Rotation controlling of spiraling elliptic beams in inhomogeneous nonlocal media

Guo Liang and Qing Wang

1 School of Electrical & Electronic Engineering, Shangqiu Normal University, Shangqiu, 476000, People’s Republic of China
2 The Key Laboratory of Weak Light Nonlinear Photonics, Ministry of Education, Nankai University, Tianjin, 300457, People’s Republic of China
3 School of Science and Key Laboratory of Solid State Microstructure of Jiangxi Province, College of Science, Jiujiang University, Jiujiang, 332005, People’s Republic of China

* Author to whom any correspondence should be addressed.
E-mail: wangqingszu@sohu.com

Keywords: nonlinear optics, inhomogeneous nonlocal media, spiraling elliptic beams, rotation controlling

Abstract

The dynamics of spiraling elliptic beams in longitudinally-inhomogeneous nonlocal media with \( z \)-varied characteristic length (CL) were discussed. When the response CL of nonlocal nonlinear media gradually varies, the spiraling elliptic beams at the critical powers can evolve as rotating solitons with CL-dependent rotating velocity. This kind of dynamical evolution for spiraling elliptic beam is confirmed to exist in the actual nonlinear media, the nematic liquid crystals, when the CL varies with \( z \). We also investigate the gradient force exerted by an elliptic beam upon micro-particles from the application point of view in optical tweezers, and find that the gradient force also can be conveniently controlled by the varied CL.

1. Introduction

Currently, one of the major scientific endeavors is to modulate spatial beams exhibiting novel properties in various applications. Among the concerned optical patterns, optical beams carrying the orbital angular momentum (OAM) can exert forces and torques on the microparticles, which make them rotate [1]. The technologies associated with OAM, including spatial light modulators and hologram design, have wide applications in optical tweezers [2], optical trapping [3], imaging [4], and information processing [5, 6].

The optical patterns with the OAM are usually associated with optical vortices and related ring-shaped beams with the phase-singularity, including the Laguerre–Gauss beams [7] and the Bessel beams [8], as well as the other hollow beams [9]. However, there is another kind of OAM-carrying patterns, namely the astigmatic (elliptical Gaussian) beam [10], which exhibits the cross phase and is absolutely different from the corkscrew-like phase of the vortices. The OAM makes the elliptic beams rotate during propagations, therefore the astigmatic elliptic beams are also called the spiraling elliptic beams [11, 12]. The propagation of elliptical beam is more complicated than that of circular beams [13–15]. An elliptic beam will periodically oscillate during its propagation in isotropic media [16, 17]. In 2010, the OAM was proposed to stabilize the elliptical solitary wave [11] due to the OAM-induced anisotropic diffraction (AD), and makes the elliptical Gaussian mode exist in isotropic media [12, 18], where only the circular eigenmode is supposed to exist for a beam without OAM.

The spiraling elliptic beams exhibit novel properties during their linear and nonlinear propagations. In linear anisotropy media, the rotating velocity of spiraling elliptic beams can be readily controlled by both initial OAM and the anisotropy parameter of media [19]. When a Hermite–Gaussian beam is modulated with a cross-phase, the inter-conversion between the Hermite–Gaussian modes and the Laguerre–Gaussian modes can be achieved [20]. As for the nonlinear case, the spiraling elliptic solitons are theoretically predicted when the optical power and the initial OAM are both equal to their corresponding critical values in media with the saturable nonlinearity [11] and the nonlocal nonlinearity [12]. Recently, the spiraling elliptic soliton is observed for the first time in cylindrical lead glass [21].
In nonlocal nonlinear media, such as nematic liquid crystal [22] and lead glass [23], the evolution of spatial optical beam is governed by the nonlocal nonlinear Schrödinger equation (NNLSE) [24–26], in which the nonlinear term is expressed by a convolution between optical intensity and the response function. The characteristic length (CL) of the response function is in fact quite handy to be modulated. For the nematic liquid crystal, its CL depends closely on the bias voltage [27]. While, the CL of the lead glass is determined by sample sizes [23]. Therefore, it is possible to make the CL vary with z in the two actual nonlinear media [28]. In this paper, we will explore the dynamics of spiraling elliptic beams in such longitudinally-inhomogeneous nonlocal media.

2. Rotation of spiraling elliptic beams and transitions of soliton states

The propagation of spatial optical beam in nonlocal nonlinear media is modeled by the following NNLSE [24–26]

\[ i \frac{∂φ}{∂z} + \frac{1}{2} \left( \frac{∂²}{∂x²} + \frac{∂²}{∂y²} \right) φ + φ \int_{-∞}^{∞} \int_{-∞}^{∞} R(x-x', y-y') |φ(x', y', z)|² dx'dy' = 0, \]

where \( φ(x, y, z) \) is the complex amplitude, \( x \) and \( y \) are the transverse coordinates, \( z \) is the longitudinal coordinate, and \( R \) is the response function of the media. We take the Gaussian response for analytical discussions in this section

\[ R(x, y, z) = \frac{1}{πσ(z)^²} \exp \left[ -\frac{x² + y²}{σ(z)^²} \right], \]

where \( σ \) is the CL.

As mentioned in the introduction, it is possible to realize the \( z \)-dependent CL in actual nonlocal nonlinear media such as the nematic liquid crystal and the lead glass. In the following, we will study the evolution of a spiral elliptic beam in longitudinal nonhomogeneous nonlocal medium [12, 29]

\[ φ(x, y, 0) = \left( \frac{P₀}{πbc} \right)^{1/2} \exp \left( -\frac{x²}{2b²} - \frac{y²}{2c²} \right) \exp(iΘxy), \]

where \( P₀ = \int \int |Φ|^² dx dy \) is the optical power, \( b \) and \( c \) are the semi axes of the elliptic beam, and the last term \( \exp(iΘxy) \) is called the cross phase [12, 18–21], which contributes to the OAM [12]

\[ M = \text{Im} \int_{-∞}^{∞} \int_{-∞}^{∞} \phi^* \left( x \frac{∂φ}{∂y} - y \frac{∂φ}{∂x} \right) dx dy = \frac{P₀}{2} (b² - c²) Θ. \]

\( Θ \) is the cross phase coefficient, which can be adjusted by the angle between the major axis of the elliptically shaped beam and the cylindrical lens in experiments [11]. If the CL is a constant, the spiraling elliptic beams can evolve as solitons when the optical power and the initial OAM both equal to their critical values under the strong nonlocality [12, 28]

\[ P_c = \frac{π(b² + c²)²σ^4}{8b²c²}, \quad Θ_c = \frac{b² - c²}{2b²c²}, \quad τ_c = \frac{(b² - c²)²}{4b²c²}, \]

with \( τ_c = M_c/P_c \), where \( P_c \) and \( M_c \) are the critical power and OAM, respectively.

We have demonstrated that the ellipticity of the spiraling elliptic beam during propagations is only determined by the OAM, and is independent of the nonlinearity [21], which can be also confirmed by the equation (23) of reference [18]

\[ ρ ≡ \frac{b}{c} = \sqrt{τ_c} + \sqrt{τ_c + 1}. \]

In reference [12], it was discovered that the decaying rate of the OAM is extremely low for the spiraling elliptic solitons in nonlocal nonlinear media. Furthermore, the larger is the degree of nonlocality, the lower is the decay of the OAM. In this paper, the nonlocality is quite strong. Therefore, under the critical OAM the elliptic beam will keep its ellipticity \( ρ \) invariable during propagations. Combining equations (5) and (6), the beam width can be analytically obtained

\[ c = \sqrt{\frac{(ρ² + 1)²π}{8ρ²P_c σ}}, \quad b = \sqrt{\frac{(ρ² + 1)²π}{8P_c σ}}. \]
The critical angular velocity of soliton rotation is \( \omega_c = \frac{(b^2+c^2)}{2b^2c^2} \) [12, 28], which can be rewritten with the aid of equation (7)

\[
\omega_c = \sqrt{\frac{2P_c}{\pi \sigma^4}}.
\]

Then the total rotation angles of the spiraling elliptical soliton during propagations can be obtained

\[
\theta = \int_0^z \sqrt{\frac{2P_c}{\pi \sigma^4}} \, dz.
\]

Obviously, the beam widths, angular velocities and rotation angles all can be controlled by the CL.

Then we will simulate the evolutions of spiraling elliptic beam given by equation (3) in longitudinally-inhomogeneous nonlocal media by using split-step Fourier method. The gradually-decreasing-CL is shown in figure 1(a), and the evolution characteristics of the spiraling elliptic beams are summarized in figures 1(b) and (c). When \( P_0 = P_c \) and \( M_0 = M_c \), the semi axes \( b \) and \( c \) also decrease in a nearly linear manner like the CL (solid green lines in figure 1(b)). While, if \( P_0 \neq P_c \) the two semi axes will both oscillate around the linear lines (pink and blue lines in figure 1(b)). During propagations, the beam rotates, and the rotating velocity can smoothly speeds up when the CL decreases, which is shown in figure 1(c). The beam’s contraction-induced acceleration can be explained in the following. By an analogy between the rotating beam and a rigid body, we calculate the pattern’s moment of inertia \( I \) and the rotating velocity \( \omega \).
Figure 3. (a) Varying CL of different curves with respect to $z$, (b) the corresponding evolutions of semi axes, (c) rotating velocities, and rotating angles (d) for spiraling elliptic beams at $P_0 = P_c$ and $M_0 = M_c$. The solid and dashed lines corresponding to analytical solutions and numerical simulations, respectively.

Figure 4. (a) Varying CL of different curves with respect to $z$, (b)–(d) three-dimensional numerical evolutions of the spiraling elliptic beams in nonlocal nonlinear media with different gradual response length, the parameters are chosen as $b = 1.5$, $c = 1$, $P_0 = P_c$ and $M_0 = M_c$.

\[ J = \iint \left( x^2 + y^2 \right) |\phi|^2 \, dx \, dy, \quad \omega = M/J. \]  

(10)

The inertia of a rigid body here corresponds to the beam width. Therefore, when the optical beam contracts, the so-called moment of inertia $J$ decreases, then the rotation will become quicker.

Furthermore, we numerically found that the beam power keeps invariable, and the OAM are nearly constant during propagations when the CL decreases as shown in figures 2(a) and (d), respectively. The invariability of the ellipticity for the elliptic beam when $M_0 = M_c$ is also numerically confirmed as shown in figure 2(c). By combining equation (7) and $\delta = \sigma/b$, we can rewrite the degree of nonlocality as

\[ \delta = \left[ \frac{8P_c}{\pi (\rho^2 + 1)^2} \right]^{1/4}. \]  

(11)

Obviously, the degree of nonlocality also remains the same when the CL change with propagation distance $z$, which confirmed by the numerical solution as seen in figure 2(d).

The results above provide a way to control the rotation of an optical beam by changing the external conditions. A key point should be pointed out that the spiraling elliptic beam still can keep soliton states even when its rotating velocities change, which is shown in figure 3. When the CL changes from one value to the other, the solitons can smoothly transit to the different ones of different widths (figure 3(b)), and exhibit a stable rotation with different velocities (figure 3(c)).
![Figure 5. Orientations of spiraling elliptic beams at the exit surface (z = 17.6) of nonlocal media when the CL (=20) is modulated to different values (=17, 18, 19, 20), the parameters are chosen as b = 1.5, c = 1, P_0 = P_c and M_0 = M_c.](image)

The transitions among soliton states with different widths and rotating velocities when the CL gradually changes can be clearly confirmed by figure 4, where three dimensional evolutions of the spiraling elliptic beams under different cases of varying CL, slowly increasing in (a), keeping constant in (b), and slowly increasing in (c) are given. In addition, different rotating velocities of the beam will result in the different total rotation angles at the output plane of nonlinear media, which is shown in figure 3(d), and offers a method to engineer beams in optical signal processing. As numerical results shown in figure 5, the orientation at the exit surface can be controlled by changing the CL in nonlocal nonlinear media with fixed length.

3. Gradient force exerted by spiraling elliptic beams

The spiraling elliptic beams carrying the OAM, have potential applications in optical tweezers. In this section, we will discuss the induced variation of the gradient force exerted by spiraling elliptic beam when the CL gradually changes.

In the presence of an optical field, a particle experiences the gradient force [30]

\[ G = k \nabla |\phi|^2, \tag{12} \]

where \( k \) is a dimensionless quantity related to the refractive index, the radius, and the effective polarizability of the particle. When the refractive index of particle is larger than that of medium, \( k > 0 \), and the particle will be subjected to the gradient force directed at the maximum value of intensity. In the following calculations, we assume that \( k = 1 \) for simplicity. Substitution of equation (3) into equation (12) at the critical power yields the gradient force along principal axes of ellipse

\[ G_b = -\frac{(b^2 + c^2)^2 \sigma^4}{4b^2 c^2} x \exp\left(-\frac{x^2}{b^2}\right), \]

\[ G_c = -\frac{(b^2 + c^2)^2 \sigma^4}{4b^2 c^2} y \exp\left(-\frac{y^2}{c^2}\right). \tag{13} \]

And the maxima of the gradient force can be obtained from equation (13),
Figure 6. Dependence of the maximal gradient force on the CL. The parameters are chosen as those in figure 3 with the same color.

Figure 7. The gradient force of the spiraling elliptic soliton in nonlocal media. The parameters are chosen as the green lines in figure 3.

\[
M(G_b) = \frac{\sqrt{2}(b^2 + c^2)^2 e^{-1/2\delta^4}}{8b^2c^3} = \frac{(P_c/\pi)^{3/4}\rho((\rho^2 + 1)^{1/2}\delta^4}{2^{1/4}\rho^{1/2}\sigma^3}
\]

\[
M(G_c) = \frac{\sqrt{2}(b^2 + c^2)^2 e^{-1/2\delta^4}}{8bc^6} = \frac{(P_c/\pi)^{3/4}\rho^2((\rho^2 + 1)^{1/2}\delta^4}{2^{1/4}\rho^{1/2}\sigma^5}
\]

which reveals that the gradient force is anisotropic, \(M(G_c)/M(G_b) = \rho = b/c\). Obviously, as the CL changes, the anisotropy of the gradient force remains and the gradient force can be controlled as shown in figure 6. The analytical solutions were well confirmed by the numerical results.

Although the trapping mechanisms is the same as those mentioned in [30], which are both resulting from the gradient force. However, due to the structured beam considered in the manuscript, the gradient force is anisotropic, that is, the strength of the force is different along different directions as shown in figure 6. Furthermore, because the beam is rotating as shown in figure 7, the orientation of the maximal gradient force will also rotate. We think the direction-controllable optical tweezers can be also used to rotate the rod-shaped micro-particles.

4. Theoretical extension in the nematic liquid crystal

The discussions above are in the nonlocal nonlinear media with the Gaussian response, which can be analytically analysed. However, the actual response functions in physical nonlinear system always exhibit some singularities. For example, the response function of the nematic liquid crystals, which are typical
nonlocal nonlinear media [31, 32], can be expressed by [33]

\[ R(x, y) = \frac{1}{2\pi \sigma^2} K_0 \left( \frac{\sqrt{x^2 + y^2}}{\sigma} \right), \]  

(15)

where \( K_0 \) is the zeroth-order modified Bessel function. In fact, this response function is obtained from the simplified model rather than the complete one for the nematic liquid crystal [27, 28, 34]. However, this model can be used to reveal the main phenomena of spiraling soliton transition in nematic liquid crystal. On the other hand, the results obtained here can be used as a reference for the manipulation of widths and velocities of spiraling solitons in other real nonlocal nonlinear media which also have singular response functions.

First, we iterate the spiraling elliptic solitons by using the imaginary-time evolution method [35, 36] in nematic liquid crystal. Since the iterated solution is spiraling in the Cartesian coordinate system \((X, Y, Z)\), we must transform the NNLSE to its corresponding form in the rotating coordinate system \((x, y, z)\), we must transform the NNLSE to its corresponding form in the rotating coordinate system. The relations between the two coordinate systems are

\[ X = x \cos(\omega z) + y \sin(\omega z), \]
\[ Y = -x \sin(\omega z) + y \cos(\omega z), \]
\[ Z = z. \]  

Then equation (1) will be rewritten in the rotating coordinate system

\[ i \frac{\partial \psi}{\partial Z} + \frac{1}{2} \left( \frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} \right) + i \omega \left( Y \frac{\partial \psi}{\partial X} - X \frac{\partial \psi}{\partial Y} \right) \]
\[ + \psi \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R(X - X', Y - Y')|\psi(X', Y', Z)|^2 dX' dY' = 0. \]  

(17)

Stationary solutions of equation (1) are assumed as \( \psi = u(X, Y)e^{i\beta Z} \), where \( u(X, Y) \) is a complex function and \( \beta \) is a real propagation constant. Inserting \( \psi = u(X, Y)e^{i\beta Z} \) into equation (17), we can obtain

\[ -\beta u + \frac{1}{2} \left( \frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} \right) u + i \omega \left( Y \frac{\partial u}{\partial X} - X \frac{\partial u}{\partial Y} \right) \]
\[ + u \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R(X - X', Y - Y')|u(X', Y', Z)|^2 dX' dY' = 0. \]  

(18)

Introducing an operator \( L_{00} \)

\[ L_{00} = \frac{1}{2} \left( \frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} \right) + i \omega \left( Y \frac{\partial u}{\partial X} - X \frac{\partial u}{\partial Y} \right) \]
\[ + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R(X - X', Y - Y')|u(X', Y', Z)|^2 dX' dY'. \]  

(19)

Then equation (18) can be rewritten as \(-\beta u + L_{00}u = 0.\) And the evolution equation in imaginary time is [35, 36]

\[ \frac{\partial u}{\partial t} = M^{-1}(L_{00}u - \beta u), \]  

(20)

where \( M = c - \nabla^2, M \) is the acceleration operator, and \( c = 3 \) in general [35]. Equation (20) can be solved by Euler’s method

\[ u_{n+1} = u_n + M^{-1}(L_{00}u - \beta u)|_{u_n= u_n, \beta = \beta_n} \Delta T, \]  

(21)

where \( u_n \) is the solution of iteration \( n \) times, \( \beta_n = (L_{00}u_n, M^{-1}u_n)\langle u_n, M^{-1}u_n \rangle. \) \( \langle L_{00}u_n, M^{-1}u_n \rangle \) is the inner product of \( L_{00}u_n \) and \( M^{-1}u_n \). When \( u_{n+1} \) was obtained, it can be adjusted by [35]

\[ u_{n+1} = \left[ \frac{P}{(u_{n+1}, u_{n+1})} \right]^{1/2} u_{n+1}, \]  

(22)

where \( P \) is the given power, which is fixed [35]. In addition, the \( \sigma \) and \( \omega \) should be given first before iterations. Substituting a trial solution of elliptic Gaussian-shaped beam into equations (20)–(22), a stable soliton can be iteratively obtained, and it will rotate under the Cartesian coordinate system.
Figure 8. (a) Varying CL of different curves with respect to \( z \), (b) corresponding evolutions of semi axes, (c) rotating velocities, (d) and (e) and maximal gradient forces for spiraling elliptic beams at \( P_0 = P_c \) and \( M_0 = M_c \). The subfigures (d) and (e) correspond to the cases marked by blue and pink curves in (a).

Figure 9. Evolutions of the spiraling elliptic solitons in nematic liquid crystal with different varying CL. Rows of (a)–(c) correspond to the blue, green and pink curves in figure 8(a).

The evolutions of the iterative solitons in nematic liquid crystal are simulated by the split-step Fourier method. In addition to linear gradually change of CL, Gaussian-shaped (as shown in figure 8(a)) or even periodic varying CL also can be designed. The numerical results in figures 8(b)–(e) demonstrate that the beam widths, angular velocity and gradient force all can be controlled by modulating the CL. The transition evolutions of the iterative spiraling elliptic soliton corresponding to figure 8(a) are shown in figure 9. Solitons at \( z = 0 \) are the numerically found by means of the imaginary-time evolution method. The soliton stability can be confirmed by the direct propagations from \( z = 0 \) to \( z = 100 \). The CL begins to change at \( z = 100 \), and gradually changes along different curves to another values. Accordingly, the solitons transit to another ones with different widths at \( z = 300 \). The new solitons also can stably propagate from \( z = 300 \) to \( z = 400 \). The comparison of figures 9(a) and (c) shows that the soliton widths can be easy controlled by the varied CL.

Figures 10(a) and (b) demonstrate that the powers of spiraling elliptic solitons in nematic liquid crystal with varied CL are constant, and the OAM exhibits a small decay. Of course, the comparison of blue, green and pink lines in figure 10(b) shows that the OAM radiation when the CL changes slowly is slightly larger.
Evolutions of (a) power, (b) OAM, (c) ellipticity, and (d) degree of nonlocality for spiraling elliptic solitons at $P_n = P_c$ and $M_0 = M_c$. The parameters are chosen as those in figure 8 with the same color.

Figure 11. Same as figure 8 but with plots corresponding to ‘sin-’ varying CL.

than the case of the constant CL. In addition, both the ellipticity and the degree of nonlocality present small oscillations as shown in figures 10(c) and (d).

It is shown in figure 11 that all the predicated phenomena can be periodically controlled for the ‘sin-’ varying CL. Which means that the only requirement of the CL for the solitons transition is that the variation of the CL must be slow enough.

5. Conclusion

In longitudinally-inhomogeneous nonlocal media, we have investigated the propagation properties of the spiraling elliptic beams, which exhibit the cross phase and carry the OAM. We have demonstrated that, when the CL gradually changes, a spiraling elliptic soliton can transit to another different one of the different width. During the transition, the rotating velocity, total rotation angles, and gradient force are all changes, and their values are well controlled by the CL. This kind of rotating pattern carrying the OAM may find its potential applications in optical tweezers. Besides, the CL-dependent rotating is also useful in the field of light controlling.
Acknowledgements

This research is supported by the National Natural Science Foundation of China (11604199). Open Subject of the Key Laboratory of Weak Light Nonlinear Photonics of Nankai University (Grant No. OS 21-3)

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

ORCID iDs

Guo Liang https://orcid.org/0000-0001-9897-9306
Qing Wang https://orcid.org/0000-0002-3607-594X

References

[1] Molina-Terriza G, Torres J P and Torner L 2007 Nat. Phys. 3 305
[2] Griör D G 2003 Nature 424 810
[3] MacDonald M P, Paterson L, Volke-Sepulveda K, Arlt J, Sibbett W and Dholakia K 2002 Science 296 1101
[4] Bernet S, Jesacher A, Fürhapter S, Maurer C and Ritsch-Marte M 2006 Opt. Express 14 3792
[5] Mair A, Vaziri A, Weihs G and Zeilinger A 2001 Nature 412 213
[6] Ding D-S, Zhou Z-Y, Shi B-S and Guo G-C 2013 Nat. Commun. 4 2527
[7] Allen L, Beijersbergen M W, Spreeuw R J C and Wórdman J P 1992 Phys. Rev. A 45 8185
[8] Volke-Sepulveda K, García-Chávez V, Chávez-Cerda S, Arlt J and Dholakia K 2002 J. Opt. B: Quantum Semiclass. Opt. 4 S82
[9] Gutiérrez-Vega J C 2008 Characterization of elliptic dark hollow beams Proc. SPIE 7062 706207
[10] Courtial J, Dholakia K, Allen L and Padgett M J 1997 Opt. Commun. 144 210
[11] Desyatnikov A S, Buccoliero D, Dennis M R and Kivshar Y S 2010 Phys. Rev. Lett. 104 053902
[12] Liang G and Guo Q 2013 Phys. Rev. A 88 043825
[13] Katz O, Carmon T, Schwartz T, Segev M and Christodoulides D N 2004 Opt. Lett. 29 1248
[14] Assanto G, Peccianti M and Conti C 2003 Opt. Photonics News 14 44
[15] Peccianti M and Assanto G 2012 Phys. Rep. 516 147
[16] Barthelmy A, Freyhly C, Maneuf S and Reynaud E 1992 Opt. Lett. 17 844–6
[17] Eugenieva E D, Christodoulides D N and Segev M 2000 Opt. Lett. 25 972
[18] Liang G, Wang Y, Guo Q and Zhang H 2018 Opt. Express 26 8084
[19] Liang G, Jia T J and Ren Z M 2017 IEEE Photon. J. 9 6101608
[20] Liang G and Wang Q 2019 Opt. Express 27 10684
[21] Liang G, Zhang H, Fang L, Shou Q, Hu W and Guo Q 2020 Laser Photonics Rev. 14 2000141
[22] Conti C, Peccianti M and Assanto G 2005 Phys. Rev. E 72 066614
[23] Rothschild C, Cohen O, Manela O, Segev M and Carmon T 2005 Phys. Rev. Lett. 95 213904
[24] Krolikowski W, Bang O, Rasmussen J J and Wyller J 2001 Phys. Rev. E 64 016612
[25] Guo Q, Lu D Q and Deng D M 2015 Nonlocal spatial optical solitons Advances in Nonlinear Optics ed X F Chen, Q Guo, W She, H Zeng and G Zhang (Berlin: de Gruyter & Co) pp 277–306
[26] Zhong W P, Belic M R, Malomed B A, Zhang Y and Huang T 2016 Phys. Rev. E 94 012216
[27] Conti C, Peccianti M and Assanto G 2003 Phys. Rev. Lett. 91 073901
[28] Beeckman J, Arzarian H and Haertterman M 2009 Opt. Lett. 34 1900
[29] Liang G, Guo Q, Cheng W, Yin N, Wu P and Cao H 2015 Opt. Express 23 24612
[30] Gautam R et al 2020 Adv. Phys.: X 5 177826
[31] Assanto G and Smyth N F 2020 Physica D 402 132182
[32] Izdebskaya Y, Krolikowski W, Smyth N F and Assanto G 2016 J. Opt. 18 054006
[33] Hu W, Zhang T, Guo Q, Xuan L and Lan S 2006 Appl. Phys. Lett. 89 071111
[34] Peccianti M, Conti C and Assanto G 2005 Opt. Lett. 30 415
[35] Ya-Dong Y, Guo L, Zhan-Mei R and Qi G 2013 Acta Phys. Sin. 64 154202
[36] Yang J and Lakoba T I 2008 Stud. Appl. Math. 120 265