Locally Maximizing Metric of Width on Manifolds with Boundary

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Abstract

In this paper we use min-max theory to study the existence free boundary minimal hypersurfaces (FBMHs) in compact manifolds with boundary \((M^{n+1}, \partial M, g)\), where \(2 \leq n \leq 6\). Under the assumption that \(g\) is a local maximizer of the width of \(M\) in its conformal class, we show the existence of a sequence of almost-properly embedded equidistributed FBMHs. This work extends the result of Ambrosio-Montezuma [2].

1 Introduction

In a recent work of Ambrosio and Montezuma [2], the equidistribution phenomenon of minimal \(S^2\) in \(S^3\) is studied. With the assumption that the metric \(g_0\) on \(S^3\) is a local maximizer (in its conformal class) of the Simon-Smith width functional \(W(S^3, g)\), the authors proved the existence of equi-distributed minimal 2-spheres in measure theoretic sense. In this paper we follow their main ideas and extend the results to embedded free boundary minimal hypersurfaces in a ball of dimension \(3 \leq n + 1 \leq 7\). We shall prove the following result:

**Theorem 1.1** Given metric \(g\) on \((M^{n+1}, \partial M)\), \(2 \leq n \leq 6\), if \(g\) maximizes the normalized width \(W(M, g)\) in the conformal class of \(g\), then there exist a sequence \(\{\Sigma^n_i\}\) of free boundary minimal hypersurfaces with index zero or one and area no greater than \(W(M, g)\) for which the following holds:

\[
\lim_{k \to \infty} \frac{1}{\sum_{i=1}^{k} \text{area}(\Sigma^n_i, g)} \sum_{i=1}^{k} \int_{\Sigma^n_i} f dA_g = \frac{1}{\text{vol}(M, g)} \int_M f dV_g.
\]

Furthermore, if we assume that \((M, \partial M, g)\) contains no stable free boundary minimal hypersurface with area greater than its width \(W(M, g)\), then we can choose \(\{\Sigma^n_i\}\) so that each of them has index 1 and area equal to \(W(M, g)\):

**Theorem 1.2** Given metric \(g\) on \((M^{n+1}, \partial M)\), \(2 \leq n \leq 6\), if \(g\) maximizes the normalized width in the conformal class of \(g\) and there exists no stable free
boundary minimal hypersurface of area less than or equal to \( W(M, g) \), then there exist a sequence \( \{ \Sigma_i \} \) of free boundary minimal disks with index one and area equal to \( W(M, g) \) for which the following holds:

\[
\lim_{k \to \infty} \frac{1}{kW(M, g)} \sum_{i=1}^{k} \int_{\Sigma_i} f dA_g = \frac{1}{\text{vol}(M, g)} \int_M f dV_g.
\]

The main difference between our theorem and Proposition 1.4.1 of [2] is that in free boundary case, we can not rule out the case when \( \{ \Sigma_i \} \) is not properly embedded (\( \Sigma \cap \partial M \neq \emptyset \)), due to the lack of convexity of \( \partial M \). Readers can see [6] for a possible example of non-properly embedded free boundary minimal hypersurface in an Euclidean domain.

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2 Preliminaries

In the following let \( 2 \leq n \leq 6 \), and \((M^{n+1}, \partial M, g)\) be a Riemannian manifold with smooth boundary \( \partial M \) and metric \( g \). The notions of sweepout and width are crucial in the min-max theory of minimal hypersurfaces. In [6] a min-max theory of free boundary minimal hypersurfaces (FBMH) were developed, which is of great use here in our context. First we give a introduction to FBMH.

2.1 Free Boundary Minimal Hypersurfaces and Morse Index

Let \( (M, \partial M, g) \) be as above. A free boundary minimal hypersurface \( \Sigma \) in \((M, g)\) is a \( n \)-dimensional submanifold of \( M \) with vanishing mean curvature\( (H = 0) \) and boundary \( \partial \Sigma \) orthogonal to \( \partial M \). We can also use the first variation of area of \( \Sigma \) to characterize this property: given a smooth perturbation of \( M \) defined by \( \phi : M \times (-\epsilon, \epsilon) \to M \) with \( \phi(\cdot, 0) = \text{id}_M \) and \( \phi(\partial M, \cdot) \subset \partial M \), we have the following first variation formula:

\[
\frac{\partial}{\partial s} \text{area}[\phi(\Sigma, s)] \bigg|_{s=0} = \int_{\Sigma} -H \vec{n} \cdot \frac{\partial \phi}{\partial s} \bigg|_{s=0} dA + \int_{\partial \Sigma} \phi \eta \cdot \vec{n} ds
\]

where \( \vec{n} \) is the unit normal of \( \Sigma \) and \( \eta \) is the outward conormal along \( \partial \Sigma \). Therefore \( \Sigma \) is a critical point if and only if \( H = 0 \) on \( \Sigma \) and \( \eta \perp \vec{n} \) on \( \partial \Sigma \), as in the definition of FBMS. For variation in normal direction as \( \frac{\partial \phi}{\partial t} \bigg|_{s=0} = f \cdot \vec{n} \), we have the second variation of area:

\[
\frac{\partial^2}{\partial s^2} \text{area}[\phi(\Sigma, s)] \bigg|_{s=0} = \int_{\Sigma} \left( |\nabla f|^2 - \text{Ric}_M(\vec{n}, \vec{n}) f^2 - |A|^2 |f|^2 \right) d\mu - \int_{\partial \Sigma} h^\partial M f^2 ds
\]
After an integration by part, the right hand side of the second variation formula defines a quadratic form on $C^\infty(\Sigma)$:

$$ I(f, g) = \int_\Sigma (-f \Delta g - \text{Ric}_M(\vec{n}, \vec{n})fg - |A|^2 fg) d\mu + \int_{\partial \Sigma} (f \frac{\partial g}{\partial \vec{n}} - h \partial_M fg) ds $$

and we define the index of $\Sigma$ to be the number of negative eigenvalues of $I$. $\Sigma$ is called a stable FBMS if its index is 0, i.e. there is no variation that reduce the area of $\Sigma$ to the second order.

### 2.2 Min-max Construction

Given a manifold with boundary $(M^{n+1}, \partial M, g)$, let $\mathcal{Z}_n(M, \partial M, \mathbb{Z})$ be the space of integer rectifiable $n$-currents $T$ in $M$ with coefficients in $\mathbb{Z}$, such that $\partial T \in \partial M$, modulo the following equivalence relation:

$$ T \sim S \text{ iff } T - S \in R^n(\partial M, \mathbb{Z}) $$

where $R_n(\partial M, \mathbb{Z})$ is the space of $n$-rectifiable integral currents in a sufficiently high dimensional Euclidean space $\mathbb{R}^L$, supported on $\partial M$. (We can regard the $M$ as embedded isometrically in $\mathbb{R}^L$.) We endow $\mathcal{Z}_n(M, \partial M, \mathbb{Z})$ with the flat topology $\mathcal{F}$. Let us define the notion of 1-sweepout and 1-width.

**Definition 2.1 (cf [5])** Let $(M, \partial M)$ be defined as above. A 1-sweepout of $M$ is a one parameter family of maps $\Phi : [-1, 1] \rightarrow \mathcal{Z}_n(M, \partial M, g)$ satisfying the following conditions:

1. $\Phi$ is continuous in flat topology;
2. $\sup_{t \in [-1, 1]} M(\Phi(t)) < +\infty$;
3. there is no mass concentration on $\Phi$;
4. $F(\Pi_\Phi)$ represents a non-zero element in $H_{n+1}(M, \partial M)$.

**Definition 2.2** We define the width of a manifold with metric $g$ as

$$ W(M, \partial M, g) = \inf_{\Phi \in \Lambda} \left( \max_{t \in [-1, 1]} M(\Phi(t), g) \right) $$

where $\Phi$ is a sweepout of $(M, \partial M, g)$. The normalised width is defined by

$$ W_n(M, \partial M, g) = \frac{W(M, \partial M, g)}{\text{Vol}(M, g)^{(n+1)/n}}. $$

Let us note that by a similar argument as in [7], under a smooth variation of metric $g(t)$ with respect to the original metric, $W(\mathbb{B}, g(t))$ is a Lipshitz function of $t$. 

3
3 Proof of the Main Theorems

In this section we prove Theorem 1.1 using a perturbation method originally due to Marques-Neves-Song[7], and prove Theorem 1.2 by a calculation of derivative of width inspired by Fraser-Schoen’s work[3] on Steklov eigenvalues.

3.1 Proof of Theorem 1.1

In view of the abstract theorem 4.2, we can reduce the equi-distribution property to proving the following lemma:

**Lemma 3.1.1** Let $g$ be a Riemannian metric on $M$ that maximizes the normalized width in its conformal class. For every continuous function $f$ satisfying

$$\int_M f dV_g < 0,$$

there exists some integers $n_1, \cdots, n_N$, and disjoint embedded free boundary minimal hypersurfaces $\Sigma_1, \cdots, \Sigma_N$ in $(M, g)$ such that

$$W(M, g) = \sum_{i=1}^{N} n_i \text{area}(\Sigma_i, g), \quad \sum_{i=1}^{N} \text{Ind}_g(\Sigma_i) \leq 1$$

and

$$\sum_{i=1}^{N} n_i \int_{\Sigma_i} f dA_g \leq 0.$$

In order to associate the function $f$ with the derivative of width under a conformal change of metric, we need to perturb the conformal family of the original metric to a new family so that the width is differentiable. The following technical lemma is crucial:

**Lemma 3.1.2** Let $q \geq 4$ be an integer, and $g : [0, 1] \to \Gamma_q$ be a smooth embedding. Then there exist smooth embeddings $h : [0, 1] \to \Gamma_q$ which are arbitrarily close to $g$ in the smooth topology, and $J \subset [0, 1]$ with full Lebesgue measure such that

1. The function $W(M, h(\tau))$ is differentiable at every $\tau \in J$; and
2. For each $\tau \in J$, there exist a collection of integers $\{n_1, \cdots, n_N\}$ and a finite collection $\{\Sigma_1, \cdots, \Sigma_N\}$ of disjoint free boundary embedded minimal hypersurfaces of class $C^q$ in $(M, h(\tau))$ such that

$$W(M, h(\tau)) = \sum_{k=1}^{N} n_k \cdot \text{area}(\Sigma_k, h(\tau)), \quad \sum_{k=1}^{N} \text{ind}_{h(\tau)}(\Sigma_k) \leq 1,$$

and

$$\left. \frac{d}{dt} W(M, h(t)) \right|_{t=\tau} = \frac{1}{2} \sum_{k=1}^{N} n_k \int_{\Sigma_k} \text{Tr}(\Sigma_k, h(\tau))(\partial_t h(\tau))dA_{h(\tau)}.$$
Proof. (cf [2]) First, due to the density of bumpy metric on $M$ and Rademacher’s theorem, we can perturb the smooth family $g : [0, 1] \to \Gamma_q$ to $h : [0, 1] \to \Gamma_q$ which is arbitrarily close to $g$ in smooth topology, and a set $J \subset [0, 1]$ of full measure such that $h(\tau)$ is a bumpy metric and $W(M, h(t))$ is differentiable at $\tau$, for all $\tau \in J$.

For all $\tau \in J$, fix a sequence $t_i \to \tau$, we have
\[
\left. \frac{d}{dt} W(M, h(t)) \right|_{t=\tau} = \lim_{i \to \infty} \frac{W(M, h(t_i)) - W(M, h(\tau))}{t_i - \tau}.
\]

By Proposition 7.3 of [1], we can find a finite disjoint collection of FBMHs \( \{\Sigma_1(t_i), \ldots, \Sigma_{i_k}(t_i)\} \) and integers \( \{N_1, \ldots, N_{i_k}\} \) such that
\[
W(M, h(t_i)) = \sum_{j=1}^{k} N_j \text{area}(\Sigma_{i_j}(t_i)) - \sum_{j=1}^{k} N_j \cdot \text{Ind}(\Sigma_{i_j}(t_i)) \leq 1
\]
Now as $t_i \to \tau$, since $h$ is a smooth family we have $\text{area}(\Sigma_{i_j}(t_i))$ uniformly bounded below and above by $W(M, h(\tau))$ as $t_i$ is sufficiently close to $\tau$. Therefore by the compactness theorem we can extract a subsequence $t_{i_j}$ so that $\Sigma_{i_{j_k}}$ converges in the varifold sense to $\Sigma_k$, since the metric $h(\tau)$ is bumpy, there is no multiplicity issue in the convergence, so we can conclude that the convergence is graphical and smooth. Therefore standard calculation shows
\[
\lim_{i \to \infty} \frac{\text{area}(\Sigma_{i_{j_k}}, h(t_{i_j})) - \text{area}(\Sigma_{i_j}, h(\tau))}{t_{i_j} - \tau} = \frac{1}{2} \int_{\Sigma_k} \text{Tr}(\Sigma_{i_{j_k}, h(\tau)})(\partial h(\tau)) dA_h(\tau)
\]
and hence we have the derivative of width formula. \( \square \)

Now we can finish the proof of Theorem 1.1 by showing Lemma 1.1.1. For a continuous function $f$ with $\int_M f dV_g < 0$, we can define a conformal change of metric:
\[
g(t) = (1 + \frac{n+1}{n} t f)^{\frac{n}{n+1}} g \quad \text{for} \quad 0 \leq t \leq T.
\]
We have $\partial_t g(t)|_{t=0} = f g$, hence for small $T > 0$ we have $\text{Vol}(M, g(t))$ less than the the volume under the original metric. Since $g$ maximizes the normalised width, we have
\[
\frac{W(M, g(t))}{\text{Vol}(M, g(t))^{\frac{n}{n+1}}} \leq \frac{W(M, g(0))}{\text{Vol}(M, g(0))^{\frac{n}{n+1}}} \quad \text{for} \quad 0 \leq t \leq T.
\]
Hence
\[
W(M, g(t)) \leq W(M, g(0)) \left( \frac{\text{Vol}(M, g(t))^{\frac{n}{n+1}}}{\text{Vol}(M, g(0))^{\frac{n}{n+1}}} \right) < W(M, g(0)) \quad \text{for} \quad 0 \leq t \leq T.
\]
Fix $q \geq 4$. Now for each $i \in \mathbb{N}$ with $1/i < T$, we can find a perturbation $h_i : [0, 1/i] \to \Gamma_q$ and $J_i \subset [0, 1/i]$ with full Lebesgue measure such that
\[
W(M, h_i(1/i)) < W(M, h_i(0))
\]
and so there is $\tau_i \in J_i$ such that

$$\left. \frac{d}{dt} W(M, h_i(t)) \right|_{t=\tau_i} \leq 0$$

due to the first fundamental theorem of calculus. Hence by the previous lemma there are FBMHs $\Sigma_{i_j}$, $j = 1, 2, \cdots, n_i$ and a set of integers $\{n_{i_1}, \cdots, n_i\}$ such that

$$W(M, h_i(\tau_i)) = \sum_{k=1}^N n_{i_k} \cdot \text{area}(\Sigma_{i_k}, h_i(\tau_i)), \quad \sum_{k=1}^N \text{ind}_{h_i(\tau)}(\Sigma_{i_k}) \leq 1,$$

and

$$\left. \frac{d}{dt} \right|_{t=\tau_i} W(M, h_i(t)) = \frac{1}{2} \sum_{k=1}^N n_{i_k} \int_{\Sigma_{i_k}} \text{Tr}(\Sigma_{i_k}, h_i(\tau_i))(\partial_t h_i(\tau_i)) dA_{h_i(\tau_i)} \leq 0.$$

We can relabel these $\Sigma_{i_k}$ such that except for $\Sigma_{i_1}$, others have index 0. Now we can use the Compactness Theorem A.6 to conclude that, by picking a subsequence $\tau_i \to 0$, the FBMHs subconverges smoothly and graphically to $\{\Sigma_1, \cdots, \Sigma_N\}$ with multiplicity 1, except for $\Sigma_1$, where the multiplicity can be 2 if $\Sigma_1$ is stable. Therefore we can pass the limit of the formula above and show

$$\frac{1}{2} \sum_{k=1}^N n_k \int_{\Sigma_k} f dA_g = \frac{1}{2} \sum_{k=1}^N n_k \int_{\Sigma_k} \text{Tr}_{\Sigma_k}(\partial_t g(0)) dA_g \leq 0$$

Hence this finish the proof when $f$ is a smooth function on $(M, g(0))$. When $f$ is a continuous function we can use smooth functions to approximate $f$ uniformly and use similar arguments in the sequence picking process. Hence once Lemma 1.1.1 is proved, then the implication i) to iv) in Theorem 4.2 applies if we let $Y$ be the Radon measure induced from FBMHs in $M$, and $\mu_0$ be the Hausdorff measure on $(M, g)$.

### 3.2 Proof of Theorem 1.2

Now we prove Theorem 1.2. First we need a result that guarantees the existence of optimal sweepout in Lemma 1.2.2, and then we can compute the derivative of width under a general smooth family of metrics.

**Lemma 3.2.1** ([5] Prop. 5.4). Let $(\Sigma, \partial \Sigma) \subset (M, \partial M)$ be an orientable, almost properly embedded, free boundary minimal hypersurface with Area($\Sigma$) less than the area of the stable free boundary minimal hypersurface in $M$. Then there is a sweepout $\Psi : [-1, 1] \to Z_n(M, \partial M)$, such that:

1. $\Psi(0) = \Sigma$;
2. $\Psi(1) = M$;
3. $\Psi(t) < \text{Area}(\Sigma)$ for $t \neq 0$. 

Lemma 3.2.2 Given \( \{g(t)\}_{t \in (a, b)} \) as a one parameter family of metrics on \( M \) varying smoothly, if \( t_0 \in (a, b) \) is a point where \( W(t) := W(M, g(t)) \) is differentiable, then there is an almost properly embedded free boundary minimal hypersurface \( \Sigma \) in \( (M, g(t_0)) \) such that

\[
\text{area}(\Sigma, g(t_0)) = W(t_0) \quad \text{and} \quad \frac{d}{dt} W(M^n, g(t)) \bigg|_{t=t_0} = \frac{1}{2} \int_{\Sigma} \text{Tr}_\Sigma \left( \frac{\partial}{\partial t} g(t) \bigg|_{t=t_0} \right) dA(g(t_0)).
\]

Proof. By Lemma 1.2.1, there exist an optimal sweepout \( \{\Sigma_s\}_{s \in [-1, 1]} \) such that area(\( \Sigma_0 \)) = \( W(M, g(t_0)) \) and for all \( s \neq 0 \), area(\( \Sigma_s \)) \(<\) area(\( \Sigma_0 \)). Consider a smooth function \( F : (a, b) \times [-1, 1] \rightarrow \mathbb{R} \) defined as \( F(t, s) = \text{area}(\Sigma_s, g(t)) \), then we have \( F_s(t_0, 0) = 0 \) and \( F_{ss}(t_0, 0) < 0 \). Now let us show that there exists \( \epsilon > 0 \) such that there is a differentiable function \( s = s(t) \) for \( t \in (t_0 - \epsilon, t_0 + \epsilon) \), such that

\[
F(t, s(t)) = \max_{s \in [-1, 1]} F(t, s).
\]

Since \( F_{ss}(t_0, 0) < 0 \), the implicit function theorem guarantees that \( F_s(t, s) = 0 \) defines a smooth function \( s = s(t) \) on \((t_0 - \epsilon, t_0 + \epsilon)\). Now there is a neighborhood of \((t_0, 0)\) such that \( F_{ss} < 0 \), and therefore \( F(t, s(t)) \) is a local maximum for each fixed \( t \in (t_0 - \epsilon', t_0 + \epsilon') \). Due to the construction of sweepout(property 3) and possibly making \( \epsilon' \) even smaller we can make sure \( F(t, s(t)) \) is a strict maximum. Hence the claim is proved. Now we define a function \( h(t) = F(t, s(t)) - W(t) \) over a neighborhood of \( t_0 \). We have that \( h(t) \geq 0 \) due to the definition of width, and \( h(t_0) = 0 \) is the local minimum. Since \( W(t) \) is differentiable at \( t_0 \), \( h \) is also differentiable and \( h'(t_0) = 0 \). Hence we have

\[
W'(t_0) = \frac{\partial}{\partial t} F(t, s(t)) \bigg|_{t=t_0} = F_s(t_0, 0)s'(t_0) + F_t(t_0, 0) = \frac{1}{2} \int_{\Sigma} \text{Tr}_\Sigma \left( \frac{\partial g(t)}{\partial t} \right) dA(g(t_0))
\]

\( \square \)

Similar to the proof of Theorem 1.1, we can define a conformal change of the metric \( g \), this time with a volume preserving factor. More precisely, for a smooth function \( f \) with \( \int_B fdV_g = 0 \), we fix a small \( T > 0 \) and let

\[
g(t) = \frac{\text{Vol}(M, g)^{\frac{n}{n-2}} \text{Vol}(M, (1 + ft)g)^{\frac{2}{n-2}}}{\text{Vol}(M, g(0))} g \quad \text{for all } t \in [0, T).
\]

It is straightforward to show that \( \text{Vol}(M, g(t)) = \text{Vol}(M, g(0)) \) for all \( t \in [0, T] \), and that \( \partial_t g(0) = fg \).

Lemma 3.2.3 Let \( g(t), \ t \in [0, \epsilon) \) be a smooth family of Riemannian metrics on \( M \) that contains no stable free boundary minimal surface with area greater than \( W(M, g) \). If

\[
W(M, g(0)) \geq W(M, g(t))
\]

then there exists a free boundary minimal disk \( \Sigma \) such that

\[
\text{area}(\Sigma, g(0)) = W(\mathbb{B}^n, g(0)) \quad \text{and} \quad \int_{\Sigma} \text{Tr}_\Sigma(\partial_t g(0)) dA(g(t_0)) \leq 0.
\]
Proof. Take an \( \epsilon > 0 \). By Rademacher’s Theorem, \( W \) is differentiable at almost all \( t \in [0, \epsilon) \). Since \( W \) assumes local maximum at 0, there exists a sequence \( t_n \in [0, \epsilon) \) converging to \( t_0 \) such that \( W'(t_n) \leq 0 \) for all \( n \). Hence by the previous lemma we can find an embedded free boundary minimal disk \( \Sigma_n \) in \((B^n, g(t_n))\) with area\( (\Sigma_n, g(t_n)) = W(t_n) \) and \( \int_{\Sigma_n} \text{Tr}_{\Sigma_n} (\partial_t g(t_n)) dA_{g(t_n)} \leq 0 \). Now by the compactness theorem we see that \( \Sigma_n \) subconverges to an embedded free boundary minimal disk \( \Sigma \). By the smooth convergence we have area\( (\Sigma, g(0)) = W(0) \) and \( \int_{\Sigma} \text{Tr}_\Sigma (\partial_t g(0)) dA_{g(0)} \leq 0 \). 

Combining the Lemma 1.2.3 and the previously defined conformal change of metric, we can show the following statement:

**Proposition 3.2.4** Let \( f \) be a continuous function on \((M, g)\) with zero average, and if \((M, g)\) contains no stable free boundary minimal surface with area greater than \( W(M, g) \), we can find a almost properly embedded FBMH \( \Sigma \) in \((\mathbb{R}^n, g)\) such that \( \int_{\Sigma} \text{Tr}_\Sigma (f) dV_g \leq 0 \).

**Proof.** This statement follows when we approximate the function \( f \) uniformly by smooth functions, and use the previous conformal change of metric. 

Then as in the proof of Theorem 1, the implication ii) to iv) in Theorem 4.2 will confirm the existence of equidistributed FBMHs in \( M \), and as Lemma 3.2.3 shows, each \( \Sigma_i \) has area equal to \( W(M, g(0)) \).

### 4 Compactness Theorem and Equidistribution Theorem

In this section we collect a compactness theorem of FBMH for varying background metric, and the abstract theorem on the existence of equi-distributed sequence of measures.

**Theorem 4.1.** Let \( 2 \leq n \leq 6 \) and \( N^{n+1} \) be a compact manifold and \( \{g_k\}_{k \in \mathbb{N}} \) a family of Riemannian metrics on \( N \) converging smoothly to some limit \( g \). If \( \{M_k^n\} \subset N \) is a sequence of connected and embedded free boundary minimal hypersurface in \((N, g_k)\) with

\[
H^n(M_k) \leq \Lambda < \infty \quad \text{and} \quad \text{index}_k(M_k) \leq I,
\]

for some fixed constants \( \Lambda \in \mathbb{R}, I \in \mathbb{N} \), both independent of \( k \). Then up to subsequence, there exists a connected and free boundary embedded minimal hypersurface \( M \subset (N, g) \) where \( M_k \to M \) in the varifold sense with

\[
H^n(M) \leq \Lambda < \infty \quad \text{and} \quad \text{index}(M_k) \leq I
\]

we have that the convergence is smooth and graphical for all \( x \in M - Y \) where \( Y = \{y_i\}_{i=1}^K \subset M \) is a finite set with \( K \leq I \) and the following dichotomy holds:
• if the number of leaves in the convergence is one then $Y = \Phi$, i.e. the convergence is smooth an graphical everywhere

• if the number of sheets is $\geq 2$
  -if $N$ has $\text{Ric}_X > 0$ then $M$ cannot be one-sided
  -if $M$ is two-sided the $M$ is stable.

Proof. We know by Allard’s compactness theorem that there is an $M$ such that after passing to a subsequence, $M_k \to M$ in $\textbf{IV}_n(N)$. Let $Y \subset M$ be the singular set of $M$. First we show that $|Y| \leq I$. Suppose on the contrary that $Y$ contains at least $I + 1$ points $y_1, \cdots, y_{I+1}$. Then we can find $\{\epsilon_i\}_{i=1}^{I+1}$ such that $B(y_i, \epsilon_i) \cap B(y_j, \epsilon_j) = \emptyset$, and that $\sup \sup_{M_k \cap B(y_i, \epsilon_i)} |A|^2 = \infty$, for all $i = 1, \cdots, I + 1$. Since $g_k$ converges to $g$ smoothly, the sectional curvature of $(N^{n+1}, g_k)$ are uniformly bounded. Hence curvature estimate of [Li-Guangzhou] applies to this varying metric case, that is, in $\Sigma_k$ the second fundamental form of $\Sigma_k$ are bounded by a uniform constant $C$ that depends only on $N$. Hence we infer that for sufficiently large $k$, $M_k \cap B(y_i, \epsilon_i)$ is not stable for all $i = 1, \cdots, I + 1$. This implies that $\text{index}_k(M_k) \geq I + 1$ which contradicts with the assumption.

To show that $\text{Index}(M) \leq I$, we suppose that there are $u_1, u_2, \cdots, u_{I+1} \in C^\infty(M)$ that are $L^2$-orthogonal such that $I(u_i, u_i) < 0$ for $i = 1, 2, \cdots, I + 1$. Then we extend $u_i$ to $\tilde{u}_i \in C^1(M)$ and let $u_k^i = \tilde{u}_i|_{M_k}$. Since $M_k \to M$ as varifold, we have for sufficiently large $k$, $I_k(u_k^i, u_k^j) < 0$ for $i = 1, 2, \cdots, I + 1$. Since $\text{Index}(M_k) \leq I$, $\{u_k^i\}_{i=1}^{I+1}$ must be linearly dependent. By taking a subsequence and relabeling if necessary, we can find $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}$ and $\lambda_i$’s not all zero such that $u_{I+1}^k = \sum_{i=1}^n \lambda_i u_k^i$. By varifold convergence we have $\langle u_i^k, u_j^k \rangle \to \langle u_i, u_j \rangle = \delta_{ij}$ for $i, j = 1, 2, \cdots, n + 1$. Therefore by the varifold convergence,$$
 0 = \langle u_{n+1}, u_i \rangle_M = \lim_{k \to \infty} \langle u_{n+1}^k, u_i^k \rangle_{M_k} = \lim_{k \to \infty} \lambda_i
$$This implies that $u_{n+1} = 0$ which contradicts $I(u_{n+1}, u_{n+1}) < 0.$

Now if the multiplicity of convergence is 1, then the convergence is smooth everywhere by the regularity theorem of [8]. Hence the theorem is proved. $\square$

Here we include an abstract theorem used in the proof of Theorem 1.1 and 1.2., for the proof see Theorem B.1 of [2].

Theorem 4.2. (cf [2]) Let $Y$ be a non-empty weak-* compact subset of $M(X)$. The following assertions about a measure $\mu_0$ in $M(X)$ are equivalent to each other:

i) For every function $f \in C^0(X)$ such that $\int_X f d\mu_0 < 0$, there exists $\mu \in Y$ such that $\int_X f d\mu \leq 0$.  

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ii) For every function $f \in C^0(X)$ such that $\int_X f d\mu_0 = 0$, there exists $\mu \in Y$ such that $\int_X f d\mu_0 \leq 0$.

iii) $\mu_0$ belongs to the weak-* closure of the convex hull of the positive cone over $Y$.

iv) There exists a sequence $\{\mu_k\}$ in $Y$ such that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{\mu_i(X)} \int_X f d\mu_i = \frac{1}{\mu_0(X)} \int_X f d\mu_0$$
for all $f \in C^0(X)$.

References

[1] Ambrozio, Lucas, Alessandro Carlotto, and Ben Sharp. "Compactness analysis for free boundary minimal hypersurfaces." Calculus of Variations and Partial Differential Equations 57.1 (2018): 22.

[2] Ambrozio, Lucas, and Rafael Montezuma. "On the width of unit volume three-spheres." arXiv preprint arXiv:1809.03638 (2018).

[3] Fraser, Ailana, and Richard Schoen. "Sharp eigenvalue bounds and minimal surfaces in the ball." Inventiones mathematicae 203.3 (2016): 823-890.

[4] Guang, Qiang, et al. "Min-max theory for free boundary minimal hypersurfaces II- General Morse index bounds and applications." arXiv preprint arXiv:1907.12064 (2019).

[5] Guang, Qiang, Zhichao Wang, and Xin Zhou. "Free boundary minimal hypersurfaces with least area." arXiv preprint arXiv:1801.07036 (2018).

[6] Li, Martin, and Xin Zhou. "Min-max theory for free boundary minimal hypersurfaces I-regularity theory." arXiv preprint arXiv:1611.02612 (2016).

[7] Marques, Fernando C., André Neves, and Antoine Song. "Equidistribution of minimal hypersurfaces for generic metrics." Inventiones mathematicae 216.2 (2019): 421-443.

[8] White, Brian. "The space of minimal submanifolds for varying Riemannian metrics." Indiana University Mathematics Journal (1991): 161-200.