Critical exponents of surface-interacting
self-avoiding walks on a family
of truncated n-simplex lattices

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Abstract: We study the critical behavior of surface-interacting self-avoiding random
walks on a class of truncated simplex lattices, which can be labeled by an integer \( n \geq 3 \). Using the exact renormalization group method we have been able to obtain the exact
values of various critical exponents for all values of \( n \) up to \( n = 6 \). We also derived simple
formulas which describe the asymptotic behavior of these exponents in the limit of large \( n \)
\( (n \to \infty) \). In spite of the fact that the coordination number of the lattice tends to infinity
in this limit, we found that the most of the studied critical exponents approach certain
finite values, which differ from corresponding values for simple random walks (without
self-avoiding walk constraint).

Key words: Polymer adsorption; fractals; self-avoiding walks; critical exponents;
renormalization group.
1. Introduction

Configurational properties of a single polymer chain in the vicinity of an attractive impenetrable wall has been studied long time ago (see, for instance, [1]), as a problem of great theoretical and practical importance. The general picture that springs from these studies [2-4] reveals that, under certain conditions, polymer chain can undergo an adsorption-desorption transition. The essential physics of a polymer chain near an surface can be captured by the self-avoiding random walk (SAW) model on a semi-infinite lattice, with an energy contribution $\epsilon_w$ for each step (monomer) of the walk along the lattice boundary. This leads to an increased probability, characterized by the Boltzmann factor $w = \exp(-\epsilon_w/k_B T)$, of making a step along the attractive wall ($\epsilon_w < 0$, $w > 1$ for any finite temperature $T$).

For low temperatures, the polymer chain is basically pinned at the substrate, while at higher temperatures all polymer configurations have almost same weights and a nonadsorbed behavior prevails. The transition between these two regions is marked by a critical adsorption temperature $T_a$, with a desorbed phase for $T > T_a$ and an adsorbed phase for $T < T_a$. The asymptotic behavior of the average number $M$ of steps of the walk along the boundary can be summarized in the following way [4]

$$M \sim \begin{cases} N(T_a - T)^{1/\Phi - 1}, & T < T_a \\ N^\Phi, & T = T_a \\ (T - T_a)^{-1}, & T > T_a \end{cases}$$

(1.1)

where $N$ denotes average number of monomers, and $\Phi$ is the crossover exponent. A more complete description of the statistics of SAWs near a surface requires a knowledge of the asymptotic behavior of the numbers $C_1(N, T)$, $C_{11}(N, T)$ and $C_s(N, T)$ of different polymer configurations with one, both and no ends at the wall, respectively. It is generally believed [4] that these numbers as $N \to \infty$ follow the asymptotic laws

$$C_1(N, T) \sim \mu N^{\gamma_1-1}, \quad C_{11}(N, T) \sim \mu N^{\gamma_{11}-1}, \quad C_s(N, T) \sim \mu N^{\gamma_s-1},$$

(1.2)

where $\mu = \mu(T)$ is a continuous function of temperature, and $\gamma_1$, $\gamma_{11}$ and $\gamma_s$ are the
associated critical exponents. It turns out that these exponents assume distinct values in different temperature regions [5].

The most of theoretical efforts so far have been devoted to a study of suitable models of surface-interacting SAWs on standard homogeneous spaces. Recently, a considerable research activity have appeared in the study of SAWs placed on fractal spaces [6-11], as models of polymers in nonhomogeneous environment. Aside from being interesting in its own right, we believe that such studies may yield some insights into more difficult questions related to the behavior of polymers in disordered systems. In the present paper we exhibit the large-scale behavior of surface-interacting SAWs on truncated \( n \)-simplex lattices [12], which provide a whole family of fractals that can be labeled by an integer \( n \geq 3 \). Fractal dimension of these lattices increases indefinitely when \( n \to \infty \), while their spectral dimension remain limited to a relatively narrow region (see (2.1)). Using the renormalization-group (RG) method we obtain the exact values of the critical exponents \( \Phi, \gamma_1, \gamma_{11}, \) and \( \gamma_s \) in the case of SAWs on 4-, 5- and 6-simplex lattice. Then, applying an extension of the approach described in refs. [13-14], we have been able to determine the limiting behavior of these exponents, valid for large values of \( n \). It is interesting to note here that the limiting values of \( \gamma_1, \gamma_{11}, \) and \( \gamma_s \) do not coincide with the corresponding values of these exponents for an analogous random-walk problem [15], in spite of the fact that coordination number of these lattices tends to infinity for \( n \to \infty \).

The paper is organized as follows. In section 2, we present our approach and calculate the exact and asymptotic values of the crossover exponent \( \Phi \). This method is then extended to the case of open SAWs, in section 3, where we also display the exact values and asymptotic behavior of the exponents \( \gamma_1, \gamma_{11} \) and \( \gamma_s \). Some final remarks and an overall discussion we give in section 4.
2. Crossover exponent $\Phi$

The family of truncated $n$–simplex lattices has been introduced by Dhar [12], who also developed simple RG approach to describe critical behavior of various statistical-mechanical models on these lattices. The lattices are defined recursively: One starts with a complete graph of $n$ points and replaces each of these points by a new complete graph of $n$ points (see Fig. 1 for the case $n = 6$). The subsequent stages are constructed self-similarly, by repeating this procedure, so that the complete lattice is obtained in the limit when the number of such iterations tends to infinity. The fractal $\bar{d}$ and spectral $\tilde{d}$ dimensions of these lattices are given as

$$\bar{d} = \frac{\ln n}{\ln 2}, \quad \tilde{d} = \frac{2\ln n}{\ln(n + 2)},$$

(2.1)

respectively. In the case under study, it is assumed that the adsorption boundary of a truncated $n$–simplex is a truncated $(n - 1)$–simplex lattice. The fractal $\bar{d}_s$ and the spectral $\tilde{d}_s$ dimension of the adsorption surface can be presented by (2.1), where $n$ has to be replaced by $(n - 1)$.

The crossover exponent $\Phi$ for the SAW model was found for $n = 3, 4$ [6], and $n = 5$ [8]. Here we shall present a general framework of the RG method for studying surface-interacting SAWs on the family of $n$–simplex fractals on the example of the 6–simplex lattice.

(a) Exact renormalization group approach

In order to study influence of the adsorbing wall on the SAW statistics, in addition to $w = \exp(-\epsilon_w/k_B T)$, we introduce the Boltzmann factor $t = \exp(-\epsilon_t/k_B T)$, where $\epsilon_t$ denotes the energy of a monomer that appears in the layer adjacent to the wall. Here we should set $\epsilon_t > 0$ so as to prevent the tendency of a SAW towards being always adsorbed [6]. We assign the weight $x$ to each step in the bulk (away from the wall), the weight $wx$ to
each step on the adsorbing wall, and the weight $tx$ to each step in the layer adjacent to the wall. To calculate the critical exponent $\Phi$, it is necessary to consider only a finite number of restricted partition functions, which are defined as weighted sums over all never-starting and never-ending SAWs for the given stage $r$ of the iterative construction of the lattice. It is not difficult to see that in the case of a 6-simplex lattice there are eight such restricted partition functions, $A^{(r)}$, $A_1^{(r)}$, $A_2^{(r)}$, $B^{(r)}$, $B_1^{(r)}$, $B_2^{(r)}$, $C^{(r)}$ and $C_1^{(r)}$, which are depicted in Fig. 2. The recursive nature of the fractal under consideration implies the following general form of the recursion relations:

\[
A' = \sum_{N_A} \sum_{N_B} \sum_{N_C} A_{N_A,N_B,N_C} A^{N_A} B^{N_B} C^{N_C}, \quad (2.2a)
\]

\[
B' = \sum_{N_A} \sum_{N_B} \sum_{N_C} B_{N_A,N_B,N_C} A^{N_A} B^{N_B} C^{N_C}, \quad (2.2b)
\]

\[
C' = \sum_{N_A} \sum_{N_B} \sum_{N_C} C_{N_A,N_B,N_C} A^{N_A} B^{N_B} C^{N_C}, \quad (2.2c)
\]

\[
A_1' = A_1^2 + 3 A_1^3 + 6 A_1^4 + 6 A_1^5 + 12 A_1^3 B_1 + 30 A_1^4 B_1 + 18 A_1^2 B_1^2 + 78 A_1^3 B_1^2 + 96 A_1^2 B_1^3 + 132 A_1 B_1^4 + 132 B_1^5 + F_{A_1 A} A + F_{A_1 B} B + F_{A_1 C} C \quad (2.3a)
\]

\[
A_2' = F_{A_2 A} A + F_{A_2 B} B + F_{A_2 C} C \quad (2.3b)
\]

\[
B_1' = A_1^4 + 2 A_1^5 + 4 A_1^3 B_1 + 13 A_1^4 B_1 + 32 A_1^3 B_1^2 + 88 A_1^2 B_1^3 + 22 B_1^4 + 220 A_1 B_1^4 + 186 B_1^5 + F_{B_1 A} A + F_{B_1 B} B + F_{B_1 C} C, \quad (2.3c)
\]

\[
B_2' = F_{B_2 A} A + F_{B_2 B} B + F_{B_2 C} C, \quad (2.3d)
\]

\[
C_1' = F_{C_1 A} A + F_{C_1 B} B + F_{C_1 C} C, \quad (2.3e)
\]

where we have used the prime symbol as a superscript for the $(r+1)$-th order functions and no indices for the $r$-th order functions. The coefficients $A_{N_A,N_B,N_C}$, $B_{N_A,N_B,N_C}$ and $C_{N_A,N_B,N_C}$ are numbers of ways in which the corresponding part of the SAW path in the bulk, within an $(r+1)$-th stage fractal structure, can be comprised of the SAW paths within the fractal structures of the next lower order. The coefficients $F$ are polynomials in $A_1$, $A_2$, $B_1$ and $B_2$, and they are given in the Appendix, together with the explicit forms of the relations (2.2).
The above set of relations (2.2) and (2.3) can be considered as a set of RG equations. The bulk relations (2.2) are independent of the recursions (2.3) for surface partition functions, and they have already been analysed in [13]. It turns out that system (2.2) has several nontrivial fixed points, but only one of them can be reached starting with the SAWs initial conditions

\[ A^{(1)} = x + 4x^2 + 12x^3 + 24x^4 + 24x^5, \]
\[ B^{(1)} = x^2 + 4x^3 + 6x^4, \]
\[ C^{(1)} = x^3. \]  

That happens for the critical value of the fugacity,

\[ x = x_c = \frac{1}{\mu} \approx 0.137359, \]  

and the pertinent fixed point is given by

\[ A^* \approx 0.262352, \quad B^* \approx 0.017588, \quad C^* \approx 0.000701. \]  

We are going to discuss them separately, starting with the high temperature region.

In accord with the accepted physical picture about the interaction parameters, we assume the following initial conditions for the remaining RG parameters

\[ A_1^{(1)} = xw(1 + 3xw + 6x^2w^2 + 6x^3w^3) + x^2t^2(1 + 6xw + 18x^2w^2 + 24x^3w^3), \]
\[ A_2^{(1)} = xt(1 + 4xw + 12x^2w^2 + 24x^3w^3 + 24x^4w^4), \]
\[ B_1^{(1)} = x^2w^2(1 + 2xw) + 2x^3wt^2(1 + 3xw), \]
\[ B_2^{(1)} = x^2wt(1 + 4xw + 6x^2w^2), \]
\[ C_1^{(1)} = x^3w^2t. \]  

The numerical study of the relations (2.2) and (2.3) with the initial conditions (2.4) and (2.7) shows that for any fixed value of \( t < 1 \) there are three different temperature regions. We are going to discuss them separately, starting with the high temperature region.
(i) At high temperatures, that is, for \( w < w^*(t) \) (here \( w^*(t) \) denotes a critical \( t - \) dependent value of \( w \)), the critical fugacity is constant and equal to its bulk critical value (2.5). For all these values of temperature the bulk SAW fixed point is reached

\[
(A^*, A_1^*, A_2^*, B^*, B_1^*, B_2^*, C^*, C_1^*) = (A^*, 0, 0, B^*, 0, 0, C^*, 0).
\] (2.8)

The fraction of SAW steps in contact with the surface vanishes in this temperature region, so that the polymer is in the desorbed state. Linearization about this fixed point leads to only one relevant eigenvalue

\[
\lambda_\nu \approx 3.52148,
\] (2.9)

which yields the value of the end–to–end distance critical exponent \( \nu = \ln 2 / \ln \lambda_\nu \approx 0.55061. \)

(ii) When the temperature is lowered an adsorption transition occurs for \( w = w^*(t) \). In that case \( x_c(w^*) \) is still equal to its bulk value (2.5), but equations (2.2) and (2.3) iterate towards the symmetric fixed point

\[
(A^*, A_1^*, A_2^*, B^*, B_1^*, B_2^*, C^*, C_1^*) = (A^*, A^*, A^*, B^*, B^*, B^*, C^*, C^*),
\] (2.10)

where a balance between the attractive polymer–surface potential and an effective "entropic" repulsion sets in ("special transition" in the common language of surface phase transition). This fixed point has two relevant eigenvalues: \( \lambda_\nu \), given by (2.9), and

\[
\lambda_\Phi \approx 2.97273.
\] (2.11)

The larger of them determines the end–to–end distance critical exponent \( \nu \), which means that this index has the same value as in the high temperature region. On the other hand, it is known [16] that the crossover exponent \( \Phi \) involves both eigenvalues:

\[
\Phi = \frac{\ln \lambda_\Phi}{\ln \lambda_\nu} \approx 0.86544.
\] (2.12)
(iii) In the low temperature region, \( w > w^*(t) \), the critical fugacity \( x_c(w) \) is a decreasing function of \( w \), while recursion relations (2.2) and (2.3) iterate towards the fixed point

\[
(A^*, A_1^*, A_2^*, B^*, B_1^*, B_2^*, C^*, C_1^*) = (0, (A^*)_{n=5}, 0, 0, (B^*)_{n=5}, 0, 0, 0),
\]

where \((A^*)_{n=5} \approx 0.326491, (B^*)_{n=5} \approx 0.027930\) is the polymer bulk fixed point for the case of a 5–simplex fractal [13]. The above fixed point describes the critical properties of a polymer chain in adsorbed state. It is clear, therefore, that an adsorbed polymer on the 6–simplex fractal has the same properties as a polymer chain in the bulk of a 5–simplex lattice.

It is not feasible to pursue the above described approach for large values of \( n \), because the number of required partition functions, and especially their complexity, grow up rather quickly with \( n \). To overcome this difficulty, we shall apply here an approximate procedure which provides a direct extension of a similar technique used earlier in the study of the bulk critical behavior of SAWs on the same class of lattices [13, 14]. Although one could expect that such an approach should be valid only in the asymptotic region \( n \to \infty \), it turns out that this method gives quite accurate values of \( \Phi \) even for moderate values of \( n \) \((n \sim 5)\).

(b) An approximate approach for the critical exponent \( \Phi \)

An analysis of corresponding recursion relations reveals that critical behavior of a polymer chain, in the limit \( n \gg 1 \), can be described in terms of a small number of restricted partition functions. For example, it has been established [13] that the position of the bulk fixed point takes the asymptotic form (compare (2.6) and see Fig. 2)

\[
A^* \sim \frac{1}{n}, \quad B^* \sim \frac{1}{n^2}, \quad C^* \sim \frac{1}{n^3}, \quad \ldots.
\]

\[(2.14)\]
In the neighborhood of this fixed point, one can simplify the exact recursion relations by neglecting the variables $B, C, \ldots$, which approach zero much faster than $A$ when $n \to \infty$. In this way, the bulk critical behavior of a polymer chain can be described by using only one variable, which satisfies the simple recursion relation [13]

$$A' = \sum_{k=0}^{n-2} k! \left( \frac{n-2}{k} \right) A^{k+2}, \quad (2.15)$$

In a similar way, in order to calculate the critical exponent $\Phi$ ($n \gg 1$) one can keep only the partition functions $A, A_1$, and $A_2$ and neglect all the others (see Fig. 2). Then, in addition to the above formula, one has to consider the recursion relations for surface variables

$$A'_1 = \sum_{k=1}^{n-2} (k-1)! \left( \frac{n-3}{k-1} \right) A_1^{k-1} \left( A_1^2 + kA_2^2 \right), \quad (2.16a)$$

$$A'_2 = \sum_{k=0}^{n-2} k! \left( \frac{n-2}{k} \right) A_2 A_1^k. \quad (2.16b)$$

Indeed, critical exponent $\Phi$ is still determined by relation (2.12) with $\lambda_\nu$ and $\lambda_\Phi$ being two largest eigenvalues associated to the symmetric fixed point $A = A_1 = A_2 = A^*$,

$$\lambda_\nu = n + 1 - \frac{1}{A^*}, \quad (2.17)$$

and

$$\lambda_\Phi = \frac{5 - n - A^*(1 + 3n - n^2) + \sqrt{\Delta}}{2A^*(n-2)}, \quad (2.18a)$$

with

$$\Delta = 9 - 2n + n^2 - 2A^*(1 + 4n - 2n^2 + n^3) - A^{*2}(7 - 10n + n^2 + 2n^3 - n^4), \quad (2.18b)$$

where $A^*$ denotes the fixed point of (2.15). For some specific values of $n$ we have determined this fixed point and the corresponding eigenvalues numerically. In this way we calculated the values of the crossover critical exponent which are presented in Fig. 3. For the sake
of comparison we also presented (Fig. 3a) the exact values of this exponent for the case
\( n = 3, 4, 5 \) [6], [8] and \( n = 6 \) (see (2.12)). One can see that the exact and corresponding
approximate value of \( \Phi \) are very close to each other even for these moderate values of \( n \).
One can also notice that for larger values of \( n \) exponent \( \Phi \) approaches, almost linearly in
\( 1/n \ln n \), values that are very close to 1, (see Fig 3b). This observation can be further
corroborated, by using the earlier established [14] asymptotic formula for the position of
the bulk fixed point

\[
A^* \sim (1 + s)/(n + 1), \quad s = \sqrt{(\ln n)/n}, \quad n \gg 1. \quad (2.19)
\]

Indeed, the above formulas lead to the following estimate

\[
\Phi \sim 1 - \frac{1}{n \ln n}, \quad n \gg 1, \quad (2.20)
\]

which is in excellent agreement with our numerical results presented in Fig. 3b. The fact
that critical exponent \( \Phi \) tends to 1 as \( n \to \infty \) is not surprising, since in this limit the
number of sites on the adsorbing surface becomes comparable to the total number of the
available sites on the lattice.

It is also of some interest to compare our results with the upper (\( \Phi_u \)) and lower (\( \Phi_l \))
bounds on the crossover exponent, proposed recently [6] by Bouchaud and Vannimenus for
SAWs on fractals. One can observe (see Fig. 3) that both exact and approximate values
of \( \Phi \) satisfy their bounds

\[
\Phi_l = 1 - (\bar{d} - \bar{d}_s)\nu \leq \Phi \leq \Phi_u = \bar{d}_s/\bar{d}, \quad (2.21)
\]

with \( \bar{d} \) and \( \bar{d}_s \) being the fractal dimension of the lattice and the adsorbing wall, respectively.
Besides, it is interesting to note here that for large values of \( n \) the approximate values of
\( \Phi \) are almost equal to the values of \( \Phi_l \) (see Fig. 3b). On the other hand, it was shown
recently [15] that the lower bound \( \Phi_l \) presents, in fact, the exact value of the crossover
exponent for the simple random walks (without SAWs constraint) on the same class of lattices. This is in accordance with the expectation that in the limit of large coordination number (i.e. large $n$) statistics of SAWs should be similar to the one for random walks.

3. Critical exponents $\gamma_1$, $\gamma_{11}$ and $\gamma_s$

To calculate the critical exponents $\gamma_1$, $\gamma_{11}$ and $\gamma_s$ it is helpful to introduce the convenient generating functions

$$C_1(x, T) = \sum_{N=1}^{\infty} x^N \sum_{M, L=1}^{N} C_1(N, M, L) w^M t^L = \sum_{N=1}^{\infty} C_1(N, T) x^N, \quad (3.1a)$$

$$C_{11}(x, T) = \sum_{N=1}^{\infty} x^N \sum_{M, L=1}^{N} C_{11}(N, M, L) w^M t^L = \sum_{N=1}^{\infty} C_{11}(N, T) x^N, \quad (3.1b)$$

$$C_s(x, T) = \sum_{N=1}^{\infty} x^N \sum_{M, L=1}^{N} C_s(N, M, L) w^M t^L = \sum_{N=1}^{\infty} C_s(N, T) x^N, \quad (3.1c)$$

where $C_1(N, M, L)$ ($C_{11}(N, M, L)$) represents the averaged number of $N$-step SAWs with $M$ steps on the surface, and $L$ steps in the layer adjacent to the wall provided one (both) end(s) of the walk is (are) anchored to the wall, while $C_s(N, M, L)$ is the averaged number of SAWs with no ends anchored to the wall. The weighting factors $x, w$ and $t$ were defined in the foregoing section. Assuming that numbers $C_1(N, T), C_{11}(N, T)$ and $C_s(N, T)$ behave as in (1.2), the leading singular behavior of the generating functions, when $x$ approaches $x_c = 1/\mu(T)$ from below, is of the form

$$C_1(x, T) \sim (1 - x\mu)^{-\gamma_1}, \quad C_{11}(x, T) \sim (1 - x\mu)^{-\gamma_{11}}, \quad C_s(x, T) \sim (1 - x\mu)^{-\gamma_s}. \quad (3.2)$$

The above generating functions can be expressed in terms of a finite number of restricted partition functions [7]. In addition to the restricted partition functions defined in the preceding section, one should introduce restricted partition functions describing SAW’s ending (with one or both ends) somewhere inside of an $r$–th stage fractal lattice. As in the case of the critical exponent $\Phi$, we shall first outline the general method for calculating the exponents $\gamma_1$, $\gamma_{11}$, and $\gamma_s$ on the 6–simplex lattice example.
(a) Exact method for calculating $\gamma_1$, $\gamma_{11}$ and $\gamma_s$

To treat the statistics of open SAWs, one has to introduce a number of additional restricted partition functions. In the case $n = 6$, for example, a complete description of the above generating functions (3.1) needs 46 restricted partition functions, in addition to those presented in Fig. 2. Let us start with the simplest one $-C_{11}$, which can be expressed in terms of 15 functions $D_1, D_2, E_1, E_2, E_3, F_1, F_2, F_3, G_1, G_2, H_1, H_2, H_3, I_1,$ and $I_2$, depicted in Fig. 4. These 15 functions describe different SAW configurations with ends at the vertices on the adsorbing wall. It is not difficult to show that $C_{11}$ can be written in the form

$$C_{11}(x, T) = \sum_{r=1}^{\infty} \frac{F^{(r+1)}_{11}}{5^r},$$

(3.3)

where $F^{(r+1)}_{11}$ is a quadratic function in $D^{(r)}_1, D^{(r)}_2, E^{(r)}_1, E^{(r)}_2, E^{(r)}_3, F^{(r)}_1, F^{(r)}_2,$ and $F^{(r)}_3$, and linear in $G^{(r)}_1, G^{(r)}_2, H^{(r)}_1, H^{(r)}_2, H^{(r)}_3, I^{(r)}_1,$ and $I^{(r)}_2$, with coefficients being polynomials in $A^{(r)}, A^{(r)}_1, A^{(r)}_2, B^{(r)}, B^{(r)}_1, B^{(r)}_2, C^{(r)},$ and $C^{(r)}_1$ (see, e.g., [7] for an explicit construction). For arbitrary $r$, the self-similarity of the 6-simplex lattice imply a recursion relation of the type

$$\begin{pmatrix}
D^{(r+1)}_1 \\
D^{(r+1)}_2 \\
E^{(r+1)}_1 \\
E^{(r+1)}_2 \\
E^{(r+1)}_3 \\
F^{(r+1)}_1 \\
F^{(r+1)}_2 \\
F^{(r+1)}_3
\end{pmatrix} = \hat{A} \begin{pmatrix}
D^{(r)}_1 \\
D^{(r)}_2 \\
E^{(r)}_1 \\
E^{(r)}_2 \\
E^{(r)}_3 \\
F^{(r)}_1 \\
F^{(r)}_2 \\
F^{(r)}_3
\end{pmatrix},$$

(3.4)

where elements of the matrix $\hat{A}$ are some polynomials in $A^{(r)}, A^{(r)}_1, A^{(r)}_2, B^{(r)}, B^{(r)}_1, B^{(r)}_2, C^{(r)},$ and $C^{(r)}_1$. The recursion relations (2.2) and (2.3), together with the matrix relation (3.4), and an analogous relation involving variables $G_1, G_2, H_1, H_2, H_3, I_1$ and $I_2$, form a closed set of the recursion relations. Starting with the appropriate initial conditions for the restricted partition functions, it is possible to determine the value of the generating function $C_{11}(x, T)$ for arbitrary values of $x$ and $T$. As we are interested here only in the
asymptotic form of $C_{11}(x, T)$, the precise forms of the recursion relations for $G_1, G_2, H_1, H_2, H_3, I_1$ and $I_2$, as well as the explicit form of corresponding initial conditions, are not necessary, so we do not give them here. The elements of the matrix $\hat{A}$ are very cumbersome polynomials, so we do not present them either, but they are available upon request.

When $x$ is sufficiently close to $x_c$, so that $x_c - x \ll \varepsilon \ll 1$ ($\varepsilon > 0$), it turns out that under iterations, for $r < r_0 = \ln(\varepsilon/(x_c - x))/\ln \lambda_\nu \gg 1$, each restricted partition function either tends to certain finite constant or follows the simple power law $\sim \lambda^r$ (with a specific value of $\lambda$ for each particular function, in general). In particular, this holds for the correlation functions $A^{(r)}, A_1^{(r)}, A_2^{(r)}, B^{(r)}, B_1^{(r)}, B_2^{(r)}, C^{(r)}$ and $C_1^{(r)}$, which present some non-increasing functions of the iteration index $r$. Taking into account that $F_{11}^{(r)}$ is a quadratic function in all the other partition functions, we conclude that the summand in (3.3) follows the power law $(\lambda_{11}^2/5)^r$, where $\lambda_{11}$ describes the way in which diverges the dominant of the functions $D_1^{(r)}, D_2^{(r)}, E_1^{(r)}, E_2^{(r)}, E_3^{(r)}, F_1^{(r)}, F_2^{(r)}, F_3^{(r)}, G_1^{(r)}, G_2^{(r)}, H_1^{(r)}, H_2^{(r)}, H_3^{(r)}, I_1^{(r)}$, and $I_2^{(r)}$. For $r > r_0$ all these partition functions rapidly approach some constants, so that corresponding summands in (3.3) becomes negligible. Thus, the major contribution to the sum in (3.3) comes from the term with $r \approx r_0$, that is

$$ C_{11}(x, T) \sim (\lambda_{11}^2/5)^{r_0} . $$

(3.5)

This enables us to express the critical exponent $\gamma_{11}$ in terms of $\lambda_{11}$ and $\lambda_\nu$,

$$ \gamma_{11} = \frac{\ln \lambda_{11}^2}{\ln \lambda_\nu} . $$

(3.6)

An examination of the recursion relations (3.4) in the vicinity of relevant fixed points leads to the following:

(i) In the high temperature region, controlled by the bulk fixed point (2.8), the largest contribution to the function $F_{11}^{(r)}$ comes from the function $D_2$, which, for $r < r_0$, behaves as

$$ D_2^{(r)} \sim (5A^*)^r = \lambda_{11}^r \approx (1.31176)^r , $$

(3.7)
so that we find

\[ \gamma_{11} \approx -0.8473. \quad (3.8) \]

(ii) In the case of the adsorption transition, the relevant fixed point is the symmetric one (see (2.10)), and \( \lambda_{11} \approx 5.21757 \) is the largest eigenvalue of the matrix \( \hat{A} \), which appears in the recursion relation (3.4), leading to

\[ \gamma_{11} \approx 1.3461. \quad (3.9) \]

(iii) In the low temperature region SAW is mostly adsorbed, so the critical exponent \( \gamma_{11} \) is equal to the bulk \( \gamma \) exponent for SAW on a 5-simplex lattice [13], i.e. \( \gamma_{11} \approx 1.4875 \).

The evaluation of the critical exponent \( \gamma_1 \) goes along the same lines, except for the fact that one should introduce some additional restricted partition functions. These new functions are presented in Fig. 5, and they describe the SAW configurations that have one end somewhere in the bulk. One can show that the final formula for \( \gamma_1 \) takes the form

\[ \gamma_1 = \frac{\ln(\lambda_{11}\lambda/5)}{\ln \lambda_\nu}, \]

(3.10)

where \( \lambda \) denotes the largest eigenvalue of the corresponding matrix appearing in the recursion relations involving bulk variables \( D, E \) and \( F \). Numerically we find: \( \gamma_1 \approx 0.39499 \) for the desorbed SAW, \( \gamma_1 \approx 1.49173 \) for the SAW at the adsorption transition, and \( \gamma_1 \approx 1.4875 \) for an adsorbed polymer chain (which is equal to the value of \( \gamma \) for the SAW on the 5-simplex lattice). In a similar way, we find

\[ \gamma_s = \frac{\ln(\lambda^2/5)}{\ln \lambda_\nu}, \]

(3.11)

which means that \( \gamma_s \) has the same value at the point of the adsorption transition as well as in the high temperature phase (\( \gamma_s \approx 1.63732 \) for \( T \geq T_a \), while in the low temperature region its value coincides with the value of \( \gamma(n = 5) \approx 1.4875 \)).

The above described approach can be, in principle, applied for general \( n \)-simplex lattice. In particular, it is easy to see that critical exponents \( \gamma_1, \gamma_{11} \) and \( \gamma_s \) can be expressed
in terms of the above mentioned eigenvalues \( \lambda_{11}, \lambda_\nu \) and \( \lambda \),

\[
\begin{align*}
\gamma_{11} &= \frac{\ln \frac{\lambda_{11}^2}{\lambda_\nu}}{\ln \lambda_\nu}, \quad \gamma_1 &= \frac{\ln \frac{\lambda_{11} \lambda_\nu}{n-1}}{\ln \lambda_\nu}, \quad \gamma_s = \frac{\ln \frac{\lambda_{11}^2}{n-1}}{\ln \lambda_\nu},
\end{align*}
\tag{3.12}
\]

We have determined all these indices for 4-simplex and 5-simplex lattice (the case \( n = 3 \) has been already studied in some details [7]). Our results can be summarized in the following way: for the SAWs on the 4-simplex lattice we have obtained \( \gamma_{11} \approx -0.57558, \gamma_1 \approx 0.57514, \gamma_s \approx 1.7259 \) for \( T > T_a \), and \( \gamma_{11} \approx 1.1595, \gamma_1 \approx 1.4427, \gamma_s \approx 1.7259 \) at \( T = T_a \), whereas in the case \( n = 5 \) we have found \( \gamma_{11} \approx -0.73826, \gamma_1 \approx 0.46562, \gamma_s \approx 1.6695 \) for the high temperature phase, and \( \gamma_{11} \approx 1.30962, \gamma_1 \approx 1.4896, \gamma_s \approx 1.6695 \), at the adsorption transition. It is interesting to note here that these values of the critical exponents satisfy the usual scaling relations for surface interacting SAWs on fractals [7]

\[
\gamma_s = 2\gamma_1 - \gamma_{11} = \gamma + \nu(\bar{d} - \bar{d}_s),
\tag{3.13}
\]

where the bulk critical exponents \( \gamma \) and \( \nu \) can be written in terms of \( \lambda \) and \( \lambda_\nu \): \( \gamma = \ln(\lambda^2/n)/\ln \lambda_\nu \) and \( \nu = \ln 2/\ln \lambda_\nu \).

As it has been emphasized, the exact calculation of the critical exponents \( \gamma_1, \gamma_{11} \) and \( \gamma_s \) becomes very complicated for larger values of \( n \), so that we are going now to present an asymptotic analysis of these exponents, which is valid for \( n \gg 1 \).

(b) Approximate calculation of the critical exponents \( \gamma_1, \gamma_{11} \) and \( \gamma_s \)

To treat open walks, in the limit of large \( n \), we use here an extension of the approach proposed in ref. [13]. It turns out that in this limit we can simplify the problem considerably, and study only 5 restricted partition functions: \( D^{(r)}, D_1^{(r)}, D_2^{(r)}, D_3^{(r)} \) and \( D_4^{(r)} \), which describe the SAW configurations with one end lying somewhere inside an \( r \)-th order \( n \)-simplex. These functions have the same meaning as those presented in Figs. 4. and 5. for the case of the 6-simplex lattices and they obey the following recursion relations

\[
\begin{pmatrix}
D'_1 \\
D'_2
\end{pmatrix} = \begin{pmatrix}
F_{D_1D_1} & F_{D_1D_2} \\
F_{D_2D_1} & F_{D_2D_2}
\end{pmatrix} \begin{pmatrix}
D_1 \\
D_2
\end{pmatrix},
\tag{3.14a}
\]

15
\[
\begin{pmatrix}
D'_3 \\
D'_4
\end{pmatrix} = \begin{pmatrix}
f_{D_1D_1} & f_{D_1D_2} \\
f_{D_2D_1} & f_{D_2D_2}
\end{pmatrix} \begin{pmatrix}
D_3 \\
D_4
\end{pmatrix} + D \begin{pmatrix}
f_{D_3D} \\
1
\end{pmatrix},
\] (3.14b)

where

\[
f_{D_1D_1} = 1 + (n - 2)A_1 + (n - 2)(n - 3)A_1^2 + \cdots + (n - 2)!A_1^{n-2} + A A_2(n - 2)(n - 3)
\times (1 + 2(n - 4)A_1 + 3(n - 4)(n - 5)A_1^2 + \cdots + (n - 3)(n - 4)!A_1^{n-4}),
\]

\[
f_{D_1D_2} = A A_2(n - 2) \left(1 + (n - 3)A_1 + (n - 3)(n - 4)A_1^2 + \cdots + (n - 3)!A_1^{n-3}\right),
\]

\[
f_{D_2D_1} = A A_2((n - 1)(n - 2) + (n - 1)(n - 2)(n - 3)A_1 + \cdots + (n - 1)!A_1^{n-3})
\]

\[
f_{D_2D_2} = (n - 1)A,
\]

\[
f_{D_3D} = A_2 \left(1 + (n - 2)A_1 + (n - 2)(n - 3)A_1^2 + \cdots + (n - 2)!A_1^{n-2}\right).
\] (3.15)

In addition, we need the recursion relation for the bulk variable \(D\), which follows from (3.14) for \(D_1 = D_2 = D_3 = D_4 = D\) and \(A_1 = A_2 = A\) (see also [13]),

\[
D' = D \left(1 + (n - 1)A + (n - 1)(n - 2)A^2 + \cdots + (n - 1)!A^{n-1}\right).
\] (3.16)

The high temperature fixed point of the recursion relations (3.14) has two relevant eigenvalues

\[
\lambda_{11} = (n - 1)A^*, \quad \lambda = n,
\] (3.17)

where we use the notation of the preceding subsection. Similarly, at the point of the critical adsorption transition \((T = T_a)\) one has

\[
\lambda_{11} = n - 1, \quad \lambda = n.
\] (3.18)

It is a simple matter to obtain the numerical values of corresponding critical exponents for each \(n\), by taking into account (3.12). The values of all these exponents, obtained within this approximate approach, together with corresponding exact values (reported in subsection 3a), are displayed in Fig. 6. As in the case of the critical exponent \(\Phi\), one can notice that the exact and concomitant approximate values are quite close to each other,
even for small values of $n$, which provides a justification of the above approach. For large $n$ all these exponents tend to certain finite limiting values that can be determined (see Fig. 7, where we plotted the approximate values of $\gamma_1$, $\gamma_{11}$ and $\gamma_s$ against $\ln \ln n / \ln n$). Indeed, for $T > T_a$, one can derive the simple asymptotic forms

$$\gamma_{11} \sim -2 \left(1 - \frac{\ln \ln n}{\ln n}\right), \quad \gamma_1 \sim \frac{2}{\sqrt{n \ln n}}, \quad \gamma_s \sim 2 \left(1 - \frac{\ln \ln n}{\ln n}\right), \quad (3.19)$$

which are in good agreement with our numerical findings (see Fig. 7), obtained via the approximate approach. A similar conclusion holds for $T = T_a$, in which case we have found

$$\gamma_{11} \sim \gamma_1 \sim \gamma_s \sim 2 \left(1 - \frac{\ln \ln n}{\ln n}\right). \quad (3.20)$$

4. Conclusion

In this paper we have studied critical behavior of surface-interacting SAWs on the family of truncated $n$–simplex lattices. We have calculated the exact values of the crossover critical exponent $\Phi$ and the surface susceptibility critical exponents $\gamma_1$, $\gamma_{11}$ and $\gamma_s$, in different temperature regions, for low values of the lattice parameter $n$ (up to $n = 6$). These values are then compared with the corresponding values that we have obtained by using an approximate approach, which is expected to be valid in the limit of large $n$. It is shown that our approximate findings are quite close to the exact ones, even for these small values of $n$ (see Fig. 3. and Fig. 6).

We have found that both exact and approximate values of $\Phi$ increase as $n$ grows, becoming closer to the lower bound $\Phi_l$ (2.21) for this exponent (Fig. 3a). For larger $n$ the values of $\Phi$ (calculated approximately) monotonically approach 1 (Fig. 3b), in the same way as in the case of an analogous, recently studied [15], problem of surface interacting random walks. This is in agreement with a na"ive expectation that the critical behavior of SAWs on a lattice should be more and more similar to the one for simple random walks, when the coordination number of the underlying lattice grows up indefinitely. The limiting
behavior of the surface susceptibility critical exponents for SAWs is, however, completely
different from those for simple random walks (for \( n \to \infty \) and \( T = T_a \), for example, one
has \( \gamma_{11} = \gamma_1 = \gamma_s = 2 \) and \( \gamma_{11}^{RW} = \gamma_1^{RW} = \gamma_s^{RW} = 1 \) [15], for SAWs and random walks,
respectively). Nevertheless, the usual scaling relations (3.13) are satisfied in all these cases.
One can also notice that inequality \( \gamma_{11}(T > T_a) < \gamma_1(T > T_a) < \gamma_{11}(T = T_a) < \gamma_1(T = T_a) < \gamma_s(T \geq T_a) \) holds for small, as well as for larger values of \( n \) (see Fig. 6 and Fig. 7).
This is quite plausible, since the SAWs with at least one end attached to the adsorbing
wall certainly outnumbers the SAWs with both ends at the wall, and also the number
of the monomers at the wall is larger at the adsorption transition \( (T = T_a) \) than in the
high temperature region \( (T > T_a) \), when the SAWs are desorbed (see (1.2)). The values
of the critical exponent \( \gamma_s \) are the same in the bulk and the crossover region, and these
values are larger than the corresponding values of \( \gamma_1 \) and \( \gamma_{11} \) in both regions, which means
that, asymptotically, average number of SAWs with no ends at the adsorbing wall is larger
than the number of SAWs with ends at the wall. At the adsorption transition \( \gamma_1 \) and
\( \gamma_{11} \) become almost equal to \( \gamma_s \), when \( n \to \infty \) (Fig. 7b), monotonically approaching the
value 2, predicted for the exponent \( \gamma \) [14], which is related to all possible SAWs. On the
contrary, for the desorbed SAWs \( (T > T_a) \) the difference in the values of \( \gamma_1, \gamma_{11}, \) and \( \gamma_s \)
becomes larger when \( n \) increases (Fig. 7a), since \( \gamma_{11} \to -2 \) and \( \gamma_1 \to 0 \), when \( n \to \infty \).

At the end we can conclude that simple approximation of the exact RG approach
for calculating critical exponents of SAWs on \( n \)-simplex lattices [13] turned out to be
very fruitful in the case of the surface-interacting SAWs. Unfortunately, this approximate
method is valid only in the case of SAW model for a diluted polymer solution, whereas
the exact RG equations allow for investigating other interesting features of the polymer
chains situated on fractals, for instance, adsorption and collapse transition [6, 8]. The
corresponding analysis of the exact RG equations for the \( 6 \)-simplex lattice, when interaction
between the contiguous monomers in the bulk is taken into account, will be presented
elsewhere in the near future.
Appendix

Here we give the explicit forms of the recursion relations (2.2a – c), as well as the polynomial functions \( F \) appearing in the relations (2.3a – e), for the case of a truncated 6-simplex lattice.

\[
A' = 25008B^4C^2 + 528B^5 + 20544B^5C + 6576B^6 + 11328AB^3C^2 + 528AB^4 + 15264AB^4C + 8688AB^5 + A^2 + 36A^2B^2 + 384A^2B^3 + 4992A^2B^3C + 5544A^2B^4 + 4A^3 + 24A^3B + 312A^3B^2 + 1728A^3B^2C + 2592A^3B^3 + 12A^4 + 120A^4C^2 + 120A^4B + 480A^4BC + 960A^4B^2 + 24A^5 + 48A^5C + 216A^5B + 24A^6, \quad (A1)
\]

\[
B' = 94336B^2C^4 + 76800B^3C^3 + 22B^4 + 48160B^4C^2 + 372B^5 + 23520B^5C + 5440B^6 + 16672AB^3C^2 + 440AB^4 + 17120AB^4C + 6576AB^5 + 2832A^2B^2C^2 + 176A^2B^3 + 5088A^2B^3C + 3620A^2B^4 + 4A^3B + 64B^3C^2 + 832A^3B^2C + 1232A^3B^3 + A^4 + 26A^4B + 144A^4BC + 324A^4B^2 + 4A^5 + 16A^5C + 64A^5B + 6A^6, \quad (A2)
\]

\[
C' = 541568C^6 + 94336B^3C^3 + 43200B^4C^2 + 14448B^5C + 2940B^6 + 6252AB^4C + 2568AB^5 + 1416A^2B^3C + 954A^2B^4 + 208A^3B^3 + 54A^4B^2 + 6A^5C + 12A^5B + A^6, \quad (A3)
\]

\[
F_{A_1A} = A_2^2 + 6A_1A_2^2 + 18A_1^2A_2^2 + 24A_1A_3A_2^2 + 12A_1A_2^2B_1 + 54A_1^2A_2^2B_1 + 36A_1A_2^2B_1^2 + 12A_1A_2B_2 + 72A_1^2A_2B_2 + 144A_1^3A_2B_2 + 72A_1A_2B_1B_2 + 432A_1^2A_2B_1B_2 + 456A_1A_2B_1^2B_2 + 264A_2B_1^3B_2 + 18A_1B_2^2 + 126A_1^2B_2^2 + 312A_1^3B_2^2 + 192A_1B_1B_2^2 + 1260A_1^2B_1B_2^2 + 132B_1^2B_2^2 + 2244A_1B_1^2B_2^2 + 1908B_1^3B_2^2 + 48A_1^3A_2C_1 + 144A_1A_2B_1C_1 + 288A_1^3B_2C_1 + 1296A_1^2B_1B_2C_1 + 2496A_1B_1^2B_2C_1
\]
\[ F_{A_1B} = 6 A_2^4 + 18 A_1 A_2^4 + 12 A_2^4 B_1 + 36 A_2^3 B_2 + 168 A_1 A_2^3 B_2 + 96 A_2^3 B_1 B_2 \]
\[ + 96 A_2^2 B_2^2 + 780 A_1 A_2^2 B_2^2 + 660 A_2^2 B_1 B_2^2 + 264 A_2 B_2^3 + 2376 A_1 A_2 B_2^3 \]
\[ + 3288 A_2 B_1 B_2^3 + 396 B_2^4 + 3492 A_1 B_2^4 + 6576 B_1 B_2^4 + 48 A_1 B_2^3 C_1 \]
\[ + 432 A_1 A_2^2 B_2 C_1 + 2496 A_1 A_2 B_2^2 C_1 + 3552 A_2 B_1 B_2^2 C_1 + 8160 A_1 B_2^3 C_1 \]
\[ + 20544 B_1 B_2^3 C_1 + 8496 A_1 B_2^2 C_1^2 + 25008 B_1 B_2^2 C_1^2 , \]  
(4)

\[ F_{A_1C} = 0 , \]  
(5)

\[ F_{A_2A} = A_2 + 4 A_1 A_2 + 12 A_1^2 A_2 + 24 A_1^3 A_2 + 24 A_1 A_2^2 B_1 + 72 A_1^3 A_2 B_1 \]
\[ + 72 A_1^2 A_2 B_1^2 + 12 A_1^2 B_2 + 48 A_1^3 B_2 + 72 A_1^4 B_2 + 72 A_1^2 B_1 B_2 \]
\[ + 288 A_1^3 B_1 B_2 + 456 A_1^2 B_1 B_2^2 + 528 A_1 B_1^3 B_2 + 528 B_1^4 B_2 \]
\[ + 24 A_1^4 C_1 + 96 A_1^3 B_1 C_1 + 528 B_1^4 C_1 , \]  
(6)

\[ F_{A_2B} = 12 A_2^3 + 48 A_1 A_2^3 + 72 A_1^2 A_2^3 + 24 A_2^3 B_1 + 96 A_1 A_2^3 B_1 + 36 A_2^2 B_2 \]
\[ + 216 A_1 A_2^2 B_2 + 504 A_1^2 A_2^2 B_2 + 576 A_1^2 A_2 B_1 B_2 + 384 A_1 A_2 B_2^2 \]
\[ + 1560 A_1^2 A_2 B_2^2 + 2640 A_1 A_2 B_1 B_2^2 + 1584 A_2 B_1 B_2^2 + 528 A_1 B_2^3 \]
\[ + 2376 A_1^2 B_2^3 + 528 B_1 B_2^3 + 6576 A_1 B_1 B_2^3 + 6576 B_1^2 B_2^3 \]
\[ + 144 A_1^2 A_2^2 C_1 + 864 A_1^2 A_2 B_2 C_1 + 2496 A_1^2 B_2^2 C_1 \]
\[ + 7104 A_1 B_1 B_2^2 C_1 + 10272 B_1^2 B_2^2 C_1 , \]  
(7)

\[ F_{A_3C} = 24 A_2^5 + 240 A_2^4 B_2 + 864 A_2^3 B_2^2 + 2496 A_2^2 B_2^3 + 7632 A_2 B_2^4 \]
\[ + 10272 B_2^5 + 120 A_2^4 C_1 + 11328 A_2 B_2^3 C_1 + 25008 B_2^4 C_1 , \]  
(8)

\[ F_{B_1A} = 2 A_1^2 A_2^2 + 6 A_1^3 A_2^2 + 12 A_1^2 A_2^2 B_1 + 12 A_1^2 A_2 B_2 + 48 A_1^3 A_2 B_2 \]
\[ + 152 A_1^2 A_2 B_1 B_2 + 264 A_1 A_2 B_1^2 B_2 + 352 A_2 B_1^3 B_2 + 32 A_1^2 B_2^2 \]
\[ + 140 A_1^3 B_2^2 + 88 A_1 B_1 B_2^2 + 748 A_1^2 B_1 B_2^2 + 132 B_1^2 B_2^2 + 1908 A_1 B_1^2 B_2^2 \]
\[ + 2192 B_1^3 B_2^2 + 16 A_1^3 A_2 C_1 + 352 A_2 B_1^3 C_1 + 144 A_1^3 B_2 C_1 \]
\[ F_{B_1 C} = 0 \],
\[ F_{B_2 A} = A_1^2 A_2 + 4 A_1^3 A_2 + 6 A_1^4 A_2 + 6 A_1^2 A_2 B_1 + 24 A_1^3 A_2 B_1 + 38 A_1^2 A_2 B_1^2 + 44 A_1 A_2 B_1^3 + 44 A_2 B_1^4 \]
\[ + 3 A_1^2 B_2 + 14 A_1^3 B_2 + 26 A_1^4 B_2 + 32 A_1^2 B_1 B_2 + 140 A_1^3 B_1 B_2 + 44 A_1 B_1^2 B_2 + 374 A_1^2 B_1 B_2 + 44 B_1^3 B_2 + 636 A_1 B_1^3 B_2 + 548 B_1^4 B_2 + 12 A_1^4 C_1 + 72 A_1^3 B_1 C_1 + 208 A_1^2 B_1^2 C_1 + 592 A_1 B_1^3 C_1 + 856 B_1^4 C_1, \]  
\[ F_{B_2 B} = A_2^3 + 6 A_1 A_2^3 + 14 A_1^2 A_2^3 + 16 A_1 A_2^3 B_1 + 32 A_1 A_2^2 B_2 + 130 A_1^2 A_2^2 B_2 + 220 A_1 A_2^2 B_1 B_2 + 132 A_2^2 B_1^2 B_2 + 132 A_1 A_2 B_2^2 + 594 A_1^2 A_2 B_2^2 + 132 A_2 B_1 B_2^2 + 1644 A_1 A_2 B_1 B_2^2 + 1644 A_2 B_1^2 B_2^2 + 22 B_2^3 + 264 A_1 B_2^3 + 1164 A_1^2 B_2^3 + 372 B_1 B_2^3 + 4384 A_1 B_1 B_2^3 + 5440 B_1^2 B_2^3 + 36 A_1^2 A_2^2 C_1 + 416 A_1^2 A_2 B_2 C_1 + 1184 A_1 A_2 B_1 B_2 C_1 + 1712 A_2 B_1^2 B_2 C_1 + 2040 A_1^2 B_2^2 C_1 + 10272 A_1 B_1 B_2^2 C_1 + 17640 B_1^2 B_2^2 C_1 + 1416 A_1^2 B_2 C_1^2 + 8336 A_1 B_1 B_2 C_1^2 + 24080 B_1^2 B_2 C_1^2 + 19200 B_1^3 C_1^3, \]
\[ F_{B_2 C} = 4 A_2^5 + 36 A_2^4 B_2 + 208 A_2^3 B_2^2 + 1272 A_2^2 B_2^3 + 4280 A_2 B_2^4 + 5880 B_2^5 \]
\( + 1416 A_2^2 B_2^2 C_1 + 8336 A_2 B_2^3 C_1 + 24080 B_2^4 C_1 \)
\( + 57600 B_2^3 C_1^2 + 94336 B_2^2 C_1^3 , \) \( (A15) \)

\( F_{C_1 A} = A_1^4 A_2 + 4 A_1^3 A_2 B_1 + 22 A_2 B_1^4 + 6 A_1^4 B_2 + 36 A_1^3 B_1 B_2 + 104 A_1^2 B_1^2 B_2 \)
\( + 296 A_1 B_1^3 B_2 + 428 B_1^4 B_2 + 5 A_1^4 C_1 + 472 A_1 B_1^3 C_1 + 1042 B_1^4 C_1 , \) \( (A16) \)

\( F_{C_1 B} = 2 A_1^2 A_2^3 + 18 A_1^2 A_2^2 B_2 + 104 A_1^2 A_2 B_2^2 + 296 A_1 A_2 B_1 B_2^2 \)
\( + 428 A_2 B_1^2 B_2^2 + 340 A_1^2 B_2^3 + 1712 A_1 B_1 B_2^3 + 2940 B_1^2 B_2^3 \)
\( + 708 A_1^2 B_2^2 C_1 + 4168 A_1 B_1 B_2^2 C_1 + 12040 B_1^2 B_2^2 C_1 \)
\( + 28800 B_1^2 B_2 C_1^2 + 47168 B_1^2 C_1^3 , \) \( (A17) \)

\( F_{C_1 C} = A_2^5 + 236 A_2^2 B_2^3 + 1042 A_2 B_2^4 + 2408 B_2^5 + 14400 B_2^4 C_1 \)
\( + 47168 B_2^3 C_1^2 + 541568 C_1^5 . \) \( (A18) \)
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Figure captions

Figure 1. First two stages in the iterative construction of the truncated 6−simplex lattice; the adsorbing surface is presented by a 5−simplex lattice (shadow region).

Figure 2. Schematic representation of the eight restricted partition functions for an $r$−th stage 6-simplex fractal used in the calculation of the crossover critical exponent $\Phi$. Here $B_2$, for example, represents a part of a SAW path that enters the $r$-th order hexagon at a vertex lying on the adsorption wall, exits it at the bulk vertex, then enters the hexagon again through a surface vertex, and goes away, finally, via one of the remaining unoccupied surface vertices.

Figure 3. (a) The exact (full circles) and the approximate (open circles) values of the crossover exponent $\Phi$ for the surface interacting SAWs on the truncated $n$−simplex lattices ($n = 3, 4, 5, 6$). The full lines represent the upper and the lower bounds on the exponent $\Phi$ (see (2.21)). (b) The approximate values of $\Phi$ (open circles) against $1/n \ln n$, for $n = 10, 20, \ldots, 100$. The full lines provide the upper and the lower bounds on the exponent $\Phi$.

Figure 4. Schematic sketch of 15 restricted partition functions needed (in addition to those presented in Fig. 2.) for a complete description of the generating function $C_{11}(x, T)$ in the case of a 6−simplex fractal lattice. All here presented end points of a SAW are anchored to the attractive wall (full circles).

Figure 5. Diagrams representing some additional restricted partition functions, required for the evaluation of the critical exponents $\gamma_1$ for SAWs on the 6-simplex lattice. The open circles denote the ends of a SAW path ending somewhere in the lattice bulk.

Figure 6. The exact (full symbols) and approximate (open symbols) values of the critical exponents $\gamma_1$, $\gamma_{11}$ and $\gamma_s$ against $1/n$ ($n = 3, 4, 5, 6$), for ($T > T_a$) and ($T = T_a$), for the surface interacting SAWs on truncated $n$−simplex lattices. The lines that connect the
exact results serve merely as guides to the eye.

**Figure 7.** The approximate values of the exponents $\gamma_s$ (squares), $\gamma_1$ (circles) and $\gamma_{11}$ (triangles) against $\ln \ln n / \ln n$ ($10^2 \leq n \leq 10^4$), for the desorbed phase (a) and at the adsorption transition (b) for surface interactin SAWs on the $n$–simplex lattices. The full lines represent the asymptotic behavior described by (3.19) and (3.20).
\[ \gamma_s (T \geq T_a) \]
\[ \gamma_1 (T = T_a) \]
\[ \gamma_{11} (T = T_a) \]
\[ \gamma_1 (T > T_a) \]
\[ \gamma_{11} (T > T_a) \]

1/ \( n \)
