Nonlinear Response Approach to Cooper Pair Tunneling in Ultrasmall SIS Junctions

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Single-charge tunneling in ultrasmall voltage biased SIS junctions in a high-impedance electromagnetic environment is considered. The Cooper pair current is calculated at $T=0$ on the basis of the elementary tunnel Hamiltonian for quasiparticles. Therefore, the transfer of Cooper pairs emerges automatically as an effect of higher order perturbation theory. The supercurrent also depends on the dissipative part of the Josephson current amplitude.

I. INTRODUCTION

Effects of single-charge tunneling in ultrasmall capacitance junctions has become of much experimental and theoretical interest. For a review see for instance Ref. [1]. Modern nanolithography allows the fabrication of junctions with capacitances $C < 10^{-16}$F where the electrostatic energy differences dominate thermal fluctuations at the 1K scale. This opens the door to a new kind of electronics.

The current through a SIS junction (see Fig. 1) can be carried by quasiparticles (quasiparticle current $\langle I\rangle_{qp}$) and by Cooper pairs (supercurrent $\langle I\rangle_s$). Here we develop a nonlinear response approach to the Cooper pair tunneling in voltage biased ultrasmall SIS junctions basing on the elementary tunneling Hamiltonian for quasiparticles. Since this Hamiltonian describes only the tunneling of single quasiparticles ($1e$) the Cooper pair tunneling ($2e$) corresponds in perturbation theory to a process of higher order. In leading order this approach yields the known result of quasiparticle tunneling. Because of the voltage biasing and due to Eq. (5) the ordinary Josephson current which is also an effect of first order perturbation theory does not play any role.

The current-voltage characteristic of small tunnel junctions is essentially influenced by the external circuit. This electromagnetic environment is able to absorb energy which is for superconducting electrodes at zero temperature the only possibility to transfer the energy gain of the tunneling process. Because Cooper pairs live in the condensate they cannot absorb this energy. To simplify matters we restrict ourselves to the limiting cases of low- and high-resistance environments ($R_E \ll R_Q$ and $R_E \gg R_Q$) at $T=0$ where the quasiparticle currents are suppressed for voltages lower than the thresholds $2\Delta/e$ and $(2\Delta + E_c)/e$ respectively. $R_Q = h/e^2$ is the quantum resistance and $2\Delta$ labels the superconducting energy gap. The additional part $E_c = \hbar\omega_c = e^2/(2C)$ corresponds to the Coulomb energy. For single junctions is known that the Coulomb blockade can only be observed if the junction is sufficiently decoupled from the voltage bias by a high-resistance environment. Beyond the thresholds Cooper pairs can break up into quasiparticles and the tunnel current is carried mainly by quasiparticles.

Cooper pair tunneling is described in literature by using the model of an effective Hamiltonian $H_T = E_J \cos \Psi$ with the perturbation term

$$H_T = E_J \cos \Psi$$

where the operator $\exp(\pm i\Psi)$ changes the macroscopic charge $Q$ on the junction by the value $\pm 2e$ corresponding to the charge of a Cooper pair. This means that simultaneously tunneling of two electrons (Cooper pair) is introduced by hand. Then the calculated supercurrent reads

$$\langle I\rangle_s(V) = \frac{\pi e E_J^2}{\hbar} \left\{ P' \left( \frac{2eV}{\hbar} \right) - P' \left( -\frac{2eV}{\hbar} \right) \right\}$$

with

$$P' (\omega) = \frac{1}{2\pi} \int_\infty^{-\infty} e^{4J(\tau) + i\omega\tau} d\tau .$$

The function $J(\tau)$ contains the information about the structure of the environment ($\beta = 1/(k_B T)$).
\[ J(\tau) = \frac{1}{R_Q} \int_{-\infty}^{\infty} \frac{\text{Re}Z_1(\omega)}{\omega} \left\{ \coth \frac{\beta h \omega}{2} [\cos \omega \tau - 1] - i \sin \omega \tau \right\} d\omega, \] (4)

where \( Z_1(\omega) = 1/(i \omega C + 1/R_E) \). Note, that at \( T = 0 \) one has
\[ P'(\frac{2eV}{h}) \Rightarrow h \delta(2eV) \quad \text{for} \quad \frac{R_E}{R_Q} \to 0, \]
\[ P'(\frac{2eV}{h}) \Rightarrow h \delta(2eV - 4E_c) \quad \text{for} \quad \frac{R_E}{R_Q} \to \infty. \]

This peak structure actually has been seen in experiment.\(^4\) The result is said to be correct if the Josephson coupling energy \( E_J = \hbar/(2eI_c) \) is much smaller than \( E_c \). By use of the known formula for the critical current \( I_c \) one gets the inequality
\[ E_c \gg \frac{R_Q}{8R} \Delta. \] (5)

\( R \) is the normal tunnel resistance which obeys the relation \( R_Q \ll R \). Unfortunately, this model contains nearly no information about the superconducting electrodes. In contrast to this the quasiparticle current \( \langle I \rangle \) is expressed in terms of the quasiparticle current amplitude \( \text{Im}I_q \) which depends in a characteristic way on \( \omega \). In case of a symmetric junction the current amplitude \( \text{Im}I_q(\omega) \) reads at \( T = 0 \) according to standard BCS theory.\(^5\)

\[ \text{Im}I_q(\omega) = \frac{\Delta}{eR} \begin{cases} 0 & \text{for} \ 0 < \frac{h\omega}{\Delta} < 2 \\ \frac{\Delta}{eR} E \left( \sqrt{1 - \left( \frac{\Delta}{k_B T} \right)^2} \right) - \frac{\Delta}{k_B T} K \left( \sqrt{1 - \left( \frac{\Delta}{k_B T} \right)^2} \right) & \text{for} \ \frac{h\omega}{\Delta} > 2. \end{cases} \] (7)

The symbols \( E \) and \( K \) stand for the complete elliptic integrals of the first kind.\(^6\) The definition of \( P(\omega) \) differs from that of \( P'(\omega) \) (Eq. (6)) by the lack of the factor 4 in front of the function \( J(\tau) \). The dependence of the supercurrent on the factor \( I_c^2 \) in Eq. (1) indicates that the supercurrent has something to do with the squared Josephson current amplitude \( \text{Re}I_q \). This is motivation to express the supercurrent \( \langle I \rangle \) by means of a perturbation theory of higher order in the elementary tunneling Hamiltonian
\[ H_T = H_+ + H_-, \quad H_- = H^+_+, \]
\[ H_+ = \sum_{l,r,\sigma} T_{lr}c^\dagger_{l,r,\sigma}c_{l,\sigma}e^{i\Phi}, \] (8)

where \( c_{l,\sigma} \) and \( c^\dagger_{r,\sigma} \) stand for quasiparticle annihilation operators of the left and right electrode satisfying anticommutation relations. The spin is labeled by the subscripts \( l, r \).

In this way the special features of Cooper pair tunneling (transfer of charges \( 2e \), energy transfer only to the environment) arise automatically. In other words we do not consider tunneling particles with charge \( 2e \) from the beginning. Rather than we start with elementary particles (electrons) with charge 1e and the supercurrent arises as an effect of higher order. Furthermore, the dependence on \( \text{Re}I_q \) describes the transition to a new branch if the supplied energy is able to break Cooper pairs into quasiparticles (\( eV = 2\Delta \)).

Using the expression for the mean current in nonlinear response theory those parts which correspond to the Cooper pair tunneling can be identified and calculated in a systematic way.

**II. RESPONSE THEORY**

The dynamics of a physical system modeled by \( H = H_o + H_T \) will be described by the statistical operator \( \rho \) satisfying the von Neumann equation which is in the interaction representation (superscript \( (I) \)) equivalent to the integral equation
\[ \rho(t) = \rho_0 - \frac{i}{\hbar} \int_{-\infty}^{t} [H_T(t'), \rho(t')] dt' . \]  

(9)

\( H_0 \) is the unperturbed part of the Hamiltonian whereas the interaction part \( H_T \) reads in the interaction representation

\[ H_T(t) = e^{ \frac{i}{\hbar} H_0 t } H_T e^{- \frac{i}{\hbar} H_0 t} . \]

It is assumed that the interaction is switched on at \( t = -\infty \) adiabatically. The operator \( \rho_0 \) is given by the canonical expression

\[ \rho_0 = e^{- \beta H_0} tr\{ e^{- \beta H_0} \} . \]

The solution can be found by successive approximation. Now the mean value of the current operator reads

\[ \langle I \rangle = \frac{1}{i\hbar} \int_{-\infty}^{t} dt' \langle [I(t), H_T(t')] \rangle_o \]

\[ + \left( \frac{1}{i\hbar} \right)^3 \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \int_{-\infty}^{t''} dt''' \langle [[[I(t), H_T(t')], H_T(t'')], H_T(t''')] \rangle_o + \ldots \]

(10)

In this equation the mean values have to be calculated with respect to \( \rho_o \). A term of zeroth order is missing because in case of no interaction (tunneling) there is also no current. The first term corresponds to linear response theory and leads either to the known quasiparticle tunneling (Eq. (6)) or to the Josephson current. The second term describes the first nonlinear corrections.

III. THE MODEL

Now let us apply this theory to tunneling through a SIS junction with environment (Fig. 1). The total Hamiltonian reads

\[ H = H_0 + H_T = QV + H_{res} + H_T , \]

(11)

where the tunnel Hamiltonian is given by Eq. (8). In case of superconducting electrodes one can assume that the macroscopic phase is already contained in the phase operator \( \Phi_{12} \). Owing to this phase operator tunneling is connected with excitations in the electromagnetic environment. The operator \( H_\pm \) means e.g. tunneling from left to right in contrast to the Hermitian conjugate which describes the reverse process. \( T_{lr} \) are the tunneling matrix elements. The basic algebra ruling this approach is the following relation

\[ H_\pm \cdot F(Q) = F(Q \pm e) \cdot H_\pm , \]

(12)

where \( F \) is an arbitrary function of the junction charge \( Q \). This algebra corresponds to the elementary commutation relation

\[ [Q, \Phi] = ie . \]

(13)

The convention is chosen in such a way that a positive voltage favors tunneling from left to right which reduces the junction charge \( Q \) by \( e \).

The reservoir Hamiltonian \( H_{res} \) consists of terms corresponding to the left and right electrodes and the environment which can be described in standard way.

For the calculation of the stationary mean current in terms of Eq. (10) a current operator has to be defined. This is done in the following way

\[ I = -\frac{d}{dt} Q = -\frac{1}{i\hbar} [Q, H] = \frac{e}{i\hbar} (H_+ - H_-) . \]

(14)

Now the quasiparticle current which is the first order term in Eq. (10) reads
\[
\langle I \rangle_{qp} = -\frac{2e^2}{h^2} \text{Re} \int_{-\infty}^{t} dt' \langle [H_+^{(I)}(t), H_-^{(I)}(t')] \rangle_o .
\]  \hspace{2cm} (15)

The Cooper pair current is contained in the following terms of second nonvanishing order

\[
\langle I \rangle_s = \frac{2e^2}{h^2} \text{Re} \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \langle [H_+^{(I)}(t), H_+^{(I)}(t')], H_-^{(I)}(t''), H_-^{(I)}(t''') \rangle_o 
+ \langle [H_+^{(I)}(t), H_-^{(I)}(t')], H_+^{(I)}(t''), H_-^{(I)}(t''') \rangle_o 
+ \langle [H_-^{(I)}(t), H_+^{(I)}(t')], H_+^{(I)}(t''), H_-^{(I)}(t''') \rangle_o \right\} .
\]  \hspace{2cm} (16)

An explanation should be given why only correlations with vanishing signature (+ + − − and their permutations) are taken into account. The reason is the following. The separation of the voltage dependence in the correlation functions \( \langle \ldots \rangle_o \) without signature zero (e.g. \( \langle H_+ H_+ H_+ H_+ \rangle_o \)) leads to expressions containing time dependent \( t \) terms. Furthermore, a phase \( \phi_o \) remains indeterminated additionally. This corresponds to common Josephson physics where one has contributions proportional to \( \sin(2eVt/h + \phi_o) \) and \( \cos(2eVt/h + \phi_o) \). However, in Josephson physics the phase \( \phi_o \) is defined by current biasing. Therefore, in our case of voltage-biasing one has to average with respect to this phase factor \( \phi_o \) which yields zero.

By splitting-off the voltage dependence by means of Eq. (12) and using new time variables

\[ \tau \equiv t - t'; \quad \tau' \equiv t' - t''; \quad \tau'' \equiv t'' - t''', \]

Eq. (16) reads

\[
\langle I \rangle_s = \frac{2e^2}{h^2} \text{Re} \int_{0}^{t} d\tau \int_{0}^{\infty} dr \int_{0}^{\infty} d\tau' \int_{0}^{\infty} d\tau'' \left\{ e^{-\frac{i}{2} eV(\tau + 2\tau' + \tau'')} \kappa_1(\tau, \tau', \tau'') 
+ e^{-\frac{i}{2} eV(\tau + \tau'')} \kappa_2(\tau, \tau', \tau'') 
+ e^{-\frac{i}{2} eV(\tau - \tau'')} \kappa_3(\tau, \tau', \tau'') \right\} .
\]  \hspace{2cm} (17)

**IV. CALCULATION OF CORRELATION FUNCTIONS**

Now one has to deal with the three correlation functions \( \kappa_1, \kappa_2 \) and \( \kappa_3 \). The operators \( H_\pm(t) \) in the interaction representation can be written as \( \tilde{H}_\pm(t) \exp(\pm i\Phi(t)) \) where the operators \( \tilde{H}_\pm(t) \) carry only the time dependence with respect to the electrodes and the phase dependent operators \( \exp(\pm i\Phi(t)) \) carry those with respect to the environment. For instance the function \( \kappa_1 \) reads therefore

\[
\kappa_1 = (\tilde{H}_+(t_1)\tilde{H}_+(t_2)\tilde{H}_-(t_3)\tilde{H}_-(t_4))_o
\times \langle e^{i\Phi(t_1)}e^{i\Phi(t_2)}e^{-i\Phi(t_3)}e^{-i\Phi(t_4)} \rangle_o + 7 \text{ further terms} .
\]  \hspace{2cm} (18)

These other terms arise due to the resolution of the interlaced commutators. The decisive step of the identification of the contributions which describe Cooper pair tunneling is to reduce the 4-correlators with respect to \( \tilde{H} \) into 2-correlators containing only operators with the same signature, namely

\[
\langle \tilde{H}_+(t_1)\tilde{H}_+(t_2)\tilde{H}_-(t_3)\tilde{H}_-(t_4) \rangle_o = \langle \tilde{H}_+(t_1)\tilde{H}_+(t_2) \rangle_o \langle \tilde{H}_-(t_3)\tilde{H}_-(t_4) \rangle_o .
\]

This decomposition guarantees that only condensate states corresponding to Cooper pairs are taken into account. One can prove this from the point of view of the elementary operators \( c_{l,r}^\dagger \) and \( c_{l,r} \). Then the decomposition is equivalent to

\[
\langle c_{l}^\dagger(t_1)c_{r}^\dagger(t_3)\rangle_o \langle c_{l}(t_2)c_{r}(t_4) \rangle_o + \langle c_{l}(t_1)c_{r}(t_3)\rangle_o \langle c_{l}^\dagger(t_2)c_{r}^\dagger(t_4) \rangle_o
\]

and one can see that on both banks of the junction only the condensate properties contribute. The terms which have been neglected in this decomposition belong to quasiparticle tunneling of higher order and processes including both
quasiparticles and Cooper pairs. The correlators \( \langle \tilde{H}_+ (t_1) \tilde{H}_- (t_2) \rangle_o \) can be expressed by the current amplitude \( \text{Im} I_p \) in the following way:

\[
\kappa_\pm (\tau) = \frac{\hbar^2}{2\pi e} \int d\omega \text{Im} I_p (\omega) e^{-i\omega \tau} \frac{1}{1 - e^{-\hbar \omega / (k_b T)}} .
\]  

(19)

Note the symmetry \( \kappa_+ (\tau) = \kappa_- (\tau) \). The phase correlations can also be calculated, for instance by generalizing the method presented in Ref. 5.

Now the correlation function \( \kappa_1 (\tau, \tau', \tau'') \) can be written as

\[
\kappa_1 (\tau, \tau', \tau'') = \kappa_+ (\tau) \kappa_- (\tau') e^{J(\tau + 4\tau' + \tau'')} - \kappa_+ (-\tau) \kappa_- (-\tau'') e^{J(-\tau - 4\tau' - \tau'')}
\]

\[
- \kappa_+ (-\tau) \kappa_- (-\tau'') e^{J(3\tau + 4\tau' + \tau'')} + \kappa_+ (\tau) \kappa_- (-\tau'') e^{J(-3\tau - 4\tau' - \tau'')}
\]

\[
- \kappa_+ (\tau) \kappa_- (\tau'') e^{J(-\tau + \tau'')},
\]

(20)

For the environment function \( J(\tau) \) see Eq. (3). We remind of the property \( J(\tau) = -i\omega_c \tau \) at \( T = 0 \) and for \( R_E / R_Q \to \infty \).

V. SUPERCURRENT

Now the contribution to \( \langle I \rangle_s \) coming of \( \kappa_1 \) \( \langle I \rangle_s^{(\kappa_1)} \) can be calculated. Using the definition

\[
f(\omega) = \lim_{\tau \to 0} \frac{\text{Im} I_p (\omega)}{1 - e^{-\beta \hbar \omega}} \Theta (\omega) ,
\]

(21)

where \( \Theta \) is the unit step function as well as the definition of the function \( \delta_+ \)

\[
\delta_+ (x) = \frac{1}{2\pi} \int_0^\infty e^{i k x} dk ,
\]

the result can be written as

\[
\langle I \rangle_s^{(\kappa_1)} = \frac{2\pi}{\hbar} \text{Re} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' f(\omega) f(\omega') \left\{ \delta_+(\omega - \omega - \omega_c) \delta_+(\omega - \omega' - \omega_c) \delta_+(\omega - 2\omega_c) \right. 
\]

\[
- \delta_+(\omega + \omega + \omega_c) \delta_+(\omega + \omega' + \omega_c) \delta_+(\omega + 2\omega_c) 
\]

\[
- \delta_+(\omega - \omega - 3\omega_c) \delta_+(\omega - \omega' - \omega_c) \delta_+(\omega - 2\omega_c) 
\]

\[
+ \delta_+(\omega - \omega + 3\omega_c) \delta_+(\omega - \omega' + \omega_c) \delta_+(\omega + 2\omega_c) 
\]

\[
- \delta_+(\omega - \omega + \omega_c) \delta_+(\omega - \omega' - \omega_c) \delta_+(\omega - v) 
\]

\[
+ \delta_+(\omega + \omega - \omega_c) \delta_+(\omega + \omega' + \omega_c) \delta_+(\omega - v) 
\]

\[
+ \delta_+(\omega - \omega + \omega_c) \delta_+(\omega - \omega' - \omega_c) \delta_+(\omega - v) 
\]

\[
- \delta_+(\omega - \omega - \omega_c) \delta_+(\omega + \omega' + \omega_c) \delta_+(\omega - v) \}
\]

(22)

Here, the variable \( v = eV / \hbar \) is used. The same procedure has to be employed with respect to the terms including the other correlation functions \( \kappa_2 (\tau, \tau', \tau'') \) and \( \kappa_3 (\tau, \tau', \tau'') \). The real part of the sum of these terms can be calculated by means of the Dirac formula

\[
\delta_+ (x) = \frac{1}{2} \left( \delta (x) + \frac{\partial^2}{\partial x^2} \right) .
\]

(23)

Finally, at least one integration (e.g. with respect to \( \omega' \), see Eq. (22)) can be carried out and one gets after a lengthy calculation.
The opposite case of a low-resistance environment corresponds to the condition
\[ I_s(v) = \frac{\pi}{2\epsilon} \left\{ \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_c} \right]^2 + f(\omega_c) \right\} \delta(v - 2\omega_c) \]
\[- \frac{f(v - \omega_c)}{2\pi e} \frac{2\omega_c}{v(v - 2\omega_c)} \int_{-\infty}^{\infty} d\omega \frac{f(\omega + \omega_c)}{(\omega + v - \omega_c)(\omega - v + \omega_c)}
- [v \to -v]. \tag{24} \]

The dash in the integral sign means that one has to take the principal value of the integral. Eq. (24) is our main result.

VI. DISCUSSION

Concerning the structure of the current let us concentrate on the case of a high-resistance environment (\( \omega_c \neq 0 \)). The opposite case of a low-resistance environment corresponds to the condition \( \omega_c \equiv 0 \). One can make the following remarks:

(i) The current is an antisymmetric function of the applied voltage which reflects the expectation that a reversed voltage leads to a reversed current.

(ii) The current shows a \( \delta \)-like singularity at \( 2eV = 4E_c \), corresponding to the fact that the energy \( 2eV \) connected with the tunneling of a Cooper pair has to be transferred to the environment. Because Cooper pairs live in the condensate they cannot absorb this energy. Of course, this singular expression will be smoothed both due to finite temperatures and finite environment resistances.

(iii) There is an additional current contribution which is proportional to \( f(v - \omega_c) \). Because of Eq. (21) and of the known structure of \( \text{Im}I_p(\omega) \) in standard BCS theory (\( \text{Im}I_p(\omega) = -\text{Im}I_p(-\omega) \))

\[ \text{Im}I_p(\omega) = \frac{2}{\pi} I_c \left\{ \begin{array}{cl} 0 & \text{for } 0 < \frac{\hbar \omega}{\Delta} < 2 \\ \frac{2\Delta}{\hbar \omega} K \left( \sqrt{1 - \left( \frac{2\Delta}{\hbar \omega} \right)^2} \right) & \text{for } \frac{\hbar \omega}{\Delta} > 2 \end{array} \right. \tag{25} \]

This current contribution only exists if \( v - \omega_c \leq 2\Delta/\hbar \) or \( eV \leq 2\Delta + E_c \), which is just the condition for the onset of the quasiparticle current. Since our approach is based on higher order perturbation theory we are only interested in effects which occur in the gap region of quasiparticle tunneling. Therefore, only the first term of Eq. (24) has to be considered.

The integral in the first term of Eq. (24) reminds of the definition of \( \text{Re}I_p(\omega_c) \) according to the Kramers-Kronig relation. The only difference is the \( \Theta \)-function in the integrand. Nevertheless, it is reasonable to discuss the case \( \omega_c < 2\Delta/\hbar \) corresponding to the subgap region between 0 and the position of the Riedel peak. This means \( f(\omega_c) = 0 \) and the supercurrent reads for \( 0 < v < 2\Delta/\hbar + \omega_c \)

\[ \langle I_s \rangle(v) = \frac{\pi}{2\epsilon} \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_c} \right]^2 \delta(v - 2\omega_c). \tag{26} \]

Fig. 3, where we have plotted the expressions

\[ \text{Re}I_p(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im}I_p(\omega')}{\omega' - \omega} = \frac{2}{\pi} I_c \left\{ \begin{array}{cl} K \left( \frac{\hbar \omega}{\Delta} \right) & \text{for } 0 < \frac{\hbar \omega}{\Delta} < 2 \\ \frac{2\Delta}{\hbar \omega} K \left( \frac{\hbar \omega}{\Delta} \right) & \text{for } \frac{\hbar \omega}{\Delta} > 2 \end{array} \right. \tag{27} \]

(see Ref. [1]) and

\[ -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im}I_p(\omega') \Theta(\omega')}{\omega' - \omega}, \tag{28} \]
respectively, shows for $0 \leq \omega < 2\Delta/\hbar$ that sufficiently far from the position of the Riedel peak the approximation

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_c} \approx \frac{1}{2} \text{Re} I_p(\omega_c)$$

(29)

holds which becomes exact for $\omega_c \to 0$. Hence, for $\omega_c < 2\Delta/\hbar$ one can write

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega - \omega_c} \approx \frac{1}{2} I_c.$$

(30)

Using this approximation, the supercurrent reads

$$\langle I \rangle_s(v) = \frac{\pi}{8e} I_c^2 \{\delta(v - 2\omega_c) - \delta(-v - 2\omega_c)\}$$

(31)

which corresponds for $T \to 0$ and $R_E/R_Q \to \infty$ exactly to the result of Eq. (3). It has been shown that this formula is valid for $0 < eV < 2\Delta + E_c$ and $E_c \ll 2\Delta$. There is no contradiction to the inequality (3) because $2\Delta > E_c > R_Q/(8R)\Delta$ is satisfied provided that the relation $R_Q \ll R$ holds. However, this condition is just necessary for single-charge tunneling because it guarantees that quantum fluctuations can be neglected. Roughly speaking (cf. formula (26)), formula (26) shows indeed that the Josephson current amplitude $\text{Re} I_p(\omega_c)$ determines the strength of the $\delta$-like current peak at the $eV = 2E_c$. Note, that this strength becomes singular if the $\delta$-singularity tends to the onset position of quasiparticle tunneling because from $v = 2\omega_c \to 2\Delta/\hbar + \omega_c$ follows $\omega_c \to 2\Delta/\hbar$. Eq. (24) shows that for $T > 0$ there are also current contributions depending on the dissipative part of the Josephson current $\text{Im} I_p$ which describes pair transfer processes via thermally excited quasiparticles.

In case of a finite environment resistance the substitution $J(t) = -i\omega t$ in Eq. (25) is not possible. The investigation has shown that the origin of the resulting $\delta$-function in Eq. (31) is the integration over $\tau'$ in Eq. (17). Therefore, this integration would indeed generate the function $P'(2eV)$ known from Eq. (3). But there are also functions $J(\tau)$ and $J(\tau'')$ which are modifying the other integrations over $\tau$ and $\tau''$. To sum up it can be said that in this stricter approach the dependence on the environment is much more complicated than in the model (3) which leads to Eq. (4).

VII. RECONSTRUCTION OF AN EFFECTIVE HAMILTONIAN

Using some simple assumptions we are going to reconstruct an effective Hamiltonian which leads in first order perturbation theory to the same result (31). It can also be written as

$$\langle I \rangle_s(v) = \frac{I_c^2}{8e} \text{Re} \left\{ \int_0^\infty d\tau e^{i(v-2\omega_c)\tau} - \int_0^\infty d\tau e^{i(-v-2\omega_c)\tau} \right\}.$$

(32)

It will turn out that this effective Hamiltonian corresponds just to the Hamiltonian (3). Our starting point is the Hamiltonian (11) and the assumption that the perturbation term $H_T$ can be written as

$$H_T = H_+ + H_- = H e^{i\Psi} + H e^{-i\Psi}.$$

This ansatz with real constants $H$ seems to be very likely because the factor in front of the $\delta$-function in the supercurrent is also a constant. $\Psi$ is a phase operator which is assumed to obey the commutation relation

$$[Q, \Psi] = ike,$$

(33)

where the constant $k$ is for the time being arbitrary. One gets the algebra

$$H_\pm F(Q) = F(Q \pm k\hbar)e^{i\Psi}$$

(34)

and finds using Eq. (14) by linear response theory the mean current

$$\langle I \rangle_s = -\frac{2ke}{\hbar t} \text{Re} \int_{-\infty}^{t} dt' \langle \langle H_+^{(1)}(t), H_-^{(1)}(t') \rangle \rangle_0$$

(35)
with
\[ H^{(I)}_{\pm}(t) = H e^{\mp ikeVt} e^{\pm i\Psi(t)}. \]

The time dependence of the phase operator governed by the environment Hamiltonian can be calculated in a standard way. Finally, Eq. (35) reads
\[ \langle I \rangle_s = -\frac{2keH^2}{\hbar^2} \text{Re} \int_0^\infty d\tau e^{-\frac{i}{\hbar}keV\tau} \left[ e^{k^2J(\tau)} - e^{k^2J(-\tau)} \right], \tag{36} \]
where we already know the function \( J \) from Eq. (4). By comparing Eq. (36) with Eq. (12) in the limit \( T = 0 \) the unknown constants \( H \) and \( k \) can be determined
\[ k = 2, \quad H = \frac{\hbar}{4e} I_c = \frac{E_J}{2} \tag{37} \]
which reproduce expression (1) with \( \Psi = 2\Phi \). The value \( k = 2 \) shows that the effective Hamiltonian describes tunneling of electron pairs (Cooper pairs). In this way the transition to the effective model corresponds to the transition from \([Q, \Phi] = ie\) to \([Q, \Psi] = i2e\). However, the effective model does not contain dissipation because it depends instead of \( I_p(\omega) \) on \( I_c \) only.

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**FIG. 1.** Scheme of the circuit

**FIG. 2.** Plot of the expressions according to Eq. (27) (top) and (28) (bottom) in units of \( I_c \) versus \( \omega \) in units of \( \Delta/h \) in the subgap region \( (0 < \omega < 2\Delta/h) \)
\[ V \rightarrow R, C \rightarrow R_E \]
