Anisotropic neutrino emission during the neutron star formation can be the origin of the observed proper motions of pulsars. We derive a general expression for the momentum asymmetry in terms of the neutrino energy flux gradient, and show that a nonvanishing effect is induced at the lowest order by a deformed neutrinosphere. In particular, this result is valid for a neutrino flux transported through a spherical atmosphere with constant luminosity.

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I. INTRODUCTION

Observations show that pulsars have peculiar proper motions. They have very high translational velocities with respect to the surrounding stars, with a mean value of 450 km/s and up to a maximum of about 1000 km/s [1]. This suggests that some kind of impulse (kick) happens during the birth of the neutron star. Different mechanisms have been proposed to explain the kick, but most of them have difficulties to produce the large observed velocities.

Neutrinos carry away almost all the energy released in the gravitational collapse ($\approx 3 \times 10^{53}$ erg), taking with them a momentum $\sim 100$ times the momentum associated with the spatial motion of pulsars. Therefore, a 1% anisotropy in the momentum distribution of the outgoing neutrinos would suffice to account for the translational kick.

An interesting mechanism to explain the asymmetric neutrino emission from a cooling protoneutron star has been proposed by Kusenko and Segrè [2]. It is based on the matter neutrino oscillations in the presence of an intense magnetic field. The emission surface of the electron neutrino is located at a radius larger than the one corresponding to the muon or tau neutrino. Under suitable conditions, a resonant transformation $\nu_e \rightarrow \nu_\tau$ can take place in the region between the boundaries of the electron and the tau neutrinospheres. The $\nu_e$ are trapped by the medium, but the $\nu_\tau$ produced in this way are outside their neutrinosphere and free to escape from the protostar. Consequently, the surface of resonance acts as an effective tau neutrinosphere. If there is a magnetic field, or another non-isotropic effect, this surface of resonance becomes distorted and an anisotropy in the energy flux is generated, causing a kick to the protostar.

Doubts about the effectiveness of the above mechanism have been raised by Janka and Raffelt [3]. According to them, no effect is generated at lowest order because it is not justified to calculate the flux asymmetry from the temperature variation around the surface of resonance. The neutrino luminosity in the protoneutron star is controlled by the core emission and is not affected by local processes in the atmosphere, where the flavor transformation occurs. In support of their argument, Janka and Raffelt use the Eddington model for a plane-parallel stellar atmosphere [4] to estimate a residual asymmetry that is induced by higher-order corrections.

In this work we reconsider the problem of the neutrino oscillation mechanism for pulsar kicks. In fact, a neutron protostar is a very complex system, mainly formed by interacting nucleons, electrons, photons and neutrinos, rotating quickly and with a strong magnetic field. However, for our analysis a simplified and perturbative description is suitable. The star can be considered as constituted by two gases, the nucleon gas and the neutrino gas. From the hydrodynamic point of view electrons and photons can be ignored. The dynamics of these gases are controlled by the gravitational field, the magnetic field, the neutrino-nucleon interactions and the hydrostatic equilibrium equations. At the lowest order in this perturbative approach, and neglecting the star rotation, the dynamics of the nucleon gas is dominated by the gravitational field and the equilibrium hydrodynamic equations, and it is thus described by a spherical distribution. Besides this we have the neutrino gas, interacting with the nucleon gas and the magnetic field. This last interaction acts in particular on the neutrino mass columns, breaking their isotropy. This breaking alters the hydrostatic equilibrium equations, producing a non-isotropic distortion in the nuclear matter. In the following we...
neglect these contributions, but even so there is a non vanishing kick effect induced by the geometrical deformation of the resonance surface, provided that it acts as an effective emission surface.

We parametrize the resonance surface as \( R = R_e + \delta \cos \theta \), and develop the expressions up to first order terms in \( \delta/R_e \). From this approach we derive an expression for the fractional momentum asymmetry in terms of the spatial derivative of the energy flux. Using this result, we show that the distortion of the resonance surface by the presence of a magnetic field generates a geometrical asymmetry in the neutrino emission, even in the case of a constant luminosity. To illustrate this effect, we consider two simple self-consistent models for a spherical protostar atmosphere, which satisfy the energy flux conservation. In particular, in one of the models we use the Eddington approximation adapted to the spherical geometry. The required magnetic field results to be one order of magnitude higher than the value derived in the original articles [9].

To make the comparison with the existing literature easier, we restrict the discussion to the standard mechanism of neutrino oscillations between massive active neutrinos, and a magnetic-field induced deformation of the resonance surface. Nevertheless, our approach can be straightforwardly extended to other situations [6], and in particular to oscillations produced by a violation of the equivalence principle (VEP) [7,8,9]. In the case of VEP no magnetic field is needed to deform the resonance surface [5].

In the following section we examine the effect of the deformation of the resonance surface on the neutrino energy flux, emphasizing the relevance of the geometrical variation of flux to produce an asymmetric momentum emission. In Sections III and IV, we apply the results of Section II to the Eddington and the polytrope neutrinosphere models and estimate the magnitude of the magnetic field required to explain the observations. The last section presents some general conclusions.

II. SURFACE OF RESONANCE AND NEUTRINO ENERGY FLUX

We consider oscillations between two neutrino flavors, say \( \nu_e \) and \( \nu_x \), in the interior of a protostar. Neutrinos have an average energy \( E \cong k = |\mathbf{k}| \), which depends on the radial coordinate \( r \). In the absence of a magnetic field, or any isotropy-breaking interaction, the resonant transformation takes place on the surface of a sphere of radius \( R_e \), given by the condition

\[
\frac{\Delta m^2}{2k} \cos 2\theta = \sqrt{2} G_F N_e \tau_e ,
\]

where \( k = k(R_e) \) and \( N_e = N_e(R_e) \). Here, \( \theta \) is the vacuum mixing angle, \( G_F \) is the Fermi constant, and \( \Delta m^2 = m_2^2 - m_1^2 \) is the difference between the square mass of the neutrinos. We assume that the number density of electrons \( N_e(r) \) is proportional to the baryon density \( \rho(r) \), \( N_e(r) \approx \frac{0.1}{m_e} \rho(r) \).

In the absence of a magnetic field the emission surface is spherical, the outgoing energy flux is radial and there is no kick, but the presence of a magnetic field, described here by a uniform field \( \mathbf{B} \), distorts the surface of resonance. In the simplest case this is now defined by a function \( R(\vartheta) \approx R_e + \delta \cos \vartheta \), where \( \delta < R_e \) and \( \vartheta \) is the angle between the vector position of a point at the surface and the direction of the magnetic field of the protostar. The distortion of the resonance surface leads to a modification in the outgoing energy flux. The kick is characterized by the fractional momentum asymmetry factor

\[
\frac{\Delta k}{k} = \frac{1}{6} \int_0^\pi \mathbf{F}_e(\vartheta) \cdot \mathbf{B} \, da ,
\]

where \( \mathbf{F}_e \) is the outgoing electron neutrino energy flux at the element of area \( da \) of the emission surface. The integrals in the denominator and numerator give the total momentum lost by the protostar per unit of time and its component along the direction of the magnetic field, respectively. The factor \( 1/6 \) comes from the fact that only the electron neutrino contributes to the energy flux asymmetry.

To compute the asymmetry factor it is necessary to take into account the structure of the flux at the resonance surface, considered as an effective emission surface. The details of this analysis are presented in the appendix. The basic information is given by the distribution function for neutrinos in thermal equilibrium with the medium and satisfying the diffusion approximation

\[
f_\nu \simeq f_\nu^{eq} - \frac{1}{\Lambda} \hat{\Omega} \cdot \nabla f_\nu^{eq} = f_\nu^{eq} - \frac{1}{\Lambda} \hat{\Omega} \cdot \hat{n} \frac{df_\nu^{eq}}{dr} ,
\]
where \( \mathbf{k} = k\mathbf{\hat{\Omega}} \) is the momentum of the neutrinos, \( f_{\nu}^{eq} = (1 + e^{(k-\mu_{\nu})/T})^{-1} \) is the neutrino distribution function at equilibrium, \( \mu_{\nu} \) is the chemical potential, and the factor \( \Lambda \) comes from the differential cross section for neutrino reactions. The energy flux is defined by this distribution function according to

\[
\mathbf{F}_{\nu}(\mathbf{r}) = \int \frac{dS_{\nu}}{(2\pi)^3} \mathbf{k} f_{\nu}(\mathbf{r}, \mathbf{k}) .
\] (4)

At the interior of the resonance surface the energy flux is radial (\( \mathbf{F} = F \mathbf{\hat{r}} \)) and results from the diffusive part of the neutrino distribution. As discussed in Ref. 3, in a neutron protostar the neutrino luminosity \( L_{\nu} \) is governed by the energy loss from the core. Throughout the neutrinosphere, within the resonance surface, the luminosity can be assumed as not being dependent on the radial coordinate, and thus as satisfying \( F(r) = \frac{L_{\nu}}{4\pi r^2} \). Thus, below the limit surface, the flux is purely diffusive, and is given by the constant luminosity condition. Once the energy flux above the limit surface is computed using the expression (4), we find that it has a radial component, associated to the diffusive part of the neutrino distribution, and a normal component, due to the isotropic part of the neutrino distribution

\[
\mathbf{F}_{s} = F_{\hat{n}} \mathbf{\hat{n}} + F_{\hat{r}} \mathbf{\hat{r}},
\] (5)

where \( \mathbf{\hat{n}} \) and \( \mathbf{\hat{r}} \) are unit vectors along the normal and radial directions respectively, at the considered point on the resonance surface. The local flux conservation (\( \nabla \cdot \mathbf{F} = 0 \)) implies that, at the resonance surface, the normal outgoing flux is equal to the diffusive outgoing flux, and that both are equal to one half of the diffusive flux calculated just below the resonance surface

\[
F_{\nu}(\vartheta) = F_{\hat{n}}(\vartheta) = \frac{1}{2} F_{\nu}(R_{\nu} + \delta \cos \vartheta) \simeq \frac{1}{2} F_{\nu}(R_{\nu}) \left( 1 + h^{-1}_{F} \delta \cos \vartheta \right),
\] (6)

where \( h^{-1}_{F} = \left. \frac{dF}{dr} \right|_{R_{\nu}} \).

In the integrals of Eq. (3), \( da \) is the element of area on the distorted surface of resonance and does not coincide with the element of area \( d\Omega \) on the sphere of radius \( R_{\nu} \). These areas are related as follows:

\[
da = \left[ 1 + \left( \frac{1}{R} \frac{dR}{d\vartheta} \right)^2 \right]^{1/2} 2\pi R^2 \sin \vartheta d\vartheta d\varphi \simeq (1 + 2 \frac{\delta}{R_{\nu}} \cos \vartheta) da_{\nu} .
\] (7)

In addition, we have \( \mathbf{\hat{n}} \cdot \mathbf{\hat{B}} = \cos \vartheta \), and

\[
\mathbf{\hat{n}} \cdot \mathbf{\hat{B}} = \left[ 1 + \left( \frac{1}{R} \frac{dR}{d\vartheta} \right)^2 \right]^{-1/2} \left[ \cos \vartheta + \frac{1}{R} \frac{dR}{d\vartheta} \sin \vartheta \right] \simeq \cos \vartheta - \frac{\delta}{R_{\nu}} \sin^2 \vartheta .
\] (8)

From Eqs. (3)-(8), to first order, we observe that only \( F_{\hat{n}} \), the component of the energy flux normal to the resonance surface, gives a non null contribution to the integral in the numerator of the asymmetry factor, and it is thus responsible for the kick. It results

\[
\Delta k = \frac{1}{36} h^{-1}_{F} \delta .
\] (9)

Taking into account the energy emitted by the protostar in form of neutrinos, this ratio must have a value of the order of \( 10^{-2} \) to produce the observed kicks. This expression clearly shows that the existence of a kick requires a nonvanishing gradient of the flux, i.e. \( h^{-1}_{F} \neq 0 \). Using \( h^{-1}_{F} = -2R_{\nu}^{-1} \), we obtain

\[
\Delta k = -\frac{1}{18} \frac{\delta}{R_{\nu}} .
\] (10)

This means that \( \delta \) must be of the order of \( R_{\nu}/6 \) to produce the required kick.

To calculate \( \delta \), let us remember that the index of refraction of the neutrinos is modified by the presence of an external magnetic field [10]. This fact affects the flavor transformations of mixed neutrinos producing an anisotropic contribution to the resonance condition [11]. For neutrinos propagating through a degenerate electron gas, the resonance condition becomes

\[
\frac{\Delta m^2}{2k} \cos 2\theta = \sqrt{2} G_F N_e + \frac{e G_F}{\sqrt{2}} \left( \frac{3N_e}{\pi^2} \right)^{1/3} B \cos \vartheta ,
\] (11)

3
where $e$ is the electron charge, $B = |\mathbf{B}|$, and $k$ and $N_e$ are evaluated at $R(\vartheta)$. In Eq. (11) it has been assumed that the weak-field limit is satisfied, i.e. $2eB \ll (3\pi^2 N_e)^{2/3}$. More general features of the neutrino propagation in magnetized media, incorporating the effect of strong magnetic fields, have been considered by several authors [12]. In our case, the weak-field condition means $B \ll 5 \times 10^{16} G$ and, in fact, is marginally satisfied in the models discussed below. However, taking into account the ambiguities in the values considered for the parameters of a protoneutron star, a more accurate computation is not necessary. This could be meaningful in the context of a more precise and detailed model, which is beyond the scope of this article.

By writing $k = k_r + \delta k$ and $N_e = N_{e_r} + \delta N_e$, from Eqs. (11) and (11) we obtain the relation

$$\frac{\Delta m^2}{2k_r^2} \delta k \cos 2\theta \simeq -\sqrt{2} G_F \frac{N_e}{\pi^2} \frac{3N_e}{\pi^2} B \cos \vartheta .$$

(12)

There are two contributions to $\delta N_e$. One is due to the geometrical distortion of the surface of resonance, assuming that the different profiles of the neutrinosphere remain unchanged. The other comes from the distortion of the profiles of temperature, pressure, density, etc., induced by the geometrical distortion. The equations that define the model give the relation between the last higher-order contribution and the deformation of the surface of resonance. In this work we will only consider the first order contribution. Thus, we have

$$\delta N_e \bigg|_{R_e} = \frac{dN_e}{dr} \bigg|_{R_e} \delta R \equiv h_{N_e}^{-1} N_e \delta \cos \vartheta .$$

(13)

Analogously, we also have

$$\delta k \bigg|_{R_e} = \frac{dk}{dr} \bigg|_{R_e} \delta R \equiv h_k^{-1} k \delta \cos \vartheta .$$

(14)

Inserting Eq. (13) and (14) into Eq. (12), we get

$$\delta \simeq -\frac{e}{2} \left( \frac{3}{\pi^2} \right) \frac{1}{h_{N_e}^{-1} \rho} \frac{1}{h_k^{-1} + h_{N_e}^{-1}} \bigg|_{R_e} .$$

(15)

If we assume that the electron neutrinos are in thermal equilibrium with the stellar medium, the average energy of the emitted neutrinos is proportional to the temperature at the emission point, $k = \frac{2\pi^4}{150\hbar(3)} T \simeq 3.15 T$. In such a case, from Eq. (1) we get

$$\Delta m^2 \cos 2\theta \simeq \frac{G_F \rho \nu T_r}{m_n} .$$

(16)

To have the resonance within the electron neutrinosphere, $\rho, T_r$ must be larger than the corresponding value at the surface of the neutrinosphere, $\rho_{\nu_r}, T_{\nu_r}$. For an ideal gas this simply means that the pressure at the resonance must be larger than the pressure at the surface of the neutrinosphere. The $\nu_e$ trapping density is $\rho \gtrsim 10^{11} \text{ g cm}^{-3}$, and hence

$$\Delta m^2 \cos 2\theta > \frac{G_F \rho_{\nu} T_{\nu}}{m_n} \simeq 4.3 \times 10^{-8} \times T_{\nu_e} ,$$

(17)

where the temperature $T_{\nu_e}$ is given in $MeV$. For $T_{\nu_e} \simeq (3 - 5) MeV$, it requires $m_{\nu_e} \gtrsim 100 eV$. Now, $h_k = h_T$ and, together with $h_{N_e} = h_\rho$, we obtain

$$\delta \simeq -\frac{3eB}{2} \left( \frac{10m_n}{3\pi^2} \right)^{2/3} \rho^{-2/3} \frac{1}{h_T^{-1} + h_\rho^{-1}} \bigg|_{R_e} .$$

(18)

In general, to compute $\delta$ and $h_r^{-1}$ (or $R_r$ in the case of Eq. (10) a model for the neutrino atmosphere of a neutron protostar must be specified. This will be done in the next sections, where we examine two analytical models that are meaningful up to the neutrinosphere, where the neutrino transport equation holds. In both models the neutrino transport is in the diffusion regime and the luminosity is independent of $r$, but they differ in the assumed properties for the medium. In the first one, the Eddington model, the medium is an ideal gas of nucleons, while in the second one it is a polytrope gas.
III. THE SPHERICAL EDDINGTON MODEL

The Eddington model gives a simple and physically reasonable description of a neutrino atmosphere, locally homogeneous and isotropic. For a plane geometry the model was developed by Schinder and Shapiro [4], and here we extend it to the spherical geometry.

For neutrinos and antineutrinos in thermal equilibrium with the medium, and satisfying a transport regime consistent with the diffusion approximation, the energy density, the energy flux, and the stress tensor of neutrinos with momentum $k$ are [4]:

$$U_k = \frac{k^3 f_{\nu}^{eq}}{2\pi^2},$$  \hspace{1cm} (19)

$$F_k = -\frac{1}{3\Lambda} \frac{k^3}{2\pi^2} \nabla f_{\nu}^{eq},$$  \hspace{1cm} (20)

$$(T_k)_{ij} = \frac{1}{3} \delta_{ij} U_k.$$  \hspace{1cm} (21)

We assume that $\Lambda = \kappa \rho k^2$, with $\kappa = 5.6 \times 10^{-9}$ erg$^{-1}$ cm$^{-3}$ s$^{-2}$. In the interior of the neutrinosphere we can consider that there are two perfect fluids. One of them is constituted by nonrelativistic nucleons of mass $m_n$, with a density $\rho$, and the other by ultrarelativistic neutrinos and antineutrinos, with a vanishing chemical potential, $\mu_\nu = \mu_{\bar{\nu}} = 0$. Photons and electrons are of course present and, in fact, electrons make the relevant contribution to the effective potential in the case of matter oscillations between active neutrinos. However, we can ignore both of them for the hydrodynamic description of the system. Therefore, the relationships between pressure $P$, density of energy $\varepsilon$, and temperature $T$ are

$$P = \frac{\rho}{m_n} T + \frac{7\sigma}{24} T^4,$$  \hspace{1cm} (22)

$$\varepsilon = \rho + \frac{7\sigma}{8} T^4,$$  \hspace{1cm} (23)

where $\sigma \simeq 2.09 \times 10^{49}$ erg$^{-3}$ cm$^{-3}$ is the Stefan-Boltzmann constant. Taking into account the gravitational field $\phi$ of the star, we have for $P$

$$\nabla P = -(P + \rho) \nabla \phi.$$  \hspace{1cm} (24)

Putting all these relations together, we obtain the set of equations which describes an isotropic neutrinosphere

$$U(r) = \frac{7}{8} \sigma T^4(r),$$  \hspace{1cm} (25)

$$F(r) = -\frac{1}{36} \frac{1}{\kappa \rho(r)} \frac{d}{dr} T^2(r),$$  \hspace{1cm} (26)

$$P(r) = \frac{\rho}{m_n} T(r) + \frac{7\sigma}{24} T^4(r),$$  \hspace{1cm} (27)

$$\frac{d}{dr} P(r) = -\left[ P(r) + \rho(r) + \frac{7\sigma}{8} T^4(r) \right] \frac{GM(r)}{r^2},$$  \hspace{1cm} (28)

with $M(r) = 4\pi \int_0^r d\bar{r} \bar{r}^2 \rho_\gamma(\bar{r})$, where $\rho_\gamma$ is the total density of mass.

In the region where a resonant transformation could happen the baryon density is $\rho \simeq (10^{11} - 10^{12})$ g cm$^{-3}$ and $T \simeq (3 - 10)$ MeV. Thus, we have $\rho_{\nu_e} = \frac{7}{8} T^3 \simeq (10^{-5} - 10^{-2}) \rho$ and $P \simeq (10^{-3} - 10^{-2}) \rho$. Therefore, Eqs. (27) and (28) reduce to

$$P = \frac{\rho T}{m_n},$$  \hspace{1cm} (29)

$$\frac{dP}{dr} = -\rho \frac{GM(r)}{r^2}.$$  \hspace{1cm} (30)

These last equations together with Eq. (26) describe an ideal gas of nucleons at hydrostatic equilibrium, with an energy flux given by the transport of neutrinos.

From Eqs. (29) and (30) we have
\[ h^{-1} \rho^{-1} + h^{-1} T^{-1} \big|_{r=R_r} = - \frac{G m_r M_r}{R_r^2 T_r}, \]  

(31)

where \( M_r \) is the protostar mass enclosed by the resonance sphere. Therefore, from Eqs. (31), (18), and (10), the necessary magnetic field results

\[ B \simeq \left( \frac{10^4 m_n}{3 \pi^2 \rho_r} \right)^{-2/3} \frac{12 G m_n M_r}{e R_r T_r}. \]  

(32)

At this point, assuming that the resonance occurs near the surface of the neutrinosphere, i.e. \( R_r \simeq R_{\nu_e} \), we can make an estimation of the order of magnitude of \( B \) for a typical protoneutron star. Then, taking \( R_{\nu_e} \simeq 30 \text{ km}, M_r \simeq M_\odot, \ T_r \simeq 5 \text{ MeV}, \) and \( \rho_r \simeq 10^{11} \text{ g cm}^{-3}, \) to have \( \Delta k/k \simeq 0.01 \) the magnetic field must be \( B \simeq 10^{16} \text{ G} \). Of course, these parameters are not independent and a more careful discussion of the model is necessary.

The neutrinosphere is defined by four functions: pressure \( P(r) \), temperature \( T(r) \), baryonic density \( \rho(r) \), and energy flux \( F(r) \). Up to this point we have only three independent equations relating these functions. A complete specification of the system requires a fourth relation. A simple analytical model that satisfies the requirement of a constant luminosity is the Eddington atmosphere [4]. This model is defined by Eqs. (26), (29), and (30), plus the hypothesis of the energy flux conservation \( \nabla \cdot F = 0 \). For an isotropic flux, the additional assumption leads to:

\[ \frac{\partial (r^2 F)}{\partial r} = 0, \]  

(33)

which simply means

\[ F = \frac{L_c}{4 \pi r^2}, \]  

(34)

where \( L_c \) is the luminosity of the protostar.

From Eqs. (33) and (34) we have for the mass density

\[ \rho = - \frac{2 \pi r^2}{9} \frac{dT}{dT} \]  

(35)

and replacing this expression in Eq. (33) we arrive at

\[ \frac{dP}{dr} = \frac{2 \pi}{9} \frac{G}{\kappa L_c} M(r) T \frac{dT}{dr}. \]  

(36)

To find the solution of the structure equations for this model we use the following procedure. First we define a reduced effective mass \( m(r) \), given by

\[ \int_{R_c}^r \, dr \, M(r) T \frac{dT}{dr} = M_c m(r) \int_{R_c}^r \, dr \, T \frac{dT}{dr}, \]  

(37)

where \( M_c \) is the mass of the core. In terms of \( m(r) \) the solution of Eq. (36) is immediate. Defining \( \alpha_c = \frac{G M_c m_T}{9 \kappa L_c \rho_c} \), we have

\[ P(r) = \frac{P_c}{1 - a} \left( \frac{T^2}{T_c^2} - a \right), \]  

(38)

where

\[ a(r) = 1 - \frac{1}{\alpha_c m(r)}. \]  

(39)

From Eqs. (28) and (38) we can express the density in terms of the temperature as follows

\[ \rho(r) = \frac{\rho_c}{1 - a} \frac{T_c}{T^2} \left( 1 \right), \]  

(40)

and using this result in Eq. (33) we obtain a first order differential equation for \( T \)
\[ \frac{dT}{dr} + \frac{\lambda_c T_e R_c}{1 - a} \frac{R_c}{r^2} \left( \frac{T^2 - T_e^2}{T^2} \right) = 0, \]  
(41)

where \( \lambda_c = \frac{9}{2\pi} \frac{\kappa_{\nu e} \rho_e}{T_e^2} \). According to this equation the slope of the temperature at \( R_c \) is independent of the function \( a(r) \), \( T'_e = -\frac{\lambda_c T_e R_c}{1 - a} \). If we refer to an idealized neutrinosphere, where this model would apply in the whole space, the interesting solutions correspond to infinite protostars where the temperature has an asymptotic behavior for \( r \gg R_c \) such that the temperature tends to \( T_s \approx \sqrt{\alpha T_e} \). Thus, the function \( a(r) \) varies in the range \( 1 - \alpha^{-1} < a < (T_s/T_e)^2 \) for \( R_c < r < \infty \).

This system of equations has no analytical solution when \( a \) is a function of \( r \), and in general there is no perturbative expansion that gives a good approximate solution at every point within the neutrinosphere. To find an approximate solution let us consider the differential equation (41) with \( a \) a constant. In this case, for \( T_e > T > \sqrt{\alpha T_e} \) an analytical (implicit) solution is given by

\[ T - T_e + \frac{T_e \sqrt{a}}{2} \left[ \ln \left( \frac{T - T_e \sqrt{a}}{T + T_e \sqrt{a}} \right) - \ln \left( \frac{1 - \sqrt{a}}{1 + \sqrt{a}} \right) \right] = \frac{T_c \lambda_c}{1 - a} \left( \frac{R_c}{r} - 1 \right). \]  
(42)

If we replace the constant \( a \) for a (well behaved) function of \( r \), then the above expression still satisfies Eq. (41) at \( r = R_c \). For an infinite atmosphere, a good approximation to the exact solution is given by Eq. (42), with \( a \) now a function of \( T(r) \):

\[ a(T) = 1 - \frac{1}{\alpha_c} - A \left( 1 - \frac{T}{T_e} \right), \]  
(43)

where \( A = \frac{T_r - T_e}{T_e - T_s} \left( 1 - \frac{1}{\alpha_c} - \frac{T_e^2}{T_s^2} \right) \), and \( T_s \) denotes the temperature at the surface of the electron neutrinosphere, which is assumed to lie in the asymptotic region. From Eq. (43), we see that \( a(T_s) = (T_s/T_e)^2 \) and \( a(T_e) = 1 - \alpha^{-1} \), and thus it fits the extreme values of \( a(r) \).

The surface of the electron neutrinosphere corresponds to a density \( \rho_s \approx \rho_{\nu_e} \). Assuming that the temperature at this surface is close to \( T_s \), from Eq. (40) we obtain

\[ T_{\nu_e} \approx T_s \left( 1 + \frac{T_e}{2T_s - AT_e \rho_{\nu_e}} \rho_{\nu_e} \right), \]  
(44)

with \( \rho_{\nu_e} \ll \rho_c \). The radius of the neutrinosphere is obtained by evaluating Eq. (42) in \( T_{\nu_e} \).

To illustrate the predictions of the model, we adopt a neutron protostar with reasonable values for its parameters. The core is assumed to have \( M_c = M_c = 1.13 \times 10^{60} \text{ MeV} \), \( R_c = 10 \text{ km} \), \( L_c = 9.5 \times 10^{51} \text{ erg s}^{-1} \), \( \rho_c = 10^{14} \text{ g cm}^{-3} \), and \( T_e = 40 \text{ MeV} \). We take the surface of resonance as defined by a density \( \rho_r = 10^{11} \text{ g cm}^{-3} \), while a numerical estimation for the asymptotic value of the solution of Eq. (12) gives \( T_s = 4.8 \text{ MeV} \). Inserting these values into Eqs. (42) and (44) we get \( R_{\nu_e} \approx 2.7 R_c \), and a numerical computation of the total mass of the star from Eq. (40) gives \( M_1 = 1.4 M_\odot \). Then, for a resonance region lying near the surface of the electron neutrinosphere, from Eqs. (11), (40), and (43) we have

\[ h_{T}^{-1}|_{r=R_c} \approx \frac{T_e^2 R_c \rho_{\nu_e}}{T_e^2 R_c \rho_e} \approx -0.4 \frac{1}{R_r}, \]  
(45)

\[ h_{F}^{-1}|_{r=R_c} \approx \frac{R_e}{T_e \rho_e} \left( 2 - \frac{T_e}{T_r} \right), \]  
(46)

According to Eqs. (13) and (14), we see that a \( \Delta k/k \) of order 0.01 can be obtained with \( B \approx 3 \times 10^{16} \text{ G} \). This value is in agreement with the estimation done by means of Eq. (12). In general, for more extended and hotter Eddington protostars smaller magnetic fields are needed, as can also be seen from Eq. (12).

The strength of the magnetic field we have obtained is somewhat higher than those estimated in previous works on the subject [35]. However, it is important to note that for us \( B \) is at least an order of magnitude lower than the one given in Ref. [3]. The discrepancy can be easily understood. We have used a more realistic spherical model, where the flux varies as \( r^{-2} \) (Eq. (14)), while in Ref. [3] the resonance region was described in terms of a plane Eddington atmosphere, where \( F \) is constant. In the last case \( h_{F}^{-1} \) vanishes and no kick is generated at the lowest order.
IV. THE POLYTROPE MODEL

The conditions at the inner core of the protostar are consistent with a polytrope gas of relativistic nucleons with an adiabatic index $\Gamma = 4/3$ [13,14,15]. In this section we assume that this value of $\Gamma$ also holds in the rest of the star. This model satisfies the same equations of hydrodynamic equilibrium (30), energy transport (26) and flux conservation (34) as the Eddington model, but differs in the equation of state. It leads to a confined atmosphere, where the density becomes zero at a radius of the order of a few core radii.

The equation of state for a polytrope gas of adiabatic index $\Gamma$ is given by [16,15,17]

$$P = K \rho^{\Gamma}.$$  \hspace{1cm} (47)

From Eqs. (47) and (30), the density profile is determined by the integro-differential equation

$$\frac{d\rho^{\Gamma-1}}{dr} = -\frac{\lambda_{\Gamma} R_c \rho_c^{\Gamma-1} M(r)}{r^2},$$  \hspace{1cm} (48)

where $\lambda_{\Gamma} = \frac{GM_c}{R_c \rho_c^{\Gamma}} \frac{(\Gamma-1)}{K_{\text{eff}}}$. An extremely good approximation for $\rho$ is given by the function

$$\rho^{\Gamma-1}(r) = \rho_c^{\Gamma-1} \left[ \lambda_{\Gamma} \left( \frac{R_c}{r} - 1 \right) m(r) + 1 \right],$$  \hspace{1cm} (49)

with $m(r) = \mu + (1 - \mu) \frac{R_c}{r}$, such that $m(R_c) = 1$. The radius of the star, defined by the surface where the density becomes zero, $\rho(R_s) = 0$, determines the value of the $\mu$ parameter. The relationship between $\mu$ and $R_s$ is given by

$$\mu = \left[ \frac{R_s}{\lambda_{\Gamma} (R_s - R_c)} \right] \frac{R_c}{R^{\Gamma}_{s - c} (R_s - R_c)}.$$  \hspace{1cm} (50)

It is convenient to rewrite Eq. (49) as follows

$$\rho^{\Gamma-1}(r) = \rho_c^{\Gamma-1} \left[ a \left( \frac{R_c}{r} \right)^2 + b \frac{R_c}{r} + c \right],$$  \hspace{1cm} (51)

with $a = (1 - \mu) \lambda_{\Gamma}$, $b = (2 \mu - 1) \lambda_{\Gamma}$, and $c = 1 - \mu \lambda_{\Gamma}$.

Once the mass density is known, the temperature profile is determined by the condition of constant luminosity (34), together with (26):

$$\frac{dT^2}{dr} = -\frac{9 \kappa L_c}{\pi r^2} \rho,$$  \hspace{1cm} (52)

with $\rho$ given by Eq. (49). The solution to Eq. (52) for $\Gamma = 4/3$ can be found exactly. We write it as

$$T(r) = T_c \sqrt{\lambda_c \left[ \chi \left( \frac{R_c}{r} \right) - \chi(1) + 1 \right]},$$  \hspace{1cm} (53)

where $\chi(x)$ is a polynomial of degree seven in the variable $x$

$$\chi(x) = c^3 x^3 + \frac{3}{2} h c^2 x^2 + c(ac + b^2)x^3 + \frac{b}{4}(6ac + b^2)x^4$$
$$+ \frac{3a}{5}(ac + b^2)x^5 + \frac{ba^2}{2}x^6 + \frac{a^3}{7}x^7,$$  \hspace{1cm} (54)

where $a$, $b$, and $c$ are the parameters introduced in Eq. (51).

The inverse characteristic lengths of the temperature and density at the resonance can now be calculated from Eqs. (53) and (49), respectively

$$h_T^{-1} = -\lambda_c \rho_c \left( \frac{T_c}{T_r} \right)^2 \frac{R_c}{R_T^2},$$  \hspace{1cm} (55)

$$h_{\rho}^{-1} = -3 \left( \frac{\rho_c}{\rho_r} \right)^{1/3} \left( 2a + \frac{R_c}{R_r} \right) \frac{R_c^2}{R_r^2}.$$  \hspace{1cm} (56)
To estimate the magnitude of $B$ we use the same values for the core parameters as in the Eddington model. The constant $K$ is fixed by the condition $K = P/\rho f$, which gives $K = T_c/m_\nu \rho f^{1/3} = 5.6 \times 10^{-5} \text{ MeV}^{-4/3}$. Similarly to the previous section, we adopt here $R_s = 5.8R_c$ from a numerical estimation. The radius of the surface of resonance can be calculated from Eq. (11) and the result is $R_c = 4.2R_c$. Using these values in Eq. (58) we obtain $h_c^{-1} = -4 \times 10^{-3}R_c^{-1}$ and $h_p^{-1} = -11R_c^{-1}$, which substituted in Eqs. (18) and (19) yield $B \simeq 5 \times 10^8 \text{ G}$ in order to have $\Delta k/k \simeq 0.01$. This result is in agreement with the values for the magnitude of the magnetic field calculated in the previous section.

V. CONCLUSIONS

In this paper we revisit the resonant neutrino conversion for pulsar kicks. By expressing the momentum asymmetry in terms of the logarithmic derivative of the energy flux, we make clear that a kick is produced at order $\delta/R_c$ by the radial dependence of this flux combined with a deformation of the resonance surface. An important ingredient for this result is the existence of a component of the neutrino flux normal to the resonance surface, which acts an effective tau neutrinosphere. This is valid even though the neutrino luminosities are controlled by the core emission. In the particular case of a plane atmosphere with constant flux, $h_c^{-1}$ vanishes and there is no kick effect to lowest order. However, with a more realistic spherical geometry for the atmosphere this is not true anymore.

To estimate the neutrino flux anisotropy, we consider two simple self-consistent models for the stellar atmosphere: the spherical Eddington model and a model for a star composed of a polytropic neutron gas. Both models take into account the energy flux conservation and give reasonable profiles for temperature, density, and pressure. The results clearly show that the main effect is due to the geometrical deformation of the neutrinosphere surface, and not to the alteration of a conserved plane-parallel flux by a response of the global stellar structure \cite{3}. This last contribution is much weaker than the geometrical one. It should be noticed that since sterile neutrinos do not interact with the medium, the in-going $\nu_s$ produced at the resonance surface would not be absorbed. Thus, in this case, in the approximation we use, the resonant conversion mechanism is not able to generate a kick.

For typical values of the protoneutron star parameters, in both models the magnetic field required to generate an appropriate kick is of order of $B \sim 10^{16} \text{ G}$. A more reliable quantitative evaluation of the effect would require a detailed model and much more involved numerical calculation. Nevertheless, our simplified discussion indicates that the neutrino oscillation mechanisms can not be discarded as a possible explanation for the pulsar velocities. Here, we adopt the usual mechanism of oscillations between massive active neutrinos, which requires a large tau neutrino mass of the order of 100 $eV$. However, the same analysis can be performed in other scenarios in agreement with the present boundaries on the neutrino properties, such as oscillations induced by a violation of the equivalence principle.

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APPENDIX: ENERGY FLUX ON A DISTORTED RESONANCE SURFACE

Here we analyze the energy flux through the distorted surface of resonance, a sphere with its center shifted in $\delta$ with respect to the center of the star. At a given point on this surface characterized by a polar angle $\theta$, referred to the direction of the magnetic field, the normal forms an angle $\alpha$ with respect to the radial direction, given by
\begin{equation}
\sin \alpha = \frac{\delta}{R} \sin \theta.
\end{equation}

The expression for the neutrino distribution in the diffusive approximation takes the form
\begin{equation}
f_\nu(r, k, \hat{\Omega}) = f_\nu^0(r, k) + \frac{1}{n_{\nu} c^2} \left| \frac{df_{\nu}^0}{dr} \right| \cos \beta,
\end{equation}
where $\beta$ is the angle between $k = k\hat{\Omega}$ and the radial direction $\hat{r}$. We assume null chemical potential. Introducing local coordinates given by an azimuthal angle $\chi$ on the tangent plane (with $\hat{\chi} \cdot \hat{n} = 0$), and a polar angle $\phi$ respect to the interior normal ($-\hat{n}$), $\beta$ can be expressed as
\[
\cos \beta = \sin \chi \sin \phi \sin \alpha + \cos \phi \cos \alpha .
\] (59)

At a given point of the resonance surface the produced tau neutrinos have the same isotropic distribution as the parent electron neutrinos. The in-going tau neutrinos that fall into their own neutrinosphere are absorbed and thermalized. The remaining tau neutrinos escape from the star and contribute to the kick. In the following we will assume for simplicity that all the in-going tau neutrinos are absorbed. Thus, from Eq. (6), restricting the integration to \( \mathbf{k} \cdot \mathbf{\hat{n}} \geq 0 \), we obtain that the flux has the normal component \( (F_\parallel \mathbf{\hat{n}}) \) and the tangential one \( (F_\parallel \mathbf{\hat{t}}) \) given by

\[
F_\parallel = \frac{7}{64} \sigma T^4 + \frac{1}{144} \kappa \rho \left| \frac{dT^2}{dr} \right| \cos \alpha ,
\]

\[
F_\parallel = \frac{1}{72} \kappa \rho \left| \frac{dT^2}{dr} \right| \sin \alpha ,
\]

where all quantities are evaluated at the point of the resonance surface defined by the angle \( \vartheta \). Alternatively, we can describe this flux in terms of a normal component and a radial one, which are

\[
F_\parallel = \frac{7}{64} \sigma T^4 ,
\]

\[
F_\parallel = \frac{1}{72} \kappa \rho \left| \frac{dT^2}{dr} \right| ,
\]

respectively. The normal flux comes from the contribution of \( f_{\nu}^{\sigma} \) to the neutrino distribution function, while the radial flux comes from the term which depends on its gradient. We observe that the radial flux above the limit surface is one half of the flux immediately below that surface,

\[
F(R(\vartheta) - 0^+) = \frac{1}{72} \kappa \rho \left| \frac{dT^2}{dr} \right| ,
\]

which is purely diffusive. Because \( \cos \alpha = 1 + O(\delta^2/R^2) \), the equation \( \nabla \cdot \mathbf{F} = 0 \), integrated in an infinitesimally thin volume just above the limit surface, implies that \( F_\parallel \) is equal to \( F_\parallel \). Hence, for a given \( \vartheta \) we have

\[
F_\parallel(\vartheta) = F_\parallel(\vartheta) = \frac{1}{2} F_\nu(R_\vartheta + \delta \cos \vartheta) \approx \frac{1}{2} F_\nu(R_\vartheta) (1 + h^{-1}_F \delta \cos \vartheta) ,
\]

where \( h^{-1}_F = -2R^{-1} \).

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