INDEX PAIRING WITH ALEXANDER-SPANIER COCYCLES

ALEXANDER GOROKHOVSKY AND HENRI MOSCOVICI

To ALAIN CONNES, with admiration and deep appreciation

Abstract. We give a uniform construction of the higher indices of elliptic operators associated to Alexander–Spanier cocycles of either parity in terms of a pairing à la Connes between the $K$-theory and the cyclic cohomology of the algebra of complete symbols of pseudodifferential operators, implemented by means of a relative form of the Chern character in cyclic homology. While the formula for the lowest index of an elliptic operator $D$ on a closed manifold $M$ (which coincides with its Fredholm index) reproduces the Atiyah-Singer index theorem, our formula for the highest index of $D$ (associated to a volume cocycle) yields an extension to arbitrary manifolds of any dimension of the Helton–Howe formula for the trace of multicommutators of classical Toeplitz operators on odd-dimensional spheres. In fact, the totality of higher analytic indices for an elliptic operator $D$ amount to a representation of the Connes-Chern character of the $K$-homology cycle determined by $D$ in terms of expressions which extrapolate the Helton–Howe formula below the dimension of $M$.

1. Introduction

The intuition that the Alexander–Spanier version of cohomology was an yet untapped resource for index theory has been one of the surprising insights that made the original list of topics in Alain Connes’ master plan for noncommutative geometry [3, Introduction]. It has materialized in [4], where higher indices of elliptic operators associated to Alexander–Spanier cocycles were used to prove the Novikov conjecture for manifolds with word-hyperbolic fundamental group.

For an elliptic operator $D$ between two vector bundles over a closed manifold $M$, the higher analytic index $\text{Ind}_\phi D \in \mathbb{C}$ corresponding to an even-dimensional Alexander–Spanier cocycle $\phi$ on $M$ was constructed

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essentially by folding the Alexander–Spanier cohomology of the manifold into cyclic cohomology for pseudodifferential operators, by exploiting the fact that the parametrices of $D$ can be localized at will near the diagonal. The resulting number $\text{Ind}_D$ depends only on the cohomology class $[\phi] \in H^c(M, \mathbb{C})$; in particular $\text{Ind}_1 D$ coincides with the Fredholm index of $D$. After showing that these higher indices admit cohomological expressions akin to the Atiyah-Singer index formula \[ 1 \], Connes and Moscovici proved a generalization of the $\Gamma$-index theorem (cf. Atiyah \[ 2 \] and Singer \[ 16 \]), which was instrumental in their proof \[ 4 \] of the homotopy invariance of higher signatures.

In a different role, the pairing between Alexander–Spanier cohomology and the signature operator was used to produce local expressions for the rational Pontryagin classes of topological manifolds (quasiconformal in \[ 5 \] and Lipschitz in \[ 13 \]), and also for the Goresky-MacPherson $L$-class of Witt spaces \[ 14, 15 \].

In the aforementioned applications the odd case of the pairing, associating higher analytic indices of selfadjoint elliptic operators to odd-dimensional Alexander–Spanier cocycles, was handled in an indirect manner, essentially by reducing it via Bott suspension to the even case. The lack of a natural uniform definition irrespective of parity remained though a challenging conundrum, at least in the minds of the present authors. It is the purpose of this paper to provide a unified construction for higher analytic indices of either parity, based on recasting their definition within the conceptual framework of Connes’ pairing between the $K$-theory and the cyclic cohomology of an algebra. Instead of the algebra of functions, the appropriate algebra for the task at hand is the algebra of complete symbols of pseudodifferential operators, over which the pairing is implemented by means of a relative form of the Chern character in cyclic homology.

Perhaps the quickest way to illustrate the flavor of the present picture for the higher index pairing (described in \[ 2 \] below) is to mention its extreme cases. While $\text{Ind}_1 D$ is just the Fredholm index of the elliptic operator $D$, the higher analytic index of $D$ associated to a top-dimensional Alexander–Spanier (volume) cocycle on a manifold $M$ has an expression reminiscent of the Helton–Howe formula \[ 7, \S7 \] for the trace of the top multicommutator of Toeplitz operators on an odd-dimensional sphere. As a matter of fact our formula represents a twofold extension of the Helton–Howe formula, to arbitrary manifolds and to any dimension, regardless of parity.

It was already noticed by Connes (cf. \[ 3 \] Part I, \S7]) that in the commutative case the total antisymmetrization of his Chern character in $K$-homology yields the Helton–Howe fundamental trace form. The
cohomological formulas (proved in §3 below) for the higher analytic indices of an elliptic operator $D$ on a closed manifold $M$ show that, conversely, the full Connes-Chern character of the K-homology cycle determined by $D$ can be recovered by extrapolating the Helton–Howe formula below the dimension of $M$.

2. Higher analytic indices

2.1. Higher Alexander–Spanier traces. Let $M$ be a closed manifold. The (smooth) Alexander–Spanier complex of $M$ is the quotient complex $C^\bullet(M) = \{C^\bullet(M)/C^\bullet_0(M), \delta\}$, where $C^q(M) = C^\infty(M^{q+1})$, $C^q_0(M)$ consists of those $\phi \in C^q(M)$ which vanish in a neighborhood of the iterated diagonal $\Delta_{q+1}M = \{(x, \ldots, x) \in M^{q+1}\}$, and

$$\delta \phi(x_0, x_1, \ldots, x_{q+1}) = \sum (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{q+1}).$$

Its cohomology groups $H^\bullet_{AS}(M)$ yield the usual cohomology of $M$, alternatively computed from the Čech, the simplicial or the de Rham complex. The same cohomology is obtained from several variants of the Alexander–Spanier complex. We single out one of these variations which is particularly suited to the purposes of this paper. The specific complex consists of decomposable Alexander–Spanier cochains, i.e. finite sums of the form

$$\phi = \sum_\alpha f^\alpha_0 \otimes f^\alpha_1 \otimes \ldots \otimes f^\alpha_q, \quad f^\alpha_i \in C^\infty(M),$$

which in addition are totally antisymmetric

$$\phi(x_{\nu(0)}, x_{\nu(1)}, \ldots, x_{\nu(q)}) = \text{sgn}(\nu) \phi(x_0, x_1, \ldots, x_q), \quad \forall \nu \in S_{q+1}.$$

The decomposable and totally antisymmetric cochains give rise to a quasi-isomorphic subcomplex $C^\bullet_{\Lambda}(M) = \{C^\bullet_{\Lambda}(M)/C^\bullet_{\Lambda,0}(M), \delta\}$ of the Alexander–Spanier complex $C^\bullet(M)$.

On the other hand we consider the algebra of classical pseudodifferential operators $\Psi(M)$ on $M$, and the exact sequence

$$(1) \quad 0 \rightarrow \Psi^{-\infty}(M) \rightarrow \Psi(M) \xrightarrow{\sigma} S(M) \rightarrow 0,$$

where $\Psi^{-\infty}(M)$ is the ideal of smoothing operators and $S(M)$ is the quotient algebra of complete symbols; $\sigma$ denotes the complete symbol map.

To any cochain $\phi = \sum_i f^i_0 \otimes f^i_1 \otimes \ldots \otimes f^i_k \in C^k_{\Lambda}(M)$ we associate the multilinear form $\text{Tr}_\phi$ on $\Psi^{-\infty}(M)$, defined by

$$\text{Tr}_\phi(A_0, A_1, \ldots, A_k) = \sum_i \text{Tr} (A_0 f^i_0 A_1 f^i_1 \ldots A_k f^i_k),$$

where $\text{Tr}$ is the trace. 
The assignment $C^\bullet_A(M) \ni \phi \mapsto \text{Tr}_\phi \in CC^\bullet(\Psi^{-\infty}(M))$ satisfies the coboundary identities

$$b \text{Tr}_\phi = \text{Tr}_{\delta \phi}, \quad B \text{Tr}_\phi = 0;$$

the first one is tautological and the second follows from the total antisymmetry of $\phi$.

To extend this assignment to the full algebra of pseudodifferential operators we use zeta-regularization of the operator trace. Fix an elliptic operator $R \in \Psi^1(M)$, invertible and positive. For $A \in \Psi^p(M)$ form

$$\zeta(s) = \text{Tr} AR^{-s}, \quad \Re(s) \gg 0.$$

$\zeta(s)$ has meromorphic continuation beyond 0, around which

$$\zeta(s) = \frac{1}{s} \text{Res} A + \text{Tr} A + O(s).$$

The Wodzicki residue $\text{Res} A$ is independent of the choice of $R$, defines a trace on the algebra $\Psi(M)$ and can be computed from its symbol $\sigma(A)(x, \xi) \sim \sum_{j=0}^\infty \sigma_{p-j}(A)(x, \xi)$, with $\sigma_k(A)(x, \xi)$ homogeneous of degree $k$, by the formula (cf. [17])

$$\text{Res} A = (2\pi)^{-\dim M} \int_{S^*M} \sigma_{-\dim M,q}(A)(x, \xi)|d\xi dx|.$$

In particular, $\text{Res} A = 0$ if $A$ is trace class.

On occasion we shall need to involve the logarithm of a positive elliptic pseudodifferential operator. An appropriate such algebra is that of pseudodifferential operators with log-polyhomogeneous symbols (cf. [8]), which will be denoted here $\Psi^p_{\log}(M)$. The residue functional admits a canonical extension to $\Psi^p_{\log}(M)$, which for $A \in \Psi^p_{\log}(M)$ with symbol $\sigma(A)(x, \xi) \sim \sum_{j=0}^\infty \sum_{k=0}^q \sigma_{p-j,k}(A)(x, \xi) \log^k |\xi|$ is given by the local formula [8 Cor. 4.8],

$$\text{Res}_q A = \frac{(q+1)!}{(2\pi)^{\dim M}} \int_{S^*M} \sigma_{-\dim M,q}(A)(x, \xi)|d\xi dx|.$$

If $\phi = \sum_i f_0^i \otimes f_1^i \otimes \ldots \otimes f_k^i \in C^k(M)$, we define the multilinear form $\text{Tr}_\phi \in CC^k(\Psi(M))$ by

$$\text{Tr}_\phi(A_0, A_1, \ldots, A_k) = \sum_i \text{Tr} (A_0 f_0^i A_1 f_1^i \ldots A_k f_k^i).$$
The assignment $\phi \mapsto \hat{\text{Tr}}\phi$ fails to be a chain map, since $\hat{\text{Tr}}$ is not a trace. Instead, its Hochschild boundary is given by a local formula (see e.g. (11)):

\begin{equation}
(9) \quad b \hat{\text{Tr}}(A, B) \equiv \hat{\text{Tr}}[A, B] = \text{Res}(A[\log R, B]).
\end{equation}

**Lemma 2.1.** Let $\phi = \sum_i f_i^0 \otimes f_i^1 \otimes \ldots \otimes f_i^k \in C^k_{\wedge, 0}(M)$ and let $A_0, \ldots, A_k \in \Psi(M)$. Then

\begin{align}
(10) \quad \sum \hat{\text{Tr}}(A_0 f_i^0 A_1 f_i^1 \ldots A_k f_i^k) &= \sum \hat{\text{Tr}}(f_i^1 A_0 f_i^0 A_1 f_i^1 \ldots A_k);
\end{align}

\begin{align}
(11) \quad \sum \hat{\text{Tr}}(A_0 f_i^0 A_1 f_i^1 \ldots A_k f_i^k) &= \sum \hat{\text{Tr}}(f_i^1 A_1 f_i^1 \ldots A_k f_i^k A_0).
\end{align}

**Proof.** By equation (9), one has

\begin{equation}
\sum \hat{\text{Tr}}[A_0 f_i^0 A_1 f_i^1 \ldots A_k, f_i^k] = \text{Res} \sum_i A_0 f_i^0 A_1 f_i^1 \ldots A_k[\log R, f_i^k].
\end{equation}

In the right hand side of (11) the Res functional is applied to a classical pseudodifferential operator which is a difference of two operators in $\Psi_{\log}^1(M)$, namely

\begin{equation}
\sum_i A_0 f_i^0 A_1 f_i^1 \ldots A_k (\log R) f_i^k - \sum_i f_i^1 A_0 f_i^0 A_1 f_i^1 \ldots A_k \log R.
\end{equation}

Since $\phi$ is locally zero, its jet at the iterated diagonal $\Delta_{k+1} M$ is identically zero. The symbol multiplication formula then shows that complete symbol of each of the above two operators vanishes, and so the identity (11) follows from (9). The identity (11) is checked in a similar fashion. \(\Box\)

As a consequence of this lemma, we can now reproduce the coboundary equations (3) but only for the restriction to locally zero cochains:

\begin{equation}
(12) \quad b \hat{\text{Tr}} \phi = \hat{\text{Tr}} b \phi, \quad B \hat{\text{Tr}} \phi = 0, \quad \forall \phi \in C^*_{\wedge, 0}(M);
\end{equation}

the first follows from (11), and the second from (10) combined with the total antisymmetry of $\phi$.

To promote the above assignment to the full complex we define the map

\begin{equation}
(13) \quad \chi \phi = \hat{\text{Tr}} b \phi -(b + B) \hat{\text{Tr}} \phi, \quad \phi \in C^*_\wedge(M).
\end{equation}

**Proposition 2.2.** The map $C^*_\wedge(M) \ni \phi \mapsto \chi \phi \in CC^*_\wedge(\Psi(M))$ induces a morphism of complexes with a shift of degree $\chi : C^*_\wedge(M) \to CC^*_{\wedge+1}(S(M))$. 
Proof. Let $\phi \in C^q(M)$. By (12), $\chi_\phi = 0$ if $\phi$ is locally zero, ensuring that the map $\phi \mapsto \chi_\phi$ descends to the quotient complex $C^\bullet_\Lambda(M)$. Lemma 2.1 also implies that $\chi_\phi(A_0, A_1, \ldots, A_{q+1}) = 0$ whenever one of the operators $A_i$’s is trace class, and so $\chi_\phi \in CC^{q+1}(S(M))$. Finally, the map $\chi : C^\bullet_\Lambda(M) \to CC^{\bullet+1}(S(M))$ is a morphism of complexes since

$$(b + B)\chi_\phi = (b + B)\hat{\text{Tr}}_{\delta \phi} = -\chi_\delta \phi.$$ 

We shall mostly work with a variant of this construction, in which the full pseudodifferential extension (1) is replaced by

(14) $0 \to \mathcal{J}^0(M) \to \Psi^0(M) \xrightarrow{\sigma} S^0(M) \to 0$;

here $\mathcal{J}^0(M) = \mathcal{L}_1 \cap \Psi^0(M)$ is the ideal of trace class operators in $\Psi^0(M)$, and $S^0(M)$ is the corresponding quotient algebra of symbols. The same reasoning as above yields the chain map

(15) $\chi : C^\bullet_\Lambda(M) \to CC^{\bullet+1}(S^0(M))$, $\chi_\phi = \hat{\text{Tr}}_{\delta \phi} - (b + B)\hat{\text{Tr}}_{\phi}$, $\phi \in C^\bullet_\Lambda(M)$.

The relevance of the map $\chi$ for index theory becomes readily apparent when applied to the 0-dimensional cocycle $\phi \equiv 1$. Indeed, if $D \in M_\infty(\Psi^0(M))$ is elliptic, $u = \sigma(D)$ and $Q$ is a parametrix for $D$ with $\sigma(Q) = u^{-1}$, then

(16) $\chi_1(u^{-1}, u) = -b\hat{\text{Tr}}(Q, D) = \hat{\text{Tr}}(DQ - QD) = \text{Tr} \left( (\text{Id} - QD) - (\text{Id} - DQ) \right) = \text{ind} D$.

It will be shown below that the pairing determined by the chain map $\chi$ captures not just the Fredholm index but also its higher analogues, detecting the rational $K$-homology class of $D$.

2.2. Relative Chern character for quotient algebras. To prepare the ground for the discussion of higher analytic indices, we recall the definition of the relative Chern character in cyclic homology.

We begin by recalling some standard notation for the cyclic homology (normalized) bicomplex of a unital algebra $\mathcal{A}$. Namely, $C_q(\mathcal{A}) := \mathcal{A} \otimes (\mathcal{A}/\mathbb{C})^\otimes q$, and the boundary operators $b : C_{q+1}(\mathcal{A}) \to C_q(\mathcal{A})$ and
$B: C_{q-1}(\mathcal{A}) \to C_q(\mathcal{A})$ are defined by

$$b\varphi(a^0, a^1, \ldots, a^{q+1}) := \sum_{j=0}^{q} (-1)^j (a^0, \ldots, a^j a^{j+1}, \ldots, a^{q+1}) + (-1)^{q+1} (a^{q+1} a^0, a^1, \ldots, a^q),$$

$$B\varphi(a_0, \ldots, a_{q-1}) := \sum_{j=0}^{n-1} (-1)^{(n-1)j} (1, a_j, \ldots, a_{q-1}, a_0, \ldots, a_{j-1}).$$

The complex \( (CC_q(\mathcal{A}) = \bigoplus_{0 \leq k \leq q} C_k(\mathcal{A}), b + B) \) yields the cyclic homology groups \( HC_q(\mathcal{A}) \), while \( (CC_{ev|odd}(\mathcal{A}) = \prod_{2k|2k+1} C_k(\mathcal{A}), b + B) \) yields the periodic cyclic homology groups \( HC_{ev|odd}(\mathcal{A}) \).

The datum for relative cyclic homology consists of a unital algebra \( \mathcal{A} \) together with a two-sided ideal \( \mathfrak{J} \). With \( CC_\bullet(\mathcal{A}) \) and \( CC_\bullet(\mathcal{A}/\mathfrak{J}) \) denoting the respective cyclic homology mixed complexes, and \( CC_\bullet(\mathcal{A}, \mathfrak{J}) \) standing for the kernel complex, one has the exact sequence of complexes

$$0 \to CC_\bullet(\mathcal{A}, \mathfrak{J}) \xrightarrow{\iota_*} CC_\bullet(\mathcal{A}) \xrightarrow{\sigma_*} CC_\bullet(\mathcal{A}/\mathfrak{J}) \to 0,$$

where \( \iota: \mathfrak{J} \to \mathcal{A} \) is the inclusion map and \( \sigma: \mathcal{A} \to \mathcal{A}/\mathfrak{J} \) is the projection. The relative cyclic homology is the homology \( HC_\bullet(\mathcal{A}, \mathfrak{J}) \) of the kernel complex. If \( \mathfrak{J} \) is excisive (e.g. H-unital) these groups are are naturally isomorphic to the groups \( HC_\bullet(\mathfrak{J}) \), and in any case \( HC_{ev|odd}(\mathcal{A}, \mathfrak{J}) \) are naturally isomorphic to the groups \( HC_{ev|odd}(\mathfrak{J}) \). The standard homological cone construction (cf. e.g. [9, §1.1]) applied to the projection \( \sigma_*: CC_\bullet(\mathcal{A}) \to CC_\bullet(\mathcal{A}/\mathfrak{J}) \) gives an alternative description to the relative cyclic homology, and therefore to the cyclic homology of an excisive two-sided ideal \( \mathfrak{J} \).

For the purposes of this paper we shall need an alternative description of the cyclic homology of the quotient algebra \( \mathcal{A}/\mathfrak{J} \), which is obtained by forming the cone complex of the inclusion map \( i_*: CC_\bullet(\mathcal{A}, \mathfrak{J}) \to CC_\bullet(\mathcal{A}) \). The resulting homology groups will be denoted \( HC_\bullet(\mathcal{A}: \mathfrak{J}) \), resp. \( HC_{ev|odd}(\mathcal{A}: \mathfrak{J}) \). Explicitly,

$$\text{Cone}_k [CC_\bullet(\mathcal{A}, \mathfrak{J}) \to CC_\bullet(\mathcal{A})] := CC_{k-1}(\mathcal{A}, \mathfrak{J}) \oplus CC_k(\mathcal{A})$$

with the differential given by

$$\alpha, \beta \mapsto -(b + B)\alpha, i_* (\alpha) + (b + B)\beta$$
for \((\alpha, \beta) \in CC_{k+1}(A, J) \oplus CC_k(A)\). There is a natural quasi-isomorphism 
\(\text{Cone}[CC_\bullet(A, J) \to CC_\bullet(A)] \to CC_\bullet(A/J)\), given by the assignment 
\((\alpha, \beta) \mapsto \sigma_*(\beta)\),
which induces a canonical isomorphism \(\kappa_\bullet: HC_\bullet(A; J) \to HC_\bullet(A/J)\) and its periodic version \(\kappa_\bullet: HC^\text{per}_{ev|odd}(A; J) \to HC^\text{per}_{ev|odd}(A/J)\).

In the remainder of this subsection \(A\) is assumed to be a unital Fréchet algebra with the group of invertibles open and continuous inversion; \(J\) is a closed two-sided ideal. The topological \(K\)-theory of \(A\) is related to the periodic cyclic cohomology of \(A\) via the Chern character 
\(\text{ch}: K_\bullet(A) \to HC^\text{per}_\bullet(A)\), whose definition we proceed to recall.

If \(E \in M_\infty(A)\) is an idempotent representing a class in \(K_0(A)\), the corresponding Chern character is given by the \((B,b)\)-cocycle in \(CC^\text{per}_{ev}(A)\)
\[(17) \quad \text{ch}(E) = \text{tr}(E) + \sum_{q \geq 1} (-1)^q (\frac{2q!}{q!}) \text{tr}_\otimes (E - \frac{1}{2}) \otimes E^\otimes 2q),\]
and if \(U \in GL_\infty(A)\) is an invertible then the representing a class in \(K_1(A)\), then Chern character of its class is represented by the \((B,b)\)-cocycle in \(CC^\text{per}_{odd}(A)\)
\[(18) \quad \text{ch}(U) = \sum_{q \geq 0} (-1)^q q! \text{tr}_\otimes ((U^{-1} \otimes U)^\otimes q+1)).\]

The same formulas apply, of course, to the quotient algebra, defining \(\text{ch}: K_\bullet(A/J) \to HC^\text{per}_\bullet(A/J)\). In order to define the relative version of the latter though we shall also need the transgressed cochains that are canonically associated to \(C^1\)-paths of idempotents \(E(t)\), resp. invertibles \(U(t)\), \(t \in [t_0, t_1]\). They are defined as follows. In the case of a path of idempotents,
\[(19) \quad \text{Tch}(E, \dot{E}) = \int_{t_0}^{t_1} \varphi_0(E(t), \dot{E}(t)) \, dt, \quad \text{where} \quad \varphi_0(E, \dot{E}) = \iota((2E - 1)\dot{E}) (\text{ch}(E)) \quad \text{with} \quad \iota(B)(A_0 \otimes \ldots \otimes A_q) = \sum_{j=0}^{q} (-1)^j A_0 \otimes \ldots \otimes A_j \otimes B \otimes \ldots \otimes A_q;\]
one has 
\[\frac{d}{dt} \text{ch} E(t) = (b + B) \varphi_0(E, \dot{E}),\]
whence the transgression identity
\[(20) \quad \text{ch} E(t_1) - \text{ch} E(t_0) = (b + B) \text{Tch}(E, \dot{E}).\]
For a path of invertibles

(21)

\[ T\text{ch}(U, \dot{U}) := \int_{t_0}^{t_1} \varphi h(U(t), \dot{U}(t)) \, dt, \]

where

\[ \varphi h(U, \dot{U}) = \text{tr}(U^{-1}\dot{U}) + \sum_{q=0}^{\infty} (-1)^{q+1} q! \sum_{j=0}^{q} \text{tr} \otimes ((U^{-1} \otimes U)^{\otimes j+1} \otimes U^{-1} \dot{U} \otimes (U^{-1} \otimes U)^{\otimes q-j}), \]

satisfying

\[ \frac{d}{dt} \text{ch}(U) = (b + B) \varphi h(U, \dot{U}), \]

whence the transgression identity

(22)

\[ \text{ch}(U(t_1)) - \text{ch}(U(t_0)) = (b + B) T\text{ch}(U, \dot{U}). \]

With this notation established, we now proceed to define the relative realization of the Chern character, to be denoted \( \text{chr} : K_\bullet(A/\mathcal{J}) \to HC_{\text{per}}^\bullet(A; \mathcal{J}) \). The way in which the relative Chern character will be defined is patterned on the standard construction of the connecting map in the \( K \)-theory long exact sequence.

Starting with the odd case, let \( u \in GL_N(A/\mathcal{J}) \) be representing a class in \( K_1(A/\mathcal{J}) \). Then \( v = \begin{bmatrix} 0 & -u^{-1} \\ u & 0 \end{bmatrix} \) is in the connected component of the identity of \( GL_{2N}(A/\mathcal{J}) \), hence can be lifted to an invertible \( V \in GL_{2N}(A) \). Indeed, choosing lifts \( D \) and \( Q \) in \( M_N(A) \) such that \( \sigma(D) = u \) and \( \sigma(Q) = u^{-1} \), let \( V = \begin{bmatrix} S_0 & -(1 + S_0)Q \\ D & S_1 \end{bmatrix} \), where \( S_0 = I - QD \) and \( S_1 = I - DQ \). For the record, its inverse is \( V^{-1} = \begin{bmatrix} S_0 & (1 + S_0)Q \\ -D & S_1 \end{bmatrix} \).

Let \( e := \begin{bmatrix} I_N & 0 \\ 0 & 0 \end{bmatrix} \) and

(23)

\[ E(V) = VeV^{-1} = \begin{bmatrix} S_0 & S_0(1 + S_0)Q \\ S_1D & 1 - S_1^2 \end{bmatrix}. \]

With \( J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \), form the path of idempotents

(24)

\[ E(t) := \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} e^{-tJ} \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} e^{tJ} \begin{bmatrix} V^{-1} & 0 \\ 0 & I \end{bmatrix}, \quad t \in [0, \frac{\pi}{2}], \]

which begins at \( \tilde{E}(V) = \begin{bmatrix} VeV^{-1} & 0 \\ 0 & 0 \end{bmatrix} \) and ends at \( \tilde{e} = \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix} \).
We define
\[
\text{chr}(u) := (\text{ch} E(V) - \text{ch} e, \text{Tch}(E, \dot{E})) ,
\]
and note that by equation (20) chr(u) is a cocycle in the cone complex
\[
\text{Cone} \left[ CC^\bullet_{\text{per}}(A, J) \rightarrow CC^\bullet_{\text{per}}(A) \right].
\]

By means of secondary transgression formulas (see e.g. [9, (1.42)])
one can prove that the homology class chr[u] := [chr(u)] ∈ HC^per_{\text{odd}}(A, J)
is well-defined, i.e. only depends on the class [u] ∈ K_1(A/J).

**Theorem 2.3.** The diagram
\[
\begin{array}{ccc}
K_1(A/J) & \xrightarrow{\text{chr}} & HC^\text{per}_{\text{odd}}(A/J) \\
\downarrow \text{ch} & & \downarrow \cong \kappa^\bullet \\
HC^\text{per}_{\text{odd}}(A; J) & \xrightarrow{\text{Tch}(e, \dot{e})} & HC^\text{per}_{\text{odd}}(A/J)
\end{array}
\]
is commutative.

**Lemma 2.4.** Let \( B \) be a unital algebra and \( u \in B \) an invertible element.
Set \( v = \begin{bmatrix} 0 & -u^{-1} \\ u & 0 \end{bmatrix} \)
and with \( I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \).

\[
e(t) := \begin{bmatrix} v & 0 \\ 0 & I \end{bmatrix} \exp(-tJ) \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \exp(tJ) \begin{bmatrix} v^{-1} & 0 \\ 0 & I \end{bmatrix}, \quad t \in [0, \frac{\pi}{2}],
\]

Then \( \text{Tch}(e, \dot{e}) \in CC^\bullet_{\text{per}}(B) \) is a cycle and
\[
[\text{Tch}(e, \dot{e})] = [\text{chr}(u)] \in HC^\text{per}_{\text{odd}}(B).
\]

**Proof.** Since \( \text{ch} e(0) = \text{ch} e(\pi/2), \)
\( (b + B) \text{Tch}(e, \dot{e}) = \text{ch} e(\pi/2) - \text{ch} e(0) = 0. \)
so \( \text{Tch}(e, \dot{e}) \) is a cycle. Now, a straightforward calculation using equations
\[
e(t) = \begin{bmatrix} vev^{-1} \cos^2 t & ve \sin t \cos t \\ ev^{-1} \sin t \cos t & e \sin^2 t \end{bmatrix}, \quad t \in [0, \frac{\pi}{2}],
\]
and
\[
(2e(t) - 1)\dot{e}(t) = \begin{bmatrix} 0 & ve \\ -ev^{-1} & 0 \end{bmatrix}.
\]
shows that \( \text{Tch}(e, \dot{e}) = \frac{1}{2}(\text{ch} u - \text{ch} u^{-1}). \) The overall factor \( \frac{1}{2} \) appears
here because
\[
\int_0^{\frac{\pi}{2}} (\sin t)^\alpha (\cos t)^\beta dt = \frac{1}{2} B(\alpha + 1, \beta + 1) = \frac{(q!)^2}{2(2q + 1)!}.
\]

Since \( [\text{ch} u^{-1}] = -[\text{ch} u] \in HC^\text{per}_{\text{odd}}(B), \) the statement follows.
One can also avoid most of the computations by using the following observation. First of all, by functoriality it is sufficient to prove the statement for the algebra $B = \mathbb{C}[u, u^{-1}]$. In this case $HC_{\text{odd}}(B)$ is one dimensional and is generated by $[\text{ch} \, u]$. Hence

$$[\text{Tch}(e, \dot{e})] = C[\text{ch} \, u]$$

for some constant $C$. To determine the constant it suffices to compute the degree 1 component of $\text{Tch}(e(t), e'(t))$:

$$\int_0^{\pi/2} \text{tr}_\otimes e(t) \otimes (2e(t) - 1)\dot{e}(t) \, dt =$$

$$\int_0^{\pi/2} \text{tr}_\otimes \begin{bmatrix} vev^{-1} \cos^2 t & ve \sin t \cos t \\ ev^{-1} \sin t \cos t & ve^2 t \end{bmatrix} \otimes \begin{bmatrix} 0 & ve \\ -ve^{-1} & 0 \end{bmatrix} \, dt =$$

$$\int_0^{\pi/2} \text{tr}_\otimes (-ve \otimes ev^{-1} + ev^{-1} \otimes ve) \sin t \cos t \, dt = \frac{1}{2}(u^{-1} \otimes u - u \otimes u^{-1}) .$$

Let $\Phi \in CC^1(B)$ be the cyclic cocycle on $B$ given by

$$\Phi(u^k, u^l) = \begin{cases} l, & \text{if } k + l = 0 \\ 0, & \text{otherwise}. \end{cases}$$

Then $\Phi(\text{Tch}(e, \dot{e})) = \Phi(\text{ch} \, u) = 1$, hence $C = 1$. $

\textbf{Proof of Theorem 2.3.}$ Let $u$ be an invertible element of $A/\mathfrak{J}$. In order to prove that

$$\kappa_\ast \circ \text{chr}[u] = \text{ch}[u]$$

we will show that the path $E(t)$ constructed above has the property that

$$\sigma_\ast \left(\text{Tch}(E, \dot{E})\right) = \text{ch} \, u .$$

But $\sigma(E(t)) = e(t)$, where $e(t)$ is a path of idempotents in $A/\mathfrak{J}$ given by (26). Therefore $\sigma_\ast \left(\text{Tch}(E, \dot{E})\right) = \text{Tch}(e(t), e'(t))$, and the statement follows from (27) in Lemma 2.4. $

\textbf{Remark 2.5.}$ Note that the paths $E(t)$, $e(t)$ are polynomial in $\sin t$, $\cos t$. Therefore all the transgression formulas hold in the algebraic
cyclic complex, without any completions of tensor products. As a consequence, the same proof shows that the following diagram commutes:

\[ \begin{array}{ccc}
K^\text{alg}_1(A/J) & \xrightarrow{\text{chr}} & HC^{-1}(A/J) \\
\text{chr} & & \cong \kappa \cdot \text{ch} & \text{ch} \\
HC^{-1}(A:J) & \xrightarrow{\approx} & HC^{-1}(A/J)
\end{array} \]

Here \( K^\text{alg}_1(A/J) \) denotes the algebraic \( K \)-theory, \( HC^{-1}(A/J) \) is the negative cyclic homology and \( HC^{-1}(A:J) \) is the homology of the mapping cone \( \text{Cone}[CC^\bullet_-(A,J) \to CC^\bullet_-(A)] \).

Passing now to the even case, let \( p = p^2 \in M_\infty((A/J)^+) \). Choose a lift \( P \in M_\infty(A) \) of \( p \). Then \( U(t) = e^{2\pi it} t, \ t \in [0,1] \) lifts the loop \( e^{2\pi it}p = 1 - p + e^{2\pi it}p, \ t \in [0,1] \); in particular \( U(1) - U(0) \in M_\infty(J) \). Furthermore, one has

\[
(29) \quad \text{ch} U(1) = \text{ch} U(1) - \text{ch} U(0) = (b + B) Tch(U, \dot{U}) \quad \text{where}
\]

\[
(30) \quad Tch(U, \dot{U}) = \int_0^1 \phi(U, \dot{U})(t) dt.
\]

By definition,

\[
(31) \quad \text{chr} p := \frac{1}{2\pi i} \left( - \text{ch} U(1), \ Tch(U, \dot{U}) \right),
\]

which due to (29) is a cocycle in \( \text{Cone}[CC^{\text{per}}_-(A,J) \to CC^{\text{per}}_-(A)] \).

Again by means of secondary transgression formulas (see [9, (1.15)]), one proves that the homology class \( \text{chr}[p] := [\text{chr} p] \in HC^{\text{per}}_{\text{odd}}(A,J) \) is well-defined.

**Theorem 2.6.** The diagram

\[ \begin{array}{ccc}
K_0(A/J) & \xrightarrow{\text{chr}} & HC^{\text{per}}_{\text{ev}}(A:J) \\
\text{chr} & & \cong \kappa \cdot \text{ch} & \text{ch} \\
HC^{\text{per}}_{\text{ev}}(A/J) & \xrightarrow{\approx} & HC^{\text{per}}_{\text{ev}}(A/J)
\end{array} \]

is commutative.

**Proof.** The proof is similar to that of Theorem 2.3. We only need to verify the identity

\[
(32) \quad \sigma_* \left( Tch(U, \dot{U}) \right) = 2\pi i \text{ch} p,
\]

which in turn follows immediately from the next lemma.
Lemma 2.7. Let $B$ be an algebra and $p \in B$ an idempotent. Form a loop of invertibles $u(t) := 1 - p + e^{2\pi it}p$, $0 \leq t \leq 1$. Then

$$Tch(u, \dot{u}) = 2\pi i\text{ch } p $$

It is not difficult to verify (33) by direct computation. More elegantly, one can reduce to the case when $B$ is a unital algebra generated by an idempotent $p$, and observe that the chain $Tch(u, \dot{u})$ satisfies the cycle identity

$$(b + B)\sigma_*(Tch(u, \dot{u})) = \text{ch}(\sigma_*(u(1) - u(0))) = 0.$$ 

Since its degree 0 component is $2\pi ip$ by [6, Proposition 1.1] it must coincide with $2\pi i\text{ch } p$. □

2.3. Higher indices of elliptic operators – the even case. The higher analytic indices of a $\mathbb{Z}_2$-graded elliptic operator $D$ have been introduced in [4, §2]. For any $\phi \in C^{2q}(M)$ the corresponding index $\text{Ind}_\phi(D)$ was defined as follows:

$$\text{Ind}_\phi(D) = \text{Tr}_\phi (E(V), \ldots, E(V)),$$

where the idempotent $E(V)$ is constructed as in (33). The right hand side makes sense whenever the support of $E(V) \otimes (2q + 1)$ is contained in the support of the locally zero boundary $\delta \phi$, which can always be arranged by choosing a parametrix $Q$ of $D$ with support sufficiently close to the diagonal.

We shall give below an enhanced version of the definition, which in particular removes the support restrictions. It relies on the relative Chern character introduced in §2.2 applied to the following two cases:

(I) $A = \Psi(M)$ is the algebra of classical pseudodifferential operators, $J = \Psi^{-\infty}(M)$ is the ideal of smoothing operators, and the quotient algebra $S(M)$ is the algebra of complete symbols;

(II) $A = \Psi^0(M)$ is the algebra of bounded pseudodifferential operators, $J = J^0(M)$ is the ideal of trace class pseudodifferential operators, and $S^0(M)$ is the corresponding algebra of symbols.

A symbol $u \in M_N(S(M))$ is called elliptic if it is a complete symbol of an elliptic pseudodifferential operator, i.e. it has an inverse $u^{-1} \in M_N(S(M))$ and $\text{ord } u + \text{ord } u^{-1} = 0$.

Recall that constructions of the Section 2.2 allow to associate with an elliptic symbol $u$ the following pseudodifferential idempotents.

First choose $D$ and $Q$ in $M_N(\Psi(M))$ such that $\sigma(D) = u$ and $\sigma(Q) = u^{-1}$. Then set $V = \begin{bmatrix} S_0 & -(1 + S_0)Q \\ D & S_1 \end{bmatrix}$, where $S_0 = I - QD$ and $S_1 =$
\[ I - DQ. \] Let \( e := \begin{bmatrix} I_N & 0 \\ 0 & 0 \end{bmatrix} \) and

\[
E(V) = VeV^{-1} = \begin{bmatrix}
S_0^2 & S_0(1 + S_0)Q \\
S_1D & 1 - S_1^2
\end{bmatrix}.
\]

(35)

One also has the following path from \( \begin{bmatrix} E(V) & 0 \\ 0 & 0 \end{bmatrix} \) to \( \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix} \):

\[
E(t) = \begin{bmatrix}
VeV^{-1} \cos^2 t & Ve \sin t \cos t \\
Ve^{-1} \sin t \cos t & e \sin^2 t
\end{bmatrix}, \quad t \in [0, \frac{\pi}{2}].
\]

Definition 1. The higher analytic index of an elliptic symbol \( u \in GL_N(S(M)) \), associated to an Alexander–Spanier cocycle \( \phi \in C^{2q}_\Lambda (M) \), is defined by the formula

\[
\text{ind}_\phi(u) := \hat{\text{Tr}}_\phi \left( \text{ch} \ E(V) - \text{ch} \ e \right) + \hat{\text{Tr}}_{\delta \phi} \left( \text{Tch}(E, \dot{E}) \right)
\]

\[ \equiv \text{Tr}_\phi \left( \text{ch} \ E(V) - \text{ch} \ e \right) + \text{Tr}_{\delta \phi} \left( \text{Tch}(E, \dot{E}) \right). \]

(36)

The second identity, suggesting that both summands of the formula are independent of the regularization of the trace, requires an explanation. Firstly, since \( \delta \phi \) vanishes in a neighborhood of the iterated diagonal \( \Delta_{2q+1}M \), \( \hat{\text{Tr}} \) is applied to a trace class operator, and so the notation \( \text{Tr}_{\delta \phi} \) is justified. Secondly, we note that \( \sigma(E(V)) = \sigma(e) \). This implies that, in the second equality of the formula (36), \( \hat{\text{Tr}} \) is applied to an operator with vanishing symbol, hence to a trace class operator.

The higher index \( \text{ind}_\phi(u) \) depends only on the \( K \)-theory class \([u] \in K_1^{\text{alg}}(S(M))\) and on the cohomology class \([\phi] \in H^{2q}_\text{AS}(M)\). This can be gleaned directly from the very definition, but it also follows from the next statement which computes this index directly from the Chern character of the symbol, generalizing the equation (16).

Theorem 2.8. Let \( u \in GL_N(S(M)) \) be an elliptic symbol and let \( \phi \in C^{2q}_\Lambda (M) \) be an Alexander–Spanier cocycle. Then

\[
\text{ind}_\phi(u) = \chi_\phi(\text{ch} \ u).
\]

(37)

Proof. From the definition (13) of the map \( \chi_\phi \), the transgression identity (20) and equation (36) it follows that

\[
\text{ind}_\phi(u) = \chi_\phi \left( \sigma_*(\text{Tch}(E, \dot{E})) \right),
\]

(38)

so the proof is achieved by invoking the identity (28), see also Remark 2.5 \( \square \)
When computed on symbols of the form \( u = \sigma(D) \), where \( D \) is an elliptic pseudodifferential operator on \( M \), the higher index \( \text{ind}_\phi(u) \) coincides, up to a normalizing factor, with the original definition (34).

Indeed, if both \( D \) and its parametrix \( Q \) can be chosen to be supported sufficiently close to the the diagonal. In this case \( \text{Tch}(E, \dot{E}) \) is supported close to the diagonal as well and hence \( \text{Tr}_{\delta \phi} \left( \text{Tch}(E, \dot{E}) \right) = 0 \). Therefore (36) reduces to

\[
\text{ind}_\phi(\sigma(D)) = (-1)^q \frac{(2q)!}{q!} \text{Ind}_\phi(D).
\]

We now proceed to establish some properties of the higher index pairing which will allow to show the cases (I) and (II) mentioned above lead to the same outcome.

**Proposition 2.9.** Let \( u, v \) be two elliptic symbols and \( \phi \) an Alexander–Spanier cocycle. Then

\[
\text{ind}_\phi uv = \text{ind}_\phi u + \text{ind}_\phi v
\]

**Proof.** By Theorem 2.8

\[
\text{ind}_\phi(uv) = \chi_\phi(\text{ch}(uv)) = \chi_\phi(\text{ch} u + \text{ch} v) = \chi_\phi(\text{ch} u) + \chi_\phi(\text{ch} v) = \text{ind}_\phi(u) + \text{ind}_\phi(v).
\]

**Lemma 2.10.** Let \( \Delta \in \Psi(M) \) be a positive, invertible, elliptic operator, and let \( \psi \in C_{2q+1}^\Delta(M) \) such that \( \delta \psi = 0 \). Then

\[
\text{Tr}_\psi(\Delta^{-1}, \Delta, \ldots, \Delta^{-1}, \Delta) = 0.
\]

**Proof.** Using the identities of Lemma 2.1, extended to pseudodifferential operators in \( \Psi_{\log}(M) \), it is easily seen that for any locally zero cochain \( \psi \in C_{2q+1}^{\Delta,0}(M) \) one has

\[
\frac{d}{dz} \text{Tr}_\psi(\Delta^{-z}, \Delta^z, \ldots, \Delta^{-z}, \Delta^z) = - (q + 1) \text{Tr}_{\delta \psi}(\log \Delta, \Delta^{-z}, \Delta^z, \ldots, \Delta^{-z}, \Delta^z) = 0, \quad z \in \mathbb{C}.
\]

When \( \delta \psi = 0 \) it follows that \( \text{Tr}_\psi(\Delta^{-z}, \Delta^z, \ldots, \Delta^{-z}, \Delta^z) \) is independent of \( z \), and the proof is achieved by equating its values at \( z = 1 \) and \( z = 0 \).

**Proposition 2.11.** Let \( \Delta \in \Psi(M) \) be a positive, invertible, elliptic operator with symbol \( v = \sigma(\Delta) \), and let \( \phi \in C_{2q}^\Delta(M) \) be a cocycle. Then

\[
\text{ind}_\phi(v) = 0.
\]

**Proof.** Being invertible in \( \Psi(M) \), \( \Delta \) defines a class \( [\Delta] \in K_1^{adg}(\Psi(M)) \).

Its Chern character in negative cyclic cohomology \( \text{ch} \Delta \in CC^{-}_*(\Psi(M)) \)
(see [10] for the relevant definitions) is a lift of \( \text{ch} v \). Using Theorem 2.8 and Lemma 2.10, it follows that

\[
\text{ind}_\phi(u) = \chi_\phi(\text{ch} \Delta) = -\text{Tr}_{\delta \phi}(\text{ch} \Delta) = -(-1)^q q! \text{Tr}_{\delta \phi}(\Delta^{-1}, \Delta, \Delta^{-1}, \ldots, \Delta) = 0.
\]

\[\square\]

Combining the results of Propositions 2.9 and 2.11 we obtain the following corollary, which reduces the calculation of higher indices to the case of symbols of order 0.

**Corollary 2.12.** Let \( \Delta \) be the Laplace operator associated with a Riemannian metric on \( M \). For any elliptic symbol \( u \) of order \( m \), and any cocycle \( \phi \in C^2(M) \) one has

\[
\text{ind}_\phi(u) = \text{ind}_\phi(u \sigma(\Delta)^{-\frac{m}{2}}).
\]

We now turn our attention to the case (II), of operators of order 0 and the corresponding quotient algebra of symbols \( S^0(M) \). Any choice of a quantization map establishes a vector space isomorphism between \( S^0(M) \) and the direct sum of finitely many copies of \( C^\infty(S^*M) \), which can be used to endow \( S^0(M) \) with a structure of Fréchet space. While the isomorphism itself depends on the choice of quantization, the induced topology does not. With this topology the set of invertible elements in \( S^0(M) \) is open and the operation of inversion is continuous. The set of elliptic symbols (for a rank \( N \) trivial bundle) is all of \( GL_N(S^0(M)) \).

An elliptic symbol \( u \in GL_N(S^0(M)) \) defines an element of the topological \( K \)-group \( K_1(S^0(M)) \). It follows from Theorem 2.3 that the higher analytic index \( \text{ind}_\phi(u) \) depends only on the class \([u] \in K_1(S^0(M)) \) and the class \([\phi] \in H^*_\text{AS}(M)\).

Now the canonical projection \( \sigma_{pr}: S^0(M) \to C^\infty(S^*M) \) onto the principal part of the symbol induces an isomorphism in topological \( K \)-theory,

\[
K_1(S^0(M)) \cong K_1(C^\infty(S^*M)) = K^1(S^*M).
\]

Via this isomorphism, the higher index pairing turns into a pairing of \( K^1(S^*M) \) with \( H^*_\text{AS}(M) \).

Next, let \( \pi_*(K^1(M)) \) be the subgroup of \( K^1(S^*M) \) made of classes \([u] \) lifted from \( M \). Choosing the tautological lift \( V \) of \( v = \begin{bmatrix} 0 & -u^{-1} \\ u & 0 \end{bmatrix} \) to \( GL_2N(\Psi^0(M)) \), one has \( V e V^{-1} = e \), so \( \text{ch}(E(V)) - \text{ch} e = 0 \). On the other hand, since Schwartz kernel of components of \( \text{Tch}(E, \hat{E}) \) are distributions supported on the diagonal and \( \delta \phi \) is locally 0, \( \text{ind}_\phi(u) = 0 \).
We sum up the conclusion in the following remark.

**Remark 2.13.** The higher index map associated to any \( \phi \in H_{AS}^n(M, \mathbb{C}) \) descends to the quotient, \( \text{ind}_\phi: K^1(S^*M)/\pi^*(K^1(M)) \rightarrow \mathbb{C} \), and therefore is completely determined by its values on the principal symbols of elliptic operators.

To illustrate the computation of the higher index by means of non-localizable parametrices, we relate it to the heat operator. This, of course, applies to elliptic operators of order higher than 0.

As in [3, §2], given an elliptic operator \( D \in \Psi^1(M) \) one can form out of the heat operator the parametrix

\[
Q(D) = I - e^{-\frac{1}{2}D^*D}D^* \equiv \int_0^1 e^{-\frac{s}{2}D^*D}D^* ds.
\]

Then \( S_0(D) = e^{-\frac{1}{2}D^*D} \), \( S_1(D) = e^{-\frac{1}{2}DD^*} \), and so

\[
V(D) = \begin{bmatrix} D & e^{-\frac{1}{2}DD^*} \\ -e^{-\frac{1}{2}D^*D} & \frac{I - e^{-\frac{1}{2}D^*D}}{D^*D} \end{bmatrix}, \quad V(D)^{-1} = \begin{bmatrix} I - e^{-\frac{1}{2}D^*D}D^* & -e^{-\frac{1}{2}D^*D} \\ e^{-\frac{1}{2}DD^*} & D \end{bmatrix};
\]

the resulting idempotent \( E(D) = V(D)eV(D)^{-1} \) is

\[
E(D) = \begin{bmatrix} I - e^{-\frac{1}{2}DD^*} & -De^{-\frac{1}{2}D^*D} \\ -\frac{I - e^{-\frac{1}{2}D^*D}}{D^*D}e^{-\frac{1}{2}DD^*}D^* & e^{-\frac{1}{2}D^*D} \end{bmatrix}.
\]

**Remark 2.14.** Replacing \( D \) by \( tD \) one obtains the following extension of the McKean-Singer formula:

\[
\text{ind}_\phi(u) = \text{Tr}_\phi (\text{ch} E(V(tD)) - \text{ch} e) + \text{Tr}_{\delta \phi}(\text{Tch}(E(tD), W(tD))),
\]

for any \( t > 0 \).

In particular letting \( t \rightarrow \infty \) yields the equality

\[
(39) \quad \text{ind}_\phi(u) = \text{Tr}_\phi (\text{ch} E_{\infty} - \text{ch} e) + \text{Tr}_{\delta \phi}(\text{Tch}(E_{\infty}, W_{\infty})).
\]

The limit operators are obtained by replacing the heat kernel parametrix \( Q(D) \) with the partial inverse \( Q_\infty = D^{-1}(1 - H_1) = (D(1-H_0))^{-1} \); thus

\[
V_\infty = \begin{bmatrix} D & H_1 \\ -H_0 & Q_\infty \end{bmatrix}, \quad V_\infty^{-1} = \begin{bmatrix} Q_\infty & -H_0 \\ H_1 & D \end{bmatrix} \quad \text{and} \quad E_\infty = \begin{bmatrix} I - H_1 & 0 \\ 0 & H_0 \end{bmatrix},
\]

where \( H_0 \), resp. \( H_1 \) are the projections onto \( \text{Ker} D \), resp. \( \text{Ker} D^* \).
The first term of the index formula (39) is easy to compute. If 
\[ \phi = \sum_i f_i^0 \otimes f_i^1 \otimes \ldots \otimes f_i^{2q} \]
then
\[ \text{Tr}_\phi (\text{ch} E_\infty - \text{ch} e) = (-1)^q \frac{(2q)!}{q!} \sum_i \left( \text{Tr} \left( H_0 f_i^0 H_0 f_i^1 \ldots H_0 f_i^{2q} \right) \right. \]
\[ \left. - \text{Tr} \left( H_1 f_i^0 H_1 f_i^1 \ldots H_1 f_i^{2q} \right) \right). \]

An explicit expression for the second term is also computable but more cumbersome. However, if the operator \( D \) is invertible the index formula reduces to the second term, and has an attractive form.

**Proposition 2.15.** Let \( D \in M_N(\Psi(M)) \) be an invertible elliptic operator and let \( u = \sigma(D) \). Denoting \( S_f := D^{-1}[D, f] = f - D^{-1}fD \), one has
\[ \text{ind}_\phi(u) = (-1)^q q! \text{Tr} \left( \sum_i S_{f_i^0} S_{f_i^1} \ldots S_{f_i^{2q}} \right). \]

In terms of the phase operator \( F = D|D|^{-1} \), one has
\[ \text{ind}_\phi[u] = (-1)^q q! \text{Tr} \left( \sum_i [F, f_i^0] \ldots [F, f_i^{2q}] \right). \]

**Proof.** Since \( D \) is invertible and elliptic, its inverse \( D^{-1} \) is also pseudodifferential. The Chern character \( \text{ch} D \in CC^\bullet(\Psi^0(M)) \) is an obvious lift of \( \text{ch} u \). Using Proposition 2.8 it follows that
\[ \text{ind}_\phi(u) = \chi_\phi(\text{ch} u) = \text{Tr}_\delta(\text{ch} D) \]
\[ = (-1)^q q! \text{Tr}_\delta(D^{-1}, D, D^{-1}, \ldots, D). \]

Due to the antisymmetry of \( \phi \) and the specific form of the above expression, it is easily seen that only the extreme terms of \( \delta \phi \) contribute to the formula, and therefore
\[ \text{ind}_\phi(u) = (-1)^q q! \text{Tr} \sum_i \left( f_i^0 D^{-1} f_i^1 \ldots D f_i^{2q} - D^{-1} f_i^0 D f_i^1 \ldots f_i^{2q} D \right) \]
\[ = (-1)^q q! \text{Tr} \left( \sum_i S_{f_i^0} S_{f_i^1} \ldots S_{f_i^{2q}} \right). \]

\[ \square \]

2.4. Higher indices of elliptic operators – the odd case. Thanks to the relative Chern character mediation, the definition of the higher indices in the odd case is completely parallel to the one given in the even case.
Definition 2. Let \( p^2 = p \in M_N(S^0(M)) \), and let \( \phi \in C^\infty_\Lambda(M) \) be an Alexander–Spanier cocycle. Let \( U(t) \) be a lift of the loop \( u(t) = 1 - p + e^{2\pi it}p, t \in [0, 1] \), to a loop in \( M_N(\Psi^0(M)) \). The corresponding higher analytic index of the idempotent symbol is defined by the formula
\[
\text{ind}_\phi(p) = \frac{1}{2\pi i} \left( \text{Tr}_\phi \left( \text{ch} U(0) \right) - \text{Tr}_\phi \left( \text{ch} U(1) \right) + \text{Tr}_\delta \left( \text{Tch}(U, \dot{U}) \right) \right).
\]

As in the even case, the notation \( \text{Tr}_\phi \) and \( \text{Tr}_\delta \) makes sense because \( \sigma(U(1)) = \sigma(U(0)) = 0 \) on the one hand, and \( \delta \phi \) is locally zero on the other hand.

The odd analogue of Theorem 2.8 computes the higher index directly from the symbol.

Theorem 2.16. With the above notation one has
\[
\text{ind}_\phi(p) = \chi_\phi(\text{ch} p), \quad \forall [p] \in K_0(S^0(M)).
\]

Proof. The claimed identity is obtained by combining (32) with the transgression formula (29). □

Again, this implies that \( \text{ind}_\phi(p) \) only depends on the \( K \)-theory class \( [p] \in K_0(S^0(M)) \) and the cohomology class \( [\phi] \in H^\text{odd}_\Lambda(M) \), and via the canonical isomorphism \( K_0(S^0(M)) \cong K_0(C^\infty(S*M)) \) the higher indices are defined on \( K^0(S^*M) = K_0(C^\infty(S*M)) \). If \( p^2 = p \in M_N(C^\infty(M)) \), the loop \( e^{2\pi it}p \) admits a tautological lift and so \( \text{ch} U(1) = \text{ch}(\text{Id}) = 0 \). On the other hand \( \text{Tr}_\delta \left( \text{Tch}(U, \dot{U}) \right) = 0 \), because the components of \( \text{Tch}(U, \dot{U}) \) are all supported on the diagonal. So we arrive at the similar conclusion as in the even case.

Remark 2.17. The higher index map associated to any \([\phi] \in H^\text{odd}_\Lambda(M, \mathbb{C})\) descends to the quotient, \( \text{ind}_\phi : K^0(S^*M)/\pi^*(K^1(M)) \to \mathbb{C} \), and therefore is completely determined by its values on the principal symbols of elliptic operators.

In turn, those higher indices are explicitly expressed by the following “higher Toeplitz index” counterpart of the “higher analytic index” formula (40).

Proposition 2.18. Let \( D \) be a selfadjoint elliptic pseudodifferential operator, \( P \) the projection onto its positive spectrum, and let \( p = \sigma(P) \). If \( \phi = \sum_i f_{0} \otimes f_{1} \otimes \ldots \otimes f_{2q-1} \in C^\infty_\Lambda(M) \) is an Alexander–Spanier cocycle, then
\[
\text{ind}_\phi(p) = (-1)^q \frac{(2q)!}{q!} \text{Tr} \left( \sum_i T_{f_{0}}T_{f_{1}} \ldots T_{f_{2q-1}} \right).
\]
where $T_f := PfP$.

**Proof.** Since $P$ is a projection, $\text{ch} U(1) = \text{ch}(\text{Id}) = 0$, and so

$$2\pi i \cdot \text{ind}_\phi(p) = \text{Tr}_{\delta \phi} \left( \text{Tch}_{2q}(U, \dot{U}) \right).$$

For later use, let us record that

$$(b + B) \left( \text{Tch}(U, \dot{U}) \right) = 0,$$

which makes $\text{Tch}(U, \dot{U})$ a cocycle in $\text{CC}^{\text{per}}(\Psi^0(M))$.

Apart from the overall numerical factor, each component of the cocycle $\text{Tch}_{2q}(U, \dot{U})$ has all the factors equal to $\text{Id} - P$, except one which is $U - 1 \cdot \dot{U} = 2\pi i P$. After obvious cancelations due to antisymmetrization, a closer look reveals that the contribution to the pairing of the last $2q$ terms in the expression of $\delta \phi$ is nil, because two of the summands vanish and the rest of them successively cancel each other. The only contribution comes from the the first term of $\delta \phi$, i.e. from $\sum_i 1 \otimes f_0^i \otimes f_1^i \otimes \ldots \otimes f_{2q-1}^i$ and it yields the claimed formula. □

### 2.5. Suspended indices (the odd case)

In [12, §3] (cf. also [13, §4]) the pairing of self-adjoint elliptic operators with odd-dimensional Alexander–Spanier cohomology has been handled via suspension, by means of a suspended Chern character. We proceed to prove that the corresponding higher indices coincide with those we have defined above.

Let $D$ be a self-adjoint elliptic operator. By adding a scalar multiple of the identity, one can assume that $D$ is invertible. Let $F \in \Psi^0(M)$ be a self-adjoint operator such that $F - \frac{D}{\|D\|} \in \Psi^{-\infty}(M)$, and whose Schwarz kernel of is supported in a sufficiently small neighborhood of the diagonal, to be specified later. (For the construction of such operators $F$, see [13, Lemma 4.1]).

Consider then the loop of elliptic operators:

$$\Phi(\theta) = \begin{cases} 
\cos \theta \cdot I + i \sin \theta \cdot F, & 0 \leq \theta \leq \pi, \\
(\cos \theta + i \sin \theta) \cdot I, & \pi \leq \theta \leq 2\pi.
\end{cases}$$

Then

$$\Psi(\theta) = \Phi(\theta)^* = \begin{cases} 
\cos \theta \cdot I - i \sin \theta \cdot F, & 0 \leq \theta \leq \pi, \\
(\cos \theta - i \sin \theta) \cdot I, & \pi \leq \theta \leq 2\pi.
\end{cases}$$

is a loop of parametrices of $\Phi(\theta)$. Out of them one forms the loop of idempotents

$$E_\theta = V_\theta \begin{bmatrix} 1 & 0 \\
0 & 0
\end{bmatrix} V_\theta^{-1} = \begin{bmatrix} S(\theta)^2 & S(\theta)(1 + S(\theta))\Psi(\theta) \\
S(\theta)\Phi(\theta) & 1 - S(\theta)^2
\end{bmatrix}, \ 0 \leq \theta \leq 2\pi.$$
where \( S(\theta) = 1 - \Phi(\theta)\Psi(\theta) = 1 - \Psi(\theta)\Phi(\theta) \) and
\[
V_\theta = \begin{bmatrix} S(\theta) & (1 + S(\theta))\Psi(\theta) \\ \Phi(\theta) & S(\theta) \end{bmatrix}.
\]

The loop \( E = \{ E_\theta; 0 \leq \theta \leq 2\pi \} \) represents an element in \( K_0(S\Psi^0(M)) \), and its suspended Chern character, as defined in [12, §1.5], is given by the cycle
\[
(48) \quad \text{Sch}(E) := \text{Tch}(E, E') = \int_0^{2\pi} \varphi \left( E_\theta, \frac{dE}{d\theta} \right) d\theta;
\]
\((b + B) \text{Sch}(E) = 0\), since \( E_0 = E_{2\pi} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \), so \( \text{Sch}(E) \) is indeed a cycle in the cyclic homology bicomplex of \( \Psi^0(M) \).

If now \( \phi \in C^{2q+1}_\wedge(M) \) is an Alexander–Spanier cocycle, we define the corresponding suspended index by the formula
\[
(49) \quad \text{Sind}_\phi(D) := \frac{1}{2\pi i} \text{Tr}_\phi \left( \text{Sch}(E) \right);
\]
since \( S(\theta) \) are smoothing operators, the the right hand side is well-defined. For a fixed cocyle \( \phi \) the definition does not depend on the choice of \( F \) as above, as long as its support is sufficiently localized around the diagonal.

**Theorem 2.19.** Let \( D \) be a selfadjoint elliptic operator and let \( \phi \in C^{2q+1}_\wedge(M) \) be an Alexander–Spanier cocycle. Then
\[
\text{Sind}_\phi(D) = \text{ind}_\phi(p),
\]
where \( p \) is the symbol of the orthogonal projection onto the positive spectrum of \( D \).

**Proof.** The loop \( E \oplus 0 \) can be retracted to a constant loop via a family of loops \([0, 1] \ni t \mapsto \tilde{E}(t)\), with \( \tilde{E}(0) = E \) and \( \tilde{E}(1) = e \), where
\[
\tilde{E}_\theta(t) := \begin{bmatrix} V_\theta & 0 \\ 0 & 1 \end{bmatrix} \exp\left( \frac{\pi}{2} tJ \right) \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \exp\left( -\frac{\pi}{2} tJ \right) \begin{bmatrix} V_\theta & 0 \\ 0 & 1 \end{bmatrix}^{-1},
\]
and as before,
\[
e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]
In order to show that the suspended Chern character is well-defined on \( K \)-theory, Moscovici and Wu gave a secondary transgression formula between two homotopic loops. Specialized to our situation, [12]...
Proposition 1.15] provides an explicit chain \( \text{TSch}(\tilde{E}, \hat{E}) \) in the cyclic homology bicomplex of \( \Psi^0(M) \) satisfying

\[
(50) \quad \text{Sch}(E) \equiv \text{Sch}(E_\theta(1)) - \text{Sch}(E_\theta(0)) = (b + B) \left( \text{TSch}(\tilde{E}, \hat{E}) \right).
\]

Using the explicit expression of \( \text{TSch}(\tilde{E}, \hat{E}) \), one verifies by a straightforward calculation that

\[
\sigma \left( \text{TSch}(\tilde{E}, \hat{E}) \right) = 2\pi i \text{ch}(p),
\]

where \( p = (1 + \sigma(F))/2 \). From this and the transgression formula \([50]\), by the very definition \([13]\) of the map \( \chi_\phi \), we obtain

\[
2\pi i \chi_\phi(ch\ p) = \text{Tr}_{\delta\phi} \left( \text{TSch}(\tilde{E}, \hat{E}) \right) + \text{Tr}_\phi (\text{Sch}(E)).
\]

By choosing \( F \) with Schwartz kernel supported in a sufficiently small neighborhood of diagonal, depending on the vanishing locus of \( \delta\phi \), one can ensure that the first term of the above sum vanishes, and therefore

\[
\chi_\phi(ch\ p) = \frac{1}{2\pi i} \text{Tr}_\phi (\text{Sch}(E)) = \text{Sind}_\phi(D).
\]

In view of Theorem 2.16 this completes the proof. \( \square \)

3. Cohomological formulas for higher indices

As explained in the remarks [2.13 and 2.17], the index pairing between the \( K \)-theory groups of the symbol algebra \( S^0(M) \) and Alexander–Spanier cohomology of \( M \) is completely determined by the higher analytic indices of elliptic operators, which in turn depend only on their principal symbols. This allows to reduce the cohomological computation of the higher index pairing to finding explicit cohomological expressions for higher indices of Dirac-type operators. The latter have already been established, cf. [11, §3] for the even case, while in the odd case similar formulas can be easily derived for the suspended version of higher indices from the results in [12, §3]. In view of Theorem 2.19 it only remains to combine the formulas for higher analytic indices obtained in the previous section with the corresponding cohomological expressions.

In the even case the cohomological formula for the index of an elliptic operator \( D \) is

\[
\text{ind}_\phi(u) = \frac{(-1)^q}{(2\pi i)^q} \int_M \pi_+(\text{ch}(\sigma_{pr}(D))) \wedge \text{Td}(TM \otimes \mathbb{C}) \wedge \left( \sum_i f_0^i df_1^i \cdots df_{2q}^i \right);
\]
here \( u = \sigma(D), \sigma_{pr}(D) \in C^\infty(S^*M) \) is its principal symbol, \( \pi: S^*M \to M \) is the canonical projection, and \( \text{Td}(TM \otimes \mathbb{C}) \) is the Todd class of the complexified tangent bundle.

From [3, Theorem (3.9)] and Proposition 2.15 we obtain:

**Theorem 3.1.** Let \( D \in M_N(\Psi(M)) \) be an invertible elliptic operator. Set \( S_f := D^{-1}[D, f] = f - D^{-1}fD \). If \( \phi = \sum_i f_i^0 \otimes f_i^1 \otimes \ldots \otimes f_{2q} \in C^\wedge_{2q}(M) \) is an Alexander–Spanier cocycle, then

\[
\text{Tr} \left( \sum_i S_{f_i^0} S_{f_i^1} \ldots S_{f_{2q}} \right) = \\
\frac{1}{(2\pi i)^q q!} \int_M \pi_*(\text{ch}(\sigma_{pr}(D))) \wedge \text{Td}(TM \otimes \mathbb{C}) \wedge \left( \sum_i f_0^i df_1^i \ldots df_{2q}^i \right).
\]

In the odd case the cohomological formula for the index of a selfadjoint elliptic operator \( D \) is

\[
\text{ind}_\phi(p) = \frac{(-1)^q}{(2\pi i)^q} \int_M \pi_*(\text{ch}(\sigma_{pr}(p))) \wedge \text{Td}(TM \otimes \mathbb{C}) \wedge \left( \sum_i f_0^i df_1^i \ldots df_{2q-1}^i \right);
\]

here \( p = \sigma(P) \), where \( P \) is the orthogonal projection on the positive spectrum of \( D \).

For a Dirac-type operator \( D \) the cohomological expression of \( \text{Sind}_\phi(D) \) can be computed by means of the trace formula for the inverse Laplace transform [13, Proposition 3.6] and using Getzler’s asymptotic calculus, along the lines invoked in proving [13, Theorem 4.2]. The calculation per se mirrors the one performed in [12, §3], and yields the right hand side of the above formula. Combining this with Theorem 2.19 and Proposition 2.18 we obtain:

**Theorem 3.2.** Let \( D \) be a selfadjoint elliptic pseudodifferential operator, \( P \) the projection onto its positive spectrum. Set \( T_f := PfP \). If \( \phi = \sum_i f_i^0 \otimes f_i^1 \otimes \ldots \otimes f_{2q-1} \in C^\wedge_{2q-1}(M) \) is an Alexander–Spanier cocycle, then

\[
\text{Tr} \left( \sum_i T_{f_i^0} T_{f_i^1} \ldots T_{f_{2q-1}} \right) = \\
\frac{1}{(2\pi i)^q (2q)!} \int_M \pi_*(\text{ch}(\sigma_{pr}(P))) \wedge \text{Td}(TM \otimes \mathbb{C}) \wedge \left( \sum_i f_0^i df_1^i \ldots df_{2q-1}^i \right).
\]

The identities (51) and (52) generalize the Helton–Howe trace formula for the top multi-commutator of Toeplitz operators [7, Theorem 7.2]. The latter corresponds to the case when \( M \) is the cosphere
bundle $S^*N$ of another manifold $N$, and $D$ is the canonical Spin$^c$-
Dirac operator. However, since the Helton–Howe result applies to arbi-
trary cochains $\phi \in C^m(M)$, $m = \dim M$, we need to explain how to
modify the complex so that all the top-dimensional Alexander–Spani-
er cochains qualify as cocycles.

Recall that the Alexander–Spanier complex we used in this paper
was defined as the quotient complex $C^\bullet_{AS}(M) = \{C^\bullet_{\Lambda}(M)/C^\bullet_{\Lambda,0}(M), \delta\}$, where $C^\bullet_{\Lambda}(M)$ consists of smooth indecomposable totally antisymmetric
cochains, and and $C^q_{\Lambda,0}(M)$ consists of those $\phi \in C^\bullet_{\Lambda}(M)$ which vanish
in a neighborhood of the iterated diagonal $\Delta_{q+1}M$. We can modify this
definition by taking instead the quotient by the larger subcomplex
$$\bar{C}^q_{\Lambda,0}(M) := \{\phi \in C^q_{\Lambda}(M) \mid \text{the } m\text{-th jet of } \phi \text{ vanishes at } \Delta_{q+1}M\}$$
where, recall, $m = \dim M$. It is easy to see that $\bar{C}^q_{\Lambda,0}(M)$ is indeed a
subcomplex. Set
$$\bar{C}^\bullet_{AS}(M) := \{C^\bullet_{\Lambda}(M)/\bar{C}^\bullet_{\Lambda,0}(M), \delta\}$$
We note that, due to the total antisymmetry,
$$\bar{C}^q_{\Lambda,0}(M) = 0 \text{ for } q > m.$$ 
It is also easy to see that the canonical morphism to de Rham complex
$$\lambda: \bar{C}^\bullet_{AS}(M) \rightarrow \Omega^\bullet(M)$$
given by
$$\lambda(\sum_i f_0^i \otimes f_1^i \otimes \ldots \otimes f_q^i) := \sum_i f_0^i df_1^i \ldots df_q^i$$
is well defined and surjective.

Due to the total antisymmetry of a chain $\phi \in C^m_{\Lambda}(M)$, its $(m-1)$-jet
vanishes at $\Delta_{m+1}M$; it is easy to see then that
$$\lambda: \bar{C}^m_{AS}(M) \rightarrow \Omega^m(M)$$
is an isomorphism of vector spaces. The surjectivity of $\lambda$ implies that
the cohomology of the complex $\bar{C}^\bullet_{AS}(M)$ in degree $m$ coincides with
$H^m(M)$. Hence the canonical projection
$$C^\bullet_{AS}(M) \rightarrow \bar{C}^\bullet_{AS}(M)$$
duces an isomorphism in degree $m$ cohomology. Equivalently, every
cohomology class of $\bar{C}^\bullet_{AS}(M)$ of degree $m$ has a representative $\phi \in
C^m_{\Lambda}(M)$ such that $\delta \phi \in C^{m+1}_{\Lambda,0}(M)$.

The construction of the map $\chi$ extends without difficulty to the
new complex, provided that the target is the cyclic complex of 0-order
complete symbols, thus yielding the map of complexes
$$\chi: \bar{C}^\bullet_{AS}(M) \rightarrow CC^{\bullet+1}(S^0(M)).$$
Indeed, Lemma 2.1 remains true when $\phi \in \mathcal{C}^k_{\lambda,0}(M)$ and $A_0, \ldots, A_k \in \Psi^0(M)$, because the vanishing of the $m$-th jet of $\phi$ at $\Delta_{k+1}M$ guarantees that $\sum_i A_0f_0^i A_1f_1^i \ldots A_k[\log R, f_k^i] \in \Psi^{-m-1}(M)$ and therefore, by equation (6), $\text{Res} \sum_i A_0f_0^i A_1f_1^i \ldots A_k[\log R, f_k^i] = 0$. Thus, the proof of Proposition 2.2 remains unchanged, provided that $\chi_\phi$ is applied to 0-order pseudodifferential operators.

We finally note that the proofs of the identities (51) and (52) effectively involve only pseudodifferential operators of order 0. Since all the top degree cohomology classes in the new complex $\mathcal{C}^\bullet_{AS}(M)$ can be represented by cocycles in $\mathcal{C}^\bullet_{AS}(M)$, the identities (51) and (52) remain true for all the top degree cocycles (i.e. all degree $m$ cochains) in the new complex $\mathcal{C}^\bullet_{AS}(M)$. Recalling the notation

$$[A_1, A_2, \ldots, A_k] := \frac{1}{k!} \sum_{\tau \in S_k} \text{sgn} \tau A_{\tau(1)}A_{\tau(2)} \ldots A_{\tau(k)},$$

we can therefore conclude that the following extensions of the Helton–Howe formula hold true.

**Corollary 3.3.** Let $M$ be a connected, closed, smooth manifold of even dimension $m = 2q$. Let $D \in M_N(\Psi^0(M))$ be an invertible elliptic operator. Then, for any $f_0, f_1, \ldots, f_{2q} \in C^\infty(M)$

$$\text{Tr} [S_{f_0}, S_{f_1}, \ldots, S_{f_{2q}}] = \kappa \frac{(2q + 1)!}{(2\pi i)^q} \int_M f_0 df_1 \ldots df_{2q},$$

where $\kappa$ is the component of $\pi_* (\text{ch}(\sigma_{pr}(D)))$ in $H^0(M, \mathbb{Q})$, which is canonically identified with $\mathbb{Q}$.

Also, for any $f_0, f_1, \ldots, f_{2r} \in C^\infty(M)$ with $r > q$,

$$\text{Tr} [S_{f_0}, S_{f_1}, \ldots, S_{f_{2r}}] = 0.$$

**Corollary 3.4.** Let $M$ be a connected, closed, smooth manifold of odd dimension $m = 2q - 1$. Let $D$ be a selfadjoint elliptic pseudodifferential operator, and $P$ the projection onto its positive spectrum. Then, for any $f_0, f_1, \ldots, f_{2q-1} \in C^\infty(M)$, one has

$$\text{Tr} [T_{f_0}, T_{f_1}, \ldots, T_{f_{2q-1}}] = \kappa \frac{q!}{(2\pi i)^q} \int_M f_0 df_1 \ldots df_{2q-1},$$

where $\kappa$ is the component of $\pi_* (\text{ch}(\sigma_{pr}(P)))$ in $H^0(M, \mathbb{Q}) \cong \mathbb{Q}$.

Also, for any $f_0, f_1, \ldots, f_{2r-1} \in C^\infty(M)$ with $r > q$,

$$\text{Tr} [T_{f_0}, T_{f_1}, \ldots, T_{f_{2r-1}}] = 0.$$
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Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, USA
E-mail address: alexander.gorokhovsky@colorado.edu

Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA
E-mail address: moscovici.1@osu.edu