Nearly model-independent constraints on dense matter equation of state in a Bayesian approach

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Based on:- Phys.Rev.D 106, 043024 (2022) [arXiv:2203.08521]

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September 5, 2022
Out Lines

1. Introduction
2. Motivations
3. Methodology
4. Construction of EOSs
5. Results
6. Conclusions
Some bounds on neutron star structure

- For the first time National Radio Astronomy Observatory (NRAO) detected a massive neutron star of mass $2.01 \pm 0.04 \, M_\odot$ and the radius of $13 \pm 2 \, km$ from PSR J0348+0432 in 2007.
  
  J. Antoniadis et al., Science 340, 6131 (2013)

- Recently in 2019, Neutron Star Interior Composition Explorer (NICER) detected a most massive neutron stars of mass $2.072^{+0.067}_{-0.066} \, M_\odot$ and radius of $12.39^{+1.30}_{-0.98} \, km$ from PSR J0740+6620.
  
  T. E. Riley et al., Astrophys. J. Lett. 918, L27 (2021)
Some important aspects of neutron star structure

- The neutron star radius is primarily determined by the behavior of the pressure of matter in the vicinity of nuclear matter equilibrium density.

\[ R_M = C(n, M)[P(n)]^{0.23-0.26} \]  

1. \( P(n) \) is the total pressure
2. \( C(n, M) \) is a number that depends on the density \( n \) at which the pressure was evaluated and the stellar mass \( M \).
3. The correlations seen tighter in case of baryon density \( n = 1.5 \, n_s \) and \( 2 \, n_s \).

J. M. Lattimer & M. Prakash, Astrophys.J. 550,426 (2001)
Some important aspects of neutron star structure

- The nuclear equation of state (EOS) is very sensitivity to the neutron star tidal deformability ($\Lambda$) on 1 to 2 solar masses neutron stars.

C.Y. Tsang et al., Phys.Rev.C 102, 045808 (2020)
Some important aspects of neutron star structure

- A global power law dependence between tidal deformability and compactness parameter \((M/R)\) is verified over the mass region 1 to 2 solar masses neutron stars.

\[
\Lambda = 1.31 \times 10^{-3} \left(\frac{R}{M}\right)^{5.84} \quad (2)
\]

C.Y. Tsang et al., Phys.Rev.C 102, 045808 (2020)
Motivations

- Find out the correlations of various NS properties with the key parameter of EOS in a model-independent manner.
Objectives

- To construct large number of minimally constrained equations of state (EOSs) and study their correlations with a few selected properties of a neutron star.
- To see the nearly model-independent manner above the saturation density on the correlations of NS properties and the pressure of $\beta$-equilibrated matter.
- Formulate a parametrize form of the pressure for $\beta$-equilibrated matter, around $2\rho_0$, as a function of neutron star mass and the corresponding tidal deformability.
Equation of States

- We consider n, p, e, $\mu$ in the core part of neutron star.
- The three conditions such as:
  - Conservation of baryon no.: $\rho = \rho_n + \rho_p$
  - Charge neutrality: $\rho_p = \rho_e + \rho_\mu$
  - Beta-equilibrium: $\mu_n = \mu_p + \mu_e$

- Using these conditions, we calculate proton fraction and isospin asymmetry parameter ($\delta = \frac{\rho_n - \rho_p}{\rho}$).
- The energy per nucleon for neutron star matter $E(\rho, \delta)$ a given total nucleon density $\rho$ decomposed as,

$$
E(\rho, \delta) = E(\rho, 0) + E_{\text{sym}}(\rho)\delta^2 + \ldots, \quad (3)
$$

- where, $E(\rho, 0)$ is for symmetric nuclear matter and $E_{\text{sym}}(\rho)$ for density-dependent symmetry energy.
Models

- We have taken two different models such as Taylor and $\frac{n}{3}$ which are based on expansions of $E(\rho, 0)$ and $E_{\text{sym}}(\rho)$. 

\[ E(\rho, 0) = 4 \sum_{n=0}^{\infty} a_n n! \left( \rho - \rho_0 \right)^n, \quad (4) \]
\[ E_{\text{sym}}(\rho) = 4 \sum_{n=0}^{\infty} b_n n! \left( \rho - \rho_0 \right)^n, \quad (5) \]
\[ E(\rho, \delta) = 4 \sum_{n=0}^{\infty} \frac{1}{n!} (a_n + b_n \delta^2) \left( \rho - \rho_0 \right)^n, \quad (6) \]
\[ E(\rho, \delta) = 6 \sum_{n=2}^{\infty} \left( a'_n - 2 \right) \left( \rho - \rho_0 \right)^n, \quad (7) \]
\[ E_{\text{sym}}(\rho) = 6 \sum_{n=2}^{\infty} \left( b'_n - 2 \right) \left( \rho - \rho_0 \right)^n, \quad (8) \]
\[ E(\rho, \delta) = 6 \sum_{n=2}^{\infty} \left( a'_n - 2 + b'_n \delta^2 \right) \left( \rho - \rho_0 \right)^n. \quad (9) \]
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**Taylor’s expansion**

\begin{align*}
E(\rho, 0) &= \sum_{n=0}^{4} \frac{a_n}{n!} \left( \frac{\rho - \rho_0}{3\rho_0} \right)^n, \quad (4) \\
E_{\text{sym}}(\rho) &= \sum_{n=0}^{4} \frac{b_n}{n!} \left( \frac{\rho - \rho_0}{3\rho_0} \right)^n, \quad (5) \\
E(\rho, \delta) &= \sum_{n=0}^{4} \frac{1}{n!} (a_n + b_n\delta^2) \left( \frac{\rho - \rho_0}{3\rho_0} \right)^n, (6)
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Models

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**Taylor’s expansion**

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E(\rho, 0) = \sum_{n=0}^{4} \frac{a_n}{n!} \left(\frac{\rho - \rho_0}{3\rho_0}\right)^n,
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$$
E_{\text{sym}}(\rho) = \sum_{n=0}^{4} \frac{b_n}{n!} \left(\frac{\rho - \rho_0}{3\rho_0}\right)^n,
$$

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$$
E(\rho, \delta) = \sum_{n=0}^{4} \frac{1}{n!} (a_n + b_n \delta^2) \left(\frac{\rho - \rho_0}{3\rho_0}\right)^n,
$$

(6)   

**$\frac{n}{3}$ expansion**

$$
E(\rho, 0) = \sum_{n=2}^{6} (a'_{n-2}) \left(\frac{\rho}{\rho_0}\right)^{\frac{n}{3}},
$$

(7)   

$$
E_{\text{sym}}(\rho) = \sum_{n=2}^{6} (b'_{n-2}) \left(\frac{\rho}{\rho_0}\right)^{\frac{n}{3}},
$$

(8)   

$$
E(\rho, \delta) = \sum_{n=2}^{6} (a'_{n-2} + b'_{n-2} \delta^2) \left(\frac{\rho}{\rho_0}\right)^{\frac{n}{3}}.
$$

(9)

- $a_n, a'_n = \epsilon_0, 0, K_0, Q_0, Z_0$
- $b_n, b'_n = J_0, L_0, K_{\text{sym}, 0}, Q_{\text{sym}, 0}, Z_{\text{sym}, 0}$
Bayesian estimation

This approach is mainly based on the Bayes theorem which states that,

\[ P(\theta|D) = \frac{\mathcal{L}(D|\theta)P(\theta)}{Z}, \]  

where:

- \( \theta \) is model parameters.
- \( D \) is the data.
- \( Z \) is the evidence.
- \( P(\theta) \) is the prior for the model parameters.
- The likelihood function,

\[ \mathcal{L}(D|\theta) = \prod_j \frac{1}{\sqrt{2\pi\sigma_j^2}} e^{-\frac{1}{2} \left( \frac{d_j - m_j(\theta)}{\sigma_j} \right)^2}. \]

The marginalized posterior distribution for a parameter \( \theta_i \) can be obtained as,

\[ P(\theta_i|D) = \int P(\theta|D) \prod_{k \neq i} d\theta_k. \]
Construction of EOSs

- For construction large sets of EOSs, we use Bayesian approach. It needs,
  - Data (D)
  - Model (M(θ))
  - Prior (P(θ))

- We have considered the data as energy per neutron of PNM from Ref. K. Hebeler et al. Astrophys. J. 773, 11 (2013).
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The prior distributions of nuclear matter parameters given in Table as,

| NMPs    | Pr-Dist | μ   | σ   |
|---------|---------|-----|-----|
| ε₀      | G       | -16 | 0.3 |
| K₀      | G       | 240 | 50  |
| Q₀      | G       | -400| 400 |
| Z₀      | G       | 1500| 1500|
| J₀      | G       | 32  | 5   |
| L₀      | G       | 50  | 50  |
| Kₜₙₜ₀   | G       | -100| 200 |
| Qₜₙₜ₀   | G       | 550 | 400 |
| Zₜₙₜ₀   | G       | -2000| 2000|
Construction of EOSs

- The nuclear matter parameters are filtered by demanding,
  - pressure for the $\beta$-equilibrated matter should increase monotonically with density and symmetry energy should not be negative (thermodynamic stability).
  - speed of sound must not exceed the speed of light (causality).
  - maximum mass of neutron star must exceeds $2M_\odot$ (observational constraint).

- The causality breaks down at higher density mostly for the Taylor EOS. In such cases, we use the stiffest EOS, $P(\epsilon) = P_m + (\epsilon - \epsilon_m)$.

- Once the EOS for the core and crust are known the values of NS mass, radius and tidal deformability corresponding to given central pressure can be obtained by solving Tolman-Oppenheimer-Volkoff equations.
Fig.1: Corner plots for the nuclear matter parameters (in MeV) obtained for Taylor (left) and $\frac{n}{3}$ expansions (right). The one dimensional marginalized posterior distributions (salmon) and the prior distributions (green lines) are displayed along the diagonal plots. The vertical lines indicate 68% confidence interval of nuclear matter parameters.
### Posterior Distribution of NM Parameters

| NMPs (in MeV) | without PNM | with PNM |
|--------------|-------------|----------|
|              | **Taylor**  | $\frac{n}{3}$ | **Taylor**  | $\frac{n}{3}$ |
| $\varepsilon_0$ | $-15.99^{+0.27}_{-0.27}(0.43)$ | $-15.99^{+0.27}_{-0.27}(0.51)$ | $-16.00^{+0.27}_{-0.30}(0.42)$ | $-16.00^{+0.27}_{-0.28}(0.44)$ |
| $K_0$       | $233.38^{+48.94}_{-42.73}(76.14/83.95)$ | $237.43^{+44.24}_{-45.75}(72.25/83.22)$ | $231.96^{+44.80}_{-41.33}(72.94/76.63)$ |
| $Q_0$       | $-411.84^{+207.53}_{-210.88}(301.56/409.00)$ | $-419.81^{+262.96}_{-272.47}(437.69/531.58)$ | $-418.89^{+187.43}_{-179.25}(300.76/377.42)$ |
| $Z_0$       | $1600.07^{+1067.33}_{-1362.28}(1883.00/2615.10)$ | $1403.84^{+704.56}_{-690.82}(1133.85/1386.25)$ | $1638.14^{+1241.83}_{-1277.48}(1906.75/2244.23)$ |
| $J_0$       | $32.37^{+4.69}_{-4.71}(7.22/10.23)$ | $31.88^{+0.87}_{-0.92}(1.43/1.85)$ | $31.87^{+0.93}_{-0.82}(1.49/1.68)$ |
| $L_0$       | $55.60^{+37.59}_{-43.88}(63.89/84.62)$ | $51.25^{+13.32}_{-13.91}(21.60/25.54)$ | $52.25^{+13.55}_{-12.76}(22.73/23.04)$ |
| $K_{\text{sym},0}$ | $-40.03^{+161.60}_{-135.08}(271.89/234.67)$ | $-96.65^{+141.41}_{-127.49}(225.69/216.74)$ | $-67.44^{+127.18}_{-114.80}(206.09/200.38)$ |
| $Q_{\text{sym},0}$ | $705.36^{+311.23}_{-352.72}(511.39/727.86)$ | $699.56^{+324.38}_{-323.52}(521.95/639.30)$ | $726.49^{+300.40}_{-358.51}(510.33/631.86)$ |
| $Z_{\text{sym},0}$ | $-1390.39^{+1518.69}_{-1856.18}(2526.53/3623.74)$ | $55.34^{+1205.62}_{-782.52}(2255.28/1415.84)$ | $-1622.35^{+1606.61}_{-1911.81}(2788.70/3468.40)$ |
Fig. 2: Corner plots for the marginalized posterior distributions (salmon) of the tidal deformability $\Lambda_{1.4}$, radii $R_{1.4}$ (km) and $R_{2.07}$ (km) and the maximum mass $M_{\text{max}}$ ($M_{\odot}$) for Taylor (left) and $n/3$ (right) expansions. The green lines represent effective priors obtained using the priors for nuclear matter parameters.
Joint Probability Distributions of $M, R$

| NS properties | without PNM | with PNM |
|---------------|-------------|----------|
| $A_{1.4}$     | 527.72      | 455.85   |
| $R_{1.4}$ (km)| 14.69       | 14.15    |
| $R_{207}$ (km)| 13.24       | 12.27    |
| $M_{\text{max}}$ ($M_\odot$) | 2.45       | 2.19     |

Fig.3: Plot for joint probability distribution $P(M, R)$ as a function of mass and radius of neutron star obtained for $n/3$ expansion. The red dashed line represents the 90% confidence interval.
Correlations of NS properties with EOS

Fig. 4: The correlation coefficients $r(x, P_{BEM}(\rho))$ of both Taylor and $\frac{n}{3}$ expansions along with the mean-field theory calculations are shown.
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Fig. 5: The variations of pressure for $\beta$-equilibrated matter ($P_{BEM}(\rho)$) at selected densities versus NS properties.
Correlations of Tidal deformability with Pressure

Fig. 6: Here is the plot of correlation coefficients between tidal deformability ($\Lambda_M$) and the pressure of $\beta$-equilibrated matter ($P_{\text{BEM}}(\rho)$) on neutron star mass ($M$) and density ($\rho$).

\[ P_{\text{BEM}}(\rho) = a(M) + b(M)\Lambda_M \]

where $M_0$ is taken to be 1.4 $M_\odot$.

The values of $a_i$ and $b_i$ are estimated using a Bayesian approach with the help of $P_{\text{BEM}}(\rho)$ and tidal deformability obtained for Taylor and $n_3$ expansions.
Correlations of Tidal deformability with Pressure

- The $P_{\text{BEM}}(\rho)$ at $\rho \sim 1.5 - 2.5 \rho_0$ are strongly correlated ($r \sim 0.8 - 1$) with tidal deformability for NS masses in the range $1.2 - 2.0 M_\odot$.

Fig.6: Here is the plot of correlation coefficients between tidal deformability ($\Lambda_M$) and the pressure of $\beta$-equilibrated matter ($P_{\text{BEM}}(\rho)$) on neutron star mass ($M$) and density ($\rho$).
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Hence, $P_{\text{BEM}}(\rho)$ can be parametrized at a given $\rho$ as,

$$\frac{P_{\text{BEM}}(\rho)}{\text{MeVfm}^{-3}} = a(M) + b(M)\Lambda_M,$$

(13)

The mass-dependent coefficients $a(M)$ and $b(M)$ expanded as

$$a(M) = (a_0 + a_1(M - M_0) + a_2(M - M_0)^2),$$

(14)

$$b(M) = (b_0 + b_1(M - M_0) + b_2(M - M_0)^2),$$

(15)

where $M_0$ is taken to be $1.4M_\odot$.

- The values of $a_i$ and $b_i$ are estimated using a Bayesian approach with the help of $P_{\text{BEM}}(\rho)$ and tidal deformability obtained for Taylor and $n^3/3$ expansions.
Correlations of Tidal deformability with Pressure

- The priors for $a_i$ and $b_i$ are taken to be uniform in the range of -100 to 100.
- The calculations are performed for $\rho = 1.5, 2.0$ and $2.5 \rho_0$. 

The average deviation of $P_{BEM}(\rho_0^2)$, obtained using Eq. (13), from the actual values is about 10%. We noticed marginal improvement when the terms corresponding to quadratic in tidal deformability are included in Eq. (13).

Here, we display the variations of tidal deformability as a function of mass and pressure for $\beta$-equilibrated matter at $\rho = 1.5, 2.0$ and $2.5 \rho_0$. One can easily estimate the values of $P_{BEM}(\rho)$ for $\rho \sim 2\rho_0$ once the values of tidal deformability known in NS mass ranges $1 - 2M_\odot$. 

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QNP2022

September 5, 2022
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- One can easily estimate the values of $P_{\text{BEM}}(\rho)$ for $\rho \sim 2\rho_0$ once the values of tidal deformability known in NS mass ranges $1.2 - 2.0 M_\odot$. 
Conclusions

- A few low-order NMPs such as $\epsilon_0$, $K_0$, $J_0$ and $L_0$ constrained in narrow windows.
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- The correlations of neutron star properties over a wide range of mass with various key quantities characterizing the EOS are investigated.
  - The values of tidal deformability and radius for the NS with $1.4M_\odot$ are strongly correlated with the pressure for the $\beta$-equilibrated matter at density $\sim 2\rho_0$.
  - The radius for $2.07M_\odot$ NS is strongly correlated with the pressure for $\beta$-equilibrated matter at density $\sim 3\rho_0$.
  - The maximum mass of NS is correlated with the pressure for the $\beta$-equilibrated matter at density $\sim 4.5\rho_0$. 

These correlation systematics are similar with those obtained for unified EOSs for a diverse set of non-relativistic and relativistic mean-field models. We exploit the model independence of correlations to the pressure for $\beta$-equilibrated matter, in the density range $1.5 - 2.5\rho_0$, in terms of the mass and tidal deformability of neutron stars. This parametric form facilitates estimation of the pressure at densities around $2\rho_0$ for a given value of tidal deformability of neutron stars with mass in the range of $1.2 - 2.0 M_\odot$. 

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Acknowledgement

- The Department of Science and Technology, Ministry of Science and Technology, India, for the support of DST/INSPIRE Fellowship/2019/IF190058.

Collaborations list

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- B. K. Agrawal
- Hiranmaya Mishra
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- Sk Md Adil Imam
- Debasree Sen
Thank You!