Some new results related to Lorentz $G\Gamma$-spaces and interpolation

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The presentation is based:

Ahmed, Irshaad; Fiorenza, Alberto; Formica, Maria Rosaria; Gogatishvili, Amiran; Rakotoson, Jean Michel; Some new results related to Lorentz $G\Gamma$-spaces and interpolation. J. Math. Anal. Appl. 483 (2020), no. 2, 123623.
The original question comes from an unpublished manuscript by H. Brezis. Let $f$ be given in $L^1(\Omega, \text{dist}(x, \partial \Omega))$ ($\Omega$ bounded smooth open set of $\mathbb{R}^n$), then H. Brezis shows the existence and uniqueness of a function $u \in L^1(\Omega)$ satisfying

$$|u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega, \text{dist}(x, \partial \Omega))}$$

with

$$GD(\Omega) = \begin{cases} - \int_{\Omega} u \Delta \varphi \, dx = \int_{\Omega} f \varphi \, dx, & \forall \varphi \in C^2_0(\overline{\Omega}), \\ \text{with } C^2_0(\overline{\Omega}) = \{ \varphi \in C^2(\overline{\Omega}), \varphi = 0 \text{ on } \partial \Omega \}. \end{cases}$$

Therefore, the question of the integrability of the generalized derivative of $u$ arises in a natural way and

It was raised already in the note by H. Brezis and developed in:

J.I. Díaz, J.M. Rakotoson, On the differentiability of the very weak solution with right-hand side data integrable with respect to the distance to the boundary

*J. Functional Analysis* 257 (2009) 807-831.

J.M. Rakotoson, New hardy inequalities and behaviour of linear elliptic equations

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*AMO* 73 (2016) 153-163.

Some new results related to $G\Gamma$-spaces and interpolation
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**J.M. Rakotoson**, A sufficient condition for a blow-up on the space. Absolutely conditions functions for the very weak solution *AMO* **73**(2016) 153-163.
More generally, the question of the regularity of $u$ is arised, according to $f$.

In the papers:

**J. I. Díaz, D. Gómez-Castro, J. M. Rakotoson, R. Temam**, Linear diffusion with singular absorption potential and/or unbounded convective flow: the weighted space approach, *Discrete and Continuous Dynamical Systems* **38**, 2, (2018) 509-546

**A. Fiorenza, M.R. Formica, J.M. Rakotoson**, Pointwise estimates for $G\Gamma$-functions and applications, *Differential Integral Equations* **30**, 11-12 (2017) 809-824.

Following result have shown:
Theorem (1)

Let $\Omega$ be a bounded open set of class $C^2$ of $\mathbb{R}^n$, $|\Omega| = 1$ and $\alpha \geq \frac{1}{n'}$ where $n' = \frac{n}{n-1}$, $f \in L^1(\Omega; \delta)$, with $\delta(x) = \text{dist}(x; \partial \Omega)$.

Consider $u \in L^{n',\infty}(\Omega)$, the very weak solution (v.w.s.) of

$$-\int_{\Omega} u \Delta \varphi \, dx = \int_{\Omega} f \varphi \, dx \quad \forall \varphi \in C^2(\overline{\Omega}), \varphi = 0 \text{ on } \partial \Omega. \quad (1)$$

Then,

1. if $f \in L^1\left(\Omega; \delta\left(1 + |\text{Log} \delta|\right)\right)$ and $\alpha > \frac{1}{n'}$:

   $$u \in L^{(n',n\alpha-n+1)}(\Omega) = G\Gamma(n',1;w_\alpha), \ w_\alpha(t) = t^{-1}(1 - \text{Log} \ t)^{\alpha-1-\frac{1}{n'}}$$

   and

   $$\|u\|_{G\Gamma(n',1;w_\alpha)} \leq K_0 |f|_{L^1(\Omega;\delta(1+|\text{Log} \delta|)\alpha)} \quad (2)$$

2. if $f \in L^1\left(\Omega; \delta\left(1 + |\text{Log} \delta|\right)^{\frac{1}{n'}}\right)$ then $u \in L^{n'}(\Omega)$ and

   $$\|u\|_{L^{n'}} \leq K_1 |f|_{L^1(\Omega;\delta(1+|\text{Log} \delta|)^{1/n'})} \quad (3)$$
The natural question is *how to extend of Theorem 1 for* $\alpha < \frac{1}{n'}$ *and how to improve the estimate when* $\alpha = \frac{1}{n'}$?

Since the solution of (1) satisfies also

$$
|u|_{L^{n'}, \infty (\Omega)} \leq K_1 |f|_{L^1(\Omega; \delta)},
$$

(4)

the natural idea to obtain an estimate is to use the real interpolation method of Marcinkiewicz to derive

$$
|u|_{(L^{n'}, \infty, L^{n'})_{\alpha, 1}} \leq K_2 |f|_{L^1(\Omega; \delta (1 + |\log \delta|)^\alpha)} \quad \text{for } 0 < \alpha < 1.
$$

(5)
How to characterize the space \( \left( L^{n', \infty}(\Omega), L^{(n')}(\Omega) \right)_{\alpha,1} \)?
Question 2

*How to characterize the space* \( \left( L^{n', \infty}(\Omega), L^{n'}(\Omega) \right)_{\alpha,1} \)?

We still have not an answer to this question. Therefore, we will provide a lower estimate for the norm of \( u \) in relation (5), a particular overbound can be obtained from our work made in

**A. Fiorenza, M.R. Formica, A. Gogatishvili, T. Kopaliani, J.M. Rakotoson**, Characterization of interpolation between Grand, small or classical Lebesgue spaces, *Non Linear Analysis* **177** (2018) 422-453. DOI https://doi.org/10.1016/j.na.2017.09.005 :
Since $L^{n',\infty}(\Omega) \subset L^{n'}(\Omega)$, then we have

$$
\left( L^{n',\infty}(\Omega), L^{(n')}(\Omega) \right)_{\alpha,1} \subset \left( L^{n'}(\Omega), L^{(n')}(\Omega) \right)_{\alpha,1}
$$
Since $L^{n',\infty}(\Omega) \subset L^{n'}(\Omega)$, then we have

$$\left( L^{n',\infty}(\Omega), L^{(n')(\Omega)} \right)_{\alpha,1} \subset \left( L^{n'}(\Omega), L^{(n')(\Omega)} \right)_{\alpha,1}$$

we have shown the following

**Theorem (characterization of the interpolation between Grand and Small Lebesgue space)**

$$\left( L^{n'}(\Omega), L^{(n')(\Omega)} \right)_{\alpha,1} = G\Gamma(n'; 1; w_1; w_2)$$

*with* $w_1(t) = \frac{(1 - \log t)^{\alpha-1}}{t}$, $w_2(t) = \frac{1}{1 - \log t}$. 

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Some new results related to $G\Gamma$-spaces and interpolation
Theorem

\[
\left( L^{n'}_{\infty}(\Omega), L^{(n')_{\infty}}(\Omega) \right)_{\alpha,1} \subset G^{\Gamma}(\infty; 1; \nu_1; \nu_2)
\]

with \( \nu_1(t) = t^{-1}(1 - \log t)^{\alpha n^{-1}} \), \( \nu_2(t) = t^{\frac{1}{n'}} \).
Let $u$ be the solution of (1), $0 < \alpha < 1$. Then,

$$\|u\|_{G^\Gamma(n',1;w_1,w_2) \cap G^\Gamma(\infty,1;v_1,v_2)} \lesssim |f|_{L^1(\Omega;\delta(1+|\log \delta|)^\alpha)},$$

where $w_1(t) = t^{-1} (1 - \log t)^{\alpha-1}$, $w_2(t) = (1 - \log t)^{-1}$, $v_1(t) = t^{-1} (1 - \log t)^{\frac{\alpha}{n}-1}$ and $v_2(t) = t^{\frac{1}{n'}}$. 

Some new results related to $G\Gamma$-spaces and interpolation.
Proposition

Let \( u \) be the solution of (1), \( 0 < \alpha < 1 \). Then,

\[
\|u\|_{G\Gamma(n', 1; w_1, w_2) \cap G\Gamma(\infty, 1; v_1, v_2)} \lesssim |f|_{L^1(\Omega; \delta(1 + |\log \delta|)\alpha)},
\]

where \( w_1(t) = t^{-1}(1 - \log t)^{\alpha - 1} \), \( w_2(t) = (1 - \log t)^{-1} \),
\( v_1(t) = t^{-1}(1 - \log t)^{\frac{\alpha}{n} - 1} \) and \( v_2(t) = t^{\frac{1}{n'}} \).

\[
\|u\|_{G\Gamma(n', 1; w_1, w_2) \cap G\Gamma(\infty, 1; v_1, v_2)}
\]
\[
= \int_0^1 (1 - \log t)^{\alpha - 1} \left( \int_0^t \frac{u^*_n(x) dx}{1 - \log x} \right)^{\frac{1}{n'}} \frac{dt}{t} + \int_0^1 (1 - \log t)^{\frac{\alpha}{n} - 1} \left( \sup_{0 < x < t} x^{\frac{1}{n'}} u_*(x) \right) \frac{dt}{t}
\]
\[
\lesssim |f|_{L^1(\Omega; \delta(1 + |\log \delta|)\alpha)}.\]
For a measurable function $f : \Omega \to \mathbb{R}$, we set for $t \geq 0$

$$D_f(t) = \text{measure} \left\{ x \in \Omega : |f(x)| > t \right\}$$

and $f_*$ the decreasing rearrangement of $|f|$, $f_*(s) = \inf \left\{ t : D_f(t) \leq s \right\}$ with $s \in (0, |\Omega|)$, $|\Omega|$ is the measure of $\Omega$, that we shall assume to be equal to 1 for simplicity.
The Lorentz $G\Gamma$-space is defined as follows:

**Definition (of Generalized Gamma space with double weights (Lorentz-$G\Gamma$))**

Let $w_1, w_2$ be two weights on $(0, 1)$, $m \in [1, +\infty]$, $1 \leq p < +\infty$. We assume the following conditions:

**c1)** There exists $K_{12} > 0$ such that $w_2(2t) \leq K_{12}w_2(t) \forall t \in (0, 1/2)$. The space $L^p(0, 1; w_2)$ is continuously embedded in $L^1(0, 1)$.

**c2)** The function $\int_0^t w_2(\sigma)d\sigma$ belongs to $L^m_\frac{p}{p}(0, 1; w_1)$.

A generalized Gamma space with double weights is the set:

$$G\Gamma(p, m; w_1, w_2) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable} \right. \\
\left. \int_0^t f_\star^p(\sigma)w_2(\sigma)d\sigma \text{ is in } L^m_\frac{p}{p}(0, 1; w_1) \right\}.$$

This space was introduced in: **A. Gogatishvili, M. Krepela, L. Pick, F. Soudsky**, Embeddings of Lorentz-type spaces involving weighted integral means, *J.F.A.* 273, 9 (2017) 2939-2980.
Let $G\Gamma(p, m; w_1, w_2)$ be a Generalized Gamma space with double weights and let us define for $f \in G\Gamma(p, m; w_1, w_2)$

$$\rho(f) = \left[ \int_0^1 w_1(t) \left( \int_0^t f^p_*(\sigma)w_2(\sigma)d\sigma \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}}$$

with the obvious change for $m = +\infty$.

Then,

1. $\rho$ is a quasinorm.
2. $G\Gamma(p, m; w_1, w_2)$ endowed with $\rho$ is a quasi-Banach function space.
3. If $w_2 = 1$

$$G\Gamma(p, m; w_1, 1) = G\Gamma(p, m; w_1).$$
Proposition (1)

Consider the classical Lorentz space $\Lambda^p(w_2)$. Then it is equal to the set

$$\left\{ f : \Omega \to \mathbb{R} \text{ measurable} : \left( \int_0^1 f^p_*(\sigma)w_2(\sigma)d\sigma \right)^{\frac{1}{p}} = ||f||_{\Lambda^p(w_2)} < +\infty \right\}.$$

If $w_1$ and $w_2$ are integrable weights on $(0, 1)$ and $w_2$ satisfies c1) then

$$G\Gamma(p, m; w_1, w_2) = \Lambda^p(w_2).$$
Assume that $w_1(t) = t^{-1}(1 - \log t)^\gamma$, $w_2(t) = (1 - \log t)^\beta$, $(\gamma, \beta) \in \mathbb{R}^2$, $m \in [1, +\infty[$, $p \in [1, +\infty[$.

1. If $\gamma < -1$ then $G\Gamma(p, m; w_1, w_2) = \Lambda^p(w_2)$.

2. If $\gamma > -1$ and $\gamma + \beta \frac{m}{p} + 1 \geq 0$ then

$$G\Gamma(p, m; w_1, w_2) = G\Gamma(p, m; \overline{w}_1, 1), \quad \overline{w}_1(t) = t^{-1}(1 - \log t)^{\gamma + \beta \frac{m}{p}}.$$
Assume that $w_1(t) = t^{-1}(1 - \log t)^\gamma$, $w_2(t) = (1 - \log t)^\beta$, $(\gamma, \beta) \in \mathbb{R}^2$, $m \in [1, \infty]$, $p \in [1, \infty]$. If $\gamma > -1$ and $\gamma + \beta \frac{m}{p} + 1 < 0$, then

$$\|f\|_{\mathcal{G}^\Gamma(p,m;w_1,w_2)}^m \approx \int_0^1 (1 - \log t)^{\gamma + \beta \frac{m}{p}} \left( \int_t^1 f_*(x)^p \, dx \right)^{m/p} \frac{dt}{t}.$$
Lemma

Assume that \( w_1(t) = t^{-1}(1 - \log t)^\gamma \), \( w_2(t) = (1 - \log t)^\beta \), \((\gamma, \beta) \in \mathbb{R}^2\), \( m \in [1, \infty[, \ p \in [1, \infty[\). If \( \gamma > -1 \) and \( \gamma + \beta \frac{m}{p} + 1 < 0 \), then

\[
\|f\|_{G_\Gamma(p,m;w_1,w_2)}^m \approx \int_0^1 (1 - \log t)^{\gamma + \beta \frac{m}{p}} \left( \int_t^1 f^*_*(x)^p \, dx \right)^{m/p} \frac{dt}{t}.
\]

Lemma

Assume that \( w_1(t) = (1 - \log t)^\gamma \), \( w_2(t) = (1 - \log t)^\beta \), \((\gamma, \beta) \in \mathbb{R}^2\), \( p \in [1, \infty[\). If \( \gamma > 0 \) and \( \gamma + \frac{\beta}{p} < 0 \), then

\[
\|f\|_{G_\Gamma(p,\infty;w_1,w_2)} \approx \sup_{0 < t < 1} (1 - \log t)^{\gamma + \frac{\beta}{p}} \left( \int_t^1 f^*_*(x)^p \, dx \right)^{1/p}.
\]
Definition (of the small Lebesgue space)

The small Lebesgue space associated to the parameters $p \in ]1, +\infty[$ and $\theta > 0$ is the set

$$L^{(p, \theta)}(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable :} \right.$$

$$\left. \| f \|_{(p, \theta)} = \int_0^1 (1 - \log t)^{-\frac{\theta}{p} + \theta - 1} \left( \int_0^t f^p_*(\sigma) d\sigma \right)^{1/p} \frac{dt}{t} < +\infty \right\}. $$
Definition (of the small Lebesgue space)

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\]

\[
\|f\|_{(p,\theta)} = \int_0^1 (1 - \log t)^{-\frac{\theta}{p} + \theta - 1} \left( \int_0^t f_*^p(\sigma)d\sigma \right)^{1/p} \frac{dt}{t} < +\infty \right\}.
\]

Definition (of the grand Lebesgue space)

The grand Lebesgue space is the associate space of the small Lebesgue space, with the parameters \( p \in ]1, +\infty[ \) and \( \theta > 0 \) is the set

\[
L^{(p),\theta}(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable :} \right.
\]

\[
\|f\|_{(p),\theta} = \sup_{0 < t < 1} (1 - \log t)^{-\frac{\theta}{p}} \left( \int_t^1 f_*^p(\sigma)d\sigma \right)^{1/p} \frac{dt}{t} < +\infty \right\}.
\]
Corollary

Assume that \( w_1(t) = t^{-1}(1 - \log t)^\gamma \), \( w_2(t) = (1 - \log t)^\beta \), \((\gamma, \beta) \in \mathbb{R}^2\), \( p \in ]1, +\infty[ \). If \( \gamma + 1 + \frac{\beta}{p} > 0 \) and \( \gamma > -1 \), then

\[
G\Gamma(p, 1; w_1, w_2) = L^{(p, \theta}}, \quad \theta = p' \left( \gamma + 1 + \frac{\beta}{p} \right).
\]

\[
L^{(p, 1)}(\Omega) = L^{(p)}(\Omega).
\]
Corollary

Assume that $w_1(t) = (1 - \log t)^\gamma$, $w_2(t) = (1 - \log t)^\beta$, $(\gamma, \beta) \in \mathbb{R}^2$, $p \in ]1, +\infty[$. If $\gamma + \frac{\beta}{p} < 0$ and $\gamma > 0$, then

$$G\Gamma(p, \infty; w_1, w_2) = L^{p,\theta}(\Omega), \quad \theta = -p \left(\gamma + \frac{\beta}{p}\right).$$

$$L^{p,1}(\Omega) = L^p(\Omega).$$
We recall also the following definition of interpolation spaces. Let $(X_0, \| \cdot \|_0), (X_1, \| \cdot \|_1)$ two Banach spaces contained continuously in a Hausdorff topological vector space (that is $(X_0, X_1)$ is a compatible couple). For $g \in X_0 + X_1, \ t > 0$ one defines the so called $K$ functional $K(g, t; X_0, X_1) = K(g, t)$ by setting

$$K(g, t) = \inf_{g = g_0 + g_1} (\|g_0\|_0 + t\|g_1\|_1). \quad (6)$$

For $0 \leq \theta \leq 1, \ 1 \leq p \leq +\infty, \ \alpha \in \mathbb{R}$ we shall consider

$$(X_0, X_1)_{\theta, p; \alpha} = \left\{ g \in X_0 + X_1, \ \|g\|_{\theta, p; \alpha} = \|t^{-\theta - \frac{1}{p}}(1-\log t)^\alpha K(g, t)\|_{L^p(0, 1)} \text{ is finite} \right\}.$$

Here $\| \cdot \|_{L^p(0, 1)}$ denotes the norm in a Lebesgue space $L^p(0, 1), \ 0 < p \leq +\infty.$
We recall also the following definition of interpolation spaces.
Let \((X_0, \| \cdot \|_0), (X_1, \| \cdot \|_1)\) two Banach spaces contained continuously in a Hausdorff topological vector space (that is \((X_0, X_1)\) is a compatible couple).
For \(g \in X_0 + X_1, \ t > 0\) one defines the so called \(K\) functional
\[
K(g, t; X_0, X_1) := K(g, t)
\]
by setting
\[
K(g, t) = \inf_{g = g_0 + g_1} (\|g_0\|_0 + t\|g_1\|_1).
\]
For \(0 \leq \theta \leq 1, \ 1 \leq p \leq +\infty, \ \alpha \in \mathbb{R}\) we shall consider
\[
(X_0, X_1)_{\theta, p; \alpha} = \left\{ g \in X_0 + X_1, \ |g|_{\theta, p; \alpha} = |t^{-\theta - \frac{1}{p}} (1-\log t)^\alpha K(g, t)|_{L^p(0,1)} \text{ is finite} \right\}.
\]
Here \(| \cdot |_{L^p(0,1)}\) denotes the norm in a Lebesgue space \(L^p(0,1), 0 < p \leq +\infty\).

Our definition of the interpolation space is different from the usual one since we restrict the norms on the interval \((0,1)\).
If we consider ordered couple, i.e. \(X_1 \hookrightarrow X_0\) and \(\alpha = 0,\)
\[
(X_0, X_1)_{\theta, p; 0} = (X_0, X_1)_{\theta, p}
\]
is the interpolation space as it is defined by J. Peetre.
Theorem

Let $\varphi(t) = e^{1 - \frac{1}{tp'}}$, $0 < t \leq 1$. Then

$$K(f, t; L^p, L^{(p)}) \approx t \int_{\varphi(t)}^{1} (1 - \log \sigma)^{-\frac{1}{p}} \left( \int_{0}^{\sigma} f_{*}(x) dx \right)^{\frac{1}{p}} \frac{d\sigma}{\sigma} = K^2(t)$$

for all $f \in L^p + L^{(p)}$. 

Corollary

One has, for $r \in [1, +\infty]$ and $0 < \theta < 1$,

$$||f||_r(L^p, L^{(p)}) \approx \int_{0}^{1} (1 - \log x)^{\theta} r \left( \int_{0}^{x} f_{*}(s) ds \right)^{r'} \frac{dx}{x} (1 - \log x)^{\theta}.$$
Theorem

Let $\varphi(t) = e^{1 - \frac{1}{t^p}}, 0 < t \leq 1$. Then

$$K(f, t; L^p, L^{(p)}) \approx t \int_{\varphi(t)}^{1} (1 - \log \sigma)^{-\frac{1}{p}} \left( \int_{0}^{\sigma} f_*^p(x) \, dx \right)^{\frac{1}{p}} \frac{d\sigma}{\sigma} = K^2(t)$$

for all $f \in L^p + L^{(p)}$.

Corollary

One has, for $r \in [1, +\infty[, 0 < \theta < 1$,

$$\|f\|_{(L^p, L^{(p)})_\theta, r}^r \approx \int_{0}^{1} (1 - \log x)^{\theta r p} \left( \int_{0}^{x} f_*^p(s) \, ds \right)^{\frac{r}{p}} \frac{dx}{x(1 - \log x)}.$$
Interpolation between grand and classical Lebesgue spaces in the critical case

**Lemma**

Let $1 < p < \infty$, and let $f \in L^p)$. Then, for all $0 < t < 1$,

$$K(f, t; L^p), L^p) \approx \sup_{0<s<\varphi(t)} (1 - \log s)^{-1/p} \left( \int_s^1 f_*(x)^p \, dx \right)^{1/p},$$

where $\varphi(t) = e^{1 - \frac{1}{tp}}$. 

**Theorem**

Let $1 < p < \infty$, $0 < \theta < 1$, and $1 \leq r < \infty$. Then

$$(L^p), L^p))^{\theta, r} = G^\Gamma(p, r; w_1, w_2),$$

where $w_1(t) = t^{-1}(1 - \log t)^{\theta/p - 1}$ and $w_2(t) = (1 - \log t)^{-1}$. 

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Some new results related to $G\Gamma$-spaces and interpolation
Interpolation between grand and classical Lebesgue spaces in the critical case

**Lemma**

Let $1 < p < \infty$, and let $f \in L^p)$. Then, for all $0 < t < 1$,

$$K(f, t; L^p), L^p) \approx \sup_{0 < s < \varphi(t)} (1 - \log s)^{-1/p} \left( \int_s^1 f^*(x)^p \, dx \right)^{1/p},$$

where $\varphi(t) = e^{1 - \frac{1}{tp}}$.

**Theorem**

Let $1 < p < \infty$, $0 < \theta < 1$, and $1 \leq r < \infty$. Then

$$(L^p), L^p)_{\theta, r} = G\Gamma(p, r; w_1, w_2),$$

where $w_1(t) = t^{-1}(1 - \log t)^{r\theta/p - 1}$ and $w_2(t) = (1 - \log t)^{-1}$.
Lemma

Let $1 < p < \infty$ and $0 < \beta < \alpha$. Let $f \in L^{p),\alpha}$. Then, for all $0 < t < 1$,

$$K(f, t; L^{p),\alpha}, L^{p),\beta}) \approx \sup_{0<s<\varphi(t)} (1 - \log s)^{-\frac{\alpha}{p}} \left( \int_s^{\varphi(t)} f_*(x)^p \, dx \right)^{1/p}$$

$$+ t \sup_{\varphi(t)<s<1} (1 - \log s)^{-\frac{\beta}{p}} \left( \int_s^{1} f_*(x)^p \, dx \right)^{1/p},$$

where $\varphi(t) = e^{1-t \frac{p}{\beta-\alpha}}$. 

Interpolation between grand Lebesgue spaces in the critical case
Interpolation between grand Lebesgue spaces in the critical case

**Lemma**

Let $1 < p < \infty$ and $0 < \beta < \alpha$. Let $f \in L^{p,\alpha}$. Then, for all $0 < t < 1$,

$$
K(f, t; L^{p,\alpha}, L^{p,\beta}) \approx \sup_{0 < s < \varphi(t)} (1 - \log s)^{-\frac{\alpha}{p}} \left( \int_s^{\varphi(t)} f_*(x)^p \, dx \right)^{1/p} + t \sup_{\varphi(t) < s < 1} (1 - \log s)^{-\frac{\beta}{p}} \left( \int_s^1 f_*(x)^p \, dx \right)^{1/p},
$$

where $\varphi(t) = e^{1-t^{\frac{p}{\beta-\alpha}}}$.

**Theorem**

Let $1 < p < \infty$, $0 < \beta < \alpha$, $0 < \theta < 1$, and $1 \leq r < \infty$. Then

$$(L^{p,\alpha}, L^{p,\beta})_{\theta,r} = G\Gamma(p, r; w_1, w_2),$$

where $w_1(t) = t^{-1}(1 - \log t)^{\frac{r\theta}{p}(\alpha - \beta) - 1}$ and $w_2(t) = (1 - \log t)^{-\alpha}$.

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Some new results related to $G\Gamma$-spaces and interpolation
The $K$-functional for the couple $(L^p, \infty, L^p)$, $1 < p < +\infty$

**Theorem**

For a measurable set $E \subset [0, 1]$, we denote $|E|_\nu = \int_E \frac{dx}{x}$ and for $f \in L^{p, \infty} + L^p$, $1 < p < +\infty$, we define

$$K_p(f, t) = t \sup \left\{ \left( \int_E f^p_*(\sigma) d\sigma \right)^{\frac{1}{p}} : |E|_\nu = t^{-p} \right\} \quad t \in ]0, 1].$$

Then

$$K(f, t; L^{p, \infty}, L^p) \approx K_p(f, t)$$

and

$$K_p(f, t) = t \left[ \int_0^{t^{-p}} \psi_{*, \nu}(x)^p dx \right]^{\frac{1}{p}}$$

where $\psi(s) = s^{\frac{1}{s}} f_*(s)$, $\psi_{*, \nu}$ its decreasing rearrangement with respect to the measure $\nu$. 
Lemma

Let $1 < p < \infty$. Then for any $f \in L^{p,\infty}$ and all $0 < t < 1$,

$$\sup_{0 < s < t} s^{\frac{1}{p}} f_*(s) \lesssim K(\rho(t), f; L^{p,\infty}, L^{(p)}),$$

where $\rho(t) = (1 - \log t)^{-1 + \frac{1}{p}}$. 

Amiran Gogatishvili

Some new results related to $G\Gamma$-spaces and interpolation
**Lemma**

Let $1 < p < \infty$. Then for any $f \in L^{p,\infty}$ and all $0 < t < 1$,

$$\sup_{0 < s < t} s^{\frac{1}{p}} f_*(s) \lesssim K(\rho(t), f; L^{p,\infty}, L^{(p)}),$$

where $\rho(t) = (1 - \log t)^{-1 + \frac{1}{p}}$.

**Theorem**

Let $1 < p < \infty$, $0 < \theta < 1$, and $1 \leq r < \infty$. Then, for any $f \in (L^{p,\infty}, L^{(p)}_{\theta,r})$, one has

$$\|f\|_{\Gamma(\infty,r;v_1,v_2)} \lesssim \|f\|_{(L^{p,\infty},L^{(p)}_{\theta,r})},$$

where $v_1(t) = t^{-1}(1 - \log t)^{r\theta(1-1/p)-1}$ and $v_2(t) = t^{1/p}$. 
Theorem

Let $1 < p < \infty$, $0 < \theta < 1$, and $1 \leq r < \infty$. Then

$$\|f\|_{G\Gamma(p,r;w_1,w_2)} \lesssim \|f\|_{(L^p,\infty),(L^p)_{\theta,r}},$$

where $w_1(t) = t^{-1}(1 - \log t)^{r\theta-1}$ and $w_2(t) = (1 - \log t)^{-1}$. 

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Combination these two theorem we have

**Theorem**

Let $1 < p < \infty$, $0 < \theta < 1$ and $1 \leq r < \infty$. Then

$$
\|f\|_{G\Gamma(p,r;w_1,w_2) \cap G\Gamma(\infty,r;v_1,v_2)} \lesssim \|f\|_{(L^p,\infty,L^{(p)}_\theta,r)},
$$

where $w_1(t) = t^{-1}(1 - \log t)^{r\theta - 1}$, $w_2(t) = (1 - \log t)^{-1}$, $v_1(t) = t^{-1}(1 - \log t)^{r\theta(1-1/p) - 1}$ and $v_2(t) = t^{1/p}$. 
THANK YOU FOR ATTENTION!!