Square-root quantization: application to quantum black holes

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Introduction

In this short paper we present two different quantizations of the simple one-dimensional model - selfgravitating spherically symmetric thin shell. Fortunately enough, both ways of quantization lead to exactly solvable problems and we obtained mass spectra of the shell, analyzed them and extracted the quantum black hole spectra.

The first approach can be called a proper time quantization. By using it we obtain a finite difference Schroedinger equation. Its detailed description, the method of finding the general solution, the needed boundary conditions, the discrete spectrum of eigenvalues in the case of bound motion and the resulting quantum black hole mass spectrum were already published in [1]. Here we only pointed out the important steps that lead us to the discrete mass spectrum of quantum black holes.

Another approach can be called a Lorentzian time quantization. We show that there exists a suitable canonical transformation converting a famous square-root operator into an exponential operator. The resulting Schroedinger equation is, again, a finite difference equation of the same type as in the first approach. Then, repeating the steps we made in the first part of the paper we jump to the mass spectra of the shells and quantum black holes. The latter is similar to that found earlier. Part One As was shown in [2], the classical evolution of a selfgravitating spherically symmetric thin dust shell
is described completely by the single equation

\[ m = M \sqrt{\dot{\rho}^2 + 1 - \frac{GM^2}{2\rho}} \]  

(1)

where \( m \) is a total mass (energy) of the system, \( M \) is a bare mass of the shell, \( \rho(\tau) \) is its radius as a function of the shell’s proper time, and \( G \) is the gravitational constant.

Using Eqn.(1) as a pre-Hamiltonian we can introduce a momentum \( p \) corresponding to the proper time velocity \( \dot{\rho} \) and calculate the Hamiltonian

\[ H = M \left( \cosh p - \frac{\kappa M^2}{2x} \right) \]  

(2)

where \( x = M\rho \) is a dimensional radius. Such a Hamiltonian leads to the following Schroedinger equation in finite differences

\[ \Psi(x+i) + \Psi(x-i) = \left( 2\epsilon + \frac{\alpha}{x} \right) \Psi(x), \]  

(3)

where \( \epsilon = m/M, \alpha = GM^2 \).

The general solution to this equation in the momentum representation is

\[ \Psi_p = C \frac{z}{(z-z_0)(z-\overline{z}_0)} \left( \frac{z-\overline{z}_0}{z-z_0} \right)^\beta, \]

\[ z = e^p, \quad z_0 = e^{i\lambda}, \quad \overline{z}_0 = e^{-i\lambda}, \]

\[ \epsilon = \cos \lambda, \quad \alpha = GM^2 = 2\beta \sin \lambda. \]  

(4)

The general solution in the coordinate representation can be written in the form

\[ \Psi_{\text{gen}} = \Psi_\beta(x) \sum_{k=-\infty}^{\infty} c_k e^{-2\pi x}, \]  

(5)

\[ \Psi_\beta = (-4\pi \beta e^{-i\lambda} \sin \lambda)xe^{-\lambda x} \]

\[ F(1 - ix, 1 - \beta; 2; 1 - e^{-2i\lambda}) \]  

(6)

where \( F(a, b, c; z) \) is a Gauss’s hypergeometric function.
Due to the lack of space we will not discuss here the problem of boundary condition (for the details see [1]). The main goal of the present paper is to find a discrete mass spectrum of the bound states, and the desired spectrum comes not from the boundary conditions but from the analytical properties of the solutions. The result is

$$\beta = n, \quad n = 1, 2, \ldots$$

$$\epsilon = \sqrt{1 - \frac{\alpha^2}{4n^2}},$$

$$m = M \sqrt{1 - \frac{G^2 M^4}{4n^2}}. \quad (7)$$

It was argued in [1] that the above expression is a mass spectrum of the selfgravitating shells but not that of the black holes. To find the latter one we should look at the function \(m(M)\) more attentively. This function has two branches, the increasing and decreasing ones. The increasing branch describes the shells that do not collapse. The decreasing branch corresponds to the wormhole states. And only maxima of the curves \(m(M)\) for fixed quantum numbers \(n\) give us the black hole states. Thus, we have for the black hole mass spectrum

$$m_{BH} = \frac{2}{\sqrt{27}} \frac{\sqrt{n}}{G} \approx 0.9 \sqrt{n} M_{Pl}. \quad (8)$$

We will not discuss this formula here noting only that the same functional form was obtained by many authors, and J.Bekenstein was the first one [4].

**Part Two**

Let us now return to the second way of dealing with square-root kinetic terms.

If we will use the Lorentz time velocity \(v\) instead of the proper time velocity \(\dot{\rho}\) the pre-Hamiltonian for our shell take the more familiar form

$$m = \frac{M}{\sqrt{1 - v^2}} - \frac{GM^2}{2\rho} \quad (9)$$
Introducing now a momentum $\Pi$ corresponding to the Lorentz time velocity $v$ we arrive at the famous square-root Hamiltonian for the radial motion in the field of Coulomb potential

$$H = \sqrt{\Pi^2 + M^2} - \frac{\alpha}{2\rho}$$

(10)

The quantum analogs of the radius and the conjugate momentum are operators subject to the well known commutation relations. The kinetic part of the Hamiltonian is a nonlocal operator. To reveal this non-locality more explicitly we make the following canonical transformation before going to the quantization procedure.

$$\Pi = M \sinh p, \quad \rho = \frac{y}{M \cosh p}$$

(11)

The Hamiltonian (10) now becomes

$$H(y, p) = M \cosh p(1 - \frac{\alpha}{2y}).$$

(12)

The expression for the quantum counterpart depends on the chosen operator ordering. We will use the Hamiltonian (12) write the corresponding Schroedinger equation in the form

$$(y - \frac{\alpha}{2})(\Phi(y + i) + \Phi(y - i)) = 2\epsilon y \Phi(y),$$

$$\Phi = (1 - \frac{\alpha}{2y})\Psi.$$  

(13)

If we would choose the reverse ordering of momentum and coordinate functions we would get the same equation for $\Psi(y)$ instead of $\Phi(y)$. We will not discuss here the important question of hermiticity of the quantum Hamiltonian.

The solution to Eqn.(13) in the momentum representation is

$$\Phi_p = C \frac{z^{1+i\frac{\beta}{2}}}{(z - z_0)(\bar{z} - \bar{z}_0)} \left(\frac{z - \bar{z}_0}{z - z_0}\right)^\beta,$$

$$z = e^p, \quad z_0 = e^{i\lambda}, \quad \bar{z}_0 = e^{-i\lambda},$$

$$\epsilon = \cos \lambda, \quad \alpha = GM^2, \quad \beta = \frac{\alpha}{2} \cot \lambda.$$  

(14)
This expression differs from the solution only by the factor \( z^{i\alpha/2} \). But if we shift the argument in the corresponding solution in the coordinate representation \( y \rightarrow (y - \alpha/2) \) then the Fourier transform of this shifted function \( \tilde{\Phi}_p \) will be exactly the same as \( \Psi_p \) in Part One (Eqn.(4)). This means that the discrete spectrum in our second approach is determined by

\[
\beta = n, \quad n = 1, 2, \ldots
\]

\[
m = \frac{M}{\sqrt{1 + \frac{G^2M^4}{4n^2}}}
\]

(15)

This is the so called Sommerfeld spectrum which was obtained in [3] for the same classical model but using the Klein-Gordon Hamiltonian.

By the same procedure as in Part One we obtain the following black hole mass spectrum

\[
m_{BH} = \sqrt{n}M_{PL}.
\]

(16)

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