A VARIATION ON THE DONSKER-VARADHAN INEQUALITY FOR THE PRINCIPAL EIGENVALUE

JIANFENG LU AND STEFAN STEINERBERGER

Abstract. The purpose of this short paper is to give a variation on the classical Donsker-Varadhan inequality, which bounds the first eigenvalue of a second-order elliptic operator on a bounded domain \( \Omega \) by the largest mean first exit time of the associated drift-diffusion process via

\[
\lambda_1 \geq \frac{1}{\sup_{x \in \Omega} \mathbb{E}_x \tau_{\Omega^c}}.
\]

Instead of looking at the mean of the first exit time, we study quantiles: let \( d_{p,\partial \Omega} : \Omega \to \mathbb{R}_+ \) be the smallest time \( t \) such that the likelihood of exiting within that time is \( p \), then

\[
\lambda_1 \geq \frac{\log (1/p)}{\sup_{x \in \Omega} d_{p,\partial \Omega}(x)}.
\]

Moreover, as \( p \to 0 \), this lower bound converges to \( \lambda_1 \).

1. Introduction

We consider, for open and bounded \( \Omega \subset \mathbb{R}^n \), solutions of the equation

\[
-\text{div}(a(x)\nabla u(x)) + \nabla V \cdot \nabla u = \lambda u \quad \text{in } \Omega
\]
\[
u = 0 \quad \text{on } \partial \Omega
\]

Estimating the smallest possible value of \( \lambda_1 \) for which the equation has a solution is a problem of fundamental importance. Finding upper bounds is, in many instances, rather straightforward by testing with a family of functions – finding lower bounds is substantially more difficult. An important conceptual leap is due to Donsker & Varadhan, who take

\[
L = -\text{div}(a(x)\nabla u(x)) + \nabla V \cdot \nabla u
\]

and take \(-L\) as the infinitesimal generator of a drift-diffusion process (here and in all subsequent steps we always assume sufficient regularity on both the operator and the domain). The maximum mean exit time then serves as a lower bound of the first eigenvalue \( \lambda_1 \) (we use the formulation from [1]).

Theorem (Donsker-Varadhan [5, 6], CPAM 1976).

\[
\lambda_1 \geq \frac{1}{\sup_{x \in \Omega} \mathbb{E}_x \tau_{\Omega^c}}.
\]
Proof. The proof is simple: note that  \( w(x) = \mathbb{E}_x \tau_{\Omega^c} \) solves the equation
\[
- \text{div}(a(x) \nabla w(x)) + \nabla V \cdot \nabla w = 1 \quad \text{in} \quad \Omega \\
w = 0 \quad \text{on} \quad \partial \Omega.
\]
We also observe that, by definition, the first eigenfunction  \( u(x) \) solves
\[
- \text{div}(a(x) \nabla u(x)) + \nabla V \cdot \nabla u = \lambda_1 u \quad \text{in} \quad \Omega \\
u = 0 \quad \text{on} \quad \partial \Omega.
\]
This implies, by linearity,
\[
- \text{div} \left( a(x) \nabla \left[ \lambda_1 w(x) \max_{x \in \Omega} |u(x)| - u(x) \right] \right) + \nabla V \cdot \nabla \left( \lambda_1 w(x) \max_{x \in \Omega} |u(x)| - u(x) \right) \geq 0.
\]
The maximum principle then implies
\[
\lambda_1 w(x) \max_{x \in \Omega} |u(x)| - u(x) \geq 0,
\]
which yields
\[
\lambda_1 w(x) \geq \frac{u(x)}{\max_{x \in \Omega} |u(x)|}
\]
from which we obtain, by setting \( x \) so that  \( |u| \) assumes its maximum,
\[
\lambda_1 \max_{x \in \Omega} w(x) \geq 1.
\]
\[\square\]

The result can be interpreted in two ways: if, perhaps by symmetry considerations, it is possible to roughly predict the location that maximizes the mean first exit time, then the result allows for lower bounds on the eigenvalue  \( \lambda_1 \) and, conversely, knowledge about the eigenvalue  \( \lambda_1 \) guarantees the existence of points in the domain for which the mean first exit time is ‘large’. Among other applications, the Donsker-Varadhan estimate is crucially used in the potential theoretic analysis of metastability in [1, 2, 3] (see Lemma 2.1 in [3] where the Lemma is quoted and an improvement in Lemma 2.2 in the same paper) and in Markov state models (see e.g., [8] and references therein). We shall not focus too much on the minimal regularity of  \( L \): the reader may assume that  \( a(x) \) is uniformly elliptic and both  \( a(x) \) and  \( V(x) \) are smooth; in practice, the results will hold in much rougher situations and only relies on the Feynman-Kac formula being applicable (which even allows moderate singularities in  \( V \) ). Moreover, the arguments are versatile enough to be applicable to Graph Laplacian on Markov chain; the changes are completely obvious changes of symbols and will not be detailed in this paper.

2. The Result

The Donsker-Varadhan inequality is based on the mean value of the first exit time. We will work with quantiles of that distribution instead: for fixed  \( 0 < p < 1 \), we define the diffusion distance to the boundary  \( d_{p,\partial \Omega} : \Omega \to \mathbb{R}_+ \) implicitly as the smallest number
\[
P (\text{first exit time} \geq d_{p,\partial \Omega}(x_0)) \leq p,
\]

where the probability is taken over drift-diffusion processes generated by $-L$ and started in $x_0$. Our main result is that there is a natural relation between that quantity and the smallest eigenvalue $\lambda_1$ of the differential operator.

**Theorem.** We have

\[
d_{p,\partial\Omega}(x) \geq \frac{1}{\lambda} \log \left( \frac{1}{p} \frac{|u(x)|}{\|u\|_{L^\infty(\Omega)}} \right).
\]

We are not aware of this result being in the literature. Related statements seem to have first appeared in [7, 9], a discrete analogue was given by Cheng, Rachh and the second author in [4]. In most cases, the definition may be simplified as $P(\text{first exit time} \geq d_{p,\partial\Omega}(x_0)) = p$, however, the definition above also covers time-discrete processes on Markov chains with absorbing states where a similar estimate can be easily obtained (we leave the details to the reader).

**Corollary** (Donsker-Varadhan for Quantiles).

\[
\lambda_1 \geq \log(1/p) \sup_{x \in \Omega} d_{p,\partial\Omega}(x).
\]

Moreover, the right-hand side converges to $\lambda_1$ as $p \to 0$.

We observe two major differences that become relevant when estimating $d_{p,\partial\Omega}(x)$ with a Monte Carlo method:

1. Instead of having to compute a mean (which, especially for heavy-tail distributions, can be difficult), it suffices to estimate the likelihood of exiting within a fixed time $t$. The desired outcome is a Bernoulli variable with likelihood $p$ – the problem thus reduces to estimating the parameter in a $\{0,1\}$ Bernoulli distribution and adjusting time $t$, which is more stable.

2. By decreasing the value of $p$, the result can be arbitrarily refined – the difficulty being that estimating the parameter becomes more computationally costly as $p \to 0$, as one needs more simulations to ensure that there are enough samples in the $p-$th quantile to give a stable estimation of the Bernoulli parameter. In practice, the available amount of computation will impose a restriction on the value of $p$ that can be reasonably estimated with a certain degree of confidence.

3. **Proofs**

### 3.1. Proof of the Theorem.

**Proof.** We assume w.l.o.g. that $u(x) > 0$ and define the parameter $0 < \delta < 1$ implicitly via $\delta\|u\|_{L^\infty} = u(x)$. We use $\omega(t)$ to denote drift-diffusion process (associated to the Feynman-Kac formula) started in $x$ and running up to time $t$.

Since

\[
- \text{div}(a(x) \nabla u(x)) + \nabla V \cdot \nabla u = \lambda_1 u,
\]

we have that

\[
u(x) = e^{\lambda t} E_x (u(\omega(t)))
\]

with the convention that $u(\omega(t))$ is 0 if the drift-diffusion processes leaves $\Omega$ at some point in the interval $[0,t]$. Let now $t = d_{p,\partial\Omega}(x)$, in which case we see that

\[
E_x (u(\omega(t))) \leq p\|u\|_{L^\infty} + (1-p)0.
\]
Altogether, we obtain
\begin{equation}
\delta \|u\|_{L^\infty} = u(x) = e^{\lambda d_{p,\partial\Omega}(x)} E_x \langle u(\omega(t)) \rangle \leq e^{\lambda d_{p,\partial\Omega}(x)} p \|u\|_{L^\infty}
\end{equation}
from which the statement follows. \qed

3.2. Proof of the Corollary.

Proof. It remains to show that the lower bound is asymptotically sharp as \( p \to 0^+ \). Let \( x \in \Omega \) be arbitrary and let \( \delta_x \) be the Dirac distribution centered at \( x \). We are interested in the long-time behavior of applying the drift-diffusion process to these initial conditions; denoting the eigenpairs of the differential operator by \( (\lambda_k, \phi_k) \), we can use the spectral theorem (see e.g. \cite{10}) to estimate
\begin{equation}
\int_{\Omega} e^{(\text{div}(a(x)\nabla\cdot x) - \nabla\cdot x) t} \delta_x dz = \int_{\Omega} \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle \delta_x, \phi_k \rangle \phi_k(z) dz.
\end{equation}
The spectral gap implies that, as \( t \to \infty \),
\begin{equation}
\int_{\Omega} \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle \delta_x, \phi_k \rangle \phi_k(z) dz = \phi_1(x) e^{-\lambda_1 t} \int_{\Omega} \phi_1(z) dz + o(e^{-\lambda_1 t}).
\end{equation}
This means that, asymptotically as \( t \to \infty \), the survival probability is maximized by starting in the point in which the first eigenfunction assumes a global maximum. Conversely, the case \( p \to 0 \) is equivalent to the case \( t \to \infty \) and by locating \( x \) in the point \( x_0 \in \Omega \), where the ground state assumes its maximum, we get that
\begin{equation}
\sup_{x \in \Omega} d_{p,\partial\Omega}(x) = (1 + o(1)) d_{p,\partial\Omega}(x_0).
\end{equation}
The computation above shows, as \( p \to 0 \) and \( t \to \infty \)
\begin{equation}
e^{-\text{div}(a(x)\nabla\cdot x) + \nabla\cdot x) t} \delta_{x_0} = (1 + o(1)) \| \phi_1 \|_{L^\infty} e^{-\lambda_1 t} \int_{\Omega} \phi_1(x) dx.
\end{equation}
This implies nontrivial bounds on the logarithm of the survival probability
\begin{equation}
\log P(\text{first exit time} \geq t) = -(\lambda_1 + o(1)) t.
\end{equation}
Then, by definition,
\begin{equation}
\log p = \log P(\text{first exit time} \geq d_{p,\partial\Omega}(x_0)) = -(\lambda_1 + o(1)) d_{p,\partial\Omega}(x_0)
\end{equation}
and this then implies
\begin{equation}
(1 + o(1)) \lambda_1 \sup_{x \in \Omega} d_{p,\partial\Omega}(x) = \log (1/p).
\end{equation}
\qed

The main idea of the argument is that \( p \to 0 \) naturally corresponds to \( t \to \infty \). The spectral theorem implies that long-time asymptotics is essentially given by the first eigenvalue and the first eigenfunction via
\begin{equation}
e^{-t L} f \sim e^{-\lambda_1 t} \langle f, \phi_1 \rangle \phi_1
\end{equation}
and this is how we indirectly obtain estimates on \( \lambda_1 \). This also suggests that it might perhaps be possible to obtain estimates on the convergence speed depending on the spectral gap.
4. Numerical Examples

4.1. Unit interval. A toy example is given by

\begin{equation}
-\Delta u = \lambda u \quad \text{in } [0, 1]
\end{equation}

\begin{align*}
u(0) &= 0 = u(1).
\end{align*}

The ground state is \( u(x) = \sin \pi x \) and \( \lambda_1 = \pi^2 \sim 9.86 \). – the Donsker-Varadhan estimate requires us to solve \(-\Delta w = 1\), which easily gives \( w(x) = x/2 - x^2/2 \) and from which we get the lower bound \( \lambda \geq 8 \). In comparison, our bound for various values of \( p \) are

| \( p \) | 1/2 | 1/4 | 10^{-1} | 10^{-2} | 10^{-8} |
|-------|-----|-----|---------|---------|--------|
| lower bound | 7.28 | 8.40 | 8.92 | 9.39 | 9.74 |

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4.2. Unit interval with a quadratic potential. Let us consider a 1D example with a quadratic potential \( V = \frac{1}{2}x^2 \) on \([-1, 1] \):

\begin{equation}
-\Delta u + x \nabla u = \lambda u \quad \text{in } [-1, 1]
\end{equation}

\begin{align*}
u(-1) &= 0 = u(1).
\end{align*}

The ground state is \( u(x) = 1 - x^2 \) with \( \lambda = 2 \). The mean first exit time \( w \) solves

\begin{equation}
-\Delta w + x \nabla w = 1 \quad \text{in } [-1, 1]
\end{equation}

with Dirichlet boundary condition. Solving the equation by central difference scheme with mesh size \( h = 10^{-4} \) yields the Donsker-Varadhan estimate \( \lambda \geq 1.678 \). To use our bound for various values of \( p \), we simulate \( 10^4 \) paths using an Euler-Maruyama scheme with time step size \( t = 10^{-4} \) starting at the origin (thanks to the symmetry), the following lower bounds are obtained.

| \( p \) | 0.5 | 0.3 | 0.2 | 0.1 | 0.05 |
|-------|-----|-----|-----|-----|-----|
| lower bound | 1.522 | 1.675 | 1.740 | 1.799 | 1.834 |

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4.3. Unit disk. Finally, we estimate the ground state of the Laplacian on the unit disk in \( \mathbb{R}^2 \), which is given by the first nontrivial zero of the Bessel function \( \lambda_1 \sim 2.40 \ldots \) while the Donsker-Varadhan estimate gives

\begin{equation}
w(x) = 1/2 - (x^2 + y^2)/2 \quad \text{and thus } \lambda_1 \geq 2.
\end{equation}

Suppose we could not solve any of these equations in closed form (as is usually the case): using the symmetry of the domain, it suffices to take Brownian motion started in the origin. Discrete Brownian motion with step size (in time) \( t = 10^{-4} \) and \( 10^4 \) paths give the following estimates for a lower bound on \( \lambda_1 \)

| \( p \) | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 |
|-------|-----|-----|-----|-----|-----|
| lower bound | 1.68 | 1.85 | 2.04 | 2.19 | 2.37 |

Donsker-Varadhan | 1.96
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(Jianfeng Lu) DEPARTMENT OF MATHEMATICS, DEPARTMENT OF PHYSICS, AND DEPARTMENT OF CHEMISTRY, DUKE UNIVERSITY, BOX 90320, DURHAM NC 27708, USA
E-mail address: jianfeng@math.duke.edu

(Stefan Steinerberger) DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06510, USA
E-mail address: stefan.steinerberger@yale.edu