Galaxy rotation curves in $f(R, \phi)$-gravity

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We investigate the possibility to explain theoretically the galaxy rotation curves by a gravitational potential in total absence of dark matter. To this aim an analytic fourth-order theory of gravity, nonminimally coupled with a massive scalar field is considered. Specifically, the interaction term is given by an analytic function $f(R, \phi)$ where $R$ is the Ricci scalar and $\phi$ is the scalar field. The gravitational potential is generated by a point-like source and compared with the so called Sanders’s potential that can be exactly reproduced in this case. This result means that the problem of dark matter in spiral galaxies could be fully addressed by revising general relativity at galactic scales and requiring further gravitational degrees of freedom instead of new material components that have not been found out up to now.

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I. INTRODUCTION

J.H. Oort first pointed out the missing matter problem in the 30’s of last century$^{[1, 2]}$. The issue came out by observing the Doppler shift of stars moving near the plane of our Galaxy and calculating the star velocities. The result was that there had to be a large amount of matter inside the galaxy to prevent the stars from escaping. Such a "matter" should give rise to a central gravitational force much larger than Sun’s gravitational pull to keep a planet in its orbit. However it turned out that there was not enough luminous mass in the Galaxy to account for this dynamics. The discrepancy was very large and the Galaxy had to be at least two or three times more massive than the sum of all its luminous components in order to match the result. Later on, the tangential velocity of stars in orbits around the Galactic center was calculated as a function of distance from the center. Surprisingly it was found that far away from the Galactic Center, stars move with the same velocity independent of their distance out from the Galactic Center. These results strongly posed the problem that either luminous matter was not able to reliably trace the radial profile of the Galaxy or the Newtonian potential was not able to describe dynamics far from the Galactic center.

Soon after, other dark matter issues came out from dynamical descriptions of self-gravitating astrophysical systems like stellar clusters, galaxies, groups and clusters of galaxies. In all these cases, there is more matter dynamically inferred than that can be accounted for by luminous matter components. The mass discrepancy comes out assuming the validity of Newton law at any astrophysical scales. Problems emerged also at larger scales. F. Zwicky discovered anomalous motions of galaxies in the Coma cluster finding that the visible mass was too little to produce enough gravitational force to hold the cluster together$^{[3]}$.

At the beginning, the only possibility considered was to assume the Newton law holding at all scales and postulating some non-luminous component to make up the missing mass. Many names have been coined to define these invisible components. For example, the MAssive Compact Halo Objects (MACHOs) are objects like black holes and neutron stars (in general sub-luminous objects) that populate the outer reaches of galaxies like the Milky Way. There are the Weakly Interacting Massive Particles (WIMPs) which do not interact with standard matter (constituted by baryons as protons and neutrons): they are supposed to be particles out of the Standard Model of Particles but, up to now, there is no final indication for their existence$^{[4]}$. In general, dark matter is assumed to come in two flavors, hot (HDM) and cold (CDM) dark matter. The CDM should be in dead stars, planets, brown dwarfs etc., while HDM should be constituted by fast moving relativistic particles. It should be neutrinos, tachyons etc. However, there is still no definitive proof that WIMPs exist, or that MACHOs will ever make up more than five percent of the total amount of missing matter.

On the other hand, the need of unknown components as dark energy (coming from cosmology) and dark matter

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could be considered nothing else but as a signal of the breakdown of Einstein General Relativity (GR) at astrophysical (galactic and extragalactic) and cosmological scales.

In this context, Extended Theories of Gravity (ETGs) could be, in principle, an interesting alternative to explain cosmic acceleration and large scale structure without any dark components. In their simplest version, the Ricci curvature scalar $R$, linear in the Hilbert-Einstein action, could be replaced by a generic function $f(R)$ whose true form could be "reconstructed" by the data. In fact, there is no a priori reason to consider the gravitational Lagrangian linear in the Ricci scalar while observations and experiments could contribute to define and constrain the "true" theory of gravity (see [5–11]).

Coming to the weak-field limit, any alternative relativistic theory of gravity is expected to reproduce GR results which, in any case, are firmly tested only at Solar System scales in the Newtonian limit [12]. Even this limit is matter of debate since several relativistic theories do not reproduce it. For example, Yukawa-like corrections to the Newtonian potential easily comes out [13] with interesting physical consequences. For example, it is claimed by some authors that the flat rotation curves of galaxies can be explained by such terms [14]. Other authors have shown that a conformal theory of gravity is nothing else but a fourth order theory containing such terms in the Newtonian limit.

In general, any relativistic theory of gravitation yields corrections to the weak-field gravitational potentials (e.g., [15]) which, at the post-Newtonian level and in the Parametrized Post-Newtonian formalism, could constitute a test of these theories [12].

This point deserves a deep discussion. Beside the fundamental physics motivations coming from Quantum Gravity and unification theories (see [5–10]), ETGs pose the problem that there are further gravitational degrees of freedom (related to higher order terms, non-minimal couplings and scalar fields in the field equations) and gravitational interaction is not invariant at any scale. This means that, besides the Schwarzschild radius, other characteristic gravitational scales could come out from dynamics. Such scales, in the weak field approximation, should be responsible of characteristic lengths of astrophysical structures that should result confined in this way [16].

In this paper, without claiming for completeness, we will try to address the problem of describing galaxy rotation curves without dark matter but asking for corrections to the Newtonian potential that could fit data and reproduce dynamics. These corrections are not phenomenological but come out from the weak field limit of general relativistic theories of gravity that predict the existence of corrections (e.g. Yukawa-like corrections) to the Newtonian potential. The only exception is GR where the action is chosen to be $R$, that is linear in the Ricci curvature scalar and does not contain corrections to the Newtonian potential in the weak field limit. Relaxing such a hypothesis, it is possible to show that any analytic ETG presents Yukawa corrections in the weak-field limit (see also [15] for a detailed calculation). From an astrophysical point of view, these corrections means that further scales have to be taken into account and that their effects could be irrelevant at local scales as Solar System. With this scheme in mind, we will give a summary of ETGs in Sec. II discussing also their conformal properties. In fact any ETG can be conformally transformed to the Einstein one plus scalar fields representing the further gravitational degrees of freedom. This feature is extremely important to select characteristic length scales (related to the effective masses of scalar fields) that could account for dynamics. In this sense, considering $f(R)$ gravity means to take into account an Einstein theory plus a scalar field; considering $f(R, \Box R)$-gravity means to assumes Einstein + two scalar fields and so on. The emergence of Yukawa-like corrections to the Newtonian potential is discussed in Sec. IIII where the weak-field limit of $f(R)$-gravity, the simplest ETG, is worked out. Here, $f(R)$ is a generic analytic function of the Ricci curvature scalar $R$. Furthermore, we discuss the case of $f(R, \phi)$-gravity, corresponding to $f(R, \Box R)$-gravity, i.e. Einstein plus two scalar fields, showing that a further free parameter in needed to better model dynamics. Sec. IV is devoted to discussion and conclusions.

II. EXTENDED GRAVITY AND CONFORMAL TRANFORMATIONS

Higher-order and scalar-tensor gravities are examples of ETGs. For a comprehensive discussion, see [5–10]. Essentially these theories can be characterized by two main feature: the geometry can non-minimally couple to some scalar field; derivatives of the metric components of order higher than second may appear. In the first case, we say that we have scalar-tensor gravity, and in the second case we have higher-order theories. Combinations of non-minimally coupled and higher order terms can also emerge in effective Lagrangians, producing mixed higher order/scalar-tensor gravity. The physical foundation of such models can be found at fundamental level by considering effective actions coming from quantum fields in curved space-times, string/M theory and so on [10]. A general class of higher-order-scalar-tensor theories in four dimensions is given by the effective action

$$S = \int d^4x \sqrt{-g} \left[ f(R, \Box R, \Box^2 R, \ldots, \Box^k R, \phi) + \omega(\phi)\phi^{\alpha\beta} \phi^\alpha \phi^\beta + \mathcal{N} \mathcal{L}_m \right],$$  \hspace{1cm} (1)
where \( f \) is an unspecified function of curvature invariants and scalar field \( \phi \) and \( \mathcal{X} = 8\pi G^1 \). The convention for Ricci’s tensor is \( R_{\mu\nu} = R^\sigma_{\mu\sigma\nu} \), while for the Riemann tensor is \( R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\mu\nu} + \ldots \). The affinities are the usual Christoffel symbols of the metric: \( \Gamma^\alpha_{\mu\beta} = \frac{1}{2}g^{\alpha\sigma}(g_{\mu\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}) \). The adopted signature is \((+−−−)\).

The affinities are the usual Christoffel symbols of the metric: \( \Gamma^\alpha_{\mu\beta} = \frac{1}{2}g^{\alpha\sigma}(g_{\mu\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}) \). The term \( \mathcal{L}_m \) is the minimally coupled ordinary matter contribution, considered as a perfect fluid; \( \omega(\phi) \) is a function of scalar field which specifies the theory. Actually its values can be \( \omega(\phi) = ±1, 0 \) fixing the nature and the dynamics of the scalar field which can be a canonical scalar field, a phantom field or a field without dynamics (see [17–19] for details). In the metric approach, the field equations are obtained by varying \( \mathcal{S} \) with respect to \( g_{\mu\nu} \). By introducing the Einstein tensor \( G_{\mu\nu} \) we get

\[
G_{\mu\nu} = \mathcal{X} T_{\mu\nu} + \frac{f - \frac{1}{2}fR}{2} g_{\mu\nu} - g_{\mu\nu} \Box \mathcal{G} - \omega(\phi) \left( \phi_{\mu\nu} - \frac{\phi\alpha\phi^\alpha}{2} g_{\mu\nu} \right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{i} (g_{\mu\nu} g^{\lambda\sigma} + g_{\mu}^{\lambda} g_{\nu}^{\sigma}) \left( \Box^{j-1} R \right)_{\lambda} \times \left( \Box^{i-j} \frac{\partial f}{\partial \Box^{j} R} \right)_{\mu\nu} - g_{\mu\nu} \left( \left( \Box^{j-1} R \right)_{\alpha} \Box^{i-j} \frac{\partial f}{\partial \Box^{j} R} \right)_{\sigma} ,
\]

(2)

where we have introduced the quantity

\[
\mathcal{G} = \sum_{j=0}^{n} \Box (\frac{\partial f}{\partial \Box^{j} R}) ,
\]

(3)

the energy-momentum tensor of matter

\[
T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \left( \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}} \right)
\]

(4)

and \( \Box = \ldots \times \) is the d’Alembert operator. The differential Eqs. (2) are of order at most \((2k + 4)\). The (possible) contribution of a self-interaction potential \( V(\phi) \) is contained in the definition of \( f \). By varying with respect to the scalar field \( \phi \), we obtain the generalized Klein-Gordon equation

\[
2\omega(\phi) \Box \phi + \omega_\phi(\phi) \phi\alpha\phi^\alpha - f_\phi = 0 ,
\]

(5)

where \( f_\phi = \frac{df}{d\phi} \) and \( \omega_\phi(\phi) = \frac{d\omega(\phi)}{d\phi} \). Several interesting cases can be worked out starting from the action \( \mathcal{S} \). Below, we give some significant examples that will result useful for the astrophysical applications of this paper.

**A. The case of \( f(R) \)-gravity**

The simplest extension of GR is achieved assuming,

\[
R \to f(R) , \quad \omega(\phi) = 0
\]

(6)

and the action \( \mathcal{S} \) becomes

\[
\mathcal{S} = \int d^4x \sqrt{-g} \left[ f(R) + \mathcal{X} \mathcal{L}_m \right] .
\]

(7)

Then the field equations (2) become

\[
f_R G_{\mu\nu} = \mathcal{X} T_{\mu\nu} + \frac{f - \frac{1}{2}fR}{2} g_{\mu\nu} + f_{R\mu\nu} - g_{\mu\nu} \Box f_R \equiv \mathcal{X} T_{\mu\nu} + f_R T_{\mu\nu}^{f(R)}
\]

(8)

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\(^1\) Here we use the convention \( \hbar = 1 \).
where $f_R = \frac{df}{dR}$. The gravitational contribution due to higher-order terms can be reinterpreted as a stress-energy tensor contribution $T^{(R)}_{\mu\nu}$. This means that additional and higher-order terms in the gravitational action act, in principle, as a stress-energy tensor, related to the form of $f$. In the case of GR, $T^{(R)}_{\mu\nu}$ identically vanishes while the standard, minimal coupling is recovered for the matter contribution.

The peculiar behavior of $f(R) = R$ is due to the particular form of the Lagrangian itself which, although it is a second order Lagrangian, can be non-covariantly rewritten as the sum of a first order Lagrangian plus a pure divergence term. The Hilbert-Einstein Lagrangian can be in fact recast as follows:

$$L_{HE} = L_{HE} \sqrt{-g} = \left[p^{\alpha\beta}(\Gamma^\rho_{\alpha\sigma} \Gamma^\sigma_{\rho\beta} - \Gamma^\rho_{\rho\sigma} \Gamma^\sigma_{\alpha\beta}) + \nabla_\sigma(p^{\alpha\beta} u^\sigma_{\alpha\beta})\right],$$

where:

$$p^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} = \frac{\partial L}{\partial R^{\alpha\beta}},$$

$\Gamma$ is the Levi-Civita connection of $g$ and $u^\sigma_{\alpha\beta}$ is a quantity constructed out with the variation of $\Gamma$. Since $u^\sigma_{\alpha\beta}$ is not a tensor, the above expression is not covariant; however standard procedures can be used to recast covariance. This clearly shows that the field equations has to be of second order and the Hilbert-Einstein Lagrangian is thus degenerate.

B. The case of scalar-tensor gravity

From the action (1), it is possible to obtain another interesting case by choosing

$$f = F(\phi)R + V(\phi), \quad \omega(\phi) = 1/2,$$

then

$$S = \int d^4x \sqrt{-g} \left[ F(\phi)R + V(\phi) + \frac{\phi ;\alpha \phi ;\alpha}{2} + X L_m \right],$$

where $V(\phi)$ and $F(\phi)$ are generic functions describing respectively the potential and the coupling of a scalar field $\phi$. The Brans-Dicke theory of gravity is a particular case of the action (12) for $V(\phi) = 0$ [10]. The variation with respect to $g_{\mu\nu}$ gives now the second-order field equations (particular form of field equations (2))

$$F(\phi) G_{\mu\nu} = XT_{\mu\nu} + \frac{V(\phi)}{2} g_{\mu\nu} + F(\phi) ;\mu ;\nu - g_{\mu\nu} \Box F(\phi) - \frac{1}{2} \left( \phi ;\mu \phi ;\nu - \frac{\phi ;\alpha \phi ;\alpha}{2} g_{\mu\nu} \right) \equiv XT_{\mu\nu} + F(\phi) T^{(\phi)}_{\mu\nu}$$

where $T^{(\phi)}_{\mu\nu}$ is the energy-momentum tensor relative to the scalar field $\phi$. The variation with respect to $\phi$ provides the Klein-Gordon equation, i.e. the field equation for the scalar field

$$\Box \phi - F_{\phi}(\phi) R - V_{\phi}(\phi) = 0,$$

where $F_{\phi}(\phi) = \frac{dF(\phi)}{d\phi}$, $V_{\phi}(\phi) = \frac{dV(\phi)}{d\phi}$. This last equation is equivalent to the Bianchi contracted identity [21].

C. Conformal transformations

These models, and, in general, any theory of the class (1), can be conformally reduced to the Einstein theory plus scalar fields. Conformal transformations are mathematical tools very useful in ETGs in order to disentangle the further gravitational degrees of freedom coming from general actions [10, 22, 24]. The idea is to perform a conformal
rescaling of the space-time metric \( g_{\mu\nu} \to \tilde{g}_{\mu\nu} \). Often a scalar field is present in the theory and the metric rescaling is accompanied by a (nonlinear) redefinition of this field \( \phi \to \tilde{\phi} \). New dynamical variables \( \{\tilde{g}_{\mu\nu}, \tilde{\phi}\} \) are thus obtained. The scalar field redefinition serves the purpose of casting the kinetic energy density of this field in a canonical form. The new set of variables \( \{\tilde{g}_{\mu\nu}, \tilde{\phi}\} \) is called the Einstein conformal frame, while \( \{g_{\mu\nu}, \phi\} \) constitute the Jordan frame.

When a scalar degree of freedom \( \phi \) is present in the theory, as in scalar tensor or \( f(R) \) gravity, it generates the transformation to the Einstein frame in the sense that the rescaling is completely determined by a function of \( \phi \). In principle, infinitely many conformal frames could be introduced, giving rise to as many representations of the theory.

Let the pair \( \{\mathcal{M}, g_{\mu\nu}\} \) be a space-time, with \( \mathcal{M} \) a smooth manifold of dimension \( n \geq 2 \) and \( g_{\mu\nu} \) a (pseudo)-Riemannian metric on \( \mathcal{M} \). The point-dependent rescaling of the metric tensor

\[
g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu},
\]

where the conformal factor \( \Omega \) is a nowhere vanishing, regular function, is called a Weyl or conformal transformation. Due to this metric rescaling, the lengths of space-like and time-like intervals and the norms of space-like and time-like vectors are changed, while null vectors and null intervals of the metric \( g_{\mu\nu} \) remain null in the rescaled metric \( \tilde{g}_{\mu\nu} \).

The light cones are left unchanged by the transformation (15) and the space-times \( \{\mathcal{M}, g_{\mu\nu}\} \) and \( \{\mathcal{M}, \tilde{g}_{\mu\nu}\} \) exhibit the same causal structure; the converse is also true \([25]\). A vector that is time-like, space-like, or null with respect to \( g_{\mu\nu} \) has the same character with respect to \( \tilde{g}_{\mu\nu} \), and vice-versa.

Conformal invariance corresponds to the absence of a characteristic length (or mass) scale in the physics. In general, the effective potential \( V(\phi) \) coming from conformal transformations contains dimensional parameters (such as a mass \( m \), that is a further ”characteristic gravitational length”). This means that the further degrees of freedom coming from ETGs give rise to features that could play a fundamental role in the dynamics of astrophysical structures. In what follows, we will see that these further gravitational lengths could solve, in principle, the dark matter problem.

### D. Conformal transformations and higher-order gravity

Performing the conformal transformation for \( f(R) \)-gravity with \( \Omega^2 = f_R \) we have

\[
\int d^4x \sqrt{-\tilde{g}}[f(R) + \mathcal{X} \mathcal{L}_m] = \int d^4x \sqrt{-g} \left( \tilde{R} + W(\tilde{\phi}) - \frac{\tilde{\phi}_\alpha \tilde{\phi}^\alpha}{2} + \mathcal{X} \tilde{\mathcal{L}}_m \right)
\]

where \( \tilde{\phi} = \sqrt{\frac{3}{2}} \ln f_R \) while the potential \( W \) and the nonminimally coupled lagrangian of ordinary matter \( \tilde{\mathcal{L}}_m \) are given by

\[
W(\tilde{\phi}) = e^{-2\tilde{\phi}/\sqrt{\mathcal{X}}} V(e^{\tilde{\phi}/\sqrt{\mathcal{X}}})
\]

\[
\tilde{\mathcal{L}}_m = e^{-2\tilde{\phi}/\sqrt{\mathcal{X}}} \mathcal{L}_m \left( e^{-\tilde{\phi}/\sqrt{\mathcal{X}} g_{\rho\sigma}} \right)
\]

The function \( V \) is defined by the analogy between the \( f(R) \)-gravity and the scalar-tensor gravity (the so-called O’Hanlon lagrangian)

\[
V(\phi) = f(R) - R f_R(R)
\]

where \( \phi = f_R \). The field equations in standard form are given in the Einstein frame as follow

\[
\tilde{G}_{\mu\nu} = \mathcal{X} \tilde{T}_{\mu\nu} + \frac{W(\tilde{\phi})}{2} \tilde{g}_{\mu\nu} + \frac{1}{2} \left( \tilde{\phi}_\mu \tilde{\phi}_\nu - \frac{\tilde{\phi}_\alpha \tilde{\phi}^\alpha}{2} \tilde{g}_{\mu\nu} \right)
\]

\[
\tilde{\Box} \tilde{\phi} + W_\delta(\tilde{\phi}) = -\mathcal{X} \frac{\delta \tilde{\mathcal{L}}_m}{\delta \tilde{\phi}}
\]
However, the problem is completely solved if $\phi = f_R$ can be analytically inverted. In summary, a fourth-order theory is conformally equivalent to the standard second-order Einstein theory plus a scalar field (see also [26, 27]).

If the theory is higher than fourth order, we have Lagrangian densities of the form [28–30],

$$L = L(R, \Box R, \ldots, \Box^k R).$$

(21)

Every $\Box$ operator introduces two further terms of derivation into the field equations. For example a theory like

$$L = R \Box R,$$

(22)

is a sixth-order theory, and the above approach can be pursued considering a conformal factor of the form

$$\Omega^2 = \frac{\partial L}{\partial R} + \Box \frac{\partial L}{\partial \Box R}.$$  

(23)

In general, increasing two orders of derivation in the field equations (i.e. every term $\Box R$), corresponds to add a scalar field in the conformally transformed frame [29]. A sixth-order theory can be reduced to an Einstein theory with two minimally coupled scalar fields; a $2n$-order theory can be, in principle, reduced to an Einstein theory + $(n-1)$-scalar fields. On the other hand, these considerations can be directly generalized to higher-order-scalar-tensor theories in any number of dimensions as shown in [10]. With these considerations in mind, we can easily say that a higher order theory like $f(R, \Box R)$ is dynamically equivalent to $f(R, \phi)$. This feature, as we will show, gives the minimal ingredients to reproduce the rotation curves of galaxies since two Yukawa like corrections come out. For a detailed derivation see [15].

III. YUKAWA-LIKE CORRECTIONS TO THE GRAVITATIONAL POTENTIAL

In order to deal with standard self-gravitating systems, any theory of gravity has to be developed to its Newtonian or post-Newtonian limit depending on the order of approximation of the theory in terms of power of velocity $v^2$ [21, 31]. The paradigm of the Newtonian limit starts from the development of the metric tensor (and of all additional quantities in the theory) with respect to the dimensionless velocity $v^2$ of the moving massive bodies embedded in the gravitational potential. The perturbative development takes only first term of $0$, $0$- and $i, j$-component of metric tensor $g_{\mu\nu}$ (for details, see [32, 33]). The metric assumes the following form

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Psi)\delta_{ij}dx^i dx^j$$

(24)

where the gravitational potentials $\Phi$, $\Psi < 1$ are proportional to $v^2$. The adopted set of coordinates, the so-called isotropic coordinates, is $x^\mu = (t, x) = (t, x^1, x^2, x^3)$. The Ricci scalar is approximated as $R = R^{(1)} + R^{(2)} + \ldots$ where $R^{(1)}$ is proportional to $\Phi$, and $\Psi$, while $R^{(2)}$ is proportional to $\Phi^2$, $\Psi^2$ and $\Phi \Psi$.

Here we show as a general gravitational potential, with a Yukawa correction, can be obtained in the Newtonian limit of any analytic $f(R)$-gravity model. From a phenomenological point of view, this correction allows to consider as viable this kind of models even at small distances, provided that the Yukawa correction turns out to be not relevant in this approximation as in the so called "chameleon mechanism" [34].

A. Yukawa-like corrections in $f(R)$-gravity

Starting from the action $\mathcal{L}$ for the case $f(R, \Box R, \Box^2 R, \ldots, \Box^k R, \phi) + \omega(\phi)\phi, \phi^{\alpha\beta}$ reduced to $f(R)$, the field equations are [35]. In principle, the following analysis can be developed for any ETGs. Let us now start with the $f(R)$ case. As discussed in [3, 33, 36], we can deal with the Newtonian limit of $f(R)$ gravity adopting the spherical symmetry. By introducing the radial coordinate $r = |x|$ the metric (24) can be recast as follows

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2 The velocity $v$ is expressed in unit of light speed.

3 The Greek index runs from 0 to 3; the Latin index runs from 1 to 3.
\[ ds^2 = [1 + g_{tt}^{(1)}(t, r)] \, dt^2 - [1 - g_{rr}^{(1)}(t, r)] \, dr^2 - r^2 \, d\Omega , \]  
(25)

where \( d\Omega = d\theta^2 + \sin^2 \theta \, d\phi^2 \) is the angular distance, \((t, r, \theta, \phi)\) are standard coordinates. Since we want to obtain the most general result, we do not provide any specific form for the \( f(R) \). We assume, however, analytic Taylor expandable \( f(R) \) functions with respect to the value \( R = 0 \) (Minkowskian background):

\[
f(R) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} R^n = f_0 + f_1 R + \frac{f_2}{2} R^2 + \frac{f_3}{6} R^3 + ... \]  
(26)

In order to obtain the weak field approximation, one has to insert expansions (26) and (27) into field Eqs. (5) and expand the system up to the orders \( \mathcal{O}(0) \) and \( \mathcal{O}(1) \). This approach provides general results and specific (analytic) theories are selected by the coefficients \( f_i \) in Eq. (26). It is worth noticing that, at the order \( \mathcal{O}(0) \), the field equations give the condition \( f_0 = 0 \) and then the solutions at further orders do not depend on this parameter as we will show below. If we now consider the \( \mathcal{O}(1) \) - order approximation, the field equations in vacuum \((T_{\mu\nu} = 0)\), results to be

\[
f_1 r R^{(1)} - 2 f_1 g^{(1)}_{tt, r} + 4 f_2 R^{(1)} - f_1 r g^{(1)}_{tt, rr} + 2 f_2 r R^{(1)} = 0 ,
\]
\[
f_1 r R^{(1)} - 2 f_1 g^{(1)}_{rr, r} + 4 f_2 R^{(1)} - f_1 r g^{(1)}_{rr, rr} = 0 ,
\]
\[
2 f_1 g^{(1)}_{rr} - r \left[ f_1 r R^{(1)} - f_1 g^{(1)}_{tt, r} - f_1 g^{(1)}_{rr, r} + 2 f_2 R^{(1)} + 2 f_2 r R^{(1)} \right] = 0 ,
\]
\[
f_1 r R^{(1)} + 3 f_2 \left[ 2 R^{(1)} + r R^{(1)} \right] = 0 ,
\]
\[
2 g^{(1)}_{rr} + r \left[ 2 g^{(1)}_{tt, r} - r R^{(1)} + 2 g^{(1)}_{rr, r} + r g^{(1)}_{tt, rr} \right] = 0 .
\]

It is evident that the trace equation (the fourth in the system (27)), provides a differential equation with respect to the Ricci scalar which allows to solve exactly the system (27) at \( \mathcal{O}(1) \) - order. Finally, one gets the general solution:

\[
g^{(1)}_{tt} = \delta_0 - \frac{Y}{f_1 r} + \frac{\delta_1(t) e^{-mr}}{3 m^2 r} + \frac{\delta_2(t) e^{mr}}{6 m^3 r} ,
\]
\[
g^{(1)}_{rr} = - \frac{Y}{f_1 r} - \frac{\delta_1(t) [1 + mr] e^{-mr}}{3 m^2 r} - \frac{\delta_2(t) [1 - mr] e^{mr}}{6 m^3 r} ,
\]
\[
R^{(1)} = \frac{\delta_1(t) e^{-mr}}{r} + \frac{\delta_2(t) e^{mr}}{2 mr} ,
\]
(28)

where \( m^2 \equiv -\frac{f_2}{f_1} \), \( \delta_0 \) and \( Y \) are arbitrary constants, while \( \delta_1(t) \) and \( \delta_2(t) \) are arbitrary time functions. When we consider the limit \( f(R) \to R \) then \( m \to \infty \) and \( f_1 \to 1 \), in the case of a point-like source of mass \( M \), we recover the standard Schwarzschild solution if we set \( \delta_0 = 0 \) and \( Y = 2GM \). Let us notice that the integration constant \( \delta_0 \) is dimensionless, while the two functions \( \delta_1(t) \) and \( \delta_2(t) \) have respectively the dimensions of length\(^{-1}\) and length\(^{-2}\). These functions are completely arbitrary since the differential equation system (27) contains only spatial derivatives and can be fixed to constant values. Besides, the condition \( \delta_0 = 0 \) is available since it represents an unessential additive quantity for the potential.

The solutions (28) are valid if \( m^2 > 0 \) i.e. \( f_1 \) and \( f_2 \) are assumed to have different signs in Eq. (26). If the algebraic signs are equal we find an oscillating solution where the correction term to the newtonian component \((\propto 1/r)\) is proportional to the \((\cos mr + \sin mr)/r \) \( [32] \). In this paper we consider only the correction Yukawa-like.

It is possible now, to write the general solution of the problem considering the previous expressions (28). In order to match at infinity the Minkowskian prescription for the metric, one can discard the Yukawa growing mode in (28) and then we obtain:
\[
\begin{align*}
R &= \frac{\delta_1(t)e^{-mr}}{r} .
\end{align*}
\]

At this point, one can provide the solution in terms of gravitational potentials. The first of (28) gives the first order solution in term of the metric expansion (see the definition (25)). This term coincides with the gravitational potential at the Newton order. In particular, since \( g_{tt} = 1 + 2\Phi_{grav} = 1 + g_{tt}^{(1)} \), the gravitational potential of \( f(R) \)-gravity, analytic in the Ricci scalar \( R \), is

\[
\Phi_{grav} = -\left( \frac{GM}{f_1r} - \frac{\delta_1(t)e^{-mr}}{6m^2r} \right) .
\]

This general result means that the standard Newton potential is achieved only in the particular case \( f(R) = R \) while it is not so for analytic \( f(R) \) models up to exceptions of measure zero. Specifically all models with \( f_1/f_2 > 0 \) are excluded by hand. Eq. (30) deserves some comments. The parameters \( f_1, m \) and the function \( \delta_1(t) \) represent the deviations with respect the standard Newton potential. To test these theories of gravity inside the Solar System, we need to compare such quantities with respect to the current experiments, or, in other words, Solar System constraints should be evaded fixing such parameters [34]. On the other hand, these parameters could acquire non-trivial values (e.g. \( f_1 \neq 1, \delta_1(t) \neq 0, m < \infty \)) at scales different from the Solar System ones. Since the parameter \( m \) can be related to an effective length \( L^{-1} \), Eq. (30) can be recast as

\[
\Phi_{grav} = -\left( \frac{GM}{1+\delta} \right) \frac{1+\delta e^{-r/L}}{r} ,
\]

where the first term is the Newtonian-like part of the potential for a point-like mass \( \frac{M}{1+\delta} \) and the second term is a modification of gravity including a new scale length, \( L \) associated to the coefficients of the Taylor expansion. If \( \delta = 0 \) the Newtonian potential and the standard gravitational coupling are recovered. Comparing Eqs. (30) and (31), we assumed \( f_1 = 1 + \delta \) and \( \delta_1(t) = -\frac{6GM}{L^2} \left( \frac{\delta}{1+\delta} \right) \) where \( \delta \) can be chosen quasi-constant. Under this assumption, the scale length \( L \) could naturally arise and reproduce several phenomena that range from Solar System to large scale structure [14]. Understanding on which scales the modifications to GR are working or what is the weight of corrections to Newton potential is a crucial point that could confirm or rule out these extended approaches to gravitational interaction.

### B. Yukawa-like corrections in \( f(R, \phi) \)-gravity

A further step is to analyze the Newtonian limit starting from the action (1) and considering a generic function of Ricci scalar and scalar field. Then the action becomes

\[
A = \int d^4x \sqrt{-g} \left[ f(R, \phi) + \omega(\phi) \phi, \phi + X \mathcal{L}_m \right] .
\]

The field equations are obtained from (2) by setting \( f(R, \Box R, \Box^2 R, \ldots, \Box^k R, \phi) \rightarrow f(R, \phi) \). As discussed in Sec. II, this case can be considered from a purely geometric point of view assuming \( f(R, \Box R) \) theories where the terms \( \Box R \) gives a further scalar field contribution [15]. We get

\[
2\omega(\phi) \Box \phi + \omega(\phi) \phi, \phi - f_\phi = 0
\]
A further equation is the trace of field equation with respect to the metric tensor $g_{\mu \nu}$

$$f_R R - 2f - \omega(\phi) \phi,_{\alpha} \phi^{\alpha} + 3 \Box f_R = \chi T$$ \hspace{1cm} (34)

where $T = T^\sigma_\sigma$ is the trace of energy-momentum tensor.

Let us consider a point-like source with mass $M$. The energy-momentum tensor is

$$T_{\mu \nu} = \rho u_\mu u_\nu, \quad T = \rho$$ \hspace{1cm} (35)

where $\rho$ is the mass density and $u_\mu$ satisfies the condition $g^{\mu \nu}u_\mu u_\nu = 1$ and $u_i = 0$. Here, we are not interested to the internal structure. It is possible to analyze the problem in the more general case by using the isotropic coordinates $(t, x^1, x^2, x^3)$, then the metric is expressed as in Eq. \[3^1\]. In this framework, also the scalar field $\phi$ is approximated as the gravitational potentials $\Phi$ and $\Psi$ are given by

$$\Phi - \frac{f_{R\phi}(0, \phi(0))}{f_R(0, \phi(0))} \phi(1) + f_{\phi\phi}(0, \phi(0)) \phi(0) \phi(1) + \ldots$$

and analogous relations for the derivatives are obtained. From the lowest order of field Eqs. \[3^3\] we have

$$f(0, \phi(0)) = 0, \quad f_{\phi}(0, \phi(0)) = 0$$ \hspace{1cm} (37)

and also in this modified fourth order gravity a missing cosmological component in the action (1) implies that the space-time is asymptotically Minkowskian (the same outcome of previous section); moreover the ground value of scalar field $\phi$ must be a stationary point of potential. In the Newtonian limit, we have

$$\Delta \left[ \Phi - \frac{f_{R\phi}(0, \phi(0))}{f_R(0, \phi(0))} R(1) \right] - \frac{R(1)}{2} = \frac{\chi \rho}{f_R(0, \phi(0))}$$

$$\left\{ \Delta \left[ \Psi + \frac{f_{R\phi}(0, \phi(0))}{f_R(0, \phi(0))} R(1) \right] + \frac{R(1)}{2} \right\} \delta_{ij} +$$

$$\left\{ \Psi - \Phi - \frac{f_{RR}(0, \phi(0))}{f_R(0, \phi(0))} R(1) - \frac{f_{R\phi}(0, \phi(0))}{f_R(0, \phi(0))} \phi(1) \right\}_{,ij} = 0$$ \hspace{1cm} (38)

$$\Delta \phi(1) + \frac{f_{\phi\phi}(0, \phi(0))}{2 \omega(\phi(0))} \phi(1) = -\frac{f_{R\phi}(0, \phi(0))}{2 \omega(\phi(0))} R(1)$$

$$\Delta R(1) + \frac{f_{R\phi}(0, \phi(0))}{3 f_{RR}(0, \phi(0))} R(1) = -\frac{\chi \rho}{3 f_{RR}(0, \phi(0))} - \frac{f_{R\phi}(0, \phi(0))}{f_R(0, \phi(0))} \Delta \phi(1)$$

where $\Delta$ is the Laplacian in the flat space. The last equation in \[3^2\] is the trace coming from Eq. \[3^3\]. These equations are not simply the merging of field equations of $f(R)$-gravity and a further massive scalar field, but are due to the fact that the model $f(R, \phi)$ generates a coupled system of equations with respect to Ricci scalar $R$ and scalar field $\phi$. By supposing that $f_{\phi\phi} \neq 0$ and obviously $f_{RR} \neq 0$, we can introduce the two characteristic length scales

$$m_R^2 \doteq -\frac{f_R(0, \phi(0))}{3 f_{RR}(0, \phi(0))}, \quad m_\phi^2 \doteq -\frac{f_{\phi\phi}(0, \phi(0))}{2 \omega(\phi(0))}$$ \hspace{1cm} (39)

where the two masses are assumed to be real and this gives further restrictions to the set of viable models. The gravitational potentials $\Phi$ and $\Psi$ are given by

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4 The potential $\Psi$ can be found also as $\Psi(x) = \frac{1}{M} \int d^3 x' \frac{R^{(1)}(x', y')}{|x' - x|^3} + \frac{R^{(1)}(x)}{4 \pi m_R^2} - \frac{f_{R\phi}(0, \phi(0))}{f_R(0, \phi(0))} \phi(1)(x)$. 
\[ \Phi(x) = -\frac{\mathcal{X}}{4\pi f_R(0, \phi^{(0)})} \int d^3 \mathbf{x'} \frac{\rho(\mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|} - \frac{1}{8\pi} \int d^3 \mathbf{x'} \frac{R^{(1)}(\mathbf{x'})}{|\mathbf{x} - \mathbf{x'}|} - \frac{R^{(1)}(\mathbf{x})}{3m_R^2} + \frac{f_{R\phi}(0, \phi^{(0)})}{f_R(0, \phi^{(0)})} \phi^{(1)}(\mathbf{x}) \]

\[ \Psi(x) = \Phi(x) - \frac{R^{(1)}(x)}{3m_R^2} + \frac{f_{R\phi}(0, \phi^{(0)})}{f_R(0, \phi^{(0)})} \phi^{(1)}(x) \]

while, for the Ricci scalar and the scalar field, we have the coupled system of equations

\[ \left[ \triangle - m_\phi^2 \right] \phi^{(1)}(x) = -\frac{f_{R\phi}(0, \phi^{(0)})}{2\omega(\phi^{(0)})} \frac{R^{(1)}}{f_R(0, \phi^{(0)})} \]

\[ \left[ \triangle - m_R^2 \right] R^{(1)}(x) = \frac{m_R^2 \mathcal{X} \rho}{f_R(0, \phi^{(0)})} + \frac{3m_R^2 f_{R\phi}(0, \phi^{(0)})}{f_R(0, \phi^{(0)})} \triangle \phi^{(1)} \]

The definition of \( m_R^2 \) is the generalization of \( m^2 \) in the case of pure \( f(R) \)-gravity. By using the Fourier transformation the system (11) has the following solutions

\[ \phi^{(1)}(x) = -\frac{m_R^2 f_{R\phi}(0, \phi^{(0)}) \mathcal{X}}{2\omega(\phi^{(0)}) f_R(0, \phi^{(0)})} \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \frac{\hat{\rho}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}}{(k^2 + k_1^2)(k^2 + k_2^2)} \]

\[ R^{(1)}(x) = -\frac{m_R^2 \mathcal{X}}{f_R(0, \phi^{(0)})} \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \frac{\hat{\rho}(\mathbf{k}) (k^2 + m_\phi^2) e^{i\mathbf{k} \cdot \mathbf{x}}}{(k^2 + k_1^2)(k^2 + k_2^2)} \]

where

\[ 2k_{1,2}^2 = m_R^2 + m_\phi^2 - \frac{3f_{R\phi}(0, \phi^{(0)})^2 m_R^2}{2\omega(\phi^{(0)}) f_R(0, \phi^{(0)})} \pm \sqrt{\left[ m_R^2 + m_\phi^2 - \frac{3f_{R\phi}(0, \phi^{(0)})^2 m_R^2}{2\omega(\phi^{(0)}) f_R(0, \phi^{(0)})} \right]^2 - 4m_R^2 m_\phi^2} \]

In order to understand the relevant physical consequences of solutions (10), it is sufficient to analyze the point-like source framework. Then, if we consider \( \hat{\rho}(\mathbf{k}) = M/(2\pi)^{3/2} \), where \( M \) is the mass, and \( k_{1,2}^2 > 0 \) Eqs. (12) become

\[ \phi^{(1)}(x) = \frac{f_{R\phi}(0, \phi^{(0)})}{2\omega(\phi^{(0)}) f_R(0, \phi^{(0)})} \frac{r_g e^{-m_R k_1 |x|} - e^{-m_R k_2 |x|}}{k_1^2 - k_2^2} \]

\[ R^{(1)}(x) = -\frac{m_R^2}{f_R(0, \phi^{(0)})} \frac{r_g (\tilde{k}_1^2 - \eta^2) e^{-m_R k_1 |x|} - (\tilde{k}_2^2 - \eta^2) e^{-m_R k_2 |x|}}{k_1^2 - k_2^2} \]

where \( r_g \) is the Schwarzschild radius. Furthermore, we introduced the dimensionless quantities

\[ 2k_{1,2}^2 = \frac{2k_{1,2}^2}{m_R^2} = 1 - \xi + \eta^2 \pm \sqrt{(1 - \xi + \eta^2)^2 - 4\eta^2} \]

and \( \eta = \frac{m_\phi}{m_R}, \xi = \frac{3f_{R\phi}(0, \phi^{(0)})^2}{2\omega(\phi^{(0)}) f_R(0, \phi^{(0)})} \). These two parameters have to ensures two different conditions for the roots \( k_{1,2}^2 \) that have to be both real and positive i.e. \( k_{1,2}^2 > 0 \). Such conditions can be reformulated as \( \xi \leq (\eta - 1)^2 \) and this fact restrict the class of viable Lagrangians. In fact we have

\[ \xi = \frac{(k_1^2 - m_R^2)(k_2^2 - m_R^2)}{m_R^4}, \quad \eta^2 = \frac{k_1^2 k_2^2}{m_R^4} \]
where $\xi$ and $\eta$ are given in terms of $k_{1,2}^2$ and $m_R$ which are the parameters defining the form of Yukawa-like terms in the potentials. Specifically the conditions

$$f_{RR}(0, \phi^{(0)}) = -\frac{f_R(0, \phi^{(0)})}{3m_R^2}, \quad f_{\phi\phi}(0, \phi^{(0)}) = -2\omega(\phi^{(0)}) \frac{k_1^2k_2^2}{m_R^2},$$

which, together with conditions (37), give the form of possible Lagrangians. It is worth noticing that $f_R(0, \phi^{(0)})$ can be assumed equal to 1 in standard units, while $\omega(\phi^{(0)})$ fixes the form of scalar field kinetic term that is equal to $1/2$ in the canonical case. For $\omega(\phi^{(0)}) < 0$ a ghost scalar field is possible.

The potentials (40) become

$$\Phi(x) = -\frac{GM}{f_R(0, \phi^{(0)})} \left\{ 1 + g(\xi, \eta) e^{-m_Rk_1|x|} + \left[ \frac{1}{3} - g(\xi, \eta) \right] e^{-m_Rk_2|x|} \right\}$$

$$\Psi(x) = -\frac{GM}{f_R(0, \phi^{(0)})} \left\{ 1 - g(\xi, \eta) e^{-m_Rk_1|x|} - \left[ \frac{1}{3} - g(\xi, \eta) \right] e^{-m_Rk_2|x|} \right\}$$

(48)

where $g(\xi, \eta) = \frac{\tilde{k}_2^2(2\eta^2 - 2k_2^2 - 2\xi + 3) - 3\eta^2}{3k_1^2(k_1^2 - k_2^2)}$. In Figs. (1), (2), (3) and (4) we show respectively the spatial behaviors of the scalar field, the Ricci scalar and the potentials $\Phi$, $\Psi$.

![FIG. 1: The spatial behavior of scalar field $\phi^{(1)}$ generated by a point-like source \cite{44} for $\eta = 0.1$ and $\xi = -2$.](image)

The solutions (44), (48) are the generalization of solutions obtained in $f(R)$-gravity and scalar tensor gravity. In fact, we can easily obtain the outcomes that in the case of minimally coupled scalar field i.e. $f_{R\phi} = 0 \rightarrow \xi = 0$, $k_{1,2} = 1$, $\eta^2$, $g(\xi, \eta) = 1/3$ we find the point-like solutions of $f(R)$-gravity \cite{32, 33},

$$\Phi_{f(R)}(x) = -\frac{GM}{f_R(0)|x|} \left\{ 1 + \frac{1}{3} e^{-m_R|x|} \right\}$$

$$\Psi_{f(R)}(x) = -\frac{GM}{f_R(0)|x|} \left\{ 1 - \frac{1}{3} e^{-m_R|x|} \right\}$$

(49)

while in the case of the Brans-Dicke theory i.e. $f_{R\phi} = 1$, $f_R = \phi$, $\omega(\phi) = -\omega_0/\phi$, $f_{RR} = 0$, $f_{\phi\phi} = 0$, from the
FIG. 2: The spatial behavior of scalar field $R^{(1)}$ (dashed line) generated by a point-like source compared with respect to the same quantity (dotted line) in the $f(R)$-gravity. In both cases we set $\eta = 0.1$ and $\xi = -2$.

FIG. 3: Comparison among the potentials $\Phi$ generated by a point-like source in three frameworks: the first one (48) induced by the action (32) (dashed line), the second one (dotted line) induced by $f(R)$-gravity (49) and the last one (solid line) is the Newtonian limit of GR. In the two alternative theories, we set $\eta = 0.1$ and $\xi = -2$.

field Eqs. (38), we find the classical solutions of Brans-Dicke gravity

$$\Phi_{BD}(x) = -\frac{GM}{\phi^{(0)}|x|} \frac{2(2 + \omega_0)}{2 \omega_0 + 3}$$

$$\Psi_{BD}(x) = -\frac{GM}{\phi^{(0)}|x|} \frac{2(1 + \omega_0)}{2 \omega_0 + 3}$$

(50)

The Brans-Dicke behavior is also present in the solutions (48). In fact we recover the solutions (50) when $m_R \to \infty$, $m_\phi = 0$, $\xi = -\frac{3}{2 \omega_0}$. Also solutions (49) have been found in vacuum, but here also the boundary conditions for the Yukawa term in the origin (where the mass is placed) have been inserted. In this case, the arbitrary time-function $\delta_1(t)$ in Eq. (39) is fixed to the value $-\frac{2m^2}{f_1}$ and then we can define $\Phi_{grav} = \Phi_{f(R)}$, while the expression for $\Psi_{grav}$ in the Eq. (29) becomes

$$\Psi_{grav} = \frac{g_{rr}^{(1)}}{2} = -\frac{GM}{f_1 r} \left\{1 - \frac{1 + mr}{3} e^{-mr}\right\}$$

(51)

It is possible to show that the potential $\Psi_{grav}$ is equal to the potential $\Psi_{f(R)}$ of Eqs. (49) if we assume standard coordinates. The passage from the isotropic coordinates $(t, x^1, x^2, x^3)$ to the standard ones $(t, r, \theta, \phi)$ is given by the
transformation \[ 1 - 2 \Psi(f(R)(|x|)) \] \(|x|^2 = r^2 \) where \(|x|^2 = x_1x^1\), then, at first order with respect to the quantity \(r_g/r \) (or \(r_g/|x|\)), the metrics

\[
d s^2 = \left[ 1 - \frac{r_g}{f_R(0)|x|} \left( 1 + \frac{1}{3} e^{-m_R|x|} \right) \right] d t^2 - \left[ 1 + \frac{r_g}{f_R(0)|x|} \left( 1 - \frac{1}{3} e^{-m_R|x|} \right) \right] \delta_{ij} dx^i dx^j
\]

\[ ds^2 = \left[ 1 - \frac{r_g}{f_{R1} r} \left( 1 + \frac{1}{3} e^{-m r} \right) \right] d t^2 - \left[ 1 + \frac{r_g}{f_{R1} r} \left( 1 - \frac{1}{3} e^{-m r} \right) \right] d r^2 - r^2 d \Omega \]

coincide and, obviously, for \(f(R, \phi) \to f(R)\), we have \(m_R = m\) and \(f_R(0) = f_1\).

It is interesting to note that, in the case of minimally coupled scalar field \((f_{R\phi} = 0)\), the Newtonian level of the field Eqs. \[52\] for the metric tensor is unaffected by the presence of the scalar field \(\phi\). Moreover \(\phi\) is not linked to the energy-momentum tensor via the Ricci scalar and must satisfy only the boundary conditions at infinity, while the amplitude of scalar field is generic and depending only on time. In fact, the solution of the first of Eqs. \[51\] is

\[
\phi^{(1)}(t, x) = \frac{K(t)}{2 \omega(\phi^{(0)})} \frac{e^{-m_R|x|}}{|x|},
\]

where \(K(t)\) is a generic function depending on time. The evolution of \(K(t)\) is fixed by the post-Newtonian level of field equations. By considering \(f_{R\phi} \neq 0\), we find a further contribution in the energy-momentum tensor. Another interesting case is the generalization of Brans-Dicke theory. In fact, we can consider a scalar-tensor theory, but the geometric sector is given only by the Ricci scalar. Without losing generality, we can set the interaction term in the action \[53\] as \(\phi R\) and not as \(f(\phi) R\) because by introducing a new scalar field we obtain formally the same equations \[53\]. By setting \(f_{R,\phi} = 1, f_{RR} = 0, f_R = \phi\) the field Eqs. \[53\] become

\[\]

\[5\] Also in this case we can find the solution of \(\Psi\) as \(\Psi(x) = \frac{1}{4\pi} \int d^3x' R^{(1)}(x') \frac{\phi^{(1)}(x)}{|x-x'|},\)
\[ \frac{\Delta \Phi}{\phi(0)} + \frac{\Delta \phi(1)}{\phi(0)} + \frac{R(1)}{2} \]

\[ \Psi = \Phi + \frac{\phi(1)}{\phi(0)} \]

\[ \left[ \Delta - m_\phi^2 \right] \phi(1) = -\frac{R(1)}{2 \omega(\phi(0))} \]

\[ R(1) = \frac{\mathcal{X} \rho(0)}{\phi(0)} \frac{3 \Delta \phi(1)}{\phi(0)} \]

and their solutions are

\[ \phi(1)(x) = -\frac{1}{2 \omega(\phi(0)) \phi(0) - 3 |x|} \frac{r_g}{e^{-\sqrt{\frac{2 \omega(\phi(0)) \phi(0)}{2 \omega(\phi(0)) \phi(0) - 3} m_\phi |x|}} \]

\[ R(1)(x) = -\frac{4 \pi r_g}{\phi(0)} \frac{3 \omega(\phi(0)) m_\phi^2}{\phi(0) - 3 |x|} \frac{r_g}{e^{-\sqrt{\frac{2 \omega(\phi(0)) \phi(0)}{2 \omega(\phi(0)) \phi(0) - 3} m_\phi |x|}} \]

\[ \Phi_{ST}(x) = \frac{-GM}{\phi(0)|x|} \left\{1 - e^{-\sqrt{\frac{2 \omega(\phi(0)) \phi(0)}{2 \omega(\phi(0)) \phi(0) - 3} m_\phi |x|}} \right\} \]

\[ \Psi_{ST}(x) = \frac{-GM}{\phi(0)|x|} \left\{1 + e^{-\sqrt{\frac{2 \omega(\phi(0)) \phi(0)}{2 \omega(\phi(0)) \phi(0) - 3} m_\phi |x|}} \right\} \]

which, in the case of massless scalar field, become the typical solution of Brans-Dicke theory.

In the cases that we have shown, the Newtonian contribution \(|x|^{-1}\) to the potential is ever present. We can find a difference in the definition of gravitational constant \(G\), since in these theories we have a multiplying factor \(f_R(0, \phi(0))^{-1}\), while the additional terms are depending on the form of the Lagrangian. It is important to stress that, in all cases that we have considered, the limit and results of GR are fully recovered.

If we have a generic matter source distribution \(\rho(x)\), it is sufficient to use the superposition principle by starting from point-like solutions. Then we substitute to the solutions (48) the integral expression: \(\Phi \rightarrow \int \Phi\). This approach is correct only in the Newtonian limit since such a limit correspond also to the linearized version of the theory.

**IV. ROTATION CURVES OF GALAXIES**

At astrophysical level, the probe for the validity of alternative theories of gravity is the correct reproduction of rotation curves of spiral galaxies [16]. As discussed above, the foundation of the dark matter issue lies on this observational evidence. In order to face such a problem, one has to discuss the motion of a body embedded in a gravitational field. Let us take into account the geodesic equation

\[ \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \]

where \(ds = \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}\) is the relativistic distance. In the Newtonian limit, from Eq.(56), we obtain the equation of motion equation

\[ \frac{d^2 x}{dt^2} = -\nabla \Phi(x). \]
In our case, the gravitational potentials are given by (48). The study of motion is very simple if we consider a particular symmetry of mass distribution $\rho$, otherwise the analytical solutions are not available. Our aim is to evaluate the corrections to the classical motion in the easiest situation: the circular motion. In this case we do not consider the radial and vertical motions. The condition of stationary motion on the circular orbit is

$$v_c(|x|) = \sqrt{|x| \frac{\partial \Phi(x)}{\partial |x|}}$$

(58)

where $v_c$ is the velocity. Generally the correction terms do not satisfy the Gauss theorem [39] and this aspect implies that a sphere cannot be simply reduced to a point. In fact the gravitational potential generated by a sphere (also with constant density) is depending also on the Fourier transform of the sphere [39]. Only in the limit case, where the radius of the sphere is small with respect to the distance (point-like source), we obtain the simple expression (48).

A further remark on Eqs. (48) is needed. The structure of solutions is mathematically similar to the one of fourth-order gravity generated by $f(R, R_{\alpha\beta}R^{\alpha\beta})$. However there is a fundamental difference with the present case: the two Yukawa corrections have different algebraic sign. In particular, the Yukawa correction induced by a generic function of Ricci scalar implies a stronger attractive gravitational force, while the second one, induced by squared Ricci tensor, implies a repulsive force [32, 40]. In the present paper, the Yukawa corrections are induced by a generic function of Ricci scalar and a non-minimally coupled scalar field. Both corrections have a positive coefficient. In fact in Fig. 5, we show the coefficient $g(\xi, \eta)$ with respect to $\xi$ for a given values of $\eta$. The function $g(\xi, \eta)$ assumes the maximum value ($= 1/3$) when $\xi = 0$ (we have a pure $f(R)$-gravity and the scalar field does not contribute to the gravitational potential in this case), otherwise we have two Yukawa corrections with positive coefficients. The scalar field gives rise to a more attractive force than in $f(R)$-gravity. The interesting range of values of $\eta$ is between 0 and 1. In the case $\eta > 1 \to m_\phi > m_R$, the correction induced by scalar field is suppressed with respect to the other one.

![Figure 5](image.png)

FIG. 5: Plot of coefficient $g(\xi, \eta)$ with respect to quantity $\xi$ for $0 \leq \eta \leq 0.99$ with step 0.33.

From this analysis, the choice of $f(R, \phi)$-gravity is better than $f(R, R_{\alpha\beta}R^{\alpha\beta})$-gravity, but we have a problem in the limit for $|x| \to \infty$: the interaction is, of course, scale-depending (the scalar fields are massive) and in the vacuum the corrections turn off. For this reason, at large distances, we recover only the classical Newtonian contribution. Therefore the presence of scalar fields makes the profile smooth. This behavior is very clear in the study of rotation curves [58]. Let us assume a phenomenological point-like gravitational potential as supposed by Sanders [14, 41]

$$\Phi_{SP}(x) = -\frac{GM}{|x|}(1 + \alpha e^{-m|x|})$$

(59)

where $\alpha$ and $m$ are free parameters that, following Sanders [41], can be assumed to be $\alpha \simeq -0.92$ and $r_0 = 1/m \simeq 40$ Kpc to fit the galactic rotation curves. This potential has been introduced to explain the rotation curves of spiral galaxies [41, 42], however the theoretical framework generating it is purely phenomenological. Recently by using the same potential it has been possible to fit elliptical galaxies [43]. In both cases by setting a negative value to $\alpha$ an almost constant profile of rotation curve is recovered. Such a rotation curve is obviously possible but there are two problems: the first one consists that no $f(R, \phi)$-gravity, by imposing all boundary conditions at origin and at
infinity, gives that negative value of $\alpha$. The second one is linked to the value of gravitational constant $G$. In fact in presence of Yukawa-like correction with negative coefficient, we find a lower rotation curve and only by resetting $G$ (or the point-like mass) we can fit the experimental data. In Fig. 6 we compare the profiles derived in the Newtonian limit of GR, $f(R)$- and $f(R, \phi)$-gravity and the potential (59). It is extremely interesting to note that the presence of scalar the field $\phi$, in the case $m_\phi \sim m_R$, guarantees a rotation curve higher than the other ones but also in this case we find the asymptotic flatness are derived from observations. Only if we consider a massive scalar tensor theory non-minimally coupled, we get a potential with negative coefficient in Eq.(55). In fact by setting the gravitational constant as $G_0 = \frac{2 \omega(\phi^{(0)}) \dot{\phi}^{(0)}}{2 \omega(\phi^{(0)}) \phi^{(0)} - 3 \phi^{(0)}}$, where $G_\infty$ is the gravitational constant as measured at infinity and by imposing $\alpha^{-1} = 3 - 2 \omega(\phi^{(0)}) \phi^{(0)}$, the potential $\Phi_{\text{ST}}$ in the (55) becomes

$$\Phi_{\text{ST}}(x) = -\frac{G_\infty M}{|x|} \left\{1 + \alpha e^{-\sqrt{1 - 3\alpha m_\phi|x|}}\right\}$$

and then the Sanders potential (59) is fully recovered.

V. DISCUSSION AND CONCLUSIONS

The dark matter issue, together with dark energy, can be considered the major problem of modern astrophysics and cosmology. Beside the huge amount of observations confirming its effects, practically at all astrophysical scales, no final answer exists, at fundamental level, definitively confirming one (or more than one) candidate supposed to explain the phenomenology. Furthermore, GR has been firmly tested only up to Solar System scale and then its features have been inferred at larger scales. In this situation, dark matter and dark energy could be nothing else but the manifestation that GR does not work at IR scales.

A similar disturbing situation is found at UV scales where no Quantum Gravity theory is up to now definitely available. Alternative gravities (in particular ETGs) could represent a way out to this puzzle being effective theories of gravity representing a reliable picture of quantum fields in high curvature regimes and an approach to overcome the dark side problem at larger scales.

In this paper, we have discussed the weak field limit (in particular the Newtonian limit) of some classes of ETGs in view to explain the almost flat rotation curves of spiral galaxies. In particular, we have shown that ETGs, in general, present Yukawa-like corrections in the gravitational potential. In particular, we have analyzed the case of $f(R)$ and $f(R, \phi)$. The latter are known to be analogue to $f(R, \Box R)$.

After a discussion of the mathematical features of the emerging corrections, we have confronted the results with the phenomenological Sanders potential, assumed as a possible dynamical explanation of flat rotation curves. The suitable value of phenomenological parameters can be exactly reproduced in the framework of $f(R, \phi)$-gravity since the concurring Yukawa corrections allow to recover attractive and repulsive components of potential. In this case, no
dark matter is required to fit dynamics like in the case discussed in [42] where only a Yukawa-like correction was not sufficient to reproduce realistic rotation curves.

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