GORENSTEIN VON NEUMANN REGULAR RINGS

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Abstract. In this paper, we study the rings with zero Gorenstein weak dimensions, which we call them Gorenstein Von Neumann regular rings.

1. Introduction

Throughout this paper, all rings are commutative with identity element, and all modules are unital.

Let $R$ be a ring, and let $M$ be an $R$-module. As usual we use $\text{pd}_R(M)$, $\text{id}_R(M)$ and $\text{fd}_R(M)$ to denote, respectively, the classical projective, injective and flat dimensions of $M$. By $\text{gldim}(R)$ and $\text{wdim}(R)$ we denote, respectively, the classical global and weak dimensions of $R$. It is by now a well-established fact that even if $R$ is non-Noetherian, there exists Gorenstein projective, injective and flat dimensions of $M$, which are usually denoted by $\text{Gpd}_R(M)$, $\text{Gid}_R(M)$ and $\text{Gfd}_R(M)$, respectively. Some references are [2, 3, 6, 7, 10, 11, 13, 16].

Recently in [3], the authors started the study of global Gorenstein projective (resp. global Gorenstein injective, global weak) dimensions of $R$, denoted by $\text{Gpd}(R)$ (resp. $\text{Gid}(R)$, $\text{Gwdim}(R)$) and defined as the supremum of Gorenstein projective (resp. Gorenstein injective, Gorenstein flat) dimensions of $M$ where $M$ ranges over all $R$-modules.

It is known that for any ring $R$, $\text{Gwdim}(R) \leq \text{Gid}(R) = \text{Gpd}(R)$, see [3, Theorems 1.1 and Corollary 1.2(1)]. So, according to the terminology of the classical theory of homological dimensions of rings, the common value of $\text{Gpd}(R)$ and $\text{Gid}(R)$ is called Gorenstein global dimension of $R$, and denoted by $G.gldim(R)$. The Gorenstein global (resp. weak) dimension is refinement of the classical global (resp. weak) dimension of rings. That is $G.gldim(R) \leq \text{gldim}(R)$ (resp. $G.gldim(R) \leq \text{wdim}(R)$) with equality holds if $\text{gldim}(R)$ (resp. $\text{wdim}(R)$) is finite, see [3, Corollary 1.2(2 and 3)]).

The rings with zero global Gorenstein dimension (i.e. every module is Gorenstein projective module) are studied in [4], where the authors called them Gorenstein semisimple.

In this paper, motivating by the work in [4], we study the rings with zero Gorenstein weak dimension. Analogy to the classical ones we call them Gorenstein Von Neumann regular rings.

It is known result that in a Von Neumann regular ring, every finitely generated projective submodule of a projective $R$-module $P$ is a direct summand of $P$. One
of the result in this paper is an analog of this classical one. It is shown that \( R \) is Gorenstein Von Neumann regular ring if and only if \( R \) is Gorenstein semihereditary ring that every finitely generated projective submodule of a projective \( R \)-module \( P \) is a direct summand of \( P \) (recall that a ring \( R \) is called to be Gorenstein semihereditary if it is coherent and every submodule of a projective module is Gorenstein projective).

It is known that a semisimple ring is exactly a Noetherian Von Neumann regular ring. The second main result of this paper is an analog of this classical one. We show that \( R \) is Gorenstein semisimple ring if and only if \( R \) is Noetherian Gorenstein Von Neumann regular.

Above we have only mentioned the Gorenstein Von Neumann regular rings. One can also define the strongly Gorenstein Von Neumann regular rings by using the notions of strongly Gorenstein global dimension, cf. [2]. All the results concerning Gorenstein Von Neumann regular rings, have a strongly Gorenstein Von Neumann regular part. We do not state or prove these strongly Gorenstein Von Neumann regular rings results. This is left to the reader.

2. MAIN RESULTS

**Definitions 2.1.** A ring \( R \) is called Gorenstein Von Neumann regular (\( G \)-VNR for short) if every \( R \)-module is \( G \)-flat (i.e., \( G - \text{wdim}(R) = 0 \)).

**Remarks 2.2.**
(1) From [3, Corollary 1.2(1)], we can see that every \( G \)-semisimple ring is \( G \)-Von Neumann regular with equivalence if the ring is Noetherian ([12, Theorem 1.2.3.1]).

(2) Every Von Neumann regular ring is \( G \)-Von Neumann regular with equivalence if \( \text{wdim}(-) \) is finite ([3, Corollary 1.2]).

**Definition 2.3.** ([18] and [14]) Let \( R \) be a ring and \( M \) an \( R \)-module.

(1) We say that \( M \) is \( FP \)-injective (or absolutely pure) if \( \text{Ext}^{1}_{R}(P, M) = 0 \) for every finitely presented \( R \)-module \( P \).

(2) \( R \) is said to be a weakly quasi-Frobenius ring (or also \( FC \)-ring) if it is coherent and it is self-\( FP \)-injective (i.e., \( R \) is \( FP \)-injective as an \( R \)-module). \( R \) is called IF-ring if every injective \( R \)-module is flat [8]. Over a commutative rings, the IF-rings and the weakly quasi-Frobenius rings are the same ([15, Proposition 4.2]).

**Lemma 2.4.** The following are equivalent:

(1) Every injective \( R \)-module is flat (i.e., IF-ring).

(2) Every Gorenstein injective \( R \)-module is Gorenstein flat (i.e., GIF ring).

**Proof.** 1 \( \Rightarrow \) 2. Consider an arbitrary complete injective resolution:

\[ I : \ldots \rightarrow I_{1} \rightarrow I_{0} \rightarrow I^{0} \rightarrow I^{1} \rightarrow \ldots \]

Since every injective module is flat, \( I \) is also an exact flat resolution. Moreover, for every injective \( J, I \otimes_{R} J \) still exact (since \( J \) is also flat). Thus, \( I \) is a complete flat resolution. Then, every Gorenstein injective \( R \)-module is Gorenstein flat, as desired.

2 \( \Rightarrow \) 1. Let \( I \) be an arbitrary injective \( R \)-module (and so Gorenstein injective).

\[ I : \ldots \rightarrow I_{1} \rightarrow I_{0} \rightarrow I^{0} \rightarrow I^{1} \rightarrow \ldots \]
Then, by hypothesis, $I$ is Gorenstein flat. Thus, it can be embedded in a flat module $F$. So, $I$ is a direct summand of $F$ and so flat, as desired. □

**Proposition 2.5.** Let $R$ be a coherent commutative ring. The following are equivalent:

1. $\text{wGgldim}(R) \leq n$.
2. $\text{fd}_R(I) \leq n$ for every injective $R$-module $I$ (ie, n-IF).
3. $\text{Gfd}_R(I) \leq n$ for every Gorenstein injective $R$-module $I$ (ie., n-GIF).

**Proof.** The equivalence (1 \iff 2) follows by combining the equivalence [9, Theorem 7(1 \iff 2)] and the equality [5, Theorem 3.7(1=2)].

1 \implies 3. Obvious.

3 \implies 2. Let $I$ be an arbitrary injective $R$-module. By hypothesis, $\text{Gfd}_R(I) \leq n$. Thus, from [10, Theorem 2.10], there is an exact sequence $0 \to K \to G \to I \to 0$ where $G$ is Gorenstein flat and $\text{fd}_R(K) \leq n - 1$. For the $R$ module $G$ we can pick an short exact sequence $0 \to G \to F \to G' \to 0$ where $F$ is flat and $G'$ is Gorenstein flat. We have the following pushout diagram:

$$
\begin{array}{c}
0 & 0 \\
\downarrow & \downarrow \\
K = K & \\
\downarrow & \downarrow \\
0 & G & F & G' & 0 \\
\downarrow & \downarrow & | & || \\
0 & I & D & G' & 0 \\
\downarrow & \downarrow & \\
0 & 0
\end{array}
$$

Clearly, $\text{fd}_R(D) \leq n$. On the other hand, $I$ is a direct summand of $D$. Thus, $\text{fd}_R(I) \leq n$ as desired. □

**Theorem 2.6.** The following conditions are equivalent:

1. $R$ is $G$-Von Neumann regular.
2. Every finitely presented $R$-module is $G$-flat.
3. Every finitely presented $R$-module is $G$-projective.
4. Every finitely presented $R$-module embeds in a flat $R$-module.
5. $R$ is coherent and self-FP-injective ring (i.e., weakly quasi-Frobenius).
6. Every injective $R$-module is flat (i.e., $R$ is IF-ring).

**Proof.** The equivalences (1 \iff 2 \iff 3) is an other way to see [5, Theorem 6].

1 \implies 4. Easy by definition of a Gorenstein flat modules.

4 \iff 5 \iff 6. By [13, Proposition 2.5 and Theorem 2.8].

4 \implies 1. For any $R$-module $M$, assemble any flat resolution of $M$ with its any injective resolution, we get an exact sequence of flat $R$-modules (by hypothesis), which is also by hypothesis a complete flat resolution. This means that $M$ is Gorenstein flat. □

**Remark 2.7.** From Theorem 2.6, we deduce that a $G$-Von Neumann regular ring is always coherent.

**Proposition 2.8.** If $R$ is a coherent ring. Then, the following conditions are equivalent:
(1) \( R \) is a G-Von Neumann regular ring.
(2) For every finitely generated ideal \( I \), the module \( R/I \) is G-flat.
(3) Every injective \( R \)-module is flat.
(4) Every flat \( R \)-module is FP-injective.
(5) Every FP-injective \( R \)-module is flat.

Proof. 1 \( \Rightarrow \) 2. Easy.
2 \( \Rightarrow \) 3. Follows from \[16\] Theorem 3.4 and \[15\] Theorem 1.3.8.
3 \( \Rightarrow \) 4 \( \Rightarrow \) 5 \( \Rightarrow \) 1. Follows from \[9\] Theorem 3.5 and 3.8] and Theorem 2.6.

Recall that a ring \( R \) is called to be Gorenstein semihereditary if it is coherent and every submodule of a projective module is Gorenstein projective. In other words, the ring \( R \) is Gorenstein semihereditary if \( R \) is coherent and \( G.wdim(R) \leq 1 \) (\[17\]).

The following two properties “every finitely generated proper ideal of \( R \) has nonzero annihilator” and “every finitely generated projective submodule of a projective \( R \)-module \( P \) is a direct summand of \( P' \) are equivalent by \[1\] Theorem 5.4]. For short, we call such ring a CH-ring. It is known that a Von Neumann regular ring is exactly a semihereditary ring which is a CH-ring. The first main result of this paper is an analog of this classical one.

**Theorem 2.9.** The following statements are equivalents:

1. \( R \) is Gorenstein Von Neumann regular;
2. \( R \) is Gorenstein semihereditary which is CH-ring.

The proof of the theorem involves the following Lemma which is a new characterization of a CH-ring.

**Lemma 2.10.** Let \( R \) be a ring. The following conditions are equivalents:

1. Every finitely generated projective submodule of a G-projective \( R \)-module is a direct summand.
2. \( R \) is a CH-ring.

Proof. 1 \( \Rightarrow \) 2 Obvious since every projective module is SG-projective and every SG-projective module is G-projective.
2 \( \Rightarrow \) 1 Assume that \( R \) satisfied the (CH)-property and let \( P \) be a finitely presented projective submodule of a G-projective module \( G \). We claim that \( P \) is a direct summand of \( G \). Pick a short exact sequence \( 0 \rightarrow G 
\rightarrow Q 
\rightarrow G' 
\rightarrow 0 \) where \( Q \) is a projective \( R \)-module and \( G' \) is a G-projective \( R \)-module. Thus, we have the following pull-back diagram with exact rows and columns:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
P &=& P \\
\downarrow & & \downarrow \\
0 & \rightarrow & G 
\rightarrow Q 
\rightarrow G' 
\rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & G/P 
\rightarrow K 
\rightarrow G' 
\rightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

From the middle vertical sequence we conclude that \( P \) is isomorphic to a direct summand of \( Q \) (since \( R \) is a CH-ring). So, \( K \) is also isomorphic to a direct summand of \( Q \). Consequently, it is projective. This implies, from \[10\] Theorem 2.5], that
$G/P$ is $G$-projective and so, $\text{Ext} (G/P, P) = 0$. Therefore, the short exact sequence $0 \rightarrow P \rightarrow G \rightarrow G/P \rightarrow 0$ splits, that means that $G = P \oplus G/P$, as desired. \hfill \Box

Proof of Theorem 2.9
Firstly assume that $R$ is a Gorenstein Von Neumann regular ring. Clearly, $R$ is Gorenstein semihereditary. So, we have to prove that $R$ satisfied the $(CH)$-property. Let $P$ be a finitely generated projective submodule of a projective module $Q$. We claim that $P$ is a direct summand of $Q$. Let $Q'$ be a projective $R$-module such that $Q \oplus Q'$ is a free $R$-module. We have the short exact sequence

$$0 \rightarrow P \rightarrow Q \oplus Q' \rightarrow Q/P \oplus Q' \rightarrow 0$$

We can identify $P$ to be a submodule of $L = Q \oplus Q'$. Now, let $L_0$ be a finitely generated free direct summand of $L$ such that $P \subset L_0$ (that exists since $P$ is finitely generated submodule of $L$) and let $L_1$ be a free module such that $L = L_0 \oplus L_1$. The $R$-module $L_0/P$ is finitely presented and so it is $G$-projective by Theorem 2.6. Therefore, $L_0/P$ is projective by [16, Proposition 2.27] (since $\text{pd}_R(L_0/P) \leq 1$).

Now, we consider the following pull-back diagram with exact rows and columns:

$$
\begin{array}{cccccc}
0 & \rightarrow & P & \rightarrow & Q \oplus Q' & \rightarrow & Q/P \oplus Q' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L_0 & \rightarrow & L_0/P & \rightarrow & 0 \\
\parallel & & \downarrow & & \downarrow & & \\
0 & \rightarrow & P & \rightarrow & L & \rightarrow & Q/P \oplus Q' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_1 & = & L_1 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 \\
\end{array}
$$

From the right exact sequence we deduce that $Q/P \oplus Q'$ is projective (since $L_0/P$ and $L_1$ are projective). Then, $Q/P$ is projective and so $Q = Q/P \oplus P$, as desired. Conversely, assume that $R$ is a $G$-semihereditary ring which satisfied the $(CH)$-property. From Theorem 2.6 to prove that $R$ is $G$-Von Neumann regular, we have to prove that every finitely presented $R$-module is $G$-projective. So, let $M$ be a finitely presented $R$-module and pick a short exact sequence of $R$-modules $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$, where $P$ is a finitely generated projective $R$-module and $K$ is a finitely generated $R$-module. By [5, Theorem 7], the $R$-module $K$ is $G$-projective (since $R$ is $G$-semihereditary then is also coherent). Then, there is an exact sequence of $R$-modules $0 \rightarrow K \rightarrow Q \rightarrow K' \rightarrow 0$ where $Q$ is projective and $K'$ is $G$-projective. Now, consider the pull-back diagram with exact rows and columns:

$$
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & P & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \parallel & & \\
0 & \rightarrow & Q & \rightarrow & Z & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K' & = & K' \\
\end{array}
$$
The middle vertical short exact sequence $0 \to P \to Z \to K' \to 0$ splits since $K'$ is $G$-projective and $P$ is projective (by [16, Theorem 2.20]). Then $Z$ is $G$-projective ($Z = P \oplus K'$). Now, from Lemma 2.10, $Q$ is isomorphic to a direct summand of $Z$ (since $R$ satisfies the $(CH)$-property). Then, $M$ is also isomorphic to a direct summand of $Z$ and so $M$ is $G$-projective by [16, Theorem 2.5], as desired.

The strongly special cases is from the fact that the $G$-semihereditary ring (resp. the $G$-Von Neumann regular ring) is $SG$-semihereditary ring (resp. the $SG$-Von Neumann regular ring) if, and only if, every $G$-flat module is $SG$-flat. □

We know that a semisimple ring is exactly a Noetherian Von Neumann regular ring. The second main result of this paper is an analog of this classical one.

**Proposition 2.11.** Let $R$ be a ring. The following statement are equivalents.

1. $R$ is Gorenstein semisimple;
2. $R$ is Noetherian Gorenstein Von Neumann regular.

**Proof.** From [3, Proposition 2.6], the quasi Frobenius rings and the Gorenstein semisimple rings are the same; then, they are Noetherian. On the other hand, by [12, Theorem 12.3.1], if $R$ is Noetherian we have $G.wdim(R) = G.gldim(R)$. Thus, $R$ is Gorenstein semisimple if, and only if, $R$ is a Noetherian Gorenstein Von Neumann regular.

So, to finish the proof we have to prove the strongly particular cases.

Assume that $R$ is Gorenstein semisimple and we claim that $R$ is Gorenstein Von Neumann regular. Let $M$ be an arbitrary $R$-module. We claim that $M$ is Gorenstein flat. From above, $R$ is Gorenstein Von Neumann regular and so $M$ is Gorenstein flat; hence, for any injective module $I$, $Tor(M, I) = 0$. On the other hand since $R$ is Gorenstein semisimple, $M$ is Gorenstein projective and so, from [2, Proposition 2.9], there is an exact sequence $0 \to M \to P \to M \to 0$ where $P$ is projective (then flat). Consequently, by [2, Proposition 3.6], $M$ is Gorenstein flat, as desired.

Conversely, assume that $M$ is a Noetherian Gorenstein Von Neumann regular ring and let $M$ an arbitrary $R$-module. We claim that $M$ is Gorenstein projective. From the first part of the proof, $R$ is Gorenstein semisimple ring. Thus, $M$ is Gorenstein projective and so for any projective module $P$ we have $Ext(M, P) = 0$. On the other hand, $M$ is Gorenstein flat module and so, from [2, Proposition 3.6], there is an exact sequence $0 \to M \to F \to M \to 0$ where $F$ is flat. Using [3, Corollary 2.7] and the fact that $R$ is Gorenstein semisimple, $F$ is also projective. Hence, $M$ is Gorenstein projective module ([2, Proposition 2.9]), as desired. □

The final result of this paper is an analog of the classical one which say that a Von Neumann regular domain is a field.

**Proposition 2.12.** Every Gorenstein Von Neumann regular domain is a field.

**Proof.** Assume that $R$ is a Gorenstein Von Neumann regular domain. We claim that $R$ is a field. So, let $x$ be a nonzero element of $R$. We have to prove that $x$ is invertible. Clearly, $pd(R/xR) \leq 1$ since $R$ is a domain. On the other hand, from Proposition 2.6, $R/xR$ is $G$-projective since it is finitely presented and $R$ is Gorenstein Von Neumann regular. Therefore, from [16, Proposition 2.27], $R/xR$ is projective. Thus, the short exact sequence $0 \to xR \to R \to R/xR \to 0$ splits and so $R \cong xR \oplus R/xR$. Localizing this isomorphism with a maximal ideal $\mathcal{M}$ of $R$, we obtain $R_{\mathcal{M}} \cong (x/1)R_{\mathcal{M}} \oplus (R/xR)_{\mathcal{M}}$. But, $(x/1)R_{\mathcal{M}} \neq \{0\}$ since $R$ is a domain.
So, \((R/xR)_M = \{0\}\) for every maximal ideal \(M\) since \(R_M\) is local. This implies that \(R/xR = 0\) which means that \(x\) is invertible, as desired. \(\square\)

**Remark 2.13.** A local G-Von Neumann regular (or \(G\)-semisimple) ring is not necessarily a field. For example, we can take \(R = K[X]/(X^2)\) which is a local \(G\)-semisimple (then G-Von Neumann regular) ring but not a field.

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