Interacting Electrons on a Fluctuating String

Dror Orgad
Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel
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We consider the problem of interacting electrons constrained to move on a fluctuating one-dimensional string. An effective low-energy theory for the electrons is derived by integrating out the string degrees of freedom to lowest order in the inverse of the string tension and mass density, which are assumed to be large. We obtain expressions for the tunneling density of states, the spectral function and the optical conductivity of the system. Possible connections with the phenomenology of the cuprate high temperature superconductors are discussed.

I. INTRODUCTION

The problem of interacting electrons moving in one dimension has attracted considerable attention since the early theoretical formulations of Tomonaga1 and Luttinger2. The motivation for the continuous activity in this field has been twofold. First, this system offers a concrete realization of various non-Fermi liquid phenomena and is amenable to controlled theoretical treatments. As such it constitutes a unique theoretical laboratory for studying strong correlations. Secondly, in addition to the existence of various quasi-one-dimensional compounds such as organic Bachgaard salts3 and inorganic purple bronzes4, there is a growing set of higher dimensional systems whose experimentally observed behavior appears to be, nevertheless, quasi-one-dimensional. Among them one finds the high temperature superconductors5-9, the manganites10 and quantum Hall systems11-14. It is possible that the electronic structure of these materials is actually quasi-one-dimensional on a local scale.

The overwhelming majority of the studies of one-dimensional systems that have been carried out so far have assumed that the electrons move along static straight chains. A notable exception is the study of the smectic phase in quantum Hall samples14-18. This theory presupposes the existence of a unidirectional charge density wave in the sample and focuses on the quantum mechanical description of the fluctuating chiral edge channels that develop on the boundaries between regions of different filling fraction. Each channel is characterized by a displacement field $\mathbf{Y}$, representing the transverse displacement of the edge from its classical ground state position (taken to be along the $x$ axis), and a Luttinger field $\phi$, describing the phase of the electronic charge density modulation along the edge. In this problem the two are coupled through the term $\mathbf{J} \cdot \mathbf{A}$, which in the gauge $\mathbf{A} = By\mathbf{x}$, and using the fact that the edge current is $e\partial_t \phi$ translates into $(eB)Y\partial_x \phi$. As a result $Y$ and $\phi$ are canonically conjugated and thus do not constitute independent degrees of freedom. This is a fundamental difference between the quantum Hall edge and the problem of spinful interacting electrons on a fluctuating string which this paper explores.

While the model of interacting electrons on a dynamical one-dimensional geometry possesses an independent theoretical appeal, we are particularly motivated by its possible relevancy to the physics of doped Mott insulators19. There is evidence that over a wide range of the phase diagram of these systems the doped holes spontaneously segregate into one-dimensional charged stripes which also form domain walls across which the background spin texture undergoes a $\pi$-phase shift.

In some instances the observed stripe order is static, at least on the time scale of the experimental probe. Elastic neutron diffraction measurements revealed static spin and charge stripe order in the quasi-two-dimensional insulating nickelate La$_2$NiO$_4$125 and in the isostructural system La$_{2-x}$Sr$_x$NiO$_4$20,21. Similar results were obtained22 for the non-superconducting relative La$_{1.4-x}$Nd$_{0.6}$Sr$_x$CuO$_4$ of the high-temperature superconductors. The charge order has since been confirmed by high-energy x-ray diffraction23.

Systems that exhibit static stripe order are typically insulators or bad conductors and stripe fluctuations seem to play an important role in turning them into superconductors. In the superconducting material La$_{1.6-x}$Nd$_{0.4}$Sr$_x$CuO$_4$ the magnetic ordering temperature, $T_m$, and the superconducting transition temperature, $T_c$, are anti-correlated24. For example, around $x = 1/8$ where stripes are particularly stable, $T_m$ reaches a maximum while $T_c$ exhibits a substantial dip. The same effect has been observed in La$_{0.875}$Ba$_{0.125}$Sr$_x$CuO$_4$25. Static magnetic stripe signatures exist in the insulating spin-glass phase of the Nd-free La$_{2-x}$Sr$_x$CuO$_4$ with $0.02 \leq x \leq 0.0526$ and also in the underdoped superconducting region $0.05 < x \leq 0.12$27,28. However, near optimal doping and in the overdoped regime $(0.12 < x \leq 0.25)$, where quantum fluctuations are presumably larger, static ordering is averted and only dynamical stripe correlations have been found by inelastic neutron scattering experiments29,30. Evidence for slowly fluctuating spin and charge order has also been detected in underdoped superconducting YBa$_2$Cu$_3$O$_{7-\delta}$31-34.

Qualitative arguments and several estimates of the impact of stripes fluctuations on the inter-stripe couplings and the resulting phase diagram of the many-stripe system have been presented in Ref. 35. While the present paper does not deal with questions concerning the physics
that emerges from such couplings its purpose is to establish a quantum description of the single fluctuating one-dimensional electron gas as a basis for tackling these issues at a later stage.

In Sec. II we introduce a model of interacting electrons which are constrained to move on a fluctuating elastic string. We show how this constraint leads to a change in the metric and to the coupling of the electronic dynamics to effective scalar and vector potentials, which are determined by the string fluctuations. The model is then quantized and bosonized in Sec. III.

The electronic correlation functions of the resulting low-energy theory are calculated in Sec. IV by integrating out the string degrees of freedom under the assumption that it is stiff and massive. This assumption enables us to generate a perturbative expansion in the inverse string tension and mass density.

As long as one is concerned with correlations of order parameters which are projected on the line defined by the string equilibrium configuration the results are of the usual Luttinger type. The effects of fluctuations enter through the renormalization of the exponents of the power-laws that describe such correlations. If the number of particles is held fixed this renormalization is mainly due to the increase of the average length of the string by its fluctuations. Consequently the relative strength of the interactions between particles is increased compared to their kinetic energy. If on the other hand a constant particle density along the string is maintained this effect is absent and the renormalization is dominated by the attractive interaction induced by the exchange of elastic string waves.

More generally, however, the correlation functions depend on the position in the embedding plane, or equivalently, in Fourier space, on both the wave-vector component along the string axis, $k_x$, and its component in the perpendicular direction $k_y$. An example is the single-hole spectral function. Under certain conditions we find that it retains its Luttinger liquid structure in terms of $k_x$ and the frequency $\omega$ but with interaction parameters which are increasing functions of the momentum component $k_y$ along the one-dimensional Fermi surface. This effect also leads to a weak logarithmic suppression of the tunneling density of states in addition to the renormalization of its power-law exponent as discussed above.

When the string fluctuates electrical current may flow in the $y$ direction either because electrons are being dragged by the string or because charge flows along segments of the string which are at an angle to the $x$ axis. The first process gives rise to a Drude-peak in the $y$ component of the optical conductivity whose weight is smaller relative to the corresponding peak in $\sigma_{xx}$, by the ratio of the electronic and string mass densities. The second process results in a contribution which is linear in $\omega$ and whose oscillator strength is further reduced in comparison to the Drude peak. However, this contribution survives even if the string is pinned and can not execute rigid translations, while the peak does not.

We end Section IV with a brief consideration of the effects of placing the string inside a harmonic confining potential. As expected the potential predominantly sets a cross-over scale below which correlations are the same as of a regular Luttinger liquid.

Finally, in Sec. V, we discuss few possible applications of our findings to measurements of the cuprate high temperature superconductors.

II. THE MODEL

We consider electrons that are constrained to move on a stretchable one-dimensional fluctuating string, embedded in a two-dimensional plane. We assume that the projection of the string on the $x$ axis is of fixed length, $L_x$, and that it obeys periodic boundary conditions along this direction. We consider the limit of a stiff and massive string whose characteristic quantum and thermal fluctuations are smooth on the length-scale of any short distance cutoff associated with its dynamics, for example the lattice spacing, $a$, in the case of stripes in doped Mott insulators. This assumption allows us to work in the continuum limit and to ignore overhangs. Consequently we describe the string using a function, $Y(x,t)$, with $0 \leq x \leq L_x$, which represents its transverse oscillations relative to the string classical equilibrium configuration, which we take to be along the $x$ axis.

The string is characterized by two parameters: the linear mass density, $\rho$, and the tension, $\sigma$. In the case where the string models a stripe, $\rho$ is determined by the kinetic energy per unit length of the domain wall while $\sigma$ is given by the energy to create a unit length of the domain wall inside the Mott insulator. The string Lagrangian is therefore

$$L_S = \int_0^{L_x} dx \left[ \frac{\rho}{2} \left( \frac{\partial Y}{\partial t} \right)^2 - \sigma \sqrt{1 + \left( \frac{\partial Y}{\partial x} \right)^2} \right]$$

$$\approx \int_0^{L_x} dx \sigma \left[ \frac{1}{u^2} \left( \frac{\partial Y}{\partial t} \right)^2 - \left( \frac{\partial Y}{\partial x} \right)^2 \right],$$

where

$$u = \sqrt{\frac{\sigma}{\rho}},$$

is the sound velocity on the string. The linearized approximation of the action is valid as long as the short distance cutoff for the string fluctuations, which we denote by, $a$, is large compared to the length scale $(\sigma \rho)^{-1/4}$ set by its parameters (here and throughout the paper we take $h = 1$.) Assuming that this condition holds we will use it below to carry out a perturbative calculation of the effective electronic action in the small dimensionless parameter.
\[ \epsilon = \left( \frac{\partial Y}{\partial x} \right)^2 = \frac{1}{2\pi a^2 \sqrt{\rho}}. \]  

This can be interpreted as the average (squared) slope of the string relative to its equilibrium configuration or as the average fluctuations-induced local dilatation of the string, since \( (\partial Y/\partial x)^2 = (ds)^2/(dx)^2 - 1 \), where \( ds \) is the length element along the string.

Next we consider the \( N_e \) electrons that move on the string. The position of the \( i \)-th electron is given by its Cartesian coordinates in the plane

\[ r_i(t) = [x_i(t), y_i(t)] = [x_i(t), Y(x_i(t), t)] , \]

where the last equality expresses the constraint that confines the particles to the string. Its velocity is given by

\[ \dot{v}_i = \frac{dx_i}{dt} = \left[ \frac{dx_i}{dt}, \frac{\partial Y}{\partial x} \right] \left. dx_i \over \partial Y \right|_{x=x_i} dt + \frac{\partial Y}{\partial \ell} \right|_{x=x_i} dt , \]

which demonstrates that the electrons change their position in the plane either by moving along the string or by being dragged by it in the \( y \)-direction.

Including the coupling of the particles to an external electromagnetic field, with scalar potential \( A_0 \) and vector potential \( \vec{A} \), the electronic Lagrangian is given by

\[
L_e = \sum_{i=1}^{N_e} \left[ \frac{1}{2} m v_i^2 + e A_0(r_i, t) - \frac{e}{c} \dot{v}_i \cdot \vec{A}(r_i, t) \right] - \frac{1}{2} \sum_{i \neq j=1}^{N_e} V(|r_i - r_j|) ,
\]

where \( m \) and \( -e < 0 \) are the electronic mass and charge, respectively, and \( V \) is the pair interaction, which in the following we take to be short ranged. Using relations (4) and (5) this Lagrangian can also be written as

\[
L_e = \sum_{i=1}^{N_e} g(x_i, t) \left( \frac{dx_i}{dt} \right)^2 \left[ e A_0(x_i, t) - \frac{e}{c} \frac{dx_i}{dt} A_1(x_i, t) \right] - \frac{1}{2} \sum_{i \neq j=1}^{N_e} V(x_i, x_j) ,
\]

where

\[
g(x, t) = 1 + \left( \frac{\partial Y}{\partial x} \right)^2 ,
\]

is the metric induced by the string in the coordinate system of the \( x_i \)s. The string fluctuations also participate in determining the effective gauge potentials for the \( x_i \) degrees of freedom

\[
A_0(x, t) = \frac{m}{2e} \left( \frac{\partial Y}{\partial t} \right)^2 + A_0[x, Y(x, t), t] - \frac{1}{c} \frac{\partial Y}{\partial t} A_y[x, Y(x, t), t] ,
\]

\[
A_1(x, t) = -\frac{mc}{e} \frac{\partial Y}{\partial \ell} \partial Y \left. \frac{dx_i}{\partial Y} \right|_{x=x_i} dt + A_x[x, Y(x, t), t] + \partial Y \left. A_y[x, Y(x, t), t] \right. .
\]

Finally the pair interaction is

\[
V(x, x') = V \left( \sqrt{(x - x')^2 + (Y(x, t) - Y(x', t))^2} \right) .
\]

Equations (7)-(10) establish a description of the electronic physics in Cartesian coordinates. Although simple in appearance this formulation poses difficulties when one attempts to quantize the model. The source of the problem is the space-time dependence of the metric that appears in the kinetic term of the Lagrangian. Upon quantization it leads to operator ordering ambiguities. In order to circumvent this problem we parameterize the positions of the particles on the string using the arc-length variable

\[
\ell(x, t) = \int_0^x dx' \sqrt{g(x', t)} .
\]

Denoting by \( \ell_i = \ell(x_i, t) \) the position of the \( i \)-th particle along the string we find

\[
\frac{dx_i}{dt} = \frac{1}{\sqrt{g(x_i, t)}} \left[ \frac{d\ell_i}{dt} - \frac{\partial \ell}{\partial t}(x_i, t) \right] ,
\]

where

\[
\frac{\partial \ell}{\partial t}(x, t) = \int_0^x dx' \frac{1}{2 \sqrt{g(x', t)}} \frac{\partial g(x', t)}{\partial t} .
\]

In terms of the new coordinates the electronic Lagrangian appears as

\[
L_e = \sum_{i=1}^{N_e} \left[ \frac{m}{2} \left( \frac{d\ell_i}{dt} \right)^2 + e \tilde{A}_0(\ell_i, t) - \frac{e}{c} \frac{d\ell_i}{dt} \tilde{A}_1(\ell_i, t) \right] - \frac{1}{2} \sum_{i \neq j=1}^{N_e} V(x(\ell_i), x(\ell_j)) ,
\]

and the gauge potentials are

\[
\tilde{A}_0(\ell, t) = A_0[x(\ell), t] + \frac{m}{2e} \left( \frac{\partial \ell}{\partial t}[x(\ell), t] \right)^2 + \frac{1}{c} \frac{1}{\sqrt{g[x(\ell), t]}} \frac{\partial \ell}{\partial t}[x(\ell), t] A_1[x(\ell), t] ,
\]

\[
\tilde{A}_1(\ell, t) = \frac{mc}{e} \frac{\partial \ell}{\partial t}[x(\ell), t] + \frac{1}{\sqrt{g[x(\ell), t]}} A_1[x(\ell), t] .
\]
In this form the problem reduces to a system of particles in flat 1+1 space-time (whose spatial extent is time dependent) which interact with each other via the pair potential $V$ and couple to the gauge potentials $\mathbf{A}_\mu$, which are functions of the string configuration $Y[x(\ell),t]$. The price we pay for the simple form of the kinetic term is the fact that these gauge potentials are non-local since they are functions of $\partial \ell / \partial t$ defined in Eq. (13).

As a last step towards the quantization of the problem we identify the electronic Hamiltonian in the arc-length coordinate. The conjugate momentum to $\ell_i$ is $p_i = \partial L_e / \partial (d\ell_i / dt) = m(d\ell_i / dt) - (e/c)A_1(\ell_i, t)$ and the classical electronic Hamiltonian is

$$H_e = \sum_{i=1}^{N_e} \left[ \frac{1}{2m} \left( p_i + \frac{e}{c} A_1(\ell_i, t) \right)^2 - e A_0(\ell_i, t) \right] + \frac{1}{2} \sum_{i \neq j=1}^{N_e} V[x(\ell_i), x(\ell_j)] .$$

(16)

III. QUANTIZATION AND BOSONIZATION

Quantizing the model defined by Eq. (16) is a straightforward task, analogous to the quantization of a conventional electron gas. The resulting second quantized Hamiltonian for the many body-system, written in terms of the electronic fields $\tilde{\psi}_\sigma(\ell, t)$ for the two possible spin polarizations, is

$$H_e = \int_0^{L(t)} d\ell \sum_{\sigma = \pm} \left[ \tilde{\psi}_\sigma \frac{1}{2m} \left( -i \partial_\ell + \frac{e}{c} A_1 \right)^2 \tilde{\psi}_\sigma - e A_0 \tilde{\psi}_\sigma^\dagger \tilde{\psi}_\sigma \right] + \frac{1}{2} \int_0^{L(t)} d\ell d\ell' \sum_{\sigma, \sigma' = \pm} V_{\ell, \ell'}^{\sigma, \sigma'} \tilde{\psi}_\sigma(\ell) \tilde{\psi}_{\sigma'}(\ell') \tilde{\psi}_{\sigma}(\ell') \tilde{\psi}_{\sigma'}(\ell) ,$$

(17)

where $L(t) = \int_0^{L(t)} dx \sqrt{g(x,t)}$ is the instantaneous length of the string and where the spatial part of $V_{\ell, \ell'}^{\sigma, \sigma'} = V_{\sigma, \sigma'} V[x(\ell), x(\ell')]$ takes the form Eq. (10). At this stage we are still considering the string degrees of freedom as a classical field.

Despite its apparent simplicity the formulation of the problem in the arc-length parameterization, Eq. (17), suffers from two significant shortcomings. The first is the non-local nature of the gauge potentials $\mathbf{A}_\mu$. The second problem stems from the fact that experiments measure the electronic correlation functions as functions of the coordinates in the embedding plane and not of the arc-length along the string. It is therefore desirable to rewrite the theory in terms of the projected coordinate $x$. It turns out that by doing so one also remedy the first difficulty.

To this end we note that the spatial and temporal derivatives transform as

$$\frac{\partial}{\partial \ell} \bigg|_t = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \bigg|_t ,$$

$$\frac{\partial}{\partial t} \bigg|_\ell = \frac{\partial}{\partial t} \bigg|_x - \frac{1}{\sqrt{g}} \frac{\partial}{\partial x} \bigg|_x \frac{\partial}{\partial t} ,$$

(18) (19)

and the integration with its measure according to

$$\int_0^{L(t)} d\ell \to \int_0^{L(x)} dx \sqrt{g} .$$

(20)

Since the fermionic fields in the arc-length parameterization obey the anti-commutation relations

$$\left\{ \tilde{\psi}_\sigma(\ell, t), \tilde{\psi}_{\sigma'}^\dagger(\ell', t) \right\} = \delta(\ell - \ell') \delta_{\sigma, \sigma'} ,$$

$$= \frac{1}{\sqrt{g(x,t)}} \delta(x - x') \delta_{\sigma, \sigma'} ,$$

(21)

we rescale the transformed fermionic fields according to

$$\psi_\sigma(x, t) = [g(x, t)]^{1/4} \tilde{\psi}_\sigma(\ell(x, t), t) ,$$

(22)

such that the new fields obey the canonical anti-commutation relations. In this way we also have $\rho_\sigma(x, t) = \psi_\sigma^\dagger(x, t) \psi_\sigma(x, t)$ as the projected electronic densities in the length element $dx$.

The result for the transformed electronic Lagrangian is then given by

$$L_e = \int_0^{L(x)} dx \sum_\sigma \left[ i \psi_\sigma^\dagger \partial_\ell \psi_\sigma - \frac{1}{2m} \psi_\sigma^\dagger g^{-1/2} \left( -i \partial_x + \frac{e}{c} A_1 \right) g^{-1/2} \left( -i \partial_\ell + \frac{e}{c} A_1 \right) g^{1/2} \psi_\sigma + e A_0 \rho_\sigma - \frac{1}{2} g^{-1/2} \sum_{\sigma'} V_{\sigma, \sigma'} \rho_\sigma \rho_{\sigma'} \right] ,$$

(23)

We are interested in obtaining a low-energy effective description of the system under the assumption that our stiff string fluctuates over typical length and time scales which are long compared to the inverse of the Fermi wave-
vector and Fermi energy of the one-dimensional electron gas. In order to separate the fast and slow degrees of freedom we decompose the fermionic fields into left ($\eta = -1$) and right ($\eta = 1$) moving fields, $\psi_{\eta, \sigma}$, which describe the slow long-wavelength variations around the respective Fermi points

$$\psi_{\eta}(x, t) = e^{-i k_F x} \psi_{-1, \sigma}(x, t) + e^{i k_F x} \psi_{1, \sigma}(x, t).$$

(24)

Here we have also used the assumption that the number of electrons on the string, $N_e$, is fixed and hence the average projected electronic density, $\bar{n} = N_e/L_x$, and with it the projected Fermi wave-vector, $k_F = \pi \bar{n}/2$, are time independent and equal their values in the straight static string\(^3\).

The low-energy theory is obtained by using the decomposition, Eq. (24), keeping leading order terms in $k_F$, which is assumed to be large, and neglecting irrelevant terms in the renormalization group sense. We also assume that the $2k_F$ components of the gauge potentials and the metric vanish thus allowing us to disregard single-particle backscattering.

The theory can then be bosonized in the standard way\(^3\) with attention to the metric factors appearing in the kinetic term of Eq. (23). The fermionic and bosonic representations are related via the identity

$$\psi_{\eta, \sigma}(x, t) = \frac{1}{\sqrt{2\pi \alpha_e}} F_{\eta, \sigma} \exp[-i\Phi_{\eta, \sigma}(x, t)],$$

(25)

where the self-dual fields $\Phi_{\eta, \sigma}$ are themselves combinations of the bosonic fields $\phi_c$ and $\phi_s$ and their conjugated momenta $\partial_x \phi_c$ and $\partial_x \phi_s$

$$\Phi_{\eta, \sigma} = \frac{\pi}{2} [\theta_c - \eta \phi_c] + \sigma (\theta_s - \eta \phi_s).$$

(26)

The Klein factors $F_{\eta, \sigma}$ in Eq. (25) are responsible for reproducing the correct anti-commutation relations between different fermionic species and $\alpha_e \sim k_F^{-1}$ is a short distance cutoff.

The result of these manipulations is the bosonic representation of the effective electronic Lagrangian

$$L_c = \int_0^{L_c} dx \left\{ \sum_{\alpha = c, s} \left[ \partial_t \phi_{\alpha} \partial_x \phi_{\alpha} - \frac{\bar{\nu}_c K_{\alpha}}{2} (\partial_x \phi_{\alpha})^2 - \frac{\bar{\nu}_s}{2 K_{\alpha}} (\partial_x \phi_{\alpha})^2 \right] + \left[ e A_0 - g^{-1} \bar{E}_F - \frac{e^2}{2mc^2} g^{-1} A_1^2 - \frac{1}{32m} g^{-3} (\partial_x g)^2 \right] \left( \bar{n} + \sqrt{\frac{2}{\pi}} \partial_x \phi_c \right) + \bar{\nu}_F e \sqrt{\frac{2}{\pi}} g^{-1} A_1 \partial_x \phi_c \right\},$$

(27)

where $\bar{E}_F = k_F^2/(2m) = m \bar{v}_F^2/2$ is the Fermi energy of the non-interacting electron gas on the static system. The $A_1^2$ term was kept since its diamagnetic contribution to the conductivity is essential if one is to obtain the correct response when the electric field is introduced via a time-dependent potential, as we do later on. It also renormalizes the string dependent part of the scalar potential.

The first two terms in Eq. (27) involving the average projected electronic density, $\bar{n}$, are responsible for the renormalization of the bare string parameters. This becomes evident if we expand them to second order in derivatives of $Y$ in the absence of external electromagnetic fields. The first term $(e A_0 \bar{n})$ is identically $m \bar{v}_0 \int_0^{L_c} dx (\partial_x Y)^2/2$. It expresses the fact that due to their constrained dynamics the electrons are dragged with the string as it moves, thereby increasing its effective mass density

$$\rho \rightarrow \rho + m \bar{n}.$$ 

(28)

The second term reads $-\bar{E}_F \bar{n} + 2 \bar{E}_F \bar{n} \int_0^{L_c} dx (\partial_x Y)^2/2$. It manifestly tells us that the electronic kinetic energy is lowered by the fluctuations. This gain in kinetic energy favors a more flexible string through the renormalization of its tension according to

$$F_{\eta, \sigma} = \frac{1}{2\pi} \sqrt{2(2\pi \bar{v}_F g^{-\frac{1}{2}} + 2V_4)^2 - (2V_2 - V_1||)^2},$$

$$\bar{v}_c(x, t) = \frac{g^{-\frac{1}{2}}}{2\pi} \sqrt{(2\pi \bar{v}_F g^{-\frac{1}{2}})^2 - V_1^2||},$$

(30)

and

$$\bar{v}_s(x, t) = \frac{g^{-\frac{1}{2}}}{2\pi} \sqrt{(2\pi \bar{v}_F g^{-\frac{1}{2}})^2 - V_1^2||},$$

(30)
\[
\bar{K}_c(x,t) = \frac{2\pi v_F g - \frac{1}{2} + 2V_4 - 2V_2 + V_{1\parallel}}{2\pi \bar{v}_F g - \frac{1}{2} + 2V_4 + 2V_2 - V_{1\parallel}},
\]
\[
\bar{K}_s(x,t) = \frac{2\pi \bar{v}_F g - \frac{1}{2} + V_{1\parallel}}{2\pi \bar{v}_F g - \frac{1}{2} - V_{1\parallel}}.
\]

Note, however, that the model still exhibits spin-charge separation.

At this point we quantize the string. The effective low energy theory is then described by a path integral over the string and electronic configurations with respect to the combined action \( S_s + S_e = \int dt (L_s + L_e) \), defined by Eqs. (1) and (27).

**IV. CORRELATION FUNCTIONS**

A. Effective Action for Projected Correlation Functions

As long as one is interested in calculating correlation functions of electronic order parameters which can be expressed solely in terms of the \( \psi_{\alpha,s}(x) \), such as the projected singlet pair annihilation operator \( O_{SS}(x) = \sum_\sigma \sigma \psi_{1,\sigma}(x) \psi_{-1,-\sigma}(x) \), one may integrate out the string from the problem. Our assumption regarding the smallness of the string fluctuations allows us to carry out this procedure perturbatively in the small dimensionless parameter, \( \epsilon \), defined in Eq. (3). We begin by doing so in the absence of external electromagnetic fields.

Using a cumulant expansion for the electronic action we obtain a power series in derivatives of \( Y(x,t) \), which can be readily averaged over the string dynamics, as described by the Lagrangian, Eq. (1). To first order in \( \epsilon \), and for periodic boundary conditions along the \( x \) direction, the resulting effective electronic Lagrangian is of the ordinary Luttinger type. In imaginary time it reads

\[
L_{\text{eff}}^c = \int_0^{L_x} dx \sum_{\alpha = c,s} \left[ -i \partial_x \phi_\alpha \partial_x \theta_\alpha + \frac{\tilde{v}_\alpha K_\alpha}{2} (\partial_x \theta_\alpha)^2 + \frac{v_\alpha}{2 K_\alpha} (\partial_x \phi_\alpha)^2 \right],
\]

where the velocities are renormalized away from their values \( \tilde{v}_\alpha \) in the straight static string of length \( L_x \) according to

\[
v_\alpha = \tilde{v}_\alpha - \epsilon \left[ \frac{\tilde{v}_\alpha}{2} + \frac{\bar{v}_F}{4} \left( \frac{1}{K_\alpha} + \bar{K}_\alpha \right) \right],
\]

and the Luttinger parameters acquire the values

\[
K_\alpha = \bar{K}_\alpha - \epsilon \left[ \frac{\bar{v}_F}{4} \left( 1 - \bar{K}_\alpha^2 \right) \right].
\]

Following our notation, the barred quantities characterize the system with no fluctuations and \( \alpha = c, s \).

The Luttinger parameters constitute a measure of the relative size of the kinetic and potential energies of the one-dimensional electron gas. \( K_c \) progressively deviates from 1 as the strength of the interactions in the system is increased compared to its Fermi velocity. The deviation is negative for repulsive interactions and positive in the presence of attraction between the electrons. \( K_s \) behaves in a similar manner but with an opposite sign for the deviation. As seen from Eq. (34) the string fluctuations have the effect of increasing the relative size of the interactions in the system. The origin of this effect can be easily traced back to the different dependence of the kinetic and potential energies of the gas on the length, \( L_s \), of the string. While the kinetic energy scale as \( L^{-2} \) the potential energy behaves as \( L^{-1} \). Since the fluctuations increase the average length of string \( \langle L \rangle = \left( 1 + \frac{1}{2} \epsilon - \frac{1}{2} \epsilon^2 + \cdots \right) L_x \) the above mentioned changes in \( K_c \) and \( K_s \) follow.

It is interesting to understand the effects of fluctuations beyond the simple elongation of the string which leads to Eqs. (33) and (34). To this end we express \( v_\alpha \) and \( K_\alpha \) in terms of the corresponding quantities \( \bar{v}_\alpha \) and \( \bar{K}_\alpha \) of a straight static string of length \( \langle L \rangle \) - the average length of the fluctuating string, and with the same number of particles \( N_c \). This is also useful if one wishes to study the case where the average density along the string is held fixed rather than the projected density. To lowest order in \( \epsilon \) we find

\[
v_\alpha = \bar{v}_\alpha \left( 1 - \frac{\epsilon}{2} \right),
\]

where the proportionality factor is still geometrical in nature and stems from the fact that \( v_\alpha \) expresses the \( x \)-component of the velocities on the fluctuating string.

As expected the first order difference between \( K_\alpha \) and \( \bar{K}_\alpha \) vanishes and one needs to obtain the effective action to second order in \( \epsilon \). Ignoring non-local terms that are generated in the process which are irrelevant in the renormalization group sense we find that the effective action retains the same form as Eq. (32) with

\[
K_\alpha = \bar{K}_\alpha + \epsilon^2 \left[ \frac{1}{2} \bar{v}_F \left( 1 - K_\alpha^2 \right) \right]
\]

\[+ \frac{1}{2} \frac{v_\alpha}{\bar{v}_\alpha} \left( K_\alpha^2 \right)^2 \right] \delta_{\alpha,c} \right).
\]

The first term in the curly brackets is similar to the first order term in Eq. (34) with the exception that it has the effect of decreasing the relative strength of the interactions. It too can be removed by an appropriate choice of the length of the straight static reference system. More importantly, the charge Luttinger parameter acquires an additional contribution which tends to increase its value. It originates from the induced attractive interaction due to the exchange of string waves between particles.
Since the effective action is that of an ordinary interacting one-dimensional electron liquid the projected correlation functions take, at zero temperature, the familiar power-law form, with exponents which are determined by the effective parameters, Eqs. (34,36). In some cases, however, one is interested in calculating the correlation functions as functions of both coordinates in the embedding plane, not least since these are the quantities measured in experiments. We therefore consider this issue next.

B. The Single Hole Green Function and the Tunneling Density of States

The single hole Green function in the two-dimensional plane is defined as

$$G^<(\vec{r}_1, \vec{r}_2, t) = \langle \Psi_\sigma^\dagger(\vec{r}_1, t)\Psi_\sigma(\vec{r}_2, 0) \rangle.$$  \hspace{1cm} (37)

The operator $\Psi_\sigma(\vec{r})$ annihilates an electron with spin $\sigma$ at point $\vec{r}$ in a plane of dimensions $L_x \times L_y$ and the average is with respect to the string and electronic actions, $S_\sigma + S_e = \int dt (L_s + L_e)$, Eqs. (1) and (27).

In the following we will consider the case where the string is fixed at a point such that its rigid translations are eliminated. In order to restore translational invariance of the Green function along the $y$-direction we average it over $\bar{y} = (y_1 + y_2)/2$. The result

$$\bar{G}^<(x, y, t) = \frac{1}{L_y} \int d\bar{y} G^<(\vec{r}_1, \vec{r}_2, t)$$  \hspace{1cm} (38)

is a function of the relative coordinates $x = x_1 - x_2$ and $y = y_1 - y_2$ only.

Since the electrons are confined to the string, it must pass through the space-time point where the electron is to be created or annihilated in order for such a process to be possible. Therefore

$$\Psi_\sigma(\vec{r}, t) = \psi_\sigma(x, t)\sqrt{δ}[y - Y(x, t)].$$  \hspace{1cm} (39)

Applying the decomposition, Eq. (24), and expressing the $δ$-function constraints through their integral representation the Green function of a right moving hole (the spin polarization is irrelevant here) can be written as

$$\bar{G}^<(x, y, t) = \frac{a}{L_y} e^{-ikF x} \int d\bar{y} \int_{-∞}^{∞} dλ_1 \frac{dλ_2}{2π} \frac{dλ}{2π}$$  \hspace{1cm} (40)

$$× \langle e^{iλ_1[Y(x, t) - y_1]} \psi^{\dagger}_{11}(x, t) e^{iλ_2[Y(0, 0) - y_2]} \psi_{11}(0, 0) \rangle$$

$$= \frac{a}{L_y} e^{-ikF x} \int_{-∞}^{∞} dλ \frac{dλ}{2π}$$  \hspace{1cm} (41)

$$× \langle e^{-iλy} e^{iλY(x, t)} \psi^{\dagger}_{11}(x, t) e^{-iλY(0, 0)} \psi_{11}(0, 0) \rangle.$$  \hspace{1cm} (42)

In the last step we transformed to new integration variables $λ = (λ_1 - λ_2)/2$ and $\Lambda = λ_1 + λ_2$ and carried out the $λ$ and $\bar{y}$ integrals.

We will calculate the Green function $G^<$ for finite temperatures from the imaginary-time-ordered correlation function

$$G^<(x, y, τ) = \frac{a}{L_y} e^{-ikF x} \int_{-∞}^{∞} dλ \frac{dλ}{2π}$$  \hspace{1cm} (43)

$$× e^{-iλy} \psi^{\dagger}(x, τ) \psi(0, 0) e^{iλ[Y(x, τ) - Y(0, 0)]} e^{[S_\sigma + S_e]}.$$  \hspace{1cm} (44)

where $Z$ is the partition function. Concentrating on the $Y$ functional integration we have

$$\frac{1}{Z} \int DYe^{iλ[Y(x, τ) - Y(0, 0)]} e^{-[S_\sigma + S_e]}$$  \hspace{1cm} (45)

$$= \frac{1}{Z} \int DYe^{iλ[Y(x, τ) - Y(0, 0)]} e^{-S_\sigma(Y, ψ)} e^{-S_e(Y, ψ)}$$  \hspace{1cm} (46)

with $J(λ, ν) = λ[e^{-i(λx - ντ)} - 1]$, and where the sums run over the Matsubara frequencies $ν_n = \frac{2π}{β} \ln(2πTn)$, $(k_B = 1)$, and the allowed wave-vectors $q_n = \frac{2π}{L_x} n \neq 0$. In the last line of Eq. (42) we have shifted $Y(x, ν) \rightarrow Y(q, ν) + \frac{1}{βL_x} \int dτ e^{iλ[q_τ]} J(λ, ν)$, and calculated the effective fermionic action to first order in $n$. The result, to this order, is identical to the effective action, Eq. (32), and $Z' = \int Dψe^{-\int_0^∞ dτ L_{eff}^"}$.

Carrying out the remaining functional integral over $ψ$ is a simple matter that gives $G_{1d}(x, τ) -$ the finite temperature single hole Green function of a one-dimensional electron gas with velocities $v_ν$, Eq. (33), and Luttinger parameters $K_n$, Eq. (34). Finally, performing the Gaussian integral over $λ$, taking $τ > 0$, and analytically continuing to the real time axis we obtain in the limit $L_x \rightarrow ∞$

$$G^<(x, y, t) = F(x, y, t)G^<_{1d}(x, t)$$  \hspace{1cm} (47)

where

$$F(x, y, t) = \frac{a}{L_y} \frac{\exp \left[ -y^2 / f(x, t) \right]}{\sqrt{πf(x, t)}}$$  \hspace{1cm} (48)

$$f(x, t) = 2ea^2 \left\{ \ln \left[ -i \left( \frac{λT_u}{a} \right) \sinh \left( \frac{x - ut + ia}{λT_u} \right) \right] \right. $$  \hspace{1cm} (49)

$$+ \ln \left[ i \left( \frac{λT_u}{a} \right) \sinh \left( \frac{x + ut - ia}{λT_u} \right) \right] \right\},$$  \hspace{1cm} (50)

and where the one-dimensional Green function is given by
\[ G_{id}(x, t) = \frac{1}{2\pi a_x} e^{-iK_x x} \left( \frac{a_x}{\lambda_{T,c}} \right)^{2\gamma_c+\frac{1}{2}} \left( \frac{a_x}{\lambda_{T,s}} \right)^{2\gamma_s+\frac{1}{2}} \times \prod_{\alpha=c,s} \left[ -i \sinh \left( \frac{x-v_{\alpha} t + ia_x}{\lambda_{T,\alpha}} \right) \right]^{-\gamma_\alpha} \times \left[ i \sinh \left( \frac{x+v_{\alpha} t - ia_x}{\lambda_{T,\alpha}} \right) \right]^{-\gamma_\alpha}. \] (46)

In the above we have introduced the thermal lengths
\[ \lambda_{T,u} = \frac{u}{\pi T}, \quad \lambda_{T,c,s} = \frac{v_{c,s}}{\pi T}, \] (47)
and the exponents \( \gamma_{c,s} \), which are defined as
\[ \gamma_{c,s} = \frac{1}{8}(K_{c,s} + K_{c,s}^{-1} - 2). \] (48)

Note that to first order in \( \epsilon \) these exponents are larger by \( \frac{1}{2} (\gamma_{c,s} - 1)^2 \) than their values on the static string. However, they still obey \( \gamma_{c,s} = 0 \) for non-interacting electrons.

The effects of the string fluctuations, as given by the factor \( F(x, y, t) \), introduce a Gaussian decay in \( y \) to the Green function and reduce its amplitude at long times or large \( x \) separations. In particular it means that in the presence of fluctuations the low energy \( (T < \omega \ll u/a) \) tunneling density of states
\[ \rho^<(\omega) = \int \frac{dt}{2\pi} e^{i\omega t} G^<(0, 0, t) \propto \frac{\omega^{2(\gamma_c+\gamma_s)}}{\sqrt{\ln(a\omega/u)}} \Theta(\omega), \] (49)
is logarithmically suppressed compared to its behavior in the static Luttinger liquid, on top of the renormalization of its power-law exponent. Here and in the following the energy is measured relative to the Fermi energy on the straight string \( E_F \). (41)

### C. The Spectral Function

Angle resolved photoemission spectroscopy (ARPES) measures the single hole spectral function
\[ A^< (\tilde{k}, \omega) = \frac{1}{L_x L_y} \int d^2 r d^2 r' d^2 \omega' e^{i(\tilde{k} - \tilde{r}') - \omega t} G^<(\tilde{r}, \tilde{r}', t). \] (50)
The integrals over the \( y \) coordinates are immediate owing to the \( \delta \)-function factors in \( G^<(\tilde{r}, \tilde{r}', t) \). One therefore finds
\[ A^< (\tilde{k}, \omega) = \frac{a}{L_y} \int dx dt e^{i(k_x - \tilde{k} x - \omega t)} \langle e^{iK_y Y(x, t)} \psi_1^+(x, t) e^{-iK_y Y(0, 0)} \psi_1(0, 0) \rangle. \] (51)
The average that appears in Eq. (51), which we denote by \( A^<(x, t; k_y) \), can be evaluated for finite temperatures using similar manipulations as the ones indicated following Eqs. (41) and (42). The result
\[ A^<(x, t; k_y) = F(x, t; k_y) G^<_{id}(x, t), \] (52)
where
\[ F(x, t; k_y) = \frac{a}{L_y} \left( \frac{a}{\lambda_{T,u}} \right)^{2\Delta(k_y)} \times \left[ -i \sinh \left( \frac{x - ut - ia}{\lambda_{T,u}} \right) \right]^{-\Delta(k_y)} \times \left[ i \sinh \left( \frac{x + ut - ia}{\lambda_{T,u}} \right) \right]^{-\Delta(k_y)}, \] (53)
depends on \( k_y \) through the exponent
\[ \Delta(k_y) = \frac{\epsilon}{2}(ak_y)^2. \] (54)

To find the spectral function \( A^<(\tilde{k}, \omega) \) we still need to perform the Fourier transform over \( A^<(x, t; k_y) \), which can be written as a convolution of the Fourier transform of \( F(x, t; k_y) \) and the one-dimensional spectral function \( A^<_{1d}(k_x, \omega) \)
\[ A^< (\tilde{k}, \omega) = \frac{1}{(2\pi)^2} \int dq d\nu F(q, \nu; k_y) A^<_{1d}(k_x - q, \omega - \nu). \] (55)
The finite-temperature one-dimensional spectral function has been evaluated in Ref. 42. Using a similar technique and in terms of the dimensionless scaling variables
\[ \tilde{k}_x = \frac{uk_x}{\pi T}, \quad \tilde{\omega} = \frac{\omega}{\pi T}, \] (56)
and the Fourier transform of \( \lim_{a \to 0} [-i \sinh(x + ia)]^{-\gamma} \)
\[ h_{\gamma}(k) = \text{Re} \left[ (2i)^\gamma B \left( \frac{\gamma - ik}{2}, 1 - \gamma \right) \right] \] (57)
where \( B(x, y) \) is the beta function, one obtains
\[ F(k_x, \omega; k_y) = \frac{a^3}{2uL_y} \left( \frac{a}{\lambda_{T,u}} \right)^{2\Delta(k_y) - 2} \times h_{\Delta(k_y)} \left( \frac{\tilde{\omega} + \tilde{k}_x}{2} \right) h_{\Delta(k_y)} \left( \frac{\tilde{\omega} - \tilde{k}_x}{2} \right). \] (58)

Evaluating the convolution is difficult in the general case, especially as \( A^<_{1d} \) itself is a convolution of charge and spin pieces (42). However, some features of the spectral function may be deduced from general kinematical considerations without resorting to detailed calculations (8, 43).
It is non-vanishing at \( T \). When the string itself is a dynamic entity this creation of at least one charge excitation and one spin excitation of an electron to the system necessarily involves the separation that takes place in one dimension the additional forces.

\[
|\omega| = v_s k_x
\]

\[
|\omega| = v_c k_x
\]

\[
|\omega| = u k_x
\]

\[
|\omega| = u k_x
\]

\[
|\omega| = u k_x
\]

\[
|\omega| = v_s k_x
\]

\[
|\omega| = v_s k_x
\]

An example is the the shape of the zero temperature support of \( A^<(k, \omega) \), \( i.e. \) the region in \((k, \omega)\) space where it is non-vanishing at \( T = 0 \). Because of the spin-charge separation that takes place in one dimension the addition of an electron to the system necessarily involves the creation of at least one charge excitation and one spin excitation. When the string itself is a dynamic entity this process also involves the emission of string waves. The momentum \( k_x \) and energy \( \omega \) of the added electron are distributed among these excitations

\[
k_x = v_c + k_s + k_u , \quad \omega = v_c |k_c| + v_s |k_s| + u |k_u|
\]  

(59)

It is easy to check that any point in the \((k_x, \omega)\) plane above the dispersion curve of the slowest excitation branch may be reached by appropriately distributing the momentum and energy of the added electron in a manner consistent with the conservation laws, Eq. (59). If the system possesses additional symmetries, such as spin-rotation symmetry that inhibits the decay of a right moving electron into left moving spin excitations, \( i.e. \) enforces \( k_x > 0 \), the region of support may be further constrained. The spectral function of a hole is subject to a similar consideration with results which are presented in Fig. 1. Note that in this figure and in the following \( k_x \) and \( \omega \) are measured relative to \( k_F \) and \( E_F \) respectively.

While the support is governed by kinematics the distribution of the spectral weight inside this region is determined by the matrix elements which connect the single hole state with various multi-excitation states which form the continuum. For weakly interacting electrons much of the weight is concentrated around the non-interacting dispersion line \( |\omega| = v_s k_x \approx v_c k_x \). It progressively spreads throughout the region of support with increasing strength of the electron-electron interactions. It is also smeared due to the excitation of string waves. This smearing is more pronounced for larger values of \( k_y \). It is a result of the fact that the \( y \) dependent parts of the electronic operators in the spectral function, Eq. (51), shift the momentum of the string by \( k_y \), thus increasing the number of string waves accordingly.

Analytical progress in evaluating the spectral function is possible for several special cases:

1. The case \( u = v_c = v_s \)

Denoting \( \lambda_T = \lambda_T u \) and \( \gamma = \gamma_c + \gamma_s \) one finds

\[
A^<(k, \omega) = \frac{1}{4 \pi^2} \frac{1}{T L y} \left( \frac{\omega}{\lambda_T} \right)^2 \left( \frac{\alpha}{\lambda_T} \right)^{2 \Delta(k_y)}
\]

\[
\times \frac{\hbar}{2} \gamma + \Delta(k_y) + 1 \left( \frac{\omega + k_x}{2} \right) \frac{\hbar}{2} \gamma + \Delta(k_y) \left( \frac{\omega - k_x}{2} \right).
\]

(60)

The zero temperature limit of this result is easily obtained using

\[
\frac{\hbar}{\Gamma(\gamma)} \Theta(-k)(-k)^{-1}.
\]

(61)
2. The case \( u = v_c \neq v_s \) and \( \gamma_s = 0 \)

This case is of importance since \( \gamma_s = 0 \) for a spinrotationally invariant system. Denoting by \( r = v_s/v_c \) the ratio between the spin and charge velocities one finds

\[
A^c(\vec{k}, \omega) = \frac{1}{(2\pi)^3 \sqrt{rT}} L_y \left( \frac{a_c}{\lambda T} \right)^{2\gamma_c} \left( \frac{a}{\lambda T^2} \right)^{2\Delta(k_y)}
\times \int_{-\infty}^{\infty} dq h_{1/2}(q) h_{\gamma_c + \Delta(k_y)} \left[ \frac{\omega - \bar{k}_x}{2} - \left( 1 - \frac{1}{r} \right) q \right]
\times h_{\gamma_c + \Delta(k_y)} + \frac{1}{2} \left[ \frac{\omega + \bar{k}_x}{2} - \left( 1 + \frac{1}{r} \right) q \right].
\]

(62)

The results, Eqs. (60) and (62), are similar to the spectral functions of a static one-dimensional electron gas under the same conditions with the exception that the exponents that govern their behavior are now functions of the strength of the string fluctuations and, more importantly, of \( k_y \), as given by the definition of \( \Delta(k_y) \), Eq. (54). We expect this similarity to continue to exist as long as \( u \) is not very different from either \( v_c \) or \( v_s \).

If one relaxes the constraint that confines the electronic wave function to live strictly on the one-dimensional string and allows some leakage of it into the surrounding environment the above spectral functions are modified. In the simple case where one can replace the electronic operator, Eq. (39), by \( \Psi(t) = \psi(x, t) \varphi(y) \), they turn, in real space, into a convolution of the results obtained with the \( \delta \)-function constraint and the wave function in the transverse direction \( \varphi(y) \). Consequently, in Fourier space the spectral functions are multiplied by \( |a|^{-1/2} \int dy e^{-ik_y y} \varphi(y)^2 \).

### D. Optical Conductivity

The effective electronic Lagrangian, Eq. (27), also contains the electromagnetic response of the system. In particular, we are interested in its optical conductivity, i.e., the manner in which it responds to the application of a time-dependent spatially-uniform electric field.

To this end we consider a free string (in this section we allow the string to perform rigid translations) and introduce the electric field via a time dependent vector potential \( \vec{A} \) which enters the electronic Lagrangian, Eq. (27), through the effective vector potential \( A_1 \), Eq. (9). We evaluate the (imaginary time) generating functional \( e^{-S[\vec{A}(\omega)]} \) by integrating out the string and electronic degrees of freedom using a cumulant expansion to first order in \( \epsilon \) and second order in \( \vec{A} \). The optical conductivity is then given by

\[
\sigma_{\mu\nu}(i\omega) = \frac{c^2}{\beta L_y L_y} \omega dA_{\mu}(-i\omega)dA_{\nu}(i\omega).
\]

(63)

By analytically continuing \( i\omega \to \omega + i\delta \) we find in the limit of zero temperature and \( a \to 0 \) (while keeping \( \epsilon \) and \( na \) finite)

\[
\sigma_{xx}(\omega) = \frac{2 \epsilon^2}{\pi L_y} \bar{v}_F (1 - \epsilon) \frac{i}{\omega + i\delta},
\]

(64)

\[
\sigma_{yy}(\omega) = \frac{2 \epsilon^2}{\pi L_y} (\bar{v}_c + \bar{v}_F) \left( \bar{v}_c - \bar{v}_F \right) \frac{i}{\omega + i\delta} + \frac{2 \epsilon^2}{\pi L_y} u \bar{K}_c (u + \bar{v}_c) \frac{i}{\omega + i\delta}.
\]

(65)

In a system with momentum-independent interactions\(^{44}\) \( \bar{v}_F = \bar{v}_c \bar{K}_c \) and as a result

\[
\sigma_{xx}(\omega) = \frac{2 \epsilon^2}{\pi L_y} v_c \bar{K}_c \frac{i}{\omega + i\delta},
\]

(66)

\[
\sigma_{yy}(\omega) = \frac{8 \epsilon^2}{\pi L_y} (\bar{K}a)^2 u \frac{i}{\omega + i\delta}.
\]

(67)

The expression for \( \sigma_{xx} \) is similar to the optical conductivity of a static Luttinger liquid\(^{45}\) but with renormalized parameters owing to the effects of string fluctuations.

The result for \( \sigma_{yy} \) is the conductivity of a classical string charged with immobile charges of linear density \( \bar{n} \). One can simply derive it by solving the classical equation of motion of a charged string in the presence of an electric field: \( \rho d^2Y/dt^2 - \sigma d^2Y/dx^2 + \bar{\bar{n}} e E_y(t) = 0 \) and using the fact that the current density in the y-direction is given by \( J_y(t) = (-e\bar{n}/L_y)(dY/dt) \).

The effects coming from the motion of the charges along the string as it fluctuates vanish in the limit \( a \to 0 \) and \( \epsilon \) fixed. However, in case the string is fixed at a point such that it can not execute rigid translations in the y-direction the result for \( \sigma_{yy} \), Eq. (67), vanishes while the contributions to \( \sigma_{yy} \) coming from the motion of charge along the string remain and dominate. To lowest order in \( \epsilon \) and \( a \) we find them to be (in the case of momentum-independent interactions)

\[
\sigma_{yy}(\omega) = \frac{1}{\pi L_y} \bar{K}_c e^2a \left[ \pi |\omega| - 2i\omega \ln \left( \frac{u + \bar{v}_c}{e^{\gamma a/|\omega|}} \right) \right],
\]

(68)

where here \( \gamma \) is the Euler constant. In this case the real part of \( \sigma_{yy} \), originating from the excitation of string waves\(^{46}\), is linear in \( \omega \) with an oscillator strength (integrated up to the cut-off \( u/a \)) which is smaller by a factor \( \epsilon(u/v_c)^2 \) than the weight of the Drude-peak in \( \sigma_{xx} \).

### E. A String in a Confining Potential

Next we consider the case in which the string is placed inside a parabolic confining potential. The string Lagrangian becomes
\[ L_s = \int_0^{L_y} dx \frac{\sigma}{2} \left[ \frac{1}{u^2} \left( \frac{\partial Y}{\partial t} \right)^2 - \left( \frac{\partial Y}{\partial x} \right)^2 - \kappa^{-2}Y^2 \right] . \] (69)

Since the renormalization of the parameters of the effective action for calculating projected correlations is governed by the short-wavelength string fluctuations the results, Eqs. (32)-(36), are unchanged by the long-wavelength cutoff \( \kappa^{-1} \) set by the confining potential (as long as \( \kappa^{-1} \gg a \)).

The modified string propagator does affect the result for the single hole Green function [see Eq. (42).] At zero temperature we find that the function \( f(x,t) \) which enters the factor \( F(x,y,t) \) in Eq. (44) is changed into

\[ f(x,t) = 4\pi a^2 \times \left[ K_0(\kappa a) - K_0(\kappa \sqrt{(x-ut+ia)(x+ut-ia)}) \right] , \] (70)

where \( K_0 \) is the modified Bessel function. Consequently the low energy tunneling density of states exhibits a crossover from the Luttinger liquid behavior \( \rho(\omega) \propto \omega^{2(\gamma_a+\gamma_s)} \) for \( \omega < \kappa u \) to the logarithmically suppressed form, Eq. (49), at higher frequencies.

Similarly, after performing the \( y \) Fourier transform, one obtains

\[ A^\omg(x,t;ky) = \frac{a}{2\pi L_y} e^{-ik_y x} \alpha e^{2(\gamma_a+\gamma_s)} e^{-2\Delta(k_y)K_0(\kappa a)} \times \exp \left[ 2\Delta(k_y)K_0(\kappa \sqrt{(x-ut+ia)(x+ut-ia)}) \right] \times \prod_{\alpha=c,s} \left[ -i(x-v_\alpha t + ia_\alpha) \right]^{-\gamma_\alpha} \times \left[ i(x+v_\alpha t - ia_\alpha) \right]^{-\gamma_\alpha} , \] (71)

where \( \gamma_c, \gamma_s \) and \( \Delta(k_y) \) retain their previously found values, Eqs. (48,54). Although we are unable to provide a closed expression for \( A^\omg(k,\omega) \) we expect it to exhibit a crossover around \( \omega \sim -u\sqrt{k_y^2 + \kappa^2} \) where the exponent of its approximate power-law behavior increases by \( \Delta(k_y) \) relative to its value in the region \( \omega \approx -u|k_y| \).

Finally, while the result for \( \sigma_{xx}(\omega) \), Eq. (66), is unchanged by the presence of the confining potential the optical conductivity along the \( y \) direction becomes

\[ \sigma_{yy}(\omega) = \frac{8}{\pi L_y} \int_0^{L_y} e^{2} (\tilde{k} \tilde{a})^2 u \frac{i\omega}{(\omega + i\delta)^2 - (u\kappa)^2} . \] (72)

This is again just the classical conductivity of a charged string fluctuating in a parabolic well and can be easily derived by solving the classical equation of motion in a similar fashion to the one we have outlined following Eq. (67).

V. DISCUSSION

As we mentioned in the Introduction our model was motivated by the behavior of data which suggests that striped inhomogeneous states appear over a wide range of the phase diagram of the cuprate high temperature superconductors\(^{19}\). Our aim here is to explore possible connections between the model we have studied and a few aspects of the phenomenology of these systems. There is no doubt that the physics of stripes in doped Mott insulators is more involved than the one we have considered. The dynamics of stripes as loci of sites visited by holes is intricately and self-consistently determined by the dynamics of the holes, their interactions with the spin background and with each other. In our model we have concentrated on a single string and assumed that rotational invariance has been completely broken, possibly due to a strong orienting field originating from the underlying lattice. A more realistic treatment of smectic and nematic stripe phases in the cuprates should include the interactions between stripes, which are expected to be invariant under small rotations\(^{14}\). We defer the study of these issues to the future. Notwithstanding, we hope that the model captures at least some of the features associated with the relevant stripe physics.

A. The Pseudogap

One such feature, which is revealed by the model, is the tendency of the holes to enhance the flexibility of the string in order to increase its fluctuations and by that the gain in their kinetic energy. We believe that this is also the driving force behind stripe fluctuations in the copper-oxygen planes, where holes hop in the transverse direction to the stripe in order to lower their kinetic energy.

As we have shown, as long as the number of electrons is fixed, an increase in the string fluctuations leads to an increase in the relative strength of the electron-electron interactions as reflected by the Luttinger parameters \( K_a \), Eq. (34). This fact may carry with it consequences to the size of the pseudogap which is observed, especially in underdoped samples\(^{47,48}\).

A possible explanation of the pseudogap phenomenon is the so-called “spin gap proximity effect” suggested in Ref. 49. In this scenario two coupled one-dimensional systems with different Fermi wave-vectors are considered, modeling the stripe and its environment. Single particle tunneling between the systems is suppressed owing to the difference in their Fermi wave-vectors. However, under appropriate circumstances singlet pair-tunneling processes can become relevant. When this happens the coupled system scales to a new strong coupling fixed point which exhibits a total spin gap and strong global superconducting fluctuations. The physics is analogous to the proximity effect in conventional superconductors since both are driven by the gain in the zero point kinetic energy of the pairs which outweighs the cost of pairing.

The relevancy of the pair-tunneling term is governed by its scaling dimension.
\[ \delta_{\text{pair}} = \frac{1}{2} \left( \frac{A}{K_c} + \frac{B}{K^{(c)}_c} + K_s + K^{(c)}_s \right), \]

where \( K_a, K^{(c)}_a \) are the Luttinger parameters of the stripe and the environment respectively. \( A = 1 \) and \( B = 1 \) in the absence of forward scattering inter-system density-density and current-current interactions but are in general complicated functions of the coupling constants. In particular they are decreasing functions of \( K_c \). Pair tunneling is perturbatively relevant if \( \delta_{\text{pair}} < 2 \) and irrelevant otherwise.

Intra-stripe repulsive interactions \((K_c < 1)\) increase the value of \( \delta_{\text{pair}} \), thus making it less relevant. This is physically reasonable since repulsive interactions within the stripe are unfavorable for pairing and therefore for pair tunneling. Since stripe fluctuations have the effect of decreasing the value of \( K_c \), see Eq. (34), they tend to increase the scaling dimension \( \delta_{\text{pair}} \) and thus reduce the effectiveness of the spin-gap proximity effect. Since there is experimental evidence that correlates between higher levels of doping and a greater degree of stripe fluctuations this would mean that fluctuations may account, at least partially, for the decrease in the pseudogap with doping.

Here it should be noted, however, that fluctuations have additional effects on the Josephson coupling between stripes. As this coupling involves tunneling of pairs from one system to another it is governed by the points of closest approach between them. Since fluctuations increase the probability of having the two systems in close proximity the amplitude of such tunneling processes is exponentially enhanced by them\(^5\). In addition it has been shown that inter-stripe forward scattering interactions tend to increase the relevance of pair tunneling\(^5\). Such interactions arise naturally when integrating out the shape fluctuations in coupled chain systems.

Which of the three effects just mentioned dominates the physics is a matter of details. It is possible that in the spin-gap proximity effect, in which pairs do not have to tunnel through an intervening barrier and where the forward scattering interactions induced by the fluctuations are second order in \( \epsilon \), the reduction in the kinetic energy along the stripe is the most important, at least for small fluctuations. It is also likely that for the establishment of global phase coherence in a quasi-one-dimensional superconductor through Josephson tunneling between stripes the increase of the tunneling amplitude due to fluctuations is the major effect.

B. Angle Resolved Photoemission Spectroscopy

ARPES measurements provide indirect evidence for the existence of stripes in the cuprate compounds\(^5\). For instance, measurements\(^5\) of \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \) and the related materials\(^5\) \( \text{La}_{1.6-x}\text{Nd}_{0.4}\text{Sr}_x\text{CuO}_4 \) and \( \text{La}_{1.4-x}\text{Nd}_{0.6}\text{Sr}_x\text{CuO}_4 \) have revealed that the frequency-integrated spectral weight is confined inside one-dimensional segments in momentum space. These segments can be interpreted as the Fermi surfaces of quarter-filled one-dimensional stripes running along the \( a \) and \( b \) directions in the planes.

The spectral function \( A^<(\mathbf{k},\omega) \), which is measured by ARPES, is often a broad function of the frequency \( \omega \), especially in the vicinity of the anti-nodal points \((0,\pm\pi), (\pm\pi,0)\). This behavior is naturally explained\(^8\) if one considers the signal as coming from a collection of one-dimensional systems which are capable of producing such wide spectra for a large enough interaction strength [large enough values of \( \gamma \), see Eq. (48)].

The width of any \( \omega \) structure in \( A^<(\mathbf{k},\omega) \) typically becomes smaller as one moves from the anti-nodal regions toward the \( \Gamma \)-point \((0,0)\) or the nodal regions around \((\pm\pi/2, \pm\pi/2)\). This tendency is ubiquitous and appears in various systems, for example\(^5\) \( \text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta} \). Within the interpretation of the two-dimensional Fermi surface as being a superposition of two Fermi surfaces of one-dimensional stripes, the anti-nodal regions correspond to wave-vectors with a large component in the direction perpendicular to the stripes. We have seen that the effects of stripes fluctuations tend to increase, via the term \( \Delta(k_y) = \epsilon(ak_y)^2/2 \), the value of the exponent that governs the behavior of the spectral function, Eqs. (60,62). This enhancement grows with the magnitude of the transverse wave-vector component, \( k_y \), and leads to broader spectra. It may constitute at least a partial reason for the observed broad spectra in the anti-nodal regions.

C. Anisotropies in the Optical Conductivity

We have shown that stripe fluctuations lead to an anisotropy in the optical conductivity, Eqs. (66,67), whose size is inversely proportional to the degree of fluctuations as given by the parameter \( \epsilon \). Such anisotropies of the far infra-red conductivity in the \( a-b \) plane have been observed\(^6\) in detwinned samples of \( \text{YBa}_2\text{Cu}_3\text{O}_{7-\delta} \) and \( \text{YBa}_2\text{Cu}_3\text{O}_8 \). More recently similar anisotropies have been found\(^7\) in the infra-red conductivity of a detwinned \( x = 0.03 \) crystal of \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \). Although it is likely that part of the anisotropies observed in YBCO are due to the Cu-O chains present in this system it is nevertheless probable, particularly in view of the neutron scattering data, that at least some of the effect comes from stripes. If this is the case then the fact that the anisotropy is about a factor of 2-3 at most, teaches us that in the measured samples there is a large degree of stripe orientational disorder and fluctuations.

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One can gain some semiclassical understanding of this result by considering the change in the velocity $v = dℓ/dt$ of a classical particle moving on the string. If scattering from the fluctuations-induced potentials is ignored the only change in the velocity is due to the component of the electric field along the string. For small fluctuations this change is $dv/dt = (-e/m)E_y(t) \frac{dY}{dx}(x,t)$. Assuming that initially the particle was moving with velocity $\bar{v}$ one finds, to lowest order in $E_y$, that the average $y$-component of the velocity at time $t$ is given by $\langle v_y(t) \rangle = -(e/m)E_y(t) \int_{-\infty}^{0} dt' e^{-i\omega t'} \left\langle \frac{dY}{dx}(0,0) \frac{dY}{dx}(\bar{v}_F t', t') \right\rangle$. By taking the average with respect to the quantum dynamics of the string, i.e. using $\frac{1}{\pi} \left\langle \left( \frac{dY}{dx}(0,0), \frac{dY}{dx}(\bar{v}_F t, t) \right) \right\rangle_s$, a similar $\omega$ dependence to that of Eq. (68) is obtained.