Modular invariance of bosonic string on orbifolds

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Abstract
I construct a complete 1-loop partition function of a bosonic closed string on orbifolds. Furthermore, I derive sufficient conditions for the modular invariance of the partition function.

1 Introduction
Orbifold compactification models of the heterotic string [1] are candidates for unified theories. In the string theory, modular invariance is necessary for consistency. But modular invariance of orbifold models is non-trivial. A modular invariance condition of orbifold models is naively the level matching [2, 3]. In contrast, a free fermion model was built as another possibility [4]. Although the free fermion model is given by fermionizing all internal coordinates, the fermionization on orbifolds is restricted on the \( \mathbb{Z}_2 \) orbifold. Therefore we should consider the internal coordinates as bosonic variables so as to generalize the model to include general orbifolds. A 1-loop partition function of a bosonic closed string on orbifolds was constructed in the ref.[5]. But I think that this partition function is not complete in the points explained in the text. In this paper I construct a complete 1-loop partition function of a bosonic closed string on orbifolds. Furthermore, I derive sufficient conditions for the modular invariance of the partition function.

2 Preliminaries
In this section I set up the framework for this paper. I consider a heterotic string compactified to four space-time dimensions. In the light-cone gauge, there are the following world-sheet degrees of freedom: eight transverse bosons \( X^i(z, \bar{z}) \), eight right-moving transverse fermions \( \psi^i(\bar{z}) \) where \( i = 2, 3, \ldots, 9 \), and sixteen left-moving bosons \( X_f^I(z) \) where \( I = 10, 11, \ldots, 25 \). Here the world-sheet coordinates are \( z = e^{-i(\sigma^1+i\sigma^2)} \) and \( \bar{z} = e^{i(\sigma^1-i\sigma^2)} \). In this paper I particularly pay attention to the bosons \( X^n \) where \( n = 4, 5, \ldots, 9 \) (corresponding to six internal coordinates).

An orbifold is obtained from flat space by the following identification under a discrete group \( G \). An element \( g \) of \( G \) acts on the coordinates as a rotation \( \theta \) and a translation \( 2\pi R \),

\[
g : X^n \to \theta^m X^m + 2\pi R^n, \tag{2.1}
\]

where \( m, n = 4, 5, \ldots, 9 \). Let \( N \) be the smallest integer such as \( \theta^N = 1 \), then this orbifold is called a \( \mathbb{Z}_N \) orbifold. It is convenient to change the basis of the coordinates so that the

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rotation $\theta$ becomes a diagonal matrix. Then the real bosons $X^a$ become complex bosons $\phi^a$ where $a = 1, 2, 3$. In this basis, the action of $g$ becomes

$$ g : \phi^a \rightarrow e^{2\pi i \alpha v^a} \phi^a + 2\pi \ell^a, $$

where $\alpha$ is an integer, $v^a$ is a multiple of $1/N$ in the $\mathbb{Z}_N$ orbifold, and $\ell$ is an element of a 3-dimensional complex lattice $\Lambda$. The lattice $\Lambda$ must be invariant under the rotation $\text{diag}(e^{2\pi i v^1}, e^{2\pi i v^2}, e^{2\pi i v^3})$ so that the orbifold is well-defined. $\alpha v^a$ is defined by $\alpha v^a = \alpha v^a \pmod{1}$ and $0 \leq \alpha v^a < 1$. Then the elements of the group $G$ are in one-to-one correspondence with the parameters $\alpha v^a$ and $\ell$, which are denoted by $g(\alpha v^a, \ell)$. A representation of $g(\alpha v^a, \ell)$ is defined by

$$ g^{-1}(\alpha v^a, \ell) \phi^a g(\alpha v^a, \ell) = e^{2\pi i \alpha v^a} \phi^a + 2\pi \ell^a, $$

and $g(\alpha v^a, \ell)$ is a unitary operator.

### 3 Partition function

In this section I construct the 1-loop partition function of the bosonic field $\phi$ on the orbifold. The heterotic string theory contains only the closed string. The 1-loop world-sheet of a closed string is a torus. The torus is described by the identifications $z \cong e^{2\pi i z}$ and $z \cong e^{2\pi i \bar{z}}$, where $\tau$ is a complex number. Therefore the field $\phi^a$ satisfies the periodic boundary conditions $\phi^a(e^{2\pi i z}, e^{-2\pi i \bar{z}}) = \phi^a(z, \bar{z})$ and $\phi^a(e^{2\pi i \bar{z}}, e^{-2\pi i z}) = \phi^a(z, \bar{z})$. On the orbifold, because of the identification under the group $G$, the following boundary conditions are allowed,

$$ \phi^a(e^{2\pi i z}, e^{-2\pi i \bar{z}}) = g^{-1}(\alpha v, \ell) \phi^a(z, \bar{z}) g(\alpha v, \ell), \quad (3.1a) $$

$$ \phi^a(e^{2\pi i \bar{z}}, e^{-2\pi i z}) = g^{-1}(\beta v, \ell') \phi^a(z, \bar{z}) g(\beta v, \ell'), \quad (3.1b) $$

where $\alpha$ and $\beta$ are integers and $\ell, \ell' \in \Lambda$. Then the partition function is given by

$$ Z(\tau) = \frac{1}{N} \sum_{\alpha=0}^{N-1} \sum_{\beta=0}^{N-1} Z_{\alpha v^a, \beta v^a}(\tau), \quad (3.2) $$

with

$$ Z_{\alpha v^a, \beta v^a}(\tau) = \sum_{\ell} \sum_{\ell'} \text{Tr} \left\{ q^{H_{\alpha v^a, \beta v^a}(\tau)} g(\alpha v, \ell) \right\}, \quad (3.3) $$

where $q = e^{2\pi i \tau}$, and $H_{\alpha v^a, \beta v^a}$ and $\tilde{H}_{\alpha v^a, \beta v^a}$ are left- and right-moving Hamiltonians respectively.

Let $\phi_{\alpha v^a, \beta v^a}^a$ be a field which satisfies the boundary condition $(3.1a)$. Then I obtain

$$ \phi_{\alpha v^a, \beta v^a}^a(e^{2\pi i z}, e^{-2\pi i \bar{z}}) = e^{2\pi i \alpha v^a} \phi_{\alpha v^a, \beta v^a}^a(z, \bar{z}) + 2\pi \ell^a. \quad (3.4) $$

In case of $\alpha v^a = 0$, this field has the mode expansion

$$ \phi_{0,0}^a(z, \bar{z}) = \chi^a - i\rho_L^a \log z - i\rho_R^a \log \bar{z} + i \sum_{n=1}^{\infty} \frac{1}{n} (\beta_n z^{-n} + \gamma_n^a z^n + \tilde{\beta}_n \bar{z}^{-n} + \tilde{\gamma}_n^a \bar{z}^n), \quad (3.5) $$

where I defined the left- and right-moving momenta respectively by

$$ \rho_L^a = \rho^a + \frac{\ell^a}{2}, \quad \rho_R^a = \rho^a - \frac{\ell^a}{2}. $$

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\( \chi \) is the center of mass position, \( \gamma_n^+ \) and \( \bar{\gamma}_n^+ \) are the creation operators, and \( \beta_n \) and \( \bar{\beta}_n \) are the annihilation operators. The commutation relations of these operators are

\[
\begin{align*}
[\chi^a, \rho^{ib}] &= [\chi^a_L, \rho^{ib}_L] = [\chi^a_R, \rho^{ib}_R] = i\delta^{ab}, \quad (3.7a) \\
[\beta^a_m, \beta^{ib}_n] &= [\bar{\gamma}^a_m, \gamma^{ib}_n] = [\bar{\beta}^a_m, \bar{\beta}^{ib}_n] = [\bar{\gamma}^{a}_m, \bar{\gamma}^{ib}_n] = m\delta_{mn}\delta^{ab}, \quad (3.7b)
\end{align*}
\]

where \( \chi^a_L \) and \( \chi^a_R \) are the left- and right-moving parts of \( \chi^a \), respectively. In case of \( \alpha \overline{\alpha}^{\mu} \neq 0 \), on the other hand, the field \( \phi^a_{\alpha \overline{\alpha}, \ell} \) has the mode expansion

\[
\phi^a_{\alpha \overline{\alpha}, \ell}(z, \overline{z}) = \chi^a + i \sum_{n=1}^{\infty} \left( \frac{\beta^a_n - \bar{\gamma}^a_n}{n - \alpha \overline{\alpha}^{\mu}} z^{-n + \alpha \overline{\alpha}^{\mu} - 1} + \frac{\gamma^a_{n+\alpha \overline{\alpha}^{\mu} - 1}}{n + \alpha \overline{\alpha}^{\mu} - 1} z^{n + \alpha \overline{\alpha}^{\mu} - 1} \\
+ \frac{\bar{\gamma}^a_{n+\alpha \overline{\alpha}^{\mu} - 1}}{n + \alpha \overline{\alpha}^{\mu} - 1} z^{-n - \alpha \overline{\alpha}^{\mu} + 1} + \frac{\gamma^a_{n+\alpha \overline{\alpha}^{\mu} - 1}}{n - \alpha \overline{\alpha}^{\mu}} z^{n - \alpha \overline{\alpha}^{\mu}} \right). \quad (3.8)
\]

The commutation relations of the creation and annihilation operators are similar to (3.7). Then, to satisfy the boundary condition (3.4), \( \chi^a \) must be a fixed point which satisfies

\[
e^{2\pi \alpha \overline{\alpha}^{\mu}} \chi^a + 2\pi \ell^a = \chi^a. \quad (3.9)
\]

Now I decompose the left-moving Hamiltonian \( H_{\alpha \overline{\alpha}, \ell} \) into \( H_{\rho_{\alpha \overline{\alpha}, \ell}} \) and \( H_{N_{\alpha \overline{\alpha}, \ell}} \), where \( H_{\rho_{\alpha \overline{\alpha}, \ell}} \) depends on the momentum \( \rho_L \). Let the right-moving Hamiltonian \( \tilde{H}_{\alpha \overline{\alpha}, \ell} \) be similarly divided.

\[
H_{\alpha \overline{\alpha}, \ell} = H_{\rho_{\alpha \overline{\alpha}, \ell}} + H_{N_{\alpha \overline{\alpha}, \ell}}, \quad \tilde{H}_{\alpha \overline{\alpha}, \ell} = \tilde{H}_{\rho_{\alpha \overline{\alpha}, \ell}} + \tilde{H}_{N_{\alpha \overline{\alpha}, \ell}}. \quad (3.10)
\]

Here

\[
\begin{align*}
H_{\rho_{\alpha \overline{\alpha}, \ell}} &= \rho^a_L \circ \rho_L = (\rho^a_L + 7/2) \circ (\rho + \ell/2), \quad (3.11a) \\
\tilde{H}_{\rho_{\alpha \overline{\alpha}, \ell}} &= \rho^a_R \circ \rho_R = (\rho^a_R - 7/2) \circ (\rho - \ell/2), \quad (3.11b)
\end{align*}
\]

(the symbol \( \circ \) is defined by \( x \circ y = \sum_{a=1}^{3} \delta_{\alpha \overline{\alpha}^{\mu}} x^a y^a \))

\[
\begin{align*}
H_{N_{\alpha \overline{\alpha}, \ell}} &= \sum_{n=1}^{\infty} \left\{ (n - \alpha \overline{\alpha}^{\mu}) \cdot N_{n-\alpha \overline{\alpha}^{\mu}} + (n + \alpha \overline{\alpha}^{\mu} - 1) \cdot N'_{n+\alpha \overline{\alpha}^{\mu} - 1} \right\} + \frac{1}{2} \alpha \overline{\alpha}^{\mu} \cdot (1 - \alpha \overline{\alpha}^{\mu}) - \frac{3}{12}, \quad (3.11c) \\
\tilde{H}_{N_{\alpha \overline{\alpha}, \ell}} &= \sum_{n=1}^{\infty} \left\{ (n + \alpha \overline{\alpha}^{\mu} - 1) \cdot \tilde{N}_{n+\alpha \overline{\alpha}^{\mu} - 1} + (n - \alpha \overline{\alpha}^{\mu}) \cdot \tilde{N}'_{n-\alpha \overline{\alpha}^{\mu}} \right\} + \frac{1}{2} \alpha \overline{\alpha}^{\mu} \cdot (1 - \alpha \overline{\alpha}^{\mu}) - \frac{3}{12}. \quad (3.11d)
\end{align*}
\]

\( N_r^a, N_r^{a\prime}, \tilde{N}_r^a \) and \( \tilde{N}_r^{a\prime} \) are the occupation numbers defined as \( N_r^a = \beta^a_r / r^a \), \( N_r^{a\prime} = \gamma^a_r / r^a \), \( \tilde{N}_r^a = \tilde{\beta}^a_r / r^a \) and \( \tilde{N}_r^{a\prime} = \tilde{\gamma}^a_r / r^a \), but \( N_{r_0}^a = N_{r_0}^{a\prime} = 0 \).

Now an element \( g_{\alpha \overline{\alpha}, \ell}(\beta^a, \ell') \) of the group \( G \) is defined by

\[
g_{\alpha \overline{\alpha}, \ell}^{-1}(\beta^a, \ell') \phi^a_{\alpha \overline{\alpha}, \ell}(z, \overline{z}) g_{\alpha \overline{\alpha}, \ell}(\beta^a, \ell') = e^{2\pi i \beta^a} \phi^a_{\alpha \overline{\alpha}, \ell}(z, \overline{z}) + 2\pi \ell'^a. \quad (3.12)
\]

up to an arbitrary constant \( C_{\alpha \overline{\alpha}, \ell} \). This constant is a phase factor \( \left(C_{\alpha \overline{\alpha}, \ell}^2 = 1\right) \), because \( g_{\alpha \overline{\alpha}, \ell}(\beta^a, \ell') \) is a unitary operator. In ref. [8], this constant was chosen to be independent of \( \ell \) and \( \ell' \). But \( \ell \) and \( \ell' \) dependence of this constant is necessary for convergence of the partition function (3.3) summed over all \( \ell \) and \( \ell' \). In this paper, therefore, I consider \( \ell \) and \( \ell' \) dependence
of this constant. Now I divide \( g_{\beta v, \ell} (\beta v, \ell') \) except this constant into a factor \( h_{\rho; \beta v}(\beta v, \ell') \) which depends the momenta and the other factor \( h_{N; \beta v}(\beta v), \)

\[
g_{\beta v, \ell}(\beta v, \ell') = C_{\beta v, \ell} h_{\rho; \beta v}(\beta v, \ell') h_{N; \beta v}(\beta v), \tag{3.13}
\]

where

\[
h_{\rho; \beta v}(\beta v, \ell') = \exp \left\{ -2\pi i (\ell' \circ \rho + \ell' \circ \rho') \right\} \exp \left\{ 2\pi i \beta v \circ (K_L + K_R) \right\}, \tag{3.14a}
\]

\[
h_{N; \beta v}(\beta v) = \exp \left\{ 2\pi i \beta v \cdot (J_{\beta v} + \tilde{J}_{\beta v}) \right\}, \tag{3.14b}
\]

\[
K^a_L = -i \left( \chi^a_L \rho^a_L - \rho^a_L \chi^a_L \right), \quad K^a_R = -i \left( \chi^a_R \rho^a_R - \rho^a_R \chi^a_R \right), \tag{3.14c}
\]

\[
J^a_{\beta v} = \sum_{n=1}^{\infty} (N^a_n - N^a_{n+1}), \quad \tilde{J}^a_{\beta v} = \sum_{n=1}^{\infty} (\tilde{N}^a_n - \tilde{N}^a_{n+1}). \tag{3.14d}
\]

In case of \( \alpha \beta v \neq 0 \), \( \chi^a \) must be a fixed point which satisfies

\[
e^{2\pi i \beta v} \chi^a + 2\pi \ell^a = \chi^a, \tag{3.15}
\]

so that the field \( \chi^a \) is transformed according to \( \beta v \), because there is no noncommutative operator to \( \chi^a \).

Thus, the partition function \( Z_{\beta v} \) is given by product of two functions \( Z_{\rho; \beta v}(\tau) \) and \( Z_{N; \beta v}(\tau), \)

\[
Z_{\beta v}(\tau) = Z_{\rho; \beta v}(\tau) Z_{N; \beta v}(\tau), \tag{3.16}
\]

with

\[
Z_{\rho; \beta v}(\tau) = \sum_\ell \sum_{\ell'} C_{\beta v, \ell} \text{Tr} \left\{ q^{H_{\delta; \beta v} / \beta v} \delta^{H_{\delta; \beta v} / \beta v} h_{\rho; \beta v}(\beta v, \ell') \right\}, \tag{3.17a}
\]

\[
Z_{N; \beta v}(\tau) = \text{Tr} \left\{ q^{H_{\delta; \beta v} / \beta v} \delta^{H_{\delta; \beta v} / \beta v} h_{N; \beta v}(\beta v) \right\}. \tag{3.17b}
\]

First, I calculate the function \( Z_{\rho; \beta v}(\tau) \). I consider common eigenstates of the operators \( H_{\rho; \beta v}, \tilde{H}_{\rho; \beta v} \) and \( h_{\rho; \beta v}(\beta v, \ell') \) so as to calculate the function \( Z_{\rho; \beta v}(\tau) \). In ref. [3], these eigenstates are not discussed. But, if these eigenstates are not discussed then the function \( Z_{\rho; \beta v}(\tau) \) cannot be exactly derived. In this paper, therefore, I derive these eigenstates and their eigenvalues. Let \( | \rho_L, \rho_R \rangle_{\beta v} \) be a common eigenstate of both operators \( \rho_L \) and \( \rho_R \). This state is an eigenstate of both \( H_{\rho; \beta v} \) and \( \tilde{H}_{\rho; \beta v} \). But this state is not an eigenstate of \( h_{\rho; \beta v}(\beta v, \ell') \), because the operator \( e^{2\pi i \beta v (K_L + K_R)} \) in \( h_{\rho; \beta v}(\beta v, \ell') \) rotate the momenta \( \rho_L \) and \( \rho_R \) of this state,

\[
\rho^a_L \left( e^{2\pi i \beta v (K_L + K_R)} | \rho_L, \rho_R \rangle_{\beta v} \right) = e^{2\pi i \beta v} \rho^a_L \left( e^{2\pi i \beta v (K_L + K_R)} | \rho_L, \rho_R \rangle_{\beta v} \right), \tag{3.18a}
\]

\[
\rho^a_R \left( e^{2\pi i \beta v (K_L + K_R)} | \rho_L, \rho_R \rangle_{\beta v} \right) = e^{2\pi i \beta v} \rho^a_R \left( e^{2\pi i \beta v (K_L + K_R)} | \rho_L, \rho_R \rangle_{\beta v} \right). \tag{3.18b}
\]

Therefore I consider linear combinations of the states,

\[
| \rho_L \circ \rho_L, \rho_R \circ \rho_R \rangle_{\beta v, \ell'} = \frac{1}{\sqrt{m}} \sum_{n=0}^{n-1} \exp \left\{ 2\pi i \chi^a_n \right\} \exp \left\{ 2\pi i \beta v \circ (K_L + K_R) \right\} | \rho_L, \rho_R \rangle_{\beta v}, \tag{3.19}
\]
where \( m \) is the smallest positive integer such that \( \overline{m\beta^a v^a} = 0 \) for \( \forall a \) (\( \alpha v^a = 0 \), (\( \rho_L^a \neq 0 \) or \( \rho_R^a \neq 0 \)), and \( k = 0, 1, \ldots, m - 1 \). Let \( \lambda_0^k \) be zero. In addition, let \( \lambda_n^k \) satisfy

\[
\frac{1}{m} \sum_{n=0}^{m-1} \exp \left( -2\pi i \lambda_n^k \right) \exp \left( 2\pi i \lambda_n^{k'} \right) = \delta^{kk'},
\]

so that these states \([51.19]\) belong to the orthonormal system. These states are eigenstates of both \( H_{\rho, \overline{\alpha v}} \ell = \rho_L \circ \rho_L \) and \( \tilde{H}_{\rho, \overline{\alpha v}} \ell = \rho_R \circ \rho_R \) because of \([3.18a]\) and \([3.18b]\). The operation of \( h_{\rho, \overline{\alpha v}} (\beta v, \ell') \) to these states becomes

\[
h_{\rho, \overline{\alpha v}} (\beta v, \ell') | \rho_L \circ \rho_L, \rho_R \circ \rho_R, k \rangle = \frac{1}{\sqrt{m}} \sum_{n=0}^{m-1} \exp \left( 2\pi i \lambda_n^k \right) \exp \left\{ -2\pi i (\overline{\ell'} \circ \rho + \ell' \circ \rho) \right\} \exp \left\{ 2\pi i (\overline{\beta v} \circ (K_L + K_R)) \right\} \times \exp \left\{ 2\pi i (n + 1) \beta v \circ (K_L + K_R) \right\} | \rho_L, \rho_R \rangle
\]

where \( \lambda_n^k \) are

\[
\lambda_n^k = n \lambda_k^0 - \left\{ \overline{\ell'} \circ \left( e^{2\pi i \alpha v} + e^{2\pi i \beta v} + \cdots + e^{2\pi i m \beta v} \right) \rho + c.c. \right\} \mod 1,
\]

so that these states are eigenstates of \( h_{\rho, \overline{\alpha v}} (\beta v, \ell') \). Then the eigenvalue of \( h_{\rho, \overline{\alpha v}} (\beta v, \ell') \) is \( \exp \left( -2\pi i \lambda_k^0 \right) \). If \( \rho_a^\ell \neq 0 \) then \( \rho_L^a \neq 0 \) or \( \rho_R^a \neq 0 \) because of the relations \([3.6]\). Therefore, if \( \rho_a^\ell \neq 0 \) then \( \overline{m\beta^a v^a} = 0 \) owing to the definition of \( m \). Then, if \( \overline{3\beta v^a} \neq 0 \) then the summation \( e^{2\pi i \alpha v} + e^{2\pi i \beta v} + \cdots + e^{2\pi i m \beta v} \) is zero. On the other hand, if \( \overline{3\beta v^a} = 0 \) then this summation is \( m \). Thus \( \lambda_m^k \) are

\[
\lambda_m^k = m \lambda_k^0 - m (\overline{\ell'} \circ \rho + \ell' \circ \rho) \mod 1,
\]

where the symbol \( \circ \) is defined by \( x \circ y = \sum_{a=1}^{3} \epsilon_{\alpha v a}^\ell \delta_{\beta^a v}^\ell x^a y^a \). \( \lambda_m^k \) must be equal to \( \lambda_0^k \) modulo 1 (\( \lambda_0^0 = 0 \)). Hence \( \lambda_k^0 \) are

\[
\lambda_k^0 = \overline{\ell'} \circ \rho + \ell' \circ \rho + \frac{k}{m},
\]

so that \( \lambda_k^0 \) satisfy the orthonormal condition \([5.20]\).

In the function \( Z_{\rho, \overline{\alpha v}} (\tau) \), the parameter \( k \) is included in the eigenvalue of only \( h_{\rho, \overline{\alpha v}} (\beta v, \ell') \). Therefore, in the trace in \( Z_{\rho, \overline{\alpha v}} (\tau) \), the eigenvalues of only \( h_{\rho, \overline{\alpha v}} (\beta v, \ell') \) are summed over \( k \),

\[
\sum_{k=0}^{m-1} \exp \left\{ -2\pi i \left( \overline{\ell'} \circ \rho + \ell' \circ \rho + \frac{k}{m} \right) \right\} = \begin{cases} \exp \left\{ -2\pi i \left( \overline{\ell'} \circ \rho + \ell' \circ \rho \right) \right\} & (m = 1) \\ 0 & (m > 1) \end{cases}
\]

Hence only the states of \( m = 1 \) contribute to \( Z_{\rho, \overline{\alpha v}} (\tau) \). In the states of \( m = 1 \), \( \rho_L^a = \rho_R^a = 0 \) for \( \forall a \) (\( \alpha v^a = 0 \), \( \beta^a v^a \neq 0 \)) because of the definition of \( m \). If \( \rho_L^a = \rho_R^a = 0 \) then \( \rho_a^\ell = 0 \) owing to the relations \([3.6]\). Therefore only the states of \( \rho_a^\ell = \ell^a = 0 \) for \( \forall a \) (\( \alpha v^a = 0 \), \( \beta^a v^a \neq 0 \)) contributes to \( Z_{\rho, \overline{\alpha v}} (\tau) \). Since \( \rho \) is the vector in \( \overline{\alpha v^a} = 0 \) space, only \( \rho \) in \( \overline{\alpha v^a} = \beta^a v^a = 0 \) space
contributes to \( Z_{\rho_{\mathbf{\beta}^a}}(\tau) \). I write the momentum in \( \mathbf{\alpha} \mathbf{v} = \mathbf{\beta} \mathbf{v} = 0 \) space as \( \rho_0 \). In addition, the set of such \( \ell \) as contribute to \( Z_{\rho_{\mathbf{\beta}^a}}(\tau) \) is

\[
\Lambda_{\ell_{\mathbf{\beta}^a}} = \{ \ell \mid \ell \in \Lambda, \forall a (\mathbf{\alpha} v^a = 0, \mathbf{\beta} v^a \neq 0), \ell^a = 0 \} .
\]  

(3.26)

In case of \( \mathbf{\alpha} v^a \neq 0 \), if \( \mathbf{\beta} v^a = 0 \) then \( \ell^a = 0 \) because of the fixed point condition (3.16). Therefore the set of such \( \ell' \) as contribute to \( Z_{\rho_{\mathbf{\beta}^a}}(\tau) \) is

\[
\Lambda_{\ell'_{\mathbf{\beta}^a}} = \{ \ell' \mid \ell' \in \Lambda, \forall a (\mathbf{\alpha} v^a \neq 0, \mathbf{\beta} v^a = 0), \ell'^a = 0 \} .
\]  

(3.27)

But all elements of \( \Lambda_{\ell_{\mathbf{\beta}^a}} \) and \( \Lambda_{\ell'_{\mathbf{\beta}^a}} \) do not necessarily contribute to \( Z_{\rho_{\mathbf{\beta}^a}}(\tau) \), because \( \chi \) satisfied to the both conditions (3.9) and (3.15) must exist. In ref. [5], it is not considered that \( \chi \) must satisfy the both conditions (3.9) and (3.15). But, if it is not considered then the function \( Z_{\rho_{\mathbf{\beta}^a}}(\tau) \) cannot be exactly derived. In this paper, therefore, I introduce the factor

\[
\xi_{\mathbf{\alpha}v,\ell,\ell'} = \begin{cases} 1 & \left( \exists \chi, \forall a (\mathbf{\alpha} v^a \neq 0, \mathbf{\beta} v^a \neq 0), e^{2\pi i \mathbf{\alpha} v^a} \chi^a + 2\pi \ell^a = e^{2\pi i \mathbf{\beta} v^a} \chi^a + 2\pi \ell'^a = \chi^a \right) \nonumber \\ 0 & \text{(otherwise)} \end{cases}
\]  

(3.28)

In addition, \( H_{\rho,\mathbf{\alpha}v,\ell} = (\rho_0 + \ell/2) \bullet (\rho_0 + \ell/2) \) and \( \tilde{H}_{\rho,\mathbf{\alpha}v,\ell} = (\rho_0 - \ell/2) \bullet (\rho_0 - \ell/2) \), because \( \mathbf{\alpha} v^a = 0, \mathbf{\beta} v^a \neq 0 \) components of \( \rho \) and \( \ell \) are zero. Thus the function (3.17a) becomes

\[
Z_{\rho_{\mathbf{\beta}^a}}(\tau) = \sum_{\ell \in \Lambda_{\mathbf{\beta}^a}} \sum_{\ell' \in \Lambda_{\mathbf{\beta}^a}} \xi_{\mathbf{\alpha}v,\ell,\ell'} C_{\mathbf{\alpha}v,\ell}^{\mathbf{\beta}v,\ell'} 
\times \int d^d \rho_0 d^d \rho_0' q^{(\rho_0 + \ell/2)} q^{(\rho_0 - \ell/2)} e^{-2\pi i (\ell \bullet \rho_0 + \ell' \bullet \rho_0')} ,
\]  

(3.29)

where \( d \) is the dimension of \( \mathbf{\alpha} v^a = \mathbf{\beta} v^a = 0 \) space. The \( \mathbf{\alpha} v^a \neq 0 \) or \( \mathbf{\beta} v^a \neq 0 \) components of \( \ell \) and \( \ell' \) contribute to only the factor \( \xi_{\mathbf{\alpha}v,\ell,\ell'} C_{\mathbf{\alpha}v,\ell}^{\mathbf{\beta}v,\ell'} \) in \( Z_{\rho_{\mathbf{\beta}^a}}(\tau) \). Therefore I sum up this factor over \( \ell \) and \( \ell' \) whose \( \mathbf{\alpha} v^a = \mathbf{\beta} v^a = 0 \) components are fixed.

\[
B_{\mathbf{\alpha}v,\ell_0,\ell'_0}^{\mathbf{\beta}v,\ell_0} = \sum_{\ell_0 \in \Lambda_{\mathbf{\beta}^a}} \sum_{\ell'_0 \in \Lambda_{\mathbf{\beta}^a}} \xi_{\mathbf{\alpha}v,\ell_0,\ell'_0} C_{\mathbf{\alpha}v,\ell_0}^{\mathbf{\beta}v,\ell'_0},
\]  

(3.30)

where \( \ell_0 \) and \( \ell'_0 \) are the vectors of \( \ell \) and \( \ell' \) respectively projected onto \( \mathbf{\alpha} v^a = \mathbf{\beta} v^a = 0 \) space,

\[
\Lambda_{\mathbf{\alpha}v,\ell_0} = \{ \ell \mid \ell \in \Lambda_{\mathbf{\beta}^a}, \forall a (\mathbf{\alpha} v^a = \mathbf{\beta} v^a = 0), \ell^a = \ell_0^a \} ,
\]

(3.31a)

\[
\Lambda_{\mathbf{\alpha}v,\ell'_0} = \{ \ell' \mid \ell' \in \Lambda_{\mathbf{\beta}^a}, \forall a (\mathbf{\alpha} v^a = \mathbf{\beta} v^a = 0), \ell'^a = \ell'_0^a \} .
\]

(3.31b)

\( C_{\mathbf{\alpha}v,\ell}^{\mathbf{\beta}v,\ell'} \) must be such constants as \( B_{\mathbf{\alpha}v,\ell_0,\ell'_0}^{\mathbf{\beta}v,\ell_0} \) is convergent. Therefore \( \ell \) and \( \ell' \) dependence of this constant is necessary for convergence of \( Z_{\rho_{\mathbf{\beta}^a}}(\tau) \). Then the function (3.29) becomes

\[
Z_{\rho_{\mathbf{\beta}^a}}(\tau) = \sum_{\ell_0 \in \Lambda_{0_{\mathbf{\beta}^a}}} \sum_{\ell'_0 \in \Lambda_{0_{\mathbf{\beta}^a}}} B_{\mathbf{\alpha}v,\ell_0,\ell'_0}^{\mathbf{\beta}v,\ell_0} \int d^d \rho_0 d^d \rho_0' q^{(\rho_0 + \ell_0/2)^2} q^{(\rho_0 - \ell_0/2)^2} e^{-2\pi i (\ell_0 \bullet \rho_0 + \ell'_0 \bullet \rho_0')},
\]

(3.32)

where \( \Lambda_{0_{\mathbf{\beta}^a}} \) and \( \Lambda_{0_{\mathbf{\beta}^a}} \) are the lattices of \( \Lambda_{\mathbf{\alpha}v,\ell} \) and \( \Lambda_{\mathbf{\alpha}v,\ell'} \) respectively projected onto \( \mathbf{\alpha} v^a = \mathbf{\beta} v^a = 0 \) space. Since \( \ell_0, \ell'_0 \) and \( \rho_0 \) are the vectors in \( \mathbf{\alpha} v^a = \mathbf{\beta} v^a = 0 \) space, the product of these
is $x_0 \cdot y_0 = x_0 \cdot y_0$. In particular, in case of $\alpha \alpha^{\alpha \varepsilon} \neq 0$ or $\beta \beta^{\alpha \varepsilon} \neq 0$ for all $\alpha$, the function\(3.32\) reads

$$Z^{\alpha \alpha \varepsilon}_{\beta \beta \varepsilon}(\tau) = B_{\beta \beta \varepsilon,0}^{\alpha \alpha \varepsilon},$$

(3.33)

because $\rho_0$, $\ell_0$ and $\ell'_0$ do not exist. Thus I have derived the function $Z^{\alpha \alpha \varepsilon}_{\beta \beta \varepsilon}(\tau)$.

In case of the toroidal compactification ($\alpha = \beta = 0$), $\Lambda \rho^{0}_{00} = \Lambda$ because of\(3.27\). Therefore, if $B_{0,0}^{0,0}$ do not depend on $\ell'_0$ then the summation over $\ell'_0$ in the function\(3.32\) is

$$\sum_{\ell'_0 \in \Lambda} e^{-2\pi i (\ell'_0 \rho_0 + \ell'_0 \rho_0)} = V_{\Lambda^*} \sum_{\ell'_0 \in \Lambda^*} \delta(\rho_0 - \ell'_0),$$

(3.34)

where $\Lambda^*$ is the dual-lattice of $\Lambda$, and $V_{\Lambda^*}$ is the volume of a unit cell of $\Lambda^*$. Hence, only $\rho_0$ included in $\Lambda^*$ contribute to the function $Z_{\rho_0}^{0,0}(\tau)$. Therefore only the invariant states under $h_{\rho,\pi\varepsilon}(\beta \varepsilon, \ell')$ contribute to the partition function, because if $\rho_0 \in \Lambda^*$ then the eigenvalue of $h_{\rho,\pi\varepsilon}(\beta \varepsilon, \ell')$ is $e^{-2\pi i (\ell'_0 \rho_0 + \ell'_0 \rho_0)} = 1$. In the orbifold compactification, however, we must also consider states which is not invariant under $h_{\rho,\pi\varepsilon}(\beta \varepsilon, \ell')$. The factor $e^{-2\pi i (\ell'_0 \rho_0 + \ell'_0 \rho_0)}$ is necessary for modular invariance of the partition function (see the next section).

Second, I calculate the function $Z_{\beta \beta \varepsilon}^{\alpha \alpha \varepsilon}(\tau)$. The function $Z_{\beta \beta \varepsilon}^{\alpha \alpha \varepsilon}(\tau)$ can be derived by calculating the trace in this function\(3.17\b) by attention to $N_{0}^{\alpha} = N^{\alpha}_{0} = 0$,

$$Z_{\beta \beta \varepsilon}^{\alpha \alpha \varepsilon}(\tau) = (q \bar{q})^{\alpha \varepsilon} (1 - \alpha \varepsilon)/2 \cdot 3/12 \prod_{a=1}^{3} \prod_{n=1}^{\infty} \left\{ \delta_{\alpha \varepsilon,0} \left| 1 - q^{n} e^{2\pi i \alpha \varepsilon} \right|^{-1} \left| 1 - q^{n} e^{-2\pi i \alpha \varepsilon} \right|^{-1} \right\}^{2} + (1 - \delta_{\alpha \varepsilon,0}) \left| 1 - q^{n} e^{2\pi i \alpha \varepsilon} \right|^{-1} \left| 1 - q^{n} e^{-2\pi i \alpha \varepsilon} \right|^{-1} \right\}^{2}. \tag{3.35}\$$

Thus I have derived the function $Z_{\beta \beta \varepsilon}^{\alpha \alpha \varepsilon}(\tau)$.

I have shown that the partition function $Z^{\alpha \alpha \varepsilon}_{\beta \beta \varepsilon}(\tau)$ is given by product of the functions\(3.32\) and\(3.35\). Finally, I conclude that the complete total partition function $Z(\tau)$ is given by summation of the function $Z^{\alpha \alpha \varepsilon}_{\beta \beta \varepsilon}(\tau)$.

4 Modular invariance conditions

In this section, I derive modular invariance conditions of the total partition function $Z(\tau)$. The modular transformation is composed of the following two transformations: the one is $\tau \rightarrow \tau + 1$ called T-transformation, the other is $\tau \rightarrow -1/\tau$ called S-transformation.

4.1 T-transformation

First, I calculate the T-transformation of the functions $Z^{\alpha \alpha \varepsilon}_{\beta \beta \varepsilon}(\tau)$, $Z_{\beta \beta \varepsilon}^{\alpha \alpha \varepsilon}(\tau)$ and $Z^{\alpha \alpha \varepsilon}_{\beta \beta \varepsilon}(\tau)$. In case of $\alpha \alpha^{\alpha \varepsilon} = \beta \beta^{\alpha \varepsilon} = 0$ for some $a$, the T-transformation of the function\(3.32\) becomes

$$Z^{\alpha \alpha \varepsilon}_{\beta \beta \varepsilon}(\tau + 1) = \sum_{\ell_0 \in \Lambda_{\alpha \alpha \varepsilon}} \sum_{\ell'_0 \in \Lambda_{\beta \beta \varepsilon}} B_{\beta \beta \varepsilon,0}^{\alpha \alpha \varepsilon} \int d^d \rho_0 d^d \rho_0' \cdot e^{2\pi i (\tau + 1) |\rho_0 + \ell_0/2|^2 e^{-2\pi i (\tau + 1) |\rho_0 - \ell_0/2|^2 e^{-2\pi i (\ell_0' - \ell_0) \rho_0}$$

$$= \sum_{\ell_0 \in \Lambda_{\alpha \alpha \varepsilon}} \sum_{\ell'_0 \in \Lambda_{\beta \beta \varepsilon}} B_{\beta \beta \varepsilon,0}^{\alpha \alpha \varepsilon} \int d^d \rho_0 d^d \rho_0' q^{2 |\rho_0 + \ell_0/2|^2} q^{2 |\rho_0 - \ell_0/2|^2} e^{-2\pi i (\ell_0' - \ell_0) \rho_0 + (\ell'_0 - \ell_0) \rho_0}. \tag{4.1}\$$
If $\xi_{\beta v, \ell}^\alpha = 1$ then $\ell_0^a - \ell^a = 0$ at $\alpha_0^a = 0$, $\beta_0^a - \alpha_0^a = 0$ because of (3.28). Therefore, if $
exists_{\beta v, \ell}^\alpha = 1$ then $\ell' - \ell \in \Lambda_{\beta v}^{\alpha_0}$ owing to (3.27). Hence, let $\ell_0' = \ell_0 - \ell_0$, then $\ell_0' \in \Lambda_{\beta v}^{\alpha_0}$. In addition, $\Lambda_{\beta v}^{\alpha_0} = \Lambda_{\beta v}^{\alpha_0}$ because of (3.26). Thus this T-transformation becomes

$$Z_{\rho v v}^\beta(\tau + 1) = \sum_{\ell_0 \in \Lambda_{\rho v v}^{\beta v v} - \beta_v v} \sum_{\ell_0' \in \Lambda_{\rho v v}^{\beta v v} - \beta_v v} B_{\rho v v}^{\beta v v} \epsilon_0 + \epsilon_0' \int d\tau_0 d\tau_0' q^{|\rho_0 + \epsilon_0'/2|^2} \eta^{\epsilon_0'/2} e^{-2\pi i (\epsilon_0' + \epsilon_0 + \tau_0')}. \tag{4.2}$$

I assume the condition $B_{\rho v v}^{\beta v v} + \epsilon_0 = F_\tau B_{\rho v v}^{\beta v v} + \epsilon_0$, for all $\ell_0$ and $\ell_0'$, where $F_\tau$ is a constant. Then this T-transformation is $Z_{\rho v v}^\beta(\tau + 1) = F_\tau Z_{\rho v v}^\beta(\tau)$. This constant is $F_\tau = B_{\rho v v}^{\beta v v} / B_{\rho v v}^{\beta v v - \alpha v v}$, because if $\ell_0 = \ell_0'$ then $B_{\rho v v}^{\beta v v} = F_\tau B_{\rho v v}^{\beta v v - \alpha v v}$. Then this T-transformation is given by

$$Z_{\rho v v}^\beta(\tau + 1) = \frac{F_{\rho v v}^{\beta v v}}{B_{\rho v v}^{\beta v v - \alpha v v}} Z_{\rho v v}^\beta(\tau). \tag{4.3}$$

The condition $B_{\rho v v}^{\beta v v} + \epsilon_0 = F_\tau B_{\rho v v}^{\beta v v} + \epsilon_0$, becomes

$$\frac{B_{\rho v v}^{\beta v v} + \epsilon_0}{B_{\rho v v}^{\beta v v - \alpha v v}} = \frac{F_{\rho v v}^{\beta v v}}{B_{\rho v v}^{\beta v v - \alpha v v}} \epsilon_0. \tag{4.4}$$

In case of $\alpha_0^a = 0$ or $\beta_0^a = 0$ for all $a$, the function (3.33) are transformed equally as (4.3). Thus I have derived the T-transformation of the function $Z_{\rho v v}^\beta(\tau)$.

The T-transformation of the function (8.35) can be easily calculated,

$$Z_{\rho v v}^\beta(\tau + 1) = Z_{\rho v v}^\beta(\tau). \tag{4.5}$$

Thus I have derived the T-transformation of the function $Z_{\rho v v}^\beta(\tau)$.

By using the transformations (4.3) and (4.5), the T-transformation of the partition function $Z_{\rho v v}^\beta(\tau)$ is given by

$$Z_{\rho v v}^\beta(\tau + 1) = \frac{F_{\rho v v}^{\beta v v}}{B_{\rho v v}^{\beta v v - \alpha v v}} Z_{\rho v v}^\beta(\tau). \tag{4.6}$$

Thus I have derived the T-transformation of the partition function $Z_{\rho v v}^\beta(\tau)$.

### 4.2 S-transformation

Second, I calculate the S-transformation of the function $Z_{\rho v v}^\beta(\tau)$, $Z_{\rho v v}^\beta(\tau)$ and $Z_{\rho v v}^\beta(\tau)$. In case of $\alpha_0^a = \beta_0^a = 0$ for some $a$, by integrating over $\rho_0$, the function (3.32) becomes

$$Z_{\rho v v}^\beta(\tau) = \frac{1}{(2\pi)^d} \sum_{\ell_0 \in \Lambda_{\rho v v}^{\beta v v}} \sum_{\ell_0' \in \Lambda_{\rho v v}^{\beta v v}} B_{\rho v v}^{\beta v v} \epsilon_0 \epsilon_0' \times \exp \left\{ -\frac{\pi}{\tau_2} (|\tau|^2 \epsilon_0^2 - \tau_1 \epsilon_0 \epsilon_0' + \tau_1 \epsilon_0^2 + \epsilon_0'^2) \right\}, \tag{4.7}$$
where \( \tau = \tau_1 + i\tau_2 \). The S-transformations of \( \tau_1 \) and \( \tau_2 \) are \( \tau_1 \to -\tau_1/|\tau|^2 \) and \( \tau_2 \to \tau_2/|\tau|^2 \). Therefore the S-transformation of the function \( \beta^{(m)} \) becomes

\[
Z_{\rho \beta^{(m)}(1/\tau)} = \left( \frac{|\tau|^2}{2\tau_2} \right)^d \sum_{\ell_0 \in \Lambda_{\rho \beta^{(m)}}} \sum_{\ell'_0 \in \Lambda_{\rho \beta^{(m)}}} B_{\beta^{(m)},0}^{\rho \beta^{(m)},0} \times \exp \left\{ -\frac{\pi}{\tau_2} \left( |\ell'_0|^2 + \tau_1 \ell_0 \cdot \ell'_0 + \tau_1 \ell_0 \cdot \ell'_0 + |\ell|^2 |\ell'_0|^2 \right) \right\}.
\]  

(4.8)

I change \( \ell_0 \) and \( \ell'_0 \) to \(-\ell'_0 \) and \( \ell_0 \), respectively. Then \( \ell_0 \in \Lambda_{\rho \beta^{(m)}} \) and \( \ell'_0 \in \Lambda_{\rho \beta^{(m)}} \) because of (3.20) and (3.27). Thus this S-transformation becomes

\[
Z_{\rho \beta^{(m)}(-1/\tau)} = \left( \frac{|\tau|^2}{2\tau_2} \right)^d \sum_{\ell_0 \in \Lambda_{\rho \beta^{(m)}}} \sum_{\ell'_0 \in \Lambda_{\rho \beta^{(m)}}} B_{\beta^{(m)},0}^{\rho \beta^{(m)},0} \times \exp \left\{ -\frac{\pi}{\tau_2} \left( |\ell|^2 |\ell'_0|^2 - \tau_1 \ell_0 \cdot \ell'_0 - \tau_1 \ell_0 \cdot \ell'_0 + |\ell'_0|^2 \right) \right\}.
\]  

(4.9)

I assume the condition \( B_{\beta^{(m)},0}^{\rho \beta^{(m)},0} = F_S B_{\beta^{(m)},0}^{\rho \beta^{(m)},0} \) for all \( \ell_0 \) and \( \ell'_0 \), where \( F_S \) is a constant. Then this S-transformation is \( Z_{\rho \beta^{(m)}(-1/\tau)} = F_S |\tau|^{2d} Z_{\beta^{(m)}(\tau)} \). This constant is \( F_S = \frac{B_{\beta^{(m)},0}^{\rho \beta^{(m)},0}}{B_{\beta^{(m)},0}^{\rho \beta^{(m)},0}} \). Then this S-transformation is given by

\[
Z_{\rho \beta^{(m)}(-1/\tau)} = \frac{B_{\beta^{(m)},0}^{\rho \beta^{(m)},0}}{B_{\beta^{(m)},0}^{\rho \beta^{(m)},0}} |\tau|^{2d} Z_{\rho \beta^{(m)}(\tau)}.
\]  

(4.10)

The condition \( B_{\beta^{(m)},0}^{\rho \beta^{(m)},0} = F_S B_{\beta^{(m)},0}^{\rho \beta^{(m)},0} \) becomes

\[
\frac{B_{\beta^{(m)},0}^{\rho \beta^{(m)},0}}{B_{\beta^{(m)},0}^{\rho \beta^{(m)},0}} = \frac{B_{\beta^{(m)},0}^{\rho \beta^{(m)},0}}{B_{\beta^{(m)},0}^{\rho \beta^{(m)},0}}.
\]  

(4.11)

In case of \( \alpha^{(m)} \neq 0 \) or \( \beta^{(m)} \neq 0 \) for all \( a \), the function \( \beta^{(m)} \) are transformed equally as (4.10) because of \( d = 0 \). Thus I have derived the S-transformation of the function \( Z_{\rho \beta^{(m)}(\tau)} \).

Now I rewrite the function (3.23) with the Dedekind eta function \( \eta(\tau) \) and the theta function \( \vartheta(\nu, \tau) \). The Dedekind eta function and the theta function are defined as

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\]  

(4.12a)

\[
\vartheta(\nu, \tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 + e^{2\pi i \nu q^{n-1/2}})(1 + e^{-2\pi i \nu q^{n-1/2}}).
\]  

(4.12b)

Then the function \( Z_{N \beta^{(m)}(\tau)} \) becomes

\[
Z_{N \beta^{(m)}(\tau)} = \prod_{a=1}^{3} \left\{ \frac{\delta_{\alpha^{(m)},0} \delta_{\beta^{(m)},0}}{|\vartheta(\nu, \tau)|^2} + (qq)^{-\alpha^{(m)} - 1/2} \left( \delta_{\alpha^{(m)},0}(1 - \delta_{\beta^{(m)},0}) \frac{4 \pi \beta^{(m)}}{\eta(\tau)} + (1 - \delta_{\alpha^{(m)},0}) \right) \right\}.
\]  

(4.11)
The S-transformation of the function (4.13) can be calculated by using the formulas \( \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \), \( \vartheta(-\nu, \tau) = \vartheta(\nu, \tau) \) and \( \vartheta(\nu/\tau, -1/\tau) = (-i\tau)^{1/2}e^{\pi\nu^2/2\tau}\vartheta(\nu, \tau) \),

\[
Z_{N_{\alpha v}}^{\beta v}(-1/\tau) = \prod_{a=1}^{3} \left\{ \frac{\delta_{\alpha v, 0}(1 - \delta_{\beta v, 0})(1 - \delta_{\beta v, 0})}{4\sin^2 \pi \beta v} + \delta_{\alpha v, 0}\delta_{\beta v, 0} + (1 - \delta_{\alpha v, 0})(1 - \delta_{\beta v, 0}) \right\} Z_{N_{-\alpha v}}^{\beta v}(\tau)
\]

Thus I have derived the S-transformation of the total partition function (4.14)

\[
= \frac{F_{\alpha v}^{\beta v}}{F_{-\alpha v}^{\beta v}} \frac{1}{|\tau|^2} Z_{N_{-\alpha v}}^{\beta v}(\tau),
\]

(4.14)

Thus I have derived the S-transformation of the function \( Z_{N_{\alpha v}}^{\beta v}(\tau) \).

By using the transformations (4.10) and (4.14), the S-transformation of the partition function \( Z_{N_{\alpha v}}^{\beta v}(\tau) \) is given by

\[
Z_{\alpha v}^{\beta v}(1/\tau) = \frac{B_{\beta v, 0}^{\alpha v, 0}}{B_{\beta v, -\alpha v, 0}^{\alpha v, 0}} \frac{F_{\alpha v}^{\beta v}}{F_{-\alpha v}^{\beta v}} Z_{N_{-\alpha v}}^{\beta v}(\tau).
\]

(4.16)

Thus I have derived the S-transformation of the partition function \( Z_{\alpha v}^{\beta v}(\tau) \).

### 4.3 Modular invariance conditions

Finally, I derive the modular invariance conditions of the total partition function (3.2). By using the T-transformation (4.10), the T-transformation of the total partition function \( Z(\tau) \) becomes

\[
Z(\tau + 1) = \frac{1}{N} \sum_{\alpha=0}^{N-1} \sum_{\beta=0}^{N-1} B_{\beta v, 0}^{\alpha v, 0} Z_{N_{-\alpha v}}^{\beta v}(\tau).
\]

(4.17)

I find that the condition \( B_{\beta v, 0}^{\alpha v, 0} = B_{\beta v, -\alpha v, 0}^{\alpha v, 0} \) implies T-invariance of \( Z(\tau) \): \( Z(\tau + 1) = Z(\tau) \). By using this condition, the condition (4.11) becomes \( B_{\beta v, \ell_0 + \ell_0'}^{\alpha v, 0} = B_{\beta v, 0}^{\alpha v, 0} \). In fact, the condition \( B_{\beta v, \ell_0 + \ell_0'}^{\alpha v, 0} = B_{\beta v, 0}^{\alpha v, 0} \) includes the condition \( B_{\beta v, 0}^{\alpha v, 0} = B_{\beta v, 0}^{\alpha v, 0} \) itself as a special case.

Similarly, by using the S-transformation (4.10), the S-transformation of the total partition function \( Z(\tau) \) becomes

\[
Z(-1/\tau) = \frac{1}{N} \sum_{\alpha=0}^{N-1} \sum_{\beta=0}^{N-1} B_{\beta v, \ell_0}^{\alpha v, \ell_0} \frac{F_{\alpha v}^{\beta v}}{F_{-\alpha v}^{\beta v}} Z_{N_{-\alpha v}}^{\beta v}(\tau).
\]

(4.18)

I find that the condition \( B_{\beta v, 0}^{\alpha v, 0} F_{\alpha v}^{\beta v} = B_{\beta v, 0}^{\alpha v, 0} F_{\beta v, 0}^{\alpha v} \) implies S-invariance of \( Z(\tau) \): \( Z(-1/\tau) = Z(\tau) \). By using this condition, the condition (4.11) becomes \( B_{\beta v, \ell_0}^{\alpha v, \ell_0} F_{\alpha v}^{\beta v} = B_{\beta v}^{\alpha v, \ell_0} F_{\alpha v}^{\beta v} \). In fact, the condition \( B_{\beta v, \ell_0}^{\alpha v, \ell_0} F_{\alpha v}^{\beta v} = B_{\beta v}^{\alpha v, \ell_0} F_{\alpha v}^{\beta v} \) includes the condition \( B_{\beta v, 0}^{\alpha v, 0} F_{\alpha v}^{\beta v} = B_{\beta v, 0}^{\alpha v, 0} F_{\beta v, 0}^{\alpha v} \) itself as a special case.
Thus I have derived sufficient conditions for the modular invariance of the total partition function are

\[
B_{\beta v,\ell_0}^{\alpha v, \ell_0 + \ell_0} = B_{\beta v, -\alpha \ell_0}^{\alpha v, \ell_0} F_{\alpha v, \beta v}^{\beta v, \ell_0} = B_{-\alpha v, \ell_0}^{\beta v, \ell_0 - \alpha \ell_0} F_{\alpha v, \beta v}^{\beta v, \ell_0} \quad (4.19)
\]

5 Summary and remarks

I have constructed the complete 1-loop partition function of a bosonic closed string on orbifolds. In particular, I have paid attention to the following points: the \( \ell \) and \( \ell' \) dependence of the constant \( C_{\beta v, \ell_0}^{\alpha v} \), the derivation of the eigenstates and eigenvalues of the operator \( h_{\rho, \pi v(\beta v, \ell')} \), and existence of the fixed points which satisfy both conditions (3.9) and (3.15). Furthermore, I have derived sufficient conditions (4.19) for the modular invariance of the total partition function.

In this paper, I particularly discussed the six internal coordinates in the heterotic string. If this argument is adapted to sixteen left-moving bosons (corresponding to sixteen internal coordinates) in the heterotic strings, it is expected to derive higher level current algebras naturally [5]. The higher level current algebras are necessary for grand unified models in the 4-dimensional heterotic string [6]. I think that the \( \ell \) and \( \ell' \) dependence of the constant \( C_{\beta v, \ell_0}^{\alpha v} \) is necessary for modular invariance of the left-moving bosons on orbifolds.

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