The Schwartz–Soffer and more inequalities for random fields

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Abstract
A new series of correlation inequalities for random field spin systems is proven rigorously. First one corresponds to the well-known Schwartz–Soffer inequality. These are expected to rule out incorrect results calculated in effective theories and numerical studies. The large $N$ expansion with the replica method for random field systems as an example is checked by these inequalities. It is shown that several critical exponents of multiple-point correlation functions at critical point satisfy obtained inequalities.

Keywords: disordered systems, correlation inequalities, multiple-point correlation functions, Gaussian interpolation, large $N$ expansion, critical exponents

1. Introduction
The Schwartz–Soffer inequality is a well-known useful inequality for random field spin systems [1]. This inequality claims that the Fourier transformed connected correlation function is bounded from the above by the square root of the corresponding disconnected one. This relation is quite useful to rule out incorrect results obtained by effective theories and numerical studies for random field spin systems. It is proven in a simple way with integration by parts over the Gaussian random fields and the Cauchy–Schwarz inequality. The Schwartz–Soffer inequality can check several approximation theories. Critical exponents calculated in the functional renormalization group [2–14], the large $N$ expansion with the replica method [4, 15] and recent numerical methods [16–20] satisfy the Schwartz-Soffer inequality. A few studies discuss generalization of this inequality [21].

In the present paper, we provide a new series of inequalities for multiple-point correlation functions in spin systems with Gaussian random fields. These inequalities are obtained in a square interpolation method, which is a rigorous mathematical method used for Gaussian random spin models extensively [22–30]. This method was used for the first time to obtain Guerra’s replica symmetric bound on the free energy density of the Sherrington–Kirkpatrick
model [22]. The usefulness of this method has attracted attention of many mathematicians and physicists since Talagrand proved the validity of the Parisi formula [31] for the replica symmetry breaking free energy density in the Sherrington–Kirkpatrick model [32] rigorously [26, 27]. Chatterjee generalized this method to evaluate bounds on correlation functions as well as on the free energy density in random spin systems [25]. Chatterjee’s generalization is quite useful to evaluate several observables [28–30]. We use his method to obtain a series of inequalities. The first one in this series is the Schwartz–Soffer inequality which gives a lower bound on a disconnected correlation function. The second one in the series gives an upper bound on the disconnected correlation function. These and other inequalities for multiple-point correlation functions are expected to rule out incorrect results furthermore.

This paper is organized as follows. Section 2 gives a definition of random field spin models and our main theorem. In section 3, two lemmas are proven, and these enable us to prove our main theorem. In section 4, several multiple-point correlation functions are calculated in large $N$ expansion with the replica method. It is shown that critical exponents calculated in the large $N$ expansion with the replica method satisfy these inequalities. Section 5 summarizes our results.

2. Definitions and main theorem

First, we define the model and functions. Coupling constants in a system with quenched disorder are given by independent and identically distributed (i.i.d.) random variables. We can regard a given disordered sample as a system obtained by a random sampling of these variables. All physical quantities in such systems are functions of these random variables. Consider a random field $O(N)$ invariant Ginzburg–Landau model on a $d$-dimensional hypercubic lattice $\Lambda_L : = [1, L]^d \cap \mathbb{Z}^d$ whose volume is $|\Lambda_L| = L^d$. Let $J = (J_{x, y})_{x, y \in \Lambda_L}$ be a real symmetric matrix such that $J_{x, y} \equiv 1$, if $|x - y| = 1$, otherwise $J_{x, y} = 0$. Define Hamiltonian as a function of $N$ dimensional spin vector configurations $\phi = (\phi^i_n)_{x \in \Lambda_L, n = 1, 2, \ldots, N} \in (\mathbb{R}^N)^{N_L}$ and i.i.d. standard Gaussian random variables $g = (g^i_n)_{n = 1, 2, \ldots, N}$ by

$$H(\phi, g) := - \sum_{x, y \in \Lambda_L} J_{x, y} \phi_x \cdot \phi_y - h \sum_{x \in \Lambda_L} g_x \cdot \phi_x,$$  \hspace{1cm} (1)$$

with a real constant $h$. Here, we define Gibbs state for the Hamiltonian. For a positive $\beta$, the partition function is defined by

$$Z_\beta(\beta, h, g) := \int_{\mathbb{R}^N|\Lambda_L|} D\phi \ e^{-\beta H(\phi, g)}.$$  \hspace{1cm} (2)$$

The measure $D\phi$ is $O(N)$ invariant, for example, it is defined by

$$D\phi := C \prod_{x \in \Lambda_L} \prod_{n = 1}^N d\phi_n^x e^{-\alpha (\phi_n^x - \bar{\phi}_x)^2},$$  \hspace{1cm} (3)$$

where $\alpha > 0$ and $C$ is a normalization constant satisfying

$$C^{-1} = \int_{\mathbb{R}^N|\Lambda_L|} D\phi.$$
The following is also possible
\[
D\phi = \prod_{x \in \Lambda_L} \prod_{n=1}^{N} \prod_{x \in \Lambda_L} \delta (\phi_n \cdot \phi_n - 1),
\] (4)
which can be obtained by the limit \( u \to \infty \) of (3).

The expectation of a function of spin configuration \( f(\phi) \) in the Gibbs state is given by
\[
(f(\phi))_\beta = \frac{1}{Z_L(\beta, h, g)} \int \mathbb{R}^{N|\Lambda_L} D\phi f(\phi) e^{-\beta H(\phi,g)}.
\] (5)

Define the following function of \( (\beta, h) \in [0, \infty) \times \mathbb{R} \) and randomness \( g = (g^n_{ij})_{n \in \Lambda_L} \)
\[
\psi_L(\beta, h, g) := \frac{1}{|\Lambda_L|} \log Z_L(\beta, h, g).
\] (6)

\(-\frac{\beta}{\Lambda_L} \psi_L(\beta, h, g)\) is called free energy in statistical physics. We define a function \( p_L : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) by
\[
p_L(\beta, h) := E \psi_L(\beta, h, g),
\] (7)
where \( E \) stands for the expectation over the random variables \( (g^n_{ij})_{n \in \Lambda_L, i=1, \ldots, N} \).

Impose the periodic boundary condition for all variables on the lattice \( \Lambda_L \) and define their Fourier transformation
\[
\tilde{g}_q^p := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} e^{-i q \cdot x} g^p_x, \quad \tilde{g}_q^p := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} e^{-i q \cdot x} g^p_x, \quad \tilde{g}_q^p := \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} e^{-i q \cdot x} g^p_x.
\] (8)

Define a connected correlation function for arbitrary operators \( A_1, \ldots, A_j \) for a positive integer \( j \) by
\[
\langle A_1; \ldots; A_j \rangle_g = \left[ -\frac{\partial^j}{\partial b_1 \ldots \partial b_j} \log Z_L(\beta, h, g) \right]_{b_1=\ldots=b_j=0},
\]
where the generating function is defined for \( b := (b_1, \ldots, b_j) \in \mathbb{R}^j \) by
\[
Z_L(b, \beta, h, g) = \int_{\mathbb{R}^{N|\Lambda_L}} D\phi e^{-\beta H(\phi,g)+\sum_{i=1}^{j} b_i h_i}.
\]

Note that \( Z_L(0, \ldots, 0, \beta, h, g) = Z_L(\beta, h, g), \) and \( \langle A_1; A_2 \rangle_g := \langle A_1 A_2 \rangle_g - \langle A_1 \rangle_g \langle A_2 \rangle_g \) for \( j = 2 \).

Then, we have the following main theorem.

**Theorem.** Consider the random field \( O(N) \) invariant ferromagnetic spin model. For a non-negative integer \( k, l \), the following inequalities for the variance of \( k \)-point correlation functions
are valid
\[
\frac{\beta^2 n^2}{n!} \sum_{p_1 \in \Lambda^*_L} \cdots \sum_{p_N \in \Lambda^*_L} \sum_{n_1 = 1}^N \cdots \sum_{n_N = 1}^N |\mathbb{E} W_{p_1, \ldots, p_N}^n|^2 \]
\[
\leq \text{Var}(f_1(\phi); \ldots; f_s(\phi))_g \leq \beta^2 n^2 \sum_{p \in \Lambda^*_L} \mathbb{E}|W_p|^2,
\]
where \((k + l)\)-point correlation function for a sequence of complex valued functions \(\tilde{f} := (f_1(\phi), \ldots, f_s(\phi))\) of a spin configuration, is defined by
\[
W_{p_1, \ldots, p_N}^n := \langle \tilde{\phi}_{p_1}^n; \ldots; \tilde{\phi}_{p_N}^n; f_1(\phi); \ldots; f_s(\phi) \rangle_g.
\]
and the sample variance is defined by
\[
\text{Var} F(g) := \mathbb{E}[F(g)]^2 - \mathbb{E}[F(g)]^2,
\]
for a complex valued function \(F\) of the sequence of random fields \(g\).

Note. For \(k = l = 1\) and \(f_1(\phi) = \tilde{\phi}_{q}^m\), theorem gives the following upper and lower bounds on the disconnected correlation function
\[
\left( \mathbb{E} \langle \tilde{\phi}_q^m; \tilde{\phi}_{q'}^{m'} \rangle_g \right)^2 \leq \beta^{-2} \mathbb{E} \mathbb{E} \langle \tilde{\phi}_q^m \rangle_g^2 \leq \sum_{p \in \Lambda^*_L} \sum_{n=1}^N \mathbb{E} \mathbb{E} \langle \tilde{\phi}_{p}^n \rangle_g^2 .
\]
Note that \(\mathbb{E} \langle \tilde{\phi}_q^m; \tilde{\phi}_{q'}^{m'} \rangle_g \leq \sum_{p \in \Lambda^*_L} \sum_{n=1}^N |\mathbb{E} \langle \tilde{\phi}_{p}^n \rangle_g|^2\) and \(\mathbb{E} \langle \tilde{\phi}_q^m \rangle_g = 0\) for any \(m, q \in \Lambda^*_L\), because of the \(O(N)\) symmetry. The lower bound (10) implies the Schwartz–Soffer inequality
\[1\].

3. Proof

Theorem can be proven in terms of the square root interpolation method used in disordered systems \([24, 25, 27, 28]\). Let \(g = (g^i_q)_{q \in \Lambda_L, i = 1, \ldots, N}, g' = (g'^i_q)_{q \in \Lambda_L, i = 1, \ldots, N}\) be two sequences of i.i.d. standard Gaussian variables, and their Fourier transformed sequences \(\tilde{g} = (\tilde{g}^i_q)_{q \in \Lambda_L, i = 1, \ldots, N}, \tilde{g}' = (\tilde{g}'^i_q)_{q \in \Lambda_L, i = 1, \ldots, N}\) and define a function of \(t \in [0, 1]\) by
\[
G(t) := \sqrt{t} \tilde{g} + \sqrt{1 - t} \tilde{g}',
\]
and its Fourier transform
\[
\tilde{G}(t) := \sqrt{t} \tilde{g} + \sqrt{1 - t} \tilde{g}'.
\]
For a sequence of functions \(\tilde{f} := (f_1(\phi), \ldots, f_s(\phi))\) of a spin configuration, define a generating function \(\gamma_{\tilde{f}}(t)\) of a parameter \(s \in [0, 1]\) by
\[
\gamma_{\tilde{f}}(t) = \mathbb{E} |\mathbb{E} f_1(\phi); \ldots; f_s(\phi)_g)|^2 .
\]
\[11\]
Lemma 1. For any $\beta, h \in [0, \infty) \times \mathbb{R}$, any positive integers $k, l$ and a sequence of complex valued functions $\vec{f} = (f_1(\phi), \ldots, f_k(\phi))$ of a spin configuration, $l$th order derivative of $\gamma_f(t)$ is represented in the following connected correlation function for an arbitrary $t \in [0, 1]$:

$$
\gamma^{(0)}_f(t) = \beta^2 h^2 \sum_{q_1 \in \Lambda^*_L} \cdots \sum_{q_l \in \Lambda^*_L} \sum_{n_1=1}^{N} \cdots \sum_{n_l=1}^{N} \mathbb{E}[\mathbb{E}' \mathbb{W}^n_{q_1, \ldots, q_l ; \vec{f}(\vec{q}) \gamma \vec{g}'}]^2.
$$

Proof. For shorthand notation, denote

$$(\vec{f})_{\vec{G}^{(0)}} := \langle f_1(\phi); \cdots ; f_k(\phi) \rangle_{\vec{G}^{(0)}}.$$ 

The first derivative of $\gamma_f$ is calculated in integration by parts

$$
\gamma'_f(t) = \mathbb{E} \left[ E'_{\vec{f}(\vec{G}^{(0)})} E' \sum_{p \in \Lambda^*_L} \left( \frac{\vec{g}_p^m}{2\sqrt{1 - t}} - \frac{\vec{g}_p^m}{2\sqrt{1 - t}} \right) \frac{\partial (\vec{f})_{\vec{G}^{(0)}}}{\partial G_p^m} + \text{c.c.} \right]
$$

$$
= \mathbb{E} \sum_{p \in \Lambda^*_L} \sum_{n=1}^{N} \left[ \frac{1}{2\sqrt{1 - t}} \frac{\partial}{\partial G_p^m} \mathbb{E}' (\vec{f})_{\vec{G}^{(0)}} \mathbb{E}' \frac{\partial (\vec{f})_{\vec{G}^{(0)}}}{\partial G_p^m} \right]
$$

$$
- \mathbb{E}' (\vec{f})_{\vec{G}^{(0)}} \mathbb{E}' \frac{1}{2\sqrt{1 - t}} \frac{\partial}{\partial G_p^m} \frac{\partial (\vec{f})_{\vec{G}^{(0)}}}{\partial G_p^m} + \text{c.c.} \right] = \sum_{p \in \Lambda^*_L} \sum_{n=1}^{N} \mathbb{E} \left| \mathbb{E}' \frac{\partial (\vec{f})_{\vec{G}^{(0)}}}{\partial G_p^m} \right|^2
$$

$$
= \beta^2 h^2 \sum_{p \in \Lambda^*_L} \sum_{n=1}^{N} \mathbb{E} \left| \mathbb{E}' \frac{\partial (\vec{f})_{\vec{G}^{(0)}}}{\partial G_p^m} \right|^2.
$$

The formula for the $l$th order derivative $\gamma^{(l)}_f(t)$ is proven by a mathematical induction. □

Note the positive semi-definiteness of an arbitrary order derivative $\gamma^{(l)}_f(t)$ which implies that all order derivative functions are monotonically increasing and convex.

Lemma 2. For any $t_1 < t_2$, any non-negative integers $j, k, l$, and a sequence $\vec{f} = (f_1(\phi), \ldots, f_k(\phi))$, the following inequality is valid

$$(t_2 - t_1)^j \gamma^{(l+j)}_f(t_1) \leq (t_2 - t_1)^j \gamma^{(l+j)}_f(t_2).$$

Proof. Taylor’s theorem implies that there exists $t \in (t_1, t_2)$, such that

$$
\gamma^{(l)}_f(t_2) = - \sum_{j=0}^{l-1} \frac{1}{l!} (t_2 - t_1)^{j+1} \gamma^{(l+j)}_f(t_1) + \frac{1}{l!} (t_2 - t_1)^{j+1} \gamma^{(l+j)}_f(t).
$$

The inequality is obvious, since each Taylor coefficient is positive semi-definite. □

Proof of theorem. Note the following representation of the variance of the $k$-point connected correlation function

$$
\text{Var}(f_1(\phi); \cdots ; f_k(\phi))_g = \gamma_f(1) - \gamma_f(0).
$$
Lemma 2 and monotonicity of $\gamma_f^r(t)$ gives
\[
\frac{\gamma_f^r(0)}{r} \leq \gamma_f^r(0) \int_0^1 \frac{t^{l-1}}{(l-1)!} \, dt \leq \int_0^1 \gamma_f^r(t) \, dt \leq \gamma_f^r(1),
\]
for a positive integer $l$. This enables us to prove theorem. \hfill \Box

4. Large $N$ expansion with the replica method

Here, we check whether a critical exponent calculated in the large $N$ expansion with the replica method obtained in reference [4] satisfies the correlation inequality (10). Consider the model defined by the partition function (2) with $O(N)$ invariant Hamiltonian (1) and measure (4) for $d > 4$ to study critical phenomenon in random field $O(N)$ spin model. Here, for the purpose of the field-theoretical description, we redefine the lattice as $\Lambda_L := (-L/2, L/2)^d \cap \mathbb{Z}^d$. The boundary condition for all variables on the lattice $\Lambda_L$ remains periodic, and their Fourier transformation are redefined as
\[
\tilde{\phi}_q^r := \sum_{x \in \Lambda_L} e^{-i q \cdot x} \phi_x^r, \quad \tilde{g}_q^r := \sum_{x \in \Lambda_L} e^{-i q \cdot x} g_x^r,
\]
where $q \in \frac{2\pi}{L} \Lambda_L =: \Lambda^*_L$. In order to compute the expectation over the Gaussian random variables $(g_{x,a})_{x \in \Lambda_L, a=1,2,...,N}$ in (7) by use of the replica method, we introduce $r$ copies of the original spins: $(\phi_{x,a}^r)_{x \in \Lambda_L, a=1,2,...,N, r=1,2,...,r}$. The expectation in (7) is calculated via the following relation:
\[
\mathbb{E}[\psi(\beta,h,g)] = \frac{1}{|\Lambda_L|} \mathbb{E}[\log Z_L(\beta,h,g)] = \frac{1}{|\Lambda_L|} \lim_{r \downarrow 0} \frac{\mathbb{E}Z_L(\beta,h,g)^r - 1}{r},
\]
where $\mathbb{E}Z_L(\beta,h,g)^r$ and the limit $r \downarrow 0$ are called the replica partition function and the replica limit, respectively. The expectation value of the function $f(\phi)$ of spin configurations can be calculated in the following replica limit,
\[
\mathbb{E}(f(\phi))^r = \frac{1}{Z_L(\beta,h,g)^r} \int_{\mathbb{R}^{|\Lambda_L|}^r} D\phi f(\phi) e^{-rH(\phi,g)}
= \lim_{r \downarrow 0} \frac{1}{Z_L(\beta,h,g)^r} \int_{\mathbb{R}^{|\Lambda_L|}^r} D\phi_a f(\phi_a) e^{-\beta \sum_{a=1}^r H(\phi_a,g)}.
\]
Two different replicas of spin configurations $\phi_a, \phi_b$ enable us to calculate the sample expecta- tion of the product of two Gibbs expectations of two functions $f_1(\phi), f_2(\phi)$ of spin configuration.
\[
\mathbb{E}(f_1(\phi) f_2(\phi))^r = \frac{1}{Z_L(\beta,h,g)^r} \int_{\mathbb{R}^{|\Lambda_L|}^r} D\phi f_1(\phi) e^{-rH(\phi,g)} \frac{1}{Z_L(\beta,h,g)^r} \int_{\mathbb{R}^{|\Lambda_L|}^r} D\phi f_2(\phi) e^{-rH(\phi,g)}
\times \int_{\mathbb{R}^{|\Lambda_L|}^r} D\phi f_2(\phi) e^{-rH(\phi,g)}
= \lim_{r \downarrow 0} \frac{1}{Z_L(\beta,h,g)^r} \int_{\mathbb{R}^{|\Lambda_L|}^r} D\phi_a f_1(\phi_1) f_2(\phi_2) e^{-\beta \sum_{a=1}^r H(\phi_a,g)}.
\]
This replica method is believed to be useful, and has been employed to calculate observables extensively in statistical physics, although the replica limit is not mathematically rigorous. There have been many results calculated in effective theories using the replica method. At least, we have to check them in rigorous inequalities. After integrating over the Gaussian random variables \((g^n_\alpha)_\alpha\in\Lambda_L\), \(n=1,2,...,N\), we obtain the following expression for the replica partition function:

\[
\mathbb{E}Z_L(\beta, h, g)' = e^{i|\Lambda_L|^2dd} \int \left( \prod_{\alpha \in \Lambda_L} \prod_{a=1}^r \sqrt{N} d\phi_{\alpha,a} \delta \left( \phi_{\alpha,a}^2 - 1 \right) \right) e^{-\beta H_{\text{rep}}},
\]

\[
\beta H_{\text{rep}} = \frac{\beta}{2} \sum_{x \in \Lambda_L} \sum_{a,b=1}^r \phi_{x,a}(-J \hat{\Delta} \delta_{a,b} - \beta \Delta_G) \phi_{x,b}.
\]

Here, we have redefined \(J_{x,y} = J/2 (J > 0)\), if \(|x - y| = 1\), otherwise \(J_{x,y} = 0\). \(\hat{\Delta}\) denotes the lattice Laplacian in the Euclidean space, which is represented by \(\hat{\Delta} = 2\sum_{\mu=1}^q (\cos q_\mu - 1)\) in the momentum representation. \(\Delta_G\) denotes the strength of the Gaussian random variables \((g^n_\alpha)_\alpha\in\Lambda_L\), \(n=1,2,...,N\) such that \(h^2 g^n_\alpha = \Delta_G \delta_{\alpha,\alpha'}\). Integrating over the replicated spin variables \((\phi^n_\alpha)_\alpha\in\Lambda_L\) after introducing the auxiliary variable \(\lambda_{ax} \in \mathbb{R}\) to rewrite \(\delta(\phi_{\alpha,a}^2 - 1)\) as

\[
\delta(\phi_{\alpha,a}^2 - 1) = \int_{-\infty}^{\infty} \frac{d\lambda_{ax}}{4\pi} e^{-\beta \lambda_{ax} (\phi_{\alpha,a}^2 - 1)/2},
\]

the replica partition function becomes

\[
\mathbb{E}Z_L(\beta, h, g)' = e^{N|\Lambda_L|^2dd \left( \frac{N\beta}{4\pi} \right)^{r|\Lambda_L|} \left( \frac{2\pi}{\beta} \right)^{N|\Lambda_L|^2/2} \int \left( \prod_{\alpha \in \Lambda_L} \prod_{a=1}^r d\lambda_{ax} \right) e^{-S_{\text{eff}}}},
\]

\[
S_{\text{eff}} = \frac{N}{2} \sum_{x \in \Lambda_L} \langle x | \text{Tr} \ln \left( -J \hat{\Delta}_r + \chi \right) | x \rangle - \frac{N\beta}{2} \sum_{x \in \Lambda_L} \sum_{a=1}^r i\lambda_{ax},
\]

where \(\mathbf{1}_r\) is an \(r \times r\) unit matrix, \(\chi\) is an \(r \times r\) symmetric matrix with

\[
\chi_{ax} = \delta_{ax} - \beta \Delta_G.
\]

Here we have redefined the parameters as follows:

\[
\frac{\beta}{N} \rightarrow \beta,
\]

\[
N\Delta_G \rightarrow \Delta_G,
\]

to keep \(\beta/N\) and \(N\Delta_G\) with finite in the large \(N\) limit. We should note that the relation between \(\beta^2 h^2\) which is appeared in the previous sections and \(\beta \Delta_G\) which is introduced in this section is given by \(\beta^2 h^2 \leftrightarrow \beta \Delta_G\).

### 4.1. Replica-symmetric saddle-point equation and expansion of \(S_{\text{eff}}\) around replica-symmetric saddle point

Note that the replica symmetry breaking (RSB) does not occur in the random field Ising model for almost all \((\beta, h) \in (0, \infty) \times \mathbb{R}\) [33]. It is believed that RSB does not occur either in the
random field $O(N)$ spin model [2, 4–11, 13, 14]. Here, we assume replica symmetry to calculate correlation functions. The saddle-point equation is obtained by differentiating $S_{\text{eff}}$ by $i\lambda_{ax}$ as follows:

$$
\frac{\delta S_{\text{eff}}}{\delta i\lambda_{ax}} = \frac{N}{2} \left\langle x \left[ \left( -J \Delta_1 \mathbf{1}_r + \chi \right)_a \right] \right\rangle - \frac{N\beta}{2} = 0.
$$

We assume the replica symmetry

$$
i\lambda_{ax} = m^2,
$$

and write the replica-symmetric solution formally as

$$
\bar{\chi}_{ab} = m^2 \delta_{a,b} - \beta \Delta_G.
$$

In this assumption, the propagator is written in the momentum representation as

$$
\left\langle k \left| \left( -J \Delta_1 \mathbf{1}_r + \bar{\chi} \right)_{ab} \right| k \right\rangle = \frac{1}{-J \Delta_k + m^2 \delta_{a,b} + \beta \Delta_G} = G^S_{ok} \bar{\delta}_{a,b} + (\beta \Delta_G) G^d_{ok} = G_{ab}.
$$

Then, the saddle-point equation becomes

$$
\beta = \frac{1}{\left| \Lambda_L \right|} \sum_{k \in \Lambda_L^*} \frac{1}{-J \Delta_k + m^2 + \beta \Delta_G} \frac{1}{\left| \Lambda_L \right|} \sum_{k \in \Lambda_L^*} \frac{1}{-J \Delta_k + m^2}.
$$

We put

$$
\chi_{abx} = \bar{\chi}_{ab} + i\epsilon_{a,x} \delta_{a,b} =: \bar{\chi}_{ab} + \delta \chi_{abx},
$$

and expand $S_{\text{eff}}$ up to the second order of $\delta \chi_{abx}$. The second-order term of $\delta \chi_{abx}$ for $S_{\text{eff}}$ becomes

$$
\delta^2 S_{\text{eff}} = -\frac{N}{4} \sum_{x \in \Lambda_L} \left\langle x \left[ \text{Tr} \left( \frac{1}{-J \Delta_1 \mathbf{1}_r + \chi} \right) \frac{1}{-J \Delta_1 \mathbf{1}_r + \chi} \delta \chi \right] \right\rangle
$$

$$
= -\frac{N}{4} \left| \Lambda_L \right| \sum_{k \in \Lambda_L^*} \sum_{a,b=1}^r \epsilon_{a,k} \epsilon_{b,-k} \Pi_{abk}.
$$

Here, $\Pi_{abk}$ is

$$
\Pi_{abk} := \frac{1}{\left| \Lambda_L \right|} \sum_{q \in \Lambda_L^*} G_{ok}^{ab} G_{0q}^{ba}
$$

$$
= [(A + A)_k + (A \ast B)_k + (B \ast A)_k] \delta_{a,b} + (B \ast B)_k,
$$

$$
(A + A)_k := \frac{1}{\left| \Lambda_L \right|} \sum_{q \in \Lambda_L^*} G_{0k}^{ab} G_{0q}^{ba},
$$

$$
(A \ast B)_k := (\beta \Delta_G) \frac{1}{\left| \Lambda_L \right|} \sum_{q \in \Lambda_L^*} G_{0k}^{ab} G_{0q}^{ba}.
$$



\[
(B + A)_k := (βΔ_G) \frac{1}{|A_l|} \sum_{q \in A_l} G^l_{0q} G_{0q},
\]

\[
(B + B)_k := (βΔ_G)^2 \frac{1}{|A_l|} \sum_{q \in A_l} G^d_{0q} G^d_{0q},
\]

\[
4.2. \text{Calculation of multi-point correlation functions}
\]

Here, we calculate the multi-point correlation functions up to the second order of the perturbation. For simplicity, we put \( J = 1 \). In the thermodynamic limit \( L \to \infty \), the lattice Laplacian at criticality and the summation over \( q \in A_l \) become

\[
\tilde{Δ}_x = \sum_{\mu=1}^d \frac{∂^2}{∂x^2}_μ =: \partial^2, \quad \tilde{Δ}_k = -k^2,
\]

\[
\frac{1}{|A_l|} \sum_{q \in A_l} \rightarrow \prod_{\mu=1}^d \left( \int_{-\pi}^{\pi} \frac{dq_μ}{2\pi} \right) =: \int_{[-\pi,\pi]^d} \frac{d^d q}{(2\pi)^d} =: \int_q.
\]

Then, (31) is

\[
Π_{abh} = (c_0 + c_1 k^d - 4 + c_2 k^{d-6}) \delta_{a,b} + c_3 k^{d-8},
\]

for \( m^2 = 0 \), where

\[
c_0 = \int_q \frac{1}{|q|^4},
\]

\[
c_1 \simeq \frac{1}{(4\pi)^{d/2}} \frac{\Gamma \left( \frac{d-4}{2} \right) 2 \Gamma \left( \frac{d-2}{2} \right)^2 - \frac{1}{4} \Gamma \left( \frac{d-2}{2} \right)^2}{\Gamma(d-4)}.
\]

\[
c_2 \simeq \frac{2(βΔ_G) \Gamma \left( \frac{d-4}{2} \right) \Gamma \left( \frac{d-2}{2} \right) \Gamma \left( \frac{d-4}{2} \right)}{(4\pi)^{d/2} \Gamma(d-3)},
\]

\[
c_3 \simeq \frac{(βΔ_G)^2 \Gamma \left( \frac{d-4}{2} \right) \Gamma \left( \frac{d-2}{2} \right)^2}{(4\pi)^{d/2} \Gamma(d-4)}.
\]

\[
4.2.1. \text{Two-point correlation function.} \quad \text{In order to evaluate the two-point correlation functions} \quad \mathbb{E} \langle \tilde{φ}^a_q, \tilde{φ}^a_{-q} \rangle_k \quad \text{and} \quad \mathbb{E} \langle \tilde{φ}^a_q, \tilde{φ}^a_{-q} \rangle_k, \quad \text{we compute the following correlation function up to the second order of} \quad \epsilon_{a,k}:\]

\[
G^q_{ab12} \delta_{mn} := \frac{δ_{mn}}{Z_r} \left( \int \prod_{k \in A_l} \prod_{α=1}^r d\tilde{ε}_{a,k} \right) \left\langle q \left| \frac{1}{-\partial^2 L + \chi} \right| a_{12} \right\rangle e^{-S_{\text{eff}}}
\]

\[
\simeq \left[ G^q_{ab1} = \sum_{b_1,b_2=1}^r G^q_{0b_1} G_{0b_2} \int_k G_{0q-2k} \delta_{b_1 k} \tilde{ε}_{b_2,-k} G_{0q} \right] \delta_{mn}, \quad (43)
\]
where \( \mathcal{Z}_< \) and \( \langle \tilde{e}_{b_1, k} \tilde{e}_{b_2, -k} \rangle \) are defined by

\[
\mathcal{Z}_< := \left( \int \prod_{k \in \Lambda_3^*} \prod_{a=1}^r d\tilde{e}_{a,k} \right) e^{-\beta S_{\text{eff}}}, \tag{44}
\]

\[
\langle \tilde{e}_{b_1, k} \tilde{e}_{b_2, -k} \rangle := \frac{1}{\mathcal{Z}_<} \left( \int \prod_{k \in \Lambda_3^*} \prod_{a=1}^r d\tilde{e}_{a,k} \right) \tilde{e}_{b_1, k} \tilde{e}_{b_2, -k} e^{-\beta S_{\text{eff}}}. \tag{45}
\]

In \( 4 < d < 6 \) and in low momentum, \( \langle \tilde{e}_{b_1, k} \tilde{e}_{b_2, -k} \rangle \) becomes

\[
\langle \tilde{e}_{b_1, k} \tilde{e}_{b_2, -k} \rangle \simeq \frac{2}{N c_2} \left( \frac{1}{k^{d-6}} \delta_{b_1, b_2} - (\beta \Delta_G) \frac{6 - d}{2} \right) =: (\Pi^{-1})_{b_1 b_2}. \tag{46}
\]

Thus, we get the following expression for \( G_q^{a_1 a_2} \):

\[
G_q^{a_1 a_2} \simeq \frac{1}{q^2} \left( 1 + \frac{d - 4}{N} \log q \right) \delta_{a_1, a_2} + \frac{\beta \Delta_G}{q^4} \left( 1 + \frac{d - 4}{N} \log q \right). \tag{47}
\]

### 4.2.2. Disconnected four-point correlation function

In order to evaluate the correlation function \( \sum_{p \in \Lambda_4^*} \mathbb{E} \left( \langle \tilde{\phi}_m^{(m)} \tilde{\phi}_p^{(p)} \rangle \right)^2 \), we compute the following correlation function up to the second order of \( \tilde{e}_{a,k} \):

\[
G_q^{a_1 a_2 a_3 a_4} \delta_{m,n} := \frac{\delta_{m,n}}{\mathcal{Z}_<} \left( \int \prod_{k \in \Lambda_3^*} \prod_{a=1}^r d\tilde{e}_{a,k} \right) \int_p \left\langle q \left| \left( \frac{1}{-\partial^2 1_r + \chi} \right)_{a_1 a_2} \right| p \right\rangle \\
\times \left\langle p \left| \frac{1}{-\partial^2 1_r + \chi} \right| a_3 a_4 \right\rangle e^{-\beta S_{\text{eff}}} \\
\simeq \left[ G_q^{a_1 a_2} G_q^{a_3 a_4} - \sum_{b_1, b_2 = 1}^r G_q^{a_1 b_1} G_q^{b_2 a_4} \int_k G_k^{b_1 b_2} (\Pi^{-1})_{b_1 b_2} G_q^{b_2 a_4} \right] \delta_{m,n}. \tag{48}
\]

The contribution of \( G_q^{a_1 a_2 a_3 a_4} \) to the correlation function \( \sum_{p \in \Lambda_4^*} \mathbb{E} \left( \langle \tilde{\phi}_m^{(m)} \tilde{\phi}_p^{(p)} \rangle \right)^2 \) originates from the terms which are proportional to \( \delta_{a_1, a_2} \delta_{a_3, a_4} \). Thus, we get the following expression for \( G_q^{a_1 a_2 a_3 a_4} \):

\[
G_q^{a_1 a_2 a_3 a_4} \simeq \frac{1}{q^4} \left( 1 + \frac{d - 4}{N} \log q \right) \delta_{a_1, a_2} \delta_{a_3, a_4}. \tag{49}
\]
4.2.3. Connected four-point correlation function. In order to evaluate the correlation function

\[ \sum_{p_1,p_2,p_3} \sum_{N} \left| \mathcal{E} \left\{ \tilde{\phi}_{p_1}^{a_1}, \tilde{\phi}_{p_2}^{a_2}, \tilde{\phi}_{p_3}^{a_3}, \tilde{\phi}_{q}^{a_4} \right\} \right|^2, \]

we compute the following correlation function up to the second order of \( \bar{\epsilon}_{a,k} \):

\[ \sum_{p_1,p_2,p_3} \sum_{N} G_{p_1,p_2,p_3}^{a_1,a_2,a_3} G_{q,p_2,p_3}^{a_4,a_2,a_3} \delta_{p_1,n_{a_1}} \delta_{p_2,n_{a_2}} = N \sum_{p_1,p_2,p_3} G_{p_1,p_2,p_3}^{a_1,a_2,a_3} G_{q,p_2,p_3}^{a_4,a_2,a_3}. \tag{50} \]

where

\[ G_{p_1,p_2,p_3}^{a_1,a_2,a_3} = \frac{1}{Z_c} \left( \int \prod_{k \in \Lambda_L^3} d\bar{\epsilon}_{a,k} \right) \langle p_1 \left| \frac{1}{-\partial^2 + \chi} \right| a_{1}^{a_2} \rangle \times \langle p_3 \left| \frac{1}{-\partial^2 + \chi} \right| a_{3}^{a_4} \rangle e^{-\Delta_{\text{eff}}} \]

\[ \approx \sum_{b_{0p_1}} G_{b_{0p_1}}^{a_1,a_2,a_4} \delta_{p_1,n_{b_{0p_1}}} - \sum_{b_1,b_2} G_{b_{1p_1}}^{a_1,a_2} G_{b_{2p_3}}^{a_4} \times \left( \prod_{k} G_{b_{k}^{b_1,b_2}} \right) \delta_{p_1,n_{b_{k}}} \delta_{p_3,n_{b_{k}}} - \sum_{b_1,b_2} G_{b_{1p_1}}^{a_1,a_2} G_{b_{2p_2}}^{a_4,a_2} \delta_{p_1,n_{b_{2}}} \delta_{p_2,n_{b_{1}}} - \sum_{b_1,b_2} G_{b_{1p_1}}^{a_1,a_2} G_{b_{2p_2}}^{a_4,a_2} \delta_{p_1,n_{b_{1}}} \delta_{p_2,n_{b_{2}}}. \tag{51} \]

The leading contribution of \( G_{p_1,p_2,p_3}^{a_1,a_2,a_3} G_{q,p_2,p_3}^{a_4,a_2,a_3} \) to the correlation function

\[ \sum_{p_1,p_2,p_3} \sum_{N} \left| \mathcal{E} \left\{ \tilde{\phi}_{p_1}^{a_1}, \tilde{\phi}_{p_2}^{a_2}, \tilde{\phi}_{p_3}^{a_3}, \tilde{\phi}_{q}^{a_4} \right\} \right|^2 \]

originates from the terms which are proportional to \( \delta_{a_1,a_2} \delta_{a_1,a_3} \delta_{a_3,a_4}, \delta_{p_1,p_2,p_3} \) and \( (\beta \Delta_{\text{eff}})^{-2} \).

Calculating

\[ \sum_{p_1,p_2,p_3} G_{p_1,p_2,p_3}^{a_1,a_2,a_3} G_{q,p_2,p_3}^{a_4,a_2,a_3}, \]

we have

\[ \sum_{p_1,p_2,p_3} G_{p_1,p_2,p_3}^{a_1,a_2,a_3} G_{q,p_2,p_3}^{a_4,a_2,a_3} \approx \left( \frac{4 - d}{4(\beta \Delta_{\text{eff}})^2 N^2} \log q \right) \delta_{a_1,a_2} \delta_{a_1,a_3} \delta_{a_3,a_4}. \tag{52} \]
4.3. Critical exponents and check of theorems and inequalities

Assume the following asymptotic form of correlation functions for small wave number $q$

$$\mathbb{E}\langle \tilde{\phi}_m^q, \tilde{\phi}_n^{-q} \rangle_k \approx \frac{\delta_{m,n}}{q^{2-\eta}},$$

$$\mathbb{E}\langle \tilde{\phi}_m^q \rangle_k \langle \tilde{\phi}_n^{-q} \rangle_k \approx \frac{\delta_{m,n}}{q^{4-\bar{\eta}}}. \quad (53)$$

The Schwartz–Soffer inequality (10) imposes

$$2\eta \geq \bar{\eta}. \quad (54)$$

These critical exponents $\eta$ and $\bar{\eta}$ calculated in several functional renormalization group calculations [2–14], large $N$ expansion studies with the replica method [4, 15] and recent numerical studies [16–20] satisfy this inequality. According to (47) in the leading order of the large $N$ expansion with the replica method [4], these are

$$\mathbb{E}\langle \tilde{\phi}_m^q, \tilde{\phi}_n^{-q} \rangle_k \approx \frac{\delta_{m,n}}{q^{2}} \left(1 + \frac{d-4}{N} \log q \right), \quad (55)$$

$$\mathbb{E}\langle \tilde{\phi}_m^q \rangle_k \langle \tilde{\phi}_n^{-q} \rangle_k \approx \frac{\delta_{m,n}}{q^{4}} \left(1 + \frac{d-4}{N} \log q \right). \quad (56)$$

Then, this expansion gives the correlation exponents $\eta$ and $\bar{\eta}$

$$\eta = \frac{d-4}{N}, \quad \bar{\eta} = \frac{d-4}{N}, \quad (57)$$

for the $d$-dimensional random field O($N$) spin model [4]. These values in (57) are consistent with results obtained in the functional renormalization group [4–9, 11, 14]. Note that $\eta = \bar{\eta}$ satisfies the Schwartz-Soffer inequality (10). This identity is well-known as the dimensional reduction which claims that the critical exponents of a random field spin system in dimension $d$ are identical to those of the corresponding spin system without random field in dimension $d-2$ [34, 35]. Parisi and Sourlas conjectured the dimensional reduction in the argument of the hidden supersymmetry [35]. Although the dimensional reduction and the supersymmetry conjectures fail in dimensions less than four, its validity near six dimensions or for large $N$ is discussed still [2–9, 11, 14–20, 36]. In addition to this result, consider another critical exponent $\eta'$ of the following correlation function

$$\sum_{p \in \Lambda^*_L} \mathbb{E}|\tilde{\phi}_m^q(p); \tilde{\phi}_n^q(p')|^2 \approx \frac{\delta_{m,n}}{q^4-q'^4}. \quad (58)$$

According to the (49) in the large $N$ expansion with the replica method,

$$\eta' = \frac{d-4}{N}$$

is obtained. This result satisfies another inequality (10).

The connected four-point correlation function satisfies the inequality for $k = 1, l = 3$ and $f_{1}(\phi) = \tilde{\phi}_m^q$ given by theorem.

$$\sum_{p_1 \in \Lambda^*_L} \sum_{p_2 \in \Lambda^*_L} \sum_{p_3 \in \Lambda^*_L} \sum_{p_4 \in \Lambda^*_L} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \sum_{n_3=1}^{N} \sum_{n_4=1}^{N} \sum_{m_1}^{N} \sum_{m_2}^{N} \sum_{m_3}^{N} \sum_{m_4}^{N} \mathbb{E}|\tilde{\phi}_{p_1}^{n_1}; \tilde{\phi}_{p_2}^{n_2}; \tilde{\phi}_{p_3}^{n_3}; \tilde{\phi}_{p_4}^{n_4}|^2 \leq 3! \beta^{-6} h^{-6} \mathbb{E}|\tilde{\phi}_m^q|^2. \quad (59)$$
According to (52), the left hand side can be calculated in the large $N$ expansion
\begin{equation}
\sum_{p_1 \in \Lambda^*_L} \sum_{p_2 \in \Lambda^*_L} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \left| \mathbb{E} \langle \tilde{\phi}^{m}_{p_1}; \tilde{\phi}^{n}_{p_2}; \tilde{\phi}^{m}_{q}; \tilde{\phi}^{n}_{-q} \rangle \right| \simeq \frac{4 - d}{4 \beta^4 h^2 N} \log q. \tag{60}
\end{equation}

Since the right hand side in the large $N$ expansion is
\[3! \beta^{-6} h^{-6} \mathbb{E} \langle \tilde{\phi}^{m}_{q} \rangle \simeq \frac{3!}{\beta^4 h^2 q^4} \left( 1 + \frac{d - 4}{N} \log q \right),\]
these satisfy the inequality (59).

The wave number dependent susceptibility can be represented in terms of correlation function
\[\chi^{mn}(q, g) := \langle \tilde{\phi}^{m}_{q}; \tilde{\phi}^{n}_{-q} \rangle. \tag{61}\]

In the large $N$ expansion, the variance of the susceptibility and a correlation function are obtained
\begin{align}
\text{Var} \chi^{mn}(q, g) &= \mathbb{E} \langle \tilde{\phi}^{m}_{q}; \tilde{\phi}^{n}_{-q} \rangle^2 - \left| \mathbb{E} \langle \tilde{\phi}^{m}_{q}; \tilde{\phi}^{n}_{-q} \rangle \right|^2 \simeq \frac{\delta_{mn}}{q^2} (\eta' - 2 \eta) \log q, \tag{62}
\end{align}
\begin{align}
\frac{\beta^4 h^4}{2} \sum_{p_1 \in \Lambda^*_L} \sum_{p_2 \in \Lambda^*_L} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \left| \mathbb{E} \langle \tilde{\phi}^{n}_{p_1}; \tilde{\phi}^{m}_{p_2}; \tilde{\phi}^{m}_{q}; \tilde{\phi}^{n}_{-q} \rangle \right|^2 \simeq \frac{4 - d \delta_{mn}}{8N} q^4 \log q. \tag{63}
\end{align}

The following inequalities are obtained by $j = 0, k = 2, l = 1$ $f_1 = \tilde{\phi}^{m}_{q}$, $f_2 = \tilde{\phi}^{n}_{-q}$ in theorem and lemma 2. These give variance inequalities for the susceptibility
\[\frac{\beta^4 h^4}{2} \sum_{p_1 \in \Lambda^*_L} \sum_{p_2 \in \Lambda^*_L} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \left| \mathbb{E} \langle \tilde{\phi}^{n}_{p_1}; \tilde{\phi}^{m}_{p_2}; \tilde{\phi}^{m}_{q}; \tilde{\phi}^{n}_{-q} \rangle \right|^2 \leq \text{Var} \chi^{mn}(q, g) \leq \mathbb{E} \langle \tilde{\phi}^{m}_{q} \rangle^2. \tag{64}\]

These results calculated in the large $N$ expansion with the replica method agree with these inequalities.

5. Summary

A new series of inequalities for correlation functions in random field systems has been obtained systematically in the square interpolation which is a mathematically rigorous method. The first inequality is the Schwartz–Soffer inequality which gives the relation between connected and disconnected two-point functions. This is well-known as a useful inequality to check critical exponents of two-point correlation functions calculated in effective theories and numerical studies [1]. Other inequalities give new relations among multiple-point correlation functions. These relations enable us to examine several critical exponents calculated in large $N$ expansion with the replica method [4]. All obtained results satisfy these inequalities.

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