Hawking–Page phase transitions of the black holes in different extended phase spaces

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ABSTRACT: The Hawking–Page (HP) phase transitions of the $d$-dimensional Schwarzschild and charged black holes are explored in two different extended phase spaces. One is to enclose the black hole in the anti-de Sitter (AdS) space, and the other is to confine the black hole in a spherical cavity. The phase transition temperature $T_{\text{HP}}$, minimum black hole temperature $T_0$, and Gibbs free energy $G$ are systematically calculated in an analytical way. There are remarkable similarities in the two extended phase spaces. Especially, for the Schwarzschild black holes, a dual relation of $T_{\text{HP}}(d) = T_0(d + 1)$ in successive dimensions exactly valid in the AdS case is found to be also approximately valid in the cavity case with a high precision. Moreover, this relation can be further generalized to the charged black holes in a suitable form. At the same time, significant dissimilarities also exist between the two extended phase spaces, like the notable terminal points in the $G$–$T$ curves of the charged black holes in a cavity. Our work helps to establish the universal properties of the black holes in different extended phase spaces, and simultaneously motivates further studies on their thermodynamic behaviors that are sensitive to specific boundary conditions.

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1 Introduction

In the recent half century, black hole physics, both in theoretical and experimental aspects, has been one of the most fruitful areas in physics. The establishment of the four laws of black hole thermodynamics revealed the striking resemblance between a black hole and a traditional thermodynamic system [1]. Furthermore, the Hawking mechanism indicated that a black hole can not only absorb, but also radiate matter and energy to the environment [2]. All these remarkable achievements indicated that a black hole is not merely a mathematical singularity, but should be regarded as a complicated physical system with temperature and entropy [3].

However, despite the glorious successes in black hole physics, there still remain two obvious differences between black hole thermodynamics and traditional thermodynamics. The first and apparent discrepancy is that there is no $p-V$ term in the first law of black hole thermodynamics, making a black hole slightly different from a typical $p-V-T$ system. The second and more essential one is that a black hole in the asymptotically flat space has negative heat capacity and is thus thermodynamically unstable. As a result, when a black hole radiates via the Hawking mechanism, it can no longer maintain thermal equilibrium with the environment and will evaporate eventually. Actually, these two issues can be dealt with together. The basic idea is to impose new appropriate boundary conditions instead of the asymptotically flat one, and simultaneously introduce an effective pressure $p$ and an effective volume $V$ to the black hole system. In this way, two new dimensions, $p$ and $V$,
are added into the black hole phase space, and such a theory is thus named as black hole thermodynamics in the “extended phase space” [4], in which the $p$–$V$ term reappears in the first law, and a black hole possesses a stable branch with positive heat capacity, more consistent with traditional thermodynamics.

As the gravitational potential of a black hole in the asymptotically flat space tends to vanish at infinity, the basic function of an appropriate boundary condition is to alter the space-time structure and to increase the gravitational potential at large distances. By this means, the boundary plays a role of a reflecting wall against the Hawking radiation and thus stabilizes the black hole. In principle, any boundary condition meeting such requirements will work and will restore the $p$–$V$ term at the same time. Amongst them, there are two natural choices, corresponding to the two different kinds of extended phase spaces discussed in this paper.

The first one is to enclose the black hole in the anti-de Sitter (AdS) space, and a positive effective pressure $p$ can be introduced from the negative cosmological constant $\Lambda$ as [5, 6]

$$p = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2},$$

(1.1)

where $d$ is the dimension of space-time, and $l$ is the AdS curvature radius. If $\Lambda$ is running, $p$ can be regarded as a thermodynamic quantity, and then an effective volume of the black hole can be defined as the conjugate variable of $p$ as $V = (\partial M/\partial p)_S$, with $M$ being the black hole mass [4]. Moreover, it is found that the $p$–$V$ term in the first law of black hole thermodynamics is $V dp$ but not the usual work term $-p dV$. Therefore, $M$ should be considered as the enthalpy rather than the thermal energy of the black hole, which is the most notable feature of this kind of extended phase space.

The second appropriate boundary condition is to place the black hole still in the asymptotically flat space, but to confine it in a spherical cavity, on the wall of which the black hole metric is fixed. Hence, the gravitational potential on the wall is infinite (like a hard-sphere potential) and can thus play a similar role of the AdS space. In the cavity case, there is a new characteristic length: the cavity radius $r_B$, so an effective volume $V$ of the black hole can be formally defined as the Euclidean volume. In Ref. [7], the authors set $V = 4\pi r_B^3/3$ in 4-dimensional space-time. Now, we generalize it to $d$ dimensions as

$$V = \frac{\Omega r_B^{d-1}}{d-1},$$

(1.2)

where $\Omega = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$ is the $(d-2)$-dimensional total solid angle. If $r_B$ is allowed to vary, $V$ becomes a thermodynamic quantity, and an effective pressure can be introduced as the conjugate variable of $V$ as $p = -(\partial E/\partial V)_S$, with $E$ being the thermal energy of the black hole [7]. In this way, another kind of extended phase space can be constructed. (For more relevant works on black hole thermodynamics in a cavity, see [8–35].)

It is interesting to see that the orders of the introduction of the $p$–$V$ term are exactly the opposite in the AdS space and in a cavity. Consequently, there should be remarkable similarities and dissimilarities in these two different extended phase spaces simultaneously. For instance, there exists a minimum temperature $T_0$, above which there are two black
hole solutions: a stable large black hole with positive heat capacity and an unstable small black hole with negative heat capacity, but below which there is no black hole solution any longer. The stable large black hole possesses an equation of state like that of a non-ideal fluid, and has rich phase transitions and critical phenomena. These topics have been extensively studied in the AdS case (see Ref. [36] for a recent review), but the relevant studies on the cavity case are just starting [7, 37].

One of the most important issues in black hole thermodynamics is the famous Hawking–Page (HP) phase transition that occurs between the stable large black hole and the thermal gas at the HP temperature $T_{HP}$ [38]. This process was first investigated for the Schwarzschild black hole in Ref. [38] and then for the charged black hole in different ensembles in Refs. [39–41]. Later on, the HP phase transition has been broadly studied in the literature, especially under the influence of the AdS/CFT duality [42–54]. After the introduction of the extended phase space in the AdS space, this topic was scrutinized again with fruitful results [55–71]. However, the corresponding discussions in the extended phase space in a cavity are far from sufficient [7, 37], and this will be one of the aims of this paper. Currently, the studies on the HP phase transition are mainly focused on various gravity theories beyond the Einstein gravity.

The most straightforward generalization of the Einstein gravity is naturally the high-dimensional general relativity [72]. However, the extension to the $d$-dimensional spacetime is not a direct task. Many familiar results will receive nontrivial modifications in higher dimensions, with the most obvious example being the violation of the uniqueness theorem in 4 dimensions. For instance, in 5 dimensions, besides the ordinary Schwarzschild–Tangherlini black hole solution [73], there exist the more complicated Myers–Perry black hole solution [74] and even the black ring solutions with one or two angular momenta [75, 76]. Consequently, it is also highly necessary to investigate the HP phase transitions in the general $d$-dimensional extended phase spaces. As an example, in Ref. [77], the authors discovered an interesting dual relation of the HP temperature $T_{HP}(d)$ in $d$ dimensions and the minimum temperature $T_0(d+1)$ in $d+1$ dimensions for the Schwarzschild black hole in the AdS space,

$$T_{HP}(d) = T_0(d+1),$$

(1.3)

Some possible explanation was presented in Ref. [77], such as the duality between the ground and excited states of the black holes in two successive dimensions. However, the deeper physical meaning of Eq. (1.3) is still unknown, since $T_{HP}$ and $T_0$ are both for the gravitational system, unlike the usual duality with the holographic interpretation. Therefore, it is also worthy of further exploring this dual relation in other extended phase space, in order to check its universality.

The purpose of this paper is to study the HP phase transitions in two different $d$-dimensional extended phase spaces. This paper is a successive research of our previous work on the HP phase transitions of the 4-dimensional black holes in the AdS space and in a cavity [37]. The main improvements in our present work are threefold. First, we will derive all the analytical expressions of the HP temperature $T_{HP}$, the minimum black hole temperature $T_0$, and the Gibbs free energy $G$ in arbitrary dimension, which were usually
obtained only by numerical methods in previous literature. Second, we will study the
dimension-dependence of $T_{HP}$ and $T_0$ for both the neutral and charged black holes, and will
show that the dual relation in Eq. (1.3) is still approximately valid in the cavity case. For
the charged black holes, we will reformulate Eq. (1.3) in a more concise and convenient
way, suitable for both two extended phase spaces. Third, we will discuss the discrepancies
between the two extended phase spaces, with emphasis on the terminal points in the $G$–$T$
curves of the charged black holes in a cavity. Altogether, we will systematically investigate
the similarities and compare the dissimilarities between the AdS and cavity cases, and wish
to present a whole picture and thorough understanding of the HP phase transitions in
different extended phase spaces to the most general extent.

This paper is organized as follows. In Sect. 2, we list the thermodynamic properties of
the black holes in two different extended phase spaces. In Sects. 3 and 4, we study the HP
phase transitions of the neutral and charged black holes in order, both in the AdS space and
in a cavity, with special attention to the universality of the dual relation. We conclude in
Sect. 5. In this paper, we work in the natural system of units and set $c = G_d = \hbar = k_B = 1$.

2 Black hole thermodynamics in the extended phase spaces

In this section, we first explain the thermodynamic properties of the black holes in two
different extended phase spaces, and then discuss the HP phase transition in more detail.

2.1 Black hole thermodynamics in the AdS space

The action of the $d$-dimensional charged black hole in the AdS space reads [36]

$$ I = \frac{1}{16\pi} \int_M \sqrt{-g} \left[ R - F_{\mu\nu} F^{\mu\nu} + \frac{(d-1)(d-2)}{l^2} \right], $$

(2.1)

where $R$ is the Ricci scalar, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and $A_\mu$
is electromagnetic potential vector. The metric of the spherically symmetric charged black
hole in the AdS space is

$$ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, $$

where

$$ f(r) = 1 - \frac{16\pi M}{(d-2)\Omega^{d-3}} + \frac{32\pi^2 Q^2}{(d-2)(d-3)\Omega^2 r^{2d-6}} + \frac{16\pi p r^2}{(d-1)(d-2)}, $$

(2.2)

with $M$ and $Q$ being the mass and electric charge of the black hole, and $p$ being the effective
pressure introduced in Eq. (1.1). The event horizon radius $r_+$ is determined as the largest
root of $f(r) = 0$, in terms of which the black hole mass can be reexpressed as

$$ M = \frac{(d-2)\Omega^{d-3}}{16\pi} \left[ 1 + \frac{32\pi^2 Q^2}{(d-2)(d-3)\Omega^2 r_+^{2d-6}} + \frac{16\pi p r_+^2}{(d-1)(d-2)} \right]. $$

(2.3)
Now, we turn to black hole thermodynamics in the AdS space. First, the Hawking temperature can be calculated as

\[
T_H = \frac{f'(r_+)}{4\pi} = \frac{d-3}{4\pi r_+} - \frac{8\pi Q^2}{(d-2)\Omega^2 r_+^{d-5}} + \frac{4pr_+}{d-2}.
\]

Next, the black hole entropy is one quarter of the \((d-2)\)-dimensional horizon volume,

\[
S = \frac{\Omega_{d-2} r}{4}.
\]

With these preparations, a direct differentiation of \(M\) in Eq. (2.3) yields the first law of black hole thermodynamics in the AdS space,

\[
dM = T\,dS + V\,dp + \Phi\,dQ.
\]

Here, we clearly find the \(p-V\) term as \(V\,dp\) but not \(-p\,dV\), so the black hole mass \(M\) should be regarded as its enthalpy, instead of the internal energy. Then, the black hole temperature, effective volume, and electric potential at horizon can be obtained in order,

\[
T = \left(\frac{\partial M}{\partial S}\right)_{p,Q} = \frac{d-3}{4\pi r_+} - \frac{8\pi Q^2}{(d-2)\Omega^2 r_+^{d-5}} + \frac{4pr_+}{d-2},
\]

\[
V = \left(\frac{\partial M}{\partial p}\right)_{S,Q} = \frac{\Omega_{d-1}^2}{d-1},
\]

\[
\Phi = \left(\frac{\partial M}{\partial Q}\right)_{S,p} = \frac{4\pi Q}{(d-3)\Omega r^{d-3}}.
\]

It is interesting to see that \(T\) is the same as the Hawking temperature \(T_H\) in Eq. (2.4), and \(V\) coincides with the volume of the \((d-1)\)-dimensional ball with a radius \(r_+\) in the Euclidean space. In addition, combining the expressions in Eqs. (2.3) and (2.5)–(2.8), we arrive at the Smarr relation [78] (i.e., the Gibbs–Duhem relation in traditional thermodynamics) in the extended phase space in the \(d\)-dimensional AdS space [36],

\[
M = \frac{d-2}{d-3} TS - \frac{2}{d-3}pV + \Phi Q.
\]

Furthermore, from Eq. (2.6), the equation of state of the charged black hole in the AdS space can be extracted as

\[
p = \frac{(d-2)T}{4r_+} - \frac{(d-2)(d-3)}{16\pi r_+^2} + \frac{2\pi Q^2}{\Omega^2 r_+^{2d-4}} = \frac{T}{v} - \frac{d-3}{(d-2)\pi v^2} + \frac{2^{d-7}\pi Q^2}{(d-2)^{2d-4}\Omega^2 v^{2d-4}},
\]

where

\[
v = \frac{4r_+}{d-2} = \frac{4}{d-2} \left[ \frac{(d-1)V}{\Omega} \right]^{\frac{1}{d-1}}
\]

is the specific volume of the black hole. Therefore, the charged black hole in the AdS space possesses an equation of state similar to that of a non-ideal fluid, and will thus exhibit rich
thermodynamic behaviors, such as phase transitions and critical phenomena. For example, the critical point can be determined by

\[
\left( \frac{\partial p}{\partial r} \right)_T = \left( \frac{\partial^2 p}{\partial r^2} \right)_T = 0,
\]

and the critical values of event horizon radius, pressure, specific volume, and temperature are obtained as

\[
r_c = \left[ \frac{32(2d-5)\pi^2 Q^2}{(d-3)\Omega^2} \right]^{2/(d-3)}, \quad p_c = \frac{(d-3)^2}{16\pi r_c^2}, \quad v_c = \frac{4r_c}{d-2}, \quad T_c = \frac{(d-3)^2}{(2d-5)\pi r_c}.
\]

2.2 Black hole thermodynamics in a cavity

The action of the \(d\)-dimensional charged black hole in a cavity consists of two parts [31],

\[
I = I_{\text{bulk}} + I_{\text{surface}},
\]

and the bulk and surface actions read

\[
I_{\text{bulk}} = \frac{1}{16\pi} \int_{\mathcal{M}} d^d x \sqrt{-g} (R - F_{\mu \nu} F^{\mu \nu}),
\]

\[
I_{\text{surface}} = -\frac{1}{16\pi} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{-\gamma} \left[ 2(K - K_0) + F_{\mu \nu} A^\mu n^\nu \right],
\]

where \(\partial \mathcal{M}\) is the boundary of the cavity with a radius \(r_B\), \(\gamma\) is the determinant of the induced metric on \(\partial \mathcal{M}\), \(n^\nu\) is the normal vector, \(K\) is the trace of the extrinsic curvature tensor, and \(K_0\) is its corresponding value when \(\partial \mathcal{M}\) is embedded in a flat space-time. Then, the metric function \(f(r)\) of the spherically symmetric charged black hole in a cavity is

\[
f(r) = 1 - \frac{r^d-3}{r'^d-3} \left[ 1 - \frac{32\pi^2 Q^2}{(d-2)(d-3)\Omega^2 r^d-5} \right]. \tag{2.9}
\]

In fact, it is the same as \(f(r)\) in Eq. (2.2), only without the last term \(16\pi p r^2 / [(d-1)(d-2)]\) caused by the effective pressure \(p\) in the AdS space. This is because the bulk action \(I_{\text{bulk}}\) is just the action \(I\) in Eq. (2.1), without the last term \((d-1)(d-2)/l^2\). Again, by the same trick in Sect. 2.1, \(f(r)\) can be reexpressed in terms of the event horizon radius \(r_+\),

\[
f(r) = \left( 1 - \frac{r'^d-3}{r^d-3} \right) \left[ 1 - \frac{32\pi^2 Q^2}{(d-2)(d-3)\Omega^2 r'^d-3 r^d-3} \right]. \tag{2.10}
\]

The thermodynamic properties of the charged black hole in a cavity can be studied by the Euclidean action \(I_E\), which is obtained via the analytic continuation method as \(I_E = iI\). By this means, the Helmholtz free energy of the black hole is obtained as \(F = -T \ln Z = T I_E = F(T,Q,r_B; r_+)\). From Eq. (2.10), calculating the minimum of \(F\) with respect to \(r_+\), we have \(f'(r_+) = 4\pi T \sqrt{f(r_B)}\). Hence, the temperature of the black hole in a cavity is [9]

\[
T = \frac{f'(r_+)}{4\pi \sqrt{f(r_B)}} = \frac{T_H}{\sqrt{f(r_B)}} = \frac{1}{\sqrt{f(r_B)}} \left[ \frac{d-3}{4\pi r_+} - \frac{8\pi Q^2}{(d-2)\Omega^2 r_+^{2d-5}} \right], \tag{2.11}
\]

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indicating that the temperature $T$ measured at the cavity radius $r_B$ is blue-shifted from the Hawking temperature $T_H$ measured at infinity by a factor $1/\sqrt{f(r_B)}$. Moreover, the black hole entropy remains one quarter of the $(d - 2)$-dimensional horizon volume,

$$S = \frac{\Omega r_B^{d-2}}{4}. \tag{2.12}$$

Therefore, the thermal energy $E$ of the black hole can be derived from the Gibbs–Helmholtz equation,

$$E = F + TS = -T^2 \left( \frac{\partial(F/T)}{\partial T} \right)_{r_B} = \frac{(d - 2) \Omega r_B^{d-3}}{8\pi} \left[ 1 - \sqrt{f(r_B)} \right]. \tag{2.13}$$

Here, we should mention that $E$ is not simply the same as the black hole mass $M$. From Eq. (2.9), their relation is

$$E = \frac{(d - 2) \Omega r_B^{d-3}}{8\pi} \left[ 1 - \sqrt{1 - \frac{16\pi M}{(d - 2) \Omega r_B^{d-3} + \frac{32\pi^2 Q^2}{(d - 2)(d - 3) \Omega^2 r_B^{2d-6}}} \right],$$

but in the limit of $r_B \to \infty$, they naturally coincide.

Complementary to the way of introducing the $p$–$V$ term in the AdS space, the order of the introduction of the $p$–$V$ term is now exactly opposite. As shown in Eq. (1.2), we formally define an effective volume $V$ of the spherically symmetric black hole in a cavity (no matter neutral or charged) as the volume of the $(d - 1)$-dimensional ball with a radius $r_B$ in the Euclidean space. Obviously, the cavity radius $r_B$ should always be larger than the event horizon radius $r_+$,

$$r_B > r_+. \tag{2.15}$$

This constraint will have significant effects on the HP phase transition in a cavity, especially for the charged black holes, to be explained in Sect. 4.2. Furthermore, the effective pressure $p$ in a cavity is defined as the conjugate variable of $V$,

$$p = -\left( \frac{\partial E}{\partial V} \right)_{S,Q} = \frac{(d - 2)(d - 3)}{8\pi r_B^2} \left[ \frac{2 - \frac{r_B^{d-3}}{r_+^{d-3}} - \frac{32\pi^2 Q^2}{(d - 2)(d - 3) \Omega^2 r_+^{2d-6}}}{2 \sqrt{f(r_B)}} - 1 \right]. \tag{2.14}$$

Next, the electric potential at horizon reads

$$\Phi = \left( \frac{\partial E}{\partial Q} \right)_{S,V} = \frac{4\pi Q \left( 1 - \frac{r_B^{d-3}}{r_+^{d-3}} \right)}{(d - 3) \Omega r_B^{d-3} \sqrt{f(r_B)}}. \tag{2.15}$$

The explicit expressions of $E$, $T$, $p$, and $\Phi$ will be shown in due time in Sects. 3.2 and 4.2 when necessary.

From Eqs. (1.2) and (2.11)–(2.15), the first law of black hole thermodynamics in the extended phase space in a cavity can be confirmed as

$$dE = T \, dS - p \, dV + \Phi \, dQ.$$
Again, we stress that the variable in the first law is the thermal energy $E$ in a cavity, instead of the enthalpy $M$ in the AdS space. Moreover, the Smarr relation in a cavity reads \cite{7}

$$E = \frac{d-2}{d-3}TS - \frac{d-1}{d-3}pV + \Phi Q,$$

which is also apparently different from but essentially consistent with that in the AdS space.

Last, the equation of state and the critical point of the charged black hole in a cavity can also be obtained in the same way as that in Sect. 2.1. However, the specific expressions are rather tedious and will not be shown here, as they are not very relevant to the following discussions.

### 2.3 HP phase transition

In the asymptotically flat space-time, the black hole temperature is allowed to reach zero (at least for the Schwarzschild case). However, due to the increasing gravitational potential at large distances in the extended phase space, there exists a minimum temperature $T_0$ for both the neutral and charged black holes. Below $T_0$, there is no black hole solution at all; above $T_0$, there are two black hole branches: a stable large black hole with larger event horizon radius and positive heat capacity and an unstable small one on the contrary. On account of the Hawking mechanism, the large black hole can exchange matter and energy with the thermal gas in the extended phase space, and thus establish thermal equilibrium with the environment. Due to the conservation of charge, the phase transition in the black hole–thermal gas system should be studied in the grand canonical ensemble with fixed electric potential and varying electric charge, as the thermal gas is electrically neutral. Consequently, the thermodynamic potential of interest in the HP phase transition is the Gibbs free energy $G$.

In the following sections, we will show that the Gibbs free energy of the large black hole decreases with temperature. Therefore, at a certain temperature (i.e., the HP temperature $T_{\text{HP}}$), the Gibbs free energy of the large black hole becomes zero, and the HP phase transition occurs between the large black hole and the thermal gas that always has vanishing Gibbs free energy due to the non-conservation of particle number. Because the phase transition point corresponds to the global minimum of $G$, the criterion of the HP phase transition is that the Gibbs free energy of the black hole–thermal gas system vanishes,

$$G(T_{\text{HP}}) = 0,$$  \hspace{1cm} (2.16)

and the HP temperature $T_{\text{HP}}$ can thus be determined. Below $T_{\text{HP}}$, the thermal gas phase with vanishing $G$ is globally preferred; above $T_{\text{HP}}$, the large black hole phase with negative $G$ is thermodynamically preferred, and the thermal gas will collapse into the black hole. In Sects. 3 and 4, all the expressions of $T_{\text{HP}}$ and $T_0$ for neutral and charged black holes in different extended phase spaces will be analytically calculated in $d$ dimensions, and special attention will be paid to their ratio and the dual relation.
3 HP phase transitions of the Schwarzschild black holes

In this section, we discuss the HP phase transitions of the \(d\)-dimensional Schwarzschild black holes in the AdS space and in a cavity respectively, and compare the similarities and dissimilarities between these two extended phase spaces, with special attention to the dual relation of \(T_{\text{HP}}(d)\) and \(T_0(d+1)\).

3.1 HP phase transition of the Schwarzschild black hole in the AdS space

From Eqs. (2.3) and (2.6), the mass and temperature of the \(d\)-dimensional Schwarzschild black hole in the AdS space are

\[
M = \frac{(d-2)\Omega r_+^{d-3}}{16\pi} + \frac{\Omega p r_+^{d-1}}{d-1},
\]

\[
T = \frac{d-3}{4\pi r_+} + \frac{4pr_+}{d-2}.
\]

Therefore, from Eqs. (2.5), (3.1), and (3.2), its Gibbs free energy reads

\[
G = M - TS = \frac{[(d-1)(d-2) - 16\pi pr_+^3] \Omega r_+^{d-3}}{16\pi(d-1)(d-2)}.
\]

When the HP phase transition occurs, from Eq. (3.3) and the criterion in Eq. (2.16), the event horizon radius \(r_+\) can be solved as \(r_+ = \sqrt{(d-1)(d-2)/(16\pi p)}\). Substituting it into Eq. (3.2), we obtain the HP temperature \(T_{\text{HP}}\) as a function of pressure \(p\) and dimension \(d\) (i.e., the equation of coexistence line),

\[
T_{\text{HP}} = 2\sqrt{\frac{d-2}{d-1}} \sqrt{\frac{p}{\pi}}.
\]

The coexistence lines of the Schwarzschild black holes in the AdS space in different dimensions are shown in Fig. 1. We find that \(T_{\text{HP}}\) increases with \(p\) and \(d\). Since there is no terminal point (i.e., the critical point) in the coexistence line, the HP phase transition can occur at all pressures, more like a solid–liquid phase transition rather than a liquid–gas one. Interestingly, the thermal AdS phase even lies below the coexistence line, playing a role of a solid instead of a liquid [57].

For the minimum temperature \(T_0\) of the Schwarzschild black hole in the AdS space, setting \((\partial T/\partial r_+)_p = 0\) in Eq. (3.2), we have \(r_+ = \sqrt{(d-2)(d-3)/(16\pi p)}\). Substituting it into Eq. (3.2), we obtain

\[
T_0 = 2\sqrt{\frac{d-3}{d-2}} \sqrt{\frac{p}{\pi}}.
\]

Let us discuss the relation of \(T_{\text{HP}}\) and \(T_0\) in more detail. First, comparing Eqs. (3.4) and (3.5), we have

\[
\frac{T_{\text{HP}}(d)}{T_0(d)} = \frac{d-2}{\sqrt{(d-1)(d-3)}}.
\]
Figure 1. The HP temperature $T_{\text{HP}}$ of the Schwarzschild black holes in the AdS space as a function of pressure $p$, with different dimensions $d$. At a given $p$, $T_{\text{HP}}$ increases with $d$. There is no terminal point in the coexistence line, and the HP phase transition can occur at all pressures. The thermal AdS phase lies below the coexistence line, behaving as a solid instead of a liquid.

Hence, in the limit of $d \to \infty$, it is easy to learn

$$\lim_{d \to \infty} \frac{T_{\text{HP}}(d)}{T_0(d)} = 1,$$

meaning that there will be no metastable large black hole phase when $d \to \infty$.

Second, in Ref. [77], the authors discovered an interesting dual relation of the HP temperature $T_{\text{HP}}(d)$ in $d$ dimensions and the minimum temperature $T_0(d + 1)$ in $d + 1$ dimensions for the Schwarzschild black holes in the AdS space,

$$T_{\text{HP}}(d) = T_0(d + 1),$$

and this relation is independent of pressure $p$. However, unfortunately, the underlying interpretation of this dual relation is still absent. Therefore, both the results in Eqs. (3.7) and (3.8) will be explored in the cavity case in the following sections, in order to check whether these relations are purely coincidental or physically universal.

Next, from Eqs. (3.2) and (3.3), the $G$–$T$ curves of the Schwarzschild black holes in the AdS space in different dimensions are shown in Fig. 2. The two branches of the curves correspond to the large and small black holes respectively, meeting at $(T_0, G(T_0))$ with a cusp. The $G$–$T$ curves of the small black holes are concave and will never reach the $T$-axis, so there is no HP phase transition for them at all (not to mention that they are even unstable). On the contrary, the $G$–$T$ curves of the large black holes intersect the $T$-axis at the HP temperature $T_{\text{HP}}$. As the derivative of the Gibbs free energy of the black hole–thermal AdS system is discontinuous at $T_{\text{HP}}$, the HP phase transition is of first-order. With $d$ increasing, $T_{\text{HP}}$ moves rightward, indicating that $T_{\text{HP}}$ increases in high dimensions, consistent with Eq. (3.4). From Fig. 2, we clearly observe that, with $d$ increasing, the slopes of the $G$–$T$ curves of the large black holes become larger and larger. This means that the metastable large black hole phase [i.e., the segment of the $G$–$T$ curve in the temperature
interval \((T_0, T_{\text{HP}}]\) tends to vanish when \(d\) increases, confirming the result in Eq. (3.7). Moreover, the HP temperature \(T_{\text{HP}}\) in \(d\) dimensions are exactly equal to the minimum temperature \(T_0\) in \(d + 1\) dimensions, as shown in the dual relation in Eq. (3.8).

Figure 2. The Gibbs free energies \(G\) of the Schwarzschild black holes in the AdS space as a function of temperature \(T\), with pressure \(p = 0.2\) and different dimensions \(d\) (LBH and SBH stand for large and small black hole respectively). The \(G-T\) curves of the large black hole phase and the thermal AdS phase intersect at the HP temperature \(T_{\text{HP}}\), corresponding to the first-order HP phase transition. When \(d\) increases, the slopes of the \(G-T\) curves of the large black holes become larger and larger, confirming the limit in Eq. (3.7). In addition, the HP temperature \(T_{\text{HP}}\) in \(d\) dimensions are exactly equal to the minimum black hole temperature \(T_0\) in \(d + 1\) dimensions, as indicated in the dual relation in Eq. (3.8). There is no HP phase transition for the small black holes, as their \(G-T\) curves are always above the \(T\)-axis.

3.2 HP phase transition of the Schwarzschild black hole in a cavity

Now, we continue to discuss the HP phase transitions of the \(d\)-dimensional Schwarzschild black holes in a cavity. First, from Eqs. (2.11), (2.13), and (2.14), we have the thermal energy, temperature, and pressure as

\[
E = \frac{(d - 2)\Omega r_B^{d-3}}{8\pi} \left(1 - \sqrt{1 - \frac{r_+^{d-3}}{r_B^{d-3}}} \right),
\]

\[
T = \frac{d - 3}{4\pi r_+\sqrt{1 - \frac{r_+^{d-3}}{r_B^{d-3}}}},
\]

\[
p = \frac{(d - 2)(d - 3)}{8\pi r_B^2} \left(\frac{2 - \frac{r_+^{d-3}}{r_B^{d-3}}}{\sqrt{1 - \frac{r_+^{d-3}}{r_B^{d-3}}}} - 1 \right).
\]
As a result, from Eqs. (1.2), (2.12), and (3.9)–(3.11), the Gibbs free energy of the Schwarzschild black hole in a cavity reads

\[ G = E - TS + pV = \frac{(d - 2)\Omega r_B^{d-3}}{4\pi(d-1)} \left[ 1 - \frac{4(d - 2) - (3d - 5)r_B^{d-3}}{4(d - 2)\sqrt{1 - r_B^{d-3}}} \right]. \quad (3.12) \]

At the HP phase transition point, from Eqs. (2.16) and (3.12), we have

\[ r_B = \left[ \frac{(3d - 5)^2}{8(d - 1)(d - 2)} \right]^{\frac{1}{d-3}} r_+. \]

Substituting \( r_B \) into Eqs. (3.10) and (3.11), we obtain \( T_{\text{HP}} \) and \( p \) in terms of \( r_+ \) respectively,

\[ T_{\text{HP}} = \frac{3d - 5}{4\pi r_+}, \quad p = \left[ \frac{(d - 1)^{2d-4}(d - 2)^{d-1}}{4^{d-6}(3d - 5)^{d+1}} \right]^{\frac{1}{d-3}} \frac{1}{\pi r_+^2}, \]

so the HP temperature \( T_{\text{HP}} \) can be expressed as a function of pressure \( p \) and dimension \( d \),

\[ T_{\text{HP}} = \left[ \frac{(3d - 5)^{3d-5}}{4^d(d - 1)^{2d-4}(d - 2)^{d-1}} \right]^{\frac{1}{d-3}} \sqrt{\frac{p}{\pi}}. \quad (3.13) \]

The coexistence lines of the Schwarzschild black holes in a cavity in different dimensions are shown in Fig. 3. Similar to the AdS case, the HP phase transition can occur at all pressures, and \( T_{\text{HP}} \) increases with both \( p \) and \( d \).

**Figure 3.** The HP temperature \( T_{\text{HP}} \) of the Schwarzschild black holes in a cavity as a function of pressure \( p \), with different dimensions \( d \). The HP phase transition can occur at all pressures, and \( T_{\text{HP}} \) increases with \( d \) at a given \( p \). These characteristics are analogous to those in the AdS case.

Next, we move on to the minimum temperature \( T_0 \) of the Schwarzschild black hole in a cavity. We should stress that the calculation is now quite different from that in the AdS case, because from Eqs. (3.10) and (3.11), \( T \) and \( p \) are expressed in the parametric...
equations, and the derivation of $T_0$ is thus more difficult. For this reason, from Eq. (3.10), we first rewrite $r_B$ as

$$r_B = \left[ \frac{u^2}{u^2 - (d - 3)^2} \right]^{\frac{1}{d-3}} r_+,$$

where $u = 4\pi r + T$. By this means, we can reexpress Eq. (3.11) as

$$\frac{p}{(d-2)\pi T^2} = \left\{ \frac{[u+(d-3)]^2[u-(d-3)]^{2d-4}}{u^{3d-5}} \right\}^{\frac{1}{d-3}}. \tag{3.14}$$

It is straightforward to find that, when $u = d - 3 + 2\sqrt{(d-2)(d-3)}$, the right-hand-side of Eq. (3.14) has a maximum,

$$\frac{p}{(d-2)\pi T^2} \leq \left\{ \frac{4^{d-1}(d-2)^{d-2}(d-3)^{3d-5} [2d - 5 + 2\sqrt{(d-2)(d-3)}]}{[d - 3 + 2\sqrt{(d-2)(d-3)}]^{3d-5}} \right\}^{\frac{1}{d-3}}.$$

Consequently, the minimum temperature $T_0$ reads

$$T_0 = \left\{ \frac{4^{d-1}(d-2)^{d-2}(d-3)^{3d-5} [2d - 5 + 2\sqrt{(d-2)(d-3)}]}{4(d-1)(d-2)^{d-4}[d - 3 + 2\sqrt{(d-2)(d-3)}]^{3d-5}} \right\}^{\frac{1}{2(d-3)}} \sqrt{\frac{p}{\pi}}. \tag{3.15}$$

and $T_0$ also monotonically increases with both $p$ and $d$.

To compare with the results in the AdS case, we first calculate the ratio of $T_{HP}$ to $T_0$ in a cavity in $d$ dimensions,

$$\frac{T_{HP}(d)}{T_0(d)} = \left\{ \frac{(d-2)^{d-4}(d-3)^{d-1}(3d-5)^{3d-5} [2d - 5 + 2\sqrt{(d-2)(d-3)}]}{4(d-1)^{2d-4}[d - 3 + 2\sqrt{(d-2)(d-3)}]^{3d-5}} \right\}^{\frac{1}{2(d-3)}}. \tag{3.16}$$

Apparently, Eq. (3.16) is much more complicated than Eq. (3.6), but by a direct power counting, we easily find in the numerator and denominator that the powers of $d$ are both $5d - 9$, and the numerical factors are both $4 \cdot 3^{3d-5}$. Hence, in the limit of $d \to \infty$, we still have the same result as that in Eq. (3.7),

$$\lim_{d \to \infty} \frac{T_{HP}(d)}{T_0(d)} = 1.$$

Second, it seems not straightforward at all to reproduce the simple dual relation $T_{HP}(d) = T_0(d + 1)$ in Eq. (3.8). Nevertheless, we should state that this interesting relation does hold approximately in the cavity case, only with slight deviation. From Eqs. (3.13) and (3.15), we have

$$\frac{T_{HP}(d)}{T_0(d + 1)} = \left\{ \frac{(3d - 5)^{3d^2-11d+10}[2d - 3 + 2\sqrt{(d-1)(d-2)}]^{d-3}}{4(d-1)^{d-1}(d-2)^2[2d - 2 + 2\sqrt{(d-1)(d-2)}]^{3d^2-11d+6}} \right\}^{\frac{1}{2(d-2)(d-3)}}. \tag{3.17}$$
Despite the apparent complicity, Eq. (3.17) is still independent of pressure $p$. Again, by a power counting, we find that the ratios of $T_{\text{HP}}(d)/T_0(d + 1)$ are very close to 1, even when $d$ is only 4. The detailed values of $T_{\text{HP}}(d)/T_0(d + 1)$ for both the Schwarzschild black holes in the AdS space and in a cavity are listed in Tab. 1. From the last column, we clearly observe that the ratios of $T_{\text{HP}}(d)/T_0(d + 1)$ in the cavity case are precisely close to 1 and approach to 1 when $d \to \infty$. Therefore, we are allowed to conclude that the dual relation in Eq. (3.8) is not merely a mathematical coincidence, but is a remarkable and universal character of the HP phase transitions in different extended phase spaces, insensitive to the boundary conditions of the black holes.

| $d$   | $T_{\text{HP}}/\sqrt{p}$ | $T_0/\sqrt{p}$ | $T_{\text{HP}}/\sqrt{p}$ | $T_0/\sqrt{p}$ | $T_{\text{HP}}(d)/T_0(d + 1)$ |
|-------|--------------------------|----------------|--------------------------|----------------|-------------------------------|
| $d = 4$ | 0.92132 | 0.79789 | 1.25707 | 1.13393 | 0.99853                      |
| $d = 5$ | 0.97721 | 0.92132 | 1.31413 | 1.25892 | 0.99930                      |
| $d = 6$ | 1.00925 | 0.97721 | 1.34658 | 1.31505 | 0.99959                      |
| $d = 7$ | 1.03006 | 1.00925 | 1.36756 | 1.34713 | 0.99974                      |
| $d = 8$ | 1.04468 | 1.03006 | 1.38225 | 1.36792 | 0.99982                      |
| $d = 9$ | 1.05550 | 1.04468 | 1.39311 | 1.38250 | 0.99986                      |
| $d = 10$ | 1.06385 | 1.05550 | 1.40147 | 1.39330 | 0.99989                      |
| $d = \infty$ | 1.12838 | 1.12838 | 1.46581 | 1.46581 | 1.00000                      |

Table 1. The detailed values of the HP temperature $T_{\text{HP}}$ and the minimum temperature $T_0$ of the Schwarzschild black holes in the AdS space and in a cavity, with different dimensions $d$. The ratios of $T_{\text{HP}}(d)/T_0(d + 1)$ in the cavity case are listed explicitly, which are precisely close to 1 even when $d$ is only 4, and the deviation from 1 decreases as $d \to \infty$.

Finally, the $G$–$T$ curves of the Schwarzschild black holes in a cavity in different dimensions are shown in Fig. 4. Again, for the small black holes in a cavity, they do not have the HP phase transition, as in the AdS case. For the large black holes, when $d$ increases, the ratios of $T_{\text{HP}}(d)/T_0(d)$ tend to 1. Moreover, the HP temperatures $T_{\text{HP}}(d)$ in $d$ dimensions are still approximately equal to the minimum temperatures $T_0(d + 1)$ in $d + 1$ dimensions, indicating the similarity between the two different extended phase spaces.

4 HP phase transitions of the charged black holes

In this section, we discuss the HP phase transitions and the dual relations of the charged black holes in the AdS space and in a cavity respectively, and the influences of the electric potential on the HP temperature and the dual relation will be studied carefully.

4.1 HP phase transition of the charged black hole in the AdS space

As the thermal gas is electrically neutral, due to the conservation of charge, the HP phase transition of the charged black hole should be investigated in a grand canonical ensemble rather than a canonical one. In other words, the electric potential $\Phi$ at horizon should be
Figure 4. The Gibbs free energies $G$ of the Schwarzschild black holes in a cavity as a function of temperature $T$, with pressure $p = 0.5$ and different dimensions $d$. Similar to the AdS case, the ratios of $T_{\text{HP}}(d)/T_0(d)$ tend to 1 as $d$ increases, and the HP temperatures $T_{\text{HP}}(d)$ in $d$ dimensions are still approximately equal to the minimum temperatures $T_0(d+1)$ in $d+1$ dimensions, analogous to the dual relation in the AdS case.

fixed, and the electric charge $Q$ is not a conserved quantity but is allowed to vary. The explicit relation of $Q$ and $\Phi$ in the AdS space can be obtained from Eq. (2.8) as

$$Q = \frac{(d - 3)\Omega r^{d-3}_+ \Phi}{4\pi}.$$  \hspace{1cm} (4.1)

Substituting Eq. (4.1) into Eq. (2.6), we have the temperature of the charged black hole as

$$T = \frac{(d - 3)[d - 2 - 2(d - 3)\Phi^2] + 16\pi pr^2_+}{4\pi(d - 2)r_+}.$$  \hspace{1cm} (4.2)

Hence, the electric potential $\Phi$ reduces the black hole temperature $T$. Then, from Eqs. (2.3), (2.5), (4.1), and (4.2), the Gibbs free energy of the $d$-dimensional charged black hole in the AdS space reads

$$G = M - TS - \Phi Q = \frac{\{(d - 1)[d - 2 - 2(d - 3)\Phi^2] - 16\pi pr^2_+\}r^{d-3}_+}{16\pi(d - 1)(d - 2)}.$$

By the same method in Sect. 3.1, it is direct to obtain the HP temperature $T_{\text{HP}}$ of the charged black hole,

$$T_{\text{HP}} = 2\sqrt{\frac{d - 2 - 2(d - 3)\Phi^2}{d - 1} \sqrt{\frac{p}{\pi}}}.$$  \hspace{1cm} (4.3)

Here, an important point should be stressed. If the HP phase transition occurs, as $T_{\text{HP}}$ is positive, from Eq. (4.3), the electric potential $\Phi$ must have an upper bound as

$$\Phi < \sqrt{\frac{d - 2}{2(d - 3)}}.$$  \hspace{1cm} (4.4)
Obviously, this upper bound has a maximum of 1 when $d = 4$, and then decreases with $d$, reaching a minimum of $\sqrt{2}/2$ in the limit of $d \to \infty$.

The $T_{\text{HP}} - p$ curves of the charged black holes in the AdS space are shown in Fig. 5 with different electric potentials $\Phi$ and in Fig. 6 with different dimensions $d$ respectively. We observe that $T_{\text{HP}}$ monotonically decreases with $\Phi$, consistent with our previous studies on various charged black holes in Refs. [37, 64, 69, 71]. However, the influences of $d$ on $T_{\text{HP}}$ are somewhat complicated. From Eq. (4.3), it is not difficult to find that, when $\Phi < 1/2$, $T_{\text{HP}}$ increases with $d$, but when $\Phi > 1/2$, $T_{\text{HP}}$ decreases with $d$. More interestingly, when $\Phi = 1/2$, $T_{\text{HP}}$ is always $\sqrt{2p/\pi}$, independent of dimension.

**Figure 5.** The HP temperature $T_{\text{HP}}$ of the charged black holes in the AdS space as a function of pressure $p$, with dimension $d = 6$ and different electric potentials $\Phi$. $T_{\text{HP}}$ monotonically decreases with $\Phi$ at a given pressure.

**Figure 6.** The HP temperature $T_{\text{HP}}$ of the charged black holes in the AdS space as a function of pressure $p$, with different dimensions $d$ and electric potentials $\Phi$. When $\Phi < 1/2$, $T_{\text{HP}}$ increases with $d$ (left panel); when $\Phi > 1/2$, $T_{\text{HP}}$ decreases with $d$ (right panel).

Furthermore, from Eq. (4.2), we obtain the minimum temperature $T_0$ of the charged
black hole in the AdS space,
\[
T_0 = \frac{2\sqrt{(d-3)(d-2-2(d-3)\Phi^2)}}{d-2} \sqrt{\frac{p}{\pi}}.
\] (4.5)

Similar to \(T_{HP}\), \(T_0\) also monotonically decreases with \(\Phi\) in all dimensions, but the effects of \(d\) on \(T_0\) are more complicated. Taking a partial derivative, we have
\[
\frac{\partial T_0}{\partial d} = \frac{d - 2 - 2(d-3)\Phi^2}{(d-2)^2\sqrt{(d-3)(d-2-2(d-3)\Phi^2)}} \sqrt{\frac{p}{\pi}}.
\]

Thus, when \(\Phi < 1/2\), \(\partial T_0/\partial d\) is positive definite, and \(T_0\) increases with \(d\); when \(\Phi > \sqrt{2}/2\), \(\partial T_0/\partial d\) is negative definite, and \(T_0\) decreases with \(d\). However, when \(1/2 < \Phi < \sqrt{2}/2\), the sign of \(\partial T_0/\partial d\) is indefinite, and \(T_0\) first increases and then decreases with \(d\).

Now, we turn to the relation of \(T_{HP}\) and \(T_0\). First, from Eqs. (4.3) and (4.5), we have
\[
\frac{T_{HP}(d)}{T_0(d)} = \frac{d - 2}{\sqrt{(d-1)(d-3)}}.
\] (4.6)

Notably, this ratio is exactly the same as that for the Schwarzschild black hole in Eq. (3.6), because the dependence on the electric potential cancels out in Eqs. (4.3) and (4.5), meaning that \(\Phi\) does not affect the ratio of \(T_{HP}(d)/T_0(d)\), although it decreases \(T_{HP}\) and \(T_0\) individually. Thus, demanding the minimum temperature \(T_0\) to be positive will not provide further constraint on \(\Phi\). Furthermore, when \(d \to \infty\), we still have
\[
\lim_{d \to \infty} \frac{T_{HP}(d)}{T_0(d)} = 1.
\]

Second, the simple dual relation \(T_{HP}(d) = T_0(d+1)\) for the Schwarzschild black hole in Eq. (3.8) no longer holds in the current circumstance. At present, the ratio of \(T_{HP}(d)\) to \(T_0(d+1)\) depends on both \(d\) and \(\Phi\),
\[
\frac{T_{HP}(d)}{T_0(d+1)} = \sqrt{\frac{(d-1)(d-2-2(d-3)\Phi^2)}{(d-2)(d-1-2(d-2)\Phi^2)}}.
\] (4.7)

In Ref. [77], the authors generalized the dual relation from Eq. (3.8) to the case of the charged black hole in the AdS space as
\[
T_{HP}(d, \Phi) = T_0 \left( d + 1, \frac{\sqrt{(d-1)(d-3)}}{d-2} \Phi \right). \tag{4.8}
\]

Now, we point out that, from Eq. (4.6), the dual relation in Eq. (4.8) can actually be rewritten in a more concise form,
\[
T_{HP}(d, \Phi) = T_0 \left( d + 1, \frac{T_0(d)}{T_{HP}(d)} \Phi \right). \tag{4.9}
\]

Here, we must emphasize that the essential reason why we reformulate Eq. (4.8) as Eq. (4.9) is not merely for mathematical tidiness, but for deeper physical consideration. It will
be shown in Sect. 4.2 that Eq. (4.9) still holds approximately for the charged black hole in a cavity, albeit the ratio of $T_0(d)/T_{HP}(d)$ becomes much more complicated [see Eq. (4.16)]. In other words, it is the dual relation expressed only in the form of Eq. (4.9) that can be applied to the cavity case, not the original one in Eq. (4.8) presented in Ref. [77]. Moreover, we do not need to distinguish the values of $T_0(d)/T_{HP}(d)$ for the neutral and charged black holes in the AdS space, since they are equal in Eqs. (3.6) and (4.6).

Last, the $G$–$T$ curves of the charged black holes in the AdS space are shown in Fig. 7 with different electric potentials $\Phi$ and in Fig. 8 with different dimensions $d$ respectively. It can be found that both the HP temperature $T_{HP}$ and minimum temperature $T_0$ decrease with $\Phi$ at a given $d$. Also, when $\Phi < 1/2$, both $T_{HP}$ and $T_0$ increase with $d$; when $\Phi > \sqrt{2}/2$, they both decrease. However, when $1/2 < \Phi < \sqrt{2}/2$, $T_{HP}$ monotonically decreases with $d$, but $T_0$ first increases and then decreases with $d$.

**Figure 7.** The Gibbs free energies $G$ of the charged black holes in the AdS space as a function of temperature $T$, with pressure $p = 0.2$, dimension $d = 6$, and different electric potentials $\Phi$. It can be seen that both $T_{HP}$ and $T_0$ monotonically decrease with $\Phi$.

### 4.2 HP phase transition of the charged black hole in a cavity

Finally, let us discuss the HP phase transition of the charged black hole in a cavity. This is not just a trivial extension of the results in Sect. 3.2, but there exist significant differences, especially in the $G$–$T$ curves, to be explained in depth.

Following Sect. 4.1, we utilize the grand canonical ensemble, and first reexpress the electric charge $Q$ in terms of the electric potential $\Phi$ by Eq. (2.15),

$$Q = \frac{(d - 3)\Omega r^{d-3}\Phi}{4\pi} \sqrt{\frac{d - 2}{d - 2 - [d - 2 - 2(d - 3)\Phi^2]^2}}.$$  \hspace{1cm} (4.10)

Then, substituting Eq. (4.10) into Eqs. (2.11), (2.13), and (2.14), we obtain the thermal
The Gibbs free energies $G$ of the charged black holes in the AdS space as a function of temperature $T$, with pressure $p = 0.2$ and different dimensions $d$ and electric potentials $\Phi$. When $\Phi < 1/2$, both $T_{HP}$ and $T_0$ increase with $d$ (left panel); when $\Phi > \sqrt{2}/2$, both $T_{HP}$ and $T_0$ decrease with $d$ (right panel). The case with $1/2 < \Phi < \sqrt{2}/2$ is not shown due to its unnecessary complicity.

energy, temperature, and pressure as

$$E = \frac{(d-2)\Omega r_B^{d-3}}{8\pi} \left\{ 1 - \left( 1 - \frac{r_+^{d-3}}{r_B^{d-3}} \right) \sqrt{\frac{d-2}{d-2 - [d-2-2(d-3)\Phi]^2} \frac{r_B^{d-3}}{r_+^{d-3}}} \right\}, \quad (4.11)$$

$$T = \frac{4\pi r_+}{(d-3)[d-2-2(d-3)\Phi]} \left\{ (d-2) - [d-2-2(d-3)\Phi]^2 \frac{r_B^{d-3}}{r_+^{d-3}} \right\}, \quad (4.12)$$

$$p = \frac{(d-2)(d-3)}{8\pi r_B^2} \left\{ \frac{2(d-2) - (d-2-2(d-3)\Phi)^2 \frac{r_B^{d-3}}{r_+^{d-3}}}{2\sqrt{(d-2) - (d-2-2(d-3)\Phi)^2 \frac{r_B^{d-3}}{r_+^{d-3}}}} - 1 \right\}. \quad (4.13)$$

Accordingly, from Eqs. (1.2), (2.12), and (4.10)–(4.13), the Gibbs free energy of the $d$-dimensional charged black hole in a cavity reads

$$G = E + pV - TS - \Phi Q = \frac{(d-2)\Omega r_B^{d-3}}{4\pi(d-1)} \times$$

$$\left\{ 1 - \left[ \frac{d-2}{d-2 - (d-2-2(d-3)\Phi)^2 \frac{r_B^{d-3}}{r_+^{d-3}}} \right] \frac{\left( 3d-5)(d-2-2(d-3)\Phi)^2 \frac{r_B^{d-3}}{r_+^{d-3}} \right)}{4(d-2)^2} \right\}. \quad (4.14)$$

If the HP phase transition occurs, setting $G = 0$ in Eq. (4.14), we can first solve the cavity radius $r_B$ in terms of the event horizon radius $r_+$,

$$r_B = \left\{ \frac{(3d-5)^2[d-2-2(d-3)\Phi]^2}{8(d-1)(d-2)^2} \right\}^{\frac{1}{4d-5}} r_+.$$

---

**Figure 8.** The Gibbs free energies $G$ of the charged black holes in the AdS space as a function of temperature $T$, with pressure $p = 0.2$ and different dimensions $d$ and electric potentials $\Phi$. When $\Phi < 1/2$, both $T_{HP}$ and $T_0$ increase with $d$ (left panel); when $\Phi > \sqrt{2}/2$, both $T_{HP}$ and $T_0$ decrease with $d$ (right panel). The case with $1/2 < \Phi < \sqrt{2}/2$ is not shown due to its unnecessary complicity.
and the constraint $r_B > r_+$ sets an upper bound of the electric potential $\Phi$ in a cavity,

$$\Phi < \sqrt{\frac{(d - 2)(d - 3)}{2(3d - 5)^2}}. \quad (4.15)$$

Comparing the results in Eqs. (4.4) and (4.15), we find that, opposite to the AdS case, the upper bound of $\Phi$ now has a minimum of $1/7$ when $d = 4$, and then increases with $d$, reaching a maximum of $\sqrt{2}/6$ in the limit of $d \to \infty$. Moreover, this upper bound in the cavity case is even stricter than that in the AdS case.

Following the same procedure in Sect. 3.2, after some algebra, the HP temperature is obtained as

$$T_{\text{HP}} = \left\{ \frac{(3d - 5)^{3d-5}[d - 2 - 2(d - 3)\Phi^2]^{2d-4}}{4^d(d - 1)^{2d-4}(d - 2)^{3d-5}} \right\}^{\frac{1}{2(d-3)}} \sqrt{\frac{p}{\pi}}.$$

Here, we mention that the requirement $T_{\text{HP}} > 0$ sets the same loose constraint on $\Phi$ as that in Eq. (4.4), so the tighter upper bound in Eq. (4.15) will not be changed. The $T_{\text{HP}}$ curves of the charged black holes in a cavity with different $\Phi$ and $d$ are shown in Fig. 9.

First, similar to the AdS case, $T_{\text{HP}}$ decreases with $\Phi$ with a given dimension $d = 6$ (left panel), and increases with $d$ with a given electric potential $\Phi = 1/7$ (right panel).

Furthermore, the minimum temperature $T_0$ can also be obtained as

$$T_0 = \left\{ \frac{[d - 3 + 2\sqrt{(d - 2)(d - 3)}]^{3d-5}[d - 2 - 2(d - 3)\Phi^2]^{2d-4}}{4^{d-1}(d - 2)^{4d-9}(d - 3)^{d-1}[2d - 5 + 2\sqrt{(d - 2)(d - 3)}]} \right\}^{\frac{1}{2(d-3)}} \sqrt{\frac{p}{\pi}}.$$

Similar to $T_{\text{HP}}$, $T_0$ also decreases with $\Phi$ and increases with $d$.

Now, we discuss the relation between $T_{\text{HP}}$ and $T_0$ for the charged black hole in a cavity. First, despite the formal complicities of $T_{\text{HP}}(d)$ and $T_0(d)$, it is interesting to find that their
ratio is exactly the same as that in Eq. (3.16) for the Schwarzschild black hole in a cavity,

\[
\frac{T_{HP}(d)}{T_0(d)} = \left\{ \frac{(d-2)^{d-4}(d-3)^{d-1}(3d-5)^{3d-5}[2d-5 + 2\sqrt{(d-2)(d-3)}]}{4(d-1)^{2d-4}[d-3 + 2\sqrt{(d-2)(d-3)}]^{3d-5}} \right\}^{\frac{1}{2(d-3)}}. \tag{4.16}
\]

Consequently, we are allowed to conclude that the independence of \(T_{HP}(d)/T_0(d)\) on the electric potential \(\Phi\) is universal, valid for both the AdS and cavity cases. In addition, although the ratio in Eq. (4.16) is much more complicated than that in Eq. (4.6) for the AdS case, in the limit of \(d \to \infty\), we still have

\[
\lim_{d \to \infty} \frac{T_{HP}(d)}{T_0(d)} = 1.
\]

Second, for the dual relation between \(T_{HP}(d)\) and \(T_0(d+1)\), we first point out that the original one in Eq. (4.8) suggested in Ref. [77] is no longer suitable for our current discussion, as the ratio of \(\sqrt{(d-1)(d-3)}/(d-2)\) in the AdS case has no explicit physical interpretation in the cavity case. However, we will show that the dual relation rewritten in the form of Eq. (4.9) does apply to the charged black hole in a cavity rather well. In Fig. 10, we show in the counter map the influences of \(d\) and \(\Phi\) on the ratio of

\[
\frac{T_{HP}(d, \Phi)}{T_0(d + 1, \frac{T_0(d + 1)}{T_{HP}(d, \Phi)})}.
\]

The explicit expression is not shown here due to the unnecessary mathematical tediousness. We clearly see that this ratio is always approximately equal to 1, and will be even closer to 1 with larger \(d\) and smaller \(\Phi\). These observations further support that our new dual relation in Eq. (4.9) is suitable for both the AdS and cavity cases, confirming its universality in different extended phase spaces.

Last, we discuss the \(G-T\) curves. Before doing so, we should state that there exists a significant difference between the \(G-T\) curves of the charged black holes in different extended phase spaces. There is no terminal point in the \(G-T\) curves in the AdS case in Fig. 8. However, due to the extra constraint \(r_B > r_+\) in a cavity, there will be terminal points in the \(G-T\) curves now.

To figure out the terminal points in the \(G-T\) curves, from Eq. (4.13), we first solve \(r_+\) in terms of \(r_B\) as

\[
r_+ = \left\{ \frac{8\sqrt{\pi p r_B^2} [4\pi p r_B^2 + (d-2)(d-3)]}{(d-2)(d-3)^2(d-2 - 2(d-3)\Phi^2)} \left[ \sqrt{4\pi p r_B^2} + (d-2)(d-3) - \sqrt{4\pi p r_B^2} \right]^{2d-3} \right\} r_B.
\]

Then, considering the constraint \(r_+ < r_B\), we obtain the upper bound of \(r_B\) as

\[
r_B < \frac{[2(d-2)(d-3)]^{\frac{1}{2}} [\sqrt{d-2} - \sqrt{2(d-3)}\Phi]}{4\sqrt{2\pi p \Phi}}. \tag{4.18}
\]
The ratio of $\frac{T_{HP}(d, \Phi)}{T_0(d + 1, \frac{T_0(d)}{r_B(d)^2}, \Phi)}$ of the charged black hole in a cavity as a function of dimension $d$ and electric potential $\Phi$. The upper bound of $\Phi$ is also plotted according to Eq. (4.15). The dual relation rewritten in the form of Eq. (4.9) is still precisely valid in the cavity case. The ratio is already 0.99092 when $d = 4$ and $\Phi = 1/7$, and will be even closer to 1 with larger $d$ and smaller $\Phi$.

This result is evidently different from the case of the Schwarzschild black hole in a cavity, in which $r_B$ can take any value. Nevertheless, in the limit of $\Phi \to 0$, the upper bound of $r_B$ naturally tends to infinity. Substituting Eq. (4.17) into Eqs. (4.12) and (4.14), we are able to reexpress the temperature and Gibbs free energy in the parametric equations as $T = T(d, p, \Phi, r_B)$ and $G = G(d, p, \Phi, r_B)$. Taking into account the upper bound of $r_B$ in Eq. (4.18), we can eventually determine the terminal points in the $G$-$T$ curves of the charged black holes in a cavity. Here, we only give their coordinates in 4 dimensions,

$$(T, G) = \left(1 + \Phi\right)\sqrt{\frac{p}{2\pi\Phi}} - \frac{(1 - \Phi^2)(1 - 7\Phi)}{24\sqrt{2\pi p\Phi^3}}.$$ 

and the general results in $d$ dimensions will not be shown for their lengthy expressions.

The $G$-$T$ curves of the charged black holes in a cavity are plotted in Fig. 11, with different dimensions $d$ and electric potentials $\Phi$. Because of the terminal point, the temperature of the large charged black hole has an upper bound. Especially, in the right panel, as $\Phi$ increases, the terminal points move towards the upper left corner of the $G$-$T$ plane, and finally when $\Phi$ reaches its upper bound, the terminal point is located exactly on the $T$-axis at $(T_{HP}, 0)$. Moreover, when $\Phi \to 0$, the terminal points move towards the lower right corner of the $G$-$T$ plane, and finally disappear when $\Phi = 0$, consistent with the case of the Schwarzschild black hole in a cavity in Fig. 4. Another notable character in the right panel is that all the cusps of the $G$-$T$ curves have the same height. This is because when we substitute Eq. (4.17) into Eq. (4.14), the electric potential $\Phi$ will cancel out. As a result, when $d$, $p$, and $r_B$ are fixed, except the different terminal points caused by the different values of $\Phi$, all the $G$-$T$ curves distinguish one another only by a shift in the $T$-direction.
5 Conclusion

In recent years, black hole thermodynamics in the extended phase space is receiving increasing research interests. The basic aims of the relevant studies are twofold: to stabilize the black hole from inevitable evaporation and to restore the nonexistent $p-V$ term in the first law of black hole thermodynamics. Naturally, there are two complementary ways of introducing the effective pressure $p$ and effective volume $V$ for the black hole. One is to introduce $p$ from the cosmological constant $\Lambda$ in the AdS space, and then to define $V$ as its conjugate variable; the other is to introduce $V$ as the Euclidean volume of a spherical cavity, and then to define $p$ as its conjugate variable. Consequently, we achieve two extended phase spaces, and both possess increasing gravitational potential at large distances, but with different boundary conditions from a mathematical point of view. Due to these factors, there will be similarities and dissimilarities simultaneously in these two extended phase spaces, and the exploration of all these issues is the main purpose of this paper.

In our present work, we systematically investigate one of the most important topics in black hole thermodynamics: the HP phase transition in the black hole–thermal gas system, in the AdS space and in a cavity. The HP temperature $T_{HP}$, minimum black hole temperature $T_0$, and Gibbs free energy $G$ for both the neutral and charged black holes are calculated and illustrated in arbitrary dimension. For the charged black holes, we work in the grand canonical ensemble with fixed electric potential at horizon. Especially, the dual relation of $T_{HP}$ and $T_0$ is explored in the cavity case with great care. The basic conclusions of this paper can be drawn as follows.

1. We find the evident similarities in the two extended phase spaces. (1) The HP phase transition can occur at all pressures, and both $T_{HP}$ and $T_0$ increase with pressure $p$ in all
circumstances. (2) $T_{HP}$ and $T_0$ increase with dimension $d$ for the Schwarzschild black holes. (3) $T_{HP}$ and $T_0$ decrease with electric potential $\Phi$ if the black holes are charged.

2. As for the relations of $T_{HP}$ and $T_0$, the main aspects are the similarities in the two extended phase spaces. (1) The ratios of $T_{HP}(d)/T_0(d)$ are always independent of $p$ and $\Phi$, and approach to 1 in the limit of $d \to \infty$, meaning that the metastable large black hole phase tends to disappear in higher dimensions. (2) For the Schwarzschild black holes, we find that the simple dual relation $T_{HP}(d) = T_0(d+1)$ for the AdS case is also approximately valid for the cavity case to a rather high precision, and the deviation of $T_{HP}(d)/T_0(d+1)$ from 1 tends to vanish as $d$ increases (see Tab. 1). (3) For the charged black holes, we reformulate the dual relation from Eq. (4.8) suggested in Ref. [77] to Eq. (4.9). Besides mathematical conciseness, this new form can be further applied to the cavity case in a more physically reasonable sense, also with considerable precision (see Fig. 10).

3. For the charged black holes, there are significant dissimilarities in the two extended phase spaces. (1) If the HP phase transition occurs, the upper bound of $\Phi$ in the cavity case is much tighter than that in the AdS case, due to the extra constraint of $r_+ < r_B$. Moreover, this upper bound decreases with $d$ in the AdS case but increases with $d$ in the cavity case. (2) The dimension-dependence of $T_{HP}$ and $T_0$ is complicated in the AdS case. Simply speaking, $T_{HP}$ and $T_0$ increase with $d$ when $\Phi$ is small but decrease with $d$ when $\Phi$ is large, also with some non-monotonic intermediate interval. However, both $T_{HP}$ and $T_0$ only monotonically increase with $d$ in the cavity case, as the upper bound of $\Phi$ is now rather small. (3) There are terminal points in the $G$--$T$ curves in the cavity case. These points are located at $(T_{HP},0)$ when $\Phi$ reaches its upper bound, and will be absent when $\Phi \to 0$.

4. A side remark is that all the results in this paper are analytical, which were usually obtained only numerically in previous literature, such that we are able to have a clear and quantitative understanding of the HP phase transitions in different extended phase spaces.

In summary, on the one hand, we find remarkable analogies in the HP phase transitions in the AdS space and in a cavity. These similarities indicate the universal property in different extended phase spaces. On the other hand, we also observe some notable differences at the same time. Generally speaking, all these dissimilarities stem from the fact that the cavity radius $r_B$ must be always larger than the event horizon radius $r_+$ of the black hole. Of course, in the limits of $\Lambda \to 0$ and $r_B \to \infty$, the two extended phase spaces tend to be equivalent, but for finite values of $\Lambda$ and $r_B$, there are still some discrepancies that are sensitive to the boundary conditions. Altogether, we wish to provide in the paper a whole picture of the HP phase transitions in the AdS space and in a cavity, and to motivate further studies on other thermodynamic properties of different extended phase spaces, such as the underlying physical interpretation of the dual relation of $T_{HP}(d)$ and $T_0(d+1)$.

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