On the Structure of the Magnetic Field Near a Black Hole in Active Galactic Nuclei

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**Abstract** – Using the Grad-Shafranov equation, we consider a new analytical model of the black hole magnetosphere based on the assumption that the magnetic field is radial near the horizon and uniform (cylindrical) in the jet region. Within this model, we have managed to show that the angular velocity of particles $\Omega_F$ near the rotation axis of the black hole can be smaller than $\Omega_H/2$. This result is consistent with the latest numerical simulations.

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1. INTRODUCTION

The main model responsible for the energy release in active galactic nuclei (AGNs) is presently known to be the electrodynamic model dating back to the paper by Blandford and Znajek (1977). Within this model, the energy losses of a rotating black hole are related to the flux of electromagnetic energy flowing along the magnetic field lines from the black hole surface in the direction of the jets. The question about the magnetic field structure in the vicinity of a black hole (which should be generated in an accretion disk) still remains an open one. This question becomes particularly topical in connection with the latest observations of the inner jet regions (see, e.g., Doeleman et al. 2012) and with the successful launch of the Spectrum-R (Radioastron) Space Observatory. The latter allows spatial scales comparable to the size of the central black hole to be resolved (Kardashev 2009).

By now, a wide variety of magnetic field geometries near a black hole have been considered in the force-free approximation within an analytical approach. These include a split monopole field near the horizon and far from the black hole (Blandford and Znajek 1977), a parabolic field near the horizon and far from the black hole (Blandford and Znajek 1977; Ghosh and Abramowicz 1997), and a uniform magnetic field near the horizon and a split monopole field at large distances (Beskin et al. 1992). In all cases, the angular velocity of the plasma $\Omega_F(0)$ (which is known to depend uniquely on the magnetic field geometry) near the rotation axis has always been exactly half the angular velocity of the black hole $\Omega_H$ (see Fig. 1). However, the latest numerical simulations show that the condition $\Omega_F(0) = 0.5 \Omega_H$ can be violated. In particular, McKinney et al. (2012) argue that their angular velocity profile $\Omega_F(\theta)$ near the black hole horizon more closely corresponds to a parabolic field, for which actually drops to $0.3 \Omega_H$ at $\theta = \pi/2$. In this case, however, some of the magnetic field lines should be connected not with the black hole horizon but with the accretion disk near the ergosphere (Punsly 2001).

In this paper, we study an analytical model of the black hole magnetosphere based on a previously unconsidered magnetic field geometry: a radial field near the horizon and a vertical field far from the black hole. In the second section, we give a brief overview of the Grad-Shafranov equation and other models of the black hole magnetosphere based on it. In
Fig. 1: $\Omega_F/\Omega_H$ versus $\Psi/\Psi_*$ for a split monopole field near the horizon and far from the black hole (dashed line), a parabolic field near the horizon and far from the black hole (dash-dotted line), and an uniform magnetic field near the horizon and a split monopole field at large distances (solid line). Here $\Psi_*$ is the total magnetic flux through the horizon.

In the third section, we consider the model itself and compare it with the results of numerical simulations of the black hole magnetosphere (McKinney et al. 2012). We show that the derived angular velocity profile can be easily explained in terms of this model.

2. THE GRAD-SHAFRANOV EQUATION

The Grad-Shafranov equation describes axisymmetric stationary flows in terms of ideal magnetohydrodynamics (MHD). This approximation is based on the assumption about a good conductivity of the plasma filling the magnetosphere of a compact astrophysical object. In the vicinity of a rotating black hole (whose metric is also axisymmetric and stationary), this is provided by an efficient production of electron-positron pairs (Blandford and Znajek 1977). This approach is convenient in that quite a few integrals of motion, i.e., quantities conserved along the particle trajectory, exist in the case of stationary ideal MHD. This allows the MHD equations to be reduced to one second-order equation for the magnetic flux function $\Psi(r, \theta)$,
defining the magnetic field:

$$
\mathbf{B} = \frac{\nabla \Psi \times \hat{e}_\varphi}{2\pi \varpi} - \frac{2I}{c\varpi} \hat{e}_\varphi.
$$

(1)

Here, \( \varpi = \sqrt{g_{\varphi \varphi}} \) is the distance to the rotation axis. For this choice of designations, the function \( \Psi(r, \theta) \) coincides with the magnetic flux passing through a circle \( r, \theta, 0 < \varphi < 2\pi \) while the function \( I(r, \theta) \) is the total current flowing through the same circle.

In addition, the following important properties hold.

(1) The equation \( \nabla \cdot \mathbf{B} = 0 \) holds automatically. As a result, the three magnetic field components are defined by two scalar functions, \( \Psi(r, \theta) \) and \( I(r, \theta) \).

(2) Since the equation \( \mathbf{B} \cdot \nabla \Psi = 0 \) holds automatically, the \( \Psi(r, \theta) = \text{const} \) lines specify the shape of the magnetic surfaces.

Next, using the freezing-in condition \( \mathbf{E} + \mathbf{v} \times \mathbf{B}/c = 0 \) and the assumption about axisymmetry, we can determine the electric field as follows (for more details, see Beskin 2010):

$$
\mathbf{E} = -\frac{(\Omega_F - \omega)}{2\pi c} \nabla \Psi,
$$

(2)

where \( \omega \) is the Lense-Thirring angular velocity. As a result, the Maxwell equation \( \nabla \times \mathbf{E} = 0 \) leads to the relation \( \nabla \Omega_F \times \nabla \Psi = 0 \), whence it follows that

$$
\Omega_F = \Omega_F(\Psi).
$$

(3)

The function \( \Omega_F \) introduced in this way means the angular velocity of the particles moving in a plasma filled magnetosphere, while condition (3) is Ferraro isorotation law, according to which the angular velocity of the particles on axisymmetric magnetic surfaces must be constant (Ferraro 1937). Similarly, from the Maxwell equations we can deduce that \( \nabla I \times \nabla \Psi = 0 \) and, consequently,

$$
I = I(\Psi).
$$

(4)

This means that the total electric current within a magnetic flux tube is also conserved.

It is important to emphasize that in contrast to the nonrelativistic problem, there is a second family of singular surfaces associated with the accreting matter in the black hole magnetosphere. As a result, the additional critical condition allows an additional relation
between the current \( I(\Psi) \) and angular velocity \( \Omega_F(\Psi) \) to be determined. In the force-free approximation, this relation can be written as (Thorne and MacDonald 1982)

\[
4\pi I(\Psi) = [\Omega_H - \Omega_F(\Psi)] \sin \theta \frac{r_g^2 + a^2}{r_g^2 + a^2 \cos^2 \theta} (\frac{d\Psi}{d\theta}),
\]

(5)

where \( r_g \) is the black hole radius and \( a \) is the rotation parameter. Recall that the true meaning of Eq. (5) is the critical condition on the inner fast magnetosonic surface that coincides with the black hole horizon in the force-free approximation (Beskin 2010). As a result, this condition allows not only the longitudinal current \( I(\Psi) \) but also the angular velocity \( \Omega_F(\Psi) \) to be determined.

3. THE PLASMA ANGULAR VELOCITY PROFILE

As has already been said, several analytical models of the black hole magnetosphere were proposed in the literature. The first of them was constructed by Blandford and Znajek (1977). They considered a slowly rotating black hole for which a nonrotating black hole with a split monopole field was chosen as the zeroth approximation. Such a geometry can be easily realized in the presence of a thin accretion disk. In this case, the flux function \( \Psi = \Psi_0 (1 - \cos \theta) \) for \( \theta < \pi/2 \) and \( \Psi = \Psi_0 (1 + \cos \theta) \) for \( \theta > \pi/2 \) will be an exact solution of the Grad-Shafranov equation for a nonrotating black hole. The same authors considered a model magnetosphere with a parabolic magnetic field in the vicinity of a slowly rotating black hole. The shape of the field lines for \( \theta < \pi/2 \) at large distances is described by the flux function \( \Psi = \Psi_0 (r/r_g)(1 - \cos \theta) \). Since \( \Psi(r, \pi) \neq \text{const} \) for it, this implies the presence of sources or sinks in the volume (and not only in the gravitating center or at infinity). Such sources can also be realized in an accretion disk. Finally, Beskin et al. (1992) investigated the case where the black hole is in the center of a well-conducting disk bounded by an inner radius \( b \). In this case, the magnetic field was almost uniform near the black hole and still remained a split monopole one at large distance (\( r \gg b \)). As can be seen from Fig. 1, the angular velocity \( \Omega_F(\Psi) \) near the rotation axis is \( \Omega_H/2 \) in all these cases.

On the other hand, as is shown in Fig. 2, in their recent paper devoted to numerical simulations of the black hole magnetosphere, McKinney et al. (2012) obtained a profile of
Fig. 2: The plot of $\Omega_F/\Omega_H$ on the horizon versus polar angle derived during the numerical simulations of the black hole magnetosphere (McKinney et al. 2012). The dashed and dash-dotted lines correspond to monopole and parabolic fields, respectively.

the angular velocity $\Omega_F$ that not only differed from $\Omega_H/2$ near the axis but also even became negative here. The authors conclude that the derived profile is nevertheless closer to the parabolic solution, especially since the outer magnetic surfaces actually have such a shape. However, a significant fraction of the magnetic field lines in the parabolic solution must pass through the accretion disk. Consequently, not the rotating black hole but the equatorial region of the ergosphere will be an energy source for the corresponding magnetic surfaces (such a model was developed, for example, in Punsly’s works; see Punsly 2001).

Below, we will show that the results obtained by McKinney et al. (2012) are in best agreement with the previously unconsidered model of a black hole magnetosphere with a (split) monopole magnetic field near the black hole horizon and a cylindrical field far from the black hole. In this model, in which the bulk of the magnetic flux now passes through the black hole horizon, not only a collimation of the magnetic surfaces (it will be connected with the fairly high density of the ambient medium) but also angular velocities smaller than $\Omega_H/2$ near the axis can be obtained.

In our model, we will use the assumption that the flow is cylindrical near the rotation axis far from the black hole, as is obtained in numerical simulations, so that all quantities depend only on the cylindrical radius $\varpi$. In this case, the Grad-Shafranov equation is one-dimensional and can be integrated. In the force-free approximation, the solution of the Grad-Shafranov
equation takes the form (see, e.g., Istomin and Pariev 1994)

$$\Omega_F^2(\Psi)\varpi^2 B_z^2 c^{-2} = \varpi^2 B_\varphi^2 + \int_0^\varpi x^2 \frac{d}{dx} (B_z)^2 \, dx. \quad (6)$$

Substituting the expression of the toroidal field in terms of the total current $B_\varphi = -2I/\varpi c$, we can rewrite Eq. (6) as

$$\Omega_F^2(\Psi)A_1^2(\Psi) = 4I^2(\Psi) + A_2(\Psi), \quad (7)$$

where the following notation is used:

$$A_1(\Psi) = \varpi^2 B_z; \quad (8)$$

$$A_2(\Psi) = c^2 \int_0^\varpi x^2 \frac{d}{dx} (B_z)^2 \, dx. \quad (9)$$

As regards the ”boundary condition on the horizon” (5), it can be rewritten as

$$2I(\Psi) = [\Omega_H - \Omega_F(\Psi)] A_3(\Psi), \quad (10)$$

where

$$A_3 = \frac{1}{2\pi} \sin \theta \frac{r_g^2 + a^2}{r_g^2 + a^2 \cos^2 \theta} \left( \frac{d\Psi}{d\theta} \right). \quad (11)$$

Substituting the latter expression for $2I(\Psi)$ into Eq. (7) yields a quadratic equation for $\Omega_F(\Psi)$:

$$\Omega_F^2(A_1^2 - A_3^2) + 2\Omega_F\Omega_H A_3^2 - \Omega_F^2 A_3^2 - A_2 = 0. \quad (12)$$

Hence, the general expression for the angular velocity can be written as

$$\Omega_F = \Omega_H \left[ \frac{A_3}{A_3 + A_1} + \frac{A_2}{\Omega_H^2 A_1 A_3 \left( 1 + \sqrt{1 - \frac{A_2(A_1^2 - A_3^2)}{\Omega_H^2 A_1^2 A_3^2}} \right)} \right]. \quad (13)$$

This form stems from the fact that the relation $A_1 = A_3$ holds for the previously considered solutions on the rotation axis. Therefore, we tried to avoid the quantities $(A_1 - A_3)$ in the denominators of the corresponding expressions. Let us first consider the case where the magnetic field is vertical and uniform far from the black hole ($\Psi = \pi \varpi^2 B_0$ at $r \gg r_g$) and
exactly radial on the horizon ($\Psi = \Psi_\ast (1 - \cos \theta)$ at $r = r_g$). Substituting the corresponding flux functions into Eqs. (8), (9), and (11), we will then obtain

$$
A_1(\Psi) = \frac{\varpi^2}{2\pi \varpi} \frac{1}{d\varpi} \frac{d\Psi}{d\varpi} = \frac{\Psi}{\pi},
$$

$$
A_2(\Psi) = c^2 \int_0^x x^2 \frac{d}{dx} (B_0)^2 \, dx = 0,
$$

$$
A_3(\Psi) = \frac{\Psi}{\pi} \cdot \frac{r_g^2 + a^2}{r_g^2 + a^2 (1 - \Psi / \Psi_\ast)^2}.
$$

Consequently, $\Omega_F = \Omega_H/2$ on the rotation axis. On the other hand, both analytical (Beskin and Nokhrina 2009) and numerical (Komissarov et al. 2006; Tchekhovskoy et al. 2009; Porth et al. 2011) calculations show that a central core, which, as will be shown below, can change significantly the situation, can exist in the jet. Consider the case where the magnetic field is still exactly radial on the horizon and vertical far from the black hole, but now a denser core of radius $r_{\text{core}}$ exists near the rotation axis. As has been shown, such a core must actually be formed at a sufficiently low pressure of the ambient medium, with (see, e.g., Beskin and Nokhrina 2009)

$$
r_{\text{core}} = k \frac{c}{\Omega_F(0)}.
$$

Here, $k \approx \gamma_{\text{in}}$, where $\gamma_{\text{in}}$ is the characteristic Lorentz factor of the particles flowing along the jet axis. At distances $\varpi \leq r_{\text{core}}$ from the rotation axis, we can then write

$$
B_z = B_0 - B_0 \frac{\varpi^2}{r_{\text{core}}^2}.
$$

The following flux function corresponds to this field:

$$
\Psi = \pi \varpi^2 B_0 - \frac{1}{2} \pi B_0 \frac{\varpi^4}{r_{\text{core}}^2}.
$$

As a result, we still have $A_1(\Psi) \approx A_3(\Psi) \approx \Psi / \pi$ near the rotation axis in the first order in $\Psi$.

Now, however, $A_2$ will be nonzero,

$$
A_2 \approx -\frac{\Psi^2}{\pi^2 r_{\text{core}}^2}.
$$

In the upshot, substituting the expressions for $A_1, A_2$ and $A_3$ into the general formula (13), we have

$$
\Omega_F(0) \approx \frac{\Omega_H}{2} \left(1 - \frac{c^2}{\Omega_H^2 r_{\text{core}}^2}\right).
$$
If, however, we express here $r_{\text{core}}$ using Eq. (15), then

$$\Omega_F(0) \approx \frac{\Omega_H}{1 + \sqrt{1 + 1/k^2}}. \quad (20)$$

As we see, in the presence of a dense core, the angular velocity on the jet axis $\Omega_F(0)$ can be smaller than $\Omega_H/2$. In particular, for a mildly relativistic flow, $k = 1$, we obtain $\Omega_F(0) = 0.41 \Omega_H$.

Let us now use our model to analyze the results of the numerical simulations performed by McKinney et al. (2012), in which a dense core also takes place. According to these results, the magnetic field near the black hole horizon may be considered radial with a good accuracy. From the plot of $B_r(r_H, \theta)$, we can then derive the plot of the magnetic flux function $\Psi(r_H, \theta)$ and subsequently the plot of $A_3(\Psi)$. Since the flow near the axis at large distances may be considered cylindrical with a good accuracy, we may set $B_z \approx B_r$ in this region and use the plot of the radial magnetic field at $r = 30r_g$. However, the magnetic field directly on the axis is overestimated due to the peculiarities of the numerical method. Let us now consider a model magnetic field

$$B_z = \frac{B_0}{1 + \varpi^2/r_{\text{core}}^2} + B_1, \quad (21)$$

where $B_0$, $B_1$ and $r_{\text{core}}$ are the parameters of the problem. Let us choose them so that, first, the plot of the function $B_z(\theta)$ at small $\theta$ is close to the plot of $B_r(30r_g, \theta)$ from McKinney et al. (2012), and, second, the total magnetic flux corresponding to this $B_z$ should coincide with the total magnetic flux on the horizon. The latter condition is based on the property of magnetic flux conservation and the fact that much of the magnetic flux emerging from the black hole horizon is subsequently concentrated inside the jet, i.e., near the axis. The following flux function corresponds to the model magnetic field (21):

$$\Psi = \pi r_{\text{core}}^2 B_0 \ln(1 + \varpi^2/r_{\text{core}}^2) + \pi B_1 \varpi^2. \quad (22)$$

Such a flux function does not allow the inverse dependence $\varpi(\Psi)$ and, consequently, the dependences $A_1(\Psi)$ and $A_2(\Psi)$ to be derived analytically. However, these dependences can be derived numerically and the angular velocity profile $\Omega_F(\Psi(\varpi))$ can be found using Eq. (13).

Fig. 3 presents a plot where $\Omega_F/\Omega_H$ on the black hole horizon is along the vertical axis and the polar angle $\theta$ is along the horizontal axis. The range of angles was chosen from
Fig. 3: The plot of $\Omega_F/\Omega_H$ on the horizon versus polar angle $\theta$ derived from the analytical model described here (solid line) and the plot from McKinney et al. (2012) (dashed line).

the following considerations. At very small $\theta$, the magnetic field obtained in the numerical simulations diverges, most likely due to the peculiarities of the numerical method. On the other hand, at large $\theta$, the assumption that the magnetic field is vertical will break down. As we see, the proposed model is in excellent agreement with the numerical simulations. Note that $\Omega_F$ is negative near the axis, which is also in agreement with the work by McKinney et al. (2012).

4. CONCLUSIONS

We investigated a new analytical model of the black hole magnetosphere based on a previously unconsidered geometry of magnetic surfaces: a radial magnetic field near the horizon and a vertical field far from the black hole. We showed that in the presence of a dense core near the jet axis, there is excellent agreement of this model with the numerical simulations. And this is despite the fact that the analytical calculations were performed within the simplest force-free approximation and under the assumption that the flow was axisymmetric and stationary, while McKinney et al. (2012) carried out their 3D numerical simulations in the
full MHD version by taking into account the fact that the flows under consideration were nonstationary.

We emphasize that the negative values of the angular velocity $\Omega_F$ are most likely associated with the difficulties of the numerical procedure near the rotation axis. Therefore, actually one should not expect the appearance of a region with counter-rotation near the jet axis. For us it was important here to only show that given the magnetic field structure near the black hole horizon and in the jet region, the angular velocity profile obtained in a self-consistent way could be reproduced using a simple analytical model.

Good agreement between the theory and numerical simulations once again shows that axisymmetric stationary flows, for which quite a few analytical results have been obtained in the last three decades, remain a good basis for analyzing the processes occurring in real astrophysical sources. One of such properties is that despite the turbulent nature of the flow in the region above the accretion disk, the flow near the rotation axis remains fairly regular. Therefore, there is hope that the previously formulated simple analytical asymptotics (and, in particular, the assertion that the magnetic field structure near the horizon should be nearly radial) will also be needed in the future.

Finally, note that a parabolic field such that a significant fraction of the magnetic field lines cross the equator within the ergosphere would require the existence of an energy source directly in the accretion disk. In the steady-state problem, such a situation is unlikely to be possible. In our view, the fact that the 3D simulations discussed above lead to a quasiparabolic structure of magnetic surfaces is related to a fairly high external pressure. As a result, the jet radius exceeds the black hole horizon radius only by several times. In fact, however, as can be clearly seen from the structure of the innermost magnetospheric regions in McKinney et al. (2012), only a very small fraction of the magnetic field lines pass through the equator.

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