DEFINABLE DISCRETE SETS WITH LARGE CONTINUUM

DAVID SCHRITTESSER

Abstract. Let $R$ be a $\Sigma^1_1$ binary relation and call a set $R$-discrete iff no two distinct of its elements are $R$-related. We show that in the extension of $L$ by iterated Sacks forcing, there is a $\Delta^1_2$ maximal $R$-discrete set, and thus the existence of such sets is compatible with the negation of the continuum hypothesis. As an application we find a $\Pi^1_2$ maximal orthogonal family of Borel probability measures in said extension. The basis of this is a new Ramsey theoretic result.

1. Introduction

A. A map $c$ is a coloring of pairs from a set $X$ iff $c: [X]^2 \rightarrow \{0, 1\}$ where $[X]^2 = \{ \{x_0, x_1\} \mid x_0, x_1 \in X, x_0 \neq x_1 \}$ is the set of unordered pairs of $X$. A set $H \subseteq X$ is homogeneous for $c$ iff $c$ is constant on $[H]^2$. A major theme in Ramsey theory is to find homogeneous sets which are in some sense large assuming that $c$ is regular in some sense: A prime example is the following theorem due to Galvin (unpublished) that every Baire measurable coloring of an uncountable Polish space has a perfect homogeneous subset (see [12][19.6, p. 130]). This is easily seen to be equivalent to the following statement about Sacks forcing $S$:

Theorem 1.1 (Galvin’s Theorem for Sacks forcing). For any $p \in S$ and any Baire measurable coloring $c$ of pairs from $[p]$ there is $q \in S$ such that $q \leq p$ and $[q]$ is homogeneous for $c$.

Recall here that a condition $p$ in $S$ is just a perfect tree on $2 = \{0, 1\}$, conditions are ordered by inclusion, and that the branch sets $[p]$ of conditions $p \in S$,

$$[p] = \{ x \in 2^\omega \mid (\forall n \in \omega) x \upharpoonright n \in p \}$$

are exactly the perfect subspaces of Cantor space $C$ (see [2]).

Can Sacks forcing be replaced by iterated Sacks forcing in [11]? Using notation discussed just below, we provide an answer in the following theorem; we argue in Fact 3.20 and Example 3.21 below that this is optimal.

Theorem 1.2. For every $\bar{p}$ from a dense subset of $P$ and every $C$-universally Baire coloring $c$ of pairs from $[\bar{p}]$ there is $\bar{q} \in P$ such that $\bar{q} \leq \bar{p}$ and for each $\xi \in \supp(\bar{q})$, $c \upharpoonright \Delta_\xi$ can be extended to a continuous map on $[[\bar{q}]]^2$.

Above, $P$ denotes an iteration of Sacks forcing of arbitrary length. In §3 we associate to every $\bar{p}$ in a dense subset $D^K$ of $P$—the set of topologically determined conditions—a topological branch space $[\bar{p}]$. It will be a subspace of $\omega^{(\omega^2)}$, where $\omega$ denotes the length of the iteration $P$.

As is frequently the case in Ramsey theory, several obstructions to finding homogeneous sets have to be taken into account. Consider the following family of
colorings of pairs from $\lambda(\omega^2)$: for $\xi < \lambda$, let
\[
c_q(\bar{x}, \bar{y}) = \begin{cases} 1 & \iff \bar{x}(\xi) = \bar{y}(\xi), \\ 0 & \text{otherwise}, \end{cases}
\]
In Fact 3.20 we shall see that this family represents a fundamental obstruction to finding homogeneous branch sets, leading us to partition $\lambda(\omega^2)$:

**Definition 1.3.** Given $\bar{x}, \bar{y} \in \lambda(\omega^2)$, write $\Delta(\bar{x}, \bar{y})$ for the least $\xi < \lambda$ such that $\bar{x}(\xi) \neq \bar{y}(\xi)$. Moreover, for $\xi < \lambda$ let
\[
\Delta_\xi = \{ (\bar{x}, \bar{y}) \in [\lambda(\omega^2)]^2 \mid \Delta(\bar{x}, \bar{y}) = \xi \}.
\]

The class of $\mathcal{C}$-universally Baire sets \[\] is found to be the appropriate notion of ‘being regular’ (see the discussion after Fact 3.20). This is a well-behaved pointclass of Baire measurable sets which includes all analytic sets and more; see \[\].

**B.** We apply Theorem 1.2 to study definable maximal discrete sets. We use this term in the sense introduced by \[\].

**Definition 1.4.** Let $\mathcal{R}$ be a binary relation on a set $X$. A set $A \subseteq X$ is called $\mathcal{R}$-discrete iff $A$ contains no two distinct $\mathcal{R}$-related elements. By a maximal $\mathcal{R}$-discrete set we mean an $\mathcal{R}$-discrete set which is not a proper subset of any $\mathcal{R}$-discrete set.

Discrete sets are familiar from many contexts: e.g. in the context of graphs (i.e. symmetric irreflexive relations) as independent sets, or that of equivalence relations, as transversals. Another context in which maximal discrete sets are of interest is when $\mathcal{R}$ arises as a compatibility relation from a preorder: if $\preceq$ is a preorder, define the associated compatibility relation $\mathcal{R}_{\preceq}$ by
\[
x \mathcal{R}_{\preceq} y \iff (\exists z) z \preceq x \land z \preceq y.
\]
In these contexts, $\mathcal{R}_{\preceq}$ is always reflexive and symmetric and $\mathcal{R}_{\preceq}$-discrete sets are also called antichains.

Using the axiom of choice, one can find a maximal discrete sets exist for an arbitrary binary relation $\mathcal{R}$, but the existence of definable such sets may be contentious. In Gödel’s constructible universe $\mathbf{L}$, any $\Sigma^1_1$ (i.e. effectively analytic) binary relation admits a $\Delta^1_2$ maximal discrete set. On the other hand, consider the equivalence relation $E_0$ on $\mathcal{C}$, where $x E_0 y$ iff $y(n) = x(n)$ for all but finitely many $n$. It can be shown that no Baire measurable and hence no analytic or co-analytic maximal $E_0$-discrete set exists.

In this paper, we show that nevertheless, after forcing to add $\omega_2$-many Sacks reals to $\mathbf{L}$, we can still find maximal discrete sets which are definable without parameters:

**Theorem 1.5.** Let $\mathcal{R}$ be a $\Sigma^1_1$ binary relation on an effectively presented Polish space, and let $\mathcal{G}$ be generic for an iteration of Sacks forcing of length $\omega_2$. Then there is a $\Delta^1_2$ maximal $\mathcal{R}$-discrete set in $\mathbf{L}[\mathcal{G}]$.

In a previous joint work with A. Törnquist \[\], we proved the above for ordinary Sacks forcing (i.e. iterations of length 1). The more general Theorem 1.5 yields:

**Corollary 1.6.** The existence of $\Delta^1_2$ maximal discrete sets for $\Sigma^1_1$ binary relations is consistent with the negation of the continuum hypothesis.

Our main application concerns the compatibility relation associated to universal continuity of measures. Recall that if $\mu$ and $\mu$ are (non-trivial) measures on a measurable space $X$, then we write $\mu \ll \nu$ just in case every set which is null for $\nu$
is also null for $\mu$. Two measures $\mu$ and $\nu$ that are not compatible in $\ll$ are called orthogonal, written $\mu \perp \nu$. Orthogonal families of measures in the Polish space $P(X)$ of Borel probability measures on a Polish space $X$ show up in many different contexts including representation theory, ergodic theory and operator algebras.

Interest in the definability of maximal orthogonal families (or short, mofs) originated in the following question posed by Mauldin: If $X$ is a perfect Polish space, is there an analytic mof in $P(X)$? This was answered negatively by Preiss and Rataj [15] (Kechris and Sofronidis [11] later gave a proof based on Hjorth’s theory of turbulence). It was shown in [7] that on the other hand, in $L$ there is a $\Pi^1_1$ (i.e. effectively co-analytic) mof.

Maximality of an orthogonal family in $P(X)$ never persists when passing to an outer model with a new real (observation due to B. Miller, see §5), and if there are Cohen, Random or Mathias reals over $L$, there is no $\Delta^1_1$ (equivalently, no $\Sigma^1_1$) mof [6, 7, 20]. This makes it plausible that the existence of $\Pi^1_1$ mofs is essentially limited to $L$. In joint work with A. Törnquist, we showed this not to be the case [20]. From Theorem [14] and the work in [20] follows immediately the following strong version of this result:

**Theorem 1.7.** The existence of a $\Pi^1_1$ mof is consistent with the negation of the continuum hypothesis.

C. The paper is organized as follows. In §2 we fix some notation regarding functions and trees and review definitions and facts regarding the universally Baire sets, Ramsey theory and colorings, as well as fusion for Sacks forcing and its iteration.

We prove our Ramsey theoretic result in §3. To this end, we define the set $D^t$ of topologically determined conditions and the branch space $[\bar{p}]$ in §3.1. We then introduce the dense set of simple conditions $D^s \subseteq D^t$, prove that $D^s$ is dense in $P$ and at the same time continuous reading of names for $P$ in §3.2. Much of this material is in some sense implicit in [2] and [10] (again, also compare [14]).

We establish a weak connection between products and iterations of Sacks forcing in §3.3 and prove a Ramsey theoretic result (for meager relations and such products) in §3.4. Finally, we prove Theorem [1.2] in §3.5 in a three step argument: we show $C$-universal sets have an auxiliary property we call $Y^p$ measurability; we then show colorings of such sets can be made continuous in strong sense, and finally the theorem. All three steps take the form of a fusion argument.

§4 is devoted to our main result about maximal discrete sets Theorem [1.5] and §5 quickly states how to obtain Theorem [1.7] about mofs using methods from [7] and presents the aforementioned result due to B. Millers (included with his permission). We conclude with §6 listing a few questions which remain open.

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2. Preliminaries

If $s$ is a sequence (i.e. a function whose domain is an ordinal), we write $\text{lh}(s)$ for its length (said ordinal). Where $f: A \to B$ and $A_0 \subseteq A$ we write $\{f(x) \mid x \in A_0\}$ as $f[A_0]$ or as $f''A_0$ to avoid confusion with the branch space.

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*Determined conditions* as well as $[\bar{p}]$ for determined $\bar{p} \in P$ have been defined independently (and at least on the surface, differently) in [13], where also a result that follows from our Corollary [5.12] is proved. We thank A. W. Miller for calling this to our attention.
We write $A B$ for the set of functions from $A$ to $B$. Throughout, we shall freely identify $A(B C)$ with $A \times B C$ (in the sense that we leave it to the reader to insert the obvious identifying map when needed). For instance when $x \in A(B C)$ and $B_0 \subseteq B$ we write $x \upharpoonright A \times B_0$ and treat this as an element of $A(B_0 C)$. We write $\par_{<\omega}(A, B)$ for the set of finite partial functions from $A$ to $B$. We make the same identifications for partial functions.

Throughout this paper, let $\lambda$ be an ordinal. We also identify $\lambda \times \omega 2$ and $\lambda(\omega 2)$ as topological spaces (carrying the product topology). Given $s \in \par_{<\omega}(\lambda \times \omega, 2)$ (or $s \in \par_{<\omega}(\lambda, \par_{<\omega}(\omega, 2))$, identical in the above sense) we write

$$N_s = \{ x \in \lambda \times \omega 2 | \bar{s} \subseteq \bar{x} \}$$

for the basic open neighborhood in $\lambda(\omega 2)$ defined by $s$.

We write PTrees for the effective Polish space of perfect trees on $2$ (the subspace of $C$ consisting of characteristic functions of such trees; see $[12, 4.32]$).

If $X$ is a topological space with basis $B$ and $F \in C(X, \omega 2)$, we can consider as a code for $F$ the function $f : B \rightarrow \subset \omega 2$ defined by letting

$$f(b) = s \iff (\forall x \in b) F(x) \in N_s.$$  

We write $f^*$ for the unique $F$ which $f$ codes, if there is such $F$. Provided $X$ is an effective Polish space, being the the code of some $F \in C(X, \omega 2)$ is a $\Pi_1^1$ property (intuitively stating convergence everywhere; see $[12, 2.6]$).

In the cases arising in this article, $\{ b \in B | x \in b \}$ will be linearly ordered by $\subseteq$ for every $x \in X$ so that $f$ is a code iff

$$f^*(x) = \bigcup \{ f(b) | x \in b \in B \}$$

defines a total function $f^* : X \rightarrow \subset \omega 2$.

We now review basic facts concerning the pointclass of $\mathcal{C}$-universally Baire sets, where $\mathcal{C}$ denotes Cantor space $\omega 2$. For this purpose, let $X$ be any Polish space. For a (non-meager) topological space $\Omega$, $A \subseteq X$ is $\Omega$-universally Baire iff for every continuous $f: \Omega \rightarrow X$, $f^{-1}[A]$ has the property of Baire (in $\Omega$); a set $A \subseteq X$ is called universally Baire iff $X$ is $\Omega$-universally Baire for every compact Hausdorff space $\Omega$. The pointclass of universally Baire sets is a $\sigma$-algebra which is also closed under continuous pre-images. The same holds for the $\mathcal{C}$-universally Baire sets. Moreover, every absolutely $\Delta_1^1$ and thus every $\sigma(\Sigma_1^1)$ set is universally Baire, and thus $\mathcal{C}$-universally Baire. See $[3, 21]$ 10.110, p. 795 and $[8]$ for more details.

A map $g : X \rightarrow Y$, where $Y$ is a topological space, is said to be $\mathcal{C}$-universally Baire precisely if the pre-images of open sets under $g$ are $\mathcal{C}$-universally Baire.

We review some Ramsey theoretic terminology regarding colorings of pairs. Pairs will always be from a subspace $X$ of $\lambda(\omega 2)$. Let $<\bar{}$ denote the lexicographic ordering of $\lambda(\omega 2)$—by this we mean that $\bar{x} < \bar{y}$ holds iff for the least $\xi < \lambda$ such that $\bar{x}(\xi)$ differs from $\bar{y}(\xi)$ and for the least $n < \omega$ such that $\bar{x}(\xi)(n)$ differs from $\bar{y}(\xi)(n)$ we have $\bar{x}(\xi)(n) < \bar{y}(\xi)(n)$ (i.e. $\bar{x}(\xi)$ comes before $\bar{y}(\xi)$ in the lexicographic ordering of $\omega 2$). Let $\text{lex}_0(\{x_0, x_1\})$ denote the lexicographically smaller element of the unordered pair $\{x_0, x_1\} \in [X]^2$ and $\text{lex}_1(\{x_0, x_1\})$ the lexicographically greater element.

Equip $[X]^2$ with the initial topology with respect to the inclusion $\iota : [X]^2 \rightarrow X^2$, $\iota(z) = (\text{lex}_0(z), \text{lex}_1(z))$ (i.e. identify $[X]^2$ with the ‘lower half’ of $X^2$).

Note that it is in some cases more natural to work with $X^2$ directly. In this case, we talk about symmetric maps on $X^2$ instead of colorings of pairs from $X$: Consider a symmetric map $c^* : X^2 \rightarrow [0, 1]$. Any such $c^*$ clearly induces a map $c \circ \iota : [X]^2 \rightarrow [0, 1]$. Vice versa, if $c : [X]^2 \rightarrow [0, 1]$, consider the symmetric map $c^* : X^2 \rightarrow [0, 1]$.
defined by \( c^*(x_0, x_1) = c\{x_0, x_1\} \); let \( c^* \) take value 0 by convention on \( \text{diag}(X) \).

The reader is invited to check that Baire measurability and \( \Omega \)-universally Baire measurability is preserved by both of these translations. The same is not true for continuous colorings: a continuous coloring of pairs from \( X \) corresponds to a symmetric map which is continuous on \( X^2 \setminus \text{diag}(X) \). We shall also use the analogue of the partition from Definition 1.3 in the context of ordered pairs:

**Definition 2.1.** For \( \xi < \lambda \) let \( \Delta_\xi^c = \{(\bar{x}, \bar{y}) \in (\lambda^2)^2 \mid \Delta(\bar{x}, \bar{y}) = \xi\} \).

All terminology regarding trees and sequences not explicitly introduced below is taken from [12]. For a tree \( T \subseteq 2^{<\omega} \) and \( t \in 2^{<\omega} \),

- For \( a \in 2 \), \( t \upharpoonright a \) is the element \( t' \) of \( 2^{<\omega} \) such that \( t' \upharpoonright \text{lh}(t) = t \) and \( t'(\text{lh}(t)) = a \).
- \( t \) is a splitting node (of \( T \)) iff there are \( a, a' \in 2 \) such that \( a \neq a' \) and both \( t \upharpoonright a \) and \( t \upharpoonright a' \) are in \( T \);
- \( T \) is perfect iff every node in \( T \) can be extended to a splitting node of \( T \);
- Write \( T_t \) for the set \( \{s \in T \mid s \subseteq t \text{ or } t \subseteq s\} \).
- \([T]\) denotes \( \{x \in \omega^2 \mid (\forall n \in \omega) x \upharpoonright n \in T\} \), the set of branches through \( T \).
- For \( n > 0 \), we say \( t \) is an \( n \)th splitting node (of \( T \)) if and only if \( t \) is a splitting node of \( T \) with exactly \( n - 1 \) splitting nodes strictly below it.

We introduce our own notation for two concepts that we use frequently:

**Definition 2.2.**

- For \( n \in \omega \setminus \{0\} \), write \( (T)_n^* \) for the set of \( n \)th splitting nodes and \( (T)_{\leq n}^* \) for \( \{t' \in T \mid (\exists t) t' \subseteq t \wedge t \in (T)_n^* \} \). We also write \( (T)_0^* = \{\emptyset\} \) for consistency.
- For \( n \in \omega \), write \( (T)_n \) for the set \( t \in T \) such that \( t = t' \upharpoonright a \) for some \( a \in A \) and \( t' \in (T)_n^* \) (i.e. the immediate \( T \)-successors of \( n \)th splitting nodes) and write \( (T)_{\leq n} \) for \( \{t' \in T \mid (\exists t) t' \subseteq t \wedge t \in (T)_n \} \). We also write \( (T)_0 = \{\emptyset\} \) for consistency. When there is no danger of misinterpretation, we leave out the brackets and write \( T_n \) and \( T_{\leq n} \).

Note that if \( T \) and \( T' \) are perfect trees, then \( T \) and \( T' \) have the same \( n \)th splitting level if and only if \( (T)_{\leq n} = (T')_{\leq n} \) (i.e. they agree just past the \( n \)th splitting level).

Sacks forcing \( S \) is the set of perfect subtrees of \( 2^{<\omega} \), ordered by \( q \subseteq p \iff q \subseteq \bar{p} \). It admits a Fusion Lemma, which we review below using the following terminology:

**Definition 2.3.**

- For \( n \in \omega \) and \( p, q \in S \), define \( q \leq_n p \) to mean that \( q \leq p \) and \( q)_{\leq n} = (p)_{\leq n} \) (or equivalently, \( q)_{= n} = (p)_{= n} \).
- We say a sequence \( \langle p_n : n \in \omega \rangle \) of Sacks conditions is a fusion sequence iff for any \( m \in \omega \) there is \( n_0 \in \omega \) such that for \( (\forall n \geq n_0) p_n \leq_m p_n \).

**Lemma 2.4** (Fusion for \( S \)). Any fusion sequence of Sacks conditions has a lower bound in \( S \).

A second important fact about Sacks forcing is that it satisfies the property in the following lemma, often referred to as continuous reading of names. For a proof see e.g. [20, 3.3].

**Lemma 2.5** (Continuous reading of names for \( S \)). Let \( \dot{x} \) be a \( S \)-name for an element of \( \omega^\omega \) and let \( p \in S \). Then there is \( q \in S \) stronger than \( p \) and a continuous function \( F^* : [q] \rightarrow <\omega^\omega \) such that \( \dot{q} \Vdash S \ F^*(\dot{q}) = \dot{x} \).

Our main focus will be iterations of Sacks forcing. Throughout, let \( \lambda \) be an ordinal and let \( P \) be an iteration of Sacks forcing with countable support of length \( \lambda \).
We denote by $P_\xi$ the initial segment of length $\xi$. Recall that $P$ is the set of sequences $p: \lambda \to V_\kappa$ (where $\kappa$ is some fixed, large enough ordinal) such that for each $\xi \in \lambda$, $p(\xi)$ is a $\beta_\xi$-name which is forced by $1_{\beta_\xi}$ (the trivial condition) to be a Sacks condition. Thus if $\bar{p} \in P$, we have $\bar{p} | \xi \in P_\xi$. If $G$ is a filter on $P$, $G | \xi$ denotes $\{ \bar{p} | \xi : \bar{p} \in G \}$.

For such $G | \xi$ and any $\bar{p} \in P$, the ‘tail segment’ $\bar{p} | (\lambda \setminus \xi)$ determines a condition in $P^{G | \xi}$, where we identify $P^{G | \xi}$ with the countable support iteration of Sacks forcing of length $\text{otp}(\lambda \setminus \xi)$ in $V[G | \xi]$ (see [1]). We denote the natural name for this condition by $\bar{p}^T$, but shall sometimes omit the superscript $\xi$, as $\bar{p}$ can be naturally identified with a $P_\xi$-name for a condition in $P^{G | \xi}$.

Any filter $G$ which is $P$-generic over $V$ determines a sequence of length $\lambda$ of element of $\omega_2$, denoted by $s_G = (s_G(\xi): \xi < \lambda)$, such that for each $\xi < \lambda$, $s_G(\xi)$ is a Sacks real over $V[G \upharpoonright \xi]$. Vice versa, every such sequence gives rise to a $(V, P)$-generic filter.

Recall from [2] that whenever convenient, we consider $s_G$ to be an element of $\lambda \times \omega_2$ via the identification of $\lambda(\omega_2)$ with $\lambda \times \omega_2$.

Recall that like Sacks forcing, $P$ admits a Fusion Lemma:

**Definition 2.6.** We say a sequence $\langle p_n: n \in \omega \rangle$ of conditions in $P$ is a fusion sequence iff for any $\xi \in \bigcup_{n<\omega} \text{supp}(p_n)$ and any $m \in \omega$ there is $n_0 \in \omega$ such that for $n \geq n_0$, $p_n | \xi \models p_n(\xi) \leq m p_{n_0}(\xi)$.

**Lemma 2.7.** Any fusion sequence of conditions in $P$ has a lower bound in $P$.

Similar to [2] p. 274, given $\bar{p} \in P$ and $\bar{t} \in \text{par}(\lambda \times \omega, 2)$, we write $\bar{p}_\bar{t}$ for the sequence of names defined inductively such that for each $\xi < \lambda$,

$$(\bar{p}_\bar{t}) | \xi \models p_{\bar{t}(\xi)}(\xi) = \bar{p}(\xi)^T.$$ 

Note that it is not necessarily the case that $\bar{p}_\bar{t} \in P$. Therefore, we say $\bar{p}$ is accepts $\bar{t}$ precisely if $\bar{p}_\bar{t} \in P$, or equivalently, precisely if $(\forall \xi < \lambda) (\bar{p}_\bar{t}) | \xi \models \bar{t} \in \bar{p}(\xi)$. In the next [3,2] we will see that for $\bar{p} \in D^\kappa$ the $\bar{t}$ accepted by $\bar{p}$ essentially form a tree which we shall call $\text{init}(\bar{p})$ ($\bar{t}$ is accepted iff it can be extended into $\text{init}(\bar{p})$).

3. Ramsey theory of iterated Sacks forcing

3.1. Topologically determined conditions and their branch space. We now define the set $D^\kappa \subseteq P$ of topologically determined conditions: for $\bar{p} \in D^\kappa$, the branch space $|\bar{p}|$ can be usefully defined. That $\bar{p} \in D^\kappa$ intuitively means that $\bar{p} \in P$ is described simply by giving each $\bar{p}(\xi)$ as a continuous function of finitely many $s_G(\sigma_0), \ldots, s_G(\sigma_k)$ (where $G$ is $P$-generic). We will see in [3,2] that $D^\kappa$ is dense.

**Definition 3.1** (Topologically determined conditions).

(A) Given $\bar{p} \in P$ a standard enumeration of $\text{supp}(\bar{p})$ is a sequence

$$\Sigma = \langle \sigma_l | l < \alpha \rangle$$

where $\alpha \leq \omega$, $\{ \sigma_l | l < \alpha \} = \text{supp}(\bar{p})$ and $\sigma_0 = 0$.

(B) For $\bar{p} \in P$ and $\Sigma = \langle \sigma_l | l < \alpha \rangle$ we say that $F$ topologically determines $\bar{p}$ with respect to $\Sigma$, abbreviated by $\bar{p} \in D^\alpha_{\Sigma,F}$, iff $\Sigma$ is a standard enumeration of $\text{supp}(\bar{p})$ and $F$ is a sequence $F = \langle F_k | k < \alpha \setminus \{0\} \rangle$ of functions such that for each $k \in \omega \setminus \{0\}$ $F_k: X_k \to \text{PTREES}$ where

1. $X_k = \{ x: \{ \sigma_l | l < k \} \cap \sigma_k \to \omega_2 | (\forall l < k) \bar{x}(\sigma_l) \in [F_l(\bar{x} | \{\sigma_0, \ldots, \sigma_{l-1} \cap \sigma_l\})] \}$;

and for any $\bar{x} \in X_k$ we have

$$\bar{p} | \sigma_k \models F_k(\bar{s}_G | \{ \sigma_l | l < k \} \cap \sigma_k) = \bar{p}(\sigma_k).$$
(C) We write $\bar{p} \in D^k_\Sigma$ iff there exists $\bar{F}$ such that $\bar{p} \in D^k_{\Sigma,F}$ and we write $\bar{p} \in D^k$ iff there exists $\Sigma$ such that $\bar{p} \in D^k_\Sigma$. We say $\bar{p}$ is \textit{topologically determined} iff $\bar{p} \in D^k$.

Given $\bar{F}$, $\Sigma$ and $\bar{p} \in D^k_{\Sigma,F}$ we let for notational convenience $F_0$ be the function with $\text{dom}(F_0) = \{\emptyset\}$ defined by

$$F_0(\emptyset) = [\bar{p}(0)]$$

and we shall also write $\bar{p} \in D^k_{\Sigma,F}$ where $\bar{F} = \{F^k \mid k \in \omega\}$.

We can now define the branch space for $\bar{p} \in D^k$. Note that it depends on the choice of a particular standard enumeration and accordingly has a parameter $\Sigma$—nonetheless, as in many arguments $\Sigma$ remains the same for a given $\bar{p}$, we often will suppress its mention.

\textbf{Definition 3.2.} Given $\Sigma$ and $\bar{F}$ and $\bar{p} \in D^k_{\Sigma,F}$, define the \textit{branch space of $\bar{p}$ (with respect to $\Sigma$)} to be

$$[\bar{p}]_\Sigma = \{\bar{x} \in \text{supp}(\bar{p})(\omega) \mid \forall k \in \text{lh}(\Sigma) \; \bar{x}(\sigma_k) \in [F_k(\bar{x}(\{\sigma_0, \ldots, \sigma_{k-1}\} \cap \sigma_k))],$$

with the topology inherited as a subspace of $\omega^\omega$. We write $[\bar{p}]$ for $[\bar{p}]_\Sigma$ if $\Sigma$ can be inferred from the context.

Note that at $k = 0$, correctly interpreting $\bar{x} \upharpoonright \emptyset$ as the empty function $\emptyset$, the condition on the right just reads $\bar{x}(\sigma_0) \in [\bar{p}(0)]$.

The space $[\bar{p}]_\Sigma$ (for $\bar{p} \in D^k_{\Sigma}$) is closed in $\text{supp}(\bar{p})^{\omega}$, making it a perfect Polish compact 0-dimensional space; in fact, it is isomorphic to a space which is \textit{effectively} Polish relative to a parameter (see Definition 3.2). Furthermore, one can show $\{N_s \cap [\bar{p}]_\Sigma \mid \bar{p} \text{ accepts } s\}$ is a topological basis for $[\bar{p}]_\Sigma$ and $\bar{p} \Vdash s_G \in [\bar{p}]^{V[G]}$.

\textbf{3.2. A simpler topological description of iterated Sacks conditions.} We now define the set $D^s \subseteq D^k$ of \textit{simple conditions} for which $[\bar{p}]$ will be even easier to describe; in Lemma 3.4 we show that $D^s$ and hence $D^k$ is dense.

Just as a Sacks condition is a tree consisting of finite sequences from $\{0, 1\}$, we define a set $\text{init}(\bar{p}) \subseteq \text{par}_{<\omega}(\lambda \times \omega, 2)$ of ‘finite approximations’ to conditions $\bar{p} \in \mathbb{P}$, which will at the same time form a basis consisting of clopen sets for the topology of $[\bar{p}]$ whenever $\bar{p} \in D^k$. These finite approximations generalize the sets $(T)_n$; recall that $(T)_0 = \emptyset$ and $(T)_n$ consists of the successors of $n$th splitting nodes (i.e. $(T)_n$ has size $2^n$).

\textbf{Definition 3.3.} Given $n \in \omega$, $\bar{p} \in \mathbb{P}$ and a finite or infinite sequence $\Sigma = (\sigma_l \mid l < \alpha)$ of ordinals in $\text{supp}(\bar{p})$ such that $n < \alpha \leq \omega$ and $\sigma_0 = 0$, let

$$\text{init}^n(\bar{p}, \Sigma) = \{\bar{l} \mid \{\sigma_0, \ldots, \sigma_{n-1}\} \cap \sigma_n \rightarrow <\omega 2 \mid \forall l \leq n \; \bar{p}_l \upharpoonright \bar{l}(\sigma_l) \in \bar{p}(\sigma_l)_n\},$$

and for $k$ such that $0 < k \leq n$ let

$$\text{init}^n_k(\bar{p}, \Sigma) = \{\bar{l} \mid \{\sigma_0, \ldots, \sigma_{k-1}\} \cap \sigma_k \upharpoonright \bar{l} \in \text{init}^n(\bar{p}, \Sigma)\}.$$

We also let $\text{init}(\bar{p}, \Sigma) = \bigcup_{n \in \omega} \text{init}^n(\bar{p}, \Sigma)$. When $\Sigma$ can be inferred from the context unambiguously, we write $\text{init}^n(\bar{p})$, $\text{init}^n_k(\bar{p})$, and $\text{init}(\bar{p})$.

In the definition of $\text{init}^n(\bar{p})$, observe by induction on $l$ that $\bar{p}$ accepts $\bar{l} \upharpoonright \sigma_l$ so that the definition makes sense. Note that since we assume $\sigma_0 = 0$, for any $n$ and $k$ we have $\text{init}^n_0(\bar{p}, \Sigma) = \text{init}^n(\bar{p}, \Sigma) = \text{init}^0(\bar{p}, \Sigma) = \emptyset$; moreover, $\text{init}^n(\bar{p}, \Sigma) = \text{init}^n(\bar{p}, \Sigma)$, $\text{init}^n_k(\bar{p})$ only depends on $\Sigma \upharpoonright k$ and

$$\text{init}^n_k(\bar{p}, \Sigma) = \{\bar{l} \mid \{\sigma_0, \ldots, \sigma_{k-1}\} \cap \sigma_k \rightarrow <\omega 2 \mid \forall l < k \; \bar{p}_l \upharpoonright \bar{l}(\sigma_l) \in \bar{p}(\sigma_l)_n\}.$$
We will only be interested in $\init(\bar{p})$ when $\bar{p} \in D^\omega$; in fact, for arbitrary $\bar{n} \in \mathbb{P}$, $\init(\bar{n})$ may be uninteresting—for instance, it may well be that $\init(\bar{n})$ contains only the trivial sequence. On the other hand, it will be convenient to be able to talk about $\init(\bar{p}, \sigma_0, \ldots, \sigma_{k-1})$ for arbitrary $\bar{p}$ when we prove that $D^\omega$ is dense.

**Definition 3.4.**

1. For $\bar{p} \in \mathbb{P}$ and $\Sigma = (\sigma_l \mid l < \alpha)$ we say that $\bar{F}$ describes $\bar{p}$ simply, abbreviated by $\bar{p} \in D^\omega_{\Sigma, \bar{F}}$, iff $\Sigma$ is a standard enumeration of $\supp(\bar{p})$ and $\bar{F}$ is a sequence $\bar{F} = (F^\alpha_k \mid k \leq \alpha)$ of functions such that for any $k \leq n < \alpha$ it holds that $\init(\bar{p}, \Sigma) = \dom(F^\alpha_k)$ and for any $t \in \init(\bar{p}, \Sigma)$

$$\init(\bar{p}, \Sigma) = \supp(\bar{F}) \quad \text{(1)}$$

$$\init(\bar{p}, \Sigma) = \supp(\bar{F}) \quad \text{(2)}$$

2. We say that $\bar{p}$ is simple with respect to $\Sigma$, abbreviated by $\bar{p} \in D^\omega_{\Sigma, \bar{F}}$, iff there exists a sequence $\bar{F}$ such that $\bar{n} \in D^\omega_{\Sigma, \bar{F}}$. We say that $\bar{p}$ is simple, abbreviated by $\bar{n} \in D^\omega$, if and only if there exists $\Sigma$ such that $\bar{n} \in D^\omega_{\Sigma}$.

We pause to note that when $k = 0$, as $\mathbb{P}_{\sigma_0}$ is just the trivial forcing and $\init(\bar{p}, \Sigma) = (\emptyset)$, trivially $F^\alpha_0$ is the constant function with domain $(\emptyset) = \init(\bar{p}, \Sigma)$ taking the value $\bar{n}(0) = n$.

**Lemma 3.5.** For any $\Sigma$ it holds that $D^\omega_{\Sigma, \bar{F}} \subseteq D^\omega_{\Sigma}$.

**Proof.** This is obvious, but for the readers convenience we provide some detail. Given $\Sigma$, $\bar{F}$ and $\bar{p} \in D^\omega_{\Sigma, \bar{F}}$, we define a partial function

$$F_k : \{\sigma_0, \ldots, \sigma_{k-1}\} \cap \sigma_k (\omega)^2 \rightarrow \text{PTrees}$$

(noting $\emptyset X = \{\emptyset\}$) by

$$F_k(\bar{x}) = \bigcup\{F^\alpha_k(\bar{t}) \mid k \leq n < \lh(\Sigma) \land \bar{t} \in \init(\bar{p}, \Sigma) \land \bar{t} \subseteq \bar{x}\}.$$ 

Clearly, $F_k$ is defined on $X_k$ as in (1) and is $(\bigcup_n F^\alpha_n)^*$ in the sense of (2) there; moreover $\bar{n} \in D^\omega_{\Sigma, \bar{F}}$ where $\bar{F} = (F_k \mid k < \lh(\Sigma))$. \(\square\)

Note that for any $\bar{n} \in D^\omega_{\Sigma}$ we have

$$[\bar{n}]_\Sigma = \{\bar{x} \in \supp(\bar{n}) \times 2 \mid \text{For any } s \in \par_{\omega}(\supp(\bar{n}) \times \omega, 2) \text{ such that } s \subseteq \bar{x},$$

$$\text{there exists } n \in \omega \text{ and } \bar{t} \in \init(n, \bar{n}) \text{ such that } s \subseteq \bar{t} \text{ and } \bar{t} \subseteq \bar{x}\}.$$ 

Thus, when $\bar{n} \in D^\omega$, there is a one-to-one correspondence between $[\bar{n}]$ and the set of branches through $\bigcup_n \init(n, \bar{n})$, where the latter is considered as a tree under the ordering given by $\subseteq$. Furthermore, clearly $\{N \in [\bar{n}]_\Sigma \mid s \in \init(n, \bar{n}), n \in \omega\}$ is a topological basis for the space $[\bar{n}]_\Sigma$.

We now give a version of continuous reading for names for iterated Sacks forcing; at the same time we prove that $D^\omega$ (and thus $D^k$) is dense in $\mathbb{P}$.

**Lemma 3.6.** Let $\bar{x}$ be a $\mathbb{P}$-name for an element of $\omega^\omega$ and $\bar{n} \in \mathbb{P}$. Then we can find $\bar{q} \in D^\omega$ such that $\bar{q} \leq \bar{n}$ together with a map $f : \init(\bar{q}) \rightarrow \omega^\omega$ such that for every $n \in \omega$ and $s \in \init(n, \bar{n})$

$$(\bar{q})_s \forces \bar{x} \mid n = f(s).$$

In particular, $f$ codes a continuous map $F : [\bar{q}] \rightarrow \omega^\omega$ such that $\bar{q} \forces \bar{x} = F(\bar{s}_G)$, where we use $F$ to denote the function coded by $f$ in both the ground model and the extension via $\mathbb{P}$.

We introduce some more terminology that will help us build fusion sequences in the proof of Lemma 3.6.
Definition 3.7. Given conditions \( \bar{p}, \bar{q} \in \mathbb{P} \), \( n \in \omega \) and a finite or infinite sequence \( \Sigma = \langle \sigma_k \mid k \in \alpha \rangle \) of ordinals in \( \text{supp}(\bar{p}) \) with length \( \alpha \geq n \), we write \( \bar{q} \succeq_n \bar{p} \) exactly if \( \bar{q} \leq \bar{p} \) and for every \( k < n \),

\[
\bar{q} \upharpoonright | \sigma_k | \vdash q(\sigma_k)_n = \bar{p}(\sigma_k)_n.
\]

We also write \( \succeq_n^{\sigma_0, \ldots, \sigma_{n-1}} \) for \( \succeq_n \) when \( \Sigma | n = \langle \sigma_k \mid k < n \rangle \).

The reader should note that \( \succeq_n \) is just \( \leq \) and that the relation \( \succeq_n \) only depends on \( \Sigma | n \).

Now we are ready to give the fusion argument that proves the lemma.

Proof of Lemma 3.7. Let \( \bar{p} \in \mathbb{P} \) and \( \mathbb{P} \) be an \( \mathbb{P} \)-name \( \dot{x} \) for an element of \( ^\omega \omega \) be given; we shall find a stronger condition \( \bar{q} \) together with \( \Sigma \) and \( \bar{F} = \langle F_k \mid k, n \in \omega, k \leq n \rangle \) and a map \( f \) such that \( \bar{q} \in D_{\Sigma, \bar{F}} \) and \( f \) satisfies the statement of the lemma.

Let \( p_0 = \bar{p} \) and \( f(\emptyset) = \emptyset \) and build a fusion sequence of conditions \( \langle p_n \mid n \in \omega \rangle \) such that \( p_0 \geq p_1 \geq p_2 \ldots \), as follows: Fix a standard enumeration \( \Sigma^0 = \langle \sigma_0^0, \sigma_1^0, \sigma_2^0, \ldots \rangle \) of \( \text{supp}(p_0) \). For every further step \( n > 0 \) in the construction of the fusion sequence, after having obtained \( p_n \) we shall also fix an enumeration \( \Sigma^n = \langle \sigma_k^n \mid n \in \omega \rangle \) of \( \text{supp}(p_n) \setminus \text{supp}(p_{n-1}) \). Note that the choice of enumeration at each step is essentially arbitrary.

At the end of the construction, to argue that the greatest lower bound \( \bar{q} \) of this sequence is in \( D^\mathbb{P} \), we use the \( \Sigma \), for \( n \in \omega \) to obtain a standard enumeration \( \Sigma \) for \( \bar{q} \) as follows: Let \( f : \omega \times \omega \to \omega \) be the well-known bijection given by:

\[
f(n, k) = \frac{(n + k + 1)(n + k)}{2} + n.
\]

Then we shall let \( \Sigma = \langle \sigma_l \mid l \in \omega \rangle \) be defined by

\[
\sigma_{f(n, k)} = \sigma_k^n.
\]

In other words, \( \Sigma \) will enumerate \( \{ \sigma_k^n \mid n, k \in \omega \} \) by the well-known diagonal counting procedure.

Most importantly, our construction will be set up so as to guarantee that for \( n > 0 \)

\[
\bar{p}_0 \succeq_1 \bar{p}_1 \succeq_2 \bar{p}_2 \succeq_3 \ldots \bar{p}_{n-1} \succeq_n \bar{p}_n \ldots
\]

We now give the details of the successor step of the construction. Assume we have already constructed \( p_n \) and \( \Sigma^n \) for \( n' \leq n \). The reader should note that we have already determined the first \( n+1 \) elements \( \langle \sigma_0, \ldots, \sigma_n \rangle \) of \( \Sigma \). We have determined \( \sigma_{f(n', k)} \) for all \( k \in \omega \) and all \( n' \leq n \), so the first index which has not yet been assigned a value in \( \Sigma \) is \( f(n+1, 0) \) and \( n+1 < f(n+1, 0) \) (determining in fact \( \langle \sigma_0, \ldots, \sigma_{n+1} \rangle \)).

We also assume by induction that we have already defined \( F^n_{n'} \) for every pair \( n', k \in \omega \) with \( n' \leq n \) and \( k \leq n' \) as well as a \( f(\bar{s}) \) for \( \bar{s} \in \bigcup_{n' \leq n} \text{init}_{n'}(p_{n'}) \) (these assumptions are trivial if \( n = 0 \)). We will now find \( p_{n+1} \), \( F^n_{n'} \) for \( k \leq n' \leq n + 1 \) and define \( f \) on \( \text{init}_{n+1}(p_{n+1}) \).

Claim 3.8. Let \( \bar{q} \in \mathbb{P} \) such that \( \bar{q} \succeq_n^{\sigma_0, \ldots, \sigma_n} \bar{p}_n \), let \( k \in \omega \) such that \( k \leq n + 1 \) and let \( \bar{q} \in \text{init}_{n+1}^{k}(\bar{q}(\sigma_0, \ldots, \sigma_k)) \). There is \( \bar{q}' \in \mathbb{P} \) such that \( \bar{q}' \succeq_{n+1}^{\sigma_0, \ldots, \sigma_n} \bar{q} \) \( \bar{q} \) such that for some tree \( T \subseteq 2^\omega \),

\[
(q \upharpoonright | \sigma_k)|_T \vdash q(\sigma_k)_{n+1} = T
\]

In case \( k = n+1 \), i.e. \( \bar{q} \in \text{init}_{n+1}^{k}(\bar{q}) \) we can moreover demand that for some \( s \in \omega \),

\[
(q \upharpoonright | \sigma_k)|_T \vdash \bar{x} \upharpoonright \bar{n} = \bar{s}
\]

where \( \bar{x} \) is the trace of \( \bar{q} \) through the tree \( \bar{T} \).
Likewise, it follows that for every \( n \in \mathbb{N} \), \( f \) such that \( k \in \mathbb{N} \), a standard enumeration for \( F \). By Lemma 2.7, let \( \Sigma \) with the standard enumeration \( p \sigma \). The induction hypothesis.

To see that such a condition \( q \) indeed exists, construct \( q' \) by induction on \( \xi < \lambda \) as follows: Assume we have constructed \( q' \) such that \( q' \models \xi \). Find \( \xi \) such that \( q' \models \xi \) and whenever \( \xi \in \Pi(\xi) \), we have \( \xi \models \xi \). Clearly, \( q' \models \xi + 1 \) accepts \( \xi \) for every \( \xi < \lambda \) for the induction hypothesis.

Starting with \( k = 1 \), apply the claim for every \( \xi \in \text{init}^k(\bar{p}_n, \Sigma \mid n) \), finding \( q \) such that

\[
(\text{init}^k(\bar{p}_n, \Sigma \mid n), q) \models \xi \models \xi \]

for every such \( \xi \). Repeating for each \( k \in \{2, \ldots, n\} \) successively strengthen \( q \) to ensure (2) for each \( \xi \in \text{init}^k(\bar{p}_n, \Sigma \mid n) \) in finitely many steps. Finally apply the claim again finitely many times, for each \( \xi \in \text{init}^k(\bar{p}_n, \Sigma \mid n) \) to obtain a condition \( \bar{p}_{n+1} \in \mathbb{P} \) so that \( \bar{p}_{n+1} \leq^{\sigma_0, \ldots, \sigma_n} \bar{p}_n \) and for any \( \xi \in \text{init}^n(\bar{p}_n, \Sigma \mid n) \), there is \( s = s(\xi) \) such that

\[
(\bar{p}_{n+1} \mid \sigma_n) \models \xi \models \xi \]

By construction for any \( k \leq n \) and \( \xi \in \text{init}^k(\bar{p}_n, \Sigma \mid n) \), there is \( T = T(k, \bar{p}) \) such that

\[
(\text{init}^k(\bar{p}_n, \Sigma \mid n), q) \models \xi \models \xi \]

Define \( F^k \), with domain \( \text{init}^k(\bar{p}_n, \Sigma \mid n) \) by \( F^k(\bar{p}) = T(k, \bar{p}) \) and for \( \xi \in \text{init}^n(\bar{p}_n, \Sigma \mid n) \), define \( f(\bar{p}) = s(\bar{p}) \).

It is clear that the sequence \( (\bar{p}_n \mid n \in \omega) \) is a fusion sequence in the sense of Definition 2.6. By Lemma 2.7 let \( q \) be its greatest lower bound. As promised, \( \Sigma \) is a standard enumeration for \( q \).

For each \( n \in \omega \) we have \( q \leq^{\sigma_n} \bar{p}_n \), i.e. for each \( k \in \omega \) such that \( k < n + 1 \),

\[
q \models \sigma_k \models q \models \sigma_k \models \Xi \models \Xi \]

It follows that for every \( n \in \omega \), dom(\( F^k \)) = \( \text{init}^k(\bar{p}_n, (\sigma_0, \ldots, \sigma_k)) = \text{init}^k(\bar{p}_n, \Sigma) \).

Likewise, dom(\( f \)) = \( \text{init}(\bar{q}) \). By construction of the functions \( F^k \), we have \( \bar{q} \in D^k_{\Sigma, \bar{F}} \) where \( \bar{F} = (F^k \mid n, k \leq n \leq k \leq n) \).

Remark 3.9. Suppose that \( \bar{p} \in \mathbb{P} \) in the statement of Lemma 3.6 is already simple (or just topologically determined) and fix \( \Sigma^* = \{\sigma^* \mid k < \lambda(\alpha^*)\} \) such that \( \bar{p} \in D^k_{\Sigma^*} \). It is now worth noting that in the above construction, \( \Sigma^0 = \Sigma^* \) so that we obtain a function \( f : \omega \to \omega \) such that \( \sigma^*_f(k) < \sigma^*_f(k' \downarrow \sigma^*_f(k)) \iff \sigma^*_f(k) < \sigma^*_f(k') \). The standard enumeration \( \Sigma^0 \) of \( \text{supp}(\bar{q}) \) is in a sense compatible with the standard enumeration \( \Sigma^* \) of \( \text{supp}(\bar{p}) \) we started with.

We shall need the following variant of Lemma 3.6 in the proof of Lemma 3.8 below.

Lemma 3.10. Let \( \bar{p}^0, \bar{p}^1 \in \mathbb{P} \) and suppose \( \bar{p}^0 \models \delta = \bar{p}^1 \mid \delta \), where \( \delta < \lambda \). Then we can find \( \delta' \), \( \delta^1 \in D^* \) such that for each \( i \in \{0, 1\} \), \( \delta' \leq \delta^i \) and \( \delta^0 \models \delta = \delta^1 \mid \delta \).

Proof. Adapt the previous argument to construct two fusion sequences \( (\bar{p}^n \mid n \in \omega) \) side-by-side, simultaneously, such that \( \bar{p}^n_i = \bar{p}^i \) for each \( i \in \{0, 1\} \) and for each \( n \in \omega \) we have \( \bar{p}^n_i \models \delta = \bar{p}_n^i \mid \delta \). We must check that at the successor step, we can preserve equality up to \( \delta \). To this end, chose the standard enumerations \( \Sigma^* \) of \( \text{supp}(\bar{p}_n^i) \) for \( i \in \{0, 1\} \) so that they agree below \( \delta \). It is straightforward to adapt the proof of Claim 3.8 in the appropriate manner. □
3.3. The ‘Push-forward’ Lemma. It is rather obvious that for \( \bar{p} \in D^\omega \), \( [\bar{p}] \) and \( \omega^{(\omega_2)} \) are homeomorphic. Since we shall make extensive use of a particular homeomorphism, let us consider it in some detail.

**Lemma 3.11.** Let \( \Sigma \) and \( \bar{p} \in D^\omega_\Sigma \) be given. \( [\bar{p}]_\Sigma \) and \( \omega^{(\omega_2)} \) are homeomorphic.

**Proof.** We define a homeomorphism

\[
F_{\bar{p}} : \omega^{(\omega_2)} \to [\bar{p}].
\]

For each \( p \in \mathcal{S} \), let \( f_p : <\omega_2 \to <\omega_2 \) be the map such that for each \( n \in \omega \), \( f(\omega_2) = (p)_n \) and \( f(p \upharpoonright \omega_2) \) preserves the lexicographic ordering. Thus, \( f_p \) is an embedding of \( <\omega_2 \) into \( p \). Note that to determine \( f_p(s) \) for \( s \in \omega_2 \), it is enough to know \( (p)_n \). Thus we are justified in writing \( f_{(p)_n}(s) \) in this case.

Now let \( \bar{p} \in \mathbb{P} \) be given, and fix \( \Sigma = (\sigma_n \mid n \in \omega) \) and \( \bar{F} = (\bar{P}_n^k \mid n, k \in \omega, k \leq n) \) such that \( \bar{p} \in D_{\Sigma, \bar{F}} \). We define another map

\[
f_{\bar{p}} : \bigcup_{n \in \omega} n(\omega_2) \to \text{init}(\bar{p})
\]

as follows: Given \( \bar{s} \in n(\omega_2) \), let \( f_{\bar{p}}(\bar{s}) \) be the unique \( \bar{t} \in \text{init}(\bar{p}) \) such that

\[
\begin{align*}
\bar{t}(\sigma_0) &= f_{p(0)}(\bar{s}(0)), \\
\bar{t}(\sigma_1) &= f_{p(n(0))}(\bar{s}(1)), \\
&\vdots \\
\bar{t}(\sigma_{n-1}) &= f_{p(\omega_2)}(\bar{s}(n-1)).
\end{align*}
\]

The map \( f_{\bar{p}} \) induces a map

\[
F_{\bar{p}} : \omega^{(\omega_2)} \to [\bar{p}]
\]

as follows: Given \( \bar{x} \in \omega^{(\omega_2)} \), let

\[
F_{\bar{p}}(\bar{x}) = \bigcup \{ f_{\bar{p}}(\bar{s}) \mid \bar{s} \in n(\omega_2) \wedge (\forall k < n) \bar{s}(k) = \bar{x}(k) \upharpoonright n \}.
\]

The map \( F_{\bar{p}} \) is a homeomorphism.

It is easy to see that whenever \( q = (q_n \mid n \in \omega) \) is a condition in \( \mathcal{S}^{\omega_2} \) (the full support \( \omega \)-fold product of Sacks forcing) then \( q \) gives rise via \( F_{\bar{p}} \) to a condition \( q \in \mathbb{P} \) such that \( q \leq \bar{p} \) by coordinate-wise pushing forward the components: let \( \bar{q}(0) = f_{p(0)}(q_0) \); let \( \bar{q}(1) \) be a \( \mathbb{P}_1 \)-name for a perfect tree such that \( \bar{q}(0) \geq \bar{q}(1) = f_{p(\omega_2)}(\bar{s}(1)) \) and inductively, let \( \bar{q}(\sigma_k) \) be a \( \mathbb{P}_{\sigma_k} \)-name for a perfect tree such that \( \bar{q} \upharpoonright \sigma_k \geq \bar{q}(\sigma_k) = f_{p(\omega_2)}(\bar{s}(\sigma_k)) \). Call the map \( q \mapsto \bar{q} \) we just described \( \iota_{\Sigma, \bar{p}} : \mathcal{S}^{\omega_2} \to \mathbb{P} \).

Note that \( \iota_{\Sigma, \bar{p}} \) is order preserving (and also preserves incompatibility of conditions) but emphatically not a complete embedding of preorders (see e.g. \( \mathbb{P} \)). Moreover, it is easy to see \( \iota_{\Sigma, \bar{p}}(q) \in D_{\bar{F}}^\omega \). In particular, letting \( C = \prod_{n \in \omega} [q_n] \) and \( \bar{q} = \iota_{\Sigma, \bar{p}}(q) \), it holds that \( F_{\bar{p}}[C] = [\bar{q}]_\Sigma \).

In fact, \( F_{\bar{p}} \) and \( \iota_{\Sigma, \bar{p}} \) have the following stronger property, which will be very convenient for us:

**Lemma 3.12 (Push-forward).** Let \( \Sigma \) and \( \bar{p} \in D^\omega_\Sigma \) be given. For any \( q = (q_n \mid n \in \omega) \in \mathcal{S}^{\omega_2} \) there is \( \bar{q}' = (q_n' \mid n \in \omega) \in \mathcal{S}^{\omega_2} \) stronger than \( q \) such that \( \iota_{\Sigma, \bar{p}}(q') \in D_{\bar{F}}^\omega \).

**Proof.** Suppose \( \bar{p} \in D_{\mathcal{S}, \bar{F}}^\omega \), and we are given \( \langle q_n \mid n \in \omega \rangle \). For arbitrary \( n \in \omega \), we call a sequence of natural numbers \( \langle l_0, \ldots, l_n \rangle \) good if for each \( k \in \omega \setminus \{0\} \) whenever \( s \) is a \( m_k \)-th splitting node in \( q_k \) and \( s' \) is a \( m_{k-1} \)-th splitting node in \( q_{k-1} \), then \( \text{lh}(s') \geq \text{lh}(s) \).
Define by induction on \( n \) a sequence of trees \((T_n^k)_{k \leq n < \omega}\), where \( T^0_n = \emptyset \) for all \( n \) and in the end \( T^k_n \) will be the \( n \)th splitting level of \( q^k_{n-1} \) when \( n, k > 0 \) (the role of this definition when \( k = 0 \) will become clear when we discuss the relation to \( \tau_{\Sigma, p}(q') \)).

At stage \( n + 1 \), suppose we have defined \( T^k_n \) for each \( l \leq n \). Let \( T^{n+1}_{n+1} \) the initial segment of \( q_n \) up to and including the \( n + 1 \)-st splitting level of \( q_n \) and let
\[
m^n = \max\{ \lh(t) \mid t \in T^{n+1}_{n+1} \}.
\]
Fix some good sequence \( m_0^\sigma, \ldots, m^\sigma_n \) with \( m^n = m^\sigma \)—this is easy; for instance, let
\[
m_k^n = \max\{ \lh(t) \mid t \text{ is a } k + 1\text{-th splitting node in } q_k \}
\]
for by finite (reverse) on induction \( k \leq n \). For each \( k \leq n \) and each \( t \in T^k_n \) let \( s(t) \in q_k \) be some node such that \( t(k) \subseteq s(t)(k) \) and \( t(k) \) is a \( m_k^n\)-th splitting node in \( q_k \). Finally, for each \( k \leq n \) let \( T^{n+1}_k = \{ s(t) \mid t \in T^k_n \} \), let \( q_k' = \bigcup_{n \geq k+1} T^{n+1}_k \), let \( q' = \{ q_k' \mid k \leq \omega \} \) and consider \( \bar{q} = \tau_{\Sigma, p}(q') \).

For any \( s \in (q_0')_n \times \ldots \times (q_n')_n \) letting \( \bar{t} = f_p(\bar{s}) \) \( \{ \sigma_0, \ldots, \sigma_n \} \) show by induction on \( k \leq n \) that \( t \mid \sigma_k \) decides (relative to \( \bar{p} \mid \sigma_k \)) the entire level of \( \bar{l}(\sigma_k) \), namely \( p(\sigma_k)m^\sigma_k \). To be precise, \( (\bar{p})_t \mid \sigma_k \models p(\sigma_k)m^\sigma_k = F_{m^\sigma_k}^{\sigma_k} \) and \( (\bar{p})_t \mid \sigma_k \models \bar{l}(\sigma_k) \in (p(\sigma_k))_{m^\sigma_k} \). It follows that \( f_p(\bar{s}) \in \init^n(\Sigma, \bar{q}) \), since \( \bar{q} = \tau_{\Sigma, p}(q') \) and by choice of \( s \). Likewise, letting \( G_k^n = F_{m_k}^{n_k} \) and \( \bar{G} = \langle G_k^n \mid k \leq n < \omega \rangle \) we conclude that \( \bar{G} \) witnesses that \( \bar{q} \in D_{\Sigma, G}^\bar{G} \), i.e. \( \bar{q} \in D_{\Sigma, G} \).

\section{The Polarized Mycielski’s Theorem for product of Sacks forcing}
Before we prove Galvin’s Theorem for iterated Sacks forcing, we go back to product Sacks forcing: The following result generalizes of Mycielski’s Theorem (see [17, Theorem 1, p. 141] or [12, 19.1, p. 129]) to infinite sequences of elements of Cantor space. It will be the main tool in proving Galvin’s Theorem for iterated Sacks forcing. Ramsey theory for infinite sequences was also studied in [19] and [13].

\begin{theorem}
Let \( B \) be a comeager subset of \( X = \omega^\omega \). Then there is a sequence \( C_n \), for \( n \in \omega \), of perfect subsets of \( \omega^\omega \) such that \( \bigcap_{n \in \omega} C_n \cap B = \emptyset \).
\end{theorem}

The proof is an elaboration of the argument from [3]; a proof could also be based using methods from [19].

\begin{proof}
Let \( \bar{Q} \) be the partial order consisting of pairs \( q = (n_q, f_q) \) with \( n_q \in \omega \) and \( f_q : n^\omega_2 \to \omega^\omega \), ordered by \( q \leq p \) if \( n_q \geq n_p \) and for each \( s \in n^\omega_2, f_q(s) \supseteq f_p(s \upharpoonright n_p) \).

Clearly, \( \bar{Q} \) as a forcing adds a perfect subset of Cohen reals to \( \omega^\omega \). Further, let \( \bar{Q} = \prod_{q \in \omega} Q_q \) the finite support product of \( \omega \) many copies of \( Q \). We shall presently find a filter for \( \bar{Q} \) in the ground model, meeting countably many dense sets, to give us the desired sequence of perfect sets.

Let \( \bar{Q}^* \) denote the set \( p \in \bar{Q} \) such that for some \( n, m \in \omega \), we have \( n \leq m \) and
\[
m = \supp(p),
\]
\begin{align*}
(\forall k \in n) & \ n_p(k) = n, \\
(\forall k \in m \setminus n) & \ n_p(k) = 1.
\end{align*}

Clearly, \( \bar{Q}^* \) is dense in \( \bar{Q} \).

For \( p \in \bar{Q}^* \), write \( n_p \) and \( m_p \) for the unique \( n \) and \( m \) such that (3) holds.

Moreover, for \( p \in \bar{Q}^* \) and \( s \in n(n^\omega_2) \), where \( n = n_p \), we shall write \( f_p(s) \) for the sequence of length \( n \) given by
\[
\langle f_p(0)(s(0)), \ldots, f_p(n-1)(s(n-1)) \rangle.
\]

For the following two claims, let \( O \) be open dense in \( X \).
Claim 3.14. Suppose $\bar{p} \in \mathcal{Q}^*$, $n = n_p$ and $\bar{s} \in n^{(n+2)}$. Then there is $\bar{q} \leq \bar{p}$ in $\mathcal{Q}^*$ with $n_q = n_p$ such that

$$[f_q(\bar{s})] \subseteq O.$$  

Proof. As $O$ is open dense, we may pick an extension $\ell'$ of $f_p(\bar{s})$ such that $[\ell'] \subseteq O$. Now let $\bar{q}$ be any extension of $\bar{p}$ in $\mathcal{Q}^*$ such that $n_q = n_p$, $f_q(\bar{s}) = \ell' \upharpoonright n$ and $f_{\bar{q}(k)}(\theta) = \ell'(k)$ for all $k \in \text{dom}(\ell') \setminus n_p$. \qed

Claim 3.15. The set $D_O = \{ \bar{q} \in \mathcal{Q} \mid (\forall \bar{s} \in n^{(n+2)}) f_q(\bar{s}) \subseteq O \}$ is dense in $\mathcal{Q}$.

Proof. Suppose $\bar{p} \in \mathcal{Q}^*$. Repeatedly strengthen $\bar{p}$, applying the previous claim for each $s \in n^{(n+2)}$ in turn. After finitely many steps we arrive at a condition $\bar{q} \in D_O$. \qed

Now let $B = \bigcap_{n \in \omega} O_n$, where each $O_n$ is open dense in $X$. By standard arguments, we may find a filter $G$ on $\mathcal{Q}^*$ meeting every $D_{O_n}$ for $n \in \omega$, and such that for every $k \in \omega$ there is $\bar{p} \in G$ with $n_p \geq k$. We leave it to the reader to derive, for each $k \in \omega$, a perfect tree from $\{ f_{\bar{p}(k)} \mid \bar{p} \in G \}$; let $C_k$ be the set of branches through this tree. \qed

Note that the previous theorem has the following trivial (but useful!) corollary; following [13], one might call this a ‘polarized version’ of Mycielski’s Theorem.

Corollary 3.16. If $R$ is a meager $k$-ary relation on $\omega^{\omega^2}$, there is for each $i \in k$ a sequence $(C_{i} : n \in \omega)$ perfect subsets of $\omega^2$ such that $\prod_{i \in k} \prod_{n \in \omega} C_{i}^{n}$ is $R$-discrete.

3.5. A Galvin-type theorem for iterated Sacks forcing. Building upon the work in the previous section, we are now ready to give a topological proof of Galvin’s Theorem for iterated Sacks forcing. We will prove Theorem 1.2 in the following, equivalent form.

Theorem 3.17 (Galvin’s Theorem for iterated Sacks forcing, 2nd form). Let $\bar{p} \in \mathcal{P}$ and $c : \lambda^{\omega^2} \rightarrow 2$ be $\mathcal{C}$-universally Baire. Then there is $\Sigma$ and $\bar{q} \in D^*_\Sigma$ such that $\bar{q} \leq \bar{p}$ and for each $\xi \in \text{supp}(\bar{q})$, there is $n = n(\xi) \in \omega$ such that for each $\bar{s}, \bar{t} \in \text{init}^n(\bar{q})$, $c$ is constant on $[\bar{s}] \cap \Delta_{\xi} \cap N_{\bar{s}} \times N_{\bar{t}}$.

Moreover if we assume that $\bar{p} \in D^*_{\Sigma^*}$ in the above, we can demand that $\Sigma = \Sigma^*$ so that $\bar{q} \in D^*_{\Sigma^*}$.

The second form clearly implies the first (i.e. Theorem 1.2); the reverse direction can be seen from the fact that $[\bar{q}]^2$ is compact; we shall leave this to the reader to prove.

We also point that Theorem 1.2 has a simply corollary for sets (this was shown in [14] for Borel sets) which we shall use in [14].

Corollary 3.18. Let $\bar{p} \in D^*$ and $B \subseteq [\bar{p}]$ be $\sigma(\Sigma^1_1)$ (absolutely $\Delta^1_2$, $\mathcal{C}$-universally Baire, $\mathcal{P}$ measurable). Then there is $\bar{q} \leq \bar{p}$ such that $\bar{q} \in D^*$ and $[\bar{q}] \subseteq B$ or $[\bar{q}] \cap B = \emptyset$.

For this, simply apply Theorem 1.2 for

$$c(\{x_0, x_1\}) = \begin{cases} 1 & \text{if } \text{lex}_0(\{x_0, x_1\}) \in B, \\ 0 & \text{otherwise} \end{cases}$$

and restrict to a basic open set. If we assume that $\bar{p} \in D^*_{\Sigma^*}$, then we can additionally demand that $\bar{q} \in D^*_{\Sigma^*}$ (but we shall have no use for this).

As shall become clear, this solution is close to being optimal and can be arrived at by taking into account obstructions to finding homogeneous conditions. As
our point of departure, we state the following theorem for continuous colorings (equivalently, clopen partitions) due to Geschke, Kojman, Kubiś and Schipperus:

**Theorem 3.19** (Claim 31 from [9]). Let \( \bar{p} \in \mathbb{P} \) and let \( c: [\bar{p}]^2 \to 2 \) be a continuous coloring. Then there is \( \bar{q} \in D^s \), \( \bar{q} \preceq \bar{p} \) such that \( c \) is constant on \( [\bar{q}]^2 \).

Note that in the language of symmetric maps \( c^*: [\bar{p}]^2 \to 2 \), the requirement on the coloring becomes that \( c^* \mid ([\bar{p}]^2 \setminus \text{diag}([\bar{p}]^2)) \) be continuous. Clearly, it would be desirable to weaken the requirement of being continuous in the above, e.g. to being Borel or even Baire measurable. The family of colorings \( c_\xi \) from [11] represent a fundamental obstruction to such a result:

**Fact 3.20.** Assume momentarily that \( \lambda \geq 2 \). Then for any \( \Sigma \) and \( \bar{q} \in D_\Sigma \) whose support has size at least \( 2 \) and any \( \xi \in \text{supp}(\bar{q}) \), \( c_\xi \) takes both colors on pairs from \( [\bar{q}]^2 \).

This explains the role of \( \Delta_\xi \). Moreover, note that \( \Delta_0 \) is open dense and \( \Delta_\xi \) is meager for \( \xi > 0 \). Therefore, it is impossible to replace \( \mathcal{C} \)-universally Baire by Baire measurable in said theorem—as a Baire measurable map \( c \) can take completely arbitrary values on a meager set.

An elaboration of the argument from Fact 3.20 uncovers yet another, more subtle obstruction:

**Example 3.21.** Let \( \bar{p} \in D_{\Sigma}^c \); we define a coloring \( c \) on \( [\bar{p}]^2 \) with two colors. Fix a surjection \( g: \text{supp}(\bar{p}) \to \text{supp}(\bar{p}) \times \omega \). Also fix functions \( \lambda \) and \( n \) so that for all \( \xi \in \text{supp}(\bar{p}) \), \( g(\xi) = (\lambda(\xi), n(\xi)) \).

Let \( \{ \bar{x}_0, \bar{x}_1 \} \in [\bar{p}]^2 \) be given and let \( \xi = \Delta(\bar{x}_0, \bar{x}_1) \). Suppose further that \( \bar{x}_0(\xi) \) comes before \( \bar{x}_1(\xi) \) in the lexicographical ordering. Define

\[
c(\{\bar{x}_0, \bar{x}_1\}) = \bar{x}_1(\lambda(\xi))(n(\xi)).
\]

Now suppose that \( \bar{q} \in \mathbb{P} \) is stronger than \( \bar{p} \) and that for \( \{ \bar{x}, \bar{y} \} \in [\bar{q}]^2 \), \( c(\{\bar{x}, \bar{y}\}) \) is simply a function of \( \Delta(\bar{x}, \bar{y}) \); that is, suppose we can find \( F: \text{supp}(\bar{p}) \to \{0, 1\} \) such that for any \( \{ \bar{x}, \bar{y} \} \in [\bar{p}]^2 \),

\[
c(\{\bar{x}, \bar{y}\}) = F(\Delta(\bar{x}, \bar{y})).
\]

This leads to a contradiction: Let \( \bar{z} \in \text{supp}(\bar{p})^{\leq 2} \) be given by

\[
\bar{z}(\nu)(k) = F(g^{-1}(\nu)(k)).
\]

Let \( \bar{x} \neq \bar{z} \) be such that for each \( \xi, \bar{x}(\xi) \) is the not the right-most branch of \( [\bar{q}] \). But for any \( \xi \), we may chose \( \bar{y} \in [\bar{q}] \) such that \( \Delta(\bar{x}, \bar{y}) = \xi \) and \( \bar{y}(\xi) \) comes before \( \bar{x}(\xi) \) lexicographically, and thus

\[
\bar{x}(\lambda(\xi))(n(\xi)) = c(\{\bar{x}, \bar{y}\}) = F(\xi) = \bar{z}(\lambda(\xi))(n(\xi));
\]

contradicting the choice of \( \bar{x} \).

Taking into account this last obstruction, we arrive at the formulation of Theorem 3.19.

To clean up the proof, we shall work with \( c: [\bar{p}]^2 \to 2 \), i.e. consider the coloring to be defined on the product space rather than \( [\bar{p}]^2 \). It will be practical to use not \( \mathcal{C} \)-universally Baire measurability of \( c \), but a slightly weaker notion of measurability, crafted to suit its precise employment in the argument. This quite likely defines the largest class of colorings for which Theorem 3.19 can be shown to hold (in ZFC).

---

\(^3\)Note that on the other hand, the proof of Theorem 1.2 goes through for Baire measurable \( c \), if we restrict to the case \( \xi = 0 \); in fact, thus we prove (a statement which directly implies) Galvin’s original theorem, and the proof is similar to [12].
Definition 3.22. Let $\tilde{p} \in \mathbb{P}$ and $\xi \in \text{supp}(\tilde{p})$. We say a set $B \subseteq [\tilde{p}]^2$ is $\mathbb{Y}^p_\xi$ measurable if and only if for any any $\tilde{p}_0, \tilde{p}_1 \in \mathbb{P}$ stronger than $\tilde{p}$ such that $\tilde{p}_0 \upharpoonright \xi = \tilde{p}_1 \upharpoonright \xi$, there are $\tilde{q}_i \in \mathbb{P}$ with $\tilde{q}_i \leq \tilde{p}_i$ for each $i \in \{0, 1\}$ such that $\tilde{q}_0 \upharpoonright \xi = \tilde{q}_1 \upharpoonright \xi$ and
\[
(\{\tilde{q}_0\} \times \{\tilde{q}_1\}) \cap \Delta^*_\xi \subseteq B
\]
or
\[
(\{\tilde{q}_0\} \times \{\tilde{q}_1\}) \cap \Delta^*_\xi \cap B = \emptyset.
\]
Moreover, we say a set $B \subseteq [\tilde{p}]^2$ is $\mathbb{Y}^p$ measurable just if $B$ is $\mathbb{Y}^p_\xi$ measurable for every $\xi \in \text{supp}(\tilde{p})$. We say a map $c^* : [\tilde{p}]^2 \to X$, for a topological space $X$, is $\mathbb{Y}^p$ measurable (resp. $\mathbb{Y}^p_\xi$ measurable) just if its image is under the map $i$ which identifies $[\tilde{p}]^2$ with the lower half of $[\tilde{p}]^2$; the definition carries over as usual for maps $c : [\tilde{p}]^2 \to X$ into a topological space $X$. Note that $c : [\tilde{p}]^2 \to X$ is $\mathbb{Y}^p$ measurable (resp. $\mathbb{Y}^p_\xi$ measurable) if and only if this holds for the induced symmetric map $c^*$. We may also assume that for any $\xi \in \text{supp}(\tilde{p})$.

We show this is indeed a weaker property than being universally Baire.

Theorem 3.23. Let $\tilde{p} \in D^\mathbb{P}$. Every universally Baire (or just $\mathcal{C}$-universally Baire) set $B \subseteq [\tilde{p}]^2$ is $\mathbb{Y}^p$ measurable.

The proof depends on the ‘Polarized Mycielski’ Theorem 3.13 and on the ‘Push-forward’ Lemma 3.12.

Proof. We phrase the proof for $B \subseteq [\tilde{p}]^2$, but it works verbatim for $B \subseteq [\tilde{p}]^2$. Let $\xi \in \text{supp}(\tilde{p})$ and for $i \in \{0, 1\}$, let $\tilde{p}_i \in \mathbb{P}$ such that $\tilde{p}_i \leq \tilde{p}$ and satisfying $\tilde{p}_0 \upharpoonright \xi = \tilde{p}_1 \upharpoonright \xi$ be given. We may assume by Lemma 3.10 that for each $i \in \{0, 1\}$ we have $\tilde{p}_i \in D^p_{\Sigma^i}$, for standard enumerations $\Sigma^i = \langle \sigma^i_k \rangle$ satisfying $\langle \sigma^i_n \rangle \upharpoonright n \in I = \{ k \in \omega \mid \sigma^i_k < \xi \} = \langle \sigma^i_1 \rangle \upharpoonright \sigma^i_1 < \xi \rangle$. We may also assume that for any pair $(\tilde{x}_0, \tilde{x}_1) \in [\tilde{p}_0] \times [\tilde{p}_1]$ it holds that $\tilde{x}_0(\xi)$ come lexicographically before $\tilde{x}_1(\xi)$ (perhaps exchanging the indices of $\tilde{p}_0$ and $\tilde{p}_1$).

Let $X = \omega^2$ and let $F^* = F_{\omega^2, \omega^2} : X \to [\tilde{p}]$ be the continuous map given as in the proof of Lemma 3.11. Consider further the homeomorphism
\[
H : l(\omega^2) \times \omega^\omega \to X
\]
given by
\[
H^i(u, v)(n) = \begin{cases} u(n) & \text{if } n \in I, \\ v(n) & \text{if } n \notin I\end{cases}
\]
and the homeomorphism
\[
H' : l(\omega^2) \times \omega^\omega \times \omega^\omega \to [\tilde{p}_0] \times [\tilde{p}_1]
\]
given by
\[
H'(u, v_0, v_1) = (F_{\rho_0} \circ H(u, v_0), F_{\rho_1} \circ H(u, v_1)).
\]
As $c^{-1}(\{0\})$ is $\mathcal{C}$-universally Baire, $B = (H' \circ c)^{-1}(\{0\})$ has the property of Baire in $X^3$. Thus, by Corollary 3.16, we can find sequences $r^i = \langle r^i_n \rangle \upharpoonright n \in \omega$ (for $i \in \{0, 1\}$) of Sacks conditions such that $\bar{r}^0_n = \bar{r}^1_n$ for $n \in I$ and $B$ is clopen in the subspace
\[
C = \prod_{n \in I} [r^0_n] \times \prod_{n \in \omega \setminus I} [r^0_n] \times \prod_{n \in \omega \setminus I} [r^1_n].
\]
Indeed, taking the intersection with a basic clopen set, we may assume that $C$ is either completely contained in or disjoint from $B$. 

By the proof of the Push-forward Lemma 3.12 we can further demand that letting
\[ \bar{q}_i = \iota_{\mu_i}(s^i) \]
we obtain a pair of conditions \( \bar{q}_i \in D_{\Sigma_i} \) \((i \in \{0, 1\})\). By construction, \([\bar{q}_0] \times [\bar{q}_1] \subseteq \Delta^*_\xi \) and is monochromatic.

In the following lemma, we show \( Y^\xi \) measurability is equivalent to what perhaps appears to be a stronger property. This lemma encapsulates the most involved part of the proof of Theorem 3.17.

**Lemma 3.24.** Let \( \hat{p} \in \mathbb{P} \) and \( \xi \in \text{supp}(\hat{p}) \). A set \( B \subseteq [\hat{p}]^2 \) is \( Y^\xi \) measurable if and only if it has the following property: For any any \( \hat{p}_0, \hat{p}_1 \in \mathbb{P} \) stronger than \( \hat{p} \) and any \( \delta \geq \xi \), if \( \hat{p}_0 \upharpoonright \delta = \hat{p}_1 \upharpoonright \delta \), there are \( \bar{q}_i \in \mathbb{P} \) with \( \bar{q}_i \leq \hat{p}_i \) for each \( i \in \{0, 1\} \) such that \( \bar{q}_0 \upharpoonright \delta = \bar{q}_1 \upharpoonright \delta \) and

\[ ([\bar{q}_0] \times [\bar{q}_1]) \cap \Delta^*_\xi \subseteq B \]

or

\[ ([\bar{q}_0] \times [\bar{q}_1]) \cap \Delta^*_\xi \cap B = \emptyset. \]

This stronger version of tail-measurability will be used in the successor step of our fusion argument. Note that letting \( \delta = \lambda \), the lemma immediately allows us to construct conditions which are homogeneous on a single subspace \( \Delta^*_\xi \), for one particular \( \xi \in \text{supp}(\hat{p}) \).

**Proof.** The proof proceeds in two steps: First, we find \( \mathbb{P} \)-conditions \( \hat{p}_i^0 \leq \hat{p}_i^1 \) for each \( i \in \{0, 1\} \) such that \( \hat{p}_0^0 \upharpoonright \delta = \hat{p}_1^1 \upharpoonright \delta \) and \( c \) is continuous, in a strong sense, when restricted to \( \Delta^*_\xi \cap ([\bar{q}_0^1] \times [\bar{q}_1^1]) \). Second, we thin out \( \hat{q}_0^0 \) and \( \hat{q}_1^1 \) level by level, restricting to open sets with the same color \( k \), with \( k \) chosen appropriately (the color that occurs “densely”, in an appropriate sense). Both steps take the form of a fusion argument.

For each \( i \in \{0, 1\} \), suppose \( \hat{p}_i^0 \in \mathbb{P} \) accepts \( s^i \in \text{par}_{<\omega}(\omega \times \text{supp}(\hat{p}_i^0), 2) \). We say \( (\hat{p}_0^0, \hat{p}_1^1) \) is k-good at \((s^0, s^1)\) if and only if

\[ \left( \forall x^0, x^1 \right) \left( (x^0, x^1) \in [\hat{p}_0^0] \times [\hat{p}_1^1] \Rightarrow c(\{x^0, x^1\}) = k \right). \]

We say \( (\hat{p}_0^0, \hat{p}_1^1) \) is good at \((s^0, s^1)\) if any only if it is k-good at \((s^0, s^1)\) for some \( k \in \{0, 1\} \).

Further, we say \( (\hat{p}_0^0, \hat{p}_1^1) \) is good up to \( n \) if and only if for all \( l \leq n \) and every \((s^0, s^1) \in \text{init}_l(\hat{p}_0^0) \times \text{init}_l(\hat{p}_1^1) \) such that \( s^0(\xi) \neq s^1(\xi) \), we have that \( (\hat{p}_0^0, \hat{p}_1^1) \) is good at \((s^0, s^1)\).

**Step 1.** A in Lemma 3.6 we build, for each \( i \in \{0, 1\} \), a fusion sequence \( \hat{p}_0^0 \geq \Sigma^0_{n} \hat{p}_1^1 \geq \Sigma^0_{n} \hat{p}_1^2 \ldots \). At the end, we shall define \( \hat{p}_i^1 \) to be the greatest lower bound of \((\hat{p}_0^0, \hat{p}_1^1, \ldots) \) and proceed to step 2. In fact, we can take \( \Sigma^0_n = \Sigma^0 \), as our construction will never change the support; note that this convenient, but not strictly essential and the construction of Lemma 3.6 would go through just as well. For the remainder of the proof, drop the superscripts from \( \geq_n \) and suppress all mention of \( \Sigma \).

Let \( \hat{p}_0^0 = \hat{p}_i^0 \) for each \( i \in \{0, 1\} \). Assuming we have built \( \hat{p}_0^0 \) and \( \hat{p}_1^1 \), we specify how to obtain \( \hat{p}_{n+1}^0 \leq \hat{p}_n^0 \), for each \( i \in \{0, 1\} \). Fix, momentarily, \( s^i \in \text{init}_{n+1}(\hat{p}_n^i) \) for \( i \in \{0, 1\} \) such that \( s^0 \upharpoonright \xi + 1 = s^1 \upharpoonright \xi + 1 \). We also assume that \( s^0(\xi) \) and \( s^1(\xi) \) agree up to their last value.

As \( c^* \) is \( Y^\xi \) measurable, we can find \( k \in \{0, 1\} \) and \( \bar{q}_i^* \in D_{\Sigma^*} \) such that \( \bar{q}_i^* \leq (\hat{p}_n)_s^i \) for \( i \in \{0, 1\} \) such that \( q_i^* \upharpoonright \xi = q_i^* \upharpoonright \xi \) and \( c^* \) has constant value \( k \) on \([\bar{q}_0^*] \times [\bar{q}_1^*] \cap \Delta^*_\xi \). Let’s denote this \( k \) by \( k(n + 1, s^0, s^1) \) for later reference.

Now we must “thin out” \( \hat{p}_0^0 \) and \( \hat{p}_1^1 \) to agree with “the sum” of \( \bar{q}_0^* \) and \( \bar{q}_1^* \). We first take care of their common part: Let \( \hat{p}_n \) denote \( \hat{p}_n^0 \upharpoonright \delta \) (which equals \( \hat{p}_n^1 \upharpoonright \delta \), let
\(\overline{q}_i\) denote \(\overline{q}_i^0 \restriction \xi\) (which equals \(\overline{q}_i^1 \restriction \xi\)), let \(\overline{s}\) denote \(\overline{s}^0 \restriction \xi\) (which equals \(\overline{s}^1 \restriction \xi\)) and let \(i\) denote the longest common initial segment of \(\overline{s}^0(\xi)\) and \(\overline{s}^1(\xi)\) (so that \(i\) is forced to be an \(n\)th splitting node by assumption). Find a condition \(\overline{r} \in P_{\aleph_1^n+1}\) such that 

\[\overline{r} \leq_{\aleph_1^n+1} \overline{p}_n \xi, \overline{r} \restriction \xi = \overline{q}_i^0 \restriction \xi\]  

and 

\(\overline{r} \restriction \xi \models \overline{r}(\xi)\xi = \overline{q}_i^0(\xi) \cup \overline{q}_i^1(\xi)\).

Now find \(\overline{r}_* \in P_\delta\) such that \(\overline{r}_* \restriction \xi + 1 = \overline{r}, \overline{r}_* \leq_{\aleph_1^n+1} \overline{p}_n\) and for each \(\sigma \in [\xi + 2, \delta]\), 

\(\overline{r}_* \restriction [\sigma, \nu]_{\overline{p}_i^0 + \sigma} \models \overline{r}(\sigma)_{\overline{p}_i^0(\sigma)} = \overline{q}_i^0(\sigma)\).

This is possible \((\overline{r}_* \restriction \nu)_{\overline{p}_i^0 + \sigma} \) and \((\overline{r}_* \restriction \nu)_{\overline{p}_i^1 + \sigma}\) are incompatible since \(\overline{s}^0(\xi)\) and \(\overline{s}^1(\xi)\) are.

Finish “thinning out” by finding \(\overline{r}_*^0 \in P\) such that 

\(\overline{r}_*^0 \restriction \delta = \overline{r}_*, \overline{r}_*^0 \leq_{\aleph_1^n+1} \overline{p}_n\) for each \(i \in \{0, 1\}\), and so that for any \(\sigma \in \text{supp}(\overline{r}_*^0) \setminus \delta\) we have 

\((\overline{r}_*^0)_{\sigma} \models \overline{r}_*^0(\sigma)_{\overline{p}_i^0(\sigma)} = \overline{q}_i^0(\sigma)\) 

for each \(i \in \{0, 1\}\).

Repeat the above for every pair \(\overline{s}_0, \overline{s}_1 \in \text{init}_{\aleph_1^n+1}(\overline{p}_n)\), obtaining a \(\leq_{\aleph_1^n+1}\)-decreasing finite sequence of conditions; define \(\overline{p}_n+1\) to be the last condition of this sequence.

Note that \(\overline{p}_{n+1}\) has the following property: For any pair \(\overline{s}_i \in \text{init}^{\aleph_1^n+1}(\overline{p}_{n+1})\) (where \(i \in \{0, 1\}\) such that \(\overline{s}_0 \restriction \xi = \overline{s}_1 \restriction \xi\) and \(\overline{s}_0(\xi)\) only differs from \(\overline{s}_1(\xi)\) at its final value, and any \(\{\overline{x}_0, \overline{x}_1\} \subseteq [\overline{p}^0]^{\aleph_1}\) such that \(\overline{s}_i \subseteq \overline{x}_i\) and such that \(\overline{s}_i(\xi)\) is the maximal common initial segment of \(\overline{x}_0(\xi)\) and \(\overline{x}_1(\xi)\) for each \(i \in \{0, 1\}\), we have 

\(c(\overline{x}_0, \overline{x}_1) = k(n + 1, \overline{s}_0, \overline{s}_1)\).

Finally, we take the greatest lower bound of the sequence \(\overline{p}_0, \overline{p}_1, \ldots\), to obtain a condition \(\overline{p}_*\) with the following property:

For any \(n \in \omega \setminus \{0\}\), any pair \(\overline{s}_0, \overline{s}_1 \in \text{init}^{\aleph_1^n}(\overline{p}_*)\) such that 

\(\overline{s}_0 \restriction \xi = \overline{s}_1 \restriction \xi\) and such that 

\(\overline{s}_0(\xi) \restriction k = \overline{s}_1(\xi) \restriction k\) where 

\(k = \text{lh}(\overline{s}_0(\xi)) = \text{lh}(\overline{s}_1(\xi))\), for any \(\{\overline{x}_0, \overline{x}_1\} \subseteq [\overline{p}^0]^{\aleph_1}\) such that \(\overline{s}_i \subseteq \overline{x}_i\) and such that \(\overline{s}_i(\xi)\) (equivalently, \(\overline{s}_i(\xi)\)) is the maximal common initial segment of \(\overline{x}_0(\xi)\) and \(\overline{x}_1(\xi)\), we have 

\(c(\overline{x}_0, \overline{x}_1) = k(n, \overline{s}_0, \overline{s}_1)\).

Step 2. For the remainder of this proof, let \(Q\) denote the set of tuples \(\langle s^0, s^1 \rangle \in \bigcup_{n \in \omega} \text{init}^n(\overline{p}_*) \times \text{init}^n(\overline{p}_*)\) such that 

\(s^0 \restriction \delta = \overline{s}^1 \restriction \delta\) and consider the partial order \((Q, \leq_{Q})\) where \(Q\) is ordered in the obvious way, i.e. \(\langle 0^0, 0^1 \rangle \leq_{Q} \langle s^0, s^1 \rangle\) exactly if \(\overline{s}^0 \subseteq s^1\) for each \(i \in \{0, 1\}\).

Further, we say \(\langle s^0, s^1 \rangle \in Q\) has type \(k\), for \(k \in \{0, 1\}\) exactly if there exists \(n\) and a pair \(\overline{l} \in \text{init}^n(\overline{p}_*)\) (where \(i \in \{0, 1\}\) with \(s^i \subseteq \overline{l}\) for each \(i \in \{0, 1\}\) such that 

\(\overline{l}^0 \restriction \xi = \overline{l} \restriction \xi\) and \(\overline{l}^0 \restriction l = \overline{l} \restriction l\) for \(l = \text{lh}(\overline{l}^0(\xi)) = \text{lh}(\overline{l}(\xi))\) and \(k(n, \overline{l}^0, \overline{l})\) has been defined and is equal to \(k\). We say \(\langle \overline{l}^0, \overline{l}^0 \rangle\) witnesses type \(k\) for \(\langle s^0, s^1 \rangle\) if the above holds.

Also define \(\xi\)-join\((\overline{l}^0, \overline{l}^0)\) in this situation to denote the unique pair \(\langle s^0, s^1 \rangle\) such that for each \(i \in \{0, 1\}\) we have 

\(\text{dom}(s^i) = \text{dom}(\overline{l})\), \(s^i_*(\xi) = u^{-1} s^i \text{ and } s^i(\sigma) = s^i_*(\sigma)\) for \(\sigma \in \text{dom}(s^i) \setminus \{\xi\}\)

Note that every pair in \(Q\) has a type (since every pair in \(Q\) can be extended to a pair \(\langle \overline{l}^0, \overline{l}^0 \rangle\)) such that \(\overline{l}^i \in \text{init}^n(\overline{p}_*)\) for some \(n\) and each \(i \in \{0, 1\}\) and then 

\(k(n, \overline{l}^0, \overline{l})\) is defined by construction.

Thus by a simple argument density argument on \((Q, \leq_{Q})\) which we leave to the reader, we can find a pair \(\langle \overline{l}^0, \overline{l}^1 \rangle \in Q\) and a \(k \in \{0, 1\}\) such that the set of triples of type \(k\) is dense below \(\langle \overline{l}^0, \overline{l}^1 \rangle\) in \(Q\). Moreover, without loss of generality \(\langle \overline{l}^0, \overline{l}^1 \rangle\) is itself of type \(k\).

We make use of our standard fusion argument a second time: We build a sequences \(s^0 \geq_{0} q^0_1 \geq_{1} q^2_1 \geq_{2} \ldots\), as before dropping the superscripts from \(\geq_{n}\) (one can easily check that the supports never change; or otherwise supply the missing
notational details). Taking greatest lower bounds of these two sequences, we obtain the pair \( \bar{q}^0, \bar{q}^1 \) satisfying the conclusion of the theorem.

For each \( i \in \{0, 1\} \), let \( \bar{q}^i_n = (p^i_n)_{i \geq n} \). Assuming we have built \( \bar{q}^i_n \), we specify how to obtain \( \bar{q}^{i+1}_{n+1} \leq_{n+1} \bar{q}^i_n \).

Fix, for the moment, \( \langle s^0_i, s^1_i \rangle \in Q \) such that \( s^i \in \text{init}^{n+1}(\bar{q}^i_n) \) for each \( i \in \{0, 1\} \).

By assumption, we can find \( \langle s^0_i, s^1_i \rangle \in Q \) of type \( k \) which \( \leq_{\Sigma} \)-extends \( (l^0, l^1) \) and thus by definition of type, we can find a \( l^i \in \text{init}^n(p^i_n) \) which witnesses type \( k \) for \( \langle s^0_i, s^1_i \rangle \).

For each \( i \in \{0, 1\} \), we assume by induction that \( \bar{q}^i_n \) accepts any \( s \in \text{init}(p^i_n) \) for which there is \( s^i \in \text{init}^n(\bar{q}^i_n) \) satisfying \( s^i \subseteq s \). By this assumption, \( \bar{q}^i_n \) accepts \( l^i \).

Now we must "thin out \( \bar{q}^i_n \)" to a condition \( q^i_{n+1} \) so that \( s^i \) trivially extends in \( q^i_{n+1} \) to \( l^i \), preserving equality on \( \delta \) (so that \( q^i_{n+1} \upharpoonright \delta = q^i_0 \upharpoonright \delta \)). To this end, let \( q = q^0 \upharpoonright \delta \) (which equals \( q^1 \upharpoonright \delta \)), let \( t = l^0 \upharpoonright \xi \) (which equals \( l^1 \upharpoonright \xi \)) and let \( u = l^0(\xi) \upharpoonright l \) where \( l = \text{lh}(l^0(\xi)) - 1 \) (so that also \( u = l^1(\xi) \upharpoonright l \) and \( u \) is forced by \( \bar{q}^i_n \) to be an \( n \)th splitting node).

Argue as in the previous step to obtain \( q^i_{n+1} \leq_{n+1} \bar{q}^i_n \) satisfying \( q^0 \upharpoonright \delta = q^1 \upharpoonright \delta \) and such that the following hold for each \( i \in \{0, 1\} \):

\[
(q^i_{n+1} \upharpoonright \xi)_{s_{\xi \xi}} = (q \upharpoonright \xi)_{t_i}
\]

while

\[
(q^i \upharpoonright \xi)_{s_{\xi \xi}} \vdash q^i_0(\xi)_{s_{\xi \xi}} = q(\xi)_{u_i},
\]

and letting \( \langle s^0_i, s^1_i \rangle = \xi \)-join\((l^0, l^1) \) we have for \( \sigma \in [\xi + 1, \delta) \)

\[
(q^i_{n+1} \upharpoonright \sigma)_{s^i_{\xi \sigma}} \vdash q^i_{n+1}(\sigma)_{s^i_{\xi \sigma}} = q(\sigma)_{t_i(\sigma)}
\]

and when \( \sigma \in [\xi + 1, \lambda) \),

\[
(q^i_{n+1} \upharpoonright \sigma)_{s^i_{\xi \sigma}} \vdash q^i_{n+1}(\sigma)_{s^i_{\xi \sigma}} = q^i_0(\sigma)_{s^i_{\xi \sigma}}.
\]

This definition makes sense since \( s^i \) is accepted by \( \bar{q}^i_n \) for each \( i \in \{0, 1\} \) by construction. The induction hypothesis is clearly satisfied, as above each element of \( \text{init}^{n+1}(\bar{q}^i_{n+1}) \), we have preserved the full tree.

Repeat this for all \( \langle s^0, s^1 \rangle \in Q \) such that \( s^i \in (\bar{q}^i_{n+1})_{n+1} \) for each \( i \in \{0, 1\} \), building a finite \( \leq_{n+1} \)-descending sequence of conditions; the last condition is \( \bar{q}^i_{n+1} \).

By construction, the greatest lower bound \( q \) of the sequence \( \bar{q}^0, \bar{q}^1, \ldots \) has the required property.

The reader will find that by the proof of Theorem 3.23 instead of tail-measurability, we could have required that \( c \) is Baire measurable on the subspace \( \Delta_{\xi} \), when restricted to that space (note that \( \Delta_{\xi} \) is itself meager in \([p]\)^2).

Now, having dealt with the most intricate aspect of the proof in Lemma 3.24, we can quickly prove the main theorem of this section. Together with Theorem 3.23, Theorem 3.24 follows immediately.

Theorem 3.25. Theorem 3.17 holds for \( \bar{p} \in D^k \) and \( c: [\bar{p}] \to \{0, 1\} \) which are \( \Upsilon^\bar{p} \) measurable.

Proof. We work with the symmetric map \( c^* \) induced by \( c \) on \([\bar{p}]^2 \). The lemma follows from the previous lemmas by our standard fusion argument: As in the previous lemma, let \( \bar{p}_0 = \bar{p} \) and build a sequence \( \bar{p}_0 \geq_0 \bar{p}_1 \geq_1 \bar{p} \geq_2 \bar{p}_2 \geq_2 \ldots \) fixing standard enumerations \( \Sigma^i = \{\sigma^i_0, \sigma^i_1, \ldots\} \) of \( [\bar{p}]^2 \) and letting \( \Sigma \) be obtained as in said lemma. As a convenience, let us assume \( \bar{p} \in D^k_{\xi} \) and \( \Sigma^k = \Sigma \) for all \( k \in \omega \).

Assuming we have constructed \( \bar{p}_n \), we specify how to obtain \( \bar{p}_{n+1} \leq_{n+1} \bar{p}_n \). Let \( \xi = \sigma_n \). Observe the following:
Claim 3.26. Let $\bar{s}, \bar{s}' \in \text{init}^{\Sigma^1_n}_{n+1}(\bar{p}_n)$ such that $\bar{s} \upharpoonright \xi = \bar{s}' \upharpoonright \xi$ and $\bar{q} \in \mathbb{P}$ such that $\bar{q} \preceq^{\Sigma^1_{n+1}} \bar{p}_n$ be given. There is $\bar{q}^* \preceq^{\Sigma^1_{n+1}} \bar{q}$ such that $c^*$ is constant on $[\{\bar{q}^*_n\} \times \{\bar{q}^*_n\}] \cap \Delta^*_\xi$.

Clearly, the statement in the claim is only meaningful for $\bar{s}, \bar{s}'$ such that $\bar{s} \upharpoonright \xi = \bar{s}' \upharpoonright \xi$ (as otherwise, the intersection with $\Delta^*_\xi$ is empty).

Proof. Let $\bar{\delta} = \lambda$ if $\bar{s} = \bar{s}'$; otherwise, let $\bar{\delta}$ be least such that $\bar{s}(\bar{\delta}) \neq \bar{s}'(\bar{\delta})$. Using the strong consequence of $Y^\mathbb{P}$ measurability from Lemma 3.24, find $\bar{r}, \bar{r}' \in \mathbb{P}$ such that $\bar{r} \leq \bar{q}_* \leq \bar{q} \leq \bar{q}' \leq \bar{r}'$ and $c^*$ is constant on $[\{\bar{r}\} \times \{\bar{r}'\}] \cap \Delta^*_\xi$. This clearly proves the claim in this case, as we may thin out $\bar{q}$ above $\bar{s}$ and $\bar{s}'$, to obtain $\bar{q}^*$ as in the claim: As in the proof of Claim 3.26, find $\bar{q}^* \in \mathbb{P}$ be such that $\bar{q}^*_n \leq \bar{q}_n$. Let $\bar{p}_{n+1}$ be the last element of the sequence.

We let $\bar{q}$ be the greatest lower bound of the sequence $\{\bar{p}_n \mid n \in \omega\}$; by construction, this finishes the proof of the theorem. \qed

4. Definable discrete sets in the iterated Sacks extension

We shall now prove Theorem 1.5 in slightly more general form:

Theorem 4.1. Let $\mathcal{R}$ be a $\Sigma^1_1[a]$ binary relation on $\omega^\omega$, and let $\bar{s}$ be generic sequence of reals for an iteration of Sacks forcing over $L[a]$. Then there is $\Sigma^2_1[a]$ formula $\phi$ which defines a maximal $\mathcal{R}$-discrete set in both $L[a]$ and $L[a][\bar{s}]$. In fact, in either case the maximal $\mathcal{R}$-discrete set defined by $\phi$ is $\Delta^1_3[a]$.

It is vital for this descriptive set theoretic result that the iterated Sacks conditions $\bar{p} \in D^\mathbb{P}$ (or even just in $D^\mathbb{L}$) and functions defined on their branch space can be coded by reals. We fix the following coding mechanism.

Its definition and as well as Fact 4.2 and Lemma 4.3 are phrased for $D^\mathbb{P}$—we wish to call to the readers attention that with certain adjustments, it is possible to replace $D^\mathbb{P}$ with $D^\mathbb{L}$ everywhere.

Definition 4.2. Let $\Sigma = \langle \sigma_k \mid k < \text{lh}(\Sigma) \rangle$ and $\bar{p} \in D^\mathbb{L}_\Sigma$ be given.

1. The code for $(\bar{p}, \Sigma)$—also called the code for $\bar{p}$ (relative to $\Sigma$)—is the pair $(\Sigma^*, F^*)$, where
   (a) $\Sigma^*$ is the binary relation on $\omega$ given by $l \Sigma^* k \iff \sigma_l \leq \sigma_k$.
   (b) We may naturally write $\Sigma$ for the function $k \mapsto \sigma_k$, so that $\Sigma : \omega \to \lambda$.

   For each $\bar{t} \in \text{init}^n(\bar{p})$, let $\bar{t}^* = \bar{t} \circ \Sigma$, so that $\bar{t}^*(k) = l(\sigma_k)$ for all $k \in \text{dom}(\bar{t}^*)$, the latter being $\{k \mid \sigma_k \in \text{dom}(\bar{t})\}$. Then it holds that $F^*$ is the function such that for each $k < n < \text{lh}(\Sigma)$ and each $\bar{t} \in \text{init}^n(\bar{p})$

   \[ F^*(\bar{t}^*) = F^*_k(\bar{t}). \]

   and whose domain is $\{\bar{t}^* \mid \bar{t} \in \text{init}^n(\bar{p}), k \leq n < \text{lh}(\Sigma)\}$.

2. We say $c = (\Sigma^*, F^*)$ is a code (for a simple iterated Sacks condition) iff $c$ is the code of $\bar{p}$ relative to $\Sigma$ for some $\Sigma$ and $\bar{p} \in D^\mathbb{L}_\Sigma$.

3. Now fix a code $c = (\Sigma^*, F^*)$. We say $G^*$ codes $G$ w.r.t. $(\Sigma^*, F^*)$ iff for any $\Sigma$ and $\bar{p}$ whose code is $c$,
   (a) $\text{dom}(G^*) = \{\bar{t}^* \mid \bar{t} \in \text{init}(\Sigma, \bar{p})\}$.
   (b) $G : \text{init}(\bar{p}) \to \omega^\omega$ codes a continuous function in the sense of $\mathfrak{B}$ where $G$ is given by $G(\bar{t}) = G^*(\bar{t}^*)$ for $\bar{t} \in \text{init}(\Sigma, \bar{p})$. \unskip
(4) We also say more suggestively \( G^* \) codes the function \( G : [\bar{p}]_\Sigma \to \omega^\omega \) or codes the function \( G \) on \([\bar{p}]_\Sigma \) instead of \( G^* \) codes \( G \) w.r.t. \((\Sigma^*, F^*)\), where \((\bar{p}, \Sigma)\) is arbitrary with code \( c \)—this is meaningful because the property either holds for all such \((\bar{p}, \Sigma)\) or for none, that is it only depends on the code \( c \).

(5) Let \((\Sigma', F')\) be the code of \( p \) relative to \( \Sigma \). We say \( F^* \) is a code for a continuous function (on \([\bar{p}]_\Sigma\)) iff there is \( F : [\bar{p}] \to \omega^\omega \) such that \( F^* \) codes \( F \).

It is implicit in the above that \( \Sigma \) and \( \bar{p} \in D^p_\Sigma \) uniquely determine a code \( c(\Sigma, \bar{p}) \) for \( \bar{p} \). Conversely, for any code \( c = (\Sigma, F^*) \) there is precisely one condition \( \bar{p}(c) \) such that \( \text{supp}(\bar{p}) \in \omega_1 \) which has code \( c \), but without this restriction, many conditions share a code. If \( c = c(\bar{p}, \Sigma) = c(\bar{p}', \Sigma') \), the homeomorphism of \([\bar{p}]_\Sigma \) with \([\bar{p}']_{\Sigma'}\) arising from this fact respects the interpretation on each respective space of a code \( G^* \) for a function w.r.t. to \( c \); in this sense, \( G^* \) codes a unique function.

The following lemma is pivotal in the proof of Theorem 4.1. It has subtle consequences regarding absoluteness between two Sacks extensions of different length which we do not have space to explore here (but the phenomenon is exploited heavily in the proof of Theorem 4.4).

**Lemma 4.3.** Suppose \( \bar{p} \in D^p_\Sigma \), \( \bar{p}' \in D^{p'}_{\Sigma'} \) and both conditions have the same code (relative to \( \Sigma \) and \( \Sigma' \) respectively). For any \( q \leq \bar{p} \) such that \( q \in D^q_\Sigma \), there is \( q' \in D^{q'}_{\Sigma'} \) such that \( q' \leq \bar{p}' \) and \( [q]_\Sigma \cong [q']_{\Sigma'} \).

**Proof.** This is proved similarly to Lemma 4.12 but is notationally much more involved. Suppose \( \bar{p} \in D^p_{\Sigma, F} \) where \( \Sigma = \langle \sigma_k \mid k < \text{lh}(\Sigma) \rangle \) and \( \bar{p}' \in D^{p'}_{\Sigma', F'} \) where \( \Sigma' = \langle \sigma'_k \mid k < \text{lh}(\Sigma) \rangle \). Let \( \mathcal{S}_k = \{\sigma_0, \ldots, \sigma_{k-1}\} \cap \sigma_k \) and \( \mathcal{S}'_k = \{\sigma'_0, \ldots, \sigma'_{k-1}\} \cap \sigma'_k \). Moreover write as a shorthand
\[
\langle \bar{p} \rangle_k = \{\bar{x} \upharpoonright \mathcal{S}_k \mid \bar{x} \in [\bar{p}]_\Sigma\}
\]
and similarly for \( \bar{p}', \mathcal{S}' \) and \( F' \). For \( F = \langle F_k \mid k < \text{lh}(\Sigma) \rangle \) we introduce the more suggestive notation
\[
\bar{p}(\sigma_k)|_{\bar{x}} = F_k(\bar{x}),
\]
where \( \bar{x} \in [\bar{p}]_k \). Similarly for \( \bar{q}, \Sigma \) and \( \bar{p}', \Sigma' \).

We now define by induction on \( k \in \omega \) maps
\[
G_k : [\bar{p}]_k \to [\bar{p}']_k
\]
In the end we obtain a homeomorphism \( G : [\bar{p}]_\Sigma \to [\bar{p}']_{\Sigma'} \) by letting
\[
G(\bar{x}) = \bigcup_{k \in \omega} G_k(\bar{x} \upharpoonright \mathcal{S}_k).
\]
Simultaneously we define (in the same induction on \( k \in \omega \))
\[
H_k : [\bar{p}]_k \to V
\]
such that for each \( \bar{x} \in [\bar{p}]_k \),
\[
H_k(\bar{x}) : \bar{p}(\sigma_k)|_{\bar{x}} \to \bar{p}'(\sigma'_k)|_{G_k(\bar{x})}
\]
For any pair of perfect trees \( T, T' \) on 2, let \( H^{F,F'}_{T,T'} \) be the unique homeomorphism from \( T \) to \( T' \) which preserves length and the lexicographic ordering. Noting once more that \( [\bar{p}]_0 = \{\emptyset\} \), let \( H_0(\emptyset) = H^{p(\emptyset)}_{p'(\emptyset)} \). The map \( G_0 \) arises purely as a notational artifact and can only be the identity on \( \{\emptyset\} \).

Assume we have constructed \( G_k \) and let \( \bar{x} \in [\bar{p}]_{k+1} \) be given. Write \( \bar{x}_0 \) for \( \bar{x} \upharpoonright \mathcal{S}_k \) and \( \bar{y}_0 \) for \( G_k(\bar{x}_0) \). Define
\[
H_{k+1}(\bar{x}) = H^{p(\sigma_k)|_{\bar{x}_0}}_{p'(\sigma_k)|_{\bar{y}_0}}
\]
and define $G_{k+1}(\bar{x})$ to be the unique $\bar{y}$ such that $\bar{y} \mid S_k = \bar{y}_0$ and
\[ \bar{y}(\sigma_k) = (H_{k+1}(\bar{x}_0))^\ast, \]
the map coded by $H_{k+1}(\bar{x}_0)$ in the sense of \ref{def:homeo}. By construction, $G$ as above is indeed a homeomorphism.

Now let $\bar{q} \in D_{\Sigma,\bar{Q}}$ where $\bar{Q} = (Q_k \mid k < \text{lh}(\Sigma))$. We show how to find $\bar{Q}' = (Q'_k \mid k < \text{lh}(\Sigma))$ such that the unique $\bar{q}' \in D_{\Sigma,\bar{Q}'}$ is as required in the theorem; again we use the suggestive notation
\[ [\bar{q}]_k = \{ \bar{x} \mid \bar{x} \in [\bar{q}]_\Sigma \} \]
and again, similarly for $\bar{q}'$, $\Sigma'$ and $\bar{Q}'$.

The proof is by induction on $k \in \omega$. Let $\bar{q}'(0) = G(\emptyset)''\bar{q}(0)$. For $k > 0$ and $\bar{y} \in [\bar{q}]_k$ let
\[ \bar{q}'(\sigma_k)_{\bar{y}} = H(\bar{x})''\bar{q}(\sigma_k)|_{G_{k+1}^{-1}(\bar{y})}. \]
The proof that $\bar{q}' \leq \bar{p}'$ and that $G \upharpoonright [\bar{q}]$ is the a homeomorphism is left to the reader. \hfill $\square$

Any code $(\Sigma^*, \bar{F}^*)$ is an element of $H(\omega_1)$ and may easily be identified with a real. The of such reals is $\Pi^1_1$: one only has to express that $\Sigma^*$ is well-founded.

**Fact 4.4.** $(\Sigma^*, \bar{F}^*)$ is a code iff

1. $\Sigma^* \subseteq \omega^2$ is a well-ordering of $\omega$.
2. $\bar{F}^*$ is a partial function from $H(\omega)$ to itself.
3. Let let $\sigma$ be the order-type of $\Sigma^*$ and let $\sigma_k$ be the order-type of the initial segment of $\Sigma^*$ determined by $\{ l \in \omega \mid l \Sigma^* k \}$. Then there is a unique $\bar{p} \in \bar{P}_\sigma$ and such that $\bar{p} \in D^k_{\Sigma^*}$ and letting $F^k_l$ be the function such that for $k \leq n < \text{lh}(\Sigma)$ and $t \in \text{init}_\Sigma^k(\bar{p}, \Sigma)$
\[ F^k_l(t) = F^*(l^*) \]
and letting $\bar{F} = (F^k_l \mid k \leq n < \alpha)$, it hold that $(\Sigma^*, \bar{F}^*)$ is the code of the unique $\bar{p} \in D^k_{\Sigma, \bar{F}}$.

It is straightforward to verify that the last clause \ref{item:fact:4.4:3} is arithmetical (that \ref{item:fact:4.4:1} and \ref{item:fact:4.4:2} holds (one only has to express that the system of finite trees coded by $F^*$ is well-behaved enough to yield $\bar{p}$). Likewise, that $G^*$ codes a continuous functions on $[\bar{p}]_\Sigma$ is $\Pi^1(\Sigma^*, \bar{F}^*)$, whenever $\bar{p}$ has code $(\Sigma^*, \bar{F}^*)$ relative to $\Sigma$ (one only has to express the coded function is total).

We start by sketching a simplified, but incomplete version of the argument. For this we also assume that $\lambda = \omega_1$, $R$ is Borel and $a = \emptyset$. We work with an enumeration $\langle (\bar{p}_\xi, \bar{F}_\xi) \mid \xi < \omega_1 \rangle$ of essentially all pairs $(\bar{p}, \bar{F})$ such that $p_\xi \in \bar{P}$ and $\bar{F}$ is a $\bar{P}$ is name for an element of Baire space given as a continuous function on $[\bar{p}]$. Using the appropriate coding, we can assume this is a good $\Sigma^1_2$ enumeration, i.e. the enumeration has a $\Sigma^1_2$ coding of initial segments (see \ref{fact:4.3}).

We shall construct by recursion in $L$ a $\Sigma^1_2$ sequence $\bar{a} = (\bar{q}_\xi, z_\xi \mid \xi < \omega_1)$, where either $\bar{q}_\xi \in \bar{P}$ and $\bar{q}_\xi \leq \bar{p}_\xi$ or $\bar{q}_\xi = \emptyset$, and $\bar{z}_\xi \in [\bar{p}_\xi]$ or $\bar{z}_\xi = \emptyset$; we will do this so that
\[ A = \{ y \in \omega \mid (\exists \xi < \omega_1) [y = F(z_\xi) \vee (\exists x \in [\bar{q}_\xi]) y = F_\xi(x)] \} \]
is $R$ discrete—by absoluteness, this will hold when interpreting this definition $L[8]$ just if it holds when interpreting it in $L$, so it is enough to ensure the latter. Note that $A$ is defined only from the definable sequence $\bar{a}$ in a way that makes no reference to the generic or any other parameter, which is why $A^{L[8]}$ is lightface $\Sigma^1_2$. 

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We describe the successor step in the recursive construction: assume we have already chosen \( \langle q_\nu, z_\nu : \nu < \xi \rangle \) and thus a part of \( A \):

\[
A_\xi = \left\{ y \in \omega : \left( \exists \nu < \xi \right) \left[ y = F_\nu(z_\nu) \lor \left( \exists x \in [q_\nu] \ y = F_\nu(z_\nu) \right) \right] \right\}
\]

We sketch how to pick \( \langle q_\xi, z_\xi \rangle \).

**Step 1:** Firstly, for the purpose of ensuring maximality we can simply ignore \( p_\xi \) which don’t satisfy

\[
\text{\( p_\xi \models \left\{ F(s_C) \cup A_\xi \right\} \text{ is \( \mathcal{R}\)-discrete}; \)
\]

so if \( \text{\( \mathcal{R}\)} \) fails we simply set \( \tilde{q}_\xi = \tilde{z}_\xi = \emptyset \) (in effect, \( A_{\xi+1} = A_\xi \)). If on the other hand \( \text{\( \mathcal{R}\)} \) holds, we make a step towards ensuring discreteness of \( A \) by finding \( p_\xi' \leq p_\xi \) so that

\[
\left( \forall \bar{x} \in [p_\xi'] \right) \{ x \} \cup A_\xi \text{ is \( \mathcal{R}\)-discrete}.
\]

**Step 2:** Secondly, it will become clear that (by density) if \( \tilde{p}_\xi \) isn’t a Galvin witnesses for \( \mathcal{R} \), we can also set \( \tilde{q}_\xi = \tilde{z}_\xi = \emptyset \) without jeopardizing maximality of \( A^{L(\xi)} \). If \( \tilde{p}_\xi \) is \( \mathcal{R}\)-discrete, so is \( p_\xi' \) and we can simply set \( \tilde{q}_\xi = p_\xi' \) and obtain

\[
F([\tilde{q}_\xi]) \cup A_\xi \text{ is \( \mathcal{R}\)-discrete}.
\]

Otherwise, as \( \tilde{p}_\xi \) is a Galvin witness there are \( t_0 \) and \( t_1 \) such that every \( \bar{x}_0 \in ([p_\xi']_{t_0}) \) is \( \mathcal{R}\)-related to every \( \bar{x}_1 \in ([p_\xi']_{t_1}) \). We let \( q_\xi = \emptyset \) and \( z_\xi \) some effectively chosen branch through \( (\tilde{p}_\xi')_{t_0} \), say.

Maximality is ensured by a density argument; it is pivotal that \( \tilde{q}_\xi \models F_\xi(s_C) \in A \) in the first (discrete) case in Step 2, and \( (\tilde{p}_\xi')_{t_1} \models F_\xi(s_C) \mathcal{R} F_\xi(\tilde{z}_\xi) \) and \( \tilde{z}_\xi \in A^\omega \) in the second (complete) case in Step 2.

There are qualifications to this: distinguishing cases according to whether \( \tilde{p}_\xi \) require that \( \text{\( \mathcal{R}\)} \) be \( \Delta^1_1 \). While we could show that in fact \( \text{\( \mathcal{R}\)} \) is a \( \Sigma^1_1 \) (using a version of a Cantor-Bendixon analysis for our branch spaces), we find it interesting to note that this can be circumvented: To replace \( \text{\( \mathcal{R}\)} \) and its negation by a pair of stronger statements whose disjunction holds on a dense set of conditions—in fact, \( \text{\( \mathcal{R}\)} \) is replaced directly by \( \text{\( \mathcal{R}\)} \) (see \( \text{\( \mathcal{R}\)} \) below); in order to not miss any conditions we likewise replace the negation of \( \text{\( \mathcal{R}\)} \) by a \( \Sigma^1_3 \) condition implying it (namely \( \text{\( \mathcal{R}\)} \), below), and which can be made to hold by strengthening \( p_\xi \).

A similar problem occurs for non-Borel \( \mathcal{R} \) in Step 2: here it is the clause that \( p_\xi \) is \( \mathcal{R}\)-complete which cannot be suitably expressed by a \( \Sigma^1_2 \) condition (in fact, unlike in the first case, there is no reason to assume this should be possible). The solution follows the same strategy. Lastly, we have to deal with the possibility that \( \lambda > \omega_1 \) using Lemma \( \text{\( \mathcal{R}\)} \).

Having given a sketch for the readers orientation, we are ready to give the detailed proof.

**Proof of Theorem**

We first show the theorem under the assumption that \( \mathcal{R} \) is a \( \Delta^1_2[a] \) binary relation. The proof relativizes easily to the parameter \( a \), so we suppress it below.

Let \( \langle \bar{p}_\xi, F_\xi : \xi < \omega_1 \rangle \) enumerate all triples \( \langle p_\xi, F \rangle \) where \( p_\xi \in D^2_{\omega_1} \) and \( \text{\( \mathcal{R}\)}(p_\xi) \in \omega_1 \) and \( F : [p_\xi] \to \omega\) is continuous. Identifying conditions and continuous functions with their codes (as established in Definition \( \text{\( \mathcal{R}\)} \)), we can take this sequence to be a good \( \Sigma^1_3 \) enumeration.

We will construct a sequence \( \langle (\bar{p}_\xi, H_\xi, \tilde{q}_\xi, \tilde{z}_\xi) : \xi < \omega_1 \rangle \) such that for each \( \xi < \omega_1 \):

1. \( \bar{p}_\xi \in D^2_{\Sigma_3} \) such that \( p_\xi' \leq p_\xi \), \( H_\xi : [p_\xi']_{\Sigma_3} \to \bigcup_{\nu < \xi} [q_\nu]_{\Sigma_\omega} \) is continuous and one of the following holds:
(a) \((\forall x \in [p']_{\xi}) (\forall \nu < \xi)(\forall y \in [q_{\xi}]) \neg(F_{\xi}(y) \mathcal{R} \bar{F}_{\xi}(x))\),
(b) \((\forall x \in [p']_{\xi})(\exists \nu < \xi) (H_{\xi}(x) \in [q_{\xi}]) \land (F_{\xi} \circ \bar{H}_{\xi})(x) \mathcal{R} \bar{F}_{\xi}(x)\).

(2) \((p'_{\xi}, \bar{H}_{\xi})\) is \(\leq_{L}\)-least such that (1) holds.

(3) \(\bar{q}_{\xi} = 0\) or \(\bar{q}_{\xi} \in D_{\Sigma}^{\bar{\xi}}\) such that \(\bar{q}_{\xi} \leq p'_{\xi}\).

(4) \(\bar{z}_{\xi} = 0\) or \(\bar{z}_{\xi} \in [p'_{\xi}]\).

(5) If (11) holds or \(p'_{\xi}\) is not a Galvin witness, \(\bar{q}_{\xi} = \bar{z}_{\xi} = 0\).

(6) If (1a) holds and \([p'_{\xi}]\) is \(\mathcal{R}\)-discrete, \(\bar{q}_{\xi} = p'_{\xi}\) and \(z_{\xi} = 0\).

(7) If (1a) holds and for some \(n \in \omega\), \(t_{0}, t_{1} \in \text{init}_{n}(p'_{\xi})\) and \(\sigma \in \text{supp}(p'_{\xi})\) we have

\((\exists x_{0}, x_{1}) ((x_{0}, x_{1}) \in \Delta_{\Sigma}^{\bar{\xi}} \cap \left([([p'_{\xi}]_{t_{0}}) \times ([p'_{\xi}]_{t_{1}})\) \Rightarrow \bar{F}_{\xi}(x_{0}) \mathcal{R} \bar{F}_{\xi}(x_{1})\]);

then \(\bar{z}_{\xi}\) is the coordinate-wise left-most branch of \((p'_{\xi})_{t_{1}}\) and \(\bar{q}_{\xi} = 0\).

(8) Identifying conditions and continuous functions with their respective \emph{codes} (as discussed in \S 2 and Definition 4.2), the set

\[\{(p'_{\xi}, \bar{H}_{\xi}, \bar{q}_{\xi}, \bar{z}_{\xi}) \mid \xi < \omega_{1}\}\]

is \(\Sigma_{2}^{1}\).

Suppose for the moment that such a sequence can be found and let \(\phi(x)\) be the disjunction of the \(\Sigma_{2}^{1}\) formulas

\[\phi_{0}(x) \equiv (\exists \xi < \omega_{1}) (\exists \bar{z} \neq \emptyset \land x = F_{\xi}(\bar{z}_{\xi})),\]

\[\phi_{1}(x) \equiv (\exists \xi < \omega_{1})(\exists x \in [q_{\xi}] x = F_{\xi}(\bar{z}_{\xi})).\]

By (1), (5), (6) and by \(\Pi_{1}^{1}\)-absoluteness \(\phi(x)\) defines an \(\mathcal{R}\)-discrete set in any model.

We now show that this set is maximal in \(L[s]\), where \(s = (\mathbb{P}, L)\)-generic. The reader will notice that the same proof shows that \(\phi(x)\) defines a maximal \(\mathcal{R}\)-discrete set in \(L\), as well.

Towards a contradiction, suppose that for some \(\bar{p} \in \mathbb{P}\) and some \(\mathbb{P}\)-name \(\dot{y}\) for an element of \(\omega\), we have

\((\exists x \in \omega) (\forall x)(\phi(x) \Rightarrow \neg(x \mathcal{R} \dot{y})).\)

We can assume that \(\bar{p} \in D^{\emptyset}\) and

\((\exists x \in \omega) \phi(x) \Rightarrow \neg(x \mathcal{R} \dot{y})\)

for some continuous map \(\bar{F} : [\bar{p}] \to \omega\) (that is, of course with respect to some standard enumeration of \(\text{supp}(\bar{p})\), which we suppress). We can also assume that \(\bar{p}\) is a Galvin witness for \((\mathcal{R}, \bar{F})\).

Pick \(\xi < \omega_{1}\) so that \(\bar{p}_{\xi}\) has the same code \(c\) as \(\bar{p}\) (relative to \(\Sigma\)) and \(\bar{F}_{\xi}\) has the same code as \(\bar{F}\) (w.r.t. \(c\)). By (9) and as \([\bar{p}]_{\Sigma}\) and \([\bar{p}_{\xi}]_{\Sigma_{\xi}}\) are homeomorphic, (11) cannot possibly hold; so (1a) holds and \(\bar{q}_{\xi} \neq 0\).

As \(\bar{p}\) was chosen to be a Galvin witness and again as \([\bar{p}]\) is homeomorphic to \([\bar{p}_{\xi}]\) either the latter is \(\mathcal{R}\)-discrete, or the condition in (7) obtains. Using the method from Lemma 4.3 find a condition \(\bar{r} \leq \bar{p}\) such that \(\bar{r} \in D_{\Sigma}\) and \([\bar{r}]_{\Sigma}\) is homeomorphic to \([\bar{q}]_{\Sigma_{\xi}}\). Thus, \([\bar{r}]\) is also \(\mathcal{R}\)-discrete, or the condition in (7) obtains with \([\bar{q}]_{\Sigma_{\xi}}\) replaced by \([\bar{r}]_{\Sigma}\) everywhere.

In the first case, by (10) and as \(\bar{r} \models \bar{s}_{G} \in [\bar{r}]\) we clearly have \(\bar{r} \models \phi_{1}(\dot{y})\) which contradicts (9) since \(\bar{r} \leq \bar{p}\).

In the second case, by (7) and (10) similarly, for some \(\bar{t}_{0}\) it must be the case \((\bar{r})_{\bar{t}_{0}} \models \dot{y} \mathcal{R} \bar{s}_{\bar{z}}\). As \(\nu \models \phi_{0}(\bar{z}_{\xi})\) this too contradicts (9), finishing the proof that the \(\mathcal{R}\)-discrete set defined by \(\phi(x)\) is maximal.
Next, we show that a sequence \((\langle p'_\xi, \hat{H}_\xi, \check{q}_\xi, z_\xi \rangle \mid \xi < \omega_1)\) satisfying (11)–(13) can be found. Assume by induction that we have constructed \(\check{s} = \langle (\langle p'_\xi, \hat{H}_\xi, \check{q}_\xi, z_\xi \rangle \mid \nu < \xi)\rangle\). We must show \(p'_\xi\) satisfying (11) exists.

To see this, apply Corollary 3.13 for the \(\Pi^1_1\) set
\[
B = \{ \check{x} \in [\check{p'}_\xi]_{\Sigma^0_1} \mid (\forall \nu < \xi)(\forall \check{y} \in [\check{q}_\nu]) \neg (\check{F}_\nu(\check{y}) \cap \check{F}_\xi(\check{x}_0)) \}
\]
to find \(p \leq \check{p}_\xi\) such that \(p \in D^\check{\Sigma}_1\) and either \([p] \leq B\) or \([p] \cap B = \emptyset\). In the first case, we are done, so suppose \([p] \cap B = \emptyset\). Let \(\sigma = \supp(\check{p}_\xi)\). By absoluteness of \(\Sigma^1_1\) formulas between \(L[G \upharpoonright \sigma]\) and \(L[G]\) for any \(\mathbb{P}\)-generic \(G\), using Lemma 4.10 find \(\check{p}' \in D^\check{\Sigma}_1\) such that \(\check{p}' \leq \check{p}\) and a continuous function \(H : [\check{p}'] \to \bigcup_{\nu < \xi} [\check{q}_\nu]\) such that
\[
(\forall \check{x} \in [\check{p}'](\exists \nu < \xi) \left(H(\check{x}) \in [\check{q}_\nu]\right) \land (\check{F}_\nu \circ H)(\check{x}) \in \check{F}_\xi(\check{x}).
\]
This shows \(p'_\xi\) is well-defined.

Then if the right-hand side in (6) obtains argue as in the previous using Theorem 3.17 to find a Galvin witness \(\check{q} \leq p'_\xi\) for \((\mathcal{R}, \check{F}_\xi)\). Thus \(\check{q}_\xi\) is well-defined. Moreover, as \(\check{q}_\xi\) is a Galvin witness for \((\mathcal{R}, \check{F}_\xi)\), either \(\check{F}_\xi([\check{q}_\xi])\) is \(\mathcal{R}\)-discrete, or we can find \(n, t_0, t_1\) and \(\sigma\) as in (7). In either case, \(z_\xi\) is uniquely determined.

It remains to show:

**Claim 4.5.** The conditions (11)–(13) are uniformly \(\Sigma^1_2\).

Proof of claim. In (11) \(p'_\xi \leq \check{p}_\xi\) is arithmetic in the codes; that \(\hat{H}_\xi\) codes a function as required is \(\Pi^1_1\) in the codes. Clauses (11) and (12) are \(\Pi^1_1\) in the codes. Thus (11) is \(\sigma(\Sigma^1_1)\) (and \(\Sigma^1_1\) would suffice for the purpose of this proof). Clause (2) is therefore \(\Sigma^1_1\) by “goodness” of our enumeration (see (14)).

By the same argument as for (11), clause (3) is \(\sigma(\Sigma^1_1)\); (4) is arithmetic.

Being \(\mathcal{R}\)-discrete on some basic neighborhood is \(\Pi^1_1\) even when \(\mathcal{R}\) is \(\Sigma^1_1\); being \(\mathcal{R}\)-complete is \(\Pi^1_1\) for Borel \(\mathcal{R}\) (but \(\Pi^1_1\) when only assumed to be \(\mathcal{R}\) is \(\Sigma^1_1\)); so the property of being a Galvin witness is \(\Pi^1_1\) when \(\mathcal{R}\) is \(\Delta^1_1\) (all of this relative to the appropriate codes). Thus (5), (6) and (7) are \(\sigma(\Sigma^1_1)\) in the appropriate codes.

To show the theorem for a \(\Sigma^1_1\) binary relation \(\mathcal{R}\), we change the definition of the sequence \((\langle p'_\xi, \Sigma^0_1, \hat{H}_\xi, \check{q}_\xi, z_\xi \rangle \mid \xi < \omega_1)\) slightly: We replace (5) by the simpler

(5') If (11) holds, \(\check{q}_\xi = z_\xi = \emptyset\).

We replace (7) by the following, in which \(\check{p}\) denotes \(\check{p}_\xi\) for simplicity:

(7') If for some \(n \in \omega\) and some \(t_0, t_1 \in \text{init}_n(\check{p})\) we have that for every \(\langle p'_0, p'_1 \rangle \in \text{init}(\check{p})\) if \(t_i \subseteq t'_i\) for each \(i \in \{0, 1\}\) then there exists a pair \((\check{x}_0, \check{x}_1) \in [\check{p}'_\xi] \times [\check{p}'_\xi]\) such that \(\check{x}_i \in \Delta^*_n\) and \(\check{F}_\xi(\check{x}_0) \cap \check{F}_\xi(\check{x}_1)\); then \(z\) is the component-wise left-most branch of \(\check{q}'_\xi\).

The proof goes through almost entirely unchanged. One must check in the proof of maximality in the case when \(\check{p}_\xi\) (which we denote by \(\check{p}\) for the remainder of this proof) is not \(\mathcal{R}\)-discrete, that in (7) \(z_\xi\) and \(t_0, t_1\) are picked correctly at stage \(\xi\) so that we do indeed have that

\[(\forall \check{x}_0, \check{x}_1) \left[(\check{x}_0, \check{x}_1) \in \Delta^*_n \cap ([\check{p}_0] \times [\check{p}_1]) \Rightarrow \check{F}_\xi(\check{x}_0) \cap \check{F}_\xi(\check{x}_1)\right].\]

This would then allow us to conclude as before that \(\langle \check{q}\rangle_{\check{K}_0} \models \check{F}_\xi(\check{s}_G) \cap \check{F}_\xi(\check{z}_\xi)\). To see (12) holds, note that as \(\check{p}_\xi\) is a Galvin witness, we have that for some \(n\) and

4But note here that the property of being \(\mathcal{R}\)-complete and thus (6) can be \(\Pi^1_1\) when \(\mathcal{R}\) is only assumed to be \(\Sigma^1_1\) (all of this relative to the appropriate codes).
any pair
\[(s_0, s_1) \in \left( \bigcup_{m \geq n} \text{init}_m(\bar{p}) \right)^2\]
either
\[(\forall \bar{x}_0, \bar{x}_1) \left[ (\bar{x}_0, \bar{x}_1) \in \Delta^*_\sigma_n \cap ([\bar{p}_{\ell_0}] \times [\bar{p}_{\ell_1}]) \Rightarrow F_\xi(\bar{x}_0) \mathcal{R} F_\xi(\bar{x}_1) \right].
\]
or
\[(\forall \bar{x}_0, \bar{x}_1) \left[ (\bar{x}_0, \bar{x}_1) \in \Delta^*_\sigma_n \cap ([\bar{p}_{\ell_0}] \times [\bar{p}_{\ell_1}]) \Rightarrow \neg (F_\xi(\bar{x}_0) \mathcal{R} F_\xi(\bar{x}_1)) \right].
\]
But \[(7)\] precludes the second possibility for any \((s_0, s_1)\) satisfying \((13)\) such that \(s_i \subseteq \ell_i\) for each \(i \in \{0, 1\}\); hence \((12)\) holds.

It remains to show that the set obtained is \(\Sigma^1_1\); this is a straightforward simplification of the previous argument, as \((5)\) has been deleted and \((7)\) has merely been replaced by \((6)\), which is easily seen to be \(\Sigma^1_1\) requirement on \(\bar{q}\). \(\square\)

5. A co-analytic maximal orthogonal family in the iterated Sacks extension

In \((6)\), the authors show that the space \(P(X)\) of Borel probability measures on a perfect effective Polish space \(X\) is itself an effective Polish space, and that the orthogonality relation is Borel. The same holds for \(P^*(X)\), the Borel subset of \(P(X)\) consisting of atomless Borel probability measures. It follows immediately from Theorem \((15)\) that in the iterated Sacks extension of length \(\omega_1\) there is a \(\Delta^1_2\) mof in \(P^*(X)\). As any measure with an atom is non-orthogonal to a Dirac measure we immediately obtain Theorem \((17)\) by quoting the following result from \((20)\):

**Theorem 5.1.** If \(X\) is a perfect effective Polish space and there is a \(\Delta^1_2\) (equivalently, a \(\Sigma^1_2\)) mof in \(P^*(X)\) then there is a \(\Pi^1_2\) mof in \(P^*(X)\).

We round off the results from the present paper as well as from \((20)\) with an observation due to B. Miller. We see this as evidence that there is no substantially simpler way of finding mofs in forcing extension with new reals, e.g. one cannot find a mof in \(L\) that is indestructible by forcing.

**Theorem 5.2.** Suppose \(M\) be an inner model of \(V\) and suppose \(\mathcal{P}(\omega)^M \neq \mathcal{P}(\omega)\) and let \(X\) be a perfect effective Polish space. If \(\mathcal{A} \subseteq M\) and \(\mathcal{A} \models \mathcal{A}\) be a maximal orthogonal families of measures on \(P(X)\) then \(\mathcal{A}\) is not maximal in \(V\).

**Proof.** For the purpose of this proof, identify \(\mathcal{C}\) with a subset of \(X\). Consider the map \(F: \mathcal{C}^2 \rightarrow \mathcal{C}\) given by
\[F(x, y)(n) = \begin{cases} x(\frac{n}{2}) & \text{if } n \text{ is even,} \\ y(\frac{n-1}{2}) & \text{if } n \text{ is odd.} \end{cases}\]
for \(n \in \omega\). This gives rise to a perfect family of perfect sets \(\langle P_x \mid x \in \mathcal{C} \rangle\), where \(P_x = \{F(x, y) \mid y \in \mathcal{C}\}\).

Then for a \(\mu \in P(X)\), define an ordering on the set \(B_\mu = \{x \mid \mu(P_x) > 0\}\) by letting
\[x <_\mu x' \iff \left[ \mu(P_x) > \mu(P_{x'}) \lor (\mu(P_x) = \mu(P_{x'}) \land x' < x) \right].\]
The ordertype of \(<_\mu\) on \(B_\mu\) is \(\omega\). Thus for any \(\nu \in P(X)^M\), since the measure algebra has the ccc, there is a countable \(H \subseteq \mathcal{C} \subseteq M\) such that
\[(\forall x \in \mathcal{C}) \ x \in H \iff \nu(P_x) > 0.\]
This statement is \(\Delta^1_1(H)\) and hence absolute between \(M\) and \(V\). In particular, if \(y \in \mathcal{C}^V \setminus M\) and \(\nu \in M\) it must be the case that \(\nu(P_y) = 0\).
Now let $A$ as in the Theorem be given and fix $y \in C \setminus M$. Find by the product measure construction $\mu \in P(X)$ such that $\mu(P_y) > 0$. By the above, for any $\nu \in M$ we have $\nu \perp \mu$. □

6. Questions

1. Does Galvin’s Theorem have an analogue for iterated Miller forcing?
2. In which forcing extensions other than iterated Sacks extensions of $L$ do analytic relations have (lightface) projective maximal discrete sets? Is there a $\Pi^1_1$ (or a $\Delta^1_2$) $\mathbb{M}$ in the Laver or Silver extension?
3. What is the consistency strength of “Every projective binary relation on a Polish space has a projective transversal?”

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Department of Mathematical Sciences, University of Copenhagen, Universitetsparken
5, 2100 Copenhagen, Denmark

E-mail address: david.s@math.ku.dk