An isoperimetric result for an energy related to the $p$-capacity

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Abstract

In this paper, we generalize the notion of relative $p$-capacity of $K$ with respect to $\Omega$, by replacing the Dirichlet boundary condition with a Robin one. We show that, under volume constraints, our notion of $p$-capacity is minimal when $K$ and $\Omega$ are concentric balls. We use the $H$-function (see [4, 8]) and a derearrangement technique.

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1 Introduction

Let $p > 1$, $\beta > 0$ be real numbers. For every open bounded sets $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, and every compact set $K \subseteq \overline{\Omega}$ with Lipschitz boundary, we define

$$E_{\beta,p}(K, \Omega) = \inf_{v \in W^{1,p}(\Omega), v = 1 \text{ in } K} \left( \int_{\Omega} |\nabla v|^p \, dx + \beta \int_{\partial \Omega} |v|^p \, d\mathcal{H}^{n-1} \right).$$

(1.1)

We notice that it is sufficient to minimize among all functions $v \in H^1(\Omega)$ with $v = 1$ in $K$ and $0 \leq v \leq 1$ a.e., moreover if $K, \Omega$ are sufficiently smooth, a minimizer $u$ satisfies

$$\begin{cases}
  u = 1 & \text{in } K, \\
  \Delta_p u = 0 & \text{in } \Omega \setminus K, \\
  |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial \Omega \setminus \partial K,
\end{cases}$$

(1.2)

where $\Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right)$ is the $p$-Laplacian of $u$ and $\nu$ is the outer unit normal to $\partial \Omega$. If $\hat{K} = \Omega$, equation (1.2) has to be intended as $u = 1$ in $\Omega$, and the energy is

$$E_{\beta,p}(\Omega, \Omega) = \beta \mathcal{H}^{n-1}(\partial \Omega).$$

In general, equation (1.2) has to be intended in the weak sense, that is: for every $\varphi \in W^{1,p}(\Omega)$ such that $\varphi \equiv 0$ in $\hat{K}$,

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi \, d\mathcal{L}^n + \beta \int_{\partial \Omega} u^{p-1} \varphi \, d\mathcal{H}^{n-1} = 0.$$ 

(1.3)

In particular if $u$ is a minimizer, letting $\varphi = u - 1$, we have that

$$E_{\beta,p}(K, \Omega) = \int_{\Omega} |\nabla u|^p \, dx + \beta \int_{\partial \Omega} u^p \, d\mathcal{H}^{n-1} = \beta \int_{\partial \Omega} u^{p-1} \, d\mathcal{H}^{n-1}. $$

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Moreover from the strict convexity of the functional, the minimizer is the unique solution to (1.3).

This problem is related to the so-called relative $p$-capacity of $K$ with respect to $\Omega$, defined as
\[ \text{Cap}_p(K, \Omega) := \inf_{v \in W^{1,p} \Omega, 0} \left( \int_\Omega |\nabla v|^p \, dx \right). \]

In the case $p = 2$ it represents the electrostatic capacity of an annular condenser consisting of a conducting surface $\partial \Omega$, and a conductor $K$, where the electrostatic potential is prescribed to be 1 inside $K$ and 0 outside $\Omega$. Let $\omega_n$ be the measure of the unit sphere in $\mathbb{R}^n$, and let $M > \omega_n$, then it is well known that there exists some $r \geq 1$ such that
\[ \min_{|K| = \omega_n, |\Omega| \leq M} \text{Cap}_p(K, \Omega) = \text{Cap}_p(B_1, B_r). \]

This is an immediate consequence of the Pólya-Szegő inequality for the Schwarz rearrangement (see for instance [11, 9]). We are interested in studying the same problem for the energy defined in (1.1), which corresponds to changing the Dirichlet boundary condition on $\partial \Omega$ into a Robin boundary condition, namely, we consider the following problem
\[ \text{inf}_{|K| = \omega_n, |\Omega| \leq M} E_{\beta,p}(K, \Omega). \] (1.4)

In this case, the previous symmetrization techniques cannot be employed anymore.

Problem (1.4) has been studied in the linear case $p = 2$ in [6], with more general boundary conditions on $\partial \Omega$, namely
\[ \frac{\partial u}{\partial \nu} + \frac{1}{2} \Theta'(u) = 0, \]
where $\Theta$ is a suitable increasing function vanishing at 0. This problem has been addressed to thermal insulation (see for instance [7, 1]). Our main result reads as follows.

**Theorem 1.1.** Let $\beta > 0$ such that
\[ \beta \frac{1}{p-1} > n - \frac{p}{p-1}. \]
Then, for every $M > \omega_n$ the solution to problem (1.4) is given by two concentric balls $(B_1, B_r)$, that is
\[ \min_{|K| = \omega_n, |\Omega| \leq M} E_{\beta,p}(K, \Omega) = E_{\beta,p}(B_1, B_r), \]
in particular we have that either $r = 1$ or $M = \omega_n r^n$.

Moreover, if $K_0 \subseteq \Omega_0$ are such that
\[ E_{\beta,p}(K_0, \Omega_0) = \min_{|K| = \omega_n, |\Omega| \leq M} E_{\beta,p}(K, \Omega), \]
and $u$ is the minimizer of $E_{\beta,p}(K_0, \Omega_0)$, then the sets $\{ u = 1 \}$ and $\{ u > 0 \}$ coincide with two concentric balls up to a $\mathcal{H}^{n-1}$-negligible set.
Remark 1.2. In the case
\[ \beta \frac{1}{p-1} \leq \frac{n-p}{p-1}, \]
adapting the symmetrization techniques used in [6], it can be proved that a solution to problem (1.4) is always given by the pair \((B_1, B_1)\).

We point out that the proof of the theorem relies on the techniques involving the \(H\)-function introduced in [4, 8].

The case in which \(\Omega\) is the Minkowski sum \(\Omega = K + B_r(0)\), the energy \(E_{\beta,p}(K, \Omega)\), has been studied in [3] under suitable geometrical constraints.

2 Proof of the theorem

In order to prove Theorem 1.1, we start by studying the function

\[ R \mapsto E_{\beta,p}(B_1, B_R). \]

A similar study of the previous function can also be found in [3]. Let

\[ \Phi_{p,n}(\rho) = \begin{cases} \log(\rho) & \text{if } p = n, \\ -\frac{p-1}{n-p} \frac{1}{\rho^{\frac{n-p}{p-1}}} & \text{if } p \neq n. \end{cases} \]

For every \(R > 1\), consider

\[ u^*(x) = 1 - \frac{\beta \frac{1}{p-1} (\Phi_{p,n}(|x|) - \Phi_{p,n}(1))}{\Phi'_{p,n}(R) + \beta \frac{1}{p-1} (\Phi_{p,n}(R) - \Phi_{p,n}(1))}, \tag{2.1} \]

the solution to

\[ \begin{aligned} u^* &= 1 & & \text{in } B_1, \\ \Delta_p u^* &= 0 & & \text{in } B_R \setminus B_1, \\ |\nabla u^*|^{p-2} \frac{\partial u^*}{\partial \nu} + \beta |u^*|^{p-2} u^* &= 0 & & \text{on } \partial B_R. \end{aligned} \]

We have that

\[ E_{\beta,p}(B_1, B_R) = \int_{B_R} |\nabla u^*|^p \, dx + \beta \int_{\partial B_R} |u^*|^p \, dH^{n-1} \]

\[ = \frac{n \omega_n \beta}{\left[ \Phi'_{p,n}(R) + \beta \frac{1}{p-1} (\Phi_{p,n}(R) - \Phi_{p,n}(1)) \right]^{p-1}}. \tag{2.2} \]

Notice that \(E_{\beta,p}(B_1, B_R)\) is decreasing in \(R > 0\) if and only if

\[ \frac{d}{dR} \left( \Phi'_{p,n}(R) + \beta \frac{1}{p-1} \Phi_{p,n}(R) \right) \geq 0. \]
that is, if and only if
\[ R \geq \frac{n-1}{p-1} \frac{1}{\beta^{p-1}} =: \alpha_{\beta,p}. \]
Moreover
\[ E_{\beta,p}(B_1, B_1) = n\omega_n \beta, \]
\[ \lim_{R \to \infty} E_{\beta,p}(B_1, B_R) = \begin{cases} n\omega_n \left(\frac{n-p}{p-1}\right)^{p-1} & \text{if } p < n, \\ 0 & \text{if } p \geq n. \end{cases} \]
Therefore, there are three cases:

- if
  \[ \beta^{p-1} \geq \frac{n-1}{p-1}, \]
  \( R \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_R) \) is decreasing;
- if
  \[ \frac{n-p}{p-1} < \beta^{p-1} < \frac{n-1}{p-1}, \]
  \( R \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_R) \) increases on \([1, \alpha_{\beta,p}]\) and decreases on \([\alpha_{\beta,p}, +\infty)\), with the existence of a unique \( R_{\beta,p} > \alpha_{\beta,p} \) such that \( E_{\beta,p}(B_1, B_{R_{\beta,p}}) = E_{\beta,p}(B_1, B_1) \);
- if
  \[ \beta^{p-1} \leq \frac{n-p}{p-1}, \]
  \( R \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_R) \) reaches its minimum at \( R = 1 \).

See for instance Figure 1, where
\[ \beta_1 = \left(\frac{n-p}{p-1}\right)^{p-1}, \quad \beta_2 = \left(\frac{n-1}{p-1}\right)^{p-1}, \quad p = 2.5, \quad n = 3. \]

In the following, we will need

**Lemma 2.1.** Let \( R > 1, \beta > 0 \) and let \( u^* \) be the solution of the problem on \((B_1, B_R)\). Then
\[ \frac{|\nabla u^*|}{u^*} \leq \beta^{\frac{1}{p-1}} \]
in \( B_R \setminus B_1 \), if and only if
\[ E_{\beta,p}(B_1, B_1) \geq E_{\beta,p}(B_1, B_R) \]
for every \( \rho \in [1, R] \).

**Proof.** Recalling the expressions of \( u^* \) in (2.1), by straightforward computations we have that
\[ \frac{|\nabla u^*|}{u^*} \leq \beta^{\frac{1}{p-1}} \]
in $B_R \setminus B_1$ if and only if

$$\Phi'_{p,n}(R) + \beta \frac{1}{p-1} (\Phi_{p,n}(R) - \Phi_{p,n}(1)) \geq \Phi'_{p,n}(\rho) + \beta \frac{1}{p-1} (\Phi_{p,n}(\rho) - \Phi_{p,n}(1))$$

(2.3)

for every $\rho \in [1, R]$, using the expression of $E_{\beta,p}(B_1, B_\rho)$ in (2.2), (2.3) is equivalent to

$$E_{\beta,p}(B_1, B_\rho) \geq E_{\beta,p}(B_1, B_R)$$

for every $\rho \in [1, R]$.

**Definition 2.2.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $U \subseteq \Omega$ be another set. We define the internal boundary of $U$ as

$$\partial_i U = \partial U \cap \Omega,$$

and the external boundary of $U$ as

$$\partial_e U = \partial U \cap \partial \Omega.$$

Let $K \subseteq \overline{\Omega} \subseteq \mathbb{R}^n$ be open bounded sets, and let $u$ be the minimizer of $E_{\beta,p}(K, \Omega)$. In the following, we denote by

$$U_t = \{ x \in \Omega \mid u(x) > t \}.$$

**Definition 2.3 (H-function).** Let $\varphi \in W^{1,p}(\Omega)$. We define

$$H(t, \varphi) = \int_{\partial_i U_t} |\varphi|^{p-1} d\mathcal{H}^{n-1} - (p-1) \int_{U_t} |\varphi|^p d\mathcal{L}^n + \beta \mathcal{H}^{n-1}(\partial_e U_t).$$

Notice that this definition is slightly different from the one given in [5].

**Lemma 2.4.** Let $K \subseteq \Omega \subseteq \mathbb{R}^n$ be an open, bounded sets, and let $u$ be the minimizer of $E_{\beta,p}(K, \Omega)$. Then for a.e. $t \in (0, 1)$ we have

$$H \left( t, \frac{\nabla u}{u} \right) = E_{\beta,p}(K, \Omega).$$
Proof. Recall that
\[ E_{\beta,p}(K,\Omega) = \int_{\Omega} |\nabla u|^p \, d\mathcal{L}^n + \beta \int_{\partial\Omega} u^p = \beta \int_{\partial\Omega} u^{p-1} \, d\mathcal{H}^{n-1}. \]  
(2.4)

Let \( t \in (0,1) \), we construct the following test functions: let \( \varepsilon > 0 \), and let
\[ \varphi_{\varepsilon}(x) = \begin{cases} 
-1 & \text{if } u(x) \leq t, \\
\frac{u(x) - t}{\varepsilon u(x)^{p-1}} - 1 & \text{if } t < u(x) \leq t + \varepsilon, \\
\frac{1}{u(x)^{p-1}} - 1 & \text{if } u(x) > t + \varepsilon,
\end{cases} \]
so that \( \varphi_{\varepsilon} \) is an approximation the function \( (u^{1-p}\chi_{U_t} - 1) \), and
\[ \nabla \varphi_{\varepsilon}(x) = \begin{cases} 
0 & \text{if } u(x) \leq t, \\
\frac{1}{\varepsilon} \left( \frac{\nabla u(x)}{u(x)^{p-1}} - (p-1) \frac{\nabla u(x)(u(x) - t)}{u(x)^p} \right) & \text{if } t < u(x) \leq t + \varepsilon, \\
-(p-1) \frac{\nabla u(x)}{u(x)^p} & \text{if } u(x) > t + \varepsilon.
\end{cases} \]

We have that \( \varphi_{\varepsilon} \) is an admissible test function for the Euler-Lagrange equation (1.3), which entails
\[ 0 = \frac{1}{\varepsilon} \int_{\{t < u \leq t + \varepsilon\} \cap \Omega} |\nabla u|^{p-1} |\nabla u| \, d\mathcal{L}^n - (p-1) \int_{\{t < u \leq t + \varepsilon\} \cap \Omega} \frac{|\nabla u|^p u - t}{u^p} \, d\mathcal{L}^n \]
\[ - (p-1) \int_{\{u > t + \varepsilon\} \cap \Omega} \frac{|\nabla u|^p}{u^p} \, d\mathcal{L}^n + \beta \int_{\{t < u \leq t + \varepsilon\} \cap \partial\Omega} \frac{u - t}{\varepsilon} \, d\mathcal{H}^{n-1} \]
\[ + \beta \mathcal{H}^{n-1}(\partial\Omega \cap \{u > t + \varepsilon\}) - \beta \int_{\partial\Omega} u^{p-1} \, d\mathcal{H}^{n-1}. \]

Letting now \( \varepsilon \) go to 0, by coarea formula we get that for a.e. \( t \in (0,1) \)
\[ \beta \int_{\partial\Omega} u^{p-1} \, d\mathcal{H}^{n-1} = \int_{\partial U_t} \left( \frac{|\nabla u|}{u} \right)^{p-1} \, d\mathcal{H}^{n-1} - (p-1) \int_{U_t} \left( \frac{|\nabla u|}{u} \right)^p \, d\mathcal{L}^n \]
\[ + \beta \mathcal{H}^{n-1}(\partial U_t). \]  
(2.5)

Joining (2.4) and (2.5), the lemma is proven. \( \square \)

Remark 2.5. Notice that if \( K \) and \( \Omega \) are two concentric balls, the minimizer \( u \) is the one written in (2.1), for which the statement of the above Lemma holds for every \( t \in (0,1) \).

Lemma 2.6. Let \( \varphi \in L^\infty(\Omega) \). Then there exists \( t \in (0,1) \) such that
\[ H(t,\varphi) \leq E_{\beta,p}(K,\Omega). \]
Proof. Let 
\[ w = |\varphi|^{p-1} - \left(\frac{|\nabla u|}{u}\right)^{p-1}, \]
then we evaluate

\[
\begin{align*}
H(t, \varphi) - H\left(t, \frac{|\nabla u|}{u}\right) &= \int_{\partial U_t} w \, d\mathcal{H}^{n-1} - (p-1) \int_{U_t} \left(|\varphi|^p - \left(\frac{|\nabla u|}{u}\right)^p\right) \, d\mathcal{L}^n \\
&\leq \int_{\partial U_t} w \, d\mathcal{H}^{n-1} - p \int_{U_t} \frac{|\nabla u|}{u} w \, d\mathcal{L}^n \\
&= -\frac{1}{t^{p-1}} \frac{d}{dt} \left(t^p \int_{U_t} \frac{|\nabla u|}{u} w \, d\mathcal{L}^n\right),
\end{align*}
\]
where we used the inequality
\[ a^p - b^p \leq \frac{p}{p-1} a^{p-1} (a^{p-1} - b^{p-1}) \quad \forall a, b \geq 0. \tag{2.6} \]
Multiplying by \( t^{p-1} \) and integrating, we get
\[
\int_0^1 t^{p-1} \left( H(t, \varphi) - H\left(t, \frac{|\nabla u|}{u}\right) \right) \, dt \leq -\left[ t^p \int_{U_t} \frac{|\nabla u|}{u} w \, d\mathcal{L}^n\right]_0^1 = 0, \tag{2.7}
\]
from which we obtain the conclusion of the proof. \( \square \)

Remark 2.7. Notice that the inequality (2.6) holds as equality if and only if \( a = b \). Therefore, if \( \varphi \neq \frac{|\nabla u|}{u} \) on a set of positive measure, then the inequality in (2.7) is strict, since
\[
\left|\left\{ \varphi \neq \frac{|\nabla u|}{u} \right\} \cap U_t \right| > 0
\]
for small enough \( t \). Therefore, there exists \( S \subset (0, 1) \) such that \( \mathcal{L}^1(S) > 0 \) and for every \( t \in S \)
\[ H(t, \varphi) < E_{\beta,p}(K, \Omega). \]

In the following, we fix a radius \( R \) such that \( |B_R| \geq |\Omega| \), \( u^* \) the minimizer of \( E_{\beta,p}(B_1, B_R) \), and
\[
H^*(t, \varphi) = \int_{\partial\{u^* > t\} \cap B_R} |\varphi|^{p-1} \, d\mathcal{H}^{n-1} - (p-1) \int_{\{u^* > t\}} |\varphi|^p \, d\mathcal{L}^n \\
+ \beta \mathcal{H}^{n-1}(\partial\{u^* < t\} \cap \partial B_R).
\]

Proposition 2.8. Let \( \beta > 0 \). Assume that
\[ \frac{|\nabla u^*|}{u^*} \leq \beta^{\frac{1}{p-1}}. \tag{2.8} \]
Then we have that
\[ E_{\beta,p}(K, \Omega) \geq E_{\beta,p}(B_1, B_R). \]
**Proof.** In the following, if \( v \) is a radial function on \( B_R \) and \( r \in (0, R) \), we denote with abuse of notation
\[
v(r) = v(x),
\]
where \( x \) is any point on \( \partial B_r \). By Lemma 2.4 we know that for every \( t \in (0, 1) \)
\[
H^* \left( t, \frac{\| \nabla u^* \|}{u^*} \right) = E_{\beta,p}(B_1, B_R), \tag{2.9}
\]
while by Lemma 2.6, for every \( \varphi \in L^\infty(\Omega) \) there exists a \( t \in (0, 1) \) such that
\[
E_{\beta,p}(K, \Omega) \geq H(t, \varphi). \tag{2.10}
\]
We aim to find a suitable \( \varphi \) such that, for some \( t \),
\[
H(t, \varphi) \geq H^* \left( t, \frac{\| \nabla u^* \|}{u^*} \right), \tag{2.11}
\]
so that combining (2.10), (2.11), and (2.9) we conclude the proof. In order to construct \( \varphi \), for every \( t \in (0, 1) \) we define
\[
r(t) = \left( \frac{|U_t|}{\omega_n} \right)^{\frac{1}{n}}, \tag{2.12}
\]
then we set, for every \( x \in \Omega \),
\[
\varphi(x) = \frac{\| \nabla u^* \|}{u^*}(r(u(x))).
\]

**Claim** The functions \( \varphi \chi_{U_t} \) and \( \frac{\| \nabla u^* \|}{u^*}\chi_{B_r(t)} \) are equi-measurable, in particular
\[
\int_{U_t} \varphi^p \, d\mathcal{L}^n = \int_{B_r(t)} \left( \frac{\| \nabla u^* \|}{u^*} \right)^p \, d\mathcal{L}^n. \tag{2.13}
\]
Indeed, let \( g(r) = \frac{\| \nabla u^* \|}{u^*}(r) \), and by coarea formula,
\[
|U_t \cap \{ \varphi > s \}| = \int_{U_t \cap \{ g(r(u(x))) > s \}} d\mathcal{L}^n
\]
\[
= \int_t^{+\infty} \int_{\partial^* U_t \cap \{ g(r(\tau)) > s \}} \frac{1}{\| \nabla u(x) \|} \, d\mathcal{H}^{n-1}(x) \, d\tau \tag{2.14}
\]
\[
= \int_0^{r(t)} \int_{\partial^* U_{r^{-1}(\sigma)}} \frac{1}{\| \nabla u(x) \| r'(r^{-1}(\sigma))} \, \chi_{\{ g(\sigma) > s \}} \, d\mathcal{H}^{n-1}(x) \, d\sigma.
\]

Notice now that, since
\[
\omega_n r(\tau)^n = |U_\tau|,
\]
then
\[
r'(\tau) = -\frac{1}{n \omega_n r(\tau)^{n-1}} \int_{\partial^* U_r} \frac{1}{\| \nabla u(x) \|} \, d\mathcal{H}^{n-1}(x). \tag{2.15}
\]
Therefore, substituting in (2.14), we get
\[
|U_t \cap \{ \varphi > s \}| = \int_0^{r(t)} n \omega_n \sigma^{n-1} \chi_{\{ g(\sigma) > s \}} \, d\sigma = |B_r(t) \cap \{ \frac{\| \nabla u^* \|}{u^*} > s \}|;
\]
where we have used polar coordinates to get the last equality. Thus, the claim is proved.

Recalling the definition of $\varphi$, (2.8) reads

$$\beta \geq \varphi^{p-1},$$

then using (2.13) and the definition of $H$ (see Definition 2.3), we have

$$H(t, \varphi) = \beta H^{n-1}(\partial_c U_t) + \int_{\partial U_t} \varphi^{p-1} dH^{n-1} - (p - 1) \int_{U_t} \varphi^p \, d\mathcal{L}^n$$

$$\geq \int_{\partial U_t} \varphi^{p-1} dH^{n-1} - (p - 1) \int_{B_{r(t)}} \left( \frac{\nabla u^*}{u^*} \right)^p \, d\mathcal{L}^n$$

$$\geq \int_{\partial B_{r(t)}} \left( \frac{\nabla u^*}{u^*} \right)^{p-1} dH^{n-1} - (p - 1) \int_{B_{r(t)}} \left( \frac{\nabla u^*}{u^*} \right)^p \, d\mathcal{L}^n$$

$$= H^* \left( u^*(r(t)), \frac{\nabla u^*}{u^*} \right)$$

$$= E_{\beta,p}(B_1, B_R),$$

where in the last inequality we have used the isoperimetric inequality and the fact that $\varphi$ is constant on $\partial U_t$. \hfill \Box

**Remark 2.9.** By Remark 2.7, we have that if $K$ and $\Omega$ are such that

$$E_{\beta,p}(K, \Omega) = E_{\beta,p}(B_1, B_R),$$

then

$$\varphi = \frac{\nabla u}{u} \quad \text{for a. e. } x \in \Omega,$$

so that, by Lemma 2.4, we have equality in (2.16) for a.e. $t \in (0, 1)$. Thus, by the rigidity of the isoperimetric inequality, we get that $U_t$ coincides with a ball up to a $H^{n-1}$-negligible set for a.e. $t \in (0, 1)$. In particular, $\{ u > 0 \} = \bigcup_t U_t$ and $\{ u = 1 \} = \bigcap_t U_t$ coincide with two balls up to a $H^{n-1}$-negligible set.

**Proof of Theorem 1.1.** Fix $M = \omega_n R^n$ with $R > 1$. We divide the proof of the minimality of balls into two cases, and subsequently, we study the equality case.

Let us assume that

$$\beta \frac{1}{p-1} \geq \frac{n-1}{p-1},$$

and recall that in this case the function

$$\rho \in [1, +\infty) \mapsto E_{\beta,p}(B_1, B_{\rho})$$

is decreasing. Let $u^*$ be the minimizer of $E_{\beta,p}(B_1, B_R)$, by Lemma 2.1 condition (2.8) holds and, by Proposition 2.8, we have that a solution to (1.4) is given by the concentric balls $(B_1, B_R)$.

Assume now that

$$\frac{n-p}{p-1} < \beta \frac{1}{p-1} < \frac{n-1}{p-1},$$
then, in this case, letting
\[
\alpha_{\beta,p} = \frac{(n-1)}{(p-1)\beta^{\frac{1}{p-1}}},
\]
the function
\[
\rho \in [1, +\infty) \mapsto E_{\beta,p}(B_1,B_\rho)
\]
increases on \([1, \alpha_{\beta,p}]\) and decreases on \([\alpha_{\beta,p}, +\infty)\), and there exist a unique \(R_{\beta,p} > \alpha_{\beta,p}\) such that \(E_{\beta,p}(B_1,B_{R_{\beta,p}}) = E_{\beta,p}(B_1,B_1)\). If \(R \geq R_{\beta,p}\) the function \(u^*\), minimizer of \(E_{\beta,p}(B_1,B_R)\), still satisfies condition (2.8) and, as in the previous case, a solution to (1.4) is given by the concentric balls \((B_1,B_R)\). On the other hand, if \(R < R_{\beta,p}\), we can consider \(u^*_{\beta,p}\) the minimizer of \(E_{\beta,p}(B_1,B_{R_{\beta,p}})\). By Lemma 2.1 we have that, for the function \(u^*_{\beta,p}\), condition (2.8) holds and, by Proposition 2.8, we have that if \(K\) and \(\Omega\) are open bounded Lipschitz sets with \(K \subseteq \Omega\), \(|K| = \omega_n\), and \(|\Omega| \leq M\), then
\[
E_{\beta,p}(K,\Omega) \geq E_{\beta,p}(B_1,B_{R_{\beta,p}}) = E_{\beta,p}(B_1,B_1)
\]
and a solution to (1.4) is given by the pair \((B_1,B_1)\).

For what concerns the equality case, we will follow the outline of the rigidity problem given in [10, Section 3] (see also [2, Section 2]). Let \(K_0 \subseteq \Omega_0\) be such that
\[
E_{\beta,p}(K_0,\Omega_0) = \min_{|K|=\omega_n, |\Omega| \leq M} E_{\beta,p}(K,\Omega),
\]
let \(u\) be the minimizer of \(E_{\beta,p}(K_0,\Omega_0)\). If \(K_0 = \Omega_0\), then \(|\Omega_0| = |B_1|\) and isoperimetric inequality yields
\[
\mathcal{H}^{n-1}(\partial \Omega_0) \geq \mathcal{H}^{n-1}(\partial B_1),
\]
while, from the minimality of \((K_0,\Omega_0)\) we have that
\[
E_{\beta,p}(K_0,\Omega_0) = \beta \mathcal{H}^{n-1}(\partial \Omega_0) \leq E_{\beta,p}(B_1,B_1) = \beta \mathcal{H}^{n-1}(\partial B_1),
\]
so that \(\mathcal{H}^{n-1}(\Omega_0) = \mathcal{H}^{n-1}(\partial B_1)\). Hence, by the rigidity of the isoperimetric inequality we have that \(K_0 = \Omega_0\) are balls of radius 1. On the other hand, if \(K_0 \neq \Omega_0\), from the first part of the proof, there exists \(R_0 > 1\) such that \(|B_{R_0}| \geq M\) and
\[
E_{\beta,p}(K_0,\Omega_0) = E_{\beta,p}(B_1,B_{R_0}).
\]
Therefore, by Remark 2.9, we have that for a.e. \(t \in (0,1)\), the superlevel sets \(U_t\) coincide with balls up to \(\mathcal{H}^{n-1}\)-negligible sets, and \(\{u = 1\}\) and \(\{u > 0\}\) coincide with balls, up to \(\mathcal{H}^{n-1}\)-negligible sets, as well. We only have to show that \(\{u = 1\}\) and \(\{u > 0\}\) are concentric balls. To this aim, let us denote by \(x(t)\) the center of the ball \(U_t\) and by \(r(t)\) the radius of \(U_t\), as already done in (2.12). In addition, we also have that
\[
\frac{\nabla u^*}{u^*}\left(r(u(x))\right) = \varphi(x) = \frac{\nabla u}{u^*}(x),
\]
so that, if \(u(x) = t\), then \(\nabla u(x) = C_t > 0\). This ensures that we can write
\[
x(t) = \frac{1}{|U_t|} \int_{U_t} x d\mathcal{L}^n(x) = \frac{1}{|U_t|} \left( \int_{t}^{1} \int_{\partial U_s} \frac{x}{|\nabla u(x)|} d\mathcal{H}^{n-1}(x) ds + \int_{K} x d\mathcal{L}^n(x) \right),
\]
and we can infer that $x(t)$ is an absolutely continuous function, since $|\nabla u| > 0$ implies that $|U_t|$ is an absolutely continuous function as well. Moreover, on $\partial U_t$ we have that for every $\nu \in S^{n-1}$,

$$u(x(t) + r(t)\nu) = t,$$

from which

$$\nabla u(x(t) + r(t)\nu) = -C_t\nu.$$  \hfill (2.18)

Differentiating (2.17), and using (2.18), we obtain

$$-C_t x'(t) \cdot \nu - C_t r'(t) = 1.$$ \hfill (2.19)

Finally, joining (2.19) and (2.15), and the fact that $|\nabla u| = C_t$ on $\partial U_t$, we get

$$x'(t) \cdot \nu = 0$$

for every $\nu \in S^{n-1}$, so that $x(t)$ is constant and $U_t$ are concentric balls for a.e. $t \in (0, 1)$. In particular, $\{ u = 1 \} = \bigcap_t U_t$ and $\{ u > 0 \} = \bigcup_t U_t$ share the same center.

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