Comments on quasiparticle models of quark-gluon plasma

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Abstract

Here we comment on the thermodynamic inconsistency problem and the reformulation of statistical mechanics of widely studied quasiparticle models of quark-gluon plasma. Their starting relation, the expression for pressure itself is a wrong choice and lead to thermodynamic inconsistency and the requirements of the reformulation of statistical mechanics. We propose a new approach to the problem using the standard statistical mechanics and is thermodynamically consistent. We also show that the other quasiparticle models may be obtained from our general formalism as a special case under certain restrictive condition. Further, as an example, we have applied our model to explain the nonideal behaviour of gluon plasma and obtained a remarkable good fit to the lattice results by adjusting just a single parameter.

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1 Introduction:

The quasiparticle quark-gluon plasma (qQGP) is a phenomenological model, with few fitting parameters, widely used to describe the nonideal behaviour of quark-gluon plasma (QGP). It was first proposed by Goloviznin and Satz [1] and then by Peshier et. al. [2] to explain the equation of state (EoS) of QGP from lattice gauge theory (LGT) simulation of quantum chromodynamics (QCD) at finite temperature [3]. The model, however, failed [5] to explain the more accurate, recent lattice data [4]. Further, Gorenstein and Yang [6] pointed out that the model is thermodynamically inconsistent. This thermodynamically inconsistency problem was remedied by them by introducing a temperature dependent vacuum energy and forced it to cancel the thermodynamically inconsistent term, which was named as the reformulated statistical mechanics. It is still not clear what is the physics or origin of this constraint which was called as thermodynamic consistency check in Ref. [6, 7, 8, 9] Here we show that the whole exercise is unnecessary and following the standard statistical mechanics (SM), we propose a new qQGP model which contains a single phenomenological parameter. Our model is thermodynamically consistent and explains lattice data very well.

2 Our model of qQGP:

Let us start with the work of Peshier et. al. [2] on gluon plasma. All thermodynamic quantities were derived from the pressure, \( P \), which was assumed as

\[
\frac{PV}{T} = -\sum_{k=0}^{\infty} \ln(1 - e^{-\beta \epsilon_k}) ,
\]

where the right hand side is the logarithm of the grand partition function, \( Q_G(T) \), and \( \epsilon_k \) is the single particle energy of quasi-gluon, i.e, gluon with temperature dependent mass, given by,

\[
\epsilon_k = \sqrt{k^2 + m^2(T)} ,
\]

where \( k \) is momentum and \( m \) is mass. \( \beta \) is defined as \( 1/T \). The expression for pressure is similar to that of ideal gas with temperature dependent mass and hence let us denote it as \( P_{id} \). We want to stress that this assumption itself is the root cause of thermodynamic inconsistency and
hence the reformulation of SM by Gorenstein and Yang [6]. Generally, in grand canonical ensemble (GCE), energy \(E_r\) and number of particles \(N_s\) fluctuate, but temperature \(T\) and the chemical potential \(\mu\) are fixed. Hence, the average energy \(U\) and average number of particles \(N\) are defined and may be related to the grand partition function or q-potential,

\[
q \equiv \ln Q_G = \ln\left(\sum_{s,r} e^{-\beta E_r - \alpha N_s}\right) = \mp \sum_{k=0}^{\infty} \ln(1 \mp e^{-\beta \epsilon_k}) ,
\]

where \(\mp\) for bosons and fermions and \(z \equiv e^{\mu/T} = e^{-\alpha}\) is called fugacity. The average energy \(U\) is defined as,

\[
U \equiv <E_r> = \frac{\sum_{r,s} E_r e^{-\beta E_r - \alpha N_s}}{Q_G} = -\frac{\partial}{\partial \beta} \ln Q_G = \sum_k \frac{z \epsilon_k e^{-\beta \epsilon_k}}{1 \mp z e^{-\beta \epsilon_k}} .
\]

Note that the partial differentiation with respect to \(\beta\) above is just a mathematical method to express \(U\) in terms of sum over single particle energy levels, \(\epsilon_k\), making use of Eq. (2). While differentiating, indirect dependence of \(\beta = 1/T\) in the fugacity, \(z\), and mass, \(m(T)\), must be ignored by definition. Otherwise, we will not get back \(<E_r>\). Similarly, the average density \(N\) is defined as,

\[
N \equiv <N_s> = \frac{\sum_{r,s} N_s e^{-\beta E_r - \alpha N_s}}{Q_G} = -\frac{\partial}{\partial \alpha} \ln Q_G = z \frac{\partial}{\partial z} \ln Q_G = \sum_k \frac{z \epsilon_k e^{-\beta \epsilon_k}}{1 \mp z e^{-\beta \epsilon_k}} ,
\]

These (Eqs. (3), (4)) are the standard relations [10] of \(U\) and \(N\) to the partition function, which is valid even for quasiparticle with \((T, \mu)\) dependent mass by the definition of averages. Here, for gluon plasma, we have \(\mu = 0\) or \(z = 1\). Next, pressure may be obtained by two methods. In method-I, one starts from \(U\) and using thermodynamic relation,

\[
\varepsilon \equiv \frac{U}{V} = T\frac{\partial P}{\partial T} - P ,
\]

and on integration, one gets pressure which is the procedure that we follow here. In method II, again following the standard text books on SM [10], we can relate \(P\) to q-potential as follows. The variation in q-potential due to variations in it’s dependence, namely \(T\), \(\mu\) and volume \(V\), specifying the macro-state of GCE system, is,

\[
\delta q = \frac{1}{Q_G} \left[ \sum_{r,s} e^{-\beta (E_r - \mu N_s)} (-E_r \delta \beta - \beta \delta E_r + N_s \delta (\beta \mu)) \right] .
\]
Now, when compared with the textbook results, we have an extra term coming from $\delta E_r$ due to temperature dependent mass and then using the definition of averages, we get,

$$\frac{PV}{T} = \pm \sum_{k=0}^{\infty} \ln(1 \mp z e^{-\beta \epsilon_k}) + \int d\beta \frac{\partial m}{\partial \beta} < \frac{\partial E_r}{\partial m} > .$$

(7)

Therefore we see that $P$ is not just equal to $P_{id}$, but there is an extra term. This extra term ensure thermodynamic consistency of the relation as follows. From above $P$, on differentiating with respect to $T$ for a system with $\mu = 0$ or $z = 1$,

$$\frac{\partial P}{\partial T} = \frac{P}{T} + \frac{\varepsilon}{T} - \frac{1}{V} < \frac{\partial \epsilon_k}{\partial T} > + \frac{1}{V} < \frac{\partial E_r}{\partial T} >$$

(8)

where the last two terms exactly cancels (following the procedure used in deriving Eq. (3)) and hence the thermodynamic relation, Eq. (5), is obeyed as expected.

Further more, this $P$ is also consistent with the $P$ obtained from $U$ through thermodynamic relations which may be shown as follows. The Eq. (7) may be simplified by evaluating $< \frac{\partial E_r}{\partial m} >$ and taking continuum limit ($V \to \infty$), for a system with $\mu = 0$, as,

$$\frac{P}{T} = \pm \frac{g_f}{2\pi^2} \int_0^{\infty} dk k^2 \ln(1 \mp e^{-\beta \epsilon_k}) + \int d\beta \beta \frac{g_f}{2\pi^2} m \frac{dm}{d\beta} \int_0^{\infty} dk \frac{k^2}{\epsilon_k (e^{\beta \epsilon_k} + 1)} ,$$

(9)

which on simplification, reduces to,

$$\frac{P}{T} = \frac{g_f}{2\pi^2} \left[ T^3 \sum_{l=1}^{\infty} \frac{(\pm 1)^{l-1}}{l^4} (\beta m l)^2 K_2(\beta m l) 
+ \int d\beta \frac{\beta}{m} \frac{\partial m}{\partial \beta} \frac{1}{\beta^4} \sum_{l=1}^{\infty} (\pm 1)^{l-1} \frac{1}{l^4} (\beta m l)^3 K_1(\beta m l) \right] ,$$

(10)

where $g_f$ is the internal degrees of freedom and $K_1$, $K_2$ are modified Bessel functions. Using the recurrence relations of Bessel functions and on integration by parts, above equation may be further simplified to get,

$$\frac{P}{T} = \frac{P_0}{T_0} - \int_{\beta_0}^{\beta} d\beta \varepsilon ,$$

(11)

where $\varepsilon$ is the energy density and $P_0$ is the pressure at some temperature $T_0$ or $\beta_0$. This equation is nothing but the thermodynamic relation, Eq. (5). Therefore, Gorenstein and Yang’s starting argument that above two methods give different $m(T)$ does not exist now by using our derived expression for $P$, instead of the assumption [2, 6].
3 Question of vacuum energy $B(T)$:

After the reformulation of SM by Gorenstein and Yang, almost all study in qQGP is based on the thermodynamic consistency relation, related to vacuum energy $B(T)$. Different authors call and interpret $B(T)$ in a different way, like vacuum energy, background field or bag pressure. But, by definition of quasiparticle, whole thermal energy is used to excite these quasiparticles. So quasiparticles are excitations above the ground state or vacuum state which may not depend on temperature or chemical potential. This is our assumption. As noted earlier, we also don’t have any thermodynamic inconsistency in our model.

In fact, when we redo our derivation of pressure with vacuum or zero point energy in single particle energy, like in Ref. [6], Eq. (9) is modified as,

$$P = P_{id} - B(T) + T \left( \int_{T_0}^{T} \frac{dT}{\tau} \left[ \frac{g_f}{2\pi^2} m \frac{dm}{d\tau} \int_{0}^{\infty} dk \frac{k^2}{\epsilon_k (e^{\epsilon_k/T} - 1)} + \frac{\partial B}{\partial \tau} \right] \right), \quad (12)$$

and the energy density,

$$\varepsilon = \varepsilon_{id} + B(T). \quad (13)$$

where $\varepsilon_{id}$ is the expression for $\varepsilon$ similar to ideal gas. Again it is easy to show that above $P$ and $\varepsilon$ obey thermodynamic relation Eq. (5). The thermodynamic consistency relation [6], used in other qQGP models, is nothing but a restrictive condition that the terms inside the square bracket in Eq. (12) is zero. At present it is not clear what is the physical origin of this constraint. Note that without this constraint, so called thermodynamic consistency relation, our system is thermodynamically consistent even with the zero-point energy contribution, $B(T)$. One may model $B(T)$ based on other effects of strongly interacting QCD system, like hadronic states, resonances and may be relevant at the transition point. In our study of gluon plasma here, we neglect all these effects and consider only the thermal properties of gluons. Hence we take $B(T) = 0$ and we get a very good fit to lattice results except at very close to the transition temperature, i.e, for $T < 1.2T_c$. 
4 EoS of gluon plasma:

As an example, let us apply our model to gluon plasma which is a QCD plasma without quarks. We first calculate the energy density, expressed in terms of \( e(T) \equiv \varepsilon/\varepsilon_s \), and then obtain \( P \) from thermodynamic relation, Eq. (5). So we have, from Eq. (3) after some algebra,

\[
e(T) = \frac{15}{\pi^4} \sum_{l=1}^{\infty} \frac{1}{l^4} \left[ \left( \frac{m_g l}{T} \right)^3 K_1 \left( \frac{m_g l}{T} \right) + 3 \left( \frac{m_g l}{T} \right)^2 K_2 \left( \frac{m_g l}{T} \right) \right]
\]

where \( \varepsilon_s \) is the Stefan-Boltzmann gas limit of QGP, \( m_g \) is the temperature dependent mass and \( K_1, K_2 \) are modified Bessel functions. The results are plotted in Fig. 1, for two different mass terms, \( m_g^2(T) = \omega_p^2 = g^2(T) T^2 / 3 \) (our model) and \( m_g^2(T) = g^2(T) T^2 / 2 \) (other qQGP models). \( g^2(T) \) is related to the two-loop order running coupling constant, given by,

\[
\alpha_s(T) = \frac{6\pi}{(33 - 2n_f) \ln(T/\Lambda_T)} \left( 1 - \frac{3(153 - 19n_f)}{(33 - 2n_f)^2} \frac{\ln(2 \ln(T/\Lambda_T))}{\ln(T/\Lambda_T)} \right)
\]

where \( \Lambda_T \) is a parameter related to QCD scale parameter. This choice of \( \alpha_s(T) \) is an approximate expression of the running coupling constant used in lattice simulations [4]. Then the pressure is obtained from the thermodynamic relation Eq. (5) or Eq. (11). Since we have only one parameter to adjust, we don’t get good fit for the generally used second choice of quasi-gluon mass. The best fitted parameter is \( \Lambda_T/T_c = 0.3 \). But a remarkably good fit may be obtained for our choice of gluon mass which is motivated from the fact that the quasi-gluons are the thermal excitations of plasma waves with mass equal to the plasma frequency [11]. The value of the fitted parameter is \( \Lambda_T/T_c = 0.65 \).

Let us now compare our results with the results from other qQGP models, for example Ref. [7], where \( B(T) \) is not zero, but is determined by thermodynamic consistency relation. From the Fig. 2, we see that only at large \( T/T_c \) both the results almost match, but differ near to \( T/T_c = 1 \). We used the same \( \alpha_s(T) \) with \( \Lambda_T/T_c = .65 \) for both the cases. Further, results from our model with \( B(T) = 0 \) fits well the lattice data. A very good fit to lattice data was also obtained in Ref. [7], but with a different expression for \( \alpha_s(T) \), having two free parameters, and an additional parameter related to degrees of freedom.
5 Conclusions:

Here we have pointed out the basic reason for the thermodynamic inconsistency of the extensively studied quasiparticle QGP models [2] and its consequence of the reformulation of statistical mechanics [6]. To revise it, we have proposed a new qQGP model which follows from the standard SM and has no thermodynamic inconsistency. When we extend our formalism to a system with temperature dependent vacuum energy, again, we get a thermodynamically consistent general model and we obtained other widely studied qQGP models as a special case of our model under certain restrictive condition which was called as thermodynamic consistency relation in Ref. [6, 7, 8, 9]. As an example, we studied the gluon plasma using our model. A remarkable good fit to the LGT data was obtained by adjusting just one parameter and without the temperature dependent vacuum energy $B(T)$. Whereas we know that the other qQGP models has 3 or more parameters. Further extension of our model to flavored QGP without and with masses, and also without and with chemical potential, fit remarkably well the lattice results and were reported in [12, 13].

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Figure 1: Plots of $P/T^4$ (lower set of graphs) and $\varepsilon/T^4$ (upper set of graphs) as a function of $T/T_c$ from our model and lattice results (symbols) [4] for gluon plasma with two different models for mass, $m_g^2(T) = g^2 T^2/3$ (dashed line) and $m_g^2(T) = g^2 T^2/2$ (dashed-dotted line).

Figure 2: Plots of $P/T^4$ (lower set of graphs) and $\varepsilon/T^4$ (upper set of graphs) as a function of $T/T_c$ from our model and lattice results (symbols) [4] for gluon plasma with $B(T) = 0$ (dashed line) and using thermodynamic consistency relation (dashed-dotted line).
Fig. 1
Fig. 2