SUMS OF KLOOSTERMAN SUMS IN THE PRIME GEODESIC THEOREM

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Abstract. We develop a new method for studying sums of Kloosterman sums related to the spectral exponential sum. As a corollary, we obtain a new proof of the estimate of Soundararajan and Young for the error term in the prime geodesic theorem.

Contents

1. Introduction 1
2. Notation and preliminary results 7
3. Sums of Kloosterman sums 12
4. Proof of Theorems 1.1 and 1.2 24
Acknowledgements 25
References 26

1. Introduction

The goal of this paper is to prove the following estimate for the sums of Kloosterman sums.

**Theorem 1.1.** Let $\theta$ be a subconvexity exponent for Dirichlet $L$-functions of real primitive characters and let $S(n,n,q)$ denote the Kloosterman

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sum. For $X,T \gg 1$ the following estimate holds

$$\frac{1}{N} \sum_n h(n) \sum_{q=1}^{\infty} \frac{S(n,n;q)}{q} \varphi\left(\frac{4\pi n}{q}\right) \ll \max(X^{1/4+\theta/2}T^{3/2},X^{\theta/2}T^2)\log^2(XT) + \frac{X^{1/4+\theta}T^{3/2}}{N^{1/2}} \left(1 + \frac{T}{X^{1/2}}\right),$$

where

$$\varphi(x) = \frac{\sinh^2 \beta}{2\pi} x^2 \exp(ix \cosh \beta)$$

with

$$\beta = \frac{1}{2} \log X + \frac{i}{2T}$$

and $h(x)$ is a smooth function supported in $[N,2N]$ for some $N > 1$ such that

$$|h^{(j)}(x)| \ll N^{-j}, \text{ for } j = 0, 1, 2, \ldots \int_{-\infty}^{\infty} h(x)dx = N.$$

**Remark.** Throughout the paper we use the standard notation $A \ll B$ meaning that there exists a constant $c > 1$ such that $A \leq cB$. On the other hand, by writing $E \gg F$ we mean that there exists a constant $c > 1$ such that $E \geq cF$.

**Remark.** Applying the Weil bound for Kloosterman sums, we obtain (for $N \gg TX^{-1/2}$) the estimate

$$\frac{1}{N} \sum_n h(n) \sum_{q=1}^{\infty} \frac{S(n,n;q)}{q} \varphi\left(\frac{4\pi n}{q}\right) \ll N^{1/2}X^{1/4}T^{3/2}\log(NX).$$

Therefore, (1.1) improves (1.5) if

$$N \gg X^\theta \left(1 + TX^{-1/2}\right).$$

The sum (1.1) is particularly interesting because it is ultimately related to the prime geodesic theorem, as we now explain.

The prime geodesic theorem gives an asymptotic formula as $X \to \infty$ for the number $\pi_\Gamma(X)$ of primitive hyperbolic classes $\{P\}$ in $PSL_2(\mathbb{Z})$ with norm $NP$ less than or equal to $X$. In direct analogy with prime numbers, it is convenient to study the weighted counting function

$$\Psi_\Gamma(X) = \sum_{NP \leq X} \Lambda(P),$$

where the sum is over all hyperbolic classes and $\Lambda(P) = \log NP_0$ if $\{P\}$ is a power of the primitive hyperbolic class $\{P_0\}$. 
Iwaniec [10, Lemma 1] proved for \( \frac{1}{2} \leq T \leq X^{1/2} \log^{-2} X \) that

\[
\Psi_{\Gamma}(X) = X + 2X^{1/2}\Re \left( \sum_{0 < t_j \leq T} \frac{X^{it_j}}{1/2 + it_j} \right) + O \left( \frac{X}{T} \log^2 X \right),
\]

where \( \kappa_j = 1/4 + t_j^2 \) are the eigenvalues of the hyperbolic Laplacian for \( \text{PSL}_2(\mathbb{Z}) \). Exact formula (1.6) provides a connection between \( \Psi_{\Gamma}(X) \) and the spectral exponential sum

\[
S(T, X) = \sum_{0 < t_j \leq T} X^{it_j}.
\]

It follows from Weyl’s law that \( S(T, X) \ll T^2 \), and correspondingly

\[
\Psi_{\Gamma}(X) = X + O(X^{3/4} \log X).
\]

The first non-trivial estimate

\[
S(T, X) \ll TX^{11/48+\epsilon}
\]

was obtained by Iwaniec in [10] for \( 1 \leq T \leq X^{1/2} \log^{-2} X \). Using (1.8) and taking \( T = X^{13/48} \) in (1.6), we have

\[
\Psi_{\Gamma}(X) = X + O(X^{13/48+\epsilon}).
\]

In order to prove (1.8), Iwaniec showed that the problem can be reduced to the investigation of the smoothed sum

\[
\sum_{j} X^{it_j} \exp(-t_j/T).
\]

Consequently, using properties of the Rankin-Selberg \( L \)-function and introducing the additional parameter \( N \), Iwaniec obtained the following decomposition

\[
\sum_{j} X^{it_j} \exp(-t_j/T) = A + B + O \left( T \log^2 T + \frac{N^{1/2} \log^2 N}{X^{1/2}} \right)
\]

with

\[
A = \frac{\pi^2}{12N} \sum_{n} h(n) \sum_{q=1}^{\infty} \frac{S(n, n; q)}{q} \phi \left( \frac{4\pi n}{q} \right),
\]

\[
B = -\frac{\zeta(2)}{2N} \int_{1/2}^{1} \tilde{h}(s) \sum_{j} \phi(t_j) \frac{L(u_j \otimes u_j, s)}{\cosh(\pi t_j)} ds,
\]

where \( h(x) \) is as in Theorem 1.1, \( \tilde{h}(s) \) is the Mellin transform of \( h(x) \),

\[
\phi(x) = \frac{\sinh \beta}{\pi} x \exp(ix \cosh \beta), \quad \beta = \frac{1}{2} \log X + \frac{i}{2T}.
\]
and

\begin{equation}
\hat{\phi}(t) = \frac{\sinh(\pi t + 2i\beta t)}{\sinh(\pi t)} = X^it \exp(-t/T) + O(\exp(-\pi t)).
\end{equation}

We remark that the original choice of \( \phi(x) \) made by Iwaniec was different. However, the function (1.14) produces a smaller error term in the approximation (1.15), as has been shown by Deshouillers and Iwaniec in [6, Lemmas 7 and 9].

The next step is to optimise the choice of the parameter \( N \) by proving the sharpest possible estimates on \( A \) and \( B \). To estimate the part \( B \), Iwaniec studied the first moment of Rankin-Selberg \( L \)-functions and proved an upper bound for this moment that is slightly weaker than the mean Lindelöf estimate. Consequently,

\begin{equation}
B \ll N^{-1/2}T^{5/2} \log^2 T.
\end{equation}

Note that in order to break the 3/4 barrier in the prime geodesic theorem, it is insufficient to combine (1.16) with the trivial bound for the part \( A \), namely

\begin{equation}
A \ll N^{1/2}T^{1/2}X^{1/4} \log T.
\end{equation}

For this reason, Iwaniec analysed sums of Kloosterman sums in the part \( A \) using the Burgess bound for character sums, and taking \( N \) sufficiently large \((N \gg T^{1+\epsilon} \gg X^{11/48+\epsilon})\), he finally proved (1.8).

The estimate (1.8) was further improved by Luo and Sarnak [12] as follows

\begin{equation}
S(T, X) \ll X^{1/8}T^{5/4} \log^2 T, \quad T, X \gg 1.
\end{equation}

More precisely, by proving the mean Lindelöf estimate for the first moment in the part \( B \), Luo and Sarnak showed that

\begin{equation}
B \ll N^{-1/2}T^2 \log^2 T.
\end{equation}

Substituting (1.17) and (1.19) to (1.11) yields that the optimal choice of \( N \) is \( N = T^{3/2}X^{-1/4} \). This proves (1.18) provided that \( T > X^{1/6} \). Finally, applying (1.18) to evaluate (1.6), it turns out that the optimal choice of \( T \) is \( T = X^{3/10} \), and this gives

\begin{equation}
\Psi_{\Gamma}(X) = X + O(X^{7/10+\epsilon}).
\end{equation}

Consequently, \( N = X^{1/5} = T^{2/3} \). Such a small value of \( N \), on the one hand, makes it more difficult to show cancellations in sums of Kloosterman sums in the part \( A \), but on the other hand, it suffices to apply the trivial estimate (1.17) based on the Weil bound in order to prove (1.18).
The next improvement is due to Cai [4], who showed that for \( X^{1/10} \leq T \leq X^{1/3} \) the following estimate holds
\[
S(T, X) \ll T^{2/5} X^{11/30 + \epsilon}.
\]

The approach of [4] combines the estimate \((1.19)\) for the part \( B \) and non-trivial analysis of the part \( A \) via the Burgess bound for character sums, which is possible because the parameter \( N \) is sufficiently large, namely \( N = T^{1+\epsilon} \). Note that \((1.21)\) improves \((1.18)\) if \( T > X^{29/102+\epsilon} \). This means that Cai obtained a non-trivial estimate on the part \( A \) for \( N > X^{29/102+\epsilon} \). It follows from \((1.6)\), \((1.21)\) and \((1.18)\) that
\[
(1.22) \quad \Psi_{\Gamma}(X) = X + O(X^{71/102+\epsilon}).
\]

Finally, Soundararajan and Young [14] proved the prime geodesic theorem in the strongest known form
\[
(1.23) \quad \Psi_{\Gamma}(X) = X + O\left(X^{2/3 + \theta/6 \log \frac{3}{3}}\right), \quad \theta = 1/6 + \epsilon.
\]

The proof is based on the estimate \((1.18)\) and a formula relating \( \Psi_{\Gamma}(X) \) with sums of generalized Dirichlet \( L \)-functions.

Another interesting question related to the prime geodesic theorem is the correct order of magnitude of the spectral exponential sum \((1.7)\). In [13] Conj. 2.2 Petridis and Risager conjectured that for \( X, T \gg 1 \)
\[
(1.24) \quad S(T, X) \ll T(TX)^{\epsilon}.
\]

For a fixed \( X \) and \( T \to \infty \) the conjecture was proved by Laaksonen in the appendix of [13].

The following estimates proved for \( X, T \gg 1 \) in [1] confirm the conjecture of Petridis and Risager in some ranges
\[
(1.25) \quad S(T, X) \ll \max \left( X^{1/4 + 3/2 + \theta/6 T^{1/2}}, X^{\theta/2 T} \right) \log^3 T,
\]
\[
(1.26) \quad S(T, X) \ll T \log T \quad \text{if} \quad T > \frac{X^{1/2 + \theta/6}}{\kappa(X)},
\]
where
\[
(1.27) \quad \kappa(X) = \| X^{1/2} + X^{-1/2} \|
\]
and \( \| x \| \) denotes the distance from \( x \) to the nearest integer. Furthermore, combining \((1.25)\) with \((1.18)\), we obtain a new proof of \((1.23)\).
Estimates (1.25) and (1.26) follow from the nontrivial bound for the part $B$

$$B \ll \frac{T \log^3 T}{N^{1/2}} + \frac{X^\theta}{X^{1/4} N^{1/2}} \left( X^{1/2} \min \left( T, \frac{X^{1/2}}{\kappa(X)} \right) \right)^{1/2} + \min \left( T, \frac{X^{1/2}}{\kappa(X)} \right)^{3/2},$$

while the part $A$ is estimated using Weil’s bound as in (1.17). Consequently, the optimal choice of $N$ is $N = X^\theta$.

An obvious idea is to try to improve also the trivial estimate on $A$. Nevertheless, it turns out that the Weil bound gives a stronger result in the required range than Iwaniec’s method based on the Burgess bound for character sums. Indeed, the estimate (1.25) is better than (1.18) only if $T > X^{1/6 + 2\theta/3 + \epsilon}$. In order to establish cancellations in character sums using the Burgess bound, it is required that $N \gg T$. Thus $N \gg T > X^{1/6 + 2\theta/3 + \epsilon}$. However, the optimal choice of $N$ is $N = X^\theta$, which is much less than $X^{1/6 + 2\theta/3 + \epsilon}$.

The aim of the present paper is to develop a new method for estimating the part $A$, which is based on some ideas of Kuznetsov [11]. As a result, we obtain Theorem 1.1 and a new proof of (1.23).

Due to some problems with convergence, it is required to replace the function $\phi(x)$ defined by (1.14) with $\varphi(x)$ (see (1.2)), which decays faster at $x = 0$. As will be shown later, in this case

$$\hat{\varphi}(t) = t \frac{\cosh(\pi t + 2i\beta t)}{\sinh(\pi t)} + \frac{i \sinh(\pi t + 2i\beta t)}{2 \tanh \beta \sinh(\pi t)} = tx^it \exp(-t/T) + \frac{i \cosh \beta}{2 \sinh \beta} X^it \exp(-t/T) + O(\exp(-\pi t)).$$

Consequently, we study the sum

$$\sum_{0 < t_j \leq T} t_j X^{it_j}$$

instead of (1.7). The approach of Iwaniec [10] (see also [12, Section 6] for more details) yields

$$\sum_j \hat{\varphi}(t_j) = A_1 + B_1 + O \left( T^2 \log^2 T + \frac{N^{1/2} \log^2 N}{X^{1/2}} \right)$$

with

$$A_1 = \frac{\pi^2}{12N} \sum_n h(n) \sum_{q=1}^\infty S(n, n; q) \frac{\varphi \left( \frac{4\pi n}{q} \right)}{q},$$
\[ B_1 = -\frac{\zeta(2)}{2\pi N} \frac{1}{\pi} \int_{(1/2)} \hat{h}(s) \sum_j \hat{\varphi}(t_j) \frac{L(u_j \otimes u_j, s)}{\cosh(\pi t_j)} ds. \]

Note that \( B_1 \ll TB \), and therefore, all previous estimates on the part \( B \) are valid for \( B_1 \) being multiplied by \( T \).

In fact, the estimate (1.1) is very strong as it allows us to prove the following theorem even by using the original estimate (1.16) of Iwaniec on the part \( B \), while the proofs given in [13] and [1] rely crucially on the mean Lindelöf estimate (1.19) by Luo and Sarnak.

**Theorem 1.2.** For \( X, T \gg 1 \) the following holds
\[ \sum_{t_j \leq T} t_j X^{it_j} \ll \max \left( X^{1/4+\theta/2}T^{3/2}, X^{\theta/2}T^2 \right) \log^3(XT). \]

Note that (1.25) (multiplied by \( T \)) and (1.33) are of the same quality. Therefore, as a consequence, we obtain one more proof of the prime geodesic theorem in the strongest known form (1.23). We remark that it is not possible to improve (1.23) further by combining (1.1) with the strongest known estimate for the part \( B \). The reason is that the largest contribution to the final result comes from the first summand on the right-hand side of (1.1), and this summand does not depend on \( N \). This eliminates the possibility of improvement by optimising the additional parameter \( N \), which was the main idea of Iwaniec’s approach.

### 2. Notation and Preliminary Results

Introduce the following notation
\[ \sum_{n=0}^{\infty} a_n = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n. \]

For a function \( f(x) \) let
\[ \tilde{f}(s) = \int_0^{\infty} f(x)x^{s-1}dx \]

be its Mellin transform. Let \( e(x) = \exp(2\pi ix) \). The classical Kloosterman sum
\[ S(n, m; c) = \sum_{\substack{a \pmod{c} \\ (a,c) = 1}} e \left( \frac{an + a^*m}{c} \right), \quad aa^* \equiv 1 \pmod{c}, \]

satisfies Weil’s bound (see [9, Theorem 4.5])
\[ |S(m, n; c)| \leq \tau_0(c) \sqrt{(m, n, c)} \sqrt{c}. \]
Consider the generalized Dirichlet $L$-function

$$
\mathcal{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{\rho_q(n)}{q^s} = \sum_{q=1}^{\infty} \frac{\lambda_q(n)}{q^s}, \quad \Re s > 1,
$$

where

$$
\rho_q(n) := \# \{ x \pmod{2q} : x^2 \equiv n \pmod{4q} \},
$$

$$
\lambda_q(n) := \sum_{q_1q_3 = q} \mu(q_2)\rho_{q_1}(n).
$$

Zagier [16, Proposition 3] showed that (2.2) can be meromorphically continued to the whole complex plane. Furthermore, it was proved in [16, Proposition 3] that $\mathcal{L}_n(s)$ is identically zero if $n \equiv 2, 3 \pmod{4}$.

For $n = 0$ we have

$$
\mathcal{L}_0(s) = \zeta(2s - 1).
$$

Otherwise, for $n = Dl^2$ with $D$ fundamental discriminant the following decomposition holds

$$
\mathcal{L}_n(s) = l^{1/2-s}T_{l}(D)(s)L(s, \chi_D),
$$

where $L(s, \chi_D)$ is a Dirichlet $L$-function for primitive quadratic character $\chi_D$ and

$$
T_{l}(D)(s) = \sum_{l_1l_2 = l} \chi_D(l_1)\frac{\mu(l_1)}{\sqrt{l_1}}\tau_{s-1/2}(l_2).
$$

It follows from (2.6) that if $n$ is non-zero and is not a full square, then $\mathcal{L}_n(s)$ is an entire function. Another consequence of (2.6) is that for any $n$ and some constant $A > 0$ one has

$$
\mathcal{L}_n(1/2 + it) \ll (1 + |n|)^\theta(1 + |t|)^A,
$$

where $\theta$ is the best known result towards the Lindelöf hypothesis for Dirichlet $L$-functions of real primitive characters. Conrey and Iwaniec [5] showed that $\theta = 1/6 + \epsilon$ is admissible for any $\epsilon > 0$ and Young [15] proved the hybrid bound with $\theta = A = 1/6 + \epsilon$.

For $V \geq 1$ define the series

$$
S_V(n^2 - 4) := \sum_{q=1}^{\infty} \frac{\lambda_q(n^2 - 4)}{q} \exp(-q/V).
$$

It was shown in [3, Eqs. (1.6)-(1.8), p. 723], [14, p. 116, line 2] that for $V > 0$ and $n \neq 2$ one has

$$
\mathcal{L}_{n^2-4}(1) = S_V(n^2 - 4) - \frac{1}{2\pi i} \int_{(-1/2)} \mathcal{L}_{n^2-4}(1 + s)V^s\Gamma(s)ds.
$$
We will also use [14, Lemma 2.3], which states that for $q = a^2b$ with $b$ square-free

\begin{equation}
\sum_{2<n\leq z} \lambda_q(n^2 - 4) = z \frac{\mu(b)}{b} + O(q^{1/2+\epsilon}) \text{ for any } z \geq 2.
\end{equation}

**Lemma 2.1.** For $z \geq 2, Q \geq 1$ the following estimate holds

\begin{equation}
\sum_{q \leq Q} \left( \sum_{2<n\leq z} \lambda_q(n^2 - 4) - z \frac{\mu(b)}{b} \right) \ll Q^{3/2} \log^2 Q.
\end{equation}

**Proof.** It follows from the proof of [14, Lemma 2.3] that

\[
\sum_{q \leq Q} \left( \sum_{n \leq z} \lambda_q(n^2 - 4) - z \frac{\mu(b)}{b} \right) = \\
\sum_{q \leq Q} \sum_{q_1^2 q_2 = q} \frac{1}{q_2} \sum_{k(q_2) \neq 0} S(k^2, 1; q_2) \sum_{n \leq z} e \left( \frac{k}{q_2} \right) + O(Q).
\]

Applying the estimate \( \sum_{n \leq z} e \left( \frac{k}{q_2} \right) \ll \|k/q_2\|^{-1} \) and Weil’s bound \(2.1\) we obtain

\[
\sum_{q \leq Q} \left( \sum_{n \leq z} \lambda_q(n^2 - 4) - z \frac{\mu(b)}{b} \right) \ll \\
\sum_{q_1^2 q_2 \leq Q} q_2^{1/2} \tau_0(q_2) \log q_2 \ll Q^{3/2} \log^2 Q.
\]

\[
\square
\]

Let \( \varphi(x) \) be a smooth function on \([0, \infty)\) such that

\[
\varphi(0) = 0, \quad \varphi^{(j)}(x) \ll (1 + x)^{-2-\epsilon}, \quad j = 0, 1, 2.
\]

Define

\begin{equation}
\varphi_0 = \frac{1}{2\pi} \int_{0}^{\infty} J_0(y) \varphi(y) dy,
\end{equation}

\begin{equation}
\varphi_B(x) = \int_{0}^{1} \int_{0}^{\infty} \xi x J_0(\xi x) J_0(\xi y) \varphi(y) dy d\xi,
\end{equation}

\begin{equation}
\varphi(x) = \int_{0}^{1} \int_{0}^{\infty} \xi x J_0(\xi x) J_0(\xi y) \varphi(y) dy d\xi.
\end{equation}
\[ \hat{\varphi}(t) = \frac{\pi i}{2 \sinh(\pi t)} \int_0^\infty (J_{2it}(x) - J_{-2it}(x))\varphi(x) \frac{dx}{x} \]

In order to avoid convergence problems, we modify slightly the choice of \( \varphi(x) \) that was made in [6] and [12]. For \( X, T \gg 1 \) let

\[ \varphi(x) := \frac{\sinh^2 \beta}{2\pi} x^2 \exp(ix \cosh \beta) \]

with

\[ \beta := \frac{1}{2} \log X + \frac{i}{2T}. \]

It is convenient to introduce the following notation

\[ c := -i \cosh \beta = a - ib, \]

\[ \begin{cases} a := \sinh(\log \sqrt{X}) \sin((2T)^{-1}), \\ b := \cosh(\log \sqrt{X}) \cos((2T)^{-1}). \end{cases} \]

Note that

\[ \arg c = -\pi/2 + \gamma, \quad 0 < \gamma, \quad T^{-1} \ll \gamma \ll T^{-1}. \]

**Lemma 2.2.** For \( X, T \gg 1, t > 0 \) the following holds

\[ \hat{\varphi}(t) = t \frac{\cosh(\pi t + 2i\beta t)}{\sinh(\pi t)} + \frac{i \sinh(\pi t + 2i\beta t)}{2 \tanh(\beta) \sinh(\pi t)} \]

\[ tX^{it} \exp(-t/T) + \frac{i \cosh \beta}{2 \sinh \beta} X^{it} \exp(-t/T) + O(\exp(-\pi t)), \]

\[ \varphi_0 = \frac{-i}{4\pi^2} \left( \frac{2}{\sinh \beta} + \frac{3}{\sinh^3 \beta} \right) \ll X^{-1/2}, \]

\[ \varphi_B(x) = \frac{-i \sinh^2 \beta}{2\pi} \int_0^1 \xi x J_0(\xi x) \]

\[ \times \left( \frac{2}{(\cosh^2 \beta - \xi^2)^{3/2}} + \frac{3\xi^2}{(\cosh^2 \beta - \xi^2)^{5/2}} \right) d\xi \ll X^{-1/2} \min(x, x^{1/2}). \]

\(^1\)Note that there is a typo (the imaginary unit \( i \) is placed in the denominator instead of the numerator) in [6, p. 68, line -1] and [12, p. 233, line -3] in the definition of \( \hat{\varphi}(t) \).
Proof. The proof is similar to [6, Lemma 7]. It follows from [7, Eq. 6.6621.1], [8, Eq. 15.4.18] that

\[ \int_0^\infty J_{2it}(x) \exp(ix \cosh \beta) dx = -\frac{\exp(-(\pi + 2i\beta)t)}{i \sinh \beta}. \]

Differentiating equation (2.24) with respect to \( \beta \), we have

\[ \int_0^\infty J_{2it}(x) \exp(ix \cosh \beta) dx = -\exp(-(\pi + 2i\beta)t) \left( \frac{2it}{\sinh^2 \beta} + \frac{\cosh \beta}{\sinh^3 \beta} \right). \]

Substituting (2.16) to (2.15) and using (2.25), we obtain (2.21).

Let us now prove (2.22). Differentiating equation (2.24) with respect to \( \beta \) and taking \( t = 0 \) yields

\[ \int_0^\infty J_0(x) \exp(ix \cosh \beta) dx = -\frac{\cosh \beta}{\sinh \beta}. \]

Differentiating equation (2.24) with respect to \( \beta \) twice, taking \( t = 0 \) and using (2.26), we show that

\[ \int_0^\infty J_0(x) \exp(ix \cosh \beta) x^2 dx = -i \left( \frac{2}{\sinh^3 \beta} + \frac{3}{\sinh^5 \beta} \right). \]

Substituting (2.16) to (2.13) and using (2.27), we prove (2.22).

Finally, the first equality in (2.23) can be proved similarly to [6, Eq. 7.5]. The only difference is that we now use (2.27) instead of (2.26). The final estimate on \( \varphi_B(x) \) in (2.23) can be proved in the same way as [6, Lemma 11].

Now we are ready to prove (1.30). As it was mentioned in the introduction, our arguments are similar to Iwaniec’s proof of (1.11) in [10] (see also [12, Section 6] for more details). For completeness we sketch the proof below.

Consider the following sum

\[ \sum_n h(n) \sum_j \frac{\rho_j(n)}{\cosh(\pi t_j)} \hat{\varphi}(t_j), \]

where \( h(x) \) is as in Theorem 1.1 and \( \rho_j(n) \) is the \( n \)-th Fourier coefficient of the Hecke-Maaß cusp forms. Applying the Mellin inversion formula

\[ 2 \text{Note that there is the typo in [6, Eq. 7.7]: the minus sign is missed.} \]

\[ 3 \text{Correcting the typo in [6, Eq. 7.7], the formula (2.26) has an additional minus sign compared to [6, Eq. 7.8].} \]
to $h(n)$ and using the properties of the Rankin-Selberg $L$-function, we have

$$
(2.29) \quad \sum_{j} \hat{\varphi}(t_j) = \frac{\zeta(2)}{2N} \sum_{n} h(n) \sum_{j} \frac{|\rho_j(n)|^2}{\cosh(\pi t_j)} \hat{\varphi}(t_j) + B_1,
$$

where $B_1$ is defined by (1.32). Applying the Kuznetsov trace formula [12, p. 234] to the sum over $j$ on the right-hand side of equation (2.29) and estimating some of the arising terms using (2.22) and (2.23) in the same way as in the paper of Luo-Sarnak [12, p. 234], we prove (1.30).

### 3. Sums of Kloosterman sums

In order to analyse sums of Kloosterman sums in (1.31) we apply the following result of Kuznetsov [11]. Let

$$
(3.1) \quad Z_\psi(s) := \sum_{q=1}^{\infty} \frac{1}{q} \sum_{n=1}^{\infty} \frac{S(n, n; q)}{n^s} \psi \left( \frac{4\pi n}{q} \right).
$$

**Lemma 3.1.** Assume that for $\Delta > 3/4$ we have

$$
\psi(x) \ll x^{2\Delta} \quad \text{as } x \to +0,
$$

$$
\psi(x) \ll 1 \quad \text{as } x \to +\infty.
$$

Then for $1 + 2\Delta > \Re s > 3/2$

$$
(3.2) \quad Z_\psi(s) = \frac{2\zeta(s)}{\zeta(2s)} \sum_{n=0}^{\infty} L_{n^2-4}(s) \Psi(n, s),
$$

with

$$
(3.3) \quad \Psi(n, s) = (4\pi)^{s-1} \int_{0}^{\infty} \psi(x) \cos \left( \frac{nx}{2} \right) x^{-s} dx,
$$

provided that the function $\psi(x)$ is such that the series on the right-hand side of (3.2) is absolutely convergent.

**Remark.** Lemma 3.1 was proved by Kuznetsov in [11]. We provide a proof below because this reference is hard to find.

**Proof.** Using the conditions on $\psi(x)$ and Weil’s bound (2.1), we find that the series (3.1) is absolutely convergent for $\Re s > 3/2$.

On the one hand, we have

$$
(3.4) \quad Z_\psi(s) = \sum_{q=1}^{\infty} \frac{1}{q} \sum_{l=1}^{q} S(l, l; q) \sum_{n=0}^{\infty} (l + nq)^{-s} \psi \left( \frac{4\pi(l + nq)}{q} \right).
$$
On the other hand, since \( S(-l, -l; q) = S(l, l; q) \), it follows that

\[
(3.5) \quad Z_\psi(s) = \sum_{q=1}^{\infty} \frac{1}{q} \sum_{l=0}^{q-1} S(l, l; q) \sum_{n=0}^{\infty} (q-l+nq)^{-s} \psi \left( \frac{4\pi(q-l+nq)}{q} \right).
\]

Let

\[
(3.6) \quad F_\psi(x, s) = \sum_{n=0}^{\infty} (n+x)^{-s} \psi \left( 4\pi(n+x) \right).
\]

Then (3.4) and (3.5) can be rewritten as

\[
(3.7) \quad Z_\psi(s) = \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{l=1}^{q} S(l, l; q) F_\psi \left( \frac{l}{q}, s \right),
\]

\[
(3.8) \quad Z_\psi(s) = \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{l=0}^{q-1} S(l, l; q) F_\psi \left( 1 - \frac{l}{q}, s \right).
\]

Let us assume that \( 3/2 < \Re s < 2\Delta \). Then

\[
\lim_{x \to 0} \psi(x)x^{-s} = 0,
\]

and consequently \( F_\psi(1, s) = F_\psi(0, s) \). Therefore, (3.8) can be written as

\[
(3.9) \quad Z_\psi(s) = \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{l=1}^{q} S(l, l; q) F_\psi \left( 1 - \frac{l}{q}, s \right).
\]

Applying (3.7) and (3.9) we have

\[
(3.10) \quad Z_\psi(s) = \frac{1}{2} \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{l=1}^{q} S(l, l; q) \left( F_\psi \left( \frac{l}{q}, s \right) + F_\psi \left( 1 - \frac{l}{q}, s \right) \right).
\]

Expanding the function \( F_\psi(x, s) + F_\psi(1-x, s) \) in the Fourier series gives

\[
(3.11) \quad \frac{F_\psi(x, s) + F_\psi(1-x, s)}{2} = 2 \sum_{n=0}^{\infty} \Psi(n, s) \cos(2\pi nx).
\]

Substituting (3.11) in (3.10) yields

\[
(3.12) \quad Z_\psi(s) = 2 \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{n=0}^{\infty} \Psi(n, s) \sum_{l=1}^{q} S(l, l; q) \cos \left( \frac{2\pi ln}{q} \right).
\]

The change of the orders of summation is justified due to the absolute convergence of the right-hand side of (3.12). This follows from Weil’s
bound (2.1) and the conditions of the lemma, namely that \( \Re s > 3/2 \) and that the function \( \Psi(n, s) \) is of rapid decay.

The last step of the proof is to show that

\[
\sum_{l=1}^{q} S(l, l; q) \cos \left( \frac{2\pi l n}{q} \right) = q \rho_q (n^2 - 4).
\]

Since Kloosterman sums are always real we obtain

\[
\sum_{l=1}^{q} S(l, l; q) \cos \left( \frac{2\pi l n}{q} \right) = q \sum_{d+d^*+n \equiv 0(q)} \sum_{d=1}^{q} 1 = q \rho_q (n^2 - 4),
\]

where \( dd^* \equiv 1(q) \). For the proof of the last equality see [14, Lemma 2.3].

**Lemma 3.2.** The following exact formula holds

\[
\sum_{n} h(n) \sum_{q=1}^{\infty} \frac{S(n, n; q)}{q} \varphi \left( \frac{4\pi n}{q} \right) = \frac{2\hat{h}(1)}{\zeta(2)} \sum_{n=0}^{\infty} {\mathcal{L}}_{n^2-4}(1) \Phi(n, 1) + 2 \text{res}_{s=1} \left( \frac{\zeta(s)\zeta(2s-1)}{\zeta(2s)} \Phi(2, s) \right) + \frac{1}{2\pi i} \int_{(1/2)} \frac{\hat{h}(s)}{\zeta(2s)} \sum_{n=0}^{\infty} {\mathcal{L}}_{n^2-4}(s) \Phi(n, s) ds,
\]

where

\[
\Phi(n, s) = \frac{\sinh^2 \beta}{2\pi} \frac{(4\pi)^{s-1} \Gamma(3-s)}{(c^2 + n^2/4)^{3/2-s/2}} \cos \left( (3-s) \arctan \frac{n}{2c} \right),
\]

**Proof.** With the goal of using Lemma 3.1 in order to evaluate (3.31), we apply the Mellin inversion formula to the function \( h(n) \), getting

\[
\sum_{n} h(n) \sum_{q=1}^{\infty} \frac{S(n, n; q)}{q} \varphi \left( \frac{4\pi n}{q} \right) = \frac{1}{2\pi i} \int_{(\sigma)} \hat{h}(s) Z_\varphi(s) ds,
\]

where \( 3 > \sigma > 3/2 \). The function \( \varphi(x) \) defined by (2.10) satisfies the conditions of Lemma 3.1 because it behaves like \( x^2 \) when \( x \to 0 \) and it
decays exponentially when \(x \to +\infty\). Then Lemma \([3.1]\) yields

\[
(3.16) \quad \sum_{n} h(n) \sum_{q=1}^{\infty} \frac{S(n, n; q)}{q} \varphi \left( \frac{4\pi n}{q} \right) =
\]

\[
\frac{1}{2\pi i} \int_{(\sigma)} \frac{2\zeta(s)\check{h}(s)}{\zeta(2s)} \sum_{n=0}^{\infty} \mathcal{L}_{n^2-4}(s) \Phi(n, s) ds,
\]

where

\[
(3.17) \quad \Phi(n, s) = \left(4\pi\right)^{s-1} \int_{0}^{\infty} \varphi(x) \cos \left( \frac{nx}{2} \right) x^{-s} dx =
\]

\[
\frac{\sinh^2 \beta}{2\pi} \left(4\pi\right)^{s-1} \int_{0}^{\infty} \cos \left( \frac{nx}{2} \right) \exp(ix \cosh \beta) x^{2-s} dx.
\]

To evaluate \((3.17)\) we apply \([7, \text{Eq. 3.944.6}]\) and obtain \((3.15)\). As explained in Section \(2\) the function \(\mathcal{L}_{n^2-4}(s)\) has a pole at \(s = 1\) only if \(n = 2\). In the later case \(\mathcal{L}_{0}(s) = \zeta(2s - 1)\). Finally, moving the line of integration in \((3.16)\) to \(\Re s = 1/2\) we obtain \((3.14)\).

To derive an asymptotic formula from \((3.14)\) we need the following lemma.

**Lemma 3.3.** Let \(c = a - ib\) being defined in \((2.18)\) and

\[
(3.18) \quad z_{\pm}(n) := 2ci \pm n = 2b \pm n + 2ai.
\]

Then for \(n \geq 0\) and any real \(t\) the following inequality holds

\[
(3.19) \quad -\pi |t| - t \arg (n^2/4 + c^2) \pm t(\arg z_{+}(n) - \arg z_{-}(n)) \leq 0.
\]

**Proof.** There are several cases to consider due to the presence of \(\pm\) and \(t \leq 0\). For \(t < 0\) it is required to prove that

\[
(3.20) \quad \arg (n^2/4 + c^2) \pm (\arg z_{+}(n) - \arg z_{-}(n)) \leq \pi.
\]

Let us consider the \(-\) case in \((3.20)\). Then we need to show that

\[
(3.21) \quad \arg (n^2/4 + c^2) - \arg z_{+}(n) + \arg z_{-}(n) \leq \pi
\]

This is satisfied because

\[
\arg z_{-}(n) < \pi, \quad \arg z_{+}(n) > 0, \quad \arg (n^2/4 + c^2) < 0.
\]

Let us consider the \(+\) case in \((3.20)\). Then it is required to prove that

\[
(3.22) \quad \arg (n^2/4 + c^2) + \arg z_{+}(n) - \arg z_{-}(n) \leq \pi.
\]

This inequality holds because

\[
\arg z_{-}(n) > 0, \quad \arg z_{+}(n) \ll T^{-1}, \quad \arg (n^2/4 + c^2) < 0.
\]
As a result, (3.19) is proved for $t < 0$.

Now we assume that $t > 0$. Then we need to show that

$$- \arg \left( \frac{n^2}{4} + c^2 \right) \pm (\arg z_+(n) - \arg z_-(n)) \leq \pi.$$

If we consider the + case in (3.23), then our goal is to prove that

$$- \arg \left( \frac{n^2}{4} + c^2 \right) + \arg z_+(n) - \arg z_-(n) \leq \pi,$$

or equivalently,

$$\arg \left( \frac{n^2}{4} + a^2 - b^2 + 2abi \right) + \arg (2b + n + 2ai) - \arg (2b - n + 2ai) \leq \pi.$$

If $n > 2(b^2 - a^2)^{1/2}$, then

$$\arg \left( \frac{n^2}{4} + a^2 - b^2 + 2abi \right) < \pi/2, \quad \arg z_+(n) \ll T^{-1}, \quad \arg z_-(n) > 0,$$

and consequently, (3.25) is satisfied.

So we are left to analyse the case $n \leq 2(b^2 - a^2)^{1/2}$. In this case

$$\arg \left( \frac{n^2}{4} + a^2 - b^2 + 2abi \right) = \frac{\pi}{2} + \arctan \frac{b^2 - a^2 - n^2/4}{2ab}.$$

Furthermore,

$$\arg (2b + n + 2ai) = \arctan \frac{2a}{2b + n},$$
$$\arg (2b - n + 2ai) = \arctan \frac{2a}{2b - n}.$$

Therefore, in order to prove (3.23) it is sufficient to show that

$$\arctan \frac{b^2 - a^2 - n^2/4}{2ab} + \arctan \frac{2a}{2b + n} \leq \pi/2 + \arctan \frac{2a}{2b - n}.$$

Let us prove that the left-hand side of (3.27) is bounded by $\pi/2$. With this goal, we evaluate the tangent of the left-hand side of (3.27), which turns out to be positive because

$$\frac{b^2 - a^2 - n^2/4}{2ab} \times \frac{2a}{2b + n} < 1.$$

Thus (3.27) is proved.

The + case in (3.23) requires proving that

$$- \arg \left( \frac{n^2}{4} + c^2 \right) - \arg z_+(n) + \arg z_-(n) \leq \pi,$$

or equivalently,

$$\arg \left( \frac{n^2}{4} + a^2 - b^2 + 2abi \right) - \arg (2b + n + 2ai) + \arg (2b - n + 2ai) \leq \pi.$$
First, assume that \( n \leq 2(b^2 - a^2)^{1/2} \). Using (3.26) and
\[
\arg{(2b - n + 2ai)} = \frac{\pi}{2} - \arctan{\frac{2b - n}{2a}},
\]
the inequality (3.29) can be written as
\[
(3.30) \quad \arctan{\frac{b^2 - a^2 - n^2/4}{2ab}} \leq \arctan{\frac{2b - n}{2a}} + \arctan{\frac{2a}{2b + n}}.
\]
This is equivalent to
\[
(3.31) \quad \frac{b^2 - a^2 - n^2/4}{2ab} \leq \frac{(2b - n)/(2a) + 2a/(2b + n)}{1 - (2b - n)/(2b + n)}.
\]
Simplifying we obtain
\[
(3.32) \quad (b^2 - a^2 - n^2/4)(2n - 4b) \leq 8a^2b.
\]
Inequality (3.32) always holds since the left-hand side is negative. This yields (3.29).

Second, assume that \( 2(b^2 - a^2)^{1/2} < n \leq 2b \). Then (3.29) can be rewritten as
\[
\arctan{\frac{2ab}{n^2/4 - b^2 + a^2}} + \arctan{\frac{2a}{2b - n}} - \arctan{\frac{2a}{2b + n}} \leq \pi.
\]
This can be simplified to
\[
(3.33) \quad - \arctan{\frac{n^2/4 - b^2 + a^2}{2ab}} - \arctan{\frac{2b - n}{2a}} - \arctan{\frac{2a}{2b + n}} \leq 0.
\]
The inequality (3.33) is always satisfied and so is the inequality (3.29).

Third, assume that \( n > 2b \). In this case (3.29) can be formulated in the following form
\[
(3.34) \quad \arctan{\frac{2ab}{n^2/4 - b^2 + a^2}} \leq \arctan{\frac{2a}{n - 2b}} + \arctan{\frac{2a}{2b + n}},
\]
which is true because
\[
(3.35) \quad \arctan{\frac{2ab}{n^2/4 - b^2 + a^2}} + \arctan{\frac{2a}{2b + n}} = \arctan{\frac{2a}{n - 2b}}.
\]
The last equation can be proved by taking the tangent of both sides. Therefore, inequality (3.29) is satisfied.
Lemma 3.4. For $N, X, T \gg 1$ the following asymptotic formula holds

\begin{equation}
\sum_{n} h(n) \sum_{q=1}^{\infty} S(n, n; q) \frac{4\pi n}{q} \varphi \left( \frac{4\pi n}{q} \right) = \frac{2\tilde{h}(1)}{\zeta(2)} \sum_{n=3}^{\infty} \mathcal{L}_{n^2-4}(1) \Phi(n, 1) \\
+ O\left( N \log(NX) \right) + O \left( N^{1/2}X^{1/4+\theta}T^{3/2} \left( 1 + \frac{T}{X^{1/2}} \right) \right),
\end{equation}

where for $n > 0$

\begin{equation}
\Phi(n, 1) = \frac{\sinh^2 \beta}{2\pi c^2} \frac{1 - n^2/(4c^2)}{(1 + n^2/(4c^2))^2}.
\end{equation}

Proof. According to (2.17), (2.18) we have $|\sinh \beta|, |c|^2 \sim X$. Then it follows from (3.15) that for $n = 0, 1, 2$

$$|\Phi(n, 1)| \ll 1, \quad |\Phi'(2, 1)| \ll \log X,$$

where the derivative is taken with respect to $s$. Since $\tilde{h}(1) \ll N$ and $\tilde{h}'(1) \ll N \log N$ we obtain

$$\tilde{h}(1) (\mathcal{L}_4(1) \Phi(0, 1) + \mathcal{L}_{-3}(1) \Phi(1, 1)) + 2 \text{ res}_{s=1} \left( \tilde{h}(s) \zeta(s) \zeta(2s) - 1 \right) \Phi(2, s) = O \left( N \log(NX) \right).$$

Thus, it remains to estimate the integral on the right-hand side of (3.14). Note that

$$\tilde{h}(1/2 + it) \ll N^{1/2}(1 + |t|)^{-A} \text{ for any } A > 0.$$

Applying (2.8), it is sufficient to prove the following estimate

\begin{equation}
\sum_{n=0}^{\infty} (1 + n) 2^\theta |\Phi(n, 1/2 + it)| \ll (1 + |t|)^B X^{1/4+\theta}T^{3/2} \left( 1 + \frac{T}{X^{1/2}} \right).
\end{equation}

As a consequence of (3.15), (2.20) and the Stirling formula we obtain

\begin{equation}
|\Phi(0, 1/2 + it)| \ll (1 + |t|)^B \frac{X}{|c|^{5/2}} \exp(-\pi |t|/2 - t \arg c)
\ll (1 + |t|)^B X^{-1/4},
\end{equation}

since $|c|^2 \sim X$. Hence it remains to prove that

\begin{equation}
\sum_{n=1}^{\infty} n 2^\theta |\Phi(n, 1/2 + it)| \ll (1 + |t|)^B X^{1/4+\theta}T^{3/2} \left( 1 + \frac{T}{X^{1/2}} \right)
\end{equation}
for some constant $B > 0$. Using (3.15) we obtain

$$
|\Phi(n, 1/2 + it)| \ll \frac{(1 + |t|)^2 X \exp(-\pi|t|/2)}{|n^2/4 + c^2|^{5/4}} \times
\left|\left(n^2/4 + c^2\right)^{it/2} \cos \left((5/2 - it) \arctan \frac{n}{2c}\right)\right|.
$$

It follows from [8, Eq. 4.23.26] that

$$
\arctan \frac{n}{2c} = \frac{i}{2} \log \frac{2ci + n}{2ci - n} = \frac{i}{2} \left(\log z_+(n) - \log z_-(n)\right),
$$

where for $c = a - ib$ we have

$$
z_\pm(n) = 2ci \pm n = 2b \pm n + 2ai.
$$

Therefore,

$$
|\cos \left((5/2 - it) \arctan \frac{n}{2c}\right)| \ll \sum_{j = \pm 1} \left|\frac{z_+(n)}{z_-(n)}\right|^{5j/4} \exp \left(jt \left(\arg z_+(n) - \arg z_-(n)\right)/2\right).
$$

Substituting (3.42) to (3.41) gives

$$
|\Phi(n, 1/2 + it)| \ll \frac{(1 + |t|)^2 X}{|n^2/4 + c^2|^{5/4}} \sum_{j = \pm 1} \left|\frac{z_+(n)}{z_-(n)}\right|^{5j/4} \times \exp \left(-\pi|t|/2 - t \arg (n^2/4 + c^2)/2 + jt \left(\arg z_+(n) - \arg z_-(n)\right)/2\right).
$$

By Lemma 3.3 for $j = \pm 1$ we obtain

$$
|\Phi(n, 1/2 + it)| \ll \frac{(1 + |t|)^2 X}{|n^2/4 + c^2|^{5/4}} \sum_{j = \pm 1} \left|\frac{z_+(n)}{z_-(n)}\right|^{5j/4} \times \exp \left(-\pi|t|/2 - t \arg (n^2/4 + c^2)/2 + jt \left(\arg z_+(n) - \arg z_-(n)\right)/2\right).
$$

According to (2.20) we have

$$
|z_+(n)|^2 = (2|c| \cos \gamma + n)^2 + (2|c| \sin \gamma)^2 = (2|c|)^2 \left(1 - \frac{n}{2|c|}\right)^2 + 4n^2/\left|\frac{2|c|}{\cos^2 \gamma}\right)^2,
$$

$$
|z_-(n)|^2 = (2|c|)^2 \left(1 - \frac{n}{2|c|}\right)^2 + 4n^2/\left|\frac{2|c|}{\sin^2 \gamma}\right)^2,
$$

$$
|n^2/4 + c^2|^2 = (n^2/4 - |c|^2 \cos 2\gamma)^2 + (|c|^2 \sin 2\gamma)^2 = |c|^4 \left(1 - \left(\frac{n}{2|c|}\right)^2\right)^2 + 4\left(\frac{n}{2|c|}\right)^2 \sin^2 \gamma.
$$
It follows from (3.45), (3.46), (3.47) and the fact that $|c|^2 \sim X$ that
\[
(3.48) \quad n^{2\theta} |\Phi(n, 1/2 + it)| \ll (1 + |t|)^2 X^{-1/4 + \theta} \sum_{j=\pm 1} f_j \left( \frac{n}{2|c|} \right),
\]
where for $x > 0$
\[
(3.49) \quad f_j(x) = \left( \frac{(1 - x)^2 + 4x \cos \gamma/2}{(1 - x)^2 + 4x \sin \gamma/2} \right)^{5j/8} \frac{x^{2\theta}}{((1 - x)^2 + 4x^2 \sin^2 \gamma)^{5/8}}.
\]
Note that since $0 < \gamma \ll T^{-1}$ (see (2.20)), we have $f_{-1}(x) < f_1(x)$. Note that the function $f_1(x)$ is continuous and has finitely many monotonic segments. As a result,
\[
(3.50) \quad \sum_{n=1}^{\infty} n^{2\theta} |\Phi(n, 1/2 + it)| \ll (1 + |t|)^2 X^{-1/4 + \theta} \times\left( X^{1/2} \int_0^\infty f_1(x) dx + \max_{x > 0} |f_1(x)| \right).
\]
Since $T^{-1} \ll \gamma \ll T^{-1}$, for $x > 0$ we have
\[
(3.51) \quad f_1(x) \ll \min \left( T^{5/2}, \frac{x^{2\theta}}{|1 - x|^{5/2}} \right).
\]
Finally, using (3.51) to estimate the right-hand side of (3.50) we obtain
\[
(3.52) \quad \sum_{n=1}^{\infty} n^{2\theta} |\Phi(n, 1/2 + it)| \ll (1 + |t|)^2 X^{1/4 + \theta} T^{3/2} \left( 1 + \frac{T}{X^{1/2}} \right).
\]

**Lemma 3.5.** For $N, X, T \gg 1$ the following estimate holds
\[
(3.53) \quad \sum_{n=3}^{\infty} L_{n^2 - 4}(1) \Phi(n, 1) \ll \max(X^{1/4 + \theta/2} T^{3/2}, X^{\theta/2} T^2) \log^2 X.
\]

**Proof.** Applying (2.10) we obtain
\[
(3.54) \quad \sum_{n=3}^{\infty} L_{n^2 - 4}(1) \Phi(n, 1) = \sum_{n=3}^{\infty} \Phi(n, 1) S_V(n^2 - 4) -\frac{1}{2\pi i} \int_{(-1/2)} \sum_{n=3}^{\infty} \Phi(n, 1) L_{n^2 - 4}(1 + s)V^* \Gamma(s) ds.
\]
Let us first estimate the integral. Using (2.8) and (3.37) we have

\begin{align*}
(3.55) \quad & \frac{1}{2\pi i} \int_{(-1/2)}^\infty \sum_{n=3}^\infty \Phi(n,1) \mathcal{L}_{n^2-4}(1+s)V^s \Gamma(s) ds \ll \\
& V^{-1/2} \sum_{n=3}^\infty |\Phi(n,1)| n^{2\theta} \ll V^{-1/2} \sum_{n=3}^\infty \left| \frac{1-n^2/(4c^2)}{(1+n^2/(4c^2))^2} \right| n^{2\theta}.
\end{align*}

It follows from (2.20) that

\begin{align*}
(3.56) \quad & \left| 1 - \frac{n^2}{4c^2} \right|^2 = 1 + \left( \frac{n^2}{4|c|^2} \right)^2 + 2 \frac{n^2}{4|c|^2} \cos(2\gamma) \\
& = \left( 1 - \frac{n^2}{4|c|^2} \right)^2 + 4 \frac{n^2}{4|c|^2} \cos^2 \gamma,
\end{align*}

\begin{align*}
(3.57) \quad & \left| 1 + \frac{n^2}{4c^2} \right|^2 = \left( 1 - \frac{n^2}{4|c|^2} \right)^2 + 4 \frac{n^2}{4|c|^2} \sin^2 \gamma.
\end{align*}

Therefore,

\begin{align*}
(3.58) \quad & \sum_{n=3}^\infty \left| \frac{1-n^2/(4c^2)}{(1+n^2/(4c^2))^2} \right| n^{2\theta} \ll X^\theta \sum_{n=3}^\infty g \left( \frac{n}{2|c|} \right),
\end{align*}

where

\begin{equation}
(3.59) \quad g(y) = \frac{(1-y^2)^2 + 4y^2 \cos^2 \gamma}{(1-y^2)^2 + 4y^2 \sin^2 \gamma} y^{2\theta}.
\end{equation}

Since the function $g(x)$ is continuous and has finitely many monotonic segments, we obtain

\begin{align*}
(3.60) \quad & \sum_{n=3}^\infty \left| \frac{1-n^2/(4c^2)}{(1+n^2/(4c^2))^2} \right| n^{2\theta} \ll X^\theta \left( X^{1/2} \int_0^\infty g(y) dy + \max_{y>0} |g(y)| \right).
\end{align*}

Note that we used the fact that $|c| \sim X^{1/2}$. Since $T^{-1} \ll \gamma \ll T^{-1}$, for $y > 0$ the following holds

\begin{equation}
(3.61) \quad g(y) \ll \min \left( T^2, \frac{y^{2\theta}}{1-y^2} \right).
\end{equation}

Using (3.61) to estimate the right-hand side of (3.60), we obtain

\begin{align*}
(3.62) \quad & \sum_{n=3}^\infty \left| \frac{1-n^2/(4c^2)}{(1+n^2/(4c^2))^2} \right| n^{2\theta} \ll X^\theta T^2 + X^{1/2+\theta} T.
\end{align*}
Applying (3.58), (3.55) and (3.54) we prove

\[(3.63) \sum_{n=3}^{\infty} L_{n^2-4}(1) \Phi(n, 1) = \sum_{n=3}^{\infty} \Phi(n, 1) S_V(n^2 - 4) + O \left( V^{-1/2} \left( X^{\theta_T^2} + X^{1/2+\theta_T} \right) \right). \]

To estimate the sum in (3.63) we apply (2.9) and obtain

\[(3.64) \sum_{n=3}^{\infty} \Phi(n, 1) S_V(n^2 - 4) = \sum_{q=1}^{\infty} \frac{\exp(-q/V)}{q} \sum_{n=0}^{\infty} \Phi(n, 1) \lambda_q(n^2 - 4) a(n), \]

where \(a(n) = 0\) for \(n = 0, 1, 2\) and \(a(n) = 1\) for \(n > 2\). Abel’s summation formula yields

\[(3.65) \sum_{n=0}^{\infty} \Phi(n, 1) \lambda_q(n^2 - 4) a(n) = - \int_0^{\infty} \Phi'(x, 1) \sum_{0 \leq n \leq x} \lambda_q(n^2 - 4) a(n) dx. \]

We remark that for a real positive \(x\) the function \(\Phi(x, 1)\) is still defined by (3.37). Let \(q = a^2 b\). Then

\[(3.66) \sum_{n=0}^{\infty} \Phi(n, 1) \lambda_q(n^2 - 4) a(n) = - \int_0^{\infty} \Phi'(x, 1) x \frac{\mu(b)}{b} dx - \int_0^{\infty} \Phi'(x, 1) \left( \sum_{0 \leq n \leq x} \lambda_q(n^2 - 4) a(n) - x \frac{\mu(b)}{b} \right) dx = \]

\[\frac{\mu(b)}{b} \int_0^{\infty} \Phi(x, 1) dx - \int_0^{\infty} \Phi'(x, 1) \left( \sum_{0 \leq n \leq x} \lambda_q(n^2 - 4) a(n) - x \frac{\mu(b)}{b} \right) dx. \]

Applying Abel’s summation formula, (2.12) and [7, Eq. 4.358.2], we have

\[(3.67) \sum_{q=1}^{\infty} \frac{\exp(-q/V)}{q} \left( \sum_{n \leq z} \lambda_q(n^2 - 4) a(n) - z \frac{\mu(b)}{b} \right) \ll V^{1/2} \log^2 V. \]
Substituting (3.66) to (3.64) and using (3.67), we obtain

\[
(3.68) \quad \sum_{n=3}^{\infty} \phi(n,1)S_V(n^2 - 4) = \sum_{q=1}^{\infty} \frac{\exp(-q/V)\mu(b)}{q^b} \int_0^\infty \Phi(x,1)dx + O \left(V^{1/2}\log V \int_0^\infty |\Phi'(x,1)|dx \right).
\]

It follows from (3.37) that

\[
(3.69) \quad \Phi(x,1) = \frac{\sinh^2 \beta}{2\pi c^2} \frac{1 - x^2/(4c^2)}{(1 + x^2/(4c^2))^2}.
\]

Hence using the fact that \(|\sinh^2 \beta| \sim |c|^2 \sim X\), we show the following inequality

\[
(3.70) \quad |\Phi'(x,1)| \ll \frac{x}{|c|^2} \frac{|3/2 - (x/(2c))^2|}{|1 + (x/(2c))^2|^3} = \frac{x}{|c|^2} \left(\frac{3}{2} - \left(\frac{x}{2|c|}\right)^2\right)^2 + 6 \left(\frac{x}{2|c|}\right)^2 \cos^2 \gamma \right)^{1/2} \times \left(1 - \left(\frac{x}{2|c|}\right)^2\right)^2 + 4 \left(\frac{x}{2|c|}\right)^2 \sin^2 \gamma \right)^{-3/2}.
\]

Integrating (3.70) with respect to \(x\) gives

\[
(3.71) \quad \int_0^\infty |\Phi'(x,1)|dx \ll \int_0^\infty \frac{(3/2 - x^2)^2 + 6x^2 \cos^2 \gamma)^{1/2}}{(1 - x^2)^2 + 4x^2 \sin^2 \gamma)^{3/2}}xdx \ll \int_0^\infty \min \left(T^3, \frac{1}{|1 - x|^3}\right)dx \ll T^2.
\]

Now let us consider the first summand on the right-hand side of (3.68). It follows from (3.69) that

\[
(3.72) \quad \int_0^\infty \Phi(x,1)dx = \frac{\sinh^2 \beta}{2\pi c^2} \int_0^\infty \frac{1 - x^2/(4c^2)}{(1 + x^2/(4c^2))^2}dx.
\]

In order to evaluate the integral above we use [2, page 311, Eq. (30)], namely

\[
(3.73) \quad \int_0^\infty \frac{x^{s-1}}{(1 + \alpha x^h)^\nu}dx = h^{-1} \alpha^{-s/h} B(s/h, \nu - s/h),
\]
where $B(x, y)$ is the beta function, $|\arg \alpha| < \pi$, $h > 0$, $0 < \Re s < h\Re \nu$.

Applying (3.73) twice we show that

\[(3.74)\quad \int_{0}^{\infty} \Phi(x, 1)dx = 0.\]

Substituting (3.71) and (3.74) in (3.68) gives

\[(3.75)\quad \sum_{n=3}^{\infty} \Phi(n, 1)S_V(n^2 - 4) \ll V^{1/2}T^2 \log^2 V.\]

Substituting (3.75) to (3.63) yields

\[(3.76)\quad \sum_{n=3}^{\infty} \mathcal{L}_{n^2-4}(1)\Phi(n, 1) \ll V^{1/2}T^2 \log^2 V + V^{-1/2}\left(X^\theta T^2 + X^{1/2+\theta}T\right).\]

Now (3.53) follows from (3.76) by letting

\[V = X^\theta (1 + X^{1/2}/T).\]

4. PROOF OF THEOREMS 1.1 AND 1.2

Theorem 1.1 follows directly from Lemma 3.4 and Lemma 3.5.

Applying Theorem 1.1 to estimate the term $A_1$ in (1.30) we obtain

\[(4.1)\quad \sum_{j} \hat{\phi}(t_j) \ll B_1 + \max(X^{1/4+\theta/2}T^{3/2}, X^{\theta/2}T^2) \log^2 (XT) + O\left(N^{-1/2}X^{1/4+\theta}T^{3/2}\left(1 + \frac{T}{X^{1/2}}\right)\right) + O\left(T^2 \log^2 T + \frac{N^{1/2} \log^2 N}{X^{1/2}}\right).\]

The part $B_1$, defined by (1.32), can be bounded, for example, using (2.21) and the estimate of Luo and Sarnak [12]. This yields

\[(4.2)\quad B_1 \ll N^{-1/2}T^3 \log^2 T.\]

Nevertheless, as mentioned in the introduction, it is sufficient to use an analogue of the original result of Iwaniec (1.16), namely

\[(4.3)\quad B_1 \ll N^{-1/2}T^{7/2} \log^2 T.\]
Accordingly, substituting (4.3) to (4.1), we obtain

\[
\sum_j \hat{\phi}(t_j) \ll \max(X^{1/4+\theta/2}T^{3/2}, X^{\theta/2}T^2) \log^2(XT) + \\
\frac{T^{7/2}}{N^{1/2}} \log^2 T + N^{-1/2} X^{1/4+\theta}T^{3/2} \left(1 + \frac{T}{X^{1/2}}\right) + \\
T^2 \log^2 T + \frac{N^{1/2} \log N}{X^{1/2}}.
\]

Hence the optimal choice of \(N\) is

\[
N = T^{7/2} X^{1/2} + X^{3/4+\theta}T^{3/2} \left(1 + \frac{T}{X^{1/2}}\right).
\]

It should be noted that \(N\) is supposed to be an integer. Another important observation is that the summand \(X^{3/4+\theta}T^{5/2}X^{-1/2}\) is negligible in comparison with the other two summands in (4.5). Thus we can simplify the choice of \(N\) as follows

\[
N = \left[ T^{7/2} X^{1/2} + X^{3/4+\theta}T^{3/2} \right],
\]

where for a real number \(x\), \([x]\) denotes the integral part of \(x\). Substituting (4.6) to (4.8), we obtain

\[
\sum_j \hat{\phi}(t_j) \ll \max(X^{1/4+\theta/2}T^{3/2}, X^{\theta/2}T^2) \log^2(XT) + \\
T^2 \log^2 T + \left( T^{7/4} X^{-1/4} + X^{-1/8+\theta/2}T^{3/4} \right) \log^2(XT) \ll \\
\ll \max(X^{1/4+\theta/2}T^{3/2}, X^{\theta/2}T^2) \log^2(XT).
\]

Applying (2.21) and the trivial bound

\[
\sum_{t_j} X^{it_j} \exp(-t_j/T) \ll T^2,
\]

we prove that

\[
\sum_{t_j} t_j X^{it_j} \exp(-t_j/T) \ll \max \left( X^{1/4+\theta/2}T^{3/2}, X^{\theta/2}T^2 \right) \log^2(XT).
\]

Now (1.33) follows from (4.8). See [12, p. 235-236] for details.

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