UNFOLDED FORM OF CONFORMAL EQUATIONS IN M DIMENSIONS AND \( \mathfrak{o}(M+2) \)-MODULES

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Abstract. A constructive procedure is proposed for formulation of linear differential equations invariant under global symmetry transformations forming a semi-simple Lie algebra \( \mathfrak{f} \). Under certain conditions \( \mathfrak{f} \)-invariant systems of differential equations are shown to be associated with \( \mathfrak{f} \)-modules that are integrable with respect to some parabolic subalgebra of \( \mathfrak{f} \). The suggested construction is motivated by the unfolded formulation of dynamical equations developed in the higher spin gauge theory and provides a starting point for generalization to the nonlinear case. It is applied to the conformal algebra \( \mathfrak{o}(M,2) \) to classify all linear conformally invariant differential equations in Minkowski space. Numerous examples of conformal equations are discussed from this perspective.

Contents

1. Background and Introduction 2
2. The Simplest Conformal Systems 9
2.1. Conformal Scalar 9
2.2. Conformal Spinor 11
2.3. Conformal \( p \)-Forms 12
2.4. \( M = 4 \) Electrodynamics 14
3. General Construction 19
4. Conformal Systems of Equations 28
4.1. Irreducible Tensors and Spinor–tensors 29
4.2. Generalized Verma Modules 31
4.3. Contragredient Modules 33
4.4. Structure of \( \mathfrak{o}(M + 2) \)-Generalized Verma Modules 34
4.5. Cohomology of Irreducible \( \mathfrak{o}(M + 2) \)-Modules 38
4.6. Examples of Calculating Cohomology of Reducible \( \mathfrak{o}(M + 2) \)-Modules 42
4.7. Conformal Equations 43
5. Conclusions 52
Acknowledgments 54
Appendix A. Relevant facts from representation theory 54
Appendix B. Homomorphism diagrams 60
References 62

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1. Background and Introduction

In this paper we apply a method of the analysis of dynamical systems called *unfolded formulation* to classify all conformally invariant linear differential equations in any space-time dimension $M > 2$. This method, suggested originally for the analysis of higher spin dynamical systems [1]-[6], proved to be useful for the analysis of problems of deformation quantization [7, 8].

Unfolded formulation of a system of partial differential equations in a space-time with coordinates $x^m$ ($m = 0, \ldots, M-1$) consists of its reformulation in the first-order form with respect to all coordinates. As such, it is a generalization of the first-order form of ordinary (i.e. $M = 1$) differential equations $\dot{q}^i = G^i(q)$. More precisely, unfolded equations have the form

$$dU^\Omega(x) = G^\Omega(U(x))$$ \hspace{1cm} (1.1)

Here $d = \xi^m \frac{\partial}{\partial x^m}$ is the exterior differential\(^1\). $U^\Omega(x)$ denotes a set of variables being differential forms (i.e., polynomials in $\xi^m$). The condition

$$G^\Lambda(U(x)) \wedge \frac{\delta G^\Lambda(U(x))}{\delta U^\Omega(x)} = 0$$ \hspace{1cm} (1.2)

is imposed on $G^\Lambda(U(x))$ to guarantee that the system is formally consistent. (It is assumed that only wedge products of differential forms appear in (1.1) and (1.2), i.e. $G^\Omega(U(x))$ is a polynomial of $U^\Omega(x)$ containing no derivatives in $\xi^m$ and $x^m$.)

In the case of ordinary differential equations the variables $q^i(t)$ taken at any $t = t_0$ provide a full set of initial data. For an $M > 1$ unfolded field-theoretical system the knowledge of the fields $U(x)$ at any $x^m = x^m_0$ also reconstructs $U(x)$ in some neighborhood of $x^m_0$. Therefore, to unfold a field-theoretical system with infinitely many degrees of freedom it is necessary to introduce infinitely many auxiliary fields. The latter identify with all derivatives of dynamical fields (i.e. with infinitely many generalized momenta).

Unfolded formulation, which is available for any dynamical system, has a number of properties proved to be useful for the analysis of various aspects of linear and nonlinear dynamics (see [9] for a recent review). The property of the unfolded formulation which is of particular importance for the analysis of this paper is that it makes symmetries of a model manifest. In particular, unfolded formulation of any dynamical system possessing one or another linearly realized global symmetry $g$ is formulated in terms of some $g$-module. This simple observation makes it trivial to list unfolded dynamical systems of a given symmetry. The nontrivial part of the problem is to single out nontrivial dynamical systems in this list that result

\(^1\)Throughout this paper we use the notation $\xi^m$ for the basis 1-forms conventionally denoted $dx^m$.\n
Unfolded form of linear conformal equations in $M$-dimensions and \ldots

from unfolding of certain differential equations. (Note that, generally, unfolded equations may describe an infinite set of constraints with no differential equations among them.) As we show in this paper, nontrivial $g$-invariant differential equations are associated with the unfolded equations based on $g$-modules resulting from factorization of generalized Verma $g$-modules over singular submodules. Our scheme is quite general and can be applied to the analysis of various dynamical systems. In this paper we apply this analysis to classification of conformally invariant linear differential equations.

Let us now analyze relevant properties of unfolded equations more carefully. Due to (1.2) the system (1.1) is invariant under the gauge transformations

$$
\delta U^\Omega(x) = d\epsilon^\Omega(x) + \epsilon^A(x) \wedge \frac{\delta G^\Omega(U(x))}{\delta U^A(x)},
$$

where the gauge parameters $\epsilon^\Omega(x)$ are arbitrary functions of the coordinates $x^m$. Let $\omega^A(x) = \xi^m \omega^A_m(x)$ be the set of 1-forms in $U^\Omega(x)$. The requirement that the restriction $G^A(U(x))|_{\omega(x)} = -G^A_{BC}\omega^B(x) \wedge \omega^C(x)$ to the sector of 1-forms $\omega^A(x)$ is compatible with (1.2) implies that $G^A_{BC}$ satisfy (super)Jacobi identities thus being structure coefficients of some Lie (super)algebra. As a result, the restriction of equation (1.1) to the sector of 1-forms amounts to the flatness condition on $\omega^A(x)$. In higher spin theories, $\mathfrak{h}$ is some infinite dimensional higher spin symmetry algebra \cite{10-14, 5, 6, 15}, which contains one or another finite dimensional space–time symmetry subalgebra $\mathfrak{f}$. For example, $\mathfrak{f} = \mathfrak{o}(n, 2)$ appears either as anti-de Sitter ($n = M - 1$) or as conformal ($n = M$) algebra in $M$ dimensions.

Let $\omega_0^\Omega(x)$ be a fixed 1-form taking values in $\mathfrak{f}$, i.e. $\omega_0(x) = \omega_0^i(x)e_i$, where $e_i$ is a basis in $\mathfrak{f}$. Equation (1.1) for $U^\Omega(x) = \omega_0^\Omega(x)$ is equivalent to the zero curvature condition

$$
d\omega_0^i(x) + \omega_0^j(x) \wedge \omega_0^k(x)f^i_{jk} = 0,
$$

where $f^i_{jk}$ are structure coefficients of $\mathfrak{f}$. For $\mathfrak{f}$ isomorphic to Poincaré algebra, $\omega_0^0(x)$ is usually identified with the flat space gravitational field with co-frame and Lorentz connection corresponding to generators of translations $\mathcal{P}_n$ and Lorentz rotations $\mathcal{L}_{mn}$, respectively. The components of the co-frame part of the connection are required to form a non-degenerate $M \times M$ matrix in which case we call connection non-degenerate. For example, Minkowski space–time in Cartesian coordinates is described by zero Lorentz connection and co-frame $\xi^\mu \mathcal{P}_n$ so that the components of the co-frame 1-form $e^m_n = \delta^m_n$ form a non-degenerate matrix. The freedom in the choice of a non-degenerate $\omega_0^0(x)$ encodes the coordinate choice ambiguity.

One can analyse equation (1.1) perturbatively by setting

$$
U^\Omega(x) = \omega_0^\Omega(x) + U_1^\Omega(x),
$$

\footnote{To introduce superalgebraic structure it is enough to let the 1-forms $\omega^A(x)$, which correspond to the even(odd) elements of superalgebra $\mathfrak{h}$, be Grassmann even(odd).}
where $U_1^0(x)$ describes first-order fields (fluctuations), while $\omega_0^0(x)$ is zero-order. Let $|\Phi^p(x)\rangle^\lambda$ be the subset of $p$-forms contained in $U_1^0(x)$ (we use Dirac ket notation for the future convenience). The linearized part of equations (1.1) associated with the $p$-forms reduces to some equations of the form

$$D|\Phi^p(x)\rangle^\lambda = 0,$$

with

$$D|\Phi^p(x)\rangle^\eta = (d\delta^\eta_\lambda + \omega^\eta_0(x)t^n_{\eta\lambda})|\Phi^p(x)\rangle^\lambda.$$  

(1.7)

The identity (1.2) implies that the matrices $t^n_{\eta\lambda}$ form a representation of $f$ (i.e. $[t^j, t^k] = 2f^j_{jk}t_i$). Let $\mathfrak{M}$ be the $f$-module associated with $|\Phi^p(x)\rangle^\lambda$, i.e. $|\Phi^p(x)\rangle^\lambda$ be a section of the trivial bundle $\mathcal{B} = \mathfrak{M} \times \mathbb{R}^M$ with the fiber $\mathfrak{M}$ and the $M$-dimensional Minkowski base space $\mathbb{R}^M$. The covariant derivative $D$ (1.7) in $\mathcal{B}$ is flat,

$$DD = 0$$

(1.8)

as a consequence of (1.4).

Let the associative algebra $A_{2\mathfrak{M}}$ be the quotient of the universal enveloping algebra of $f$ over the ideal Ann$(\mathfrak{M})$ that annihilates the representation $\mathfrak{M}$, i.e. $A_{2\mathfrak{M}} = U(f)/Ann(\mathfrak{M})$. Let $E_i$ be a basis of $A_{2\mathfrak{M}}$ and $T^t_{\eta\lambda}$ be the representation of $A_{2\mathfrak{M}}$ induced from the representation $t^n_{\eta\lambda}$. If $\omega^t_0(x)e_i$ satisfying equation (1.4) is (locally) represented in a pure gauge form

$$\omega^t_0(x)e_i = g(x)dg^{-1}(x)$$

(1.9)

with an invertible element $g(x) = g^t(x)E_i \in A_{2\mathfrak{M}}$, the generic local solution of equation (1.6) gets the form

$$|\Phi^p(x)\rangle^\eta = g^t(x)T^t_{\eta\lambda}|\Phi^p(x_0)\rangle^\lambda.$$  

(1.10)

We see that $|\Phi^p(x_0)\rangle^\lambda$ plays a role of initial data for equation (1.6), fixing $|\Phi^p(x)\rangle^\eta \bigg|_{x \in \varepsilon(x_0)}$ in a neighborhood $\varepsilon(x_0)$ of a point $x_0$ such that $g(x_0) = 1$. As a result, solutions of equation (1.6) are parameterized by elements of the $f$-module $\mathfrak{M}$. If the $f$-module $\mathfrak{M}$ is finite dimensional, we will call the corresponding equation (1.6) topological because it describes at most $\text{dim}(\mathfrak{M})$ degrees of freedom.

The system (1.4), (1.6) is invariant under the gauge transformations (1.3)

$$\delta\omega^t_0(x) = d\varepsilon^t(x) - 2\varepsilon^t(x)\omega^k_0(x)f^t_{jk},$$

$$\delta|\Phi^p(x)\rangle^\eta = d\varepsilon(x) - (\varepsilon^t(x)\omega^t_0(x)t^n_{\eta\lambda})^\lambda - \varepsilon^t(x)t^n_{\eta\lambda}|\Phi^p(x)\rangle^\lambda,$$

(1.11)

(1.12)

where the $(p-1)$-form $|\varepsilon(x)\rangle^\eta$ and 0-form $\varepsilon^t(x)$ are infinitesimal gauge symmetry parameters.

(Note that if $p = 0$ then $|\varepsilon(x)\rangle^\eta \equiv 0$.) Any fixed solution $\omega^t_0(x)$ of equation (1.4) (called vacuum solution) breaks the local $f$ (super)symmetry associated with $\varepsilon^t(x)$ to its stability
subalgebra with the infinitesimal parameter \( \epsilon^i_0(x) \) satisfying equation
\[
d\epsilon^i_0(x) - 2\epsilon^i_0(x)\omega_0^k(x)f^j_{ik} = 0.
\]
This equation is consistent due to the zero curvature equation (1.4), and its generic (local) solution is parameterized by the values of \( \epsilon^i_0(x_0) \),
\[
\epsilon^i_0(x)e_i = \epsilon^i_0(x_0)g(x)e_ig^{-1}(x).
\]
The leftover global symmetry
\[
\delta|\Phi^p(x)|^n = \epsilon^i_0(x_0)\left(g(x)t_ig^{-1}(x)\right)^\eta_\lambda|\Phi^p(x)|^\lambda,
\]
with the symmetry parameters \( \epsilon^i_0(x_0) \) forms the Lie (super)algebra \( \mathfrak{f} \). From the Poincaré lemma it follows that the gauge symmetries (1.12) of \( |\Phi^p(x)|^n \) associated with the parameters \( |\epsilon(x)|^n \), which are \( p - 1 > 0 \) forms, do not give rise to additional global symmetries of (1.4) and (1.6) in the topologically trivial situation.

In fact, equations (1.4) and (1.6) have a larger symmetry \( \mathfrak{g}_{2R} \supset \mathfrak{f} \) manifest. Let \( \mathfrak{g}_{2R} \) be the Lie (super)algebra built from \( A_{2R} \) via (super)commutators. One can extend (1.4) and (1.6) to
\[
dw^I(x) + w^J(x)w^K(x)h^I_{JK} = 0,
\]
\[
D|\Phi^p(x)|^n = (d\delta_\lambda^\eta + w^I(x)T_I^{\eta_\lambda})|\Phi^p(x)|^\lambda = 0,
\]
where \( \zeta^m w^I_m(x) \) are the gauge fields of \( \mathfrak{g}_{2R} \), and \( h^I_{JK} \) are the structure coefficients of \( \mathfrak{g}_{2R} \). The system (1.16), (1.17) is consistent in the sense of (1.2) and has global symmetry \( \mathfrak{g}_{2R} \) for any \( w^I(x) \), which solves (1.16). Since \( \mathfrak{f} \) is canonically embedded into \( \mathfrak{g}_{2R} \), setting \( w^I(x)E_I = \omega_0^i(x)e_i \) one recovers the system (1.4), (1.6) thus proving invariance of the system (1.4), (1.6) under the infinite dimensional global symmetry \( \mathfrak{g}_{2R} \). Infinite dimensional symmetries of this class appear in the field-theoretical models as higher spin symmetries.

This approach is universal: any system of \( \mathfrak{f} \)-invariant linear differential equations can be reformulated in the form (1.4), (1.6) by introducing auxiliary variables associated with the appropriate (usually infinite dimensional) \( \mathfrak{f} \)-module \( \mathcal{M} \) (also see examples below). As a result, classification of \( \mathfrak{f} \)-invariant linear systems of differential equations is equivalent to classification of \( \mathfrak{f} \)-modules \( \mathcal{M} \) of an appropriate class. More precisely, let \( \mathfrak{f} , \mathfrak{p}_{\Pi} \subset \mathfrak{f} \) and \( \mathcal{M} \) be, respectively, some semi-simple Lie algebra, its parabolic subalgebra and \( \mathfrak{f} \)-module integrable with respect to \( \mathfrak{p}_{\Pi} \) (for necessary definitions see section 3). We show that, for a non-degenerate flat connection 1-form \( \omega^i_0(x) \), the covariant constancy equation (1.6) on a \( p \)-form \( |\Phi^p(x)|^\lambda \) taking values in \( \mathcal{M} \) encodes an \( \mathfrak{f} \)-invariant system of differential equations \( R_{2R}|\Phi^p(x)|^\lambda = 0 \) on a \( p \)-form \( |\Phi^p(x)|^\lambda \) from the \( p \)-th cohomology \( H^p(\mathfrak{r}_\Pi, \mathcal{M}) \) of the radical \( \mathfrak{r}_\Pi \subset \mathfrak{p}_{\Pi} \) with coefficients in \( \mathcal{M} \). For Abelian radical \( \mathfrak{r}_\Pi \) we prove that each differential operator from
$R_{\mathfrak{g} \mathfrak{r}}$ corresponds to an element of $H^{p+1}(\mathfrak{p}_{\Pi}, \mathfrak{m})$ and vice versa. We introduce classification of $\mathfrak{f}$-invariant systems of equations $R_{\mathfrak{g} \mathfrak{r}}$ by reducibility of $\mathfrak{f}$-modules $\mathfrak{m}$. $\mathfrak{f}$-invariant systems that correspond to (reducible) irreducible $\mathfrak{f}$-modules $\mathfrak{m}$ are called (non-)primitive. Non-primitive systems contain nontrivial subsystems and can be described as extensions of the primitive ones.

This general construction is applied to classification of linear homogeneous conformally invariant equations on $|\phi^0(x)\rangle \in H^0(\mathfrak{p}_{\Pi}, \mathfrak{m})$, where we set $\mathfrak{f} = \mathfrak{o}(M, 2)^3$, $\mathfrak{p}_{\Pi} = \mathfrak{t}(\mathfrak{m})$ (the algebra of translations) and $\mathfrak{p}_{\Pi} = \mathfrak{iso}(M) \oplus \mathfrak{o}(2)$ (i.e., the direct sum of Poincaré algebra and the algebra of dilatations). Conformally invariant equations are determined by $H^1(\mathfrak{t}(\mathfrak{m}), \mathfrak{m})$. Examples of primitive equations include Klein–Gordon and Dirac equations and their conformal generalizations to higher (spinor-)tensor fields, conformal equations on $p$-forms and, in particular, (anti)selfduality equations. Examples of non-primitive equations correspond to reducible $\mathfrak{m}$ and include $\mathfrak{m} = 4$ electrodynamics with and without external current and its higher spin generalization to higher tensors in the flat space of any even dimension. Note that our construction allows us to write these systems both in gauge invariant and in gauge fixed form. In the latter case we automatically obtain conformally invariant gauge conditions. A number of examples of conformal systems are considered in sections 2 and 4.

To find $H^1(\mathfrak{t}(\mathfrak{m}), \mathfrak{J})$ with coefficients in an irreducible integrable with respect to $\mathfrak{iso}(M) \oplus \mathfrak{o}(2)$ conformal module $\mathfrak{J}$ we consider a generalized Verma module $\mathfrak{V}$ of $\mathfrak{o}(M + 2)$ such that $\mathfrak{J}$ is its irreducible quotient. We calculate $H^1(\mathfrak{t}(\mathfrak{m}), \mathfrak{I})$ for any $\mathfrak{I}$. As an $\mathfrak{iso}(M) \oplus \mathfrak{o}(2)$-module, $H^1(\mathfrak{t}(\mathfrak{m}), \mathfrak{J})$ is shown to be isomorphic to the space of certain systems of singular and subsingular vectors in $\mathfrak{V}$. As a result, the form of a primitive system of conformal differential equations $R_{\mathfrak{g} \mathfrak{r}}$ encoded by the covariant constancy equation (1.6) is completely determined by these systems of singular and subsingular vectors in $\mathfrak{V}$. Since any reducible integrable with respect to $\mathfrak{iso}(M) \oplus \mathfrak{o}(2)$ module $\mathfrak{m}$ is an extension of some irreducible modules $\mathfrak{J}$, $H^1(\mathfrak{t}(\mathfrak{m}), \mathfrak{m})$ can be easily calculated in terms of $H^1(\mathfrak{t}(\mathfrak{m}), \mathfrak{J})$, thus allowing classification of all possible conformal differential equations.

Practical calculating of $H^p(\mathfrak{p}_{\Pi}, \mathfrak{m})$ may be difficult for a general pair $\mathfrak{p}_{\Pi} \subset \mathfrak{f}$ because the structure of generalized Verma modules is not known in the general case. In the relatively simple case where $\mathfrak{p}_{\Pi} = \mathfrak{iso}(M) \oplus \mathfrak{o}(2)$ and $\mathfrak{f} = \mathfrak{o}(M + 2)$ we calculate the structure of generalized Verma modules using the results of [17, 18]. This allows us to calculate $H^p(\mathfrak{t}(\mathfrak{m}), \mathfrak{m})$ for any integrable with respect to $\mathfrak{iso}(M) \oplus \mathfrak{o}(2)$ module $\mathfrak{m}$.

---

3In fact we consider only complex case. Thus $\mathfrak{o}(M, 2) \sim \mathfrak{o}(M + 2)$.
Let us note that our approach has significant parallels with important earlier works. In particular, the relation between conformally quasi-invariant\(^4\) differential operators and singular vectors in the generalized Verma modules of the conformal algebra was originally pointed out in [19] for a particular case. For any semi-simple Lie algebra \(\mathfrak{g}\) and some its parabolic subalgebra \(\mathfrak{p}_\Pi\), a correspondence between homogeneous \(\mathfrak{g}\)-(quasi)invariant linear differential operators acting on a finite set of \(\mathfrak{p}_\Pi\)-covariant fields and jet bundle \(\mathfrak{p}_\Pi\)-homomorphisms was studied in [20]. Namely, let the Lie groups \(\mathcal{A}\) and \(\mathcal{P} \subset \mathcal{A}\) correspond to the Lie algebras \(\mathfrak{g}\) and \(\mathfrak{p}_\Pi\), respectively, and \(\mathcal{E}\) and \(\mathcal{F}\) be homogeneous vector bundles with the base \(\mathcal{A}/\mathcal{P}\) and, respectively, the fibers \(\mathcal{E}\) and \(\mathcal{F}\) being some finite dimensional \(\mathfrak{p}_\Pi\)-modules. \(J^k\mathcal{E}\) is the \(k\)th associated jet bundle of \(\mathcal{E}\). By taking the projective limit

\[
\cdots \to J^k\mathcal{E} \to J^{k+1}\mathcal{E} \to J^k\mathcal{F} \to \cdots \to J^1\mathcal{E} \to E
\]

one finds [20] that there exists a class of \(\mathfrak{g}\)-(quasi)invariant linear differential operators corresponding to \(\mathfrak{g}\)-homomorphisms \(J^\infty\mathcal{E} \to J^\infty\mathcal{F}\). To establish relation with our approach one observes that the \(\mathfrak{g}\)-module dual to the module \(J^\infty\mathcal{E}\) identifies with the generalized Verma module induced from the \(\mathfrak{p}_\Pi\)-module \(\mathcal{E}\), i.e. \(\mathcal{G} = (J^\infty\mathcal{E})^\#\), where \((J^\infty\mathcal{E})^\#\) is the contragredient module to \(J^\infty\mathcal{E}\). The image of the highest–weight subspace of \((J^\infty\mathcal{F})^\#\) in \((J^\infty\mathcal{E})^\#\) under the dual mapping \((J^\infty\mathcal{F})^\# \rightarrow (J^\infty\mathcal{E})^\#\) is spanned by singular vectors. We expect that \(\mathbb{R}^M\) in our construction corresponds to the big cell of \(\mathcal{A}/\mathcal{P}\) and the sections of the bundle \(\mathcal{G} \times \mathbb{R}^M\) satisfying (1.6) along with appropriate boundary conditions coincide with sections of the bundle \(J^\infty\mathcal{E}\) over \(\mathcal{A}/\mathcal{P}\).

The approach developed in this paper allows one to classify all \(\mathfrak{g}\)-invariant homogeneous differential equations on a finite number of fields that form finite dimensional modules of a parabolic subalgebra \(\mathfrak{p}_\Pi \subset \mathfrak{g}\) with the Abelian radical \(\mathfrak{r}_\Pi \subset \mathfrak{p}_\Pi\). Equations of this class are referred to as \(\mathfrak{g}_{\mathfrak{p}_\Pi}\)-invariant equations in the rest of this paper. In particular we give the full list of conformally invariant equations in Minkowski space. In the case of even space–time dimension this list is broader than that of [20] because of taking into account the equations resulting from subsingular vectors.

Apart from giving a universal tool for classification of various \(\mathfrak{g}\)-invariant linear equations, the unfolded formulation is particularly useful for the study of their nonlinear deformations [1]. Once some set of linear equations is formulated in the unfolded form (1.4) (1.6), the problem is to check if there exists a nonlinear unfolded system (1.1), which gives rise to the linear equations in question in the free field limit. In particular, nonlinear dynamics of higher spin gauge fields in various dimensions was formulated this way in [2, 6]. This paper is the first step towards realization of the full scale program of the study of nonlinear equations.

\(^4\)An operator \(g\) is called \(\mathfrak{g}\)-quasi-invariant for a Lie algebra \(\mathfrak{g}\) if for any \(f \in \mathfrak{g}\) there exists an operator \(h\) such that \([g, f] = hg\).
deformations of $\mathfrak{f}$-invariant equations. In fact, the analysis of this paper clarifies some ways towards nonlinear deformation. In particular, one can consider extensions of the modules $\mathcal{M}$ associated with the free fields of the model by the “current” modules contained in the tensor products of $\mathcal{M}$.

Let us note that the unfolded equations (1.1) can be thought of as a particular $L_\infty$ algebra \cite{21, 22} (and references therein). The specific property of the system (1.1), extensively used in the analysis of higher spin models \cite{1, 2, 6}, is that it is invariant under diffeomorphisms and, therefore, is ideally suited for the description of theories which contain gravity. It is important to note that in this case a nonlinear deformation within the system (1.1) may deform the $\mathfrak{f}$-symmetry transformations by some field-dependent terms originating from (1.3), that may complicate the description of this class of deformations within the manifestly $\mathfrak{f}$-symmetric schemes. For example this happens when gravity or (conformal gravity) is described in this formalism with the Weyl tensor 0-form interpreted as a particular dynamical field of the system, added to the right hand side of (1.16) \cite{1, 23}. Note that such a deformation is inevitable in any theory of gravitation because no global symmetry $\mathfrak{f}$ is expected away from a particular $\mathfrak{f}$-symmetric vacuum. Within unfolded formulation deformations of this class also admit a natural module extension interpretation.

The content of the rest of the paper is as follows. In section 2 we consider unfolded formulation of some simple conformal systems. In particular, conformal scalar is considered in section 2.1, conformal spinor is considered in section 2.2, conformal $p$-forms are considered in section 2.3 and $M = 4$ electrodynamics is considered in section 2.4. The general construction, which allows us to classify $\mathfrak{f}_{\Pi}$-invariant linear differential equations for any semi-simple Lie algebra $\mathfrak{f}$ and $\mathfrak{p}_\Pi \subset \mathfrak{f}$ with Abelian radical $\mathfrak{r}_\Pi$ is given in section 3. In section 4 we apply this construction to the conformal algebra $\mathfrak{o}(M, 2)$. Irreducible finite-dimensional representations of the Lorentz algebra are considered in section 4.1. Conformal modules (in particular generalized Verma modules and contragredient to generalized Verma modules) are discussed in sections 4.2 and 4.3 respectively. In section 4.4 we collect relevant facts about submodule structure of conformal generalized Verma modules for the cases of odd (section 4.4.1) and even (section 4.4.2) space–time dimensions. Cohomology with coefficients in irreducible conformal modules is calculated in section 4.5. Examples of calculating cohomology with coefficients in reducible conformal modules are given in section 4.6. In section 4.7 we formulate an algorithm that permits us to obtain explicit form of any conformal equation thus completing the analysis of conformally invariant equations. Conformal generalizations of the Klein–Gordon and the Dirac equations to the fields with block–type (rectangular) Young symmetries are given in section 4.7.1. Generalization of $M = 4$ equations for massless higher spin fields to a broad class of tensor fields in the flat space of arbitrary even dimension is
given in section 4.7.2. Fradkin–Tseytlin conformal higher spin equations in even dimensions are considered in section 4.7.3. In section 5 we conclude our results. In Appendix A we sketch the analysis of submodule structure of generalized Verma modules for odd and even dimensions. Corresponding homomorphism diagrams are given in Appendix B.

2. The Simplest Conformal Systems

The nonzero commutation relations of the conformal algebra $\mathfrak{o}(M, 2)$ are

\begin{align*}
[\mathcal{L}_m^m, \mathcal{L}_r^r] &= \eta_m^r \mathcal{L}^s_n + \eta_m^s \mathcal{L}^r_n - \eta_r^s \mathcal{L}^m_n - \eta_m^n \mathcal{L}^r_s, \\
[\mathcal{L}_m^m, \mathcal{P}^s] &= \eta_r^m \mathcal{P}^n - \eta_n^m \mathcal{P}^r, \\
[\mathcal{L}_m^m, \mathcal{K}^s] &= \eta_r^m \mathcal{K}^n - \eta_n^m \mathcal{K}^r, \\
[D, \mathcal{P}^n] &= -\mathcal{P}^n, \\
[D, \mathcal{K}^m] &= \mathcal{K}^m, \\
[\mathcal{P}^n, \mathcal{K}^m] &= 2\eta_n^m \mathcal{D} + 2\mathcal{L}_m^m, \\
(2.1)
\end{align*}

where $\eta_m^m$ is an invariant metric of the Lorentz algebra $\mathfrak{o}(M - 1, 1)$ and $\mathcal{L}_m^m, \mathcal{P}^n, \mathcal{K}^n,$ and $D$ are generators of $\mathfrak{o}(M - 1, 1)$ Lorentz rotations, translations, special conformal transformations and dilatation, respectively. Minkowski metric $\eta_m^m$ and its inverse $\eta_m^n$ are used to raise and lower Lorentz indices.

Let $|\Phi(\mathbf{x})\rangle = e_\eta|\Phi(\mathbf{x})\rangle^\eta$ be a 0-form section of the trivial bundle $\mathbb{R}^M \times \mathfrak{M}$. Here $\mathfrak{M}$ is some $\mathfrak{o}(M, 2)$-module. In most examples in this section we consider the case with an irreducible module $\mathfrak{M} \sim \mathfrak{I}_\Delta$ where $\mathfrak{I}_\Delta$ is a quotient of the generalized Verma module $\mathfrak{V}_\Delta$ freely generated by $\mathcal{K}^n$ from a vacuum Lorentz representation $|\Delta\rangle^A$ having a definite conformal weight $\Delta \in \mathbb{C}$

\begin{equation}
D|\Delta\rangle^A = \Delta|\Delta\rangle^A
\end{equation}

and annihilated by $\mathcal{P}^n$

\begin{equation}
\mathcal{P}^n|\Delta\rangle^A = 0.
\end{equation}

To describe Minkowski space in Cartesian coordinates, we choose the flat connection

\begin{equation}
D = \xi_n(\partial_n + \mathcal{P}_n).
\end{equation}

2.1. Conformal Scalar. In order to describe a conformal scalar field let us consider the generalized Verma module $\mathfrak{V}_{\Delta,0}$ induced from the trivial Lorentz representation with the basis vector $|\Delta, 0\rangle$ satisfying $\mathcal{L}_m^m|\Delta, 0\rangle = 0$. The generic element of $\mathfrak{V}_{\Delta,0}$ is

\begin{equation}
\sum_{l=0}^{\infty} \frac{1}{l!} C_{n_1...n_l} \mathcal{K}^{n_1} \ldots \mathcal{K}^{n_l}|\Delta, 0\rangle,
\end{equation}

where $C_{n_1...n_l} \in \mathbb{C}$ are totally symmetric tensor coefficients.

Let $|\Phi_{\Delta,0}(\mathbf{x})\rangle$ be a section of the trivial bundle $\mathbb{R}^M \times \mathfrak{V}_{\Delta,0}$, i.e.,

\begin{equation}
|\Phi_{\Delta,0}(\mathbf{x})\rangle = \sum_{l=0}^{\infty} \frac{1}{l!} C_{n_1...n_l}(x) \mathcal{K}^{n_1} \ldots \mathcal{K}^{n_l}|\Delta, 0\rangle,
\end{equation}
where $C_{n_1...n_l}(x)$ are some functions on $\mathbb{R}^M$. The covariant constancy condition (1.6) for the field $|\Phi_{\Delta,0}(x)\rangle$

$$D|\Phi_{\Delta,0}(x)\rangle = 0$$

(2.7)
is equivalent to the infinite system of equations

$$\partial_n|\Phi_{\Delta,0,l-1}(x)\rangle + P_n|\Phi_{\Delta,0,l}(x)\rangle = 0, \quad l \geq 1,$$

(2.8)

where

$$|\Phi_{\Delta,0,l}(x)\rangle = \frac{1}{l!} C_{n_1...n_l}(x) \mathcal{K}^{n_1} \cdots \mathcal{K}^{n_l} |\Delta,0\rangle.$$  

(2.9)

With the definition

$$\partial_n|\Delta,0\rangle = 0,$$

(2.10)

(2.8) amounts to the system of equations

$$\partial_n C_{m_1...m_{l-1}}(x) + 2(\Delta + l - 1) C_{nm_1...m_{l-1}}(x) - (l - 1) \eta_{n(m_1} C_{k m_2...m_{l-1})}(x) = 0$$

(2.11)

for $l \geq 1$, where parentheses imply symmetrization over the indices denoted by the same letter, i.e.,

$$\eta_{nm_1} C_{k m_2...m_{l-1}} = \frac{1}{l!} \eta_{nm_1} C_{k m_2...m_{l-1}} + \eta_{nm_2} C_{k m_1 m_3...m_{l-1}} + \ldots.$$

(2.12)

For $\Delta \not\in \frac{1}{2} \mathbb{Z}$, (2.11) expresses all tensors $C_{m_1...m_l}(x)$ via the derivatives of $C(x)$ imposing no differential conditions on the latter. For half-integer $\Delta$ the situation is more interesting. For example, for $\Delta = \frac{1}{2} M - 1$ system (2.11) imposes the Klein–Gordon equation on $C(x)$ and expresses all higher rank tensors in terms of the higher derivatives of $C(x)$ and $C^{mn}(x) \eta_{mn}$. Indeed, the first two equations in (2.11) are

$$\partial_n C(x) + 2\Delta C_n(x) = 0,$$

(2.13)

$$\partial_n C_m(x) + 2(\Delta + 1) C_{nm}(x) - \eta_{mn} C^k(x) = 0.$$  

(2.14)

Contracting (2.14) with $\eta^{nm}$ and substituting $C_n(x)$ from (2.13) we obtain

$$-\frac{1}{2\Delta} \Box C(x) + (2\Delta + 2 - M) C^k(x) = 0.$$  

(2.15)

Thus, for $\Delta = \frac{1}{2} M - 1$, $\Delta \neq 0$ (i.e., $M \neq 2$) (2.13) is equivalent to the Klein–Gordon equation for $C(x)$

$$\Box C(x) = 0.$$  

(2.16)

Algebraically, the situation is as follows. Whenever $\Delta$ is not half-integer $P_n|\Phi_{\Delta,0,l}(x)\rangle \neq 0$ for any $|\Phi_{\Delta,0,l}(x)\rangle$ with $l \geq 1$ and the module $\mathfrak{H}_{\Delta,0}$ is irreducible. This means that it is possible to solve the chain (2.11) by expressing each $|\Phi_{\Delta,0,l}(x)\rangle$ via derivatives of $|\Phi_{\Delta,0,l-1}(x)\rangle$.
for \( l \geq 1 \). Abusing notations, \(|\Phi_{\Delta,0,l}(x)\rangle = -(P^{-1})^n \partial_n |\Phi_{\Delta,0,l-1}(x)\rangle\), \( l \geq 1 \). For \( \Delta = \frac{1}{2} M - 1 \), the module \( \mathfrak{V}_{\Delta,0} \) is reducible because the identity

\[
P_n |s\rangle = 0, \quad |s\rangle = K_m K^m |\Delta, 0\rangle
\]

(2.17)

implies that \(|s\rangle\) is a singular vector, i.e. it is a vacuum vector of the submodule \( \mathfrak{V}_{\Delta,0} \subset \mathfrak{V}_{\Delta,0} \) generated from \(|s\rangle\) by \( K^n \). Effectively, the algebraic condition \( 2.17 \) imposes the Klein–Gordon equation on \(|\Phi_{\Delta,0,0}(x)\rangle = C(x)|\Delta, 0\rangle\). The same time, since the coefficient in front of \( C_n^m K_m K^n |\Delta, 0\rangle \in |\Phi_{\Delta,0,2}(x)\rangle \) in equation \( 2.11 \) with \( l = 2 \) vanishes, \( C_n^m(x) \) cannot be expressed in terms of derivatives of \( |\Phi_{\Delta,0}(x)\rangle \), thus becoming an independent field. Setting \( C_n^m(x) = 0 \) is equivalent to restriction of \( \mathbb{R}^M \times \mathfrak{V}_{\Delta,0} \) to the bundle \( \mathbb{R}^M \times \mathfrak{V}_{\Delta,0} \) with the irreducible fiber \( \mathfrak{V}_{\Delta,0} = \mathfrak{V}_{\Delta,0}/\mathfrak{V}_{\Delta,0} \). As a result, the conformally invariant equation \( 2.16 \) corresponds to the irreducible \( \mathfrak{o}(M,2) \)-module \( \mathfrak{V}_{\Delta,0} \), thus being primitive.

More generally, the generalized Verma module \( \mathfrak{V}_{\Delta,0} \) is reducible for \( \Delta = \frac{1}{2} M - n \). Starting from \( \mathfrak{V}_{\Delta,0} \) one obtains the conformal equation \( \Box^n C(x) = 0 \) associated with \( \mathfrak{V}_{\Delta,0} = \mathfrak{V}_{\Delta,0}/\mathfrak{V}_{\Delta,0} \).

### 2.2. Conformal Spinor.

Massless Dirac equation admits an analogous reformulation. Let the module \( \mathfrak{V}_{\Delta,1/2} \) be generated by \( \mathfrak{K}^n \) from the spinor module of the \( \mathfrak{o}(M, 1 - 1) \) subalgebra with the basis elements \( |\Delta, 1/2\rangle^\alpha (\alpha = 1, \ldots, 2^{[M/2]} \) is the spinor index)

\[
|L^{nm}|\Delta, 1/2\rangle^\alpha = \frac{1}{4} (\gamma^n \gamma^m - \gamma^m \gamma^n)^\alpha _\beta |\Delta, 1/2\rangle^\beta .
\]

(2.18)

Here \( \gamma^{\alpha \beta} \) are gamma matrices

\[
\gamma^{n \beta} \gamma^{m \gamma} + \gamma^{m \beta} \gamma^{n \gamma} = (\gamma^n \gamma^m + \gamma^m \gamma^n)\alpha _\beta = 2 \eta^{nm} \delta^\alpha _\beta .
\]

(2.19)

The covariant constancy condition \( 1.0 \) imposed on the field

\[
|\Phi_{\Delta,1/2}(x)\rangle = \sum_{l=0}^\infty \frac{1}{l!} C_{m_1 \ldots m_l, \alpha}(x) \mathfrak{K}^{m_1} \ldots \mathfrak{K}^{m_l} |\Delta, 1/2\rangle^\alpha ,
\]

(2.20)

(i.e. on the section of the bundle \( \mathbb{R}^M \times \mathfrak{V}_{\Delta,1/2} \)) is equivalent to the system of equations

\[
\partial_n C_{m_1 \ldots m_{l-1}, \alpha}(x) + 2(\Delta + l - 1) C_{nm_{l-1}, \alpha}(x) - (l - 1) \eta_{nm} C_{k_{m_2 \ldots m_{l-1}}, \alpha}(x) + \\
+ \frac{1}{2} (\gamma^n \gamma_n - \gamma_n \gamma_n)^\beta _\alpha C_{qm_1 \ldots m_{l-1}, \beta}(x) = 0 , \quad l \geq 1 .
\]

(2.21)

Whenever \( \Delta \) is not half-integer, the system \( 2.21 \) just expresses all higher rank spinor–tensors in terms of higher derivatives of \( C_\alpha(x) \). For example, from \( 2.21 \) it follows that \( l = 1 \)

\[
\gamma^{\alpha \beta} \left( \partial_n C_\alpha(x) + (2\Delta - M + 1) C_{\alpha, \alpha}(x) \right) = 0 .
\]

(2.22)

For \( \Delta = (M-1)/2 \) the coefficient in front of \( C_{n, \alpha}(x) \) vanishes and we arrive at the massless Dirac equation for \( C_\alpha(x) \)

\[
\gamma^{\alpha \beta} \partial_n C_\alpha(x) = 0 .
\]

(2.23)
Other equations of the system (2.21) with $\Delta = (M - 1)/2$ express higher rank spinor–tensors in terms of higher derivatives of $C_{\alpha}(x)$ and $\gamma^{\alpha\beta}C_{\alpha\beta}(x)$.

Algebraically, the situation is analogous to the case of the Klein–Gordon equation. For $\Delta = (M - 1)/2$ the module $\mathfrak{V}_{\Delta,1/2}$ is reducible. It contains the submodule $\mathfrak{V}_{(M-1)/2,1/2} \subset \mathfrak{V}_{(M-1)/2,1/2}$ generated by $\mathcal{K}_n$ from the singular vectors

$$|s\rangle^\alpha = \gamma_m \alpha \mathcal{K}^m |(M - 1)/2, 1/2\rangle^\beta.$$  \hspace{1cm} (2.24)

Setting $\gamma^{\alpha\beta}C_{\alpha\beta}(x) = 0$ is equivalent to the restriction to the subbundle $\mathbb{R}^M \times \mathfrak{V}_{(M-1)/2,1/2}$, where the irreducible module $\mathfrak{V}_{(M-1)/2,1/2} = \mathfrak{V}_{(M-1)/2,1/2}/\mathfrak{V}_{(M-1)/2,1/2}$ corresponds to the primitive conformal equation (2.23).

2.3. **Conformal p-Forms.** Consider a trivial bundle $\mathbb{R}^M \times \mathfrak{V}_{\Delta,p}$, where the module $\mathfrak{V}_{\Delta,p}$ is induced from the rank $p$ ($p \leq M$) totally antisymmetric tensor module of $\mathfrak{o}(M - 1, 1)$ with the basis $|\Delta, p\rangle^{k_1...k_p}$

$$\mathcal{L}_{nm}|\Delta, p\rangle^{k_1...k_p} = p\delta^{[k_1}_{n} |\Delta, p\rangle_{m}^{k_2...k_p} - p\delta^{[k_1}_{m} |\Delta, p\rangle_{n}^{k_2...k_p}.$$  \hspace{1cm} (2.25)

Here square brackets imply antisymmetrization over indices denoted by the same letter

$$\delta^{[k_1}_{n} |\Delta\rangle_{m}^{k_2...k_p} = \frac{1}{p} (\delta^{[k_1}_{n} |\Delta\rangle_{m}^{k_2...k_p} - \delta^{[k_2}_{n} |\Delta\rangle_{m}^{k_1...k_p} + \ldots).$$  \hspace{1cm} (2.26)

Consider a section $|\Phi_{\Delta,p}(x)\rangle$ of the bundle $\mathbb{R}^M \times \mathfrak{V}_{\Delta,p}$

$$|\Phi_{\Delta,p}(x)\rangle = \sum_{l=0}^{1} \frac{1}{l!} C_{m_1...m_l;k_1...k_p}(x) \mathcal{K}^{m_1} \cdots \mathcal{K}^{m_l} |\Delta, p\rangle^{k_1...k_p},$$  \hspace{1cm} (2.27)

where the tensor $C_{m_1...m_l;k_1...k_p}(x)$ is totally symmetric in the indices $m$ and totally antisymmetric in the indices $k$. (The semicolon separates the groups of totally symmetric and antisymmetric indices).

Equation (1.6) for the field $|\Phi_{\Delta,p}(x)\rangle$ amounts to

$$\partial_n C_{m_1...m_{l-1};k_1...k_p}(x) + 2(\Delta + l - 1)C_{n;m_{l-1};k_1...k_p}(x) - (l - 1)\eta_{n}(m_{l}C^{q} m_{l-1};k_1...k_p)(x) + \frac{2}{l}C_{m_1...m_{l-1};k_1...k_p}(x) - 2p\eta_{n}[k_1 C_{m_1...m_{l-1};k_1...k_p}^{q} k_2...k_p](x) = 0, \quad l \geq 1.$$  \hspace{1cm} (2.28)

The differential equations imposed by the system (2.28) depend on the conformal weight $\Delta$.

1. $\Delta \notin \frac{1}{2}\mathbb{Z}$.

(2.28) imposes no differential restrictions, just expressing all higher rank tensor fields in terms of derivatives of the field $C_{;k_1...k_p}(x)$.

2. $M$ is odd, $\Delta = p = 0, 1, \ldots, M - 1$ or $M$ is even, $\Delta = p = 0, \frac{M}{2} + 1, \frac{M}{2} + 2, \ldots, M - 1$.

In this case (2.28) imposes the closedness condition on the $\Delta$-form $\tilde{C}_{;k_1...k_{\Delta}}(x)$

$$\partial_{[k_{\Delta+1}} C_{;k_1...k_{\Delta}]}(x) = 0.$$  \hspace{1cm} (2.29)
Unfolded form of linear conformal equations in $M$-dimensions and . . .

and expresses all higher rank tensor fields in terms of derivatives of $C_{;k_1...k_\Delta}(x)$ and $C_{[k_{\Delta+1};k_1...k_\Delta]}(x)$. Actually, consider (2.28) at $l = 1$. We have

$$\partial_{n}C_{;k_1...k_p}(x) + 2\Delta C_{;n;k_1...k_p}(x) + 2pC_{[k_1;k_2...k_p]}(x) - 2p\eta_{[k_1}C_{q;k_2...k_p]}(x) = 0. \quad (2.30)$$

Total antisymmetrization of indices in (2.30) gives

$$\partial_{[k_p+1}C_{;k_1...k_p]}(x) + 2(\Delta - p)C_{;[k_p+1;k_1...k_p]}(x) = 0, \quad (2.31)$$

For $\Delta = p$ we obtain (2.29).

3. $M$ is odd, $\Delta = M - p = 0, 1, \ldots, M - 1$ or $M$ is even, $\Delta = M - p = 0, M/2 + 1, M/2 + 2, \ldots, M - 1$.

In this case (2.28) imposes the dual form of equation (2.29) implying that the polyvector $C^{;k_1...k_M-\Delta}(x)$ conserves

$$\partial_nC^{;nk_2...k_M-\Delta}(x) = 0. \quad (2.32)$$

Also (2.28) expresses all higher rank tensor fields in terms of derivatives of the fields $C^{;k_1...k_M-\Delta}(x)$ and $C_{q;k_2...k_M-\Delta}(x)$. Indeed, contracting indices in (2.30) with $\eta^{nk_1}$, one obtains (2.32) from

$$\partial^nC^{;nk_2...k_p}(x) + 2(\Delta + p - M)C^{;nk_2...k_p}(x) = 0. \quad (2.33)$$

4. $M$ is even, $\Delta = p = 1, 2, \ldots, M/2 - 1$.

In this case, (2.28) imposes on $C_{;k_1...k_\Delta}(x)$ equation (2.29) along with equation

$$\Box^{M/2-\Delta}\partial^nC^{;nk_2...k_\Delta}(x) = 0 \quad (2.34)$$

and expresses all higher rank tensor fields in terms of derivatives of the fields $C_{;k_1...k_\Delta}(x)$, $C_{[k_{\Delta+1};k_1...k_\Delta]}(x)$, and $C^{m_1...n_{M/2-\Delta}q_1...q_{k_2...k_{M-\Delta}}}(x)$.

5. $M$ is even, $\Delta = M - p = 1, 2, \ldots, M/2 - 1$.

Now (2.28) imposes on $C_{;k_1...k_{M-\Delta}}(x)$ equation (2.32) along with

$$\Box^{M/2-\Delta}\partial_{[k_{M-\Delta+1}}C_{;k_1...k_{M-\Delta}]}(x) = 0 \quad (2.35)$$

and expresses all higher rank tensor fields in terms of derivatives of the fields $C_{;k_1...k_{M-\Delta}}(x)$, $C^{m_1...n_{M/2-\Delta}q_1...q_{k_2...k_{M-\Delta}}}(x)$, and $C^{;qk_2...k_{M-\Delta}}(x)$. Note that system (2.32), (2.35) is dual to system (2.29), (2.34).

6. $M$ is even, $\Delta = p = \frac{M}{2}$.

In this case, the vacuum vectors $|M/2, M/2\rangle_{k_1...k_{M/2}}$ form a reducible $\mathfrak{o}(M,2)$-module. The irreducible parts are singled out by the additional (anti)selfduality conditions

$$|M/2, M/2\rangle_{\pm} \ = \ \pm \ i^{M^2/4} \ (M/2)! \ e_{p_1...p_{M/2}}^{k_1...k_{M/2}} |M/2, M/2\rangle_{\pm}^{p_1...p_{M/2}}, \quad (2.36)$$
which in the complex case can be imposed for any even space–time dimension. Equation (2.28) imposes primitive equation
\[ \partial^p C_{nk_1...k_{M/2}}(x) = 0 \] (2.37)
on the (anti)selfdual field \( C_{k_1...k_{M/2}}(x) \)
\[ C_{k_1...k_{M/2}}(x) = \pm \frac{i^{M/4}}{(M/2)!} e^{p_1...p_{M/2}}_{k_1...k_{M/2}} C_{p_1...p_{M/2}}(x) \] (2.38)
and expresses all higher rank tensor fields in terms of derivatives of the fields \( C_{k_1...k_{M/2}}(x) \) and \( C^{q k_1...k_{M/2}}(x) \).

Vanishing coefficients in front of higher tensors in (2.31) and (2.33) imply the appearance of the singular vectors
\[ |s\rangle^{k_1...k_{\Delta+1}} = \mathcal{K}^{k_1|\Delta, \Delta}^{k_2...k_{\Delta+1}} \] (2.39)
\[ |s\rangle^{k_1...k_{M-\Delta-1}} = \mathcal{K}^{k_1\Delta, M-\Delta}^{nk_1...k_{M-\Delta-1}} \] (2.40)
in \( \mathfrak{V}_{\Delta,p} \) for \( \Delta = p = 0, \ldots, M-1 \) and \( \Delta = M - p = 0, \ldots, M-1 \), respectively. These singular vectors induce proper submodules \( \mathfrak{V}'_{\Delta,\Delta} \subset \mathfrak{V}_{\Delta,\Delta} \) and \( \mathfrak{V}'_{\Delta,M-\Delta} \subset \mathfrak{V}_{\Delta,M-\Delta} \). In the cases 2 and 3 the quotients \( \mathfrak{Q}_{\Delta,\Delta} = \mathfrak{V}_{\Delta,\Delta}/\mathfrak{V}'_{\Delta,\Delta} \) and \( \mathfrak{Q}_{\Delta,M-\Delta} = \mathfrak{V}_{\Delta,M-\Delta}/\mathfrak{V}'_{\Delta,M-\Delta} \) are irreducible and, therefore, equations (2.29) and (2.32) are primitive. In the cases 4 and 5 the modules \( \mathfrak{Q}_{\Delta,\Delta} \) and \( \mathfrak{Q}_{\Delta,M-\Delta} \) are reducible. They contain submodules \( \mathfrak{V}'_{\Delta,\Delta} \subset \mathfrak{Q}_{\Delta,\Delta} \) and \( \mathfrak{V}'_{\Delta,M-\Delta} \subset \mathfrak{Q}_{\Delta,M-\Delta} \) generated from the subversors
\[ |s'\rangle^{k_1...k_{\Delta-1}} = (\mathcal{K}_n \mathcal{K}^n)^{M/2-\Delta} \mathcal{K}^{m\Delta, \Delta}^{mk_1...k_{\Delta-1}} \] (2.41)
\[ |s'\rangle^{k_1...k_{M-\Delta+1}} = (\mathcal{K}_n \mathcal{K}^n)^{M/2-\Delta} \mathcal{K}^{k_1|\Delta, \Delta}^{k_2...k_{M-\Delta+1}} \] (2.42)
respectively. The quotients \( \mathfrak{Q}'_{\Delta,\Delta} = \mathfrak{Q}_{\Delta,\Delta}/\mathfrak{V}'_{\Delta,\Delta} \) and \( \mathfrak{Q}'_{\Delta,M-\Delta} = \mathfrak{Q}_{\Delta,M-\Delta}/\mathfrak{V}'_{\Delta,M-\Delta} \) are irreducible and systems (2.29), (2.31) and (2.32), (2.35) are primitive. Note that in the cases 4 and 5 the systems (2.29) and (2.32) alone are also conformally invariant but non-primitive.

In the case 6 the singular vector (2.39) coincide (modulo a sign) with the singular vector (2.40). This vector contained in both generalized Verma modules \( \mathfrak{V}_{M/2,2}^\pm \) and \( \mathfrak{V}_{M/2,2}^- \) generated from the selfdual and the antiselfdual vacuum Lorentz representations correspondingly. The quotients \( \mathfrak{Q}_{M/2,2} = \mathfrak{V}_{M/2,2}^+/\mathfrak{V}_{M/2,2}^- \) are irreducible and, therefore, system (2.37), (2.38) is primitive.

2.4. \( M = 4 \) Electrodynamics. Primitive conformally invariant equations constructed with the use of irreducible conformal modules are the simplest ones in the sense that it is impossible to impose any stronger conformally invariant equations that admit nontrivial solutions.
Unfolded form of linear conformal equations in $M$-dimensions and . . .

As follows from the examples 4 and 5 in section 2.3, non-primitive equations not necessarily reduce to a set of independent primitive subsystems.

A somewhat trivial example of a non-primitive system is provided by the case 6 in section 2.3 with the relaxed (anti)selfduality condition (2.36). Namely, consider the module $\mathfrak{V}_{M/2,M/2}$ induced from the reducible vacuum $|M/2,M/2\rangle^{k_1,...,k_{M/2}}$. It contains both singular vectors (2.39) and (2.41). Thus the equation (2.28) imposes the system (2.29), (2.32) on the field $C_{\mathfrak{M}/2,M/2}(x)$ and expresses all higher rank tensor fields in terms of derivatives of the fields $C_{k_1,...,k_{M/2}}(x)$, $C_{[k_{M/2+1};k_1,...,k_{M/2}]}(x)$, and $C^{\eta|q}k_2...k_{M/2}(x)$. This system is non-primitive because it reduces to the combination of the independent subsystems for selfdual and anti-selfdual parts. For $M = 4$ it coincides with the free Maxwell equations formulated in terms of field strengths.

A less trivial important example of a nontrivial non-primitive system, which allows us to illustrate the idea of the general construction is provided by the potential formulation of $4$ electrodynamics. Consider the module $\mathfrak{V}_{4}$ generated from the vacuum vectors $|\Phi_A(x)\rangle = \sum_{l=0}^{1} \frac{1}{l!} A_{m_1...m_l;k}(x)K^{m_1}...K^{m_l}|A\rangle^k$, $m,k = 1,...,4$ (2.43) encodes the following differential equations on $A_{k}(x)$:

\[
\begin{align*}
\partial_{[n}A_{k]}(x) &= 0, \\
\Box \partial^{k}A_{k}(x) &= 0. 
\end{align*}
\] (2.44)

(2.45)

Let us extend the irreducible module $\mathfrak{J}_A$ to a module $\mathfrak{E}_{A,F}$ by “gluing” the module $\mathfrak{K}_{F} = \mathfrak{Q}_{2,2+} \oplus \mathfrak{Q}_{2,2-}$ (see explanation to the case 6 at the end of section 2.3) to $\mathfrak{J}_A$ as follows. The module $\mathfrak{E}_{A,F}$ is generated from the vacuum vectors $|A\rangle^k$ and $|F\rangle^{k_1k_2}$ of the modules $\mathfrak{V}_A = \mathfrak{Q}_{1,1}$ and $\mathfrak{V}_F = \mathfrak{Q}_{2,2}$, respectively, with the following additional relations imposed

\[
\begin{align*}
K^{[n}|A\rangle^k &= 0, & K^{m}K^{m}K^{k}|A\rangle^k &= 0, \\
K^{[n}|F\rangle^{k_1k_2} &= 0, & K^{n}|F\rangle^{nk} &= 0, \\
\mathcal{P}^{n}|F\rangle^{k_1k_2} &= -\gamma^{n[k_1}|A\rangle^{k_2].
\end{align*}
\] (2.46)

(2.47)

(2.48)

Here the conditions (2.46) and (2.47) single out $\mathfrak{J}_A$ and $\mathfrak{K}_{F}$ from the generalized Verma modules $\mathfrak{V}_A$ and $\mathfrak{V}_F$, respectively. The condition (2.48) “glues” the modules $\mathfrak{J}_A$ and $\mathfrak{K}_{F}$ into $\mathfrak{E}_{A,F}$.

Consider the section

\[
|\Phi_{A,F}(x)\rangle = \sum_{l=0}^{1} \frac{1}{l!} A_{m_1...m_l;k}(x)K^{m_1}...K^{m_l}|A\rangle^k + \sum_{l=0}^{1} \frac{1}{l!} F_{m_1...m_l;k_1k_2}(x)K^{m_1}...K^{m_l}|F\rangle^{k_1k_2}
\] (2.49)
of the bundle $\mathbb{R}^4 \times \mathfrak{E}_{A,F}$. The covariant constancy condition $\mathcal{D}|\Phi_{A,F}(x)| = 0$ amounts to the infinite differential system

$$
\partial_n A_{m_1...m_{l-1};k}(x) + 2l A_{m_1...m_{l-1};k}(x) - (l - 1) \eta_{n(m_1 A_q m_2...m_{l-1});k}(x) + 2 A_{m_1...m_{l-1};kn}(x) - 2 \eta_{nk} A_{m_1...m_{l-1};q}(x) - F_{m_1...m_{l-1};nk}(x) = 0,
$$

(2.50)

$$
\partial_n F_{m_1...m_{l-1};k_{1}k_{2}}(x) + 2(l + 1) F_{m_1...m_{l-1};k_{1}k_{2}}(x) - (l - 1) \eta_{n(m_1 F_q m_2...m_{l-1});k_{1}k_{2}}(x) + 4 F_{m_1...m_{l-1};k_{1}k_{2}}(x) - 4 \eta_{nk} F_{m_1...m_{l-1};q,k}(x) = 0
$$

(2.51)

for $l = 1, 2, \ldots$. The subsystem (2.51) coincides with the system (2.28) for $M = 4$ and $\Delta = p = 2$. It expresses all higher components $F_{m_1...m_{l};k_{1}k_{2}}(x)$ via the higher derivatives of the field $F_{k_{1}k_{2}}(x)$ (note that components $F^q_{t_{l}k_{2}}(x)$ and $F_{[k_{1}k_{2}]}(x)$ are set to zero in the bundle $\mathbb{R}^4 \times \mathfrak{E}_{A,F}$ due to the relation (2.47)) and imposes Maxwell equations on the field strength 2-form $F_{k_{1}k_{2}}(x)$

$$
\partial_n F_{k_{1}k_{2}}(x) = 0,
$$

(2.52)

$$
\partial^a F_{nk}(x) = 0.
$$

(2.53)

The subsystem (2.50) is a deformation of the system (2.28) for $\mathfrak{J}_A$ by the additional terms containing the fields $F_{m_1...m_{l};k_{1}k_{2}}(x)$ resulting from the “gluing” condition (2.28) which links the vacuums $|A|^k$ and $|F|^k_{k_{1}k_{2}}$. The system (2.50) expresses all higher fields $A_{m_1...m_{l};k}(x)$ ($l \geq 1$) via the higher derivatives of $A_{k}(x)$ (in $\mathbb{R}^4 \times \mathfrak{E}_{A,F}$ components $A_{[k_{2};k_{1}]}(x) = 0$ and $A^n_{q,q}(x) = 0$ due to (2.46)) and also imposes the differential equation (2.47) on $A_{k}(x)$ and the constraint

$$
\partial_{[k_{1}} A_{k_{2}]}(x) = F_{k_{1}k_{2}}(x)
$$

(2.54)

on $F_{k_{1}k_{2}}(x)$. The constraint (2.54) replaces the closedness condition (2.44) for the potential 1-form $A_{k}(x)$. The point is that the singular vector $|s|^k_{k_{1}k_{2}} = K^{[k_{1}][1,1]}_{k_{2}}$ from the module $\mathfrak{M}_A$ responsible for (2.44) is “glued” in the module $\mathfrak{E}_{A,F}$ by the field $F_{k_{1}k_{2}}(x)$ in (2.48). As a result, the field $F_{k_{1}k_{2}}(x)$ replaces zero on the right hand side of (2.44) giving rise to the constraint (2.54), which identifies $A_{k}(x)$ with the potential for the field strength $F_{k_{1}k_{2}}(x)$.

Thus the infinite system (2.50) and (2.51) provides the potential formulation of $M = 4$ electrodynamics (2.52), (2.53) and (2.54) and (2.48) along with infinitely many constraints on the auxiliary fields $A_{m_1...m_{l};k}(x)$ and $F_{m_1...m_{l};k_{1}k_{2}}(x)$ for $l \geq 1$. Equation (2.45) is the conformally invariant gauge condition, considered originally in [21, 25]. The system (2.52), (2.53), (2.54) and (2.48) is non-primitive. Its primitive reduction results from the condition $F_{k_{1}k_{2}}(x) = 0$.

The module $\mathfrak{E}_{A,F}$ can be further extended by the module $\mathcal{J}_J = \mathfrak{M}_{3,1}/\mathfrak{P}_{3,1}$ (see explanation to the case 3 at the end of section 2.3) to a module $\mathfrak{E}_{A,F,J}$ as follows. $\mathfrak{E}_{A,F,J}$ is generated from the totally antisymmetric vacua $|A|^k$, $|F|^k_{k_{1}k_{2}}$ and $|J|^k$ with the properties (2.46), (2.47),
Unfolded form of linear conformal equations in $M$-dimensions and 

\[ \mathcal{K}_k | J \rangle^k = 0, \]  
\[ \mathcal{P}^n | J \rangle^k = -\frac{2}{3} | F \rangle^{nk}. \]

The covariant constancy condition for the section

\[ | \Phi_{A,F,J}(x) \rangle = \sum_{l=0}^{1} \frac{1}{l!} A_{m_1 \ldots m_l;k}(x) \mathcal{K}^{m_1} \ldots \mathcal{K}^{m_l} | A \rangle^k + \]
\[ + \sum_{l=0}^{1} \frac{1}{l!} F_{m_1 \ldots m_l;k_1k_2}(x) \mathcal{K}^{m_1} \ldots \mathcal{K}^{m_l} | F \rangle^{k_1k_2} + \sum_{l=0}^{1} \frac{1}{l!} J_{m_1 \ldots m_l;k}(x) \mathcal{K}^{m_1} \ldots \mathcal{K}^{m_l} | J \rangle^k \]

of the trivial bundle $\mathbb{R}^4 \times \mathfrak{E}_{A,F,J}$ contains several parts. The first one is the system (2.50), which gives rise to equations (2.54), (2.45). The second one is the system for the fields $J_{m_1 \ldots m_l;k}(x)$ of the form (2.58) with $M = 4$ and $\Delta = M - p = 3$. This system encodes equation

\[ \partial^k J_{ik}(x) = 0 \]  
(2.58)

on the field $J_{ik}(x)$ and expresses all the higher fields $J_{m_1 \ldots m_l;k}(x)$ ($l \geq 1$) in terms of higher derivatives of $J_{ik}(x)$ (in $\mathbb{R}^4 \times \mathfrak{E}_{A,F,J}$ component $J^q;_q(x) = 0$ due to (2.55)). The third part reads

\[ \partial_n F_{m_1 \ldots m_{l-1};k_1k_2}(x) + 2(l+1) F_{nm_1 \ldots m_{l-1};k_1k_2}(x) - (l-1) \eta_{n(m_1} F^q_{m_2 \ldots m_{l-1});k_1k_2}(x) + \]
\[ + 4 F_{m_1 \ldots m_{l-1};k_1nk_2}(x) - 4 \eta_{n[k_1} F_{m_1 \ldots m_{l-1};q} ; k_2](x) - \frac{2}{3} \eta_{n[k_1} J_{m_1 \ldots m_{l-1};k_2]}(x) = 0 \]  
(2.59)

for $l = 1, 2, \ldots$. It is a deformation of the system (2.51) with the additional terms containing $J_{m_1 \ldots m_l;k}(x)$, which result from the “gluing” condition (2.50). This system encodes the Bianchi identities (2.52) along with the second pair of Maxwell equations with external current

\[ \partial^n F_{nk}(x) = J_{ik}(x) \]  
(2.60)

and expresses $F_{m_1 \ldots m_l;k_1k_2}(x)$ for $l \geq 1$ via the derivatives of $F_{ik}(x)$. Thus the covariant constancy condition (1.6) for the bundle $\mathbb{R}^4 \times \mathfrak{E}_{A,F,J}$ encodes the non-primitive system of differential equations (2.52), (2.51), (2.45), (2.60) and (2.58). Note that analogous differential system was derived in [26] in terms of a 5-potential that transforms according to a non-decomposable representation of $SU(2, 2)$ (see also [24] and references therein).

This system admits two interpretations. The first one with $J_{m_1 \ldots m_l;k}(x)$ treated as independent fields restricted only by equations (1.6) is that it provides the off-mass-shell version of the Maxwell electrodynamics, which accounts for all differential consequences of the Bianchi identities. Another interpretation comes out when the field $J_{ik}(x)$ is a nonlinear combination of some other “matter” fields. In that case, equations (1.6) should be treated as Maxwell equations describing electromagnetic interactions of the matter fields. Clearly, for this to be
possible it is necessary to single out the module $\mathcal{J}_J$ from the tensor product of some other “matter modules” that leads to a nonlinear system describing electromagnetic interactions of matter fields from which the current $J_k(x)$ is built. The equation $(2.58)$ imposes the conservation condition on this current.

Finally let us note that to have a gauge invariant form of the Maxwell equations (i.e. to relax the gauge condition $(2.45)$) one has to consider the further extension $\mathcal{E}_{A,F,J,G}$ of the module $\mathcal{E}_{A,F,J}$ with the module $\mathcal{J}_G = \mathcal{Y}_{4,0}$. The module $\mathcal{E}_{A,F,J,G}$ is defined by the relations $(2.46)$, $(2.47)$, $(2.48)$, $(2.55)$, $(2.56)$ along with

$$P^n|G\rangle = -\frac{1}{16}K^mK_m|A\rangle^n,$$  \hspace{1cm} (2.61)$$

where $|G\rangle$ is the vacuum of the module $\mathcal{J}_G$. Consider a section

$$|\Phi_{A,F,J,G}(x)\rangle = \sum_{l=0}^{\infty} \frac{1}{l!} A_{m_1...m_l;k}(x)K^{m_1}...K^{m_l}|A\rangle^k + \sum_{l=0}^{\infty} \frac{1}{l!} F_{m_1...m_l;k_1k_2}(x)K^{m_1}...K^{m_l}|F\rangle^{k_1k_2} +$$ $$+ \sum_{l=0}^{\infty} \frac{1}{l!} J_{m_1...m_l;k}(x)K^{m_1}...K^{m_l}|J\rangle^k + \sum_{l=0}^{\infty} \frac{1}{l!} G_{m_1...m_l}(x)K^{m_1}...K^{m_l}|G\rangle$$  \hspace{1cm} (2.62)$$

of the bundle $\mathbb{R}^4 \times \mathcal{E}_{A,F,J,G}$. The consequences of the covariant constancy condition imposed on $(2.62)$ are analogous to those for the section $|\Phi_{A,F,J}(x)\rangle$ but with subsystem $(2.50)$ replaced with

$$\partial_n A_{m_1...m_{l-1};k}(x) + 2lA_{m_1...m_{l-1};k}(x) - (l-1)\eta_n(m_1A_{q}m_2...m_{l-1};k)(x) +$$ $$+ 2A_{m_1...m_{l-1};k;n}(x) - 2\eta_{nk}A_{m_1...m_{l-1};q}(x) - F_{m_1...m_{l-1};nk}(x) -$$ $$- \frac{1}{16}(l-1)(l-2)\eta_{nk}\eta_{m_1m_2G_{m_3...m_{l-1}}} = 0,$$  \hspace{1cm} (2.63)$$

and additional subsystem of the form $(2.28)$ with $M = 4$, $\Delta = M - p = 4$ for the fields $G_{m_1...m_l}(x)$. $G$-dependent terms in $(2.63)$ modify equation $(2.45)$ to

$$\Box \partial^k A_{j,k}(x) = G(x).$$  \hspace{1cm} (2.64)$$

Subsystem for the fields $G_{m_1...m_l}(x)$ expresses higher components of $G_{m_1...m_l}(x)$ ($l \geq 1$) in terms of derivatives of $G(x)$.

In section 4.7.2 we consider a generalization of this construction to a case of an almost arbitrary tensor structure of the field strength in any even space–time dimension $M > 2$. 

3. General Construction

Let \( \mathfrak{f} \) be a complex semi-simple\(^5\) Lie algebra with simple roots \( \Pi = (\alpha_0, \alpha_1, \ldots, \alpha_q) \). Then \( \mathfrak{f} \) is generated by elements \( \mathcal{H}_i, \mathcal{E}_i \) and \( \mathcal{F}_i \), \( 0 \leq i \leq q \) with the relations

\[
\begin{align*}
[\mathcal{H}_i, \mathcal{E}_j] &= A_{ij} \mathcal{E}_j, & [\mathcal{H}_i, \mathcal{F}_j] &= -A_{ij} \mathcal{F}_j, \\
[\mathcal{E}_i, \mathcal{F}_j] &= \delta_{ij} \mathcal{H}_j, & (\text{ad} \mathcal{E}_i)^{1-A_{ij}} \mathcal{E}_j &= 0, & (\text{ad} \mathcal{F}_i)^{1-A_{ij}} \mathcal{F}_j &= 0, \quad i \neq j,
\end{align*}
\]

(3.1) (3.2) (3.3)

where no summation over repeated indices is assumed and

\[
A_{ij} = \alpha_j(\mathcal{H}_i), \quad A_{i,j\neq 0} \leq 0, \quad A_{ii} = 2
\]

(3.4)
is the Cartan matrix. The transformation \( \tau \)

\[
\tau(\mathcal{E}_i) = \mathcal{F}_i, \quad \tau(\mathcal{F}_i) = \mathcal{E}_i, \quad \tau(\mathcal{H}_i) = \mathcal{H}_i
\]

(3.5)
generates the involutive antilinear antiautomorphism of \( \mathfrak{f} \) called the Chevalley involution.

Choose a subset of the set of simple roots \( \overline{\Pi} \subset \Pi \). Let \( \mathfrak{a}_{\overline{\Pi}} \subset \mathfrak{f} \) denote the semi-simple subalgebra generated by elements \( \mathcal{E}_i, \mathcal{F}_i, \mathcal{H}_i \) such that \( \alpha_i \in \overline{\Pi} \). \( \mathfrak{h}_{\overline{\Pi}} \) is the Cartan subalgebra of \( \mathfrak{a}_{\overline{\Pi}} \). Let \( \mathfrak{p}_{\overline{\Pi}} \) be the parabolic subalgebra with respect to \( \overline{\Pi} \), i.e. \( \mathfrak{p}_{\overline{\Pi}} \) is generated by \( \mathcal{H}_i, \mathcal{E}_i, \mathcal{F}_i \) with \( 0 \leq i \leq q \) and \( \mathcal{F}_i \) corresponding to simple roots in \( \overline{\Pi} \). Evidently, \( \mathfrak{a}_{\overline{\Pi}} \subset \mathfrak{p}_{\overline{\Pi}} \subset \mathfrak{f} \) for any \( \overline{\Pi} \). The parabolic subalgebra \( \mathfrak{p}_{\overline{\Pi}} \) admits the Levi–Maltsev decomposition \( \mathfrak{p}_{\overline{\Pi}} = \mathfrak{f}_{\Pi} \oplus \mathfrak{r}_{\overline{\Pi}} \), where \( \mathfrak{f}_{\Pi} = \mathfrak{h}_{\Pi} \oplus \mathfrak{a}_{\Pi} \) is the Levi factor of \( \mathfrak{p}_{\overline{\Pi}} \) and \( \mathfrak{r}_{\Pi} \) is the radical of \( \mathfrak{p}_{\overline{\Pi}} \). The linear space \( \mathfrak{f} \) can thus be decomposed into the direct sum \( \mathfrak{f} = \mathfrak{a}_{\Pi} \oplus \mathfrak{h}_{\Pi} \oplus \mathfrak{r}_{\Pi} \oplus \mathfrak{f}/\mathfrak{p}_{\overline{\Pi}} \). Let us choose a basis \( \left( \mathcal{L}_\beta, \mathcal{D}_I, \mathcal{P}_a, \mathcal{K}_a \right) \) of \( \mathfrak{f} \) such that the elements \( \mathcal{L}_\beta, \mathcal{D}_I, \mathcal{P}_a \) and \( \mathcal{K}_a \) form some bases in \( \mathfrak{a}_{\Pi}, \mathfrak{h}_{\Pi}, \mathfrak{r}_{\Pi} \) and \( \mathfrak{f}/\mathfrak{p}_{\overline{\Pi}} \) respectively. Note that the involution \( \tau \) maps \( \mathfrak{r}_{\Pi} \) to \( \mathfrak{f}/\mathfrak{p}_{\overline{\Pi}} \) and vice versa. Therefore, both for \( \mathcal{P}_a \) and for \( \mathcal{K}_a \) the index \( a \) takes values \( a = 0, \ldots, M - 1 \), where \( M = \dim(\mathfrak{r}_{\Pi}) = \dim(\mathfrak{f}/\mathfrak{p}_{\overline{\Pi}}) \).

Note that the commutation relations of \( \mathfrak{f} \) in the basis \( \left( \mathcal{L}_\beta, \mathcal{D}_I, \mathcal{P}_a, \mathcal{K}_a \right) \) have the following structure

\[
\begin{align*}
[\mathcal{L}, \mathcal{L}] &= \mathcal{L}, & [\mathcal{P}, \mathcal{P}] &= \mathcal{P}, & [\mathcal{K}, \mathcal{K}] &= \mathcal{K}, \\
[\mathcal{D}, \mathcal{L}] &= \mathcal{L}, & [\mathcal{L}, \mathcal{P}] &= \mathcal{P}, & [\mathcal{L}, \mathcal{K}] &= \mathcal{K}, \\
[\mathcal{P}, \mathcal{K}] &= \mathcal{L} + \mathcal{D} + \mathcal{P} + \mathcal{K}, & [\mathcal{D}, \mathcal{P}] &= \mathcal{P}, & [\mathcal{D}, \mathcal{K}] &= \mathcal{K}, \\
[\mathcal{D}, \mathcal{D}] &= 0,
\end{align*}
\]

(3.6)

where \( \mathcal{L}_\beta, \mathcal{D}_I, \mathcal{P}_a, \mathcal{K}_a \) are operators of generalized Lorentz transformations, dilatations, translations and special conformal transformations, respectively.

\(^5\)In fact the following consideration remains essentially the same for any Kac–Moody algebra.
Let $\mathfrak{M}$ be some (usually infinite dimensional) $\mathfrak{f}$-module with the following properties. $\mathfrak{M}$ decomposes into the direct sum of irreducible finite dimensional modules of $\mathfrak{l}_\Pi$. The action of the Cartan subalgebra $\mathfrak{h}_\Pi \subset \mathfrak{f}$ is diagonalizable in $\mathfrak{M}$. The action of the radical $\mathfrak{r}_\Pi$ is locally nilpotent in $\mathfrak{M}$, i.e. $\mathfrak{M}$ admits a filtration by $\mathfrak{l}_\Pi$-modules

$$\mathfrak{M}(0) \subset \mathfrak{M}(1) \subset \cdots \subset \mathfrak{M}(f) \subset \cdots \subset \mathfrak{M},$$

(3.7)

where a $\mathfrak{l}_\Pi$-module $\mathfrak{M}(f)$ is such that

$$(\mathfrak{r}_\Pi)^{f+1}\mathfrak{M}(f) \equiv 0,$$

(3.8)
i.e. a product of any $f+1$ elements from $\mathfrak{r}_\Pi$ annihilates any vector from $\mathfrak{M}(f)$.

The filtration (3.7) gives rise to the grading on $\mathfrak{M}$

$$\mathfrak{M} = \bigoplus_{l=0}^\infty \mathfrak{M}[l].$$

(3.9)

Here $\mathfrak{M}[0] = \mathfrak{M}(0)$ and $\mathfrak{M}[l]$ ($l \geq 1$) is the preimage of the quotient morphism

$$q : \quad \mathfrak{M}(l) \rightarrow \mathfrak{M}(l)/\mathfrak{M}(l-1)$$

(3.10)

$\mathfrak{M}[l] = q^{-1}\left(\mathfrak{M}(l)/\mathfrak{M}(l-1)\right)$, where $q^{-1}$ is a homomorphism of $\mathfrak{l}_\Pi$ modules satisfying $qq^{-1} = 1$. $q^{-1}$ is fixed uniquely provided that $\mathfrak{M}(l-1)$ does not contain $\mathfrak{l}_\Pi$-irreducible submodules isomorphic to some of the $\mathfrak{l}_\Pi$-irreducible submodules of $\mathfrak{M}(l)/\mathfrak{M}(l-1)$. Otherwise, to fix the arbitrariness in $q^{-1}$, an appropriate additional prescription is needed. We demand every $\mathfrak{M}[l]$, which is called level $l$ submodule of $\mathfrak{M}$, to form a finite dimensional module of $\mathfrak{l}_\Pi$. An element $r \in \mathfrak{r}_\Pi$ decreases the grading

$$r : \quad \mathfrak{M}[l] \rightarrow \mathfrak{M}[l-n(r)],$$

(3.11)

where $n(r) \geq 1$ is an integer. Note that if $\mathfrak{r}_\Pi$ is Abelian then $n(r) = 1$ for any $r \in \mathfrak{r}_\Pi$.

Let $\Xi$ be the Grassmann algebra on $\xi^n$, $n = 0, 1, \ldots, M-1$, $\xi^n \xi^m = -\xi^m \xi^n$ and $\xi^n$ are identified with space–time basis 1-forms. Consider the tensor product $\mathfrak{F} = \mathfrak{M} \otimes \Xi$. $\mathfrak{F}$ is bi-graded by the level of $\mathfrak{M}$ (3.9) and by the exterior form degree of $\Xi$

$$\mathfrak{F} = \bigoplus_{p=0}^{M} \bigoplus_{l=0}^\infty \mathfrak{F}[l],$$

(3.12)

where $\mathfrak{F}[l]$ is the space of $p$-forms taking values in $\mathfrak{M}[l]$. $\mathfrak{F}^p$ is the space of $p$-forms taking values in the whole module $\mathfrak{M}$.

Consider the trivial vector bundle $\mathcal{B} = \mathbb{R}^M \times \mathfrak{F}$ over $\mathbb{R}^M$

$$\mathfrak{F} \rightarrow \mathcal{B} \quad \downarrow$$

(3.13)
with the fiber $\mathcal{F}$. Let $\Gamma(\mathcal{B})$ denote the space of sections of $\mathcal{B}$. We define the covariant derivative in $\mathcal{B}$

$$
D = \xi^n \partial_n + \xi^n \omega_n^\beta(x) L_\beta + \xi^n \omega_n^a(x) \mathcal{P}_a + \xi^n \omega_n^I(x) \mathcal{D}_I,
$$

where $x^n$, $n = 0, 1, \ldots, M - 1$ are the space–time coordinates in $\mathbb{R}^M$. The connection 1-forms $\omega_n^\beta(x)$, $\omega_n^a(x)$ and $\omega_n^I(x)$ are chosen to satisfy the zero curvature equation (1.8). We require $\omega_n^a(x)$ to be non-degenerate

$$
\det |\omega_n^a(x)| \neq 0.
$$

In the rest of this paper we focus on the case of Abelian $\mathfrak{u}_\Pi$,

$$
[\mathcal{P}_a, \mathcal{P}_b] = 0.
$$

In this case (1.8) and (3.15) admit the simple solution

$$
D = \xi^n \partial_n + \xi^n \delta_n^a \mathcal{P}_a,
$$

with $\omega_n^a(x) = \omega_n^I(x) = 0$ and $\omega_n^a(x) = \delta_n^a$, where $\delta_n^a$ is identified with the flat space co-frame in Cartesian coordinates. Choosing different solutions of (1.8) allows one to analyse the problem in any other coordinates. Having fixed the flat frame in the form of Kronecker delta, in what follows we will not distinguish between the base and the fiber indices.

Let us introduce the exterior differential

$$
d = \xi^n \partial_n : \mathcal{F}^p \to \mathcal{F}^{p+1}
$$

and the operator

$$
\sigma_- = \xi^n \mathcal{P}_n : \mathcal{F}^p \to \mathcal{F}^{p+1}_{[p-1]}.
$$

We have

$$
D = d + \sigma_-.
$$

From (1.8), (3.18) and (3.19) it follows that the operators $d$ and $\sigma_-$ are nilpotent and anticommutative

$$
dd = 0, \quad \sigma_- \sigma_- = 0,
$$

$$
d\sigma_- + \sigma_- d = 0.
$$

Let $\mathfrak{c} \subset \mathcal{F}$ and $\mathfrak{c} \subset \mathfrak{c} \subset \mathcal{F}$ be the spaces of $\sigma_-$-closed and $\sigma_-$-exact forms, respectively,

$$
\sigma_- \mathfrak{c} = 0, \quad \mathfrak{c} = \sigma_- \mathcal{F}.
$$

The cohomology $H(\mathfrak{u}_\Pi, \mathcal{M})$ of $\mathfrak{u}_\Pi$ is the quotient $\mathfrak{c}/\mathfrak{c}$. Let $p$ be the quotient mapping

$$
p : \mathfrak{c} \to H(\mathfrak{u}_\Pi, \mathcal{M}).
$$

This mapping is a $\mathfrak{u}_\Pi$-homomorphism. We define the mapping

$$
p^{-1} : H(\mathfrak{u}_\Pi, \mathcal{M}) \to \mathfrak{c}
$$
such that $pp^{-1} = 1$ and $p^{-1}$ is a $L$-homomorphism. These requirements fix $p^{-1}$ uniquely provided that $e$ does not contain $L$-irreducible submodules isomorphic to some of the $L$-irreducible submodules of $c/e$. Otherwise, to fix the arbitrariness in $p^{-1}$, an appropriate additional prescription is needed. The space $\mathfrak{g}$ decomposes into the direct sum of $L$-modules

$$\mathfrak{g} = H \oplus e \oplus F. \quad (3.25)$$

Here $H$ denotes $p^{-1}(H(\mathfrak{m}; \mathfrak{m}))$, $e$ complements $H$ to $\mathfrak{g}$ and $F$ complements $e$ to $\mathfrak{g}$. The gradings $[3.12]$ of $\mathfrak{g}$ induces the gradings of $H$, $e$ and $F$. Let $H^p_\ell$, $e^p_\ell$ and $F^p_\ell$ denote corresponding homogeneous subspaces. Note that $H^0 = e^0 = \mathfrak{g}^0_{[0]}$ and thus $p^{-1}$ is identical in the sector of 0-forms.

Introduce the subbundle $b = \mathbb{R}^M \times H$ of the bundle $B$

$$H \longrightarrow b \quad \text{down to} \quad \mathbb{R}^M \quad (3.26)$$

with the fiber $H \subset \mathfrak{g}$. Let $\Gamma(b)$ denote the space of sections of $b$. Let a $p$-form $|\phi^p(x)\rangle \in \Gamma(b)$ be a section of $b$. Now we are in a position to formulate $L$-invariant differential equations on $|\phi^p(x)\rangle$ as the conditions for $|\phi^p(x)\rangle$ to admit a lift to a $p$-form $|\Phi^p(x)\rangle \in \Gamma(B)$ such that

$$D|\Phi^p(x)\rangle = 0,$$

$$|\Phi^p(x)\rangle|_b = |\phi^p(x)\rangle. \quad (3.27)$$

Here $|\Phi^p(x)\rangle|_b$ is the projection of $\mathfrak{g}$ to $H$ in the decomposition $[3.25]$. Call a section $|\Phi^p(x)\rangle \in \Gamma(B)$ D-horizontal if $D|\Phi^p(x)\rangle = 0$. Call a section $|\Phi^p(x)\rangle \in \Gamma(B)$ D-horizontal lift of $|\phi^p(x)\rangle \in \Gamma(b)$ if it satisfies $[3.27]$. Taking into account $[1.8]$, the equation $D|\Phi^p(x)\rangle = 0$ is invariant under the gauge transformation

$$\delta|\Phi^p(x)\rangle = D|e^{p-1}(x)\rangle,$$  

$$\delta|e^{p-1}(x)\rangle = D|\chi^{p-2}(x)\rangle,$$  

where $e^{p-1} \in \Gamma(B)$ is an arbitrary $p-1$-form. Note that for $p \geq 2$ $[3.28]$ is invariant under the second order gauge transformation

$$\delta|\chi^{p-2}(x)\rangle = D|\chi^{p-2}(x)\rangle,$$  

where $|\chi^{p-2}(x)\rangle$ is an arbitrary $p-2$-form. For $p \geq 3$ $[3.29]$ is invariant under the third order gauge transformation and so on.

We will distinguish between $T$ (trivial), $D$ (differential) and $A$ (algebraic) classes of gauge transformations with the gauge parameters $|e^{p-1}_T(x)\rangle = |\psi^{p-1}_T(x)\rangle + D|\chi^{p-2}_T(x)\rangle$, $|e^{p-1}_D(x)\rangle = |\psi^{p-1}_D(x)\rangle + D|\chi^{p-2}_D(x)\rangle$ and $|e^{p-1}_A(x)\rangle = |\psi^{p-1}_A(x)\rangle + D|\chi^{p-2}_A(x)\rangle$, respectively, with some $p-1$-forms $|\psi^{p-1}_T(x)\rangle \in e$, $|\psi^{p-1}_D(x)\rangle \in H$, $|\psi^{p-1}_A(x)\rangle \in F$. The ambiguity in the second-order gauge parameters $|\chi^{p-2}_T(x)\rangle$, $|\chi^{p-2}_D(x)\rangle$ and $|\chi^{p-2}_A(x)\rangle$ manifests the fact that the decomposition into the $T$, $D$, and $A$ gauge transformations is not unique. One can see, in particular,
that any $T$-transformation reduces to a linear combination of some $A$-transformation and $D$-transformation and can therefore be discarded. Indeed, let $|\epsilon^{p-1}_{T[l]}(x)| = \sigma_-|\chi^{p-2}_{T[l+1]}(x)|$ be a level-$l$ $T$-transformation parameter. Taking into account (3.21) one gets

$$\delta_T|\Phi^p(x)| = d|\epsilon^{p-1}_{T[l]}(x)| = -\sigma_-d|\chi^{p-2}_{T[l+1]}(x)| = -Dd|\chi^{p-2}_{T[l+1]}(x)|. \quad (3.30)$$

Decompose $-d|\chi^{p-2}_{T[l+1]}(x)|$ into a combination of level $l + 1$ $D$, $A$, and $T$ gauge parameters. If the resulting level $l + 1$ $T$-parameter is nonzero one applies the same procedure, and so on.

The roles of the $D$ and $A$ gauge transformations are as follows. The variation of $|\Phi^p(x)|$ under $D$-transformations is purely differential

$$\delta_D|\Phi^p(x)| = d|\epsilon^{p-1}_D(x)|. \quad (3.31)$$

$D$-transformations generalize the gradient transformations in electrodynamics and linearized diffeomorphisms in gravity. $A$-transformations are gauge transformations of the form

$$\delta_A|\Phi^p(x)| = d|\epsilon^{p-1}_A(x)| + \sigma_-|\epsilon^{p-1}_A(x)| \quad (3.32)$$

with a nonzero second term. These are analogous to the linearized local Lorentz transformations in gravity.

Now, following to [13], we prove that the existence of a $D$-horizontal lift (see (3.27)) is governed by $H^{p+1}(\mathfrak{m}, \mathfrak{m})$.

**Theorem 3.1.**

1. Let $|\phi^p(x)| \in \Gamma(\mathfrak{b})$ and let there exist $|\Phi^p(x)|_1$ and $|\Phi^p(x)|_2 \in \Gamma(\mathcal{B})$ that are $D$-horizontal lifts of $|\phi^p(x)|$. Then $|\Phi^p(x)|_1 - |\Phi^p(x)|_2 = \delta_A|\chi^{p-1}(x)|$ for some $|\chi^{p-1}(x)| \in \Gamma(\mathcal{B})$ (see (3.32)).

2. The two statements are equivalent
   
   (a) any section $|\phi^p(x)| \in \Gamma(\mathfrak{b})$ has a $D$-horizontal lift to a $|\Phi^p(x)| \in \Gamma(\mathcal{B})$

   (b) $H^{p+1}(\mathfrak{m}, \mathfrak{m}) = 0$.

3. If $H^{p+1}(\mathfrak{m}, \mathfrak{m}) \neq 0$, there exists a system of differential equations

$$R|\phi^p(x)| = 0 \quad (3.33)$$

such that any solution of (3.33) admits a $D$-horizontal lift to a $|\Phi^p(x)| \in \Gamma(\mathcal{B})$ and all $|\phi^p(x)| \in \Gamma(\mathfrak{b})$ admitting such a lift satisfy (3.33).

**Proof.** Let us look for a lift $|\Phi^p(x)|$ in the form

$$|\Phi^p(x)| = |\varphi[0](x)| + |\varphi[1](x)| + |\varphi[2](x)| + \cdots, \quad (3.34)$$

where $|\varphi[0](x)| \in \mathfrak{F}^p_{[0]}$. The condition $|\Phi^p(x)|_b = |\phi^p(x)|$ fixes the first term in this decomposition $|\varphi[0](x)|_b = |\phi^p(x)| \cap \mathfrak{F}^p_{[0]}$ modulo a $\sigma_-$-exact form $|\varphi[0](x)|_e \in \mathfrak{e}^p_{[0]}$. The freedom in
\( |\varphi_0(x)\rangle \in \mathfrak{c}^p_{[0]} \) is a consequence of \( A \)-gauge symmetry, i.e. \( |\varphi_0(x)\rangle \) is reconstructed modulo an \( A \)-gauge part (which, of course, also contributes to \( |\varphi_1(x)\rangle_H \)).

Suppose that \( H^{p+1}(\mathfrak{c}_M) \) is trivial, i.e. \( \mathfrak{c}^{p+1} = \mathfrak{c}^{p} \). To reconstruct \( |\Phi^p(x)\rangle \) we use the following step-by-step procedure. The zero level part of (3.27) reads

\[
\frac{d}{dx}|\varphi_0(x)\rangle + \sigma_- |\varphi_1(x)\rangle = 0, \tag{3.35}
\]
\[
|\varphi_1(x)\rangle_b = |\phi^p(x)\rangle \cap \mathfrak{f}^p_{[1]}. \tag{3.36}
\]

Since \( |\varphi_0(x)\rangle \) has the lowest grading, it is \( \sigma_- \)-closed. \( \frac{d}{dx}|\varphi_0(x)\rangle \) is also \( \sigma_- \)-closed because \( d\sigma_- + \sigma_- d = 0 \). Since \( H^{p+1}(\mathfrak{c}_M) \) is trivial, \( \frac{d}{dx}|\varphi_0(x)\rangle \) is \( \sigma_- \)-exact

\[
\frac{d}{dx}|\varphi_0(x)\rangle = \sigma_- |\chi_1(x)\rangle \tag{3.37}
\]

for some \( |\chi_1(x)\rangle \). Setting \( |\varphi_1(x)\rangle = -|\chi_1(x)\rangle \) we solve the equation (3.35) modulo an arbitrary \( \sigma_- \)-closed form \( |\varphi_1(x)\rangle \_\in \mathfrak{c}^p_{[1]} \). The condition (3.36) fixes \( |\varphi_1(x)\rangle \_ \) modulo an arbitrary \( \sigma_- \)-exact form \( |\varphi_1(x)\rangle \_ \in \mathfrak{c}^p_{[1]} \), which parameterizes the level 1 restriction of some \( A \)-gauge part with level 2 gauge parameter. As a result, \( |\varphi_1(x)\rangle \_ \in \mathfrak{f}^p_{[1]} \) is expressed via the first derivatives of \( |\phi^p(x)\rangle \cap \mathfrak{f}^p_{[0]} \) and via \( |\phi^p(x)\rangle \cap \mathfrak{f}^p_{[1]} \) modulo an arbitrary \( A \)-gauge part. The first level part of (3.27)

\[
\frac{d}{dx}|\varphi_1(x)\rangle + \sigma_- |\varphi_2(x)\rangle = 0, \tag{3.38}
\]
\[
|\varphi_2(x)\rangle_b = |\phi^p(x)\rangle \cap \mathfrak{f}^p_{[2]} \tag{3.39}
\]

is considered analogously. \( \frac{d}{dx}|\varphi_1(x)\rangle \) is \( \sigma_- \)-closed because \( \sigma_- \frac{d}{dx}|\varphi_1(x)\rangle = -d\sigma_- |\varphi_1(x)\rangle = d^2|\varphi_0(x)\rangle = 0 \). Introducing \( |\chi_2(x)\rangle \_ \in \mathfrak{f}^p_{[2]} \) such that \( \frac{d}{dx}|\varphi_1(x)\rangle = \sigma_- |\chi_2(x)\rangle \) and setting \( |\varphi_2(x)\rangle = -|\chi_2(x)\rangle \_ \) we solve equation (3.38) modulo an arbitrary \( \sigma_- \)-closed form \( |\varphi_2(x)\rangle \_ \in \mathfrak{c}^p_{[2]} \). The condition (3.39) fixes \( |\varphi_2(x)\rangle \_ \) modulo an arbitrary \( \sigma_- \)-exact form \( |\varphi_2(x)\rangle \_ \in \mathfrak{c}^p_{[2]} \), which parameterizes the level 2 restriction of some \( A \)-gauge part with level 3 gauge parameter. As a result \( |\varphi_2(x)\rangle \_ \) is expressed via the second derivatives of \( |\phi^p(x)\rangle \cap \mathfrak{f}^p_{[0]} \), via the first derivatives of \( |\phi^p(x)\rangle \cap \mathfrak{f}^p_{[1]} \) and via the \( |\phi^p(x)\rangle \cap \mathfrak{f}^p_{[2]} \) modulo some \( A \)-gauge terms. Repetition of this procedure reconstructs the lift \( |\Phi^p(x)\rangle \) in the form (3.34) with \( |\varphi_1(x)\rangle \_ \) expressed in terms of derivatives of \( |\phi^p(x)\rangle \) modulo an \( A \)-gauge part.

Suppose now that \( H^{p+1}(\mathfrak{c}_M) \) is nontrivial. Then it decomposes into a sum of some definite grade nonzero subspaces

\[
H^{p+1}(\mathfrak{c}_M) = H^{p+1}_{[1]}(\mathfrak{c}_M) \oplus H^{p+1}_{[2]}(\mathfrak{c}_M) \oplus \cdots, \tag{3.40}
\]

where \( 0 \leq l_1 < l_2 < \cdots \). Carrying out the first \( l_1 \) steps of the described procedure we solve (3.27) up to the \( l_1 - 1 \)-th level, expressing all \( |\varphi_l(x)\rangle \) with \( 1 \leq l \leq l_1 \) via derivatives. 

In addition, equation (3.41), (3.42) expresses $A^\parallel$ to vanish. This imposes some differential equations on us construct a section $A^\parallel$. Because $\sigma$ and impose some additional differential equations of orders not higher than Equation (3.41) imposes however a stronger condition that $d\varphi_{l1}(x)$ be $\sigma$-exact thus requiring those combinations of $d\varphi_{l1}(x)$ that belong to the cohomology class $H^{p+1}(\mathcal{M})$ to vanish. This imposes some differential equations on $|\phi^p(x)|$ of orders not higher than $l_1$

$$R_{[l1]}|\phi^p(x)| = 0.$$ (3.43)

In addition, equation (3.41), (3.42) expresses $|\varphi_{l1+1}(x)|$ via derivatives of $|\phi^p(x)|$ modulo an arbitrary $A$-gauge part $|\varphi_{l1+1}(x)| \in e_{l1+1}^p$.

Solving further (3.27) level by level we fix $|\varphi_{l1+1}(x)|, \ldots, |\varphi_{l2}(x)|$ modulo an arbitrary $A$-gauge part. At the level $l_2$, equations

$$d|\varphi_{l2}(x)| + \sigma_-|\varphi_{l2+1}(x)| = 0$$ (3.44)

$$|\varphi_{l2+1}(x)| \bigr|_b = |\phi^p(x)| \cap \mathcal{F}^p_{l2+1}$$ (3.45)

fix $|\varphi_{l2+1}(x)|$ in terms of derivatives of $|\phi^p(x)|$ modulo an $A$-gauge part $|\varphi_{l2+1}(x)| \in e_{l2+1}^p$ and impose some additional differential equations of orders not higher than $l_2$

$$R_{[l2]}|\phi^p(x)| = 0.$$ (3.46)

Repetition of this procedure reconstructs modulo an $A$-gauge part a lift $|\Phi^p(x)|$ in the form (3.41) for $|\phi^p(x)|$ satisfying the system of differential equations

$$R_{[l1]}|\phi^p(x)| = 0,$$

$$R_{[l2]}|\phi^p(x)| = 0,$$

$$\ldots$$ (3.47)

To show that the system (3.47) is necessarily nontrivial if $H^{p+1}(\mathcal{M})$ is nonzero, let us construct a section $|\phi^p(x)| \in \Gamma(\mathcal{F})$ that does not satisfy (3.47). Let us choose some $|\tilde{\psi}_{l1}(x)| \in H_{[l1]}^{p+1}$ such that $d|\tilde{\psi}_{l1}(x)| = 0$ (for example one can choose $|\tilde{\psi}_{l1}(x)| \in H_{[l1]}^{p+1}$ to be $x$-independent). Then $|\tilde{\psi}_{l1}(x)| = d|\tilde{\varphi}_{l1}(x)|$ for some $|\tilde{\varphi}_{l1}(x)|$. Decompose $|\tilde{\varphi}_{l1}(x)| = |\tilde{\varphi}_{l1}(x)|_H + |\tilde{\varphi}_{l1}(x)|_e + |\tilde{\varphi}_{l1}(x)|_F$ in accordance with (3.27). Consider now the $l_1-1$-th level part of equation (3.27)

$$d|\tilde{\varphi}_{l1-1}(x)| + \sigma_-|\tilde{\varphi}_{l1}(x)| = 0.$$ (3.48)

Because $\sigma_-|\tilde{\varphi}_{l1}(x)|$ is $d$-closed ($d\sigma_-|\tilde{\varphi}_{l1}(x)| = -\sigma_-d|\tilde{\varphi}_{l1}(x)| = 0$) we can solve it for $|\tilde{\varphi}_{l1-1}(x)| = |\tilde{\varphi}_{l1-1}(x)|_H + |\tilde{\varphi}_{l1-1}(x)|_e + |\tilde{\varphi}_{l1-1}(x)|_F$. Repeating this "inverse" procedure we find $|\tilde{\varphi}_{l1}(x)|_H, \ldots, |\tilde{\varphi}_0(x)|_H$, arriving at the field $|\phi^p(x)| = |\tilde{\varphi}_0(x)|_H + \cdots + |\tilde{\varphi}_{l1}(x)|_H$
that solves (3.27) for the levels 0, 1 \ldots l_1 - 1 but satisfies the modified equation (3.27) with the nonzero right hand side proportional to \( \tilde{\psi}_{[l_1]}(x) \) \( \in H^{p+1}_{[l_1]} \) at the level \( l_1 \), thus violating (3.47).

**Remark 3.2.** If there exists a \( D \)-horizontal lift of 0-form \( |\psi^0(x)\rangle \) to a 0-form \( |\Phi^0(x)\rangle \in \Gamma(B) \) then it is unique.

**Proof.** Gauge symmetries (3.28) trivialize in the sector of 0-forms. \( \blacksquare \)

**Remark 3.3.** Consider the subbundle \( b' = \mathbb{R}^M \times H \oplus c \) of the bundle \( B \)

\[
H \oplus c \quad \longrightarrow \quad b' \\
\downarrow \\
\mathbb{R}^M
\]

(3.49)

with the fiber \( H \oplus c = c \subset \mathfrak{g} \). If there exists a \( D \)-horizontal lift of \( |\phi^p(x)\rangle \) to a \( p \)-form \( |\Phi(x)\rangle \) \( \in \Gamma(B) \) then it is unique.

**Proof.** Restriction to \( b' \) fixes some \( A \)-gauge. \( \blacksquare \)

**Remark 3.4.** Theorem 3.1 allows the following interpretation. Given equation (3.27), a section \( |\Phi^p(x)\rangle \) decomposes into \( |\Phi^p(x)\rangle = |\Phi^p(x)\rangle_H + |\Phi^p(x)\rangle_c + |\Phi^p(x)\rangle_F \). The subsection \( |\Phi^p(x)\rangle_H \) describes dynamical fields subject to some differential equations (3.33). Solutions of these differential equations are moduli of solutions of the equation (3.27). The part \( |\Phi^p(x)\rangle_F \) describes (usually infinite) set of fields expressed by the equation (3.27) via derivatives of the dynamical fields. The fields of this class are called auxiliary fields and the equations that express them are called constraints. The \( A \)-gauge symmetry (3.32) (generalized local Lorentz symmetry) allows one to get rid of \( \sigma_- \)-exact terms \( |\phi^p(x)\rangle_c \). The \( D \)-gauge symmetry (3.31) with the parameters in \( H^{p-1}(\mathfrak{g}, \mathfrak{m}) \) acts on the dynamical fields \( |\Phi^p(x)\rangle_H \) and is the gauge symmetry of equations (3.33).

**Remark 3.5.** According to (1.10) solutions of (3.27) are parameterized by the values of \( |\Phi^p(x)\rangle|_{x \in \epsilon(x_0)} \) at a neighborhood \( \epsilon(x_0) \) of any point \( x_0 \). This is because the equation (3.27) expresses all higher level \( (l \geq 1) \) components of \( |\Phi^p(x)\rangle \) via higher derivatives of \( |\phi^p(x)\rangle \). As a result, the fields \( |\phi^p(x)\rangle \) can be expressed modulo gauge symmetries in terms of \( |\Phi^p(x)\rangle|_{x \in \epsilon(x_0)} \) by virtue of the Taylor expansion.

In the rest of this paper we mostly confine ourselves to the sector of 0-forms, which turns out to be reach enough to reformulate any \( f_{\mathfrak{m}} \)-invariant linear differential system in the unfolded form by virtue of introducing appropriate auxiliary fields. In other words, for any \( f_{\mathfrak{m}} \)-invariant linear differential system \( R|\phi^0(x)\rangle = 0 \), there exists some \( f \)-module \( \mathfrak{m}_R \), which
gives rise to $R|\phi^0(x)\rangle = 0$ by virtue of the procedure described above\(^6\). Thus the problem of listing all linear $f_{\Pi}$-invariant differential systems is equivalent to the problem of calculating the cohomology $H^0(\tau_{\Pi}, M)$ and $H^1(\tau_{\Pi}, M)$ for any $f$-module $M$. An important subclass of such systems is formed by those associated with irreducible $M$.

**Definition 3.6.** A system of $f_{\Pi}$-invariant linear differential equations

$$
R|\phi^0(x)\rangle = 0
$$

(3.50)

is called primitive if the $f$-module $M_R$ corresponding to (3.50) as in Theorem 2.1 is irreducible.

Reducible modules can be treated as extensions of the irreducible ones. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be some irreducible $f$-modules. Consider a module $M$ defined by the exact sequence

$$
0 \longrightarrow \mathcal{I}_1 \longrightarrow M \longrightarrow \mathcal{I}_2 \longrightarrow 0. \quad (3.51)
$$

A trivial possibility is $M = \mathcal{I}_1 \oplus \mathcal{I}_2$. The non-primitive system corresponding to $\mathcal{I}_1 \oplus \mathcal{I}_2$ decomposes into two independent primitive subsystems

$$
R_{3,1}|\phi^0_{3,1}(x)\rangle = 0, \quad (3.52)
$$

$$
R_{3,2}|\phi^0_{3,2}(x)\rangle = 0, \quad (3.53)
$$

where $R_{3,1}, |\phi^0_{3,1}(x)\rangle$ and $R_{3,2}, |\phi^0_{3,2}(x)\rangle$ correspond to $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively. For some particular irreducible $\mathcal{I}_1$ and $\mathcal{I}_2$, a module $M = \mathcal{E}_{\mathcal{I}_1,\mathcal{I}_2}$ non-isomorphic to $\mathcal{I}_1 \oplus \mathcal{I}_2$ may also exist however. The non-primitive system corresponding to $\mathcal{E}_{\mathcal{I}_1,\mathcal{I}_2}$

$$
R_{\mathcal{E}_{\mathcal{I}_1,\mathcal{I}_2}}|\phi^0_{\mathcal{E}_{\mathcal{I}_1,\mathcal{I}_2}}(x)\rangle = 0
$$

(3.54)

contains the system (3.53) for the dynamical fields $|\phi^0_{\mathcal{I}_2}(x)\rangle$ associated with $M = \mathcal{I}_2$. The system (3.52) results from (3.54) at $|\phi^0_{\mathcal{I}_2}(x)\rangle = 0$, which means that the space of solutions of the non-primitive system (3.54) contains the invariant subspace of solutions of the system (3.52). In other words, the equations that contain $d|\phi^0_{\mathcal{I}_2}(x)\rangle$ are $|\phi^0_{\mathcal{I}_1}(x)\rangle$ independent, while those, that contain $d|\phi^0_{\mathcal{I}_1}(x)\rangle$, contain some terms with $|\phi^0_{\mathcal{I}_2}(x)\rangle$.

Further extensions of the types

$$
0 \longrightarrow \mathcal{I}_3 \longrightarrow M' \longrightarrow \mathcal{E}_{\mathcal{I}_1,\mathcal{I}_2} \longrightarrow 0
$$

(3.55)

or

$$
0 \longrightarrow \mathcal{E}_{\mathcal{I}_1,\mathcal{I}_2} \longrightarrow M'' \longrightarrow \mathcal{I}_3 \longrightarrow 0
$$

(3.56)

with indecomposable modules $M'$ and $M''$ can also be considered. As a result, all possible $f_{\Pi}$-invariant linear differential equations can be classified in terms of extensions of the primitive

---

\(^6\)Note that any equation $R|\phi^0(x)\rangle = 0$ can be rewritten in terms of 0-forms by converting indices of forms into tangent indices with the aid of the frame field. The formulation in terms of higher forms may be useful however for the analysis of nonlinear dynamics and will be discussed elsewhere.
equations. Some examples of nontrivial extensions are considered in section 2.4.2 and 4.7.3.

To summarize, the construction is as follows. To write down all \( f_{\Pi} \)-invariant homogeneous equations on a finite number of fields for a semi-simple Lie algebra \( f \) one has to classify all \( f \)-modules that are integrable with respect to parabolic subalgebra \( p_{\Pi} \subset f \) with the Abelian radical \( r_{\Pi} \). These consist of irreducible \( f \)-modules of this class and all their extensions. The unfolded form of the \( f_{\Pi} \)-invariant homogeneous equations has the form of the covariant constancy equation (1.6) for the 0-form section \( |\Phi_0(x)\rangle \) of the bundle \( B \). Dynamical fields form the 0-form section \( |\phi_0(x)\rangle \) of \( b \). Differential field equations on the dynamical fields are characterized by the cohomology \( H^1(r_{\Pi}, M) \), which is the linear space where the nontrivial left hand sides of the equations \( R|\phi_0(x)\rangle = 0 \) take their values. Since equation (1.6) is \( f \)-invariant, the equation \( R|\phi_0(x)\rangle = 0 \) is \( f \)-invariant as well, i.e. \( f \) maps its solutions to solutions. The construction is universal because any differential equations can be “unfolded” to some covariant constancy equation by adding enough (usually infinitely many) auxiliary fields expressed by virtue of the unfolded equations through derivatives of the dynamical fields \( |\phi_0(x)\rangle \). If the original system of differential equations is \( f \)-invariant, the corresponding unfolded equation is also \( f \)-invariant, and auxiliary fields together with the dynamical fields, span the space of sections of \( B \).

Now we are in a position to give the full list of conformally invariant systems of differential equations in \( \mathbb{R}^M \) (\( M \geq 3 \)).

4. CONFORMAL SYSTEMS OF EQUATIONS

We set \( f = \mathfrak{o}(M+2) \) with the commutation relations (2.1) (\( \mathfrak{o}(M+2) \sim \mathfrak{o}(M,2) \) for the complex case we focus on). The structure of simple roots \( \Pi \) for \( \mathfrak{o}(M+2) \) depends on whether \( M \) is odd or even. For \( M = 2q \), \( \mathfrak{o}(M+2) = D_{q+1} \) and \( \Pi \) is described by the Dynkin diagram

\[
\begin{array}{c}
\circ \circ \circ \circ \ldots \circ \\
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_{q-2} & \alpha_q
\end{array}
\]  \( (4.1) \)

For odd \( M = 2q + 1 \), \( \mathfrak{o}(M+2) = B_{q+1} \) and \( \Pi \) is described by the Dynkin diagram

\[
\begin{array}{c}
\circ \circ \circ \circ \circ \\
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_{q-1} & \alpha_q
\end{array}
\]  \( (4.2) \)

In both cases we choose \( \Pi = (\alpha_1, \ldots, \alpha_q) \) and hence \( p_{\Pi} = \mathfrak{i}o(M) \oplus \mathfrak{o}(2) = \mathfrak{o}(M) \oplus \mathfrak{o}(2) \oplus \mathfrak{t}(M) \) where \( l_{\Pi} = \mathfrak{o}(M) \oplus \mathfrak{o}(2) \) is the direct sum of the Lorentz algebra and the dilatation while \( v_{\Pi} = \mathfrak{r}(M) \oplus \mathfrak{r}(2) \).
We choose the highest weight of $N$ is known, the cohomology $H_{\mathfrak{A}}$ labelled by $N$ isomodules module 4.1.

For the conformal algebra $\mathfrak{o}(M + 2)$ and its parabolic subalgebra $\mathfrak{iso}(M + 2) \oplus \mathfrak{o}(2)$, we calculate the cohomology $H^p(\mathfrak{t}(M), \mathfrak{J})$ for any $p$ and any irreducible module $\mathfrak{J}$ using the information on the structure of the generalized Verma modules obtained by the methods developed in [17] [18] [28]. Once the cohomology $H^p(\mathfrak{t}(M), \mathfrak{J})$ for any irreducible module $\mathfrak{J}$ is known, the cohomology $H^p(\mathfrak{t}(M), \mathfrak{E})$ for any extension $\mathfrak{E}$ of the irreducible modules can also be easily found.

4.1. Irreducible Tensors and Spinor–tensors. Consider an irreducible finite dimensional module $\mathfrak{M}_{(\lambda)}$ of $\mathfrak{iso}(M) \oplus \mathfrak{o}(2)$ with some basis elements $|\lambda\rangle^A$ of the carrier space, labelled by $A$,

$$
(\mathcal{L}^{nm}|(\lambda)\rangle)^A = \mathcal{L}_0^{nm}B|\lambda\rangle^B, \quad \mathcal{D}|\lambda\rangle^A = \Delta|\lambda\rangle^A, \quad \mathcal{P}^n|\lambda\rangle^A = 0.
$$

(4.3)

We choose the highest weight of $\mathfrak{M}_{(\lambda)}$ in the form $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_q)$, where $\lambda_0 = -\Delta$ is the highest weight of $\mathfrak{o}(2)$ and $(\lambda_1, \ldots, \lambda_q)$ is the highest weight of $\mathfrak{o}(M)$. The condition that $\mathfrak{M}_{(\lambda)}$ is finite dimensional demands

$$
2\lambda_1 \equiv \cdots \equiv 2\lambda_q \mod 2,
$$

(4.4)

$$
\lambda_1 \geq \lambda_2 \geq \ldots \geq |\lambda_q| \geq 0, \quad M \text{ is even},
$$

(4.5)

$$
\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_q \geq 0, \quad M \text{ is odd}.
$$

(4.6)

It is customary in physics to describe finite dimensional representations of the Lorentz algebra as appropriate irreducible spaces of traceless tensors or $\gamma$-transversal spinor–tensors. One possible realization is as follows. Let $2\lambda_1 \equiv \cdots \equiv 2\lambda_q \equiv 0 \mod 2$. Consider the space of traceless tensors

$$
T^{n_1(\lambda_1), n_2(\lambda_2), \ldots, n_q(\lambda_q)}, \quad \eta_{n_1n_2}T^{n_1(\lambda_1), n_2(\lambda_2), \ldots, n_q(\lambda_q)} = 0, \quad 1 \leq i, j \leq q.
$$

(4.7)

where, following [29], we write $n^i(\lambda_i)$ instead of writing a set of $\lambda_i$ totally symmetrized indices $n_1^i, n_2^i, \ldots, n_{\lambda_i}^i$, i.e. we indicate in parentheses how many indices are subject to total symmetrization. For example, we write $T^{n(\lambda)}$ instead of rank-$\lambda$ symmetric tensor $T^{n_1 \ldots n_\lambda}$. We use the convention that upper(lower) indices denoted by the same latter inside parentheses are symmetrized. For example, $T^{n_1P^{n_2}}$ is equivalent to $\frac{1}{2}(T^{n_1 P^{n_2}} + T^{n_2 P^{n_1}})$. The tensor $T^{n_1(\lambda_1), n_2(\lambda_2), \ldots, n_q(\lambda_q)}$ is totally symmetric within each group of $\lambda_i$ indices $n^i$. We impose the condition that the total symmetrization of indices $n^i(\lambda_i)$ with any index from some
set \( n^j(\lambda_j) \) with \( j > i \) gives zero. Such symmetry properties are described by the Young tableau \( \Lambda \) composed of rows of length \( \lambda_1, \lambda_2, \ldots, |\lambda_q| \). Such tensors span the irreducible representation \( \mathfrak{H}(\Lambda) \) whenever \( M \) is odd or \( \lambda_q = 0 \). For even \( M \) and \( \lambda_q \neq 0 \) this space is \( \mathfrak{H}(\lambda_0,\lambda_1,\ldots,\lambda_q) \oplus \mathfrak{H}(\lambda_0,\lambda_1,\ldots,\lambda_q) \), where the direct summands are the selfdual and antiselfdual parts of the tensors (see below).

Let \( \sigma_1, \ldots, \sigma_p \) be the heights of the columns of \( \Lambda \). Another basis in \( \mathfrak{H}(\Lambda) \) with explicit antisymmetrizations consists of the traceless tensors
\[
T^{m_1[\sigma_1],m_2[\sigma_2],\ldots,m_p[\sigma_p]}, \quad \eta_{m_1 m_3} T^{m_1[\sigma_1],m_2[\sigma_2],\ldots,m_p[\sigma_p]} = 0, \quad 1 \leq i, j \leq p,
\]
where \( m_i[\sigma_i] \) denotes a set of totally antisymmetrized indices \( m_{\sigma_1}^1, m_{\sigma_2}^2, \ldots, m_{\sigma_1}^p \). We use the convention that upper(lower) indices denoted by the same latter inside square brackets are antisymmetrized \([22]\). For example, \( T^{[m_1]P^{m_2}} \) is equivalent to \( \frac{1}{2}(T^{m_1 P^{m_2}} - T^{m_2 P^{m_1}}) \). For a tensor associated with the Young tableau \( \Lambda \) the condition is imposed that the total antisymmetrization of the indices \( m_i[\sigma_i] \) with any index from some set \( m_j[\sigma_j] \) with \( j > i \) gives zero.

From the formula
\[
\epsilon_{n_1 \ldots n_M} \epsilon_{m_1 \ldots m_M} = \sum_p (-1)^{\pi(p)} \epsilon_{m_{p(1)} \ldots m_{p(M)}} \eta_{n_1 m_{p(1)}} \ldots \eta_{n_M m_{p(M)}} ,
\]
where summation is over all permutations \( p \) of indices \( m_i \), and \( \pi(p) = 0 \) or 1 is the oddness of the permutation \( p \), it follows for traceless tensors that
\[
T_{\ldots, m_i[\sigma_i], \ldots, m_j[\sigma_j], \ldots} = 0
\]
if \( \sigma_i + \sigma_j > M \) for some \( i \neq j \). From (4.10) along with the property that
\[
T_{\ldots, m_i[\sigma], \ldots, m_j[\sigma], \ldots} = T_{\ldots, m_j[\sigma], \ldots, m_i[\sigma], \ldots},
\]
it follows that there is essentially one way to define the Hodge conjugation operation * for such tensors,
\[
(*T)^{k[M-\sigma_1],m_2[\sigma_2],\ldots,m_p[\sigma_p]} = \frac{(i)^{\sigma_1(M-\sigma_1)}}{\sigma_1!} T^{m_1[\sigma_1],m_2[\sigma_2],\ldots,m_p[\sigma_p]} \epsilon_{m_1[\sigma_1]}^{k[M-\sigma_1]},
\]
where the normalization factor is fixed such that
\[
(*\ast T)^{m_1[\sigma_1],\ldots,m_p[\sigma_p]} = T^{m_1[\sigma_1],\ldots,m_p[\sigma_p]}.
\]
For \( M = 2q \) and \( \lambda_q \neq 0 \), to single out the irreducible part of the \( \mathfrak{o}(2q) \) tensor representation \( T^{m_1[q],m_2[\sigma_2],\ldots,m_p[\sigma_p]} \), we impose the (anti)selfduality condition
\[
* T^{m_1[q],m_2[\sigma_2],\ldots,m_p[\sigma_p]} = \pm T^{m_1[q],m_2[\sigma_2],\ldots,m_p[\sigma_p]} .
\]
When $2\lambda_1 \equiv \cdots \equiv 2\lambda_q \equiv 1 \mod 2$, the basis $|\lambda\rangle^A$ of the module $\mathfrak{V}_\lambda$ can be realized by spinor–tensors

$$T^{m_1}(\lambda_1-\frac{1}{2}),n^2(\lambda_2-\frac{1}{2}),\ldots,n^q(\lambda_q-\frac{1}{2}),\alpha \quad \text{or} \quad T^{m_1[1],m^2[2],\ldots,m^p[\sigma],\alpha},$$

(4.15)

where $\alpha = 1,\ldots,2^{[M/2]}$ is the spinor index. They satisfy analogous (anti)symmetry conditions and are $\gamma$-transversal, i.e.

$$\gamma_{\alpha}^{\beta} T^{m_1}(\lambda_1-\frac{1}{2}),n^2(\lambda_2-\frac{1}{2}),\ldots,n^q(\lambda_q-\frac{1}{2}),\alpha = 0, \quad 1 \leq i \leq q,$$

(4.16)

$$\gamma_{m}^{\beta} T^{m_1[1],m^2[2],\ldots,m^p[\sigma],\alpha} = 0, \quad 1 \leq j \leq p,$$

where $\gamma$ matrices satisfy (2.19). From (4.16) it follows that $T^{m_1}(\lambda_1-\frac{1}{2}),n^2(\lambda_2-\frac{1}{2}),\ldots,n^q(\lambda_q-\frac{1}{2}),\alpha$ and $T^{m_1[1],m^2[2],\ldots,m^p[\sigma],\alpha}$ are traceless. A counterpart of the identity (4.10) for $\gamma$-transversal spinor–tensors is

$$T^{m_1[1],m^2[2],\ldots,m^p[\sigma],\alpha} = 0$$

(4.17)

if $2\sigma_i > M$ for some $i$.

For $M = 2q$, to single out the irreducible part of a spinor–tensor $\mathfrak{o}(2q)$ module, one imposes the additional chirality condition

$$\Gamma_{\alpha}^{\beta} T^{m_1}(\lambda_1-\frac{1}{2}),n^2(\lambda_2-\frac{1}{2}),\ldots,n^q(\lambda_q-\frac{1}{2}),\alpha = \pm T^{m_1(\lambda_1-\frac{1}{2}),n^2(\lambda_2-\frac{1}{2}),\ldots,n^q(\lambda_q-\frac{1}{2}),\beta},$$

(4.18)

$$\Gamma_{\alpha}^{\beta} T^{m_1[1],m^2[2],\ldots,m^p[\sigma],\alpha} = \pm T^{m_1[1],m^2[2],\ldots,m^p[\sigma],\beta},$$

where

$$\Gamma_{\alpha}^{\beta} = (-i)^{(\gamma^1 \cdots \gamma^{2q})_{\alpha}^{\beta}}$$

(4.19)

is normalized to have unit square

$$\Gamma^{\gamma}_{\gamma} \Gamma_{\beta}^{\gamma} = \delta_{\beta}^{\alpha}.$$  

(4.20)

(Note that for odd $M$, $\Gamma$ is the central element, which is required to be $\pm \mathbb{I}$ in a chosen spinor representation and hence (4.18) is automatically satisfied.) For even $M$, a $\gamma$-transversal chiral spinor–tensor $T^{m_1[1],m^2[2],\ldots,m^p[\sigma],\alpha}$ that has definite Young properties, is automatically (anti)selfdual because

$$*T^{m_1[1],m^2[2],\ldots,m^p[\sigma],\beta} = \Gamma_{\alpha}^{\beta} T^{m_1[1],m^2[2],\ldots,m^p[\sigma],\alpha}.$$

(4.21)

4.2. Generalized Verma Modules. The generalized Verma $\mathfrak{o}(M+2)$-module $\mathfrak{V}_\lambda$ is freely generated from a vacuum module $\mathfrak{V}_\lambda$ (see section 4.1) by the operators $\mathcal{K}^n$. Recall that $(\lambda) = (\lambda_0,\ldots,\lambda_q)$ satisfy (4.4), (4.5), (4.6). It is convenient to represent the action of $\mathcal{K}^n$ as a multiplication by an independent variable $y^n$. Basis elements of $\mathfrak{V}_\lambda$ are formed by homogeneous polynomials

$$|l\rangle^{(n)} = \left(\underbrace{y^n \cdots y^n}_{l}\right) |(\lambda)\rangle^A, \quad l = 0, 1, 2, \ldots$$

(4.22)
A special universality property of generalized Verma modules that makes them important for our analysis is that any irreducible $\mathfrak{a}(M + 2)$-module $\mathfrak{V}(\lambda)$ with the highest weight $(\lambda)$ integrable with respect to the parabolic subalgebra $\mathfrak{iso}(M) \oplus \mathfrak{o}(2)$ is a quotient of $\mathfrak{V}(\lambda)$.

The subspace $\mathfrak{V}(\lambda)_l \subset \mathfrak{V}(\lambda)$ spanned by degree $l$ monomials ($4.22$) is called the $l$-th level of $\mathfrak{V}(\lambda)$. The associated grading in $\mathfrak{V}(\lambda)$ is

$$\mathfrak{V}(\lambda) = \bigoplus_{l=0}^{\infty} \mathfrak{V}(\lambda)_l . \tag{4.23}$$

The representation of the conformal algebra in $\mathfrak{V}(\lambda)$ is

$$\mathcal{L}^{mk} |v\rangle = \left(y^k \frac{\partial}{\partial y_m} - y^m \frac{\partial}{\partial y_k} + \mathcal{L}_0^{mk}\right) |v\rangle , \tag{4.24}$$

$$\mathcal{D} |v\rangle = \left(-\lambda_0 + y^j \frac{\partial}{\partial y^j}\right) |v\rangle , \tag{4.25}$$

$$\mathcal{K}^m |v\rangle = y^m |v\rangle , \tag{4.26}$$

$$\mathcal{P}^m |v\rangle = \left(2 \left(-\lambda_0 + y^j \frac{\partial}{\partial y^j}\right) \frac{\partial}{\partial y_m} - y^m \frac{\partial}{\partial y^j} \frac{\partial}{\partial y_j} + 2\mathcal{L}_0^{mj} \frac{\partial}{\partial y^j}\right) |v\rangle , \tag{4.27}$$

where $|v\rangle \in \mathfrak{V}(\lambda)$ and $\mathcal{L}_0^{mk}$ acts in the vacuum module ($4.3$). $\mathcal{L}^{mk}$ and $\mathcal{D}$ preserve level $l$. $\mathcal{D}$ is the grading operator, i.e. $\mathfrak{V}(\lambda)_l$ is the eigenspace of $\mathcal{D}$ with the eigenvalue $-\lambda_0 + l$. $\mathcal{K}^m$ and $\mathcal{P}^m$ increase and decrease a level by one unit, respectively.

Every level $\mathfrak{V}(\lambda)_l$ decomposes into a direct sum of $\mathfrak{o}(M) \oplus \mathfrak{o}(2)$ irreducible modules,

$$\mathfrak{V}(\lambda)_l = \bigoplus_{i=0}^{[l/2]} \mathfrak{M}(\lambda) \otimes \mathfrak{M}(\lambda) \otimes \mathfrak{M}(-l-2i,0,...,0) = \bigoplus_{(\mu) \in \Lambda(\lambda)_l} \mathfrak{M}(\mu) , \tag{4.28}$$

where $\Lambda(\lambda)_l$ is the set of highest weights in this decomposition. A $\mathfrak{o}(M) \oplus \mathfrak{o}(2)$-module $\mathfrak{G}(\mu)$ in decomposition ($4.28$) with $l \geq 1$ is called singular module if

$$\mathcal{P}^n \mathfrak{G}(\mu) = 0 . \tag{4.29}$$

Any vector from $\mathfrak{G}(\mu)$ is called singular vector. Let singular vectors $|s\rangle^A$ form a basis of $\mathfrak{G}(\mu)$. Any singular module $\mathfrak{G}(\mu) \subset \mathfrak{V}(\lambda)_l$ induces the proper submodule $\mathfrak{V}(\lambda)_{l,\mu}$ of $\mathfrak{V}(\lambda)$ with the homogeneous elements of the form

$$|m\rangle^{n(m):A} = \underbrace{y^{(n \cdots y^n)}}_m |s\rangle^A , \quad m \geq 0 . \tag{4.30}$$

Note that $\mathfrak{V}(\lambda)_{l,\mu}$ is not freely generated from $\mathfrak{G}(\mu)$, i.e., the elements $|m\rangle^{n(m):A}$ are not necessarily linearly independent. Also note that the grading ($4.23$) defined for generalized Verma modules differs from the grading ($3.39$) defined in section 3 for arbitrary $\mathfrak{fr}$-integrable modules. Namely, $\mathfrak{V}(\lambda)_{[0]}$ consists of $\mathfrak{V}(\lambda)_{0}$ along with all singular subspaces of $\mathfrak{V}(\lambda)$. In what follows we use the grading ($4.28$).
If \( \mathfrak{V}_\lambda \) is irreducible it does not contain singular modules. For reducible \( \mathfrak{V}_\lambda \), let \( \mathfrak{G}_{(\mu_1)} \), \( \mathfrak{G}_{(\mu_2)} \), ... list all singular modules of \( \mathfrak{V}_\lambda \). Let \( \mathfrak{V}_\lambda \) be the image in \( \mathfrak{V}_\lambda \) of the module induced from \( \mathfrak{G}_{(\mu_1)} \oplus \mathfrak{G}_{(\mu_2)} \oplus \ldots \). Consider the quotient \( \mathfrak{O}_\lambda = \mathfrak{V}_\lambda / \mathfrak{V}_\lambda \). A singular module \( \mathfrak{O}_\lambda \) of \( \mathfrak{O}_\lambda \) is called a subsingular module of \( \mathfrak{V}_\lambda \). Its elements are called subsingular vectors. A singular module of the quotient \( \mathfrak{O}_\lambda = \mathfrak{V}_\lambda / \mathfrak{V}_\lambda \) is called a subsingular module \( \mathfrak{S}_\mu \) of \( \mathfrak{V}_\lambda \) and so on. For generalized Verma modules \( \mathfrak{V}_\lambda \) of the conformal algebra the situation is relatively simple because \( \mathfrak{V}_\lambda \) can have only singular and subsingular modules for \( M \) even and only singular modules for \( M \) odd (see section 4.3 and Appendix A for more details).

4.3. **Contragredient Modules.** Let \( \mathfrak{M} \) be an \( \mathfrak{f} \)-module. The module \( \mathfrak{M}^\natural \) contragredient to module \( \mathfrak{M} \) is the graded dual to \( \mathfrak{M} \) vector space with the action of the algebra \( \mathfrak{f} \) defined as

\[
f\alpha(v) = \alpha(\tau(f)v),
\]

where \( f \in \mathfrak{f}, v \in \mathfrak{M}, \alpha \in \mathfrak{M}^\natural \) and \( \tau \) is the Chevalley involution . Note that for any irreducible module \( \mathfrak{J}_\lambda \) with the highest weight \( \lambda \), the contragredient module \( \mathfrak{J}_\lambda^\natural \) is also irreducible with the same highest weight and, thus, \( \mathfrak{J}_\lambda \sim \mathfrak{J}_\lambda^\natural \).

The module \( \mathfrak{V}_\lambda^\natural \) contragredient to the generalized Verma module \( \mathfrak{V}_\lambda \) can be realized as follows. Consider \( \mathfrak{M}_\lambda^\natural \sim \mathfrak{V}_\lambda \) with the basis \( A(\langle \lambda \rangle) \) dual to \( |\langle \lambda \rangle \rangle^A \)

\[
B(\langle \lambda \rangle)\langle \langle \lambda \rangle \rangle^A = \delta_B^A
\]

and the following action of the \( \mathfrak{iso}(M) \oplus \mathfrak{o}(2) \) algebra

\[
A(\langle \lambda \rangle |\mathcal{L}^{nm}) = B(\langle \lambda \rangle)\mathcal{L}^{nm} A_B^B, \quad A(\langle \lambda \rangle)\mathcal{D} = -A(\langle \lambda \rangle)|\lambda_0, \quad A(\langle \lambda \rangle)\mathcal{P}^n = 0.
\]

The vector space \( \mathfrak{V}_\lambda^\natural \) can be realized as the space of polynomials of \( y^n \) with coefficients in \( \mathfrak{M}_\lambda^\natural \). It is convenient to extend the definition of the Chevalley involution to this realization as follows:

\[
\tau(y^n) = \frac{\partial}{\partial y_n}, \quad \tau\left( \frac{\partial}{\partial y_n} \right) = y^n.
\]

The \( l \)-th level \( \mathfrak{V}_\lambda^\natural(l) \) of \( \mathfrak{V}_\lambda^\natural \) is spanned by the monomials

\[
n(\langle l \rangle; A)\langle l \rangle = \frac{1}{l!}A(\langle \lambda \rangle)\underbrace{y(y \cdots y)}_{l}.
\]

From

\[
\tau(\mathcal{L}^{nm}) = -\mathcal{L}^{nm}, \quad \tau(\mathcal{D}) = \mathcal{D}, \quad \tau(\mathcal{K}^n) = \mathcal{P}^n, \quad \tau(\mathcal{P}^n) = \mathcal{K}^n
\]

\( \text{Graded dual vector space to the graded space } V = \oplus_i V_i \text{ with finite-dimensional homogeneous components } V_i \text{ is defined as } V^* = \oplus_i V_i^*, \) where each \( V_i^* \) is dual to the corresponding \( V_i \).
it follows that the action \( \mathfrak{o}(M+2) \) of \( \mathfrak{o}(M+2) \) on \( \mathfrak{v}^\lambda \) is

\[
\langle \alpha \mid L^{mk} = \langle \alpha \mid \left( \frac{\partial}{\partial y_m} y^k - \frac{\partial}{\partial y_k} y^m + L^{mk}_0 \right) ,
\]

(4.37)

\[
\langle \alpha \mid D = \langle \alpha \mid \left( -\lambda_0 + \frac{\partial}{\partial y_j} y^j \right) ,
\]

(4.38)

\[
\langle \alpha \mid K^m = \langle \alpha \mid \left( 2(-\lambda_0 + \frac{\partial}{\partial y_j} y^j) y^m - \frac{\partial}{\partial y_m} y^j y_j + 2L^{mj}_0 y_j \right) ,
\]

(4.39)

\[
\langle \alpha \mid P^m = \langle \alpha \mid \frac{\partial}{\partial y_m} ,
\]

(4.40)

for \( \langle \alpha \mid \in \mathfrak{v}^\lambda \). Note that the elements \( P^n \) act co-freely in \( \mathfrak{v}^\lambda \), i.e. any vector in \( \mathfrak{v}^\lambda \) has a preimage under the action of \( P^n \) for every \( n \).

4.4. **Structure of \( \mathfrak{o}(M+2) \) Generalized Verma Modules.** In this section we describe the structure of \( \mathfrak{o}(M+2) \) generalized Verma modules. Singular modules in \( \mathfrak{o}(M+2) \) generalized Verma modules were completely investigated in [30, 31, 28]. To find subsingular modules we use general results from [17, 18]. This analysis is sketched in Appendix A.

4.4.1. \( M = 2q + 1 \). It turns out that for odd \( M = 2q + 1 \), \( M \geq 3 \) any \( \mathfrak{o}(M+2) \) generalized Verma module \( \mathfrak{v}^\lambda \) does not have subsingular modules\(^8\). This means that the maximal submodule \( \mathfrak{p}^\lambda \subset \mathfrak{v}^\lambda \) such that the quotient \( \mathfrak{q}^\lambda = \mathfrak{v}^\lambda / \mathfrak{p}^\lambda \) is irreducible, is induced from singular modules. For generic \( \lambda \), \( \mathfrak{v}^\lambda \) is irreducible. There are two series of reducible generalized Verma modules.

Let \( (\lambda)_0 \) be an arbitrary dominant integral weight, i.e. \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_q \) and \( 2\lambda_0 \equiv \cdots \equiv 2\lambda_q \mod 2 \). The first series consists of the modules with the following highest weights

\[
(\lambda)_0 = (\lambda_0, \lambda_1, \ldots, \lambda_q) ,
\]

\[
(\lambda)_1 = (\lambda_1 - 1, \lambda_0 + 1, \lambda_2, \ldots, \lambda_q) ,
\]

\[
(\lambda)_N = (\lambda_N - N, \lambda_0 + 1, \ldots, \lambda_N - 1 + 1, \lambda_{N+1}, \ldots, \lambda_q) , \quad N = 0, \ldots, q ,
\]

\[
(\lambda)_q = (\lambda_q - q, \lambda_0 + 1, \ldots, \lambda_{q-1} + 1) ,\quad (4.41)
\]

\[
(\lambda)_{q+1} = (-\lambda_q - q - 1, \lambda_0 + 1, \ldots, \lambda_{q-1} + 1) ,
\]

\[
(\lambda)_{q+K} = (-\lambda_{q+K} - q - K, \lambda_0 + 1, \ldots, \lambda_{q-K} + 1, \lambda_{q-K+2}, \ldots, \lambda_q) , \quad K = 1, \ldots, q ,
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Unfolded form of linear conformal equations in $M$-dimensions and 

\[(\lambda)_{2q-1} = (-\lambda_2 - 2q + 1, \lambda_0 + 1, \lambda_1 + 1, \lambda_3, \ldots, \lambda_q), \]
\[(\lambda)_{2q} = (-\lambda_1 - 2q, \lambda_0 + 1, \lambda_2, \ldots, \lambda_q). \]

The generalized Verma modules with highest weights from (4.41) have the structure described by the following short exact sequences

\[0 \to \mathfrak{J}(\lambda)_1 \to \mathfrak{V}(\lambda)_0 \to \mathfrak{J}(\lambda)_0 \to 0; \quad (4.42)\]
\[0 \to \mathfrak{J}(\lambda)_{N+1} \to \mathfrak{V}(\lambda)_N \to \mathfrak{J}(\lambda)_N \to 0, \quad N = 0, \ldots, 2q; \quad (4.43)\]
\[0 \to \mathfrak{J}(\lambda)_{2q+1} \to \mathfrak{V}(\lambda)_{2q} \to \mathfrak{J}(\lambda)_{2q} \to 0; \quad (4.44)\]
\[0 \to \mathfrak{V}(\lambda)_{2q+1} \to \mathfrak{J}(\lambda)_{2q+1} \to 0, \quad (4.45)\]

where $(\lambda)_{2q+1} = (-\lambda_0 - 2q - 1, \lambda_1, \ldots, \lambda_q)$ and all $\mathfrak{J}(\lambda)$ are irreducible. Equation (4.45) means that $\mathfrak{V}(\lambda)_{2q+1} = \mathfrak{J}(\lambda)_{2q+1}$ is irreducible. Equation (4.44) means that $\mathfrak{J}(\lambda)_{2q+1}$ is the maximal submodule of $\mathfrak{V}(\lambda)_{2q}$ and the quotient $\mathfrak{J}(\lambda)_{2q} = \mathfrak{V}(\lambda)_{2q}/\mathfrak{V}(\lambda)_{2q+1}$ is irreducible. The maximal submodule of $\mathfrak{V}(\lambda)_{2q-1}$ is $\mathfrak{J}(\lambda)_{2q-1}$ and the quotient $\mathfrak{J}(\lambda)_{2q-1} = \mathfrak{V}(\lambda)_{2q-1}/\mathfrak{J}(\lambda)_{2q}$ is irreducible, and so on.

The second series consists of reducible generalized Verma modules with non-integral highest weights. Let $\mu_1 \geq \cdots \geq \mu_q$ and $2\mu_1 \equiv \cdots \equiv 2\mu_q \mod 2$. Consider the highest weight

\[(\mu) = (\mu_0, \mu_1, \ldots, \mu_q), \]
\[\mu_0 = -q + \frac{1}{2} + N_0 \quad \text{if } 2\mu_1 \equiv 2\mu_q \equiv 0 \mod 2, \quad (4.46)\]
\[\mu_0 = -q + N_0 \quad \text{if } 2\mu_1 \equiv 2\mu_q \equiv 1 \mod 2. \]

We have

\[0 \to \mathfrak{V}((\mu))' \to \mathfrak{V}(\mu) \to \mathfrak{J}(\mu) \to 0, \quad (4.47)\]

where

\[(\mu)' = (-\mu_0 - 2q - 1, \mu_1, \ldots, \mu_q). \quad (4.48)\]

The modules $\mathfrak{J}(\mu) = \mathfrak{V}(\mu)/\mathfrak{V}(\mu)'$ and $\mathfrak{V}(\mu)'$ are irreducible.

The described two series give the full list of reducible $\mathfrak{o}(M+2)$ generalized Verma modules for odd $M$.

4.4.2. $M = 2q$. The structure of $\mathfrak{o}(M+2)$ generalized Verma modules $\mathfrak{V}(\lambda)$ for even $M$ is more complicated because in the even dimensional case some $\mathfrak{V}(\lambda)$ have subsingular modules.
(no subsubsingular modules, however\(^9\)). Again, there are two series of reducible generalized Verma modules.

Let \((\lambda)_{-q}\) be an arbitrary dominant integral weight, i.e. \(\lambda_0 \geq \lambda_1 \geq \cdots \geq |\lambda_q|\) and \(2\lambda_0 \equiv \cdots \equiv 2\lambda_q \mod 2\). Consider the set of highest weights

\[
(\lambda)_{-q} = (\lambda_0, \lambda_1, \ldots, \lambda_q),
\]

\[
(\lambda)_{-q+1} = (\lambda_1 - 1, \lambda_0 + 1, \lambda_2, \ldots, \lambda_q),
\]

\[
\vdots
\]

\[
(\lambda)_{-q+N} = (\lambda_N - N, \lambda_0 + 1, \ldots, \lambda_{N-1} + 1, \lambda_{N+1}, \ldots, \lambda_q),
\]

\[
N = 0, \ldots, q - 1,
\]

\[
\vdots
\]

\[
(\lambda)_{-1} = (\lambda_{q-1} - q + 1, \lambda_0 + 1, \ldots, \lambda_{q-2} + 1, \lambda_q),
\]

\[
(\lambda)_0 = (\lambda_q - q, \lambda_0 + 1, \ldots, \lambda_{q-1} + 1), \quad (\lambda)_{0'} = (-\lambda_q - q, \lambda_0 + 1, \ldots, \lambda_{q-2} + 1, -\lambda_{q-1} - 1),
\]

\[
(\lambda)_1 = (-\lambda_{q-1} - q - 1, \lambda_0 + 1, \ldots, \lambda_{q-2} + 1, -\lambda_q),
\]

\[
\vdots
\]

\[
(\lambda)_K = (-\lambda_{q-K} - q - K, \lambda_0 + 1, \ldots, \lambda_{q-K-1} + 1, \lambda_{q-K+1}, \ldots, \lambda_{q-1}, -\lambda_q), \quad K = 1, \ldots, q - 1,
\]

\[
\vdots
\]

\[
(\lambda)_{q-2} = (-\lambda_2 - 2q + 2, \lambda_0 + 1, \lambda_1 + 1, \lambda_3, \ldots, \lambda_{q-1}, -\lambda_q),
\]

\[
(\lambda)_{q-1} = (-\lambda_1 - 2q + 1, \lambda_0 + 1, \lambda_2, \ldots, \lambda_{q-1}, -\lambda_q).
\]

The structure of the generalized Verma modules with the highest weights \((4.49)\) is described by the following short exact sequences

\[
0 \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{-q+1}} \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{-q}} \rightarrow \mathfrak{J}_{\lambda}^{(\lambda)_{-q}} \rightarrow 0; \quad (4.50)
\]

\[
0 \rightarrow \mathfrak{Y}_{\lambda}^{(\lambda)_{-q}} \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{-q+1}} \rightarrow \mathfrak{J}_{\lambda}^{(\lambda)_{-q+1}} \rightarrow 0; \quad (4.51)
\]

\[
0 \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{-q+2}} \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{-q+1}} \rightarrow \mathfrak{J}_{\lambda}^{(\lambda)_{-q+1}} \rightarrow 0; \quad (4.52)
\]

\[
0 \rightarrow \mathfrak{Y}_{\lambda}^{(\lambda)_{q-1}} \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{-q+2}} \rightarrow \mathfrak{J}_{\lambda}^{(\lambda)_{-q+2}} \rightarrow 0; \quad (4.53)
\]

\[
\vdots
\]

\[
0 \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{N+1}} \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{N}} \rightarrow \mathfrak{J}_{\lambda}^{(\lambda)_{N}} \rightarrow 0, \quad N = -q, -q + 1, \ldots, -2, \quad (4.54)
\]

\[
0 \rightarrow \mathfrak{Y}_{\lambda}^{(\lambda)_{N}} \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{N+1}} \rightarrow \mathfrak{J}_{\lambda}^{(\lambda)_{N+1}} \rightarrow 0; \quad (4.55)
\]

\[
\vdots
\]

\[
0 \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{-1}} \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{-2}} \rightarrow \mathfrak{J}_{\lambda}^{(\lambda)_{-2}} \rightarrow 0; \quad (4.56)
\]

\[
0 \rightarrow \mathfrak{Y}_{\lambda}^{(\lambda)_{-2}} \rightarrow \mathfrak{V}_{\lambda}^{(\lambda)_{-1}} \rightarrow \mathfrak{J}_{\lambda}^{(\lambda)_{-1}} \rightarrow 0; \quad (4.57)
\]

\(^9\)The fact that the homogeneous space \(SO(M+2)/ISO(M) \otimes SO(2)\) does not contain three cells of the equal dimension forbids appearance of subsubsingular modules \[17\].
Unfolded form of linear conformal equations in $M$-dimensions and...

\[\begin{align*}
0 & \to J_{(\lambda)_0} \to V_{(\lambda)_0} \to J_{(\lambda)_-1} \to 0, \\
0 & \to V^2_{(\lambda)_1} \to J_{(\lambda)_0} \to J_{(\lambda)_0} \oplus J_{(\lambda)_0} \to 0; \\
0 & \to J_{(\lambda)_1} \to V_{(\lambda)_0} \to J_{(\lambda)_0} \to 0; \\
0 & \to J_{(\lambda)_1} \to V_{(\lambda)_0} \to J_{(\lambda)_0} \to 0; \\
0 & \to J_{(\lambda)_1} \to V_{(\lambda)_0} \to J_{(\lambda)_0} \to 0; \\
\vdots \\
0 & \to J_{(\lambda)_{N+1}} \to V_{(\lambda)_N} \to J_{(\lambda)_N} \to 0, \quad N = 1, \ldots, q - 1; \\
\vdots \\
0 & \to J_{(\lambda)_q} \to V_{(\lambda)_{q-1}} \to J_{(\lambda)_{q-1}} \to 0; \\
0 & \to V_{(\lambda)_q} \to J_{(\lambda)_q} \to 0.
\end{align*}\]

Here $(\lambda)_q = (-\lambda_0 - 2q, \lambda_1, \ldots, \lambda_{q-1}, -\lambda_q)$, and all $J_{(\lambda)}$ are irreducible. Analogously to the odd dimensional case, $(4.65)$ means that $V_{(\lambda)_q} = J_{(\lambda)_q}$ is irreducible. From $(4.64)$ it follows that $J_{(\lambda)_q}$ is the maximal submodule of $V_{(\lambda)_{q-1}}$ and the quotient $J_{(\lambda)_{q-1}}$ is irreducible, which in its turn is the maximal submodule of $V_{(\lambda)_{q-2}}$ and so on. Continuing the same way one finally arrives at $J_{(\lambda)_1} = V_{(\lambda)_1}/J_{(\lambda)_2}$ $(4.62)$. The structure of the modules $V_{(\lambda)_1}, \ldots, V_{(\lambda)_{q-1}}$ is analogous to that of the odd dimensional case.

The modules $V_{(\lambda)_0}$ and $V_{(\lambda)_0}$ have the common maximal submodule $J_{(\lambda)_1}$ (see $(4.60)$, $(4.61)$) and the quotients $J_{(\lambda)_0} = V_{(\lambda)_0}/J_{(\lambda)_1}$ and $J_{(\lambda)_0}/J_{(\lambda)_1}$ are irreducible. The module $V_{(\lambda)_{-1}}$ has the most complicated structure of submodules. Equation $(4.59)$ describes the structure of the maximal submodule $V_{(\lambda)_0}$ of $V_{(\lambda)_{-1}}$. The appearance of the contragredient module $V^2_{(\lambda)_1}$ in $(4.59)$ means that the maximal submodule of $V_{(\lambda)_{-1}}$ cannot be generated from singular modules because the module contragredient to a generalized Verma module is not (unless it is irreducible) a highest-weight module and therefore $V_{(\lambda)_{-1}}$ contains a subsingular module. Analogously the modules $V_{(\lambda)_{-2}} \ldots V_{(\lambda)_{-q+1}}$ contain singular and subsingular modules as described by $(4.56)$ $(4.57)$ $(4.52)$ $(4.53)$. Finally, the module $V_{(\lambda)_{-q}}$ contains the submodule $V^2_{(\lambda)_q}$ but in this case subsingular modules do not appear because $V^2_{(\lambda)_q}$ is isomorphic to $V_{(\lambda)_q} = J_{(\lambda)_q}$, and therefore the maximal submodule of $V_{(\lambda)_{-q}}$ is generated from singular modules.
Let $\mu_1 \geq \cdots \geq \mu_{q-1} \geq |\mu_q|$ and $2\mu_1 \equiv \cdots \equiv 2\mu_q \mod 2$. The second series of reducible generalized Verma $\mathfrak{o}(M + 2)$ modules with even $M$ contains the modules with the singular highest weights $(\mu) = (\mu_0, \mu_1, \ldots, \mu_q)$ such that

\begin{align*}
\mu_0 &= \mu_N - N \quad \text{for some } N = 1, \ldots, q, \\
\mu_0 &\neq -q, \\
\mu_0 + \mu_q + q &\in \mathbb{N}_0, \\
\mu_0 - \mu_q + q &\in \mathbb{N}_0.
\end{align*}

(4.66)

The structure of $\mathfrak{V}(\mu)$ is described by the short exact sequence

\begin{equation}
0 \rightarrow \mathfrak{V}(\mu)' \rightarrow \mathfrak{V}(\mu) \rightarrow \mathfrak{J}(\mu) \rightarrow 0,
\end{equation}

where $(\mu)' = (-\mu_0 - 2q, \mu_1, \ldots, \mu_{q-1}, -\mu_q)$ and $\mathfrak{J}(\mu) = \mathfrak{V}(\mu)/\mathfrak{V}(\mu)'$ is irreducible.

### 4.5. Cohomology of Irreducible $\mathfrak{o}(M + 2)$-Modules.

Any irreducible $\mathfrak{o}(M + 2)$-module $\mathfrak{J}(\lambda)$ with the highest weight $(\lambda)$ integrable with respect to the parabolic subalgebra $\mathfrak{iso}(M) \oplus \mathfrak{o}(2)$ is a quotient of an appropriate generalized Verma $\mathfrak{o}(M + 2)$-module $\mathfrak{V}(\lambda)$. (Recall that $(\lambda)$ is required to satisfy (4.4)-(4.6).) In this section we show that once the structure of all generalized Verma modules is known, one can calculate $H^p(t(M), \mathfrak{J}(\lambda))$ (i.e. the cohomology of $t(M)$ with coefficients in $\mathfrak{J}(\lambda)$) for any $p$ and irreducible $\mathfrak{J}(\lambda)$. Recall that $t(M)$ is the subalgebra of $\mathfrak{o}(M + 2)$ generated by the momenta $P^n$.

Let us start with the following Lemma.

**Lemma 4.1.** Let $\mathfrak{V}(\lambda)$ be the generalized Verma $\mathfrak{o}(M + 2)$-module induced from $\mathfrak{N}(\lambda)$. Then

\begin{align*}
H^0(t(M), \mathfrak{V}(\lambda)) &= \mathfrak{N}(\lambda), \\
H^p(t(M), \mathfrak{V}(\lambda)) &= 0 \quad \text{for } p = 1, \ldots.
\end{align*}

(4.68) (4.69)

**Proof.** From (4.40) it follows that $\sigma_- = \xi^n \frac{\partial}{\partial y^n}$ (see (3.19)) for any $\mathfrak{V}(\lambda)$. Equations (4.68) and (4.69) follow from the standard Poincaré Lemma.

The following two Theorems describe the $\sigma_-$ cohomology $H^p(t(M), \mathfrak{J}(\lambda))$ with coefficients in $\mathfrak{J}(\lambda)$. Recall that any $\mathfrak{J}(\lambda)$ is a quotient of the generalized Verma module $\mathfrak{V}(\lambda)$ induced from $\mathfrak{N}(\lambda)$ as described in Sec. 4.2.

**Theorem 4.2.** Let $M$ be odd

1. If $\mathfrak{V}(\lambda)$ is irreducible then

\begin{align*}
H^0(t(M), \mathfrak{J}(\lambda)) &= \mathfrak{N}(\lambda), \\
H^p(t(M), \mathfrak{J}(\lambda)) &= 0, \quad p = 1, 2, \ldots.
\end{align*}

(4.70) (4.71)
2. If $\mathfrak{V}(\lambda)$ is reducible and $(\lambda) = (\lambda)_N$ $(N = 0, \ldots, 2q)$ belongs to the series (4.42) then

\[
H^p(t(M), J_{(\lambda)_N}) = \mathfrak{N}(\lambda)_{p+N}, \quad p = 0, \ldots, 2q + 1 - N, \quad (4.72)
\]
\[
H^p(t(M), J_{(\lambda)_N}) = 0, \quad p = 2q + 2 - N, \ldots. \quad (4.73)
\]

3. If $\mathfrak{V}(\lambda)$ is reducible and $(\lambda) = (\mu)$ belongs to the series (4.46) then

\[
H^0(t(M), J_{(\mu)}) = \mathfrak{N}(\mu), \quad (4.74)
\]
\[
H^1(t(M), J_{(\mu)}) = \mathfrak{N}(\mu)^\prime, \quad (4.75)
\]
\[
H^p(t(M), J_{(\mu)}) = 0, \quad p = 2, \ldots. \quad (4.76)
\]

**Proof.** Item 1 follows from Lemma 4.1 and the observation that $\mathfrak{V}(\lambda)$ is isomorphic to $\mathfrak{V}^\sharp(\lambda)$ whenever $\mathfrak{V}(\lambda)$ is irreducible.

Items 2 and 3 follow from Lemma 4.1 and long cohomological sequences corresponding to short exact sequences contragredient to (4.42)-(4.45) and (4.47). □

**Theorem 4.3.** Let $M$ be even

1. If $\mathfrak{V}(\lambda)$ is irreducible then

\[
H^0(t(M), J_{(\lambda)}) = \mathfrak{N}(\lambda), \quad (4.77)
\]
\[
H^p(t(M), J_{(\lambda)}) = 0, \quad p = 1, 2, \ldots. \quad (4.78)
\]

2. If $\mathfrak{V}(\lambda)$ is reducible and $(\lambda) = (\lambda)_N$ $(N = -q, -q + 1, \ldots, -1, 0, 0', \ldots, q)$ belongs to the series (4.49) then

\[
H^0(t(M), J_{(\lambda)_N}) = \mathfrak{N}(\lambda)_N, \quad N = -q, -q + 1, \ldots, -1, 0, 0', 1, \ldots, q, \quad (4.79)
\]
\[
H^p(t(M), J_{(\lambda)_N}) = \widetilde{\mathfrak{N}}(\lambda)_{p+N} \oplus \widetilde{\mathfrak{N}}(\lambda)_{p-N}, \quad p = 1, \ldots, N = -q + 1, -q + 2, \ldots - 1, \quad (4.80)
\]
\[
H^p(t(M), J_{(\lambda)_N}) = \widetilde{\mathfrak{N}}(\lambda)_{p+N}, \quad p = 1, \ldots, N = -q, 0, 0', 1, 2, \ldots q, \quad (4.81)
\]

where

\[
\widetilde{\mathfrak{N}}(\lambda)_N = \mathfrak{N}(\lambda)_N \quad \text{for } N = -q, -q + 1, \ldots, q \text{ and } N \neq 0, \quad (4.82)
\]
\[
\widetilde{\mathfrak{N}}(\lambda)_0 = \mathfrak{N}(\lambda)_0 \oplus \mathfrak{N}(\lambda)_0', \quad (4.83)
\]
\[
\widetilde{\mathfrak{N}}(\lambda)_N = 0 \quad \text{for } N = q + 1, \ldots, \quad (4.84)
\]

and $p + 0 = p + 0' = p$.

3. If $\mathfrak{V}(\lambda)$ is reducible and $(\lambda) = (\mu)$ belongs to the series (4.67) then

\[
H^0(t(M), J_{(\mu)}) = \mathfrak{N}(\mu), \quad (4.85)
\]
\[
H^1(t(M), J_{(\mu)}) = \mathfrak{N}(\mu)^\prime, \quad (4.86)
\]
\[
H^p(t(M), J_{(\mu)}) = 0, \quad p = 2, \ldots. \quad (4.87)
\]
The standard spectral sequence technique together with the definition of $t$-invariants of $\mathfrak{g}$ and for the modules $\mathfrak{g}$ allows us to calculate the cohomology of the irreducible modules for $N = -q, 0, 0', 1, 2, \ldots, q$. Using these we have

$$H^0 (t(M), \mathfrak{g}(\lambda)_N) = \mathfrak{f}(\lambda)_N \quad \text{for } N = -q, -q + 1, \ldots, -1, 0, 0', 1, 2, \ldots, q, \quad (4.88)$$

$$H^p (t(M), \mathfrak{g}(\lambda)_N) = \sim \mathfrak{f}(\lambda)_{N+p} \quad \text{for } p = 1, \ldots, N = -q, 0, 0', 1, 2, \ldots, q. \quad (4.89)$$

This proves (4.79) and (4.81). In order to prove (4.80) we consider the short exact sequences contragredient to (4.60)–(4.69)

$$0 \to \tilde{\mathfrak{g}}(\lambda)_{N+1} \to \mathfrak{g}(\lambda)_N \to \mathfrak{g}(\lambda)_N^2 \to 0, \quad (4.90)$$

$$0 \to \tilde{\mathfrak{g}}(\lambda)_{N+1} \to \mathfrak{g}(\lambda)_{N+1} \to \mathfrak{g}(\lambda)_{N+1} \to 0, \quad (4.91)$$

where $N = -q, -q + 1, \ldots, -1$ and $\tilde{\mathfrak{g}}(\lambda)_N = \mathfrak{g}(\lambda)_N$ for $N \neq 0$ and $\tilde{\mathfrak{g}}(\lambda)_{-N} = \mathfrak{g}(\lambda)_0 \oplus \mathfrak{g}(\lambda)_{-N}$. The long cohomological exact sequence corresponding to (4.90) gives

$$H^p (t(M), \mathfrak{g}(\lambda)_{N+1}) = H^{p+1} (t(M), \tilde{\mathfrak{g}}(\lambda)_N). \quad (4.92)$$

Then substituting this into the long cohomological exact sequence corresponding to (4.91)

$$\ldots \xrightarrow{g_{N+1}^p} H^p (t(M), \tilde{\mathfrak{g}}(\lambda)_{N+1}) \xrightarrow{f_N^p} H^p (t(M), \mathfrak{g}(\lambda)_{N+1}) \xrightarrow{f_N^p} H^p (t(M), \mathfrak{g}(\lambda)_{N+2}) \xrightarrow{g_N^p} \ldots \quad (4.93)$$

we obtain the long exact sequence

$$\ldots \xrightarrow{g_{N+1}^p} H^p (t(M), \tilde{\mathfrak{g}}(\lambda)_{N+1}) \xrightarrow{f_N^p} H^p (t(M), \mathfrak{g}(\lambda)_N) \xrightarrow{f_N^p} H^p (t(M), \mathfrak{g}(\lambda)_{N+1}) \xrightarrow{f_N^p} \ldots \quad (4.94)$$

Using (4.89), (4.88) and short exact sequences (4.60)–(4.65) we calculate the cohomology of the generalized Verma modules $\mathfrak{g}(\lambda)_N$ for $N = 0, 0', 1, 2, \ldots, q$

$$H^0 (t(M), \mathfrak{g}(\lambda)_N) = \mathfrak{f}(\lambda)_{N+1} \oplus \tilde{\mathfrak{g}}(\lambda)_{N+1} \quad \text{for } N = 0, 0', 1, 2, \ldots, q, \quad (4.95)$$

$$H^p (t(M), \mathfrak{g}(\lambda)_N) = \tilde{\mathfrak{g}}(\lambda)_{N+p} \oplus \tilde{\mathfrak{g}}(\lambda)_{N+p+1} \quad \text{for } p = 1, \ldots, N = 0, 0', 1, 2, \ldots, q. \quad (4.96)$$
Substituting this into (4.94) we have
\[
\ldots \xrightarrow{g_{N}^{-1}} H^{p}(t(M), \tilde{\mathcal{J}}_{(\lambda)N+1}) \longrightarrow H^{p+1}(t(M), \tilde{\mathcal{J}}_{(\lambda)N}) \xrightarrow{f_{N}^{p}} \tilde{\mathcal{H}}_{(\lambda)N+p} \oplus \tilde{\mathcal{H}}_{(\lambda)N+p+1} \xrightarrow{g_{N}^{p}} \ldots \tag{4.97}
\]
whence we can obtain the following recurrent relation between cohomology
\[
H^{1}(t(M), \tilde{\mathcal{J}}_{(\lambda)N}) = H^{0}(t(M), \tilde{\mathcal{J}}_{(\lambda)N+1}) \oplus \im f_{N}^{0} \quad \text{for } N = -q, -q + 1, \ldots, -1 \tag{4.98}
\]
\[
H^{p}(t(M), \tilde{\mathcal{J}}_{(\lambda)N}) = (H^{p-1}(t(M), \tilde{\mathcal{J}}_{(\lambda)N+1})/\im g_{N}^{p-2}) \oplus \im f_{N}^{p-1} \quad \text{for } N = -q, -q + 1, \ldots, -1 \text{ and } p \geq 2. \tag{4.99}
\]
These relations interpolate between $H^{p}(t(M), \mathcal{J}_{(\lambda)q})$ and $H^{p}(t(M), \tilde{\mathcal{J}}_{(\lambda)0})$ calculated above. This allows us to calculate
\[
\im f_{N}^{p} = \tilde{\mathcal{H}}_{(\lambda)N+p+1} \tag{4.100}
\]
\[
\im g_{N}^{p} = \tilde{\mathcal{H}}_{(\lambda)N+p}. \tag{4.101}
\]
Then we have
\[
H^{1}(t(M), \tilde{\mathcal{J}}_{(\lambda)N}) = H^{0}(t(M), \tilde{\mathcal{J}}_{(\lambda)N+1}) \oplus \tilde{\mathcal{H}}_{(\lambda)N+1} \quad \text{for } N = -q, -q + 1, \ldots, -1 \tag{4.102}
\]
\[
H^{p}(t(M), \tilde{\mathcal{J}}_{(\lambda)N}) = (H^{p-1}(t(M), \tilde{\mathcal{J}}_{(\lambda)N+1})/\tilde{\mathcal{H}}_{(\lambda)N+p+2}) \oplus \tilde{\mathcal{H}}_{(\lambda)N+p} \quad \text{for } N = -q, -q + 1, \ldots, -1 \text{ and } p \geq 2. \tag{4.103}
\]
Finally these recurrent relations give (4.30).

Item 3 is analogous to that of Theorem 4.2.

According to Sec. 4.4.2, items 2 and 3 in Theorems 4.2 and 4.3 describe all reducible $\mathcal{V}_{(\lambda)}$.

Let us summarize the results for $H^{0}(t(M), \mathcal{J}_{(\lambda)})$ and $H^{1}(t(M), \mathcal{J}_{(\lambda)})$, which are most important for this paper:
\[
H^{0}(t(M), \mathcal{J}_{(\lambda)}) = \mathcal{H}_{(\lambda)}, \tag{4.104}
\]
\[
H^{1}(t(M), \mathcal{J}_{(\lambda)}) = 0 \quad \text{if } \mathcal{J}_{(\lambda)} \sim \mathcal{V}_{(\lambda)}; \tag{4.105}
\]
\[
H^{1}(t(M), \mathcal{J}_{(\lambda)}) = \mathcal{H}_{(\mu)}' \quad \text{if } (\lambda) = (\mu) \text{ from (4.46) or (4.66)}; \tag{4.106}
\]
\[
H^{1}(t(M), \mathcal{J}_{(\lambda)N}) = \mathcal{H}_{(\lambda)N+1} \quad \text{if } M = 2q + 1, N = 0, \ldots, 2q \text{ and } (\lambda)_{N} \text{ belongs to (4.41)}; \tag{4.107}
\]
\[
\text{or if } M = 2q, N = -q, 1, \ldots, q - 1 \text{ and } (\lambda)_{N} \text{ belongs to (4.49)}. \]
In addition, for $M = 2q$

$$H^1(t(M), \mathcal{J}_{(\lambda)_0}) = H^1(t(M), \mathcal{J}_{(\lambda)_0'}) = \mathcal{N}_{(\lambda)_1}, \quad (4.108)$$

$$H^1(t(M), \mathcal{J}_{(\lambda)_{-1}}) = \mathcal{N}_{(\lambda)_0} \oplus \mathcal{N}_{(\lambda)_0'} \oplus \mathcal{N}_{(\lambda)_2}, \quad (4.109)$$

$$H^1(t(M), \mathcal{J}_{(\lambda)_N}) = \mathcal{N}_{(\lambda)_{N+1}} \oplus \mathcal{N}_{(\lambda)_{-N+1}} \quad \text{if } (\lambda)_N \text{ with } N = -2, \ldots, -q + 1 \quad (4.110)$$

Remark 4.4. For any irreducible module $\mathcal{J}_{(\lambda)}$, $H^1(t(M), \mathcal{J}_{(\lambda)})$ is equal to the direct sum of those singular and subsingular modules of the generalized Verma module $\mathfrak{V}_{(\lambda)}$, that are not descendants of some other singular module in $\mathfrak{V}_{(\lambda)}$. This property is expected because, as one can see from the examples in sections 2, 4.7, both $H^1(t(M), \mathcal{J}_{(\lambda)})$ and singular and subsingular modules determine the structure of differential equations on the dynamical fields.

4.6. Examples of Calculating Cohomology of Reducible $\mathfrak{o}(M + 2)$-Modules. Using Theorems 4.2 and 4.3 one can easily calculate $H^p(t(M), \mathfrak{M})$ for any integrable module $\mathfrak{M}$. Let $\mathfrak{E}_{3_1, 3_2}$ be the first extension of the irreducible modules $\mathcal{J}_1, \mathcal{J}_2$ given by the nonsplittable short exact sequence

$$0 \rightarrow \mathcal{J}_1 \rightarrow \mathfrak{E}_{3_1, 3_2} \rightarrow \mathcal{J}_2 \rightarrow 0. \quad (4.111)$$

From the long exact sequence for cohomology

$$0 \rightarrow H^0(t(M), \mathcal{J}_1) \rightarrow H^0(t(M), \mathfrak{E}_{3_1, 3_2}) \rightarrow H^0(t(M), \mathcal{J}_2) \rightarrow H^1(t(M), \mathcal{J}_1) \rightarrow \cdots, \quad (4.112)$$

where $H^p(t(M), \mathcal{J}_1)$ and $H^p(t(M), \mathcal{J}_2)$ are given by Theorems 4.2 and 4.3, one obtains $H^p(t(M), \mathfrak{E}_{3_1, 3_2})$.

Using Theorem 4.2 it is not hard to see that in the case $M = 2q + 1$ any extension of an irreducible conformal module is isomorphic to a contragredient generalized Verma module. This means that any odd dimensional conformal system of equations is either primitive or decomposes into independent primitive subsystems. We therefore focus on the even dimensional case.

As an example, let us calculate cohomology of the module $\mathfrak{E}_{A,F}$ which corresponds to the case of $M = 4$ electrodynamics considered in section 2.4. The module $\mathfrak{E}_{A,F}$ is defined by the short exact sequence

$$0 \rightarrow \mathcal{J}_A \rightarrow \mathfrak{E}_{A,F} \rightarrow \mathcal{R}_F \rightarrow 0, \quad (4.113)$$

where $\mathcal{J}_A = \mathcal{J}_{(\lambda)_{-1}}$ and $\mathcal{R}_F = \mathcal{J}_{(\lambda)_0} \oplus \mathcal{J}_{(\lambda)_0'}$ belong to the series (4.49) that starts from the dominant highest weight $(\lambda)_{-2} = (0, 0, 0)$, $M = 2q = 4$. From Theorem 4.3 we obtain the
long exact cohomology sequence

\[ 0 \to \mathfrak{n}_{(\lambda)_{-1}} \to H^0(t(M), \mathcal{E}_{A,F}) \to \mathfrak{n}_{(\lambda)_{0}} \oplus \mathfrak{n}_{(\lambda)_{1}} \to \mathfrak{n}_{(\lambda)_{0}} \oplus \mathfrak{n}_{(\lambda)_{1}} \to H^1(t(M), \mathcal{E}_{A,F}) \to \mathfrak{n}_{(\lambda)_{1}} \oplus \mathfrak{n}_{(\lambda)_{2}} \to \mathfrak{n}_{(\lambda)_{1}} \oplus \mathfrak{n}_{(\lambda)_{2}} \to \ldots \]

whence

\[ H^0(t(M), \mathcal{E}_{A,F}) = \mathfrak{n}_{(\lambda)_{-1}}, \quad H^1(t(M), \mathcal{E}_{A,F}) = \mathfrak{n}_{(\lambda)_{1}} \oplus \mathfrak{n}_{(\lambda)_{2}}. \]  

(4.114)

As a generalization of (4.113) let us consider the module \( \mathfrak{E}_{\beta(\lambda)_{-N} \beta(\lambda)_{-N+1}} \) defined by the short exact sequence

\[ 0 \longrightarrow \mathfrak{i}_{(\lambda)_{-N}} \longrightarrow \mathfrak{E}_{\beta(\lambda)_{-N} \beta(\lambda)_{-N+1}} \longrightarrow \tilde{\mathfrak{i}}_{(\lambda)_{-N+1}} \longrightarrow 0, \]  

(4.116)

where \((\lambda)_{-N}\) and \((\lambda)_{-N+1}\) with \(N = 1, \ldots, q - 1\) belong to (4.49), \(\tilde{\mathfrak{i}}_{(\lambda)_{N}} = \mathfrak{i}_{(\lambda)_{N}}\) for \(N \neq 0\) and \(\tilde{\mathfrak{i}}_{(\lambda)_{0}} = \mathfrak{i}_{(\lambda)_{0}} \oplus \mathfrak{i}_{(\lambda)_{1}}\). Cohomology of \( \mathfrak{E}_{\beta(\lambda)_{-N} \beta(\lambda)_{-N+1}} \) is calculated from

\[ 0 \to \mathfrak{n}_{(\lambda)_{-N}} \to H^0(t(M), \mathfrak{E}_{\beta(\lambda)_{-N} \beta(\lambda)_{-N+1}}) \to \tilde{\mathfrak{n}}_{(\lambda)_{-N+1}} \to \tilde{\mathfrak{n}}_{(\lambda)_{-N+1}} \oplus \mathfrak{n}_{(\lambda)_{N+1}} \to \]

\[ H^1(t(M), \mathfrak{E}_{\beta(\lambda)_{-N} \beta(\lambda)_{-N+1}}) \to \tilde{\mathfrak{n}}_{(\lambda)_{-N+2}} \oplus \mathfrak{n}_{(\lambda)_{N}} \to \tilde{\mathfrak{n}}_{(\lambda)_{-N+2}} \oplus \mathfrak{n}_{(\lambda)_{N+2}} \to \ldots \]  

(4.117)

where \(\tilde{\mathfrak{n}}_{(\lambda)_{N}}\) is defined in (4.82). From (4.117) we have that

\[ H^0(t(M), \mathfrak{E}_{\beta(\lambda)_{-N} \beta(\lambda)_{-N+1}}) = \mathfrak{n}_{(\lambda)_{-N}}, \quad H^1(t(M), \mathfrak{E}_{\beta(\lambda)_{-N} \beta(\lambda)_{-N+1}}) = \mathfrak{n}_{(\lambda)_{N}} \oplus \mathfrak{n}_{(\lambda)_{N+1}}. \]  

(4.118)

Equations corresponding to \( \mathfrak{E}_{\beta(\lambda)_{-N} \beta(\lambda)_{-N+1}} \) are considered for \(N = 1\) in subsection 4.7.2 and for \(N = q - 1\) in subsection 4.7.3. An important general property of the dynamical systems associated with the module \( \mathfrak{E}_{\beta(\lambda)_{-N} \beta(\lambda)_{-N+1}} \) in (4.116) is that the Lorentz algebra representations of the dynamical fields and dynamical equations are isomorphic while the sum of their conformal dimensions is \(2q\) which is the canonical dimension of a Lagrangian density. We therefore expect that all these dynamical systems are Lagrangian.

4.7. Conformal Equations. Now it is straightforward to write down conformal equations \( \mathfrak{R}_M(\phi^0(x)) = 0 \) corresponding to any conformal module \( \mathfrak{M} \). First, one represents \( \mathfrak{M} \) as an extension of irreducible conformal modules. Then (as explained in section 4.6) the results of Theorem 4.2 (for odd \(M\)) and Theorem 4.3 (for even \(M\)) are used to calculate \( H^0(t(M), \mathfrak{M}) \) and \( H^1(t(M), \mathfrak{M}) \). Finally, along the lines of the proof of Theorem 3.1 one expresses auxiliary fields contained in \( |\Phi^0(x)| \) (see Remark 3.4) in terms of derivatives of the dynamical field \( |\phi^0(x)| \) and reconstructs the nontrivial equations \( \mathfrak{R}_M(\phi^0(x)) = 0 \) on the latter. These equations are associated with \( H^1(t(M), \mathfrak{M}) \). In practice, it is most useful to use Remark 4.4 which identifies the left hand sides of the field equations with the singular and subsingular modules of \( \mathfrak{S}_{(\lambda)} \). In those cases where \( \mathfrak{S}_{(\lambda)} \) does not contain modules of the Levi factor \( \mathfrak{l}_{M} \) equivalent to (but different from) the singular and subsingular modules, the explicit form of
conformal equations corresponding to the irreducible conformal module \( \mathcal{J}_{(\lambda)} \) can be obtained by replacing \( \mathcal{K}_n \) by \( \frac{\partial}{\partial \sigma^n} \) in the expressions for a basis of the singular and subsingular modules.

The examples given in section 2 and in the rest of this section result from the application of this general scheme to the following modules (here \( \mathcal{I} \) denotes an irreducible module and \( \mathcal{E} \) denotes an extension).

1. \( \mathcal{J}_{(2-M)/2,0,\ldots,0} \) corresponds to Klein–Gordon equation (2.16) (primitive)
2. \( \mathcal{J}_{(1-M)/2,1/2,\ldots,1/2} \) corresponds to Dirac equation (2.23) (primitive)
3. \( \mathcal{J}_{(-p,1,\ldots,0,\ldots,0)} \) for odd \( M \) or for even \( M \) and \( p = 0 \) corresponds to closedness equation (2.29) on a \( p \)-form or equivalent conservation equation (2.32) on a \((M-p)\)-polyvector (primitive);
   for even \( M \) and \( p > 0 \), \( \mathcal{J}_{(-p,1,\ldots,0,\ldots,0)} \) corresponds to the system (2.29), (2.34) on a \( p \)-form or the equivalent system (2.32), (2.35) on a \((M-p)\)-polyvector (primitive)
4. \( \mathcal{J}_{(p-M,1,\ldots,1,0,\ldots,0)} \) \( p > 0 \) corresponds to conservation equation (2.32) on a \( p \)-polyvector or the equivalent closedness equation (2.29) on a \((M-p)\)-form (primitive)
5. \( \mathcal{J}_{(M/2,1,\ldots,1,\pm 1)} \) for even \( M \) corresponds to (anti)selfduality equation (2.37), (2.38) (primitive)
6. \( \mathcal{E}_{A,F} \) corresponds to the potential form of Maxwell equations (2.52), (2.53) in conformal gauge (2.45) (non-primitive)
7. \( \mathcal{E}_{A,F,J} \) corresponds to the off-mass-shell version of Maxwell electrodynamics (2.52), (2.54), (2.60), (2.58) in conformal gauge (2.45) (non-primitive)
8. \( \mathcal{E}_{A,F,J,G} \) corresponds to the off-mass-shell gauge invariant version of Maxwell electrodynamics (2.52), (2.54), (2.60), (2.58), (2.64) (non-primitive)
9. \( \mathcal{J}_{((2-M)/2,\lambda,\ldots,\lambda,0,\ldots,0)} \) for odd \( M \) or for even \( M \) with either \( \nu \leq q - 2 \) or \( \nu = q \), \( \lambda = 1 \) corresponds to Klein–Gordon–like equation (4.122) on a tensor field described by the \( \lambda \times \nu \)-rectangular Young tableau (primitive)
10. \( \mathcal{J}_{((1-M)/2,\lambda+1/2,\ldots,\lambda+1/2,1/2,\ldots,\pm 1/2)} \) for odd \( M \) or for even \( M \) with \( \nu \leq q - 1 \) corresponds to Dirac–like equation (1.129) on a spinor–tensor field described by the \( \lambda \times \nu \)-rectangular Young tableau (primitive)
11. \( \mathcal{J}_{((1-M)/2,\lambda+1/2,\ldots,\lambda+1/2,1/2,\ldots,\pm 1/2)} \) for odd \( M \) or for even \( M \) with \( \nu \leq q - 1 \) corresponds to
12. \( \mathcal{K}_{(\lambda)} = \mathcal{J}_{(\lambda)+} \oplus \mathcal{J}_{(\lambda)-} \) for even \( M \) corresponds to the field strength form of conformal higher spin equations (4.35), (4.36) (non-primitive)
13. $\mathcal{E}_{3(\lambda), J}(\phi)$ for even $M$ corresponds to the gauge fixed potential form of conformal higher spin equations (4.135), (4.136), (4.144), (4.150) (non-primitive)

14. $\mathcal{E}_{3(\lambda), J}(\phi, \lambda)$ for even $M$ corresponds to the gauge fixed off-mass-shell version of conformal higher spin equations (4.135), (4.144), (4.145), (4.150), (4.154) (non-primitive)

15. $\mathcal{E}_{3(\lambda), J}(\phi, \lambda)$ for even $M$ corresponds to the gauge invariant off-mass-shell version of conformal higher spin equations (4.135), (4.144), (4.150), (4.151), (4.154), (4.155) (non-primitive)

16. $\mathcal{J}_C$ for even $M$ corresponds to the condition that the generalized Weyl tensor for spin $\lambda \geq 1/2$ symmetric tensor field equals to zero (4.155) supplemented with the gauge fixing condition (4.157) (primitive)

17. $\mathcal{E}_{3(\lambda), J}(\phi)$ for even $M$ corresponds to gauge fixed spin $\lambda \geq 1/2$ Fradkin–Tseytlin conformal higher spin equation (4.157), (4.159), (4.162) (non-primitive)

18. $\mathcal{E}_{3(\lambda), J}(\phi)$ for even $M$ corresponds to gauge invariant spin $\lambda \geq 1/2$ Fradkin–Tseytlin conformal higher spin equation (4.157), (4.159), (4.162) (4.164) (non-primitive)

Note that flat limits of the most non-flat conformal equations considered in [20], [32]–[39] belong to the case 10. The system of conformal equations considered in [20] corresponds to the case 8.

4.7.1. Conformal Klein–Gordon and Dirac-like equations for a block. Let $(\lambda) = (-(M - 2)/2, \ldots, \lambda, 0, \ldots, 0), \lambda \in \mathbb{N}$, and $\mathfrak{J}(\lambda)$ be the irreducible conformal module with the highest weight $(\lambda)$. It is represented by the short exact sequence (4.147) for odd $M$ and by (4.167) for even $M$. Let us consider the bundle $\mathcal{B}(\lambda) = \mathbb{R}^M \times \mathfrak{J}(\lambda)$ and its subbundle $\mathcal{B}(\lambda) \supset \mathfrak{b}(\lambda) = \mathbb{R}^M \times \mathfrak{M}(\lambda)$. Consider a section

$$|\phi(x)\rangle = C_{n^1(\lambda), n^2(\lambda), \ldots, n^\nu(\lambda)}(\lambda)(\lambda)^{n^1(\lambda), n^2(\lambda), \ldots, n^\nu(\lambda)}$$

(4.119)

of $\mathfrak{b}(\lambda)$ and a section $|\Phi(x)\rangle$ of $\mathcal{B}(\lambda)$ such that, $|\Phi(x)\rangle|_{\mathfrak{b}(\lambda)} = |\phi(x)\rangle$,

$$|\Phi(x)\rangle = \sum_{l=0}^{\lfloor M/2 \rfloor} \frac{1}{l!} C_{n^1(\lambda), n^2(\lambda), \ldots, n^\nu(\lambda); m(l)}(\lambda) \phi^{m_1 \ldots m_l}(\lambda)^{n^1(\lambda), n^2(\lambda), \ldots, n^\nu(\lambda)}.$$ (4.120)

Here $|\langle \lambda \rangle^{n^1(\lambda), n^2(\lambda), \ldots, n^\nu(\lambda)}$ form a basis of $\mathfrak{M}(\lambda)$. The symmetry properties of $|\langle \lambda \rangle^{n^1(\lambda), n^2(\lambda), \ldots, n^\nu(\lambda)}$ imply that symmetrization over any $\lambda + 1$ indices gives zero. The corresponding Young tableau is a rectangle of length $\lambda$ and height $\nu$ and is referred to as a block. Note that fields that appear in most of physical applications belong to this class.

As shown in section 3 the covariant constancy equation (3.27) encodes the differential equations on the dynamical variables that take values in $H^0(t(M), \mathfrak{J}(\lambda))$. The form of these
differential equations is determined by $H^1(t(M), \mathfrak{J}(\lambda))$. These cohomology groups are determined in (4.101) and (4.106). Using the symmetry properties of the block Young tableau it can be easily seen that $H^1(t(M), \mathfrak{J}(\lambda))$ corresponds to the singular module $\mathfrak{S}(\lambda)$ of $\mathfrak{Y}(\lambda)$ described by the block tableau with the conformal weight $M/2 + 1$, i.e. it has the weights $\nu = (-(M + 2)/2, \lambda, \ldots, \lambda, 0, \ldots, 0)$. It is easy to see that $|s\rangle \in \mathfrak{S}(\lambda)$ has the form

$$|s\rangle = \psi^{\nu_1(\lambda), \ldots, \nu_s(\lambda)} \left( y^m y_n \delta_{\nu_m}^{\nu_n} - \frac{4\lambda\nu}{2\lambda - 2\nu + M} y_n y^k \right) |\lambda\rangle_{\nu_1(\lambda), \ldots, \nu_s(\lambda - 1)\lambda},$$

where $\psi^{\nu_1(\lambda), \ldots, \nu_s(\lambda)}$ is an arbitrary parameter taking values in the $\lambda \times \nu$ traceless block tableau. In fact, $\psi^{\nu_1(\lambda), \ldots, \nu_s(\lambda)}$ can be thought of as an arbitrary element of the dual space of $H^1(t(M), \mathfrak{J}(\lambda))$. The conformal equation associated with $H^1(t(M), \mathfrak{J}(\lambda))$ is

$$\psi^{\nu_1(\lambda), \ldots, \nu_s(\lambda)} \left( \square C_{\nu_1(\lambda), \ldots, \nu_s(\lambda)}(x) - \frac{4\lambda\nu}{2\lambda - 2\nu + M} \partial_{\nu_m} \partial^m C_{\nu_1(\lambda), \ldots, \nu_s(\lambda - 1)\nu}(x) \right) = 0. \quad (4.122)$$

This is the Klein–Gordon type conformal equation for a field with the block symmetry properties and conformal weight $M/2 - 1$.

Note that for even $M$, (4.66) requires either $\nu \leq q - 2$ or $\nu = q, \lambda = 1$. This is in accordance with our analysis because, although being conformally invariant, the equations (4.122) with $\nu = q - 1, M = 2q$ are non-primitive (see section 4.7.2). Also one can see for even $M$ and $\nu = q, \lambda \geq 2$ that the singular vector (4.121) is zero\(^{10}\) and equation (4.122) becomes the identity $0 = 0$.

For the particular cases of $\nu = 1, \lambda = 0, 1, 2$ equation (4.122) reads

$$\square C(x) = 0, \quad (4.123)$$

$$\psi^n \left( \square C_n(x) - \frac{4}{M} \partial_n \partial^m C_n(x) \right) = 0, \quad (4.124)$$

$$\psi^{n_1 n_2} \left( \square C_{n_1 n_2}(x) - \frac{8}{2 + M} \partial_{n_1} \partial^m C_{n_2 m}(x) \right) = 0. \quad (4.125)$$

Equation (4.123) is the usual Klein–Gordon equation. Equation (4.124) for $M = 4$ corresponds to Maxwell electrodynamics formulated in terms of potential. Equation (4.124) for $M \neq 4$ and equation (4.125) correspond to non-unitary field-theoretical models.

The Dirac–like equations are associated with the bundles $\mathfrak{b}(\lambda)$ and $\mathfrak{B}(\lambda)$ with

$$\lambda = (- (M - 1)/2, \lambda + \frac{1}{2}, \ldots, \lambda + \frac{1}{2}, \frac{1}{2}, \ldots, \pm \frac{1}{2}), \quad \lambda \in \mathbb{N}$$

\(^{10}\)One way to see this is to observe that for the case of $\nu = q$ the tensor contracted with $\psi^{\nu_1(\lambda), \ldots, \nu_s(\lambda)}$ on the left hand side of (4.122) has opposite (anti)selfduality properties for the first and last columns of the corresponding rectangular Young tableau, that is only possible when it is zero.
and their sections
\[ |\phi(x)\rangle = C_{n^1,\ldots,n^\nu}(\lambda) \gamma_\mu^n \gamma_\nu^{\alpha\beta} \delta_{\nu}^{\lambda \nu} \frac{2\lambda \nu}{2\lambda - 2\nu + M} \gamma^{\nu\lambda\beta} |\lambda\rangle, \quad \text{for } \nu = 1, \lambda = 0,1 \]

(4.126)

and
\[ |\Phi(x)\rangle = \sum_{l=0}^{\infty} \frac{1}{l!} C_{\lambda, n^2(\lambda),\ldots,n^\nu(\lambda),\alpha, m(l)}(x) \gamma_\mu^n \gamma_\nu^{\alpha\beta} \delta_{\nu}^{\lambda \nu} \frac{2\lambda \nu}{2\lambda - 2\nu + M} \gamma^{\nu\lambda\beta} |\lambda\rangle, \quad \text{for } \nu = 1, \lambda = 0,1 \]

(4.127)

where \( |\Phi(x)\rangle \big|_b = |\phi(x)\rangle \). Here \( \alpha = 1, \ldots, 2[\frac{M}{2}] \) is a spinorial index. \( C_{n^1,\ldots,n^\nu,\alpha}(x) \) is a \( \gamma \)-transversal block spinor–tensor with definite chirality. The cohomology groups \( H^0(t(M), \mathfrak{F}_{(\lambda)}) \) and \( H^1(t(M), \mathfrak{F}_{(\lambda)}) \) are given in (4.104) and (4.106), respectively. \( H^1(t(M), \mathfrak{F}_{(\lambda)}) \) corresponds to the singular module \( \mathcal{H}_{(\lambda)} \) in \( \mathfrak{V}_{(\lambda)} \) with the general element
\[ |s\rangle = \psi_{n^1,\ldots,n^\nu,\alpha}(x) \left( \gamma_\mu^n \gamma_\nu^{\alpha\beta} \delta_{\nu}^{\lambda \nu} \frac{2\lambda \nu}{2\lambda - 2\nu + M} \gamma^{\nu\lambda\beta} |\lambda\rangle \right), \quad |s\rangle = 1, \lambda = 0,1 \]

(4.128)

Here \( \psi_{n^1,\ldots,n^\nu,\alpha}(x) \) is an arbitrary \( \gamma \)-transversal chiral spinor–tensor parameter taking values in the \( \lambda \times \nu \)-block tableau. The conformal equation encoded by the covariant constancy equation (3.27) is
\[ \psi_{n^1,\ldots,n^\nu,\alpha}(x) \left( \partial_m^\alpha \gamma_\mu^n \gamma_\nu^{\alpha\beta} C_{m^1,\ldots,n^\nu,\beta}(x) - \frac{2\lambda \nu}{2\lambda - 2\nu + M} \gamma^{\nu\lambda\beta} \partial_m C_{n^1,\ldots,n^\nu,\beta}(x) \right) = 0. \]

(4.129)

This is the conformally invariant generalization of the Dirac equation to a block spinor–tensor with conformal weight \( (M - 1)/2 \). For the particular cases of \( \nu = 1, \lambda = 0,1 \) we get
\[ \partial_m^\alpha \gamma_\mu^n \gamma_\nu^{\alpha\beta} C_{n^1,\ldots,n^\nu,\beta}(x) = 0, \]

(4.130)
\[ \psi_{n^1,\alpha}(x) \left( \partial_m^\alpha \gamma_\mu^n \gamma_\nu^{\alpha\beta} C_{n^1,\ldots,n^\nu,\beta}(x) - \frac{2\lambda \nu}{M} \gamma^{\nu\lambda\beta} \partial_m C_{n^1,\ldots,n^\nu,\beta}(x) \right) = 0. \]

(4.131)

Equation (4.130) is the usual Dirac equation. Note that conditions (4.66) require \( \nu \leq q - 1 \) for even \( M \). Analogously to the case of Klein–Gordon type equations one can prove that singular vector (4.128) is zero for even \( M, \nu = q \), and corresponding equation (4.129) becomes identity \( 0 = 0 \).

Analogous conformally invariant generalizations of the Klein–Gordon and Dirac equations exist for tensor fields of other symmetry types. They correspond to other irreducible modules \( \mathfrak{F}_{(\lambda)} \) from the series (4.47) for odd \( M \) and (4.66) for even \( M \). All these systems however are not expected to correspond to unitary field-theoretical models in accordance with the general fact (40) (41) (42) that conformal field equations compatible with unitarity are exhausted by the massless equations for a scalar, a spinor and blocks of the height \( [(M - 1)/2] \).
4.7.2. Conformal higher spins in even dimensions. Here we describe a generalization of the equations for $M = 4$ massless higher spin fields to a broad class of conformal field equations for tensor fields in $M = 2q$ dimensions (the following construction can be easily formulated also for spinor–tensor fields). Let $(\lambda)_{\pm} = (\mp q, \lambda_1, \lambda_2, \ldots, \lambda_{q-1}, \pm 1)$, where $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{q-1} \geq 1$. Let $q = \mu_1 > \mu_2 \geq \mu_3 \geq \cdots \geq \mu_p$ be the heights of the columns in the Young tableau corresponding to $\mathfrak{g}_{(\lambda)_{\pm}}$. (Note that the first column is required to have the maximal height $q$, while the second one is required to be smaller.) Let us denote $\mathfrak{g}_{(\lambda)_{+}} = \mathfrak{g}_{(\lambda)} + \mathfrak{g}_{(\lambda)-}$. Consider the bundle $\mathcal{B}_F = \mathbb{R}^M \times \mathfrak{g}_{(\lambda)_{+}}$ and its subbundle $\mathfrak{b}_F = \mathbb{R}^M \times (\mathfrak{g}_{(\lambda)} + \mathfrak{g}_{(\lambda)-})$. Irreducible modules $\mathfrak{g}_{(\lambda)_{+}}$ and $\mathfrak{g}_{(\lambda)-}$ are defined by the short exact sequences (4.60) and (4.61), respectively. Choose a section of $\mathfrak{b}_F$

$$|\phi_F(x)\rangle = F_{n_1[2],n_2[2],\ldots,n_p[\mu_p]}(x)|\lambda_F\rangle^\dagger n_1[2],n_2[2],\ldots,n_p[\mu_p],$$

(4.132)

where $|\lambda_F\rangle^\dagger$ is a basis in $\mathfrak{g}_{(\lambda)_{+}} + \mathfrak{g}_{(\lambda)-}$, i.e. it contains both selfdual and antiselfdual parts. We treat $|\phi_F(x)\rangle$ as a higher spin field strength. Let $|\Phi_F(x)\rangle$ be a section of $\mathcal{B}_F$ such that $|\Phi_F(x)\rangle|_{\mathfrak{b}_F} = |\phi_F(x)\rangle$

$$|\Phi_F(x)\rangle = \sum_{l=0}^{1} \frac{1}{l!} F_{n_1[2],n_2[2],\ldots,n_p[\mu_p];m(\ell)}(x) \prod_{\ell=1}^{m} (4.133)$$

As follows from (4.104) and (4.108), the condition

$$\mathcal{D}|\Phi_F(x)\rangle = 0$$

(4.134)

implies the equations

$$\psi^{n_1[2]-1,n_2[2],\ldots,n_p[\mu_p]} \partial^m (\ast F)_{mn_1[2]-1,n_2[2],\ldots,n_p[\mu_p]}(x) = 0,$$

(4.135)

$$\psi^{n_1[2]-1,n_2[2],\ldots,n_p[\mu_p]} \partial^m F_{mn_1[2]-1,n_2[2],\ldots,n_p[\mu_p]}(x) = 0,$$

(4.136)

where an arbitrary element $\psi^{n_1[2]-1,n_2[2],\ldots,n_p[\mu_p]}$ of the irreducible $\mathfrak{o}(M)$-module associated with the Young tableau with columns of heights $q - 1, \mu_2, \ldots, \mu_p$ is introduced to avoid complicated projection operators. For the particular case of the block with $\mu_2 = \mu_3 = \cdots = q - 1$ these are equations of motion (formulated in terms of field strengths) for the conformal fields that respect unitarity [40 41 42]. For $q = 2$ one recovers the usual equations of motion for massless fields in four dimensions formulated in terms of field strengths. For $q = 3$ the conformal massless higher spins of this type were discussed in [43].

The system (4.135), (4.136) admits extensions analogous to that of the system (2.52), (2.53). In particular, one can introduce potentials to the field strength $F_{n_1[2],n_2[2],\ldots,n_p[\mu_p]}(x)$ in both gauge invariant and conformal gauge fixed forms. To this end we consider the nontrivial extension $\mathfrak{e}_{\mathfrak{g}_{(\lambda)_{+}}}$ of the module $\mathfrak{g}_{(\lambda)_{+}}$ by the module $\mathfrak{g}_{(\lambda)_{+}}$ where $(\lambda)_{A = \ldots}$
(−q + 1, λ₁, ..., λ_{q−1}, 0).  \( \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \) is defined by the short exact sequence

\[
0 \rightarrow \mathcal{J}_{\lambda(A)} \rightarrow \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \rightarrow \mathfrak{R}_{\lambda(F)} \rightarrow 0 .
\]  

(4.137)

The module \( \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \) can be described as follows. Let \( (\lambda)_{A} \) be the basis in \( \mathfrak{R}_{\lambda(A)} \). Impose the following relations

\[
\mathcal{J}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \mathfrak{R}_{\lambda(F)} = 0 ,
\]

(4.138)

\[
\mathcal{J}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \mathfrak{R}_{\lambda(F)} = 0 ,
\]

(4.139)

\[
\mathfrak{R}_{\lambda(F)} = 0 ,
\]

(4.140)

which single out the modules \( \mathfrak{R}_{\lambda(F)} \) and \( \mathcal{J}_{\lambda(A)} \), respectively. The nontrivial extension is defined by the condition

\[
\mathcal{J}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \mathfrak{R}_{\lambda(F)} = 0 ,
\]

(4.141)

The module \( \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \) is generated by \( y_{n} \) from \( (\lambda)_{F} \) and \( (\lambda)_{A} \).

Consider the bundle \( \mathcal{B}_{A,F} = \mathbb{R}^{M} \times \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \). \( \mathcal{B}_{F} \) and \( \mathcal{B}_{A} = \mathbb{R}^{M} \times \mathfrak{R}_{\lambda(A)} \) are its subbundles. Consider a section \( \Phi_{A,F}(x) \) of \( \mathcal{B}_{A,F} \)

\[
|\Phi_{A,F}(x)| = \sum_{l=0}^{1} \frac{1}{l!} F_{n[1],n[2],...,n[p][\mu]}(m)_{l}(x) y^{m_{1}} \cdot \cdot \cdot y^{m_{l}} (\lambda)_{F}^{n[1],n[2],...,n[p][\mu]}
\]

+ \[
\sum_{l=0}^{1} \frac{1}{l!} A_{n[1],n[2],...,n[p][\mu]}(m)_{l}(x) y^{m_{1}} \cdot \cdot \cdot y^{m_{l}} (\lambda)_{A}^{n[1],n[2],...,n[p][\mu]} .
\]

(4.142)

Cohomology \( H^{0}(t(M), \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)}) \), \( H^{1}(t(M), \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)}) \) is given in (4.148) for \( N = 1 \).

Condition \( D|\Phi_{A,F}(x)| = 0 \) implies

\[
\mathcal{J}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \mathfrak{R}_{\lambda(F)} = 0 ,
\]

(4.143)

\[
\mathcal{J}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \mathfrak{R}_{\lambda(F)} = 0 ,
\]

(4.144)

\[
\mathfrak{R}_{\lambda(F)} = 0 ,
\]

(4.145)

together with (4.135) and (4.136). This extension introduces gauge potentials \( A_{n[1],n[2],...,n[p][\mu]}(x) \) to the field strength, along with the conformally invariant gauge condition (4.145).

Now we introduce the module \( \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \mathfrak{J}_{\lambda(A)} \) that extends \( \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \) by the module \( \mathfrak{J}_{\lambda(A)} \), where \( (\lambda)_{J} = (−q − 1, λ₁, ..., λ_{q−1}, 0) \). \( \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \mathfrak{J}_{\lambda(A)} \) is described by the short exact sequence

\[
0 \rightarrow \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \rightarrow \mathfrak{E}_{\lambda(A)} \mathfrak{R}_{\lambda(F)} \mathfrak{J}_{\lambda(A)} \rightarrow \mathfrak{J}_{\lambda(A)} \rightarrow 0 .
\]

(4.146)
The module $\mathfrak{E}_{\mathfrak{E}_A,\mathfrak{E}_F,\mathfrak{E}_\lambda}$ is generated by $Y^\mu$ from $| (\lambda) F \rangle ^{n_1[q],n_2[m_2],...,n_p[\mu_p]}$, $| (\lambda) A \rangle ^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]}$ and $| (\lambda) J \rangle ^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]}$ satisfying conditions \ref{4.138}--\ref{4.142} along with

\[
\psi^{n_1[q-2],n_2[m_2-1],...,n^\lambda_{q-1} \mu_{\lambda_{q-1} -1},n^\lambda_{q-1} \mu_{\lambda_{q-1} +1} ,...,n_p[\mu_p]} y^{n_1} \cdots y^{n_1}\ | (\lambda) J \rangle ^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]} = 0
\]

and

\[
\psi^{mn_1[q-1],n_2[m_2],...,n_p[\mu_p]} F^m | (\lambda) A \rangle ^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]} = -\frac{q}{3} \psi^{mn_1[q-1],n_2[m_2],...,n_p[\mu_p]} | (\lambda) F \rangle ^{mn_1[q-1],n_2[m_2],...,n_p[\mu_p]}
\]

Consider a section $| \Phi_{A,F,J}(x) \rangle$ of the bundle $\mathbb{R}^M \times \mathfrak{E}_{\mathfrak{E}_A,\mathfrak{E}_F,\mathfrak{E}_\lambda}$

\[
| \Phi_{A,F,J}(x) \rangle = \sum_{l=0}^{1} \frac{1}{l!} F^{n_1[q],n_2[m_2],...,n_p[\mu_p];m(t)(x) y^{(m \cdots y^m)} | (\lambda) F \rangle ^{n_1[q],n_2[m_2],...,n_p[\mu_p])
\]

\[
+ \sum_{l=0}^{1} \frac{1}{l!} A^{n_1[q-1],n_2[m_2],...,n_p[\mu_p];m(t)(x) y^{(m \cdots y^m)} | (\lambda) A \rangle ^{n_1[q-1],n_2[m_2],...,n_p[\mu_p])
\]

\[
+ \sum_{l=0}^{1} \frac{1}{l!} J^{n_1[q-1],n_2[m_2],...,n_p[\mu_p];m(t)(x) y^{(m \cdots y^m)} | (\lambda) J \rangle ^{n_1[q-1],n_2[m_2],...,n_p[\mu_p])
\]

Calculating the cohomology $H^p(\mathfrak{t}(M), \mathfrak{E}_{\mathfrak{E}_A,\mathfrak{E}_F,\mathfrak{E}_\lambda})$ from \ref{4.146} one obtains that the condition $D| \Phi_{A,F,J}(x) \rangle = 0$ implies equations \ref{4.135}, \ref{4.144}, \ref{4.135} along with equations

\[
\psi^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]} \left( \partial^{n_1} F^{n_1[q],n_2[m_2],...,n_p[\mu_p]}(x) - J^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]}(x) \right) = 0,
\]

\[
\psi^{n_1[q-2],n_2[m_2-1],...,n^\lambda_{q-1} \mu_{\lambda_{q-1} -1},n^\lambda_{q-1} \mu_{\lambda_{q-1} +1} ,...,n_p[\mu_p]} \partial^{n_1} \cdots \partial^{n_1} J^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]}(x) = 0.
\]

For $\lambda_{q-1} = 1$ (equivalently $\mu_2 \leq q - 2$) the system \ref{4.135}, \ref{4.144}, \ref{4.135}, \ref{4.131} generalizes the ordinary $M = 4$ electrodynamics to any even space–time dimension and arbitrary tensor structure of fields. Here \ref{4.144} defines the generalized field strength $F^{mn_1[q-1],n_2[m_2],...,n_p[\mu_p]}(x)$ via the generalized potential $A^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]}(x)$. Equation \ref{4.135} is the Bianchi identity for generalized field strength. Equation \ref{4.135} describes “interaction” with the “current” (see section \ref{2.2}) $J^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]}(x)$, which conserves due to equation \ref{4.151}. The system \ref{4.135}, \ref{4.144}, \ref{4.130}, \ref{4.131} is gauge invariant under the generalized gradient transformations

\[
\psi^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]} \delta A^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]}(x) = \psi^{n_1[q-1],n_2[m_2],...,n_p[\mu_p]} \partial^{n_1} \epsilon^{n_1[q-2],n_2[m_2],...,n_p[\mu_p]}(x)
\]

with an arbitrary parameter $\epsilon^{n_1[q-2],n_2[m_2],...,n_p[\mu_p]}(x)$. Equation \ref{4.135} fixes conformal gauge, generalizing equation \ref{2.245}.
Analogously to the example in section 2.4, one can relax the gauge fixing condition by considering the module \( \mathfrak{e}_{(\lambda)_A,\mathcal{R}(\lambda)_F,\mathcal{R}(\lambda)_J,\mathcal{J}(\lambda)_G} \) defined by the short exact sequence
\[
0 \to \mathfrak{e}_{(\lambda)_A,\mathcal{R}(\lambda)_F,\mathcal{R}(\lambda)_J,\mathcal{J}(\lambda)_G} \to \mathfrak{e}_{(\lambda)_A,\mathcal{R}(\lambda)_F,\mathcal{R}(\lambda)_J,\mathcal{J}(\lambda)_G} \to \mathcal{J}(\lambda)_G \to 0,
\]
where \((\lambda)_G = (-\lambda_q - q - 1, \lambda_1, \ldots, \lambda_{q-2}, 0, 0)\). The covariant constancy condition for the section \( |\Phi_{A,F,J,G}^2| \) implies equations (4.153), (4.144), (4.150), (4.151) along with the equation
\[
\psi^{n[1]+q-2,m[2],n[2]}(\Box \partial^{n[1]} A_{n[1]}[q-1,n[2]],n[2]) = C_{n[1]}[q-2,n[2]],n[2] \partial^{n[1]} \cdots \partial^{n[1]} C_{n[1]}[q-2,n[2]],n[2] = 0.
\]
Instead of (4.155), the field \( G_{n[1]}[q-1,n[2]],n[2] \partial^{n[1]} \cdots \partial^{n[1]} C_{n[1]}[q-2,n[2]],n[2] \partial^{n[1]} \cdots \partial^{n[1]} C_{n[1]}[q-2,n[2]],n[2] = 0 \).

4.7.3. Fradkin–Tseytlin conformal higher spins in even dimensions. Consider highest weight \((\lambda)_C = (\lambda - 2, \lambda, 0, \ldots, 0)\), \( \lambda_i \in \mathbb{N} \) (the case of half-integer \( \lambda_i \) can be considered analogously). Let \( \mathcal{J}(\lambda)_C \) be irreducible conformal module with the highest weight \((\lambda)_C\). Using Theorems 3.1 and 4.3 we obtain primitive conformal system corresponding to the module \( \mathcal{J}(\lambda)_C \). It has the form
\[
\psi^{n(\lambda),m(\lambda)} \partial_{m} \cdots \partial_{m} C_{n(\lambda)}(x) = 0,
\]
\[
\psi^{n(\lambda-1)}(\partial \cdot \partial)^{\lambda+q-1} \partial^{n} C_{n(\lambda)}(x) = 0.
\]
Here \( C_{n(\lambda)}(x) \) is a symmetric traceless tensor field, \( \psi^{n(\lambda),m(\lambda)} \) is an arbitrary traceless tensor parameter corresponding to the \( \lambda \times 2 \)-block Young tableau. \( (\partial \cdot \partial)^{\lambda+q-1} \) is an order \( 2(\lambda+q-1) \) differential operator
\[
(\partial \cdot \partial)^{\lambda+q-1} C_{n(\lambda)}(x) = \sum_{p+r=\lambda+q-1} a(p,r) \partial^{p} \partial_{m} \cdots \partial_{m} C_{n(\lambda-r)m(r)}(x)
\]
for some \( a(p,r) \).

The left hand side of the equation (4.156) can be interpreted as the generalized Weyl tensor for the field \( C_{n(\lambda)}(x) \)
\[
\psi^{n(\lambda),m(\lambda)} \partial_{m} \cdots \partial_{m} C_{n(\lambda)}(x) = \psi^{n(\lambda),m(\lambda)} W_{n(\lambda),m(\lambda)}(x).
\]

It is gauge invariant under the gauge transformations
\[
\psi^{n(\lambda)} \delta C_{n(\lambda)}(x) = \psi^{n(\lambda)} \partial_{n} \epsilon_{n(\lambda-1)}(x),
\]
where \( \epsilon_{n(\lambda-1)}(x) \) is a gauge parameter. Equation (4.160) sets \( W_{n(\lambda),m(\lambda)}(x) \) to zero and is dynamically trivial (i.e. describes pure gauge degrees of freedom). Equation (4.157) is the conformal gauge condition for \( C_{n(\lambda)}(x) \). (Note that, as any covariant gauge condition, it is incomplete.)
A non-trivial dynamical system with nonzero Weyl tensor is non-primitive and results from the reducible module $\mathcal{E}_{\gamma_{\lambda}C,\gamma_{\lambda}W}$ defined by the short exact sequence

$$0 \longrightarrow J_{\lambda}C \longrightarrow \mathcal{E}_{\gamma_{\lambda}C,\gamma_{\lambda}W} \longrightarrow J_{\lambda}W \longrightarrow 0,$$

where $J_{\lambda}W$ is the irreducible conformal module with the highest weight $(\lambda)_W = (-2, \lambda, \lambda, 0, \ldots, 0)$ corresponding to the Weyl tensor $W_{n(\lambda),m(\lambda)}(x)$. Cohomology of $\mathcal{E}_{\gamma_{\lambda}C,\gamma_{\lambda}W}$ is given in (4.118) for $N = q - 1$. The module $\mathcal{E}_{\gamma_{\lambda}C,\gamma_{\lambda}W}$ gives rise to the gauge fixing equation (4.157) along with the definition of the Weyl tensor (4.159) and the equation

$$\psi^{n(\lambda)} \Box^{2q-4} \partial^m \cdots \partial^m W_{n(\lambda),m(\lambda)} = 0.$$

This class of conformal equations was found by Fradkin and Tseytlin in [44] along with the analogous equations for spinor–tensors for $M = 4$ and generalized to arbitrary even $M = 2q$ in [45].

Gauge invariant form of the same system (i.e., without equation (4.157)) results from our construction applied to the module $\mathcal{E}_{\gamma_{\lambda}C,\gamma_{\lambda}W,\gamma_{\lambda}G}$ defined by the short exact sequence

$$0 \longrightarrow \mathcal{E}_{\gamma_{\lambda}C,\gamma_{\lambda}W} \longrightarrow \mathcal{E}_{\gamma_{\lambda}C,\gamma_{\lambda}W,\gamma_{\lambda}G} \longrightarrow J_{\lambda}G \longrightarrow 0.$$

Here $J_{\lambda}G$ is the irreducible conformal module with the highest weight $(\lambda)_G = (-\lambda - 2q + 1, \lambda - 1, 0, \ldots, 0)$. Module $\mathcal{E}_{\gamma_{\lambda}C,\gamma_{\lambda}W,\gamma_{\lambda}G}$ gives rise to the system containing equations (4.159), (4.162) and the equation

$$\psi^{n(\lambda-1)} (\partial \cdot \partial)^{\lambda+q-1} \partial^m C_{n(\lambda)}(x) = \psi^{n(\lambda-1)} G_{n(\lambda-1)},$$

which relaxes the gauge fixing equation (4.157).

5. Conclusions

In this paper we study a general framework, which allows us to classify and obtain the explicit form of all linear homogeneous $\mathfrak{m}_p$–invariant $M$-dimensional equations for an arbitrary semi-simple Lie algebra $\mathfrak{f}$ which has a parabolic subalgebra $p_\Pi$ with an $M$-dimensional Abelian radical $\mathfrak{r}_\Pi$. These equations are written in the form of the covariant constancy conditions

$$D|\Phi^p(x) = (d + \omega_0(x))|\Phi^p(x) = 0.$$ (5.1)

Here the connection 1-form $\omega_0(x)$ takes values in $\mathfrak{f}$ and is flat, i.e. $(d + \omega_0(x))^2 = 0$. A particularly useful choice of the connection is $\omega_0(x) = \sigma_-$, where $\sigma_-$ takes values in radical $\mathfrak{r}_\Pi$ and is $x$-independent, i.e., $d\sigma_- + \sigma_- d = 0$, $\sigma_-^2 = 0$. The $p$-forms $|\Phi^p(x)\rangle$ take values in an $\mathfrak{f}$-module $\mathfrak{M}$ that is required to be $p_\Pi$-integrable. We prove that (5.1) leads to a linear homogeneous $\mathfrak{f}$-invariant equation

$$R_{\mathfrak{M}}|\Phi^p(x)\rangle = 0$$ (5.2)
on the set of dynamical fields $|\phi^p(x)\rangle$ that are elements of the $p$-th cohomology of $\sigma_-$ (see Remark 3.4). All other fields from the set $|\Phi^p(x)\rangle$ are either pure gauge or auxiliary fields expressed in terms of derivatives of the dynamical fields. The form of equations (5.2) is determined by the $p + 1$-th cohomology of $\sigma_-$. $\mathfrak{f}_{\Pi}$-invariant equations (5.2) are classified by the modules $\mathcal{M}$. This classification is complete because any equation can be unfolded to the form (5.1) by introducing auxiliary fields. A constructive procedure is described, which allows one to obtain the explicit form of the $\mathfrak{f}_{\Pi}$-invariant equation associated with $\mathcal{M}$. In this paper, the proposed general construction is applied to obtain the complete classification of conformally invariant differential equations in terms of singular and subsingular modules of generalized Verma modules of the conformal algebra in $M$ dimensions.

The approach proposed in this paper can be further applied to several problems. The most straightforward application is to study free (i.e., linear) equations invariant under symmetries different from the usual conformal symmetry. A particularly interesting example is that of the symplectic algebra $\mathfrak{sp}(m)$ which was shown [5] to be a proper extension of the usual conformal algebra, acting on the infinite systems of fields of higher spins. More examples of $\mathfrak{sp}(m)$-invariant equations were obtained recently in [46]. It is also tempting to apply our approach to the study of $M = 2$ conformal systems starting with the related infinite-dimensional symmetries.

Another interesting generalization to be studied consists in relaxing the requirement that the radical $\mathfrak{r}_{\Pi}$ is Abelian. In this case one can still formulate invariant equations in the form (5.1). The resulting equations are not translationally invariant because $\omega_0(x)$ is necessarily $x$-dependent. Also it is not clear how to implement the analysis of the dynamical content of the invariant equations in terms of cohomology. Let us note that this case is not of a purely “academic” interest. An important class of equations of this type is provided by superfield equations for supersymmetric systems, which are known to contain an explicit dependence on anticommuting variables through the supercovariant derivatives. It is well known that it is sometimes difficult to distinguish between constraints and “true” field equations in superspace. As mentioned in [5], the origin of this difficulty can be traced back to the absence of a distinct $\sigma_-$ cohomology description.

One of the most important problems is to go beyond the class of linear equations. A suggestive feature of our approach mentioned in section 2.4 is that it allows a natural definition of current modules. As a result, the interaction problem admits a reformulation in terms of the realization of current modules as tensor products (i.e., nonlinear combinations) of modules associated with matter fields.

By analogy with higher spin theory, to put interacting theory in the framework of gravity with the gravitational field being one of the dynamical fields (i.e., not just a background one
as in this paper) it is important to extend the formalism to (extensions of) field equations formulated in terms of differential \( p \)-forms with \( p > 0 \). Among other things, this requires clarifying the relationship between the dynamical equations formulated in terms of 0-forms as in this paper and those formulated in terms of higher differential forms (in particular, 1-forms) as in higher spin gauge theory \([6, 3]\). In this respect Theorems 4.2, 4.3 in this paper and their generalizations to other Lie algebras to be worked out are likely to play the key role because they link together cohomology groups which determine dynamical fields and field equations in terms of various differential forms.

Finally, it would be very instructive to make contact with other cohomological approaches such as developed e.g. in \([21, 47, 48]\).

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Appendix A. Relevant facts from representation theory

The structure of generalized Verma modules can be investigated using methods developed in \([17, 18, 30, 31, 49, 50]\). Let us first recall some notations. Let \( \mathfrak{h} \) be the Cartan subalgebra and \( \mathfrak{h}^* \) is its dual space. Let simple roots be denoted \( \alpha_0, \alpha_1, \ldots, \alpha_q \) and \( \Pi \) consists of \( \alpha_1, \ldots, \alpha_q \) (see section \( \text{[8]} \)). The Weyl group \( W^{q+1} \) is generated by reflections \( r_{\alpha_i} \equiv \mathbf{r}_i \) \((0 \leq i \leq q)\) of \( \mathfrak{h}^* \) over the hyperplane orthogonal to the simple root \( \alpha_i \)

\[
\mathbf{r}_i \lambda = \lambda - 2 \frac{\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i, \tag{A.1}
\]

\( \lambda \in \mathfrak{h}^* \). The action \( r_\alpha \cdot \lambda \) (nonlinear representation) of \( W^{q+1} \) in \( \mathfrak{h}^* \) is defined by the formula

\[
r_\alpha \cdot \lambda = \lambda - 2 \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \tag{A.2}
\]

for any \( \alpha, \lambda \in \mathfrak{h}^* \). Here \( \rho \) is half of the sum of positive roots\(^{11}\). Let \( W^q \) be the subgroup of the Weyl group generated by simple reflections \( \mathbf{r}_i \) with \( 1 \leq i \leq q \). Denote by \( Q \) the

\(^{11}\)Note that this formula is universal: given linear representation of a group \( G \) in a linear space \( V \) and a fixed vector \( \rho \in V \), the transformations \( A \cdot \lambda = A\lambda + (A - I)\rho \) for \( A \in G \) and \( \lambda \in V \) define the (nonlinear) action of \( G \) in \( V \).
root lattice \( \{ \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_q \} \). For any highest weight \( \lambda \), let \( W_\lambda^{q+1} \) be the subgroup constituted by such elements \( w \in W^{q+1} \) that
\[
w \cdot \lambda \in \lambda + \mathbb{Q}.
\] (A.3)

Let \( S_\lambda \subset W_\lambda^{q+1} \) be the stability subgroup of \( \lambda \)
\[
s \cdot \lambda = \lambda, \quad s \in S_\lambda.
\] (A.4)

Consider the quotient
\[
T_\lambda = (W^q \cap W_\lambda^{q+1})\backslash W_\lambda^{q+1} / (W^q \cap S_\lambda).
\] (A.5)

Denote by \( \mathbb{L} \) the set of highest weights of the form \( \lambda = (\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_q) \) where \( (\lambda_1, \lambda_2, \ldots, \lambda_q) \) is a dominant integral highest weight of \( B_q \) (i.e. \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q \) and \( 2\lambda_i \) are all even or odd simultaneously). For any equivalence class from \( T_\lambda \) one can choose a representative \( t \) such that \( t \cdot \lambda \in \mathbb{L} \) whenever \( \lambda \in \mathbb{L} \). Let \( T_\lambda \subset T_\lambda \) denote the set of all such representatives. For any weight \( \nu \in \mathbb{L} \), the set of elements \( T_\lambda \) generates the set of highest weights \( \{ t \cdot \nu \}_{t \in T_\lambda} \).

Elements \( t \in T_\lambda \) are ordered with respect to their \textit{length}(\( t \)), where the \textit{length}(\( t \)) is the number of the multipliers in the reduced (i.e. minimal) decomposition of \( t \) into a product of the elementary reflections generated by the simple roots. The reduced decomposition is unique. We write \( t_1 \prec t_2 \) whenever \textit{length}(\( t_1 \)) < \textit{length}(\( t_2 \)). Note that such defined order is partial because any two elements with the same length can not be compared. The main point is that the generalized Verma module \( \mathcal{V}_{t_2, \nu} \) admits a nontrivial homomorphism into the generalized Verma module \( \mathcal{V}_{t_1, \nu} \) whenever \( t_1 \prec t_2 \) \([50]\). Applying this general method to the conformal algebra one obtains the structure of singular modules in \( \mathcal{V}_\lambda \) in the cases \( B_{q+1} \) and \( D_{q+1} \), which was completely studied in \([30, 31]\) (see also \([28]\) for a textbook). This exhausts the case of \( B_{q+1} \). In the case of \( D_{q+1} \) subsingular modules exist and their structure should be investigated separately. Let us sketch the final results separately for the cases \( B_{q+1} \) (i.e. \( M = 2q + 1 \)) and \( D_{q+1} \) (i.e. \( M = 2q \)).

Let \( M = 2q + 1 \). The Dynkin diagram of the algebra \( B_{q+1} \) is \([12]\). Choose an orthogonal basis \( \epsilon_i \) \( 0 \leq i \leq q \) in \( \mathfrak{h}^* \). Then
\[
\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad 0 \leq i \leq q - 1, \quad \alpha_q = \epsilon_q.
\] (A.6)

Introduce the basis in \( \mathfrak{h} \) dual to \( \epsilon_i \) (i.e. \( \epsilon_i(\epsilon^j) = \delta^j_i \))
\[
\epsilon^0 = -\mathcal{D}, \quad \epsilon^1 = \mathcal{L}_{12}, \quad \epsilon^i = \sqrt{-1}\mathcal{L}_{2i-1,2i}, \quad 1 \leq i \leq q - 1, \quad \epsilon^q = \sqrt{-1}\mathcal{L}_{2q,2q+1}.
\] (A.7)

Then
\[
\mathcal{H}_i = \epsilon^i - \epsilon^{i+1}, \quad 0 \leq i \leq q - 1, \quad \mathcal{H}_q = 2\epsilon^q.
\] (A.8)
Half the sum of all positive roots is in this case

$$\rho = \sum_{i=0}^{q} (q - i + \frac{1}{2}) \varepsilon_i.$$  \hspace{1cm} \text{(A.9)}

Recall that \( r_i \) denote the simple reflections \( r_i = r_{\alpha_i} = r_{\epsilon_i-\epsilon_{i+1}} \) for \( 0 \leq i \leq q - 1 \) and \( r_q = r_{\alpha_q} = r_{\epsilon_q} \).

In the case of dominant integral \( \lambda \) the stability subgroup is trivial and the set \( \mathcal{T}_\lambda \) consists of the following elements \cite{28}

\[
e \prec r_0 \prec r_1 r_{\epsilon_0 - \epsilon_2} \prec r_1 r_2 r_{\epsilon_0 - \epsilon_3} \prec \cdots \prec r_1 r_2 \cdots r_q r_{\epsilon_0 - \epsilon_q} \prec r_1 r_2 \cdots r_q r_{\epsilon_0 + \epsilon_q} \prec \]

\[
\prec r_1 r_2 \cdots r_q r_{\epsilon_{q-1} + \epsilon_{q-1}} \prec \cdots \prec r_1 r_2 r_{\epsilon_0 + \epsilon_2} \prec r_1 r_2 r_{\epsilon_0 + \epsilon_1} \prec r_{\epsilon_0}. \] \hspace{1cm} \text{(A.10)}

Note that these elements are written in the non-reduced form, which, however, is more convenient for calculations. This gives rise to the diagram \cite{B11} (see the end of the paper) of homomorphisms of modules \( \mathcal{W}_\lambda \), where \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_q \geq 0 \) and \( 2\lambda_i \) are either all even or all odd \( 0 \leq i \leq q \). Composition of any two homomorphisms (arrows) in the diagram is zero.

For non-integral \( \lambda \), homomorphisms are associated with \( \mathcal{T}_\lambda = \{ e \prec r_{\epsilon_0} \} \). In this case the parameters of the highest weight should satisfy

\[
\begin{align*}
\lambda_0 &= -q - \frac{1}{2} + n, \quad n \in \mathbb{N}, \quad \lambda_i \in \mathbb{N}, \quad 1 \leq i \leq q, \quad \text{(A.11)} \\
\lambda_0 &= -q + n, \quad n \in \mathbb{N}_0, \quad \lambda_i \in \frac{1}{2} + \mathbb{N}_0, \quad 1 \leq i \leq q. \quad \text{(A.12)}
\end{align*}
\]

This leads to the following diagram of homomorphisms

\[
\mathcal{W}_{(\lambda_0, \lambda_1, \ldots, \lambda_q)} \xleftarrow{(r_{\epsilon_0}, l)} \mathcal{W}_{(-\lambda_0 - 2q - 1, \lambda_1, \ldots, \lambda_q)}. \hspace{1cm} \text{(A.13)}
\]

For the case \( \text{(A.11)} \) \( l = 2n \). For the case \( \text{(A.12)} \) \( l = 2n + 1 \).

Let \( M = 2q \). The Dynkin diagram of the algebra \( D_{q+1} \) is \cite{L}. Choose an orthogonal basis \( \varepsilon_i \) in \( h^* \). Then

\[
\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad 0 \leq i \leq q - 1, \quad \alpha_q = \varepsilon_{q-1} + \varepsilon_q. \hspace{1cm} \text{(A.14)}
\]

The half of the sum of all positive roots is

\[
\rho = \sum_{i=0}^{q-1} (q - i) \varepsilon_i. \hspace{1cm} \text{(A.15)}
\]
The analysis analogous to that of the odd dimensional case gives that $\mathcal{T}_\lambda$ with a dominant integral $\lambda$ consists of the following elements $\lambda$-generated from all singular submodules of $V$

\[ e < r_0 < r_1 r_{\varepsilon_0 - \varepsilon_1} < r_1 r_2 r_{\varepsilon_0 - \varepsilon_2} < \cdots < r_1 r_2 \cdots r_{q-2} r_{\varepsilon_0 - \varepsilon_{q-1}} < \]
\[ r_1 r_2 \cdots r_{q-1} r_{\varepsilon_0 - \varepsilon_q} < r_1 r_2 \cdots r_{q-1} r_{\varepsilon_0 + \epsilon_q} < \]
\[ r_1 r_2 \cdots r_{q-3} r_{\varepsilon_2 - \varepsilon_q} r_{\varepsilon_0 + \epsilon_q} < r_1 r_2 \cdots r_{q-4} r_{\varepsilon_3 - \varepsilon_q} r_{\varepsilon_0 + \epsilon_q} < \cdots < \]
\[ r_1 r_{\varepsilon_2 - \varepsilon_q} r_{\varepsilon_0 + \epsilon_q} < r_{\varepsilon_1 - \varepsilon_q} r_{\varepsilon_0 + \epsilon_q} < r_{\varepsilon_0 - \varepsilon_q} r_{\varepsilon_0 + \epsilon_q} \quad (A.16) \]

The diagram of $\mathcal{W}_\lambda$-homomorphisms is $\text{(B.2)}$ (see the end of the paper), where $\lambda_0 \geq \lambda_1 \geq \cdots \geq |\lambda_q|$ and $2\lambda_i$ are either all even or all odd $0 \leq i \leq q$. Here, the composition of any two homomorphisms, except for those in the central rhombus and those that are labeled by NS, is zero. There exist also $q-1$ nonstandard homomorphisms $\text{[30]}$ (they are labeled by the symbol NS in the diagram $\text{(B.2)}$) between modules in this diagram that correspond to the element $r_{\varepsilon_0 - \varepsilon_q} r_{\varepsilon_0 + \epsilon_q}$ from $\mathcal{T}_\lambda$

\[ \mathcal{W}(\lambda_{N-1}, \lambda_0 + 1, \lambda_1 + 1, \ldots, \lambda_N - 1 + 1, \lambda_{N+1}, \ldots, \lambda_q) \leftrightarrow \mathcal{W}(-\lambda_{N-2}, \lambda_0 + 1, \lambda_1 + 1, \ldots, \lambda_N - 1 + 1, \lambda_{N+1}, \ldots, -\lambda_q) \quad (A.17) \]

for $12 \leq N < q - 1$.

There are also nonstandard homomorphisms in the case when $\lambda$ is singular i.e. $\lambda + \rho$ lies on a wall of the Weyl chamber. Then $\mathcal{T}_\lambda = \{ e < r_{\varepsilon_0 - \varepsilon_q} r_{\varepsilon_0 + \epsilon_q} \}$ and the parameters of the highest weight satisfy the following relations

\[ \lambda_0 - \lambda_N + N = 0 \quad \text{for some } N = 1, 2, \ldots, q , \quad (A.18) \]
\[ \lambda_0 + \lambda_q + q = n \in \mathbb{N}_0 , \quad (A.19) \]
\[ \lambda_0 - \lambda_q + q = m \in \mathbb{N}_0 \quad \text{and } m + n \neq 0 . \quad (A.20) \]

Here $\text{(A.18)}$ is the condition that the highest weight is singular and $\text{(A.19)}$, $\text{(A.20)}$ are conditions that $r_{\varepsilon_0 - \varepsilon_q} r_{\varepsilon_0 + \epsilon_q} \lambda$ belongs to the weight lattice. These homomorphisms are

\[ \mathcal{W}(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_q) \leftrightarrow \mathcal{W}(-\lambda_0 - 2q, \lambda_1, \lambda_2, \ldots, -\lambda_q) . \quad (A.21) \]

The quotient of an arbitrary generalized Verma module $\mathcal{W}_\lambda$ over the submodule $\mathcal{V}(\lambda)$ generated from all singular submodules of $\mathcal{W}_\lambda$ is not necessarily irreducible. In fact, the module $\mathcal{W}_\lambda$ can have subsingular submodules (those that are singular in $\mathcal{W}_\lambda/\mathcal{V}(\lambda)$), subsubsingular submodules etc. In the conformal algebra case subsingular submodules do not appear.

To describe the structure of $\mathcal{W}_\lambda$ for the highest weight $\lambda$ belonging to series $\text{(4.49)}$ we start with the case of $\lambda_{-q} = (0, 0, \ldots, 0)$. All other cases can be obtained from this one by

\[ \text{For } N = q - 1, \text{ this homomorphism amounts to the composition of the homomorphisms that constitute the rhombus.} \]
application of the shift functor \[17\] to modules belonging to the case \((\lambda)_{-q} = (0, 0, \ldots, 0)\).
So let us consider the case
\[
(\lambda)_{-q} = (0, 0, \ldots, 0),
\]
\[
(\lambda)_{-q+1} = (-1, 1, 0, \ldots, 0),
\]
\[
\vdots
\]
\[
(\lambda)_{-q+N} = (-N, 1, 1, \ldots, 0, \ldots, 0), \quad N = 0, \ldots, q - 1,
\]
\[
\vdots
\]
\[
(\lambda)_{-1} = (-q + 1, 1, \ldots, 1, 0),
\]
\[
(\lambda)_0 = (-q, 1, \ldots, 1), \quad (\lambda)_0' = (-q, 1, \ldots, 1, -1),
\]
\[
(\lambda)_1 = (-q - 1, 1, \ldots, 1, 0),
\]
\[
\vdots
\]
\[
(\lambda)_K = (-q - K, 1, \ldots, 1, 0, \ldots, 0), \quad K = 1, \ldots, q - 1,
\]
\[
\vdots
\]
\[
(\lambda)_{q-2} = (-2q + 2, 1, 1, 0, \ldots, 0),
\]
\[
(\lambda)_{q-1} = (-2q + 1, 1, 0, \ldots, 0),
\]
\[
(\lambda)_q = (-2q, 0, \ldots, 0).
\]

The structure of generalized Verma modules with these highest weights can be elaborated by the direct calculation. Solving explicitly the system of equations
\[
P_n F_A(y^m)|((\lambda)_N)^A = 0 \quad (A.23)
\]
for the polynomials \(F_A(y^m)\), where \(P_n\) are differential operators \([127]\) we obtain that the module \(\mathfrak{V}_{(\lambda)_{-q}}\) contains singular vectors
\[
|s^1_{(\lambda)_{-q}}\rangle^m = y^m|((\lambda)_{-q})\rangle,
\]
\[
|s^2_{(\lambda)_{-q}}\rangle = (y^2)^q|((\lambda)_{-q})\rangle. \quad (A.24)
\]
The modules \(\mathfrak{V}_{(\lambda)_N}\) for \(N = -q + 1, \ldots, -1\) contain singular vectors
\[
|s^1_{(\lambda)_N}\rangle^{m[N+q+1]} = y^m|((\lambda)_N)^{m[N+q]}\rangle,
\]
\[
|s^2_{(\lambda)_N}\rangle^{m[N+q]} = (y^2)^{-N}|((\lambda)_N)^{m[N+q]}\rangle - (N + q)(y^2)^{-N-1}y_n y^m|((\lambda)_N)^{nm[N+q-1]}\rangle. \quad (A.26)
\]
and subsingular vectors
\[
|subs_{(\lambda)_N}\rangle^{m[N+q-1]} = (y^2)^{-N}y_n|((\lambda)_N)^{nm[n+q-1]}\rangle. \quad (A.27)
\]
The modules $\mathcal{U}_{(\lambda)N}$ for $N = 0, 0', 1, \ldots, q - 1$ contain singular vectors

$$|s_{(\lambda)N}\rangle^{m[q-N-1]} = y_m|\langle(\lambda)N\rangle^{m[q-N]}.$$  \hspace{1cm} (A.29)

The completeness of this list of singular and subsingular modules follows from the theory of intersection cohomology sheaves \cite{18}. 

Unfolded form of linear conformal equations in $M$-dimensions and ... 59
Appendix B. Homomorphism diagrams

\[ \mathcal{V}(\lambda)(0) = \mathcal{V}(\lambda_0, \lambda_1, \lambda_2, ..., \lambda_q) \]
\[ \uparrow (r_{0}, -\lambda_{0} - \lambda_{1} + 1) \]
\[ \mathcal{V}(\lambda)(1) = \mathcal{V}(\lambda_1 - 1, \lambda_0 + 1, \lambda_2, ..., \lambda_q) \]
\[ \uparrow (r_{0} - \lambda_{1}, -\lambda_{2} + 1) \]
\[ \vdots \]
\[ \uparrow (r_{0} - \lambda_{N}, \lambda_{N-1} - \lambda_{N} + 1) \]
\[ \mathcal{V}(\lambda)(N) = \mathcal{V}(\lambda_{N-N}, \lambda_{0} + 1, \lambda_{1} + 1, ..., \lambda_{N-1} + 1, \lambda_{N+1}, ..., \lambda_q) \]
\[ \uparrow (r_{0} - \lambda_{N+1}, \lambda_{N} - \lambda_{N+1} + 1) \]
\[ \mathcal{V}(\lambda)(N+1) = \mathcal{V}(\lambda_{N+1-N-1}, \lambda_{0} + 1, \lambda_{1} + 1, ..., \lambda_{N+1}, \lambda_{N+2}, ..., \lambda_q) \]
\[ \uparrow (r_{0} - \lambda_{N+2}, \lambda_{N+1} - \lambda_{N+2} + 1) \]
\[ \vdots \]
\[ \uparrow (r_{0} - \lambda_{q}, \lambda_{q-1} - \lambda_{q} + 1) \]
\[ \mathcal{V}(\lambda)(q) = \mathcal{V}(\lambda_{q-q}, \lambda_{0} + 1, \lambda_{1} + 1, ..., \lambda_{q-1} + 1) \]
\[ \uparrow (r_{0}, 2\lambda_{q} + 1) \]
\[ \mathcal{V}(\lambda)(q+1) = \mathcal{V}(\lambda_{q+1-q}, \lambda_{0} + 1, \lambda_{1} + 1, ..., \lambda_{q-1} + 1) \]
\[ \uparrow (r_{0} + \lambda_{q}, \lambda_{q-1} - \lambda_{q} + 1) \]
\[ \vdots \]
\[ \uparrow (r_{0} + \lambda_{q+N}, \lambda_{N} - \lambda_{N+1} + 1) \]
\[ \mathcal{V}(\lambda)(2q+1-N) = \mathcal{V}(\lambda_{N-2q-1-N}, \lambda_{0} + 1, \lambda_{1} + 1, ..., \lambda_{N-1} + 1, \lambda_{N+1}, ..., \lambda_q) \]
\[ \uparrow (r_{0} + \lambda_{N}, \lambda_{N-1} - \lambda_{N} + 1) \]
\[ \mathcal{V}(\lambda)(2q+2-N) = \mathcal{V}(\lambda_{N-2q-2-N}, \lambda_{0} + 1, \lambda_{1} + 1, ..., \lambda_{N-2} + 1, \lambda_{N}, ..., \lambda_q) \]
\[ \uparrow (r_{0} + \lambda_{N-1}, \lambda_{N-2} - \lambda_{N-1} + 1) \]
\[ \vdots \]
\[ \uparrow (r_{0} + \lambda_{2q+1}, \lambda_{0} - \lambda_{1} + 1) \]
\[ \mathcal{V}(\lambda)(2q+1) = \mathcal{V}(\lambda_{2q+1-q}, \lambda_{0} - \lambda_{q-1} - \lambda_{q}, ..., \lambda_{q}) \]

The label \((r, l)\) at a homomorphism arrow has the following meaning. \(r\) denotes the reflection that connects highest weights of the two modules. \(l\) is the level at which a singular module resulting from the arrow homomorphism is situated.
The label \((r, l)\) at a homomorphism arrow has the following meaning. \(r\) denotes the reflection that connects highest weights of the two modules. \(l\) is the level at which a singular module resulting from the arrow homomorphism is situated.
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