Kirchhoff equations with Choquard exponential type nonlinearity involving the fractional Laplacian

Sarika Goyal∗

Department of Mathematics,
Bennett University Greater Noida, Uttar Pradesh-201310, India,

Tuhina Mukherjee†

T.I.F.R. Centre for Applicable Mathematics,
Post Bag No. 6503, Sharadanagar, Yelahanka New Town, Bangalore 560065.

Abstract

In this article, we deal with the existence of non-negative solutions of the class of following non local problem

\[
\begin{aligned}
-\mathcal{M} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^\frac{n}{s}}{|x - y|^{2n}} \, dx \, dy \right) (\Delta)^s_{n/s} u &= \left( \int_{\Omega} \frac{G(y, u)}{|x - y|^\mu} \, dy \right) g(x, u) \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

where \((\Delta)^s_{n/s}\) is the \(n/s\)-fractional Laplace operator, \(n \geq 1, s \in (0, 1)\) such that \(n/s \geq 2\), \(\Omega \subset \mathbb{R}^n\) is a bounded domain with Lipschitz boundary, \(\mathcal{M} : \mathbb{R}^+ \to \mathbb{R}^+\) and \(g : \Omega \times \mathbb{R} \to \mathbb{R}\) are continuous functions, where \(g\) behaves like \(\exp(|u|^{\frac{n}{2-n}})\) as \(|u| \to \infty\).

Key words: Doubly non local problems, Kirchhoff equation, Choquard nonlinearity, Trudinger-Moser nonlinearity.

2010 Mathematics Subject Classification: 35R11, 35J60, 35A15

1 Introduction

Let \(n \geq 1, s \in (0, 1)\) such that \(n/s \geq 2\) and \(\Omega \subset \mathbb{R}^n\) be a bounded domain with Lipschitz boundary then we intend to study the existence of a non negative solutions of following fractional Kirchhoff type problem with Trudinger-Moser type Choquard nonlinearity

\[
\begin{aligned}
\mathcal{M} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^\frac{n}{s}}{|x - y|^{2n}} \, dx \, dy \right) (\Delta)^s_{n/s} u &= \left( \int_{\Omega} \frac{G(y, u)}{|x - y|^\mu} \, dy \right) g(x, u) \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

∗email: sarika.goyal@bennett.edu.in
†email: tuhina@tifrbng.res.in
where \((-\Delta)^{n/s}_{n/s}\) is the \(n/s\)-fractional Laplace operator which, up to a normalizing constant, is defined as
\[
(-\Delta)^{n/s}_{n/s}u(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{\frac{n}{2} - 2} (u(x) - u(y))}{|x - y|^{2n}} dy, \quad x \in \mathbb{R}^n, \ u \in C_0^\infty(\mathbb{R}^n).
\]
The functions \(M: \mathbb{R}^+ \to \mathbb{R}^+\) and \(g: \Omega \times \mathbb{R} \to \mathbb{R}\) are continuous satisfying some appropriate conditions which will be stated later on.

Our problem \((M)\) is basically driven by the Hardy-Littlewood-Sobolev inequality and the Trudinger-Moser inequality. Let us first recall the following well known Hardy-Littlewood-Sobolev inequality [Theorem 4.3, p.106] [13].

**Proposition 1.1 (Hardy-Littlewood-Sobolev inequality)** Let \(t, r > 1\) and \(0 < \mu < n\) with \(1/t + \mu/n + 1/r = 2\), \(g \in L^t(\mathbb{R}^n)\) and \(h \in L^r(\mathbb{R}^n)\). Then there exists a sharp constant \(C(t, n, \mu, r)\), independent of \(g, h\) such that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{g(x)h(y)}{|x - y|^{\mu}} dx dy \leq C(t, n, \mu, r) \|g\|_{L^t(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}.
\]
(1.1)

If \(t = r = \frac{2n}{2n - \mu}\) then
\[
C(t, n, \mu, r) = C(n, \mu) = \pi^{\frac{n}{2}} \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} - \frac{\mu}{2} \right) \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n}{2} - \frac{\mu}{2} \right)} \right)^{-1 + \frac{\mu}{n}}.
\]

In this case there is equality in (1.1) if and only if \(g \equiv (\text{constant})h\) and
\[
h(x) = A (\gamma^2 + |x - a|^2)^{-\frac{(2n-\mu)}{2}}
\]
for some \(A \in \mathbb{C}, 0 \neq \gamma \in \mathbb{R}\) and \(a \in \mathbb{R}^n\).

The study of Choquard equations originates from the work of S. Pekar [19] and P. Choquard [12] where they used elliptic equations with Hardy-Littlewood-Sobolev type nonlinearity to describe the quantum theory of a polaron at rest and to model an electron trapped in its own hole in the Hartree-Fock theory, respectively. For more details on the application of Choquard equations, we refer [17]. On the other hand, the boundary value problems involving Kirchhoff equations arise in several physical and biological systems. These type of non-local problems were initially observed by Kirchhoff in 1883 in the study of string or membrane vibrations to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations.

Lü [14] in 2015 studied the following Kirchhoff problem with Choquard nonlinearity
\[
-\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + (1 + \mu g(x))u = (|x|^{-\alpha} * |u|^p)u^{p-2}u \quad \text{in} \ \mathbb{R}^3
\]
for \(a > 0, \ b \geq 0, \ \alpha \in (0, 3), \ p \in (2, 6 - \alpha), \ \mu > 0\) is a parameter and \(g\) is a nonnegative continuous potential with some growth assumptions. He proved the existence of solution to
the above problem for \( \mu \) sufficiently large and also showed their concentration behavior when \( \mu \) approaches \(+\infty\). In [11], authors discuss the existence and concentration of sign-changing solutions to a class of Kirchhoff-type systems with Hartree-type nonlinearity in \( \mathbb{R}^3 \) by the minimization argument on the sign-changing Nehari manifold and a quantitative deformation lemma. In the nonlocal case that is problems involving the fractional Laplace operator, Kirchhoff problem with Choquard nonlinearity has been studied by Pucci et al. in [21] via variational techniques.

The study of elliptic equations involving nonlinearity with exponential growth are motivated by the following Trudinger-Moser inequality in [15], namely

**Theorem 1.2** let \( \Omega \) be a open bounded domain then we define \( \tilde{W}_{0}^{s,n/s}(\Omega) \) as the completion of \( C_{c}^{\infty}(\Omega) \) with respect to the norm \( \| u \|_{n,s} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^n}{|x-y|^{2n}} \, dx \, dy \right)^{1/n} \). Then there exists a positive constant \( \alpha_{n,s} \) given by

\[
\alpha_{n,s} = \frac{n}{\omega_{n-1}} \left( \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)2^{s/2}\pi^{n/2}} \right)^{\frac{n}{n-s}}
\]

where \( \omega_{n-1} \) be the surface area of the unit sphere in \( \mathbb{R}^n \) and \( C_{n,s} \) depending only on \( n \) and \( s \) such that

\[
\sup_{u \in \tilde{W}_{0}^{s,n/s}(\Omega), \| u \|_{n,s} \leq 1} \int_{\Omega} \exp\left( \alpha \| u \|_{n,s}^{n-s} / n \right) \, dx \leq C_{n,s} |\Omega| \quad (1.2)
\]

for each \( \alpha \in [0, \alpha_{n,s}] \). Moreover there exists a \( \alpha_{n,s}^* \geq \alpha_{n,s} \) such that the right hand side of (1.2) is \(+\infty\) for \( \alpha > \alpha_{n,s}^* \).

It is proved in [15] (see Proposition 5.2) that

\[
\alpha_{n,s}^* = n \left( \frac{(2n\mathcal{W}_n)^2\Gamma\left(\frac{n}{2}\right) + 1}{n!} \sum_{i=0}^{\infty} \frac{(n+i-1)!}{i!(n+2i)^{\frac{n}{2}}} \right)^{\frac{n}{n-s}},
\]

where \( \mathcal{W}_n = \frac{w_{n-1}}{n} \) is the volume of the unit sphere in \( \mathbb{R}^n \). It is still unknown whether \( \alpha_{n,s}^* = \alpha_{n,s} \) or not.

The \( p \)-fractional Kirchhoff problems involving the Trudinger-Moser type nonlinearity has been recently addressed in [16, 23]. We also refer [6, 7] to the readers, in the linear case i.e. when \( p = 2 \). The Choquard equations with exponential type nonlinearities has been comparatively less attended. In this regard, we cite [1] where authors studied a singularly perturbed nonlocal Schrödinger equation via variational techniques. We also refer [2] for reference. On a similar note, there is no literature available on Kirchhoff problems involving the Choquard exponential nonlinearity except the very recent article [3] where authors studied the existence of positive solutions to the following problem

\[
-m \left( \int_{\Omega} |\nabla u|^n \, dx \right) \Delta_n u = \left( \int_{\Omega} \frac{F(y,u)}{|x-y|^n} \, dy \right) f(x,u), \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \partial \Omega
\]
Doubly nonlocal problems with Trudinger-Moser type Choquard nonlinearity

where $-\Delta_n = \nabla.(|\nabla u|^{n-2}\nabla u)$, $\mu \in (0, n)$, $n \geq 2$, $m$ and $f$ are continuous functions satisfying some additional assumptions, using the concentration compactness arguments. They also established multiplicity result corresponding to a perturbed problem via minimization over suitable subsets of Nehari manifold. Whereas in the $p$-fractional laplacian case, motivated by above research, our paper represents the first article to consider the Kirchhoff problem with Choquard exponential nonlinearity.

The problem of the type $(M)$ are categorized under doubly nonlocal problems because of the presence of the term $M \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^\frac{2}{n}}{|x-y|^{2n}} \, dx \, dy \right)$ and $\left( \int_{\Omega} \frac{G(y, u)}{|x-y|^{\mu}} \, dy \right) g(x, u)$ which does not allow the problem $(M)$ to be a pointwise identity. Additionally, we also deal with the degenerate case of Kirchhoff problem which is a new result even in the case of $s = 1$. This phenomenon arises mathematical difficulties which makes the study of such a class of problem interesting. Generally, the main difficulty encountered in Kirchhoff problems is the competition between the growths of $M$ and $g$. Precisely, mere weak limit of a Palais Smale (PS) sequence is not enough to claim that it is a weak solution to $(M)$ because of presence of the function $M$, which holds in the case of $M \equiv 1$. Next technical hardship emerge while proving convergence of the Choquard term with respect to (PS) sequence. We use delicate ideas in Lemma 3.3 and Lemma 3.5 to establish it. Following a variational approach, we prove that the corresponding energy functional to $(M)$ satisfies the Mountain pass geometry and the Mountain pass critical level stays below a threshold (see Lemma 3.3) using the Moser type functions established by Parini and Ruf in [18]. Then we perform a convergence analysis of the Choquard term with respect to the (PS)-sequences in Lemma 3.4. This along with the higher integrability Lemma 2.5 benefited us to get the weak limit of (PS)-sequence as a weak solution of $(M)$ leading to build the proof of our main result. The approach although may not be completely new but the problem is comprehensively fresh.

Our article is divided into 3 sections- Section 2 illustrates the functional set up to study $(M)$ and contains the main result that we intend to establish. Section 3 contains the proof of our main result.

2 Functional Setting and Main result

Let us consider the usual fractional Sobolev space

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega); \frac{(u(x) - u(y))}{|x-y|^{\frac{n+s}{p}}} \in L^p(\Omega \times \Omega) \right\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+ps}} \, dx \, dy \right)^{\frac{1}{p}}$$
where Ω ⊂ \mathbb{R}^n is an open set. We denote \( W^{s,p}_0(\Omega) \) as the completion of the space \( C_0^\infty(\Omega) \) with respect to the norm \( \| \cdot \|_{W^{s,p}(\Omega)} \). To study fractional Sobolev spaces in details we refer to [5].

Now we define

\[
X_0 = \{ u \in W^{s,n/s}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \}
\]

with respect to the norm

\[
\| u \|_{X_0} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^\frac{n}{s}}{|x - y|^{2n}} \, dx \, dy \right)^\frac{s}{n},
\]

where \( Q = \mathbb{R}^{2n} \setminus (C \Omega \times C \Omega) \) and \( C \Omega := \mathbb{R}^n \setminus \Omega \). Then \( X_0 \) is a reflexive Banach space and continuously embedded in \( W^{s,p}_0(\Omega) \). Also \( X_0 \hookrightarrow L^q(\Omega) \) compactly for each \( q \in [1, \infty) \). Note that the norm \( \| \cdot \|_{X_0} \) involves the interaction between \( \Omega \) and \( \mathbb{R}^n \setminus \Omega \). We denote \( \| \cdot \|_{X_0} \) by \( \| \cdot \| \) in future, for notational convenience. This type of functional setting was introduced by Servadei and Valdinoci for \( p = 2 \) in [22] and for \( p \neq 2 \) in [8].

Moreover, we define the space

\[
\tilde{W}^{s,p}_0(\Omega) = \overline{C_0(\Omega)}^{\| \cdot \|_{W^{s,p}(\mathbb{R}^n)}}.
\]

The space \( \tilde{W}^{s,p}_0(\Omega) \) is equivalent to the completion of \( C_0^\infty(\Omega) \) with respect to the semi norm \( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^\frac{n}{s}}{|x - y|^{2n}} \, dx \, dy \) (see for example [9], Remark 2.5). If \( \partial \Omega \) is Lipschitz, then \( \tilde{W}^{s,p}_0(\Omega) = X_0 \), (see [10], Proposition B.1). The embedding \( \tilde{W}^{s,p}_0(\Omega) \ni u \mapsto \exp(|u|^\beta) \in L^1(\Omega) \) is compact for all \( \beta \in \left( 1, \frac{n}{n-s} \right) \) and is continuous when \( \beta = \frac{n}{n-s} \).

We now state our assumptions on \( M \) and \( g \). The function \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous function which satisfies the following assumptions:

(M1) For all \( t, s \geq 0 \), it holds

\[
\hat{M}(t + s) \geq \hat{M}(t) + \hat{M}(s),
\]

where \( \hat{M}(t) = \int_0^t M(s) \, ds \), the primitive of \( M \).

(M2) There exists a \( \gamma > 1 \) such that \( t \mapsto \frac{M(t)}{t^{\gamma}} \) is non increasing for each \( t > 0 \).

(M3) For each \( b > 0 \), there exists a \( \kappa := \kappa(b) > 0 \) such that \( M(t) \geq \kappa \) whenever \( t \geq b \).

The condition (M3) asserts that the function \( M \) has possibly a zero only when \( t = 0 \).

**Remark 2.1** From (M2), we can easily deduce that \( \gamma \hat{M}(t) - M(t)t \) is non decreasing for \( t > 0 \) and

\[
\gamma \hat{M}(t) - M(t)t \geq 0 \quad \forall \, t \geq 0. \tag{2.1}
\]

We also have the following remark as a consequence of (2.1).
Remark 2.2 For each $t \geq 0$, by using (2.1) we have
\[ \frac{d}{dt} \left( \frac{\hat{M}(t)}{t^\gamma} \right) = \frac{M(t)}{t^\gamma} - \frac{\gamma \hat{M}(t)}{t^{\gamma+1}} \leq 0. \]
So the map $t \mapsto \frac{\hat{M}(t)}{t^\gamma}$ is non increasing for $t > 0$. Hence
\[ \hat{M}(t) \geq \hat{M}(1)t^\gamma \quad \text{for all } t \in [0, 1], \quad (2.2) \]
and
\[ \hat{M}(t) \leq \hat{M}(1)t^\gamma \quad \text{for all } t \geq 1. \quad (2.3) \]
We note that the condition $(M1)$ is valid whenever $M$ is non decreasing.

Example 1 Let $M(t) = m_0 + at^\gamma - 1$, where $m_0, a \geq 0$ and $\gamma > 1$ such that $m_0 + a > 0$ then $M$ satisfies the conditions $(M1) - (M3)$. If $m_0 = 0$, this forms an example of the degenerate case whereas of the non degenerate case if $m_0 > 0$.

The nonlinearity $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(x, t) = h(x, t) \exp(|t|^{\frac{n}{n-s}})$, where $h(x, t)$ satisfies the following assumptions:

$(g1)$ $h \in C^1(\Omega \times \mathbb{R})$, $h(x, t) = 0$, for all $t \leq 0$, $h(x, t) > 0$, for all $t > 0$.

$(g2)$ For any $\epsilon > 0$,
\[ \lim_{t \to \infty} \sup_{x \in \Omega} h(x, t) \exp(-\epsilon |t|^{\frac{n}{n-s}}) = 0, \]
\[ \lim_{t \to \infty} \inf_{x \in \Omega} h(x, t) \exp(\epsilon |t|^{\frac{n}{n-s}}) = \infty. \]

$(g3)$ There exist positive constants $T, T_0$ and $\gamma_0$ such that
\[ 0 < t^{\gamma_0} G(x, t) \leq T_0 g(x, t) \quad \text{for all } (x, t) \in \Omega \times [t_0, +\infty). \]

$(g4)$ For $\gamma > 1$ (defined in $(M2)$), there exists a $l > \frac{2n}{2s} - 1$ such that the map $t \mapsto \frac{g(x, t)}{t^l}$ is increasing on $\mathbb{R}^+ \setminus \{0\}$, uniformly in $x \in \Omega$.

Remark 2.3 Condition $(g4)$ implies that for each $x \in \Omega$,
\[ t \mapsto \frac{g(x, t)}{t^{\frac{n}{2s} - 1}} \quad \text{is increasing for } t > 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{g(x, t)}{t^{\frac{n}{2s} - 1}} = 0, \]
uniformly in $x \in \Omega$. Also, for each $(x, t) \in \Omega \times \mathbb{R}$ we have
\[ (l + 1)G(x, t) \leq tg(x, t). \]

Example 2 Let $g(x, t) = h(x, t)e^{\frac{\alpha |t|^{\frac{n}{n-s}}}{t^{\alpha + (\frac{n}{2s} - 1)}}}$, where $h(x, t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^{\alpha + (\frac{n}{2s} - 1)} \exp(dt^\beta) & \text{if } t > 0 \end{cases}$ for some $\alpha > 0$, $0 < d \leq \alpha_{n,s}$ and $1 \leq \beta < \frac{n}{n-s}$. Then $g$ satisfies all the conditions from $(g1) - (g4)$. 
Definition 2.4 We say that \( u \in X_0 \) is a weak solution of \((M)\) if, for all \( \phi \in X_0 \), it satisfies
\[
M(\|u\|^{\frac{n}{s}}) \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{\frac{n}{s} - 2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{2n}} \, dx \, dy = \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u)}{|x - y|^{n}} \, dy \right) g(x, u) \phi \, dx.
\]

Before stating our main Theorem, we recall a result of [18] which will be used to find an upper bound for the Mountain Pass critical level. Assume that \( 0 \in \Omega \) and \( B_1(0) \subset \Omega \). Then we consider the following Moser type functions which is given by equation (5.2) of [18]. For each \( x \in \mathbb{R}^n \) and \( k \in \mathbb{N} \),
\[
\tilde{w}_k(x) = \begin{cases} 
|\log k|^{\frac{n}{s}} & \text{if } 0 \leq |x| \leq \frac{1}{k}, \\
\frac{|\log(|x|)|}{|\log(1/k)|^{s/n}} & \text{if } \frac{1}{k} \leq |x| \leq 1, \\
0 & \text{if } |x| \geq 1,
\end{cases}
\]
then \( \text{supp}(\tilde{w}_k) \subset B_1(0) \subset \Omega \) and \( \tilde{w}_k|_{B_1(0)} \in W^{s,\frac{n}{s}}_0(B_1(0)) \).

Now by Proposition 5.1 of [18] we know that
\[
\lim_{k \to \infty} \|\tilde{w}_k\|^{\frac{n}{s}} = \lim_{k \to \infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\tilde{w}_k(x) - \tilde{w}_k(y)|^{\frac{n}{s}}}{|x - y|^{2n}} \, dx \, dy = \gamma_{n,s},
\]
where
\[
\gamma_{n,s} := \frac{2(nW_n)^2 \Gamma\left(\frac{n}{2} + 1\right)}{n!} \sum_{i=0}^{\infty} \frac{(n + i - 1)!}{i!(n + 2i)^{\frac{n}{2}}},
\]
where \( W_n \) denotes the volume of \( n \)-dimensional unit sphere. We also recall the following result of Lions known as higher integrability Lemma in case of fractional Laplacian, proved in [20].

Lemma 2.5 Let \( \{v_k : \|v_k\| = 1\} \) be a sequence in \( W^{s,\frac{n}{s}}_0(\Omega) \) converging weakly to a non-zero function \( v \). Then for every \( p \) such that \( p < \alpha_{n,s}(1 - \|v\|^{\frac{n}{s}})\),
\[
\sup_k \int_{\Omega} \exp(p\|v_k\|^{\frac{n}{s}}) < +\infty.
\]

Now we state our main result:

Theorem 2.6 Suppose \((M1) - (M3)\) and \((g1) - (g4)\) hold. Assume in addition that for \( \beta > \frac{2\alpha_{n,s}}{\alpha_{n,s}} \),
\[
\lim_{t \to +\infty} \frac{tg(x,t)G(x,t)}{\exp\left(\beta t^{\frac{n}{n-2}}\right)} = \infty \text{ uniformly in } x \in \overline{\Omega}.
\]

Then, problem \((M)\) admit a non negative non trivial solution.
3 Proof of Main result

We begin this section with the study of mountain pass structure and Palais-Smale sequences corresponding to the energy functional $J : X_0 \to \mathbb{R}$ associated to the problem (M) which is defined as

$$J(u) = \frac{s}{n} \tilde{M} \left( \|u\|_{\frac{n}{2}} \right) - \frac{1}{2} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u)}{|x - y|^\mu} dy \right) G(x, u) \, dx.$$ 

From the assumptions, $(g1) - (g4)$, we obtain that for any $\epsilon > 0$, $r \geq 1, 1 \leq \alpha < l + 1$ there exists $C(\epsilon) > 0$ such that

$$|G(x, t)| \leq \epsilon |t|^{\alpha} + C(\epsilon)|t|^{\rho} \exp((1 + \epsilon)|t|^{\frac{n}{n-\alpha}}), \text{ for all } (x, t) \in \Omega \times \mathbb{R}. \quad (3.1)$$

Now by Proposition $\text{(1.1)}$ for any $u \in X_0$ we obtain

$$\int_{\Omega} \left( \int_{\Omega} \frac{G(y, u)}{|x - y|^\mu} dy \right) G(x, u) \, dx \leq C(n, \mu) \|G(\cdot, u)\|^2_{L^{2n/(n-\mu)}(\Omega)}. \quad (3.2)$$

This implies that $J$ is well defined using Theorem $\text{(1.2)}$. Also one can easily see that $J$ is Fréchet differentiable and the critical points of $J$ are the weak solutions of (M).

Lemma 3.1 Assume that the conditions $(M1)$ and $(g1) - (g4)$ hold. Then $J$ satisfies the Mountain Pass geometry around 0.

Proof. From $(3.1)$, $(3.2)$, Hölder inequality and Sobolev embedding, we have

$$\int_{\Omega} \left( \int_{\Omega} \frac{G(y, u)}{|x - y|^\mu} dy \right) G(x, u) \, dx$$

$$\leq C(n, \mu) 2^2 \left( \epsilon^{2n/\mu} \int_{\Omega} |u|^{2n/\mu} + (C(\epsilon))^{2n/\mu} \int_{\Omega} |u|^{2n/\mu} \exp \left( \frac{2n(1 + \epsilon)}{2n - \mu} |u|^{\frac{n}{n-\alpha}} \right) \right)^{\frac{2n-\mu}{2n}}$$

$$\leq C \left( \epsilon^{2n/\mu} \int_{\Omega} |u|^{2n/\mu} + C_1(\epsilon) \|u\|_{2n/\mu}^{2n/\mu} \left( \int_{\Omega} \exp \left( \frac{4n(1 + \epsilon)}{2n - \mu} \frac{|u|^{\frac{n}{n-\alpha}}}{\|u\|_{2n/\mu}} \right) \right)^{\frac{1}{2}} \right)^{\frac{2n-\mu}{n}}. \quad (3.3)$$

So if we choose $\epsilon > 0$ small enough and $u$ such that $\frac{4n(1 + \epsilon)\|u\|_{2n/\mu}^{2n/\mu}}{2n - \mu} \leq \alpha_{n,s}$ then using the fractional Trudinger-Moser inequality $\text{(1.2)}$ in $(3.3)$, we obtain

$$\int_{\Omega} \left( \int_{\Omega} \frac{G(y, u)}{|x - y|^\mu} dy \right) G(x, u) \, dx \leq C_2(\epsilon) \left( \|u\|_{2n/\mu}^{2n/\mu} + \|u\|_{2n/\mu}^{2n/\mu} \right)^{2n-\mu}$$

$$\leq C_3(\epsilon) \left( \|u\|^{2\alpha} + \|u\|^{2r} \right).$$

Using $(3.2)$ and above estimate, we have

$$J(u) \geq \frac{s}{n} \tilde{M}(1) \|u\|_{\frac{n}{2}}^{\frac{n}{2}} - C_3(\epsilon) \left( \|u\|^{2\alpha} + \|u\|^{2r} \right),$$

when $\|u\| \leq 1$. Choosing $\alpha > \frac{m}{2s}, r > \frac{m}{2s}$ and $\rho > 0$ such that $\rho \leq \min \left\{ 1, \left( \frac{\alpha_{n,s}(2n-\mu)}{4n(1+\epsilon)} \right)^{\frac{n-\mu}{n}} \right\}$ we obtain $J(u) \geq \sigma > 0$ for all $u \in X_0$ with $\|u\| = \rho$ and for some $\sigma > 0$ depending on $\rho$. 

\[ \text{Proof end.} \]
The condition (g4) implies that there exist some positive constants $C_1$ and $C_2$ such that
\begin{equation}
G(x,t) \geq C_1 t^{l+1} - C_2 \quad \text{for all } (x,t) \in \Omega \times [0,\infty).
\end{equation}

Let $\phi \in X_0$ such that $\phi \geq 0$ and $\|\phi\| = 1$ then by (3.4) we obtain
\begin{align*}
\int_\Omega \left( \int_\Omega \frac{G(y,t\phi)}{|x-y|^\mu} dy \right) G(x,t\phi) \, dx & \geq \int_\Omega \int_\Omega \frac{(C_1(t\phi)^{l+1}(y) - C_2)(C_1(t\phi)^{l+1}(x) - C_2)}{|x-y|^\mu} \, dx \, dy \\
& = C_1^2 t^{2(l+1)} \int_\Omega \int_\Omega \frac{|\phi^{l+1}(y)\phi^{l+1}(x)|}{|x-y|^\mu} \, dx \, dy \\
& \quad - 2C_1C_2 t^{l+1} \int_\Omega \int_\Omega \frac{|\phi^{l+1}(y)|}{|x-y|^\mu} \, dx \, dy + C_2^2 \int_\Omega \int_\Omega \frac{1}{|x-y|^\mu} \, dx \, dy.
\end{align*}

This together with (2.3), we obtain
\begin{align*}
J(t\phi) & \leq \frac{s}{n} M(1)\|t\phi\|^{2n} - \frac{1}{2} \int_\Omega \left( \int_\Omega \frac{G(y,t\phi)}{|x-y|^\mu} dy \right) G(x,t\phi) \, dx \\
& \leq C_3 + C_4 t^{\frac{2n}{1-\mu}} - C_5 t^{2(l+1)} + C_6 t^{l+1},
\end{align*}

where $C_i's$ are positive constants for $i = 3, 4, 5, 6$. This implies that $J(t\phi) \to -\infty$ as $t \to \infty$, since $l+1 > \frac{2n}{1-\mu}$. Thus there exists a $v_0 \in X_0$ with $\|v_0\| > \rho$ such that $J(v_0) < 0$. Therefore, $J$ satisfies Mountain Pass geometry near zero.

Let $\Gamma = \{ \gamma \in C([0,1], X_0) : \gamma(0) = 0, J(\gamma(1)) < 0 \}$ and define the Mountain Pass critical level $c_* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$. Then by Lemma 3.1 and the Mountain pass theorem we know that there exists a Palais Smale sequence $\{u_k\} \subset X_0$ for $J$ at $c_*$ that is
\[ J(u_k) \to c_* \text{ and } J'(u_k) \to 0 \text{ as } k \to \infty. \]

**Lemma 3.2** Every Palais-Smale sequence of $J$ is bounded in $X_0$.

**Proof.** Let $\{u_k\} \subset X_0$ denotes a $(PS)_c$ sequence of $J$ that is
\[ J(u_k) \to c \text{ and } J'(u_k) \to 0 \text{ as } k \to \infty \]
for some $c \in \mathbb{R}$. This implies
\begin{align*}
\frac{sM(\|u_k\|_2^n)}{n} & - \frac{1}{2} \int_\Omega \left( \int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) G(x,u_k) \, dx \to c \text{ as } k \to \infty, \\
|\lambda(\|u_k\|_2^n)\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u_k(x) - u_k(y)|^{\frac{2n}{1-\mu}}(u_k(x) - u_k(y))(\phi(x) - \phi(y)) |x-y|^{2n} \, dx \, dy & \quad - \int_\Omega \left( \int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} dy \right) g(x,u_k) \phi \, dx \leq \epsilon_k \|\phi\| \tag{3.5}
\end{align*}
where $\epsilon_k \to 0$ as $k \to \infty$. In particular, taking $\phi = u_k$ we get

$$\left| M\left( \|u_k\|^{\frac{2}{s}} \right) \|u_k\|^{\frac{2}{s}} - \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) g(x, u_k) \, dx \right| \leq \epsilon_k \|u_k\|. \quad (3.6)$$

Now Remark (2.3) gives us that

$$(l + 1) \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) G(x, u_k) \, dx \leq \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) g(x, u_k) u_k \, dx. \quad (3.7)$$

Then using (3.5), (3.6) along with (3.7) and (2.1), we get

$$J(u_k) - \frac{1}{2(l+1)} \langle J'(u_k), u_k \rangle = \frac{s}{n} \tilde{M}(\|u_k\|^{\frac{2}{s}}) - \frac{1}{2(l+1)} M(\|u_k\|^{\frac{2}{s}}) \|u_k\|^{\frac{2}{s}}$$

$$\quad - \frac{1}{2} \left[ \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) G(x, u_k) \, dx \right] - \frac{1}{(l+1)} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) g(x, u_k) u_k \, dx$$

$$\geq \frac{s \tilde{M}(\|u_k\|^{\frac{2}{s}})}{n} - \frac{1}{2(l+1)} M(\|u_k\|^{\frac{2}{s}}) \|u_k\|^{\frac{2}{s}}$$

$$\geq \left( \frac{s}{n\gamma} - \frac{1}{2(l+1)} \right) M(\|u_k\|^{\frac{2}{s}}) \|u_k\|^{\frac{2}{s}}. \quad (3.8)$$

To prove the Lemma, we assume by contradiction that $\{\|u_k\|\}$ is an unbounded sequence. Then without loss of generality, we can assume that, up to a subsequence, $\|u_k\| \to \infty$ and $\|u_k\| \geq \alpha > 0$ for some $\alpha$ and for all $k$. This along with (3.8) and (M3) gives us

$$J(u_k) - \frac{1}{2(l+1)} \langle J'(u_k), u_k \rangle \geq \left( \frac{s}{n\gamma} - \frac{1}{2(l+1)} \right) \kappa \|u_k\|^{\frac{2}{s}} \quad (3.9)$$

where $\kappa$ depends on $\alpha$. Also from (3.5) and (3.6) it follows that

$$J(u_k) - \frac{1}{2(l+1)} \langle J'(u_k), u_k \rangle \leq C \left( 1 + \frac{\epsilon_k}{2(l+1)} \right) \frac{\|u_k\|}{2(l+1)} \quad (3.10)$$

for some constant $C > 0$. Therefore from (3.9) and (3.10) we get that

$$\left( \frac{s}{n\gamma} - \frac{1}{2(l+1)} \right) \kappa \|u_k\|^{\frac{2}{s}} \leq C \left( 1 + \frac{\epsilon_k}{2(l+1)} \right)$$

which gives a contradiction because $l + 1 > \frac{7n}{2s}$ and $\frac{n}{s} > 1$. This implies that $\{u_k\}$ must be bounded in $X_0$. □

Assume that $0 \in \Omega$ and $\rho > 0$ be such that $B_\rho(0) \subset \Omega$. Then for $x \in \mathbb{R}^n$, we define $w_k(x) := \tilde{w}_k \left( \frac{x}{\rho} \right)$, where $\tilde{w}_k$ is same as (2.4) then supp$(w_k) \subset B_\rho(0) \subset \Omega$. We note that $w_k \in W_0^{s, \frac{2}{s}}(\mathbb{R}^n)$ and by (2.5), we have

$$\lim_{k \to \infty} \|w_k\|_{\frac{2}{s}} = \lim_{k \to \infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\tilde{w}_k(x) - \tilde{w}_k(y)|^{\frac{2}{s}}}{|x-y|^{2n}} \, dx \, dy = \gamma_{n,s}. \quad (3.11)$$

Next, we use $w_k$’s efficiently to obtain the following bound on $c_\ast$. 

\[ \]
Lemma 3.3  It holds that

\[ 0 < c_* < \frac{s}{n} \hat{M} \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right)^{\frac{n-s}{n}}. \]

**Proof.** Using Lemma 3.1 we deduce that \( c_* > 0 \) and \( J(t\phi) \to -\infty \) as \( t \to \infty \) if \( 0 \leq \phi \in X_0 \setminus \{0\} \) with \( \|\phi\| = 1 \). Also by definition of \( c_* \), we have \( c_* \leq \max_{t \in [0,1]} J(t\phi) \) for each non negative \( \phi \in X_0 \setminus \{0\} \) with \( J(\phi) < 0 \) which assures that it is enough to prove that there exists a non negative \( w \in X_0 \setminus \{0\} \) such that

\[ \max_{t \in [0,\infty)} J(tw) < \frac{s}{n} \hat{M} \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right)^{\frac{n-s}{n}}. \]

To prove this, we consider the sequence of non negative functions \( \{w_k\} \) (defined before this Lemma) and claim that there exists a \( k \in \mathbb{N} \) such that

\[ \max_{t \in [0,\infty)} J(tw_k) < \frac{s}{n} \hat{M} \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right)^{\frac{n-s}{n}}. \]

Suppose this is not true, then for all \( k \in \mathbb{N} \) there exists a \( t_k > 0 \) such that

\[ \max_{t \in [0,\infty)} J(tw_k) = J(tkw_k) \geq \frac{s}{n} \hat{M} \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right)^{\frac{n-s}{n}} \]

and \( \frac{d}{dt} (J(tw_k))|_{t=t_k} = 0. \) (3.12)

From the proof of Lemma 3.1, \( J(tw_k) \to -\infty \) as \( t \to \infty \) for each \( k \). Then we infer that \( \{t_k\} \) must be a bounded sequence in \( \mathbb{R} \) which implies that there exists a \( t_0 \) such that, up to a subsequence which we still denote by \( \{t_k\} \), \( t_k \to t_0 \) as \( k \to \infty \). From (3.12) and definition of \( J(tkw_k) \) we obtain

\[ \frac{s}{n} \hat{M} \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right)^{\frac{n-s}{n}} < \frac{s}{n} \hat{M} (\|tw_k\|_n^{\frac{n}{n}}). \]

(3.13)

Since \( \hat{M} \) is monotone increasing, from (3.13) we get that

\[ \|tw_k\|_n^{\frac{n}{n}} \geq \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right)^{\frac{n-s}{n}}. \]

(3.14)

From (3.14) and since (3.11) holds, we infer that

\[ t_k (\log k)^{\frac{n-s}{n}} \to \infty \text{ as } k \to \infty. \]

(3.15)

Furthermore from (3.12), we have

\[ M(\|tw_k\|_n^{\frac{n}{n}}) \|tw_k\|_n^{\frac{n}{n}} = \int_\Omega \left( \int_\Omega \frac{G(y, tkw_k)}{|x-y|^\mu} dy \right) g(x, tkw_k) tkw_k \ dx \]

\[ \geq \int_{B_{\rho/k}} g(x, tkw_k) tkw_k \int_{B_{\rho/k}} \frac{G(y, tkw_k)}{|x-y|^\mu} dy \ dx. \]

(3.16)
In addition, as in equation (2.11) p. 1943 in [1], it is easy to get that
\[
\int_{B_{\rho/k}} \int_{B_{\rho/k}} \frac{dx\,dy}{|x-y|^\mu} \geq C_{\mu,n} \left( \frac{\rho}{k} \right)^{2n-\mu},
\]
where \( C_{\mu,n} \) is a positive constant depending on \( \mu \) and \( n \). From (2.14), it is easy to deduce that for \( \beta > \frac{2\alpha_{n,s}}{\alpha_{n,s}} \) and for each \( d > 0 \) there exists a \( r_d \in \mathbb{N} \) such that
\[
gr(x,r)G(x,r) \geq d \exp \left( \beta r \left( \frac{n}{n-s} \right) \right)
\]
whenever \( r \geq r_d \).

Since (3.15) holds, we can choose a \( N_d \in \mathbb{N} \) such that
\[
t_k (\log k)^{\frac{n-s}{n}} \geq r_d \text{ for all } k \geq N_d.
\]
Using these estimates in (3.16) and from (3.14), for \( d \) large enough we get that
\[
M(\|t_k w_k\|^{\frac{n}{n-s}}) \|t_k w_k\|^{\frac{n}{n-s}} \geq d \exp \left( \beta t_k^{\frac{n}{n-s}} \log k \right) C_{\mu,n} \left( \frac{\rho}{k} \right)^{2n-\mu}
= dC_{\mu,n} \rho^{2n-\mu} \exp \left( \left( \frac{\beta t_k^{\frac{n}{n-s}}}{\rho} - (2n-\mu) \right) \log k \right)
\geq dC_{\mu,n} \rho^{2n-\mu} \exp \left( \log k \left( \frac{(2n-\mu)\beta_{\alpha_{n,s}}}{2n\|w_k\|^{\frac{n}{n-s}}} - (2n-\mu) \right) \right)
\]
Since \( \beta > \frac{2\alpha_{n,s}}{\alpha_{n,s}} = \frac{2n\gamma_{n,s}}{\alpha_{n,s}} \) and (3.11) hold, the R.H.S. of (3.17) tends to \(+\infty\) as \( k \to \infty \). Whereas from continuity of \( M \) it follows that
\[
\lim_{k \to \infty} M(\|t_k w_k\|^{\frac{n}{n-s}}) \|t_k w_k\|^{\frac{n}{n-s}} = M \left( t_k^{\frac{n}{n-s}} \gamma_{n,s} \right) \left( t_k^{\frac{n}{n-s}} \gamma_{n,s} \right),
\]
which is a contradiction. This establishes our claim and we conclude the proof of Lemma. \( \square \)

In order to prove that a Palais-Smale sequence converges to a weak solution of problem (\( M \)), we need the following convergence Lemma. The idea of proof is borrowed from Lemma 2.4 in [1].

**Lemma 3.4** If \( \{u_k\} \) is a Palais Smale sequence for \( J \) at \( c \) then there exists a \( u \in X_0 \) such that, up to a subsequence,
\[
\left( \int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} \, dy \right) G(x,u_k) \rightharpoonup \left( \int_\Omega \frac{G(y,u)}{|x-y|^\mu} \, dy \right) G(x,u) \text{ in } L^1(\Omega)
\]
**Proof.** From Lemma 3.2, we know that the sequence \( \{u_k\} \) must be bounded in \( X_0 \). Consequently, up to a subsequence, there exists a \( u \in X_0 \) such that \( u_k \to u \) weakly in \( X_0 \) and strongly in \( L^q(\Omega) \) for any \( q \in [1,\infty) \) as \( k \to \infty \). Also, still up to a subsequence, we can assume that \( u_k(x) \to u(x) \) pointwise a.e. for \( x \in \Omega \).

From (3.14), (3.15) and (3.17) we get that there exists a constant \( C > 0 \) such that
\[
\int_\Omega \left( \int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} \, dy \right) G(x,u_k) \, dx \leq C,
\int_\Omega \left( \int_\Omega \frac{G(y,u_k)}{|x-y|^\mu} \, dy \right) g(x,u_k) \, dx \leq C.
\]
Now, it is well known that if \( f \in L^1(\Omega) \) then for any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that
\[
\left| \int_U f(x) \, dx \right| < \epsilon,
\]
for any measurable set \( U \subset \Omega \) with \( |U| \leq \delta \). Also \( f \in L^1(\Omega) \) implies that for any fixed \( \delta > 0 \) there exists \( M > 0 \) such that
\[
|\{ x \in \Omega : |f(x)| \geq M \}| \leq \delta.
\]
Now using (3.19), we have
\[
\left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) G(\cdot, u_k) \in L^1(\Omega)
\]
and also by (3.2)
\[
\left( \int_{\Omega} \frac{G(y, u)}{|x-y|^\mu} \, dy \right) G(\cdot, u) \in L^1(\Omega).
\]
Now we fix \( \delta > 0 \) and choose \( M > \max \left\{ \left( \frac{C_{\mu}}{\delta} \right)^{\frac{1}{\mu+1}}, t_0 \right\} \). Then we use (g3) to obtain
\[
\int_{\Omega \cap \{ u_k \geq M \}} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) G(x, u_k) \, dx \leq T_0 \int_{\Omega \cap \{ u_k \geq M \}} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) \frac{g(x, u_k)}{u_k^{\gamma}} \, dx
\]
\[
\leq \frac{T_0}{M^{\gamma+1}} \int_{\Omega \cap \{ u_k \geq M \}} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) g(x, u_k) u_k \, dx < \delta.
\]
Next we consider
\[
\left| \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) G(x, u_k) \, dx - \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u)}{|x-y|^\mu} \, dy \right) G(x, u) \, dx \right|
\]
\[
\leq 2\delta + \int_{\Omega \cap \{ u_k \leq M \}} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) G(x, u_k) \, dx - \int_{\Omega \cap \{ u_k \leq M \}} \left( \int_{\Omega} \frac{G(y, u)}{|x-y|^\mu} \, dy \right) G(x, u) \, dx
\]
To prove the result, it is enough to establish that as \( k \to \infty \)
\[
\int_{\Omega \cap \{ u_k \leq M \}} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} \, dy \right) G(x, u_k) \, dx \to \int_{\Omega \cap \{ u \leq M \}} \left( \int_{\Omega} \frac{G(y, u)}{|x-y|^\mu} \, dy \right) G(x, u) \, dx. \quad (3.20)
\]
Since \( \left( \int_{\Omega} \frac{G(y, u)}{|x-y|^\mu} \, dy \right) G(\cdot, u) \in L^1(\Omega) \), so by Fubini’s theorem we get
\[
\lim_{K \to \infty} \int_{\Omega \cap \{ u_k \leq M \}} \left( \int_{\Omega \cap \{ u \geq K \}} \frac{G(y, u)}{|x-y|^\mu} \, dy \right) G(x, u) \, dx
\]
\[
= \lim_{K \to \infty} \int_{\Omega \cap \{ u \geq K \}} \left( \int_{\Omega \cap \{ u \leq M \}} \frac{G(y, u)}{|x-y|^\mu} \, dy \right) G(x, u) \, dx = 0.
\]
Thus we can fix a $K > \max \left\{ \left( \frac{C_M}{\delta} \right)^{\frac{1}{\gamma+1}}, t_0 \right\}$ such that
\[
\int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \geq K\}} \frac{G(y, u_k)}{|x-y|^{\mu}} dy \right) G(x, u) \, dx \leq \delta.
\]
From (g3), we get
\[
\int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \geq K\}} \frac{G(y, u_k)}{|x-y|^{\mu}} dy \right) G(x, u_k) \, dx
\leq \frac{1}{K^{\gamma+1}} \int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \geq K\}} \frac{u_k^{\gamma+1}(y)G(y, u_k)}{|x-y|^{\mu}} dy \right) G(x, u_k) \, dx
\leq \frac{T_0}{K^{\gamma+1}} \int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \geq K\}} \frac{u_k(y)g(y, u_k)}{|x-y|^{\mu}} dy \right) G(x, u_k) \, dx
\leq \frac{T_0}{K^{\gamma+1}} \int_{\Omega} \left( \int \frac{G(y, u_k)}{|x-y|^{\mu}} dy \right) g(x, u_k) u_k \, dx \leq \delta.
\]
Thus we have proved that
\[
\left| \int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \geq K\}} \frac{G(y, u)}{|x-y|^{\mu}} dy \right) G(x, u) \, dx
- \int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \geq K\}} \frac{G(y, u_k)}{|x-y|^{\mu}} dy \right) G(x, u_k) \, dx \right| \leq 2\delta
\]
Finally, to complete the proof of Lemma, we need to verify that as $k \to \infty$
\[
\left| \int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \leq K\}} \frac{G(y, u_k)}{|x-y|^{\mu}} dy \right) G(x, u_k) \, dx
- \int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \leq K\}} \frac{G(y, u)}{|x-y|^{\mu}} dy \right) G(x, u) \, dx \right| \to 0
\] (3.21)
for fixed positive $K$ and $M$. It is easy to see that
\[
\left( \int_{\Omega \cap \{u \leq K\}} \frac{G(y, u_k)}{|x-y|^{\mu}} dy \right) G(x, u_k) \chi_{\{u \leq K\}} \to \left( \int_{\Omega \cap \{u \leq K\}} \frac{G(y, u)}{|x-y|^{\mu}} dy \right) G(x, u) \chi_{\{u \leq K\}}
\]
pointwise a.e. as $k \to \infty$. Now choose $r = \alpha$ in (5.1), which gives us that there exist a constant $C_{M,K} > 0$ depending on $M$ and $K$ such that
\[
\int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \leq K\}} \frac{G(y, u_k)}{|x-y|^{\mu}} dy \right) G(x, u_k) \, dx
\leq C_{M,K} \int_{\Omega \cap \{u \leq M\}} \left( \int_{\Omega \cap \{u \leq K\}} \frac{|u_k(y)|^r}{|x-y|^{\mu}} dy \right) |u_k(x)|^r \, dx
\leq C_{M,K} \int_{\Omega} \left( \int \frac{|u_k(y)|^r}{|x-y|^{\mu}} dy \right) |u_k(x)|^r \, dx
\leq C_{M,K} \|u_k\|_{L^{2r\mu}}^2 \to C_{M,K} \|u\|_{L^{2r\mu}}^2 \text{ as } k \to \infty,
\]
where we used the Hardy-Littlewood-Sobolev inequality in the last inequality and then used the fact that \( u_k \to u \) strongly in \( L^q(\Omega) \) for each \( q \in [1, \infty) \). This implies that, using Theorem 4.9 of [4], there exists a constant \( h \in L^1(\Omega) \) such that, up to a subsequence, for each \( k \)

\[
\left| \left( \int_{\Omega \cap \{u_k \leq K\}} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) G(x, u_k) \chi_{\Omega \cap \{u_k \leq M\}} \right| \leq |h(x)|
\]

This helps us to employ the Lebesgue dominated convergence theorem and conclude (3.21).

\[\Box\]

**Lemma 3.5** Let \( \{u_k\} \subset X_0 \) be a Palais Smale sequence of \( J \). Then there exists a \( u \in X_0 \) such that, up to a subsequence, for all \( \phi \in X_0 \)

\[
\int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) \phi \, dx \to \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u)}{|x - y|^\mu} dy \right) g(x, u) \phi \, dx \quad \text{as} \quad k \to \infty . \tag{3.22}
\]

**Proof.** As we argued in previous Lemma, we have that there exists a \( u \in X_0 \) such that, up to a subsequence, \( u_k \rightrightarrows u \) weakly in \( X_0 \), \( u_k \to u \) pointwise a.e. in \( \mathbb{R}^n \), \( \|u_k\| \rightarrow \tau \) as \( k \to \infty \), for some \( \tau \geq 0 \) and \( u_k \to u \) strongly in \( L^q(\Omega) \), \( q \in [1, \infty) \) as \( k \to \infty \).

Let \( \Omega' \subset \subset \Omega \) and \( \varphi \in C_c^\infty(\Omega') \) such that \( 0 \leq \varphi \leq 1 \) and \( \varphi \equiv 1 \) in \( \Omega' \). Then by taking \( \varphi \) as a test function in (3.5), we obtain the following estimate

\[
\int_{\Omega'} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) \, dx \leq \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) \varphi \, dx \\
\leq \varepsilon_k \|\varphi\| + M(\|u_k\|^{\frac{\mu}{2}}) \int_{\mathbb{R}^n} \frac{|u_k(x) - u_k(y)|^{\frac{\mu}{2}} - 2(u_k(x) - u_k(y))(\varphi(x) - \varphi(y))}{|x - y|^{2\mu}} \, dx \, dy \\
\leq \varepsilon_k \|\varphi\| + C \|u_k\| \|\varphi\| \leq C,
\]

since \( \|u_k\| \leq C_0 \) for all \( k \). This implies that the sequence \( \{\mu_k\} := \left\{ \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right\} g(x, u_k) \) is bounded in \( L^1_{\text{loc}}(\Omega) \) which implies that up to a subsequence, \( \mu_k \to \mu \) in the weak\(^*\)-topology as \( k \to \infty \), where \( \mu \) denotes a Radon measure. So for any \( \phi \in C_c^\infty(\Omega) \) we get

\[
\lim_{k \to \infty} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) \phi \, dx = \int_{\Omega} \phi \, d\mu, \quad \forall \phi \in C_c^\infty(\Omega).
\]

Since \( u_k \) satisfies (3.5), for any measurable set \( E \subset \Omega \), taking \( \phi \in C_c^\infty(\Omega) \) such that \( \text{supp} \phi \subset E \), we get that

\[
\mu(E) = \int_E \phi \, d\mu = \lim_{k \to \infty} \int_{E} \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right\} g(x, u_k) \phi(x) \, dx \\
= \lim_{k \to \infty} \int_{\Omega} \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right\} g(x, u_k) \phi(x) \, dx \\
= \lim_{k \to \infty} M(\|u_k\|^{\frac{\mu}{2}}) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u_k(x) - u_k(y)|^{\frac{\mu}{2}} - 2(u_k(x) - u_k(y))(\phi(x) - \phi(y))}{|x - y|^{2\mu}} \, dx \, dy \\
= M(\tau^{\frac{\mu}{2}}) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{\frac{\mu}{2}} - 2(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{2\mu}} \, dx \, dy,
\]
where we used the continuity of $M$ and weak convergence of $u_k$ to $u$ in $X_0$. This implies that
\[ \mu \text{ is absolutely continuous with respect to the Lebesgue measure.} \]
Theorem 1.2 establishes that there exists a function $h \in L^1_{\text{loc}}(\Omega)$ such that for any $\phi \in C^\infty_c(\Omega)$,
\[ \int_{\Omega} \phi \, d\mu = \int_{\Omega} \phi h \, dx. \]
Therefore for any $\phi \in C^\infty_c(\Omega)$ we get
\[ \lim_{k \to \infty} \int_{\Omega} \left( \int_{\Omega} G(y, u_k) \, dy \right) g(x, u_k) \phi \, dx = \int_{\Omega} \phi h \, dx = \int_{\Omega} \left( \int_{\Omega} G(y, u) \, dy \right) g(x, u) \phi \, dx \]
and the above holds for any $\phi \in X_0$ using the density argument. This completes the proof. \[ \square \]

Now we define the Nehari manifold associated to the functional $J$, as
\[ \mathcal{N} := \{ 0 \neq u \in X_0 : \langle J'(u), u \rangle = 0 \} \]
and let $b := \inf_{u \in \mathcal{N}} J(u)$. Then we need the following Lemma to compare $c_s$ and $b$.

**Lemma 3.6** If condition (g4) holds, then for each $x \in \Omega$, $tg(x, t) - \frac{\gamma n}{2s} G(x, t)$ is increasing for $t \geq 0$. In particular $tg(x, t) - \frac{\gamma n}{2s} G(x, t) \geq 0$ for all $(x, t) \in \Omega \times [0, \infty)$ which implies $\frac{G(x, t)}{t^{\frac{n}{2s}}}$ is non-decreasing for $t > 0$.

**Proof.** Suppose $0 < t < r$. Then for each $x \in \Omega$, we obtain
\[ tg(x, t) - \frac{\gamma n}{2s} G(x, t) \]
\[ \leq \frac{g(x, t)}{t^l} t^{l+1} - \frac{\gamma n}{2s} G(x, r) + \frac{\gamma n}{2s} \int_t^r g(x, \tau) \, d\tau \]
\[ \leq r g(x, r) - \frac{\gamma n}{2s} G(x, r), \]
using (g4). This completes the proof. \[ \square \]

**Lemma 3.7** Under the assumptions (M2) and (g4), it holds $c_s \leq b$.

**Proof.** Let $u \in \mathcal{N}$ be non negative and we define $h : (0, \infty) \to \mathbb{R}$ by $h(t) = J(tu)$. Then for all $t > 0$
\[ h'(t) = \langle J'(tu), u \rangle = M(t^\frac{\mu}{\gamma} \|u\|_{\gamma}^\frac{\mu}{\gamma}) t^{\frac{n}{2s} - 1} \|u\|_{\frac{n}{2s}} - \int_{\Omega} \left( \int_{\Omega} G(y, tu) \, dy \right) g(x, tu) u \, dx. \]

Since $\langle J'(u), u \rangle = 0$ and $t \mapsto \frac{g(x, tu)}{t^{\frac{n}{2s} - 1}}$ is increasing for $t > 0$, we have
\[ h'(t) = \|u\|_{\frac{n}{2s}}^\frac{\mu}{\gamma} t^{\frac{n}{2s} - 1} \left( \frac{M(t^\frac{\mu}{\gamma} \|u\|_{\gamma}^\frac{\mu}{\gamma})}{t^{(\gamma - 1)\frac{\mu}{\gamma}} \|u\|_{\gamma}^{(\gamma - 1)\frac{\mu}{\gamma}}} - \frac{M(\|u\|_{\gamma}^\frac{\mu}{\gamma})}{\|u\|_{\gamma}^{(\gamma - 1)\frac{\mu}{\gamma}}} \right) \]
\[ + t^{\frac{n}{2s} - 1} \int_{\Omega} \left( \int_{\Omega} G(y, tu) \, dy - \int_{\Omega} G(y, tu) \frac{G(y, tu)}{(tu)^{\frac{n}{2s} - 1}(x)} \, dy \right) u^{\frac{\mu}{\gamma}}(x) \, dx \]
\[ \geq \|u\|_{\frac{n}{2s}}^\frac{\mu}{\gamma} t^{\frac{n}{2s} - 1} \left( \frac{M(t^\frac{\mu}{\gamma} \|u\|_{\gamma}^\frac{\mu}{\gamma})}{t^{(\gamma - 1)\frac{\mu}{\gamma}} \|u\|_{\gamma}^{(\gamma - 1)\frac{\mu}{\gamma}}} - \frac{M(\|u\|_{\gamma}^\frac{\mu}{\gamma})}{\|u\|_{\gamma}^{(\gamma - 1)\frac{\mu}{\gamma}}} \right) \]
\[ + t^{\frac{n}{2s} - 1} \int_{\Omega} \left( \int_{\Omega} G(y, u) - \frac{G(y, tu)}{t^\frac{\mu}{\gamma}} \right) \frac{1}{|x - y|^\mu} \, dy \right) \frac{g(x, tu)}{(tu)^{\frac{n}{2s} - 1}(x)} u^{\frac{\mu}{\gamma}}(x) \, dx. \]
when \(0 < t < 1\). So using Lemma 3.6 and (M2) we have \(h'(1) = 0, h'(t) \geq 0\) for \(0 < t < 1\) and \(h'(t) < 0\) for \(t > 1\). Hence \(J(u) = \max_{t \geq 0} J(tu)\). Now define \(f : [0, 1] \to X_0\) as \(f(t) = (t_0u)t\), where \(t_0 > 1\) is such that \(J(t_0u) < 0\). Then we have \(f \in \Gamma\) and therefore

\[
c_* \leq \max_{t \in [0,1]} J(f(t)) \leq \max_{t \geq 0} J(tu) = J(u) \leq \inf_{u \in \mathcal{N}} J(u) = b.
\]

Hence the proof is complete. \(\square\)

**Definition 3.8** A solution \(u_0\) of (M) is a ground state if \(u_0\) is a weak solution of (M) and satisfies \(J(u_0) = \inf_{u \in \mathcal{N}} J(u)\).

Since \(c_* \leq b\) in order to obtain a ground state solution \(u_0\) for (M), it is enough to show that there exists a weak solution of (M) such that \(J(u_0) = c_*\).

**Lemma 3.9** Any nontrivial solution of problem (M) is nonnegative.

**Proof.** Let \(u \in X_0 \setminus \{0\}\) be a critical point of functional \(J\). Clearly \(u^- = \max\{-u, 0\} \in X_0\). Then \(\langle J'(u), u^- \rangle = 0\), i.e.

\[
M(||u||^\frac{2}{\mu}) \int_{\mathbb{R}^n} \left| u(x) - u(y) \right|^\frac{\mu-2}{\mu} (u(x) - u(y))(u^- (x) - u^- (y)) \frac{dx dy}{|x-y|^{2n}}
\]

\[
= \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u)}{|x-y|^\mu} dy \right) g(x, u) u^- dx.
\]

For a.e. \(x, y \in \mathbb{R}^n\), using \(\left| u^- (x) - u^- (y) \right| \leq |u(x) - u(y)|\), we have

\[
\left| u(x) - u(y) \right|^\frac{\mu-2}{\mu} (u(x) - u(y))(u^- (x) - u^- (y))
\]

\[
= -\left| u(x) - u(y) \right|^\frac{\mu-2}{\mu} (u^+ (x) u^- (y) + u^- (x) u^+ (y) + u^- (x) - u^- (y))^2
\]

\[
\leq -\left| u^- (x) - u^- (y) \right|^\frac{\mu}{\mu}
\]

and \(g(x, u)u^- = 0\) a.e. \(x \in \Omega\) by assumption. Hence,

\[
0 \leq -M(||u||^\frac{2}{\mu}) ||u^-||^\frac{\mu}{\mu} \leq 0.
\]

So, \(u^- \equiv 0\) since \(||u|| > 0\) and (M3) holds. Hence \(u \geq 0\) a.e. in \(\Omega\). \(\square\)

**Proof of Theorem 2.6** Since \(J\) satisfies the Mountain Pass geometry (refer Lemma 3.1), by Mountain Pass Lemma we know that there exists a Palais Smale \(\{u_k\}\) sequence for \(J\) at \(c_*\). Then by Lemma 3.7 \(\{u_k\}\) must be bounded in \(X_0\) so that, up to a subsequence, \(u_k \rightharpoonup u_0\) weakly in \(X_0\), strongly in \(L^q(\Omega)\) for \(q \in [1, \infty)\), pointwise a.e. in \(\Omega\), for some \(u_0 \in X_0\) and \(||u_k|| \to \rho_0 \geq 0\) as \(k \to \infty\).

**Claim 1:** \(u_0 \equiv 0\) in \(\Omega\).

**Proof.** We argue by contradiction. Suppose that \(u_0 \equiv 0\). Then using Lemma 3.4 we obtain

\[
\int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^\mu} dy \right) G(x, u_k) dx \to 0 \text{ as } k \to \infty.
\]

(3.23)
This together with \( \lim_{k \to \infty} J(u_k) = c_\star \) gives that
\[
\lim_{k \to \infty} \frac{s}{n} \hat{M}(\|u_k\|^{\frac{n}{s}}) = c_\star < \frac{s}{n} \hat{M} \left( \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right)^{\frac{n}{s-2}} \right).
\]

Thus \( \hat{M} \) being increasing function gives that there exists a \( k_0 \in \mathbb{N} \) such that \( \|u_k\|^{\frac{n}{s}} \leq \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right)^{\frac{n}{s-2}} \) for all \( k \geq k_0 \). We fix \( k \geq k_0 \) and choose \( p > 1 \) close to 1 and \( \epsilon > 0 \) small enough such that
\[
\frac{2np(1 + \epsilon)}{2n - \mu} \|u_k\|^{\frac{n}{s}} < \alpha_{n,s}.
\]

Using the growth assumptions on \( g \) and Theorem 1.2 we have
\[
\|g(\cdot, u_k)u_k\|_{L^{\frac{2n}{n-\mu}}(\Omega)} \leq C(\epsilon) \left( \int_{\Omega} |u_k|^{\frac{2n \mu}{n-\mu}} dx + \int_{\Omega} |u_k|^{\frac{2n \mu}{n-\mu}} \exp \left( \frac{2n(1 + \epsilon)}{2n - \mu} \|u_k\|^{\frac{n}{s}} \right) dx \right)
\]
\[
\leq C(\epsilon) \left( \int_{\Omega} |u_k|^{\frac{2n \mu}{n-\mu}} dx + \left( \int_{\Omega} |u_k|^{\frac{2n \mu'}{n-\mu'}} dx \right)^{\frac{1}{p'}} \right.
\]
\[
\left. \left( \int_{\Omega} \exp \left( \frac{2np(1 + \epsilon)}{2n - \mu} \|u_k\|^{\frac{n}{s}} \left( \frac{|u_k|}{\|u_k\|} \right)^{\frac{n}{s-2}} \right) dx \right)^{\frac{1}{p'}} \right)
\]
where \( 1 < \alpha < l + 1 \) and \( 1 < r \). Thus,
\[
\|g(\cdot, u_k)u_k\|_{L^{\frac{2n}{n-\mu}}(\Omega)} \leq C(\epsilon) \left( \|u_k\|_{L^{\frac{2n \mu}{n-\mu}}(\Omega)} + \|u_k\|_{L^{\frac{2n \mu}{n-\mu'}}(\Omega)} \right) \to 0 \text{ as } k \to \infty,
\]
(3.24)

where \( p' \) denotes the Hölder conjugate of \( p \) and \( C(\epsilon) > 0 \) is a constant depending on \( \epsilon \) which may change value at each step. From the semigroup property of the Riesz potential and Hardy-Littlewood-Sobolev inequality we get that
\[
\left| \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) u_k dx \right|
\]
\[
\leq \left( \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) G(x, u_k) dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \left( \int_{\Omega} \frac{g(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) u_k dx \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) G(x, u_k) dx \right)^{\frac{1}{2}} C_{n,\mu} \|g(\cdot, u_k)u_k\|_{L^{\frac{2n}{n-\mu}}(\Omega)} \to 0
\]
as \( k \to \infty \) using (3.23) and (3.24). This together with \( \langle J'(u_k), u_k \rangle = 0 \) implies that
\[
M(\|u_k\|^{\frac{n}{s}})\|u_k\|^{\frac{n}{s}} \to 0.
\]
From (M3), we deduce that \( \|u_k\| \to 0 \). Furthermore, we obtain
\[
\lim_{k \to \infty} J(u_k) = 0 = c_\star,
\]
which is a contradiction to the fact that \( c_\star > 0 \). Hence, we must have \( u_0 \not\equiv 0 \).

**Claim 2:** \( M(\|u_0\|^{\frac{n}{s}})\|u_0\|^{\frac{n}{s}} \geq \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_0)}{|x - y|^\mu} dy \right) g(x, u_0) u_0 dx \).

**Proof.** Suppose by contradiction that
\[
M(\|u_0\|^{\frac{n}{s}})\|u_0\|^{\frac{n}{s}} < \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_0)}{|x - y|^\mu} dy \right) g(x, u_0) u_0 dx.
\]
That is, \( \langle J'(u_0), u_0 \rangle < 0 \).
It is easy to see, using (M2), that \( M(t)t \geq M(1)t^{\gamma} \) when \( t \in [0,1] \). So for \( 0 < t < \frac{1}{\|u_0\|} \), using Lemma 3.6 and Hardy-Littlewood-Sobolev inequality we have that

\[
\langle J'(tu_0), u_0 \rangle \geq M(1)t^{\frac{2n\gamma}{n-\mu}} \|u_0\|^{\frac{2n\gamma}{n-\mu}} - \frac{2s}{\gamma n \mu} \int_{\Omega} \left( \int_{\Omega} \frac{g(y, tu_0)u_0(y)}{|x-y|^{\mu}} \, dy \right) g(x, tu_0)u_0(x) \, dx
\]

\[
\geq M(1)t^{\frac{2n\gamma}{n-\mu}} \|u_0\|^{\frac{2n\gamma}{n-\mu}} - \frac{C}{t} \left( \int_{\Omega} |g(x, tu_0)|^{\frac{2n\gamma}{2n-\mu}} \, dx \right)^{\frac{2n-\mu}{n-\mu}}.
\]

But from the growth assumptions on \( g \) we already know that for \( \epsilon > 0, \alpha > \frac{\gamma n}{2s} \) and \( r > \frac{\gamma n}{2s} \),

\[
\left( \int_{\Omega} |g(x, tu_0)|^{\frac{2n\gamma}{2n-\mu}} \, dx \right)^{\frac{2n-\mu}{n-\mu}} \leq C(\epsilon) \left( \int_{\Omega} \left( \frac{4n(1+\epsilon)\|u_0\|^{\frac{n}{n-\mu}}}{2n-\mu} \right)^{\frac{n\gamma}{2}} \right)^{\frac{1}{2}}
\]

by choosing \( t < \left( \frac{(2n-\mu)\alpha\|u_0\|^{\frac{n}{n-\mu}}}{4n(1+\epsilon)\|u_0\|^{\frac{n}{n-\mu}}} \right)^{\frac{n-\mu}{2n-\mu}} \) and using Trudinger-Moser inequality. Therefore for \( t > 0 \) small enough as above, we obtain

\[
\langle J'(tu_0), u_0 \rangle \geq M(1)t^{\frac{2n\gamma}{n-\mu}} \|u_0\|^{\frac{2n\gamma}{n-\mu}} - C(\epsilon) \left( t^{2\alpha-1}\|u_0\|^{2\alpha} + t^{2r-1}\|u_0\|^{2r} \right)
\]

which suggests that \( \langle J'(tu_0), u_0 \rangle > 0 \) when \( t \) is sufficiently small. Thus there exists a \( \sigma \in (0,1) \) such that \( \langle J'(\sigma u_0), u_0 \rangle = 0 \) that is, \( \sigma u_0 \in \mathcal{N} \). Thus from Lemmas 3.6 3.7 and Remark 2.1 it follows that

\[
c_* \leq b \leq J(\sigma u_0) = J(\sigma u_0) - \frac{s}{n\gamma} \langle J'(\sigma u_0), \sigma u_0 \rangle
\]

\[
= \frac{s}{n} \bar{M}(\|\sigma u_0\|^{\frac{2}{\gamma}}) - \frac{s}{n\gamma} M(\|\sigma u_0\|^{\frac{2}{\gamma}})\|\sigma u_0\|^{\frac{2}{\gamma}} + \frac{s}{n\gamma} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, \sigma u_0)}{|x-y|^{\mu}} \, dy \right) g(x, \sigma u_0)\sigma u_0 - \frac{n\gamma}{2s} G(x, \sigma u_0)
\]

\[
< \frac{s}{n} \bar{M}(\|u_0\|^{\frac{2}{\gamma}}) - \frac{s}{n\gamma} M(\|u_0\|^{\frac{2}{\gamma}})\|u_0\|^{\frac{2}{\gamma}} + \frac{s}{n\gamma} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_0)}{|x-y|^{\mu}} \, dy \right) g(x, u_0)u_0 - \frac{n\gamma}{2s} G(x, u_0)
\]

Also by lower semicontinuity of norm and Fatou’s Lemma, we obtain

\[
c_* \leq b \leq \liminf_{k \to \infty} \left( \frac{s}{n} \bar{M}(\|u_k\|^{\frac{2}{\gamma}}) - \frac{s}{n\gamma} M(\|u_k\|^{\frac{2}{\gamma}})\|u_k\|^{\frac{2}{\gamma}} \right)
\]

\[
+ \liminf_{k \to \infty} \frac{s}{n\gamma} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^{\mu}} \, dy \right) [g(x, u_k)u_k - \frac{n\gamma}{2s} G(x, u_k)] \, dx
\]

\[
\leq \lim_{k \to \infty} \left[ J(u_k) - \frac{s}{n\gamma} \langle J'(u_k), u_k \rangle \right] = c_*
\]

which is a contradiction. Hence Claim 2 is proved.

**Claim 3:** \( J(u_0) = c_* \).

**Proof.** Using \( \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x-y|^{\mu}} \, dy \right) G(x, u_k) \, dx \to \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_0)}{|x-y|^{\mu}} \, dy \right) G(x, u_0) \, dx \) and lower
semicontinuity of norm we have $J(u_0) \leq c_*$. Now we are going to show that the case $J(u_0) < c_*$ cannot occur. Indeed, if $J(u_0) < c_*$ then $\|u_0\|^{\frac{n}{s}} < \rho_0^{\frac{n}{s}}$. Moreover,

$$\frac{\dot{M}}{n} (\rho_0^{\frac{n}{s}}) = \lim_{k \to \infty} \frac{\dot{M}}{n} (\|u_k\|^{\frac{n}{s}}) = c_* + \frac{1}{2} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_0)}{|x-y|^\mu} dy \right) G(x, u_0) dx,$$

(3.25)

This gives that

$$\rho_0^{\frac{n}{s}} = \dot{M}^{-1} \left( \frac{n}{s} c_* + \frac{n}{2s} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_0)}{|x-y|^\mu} dy \right) G(x, u_0) dx \right).$$

Next defining $v_k = \frac{u_k}{\|u_k\|}$ and $v_0 = \frac{u_0}{\|u_0\|}$, we have $v_k \to v_0$ in $X_0$ and $\|v_0\| < 1$. Thus by Lemma 2.5

$$\sup_{k \in \mathbb{N}} \int_{\Omega} \exp(p|v_k|^{\frac{1}{p}}) \, dx < \infty \text{ for all } 1 < p < \frac{\alpha_1 s}{(1 - \|v_0\|^{\frac{n}{s}})^{\frac{n-s}{n}}}.$$

(3.26)

On the other hand, by Claim 2, (2.11) and Lemma 3.6 we have

$$J(u_0) \geq \frac{\dot{M}}{n} (\|u_0\|^{\frac{n}{s}}) - \frac{\dot{M}}{n^\gamma} (\|u_0\|^{\frac{n}{s}}) \|u_0\|^{\frac{n}{s}}$$

$$+ \frac{\dot{M}}{n^\gamma} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_0)}{|x-y|^\mu} dy \right) \left( g(x, u_0) u_0 - \frac{n\gamma}{2s} G(x, u_0) \right) dx \geq 0.$$

Using this together with Lemma 3.3 and the equality, $\frac{n}{s} (c_* - J(u_0)) = \dot{M} \left( \frac{n}{s} \rho_0 \right) - \dot{M} \left( \|u_0\|^{\frac{n}{s}} \right)$ we obtain

$$\dot{M} \left( \frac{n}{s} \rho_0 \right) \leq \frac{n}{s} c_* + \dot{M} (\|u_0\|^{\frac{n}{s}}) < \dot{M} \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right) \left( \frac{n-s}{s} \right) + \dot{M} (\|u_0\|^{\frac{n}{s}})$$

and therefore by (M1)

$$\frac{n}{s} \rho_0 < \dot{M} \left( \frac{2n - \mu}{2n} \alpha_{n,s} \right) \left( \frac{n-s}{s} \right) + \|u_0\|^{\frac{n}{s}}. \quad (3.27)$$

Since $\rho_0^n (1 - \|v_0\|^{\frac{n}{s}}) = \rho_0^n - \|u_0\|^\frac{n}{s}$, from (3.27) it follows that

$$\rho_0^{\frac{n}{s}} < \frac{(2n - \mu) \alpha_{n,s} \|u_0\|^\frac{n-s}{s}}{1 - \|v_0\|^\frac{n}{s}}.$$

Thus, there exists $\beta > 0$ such that $\|u_k\|^\frac{n}{s-s} < \beta < \frac{\alpha_{n,s} (2n-\mu)}{2n(1-\|v_0\|^\frac{n}{s})^{n-s}}$ for $k$ large. We can choose $q > 1$ close to 1 such that $q\|u_k\|^\frac{n}{n-s} \leq \beta < \frac{(2n-\mu) \alpha_{n,s}}{2n(1-\|v_0\|^\frac{n}{s})^{n-s}}$ and using (3.26), we conclude that for $k$ large

$$\int_{\Omega} \exp \left( \frac{2n q |u_k|^{n-s}}{2n - \mu} \right) dx \leq \int_{\Omega} \exp \left( \frac{2n \beta |v_k|^{n-s}}{2n - \mu} \right) dx \leq C.$$

Doubly nonlocal problems with Trudinger-Moser type Choquard nonlinearity
Let us recall (2.3) and (3.24) to get that
\[
\left| \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) u_k \: dx \right| \leq C \left( \|u_k\|_{\lambda_0, 2}^{2n-\mu} L^{2n-\mu} (\Omega) + \|u_k\|_{\lambda_0, q' L^{2n-\mu}} (\Omega) \right) \rightarrow C \left( \|u_0\|_{\lambda_0, 2}^{2n-\mu} L^{2n-\mu} (\Omega) + \|u_0\|_{\lambda_0, q' L^{2n-\mu}} (\Omega) \right)
\]
as \( k \to \infty \). Then the pointwise convergence of \( \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) u_k \) to \( \left( \int_{\Omega} \frac{G(y, u_0)}{|x - y|^\mu} dy \right) g(x, u_0) u_0 \) as \( k \to \infty \) asserts that
\[
\lim_{k \to \infty} \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) u_k \: dx = \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_0)}{|x - y|^\mu} dy \right) g(x, u_0) u_0 \: dx
\]
while using the Lebesgue dominated convergence theorem. Now Lemma 3.9 we get
\[
\int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_k) (u_k - u_0) \: dx \rightarrow 0 \text{ as } k \to \infty.
\]
Since \( (J'(u_k), u_k - u_0) \to 0 \), it follows that
\[
M(\|u_k\|_{\lambda_0}^\frac{\mu}{2}) \int_{\mathbb{R}^{2n}} \frac{|u_k(x) - u_k(y)|^{\frac{\mu}{2} - 2} (u_k(x) - u_k(y))(u_k(x) - u_k(y))}{|x - y|^{2n}} \: dx \: dy \to 0. \tag{3.28}
\]
We define \( U_k(x, y) = u_k(x) - u_k(y) \) and \( U_0(x, y) = u_0(x) - u_0(y) \) then using \( u_k \to u_0 \) weakly in \( X_0 \) and boundedness of \( M(\|u_k\|_{\lambda_0}^\frac{\mu}{2}) \), we have
\[
M(\|u_k\|_{\lambda_0}^\frac{\mu}{2}) \int_{\mathbb{R}^{2n}} \frac{|U_0(x, y)|^{\frac{\mu}{2} - 2} U_0(x, y)(U_k(x, y) - U_0(x, y))}{|x - y|^{2n}} \: dx \: dy \to 0 \text{ as } k \to \infty. \tag{3.29}
\]
Subtracting (3.29) from (3.28), we get
\[
M(\|u_k\|_{\lambda_0}^\frac{\mu}{2}) \int_{\mathbb{R}^{2n}} \frac{|U_k(x, y)|^{\frac{\mu}{2} - 2} U_k(x, y) - |U_0(x, y)|^{\frac{\mu}{2} - 2} U_0(x, y)(U_k(x, y) - U_0(x, y))}{|x - y|^{2n}} \: dx \: dy \to 0 \text{ as } k \to \infty.
\]
as \( k \to \infty \). Now using this and the following inequality
\[
|a - b|^p \leq 2^{p-2}(|a|^{p-2} a - |b|^{p-2} b)(a - b) \text{ for all } a, b \in \mathbb{R} \text{ and } p \geq 2, \tag{3.30}
\]
with \( a = u_k(x) - u_k(y) \) and \( b = u_0(x) - u_0(y) \), we obtain
\[
M(\rho_0^\frac{\mu}{2}) \int_{\mathbb{R}^{2n}} \frac{|U_k(x) - U_0(x)|^\frac{\mu}{2}}{|x - y|^{2n}} \: dx \: dy \to 0 \text{ as } k \to \infty.
\]
This implies that \( u_k \to u \) strongly in \( X_0 \) and hence \( J(u) = c_* \) which is a contradiction. Therefore, claim 3 holds true. Hence \( J(u) = c_* = \lim_{k \to \infty} J(u_k) \) and \( \|u_k\| \to \rho_0 \) gives that \( \rho_0 = \|u_0\| \). Finally we have
\[
M(\|u_0\|_{\lambda_0}^\frac{\mu}{2}) \int_{\Omega} \frac{|u_0(x) - u_0(y)|^{\frac{\mu}{2} - 2} (u_0(x) - u_0(y))(\phi(x) - \phi(y))}{|x - y|^{2n}} \: dx \: dy
\]
\[
= \int_{\Omega} \left( \int_{\Omega} \frac{G(y, u_k)}{|x - y|^\mu} dy \right) g(x, u_0) \phi \: dx,
\]
for all \( \phi \in X_0 \). Thus, \( u_0 \) is a non trivial solution of \( (\mathcal{M}) \). By Lemma 3.9 we obtain that \( u_0 \) is the required nonnegative solution of \( (\mathcal{M}) \) which completes the proof. \( \Box \)
Acknowledgements: This research is supported by Science and Engineering Research Board, Department of Science and Technology, Government of India, Grant number: ECR/2017/002651. The second author wants to thank Bennett University for its hospitality during her visit there.

References

[1] Claudianor O. Alves, Daniele Cassani, Cristina Tarsi, and Minbo Yang. Existence and concentration of ground state solutions for a critical nonlocal Schrödinger equation in $\mathbb{R}^2$. *J. Differential Equations*, 261(3):1933–1972, 2016.

[2] Claudianor O. Alves and Minbo Yang. Existence of solutions for a nonlocal variational problem in $\mathbb{R}^2$ with exponential critical growth. *J. Convex Anal.*, 24(4):1197–1215, 2017.

[3] R. Arora, J. Giacomoni, T. Mukherjee, and K. Sreenadh. $n$-Kirchhoff-Choquard equations with exponential nonlinearity. *Nonlinear Anal.*, 108:113–144, 2019.

[4] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[5] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[6] J. Giacomoni, Pawan Kumar Mishra, and K. Sreenadh. Fractional elliptic equations with critical exponential nonlinearity. *Adv. Nonlinear Anal.*, 5(1):57–74, 2016.

[7] J. Giacomoni, Pawan Kumar Mishra, and K. Sreenadh. Fractional Kirchhoff equation with critical exponential nonlinearity. *Complex Var. Elliptic Equ.*, 61(9):1241–1266, 2016.

[8] Sarika Goyal and K. Sreenadh. Nehari manifold for non-local elliptic operator with concave-convex nonlinearities and sign-changing weight functions. *Proc. Indian Acad. Sci. Math. Sci.*, 125(4):545–558, 2015.

[9] E. Parini L. Brasco, E. Lindgren. The fractional Cheeger problems. *Interfaces Frr Bound.*, 16:419–458, 2014.

[10] M. Squassina L. Brasco, E. Parini. Stability of variational eigenvalues for the fractional $p$-laplacian. *Discrete Contin. Dyn. Syst.*, 36:439–455, 2016.

[11] Fuyi Li, Chunjuan Gao, and Xiaoli Zhu. Existence and concentration of sign-changing solutions to Kirchhoff-type system with Hartree-type nonlinearity. *J. Math. Anal. Appl.*, 448(1):60–80, 2017.

[12] E. H. Lieb. Existence and uniqueness of the minimizing solution of Choquard nonlinear equation. *Stud. APPL. Math.*, 57:93–105, 1976/77.
[13] Elliott H. Lieb and Michael Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[14] Dengfeng Lü. A note on Kirchhoff-type equations with Hartree-type nonlinearities. *Nonlinear Anal.*, 99:35–48, 2014.

[15] Luca Martinazzi. Fractional Adams-Moser-Trudinger type inequalities. *Nonlinear Anal.*, 127:263–278, 2015.

[16] Xiang Mingqi, Vicenţiu D. Rădulescu, and Binlin Zhang. Fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity. *Calc. Var. Partial Differential Equations*, 58(2):Art. 57, 27, 2019.

[17] Vitaly Moroz and Jean Van Schaftingen. A guide to the Choquard equation. *J. Fixed Point Theory Appl.*, 19(1):773–813, 2017.

[18] Enea Parini and Bernhard Ruf. On the Moser-Trudinger inequality in fractional Sobolev-Slobodeckij spaces. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 29(2):315–319, 2018.

[19] S. Pekar. *Untersuchung über die Elektronentheorie der Kristalle*. Akademie Verlag, Berlin. 1954.

[20] Kanishka Perera, Marco Squassina, and Yang Yang. Bifurcation and multiplicity results for critical fractional p-Laplacian problems. *Mathematische Nachrichten*, 289(2-3):332–342, 2016.

[21] P. Pucci, M. Xiang, and B. Zhang. Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional p-Laplacian. *Adv. Calc. Var.*, 12(3):253–275, 2019.

[22] Raffaella Servadei and Enrico Valdinoci. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.*, 389(2):887–898, 2012.

[23] M. Xiang, B. Zhang, and D Repovs. Existence and multiplicity of solutions for fractional Schrödinger-Kirchhoff equations with Trudinger-Moser nonlinearity. *Nonlinear Anal.*, 186:74–98, 2018.