A new geometric description for Igusa’s modular form \((azy)\)

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Abstract

The modular form \((azy)_5\) notably appears in one of Igusa’s classic structure theorems as a generator of the ring of full modular forms in genus 2, being exhibited by means of a complicated algebraic expression. In this work a different description for this modular form is provided by resorting to a peculiar geometrical approach.

1. Definitions and Notations

The symplectic group \(\text{Sp}(2g, \mathbb{R})\) acts biholomorphically and transitively on the Siegel upper half-plane \(\mathbb{H}_g\), namely the tube domain of complex symmetric \(g \times g\) matrices with positive definite imaginary part; its action is defined by:

\[
\gamma \cdot \tau := \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot \left( \begin{array}{c} a \tau + b \\ c \tau + d \end{array} \right)^{-1} \quad \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{Sp}(2g, \mathbb{R})
\]

Furthermore, for each \(k \in \mathbb{Z}\) the non-vanishing function:

\[
D(\gamma, \tau)^k := \det(c \tau + d)^k \quad \forall (\gamma, \tau) \in \text{Sp}(2g, \mathbb{R}) \times \mathbb{H}_g
\]

is a factor of automorphy, i.e. a holomorphic function on \(\mathbb{H}_g\) satisfying the cocycle condition:

\[
D(\gamma \gamma', \tau)^k = D(\gamma, \gamma' \tau)^k D(\gamma', \tau)^k \quad \forall \gamma, \gamma' \in \text{Sp}(2g, \mathbb{R})
\]

Thanks to this property, an action of \(\text{Sp}(2g, \mathbb{R})\) is also well defined on the space of holomorphic functions on \(\mathbb{H}_g\) for each \(k \in \mathbb{Z}\) by means of the rule:

\[
(\gamma k f) (\tau) := D(\gamma^{-1}, \tau)^{-k} f(\gamma^{-1} \tau)
\]

The action of the so-called Siegel modular group \(\Gamma_g := \text{Sp}(2g, \mathbb{Z})\) is particularly central to the theory of modular forms; a Siegel modular form, or simply a modular
form, of weight $k \in \mathbb{Z}^+$ with respect to a subgroup $\Gamma$ of finite index in $\Gamma_g$ is a holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ satisfying the property

$$\gamma^{-1}k f = f \quad \forall \gamma \in \Gamma$$

(4)

Modular forms with respect to a subgroup $\Gamma$ of finite index in $\Gamma_g$ form a ring $A(\Gamma)$ which is positively graded by the weights.

More generally, whenever $\chi$ is a character of $\Gamma$, a modular form with weight $k \in \mathbb{Z}^+$ and character $\chi$ is a holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ satisfying:

$$\gamma^{-1}k f = \chi(\gamma) f \quad \forall \gamma \in \Gamma$$

As for subgroups of finite index in $\Gamma_g$, they are, in fact, characterized by containing for some $n \in \mathbb{N}$ the so-called principal congruence subgroup of level $N$:

$$\Gamma_g(n) = \{ \gamma \in \Gamma_g \mid \gamma \equiv 1 \mod n \}$$

which is of finite index itself and normal in $\Gamma_g$; remarkable families of such subgroups are:

$$\Gamma_g(n, 2n) = \{ \gamma \in \Gamma_g(n) \mid \text{diag}(a' b) \equiv \text{diag}(c' d) \equiv 0 \mod 2n \}$$

$$\Gamma_g(0)(n) = \{ \gamma \in \Gamma_g \mid c \equiv 0 \mod n \}$$

An outstanding role in constructing modular forms is actually played by Riemann Theta functions with characteristics; for each $m = (m', m'') \in \mathbb{Z}^g \times \mathbb{Z}^g$ they are defined on $\mathbb{H}_g \times \mathbb{C}^g$ by the series:

$$\theta_m(\tau, z) = \sum_{m' \in \mathbb{Z}^g} \exp \left\{ i \left( n + \frac{m'}{2} \right) \tau \left( n + \frac{m'}{2} \right) + 2i \left( n + \frac{m'}{2} \right) \left( z + \frac{m''}{2} \right) \right\}$$

where $\exp(w)$ stands for $e^{\pi iw}$. Since $\theta_{m+2n}(\tau, z) = (-1)^{m' m''} \theta_m(\tau, z)$, these functions are parametrized merely by means of reduced $g$-characteristics, namely vector columns $[m' \ m'']$ with $m', m'' \in \mathbb{Z}^g_2$.

Throughout this paper reduced $g$-characteristics will be simply referred to as $g$-characteristics, their set being henceforward conventionally denoted by the symbol $C^g$. For each $m \in C^g$ the Theta constant $\theta_m$ with $g$-characteristic $m$ is defined by setting:

$$\theta_m(\tau) := \theta_m(\tau, 0)$$

The only non-vanishing Theta constants are plainly seen to be those related to even characteristics, namely characteristics $m$ satisfying $(-1)^{m' m''} = 1$. More precisely, a parity function is defined on $g$-characteristics by:

$$e(m) := (-1)^{m' m''} \quad \forall m = [m' \ m''] \in C^g$$

$^1$When $g = 1$ a modular form is also demanded to be holomorphic on the cusp $\infty$. 

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thus dividing them into even and odd respectively if \( e(m) = 1 \) or \( e(m) = -1 \); there are then \( 2^{k-1}(2^k + 1) \) distinct non-vanishing Theta constants for each \( g \geq 1 \), each being related to an even \( g \)-characteristic, while the remaining \( 2^{k-1}(2^k - 1) \) Theta constants that are associated to the odd \( g \)-characteristics are trivial functions. Henceforward, the symbols \( C_{even}^{(g)} \) and \( C_{odd}^{(g)} \) will stand respectively for the set of even \( g \)-characteristics and the set of odd \( g \)-characteristics.

An action of the group \( \Gamma_g \) is well defined on \( C^{(g)} \) by:

\[
\gamma \begin{bmatrix} m' \\ m'' \end{bmatrix} := \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{bmatrix} m' \\ m'' \end{bmatrix} + \begin{pmatrix} \text{diag}(c'd) \\ \text{diag}(d'b) \end{pmatrix} \text{mod} 2 \quad (5)
\]

\( \Gamma_g \) acts both on \( C_{even}^{(g)} \) and on \( C_{odd}^{(g)} \), the parity being preserved, and the action of the subgroup \( \Gamma_g(2) \) is, in particular, trivial. The action of \( \Gamma_g \) on \( k \)-plets of even 2-characteristics is also worth being briefly highlighted; when \( g = 2 \), a group isomorphism between \( \Gamma_2/\Gamma_2(2) \) and the symmetric group \( S_6 \) is naturally defined:

\[
\psi_p : \Gamma_2/\Gamma_2(2) \mapsto S_6 \quad (6)
\]

by merely focusing on the action of \( \Gamma_2/\Gamma_2(2) \) on the six elements of \( C_{odd}^{(2)} \). By pointing to the action of \( S_6 \) on non ordered \( k \)-plets of even 2-characteristics, the set of non-ordered 3-plets of even 2-characteristics, in particular, is found to decompose into two orbits (cf. [GS93]):

\[
C_3^- = \{ \{m_1, m_2, m_3\} \mid e(m_1 + m_2 + m_3) = -1 \}
\]

\[
C_3^+ = \{ \{m_1, m_2, m_3\} \mid e(m_1 + m_2 + m_3) = 1 \}
\]

with \( \text{card}(C_3^-) = \text{card}(C_3^+) = 60 \), while non-ordered 4-plets of even 2-characteristics decompose into three orbits \( C_4^- \), \( C_4^+ \) and \( C_4^* \) (cf. [GS93]), where, in particular:

\[
C_4^- = \{ \{m_1, \ldots, m_4\} \mid \{m_i, m_j, m_k\} \in C_3^- \forall \{m_i, m_j, m_k\} \subset \{m_1, \ldots, m_4\} \}
\]

\[
C_4^+ = \{ \{m_1, \ldots, m_4\} \mid \{m_i, m_j, m_k\} \in C_3^+ \forall \{m_i, m_j, m_k\} \subset \{m_1, \ldots, m_4\} \}
\]

with \( \text{card}(C_4^-) = \text{card}(C_4^+) = 15 \).

With reference to the actions described in (1) and (5), the transformation law for Riemann Theta function can be now outlined; for each \( m \in C^{(g)} \) and \( \gamma \in \Gamma_g \), one has (cf. [G66] or [G83]):

\[
\theta_{\gamma m}(\gamma \tau, (c\tau + d)^{-1}z) = \psi \chi_m(\gamma) \phi(\gamma, \tau, z) \det(c\tau + d)^\tau \theta_m(\tau, z)
\]

\( \forall \tau \in \mathbb{H}_g \quad \forall z \in \mathbb{C}^g \)

where:
1. \( \kappa \) is such a function that \( \kappa(\gamma)^4 = \exp\{\text{Tr}(bc)\} \) whenever \( \gamma \in \Gamma_g \) and, more particularly, \( \kappa(\gamma)^2 = \exp\left\{ \frac{1}{2} \text{Tr}(a - 1) \right\} \) whenever \( \gamma \in \Gamma_2(2) \).

2. \( \chi_m(\gamma) = \exp(2\xi_m(\gamma)) \) with:
   \[
   \xi_m(\gamma) = -\frac{1}{8}(m''bdm' + 'm'acm'' - 2'm'bcm'') + \\
   -\frac{1}{4}\text{diag}(a'b)(d'm' - cm'')
   \]

3. \( \phi(\gamma, \tau, z) = \exp\left\{ \frac{1}{4}z \left[ (c\tau + d)^{-1}c \right] z \right\} \)

4. The branch of \( \text{det}(c\tau + d)^n \) is the one whose sign is positive whenever \( R\tau = 0 \).

Since \( \phi|_{z=0} = 1 \), Theta constants transform as follows:

\[
\theta_{ym}(\gamma\tau) = \kappa(\gamma)\chi_m(\gamma) \text{det}(c\tau + d)^{\frac{1}{2}} \theta_m(\tau)
\]

the product \( \theta_m\theta_n \) of two Theta constants being thus found to be a modular form of weight 1 with respect to \( \Gamma_g(4,8) \).

Second order Theta constants can be also defined by setting for each \( m' \in \mathbb{Z}_2 \):

\[
\Theta_{nr}(\tau) := \theta_{[m']}(2\tau, 0) = \theta_{[m']}(2\tau)
\]

Whenever \( \gamma \in \Gamma_{g,0}(2) \), a remarkable transformation law holds for second order Theta constants:

\[
\Theta_{nr}(\gamma\tau) = \theta_{[m']}(\gamma2\tau) = \kappa(\gamma)\chi_{[m']}(\gamma) \text{det}(c\tau + d)^{\frac{1}{2}} \Theta_{nr}(\tau)
\]

where \( \gamma := \begin{pmatrix} a & 2b \\ c & d \end{pmatrix} \in \Gamma_g \).

By virtue of this formula the product \( \Theta_{m}\Theta_{n} \) of two second order Theta constants is likewise found to be a modular form of weight 1 with respect to \( \Gamma_{g}(2,4) \).

The product of all the non-vanishing Theta constants is a modular form of weight \( 2^{g-2}(2^g + 1) \) with respect to \( \Gamma_g \) itself whenever \( g \geq 3 \). The case \( g = 2 \) is instead a special one, the character appearing in the corresponding transformation formula being not trivial; in this case a modular form of weight 5 only with respect to \( \Gamma_2(2) \) is actually gained:

\[
\chi_5 := \prod_{m \in \mathbb{C}_g^{2^g}} \theta_m = \mu \cdot \det \begin{pmatrix} \Theta_{[0]} & \Theta_{[1]} & \Theta_{[2]} & \Theta_{[3]} \\ \frac{\partial \Theta_{[0]}}{\partial \tau_{11}} & \frac{\partial \Theta_{[1]}}{\partial \tau_{11}} & \frac{\partial \Theta_{[2]}}{\partial \tau_{12}} & \frac{\partial \Theta_{[3]}}{\partial \tau_{12}} \\ \frac{\partial \Theta_{[0]}}{\partial \tau_{12}} & \frac{\partial \Theta_{[1]}}{\partial \tau_{12}} & \frac{\partial \Theta_{[2]}}{\partial \tau_{22}} & \frac{\partial \Theta_{[3]}}{\partial \tau_{22}} \\ \frac{\partial \Theta_{[0]}}{\partial \tau_{22}} & \frac{\partial \Theta_{[1]}}{\partial \tau_{22}} & \frac{\partial \Theta_{[2]}}{\partial \tau_{22}} & \frac{\partial \Theta_{[3]}}{\partial \tau_{22}} \end{pmatrix}
\]
where \( \mu \in \mathbb{C}^* \) is a suitable non-zero constant. The square:

\[
\chi_{10} \ := \ \chi_5^2 = \prod_{m \in \mathbb{C}^2} \theta_m^2
\]  

(8)

is, though, a modular form of weight 10 with respect to \( \Gamma_2 \). As concerns the product of all the second order Theta constants:

\[
P_g \ := \ \prod_{m' \in \mathbb{Z}^g} \Theta_{m'}
\]  

(9)

this one is seen to be a modular form of weight \( 2g-1 \) with respect to \( \Gamma_g(2) \) whenever \( g \geq 2 \).

The ring \( A(\Gamma_1) \) of modular forms with respect to \( \Gamma_1 \) is classically known to be generated as a \( \mathbb{C} \)-algebra by the Eisenstein series \( E^{(1)}_4 \) and \( E^{(1)}_6 \) respectively of weight 4 and 6. Regarding the \( g = 2 \) case, Igusa’s structure theorem provides a set of generators (cf. [Ig64] and [Ig67]):

**Theorem 1.** \( A(\Gamma_2) = \mathbb{C}[E^{(2)}_4, E^{(2)}_6, \chi_{10}, \chi_{12}, \chi_{35}] \)

where \( E^{(2)}_4 \) and \( E^{(2)}_6 \) are the Eisenstein series respectively of weight 4 and 6, \( \chi_{10} \) is the modular form described in (8), \( \chi_{12} \) is a modular form of weight 12 obtained by a suitable symmetrization:

\[
\chi_{12} = \frac{1}{2173} \sum_{\substack{\{m_1, \ldots, m_6\} \mid s.t. \\ \mathbb{C}^6=\{m_1, \ldots, m_6\} \in \mathbb{C}_4}} \pm (\theta_{m_1} \cdots \theta_{m_6})^4
\]

and \( \chi_{35} := \chi_5 \cdot (azy)_5 \), where \( \chi_5 \) is as in (7) and \( (azy)_5 \) is defined by:

\[
(azy)_5 := \frac{1}{8} \sum_{\{m_1, m_2, m_3\} \in \mathbb{C}_3} \pm (\theta_{m_1} \theta_{m_2} \theta_{m_3})^{20}
\]

where the signs are to be properly chosen in order to to gain the correct symmetrization.

Due to the isomorphism \( \psi_P \) described in (6), the group \( \Gamma_2/\Gamma_2(2) \) admits a sole non trivial irreducible representation of degree 1, the corresponding character being as follows:

\[
\chi_P(\gamma) := \begin{cases} 
1 & \text{if } \psi_P([\gamma]) \text{ is an even permutation} \\
-1 & \text{if } \psi_P([\gamma]) \text{ is an odd permutation}
\end{cases}
\]  

(10)

Then, by setting \( \Gamma_2^+ = \text{Ker} \chi_P \), the following structure theorem holds (cf. [Ig64]):

**Theorem 2.** \( A(\Gamma_2^+) = \mathbb{C}[E^{(2)}_4, E^{(2)}_6, \chi_5, \chi_{12}, (azy)_5] \).
Note 1. Amid the generators of $A(\Gamma^*_2)$ described in Theorem 2, \((azy)_5\) and \((azy)_5\) are the only ones that transform with the non-trivial character $\chi_p$ under the action of the full modular group $\Gamma_2$. The function \((azy)_5\), in particular, the unique modular form of weight 30 with respect to $\Gamma_2$ admitting a non-trivial character under the action of $\Gamma_2$.

The next section is devoted to provide a different expression for \((azy)_5\) by means of a remarkable geometrical construction already described in [GS93].

2. Geometric description of the modular form \((azy)_5\)

Second order Theta constants are related to Theta constants by Riemann’s addition formula. When $g = 2$, the following relations hold in particular between the ten non-trivial Theta constants and the four second order Theta constants:

\[
\begin{align*}
\theta_{[00]}^2 &= \theta_{[0]}^2 + \theta_{[1]}^2 + \theta_{[1]}^2 + \theta_{[1]}^2 \\
\theta_{[1]}^2 &= \theta_{[0]}^2 - \theta_{[1]}^2 - \theta_{[1]}^2 - \theta_{[1]}^2 \\
\theta_{[1]}^2 &= 2\theta_{[0]}\theta_{[1]} + 2\theta_{[0]}\theta_{[1]} \\
\theta_{[1]}^2 &= 2\theta_{[0]}\theta_{[1]} - 2\theta_{[0]}\theta_{[1]} \\
\theta_{[1]}^2 &= 2\theta_{[0]}\theta_{[1]} - 2\theta_{[0]}\theta_{[1]}
\end{align*}
\]

A quadratic form $Q_m$ in the variables $X_1, X_2, X_3, X_4$ is, therefore, associated to each even 2-characteristic $m$:

\[ m \mapsto Q_m \quad \text{where} \quad \theta_m^2 = Q_m(\Theta_1, \Theta_2, \Theta_3, \Theta_4) \]

Hence, a quadric $V_m$ in the projective space $\mathbb{P}^3$ also corresponds to each even 2-characteristic $m$:

\[ m \mapsto V_m := V(Q_m) = \{ [X_1, X_2, X_3, X_4] \in \mathbb{P}^3 \mid Q_m(X_1, X_2, X_3, X_4) = 0 \} \]

Furthermore, for each 4-plet $M \in \mathbb{C}_4^*$ the set

\[ \bigcap_{m \in M} V_m \subset \mathbb{P}^3 \]

where $M^c$ stands for the 6-plet of even 2-characteristics being complementary in $\mathbb{C}_4^*$ to $M$, contains exactly four points (cf. [GS93]). A configuration of four hyperplanes in $\mathbb{P}^3$ is thus uniquely determined for any $M \in \mathbb{C}_4^*$, each hyperplane being characterized by passing through all except one of the four points.
contained in the set $\bigcap_{m \in M} V_m$. Therefore, a collection of four linear forms
describing these four hyperplanes is found to be associated to each $M \in C_4^+:

\begin{align*}
\psi^M_1 &:= \psi^M_1(X_1, X_2, X_3, X_4) \\
\psi^M_2 &:= \psi^M_2(X_1, X_2, X_3, X_4) \\
\psi^M_3 &:= \psi^M_3(X_1, X_2, X_3, X_4) \\
\psi^M_4 &:= \psi^M_1(X_1, X_2, X_3, X_4)
\end{align*}

Hence, a tetrahedron $T_M$ in the projective space $P^3$ is uniquely determined by
each 4-plet $M = (m_1, m_2, m_3, m_4) \in C_4^+$, the set $\bigcap_{m \in M} V_m$ being in fact the set of
its vertices.

A remarkable holomorphic function can be associated to each $M \in C_4^+$ by means
of the functions $\psi^M_i$:

$$F_M(\tau) := \tilde{F}_M(\Theta_{[00]}(\tau), \Theta_{[01]}(\tau), \Theta_{[10]}(\tau), \Theta_{[11]}(\tau))$$

where $F_M := \prod_{i=1}^4 \psi^M_i$

In particular, the 4-plet

$$M_0 := \begin{bmatrix} [00] & [00] & [00] & [00] \\
[00] & [01] & [10] & [11] \end{bmatrix} \in C_4^+$$

is such that:

$$\bigcap_{m \in M_0} V_m = \{ [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1] \}$$

the faces of the corresponding tetrahedron $T_{M_0}$ being thus described by:

$$\psi^M_0 = X_1; \quad \psi^M_0 = X_2; \quad \psi^M_0 = X_3; \quad \psi^M_0 = X_4;$$

Hence:

$$F_{M_0}(\tau) = \prod_{m \in Z^2} \Theta_{m'} = P_2(\tau)$$

and $F_{M_0}$ is, therefore, a modular form of weight 2 with respect to $\Gamma_2(2)$ (cf. (9)).

**Lemma 1.** The group $\Gamma_{2,0}(2)$ is the stabilizer $S_{\text{St}_{M_0}}$ of the 4-plet $M_0$.

**Proof.** On the one hand $\Gamma_{2,0}(2) \subset S_{\text{St}_{M_0}}$ by definition; on the other hand, $\gamma \in S_{\text{St}_{M_0}}$
implies $\text{diag}(cd) - cm' = 0 \mod 2$ for each $m' \in Z^2_2$, hence $\gamma \in \Gamma_{2,0}(2)$.
As \( C_4^+ \) is an orbit, Lemma [I] self-evidently implies the following:

**Corollary 1.** \([\Gamma_2 : \Gamma_{2,0}(2)] = 15\)

The map \([\gamma] \mapsto \gamma M_0\) is thus a bijection between \(\Gamma_2/\Gamma_{2,0}(2)\) and the collection \(\{T_M| M \in C_4^+\}\) of the tetrahedrons; the product of all the images of \(F_{M_0}\) under the action of the representatives of the fifteen cosets of \(\Gamma_{2,0}(2)\) in \(\Gamma_2\) is then a proper candidate to be focused on. However, the behaviour of \(F_{M_0}\) under the action of \(\Gamma_{2,0}(2)\) is due to be investigated foremost.

**Proposition 1.** Let \(\chi_F\) be the character introduced in (10) then, with reference to the action in (3), one has:

\[
\eta^{-1}F_{M_0} = \chi_F(\eta)F_{M_0} \quad \forall \eta \in \Gamma_{2,0}(2)
\]

**Proof.** One only needs to check that \(\chi_F\) is the very character involved in the transformation law for \(F_{M_0}\) under the action of \(\Gamma_{2,0}(2)\). Since one has:

\[
F_{M_0}(\eta_0 \tau) = -F_{M_0}(\tau)
\]

for \(\eta_0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_{2,0}(2)\)

then \(F_{M_0}\) transforms with a non-trivial character \(\chi\) under the action of \(\Gamma_{2,0}(2)\); the character \(\chi\) is of course trivial on \(\Gamma_{2,0}(2)^+ = \ker \chi \supset \Gamma_{2,0}(2)\), thus extending to a non-trivial character of \(\Gamma_2\) (cf. [II91] Appendix), which is well defined on \(\Gamma_2/\Gamma_2(2)\); hence \(\chi\) extends to \(\chi_F\), this one being the sole non-trivial character of \(\Gamma_2/\Gamma_2(2)\). \(\square\)

A holomorphic function is well defined for any fixed \(\gamma \in \Gamma_2\):

\[
\varphi_F(\tau) := \chi_F(\gamma)^{-1}(\gamma^{-1})F_{M_0}(\tau)
\]

In particular, for a fixed choice \(\gamma_1, \ldots, \gamma_{15}\) of the representatives of the cosets of \(\Gamma_{2,0}(2)\) in \(\Gamma_2\) a notable holomorphic function is defined:

\[
\varphi(\tau) := \prod_{i=1}^{15} \varphi_{\gamma_i}(\tau)
\]

**Proposition 2.** The function \(\varphi\) does not depend on the choice of the coset representatives.

**Proof.** If \(\gamma_1, \ldots, \gamma_{15}\) and \(\gamma'_1, \ldots, \gamma'_{15}\) are two different choices for the representatives of the cosets of \(\Gamma_{2,0}(2)\) in \(\Gamma_2\), then there exists a permutation \(j\) of the indices such that for each \(i = 1, \ldots, 15\) one has \(\gamma'_i(0) = \eta_i \gamma_j\) with \(\eta_i \in \Gamma_{2,0}(2)\). Hence, by
and Proposition \[1\]

\[
\prod_{j=1}^{15} \varphi_{\gamma_j}^\prime(\tau) = \prod_{i=1}^{15} \varphi_{\eta_i}^\prime(\tau) = \prod_{i=1}^{15} \chi_p(\eta_i \gamma_i)^{-1}((\eta_i \gamma_i)^{-1} |_{\mathbb{F}^0}) = \\
= \prod_{i=1}^{15} \chi_p(\eta_i \gamma_i)^{-1} D(\eta_i \gamma_i, \tau)^{-2} \chi_p(\eta_i) D(\eta_i \gamma_i \tau)^2 F_{\mathbb{F}^0}(\gamma_i \tau) = \\
= \prod_{i=1}^{15} \chi_p(\gamma_i)^{-1} D(\gamma_i, \tau)^{-2} F_{\mathbb{F}^0}(\gamma_i \tau) = \prod_{i=1}^{15} \varphi_{\gamma_i}(\tau)
\]

\[\square\]

**Theorem 3.** \( \varphi \) is a modular form of weight 30 with respect to \( \Gamma_2 \) with character \( \chi_p \).

**Proof.** The transformation formula \( \varphi(\gamma \tau) = \chi_p(\gamma) D(\gamma, \tau)^{30} \varphi(\tau) \) has to be proved whenever \( \gamma \in \Gamma_2 \). Let thus \( \gamma_1, \ldots, \gamma_{15} \) be a fixed collection of coset representatives of \( \Gamma_{2,(2)} \) in \( \Gamma_2 \); then, for each \( \gamma \in \Gamma_2 \) one has:

\[
\varphi(\gamma \tau) = \prod_{i=1}^{15} \varphi_{\gamma_i}(\gamma \tau) = \prod_{i=1}^{15} \chi_p(\gamma_i)^{-1} D(\gamma_i \gamma \tau)^{-2} F_{\mathbb{F}^0}(\gamma_i \gamma \tau) = \\
= \prod_{i=1}^{15} \chi_p(\gamma_i)^{-1} D(\gamma_i \gamma \tau)^{-2} D(\gamma, \tau)^2 F_{\mathbb{F}^0}(\gamma_i \gamma \tau) = \\
= D(\gamma, \tau)^{30} \prod_{i=1}^{15} \chi_p(\gamma_i)^{30} \varphi_{\gamma_i}(\tau) = \\
= \chi_p(\gamma)^{15} D(\gamma, \tau)^{30} \prod_{i=1}^{15} \varphi_{\gamma_i}(\tau) = \chi_p(\gamma) D(\gamma, \tau)^{30} \varphi(\tau)
\]

the last equality being due to Proposition \[2\]

\[\square\]

**Corollary 2.** There exists \( \lambda \in \mathbb{C}^\ast \) such that \( \varphi = \lambda (azy)_5 \).

**Proof.** By Theorem \[3\] \( \varphi \) is a modular form of weight 30 with respect to \( \Gamma_2 \) transforming with a non-trivial character under the action of \( \Gamma_2 \); hence, the thesis follows by Theorem \[2\] and Note \[1\]

Such an expression for the modular form \( (azy)_5 \) has been recently found by Gehre and Krieg in a different way by means of quaternionic Theta constants (cf. [GK10]).

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