Operational path–phase complementarity in single-photon interferometry

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Abstract
We examine two set-ups that reveal different operational implications of path–phase complementarity for single photons in a Mach–Zehnder interferometer (MZI). In both set-ups, the which-way (WW) information is recorded in the polarization state of the photon serving as a ‘flying which-way detector’. In the ‘predictive’ variant, using a fixed initial state, one obtains duality relation between the probability to correctly predict the outcome of either a which-way (WW) or which-phase (WP) measurement (equivalent to the conventional path-distinguishability–visibility). In this set-up, only one or the other (WW or WP) prediction has operational meaning in a single experiment. In the second, ‘retrodictive’ protocol, the initial state is secretly selected for each photon by one party, Alice, among a set of initial states which may differ in the amplitudes and phases of the photon in each arm of the MZI. The goal of the other party, Bob, is to retrodict the initial state by measurements on the photon. Here, a similar duality relation between WP and WW probabilities governs their simultaneous guesses in each experimental run.

1. Introduction
The operational content of quantum complementarity is that the uncertainties in the outcomes of measurements corresponding to complementary (non-commuting) observables (on identically prepared ensembles) necessarily have a tradeoff, i.e., satisfy an uncertainty relation. If one tries to measure two complementary observables on a single system, the measurement of one observable ‘disturbs’ the complementary one, i.e., introduces uncertainty in it.

For example, in the simplest system, with a Hilbert space of dimensionality 2, the measurement of one Pauli operator, say $\sigma_z$, the which-slit observable, ‘disturbs’ the ability of the two beams to interfere at a relative phase as measured by a combination of $\sigma_x$ and $\sigma_y$. The quantification of such disturbances has led to the complementarity or duality relation [1–7] for path predictability versus fringe visibility of a particle in a balanced Mach–Zehnder interferometer (MZI) with a partly efficient which-way detector (figure 1(a)). This relation reads

$$D^2 + V^2 \leq 1.$$

Here the path distinguishability, $D$, is related to the which-way (WW) probability, $P_{WW}$, of guessing the path correctly for a known input state and a WW detector of efficiency (reliability) $E \leq 1$. The fringe visibility, $V$, is related to the which-phase (WP) probability, $P_{WP}$, of guessing correctly which MZI port the particle will exit through (for an optimal choice of the phase between the arms) [8, 9]:

$$P_{WW} = \frac{1 + D}{2}, \quad P_{WP} = \frac{1 + V}{2}. \quad (2)$$

Yet, in this set-up, the WW and WP probabilities refer to two alternative measurements [10]. Indeed, $P_{WP}$ is our probability of predicting correctly where the particle will exit (figure 1(a)). By contrast, $P_{WW}$ is operationally meaningful only in a measurement (figure 1(b)) where the exit beam splitter of the MZI is removed, because only then can the readout of the partly efficient WW detector be verified. Thus, in the scheme of figure 1 our simultaneous guesses of path and phase cannot be verified or falsified in the same predictive experiment. Rather, the duality relation in equations (1) and (2) describes a trade-off between the predictabilities of two different experiments.

In the WP experiment, we may think of the WW detector as counterfactually ‘predicting’ what would have occurred in a WW experiment, had it been performed. However, counterfactual reasoning is notoriously problematic,
as commented by Asher Peres: ‘unperformed measurements have no results’.

Is it possible to obtain a duality relation for path and phase information which has a simultaneous operational meaning for each experimental run? Such a relation can indeed be given in the context of quantum state discrimination, namely, measurements aimed at optimally guessing the initial state out of a set of possible states [11, 12].

To this end, we formulate path–phase complementarity for state discrimination: the guessing by Bob which of the alternative input states had been prepared by Alice prior to the measurements Bob performed in a single experimental run. The WW and WP information is then retrodiction by Bob concerning Alice’s alternative input states rather than the standard predictions of possible outcomes of alternative measurements. Both retrodictive and predictive protocols may be implemented using single photons in an MZI, the photons themselves carrying the WW and WP information.

This paper is structured as follows. In section 2, we describe the proposed set-up and the standard predictive protocol followed by Bob and Alice in this set-up. Sections 3 and 4 describe two possible retrodictive protocols (using the same experimental set-up), and section 5 is devoted to conclusions.

2. Predictive duality in an MZI

2.1. Predictive duality relations for a particle in an MZI

In the standard formulation of the duality relation, Alice prepares the particle in a known initial state,

\[ |\text{in} \rangle = \sqrt{w_1}|A\rangle + e^{-i\phi}\sqrt{w_2}|B\rangle, \]  

where \(|A\rangle(|B\rangle\) denotes the state of being in arm \(A(B)\) of the MZI. It is preferable to use a balanced MZI to maximize the coherence. The (variable) phase delay element in arm \(B\) adds an additional relative phase of \(\phi\). The output beam splitter effects the transformation,

\[
\begin{aligned}
|A\rangle &\mapsto |+\rangle + |-\rangle \\sqrt{2} \\
|B\rangle &\mapsto |+\rangle - |-\rangle \\sqrt{2i},
\end{aligned}
\]  

where \(|\pm\rangle\) are states that correspond to exiting via the two output ports. The probability of the (ideal) photon detector in output port \(+\) to detect the particle, \(P_+\), is a function of the initial state prepared by Alice and the phase delay \(\phi\). If we repeat the experiment many times with the same initial state but varying \(\phi\), we can measure the ‘interference pattern’ \(P_+ (\phi)\).

One can then define the visibility of this ‘fringe pattern’ as

\[ V = \frac{\max(P_+) - \min(P_-)}{\max(P_+) + \min(P_-)}. \]  

In the absence of a WW detector, the only information we have on the probabilities to find the particle in one of the paths is given by the amplitudes in the initial state, or the corresponding weights \(w_1, w_2\). If \(w_1 \neq w_2\), there are unequal probabilities of finding the particle in either arm (if such a measurement is actually performed), while the visibility is reduced. The probability of correctly guessing in which arm the particle will be found is given as

\[ P_{WW} = \frac{1 + P}{2}, \]  

where \(P\) is the predictability:

\[ P = |w_1 - w_2|. \]  

The predictability satisfies a duality relation with the visibility [3]

\[ P^2 + V^2 \leq 1, \]  

which is just equation (1) with \(D = P\).

The effect of introducing an imperfect WW detector in one or both arms can be modelled as follows [5]. Let the initial (fiducial) state of the detector be denoted by \(|0\rangle_D\). Then, after the particle interacts with it, their joint state undergoes the transformation,

\[
\left( \sqrt{w_1}|A\rangle + e^{-i\phi}\sqrt{w_2}|B\rangle \right)|0\rangle_D \mapsto \sqrt{w_1}|A\rangle|D\rangle + e^{-i\phi}\sqrt{w_2}|B\rangle|D\rangle.
\]

where \(|a\rangle_D\) and \(|b\rangle_D\) are the detector ‘pointer’ states, which are not required to be orthogonal. More generally, the detector states can be allowed to be mixed \((\rho_{aD}^0, \rho_{bD}^0, \rho_{bD}^1)\). Without loss of generality, we shall assume a WW detector to be present in arm \(A\) only (as in figure 1).

When there is no \textit{a priori} bias toward either path \((P = 0)\), but there is a WW detector, equation (1) applies with

\[
D = E = \frac{1}{2} \text{Tr} |\rho_{aD}^0 - \rho_{bD}^1|. \]  

Here \(E\) is the detector efficiency[5] expressed as the trace norm of the difference of the two detector states (the absolute value of an operator, \(O\), is defined as \(|O| = \sqrt{O^\dagger O}\)). Equation (10) then gives the probability of correctly guessing the outcome.
of a WW measurement, if we (optimally) measure the state of the detector. For pure detector ‘pointer’ states, |a⟩, |b⟩, this takes the simpler form

\[ E = \sqrt{1 - |\langle a|b\rangle|^2}. \]  

(11)

In the general case, where the initial state has a priori which-way bias and there is a WW detector, equation (1) applies, with the path distinguishability, D, given by [8]:

\[ D = \text{Tr}[w_1|P_a|^2 - w_2|P_b|^2]. \]  

(12)

For pure ‘pointer’ states, D, is determined by P and E:

\[ D = \sqrt{P^2 + E^2 - E^2 P^2}. \]  

(13)

A few special cases deserve notice: for \( E = 0 \), we have \( D = P \); for \( P = 0 \), \( D = E \); and for \( E = 1 \) or \( P = 1 \), we have \( D = 1 \).

2.2. Interferometric set-up with a TIE detector

How can a limited-efficiency WW detector be experimentally implemented? In their experiments [6, 13] Dürr et al used the internal degrees of freedom of interfering atoms to store the WW information. Intraparticle translational–internal entanglement (TIE), proposed in [9, 14–16], provides a means of simultaneously encoding both WW and WP information in a ‘flying detector’, e.g., a photon in the MZI whose translational (path) and internal (polarization) states are entangled in a four-dimensional space. Recently a single-photon complementarity experiment has been carried out using polarization [7].

The simplest variant of TIE interferometry involves a particle (photon) whose orthogonal internal states |1⟩ and |2⟩ pick up a phase while traversing an MZI. In arm A the particle propagates unchanged (up to an overall phase \( \rho_0 \)), whereas in arm B, its internal states |1⟩, |2⟩ accrue a relative phase proportional to the path length (or in the present scheme, \( 2\beta \) due to the Faraday rotator):

\[ c_1|1⟩ + c_2|2⟩ \rightarrow e^{i\rho_0}c_1|1⟩ + e^{2i\beta}c_2|2⟩. \]  

(14)

The internal states serve as the detector ‘pointer’ states. Then equation (11) gives \( E = |c_1c_2\sin\beta| \).

As in [16], we here consider a photon as the TIE detector in the MZI, and its polarizations as the detector ‘pointer’ states. Specifically, we could place a Faraday rotator and phase delay in arm A of the MZI (figure 2). The Faraday rotator rotates the polarization plane through angle \( \beta \), while the phase delay is \( \theta \). This would conform to the general TIE scheme in equation (14) with \( |1(2)⟩ \) states corresponding to the right (left) circular polarization states (\( L(R) \)). For a linearly polarized input state (\( |c_1⟩ = |c_2⟩ = \frac{1}{\sqrt{2}} \)), this gives simply

\[ E = |\sin\beta|. \]  

(15)

3. Retrodictive (state-discrimination) duality in an MZI

3.1. Protocol

Alice randomly chooses, using the set-up in figure 2, to prepare the qubit in one of the four input states:

\[ |b_{ww}, b_{wp}\rangle_{a,\phi} \equiv T(b_{ww}\alpha)|A⟩ + e^{b_{wp}\phi}T(-b_{ww}\alpha)|B⟩. \]  

(16)

Here

\[ T(\pm b_{ww}\alpha) = \cos\left(\frac{\pi}{4} \pm \frac{b_{ww}\alpha}{2}\right) \]  

(17)

are the input amplitudes of |A⟩ and |B⟩, \( e^{b_{wp}\phi} \) being their relative phase factor; both are parameterized by

\[ b_{ww} = \pm 1, \quad b_{wp} = \pm 1. \]  

(18)

In an MZI, \( b_{ww}\alpha \) affects the bias for propagating along arm A versus arm B, i.e. having the photon in path state |A⟩ or |B⟩. This bias determines the which-way (WW) probability. The parameter \( b_{wp}\phi \) affects the relative phase of these states, i.e., the which-phase (WP) probability.

Figure 3 shows the Bloch representation of the set of input states with the identification of the path states |A, B⟩ as qubit states |σc = ±1⟩. Bob receives the qubit and after performing a measurement of his choice tries to guess the values of the two bits \( b_{ww}, b_{wp} \) (which are statistically independent), i.e., guess which of the four possible input states was chosen by Alice.
The two bits specifying Alice’s choice of preparation are classical variables. Similarly, the outcome of any measurements Bob chooses to make is given by classical variables. They are all operationally meaningful for each experimental run, and the task of guessing the initial state given the measurement results is a well-defined statistical problem. On the other hand, their joint probability distribution is determined by quantum mechanics, of course. Since Bob would like to maximize the information provided by his measurements on the initial state, he faces a problem of maximizing the classical information content of quantum measurements, similar to the concept of the classical information content of a quantum channel [17].

We look for the set of WW and WP probability pairs \( (P_{WW}, P_{WP}) \) that are optimal, in the sense that it is not possible to improve one of the probabilities (e.g., by changing the detector efficiency) without diminishing the other. We distinguish these ‘retrodictive’ probabilities from the ‘predictive’ \( (P_{WW}^*, P_{WP}^*) \) introduced in equation (2).

We now prove that for a WW detector with efficiency \( E \), the complementary retrodictive probabilities are related by

\[
\frac{2P_{WW} - 1}{d_{WW}^2} = E, \quad \frac{2P_{WP} - 1}{d_{WP}^2} = \sqrt{1 - E^2}, \tag{19a}
\]

where \( d_{WW} \) and \( d_{WP} \) are the trace distances between the appropriate input states in figure 3, constrained by

\[
d_{WW}^2 + d_{WP}^2 \leq 1. \tag{19b}
\]

3.2. Duality proof

In the TIE scheme of figure 2, Alice’s input state is, in the notation of equation (17),

\[
|b_{ww}, b_{wp}\rangle_{a,\phi} \equiv (T(b_{ww}\alpha)|A\rangle + e^{i\phi} T(-b_{ww}\alpha)|B\rangle)|b_{pol}\rangle,
\]

where \( |b\rangle \) is the initial polarization state, and as before, \( |A(B)\rangle \) are the states corresponding to the particle being in arm A (respectively B) of the MZI.

After interacting with the Faraday rotator, it maps to

\[
T(b_{ww}\alpha)|A\rangle|a\rangle_{pol} + e^{i\phi} T(-b_{ww}\alpha)|B\rangle|b\rangle_{pol}, \tag{21}
\]

where \( |a\rangle_{pol} \) and \( |b\rangle_{pol} \) are the polarization states correlated to the path states \( |A\rangle, |B\rangle \), respectively.

The measurement basis \( |a\rangle, |b\rangle \) that optimizes the distinguishability of the detector ‘pointer’ states is shown graphically in figure 4.

The measurement basis \( |a\rangle, |b\rangle \) of the second beam splitter (BS2 in figure 2) and the input states is

\[
|\pm\rangle_{port} = \frac{|A\rangle \pm i|B\rangle}{\sqrt{2}}. \tag{22}
\]

The joint input–output probabilities are given by

\[
P_{b_{ww}, b_{wp}, b_{ww}', b_{wp}'} = \left| \langle b_{ww}'|b_{wp}'|b_{ww}, b_{wp}\rangle \right|^2. \tag{26}
\]

These basis states correspond to the four output ports in figure 2.

The joint input–output probabilities are given by

\[
P_{b_{ww}, b_{wp}, b_{ww}', b_{wp}'} = \left| \langle b_{ww}'|b_{wp}'|b_{ww}, b_{wp}\rangle \right|^2. \tag{26}
\]

Explicitly (using equations (23)–(25))

\[
P_{b_{ww}, b_{wp}, b_{ww}', b_{wp}'} = \frac{1}{16} (1 + (-1)^{b_{ww}' - b_{wp}} E d_{ww} \tag{27}
\]

\[
+ (-1)^{b_{wp}' - b_{wp}} \sqrt{1 - E^2} d_{wp}^2)
\]
The WW and WP marginal probability distributions are
\[ P_{b_{ww},b_{wp}} = \sum_{b_{ww},b_{wp}} P_{b_{ww},b_{wp},b_{ww}',b_{wp}'} \]
\[ = \frac{1}{4} \left( 1 + (-1)^{b_{ww} - b_{wp}} E d_{ww} \right) \tag{28} \]
and similarly
\[ P_{b_{wp},b_{ww}} = \frac{1}{4} \left( 1 + (-1)^{b_{ww} - b_{wp}} \sqrt{1 - E^2} d_{wp} \right). \tag{29} \]
The probabilities of inferring the WW and WP input bits are
\[ P_{WW} = \sum_{b_{ww}} \max_{b_{wp}} P_{b_{ww},b_{wp}} = \frac{1}{2} \left( 1 + E d_{ww} \right), \tag{30} \]
\[ P_{WP} = \frac{1}{2} \left( 1 + \sqrt{1 - E^2} d_{wp} \right), \tag{31} \]
\[ d_{ww} \text{ being the trace distance between two input states with the same value of } b_{ww}, \text{ and different } b_{wp}, \text{ and conversely for } d_{wp}: \]
\[ d_{ww} \equiv d_{\text{trace}}(b_{ww} = +1, b_{wp}), \quad |b_{ww} = -1, b_{wp}), \]
\[ d_{wp} \equiv d_{\text{trace}}(|b_{ww}, b_{wp} = +1), \quad |b_{ww}, b_{wp} = -1). \tag{32} \]

This completes the proof of equation (19a) for the set of pairs \((P_{WW}, P_{WP})\) that one achieves in the MZI with the WW detector upon varying the detector efficiency, \(E\). It is shown in [20] that this equality is a bound on all possible measurement schemes, which in general satisfy the inequality
\[ \left( \frac{2P_{WW} - 1}{d_{ww}} \right)^2 + \left( \frac{2P_{WP} - 1}{d_{wp}} \right)^2 \leq 1. \tag{33} \]

In particular, the measurement procedure described here (and in section 2) defines a generalized measurement (POVM) [19] on the initial translational (path) state of the photon, in a Hilbert space of dimension 2. A Von Neumann (VN) measurement of this state would yield two possible outcomes, yet the present scheme has four possible measurement outcomes. This comes about by first performing a joint unitary operation on the translational state and the polarization state of the photon, and then performing a VN measurement on both.

The effect of the measurement on the translational (path) state alone is then
\[ \rho_{in} \mapsto \frac{K_i \rho_{in} K_i^\dagger}{Tr \{ K_i \rho_{in} K_i^\dagger \}} \quad i = 1, \ldots, 4, \tag{34} \]
where outcome \(i\) occurs with probability \(Tr\{K_i \rho_{in} K_i^\dagger\}\), and the four Kraus operators \(K_i\) are given by
\[ K_{e,\pm} = \frac{1}{2} |\pm\rangle \{ \sqrt{1 + E} (\pm i \sqrt{1 - E} A | 1/2) \} \tag{35a} \]
\[ K_{p,\pm} = \frac{1}{2} |\pm\rangle \{ \sqrt{1 - E} (\pm i \sqrt{1 + E} B | 1/2) \}. \tag{35b} \]

It is straightforward to verify that the output states are the same as the measurement basis states (equation (24)), and that the operators \(A_i \equiv K_i^\dagger K_i\) are positive (non-orthogonal) operators whose sum is the identity, they have the properties of a POVM [19].

4. Alternative path or phase retrodiction protocol

The retrodictive guessing game in section 3 is not, however, simply a time reversal of the conventional predictive one. To illustrate this, let us consider yet another ‘guessing game’ with alternative input states, which is much closer to a time-reversed version of the latter.

The predictability–visibility complementarity relation, equation (1), can be rewritten as
\[ (2P_{WW} - 1)^2 + (2P_{WP} - 1)^2 \leq 1, \tag{36} \]
using equation (2). This would be the same as our equation (33) if we had \(d_{ww} = d_{wp} = 1\). This, however, is not possible in our scenario, since it contradicts the additional constraint (19b). Hence, our retrodictive guessing game is not simply a time-reversed version of the conventional predictive scheme: different sets of states serve for measurements in the conventional scheme and in the present one (see figure 5).

Let us therefore formulate a game which is closer to a time reversal of the conventional predictive one, and calculate the path–phase probability constraints for it. Like
the conventional ‘predictive’ guessing game, in which each experimental run involves only one type of measurement (path or phase), the state preparation now involves either alternative paths or alternative phases in each experiment. Namely, in each run Alice first randomly chooses whether to prepare a which-way (WW) or a which-phase (WP) input state. This preparation basis is the same as Bob’s measurement basis in the conventional scheme (figure 5). Bob decides on his measurement strategy, ignorant of Alice’s choice. Once Bob has performed his measurement, Alice tells which basis she has used, and then Bob has to make his guess (as in the BB84 encryption scheme). Alice’s two preparation bases correspond to the degenerate cases: \( d_{ww}, d_{wp} \) equal to \((0, 0)\) or \((0, 1)\) for WW or WP, respectively.

Now let us assume that Bob has chosen, as in the conventional case, a certain detector efficiency \( E \). Then, for Alice’s WW preparation, equation (19) reduces to

\[
(2P_{ww}^\prime - 1)^2 = E^2 \tag{37}
\]

and for Alice’s WP preparation to

\[
(2P_{wp}^\prime - 1)^2 = 1 - E^2 \tag{38}
\]

(where the primes are used to distinguish this game from that discussed in section 3). These two equations together imply, of course,

\[
(2P_{ww}^\prime - 1)^2 + (2P_{wp}^\prime - 1)^2 = 1, \tag{39}
\]

which is formally the same as equation (36) with equality sign! Hence this type of retrodictive guessing game, unlike the one discussed in section 3, manifests the same tradeoff between path and phase guess probabilities as equation (1), derived for the conventional predictive guessing game.

However, for a fair comparison with equation (19a), we note that if the WW and WP bases are degenerate rectangles, we could at each run ask Bob to guess both path and phase. Then, when Alice has chosen the WW basis his probability of guessing WW correctly would be the same as above, and would be equal to \( \frac{1}{2} \) when she has chosen the WP basis (and conversely for Bob’s WP guesses). The properly averaged probabilities would then satisfy

\[
(2P_{ww}^\prime - 1)^2 + (2P_{wp}^\prime - 1)^2 = \left( \frac{1}{2} \right)^2. \tag{40}
\]

This is worse than our bound in equation (33), e.g., for the case where \( d_{ww} = d_{wp} = \frac{1}{\sqrt{2}} \).

5. Conclusions

We have examined two protocols that reveal different operational implications of path-phase complementarity for single photons in a Mach–Zehnder interferometer (MZI). Both protocols use a set-up where the which-way (WW) information is recorded in the polarization state of the photon serving as a ‘flying which-way detector’, by virtue of its translational–internal entanglement (TIE) [9, 14–16]. In the ‘predictive’ protocol, using a fixed initial state, one obtains duality relation between the probability to correctly predict the outcome of either a which-way (WW) or which-phase (WP) measurement (governed by the conventional duality of path-distinguishability and visibility). In this set-up, only one or the other (WW or WP) prediction has operational meaning in a single experiment. In this protocol, ‘retrodictive’ protocol, the initial state is secretly selected for each photon by one party. Alice, among a set of initial states which may differ in the amplitudes and phases of the photon in each arm of the MZI. The goal of the other party, Bob, is to retrodict the initial state by measurements on the photon. Here, a similar duality relation between WP and WW probabilities governs their simultaneous guesses in each experimental run.

The restatement of complementarity as a state discrimination problem opens the way to its information-theoretic formulation, to be discussed elsewhere. On the applied side, our guessing games may be the basis of quantum cryptographic schemes.

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