4 of [75], even when such a smoothing $M^n$ ($n \geq 3$) exists, it is not probable that its sections provided by $M^n_{PL}$, suffice to approximate well enough – let alone reconstruct – the Ricci curvature of $M^n$. Simply put, “there are not enough directions” in $M^n_{PL}$ to allow us to infer from the metric curvatures of a $PL$ approximation, those of a given smooth manifold $M^n$ (in fact, not even a good approximation). On the other hand, increasing of the number of directions, i.e. of 2-dimensional sections (simplices) generates a decrease of the the precision of the approximation, due to the (possible) loss of thickness of the triangulation – a problem which we have also discussed in detail in
The remarks above indicate that, unfortunately, in higher dimensional case, no general analogue of Myers' Theorem for PL manifolds can be obtained by applying solely smoothing arguments. It is true that a Ricci curvature of the smooth manifold $M^n$ is obtained in terms of that of $M^n_{PL}$. However, it is not clear, in view of the paucity of sectional directions (i.e. possible 2-sections), how precisely is this connected to its discrete counterpart. Therefore, we can obtain, at best, an approximation result (with limits imposed by the thickness constraint – see discussion above).

Due to this problematic aspect of the approximations approach, and in accord with the general mathematical drive towards generality, we adopt another approach in the following section.

Alexandrov spaces. We first pointing out the, perhaps not known well enough fact that Wald’s curvature is essentially equivalent with the much more modern notion of Alexandrov curvature, at least for spaces in which there exists “sufficiently many” minimal geodesics (see, for instance, [67, Corollary 40]), condition that certainly is fulfilled in PL surfaces. Since Alexandrov curvature represents, by now, a quite classical and standard notion, and since introducing it formally here would take us too far afield, we will not bring here the technical definition and further details, but rather we refer the reader to, e.g. [20]. However, we should mention that, in defining Alexandrov curvature, one makes appeal to comparison triangles in the model space (i.e. gauge surface $S_\kappa$), rather than quadrangles, as in the definition of Wald curvature. In fact, the sd-quads are, “up to epsilon” one of the (equivalent) ways of defining Alexandrov curvature. The reason for which we prefer to work with the Wald curvature, is that it is computable and, moreover, that it has even simpler, more practical approximations – see [79]). (For the practical consequences of the similarities and differences between the two approaches, see [75].)

It is, however, essential to notice that one has take into account the “discrete” nature of the types of spaces considered, hence to compute solely the Wald curvature of the 1-star neighbourhood of a vertex, as already stressed above, and not to consider (ever) smaller neighbourhoods, as perhaps natural in other contexts. This, however, agrees with the method of computing discrete curvature as angular defect, as employed, for instance, in [17] and in the Chow-Luo discrete Ricci flow [28] (as well as in many other instances
– see the bibliography for some of them). A positive consequence of his fact is that any such neighbourhood becomes a region having the same Alexandrov curvature bounded from below as the computed Wald one. Moreover, by the Alexandrov-Topogonov Theorem (see, e.g. [67], Theorem 43 and its proof, pp. 837-840), the whole surface becomes a space of curvature (Wald or Alexandrov) bounded from below.

In addition, taking into account only these “discrete” neighbourhoods is extremely important when equating the Wald and Alexandrov curvature, since it allows to avoid the blow-up of Alexandrov curvature at the vertices during smoothing. However, if one still wishes to consider smaller-and-smaller neighbourhood of the vertices (motivated, perhaps, by other applications then Imaging and Graphics, such as those in Regge calculus [25]), one can resort to the basic approach of Brehm and Kühnel, that is “rounding” the edges by cylinders of radius $\varepsilon$ (without any change in curvature) and replacing the polyhedral cones at the vertices by smooth “caps”, up to a predetermined admissible error of, say, $\varepsilon_1$. Note that such a “filtration” of $K_W$ by Gaussian curvature (of the approximating smooth surfaces) is in concordance with common practices in Imaging, Vision and, indeed, in many applicative fields. Moreover, considering only this “discrete” neighbourhoods is very important when equating the Wald and Alexandrov curvature, because it also allows us to avoid the blow-up of Alexandrov curvature at the vertices during smoothing.

By making appeal to the theory of Alexandrov spaces, and by using the equivalence of Wald and Alexandrov curvatures with the above mentioned provisos, a result of the desired type follows immediately from [67], Corollary 47, p. 840 and from the fact that $M_{n}^{PL}$ is locally compact:

**Theorem 2.27** (Bonnet-Myers – Alexandrov Spaces). Let $M_{n}^{PL}$ be a complete, connected PL manifold, such that $K_W(M_{n}^{PL}) \geq K_0 > 0$.

Then $M_{n}^{PL}$ is compact and, moreover

$$\text{diam}(M_{n}^{PL}) \leq \frac{\pi}{\sqrt{K_0}}.$$

Unfortunately, determining whether a general PL complex has Wald curvature bounded from below can be, in practice, quite difficult. However, in the special case of thick complexes (see definition in Section 1) one can determine a simple criterion as follows.
Lemma 2.28. Let \( M = M^n_{PL} \) be a complete, connected PL manifold thickly embedded in some \( \mathbb{R}^N \), such that \( K_W(M^2) \geq K_0 > 0 \), where \( M^2 \) denotes the 2-skeleton of \( M \). Then there exists \( K_1 > 0 \) such that \( K_W(M^n_{PL}) \geq K_1 > 0 \).

The appropriate version of Bonnet-Myers now follows as a direct corollary:

**Theorem 2.29 (Bonnet-Myers – Thick Complexes).** Let \( M = M^n_{PL} \) be a complete, connected PL manifold thickly embedded in some \( \mathbb{R}^N \), such that \( K_W(M^2) \geq K_0 > 0 \), where \( M^2 \) denotes the 2-skeleton of \( M \). Then \( M^n_{PL} \) is compact and, moreover

\[
\text{diam}(M^n_{PL}) \leq \frac{\pi}{\sqrt{K_0}}.
\]

*Remark 2.30.* We have formulated the theorem above in terms of piecewise flat manifolds since this is the case of most interest, both for theoretical ends, as well as application oriented ones. The most natural and useful instance in which such manifolds arise is that of secant approximations to smooth manifolds, as detailed and emphasized in the previous sections. However, the proof extends – mutatis mutandis – to the case of spaces whose simplices are modelled after spherical or hyperbolic spaces.

*Remark 2.31.* This result is hardly surprising, given the fact that, by [21], Theorem 3.6, Myers’ theorem holds for general Alexandrov spaces of curvature \( \geq K_0 > 0 \), and since, as already detailed above, Wald-Berestovskii curvature is essentially equivalent to the Rinow curvature, hence to the Alexandrov curvature. However, instead of making appeal to the “full force” of the original proof, we gave, in the special case of PL manifolds a simpler and more intuitive proof of the Burago-Gromov-Perelman extension of Myers’ Theorem.

2.2.3. **Metric scalar curvature.** Note that, while we defined and discussed into some depth Ricci curvature, we have not, up to this point, defined a fitting scalar curvature \( K(c) \) of a cell \( c \). Given our preceding discussion and keeping in mind the defining Formula (3.56) in [75], the definition below is most natural:

**Definition 2.32.** Let \( M = M^n_{PL} \) be an \( n \)-dimensional PL manifold (without boundary). The *scalar metric curvature* \( \text{scal}_W \) of \( M \) is defined as

\[
\text{scal}_W(v) = \sum K_W(c),
\]

the sum being taken over all the cells of \( M^* \) incident to the vertex \( v \) of \( M^* \).
Remark 2.33. While the definition of scalar curvature of $M$ is defined, somewhat counterintuitively, by passing to its dual $M^*$, this approach is consistent with our approach to Ricci curvature (and also similar to Stone’s original ideas – see the discussion in 4.1 above).

From this definition and our definition of sectional curvature of a cell, an immediate generalization of the classical curvature bounds comparison in Riemannian geometry follows (independently, in fact, of the chosen definition for the curvature of a cell).

**Theorem 2.34** (Comparison theorem). Let $M = M^n_{PL}$ be an $n$-dimensional PL manifold (without boundary), such that $K_W(M) \geq K_0 > 0$, i.e. $K(c) \geq K_0$, for any 2-cell of the dual manifold (cell complex) $M^*$. Then

$$K_W \geq \frac{\geq}{\geq} K_0 \Rightarrow \text{Ric}_W \leq \frac{\leq}{\leq} nK_0.$$  

Moreover

$$K_W \geq \frac{\geq}{\geq} K_0 \Rightarrow \text{scal}_W \leq \frac{\leq}{\leq} n(n+1)K_0.$$  

**Remark 2.35.** (1) Inequality (2.18) can also be formulated in the seemingly weaker form:

$$\text{Ric}_W \geq \frac{\geq}{\geq} nK_0 \Rightarrow \text{scal}_W \leq \frac{\leq}{\leq} n(n+1)K_0,$$

(2) We should note that in all the inequalities above, the dimension $n$ appears, rather then $n-1$ as in the smooth, Riemannian case (hence, for instance one has in (2.18), $n(n+1)K_0$, instead of $n(n-1)K_0$, as in the classical case). This is a consequence of our definition (2.10) of Ricci (and scalar) curvature, that makes appeal to the dual complex of the given triangulation, therefore imposing standard and simple combinatorics, that allow only for such weaker bounds.

**Remark 2.36.** Before concluding this section, we should mention the fact that we also discussed previously in [75], namely that, by viewing compact PL surfaces with as Alexandrov surfaces (with curvature bounded from below), one can apply the methods and results of Richard [69], to obtain
again the sought for results. Alternatively, one can mimic his approach, itself retracing the steps of the proof in the smooth case. While this is a task that we defer for further research, we take a deeper and more detailed look of the geometric consequences of viewing surfaces of bounded Wald curvature as Alexandrov surfaces in [78].

2.3. A Different Approach: From Geodesic Curvature to Ricci Curvature. We briefly sketch below a different strategy of defining a notion of Ricci curvature for PL and more general polyhedral manifolds. To this end, we make appeal to the basic idea, stemming from Stone’s work [88], [89], that we used in [38] and reviewed here in Section 2.2 above; in conjunction with Haantjes curvature in its role of geodesic curvature. The new, non-metric ingredient in our method is a discretization of the classical local Gauss-Bonnet theorem (see, e.g. [31]).

To begin with, less us notice that, in the case of piecewise flat manifolds (i.e the “staple” input for computations in Computer Graphics), the simplest and most direct approach is to view, for each triangle $T$ adjacent to an edge $e$, its Menger curvature $\kappa_M(T)$ (see [76]) as the sectional curvature of the section (plane) $T$. Since the Menger curvature represents a measure of the “flatness” of a triangle, $\kappa_M(T)$ gives a measure for the spreading of geodesics in the plane $T$, thus it provides, indeed, a discrete sectional curvature. Equipped with this metric version of sectional curvature, and using the same notations as in [38], it is immediate to obtain the fitting versions of Ricci and scalar curvatures:

\begin{equation}
\kappa_M(e) \text{ not} = \text{Ric}_M(e) = \sum_{T_e \sim e} \kappa_M(T_e),
\end{equation}

where $T_e \sim e$ denote the triangles adjacent to the edge $e$; and

\begin{equation}
\kappa_M(v) \text{ not} = \text{scal}_M(v) = \sum_{e_k \sim v} \text{Ric}_M(e_k) = \sum_{T \sim v} \kappa_M(T);
\end{equation}

where, $e_k \sim v, T \sim v$ stand for all the edges adjacent to the vertex $v$ and all the triangles $T$ having $v$ as a vertex, respectively. Note that $\text{Ric}_M(e)$ captures, in accordance to the intuition behind $\kappa_M(T)$ the geodesic dispersion rate aspect of Ricci curvature. (See [38] for a concise overview of the different aspects of Ricci curvature.)
While this manner of extending a simple metric curvature to the context of piecewise flat manifolds (and, in fact, of PL, in general), is quite direct and intuitive, it has also two limitations that hinder its usefulness in the study of such manifolds. The first such impediment is the fact that, being a discretization of the curvature notion for planar curves, Menger curvature of triangles is, intrinsically, always positive. Therefore, it can represent only a discretization of the absolute value of sectional curvature (hence of Ricci and scalar curvatures as well). The second obstruction in extending this approach to more general types of meshes/manifolds resides in the fact that, Menger curvature, by its very definition, is restricted solely to triangles, thus is not applicable to more general cells.

However, making instead appeal to another type of metric curvature provides us with a solution for both problems. The second problem that we singled out above needs to be solved in the beginning, by considering instead of Menger curvature, Haantjes curvature. More precisely, given an edge \( e = (u, v) \) and a cell \( c, \partial c = (u = v_0, v_1, \ldots, v_n = v) \), it is natural to define

\[
K_H(c) = \kappa_H(\pi),
\]

or rather, as we shall fully justify below, as

\[
(2.22) \quad K_H(c) = -\kappa_H(\pi);
\]

where \( \pi \) denotes the path \( v_0, v_1, \ldots, v_n \), subtended by the chord \( \tilde{e} = \overline{v_0v_n} \). (Observe the slight change of notation conforming the one used in the definition of Haantjes curvature [76].) With this definition we can also better extend the notion of, say, Ricci curvature, to PL (piecewise flat) manifolds: By passing again (as in the previous section) to the dual manifold, one can ensure (except in the most degenerate case) that we deal with a polyhedral manifold, having 2-cells with more than 3 sides. This turns out to be essential here, even more than in Stone’s approach, in defining a signed metric curvature, first of cells, thence of polyhedral manifolds. Note that we have to make appeal to Haantjes curvature, and not to the more simple, and well known, Menger curvature, do to the fact that, in general, there exists no natural, unique way of subdividing a cell (not embedded in some \( \mathbb{R}^n \)) into triangles.
The essential idea behind Formula (2.22) above is to make appeal to the local Gauss-Bonnet Theorem. Recall that, in the classical context of smooth surfaces it states that

\[
\int_D K dA + \sum_0^p \int_{v_i}^{v_{i+1}} k_g dl + \sum_0^p \phi_i = 2\pi \chi(D) ;
\]

where \(D \simeq \mathbb{B}^2\) is a (simple) region in the surface \(S^2\), having as boundary \(\partial D\) a piecewise-smooth curve, of vertices (i.e. points where \(\partial D\) is not smooth) \(v_i, i = 1, \ldots, p, (v_{n+1} = v_0)\); \(\phi_i\) denotes the external angles of \(\partial D\) at the vertex \(v_i\); and \(K\) and \(k_g\) denote (as usually) the Gaussian and geodesic curvatures, respectively.

We should note first that, in the absence of a background curvature, the very notion of angle is undefinable. Therefore, for abstract (non-embedded) cells, no meaningful (“decent”) notion of angle exists. In consequence, the last term on the left side of (2.23) above has no signification, thus we should discard it. Indeed, the distances between non-adjacent vertices on the same cycle (apart from the path metric) are not defined, thus the third term in the left side of formula (2.23) disappears. (We shall, however, reconsider it for embedded piecewise-flat manifolds.)

Moreover, for “purely” combinatorial PL manifolds, the area of each cell can be prescribed (as it usually is) as being equal to 1. Moreover, one assumes (quite naturally) that curvature is constant on each cell. Therefore, the first term in the left side of (2.23) reduces simply to \(K\). In addition, given that \(D\) is a 2-cell, thus \(\chi(D) = 0\), the right hand of the same vanishes. It follows, that in such a setting we obtain

\[K = -\int_{\partial D} k_g dl .\]

It is tempting to next consider \(\partial D\) as being composed of segments (on which \(k_g\) vanishes), except at the vertices, thus rendering expression above as

(2.24) \[K = -\sum_{i=1}^n \kappa_H(v_i) .\]

However, in abstract piecewise-flat manifolds, one can not define a non-trivial Haantjes curvature for each of the vertices since, as already noted
above, no distance (apart from the one given by the path metric) between the vertices \( v_{i-1} \) and \( v_{i+1} \) can be considered. In fact, in this general case, neither can the arc (path) \( \pi = v_0v_1 \ldots v_n \) be truly viewed as smooth. Therefore, in this context, we have no choice but to replace the right term in Formula (2.24) above by \( \kappa_H(\pi) \), where it should be remembered that \( \pi \) represents the path \( v_0, v_1, \ldots, v_n \), of chord \( \bar{e} = \overline{v_0v_n} \). We have thus obtained the proposed Formula (2.22). Thus, the suggested definition for the Ricci curvature of a general metric cell-complex becomes

\[
\text{(2.25)} \quad \text{Ric}(e) = \sum_{c \sim e} \kappa_H(c),
\]

where the sum is taken over all the 2-cells \( c \) adjacent to \( e \).

Before concluding this section, we should retrace our steps in deriving the proposed formula for computing the discrete Ricci curvature based on a fitting adaptation of the local Gauss-Bonnet formula, and point out that, for piecewise-flat manifolds embedded in \( \mathbb{R}^3 \) (but not only – see for instance [1], [2]) one can establish a discrete version of Gaussian/sectional curvature much closer to the classical one\(^{11}\). Since, in this case, proper angles at the vertices \( v_i \) are defined, one can also compute the curvatures \( \kappa_H(v_i) \) and, moreover, the area of each cell can be calculated (and it is independent on the specific subdivision of the cell into triangles). We defer the implementation, to the context of Graphics, of such work for later study. Moreover, by adapting the arguments of [38], one can seemingly obtain similar results to those therein, namely regarding the convergence of this discretization of Ricci curvature, and analogues of the Bonnet-Myers and comparison theorems.

3. AN ALTERNATIVE APPROACH: FORMAN’S RICCI CURVATURE

If one is willing to sacrifice the Wald metric curvature paradigm, an alternative approach – albeit somewhat more abstract – suggest itself, namely that based on Forman’s Ricci curvature [32]. While admittedly less geometric in nature (its development being based on constructing a Bochner Laplacian on weighted \( CW \) complexes), it incorporates naturally the given

\(^{11}\)Indeed, piecewise-flat surfaces (and \( PL \) manifolds in general), can be viewed as bridging the gap between smooth manifolds and discrete (metric) spaces.
metric (as, indeed, the idea residing behind its introduction is to discretize the metric and curvature of a Riemannian manifold).

Before proceeding further to the manner one can employ this curvature to the metric setting at hand, let us first give its definition:

Given a \( p \)-dimensional cell (or \( p \)-cell, for short) \( \alpha = \alpha^p \), one can define the \textit{curvature function}:

\[
(3.1) \quad F_p = \langle F_p(\alpha), \alpha \rangle,
\]

where \( F_p : C_p \rightarrow C_p \) is being regarded as a linear function on \( p \)-chains of cells (see [32] for details), the scalar product appearing in the formula being defined (rather standardly) by \( <\alpha, \alpha> = w, <\alpha, \beta> = \alpha \neq \beta \), and where

\[
(3.2) \quad F(\alpha^p) = w(\alpha^p) \left[ \left( \sum_{\beta^{p+1} > \alpha^p} \frac{w(\alpha^p)}{w(\beta^{p+1})} \right) + \sum_{\gamma^{p-1} < e_2} \frac{w(\gamma^{p-1})}{w(\alpha^p)} \right] - 
\]

\[
- \sum_{\alpha^p || \alpha^p, \alpha^p_1 \neq \alpha^p} | \sum_{\beta^{p+1} > \alpha^p, \beta^{p+1} > \alpha^p} \frac{\sqrt{w(\alpha^p)w(\alpha^p_1)}}{w(\beta^{p+1})} - \sum_{\gamma^{p-1} < \alpha^p, \gamma^{p-1} < \alpha^p} \frac{w(\gamma^{p-1})}{\sqrt{w(\alpha^p)w(\alpha^p_1)}} | \]

the notation \( \alpha < \beta \) meaning that \( \alpha \) is a face of \( \beta \), and the notation \( \alpha_1 || \alpha_2 \) signifies that the simplices \( \alpha_1 \) and \( \alpha_2 \) are \textit{parallel}, parallelism being defined as follows:

\textbf{Definition 3} Let \( \alpha_1 = \alpha_1^p \) and \( \alpha_2 = \alpha_2^p \) be two \( p \)-cells. \( \alpha_1 \) and \( \alpha_2 \) are said to be \textit{parallel} \( (\alpha_1 || \alpha_2) \) iff either: (i) there exists \( \beta = \beta^{p+1} \), such that \( \alpha_1, \alpha_2 < \beta \); or (ii) there exists \( \gamma = \beta^{p-1} \), such that \( \alpha_1, \alpha_2 > \gamma \) holds, but not both conditions simultaneously.

In the special case \( p = 1 \), one is conducted, by analogy to the smooth case, to the following

\textbf{Definition 3.1.} Let \( \alpha = \alpha^1 \) be a 1-cell (i.e. an edge). Then the \textit{Forman Ricci curvature} of \( \alpha \) is defined as:

\[
(3.3) \quad \text{Ric}_F(\alpha) = F_1(\alpha).
\]

Therefore, it follows from Formula (3.2) above that...
\begin{equation}
\text{Ric}_F(e) = \omega(e) \left[ \left( \sum_{e \sim f} \frac{\omega(e)}{\omega(f)} + \sum_{v \sim e} \frac{\omega(v)}{\omega(e)} \right) - \sum_{e,e \sim f} \frac{\sqrt{\omega(e) \cdot \omega(e)}}{\omega(f)} - \sum_{v \sim e,v \sim \hat{e}} \frac{\omega(v)}{\sqrt{\omega(e) \cdot \omega(\hat{e})}} \right].
\end{equation}

The comprehensiveness of Formula \text{3.2} above naturally raises questions about its range of applicability and effectiveness in two regards: The possible choices of weights and the specific type of cell complex considered.

As far as the weights are considered, Forman’s work assures us (Theorems 2.6 and 3.9) that any set of weights encountered in practice can be used (or at least arbitrarily well approximated). However, both in theory and in practice, \textit{natural} (or \textit{geometric}) sets of weights, i.e. proportional the dimension of the cell (s.a. length, area, volume) are preferable. This is not due just to the specific techniques employed in [32], it is in fact ingrained in the very idea resting behind Forman’s article, namely that of studying curvatures (and Laplacians) of “good” discretizations, e.g. triangulations, cubulations, etc., of smooth Riemannian manifolds, hence of their metrics and volume elements. We have employed this type of weights in the case of complexes consisting from square grids naturally arising in the Image Processing setting, in [31], [3], [85]. However, much more general (positive) weights can be considered, as already noted above. This is extremely important in many practical applications, for instance in Medical Imaging, where, for instance, MRI images are given, basically, by proton densities,\footnote{Curvature measures also are natural occurring weights in Imaging and Graphics, and weights can be attached to textures as well} but also in other fields, e.g. Manifold Learning where weights can be arise as probabilities or importance attached to a specific region, etc. We shall shortly return to the problem of “correctly” prescribing Ricci curvature to such weighted cell complexes.

Regarding the type of underlying complex, a number of observations are mandatory: First, there are two specific types of structures that warrant special attention, since they appear naturally and often in many instances, both practical and theoretical. The first such instance is that of \textit{cube manifolds}, for which, due to the simple form of parallelism holding in their case, Formula \text{3.2} is drastically simplified – see [32], Theorem 8.1 for the formula in the \textit{n}-dimensional case, and the papers mentioned above for the weighted
2 (and 3) dimensional complexes encountered in Imaging.)  which for the
weighted square and cube grids that arise in classical and volumetric Imag-
ing become

\[
\text{Ric}_{F}(e_0) = w(e_0) \left[ \frac{w(e_0)}{w(c_1)} + \frac{w(e_0)}{w(c_2)} \right] - \left( \frac{\sqrt{w(e_0)w(e_1)}}{w(c_1)} + \frac{\sqrt{w(e_0)w(e_2)}}{w(c_2)} \right),
\]

and

\[
\text{Ric}_{F}(e_0) = w(e_0) \left[ \frac{1}{\sqrt{w(e_0)}} \right] - \sqrt{w(e_0)} \left( \frac{4}{w(c)} \right),
\]

respectively. (Here and above we took into account that, for digital images,
the vertices' weights are always 0.)

See also [77] for an application of metric curvatures to the problem of
smoothability of metrics on cube complexes and the references therein for
the theoretical setting and uses of such complexes.) The second important
case is that of PL manifolds, occurring (and, indeed, motivated) principally
as triangulation and piece-wise flat approximations of smooth surfaces and
higher dimensional manifolds, both in a theoretical setting, but most com-
monly in Graphics, CAD, Imaging and related fields. Here again a simplified
Formula 3.2 can be written (due, in this case, to the lack of parallelism for
the edges of the 2-cells adjacent to an edge \( e \)):

\[
\text{Ric}_{F}(e) = \omega(e) \left[ \sum_{t > e} \frac{\omega(e)}{\omega(t)} + \left( \frac{\omega(v_1)}{\omega(e)} + \frac{\omega(v_2)}{\omega(e)} \right) \right] - \sum_{e \sim \bar{e}} \sum_{t > e, t > \bar{e}} \frac{\sqrt{\omega(e) \omega(\bar{e})}}{\omega(t)} - \sum_{v < e, v < \bar{e}} \frac{\omega(v)}{\sqrt{\omega(e) \omega(\bar{e})}}
\]

Moreover, given that \( e \) has only two 0-dimensional faces (i.e. the vertices
\( v_1, v_2 \)) we can replace the term

\[ \sum_{t > e, t > \bar{e}} \frac{\sqrt{\omega(e) \omega(\bar{e})}}{\omega(t)} \]

with

\[ \sum_{t > e, t > v_1} \frac{\sqrt{\omega(e) \omega(\bar{e})}}{\omega(t)} + \sum_{t > e, t > v_1} \frac{\sqrt{\omega(e) \omega(\bar{e})}}{\omega(t)} \]

We should also note that, for combinatorial weights, the addition to the
complex \( X \) of each triangle \( t \) adjacent to an edge \( e \) increases the Forman
Ricci curvature by 3, that is

\[(3.8) \quad \text{Ric}_F(e|X + t) = \text{Ric}_F(e|X) + 3.\]

(For details, see [100].)

However important these particular cases might be, it is still important to emphasize here again that, since Ricci curvature is a measure attached to edges, it is determined solely by the 2-skeleton, given that edges represent the common boundaries of the 2-cells of complex. This is clearly evident in the general formula for the Forman-Ricci curvature of an edge \([3.4]\).

Thus, Forman-Ricci curvature of even more complicated structures with 2-skeleta of the types above can be easily computed, independently of the structure of the higher dimensional cells. Experimental results so far indicate that, at least for images and triangular meshes, Forman’s Ricci curvature approximates well, as a measure, the Gaussian curvature with whom Ricci curvature essentially identifies in dimension \(n = 2\). A formal analysis of the convergence properties of \(\mathcal{F}(\alpha^p)\) (and of the associated Bochner Laplacian) are currently in progress [55].

**Remark 3.2.** As we have emphasized in [82], the Forman Ricci curvature has a further inherent advantage, namely that it allows for the understanding of the algebraic topological properties (homology and first homotopy groups), rendering it as a useful tool in the intelligence of discrete structures, instead – or in complement of – Persistent Homology.

In this context one could use the vicinity (dual graph) of a given tesselation (e.g triangulation or cubulation), with node weights that concentrate the weights of the respective dual faces, and compute the graph Forman-Ricci curvature and associated flow developed for Complex Networks in [86], [98], [99]. Recall that Forman’s Ricci curvature for an edge is given by the following formula:

\[(3.9) \quad \text{Ric}_F(e) = w_e \left( \frac{w_{v_1}}{w_e} + \frac{w_{v_2}}{w_e} - \sum_{e_{v_1} \sim e, e_{v_2} \sim e} \left[ \frac{w_{v_1}}{\sqrt{w_e w_{v_1}}} + \frac{w_{v_2}}{\sqrt{w_e w_{v_2}}} \right] \right).\]

**Remark 3.3.** One could make appeal to the reduced Forman curvature defined in Formula \([3.9]\) above as a “proxy” for the computation of the “true”
Forman curvature, especially in Imaging tasks, thus simplifying and speeding up computations. However, it is to be expected that the results obtained thus restricting oneself to the “skeleton” of the image will be only very weak approximations of the result when including, via Formula (3.2), the essential information contained in the 2-skeleton (pixels, voxels, etc.) of the given image (or mesh). Still, this approach appears to be quite useful in settings where little geometric content is available, being replaced by probabilistic or even more general densities, typical cases being the analysis of MRI images or texture understanding.

The straightforward analogue of Formulas (2.) and (2.2) is then

\[(3.10) \quad \tilde{\gamma}(e) - \gamma(e) = -\text{Ric}_F(\gamma(e)) \cdot \gamma(e);\]

where \(\tilde{\gamma}(e)\) denotes the new (updated) value of \(\gamma(e)\) after one time step. Note that in our discrete setting, lengths are replaced by the (positive) edge weights. Also, for convenience and clarity, time is assumed to evolve in discrete steps, thus each “clock” has a length of 1. (However, in practice, this can and should be adapted to fit the specific circumstance of a given setup - see also \[85\].)

**Remark 3.4.** A derived scalar curvature and flow can also be considered (and easily derivable from the Ricci curvature and flow), as shown in \[98\], Formulas (10) and (9), respectively.

We should note that this simple and direct approach to the Ricci flow is excellently suited for applications where a short time flow is involved (see, e.g. \[3, 85\]), it is not appropriate in situations where a long type flow should be employed, since it is not equipped with a fitting analogue for graphs of a surface of limit geometry, as in the smooth \[42\] and combinatorial \[28\] cases.

However, for the case of higher dimensional complexes (i.e. for the original setting of Forman’s definition), and in particular for polyhedral surfaces, such a limit geometry can be considered, due to Bloch’s extension \[14\] of Forman’s work to allow for a fitting Gauss-Bonnet type theorem (which was an unfortunate lacuna in \[32\] due to the algebraic approach of the original definition of Ricci curvature therein). This is possible due to his definition of Euler characteristic for posets (and, in particular for cell complexes), of which we do not bring the details here, but rather refer the reader to
the original paper \cite{14}. Suffice to say, that this is achieved by (somewhat technically) defining a triplet of combinatorial curvatures, for each of the dimensions 0, 1 and 2, denoted by $R_1$, $R_2$ and $R_3$, respectively. While their specific definitions are quite technical, we do not detail them here, but content ourselves to noting that they satisfy a Gauss-Bonnet type theorem, more precisely one has

**Theorem 3.5** (Bloch, \cite{14}, Theorem 2.4). Let $X$ be 2-dimensional cell complex. Then:

\[
\sum_{v \in F_0} R_0(v) - \sum_{e \in F_1} R_1(e) + \sum_{f \in F_2} R_2(f) = \chi(X). 
\]  

(3.11)

The full relevance of the result above and of his importance for our study is the fact that, for polyhedral surfaces, the 1-dimensional curvature function and the Forman Ricci curvature coincide, that is

\[
R_1(e) = \text{Ric}_F(e),
\]  

(3.12)

Given that polyhedral complexes represent a special case of cell complexes, from formulas (3.11) and (3.13) above imply that, for a 2-dimensional polyhedral complex, the following holds:

\[
\sum_{v \in F_0} R_0(v) - \sum_{e \in F_1} \text{Ric}_F(e) + \sum_{f \in F_2} R_2(f) = \chi(X). 
\]  

(3.13)

Bloch’s result above not only patches the serious aforementioned gap in Forman’s work regarding his discretization of Ricci curvature, it also enables us to consider a proper long time Ricci flow, by allowing for the definition of prototype (or reference, model) networks, as minimal (in the sense of having the minimal number of 2-faces) 2-dimensional polyhedral complexes with the Euler characteristics prescribed by the Gauss-Bonnet formula (3.13). In consequence, one can define a network to be spherical, Euclidean or hyperbolic, if its Euler characteristic is $>, =$ or $< 0$, respectively. More precisely, we can thus propose

**Definition 3.6** (Prototype networks). Let $X$ be a 2-dimensional polyhedral complex with Euler characteristic $\chi$ as given by the Gauss-Bonnet formula. Then we define $\chi$ to be
(1) Spherical, if $\chi > 0$;
(2) Euclidean, if $\chi = 0$;
(3) Hyperbolic, if $\chi < 0$.

However, it is more convenient and efficient to consider, in analogy with the classical (surfaces) case, an alternative, equivalent definition of prototype complexes, as being complexes of constant curvature, i.e. such that $R_1 = \text{const.}$, thus replacing conditions (1)-(3) above with the following ones, respectively:

(1') $R_1 > 0$;
(2') $R_1 = 0$;
(3') $R_1 < 0$.

This alternative definition is not only more simple, it also facilitates the definition of the desired long term flow, which we formulate here only for the case of 2-dimensional polyhedral complexes. More precisely, we define – in analogy with the classical case [42] (see also (Equation (2.3) above)), and keeping in mind that in this case $R_1(e) = \text{Ric}_F(e)$ – the Normalized Forman-Ricci flow for networks 2-dimensional polyhedral complexes, by

\[
\frac{\partial \gamma(e)}{\partial t} = - \left( \text{Ric}_F(\gamma(e)) - \overline{\text{Ric}_F} \right) \cdot \gamma(e),
\]

where $\overline{\text{Ric}_F}$ denotes the mean Forman-Ricci curvature.

It follows therefore that, by using Bloch’s results, we are able to define a proper notion of background (limit) geometry for complexes that allows for the study of long term evolution of 2-dimensional polyhedral complexes. In consequence, it enables us to study their (topological) complexity, dispersion of geodesics, volume growth, etc. – in fact all the essential properties captured by its geometric type. To be sure, the next step would be to experimentally explore how the above defined flow compares to similar geometric flows, starting with the best known one, namely the combinatorial flow. The experiments conducted so far suggest that, indeed, the notion of limit geometry introduced here is a viable one. However, we have still to provide theoretical proofs of the basic convergence theorems similar to those in [42], [28], this being a task that for now we defer for future study.
ACKNOWLEDGEMENT

The author would like to thank his former students who helped produce Figure 1 in the text, to Jürgen Jost for his warm hosting at the Max Planck Institute for Mathematics in the Sciences, Leipzig, where a substantial part of this paper was conceived and written, to David Xianfeng Gu, whose keen interest and warm encouragement started this project. Thanks are also due to Feng Luo and Areejit Samal for useful discussion regarding the Forman-Ricci flow.

REFERENCES

[1] P. M. Alsing, J. R. McDonald and W. A. Miller, The simplicial Ricci tensor, Class. Quantum Grav., 28, 155007 (17 pp), 2011.
[2] P. M. Alsing, H. A. Blair, M. Corne, G. Jones, W. A. Miller, K. Mishaikow, V. Nanda, Topological Signatures of Singularities in Simplicial Ricci Flow, arXiv:1502.02630v1 [math.AT], 2015.
[3] E. Appleboim, E. Saucan and Yehoshua Y. Zeevi, Ricci Curvature and Flow for Image Denoising and Superresolution, Proceedings of EUSIPCO 2012, 2743-2747, 2012.
[4] D. Bakry and M. Émery, Diffusions hypercontractives, J. Azéma and M. Yor, eds., Séminaire de Probabilités XIX 1983/4 (A. Dold and B. Eckmann, eds.), Lecture Notes Math., 1123, Springer-Verlag, 177-206, 1985.
[5] D. Bakry and M. Ledoux, Lévy-Gromov’s isoperimetric inequality for an infinite dimensional diffusion generator, Invent. Math. 123 (1996), 259-281.
[6] F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi, S.-T. Yau, Li-Yau inequality on graphs, J. Differential Geometry 99, 359-405, 2015.
[7] F. Bauer, J. Jost and S. Liu, Ollivier-Ricci curvature and the spectrum of the normalized graph Laplace operator, Mathematical research letters, 19, 1185-1205, 2012.
[8] M. Berger, A Panoramic View of Riemannian Geometry, Springer-Verlag, Berlin, 2003.
[9] A. Bernig, Scalar curvature of definable Alexandrov spaces, Adv. Geom. 2, 29-55, 2002.
[10] A. Bernig, Curvature bounds of subanalytic spaces, preprint, 2003.
[11] A. Bernig, Curvature tensors of singular spaces, Differential Geometry and its Applications 24, 191-208, 2006.
[12] A. Besse, Einstein Manifolds, Springer-Verlag, Berlin, 1987.
[13] E. Bierstone. and P. D. Milman, Semianalytic and subanalytic sets, Inst. Hautes Etudes Sci. Publ. Math. 67, 5-42, 1988.
[14] E. Bloch, Combinatorial Ricci Curvature for Polyhedral Surfaces and Posets, preprint, arXiv:1406.4598v1 [math.CO], 2014.
[15] A.-I. Bonciocat and K.-T. Sturm, Mass transportation and rough curvature bounds for discrete spaces, J. Funct. Anal. 256(9), 2944-2966, 2009.
[16] V. Borrelli, F. Cazals, and J-M. Morvan, On the angular defect of triangulations and the pointwise approximation of curvatures, In Curves and Surfaces, St Malo, France, 2002. INRIA Research Report RR-4590.
[17] U. Brehm and W. Kühnel, Smooth approximation of polyhedral surfaces regarding curvatures Geometriae Dedicata 12, 435 - 461, 1982.
[18] S. Brendle, Curvature flows on surfaces with boundary, Math. Ann. 324, 491-519, 2002.
[19] S. Brendle, A family of curvature flows on surfaces with boundary, Math. Z. 241, 829-869, 2002.
[20] D. Burago, Y. Burago and S. Ivanov, Course in Metric Geometry, GSM 33, AMS, Providence, 2000.
[21] Yu. D. Burago, M. L. Gromov, G. Ya. Perel’man, A.D. Alexandrov spaces with curvature bounded below, Uspekhi Mat. Nauk, 47:2(284), 3-51, 1992; Russian Math. Surveys, 47(2), 1-58, 1992.
[22] Yu. D. Burago and V. A. Zalgaller, Isometric piecewise linear immersions of two-dimensional manifolds with polyhedral metrics into $\mathbb{R}^3$, St. Petersburg Math. J., 7(3), 369-385, 1996.
[23] J. Cheeger, A Vanishing Theorem for Piecewise Constant Curvature Spaces, Lecture Notes in Mathematics – Proc. Katata 1985 1201, 33-40, 1986.
[24] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. I, II, III, J. Differential Geom. 46 (1997), 406-480; ibid. 54 (2000), 13-35 and 37-74.
[25] J. Cheeger, W. Müller, and R. Schrader, On the Curvature of Piecewise Flat Spaces, Comm. Math. Phys. 92, 405-454, 1984.
[26] B. Chow and F. Luo, Combinatorial Ricci Flows on Surfaces, J. Differential Geom. 63(1), 97-129, 2003.
[27] D. Cohen-Steiner and J.-M. Morvan, Approximation of the Curvature Measures of a Smooth Surface endowed with a Mesh, Research Report 4867, INRIA, 2003.
[28] B. Chow and F. Luo, Combinatorial Ricci Flows on Surfaces, J. Differential Geom. 63(1), 97-129, 2003.
[29] B. Chow and D. Knopf, The Ricci Flow: An Introduction, Mathematical Surveys and Monographs, 110, AMS Providence, RI, 2004.
[30] I. Corwin, N. Hoffman, S. Hurder, V. Šešum and Y. Xu, Differential Geometry of Manifolds with Density, Rose Hulman Undergraduate Journal of Mathematics, 7(1), 15pp, (2006).
[31] M. P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, N.J., 1976.
[32] R. Forman, Bochner’s Method for Cell Complexes and Combinatorial Ricci Curvature, Discrete and Computational Geometry, 29(3), 323-374, 2003.
[33] K. Fukaya, Collapsing of Riemannian manifolds and eigenvalues of Laplace operator, Invent. Math. 87 (1987), 517-547.
[34] G. Gao, X. D. Gu, F. Luo, *Discrete Ricci flow for geometric routing*, in Encyclopedia of Algorithms. Kao, M.-Y., Ed.; Springer: Berlin, Germany, 556-563, 2016.
[35] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, 152, Birkhauser, Boston, 1999.
[36] M. Gromov, *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal. 13, 178-215, 2003.
[37] X. D. Gu, personal communication.
[38] D. X. Gu and E. Saucan, *Metric Ricci curvature for PL manifolds*, Geometry, vol. 2013, Article ID 694169, 12 pages, 2013. doi:10.1155/2013/694169.
[39] X. Gu and S.-T. Yau, *Computing Conformal Structures of Surfaces*, Communications in Information and Systems, 2(2), 121-146, 2002.
[40] X. D. Gu and S.-T. Yau, *Computational Conformal Geometry*, Advanced Lectures in Mathematics 3, International Press, Somerville, MA, 2008.
[41] J. Haantjes, *Distance geometry. Curvature in abstract metric spaces*,Proc. Kon. Ned. Akad. v. Wetenseh., Amsterdam 50, 496-508, 1947.
[42] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17, 255-306, 1982.
[43] R. Hamilton, *The Ricci Flow on Surfaces*, A.M.S. Contemp. Math., 71(1), 237-261 1986.
[44] J. F. Hudson, *Piecewise Linear Topology*, Math. Lect. Notes Series, Benjamin, N.Y., 1969.
[45] M. Jin, J. Kim, and X. D. Gu, *Discrete Surface Ricci Flow: Theory and Applications*, LNCS, 4647, 209-232, 2007.
[46] J. Jost, *Riemannian Geometry and Geometric Analysis*, Springer, 2011.
[47] J. Jost and S. Liu, *Olliviers Ricci curvature, local clustering and curvature-dimension inequalities on graphs*, Discrete & Computational Geometry, 51(2), 300-322, 2014.
[48] B. L. Kotschwar, *Backwards uniqueness for the Ricci flow*, Int. Math. Res. Not. 21, 4064-4097, 2010.
[49] W. Li, J. Gu, S. Liu, Y. Zhu, S. Deng, L. Zhao, J. Han and X. Cai *Optimal Transport in Worldwide Metro Networks*, preprint, arXiv:1403.7844, 2014.
[50] P. Li, and S.T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math., 156(1), 153 201, 1986.
[51] S. Lin, Z. Luo, Z. Wang and E. Saucan, *Generalized Ricci Curvature Based Sampling and Reconstruction of Images*, Proceedings of EUSIPCO 2015, 2015.
[52] Y. Lin, L. Lu and S.-T. Yau, *Ricci curvature of graphs*, Tohoku Mathematical Journal 63(4), 605627, 2011.
[53] B. Loisel and P. Romon, *Ricci curvature on polyhedral surfaces via optimal transportation*, Axioms, 3(1), 119-139, 2014.
[54] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. 169, (3) (2009), 903-991.
[55] F. Luo and E. Saucan, In preparation.
[56] R. Martin, *Estimation of principal curvatures from range data*, International Journal of Shape Modeling, 4, 99-111, 1998.

[57] W. A. Miller, J. R. McDonald, P. M. Alsing, D. Gu and S.-T. Yau, *Simplicial Ricci Flow*, Comm. Math. Phys., 239(2), 579-608, 2014.

[58] F. Morgan, *Manifolds with density*, Notices Amer. Math. Soc. 52, 853-858, 2005.

[59] F. Münch, *Li-Yau inequality on finite graphs via non-linear curvature dimension conditions*, arXiv:1412.3340, 2014.

[60] J. R. Munkres, *Obstructions to the smoothening of piecewise-differentiable homeomorphisms*, Annals of Math., 72(2), 521-554, 1960.

[61] J. R. Munkres, *Elementary Differential Topology*, (rev. ed.), Princeton University Press, Princeton, NJ, 1966.

[63] Y. Ollivier, *Ricci curvature of Markov chains on metric spaces*, Journal of Functional Analysis 256(3) (2009) 81-864.

[64] Y. Ollivier, *A survey of Ricci curvature for metric spaces and Markov chains*, Probabilistic approach to geometry 57 (2010) 343-381.

[65] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arxiv:math.DG/0211159, 2002.

[66] G. Perelman, *Ricci flow with surgery on three manifolds*, arxiv:math.DG/0303109, 2003.

[67] C. Plaut, *Metric Spaces of Curvature \( \geq k \)*, Handbook of Geometric Topology (Daverman, R. J. and Sher, R. B., editors), 819-898, Elsevier, Amsterdam, 2002.

[68] M. Pouryahya, J. Mathews, A. Tannenbaum, *Comparing Three Notions of Discrete Ricci Curvature on Biological Networks*, arXiv:1712.02943 [q-bio.MN], 2017.

[69] T. Richard, *Canonical smoothing of compact Alexandrov surfaces via Ricci flow*, arXiv:1204.5461, 2012.

[70] E. Saucan, *Note on a Theorem of Munkres*, Mediterranean Journal of Mathematics, 2(2), 215-229, 2005.

[71] E. Saucan, *The Existence of Quasimeromorphic Mappings in Dimension 3*, Conform. Geom. Dyn., 10 (2006), 21-40.

[72] E. Saucan, *Fat Triangulations and Differential Geometry*, preprint, (arXiv:1108.3529v1 [math.DG]), 2011.

[73] E. Saucan, *Isometric Embeddings in Imaging and Vision: Facts and Fiction*, Journal of Mathematical Imaging and Vision, 43(2), 143-155, 2012.

[74] E. Saucan, *On a construction of Burago and Zalgaller*, The Asian Journal of Mathematics, 16(4), 587-606, 2012.

[75] E. Saucan, *A Metric Ricci Flow for Surfaces and Its Applications*, Geometry, Imaging and Computing, 1(2), 259-301, 2014.

[76] E. Saucan, *Metric Curvatures and their Applications I*, Geometry, Imaging and Computing, 2(4), 257-334, 2015.
[77] E. Saucan, *Metric Ricci Curvature and Flow for PL Manifolds*, Actes des rencontres du C.I.R.M., Vol. 3 n° 1 (2013) 119-129.

[78] E. Saucan, *Triangulated Surfaces as Alexandrov Surfaces I – Geometric Aspects*, in preparation.

[79] E. Saucan and E. Appleboim, *Metric Methods in Surface Triangulation*, Lecture Notes in Computer Science, **5654**, 335-355, Springer, 2009.

[80] E. Saucan, E. Appleboim and Y. Y. Zeevi, *Sampling and Reconstruction of Surfaces and Higher Dimensional Manifolds* Journal of Mathematical Imaging and Vision, **30**(1), 105-123, 2008.

[81] E. Saucan, E. Appleboim, G. Wolansky and Y. Y. Zeevi, *Combinatorial Ricci Curvature and Laplacians for Image Processing*, Proceedings of CISP’09, Vol. 2, 992-997, 2009.

[82] E. Saucan and J. Jost, *Network Topology vs. Geometry: From Persistent Homology to Curvature*, Proceedings of NIPS LHDS 2016, [http://www.cs.utexas.edu/~rofuyu/lhds-nips16/papers/11.pdf](http://www.cs.utexas.edu/~rofuyu/lhds-nips16/papers/11.pdf) 2016.

[83] E. Saucan, A. Samal, M. Weber and J. Jost, *Discrete curvatures and network analysis*, MATCH Commun. Math. Comput. Chem., **20**(3), 605-622, 2018.

[84] W.-X. Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Diff. Geom., **8**, 369-381, 1973.

[85] E. Sonn, Emil Saucan, Eli Appelboim and Yehoshua Y. Zeevi, *Ricci Flow for Image Processing*, Proceedings of IEEEI 2014.

[86] R. P. Sreejith, K. Mohanraj, J. Jost, E. Saucan and A. Samal, *Forman curvature for complex networks*, J. Stat. Mech., (2016) 063206, [http://iopscience.iop.org/1742-5468/2016/6/063206](http://iopscience.iop.org/1742-5468/2016/6/063206).

[87] D. A. Stone, *Sectional Curvatures in Piecewise Linear Manifolds*, Bull. Amer. Math. Soc. **79**(5), 1060-1063, 1973.

[88] D. A. Stone, *A combinatorial analogue of a theorem of Myers*, Illinois J. Math. **20**(1), 12-21, 1976.

[89] D. A. Stone, *Correction to my paper: “A combinatorial analogue of a theorem of Myers”*, Illinois J. Math. **20**, 551-554, 1920.

[90] D. A. Stone, *Geodesics in Piecewise Linear Manifolds*, Trans. Amer. Math. Soc. **215**, 1-44, 1976.

[91] K.-T. Sturm, *On the geometry of metric measure spaces I and II*, Acta Math. **196** (2006), 65-131 and 133-177.

[92] R. Sandhu, T. Georgiou, E. Reznik, L. Zhu, I. Kolesov, Y. Senbabaoglu and A. Tannenbaum, *Graph curvature for differentiating cancer networks*, Scientific Reports, **5**, 2015.

[93] R. Sandhu, T. Georgiou, and A. Tannenbaum, *Market fragility, systemic risk, and Ricci curvature*, [arXiv:1505.05182](http://arxiv.org/abs/1505.05182) 2015.

[94] P. M. Topping, *Lectures on the Ricci Flow*, London Mathematical Society Lecture Note Series **325**, Cambridge University Press, Cambridge, 2006.
[95] P. M. Topping, *Uniqueness and nonuniqueness for Ricci flow on surfaces: Reverse cusp singularities*, Int Math Res Notices 2012, 2356-2376, 2012.

[96] C. Villani, *Optimal Transport, Old and New*, Grundlehren der mathematischen Wissenschaften 338, Springer, Berlin-Heidelberg, 2009.

[97] C. Wang, E. Jonckheere and R. Banirazi, *Wireless Network Capacity versus Ollivier-Ricci Curvature under Heat-Diffusion (HD) Protocol*, Proceedings of ACC 2014, 3536-341, 2014.

[98] M. Weber, J. Jost and E. Saucan, *Forman-Ricci flow for change detection in large dynamical data sets*, Axioms, f 5(4), 26; doi: 10.3390/axioms5040026.

[99] M. Weber, E. Saucan and J. Jost, *Characterizing Complex Networks with Forman-Ricci curvature and associated geometric flows*, J. Complex Netw., 5(4): 527-550, 2017.

[100] M. Weber, E. Saucan and J. Jost, *Coarse geometry of evolving networks*, J Complex Netw 6(5), 706-732, 2018.

[101] M. Zeng, W. Zhang, R. Guo, F. Luo, D. Gu, *Survey on discrete surface Ricci flow*, J. Comput. Sci. Technol. 30, 598-613, 2016.

**Department of Applied Mathematics, ORT Braude College, Karmiel and Electrical Engineering Department, Technion, Haifa**

*E-mail address: semil@braude.ac.il, semil@ee.technion.ac.il*